Plateaux Transitions from S-matrices based on $SL(2, Z)$ Invariant Field Theories

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Abstract

A scattering description is proposed for a boundary perturbation of a $c = 1$ $SL(2, Z)$ invariant conformal field theory. The bulk massless S-matrices are of the form of Zamolodchikov’s staircase model. Using the boundary version of the thermodynamic Bethe ansatz, we show that the boundary free energy goes through a series of integer valued plateaux as a function of system size.
I. INTRODUCTION

A simple model was recently proposed in connection with Quantum Hall plateaux transitions [1]. After bosonization, the model consists of a scalar field coupled to a pure gauge field with a boundary interaction incorporating a circular defect line of impurities. The $c = 1$ conformal field theory possesses an $SL(2, \mathbb{Z})$ symmetry. Though the precise connection with the complete Landau problem in the presence of disorder is not entirely clear, the model does have some promising features, and we continue to investigate it in this paper.

In the next section an exact S-matrix description of the theory is proposed. We cannot claim to give here an absolute derivation of this S-matrix since we are missing a more precise treatment of the zero-mode constraint (Eq. (3)). Rather we give a suggestive derivation based on the prescriptive treatment described in [1].

The proposed scattering description allows an exact computation of the boundary (impurity) contribution to the free energy, log $g$. It reveals an infinite series of plateaux at integer values of $g$.

II. S-MATRICES

Let us first summarize some of the features of the model described in [1]. The conformal field theory was defined by the euclidean action

$$ S = \int dt d\sigma \left( \frac{1}{8\pi} (\partial_\mu \varphi)^2 - \frac{i}{2\pi \hat{g}} \epsilon_{\mu\nu} \partial_\nu \varphi A_\mu - \frac{1}{2\pi \hat{g}^2} A_\mu^2 \right) \tag{1} $$

where $A_\mu$ is the electro-magnetic gauge field. The above theory lives on a cylinder with $0 < \sigma < 2\pi$, and in the folded boundary version which we consider here, $0 < t < \infty$. The lagrangian possesses the gauge symmetry

$$ \mathcal{L}_{\text{cft}}(\tilde{\varphi} + \frac{2}{\hat{g}} \lambda, A_\mu + \partial_\mu \lambda) = \mathcal{L}_{\text{cft}}(\tilde{\varphi}, A_\mu) \tag{2} $$

where $\partial_\mu \tilde{\varphi} = -i \epsilon_{\mu\nu} \partial_\nu \varphi$. The gauge field was taken to be a pure, singular gauge $A_\mu = \partial_\mu \chi$, and the theory was supplemented by the zero mode constraint

$$ \frac{\theta}{2\pi \hat{g}} \oint dx_\mu \partial_\mu \varphi = \oint dx_\mu \partial_\mu \chi \tag{3} $$

When $\sigma_{xx} = 0$, the parameter $\theta$ was related to the Hall conductivity as $\sigma_{xy} = 1/\theta$.

The pure gauge field can be gauged away using

$$ \mathcal{L}_{\text{cft}}(\tilde{\varphi} + \frac{2}{\hat{g}} \chi, \partial_\mu \chi) = \mathcal{L}_{\text{cft}}(\tilde{\varphi}, 0) \tag{4} $$

Using the zero mode constraint Eq. (3), this leads us to consider the gauge transformation

$$ \tilde{\varphi} \rightarrow \tilde{\varphi} + \frac{\theta}{\pi \hat{g}^2} \varphi \tag{5} $$
From the transformation (5) on the primary fields of the theory, one finds that the partition function has an $SL(2,\mathbb{Z})$ invariance acting on the modular parameter $\tau = \theta/2\pi + ig^2/2$.

Upon adding a circular defect line of impurities, in the folded theory this corresponds to a boundary term in the action $\int d\sigma \cos (\varphi(0,\sigma)/\sqrt{2})$, where the boundary is at $t = 0$. Performing the gauge transformation (5) led to the boundary field theory

$$S = S_{bcft} + \lambda \int d\sigma \cos \left( b \varphi + \frac{\theta}{\pi g^2} \varphi \right)$$

(6)

where $b = \sqrt{2}$. The boundary conformal field theory $S_{bcft}$ is that of a free scalar field, but with the unusual boundary condition (1):

$$\partial_t \varphi - \frac{i\theta}{\pi g^2} \partial_\sigma \varphi = 0, \quad (t = 0)$$

(7)

The above theory is massless in the bulk, but with a mass/length scale at the boundary. To obtain an S-matrix description of this theory we follow the ideology described in [4]. Namely, we imagine turning on an integrable bulk interaction such that the integrability is not destroyed by the boundary interactions. This selects a special basis of particles in the bulk that diagonalize the boundary interactions. We then take a massless limit in the bulk. A bulk interaction that is compatible with the boundary interaction, as far as integrability goes, is $\delta S_{\text{bulk}} = \Lambda \int dt d\sigma \cos (b \varphi)$, where again $b = \sqrt{2}$. The reason is, that since $(\partial_\mu \varphi)^2 = -(\partial_\mu \varphi)^2$, written in terms of $\tilde{\varphi}$, the above theory, without the gauge field, is equivalent to the boundary sine-Gordon model, which is known to be integrable [5]. Performing the same gauge transformation Eq. (5) leads us to consider $\delta S_{\text{bulk}} = \Lambda \int dt d\sigma \cos (b \varphi + \theta \varphi/\pi g^2)$.

Next consider the boundary condition Eq. (7). Using $\epsilon_{Lt} = -\epsilon_{t\sigma} = 1$, one has $i\partial_\sigma \varphi = \partial_t \tilde{\varphi}$. Thus the boundary condition can be written as

$$\partial_t \left( \varphi - \frac{\theta}{\pi g^2} \tilde{\varphi} \right) = 0, \quad (t = 0)$$

(8)

This is a Neumann boundary condition for the combination $\varphi - \theta \tilde{\varphi}/\pi g^2$. All of this leads us to consider the bulk theory

$$S_{\text{bulk}} = \int dt d\sigma \left[ \frac{1}{8\pi} \left( \partial_\mu \left( \varphi - \frac{\theta}{2\pi} \tilde{\varphi} \right) \right)^2 + \Lambda \cos \sqrt{2} \left( \tilde{\varphi} + \frac{\theta}{2\pi} \varphi \right) \right]$$

(9)

where we have defined $\hat{\theta} = 2\theta/g^2$. When $\Lambda = 0$, the boundary version of the above free theory leads to the boundary condition (8).

Let us now rewrite the theory in terms of $\tilde{\varphi}$. As far as the bulk theory is concerned, we can drop the topological term $\partial_\mu \varphi \partial_\mu \tilde{\varphi}$, since it is identically zero. Using $(\partial_\mu \varphi)^2 = -(\partial_\mu \varphi)^2$, and defining a rescaled field $\tilde{\varphi} \rightarrow i\tilde{\varphi}/\sqrt{1 - (\hat{\theta}/2\pi)^2}$, one finds

$$S_{\text{bulk}} = \int dt d\sigma \left( \frac{1}{8\pi} (\partial_\mu \tilde{\varphi})^2 + \Lambda \cosh (b_L \tilde{\varphi}_L + b_R \tilde{\varphi}_R) \right)$$

(10)

where we have used the left-right decompositions $\varphi = \varphi_L + \varphi_R$, $\tilde{\varphi} = \varphi_L - \varphi_R$, and
When $b_L = b_R$, the above bulk theory is the well-known sinh-Gordon model. Consider now the massless limit $\Lambda \to 0$. The theory has both left and right moving particles. For the right-movers we parameterize the energy and momentum by $E = P = \mu e^\beta$, and for the left movers $E = -P = \mu e^{-\beta}$, where $\beta$ is a rapidity and $\mu$ an arbitrary energy scale. One can describe the theory in terms of an S-matrix $S_{LL}(\beta)$ for the left-movers and $S_{RR}(\beta)$ for the right-movers \[1\]. From the form of the bulk interaction in Eq. (10) it is clear that $S_{LL}$ is the sinh-Gordon S-matrix defined by the sinh-Gordon coupling $b_L$, whereas $S_{RR}$ is defined by $b_R$. One argument is the following. In the sine-Gordon version, one can characterize the S-matrices by the non-local quantum affine symmetry \[2\]. This symmetry survives in the massless limit, and the left-moving (right-moving) quantum affine symmetry will have $q$-deformation parameter defined by $b_L (b_R)$. Using the known S-matrix for the sinh-Gordon model \[3\], the result is thus

$$S_{LL}(\beta) = \frac{\tanh \frac{1}{2} (\beta - i\pi \gamma_L)}{\tanh \frac{1}{2} (\beta + i\pi \gamma_L)}$$

(12)

where

$$\gamma_L = \frac{b_L^2}{2 + b_L^2} = \frac{1}{2} + \frac{\hat{\theta}}{4\pi}$$

(13)

Similarly, $S_{RR}$ is given by Eq. (12) with

$$\gamma_R = \frac{b_R^2}{2 + b_R^2} = \frac{1}{2} - \frac{\hat{\theta}}{4\pi} = 1 - \gamma_L$$

(14)

Using the invariance of the S-matrix under $\gamma \to 1 - \gamma$, one sees that $S_{LL} = S_{RR}$.

We discuss now the $SL(2,\mathbb{Z})$ properties of the above S-matrix. As described in \[4\], the bulk conformal field theory has an $SL(2,\mathbb{Z})$ invariance acting on the modular parameter $\tau = \theta/2\pi + i\hat{g}^2/2$. More specifically, the partition function on the torus is invariant under $SL(2,\mathbb{Z})$. Since $S_{LL}, S_{RR}$ provide a scattering description of this conformal field theory, we expect the S-matrix to be at least in part characterized by this symmetry. Under $\tau \to -1/\tau$, one has that $(\hat{g}, \theta) \to (\hat{g}', \theta')$ where

$$\hat{g}^2 = \frac{\hat{g}^2}{(\hat{g}^4/4 + (\theta/2\pi)^2)}, \quad \theta' = -\frac{\theta}{(\hat{g}^4/4 + (\theta/2\pi)^2)}$$

(15)

Note that under this transformation, $\theta/\hat{g}^2 \to -\theta/\hat{g}^2$, which implies $\hat{\theta} \to -\hat{\theta}$. Thus $\tau \to -1/\tau$ simply exchanges left and right movers:

$$\tau \to -1/\tau \quad \implies \quad b_L^2 \to b_R^2$$

(16)

Since $S_{LL} = S_{RR}$, the scattering theory has this invariance.

The other independent generator of $SL(2,\mathbb{Z})$ corresponds to $\tau \to \tau + 1$, which corresponds to $\theta \to \theta + 2\pi$. The S-matrices turn out to only be invariant under a multiple of
this transformation. Namely, using the invariance of the S-matrices under \( \gamma \rightarrow \gamma + 2 \), one verifies their invariance under \( \theta \rightarrow \theta + 4\pi \hat{g}^2 \). For the original fermion model considered in [7] with \( \hat{g} = \sqrt{2} \) this corresponds to \( \theta \rightarrow \theta + 8\pi \). The significance of this is not entirely clear.

We now consider performing the analytic continuation \( \theta \rightarrow i\theta \). There are at least two justifications for doing this. First, the Hall conductivity computed in [1] is imaginary unless one performs this continuation. Second, in order to formulate the above scattering description one needs to have identified a “time” by continuing to Minkowski space. Let us analytically continue to Minkowski space by identifying \( \sigma \) as the time (based on the boundary interaction) and letting \( \sigma \rightarrow i\sigma \). The Minkowski action \( S_M \) is obtained from the euclidean one by \( S_M = iS(\sigma \rightarrow i\sigma) \). This leads to the analytic continuation of \( \theta \rightarrow i\theta \). To see this, let us rescale \( \chi \rightarrow \theta \chi \). Then the euclidean topological term is

\[
S_{\text{top}} \propto i\theta \int dt d\sigma (\partial_t \varphi \partial_\sigma \chi - \partial_\sigma \varphi \partial_t \chi)
\]

Letting \( \sigma \rightarrow i\sigma \) and identifying \( S_M \) as above one finds that the Minkowski action is obtained by the analytic continuation \( \theta \rightarrow i\theta \). Performing this continuation in Eq. (11) one finds that \( b_R = b_L^* \), \( b_L b_R = 2 \) and that the S-matrices now have the parameters:

\[
\gamma_L = \frac{1}{2} + i\frac{\hat{\theta}}{4\pi}, \quad \gamma_R = \gamma_L^* \tag{18}
\]

The resulting S-matrix has the same form as the bulk massive staircase model of Zamolodchikov [7], where there \( \gamma \) was taken as \( \gamma_{\pm} = 1/2 \pm i\theta_0 / \pi \). However since we are here dealing with a massless bulk theory, the interpretation is different and in fact our theory doesn’t suffer from some of the problems of interpretation of the bulk massive staircase model. Namely, since there is no bulk interaction in the massless limit, just a free scalar field, the theory doesn’t have the reality problems of the massive case arising from complex values of the sinh-Gordon coupling \( b \). The above scattering theory is a limiting case of the bulk massive theories studied in [8].

Finally, the interactions at the boundary are described by a reflection S-matrix \( R(\beta) \) for reflection of the above particles off the boundary. In the massless sinh-Gordon model, this reflection S-matrix, which can be obtained by taking the explicit massless limit of the boundary sinh-Gordon model, is known to be independent of the sinh-Gordon coupling \( b_L \). (See [9]). The result should be the same in our case and we thus take

\[
R(\beta) = \tanh(\beta/2 - i\pi/4) \tag{19}
\]

The physical S-matrix for right movers is \( R(\beta - \beta_B) \), and for left movers \( R(\beta + \beta_B) \), where \( \mu e^{\beta_B} \) is defined as a physical boundary energy scale.

### III. PLATEAUX TRANSITIONS IN THE BOUNDARY ENTROPY

We consider now the theory on a semi-infinite cylinder of circumference \( L \), where \( \sigma \) runs along the circumference and \( \sigma \sim \sigma + L \) and as before \( 0 < t < \infty \). Viewing \( \sigma \) as the time, the Hilbert space lives on the semi-infinite line \( 0 < t < \infty \), and finite size \( L \) effects
can be computed from the “L-channel” thermodynamic Bethe ansatz [10][11]. Of interest is the boundary entropy \( \log g \) [12], defined as the contribution to the free energy that is independent of the length \( R \) of the cylinder, \( \log Z = \log g + \log Z_{\text{bulk}} \), where \( \log Z_{\text{bulk}} \) is proportional to \( R \). Viewing the theory as a 1 + 1 dimensional quantum system, \( g \) represents the ground state degeneracy, i.e. it counts the number of states at the boundary.

Using the formulas in [11], one has

\[
\log g = \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\beta \left[ \partial_\beta \log R(\beta) \right] \log \left( 1 + e^{-\varepsilon(\beta)} \right)
\]

(20)

where \( \varepsilon(\beta) \) is a solution of the integral equation

\[
\varepsilon(\beta) = \frac{L}{2\xi_B} e^\beta - \frac{1}{2\pi i} \int d\beta' \left[ \partial_\beta \log S_{LL}(\beta - \beta') \right] \log \left( 1 + e^{-\varepsilon(\beta')} \right)
\]

(21)

In the above equation, \( \xi_B \) is a boundary length scale \( 1/\xi_B = \mu e^\beta_B \); it defines an energy scale at the boundary \( E_B = 1/\xi_B \).

The numerical solution of \( g \) as a function of \( \xi_B/L \) is shown for the two values of \( \hat{\theta} = 200, 100 \) in figures 1 and 2. There are two important features of these figures. The first is that on the plateaux \( g \) takes on the series of integers \( g = 1, 2, 3, .. \), as anticipated in [8]. The second is that the plateaux are more clearly defined as \( \hat{\theta} \) is increased.

**IV. DISCUSSION**

We conclude with a discussion of the possible implications of our results for the proposal made in [1] in connection with the Quantum Hall transitions. The integer valued plateaux that we found in the boundary state degeneracy is a promising feature. The meaning of the boundary entropy suggests that as the scale \( L \) is decreased, the model goes through a series of transitions where at each transition one more state becomes localized at the impurities. In order to verify this picture one needs to relate the boundary entropy of our 2-dimensional model to the 2 + 1 dimensional system.

The significance of \( \hat{\theta} = \pi \), as discussed in [1], cannot be seen from what we have done in this paper since we have not studied the conductivities \( \sigma_{xx}, \sigma_{xy} \). In the conformal field theory it was argued that \( \hat{\theta} = \pi \) implies \( \sigma_{xx} = 0, \sigma_{xy} = 1/\theta \). One needs to study the conductivities in the presence of the boundary interaction. It should be possible to do this using form factors. Since the conductivities are reduced to boundary effects, it seems likely that the transitions in the boundary entropy will entail transitions in the conductivity, but only a detailed analysis will tell.

A related issue which needs clarification concerns whether the scattering theory described here implicitly treats the boundary condition in a way that is consistent with the perturbative treatment in [1], which, when \( \hat{\theta} = \pi \), led to the correlation length exponent 20/9.
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FIG. 1. Ground state degeneracy as a function of $L/\xi_B$ for $\hat{\theta} = 200$. 

\[ \log(2 \xi_B / L) \]

boundary state degeneracy (g)
FIG. 2. Ground state degeneracy as a function of $L/\xi_B$ for $\theta = 100$. 

boundary state degeneracy (g)

$\log(2\xi/L)$