Consistency relations for non-Gaussianity

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Abstract. We investigate consistency relations for non-Gaussianity. We provide a model-independent dynamical proof for the consistency relation for three-point correlation functions from the Hamiltonian and field redefinition. This relation can be applied to single-field inflation, multi-field inflation and the curvaton scenario. This relation can also be generalized to $n$-point correlation functions up to arbitrary order in perturbation theory and with arbitrary number of loops.

Keywords: cosmological perturbation theory, inflation, physics of the early universe

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1. Introduction

The CMB non-Gaussianity [1] has been extensively investigated in recent years. The three-point correlation of the curvature perturbation for a large variety of inflation models has been calculated, including the standard single-field inflation [2,3], K-inflation [4], DBI inflation [5,6], multi-field inflation [7–9], the curvaton scenario [10] and the ekpyrotic scenario [11]. The four-point correlation has been calculated in [12,13].

Maldacena found that there is a consistency relation for the three-point correlation function of single-field inflation [2]. The consistency relation states that in the limit where one of the three momenta goes to zero, the three-point correlation function should be proportional to the spectral index times the square of the power spectrum. This consistency relation has been discussed in more detail in [14]. The generalization of this relation to the four-point correlation is discussed in [12].

The consistency relation is derived in two ways in the literature. One is via the original back-reaction argument of Maldacena; another is by checking the relation in explicit models. In this paper, we shall derive the consistency relation from the Hamiltonian. The proof is model independent, and follows from the dynamics. The proof also applies for a general $n$-point correlation function up to arbitrary order in perturbation theory and arbitrary number of loops. We shall also derive a consistency relation from a local field redefinition.

The consistency relation that we derive can be used not only in single-field inflation, but also in multi-field inflation as well as in the curvaton scenario. The consistency relation in multi-field models is also discussed in [16].
The paper is organized as follows. In section 2, we discuss the consistency relation for three-point correlation functions. We first set up the basic notation and equations. Then we derive the consistency relation from the action and local field redefinition. After that, we apply our relation to single-field inflation, multi-field inflation, and the curvaton scenario. In section 3, we generalize the derivation of the consistency relation to $n$-point correlation functions. We conclude in section 4.

2. Consistency relation for the three-point function

2.1. The basic setup

We use $\phi_a$ to collectively denote fields, derivatives of the fields, and the conjugate momenta. We assume that the fields are real. A complex field can be decomposed into a doublet of real fields without losing generality. These fields can be expanded around the homogeneous background,

$$\phi_a(x, t) = \bar{\phi}_a(t) + \delta\phi_a(x, t), \quad a = 1, 2, \ldots, n. \quad (1)$$

The Hamiltonian can be perturbatively expanded as

$$H[\phi(x, t)] = \sum_{N=0}^{\infty} \frac{1}{N!} \left( \sum_{a=1}^{n} \delta\phi_a(x, t) \partial_a \right)^N \bar{H}(t) \equiv \sum_{N=0}^{\infty} H_N[\delta\phi(x, t), t], \quad (2)$$

where $\partial_a$ denotes $\delta/\delta\bar{\phi}_a(t)$, which is performed on the Hamiltonian. The quantity $\delta\phi_a(x, t)$ is canonically quantized according to the second-order Hamiltonian $H_2[\delta\phi(x, t), t]$. Terms $H_N (N \geq 3)$ are treated as interactions,

$$H_{\text{int}}[\delta\phi(x, t), t] = \sum_{N=3}^{\infty} H_N[\delta\phi(x, t), t]. \quad (3)$$

An expectation value can be calculated using

$$\langle Q(t) \rangle = \sum_{N=0}^{\infty} i^N \int_{-\infty}^{t} dt_N \int_{-\infty}^{t_N} dt_{N-1} \cdots \int_{-\infty}^{t_2} dt_1$$

$$\times \langle [H_{\text{int}}(t_1), [H_{\text{int}}(t_2), \cdots [H_{\text{int}}(t_N), Q^I(t)] \cdots] \rangle, \quad (4)$$

where $Q(t)$ is some product of the field operators $\delta\phi_a$, and $Q^I(t)$ is $Q(t)$ in the interaction picture. The $N = 0$ term in the RHS should be understood to be just $\langle Q^I(t) \rangle$. We shall omit the superscript $I$ in the following sections for simplicity.

In the following discussion, we will drop the disconnected part of the correlation functions. All correlation functions automatically denote the connected part.

2.2. Consistency relation from the Hamiltonian

The three-point correlation function can be calculated using

$$\langle \delta\phi_a(k_1, t)\delta\phi_b(k_2, t)\delta\phi_c(k_3, t) \rangle = -i \int_{-\infty}^{t} dt' \langle [\delta\phi_a(k_1, t)\delta\phi_b(k_2, t)\delta\phi_c(k_3, t), H_3(t')] \rangle. \quad (5)$$
In this paper, we normalize the fields in a box, and use the convention
\[ \delta \phi_a(x, t) = \frac{1}{\sqrt{V}} \sum_k \delta \phi_a(k, t) e^{ik \cdot x}. \] (6)

Here \( V \) is the space volume. More details about our convention can be found in the appendices.

The Fourier modes of the field can be written in terms of the creation and annihilation operators as
\[ \delta \phi_a(k, t) = \delta \phi_a^{(cl)}(k, t) a_k + \delta \phi_a^{(cl)*}(-k, t) a_k^\dagger, \] (7)
where \( \delta \phi_a^{(cl)}(k, t) \) is a \( c \)-number function and satisfies the classical equation of motion.

We use \( k_i \) to denote \(|k_i|\). Consider the limit \( k_1 \ll \min(k_2, k_3, aH) \) (we shall refer to this limit as \( k_1 \to 0 \) limit for simplicity). In the correlation function, \( \delta \phi_a(k_1, t) \) can be either to the left or to the right of the Hamiltonian \( H \), corresponding to the terms \( \langle \delta \phi^3 H \rangle \) and \( \langle H \delta \phi^3 \rangle \) respectively. In the \( \langle \delta \phi^3 H \rangle \) case, keeping in mind that the annihilation operator in \( \delta \phi_a(k_1, t) \) must be contracted with one of the \( \delta \phi_d \) in \( H_3 \), we have
\[ H_3 = \frac{1}{3!} \left( \sum_d \delta \phi_d(x, t) \partial_d \right)^3 H \to \frac{1}{2} \left( \sum_d \delta \phi_d(x, t) \partial_d \right)^2 \sum_e [\overline{\delta \phi_e} \partial_x] H \]
\[ = \frac{1}{2} \left( \sum_d \delta \phi_d(x, t) \partial_d \right)^2 \left( H(\bar{\phi} + \overline{\delta \phi}) - H(\bar{\phi}) \right) = H_2(\bar{\phi} + \overline{\delta \phi}) - H_2(\bar{\phi}). \] (8)

where \( ' \to ' \) indicates that we have made the contraction with \( \delta \phi_a(k_1, t) \), and have taken the \( k_1 \to 0 \) limit. After doing the contraction, the field is replaced by its classical part, and this replacement also applies for \( \delta \phi_a(k_1, t) \). \( \delta(a, e) \) is defined as: \( \delta(a, e) = 1 \) if \( \delta \phi_a \) and \( \delta \phi_e \) are the same fundamental field, or the derivative or the conjugate momentum of it; otherwise, \( \delta(a, e) = 0 \). And \( \overline{\delta \phi_e} \equiv \delta(a, e) V^{-1} \int_V \delta \phi_e^{(cl)*}(x, t) \, dV \). Finally, we have neglected the high order terms of \( \overline{\delta \phi_e} \), because we are considering in this section the leading order perturbation. In the next section, we shall show that the consistency relation is respected order by order in the perturbation theory.

With the help of equation (8), the term \( \langle \delta \phi^3 H \rangle \) can be written as
\[ \langle \delta \phi_a(k_1, t) \delta \phi_b(k_2, t) \delta \phi_c(k_3, t) H_3(\bar{\phi}) \rangle \]
\[ \overset{k_1 \to 0}{\longrightarrow} \delta \phi_a^{(cl)}(k_1, t) \langle \delta \phi_b(k_2, t) \delta \phi_c(k_3, t)(H_2(\bar{\phi}) - H_2(\bar{\phi} + \overline{\delta \phi})) \rangle. \] (9)

The other term \( \langle H \delta \phi^3 \rangle \) can be written similarly. One difference is that the terms \( \delta \phi_a^{(cl)}(k_1, t) \) and the corresponding terms in the Hamiltonian should be replaced by their complex conjugates. However, when \( k_1 \ll aH \), this difference can be neglected. This is because after horizon crossing, the perturbations become classical, and the mode functions can be made real by a time-independent phase rotation [15]. So we have
\[ \delta(a, e) \delta \phi_a^{(cl)}(k_1, \tau) \delta \phi_e^{(cl)}(-k_1, \tau')^* = \delta(a, e) \delta \phi_a^{(cl)}(-k_1, \tau)^* \delta \phi_e^{(cl)}(k_1, \tau'). \] ¹

¹ Note that \( \tau' \) can run from \( -\infty \) to \( 0 \). \( k_1 \tau' \) is not always small along the \( \int d\tau' \) integration. However, given a lower cutoff on \( \tau' \) (which is due to the choice of the interacting vacuum), the integral uniformly converges when we take \( k_1 \to 0 \), so we can interchange the \( k_1 \to 0 \) limit and the integration to obtain
\[ \delta(a, e) \delta \phi_a^{(cl)}(k_1, \tau) \delta \phi_e^{(cl)}(-k_1, \tau')^* = \delta(a, e) \delta \phi_a^{(cl)}(-k_1, \tau)^* \delta \phi_e^{(cl)}(k_1, \tau'). \] This can be verified in models with either standard or generalized kinetic terms.
also be checked explicitly in inflation models with either standard or generalized kinetic terms.

Note that the condition \( k_1 \ll aH \) is essential in proving the consistency relation. If this condition is not satisfied, we can check explicitly that the consistency relation is not satisfied even in the simplest single-field inflation model with standard kinetic terms.

Note that the field in the interaction picture evolves according to the free Hamiltonian \( H_2(\bar{\phi}) \), not \( H_2(\phi + \delta \phi) \). The difference between these two should also be treated as the interaction Hamiltonian. So in the \( k_1 \to 0 \) limit we have

\[
\langle \delta \phi_a(k_1, t) \delta \phi_b(k_2, t) \delta \phi_c(k_3, t) \rangle
= -i \delta \phi_a^{(cl)}(k_1, t) \int_{-\infty}^{t} dt' \langle [\delta \phi_b(k_2, t) \delta \phi_c(k_3, t), (H_2(\bar{\phi} + \delta \phi) - H_2(\bar{\phi}))] \rangle
= \delta \phi_a^{(cl)}(k_1, t) \langle \delta \phi_b(k_2, t) \delta \phi_c(k_3, t) \rangle_{\delta \phi = 0} - \langle \delta \phi_b(k_2, t) \delta \phi_c(k_3, t) \rangle_{\delta \phi = 0} \langle \delta \phi_a(k_1, t) \rangle_{\delta \phi = 0},
\]

where in the last line, one should note that for the two-point correlation function, the \( N = 0 \) term in equation (4) is also present. This equation can be rewritten as

\[
\langle \delta \phi_a(k_1, t) \delta \phi_b(k_2, t) \delta \phi_c(k_3, t) \rangle \propto \delta \phi_a^{(cl)}(k_1, t) \sum_e \overline{\delta \phi_e \partial_e \langle \delta \phi_b(k_2, t) \delta \phi_c(k_3, t) \rangle}.
\]

The classical field and the quantum expectation can be related as

\[
\delta \phi_a^{(cl)}(k_1, t) \overline{\delta \phi_e} = \frac{\delta(a, e)}{V} \delta \phi_a^{(cl)}(k_1, t) \delta \phi_e^{(cl)*} = \frac{\delta(a, e)}{V^{3/2}} \langle \delta \phi_a(k_1, t) \delta \phi_e^{*}(k_1, t) \rangle.
\]

So equation (11) can be recast as

\[
\langle \delta \phi_a(k_1, t) \delta \phi_b(k_2, t) \delta \phi_c(k_3, t) \rangle
= \propto \sum_e \frac{\delta(a, e)}{V} \langle \delta \phi_a(k_1, t) \delta \phi_e^{*}(k_1, t) \rangle \overline{\partial_{\delta \phi_e^{(cl)*}}(k_1, t)} \langle \delta \phi_b(k_2, t) \delta \phi_c(k_3, t) \rangle.
\]

One should note that in the above proof, we have used the condition that \( \delta \phi_a \) is a perturbation of the background field \( \phi_a \). There are usually two such variables: \( \zeta \), as the perturbation of the logarithm of the scale factor in the uniform density slice, and \( Q_a \), as the perturbation of the inflaton fields in the flat slice. The resulting equation (13) cannot be directly used for composite perturbation variables beyond this condition. The consistency relation for induced variables can be derived from a field redefinition, discussed in the next subsection. For single-field inflation, the consistency relation for \( \zeta \) and the consistency relation for \( Q \) can also be related by such a field redefinition (see appendix B).

### 2.3. Consistency relation from local field redefinition

Sometimes, the field that we use in the Hamiltonian and the field that we use in the correlation function are up to a local non-linear redefinition of the fields. In this case, we need to consider the contribution to the consistency relation from the field redefinition. Sometimes, the non-Gaussianity is dominated by this redefinition, such as in the curvaton scenario. In this subsection, we shall derive the consistency relation from the field redefinition.
Firstly, let us consider the ansatz of local non-Gaussianity. This case can be directly generalized to a more general local field redefinition. In the local ansatz, the field redefinition takes the form

$$\zeta(x, t) = \zeta_g(x, t) + \frac{3}{5} f_{NL} \zeta_g(x, t)^2,$$  \hspace{1cm} (14)

where \(\zeta_g(x, t)\) is the Gaussian part of \(\zeta(x, t)\). In the Fourier space, equation (14) becomes

$$\zeta_k = \zeta_{gk} + \frac{3}{5} f_{NL} \frac{1}{V} \sum_{k'} \zeta_{g k' k} \zeta_{g k - k'}.$$  \hspace{1cm} (15)

In the \(k_1 \to 0\) limit, the three-point function takes the form

$$\langle \zeta_{k_1} \zeta_{k_2} \zeta_{k_3} \rangle \quad \xrightarrow{k_1 \to 0} \quad \frac{3}{5} f_{NL} \frac{1}{V} \sum_{k_1} \langle \zeta_{g k_1} \zeta_{g k_2} \zeta_{g k_3} \rangle + (k_2 \leftrightarrow k_3)$$

$$= \frac{12}{5} f_{NL} \frac{1}{V} \langle \zeta_{g k_1} \zeta_{g k_2} \rangle \langle \zeta_{g k_3} \rangle,$$  \hspace{1cm} (16)

where in the first line, we have used equation (15) to convert \(\zeta_k\) into \(\zeta_{g}\) for the first term, and \(\zeta_{k_3}\) into \(\zeta_{g}\) for the \((k_2 \leftrightarrow k_3)\) term. The above two terms are proportional to \(k_1^{-3} k_3^{-3}\) and \(k_1^{-3} k_2^{-3}\) respectively. We have neglected the term which inserts equation (15) for \(\zeta_{k_1}\), because this term is proportional to \(k_2^{-3} k_3^{-3}\), which can be neglected in the \(k_1 \to 0\) limit.

On the other hand, keeping in mind that a Gaussian random mode of the field is not affected by other modes, the derivative of \(\zeta_{k_1} \zeta_{k_3}\) takes the form

$$\frac{d}{d \zeta_{k_1}} (\zeta_{k_2} \zeta_{k_3}) \xrightarrow{k_1 \to 0} \frac{3}{5} f_{NL} \frac{1}{V} \sum_{k'} \frac{d}{d \zeta_{g k_1}} (\zeta_{g k' k_2} \zeta_{g k_3}) + (k_2 \leftrightarrow k_3) = \frac{12}{5} f_{NL} \frac{1}{V} \zeta_{g k_2} \zeta_{g k_3},$$  \hspace{1cm} (17)

where the derivative is taken directly on the operators. So we have

$$\langle \zeta_{k_1} \zeta_{k_2} \zeta_{k_3} \rangle \quad \xrightarrow{k_1 \to 0} \quad \frac{1}{V} \langle \zeta_{k_1} \zeta_{g k_1} \rangle \left\langle \frac{d}{d \zeta_{g k_1}} (\zeta_{k_2} \zeta_{k_3}) \right\rangle.$$  \hspace{1cm} (18)

Generally, when the field before the field redefinition is also non-Gaussian, one can combine the result in the previous subsection and this subsection to obtain the full consistency relation. To be explicit, if \(\delta \phi\) is the quantity in the Hamiltonian, and \(\widetilde{\delta \phi}\) is the quantity that we want to calculate in the correlation function, related by

$$\widetilde{\delta \phi}_a(x, t) = \sum_b g_{ab} \delta \phi_b(x, t) + \sum_{bc} f_{abc} \delta \phi_b(x, t) \delta \phi_c(x, t),$$  \hspace{1cm} (19)

then we have

$$\langle \widetilde{\delta \phi}_a(k_1, t) \widetilde{\delta \phi}_b(k_2, t) \widetilde{\delta \phi}_c(k_3, t) \rangle \quad \xrightarrow{k_1 \to 0} \quad \sum_e \delta(a, e) \frac{1}{V} \langle \widetilde{\delta \phi}_a(k_1, t) \delta \phi_e^*(k_1, t) \rangle$$

$$\times \left\{ \sum_{fg} g_{bf} g_{cg} \left. \frac{\partial}{\partial \delta \phi_e^*(k_1, t)} \right| \langle \delta \phi_f(k_2, t) \delta \phi_g(k_3, t) \rangle \right\}.$$  \hspace{1cm} (20)
Note that in the second line of equation (20), the derivative, in the first term, corresponds to a variation in the Hamiltonian, while in the second term (third line), the derivative acts as an operator derivative directly on the field operators \( \tilde{\delta}_\phi(k_2, t) \delta_\phi(k_3, t) \). Loosely speaking, we can combine these two derivatives to write this formula in a more compact form. However, as the meanings of the two derivatives are not the same, we shall take this more explicit form as our final result of this subsection.

### 2.4. Application to single-field inflation

In this subsection, we shall write the consistency relation (13) in an explicit form in the context of single-field inflation. We shall derive a different expression for the consistency relation from Maldacena’s relation. The new consistency relation is expressed in terms of the scalar field perturbation in the flat slice, and it is more convenient for the generalization to the multi-field case. The derivation for Maldacena’s consistency relation [2] is given in the appendix. Maldacena’s relation can also be derived from our consistency relation and a field redefinition as discussed in the previous subsection (see appendix B).

In the flat slice, the inflaton field can be written as

\[
\varphi(x, t) = \varphi(t) + Q(x, t),
\]

where \( Q \) can be made gauge invariant in a general slice, known as the Mukhanov–Sasaki variable [17].

In this case, equation (13) takes the form

\[
\langle Q_{k_1} Q_{k_2} Q_{k_3} \rangle \rightarrow -\frac{1}{V} \langle \dot{\varphi} \rangle \langle Q_{k_1} Q_{k_1} \rangle \langle Q_{k_2} Q_{k_3} \rangle.
\]

This relation is different from Maldacena’s consistency relation, because the transformation from \( Q \) to \( \zeta \) is non-linear, and the shapes of \( Q \) and \( \zeta \)’s three-point correlation functions are different.

Equation (22) can be checked explicitly. For example, in the single-field inflation model with standard kinetic term, the three-point function in the \( k \rightarrow 0 \) limit takes the form

\[
\langle Q_{k_1} Q_{k_2} Q_{k_3} \rangle \rightarrow -\frac{1}{V} \langle \dot{\varphi} \rangle \langle Q_{k_1} Q_{k_1} \rangle \langle Q_{k_2} Q_{k_3} \rangle.
\]

On the other hand, the two-point function takes the form

\[
\langle Q_{k_2} Q_{k_3} \rangle = VH^22k_3^2 \delta(k_2, -k_3).
\]

And from \( \dot{H} = -\frac{1}{2} \dot{\varphi}^2 \), the derivative can be written as

\[
\frac{d}{dQ} = \frac{d}{d\varphi} = -\frac{1}{2} \frac{\dot{\varphi}}{H} \frac{d}{dH}.
\]

Combining (23)–(25), we obtain the desired relation (22).

\footnote{Strictly speaking, the variation should also include one more term proportional to \( \dot{Q} \). This is because the variation of the background that we consider is not equal to a time variation \( t \rightarrow t + \delta t \). The correction can be calculated using \( \dot{\zeta} = 0 \), so that \( \dot{Q} = -Q \delta\varphi(H/\dot{\varphi})/(H/\dot{\varphi}) \). So we should also set \( \dot{\varphi} \rightarrow \dot{\zeta} + \dot{\varphi} \). However, this correction is of higher order in the slow roll approximation. So we neglect this correction in this paper.}
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As a more non-trivial test, let us consider the case of generalized kinetic term. We use the result [6] for the three-point function. To use this result, we need first to know how $\zeta$ corresponds to $Q$ beyond the first-order perturbation theory. The correspondence between $\zeta$ and $Q$ follows directly from the gauge transformations, so it is not changed after the kinetic term is generalized. Maldacena’s correspondence [2] between $\zeta$ and $Q$

\[ \zeta = \zeta_n + \frac{1}{2} \left( \frac{\ddot{\varphi}}{H} - \frac{\dot{H}}{H^2} \right) \zeta_n^2 + \text{derivatives}, \quad \zeta_n \equiv -\frac{HQ}{\dot{\varphi}}, \]  

(26)

where the derivatives can be neglected outside the horizon.

We would like to make two remarks at this point. Firstly, In Maldacena’s paper [2], the equation of motion has been used for the $\dot{H}$ term, so the relation in Maldacena’s paper only applies for the standard kinetic term. Here, we do not use any equations of motion in equation (26). All the terms follow directly from the gauge transformation. So equation (26) holds very generally. Secondly, the notation $\zeta_n$ is also used in [6]. However, the $\zeta_n$ in [6] is a different notation, having nothing to do with the relation between $\zeta$ and $Q$.

In the context of generalized kinetic term, using the equation $2\ddot{H} = -\dot{\varphi}^2 P_X$, equation (26) can be written as

\[ \zeta = \zeta_n + \frac{1}{4} \left( \eta - \frac{\dot{P}_X}{HP_X} \right) \zeta_n^2, \quad \eta \equiv \frac{\dot{\epsilon}}{H\epsilon}, \quad \epsilon \equiv -\frac{\dot{H}}{H^2}. \]  

(27)

With the above equation in mind, the three-point correlation function in [6] takes the form

\[ \langle Q_{k_1} Q_{k_2} Q_{k_3} \rangle^{k_1 \rightarrow 0} \frac{1}{V} \frac{H}{\dot{\varphi}} \left( 2\epsilon + s + \frac{\dot{P}_X}{HP_X} \right) \langle Q_{k_1} Q_{k_1}^* \rangle \langle Q_{k_2} Q_{k_3} \rangle, \]  

(28)

where $s \equiv \dot{c}_s/(Hc_s)$.

On the other hand, the two-point correlation function is

\[ \langle Q_{k_2} Q_{k_3} \rangle = \frac{V}{2k_2^2 c_s P_X} \delta_{k_2,-k_3}. \]  

(29)

Keeping in mind that $\partial_{\varphi} = \dot{\varphi}^{-1} \partial_t$, we can check that the consistency relation equation (22) is respected.

2.5. Application to multi-field inflation

The consistency relation in the above subsection can be straightforwardly generalized to the multi-field case. In this case, the relation takes the form

\[ \langle Q_{k_1}^I Q_{k_2}^J Q_{k_3}^K \rangle^{k_1 \rightarrow 0} \frac{1}{V} \langle Q_{k_1}^I Q_{k_1}^J \rangle \partial_{\varphi^I} \langle Q_{k_2}^J Q_{k_3}^K \rangle \]  

(30)

where index $I$ in the RHS is not summed over. We write $\partial/\partial\varphi^I$ to denote that the derivative is taken while other $\varphi^J (J \neq I)$ are fixed.
To check this relation in the multi-field inflation models with standard kinetic terms (so that the field space metric is also flat), we note that the three-point correlation function in the $k \to 0$ limit becomes

$$\langle Q_{k_1} Q_{k_2} Q_{k_3}^* \rangle \to -\frac{1}{V H^2} \langle Q_{k_1}^I \rangle \langle Q_{k_2}^J \rangle \langle Q_{k_3}^K \rangle.$$  \hspace{1cm} (31)

The two-point correlation function takes the form

$$\langle Q_{k_2}^J Q_{k_3}^K \rangle = V H^2 \frac{2\delta k_2 \delta k_3 \delta ^{JK}}{2k_2^2}.$$ \hspace{1cm} (32)

From $dH = -\frac{1}{H} \sum_I \dot{\varphi}^I \, d\varphi^I$, we have

$$\frac{\partial}{\partial \varphi^I} = \frac{\partial H}{\partial \varphi^I} \, dH.$$  \hspace{1cm} (33)

Combining (31)–(33), we find that the relation (30) is indeed satisfied.

In the case with generalized kinetic term, the $1/c_s^2$ order result is applicable. In this order, our consistency relation is satisfied trivially. We hope that the above consistency relation can be checked in some future work for a general sound speed.

In multi-field models, $\zeta$ is usually not conserved. To relate the quantities at horizon exit with observations, the transfer function method is usually used. We take double-field inflation as an example [18].

In double-field inflation, the inflaton fields can be decomposed into the inflaton direction $\sigma$ and the perpendicular direction $s$. The perturbation $Q^I$ can be defined as the perturbation parallel to the inflation direction ($Q_\sigma$), and perpendicular to the inflation direction ($Q_s$). These two fields can be further related to the comoving curvature perturbation $\mathcal{R}$ and the entropy perturbation $S$ as

$$\mathcal{R} = \frac{H Q_\sigma}{\sigma}, \quad S = \frac{H Q_s}{\sigma}.$$ \hspace{1cm} (34)

As shown in [18], the entropy perturbation is sourceless, and the comoving curvature perturbation is conserved if there is no entropy perturbation. So the comoving curvature perturbation in the late times can be written as

$$\mathcal{R} = \mathcal{R}_s + T_{RS} S_s + \delta \mathcal{R} = A_\sigma Q_\sigma s + A_s Q_s s + \delta \mathcal{R}, \quad A_\sigma \equiv H \sigma, \quad A_s \equiv T_{RS} \frac{H}{\sigma},$$ \hspace{1cm} (35)

where $T_{RS}$ can be (at least in principle) determined by experiments. The term $\delta \mathcal{R} \sim \mathcal{O}(\mathcal{R}^2)$ denotes the error in the approximation of taking the first-order comoving curvature perturbation as conserved, as well as using the linear transfer function beyond the linear perturbation theory. Here the situation is similar to (but more difficult to handle than) the difference between $\zeta$ and $\zeta_n$ in the single-field case. To the best of our knowledge, this $\delta \mathcal{R}$ correction is not noticed in the literature. We shall leave this correction uncalculated in this paper.

3 The generalization to non-standard kinetic terms can be found in [9]. We also follow the notation in this paper, but consider only the standard kinetic term case. When considering the modified kinetic terms, equations (34) and (35) are modified; other discussion still goes through.
The three-point function of $\mathcal{R}$ takes the form
\[
\langle \mathcal{R}_{k_1} \mathcal{R}_{k_2} \mathcal{R}_{k_3} \rangle = \left( (\mathcal{A}_\sigma Q_{\sigma k_1} + \mathcal{A}_s Q_{sk_1}) (\mathcal{A}_\sigma Q_{\sigma k_2} + \mathcal{A}_s Q_{sk_2}) (\mathcal{A}_\sigma Q_{\sigma k_3} + \mathcal{A}_s Q_{sk_3}) \right)_s + f(\delta \mathcal{R}),
\]
where the correlation functions of $Q$ are calculated a few e-folds after the horizon crossing, and $f(\delta \mathcal{R})$ denotes the $\delta \mathcal{R}$ correction.

The consistency relation can be written as
\[
\langle \mathcal{R}_{k_1} \mathcal{R}_{k_2} \mathcal{R}_{k_3} \rangle \overset{\text{B.2}}{=} \frac{V^{-1}}{1 \sigma} \left. \frac{\partial}{\partial \sigma} \right|_{\mathcal{R}} + A_s \langle Q_{sk_1} Q_{sk_2} \rangle \left. \frac{\partial}{\partial \sigma} \right|_{\mathcal{R}} \langle \mathcal{A}_\sigma Q_{\sigma k_3} + \mathcal{A}_s Q_{sk_3} \rangle_s.
\]
It can be shown with the standard kinetic terms that the two-point cross correlation between the curvature and entropy perturbation just after the horizon crossing is suppressed by one more order of slow roll parameters \cite{18}. In more general cases, it is also usually assumed that this correlation can be neglected. And we note that the entropy direction has the property $\dot{s} = 0$. Considering the above two conditions, equation (B.2) can be simplified to be
\[
\langle \mathcal{R}_{k_1} \mathcal{R}_{k_2} \mathcal{R}_{k_3} \rangle \overset{\text{B.2}}{=} \frac{V^{-1}}{1 \sigma} A_s \langle Q_{sk_1} Q_{sk_2} \rangle \left. \frac{\partial}{\partial \sigma} \right|_{\mathcal{R}} \langle \mathcal{A}_\sigma Q_{\sigma k_3} + \mathcal{A}_s Q_{sk_3} \rangle_s + f(\delta \mathcal{R}).
\]

This relation is also derived in \cite{16}. The above relation can be further simplified in some models. When the kinetic term is standard, the $Q_\sigma$ power spectrum and the $Q_s$ power spectrum are equal. When the kinetic term is modified and $c_s \ll 1$, the $Q_s$ power spectrum can be neglected. The relation can be tested by experiments, on condition that the $f(\delta \mathcal{R})$ term is calculated.

### 2.6. Application to the curvaton model

In the curvaton model, the leading order non-Gaussianity comes from the field redefinition. In the flat slice, $\zeta$ can be expressed as
\[
\zeta = \frac{1}{3} \frac{\delta \rho_\sigma}{\rho_\sigma} = \frac{2r}{3} \frac{\delta \sigma}{\sigma} + \frac{3}{4r} \left( \frac{2r}{3} \frac{\delta \sigma}{\sigma} \right)^2 = \zeta_g + \frac{3}{4r} \zeta_g^2,
\]
where $r = (3 \rho_\sigma) / (4 \rho_r + 3 \rho_\sigma)$, which is calculated when the curvaton decays. So we have $f_{NL} = 5/(4r)$.

The discussion from equations (14) to (18) can be directly applied to the curvaton scenario. The consistency relation (18) is satisfied.

### 3. Consistency relation for the general $n$-point function

In this section, we generalize the discussion in the previous section to $n$-point correlation functions ($n \geq 3$). The generalization for the leading order ($N = 1$ in equation (4)) is straightforward, so we shall not repeat it here. We consider in this section the general case, including $N \geq 2$. 

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To save some writing, we use $C(n, N, L)$ to denote the contribution to the $n$-point correlation function with $L$ loops and order $N$ in equation (4). We have

$$\langle \delta \phi_{a_1}(k_1, t) \cdots \delta \phi_{a_n}(k_n, t) \rangle = \sum_{N=0}^{\infty} \sum_{L=0}^{\infty} C(n, N, L).$$  \hspace{1cm} (40)$$

Note that the number of vertices ($N$), the number of initial lines ($I$) and the number of loops ($L$) satisfy $I = L + N - 1$. $C(n, N, L)$ takes the form

$$C(n, N, L) = \sum_{\text{PART}} \int_{-\infty}^{t_1} dt_1 \int_{-\infty}^{t_2} dt_2 \cdots \int_{-\infty}^{t_N} dt_N \times \langle [H_{n_1}(t_1), [H_{n_2}(t_2), \cdots [H_{n_N}(t_N), \delta \phi_{a_1}(k_1, t) \cdots \delta \phi_{a_n}(k_n, t)] \cdots \rangle \rangle,$$  \hspace{1cm} (41)

where the ‘PART’ in the summation denotes all the partitioning satisfying $n_1 + \cdots + n_N = n + 2(L + N - 1)$.

Equation (8) can be directly generalized to

$$H_{n_i} \left\rightarrow \delta \phi \rightarrow \sum_{e} \delta(a_1, e) V \langle \delta \phi_{a_1}(k_1, t) \delta \phi^*_e(k_1, t) \rangle \partial_{\delta \phi^{(cl)}_e}(k_1, t) \langle \delta \phi_{a_2}(k_2, t) \cdots \delta \phi_{a_n}(k_3, t) \rangle.$$  \hspace{1cm} (42)

Equations (41) and (42) lead to

$$C(n, N, L) \left\rightarrow \delta \phi \rightarrow \Delta C(n - 1, N, L),$$  \hspace{1cm} (43)

where $\Delta$ acts on the Hamiltonian and satisfies the Leibnitz law. From equation (43), we obtain the consistency relation

$$\langle \delta \phi_{a_1}(k_1, t) \cdots \delta \phi_{a_n}(k_3, t) \rangle \left\rightarrow \delta \phi \rightarrow \sum_{e} \delta(a_1, e) V \langle \delta \phi_{a_1}(k_1, t) \delta \phi^*_e(k_1, t) \rangle \partial_{\delta \phi^{(cl)}_e}(k_1, t) \langle \delta \phi_{a_2}(k_2, t) \cdots \delta \phi_{a_n}(k_3, t) \rangle.$$  \hspace{1cm} (44)

This relation is respected order by order and loop by loop in the perturbation theory. The consistency relation can also be iteratively used to obtain the limit in which several momenta go to zero.

Now consider the case where the quantity in the correlation function (denoted by $\tilde{\delta \phi}$) is up to a local field redefinition compared with the quantity in the Hamiltonian (denoted by $\delta \phi$). Let the field redefinition be

$$\tilde{\delta \phi}_a = \sum_{m=1}^{\infty} \sum_{b_1 \cdots b_m} f_{ab_1 \cdots b_m} \delta \phi_{b_1} \ast \cdots \ast \delta \phi_{b_m},$$  \hspace{1cm} (45)

where $\ast$ denotes the usual product in the position space and convolution in the momentum space. One can calculate directly the LHS and RHS of equation (44), replacing
\( \delta \phi \) with \( \tilde{\delta} \phi \), so that

\[
\langle \tilde{\delta} \phi_{a_1}(k_1, t) \cdots \tilde{\delta} \phi_{a_n}(k_3, t) \rangle \xrightarrow{k_1 \to 0} \sum_e \frac{\delta(a_1, e)}{V} \langle \delta \phi_{a_1}(k_1, t) \delta \phi_e^*(k_1, t) \rangle \\
\times \left\{ \sum_{m_2 \cdots m_n} \sum_{b_{21} \cdots b_{nmn}} f_{a_2 b_{21} \cdots b_{2m_2}} \cdots f_{a_n b_{n1} \cdots b_{nmn}} \partial \frac{\delta \phi_e}{\partial \delta \phi_e^{(cl)}(k_1, t)} \langle (\tilde{\delta} \phi_{b_{21}} * \cdots * \tilde{\delta} \phi_{b_{2m_2}})(k_1, t) \cdots \rangle \\
+ \left( \frac{\partial}{\partial \delta \phi_e^{(cl)}(k_1, t)} \langle \tilde{\delta} \phi_{a_2}(k_2, t) \cdots \tilde{\delta} \phi_{a_n}(k_3, t) \rangle \right) \right\}. \tag{46}
\]

The meaning of the derivatives is the same as that in equation (20).

4. Conclusion

To conclude, in this paper, we have investigated the consistency relations for non-Gaussianity.

We have proved the consistency relation for \( n \)-point correlation functions dynamically from the Hamiltonian. The proof is model independent, and valid to all orders of perturbation theory and loop corrections. We have also derived a consistency relation for local field redefinitions.

As applications, we have applied the consistency relation to single-field inflation, multi-field inflation, and the curvaton scenario.

For single-field inflation, we have got a relation in terms of the field perturbation \( Q \). This relation is different from (but consistent with) Maldacena’s relation. Experimentally, this relation is not as convenient to use as Maldacena’s relation. However, it has two applications. Firstly, it provides one more theoretical tool for checking the calculation. Secondly, this relation can be straightforwardly generalized to the multi-field case. The single-field three-point consistency relation is checked explicitly at the first non-trivial order of slow roll parameters for models with standard and generalized kinetic terms. We expect the consistency relation for the higher order correlation functions to be checked in future work.

For multi-field inflation, we have also derived a consistency relation. The relation is checked explicitly both in the standard kinetic term case, and the small \( c_s \) limit in the generalized kinetic term case. Future calculation beyond the small \( c_s \) limit should also satisfy this condition. We also point out that one should note the \( f(\delta \mathcal{R}) \) correction in the calculation.

For the curvaton scenario, we have shown that the consistency relation is also satisfied, following from a field redefinition.

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Appendix A. Perturbations in a box

In this appendix, we list the relations between the usually used $\delta$-function normalization and the box normalization used in this paper. The equations in the left column are for the box normalization, and the right column corresponds to the $\delta$-function normalization.

$$
\delta \phi_a(x, t) = \frac{1}{\sqrt{V}} \sum_k \delta \phi_a(k, t) e^{ik \cdot x} \iff \delta \phi_a(x, t) = \int \frac{d^3 k}{(2\pi)^3} \delta \phi_a(k, t) e^{i k \cdot x},
$$

$$
\delta \phi_a(k, t) = \frac{1}{\sqrt{V}} \int d^3 x \delta \phi_a(x, t) e^{-i k \cdot x} \iff \delta \phi_a(k, t) = \int d^3 x \delta \phi_a(x, t) e^{-i k \cdot x},
$$

(A.1)

Using $d \delta \phi_a(x, t) / (d \ln a_*)$ (‘*’ denotes the time of horizon crossing) appears because the rescaling of the scale factor only affects the horizon crossing time. The local Hubble constant is not affected. So a positive $\zeta_{k_1}$ corresponds to an earlier horizon crossing time.

In this context, equation (13) becomes

$$
\langle \zeta_{k_1} \zeta_{k_2} \zeta_{k_3} \rangle \overset{k_1 \to 0}{\longrightarrow} \frac{1}{V} \langle \zeta_{k_1} \rangle \frac{d}{d \ln a_*} \langle \zeta_{k_2} \zeta_{k_3} \rangle.
$$

(B.2)

This is the well-known consistency relation for single-field inflation.

This consistency relation can also be derived from equation (22) and a field redefinition. Up to second order, $\zeta$ and $Q$ are related by equation (26). Using equations (20) and (22), we have

$$
\langle \zeta_{k_1} \zeta_{k_2} \zeta_{k_3} \rangle \overset{k_1 \to 0}{\longrightarrow} \frac{1}{V} \left( -\frac{H}{\dot{\varphi}} \right) \langle \zeta_{k_1} \rangle \frac{d}{d Q_{k_1}^{(cl)\ast}} \langle Q_{k_2} Q_{k_3} \rangle + \frac{2}{V} \left( \frac{\ddot{\varphi}}{H^2} - \frac{\dot{H}}{H^2} \right) \langle \zeta_{k_1} \rangle \langle \zeta_{k_2} \rangle \langle \zeta_{k_3} \rangle.
$$

(B.3)

Using $d / (d Q_{k_1}^{(cl)\ast}) = d / (d \varphi) = d / (\dot{\varphi} \, dt)$, we find that the two terms in the RHS of (B.3) can be combined to obtain the consistency relation (B.2).

Similarly, we can recover the consistency relation for $\langle \zeta_{k_1} \gamma_{k_2 \ast}^{\omega} \gamma_{k_3}^{\omega} \rangle$, where $\gamma_{k}^{\omega}$ denotes the perturbation of gravitational waves, and $s$ denotes the polarization ($s = 1, 2$). The consistency relation takes the form [2]

$$
\langle \zeta_{k_1} \gamma_{k_2 \ast}^{\omega} \gamma_{k_3}^{\omega} \rangle \overset{k_1 \to 0}{\longrightarrow} \frac{1}{V} \langle \zeta_{k_1} \rangle \frac{d}{d \ln a_*} \langle \gamma_{k_2 \ast}^{\omega} \gamma_{k_3}^{\omega} \rangle.
$$

(B.4)

Appendix B. Maldacena’s consistency relation

In the single-field case, the quantity that one usually calculates in a correlation function is the curvature perturbation in the uniform density slice. This curvature perturbation $\zeta$ can be thought of as a local rescaling of the scale factor. When calculating the two-point function, the derivative with respect to $\zeta$ can be translated to a derivative with respect to $\dot{a}$ as

$$
\frac{1}{\sqrt{V}} \frac{d}{d \zeta} = \frac{d}{d \zeta_{k_1}} = -\frac{d}{d \ln a_*},
$$

(B.1)

The minus sign before $d / (d \ln a_*)$ (‘*’ denotes the time of horizon crossing) appears because the rescaling of the scale factor only affects the horizon crossing time. The local Hubble constant is not affected. So a positive $\zeta_{k_1}$ corresponds to an earlier horizon crossing time. In this context, equation (13) becomes

$$
\langle \zeta_{k_1} \zeta_{k_2} \zeta_{k_3} \rangle \overset{k_1 \to 0}{\longrightarrow} \frac{1}{V} \langle \zeta_{k_1} \rangle \frac{d}{d \ln a_*} \langle \zeta_{k_2} \zeta_{k_3} \rangle.
$$

This is the well-known consistency relation for single-field inflation.
Consistency relations for non-Gaussianity

To derive the consistency relation for \( \langle \gamma_k^2 \zeta_k \zeta_k \rangle \), one needs to know what the universe locally looks like with the existence of a long wavelength gravitational wave. As discussed in [2], this effect corresponds to the change \( k^2 \rightarrow k^2 - \gamma_i^j k_i k^j \). So the consistency relation becomes

\[
\langle \gamma_k^2 \zeta_k \zeta_k \rangle \xrightarrow{k \to 0} - \frac{1}{V} \langle \gamma_k^2 \gamma_k^2 \rangle \xi_k^i k_k^j \xi_k^j k_k^i \frac{d}{dk} \langle \zeta_k \zeta_k \rangle. \tag{B.5}
\]

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