Statistical Mechanics of Vortices in Type-II Superconductors

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Abstract

Thermal fluctuations and disorder play an essential role in high-\(T_c\) superconductors. After reviewing the mean-field phase diagram we describe significant modifications that result when the effects of finite temperature and disorder are incorporated. Thermal fluctuations cause the melting of the Abrikosov flux line lattice into a vortex liquid at high temperatures, and disorder produces novel glass phases at low temperatures. We analyze phase transitions between these new phases and describe transport properties in various regimes.
1 INTRODUCTION

The microscopic mechanism of superconductivity in the new high-T$_c$ materials has remained a mystery despite much vigorous recent activity [1]. However significant progress in our understanding of the behavior of these materials has been achieved through phenomenological approaches. The phenomenological theory of superconductors was introduced in the 1950’s by Landau and has been successful in describing the conventional superconductors [2]. Many of the key properties and phenomena in high-T$_c$ materials can also be described entirely with the conventional Ginzburg-Landau theory, albeit with unconventional values of parameters. In this new range of parameters an understanding of the interplay of strong thermal fluctuations and disorder, both in statics and dynamics, is required.

Within a mean-field treatment, appropriate to conventional (low-T$_c$) type II superconductors, the Ginzburg-Landau theory leads to the celebrated Abrikosov vortex lattice [3] and the Meissner superconducting phases, with well-defined continuous transitions occurring at the upper and lower critical fields $H_{c2}(T)$ and $H_{c1}(T)$, respectively (see Fig.1(a)). However, unlike these conventional type II materials, where in the absence of disorder the flux lines always arrange themselves into an ordered, static triangular lattice, high-T$_c$ superconductors exhibit much richer vortex-line states. Because of the combination of high T$_c$ ($\sim 100$ K), short coherence length $\xi$, large penetration length $\lambda$, and large anisotropy, the Abrikosov lattice of flux lines is destabilized at finite temperatures by the thermal fluctuations and disorder.

The resulting magnetic field versus temperature (HT) phase diagram for high-T$_c$ materials is drastically modified by the fluctuations and disorder. In Fig.1(b) we have schematically illustrated a resulting phase diagram for the high-T$_c$ materials, with various new novel phases and regimes that have been proposed. The fluctuation and disorder-corrected phase diagram is remarkably rich and is to be contrasted with mean-field phase diagram characterizing the conventional superconductors, displayed in Fig.1(a).

At low temperatures, the translational order of the Abrikosov lattice is lost due to disorder that appears in the form of oxygen vacancies and interstitials, and grain and twin boundaries [4]. Vortex glass [5], Bose glass [6] and polymer glass [7] are some of the new low-temperature phases that have been proposed to replace the traditional Abrikosov flux lattice. Even in pure crystals thermal fluctuations have been proposed to lead to a novel supersolid vortex phase [8, 9], which unlike the 3d Abrikosov lattice, is a quasi-2d lattice with freely wandering interstitials. At even higher temperatures or higher fields the flux lattice melts and is replaced by a vortex liquid of highly mobile and flexible vortex lines [10, 11]. Motivated by recent experiments, in which a resistive “shoulder” was observed, proposals have been made that the vortex liquid phase could possibly be further subdivided into distinct liquid
phases, and can undergo transitions between hexatic \[12, 13, 14\] and isotropic, and entangled and disentangled vortex liquids \[10\].

To study the transport properties in these new novel phases the interaction between the transport current, the screening supercurrents and the magnetic field must be understood. These interactions lead to a Magnus-Lorentz force exerted on the vortex line perpendicular to the transport current. A motion of vortices in response to this force generates finite voltages and resistivity, and therefore leads to a breakdown of true superconductivity.

The Meissner phase, in which the flux is completely expelled, is fully superconducting up to a critical surface current density \(j_c\) beyond which thermal fluctuations can produce vortex loops, which expand under the influence of the transport current. In the ideal situation an unpinned vortex lattice can move as a whole in response to a uniform current and therefore is not truly superconducting. However, the vortex lattice has no resistive linear response for periodic current patterns with a finite wavelength because of the finite shear modulus of the lattice. Therefore the vortex lattice exhibits a novel fully nonlocal linear resistivity. Of course in practice the vortex lattice will have a superconducting linear response even to a uniform current, since the sample boundaries and even weak disorder will pin the lattice in place. Newly proposed vortex glass phases are also believed to be fully superconducting. The vanishing of linear resistivity in the Meissner and the putative glassy solid phases (vortex and Bose glass) is related to the divergence of energy barriers (with vanishing current) separating the ground state and the low-lying excited states.

In the vortex liquid phase the flux lines can more easily respond to the transport current-induced forces, although interactions with other lines are believed to lead to a much higher viscosity than that of the conventional isotropic liquids \[10, 7\]. A phenomenological hydrodynamic theory can be used to describe the resulting flux-flow resistivity which is nonlocal with properties intermediate between that of the fully superconducting vortex lattice and the normal phase \[15, 16\].

The mean-field phase transitions are also modified by strong thermal fluctuations. In contrast to low-\(T_c\) materials, in the \(CuO_2\) based superconductors there is no well defined thermodynamic transition at the upper critical field, \(H_{c2}\). The mean-field transition is replaced by a smooth crossover (indicated by a dashed line in Fig.4(b)) from the normal state to the non-superconducting, but more conductive vortex liquid regime. Therefore the distinction between the vortex liquid and the normal state is only quantitative. As \(H_{c2}^{MF}(T)\) is passed from above the resistivity decreases, the vortices begin to form as the fluctuations in the superconducting order parameter grow to a nonlinear regime. However, because of the motion of vortex lines, which destroys phase coherence, the average value of the superconducting order parameter \(\psi(\vec{r})\) is still zero in vortex liquid regime as it is in the normal state.

The real transition occurs only at lower fields and temperatures, where supercon-
ducting vortex solid phases form. For clean samples a first order freezing transition transforms the liquid into the vortex lattice. Despite the lack of a full theory of this 3d melting transition the location of the melting curve $B_m(T)$ can be estimated by the Lindemann criterion [10, 17, 18]. However, in the presence of disorder and strong vortex line interactions physical arguments have predicted a transition to the highly correlated fully superconducting isotropic and anisotropic (Bose glass) vortex glass phases. The vortex glass transition is believed to replace the melting transition of pure samples. Although no fundamental analytical description is available to date, this continuous glass transition can be analyzed using scaling theory that is in good agreement with transport experiments.

Recently, there have been several experimental observations of the transition into a glassy vortex phase that agree with predictions of the scaling theory [19, 20, 21, 22]. The impressive data collapse over several decades, from which the critical exponents characterizing the transition can be determined, provides good evidence for the existence of the vortex glass phase. This data and analytical analysis is further supported by various numerical studies [23, 24, 25].

In these lectures we will review some of the highlights of the phenomenology of high-$T_c$ superconductors. We begin in Sec. 2 by reviewing the picture that emerges from the mean-field analysis, describing the equilibrium and non-equilibrium properties of the resulting phases and the mean-field transitions. As we described above, effects of thermal fluctuations and disorder are very important and in Sections 3 and 4 we describe how these effects modify the HT phase diagram. After describing equilibrium properties of the new vortex phases, we proceed to analyze their transport properties. We apply a scaling theory to predict the form of the resistivity in the superconducting phases. In Sec. 5 we describe the nature of the fluctuation-corrected phase transition and again use scaling theory to summarize its properties in terms of a small set of universal critical exponents, which can be determined through numerical and experimental means. In each section we compare theoretical results with experiments and numerical studies, and for the most part find a coherent and consistent picture emerging.

2 MEAN-FIELD THEORY

Irrespective of the microscopic mechanism responsible for the attractive interaction that leads to the binding of the electrons into Cooper pairs, the Ginzburg-Landau theory postulates the existence of an order parameter, a complex scalar field $\psi(\vec{r})$ that is the condensate “wavefunction” of the bosonic Cooper pairs of electric charge $2e$. The long wavelength properties of the superconductors are then encoded in the Ginzburg-Landau free energy functional that is assumed to have a local expansion
in powers of $\psi(\vec{r})$ and its spatial gradients. Near the superconducting transition, $T_c$, the order parameter $\psi(\vec{r})$ is small, and it is sufficient to retain only the lowest order terms in the expansion of the free energy $[26]$

$$F_{GL} = \int d^3r \left[ \frac{\hbar^2}{2m_z} \left| \left( \frac{\partial}{\partial z} - \frac{i2e}{\hbar c} A_z(\vec{r}) \right) \psi(\vec{r}) \right|^2 + \frac{\hbar^2}{2m} \left| \left( \vec{\nabla}_\perp - \frac{i2e}{\hbar c} A_\perp(\vec{r}) \right) \psi(\vec{r}) \right|^2 + \alpha |\psi(\vec{r})|^2 + \frac{1}{2} \beta |\psi(\vec{r})|^4 + \frac{1}{8\pi} \left( \vec{H} - \vec{\nabla} \times \vec{A}(\vec{r}) \right)^2 \right].$$  

(1)

The first two terms are the supercurrent kinetic energy. The sum of the third and fourth terms is the local pairing energy, approximated at quartic order. The last term is the magnetic field energy that couples the external magnetic field to the magnetic induction $\vec{B}(\vec{r}) = \vec{\nabla} \times \vec{A}(\vec{r})$. Physically the parameters $m_z$ and $m$ in the above expansion can be identified with the effective mass of the Cooper pair along the $c$-axis (chosen in $z$-direction) and in the $ab$-plane. With the anisotropy parameter $\gamma \equiv \sqrt{m/m_z} \ll 1$ we can describe the large material anisotropy of the high-$T_c$ layered materials, which are much stronger superconductors in the $ab$-plane than along the $c$-axis $[27]$. The $\alpha$ and $\beta$ are the material parameters that depend on temperature $T$. Generally $\alpha(T) \approx a(\bar{T} - T_{cMF})/T \equiv t$ while the other parameters have only a smooth dependence on $T$ without sign changes. This dependence and the general validity of above description has been verified by Gorkov $[28]$ through his derivation of the Ginzburg-Landau theory from the microscopic Bardeen-Cooper-Schreiffer (BCS) theory of conventional superconductors $[2]$.

The mean-field description which neglects the effects of fluctuations is valid only outside the critical region around the normal-superconducting (NS) transition. For conventional, low-temperature bulk superconductors in zero magnetic field, this critical region is unmeasurably small with fluctuations only becoming important for $(T - T_c)/T_c < 10^{-7}$. On the other hand the combination of high-$T_c$, strong anisotropy, short coherence length $\xi$ and large magnetic penetration length $\lambda$ of high-$T_c$ materials expands the critical region by several orders of magnitude. As we will see, thermal fluctuations will lead to significant modifications of the mean-field description that we first present below.
2.1 Mean-field statics

The mean-field theory description is obtained by simply minimizing $F_{GL}[\psi, \vec{A}]$ with respect to $\psi(\vec{r})$ and $\vec{A}(\vec{r})$. The resulting mean-field phase diagram is displayed in Fig. 1(a), with three distinct phases. For $T > T_c$, i.e. $\alpha(T) > 0$ the pairing and magnetic terms are minimized by $\psi(\vec{r}) = 0$ and $\vec{B}(\vec{r}) = \vec{H}$ characteristic of the normal state. In the Meissner phase the $U(1)$ original symmetry of $F_{GL}$ ($\psi(\vec{r}) \rightarrow \psi(\vec{r})e^{i\phi}$) is spontaneously “broken” by

$$\langle \psi^*(\vec{r})\psi(\vec{r} + \vec{R}) \rangle \rightarrow \rho_s, \quad \text{for } |\vec{R}| \rightarrow \infty,$$

which leads to “off-diagonal” long-range order that sets in exponentially over the coherence length $\xi = \hbar/\sqrt{2m|\alpha|}$ in the $ab$-plane. Along the $z$-axis the coherence length is much shorter, $\xi_z = \gamma\xi$, scaled down by the anisotropy parameter, which can be as small as 1/100 in $BSCCO$.

Borrowing the notation from superfluids we associate the square of the condensate amplitude with the density of the superconducting pairs. It plays the role of stiffness for the phase $\phi(\vec{r})$ variations, as we will see below. The coherence length over which $\psi(\vec{r})$ varies (due to for example imposed boundary conditions) is set by the competition between the kinetic energy and the pairing energy and reflects the size of the Cooper pair.

The magnetic field $\vec{B}(\vec{r}) = 0$ is expelled everywhere in the sample except for the boundary where it leaks in a distance $\lambda = \sqrt{mc^2\beta/16\pi|\alpha|e^2}$ along the $ab$-plane. The superconducting screening currents along the $z$-direction are less effective in keeping the applied field from penetrating along the $c$-axis, with $\lambda_z = \lambda/\gamma$. The London penetration length $\lambda$ is set by the competition between the magnetic field term and the kinetic energy due to the screening currents.

For vanishing applied field the transition is in the same universality class as that of paramagnetic-ferromagnetic transition described by the XY model, with disordered paramagnetic state and spin aligned correlated state corresponding to the normal and Meissner states, respectively.

As was predicted by Abrikosov, for type II superconductors, upon increasing the applied field beyond the lower critical field $H_{c1}$ a type II superconductor undergoes a transition from the Meissner phase to the Abrikosov vortex lattice phase. In the vortex lattice phase the magnetic field penetrates the material and induces elementary
topological defects in the order parameter $\psi(\vec{r})$, which in three dimensions are vortex lines. In the Abrikosov phase the long-range order persists with $\psi(\vec{r}) = |\psi(\vec{r})| e^{i\phi(\vec{r})}$. However, instead of the phase being constant everywhere, it “winds” by $2\pi$ around a vortex line. The amplitude of $\psi(\vec{r})$ is nearly constant everywhere away from the vortex core but is substantially suppressed inside the core of size $\xi$ or $\xi_z$, for fields applied perpendicular to the $ab$-plane or $z$-axis, respectively. The magnetic flux per vortex is the flux quantum, $\phi_0 = \hbar c/2e$. When the vortices are well separated this flux is confined by circulating screening currents within a London penetration length $\lambda$ of the core. For an isolated straight flux line running parallel to an axis of cylindrical symmetry of the material the field is of the form.

$$\vec{B}(\vec{r}) \approx \begin{cases} e^{-r/\lambda} & \text{for } r \gg \lambda \\ -\ln(r/\lambda) & \text{for } \xi \ll r \ll \lambda \end{cases}.$$  \hfill (3)

For the most common situation of magnetic field applied along the $c$-axis a vortex line has structure on two length scales, $\xi$ and $\lambda$, and its crosssection in $ab$-plane is illustrated in Fig. 2.

In the Meissner phase and in the Abrikosov phase far away from the core the amplitude of the order parameter is a constant $|\psi(\vec{r})| = \sqrt{\rho_s}$ that minimizes the pairing energy. Only the phase $\phi(\vec{r})$ varies substantially. Using this Ansatz for the order parameter $\psi(\vec{r}) = \sqrt{\rho_s} e^{i\phi(\vec{r})}$ inside the Ginzburg-Landau free energy we obtain the London free energy which governs the phase and magnetic field variations,

$$F_L = \int d^3r \left[ \frac{\hat{\rho}_i^s}{2} \left( \frac{\partial \phi(\vec{r})}{\partial z} - \frac{2\pi}{\phi_0} \vec{A}_z(\vec{r}) \right)^2 \\
+ \frac{\hat{\rho}_s^x}{2} \left( \vec{\nabla}_\perp \phi(\vec{r}) - \frac{2\pi}{\phi_0} \vec{A}(\vec{r}) \right)^2 \\
+ \frac{1}{8\pi} \left( \vec{H} - \vec{\nabla} \times \vec{A}(\vec{r}) \right)^2 \right],$$  \hfill (4)

where $\rho^s$ and $\rho_z^s$ are the normalized eigenvalues of the superfluid density tensor.

$$\hat{\rho}_i^s = \left( \frac{\hbar^2}{2m_i} \right) \rho_s = \frac{\phi_0^2}{16\pi^3 \lambda_i^2},$$  \hfill (5)

where index $i$ labels $x, y, z$ and $\lambda_x = \lambda_y = \lambda$.

The ratio $\kappa = \lambda/\xi$ determines whether the superconductor is type I (for which the Abrikosov phase does not exist) or type II according to whether $\kappa < 1/\sqrt{2}$ or $\kappa >$
respectively. The high-$T_c$ $CuO_2$ materials are strongly type II, characterized by very large $\kappa$. At low temperatures the typical values of $\xi < 15\text{Å}$ and $\lambda > 1000\text{Å}$ leading to $\kappa > 100$.

The excess Gibb’s free energy per unit of length for putting in $N$ straight, noninteracting vortex/flux lines along the $z$-direction is given by

$$\delta F_{\text{line}} = N \left[ \left( \frac{\phi_0}{4\pi \lambda} \right)^2 \ln \kappa - \frac{H \phi_0}{4\pi} \right],$$

(6)

where $H$ is the $z$-component of applied field $\vec{H}$. This Gibb’s energy vanishes at the lower critical field $H = H_{c1}$

$$H_{c1} = \frac{\phi_0}{4\pi \lambda^2} \ln \kappa.$$

(7)

In mean-field theory both the coherence and the penetration length diverge as $T_c$ is approached from below, $\xi(T) \sim \lambda(T) \sim (T_c - T)^{-1/2}$. Since this leads to $\kappa$ that has a smooth behavior near $T_c$ we observe from Eq.(6) that $H_{c1}$ vanishes linearly as $T$ approaches $T_c$.

For $H_{c1} < H < H_{c2}$ vortices parallel to $\vec{H}$ begin to enter the sample with increasing applied field. The equilibrium number of lines is determined by the balance between the noninteracting contribution to Gibb’s free energy, Eq.(6), which is linear in $N$, and the contribution nonlinear in $N$, coming from the interaction between flux lines. The vortex-vortex interaction between two parallel straight lines separated by distance $r > \xi$ is repulsive and is given by

$$U(\vec{r}) = \frac{\phi_0^2}{8\pi \lambda^2} [K_0(r/\lambda) - K_0(r/\xi)] ,$$

(8)

where $K_0(x)$ is the modified Bessel function with the asymptotics,

$$K_0(x) \approx \begin{cases} (\frac{\pi}{2x})^{1/2} e^{-x} & \text{for } x \to \infty \\ -\ln(x) & \text{for } x \to 0 \end{cases}.$$

(9)

In Eq.8 we have introduced a short distance cutoff for $r < \xi$, with the interaction energy saturating at the value of excess energy for creating a doubly quantized vortex.

In the absence of thermal fluctuations the flux lines crystallize into a hexagonal Abrikosov vortex lattice, a 3-d crystal of infinitely long parallel vortex lines. The vortex lattice breaks continuous translational symmetry perpendicular to $\vec{H}$ and rotational symmetry about $\vec{H}$, as well as the global $U(1)$ symmetry. The flux line density $n$ is directly related to the average magnetic field $n \approx 1/a^2 = \langle \vec{B}(\vec{r}) \rangle / \phi_0$. Several values of lattice constants $a$, and the corresponding magnetic field strengths
Figure 3: Bitter pattern showing flux lines as they emerge from the surface of YBCO, at \( T = 4.2K \). The applied magnetic field \( H_{\perp} = 20 \) Gauss corresponds to 1\( \mu \)m separation between vortices. The very straight, somewhat smeared lines cutting across the lower two corners of the image are twin planes in the YBCO crystal. These twins locally orient the Abrikosov flux lattice. However, since the twins run at right angles to one another, the two resulting orientations of the hexagonal flux lattice are incompatible. This results in the grain boundary in the flux lattice which runs from the lower center of the picture toward the upper left corner.

are displayed in Table 1. For weak magnetic fields resulting in flux line separation \( a \gg \lambda, \xi \) the amplitude of the order parameter is \( \sqrt{\rho} \) and \( \vec{B} = 0 \) away from the vortex cores. However as \( H \rightarrow H_{c2} \), the amplitude \( |\psi| \) is suppressed while \( \vec{B} \) grows continuous between vortices, until at \( H_{c2} \) the vortex cores overlap and in the absence of fluctuations a mean-field transition occurs from the Abrikosov superconducting phase to the normal phase.

| \( B \)  | 20 Gauss | 2 kGauss | 20 Tesla |
|--------|----------|----------|----------|
| \( a \) | 1 \( \mu \)m | 1000\( \AA \) | 100\( \AA \) |

The existence of the Abrikosov vortex lattice for conventional superconductors has been demonstrated experimentally many years ago through various techniques that have now been applied to study vortex states in high-\( T_c \) superconductors [32, 33]. The lattice can be imaged in real space through the Bitter decoration technique in which fine magnetic dust shows the location of flux lines as they emerge from the surface of the superconductor. Magnetic neutron scattering off \( \vec{B}(\vec{r}) \) due to the flux line lattice is another technique that has been successfully used. Figure 3 clearly shows an image of flux pattern on the surface of \( YBCO \) at \( T = 4.2K \). A 20 Gauss field corresponding to 1 vortex/\( \mu m^2 \) was used to obtain this Bitter pattern [34]. Figure 4 shows the reciprocal-space image of the flux line lattice in \( NbSe_2 \), a strongly type-II layered low-\( T_c \approx 7K \) superconductor. The image was obtained via small angle \( \approx 1^\circ \) neutron scattering with the applied field of 8 KGauss at 5.2K. [35]

Also, recently, new revolutionary electron holographic techniques have been used to image the motion of flux lines in thin Pb films. This probe, which can image flux lines in real time, can be used to study mixed states also in high-\( T_c \) superconductors, and can provide valuable information about the statics and dynamics of vortex lines [36].
Figure 4: Reciprocal space image of the Abrikosov flux lattice, obtained using neutron small angle (\(\sim 1^\circ\)) scattering from NbSe\(_2\), a strongly type II low-\(T_c\) superconductor (\(T_c \approx 7K\)). The image was obtained with 8 KGauss magnetic field applied along the c-axis, at \(T = 5.2K\). The six hexagonally placed small spots are the magnetic Bragg scattering from the flux lattice. The large spot at the center is a nonmagnetic background that includes small angle scattering from imperfections in the sample. The scattering at the higher-order Bragg spots is too weak to see under these conditions.

Because we are interested in high-\(T_c\) superconductors we will consider strongly type II materials, \(\lambda \gg \xi\), which means that \(\vec{A}(\vec{r})\) is much “stiffer” than \(\psi(\vec{r})\), corresponding to low carrier density or large effective mass. For \(H \gg H_{c1}\) the \(\vec{B}\)-field of a vortex overlaps many other vortices, and one can often make a uniform field approximation, \(\vec{B}(\vec{r}) = \vec{B} = \text{constant}\), and concentrate on the dynamics and variations of \(\psi(\vec{r})\) only. This regime can be formally obtained by taking limits \(e \to 0\) (\(\phi_0 \to \infty\)) and \(H \to \infty\) with fixed vortex density \(n = \langle \vec{B} \rangle / \phi_0\). This limit corresponds to uncharged rotating superfluid \(^4He\).

### 2.2 Mean-field dynamics

The mean-field dynamics within the infinite penetration length approximation is assumed to be accurately described by the time-dependent Ginzburg-Landau theory,

\[
\frac{\partial \psi(\vec{r}, t)}{\partial t} = \frac{i2eV(\vec{r}, t)}{\hbar} \psi(\vec{r}, t) - \Gamma \frac{\delta F_{GL}}{\delta \psi^*(\vec{r}, t)} ,
\]

where for simplicity we have left out the Hall effect contribution. The first term in (10) is the Josephson term which leads to “rotation” of \(\psi(\vec{r}, t)\) in the complex plane as the voltage gradient \(\vec{E}(\vec{r}, t) = \nabla V(\vec{r}, t)\) accelerates the supercurrent \(\vec{j}_s(\vec{r}, t) = \delta F_{GL}/\delta \vec{A}(\vec{r}, t)\)

\[
\vec{j}_s = Re \left[ \frac{2e\hbar}{m} \psi^*(-i\nabla - \frac{2e}{\hbar c} \vec{A})\psi \right] ,
\]

\[= \rho_s(\nabla \phi - \frac{2e}{\hbar c} \vec{A}) .
\]

We observe that the supercurrent is proportional to the net phase gradient. In the spirit of the two-fluid model of superfluids the normal current is simply,

\[
\vec{j}_n(\vec{r}, t) = \sigma_n \vec{E}(\vec{r}, t) ,
\]

where \(m\) and \(\sigma_n\) must be replaced by their tensor equivalents for transport in an arbitrary direction, reflecting the intrinsic material anisotropy [27, 37].
Figure 5: Illustration of the origin of the Magnus force, that the transport current $\vec{j}$ (shown flowing in the $xy$-plane) exerts on the vortex line emerging from the plane.

In the Meissner phase there are no vortices present, and therefore the phase is a true linear superconductor below critical current density $j_c$. The transport current is however confined within a penetration length of the surface, $\vec{j}_s(\vec{r}) \sim e^{-r/\lambda}$, which limits the potential applications. For $j_s > j_c$ the resulting surface $\vec{B}$ field exceeds the local $H_{c1}$, vortices begin to enter the sample. The ensuing vortex motion (in response the current-induced forces) generates dissipation and breaks down superconductivity as we explain in more detail below for the Abrikosov phase.

To understand transport properties of the superconductor in the presence of vortices we must first examine the interaction between the supercurrent and the vortex line. The nature of this interaction is still a topic of much current research $[38]$. It is generally believed that there are two sources of interaction. One is the Lorentz interaction of the supercurrent $j_s$ with the magnetic flux carried by the vortex line, resulting in the force $\vec{f}_L = \frac{\Phi_0}{c} \vec{\tau}(z) \times \vec{j}$, where $\vec{\tau}(z)$ is a tangent unit vector to the vortex line. Since we are working in the large magnetic penetration length regime of nearly uniform $\vec{B}$, the Lorentz interaction should not be very important. A more significant contribution in this regime is the interaction of the transport current with the screening supercurrents around the vortex. As is illustrated in Fig. 5, for the counterclockwise supercurrent, currents add below and subtract above the vortex resulting in higher and lower kinetic energy, respectively. The vortex therefore experiences a Magnus force (analogous to the force that gives an airplane its lift) $\vec{f}_M = \frac{\Phi_0}{c} \vec{\tau}(z) \times \vec{j}$ which moves it perpendicular to $\vec{j}$, thereby reducing the kinetic energy and the phase difference.

The evolution of the net phase difference between two points can be obtained from the dynamic Ginzburg-Landau equation $(10)$ by assuming that the amplitude of $\psi(\vec{r})$ is constant,

$$\frac{d\Delta\phi}{dt} = \frac{2e}{h} A - 2\pi \times (\text{net rate at which vortices pass between the points}) . \quad (14)$$

To maintain steady state therefore requires an applied voltage difference,

$$\Delta V = \frac{h}{2e} \times (\text{net rate at which vortices pass by}) , \quad (15)$$

which is an expression of the Josephson effect.

The vortex motion in response to the uniform transport current is viscously damped by interaction of normal electrons in the vortex core with the ions, which in
Figure 6: Illustration of vortex lattice distortion due to the presence of non-uniform currents. As described in the text, since the total force and torque vanish, the lattice distorts but does not move as a whole. This therefore illustrates the vanishing of linear resistivity at finite wavevectors.

\[ \vec{j}(\vec{r}) = \int d^3r e^{i\vec{k} \cdot \vec{r}} \vec{j}(\vec{k}) , \]

(17)

and ask about the electric field response due to such a plane wave of current,

\[ \hat{E}_j(\vec{k}) = \hat{\rho}_{ij}(\vec{k})\hat{j}_i(\vec{k}) , \]

(18)

where the summation over spatial indices is implied.

As we already mentioned for the current parallel to the vortices,

\[ \hat{\rho}^{||}_{ij}(\vec{k}) = 0 . \]

(19)

The current perpendicular to the vortices couples to “phonon” modes of the vortex lattice, imposing static elastic distortions but no steady motion for \( \vec{k} \neq 0 \). Therefore
we obtain a highly nonlocal resistivity with $\mathcal{E}(\vec{r})$ determined by $\vec{j}(\vec{r'})$ over the entire sample.

$$\hat{\rho}^\perp_{ij}(\vec{k}) \sim \delta(\vec{k}).$$

(20)

We therefore conclude that in the absence of fluctuations, the vortex lattice superconducts except for $\vec{k} = 0$ and currents perpendicular to the vortices. Since in the normal state the resistivity is fully local, dissipating at all wavevectors $\vec{k}$, $\hat{\rho}_{ij}(\vec{k}) = \rho_n$, in mean-field theory $\rho_{ij}$ is discontinuous at the NS transition (at $H_{c2}(T)$).

3 THERMAL FLUCTUATIONS

We now examine how the mean-field results discussed in the previous sections are modified by thermal fluctuations. In low temperature superconductors these fluctuations are not very important except very close to phase boundaries and in thin films or wires. The small size of the critical region in bulk low-$T_c$ materials is associated with the fact that the thermal length

$$\Lambda_T = \frac{\phi_0^2}{16\pi^2k_B T} \approx \frac{2 \times 10^8 \AA}{T},$$

(21)

is much larger than any other length in the problem except very close to $T_c$ where other lengths diverge.

For the high-$T_c$ superconductors simple considerations also allow us to construct the “Ginzburg criterion”, which determines when the effects of thermal fluctuations become important. This occurs when the ordering condensation energy in a coherence volume is equal to the thermal disordering energy, $\alpha^2 \xi^2 \xi_z / 2\beta \approx k_B T$. Stated equivalently in terms of material parameters,

$$\lambda \approx \frac{\gamma \phi_0^2}{48\pi^3k_B T \kappa} \Lambda_T = \frac{\gamma \Lambda_T}{4\pi \kappa}.$$  

(22)

We therefore observe that for the highly anisotropic high-$T_c$ superconductors the effective thermal length is reduced by more than 4 orders of magnitude by the combination of high $T_c$, large anisotropy $\gamma^{-1} \approx 10^2$ and large $\kappa \approx 10^2$. This therefore greatly expands the critical region in which mean-field theory no longer applies and fluctuation-corrected theory must be used.

We first consider fluctuations in the ideal case of disorder-free superconductors and then go on to study the effects of disorder at finite temperatures.

3.1 Disorder-free samples

The finite temperature effects can be incorporated into the statics by summing over all the configurations of the system weighted by Boltzmann weight $e^{-F_{GL}/k_BT}$, instead of
just simply minimizing the Ginzburg-Landau free energy $F_{GL}[\psi, \vec{A}]$, Eq.1. The effects of thermal fluctuations can equivalently be incorporated in the dynamic Ginzburg-Landau equation (10) by the addition of white noise $\zeta(\vec{r}, t)$ of intensity proportional to $k_B T$, as set by the fluctuation-dissipation theorem.

Near the phase boundaries the fluctuations are most effective in disrupting the ordered phases, because there the stiffness of the order parameter vanishes. For example, for the Meissner phase $\rho_{c2}^{MF}(T) \sim (T_{c2}^{MF} - T)$. Similarly, for the vortex lattice phase the shear modulus plays the role of the stiffness, which vanishes near the $H_{c2}^{MF}$ as

$$\mu(H) \sim \frac{(H_{c2}^{MF} - H)^2}{\lambda^2}.$$  \hfill (23)

The idea that the flux line lattice in high-$T_c$ materials might melt into a liquid ($\mu = 0$) of freely moving vortex lines was first suggested by Nelson [10]. Recently there has been much experimental evidence that in fact a large portion of the Abrikosov lattice is replace by the vortex liquid, all the way up to the mean-field $H_{c2}^{MF}$ boundary where the superconductor crosses over to normal behavior [11] (see figures 1(b), 4). Although a full theory of the melting transition does not exist, an estimate of the location of the melting boundary can be made using the Lindemann criterion. Much work has been directed to accurately describe the melting boundary, carefully taking into account the nonlocality of the elastic moduli of the vortex lattice [17, 18]. However to avoid drowning in technical details we illustrate the idea ignoring the wavevector dependence of the elastic moduli.

As is true for real ionic lattices, the phonon modes of the vortex lattice are thermally excited with mean elastic energy $k_B T/2$. The mean square displacement of a vortex is a rough measure of the disruption of crystalline order by thermal fluctuations,

$$\langle |\vec{u}(\vec{r}, t)|^2 \rangle = \int d^3k \langle |\vec{u}(\vec{k}, t)|^2 \rangle ,$$

$$\sim \int_{k \sim 1/a, k_z \sim 1/\xi_z} d^3k \frac{k_B T}{\mu k^2} ,$$

$$\sim \frac{\lambda^2 k_B T (H_{c2}^{MF})^2}{\xi_z (H_{c2}^{MF} - H)^2} .$$

As expected the displacement increases with temperature and as $H_{c2}^{MF}$ is approached. Melting occurs when the displacement becomes some significant fraction of the lattice spacing $a$,

$$\langle |\vec{u}(\vec{r}, t)|^2 \rangle^{1/2} \approx c_L a ,$$

where $c_L \approx 0.1$ is the empirically determined Lindemann ratio.
The resulting vortex liquid has local short-scale superconductivity, but the highly mobile disordered vortices do not allow global phase coherence to set in. In this sense the vortex liquid is the same disordered thermodynamic phase as the normal state, exhibiting only the quantitative difference of having much larger fluctuations of the superconducting order parameter. In both of these normal-state regimes $\langle \psi \rangle = 0$, unlike in the truly superconducting Abrikosov lattice and Meissner phases. At $H_{c2}^{MF}$ a gradual crossover occurs as well-defined vortices form, and conductivity and diamagnetism increase due to the local superconductivity. However, no thermodynamic singularity occurs since no symmetry is broken as $H_{c2}^{MF}$ is crossed.

The interaction between flux lines becomes very weak and falls off exponentially for separations $r > \lambda$. Recently it has been argued that because of the weak interactions, at weak fields a sliver of vortex liquid will also intrude right above $H_{c1}$ [10, 5], as is illustrated in Fig. 1(b). However the verification of the existence of the flux-line liquid in this portion of phase diagram has up to now alluded experimentalists.

A variety of liquid phases have been suggested such as the entangled and disentangled vortex liquid [10] as well as the intermediate hexatic phase [12, 13] characterized by the quasi-long-range orientational order and short-range translational order in analogy with the 2d-melting theory [14]. Recently, there have also been suggestions that in layered superconductors there is a possibility of two thermodynamically distinct vortex lattice phases. One is the conventional 3d Abrikosov lattice with vanishing linear resistivity. The second is a quasi-2d vortex lattice phase, with finite linear resistivity, also known as the supersolid phase [8, 9]. In Fig. 7 we have presented a schematic phase diagram showing the expected location of these novel vortex phases.

In the $\langle B(\vec{r}) \rangle$ versus $T$ phase diagram of Fig. 1, the Meissner phase is collapsed down to the $T$-axis. The nature of the novel hexatic and entangled liquid phases has been extensively discussed [10, 12, 13]. We now briefly describe the less known putative novel quasi-2d lattice phase.

In a usual molecular crystal an interstitial is a point defect that costs a finite amount of energy $E_i$ and in equilibrium exists at densities $\sim e^{-E_i/k_BT}$. On the other hand in a three-dimensional vortex lattice a vortex cannot end inside a superconductor (except on a magnetic monopole, which is highly unlikely) so an interstitial is a line defect with a constant free energy per unit of length. Its free energy is therefore proportional to the thickness of the sample. Since in the thermodynamic limit this defect energy diverges, at equilibrium we expect no interstitials in the vortex lattice.
and magnetic flux per unit cell is exactly $\phi_0$. This situation is to be contrasted with thin film superconductors where the interstitials and vacancies are point defects, of finite energy and therefore are present in finite density at finite temperature. Concomitantly the flux per unit cell does not equal to $\phi_0$, and hence the number of flux points per unit cell fluctuates as the interstitial and vacancies move around.

The crossover between the 2$d$ and 3$d$ behavior in layered $CuO_2$ superconductors is described by the Lawrence-Doniach model [43] with free energy,

$$F = \sum_i \left\{ F_{2d} [\psi_i(\vec{r})] - \int d^2r \text{Re} [J \psi_i^*(\vec{r}) \psi_{i+1}(\vec{r})] \right\},$$

(28)

where we have taken $\vec{B}$ to be perpendicular to layers. The first term is the free energy of independent 2$d$ layers labeled by index $i$, and the second term gives local coupling between adjacent layers.

At $J = 0$, the 2$d$ uncoupled layers allow for vacancies and the interstitials with entropy dominating over energy. As the coupling is increased at some critical value $J_c = J(T_c)$ the interstitial free energy per unit of length (tension) becomes positive, the system makes a transition to a 3$d$-dominated behavior and the net interstitial density vanishes. Since the effective interlayer coupling is actually $\hat{J} = J a^2 \sim J/\langle B \rangle$ the system becomes more 2$d$-like at higher fields. At large fields (small $\hat{J}$) the interstitials form a highly flexible and mobile line-liquid existing inside a 3$d$ vortex lattice with a true crystalline order. Since the interstitials are free to move, this supersolid is not superconducting for the reasons explained in previous sections. The resulting phase diagram is illustrated in Fig. 7 where we have plotted $\langle B \rangle$ versus $T$.

### 3.2 Nonlocal resistivity at finite temperature

As we already discussed in Sec.2.2, the transverse flux-flow resistivity of a vortex lattice is highly nonlocal,

$$\bar{\rho}(\vec{r}) = \int d^3r' \rho(\vec{r} - \vec{r}') \vec{j}(\vec{r}'),$$

(29)

and in fact in the absence of plastic flow (possible in very clean samples) we expect $\rho_{\text{lattice}}(\vec{r}) \sim 1/\text{volume}$. This form is to be contrasted with the completely local resistivity in the normal state, $\rho_{\text{normal}} \sim \delta^{(3)}(\vec{r})$. In the presence of thermal fluctuations the flux-line lattice melts over a large portion of the HT phase diagram. The resulting vortex-line liquid consists of flexible, highly mobile flux lines that can easily respond to the transport current and therefore generate finite flux-flow resistivity. Unlike the vortex lattice, in the vortex/flux-line liquid phase there is no perfect correlation in the motion between different, separated vortex lines. As the lines traverse the sample
along the applied magnetic field (say in the z-direction) in a distance \( z \) they will wander transversely a distance \( (zk_BT/\hat{\epsilon})^{1/2} \), where the line tension \( \hat{\epsilon} = \phi_0 H_{c1}/4\pi \). Even in thin samples a typical line will deviate away from its average direction much further than the average interline separation set by the applied magnetic field (see Table 1). Therefore, the flux lines experience many collision with their neighbors entangling with many distant ones, much like directed polymers in a solution. Although no rigorous calculation of the flux-line crossing barrier energy is available, simple estimates give \( E_x \approx 50k_BT \) [10], far away from \( H_{c2} \), where it is expected to vanish as \( \sim \ln[H_{c2}^{MF}/H]/\ln[H_{c2}^{MF}/H_{c1}] \) [7]. We expect that these frequent flux-line interactions and entanglements will lead to a large transverse viscosity of the vortex-line liquid [16] which replaces the perfect correlations controlled by the finite shear modulus in the Abrikosov lattice. Besides these transverse viscous correlations, the connectivity of the vortex lines leads to an even stronger nonlocality in the viscosity of the line-liquid along the direction of the applied field. Since the viscosity is a measure of the response of velocity to an applied force, it is directly related to the flux-flow resistivity. The flux-flow resistivity of vortex liquid should therefore exhibit a nonlocal behavior in-between that of a completely local one of the normal state and a completely nonlocal one of the vortex lattice. As the superconductor is cooled into the vortex liquid, \( \rho(\vec{r}) \) becomes more and more extended, especially parallel to vortices.

The nonlocality of the resistivity in the liquid phase has been recently studied experimentally [44, 45, 46] and theoretically [47] in the geometry depicted in Fig. 8. The transport current at the top surface of the superconductor exerts a transverse force on top ends of vortex lines. Since the vortex lines are well connected and rarely break, the current at the top of the sample pulls on the whole line. This results in the vortex motion and therefore electric field far away from the region where the current actually flows (top). Since the voltmeter measures the average rate at which vortex lines pass between the two contacts, excluding the possibility of line breaking we have \( V_{top} = V_{bottom} \), with the vortex velocity proportional to the total current integrated from the top to the bottom of the sample. This situation is to be contrasted with the normal state local resistivity where the \( V_{top} > V_{bottom} \) because current density is higher at the top. In experiments of Safar, et al. [46] on 35µm thick \( Y_1Ba_2Cu_3O_7 \) crystal, in a regime where vortices have a good integrity over a distances larger than the sample thickness, the effect of nonlocal resistivity has been observed.

Supposing that the Fourier transform of the conductivity is an analytical function
of \( \vec{k} = (\vec{k}_\perp, k_z) \) for small \( k \). Nonlocal resistivity of the vortex liquid can be phenomenologically described as

\[
\rho(\vec{k}) \sim \frac{1}{\gamma + \eta_{ab}|\vec{k}_\perp|^2 + \eta_z k_z^2},
\]

where \( \gamma \) is the local friction (related to the local conductivity of Bardeen and Stephen), and \( \eta_{ab}, \eta_z \) are the vortex liquid viscosities leading to nonlocality in the flux-flow resistivity. This model can be used to solve for current and voltage patterns in a realistic experimental geometry by solving 4th order partial differential equations.

### 3.3 Static disorder

In real high-temperature superconductors disorder plays an essential role. As we have already briefly described, it modifies the HT phase diagram, but more importantly, the disorder tends to pin vortices in place, drastically reducing the flux-flow resistivity, as we now describe in more detail.

The most important contribution of static disorder leads to a spatially varying transition temperature, \( T_c(\vec{r}) \). The disorder effects can therefore be incorporated into the Ginzburg-Landau description by allowing the \( \alpha \) parameter to be a function of position,

\[
\alpha(\vec{r}, T) \approx a(\vec{r}) \left( T - T_c^{MF}(\vec{r}) \right),
\]

where the properties of the function depend on the nature of disorder. Chemical and structural imperfections in the material naturally lead to random \( \alpha(\vec{r}) \) with only short-range correlations. Correlated random static disorder is also possible and might arise from grain or twin boundaries, screw dislocations, artificially created ion columnar tracks \[43, 50\], or from an epitaxial multilayers and e-beam written patterns. Finally the crystal structure itself, such as for example the \( CuO_2 \) layers of the high-\( T_c \) superconductors, provides a natural modulation of \( T_c(\vec{r}) \) describable by a periodic \( \alpha(\vec{r}) \).

Since the core of a vortex line is in a normal state, the energy is minimized when the vortex is located in the region of weakest superconductivity, where \( \alpha(\vec{r}) \) is greatest and minimum amount of condensation energy is lost. This leads to pinning of a vortex line by normal impurities, dislocations, grain/twin boundaries and to the intrinsic pinning in between the \( CuO_2 \) planes (where the superconductivity is weakest).

The phase of the wavefunction cannot be measured and therefore the disorder cannot directly couple to the phase \( \phi(\vec{r}) \). We therefore conclude that the Meissner phase is stable to disorder, except near its phase boundaries.

Both the strong and weak pinning limits have been previously studied. In the strong pinning limit the pinning energy exceeds the vortex-vortex interaction energy
and leads to a fully disrupted vortex lattice. The Ginzburg-Landau free energy is minimized by a random complex function $\psi(\vec{r}) = \psi_0(\vec{r}) e^{i\phi(\vec{r})}$.

It has been known for some time that the Abrikosov phase is unstable to introduction of disorder [4] and the long-range translational order is destroyed beyond a disorder-determined Larkin-Ovchinnikov length $L_{LO}$. It has been recently proposed that the resulting new low temperatures and fields phase possesses long-range order with a sample and disorder-specific random vortex pattern. For the case of point (uncorrelated) disorder the equilibrium phase is the vortex glass with the name chosen by the analogy with spin glass, where the ground state order parameter is random but frozen with long-time correlations [51, 5]. For correlated disorder such as columnar pins, an anisotropic version of the vortex glass has been proposed and dubbed Bose glass [6] by the isomorphism of the resulting theory with bosons on random substrate [52]. Here we will confine our review to the isotropic vortex glass, with the analysis easily extendible to Bose glass.

The stability of the vortex glass phase to thermal fluctuations depends on the nature of the low-lying states. By analogy with the scaling theory of spin glasses [53], the excitations are “bubbles” inside a ground state and correspond to the displacement of one or more vortices a distance $L$ in a region of linear size $L$. The average energy of the low-lying excitation is expected to scale with a power of $L$

$$E_e \sim \Upsilon L^\theta.$$  
(32)

Since for a circular loop, $\theta_l = 1$, characteristic of excitations in the Meissner phase, we expect that $\theta \leq 1$.

The stability of the vortex glass phase clearly depends on the sign of the $\theta$ exponent. If $\theta < 0$ then the large-scale excitations cost very little energy and will be present at $T > 0$ at equilibrium. This turns out to be the case in two dimensions ($d = 2$). The correlation length then scales as a power law with temperature

$$\xi_{VG} \sim T^{-1/|\theta|},$$  
(33)

with the long-range order only present strictly at $T = 0$.

In three dimensions no satisfactory analytical analysis is available to date, primarily because of the difficulty associated with treating the dislocations (although some arguments have proposed that the dislocations might be irrelevant). Numerical studies of highly simplified models treat the 3d vortex glass phase in the limit $\lambda \to \infty$ and find $0 \leq \theta \leq 0.3$ [23, 24, 25]. These works point to the stability of the vortex glass phase in $d = 3$, for $T > 0$.

The stability of the vortex glass phase in three dimensions to thermal fluctuations in the presence of screening ($\lambda$ finite) is still an open question. Since the interactions crucial for the formation of the vortex glass phase become exponentially weak for
$r > \lambda$, it appears that screening tends to weaken the stability of the vortex glass. Experimentally this regime is only probed when $\xi_{VG} > \lambda$ which corresponds to resistivities well below those studied in experiments to date.

In the case of weak disorder a topological glass phase might result, with properties intermediate between that of the vortex lattice and the vortex glass. As we have seen vortex glass is a phase highly correlated in time, but spatially it looks like a frozen vortex liquid with many dislocations present. In the vortex glass the strong disorder destroys the topological order and dislocations proliferate. On the other hand, in the absence of disorder, the vortex lattice has both positional and orientational long-range order and is therefore dislocation-free at large scales. It is possible that in the presence of weak disorder an intermediate topological glass forms in which the topological order of the vortex lattice survives.

The resulting topological glass phase is then describable as a dislocation-free elastic medium (lattice) in a random potential. The displacement of a vortex from a position $\vec{r} = (x, z)$ due to disorder is described by $u(\vec{r}) \perp \hat{z}$, with vortices parallel to $\hat{z}$. The Hamiltonian in $d$ dimensions is then given by

$$H = \int d^d r \, \vec{\nabla} u(\vec{r}) \cdot \vec{E} \cdot \vec{\nabla} u(\vec{r})$$

$$+ \sum_i \int dz \, V(\{x_i + u(x_i, z)\}, z),$$

(34)

where $x_i$ is the $d - 1$ dimensional position of the $i$-th line in the space transverse to $\hat{z}$ and the sum is over discrete vortex lines. The first term, above, is the elastic energy, quadratic in the displacements $u(\vec{r})$, described in terms of the elasticity tensor $\vec{E}$. The second term is the contribution due to the pinning disorder, described by a random potential $V(\vec{r})$ with spatial correlations that depend on the type of disorder.

The ground state without dislocations has been studied analytically and is characterized by displacement correlations that are logarithmic in position

$$|u(\vec{r}_1) - u(\vec{r}_2)|^2 \sim \ln |\vec{r}_1 - \vec{r}_2|,$$

(35)

for large $|\vec{r}_1 - \vec{r}_2|$. Weak disorder destroys long-range positional order replacing the Bragg peaks by power-law singularities characteristic of the quasi-long-range order. Because of the weakness of the logarithmic growth of displacements, the dislocation density vanishes and the topological order remains, consistent with the assumption of the model. Full long-range orientational order, however, still survives in this randomly pinned vortex lattice.

For the topological glass to exist as a distinct phase from the strongly disordered vortex glass it must be stable to dislocations. To establish this stability we look
Figure 9: Illustration of a domain wall of nonlinear distortion resulting from shifting a patch of topological vortex glass by a lattice vector $\mathbf{a}$. The single valued vortex displacement $\mathbf{u}_i(\vec{r})$ inside the wall differ from the distortions outside the patch, $\mathbf{u}_o(\vec{r})$ by $\mathbf{a}$. The displacement field jump of $\mathbf{a}$ is concentrated on the domain wall.

Figure 10: An addition of a dislocation line together with a negative energy domain wall might lower the energy of the topological glass, depending on the value of the $\theta_w$ exponent. This would result in distabilization of the topological glass toward a fully disorder vortex glass.

at the low-lying excited states that disrupt the order of the topological glass. An appropriate low-energy excitation is a patch of size $L$ moved over by one lattice spacing $\mathbf{a}$ with respect to the rest of the vortex lattice. The elastic nonlinear distortion is $\mathbf{u}_o(\vec{r})$, outside the patch and $\mathbf{u}_i(\vec{r})$ inside the patch, with the displacement jump $|\mathbf{u}_i(\vec{r}_{wall}) - \mathbf{u}_o(\vec{r}_{wall} + \mathbf{a} \hat{n})| = \mathbf{a}$, concentrated on the wall bounding the patch (see Fig. 9). The elastic energy is unchanged in the interior and exterior of the patch. The average total energy is increased due to the elastic strain by $\mathbf{a}$ and for optimized wall position scales as

$$E_e(L) \sim \Upsilon L^{\theta_w}.$$  \hspace{1cm} (36)

Since the elastic energy is concentrated on the wall, the average total energy can at most scale as the area of the wall, which puts an upper bound on $\theta_w$,

$$\theta_w \leq 2,$$  \hspace{1cm} (37)

where the optimization of the wall position can lower $\theta_w$ below 2.

Although the total average energy of the wall is positive, large sections of the wall can have a negative energy that scales in the same way as the average energy in Eq. 36. A section of the wall with negative energy can be created by an addition of a dislocation line, as illustrated in Fig. 10. Upon encircling the dislocation line $\mathbf{u}_i - \mathbf{u}_0$ must change by a Burger’s vector $\mathbf{b}$, a change that is concentrated on the domain wall. Therefore the energy of adding a dislocation equals to the local core energy plus the wall energy. On scale $L$ the system might prefer to create a dislocation line with the negative energy domain wall compensating the core energy cost

$$E_{\text{dislocation}} \sim L - \Upsilon L^{\theta_w}.$$  \hspace{1cm} (38)

Hence we expect that the system can lower its energy by adding dislocations on large scales if $\theta_w > 1$ and the topological glass phase will always be unstable to the vortex glass without topological order. On the other hand if $0 < \theta_w < 1$ the topological glass phase should be stable against dislocations for weak disorder. To date
the precise range into which $\theta_w$ falls, and therefore the stability of the topologically
ordered, dislocation-free vortex glass phase is an open question.

4 NONLINEAR RESISTIVITY IN SUPERCONDUCTING PHASES

Vanishing linear resistivity is probably the most natural and practical distinction
between a superconducting and normal phase. As we have already seen, the vortex
liquid has nonlocal but nonvanishing linear resistivity. On the other hand a vortex
lattice can be easily pinned with few strong inhomogeneities or even naturally pinned
by the boundaries. In this pinned state we have seen that the vortex lattice has a
true superconducting response to a uniform current.

Conventional theory of transport in disordered vortex states, based on the ideas
of Anderson and Kim [56], describes the resistivity in terms of thermally activated
motion of independent vortex bundles over the finite energy barriers $V_p$, introduced
by the disorder [57, 58]. The size of the bundles is assumed to be finite, on the or-
der of Larkin-Ovchinnikov length $L_{LO}$, beyond which the lattice order is destroyed
by disorder [4]. The transport current introduces an average tilt to the random
landscape potential and therefore an overall thermal drift of vortex bundles in the
direction of the tilt. The vortex drift destroys the phase coherence of superconduct-
ing wavefunction, resulting in finite vortex-flow linear resistivity of Arrhenius form,
$\rho_l \sim \exp(-V_p/k_BT)$. Therefore this conventional theory maintains that at finite tem-
peratures, the disordered low temperature vortex phase is not really superconducting,
so is qualitatively identical to the non-superconducting vortex liquid regime.

The difference between the conventional picture and the transport in the vor-
tex glass phase comes from the strong inter-vortex interactions not included in the
Anderson-Kim picture. Upon cooling a sharp thermodynamic transition takes place
from a non-superconducting vortex liquid to a distinct superconducting vortex glass
phase [13]. The vortex motion in the vortex glass phase is still described by the
thermally activated motion, except that the strong correlations lead to bundle and
therefore barrier sizes that diverge as $j \to 0$. We now examine the vortex glass
phase along with Meissner phase, and show that these phases possess only nonlinear
resistivity and are therefore true linear superconductors.

In the Meissner phase it is the nucleation and growth of vortex loops, and in the
vortex glasses phases it is the collective motion of already present vortices in response
to a transport current that are the excitations that lead to energy dissipation. Since
the motion of a bundle of vortices can be described by the creation and growth of
vortex loops, in both phases the low-lying excitations are vortex loops superimposed
onto the ground state background of each phase. We can use the scaling theory of
these low-lying excitations to analyze the resistivity both in the flux-free Meissner phase and the frozen vortex glass phase.

For the well understood Meissner phase we know the nature of the excited states (vortex loops) and therefore can predict the exponents in addition to the scaling form of resistivity. Unfortunately, for the vortex glass neither the ground state nor the low-lying excited states are well understood. This lack of understanding however, can be neatly packaged into a set of exponents that enter the scaling theory of the nonlinear resistivity. Some attempts have been made to apply a collective pinning theory to calculate these exponents. However, these theories ignore the existence of dislocations and therefore breakdown of long-range translational order on length scales $L > L_{LO}$ [59]. It is likely that dislocations will have an affect on the value of these exponents.

As was described in the previous section, the energy of low-lying excitations scales as $\Upsilon L^\theta$, with $\theta$ being specific to the type and shape of the excitations and the phase-background in which they are created. For example for the isotropic Meissner phase the lowest energy excitations are circular loops characterized by superfluid density stiffness $\Upsilon \sim \rho_s$ and $\theta = 1$. The transport current-generated Magnus force couples to the area $A$ projected onto the plane perpendicular to the current

$$A(L) \sim A_0 L^\tau.$$ (39)

The exponent $\tau$ has a lower bound of 2, saturated by a circular loop in the Meissner phase with $A_0 = 1$ and $\tau = 2$. The energy supplied by the Magnus force is

$$E_c(L) \sim f_M A(L),$$

$$\sim j L^\tau,$$ (41)

where $j$ is the transport current density. Balancing the elastic energy of the excitation with the current energy we obtain the average size $L_j$ of the lowest energy excitation

$$L_j \sim j^{-1/(\tau-\theta)},$$ (42)

that are produced when $\tau > \theta$.

In order to reach these low-lying states the system might have to overcome barriers by moving through less favorable regions. In general the size of the energy barriers will scale as

$$E_b = \Delta L^\psi,$$ (43)

where for loops in the Meissner phase $\psi = \theta = 1$. In general, however, the barriers are usually larger than the final energy state, and therefore $\theta$ is a lower bound for $\psi$, i.e. $\psi \geq \theta$. These two energy scales are schematically depicted in Fig.11.
Figure 11: Local random potential due to impurity disorder is characterized by the typical size of its barriers, scaling as $L^\psi$, and by the energy difference between ground state and next low-lying state, scaling as $L^\theta$.

The rate of energy dissipation via the production of the low-lying excitations is governed by the Arrhenius law $\sim \exp\left[-\Delta L^\psi_j/k_B T\right]$. The resulting nonlinear resistivity due to the vortex motion over the barriers is

$$\frac{\mathcal{E}}{j} = \rho_{nl}(j) \sim \exp\left(-\frac{c}{k_B T j\mu}\right),$$

where $\mu = \psi/(\tau - \theta)$. The excitations with the smallest $\mu$ dominate with the upper bound again set by the vortex loops as in the Meissner phase where $\mu_{\text{Meissner}} = 1$. We immediately observe that for $\tau > \theta$ the barriers diverge as $j \to 0$ and the linear dc resistivity vanishes

$$\frac{d\mathcal{E}}{dj}_{j \to 0} = 0 = \rho_l.$$ \hspace{1cm} (45)

Another consequence of Eq.\ref{eq:44} is a slow decay of non-equilibrium screening supercurrents that can be detected by measuring the relaxation of the associated magnetization $M(t)$,

$$\frac{dM(t)}{dt} \propto \frac{dj(t)}{dt} \propto \mathcal{E}(j),$$

where $\mathcal{E}(j)$ is the field needed to prevent $j(t)$ from decaying. Combining (46) with (44) leads to a slow logarithmic decay at long times,

$$j(t) \approx j_0 \left[\ln(t/t_0)\right]^{-1/\mu},$$

with $t_0$ as the microscopic time of order $10^{-9} - 10^{-12}$ seconds.

Recent transport experiments on $YBa_2Cu_3O_7$ thick films (in 3d regime) find nonlinear resistivity behavior described above with $\mu \approx 0.2 - 0.3$ which sets in when the temperature is lowered past a well defined transition temperature $T_{vg}$ (see next section) \cite{19,20}. Later much more sensitive magnetization decay experiments on $YBCO$ films have found a scaling behavior in agreement with Eq.\ref{eq:47}, with $\mu \approx 1/3$ \cite{21,22}. These experiments therefore provide confirmation of the vortex glass picture.

5 PHASE TRANSITIONS

Having discussed the nature of the phases that occur as a result of thermal fluctuations and disorder we now turn our discussion to the phase transitions between these new phases.
5.1 First-order vortex lattice melting transition in pure crystals

In the absence of disorder the low temperature and low field phase is the vortex lattice, which melts into the vortex liquid upon increasing field and/or temperature. This transition has been extensively studied experimentally, numerically and analytically. Transport experiments on clean $Y_1Ba_2Cu_3O_7$ crystals show an abrupt (with milliKelvin resolution), hysteretic resistance drop to zero at the melting transition $[60]$. This experimental work provides solid evidence for the vortex lattice melting transition being first-order. The melting transition has not yet been detected thermodynamically (i.e. by measuring latent heat or magnetization jumps) due to smallness of the vortex latent heat, estimated to be $\leq k_B T_M$ per vortex per $CuO$ layer. The vortices are very dilute and the specific heat due to electrons and phonons of the underlying solid are quite large at $T_M \approx 70K$ in comparison and dominate over the vortex lines’ contribution. Also since the high-$T_c$ materials are highly anisotropic and therefore are in the quasi-2d regime, the first-order transition is expected to be weak.

Evidence from the simulations of London theory (amplitude of $\psi(\vec{r})$ is a fixed constant) on a lattice also clearly points toward the first-order transition $[61]$. Further evidence is provided by the analytical work of Brezin, et al. $[62]$. There an $\epsilon = 6-d$ expansion is carried out which shows that the mean-field theory second-order transition at $H_{c2}^{MF}$ is destabilized by fluctuations. The resulting runaway of renormalization-group flow is taken as an indication of fluctuation driven first-order transition $[63]$. Finally based on general symmetry grounds Landau theory for melting has a term cubic in the order parameter and therefore leads to first-order melting transition in agreement with usual melting of 3d crystals $[64]$. 

5.2 Stability of first-order transition to disorder

We have previously described the strong influence disorder has on vortex phases, for example converting the vortex lattice to a vortex glass. We now examine the stability of the first-order melting transition to weak disorder using Imry-Wortis general scaling arguments $[65]$.

Near the melting transition, disorder’s local and random preference for the solid (vortex glass) or vortex liquid phase can be described via Landau theory with a random local variations in $T_M$ (see Eq.[31]). An increase in free energy due to a creation of an island of one phase inside the other will have a positive interface contribution and a favorable negative bulk contribution due to the disorder

$$\delta F \sim \sigma L^{d-1} - \Delta L^{d/2},$$ (48)

where $\sigma$ is the tension of the $d-1$-dimensional interface and $\Delta$ is the disorder strength.
The negative disorder contribution is a sum of random energies (with zero mean) and therefore by central limit theorem scales as a square root of number of impurities in the region of size $L$. For $d > 2$ and weak disorder (small $\Delta$), the free energy of the island is positive and their density will be strongly suppressed. Therefore, there will be a stable two-phase coexistence with the phase transition remaining first-order. On the other hand for $d \leq 2$ the disorder contribution will dominate the interfacial energy and system will create interpenetrating islands of one phase inside the other. These arguments therefore suggest that the melting transition of the vortex glass into a vortex liquid will become a continuous transition as the effective dimensionality decreases and the strength of disorder increases.

Recent experiments on untwinned, single crystal $YBa_2Cu_3O_7$ by Safar and coworkers [66] find a behavior that can be reconciled with the above theoretical picture. They find a hysteretic, abrupt, first-order transition at weak applied fields $H$ which becomes a continuous transition at higher $H$. It is also observed that the first-order regime increases as pinning decreases with the tricritical point moving in a range $5 - 15$ Tesla. Because increasing $H$ is equivalent to lowering $T_m$ and making the system more two-dimensional, it is believed that increase in $H$ corresponds to an increase in the effective strength of disorder. A schematic of experimentally observed phase diagram is illustrated in Fig. 12.

5.3 Scaling theory near continuous transition

The transition from the superconducting Meissner phase to the non-superconducting vortex liquid is continuous since the density of vortex lines (related to the order parameter) increases smoothly at $H_{c1}$. Also we have argued in the previous sections that at high applied fields and/or strong disorder the transition from the superconducting vortex glass phase to the vortex liquid is second order. As a continuous transition is approached a correlation length $\xi(T)$, characterizing the length scale over which correlations in the order parameter exist, begins to grow, diverging exactly at the transition $T = T_c$ with a critical exponent $\nu$

$$\xi(T) = \xi_0 |(T - T_c)/T|^{-\nu}$$  \hspace{1cm} (49)
To the correlation length corresponds a correlation time $\tau$ over which the correlations of size $\xi$ decay. In the critical region the correlation length is usually larger than any other physical length in the problem and $\tau$ scales with $\xi$ as $\tau \sim \xi^z$. Based on these observation a continuous transition can be studied using a scaling theory, where our ignorance about the transition can be packaged into a small set of unknown universal critical exponents. Above we have confined our analysis to an isotropic situation with a single diverging correlation length. For anisotropic transitions, such as for example the transition to anisotropic Bose glass (due to correlated disorder) our analysis can be easily generalized by keeping track of scaling with correlation lengths in all inequivalent directions.

The scaling theory of nonlinear resistivity can be established by identifying the scaling of the characteristic current $j$ and the electric field $\mathcal{E}$ with the correlation length $\xi$, and constructing a dimensionally-correct combination from these physical quantities. By definition, in the free-energy a supercurrent couples to the gauge invariant superfluid velocity and therefore the free-energy of the superconductor will have an additional current contribution,

$$\delta F_j = -\int d^d r \frac{\phi_0}{c} j \cdot \left( \nabla \phi - \frac{2\pi A}{\phi_0} \right). \tag{50}$$

The long-range correlations in the superconducting order parameter $\psi$ are lost beyond the coherence length scale. Equivalently, the phase $\phi$ changes by $\approx 2\pi$ on the scale $\xi$ and therefore $|\nabla \phi| \sim 2\pi/\xi$. From above equation (and dimensional analysis) the characteristic value of the vector potential $A$ is

$$|A| \sim \frac{\phi_0}{\xi}, \tag{51}$$

which when combined with equation $\mathcal{E} = -c^{-1} \partial A/\partial t$ gives the scaling of the characteristic electric field

$$\mathcal{E} \sim \frac{\phi_0}{c \xi^{1+z}}. \tag{52}$$

Similarly, from Eq.50 we find that the characteristic supercurrent density in the correlation volume is

$$j \sim \frac{ck_B T}{\phi_0 \xi^{d-1}}. \tag{53}$$

In three-dimensions above equation has a simple interpretation as a requirement that in the critical region the Magnus energy, in which current $\mathcal{J}$ couples to the projected area $\xi^2$, be of the order $k_B T$. \[27\]
Combining above equations we obtain a scaling form of the nonlinear resistivity for uniform current, valid near the NS transition

\[
\rho_{nl}(T,j) = \frac{\mathcal{E}}{j} \approx \xi^{d-z-2} F_{\pm} \left( \frac{\phi_0}{c k_B T} j \xi^{d-1} \right),
\]

(54)

where the scaling functions \( F_{\pm} \) describe the nonlinearity in resistivity for \( T > T_c \) and \( T < T_c \), respectively. Above scaling equation can be recast in a more convenient form in terms of \( T \) and \( j \) using Eq. [49]

\[
\rho_{nl}(T,j) = \frac{\mathcal{E}}{j} \approx |T - T_c|^s G_{\pm} \left( \frac{j}{|T - T_c|^\zeta} \right),
\]

(55)

where,

\[
s = \nu (2 + z - d),
\]

(56)

\[
\zeta = \nu (d - 1),
\]

(57)

and \( G_{\pm}(x) \) are some other scaling functions.

The frequency dependent linear resistivity is finite everywhere and the scaling analysis similar to the above leads to

\[
\rho_l(T,\omega) \approx \xi^{d-z-2} \tilde{G}_{\pm} (\omega \xi^z).
\]

(58)

The properties of the phases above and below \( T_c \) gives us the limiting forms of these scaling functions. Above \( T_c \) the material is normal and the behavior is Ohmic at small \( j \). Hence,

\[
G_{\pm}(x) \to \text{constant}, \quad \text{for} \quad x \to 0,
\]

(59)

with the dissipation becoming nonlinear for \( j \approx j_{nl} \sim (T - T_c)^\zeta \). The linear resistivity for \( j \ll j_{nl} \) is therefore expected to vanish as the transition is approached

\[
\rho_l(T) \sim (T - T_c)^s.
\]

(60)

On the other hand for \( j \ll j_{nl} \), i.e. in the limit of \( T \to T_c \) at \( j \neq 0 \), we expect nonlinear resistivity to be finite at \( T_c \). This requires

\[
G_{\pm}(x) \to x^{s/\zeta}, \quad \text{for} \quad x \to \infty,
\]

(61)

so that the divergences in \( (T - T_c) \) cancel out. At \( T_c \) we find nonlinear current-voltage relation

\[
\mathcal{E}(j) \sim j^{1+s/\zeta},
\]

(62)

that is a power-law just like in the Kosterlitz-Thouless transitions.
Finally, below $T_c$ the power law iv-characteristics is replaced by a stretched exponential, described in Sec.4, Eq.(44),

$$G_-(x) \to e^{-a/x^\mu}, \text{ for } x \to 0.$$  \hfill (63)

The scaling behavior emerging from above equations have been seen in dirty, heavily twinned $YBa_2Cu_3O_7$ samples [19, 20] with an impressive scaling over 4 decades for the linear resistivity above $T_c$ and over 2 decades for $\rho_{nl}(j)$ at the vortex glass-liquid NS transition. They find $s \approx 6.5$ and $2\nu \approx 4$, critical exponents that are consistent with the values obtained from numerical studies [23, 24].

Just as for normal metals skin depth for magnetic fluctuations (photons) at frequency $\omega$ is

$$\lambda(\omega) \sim \left(\frac{|\rho_l(\omega)|}{\omega}\right)^{1/2}.$$  \hfill (64)

In the superconducting phase $\rho_l(\omega) \sim i\omega/\rho_s$ and therefore

$$\lambda(\omega \to 0) \sim \frac{1}{\rho_s^{1/2}}$$  \hfill (65)

Since near $T_c$ we found that $\rho_l \sim \xi^{1-z}$ and the characteristic frequency $\omega_\xi \sim \xi^{-z}$, we conclude that $\lambda \sim \xi^{1/2}$ and the Ginzburg-Landau parameter is

$$\kappa = \frac{\lambda}{\xi} \sim \xi^{-1/2}.$$  \hfill (66)

This means that even if the superconductor is characterized by type II “bare” properties, as the transition is approached and $\xi(T) \to \infty$, the superconductor becomes effectively type I. This crossover was first investigated analytically by Halperin and coworkers [38] who found that the charge is strongly relevant at the NS transition. Although initial indications based on renormalization group calculations suggested that the fluctuations drive the transition to be first-order, subsequent numerical analysis points to the transition being that of the inverted XY-model [69], although the story is not fully settled. Recent self-consistent analytical methods also suggest a continuous second-order transition, but with different exponents [70]. If the bare $\kappa_{MF} = 100$, the crossover to this new critical behavior will occur very close to the transition, when $\xi(T)$ has grown by a factor of $10^4$ from what it was when the critical regime was first entered. Unfortunately, this is experimentally too close to the transition to be detectable by the presently available techniques.
6 CONCLUSION

As we have seen, the vortex states of high-T\textsubscript{c} superconductors are highly correlated and strongly fluctuating disordered systems that therefore share much in common with other systems well studied and of much interest in condensed matter physics. Many analytical tools have been taken over from widely different fields of polymer physics, magnetic spin-glass systems, strongly interacting electrons and bosons in random environment, to name a few. As we described in these lectures, much progress in understanding these vortex states has already been accomplished, and a theoretical picture consistent with experiments and numerical simulations has emerged. Because the physics of the novel high temperature superconductors incorporates the most developed and exciting branches of modern condensed matter physics we expect many further exciting developments and surprises in our understanding of the phenomenology of high-T\textsubscript{c} superconductors.

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