Renormalization group for network models of Quantum Hall transitions

Denis Bernard and André LeClair

Service de Physique Théorique de Saclay F-91191, Gif-sur-Yvette, France.
Newman Laboratory, Cornell University, Ithaca, NY 14853.

(July 2001)

Abstract

We analyze in detail the renormalization group flows which follow from the recently proposed all orders \( \beta \) functions for the Chalker-Coddington network model. The flows in the physical regime reach a true singularity after a finite scale transformation. Other flows are regular and we identify the asymptotic directions. One direction is in the same universality class as the disordered \( XY \) model. The all orders \( \beta \) function is computed for the network model of the spin Quantum Hall transition and the flows are shown to have similar properties. It is argued that fixed points of general current-current interactions in 2d should correspond to solutions of the Virasoro master equation. Based on this we identify two coset conformal field theories \( osp(2N|2N)_1/u(1)_0 \) and \( osp(4N|4N)_1/su(2)_0 \) as possible fixed points and study the resulting multifractal properties. We also obtain a scaling relation between the typical amplitude exponent \( \alpha_0 \) and the typical point contact conductance exponent \( X_t \) which is expected to hold when the density of states is constant.

PACS numbers: 73.43.-f, 11.25.Hf, 73.20.Fz, 111.55.Ds
I. INTRODUCTION

The Chalker-Coddington network model is perhaps the simplest model believed to be in the universality class of the Quantum Hall transition \[1\]. The model can be mapped to a free Dirac fermion theory with three independent random potentials \[2\] \[3\]. The renormalization group (RG) fixed point of the disorder averaged effective action is expected to capture the critical properties of the transition.

Recently, an all orders $\beta$ function was proposed for the network model \[4\]. The main difficulty encountered in interpreting the resulting RG flows is the occurrence of singularities in the $\beta$ function. For the case of anisotropic $su(2)$, by using duality and certain topological identifications it was shown in \[3\] that all RG trajectories were regular in the sense that all flows could be extended to arbitrarily large and small length scales without encountering any true singularities. Further checks of the $\beta$ function in the case of anisotropic $su(2)$ described in \[3\] provided additional support that the $\beta$ function could be analytically continued beyond the domain of convergence of the perturbation series.

The main purpose of this work is to analyze in detail the RG flows of the Chalker-Coddington network model and also the network model for the spin Quantum Hall transition \[6–8\] and to attempt to resolve the possible singular flows using the ideas in \[3\]. There are large classes of flows that are regular, and for these we identify two universal attractive directions at large distances, referred to as $dXY$ and $gX$ below. (Some of the phases listed as distinct in \[4\] become identified since we now understand how to cross the singularities.) The $dXY$ phase is in the same universality class as a disordered $XY$ model or Gade-Wegner class \[9\] and is equivalent to the model studied in \[10\]. Unfortunately the flows in the physical domain of the network model with all couplings (which are variances of disordered potentials) positive are pathological in the sense that they encounter a true singularity after a finite scale transformation and the flow cannot be continued to larger length scales. These singular flows seem to be related to the fact that there are level 0 current algebras in these theories which arise naturally in the supersymmetric method for disorder averaging.

Putting aside the $\beta$ function, we argue that the possible fixed points of general anisotropic left/right current-current interactions should correspond to the solutions of the Virasoro master equation \[11\] which are built upon the same tensors that define the current interactions and thus preserve the global symmetries. We classify these solutions for both network models and show that they correspond to current-algebra cosets. Since this argument is independent of the $\beta$ function, it remains a possibility that a proper resolution of the singular flows could lead to these fixed points. It was argued in \[4\] that such cosets can arise as fixed points of the RG when the coupling to a current subalgebra flows to infinity and the current subalgebra is gapped out. Some of the regular flows actually realize these fixed points, in particular phase $C$ in the unphysical regime of spin network model flows to $osp(4|4)_{1}/osp(2|2)_{-2}$. By doubling the degrees of freedom as in \[10\], this may correspond to a physical model of interest. For the spin network model one solution corresponds to the coset $osp(4N|4N)_{1}/su(2)_{0}$, where $N$ is the number of copies, and this fixed point gives some of the same exponents as percolation \[8\] though the equivalence of the two theories is still a matter of debate.

For the $u(1)$ network model the only zeros of the $\beta$ function correspond to a purely random $u(1)$ gauge field. We point out that the stress tensor of this line of fixed points
corresponding to the strength of the disorder does not satisfy the Virasoro algebra, which apparently has not been noticed before. There is a special point on this line that leads to a constant density of states, and can formally be viewed as \(\text{osp}(2N|2N)_1/u(1)_0\) as far as the computation of critical exponents is concerned. (This is not a true coset since the \(u(1)_0\) does not commute with the stress tensor, however we find the coset notation convenient and will use it in the sequel.) This \(c = 0\) conformal field theory is different from the \(c = -2\) theory \(\text{osp}(2N|2N)_1/u(1) \otimes u(1)\) considered in \([12]\), nor is it the same as the theories considered in \([13,14]\).

We wish to emphasize that due to the singularities of the RG flow in the physical regime of the network models we cannot show that the above cosets genuinely arise as fixed points. However, based on our analysis of the Virasoro master equation, they remain as interesting candidates for the correct fixed point and are worth further investigation, even with the aim of ruling them out. In this paper we thus also explore the multifractal properties of the above coset theories, and show that these cosets lead to multifractality in a very natural way. This multifractality is an important constraint on the critical theory and since it is not a generic feature of the kinds of fixed points that can arise, it can serve as a criterion for ruling out some theories. In particular we show that the coset \(\text{osp}(2N|2N)_1/u(1) \otimes u(1)\) does not exhibit multifractality, whereas \(\text{osp}(2N|2N)_1/u(1)_0\) does. For the coset \(\text{osp}(4N|4N)_1/su(2)_0\) we show that the typical amplitude exponent \(\alpha_0 = 9/4\), and the typical point contact conductance exponent is \(X_t = 1\). Since the map to percolation \([8]\) does not apply to averages of higher moments of correlation functions, this result can serve as a useful test of the validity of the fixed point \(\text{osp}(4N|4N)_1/su(2)_0\). For the \(u(1)_0\) coset, it turns out \(\alpha_0 = 3\). However if one associates \(\alpha_0\) with the twist fields which modify the boundary conditions then \(\alpha_0\) is again \(9/4\) and this is close to numerical estimates of 2.26 \([15,16]\). The small errors in the most recent simulations however seem to rule out 9/4 \([16]\). We also obtain a scaling relation between \(\alpha_0\) and the typical point-contact conductance exponent \(X_t\) which we expect to hold in a theory with a constant density of states: \(X_t = 2(\alpha_0 - 2)\) \([17]\).

Our results are presented as follows. In section II we extend the computation in \([4]\) to \(N\) copies, and show, as expected, that the \(\beta\) functions are independent of \(N\). We present a remarkable strong-weak coupling duality of the \(\beta\) function. We then describe all possible flows, and describe the nature of the singular flows. In section III we extend this analysis to the network model for the spin Quantum Hall transition and find similar properties. In section IV we present our results on the multifractality. The subsections on the Virasoro master equation and multifractality are logically independent of the sections describing the RG flows and can be read separately.

**II. THE CHALKER-CODDINGTON NETWORK MODEL**

The Chalker-Coddington network model \([1]\) is expected to be in the same universality class as the Dirac Hamiltonian with random mass, potential and abelian gauge fields \([3]\):

\[
H = \begin{pmatrix}
V + m & -i\partial_\tau + A_\tau \\
-i\partial_\tau + A_\tau & V - m
\end{pmatrix}
\]  

\(\text{(2.1)}\)

where \(A, m, V\) are centered gaussian random potentials with variance \(g_a, g_m, g_v\) respectively. The physical regime of the network model thus corresponds to all \(g\) positive.
couplings \( g_a, g_v, g_m \) correspond to randomness in the link phases, the flux per plaquette and the tunneling at the nodes respectively. This model also arose in the work [3]. The case \( g_m = g_v = 0 \) was studied in [4,13,19], and the case \( g_m + g_v = 0 \) in [10]. The one-loop \( \beta \) function for all \( g \)'s non-zero was computed in [19]. The all-orders \( \beta \) function was proposed in [4].

A. N-Copies effective action.

Following the conventions in [19,4] we study the action

\[
S = \int \frac{d^2x}{2\pi} \left[ \bar{\psi}_- (\partial_z - iA_z) \psi_+ + \psi_- (\partial_\bar{z} - iA_\bar{z}) \psi_+ 
- iV (\bar{\psi}_- \psi_+ + \psi_- \bar{\psi}_+) 
- i m (\bar{\psi}_- \psi_+ + \psi_- \bar{\psi}_+) \right]
\] (2.2)

where \( z = x + iy, \bar{z} = x - iy \). The above action corresponds to \( S = i \int \psi^* H \psi \) where \( H \) is the Hamiltonian of the network model. To study the Green functions at energy \( \mathcal{E} \), one lets \( H \to H - \mathcal{E} \) which leads to a term in the action

\[
S_\mathcal{E} = i \int \frac{d^2x}{2\pi} \mathcal{E} \Phi_\mathcal{E}, \quad \Phi_\mathcal{E} = \bar{\psi}_- \psi_+ + \psi_- \bar{\psi}_+ \] (2.3)

The retarded and advanced Green functions then correspond to \( \mathcal{E} = E \pm i\epsilon \) with \( \epsilon \) small and positive.

To study the delocalization transition one is interested in the critical points of the disorder averaged theory at \( \mathcal{E} = 0 \). Since \( \Phi_\mathcal{E} \) corresponds to various mass terms, the theory cannot be critical when \( \mathcal{E} \neq 0 \). The \( \beta \) function we determine below is for \( \mathcal{E} = 0 \), and for this reason it is independent of whether the various copies are retarded or advanced. One can in fact convert advanced copies into retarded ones by flipping the sign of the right-moving fields \( \bar{\psi}_\pm \to -\bar{\psi}_\pm \). Since this does not change the \( \bar{\psi}_\pm \) operator products, the \( \beta \) function is unchanged. Once one flows to a critical point, one needs to reintroduce \( \mathcal{E} \) in order for instance to compute conductivities. This can be compared with the sigma model approach. The latter approach was applied to the coupling sub-manifold \( g_m + g_v = 0 \) of the network model in [10]. An important point is that the saddle point approximation that leads to a sigma model occurs in the \( \mathcal{E} = 0 \) theory. Thus one expects that our RG flows should in principle capture these saddle points.

In order to study disorder averages of mixed products of retarded and advanced Green functions we introduce \( N \) copies of the action (2.2), with fields \( \psi_\alpha^\pm, \bar{\psi}_\alpha^\pm \) where \( \alpha = 1, \ldots, N \) is a copy, or flavor index. The disorder averaging is then performed using the supersymmetric method by introducing ghosts \( \beta_\alpha^\pm, \bar{\beta}_\alpha^\pm \). To present the effective action in a compact form, we introduce the 2N component vectors \( \Psi_{\pm}^a, a = 1, \ldots, 2N \), and we let \( \Psi_{\pm} = (\psi_{\pm}^1, \ldots, \psi_{\pm}^N, \beta_{\pm}^1, \ldots, \beta_{\pm}^N) \) and similarly for \( \bar{\Psi}_{\pm} \). We define a grade \([a] = 0, 1\) where \([a] = 1\) for the fermionic components \( \psi \) and \([a] = 0\) for the bosonic components \( \beta \). The effective (disorder averaged) action is then

\[
S = S_{\text{tree}} + \int \frac{d^2x}{2\pi} \left( g_+ \mathcal{O}^+ + g_- \mathcal{O}^- + g_\alpha \mathcal{O}^\alpha \right)
\] (2.4)
where \( g_\pm = g_v \pm g_m \), and \( S_{\text{free}} \) the free \( c = 0 \) conformal field theory of the \( \psi, \beta \) fields,

\[
S_{\text{free}} = \int \frac{d^2 \tau}{2\pi} \left( \tau_- \partial_\psi \tau_+ + \overline{\tau}_- \partial_{\overline{\psi}} \overline{\tau}_+ \right)
\]

(2.5)

where \( \tau_- \psi_+ = \sum_a \overline{\tau}_- \tau^a_+ \), etc. The operator product expansion (OPE) implied by the free action (2.5) is

\[
\tau^a_+ (z) \tau_-^b (0) \sim \frac{1}{z} \delta^{ab}, \quad \tau^a_- (z) \tau^b_+ (0) \sim -\left(\frac{-1}{z}\right)^{[a]} \delta^{ab},
\]

(2.6)

and similarly for \( \overline{\tau} \). In order to compute the \( \beta \) function we express the perturbing operators in eq. (2.4) as left-right current-current interactions. Define the left-moving currents

\[
J^{ab}_\pm = \tau^a_\pm \tau^b_\pm \quad \text{and} \quad H^{ab}_\pm = \tau^a_\pm \tau^b_+,\n\]

(2.7)

and similarly for the right movers \( J^{ab}_\pm \) and \( H^{ab}_\pm \). The perturbing operators can then be written as

\[
\mathcal{O}^+ = \frac{1}{2} \left( (\tau_- \psi_+)^2 + (\overline{\tau}_- \overline{\psi}_+)^2 \right) = \frac{1}{2} \sum_{a,b=1}^{2N} (-)^{[a]} \left( J^{ab}_+ \overline{H}^{ba} + J^{ba}_- H^{ab} \right)
\]

\[
\mathcal{O}^- = (\tau_- \psi_+)(\overline{\tau}_- \overline{\psi}_+) = \sum_{a,b=1}^{2N} (-)^{[a]} H^{ab} \overline{H}^{ba}
\]

\[
\mathcal{O}^a = (\tau_- \psi_+)(\overline{\tau}_- \overline{\psi}_+) = \sum_{a,b=1}^{2N} (-)^{[a]+[b]} H^{ab} \overline{H}^{ba}
\]

(2.8)

The currents (2.7) satisfy the \( osp(2N|2N)_k \) current algebra with level \( k = 1 \).

B. \textbf{Beta functions and duality.}

We now extend the computation in [4] to \( N \)-copies by using the previous compact notation. The resulting \( \beta \) function is independent of \( N \). We then describe the strong-weak coupling duality of the \( \beta \) function that is important for extending the flows to all scales.

The effective action can be expressed as

\[
S = S_{\text{wzw}}^G + \int \frac{d^2 \tau}{2\pi} \sum_A g_A \mathcal{O}^A \quad \text{with} \quad \mathcal{O}^A = d^{ab}_{\alpha \beta} T^\alpha T^\beta
\]

(2.9)

where \( S_{\text{wzw}}^G \) is the action for the conformal WZW model with current algebra symmetry \( G_k \), where \( k \) is the level, and \( J^a \) are the \( G_k \) currents. For the network model \( G_k = osp(2N|2N)_1 \), and the tensors \( d^{ab}_{\alpha \beta} \) are implicitly defined by eqs.(2.8). The \( \beta \) function proposed in [20] is expressed in terms of some OPE coefficients \( C, D, \tilde{C} \). Let \( T^A \) be the left-moving operator,

\[
T^A (z) = d^{ab}_{\alpha \beta} J^a (z) J^b (z)
\]

(2.10)

Then the RG data can be computed from the OPE’s.
\[ O^A(z, \overline{z})O^B(0) \sim \frac{1}{z^2} C_C^{AB} \ O^C(0) \] (2.11)

\[ T^A(z)O^B(0) \sim \frac{1}{z^2} \left( 2kD_C^{AB} + \tilde{C}_C^{AB} \right) O^C(0) \]

Specializing to the $N$-copy network model one finds that $C, D, \tilde{C}$ are independent of $N$ due to the fact that $\sum_{a=1}^{2N} (-)^a = 0$, which is equivalent to the statement that the superdimension of osp$(2N|2N)$ is zero. As expected, the $\beta$ function is then identical to the $N = 1$ result in [4]:

\[
\beta_{g_+} = \frac{8g_+(2g_+(g_+^2 + 4) + (2 - g_-(g_+^2 + 2g_-))}{(4 - g_+^2)(2 - g_-)^2}
\]

\[
\beta_{g_-} = \frac{8g_-^2(2 + g_-)^2}{(4 - g_-^2)^2}
\]

\[
\beta_{g_a} = \frac{4((g_+^2 - g_-^2)(16 - g_+^2 g_-^2) + 4g_+g_-^2(2 + g_-)(2 - g_-)^2)}{(4 - g_-^2)^2(2 - g_-)^2}
\] (2.12)

Here $\beta = dg/d\tau$ where $\tau = \log r$ is the RG ‘time’ and $r$ is a length scale. The only zero of the $\beta$ function is at $g_+ = 0$ with $g_a$ arbitrary. We remark that $\beta_{g_-}$ is always positive for real couplings so that $g_-$ is always increased by RG transformations.

The strong-weak coupling duality of the $\beta$ function is of the following kind. Let $g^*$ denote dual couplings which are functions of $g$. Suppose

\[
\beta^*(g^*) = \frac{\partial g^*}{\partial g} \beta(g) = -\beta(g \rightarrow g^*)
\] (2.13)

The above relation implies that if $g(r)$ is a solution of the RG equations, then so is $g^*(r_0/r)$ for some $r_0$. Furthermore, if $g$ is self-dual at some scale $r_0$, i.e. $g = g^*$, then the ultra-violet (UV) and infra-red (IR) values of $g$ are related by duality: $g_{IR} = g^*_{UV}$.

The $\beta$ function (2.12) remarkably has such a duality with

\[
g^*_+ = \frac{4}{g_+}, \quad g^*_- = \frac{4}{g_-}, \quad g^*_a = -\frac{4g_a}{g_-^2}
\] (2.14)

The self-dual points $g = g^*$ are $(g_+, g_-, g_a) = (\pm 2, \pm 2, 0)$.

The above duality is one ingredient we will use to resolve some difficulties encountered in [4] in completely interpreting the flows, as was done for anisotropic $su(2)$ in [3]. But as we shall see not all RG flows may be resolved in this way. Also, unlike the $su(2)$ case, we have not found an RG invariant for the above $\beta$ functions.

**C. Virasoro master equation.**

The action (2.9) has a well-defined classical stress tensor $T^{\text{class}}_{\mu\nu}$. Introducing a 2d metric $g^{\mu\nu}$, then $T^{\text{class}}_{\mu\nu} = -4\pi \delta S/\delta g^{\mu\nu}$. One finds that this classical stress tensor is traceless, i.e. $g^{\mu\nu}T^{\text{class}}_{\mu\nu} = 0$ and the left-moving conformal stress tensor $T = T_{zz}$ is

\[
T = T_{\text{class}} - g_A T^A
\] (2.15)
where the $T^A$ are defined in eq. (2.10), and $T^\text{class}_{\text{wzw}}$ is the classical stress tensor for the WZW model which is the $k \to \infty$ limit of the affine Sugawara stress tensor.

In the quantum theory there are order $g^2$ corrections to the trace of the stress tensor and the conformal invariance is thus broken. Based on the form (2.13) it is natural to suppose that if the theory flows under RG to a conformally invariant fixed point, then at the fixed point the stress-tensor takes the form

$$T_{\text{fixed point}} = T_{Gk} - \delta T,$$

with $\delta T = h_A T^A$ (2.16)

where $T_{Gk}$ is the affine Sugawara stress-tensor and $h_A$ are constants which are unknown functions of the fixed point values of $g_A$.

The condition that a stress tensor $T$ satisfies the Virasoro algebra, i.e.

$$T(z)T(0) \sim \frac{c/2}{z^4} + \frac{2}{z^2} T(0) + ....$$

(2.17)

is the so-called Virasoro master equation studied extensively in [11]. It is known that solutions come in pairs (K-conjugation), such that the sum of the stress tensors for a given pair is the affine-Sugawara stress tensor $T_{Gk}$. Thus $\delta T$ (and consequently $T_{\text{fixed point}}$) must be a solution of the master equation, which reads

$$2k D_{AB}^C h_A h_B + C_{AB}^C h_A h_B + 2 \tilde{C}_{AB}^C h_A h_B = h_C$$

(2.18)

where the coefficients $D^{AB}_C, C_{AB}^C$ and $\tilde{C}_{AB}^C$ are the same as those determining the $\beta$ function defined in eqs. (2.11). Though it is intriguing that the same coefficients $C, \tilde{C}, D$ determine both the $\beta$ function and the Virasoro master equation, we cannot show in complete generality that solutions of the master equation correspond to zeros of the $\beta$ function.

Using the $C, \tilde{C}, D$ computed in Ref. [4] one finds 2 pairs of solutions to eq. (2.18). The fact that the solutions have a simple interpretation serves as a check of the coefficients $C, \tilde{C}, D$. The first pair of solutions is

$$T_{\text{osp}(2N|2N)_1} = \frac{1}{2} \left( T^a - T^+ \right), \quad T_0 = 0$$

(2.19)

Here $T_{\text{osp}(2N|2N)_1}$ is the affine-Sugawara stress tensor for the full current algebra and $T_0$ for the empty coset $osp(2N|2N)_1/osp(2N|2N)_1$. The other pair of solutions is

$$T_{\text{gl}(N|N)_1} = \frac{1}{2} \left( T^a - T^- \right), \quad T_{\text{osp}(2N|2N)_1/gl(N|N)_1} = T^- - \frac{1}{2} T^a - \frac{1}{2} T^+$$

(2.20)

corresponding to the sub-current algebra $gl(N|N)_1$ of $osp(2N|2N)_1$ and the indicated coset. ($T_{G/H} = T_{G} - T_{H}$). The two terms in $T_{\text{gl}(N|N)_1}$ correspond to the two independent quadratic Casimirs of $gl(N|N)$ [21].

One can form a level-1 representation of both $osp(2N|2N)$ and $gl(N|N)$ with the same field content, thus both $T_{\text{osp}(2N|2N)_1}$ and $T_{\text{gl}(N|N)_1}$ are equivalent to the free-field stress tensor $T_{\text{free}}$ of the $\psi, \beta$ fields. This leads to the identity

$$T_{\text{free}} = T_{\text{gl}(N|N)_1} = T_{\text{osp}(2N|2N)_1}, \quad \Rightarrow T^- - \frac{1}{2} T^a - \frac{1}{2} T^+ = 0$$

(2.21)
This is equivalent to the statement $T_{\text{osp}(2N|2N)_{1/\text{gl}(N|N)_{1}}}=0$.

It is important to note that the purely random $u(1)$ with only $g_a$ not equal to zero does not appear as a solution of the master equation and is thus not a consistent stress-energy tensor. In terms of currents,

$$T_{u(1)_0} \equiv \frac{1}{2} T^a = \frac{1}{2} H^2, \quad H \equiv \sum_{a=1}^{2N} (-)^{|a|} H^{aa} = H_\beta - H_\psi$$

(2.22)

where

$$H_\beta = \sum_{\alpha=1}^{N} \beta_+^\alpha \beta_-^\alpha, \quad H_\psi = \sum_{\alpha=1}^{N} \psi_+^\alpha \psi_-^\alpha$$

(2.23)

These currents satisfy two $u(1)$ subalgebras:

$$H_\beta(z) H_\beta(0) \sim -H_\psi(z) H_\psi(0) \sim -\frac{N}{z^2}$$

(2.24)

Note that $H(z)H(0) \sim \text{reg}$., i.e. since $H$ satisfies a $u(1)$ current algebra at level-0, and $T_{u(1)_0}$ is formally the Sugawara form for $u(1)_0$. The level being zero is what prevents $T_{u(1)_0}$ from being a good stress-tensor.

Consider the class of stress-tensors

$$T_h = T_{\text{free}} - h T_{u(1)_0} = -\frac{1}{2} T^z + (1-h) T_{u(1)_0}$$

(2.25)

The stress-tensor for the pure random $u(1)$ is of this form with $h = 2g_a$, and corresponds to the only zero of the $\beta$ function. This stress tensor satisfies:

$$T_h(z) T_h(0) \sim \frac{2}{z^2} \left( T_h - h T_{u(1)_0} \right), \quad T_h(z) T_{u(1)_0}(0) \sim \frac{2}{z^2} T_{u(1)_0}(0), \quad T_{u(1)_0}(z) T_{u(1)_0}(0) \sim \text{reg}.$$  

(2.26)

The Virasoro algebra is spoiled by the $h$ term in the first equation. We should thus include this $c=0$ generalization of Virasoro as a possible fixed point.

As far as the computation of conformal scaling dimensions is concerned we can formally view the stress tensors (2.25) as the coset “$\text{osp}(2N|2N)_{1/u(1)_0}$”. However it is not a true coset since the $u(1)_0$ does not commute with $T_h$. The point $h = 1$ is special, in that $T_{u(1)_0}$ is precisely canceled and $T_{h=1}$ reduces to $-T^z/2$, and this leads to a constant density of states. (See section IV.) Since the notation is convenient and descriptive we will refer to the $T_{h=1}$ theory as $\text{osp}(2N|2N)_{1/u(1)_0}$. We emphasize that the latter theory is not the same as the $c = -2$ theories $\text{osp}(2N|2N)_{1/u(1)} \otimes u(1)$ considered in [13,12,10,4] where the $u(1)$’s are generated by $H_\psi$ and $H_\beta$.

D. Singular RG trajectories in the Chalker-Coddington network model.

There exists a small category of RG trajectories which cannot be extended consistently to arbitrarily large RG time. This is probably linked to the level zero current algebras appearing in these disordered systems.
These pathologies concern trajectories, or their duals, which enter the perturbative physical domain, the latter defined by requiring that the variances \( g_v, \ g_m, \ g_a \) are all positive. In a finite RG time these singular trajectories converge to the singular locii \( g_\pm = 2 \) with \( g_a > 0 \).

Let us first argue that the perturbative physical domain is stable under renormalization until reaching the singularities at \( g_\pm = \pm 2 \). The boundaries of this domain are \( g_a = 0, \ g_+ = 2 \) and \( g_+ \pm g_- = 0 \). From eq.\((2.12)\), one verifies that \( \beta_{g_a} > 0 \) for \( g_a > 0 \) and \( 0 < |g_\pm| < 2 \). Hence, \( g_a \) remains positive as long as the RG trajectories are in this domain. Similarly, one verifies that the scalar product \( \beta_g \cdot n \), with \( n \) the normals to the boundaries pointing inwards, is positive on the boundaries. Thus, RG flows which are initially inside the domain cannot cross the boundaries and stay within it.

Since \( g_- \) increases, the only direction in which trajectories entering the physical domain from elsewhere in the phase space could go is towards the singular point \( g_\pm = 2 \). Furthermore, since in this domain \( g_a \) is always increasing these physical trajectories reach \( g_\pm = 2 \) with \( g_a > 0 \). This typical behavior may be checked by looking at the asymptotic behavior in the vicinity of the pole \( (g_+, g_-) = (2 + \epsilon_+, 2 + \epsilon_-) \). Here the dominant terms in the \( \beta \) functions are

\[
\beta_{\epsilon_+} \simeq -\frac{64 g_a}{\epsilon_+ \epsilon_-^2}, \quad \beta_{\epsilon_-} \simeq \frac{32}{\epsilon_+^2}, \quad \beta_{g_a} \simeq \frac{16 g_a}{\epsilon_-^2}
\]

Integrating these one finds

\[
g_a \simeq C \exp \frac{\epsilon_-}{2}, \quad \epsilon_+ \simeq C' \exp 2C/\epsilon_-
\]

with \( C \) and \( C' \) two integration constants. Eq.\((2.28)\) shows that \( g_a \) flows to a non-vanishing constant as \( \epsilon_- \to 0 \), while \( \epsilon_+ \) vanishes exponentially. Since eq.\((2.28)\) involves two free parameters these trajectories are generic enough in this domain.

Unfortunately, because these physical flows reach the singularity at points distinct from the self-dual point \( (g_\pm = 2, \ g_a = 0) \), they cannot be extended smoothly beyond this point using duality and possible additional identifications as explained in [5]. Thus these RG trajectories are singular at a finite RG time, i.e. after a finite scale transformation. This pathological behavior is in contradiction to what is expected for a renormalizable field theory. The singular flows are depicted in figure 3. Trajectories can enter the physical domain either from the \( g_+ = 0 \) plane or from the point \( g_\pm = \pm 2, \ g_a = 0 \).

E. Regular RG trajectories in the Chalker-Coddington network model and phase diagram.

Most of the flows are actually not singular. As was done for the anisotropic \( su(2) \) model in ref. [3], one may use duality arguments to construct smooth extensions such that these flows may be extended to arbitrarily large scale. The main steps in resolving these flows are:

(i) RG flows to the singular self-dual points are extended beyond it using the duality and, (ii) flows where the couplings blow up in finite RG time but do not cross self-dual points are extended with certain topological constructions with the \( \beta \) function serving as a consistent patching condition. In this section we describe the typical behaviors of these trajectories and identify the resulting asymptotic phases. As we will show, for this class of RG flows, there are only two such directions referred to as \( dXY \) and \( gX \) below.
To begin describing these regular flows, consider the $\beta$ function when $g_+ = 0$. There,

$$
\beta_{g_-} = 0, \quad \beta_{g_a} = -4g_-^2/(2 - g_-)^2 \tag{2.29}
$$

Since $\beta_{g_-} = 0$, we can consider the flows as originating or terminating at $g_+ = 0$. On the $g_+ = 0$ plane, the flows all originate from $g_a = +\infty$ since there $g_a$ always decreases. Near $g_+ = 0$, one has

$$
\beta_{g_+} \simeq \frac{16g_+(g_a - g_{a0})}{(2 - g_-)^2}, \quad g_{a0} \equiv g_-(2) - 2/4 \tag{2.30}
$$

Thus, if $g_a > g_{a0}$ the flows are away from the $g_+ = 0$ plane, otherwise they are attracted to it. Since $\beta_{g_a} < 0$ and $g_a$ decreases in this domain, trajectories which are attracted to the plane $g_+ = 0$ are more and more attracted to it, while those which flow away from it may flow back toward it if $g_a - g_{a0}$ is not large enough. This implies that some trajectories form loops, in the sense that they begin and end at different points on the $g_+ = 0$ plane but with different values of $g_-$ and $g_a$. (See Figure 1.) Notice that $g_a^* (g_-) = g_{a\infty} (g_-)$ in agreement with duality.

Near $g_+ = \infty$ one finds $\beta_{g_-} = 0$, $\beta_{g_a} = -4g_-^2/(2 - g_-)^2$ and

$$
\beta_{g_+} \simeq -\frac{16g_+(g_a - g_{a\infty})}{(2 - g_-)^2}, \quad g_{a\infty} \equiv (g_- - 2)/2 \tag{2.31}
$$

Thus if $g_a > g_{a\infty}$, $g_+$ decreases away from $\infty$, otherwise it continues to flow off to infinity. Since $\beta_{g_a} < 0$, $g_a$ decreases there and trajectories flowing off will escape more and more to infinity, while those with $g_+$ decreasing may be repelled back to infinity if $g_a - g_{a\infty}$ is not large enough, again forming loops that are simply duals of the loops near $g_+ = 0$. (See Figure 1.)

Flows that do not loop back to $g_+ = 0, \infty$ eventually encounter the poles in the $\beta$ function in a finite RG time. Let us consider the ratios $\partial g_+ / \partial g_- = \beta_{g_+} / \beta_{g_-}$ and $\partial g_+ / \partial g_a = \beta_{g_+} / \beta_{g_a}$. They determine the direction of the flows. One finds that, except at the self dual points, both $\partial g_+ / \partial g_- = 0$ and $\partial g_+ / \partial g_a = 0$ when $g_+$ approaches the singularities at $g_+ = \pm 2$. This implies that the flows near the poles at $g_+ = \pm 2$ are tangent to the $g_+ = \pm 2$ planes since $\beta_+ \ll \beta_-, \beta_a$. Similarly $\partial g_- / \partial g_+ = 0$ and $\partial g_- / \partial g_a = 0$ when $g_- = \pm 2$. Thus $\beta_- \ll \beta_+, \beta_a$ and the flows are tangent to the $g_- = \pm 2$ planes near $g_- = \pm 2$. Consequently if the flows indeed cross the singular planes, they must do so at the self-dual points of $g_\pm$. In order for the flows to continue smoothly through the poles one also needs $\partial g_+ / \partial g_-$ and $\partial g_+ / \partial g_a$ to be finite at the poles. Examining the latter one finds that this can occur only when $g_a = 0$. We thus conclude that if the flows indeed cross the singular planes at finite angles, they must do so at the self-dual points $g = g^*$ for all $g$ simultaneously.

The crossing of the singularities may be described more precisely. Consider first the pole at $g_\pm = \pm 2$. Looking for flows approaching the self-dual point $g_\pm = \pm 2$, $g_a = 0$, one may search for an asymptotic expansion of $2 + g_- $ and $g_a$ as functions of $\epsilon_+ = g_+ - 2$. To lowest order, the equations for the trajectories are:

$$
ge_- = -2 - \epsilon_+ + 2Q_- \epsilon_+^4 - Q_- \epsilon_+^5 + O(\epsilon_+^6) \tag{2.32}
ge_a = Q_- \epsilon_+^4 - Q_- \epsilon_+^5 / 2 + O(\epsilon_+^6)
$$
with $Q_-$ a free parameter. These flows form a continuous set of trajectories. However since this is only a one parameter family, they are not the most generic ones. The $\beta$ function for $\epsilon_+$ is smooth, $\beta_{g_+} = -2 - \epsilon_+ + \cdots$ implying that $\epsilon_+$ decreases. So these flows, coming from the domain $g_- < 2, g_+ > 2$, reach the self-dual point in a finite RG time and then cross over. (See Figure 2).

The flows which cross the upper self-dual point at $g_+ = 2, g_a = 0$ either start with $g_+ < 2$ or $g_+ > 2$. In the first case, their trajectories are described by

$$g_- = 2 + \epsilon_+ + Q_+ \epsilon_+^2 + Q_+^2 \epsilon_+^3 + O(\epsilon_+^4)$$
$$g_a = Q_+ \epsilon_+^2/2 + Q_+^2 \epsilon_+^3/2 + O(\epsilon_+^4)$$

with $\epsilon_+ = g_+ - 2$ and $Q_+$ a free parameter labeling them. The $\beta$ function $\beta_{g_+}$ possesses a double pole, $\beta_{g_+} = 32(1 + (Q_+ - 2)\epsilon_+ + \cdots)/\epsilon_+^2$ so that $\epsilon_+^2 = 96(\tau - \tau_*) + \cdots$. These flows reach the self-dual point from $g_+ < 2$ in a finite RG time but admit an extension beyond that point. As depicted in Figure 3 they then flow off to infinity. In the second case, the trajectory equations are

$$g_- = 2 - \epsilon_+ + (4\tilde{Q}_+ + 1)\epsilon_+^2/2 + \cdots$$
$$g_a = \tilde{Q}_+ \epsilon_+^2 + \cdots$$

with $g_+ = 2 + \epsilon_+$. Again the $\beta$ function for $\epsilon_+$ has a double pole $\beta_{g_+} = -32(1 + (4\tilde{Q}_+ + 1)\epsilon_+ + \cdots)/\epsilon_+^2$ but with opposite sign. Hence, $\epsilon_+$ now decreases and, after going through the self-dual point, these trajectories flow towards the axis $g_+ = 0$. (See Figure 2.)

Let us now show how one may resolve the blow-ups in finite RG time. This concerns flows toward $g_-, g_a = \infty$ that are dual to those crossing the $g_+$ axis. Indeed, any trajectory that passes through $g_- = 0$ is mapped by duality to two separate trajectories at $g_- = \pm \infty$. Their asymptotic at infinity are

$$g_+ \to \text{const.}, \quad g_a \simeq \text{const}'.g_+^2, \quad g_- \simeq \frac{(4 - g_+^2)^2}{8g_+^2(\tau - \tau_*)}$$

where $\tau_*$ is a constant. Eq. (2.33) shows that in a finite time $\tau_*$ the flow is to $(g_+, g_-, g_a) \to (\text{const.}, \infty, \infty)$. We have to extend the flows to larger scales. Since the trajectories are continuous at $g_- = 0$, it is natural to demand that their dual be continuous at $g_- = \pm \infty$ so that flows towards $(g_-, g_a) = (\infty, \infty)$ continue at $(g_-, g_a) = (-\infty, \infty)$. Imposing this continuity requires identifying points at infinity by changing the topology of the phase space in a way consistent with the $\beta$ functions. To be more precise let us map points at infinity to finite distance by defining new coordinates $h_+, h_a$ by $h_+ = g_+, h_- = 1/g_-$ and $h_a = -g_a/g_-^2$. We identify the points with coordinates $(h_+, h_a, h_- = 0^+)$ and $(h_+, h_a, h_- = 0^-)$ with $h_a \neq 0$, or alternatively

$$(g_+, g_-, g_a) \equiv (g_+, -g_-, g_a), \quad \text{as} \ |g_-| \to \infty, \ g_a/g_+^2 \neq 0 \ \text{fixed}$$

This is consistent with RG flows as the $\beta$ functions $\beta_h = \beta_g(\partial h/\partial g)$ in the new coordinates are continuous when $h_-$ crosses the origin. This proposal was described in greater detail for the $su(2)$ case in [3]. The identification (2.36) changes the topology of the space of couplings.
We can view this as gluing multiple patches of coupling constant space along \( g_- = \pm \infty \), and we will draw the resulting diagrams with this in mind.

Having determined how regular flows navigate around singularities, we can list the trajectories which can consistently be extended to infinite RG time. These have been checked numerically, implying that they are generic enough. The flows are depicted in Figures 1-3. We classify them according to where they start from and where they terminate. The initial and final locations will either be \( g_+ = 0 \) or \( g_+ = \infty \) as it takes an infinite time for the flows to reach these locii. For each flow there is a dual one, and we only list one of the two flows in the dual pairs. It is possible to draw the flows unambiguously in the \( g_+, g_- \) plane since \( g_a \) flows in a simple manner: in the UV \( g_a = +\infty \) and in the IR \( g_a = -\infty \).

**Flows originating from the \( g_+ = 0 \) plane:**
(a-1) The trajectory returns to \( g_+ = 0 \) at some other point without crossing the singularities and without \( g_- \) going to \( \infty \). (Figure 1.)
(a-2) The flow is to \( g_-, g_a = \infty \) with \( g_+ < 2 \) without crossing the self-dual points. After identification of points at infinity as in eq. (2.36), the flow then continues at \( g_- = -\infty \) and again eventually returns to the \( g_+ = 0 \) plane. (Figure 2.)
(a-3) The flow begins with \( 0 < g_- < 2 \) and first flows towards the self-dual point \( g_\pm = 2, g_a = 0 \). It goes through this point according to eq. (2.34). After crossing the trajectory is again determined by duality, and flows to \( g_+ = \infty \). (Figure 3.)

**Flows originating from the \( g_+ = \infty \) plane, dual to those terminating at \( g_+ = 0 \):**
(b-1) The flow starts at \( g_+ = \infty \) and loops back to \( g_+ = \infty \) without crossing singularities. (Figures 1,3) The flow of this kind in Figure 3 is the dual of (a-2), and those in figure 1 the dual of (a-1).
(b-2) The flow goes to \( g_-, g_a = \infty \) without crossing singularities. After the identification (2.36) the flow goes through the \( g_\pm = \pm 2 \) poles according to eq. (2.32) and then to the \( g_+ = 0 \) plane. (Figure 2.)
(b-3) The flow starts at infinity with \( 0 < g_- < 2 \), goes up to the self-dual point \( g_\pm = 2, g_a = 0 \), crosses it according to (2.34) and terminates at \( g_+ = 0 \). (Figure 2.)
(b-4) The flow starts at infinity with \( g_- < -2 \), goes through the lower self-dual point at \( g_\pm = \pm 2, g_a = 0 \), according to (2.32), and then continues up to \( g_+ = 0 \). It is a self-dual trajectory. (Figure 2.)

All of the above flows asymptotically reach either \( g_+ = 0 \) or \( g_+ = \infty \) at large scales. They take an infinite time to arrive at these locii. We shall refer to these phases as \( dXY \) and \( gX \) phases, respectively. The flows for \( g_+ < 0 \) are the reflection of the flows for \( g_+ > 0 \) about the \( g_- \) axis due to the symmetry of the \( \beta \) function.

**\( dXY \) phase:** \( g_+ \to 0, g_- \) flows to a non-universal constant, and \( g_a \) flows to \( -\infty \) logarithmically, \( g_a \approx -a \log r \) with \( a = 4g_2^2/(2 - g_-)^2 \). Since the non-zero couplings \( (g_-, g_a) \) in the deep IR couple only the currents of the \( gl(N) \) sub-current algebra this phase is in the same universality class as the disordered XY model, or Gade-Wegner class, as described in [10]. Due to the structure of the \( \beta \) functions (2.29) since \( g_- \) does not flow to \( \infty \), one cannot argue that this phase flows to the fixed point \( osp(2N|2N)_1/gl(N) |N \) as the solution of the Virasoro master equation would suggest, and it remains unclear whether one can identify a true IR fixed point to this flow. We point out that it was shown in [10] that the
c = −2 coset conformal field theory \( osp(2N|2N)_1/u(1) \otimes u(1) \) describes a conformal sector but precise arguments identifying this with the fixed point are missing.

\( g_X \) phase: Here, \( g_- \) flows to a constant, \( g_0 \) flows logarithmically to \(-\infty\), \( g_0 \approx -a \log r \) with \( a = 4g_0/(2 - g_-)^2 \) whereas \( g_+ \) grows much faster as \( g_+ \approx \exp(2g_0^2/g_-^2) \). Since all the \( osp(2N|2N) \) currents are involved in this flow, this is probably a massive phase.
FIG. 1. RG trajectories that loop back to $g_+ = 0, \infty$ and flow to the $dXY$ and $gX$ phases respectively.
FIG. 2. RG trajectories that cross singularities and flow to the $dXY$ phase.
FIG. 3. RG trajectories that either cross singularities and flow to $gX$ phase or are singular. The physical domain is the $90^\circ$ cone opening to the right. (Singular flows shown in red.)

III. THE SPIN QUANTUM HALL NETWORK MODEL

The hamiltonian of the network model for the spin quantum Hall transition is

$$H = \left( \begin{array}{cc} 2\vec{\alpha} \cdot \vec{\sigma} + m & -i\partial_z + A_z \\ -i\partial_z + A_z & 2\vec{\alpha} \cdot \vec{\sigma} - m \end{array} \right)$$  \hspace{1cm} (3.1)

where $\vec{\alpha}, m$ are random potentials and $A$ are random $su(2)$ gauge potentials \cite{3,8}. Our conventions for the Pauli matrices are

$$\sigma^3 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma^+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma^- = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$  \hspace{1cm} (3.2)
With the above convention, $T r \sigma^a \sigma^b = \delta^{ab}/2$ and $f^{abc} f^{abd} = -2 \delta^{cd}$ where $[\sigma^a, \sigma^b] = f^{abc} \sigma^c$. The random potentials, $m, \bar{\alpha}, A$ are taken to have the centered gaussian distributions with variance $g_m, g_\alpha, g_s$.

A. Effective action and its symmetries

Introducing bosonic ghost partners $\beta_+, \beta_+$ to the fermions, the effective action upon disorder averaging is

$$S_{\text{eff}} = S_{\text{free}} + \int \frac{d^2 x}{2\pi} \left( g_\alpha \mathcal{O}^\alpha + g_m \mathcal{O}^m + g_s \mathcal{O}^s \right)$$

(3.3)

where

$$\mathcal{O}^\alpha = \left( \psi_+ \bar{\sigma} \bar{\psi}_+ + \bar{\psi}_- \bar{\sigma} \psi_+ + \beta_- \bar{\sigma} \beta_+ + \bar{\beta}_- \bar{\sigma} \beta_+ \right)^2 \quad \mathcal{O}^m = \frac{1}{4} \left( \psi_+ \bar{\psi}_+ - \bar{\psi}_- \psi_+ + \beta_- \beta_+ - \bar{\beta}_- \beta_+ \right)^2 \quad \mathcal{O}^s = (\psi_+ \bar{\sigma} \psi_+ + \beta_- \bar{\sigma} \beta_+) \cdot \left( \bar{\psi}_- \bar{\sigma} \bar{\psi}_+ + \bar{\beta}_- \bar{\sigma} \beta_+ \right)$$

(3.4)

In order to more clearly display the sub-algebraic structure, let us define new couplings

$$g_\alpha \mathcal{O}^\alpha + g_m \mathcal{O}^m = g_c \mathcal{O}^c + g_s \mathcal{O}^s, \quad g_\alpha = g_c + \frac{1}{2} g_8, \quad g_m = \frac{3}{2} g_8 - g_c$$

(3.5)

where $\mathcal{O}^c = \mathcal{O}^\alpha - \mathcal{O}^m$, and $\mathcal{O}^s = \frac{1}{2} \mathcal{O}^\alpha + \frac{3}{2} \mathcal{O}^m$. The physical domain is the intersection of $g_c > -g_8/2$ and $g_c < 3g_8/2$ with $g_s > 0$. The free conformal field theory has a maximal $osp(4|4)_{k=1}$ current algebra symmetry. The Lie superalgebra $osp(4|4)$ has two commuting subalgebras $osp(2|2)$ and $su(2)$. The above notation reflects the fact that $g_s$ ($g_c$) couples the spin (charge) degrees of freedom. Let us describe the $osp(4|4)$ currents in a notation that clarifies these subalgebras. Define bosonic and fermionic $su(2)$ currents

$$L^a_\psi = \psi_+ \sigma^a \psi_+ \quad L^a_\beta = \beta_- \sigma^a \beta_+$$

(3.6)

These satisfy level $k = 1$ and level $k = -1$ current algebra respectively

$$L^a_\psi(z) L^b_\psi(0) \sim \frac{1}{z^2} \frac{\delta^{ab}}{2} + \frac{1}{z} f^{abc} L^c_\psi(0) \quad L^a_\beta(z) L^b_\beta(0) \sim -\frac{1}{z^2} \frac{\delta^{ab}}{2} + \frac{1}{z} f^{abc} L^c_\beta(0)$$

(3.7)

(3.8)

The operator $\mathcal{O}^s$ is the $su(2)$ invariant built from the sum:

$$\mathcal{O}^s = \sum_a L^a \bar{L}^a \quad \mathcal{L} = L_\psi + L_\beta$$

(3.9)

The current $L^a$ satisfies a level $k = 0$ current algebra.

The currents in $\mathcal{O}^c$ generate an $osp(2|2)$ subalgebra. Define
\[ J = \sum_i \psi^i_+ \psi^i_-, \quad J_\pm = \sum_{ij} \epsilon_{ij} \psi^i_\pm \psi^j_\mp \]
\[ H = \sum_i \beta^i_+ \beta^i_-, \quad S_\pm = \sum_i \psi^i_\pm \beta^i_\pm \]

where \( \epsilon_{ij} = -\epsilon_{ji}, \ \epsilon_{12} = 1 \), and similarly for the right-movers \( \overline{J} = \overline{\psi}_+ \overline{\psi}_- \), \( \overline{H} = \overline{\beta}_+ \overline{\beta}_- \), ……

These currents generate an \( \text{osp}(2|2)_k \) current algebra at level \( k = -2 \) \[22\]. The currents \( J \) and \( J_\pm \) generate a charged \( \text{su}(2) \) subalgebra under which the fermions \( \psi^i_\pm \) transform as doublets.

The operator \( \mathcal{O}^c \) is the \( \text{osp}(2|2) \) invariant bilinear:
\[ \mathcal{O}^c = -J\overline{J} + H\overline{H} + \frac{1}{2} (J_+ \overline{J}_+ + J_- \overline{J}_-) + S_- \overline{S}_+ - S_+ \overline{S}_- + \hat{S}_+ \overline{\hat{S}}_- - \hat{S}_- \overline{\hat{S}}_+ \]  
(3.10)

To show this we have used the identity \([5,1]\) of the Appendix.

The currents in \( \mathcal{O}^8 \) are orthogonal to the above ones. To better reveal the \( \text{su}(2) \) properties of these currents, let us introduce intertwiners between the symmetric tensor product of two spin 1/2 representations and the spin 1 representation. These are:
\[ \rho^a = \sigma^a \epsilon, \quad \hat{\rho}^a = \epsilon \sigma^a, \quad \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \]  
(3.11)

We therefore define
\[ B_+^a = \frac{1}{\sqrt{2}} \beta_+ \hat{\rho}^a \beta_+, \quad B_-^a = \frac{1}{\sqrt{2}} \beta_- \rho^a \beta_- \]  
(3.12)
\[ U_+^a = \beta_+ \hat{\rho}^a \psi_+, \quad U_-^a = \beta_- \rho^a \psi_- \]  
(3.13)
\[ V_+^a = \psi_- \sigma^a \beta_+, \quad V_-^a = \beta_- \sigma^a \psi_+ \]  
(3.14)

They transform in the spin 1 representation of \( \text{su}(2) \). The operator \( \mathcal{O}^8 \) can then be written as
\[ \mathcal{O}^8 = -2 \left( B_- \overline{B}_+ + B_+ \overline{B}_- + U_- \overline{U}_+ - U_+ \overline{U}_- + V_- \overline{V}_+ - V_+ \overline{V}_- + L_\beta L_\beta - L_\psi L_\psi \right) \]  
(3.15)

where e.g. \( B_- \overline{B}_+ = \sum_a B^a_- \overline{B}^a_+ \).

We can now describe the \( \text{osp}(2|2) \otimes \text{su}(2) \) transformation properties of the above currents. The \( \text{osp}(2|2) \) currents \([3.10]\) commute with the \( \text{su}(2) \) currents \( \mathcal{L}^a \). The currents in \( \mathcal{O}^8 \) transform in the spin 1 representation of \( \text{su}(2) \). The fields \( L_{\psi,\beta} \) are not primary however:
\[ \mathcal{L}^a(z) L^b_\psi(0) \sim \frac{1}{z^2} \frac{\delta^{ab}}{2} + \frac{1}{z} f^{abc} L^c_\psi \]  
(3.16)
\[ \mathcal{L}^a(z) L^b_\beta(0) \sim \frac{1}{z^2} \frac{\delta^{ab}}{2} + \frac{1}{z} f^{abc} L^c_\beta \]

Under the \( \text{osp}(2|2) \) symmetry the currents in \( \mathcal{O}^8 \) transform in the 8 dimensional indecomposable representation. This can be expressed as the OPE
\[ S^a(z) J^i_{8,\alpha}(0) \sim \frac{1}{z} (t^\alpha)^i_j J^j_{8,\alpha} \]  
(3.17)

where \( S^a \) are the \( \text{osp}(2|2) \) currents, \( J^i_{8,\alpha}, \ i = 1,..,8, \ a = 1..3 \) are the 24 currents in \( \mathcal{O}^8 \) with \( a \) the \( \text{su}(2) \) index, and \( t^\alpha \) is an 8 dimensional matrix representation of \( \text{osp}(2|2) \).
IV. THE $\beta$ETA FUNCTION FOR THE SPIN NETWORK MODEL

The interactions that perturb away from the conformal field theory are of the form (2.9) where $J^a$ are the $osp(4|4)$ currents at level $k = 1$ described above, and $d^A_{ab}$ are quadratic forms that define the operators $O^A = \{O^s, O^c, O^8\}$. As before, the beta function proposed in [20] is expressed in terms of the data $C'^{AB}_{C}, D'^{AB}_{C}, \tilde{C}'^{AB}$ which can be computed from the OPE’s (3.3). One finds the non-zero values

$$
C^{ss} = C^8 = C^{s8} = -2
$$

$$
C^{cc} = C^8 = 4, \quad C^{c8} = -3
$$

$$
C^{c8} = C^{sc} = C^s = -8
$$

(4.1)

The coefficients $C^{ss}$ and $C^{cc}$ are easily understood as minus the quadratic Casimir in the adjoint representation of $su(2)$ and $osp(2|2)$ respectively.

The other RG data involves OPE’s of the purely left-moving operators (2.10). One can easily disentangle the $D$ from the $C$ term by noting that the $D$ term only arises from central terms proportional to $k$ in the OPE. One finds

$$
D^{cc} = -2, \quad D^{s8} = D^{c8} = D^8 = 1
$$

(4.2)

Some of the $\tilde{C}$ coefficients follow from simple group theory. Consider first OPE’s with $T^s$. Since $T^s$ has the form of the quadratic Casimir for $su(2)$, we have $T^sO^s \sim 2O^s, T^sO^8 \sim 2O^8$ since 2 is the quadratic Casimir for the spin 1 representation of $su(2)$. Thus $\tilde{C}^{ss} = \tilde{C}^{c8} = 2$. Similarly $T^cO^c \sim -4O^c$ where -4 is the Casimir for the adjoint representation of $osp(2|2)$. The only non-zero OPE’s of $T^c$ with $O^8$ currents are

$$
T^c(z)L_\psi \sim -T^c(z)L_\beta \sim \frac{4}{z^2} \mathcal{L}
$$

(4.3)

which implies $\tilde{C}^{cs} = 8$. The remaining $\tilde{C}$ are computed tediously but straightforwardly using the free-field OPE’s (2.6). For convenience we collect all non-zero $\tilde{C}$

$$
\begin{align*}
\tilde{C}^{ss} &= \tilde{C}^{c8} = 2 \\
\tilde{C}^{s8} &= -\tilde{C}^{ss} = -\tilde{C}^{cc} = -\tilde{C}^{c8} = 4 \\
\tilde{C}^{c8} &= 8, \quad \tilde{C}^{cs} = -6
\end{align*}
$$

(4.4)

Using the general expression in [20], the result for the beta functions is

$$
\begin{align*}
\beta_g &= \frac{-8g_8}{(g_8 + 2)^3(g_8 - 2)} \left( g_8^3 + g_8^2g_8 - 4g_8 + 4g_8 \right) \\
\beta_c &= \frac{2}{(g_8^2 - 4)(g_8 - 1)^2} \left( 12g_8^2g_c - g_8^2g_c^2 - 16g_8^2g_c - 16g_c^2 + 12g_c^2 \right) \\
\beta_s &= \frac{-4}{(g_c - 1)(g_8 - 2)^2(g_8 + 2)} \left( g_8^3(g_c - 1)(3g_8^4 + 4g_8^3 - 8g_8^2 + 16g_8 - 16) ight. \\
&\quad \left. + 4g_8^2g_8(g_8 - 2)(g_8 + 2)(g_c - 1) - 2g_8(g_8 - 2)(g_8 + 2)(g_8 + 4)(g_c + 2)(g_c + 2) \right)
\end{align*}
$$

(4.5)

The $\beta$ function remarkably also has the duality (2.13) with
\[ g_c^* = \frac{1}{g_c}, \quad g_s^* = \frac{4}{g_8}, \quad g_8^* = -\frac{4g_s}{g_8^2} \]  

(4.6)

The self-dual points are \((g_s, g_c, g_8) = (0, \pm 1, \pm 2)\). As for the \(u(1)\) network model, we have not found any RG invariants, unlike the \(su(2)\) case.

**A. Virasoro master equation and critical exponents**

As for the \(u(1)\) network model, let us describe the solutions of the Virasoro master equation (2.18). We obtain four pairs of solutions. First one is the trivial one:

\[ T_0 = 0 \quad \text{and} \quad T_{osp(4|4)_1} = \frac{1}{4}T^c - \frac{1}{2}T^s \]

Two other pairs of solutions are given by:

\[ T_{osp(2|2)_2} = -\frac{1}{8}T^c \quad \text{and} \quad T_{osp(4|4)_1/osp(2|2)_2} = \frac{3}{8}T^c - \frac{1}{2}T^s \]

\[ T_{su(2)_0} = \frac{1}{2}T^s \quad \text{and} \quad T_{osp(4|4)_1/su(2)_0} = -\frac{1}{2}T^s + \frac{1}{4}T^c - \frac{1}{2}T^8 \]

The last pair of solutions,

\[ T = \frac{1}{2}T^s - \frac{1}{8}T^c \quad \text{and} \quad T' = -\frac{1}{2}T^s + \frac{3}{8}T^c - \frac{1}{2}T^8 \]

corresponds to \(osp(2|2)_2 \otimes su(2)_0\) and to the coset \(osp(4|4)_1/osp(2|2)_2 \otimes su(2)_0\).

Let us recall the spin-charge separation of the free conformal field theory of the bosons and fermions \(\psi, \beta\). This free theory has \(osp(4|4)_1\) current algebra symmetry with stress tensor \(T_{osp(4|4)_1}\). It was shown in Ref. [22] that the latter stress tensor decomposes as follows:

\[ T_{osp(4|4)_1} = T_{osp(2|2)_2} + T_{su(2)_0} \]  

(4.7)

where \(T_{osp(2|2)_2}\) and \(T_{su(2)_0}\) are the affine-Sugawara stress tensors for the \(osp(2|2)_2\) and \(su(2)_0\) current algebras. This lead to the identity \(T' = 0\):

\[ 3T^c - 4T^8 - 4T^s = 0 \]  

(4.8)

It was argued in Ref. [22] that though the stress tensor decomposes as (4.7), the Hilbert space does not exactly factorize due to logarithmic pairs such as \(O^8, O^s\). This means that the coset \(osp(4|4)_1/su(2)_0\) is not identical to the current algebra \(osp(2|2)_2\) even though it has a global \(osp(2|2)_2\) symmetry and the scaling dimensions are correctly given by \(osp(2|2)_2\). However recent work shows that certain 4-point correlation functions can be factorized [23]. A comparison with other approaches was studied in [24].

The interesting possible fixed points are thus the cosets \(osp(4|4)_1/su(2)_0\) and \(osp(4|4)_1/osp(2|2)_2\). For each fixed point, let us compute some of the critical exponents. The density of states is proportional to the one-point function of \(\Phi_E\),

\[ \rho(E) \propto \langle \Phi_E \rangle \]  

(4.9)
where $\Phi_E$ is defined in eq. (2.3). Let $\Gamma_E$ denote the scaling dimension of $\Phi_E$ in the IR. Then

$$\rho(E) \propto E^{\Gamma_E/(2-\Gamma_E)}, \quad \text{as } E \to 0$$

(4.10)

Other exponents are generally related to other perturbations $\Phi_\delta$:

$$\delta S = \delta \int d^2 x \Phi_\delta$$

(4.11)

where $\delta = 0$ is the critical point. Letting $\Gamma_\delta$ denote the scaling dimension of $\Phi_\delta$ in the IR, the mass dimension of $\delta$ is then $2 - \Gamma_\delta$. As $\delta \to 0$, there is thus a diverging length $\xi$,

$$\xi \propto \delta^{-\nu}, \quad \nu = \frac{1}{2 - \Gamma_\delta}$$

(4.12)

For a coset fixed point $G_k/H_k'$, the dimensions of the fields in the IR are given by

$$\Gamma = 2\Delta = 2 \left( \Delta^G_k - \Delta^H_k' \right)$$

(4.13)

The spin-charge separation implies that the conformal scaling dimensions $\Delta$ satisfy the sum rule:

$$\Delta^{osp(4|4)}_1 = \Delta^{osp(2|2)-2} + \Delta^{su(2)_0}$$

(4.14)

The $osp(4|4)$ scaling dimensions are simply given by the free theory and are thus integer or half-integer, where $\psi, \beta$ have $\Delta^{osp(4|4)}_1 = 1/2$. The primary fields of the $osp(2|2)-2$ can be labeled $(j, b)$ where $j$ is the spin of the $su(2)$ generated by $J, J^\pm$ and $b$ is the $u(1)$ charge $H/2$. These fields form $8j$-dimensional multiplets and have left/right scaling dimension $[25]$

$$\Delta_{(j,b)}^{osp(2|2)-2} = \frac{j^2 - b^2}{2}$$

(4.15)

The primary fields of $su(2)_0$ on the other hand are labeled by a spin $j$ and have scaling dimension $[26]$

$$\Delta_j^{su(2)_0} = \frac{j(j+2)}{2}$$

(4.16)

$osp(4|4)_1/osp(2|2)-2$ The $osp(2|2)-2$ dimension of $\Phi_E$ is 1/4 since it transforms in the $(1/2, 0)$ representation with $\Delta^{osp(2|2)} = 1/8$. Thus $\Gamma_E = 1 - 1/4 = 3/4$ and one finds

$$\rho(E) \propto E^{3/5}$$

(4.17)

In this phase it is the charge degrees of freedom that are massive, i.e. localized.

$osp(4|4)_1/su(2)_0$. The field $\Phi_E$ has $j = 1/2$ quantum numbers under the $su(2)_0$, and thus has $\Delta^{su(2)} = 3/8$. This leads to $\Gamma_E = 1 - 3/4 = 1/4$ and

$$\rho(E) \propto E^{1/7}$$

(4.18)

Let us compare the latter coset with percolation. The map to percolation involves averaging over $su(2)$ matrices with the flat Haar measure, which is expected to correspond
to \( g_s = \infty \). The \( g_s = \infty \) theory was described in [27]. This averaging projects out the \( su(2) \) degrees of freedom and in our approach this is analogous to arguing that the RG flow gaps out the \( su(2)_0 \) sub-current algebra. We do not have an argument leading to the identification of \( \Phi_\delta \) for the localization length exponent. However there does exist a field in our theory with the correct dimension to lead to the percolation exponent. Consider a field \( \Phi_\delta \) of \( \Phi \) with the correct dimension to lead to the percolation exponent. Consider a field \( \Phi_\delta \) in the free theory of the form \( \Phi_\delta = \phi_\delta \phi_\delta \) with \( \phi_\delta \sim (\psi\psi/\beta) \). In the free theory this field has \( \Delta = 5/2 \). Since \( \psi/\beta \) have \( j = 1/2 \) under the \( su(2)_0 \), we can take \( \phi_\delta \) to be in the \( j = 3/2 \) representation of \( su(2)_0 \) with \( \Delta_{su(2)_0} = 15/8 \). Under the \( osp(2|2) \) this field transforms in the \( (j, b) = (3/2, 1) \) representation with \( \Delta_{osp(2|2)} = 5/8 \). Note that the sum rule is satisfied: \( 15/8 + 5/8 = 5/2 \). Since \( \Gamma_\delta = 5/4 \) this leads to \( \nu = 4/3 \).

**B. RG flows**

In this section we describe the RG flows based on the \( \beta \) function (4.3). As in the \( u(1) \) network model, there exist both well-behaved and singular flows.

**General properties.**

When \( g_s = 0 \) the spin and charge degrees of freedom decouple and the \( \beta \) function reduces to

\[
\beta_{g_s} = g_s^2, \quad \beta_{g_c} = -\frac{2g_c^2}{(g_c - 1)^2}
\]

The fact that the one-loop result for \( \beta_{g_s} \) is exact is due to the level \( k \) being zero for the \( su(2) \) currents \( L^a \). If \( g_s > 0 \), \( g_s \) flows to infinity after a finite scale transformation, whereas if \( g_s < 0 \) it flows to zero, i.e. \( g_s > 0 \) is marginally relevant and \( g_s < 0 \) is marginally irrelevant. On the other hand \( g_c < 0 \) is marginally relevant with \( g_c \) flowing to \( -\infty \) and \( g_c > 0 \) is marginally irrelevant with \( g_c \) flowing to zero.

When \( g_s \neq 0 \), we can consider the flows as originating from the \( g_s - g_c \) plane. On this plane the flows originate in the UV from \( g_s = 0, -\infty \) or \( g_c = 0, +\infty \). Near \( g_s = 0 \), \( \beta_{g_s} = 2g_sg_s \). Thus if \( g_s > 0 \) (\( g_s < 0 \)) one flows away (toward) the \( g_s - g_c \) plane. If \( g_s \) is not large enough, the flows can turn around and form a loop returning to the \( g_s = 0 \) plane. Otherwise it continues away from the plane until it reaches a singularity. By duality there are similar properties at \( g_s = \infty \).

At generic points, the slope of the trajectories \( \partial g_s/\partial g_s = \beta_{g_s}/\beta_{g_s} \) is zero at \( g_c = \pm 2 \), whereas the slope \( \partial g_s/\partial g_c = \beta_{g_s}/\beta_{g_c} \) is zero at \( g_s = 2 \) but equals \( \infty \) at \( g_s = -2 \). Thus the flows are generically tangent to the \( g_s = 2 \) plane, since \( \beta_{g_s} \gg \beta_{g_s} \) near \( g_s = 2 \). Furthermore, \( \beta_{g_c} \approx 3(g_c^2 - 1)^2/(g_c - 2)^2 \), so that \( g_c \) increases in the vicinity of the plane \( g_s = 2 \). On the other hand, since \( \beta_{g_c} \ll \beta_{g_s} \ll \beta_{g_s} \) near \( g_s = -2 \) on the other hand, the flows are generically attracted or repealed by the \( g_s = -2 \) plane. More precisely, because of the odd order pole, \( \beta_{g_s} \approx -32g_s/(g_s + 2)^3 \), the flows are attracted to \( g_s = -2 \) for \( g_s > 0 \), whereas for \( g_s < 0 \) flows are away from \( g_s = -2 \). One also has \( \beta_{g_s} \approx 16g_s^2/(g_s + 2)^4 \) near \( g_s = -2 \), the coupling \( g_c \) very rapidly increases.

The behavior near the poles \( g_c = 1 \) is somewhat simpler. There, the slopes are such that \( \partial g_s/\partial g_c \) and \( \partial g_c/\partial g_c \) both vanish since \( \beta_{g_s} \ll \beta_{g_s} \ll \beta_{g_c} \) at generic points. Near \( g_c = 1 \) and away from \( g_s = \pm 2 \) the beta function \( \beta_{g_c} \approx -2/(g_c - 1)^2 \) has double pole. Thus \( g_c \) always
decreases and flows can cross the $g_c = 1$ plane. Note that $g_c = -1$ is not a singular point. Near $g_c = -1$ one finds $\beta_{g_c} = -1/2$, so again the flows easily cross the $g_c = -1$ plane.

The non generic singular points are those for which the ratio of the beta functions is finite and non-vanishing. These are at $(g_0 = 2, g_c = \pm 1)$. At $g_c = -1, g_8 = 2$ and $g_s = 0$ the ratio $\partial g_8 / \partial g_c$ and $\partial g_8 / \partial g_s$ are finite. Thus one can flow through this self-dual point and extend the flow by duality. We have numerically verified such flows. As analyzed below, the flows toward $(g_c, g_8) = (1, 2)$ on the other hand are singular.

As in previous examples, blow-ups in finite RG time $\tau$ are resolved by certain topological identifications. When $g_8 \neq 0$, there are flows that blow up in finite RG time in the direction $g_c \to \infty$. As can be seen numerically, in this domain $g_c$ increases rapidly while $g_8$ flows close to the value $g_8 = 2$. This can be seen by expanding near $g_8 = 2 + \epsilon_8$ with $\epsilon_8 \to 0$. The beta-functions asymptotically reduce to

$$
\beta_s \simeq -\frac{g_s^2}{\epsilon_0}, \quad \beta_c \simeq \frac{6g_c^2}{\epsilon_8}, \quad \beta_8 \simeq -\frac{2g_8}{\epsilon_8}
$$

The solutions are

$$
g_s \simeq \sqrt{\epsilon_8 + \text{const.}}, \quad \epsilon_8 \simeq 2c\sqrt{(\tau_\ast - \tau)}, \quad g_c^{-1} \simeq \frac{3}{2c^2} \log((\tau_\ast - \tau)b)
$$

showing that $(g_s, g_c, g_8) \to (\text{const.}, \infty, 2)$. Note also that $g_c$ blows up before $\epsilon_8$ vanishes. To continue the flows note that the asymptotic behavior (4.20) satisfies $\beta(g_s, g_c, g_8) = \beta(g_s, -g_c, g_8)$ as $|g_c| \to \infty$. Viewing this as a consistent patching condition, we can thus identify $g_c = +\infty$ with $g_c = -\infty$ when $g_8 \neq 0$, such that flows toward $g_c = \infty$ continue at $g_c = -\infty$.

**Singular flows.** There are two classes of singular flows:

(i) Consider the flows toward the self-dual point $g_c = 1, g_8 = 2$. Letting $g_c = 1 + \epsilon_c, g_8 = 2 + \epsilon_8$, asymptotically the beta function behaves as

$$
\beta_{g_s} \simeq -\frac{g_s^2}{\epsilon_8} + \frac{32}{\epsilon_c \epsilon_8} + \frac{16 - 4g_s - g_s^2}{\epsilon_8} + \frac{24}{\epsilon_c} + \cdots
$$

$$
\beta_{g_c} \simeq \frac{24}{\epsilon_8^2} - \frac{2}{\epsilon_c^2} + \frac{12}{\epsilon_8} - \frac{4}{\epsilon_c} + \cdots
$$

$$
\beta_{g_8} \simeq -\frac{2g_8}{\epsilon_8} + \cdots
$$

The trajectories fine tune themselves coherently in order to converge toward the singularity and to cancel the double poles of the beta-functions. The RG trajectories may be found as series expansion in terms of $\epsilon_8$, with $\epsilon_8 \to 0$. We obtain

$$
\epsilon_c = \epsilon_8 (1 + a_c \epsilon_8 + \cdots) / 2\sqrt{3}
$$

with $a_c$ solution of $3(4\sqrt{3}a_c + \sqrt{3} - 1)/2 = -3^{1/4}$ and

$$
g_8 = 8 \cdot 3^{1/4} (1 + \frac{1}{6} \cdot 3^{1/4} \epsilon_8 \log \epsilon_8 + \cdots)
$$
The RG equation for $\epsilon_8$, $\dot{\epsilon}_8 = \beta_8$, then gives

$$\epsilon_8 \simeq -3^{1/8} \sqrt{32(\tau_s - \tau)} + \cdots$$

showing that the trajectories are effectively attracted by the pole. Since $g_s$ flows to $8 \cdot 3^{1/4}$ rather than zero, this flow cannot be continued through the singularity by duality. These singular flows are in the physical domain.

(ii) Consider the flows toward $g_8 = -2$. Letting $g_8 = -2 + \epsilon_8$, the $\beta$ function reduces to

$$\beta_{g_s} \simeq \frac{16g^2}{\epsilon_8^4}, \quad \beta_{g_c} \simeq \frac{6(g_c + 1)^2}{\epsilon_8^2}, \quad \beta_{\epsilon_8} \simeq -\frac{32g_s}{\epsilon_8^3} \quad (4.24)$$

Integrating these equations one finds the asymptotic behavior

$$g_s \simeq c\epsilon_8^{-1/2}, \quad \frac{1}{g_c + 1} \simeq \frac{3}{40c} \epsilon_8^{5/2} + c', \quad \epsilon_8 \simeq (144c(\tau_s - \tau))^{2/9} \quad (4.25)$$

for some constants $c, c', \tau_s$. Thus in a finite RG time $(g_s, g_c, g_8) \to (\infty, \text{const.}, -2)$. Again, there is neither the duality nor a consistent patching condition allows the flows to extend beyond this singularity.

Regular flows All other flows can be extended to arbitrarily large or small length scales. These well-behaved flows either don’t encounter singularities or can be extended consistently by passing between patches patches glued at $g_c = \pm \infty$, and/or flowing through the self-dual points $(g_s, g_c, g_8) = (0, -1, 2)$. The flows that do not loop back to $g_8 = 0$ or $\infty$ are shown in Figures 4,5.

Below we list the asymptotic directions, or phases, that can be continued to arbitrarily large length scales.

**phase $O$**  In this phase all disorder is marginally irrelevant. The flows are attracted to $g_8 = 0$ with $g_s < 0, g_c > 0$ and then $g_s, g_c$ continue to flow to zero according to the $\beta$ function (4.19).

**phase $C$**  This phase is similar to phase $O$ except that $g_c < 0$ and thus marginally relevant. Thus $g_s, g_8 \to 0$ and $g_c \to -\infty$.

**phases $S^\pm$**  A large set of flows, some with rather different UV properties, asymptotically flow in the direction $S^+$. Some flows originate in the UV with $g_c > 0$. Flows leaving the $g_c - g_s$ plane with $g_c > 0$ flow to $g_c = \infty$, continue at $g_c = -\infty$, through the self-dual point $(g_s, g_c, g_8) = (0, -1, 2)$ and then in the $S^+$ direction. These flows originate from $(g_s, g_c, g_8)_{\text{UV}} = (0^+, -\infty, 0^+)$, and then by duality should flow to $g^*$, i.e. in the IR we expect $(g_s, g_c, g_8)_{\text{IR}} = (-\infty, 0, +\infty)$. As shown below, the asymptotic behavior of $S^+$ supports this. Flows originating from $(g_s, g_c, g_8) = (+\infty, 0, +\infty)$ are also attracted to $S^+$ but do not pass through self-dual points.

Assuming $g_8g_c \gg 1$, the $\beta$ function asymptotically reduces to

$$\beta_s \simeq -8g_s/g_8, \quad \beta_c \simeq 2g_c^2, \quad \beta_s \simeq -12g_s^2/g_8^2 - 8g_8$$

The equation for $g_c$ decouples and $g_c \simeq 1/\tau$ with $\tau$ the RG time. Integrating the RG equations in the $g_s, g_c$ plane gives the equations for the asymptotic trajectories,

$$g_s^2 \simeq 2g_8^3 \log g_8, \quad g_s \propto \tau^2 \log \tau, \quad g_c \propto 1/\tau$$
for large $\tau$. As a consistency check one verifies that $g_8^2 \ll g_s^2$ and $g_s g_c \gg 1$. The phase $S^-$ has the same properties as $S^+$ with $g_8 \to -\infty$.

The trajectories read

$$g_s^2 \simeq c g_8^3, \quad g_8 \simeq 16 c^2 \tau^2, \quad g_c \simeq -2 \tau$$

for large $\tau$ where $c$ is some constant. The phase $SC^-$ is very similar to $SC^+$ but with $g_8 < 0$. 

These flows originate from the $g_c - g_s$ plane with $g_c < 0$. In the deep UV one has $(g_s, g_c, g_8)_{UV} = (0^+, 0^-, 0^+)$. Since the flow is through a self-dual point, one expects from duality that $(g_s, g_c, g_8)_{IR} = (+\infty, -\infty, +\infty)$. Assuming $|g_s| \ll |g_8|$ and $|g_8^3| \ll |g_c g_s^2|$, the $\beta$-functions reduce to

$$\beta_s \simeq -\frac{8g_s}{g_8}, \quad \beta_c \simeq -2, \quad \beta_s \simeq -\frac{12g_s^2}{g_8^2}$$
FIG. 4. Flows away from \( g_8 = 0 \) in the spin network model.
FIG. 5. Flows toward $g_8 = 0$ in the spin network model.

C. Identifying the fixed points

It is possible to interpret the asymptotic directions listed in the previous subsection as some of the above coset solutions of the master equation. The main hypothesis in making
this identification is the following. Suppose a coupling $g$ flows to infinity much faster than the other couplings and $g$ couples a sub-current algebra $\mathcal{H}_{k'}$. Then we argue that the $\mathcal{H}_{k'}$ currents are gapped out in the flow leading to the fixed point $G_k/\mathcal{H}_{k'}$.

Phase C. Since the only non-zero coupling is $g_c$ and $g_c \to -\infty$, this phase clearly has the interpretation as the fixed point:

\[
\text{Phase C IR fixed point : } \frac{osp(4|4)_1}{osp(2|2)-2} \quad (4.26)
\]

Phases $S^\pm$. In the phases $S^\pm$ the only couplings flowing to $\infty$ are $g_8, g_s$. It was shown in Ref. [22] that the operators $O^8, O^s$ are a logarithmic pair. One can see explicitly from the asymptotics that $g_8/g_s$ flows very rapidly to zero. We can thus interpret this as $g_8$ effectively being zero in comparison to $g_s$. Since $g_s$ is flowing to $\pm\infty$, we propose the fixed point is the coset $osp(4|4)_1/su(2)_0$.

Phases $SC^\pm$. In this phase $g_c$ decouples in the RG equations and flows to $-\infty$ with the same asymptotic as phase C. Hence both subalgebras $su(2)_0$ and $osp(2|2)-2$ are massive, and we propose the fixed point $osp(4|4)_1/su(2)_0 \otimes osp(2|2)-2 = 0$. Since this theory is empty, this is a massive phase. This leads to $\Gamma_E = 0$ and a constant density of states as $E \to 0$.

V. MULTIFRACTAL EXponents FROM CONFORMAL COSETS

Critical exponents related to the multi-fractality of the wave functions have been the subject of numerous studies [15,28–30,16]. The multi-fractality itself is an important constraint on the critical theory. These exponents and the correlation length exponent $\xi \propto E^{-\nu}$ where $\nu \approx 2.35$ [31] are both important benchmarks for the correct critical theory. The multi-fractal exponents are more easily related to the conformal scaling dimensions of field operators in the critical theory [2] [18].

In this section we show that the critical theories considered above lead to multifractality in a very natural way. For the original $u(1)$ network model we will consider the $osp(2N|2N)_1/u(1)_0$ conformal field theory defined in section IIC, with stress tensor

\[
T = T_{osp(2N|2N)_1} - T_{u(1)_0} \quad (5.1)
\]

For the spin network model we will consider the coset $osp(4N|4N)_1/su(2)_0$.

For any field $\Phi(x)$ in an effective field theory of disorder we can define the quantity

\[
P(q) = \frac{\int d^2x (\Phi(x))^q}{\left(\int d^2x (\Phi(x))^q\right)^{q}} \quad (5.2)
\]

where $(\Phi(x))$ denotes the disorder average of the correlation function $(\Phi(x))$, and $(\Phi(x))^q$ denotes the disorder average of its $q$-th moment in the effective theory with $N \geq q$ copies. The above scales as

\[
P(q) \sim L^{-\tau(q)} \quad (5.3)
\]
where $L$ is a large length scale. From simple dimensional analysis one finds

$$\tau(q) = \Gamma(q) - q\Gamma(1) + 2(q-1) \tag{5.4}$$

where $\Gamma(q)$ is the scaling dimension of $\langle \Phi \rangle^q$.

If $\tau(q)$ is quadratic in $q$, then imposing $\tau(0) = -2$ and $\tau(1) = 1$ leads to a one-parameter family

$$\tau(q) = (2 - \alpha_0)q^2 + \alpha_0 q - 2 \tag{5.5}$$

It is known that usually $\tau(q)$ changes form for $q$ above some threshold [32]. However we are interested in the parameter $\alpha_0$ which is determined near $q = 0$ since $\alpha_0 = d\tau/dq|_{q=0}$, so we do not consider the phenomenon of multi-fractality termination. It is customary to define the Legendre transform with $\alpha = d\tau/dq$, which defines $q(\alpha)$, and

$$f(\alpha) = \alpha q - \tau(q) = -\frac{(\alpha - \alpha_0)^2}{4(\alpha_0 - 2)} + 2 \tag{5.6}$$

The fact that the numerical studies show a close fit to the above parabolic form indicates that $\tau(q)$ is indeed quadratic in $q$ for small $q$. The exponent $\alpha_0$ determines the typical amplitude

$$P_{\text{typical}} = \exp(\log(\Phi)) \sim L^{-\alpha_0} \tag{5.7}$$

The multi-fractal dimensions $\Gamma(q)$ follow simply from the conformal dimensions with respect to the current algebras using eq. (4.13) where $G = osp(2N|2N)_1$ and $H = u(1)_0$ or $su(2)_0$, and $\Delta^G, \Delta^H$ are conformal scaling dimensions which follow from the operator product expansion with the stress tensor.

It is convenient to bosonize the fields, where for each copy

$$\psi_\pm = e^{\pm i\varphi}, \quad \beta_+ = e^{\varphi'} \eta, \quad \beta_- = e^{-\varphi'} \partial_x \xi \tag{5.8}$$

where $\varphi, \varphi'$ are the $z$-dependent parts of scalar fields $\phi = \varphi(z) + \varphi'(z)$, and $\phi' = \varphi'(z) + \varphi'(z)$ and $(\eta, \xi)$ is a conformal dimension $(1,0)$ fermionic system with central charge $c = -2$ [33].

The bosonized action is then

$$S_{\text{free}} = \int \frac{d^2 x}{8\pi} \left( \sum_{i=1}^N \partial_i \phi_i \partial_i \phi_i - \partial_i \phi_i \partial_i \phi_i + \eta^i \partial_x \xi^i + \eta^i \partial_x \xi^i \right) \tag{5.9}$$

The $u(1)$ currents are

$$H_\psi = \sum_{i=1}^N \psi^i_+ \psi^i_- = i \sum_{i} \partial_x \phi_i, \quad H_\beta = \sum_{i=1}^N \beta^i_+ \beta^i_- = i \sum_{i} \partial_x \phi_i' \tag{5.10}$$

and similarly for $H$.

Consider first the fields $\Phi = \exp(i a \phi)$ in the $u(1)$ theory labeled by a parameter $a$. $\Gamma(q)$ is the dimension of

$$[\Phi]_q \equiv \Phi^1(x) \Phi^2(x) \cdots \Phi^q(x) \tag{5.11}$$
where $\Phi^i$ is the $i$-th copy of the at least $N = q$ copy effective theory. The operator $[\Phi]^q$ has dimension $qa^2$ with respect to the $osp(2N|2N)_1$ since each copy has dimension $a^2$. Since the $H_u(1)$ charge is $aq$, eq. (5.1) implies

$$\Gamma(q) = a^2 q(1 - q), \quad \Rightarrow \quad \alpha_0 = 2 + a^2; \quad (5.12)$$

for $\Phi = \exp(i a \phi)$.

Note that it is important for the denominator of the coset to be independent of $N$ in order to obtain this result. This is equivalent to the statement that $\Gamma(q)$ can be computed at arbitrary $N \geq q$ and should give the same result independent of $N$. In particular the $c = -2$ theories $osp(2N|2N)_1/u(1) \otimes u(1)$ where the $u(1)$’s are generated by $H_\psi, H_\beta$ do not have multifractal properties due to the $N$-dependence in eq. (2.24).

The multi-fractal exponents for the wavefunctions $\Psi(x)$ correspond to choosing $\Phi$ as the density operator

$$\rho = \overline{\psi}_-\psi_+ + \psi_-\overline{\psi}_+ = \cos \phi \quad (5.13)$$

The reason is that $|\Psi|^2 \propto G_{ret}(x, x) \propto \langle \rho \rangle$. Since $\Gamma(1) = 0$, this implies a constant density of states at zero energy. Since $\rho$ corresponds to $a = 1$, this naively gives $\alpha_0 = 3$. However one should perhaps bear in mind that the numerical simulations of the network model used periodic boundary conditions on the wavefunctions in a cylindrical geometry. It is well-known that periodic boundary conditions on the cylinder are mapped to anti-periodic boundary conditions on the plane. The boundary conditions of free fermion fields are modified in the presence of spin, or twist, fields, which here are $\sigma_\pm = \exp(\pm i \phi/2)$. This can be seen from the OPE

$$\psi_\pm(z)e^{\pm i \phi(0)/2} \sim 1/\sqrt{z} e^{\pm i \phi(0)/2} \quad (5.14)$$

which shows that $\psi_\pm$ picks up a phase $-1$ as $z$ encircles the origin on the plane. Since the more relevant operator in the OPE of $\rho$ with the twist fields is again a twist field, we suggest that $\alpha_0$ may correspond to $a = 1/2$, which gives $\alpha_0 = 9/4$. This is quite close to the published simulations on the network model [13][14], which report $\alpha_0 = 2.26 \pm 0.01$ and $2.260 \pm 0.003$. The small errors in the latter measurement seem to rule out $9/4$. Simulations of other models believed to be in the same universality class give the slightly higher result $\alpha_0 = 2.28 \pm 0.03$.

The point-contact conductance is another multi-fractal exponent that has been studied numerically for the network model [29][30]. Define higher moments of the density-density correlation as

$$G^{(q)} = \langle \rho(r)\rho(0) \rangle^q \sim r^{-\tau_G(q)} \quad (5.15)$$

where $\rho$ is the density operator. Let us assume that in the flow to the IR there is no mixing between the various copies. Simple dimensional analysis then gives

$$\tau_G(q) = 2 \Gamma(q) \quad (5.16)$$

where $\Gamma$ is defined the same way as above, i.e. as the dimension of $\overline{\langle \Phi \rangle^q}$. 

30
It is possible to derive a general scaling relation between $\alpha_0$ and the typical point contact conductance exponent $X_t$. Let us generally suppose that $\Gamma(q) = Aq + Bq^2$ for some constants $A, B$. If the density of states is constant, this implies $\Gamma(1) = 0$, or $A = -B$. Then, for the point-contact conductance, eq. (5.16) leads to an $f(\alpha)$ spectrum

$$f_G(\alpha) = -\frac{(\alpha - X_t)^2}{4X_t}$$

where $X_t = 2A$ is the typical point-contact conductance

$$\exp(\log G) \sim r^{-X_t}$$

Comparing with eqs. (5.4,5.5) we have $\alpha_0 = 2 + A$, which implies the scaling relation:

$$X_t = 2(\alpha_0 - 2)$$

We expect the above relation to hold whenever the density of states is constant [17]. The above relation works reasonably well using the most recent values of $X_t$ of .54, .57 reported in [10] and the numerical values of $\alpha_0$ [28,13,16]. For $\alpha_0 = 9/4$, one has $X_t = 1/2$.

Let us turn next to the coset $osp(2N|2N)/su(2)$. The fermion fields transform as spin $j = 1/2$ doublets under the $su(2)$. Consider again the $su(2)$ invariant density operator $\Phi = \rho = \bar{\psi}\psi$, with dimension 1 in the $osp(4|4)$ theory. Since $su(2)$ primary fields of spin $j$ have dimension $\Delta_{su}^{(2)} = j(j + 1)/2$ when $k = 0$ [28], one finds $\Gamma(1) = 1/4$. Since $[\Phi]_q$ has spin $j = q/2$, we thus find

$$\Gamma(q) = q - q(q + 2)/4$$

This agrees with ref. [18] at small $q$. Interestingly, using eqs. (5.4,5.5), the above $\Gamma(q)$ also leads to $\alpha_0 = 9/4$. For the point-contact conductance, here the density of states is not constant, i.e. $\Gamma(1) \neq 0$, so eq. (5.18) does not apply. Rather, eq. (5.16) leads to $X_t = 1$.

If we adopt the viewpoint that the above computations are a positive indication, it is tempting to try and understand the correlation length exponent $\nu$ for the quantum Hall transition. This is less straightforward since this exponent is not associated with the density operator. Rather, it is expected to be related to a perturbation of the action $\delta S = \delta \int d^2x \Phi_\delta$ where tuning $\delta$ through zero corresponds to tuning the network model to its critical probability. As described in section IVA, this leads to a diverging length $\xi \propto \delta^{-\nu}$, where $\nu = 1/(2 - \Gamma_\delta)$.

The challenge is to determine $\Phi_\delta$ from first principles and this is beyond the scope of this paper. Even for the case of the spin quantum Hall transition the arguments leading to the identification of the operator $\Phi_\delta$ which gives the percolation exponent $\nu = 4/3$ in this approach are missing, though there does exist an operator of the right dimension in the coset $osp(4|4)/su(2)$ [22]. (See section IVA.) In the percolation picture, $\Phi_\delta$ is the two-hull operator. In the remainder of this section we just explore the kinds of exponents that can arise.

Since the 1-copy theory has a non-critical density of states we must consider at least the 2-copy theory. Since the density operator is $cos\phi$, it is natural to consider

$$\Phi_\delta = e^{i\phi_1} e^{-i\phi_2} \Phi_{\eta/\xi}^1 \Phi_{\eta/\xi}^2$$

(5.20)
where $\Phi_{\eta/\xi}^i$ is a field in the $i$-th $\eta/\xi$ copy. Each $\eta/\xi$ copy describes dense polymers with a variety of non-trivial dimensions \[34\], and dividing by $u(1)_0$ does not affect these scaling dimensions. Interestingly, there have been some recent attempts at describing the Quantum Hall transition starting from polymers rather than percolation \[33\]. See also ref. \[12,24\].

Dividing by $u(1)_0$ endows the combined $\exp i\phi$ factors with dimension 2. If we simply take $\Phi_{\eta/\xi}^1$ as the 1-leg operator with $\Gamma = -3/16$ and $\Phi_{\eta/\xi}^2$ as a 0-leg, or twist, operator with $\Gamma = -1/4$, this gives $\Gamma_\delta = 25/16$. This leads to $\nu = 16/7 \approx 2.29$. This appears to be the possibility that is closest to the numerical value of 2.35 in this scheme \[4\].

VI. CONCLUSIONS

We have described how the RG flows based on the all-orders $\beta$ function for both the Chalker-Coddington and spin network models are attracted to a true singularity after a finite scale transformation. The nature of these singularities remains to be understood. The RG flows in other domains are regular and can be continued to arbitrarily large length scales.

Using an argument that is independent of the $\beta$ function, we proposed that the fixed points of general current-current interactions correspond to solutions of the Virasoro master equation. For the network models these correspond to coset conformal field theories, the most promising being $osp(2N|2N)_1/u(1)_0$ for the $u(1)$ network and $osp(4N|4N)_1/su(2)_0$ for the spin network. Both of these theories lead to multifractal properties that we have analyzed.

For the spin Quantum Hall transition, it remains an open question whether the $osp(4N|4N)_1/su(2)_0$ for $N = 1$ is equivalent to percolation. This can perhaps be answered by comparing with the recent partition functions in \[36\]. It would also be interesting to perform numerical simulations of the multifractal properties of the spin network model and compare them with the coset prediction of $\alpha_0 = 9/4$, since this result relies on averages of all higher moments and is thus not accessible from percolation.

VII. ACKNOWLEDGMENTS

We would like to thank F. Evers, M. Halpern and D. Khmelnitskii, A. Ludwig, A. Mirlin and N. Read for discussions. A.L. would like to thank the group at LPTHE, Jussieu, Paris for their hospitality. This work is in part supported by the NSF and the CNRS.

VIII. PAULI MATRIX AND INTERTWINNER IDENTITIES

Using the relations $[\sigma^a, \sigma^b] = f^{abc}\sigma^c$, $\{\sigma^a, \sigma^b\} = \frac{1}{2}\delta^{ab}$ and $\epsilon^a = -1$, $(\sigma^a)\dagger = \sigma^a$, and $\epsilon\sigma^c = -\sigma^c\epsilon$, one can establish the following identity

$$\sigma^{a}_{ij}\sigma^{a}_{nm} = \frac{1}{2}\delta_{im}\delta_{jn} - \frac{1}{4}\delta_{ij}\delta_{nm} \quad (8.1)$$

The intertwiners satisfy the following identities
\[
\rho^a \rho^b = \frac{1}{2} f^{abc} \sigma^c - \frac{1}{2} \delta^{ab}, \quad \rho^a \sigma^b = \frac{1}{2} f^{abc} \sigma^c + \frac{1}{4} \delta^{ab} \epsilon \\
\sigma^a \rho^b = \frac{1}{2} f^{abc} \rho^c + \frac{1}{4} \delta^{ab} \epsilon, \quad \rho^a \sigma^b = -\frac{1}{2} f^{abc} \sigma^c - \frac{1}{4} \delta^{ab} \epsilon \\
\rho^a \sigma^b = -\frac{1}{2} f^{abc} \sigma^c - \frac{1}{4} \delta^{ab} \epsilon
\]

(8.2)
REFERENCES

[1] J. T. Chalker and P. D. Coddington, J. Phys. C 21 (1988) 2665.
[2] A. W. W. Ludwig, M. P. A. Fisher, R. Shankar and G. Grinstein, Phys. Rev. B 50 (1994) 7526.
[3] C.-M. Cho and J. T. Chalker, Phys. Rev. B 54, 8708 (1996).
[4] A. LeClair, to appear in Phys. Rev. B, cond-mat/0011413.
[5] D. Bernard and A. LeClair, to appear in Phys. Lett. B, hep-th/0103096.
[6] V. Kagalovksy, B. Horovitz, Y. Avishai and J. T. Chalker, Phys. Rev. Lett 82 (1999) 3516.
[7] T. Senthil, J. B. Marston and M. P. A. Fisher, Phys. Rev. B 60 (1999) 4245.
[8] I. A. Gruzberg, A. W. W. Ludwig and N. Read, Phys. Rev. Lett. 82 (1999) 4524.
[9] R. Gade and F. Wegner, Nucl. Phys. B 360 (1991) 213; R. Gade, Nucl. Phys. B 398 (1993) 499.
[10] S. Guruswamy, A. LeClair and A. W. W. Ludwig, Nucl. Phys. B 583, 475 (2000).
[11] M. B. Halpern, E. Kiritsis, N. A. Obers and K. Clubok, Phys. Rep. 265, 1 (1996).
[12] V. Gurarie, cond-mat/9907502.
[13] M. Zirnbauer, hep-th/9905054.
[14] M. Bhaseen, I. I. Kogan, O. A. Soloviev, N. Taniguchi and A. M. Tsvelik, Nucl.Phys. B 580 (2000) 688-720 (cond-mat/9912060).
[15] R. Klesse and M. Metzler, Europhys. Lett. 32 (1995) 229.
[16] F. Evers, A. Mildenberger and A. D. Mirlin, cond-mat/0105297.
[17] After this work was completed, it was pointed out to us that this scaling relation appeared in a footnote in the paper [13]. There, the derivation relies on a more general result $\tau_G(q) = 2 + \tau(q) + \tau(1-q)$ and the form of $\tau(q)$ coming from the theories in [13,14]. (See section V for definitions.) Under our assumptions this is equivalent to eq. (5.16).
[18] C. Mudry, C. Chamon and X.-G. Wen, Nucl. Phys. B 466 (1996) 383.
[19] D. Bernard, (Perturbed) conformal field theory applied to 2d disordered systems: an introduction, Cargese lectures, NATO Science Series: Physics B, vol. 362 (1997), L. Baulieu et. al. (eds); hep-th/9509137.
[20] B. Gerganov, A. LeClair and M. Moriconi, Phys.Rev.Lett. 86 (2001) 4753.
[21] L. Rozansky and H. Saleur, Nucl. Phys. B346 (1992) 461.
[22] D. Bernard and A. LeClair, cond-mat/0003073, to appear in Phys. Rev. B.
[23] M. J. Bhaseen, Nucl.Phys. B604 (2001) 537-550.
[24] M. J. Bhaseen, J.-S. Caux, I. I. Kogan and A. M. Tsvelik, cond-mat/0012240.
[25] Z. Maassarani and D. Serban, Nucl.Phys. B489 (1997) 603-625.
[26] V.G. Knizhnik and A. B. Zamolodchikov, Nucl. Phys. B247 (1984) 83.
[27] A. Nersesyan, A. Tsvelik and F. Wegner, Phys. Rev. Lett. 72 (1994) 2628.
[28] K. Pracz, M. Janssen and P. Freche, J. Phys. Condensed Matter 8 (1996) 7147.
[29] M. Janssen, M. Metzler and M. R. Zirnbauer, Phys. Rev. B 59 (1999) 15836.
[30] R. Klesse and M. R. Zirnbauer, Phys. Rev. Lett. 86 2094 (2001).
[31] B. Huckestein, Rev. Mod. Phys. 67 (1995) 357.
[32] H. E. Castillo et. al., Phys. Rev. B56 (1997) 10668; J. S. Caux, N. Taniguchi and A. M. Tsvelik, Phys. Rev. Lett. 80, 1276 (1998); J. S. Caux, Phys.Rev.Lett. 81 (1998) 4196-4199.
[33] D. Friedan, E. Martinec and S. Shenker, Nucl. Phys. B271 (1986) 93.
[34] H. Saleur, Nucl. Phys. B382 (1992) 486; hep-th/9111007.
[35] J. E. Moore, cond-mat/0104033.
[36] N. Read and H. Saleur, hep-th/0106124.