ON EVOLUTION EQUATIONS GOVERNED BY NON-AUTONOMOUS FORMS

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Abstract. We consider a linear non-autonomous evolutionary Cauchy problem
\[ \dot{u}(t) + A(t)u(t) = f(t) \quad \text{for a.e. } t \in [0, T], \quad u(0) = u_0, \]
where the operator \( A(t) \) arises from a time depending sesquilinear form \( a(t, \cdot, \cdot) \) on a Hilbert space \( H \) with constant domain \( V \). Recently a result on \( L^2 \)-maximal regularity in \( H \), i.e., for each given \( f \in L^2(0, T, H) \) and \( u_0 \in V \) the problem \( (0.1) \) has a unique solution \( u \in L^2(0, T, V) \cap H^1(0, T, H) \), is proved in [10] under the assumption that \( a \) is symmetric and of bounded variation. The aim of this paper is to prove that the solutions of an approximate non-autonomous Cauchy problem in which \( a \) is symmetric and piecewise affine are closed to the solutions of that governed by symmetric and of bounded variation form. In particular, this provides an alternative proof of the result in [10] on \( L^2 \)-maximal regularity in \( H \).

1. Introduction

In this work we are interested by evolutionary linear equations of the form
\[ \dot{u}(t) + A(t)u(t) = f(t), \quad u(0) = u_0, \]
where the operators \( A(t), t \in [0, T] \) arise from time dependent sesquilinear forms. More precisely, let \( H \) and \( V \) denote two separable Hilbert spaces such that \( V \) is continuously and densely embedded into \( H \) (we write \( V \hookrightarrow_d H \)). Let \( V' \) be the antidual of \( V \) and denote by \( \langle \cdot, \cdot \rangle \) the duality between \( V' \) and \( V \). As usual, we identify \( H \) with \( H' \) and we obtain that \( V \hookrightarrow_d H \cong H' \hookrightarrow V' \). These embeddings are continuous and dense (see e.g., [9]). Let
\[ a : [0, T] \times V \times V \to \mathbb{C} \]
be a closed non-autonomous form, i.e., \( a(t, \cdot, \cdot) \) is sesquilinear for all \( t \in [0, T] \), \( a(\cdot, u, v) \) is measurable for all \( u, v \in V \),
\[ |a(t, u, v)| \leq M \|u\|_V \|v\|_V \quad (t \in [0, T], u, v \in V) \]
and
\[ \text{Re} \ a(t, u, u) + \omega \|u\|_V^2 \geq \alpha \|u\|_V^2 \quad (t \in [0, T], u \in V) \]
for some \( \alpha > 0, M > 0 \) and \( \omega \in \mathbb{R} \). The operator \( A(t) \in \mathcal{L}(V, V') \) associated with \( a(\cdot, \cdot) \) on \( V' \) is defined for each \( t \in [0, T] \) by
\[ \langle A(t)u, v \rangle = a(t, u, v) \quad (u, v \in V). \]
Seen as an unbounded operator on \( V' \) with domain \( D(A(t)) = V \), the operator \( -A(t) \) generates a holomorphic \( C_0 \)-semigroup \( T_t \) on \( V' \). The semigroup is bounded

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on a sector if $\omega = 0$, in which case $\mathcal{A}$ is an isomorphism. We denote by $A(t)$ the part of $\mathcal{A}(t)$ on $H$; i.e.,

$$
D(A(t)) := \{ u \in V : \mathcal{A}(t)u \in H \}
$$

$$
A(t)u = \mathcal{A}(t)u.
$$

It is a known fact that $-\mathcal{A}(t)$ generates a holomorphic $C_0$-semigroup $T$ on $H$ and $T = T|_H$ is the restriction of the semigroup generated by $-\mathcal{A}$ to $H$. Then $A(t)$ is the operator induced by $a(t, \cdot, \cdot)$ on $H$. We refer to [1, 10, 23, Chap. 2] and also by Bardos [8] under additional regularity assumptions on $a$. In 1961 J. L. Lions proved that the non-autonomous Cauchy problem

$$
(1.2) \quad \dot{u}(t) + A(t)u(t) = f(t), \quad u(0) = u_0.
$$

has $L^2$-maximal regularity in $V'$:

**Theorem 1.1. (Lions)** For all $f \in L^2(0, T; V')$ and $u_0 \in H$, the problem $(1.2)$ has a unique solution $u \in MR(V, V') := L^2(0, T; V) \cap H^1(0, T; V')$.

Lions proved this result in [18] (see also [24, Chapter 3]) using a representation theorem of linear functionals due to himself and usually known in the literature as Lions's representation Theorem and using Galerkin's method in [12, XVIII, Chapter 3, p. 620]. We refer also to an alternative proof given by Tanabe [23, Section 5.5].

In Theorem 1.1 only measurability of $a : [0, T] \times V \times V \to \mathbb{C}$ with respect to the time variable is required to have a solution $u \in MR(V, V')$. Nevertheless, in applications to boundary value problems, like heat equations with non-autonomous Robin-boundary-conditions or Schrödinger equations with time-dependent potentials, this is not sufficient. One is more interested in $L^2$-maximal regularity in $H$ rather than in $V'$, i.e., in solutions which belong to

$$
(1.3) \quad MR(V, H) := L^2(0, T; V) \cap H^1(0, T; H)
$$

rather than in $MR(V, V')$. Lions asked a long time before in [18, p. 68] whether the solution $u$ of $(1.2)$ belongs to $MR(V, H)$ in the case where $a(t; u, v) = a(t; u, v)$ and $t \to a(t; u, v)$ is only measurable.

Dier [14] has recently showed that in general the unique assumption of measurability is not sufficient to have $u \in MR(V, H)$. However, several progress are already has been done by Lions [18, p. 68, p. 94,], [18, Theorem 1.1, p. 129] and [18, Theorem 5.1, p. 138] and also by Bardos [8] under additional regularity assumptions on the form $a$, the initial value $u_0$ and the inhomogeneity $f$. More recently, this problem has been studied with some progress and different approaches by Arendt, Dier, Laasri and Ouhabaz [5], Arendt and Monniaux [6], Ouhabaz [20], Dier [14], Haak and Ouhabaz [10], Ouhabaz and Spina [21]. Results on multiplicative perturbations are also established in [5, 11, 17].

In [15] we proved Theorem 1.1 by a completely different approach developed in [14] and [17]. The method uses an appropriate approximation of the $\mathcal{A}(\cdot)$. Namely, let $\Lambda := (0 = \lambda_0 < \lambda_1 < \ldots < \lambda_{n+1} = T)$ be a subdivision of $[0, T]$. Consider the following approximation $A^S_{\Lambda} : [0, T] \to \mathcal{L}(V, V')$ of $\mathcal{A}$ given by

$$
A^S_{\Lambda}(t) := \begin{cases}
A_k & \text{for } \lambda_k \leq t < \lambda_{k+1}, \\
A_n & \text{for } t = T,
\end{cases}
$$

with

$$
A_k u := \frac{1}{\lambda_{k+1} - \lambda_k} \int_{\lambda_k}^{\lambda_{k+1}} \mathcal{A}(r)udr \quad (u \in V, k = 0, 1, \ldots, n).
$$

("S" stands for step). The integral above makes sense since $t \mapsto \mathcal{A}(t)u$ is Bochner integrable on $[0, T]$ with values in $V'$ for all $u \in V$. Note that $\|A(t)u\|_{V'} \leq M\|u\|_V$.
for all \( u \in V \) and all \( t \in [0, T] \). It is worth to mention that the mapping \( t \mapsto \mathcal{A}(t) \)

is strongly measurable by the Dunford-Pettis Theorem [2] since the spaces are assumed to be separable and \( t \mapsto \mathcal{A}(t) \) is weakly measurable.

It has been proved in [15] Theorem 3.2 that for all \( u_0 \in H \) and \( f \in L^2(0, T; V') \), the non-autonomous problem

\[
\dot{u}_\Lambda(t) + A_{\Lambda}(t)u_{\Lambda}(t) = f(t), \quad u_{\Lambda}(0) = u_0
\]

has an (explicit) unique solution \( u_{\Lambda} \in MR(V, V') \), and \( (u_{\Lambda}) \) converges weakly in \( MR(V, V') \) as \( |\Lambda| \to 0 \) to the unique solution \( u \) of (1.22). If we consider \( u_0 \in V \) and \( f \in L^2(0, T; H) \) then the solution \( u_{\Lambda} \) of (1.4) belongs to \( MR(V, H) \cap C([0, T]; V) \) (see [17], [15]). If moreover, \( a \) is assumed to be piecewise Lipschitz-continuous on \([0, T]\) then we obtain the convergence of \( u_{\Lambda} \in MR(V, H) \) [15] (see also [5]).

In this paper we are concerned with the recent result obtained in [11]. Instead of (see [17], [15]). If moreover, \( a \) is assumed to be separable and is strongly measurable by the Dunford-Pettis Theorem [2] since the spaces are everywhere to \( p \)-norm \((\Lambda) \) follows.

2. Preliminary

Let \( X \) be a Banach space and \( T > 0 \). Recall that a point \( t \in [0, T] \) is said to be a Lebesgue point of a function \( f : [0, T] \to X \) if

\[
\lim_{h \to 0} \frac{1}{t+h} \int_t^{t+h} \| f(s) - f(t) \|_X \, ds = 0.
\]

Clearly each point of continuity of \( f \) is a Lebesgue point. By [2] Proposition 1.2.2 if \( f \) is Bochner integrable then almost all point are Lebesgue points.

Let \( D \) be another Banach space such that \( D \) is continuously and densely embedded into \( X \) and let \( A : [0, T] \to \mathcal{L}(D, X) \) be a bounded and strongly measurable function, i.e., for each \( x \in D \) the function \( A(.) : [0, T] \to X \) is measurable and bounded.

Let \( \Lambda := (0 = \lambda_0 < \lambda_1 < \ldots < \lambda_{n+1} = T) \) be a subdivision of \([0, T]\). We consider the following approximations of \( A : [0, T] \to \mathcal{L}(D, X) \) by step operator function \( A^S_\Lambda : [0, T] \to \mathcal{L}(D, X) \) and piecewise linear operator function \( A^L_\Lambda : [0, T] \to \mathcal{L}(D, X) \) given by

\[
A^S_\Lambda(t) := \begin{cases} A_k & \text{for } \lambda_k \leq t < \lambda_{k+1}, \\ A_n & \text{for } t = T, \end{cases}
\]

and

\[
A^L_\Lambda(t) := \frac{\lambda_{k+1} - t}{\lambda_{k+1} - \lambda_k} A_k + \frac{t - \lambda_k}{\lambda_{k+1} - \lambda_k} A_{k+1}, \quad t \in [\lambda_k, \lambda_{k+1}],
\]

where

\[
A_k x := \frac{1}{\lambda_{k+1} - \lambda_k} \int_{\lambda_k}^{\lambda_{k+1}} A(r) x \, dr \quad (x \in D, k = 0, 1, \ldots, n).
\]

Let \( |\Lambda| := \max_{j=0,1,\ldots,n} (\lambda_{j+1} - \lambda_j) \) denote the mesh of the subdivision \( \Lambda \). Assume that the subdivision \( \Lambda \) is uniform, i.e., \( \lambda_{k+1} - \lambda_k = T/n = |\Lambda| \) for all \( k = 0, 1, \ldots, n \).

In the following Lemma, we show that \( A^S_\Lambda \) and \( A^L_\Lambda \) converge strongly and almost everywhere to \( A \) as \( |\Lambda| \to 0 \), from which the strong convergence with respect to \( L^p \)-norm (\( p \in [1, \infty) \)) follows.

**Lemma 2.1.** Let \( A^S_\Lambda : [0, T] \to \mathcal{L}(D, X) \) be given as above. Then:

i) For all \( x \in D \) we have \( A^S_\Lambda(t)x \to A(t)x \) \( t \)-a.e. on \([0, T]\) as \( |\Lambda| \to 0 \).

ii) \( A^L_\Lambda(.)(u_{\Lambda}(.)) \to A(.)u(.) \) in \( L^p(0, T; X) \) as \( |\Lambda| \to 0 \) if \( u_{\Lambda} \in L^p(0, T; D) \) such that \( u_{\Lambda} \to u \) in \( L^p(0, T; D) \).
Proof. Let \( C \geq 0 \) such that \( \|A(t)x\|_X \leq C\|x\|_D \) for all \( x \in D \) and for almost every \( t \in [0, T] \). We have \( \|A_kx\|_X \leq C\|x\|_D \) for all \( x \in D \) and \( k = 0, 1, \ldots, n \). Let \( t \) be any Lebesgue point of \( A(\cdot)x \). Let \( k \in \{0, 1, \ldots, n\} \) such that \( t \in [\lambda_k, \lambda_{k+1}) \). Then

\[
A^S_{\lambda(t)}x - A(t)x = \frac{1}{\lambda_{k+1} - \lambda_k} \int_{\lambda_k}^{\lambda_{k+1}} (A(r)x - A(t)x)dr
\]

\[
= \frac{1}{\lambda_{k+1} - \lambda_k} \int_{\lambda_k}^{t} (A(r)x - A(t)x)dr + \frac{1}{\lambda_{k+1} - \lambda_k} \int_{t}^{\lambda_{k+1}} (A(r)x - A(t)x)dr
\]

\[
= \left( \frac{t - \lambda_k}{\lambda_{k+1} - \lambda_k} \right) \frac{1}{t - \lambda_k} \int_{\lambda_k}^{t} (A(r)x - A(t)x)dr + \left( \frac{\lambda_{k+1} - t}{\lambda_{k+1} - \lambda_k} \right) \frac{1}{\lambda_{k+1} - \lambda_k} \int_{t}^{\lambda_{k+1}} (A(r)x - A(t)x)dr.
\]

It follows that \( A^S_{\lambda(t)}x - A(t)x \to 0 \) as \( |\Lambda| \to 0 \). Since almost all points of \([0, T]\) are Lebesgue points of \( A(\cdot)x \) the first assertion follows.

For the second assertion let \( x \in D \) and let \( \Omega \) be a measurable subset of \([0, T]\). We set \( w = x \otimes 1_{\Omega} \). Then \( \|A^S_{\lambda(t)}w - Aw\|_{L^p(0,T;X)} = \int_{\Omega} \|A^S_{\lambda(t)}(x) - A_t(x)\|_X^p dt \to 0 \) as \( |\Lambda| \to 0 \) by i) and Lebesgue’s Theorem. From which follows that \( \|A^S_{\lambda(t)}w - Aw\|_{L^p(0,T;X)} \to 0 \) as \( |\Lambda| \to 0 \) for all simple function \( w \) and thus for all \( w \in L^p(0,T;D) \). Let now \( w_\Lambda \in L^p(0,T;D) \) such that \( w_\Lambda \to w \) in \( L^p(0,T;D) \). Then

\[
\|A^S_{\lambda(t)}w_\Lambda - Aw\|_{L^p(0,T;X)} \leq C\|w_\Lambda - w\|_{L^p(0,T;D)} + \|A^S_{\lambda(t)}w - Aw\|_{L^p(0,T;X)}.
\]

Thus (ii) holds.

Instead of functions that are constant on each subinterval \([\lambda_k, \lambda_{k+1}[\), we consider now those that are linear.

**Lemma 2.2.** Let \( A : [0, T] \to \mathcal{L}(D, X) \) be a bounded and strongly measurable function. Then the following statements hold:

1. For all \( x \in D \) we have \( A^L_{\lambda(t)}x \to A(t)x \) \( t-a.e. \) on \([0, T]\) as \( |\Lambda| \to 0 \).

2. \( A^L_{\lambda(t)}u_\Lambda(\cdot) \to A(\cdot)u_\Lambda \) in \( L^p(0,T;X) \) as \( |\Lambda| \to 0 \) if \( u_\Lambda \in L^p(0,T;D) \) such that \( u_\Lambda \to u \) in \( L^p(0,T;D) \).

**Proof.** Let \( x \in D \) and let \( t \in [0, T] \) be an arbitrary Lebesgue point of \( A(\cdot)x \) and \( k \in \{0, 1, \ldots, n\} \) be such that \( t \in [\lambda_k, \lambda_{k+1}) \). Then

\[
A^L_{\lambda(t)}x - A(t)x = \frac{\lambda_{k+1} - t}{\lambda_{k+1} - \lambda_k} (A_kx - A(t)x) + \frac{t - \lambda_k}{\lambda_{k+1} - \lambda_k} (A_{k+1}x - A(t)x)
\]

\[
= I + II
\]

For the first term \( I \) we have

\[
I = \frac{\lambda_{k+1} - t}{\lambda_{k+1} - \lambda_k} \left( A^S_{\lambda(t)}x - A(t)x \right)
\]
which converges to zero as $|\Lambda| \to 0$ by Lemma 2.1. Now we show that $II$ converges also to zero as $|\Lambda|$ goes to $0$. Indeed, we have
\begin{equation}
A_{k+1}x - A(t)x = \frac{1}{\lambda_{k+2} - \lambda_{k+1}} \int_t^{\lambda_{k+2}} (A(r) - A(t))xdr
- \frac{1}{\lambda_{k+2} - \lambda_{k+1}} \int_t^{\lambda_{k+2}} (A(r) - A(t))xdr
= (\frac{\lambda_{k+2} - t}{\lambda_{k+2} - \lambda_{k+1}}) \frac{1}{\lambda_{k+2} - \lambda_{k+1}} t \int_t^{\lambda_{k+2}} (A(r) - A(t))xdr
\end{equation}
(2.2)
\begin{equation}
- (\frac{\lambda_{k+2} - t}{\lambda_{k+2} - \lambda_{k+1}}) \frac{1}{\lambda_{k+2} - \lambda_{k+1}} t \int_t^{\lambda_{k+2}} (A(r) - A(t))xdr
\end{equation}
(2.3)
Using again [2, Proposition 1.2.2] we obtain that both terms in (2.2) and (2.3) converge to 0 as $|\Lambda| \to 0$. Consequently $II$ converges to 0. The claim follows since $t$ is arbitrary Lebesgue point of $A(.)x$. The proof of (2) is the same as the proof of (ii) in Lemma 2.1.

3. APPROXIMATION AND CONVERGENCE

In this section $H, V$ are complex separable Hilbert spaces such that $V \hookrightarrow d H$. Let $T > 0$ and let
\[ a : [0, T] \times V \times V \to \mathbb{C} \]
be a non-autonomous closed form. This means that $a(t, ., .)$ is sesquilinear for all $t \in [0, T]$, $a(t, u, v)$ is measurable for all $u, v \in V$,
\begin{equation}
|a(t, u, v)| \leq M||u||_V||v||_V \quad (t \in [0, T], u, v \in V)
\end{equation}
and
\begin{equation}
\text{Re} a(t, u, u) + \omega||u|| \geq \alpha||u||^2_V \quad (t \in [0, T], u \in V)
\end{equation}
for some $\alpha > 0, M \geq 0$ and $\omega \in \mathbb{R}$. We assume in addition that $a$ is symmetric; i.e.,
\[ a(t, u, v) = \overline{a(t, v, u)} \quad (t \in [0, T], u, v \in V). \]

For almost every $t \in [0, T]$ we denote by $A(t) \in \mathcal{L}(V, V')$ the operator associated with the form $a(t, ., .)$ in $V'$. The non-autonomous Cauchy problem (1.2) has $L^2$-maximal regularity in $V'$, i.e., for given $f \in L^2(0, T; V')$ and $u_0 \in H$, (1.2) has a unique solution $u$ in $M_R(V, V') = L^2(0, T; V) \cap H^1(0, T; V')$. The maximal regularity space $M_R(V, V')$ is continuously embedded into $C([0, T], H)$ and if $u \in M_R(V, V')$ then the function $||u(.)||^2$ is absolutely continuous on $[0, T]$ and
\begin{equation}
\frac{d}{dt}||u(.)||^2_H = 2 \text{Re}(\dot{u}(.), u(.))
\end{equation}
see e.g., [22, Chapter III, Proposition 1.2] or [23, Lemma 5.5.1].

For simplicity we may assume without loss of generality that $\omega = 0$ in (1.2). In fact, let $u \in M_R(V, V')$ and let $v := e^{-w}u$. Then $v \in M_R(V, V')$ and it satisfies
\[ \dot{v}(t) + (\omega + A(t))v(t) = e^{-wt}f(t) \quad t-a.e. \text{ on } [0, T], \quad v(0) = 0 \]
if and only if $u$ satisfies (1.2). Throughout this section $\omega = 0$ will be our assumption.

Let $\Lambda = (0 = \lambda_0 < \lambda_1 \ldots < \lambda_{n+1} = T)$ be a uniform subdivision of $[0, T]$. Let
\[ a_k : V \times V \to \mathbb{C} \quad \text{for } k = 0, 1, \ldots, n \]
be the family of sesquilinear forms given for all $u, v \in V$ and $k = 0, 1, \ldots, n$ by
\begin{equation}
a_k(u, v) := \frac{1}{\lambda_{k+1} - \lambda_k} \int_{\lambda_k}^{\lambda_{k+1}} a(r, u, v)dr.
\end{equation}
Remark that $a_k$ satisfies (3.1) and (3.2) for all $k = 0, 1, ..., n$. The associated operators are denoted by $A_k \in \mathcal{L}(V, V')$ and are given for all $u \in V$ and $k = 0, 1, ..., n$ by

$$A_k u := \frac{1}{\lambda_{k+1} - \lambda_k} \int_{\lambda_k}^{\lambda_{k+1}} A(r) u dr.$$  

This integral is well defined. Indeed, the mapping $t \mapsto A(t)$ is strongly measurable by the Pettis Theorem [2] since $t \mapsto A(t)$ is weakly measurable and the spaces are assumed to be separable. On the other hand, $\|A(t)u\|_{V'} \leq M\|u\|_V$ for all $u \in V$ and $t \in [0, T]$. Thus $t \mapsto A(t)u$ is Bochner integrable on $[0, T]$ with values in $V'$ for all $u \in V$.

The function $V$ is defined for $t \in [0, T]$ by

$$\mathbb{a}_k^L : [0, T] \times V \times V \to \mathbb{C}$$

defined for $t \in [\lambda_k, \lambda_{k+1}]$ by

$$\mathbb{a}_k^L(t; u, v) := \frac{\lambda_{k+1} - t}{\lambda_{k+1} - \lambda_k} a_k(u, v) + \frac{t - \lambda_k}{\lambda_{k+1} - \lambda_k} a_{k+1}(u, v) \quad (u, v \in V),$$

is a symmetric non-autonomous closed form and Lipschitz continuous with respect to the time variable $t \in [0, T]$. The associated time dependent operator is denoted by

$$A_k^L(t) : [0, T] \to \mathcal{L}(V, V')$$

and is given by

$$A_k^L(t) := \frac{\lambda_{k+1} - t}{\lambda_{k+1} - \lambda_k} A_k + \frac{t - \lambda_k}{\lambda_{k+1} - \lambda_k} A_{k+1} \text{ for } t \in [\lambda_k, \lambda_{k+1}]$$

Since $a_k, k = 0, 1, ..., n$ are symmetric, the function $\mathbb{a}_k(v, t)$ belongs to $W^{1, 1}(a, b)$ and the following rule formula

$$\dot{a}_k(v(t)) := \frac{d}{dt} a_k(v(t)) = 2(A_k v(t) \mid \dot{v}(t))$$

holds whenever $v \in H^1(a, b, H) \cap L^2(a, b, D(A_k))$, for all $[a, b], k = 0, 1, ..., n$ where $A_k$ is the part of $A_k$ in $H$. For the proof we refer to [3] Lemma 3.1.

**Theorem 3.1.** Given $f \in L^2(0, T; H)$ and $u_0 \in V$, there is a unique solution $u_\Lambda \in MR(V, H)$ of

$$\dot{u}_\Lambda(t) + A_k^L(t) u_\Lambda(t) = f(t), \quad u_\Lambda(0) = u_0.$$  

Moreover, $t \mapsto a_\Lambda(t, u_\Lambda(t)) \in W^{1, 2}(0, T)$ and

$$2 \text{Re}(A_k^L(t) u_\Lambda(t) \mid \dot{u}_\Lambda(t)) = \frac{d}{dt} (\mathbb{a}_k^L(t; u_\Lambda(t)) - \dot{a}_k^L(t; u_\Lambda(t)) \quad t.a.e.$$

**Proof.** The first part of the theorem follows from [18], [5] Theorem 4.2, [15] since $t \mapsto \mathbb{a}_k^L(t, u, v)$ is piecewise $C^1$ for all $u, v \in V$. The rule product follows also from [5] Theorem 3.2], but it can be also seen directly from

$$\mathbb{a}_k^L(t; u_\Lambda(t)) = \int_0^t 2 \text{Re}(A_\Lambda(s) u_\Lambda(s) \mid \dot{u}(s)) ds$$

$$+ \int_0^t \dot{a}_k^L(r, u_\Lambda(r)) dr + \mathbb{a}_k^0(0, u_0) \quad (t \in [0, T])$$

which holds for all $t \in [0, T]$. In fact, let $\delta > 0$, $t \in [0, T]$ be arbitrary and let $l \in \{0, 1, ..., n\}$ be such that $t \in [\lambda_l, \lambda_{l+1}]$. In order to apply the classical product
rule (3.10), we seek regularizing $u_A$ by multiplying with $e^{-\delta A_k}$ and $e^{-\delta A_{k+1}}$. Then
\[
\int_{\lambda_k}^{\lambda_{k+1}} (\mathcal A(s)u_A(s)\tilde u_A(s))_H ds
\]
\[
= \lim_{\delta \to 0} \int_{\lambda_k}^{\lambda_{k+1}} \left( \frac{\lambda_{k+1} - \tau}{\lambda_{k+1} - \lambda_k} (\mathcal A_k e^{-\delta A_k} u_A(s)\tilde u_A(s))_H + \frac{\tau - \lambda_k}{\lambda_{k+1} - \lambda_k} (\mathcal A_{k+1} e^{-\delta A_{k+1}} u_A(s)\tilde u_A(s))_H \right) ds
\]
for $k = 0, 1, ..., l-1$. Using (3.10) and integrating by part we obtain by an easy calculation
\[
2 \Re \int_{\lambda_k}^{\lambda_{k+1}} (\mathcal A(s)u_A(s)\tilde u_A(s))_H ds
\]
\[
= \lim_{\delta \to 0} \int_{\lambda_k}^{\lambda_{k+1}} \frac{1}{\lambda_{k+1} - \lambda_k} \left[ a_k+1(e^{-\delta A_{k+1}} u_A(\lambda_{k+1})) - a_k(e^{-\delta A_k} u_A(\lambda_k)) \right] ds
\]
\[
= a_{k+1}(u_A(\lambda_{k+1})) - a_k(u_A(\lambda_k)) - \int_{\lambda_k}^{\lambda_{k+1}} \frac{1}{\lambda_{k+1} - \lambda_k} \left[ a_k+1(u_A(s)) - a_k(u_A(s)) \right] ds
\]
\[
= a_{k+1}(u_A(\lambda_{k+1})) - a_k(u_A(\lambda_k)) - \int_{\lambda_k}^{\lambda_{k+1}} a_A(s,u_A(s)) ds
\]
for $k = 0, 2, ..., l-1$, here we have use that the restriction of $(e^{-tA_k})_{t \geq 0}$ on $V$ is a $C_0$-semigroup. By a similar argument as above we obtain for the integral over $(\lambda_l, t)$
\[
2 \Re \int_{\lambda_l}^{t} (\mathcal A(s)u_A(s)\tilde u_A(s))_H ds
\]
\[
= \frac{\lambda_{l+1} - t}{\lambda_{l+1} - \lambda_l} a_l(u_A(t)) + \frac{t - \lambda_l}{\lambda_{l+1} - \lambda_l} a_{l+1}(u_A(t)) - a_l(u_A(\lambda_l))
\]
\[
- \int_{\lambda_l}^{t} \frac{1}{\lambda_{l+1} - \lambda_l} \left[ a_{l+1}(u_A(s)) - a_l(u_A(s)) \right] ds
\]
\[
= a^L_l(t,u_A(t)) - a_l(u_A(\lambda_l)) - \int_{\lambda_l}^{t} \hat a_A(s,u_A(s)) ds
\]
Consequently
\[
2 \Re \int_{0}^{t} (\mathcal A(s)u_A(s)\tilde u_A(s))_H ds
\]
\[
= 2 \Re \sum_{k=0}^{l-1} \int_{\lambda_k}^{\lambda_{k+1}} (\mathcal A(s)u_A(s)\tilde u_A(s))_H ds + 2 \Re \int_{\lambda_l}^{t} (\mathcal A(s)u_A(s)\tilde u_A(s))_H ds
\]
\[
= -a_0(u_0) + a^L_l(t,u_A(t)) - \int_{0}^{t} \hat a^L_A(r,u_A(r)) dr
\]
This completes the proof. \qed

The next proposition shows that $u_A$ from Theorem 3.1 approximates the solution of (1.2) with respect to the norm of $MR(V, V')$. 

Proposition 3.1. Let \( f \in L^2(0, T; H) \) and \( u_0 \in V \) and let \( u_\Lambda \in MR(V, H) \) be the solution of (1.2). Then \( u_\Lambda \) converges strongly in \( MR(V, V') \) as \( |\Lambda| \to 0 \) to the solution of (1.2).

Proof. Let \( f \in L^2(0, T; H) \) and \( u_0 \in V \). Let \( u, u_\Lambda \in MR(V, V') \) be the solution of (1.2) and (3.11) respectively. Set \( w_\Lambda := u_\Lambda - u \) and \( g_\Lambda := (A - A_\Lambda^k)u \). Then \( w_\Lambda \in MR(V, V') \) and satisfies

\[
\dot{w}_\Lambda(t) + A^k_\Lambda(t)w_\Lambda(t) = g_\Lambda(t), \quad w_\Lambda(0) = 0.
\]

From the product rule (3.3) it follows

\[
\frac{d}{dt} \|w_\Lambda(t)\|_H^2 = 2 \text{Re} \{g_\Lambda(t) - A^k_\Lambda(t)w_\Lambda(t), w_\Lambda(t)\}
\]

\[
= -2 \text{Re} \{a^k_\Lambda(t, w_\Lambda(t), w_\Lambda(t)) + 2 \text{Re} \{g_\Lambda(t), w_\Lambda(t)\}\}
\]

for almost every \( t \in [0, T] \). Integrating this equality on \((0, t)\), we obtain

\[
\alpha \int_0^t \|w_\Lambda(s)\|_V^2 ds \leq \int_0^t \|g_\Lambda(s)\|_V^2 \|w_\Lambda(s)\|_V ds.
\]

This estimate and the Young’s inequality

\[
ab \leq \frac{1}{2} (a^2 + \varepsilon b^2) \quad (\varepsilon > 0, \ a, b \in \mathbb{R}).
\]

yield the estimate

\[
\alpha \|w_\Lambda\|_{L^2(0, T; V)}^2 \leq 1/\alpha \|g_\Lambda\|_{L^2(0, T; V')}^2.
\]

The term of the right hand side of this inequality converges by Proposition 2.2 to 0 as \( |\Lambda| \to 0 \). It follows that \( u_\Lambda \to u \) strongly in \( L^2(0, T; V) \). Again from the second assertion of Proposition 2.2 follows that \( A_\Lambda u_\Lambda \to Au \) in \( L^2(0, T; V') \). Letting \( |\Lambda| \) go to 0 in

\[
\dot{w}_\Lambda = \dot{u}_\Lambda - \dot{u} = f - A_\Lambda^k u_\Lambda - \dot{u}
\]

and recalling the continuous embedding of \( MR(V, V') \) into \( C([0, T]; H) \) imply the claim. \( \square \)

Next we assume additionally, as in [10] or [11], that there exists a bounded and non-decreasing function \( g : [0, T] \to L(H) \) such that

\[
|a(t; u, v) - a(s; u, v)| \leq (g(t) - g(s)) \|u\|_V \|v\|_V
\]

for \( u, v \in V, s, t \in [0, T], s \geq t \). Our aim is the show that under this assumption the solution \( u_\Lambda \) of (3.11) converges weakly in \( MR(V, H) \) as \( |\Lambda| \to 0 \) and that the limit satisfies (1.2). Without loss of generality, we will assume that \( g(0) = 0 \). Thus \( g \) is positive. Let

\[
g^k_\Lambda : [0, T] \to [0, \infty[\]

denote the analogous function to (3.8) and (3.9) for \( g \). Assume that the subdivision \( \Lambda \) is uniform, i.e., \( \lambda_{k+1} - \lambda_k = T/n = |\Lambda| \) for all \( k = 0, 1, ..., n \).

Lemma 3.2.

\[
|A^k_\Lambda(t; u, v) - A^k_\Lambda(s; u, v)| \leq |g^k_\Lambda(t) - g^k_\Lambda(s)| \|u\|_V \|v\|_V
\]

for all \( u, v \in V \) and \( t, s \in [0, T] \) with \( s \leq t \).
Proof. It suffices to show (3.11) for \( t, s \in [\lambda_k, \lambda_{k+1}] \) for some \( k \in \{0, 1, \ldots, n\} \). The general case where \( t, s \) belong to two different subintervals follows immediately. Let \( u, v \in V \), then

\[
\begin{align*}
\alpha^L(t; u, v) - \alpha^L(s; u, v) &= \frac{t - s}{\lambda_{k+1} - \lambda_k} a_{k+1}(u, v) - \frac{t - s}{\lambda_{k+1} - \lambda_k} a_k(u, v) \\
&= \frac{t - s}{\lambda_{k+1} - \lambda_k} \frac{n}{T} \int_0^T |a(r + \lambda_{k+1}; u, v) - a(r + \lambda_k; u, v)| dr.
\end{align*}
\]

Thus (3.13) implies

\[
|\alpha^L(t; u, v) - \alpha^L(s; u, v)|
\leq \frac{t - s}{\lambda_{k+1} - \lambda_k} \frac{n}{T} \int_0^T |g(r + \lambda_{k+1}) - g(r + \lambda_k)| dr ||u||_V ||v||_V
\]

\[
= \frac{t - s}{\lambda_{k+1} - \lambda_k} \frac{\lambda_{k+2} - \lambda_{k+1}}{\lambda_{k+1}} \int_{\lambda_{k+1}}^{\lambda_k} g(r) dr
\]

\[
= \frac{t - s}{\lambda_{k+1} - \lambda_k} \frac{1}{\lambda_{k+2} - \lambda_{k+1}} \int_{\lambda_{k+1}}^{\lambda_k} g(r) dr ||u||_V ||v||_V
\]

\[
= |g^L(t) - g^L(s)||u||_V ||v||_V
\]

\[
\boxed{= \|u\|_V \|v\|_V}
\]

\[
\square
\]

The main result of this section is the following

Theorem 3.3. Assume that the non-autonomous closed form \( a \) is symmetric and satisfies (3.12). Let \( f \in L^2(0,T; H) \) and \( u_0 \in V \) and let \( u_A \in MR(V,H) \) be the solution of (3.11). Then \( (u_A) \) converges weakly in \( MR(V,H) \) as \( |\Lambda| \rightarrow 0 \) and \( u = \lim_{|\Lambda| \rightarrow 0} u_A \) satisfies (3.3).

Proof. a) First since \( u_A \) satisfies (3.11) then

\[
\|\dot{u}_A(t)\|_H + (A^L(t)u_A(t) \mid \dot{u}_A(t))_H = (f(t) \mid \dot{u}_A(t))_H \ 	ext{a.e}
\]

The product rule (3.12), Cauchy-Schwartz inequality and Young’s inequality imply that for almost every \( t \in [0,T] \)

\[
\|\dot{u}_A(t)\|_H^2 + \frac{d}{dt} (A^L(t; u_A(t))) \leq \|f(t)\|_H^2 + \dot{a}_A^L(t; u_A(t)).
\]

Integrating now this inequality on \([0, t]\), it follows that

\[
\int_0^t \|\dot{u}_A(r)\|_H^2 dr + \alpha \|u_A(t)\|_V^2 \leq M \|u_0\|_V^2 + \int_0^t \|f(r)\|_H^2 dr + \int_0^t \dot{a}_A^L(r; u_A(r)) dr
\]

where \( \alpha \) and \( M \) are the constants in (5.1)-(5.2).

b) Note that by construction the derivative \( \dot{a}_A^L \) of \( a_A^L \) equals

\[
\dot{a}_A^L(r; u) = \frac{a_{k+1}(u) - a_k(u)}{\lambda_{k+1} - \lambda_k} \ 	ext{for a.e.} r \in [\lambda_k, \lambda_{k+1}], \ u \in V.
\]
Now, let $t \in [0, T]$ be arbitrary and let $l \in \{0, 1, \ldots, n\}$ be such that $t \in [\lambda_l, \lambda_{l+1}]$. Then

\[
\int_0^t \dot{a}_X^k(r; u_A(r))dr = \sum_{k=1}^{l-1} \int_{\lambda_k}^{\lambda_{k+1}} a^k(r; u_A(r))dr + \int_{\lambda_l}^t \dot{a}_X^k(r; u_A(r))dr
\]

\[
= \sum_{k=1}^{l-1} \int_{\lambda_k}^{\lambda_{k+1}} \frac{a_{k+1}(u_A(r)) - a_k(u_A(r))}{\lambda_{k+1} - \lambda_k} dr + \int_{\lambda_l}^t \frac{a_{l+1}(u_A(r)) - a_l(u_A(r))}{\lambda_{l+1} - \lambda_l} dr
\]

\[
= \sum_{k=1}^{l-1} \int_{\lambda_k}^{\lambda_{k+1}} \frac{a^k_X(\lambda_{k+1}; u_A(r)) - a^k_X(\lambda_k; u_A(r))}{\lambda_{k+1} - \lambda_k} dr + \int_{\lambda_l}^t \frac{a^k_X(\lambda_{l+1}; u_A(r)) - a^k_X(\lambda_l; u_A(r))}{\lambda_{l+1} - \lambda_l} dr.
\]

By Lemma 3.2 it follows that

\[
\int_0^t \dot{a}_X^k(r; u_A(r))dr \leq \sum_{k=1}^{l-1} \int_{\lambda_k}^{\lambda_{k+1}} \frac{g^k_X(\lambda_{k+1}) - g^k_X(\lambda_k)}{\lambda_{k+1} - \lambda_k} \|u_A(r)\|_V^2 dr + \int_{\lambda_l}^t \frac{g^k_X(\lambda_{l+1}) - g^k_X(\lambda_l)}{\lambda_{l+1} - \lambda_l} \|u_A(r)\|_V^2 dr
\]

\[
= \sum_{k=1}^{l-1} \int_{\lambda_k}^{\lambda_{k+1}} \dot{g}^k_X(r) \|u_A(r)\|_V^2 dr + \int_{\lambda_l}^t \dot{g}^k_X(r) \|u_A(r)\|_V^2 dr
\]

\[
= \int_0^t \dot{g}^k_X(r) \|u_A(r)\|_V^2 dr
\]

c) Using an analogous calculus as in part b) and the fact that

\[
g^k_X(r) = \frac{g_{k+1} - g_k}{\lambda_{k+1} - \lambda_k}
\]

for a.e. $r \in [\lambda_k, \lambda_{k+1}]$

we can easily see that

\[
(3.16)
\]

\[
\int_0^t \dot{g}^k_X(r)dr \leq g(T)
\]

since the function $g$ is positive and non-decreasing.

d) As a consequence of (3.15), the parts b)-c) and Gronwall’s lemma it follows that

\[
\sup_{r \in [0, T]} \|u_A(t)\|_V^2 \leq 1/\alpha [M\|u_0\|_V + \int_0^T \|f(r)\|_H dr] \exp(g(T)/\alpha).
\]

Inserting this estimate into (3.15), we find that there exists $c = c(\alpha, g(T), M) \geq 0$ such that

\[
(3.17)
\]

\[
\int_0^T \|\dot{u}_A(s)\|_H^2 ds \leq c \left[\|u_0\|_V^2 + \|f\|_{L^2(0,T; H)}^2\right]
\]

Since $u_A(t) = u_A(0) + \int_0^t \dot{u}_A(s)ds$, there exists a constant $c = c_H, T)$ with

\[
(3.18)
\]

\[
\int_0^T \|u_A(s)\|_H^2 ds \leq c \left[\|u_0\|_V^2 + \|\dot{u}_A\|_{L^2(0,T; H)}^2\right],
\]

where $c_H$ is the embedding constant of the embedding of $V$ into $H$. This estimate and (3.17) yield the estimate

\[
(3.19)
\]

\[
\|u_A\|_{L^2(0,T; H)}^2 \leq c \left[\|u_0\|_V^2 + \|f\|_{L^2(0,T; H)}^2\right]
\]
for some constant $c = c(\alpha, M, cH, g(T), T) > 0$ independent of the subdivision $\Lambda$.

e) It follows from the parts a) − d) that $u_\Lambda$ is bounded in $H^1(0, T; H)$. On other hand and as mentioned, Problem (1.2) has a unique solution $u$ in $MR(V, V')$ and we have seen in Proposition 3.1 that $MR(V, H) \ni u_\Lambda \to u$ in $MR(V, V')$. As a consequence $u \in MR(V, H)$. This completes the proof. □

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