Where-When-What: the General Relativity of Space-Time-Property

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We develop the general relativity of extended spacetime-property for describing events including their properties. The anticommuting nature of property coordinates, augmenting space-time \((x,t)\), allows for the natural emergence of generations and for the simple incorporation of gauge fields in the spacetime-property sector. With one electric property this results in a geometrical unification of gravity and electromagnetism, leading to a Maxwell-Einstein Lagrangian plus a cosmological term. Addition of one neutrinic and three chromic properties should lead to unification of gravity with electroweak and strong interactions.

1. A full description of events

In a static universe nothing happens. All systems continue inertially, never change and never interact; that sort of universe is the ultimate non-event. Even speaking of an ‘observer’ is a contradiction in terms, because every observer would be incommunicado and unaware of anything and everything. So, more to the point, a static universe is purely hypothetical and logically inconceivable. On the contrary the real universe is in a state of flux. It is punctuated by a succession of events from which we gain the notion of space and time, as the scenario unfolds. Historically the spacetime arena has held centre stage ever since the ideas of relativity took hold about a hundred years ago and we are accustomed to characterising an event by its spacetime location, even to the extent of describing events geometrically. Indeed the geometry of curved spacetime has revolutionised our ideas about gravity ever since Einstein’s development of general relativity.

When an event occurs, it amounts to a change in circumstances, whereby an object alters its motion and possibly character as it engages with others. (It is the impact of these changes in spacetime which underpins the process of observation and the Heisenberg uncertainty principle.) Thus when a photon is emitted and reabsorbed by two charged objects we interpret this as a succession of two events resulting in an electric interaction between the charged objects. And since quantum field theory came into being, we recognise this through a trilinear interaction between the charged object and a photon with the conservation of total energy-momentum at the event vertex, which is ensured by taking an integral over all spacetime of the interacting fields. So much is well understood. However saying that an event has
happened *there and then* does not fully specify the event; we must in addition specify *what* exactly has happened: what sort of transaction has taken place and what properties may have been exchanged. For instance when a proton emits a neutron and charged pion virtually we have to add that information and then the event becomes fully explicated. This is normally done by introducing quantum fields with particular labels and interacting in a manner that usually conserves some quantum number, such as isospin for the nuclear example.

As physicists, we are well aware that particle properties do proliferate but they can be systematised using group theory of some “internal group”, resulting in certain types of representations of various Lie algebras. These can be constructed from some fundamental representation as a result of some basic dynamics connected with primitive constituents like quarks and leptons. As new particles are discovered it occasionally becomes necessary to enlarge the group to accommodate features or properties that do not fall comfortably within the conventional picture; this is how progress in quantum field theory has been achieved and it has led to its ultimate incarnation as the standard model, wherein $\text{U}(1) \times \text{SU}(2) \times \text{SU}(3)$ reigns supreme. Nevertheless some issues remain unresolved, such as the generation problem, where repetitions of particle families lead to further concepts about “horizontal groups” for systematising them – although what the correct group and representations are is still unsettled. Most worrying of all is the number of parameters needed to characterise the interactions and masses of the (light) three generations in the standard model, though it must be admitted that those parameters are sufficient to describe a vast number of experimental facts. This has stimulated research into looking for some kind of grand unification for reducing the number of parameters and it has spawned a large number of interesting new concepts. Supersymmetry and superstrings, based on enlarged dimensions, are the foremost amongst these as they automatically solve the fine-tuning problem and naturally incorporate gravity, but we should not discount other ideas such as technicolour, preons, non-commuting spacetime variables, deformation groups, and so on.

Underlying all these extensions, the question arises: how to enumerate properties and characteristics of events at a fundamental particulate level. Traditionally one specifies the associated quantum fields by attaching as many labels to them as needed to describe the properties and by ensuring that the events due to their mutual interactions conserve whatever properties remain intact overall; in other words adapting to what the experiments dictate. This approach is predictive to the extent that if there are any missing components of the group representations, then they must eventually show up experimentally and conform to the overarching symmetry. And if the interactions do not preserve the expected symmetries, they are nowadays thought to be due to symmetry breaking effects coming from a yet unknown cause, possibly spontaneously generated.

In an effort to characterise the nature of an event, we have, over a number of years, suggested that it may be possible to specify attributes or ‘properties’ of particles by connecting them with a few anticommuting Lorentz *scalar* coordinates
(and their conjugates or opposites). An event can thereby be fully described by a conglomerate there-then-what label; by analogy to momentum conservation, the conservation of overall property, such as charge, is guaranteed by integrating appropriately over all property space. To that end we have suggested that the spacetime manifold with coordinates $x$, should be enlarged to a supermanifold $\mathbb{X}$ by attaching a few complex anticommuting $\zeta$ coordinates, with fields being regarded as functions of $X = (x, \zeta, \bar{\zeta})$. The anticommuting character of fermionic quantum fields is nothing new and the use of Grassmann variables has featured in the BRST quantisation of gauge theories, and more especially in the invention of supersymmetry – although the latter makes them Lorentz spinor rather than scalar. Once a property is ascribed it cannot be doubly ascribed, corresponding to the fact the square of a given Grassmann variable vanishes. But because the conjugate property is available, one can built up property scalars that may be multiplied with other properties, and so on. This is how one might conceive of families of systems with similar net attributes. Furthermore the anticommuting character of a fixed number of properties means that we can never encounter an infinite number of states, unlike bosonic extra dimensional spaces modelled along Kaluza-Klein lines. The mathematics of graded spaces is quite well understood now and we will take full advantage of this in what follows.

Previous investigations have shown that with a minimum of five independent $\zeta$ one can readily accommodate all the known particles and their generations. We might tentatively call these properties: charge or electricity, neutrinity and chronicity. The use of five is no accident, being based solidly on economical grand unified groups such as SU(5) and SO(10); what is more interesting is that the correct representations of those algebras come out automatically. [In our case Sp(10) is the overarching internal group.] Since addition of Grassmann variables has the mathematical effect of reducing the net ‘dimension’ of the space, there is the tantalising prospect of a universe with zero total dimension. We have elaborated on some consequences of such a scheme, including the appearance of a few new particles and the possibility of reducing the number of parameters appearing in the standard model, because we have just nine Higgs fields but only one coupling constant attached to all known fermions.

The purpose of the present paper is to describe the general relativity of spacetime-property along the lines originally devised by Einstein. Our aim is quite modest: we wish to see if one can unite electromagnetism with general relativity geometrically in a way which differs from Klein-Kaluza and (spinorial) supergravity in as much as the extended coordinates are scalar but anticommuting. Having established that, it means one may contemplate generalizations which include other forces, without producing infinite towers of excited states. This paper should be seen as a first step in that direction; further avenues for research are mentioned in the conclusions and include Higgs fields and possibly ghost fields associated with gauge-fixing at the quantized level. The extended metric will consist a $x - x$ sector,
a $\zeta - x$ sector and a $\zeta - \bar{\zeta}$ sector; the space-space sector involves the gravitational field as well as some extra $\zeta\zeta$ terms, the space-property sector brings in the gauge fields connected with property propagation, and the curved property-property sector can be the source of the cosmological term, as it happens; it is no surprise that the communicators of property or gauge fields should reside where they are, but the association of property curvature with a cosmological term is perhaps more intriguing. It is important to state that fermion fields, such as the gravitino, have no place in the extended metric as they would cause a conflict with Lorentz invariance. Because the fermionic superfield $\Psi_\alpha$ carries spinor indices, it must be studied separately and in that regard our approach differs radically from conventional spinorial supersymmetry; we give a preliminary treatment of fermions towards the end of the article.

As we are dealing with a graded manifold or supermanifold, it is essential to make sure that the super-coordinate transformations carry the correct commutation factors. This is carefully explained in Sections 2 to 5, where our conventions are established and summarized to make the paper self-contained. A good notation is of course vital and the ordinary general relativity convention with its placement of indices conflicts with conventional differentiation as a left operation (as a rule for the average reader); we have had to compromise on that as some traditional ideas are not easily overthrown. Asorey and Lavrov have written a nice exposition of these ideas but they have instead chosen to take right derivatives, which makes for marginally simpler formulae but does not conform with traditional ideas about left differentiation, which we have religiously adhered to. In Section 3 we delineate transformation properties of supertensors and the supermetric. Then we pay attention to the definition of covariant derivatives with particular application to the super-Riemann tensor $\hat{R}$; its supersymmetry properties are obtained and the super-Ricci and superscalar curvature are derived. The Bianchi identities follow in Section 5. For the remainder of the paper we consider the case of just one property, such as charge, and in Section 6 we write down the most general metric, including the electromagnetic field. Unsurprisingly we show that gauge transformations can be construed as supercoordinate transformations associated with phase changes on $\zeta$. We then find the superdeterminant of this metric in Section 7, as well as the Palatini form of the superscalar curvature. In Section 8 we go on to evaluate the super-Riemann tensor components and determine the super-Ricci tensor and full supercalar curvature. This leads to the field equations and it is rather pleasing that the Einstein-Maxwell Lagrangian emerges very naturally, together with a cosmological contribution. Further, the electromagnetic stress tensor presents itself as a purely geometrical addition to the extended Einstein tensor. All such calculations are greatly assisted by an algebraic computer program for handling anticommuting variables as well as ordinary ungraded ones, which has been developed by one of us (PDS) using Mathematica. A penultimate Section 9 details the inclusion of matter fields, but our treatment of fermions there is to be regarded as preliminary at this
stage. The conclusions close the paper and an Appendix collates a list of generalised Christoffel symbols, curvature components and super-vielbeins that are needed at intermediate steps in the main text.

2. Extended transformations and notation

The addition of extra anticommutative coordinates to space time results in a graded manifold, where the standard spacetime is even and the property sector is odd. The notation used in this work will be to define uppercase Roman indices (M, N, L, etc) to run over all the dimensions of spacetime-property and hence have mixed grading. Lower case roman indices (m, n, l, etc) will correspond to even graded spacetime, and Greek characters (µ, ν, λ, etc) will correspond to the odd graded property sector. The grading of an index given by [M] is [m] = 0 and [µ] = 1. Later on we reserve early letters of the alphabet (a, α, etc.) to signify flat or tangent space.

Our starting point is the transformation properties of contravariant and covariant vectors; from these we can build up how a general tensor should transform. We will make use of Einstein summation convention but it has to be done carefully. We pick a convention of always summing a contravariant index followed immediately by a covariant index (up then down). This results in contravariant and covariant vectors transforming as follows:

\[ V'^{M} = V^{N} \frac{\partial X^{M}}{\partial X'^{N}}, \quad V_{M}' = \frac{\partial X^{N}}{\partial X^{M}} V_{N}. \]  

(1)

The scalar \( V^{M} V_{M} \) then correctly transforms into itself:

\[ V'^{M} V'_{M} = V^{N} \frac{\partial X^{M}}{\partial X'^{N}} \frac{\partial X^{L}}{\partial X'^{M}} V_{L} = V^{N} \delta^{L}_{N} V_{L} = V^{N} V_{N}, \]  

(2)

since the (left) chain rule given by

\[ \frac{\partial X^{M}}{\partial X^{N}} \frac{\partial X^{L}}{\partial X^{M}} = \delta^{L}_{N}. \]  

(3)

From (1) one can build up the transformation properties of any tensor, by taking it to behave like a corresponding product of vectors. For example a rank two covariant tensor \( T_{MN} \) has to transform like \( V_{M} V_{N} \). Then

\[ V'_{M} V_{N}' = \frac{\partial X^{R}}{\partial X'^{M}} \frac{\partial X^{S}}{\partial X'^{N}} \frac{\partial X^{R}}{\partial X^{N}} V_{R} V_{S} \]

\[ = (-1)^{|R|(|S|+|N|)} \frac{\partial X^{R}}{\partial X'^{M}} \frac{\partial X^{S}}{\partial X^{N}} V_{R} V_{S}. \]  

(4)

In these manipulations we have adhered to the traditional convention of writing derivatives on the left, so the the sign factor arising in (4) is due to permuting \( V_{R} \)

\[ ^{a}\text{Asorey and Lavrov use the notation } \epsilon_{M} \text{ instead of } [M]. \]
through the partial derivative. In this way we find how \( T_{MN} \) transforms:

\[
T_{MN}' = (-1)^{|[N]|+[M]|} \frac{\partial X^R}{\partial X^M} \frac{\partial X^S}{\partial X^N} T_{RS}.
\]

(5)

Thus in (5) we do not have an immediate, direct up-down summation; the sign factor is introduced to compensate for this. In this manner it is not hard to derive sign factors for any sort of tensor.

3. On metrics and supertensors

The metric supertensor \( G_{MN} \) is chosen to be graded symmetric, \( G_{MN} = (-1)^{|M||N|} G_{NM} \) because it is associated with the generalised spacetime-property separation \( ds^2 = dX^N dX^M G_{MN} \), which is overall bosonic. As with standard general relativity the metric can be used to raise and lower indices; however the direct up-down summation rule must be strictly obeyed. This means that for vectors:

\[
V^M G_{MN} = V_N, \quad G^{MN} V_N = V^M
\]

(6)

If this order is not followed then the resulting vector (or tensor) will not transform correctly according to the rules in Section 2. When raising or lowering indices of a supertensor an adjoining up-down summation is sometimes impossible; in that event a sign factor like in (4) must again be included to compensate for this, using the same argument.

For illustration, consider a tensor \( T_{MN} \) whose second index \( N \) we wish to raise to get \( T^M_N \). To work out the sign factor required, look at a product of vectors instead, say \( T_{MN} = U_M V_N \); then

\[
T^M_N = U_M V^N = U_M G^{NL} V_L = (-1)^{|M|(|N|)+|L|} G^{NL} T_{ML}
\]

(7)

The sign factor ensures that \( (-1)^{|M|(|N|)+|L|} G^{NL} T_{ML} \) behaves like \( T^M_N \). This procedure extends to any tensor.

The inverse metric multiplies the covariant metric as follows:

\[
G^{MN} G_{NL} = \delta^M_L = (-1)^{|M|} \delta^M_L
\]

(8)

\[
(-1)^{|N|} G_{MN} G^{NL} = \delta^L_M
\]

(9)

These equations are consistent with each other, (the metric and its inverse being graded symmetric) and with the transformation properties of the singlet \( \delta^M_L \). They are also consistent with Asorey and Lavrov. In particular notice from (8) that the trace operation introduces a negative sign where the fermionic sector is concerned, as is well-known.

\[b\text{Note that in ref.7 the } G \text{ with raised indices differs from the present } G \text{ by a factor } (-1)^{|N|}.\]
4. Covariant derivatives and the Riemann supertensor

The connection coefficients of standard general relativity in the case of zero torsion are defined to be:

\[ \Gamma^k_{mn} = (g_{lm,n} + g_{ln,m} - g_{mn,l}) g^{lk}/2 \]  \hspace{1cm} (10)

We take this as the starting point for our covariant derivative, extending it to a graded manifold by allowing for sign factors.

\[ \Gamma^{MK}_{MN} = \left( (-1)^X G_{LM,N} + (-1)^Y G_{LN,M} - (-1)^Z G_{MN,L} \right) G^{LK}/2, \] \hspace{1cm} (11)

where \( X_{LMN}, Y_{LMN} \) and \( Z_{LMN} \) need to be determined so as to guarantee that the covariant derivative of a covariant vector transforms correctly as a rank 2 covariant tensor. We write the covariant derivative in semicolon notation as

\[ A_{M;N} = (-1)^W A_{M,N} - \Gamma^{MN}_{K} A_{K}, \] \hspace{1cm} (12)

where again \( W_{MN} \) is a sign factor to be found. Expanding this out and finding the conditions on the signs so that all second derivatives cancel and the remaining terms transform as a rank 2 covariant tensor we arrive at

\[ A_{M;N} = (-1)^{[M][N]} A_{M,N} - \Gamma^{MN}_{K} A_{K}, \] \hspace{1cm} (13)

with

\[ \Gamma^{MN}_{K} = \left[ (-1)^{[M][N] + [L]} G_{LM,N} + (-1)^{[L]} G_{NL,M} - (-1)^{[L][N] + [L]} G_{MN,L} \right] G^{LK}/2. \] \hspace{1cm} (14)

In a similar manner one may establish that

\[ A^M_{;N} = (-1)^{[M][N]} (A^M_{;N} + A^{L} \Gamma^{LN}_{M}) \] \hspace{1cm} and \hspace{1cm} (15)

\[ T_{LM;N} = (-1)^{N}[L]+[M]) T_{LM,N} - \Gamma_{NL,K} T_{K,M} - (-1)^{[L][N]+[K]} \Gamma_{NM,K} T_{LK}. \] \hspace{1cm} (16)

The curious factors of \((-1)^{[M][N]}\), etc. arise from the mismatch between left derivatives clashing with the convention of placing subscript such as \( _{,M} \) on the right and we are stuck with this inappropriateness. Anyhow, with these constructions of the covariant derivative, it is pleasing to check that \( G_{MN;N} \) vanishes, as expected. And from equations such as (13), (15), (16), one may deduce the rules for covariant derivatives of any supertensor.

The Riemann curvature tensor arises in the normal way (with suitable sign factors):

\[ (-1)^{[L]} A J R^J_{KLM} = A_{K;L,M} - (-1)^{[L][M]} A_{K;M,L} \] \hspace{1cm} (17)

\[ \text{with } (-1)^{[M][N]} = 1 \text{ for all } M,N \]
Carrying out the algebraic manipulations, we obtain
\[
R_{JKLM}^J = (-1)^{(|K|+|L|+|M|)} \left[ (-1)^{|K|} \Gamma_{KM}^I J, L - (-1)^{|K|+|M|} \Gamma_{KL}^I J, M \right.
\]
\[
+ (-1)^{|L|+|M|} \Gamma_{KM}^I R_{KL}^J - \Gamma_{KL}^R \Gamma_{RM}^J \right].
\]
\[
(18)
\]
This is the graded version of the standard Riemann curvature tensor.

It only remains to work out the fully covariant Riemann curvature tensor if only to check its graded symmetry properties. Thus we lower with the metric,
\[
R_{JKLM}^I = (-1)^{(|I|+|J|)(|K|+|L|+|M|)} R_{IJ} G_{JLM}^I,
\]
resulting in
\[
R_{JKLM}^I = (-1)^{(|I|+|J|)(|K|+|L|+|M|)} (-1)^{|K|} \Gamma_{KM}^I J, L - (-1)^{|K|+|M|} \Gamma_{KL}^I J, M
\]
\[
+ (-1)^{|L|+|M|} \Gamma_{KM}^I R_{KL}^J - \Gamma_{KL}^R \Gamma_{RM}^J \right] G_{IJ}.
\]
\[
(20)
\]
It is then not too hard to discover the expected graded symmetry relations,
\[
R_{KJLM} = -(-1)^{|K|} R_{JKLM},
\]
\[
(21)
\]
\[
R_{JKML} = -(-1)^{|L|} R_{JKLM},
\]
\[
(22)
\]
\[
R_{LMJK} = -(-1)^{(|J|+|K|)(|L|+|M|)} R_{JKLM}.
\]
\[
(23)
\]

5. Bianchi identities

The Riemann curvature tensor also satisfies the Bianchi identities. The first cyclic identity is readily established from (21) and (22):
\[
(-1)^{|K|} R_{JKLM} + (-1)^{|M|} R_{JMKL} + (-1)^{|L|} R_{JMLK} = 0.
\]
\[
(24)
\]
The second (differential) Bianchi identity, involving the covariant derivative of the curvature tensor, is most easily uncovered by proceeding to a “local frame” wherein the Christoffel symbol (but not its derivative) vanishes; in that case the tensor reduces to \( R_{JKLM;N} = (-1)^{|N|} R_{JKNM}^I G_{JLN}^I \). With this simplification there emerges the identity
\[
(-1)^{|L|} R_{JKLM;N} + (-1)^{|N|} R_{JKNL;M} + (-1)^{|M|} R_{JKMN;L} = 0.
\]
\[
(25)
\]
To get the contracted version of the second Bianchi identity, involving the Ricci tensor, we look at
\[
G^L J \left[ (-1)^{|L|} R_{JKLM;N} + (-1)^{|N|} R_{JKNL;M} + (-1)^{|M|} R_{JKMN;L} \right] = 0.
\]
\[
(26)
\]
This results in
\[
R_{KM;N} - (-1)^{|M|} R_{KN;M} + (-1)^{|M|+|L|+|N|} G^L J \Gamma_{JKLM} = 0,
\]
\[
(27)
\]
wherein the Ricci tensor has the graded symmetry, \( R_{KM} = (-1)^{|M|} R_{MK} \). One last contraction with \( G^{MK} \) gives
\[
R_{;N} = 2(-1)^{|M|} R_{M;N},
\]
\[
(28)
\]
which can be written in the form $G_{MN;M} = 0$ where
\[ G_{MN} = R_{MN} - \delta_{MN} / 2 \]
(29)
is the graded version of the Einstein tensor. Having established all the necessary equations with the requisite sign factors, we are in a position to tackle a simple but important case, featuring one property, namely charge, and ensuing electromagnetism.

6. Gauge changes as property transformations
To begin tackling the case of one property, an ansatz for the metric has to be made which incorporates the property coordinates. With everything flat the metric distance in the manifold $X = (x, \zeta, \bar{\zeta})$ is given by
\[ ds^2 = dX^A dX^B \eta_{BA} = dx^a dx^b \eta_{ba} + d\zeta d\bar{\zeta} \eta_{\zeta \bar{\zeta}} + d\bar{\zeta} d\zeta \eta_{\zeta \bar{\zeta}}, \]
where $\eta_{\zeta \bar{\zeta}} = -\eta_{\bar{\zeta} \zeta} = \ell^2 / 2$ and $\eta_{ba}$ is Minkowskian. Notice that we are obliged to introduce a fundamental length $\ell$ so as to ensure that the separation has the correct physical dimensions of length$^2$ because the $\zeta$ are being taken as dimensionless. This should be construed as the tangent space. We easily spot that it is invariant under Lorentz transformations and global phase transformations on $\zeta$. However it is not invariant under local $x$-dependent phase transformations and we are obliged to introduce the gauge field to correct for this, as we shall soon show.

To proceed to curved space we follow the standard method of invoking the tetrad formalism, but generalised to a graded space. (The metric is of course a product of appropriate frame vectors $E$, which provide the curvature.) We have also been guided by the Kaluza-Klein metric: the standard general relativistic metric is to be contained in the spacetime-spacetime sector and gauge fields must reside in the spacetime-property sector, but we have allowed for $U(1)$ invariant property curvature coefficients, denoted by $c_i$. The results below are not as adhoc as they may seem; for now we are just interested in seeing how far one may mimic Klein-Kaluza by using an anticommuting extension to space-time rather than a commuting one. [The various terms which arise in the metric below do not allow for fermion contributions as they would carry a Lorentz spinor index and would conflict with Lorentz invariance.] We envisage that the $c_i$ are expectation values of chargeless Higgs or dilaton fields which ought to be considered in the most general situation, left for future research.

The frame vectors $E_M^A$ that curve the space are stated in Appendix 2 and provide the cure for local phase invariance. They generate the metric in the usual manner via
\[ G_{MN} = (-1)^{|B|+|B|}[B][N] E_M^A \eta_{AB} E_N^B. \]
(30)

We have chosen to use the complex $\zeta, \bar{\zeta}$ description rather than the real coordinates $\xi, \eta$ (where $\zeta = \xi + i\eta$) because it lends itself more easily to group analysis when one enlarges the number of property coordinates.
The entries are tightly constrained by the fact that $G_{mn}$ and $G_{\zeta\bar{\zeta}}$ have to be bosonic while $G_{m\zeta}$ has to be fermionic in a commutational sense. Further they only admit expansions up to $\zeta\bar{\zeta}$; that is why the electromagnetic field $A_m$ multiplied by $\zeta$ appears in $G_{m\zeta}$; in principle one could also include in that sector an anticommuting ghost field $C_m$ times $\bar{\zeta}$, as one encounters in quantum gravity, but at a semiclassical level we are ignoring this aspect of the problem. Putting this all together results in the following metric

$$G_{MN} = \begin{pmatrix} G_{mn} & G_{m\zeta} & G_{m\bar{\zeta}} \\ G_{\zeta m} & 0 & G_{\zeta\zeta} \\ G_{\bar{\zeta} m} & G_{\bar{\zeta}\bar{\zeta}} & 0 \end{pmatrix}$$

(31)

where

$$
\begin{align*}
G_{mn} &= g_{mn}(1 + 2c_1\zeta\bar{\zeta}) + e^2\ell^2 A_mA_n\zeta\bar{\zeta}, \\
G_{m\zeta} &= G_{\zeta m} = -ie\ell^2 A_m\zeta/2, \\
G_{m\bar{\zeta}} &= G_{\bar{\zeta} m} = -ie\ell^2 A_m\bar{\zeta}/2, \\
G_{\zeta\zeta} &= -G_{\bar{\zeta}\zeta} = \ell^2(1 + 2c_2\zeta\bar{\zeta})/2.
\end{align*}
$$

A couple of general observations: the charge coupling $e$ accompanies the e.m. potential $A$ and the constants $c_i$ are allowed in the frame vectors to provide phase invariant property curvature rather like mass enters the Schwarzschild metric; the space-property metric is guaranteed to be anticommuting through the factor $\zeta$.

This inverse metric stays graded symmetric, $G^{MN} = (-1)^{|M||N|}G^{NM}$, and transforms correctly as a rank 2 covariant tensor. It can be derived from (8) or (9). Its elements are

$$
\begin{align*}
G_{mn}^m &= g_{mn}(1 - 2c_1\zeta\bar{\zeta}), \\
G_{m\zeta}^m &= G_{m\zeta}^m = ieA_m^m\zeta, \\
G_{m\bar{\zeta}}^m &= G_{m\bar{\zeta}}^m = -ieA_m^m\bar{\zeta}, \\
G_{\zeta\zeta}^m &= -G_{\bar{\zeta}\zeta} = 2(1 - 2c_2\zeta\bar{\zeta})/\ell^2 - e^2\ell^2 A_m\zeta\bar{\zeta}.
\end{align*}
$$

(32)

Now suppose that we make a spacetime dependent $U(1)$ phase transformation in the property sector:

$$
\begin{align*}
x' &= x; & \zeta' &= e^{i\theta(x)}\zeta; & \bar{\zeta}' &= e^{-i\theta(x)}\bar{\zeta}.
\end{align*}
$$

(34)

Then from the general transformation rules such as (5) and its contravariant counterpart we readily find that

$$eA'_m = eA_m + \partial_m\theta,$$

(35)

which shows the field $A_m$ acts as a gauge field under variations in charge phase. This can be checked for all components of the metric $G_{MN}$ from the transformation rule (5). On the other hand $G_{mn}^m$ remains unaffected and thus is gauge-invariant in the sense of (31) and (32). The same comments apply to $\mathcal{R}_{mn}$ and $\mathcal{R}^{mn}$; the former varies with gauge but the latter does not.
7. Metric superdeterminant and Palatini form

To produce the field equations we require the superdeterminant or Berezinian of the metric, which is given by Berezin and DeWitt to be

\[ s \det(X) = \det(A - BD^{-1}C) \det(D)^{-1}, \]  

(36)

for a graded matrix of the form:

\[ X = \begin{bmatrix} A & B \\ C & D \end{bmatrix}. \]

(37)

Given our metric this turns out to be:

\[ s \det(G_{MN}) = \frac{4}{\ell^4} \det(g_{mn}) \left[ 1 + (8c_1 - 4c_2)\bar{\zeta}\zeta \right] \]

(38)

or for short,

\[ \sqrt{-G} = \frac{2}{\ell^2} \sqrt{-g} \left[ 1 + (4c_1 - 2c_2)\bar{\zeta}\zeta \right]. \]

(39)

The absence of the gauge potential should be noted.

While on the subject of the super-determinant we note that in general, \( (\sqrt{G}.),_M = \sqrt{G}.(-1)^{[N]}\Gamma_{MN}^{N} \), and not just for our particular \( G \). As a direct consequence, \( [\sqrt{G}.A^M],_M = [\sqrt{G}.(-1)^{[M]}A^M],_M \), which impacts on the graded Gauss' theorem. Further, using the derivative identity,

\[ G^{LK},_M = \frac{1}{(-1)^{|M|}}G^{MN}\Gamma_{NM}^{K} - \frac{1}{(-1)^{|K|}}G^{KN}\Gamma_{NM}^{L}, \]

(40)

we can establish a useful lemma:

\[ [\sqrt{G}.G^{MK}],_L = (-1)^{|N|}\sqrt{G}.\Gamma_{LN}^{N}G^{MK} - \sqrt{G}.(-1)^{|L|}G^{MN}\Gamma_{NL}^{K} \]

\[ + (-1)^{|K|}(-1)^{|L|+[M]}\Gamma^{KN}\Gamma_{NL}^{M} \]

(41)

so that \( [\sqrt{G}.G^{LK}],_L = -\sqrt{G}.G^{LM}\Gamma_{ML}^{K} \) quite simply. Then because, under an integral sign, the total derivative terms \( [(-1)^{|L|}\sqrt{G}.G^{MK}\Gamma_{KM}^{L}],_L \), and \( [(-1)^{|L|}\sqrt{G}.G^{MN}\Gamma_{ML}^{K}],_M \) both effectively give zero, one can show that

\[ \sqrt{G}.(-1)^{|L|}G^{MK} \left[ (-1)^{|L|+|K|}G^{KN}\Gamma_{NL}^{L} - (-1)^{|L|+|K|}G^{KN}\Gamma_{NL}^{L} \right] \]

(42)

\[ = 2(-1)^{|L|}\sqrt{G}.G^{MK} \left[ \Gamma_{KL}^{N}\Gamma_{NM}^{L} - \Delta_{KM}^{N}\Gamma_{NL}^{L} \right]. \]

(43)

This means that sum of the first two (double) derivative terms in \( \sqrt{G}.R \) is exactly double the sum of the last two terms, apart from a sign change; in other words, the scalar curvature can be reduced to Palatini form, even in the graded case:

\[ \sqrt{G}.R \to (-1)^{|L|}\sqrt{G}.G^{MK} \left[ (-1)^{|L|+|K|}G^{KN}\Gamma_{NL}^{L} - \Delta_{KM}^{N}\Gamma_{NL}^{L} \right]. \]

(44)

This can help to simplify some of the calculations and it also endorses the correctness of all our graded sign factors.
8. The Ricci Tensor and Superscalar Curvature

From Eqn (14) and the metric given in (32) one may calculate the Christoffel symbols, $\Gamma_{MN}^K$. A list of these can be found in Appendix 1. Using these connections in (20) one may determine the fully covariant Riemann curvature tensor, $R_{JKLM}$. This can be a painful process and is where an algebraic computer program developed by one of us (PDS) comes in handy, for it minimises the possibility of errors. Even after making use of it and the symmetry properties of $R$ there are a large number of components. We have not bothered to list them as they are so numerous and not particularly enlightening. However the contracted Ricci tensor, 

$$R_{KM} = (-1)^{[K][L]}G^{LJ}R_{JKLM},$$

has fewer entries so we have provided a list of them and their contravariant counterparts,

$$R^{JL} = (-1)^{[M]}G^{JK}R_{KM}G^{ML},$$

in Appendix 2; the latter are gauge invariant. Finally the Ricci superscalar can be found by contraction with the metric,

$$R = G^{MK}R_{KM}. $$

In a frame that is locally flat in spacetime, the spacetime component of the contravariant Ricci tensor reduces to

$$R^{mn} = 4g^{mn}c_1[1 + (2c_2 - 6c_1)\bar{\zeta}\zeta]/\ell^2 - e^2\ell^2 F^{m1} F^{n1}\zeta/2,$$

and the curvature superscalar collapses to

$$R = 8[4c_1 - 3c_2 + c_1(8c_2 - 10c_1)\bar{\zeta}\zeta]/\ell^2 - e^2\ell^2 F_{ml} F_{nl}\bar{\zeta}\zeta/4.$$

Both expressions (48) and (49) are gauge independent. By making use of them and the superdeterminant (39) we may evaluate firstly the total Lagrangian density for electromagnetic property,

$$\mathcal{L} = \int d\zeta d\bar{\zeta} \sqrt{-G..} R \propto - \frac{1}{4} F_{mn} F^{mn} + \frac{48(c_1 - c_2)^2}{e^2\ell^4},$$

and secondly the Einstein tensor in flat spacetime:

$$\int d\zeta d\bar{\zeta} \sqrt{-G..}(R_{km} - R G_{km}/2)\times [48c_2(c_1 - c_2)g^{km}/e^2\ell^4 - (F^{kl} F_{lm} - F_{lm} F^{kl} g^{km}/4)].$$

The familiar expression for the electromagnetic stress tensor, namely $T^{km} \equiv F^{kl} F_{lm} + F_{nl} F^{n1} g^{km}/4$ emerges naturally and becomes part of the geometry. But we also recognize a cosmological constant term that is largely determined by the

---

"We are reasonably certain that those expressions, though complicated, are correct because we have checked that the differential Bianchi identity (28) is obeyed and this is a highly nontrivial test."
magnitude \((c_2 - c_1)/\ell^4\). [As an aside, we have verified that (48) and (49), remain true in a general frame, not necessarily locally flat.]

Including gravity by curving spacetime means including the standard gravitational curvature \(R\) and will render (48) and (49) generally covariant. (One has be careful here to track factors of \(\bar{\zeta}\zeta\), as one will be integrating over property.) It straightforward to see that the gravitational part of the superscalar \(R(g)\) is \(R(1-2c_1\zeta\zeta)\), while the super-Ricci tensor \(R^{(g)km}\) contains \(R^{km}(1-4c_1\zeta\zeta)\). In consequence we may evaluate the full gravitational-electromagnetic Lagrangian through the property integral:

\[
L = \int d\zeta d\bar{\zeta} \sqrt{-G} \cdot R = 2e^2 \sqrt{-g} \cdot \left[ \frac{2(c_1 - c_2)R}{e^2 \ell^2} - \frac{F_{mn}F^{mn}}{4} + \frac{48(c_1 - c_2)^2}{e^2 \ell^4} \right],
\]

wherein we recognize

\[
16\pi G_N \equiv \kappa^2 = \frac{e^2 \ell^2}{2(c_1 - c_2)}, \quad \Lambda = \frac{12(c_2 - c_1)}{\ell^2}.
\]

To verify that the entire setup is consistent and free of error we may determine the gravitational variation \(\delta G_{MN}\) which equals \(\delta g_{mn}(1+2c_1\zeta\zeta)\). Hence the gravitational field equation is obtained through

\[
0 = \int d\zeta d\bar{\zeta} \sqrt{-G} \cdot (\delta G_{MN} - G^{km}R/2)
= \sqrt{-g} \cdot \left[ \frac{4(c_1 - c_2)}{\ell^2} (R^{km} - \frac{1}{2}g^{km}R) - T^{km} - \frac{48(c_1 - c_2)^2}{e^2 \ell^4} g^{km} \right]
\]

This is just what we would have obtained from (50). In any case we see that the universal coupling of gravity to stress tensors \(T\) has a factor \(8\pi G_N \equiv \kappa^2/2 = e^2 \ell^2/(c_1 - c_2) > 0\). The result is to make the cosmological term go negative and, what is probably worse, it has a value which is inordinately larger than the tiny experimental value found by analyses of supernovae! (All cosmological terms derived from particle physics, except for exactly zero, share the same problem). Numerically speaking, \(\kappa \simeq 5.8 \times 10^{-19} \text{ (GeV)}^{-1}\) means \(\ell \sim 10^{-18} \text{ (GeV)}^{-1}\) is Planckian in scale. Of course the magnitude of the miniscule cosmological constant \(\Lambda \sim 4 \times 10^{-84} \text{ (GeV)}^2\) is at variance with Planckian expectations by the usual factor of \(10^{-120}\), which is probably the most mysterious natural ratio. So far as our scheme is concerned, we are disappointed but not particularly troubled by the wrong sign of \(\Lambda\) because it can readily be reversed by extra property curvature coefficients when we enlarge the number of properties (as we have checked when enlarging the number of properties to at least two). The magnitude of \(\Lambda\) is quite another matter because it will require some extraordinary fine-tuning, even after fixing the sign.

9. Inclusion of matter fields

The conventional results which we obtained for electromagnetism plus gravity, through the property of electricity, merely confirm the fact that our scheme is
perfectly viable and offers a novel perspective on nature. We anticipate that when one incorporates other properties, like chromicity and neutrinicity, then the usual picture of QCD plus gravity plus electroweak theory will emerge. For now we wish to exhibit some preliminary research concerning inclusion of matter fields, despite being limited to the single property of electric charge.

9.1. Scalar Field

Adhering to the tenets of the spin-statistics connection, we begin by assuming that a superscalar field $\Phi(X)$ is overall Bose and can be expanded into even powers of $\bar{\zeta}\zeta$; thus it has the general form $\Phi(x, \zeta, \bar{\zeta}) = U(x) + V(x)\bar{\zeta}\zeta$. Note that we could have included in $\Phi$ two anticommuting scalar ghost fields in the combination $\bar{\zeta}C + \bar{C}\zeta$; such ghost fields have a place in quantum theory but, with their incorrect spin-statistics, cannot be regarded as physical asymptotic states. The same comment applies to the spacetime-property sector where we could have included vector ghost fields of the type $C_m, \bar{C}_m$ multiplying $(1 + \bar{\zeta}\zeta)$. We have ignored these extras as we are only dealing with semiclassical e.m./gravity for the purpose of the present investigation, but they are sure to come in their own when quantization of the scheme is undertaken. By imposing self-duality so $U(x) = V(x) = \phi(x)$, $\Phi$ may be reduced to the form

$$\Phi(X) = \phi(x)(1 + \bar{\zeta}\zeta)/2.$$  

(52)

Necessarily $\phi$ carries zero charge and is a far cry from a Higgs field. (In fact to obtain the correct quantum numbers of the Higgs field it is imperative to attach three chromic properties to charge.) As we will be coupling this field to the supermetric, which brings in the superdeterminant $2\sqrt{-G}/\ell^2$, we shall introduce an extra factor of $\ell^2$ to eliminate this scale and we will also ignore $c_i$ curvature in what follows except from what the mixed $x - \zeta$ sector produces.

A mass term in the Lagrangian of $\mu^2\phi^2/2$ will arise through the property integral

$$(\ell^2/2)\int d\zeta d\bar{\zeta}\sqrt{-G}\mu^2\phi^2 = \int d\zeta d\bar{\zeta}\sqrt{-g_\cdot}(1 + 2\bar{\zeta}\zeta)\mu^2\phi^2/4.$$  

(53)

The kinetic term is more interesting, because it adds to the mass owing to the property components in $\Phi$; specifically this can be attributed to the $\zeta, \bar{\zeta}$ derivatives, which add a piece to the mass. Thus we consider

$$(\ell^2/2)\int d\zeta d\bar{\zeta}\sqrt{-G}G^{MN}\partial_N\phi\partial_M\phi.$$  

(54)

Upon inserting the metric from (32), we find that the contributions from the gauge field cancel out, as they must, and we are left with

$$\int d\zeta d\bar{\zeta}\sqrt{-g_\cdot}(1 + 2\bar{\zeta}\zeta)g^{mn}\partial_n\phi\partial_m\phi/4 + \bar{\zeta}\zeta\phi^2/\ell^2].$$  

(55)

The only feasible way then to cancel off mass in order to obtain a massless scalar field in this scheme is to match the $\phi^2/\ell^2$ from the property kinetic energy to the previously constructed mass term (53).
9.2. Spinor Field

In seeking a generalisation of the Dirac equation to incorporate the graded derivative, we need to bear in mind that the electromagnetic potential is embedded in the spacetime-property frame vector $E_A^M$; therefore we first need to determine the inverse vielbein, obtained via the condition $\varepsilon_M^A E_A^N = \delta_M^N$. The components are listed in the second appendix, where it will be seen that the vector potential is held in the space-property sector via $E_a^\zeta$ and $E_{\bar{a}}^{\bar{\zeta}}$. Since the Dirac operator has a natural extension from $i\gamma^a e_a^m \partial_m$ to $i \Gamma^A E_A^M \partial_M$, it is vital to include the graded derivative $\partial/\partial \zeta$ at the very least.

Just as Dirac was obliged to enlarge spinors from two to four components in order to go from non-relativistic electrons to relativistic ones, so too we are forced to extend the space in order to deal with the property derivatives. (There may be other ways to attain that goal.) Since the Dirac operator will act on a spinorial superfield, we have been led to consider an extended field $\Psi(X) = \theta \bar{\zeta} \psi(x)$ and the representation:

\[
\Gamma^a = \gamma^a, \quad \ell \Gamma^\zeta = 2i \partial/\partial \theta, \quad \ell \Gamma^{\bar{\zeta}} = 2i \theta,
\]

(56)

wherein $\theta$ is another complex scalar a-number and we eventually have to integrate over $\theta$ and $\bar{\theta}$. (The $\Gamma^\zeta, \Gamma^{\bar{\zeta}}$ act like fermionic annihilators). The action of the extended Dirac operator then yields

\[
i \Gamma^A E_A^M \partial_M \Psi = \left[ i \gamma^a e_a^m \partial_m + e \gamma^a A_a^{\bar{\zeta}} \frac{\partial}{\partial \zeta} + \frac{2}{\ell} (1 - f^\zeta \zeta) \frac{\partial^2}{\partial \theta d\bar{\theta}} \right] \theta \bar{\zeta} \psi
\]

\[= \theta \bar{\zeta} \gamma^a e_a^m (i \partial_m + e A_m) \psi - \frac{2}{\ell} (1 - f^\zeta \zeta) \psi.
\]

(57)

This means that when we include the adjoint $\bar{\Psi} = -\bar{\psi} \zeta \theta$ and integrate over the subsidiary $\theta, \zeta$, we end up with the normal gauge invariant spinorial Lagrangian density:

\[
\mathcal{L} = \int (d\zeta d\bar{\zeta}) (d\theta d\bar{\theta}) \bar{\Psi}(X) \left[ i \Gamma^A E_A^M \partial_M - \mathcal{M} \right] \Psi(X)
\]

\[= \bar{\psi}(x) \left[ \gamma^a e_a^m (i \partial_m + e A_m) - \mathcal{M} \right] \psi(x).
\]

(58)

Very likely there exists a more elegant way of reaching (58) but however this is done the coupling of the charged fermion to the spacetime-property vielbein, which contains $A$, is critical. The representation (56) will surely need revisiting in order to encompass chirality if the basic fermions are taken as left-handed, especially if we attach charge conjugate left-handed pieces, in order to encompass all spin states.

10. Conclusions

The framework underlying our research was inspired by supersymmetry, but instead of using auxiliary spinor coordinates we have made them scalar and connected them with something tangible, namely property or attribute. This point is important because all physical events are described by changes in momentum and/or property.
From this perspective, systematisation of property, with the natural occurrence of
generations, becomes a guiding principle. An obvious criticism of the approach is
that the superstructure has only led to the standard Einstein-Maxwell Lagrangian,
which is hardly an earth-shattering conclusion! True, but by geometrizing spacetime-
property we have succeeded in reinterpreting gauge fields as the messengers of prop-
erty in a larger graded curved space, besides offering a new viewpoint on the nature
of events; as a bonus we see that curvature in property space can act as a source of
a cosmological constant — this with just one property coordinate — even if its value
is ridiculous. Furthermore addition of further \(\zeta\) coordinates offers a natural path to
group theory classification without entraining infinite towers of states as one gets
with bosonic extensions to spacetime.

We foresee no intrinsic difficulty in extending the work to QCD or to electroweak
theory, though the algebraic manipulations will perforce be more intricate. Going all
the way, we anticipate an extension of our calculations to five property coordinates
(\(\zeta^0\) to \(\zeta^4\)) and the distillation of the final group to that of the standard model
will mean that property curvature coefficients \(c_{\text{col}}, c_{\text{ew}}\) are to be associated with
colour and electroweak invariants, \(\bar{\zeta}_i \zeta_i\) and \((\bar{\zeta}_0 \zeta^0 + \bar{\zeta}_4 \zeta^4)\) respectively — perhaps
engendered by expectation values of chargeless Higgs or dilaton fields. For the future
these are just the most prominent issues that come to mind with correction of the
\(\Lambda\) sign foremost among them and the quantization of the scheme in the present
framework as the next step. There are surely several research avenues to explore
in the present picture, many more than we have envisaged.

In the end, this geometrical approach of uniting gravity with other natural forces
through a larger graded space-time-property manifold may turn out to be quite
misguided. That would be disappointing as it is hard to imagine what other original
way one could unify the gravitational field with other fields. As a fallback position we
could be ultraprudent by abandoning the unification goal: just introduce gauge fields
in the time-honoured way, ensuring that differentiation is gauge covariant under
property transformations, by replacing ordinary derivatives \(\partial\) with \(D = \partial + ieA.T\),
where the generators \(T\) are represented by property rotations acting on matter
superfields; e.g. the charge operator by \((\bar{\zeta} \partial/\partial \bar{\zeta} - \zeta \partial/\partial \zeta)\). That would be a backward
step and even then, we might be confronted by insurmountable obstacles. However,
for now, the geometric scheme seems flexible enough and accords well with our
general understanding of fundamental particle physics and its content as well as

\[\text{To stress this point we mention that we have succeeded in obtaining the combined Yang-Mills-}
\text{Gravity Lagrangian for two property coordinates. The details are much more intricate than the}
\text{one \(\zeta\) case considered in this paper and will be submitted for publication separately.}
\[\text{Looking beyond the horizon of this paper, since the gravitational coupling to any stress tensor is}
\text{universal and since we have married couplings with fields it means that these interaction couplings}
\text{will need to be universal too. At low energies QCD and QED coupling constants \(e\) and \(g\) are}
\text{widely different so, unless the curvature coefficients \(c_e\) and \(c_w\) are taken to be different which}
\text{is entirely possible, we envisage a scenario where these couplings are running and unified at a}
\text{GUT-like scale \(\ell\) with \(1/\alpha \approx 40\); they only look different when we run down from}
\text{\(e^2(\ell^2) = g^2(\ell^2)\) to electroweak/strong interaction scales.}
gravitation. Should our framework fall by the wayside, there is no other recourse than to persist with variants of grand theories which are currently on the market, in the hope that experiment will, at sufficiently high energy, substantiate one of them. Failing that, we trust that one day somebody will conceive a radically new description of events that will lead to new insights with testable predictions.

11. Acknowledgments

We would like to thank Dr Peter Jarvis for much helpful advice and for his knowledgeable comments about supergroups, their representations and their dimensions.

Appendix A. Christoffel symbols, Vielbeins and Curvature components

Appendix A.1. The graded Christoffel connections

From definition and the metric elements one may derive the following components of the Christoffel symbols:

\[
\begin{align*}
\Gamma_{mn}^l &= \Gamma^{(g)}_{mn}^l + e^2\ell^2(A_n F_{mk} + A_m F_{nk})g^{kl}\zeta / 2, \\
\Gamma_{mn}^\zeta &= \frac{\zeta}{2} \left[ ie(2A^k \Gamma^{(g)}_{mnk} - A_{m,n} - A_{n,m}) - 2e^2 A_m A_n \frac{4c_1}{\ell^2} g_{mn} \right], \\
\Gamma_{mn}^\bar{\zeta} &= \frac{\bar{\zeta}}{2} \left[ ie(A_{m,n} + A_{n,m} - 2A^k \Gamma^{(g)}_{mnk}) - 2e^2 A_m A_n \frac{4c_1}{\ell^2} g_{mn} \right], \\
\Gamma_{\zeta n}^l &= \Gamma_{n\zeta}^l = \zeta \left[ ie\ell^2 F_{kn} g^{kl} / 4 - c_1 \delta_n^l \right], \\
\Gamma_{\bar{\zeta} n}^l &= \Gamma_{n\bar{\zeta}}^l = \bar{\zeta} \left[ ie\ell^2 F_{kn} g^{kl} / 4 + c_1 \delta_n^l \right], \\
\Gamma_{\zeta n}^\zeta &= \Gamma_{n\zeta}^\zeta = -ieA_n - \left[ \frac{e^2\ell^2}{4} A^k F_{kn} + ie(c_1 - 2c_2) A_n \right] \bar{\zeta}, \\
\Gamma_{\bar{\zeta} n}^\zeta &= \Gamma_{n\bar{\zeta}}^\zeta = \Gamma_{\zeta n}^{\bar{\zeta}} = \Gamma_{n\zeta}^{\bar{\zeta}} = \zeta = 0, \\
\Gamma_{\bar{\zeta} n}^{\bar{\zeta}} &= \Gamma_{n\bar{\zeta}}^{\bar{\zeta}} = \zeta, \\
\Gamma_{\zeta n}^{\bar{\zeta}} &= \Gamma_{n\zeta}^{\bar{\zeta}} = \Gamma_{\bar{\zeta} n}^{\zeta} = \Gamma_{n\bar{\zeta}}^{\zeta} = 0, \\
\Gamma_{\zeta n}^{\zeta} &= \Gamma_{n\zeta}^{\zeta} = \zeta, \\
\Gamma_{\zeta n}^{\bar{\zeta}} &= \Gamma_{n\zeta}^{\bar{\zeta}} = \zeta, \\
\Gamma_{\bar{\zeta} n}^{\zeta} &= \Gamma_{n\bar{\zeta}}^{\zeta} = \Gamma_{\zeta n}^{\bar{\zeta}} = \Gamma_{n\zeta}^{\bar{\zeta}} = 0.
\end{align*}
\]

Above, \( \Gamma^{(g)} \) signifies the purely gravitational connection and \( F_{mn} \equiv A_{n,m} - A_{m,n} \) is the standard Maxwell tensor. These connections are essential in determining the full super-Riemann components.
Appendix A.2. Ricci Tensor Components

A concise list of the Ricci supertensor components (contravariant and covariant) is as follows, where we neglect spacetime curvature:

\[
\begin{align*}
\mathcal{R}^{km} &= 4\eta^{mk} c_1 [1 + (2c_2 - 6c_1)\bar{\zeta}\zeta]/\ell^2 - \epsilon^2 \ell^2 F^{kl} F^{ml} \bar{\zeta} / 2, \\
\mathcal{R}^k\bar{\zeta} &= 4ie c_1 A^k \bar{\zeta} / \ell^2 - ie F^{kl} \bar{\zeta} / 2, \\
\mathcal{R}^\zeta\bar{\zeta} &= 8[3c_2(1-2c_2\bar{\zeta})] / \ell^4 \\
&\quad - \epsilon^2 (4A^m A_m c_1 / \ell^2 + F_{mn} F^{mn} / 4 + A_m F^{nm} / 4) \bar{\zeta}, \\
\mathcal{R}_{km} &= 4c_1 g_{km} [1 - 2(c_1 - c_2)\bar{\zeta} / \ell^2 - 4\ell^2 (2c_1 - 3c_2) A_k A_m \bar{\zeta} \\
&\quad + \epsilon^2 \ell^2 g^{ml} [A_{k,n} A_{m,l} - A_{n,m} A_{k,t}] \bar{\zeta} / 2, \\
\mathcal{R}_{k\bar{\zeta}} &= 2ie (2c_1 - 3c_2) A_k + ie \ell^2 \bar{\zeta} F_{k,l} / 4, \\
\mathcal{R}_{\zeta\bar{\zeta}} &= 2ie (2c_1 - 3c_2) A_k \bar{\zeta} + ie \bar{\zeta} \ell^2 F_{k,l} / 4, \\
\mathcal{R}_A &= [6c_2 - 4c_1 + 4(c_2 - c_1)(3c_2 - c_1) \bar{\zeta}] - \ell^4 \epsilon^2 F^{kl} F_{kl} \bar{\zeta} / 16.
\end{align*}
\]

Other components are derivable from symmetry properties of $\mathcal{R}_{MN}$.

Appendix A.3. Vielbeins

From the frame vectors, namely $\mathcal{E}_M^A$, whose components are

\[
\begin{align*}
\mathcal{E}_m^a &= (1 + c_1 \bar{\zeta}) e_m^a, \\
\mathcal{E}_m^\zeta &= -ie A_m \bar{\zeta}, \\
\mathcal{E}_m^\bar{\zeta} &= -ie A_m \zeta, \\
\mathcal{E}_\zeta^a &= 0, \\
\mathcal{E}_\zeta^\zeta &= 0, \\
\mathcal{E}_\bar{\zeta}^a &= 0, \\
\mathcal{E}_\bar{\zeta}^\zeta &= -(1 + e_2 \bar{\zeta}), \\
\mathcal{E}_\bar{\zeta}^{\bar{\zeta}} &= 0,
\end{align*}
\]

we may derive the super-vielbeins $E_A^N$, obtained via $\mathcal{E}_M^A E_A^N = \delta_M^N$. In this way we arrive at the set:

\[
\begin{align*}
E_a^m &= e_a^m (1 - c_1 \bar{\zeta}), \\
E_a^\zeta &= ie A_a \zeta, \\
E_a^{\bar{\zeta}} &= -ie A_a \bar{\zeta}, \\
E_\zeta^m &= 0, \\
E_\zeta^\zeta &= 0, \\
E_\bar{\zeta}^m &= 0, \\
E_\bar{\zeta}^{\zeta} &= -(1 + c_2 \bar{\zeta}), \\
E_\bar{\zeta}^{\bar{\zeta}} &= 0.
\end{align*}
\]

These expressions are required in Section B. As a useful check on their correctness we may ascertain that $G^{MN} = (-1)^{|A||M|} \eta^{AB} E_B^M E_A^N$, emerges properly. For example we directly arrive at

\[
\begin{align*}
G^{\zeta\bar{\zeta}} &= \eta^{ab} E_b^\zeta E_a^{\bar{\zeta}} - 2E_\zeta^{\bar{\zeta}} E_\zeta^{\bar{\zeta}} / \ell^2 + 2E_\zeta^{\bar{\zeta}} E_\bar{\zeta}^{\zeta} / \ell^2 \\
&= \eta^{mn} (ie A_n \bar{\zeta})(-i \bar{\zeta} e A_m) + 2(1 - 2c_2 \bar{\zeta}) / \ell^2 \\
&= 2(1 - 2c_2 \bar{\zeta}) / \ell^2 - \epsilon^2 A^m A_m \bar{\zeta} \zeta.
\end{align*}
\]
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