ON GLOBAL CONSERVATION LAWS AT NULL INFINITY

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ABSTRACT

The “standard” expressions for total energy, linear momentum and also angular momentum of asymptotically flat Bondi metrics at null infinity are also obtained from differential conservation laws on asymptotically flat backgrounds, derived from a quadratic Lagrangian density by methods currently used in classical field theory. It is thus a matter of taste and commodity to use or not to use a reference spacetime in defining these globally conserved quantities. Backgrounds lead to Nöther conserved currents; the use of backgrounds is in line with classical views on conservation laws. Moreover, the conserved quantities are in principle explicitly related to the sources of gravity through Einstein’s equations, while standard definitions are not. The relations depend, however, on a rule for mapping spacetimes on backgrounds.

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1. Introduction

In *Science and Hypothesis*, Poincaré (1904) imagines astronomers “whose vision would be bounded by the solar system” because of thick clouds that hide the fixed stars. There is thus no fixed frame of coordinates, only relative distances and relative angles are measurable. In those circumstances, says Poincaré, “we should be definitively led to conclude that the equations which define distances are of an order higher than the second. [...] The values of the distances at any given moment depend upon their initial values, on that of their first derivatives, and something else. What is that *something else?* If we do not want it to be merely one of the second derivatives, we have only the choice of hypotheses. Suppose, as is usually done that this something else is the absolute orientation of the universe in space, or the rapidity with which this orientation varies; this may be, it certainly is, the most convenient solution for the geometer. But it is not the most satisfactory for the philosopher, because this orientation does not exist.”

The conservation of energy, linear and angular-momentum, so useful in classical mechanics and special relativity, are related to the homogeneity and isotropy properties attributed to an absolute “background” spacetime which does indeed not exist. In general relativity, the use of backgrounds is intrinsic to the definition of pseudo-tensor conservation laws. Rosen (1958) and Cornish (1964) used backgrounds explicitly to calculate the energy-momentum in more appropriate coordinates than orthogonal ones. While backgrounds have thus proven to be useful, they nevertheless got little support and attention from the community of general relativists. On the contrary, great efforts have been spend to get rid of backgrounds, to avoid those “additional structure completely counter spirit of general relativity” (Wald 1984). True, efforts to avoid introducing background geometries generated interesting works notably by Penrose (1965), Geroch (1976) and other
mathematical physicists mentioned in Schmidt’s (1993) review on *Asymptotic flatness — a Critical Appraisal*. As a consequence, we now have a rather well understood, coordinate independant, picture of asymptotic flatness, and we have also asymptotic coordinate independant expressions for globally conserved quantities in particular at null infinity.

Conserved quantities at null infinity are given by integrals on spheres of infinite radius at fixed null time $u = t - r = \text{const.}$ They represent total energy, linear momentum, angular-momentum and the initial position of the mass center (Synge 1964) of spacetimes with isolated sources of curvature at a moment of “time” $u$. The expressions have become “standard” according to Dray and Streubel (1984) [see also Dray (1985) and Shaw (1986)], and they are invariant under coordinate transformations of the Bondi-Metzner-Sachs, or BMS, group [Sachs (1962), Newman and Penrose (1966)]. Most standard expressions draw their strength — beyond their esthetical appeal — from a few physically interpretable quantities, which have all been obtained before from the pseudo-tensors of Freud (1939) and of Landau and Lifshitz (1951). The quantities are

(a) The Schwarzschild mass $M_{\text{Schw}}$ and more generally the dominant $1/r$ term in a multipole expansion of static solutions at spatial infinity (Geroch 1970) and at null infinity (Schmidt 1993), which was obtained from Einstein’s pseudo-tensor [see Tolman 1934].

(b) The Bondi mass $M(u)$ where $u = t - r$ [see Bondi, Metzner and van der Burg (1962) see already Bondi (1960)] and Sachs’s (1962a) linear momentum $\vec{P}(u)$ at null infinity* which have been calculated by Møller (1972) using Freud’s superpotential;

(c) The Kerr angular-momentum $\vec{L}_{\text{Kerr}}$ and more generally the dominant $1/r^2$ “odd function” factor [Misner Thorne and Wheeler (1973)] in a multipole expansion of stationary spacetimes at spatial or null infinity, which is obtainable from Landau and Lifshitz

* What is actually defined in Sachs is $d\vec{P}/du$. 

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superpotential for angular momentum [Papapetrou 1965].

The angular momentum at null infinity $\vec{L}(u)$, as given for instance by Dray and Streubel (1984), has not been derived from previously known superpotential. $\vec{L}(u)$ is not a well defined physical quantity because of supertranslation freedom. However, the weak field approximation fits in well with results deduced from Landau-Lifshitz’s superpotential (Creswell and Zimmerman 1986). The LL approximation is currently used in gravitational radiation calculations (Thorne 1980).

Here we show that $M(u)$, $\vec{P}(u)$ as well as $\vec{L}(u)$ — and thus also $M_{\text{Schw}}$ and $L_{\text{Kerr}}$ — can be calculated using a single superpotential derived from a quadratic Lagrangian with an asymptotically flat background at null infinity. The derivation insures automatically Poincaré invariance, the absence of an anomalous factor of 2 in the $M/L$ ratio and zero value for all conserved quantities when the spacetime identifies with the background itself. As a result, $dM/du$ and $d\vec{P}/du$ are the standard BMS invariant fluxes while $d\vec{L}/du$, which is not BMS invariant, is nevertheless the same as the standard result. The position of the mass center, $\vec{Z}(u)$, associated with Lorentz rotations, and the corresponding flux $d\vec{Z}/du$ given here are not the same as the standard formula, at least in the non-linear approximation.

Thus, the most important standard results, energy and angular momentum, can be deduced from Noether conservation laws. One appealing aspect of Noether conservation laws is that they are closer to classical intuition than the more abstract coordinate independent definitions at scri. Moreover, Noether conserved quantities are directly related to the energy-momentum tensor of the matter through Einstein’s equations by differential conservation laws. “Standard” definitions are not.

Local differential conservation laws contain implicit definitions of local or quasi-local
conserved quantities which depend, on the local mapping of the spacetime on the background. We give here no rule for local mappings. There are also several mapping-independent definitions of quasi-local energy in the literature, but they appear to be different from each other (Berqvist 1992).

2. Superpotential and Noether Conservation Laws

Let us briefly review the elements that lead to our superpotential (Katz 1996). Full details are given in a work by Katz, Bičák and Lynden-Bell (1996) — refered to below as KBL96 — which has its roots in earlier work by Katz (1985) on flat backgrounds. KBL96 is on curved backgrounds and the superpotential obtained there is new even in the limit where the backgrounds become flat.

(i) Lagrangian density for gravitational fields on a curved background

Let $g_{\mu\nu}(x^\lambda)$, $\lambda, \mu, \nu, ... = 0, 1, 2, 3$ be the metric of a spacetime $\mathcal{M}$ with signature -2, and let $\bar{g}_{\mu\nu}(\bar{x}^\lambda)$ be the metric of the background $\bar{\mathcal{M}}$. Both are tensors with respect to arbitrary coordinate transformations. Once we have chosen a mapping so that points $P$ of $\mathcal{M}$ map into points $\bar{P}$ of $\bar{\mathcal{M}}$, then we can use the convention that $\bar{P}$ and $P$ shall always be given the same coordinates $\bar{x}^\lambda = x^\lambda$. This convention implies that a coordinate transformation on $\mathcal{M}$ inevitably induces a coordinate transformation with the same functions on $\bar{\mathcal{M}}$. With this convention, such expressions as $g_{\mu\nu}(x^\lambda) - \bar{g}_{\mu\nu}(x^\lambda)$, which can be looked at as “perturbations” of the background, become true tensors. However, if the particular mapping has been left unspecified we are still free to change it. The form of the equations for “perturbations” of the background must inevitably contain a gauge invariance corresponding to this freedom.

Let $R^\lambda_{\nu\rho\sigma}$ and $\bar{R}^\lambda_{\nu\rho\sigma}$ be the curvature tensors of $\mathcal{M}$ and $\bar{\mathcal{M}}$. These are related as follows [see Rosen (1940, 1963), see also Choquet-Bruhat (1984) for mathematical aspects}
$R^\lambda_{\nu\rho\sigma} = \bar{D}_\rho \Delta^\lambda_{\nu\sigma} - \bar{D}_\sigma \Delta^\lambda_{\nu\rho} + \Delta^\lambda_{\rho\eta} \Delta^\eta_{\nu\sigma} - \Delta^\lambda_{\sigma\eta} \Delta^\eta_{\nu\rho} + \bar{R}^\lambda_{\nu\rho\sigma}. \quad [2 - 1]$

Here $\bar{D}_\rho$ are covariant derivatives with respect to $\bar{g}_{\mu\nu}$, and $\Delta^\lambda_{\mu\nu}$ is the difference between Christoffel symbols in $\mathcal{M}$ and $\bar{\mathcal{M}}$:

$$\Delta^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \bar{\Gamma}^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} (\bar{D}_\rho g_{\mu\nu} + \bar{D}_\nu g_{\rho\mu} - \bar{D}_\rho g_{\mu\nu}). \quad [2 - 2]$$

Our quadratic Lagrangian density $\hat{\mathcal{L}}_G$ for the gravitational field is then defined as

$$\hat{\mathcal{L}}_G = \hat{\mathcal{L}} - \bar{\mathcal{L}}, \quad \hat{\mathcal{L}} = -\frac{1}{2\kappa} (\hat{R} + \partial_\mu \hat{k}^\mu), \quad \bar{\mathcal{L}} = -\frac{1}{2\kappa} \bar{R}, \quad \kappa = \frac{8\pi G}{c^4}. \quad [2 - 3]$$

The mark $\hat{}$ means multiplication by $\sqrt{-\bar{g}}$, never by $\sqrt{-g}$, and a bar over a symbol signifies that $g_{\mu\nu}, D_\rho$ etc. are replaced by $\bar{g}_{\mu\nu}, \bar{D}_\rho$ etc.. The vector density $\hat{k}^\mu$ is given by

$$\hat{k}^\mu = \frac{1}{\sqrt{-\bar{g}}} \bar{D}_\nu (\bar{g} g^{\mu\nu}) = \hat{g}^{\mu\rho} \Delta^\sigma_{\rho\sigma} - \hat{g}^{\rho\sigma} \Delta^\mu_{\rho\sigma}, \quad [2 - 4]$$

and its divergence cancels all second order derivatives of $g_{\mu\nu}$ in $\hat{R}$. $\hat{\mathcal{L}}$ is the Lagrangian used by Rosen. $\bar{\mathcal{L}}$ is $\hat{\mathcal{L}}$ in which $g_{\mu\nu}$ has been replaced by $\bar{g}_{\mu\nu}$. When $g_{\mu\nu} = \bar{g}_{\mu\nu}$, $\hat{\mathcal{L}}_G$ is thus identically zero. The intention here is to obtain conservation laws in the background spacetime so that if $g_{\mu\nu} = \bar{g}_{\mu\nu}$, conserved vectors and superpotentials will be identically zero as in Minkowski space in special relativity.

The following formula, deduced from [2-3] and [2-1], shows explicitly how $\hat{\mathcal{L}}_G$ is quadratic in the first order derivatives of $g_{\mu\nu}$ or, equivalently, quadratic in $\Delta^\mu_{\rho\sigma}$:

$$\hat{\mathcal{L}}_G = \frac{1}{2\kappa} \frac{\hat{g}^{\mu\nu}}{(-\hat{g})} (\Delta^\rho_{\mu\nu} \Delta^\sigma_{\rho\sigma} - \Delta^\rho_{\mu\sigma} \Delta^\sigma_{\rho\nu}) - \frac{1}{2\kappa} (\hat{g}^{\mu\nu} - \bar{g}^{\mu\nu}) \bar{R}_{\mu\nu}. \quad [2 - 5]$$

Notice that if $\bar{R}^\lambda_{\nu\rho\sigma} = 0$ and coordinates are such that $\bar{\Gamma}^\lambda_{\mu\nu} = 0$, $\hat{\mathcal{L}}_G$ is nothing else than the familiar “$\bar{\Gamma} \Gamma - \Gamma \Gamma$” Lagrangian density [see for instance Landau and Lifshitz (1951)].
(ii) Strong conservation laws and superpotential

If

$$\Delta x^\mu = \xi^\mu \Delta \lambda$$  \[2 - 6\]

represents an infinitesimal one parameter displacement generated by $\xi^\mu$, the corresponding changes in tensors are given in terms of the Lie derivatives with respect to the vector field $\xi^\mu$, $\Delta g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu} \Delta \lambda$, etc.. The Lie derivatives may be written in terms of ordinary partial derivatives $\partial_\mu$, covariant derivatives $\bar{D}_\mu$ with respect to $\bar{g}_{\mu\nu}$, or covariant derivatives $D_\mu$ with respect to $g_{\mu\nu}$. Thus,

$$\mathcal{L}_\xi g_{\mu\nu} = g_{\mu\lambda} \partial_\nu \xi^\lambda + g_{\nu\lambda} \partial_\mu \xi^\lambda + \xi^\lambda \partial_\lambda g_{\mu\nu}$$  \[2 - 7a\]

$$= g_{\mu\lambda} D_\nu \xi^\lambda + g_{\nu\lambda} D_\mu \xi^\lambda + \xi^\lambda D_\lambda g_{\mu\nu}$$  \[2 - 7b\]

$$= g_{\mu\lambda} D_\nu \xi^\lambda + g_{\nu\lambda} D_\mu \xi^\lambda.$$  \[2 - 7c\]

Consider now the Lie differential $\Delta \hat{\mathcal{L}}$ of $\hat{\mathcal{L}}$. With the variational principle in mind, we write $\Delta \hat{\mathcal{L}} = \mathcal{L}_\xi \hat{\mathcal{L}} \Delta \lambda$ in the form

$$\Delta \hat{\mathcal{L}} = \frac{1}{2\kappa} \hat{G}^{\mu\nu} \Delta g_{\mu\nu} + \partial_\mu \hat{A}^\mu \Delta \lambda$$  \[2 - 8\]

where Einstein’s tensor density, $\hat{G}^{\mu\nu} = \hat{R}^{\mu\nu} - \frac{1}{2} \hat{g}^{\mu\nu} R$, is the variational derivative of $\hat{\mathcal{L}}$ with respect to $g_{\mu\nu}$, $\hat{A}^\mu$ is a vector density linear in $\xi^\mu$ whose detailed form will not concern us here. The Lie derivative of a scalar density like $\hat{\mathcal{L}}$ is just an ordinary divergence $\partial_\mu (\hat{\mathcal{L}} \xi^\mu)$, Thus

$$\hat{O} \equiv \mathcal{L}_\xi \hat{\mathcal{L}} - \partial_\mu (\hat{\mathcal{L}} \xi^\mu) = 0.$$  \[2 - 9\]

Combining [2-9] with [2-8], we obtain

$$\hat{O} \equiv \frac{1}{2\kappa} \hat{G}^{\mu\nu} \mathcal{L}_\xi g_{\mu\nu} + \partial_\mu (\hat{A}^\mu - \hat{\mathcal{L}} \xi^\mu).$$  \[2 - 10\]
Bianchi identities imply $D_\nu G^{\mu\nu} = 0$ so that with [2-7c], [2-10] can be written as the divergence of a vector density $\hat{j}^\mu$, say,

$$\hat{O} = \partial_\mu \hat{j}^\mu = 0$$

where

$$\hat{j}^\mu = \frac{1}{\kappa} \hat{G}_\nu^\mu \xi^\nu + \hat{B}^\mu.$$  \[2 - 11\]

Hence, $\hat{\mathcal{L}}$ “generates” a vector density $\hat{j}^\mu$ that is identically conserved. It has been obtained without using Einstein’s field equations; [2-11] is the kind of strong conservation law introduced by Bergmann (1949). We shall, of course, assume that Einstein’s equations are satisfied, and replace $\frac{1}{\kappa} G_\nu^\mu$ by the energy-momentum of matter

$$\frac{1}{\kappa} G_\nu^\mu = T_\nu^\mu$$ \[2 - 12\]

so that our strong conservation law [2-11] reads:

$$\partial_\mu \hat{j}^\mu = \partial_\mu (\hat{T}_\nu^\mu \xi^\nu + \hat{B}^\mu) = 0.$$ \[2 - 13\]

Equations [2-13] are, strictly speaking, not identities anymore. Given $T_\mu^\nu$, [2-13] holds only for metrics that satisfy [2-12]. The vector density $\hat{j}^\mu$ is linear in $\xi^\mu$ and its derivatives up to order 2.

Since $\hat{j}^\mu$ as given by [2-11] is identically conserved whatever is $g_{\mu\nu}$, it must be the divergence of an antisymmetric tensor density that depends on the arbitrary $g_{\mu\nu}$’s as well; thus

$$\hat{j}^\mu = \partial_\nu \hat{j}^{\mu\nu}, \quad \text{where} \quad \hat{j}^{\mu\nu} = -\hat{j}^{\nu\mu}.$$ \[2 - 14\]

Indeed, $\hat{j}^{\mu\nu}$ is easy to find and is derived directly from $\hat{\mathcal{L}}$ in Katz* (1985) [see also Chruściel (1986), Sorkin (1988) and Katz and Ori (1990)]:

$$j^{\mu\nu} = \frac{1}{\kappa} D^{[\mu} \xi^{\nu]} + \frac{1}{\kappa} \xi^{[\mu} k^{\nu]}.$$ \[2 - 15\]

* In the 1985 paper the background is assumed to be flat, but the derivation of $\hat{j}^{\mu\nu}$ does not depend on that assumption.
The terms $\frac{1}{\kappa} D^{[\mu} \xi^{\nu]}$ will be recognised as $\frac{1}{2}$ Komar's (1959) superpotential. In terms of $\bar{D}_{\rho}$ derivatives,

$$D_{\rho} \xi^\mu = \bar{D}_{\rho} \xi^\mu + \Delta^\mu_{\rho\lambda} \xi^\lambda,$$  \[2 - 16\]

and, using expression [2-4] for $k^\mu$, $j^{\mu\nu}$ may be written in the form

$$\kappa j^{\mu\nu} = g^{[\mu\rho} \bar{D}_{\rho} \xi^{\nu]} + g^{[\mu\rho} \Delta^\nu_{\rho\lambda} \xi^\lambda + \xi^{[\mu} g^{\nu] \rho \Delta_{\rho\sigma} - \xi^{[\mu} \Delta_{\rho\sigma} g^{\rho\sigma},$$ \[2 - 17\]

Had we applied the identities [2-9] to $\hat{\mathcal{L}}$ instead of $\mathcal{L}$, we would have written everywhere $\bar{g}_{\mu\nu}$ instead of $g_{\mu\nu}$. We would have found strong, barred, conserved vector densities $\bar{j}^\mu$ and barred superpotentials $\bar{j}^{\mu\nu}$ with the same $\xi^\mu$'s:

$$\bar{j}^\mu = T^\mu_{\nu} \xi^\nu + B^\mu = \partial_{\nu} \bar{j}^{\mu\nu},$$ \[2 - 18\]

with

$$\bar{j}^{\mu\nu} = \frac{1}{\kappa} D^{[\mu} \bar{\xi}^{\nu]} \quad (k^\mu \equiv 0).$$ \[2 - 19\]

Strongly conserved vectors for $\hat{\mathcal{L}}_G = \hat{\mathcal{L}} - \bar{\mathcal{L}}$ are obtained by subtracting barred vectors and superpotentials from unbarred ones; in this way we define relative vectors and in particular relative superpotentials $\hat{j}^{\mu\nu} —$ relative to the background space. Setting

$$\hat{j}^\mu = j^\mu - \bar{j}^\mu, \quad \hat{j}^{\mu\nu} = j^{\mu\nu} - \bar{j}^{\mu\nu} = - \bar{j}^{\nu\mu},$$ \[2 - 20\]

we have for the strongly conserved vector $\hat{j}^\mu$ the following form:

$$\hat{j}^\mu = \hat{\theta}^\mu_{\nu} \xi^\nu + \hat{\sigma}^{\mu[\rho\sigma]} \partial_{[\rho} \xi_{\sigma]}.$$

which hold for any $\xi^\mu$ and any mapping of $\mathcal{M}$ on $\bar{\mathcal{M}}$; in [2-21]
\( \hat{\theta}_\nu^\mu, \hat{\sigma}^{\mu\rho\sigma} \) and \( \hat{\zeta}^\mu \) are given explicitly in appendix for the interested reader. The relative superpotential density \( \hat{J}^{\mu\nu} \) is now given by

\[
\hat{J}^{\mu\nu} = \frac{1}{\kappa} (D[\mu \hat{\xi}^\nu] - D[\mu \bar{\xi}^\nu] + \check{\xi}[\mu k^\nu]),
\]

and can be also written in terms of \( g_{\mu\nu}, \Delta^\mu_{\rho\sigma} \) and \( \xi^\mu \):

\[
\kappa \hat{J}^{\mu\nu} = \check{\mu}^{\rho\sigma} D^\rho_{\nu} \hat{\xi}^\sigma + \check{\xi}[\mu g_{\nu\rho}\Delta^\sigma_{\rho\sigma} - \check{\xi}[\mu \Delta^\nu_{\rho\sigma} g_{\rho\sigma}],
\]

in which

\[
\check{\mu}^{\rho\sigma} = g^{\mu\rho} - \bar{g}^{\mu\rho}.
\]

The tensors in [2-22] have a physical interpretation. On a flat background, in coordinates in which \( \bar{\Gamma}^\lambda_{\mu\nu} = 0 \), [see appendix]

\[
\hat{\theta}^\mu_\nu = \hat{T}^\mu_\nu + \hat{\iota}^\mu_\nu,
\]

and \( \hat{\iota}^\mu_\nu \) reduces to Einstein’s pseudo-tensor density. \( \hat{\theta}^\mu_\nu \) appears therefore as the energy-momentum tensor of the gravitational field with respect to the background. The second tensor in [2-22], \( \hat{\sigma}^{\mu[\rho\sigma]} \), is quadratic in the metric perturbations just like \( \iota^\mu_\nu \). It is also bilinear in the perturbed metric components \( (g_{\mu\nu} - \bar{g}_{\mu\nu}) \) and their first order derivatives. \( \hat{\sigma}^{\mu[\rho\sigma]} \) resembles, in this respect, the helicity tensor density in electromagnetism. The factor of \( \hat{\theta}_{[\rho\sigma]}^\mu \) represents thus the helicity tensor density of the perturbations with respect to the background.

It should be noted again that all the components of \( I^\mu \) and of the superpotential \( J^{\mu\nu} \) itself are identically zero if \( g_{\mu\nu} = \bar{g}_{\mu\nu} \); therefore strong conservation laws refer to “perturbations” only and not to the background.
(iii) Noether conservation laws

We now consider what happens when arbitrary $\xi^\mu$‘s are replaced by Killing vectors $\bar{\xi}^\mu$ of the background. $\hat{J}^\mu$ defined in [2-22], which contains the physics of the conservation laws [see KLB96], is not, in general, a conserved vector density since the identically conserved vector density is $\hat{I}^\mu = \hat{J}^\mu + \hat{\zeta}^\mu$ and thus

$$\partial_\mu \hat{J}^\mu = -\partial_\mu \hat{\zeta}^\mu. \quad [2 - 27]$$

However, when $\xi^\mu$ is a Killing vector of the background, $\bar{\xi}^\mu$, then [see appendix] $\hat{\zeta}^\mu = 0$ and $\hat{J}^\mu(\bar{\xi})$ is conserved.

Our $\hat{J}^\mu$ has been derived in the same way as “Noether’s theorem” in classical field theory [see for instance Schweber, Bethe and Hoffmann (1956), or Bogoliubov and Shirkov (1959)]. Thus, by replacing $\xi^\mu$ in strongly conserved currents by Killing vectors, $\bar{\xi}^\mu$, of the background we obtain Noether conserved vector densities in general relativity with mappings on curved backgrounds.

$$\hat{J}^\mu(\bar{\xi}) = \hat{\theta}^\mu_{\nu} \bar{\xi}^\nu + \hat{\sigma}^\mu_{[\rho\sigma]} \partial_{[\rho} \bar{\xi}_{\sigma]} \hat{\theta}_{\nu} = \partial^\nu \hat{J}^{\mu\nu}(\bar{\xi}), \quad \partial_\mu \hat{J}^\mu(\bar{\xi}) = 0 \quad [2 - 28]$$

with $\hat{J}^{\mu\nu}(\bar{\xi})$ given by

$$\kappa \hat{J}^{\mu\nu}(\bar{\xi}) = \hat{\eta}^{\mu\rho} \hat{D}_\rho \bar{\xi}^\nu + \hat{g}^{[\mu\rho]} \hat{\Delta}^\nu_{\rho\lambda} \hat{\xi}^\lambda + \hat{\xi}^{[\mu} \hat{g}^{\nu]\rho} \hat{\Delta}^\sigma_{\rho\sigma} - \hat{\bar{\xi}}^{[\mu} \hat{\Delta}^\nu_{\rho\sigma} \hat{g}^{\rho\sigma}. \quad [2 - 29]$$

We can now integrate [2-28], on a part $\Sigma$ of a hypersurface $S$, which spans a two-surface $\partial\Sigma$, and obtain integral conservation laws:

$$\frac{c^4}{G} \mathcal{P}(\bar{\xi}) = \int_{\Sigma} \left( \partial_\nu \hat{\xi}^\nu + \hat{\sigma}^{[\mu\rho]} \partial_{[\rho} \bar{\xi}_{\sigma]} \right) d\Sigma_\mu = \int_{\partial\Sigma} \hat{J}^{\mu\nu}(\bar{\xi}) d\Sigma_{\mu\nu}. \quad [2 - 30]$$

$\mathcal{P}(\bar{\xi})$ depends only on the gravitational field and its first derivatives on $\partial\Sigma$ and on the mapping near the boundary. The relation with the matter tensor depends, however, on the mapping all the way down to the sources of gravity.
For weak fields on a flat background, the lowest order linear approximation of \([2-30]\) is

\[
\frac{c^4}{G} \mathcal{P}(\bar{\xi}) = \int_{\Sigma} \bar{T}^\mu_{\nu} \bar{\xi}^\nu = \int_{\partial \Sigma} \bar{j}^{\mu\nu}(\bar{\xi}) d\Sigma_{\mu\nu}. \tag{2 - 31}
\]

If the spacetime is asymptotically flat, integrals over the whole hypersurface \(\mathcal{S}\) extending to infinity define globally conserved quantities. It is then appropriate to map the spacetime near infinity on a flat Minkowski space with its ten Killing vectors associated with spacetime translations, spatial rotations and Lorentz rotations. The ten Killing vectors give ten different expressions \(\mathcal{P}(\bar{\xi})\), which can be interpreted respectively as the total energy \(E\), the linear momentum vector \(\vec{P}\), the angular momentum vector \(\vec{L}\) and the initial mass center position \(\vec{Z}\) on \(\mathcal{S}\).

We shall now calculate the conserved quantities for Bondi’s asymptotic solution on a null hypersurface at infinity, using the right hand side of \([2-30]\). In what follows we intend to define all the quantities introduced. We shall not use cross-referencing for definitions which are often given with different symbols, factors of 2 or \(1/\sqrt{2}\) and other signs; most readers will appreciate this effort.

3. Elements of the Asymptotic BMS Metric in Newman-Unti Coordinates

(i) The Newman-Unti asymptotic solution

In the coordinates used by Newman and Unti (1962) \(x^\lambda = (x^0 = u, x^1 = r, x^2, x^3)\), the metric of Bondi, Metzner and Sachs (1962) has the following form:

\[
ds^2 = g_{00} du^2 + 2 du dr + 2 g_{0L} du dx^L + g_{KL} dx^K dx^L, \quad K, L = 2, 3. \tag{3 - 1}
\]

The metric components are given by Newman and Unti in their formula (41). We shall make a few changes of notations because several indices which make sense in Newman and Unti are not useful here. Thus
* The indices of the metric where shifted from \( \lambda = 1, 2, 3, 4 \) to \( \lambda = 0, 1, 2, 3 \) (\( \lambda \) or any other lower-case greek letter) so that \( x_{NU}^1 = x^0 = u, x_{NU}^2 = x^1 = r, x_{NU}^3 = x^2 \) and \( x_{NU}^4 = x^3 \).

* We have denoted real parts with a prime like \( a' \), instead of \( \text{Re}(a) \) in NU, and imaginary parts with a “second” like \( a'' \), instead of \( \text{Im}(a) \). Thus \( \sigma = \sigma' + i\sigma'' \) etc...

* Complex conjugation is denoted here by an asterisk \( a^* \), because a bar over the symbol is reserved for the background. Thus \( \bar{a}_{NU} = a^* \).

* The \( \psi_j^\circ \) and \( \sigma^\circ \) of NU are here written \( \psi_j \) and \( \sigma \). Notice that \( \psi_j \) and \( \sigma \) exist also in NU with a different meaning.

* \( O(r^{-n}) \) of NU is here written \( O_n \).

With these changes of notations, the metric components given by Newman and Unti become as follows:

\[
\begin{align*}
g^{00} &= 0 \quad g^{01} = 1 \quad g^{02} = g^{03} = 0 \\g^{11} &= -2P^2 \nabla^* \ln P - \frac{2\psi_2'}{r} + \frac{\frac{4}{3}P^2[\nabla(\psi_1^*/P)]'}{r^2} - \frac{1}{2}(\frac{|f|^2}{P^2}) + O_3 \\g^{12} + ig^{13} &= -\frac{f^*}{r^2} + \left(\frac{4}{3}P\psi_1 + 2\sigma f\right) + O_4 \\g^{23} &= \frac{4P^2\sigma''}{r^3} + \frac{4P^2|\sigma|^2\sigma''}{r^5} + O_6 \quad [3 - 2] \\g^{22} &= -\frac{2P^2}{r^2} + \frac{4P^2\sigma' - 6|\sigma|^2P^2}{r^4} + O_5 \\g^{33} &= -\frac{2P^2}{r^2} - \frac{4P^2\sigma'}{r^3} - \frac{6|\sigma|^2P^2}{r^4} + O_5.
\end{align*}
\]

\( P \) is a function of \( x^2, x^3 \) while \( \psi_1, \psi_2 \) and \( \sigma \) are complex scalar functions of \( u, x^2, x^3 \), defined in terms of the null tetrad components of the Weyl tensor. \( f \) is defined in terms of \( \sigma \) and \( P \)

\[
f \equiv 2P^4\nabla(\sigma^*/P^2), \quad [3 - 3]
\]

and

\[
\nabla \equiv \partial_2 + i\partial_3. \quad [3 - 4]
\]

Note that we have expanded $g^{23}$ to the fifth order of $1/r$.

From [3-2] we have calculated the $g_{\mu\nu}$ components:

$$g_{10} = 1 \quad g_{11} = g_{12} = g_{13} = 0$$

$$g_{00} = 2P^2 \nabla \nabla^* \ln P + \left[ \frac{2\dot{\psi}_2}{r} - \frac{2}{3} P^2 \nabla(\psi_1^*/P) + \frac{1}{r^3} \frac{4fP\psi_1 + \sigma \ddot{f}}{P^2} \right]' + O_4$$

$$g_{02} + ig_{03} = \frac{1}{2P^2} \left[ -f^* + \frac{4}{3} \frac{P\psi_1}{r} + \frac{8}{3} P \sigma \psi_1^* + 3|\sigma|^2 f^* + O_3 \right]$$

$$g_{23} = -r \sigma'' \frac{P^2}{P^2} + \frac{1}{r} \frac{|\sigma|^2 \sigma''}{P^2} + O_2$$

$$g_{22} = -\frac{r^2}{2P^2} + \frac{r \sigma'}{P^2} - \frac{|\sigma|^2}{2P^2} + O_1$$

$$g_{33} = -\frac{r^2}{2P^2} + \frac{r \sigma'}{P^2} - \frac{|\sigma|^2}{2P^2} + O_1.$$  \[3 - 5\]

A useful quantity in our calculation is the density

$$\sqrt{-g} = \frac{r^2}{2P^2} \left( 1 - \frac{|\sigma|^2}{r^2} + O_3 \right). \[3 - 6\]$$

Further useful informations taken from Newman and Unti are

(a) The $u$-derivative equations derived from the Bianchi identities, in the form given in their equations (40k,l) or (42b,c). We shall most of the time use a dot on a symbol to denote a derivative of this function with respect to $x^0 = u$; thus $\dot{\psi} \equiv \partial_0 \psi$. With this change of notation the formulas are

$$\dot{\psi}_1 - P \nabla \psi_2 - 2\sigma \psi_3 = 0 \[3 - 6\]$$

$$\dot{\psi}_2 + \sigma \ddot{\sigma}^* - P^2 \nabla \left( \frac{\psi_3}{P} \right) = 0 \[3 - 7\]$$

where

$$\psi_3 = -P^3 \nabla \left( \frac{\dot{\sigma}^*}{P^2} + \frac{1}{P} \nabla^* \nabla^* P \right). \[3 - 8\]$$

(b) The metric keeps the same form under coordinate transformations of the Bondi-Metzner-Sachs, or BMS, group (Sachs 1962b, Newman and Penrose 1966). The leading terms in powers of $1/r$ of the transformation $x^\lambda \rightarrow x^\lambda$ are given by NU in their
eq. (46):

\[ \tilde{u} = J(x^L)u + K(x^L) + O_1 \]  
\[ \tilde{r} = \frac{1}{J} r + O_0 \]  
\[ \tilde{x}^K = Y^K(x^L) + O_1. \]

\[ J \text{ and } K \text{ are arbitrary functions and the } Y^K \text{ induce conformal transformations in } (x^K) \text{ space, i.e.} \]

\[ \partial_2 Y^2 = \pm \partial_3 Y^3, \quad \partial_3 Y^2 = \mp \partial_2 Y^3. \]

From this follows that \( P(x^L) \) can be fixed with an appropriate conformal transformation [3-12], and \( r \) can be fixed by choosing a spherical boundary for the surface \( u = \text{const.} \). But \( u \) is only defined up to a supertranslation \( K(x^L) \). For a fixed \( P \) and a fixed \( r \) there are five independent scalar functions in the metric: \( \psi_1' \) and \( \psi_1'' \), \( \sigma' \) and \( \sigma'' \) and \( \psi_2' \). But they are defined up to a supertranslation \( K(x^L) \) and therefore there are actually four independent initial quantities among the ten \( g_{\mu\nu} \)'s on \( u = u_0 \). The imaginary part of \( \psi_2 \), \( \psi_2'' \), is not independent; It is defined in terms of \( P \) and \( \sigma \) [NU (40g)]:

\[ \psi_2'' = \left( P^2 \nabla^* \frac{f^*}{2P^2} + \sigma^* \dot{\sigma} \right)'' . \]

The physical interpretation of these functions has been analyzed in details in a series of papers by Bondi, Metzner and Sachs [see especially Bondi van der Burg and Metzner (VII) 1962]
4. Elements of the Asymptotic Background

The asymptotic background is flat. In Minkowski coordinates $X^\alpha = (X^0 = t, X^k)$, $k, l, \ldots = 1, 2, 3$, the metric element

$$d\bar{s}^2 = \eta_{\alpha\beta} dX^\alpha dX^\beta = dt^2 - d\vec{X}^2.$$  \[4-1\]

An arrow designates spatial 3-vectors in Minkowski space. In $X^\alpha$ coordinates, the Killing vector components of the ten translations are given by

$$\tilde{\xi}_\mu^\alpha = \delta_\mu^\alpha \quad \text{[4-2]}$$

and of rotations by

$$\tilde{\xi}_\mu^{[\alpha\beta]} = (\tilde{\xi}_\mu^\alpha \eta_{\beta\gamma} - \tilde{\xi}_\mu^\beta \eta_{\alpha\gamma}) X^\gamma. \quad \text{[4-3]}$$

In NU-coordinates $x^\lambda = (u, r, x^K)$

$$d\bar{s}^2 = \bar{g}_{\alpha\beta} dx^\alpha dx^\beta = du^2 + 2dudr - \frac{r^2}{2P^2} \left[ (dx^2)^2 + (dx^3)^2 \right], \quad \text{[4-4]}$$

where

$$r = \sqrt{\vec{X}^2} \quad \text{[4-5]}$$

and the metric of a sphere $(u = u_0, r = r_0)$ in conformal coordinates has the well known Riemann form for spaces of constant curvature [Eisenhart 1922]:

$$P = \frac{1}{2} + \frac{1}{4}[(x^2)^2 + (x^3)^2]. \quad \text{[4-6]}$$

With $P$ given by [4-6] we find that

$$2P^2 \nabla \nabla^* \ln P = 1. \quad \text{[4-7]}$$

Eq. [4-7] somewhat simplifies $g^{11}$ given in [3-2]. Moreover, since $P$ satisfies the following equation

$$\nabla^* \nabla^* P = 0, \quad \text{[4-8]}$$
ψ₃ defined in [3-8] becomes also simpler:

\[ ψ₃ = -P^3 \nabla \left( \frac{\dot{σ}^*}{P^2} \right) = -\frac{\dot{f}}{2P}, \]  

\[ [4 - 9] \]

\( f \) has been defined in [3-3].

\( x^2 \) and \( x^3 \) are not uniquely defined nor are they uniquely related to the spherical coordinates \((r, \theta, \phi)\) in \( \tilde{\mathcal{M}} \). By definition \( X^1 = r \sin \theta \cos \phi, X^2 = r \sin \theta \sin \phi \) and \( X^3 = r \cos \theta \). We shall define \( x^\mu(X^\lambda) \) as follows:

\[
\begin{align*}
    x^0 &= u = t - r, & x^1 &= r, &[4 - 10] \\
    x^2 &= \sqrt{2} \frac{P}{r} X^1 = \sqrt{2} \cot\left(\frac{1}{2} \theta\right) \cos \phi &[4 - 11] \\
    x^3 &= -\sqrt{2} \frac{P}{r} X^2 = -\sqrt{2} \cot\left(\frac{1}{2} \theta\right) \sin \phi. &[4 - 12]
\end{align*}
\]

This choice makes the connection with the formalism of Newman and Penrose (1966) simple, as we shall see below. In terms of spherical coordinates that are sometimes useful in the calculations, \( P \) becomes

\[ P = \frac{1}{2 \sin^2\left(\frac{1}{2} \theta\right)}. \]

\[ [4 - 13] \]

Let us introduce a unit vector in the radial direction in Minkowski space, denoted by \( \vec{e}_r \),

\[ \vec{e}_r = \frac{\vec{X}}{r} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta), \]

\[ [4 - 14] \]

and the complex 2-vector \( \bar{\Xi} \) on the sphere

\[ \bar{\Xi} \equiv \bar{ξ}^2 + i\bar{ξ}^3. \]

\[ [4 - 15] \]

In terms of \( \vec{e}_r \) and \( \bar{\Xi} \), the ten Killing vector 4-components \( \bar{ξ}^\mu \) or the 2 real + 1 complex components \( \bar{ξ}^\mu \equiv (\bar{ξ}^0, \bar{ξ}^1, \bar{ξ}) \) in \((u, r, x^K)\) coordinates are as follows:

(i) Time Translations \( \bar{ξ}^\mu_{(0)} \):

\[ \bar{ξ}^\mu_{(0)} = δ^\mu_0 = (1, 0, 0), \]

\[ [4 - 16] \]
(ii) 3-Space Translations $\{\bar{\xi}^\mu\} \equiv \bar{\lambda}^\mu$:

$$\bar{\lambda}^\mu = \left(-\bar{e}_r, \bar{e}_r, \frac{2P^2}{r} \nabla \bar{e}_r\right), \quad [4 - 17]$$

(iii) 3-Space rotations $\left\{\frac{1}{2} \epsilon_{m}^{kl} \bar{\xi}^\mu_{[kl]}\right\} \equiv \{\eta^\mu_m\} \equiv \bar{\eta}^\mu$:

$$\bar{\eta}^\mu = (0, 0, 2iP^2 \nabla \bar{e}_r), \quad [4 - 18]$$

(iv) 3-Spacetime (“Lorentz”) rotations $\{\bar{\xi}^\mu_{[0l]}\} \equiv \bar{\zeta}^\mu$:

$$\bar{\zeta}^\mu = \left(u \bar{e}_r, -(r + u) \bar{e}_r, -\left(1 + \frac{u}{r}\right) 2P^2 \nabla \bar{e}_r\right). \quad [4 - 19]$$

These Killing vectors satisfy the Killing equations

$$\bar{D}_\mu \bar{\xi}_\nu + \bar{D}_\nu \bar{\xi}_\mu = 0 \quad , \quad \bar{\xi}_\mu = \bar{g}_{\mu\nu} \bar{\xi}^\nu \quad [4 - 20]$$

which are very useful in further calculations; using the background metric [4-4], the Killing equations [4-20] can be written

$$\partial_1 \bar{\xi}_0 = 0 \quad , \quad \partial_0 \bar{\xi}_1 = \partial_1 \bar{\xi}_1 = -\partial_0 \bar{\xi}_0 \quad [4 - 21]$$

$$\nabla (\bar{\xi}_0 + \bar{\xi}_1) = \frac{r^2}{2P^2} \partial_0 \bar{\Xi} \quad [4 - 22]$$

$$\nabla \bar{\xi}_0 = \frac{r^2}{2P^2} \partial_1 \bar{\Xi} \quad [4 - 23]$$

$$\bar{\xi}_1 = -\frac{1}{2} P^2 \left(\nabla^* \bar{\Xi}^{P^2}\right)^\prime \quad [4 - 24]$$

$$\nabla \bar{\Xi} = 0. \quad [4 - 25]$$

From equation [4-25] applied to $\bar{\eta}^\mu$ defined in [4-18], and from equation [4-24] applied to $\bar{\zeta}^\mu$ defined in [4-19], we deduce, respectively, the following important identities

$$\nabla (P^2 \nabla \bar{e}_r) = 0 \quad [4 - 26]$$

$$P^2 \nabla^* \nabla \bar{e}_r = -\bar{e}_r. \quad [4 - 27]$$
5. Globally Conserved Quantities for Bondi’s Asymptotic Metric

The hypersurface of integration is $x^0 = \text{const}$ or $u = u_0$, with boundary $\partial \Sigma$ at infinity: $x^1 = r \to \infty$. On this boundary the conserved quantities $\mathcal{P}(\bar{\xi})$ defined in [2-30] may be written as surface integrals:

$$\mathcal{P}(\bar{\xi}) = \frac{\kappa}{8\pi} \int_{\partial \Sigma} \bar{J}^{01}(\bar{\xi}) dx^2 dx^3, \quad [5-1]$$

or in terms of the element of solid angle $d\Omega$

$$d\Omega = \sin \theta d\theta d\phi = \frac{dx^2 dx^3}{2P^2}, \quad [5-2]$$

eq 5-1 may be written as

$$\mathcal{P}(\bar{\xi}) = \lim_{r \to \infty} \oint_{\Omega} [P^2 \kappa \bar{J}^{01}(\bar{\xi})] \frac{d\Omega}{4\pi}. \quad [5-3]$$

$\kappa \bar{J}^{01}(\bar{\xi})$ can be deduced from [2-29] or from [2-20],

$$\kappa \bar{J}^{01}(\bar{\xi}) = \kappa \bar{j}^{01}(\bar{\xi}) - \kappa \bar{j}^{01}(\bar{\xi}). \quad [5-4]$$

From [2-15], one obtains

$$2\kappa \bar{j}^{01}(\bar{\xi}) = \sqrt{-g} \left( g^{1\sigma} \partial_1 g_{\sigma \lambda} \bar{\xi}^\lambda + \partial_1 \bar{\xi}^1 - g^{1\sigma} \partial_\sigma \bar{\xi}^0 \right) + \bar{\xi}^0 \hat{k}^1 - \bar{\xi}^1 \hat{k}^0, \quad [5-5]$$

which contains $\hat{k}^0$ and $\hat{k}^1$ defined in [2-4]. The quantity $\bar{j}^{01}$ is the same as [5-5] written in terms of $\bar{g}_{\mu \nu}$ rather than $g_{\mu \nu}$. Therefore, using $\bar{g}_{\mu \nu}$ defined in [4-4] and eq. [4-21], we have

$$2\kappa \bar{j}^{01}(\bar{\xi}) = -2\sqrt{-\bar{g}} \partial_0 \bar{\xi}^0 = -\frac{r^2}{P^2} \partial_0 \bar{\xi}^0, \quad [5-6]$$

remember that $\bar{k}^0 = \bar{k}^1 \equiv 0$.

We substitute the $g_{\mu \nu}$ and $g^{\mu \nu}$ components according to [3-2] and [3-5] into [5-5], and calculate $\bar{j}^{01}$ as given in [5-4]. We also use the Killing equations [4-21] and [4-24] to get
rid of $\xi^1$. We then place the resulting $\hat{J}^{01}$ into [5-3] and take the limit $r \to \infty$, to obtain the following expression for the integrant of $P(\bar{\xi})$:

\[
\lim_{r \to \infty} P^2 2\kappa \hat{J}^{01}(\bar{\xi}) = \sigma \dot{\sigma} \partial_0 \bar{\xi}^0 - \left[ 2\psi_2 + 2\dot{\sigma}^* + P^2 \nabla \left( \frac{f}{P^2} \right) \right] \dot{\xi}^0 \tag{5 - 7a}
\]

\[
+ \left[ \left( \frac{-\psi_1^*}{P} + \frac{1}{2} \nabla^* (\sigma \sigma^*) - \frac{1}{2} \frac{\sigma^* f^*}{P^2} \right) \bar{\xi} \right] \dot{\xi}^0 \tag{5 - 7b}
\]

\[
+ \left[ P^2 \nabla \left( \frac{f}{2P^2} \dot{\xi}^0 - \frac{1}{2} \frac{\sigma \sigma^*}{P^2} \bar{\xi} \right) \right] \dot{\xi}^0 \tag{5 - 7c}
\]

\[
+ r \frac{1}{2P^2} [f \bar{\xi}]'. \tag{5 - 7d}
\]

The integrant of $P(\bar{\xi})$ has been written as a sum of five terms. The term [5-7c] gives no contribution to the integral [5-3] because — see [5-2] — $P^2$ disappears from [5-7c] which becomes a pure divergence on a sphere, whose integral is zero. The term [5-7d] diverges as $r \to \infty$. However, the factor of $r$ integrated on the sphere is equal to the real part of

\[
\int \frac{1}{2P^2} f \bar{\xi}^0 d\Omega = \frac{1}{2\pi} \int \frac{f}{2P^4} \bar{\xi}^0 dx^2 dx^3 \tag{5 - 8}
\]

which becomes, using [3-3] and [4-25]

\[
\frac{1}{2\pi} \int \nabla \frac{\sigma}{P^2} \bar{\xi}^0 dx^2 dx^3 = - \frac{1}{2\pi} \int \frac{\sigma}{P^2} \nabla \bar{\xi}^0 dx^2 dx^3 = 0. \tag{5 - 9}
\]

If it is understood that one takes $r \to \infty$ after integration, the term [5-7d] does not contribute to [5-3]. Thus the integral of $P^2 2\kappa \hat{J}^{01}$ contains only the factors of $\partial_0 \bar{\xi}^0$ and $\bar{\xi}^0$ in [5-7a], and the term with $\bar{\xi}$ in [5-7b]. Further simplifications of [5-7] are still possible, but we want to keep it at present in this form.

We shall now re-write the remaining terms of $P(\bar{\xi})$ using the Newman and Penrose (1966) “edth” operators [see also Newman and Tod 1976], to enable comparisons with the formulas found in the literature of the 70’s and 80’s. More precisely, we shall denote by $\hat{\partial}$ the Newman and Penrose edth derivative times $\frac{1}{\sqrt{2}}$. Thus with formula (3.9) of their
paper we define:
\[
\dot{\eta} = \frac{1}{\sqrt{2}} \dot{\eta}_{NP66} = P^{1-s}\nabla(P^s\eta)
\]
[5 - 10]
where \( s \) is the spin weight of \( \eta \), \( SW(\eta) = s \). The complex conjugate edth derivative \( \dot{\eta}^* \) [see formula (3.17) in NP66] is defined by
\[
\dot{\eta}^* = P^{1+s}\nabla^*(P^{-s}\eta).
\]
[5 - 11]
As far as our calculation goes, we must know the following spin weights:
\[
\begin{align*}
SW(\sigma) &= 2, & SW(\sigma^*) &= -2, & SW(\dot{\sigma}\sigma) &= 1, & SW(\dot{\sigma}\sigma^*) &= -1 \\
SW(\sigma\sigma^*) &= 0, & SW(\vec{e}_r) &= 0, & SW(\dot{\sigma}\vec{e}_r) &= 1, & SW(\dot{\sigma}^*\vec{e}_r) &= -1.
\end{align*}
\]
[5 - 12]
In terms of [5-10], [5-11] and [5-12], and with [3-3], we can re-write some of the quantities appearing in [5-7a,b] as follows:
\[
\begin{align*}
\frac{f}{2P} &= P^3\nabla(P^{-2}\sigma^*) = \dot{\sigma}^* \\
P^2\nabla\frac{f}{2P^2} &= P^2\nabla\left(P^{-1}\frac{f}{2P}\right) = \dot{\sigma}\frac{f}{2P} = \dot{\sigma}^* \\
\frac{(\sigma f)^*}{2P} &= \sigma^*\dot{\sigma}^* \\
P\nabla\vec{e}_r &= \dot{\vec{e}}_r.
\end{align*}
\]
[5 - 13 - 16]
Eq. [4-26] and [4-27] can also be re-written as
\[
\dot{\sigma}^2\vec{e}_r = 0
\]
[5 - 17]
\[
\dot{\sigma}^*\dot{\vec{e}}_r = -\vec{e}_r
\]
[5 - 18]
For integration by parts, the following property [equation (3.26) in NP66] will be particularly useful
\[
SW(A) + SW(B) = -1 \quad \Rightarrow \quad \oint (\dot{\sigma}A)B\frac{d\Omega}{4\pi} = -\oint (\dot{\sigma}B)A\frac{d\Omega}{4\pi}.
\]
[5 - 19]
With [5-13] to [5-16], the integral [5-3] of the integrant [5-7] may be written, taking account of [5-8] and [5-9]

\[
\mathcal{P}(\bar{\xi}) = \frac{1}{2} \oint \sigma \sigma^* \partial_0 \bar{\xi}^0 \frac{d\Omega}{4\pi} - \oint [\psi_2 + \sigma \dot{\sigma}^* + \dot{\sigma}^2 \sigma^*] \bar{\xi}^0 \frac{d\Omega}{4\pi} \\
- \oint [\psi_1 + \sigma \dot{\sigma}^* - \frac{1}{2} \dot{\sigma}(\sigma^*)] \bar{\xi}^0 \frac{d\Omega}{4\pi}.
\]

Equation [5-20] for \( \mathcal{P}(\bar{\xi}) \) has a close resemblance with the standard expressions built on a quite different basis. For comparison see for instance Winicour’s (1980) formula given in [6-2] below. Equation [5-20] will now be simplified further. If we look at [4-17] and [4-19], we can see that the \( u \)-components of the Killing vectors \( \bar{\xi}_0 \) for space translations and Lorentz rotations are of the form \( \bar{\xi}_0 = F(u) \bar{e}_r \). For these \( \bar{\xi}_0 \)'s, and with [5-17], the integral of the \( \dot{\sigma}^2 \sigma^* \)-term in [5-20] is zero; indeed

\[
- \mathcal{F}(u) \oint [\dot{\sigma}^2 \sigma^* \bar{e}_r] \frac{d\Omega}{4\pi} = + \mathcal{F}(u) \oint [\dot{\sigma}^* \dot{\sigma} \bar{e}_r] \frac{d\Omega}{4\pi} = - \mathcal{F}(u) \oint [\sigma^* \ddot{\sigma}^2 \bar{e}_r] \frac{d\Omega}{4\pi} = 0.
\]

For the time translation the \( u \)-component \( \bar{\xi}^0_{(0)} = 1 \) (see [4-16]), and for this \( \bar{\xi}^0 \) the integral of the \( \dot{\sigma}^2 \sigma^* \)-term in [5-20] is equally zero:

\[
\frac{1}{4\pi} \oint \dot{\sigma}^2 \sigma^* d\Omega = \frac{1}{8\pi} \oint \nabla F \bar{f} dx^2 dx^3 = 0.
\]

The \( \dot{\sigma}^2 \sigma^* \)-term does not contribute to space rotations for which \( \bar{\eta}^0 \) is zero (see [4-18]). Therefore \( \dot{\sigma}^2 \sigma^* \) may be omitted from the integral [5-20], which reduces to

\[
\mathcal{P}(\bar{\xi}) = \frac{1}{2} \oint \sigma \sigma^* \partial_0 \bar{\xi}^0 \frac{d\Omega}{4\pi} - \oint [\psi_2 + \sigma \dot{\sigma}^* + \dot{\sigma}^2 \sigma^*] \bar{\xi}^0 \frac{d\Omega}{4\pi} \\
- \oint [\psi_1 + \sigma \dot{\sigma}^* - \frac{1}{2} \dot{\sigma}(\sigma^*)] \bar{\xi}^0 \frac{d\Omega}{4\pi}.
\]

Notice that \( \bar{\Xi}^* = \bar{\xi}^2 - i\bar{\xi}^3 \) contributes only through their principal parts, with \( r \rightarrow \infty \).

There is no \( \bar{\xi}_1 \) in [5-23]. If we denote the principal parts of the 0, 2, 3 components of \( \bar{\xi}^\mu \)
by $\tilde{\xi}^a = (\bar{\xi}^0, \bar{\xi}_{(r=\infty)})$ $a = 0, 2, 3$ we find, in the notations defined in [4-16] to [4-19] and in NU-coordinates that

$$\tilde{\xi}^a_{(0)} = (1, 0) \quad [5 - 24a]$$
$$\bar{\lambda}^a = (-\bar{e}_r, 0) \quad [5 - 24b]$$
$$\eta^a = (0, 2iP\dot{\bar{e}}_r) \quad [5 - 24c]$$
$$\tilde{\zeta}^a = (ue_r, -2P\dot{\bar{e}}_r). \quad [5 - 24d]$$

The term $\partial_0 \bar{\xi}^0 = 0$ except for Lorentz rotations $\tilde{\zeta}^a$, for which

$$\partial_0 \tilde{\zeta}^a = (\bar{e}_r, 0). \quad [5 - 25]$$

The $\tilde{\xi}^a$’s are the generators of the Poincaré subgroup of the BMS transformations at null infinity ($r = \infty$) [see Sachs 1962a, see also Newman and Unti 1966].

6. Detailed Comparison with Standard Results

According to Dray and Streubel (1984), the expressions for angular momentum on the cross section $S = \{u = 0\}$ given by Tamburino and Winicour (1966), Bramson (1975), Lind et al (1972), Prior (1977), Winicour (1968, 1980) and Geroch and Winicour (1981) are all of the same form, though some of these authors differ by ‘anomalous’ factors of 2 [see below]. For definiteness we shall compare [5-23] with the explicit expression for conservation laws $L_\xi(\Sigma^+) \bar{\xi}$ given in Winicour (1980) [his equation (2.16)]. The comparison is made easier with the following redefinitions of Winicour’s notations, represented here with an index $W$:

$$g_{\mu\nuW} = -g_{\mu\nu} \quad 1968 \text{ paper} \quad , \quad \psi_{jW} = 2\sqrt{2}\psi_j \quad j = 1, 2$$
$$\dot{\sigma}_w = -\sqrt{2}\dot{\sigma} \quad , \quad \sigma_w = \sqrt{2}\sigma \quad , \quad \dot{\sigma}_w = 2\dot{\sigma} \quad \text{as} \quad u_w = \frac{u}{\sqrt{2}}. \quad [6 - 1]$$
With [6-1], $L_\xi(\Sigma^+)$ can be written

$$L_\xi(\Sigma^+) = - \oint [(\psi_2 + \sigma\dot{\sigma}^* - \dot{\sigma}^*\sigma)(\xi^\mu l_\mu)]' \frac{d\Omega}{4\pi} [6 - 2a]$$

$$- 2 \oint [(\psi_1 + \sigma\dot{\sigma}^* + \frac{1}{2}\dot{\sigma}(\sigma\sigma^*))(\xi^\mu m_\mu^*)]' \frac{d\Omega}{4\pi} [6 - 2b]$$

in which

$$l_\mu = (1, 0, 0, 0) [6 - 3a]$$

and*

$$m_\mu^* = \frac{1}{2\mathcal{P}}(0, 0, 1, -i); [6 - 3b]$$

$\xi^\mu$ are the dominant parts of the asymptotic symmetry generators. Notice therefore that $\xi^\mu m_\mu^*$ in [6-2] is related to the conjugate of $\bar{\Xi}$ defined in [4-15] as follows:

$$\xi^\mu m_\mu^* = - \frac{1}{2\mathcal{P}} \bar{\Xi}^* [6 - 4]$$

(i) Energy and energy flux

For time translations (see [5-24a]), we obtain from [5-23] an expression for the total energy

$$E = \mathcal{P}(\xi(0)) = - \oint [\psi_2 + \sigma\dot{\sigma}^*]' \frac{d\Omega}{4\pi}, [6 - 5]$$

which is the expression given by Penrose (1964). Moreover, using [3-7] to eliminate $\dot{\psi}_2$ we can calculate the flux $dE/du$ of energy in terms of $\sigma$:

$$\frac{dE}{du} = - \oint \dot{\sigma}\dot{\sigma}^* \frac{d\Omega}{4\pi}. [6 - 6]$$

This is Bondi’s (Bondi, van der Burg & Metzner 1962) mass loss formula. Both $E$ and $dE/du$ are also obtained from [6-2], using [5-22].

* We are somewhat uncertain about the sign of $m_\mu^*$, and were unable to decide.
(ii) Linear momentum and linear momentum flux

For space translations (see [5-24b]), [5-23] provides an expression for the linear momentum

\[ \vec{P} = \mathcal{P}(\vec{\lambda}) = + \oint [\psi_2 + \sigma \dot{\sigma}^*] \dot{e}_r \frac{d\Omega}{4\pi}, \quad [6 - 7] \]

Which has also been proposed by Penrose (1964). Using again [3-7] and [4-7] the linear momentum flux is

\[ \frac{d\vec{P}}{du} = \oint \dot{\sigma} \dot{\sigma}^* \dot{e}_r \frac{d\Omega}{4\pi}. \quad [6 - 8] \]

This \( d\vec{P}/du \) is the BMS invariant flux first given by Sachs (1962b). Both \( \vec{P} \) and \( d\vec{P}/du \) follow also from [6-2], using [5-21].

(iii) Angular momentum and angular momentum flux

For spatial rotations in the background, defined by [5-24c], the corresponding conserved vector defined in the background is the angular momentum \( \vec{L} \)

\[ \vec{L} = \mathcal{P}(\vec{\eta}) = - \oint [\psi_1 + \sigma \dot{\sigma}^*] \dot{\sigma}^* \dot{e}_r \frac{d\Omega}{4\pi}, \quad [6 - 9] \]

Notice that the term \( \frac{1}{2} \dot{\sigma}^* \Xi \vec{e}^*/2P \), both in [5-22c] and in [6-2], drops out of the integral; indeed, using [5-19], we can write

\[ \quad \frac{1}{2} \left[ \oint (\sigma \sigma^*) \Xi \frac{d\Omega}{4\pi} \right] = \frac{1}{2} \left[ \oint (\sigma \sigma^*) \dot{\sigma} \dot{\sigma}^* \vec{e}_r \frac{d\Omega}{4\pi} \right] = \frac{1}{2} \left[ i \oint (\sigma \sigma^*) \dot{\sigma} \dot{\sigma}^* \vec{e}_r \frac{d\Omega}{4\pi} \right] \quad [6 - 10] \]

which is obviously zero because the integral is real. As a result, [6-2] gives twice (or perhaps minus twice) the value of \( \vec{L} \) defined by [6-9]. The factor 2 is the well known anomaly pointed out by Penrose (1982) which makes that the Tamburino and Winicour (1966) as well as the Geroch and Winicour (1981) definitions do not agree with what one expect in the classical interpretation of the quantized weak gravitational field (Gupta 1952). The definitions of Streubel (1978), Dray and Streubel (1984), Dray (1985), Shaw (1986) and Penrose (1982)
do not have the anomalous factor 2. The angular momentum flux can be written in the following form using [3-6] and [4-9]

\[
\frac{d\vec{L}}{du} = -\oint \left[ (\sigma^* \dot{\sigma} - \sigma \dot{\sigma}^*) \dot{\sigma}^* \vec{e}_r \right]'' \frac{d\Omega}{4\pi}.
\]

\[6 - 11\]

(iv) The mass center initial position

For Lorentz rotations in the background defined by [5-24d] and [5-25], The conserved vector is

\[
\vec{Z} = \mathcal{P} (\vec{\xi}) = -u \vec{P} + \oint \left[ (\psi_1 + \sigma \dot{\sigma}^*) \dot{\sigma}^* \vec{e}_r \right]' \frac{d\Omega}{4\pi},
\]

\[6 - 12\]

because the term with \(\partial_0 \vec{\xi}^0\) in [5-23a] cancels the term with \(\dot{\sigma}(\sigma \sigma^*)\) in [5-23c]. The flux of \(\vec{Z}\) is

\[
\frac{d\vec{Z}}{du} = -u \frac{d\vec{P}}{du} - \oint \left[ (\sigma^* \dot{\sigma} + \sigma \dot{\sigma}^*) \dot{\sigma}^* \vec{e}_r \right]' \frac{d\Omega}{4\pi}.
\]

\[6 - 13\]

Following [2-31], \(\vec{Z}\) is the expression one expects in the weak field approximation.

The conserved quantity \(\vec{Z}_W\) deduced from [6-2] is different:

\[
\vec{Z}_W = -u \vec{P} + 2 \oint \left[ (\psi_1 + \sigma \dot{\sigma}^* + \frac{1}{2} \dot{\sigma} |\sigma|^2) \dot{\sigma}^* \vec{e}_r \right]' \frac{d\Omega}{4\pi};
\]

\[6 - 14\]

remember the sign uncertainty of the integral.
7. Concluding Remarks

\( \mathcal{P}(\xi) \) defined in [2-30] is a coordinate independent expression. There are ten \( \mathcal{P}(\xi) \), one for each of the ten Killing fields. For Poincaré transformations in Minkowski space, the \( \xi \)'s and therefore also \( \mathcal{P}(\xi) \)'s transform as vectors or tensors like in Special Relativity. Bondi’s metric is, on the other hand, form invariant for supertranslations, [3-9] with \( J = 1 \) and \( \tilde{x}^K = x^K \)

\[
\frac{du}{dt} = \tilde{u} - K(x^L).
\]

[7 - 1]

Such transformations induce changes in the values of the scalar functions \( \sigma, \psi_1 \) and \( \psi_2 \). In particular

\[
\sigma(u, x^K) = \tilde{\sigma}(\tilde{u}, x^K) - \tilde{\partial}^2 K.
\]

[7 - 2]

The changes in \( d\mathcal{P}/du \) are obtained with [7-2] from [6-6], [6-8], [6-11] and [6-13]. In particular, since

\[
\frac{\partial \sigma}{\partial u} = \frac{\partial \tilde{\sigma}}{\partial \tilde{u}},
\]

[7 - 3]

we see that the fluxes of the energy and of the linear momuntum are unchanged by supertranslations, except for a re-labelling of \( u \) into \( \tilde{u} \); this is a well known result.

We have thus found that the total energy, linear and also the total angular momentum at null infinity obtained from Noether’s theorem and a Lagrangian quadratic in first order derivatives, are the same as those considered as standard. The constructions of the standard quantities are all, more or less, following the general principles outlined in Ashtekar and Winicour (1982). The use of Penrose’s conformal spaces with the BMS symmetry group and the NP formalism is in some ways more appealing and more elegant than the background formalism used here. On the other hand, the mapping independant formulation has a serious drawback pointed out by Goldberg (1990): globally conserved quantities are unrelated to the source of gravity through Einstein’s equations. While our background
metric formalism does not have this defect, it must be said that without a mapping rule the formalism remains incomplete, and we are stuck like everybody else with supertranslation ambiguities. Nevertheless, the fact that eq [2-30] relates \( P \) to the sources of gravitation is of potential value and fills, at least in principle, an important gap in the BMS invariant constructions on spheres at infinity.
Appendix

See also KBL96 for detailed calculations.

\((i)\)

\[
\dot{\hat{\gamma}}^\nu = \bar{T}^\nu - T^\nu + \frac{1}{2\kappa} \hat{\mu}^\sigma R_\rho^\sigma \delta^\mu_\nu + \hat{\nu}^\mu,
\]

in which

\[
\hat{\mu}^\nu = \hat{g}^{\mu\nu} - \bar{g}^{\mu\nu},
\]

and

\[
2\kappa \hat{\mu}^\mu = \hat{g}^{\rho\sigma} \left[ (\Delta^\lambda_\rho \Delta^\mu_\sigma + \Delta^\mu_\rho \Delta^\lambda_\sigma - 2\Delta^\mu_\rho \Delta^\lambda_\sigma) - \hat{\mu}^\nu (\Delta^\eta_\rho \Delta^\lambda_\eta - \Delta^\eta_\rho \Delta^\lambda_\eta) \right] + \hat{g}^{\mu\lambda}(\Delta^\sigma_\rho \Delta^\rho_\lambda - \Delta^\sigma_\lambda \Delta^\rho_\rho).
\]

\((ii)\)

\(\hat{\sigma}^{\mu [\rho\sigma]}\) is the antisymmetric part of \(\hat{\sigma}^{\mu\rho\sigma}\) and

\[
2\kappa \hat{\sigma}^{\mu\rho\sigma} = (g^{\mu\rho} \bar{g}^{\sigma\nu} + g^{\mu\sigma} \bar{g}^{\rho\nu} - g^{\mu\nu} \bar{g}^{\rho\sigma}) \Delta^\lambda_\nu \hat{\lambda} - (g^{\nu\rho} \bar{g}^{\sigma\lambda} + g^{\nu\sigma} \bar{g}^{\rho\lambda} - g^{\nu\lambda} \bar{g}^{\rho\sigma}) \hat{\Delta}^\mu_\lambda.
\]

The two terms containing \(\bar{g}^{\rho\sigma}\) do not contribute to \(\hat{\sigma}^{\mu [\rho\sigma]}\).

\((iii)\)

\[
4\kappa \zeta^\mu = \left( Z^\mu_\rho g^{\rho\sigma} + g^{\mu\rho} Z^\sigma_\rho - g^{\mu\sigma} Z_\rho \right) \Delta^\lambda_\sigma \lambda + (g^{\rho\sigma} Z - 2 g^{\mu\lambda} Z^\lambda_\sigma) \Delta^\mu_\rho \\
+ l^{\mu\lambda} \partial_\lambda Z + l^{\rho\sigma} (\bar{D}^\mu Z_\rho - 2\bar{D}_\rho Z^\mu_\sigma),
\]

in which

\[
Z_{\rho\sigma} \equiv \mathcal{L} \xi \bar{g}^{\rho\sigma} = \bar{D}_\rho \xi_\sigma + \bar{D}_\sigma \xi_\rho \quad , \quad Z = \bar{g}^{\rho\sigma} Z_{\rho\sigma} \quad \text{and} \quad \xi_\sigma = \bar{g}^{\rho\sigma} \xi^\mu.
\]
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