THE PROJECTIVE HEIGHT ZERO CONJECTURE

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Abstract. We propose a projective version of the celebrated Brauer’s Height Zero Conjecture on characters of finite groups and prove it, among other cases, for $p$-solvable groups as well as for (some) quasi-simple groups.

1. Introduction

Let $G$ be a finite group and let $p$ be a prime. Recent investigations of N. Rizo and the second author [17] on blocks relative to characters of normal subgroups lead us to propose Conjecture A below. Recall that if $B$ is a Brauer $p$-block of $G$, we denote by $\text{Irr}(B)$ the irreducible complex characters in $B$. If $Z$ is a normal subgroup of $G$ and $\lambda \in \text{Irr}(Z)$, then $\text{Irr}(B|\lambda)$ is the set of $\chi \in \text{Irr}(B)$ such that $\lambda$ is a constituent of the restriction $\chi_Z$. Furthermore, $\text{Irr}_0(B|\lambda)$ denotes the set of characters in $\text{Irr}(B|\lambda)$ of height 0.

Conjecture A. Let $G$ be a finite group, $p$ a prime, and $B$ be a $p$-block of $G$ with defect group $D$. Suppose that $Z \leq G$ is a central $p$-subgroup of $G$, and let $\lambda \in \text{Irr}(Z)$. Then:

$$\text{Irr}(B|\lambda) = \text{Irr}_0(B|\lambda) \iff D/Z \text{ is abelian and } \lambda \text{ extends to } D.$$ 

Of course, when $Z = 1$ then Conjecture A is Brauer’s Height Zero Conjecture. In fact, as we shall show, the “if” direction of Conjecture A follows from the Height Zero Conjecture, and therefore it is true by the main result of Kessar–Malle [9]. (Indeed, if $\lambda \in \text{Irr}(Z)$ is faithful then the condition that $D/Z$ is abelian and $\lambda$ extends to $D$ is equivalent to $D$ being abelian.) The “only if” direction, however, does not seem to be implied by neither the Height Zero Conjecture nor the inductive Alperin–McKay condition (which as we know now implies the Height Zero Conjecture [16, 10]). It is a (non-trivial) theorem of M. Murai [13] that if some $\chi \in \text{Irr}(B|\lambda)$ has height zero, then $\lambda$ extends to $D$. Hence, the essential new part of our proposed conjecture is that if all $\text{Irr}(B|\lambda)$ have height zero, then $D/Z$ is abelian.

C. Eaton has asked us if Conjecture A could still hold only assuming that $\lambda$ is a $G$-invariant character of a normal $p$-subgroup $Z$ of $G$ and that all characters in $\text{Irr}(B|\theta)$ have the same height as $\theta$, as happens with Dade’s Projective Conjecture. As a matter a fact, we believe that this is the case for the “only if” direction, but not for the “if” direction.

Our main results on Conjecture A can be summarised as follows:

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Theorem B.

(a) The “if” direction of Conjecture A is true.

(b) Conjecture A is true for $p$-solvable groups as well as for nilpotent blocks.

(c) Quasi-simple groups do not provide minimal counterexamples to Conjecture A.

After studying Conjecture A, it is tempting to conjecture that all characters in $\text{Irr}(D|\lambda)$ have the same degree implies that all characters in $\text{Irr}(B|\lambda)$ have the same height (and vice versa). However, the principal 2-block of the double cover $G$ of the alternating group $A_8$ provides an interesting counterexample: the non-trivial irreducible character $\lambda$ of $Z(G)$ is fully ramified over $P \in \text{Syl}_2(G)$, that is $\lambda^P = 8\theta$ for some irreducible $\theta \in \text{Irr}(P)$, while there are characters in $B$ over $\lambda$ of degrees 8, 24 and 48. In all the cases that we have checked, however, the minimum non-zero height of the characters in $\text{Irr}(B|\lambda)$ does coincide with the minimum non-linear degree of the characters in $\text{Irr}(D|\lambda)$. It seems interesting to discuss if the Eaton–Moretó conjecture might have a projective version.

Our paper is built up as follows. In Section 2, we prove certain cases of Conjecture A. In Section 3, we complete the proof of Theorem B by considering quasi-simple groups.

Britta Späth has raised the possibility that it might be possible to reduce Conjecture A to the inductive Alperin–McKay conjecture and a checking of Conjecture A for quasi-simple groups only; this remains open for the time being.

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2. Certain cases where Conjecture A holds

Theorem 2.1. Let $G$ be a finite group and let $p$ be a prime. Let $Z$ be a central $p$-subgroup of $G$ and $\lambda \in \text{Irr}(Z)$. Assume that $B$ is a $p$-block of $G$ with defect group $D$.

(a) If $(G, B)$ is a counterexample minimising $|G|$, to any of the two directions of Conjecture A, then $\lambda$ is faithful.

(b) The “if” direction of Conjecture A is true.

(c) If $D \in \text{Syl}_p(G)$ then Conjecture A holds if Brauer’s Height Zero Conjecture holds for $G/Z$.

(d) Conjecture A is true for $p$-solvable groups.

(e) Conjecture A is true for nilpotent blocks.

Proof. First of all, recall that if $\text{Irr}(B|\lambda)$ contains a height zero character, then $\lambda$ extends to $D$ by [13, Thm. 4.4]. Also notice that $\text{Irr}(B|\lambda)$ is not empty, because $B$ covers the principal block of $Z$ (see [14, Thm 9.2]). Since $K = \ker(\lambda)$ is a central $p$-subgroup of $G$, by [14, Thm. 9.10], there is a unique block $\overline{B}$ of $G/K$ that is contained in $B$. Furthermore, $D/K$ is a defect group of $B$. Hence the height of $\chi \in \text{Irr}(B)$ as a character of $G/K$ is the same as its height as a character in $\text{Irr}(B)$. Part (a) now easily follows.

To show (b) assume now that $D/Z$ is abelian and that $\lambda$ extends to some $\nu \in \text{Irr}(D)$. In particular, $D' \leq \ker(\nu)$. Since $D' \leq Z$, it follows that $D' \leq \ker(\lambda)$. By (a), in a minimal counterexample we can assume that $\lambda$ is faithful. Therefore we conclude that $D$ is abelian. By the main result of [9], we conclude that all irreducible characters in $B$ have height zero.
Assume now that $D \in \text{Syl}_p(G)$ and that all characters in $\text{Irr}(B|\lambda)$ have height zero. By [13, Thm. 4.4], $\lambda$ extends to $D$. By [8, Thm. 6.26 and Cor. 6.27], we have that $\lambda$ extends to $G$. Let $\mu \in \text{Irr}(G)$ be an extension of $\lambda$. Now it is easy to check that there is a $p$-block $B_1$ of $G$ such that $\text{Irr}(B_1) = \{\mu^{-1}\chi \mid \chi \in \text{Irr}(B)\}$. Let $B_1$ be the unique block of $G/Z$ contained in $B_1$. By hypothesis, we have that all the character in $B_1$ have height zero, and therefore $D/Z$ is abelian, because Brauer’s height zero conjecture is true for $G/Z$ by assumption.

Assume next that $G$ is $p$-solvable and that all irreducible characters in $\text{Irr}(B|\lambda)$ have height zero. We assume that the reader has some familiarity with the notation and results in [3]. Let $B$ be the unique block of $G/Z$ contained in $B$ (by [14, Thm. 9.10]) and let $\chi \in \text{Irr}(B)$ be of height zero. We know that $D/Z$ is a defect group of $B$, and therefore $\chi$ has height zero considered as a character of $B$. By [3, Thm. 1.2], the map $\eta \mapsto \chi \ast \eta$ defines an isometry between the ring of virtual characters of $D$ and of $B$. In fact, if $\eta \in \text{Irr}(D)$, then $\chi \ast \eta \in \text{Irr}(B)$ and has degree $\chi(1)\eta(1)$ by [3, page 125]. By the definition of the map $\chi \ast \eta$, and using that $\chi(z) = \chi(1)$, notice that $(\chi \ast \eta)(z) = \chi(1)\eta(z)$. Therefore, $\text{Irr}(B|\lambda) = \{\chi \ast \eta \mid \eta \in \text{Irr}(D|\lambda)\}$. By hypothesis, we have that $p$ does not divide $\eta(1)$ for all $\eta \in \text{Irr}(D|\lambda)$. It follows that $D/Z$ is abelian by Gallagher’s Corollary 6.17 of [8]. This proves part (e).

Finally, assume that $B$ is nilpotent and that all characters in $\text{Irr}(B|\lambda)$ have height zero. We assume that the reader has some familiarity with the notation and results in [3]. Let $\tilde{B}$ be the unique block of $G/Z$ contained in $B$ (by [14, Thm. 9.10]) and let $\chi \in \text{Irr}(\tilde{B})$ be of height zero. We know that $D/Z$ is a defect group of $\tilde{B}$, and therefore $\chi$ has height zero considered as a character of $\tilde{B}$. By [3, Thm. 1.2], the map $\eta \mapsto \chi \ast \eta$ defines an isometry between the ring of virtual characters of $D$ and of $\tilde{B}$. In fact, if $\eta \in \text{Irr}(D)$, then $\chi \ast \eta \in \text{Irr}(\tilde{B})$ and has degree $\chi(1)\eta(1)$ by [3, page 125]. By the definition of the map $\chi \ast \eta$, and using that $\chi(z) = \chi(1)$, notice that $(\chi \ast \eta)(z) = \chi(1)\eta(z)$. Therefore, $\text{Irr}(B|\lambda) = \{\chi \ast \eta \mid \eta \in \text{Irr}(D|\lambda)\}$. By hypothesis, we have that $p$ does not divide $\eta(1)$ for all $\eta \in \text{Irr}(D|\lambda)$. It follows that $D/Z$ is abelian by Gallagher’s Corollary 6.17 of [8]. This proves part (e).

We finish this section by answering Eaton’s question.

**Theorem 2.2.** Let $G$ be a finite group, let $N$ be a normal $p$-subgroup of $G$, let $\theta \in \text{Irr}(N)$ be $G$-invariant, and let $B$ be a $p$-block of $G$ with defect group $D$.

(a) Suppose that some $\chi \in \text{Irr}(B|\theta)$ has the same height as $\theta$. Then $\theta$ extends to $D$.

(b) Assume that Conjecture A holds for all finite groups. If all $\text{Irr}(B|\theta)$ have the same height as $\theta$ then $D/N$ is abelian.

**Proof.** By the proof of [15, Thm. 3.1], there exists a central extension $\pi : \tilde{G} \to G$, with kernel a finite $p$-group $E \leq C^*$, such that $\tilde{G}$ contains $N$ as a normal subgroup and $\pi^{-1}(N) = \tilde{N} = N \times E$. Also, $\pi(N) = N$. Furthermore, there exists $\tau \in \text{Irr}(\tilde{G})$ such that $\tau_N = \theta$. Now, if $\chi \in \text{Irr}(\tilde{G}|\theta)$, $\tilde{\chi} \in \text{Irr}(\tilde{G}/E)$ is the corresponding character via the isomorphism $\tilde{G}/E \to G$, then $\tilde{\chi}$ lies over $\theta$, and by Gallagher’s Corollary 6.17 of [8], there exists a unique $\chi^* \in \text{Irr}(\tilde{G}/N)$ such that $\tilde{\chi} = \tau\chi^*$. In fact, if $\lambda \in \text{Irr}(\tilde{N}/N)$ is given by $\lambda(n,z) = z^{-1}$, then the map $\ast$ defines a character triple isomorphism $\text{Irr}(\tilde{G}|\theta) \to \text{Irr}(\tilde{G}/N|\lambda)$. We assume that the reader is familiar with the properties of character triple isomorphisms [8, Definition 11.23]. Now, by [14, Thm. 9.2], notice that $\text{Irr}(B|\theta)$ is not empty.

To prove part (a), assume that $\chi \in \text{Irr}(B|\theta)$ has the same height as $\theta$. This means that $\chi(1)_p = |G : D|_p\theta(1)$. Thus $\chi^*(1)_p = |G : D|_p$. Let $B^*$ be the block of $\tilde{G}/N$ that contains $\chi^*$, and let $\tilde{D}/N$ be a defect group of $B^*$. Now if $\tilde{B}$ is the block of $\tilde{\chi} = \tau\chi^*$, then there is a defect group $P$ of $\tilde{B}$ such that $\tilde{D} \subseteq P$ by [14, Thm. 4.8] and [16, Prop. 2.5(b)]. Since

...
\( \tilde{G} \) is a central extension of \( G \), we have that \( \pi(P) = D \) by [17, Cor. 2.3(\text{c})]. Now
\[
|G : D|_p = \chi^*(1)_p = |\tilde{G} : \tilde{D}/\tilde{N}|_p\text{ht}(\chi^*) \geq |\tilde{G} : P|_p\text{ht}(\chi^*) = |G : D|_p\text{ht}(\chi^*)
\]
and we deduce that \( \tilde{D} \) is a defect group of \( \tilde{B} \) and that \( \chi^* \) has height zero in \( B^* \). By [13, Thm. 4.4], we have that \( \lambda \) extends to \( \tilde{D}/\tilde{N} \). Now, since the group \( D/N \) corresponds to \( \tilde{D}/\tilde{N} \) under the character triple isomorphism, we conclude that \( \theta \) extends to \( D \). This proves part (a).

Next, we prove (b). Since \( \text{Irr}(B|\theta) \) is not empty, fix \( \chi \in \text{Irr}(B|\theta) \), which we know has the height of \( \theta \). By part (a), we know that \( \tilde{D} = \pi^{-1}(D) \) is a defect group of the block of \( \tilde{\chi} \), and that \( \lambda \) extends to \( \tilde{D}/\tilde{N} \). Now, let \( \psi^* \in \text{Irr}(B^*|\lambda) \), where \( \psi \in \text{Irr}(G|\theta) \). Then \( \tilde{\psi} = \psi^*\tau \) and by [17, Lemma 2.4] we have that \( \tilde{\psi} \) and \( \tilde{\chi} \) belong to the same block \( \tilde{B} \) of \( \tilde{G} \). By [17, Cor. 2.3], it follows that \( \chi \) and \( \psi \) lie in \( B \). Also, by part (a), we have that \( \psi^* \) has height zero. By Conjecture A, we deduce that \( (\tilde{D}/\tilde{N})/(\tilde{N}/N) \) is abelian. This group is isomorphic to \( D/N \).

Notice that the converse of Theorem 2.2(b) is false. The group \( G = \text{GL}_2(3) \) has a unique 2-block, with defect group \( P \in \text{Syl}_2(G) \). Now, for \( N = Q_8 \) the unique character \( \theta \in \text{Irr}(N) \) of degree 2 extends to \( P \), \( P/N \) is abelian, but there are irreducible characters of \( G \) over \( \theta \) of degrees 2 and 4.

## 3. Quasi-simple groups and Conjecture A

Here, we discuss the proof of the following statement:

**Theorem 3.1.** Let \( G \) be a finite quasi-simple group. Then no \( p \)-block of \( G \) is a minimal counterexample to the “only if” direction of the Projective Height Zero Conjecture A.

Let’s start off by making a couple of trivial observations:

For the proof of Theorem 3.1 by the result of Murai, clearly we may assume that \( D/Z \) is non-abelian. We may further assume that \( Z \neq 1 \), as otherwise the result (for quasi-simple groups) is contained in Kessar–Malle [10], and that \( \lambda \) is faithful. Moreover, by Theorem 2.1(c) we need not consider the case when \( B \) is of maximal defect, again using [10]. This already drastically restricts the type of situations to consider. We will go through the various cases according to the Classification of Finite Simple Groups.

### 3.1. Sporadic groups and exceptional covering groups.

**Proposition 3.2.** Let \( G \) be a covering group of a sporadic simple group, or an exceptional covering group of a finite simple group of Lie type, or of \( A_6 \) or \( A_7 \). Then Conjecture A holds for \( G \).

**Proof.** These are only finitely many groups, with known character tables, and the assertion can readily be checked with the help of a small GAP programme [18].
3.2. Alternating groups. Here we deal with the 2-fold coverings $\tilde{A}_n = 2A_n$ of the alternating groups.

Proposition 3.3. Let $G = 2A_n$ be the 2-fold covering group of an alternating group $A_n$, $n \geq 5$. Then Conjecture A holds for all 2-blocks of $G$.

Proof. Let $B$ be a 2-block of $\tilde{A}_n$ and let $D$ denote a defect group of $B$. Then we need to show that there exists a spin character in $B$ (that is, a faithful irreducible character of $\tilde{A}_n$) of positive defect whenever $D/Z$ is non-abelian, where $Z = Z(\tilde{A}_n)$. For this let’s first consider the block $\tilde{B}$ of the double covering $\tilde{S}_n$ of the symmetric group $S_n$ covering $B$. The heights of spin characters in $\tilde{B}$ have been described by Bessenrodt and Olsson. By [1, Thm. 1.1 and Cor. 3.10] there are spin characters in $\tilde{B}$ which will be called $\ell$-blocks of weight at most 1 unless $\tilde{B}$ has weight at most 1. These characters have height at least 1 in $B$, as required. On the other hand, for blocks of weight at most 1 we have that $|D| \leq 2$ and so $D$ is abelian. This achieves the proof. □

3.3. Groups of Lie type. There are only finitely many covering groups of simple groups of Lie type in characteristic $p$ by a non-trivial $p$-group, and these have all been considered in Proposition 3.2 already. So when dealing with the finite groups of Lie type we may now assume that their underlying characteristic is different from the considered prime, which will be called $\ell$ here. This is the most difficult case in our analysis. We consider the usual setup: Let $G$ be a simple algebraic group of simply connected type over an algebraic closure of a finite field, and let $F : G \to G$ be a Frobenius endomorphism with respect to an $\mathbb{F}_q$-rational structure such that the finite group of fixed points $G := G^F$ is quasi-simple. Let $G^*$ be in duality with $G$ with corresponding Frobenius map also denoted $F$ and set $G^* := G^*F$. We will make use of Lusztig’s Jordan decomposition of characters. Let $s \in G^*$ be a semisimple element, and $E(G, s) \subseteq \text{Irr}(G)$ the corresponding Lusztig series. Then Jordan decomposition yields a bijection

$$\Psi : E(G, s) \to E(C_{G^*}(s), 1),$$

such that

$$\chi(1) = |G^* : C_{G^*}(s)|_p \Psi(\chi)(1) \quad \text{for all } \chi \in E(G, s).$$

We will also use the fact, shown by Broué–Michel, that for all semisimple $\ell'$-elements $s \in G^*$,

$$E_{\ell'}(G, s) := \bigcup_{t \in C_{G^*}(s)_{\ell'}} E(G, st),$$

where the union runs over $\ell$-elements in $C_{G^*}(s)$, is a union of $\ell$-blocks.

Lemma 3.4. Let $G$ be as above, let $s \in G^*$ be an $\ell'$-element and $B$ an $\ell$-block of $G$ such that $\text{Irr}(B) \subseteq E_{\ell}(G, s)$. Let $t \in C_{G^*}(s)$ be an $\ell$-element such that there are $\chi \in E(G, s) \cap \text{Irr}(B)$ and $\chi' \in E(G, st) \cap \text{Irr}(B)$ with $|C_{G^*}(s) : C_{G^*}(st)|\Psi(\chi')(1)/\Psi(\chi)(1)$ divisible by $\ell$. Then $\chi'$ has positive height in $B$.

Proof. Our assumptions yield

$$(\chi'(1)/\chi(1))_{\ell} = \left(\frac{|G^* : C_{G^*}(st)|\Psi(\chi')(1)}{|G^* : C_{G^*}(s)|\Psi(\chi)(1)}\right)_{\ell} = \left(\frac{|C_{G^*}(s) : C_{G^*}(st)|\Psi(\chi')(1)}{\Psi(\chi)(1)}\right)_{\ell} > 1,$$

so indeed $\chi'$ has positive height. □
Corollary 3.5. Let $G$ be as above, and let $s \in G^*$ be an $\ell'$-element such that $\mathcal{E}_\ell(G, s)$ is a single $\ell$-block. Then for every $\ell$-element $t \in C_{G^*}(s)$ with $|C_{G^*}(s) : C_{G^*}(st)|_\ell > 1$, the characters in $\mathcal{E}_\ell(G, st)$ have positive height in $B$.

Proof. The Lusztig series $\mathcal{E}(G, s)$ contains the preimages of the linear characters of $C_{G^*}(s)$ under Jordan decomposition, the so-called semisimple characters, of degree $|G^* : C_{G^*}(s)|_{\ell'}$. If $\chi$ is one of those, then any $\chi' \in \mathcal{E}(G, st)$ satisfies the assumptions of Lemma 3.4 and we conclude. \hfill $\square$

We will also require the following fact; see e.g. [12, 3.2]:

Proposition 3.6. For $G, G^*$ as above there is an isomorphism $f : Z(G) \to G^*/[G^*, G^*]$ such that for any $z \in Z(G)$, the characters in $\mathcal{E}(G, s)$ are non-trivial on $z$ for all $s$ in the coset $[G^*, G^*]f(z)$.

Thus, for example if $Z(G)$ is cyclic of prime order, then the Lusztig series of any semisimple element $s \in G^* \setminus [G^*, G^*]$ will contain only faithful characters of $G$.

We first treat groups of type $A$. Here the existence of suitable $\ell$-elements for the application of Lemma 3.4 and Corollary 3.5 is guaranteed by:

Lemma 3.7. Let $G = \text{GL}_m(q)$ and $\ell \leq m$ a prime dividing $q-1$. Write $\ell^b$ for the precise power of $\ell$ dividing $q-1$. Then there exists an $\ell$-element $t \in G$ such that $|G : C_G(t)|_\ell \geq \ell^b$, with strict inequality unless $(m, \ell) = (2, 2), (3, 2)$. If $m$ is not a power of $\ell$ then for any $b' \leq b$ we can choose $t$ to have determinant of order $\ell^{b'}$.

Proof. Let $m = \sum_{i=0}^s a_i \ell^i$ be the $\ell$-adic expansion of $m$ (so $0 \leq a_i \leq \ell - 1$ and $a_s \neq 0$). Consider the subgroup $\prod_{i=0}^s \text{GL}_{a_i}(q)$ of $G$ containing a Sylow $\ell$-subgroup of $G$. Choose $t$ inside a factor $\text{GL}_{a_i}(q)$ to be an element of order $(q^{a_i} - 1)/\ell_i$, a power of a Singer cycle. Then the centraliser of $t$ in $G$ is of the form $\text{GL}_1(q^{a_i}) \times \text{GL}_{m-a_i}(q)$ and hence has $\ell$-part $(q^{a_i} - 1)/\ell |\text{GL}_{m-a_i}(q)|_\ell$.

Now first assume that $\ell$ is odd or $q \equiv 1 \pmod{4}$. Then from the order formula we get $|\text{GL}_m(q)|_\ell = \ell^m - c_m$ with $c_m = \ell m b + \sum a_i (\ell^{i+1} - 1)/(\ell - 1)$, showing $|G : C_G(t)|_\ell \geq \ell^{(\ell^{b'} - 1)} \geq \ell^b$, with strict inequality unless $\ell = 2$ and $m \leq 3$. On the other hand, if $\ell = 2$ and $q \equiv 3 \pmod{4}$ then writing $2^d \geq 4$ for the precise power of 2 dividing $q+1$, we have that $|\text{GL}_m(q)|_2 = 2^{c_m}$ with $c_m = m + \lfloor m/2 \rfloor d + \sum a_i (2^{i+1} - 1)$. Again, the index of $C_G(t)$ in $G$ has $2$-part at least $2$, with strict inequality unless $m \leq 3$.

If $m \neq \ell$ then we can adjust $t$ by a suitable $\ell$-element in the factor $\text{GL}_{m-a_i}(q)$ to obtain an $\ell$-element with any prescribed determinant of $\ell$-power order. \hfill $\square$

Proposition 3.8. The “only if” direction of Conjecture A holds for all $\ell$-blocks of $\text{SL}_n(q)$, $\ell|(q-1)$, and of $\text{SU}_n(q)$, $\ell|(q+1)$.

Proof. Let $G = \text{SL}_n(q)$ and $\lambda$ be an irreducible character of $Z(G)$ of $\ell$-power order. (So in particular $\ell$ divides gcd($n, q-1$)). Let $N \leq Z(G)$ with $N_\ell = \ker(\lambda)$ and consider a block $B$ of $G/N$, with defect group $D$. We may assume that $D$ is non-abelian. Let $\hat{B}$ be the block of $G$ containing $B$, with defect group $\hat{D}$ such that $D = \hat{D}/N_\ell$, and $B$ a block of $\hat{G} = \text{GL}_n(q)$ covering $\hat{B}$, with defect group $\hat{D} \geq \hat{D}$. Then by the main result of [8] there exists a semisimple $\ell$-element $\hat{s} \in \text{GL}_n(q)$ such that $\text{Irr}(\hat{B}) = \mathcal{E}_\ell(\hat{G}, \hat{s})$, and $\hat{D}$ is conjugate
to a Sylow $\ell$-subgroup of $C_\hat{G}(\hat{s})$. We then have that $\text{Irr}(B) \subseteq \text{Irr}(\hat{B}) \subseteq \mathcal{E}_t(G, s)$, where $s$ is the image of $\hat{s}$ in $G^* = \text{PGL}_n(q)$. Now
\[ C_\hat{G}(\hat{s}) \cong \text{GL}_{n_1}(q^{e_1}) \times \cdots \times \text{GL}_{n_r}(q^{e_r}) \]
for suitable $n_j, e_j$ with $\sum_j n_j e_j = n$. As $D$ is non-abelian, so is $\hat{D}$, so $n_j \geq \ell$ for at least one $j$. Then by Lemma 3.7 there is an $\ell$-element $\bar{t} \in C_\hat{G}(\hat{s})$ with $|C_\hat{G}(\hat{s}) : C_\hat{G}(\hat{s}\bar{t})|_\ell > (q-1)_\ell$, unless $n_{j} \leq 3$ and $\ell = 2$. Moreover, unless $r = 1$ and $n_1 = \ell^k$ for some $k$, we can arrange for $\bar{t}$ to have any prescribed determinant of $\ell$-power order. Let $t$ be the images of $\bar{t}$ in $G^*$. Now choose the determinant of $\bar{t}$ such that $\mathcal{E}(G, st)$ lies above $\lambda$. As $|\hat{G}|/|G^*|_\ell = (q-1)_\ell$ we have that $|C_{G^*}(s) : C_{G^*}(st)|_\ell > 1$. Thus, by Corollary 3.5 we have that $\text{Irr}(B|\lambda)$ contains characters of positive height, as claimed.

It remains to discuss the exceptions, viz. either $n_j \leq 3$ ($1 \leq j \leq r$) and $\ell = 2$, or $r = 1$ and $n_1 = \ell^k$. First assume that $\ell = 2$ and $n_j \leq 3$ ($1 \leq j \leq r$). Then again by Lemma 3.7 we find a suitable $\ell$-element unless $n_2 = \ldots = n_r = 1$. If $n_1 = 3$, or if $n_1 = 2$ and $n \neq n_1$ then $|C_\hat{G}(\hat{s}) : C_\hat{G}(\hat{s}\bar{t})|_2 = |C_G(s) : C_G(st)|_2$ and we are again done.

So we are only left with the second case, namely that $r = 1$ and $n_1 = \ell^k$. Then $C_\hat{G}(\hat{s}) \cong \text{GL}_{nk}(q^e)$ with $\ell^k e = n$. Here, let $t$ be an $\ell$-element of $\text{GL}_{nk}(q^e)$ having just one non-trivial eigenvalue $z$. This can be chosen such that $\hat{t}$ has any prescribed determinant viewed as element of $\text{GL}_n(q)$. Then $C_\hat{G}(\hat{s}\hat{t}) \cong \text{GL}_{nk-1}(q^e)(q^e-1)$ has index divisible by $\ell^k$ in $C_\hat{G}(\hat{s})$, and this index remains unchanged when passing to $G^* = \text{PGL}_n(q)$, unless possibly when $\ell^k = 2$, and we are done as before. So finally assume that $\ell = 2$ and $C_\hat{G}(\hat{s}) \cong \text{GL}_2(q^e)$ with $n = 2e$. If $q^e \equiv \pm 1 \pmod{2}$ then there are 2-elements in $\text{PGL}_2(q^e)$ of order at least 4 in any non-trivial coset of $L_2(q)$ not centralised by a Sylow 2-subgroup, and we may again conclude by Corollary 3.5. Finally, if $q^e \equiv \pm 3 \pmod{2}$, then $e$ is odd, and $\gcd(n, q - 1) = \gcd(2, q^e - 1) = 2$. So $|Z| = 2$, and $D$ is a Sylow 2-subgroup of $\text{PGL}_2(q^e)$ of order 8. But then $D/Z$ is abelian.

The case of $G = \text{SU}_n(q)$ is entirely similar. 

For groups of classical type we will need the following:

**Lemma 3.9.** Let $G$ be connected reductive in odd characteristic with a Frobenius map $F : G \to G$ with respect to an $\mathbb{F}_q$-rational structure and set $G = G^F$. If $G$ is not a torus, then any coset of $G/[G, G]$ of 2-power order contains a 2-element that is not centralised by a Sylow 2-subgroup of $G$.

**Proof.** First assume that $q \equiv 1 \pmod{4}$. Let $T \leq G$ be a maximally split torus. Then $N_G(T)$ contains a Sylow 2-subgroup $S$ of $G$ (see e.g. [11, Prop. 5.20]), and $N_G(T)/T \cong W^F$, where $W$ is the Weyl group of $G$. Let $s \in W^F$ be a fundamental reflection with preimage $\hat{s} \in N_G(T)$ and $H \leq G$ the $A_1$-type subgroup corresponding to $s$. Then $\hat{s}$ acts non-trivially on $S \cap T$ and we thus find elements in $(T \cap S) \setminus Z(S)$ in any coset $G/[G, G]$ of 2-power order. Now $N_G(T)$ controls fusion in $T$ [11, Prop. 5.11], so no such element is conjugate to an element of $Z(S)$, and hence cannot be centralised by a Sylow 2-subgroup of $G$.

If $q \equiv 3 \pmod{4}$, let $T \leq G$ be the centraliser of a $\Phi_2$-torus of $G$. Then we may argue as above, except that $W^F$ has to be replaced by the reflection group $C_W(w_0F)$ where $w_0 \in W$ is the longest element. 

□
Theorem 3.10. Let $G$ be a covering group of a simple group of Lie type. Then no $\ell$-block of $G$ is a minimal counter-example to the “only if” direction of Conjecture A.

Proof. As pointed out above, we may assume by Proposition 3.2 that $G$ is not an exceptional covering group of $G/Z(G)$ and that thus $\ell$ is not the underlying characteristic of $G$. As we only need to consider the case when $\ell$ divides $|Z(G)|$, the group $G$ is not a Suzuki or Ree group. But then we may assume that $G = G^F$ as above. Furthermore, $G$ is not of type $A$ by Proposition 3.8. Then we have $\ell \leq 3$.

Let us first assume that $G$ is of classical type $B_n$, $C_n$ or $D_n$. Then $|Z(G)|$ is a 2-power, so here $\ell = 2$. Let $B$ be a 2-block of $G$. Then by Enguehard [3] Prop. 1.5] there is a semisimple element $s \in G^*$ of odd order such that Irr($B$) = $E_2(G, s)$ is a Broué–Michel union of Lusztig series, where $G^*$ denotes a group in duality with $G$. Furthermore, if $L \leq G$ denotes a Levi subgroup in duality with $C_{G^*}(s)$ (which is connected as $s$ has odd order) then any Sylow 2-subgroup of $L = L^F$ is a defect group of $B$. By Lemma 3.9 any coset of $[G^*, G^*]$ contains a 2-element $t \in C_{G^*}(s)$ such that $|C_{G^*}(st)|_2 < |C_{G^*}(s)|_2$. Then by Corollary 3.5 the characters in $E(G, st) \subseteq E_2(G, s)$ have positive height, and by Proposition 3.10) they are faithful on a chosen cyclic subgroup of the centre $Z(G)$.

Next assume that $G$ is of exceptional type. Then our assumption that $Z(G) \neq 1$ implies that either $G = E_6(q)_{sc}$ or $E_6(q)_{sc}$ with $\ell = 3$, or $G = E_7(q)_{sc}$ with $\ell = 2$. First assume that $G = E_6(q)_{sc}$ with $\ell = 3$ and $Z(G) \neq 1$, so $3|(q - 1)$. The unipotent 3-blocks of $G$ and their non-unipotent constituents are described in [4] p. 351 and Thm. B). By Theorem 2.1(c) we need not consider the principal 3-block. The only other unipotent 3-block $B$ of $G$ has defect group $D$ an extension of a homocyclic abelian group $3^n \times 3^a$, with $3^a$ the precise power of 3 dividing $q - 1$, by a cyclic group of order 3. Let $t \in G^*$ be an element of order 3 with centraliser of type $3D_4(q)\cdot 3$, not contained in the derived subgroup of $G$. Then the cuspidal character $\chi'$ in the Lusztig series $E(G, t)$ is faithful on $Z(G)$, it lies in Irr($B$) by [4] Prop. 17] and it is of positive height by Lemma 3.4. By [9] Tab. 3 the only further quasi-isolated 3-block consists of the characters in $E_3(G, s)$ for an involution $s \in G^*$ with centraliser $A_5(q)A_1(q)$. Let $t$ be a 3-element in $C_{G^*}(s)$ outside $[G^*, G^*]$ with centraliser $A_1(q)A_1(q)\Phi_1$, then we may conclude using Corollary 3.5. We postpone the discussion of non- quasi-isolated 3-blocks to the end.

The case of $E_6(q)_{sc}$ with $\ell = 3$ and $q \equiv 2$ (mod 3) is entirely similar. The relevant data are collected in Table II.

Next assume that $G = E_7(q)_{sc}$ with $\ell = 2$, so $q$ is odd. First assume that $q \equiv 1$ (mod 4). By [4] p. 354] there are three unipotent 2-blocks of $G$ with non-abelian defect, the principal block and two further ones corresponding to Harish-Chandra series of type $E_6$. Here let $t \in G^*$ be a 2-element with centraliser $E_6(q)(q - 1)$ of order $(q - 1)$. This is not contained in the derived subgroup $[G^*, G^*]$, and by order comparison it does not centralise a Sylow 2-subgroup of $G^*$, so by Lemma 3.4) the elements in $E(G, t)$ in the $E_6$-Harish-Chandra series provide faithful characters in the corresponding unipotent blocks of positive height. By [9] Tab. 4] the only further quasi-isolated 2-blocks of $G$ consist of the characters in $E_2(G, s)$ for elements $s \in G^*$ of order 3 with centraliser of type $A_5 + A_2$. For $q \equiv 1$ (mod 3), this has rational structure $A_5(q)A_2(q)$. Let $t$ be a 2-element in $C_{G^*}(s)$ outside $[G^*, G^*]$ with centraliser $A_2(q)^3\Phi_1$, then we may conclude by applying Corollary 3.5. When $q \equiv 2$ (mod 3), the centraliser of $s$ has rational structure.
the height preserving bijection $\text{Irr}(D/Z) \rightarrow \text{Irr}(\bar{b})$ coming from Jordan decomposition, and the defect group $D_b$ of $b$ is isomorphic to $D$. As we assume that $D/Z$ is non-abelian, by [10] there exist characters of positive height in the block $\bar{b}$ of $G/Z$ contained in $B$. But then the block $\bar{b}$ of $L/Z$ contained in $b$ also has characters of positive height, whence $D/Z$ is non-abelian by the proven direction [9] of the ordinary height zero conjecture. Thus, $B$ is certainly not a minimal counter-example to Conjecture A.

We have completed the proof of Theorem 3.1.

Table 1. Characters of positive height

| $G$                  | $\ell$ | $C_G^*(s)$ | HC       | $C_G^*(st)$ | $\chi(1)_\ell$ | $\chi'(1)_\ell$ |
|----------------------|--------|------------|----------|-------------|----------------|-----------------|
| $E_6(q)_{sc}$        | 3      | $G^*$      | $D_4$    | $^2D_4(q) \cdot \Phi_3, 3$ | $3^{4\ell+3}$  | $3^{\ell+2}$   |
| $(q \equiv 1 (3))$   |        |            |          |             |                |                 |
| $E_6(q)_{sc}$        | 3      | $A_5(q) \cdot A_1(q)$ | $\emptyset$ | $A_4(q) \cdot A_1(q) \cdot \Phi_1$ | $3^2$          | $3^3$           |
| $(q \equiv 2 (3))$   |        |            |          |             |                |                 |
| $E_7(q)_{sc}$        | 2      | $A_5(q) \cdot A_1(q)$ | $\emptyset$ | $^2A_4(q) \cdot A_1(q) \cdot \Phi_2$ | $3^\ell$       | $3^3$           |
| $(q \equiv 1 (4))$   |        |            |          |             |                |                 |
| $E_7(q)_{sc}$        | 2      | $A_5(q) \cdot A_2(q)$ | $\emptyset$ | $^2A_4(q) \cdot A_2(q) \cdot \Phi_2$ | $3^{3\ell+2}$  | $3^{3\ell+2}$  |
| $(q \equiv 3 (4))$   |        |            |          |             |                |                 |

$^2A_5(q), ^2A_2(q)$, and here we may take a 2-element $t$ in $C_G^*(s)$ with centraliser $^2A_2(q)^3 \cdot \Phi_2$. The case when $q \equiv 3 \pmod{4}$ is entirely similar, again see Table [1].

Now assume that $B$ is a non-quasi-isolated 3-block of $E_6(q)_{sc}, ^2E_6(q)_{sc},$ or a non-quasi-isolated 2-block of $E_7(q)_{sc}$. Then by the main result of Bonnafé–Dat–Rouquier [2], there is a Morita equivalence of $B$ to a block $b$ of a proper Levi subgroup $L$ of $G$, induced by the height preserving bijection $\text{Irr}(B) \rightarrow \text{Irr}(\bar{b})$ coming from Jordan decomposition, and the defect group $D_b$ of $b$ is isomorphic to $D$. As we assume that $D/Z$ is non-abelian, by [10] there exist characters of positive height in the block $\bar{b}$ of $G/Z$ contained in $B$. But then the block $\bar{b}$ of $L/Z$ contained in $b$ also has characters of positive height, whence $D/Z$ is non-abelian by the proven direction [9] of the ordinary height zero conjecture. Thus, $B$ is certainly not a minimal counter-example to Conjecture A.

We have completed the proof of Theorem 3.1.

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