Analyticity of the solutions to degenerate Monge-Ampère equations

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Abstract
This paper is devoted to study the following degenerate Monge-Ampère equation:
\[
\begin{aligned}
\det D^2 u &= \Lambda_q (-u)^q \quad \text{in} \quad \Omega, \\
u &= 0 \quad \text{on} \quad \partial \Omega
\end{aligned}
\] (0.1)
for some positive constant \( \Lambda_q \). Suppose \( \Omega \subset \subset \mathbb{R}^n \) is uniformly convex and analytic. Then the solution of (0.1) is analytic in \( \bar{\Omega} \) provided \( q \in \mathbb{Z}^+ \).

Keywords: Analyticity; degenerate elliptic; Monge-Ampère equations

Mathematics Subject Classification: 35A20, 35J70, 35J96

1 Introduction

In this paper, we focus on the analyticity of the solution of the following Monge-Ampère equation:
\[
\begin{aligned}
\det D^2 u &= \Lambda_q (-u)^q \quad \text{in} \quad \Omega, \\
u &= 0 \quad \text{on} \quad \partial \Omega
\end{aligned}
\] (1.1)
where \( q > 0 \) and \( \Omega \) is a bounded convex domain in \( \mathbb{R}^n \).

This problem was first studied by Lions [12]. In [12], Lions proved that for \( q = n, \) (1.1) admits a unique eigenvalue \( \Lambda_q \) and eigenfunction \( u \in C^{1,1}(\bar{\Omega}) \cap C^\infty(\Omega)(\text{up to multiplications of positive constants}) \) provided \( \Omega \) is smooth and uniformly convex.

Later, Chou [18] studied the problem (1.1) for \( q > 0 \). His approach is based on the Monge-Ampère functional
\[
J(u) = \frac{1}{n+1} \int_\Omega (-u) \det D^2 u \, dx - \frac{1}{q+1} \int_\Omega |u|^{q+1} \, dx
\] (1.2)
and the following logarithmic gradient flow
\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \ln \det(D_x^2 u) - q \ln(-u), \quad (x,t) \in \Omega \times (0, +\infty), \\
u(x,0) &= u_0(x), \quad x \in \Omega, \\
u &= 0 \quad \text{on} \quad \partial \Omega \times (0, +\infty).
\end{aligned}
\] (1.3)

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Then Chou obtained the existence of non-trivial solutions \( u \in C^{0,1}(\Omega) \cap C^\infty(\Omega) \) provided \( \Omega \) is smooth and uniformly convex. Also the uniqueness of non-trivial solution was established in [18] for \( 0 < q < n \) and \( q = n \) (up to multiplications of positive constants). Later, Hartenstine [2] extended it to bounded and strictly convex domain \( \Omega \) for \( 0 < q < n \). Recently, Le [10] proved the existence of solutions of \( \text{(1.1)} \) for \( q > 0 \) and \( q = n \) (up to multiplications of positive constants). Later, the first named author [7] proved that there exists a constant \( \varepsilon(n) > 0 \) such that \( \text{(1.1)} \) admits a unique non-trivial solution for \( q \in (n, n + \varepsilon(n)) \).

The higher order global regularity of the solutions of \( \text{(1.1)} \) remains unknown until recent 10 years. The main difficulty arises from the degeneracy of the equation \( \text{(1.1)} \) on the boundary \( \partial\Omega \). In a survey paper, Trudinger and Wang [17], P21 proposed the problem that whether \( u \) is smooth up to the boundary for \( q = n \) when \( \Omega \) is smooth and uniformly convex. Later, Hong, Huang and Wang [4] gave an affirmative answer to this problem in dimension 2. Their approach relies on an auxiliary function

\[
H = u_{22}u_1^2 - 2u_{12}u_1u_2 + u_{11}u_2^2
\]  

which is related to the Gauss curvature of the level set of \( u \). For arbitrary dimensions, Savin [16] made a first contribution on the global \( C^2 \) regularity of solutions of \( \text{(1.1)} \). Later, Le and Savin [11] completely solved the above problem in arbitrary dimensions. The key observation of the work [16] and [11] is that near the boundary,

\[
u(x) \sim \frac{1}{2} |x'|^2 + \frac{1}{(q+1)(q+2)} x_n^{q+2}
\]

which allows them to use blow-up and perturbation arguments to show \( u \in C^{2,\alpha} \). Then they can raise the regularity up to \( C^\infty \) by investigating a linear degenerate elliptic equation.

It is natural to ask that whether the solution of \( \text{(1.1)} \) is analytic up to the boundary provided the domain \( \Omega \) is analytic and uniformly convex. The analyticity of the solutions of uniformly elliptic equations(systems) are well studied. We refer readers to [15] for linear equations(systems), [13, 14] for non-linear equations(systems) and [1] for more general regularity results. There seems no unified results for degenerate elliptic cases. In the present case, the model degenerate equation is

\[
\begin{aligned}
\mathcal{L}(u) &= u_{nn} + x_n\Delta_x u = f \quad \text{in} \quad \mathbb{R}^n, \\
u(x', 0) &= g(x'), \quad x' \in \mathbb{R}^{n-1},
\end{aligned}
\]

where \( x = (x_1, \cdots, x_{n-1}, x_n) = (x', x_n) \). Usually, people call \( \text{(1.6)} \) a Grushin type degenerate elliptic equation. To the authors’ best knowledge, there seems no results considering the analyticity of solutions of fully non-linear elliptic equations with degeneracy as in \( \text{(1.6)} \).

Now we state our main results in the present paper.

**Theorem 1.1.** Suppose \( u \) is a non-trivial solution of \( \text{(1.1)} \) and \( \Omega \subset \subset \mathbb{R}^n \) is analytic and uniformly convex. Then \( u \) is analytic in \( \bar{\Omega} \) provided \( q \in \mathbb{Z}^+ \), i.e. \( u \in C^\omega(\bar{\Omega}) \).

**Remark 1.1.** The idea of the proof of Theorem 1.1 originated from [3] for semi-linear elliptic equation. For the fully non-linear uniform elliptic case, we refer the readers to [3].
The present paper is organized as follows. In Section 2, we will collect some basic estimates for the linear degenerate elliptic equation (2.1). Then we apply the ideas of [8] to show that \( u \) is analytic up to the boundary in Section 3.

2 Estimates for linear model equation

In the present section, we consider the following linear degenerate elliptic equations:

\[
\begin{aligned}
&u_{nn} + x_n^m \Delta_x u = f &\text{ in } &\mathbb{R}^+_n, \\
u(x', 0) = g(x'), & x' \in \mathbb{R}^{n-1},
\end{aligned}
\]  

(2.1)

where \( x = (x_1, \cdots, x_{n-1}, x_n) = (x', x_n) \).

Firstly, we introduce some notations. Let \( m \) be a positive integer. For a multi-index \( \alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \), we denote \( \partial_\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n} \) with \( |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n \) and \( u_i = \partial_i u = \frac{\partial u}{\partial x_i} (i = 1, 2, \cdots, n) \).

For \( k \geq 0 \), we define the following weighted Sobolev space \( \tilde{W}^{k,2}(G_1) \) as

\[
\| u \|_{\tilde{W}^{k,2}(G_1)} = \sum_{|\alpha| = k} \| \partial_\alpha^2 u \|_{L^2(G_1)} + \sum_{|\alpha| = k + 2, \alpha_n \leq 1} \| \partial_\alpha \partial_n u \|_{L^2(G_1)} + \sum_{|\alpha| = k} \| \partial_\alpha \partial_n^{2} u \|_{L^2(G_1)}
\]

where \( G_1 = \mathbb{R}^{n-1} \times [0, 1) \).

In the following, we introduce two lemmas which are the key ingredients to prove the analyticity of the solutions to Monge-Ampère equations.

Lemma 2.1. Suppose \( u \in C^\infty_c(\mathbb{R}^+_n) \) solves (2.1) with \( \text{supp } u \subset G_1 \). Then there holds

\[
\| u \|_{\tilde{W}^{0,2}(G_1)} \leq C\left( \| f \|_{L^2(G_1)} + \| g \|_{H^1(\mathbb{R}^{n-1})} \right)
\]

(2.2)

for some constant \( C \) depending only on \( n \).

Proof. We divide the proof of the present lemma into two cases.

Case 1. For \( m = 1 \), we follow the steps of standard \( W^{2,2} \)-estimates as Laplacian equation. By (2.1), one gets

\[
\int_{\mathbb{R}^+_n} (u_{nn} + x_n \Delta_x u)^2 \, dx = \int_{\mathbb{R}^+_n} f^2 \, dx.
\]

(2.3)

Integrating by parts the crossproduct term yields

\[
2 \int_{\mathbb{R}^+_n} x_n u_{nn} \Delta_x u \, dx = -2 \int_{\mathbb{R}^+_n} x_n \nabla_x u_{nn} \cdot \nabla_x u \, dx
\]

\[
= 2 \int_{\mathbb{R}^+_n} x_n |\nabla_x u_n|^2 \, dx + 2 \int_{\mathbb{R}^+_n} \nabla_x u_n \cdot \nabla_x u \, dx
\]

\[
= 2 \int_{\mathbb{R}^+_n} x_n |\nabla_x u_n|^2 \, dx - \int_{\mathbb{R}^{n-1}} |\nabla x' g|^2 \, dx'.
\]

(2.4)
Integrating by parts the other terms of (2.3), one gets
\[
\int_{\mathbb{R}^n_+} u_{nn}^2 + x_n |\nabla x' u|^2 + 2 \int_{\mathbb{R}^n_+} x_n |\nabla x'u_n|^2 dx = \int_{\mathbb{R}^n_+} f^2 dx + \int_{\mathbb{R}^{n-1}_+} |\nabla x' g|^2 dx'.
\] (2.5)

Multiplying \(-u\) on both sides of (2.1) and integrating by parts, one obtains
\[
\int_{\mathbb{R}^n_+} u_n^2 + x_n |\nabla x'u|^2 dx = -\int_{\mathbb{R}^{n-1}_+} u(x', 0)u_n(x', 0)dx' - \int_{\mathbb{R}^n_+} fudx.
\] (2.6)

Suppose
\[
\int_{\mathbb{R}^{n-1}_+} u_n^2(x', t)dx' = \inf_{x_n \in [0, 1]} \int_{\mathbb{R}^{n-1}_+} u_n^2(x', x_n)dx'.
\] (2.7)

Then we know
\[
\left| \int_{\mathbb{R}^{n-1}_+} u(x', 0)u_n(x', 0)dx' \right| \leq C_\varepsilon \int_{\mathbb{R}^{n-1}_+} g^2 dx' + \varepsilon \int_{\mathbb{R}^{n-1}_+} u_n^2(x', 0)dx'
\leq C_\varepsilon \int_{\mathbb{R}^{n-1}_+} g^2 dx' + C\varepsilon \left( \int_{\mathbb{R}^{n-1}_+} u_n^2(x', t)dx' + \int_{\mathbb{R}^n_+} u_{nn}^2 dx \right)
\leq C(||g||_{L^1(\mathbb{R}^{n-1})}^2 + ||f||_{L^2(\mathbb{R}^n_+)}^2) + \frac{1}{4} \int_{\mathbb{R}^n_+} u_n^2 dx.
\] (2.8)

In getting the last inequality, we use (2.5), (2.7) and take suitable \(\varepsilon > 0\) small. Also, one has
\[
\int_{\mathbb{R}^n_+} u_n^2 dx \leq C \int_{\mathbb{R}^{n-1}_+} g^2 dx' + \int_{\mathbb{R}^n_+} u_n^2 dx
\] (2.9)

since \(u\) has compact support. Noticing that \(\nabla x'u = \partial_n(x_n \nabla x'u) - x_n \nabla x'u_n\), one knows
\[
\int_{\mathbb{R}^n_+} |\nabla x'u|^2 dx \leq 2 \int_{\mathbb{R}^n_+} |\partial_n(x_n \nabla x'u)|^2 + x_n^2 |\nabla x'u_n|^2 dx
= 2 \int_{\mathbb{R}^n_+} x_n^2 |\nabla x'u_n|^2 dx - 2 \int_{\mathbb{R}^n_+} x_n \nabla x'u \cdot (2\nabla x'u + x_n \nabla x'u_n)dx
\] (2.10)

Combining estimates (2.5), (2.10) and the assumption \(\text{supp } u \subset G_1\) yield the present lemma for \(m = 1\).

Case 2. For \(m \geq 2\), the above method fails. We need employ some previous estimates for (2.1). One may view the solutions of (2.1) as \(u = v + w\) where \(v\) solves
\[
\begin{cases}
\partial_{nn}v + x_{nn}^2 \Delta x'v = 0 & \text{in } \mathbb{R}^n_+,
\v(x', 0) = g(x'), & x' \in \mathbb{R}^{n-1},
\end{cases}
\] (2.11)
and \( w \) solves
\[
\begin{aligned}
\partial_{nn} w + x_m^n \Delta x' w &= f & \text{in} & & \mathbb{R}^n, \\
(w(x', 0) = 0, & & x' \in \mathbb{R}^{n-1}.
\end{aligned}
\tag{2.12}
\]

Since \( g(x') \) is compactly supported in \( \mathbb{R}^{n-1} \), Hong-Wang [Lemma 3.1, [6]] proved that there exists an operator \( B \) such that \( v(x) = B(g)(x) \) solves (2.11) and satisfies the following estimates
\[
\begin{aligned}
\| \partial_{mn} (B(g))(x_n) \|_{L^2_{\infty}(\mathbb{R}^{n-1})} + \| x_n^m \Lambda \partial_n (B(g))(x_n) \|_{L^2_{\infty}(\mathbb{R}^{n-1})} \\
+ \| x_n^m \Lambda \partial_n (B(g))(x_n) \|_{L^2_{\infty}(\mathbb{R}^{n-1})} + \| \partial_n (B(g))(x', 0) \|_{L^2_{\infty}(\mathbb{R}^{n-1})} \\
+ \| x_n^m \Lambda \partial_n (B(g))(x_n) \|_{L^2_{\infty}(\mathbb{R}^{n-1})} \leq C \Lambda^{\frac{n+1}{n-1}} g \|_{L^2(\mathbb{R}^{n-1})},
\end{aligned}
\tag{2.13}
\]
where \( C \) is a universal constant depending only on \( n, \Lambda \) and \( \Lambda \) represent the singular integral operators with symbols \( |\xi| \) and \((1 + |\xi|^2)^{\frac{1}{2}} \). Also the norm \( L^2_{\infty}(\mathbb{R}^{n-1}) \) represents the \( L^2 \)-norm over \( \mathbb{R}^{n-1} \) in \( x' \) variable. Then by a simple integration over \( x_n \), one obtains
\[
\begin{aligned}
\| \partial^2_n (B(g)) \|_{L^2(G_1)} + \| x_n^m \Lambda \partial_n (B(g)) \|_{L^2(G_1)} + \| x_n^m \Lambda \partial_n (B(g)) \|_{L^2(G_1)} \\
+ \| \partial_n (B(g)) \|_{L^2(G_1)} + \| x_n^m \Lambda \partial_n (B(g)) \|_{L^2(G_1)} + \| \partial_n (B(g))(x', 0) \|_{L^2(G_1)} \\
+ \| B(g) \|_{L^2(G_1)} \leq C \Lambda^{\frac{n+1}{n-2}} g \|_{L^2(\mathbb{R}^{n-1})},
\end{aligned}
\tag{2.14}
\]

Since \( f \) is compactly supported in \( \mathbb{R}^n \), Hong-Li [Theorem 3.2, [5]] (also see Lemma 3.4 in [6]) proved that there exists an operator \( T \) such that \( w = T(f) \) solves (2.12) and satisfies the following estimates
\[
\begin{aligned}
\| \partial^2_n (T f) \|_{L^2(G_1)} + \| x_n^m \Lambda \partial_n (T f) \|_{L^2(G_1)} + \| x_n^m \Lambda (T f) \|_{L^2(G_1)} + \| \partial_n (T f) \|_{L^2(G_1)} \\
+ \| x_n^m \Lambda (T f) \|_{L^2(G_1)} + \| \partial_n (T f)(x', 0) \|_{L^2(\mathbb{R}^{n-1})} + \| T f \|_{L^2(G_1)} \leq C \| f \|_{L^2(G_1)}.
\end{aligned}
\tag{2.15}
\]

Then combining (2.14) and (2.15), one gets
\[
\begin{aligned}
\| \partial^2_n u \|_{L^2(G_1)} + \sum_{i=1}^{n-1} \| x_n^m \partial_i u \|_{L^2(G_1)} + \sum_{i,j=1}^{n-1} \| x_n^m \partial_{ij} u \|_{L^2(G_1)} + \| \partial_n u \|_{L^2(G_1)} \\
+ \sum_{i=1}^{n-1} \| x_n^m \partial_i u \|_{L^2(G_1)} + \| \partial_n (u(x', 0) \|_{L^2(\mathbb{R}^{n-1})} + \| u \|_{L^2(G_1)} \\
\leq C \| f \|_{L^2(G_1)} + \| g \|_{H^1(\mathbb{R}^{n-1})}.
\end{aligned}
\tag{2.16}
\]

The last inequality of (2.16) comes from
\[
\| \Lambda^{\frac{1}{n-1}} g \|_{L^2(\mathbb{R}^{n-1})} = \| (1 + |\xi|^2)^{\frac{1}{2}} g \|_{L^2(\mathbb{R}^{n-1})} \leq \| (1 + |\xi|^2)^{\frac{1}{2}} g \|_{L^2(\mathbb{R}^{n-1})} \leq \| g \|_{H^1(\mathbb{R}^{n-1})}
\]
because of \( m \geq 2 \). Noticing that \( m \geq m - 1 \) for \( m \geq 2 \), one knows
\[
\| x_n^m \partial_{x'} u \|_{L^2(G_1)} \geq \| x_n^{m-1} \partial_{x'} u \|_{L^2(G_1)}
\]
which implies the present lemma for \( m \geq 2 \).
Lemma 2.2. Suppose \( u \in C^{k+1}_c(\mathbb{R}^n) \) solves (2.1) with \( \text{supp } u \subset G_1 \). Then there holds
\[
\| u \|_{W^{k,2}(G_1)} \leq C_k \left( \| f \|_{H^k(G_1)} + \| g \|_{H^{k+1}(\mathbb{R}^n)} \right), \quad k \geq 0
\]
for some positive constant \( C_k \) depending only on \( k, m, n \).

Proof. For \( k = 0 \), this is just (2.2). We now prove the present lemma by induction on \( k \).

First, we consider \( \partial_x^\alpha u = \partial_\beta x^{\nu} u \), \( |\alpha| = k + 1 \), \( \alpha = 0 \). Differentiating (2.1) with respect to \( x' \) for \( k + 1 \) times, one gets
\[
\begin{aligned}
&\begin{cases}
\partial_{x'}^\alpha u + x_m^\alpha \Delta \partial_{x'}^\alpha u = \partial_{x'}^\alpha f & \text{in } \mathbb{R}^n_+,
\partial_{x'}^\alpha u(x', 0) = \partial_{x'}^\alpha g(x'), \quad x' \in \mathbb{R}^{n-1}.
\end{cases}
\end{aligned}
\]

Applying (2.2) to (2.18), one gets
\[
\| \partial_{x'}^\alpha u \|_{W^{0,2}(G_1)} \leq C(\| \partial_{x'}^\alpha f \|_{L^2(G_1)} + \| \partial_{x'}^\alpha g \|_{H^1(\mathbb{R}^{n-1})}).
\]

Especially, we obtain
\[
\sum_{|\beta| = k+1, \beta_n = 0} \| \partial_\beta \partial_{x'}^\alpha u(x', 0) \|_{L^2(\mathbb{R}^{n-1})} + \sum_{|\beta| = k+2, \beta_n = 0} \| x_m^{\beta_n} \partial_\beta \partial_{x'}^\alpha u \|_{L^2(G_1)} \leq C(\| f \|_{H^{k+1}(G_1)} + \| g \|_{H^{k+2}(\mathbb{R}^n)}).
\]

Then we consider the case \( \partial_{x'}^\alpha u = \partial_\beta x^{\nu} \partial_{x'}^\alpha u \) with \( |\alpha'| = k \). Differentiating (2.1) with respect to \( x' \) for \( k \) times and then with respect to \( x_n \), one can derive
\[
\partial_{x'}^\alpha \partial_\beta \partial_{x'}^\alpha u + x_n^\nu \Delta x \partial_\beta \partial_{x'}^\alpha u = \partial_\beta \partial_{x'}^\alpha f - m x_n^{\nu-1} \Delta x \partial_{x'}^\alpha u \quad \text{in } \mathbb{R}^n_+.
\]

For the boundary term, by (2.20), we know
\[
\| \partial_\beta \partial_{x'}^\alpha u_n(x', 0) \|_{H^1(\mathbb{R}^{n-1})} \leq C(\| f \|_{H^{k+1}(G_1)} + \| g \|_{H^{k+2}(\mathbb{R}^n)}).
\]

Applying (2.2) to (2.21) and using (2.20) and (2.22), one gets
\[
\| \partial_\beta \partial_{x'}^\alpha u \|_{W^{0,2}(G_1)} \leq C(\| f \|_{H^{k+1}(G_1)} + \| x_n^{\nu-1} \Delta x \partial_{x'}^\alpha u \|_{L^2(G_1)} + \| \partial_\beta \partial_{x'}^\alpha u_n(x', 0) \|_{H^1(\mathbb{R}^{n-1})}) \leq C(\| f \|_{H^{k+1}(G_1)} + \| g \|_{H^{k+2}(\mathbb{R}^n)}).
\]

The last case is \( \partial_{x'}^\alpha u = \partial_\beta x^{\nu} \partial_{x'}^\alpha u \), \( l \geq 2 \), \( |\beta| = k + 1 - l \). Differentiating (2.1) with respect to \( x' \) for \( k + 1 - l \) times and then with respect to \( x_n \) for \( l \) times, one can derive
\[
\partial_{x'}^\alpha \partial_\beta \partial_{x'}^\alpha u + x_n^\nu \Delta x \partial_\beta \partial_{x'}^\alpha u = \partial_\beta \partial_{x'}^\alpha f - \sum_{r=1}^{\min(m,l)} \frac{l!}{r!(l-r)!} \partial^n r(x_n^r) \partial_\beta \partial_{x'}^{l-r} \Delta x \partial_{x'}^\alpha u \quad \text{in } \mathbb{R}^n_+.
\]
We first consider the boundary term. Applying $\partial_n^{-2}\partial_x^\beta$ to (2.1), one gets
\begin{equation}
\partial_n^\beta \partial_x^\beta u = -\partial_n^{-2}\partial_x^\beta (x_n^m \Delta_x^u) + \partial_n^{-2}\partial_x^\beta f. \tag{2.24}
\end{equation}
Restricting (2.24) on $x_n = 0$, then we obtain
\begin{equation}
\|\partial_n^\beta \partial_x^\beta f(x', 0)\|_{H^1(\mathbb{R}^n-1)} \leq C\|f\|_{H^{k+1}(G_1)} \tag{2.25}
\end{equation}
by the trace theorem of Sobolev space. For the term $\partial_n^{-2}\partial_x^\beta (x_n^m \Delta_x^u)$, it matters only if $l \geq m + 2$ and equals to $m!C_n^{m-2}\partial_n^{-2}\partial_x^\beta \Delta_x^u(x', 0)$.

If $l = m + 2$, then we know
\begin{equation}
\|\partial_n^\beta \Delta_x^u(x', 0)\|_{H^1(\mathbb{R}^n-1)} \leq ||g||_{H^{k+1}(\mathbb{R}^n-1)}.
\end{equation}
If $l \geq m + 3$, then we know $\partial_n^{-2-m}\partial_x^\beta \Delta_x^u(x', 0) = \partial_n(\partial_n^{-3-m}\partial_x^\beta \Delta_x^u)(x', 0)$. This implies
\begin{equation}
\|\partial_n^{-2-m}\partial_x^\beta \Delta_x^u(x', 0)\|_{H^1(\mathbb{R}^n-1)} \leq C\|\partial_n\partial_x^\beta u(x', 0)\|_{H^1(\mathbb{R}^n-1)} \leq C_k(\|f\|_{H^k(G_1)} + \|g\|_{H^{k+1}(\mathbb{R}^n-1)}).
\end{equation}
by the induction assumption for some $\tilde{\beta}$ with $|\tilde{\beta}| = k - m \leq k - 1$.

Combining (2.25)\textendash(2.26), one obtains
\begin{equation}
\|\partial_n^\beta \partial_x^\beta u(x', 0)\|_{H^1(\mathbb{R}^n-1)} \leq C_k(\|f\|_{H^k(G_1)} + \|g\|_{H^{k+1}(\mathbb{R}^n-1)}).
\end{equation}
We only need to take care of $x_n^{m-1}\partial_n^{-1}\partial_x^\beta \Delta_x^u, x_n^{m-2}\partial_n^{-2}\partial_x^\beta \Delta_x^u (m \geq 2), l \geq 2, |\beta| = k + 1 - l$.

For the term $x_n^{m-1}\partial_n^{-1}\partial_x^\beta \Delta_x^u$:

1. If $l = 2$, i.e. $x_n^{m-1}\partial_n^{-1}\partial_x^\beta \Delta_x^u = x_n^{m-1}\partial_n\partial_x^\beta \Delta_x^u, |\beta| = k - 1$. Then by estimate (2.19), this term can be controlled by $\|f\|_{H^{k+1}(G_1)} + \|g\|_{H^{k+2}(\mathbb{R}^n-1)}$.

2. If $l \geq 3$, then by induction assumption on $k$, we have
\begin{align*}
\|x_n^{m-1}\partial_n^{-1}\partial_x^\beta \Delta_x^u\|_{L^2(G_1)} &\leq \|\partial_n^{2}(\partial_n^{-3}\partial_x^\beta \Delta_x^u)\|_{L^2(G_1)} \\
&\leq \|u\|_{W^{k,2}(G_1)} \leq C_k(\|f\|_{H^k(G_1)} + \|g\|_{H^{k+1}(\mathbb{R}^n-1)}).
\end{align*}
Similarly, for the term $x_n^{m-2}\partial_n^{-2}\partial_x^\beta \Delta_x^u (m \geq 2)$:

1. If $l = 2$, i.e. $x_n^{m-2}\partial_n^{-2}\partial_x^\beta \Delta_x^u = x_n^{m-2}\partial_x^\beta \Delta_x^u, |\beta| = k - 1$. Then by estimate (2.19), this term can be controlled by $\|f\|_{H^{k+1}(G_1)} + \|g\|_{H^{k+2}(\mathbb{R}^n-1)}$.

2. If $l \geq 3$, then by induction assumption on $k$, we have
\begin{align*}
\|x_n^{m-2}\partial_n^{-2}\partial_x^\beta \Delta_x^u\|_{L^2(G_1)} &\leq \|\partial_n^{2}(\partial_n^{-3}\partial_x^\beta \Delta_x^u)\|_{L^2(G_1)} \\
&\leq \|u\|_{W^{k,2}(G_1)} \leq C_k(\|f\|_{H^k(G_1)} + \|g\|_{H^{k+1}(\mathbb{R}^n-1)}).
\end{align*}
Overall, we obtain
\[
\|\partial_x^\alpha \partial_y^\beta u\|_{\tilde{\mathcal{W}}^{0,2}(G_1)} \leq C(\|f\|_{H^{k+1}(G_1)} + \|g\|_{H^{k+2}(\mathbb{R}^n)}) , \quad \forall l \geq 2, |\beta| = k + 1 - l.
\]
This ends the proof of present lemma.

\[\text{Lemma 2.3} \]
For any \( u, v \in \tilde{W}^{k,2}(G_1) \), there exists a constant \( C_k \) such that
\[
\|uv\|_{\tilde{W}^{k,2}(G_1)} \leq C_k \|u\|_{\tilde{W}^{k,2}(G_1)} \|v\|_{\tilde{W}^{k,2}(G_1)} \quad (2.28)
\]
provided \( k \geq n + 3 \).

\[\text{Proof.}\]
For any multi-index \( \alpha \in \mathbb{N}^n \), one has
\[
\partial_x^\alpha (uv) = \sum_{\beta + \gamma = \alpha} \frac{\alpha!}{\beta! \gamma!} \partial_x^\beta u \partial_x^\gamma v.
\]
Since the highest order of derivative in \( \tilde{W}^{k,2}(G_1) \) is \( k + 2 \), we know \( \min(|\beta|, |\gamma|) \leq \frac{k+2}{2} \). Then for the product \( \partial_x^\beta u \partial_x^\gamma v \), at least one term is in \( H^{k+\frac{4-2k}{2}}(\mathbb{R}^n) \) provided \( k > n + 2 \) by Sobolev embedding theorem. This implies the present lemma.

In the following, we give a lemma which is essentially Lemma 1 in [1].

\[\text{Lemma 2.4} \]
Let \( B_1 \times \mathbb{B}_R \) be the domain in \( \mathbb{R}^n \times \mathbb{R}^L \). Assume that \( \Phi(x, y) \) is a polynomial and \( p \) is a positive integer. Let \( \eta \in C^\infty_c(B_1) \) is a cut-off function. Then, there exist positive constants \( A_0, \tilde{A}_0 \) and \( A_1 \), depending only on \( n, L, k, \eta \) and the polynomial \( \Phi(x, y) \), such that, for any \( C^p \)-function \( y = (y_1, \ldots, y_L) : B_1 \to \mathbb{B}_R \), if for any \( x \in B_1 \) and any non-negative integer \( l \leq p \),
\[
\sum_{i=1}^L \|\eta^i \partial_x^\beta y_i(x)\|_{\tilde{W}^{k,2}(B_1^+)} \leq A_0 A_1^{(l-2)+} (l - 2)^{+}!,
\]
for some \( k > n + 2 \). Then, for any \( x \in B_1^+ \),
\[
\|\eta^i \partial_x^\beta [\Phi(x, y(x))]\|_{\tilde{W}^{k,2}(B_1^+)} \leq \tilde{A}_0 A_1^{(p-2)+} (p - 2)^{+}!.
\]
Here, if no confusion occurs, the meaning of \( l \) and \( p \) can be vary from multi-index to pure integer.

\[\text{Proof.}\]
By our assumptions, we know
\[
\Phi(x, y(x)) = \sum_{|\alpha| + |\beta| \leq d} C_{\alpha \beta} x^\alpha y(x)^\beta.
\]
Here \( d \) is the degree of the polynomial \( \Phi(x, y) \) and \( x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \). Then for \( x^\alpha y(x)^\beta = x^\alpha y_1(x) \cdots y_{|\beta|}(x) \), \( 1 \leq i_1, \ldots, i_{|\beta|} \leq L \), one knows
\[
\partial_x^p (x^\alpha y(x)^\beta) = \sum_{k_0 + k_1 + \cdots + k_{|\beta|} = p} \frac{p!}{k_0! k_1! \cdots k_{|\beta|}!} \partial_x^{k_0} (x^\alpha) \partial_x^{k_1} (y_1(x)) \cdots \partial_x^{k_{|\beta|}} (y_{|\beta|}(x)). \quad (2.29)
\]
Let $C_0, C_1$ be the positive constants such that
\[ \sum_{|\alpha| \leq d} \| \eta^\alpha \partial_x^\alpha (x^n) \|_{W^{k,2}(B_1^+(0))} \leq C_0 C_1^{(l-2)^+} (l-2)^+, \quad l \geq 0. \]

Let $\| \cdot \|$ be $\| \cdot \|_{W^{k,2}(B_1^+(0))}$. Then by Lemma 2.3 and the assumptions, we know
\[ \| \eta^\beta \partial_x^\beta (x^n y(x)^\beta) \| \leq C_k^{(\beta)+1} \sum_{k_0 + k_1 + \cdots + k_\beta = p} \frac{p!}{k_0!k_1! \cdots k_\beta!} C_0 C_1^{(k_0-2)^+} (k_0 - 2)^+ |\Pi|^{k_1} A_0 A_1^{(k_j-2)^+} (k_j - 2)^! \]  
\[ \leq (A_0 C_k)^{\beta+1} A_1^{(p-2)+} \frac{p!}{k_0!k_1! \cdots k_\beta!} \sum_{k_0 + k_1 + \cdots + k_\beta = p} (k_0 - 2)^+ (!k_1 - 2)^+ \cdots (k_\beta - 2)^! \]  
\[ \leq (A_0 C_k)^{\beta+1} A_1^{(p-2)+} \frac{p!}{k_0!k_1! \cdots k_\beta!} \sum_{k_0 + k_1 + \cdots + k_\beta = p} (k_0 - 2)^+ (!k_1 - 2)^+ \cdots (k_\beta - 2)^! \]
provided $A_0 \geq C_0, A_1 \geq C_1$. In the following, we will show that there exists a constant $C_2 > 0$ such that
\[ \sum_{k_0 + k_1 + \cdots + k_\beta = p} \frac{(k_0 - 2)^+ (!k_1 - 2)^+ \cdots (k_\beta - 2)^!}{k_0!k_1! \cdots k_\beta!} \leq C_2 (8\pi^2 (d+1))^{\beta+1} \frac{1}{(p+1)^2}, \quad \forall p \in \mathbb{N} \]
by induction for all $|\beta| \leq d$. It is easy to see that, we can choose $C_2$ large enough such that (2.31) holds for all $0 \leq p \leq 10$. Suppose (2.31) holds for $p \geq 10$. Then for $p+1$, one has
\[ \sum_{k_0 + k_1 + \cdots + k_\beta = p+1} \frac{(k_0 - 2)^+ (!k_1 - 2)^+ \cdots (k_\beta - 2)^!}{k_0!k_1! \cdots k_\beta!} \]
\[ \leq (d+1) \sum_{k_\beta = 1}^{p+1} \frac{(k_\beta - 2)^!}{k_\beta!} \sum_{k_0 + \cdots + k_{\beta-1} = p+1-k_\beta} \frac{(k_0 - 2)^+ (!k_1 - 2)^+ \cdots (k_{\beta-1} - 2)^!}{k_0!k_1! \cdots k_{\beta-1}!} \]
\[ \leq (d+1) C_2 (8\pi^2 (d+1))^{|\beta|} \sum_{k_\beta = 1}^{p+1} \frac{(k_\beta - 2)^!}{k_\beta! (p+2-k_\beta)^2} \]
\[ \leq 4(d+1) C_2 (8\pi^2 (d+1))^{|\beta|} \sum_{k_\beta = 1}^{p+1} \frac{1}{k_\beta^2 (p+2-k_\beta)^2} \]
\[ \leq 4(d+1) C_2 (8\pi^2 (d+1))^{|\beta|} \left( \frac{9}{4(p+1)^2} \left( \sum_{k_\beta = 1}^{p+1} \frac{1}{k_\beta^2} \right) + \sum_{k_\beta = 2}^{p+1} \frac{1}{(p+2-k_\beta)^2} \right) \]
\[ + \sum_{k_\beta = [\frac{p+1}{2}]}^{2\left[\frac{p+1}{2}\right]} \frac{1}{k_\beta^2 (p+2-k_\beta)^2} \]
\[ \leq C_2 (8\pi^2 (d+1))^{|\beta|+1} \frac{1}{(p+2)^2}. \]

In getting the last inequality of the above, we used
\[ \sum_{l=1}^{\infty} \frac{1}{l^2} = \frac{\pi^2}{6}, \quad \frac{27}{p+1} \leq \pi^2, \quad \frac{(p+2)^2}{(p+1)^2} \leq 2. \]
This ends the proof of (2.31). By (2.31), one knows
\[ \|\eta^p \partial^p_x y(x)\| \leq (8\pi^2 A_0 C_k(d+1))^{d+1} A_1^{(p-2)^+} (p-2)^+! \]
This implies
\[ ||\eta^p \partial^p_x [\Phi(x, y(x))]|| \leq (8\pi^2 A_0 C_k(d+1))^{d+1} A_1^{(p-2)^+} (p-2)^+! \sum_{|\alpha|+|\beta| \leq d} |C_{\alpha\beta}|. \]
By letting \( \tilde{A}_0 = (8\pi^2 A_0 C_k(d+1))^{d+1} \sum_{|\alpha|+|\beta| \leq d} |C_{\alpha\beta}| \), we finish the proof of present lemma.

3 Analyticity of the solutions

Through out this section, we let \( q = m \in \mathbb{Z}^+ \) in (1.1). Then \( u \) solves
\[
\begin{aligned}
det D^2 u &= \Lambda_m (-u)^m \quad \text{in} \quad \Omega, \\
u &= 0 \quad \text{on} \quad \partial\Omega. 
\end{aligned}
\] (3.1)
In the following, we omit the constant \( \Lambda_m \). Since (3.1) is uniformly elliptic in the interior of \( \Omega \), then applying the results of \([1, 14]\), we can get \( u \in C^\infty(\Omega) \). The remaining thing is to show \( u \) is analytic up to the boundary.

Suppose \( \Omega \subset \mathbb{R}^n_+, 0 \in \partial\Omega \) and \( x_n = 0 \) is the supporting plane of \( \Omega \) at 0. Then we know the non-trivial solution \( u \) of (3.1) satisfies
(I) \( u \in C^\infty(\Omega) \) and \( u_n(0) < 0, \nabla x'u(0) = 0 \).
(II) \( \lambda \leq D^2 u(0) \leq \Lambda \) for two positive constants \( \lambda, \Lambda \).

For the proof of (I) and (II), we refer the readers to \([\text{Theorem B}, [4]\] for \( n = 2 \) and \([\text{Theorem 1.4}, [11]\] for arbitrary dimension.

Then as in \([11]\), we take the following Hodo-graph transformation near 0,
\[
y_n = -x_{n+1}, \quad y_{n+1} = x_n, \quad y_k = x_k (1 \leq k \leq n-1). 
\]
In the new coordinates, the graph of \( u \) near 0 can be represented as
\[
y_{n+1} = v(y) \quad \text{in} \quad \{ y \in \mathbb{R}^n : y_n > 0 \} \cap B_\delta(0) 
\]
for some \( \delta > 0 \) small enough. This implies
\[
u(y', v(y)) + y_n = 0 \quad \text{in} \quad \{ y \in \mathbb{R}^n : y_n > 0 \} \cap B_\delta(0). \] (3.2)
Then, differentiating the above equation directly yields
\[
u_\alpha + u_nv_\alpha = 0, \quad u_nv_n + 1 = 0, \quad \alpha = 1, \ldots, n-1. 
\]
Since the Gauss curvature will not change, one has
\[
\begin{aligned}
det D^2 v &= K(1 + |\nabla v|^2)^{d+2} = det D^2 u^{(1+|\nabla u|^2)^{d+2}} = y_n^{m_n v_n^{n+2}} \quad \text{in} \quad B_\delta^+(0), \\
v(y', 0) &= \phi(y'), \quad |y'| \leq \delta, 
\end{aligned}
\] (3.3)
where \( y_n = \phi(y') \) represents the boundary \( \partial\Omega \) near 0. Moreover, the Hodo-graph transformation preserves the analyticity.
Lemma 3.1. Let \( v \) be given by (3.2) where \( u \) is the non-trivial solution of (3.1). Then \( v \) is analytic near 0 implies \( u \) is analytic near 0.

Proof. Let \( F(y, y_{n+1}) = v(y) - y_{n+1} \) and \( F(0) = 0 \). Then by the assumption of the present lemma, we know \( F(y, y_{n+1}) \) is analytic in \( B_{\delta}(0) \) for some \( \delta > 0 \) small. Also by (1), one has

\[
F_n(0, 0) = v_n(0) = -\frac{1}{u_n(0)} \neq 0.
\]

By the real analytic implicit function theorem (Theorem 1.8.3, [9]), we know there exists \( u \) analytic near 0 such that

\[
F(y_1, \cdots, y_{n-1}, y_1, \cdots, y_{n'}, y_{n+1}) = 0,
\]

which implies the present lemma. \( \square \)

Differentiating (3.2) with respect to \( y' \) for two times, one obtains

\[
u_{\alpha\beta} + u_{\alpha\gamma} v_{\beta} + u_{\beta\gamma} v_{\alpha} + u_{\alpha\beta} v_{\gamma} + u_{\gamma\beta} v_{\alpha} = 0, \quad 1 \leq \alpha, \beta \leq n - 1.
\]

This implies

\[
v_{\alpha\beta}(0) = -\frac{u_{\alpha\beta}(0)}{u_n(0)} \tag{3.4}
\]

is a positive definite matrix. Then we can perform the partial Legendre transformation to the solutions of (3.3):

\[
z_i = v_i(y) \quad (i \leq n - 1), \quad z_n = y_n, \quad v^*(z) = y' \cdot \nabla y - v(y). \tag{3.5}
\]

Set the partial Legendre transformation by \( z = T(y) \). Then \( v^* \) should satisfy

\[
\begin{cases}
 z_n^m (-v_n^*)^{n+2} \det D^2_y v^* + v_{n}^* = 0 & \text{in } \Sigma^+ = T(B_{\delta}^+(0)) \subset \mathbb{R}_+^n, \\
v^* = \phi^* & \text{on } \partial \Sigma^+ \cap \{z_n = 0\},
\end{cases}
\tag{3.6}
\]

where \( \phi^* \) is the Legendre transformation of \( \phi \). Also the partial Legendre transformation preserves the analyticity.

Lemma 3.2. Let \( v^* \) be the partial Legendre transformation of \( v \). Then \( v^* \) is analytic near 0 implies \( v \) is analytic near 0.

Proof. Consider

\[
F_0(y', z, t) = v^*(z) - y' \cdot z' + t, \quad F_i(y', z, t) = y_i - \partial_{z_i} v^*, \quad i = 1, \cdots, n - 1, \tag{3.7}
\]

where \( y' \in \mathbb{R}^{n-1}, z \in \mathbb{R}^n, t \in \mathbb{R} \) and \( z' = (z_1, \cdots, z_{n-1}) \). Then \( F_0(0) = F_i(0) = 0, \ i = 1, \cdots, n-1 \). By assumptions, we know \( F_0(y', z, t), F_i(y', z, t) \) are real analytic functions and

\[
\det \left( \frac{\partial (F_0, \cdots, F_{n-1})}{\partial (t, z')} \right)(0) = (-1)^{n-1} \det D^2_z v^*(0) \neq 0. \tag{3.8}
\]

Again by the real analytic implicit function theorem (Theorem 1.8.3, [9]), we know \( F_0(y', \tilde{z}, \tilde{t}) = 0, F_i(y', \tilde{z}, \tilde{t}) = 0 \) determine real analytic functions \( \tilde{t}(y', z_n), \tilde{z}'(y', z_n) \). This implies the present lemma. \( \square \)
Hence, by the conclusions of Lemma 3.1 and Lemma 3.2, we only need to consider the following type of equation:

\[
\begin{aligned}
\frac{x^m}{(−u_n)^{n+2}} \det \frac{D^2}{x} u + u_{nn} &= 0 \quad \text{in } B^+_1, \\
u = \varphi \quad \text{on } \{x_n = 0\} \cap B_1(= B'_1).
\end{aligned}
\]  

(3.9)

Theorem 3.1. Suppose \(u \in C^\infty(B^+_1)\) and \(\varphi \in C^\omega(B'_1)\) solve (3.9). Moreover, \(u\) satisfies

\[\lambda_{\min}(\frac{D^2}{x} u) \geq c_0 > 0, \quad |u_n(x', 0)| \geq c_0 > 0 \quad \text{in } \overline{B^+_1}.
\]

Then \(u \in C^\omega(B^+_2)\).

Remark 3.1. Combining Theorem 3.1, Lemma 3.1 and Lemma 3.2, we know Theorem 1.1 holds.

Since analyticity is a local property, we restrict our discussion on \(B^+_2\) for some \(r\) small enough to be determined later. Then our aim of the remaining paragraph is to show that there exist two positive constants \(A_0, A_1\) such that

\[
\|\eta^N \partial^N x u\|_{W^{k,2}} \leq A_0 A_1^{(N-4)^+} (N-4-i)^+, \quad i = 0, 1, 2.
\]  

(3.10)

Here, we fix a large enough integer \(k\) such that Lemma 2.3 holds. \(\eta\) is a cut-off function and has the following form

\[
\eta(x) = \chi(x_1) \cdots \chi(x_n),
\]

where \(0 \leq \chi \leq 1\) is a cut-off function satisfying

\[
\chi(t) \equiv 1, \quad \text{in } [-r, r], \quad \chi \equiv 0, \quad \text{in } [-2r, 2r].
\]  

(3.11)

In fact, we just need to consider \(\eta\) in \(\{x \in B_{2r} | x_n > 0\}\). And if no confusion occurs, the meaning of \(N\) can vary from multi-index to pure positive integer.

In the following, we will prove (3.10) via induction. Suppose (3.10) is true for \(0, 1, 2, \cdots, N\). We need to show it holds for \(N+1\).

Firstly, differentiating equation (3.9) with respect to \(x_l, l = 1, \cdots, n-1\), one has

\[
\begin{aligned}
\sum_{i,j=1}^{n-1} x^m_n (−u_n)^{n+2} U_{ij} \partial_i \partial_j u + \partial_{nn} \partial_l u &= −\partial_l (x^m_n (−u_n)^{n+2}) \det \frac{D^2}{x} u \quad \text{in } B^+_1, \\
\partial_l u &= \partial_l \varphi \quad \text{on } \{x_n = 0\} \cap B_1.
\end{aligned}
\]  

(3.12)

where \(U_{ij}\) is the cofactor matrix of \(\frac{D^2}{x} u\). Set

\[
G = −\partial_l (x^m_n (−u_n)^{n+2}) \det \frac{D^2}{x} u.
\]

Then we know \(G\) is a polynomial with arguments \(x, u, \nabla u, \nabla^2 u\). Without loss of generality(after a transformation of coordinates), we may assume

\[
(−u_n)^{n+2} U_{ij} (0) = \delta_{ij}.
\]
Then we can rewrite the equation of (3.12) as
\[ x_n^m \Delta_{x'}(\partial u) + \partial_{nn} \partial u = G + \sum_{i,j=1}^{n-1} a^{ij} x_n^m \partial_{ij} \partial u \quad \text{in} \quad B_1^+, \tag{3.13} \]
where
\[ a^{ij} = \delta_{ij} - (-u_n)^{n+2} U^{ij}, \quad a^{ij}(x) = O(|x|), \quad |x| << 1. \]

Differentiating (3.13) for \( N \) times with respect to \( x' \), one gets
\[ x_n^m \Delta_{x'} \partial_{x'}^{N+1} u + \partial_{nn} (\partial_{x'}^{N+1} u) = \partial_{x'}^{N} G + \sum_{i,j=1}^{n-1} \partial_{x'}^{N} (a^{ij} x_n^m \partial_{ij} \partial u) \quad \text{in} \quad B_1^+, \tag{3.14} \]

Multiplying (3.14) with \( \eta^{-1} \), one obtains
\[ x_n^m \Delta_{x'} (\eta^{-1} \partial_{x'}^{N+1} u) + \partial_{n}^2 (\eta^{-1} \partial_{x'}^{N+1} u) \]
\[ = \eta^{-1} \partial_{x'}^{N} G + \sum_{i,j=1}^{n-1} \eta^{-1} \partial_{x'}^{N} (a^{ij} x_n^m \partial_{ij} \partial u) + [\mathcal{L}, \eta^{-1}] \partial_{x'}^{N+1} u \quad \text{in} \quad B_1^+, \tag{3.15} \]

where
\[ [\mathcal{L}, \eta^{-1}] \partial_{x'}^{N+1} u = [x_n^m \Delta_{x'}, \eta^{-1} \partial_{x'}^{N+1} u] + [\partial_{n}^2, \eta^{-1} \partial_{x'}^{N+1} u] \]
\[ = x_n^m (\Delta_{x'} \eta^{-1}) \partial_{x'}^{N+1} u + 2 x_n^m \sum_{i=1}^{n-1} \partial_i (\eta^{-1}) \partial_i (\partial_{x'}^{N+1} u) \tag{3.16} \]
\[ + (\partial_{n}^2 \eta^{-1}) \partial_{x'}^{N+1} u + 2 \partial_n \eta^{-1} \partial_{n} \partial_{x'}^{N+1} u. \]

As previous, we consider three cases stated in the following three lemmas. In the following proof, the constants \( C, C_1 \ldots \) may vary from line to line, and can depend on \( r \) but is independent of \( A_0, A_1 \). In addition, we use the constant \( c \) to denote the quantities which are independent of \( r \).

We first prove (3.10) holds for \( \eta^{-1} \partial_{x'}^{N+1} u \) case.

**Lemma 3.3.** Suppose the assumptions in Theorem 3.1 are fulfilled. Suppose (3.10) holds for sufficiently large \( A_0, A_1 \). Then there holds
\[ \| \eta^{-1} \partial_{x'}^{N+1} u \|_{\overline{W}^{k-1,2}} \leq C_1 \tilde{A}_0 A_0 A_1^{N-4} (N - 3 - i)! \quad \text{for some constant } C_1 > 0 \text{ and } \tilde{A}_0 \text{ is the constant in Lemma 2.4.} \]

**Proof.** For \( i = 1 \), we know
\[ \| \eta^{-1} \partial_{x'}^{N+1} u \|_{\overline{W}^{k-1,2}} = \| \partial_{x'} (\eta^{-1} \partial_{x'}^{N} u) - (N - 1) \eta^{-2} \partial_{x'} \eta \partial_{x'}^{N} u \|_{\overline{W}^{k-1,2}} \]
\[ \leq C_1 \| \eta^{-2} \partial_{x'}^{N} u \|_{\overline{W}^{k-2}} + C_1 (N - 1) \| \eta^{-2} \partial_{x'}^{N} u \|_{\overline{W}^{k-1,2}} \]
\[ \leq CA_0 A_1^{N-4} (N - 4)!. \]
In getting the last inequality, we used (3.10). By a similar argument, we can prove the case \( i = 2 \).

It remains to prove the case \( i = 0 \). Recall that \( \eta^{N-1}\partial_{x'}^{N+1}u \) satisfies

\[
\begin{align*}
\eta^{N-1}\partial_{x'}^N G + \sum_{i,j=1}^{n-1} \eta^{N-1}\partial_{x'}^N (a_{ij}x^m_n \partial_{x'} u) + [\mathcal{L}, \eta^{N-1}]\partial_{x'}^{N+1}u
\end{align*}
\]

(3.17)

Here \( \hat{\eta}(x') = \eta(x',0) \). By Lemma 2.2 we know

\[
\|\eta^{N-1}\partial_{x'}^{N+1}u\|_{W^{k,2}} \leq C\|\mathcal{L}(\eta^{N-1}\partial_{x'}^{N+1}u)\|_{H^k} + \|\tilde{\eta}^{N-1}\partial_{x'}^{N+1}\varphi\|_{H^{k+1}}.
\]

By the analyticity of \( \varphi \), we may assume

\[
\|\tilde{\eta}^{N-1}\partial_{x'}^{N+1}\varphi\|_{H^{k+1}} \leq A_0 A_1^{N-4}(N - 3)!
\]

In the following, we only need to estimate the terms in \( \mathcal{L}(\eta^{N-1}\partial_{x'}^{N+1}u) \). By the definition of \( G \), we may denote \( G = x^m_n \hat{G} \) where \( \hat{G} \) is a polynomial of \( \nabla u, \nabla^2 u \).

Firstly, we rewrite \( \eta^{N-1}\partial_{x'}^N G \) as

\[
\begin{align*}
\eta^{N-1}\partial_{x'}^N G = & \mathcal{L}(x^m_n \eta^{N-1}\partial_{x'}^{N-2}\hat{G}) - 2(N - 1)\partial_{x'}((\partial_{x'} \eta)x^m_n \eta^{N-2}\partial_{x'}^{N-2}\hat{G}) \\
& + (N - 1)(N - 2)(\partial_{x'} \eta)(\partial_{x'} \eta)x^m_n \eta^{N-3}\partial_{x'}^{N-2}\hat{G} \\
& + (N - 1)(\partial_{x'} \eta)x^m_n \eta^{N-2}\partial_{x'}^{N-2}\hat{G}.
\end{align*}
\]

Then we estimate the terms in \( \eta^{N-1}\partial_{x'}^N G \) one by one.

\[
\begin{align*}
\|\partial_{x'}^2((\partial_{x'} \eta)x^m_n \eta^{N-2}\partial_{x'}^{N-2}\hat{G})\|_{H^k} & \leq \|x^m_n \eta^{N-1}\partial_{x'}^{N-2}\hat{G}\|_{H^{k+2}} \\
& \leq C\|\eta^{N-2}\partial_{x'}^{N-2}\hat{G}\|_{W^{k,2}} \leq C\tilde{A}_0 A_1^{N-4}(N - 4)!.
\end{align*}
\]

(3.19)

In getting the above inequality, we used Lemma 2.4 and induction assumption (3.10). And also

\[
\begin{align*}
\|\partial_{x'}((\partial_{x'} \eta)x^m_n \eta^{N-2}\partial_{x'}^{N-2}\hat{G})\|_{H^k} & \leq \|(\partial_{x'} \eta)x^m_n \eta^{N-2}\partial_{x'}^{N-2}\hat{G}\|_{H^{k+1}} \\
& \leq C\|\eta^{N-2}\partial_{x'}^{N-2}\hat{G}\|_{W^{k-1,2}} \leq C\tilde{A}_0 A_1^{N-4}(N - 5)!
\end{align*}
\]

(3.20)

Similar arguments yield that

\[
\|(\partial_{x'} \eta)(\partial_{x'} \eta)x^m_n \eta^{N-3}\partial_{x'}^{N-2}\hat{G}\|_{H^k} \leq C\tilde{A}_0 A_1^{N-4}(N - 6)!
\]

(3.21)

if we notice that \( \frac{n}{\eta} \in C^\infty_c(\mathbb{R}^n_+) \), \( j, l = 1, 2, \ldots, n - 1 \). Combining the estimates (3.19)-(3.21), one gets

\[
\|\eta^{N-1}\partial_{x'}^N G\|_{H^k} \leq C\tilde{A}_0 A_1^{N-4}(N - 4)!.
\]

(3.22)
In the following, we estimate $\eta^{-1}\partial_{x'}^N(a^{ij}x_n^m\partial_{ij}\partial_{x'}u)$. As previous, we rewrite this term as

$$\eta^{-1}\partial_{x'}^N(a^{ij}x_n^m\partial_{ij}\partial_{x'}u)$$

$$=\partial_{x'}^N(x_n^m\eta^{-1}\partial_{x'}^{N-2}(a^{ij}\partial_{ij}\partial_{x'}u)) - 2(N-1)\partial_{x'}((\partial_{x'}\eta)x_n^m\eta^{-2}\partial_{x'}^{N-2}(a^{ij}\partial_{ij}\partial_{x'}u))$$

$$+(N-1)(N-2)(\partial_{x'}\eta)(\partial_{x'}\eta)x_n^m\eta^{-3}\partial_{x'}^{N-2}(a^{ij}\partial_{ij}\partial_{x'}u)$$

$$+(N-1)(\partial_{x'}^2\eta)x_n^m\eta^{-2}\partial_{x'}^{N-2}(a^{ij}\partial_{ij}\partial_{x'}u).$$

(3.23)

Then one knows

$$\|\partial_{x'}^N(x_n^m\eta^{-1}\partial_{x'}^{N-2}(a^{ij}\partial_{ij}\partial_{x'}u))\|_{H^K} \leq \|\eta^{-1}\partial_{x'}^{N-2}(a^{ij}\partial_{ij}\partial_{x'}u)\|_{\bar{W}^{k,2}}$$

$$\leq cr\|\eta^{-1}\partial_{x'}^{N+1}u\|_{\bar{W}^{k,2}} + \sum_{l=1}^{N-2} \frac{(N-2)!}{l!(N-2-l)!} \|\eta^{-1}\partial_{x'}^{N-l}a^{ij}\partial_{x'}^{N-2-l}\partial_{ij}\partial_{x'}u\|_{\bar{W}^{k,2}}$$

(3.24)

The constant $c$ in the above inequality is independent of $r$. In getting the last inequality of (3.24), we need to use the induction assumption (3.10) and do calculations as in Lemma 2.4. Since only the order of derivative in $\eta^{-1}\partial_{x'}^{N-1}a^{ij}\partial_{x'}^{N-2-l}\partial_{ij}\partial_{x'}u$ matters to the estimate, we make a convention that all the terms $\eta^{-1}\partial_{x'}^{N-1}a^{ij}\partial_{x'}^{N-2-l}\partial_{ij}\partial_{x'}u$ are the same for a fixed $l$.

Similarly, we estimate the other terms in $\eta^{-1}\partial_{x'}^N(a^{ij}x_n^m\partial_{ij}\partial_{x'}u)$ and then get

$$\|\eta^{-1}\partial_{x'}^N(a^{ij}x_n^m\partial_{ij}\partial_{x'}u)\|_{H^K} \leq cr\|\eta^{-1}\partial_{x'}^{N+1}u\|_{\bar{W}^{k,2}} + C\tilde{A}_0A_0A_1^{-N-4}(N-3)!.$$  

(3.25)

For the last term $[L, \eta^{-1}]\partial_{x'}^{N+1}u$, the part $[x_n^m\Delta_{x'}, \eta^{-1}]\partial_{x'}^{N+1}u$ always contains the factor $x_n^m$, we can deduce the $H^k$-norm for $[x_n^m\Delta_{x'}, \eta^{-1}]\partial_{x'}^{N+1}u$ exactly the same as previous two terms in $L(\eta^{-1}\partial_{x'}^{N+1}u)$.

In the following, we estimate the remaining term $[\partial_{n}^2, \eta^{-1}]\partial_{x'}^{N+1}u$.

$$[\partial_{n}^2, \eta^{-1}]\partial_{x'}^{N+1}u = 2(N-1)\eta_n\eta^{-2}\partial_{x'}^{N+1}u$$

$$+(N-1)(N-2)\eta_n^2\eta^{-3}\partial_{x'}^{N+1}u + (N-1)\eta_n\eta^{-2}\partial_{x'}^{N+1}u.$$  

(3.26)

By the definition of $\eta$, we know

$$\text{supp } \eta_n \subset [-2r, 2r]^{n-1} \times [r, 2r](:= Q_r).$$

From the discussion at the beginning of this section, we know $u$ is analytic in $\{x_n > 0\}$. Hence, in the following, we can always assume

$$\|\eta^{-2}\partial_{x'}^{N+2}u\|_{H^K(Q_r)} \leq A_0A_1^{-N}(\tilde{N} - 4)^+!,$$  

(3.27)

Then from (3.26) and (3.27), one obtains

$$\|\eta^{-1}\partial_{x'}^{N+1}u\|_{H^K} \leq CA_0A_1^{-N-4}(N-3)!.$$  

(3.28)

Combining all the above estimates, one gets

$$\|\eta^{-1}\partial_{x'}^{N+1}u\|_{\bar{W}^{k,2}} \leq cr\|\eta^{-1}\partial_{x'}^{N+1}u\|_{\bar{W}^{k,2}} + C\tilde{A}_0A_0A_1^{-N-4}(N-3)!.$$  

By taking $r$ small enough, one proves the present lemma. 

□
Lemma 3.4. Suppose the assumptions in Theorem 3.1 are fulfilled. Suppose \((3.10)\) holds for sufficiently large \(A_0, A_1\). Then there holds

\[
\|\eta^{N-1}\partial_n\partial_x^N u\|_{W^{k-2}} \leq C_1 A_0 A_0 A_1^{N-4} (N - 3 - i)!, \quad i = 0, 1, 2.
\]

Proof. As the proof in Lemma 3.3, we only need to show it for \(i = 0\). Recall that \(\eta^{N-1}\partial_n\partial_x^N u\) satisfies

\[
x_n^m \Delta_x (\eta^{N-1}\partial_n\partial_x^N u) + \partial_n^2 (\eta^{N-1}\partial_n\partial_x^N u)
= \eta^{N-1}\partial_n\partial_x^N G + \sum_{i,j=1}^{n-1} \eta^{N-1}\partial_n\partial_x^N (a^{ij} x_n^m \partial_i \partial_j u)
+ [\mathcal{L}, \eta^{N-1}] \partial_n\partial_x^N u - m x_n^{m-1} \eta^{N-1} \Delta_x \partial_x^N u \quad \text{in} \; \mathbb{R}_+^n.
\]

We first consider the boundary term \(\eta^{N-1}\partial_n\partial_x^N u(x', 0)\).

\[
\|\eta^{N-1}\partial_n\partial_x^N u(x', 0)\|_{H^{k+1}}
= \|\partial_n \partial_x^N (\eta^{N-1}\partial_x^N u(x', 0))\|_{H^k}
\leq \|\partial_n (\eta^{N-1}\partial_x^N u(x', 0))\|_{H^k} + (N - 1)\|\eta_{x'} \partial_n (\eta^{N-2}\partial_x^N u(x', 0))\|_{H^k}
\leq C A_0 A_0 A_1^{N-4} (N - 3)!.
\]

In the calculation of \((3.30)\), the property of \(\eta(x) = \chi(x_1) \cdots \chi(x_n)\) is used in the commutation of \(\partial_n\) and \(\eta^{N-1}\) on \(x_n = 0\). In getting the last inequality of \((3.30)\), we used Lemma 3.2.

In the following, we estimate the \(H^k\)-norm for \(\eta^{N-1}\partial_n\partial_x^N G, \eta^{N-1}\partial_n\partial_x^N (a^{ij} x_n^m \partial_i \partial_j u), [\mathcal{L}, \eta^{N-1}] \partial_n\partial_x^N u, m x_n^{m-1} \eta^{N-1} \Delta_x \partial_x^N u\) due to Lemma 2.2 and \((3.29)\).

As in the proof of Lemma 3.3, we rewrite \(\eta^{N-1}\partial_n\partial_x^N G\) as

\[
\eta^{N-1}\partial_n\partial_x^N G = \eta^{N-1}\partial_n(x_n^m \partial_x^N G)
= \partial_n x_n^m (\eta^{N-2} x_n^m \partial_x^N G) - (N - 1)\partial_n (\eta_{x'} \eta^{N-2} x_n^m \partial_x^N G)
+ (N - 1)(N - 2)\eta_n \eta_{x'} \eta^{N-3} x_n^m \partial_x^N G + (N - 1)\eta_{n x'} \eta^{N-2} x_n^m \partial_x^N G.
\]

Then all the terms on the right hand side of \((3.31)\) can be estimated as in the proof of Lemma 3.3 due to the presence of \(x_n^m\). Thus we have

\[
\|\eta^{N-1}\partial_n(x_n^m \partial_x^N G)\|_{H^k} \leq C A_0 A_1^{N-4} (N - 4)!.
\]

Similarly, \(\eta^{N-1}\partial_n\partial_x^N (a^{ij} x_n^m \partial_i \partial_j u'), [\mathcal{L}, \eta^{N-1}] \partial_n\partial_x^N u\) can be estimated exactly as in Lemma 3.3. We only need to take care of \(x_n^{m-1} \eta^{N-1} \Delta_x \partial_x^N u\).

\[
x_n^{m-1} \eta^{N-1} \Delta_x \partial_x^N u
= \partial_x^N (x_n^{m-1} \eta^{N-1} \Delta_x^N u) - (N - 1)x_n^{m-1} \partial_x^N (\eta_{x'} \eta^{N-2} \partial_x^N \Delta_x u)
+ (N - 1)(N - 2)x_n^{m-1} \eta_{x'} \eta^{N-3} \partial_x^N \Delta_x u + (N - 1)\eta_{x'} x_n^{m-1} \eta^{N-2} x_n^{m-1} \partial_x^N \Delta_x u.
\]
For the first term $\partial_{x'}(x_n^{m-1} \eta^{n-1} \partial_{x'}^{N-1} \Delta_{x'} u)$, we have
\[
\|\partial_{x'}(x_n^{m-1} \eta^{n-1} \partial_{x'}^{N-1} \Delta_{x'} u)\|_{H_k} \leq \|x_n^{m-1} \eta^{n-1} \partial_{x'}^{N-1} \Delta_{x'} u\|_{H_{k+1}} \\
\leq C \|\eta^{n-1} \partial_{x'}^{N-1}u\|_{W_{k+2}} \leq C \tilde{A}_0 A_0 A_1^{N-4} (N - 3)!.
\]
(3.33)

In getting the above inequality, we used Lemma 3.3 and the definition of $W^{k,2}$. All the remaining terms in (3.32) can be estimated in a similar way.

Combining all the above estimates together yield the present lemma. □

**Lemma 3.5.** Suppose the assumptions in Theorem 3.1 are fulfilled. Suppose (3.10) holds for sufficiently large $A_0, A_1$. For $2 \leq l \leq N + 1$, there holds
\[
\|\eta^{n-1} \partial_{x'}^l \partial_{x'}^{N+1-l}u\|_{W_{k-2}} \leq C \tilde{A}_0 A_0 A_1^{N-4} (N - 3)! \quad \text{for} \quad i = 0, 1, 2.
\]

**Proof.** As previous, we only need to show the case for $i = 0$. Recall that $\eta^{n-1} \partial_{x'}^l \partial_{x'}^{N+1-l}u$ satisfies
\[
x_n^m \Delta_{x'}(\eta^{n-1} \partial_{x'}^l \partial_{x'}^{N+1-l}u) + \partial_n^2(\eta^{n-1} \partial_{x'}^l \partial_{x'}^{N+1-l}u) = \eta^{n-1} \partial_{x'}^{l-1} \partial_{x'}^{N+1-l}G + \sum_{i,j=1}^{n-1} \eta^{n-1} \partial_{x'}^{l-1} \partial_{x'}^{N+1-l} + \frac{1}{l!} \partial_{x'}^{l-m} \partial_{x'}^{N+1-l}(\Delta_{x'} \partial_{x'} u) \quad \text{in} \quad \mathbb{R}^n_+
\]

where
\[
[L, \eta^{n-1} \partial_{x'}^l \partial_{x'}^{N+1-l}u] = [x_n^m \Delta_{x'}, \eta^{n-1} \partial_{x'}^l \partial_{x'}^{N+1-l}u + \partial_n^2, \eta^{n-1} \partial_{x'}^l \partial_{x'}^{N+1-l}u].
\]

For the boundary term $\eta^{n-1} \partial_{x'}^l \partial_{x'}^{N+1-l}u(x', 0)$, we distinguish with two cases:

1. $2 \leq l \leq m + 1$. Then by the equation (3.9), we know $\eta^{n-1} \partial_{x'}^l \partial_{x'}^{N+1-l}u(x', 0) \equiv 0$.

2. $m + 2 \leq l \leq N + 1$. Differentiating (3.9) with respect to $x_n$ for $l - 2$ times and then with respect to $x'$ for $N + 1 - l$ times, one gets
\[
\eta^{n-1} \partial_{x'}^l \partial_{x'}^{N+1-l}u(x', 0)
\]
\[
= - \eta^{n-1} \partial_{x'}^{l-2} \partial_{x'}^{N+1-l}(x_n^{m} (-u_n)^{n+2} \det D_{x'}^2 u) \bigg|_{x_n=0}
\]
\[
= - \eta^{n-1} \left(\frac{(l - 2)!}{(l - 2 - m)!}\right) \partial_{x'}^{l-2-m} \partial_{x'}^{N+1-l}(x_n^{m} (-u_n)^{n+2} \det D_{x'}^2 u)(x', 0).
\]

Then following the same calculations as in Lemma 2.4 and using the induction assumptions, one obtains
\[
\|\eta^{n-1} \partial_{x'}^l \partial_{x'}^{N+1-l}u(x', 0)\|_{H^{k+1}} \leq C \tilde{A}_0 A_1^{N-4} (N - 4)!.\]

For (3.34), we can deduce the $H^k$-norm for $\eta^{n-1} \partial_{x'}^l \partial_{x'}^{N+1-l}G$, $[L, \eta^{n-1} \partial_{x'}^l \partial_{x'}^{N+1-l}u$ and $\eta^{n-1} \partial_{x'}^l \partial_{x'}^{N+1-l}u + (a_{ij} x_n^{m} \partial_{x'}^l \partial_{x'}^{N+1-l}u)$ exactly the same as previous lemmas.

Afterwards, we only need to take care of the term $\eta^{n-1} \partial_{x'}^l (x_n^m) \partial_{x'}^{l-1} \partial_{x'}^{N+1-l}(\Delta_{x'} \partial_{x'} u)$.

This term can be discussed in the following different cases:
(1) $l = 2, l' = 1$. We have
\[ \eta^{N-1}\partial_n^l(x_n^m)\partial_n^{l-l'}\partial_{x'}^{N-1-l}(\Delta_{x'}\partial_n u) = mx_{n}^{m-1}\eta^{N-1}\partial_{x'}^{N-1-l}\Delta_{x'}\partial_n u. \]

Then similar arguments as (3.32) and by the results of Lemma 3.4 one knows
\[ \|\eta^{N-1}x_{n}^{m-1}\partial_{x'}^{N-1-l}\Delta_{x'}\partial_n u\|_{H^k} \leq CA_{0}A_{1}^{N-4}(N - 3)!. \]

(2) $l \geq 3, 1 \leq l' \leq \min(l - 1, m)$. We can rewrite $\eta^{N-1}\partial_n^l\partial_n^{l-l'}\partial_{x'}^{N-1-l}(\Delta_{x'}\partial_n u)$ as
\[ \eta^{N-1}\partial_n^l\partial_n^{l-l'}\partial_{x'}^{N-1-l}(\Delta_{x'}\partial_n u) = \partial_n^2(\eta^{N-1}\partial_n^{l-l'}\partial_{x'}^{N-1-l}\Delta_{x'} u) - 2(N - 1)\partial_n(\eta \eta^{N-2}\partial_n^{l-l'}\partial_{x'}^{N-1-l}\Delta_{x'} u) + (N - 1)\eta_{nn}\eta^{N-2}\partial_n^{l-l'}\partial_{x'}^{N-1-l}\Delta_{x'} u + (N - 1)(N - 2)\eta_{n}\eta^{N-3}\partial_n^{l-l'}\partial_{x'}^{N-1-l}\Delta_{x'} u. \]

By the induction assumption (3.10), we know
\[ \|\partial_n^2(\eta^{N-1}\partial_n^{l-l'}\partial_{x'}^{N-1-l}\Delta_{x'} u)\|_{H^k} \leq \|\eta^{N-1}\partial_n^{l-l'}\partial_{x'}^{N-1-l}\Delta_{x'} u\|_{\tilde{W}^{k,2}} \leq A_{0}A_{1}^{(N-3-l')^+}(N - 3 - l')^+. \]

Similar estimates also hold for the remaining terms of $\eta^{N-1}\partial_n^{l-l'}\partial_{x'}^{N-1-l}(\Delta_{x'}\partial_n u)$ by noticing that these terms contain $\eta_n$.

Then
\[ \|\eta^{N-1}\partial_n^{l-l'}\partial_{x'}^{N-1-l}(\Delta_{x'}\partial_n u)\|_{H^k} \leq CA_{0}A_{1}^{(N-3-l')^+}(N - 3 - l')^+. \]

This implies
\[ \sum_{l' = 1}^{\min(l-1, m)} \frac{(l - 1)!}{l'!(l - 1 - l')!} \|\eta^{N-1}\partial_n^l(x_n^m)\partial_n^{l-l'}\partial_{x'}^{N-1-l}(\Delta_{x'}\partial_n)\|_{H^k} \leq CA_{0}A_{1}^{N-4}(N - 3)!. \]

Thus, applying Lemma 2.2 one proves the present lemma.

**Remark 3.2.** Combining the estimates of Lemma 3.3-3.5 and choosing $A_1 \geq C_1\tilde{A}_0$ large enough, we can get the estimate (3.10) for $N + 1$.

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