Dynamical decoherence in a cavity with a large number of

two-level atoms

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Abstract

We consider a large number of two-level atoms interacting with the mode of a cavity in the
rotating-wave approximation (Tavis-Cummings model). We apply the Holstein-Primakoff trans-
formation to study the model in the limit of the number of two-level atoms, all in their ground
state, becoming very large. The unitary evolution that we obtain in this approximation is applied
to a macroscopic superposition state showing that, when the coherent states forming the superpo-
sition are enough distant, then the state collapses on a single coherent state describing a classical
radiation mode. This appear as a true dynamical effect that could be observed in experiments with
cavities.

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I. INTRODUCTION

The foundations of quantum optics rely on the Hamiltonian of the interaction of a single radiation mode with an atom that, for all practical applications in optical regime, reduces to the well-known Jaynes-Cummings model \[1\]. The proper working of this model is due to the effectiveness of certain approximations \[2\]: The rotating wave approximation and the two-level approximation. These approximations are enough to justify the success of the Jaynes-Cummings model in the optical regime.

In experiments involving such a model a quite common effect that appears is decoherence, that is the loss of unitarity in the quantum evolution due to external environmental effects \[3\]. Decoherence can make very difficult to realize quantum computers. Indeed, our aim in this paper is to treat the problem of quantum coherence for a system obeying the Jaynes-Cummings model and being composed by \(N\) two-level atoms in the limit of \(N\) becoming infinitely large. Such a model is known in the literature as the Tavis-Cummings model \[4\].

Recent works on decoherence \[5\] have enlarged the meaning of this mechanism to a general reduction of the density matrix to the mixed form removing interference without any need of coupling to an external environment. The possible existence of such kind of decoherence without environment is, indeed, an open question.

Decoherence can also be seen as an intrinsic effect due to the unitary evolution when the number of particles becomes very large \[6, 7, 8\]. A first example in this sense has been given by Gea-Banacloche \[9\] that computed asymptotic states for the radiation field in the Jaynes-Cummings model in the limit of a large photon number. Such states have been recently observed in experiments by Haroche et al. \[10, 11\] proving the emergence of classical behavior already with very few photons. The work of Gea-Banacloche appears in some way dual to the one we present here.

In order to study the Tavis-Cummings Hamiltonian in the limit of a very large number of particles on a macroscopic superposition state we will adopt an approach firstly devised by Persico and Vetri \[12\] and recently applied in a paper by Berman et al. \[13\]. These authors use the Holstein-Primakoff \[14\] transformation to change the contribution of the two-level atoms into a bosonic field. In this way, one gets the Hamiltonian of two coupled harmonic oscillators that can be easily diagonalized. The main result of this paper will be to show how the Tavis-Cummings Hamiltonian can produce decoherence, intrinsically in
this approximation, making this effect observable in experiments with cavities. The unitary evolution is applied to a superposition state of two coherent states of radiation. When the distance between such states is large enough, as already seen in recent experiments [15, 16], decoherence sets in and the superposition disappears leaving just a coherent state.

The paper is structured in the following way. In sec.II we introduce the model and the Holstein-Primakoff transformation. In sec.III the higher order corrections to the unitary evolution are evaluated. In sec.IV the unitary evolution of a macroscopic superposition state is obtained proving that decoherence is produced when the components of the superposition state are significantly distant. In sec.V the conclusions are given.

II. TAVIS-CUMMINGS MODEL AND HOLSTEIN-PRIMAKOFF TRANSFORMATION

The N-atom Jaynes-Cummings model was firstly considered by Cummings and Tavis [4] and can be written as

\[ H = \omega a^\dagger a + \frac{\Delta}{2} \sum_{i=1}^{N} \sigma_{3i} + g \sum_{i=1}^{N} (\sigma_{+i} a + \sigma_{-i} a^\dagger) \]  

(1)

being \( \omega \) the frequency of the radiation mode, \( a \) and \( a^\dagger \) the ladder operators, \( \Delta \) the separation between the energy levels of the two level atoms, \( g \) the coupling and \( \sigma_{3i}, \sigma_{+i} \) and \( \sigma_{-i} \) the Pauli spin matrices. By introducing the operators \( S_z = \frac{1}{2} \sum_{i=1}^{N} \sigma_{3i}, \) \( S_\pm = \sum_{i=1}^{N} \sigma_{\pm i} \) we can rewrite the above Hamiltonian in the form

\[ H = \omega a^\dagger a + \Delta S_z + g(S_+ a + S_- a^\dagger). \]  

(2)

A “ferromagnetic state” is characterized by having all the two-level atoms in their ground state. For the sake of simplicity we put us at the resonance \( \Delta = \omega \). Our aim is to study the Tavis-Cummings model in this situation. This is accomplished by the so called Holstein-Primakoff transformation. This is defined by

\[ S_+ = b^\dagger \left(-2S - b^\dagger b\right)^{\frac{1}{2}} \]  

\[ S_- = \left(-2S - b^\dagger b\right)^{\frac{1}{2}} b \]  

\[ S_z = S + b^\dagger b \]  

(3)
being $S = -\frac{N}{2}$ and $b, b^+$ bosonic operators. The aim of this transformation is to obtain a series in the parameter $\frac{1}{S}$, being $|S| \gg 1$, that is pertinent to our situation. These computations are well-known in literature [12, 13] but we report it here for completeness:

\begin{align}
H_0 &= -\frac{N\omega}{2} + \omega(a^+a + b^+b) + \sqrt{Ng}(a^+b + b^+a) \\
H_1 &= -\frac{g}{2\sqrt{N}}(a^+b^1bb + ab^1b^1b) \\
\end{align}

and the second and third order corrections are given by

\begin{align}
H_2 &= -\frac{g}{8\sqrt{N^3}}(a^+b^1b^1bb + ab^1b^1b^1b) \\
H_3 &= -\frac{3g}{48\sqrt{N^5}}(a^+b^1b^1b^1bb + ab^1b^1b^1bb^1b) \\
\end{align}

from which an easy rule to obtain higher order terms is realized. So, $n$-th order term is given by the multiplicative factor $-q_n \frac{g}{N^{n-\frac{1}{2}}}$, being $q_n$ the numerical coefficient of the corresponding order in the series of $(1 - x)^{\frac{1}{2}}$, and Hamilton operator is given by $a^+ \prod_{k=1}^{n}(b^\dagger b)^k b + h.c.$.

Our approximation holds until the average number of bosonic excitations described by the operator $b$ is largely smaller than $|S|$.

The leading order Hamiltonian can be immediately diagonalized by introducing two bosonic operators $c_1$ and $c_2$ as

\begin{align}
c_1 &= \frac{a + b}{\sqrt{2}} \\
c_2 &= \frac{a - b}{\sqrt{2}} \\
\end{align}

that give

\begin{align}
H_0 = (\omega + \sqrt{Ng})c^\dagger_1 c_1 + (\omega - \sqrt{Ng})c^\dagger_2 c_2 \\
\end{align}

i.e. two independent harmonic oscillators and we have omitted the constant. This means that, at the leading order, we expect the coherence to be preserved [12]. In sec IV we will show that this conclusion can be evaded in some way. Anyhow, the unitary evolution at the leading order is given by

\begin{align}
U_0(t) = \exp \left[-it(\omega + \sqrt{Ng})c^\dagger_1 c_1 \right] \exp \left[-it(\omega - \sqrt{Ng})c^\dagger_2 c_2 \right]. \\
\end{align}

The eigenvalues are given by

\begin{align}
\epsilon_{n_1n_2}(\omega, g, N) = n_1(\omega + \sqrt{Ng}) + n_2(\omega - \sqrt{Ng}). \\
\end{align}
The eigenstates are given by

\[ |n1; n2\rangle = \frac{(c_1^\dagger)^{n_1}}{\sqrt{n_1!}} |0\rangle \frac{(c_2^\dagger)^{n_2}}{\sqrt{n_2!}} |0\rangle \] (10)

having put \( |0\rangle \) the state having all the two-level atoms in the ground state on which the \( c_2^\dagger \) operator is acting. This clarifies how the time evolution happens in the limit of \( N \) two-level atoms all in their ground state. The next order correction is \( O\left(\frac{1}{N}\right) \).

For a coherent state of the radiation mode

\[ |\psi(0)\rangle = e^{a_0^\dagger - a_0^*} |0\rangle \left| -\frac{N}{2} \right] \] (11)

we will get at the leading order

\[ |\psi(t)\rangle = |\tilde{\alpha} e^{-i(\omega + \sqrt{Ng})t}\rangle_+ + |\tilde{\alpha} e^{-i(\omega - \sqrt{Ng})t}\rangle_- \] (12)

with \( \tilde{\alpha} = \frac{\alpha}{\sqrt{2}} \), being

\[ c_1 |\tilde{\alpha} e^{-i(\omega + \sqrt{Ng})t}\rangle_+ = \tilde{\alpha} e^{-i(\omega + \sqrt{Ng})t} |\tilde{\alpha} e^{-i(\omega + \sqrt{Ng})t}\rangle_+ \] (13)

and

\[ c_2 |\tilde{\alpha} e^{-i(\omega - \sqrt{Ng})t}\rangle_- = \tilde{\alpha} e^{-i(\omega - \sqrt{Ng})t} |\tilde{\alpha} e^{-i(\omega - \sqrt{Ng})t}\rangle_- \] (14)

coherent states of the two harmonic oscillators. These equations give the useful result for our aims

\[ a|\psi(t)\rangle = \alpha e^{-i\omega t} \cos(\sqrt{Ng}t) |\psi(t)\rangle. \] (15)

From this equation it is easy to recover the well-known results [12]

\[ \langle \hat{n} \rangle = \langle \psi(t) | a^\dagger a | \psi(t) \rangle = |\alpha|^2 \cos^2(\sqrt{Ng}t) \] (16)

being \( \hat{n} \) the number operator for the radiation field and

\[ \langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2 = |\alpha|^2 \cos^2(\sqrt{Ng}t) \] (17)

that proves that at the leading order the coherence is kept. This results holds, at this order, also in the thermodynamic limit \( N \to \infty, g \to 0 \) and \( \sqrt{Ng} = constant \).
III. HIGHER ORDER CORRECTIONS TO THE UNITARY EVOLUTION

The answer to the question about the coherence being kept to higher order has been properly answered in Ref. [13]. The answer is given by computing the first order correction to the eigenvalues by the Rayleigh-Schrödinger equation. The Hamiltonian to use is given by $H_1$ that is, using the operators $c_1$ and $c_2$

$$H_1 = -\frac{g}{4\sqrt{N}}[(c_1^\dagger c_1)^2 - (c_2^\dagger c_2)^2 + c_2^\dagger c_1 - (c_1^\dagger c_1 - c_2^\dagger c_2)(c_1^\dagger c_2 + c_1 c_2^\dagger) + c_2^\dagger c_2 - c_1^\dagger c_1]$$  \hspace{1cm} (18)$$
and we obtain the correction

$$\epsilon_{n_1 n_2}^1(g, N) = -\frac{g}{4\sqrt{N}}(n_1^2 - n_2^2 + n_2 - n_1).$$ \hspace{1cm} (19)$$
that gives the following correction to the average of the number operator

$$\langle \hat{n} \rangle = |\alpha|^2 \sum_{n_1, n_2} e^{-2|\alpha|^2} \frac{\langle \tilde{\alpha} |^{2n_1} \tilde{\alpha} |^{2n_2} \rangle}{n_1! n_2!} \cos \left[ \sqrt{N}gt + \frac{g}{4\sqrt{N}}(n_1 + n_2)t \right]$$ \hspace{1cm} (20)$$
and we lose coherence if the second term in the argument of the cosine is comparable to the first one but, in the thermodynamic limit as defined above, we are granted that coherence is maintained.

It is interesting to compute also the correction to the eigenstates [10]. Rayleigh-Schrödinger series gives

$$|n_1; n_2\rangle_1 = \frac{1}{8N}(n_1\sqrt{n_1}\sqrt{n_2 + 1}|n_1 - 1; n_2 + 1\rangle + n_2\sqrt{n_2}\sqrt{n_1 + 1}|n_1 + 1; n_2 - 1\rangle)$$ \hspace{1cm} (21)$$
that is $O\left(\frac{1}{N}\right)$ while, as seen above, the correction to the eigenvalues is $O\left(\frac{1}{\sqrt{N}}\right)$. Then, we are able to compute the correction to the average of the number operator to first order. This gives

$$\delta\langle \hat{n} \rangle = \frac{1}{8N} \sum_{n_1, n_2} e^{-2|\alpha|^2} \frac{\langle \tilde{\alpha} |^{2n_1} \tilde{\alpha} |^{2n_2} \rangle}{n_1! n_2!} \cdot \left\{ (n_1 + n_2)(n_1^2 + m_1^2) - n_1^2(n_2 + 1) - n_2^2(n_1 + 1) \right\}$$ \hspace{1cm} (22)$$
+ \left\{ [(n_1 - 1)^2 + (n_2 + 1)^2]\sqrt{n_1}\sqrt{n_2 + 1} + [(n_1 + 1)^2 + (n_2 - 1)^2]\sqrt{n_1 + 1}\sqrt{n_2} \times \right.$$ \hspace{1cm} \left.$$\cos \left[ 2\sqrt{N}gt + \frac{g}{2\sqrt{N}}(n_1 + n_2 - 1)t \right] \right\}$$
+ \left\{ (n_2 + 2)\sqrt{n_1}\sqrt{n_2 + 1}\sqrt{n_1 - 1}\sqrt{n_2 + 2} + (n_1 + 2)\sqrt{n_2}\sqrt{n_1 + 1}\sqrt{n_2 - 1}\sqrt{n_1 + 2} \times \right.$$ \hspace{1cm} \left.$$\cos \left[ 4\sqrt{N}gt + \frac{g}{\sqrt{N}}(n_1 + n_2 - 1)t \right] \right\}$$
and again we have a confirmation that the thermodynamic limit grants that coherence is kept for an initial coherent state. As our aim is to apply the time evolution to a superposition of coherent states, this conclusion is crucial for our result to hold in the thermodynamic limit.
IV. UNITARY EVOLUTION OF A MACROSCOPIC SUPERPOSITION STATE

Now, let us consider a phase Schrödinger cat state \[|\psi_S(0)\rangle = \mathcal{N}(|\gamma e^{i\phi}\rangle + |\gamma e^{-i\phi}\rangle)\] (23) being \[\mathcal{N}^2 = \frac{1}{2} \frac{1}{1 + \cos(\gamma^2 \sin(2\phi))e^{-\Delta^2}},\] (24) \[\Delta^2 = 2\gamma^2 \sin^2 \phi\] the distance between the coherent states, \(\gamma\) and \(\phi\) two real numbers. The time evolution of this state is given by \[|\psi_S(t)\rangle = \mathcal{N} \left[|\tilde{\gamma} e^{-i[(\omega+\sqrt{N}g)t-\phi]}\rangle_+ + |\tilde{\gamma} e^{-i[(\omega-\sqrt{N}g)t+\phi]}\rangle_-\right] + \frac{\sqrt{2}}{\gamma} \left[|\tilde{\gamma} e^{-i[(\omega+\sqrt{N}g)t+\phi]}\rangle_+ + |\tilde{\gamma} e^{-i[(\omega-\sqrt{N}g)t-\phi]}\rangle_-\right] \] (25) with \(\tilde{\gamma} = \frac{\gamma}{\sqrt{2}}\). The computation of the averages is rather straightforward and gives using eq. (15) \[\langle \psi_S(t)| a^\dagger a |\psi_S(t)\rangle = \gamma^2 \cos^2(\sqrt{Ngt}) \frac{1 + \cos(2\phi + \gamma^2 \sin(2\phi))e^{-\Delta^2}}{1 + \cos(\gamma^2 \sin(2\phi))e^{-\Delta^2}}\] (26) that has the property to give the result of a single coherent state or \(\phi \to 0\) but, mostly important, for \(\Delta \gg 1\) one recover the same result of a single coherent state. So, more distant are the coherent states and easier the superposition is removed in the time evolution. This is confirmed by the computation:

\[\langle \psi_S(t)| a^\dagger a a^\dagger a |\psi_S(t)\rangle = \gamma^4 \cos^4(\sqrt{Ngt}) \frac{1 + \cos(4\phi + \gamma^2 \sin(2\phi))e^{-\Delta^2}}{1 + \cos(\gamma^2 \sin(2\phi))e^{-\Delta^2}}\] (27)

that gives back the result of a single coherent state both for \(\phi = 0\) and \(\Delta \gg 1\) confirming that the unitary evolution for the Tavis-Cummings model, in the thermodynamic limit, grants the disappearance of the superposition state. This situation has been encountered also for the Dicke model in Ref.[8] where it was shown an identical result in this case but resorting to the concept of singular limits. Here, the argument still relies on the thermodynamic limit, that is essential to the proof, but we have also assumed the components of the Schrödinger cat state are enough distant, \(\Delta \gg 1\).

It is important to note that, if the coherent states in the superposition are truly macroscopic, we are left with a single coherent state behaving as a classical radiation field.
V. CONCLUSIONS

In conclusion, we have shown how decoherence can be produced in the thermodynamic limit also for the N atom Jaynes-Cummings model that can be straightforwardly used to test the very existence of a somewhat different kind of decoherence, even if the N atoms can be seen as an environment. But decoherence appears dynamically.

We would like to emphasize the duality of our approach with respect to the one of Gea-Banacloche [9]. Striking evidence of classicality emerging by increasing the number of photons in a cavity, without resorting to any concept of environment, has been given recently in the experiment of Haroche’s group [11], giving full support to the analysis of Gea-Banacloche [9]. So, it would be interesting to see also states for N atoms in the thermodynamic limit as obtained here. Then, decoherence can also appear as an intrinsic effect of the unitary evolution in the thermodynamic limit.

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