Comments on the dependence between electric charge and magnetic charge\(^1\)

Kang Li\(^2\)

Department of Physics, Zhejiang University, Hangzhou, 310027, P.R. China
and
Jefferson Physical Laboratory, Harvard University, Cambridge, MA 02138, USA

Abstract

By using the two 4-dimensional potential formulation of electromagnetic (EM) field theory introduced in [1], we found that the SO(2) duality symmetric EM field theory can be reduced to the magnetic source free case by a special choice of SO(2) parameter, this special case we called nature picture of the EM field theory, the reduction condition led to a result, i.e. the electric charge and magnetic charge are no more independent. Some comments to paper [10] are also mentioned.

Keywords: Electromagnetic field theory; SO(2) duality symmetry; Magnetic charge.

PACS: 03.50.De, 11.30.-j, 14.80.Hv

\(^1\)The paper is partially supported by the National Natural Science Foundation of China and the Nature Science Foundation of Zhejiang Province under Grant Nos 102011 and 102028

\(^2\)kangli@zimp.zju.edu.cn
1 Introduction

Recently there has been a much interest in study of EM duality symmetry\cite{1}\cite{2}\cite{3}\cite{4}\cite{5}\cite{6}\cite{7}\cite{8}, because it plays a very important role in study of duality symmetry in string theory\cite{3}. In recent papers \cite{1}\cite{2}, we introduced a two 4-dimensional potential formulation to describe the duality symmetry of classical EM field theory, the theory is Lorentz covariant, and it has manifesting \textit{SO}(2) duality symmetry, moreover, we don’t have the non physical singularities around magnetic charge, i.e. we do not need the concept of Dirac string. As a matter of fact, we know that the classical and quantum electrodynamics match the experiment very well, and we never find the individual monopoles (Magnetic charges) before, it seems we don’t need the theory which include the magnetic sources, but as we know the one generator of duality group \textit{SL}(2, \mathbb{Z}) corresponds to the EM duality, if there is EM duality symmetry, magnetic sources must be introduced\cite{1}. Fortunately, in the two 4-dimensional potential formalism of the EM field theory, we find a light of hope to solve this conflict. In this formulation, because the theory invariant under the \textit{SO}(2) duality transformation, the theories are equivalent with different choice of the \textit{SO}(2) parameter. In a special case, we call it a nature picture, the equivalent magnetic charge equals to zero, and the theory returns to the usual magnetic source free electrodynamics. The condition for this nature picture led to the electric charge and magnetic charge related each other, i.e. they are dependent. The paper is organized as follows. In the next section, we will give a brief review of the two potential formulation introduced in \cite{1}, where Maxwell equations is written in both \textit{SO}(2) and Lorentz covariant way. In third section, that is the main section of this paper, we will explain how 2 potential formulation can be reduced to usual one potential formulation in the nature picture, and give out the explicit relationship between electric charge and magnetic charge, and the general expressions of Lorentz force and Aharonov-Bohm phase factor in nature picture as well as in general $\theta$ picture are also discussed. Finally there are some discussion and short comment to reference \cite{10} are given in the last section.

2 The review of two potential vector formulation

For completeness, we will give a brief review of the two potential formulation of EM field theory introduced in \cite{1}. Under this formulation, the main results of this letter can be obtained in the next section.

As we know, in the usual one potential formalism, the Maxwell equations have duality
symmetry in the source free case, but this duality symmetry will broken down immediately when the electric source switch on, in order to recover the duality symmetry, the magnetic sources must be introduced. The magnetic charge called magnetic monopole was first introduced by Dirac\[13\], but in usual one potential formulation there must exist a line around the magnetic charge on which the vector potential is singular. This is the so called Dirac string. The advantage of the two potential formulation is that the theory is $SO(2)$ duality covariant and no need to use the non-physical concept of Dirac string. Now let’s review this theory briefly.

Besides the usual definition of 4-dimensional potential which we called $A^1_\mu$, i.e.

$$A^1_\mu = (\phi_1, -A_1), \text{ or } A^{\mu 1} = (\phi_1, A_1),$$

we also introduce

$$A^2_\mu = (\phi_2, -A_2), \text{ or } A^{\mu 2} = (\phi_2, A_2),$$

where $\phi_1$ and $A_1$ are usual electric scalar potential and magnetic vector potential in electrodynamics, while the newly introduced potential $\phi_2$ is the scalar potential associated to the magnetic field and $A_2$ is a vector potential associated to the electric field. Using these potentials, the electric field strength $E$ and the magnetic induction $B$ are then expressed as:

$$E = -\nabla \phi_1 - \frac{\partial A_1}{\partial t} + \nabla \times A_2,$$  \hfill (2.3)

$$B = \nabla \phi_2 + \frac{\partial A_2}{\partial t} + \nabla \times A_1.$$  \hfill (2.4)

In the magnetic source free case, $\phi_2$ and $A_2$ are expected to be zero, so the above equation returns to the usual magnetic source free case.

By introducing two field tensors as

$$F^I_{\mu \nu} = \partial_{\mu} A^I_{\nu} - \partial_{\nu} A^I_{\mu}, \quad I = 1, 2.$$  \hfill (2.5)

and choosing Lorentz gauge $\partial_{\mu} A^I_{\mu} = 0$, then Maxwell equations in the case of existing both electric and magnetic sources, i.e.,

$$\nabla \cdot E = \rho_e, \quad \nabla \times B = j_e + \frac{\partial E}{\partial t},$$  \hfill (2.6)

$$\nabla \cdot B = \rho_m, \quad \nabla \times E = -j_m - \frac{\partial B}{\partial t},$$  \hfill (2.7)
can be recast as
\[ \partial^\mu F_{\mu\nu}^I = g^{I'I'} J_{\nu}^{I'}, \]  
(2.8)

where
\[ g^{I'I'} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

and
\[ J_1^I = J_1^e = (\rho_e, -j_e), \quad J_2^I = J_2^m = (\rho_m, -j_m). \]  
(2.9)

In this formulation the currents are manifestly conserved:
\[ \partial^\nu J_{\nu I} \propto \partial^\nu \partial^\mu F_{\mu\nu}^I = 0. \]  
(2.10)

Where the index \( I \) can be regarded as \( SO(2) \) index \([2]\), so this formulation has manifestly \( SO(2) \) duality symmetry and it related to the gauge transformations \( A_{\mu}^I \rightarrow A_{\mu}^I + \partial_{\mu} \chi^I \).

From (2.3, 2.4) and the definition (2.5) we can obtain,
\[ E_i = F_{0i}^1 + \ast F_{0i}^2, \]  
(2.11)

and
\[ B_i = \ast F_{0i}^1 - F_{0i}^2. \]  
(2.12)

So it is convenient to define a new field tensor as
\[ F_{\mu\nu} = F_{\mu\nu}^1 + \ast F_{\mu\nu}^2, \]  
(2.13)

\[ \tilde{F}_{\mu\nu} = \ast F_{\mu\nu}^1 - F_{\mu\nu}^2, \]  
(2.14)

where \( \tilde{F}_{\mu\nu} \) is exactly the Hodge star dual of \( F_{\mu\nu} \). As we shall see, using these new field tensors we can easily express the duality symmetry in a compact fashion. It is easy to see that \( F_{\mu\nu} \) is the analog to the usual electromagnetic tensor defined in classical electrodynamics, because they have exactly the same matrix form in terms of the field strengths. Since the vector potentials in our formalism have no singularities, one has \( \partial^\mu \ast F_{\mu\nu}^I = 0 \), so Maxwell’s equations can also be written as
\[ \partial^\mu F_{\mu\nu} = \partial^\mu F_{\mu\nu}^1 = J_1^\nu \]
\[ \partial^\mu \tilde{F}_{\mu\nu} = -\partial^\mu F_{\mu\nu}^2 = J_2^\nu \]  
(2.15)

we can check that the potential functions defined above satisfy the following equations,
\[ \frac{\partial^2}{\partial t^2} \phi_1 - \nabla^2 \phi_1 = \rho_e, \]
\[ \frac{\partial^2}{\partial t^2} A_1 - \nabla^2 A_1 = j_e, \]
(2.16)
\[ \frac{\partial^2}{\partial t^2} \phi_2 - \nabla^2 \phi_2 = -\rho_m, \]
\[ \frac{\partial^2}{\partial t^2} A_2 - \nabla^2 A_2 = -j_m \]

In the static case, i.e. when the sources do not depend on time \( t \), we can write

\[ \rho_I = \rho_I(x), \quad J_I = J_I(x), \quad I = 1 \text{ and } 2, \]
(2.17)

where \( I = 1, 2 \) represent \( I = e, m \) respectively. Then exactly as it is done in the standard classical electrodynamics \([13]\), the solutions of equations (2.16) are given by

\[ \phi_I = \frac{1}{4\pi} g^{I'I} \int \frac{\rho_{I'}(x')}{r} d^3x' \]
(2.18)
\[ A_I(x) = \frac{1}{4\pi} g^{I'I} \int \frac{J_{I'}(x')}{r} d^3x', \]
(2.19)

where \( r = |x - x'| \), then from equations (2.3) and (2.4) we find that the field strengths have the following representation

\[ E(x) = \frac{1}{4\pi} \int \rho_e(x') \frac{r}{r^3} d^3x' - \frac{1}{4\pi} \int J_m(x') \times \frac{r}{r^3} d^3x' \]
(2.20)
and
\[ B(x) = \frac{1}{4\pi} \int \rho_m(x') \frac{r}{r^3} d^3x' + \frac{1}{4\pi} \int J_e(x') \times \frac{r}{r^3} d^3x'. \]
(2.21)

### 3 The dependence of the electric charge and magnetic charge

This section is the main part of this letter. From the review section above, we know that the two potential formulation of EM field theory has manifestly \( SO(2) \) duality symmetry, so for different \( SO(2) \) transformation parameter \( \theta \), the theory should be equivalent. We can see from the subsequence, special choice of \( \theta \), the two potential formulation can be recast to magnetic source free one potential formulation. The special choice of \( \theta \) led to electric source and magnetic source related each other. This condition has also been discussed in reference \([14]\) in a quite different manner, but some of their results are not correct, we will comment this shortly in the conclusion section.
Let’s begin with the \( SO(2) \) duality symmetry of EM field theory. The general \( SO(2) \) dual transformation for \( F_{\mu\nu}, \tilde{F}_{\mu\nu} \), is given

\[
\begin{pmatrix}
    F'_{\mu\nu} \\
    \tilde{F}'_{\mu\nu}
\end{pmatrix} = \begin{pmatrix}
    \cos \theta & \sin \theta \\
    -\sin \theta & \cos \theta
\end{pmatrix} \begin{pmatrix}
    F_{\mu\nu} \\
    \tilde{F}_{\mu\nu}
\end{pmatrix} = R(\theta) \begin{pmatrix}
    F_{\mu\nu} \\
    \tilde{F}_{\mu\nu}
\end{pmatrix}
\] (3.1)

Where

\[
R(\theta) = \begin{pmatrix}
    \cos \theta & \sin \theta \\
    -\sin \theta & \cos \theta
\end{pmatrix}
\]

is the general \( SO(2) \) matrix. Under this transformation the Maxwell equations is invariant and the energy density and Poynting vectors of EM fields are invariant also.

As discussed in [2], we know that, under this dual transformation, the physical quantities \((E, B)\) and electromagnetic sources \(J^{\mu 1}, J^{\mu 2}\), i.e. \(\rho_e, \rho_m\) and \(J_e, J_m\), etc. must be also changed in the same manner. For example, we consider a dyon with electric charge \(q\) and magnetic charge \(g\), under the \(SO(2)\) dual transformation, the charges change like

\[
\begin{pmatrix}
    q' \\
    g'
\end{pmatrix} = R(\theta) \begin{pmatrix}
    q \\
    g
\end{pmatrix}.
\] (3.2)

that is,

\[
q' = q \cos \theta + g \sin \theta, \quad (3.3)
\]

\[
g' = -q \sin \theta + g \cos \theta. \quad (3.4)
\]

For arbitrary choice of \(\theta\), after this transformation, the electromagnetic field theory is equivalent to the theory before transformation. For examples, when \(\theta = \pi\), it is an identity transformation, when \(\theta = \frac{\pi}{2}\), the transformation reduced to the well known replacements as follows, \(q \rightarrow g\), \(g \rightarrow -q\) as well as \(F_{\mu\nu} \rightarrow \tilde{F}_{\mu\nu}, \tilde{F}_{\mu\nu} \rightarrow -F_{\mu\nu}\), \(J_{\mu 1} \rightarrow J_{\mu 2}, J_{\mu 2} \rightarrow -J_{\mu 1}\), \(E \rightarrow B, B \rightarrow -E\), etc., under this replacement, one equation changes to another equation in the Maxwell equations group, but as an equation group does not change, so the theory is equivalent. Now give a special choice of parameter \(\theta\) such that,

\[
q \sin \theta = g \cos \theta, \quad (3.5)
\]

i.e.

\[
\theta = \arctan \frac{g}{q}, \quad (3.6)
\]

Under this special case, which we call nature picture, we have,

\[
g' = 0. \quad (3.7)
\]
That is, in the nature picture, the equivalent magnetic charge is zero, and the two potential formulation given in the section above is expected returning to the usual one potential case, if this is true, we will arrive at a conclusion that the two potential formulation is equivalent to the usual one potential formulation, and the EM field theory with magnetic source is equivalent to a magnetic source free EM field theory, the condition for this equivalent is give by equation (3.5), i.e. the electric charge and magnetic charge are connected each other. In the sequel of this section, we will show that in the nature picture, the EM field theory does return to the one potential usual magnetic source free case.

Let me consider a dyon system again, in the nature picture, electric charge and magnetic charge are related by equation (3.5), i.e.

\[ g = q \tan \theta. \]  

(3.8)

because, as we know, for a dyon system we have \( \rho_e(x, t) = q\delta(x - x(t)), \) \( j_e = q\delta(x - x(t)) \frac{dx(t)}{dt}, \) and the \( \rho_m, j_m, J^2_{\mu} \) has the similar expressions just need change \( e \) to \( m \) and \( q \) to \( g \), so the condition (3.8) can also be written as

\[ \rho_m = \rho_e \tan \theta \]

(3.9)

\[ j_m = j_e \tan \theta \]

or simple,

\[ J^2_{\mu} = J^1_{\mu} \tan \theta \]

(3.10)

Because \( (\rho_e, \rho_m), (j_e, j_m) \) and \( (J^1_{\mu}, J^2_{\mu}) \) transfer same as \( (q, g) \), so in the nature picture, we have

\[ \rho'_m = 0, \quad j'_m = 0 \]  

(3.11)

and

\[ J^2_{\mu} = 0. \]  

(3.12)

So the second potential \( A^2_{\mu} \) will be vanished in the nature picture, i.e.

\[ A^2_{\mu} = -\frac{1}{4\pi} \int \frac{J^2_{\mu}(\mathbf{x}')}{r} d^3x' = 0, \]  

(3.13)

and hence, in the nature picture, the theory recovers to the usual magnetic source free case, and the Maxwell equation become

\[ \nabla \cdot \mathbf{E}' = \rho'_e, \quad \nabla \times \mathbf{B}' = j'_e + \frac{\partial \mathbf{E}'}{\partial t}, \]

\[ \nabla \cdot \mathbf{B}' = \rho'_m = 0, \quad \nabla \times \mathbf{E}' = -\frac{\partial \mathbf{B}'}{\partial t}. \]  

(3.14)
The \((E', B')\) and the original \((E, B)\) are related by

\[
E' = E \cos \theta + B \sin \theta, \tag{3.15}
\]

\[
B' = -E \sin \theta + B \cos \theta. \tag{3.16}
\]

where \(E, B\) are give by the equations (2.20) and (2.21), and \(E', B'\) have the similar expressions as,

\[
E'(x) = \frac{1}{4\pi} \int \rho'_e(x') \frac{r}{r^3} d^3x' - \frac{1}{4\pi} \int \mathbf{J}_m(x') \times \frac{r}{r^3} d^3x', \tag{3.17}
\]

and

\[
B'(x) = \frac{1}{4\pi} \int \rho'_m(x') \frac{r}{r^3} d^3x' + \frac{1}{4\pi} \int \mathbf{J}_e(x') \times \frac{r}{r^3} d^3x'. \tag{3.18}
\]

Because \(\rho'_m = 0, \mathbf{J}_m = 0\), so in nature picture we have

\[
E'(x) = \frac{1}{4\pi} \int \rho'_e(x') \frac{r}{r^3} d^3x', \tag{3.19}
\]

\[
B'(x) = \frac{1}{4\pi} \int \mathbf{J}_e(x') \times \frac{r}{r^3} d^3x'. \tag{3.20}
\]

On the other hand, we can also get equation (3.19, 3.20) from equations (3.15, 3.16) and (2.20, 2.21), for example,

\[
B' = -E \sin \theta + B \cos \theta
= \frac{1}{4\pi} \int (\rho_m(x') \cos \theta - \rho_e(x') \sin \theta) \frac{r}{r^3} d^3x' + \frac{1}{4\pi} \int (\mathbf{J}_e(x') \cos \theta + \mathbf{J}_m(x') \sin \theta) \times \frac{r}{r^3} d^3x'
= \frac{1}{4\pi} \int \mathbf{J}'_e(x') \times \frac{r}{r^3} d^3x',
\]

of course the condition (3.9) has been used to get the above result. This again shows that the nature picture correspond to the usual magnetic source free case.

The general two potential formulation, where both electric source and magnetic source are considered, can be reduced to the nature picture mentioned above, in this nature picture the EM field theory coincides with the magnetic source free usual EM field theory. The condition of the reduction (3.5) means that the real electric charge and magnetic charge are related each other, i.e. \(g = q \tan \theta\). When \(g = 0, q \neq 0\), we have to choose \(\theta = \pi\), which is a identity transformation of \(SO(2)\). Because in this case the theory is already in the nature picture, i.e. the magnetic source free case, so the reduction transformation should be identity. When \(g \neq 0, q = 0\), we have to choose \(\theta = \pi/2\), the reduction transformation is give by the follow replacement, \(g \rightarrow q, B \rightarrow E, E \rightarrow -B\), i.e. a pure magnetic source EM field theory is equivalent to a pure electric source EM field theory, the \(SO(2)\) duality transformation matrix for this two theory is given by

\[
R(\pi/2) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.21}
\]
Before finishing this section, I would like to give out the general expression of general Lorentz force and AB phase factor in the 2 potential formulation from their expression in the nature picture.

It is well know that the general Lorentz force in nature picture (magnetic source free case) is give by,

$$ F = q' (E' + v \times B'). $$

Using the transformation (3.3, 3.15, 3.16), and the reduction condition (3.5) we can get

$$ F = (q \cos \theta + g \sin \theta) [(E \cos \theta + B \sin \theta) + v \times (-E \sin \theta + B \cos \theta)] $$

$$ = qE + gB + v \times (qB - gE). $$

This is exact the general Lorentz force expression in the presence of magnetic source[16].

Now let me discuss the Aharonov-Bohm(AB) effect[9]. As it is know that the AB phase for a particle with electric charge $q'$, in nature picture, is given by[17],

$$ \Delta \phi = q' \int x A^1_{\mu} dx^\mu $$

$$ = q' \int x A^1_m \cdot d\Gamma - \int x \phi'_e(t) dt, $$

where

$$ A^1_{\mu}' = \frac{1}{4\pi} \int \frac{J^1_{\mu}'(x')}{r} d^3x'. $$

$J^1_{\mu}'$ has the similar transformation as $q'$, so using the transformation (3.3) and the reduction condition (3.5), after a simple algebra, we can get the AB phase in the two potential formulation where magnetic source is considered, which reads,

$$ \Delta \phi = q \int x A^1_\mu dx^\mu - g \int x A^2_\mu dx^\mu. $$

This equation tells us that in two potential formulation the AB phase factor contains two parts, one part is the contribution of the electric charge as we know before, another part is the contribution of the magnetic charge. In nature picture, this phase factor can be equated to a equivalent electric charge AB phase factor.

4 Conclusion remarks

Based on the $SO(2)$ duality symmetry of electromagnetic filed theory, we introduced a concept of the nature picture to EM field theory. By a special choice of $SO(2)$ parameter (reduction condition), the $SO(2)$ duality symmetric two potential formulation reduced the nature picture, in this picture the EM filed theory is the well know magnetic source
free theory. In other words, the EM field theory in the presence of magnetic source can be equated to equivalent a magnetic source free EM field theory.

Similar results of this paper have also been discussed in reference [10] with much different method, their reduction condition is same as ours, but I should comment that there exist also some mistakes in this paper. For example, in [10], if the Lorentz force is give by their equation (39), than their Maxwell equations (40 43) should contain electric source as well as magnetic source, but they didn’t, and then hence their transformation equations (46-47) are not correct. The correct transformation equation should be our equations (3.15, 3.16)\(^3\).

From the discussion above, we can conclude that in what I called Nature Picture, the real magnetic charge and electric charge are related by the reduction condition (3.8), and the electromagnetic field in the presence of magnetic source can be equivalently treated as the usual monopole free case, moreover the effect of monopoles can be measured through the change of the equivalent electric charge.

**Acknowledgements.** I would like to thank Professor Andrew Strominger for hospitality in Physics Department of Harvard University where this paper was written, and thanks are also given to Nancy Partridge for her kind helps during my research visit to Harvard.

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\(^3\)To compare their equations to ours, one should keep in mind that theirs \((E, B, \varepsilon, B)\) correspond to ours \((E', B', E, B)\).
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