NUMERICAL SOLUTION OF A LINEAR SYSTEM OF NAVIER–STOKES EQUATIONS IN AN AXISYMMETRIC DOMAIN

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The system of Navier–Stokes equations simulates the dynamics of a viscous incompressible fluid. The problem on the existence of solutions to the Cauchy–Dirichlet problem for this system is one of the most difficult mathematical problems of the present century. However, the question on the existence of solutions to the Cauchy–Dirichlet problem for the system of Navier–Stokes equations still remains unsolved. This article shows how to obtain eigenvalues for the system in the case of an axisymmetric domain.

Keywords: system of Navier–Stokes equations; Galerkin method; multipoint initial-final value condition.

Introduction

The system of Navier–Stokes equations

\[ \frac{\partial u}{\partial t} = \nu \nabla^2 u + \nabla p, \nabla \cdot u = 0 \]  \hspace{1cm} (1)

with the Dirichlet condition

\[ \vec{u}(x, t) = 0, \hspace{1cm} (x, t) \in \partial \Omega \times \mathbb{R} \]  \hspace{1cm} (2)

simulates the dynamics of a simple viscous incompressible fluid. Here \( u = (u_1, u_2, \cdots, u_n) \), \( u_i = u_i(x, t) \), \( n = 2, 3 \) is the vector function corresponding to the fluid velocity; the scalar function \( \nu \in \mathbb{R}_+ \) characterizes the viscosity of the fluid, respectively. In various aspects, equations (1) were studied in [1], [4], [5]. We present the Galerkin method for the system of the Navier–Stokes equations in the case of an axisymmetric domain with the multipoint initial-final value condition

\[ P_j(u(\tau_j) - u_j) = 0, \hspace{1cm} j = 0, n. \]  \hspace{1cm} (3)
1. Derivation of System of Navier – Stokes Equations

Consider the equation of motion of a continuous medium in the Cauchy form:

$$\rho \frac{d\vec{v}}{dt} = \text{div} \vec{\Pi} + \vec{f}. \quad (4)$$

Equation (4) can be obtained from the second Newton’s law in the d’Alembert form:

$$\int_V (\vec{F} - \vec{a}) \rho dV + \int_S \vec{p}_n dS = 0. \quad (5)$$

Here $V$ is a bounded three-dimensional volume of a continuous medium, $S$ is its sufficiently smooth surface, $\rho$ is a density of elementary volume $dV$. The vectors $\vec{F}$ and $\vec{a}$ denote external force and total acceleration per unit mass of volume $V$, respectively. The vector $\vec{p}_n$ corresponds to the normal component of the surface force acting on the surface element $dS$.

Represent the vector $\vec{p}_n$ in the form

$$\vec{p}_n = \vec{p}_1 \cos(\vec{n}, x_1) + \vec{p}_2 \cos(\vec{n}, x_2) + \vec{p}_3 \cos(\vec{n}, x_3), \quad (6)$$

where $\vec{p}_k$ is the stress vector on the elementary area, which is perpendicular to the axis $Ox_k$, and $k = 1, 2, 3$. In turn, each vector $\vec{p}_k$ can be represented as

$$\vec{p}_k = (p_{k1}, p_{k2}, p_{k3}), \quad k = 1, 2, 3. \quad (7)$$

Here $p_{kk}$ is the normal component of the vector $\vec{p}_k$, and the other two components correspond to the tangent components. Substitute (7) into (6) and obtain

$$\vec{p}_n = \vec{\Pi} \vec{n}, \quad (8)$$

where the matrix $\vec{\Pi} = ||p_{kl}||, k, l = 1, 2, 3$ is called a tensor of elastic stresses, and the vector $\vec{n} = (n_1, n_2, n_3)$, $n_k = \cos(\vec{n}, x_k)$ corresponds to the unit normal.

Consider (5) and use (8) in order to represent the second term as

$$\int_S \vec{p}_n dS = \int_S \vec{\Pi} \vec{n} dS. \quad (9)$$

Apply the Gauss-Ostrogradsky formula to (9) and obtain

$$\int_S \vec{\Pi} \vec{n} dS = \int_V \text{div} \vec{\Pi} dV. \quad (10)$$

Here the vector

$$\text{div} \vec{\Pi} = \left( \sum_{k=1}^{3} \frac{\partial p_{k1}}{\partial x_k}, \sum_{k=1}^{3} \frac{\partial p_{k2}}{\partial x_k}, \sum_{k=1}^{3} \frac{\partial p_{k3}}{\partial x_k} \right).$$

Finally, substitute (10) into (5) and obtain

$$\int_V (\vec{f} + \text{div} \vec{\Pi} - \rho \frac{d\vec{v}}{dt}) dV = 0. \quad (11)$$
Here $\vec{f} = \rho\vec{F}$ is the vector of external mass forces, and the vector
\[
\frac{d\vec{v}}{dt} = \frac{\partial\vec{v}}{\partial t} + v_k \frac{\partial\vec{v}}{\partial x_k} = \vec{a}
\]
corresponds to the dynamic full acceleration. Since the volume $V$ is arbitrary, (4) immediately follows from (11).

Next, consider the matrix
\[
\vec{D} = \left[ \begin{array}{ccc}
\frac{1}{2} \left( \frac{\partial v_k}{\partial x_l} + \frac{\partial v_l}{\partial x_k} \right)
\end{array} \right], \quad k, l = 1, 2, 3,
\]
which is called a strain rate tensor. The physical meaning of the component
\[
\frac{\partial v_k}{\partial x_k}
\]
of the tensor $\vec{D}$ is the change in the velocity of translational motion of the area element perpendicular to the axis $Ox_k$, $k = 1, 2, 3$. If the tensors $\vec{D}$ and $\vec{Π}$ are such that
\[
\vec{Π} = -p\vec{I} + 2\nu\vec{D},
\]
then the continuous medium is called Newtonian fluid.

Assume that the fluid is incompressible, i.e.
\[
\text{div } \vec{v} = 0,
\]
then we have
\[
\text{div } \vec{Π} = -\text{grad } \vec{p} +
+2\nu \left( \frac{1}{2} \sum_{k=1}^{3} \frac{\partial}{\partial x_k} \left( \frac{\partial v_k}{\partial x_1} + \frac{\partial v_1}{\partial x_k} \right), \frac{1}{2} \sum_{k=1}^{3} \frac{\partial}{\partial x_k} \left( \frac{\partial v_k}{\partial x_2} + \frac{\partial v_2}{\partial x_k} \right), \frac{1}{2} \sum_{k=1}^{3} \frac{\partial}{\partial x_k} \left( \frac{\partial v_k}{\partial x_3} + \frac{\partial v_3}{\partial x_k} \right) \right) =
= -\text{grad } \vec{p} + \nu \Delta \vec{v}. \tag{12}
\]

Assume that the density $\rho = 1$, substitute (12) into (4) and obtain the system of Navier–Stokes equations [2]
\[
\vec{v}_t = \nu\nabla^2 \vec{v} - (\vec{v} \cdot \nabla)\vec{v} - \nabla p + \vec{f},
0 = \nabla \cdot \vec{v}, \tag{13}
\]
which describes the dynamics of a viscous incompressible fluid.

Here $\nabla = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right)$ is Hamilton operator, $\vec{v}_t = \frac{\partial \vec{v}}{\partial t}$. 

2. Galerkin Method for System of Navier – Stokes Equations

Following E.N. Lorentz [3], we consider system (1) to be invariant under a shift along one of the horizontal coordinates, i.e. we consider the values \( u_i = u_i(x, t), i = 1, 2, 3, p = p(x, t) \) to be constant along the coordinate \( x_2 \), and consider the value \( u_2(x, t) \) to be constant. In this case, the incompressibility equation \( \nabla \cdot u = 0 \) takes the form

\[
\frac{\partial u_1}{\partial x_1} + \frac{\partial u_3}{\partial x_3} = 0,
\]

therefore, we can define a stream function up to an additive constant by the equations

\[
\frac{\partial \Psi}{\partial x_1} = -u_3, \quad \frac{\partial \Psi}{\partial x_3} = u_1.
\]

Replace the notation of the coordinates \( x_1 \) and \( x_3 \) with the more standard \( x = x_1 \) and \( z = x_3 \) and represent system (1) as

\[
\begin{cases}
\frac{\partial u_1}{\partial t} = \nu \nabla^2 u_1 + \nabla p, \\
\frac{\partial u_2}{\partial t} = \nu \nabla^2 u_2 + \nabla p, \\
\frac{\partial u_3}{\partial t} = \nu \nabla^2 u_3 + \nabla p,
\end{cases}
\]

(16)

\[
\begin{aligned}
\frac{\partial^2 \Psi}{\partial x_1 \partial x_3} &= \nu \left( \frac{\partial^2 \Psi}{\partial x_1^2} + \frac{\partial^2 \Psi}{\partial x_3^2} \right) - \nabla p, \\
\frac{\partial^2 \Psi}{\partial x_3 \partial x_1} &= \nu \left( \frac{\partial^2 \Psi}{\partial x_3^2} + \frac{\partial^2 \Psi}{\partial x_1^2} \right) + \nabla p, \\
\nabla^2 \frac{\partial \Psi}{\partial t} &= -\nu \nabla^4 \Psi.
\end{aligned}
\]

(17)

Consider the Dirichlet problem

\[
\begin{cases}
\Psi(x, 0, t) = \Psi(x, H, t), \\
\Psi(0, z, t) = \Psi(L, z, t), \\
\nabla^2 \Psi(0, z, t) = \nabla^2 \Psi(L, z, t)
\end{cases}
\]

(19)

for equation (18) in the rectangle \([0, L] \times [0, H]\).

Following E.N. Lorentz [3], we find the three-dimensional Galerkin approximation to problem (19) for equation (18). To this end, as the basis functions of the Galerkin method, we take the eigenfunctions of the following problem:

\[
\begin{cases}
-\nabla^2 \phi = \lambda \phi, \quad [0, L] \times [0, H], \\
\phi(x, 0) = \phi(x, H) = 0, \\
\phi(0, z) = \phi(L, z), \\
\phi'(0, z) = \phi'(L, z).
\end{cases}
\]

(20)

All non-trivial solutions to problem (20) can be divided into three families:

\[
\begin{aligned}
\alpha_{lk} &= \{ \sin \frac{\pi}{H} z \cdot \sin \frac{2\pi k}{L} x \}, \quad l, k \in \mathbb{N}, \\
\beta_{lk} &= \{ \sin \frac{\pi}{H} z \cdot \cos \frac{2\pi k}{L} x \}, \quad l, k \in \mathbb{N}, \\
\gamma_l &= \{ \sin \frac{\pi}{H} z \}, \quad l \in \mathbb{N}.
\end{aligned}
\]

(21)
We take the Galerkin approximation to the solution to problem (19) for system (20) in the form

$$\Psi = X(t)\alpha_{11}. \quad (22)$$

It is easy to see that (22) satisfies boundary conditions (19).

Substitute (22) into (18) and obtain

$$\pi^2 \left( \frac{1}{H^2} + \frac{4}{L^2} \right) \dot{X}\alpha_{11} = -\nu \pi^4 \left( \frac{1}{H^2} + \frac{4}{L^2} \right)^2 X\alpha_{11}. \quad (23)$$

The first equation (23) immediately implies

$$\dot{X} = -aX, \quad (24)$$

where

$$a = \nu \pi^2 \left( \frac{1}{H^2} + \frac{4}{L^2} \right). \quad (25)$$

### 3. Numerical Experiments

In the first experiment, we take the following values of the auxiliary parameters:

$$\lambda = 1; \beta = 2; \alpha = -1; \chi = 1; \nu = 2$$

(see Fig. 1).

In the second experiment, we take the following values of the auxiliary parameters:

$$\lambda = 1; \beta = 4; \alpha = -2; \chi = 1; \nu = 2$$

(see Fig. 2).
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Fig. 2. Graphic of the solution to problem (1) – (3) at the time $t = 1$

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ЧИСЛЕННОЕ РЕШЕНИЕ ЛИНЕЙНОЙ СИСТЕМЫ УРАВНЕНИЙ НАВЬЕ–СТОКСА В ОСЕСИММЕТРИЧНОЙ ОБЛАСТИ

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Система уравнений Навье – Стокса моделирует динамику вязкой несжимаемой жидкости. Проблема существования решений задачи Коши – Дирихле для этой системы вошла в список наиболее тяжелых математических проблем нынешнего века. Однако до сих пор не решен вопрос о существовании решений задачи Коши – Дирихле для системы уравнений Навье – Стокса. Проблема существования решений этой задачи оказалась настолько трудной, что она вошла в списки наиболее тяжелых математических проблем нынешнего века и за ее решение назначена награда в один миллион долларов. В данной статье показано как получить собственные значения для системы в случае оси симметричной области.

Ключевые слова: система уравнений Навье-Стокса; метод Галеркина; многоточечное начально-конечное условие.

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