Some sum-product type estimates for two-variables over prime fields

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Abstract

In this paper, we use a recent method given by Rudnev, Shakan, and Shkredov (2018) to improve results on sum-product type problems due to Pham and Mojarrad (2018).

1 Introduction and results

Let $A$ be a set in $\mathbb{Z}$. We define the sum and product sets as follows:

$$A + A = \{a + b: a, b \in A\},$$

$$A \cdot A = \{ab: a, b \in A\}.$$ 

A celebrated result of Erdős and Szemerédi [5] states that there is no set $A \subset \mathbb{Z}$ which has both additive and multiplicative structures. More precisely, given any finite set $A \subset \mathbb{Z}$, we have

$$\max\{|A + A|, |A \cdot A|\} \gg |A|^{1+\varepsilon}$$

for some positive constant $\varepsilon$.

Let $\mathbb{F}_q$ be an arbitrary finite field with order $q = p^r$ for some positive integer $r$ and an odd prime $p$. Bourgain, Katz, and Tao [11] showed that given any set $A \subset \mathbb{F}_p$ with $p$ prime and

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$p^\delta < |A| < p^{1-\delta}$ for some $\delta > 0$, one has

$$\max\{|A + A|, |A \cdot A|\} \geq C_\delta |A|^{1+\varepsilon},$$

for some $\varepsilon = \varepsilon(\delta) > 0$. Note that the relation between $\varepsilon$ and $\delta$ is difficult to determine.

Suppose that $|A + A| = m$ and $|A \cdot A| = n$, using Fourier analytic methods, Hart, Iosevich, and Solymosi gave an explicit bound over arbitrary finite fields as follows

$$|A|^3 \ll \frac{m^2 n |A|}{q} + q^{1/2} mn.$$

(1)

Note that this bound is only non-trivial when $|A| \gg q^{1/2}$. Using a graph theoretic method, Vinh obtained an improvement and as a consequence, derived a stronger bound for large sets, namely,

$$|A|^2 \leq \frac{mn |A|}{q} + q^{1/2} \sqrt{mn}.$$

(2)

The following are direct consequences of (2).

1. If $q^{1/2} \ll |A| < q^{2/3}$, then

$$\max\{|A + A|, |A \cdot A|\} \gg \frac{|A|^2}{q^{1/2}}.$$

(3)

2. If $q^{2/3} \leq |A| \ll q$, then

$$\max\{|A + A|, |A \cdot A|\} \gg (q|A|)^{1/2}.$$

(4)

Notice that these bounds were first proved by Garaev over prime fields by using exponential sums.

Let $G$ be a subgroup of $\mathbb{F}^*$, and an arbitrary function $g : G \to \mathbb{F}$. Define

$$\mu(g) := \max_{t \in \mathbb{F}} |\{x \in G : g(x) = t\}|.$$

Hegyvári and Hennecart studied generalizations of (3) and (4) for certain families of polynomials by using methods from spectral graph theory. The precise statements of their
results can be stated in two following theorems.

**Theorem 1.1 (Hegyvári and Hennecart, [7]).** Let $G$ be a subgroup of $F_p^*$. Consider the function $f(x,y) = g(x)(h(x) + y)$ on $G \times F_p^*$, where $g, h: G \to F_p^*$ are arbitrary functions. Define $m = \mu(g \cdot h)$. For any subsets $A \subset G$ and $B, C \subset F_p^*$, we have

$$|f(A,B)| |B \cdot C| \gg \min \left\{ \frac{|A||B|^2|C|}{pm^2}, \frac{|pB|}{m} \right\}.$$

**Theorem 1.2 (Hegyvári and Hennecart, [7]).** Let $G$ be a subgroup of $F_p^*$. Consider the function $f(x,y) = g(x)(h(x) + y)$ on $G \times F_p^*$, where $g, h: G \to F_p^*$ are arbitrary functions. Define $m = \mu(g)$. For any subsets $A \subset G$, $B, C \subset F_p^*$, we have

$$|f(A,B)||B + C| \gg \min \left\{ \frac{|A||B|^2|C|}{pm^2}, \frac{|pB|}{m} \right\}.$$

In [12], Pham and Mojarrad used a point-plane incidence bound due to Rudnev [15] to improve Theorems 1.1 and 1.2 when sets $A$, $B$, and $C$ are not too big.

**Theorem 1.3 (Pham-Mojarrad, [12]).** Let $f(x,y) = g(x)(h(x) + y)$ be a function defined on $F_p^* \times F_p^*$, where $g, h: F_p^* \to F_p^*$ are arbitrary functions. Define $m = \mu(g \cdot h)$. For any subsets $A,B,C \subset F_p^*$ with $|A|, |B|, |C| \leq p^{5/8}$, we have

$$\max \{|f(A,B)|, |B \cdot C|\} \gg \min \left\{ \frac{|A|^\frac{1}{2}|B|^\frac{1}{2}|C|^\frac{1}{2}}{m^\frac{1}{2}}, \frac{|B||C|^\frac{1}{2}}{m}, \frac{|B||A|^\frac{1}{2}}{m}, \frac{|B|^\frac{1}{2}|C|^\frac{1}{2}|A|^\frac{1}{2}}{m^\frac{1}{2}} \right\}.$$

**Theorem 1.4 (Pham-Mojarrad, [12]).** Let $f(x,y) = g(x)(h(x) + y)$ be a function defined on $F_p^* \times F_p^*$, where $g, h: F_p^* \to F_p^*$ are arbitrary functions. Define $m = \mu(g)$. For any subsets $A,B,C \subset F_p^*$ with $|A|, |B|, |C| \leq p^{5/8}$, we have

$$\max \{|f(A,B)|, |B + C|\} \gg \min \left\{ \frac{|A|^\frac{1}{2}|B|^\frac{1}{2}|C|^\frac{1}{2}}{m^\frac{1}{2}}, \frac{|B||C|^\frac{1}{2}}{m}, \frac{|B||A|^\frac{1}{2}}{m}, \frac{|B|^\frac{1}{2}|C|^\frac{1}{2}|A|^\frac{1}{2}}{m^\frac{1}{2}} \right\}.$$

It follows from Theorems 1.3 and 1.4 that for $A \subset F_p$ if $|A| \leq p^{5/8}$, then we have

$$\max \{|A + A|, |A \cdot A|\} \gg |A|^{6/5}.$$

It is worth noting that the best current bound in the literature is due to Rudnev, Shakan,
and Shkredov [17]. More precisely, they proved that for \( A \subset \mathbb{F}_p \) with \(|A| \leq p^{18/35}\), we have

\[
\max\{|A + A|, |A \cdot A|\} \gg |A|^{11/9 - o(1)}.
\]

The main purpose of this paper is to employ the method of Rudnev, Shakan, and Shkredov in [17] to improve further Theorems 1.3 and 1.4. Our first main result is as follows.

**Theorem 1.5.** Let \( f_1(x, y) = g_1(x)(h_1(x) + y), f_2(x, y) = g_2(x)(h_2(x) + y) \) be the functions defined on \( \mathbb{F}_p \times \mathbb{F}_p^* \), where \( g_1, h_1, g_2, h_2 : \mathbb{F}_p^* \to \mathbb{F}_p \) are arbitrary functions. Define \( m = \max\{|\mu(g_1), \mu(g_2)|\} \). For any subsets \( A, B, C, D \subset \mathbb{F}_p^* \), with \(|A| \leq |B| \leq p^{3/5} \) and \(|D| \leq |C| \leq p^{3/5}\), we have:

\[
\max\{|f_1(A, B)|, |f_2(D, C)|, |B - C|\} \gtrsim \frac{|B|^1/2 \cdot |C|^{1/2} \cdot |A|^{1/2} \cdot |D|^{1/2}}{m^{1/2}}.
\]

**Theorem 1.6.** Let \( f_1(x, y) = g_1(x)(h_1(x) + y), f_2(x, y) = g_2(x)(h_2(x) + y) \) be the functions defined on \( \mathbb{F}_p^* \times \mathbb{F}_p^* \), where \( g_1, h_1, g_2, h_2 : \mathbb{F}_p^* \to \mathbb{F}_p \) are arbitrary functions. Define \( m = \mu(g_1) = \mu(g_2) \). For any subsets \( A, B, C, D \subset \mathbb{F}_p^* \), with \(|A| \leq |B| \leq p^{3/5}, |D| \leq |C| \leq p^{3/5}\), we have:

\[
\max\{|f_1(A, B)|, |f_2(D, C)|, |B + C|\} \gtrsim \frac{|C|^{1/2} \cdot |B|^{1/2} \cdot |A|^{1/2} \cdot |D|^{1/2}}{m^{1/2}}.
\]

The following are consequences of theorems 1.5 and 1.6. Firstly, when \( A = B = C = D \) and \( f_1 \equiv f_2 \), we have:

**Corollary 1.7.** Let \( f(x, y) = g(x)(h(x) + y) \) be a function defined on \( \mathbb{F}_p^* \times \mathbb{F}_p^* \), where \( g, h : \mathbb{F}_p^* \to \mathbb{F}_p^* \) are arbitrary functions. Define \( m = \mu(g) \). For any subset \( A \subset \mathbb{F}_p^* \), with \(|A| \leq p^{3/5}\), we have:

\[
\max\{|f(A, A)|, |A \pm A|\} \gg |A|^{1/2 - o(1)}.
\]

Moreover, when \( A = D, f_1 \equiv f_2 \), we improved a result of Pham and Mojarrad in [12].

**Corollary 1.8.** Let \( f(x, y) = g(x)(h(x) + y) \) be a function defined on \( \mathbb{F}_p^* \times \mathbb{F}_p^* \), where \( g, h : \mathbb{F}_p^* \to \mathbb{F}_p^* \) are arbitrary functions. Define \( m = \mu(g) \). For any subsets \( A, B, C \subset \mathbb{F}_p^* \), with \(|A| \leq |B|, |C| \leq p^{3/5}\), we have:

\[
\frac{|B|^{11/18} \cdot |C|^{5/18} \cdot |A|^{2/9}}{m^{1/9}} \lesssim \max\{|f(A, B)|, |B + C|\},
\]
and
\[
\frac{|B|^{5/12}.|C|^{7/12}.|A|^{2/9}}{m^{5/9}} \lesssim \max\{|f(A, B)|, |B - C|\}.
\]

In the following theorem, we provide the multiplicative version of Theorem 1.5.

**Theorem 1.9.** Let \( f_1(x, y) = g_1(x)(h_1(x) + y), f_2(x, y) = g_2(x)(h_2(x) + y) \) be the functions defined on \( \mathbb{F}_p^* \times \mathbb{F}_p^* \), where \( g_1, h_1, g_2, h_2 : \mathbb{F}_p^* \to \mathbb{F}_p^* \) are arbitrary functions. Define \( m = \max\{\mu(g_1 \cdot h_1), \mu(g_2 \cdot h_2)\} \). For any subsets \( A, B, C, D \subset \mathbb{F}_p^* \), with \( |A| \leq |B| \leq p^{3/5}, |D| \leq |C| \leq p^{3/5} \), we have:

\[
\max\{|f_1(A, B)|, |f_2(D, C)|, |B.C|\} \gtrsim \frac{|C|^{\frac{5}{13}}.|B|^{\frac{13}{18}}.|A|^{\frac{1}{5}}.|D|^{\frac{1}{5}}}{m^{\frac{5}{9}}}. 
\]

**Corollary 1.10.** Let \( f(x, y) = g(x)(h(x) + y) \) be a function defined on \( \mathbb{F}_p^* \times \mathbb{F}_p^* \), where \( g, h : \mathbb{F}_p^* \to \mathbb{F}_p^* \) are arbitrary functions. Given \( m = \mu(g,h) \) is finite. For any subset \( A \subset \mathbb{F}_p^* \), with \( |A| \leq p^{3/5} \), we have:

\[
\max\{|f(A, A)|, |A.A|\} \gg |A|^{\frac{14}{5}} - o(1).
\]

**Corollary 1.11.** Let \( f(x, y) = g(x)(h(x) + y) \) be a function defined on \( \mathbb{F}_p^* \times \mathbb{F}_p^* \), where \( g, h : \mathbb{F}_p^* \to \mathbb{F}_p^* \) are arbitrary functions. Define \( m = \mu(g,h) \). For any subsets \( A, B, C \subset \mathbb{F}_p^* \), with \( |A| \leq |B|, |C| \leq p^{3/5} \), we have:

\[
\frac{|C|^{5/18}.|B|^{13/18}.|A|^{2/9}}{m^{8/9}} \lesssim \max\{|f(A, B)|, |B.C|\}.
\]

Given \( f_1(x, y) = x(1+y) \) and \( f_2(x, y) = x(1-y) \), we obtain a concise version of \([21],\) Theorem 2] by Warren with a stronger condition.

**Corollary 1.12.** For any subsets \( A, B, C, D \subset \mathbb{F}_p^* \), with \( |A| \leq |B| \leq p^{3/5}, |D| \leq |C| \leq p^{3/5} \), we have:

\[
\max\{|A(1 + B)|, |D(1 - C)|, |B.C|\} \gtrsim |C|^{\frac{5}{13}}.|B|^{\frac{13}{18}}.|A|^{\frac{1}{5}}.|D|^{\frac{1}{5}}. 
\]

Finally, combining Theorems 1.3 and 1.9, we are able to improve \([12],\) Theorem 1.10.

**Theorem 1.13.** Let \( f(x, y) = g(x)(h(x) + y) \) be a function defined on \( \mathbb{F}_p^* \times \mathbb{F}_p^* \), where \( g, h : \mathbb{F}_p^* \to \mathbb{F}_p^* \) are arbitrary functions. Given \( \mu(g) \) is finite. For any subsets \( A \subset \mathbb{F}_p^* \), with \( |A| \leq p^{3/5} \), satisfying:

\[
\min\{|A + A|, |A.A|\} \leq |A|^{\frac{6}{5}} - \epsilon,
\]
for some $\epsilon > 0$. Then, we have:

$$|f(A, A)| \gg |A|^{\frac{3}{5} - \frac{1}{p} + \frac{2\epsilon}{5}}.$$ 

## 2 Preliminaries

Let $G$ be an abelian group and $A$ and $B$ be two finite subsets. For any real number $n > 1$, we define the representation functions:

$$r_{A - B}(x) := \#\{(a, b) \in A \times B : x = a - b, x \in G\},$$

$$E_n(A, B) := \sum_x r_{A - B}^n(x), E_n(A) := E_n(A, A).$$

Similarly,

$$r_{A/B}(x) := \#\{(a, b) \in A \times B : x = a.b^{-1}, x \in G\},$$

$$E_n^\times(A, B) := \sum_x r_{A/B}^n(x), E_n^\times(A) := E_n^\times(A, A).$$

The initial idea of this paper is to use the Rudnev point-plane incidence’s theorem and the theory of higher order energies to optimize the bounds on $E_4(C, D), E_4^\times(C, D)$ for some small sets $C, D$.

**Theorem 2.1.** (Rudnev, [15]) Let $\mathcal{R}$ be a set of points in $\mathbb{F}_p^3$ and let $\mathcal{S}$ be a set of planes in $\mathbb{F}_p^3$, with $|\mathcal{R}| \ll |\mathcal{S}|$ and $|\mathcal{R}| \ll p^2$. Assume that there is no line containing $k$ points of $\mathcal{R}$. Then

$$I(\mathcal{R}, \mathcal{S}) \ll |\mathcal{R}|^{1/2} |\mathcal{S}| + k |\mathcal{S}|.$$ 

**Lemma 2.2.** Let $f(x, y) = g(x)(h(x) + y)$ be a function defined on $\mathbb{F}_p^* \times \mathbb{F}_p^*$, where $g, h : \mathbb{F}_p^* \to \mathbb{F}_p^*$ are arbitrary functions. Define $m = \mu(g)$. For any subsets $A, B, C \subset \mathbb{F}_p^*$, with $|A| \ll |B|$ and $|A|, |B|, |C| \leq p^{3/5}$, we have:

$$E_4(B, C) := \sum_x r_{B - C}^4(x) \ll m^4 \min\left\{\frac{|f(A, B)|^3 |C|^2}{|A|}, \frac{|f(A, B)|^2 |C|^3}{|A|}\right\} \cdot \log |A|.$$ 

**Proof.** For $1 \leq k \leq \min\{|A|, |B|, |C|\}$, let

$$n_k := |X_k := \{x \in B - C : r_{B - C}(x) \geq k\}|.$$
By a dyadic decomposition, there exist a number $k$ such that:

$$E_4(B, C) \ll n_k k^4.$$ 

We consider the following equations:

$$g(a)(x + c + h(a)) - f(a, b) = 0 \iff c = \frac{f(a, b)}{g(a)} - x - h(a), \quad (5)$$

where $a \in A, b \in B, x \in X_k, c \in C$.

By the definition of $X_k$, the number of solutions to the equation (5) is at least $k|A|n_k$.

On the other hand, by using Cauchy-Schwartz inequality on each equation in (5), we obtain:

$$M \leq \sqrt{|f(A, B)|.\sqrt{|\{(a, x, x', c, c') \in (A \times X \times C)^2 : g(a)(x + c + h(a)) = g(a')(x' + c' + h(a'))\}|}}$$

$$= \sqrt{|f(A, B)|.E_1}, \quad (6)$$

and

$$M \leq \sqrt{|C|\sqrt{|\{(a, x, x', c, c') \in (A \times X \times f(A, B))^2 : f(a, b) \frac{g(a)}{g(a')} - x - h(a) = f(a', b') \frac{g(a)}{g(a')} - x - h(a')\}|}}$$

$$= \sqrt{|C|.E_2}. \quad (7)$$

Firstly, we obtain the upper bound on $E_4(B, C)$ via $E_1$. We consider the following cases.

**Case 1:** Given $|X_k|.|A|.|C| \gg p^2$. Our assumption and definition of $X_k$ provide that $k \ll \min\{|A|, |B|, |C|\}, |A| \ll |B| \ll |f(A, B)|$, and $|X_k|.k \ll |B|.|C|$.

If $k^3 \ll \frac{|f(A, B)|^2.|C|^3}{|A||B|}$, then:

$$E_4(B, C) \ll |X_k|.k^4 \ll |B|.|C|.k^3 \ll \frac{|f(A, B)|^2.|C|^3}{|A|}.$$ 

If $k^3 \gg \frac{|f(A, B)|^2.|C|^3}{|A||B|}$, then:

$$|B|.|C|.|A|.|C| \gg |X_k|.t.|A|.|C| \gg p^2.k$$
Case 2: Given $|X_k|, |A|, |C| \ll p^2$.

Define the set of points $\mathcal{R}_1$ and the set of planes $\mathcal{S}_1$ as following:

$$ \mathcal{R}_1 := \{ (x, g(a'), g(a'), (c' + h(a')) : (a', c', x) \in A \times C \times X_k \}, $$

$$ \mathcal{S}_1 := \{ g(a) \cdot X - x' \cdot Y - Z + g(a).c + h(a)) = 0 : (a, c, x') \in A \times C \times X_k \}. $$

Note that $\mu(g) = m$ or there are at most $m$ different values of $a$ satisfying the equation $g(a) = t, \forall t$. Therefore, we must have:

$$ E_1 \leq m^2 \cdot I(\mathcal{R}_1, \mathcal{S}_1), $$

in which, $I(\mathcal{R}_1, \mathcal{S}_1)$ is the number incidences between $\mathcal{R}_1$ and $\mathcal{S}_1$.

To apply Theorem 2.1, we need to find an upper bound on the maximum number of collinear points in $\mathcal{R}_1$. The projection of $\mathcal{R}_1$ into the last two coordinates is the set $\mathcal{T} = \{ (g(a'), g(a'), (c' + h(a')) : a' \in A, c \in C \}$. The set $\mathcal{T}$ can be covered by at most $|A|$ lines of the form $X = g(a')$ with $a' \in A$, where each line contains $|C|$ points of $\mathcal{T}$. Therefore, a line in $\mathbb{F}_p^3$ contains at most $\max\{|A|, |C|\}$ points of $\mathcal{R}_1$, unless it is vertical, in which case it contains at most $|X_k|$ points. All implies the maximum number of collinear points in $\mathcal{R}_1$ is

$$ M_1 \ll \max\{|A|, |C|, |X_k|\}. $$

Case 2.1: $M_1 = \max(|X_k|, |A|, |C|) \gg (|A|, |X_k|, |C|)^{1/2}$. Since $k \ll \min(|A|, |B|, |C|)$ and $|A| \ll |B| \ll |f(A, B)|$, we obtain the followings:

2.1.1. If $M_1 = |C|$ and $|X_k| \cdot |A| \ll |C|$, then

$$ E_4(B, C) \ll |X_k| \cdot k^4 \ll |C| \cdot k^3 \ll \frac{|f(A, B)|^2}{|A|} \cdot |C|^3. $$
2.1.2. If $M_1 = |X_k| \leq |A|, |C| \ll |X_k|$, then:

$$|A|, |C| \cdot k \leq |X_k| \cdot k \ll |B|, |C| \Rightarrow |A| \cdot k \ll |B|$$

$$\Rightarrow E_4(B, C) \ll |X_k| \cdot k^4 \ll \frac{|X_k| \cdot k}{|A|} \cdot (|A| \cdot k) \cdot k^2 \ll \frac{|B||C|}{|A|} \cdot |B| \cdot k^2 \ll \frac{|f(A, B)|^2}{|A|} \cdot |C|^3.$$  

2.1.3. If $M_1 = |A|, |X_k|, |C| \ll |A|$, then:

$$E_4(B, C) \ll |X_k| \cdot k^4 \ll |A| \cdot k^3 \ll \frac{|f(A, B)|^2}{|A|} \cdot |C|^3.$$  

Case 2.2: $M_1 = \max(|X_k|, |A|, |C|) \ll (|A|, |X_k|, |C|)^{1/2}$ and $|R_1| \ll p^2$, apply Theorem 2.1, we obtain:

$$I(R_1, S_1) \ll (|A|, |X_k|, |C|)^{3/2} + \max\{|A|, |C|, |X_k|\} \cdot (|C|, |A|, |X_k|)$$

$$\ll (|A|, |X_k|, |C|)^{3/2}.$$  

The sub-cases 2.1 and 2.2 together imply that for $|X_k|, |A|, |C| \ll p^2$, then either

$$I(R_1, S_1) \ll (|A|, |X_k|, |C|)^{3/2}$$

or

$$E_4(B, C) \ll \frac{|f(A, B)|^2}{|A|} \cdot |C|^3.$$  

Therefore, collecting all above cases and (6), we get either

$$E_4(B, C) \ll \frac{|f(A, B)|^2}{|A|} \cdot |C|^3$$

or

$$k|A| \cdot n_k \leq |f(A, B)|^{1/2} m I(R_1, S_1)$$

$$\ll |f(A, B)|^{1/2} \cdot m \cdot (|A|, |C| \cdot n_k)^{3/2}$$

$$\Rightarrow n_k \ll \frac{m^4 \cdot |f(A, B)|^2 \cdot |C|^3}{k^4 \cdot |A|}.$$  

Now, we use $E_2$ to obtain another bound on $M$. Similarly, we define a set of points $R_2$ and
a set of planes $\mathcal{S}_2$ as:

$$\mathcal{R}_2 = \{(f, \frac{1}{g(a')}, h(a') + x') : (a', x', f) \in A \times X_k \times f(A, B)\},$$

$$\mathcal{S}_2 = \{\frac{1}{g(a)}X - f'Y + Z - h(a) - x = 0 : (a, x, f') \in A \times X_k \times f(A, B)\}.$$ 

Clearly, $|\mathcal{S}_2|, |\mathcal{R}_2| \ll |f(A, B)|,|A|,|X_k|$ and the maximal number of collinear points in $\mathcal{R}_2$ is at most $\max\{|f(A, B)|,|A|,|X_k|\}$.

As same as the procedure above, since $|A|,|B|,|C| \leq p^{3/5}$, applying theorem 2.1, we obtain either

$$I(\mathcal{R}_2, \mathcal{S}_2) \ll (|f(A, B)|,|A|,|X_k|)^{3/2}$$

or

$$E_4(B, C) \leq \frac{|f(A, B)|^3.|C|^2}{|A|}.$$ 

Together with (7), we get:

$$k^4.|A|^4.n_k^4 \ll M^4 \ll m^4.|C|^2.|f(A, B)|^3.|A|^3.n_k^3$$

$$\Rightarrow n_k \ll \frac{1}{k^4.m^4.|f(A, B)|^3.|C|^2}.|A|.$$ 

All above implies either:

$$E_4(B, C) \ll \min\left\{\frac{|f(A, B)|^3.|C|^2}{|A|}, \frac{|f(A, B)|^2.|C|^3}{|A|}\right\}$$

or

$$n_k \ll \frac{m^4}{k^4}\min\left\{\frac{|f(A, B)|^3.|C|^2}{|A|}, \frac{|f(A, B)|^2.|C|^3}{|A|}\right\}.|A|.$$ 

Following after dyadic summation in $k$, we finally get:

$$E_4(B, C) := \sum_x r_{B-C}(x) \ll \left(m^4.\min\left\{\frac{|f(A, B)|^3.|C|^2}{|A|}, \frac{|f(A, B)|^2.|C|^3}{|A|}\right\}\right).\log|A|.$$ 

We complete the proof of lemma 2.2.

Remark: If $B = C$ and taking the later side of the minimum, the above inequality reduces to:

$$E_4(B) \lesssim m^4. \frac{|f_1(A, B)|^2.|B|^3}{|A|},$$
for any $|A|, |B|, |C|, |D| \leq p^{3/5}$ and the functions $f_1(x, y) = g_1(x)(h_1(x) + y), f_2(x, y) = g_2(x)(h_2(x) + y)$ with $\mu(g_1), \mu(g_2) \leq m$.

Similarly, we obtain the bound on the product-representation function.

Lemma 2.3. Let $f(x, y) = g(x)(h(x) + y)$ be a function defined on $\mathbb{F}_p^* \times \mathbb{F}_p^*$, where $g, h : \mathbb{F}_p^* \to \mathbb{F}_p^*$ are arbitrary functions. Define $m = \mu(g.h)$. For any subsets $A, B, C, D \subset \mathbb{F}_p^*$, with $|A|, |B|, |C| \leq p^{3/5}$, we have:

$$E_4(C) \lesssim m^4 \cdot \frac{|f_2(D, C)|^2 \cdot |C|^3}{|D|}.$$

Similarly, we obtain the bound on the product-representation function.

Lemma 2.3. Let $f(x, y) = g(x)(h(x) + y)$ be a function defined on $\mathbb{F}_p^* \times \mathbb{F}_p^*$, where $g, h : \mathbb{F}_p^* \to \mathbb{F}_p^*$ are arbitrary functions. Define $m = \mu(g.h)$. For any subsets $A, B, C \subset \mathbb{F}_p^*$, with $|A|, |B|, |C| \leq p^{3/5}$, we have:

$$E_4^x(B, C) := \sum_x r_{B/C}^4(x) \ll \left(m^4 \cdot \min \left\{ \frac{|f(A, B)|^3 \cdot |C|^2}{|A|}, \frac{|f(A, B)|^2 \cdot |C|^3}{|A|} \right\} \right)^{1/3}. \log |A|.$$

Proof. For $1 \leq k \leq \min\{|A|, |B|, |C|\}$, let

$$s_k := |Y_k := \{ x \in B/C : r_{B/C}(x) \geq k \}|.$$

By a dyadic decomposition, there exist a number $k$ such that:

$$E_4^x(B, C) \ll |Y_k| \cdot k^4.$$

We consider the following equations:

$$g(a)(x \cdot c + h(a)) - f(a, b) = 0 \iff c = \left( \frac{f(a, b)}{g(a)} - h(a) \right)/x, \quad (8)$$

where $a \in A, b \in B, x \in Y_k, c \in C$.

Clearly, there are $M' \geq k|A| \cdot n_k$ solutions to the above equations.

Moreover, by using Cauchy-Schwarz inequality on each of them, we obtain that:

$$M' \leq \sqrt{|f(A, B)| \cdot \sqrt{|\{(a, x, c, a', x', c') \in (A \times Y_k \times C)^2 : g(a)(x \cdot c + h(a)) = g(a')(x' \cdot c' + h(a'))\}|}}$$

$$= \sqrt{|f(A, B)| \cdot E_3}, \quad (9)$$
and

\[
M' \leq \sqrt{|C|} \sqrt{\left| \left\{ (a, x, f, a', x', f') \in (A \times Y_k \times f(A, B))^2 : \left( \frac{f(a, b)}{g(a)} - h(a) \right) \frac{1}{x'} = \left( \frac{f(a', b')}{g(a')} - h(a') \right) \frac{1}{x} \right\} \right|}
\]

\[
= \sqrt{|C|} E_4.
\]  

(10)

Firstly, we obtain the upper bound on \( E_3 \). Define the set of points \( R_1 \) and the set of planes \( S_1 \) as following:

\[
R_1 = \{ (x, g(a').c', g(a').h(a')) : (a', c', x') \in A \times C \times Y_k \},
\]

\[
S_1 = \{ g(a).c \cdot X - x' \cdot Y - Z + g(a).h(a) = 0 : (a, c, x') \in A \times C \times Y_k \}.
\]

Note that \( m = \mu(g.h) \) or there are at most \( m \) different values of \( a \) satisfying the equation \( g(a).h(a) = t, \forall t \). Therefore, we must have:

\[
E_3 \leq m^2 \cdot I(R_1, S_1)
\]

in which, \( I(R_1, S_1) \) is the number of incidences between \( R_1 \), and \( S_1 \).

Similar to the lemma 2.2 and applying the theorem 2.1, we also get either

\[
s_k \ll \frac{m^4}{k^3} \frac{|f(A, B)|^2 |C|^3}{|A|},
\]

or

\[
E_4^x(B, C) \ll \frac{|f(A, B)|^2 |C|^3}{|A|}.
\]

We now use \( E_4 \) to obtain another bound on \( M' \). By the same procedure, we define a set of points \( R_2 \) and a set of planes \( S_2 \) as:

\[
R_2 = \{ (f, \frac{1}{g(a').x'}, \frac{h(a')}{x'}) : (a', x', f) \in A \times Y_k \times f(A, B) \},
\]

\[
S_2 = \{ \frac{1}{g(a).x} \cdot X - f' \cdot Y + Z - \frac{h(a)}{x} = 0 : (a, x, f') \in A \times Y_k \times f(A, B) \}.
\]
Note that if \( \frac{1}{g(a)} = u, \frac{h(a)}{x} = v \), then \( g(a)h(a) = \frac{ux}{v} \). Therefore, once again, we have:

\[
E_4 \leq m^2 I(\mathcal{R}_2, \mathcal{S}_2)
\]
in which, \( I(\mathcal{R}_2, \mathcal{S}_2) \) is the number of incidences between \( \mathcal{R}_2 \) and \( \mathcal{S}_2 \). Moreover, \( |\mathcal{S}_2|, |\mathcal{R}_2| \ll |f(A,B)|. |A|. |Y_k| \).

Again, since \( |A|, |B|, |C| \leq p^{3/5} \), we must have \( |f(A,B)|. |A|. |Y_k| \ll p^2 \), and then either

\[
E_4^\times(B,C) \ll \frac{|f(A,B)| |C|^2}{|A|},
\]
or

\[
s_k \ll \frac{m^4}{k^4} \frac{|f(A,B)| |C|^2}{|A|}.
\]

All above implies either:

\[
E_4^\times(B,C) \ll \min \left\{ \frac{|f(A,B)| |C|^2}{|A|}, \frac{|f(A,B)|^2 |C|^3}{|A|} \right\},
\]
or

\[
s_k \ll \frac{m^4}{k^4} \min \left\{ \frac{|f(A,B)| |C|^2}{|A|}, \frac{|f(A,B)|^2 |C|^3}{|A|} \right\}.
\]

Following after dyadic summation in \( k \), we finally get:

\[
E_4^\times(B,C) := \sum_x r_{B/C}(x) \ll \left( m^4 \min \left\{ \frac{|f(A,B)| |C|^2}{|A|}, \frac{|f(A,B)|^2 |C|^3}{|A|} \right\} \right) \log |A|.
\]

We complete the proof of lemma 2.3.

\[\square\]

3 Proof of Theorem 1.5

Let \( P \subset B - C \) be a set of popular differences, defined as follows: for every \( x \in P \), \( r_{B-C}(x) \geq \frac{|B||C|}{2|B-C|} \). We further obtain:

\[
| \{(b_1, c_1) \in B \times C : b_1 - c_1 \in P \}| \gg |B|.|C|.
\] (11)
Consider the equation:

\[ b - c = (a - c) - (a - b) = (d - c) - (d - b). \]  \hspace{1cm} (12)

Suppose \( x = a - c \) and \( y = d - c \) are in \( P \), while \( u = a - b \), \( v = d - b \) are both in \( B - B \) (*).

By the condition (11), equation (12) has \( N \gg (|B|, |C|)^2 \) solutions \((a, b, c, d)\).

We define an equivalent relation on \( B \times B \times C \times B \) as:

\[ (a, b, c, d) \sim (a', b', c', d') \Leftrightarrow (a, b, c, d) = (a' + t, b' + t, c' + t, d' + t), \]

for some \( t \in (B - B) \cap (C - C) \).

We denote the equivalent class of \((a, b, c, d)\) by \([a, b, c, d]\).

Clearly, if \((a, b, c, d) \sim (a', b', c', d')\) and \((a, b, c, d)\) is a solution of equation (12), then \((a', b', c', d')\) is also a solution. Thus, we can decompose \( N \) into the sum over each equivalent class, which satisfies (*).

\[ N = \sum_{[a, b, c, d]} r([a, b, c, d]), \]

in which \( r([a, b, c, d]) \) is number of elements in the \([a, b, c, d]\)-equivalent class. Applying the Cauchy-Schwartz inequality, we obtain:

\[ N^2 \leq X^* \cdot \left( \sum_{[a, b, c, d]} r^2([a, b, c, d]) \right), \]

in which, \( X^* \) is number of equivalent classes, which satisfying (*).

Moreover, each equivalent class is defined uniquely by any three of five \( x, y, u, v, w \), and each equivalent class provides a distinct solution of system:

\[ x, y \in P, u, v \in B - B, w \in B - C : x - u = y - v = w. \]

It implies:

\[ X^* \leq X = |\{x, y \in P; u, v \in B - B : x - u = y - v\}| \]

On the other hand, \((a, b, c, d) \sim (a', b', c', d')\) if and only if there exists \( t \in B - B \cap C - C \) such that: \( t = a - a' = b - b' = c - c' = d - d' \), in which \( a - a', b - b', d - d' \in B - B \) and
\(c - c' \in C - C\). Thus,

\[
\sum_{[a,b,c,d]} r^2([a,b,c,d]) \leq \sum_{x \in (B-B) \cap (C-C)} r_{B-B}^3(x).r_{C-C}(x) \leq (E_4(B))^{3/4}.(E_4(C))^{1/4}.
\]

The last inequality is obtained by Holder’s inequality.

All above leads to: \(|B|^2.|C|^2 \ll \sqrt{\sum_{x \in (B-B) \cap (C-C)} r_{B-B}^3(x).r_{C-C}(x) \cdot \sqrt{\{x, y \in P, u, v \in B - B : x - u = y - v \in B - C\}}}

\[
\Rightarrow |B|^2.|C|^2 \ll (E_4(B))^{3/4}.(E_4(C))^{1/4}.\sqrt{\mathcal{X}}. \quad (13)
\]

To bound the quantity \(\mathcal{X}\), we use popularity of the difference and dyadic localization. Namely, for some \(\Delta \geq 1\) and some \(T \subset (B - (B - C))\) one has:

\[
\mathcal{X} \ll \frac{|B - C|^2}{|B|^2|C|^2} \cdot \{|b_1, b_2 \in B, c_1, c_2 \in C; u, v \in B - B : b_1 - (c_1 - u) = b_2 - (c_2 - v) \in B - C\}| \leq \frac{|B - C|^2}{|B|^2|C|^2} \Delta^2 \cdot \{|b_1, b_2 \in B, d_1, d_2 \in T \subset (B - (B - C)) : b_1 - d_1 = b_2 - d_2 \in B - C\}| \leq \frac{|B - C|^2}{|B|^2|C|^2} \Delta^2 \cdot \sum_{w \in B - C} r_{B-T}(w)^2 \leq \frac{|B - C|^2}{|B|^2|C|^2} \Delta^2 \cdot |B - C|^{1/2}.(\sum_{w} r_{B-T}(w)^4)^{1/2} = \frac{|B - C|^{5/2}}{|B|^2|C|^2} \cdot \Delta^2 \sqrt{E_4(B, T)},
\]

where the last inequality is an application of Cauchy-Schwartz inequality.

Now, applying the Lemma \([2,2]\) one gets:

\[
\mathcal{X} \lesssim \min \left\{ \frac{|B - C|^{5/2}}{|B|^2|C|^2} \cdot \Delta^2.m^2.\frac{|f_1(A, B)|^{3/2}.|T|}{|A|^{1/2}}, \frac{|B - C|^{5/2}}{|B|^2|C|^2} \cdot \Delta^2.m^2.\frac{|f_1(A, B)|.|T|^{3/2}}{|A|^{1/2}} \right\}.
\]

Note that

\[|T| . \Delta \ll |B|.|B - C|.|T|.\Delta^2 \ll E^+(B, B - C).\]

Therefore,

\[
\mathcal{X} \lesssim \frac{|B - C|^{5/2}}{|B|^2|C|^2} m^2.\min \left\{ \frac{|f_1(A, B)|^{3/2}.E^+(B, B - C)}{|A|^{1/2}}, \frac{|f_1(A, B)|.|B - C|}{|A|^{1/2}} \right\}.
\]
Using the theorem 2.1, we can obtain the upper bound for $E^+(B, B - C)$ that:

$$|A|^2.E^+(B, B - C) = |A|^2. |\{b_1, b_2, d_1, d_2 \in B^2 \times (B - C)^2 : b_1 - d_1 = b_2 - d_2\}|$$

$$\leq |\{(a_1, a_2, f_1(a_1, b_1), f_1(a_2, b_2), d_1, d_2) \in A^2 \times f_1(A, B)^2 \times (B - C)^2 : \frac{f_1(a_1, b_1)}{g_1(a_1)} - h_1(a_1) - d_1 = \frac{f_1(a_2, b_2)}{g_1(a_2)} - h_1(a_2) - d_2\}|$$

$$\ll m^2.|f_1(A, B)|^{3/2}|A|^{3/2} |B - C|^{3/2}$$

$$\Rightarrow E^+(B, B - C) \ll m^2.|f_1(A, B)|^{3/2} |A|^{-1/2} |B - C|^{3/2}.$$ 

All implies:

$$X \lesssim \frac{m^4.|B - C|^4.|f_1(A, B)|^3}{|B|^2|C|^2|A|}.$$ \hspace{1cm} (14)

By (13), (14), and the remark of lemma 2.2 we get:

$$|B|^4.|C|^4 \ll (E_4(B))^3/4.(E_4(C))^{1/4}.m^4.|B - C|^4.|f_1(A, B)|^3$$

$$\Rightarrow |B|^4.|C|^4 \lesssim m^4.|f_1(A, B)|^{3/2}.|f_2(D, C)|^{1/2} |B|^{9/4} |C|^{3/4}$$

$$|A|^{3/4} |D|^{1/4} \Rightarrow |B|^{5/4} |C|^{21/4} |A|^{7/4} |D|^{1/4} \lesssim m^8.|B - C|^4.|f_1(A, B)|^{9/2} |f_2(D, C)|^{1/2}.$$ 

If $\alpha = \max(|f_1(A, B)|, |f_2(D, C)|, |B - C|)$, above inequality is equivalent to:

$$\alpha \gtrsim \frac{|B|^{5/12} |C|^{7/12} |A|^{7/36} |D|^{1/36}}{m^{8/9}}.$$ 

We complete the proof of theorem 1.5.

4 Proof of Theorems 1.6, 1.9, and 1.13

Proof of Theorem 1.12. Let $P$ be a set of popular sums, defined as follows. For $\epsilon = \log(C)^{-1}$:

$$P = P(C) := \left\{ x \in C + C : r_{C + C}(x) \geq \epsilon \frac{|C|^2}{|C + C|} \right\}.$$
It implies:

\[ | \{ (c, c') \in C \times C : c + c' \in P \} | \geq (1 - \epsilon)|C|^2. \]

Furthermore, let \( C' \subset C \) be

\[ C' = C'(C) := \{ c' \in C : | \{ c'' \in C : c' + c'' \in P(C) \} | \geq (1 - \epsilon)|C| \}, \]

so \( |C'| \geq (1 - \epsilon)|C| \).

Let \( P' \subset C' \) be popular by energy \( E_{4/3}(C') \). Namely \( x \in P' \) if for some \( \Delta' \geq 1, \Delta' \leq r_{C' - C'}(x) \leq 2\Delta' \), and then:

\[ E_{4/3}(C') \gtrsim |P'| \Delta'^{4/3}. \]

Applying the lemma 8 in [17], we get that:

\[ E_{4/3}(C') \gg E_{4/3}(C) \]

to be used in the end of the proof.

Now, for \((b, c) \in C \times C, (a, d) \in B \times B\), consider the following equation:

\[ -c + b = (a + b) - (a + c) = (d + b) - (d + c). \quad (15) \]

Similar to the proof of theorem [15], let us make the popularity assumption as to the variables \( a, b, c, d \). By the definition of the sets \( C' \) and \( P' \), it follows that the number of solutions \( \phi \) of the equation \((15)\), when the different \( b - c \in P' \) and all the four sums: \( x := a + b, y := a + c, u := d + c, v := d + b \in P \) is bounded from below as:

\[ \phi \geq (1 - 4\epsilon)|P'| |\Delta'| |B|^2. \]

Equation \((15)\) is invariant to a simultaneous shift of \( b, c \) by \( t \) and \( d, a \) simultaneously by \(-t\). We say \((a, b, c, d)\) is equivalent to \((a', b', c', d')\) if

\[ (a, b, c, d) = (a', b', c', d') + (t, -t, t, -t), \]

\[ \Leftrightarrow t = a - a' = b' - b = c - c' = d' - d, \]
for some \( t \in B - B \cap C - C \).

Each equivalent class \([a, b, c, d]\) yields a different solutions of the system of equations:

\[
x, y, u, v \in P, w \in P' : x - y = v - u = w.
\]

Therefore, similar to the proof of theorems 1.5 by the Cauchy-Schwartz inequality, we get:

\[
|B|^2|P'| \Delta' \lesssim \left( \sum_{x \in B - B \cap C - C} r_{B-B}^2(x).r_{C-C}^2(x). \sqrt{|x, y, u, v \in P, w \in P' : x - y = v - u = w|} \right) \lesssim \sqrt{E_4(B).E_4(C)} \frac{|B + C|^2}{|B|^2.|C|^2} \Delta. E_4(B, T)^{1/4}.
\]

There exist a popular subset \( T \in B + C - C \) where \( \forall d \in T, r_{B+C-C}(d) \approx \Delta \), for some \( \Delta \geq 1 \), such that one gets:

\[
|B|^2. \Delta'. |P'| \lesssim \sqrt{E_4(B).E_4(C)} \frac{|B + C|^2}{|B|^2.|C|^2} |P'|^{1/4} \Delta. E_4(B, T)^{1/4}.
\]

Now applying the lemma 2.2 we obtain:

\[
|B|^2. \Delta'. |P'|^{3/4} \lesssim \sqrt{E_4(B).E_4(C)} \frac{|B + C|^2}{|B|^2.|C|^2} m_n. \frac{|f_1(A, B)|^{3/4}}{|A|^{1/4}}. (|T|. \Delta)^{1/2}.
\]

Moreover,

\[
|T| \Delta^2 \leq |(b_1, b'_1) \in B, (c_1, c_2, c'_1, c'_2) \in C : b_1 + c_1 = b'_1 + c'_1, c_2 - c_2| \leq \Delta^2. |(b_1, b'_1) \in B, x_1, x_1' \in T_1 : b_1 + x_1 = b'_1 + x'_1|,
\]

where \( T_1 \subset C - C \), with \( r_{C-C}(x) \approx \Delta_1, \Delta \geq 1 \). Again, applying lemma 2.2 one gets:

\[
(|T| \Delta^2)^{1/2} \lesssim m_n. \frac{|f_1(A, B)|^{3/4}. |T_1|^{3/4}}{|A|^{1/4}}. \Delta_1 \\
\leq m_n. \frac{|f_1(A, B)|^{3/4}}{|A|^{1/4}}. (E_4(C))^3/2.
\]
Collecting all inequalities above, we get:

$$|B|^2(E_{4/3}(C'))^{3/4} \ll |B|^2 \Delta' |P'|^{3/4} \lesssim$$

$$\sqrt{E_4(B).E_4(C)} \frac{|B + C|^2}{|B|^2.|C|^2} m^2 \frac{|f_1(A, B)|^{3/2}}{|A|^{1/2}} (E_{4/3}(C))^{3/4}. $$

Since $E_{4/3}(C) \ll E_{4/3}(C')$, we can cancel $(E_{4/3}(C))^{3/4}$ and, then:

$$\Rightarrow |B|^2 \lesssim \sqrt{E_4(B).E_4(C)} \frac{|B + C|^2}{|B|^2.|C|^2} m^2 \frac{|f_1(A, B)|^{3/2}}{|A|^{1/2}}. $$

Recall that:

$$E_4(B) \lesssim m^4 \cdot \frac{|f_1(A, B)|^2 |B|^3}{|A|},$$

$$E_4(C) \lesssim m^4 \cdot \frac{|f_2(D, C)|^2 |C|^3}{|D|}.$$ 

Let $\beta = \max(|f_1(A, B)|, |f_2(D, C)|, |B + C|)$, we obtain:

$$|B|^2 \lesssim m^2 \frac{|f_1(A, B)|^{1/2} |B|^{3/4} |f_2(D, C)|^{1/2} |C|^{3/4}}{|A|^{1/4} |D|^{1/4}} \frac{|B + C|^2}{|B|^2.|C|^2} m^2 \frac{|f_1(A, B)|^{3/2}}{|A|^{1/2}},$$

$$\Leftrightarrow |B|^{13/4} |C|^{5/4} |A|^{3/4} |D|^{1/4} \lesssim m^4 |B + C|^2 |f_1(A, B)|^2 |f_2(D, C)|^{1/2}.$$ 

It is equivalent to:

$$\beta \gtrsim \frac{|B|^{13/18} |C|^{5/18} |A|^{1/6} |D|^{1/18}}{m^{8/9}}.$$ 

We complete the proof for theorem 1.6.

**Proof of theorem 1.9**

Following the proof of theorem 1.6, we replace the sum (difference) operation on $B, C$ by the product (res. quotient) operation on $B, C$. Then applying the lemma 2.3 instead of lemma 2.2, we obtain the Theorem 1.9.

**Proof of theorem 1.13.**

By corollaries 1.7 and 1.10 when $A \ll p^{3/5}$, we get:

$$|A|^{11/2} \lesssim |f(A, A)|^{5/2} |A + A|^2,$$
These inequalities imply:

\[ |A|^{11/2} \lesssim |f(A, A)|^{5/2} |A.A|^2. \]

Therefore, if

\[ |A|^{11} \lesssim |f(A, A)|^5 \left( \min \{|A + A|, |A.A|\} \right)^4. \]

one must get:

\[ |A|^{11 - 9/2 + 4\epsilon} \lesssim |f(A, A)|^5 \]

\[ \Rightarrow |A|^{13/10 + 4\epsilon} \lesssim |f(A, A)|. \]

We complete the proof of theorem 1.13.

References

[1] J. Bourgain, N. Katz, T. Tao, *A sum-product estimate in finite fields, and applications*, Geom. Funct. Anal. 14 (2004), 27–57.

[2] M. C. Chang, *Some problems in combinatorial number theory*, Integers, 8 (2008), Article A1, 1-11.

[3] C. Chen, B. Kerr, A. Mohammadi, *A new sum-product estimate in prime fields* Combinatorics (math.CO) (Jul 2018), arXiv:1807.10998.

[4] P. Erdős, E. Szemerédi, *On sums and products of integers*, Studies in pure mathematics, Birkhauser, Basel, (1983), 213-218.

[5] P. Erdős, E. Szemerédi, *On sums and products of integers*, Studies in Pure Mathematics. To the memory of Paul Turan, Basel: Birkhäuser Verlag, pp. 213-218, 1983.

[6] M. Z. Garaev, *An explicit sum-product estimate in \( \mathbb{F}_p \)*, Intern. Math. Res. Notices (2007), no 11, 1-11.

[7] N. Hegyvári, F. Hennecart, *Conditional expanding bounds for two-variable functions over prime fields*, European J. Combin., 34 (2013), 1365–1382.
[8] D. Hart, A. Iosevich, J. Solymosi, *Sum-product estimates in finite fields via Kloosterman sums*, Int. Math. Res. Not. no. 5, (2007) Art. ID rnm007.

[9] D. Koh, H. Mojarrad, T. Pham, C. Valculescu, *Four-variable expanders over the prime fields*, Proceedings of the American Mathematical Society **146**(12) (2018): 5025–5034.

[10] M. Mirzaei, *A note on conditional expanders over prime fields*, accepted in Discrete Mathematics, 2019.

[11] B. Murphy, G. Petridis, O. Roche-Newton, M. Rudnev, I. D. Shkredov, *New results on sum-product type growth over fields*, Mathematika, **65**(3) (2019): 588–642.

[12] T. Pham, H. Mojarrad, *Conditional expanding bounds for two-variable functions over arbitrary fields*, Journal of Number Theory, **186** (2018), 137-146.

[13] T. Pham, L. A. Vinh, F. de Zeeuw, *Three-variable expanding polynomials and higher-dimensional distinct distances*, Combinatorica **39**(2) (2019): 411–426.

[14] T. Pham, M. Tait, C. Timmons, L. A. Vinh, *A Szemerédi-Trotter type theorem, sum-product estimates in finite quasifields, and related results*, Journal of Combinatorial Theory, Series A, **147** (2017): 55–74.

[15] M. Rudnev, *On the number of incidences between points and planes in three dimensions*, Combinatorica (2014) 136.

[16] O. Roche-Newton, M. Rudnev, I. D. Shkredov, *New sum-product type estimates over finite fields*, Advanced in Mathematics, **293** (2016), 589-605.

[17] M. Rudnev, G. Shakan, I. D. Shkrekov, *Stronger sum-product inequalities for small sets*, accepted in Proceedings of the American Mathematical Society, 2019.

[18] M. Rudnev, I. D. Shkredov, S. Stevens, *On the energy variant of the sum-product conjecture*, accepted in Revista Matemática Iberoamericana, 2018.

[19] S. Stevens, F. de Zeeuw, *An improved point-line incidence bound over arbitrary fields*, Bulletin of the London Mathematical Society, **49**. (2017), 10.1112/blms.12077.

[20] L. A. Vinh, *The Szemerédi-Trotter type theorem and the sum-product estimate in finite fields*, Euro. J. Combin. **32** (2011), no. 8, 1177-1181.
[21] A. Warren, *On products of Shifts in Arbitrary Fields*, Mosc. J. Comb. Number Theory 8 (2019), no. 3, 247–261.