Identifying Riemannian singularities with regular non-Riemannian geometry

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Admitting non-Riemannian geometries, Double Field Theory extends the notion of spacetime beyond the Riemannian paradigm. We identify a class of singular spacetimes known in General Relativity with regular non-Riemannian geometries. The former divergences merely correspond to coordinate singularities of the generalised metric for the latter. Computed in string frame, they feature an impenetrable non-Riemannian sphere outside of which geodesics are complete with no singular deviation. Approaching the non-Riemannian points, particles freeze and strings become (anti-)chiral.

I. Introduction

Spacetime singularities urge General Relativity (GR) to evolve. If not in—still elusive—quantum gravity, the singularities in any Riemannian metric-based classical theories of gravity have several layers: i) curable, coordinate singularity of the metric, ii) genuine, curvature singularity, and iii) geodesic incompleteness, as featured prominently in Penrose theorem [1]. However, on the one hand, ii) does not necessarily imply iii) (see e.g. [2] for a recent example) and, on the other hand, one encounters an intrinsic ambiguity in choosing a frame [3, 4] when applying these notions to scalar-tensor theories, e.g. [5], since Weyl transformations simply do not leave curvatures and geodesics invariant.

From a string theory perspective, the Riemannian metric $g_{\mu\nu}$ is only a fraction of the massless sector of closed strings which should further contain a skew-symmetric $B$-field and a scalar dilaton $\phi$. The theory implies a crucial symmetry, $O(D, D)$ with spacetime dimension $D$, which transforms the trio $(g, B, \phi)$ into one another [3, 4]. It calls for an $O(D, D)$ singlet, $e^{-2d} = -g e^{-2\phi}$, and an $O(D, D)$ tensor,

$$\mathcal{H}_{AB} = \begin{pmatrix} g^{-1} & -g^{-1}B \\ Bg^{-1} & g -Bg^{-1}B \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} g^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -B \\ B & 0 \end{pmatrix},$$

which served as generalised metric in the $O(D, D)$ manifest formulations of both string worldsheet actions [8–10] and target spacetime effective descriptions currently called Double Field Theory (DFT) [17–22]. By taking not $(g, B, \phi)$ but the $O(D, D)$ multiplets themselves as the fundamental variables, DFT opens a new avenue beyond the Riemannian paradigm [23–27]. In this approach, $e^{-2d}$ is an elementary scalar density with a unit diffeomorphic weight, and $\mathcal{H}_{AB}$ satisfies its own defining properties,

$$\mathcal{H}_{AB} = \mathcal{H}_{BA}, \quad \mathcal{H}_A^C \mathcal{H}_B^D \mathcal{J}_{CD} = \mathcal{J}_{AB}. \quad (2)$$

Here $\mathcal{J}_{AB} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the $O(D, D)$ invariant metric which, with its inverse, lowers and raises the $O(D, D)$ indices, $A, B = 1, 2, \ldots, 2D$. Seen as a square matrix, $\mathcal{H}_{AB}$ squares to identity hence is invertible with $|\det \mathcal{H}| = 1$.

Remarkably, DFT can be formulated in terms of any generalised metric that satisfies the defining relations (2), and may have evolved to an alternative ‘pure gravity’ in the sense that both $d$ and $\mathcal{H}_{AB}$, i.e. the whole closed string massless sector, are taken as the fundamental, gravitational fields. It is by now fully equipped with its own Christoffel symbols $\Gamma_{ABC}$, scalar/Ricci/Einstein curvatures [27–28], and Einstein equations coupled to extra ‘matter’: $G_{AB} = 8\pi G T_{AB}$ [29]. This single expression unifies the equations of motion of the fundamental variables $(d, \mathcal{H}_{AB})$. Besides, as solutions to condition (2), all possible classical geometries which DFT is capable of describing have been classified by two non-negative integers, $(n, \bar{n})$ [24]. Only the type $(0, 0)$ is Riemannian, amounting to the well-known parametrisation [11], while others are non-Riemannian in nature: the upper left $D \times D$ block of the generalised metric is degenerate with ‘nullity’, i.e. dimension of the kernel, $n + \bar{n}$, and thus does not yield an invertible Riemannian metric. For a non-Riemannian geometry to be a consistent string background at the quantum level, it turns out necessary to put $n = \bar{n}$ in the usual critical dimensions, $D = 10$ or $26$ [26]. Non-relativistic string [30–32], or torsional/stringy Newton–Cartan gravities of recent interest [33–40], are of the type $(1, 1)$ [23, 11, 17]. The condition $n = \bar{n}$ further makes the non-Riemannian geometry compatible with type II supersymmetric DFT and superstring [42, 48] (c.f. [40, 50]): for spin group $\text{Spin}(t, s) \times \text{Spin}(s, t)$ with $t + s = 10$, the allowed range of $n = \bar{n}$ is from zero to $\text{min}(t, s)$ [24].

Formally, DFT employs a doubled coordinate system, $x^A = (\hat{x}_\mu, \hat{x}^\nu)$, and subsequently unifies diffeomorphisms and $B$-field gauge symmetry into ‘doubled’ diffeomor-
phisms. Yet, the geometry is not truly doubled: half of the coordinate dependency should be turned off e.g. by setting $\frac{\partial}{\partial x^\mu} \equiv 0$, up to $O(D,D)$ rotations. This may suggest that half of the doubled coordinates are actually "gauged" \cite{nepomecho2011}. By gauging e.g. $\bar{x}_\mu$ explicitly as $\mathrm{D}x^A = (\mathrm{d}x^\mu - A_\mu, \mathrm{d}x^\nu)$, it is possible to define an $O(D,D)$-symmetric proper length \cite{fradkin1976} by means of the corresponding pull-back of the generalised metric $\mathrm{D}x^A \mathrm{D}x^B H_{AB}$, and further construct 'doubled-yet-gauged' actions for particles \cite{fradkin1983, fradkin1984} and strings \cite{fradkin1985, fradkin1986}. When the generalised metric is $(0,0)$ Riemannian, all the components of the auxiliary gauge connection $A_\mu$ appear quadratically in the actions. After integrating them out, one recovers the conventional particle and string actions. On the other hand, when the background is non-Riemannian, $n + \bar{n}$ components come out linearly to play the role of Lagrange multipliers. Consequently, the particle is frozen with identically vanishing proper velocities over the non-Riemannian $(n + \bar{n})$-dimensions. Further, the string becomes chiral over the $n$-directions and anti-chiral over the $\bar{n}$-directions \cite{fradkin1987}, as happens in the $(1,1)$ non-relativistic string case \cite{fradkin1988}. This also implies that, at the classical level, chiral strings get frozen too \cite{fradkin1989}.

From the Riemannian perspective, all the non-Riemannian backgrounds are singular geometries, as the would-be Riemannian metric diverges. Contradistinctly, they are well-behaved regular geometries in the doubled framework. This motivates us to revisit known singular Riemannian geometries and examine their non-Riemannian regularity, c.f. earlier related discussions in \cite{fradkin1990, fradkin1991, fradkin1992}. In this Letter, we report that a class of curvature singularities in GR are regular non-Riemannian geometries of DFT. They are at worst coordinate singularities of DFT. Moreover, we show through examples that the ordinary (undoubled) geometries defined in their string frame—which descend from $O(D,D)$-symmetric (doubled) geometries—are complete and that physically measurable tidal forces do not diverge, in spite of Riemannian curvature singularities. This last statement, although motivated by the DFT perspective, holds independently within the conventional GR setup.

II. Main Idea & Results

The Riemann-wise singular geometries of our interest assume the following generic form, with $x^\mu = (t, y, x^i)$,

$$\begin{align*}
\mathrm{d}s^2 & = \frac{1}{\sigma^2} \left(-\mathrm{d}t^2 + \mathrm{d}y^2\right) + G_{ij} \mathrm{d}x^i \mathrm{d}x^j, \\
B_{(2)} & = \pm \frac{1}{2} \frac{\partial}{\partial t} \wedge \frac{\partial}{\partial y} + \frac{1}{\beta} \frac{\partial}{\partial x^\mu} \wedge \frac{\partial}{\partial x^\nu}, \\
e^{-2\phi} = F \Psi.
\end{align*}$$ (3)

Our non-exhaustive list of examples includes A. $D = 2$ black hole by Witten \cite{witten1995}, B. $D = 4$ spherical solution by Burgess et al. \cite{burgess1996}, and C. $D = 10$ black 5-brane by Horowitz and Strominger \cite{horowitz1995}. All of them feature curvature singularities where $F$ vanishes, while $G_{ij}, \beta_{\mu\nu}, \Psi$ remain harmlessly regular. Substituting (3) into (1) yields the crucial observation (\textit{c.f.} \cite{fradkin1993, fradkin1994, fradkin1995, fradkin1996} for earlier examples) that the coordinate singularity is absent in the $O(D,D)$ fundamental variables: no negative power of $F$ appears,

$$e^{-2d} = \Psi \sqrt{G}, \quad \mathcal{H} = \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix} \tilde{\mathcal{H}} \begin{pmatrix} 1 - \beta \\ 0 & 1 \end{pmatrix},$$ (4)

where, with Pauli matrices,

$$\tilde{\mathcal{H}}_{AB} = \begin{pmatrix} -F_{\sigma\gamma} & 0 & \pm \sigma_1 & 0 \\ 0 & G^{-1} & 0 & 0 \\ \pm \sigma_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & G \end{pmatrix}.$$ (5)

Clearly, at the points where $F = 0$, the DFT geometry is non-Riemannian, specifically of type $(1,1)$, as expected from the underlying Minkowskian spin group signature.

We now revisit the three aforementioned layers of singularity from the ‘doubled’ perspective.

i) Removing the coordinate singularity from generalised metric through doubled diffeomorphisms.

The singular term in $B_{(2)}$ is crucial in regularising $\mathcal{H}$. In fact, in the examples below, this term being pure gauge would not be present when writing the solutions in their simplest form. We shall nevertheless introduce such a singular pure gauge term in order to match the privileged doubled coordinate system \cite{fradkin1997}. Namely, through doubled diffeomorphisms, one can eliminate what should be construed as a coordinate singularity of the generalised metric.

ii) All the $O(D,D)$-symmetric curvatures are regular.

Once the DFT fundamental variables \{ $\mathcal{H}_{AB}, e^{-2d}$ \} are made free of any singularity, (and twice continuously differentiable), all the $O(D,D)$-symmetric curvatures are automatically regular. In fact, all our examples satisfy the equations of motion of the NS-NS supergravity, possibly with an $O(D,D)$-symmetric “cosmological constant”:

$$R + 4\Box \phi - 4 \partial_\mu \phi \partial^\mu \phi - \frac{1}{4} \mathcal{H}_{\mu\nu\rho\sigma} H^{\mu\nu} - 2\Lambda_{\text{DFT}} = 0, \quad R_{\mu\nu} + 2\nabla_\mu (\partial_\nu \phi) - \frac{1}{4} \mathcal{H}_{\mu\rho\sigma} H^{\rho\sigma} = 0, \quad d (e^{-2\phi} H_{(3)}) = 0.$$ (6)

This readily implies that the corresponding non-singular DFT ansatz (\textit{c.f.} \cite{fradkin1998}) solves the DFT Einstein equations $G_{\mu\nu} + J_{\mu\nu} \Lambda_{\text{DFT}} = 0$ everywhere even at the non-Riemannian points of $F = 0$. It also means that all the $O(D,D)$-symmetric DFT curvatures are trivially regular \cite{fradkin1999}. Similarly, while the conventional dilaton $\phi$ diverges as $F \to 0$, the DFT dilaton $d$ remains finite, \textit{c.f.} \cite{fradkin2000}, hence so does the $O(D,D)$-symmetric version of the Fradkin–Tseytlin term \cite{fradkin2001} in doubled string actions \cite{fradkin2002, fradkin2003}.

iii) Geodesics are complete in string frame: impenetrable non-Riemannian sphere.

The fact that the DFT dilaton $e^{-2d}$ carries a nontrivial diffeomorphic weight prevents it from coupling to the $O(D,D)$-symmetric doubled-yet-gauged particle and string actions \cite{fradkin2004, fradkin2005}, hence to particle and stringy
geodesics, whose equations read respectively 24, 61:

\[ e^{\frac{1}{2}}(e^{-1}H_{AB}D_rx^B) + 2\Gamma_{ABC}(P_Dr)x^B(PD_r)x^C = 0 , \]

\[ \frac{1}{\sqrt{-h}}\partial_\alpha(\sqrt{-h}H_{AB}D^\alpha x^B) + \Gamma_{ABC}(P_D\alpha)x^B(PD^\alpha x)^C = 0 , \]

(7)

where \( \epsilon, h_{\alpha\beta} \) are the worldline einbein and worldsheet metric, while \( P_{AB}, P_{\alpha B} \) stand for \( \frac{1}{2}(J_{AB} \pm H_{AB}) \) and \( D_x, D_\alpha x \) are the pull-backs of the doubled-yet-gauged differential. This rigidity naturally settles the issue of the frame ambiguity. Upon the Riemannian background of 11, an \( O(D, D) \)-symmetric particle follows geodesics defined in the string frame 62: after fixing gauges and solving for the auxiliary connection, 7 can be shown to reproduce the standard (undoubled) expressions,

\[ \ddot{x}^\mu + \gamma^{\mu\nu\rho\sigma}x^\nu\dot{x}^\rho\dot{x}_\sigma = 0 \iff \frac{d}{d\lambda}(g_{\mu\nu}\dot{x}^\nu) - \frac{1}{2}\partial_\lambda g_{\mu\nu}\dot{x}^\nu\dot{x}^\rho = 0 , \]

\[ \partial_\nu(g_{\mu\nu}\partial_\mu x^\rho) + \partial_\mu(g_{\mu\nu}\partial_\nu x^\rho) + (H_{\mu\nu\rho} - \partial_\nu g_{\mu\rho})\partial_\mu x^\rho = 0 . \]

(8)

Focusing on the ansatz 8 and by analogy with ii)), the doubled formulation 7 in terms of the generalised metric \( H_{AB} \) may still suggest that geodesics are regular in the limit \( F \to 0 \), despite the fact that its undoubled counterpart 3 involves the singular Riemannian metric \( g_{\mu\nu} \). Indeed, as we show below for the aforementioned three examples, both null and time-like geodesics are complete (at least) in the region \( F > 0 \). The non-Riemannian points of \( F = 0 \) form a sphere for \( D = 2 \) (or hyperbola for \( D = 2 \)) inside of which \( (F < 0) \) no time-like nor null geodesics can penetrate. Time-like and non-radial null geodesics cannot even come close to the sphere from outside \( (F < 0) \). Only radial null ones may approach the sphere with identically vanishing proper velocities taking infinite affine parameter. Besides, undoubled strings—as an alternative probe of the “singular” geometries—become (anti-)chiral: with light-cone coordinates \( y^\pm = y \pm t \) and also \( \partial_\lambda y^\pm = \frac{\partial}{\partial\lambda}y^\pm \) on the worldsheet, as already used in 8, we get

\[ \partial_- y^\pm = 0 = \partial_+ y^- \quad \text{and} \quad \partial_- y^+ = 0 = \partial_+ y^- \quad \text{in the limit} \quad F \to 0 . \]

Thus, the undoubled, or conventional, particle geodesics and string propagations agree with the non-Riemannian (freezing/chiral/anti-chiral) behaviors predicted from the previous doubled-yet-gauged sigma model approach 23, 24 which relied on the auxiliary gauge connection as Lagrange multiplier (c.f. Introduction).

\textbf{iii)} again: geodesic deviations are also regular.

In DFT there is no \( O(D, D) \)-symmetric completion of the Riemann curvature 27, 68, 69. Hence, the criterion of \( O(D, D) \)-symmetric curvature singularity may appear unbalanced or somewhat unfair compared to GR. As a step toward restoring the balance as well as focusing on genuine physical quantities, we further analyse the geodesic deviation and the ‘tidal force’ therein (again in string frame),

\[ \frac{D^2\xi^\mu}{d\lambda^2} = R^\mu_{\nu\rho\sigma}\dot{x}^\nu\dot{x}^\rho\xi^\sigma . \]

As the geodesics are complete and smooth for \( F > 0 \), their deviations \( \xi^\mu \) should be regular. Then, although the Riemann curvature itself diverges, its contraction with the vanishing velocities \( R^\mu_{\nu\rho\sigma}\dot{x}^\nu\dot{x}^\rho \) as well as the square norm \( |D^2\xi|^2 = g_{\mu\nu}\frac{D^2\xi^\mu}{d\lambda^2}\frac{D^2\xi^\nu}{d\lambda^2} \) can be finite, thus preventing the physically measurable quantities from being singular. This further property will be checked to hold in the following section for the examples A, B and C.

\textbf{III. Examples: geodesics and string propagation}

We now take a closer look at each example 8, 60 to analyse generic geodesics 8 supplemented by the Hamiltonian or Virasoro constraints,

\[ g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = \mathcal{E} , \quad \partial_\mu x^\rho\partial_\nu x^\sigma g_{\mu\nu} = 0 = \partial_- x^\rho\partial_- x^\sigma g_{\mu\nu} . \]

(10)

Hereafter, \( \mathcal{E} \) is to be either \(-1 \) (time-like) or \( 0 \) (null).

\textbf{A.} \( D = 2 \) black hole: non-Riemannian hyperbola

The \( D = 2 \) geometry à la Witten 88 is characterised by \( ds^2 = dy^+dy^- / F \) with \( F = -1 + y^+y^- / l^2 = \frac{-1}{2}\phi^2 e^{-2\phi} \).

It solves 5 when \( \Lambda_{\text{DFT}} = \frac{1}{2} l^2 \). From the on-shell value of the Ricci scalar \( R = -\frac{2}{l^2} \), the hyperbola \( y^+y^- = l^2 \) corresponds to a curvature singularity. We stress that though the \( H \)-flux is trivial in two dimensions, the B-field still plays a crucial role in making the generalised metric free of singularity 5.

Since the metric is invariant under scaling \( \delta y^\pm = \pm \lambda^\pm \), there are two constants of motion for every geodesic,

\[ L = \left( y^+\dot{y}^- - y^-\dot{y}^+ / F \right) , \]

\[ \mathcal{E} = \dot{y}^+\dot{y}^- / F , \]

(11)

which give, setting \( \omega = \pm \lambda^+ / y^+ \),

\[ \frac{dL}{d\lambda} = \pm \sqrt{F [4\mathcal{E} + \left( \frac{L^2}{2} + 4\mathcal{E} \right) F] } , \]

\[ \frac{d\mathcal{E}}{d\lambda} = (\omega - \frac{\mathcal{E}}{2})^2 + \left( \frac{\phi}{2} \right)^2 + 2(\phi^2 - \frac{\mathcal{E}}{2}) . \]

It follows that time-like geodesics \( (\mathcal{E} = -1) \) starting from the region \( F > 0 \) will never reach the non-Riemannian hyperbola, satisfying \( F \geq \frac{1}{(l^2)^{\frac{1}{2}}} > 0 \). Null ones \( (\mathcal{E} = 0) \) may approach only at past or future infinity as \( F = e^{\pm L / l^2} F_0 \), since the most general null geodesics are, from 11 with initial values \( y^+_0 \) and \( F_0 \) at \( \lambda = 0 \), either

\[ \begin{pmatrix} y^+(\lambda) \\ y^-(\lambda) \end{pmatrix} = \begin{pmatrix} \frac{L^2}{2y^+_0} + (y^+_0 - \frac{L^2}{2y^+_0}) e^{L\lambda/l^2} \\ y^+_0 \end{pmatrix} , \]

(12)

or \( y^+(\lambda) = y^+_0 , \quad y^-(\lambda) = \frac{L^2}{2y^+_0} + (y^+_0 - \frac{L^2}{2y^+_0}) e^{-L\lambda/l^2} \).

For these two solutions, the only nonvanishing components in 9 are respectively \( R^+_{\nu\rho\sigma}\dot{x}^\nu\dot{x}^\rho = -(L/y^+_0)^2 \) and \( R^-_{\nu\rho\sigma}\dot{x}^\nu\dot{x}^\rho = -(L/y^+_0)^2 \), which are all finite with \( |D^2\xi|^2 = 0 \). In fact, the variations of the general solutions 12 by the free parameters, \( y^+_0 \), also lead to finite deviation vectors, \( \xi^\mu \). We conclude that the space \( F > 0 \) is geodesically complete with no singular deviation. We
turn to the string dynamics \(8, 10\) which read
\[
\begin{align*}
\partial_+ \partial_y^+ - \partial_+ \partial_y^+ \partial_y^+ \frac{\ln F}{\partial y^+} &= 0, \\
\partial_\tau \partial_y^- - \partial_\tau \partial_y^- \partial_y^- \frac{\ln F}{\partial y^-} &= 0.
\end{align*}
\]
The second relation gives either \(\partial_+ \partial_y^+ = 0\) or \(\partial_+ \partial_y^- = 0\). If \(\partial_+ \partial_y^+ = 0\), the third implies \(\partial_\tau \partial_y^- = 0\) such that \(\partial_\tau \partial_y^- = FF(\sigma^-)\) for some one-function variable \(f(\sigma^-)\). Thus, \(\partial_\tau \partial_y^-\) vanishes when \(F = 0\). On the other hand, if \(\partial_+ \partial_y^- = 0\), the first implies \(\partial_\tau \partial_y^+ = 0\) and \(\partial_\tau \partial_y^+ = FF(\sigma^-)\). Thus, \(\partial_\tau \partial_y^+\) vanishes when \(F = 0\). Similar analysis holds for the last relation. We conclude that, in any case, one of \(\{\partial_+ \partial_y^+, \partial_+ \partial_y^-\}\) is chiral and the other is anti-chiral on the non-Riemannian hyperbola.

**B.** \(D = 4\) two-parameter family of spherical solutions
Our \(D = 4\) example is a spherical solution from \(53, 54\),
\[
\text{ds}^2 = \frac{1}{F(r)} \left(-dt^2 + dr^2\right) + R(r)^2 \left(d\theta^2 + \sin^2 \theta \, d\phi^2\right),
\]
where \(F(r), R(r)^2\) are, with two free parameters, \(b, h\) (see also \(21\) for their physical interpretations),
\[
\frac{1}{F(r)} = \left(1 + \frac{1}{2} - \frac{h^2}{b^2} - 2 \ln \left(\frac{r}{r_0}\right) + \frac{1}{2} - \frac{1}{2} h^2 + \frac{1}{2} h^2\right) r^b - \frac{h^2}{b^2} r^b + \frac{1}{b} r^b + h^2,
\]
\[
R(r)^2 = \left(r + \frac{b}{2} r - \frac{1}{2} h^2 r^2\right)^2 + \frac{1}{2} h^2 = \frac{r^2 + r^4}{b^2} r^b + h^2.
\]
We require \(0 < |h| \leq b\), such that the only singular source in the metric is at \(r = 0\) as \(\lim_{r \to 0} F(r) = 0\). The Ricci scalar diverges at \(r = 0\) as \(R \sim -\frac{1}{r^2}\). Nevertheless, with non-singular \(R(0)^{2\epsilon}\) due to \(h \neq 0\), the generalised metric \(H_{AB}\), dilatonic \(e^{-2\phi} = R(r)^2\) sin \(\theta\), and DFT curvatures are all finitely regular. The non-Riemannian points of \(F = 0\) form a 2-sphere with nontrivial proper area \(4\pi R(0)^2 = 2\pi b(1/2 - \sqrt{b^2 - h^2})\).

For geodesics, without loss of generality, we put \(\theta = \frac{\pi}{2}\). The conserved energy \(E\) and angular momentum \(L_\phi\) set
\[
\dot{t} = EF(r), \quad \dot{\phi} = L_\phi R(r)^{-2}.
\]
In particular, \(\dot{t}\) vanishes on the non-Riemannian sphere.

The remaining radial motion reads
\[
0 = \dot{x}^2 + V(r), \quad V(r) = \left[-E^2 F(r) + \frac{L_\phi^2}{R(r)^2}\right] F(r).
\]
From \(\lim_{r \to 0} V = 0\), it follows that \(\dot{r}\) also gets trivial at \(r = 0\). In fact, since \(\lim_{r \to 0} V'(r) = \frac{4L_\phi^2}{b^2(\sqrt{b^2 - h^2} - 1)}\) is positive for either \(E = -1\) or \(E = 0\) with \(L_\phi \neq 0\), the corresponding potential is positive close to \(r = 0^+\). Thus, both the time-like and the non-radial null geodesics cannot come close to the sphere from outside. Only the radial null ones with \(\dot{r} = 0 = L_\phi\) may do, but it takes infinite affine parameter as the integral of \(d\lambda = (EF)^{-1} dr\) diverges logarithmically. These results can be all attributed to the repulsive gravitational force around the non-Riemannian sphere, while it is attractive for large \(r \sim \mathcal{R}\) as \(g_{tt} \sim -1 + \frac{b^2(1 - h^2/2b^2)}{r^2}\).

For the radial null geodesics \(\dot{t} = EF = |\dot{r}|\), we also confirm that the deviation \(11\) is finitely regular: the only nontrivial components of \(R_{\mu\nu\rho\sigma} x^\mu x^\nu x^\rho x^\sigma\) at \(r = 0\) take the values \(\pm \frac{4E^2}{(b - \sqrt{b^2 - h^2})^2}\) for \(\mu, \sigma\) being \(t\) or \(r\), with \(\frac{d^2 F}{d\xi^2} \neq 0\).

In the limit, \(r \to 0\) hence \(F \to 0\), the string dynamics \(8, 10\) implies, with \(y^\pm = r \pm t\),
\[
\begin{align*}
\partial_+ y^+ \partial_+ y^+ F'(0) &= 0, \\
\partial_+ y^+ \partial_+ y^- F'(0) &= 0, \\
\partial_+ y^- \partial_+ y^- F'(0) &= 0, \\
\partial_+ y^- \partial_+ y^- F'(0) &= 0,
\end{align*}
\]
where \(F'(0) = \lim_{r \to 0} F'(r) = \frac{2}{b - \sqrt{b^2 - h^2}}\) is nonvanishing. This confirms that one of \(\{y^+, y^-\}\) is chiral while the other is anti-chiral on the non-Riemannian sphere.

**C.** \(D = 10\) black 5-brane
One particular black 5-brane geometry from \(61\) reads
\[
\text{ds}^2 = \frac{1}{F(r)} \left(-dt^2 + dr^2\right) + r^2 d\Omega_3^2 + d\vec{x}^2, \quad F = 1 - (r_c/r)^2 = e^{-2\phi}.
\]
The Ricci scalar diverges both at \(r = 0\) and \(r = r_c\), as \(R = -\frac{4e^\phi}{r(r_c - r)^2}\). Though the H-flux is trivial, a pure gauge \(B\)-field should be introduced, as prescribed in \(53\). The generalised metric \(53\) is then non-Riemannian regular on the 3-sphere of the radius \(r = r_c\), but still singular at \(r = 0\). We shall see soon that the non-Riemannian sphere forms the boundary of a geodesically complete space of \(F > 0\) which excludes the dangerous point \(r = 0\).

The geodesic analysis is similar to example \(B\) and reduces to \(\dot{t} = EF\) and \(\dot{\phi} + V(r) = 0\) with a potential involving non-negative constants, \(L_\Omega^2\) (total angular momentum) and \(\vec{B}^2\) (extra momentum),
\[
V(r) = \left[-E^2 F(r) + \frac{L_\phi^2}{R(r)^2} + \vec{B}^2\right] F(r).
\]
Since \(V'(r_c) = 2(-E^2 L_\Omega^2/r_c^2 + \vec{B}^2)/r_c^2\) is positive for either \(E = -1\) or \(E = 0\) with \(L_\Omega^2/r_c^2 + \vec{B}^2 \neq 0\), and \(V(r_c)\) vanishes, the corresponding potential is positive close to \(r = r_c^+\). Thus, time-like and generic null geodesics cannot reach the non-Riemannian sphere. Only the radial null ones having \(\dot{t} = EF = |\dot{\phi}|\) and \(L_\Omega^2 = 0 \neq \vec{B}^2\) can do, albeit taking infinite affine parameter with vanishing proper velocities. Besides, the deviation \(9\) is regular: the only nontrivial values of \(R_{\mu\nu\rho\sigma} x^\mu x^\nu x^\rho x^\sigma\) at \(r = r_c\) are \(\pm 2E^2/r_c^2\), for \(\mu, \sigma\) being \(t\) or \(r\), with \(\frac{d^2 F}{d\xi^2} \neq 0\).

In the limit \(r \to r_c\), hence \(F \to 0\), the chirality relations of the previous example \(11\) still hold after replacing \(F'(0)\) by \(F'(r_c) = 2/r_c^2\).

**Discussion**
We have shown that the curvature singularities featured in a large class of GR spacetimes \(49\) are mere artifacts of Riemannian geometry. In particular, we have noted the remarkable fact that physically measurable tidal forces do not diverge. Further examples
include $D = 10$ superstring \cite{1,2,7,14,56,72,73}. The corresponding generalised metric was shown to be regular in \cite{56}, with singular deviations. $D = 2$ DFT is essentially Jackiw–Teitelboim gravity \cite{75,76} as $g^{\text{JT}}_{\mu \nu} = e^{-2\phi}g_{\mu \nu}$. Witten’s solution is then mapped to flat spacetime. Constant non-Riemannian backgrounds were shown in \cite{10} to admit infinite-dimensional isometries. All of them might be realised as the asymptotic symmetries of the non-Riemannian spheres with large radius (as $F' \to 0$). The associated, infinitely many, conserved charges then might store all the information of the in-going and freezing null radial geodesics. We call for further studies.

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Supplemental Material

$D = 10$ fundamental string with negative tension

We now expand on the aforementioned example of the fundamental string \cite{70}.

$$\text{d}s^2 = \frac{\text{d}x^+ \text{d}y^-}{F(r)} + \text{d}\vec{x}^2, \quad B_2 = \frac{\text{d}y^+ \text{d}y^-}{2F(r)},$$

where $F = 1 + \frac{Q}{r} = e^{-2\phi}$ is a harmonic function and $r^2 = \vec{x} \cdot \vec{x}$. The Ricci scalar diverges at $r = 0$ as $\vec{R} = \frac{1260^2}{r^2 |Q|^{1/6}}$ and further at $r = |Q|^{1/6}$ provided that the tension is negative, $Q < 0$. The corresponding generalised metric—identified as a doubled null wave in \cite{71}—is then non-Riemannian regular at $r = |Q|^{1/6}$ \cite{43,44} but still singular at $r = 0$. Similarly to examples B and C, the non-Riemannian 7-sphere of radius $r = |Q|^{1/6}$ forms the boundary of the geodesically complete space $F \geq 0$, which thus excludes the dangerous point $r = 0$.

The longitudinal isometries give two constants of motion $P_\pm$ and set $y^\pm = P_\pm F$ which clearly vanish when $F = 0$. Moreover, time-like and also non-radial ($L^2 > 0$) null geodesics are incompatible with $F = 0$, due to the fact that $|\vec{x}|^2 = \vec{y}^2 + \frac{L^2}{F^2} = \mathcal{E} - P_\mp P_- F$ is non-negative. Hence $P_\pm P_- \leq 0$ and the acceleration $\ddot{\vec{x}} = -3(|Q| P_\pm P_- \vec{x}/r^8)$ is repulsive. (Radial ($L^2 = 0$) null ones with $P_\pm P_- < 0$ may reach the sphere only to bounce back within finite affine parameter. On the contrary, null ones with $P_\pm P_- = 0$ have $\ddot{\vec{x}} = 0$ and thus $r$ is fixed for ever. In particular, when $r = |Q|^{1/6}$, from $\dot{x}^0 = 0$, $\dot{x}^i$ should be (trivially) constant. Anyhow, time-like and null geodesics are all complete. We turn to the deviations of the bouncing null geodesics,

$$y^+(\lambda) \approx y_0^+ - 3|Q|^{-\frac{3}{2}}P_\pm P_- \lambda^2, \quad \ddot{x}(\lambda) \approx \ddot{n}_0 \left(\frac{|Q|^{3/2}}{4} - \frac{3}{2}|Q|^{-3}P_\pm P_- \lambda^2\right). \tag{SM16}$$

This expansion around the bouncing at $\lambda = 0$ comes from $F \simeq 6(r|Q|^{-1/6} - 1)$, and features all possible constants of motion: $P_\pm, y_0^+$ and a unit vector $\vec{n}_0$. Varying each of them, we acquire a deviation vector $\xi^\mu$. Contrarily to the previous examples, some of $R^\mu_{\nu \rho \sigma} \xi^\nu \xi^\rho\xi^\sigma$’s exhibit singularities as severe as $F^{-3/2} \propto \lambda^{-3} \sigma$ for $\sigma = \pm$, or $F^{-1} \propto \lambda^{-2}$ for transverse $\sigma$. Nonetheless, the full expression of the tidal force $R^\mu_{\nu \rho \sigma} \xi^\nu \xi^\rho\xi^\sigma$ is finite, once contracted with the deviation vectors of $\text{SM16}$ induced by $\delta(P_+/P_-) = 0$ (dilation, $\xi^\mu = \lambda \vec{x}^\mu$), $\delta(P_+ P_-) = 0$ (so(1,1) Lorentz), $\delta \vec{n} \cdot \vec{x} = 0$ (so(8) rotation), with the exception of $\delta y_0^+$ (translation). Translational $\xi^\mu$’s are just constant. Being longitudinal, their proper length gets singular, and subsequently so does the tidal force acting on them. Lastly, the string propagation in \cite{8} gives

$$\partial_+ (F^{-1} \partial_- y^+) = 0 = \partial_- (F^{-1} \partial_+ y^-), \tag{SM17}$$

which yields $\partial_+ y^+ = F\vec{f}_-(\sigma^-)$ and $\partial_- y^- = F\vec{f}_+(\sigma^+)$, naturally implying the (anti-)chirality at $r = |Q|^{1/6}$.

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