Existence of Random Attractors for Nonlinear Plate Equations with Thermal Memory Effect and Additive Noise

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Abstract

In this paper, the long-time dynamics of the nonlinear plate equations with thermal memory effect and additive noise caused by non-Fourier heat flow law is studied. We study the existence of random attractors for stochastic dynamical systems related to equations.

Keywords: Nonlinear plate equations, thermal memory, stochastic dynamic system, random attractor, additive noise.

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INTRODUCTION

In this paper, the research methods in the literature [4-6] consider the nonlinear Plate equation with thermal memory effect caused by the non-Fourier heat flow law with additive noise and has a random attractor in the stochastic dynamic system. The study equation is as follows

\[ \begin{align*}
\theta_t - \Delta u_t + \int_0^t k(s) [-\Delta \theta(t-s)] ds &= \sum_{j=1}^{m} \phi_j dw_j \\
u - \Delta u_t + \Delta (\Delta u + \theta) + f(u) &= 0
\end{align*} \]  

(1.1)

Boundary condition

\[ u(t) = \Delta u(t) = 0 \quad t \geq 0, x \in \Gamma \]

\[ \theta(t) = 0 \quad t \in R, x \in \Gamma \]

Initial value condition

\[ u(0) = u_0 \quad x \in U \]

\[ u_t(0) = v_0 \quad x \in U \]

\[ \theta(0) = \theta_0 \quad x \in U \]

\[ \theta(-s) = \Psi(s) \quad (x,s) \in U \times R^\ast \]

Among them, \( \sum_{j=1}^{m} \phi_j dw_j \) is an appropriate smooth boundary \( \Gamma \) in \( R^2 \), \( u = u(t,x) \) is a real-valued function on \( U \times [0, +\infty) \), \( \theta \) is the temperature change of the self-balanced reference value, \( u \) is the vertical displacement of the plate, and function \( \Psi: R^\ast \times U \mapsto R \) is called the initial history of temperature, assuming the memory kernel function \( k: R^\ast \mapsto R \). It is a positive bounded convex function and disappears at infinity. For random terms \( \phi_j (x) \in H^1_0(U) \cap H^3(U) \quad (j = 1,2, \cdots, m) \), \( \{ \omega \} \) is an independent bilateral real-valued Wiener process on probability space \( (\Omega, F, P) \). Among them,
\[ \Omega = \{ \omega = (\omega_1, \omega_2, \ldots, \omega_n) \in \mathbb{C}(R, R^n) : \omega(0) = 0 \} . \]

Given a tight open topology, \( F \) is a \( P \)-complete Borel \( \sigma \)-algebra on \( \Omega \).

There is a lot of research work on the thermoelastodynamic plate equation in the literature. The Plate equation is derived from the elastic vibration equation established by Woinowsky-Krieger ([12]) and Berger ([13]). For the Plate equation, the latest study of the attenuation estimation and asymptotic curve of the Plate equation with memory (cf.[20]); in [23], the quasi-stability properties of the system are established, and the existence of global attractors and exponential attractors with finite fractal dimensions is obtained; studied the precise controllability of the thermoelastic plate equation with memory (cf[21]); obtained the general stability results of the viscoelastic plate equation with past history and general kernel (cf[22]), etc. In these studies, we have noticed that in recent decades, the long-term behavior of the deterministic Plate equation or the solution of a single Plate equation has been studied by many scholars. Such as, the author of the literature [14] in 2005 considered the following evolution problem in the two-dimensional theory of linear genetic thermoelasticity,

\[
\begin{align*}
\varphi(t) + \Delta(\varphi(t) + \mathcal{G}(t)) &= 0 \\
\varphi_t(t) + \int_0^\infty k(s)[e\mathcal{G}(t-s) - \Delta\mathcal{G}(t-s)]ds - \Delta\varphi(t) &= 0
\end{align*}
\]

which simulates Kirchhoff thermoelastic thin plate with heat conduction memory effect. The exponential stability of the solution of the equation is considered.

The existence and uniqueness of the global solution and the existence of the global attractor are proved.

However, in real life, many systems will inevitably be affected by random factors. Therefore, this paper studies the existence of random attractors of stochastic Plate equations in stochastic dynamic systems based on [7]. This article is organized as follows. In the next section, we will review some basic concepts and properties of general stochastic dynamic systems. In Section 3, we provide some basic settings related to formula (1.1) and show that it generates a stochastic dynamic system in the appropriate function space. Section 4 is devoted to proving the existence of the unique random attractor of the stochastic dynamic system.

**Basic Definition Theorem**

**Definition 2.1** ([8]) The random dynamic system \((\Omega, F, P, (\varphi), \mathcal{G})_{\in R} \) is called RDS \( \varphi \).

\[
\begin{align*}
\varphi : T \times X \times \Omega & \rightarrow X \\
(t, x, \omega) & \mapsto \varphi(t, \omega)x,
\end{align*}
\]

where identity map \( \varphi(0, \omega) = id \) on \( X \), and has the following relationship

\[
\varphi(t + s, \omega) = \varphi(t, \mathcal{G}(\omega)) \circ \varphi(s, \omega),
\]

among them \( t, s \in T, \omega \in \Omega \), \( \circ \) represent combinations. A cluster of maps \( \varphi(t, \omega) \) satisfying (2.1) is called a cocycle, and (2.1) is also the nature of cocycle. The inverse of \( \varphi(t, \omega) \) is \( \varphi(t, \omega)^{-1} = \varphi(-t, \mathcal{G}(\omega)) \).

**Definition 2.2** ([8]) Random set \( A \) attracts another random set \( B \) if

\[
d(\varphi(t, \mathcal{G}_\omega)B(\mathcal{G}_\omega), A(\omega)) \rightarrow 0,
\]

then call \( A \) as an attractive set.

**Definition 2.3** ([8]) Random set \( K \) attracts another random set \( B \) if \( t_\mu(\omega) \) exists, for all \( t \geq t_\mu(\omega) \) there is

\[
\varphi(t, \mathcal{G}_\omega)B(\mathcal{G}_\omega) \subset K(\omega).
\]

\( t_\mu(\omega) \) is called absorption time. Then \( K \) is called absorption set.
Theorem 2.1 ([8] [9]) Let $\Phi$ be the measurable power system $(\Omega, F, P, \{\mathcal{F}_t\}_{t \in \mathbb{R}})$ and a continuous random power system in space $X$. If there is a random compact set $K(\omega)$, such that for any nonrandom bounded set $B \subset X$, there is

$$\lim_{t \to +\infty} d(\Phi(t, \mathcal{F}_t \omega)B, K(\omega)) = 0,t \to +\infty,$$

then $\Phi$ has a random attractor

$$A = \{A(\omega)\}_{\omega \in \Omega} = \bigcup_{B \subset X} \Lambda_B(\omega),$$

where $B$ is taken through the bounded subset of $X$, $\Lambda_B(\omega)$ is the $\omega$-limit set of $B$, i.e

$$\Lambda_B(\omega) = \bigcap_{t \to +\infty} \Phi(t, \mathcal{F}_t \omega)B.$$

Theorem 2.2 ([15]) Suppose $X_0, X_1, X$ are three Banach spaces, of which $X_0, X$ are reflexive and satisfy

$$X_0 \to X \to X_1 \ (\text{Embed}),$$

here embedding $X_0 \to X$ is tight. If $Y \subset L^p_0(\mathbb{R}^+, X)$ meets the following conditions:

(i) $Y$ is bounded in $L^p_0(R^+, X_0) \cap H^1_0(R^+, X_1)$;

(ii) For $s \in R^+$, there is $K_0 > 0$, so that $\sup_{\eta \in \mathcal{Y}} \|\eta(s)\|_X^2 \leq K_0$;

then $Y$ is relatively tight in $L^p_0(R^+, X)$.

Lemma 2.1 ([19]) Let $X$ be a Banach space and $Z \subset C([0, \infty), X)$. Let $E : X \to R$ be a function bounded from below such that $E(Z(0)) \leq M$ for $Z \in X$. If

$$\frac{d}{dt} E(Z(t)) + \sigma \|Z(t)\|_X^2 \leq \chi,$$

for some $\sigma \geq 0, \chi \geq 0$ independent of $Z$, then for all $\varepsilon > 0, t_0 = t_0(M, \varepsilon) > 0$, such that

$$E(Z(t)) \leq \sup_{\xi \in X} \{E(\xi) : \sigma \|\xi\|_X^2 \leq \chi + \varepsilon\}, \forall t \geq t_0.$$

Lemma 2.2 For $m < \varepsilon < l, g = 0,1, \forall B_i \in V^0, i = 1,2$, then

\begin{align*}
(1) & - \varepsilon (Au, B_i) \geq - \frac{\alpha \varepsilon}{4} \|u\|_2^2 - \frac{\varepsilon}{\alpha} \|B_i\|_2^2, \\
(2) & - \int U B_i Au dx \geq - \frac{\alpha \varepsilon}{4} \|B_i\|_2^2 - \frac{1}{\alpha} \|u\|_2^2, \\
(3) & |(A^t B_i, B_j)| \leq \|A^t B_i\|_2 \|B_j\|_2 \geq \frac{1}{\alpha} \|A^t B_i\|_2^2 + \frac{\varepsilon}{4} \|B_j\|_2^2, \\
(4) & |\int_0^r \varepsilon \mu(s) < A^t B_1, \eta(t) > ds| \leq \frac{\varepsilon \mu_{\infty}}{4} \|A^t B_1\|_2^2 + \frac{\varepsilon}{\alpha D_l^{\varepsilon/2}} \|\eta\|_m^{\varepsilon/2}.
\end{align*}

Basic Settings

Assume that the memory kernel function $\mu(s)$ and the nonlinear term $f(u)$ satisfy the following conditions:

(1) For any $s \in R^+$, there is $\delta > \frac{23D_l + 3}{6(D_l - 2)} > 0$, such that

$$\mu(s) \in C^1(R^+) \cap L^1(R^+), \mu(s) \geq 0, \mu(s) + \delta \mu(s) \leq 0. \quad (3.1)$$
(2) Assuming that \( k \) is zero at infinity, in addition, define \( \mu(s) = -k'(s) \). In recent years, various development equations under the non-Fourier heat flow law have aroused the interest of many mathematicians. Let \( q \) be the heat flow vector. According to Gurtin-Pinkin theory (cf.[16]), the linear constitutive equation of \( q \) is given by

\[
q(t) = -\int_0^\infty k(s) \nabla \theta(t - s) ds,
\]

where \( k \) is the thermal conductivity relaxation kernel, and the existence of the convolution term in the above formula means the finite propagation velocity of thermal disturbance, so in this case, the corresponding equation is hyperbolic. It is easy to see that if \( k \) is the Dirac mass at zero, the above formula can be simplified to the classic Fourier's law \( q = -\nabla \theta \). In addition,

\[
k(s) = \frac{1}{\sigma} e^{-\frac{s}{\sigma}}, \sigma > 0
\]  

Among them, \( \frac{2}{3} < k_0 < \min\{4, \frac{\delta}{2D} - \frac{\delta}{4D} - \frac{5}{4D}, 8 - \frac{10}{D} \} \).

(3) The nonlinear function \( f(u) \) satisfies:

\[
f(0) = 0, |f'(s)| \leq C_0 (1 + |s|), \forall s \in \mathbb{R},
\]

\[
F(s) = \int_0^s f(r) dr \geq C_1 (|s| - 1), \forall s \in \mathbb{R},
\]

\[
\liminf \frac{sf(s) - C_2 F(s)}{s^2} \geq 0, \forall s \in \mathbb{R}.
\]

\[
\liminf \frac{f(s)}{s} > \frac{1}{C_U}.
\]

Among them, \( 0 < C_2 \leq \frac{2C_0 \mu^2 + C_0 |u|^4}{2C_0 (|u|^4 - 1)} \), \( C_i > 0 (i = 0, 1, \cdots) \) is a normal number, in this article the constant \( C \) changes, each row or even the same row of \( C \) is not equal. \( C_U \) is the best constant that depends only on \( \tilde{U} \), then

\[
\|u\|_{L^2(U)}^2 \leq C_U \|Au\|_{L^2(U)}^2
\]

Consider the positive operator \( A \) on \( L^2(U) \), defined as \( A = -\Delta = \nabla^2 \) and bounded by \( D(A) = H^2(U) \cap H^1_0(U) \).

For \( r \in \mathbb{R} \), consider Hilbert space \( v^r = D(A^{r/2}) \), define the inner product

\[
(u_1, u_2)_r = (A^{r/2} u_1, A^{r/2} u_2), \forall u_1, u_2 \in v^r,
\]

\[
\|u\|_r^2 = (A^{r/2} u, A^{r/2} u), u \in v^r,
\]

\[
\|u\|_0^2 = (u, u), u \in v^0,
\]

where \((\cdot, \cdot)\) and \(\|\cdot\|\) represent the inner product and norm in \( v^0 = L^2(U) \). Use \((\cdot, \cdot)_r, \|\cdot\|_r\) to denote the inner product and norm of \( v^r = H^r_0(U) \), respectively. For \( r_1 > r_2 \), embedding \( v^{r_1} \rightarrow v^{r_2} \) is tight.

We introduce a weighted Hilbert memory space \( M^{1+r} = L^2_{\mu}(R^+, v^r) \), of which \( r \in \mathbb{R} \). Define the inner product as follows

\[
\langle \eta_1, \eta_2 \rangle_{m^r} = \int_0^\infty \mu(s) \langle A^{r/2} \eta_1(s), A^{r/2} \eta_2(s) \rangle ds, \forall \eta_1, \eta_2 \in M^{1+r},
\]

\[
\|\eta\|_{m^r}^2 = \int_0^\infty \mu(s) \|\eta(s)\|_0^2 ds, \forall \eta(s) \in M^{1+r},
\]

\[
\|\eta\|_{m^1}^2 = \int_0^\infty \mu(s) \|\eta(s)\|_m^2 ds, \forall \eta(s) \in M^1.
\]
Here we notice that for $r_1 > r_2$, embedding $M^{k} \rightarrow M^{\omega}$ are continuous but not compact (cf.[17,18]).

Finally, we define the inner product Hilbert space

$$V' = \mathbb{V}^{x} \times \mathbb{V}^y \times \mathbb{M}^{x+y}, r \in \mathbb{R}. $$

For all $z = (z_1, z_2, z_3, z_4)^T \in V'$, there is norm

$$\|z\|_{V'}^2 = \|A^{2\omega\nu} z_1\|^2 + \|A^{\nu\nu} z_2\|^2 + \|A^{\nu\nu} z_3\|^2 + \|z_4\|^2_{\mathbb{M}^{x+y}} ,$$

when $r = 0$, there is $V^0 = \mathbb{V}^x \times \mathbb{V}^y \times \mathbb{M}^1$.

**The Existence of Random Attractors**

It is convenient to work in the history space setting by introducing the so-called summed past history of $\theta$ which is defined as follows (cf.[7])

$$\eta'(s) = \int_0^s \theta(t - y)dy, \quad (t, s) \in [0, \infty) \times \mathbb{R}^+,$$

the variable $\eta'$ (formally) satisfies the linear equation

$$\eta'(s) + \eta''(s) = \Theta(t), \quad \text{in } \Omega, \quad (t, s) \in \mathbb{R}^+ \times \mathbb{R}^+,$$

boundaries and initial conditions

$$\eta'(0) = 0, \quad \text{in } \Omega, \quad t \geq 0,$$

$$\eta^0(s) = \eta_0(s) = \int_0^s \psi(y)dy, \quad \text{in } \Omega, \quad s \in \mathbb{R}^+.$$  

the linear operator $T$ introduced in $\mathbb{M}^1$ is defined as

$$T\eta = -\eta_s, \eta \in D(T),$$

its boundary is

$$D(T) = \{\eta \in \mathbb{M}^1 | \eta_s \in \mathbb{M}^1, \eta(0) = 0\}.$$  

here and above $\eta_s$ is the distribution derivative of $\eta$ with respect to the internal variable $s$.

We note that the time-segment integration of the convolution that occurs in the equation $\theta$ results in

$$\begin{cases}
\theta_i - \Delta u_i - \int_0^\infty \mu(s) \Delta \eta' i(s) ds = \sum_{j=1}^m \phi_j d\omega_j \\
u_i - \Delta u_i + \Delta (\Delta u + \theta) + f(u) = 0
\end{cases} \quad (4.1)$$

import the vector

$$b(t) = (u(t), v(t), \theta(t), \eta')^T,$$

define the initial value by

$$b_0 = (u_0, v_0, \Theta_0, \eta_0)^T \in V^0,$$

we can transform the studied equation into

$$b_i = Lb + G(b, t, \omega), \quad b(0) = b_0 = (u_0, u_1, \Theta_0, \eta_0) \quad (4.2)$$

among them, $G(b, t, \omega) = (0, -f(u), \sum_{j=1}^m \phi_j d\omega_j, 0)^T$ \text{let } u_j = v

$$Lb = L \begin{pmatrix}
u \\
u \\
u \\
\theta
\end{pmatrix} = \begin{pmatrix}
u \\
-Av - A(Au - \theta) \\
-Av - \int_0^\infty \mu(s) \Delta \eta' i(s) ds \\
\theta + T\eta
\end{pmatrix} \quad (4.3)$$
The existence and uniqueness of the solution

Next, explain the existence, uniqueness, and continuous dependence of the solution on the initial value of the problem (4.2), and show that the solution generates a continuous random dynamic system on probability space $(\Omega, F, P, (\mathcal{F}_t)_{t \in \mathbb{R}})$. For this, we consider the one-dimensional Ornstein-Uhlenbeck equation

$$dz_j + \zeta_j dt = d\omega_j(t), \quad j = 1, 2, \ldots, m,$$

its solution is the Ornstein-Uhlenbeck process

$$z_j(\mathcal{F}_t/\omega_j) = -\int_{-\infty}^0 e^{\zeta}(\mathcal{F}_t/\omega_j)(s)ds, \quad t \in \mathbb{R}.$$

It is known from the literature [2] that the random variable $|z_j(\omega)|$ is harmonic, and there is a measure invariant set of all measures, so that for all $\omega \in \Omega$ and $j = 1, 2, \ldots, m$, $t \mapsto z_j(\mathcal{F}_t/\omega_j)$ are continuous with respect to time $t$. Therefore, it is known from the proposition 4.3.3 in [2] that there is a harmonic function $r(\omega) > 0$ such that

$$\sum_{j=1}^m \left( |z_j(\omega)|^2 + |z_j(\omega)|^4 + |z_j(\omega)|^6 \right) \leq r(\omega)$$

among them, $r(\omega)$ satisfies P-a.e. $\omega \in \Omega$

$$r(\mathcal{F}_t/\omega_j) \leq e^{1/2\varepsilon} r(\omega), \quad t \in \mathbb{R},$$

known from (4.4) and (4.5), for P-a.e. $\omega \in \Omega$

$$\sum_{j=1}^m \left( |z_j(\mathcal{F}_t/\omega_j)|^2 + |z_j(\mathcal{F}_t/\omega_j)|^4 + |z_j(\mathcal{F}_t/\omega_j)|^6 \right) \leq e^{1/2\varepsilon} r(\omega), \quad t \in \mathbb{R}$$

let

$$z(\mathcal{F}_t/\omega) = z(x, \mathcal{F}_t/\omega) = \sum_{j=1}^m \phi_j z_j(\mathcal{F}_t/\omega_j),$$

This is a solution to the following equation

$$dz + zdt = \sum_{j=1}^m \phi_j d\omega_j,$$

**Theorem 4.1** ([10]) Variables $\rho$, $\rho^{(i)} : \Omega \mapsto \mathbb{R}^l, i = 1, 2, \ldots, l$, for all $t \in \mathbb{R}, \omega \in \Omega$

$$\|z(\mathcal{F}_t/\omega)| \leq e^{1/2\varepsilon} r(\omega), e^{1/2\varepsilon} r(\omega) \leq r(\mathcal{F}_t/\omega),$$

$$\|A^{(i)}z(\mathcal{F}_t/\omega)| \leq e^{1/2\varepsilon} r^{(i)}(\omega), e^{1/2\varepsilon} r^{(i)}(\omega) \leq r^{(i)}(\mathcal{F}_t/\omega) \leq e^{1/2\varepsilon} r^{(i)}(\omega),$$

among them, $r^{(i)}(\omega) = \sum_{j=1}^m \rho_j(\omega)|A^{(i)}h_j|$

Simplify (4.2) to the first-order evolution equation.

let

$$\omega = u + \varepsilon u - z(\mathcal{F}_t/\omega), \quad \varphi = (u, \omega, \theta, \eta)^T.$$
among them,

\[ -H(\varphi) = \begin{pmatrix} \omega - \varepsilon u \\ -A(\omega - \varepsilon u) - A(Au - \theta) \\ -A(\omega - \varepsilon u) - \int_0^\infty \mu(s)A\eta'(s)ds \\ \theta + T\eta \\ z(\partial,\omega) \\ -f(u) - Az(\partial,\omega) \\ -Az(\partial,\omega) + z(\partial,\omega) \\ 0 \end{pmatrix} = TLT^{-1}, \]

\[ F(\varphi, \partial, \omega) = \begin{pmatrix} 0 \\ z(\partial,\omega) \\ -f(u) - Az(\partial,\omega) \\ -Az(\partial,\omega) + z(\partial,\omega) \\ 0 \end{pmatrix}. \]

It is known from the literature [3] that the operator \( L \) in (4.3) is the infinitesimal generator of the compressed \( C_0 \) semigroup \( \{e^{\varepsilon t}\} \) in space \( V^0 \), because \( -H = TL T^{-1} \) and \( L \) are isomorphic operators in \( V^0 \). So operator \( -H \) also generates a compressed \( C_0 \) semigroup \( \{e^{-\varepsilon t}\} \) in \( V^0 \).

From the conditions satisfied by the nonlinear function \( f(u) \) and the embedding relationship \( H^*_\varepsilon(U) \rightarrow L^*(U) \), it is known that function \( F(\varphi, \partial, \omega, t) \) is locally Lipschitz continuous with respect to \( \varphi \) and \( \omega \in \Omega \). Moreover, the time \( t \) is bounded on a finite interval, so \( F(\varphi, \partial, \omega, t) \) is continuous with respect to \( (\varphi, t) \) and \( F \) is measurable with respect to \( \omega \), and the uniqueness of the solution of the differential equation is considered by the classic operator semigroup theory (cf.[3]), We have the following theorem.

**Theorem 4.2** If conditions (1)-(4) hold, then for any of \( \omega \in \Omega \) and \( \varphi_0 \in V^0 \), there is \( T>0 \), so that when the initial value meets \( \varphi(0, \omega, \varphi_0) = \varphi_0 \), the problem (4.8) has a unique mild solution \( \varphi(t, \omega, \varphi_0) \in C([0, T]; V^0) \) and \( \varphi(t, \omega, \varphi_0) \) satisfies

\[ \varphi(t, \omega, \varphi_0) = e^{-\varepsilon t} \varphi_0(\omega) + \int_0^t e^{-H(t-s)} F(s, \partial, \omega, \varphi(s, \omega, \varphi_0))ds, \]

further, \( \varphi(t, \omega, \varphi_0) \) is continuous with respect to \( t \) and \( \varphi_0 \) and measurable with respect to \( \omega \).

Knowing from Theorem 4.2 and the following Theorem 4.3, for any \( t \in [0, \infty) \), the solution \( \varphi(. \omega, \varphi_0) \) of equation (4.8) exists globally, therefore, \( \varphi(. \omega, \varphi_0) \in C([0, +\infty); V^0) \) defines a continuous random dynamic system from \( \Omega \) to \( (\Omega, F, P, (\partial), \varepsilon \infty) \):

\[ \Phi(t, \omega) : V^0 \rightarrow V^0 \]

\[ \varphi_0(\omega) \mapsto \varphi(t, \omega)\varphi_0. \]

In order to prove that the solution of stochastic partial differential equation (1.1) and the solution of partial differential equation (4.8) are conjugated, homeomorphic mapping is introduced

\[ R(\partial, \omega) y = (y_1, y_2 + \varepsilon y_1 - z(\partial, \omega), y_3, y_4)^T, \]

\[ y = (y_1, y_2, y_3, y_4)^T \in V^0, \]

its inverse mapping is

\[ R^{-1}(\partial, \omega) y = (y_1, y_2 - \varepsilon y_1 - z(\partial, \omega), y_3, y_4)^T, \]

then

\[ \Phi(t, \omega) = R(\partial, \omega) S(t, \omega) R^{-1}(\partial, \omega). \]

This also determines a continuous random dynamic system.

On the other hand, if

\[ a_1 = u(t), a_2 = u_1(t), a_3 = \theta(t), a_4 = \eta', \]

and

\[ a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}, G(a, \omega) = \begin{pmatrix} 0 \\ -f(u) \\ \sum_{j=1}^{m} \phi_j d\omega_j \\ 0 \end{pmatrix}. \]
then the partial differential equation (1.1) can be written as
\[\alpha + H(\alpha) = G(\alpha, \omega), G(\alpha, \omega) = (u_\alpha, u_\omega, \theta_\alpha, \eta_\omega)^T,\]
let \(T_y = (y_1, y_2, y_3, y_4)^T, y = (y_1, y_2, y_3, y_4)^T \in V^0\), its inverse map is
\[T^{-1}y = (y_1, y_2, y_3, y_4)^T,\]
so
\[\hat{S}(t, \omega) = TS(t, \omega)T^{-1}.\]

This also determines a continuous random dynamic system \((\partial, \hat{S}(t, \omega))\). Therefore, \(\hat{S}(t, \omega)\) and \(\Phi(t, \omega)\) are equivalent.

Next, the existence of the random attractor of the stochastic dynamic system \(\Phi(t, \omega)\) will be studied according to Theorem 2.1.

**Theorem 4.3** Assuming that conditions (1)-(4) hold, there is a closed sphere \((centered\ on\ 0, r_0(\omega)\ is\ radius)\), so that for any non-random set \(B = \{B(\omega)\}_{\omega \in \Omega} \subset D\), there is a \(T_{B(\omega)}(\omega) > 0\), so that for any \(\varphi \in \mathcal{B}(\partial, \omega) \in B(\partial, \omega)\), the solution \(\varphi(t, \partial, \omega, \varphi_0) = (u(t), o(t), \theta(t), \eta(t))^T\) of equation (4.8) satisfies P.a.e. \(\omega \in \Omega\), has
\[\|\varphi(t, \partial, \omega, \varphi_0)\|_{\omega} \leq r_0^2(\omega)\).

Proof: Use \(\varphi(t) = (u(t), o(t), \theta(t), \eta(t))^T\) and equation (4.8) to make the inner product in \(V^0\),
\[\frac{1}{2} \int_0^T \|u\|^2 + \|\omega\|^2 - \alpha(u'\omega, \omega) - \alpha(u, \theta) \leq \frac{1}{2} \int_0^T \mu'(s)\|\omega\|^2 ds (4.9)\]
among them,
\[\|u\|^2 \geq \gamma_{D_1\|u\|^2}, 2 < D_1 = \|u\|^2 < \frac{17 - \sqrt{65}}{2}.

We multiply the second expression in (1.1) by \(\varepsilon^2\) and add (4.9), then
\[\frac{d}{dt} \left(\frac{1}{2}\|u\|^2 + \frac{\varepsilon^2}{2}\|\omega\|^2 + \varepsilon^2 \int_0^t u\theta, dx - \alpha^2 \|u\|^2 + (\alpha + \varepsilon^2)\|u\|^2 + \|\omega\|^2\right)\]
\[-\alpha(u\omega, \omega) - \alpha(u, \theta) - \frac{1}{2} \int_0^T \mu'(s)\|\omega\|^2 ds + \varepsilon^2 \int_0^T u\theta, dx - \varepsilon^2 \int_0^T \theta u\alpha, dx (4.10)\]
substitute \(\omega = u + \varepsilon u - z(\partial, \omega)\) into the right side of equation (4.10) the inner product term on the left uses Lemma 2.2, taking the appropriate \(\alpha\) has
\[\frac{d}{dt} \left(\frac{1}{2}\|u\|^2 + \frac{\varepsilon^2}{2}\|\omega\|^2 + \varepsilon^2 \int_0^t u\theta, dx - \alpha^2 \|u\|^2 + \varepsilon^2 \int_0^t F(u) dx + \varepsilon^2 \|u\|^2 + (D_1 - \frac{\varepsilon^2}{2}) \|\omega\|^2\)
\[-\alpha(u\omega, \omega) - \alpha(u, \theta) - \frac{1}{2} \int_0^T \mu'(s)\|\omega\|^2 ds - \varepsilon^2 \int_0^T \theta u\alpha, dx + \varepsilon^2 \int_0^T u\theta, dx (4.11)\]
To estimate the right side of (4.11), from Lemma 2.2(1), take \(\alpha = \frac{1}{4}\) and have the following inequality:
\[\left((z(\partial, \omega), u)\right) \leq \frac{1}{8}\|u\|^2 + 2\|z(\partial, \omega)\|^2\]
\[(f(u), z(\partial, \omega)) \leq C_0\int_U (1 + |u|^3)|z(\partial, \omega)| dx\]
\[\leq C_0\int_U |z(\partial, \omega)| dx + C_0\left(\int_U |u|^3 dx\right)^2\|z(\partial, \omega)\|_U\]
\[ \leq C_0 \int_{\Omega} |z(\partial_t \omega)| dx + C_1 C_2^{\frac{2}{3}} \left( \frac{2}{3} \left( \epsilon_1 \left( \int_{\Omega} (F(u) + C_1) dx \right) \right)^{\frac{2}{3}} + \frac{1}{3 \varepsilon_1^3} \|z(\partial_t \omega)\|_{L^3}^3 \right) \]

\[ \leq C_0 \|z(\partial_t \omega)\| + \kappa \int_{\Omega} F(u) dx + \kappa C_1 |U| + \tau \|z(\partial_t \omega)\|_{L^3}^3, \]

among them, \( \kappa = \frac{2}{3} C_0 C_1^{\frac{2}{3}} \varepsilon_1^2, \tau = \frac{1}{3 \varepsilon_1^3}. \)

Knowing from the condition (3.5), if \( N_i > 0 \) exists, then

\[ (f(u), u) \geq C_2 \int_{\Omega} F(u) dx - \frac{\lambda_i}{8} \|u\|_{L^3}^3 - N_i. \]

Use (3.6) to know that there is \( 0 < \delta_i < \frac{1}{2}, I = I(\delta_i) > 0 \) and get \( f(s) s \geq -\frac{1 - \delta_i}{C_2} s^2, \forall s \geq N_i \). Then

\[ \int_{\Omega} f(u) u dx \geq -\|u\|_{L^3}^3 + C(\|U\|, f) \]

among them, \( C(\|U\|, f) = \|U\| \min_{|\alpha|<\omega} F(s) \).

Synthesizing the above inequalities and using Lemma 2.2, take the appropriate \( \alpha \) to get

\[ \frac{d}{dt} \left( \frac{1}{2} \|\phi\|_{L^2}^2 + \frac{\varepsilon^2}{2} \|\nabla u\|_{L^2}^2 + \varepsilon^2 \int_{\Omega} uu_t dx - \varepsilon^2 \|u\|_{L^3}^3 + \int_{\Omega} F(u) dx \right) + (D_1 - \frac{\lambda_i}{8}) \|\omega\|_{L^3}^3 \]

\[ + (D_1 - \frac{\lambda_i}{8}) \|\phi\|_{L^2}^2 \]

\[ + \left( \frac{\lambda_i}{8} - \frac{\varepsilon^2}{2} - \frac{k_0 \varepsilon}{4} \right) \|u\|_{L^3}^3 \]

\[ + \frac{1}{\varepsilon_1^3} \left( \frac{\lambda_i}{8} - \frac{\varepsilon^2}{2} - \frac{k_0 \varepsilon}{4} \right) \|\nabla z(\partial_t \omega)\|_{L^3}^3 \]

\[ + \kappa C_1 |U| + \tau \|z(\partial_t \omega)\|_{L^3}^3, \]

Let \( J(t) = -\int_0^t \mu(s) \langle \theta(t), \eta'(s) \rangle ds \), then \( J(t) \leq \|\theta(t)\| \int_0^t \mu(s) \langle \eta'(s), \eta'(s) \rangle ds \leq C \|\theta\|_{L^2}. \)

\[ \frac{d}{dt} J(t) = -\int_0^t \mu(s) \langle \theta(t), \eta'(s) \rangle ds - \int_0^t \mu(s) \langle \theta(t), \eta'(s) \rangle ds \]

\[ = -\int_0^t \mu(s) \langle -Aw, \eta'(s) \rangle ds - \int_0^t \mu(s) \langle \varepsilon Au, \eta'(s) \rangle ds + \int_0^t \mu(s) \langle Az(\partial_t \omega), \eta'(s) \rangle ds \]

\[ + \int_0^t \mu(s) \langle A\eta'(s), \eta'(s) \rangle ds - \int_0^t \mu(s) \langle z(\partial_t \omega), \eta'(s) \rangle ds - k_0 \|\theta\|_{L^2}^2 + \int_0^t \mu(s) \langle \theta(t), \eta'(s) \rangle ds \]

of the above formula has the following inequality

\[ \int_0^t \mu(s) \langle \int_0^s \mu(s) A\eta'(s), \eta'(s) \rangle ds \leq k_0 \|\eta\|_{L^2}^2, \]

At the same time using Lemma 2.2 (3)(4), take the appropriate \( \alpha \) to get

\[ \frac{d}{dt} J(t) + \frac{3 k_0}{4} \|\phi\|_{L^2}^2 \leq \frac{k_0 D_1}{4} \|\phi\|_{L^2}^2 + \frac{k_0 \varepsilon}{4} \|u\|_{L^3}^3 + \frac{1 + 4 \varepsilon}{4 D_1} + \frac{5}{4} \|\eta\|_{L^2}^2, \]

(4.13)

Next, add (4.13) and (4.12) to get

\[ \frac{d}{dt} \left( \frac{1}{2} \|\phi\|_{L^2}^2 + \frac{\varepsilon^2}{2} \|\nabla u\|_{L^2}^2 + \varepsilon^2 \int_{\Omega} uu_t dx - \varepsilon^2 \|u\|_{L^3}^3 + \int_{\Omega} F(u) dx + J(t) \right) + (D_1 - \frac{\lambda_i}{8}) \|\omega\|_{L^3}^3 \]

\[ + \left( \frac{\lambda_i}{8} - \frac{\varepsilon^2}{2} - \frac{k_0 \varepsilon}{4} \right) \|u\|_{L^3}^3 \]

\[ + \left( \frac{\lambda_i}{8} - \frac{\varepsilon^2}{2} - \frac{k_0 \varepsilon}{4} \right) \|\nabla z(\partial_t \omega)\|_{L^3}^3 \]

\[ + \kappa C_1 |U| + \tau \|z(\partial_t \omega)\|_{L^3}^3, \]

\[ -\varepsilon^2 C(\|U\|, f) + \varepsilon N_i. \]
among them
\[
\int_0^\infty \mu'(s) \|A^{1/2} \eta'(s)\|^2 ds \geq \delta \int_0^\infty \mu(s) \|A^{1/2} \eta'(s)\|^2 ds = \delta \|\eta'\|^2_{M'},
\]

Let
\[
R_0(\partial, \omega) = \left\|\nabla z(\partial, \omega)\right\|^2 + \left\|z(\partial, \omega)\right\|^2 + \left\|Az(\partial, \omega)\right\|^2 + \left\|z(\partial, \omega)\right\|^2,
\]
\[
G(t) = \frac{1}{2} \|\phi\|_{\nu^T}^2 + \frac{\epsilon^2}{2} \|\nabla u\|_{\nu^T}^2 + \epsilon^2 \int_U uu_t dx - \epsilon^2 \|u\|_{\nu^T}^2 + \int_U F(u) dx + J(t),
\]

There is
\[
\frac{d}{dt} G(t) + \beta \|\phi\|_{\nu^T}^2 \leq \frac{d}{dt} G(t) + \beta \|\phi\|_{\nu^T}^2 + (\epsilon C_2 - \kappa) \int_U F(u) dx \leq \chi
\]

among them,
\[
Q = \max\{6, C_0, 2 + k_U, \tau\},
\]
\[
\chi = QR_0(\partial, \omega) + \kappa C_1 |U| + \epsilon^2 C(\|U\|, f) + \epsilon N_0,
\]

\[
\beta = \min\{\epsilon^2 \delta - \frac{\lambda \epsilon + 1 - \delta}{8} - \frac{k_U \epsilon}{4}, D_1 - \frac{\lambda \epsilon}{4}, \frac{1}{8} - \frac{\epsilon^2}{2}, \frac{3k_U - 1}{2} - \frac{\epsilon^2}{2}, \frac{\delta + \epsilon}{2}, \frac{\delta - \epsilon}{2}, \frac{5}{4} - \frac{1}{4D_1}, \frac{k_U \epsilon}{2}, \}
\]

limited range of each parameter, each item in \(\beta\) is greater than zero).

Let \(\rho_0 = \min\{\frac{\lambda_0 + 2k_0 - \delta}{\delta + \epsilon}, \frac{\lambda_0 k_U + k_0^2 - \delta}{\delta + \epsilon}, \frac{8D_0 - 1}{2}, \frac{3k_U - 1}{2}, \frac{\delta}{2} - \frac{5}{4D_1}, \frac{k_0}{2}, \}
\)

(within the limited range of each parameter, each item in \(\rho_0\) is greater than zero).

By Theorem 4.1
\[
\int_0^\infty R_0(\partial, \omega)e^{\beta s} ds \leq \int_0^\infty R_0(\partial, \omega)e^{\beta s} ds \leq \int_0^\infty R_0(\partial, \omega)e^{\beta s} ds \leq +\infty,
\]

among them,
\[
\tilde{R}_0(\partial, \omega) = e^{2\rho_0} r^2(\omega) + e^{\frac{1}{2}\rho_0} r(\omega) + (e^{\rho_0} r(\omega))^2 + (e^{\rho_0} r(\omega))^2.
\]

By Lemma 2.1 there is \(G(\partial(\omega, \partial, \omega(\partial))) \leq \sup_{\xi\in\omega^n} [E(\xi) : \beta \|\xi\|_{\nu^T}^2 \leq \chi + \epsilon_1].\)

Use \(\partial, \omega\) instead of \(\partial\) in the above formula to get
\[
G(\partial(\omega, \partial, \omega(\partial))) \leq \sup_{\xi\in\omega^n} [E(\xi) : \beta \|\xi\|_{\nu^T}^2 \leq \chi + \epsilon_1].
\]

Then
\[
\|\phi(t, \partial, \omega, \phi(\partial, \omega))\|_{\nu^T}^2 \leq \frac{\chi + \epsilon_1}{\beta} = \rho_0^2(\omega).
\]

To sum up, the theorem is proved.

In order to obtain the regularity results of the stochastic dynamic system \(\Phi(t, \omega)\), using a method similar to [1], the solution \(\phi = (\omega, \omega, \partial, \eta')^T\) of the system is decomposed into \(\phi = \phi_L + \phi_N\), where \(\phi_L = (u_t, \omega_L, \partial_L, \eta'_L)^T\) and \(\phi_N = (u_N, \omega_N, \partial_N, \eta'_N)^T\) are the solutions of the following equations
\[
\text{L}^0 \phi_L + H(\phi_L) = 0, \phi_L(0, \omega) = (u_0, u_1 + cu_t, \partial_0, \eta'_0)^T, t \geq 0
\]
and
\[
\text{F}^0 \phi_N + H(\phi_N) = F_N(\omega), \phi_N(0, \omega) = (0, -z(\omega), 0, 0)^T, t \geq 0
\]
among them,
\[
F_N(\omega) = \begin{cases}
\begin{pmatrix}
z(\partial, \omega) \\
z(\partial, \omega) - AZ(\partial, \omega) \\
0
\end{pmatrix}
\end{cases}.
\]

For equations (4.15) and (4.16) there are regular results as follows.
Theorem 4.4 Assuming conditions (1)-(4) hold, let \( B \subset V^0 \) be a non-random bounded set, then for any \( \varphi(t, \varphi) \in (u_0, u_1 + \varepsilon u_0, \theta_0, \eta_0)^T \in B \), the following holds

\[
\|\varphi(t, \varphi)\|^2_{\infty} = \|u_0\|^2 + \|\varphi\|^2_{\infty} + \|\theta_0\|^2 + \|\eta_0\|^2_{\infty} \to 0, (t \to \infty),
\]

\( \varphi(t, \varphi) \to (u_0, u_1 + \varepsilon u_0, \theta_0, \eta_0)^T \) meet equation (4.15).

Proof: In space \( V^0 \), use \( \|u_0\|^2 + \|\varphi\|^2_{\infty} + \|\theta_0\|^2 + \|\eta_0\|^2_{\infty} \to 0 \), the following holds

\[
\frac{d}{dt} \|\varphi(t, \varphi)\|^2_{\infty} + (H(\varphi(t), \varphi(t)))_{\varphi} = 0.
\]

among them, \( (H(\varphi(t), \varphi(t)))_{\varphi} \geq \frac{\varepsilon}{2}\|u_0\|^2 + (D_1 - \varepsilon)\|\varphi\|^2_{\infty} + \frac{1}{2}\|\theta_0\|^2 + \frac{\delta}{2}\|\eta_0\|^2_{\infty} \).

Define the energy functional:

\[
K(t) = \int_U \varphi(t, \varphi) \, dx,
\]

\[
P(t) = \int_U \theta(t, \varphi) \, dx,
\]

\[
\Psi(t) = -\int_U \theta(t, \varphi(t)) \int_0^\infty \mu(s) \eta_L(s) \, ds \, dx.
\]

Then derive \( K(t), P(t) \) and \( \Psi(t) \) with respect to time \( t \), and use the Herder's inequality to get

\[
\frac{d}{dt} K(t) = \int_U \varphi(t, \varphi) \, dx + \int_U \varphi(t, \varphi) \, dx
= \int_U (\varepsilon u_0) + \int_U (\varepsilon u_0) \, dx + \int_U \varphi(t, \varphi) \, dx
\leq \frac{2}{4}\|\varphi\|^2_{\infty} + \frac{(\varepsilon(D_1 - 1) - \varepsilon)\|u_0\|^2 + \frac{1}{2}\|\theta_0\|^2}{\varepsilon} + C.
\]

\[
\frac{d}{dt} P(t) = \int_U \theta(t, \varphi) \, dx + \int_U \theta(t, \varphi) \, dx
= \int_U (\varepsilon u_0) + \int_U (\varepsilon u_0) \, dx + \int_U \theta(t, \varphi) \, dx
\leq \int_U \varepsilon u_0 \, dx + \int_U \varphi(t, \varphi) \, dx + \int_U \theta(t, \varphi) \, dx
\leq \|\varphi\|^2_{\infty} + \frac{1 + \varepsilon(D_1 + 1)}{2} \|\varphi\|^2_{\infty} + \frac{1 + \varepsilon(D_1 + 1)}{2} \|\varphi\|^2_{\infty} + \frac{2 + \varepsilon\|\eta_0\|^2_{\infty}}{4} + C.
\]

Then take the derivative of \( \Psi(t) \).

\[
\frac{d}{dt} \Psi(t) = \frac{d}{dt} \int_U \theta(t, \varphi(t)) \int_0^\infty \mu(s) \eta_L(s) \, ds \, dx
= \int_U \theta(t, \varphi(t)) \int_0^\infty \mu(s) \eta_L(s) \, ds \, dx - \int_U \theta(t, \varphi(t)) \int_0^\infty \mu(s) \eta_L(s) \, ds \, dx
= \int_U \theta(t, \varphi(t)) \int_0^\infty \mu(s) \eta_L(s) \, ds \, dx - \int_U \theta(t, \varphi(t)) \int_0^\infty \mu(s) \eta_L(s) \, ds \, dx
\]

among them,

\[
|\int_U \mu(s) \varphi(t, \varphi) \, dx| \int_0^\infty \mu(s) \eta_L(s) \, ds \leq k_0 \|\eta_0\|^2_{\infty}.
\]
\[ -\int_0^t \theta(t, s) \mu(s) \eta''(s) ds \]
\[ \leq -k_0 \| \theta(t, s) \|^2 +\int_0^t \theta(t, s) \left\| \mu'(s) \right\| \mu''(s) \left\| \eta''(s) \right\| ds + M \int_0^t \mu(s) \left\| \eta''(s) \right\| ds dt \]
\[ \leq -\left( k_0 - \frac{1}{4} \right) \| \theta(t, s) \|^2 + C \left\| \eta''(s) \right\|^2_{\mathcal{M}^1}, \]
among them, \( C = \frac{\sqrt{5}}{\lambda} \max \left\{ -\frac{1}{\mu''(s)} \int_0^t \mu'(s) ds, M \mu''_{\mathcal{M}^1} \right\}. \)

Then, \( \frac{d}{dt} \Psi(t) \leq \frac{\varepsilon}{4} \| u_1 \|^2_2 + \frac{1}{4} \| \omega_2 \|^2_2 - (k_0 - \frac{1}{4}) \| \theta(t, s) \|^2 + (C + D_1 + \frac{\varepsilon}{D_1} + k_0) \left\| \eta''(s) \right\|^2_{\mathcal{M}^1}. \)

Let, \( L(t) = MY(t) + \gamma(K(t) + P(t)) + \Psi(t), \)
among them \( \gamma < M. \)

\[ l = \min \left\{ -D_1(4k_0 - 1 - 4k_0, \varepsilon) - 2\varepsilon + 2k_0D_1 - 4\varepsilon, 16\varepsilon + 8k_0D_1 + 4D_1 + \varepsilon D_1, 2\varepsilon + 2k_0\delta D_1 - 8\varepsilon \right\}, \]
\[ \varsigma < M < \frac{4(k_0 - \gamma) - 1 - \gamma^2}{4\varepsilon}, \]
\[ \varsigma = \max \left\{ \frac{12\gamma^2 + 1 - \gamma^2}{4(D_1 - \varepsilon)}, 2(C + D_1 + k_0 + \frac{\varepsilon}{D_1} + \frac{\gamma}{2D_1}) + \frac{4\gamma + k_0}{\varepsilon} \right\}. \]

(within the limited range of each parameter, each item in \( \varsigma \) is less than \( \frac{4(k_0 - \gamma) - 1 - \gamma^2}{4\varepsilon} \))

The inequalities obtained by integrating the results of the above three functions are

\[ \frac{d}{dt} L(t) = M \frac{d}{dt} \Psi(t) \]
\[ \leq (\varepsilon M - MD_1 + 3\gamma - \frac{1 - \gamma^2}{4}) \| \omega_2 \|^2_2 + (\frac{\varepsilon}{D_1} - \frac{\delta M - C}{2D_1} + D_1 + \frac{\gamma}{2D_1} + k_0) \left\| \eta''(s) \right\|^2_{\mathcal{M}^1} + \left( \frac{\varepsilon}{2} + \frac{2\varepsilon^2}{D_1} + \frac{\gamma k_0 - \varepsilon M}{2} \right) \| u_1 \|^2_2 + (\varepsilon + \varepsilon M - k_0 + \frac{\gamma^2 + 1}{4}) \| \theta(t, s) \|^2_2, \]
let,
\[ \beta_1 = \min \left\{ MD_1 - \varepsilon M - 3\gamma - \frac{1 - \gamma^2}{4}, \frac{\varepsilon M}{2} - \frac{2\varepsilon^2}{D_1} - \frac{\gamma k_0}{2}, k_0 - \varepsilon M - \frac{\gamma^2 + 1}{4}, \frac{\delta M}{2} - \frac{\varepsilon}{D_1} - C - D_1 - \frac{\gamma}{2D_1} - k_0 \right\}, \]
\[ \rho_2 = \min \left\{ D_1, \frac{72 + D_1^3 - 15D_1 + k_0\delta D_1}{4}, \frac{\zeta D_1}{\delta} - 1 \right\} \]
the limited range of each parameter, each item in \( \beta_1 \) and \( \rho_2 \) is greater than zero.

Then
\[ \frac{d}{dt} L(t) + \beta_1 Y(t) \leq 0, \quad \text{(4.17)} \]
applying Gronwall’s lemma to the above formula
\[ L(t, \omega, \phi_{\omega}(0, \omega)) \leq Y(0, \omega, \phi_{\omega}(0, \omega) e^{-\beta_1 t}, \]
use \( \mathcal{G}_{\omega, \omega} \) instead of \( \omega \) in the above formula to get
\[ L(t, \mathcal{G}_{\omega, \omega}, \phi_{\omega}(0, \mathcal{G}_{\omega, \omega})) \leq Y(0, \mathcal{G}_{\omega, \omega}, \phi_{\omega}(0, \mathcal{G}_{\omega, \omega}) e^{-\beta_1 t}, \]
So,
\[ 2Y(t, \mathcal{G}_{\omega, \omega}, \phi_{\omega}(0, \mathcal{G}_{\omega, \omega})) \leq 2CL(t, \mathcal{G}_{\omega, \omega}, \phi_{\omega}(0, \mathcal{G}_{\omega, \omega}))) \]
\[ \leq 2C Y(0, \mathcal{G}_{\omega, \omega}, \phi_{\omega}(0, \mathcal{G}_{\omega, \omega}) e^{-\beta_1 t}) \]
\[ \leq 2C e^{-\beta_1 t} \left( \| k_0 \|^2_2 + \| u_1 \|^2_2 + \| \theta(t, s) \|^2_2 + \| \eta''(s) \|^2_{\mathcal{M}^1} \right) \rightarrow 0, \quad t \rightarrow \infty. \]
To sum up, the theorem is proved.

**Theorem 4.5** Assuming that conditions (1)-(4) hold, there is a random closed sphere $B = \{B(\omega)\}_{\omega \in \Omega} \in D$ centered on 0 and radiused $r_1(\omega) > 0$, while $B(\omega) \in D$, $\phi_n(\omega) \in B(\omega)$ and $T_{B(\omega)} > 0$ exist, so that for any $\varphi_N = (u_N, \omega_N, \theta_N, \eta_N)^T$, the system (4.16) satisfies P-a.e. $\omega \in \Omega$, there is

$$
\left\| A^{1/2}u_N \right\|_{L^2}^2 + \left\| A^{1/2}\omega_N \right\|_{L^2}^2 + \left\| A^{1/2}\theta_N \right\|_{L^2}^2 + \left\| A^{1/2}\eta_N \right\|_{L^2}^2 \leq r_1^2(\omega).
$$

**Proof:** Knowing from Theorem 4.3, Theorem 4.4 and $\varphi_N = \varphi - \varphi_L$, there is a positive random variable $M_1(\omega) > 0$, making

$$
\max \left\{ \left\| \varphi(0, \partial_t \omega, \varphi(0, \partial_t \omega)) \right\|_{L^2}, \left\| \varphi_N(0, \partial_t \omega, \varphi_N(0, \partial_t \omega)) \right\|_{L^2} \right\} \leq M_1(\omega).
$$

In space $V^0$, use $A'\varphi_N = (A'u_N, A'\omega_N, A'\theta_N, A'\eta_N)^T$ and (4.16) as the inner product

$$
\left\{ \left( \frac{1}{2} \left\| A^{1/2}u_N \right\|_{L^2}^2 + \left\| A^{1/2}\omega_N \right\|_{L^2}^2 + \left\| A^{1/2}\theta_N \right\|_{L^2}^2 + \left\| A^{1/2}\eta_N \right\|_{L^2}^2 \right) + \left( H(\varphi_N), A'\varphi_N \right) \right\}
$$

(4.18)

It can be calculated by the similar theorem 4.4

$$
\left( H(\varphi_N), A'\varphi_N \right) \geq C \left\| A^{1/2}u_N \right\|_{L^2}^2 + \left\| (D_1 - \epsilon) A^{1/2} \omega_N \right\|_{L^2}^2 - \epsilon \left\| A^{1/2} \theta_N \right\|_{L^2}^2 + \delta \left\| A^{1/2} \eta_N \right\|_{L^2}^2
$$

Next estimate each item on the right side of (4.18), using the condition (3.6), $f(u) \geq -\frac{1}{C} \frac{\delta_1}{u}$, we get

$$
\left\| (f(u), A'\omega_N) \right\| \leq \frac{(1 - \delta_1^2)C}{4C^2} r_1^2(\omega) + \left\| A^{1/2} \omega_N \right\|_{L^2}^2,
$$

where $C > 0$ is the embedding constant.

To sum up, using Lemma 2.2, take the appropriate $\alpha$ to get,

$$
\frac{d}{dt} \left\| A^{1/2} \varphi_N \right\|_{L^2}^2 + (\epsilon - \frac{1}{2}) \left\| A^{1/2}u_N \right\|_{L^2}^2 + (2D_1 - 2 \epsilon - \frac{5}{2}) \left\| A^{1/2} \omega_N \right\|_{L^2}^2 - (2 \epsilon + \frac{1}{2}) \left\| A^{1/2} \theta_N \right\|_{L^2}^2 + \delta \left\| A^{1/2} \eta_N \right\|_{L^2}^2
$$

(4.19)

$$
\leq 2 \left\| A^{1/2} \zeta(\partial_t \omega) \right\|_{L^2}^2 + \frac{(1 - \delta_1^2)C}{2C^2} \kappa_0^2(\omega) + 6 \left\| A^{1/2} \zeta(\partial_t \omega) \right\|_{L^2}^2 + 4 \left\| A^{1/2} \zeta(\partial_t \omega) \right\|_{L^2}^2.
$$

Let $A'J_N(t) = -\int_0^t \mu(s) A' \left\{ \partial_t \theta_N, \eta_N(s) \right\} ds$, then

$$
\left\| A'J_N(t) \right\| \leq \left\| A^{1/2} \theta_N \right\|_{L^2} \left\| \int_0^t \mu(s) A^{1/2} \left\{ \eta_N(s), \eta_N(s)^T \right\} ds \right\|_{L^2} \leq C \left\| A^{1/2} \varphi_N \right\|_{L^2}^2.
$$

By Theorem 4.3 there

$$
\frac{d}{dt} \left\| A^{1/2} \theta_N \right\|_{L^2}^2 \leq \frac{k_0 D_1}{4} \left\| A^{1/2} \omega_N \right\|_{L^2}^2 + \frac{k_0 \epsilon}{4} \left\| A^{1/2}u_N \right\|_{L^2}^2 + \left(1 + 4 \epsilon + 4 \delta \right) \frac{4D_1}{k_0} + \frac{k_0 + 5}{4} \left\| A^{1/2} \eta_N \right\|_{L^2}^2
$$

(4.20)

$$
+ \frac{k_0}{4} \left\| A^{1/2} \zeta(\partial_t \omega) \right\|_{L^2}^2 + \kappa_0 \left\| A' \zeta(\partial_t \omega) \right\|_{L^2}^2.
$$

Combining (4.19) and (4.20)

$$
\frac{d}{dt} \left\{ \left\| A^{1/2} \varphi_N \right\|_{L^2}^2 + A'J_N(t) + \left( \epsilon - \frac{1}{2} \right) \frac{k_0 \epsilon}{4} \left\| A^{1/2}u_N \right\|_{L^2}^2 + (2D_1 - 2 \epsilon - \frac{5}{2}) \frac{k_0 D_1}{4} \left\| A^{1/2} \omega_N \right\|_{L^2}^2
$$

$$
+ \frac{3k_0}{4} - \frac{2 \epsilon}{2} \frac{k_0 \epsilon}{4k_0} \left\| A^{1/2} \theta_N \right\|_{L^2}^2 + \left( \delta - \frac{1 + 4 \epsilon + 4 \delta}{4D_1} \right) \frac{k_0}{4} \left\| A^{1/2} \eta_N \right\|_{L^2}^2
$$

$$
\leq \left\| A^{1/2} \zeta(\partial_t \omega) \right\|_{L^2}^2 + \frac{(1 - \delta_1^2)C}{2C^2} \kappa_0^2(\omega) + 6 \left\| A^{1/2} \zeta(\partial_t \omega) \right\|_{L^2}^2 + (k_0 + 4) \left\| A^{1/2} \zeta(\partial_t \omega) \right\|_{L^2}^2 + \kappa_0 \left\| A^{1/2} \zeta(\partial_t \omega) \right\|_{L^2}^2.
$$

Let $X(t) = \left\| A^{1/2} \varphi_N \right\|_{L^2}^2 = \left\| A^{1/2}u_N \right\|_{L^2}^2 + \left\| A^{1/2} \omega_N \right\|_{L^2}^2 + \left\| A^{1/2} \theta_N \right\|_{L^2}^2 + \left\| A^{1/2} \eta_N \right\|_{L^2}^2$, there is

$$
(1 - C \epsilon) \frac{d}{dt} X(t) + \beta_1 X(t) \leq \frac{d}{dt} \left\{ \left\| A^{1/2} \varphi_N \right\|_{L^2}^2 + A'J_N(t) \right\} + \beta_2 X(t) \leq PR_1(\partial_t \omega),
$$

(4.21)
among them,
\[ R(\tilde{\omega},\omega) = \| A^{(r+1)} z(\tilde{\omega},\omega) \|_2^2 + \| A^{(r+2)} z(\tilde{\omega},\omega) \|_2^2 + \| A^{(r+3)} z(\tilde{\omega},\omega) \|_2^2 + r_0^2(\omega), \]

\[ P = \max \left\{ \beta_2 \frac{(1-\delta)^2}{2C_2}, k_0 + 4 \right\}, \]

\[ \beta_2 = \min \left\{ \frac{\varepsilon}{1-2\delta} - \frac{k_0}{4}, 2D_1 - \frac{2\varepsilon}{5} - \frac{k_0D_1}{4}, 3k_0 - \frac{2\varepsilon}{4D_1} - \frac{k_0}{4} \right\}, \]

\[ \rho_3 = \min \left\{ \frac{2}{4-k_0}, D_1 - \frac{k_0}{8}, \frac{3}{8}k_0 - \frac{1}{4} \right\}. \]

(Within the limited range of each parameter, each item in \( \beta_2 \) and \( \rho_3 \) is greater than zero).

For (4.21), using Gronwall's inequality

\[ X(t,\omega,\varphi_0(\omega)) \leq e^{-\frac{\beta_2}{1-C_n}} X(0,\omega,\varphi_0(\omega)) + \frac{P}{1-C_n} \int_0^t R(\tilde{\omega}^0(\omega))e^{-\beta_2(\omega-t)}ds, \]

use \( \tilde{\omega} \) instead of \( \omega \) in the above formula to get

\[ X(t,\tilde{\omega},\varphi_0(\tilde{\omega},\omega)) \leq e^{-\frac{\beta_2}{1-C_n}} X(0,\tilde{\omega},\varphi_0(\tilde{\omega},\omega)) + \frac{P}{1-C_n} \int_0^t \tilde{R}(s,\tilde{\omega},\varphi_0(\tilde{\omega},\omega))e^{-\beta_2s}ds, \]

by Theorem 4.1, there is

\[ \int_0^t \tilde{R}(s,\omega)e^{\beta_2s}ds \leq \int_0^t \tilde{R}(s,\tilde{\omega},\varphi_0(\tilde{\omega},\omega))e^{\beta_2s}ds \leq \int_0^t \tilde{R}(s,\omega)e^{\beta_2s}ds < +\infty, \]

among there,

\[ \tilde{R}(s,\omega) = 2(e^{\theta(t)}r^{(1)}(A^{(r/2)}\omega)^2) + 2(e^{\theta(t)}r^{(2)}(A^{(r/2)}\omega)^2) + 2r_0^2(\omega). \]

Because of \( \varphi_0(\tilde{\omega},\omega) \in \mathcal{B}(\tilde{\omega},\omega) \), so there is \( T_{\mathcal{B}(\tilde{\omega},\omega)} > 0 \), so that for all \( t \geq T_{\mathcal{B}(\tilde{\omega},\omega)} \), there is

\[ e^{-\frac{\beta_2}{1-C_n}} X(0,\tilde{\omega},\varphi_0(\tilde{\omega},\omega)) \leq \frac{4P(1-C_n)}{\beta_2} r(A^{(r/2)}\omega). \]

So,

\[ X(t) = \| A^{(r/2)} u_N \|_2^2 + \| A^{(r/2)} \omega_N \|_2^2 + \| A^{(r/2)} \varphi_0(\tilde{\omega},\omega) \|_2^2 + \| A^{(r/2)} \eta'' \|_M^2, \]

\[ \leq \frac{4P(1-C_n)}{\beta_2} r(A^{(r/2)}\omega) + \frac{P}{1-C_n} \int_0^t \tilde{R}(s,\omega)e^{\beta_2s}ds = r_1^2(\omega). \]

To sum up, the theorem is proved.

Consider the space \( M^{r+1} \) corresponding to variable \( \eta'' \), refer to [9], note that for any \( t \geq 0, \omega \in \Omega \), there are

\[ \eta''_N(t,\tilde{\omega},\omega,\varphi_0(\tilde{\omega},\omega),s) = \begin{cases} u_N(t,\tilde{\omega},\omega,\varphi_0(\tilde{\omega},\omega)) - u_N(t-s,\tilde{\omega},\omega,\varphi_0(\tilde{\omega},\omega)) & s \leq t, \\ \eta''_N(t,\tilde{\omega},\omega,\varphi_0(\tilde{\omega},\omega),s) & s \geq t. \end{cases} \]  

(4.22)

\[ \eta''_N(t,\tilde{\omega},\omega,\varphi_0(\tilde{\omega},\omega),s) = \begin{cases} u_N(t-s,\tilde{\omega},\omega,\varphi_0(\tilde{\omega},\omega)) & s \leq t, \\ 0 & s \geq t. \end{cases} \]  

(4.23)

Define collection \( \mathcal{B}(\omega) = \bigcup_{\tilde{\omega}} \varphi_0(\tilde{\omega},\omega) \in \mathcal{B}(\tilde{\omega},\omega) \bigcup_{\eta''_N} \eta''_N(t,\tilde{\omega},\omega,\varphi_0(\tilde{\omega},\omega),s), \)

where \( \varphi = (u,\omega,\theta,\eta')^T \) is the solution of equation (4.8). Then theorem 4.5, (4.22) and (4.23) can be obtained

\[ \max \left\{ \| \eta''_N(t,\tilde{\omega},\omega,\varphi_0(\tilde{\omega},\omega),s) \|_M^{r+1}, \| \eta''_N(t,\tilde{\omega},\omega,\varphi_0(\tilde{\omega},\omega),s) \|^{r+1}_M \right\} \leq 2r_1^2(\omega), \forall s \geq 0, \]

so \( \mathcal{B}(\omega) \) in \( M^{r+1} = L_{\omega}^{\infty}(R^+;V_r^{(r)}) \) is bounded.

Know by Theorem 4.3 and Theorem 4.5
Therefore, for any $\eta' \in \tilde{B}(\omega)$, there is
\[
\left\| \eta'(s) \right\|_{L^2}^2 = \int_0^\infty \mu(s) \left\| \eta'(s) \right\|_{L^2}^2 \, ds \leq 2r_1^2(\omega) \cdot \frac{1}{\sigma} \int_0^\infty e^{-\frac{s}{\sigma}} \, ds = \frac{2r_1^2(\omega)}{\sigma}.
\]
That is, $\tilde{B}(\omega) \subset L^2_\mu(R^+: H^2(U)) \cap H^1_t(U)$ is a bounded subset.

According to Theorem 2.2, $\tilde{B}(\omega)$ is tight in $L^2_\mu(R^+: H^2(U)) \cap H^1_t(U)$.

The following proves that the random power system $\Phi(t, \omega)$ has a random attractor.

**Theorem 4.6** Assuming conditions (1)-(4) hold, for any $\omega \in \Omega$, the random dynamic system $\Phi(t, \omega)$ generated by equation (4.8) has a compact random attractor set $\Lambda(\omega) \subset V^0$, and has a random attractor $A(\omega) \subseteq \Lambda(\omega) \cap B_{r_1}(\omega)$.

Proof: Any $\omega \in \Omega$, by theorem 4.5, let $B_r(\omega) = V^r \times V^r \times V^r$ be a random closed ball with a radius of $r_1(\omega)$. Definition $\Lambda(\omega) = B_r(\omega) \times \tilde{B}(\omega)$, then $\Lambda(\omega) \in D(V^0)$, because

\[
v^r \times v^r \times v^r \rightarrow H^2_t(U) \times L^2_t(U) \times L^2_t(U)
\]
and $B_r(\omega) \rightarrow H^2_t(U) \times L^2_t(U) \times L^2_t(U)$, Embed

Using theorem 2.2 again, $B(\omega)$ is compact in $M^1$, so $\Lambda(\omega)$ is compact in $V^0$. Next, prove the attractiveness of set $\Lambda(\omega)$.

For any $B(\omega) \in D(V^0)$

\[
\lim_{t \to \infty} d_\mu(B(\omega), B(\omega)) = 0.
\]

From Theorem 4.5

\[
\varphi(0, \omega, \varphi(0, \omega)) = \varphi(0, \omega, \varphi(0, \omega)) - \varphi(0, \omega, \varphi(0, \omega)) \in \Lambda(\omega),
\]

Combined with Theorem 4.4 there is

\[
d(B(\omega), B(\omega), \Lambda(\omega)) \leq e^{-2\beta_\omega} \left(\|u_0\|_2^2 + \|u_1 + e\|_2^2 + \|\theta_0\|_2^2 + \|\eta_{\omega}\|_2^2\right).
\]

Therefore, the random dynamic system $\Phi(t, \omega)$ generated by equation (4.8) has a random attractor $A(\omega) \subseteq \Lambda(\omega) \cap B_{r_1}(\omega)$.

To sum up, the theorem is proved.

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