ABOUT SOME REGGE–LIKE RELATIONS
FOR (STABLE) BLACK HOLES

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ABSTRACT

We associated, in a classical formulation of “strong gravity”, hadron constituents with suitable stationary, axisymmetric solutions of some new Einstein–type equations supposed to describe the strong field inside hadrons. These new equations can be obtained by the Einstein equations with cosmological term Λ. As a consequence, Λ and the masses M result in our theory to be scaled up, and transformed into a “hadronic constant” and into “strong masses”, respectively. Due to the unusual range of Λ and M values considered, we met a series of solutions of the Kerr–Newman–de Sitter (hereafter KNdS) type with rather interesting properties. The requirement that those solutions be stable, i.e., that their temperature (or surface gravity) be vanishingly small, implies the coincidence of at least two of their (in general, three) horizons. Imposing the stability condition of a certain horizon does yield (once chosen the values of J, q and Λ) mass and radius of the associated black–hole (hereafter BH).

In the case of ordinary Einstein equations and for stable BHs of the KNdS type, we get in particular REGGE–LIKE (hereafter RL) relations among mass M, angular momentum J, charge q and cosmological constant Λ; which did not receive enough attention in the previous literature. Besides, we show some particular and interesting cases of these relations. Another interesting point is that, with few exceptions, all such relations (among M, J, q, Λ) lead to solutions that can be regarded as (stable) cosmological models.
1 Introduction

In a purely classical approach to “strong gravity”, that is to say, in our geometric approach to hadron structure (Recami & Castorina 1976; Caldirola et al. 1978; Recami 1983; Italiano 1984; Italiano & Recami 1984), we came to associate hadron constituents with suitable stationary, axisymmetric solutions of some new Einstein–type equations, supposed to describe the strong field inside hadrons. These Einstein–type equations are nothing but the ordinary Einstein equations + Λ suitably scaled down (Recami 1982; Recami & Zanchin 1992). As a consequence, Λ and the masses $M$ result, in such a theory, to be scaled up and transformed into a “hadronic constant” and into “strong masses”, respectively.

Due to the unusual range of the values therefore assumed by Λ, $M$ and by other parameters (see §2), and even more due to our requirements, we met a series of solutions of the KNdS type, which had not received enough attention in the previous literature. In particular, the requirement that those “(strong) BH” solutions be ”stable”, i.e., that their surface temperature be vanishingly small (Recami et al. 1986; Recami & Zanchin 1986; Zanchin 1987), implies the coincidence of at least two of their (three, in general) horizons.

Aim of the present paper is putting forth for the first time such results, while “rephrasing” them in the more popular language of ordinary gravity. Some of the more important points of originality of the present approach resides in our particular point of view; i.e., in the fact that we are going to regard every black–hole studied below as a (localized) object described by an external observer living in a four-dimensional space-time asymptotically de Sitter (part of this work has been already deal by two of us, Zanchin et al. 1994).

The plane of the paper is as fellows: in §2 we describe the main properties of the horizons associated with a central, stationary body and in §3 we speak of the horizon temperature. In §4 we deal both of the stable Schwarzschild–deSitter black hole and of the mass formulas for stable KNdS black holes while in §5 we show the particular case of the triple coincidences. Finally in §6 we report our conclusions.

2 The horizons associated with a central, stationary body, and their main properties

We consider Einstein equations with cosmological term

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = -kT_{\mu\nu} \quad (1)$$

where $k \equiv \frac{8\pi G}{c^4}$. We choose, whenever convenient, units such that $G = 1$ and $c = 1$, and look for the vacuum solutions describing the stationary axisymmetric field created by a rotating charged source. This solution is the KNdS space–time, whose metric in Boyer–Lindquist (1967)–type coordinates $(t, r, \theta, \varphi)$ and being $m \equiv GM/c^2$ and $a \equiv J/Mc$, one
can write:

\[
    ds^2 = -\rho^2[dr^2/B + d\theta^2/D] - \rho^{-2}A^{-2}[(adt - (r^2 + a^2)d\phi)^2\sin^2 \theta + \\
    + BA^{-2}\rho^{-2}[dt - a\sin^2 \theta d\phi]^2]
\]  

(2)

where \(\rho, A, B \equiv B(r), D \equiv D(\theta), Q^2\) are definitited in Carter (1970, 1973) and the quantities \(M, J\) and \(q\) are mass, angular momentum and electric charge of the source, respectively. For simplicity, we here analyze only the case \(\Lambda > 0\).

One meets the event horizons of the space Eq. (2) in correspondence with the divergence of the coefficient \(g_{rr}\), i.e., when \(B(r) = 0\). This equation, \((r^2 + a^2)(1 - \frac{\Lambda r^2}{3}) - 2mr + Q^2 = 0\) (3)

admits four roots, one of which, \(r_0\), is always real and negative. The interesting case is when Eq. (3) has four real solutions; in that case we shall have three positive roots: let us call them \(r_1, r_2, r_3\), with \(r_3 \geq r_2 \geq r_1\). We shall see that at \(r = r_3\) we have a cosmological horizon Carter (1970, 1973), while at \(r = r_2\) and \(r = r_1\) we meet two BH horizons analogous to the two wellknown \(r = r_+\) and \(r = r_-\) horizons of the Kerr metric.

The three horizons 1, 2, 3, in the general case when they are all real, divide the space in the four parts I, II, III and IV (see Fig. 1). On each horizon, quantity \(g_{rr} \equiv g_{11}\) diverges, i.e. \(g^{rr} = 0\). In regions III and I it is always \(g^{rr} < 0\), as expected in the case of an ordinary Kerr BH; on the contrary, in regions II and IV it is always \(g^{rr} > 0\).

Actually, it is possible to define a Killing vector \(K^\mu\) which is simultaneously time–like in regions III and I but not in regions II and IV too (Recami 1978; Recami & Shah 1979; Pavšič & Recami 1982; Recami & Rodrigues Jr. 1982; Italiano 1986; Recami 1986; Trofimenko & Gurin 1987). Therefore one can have stationary observers (\(r = \text{const.}\)) only in regions III and I, in the sense that only there the (\(r = \text{const.}\)) trajectories are time–like. We call time–like the (ordinary type) regions III and I; and space–like the other two regions. For simplicity’s sake, let us confine ourselves to the static case (Reissner–Nordström–de Sitter metric), which is not qualitatively different. From Eq. (2), by putting \(a = 0\), we find for those geodesics:

\[
    ds^2 = F dt^2 - F^{-1} dr^2 = 0
\]  

(4)

where \(F \equiv B/r^2\). By integration of Eq. (4), after some algebra one gets:

\[
    t = \mp 3\Lambda^{-1} \sum_{m=0}^{3} \alpha_m r_m^2 \log \left|\frac{r}{r_m} - 1\right| + C_\mp, \quad [m = 0, 1, 2, 3]
\]  

(5)

where \(C_\mp\) are integration constants, and \(\alpha_m\) are “constants” whose value depends on the values of the four roots \(r_0, r_1, r_2, r_3\) of Eq. (3). In Eq. (5) the upper (lower) sign
Figure 1: Given an, almost, pointlike stationary body, generating, when $\Lambda \neq 0$, a KNdS space–time, in the Boyer–Lindquist coordinates it will in general possess three horizons 1,2,3, which divide the associated space into the four regions I, II, II, IV. Surface 3 is the cosmological horizon while surfaces 2,1 are the outer and inner BH horizons, respectively. 

The behaviour of the radial null geodesics $t = t(r)$ is given in Fig. 2 for the four regions. One has to recall that, however, such a figure does not represent in a complete manner the causal structure of our space-time. That structure can be inferred from the Penrose–Carter diagram (Carter 1966; Hawking & Ellis 1973; Gibbons & Hawking 1977; Mellor & Moss 1989; Davies 1989).

3 On the horizon temperatures

In the general (stationary, i.e. KNdS) case of metric Eq. (2), the Bekenstein–Hawking temperature $T_n$ (Bekenstein 1974; Hawking 1975) of each horizon in Fig. 1, is known to be proportional to the horizon surface gravity as follows:

$$T_n = \varepsilon \gamma_n , \quad [n = 1, 2, 3]$$
where \( \varepsilon \equiv \hbar/(2\pi k_Bc) \). On any null–surface and in particular on every horizon the surface gravity (Bardeen et al. 1973; Zheng & Yuanxing 1983) can be defined by the equation

\[
\partial_\mu (K_\nu K^\nu) = -2\gamma K_\mu
\]  

(7)

where the symbols \( \partial_\mu \) representing the covariant derivatives. To evaluate the surface gravities \( \gamma_n \), we here want remind everybody that in our metric (see Eq. (2)) does admit two Killing vector \( K_\mu^t, K_\mu^\phi \), such that one can construct Killing vectors \( K_\mu^\mu = K_\mu^t + \omega_n K_\mu^\phi \).

Then we finally get for the horizon temperatures (and for \( \Lambda \neq 0 \)) the expressions \( T_n = \varepsilon \gamma_n \) with

\[
T_n = \frac{\varepsilon\Lambda}{6A(r_n^2 + a^2)} \cdot \left| \Pi_{\ell \neq n}^0 (r_n - r_\ell) \right|, \quad [\ell = 0, 1, 2, 3]
\]  

(8)

Eq. (8) yields the result that the horizon temperature can be vanishing small only when two or more horizons tend to coincide; i.e., when two (or more) roots \( r_i \) of Eq. (3) tend to coincide. This result implies some relations among mass, radius, charge, angular momentum and \( \Lambda \) of a stable BH.

In the particular (Kerr–Newman) case when \( \Lambda = 0 \), one gets only two (or no) horizons,
corresponding to \( r_\pm = m \pm \sqrt{m^2 - a^2 - Q^2} \), and Eq. (8) has to be replaced by \( T_\pm = \varepsilon (r_+ - r_-)/(r_\pm^2 + a^2) \). We get a stable \((T = 0)\) BH solution when

\[
r_+ = r_- = m
\]

that is to say, when the RL condition does hold:

\[
m^2 = a^2 + Q^2
\]

However, since in this case region II disappeared, then the whole BH–interior is time–like!, so as the external region III. We call a solution of this type a “time–like black–hole”.

### 4 Particular cases of stable BH

An interesting case is that of the Schwarzschild–de Sitter metric (Gibbons & Hawking 1977), in which \( Q^2 = a^2 = 0 \), so that \( B = -\Lambda r^4/3 + r^2 - 2mr \) and two horizons only (with radii \( r_- \equiv r_B \), \( r_+ \equiv r_C \), respectively) are met, whose surface temperatures result to be

\[
T_\pm = \frac{\varepsilon \Lambda}{3r_\pm^2} \left( \frac{3m}{\Lambda} - r_\pm^2 \right)
\]

Once more, the requirement \( T = 0 \) implies that \( r_B = r_C \equiv r \) and that \( r = (3m/\Lambda)^{1/3} \).

The last equation can be read as

\[
r = \Lambda^{-1/2} = 3m \ , \quad [r_B = r_C \equiv r]
\]

since those two radii coincide, only, when

\[
9\Lambda m^2 = 1
\]

More interesting, here, is the observation that \( r_- \) and \( r_+ \) behave so as \( r_2 \) and \( r_3 \), respectively, of Figs. 1 and 2. For this reason we called \( r_- \equiv r_B \) and \( r_+ = r_C \) where \( B \equiv BH \) horizon and \( C \equiv cosmological \) horizon. When \( r_B \) tends to coincide with \( r_C \), the time–like regions of type III do disappear, so that we are left only with regions of type II and IV, and the BH tends to occupy the whole space inside the cosmological horizon (roughly speaking, the BH itself can be regarded as a model for a cosmos). It is worthwhile mentioning that, by choosing for \( \Lambda \) the value \( |\Lambda| \approx 10^{-52} \) m\(^{-2}\) ordinarily assumed for our cosmos, the condition given by Eq. (13) yields \( M \approx \frac{1}{2} \times 10^{53} \) kg, which is close to the estimated mass of our own cosmos.

Now, we consider the characteristics of stable BHs in the general (KNdS) case when the source is endowed also with \( J \) (stationary case) and \( q \). We have at our disposal two
equations:

\[
\begin{align*}
B(r) &= -\frac{\Lambda r^4}{3} + (1 - \frac{\Lambda a^2}{3})r^2 - 2mr + a^2 + Q^2 = 0 \\
T &= -2\frac{\Lambda r^3}{3} + (1 - \frac{\Lambda a^2}{3})r - m = 0
\end{align*}
\]  

the second equation requires the vanishing of the derivative \( B'(r) \) in correspondence with the values \( r_n \) which satisfy the first equation \( [B(r) = 0] \). Such second equation, therefore, ensures the solutions of the system to be double (or triple) “roots” of eq. \( B(r) = 0 \).

After, some algebra, we get explicitly a second equation providing us with a link among the various parameters \( m, \Lambda, a, Q \):

\[
\begin{align*}
r &= \frac{3m\sigma}{E} \\
9m^2\sigma(\delta\sigma - E) + 2\eta E^2 &= 0
\end{align*}
\]  

where \( E = 3\delta^2 + 4\Lambda\delta\eta - 18m^2\Lambda \) being

\[
\begin{align*}
\delta &= 1 - \frac{\Lambda a^2}{3} \\
\eta &= a^2 + Q^2 \\
\sigma &= \delta^2 - 4\Lambda\eta
\end{align*}
\]

Eqs. (15) do yield, of course, both the stable BH solutions resulting from the coincidence of \( r_1, r_2 \), and those resulting from the coincidence of \( r_2, r_3 \). Namely, the second of Eqs. (15) can be written as

\[
\frac{2\delta\sigma}{E} = 1 \pm \sqrt{1 - \frac{8\delta\eta}{9m^2}}
\]  

from which one can, of course, construct two independent systems.

We, starting, consider the case when \( r_1 \equiv r_2 = r_- \). In this case the regions of type II (Figs. 1 ÷ 2) do disappear and we obtain a stable Kerr–Newman–de Sitter BH, similar to the stable BH encountered in §3, in the particular Kerr–Newman case. In that case, however, the stable BH was surrounded by asymptotically flat regions of type III; whilst in the present case (we are still in presence of a cosmological \( r = r_3 \) horizon) our stable BH is surrounded by two type regions: regions of type III, and regions of type IV. In other words, both the external (III) and the internal (I) regions of the present stable BH are time–like regions, separated just by a semi–permeable membrane. In these regions, any causal observer \( O_c \) can live therein without falling into the singularity \( r = 0 \).

Now, we consider the case when \( r_2 \equiv r_3 = r_+ \). In this second case the regions of type III (see Figs. 1 ÷ 2) do disappear and we obtain a BH, bounded by a stable horizon originating from the fusion of a BH–type \( (r_2) \) surface and a cosmological–type \( (r_3) \) horizon. The stable \( r_2 \equiv r_3 \) null–surface can be regarded, therefore, both as a BH–membrane and as a
cosmological horizon. Outside such a surface, we meet regions of type IV, asymptotically de Sitter. Both the internal (II) and the external (IV) BH regions are space–like, since the time–like type III regions, where causal observers usually live, disappeared. In regions II and IV no stationary observers can exist. Inside the $r_2 = r_3$ surface, we moreover find at $r = r_1$ a null surface that can be considered the internal BH boundary, so as in the Kerr–Newman (or Kerr) case.

5 Triple coincidences

In the very special case when all the three positive roots of Eq. (3) do coincide, i.e. when $r_1 = r_2 = r_3$, we shall meet a stable BH with a single horizon, whose radius takes on a simple analytical expression. Let us write Eq. (16), after some algebra, more conveniently, as:

$$r = \frac{3m}{2\delta} \pm \sqrt{\frac{9m^2}{4\delta^2} - \frac{2\eta}{\delta}}$$

and observe that the condition of triple coincidence requires the vanishing of the square root, i.e. yields the solution:

$$r = \frac{3m}{2\delta}$$

(17)

with the two simultaneous RL constraints:

$$m^2 = \frac{8}{9}\delta(a^2 + Q^2)$$

(18)

$$m^2 = \frac{2\delta^3}{9\Lambda}$$

(19)

Eq. (19) comes from inserting Eqs. (17), (18) in either of Eqs. (14).

In the present case, all the regions II and III did disappear; and the ” type I ” interior of our stable BH is time–like whilst its ” type IV ” exterior is space–like. Such solution is therefore a “time–like black–hole”. Again, regions IV are asymptotically de Sitter. Such a BH solution is conveniently interpretable (see §4) also as a cosmological model: namely, as a model of a stable cosmos.

6 Conclusion

Let us stress, first of all, that for stable BHs we got “ RL ” relations among their $m$, $J$, $q$ and $\Lambda$. For instance, in the case $\Lambda = 0$ we got Eq. (14) which, when $q$ is negligible, can
just be written if we assume like units $G = 1$ and $c = 1$ as

$$M^2 = J. \quad (10^*)$$

On the contrary, when $J = 0$ and $q$ is still negligible, then we meet Eq. (13), which, always with $G = 1$ and $c = 1$, can read

$$M^2 = \frac{1}{9} \Lambda^{-1}. \quad (13^*)$$

In the most general case, the considered relation (among $M, J, q, \Lambda$) is involute, and was given by the second one of Eqs. (13). In the simpler case of §6, i.e. of the “triple coincidence”, we obtained two such relations, namely Eqs. (18), (19), which are still complicated. However, if $|\Lambda a^2| \ll 1$, Eqs. (18), (19) yield both

$$m^2 \simeq \frac{8}{9}(a^2 + Q^2), \quad (20)$$

to be compared with Eq. (10) and

$$M^2 \simeq \frac{2}{9} \Lambda^{-1}, \quad (21)$$

that if $c = G = 1$ can be compared with Eq. (13*).

The most interesting point is that, with the exception of Eqs. (10), (10*), all such “RL” relations can be attributed also to our stable cosmological models, i.e., to our stable “cosmoses”. Finally, let us mention that elsewhere we shall apply and interpret the results presented in this paper to the case of “strong gravity” theories and “strong BHs”: i.e., to the case of hadronic physics.

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