PWL approximation of nonlinear dynamical systems, Part–I: structural stability

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Abstract. This paper and its companion address the problem of the approximation/identification of nonlinear dynamical systems depending on parameters, with a view to their circuit implementation. The proposed method is based on a piecewise-linear approximation technique. In particular, this paper describes the approximation method and applies it to some particularly significant dynamical systems (topological normal forms). The structural stability of the PWL approximations of such systems is investigated through a bifurcation analysis (via continuation methods).

1. Introduction
This paper deals with the piecewise-linear approximation of nonlinear dynamical systems with a view to their structurally stable circuit implementation (see also [22]).

Although the proposed method could be applied to other kinds of models, we shall focus on autonomous dynamical systems described by continuous-time state-space models depending on parameters, i.e., on systems governed by the following set of differential-algebraic equations:

\[
\begin{align*}
\dot{x}(t) &= f(x(t); p) \\
y(t) &= g(x(t))
\end{align*}
\] (1)

where \(x(t) \in \mathbb{R}^n\) (state vector), \(p(t) \in \mathbb{R}^q\) (parameter vector), \(y(t) \in \mathbb{R}^m\) (output vector), \(f : S \subset \mathbb{R}^{n+q} \rightarrow \mathbb{R}^n\) (vector field), \(S\) is a bounded compact domain, \(g : \mathbb{R}^n \rightarrow \mathbb{R}^m\), and \(\dot{x}\), as usual, denotes the time derivative of \(x(t)\). All the vectors are intended as column vectors.

Here, we will address the problem of finding a piecewise-linear (PWL) approximation of either known systems or, at least, systems where a reasonable number of (noisy) measure samples of the vector field \(f\) is available (regression set). In a companion paper [6], the problem of the identification of PWL models of dynamical systems of the kind (1) starting from samples of an output variable \(y_i(t)\) will be addressed.

Generally speaking, piecewise-linear (or, more correctly, piecewise-affine) models have universal approximation properties, which essentially means that any nonlinear function can be approximated by a PWL function with arbitrary accuracy, provided that the function domain is partitioned in a large enough number of subdomains where the function is approximated by an affine system. Usually, the shape of the subdomains is triangular (for bivariate functions)
or hypertriangular (for multivariate functions). For instance, PWL approximations of surfaces in 3D spaces are widely used in computer graphics applications, as well as in the approximate solution of inverse problems such as the approximation of scattered or sampled data (e.g., electromagnetic or geomorphological) through meshless parameterizations of the 2D domains\(^1\) aimed at having multi-grid resolution and high flexibility of the resulting surface triangulations [8, 9].

Many PWL models belong to the class of function expansion models:

\[
f_{PWL}(\mathbf{z}; N) = \sum_{k=1}^{N} w_k(N) \varphi_k(\mathbf{z}; N)
\]

where \(\mathbf{z}\) is a generic (real) input vector and \(N\) is the (integer) number of basis functions \(\varphi_k(\mathbf{z}; N)\) whose sum (weighted through the coefficients \(w_k(N)\)) provides an approximation of a given scalar function \(f\). This is a very wide class of models, including for instance, kernel estimators based on Bayesian methods [27, 20] or on regularization methods [17, 3, 4], splines [28], and, in the PWL framework, wavelets and prewavelets [10, 11], fuzzy models [26, 21], and so on. The problem of finding a specific function expansion model from a given regression set is usually referred to as nonparametric regression problem.

In the field of dynamical systems approximation, a method based on mixed-integer programming for the simultaneous identification of both the number \(N\) of basis functions and the \(N\) coefficients \(w_k\) has recently been proposed in the context of the identification of hybrid control systems [18, 19]. Such a method works well for low values of \(N\) and for a limited size of the regression set (which is a good feature in the context of real-time black-box identification), but for the approximation of non-smooth functions its computational complexity can become critical.

Another PWL approach proposed in the last few years for the approximation of continuous functions [12, 13, 23, 25] is based on an \textit{a priori} domain partition through a simple type-1 triangulation (or \textit{simplicial partition}), i.e., a triangulation formed by a rectangular partition plus northeast diagonals (this can also be viewed as a three-directional box spline grid). The rectangular partition is obtained by subdividing each spatial component \(z_i\) of the domain into an integer number \(m_i\) of identically-sized segments. In this case, \(N \left(= \prod_{i=1}^{n+q} (m_i + 1)\right)\) can be fixed as a first step by some heuristic criteria, e.g., simply based on function inspection [12, 13, 23, 25]. The coefficients \(w_k\) are determined, as a second step, by minimizing a proper cost function [13, 25]. In the absence of \textit{a priori} knowledge, such a method can suffer from the \textit{curse of dimensionality} [2], since the number of elements of the regression set needed to have an accurate approximation would grow exponentially with the number of dimensions. However, if either the function to be approximated is known or we can sample it arbitrarily, it is possible to fix a reasonable number of subdivisions along each dimensional component of the domain. Another drawback of the method could be that simplicial partitions are asymmetric, as one of the possible diagonal directions is favoured over the other. For the PWL modelling of symmetric functions or regression sets, type-2 triangulations (i.e., symmetric, four-directional box spline grids) would work better.

The main advantage of the simplicial approach is in its direct circuit implementation [24, 16], which can be particularly useful whenever we aim to mimic the behaviour of dynamical systems made up of a large number of elementary units [22, 5]. Another advantage of such an approach, not shared, for instance, by the wavelets and prewavelets PWL approximations, is the simplicity of course, when the surface geometry is complex in the sense that it cannot be represented simply as the graph of a bivariate function, it is necessary to first find a homeomorphic mapping of the surface to a simply connected planar region.
of its theoretical formulation, that allows an easy application to functions defined over domains of any (at least in principle) dimensionality.

In this paper, we shall analyze the qualitative behaviour of dynamical systems characterized by PWL vector fields that approximate some topological normal forms. Such an analysis will be carried out by resorting to some packages for numerical continuation [15, 7, 14]. To guarantee that the dynamical behavior of the PWL-approximate vector field will be faithful to that of the original system for any values of some significant parameters (i.e., to verify the structural stability - in a given limited domain - of the original system to the perturbation induced by the approximation), we shall obtain a complete bifurcation scenario of the approximate system. As an essential prerequisite for using such methods as vector field smoothness, we shall replace the PWL vector field with a piecewise-smooth (PWS) version of it [22].

The rest of the paper is organized as follows. In Section 2, we shall briefly recall some basic definitions concerning the PWL approximation of continuous-time dynamical systems. In Section 3, some topological normal forms will be considered and some PWL approximations of these systems will be discussed by making reference to either equilibrium manifolds or phase portraits. Section 4 concerns the bifurcation analysis of the smoothed versions of the most significant PWL approximations to the normal forms considered. In Section 5, some concluding remarks will be made.

2. PWL approximation: basic definitions

We aim to approximate the vector field \( f \) in (1) through a proper PWL function obtained as the weighted sum of a set of \( N \) basis functions. We shall denote by \( f_{\text{PWL}} \) a continuous PWL approximation of \( f \) over the \( (n+q) \)-dimensional compact domain \( S \subset \mathbb{R}^{n+q} \), i.e., \( f_{\text{PWL}} : S \to \mathbb{R}^n \), where \( S \) is a hyperrectangle (rectangle, if \( n+q = 2 \)) of the kind

\[
S = \{ z \in \mathbb{R}^{n+q} : a_i \leq z_i \leq b_i, i = 1, \ldots, n + q \}
\]

Each dimensional component \( z_i \) of the domain \( S \) (generically denoting a component of \( x \) or \( p \)) can be subdivided into \( m_i \) subintervals of amplitude \( (b_i - a_i)/m_i \), and thus a boundary configuration \( H \) is obtained, see [12], which depends on the vector \( m = [m_1, \ldots, m_{n+q}]^T \). Each hyperrectangle contains \( (n+q)! \) non-overlapping hypertriangular (triangular, if \( n+q = 2 \)) simplices. As a result, \( S \) turns out to be partitioned (simplicial partition) into \( \prod_{i=1}^{n+q} m_i \) hyperrectangles and to contain \( N = \prod_{i=1}^{n+q} (m_i + 1) \) vertices. The domain associated with a simplicial boundary configuration \( H \) (i.e., the vector \( m \)) can be completely described by the triplets \( (a_i, b_i, m_i) \), \( i = 1, \ldots, n + q \).

As shown in [12, 13], the class of continuous PWL functions \( f_{\text{PWL}} \) that are linear over each hypertriangular simplex constitutes an \( N \)-dimensional Hilbert space \( \text{PWL}[S_H] \), which is defined by the domain \( S \), its simplicial partition \( H \), and a proper inner product (see [25] for details). Each function belonging to \( \text{PWL}[S_H] \) can be represented as a sum of \( N \) basis functions (arbitrarily organized into a vector \( \varphi(z; m) \)), weighted by an \( N \)-length coefficient vector \( w \). For a fixed \( m \), the coefficients \( w \) determine the shape of \( f_{\text{PWL}} \) uniquely.

There are many possible choices for the PWL basis functions, each of which is made up of \( N \) (linearly independent) functions belonging to \( \text{PWL}[S_H] \). For instance, there are bases more convenient for performing function interpolation or function approximation, from a computational or a circuit-synthesis point of view. However, any basis can be expressed as a linear combination of the elements of the so-called \( \beta \)-basis, which can be defined by recursively applying (up to \( n + q \) times) the following function [12, 13, 23]:

\[
\gamma(u, v) = \max(0, \min(u, v)).
\]  

In the examples given in this paper, the vector \( m \) has been fixed by function inspection. As shown in [23], once the vector \( m \) is fixed, the weighting coefficients \( w \) can easily be found by
applying optimization techniques (e.g., a least-squares criterion) to a set of \( N \) fitting samples of \( f \) uniformly distributed over the domain \( S \).

In the next section, we shall approximate three specific dynamical systems by applying the PWL technique described above. Different approximations will be characterized by different values of the triplets \((a_i, b_i, m_i)\), \( i = 1, \ldots, n + q \). In particular, for a domain \( S \) fixed for each system, we shall consider different PWL approximations by varying the numbers \( m_i \) (subdivisions) and \( n_i \) (samples of \( f \)) along any dimensional component \( z_i \) of \( S \).

3. PWL approximations of some topological normal forms

The analysis of a dynamical system can be carried out by constructing its bifurcation diagram, which represents very compactly all possible behaviors of the system and transitions (bifurcations) between them under parameter variation. The bifurcation diagram of a given dynamical system can be very complicated, but, at least locally, bifurcation diagrams of systems for many different applications can look similar (topological equivalence). The concept of topological equivalence leads to the definition of topological normal forms, i.e., polynomial forms that provide universal bifurcation diagrams and constitute one of the basic notions in bifurcation theory. Another central concept is the structural stability of a dynamical system, that is, the topological equivalence between such a system and any other system obtained by slightly changing some parameters [1, 14].

In this section, we shall find different PWL approximations of some topological normal forms (cusp, Bautin, and Bogdanov-Takens), and we shall get an idea about the structural stability of such approximations. For details concerning such normal forms, the reader is referred to [14].

3.1. Cusp bifurcation

The topological normal form for the cusp bifurcation is:

\[
\dot{x} = p_1 + p_2 x - x^3
\]

In this case, \( n = 1 \) and \( q = 2 \).

Figure 1(a) shows the equilibrium manifold \( x_{eq}(p_1, p_2) \), i.e., the locus of the points \( x \) satisfying the equation \( f(x) = 0 \), near the cusp bifurcation.

The PWL approximations \( f_{\text{PWL}} \) of the vector field \( f \) are obtained over the domain

\[
S = \{ z \in \mathbb{R}^3 : a_1 \leq x \leq b_1, \ a_2 \leq p_1 \leq b_2, \ a_3 \leq p_2 \leq b_3 \}
\]

with

\[
a_1 = -1.5, \ b_1 = 1.5; \ a_2 = -1.0, \ b_2 = 1.0; \ a_3 = -1.0, \ b_3 = 1.5
\]

The domain \( S \) is partitioned by performing \( m_1 \) subdivisions along the state component \( x \), and \( m_2 \) and \( m_3 \) subdivisions along the parameter components \( p_1 \) and \( p_2 \), respectively. The coefficients \( w \) of the \( \beta \)-basis were derived from a set of samples of \( f \) corresponding to a regular grid of \( n_1 \times n_2 \times n_3 \) points over the domain \( S \). We point out that the vector field \( f \) is linear with respect to the dimensional component \( p_1 \), and we can fix \( m_2 = 1 \). With this caveat in mind, we now focus our attention on the following three PWL approximations \( f_{\text{PWL}} \) of \( f \):

(A1) \( m_1 = 3, \ m_2 = 1, \ m_3 = 3, \ n_1 = n_2 = n_3 = 31 \);

(A2) \( m_1 = 5, \ m_2 = 1, \ m_3 = 5, \ n_1 = n_2 = n_3 = 31 \);

(A3) \( m_1 = 7, \ m_2 = 1, \ m_3 = 7, \ n_1 = n_2 = n_3 = 31 \).

Figures 1(b-d) show the equilibrium manifolds \( x_{eq}(p_1, p_2) \) for the PWL approximations (A1), (A2), and (A3), respectively. With reference to Fig. 1(a), it is easy to conclude that
- approximation (A1) is very rough: its equilibrium manifold does not exhibit qualitative similarity to the original one;

- approximation (A2) is qualitatively good: the fold bifurcation border is similar to that of the original one, even though its shape is quite saw-toothed;

- approximation (A3) is good from both the qualitative and quantitative points of view.

Figure 1. Equilibrium manifolds of (a) the cusp normal form and (b,c,d) its PWL approximations: coarse (A1), medium (A2), and fine (A3), respectively. Stable regions are shown in dark gray, unstable regions in light gray, and the fold bifurcation border is marked in black.
3.2. Bautin (generalized Hopf) bifurcation

The Bautin (generalized Hopf) bifurcation can be described by the following polynomial normal form:

\[
\begin{align*}
\dot{x}_1 &= p_1 x_1 - x_2 - x_1(x_1^2 + x_2^2)(x_1^2 + x_2^2 - p_2) \\
\dot{x}_2 &= p_1 x_2 + x_1 - x_2(x_1^2 + x_2^2)(x_1^2 + x_2^2 - p_2)
\end{align*}
\] (3)

In this case, \( n = 2 \) and \( q = 2 \).

Figure 2. Bifurcation diagram for the normal form (3). The coordinates of the three black points \( a, b, \) and \( c \) are: \( a = (-0.5, 0.9) \), \( b = (-0.15, 0.9) \), and \( c = (0.3, 0.9) \).

Figure 2 shows the bifurcation diagram for the normal form (3). The system (3) has only one equilibrium point, i.e., the origin of the plane \((x_1, x_2)\). The origin has purely imaginary eigenvalues along \( H \) and is stable for \( p_1 < 0 \) and unstable for \( p_1 > 0 \). A stable limit cycle bifurcates from the origin if we cross the half-line \( H^- \) from region \( A \) to region \( C \). On the other hand, an unstable limit cycle appears if the half-line \( H^+ \) is crossed from region \( C \) to region \( B \). Two limit cycles coexist in region \( B \), and collide and disappear on the curve \( T_c \). The codimension-two point \( DH = (0, 0) \) marks a degenerate Hopf bifurcation of the equilibrium (i.e., the first Lyapunov coefficient of the Hopf bifurcation changes its sign).

The PWL approximations \( f_{PWL} \) of the vector field \( f \) are obtained over the domain

\[ S = \{ z \in \mathbb{R}^{n+q} : a_1 \leq x_1 \leq b_1, \quad a_2 \leq x_2 \leq b_2, \quad a_3 \leq p_1 \leq b_3, \quad a_4 \leq p_2 \leq b_4 \} \]

with

\[
a_1 = a_2 = -1.5, \quad b_1 = b_2 = 1.5, \quad a_3 = a_4 = -1.0, \quad b_3 = b_4 = 1.0
\]

The domain \( S \) is partitioned by performing \( m_1 \) and \( m_2 \) subdivisions along the state components \( x_1 \) and \( x_2 \), respectively, and \( m_3 \) and \( m_4 \) subdivisions along the parameter components \( p_1 \) and \( p_2 \), respectively. The coefficients \( w \) of the \( \beta \)-basis were derived from a set of samples of \( f \) corresponding to a regular grid of \( n_1 \times n_2 \times n_3 \times n_4 \) points over the domain \( S \). We obtained three PWL approximations \( f_{PWL} \) of \( f \):

(B1) \( m_1 = m_2 = 4, \quad m_3 = m_4 = 2, \quad n_1 = n_2 = 8, \quad n_3 = n_4 = 4 \);

(B2) \( m_1 = m_2 = 6, \quad m_3 = m_4 = 2, \quad n_1 = n_2 = 8, \quad n_3 = n_4 = 4 \);

(B3) \( m_1 = m_2 = 9, \quad m_3 = m_4 = 3, \quad n_1 = n_2 = 12, \quad n_3 = n_4 = 5 \).
Figure 3 shows the point-by-point approximation errors in the state space for the first component of the vector field \( e_1 = f_{PWL1} - f_1 \) for parameter values at point \( b \) in Fig. 2 and for the PWL approximations (B1), (B2), and (B3), respectively. It is evident that the approximation accuracy increases (i.e., the overall approximation error decreases) from (B1) to (B3).

Figure 3. Plots of the point-by-point approximation errors in the state space for the first component of the vector field \( e_1 = f_{PWL1} - f_1 \) for parameter values at point \( b \) and for PWL approximations (B1), (B2), and (B3), respectively. The axis limits are the same for all plots.

Figure 4 shows some phase portraits obtained by numerically integrating the dynamical system having on the right-hand side either \( f \) or one of the three \( f_{PWL} \). The phase portraits were achieved for the parameters set to the values of points \( a \) (first row of figures), \( b \) (second row), and \( c \) (third row) in Fig. 2. The first column of figures shows the phase portraits of the original system (3), and the second, third, and fourth columns show those of the (B1), (B2), and (B3) approximations, respectively. For all the trajectories, we chose as starting points the following pairs \((x_1, x_2)\): \((\pm 1.4, \pm 1.4)\), \((0.3, 0.3)\), and \((-0.4, -0.4)\). Referring to the first column, it is easy to conclude that

- approximation (B1) is unacceptable: at the point \( a \), there is a stable limit cycle instead of the stable equilibrium point, whereas, at the other two points \( b \) and \( c \), the only attractor is an equilibrium point (hence, this PWL approximation is not qualitatively similar to the original system);

- approximation (B2) is qualitatively good: the attractors/repellors of the original system remain for all points \( a, b, \) and \( c \) in the parameter space, even if they change their shapes/positions in the state space; in particular, the presence of the unstable cycle at the point \( b \) is evident (it separates the basin of attraction of the stable focus from that of the stable limit cycle);

- approximation (B3) is very good from both the qualitative and quantitative points of view.

3.3. Bogdanov–Takens (double-zero) bifurcation

The Bogdanov–Takens (double-zero) bifurcation can be described by the following polynomial normal form:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= p_1 + p_2 x_1 + x_1^2 - x_1 x_2
\end{align*}
\]

(4)

Again, in this case, \( n = 2 \) and \( q = 2 \).
Figure 4. Phase portraits corresponding to the parameter values at points a (first row), b (second row), and c (third row). The first column shows the phase portraits for the original system (3), and the second, third, and fourth columns show those for the (B1), (B2), and (B3) PWL approximations, respectively.

Figure 5. Bifurcation diagram for the normal form (4). The coordinates of the four black points a, b, c, and d are: a = (0.2, -0.7), b = (0.05, -0.7), c = (-0.05, -0.7), and d = (-0.2, -0.7).
Figure 5 shows the bifurcation diagram for the normal form (4). To the right of \( T \) (region A in Fig. 5), system (4) does not have any equilibria, whereas two equilibria, a node \( E_- \) and a saddle \( E_+ \), appear if one crosses \( T \) from right to left. The equilibrium \( E_- \) undergoes a node-to-focus transition (which is not a bifurcation) through a curve (not shown in Fig. 5) located in region B between \( T^- \) and the vertical axis \( p_1 = 0 \). The lower half \( H \) of the axis \( p_1 = 0 \) marks a Hopf bifurcation of \( E_- \), which originates (if one crosses from right to left) a stable limit cycle in region C, whereas \( E_+ \) remains a saddle. If one further decreases \( p_1 \), the cycle grows and approaches the saddle; it turns into a homoclinic orbit when it touches the saddle. The locus of the points where the cycle turns into a homoclinic orbit is the curve \( P \). Then, in region D there are no cycles but only an unstable focus (node) and a saddle, which collide and disappear at the branch \( T^+ \) of the fold curve.

The PWL approximations \( f_{PWL} \) of the vector field \( f \) are obtained over the domain

\[
S = \{ z \in \mathbb{R}^3: a_1 \leq x_1 \leq b_1, \ a_2 \leq x_2 \leq b_2, \ a_3 \leq p_1 \leq b_3, \ a_4 \leq p_2 \leq b_4 \}
\]

with

\[
a_1 = a_2 = -1.5, \ b_1 = b_2 = 1.5; \ a_3 = a_4 = -1.0, \ b_3 = b_4 = 1.0
\]

The domain \( S \) is partitioned by performing \( m_1 \) and \( m_2 \) subdivisions along the state components \( x_1 \) and \( x_2 \), respectively, and \( m_3 \) and \( m_4 \) subdivisions along the parameter components \( p_1 \) and \( p_2 \), respectively. The coefficients \( w \) of the \( \beta \)-basis were derived from a set of samples of \( f \) corresponding to a regular grid of \( n_1 \times n_2 \times n_3 \times n_4 \) points over the domain \( S \). In particular, as the vector field \( f \) is linear in the dimensional component \( p_1 \), we can fix \( m_3 = 1 \) and \( n_3 = 4 \). By doing so, we obtained three PWL approximations \( f_{PWL} \) of \( f \):

- (C1) \( m_1 = m_2 = 3, \ m_3 = 1, \ m_4 = 2, \ n_1 = n_2 = 8, \ n_3 = 4, \ n_4 = 6 \);
- (C2) \( m_1 = m_2 = 5, \ m_3 = 1, \ m_4 = 3, \ n_1 = n_2 = 12, \ n_3 = 4, \ n_4 = 8 \);
- (C3) \( m_1 = m_2 = 9, \ m_3 = 1, \ m_4 = 4, \ n_1 = n_2 = 20, \ n_3 = 4, \ n_4 = 10 \).

Figure 6 shows the point-by-point approximation errors in the state space for the second component of the vector field \( (e_2 = f_{PWL2} - f_2) \) for parameter values at point \( b \) and for the PWL approximations (C1), (C2), and (C3), respectively. It is evident that in this case the approximation accuracy again increases (i.e., the overall approximation error decreases) from (C1) to (C3).

![Figure 6](image_url)

The phase portraits shown in Fig. 7 were obtained by fixing the parameters at the values of points a (first row of figures), b (second row), c (third row), and d (fourth row). The
last three columns show the phase portraits for the PWL approximations (C1), (C2), and (C3), respectively. The comparisons of the last three columns with the first column (phase portraits of the original system (4)) show that only approximation (C3) is both qualitatively and quantitatively very good. Approximation (C1) is already unsatisfactory from a qualitative point of view: for instance, it has a stable equilibrium point and a saddle for \( p_1 \) and \( p_2 \) fixed at \( a \), and a stable limit cycle for \( p_1 \) and \( p_2 \) fixed at \( d \). On the other hand, approximation (C2) is qualitatively good, but its phase portraits exhibit some quantitative differences, as compared with the corresponding phase portraits in the first column (e.g., see the case corresponding to point c).

![Figure 7. Phase portraits corresponding to the parameter values at points a (first row), b (second row), c (third row), and d (fourth row). The first column shows the phase portraits for the original system (4), and the second, third, and fourth columns show those for the (C1), (C2), and (C3) PWL approximations, respectively.](image-url)
4. Continuation analysis

The results presented in the previous section show that the overall qualitative behavior of a dynamical system is mimicked by a PWL approximation characterized by a relatively small number of \( m_i \)'s, i.e., a relatively small number \( N \) of basis functions. Of course, too rough a partition of the domain causes the qualitative behavior of the approximate system to differ from the original one. If one increases the number of subdivisions \( m'_i \)'s along some dimensional component of the domain \( S \) (and thus the number of basis functions), the equivalence shifts from qualitative to quantitative. This statement is further corroborated by the results of the bifurcation analysis carried out for the best PWL approximations of the Bautin and Bogdanov-Takens normal forms and presented in this section. Such results were obtained by resorting to the continuation packages CONTENT [15] and AUTO2000 [7]. In order to meet the smoothness requirements imposed by the continuation methods, we used a smoothed version of the function \( \gamma \) by first replacing the \( \max(0, \cdot) \) function in equation (2) with the function \( x + |x|^2 \), and then by replacing the absolute value with the following function:

\[
y(x) = \frac{2x}{\pi} \arctan(ax),
\]

where the parameter \( a \) controls the degree of smoothness. Of course, the smoothed (PWS) versions of the \( \beta \)-functions still form a basis, provided that the parameter \( a \) is not too small (in our continuations, we fixed \( a = 40 \)).

The transitions from smooth to PWL to PWS functions result in two main consequences [22]. The Hopf bifurcation of an equilibrium point lying within a simplex is replaced by a degenerate fold bifurcation of cycles (the more accurate the PWL approximation, the smaller the cycles). Thus, even if the bifurcation process leading to the cycle generation is locally altered, a good PWL approximation is structurally stable. On the other hand, the PWS version of \( f_{\text{PWL}} \) admits Hopf bifurcations of any equilibrium point. Then, the first consequence of the smoothing process is the possibility of finding Hopf bifurcation curves, as in the original case. This phenomenon will affect the bifurcation scenarios of approximate systems (as we shall see later on).

The second consequence is that the smoothed version of a PWL vector field can induce different bifurcation diagrams for an equilibrium point, depending on the value of the smoothing parameter \( a \). We point out that the clusters of spurious solutions generated by relatively small values of \( a \) are made up of equilibria that turn out to be numerically very close to one another. Then, the smoothing of the vector field can give rise to numerical problems in the continuation of bifurcation curves, but does not substantially affect the dynamics of the approximate system.

This kind of phenomenon can concern the bifurcations not only of equilibria but also of any invariant of the PWL vector field. In particular, the transition from stationary to cyclic regime will be marked not by a sharp bifurcation manifold in the parameter space (Hopf bifurcation) but by a transition region where Hopf bifurcations and fold bifurcations of cycles alternate.

4.1. Bautin bifurcation

The bifurcation diagrams for approximations (B2) and (B3) are shown in Fig. 8. They are superimposed, for comparison, upon the bifurcation diagram (thick dashed curves) of the original system (3). The role played by the spurious closed regions that characterize such diagrams (and that are magnified in the sketch shown in Fig. 8(c)) can easily be understood by making reference to the two consequences of the smoothing of the PWL vector field. The smoothing induces regions of coexistence of equilibria, and a transition region (around the original Hopf) from a stable equilibrium to a stable cycle, where, as discussed above, several periodic solutions close to one another coexist. Apart from these local differences, the general layout of the bifurcation diagram is preserved, at least for sufficiently large values of the smoothing parameter \( a \) and for sufficiently accurate PWL approximations. The bifurcation diagram for approximation (B1)
is not shown in Fig. 8 as the spurious closed regions become dominant in the diagram, thus resulting in a system with much richer dynamics than in the original one (the parameter space contains more regions characterized by different qualitative behaviors, such as the new equilibria and limit cycles shown in the phase portraits in the second column of Fig. 4). On the other hand, a proper approximation by relatively small numbers $m_i$'s of subdivisions along each dimensional component of the domain $S$, like (B2), not only preserves the qualitative behaviour of the original system for given parameter values (cf. third column in Fig. 4) but also keeps the qualitative partitioning of the parameter space, while maintaining the quantitative differences (cf. first diagram in Fig. 8). Finally, if we further increase some components of vector $m$ (i.e., the number $N$ of basis functions employed for the PWL approximation), as in the case (B3), we obtain an almost perfect match between the original and approximate systems in terms of simulation capability, qualitative similarity, and structural stability.

4.2. Bogdanov-Takens bifurcation

Figure 9 shows the bifurcation diagram for the PWS version of the finest PWL approximation of system (4), i.e., (C3).

The bifurcation diagram of the original system is presented for the sake of comparison (thick dashed curves). Both the homoclinic and Hopf curves are very similar to their dashed counterparts (actually, for the Hopf curve there is a transition region as in the case of the Bautin bifurcation, but it is not exhibited here for the sake of clarity). The fold curve shows some loops, which follow directly from the PWL approximation [22]. These loops induce some topological differences between the approximated and the original system, as locally, inside the loops, the approximated system has an equilibrium point not present in the original one. However, from a global point of view, the parameter space remains partitioned into two main regions, one without equilibrium points and one with two equilibrium points, as illustrated in Fig. 9. The only difference is in the border between the two regions, being a sharp line in the original system and a more irregular narrow transition region in the approximated one (as in the case of the Bautin Bifurcation). The width of such a transition region will become thinner and thinner as

![Figure 8. Bifurcation diagrams vs. $(p_1, p_2)$ for the smoothed PWL approximations (B2) (a) and (B3) (b). The bifurcation diagram of the original system (3) is shown (thick dashed curves) for comparison. The black lines mark Hopf bifurcations, the light-gray and gray lines mark tangent bifurcations of equilibria and cycles, respectively. The black dots indicate degenerate Hopf bifurcations. (c) Sketch of the bifurcations occurring at the boundaries of the spurious closed regions (here magnified) characterizing the bifurcation diagrams for a sufficiently good PWL approximation and for $a = 40$. The thick gray curves are qualitative trajectories in the phase plane corresponding to the regions A, B, and C in the parameter plane.](image)
we increase the accuracy of the approximation.

5. Concluding remarks
By combining simulations with advanced numerical continuation techniques, we have shown that a PWL approximation can be used successfully to approximate smooth dynamical systems dependent on given numbers of state variables and parameters, with a view to their circuit implementations. In this paper we have addressed the problem of the (circuit) approximation of known dynamical systems, and we have applied the proposed technique to some topological normal forms. In a companion paper [6], the problem of the identification of black-box nonlinear dynamical systems starting from noisy time series measurements will be dealt with.

We point out that the PWL approximation method adopted manages to combine reasonable mathematical features (its main weakness being the type-1 partition of the domain, its main strength the overall simplicity) with direct circuit implementations through low-power CMOS architectures [24]. The proposed bifurcation analysis procedure is a first step towards the design of structurally stable circuits that are able to imitate nonlinear dynamical systems and can be embedded in extremely small and low-power devices.

The results reported here have shown that, if we increase the approximation accuracy to a sufficient degree, the approximate dynamical systems preserve both the dynamical (trajectories) and structural-stability (bifurcations) arrangements of the original systems. This is also true in systems admitting chaotic behaviors [22]. We point out that as of now it is not possible to find a threshold for the quantitative error in the approximation of the “static” function \( f(x; p) \) necessary to achieve good qualitative results in the approximation of the corresponding dynamical system \( \dot{x} = f(x; p) \), as static and dynamic approximations are not strictly related. In fact, the dynamical system is particularly sensitive to the accuracy of the approximation of \( f(x; p) \) in some specific regions of the domain, e.g., those corresponding to invariant manifolds, as shown in some of the examples considered.

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