Non-minimal coupling inspires the Dirac cosmological model

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Abstract In the framework of the generalized Rastall theory (GRT), we study the ability of a non-minimal coupling between geometry and matter fields in order to provide a setting, which allows for a variable $G$ during the cosmic evolution. In this regard, the compatibility of this theory with Dirac hypothesis on the variations of $G$ is investigated, and additionally, the possibility of obtaining the current accelerated universe is also addressed. In summary, our study indicates that, in GRT, having in hand the $G$ profile, one may find the corresponding non-minimal coupling between the energy source and geometry and vice versa, in a compatible way with the current accelerated universe.

1 Introduction

The idea that $G$ (the Newtonian gravitational coupling) has probably experienced diverse values during the cosmic evolution has many motivations. It began with Dirac’s proposal [1–3] which states that the ubiquitousness of certain large dimensionless numbers (LDNs), arising in combinations of physical constants and cosmological quantities [4–6], was not a coincidence but an outcome of an underlying relationship between them [7,8]. In his proposal, Dirac pointed out that the electrical force between proton and electron within a hydrogen atom, i.e., $F_e = e^2/4\pi \varepsilon_0 r^2$, is a large number being 40 orders of magnitude greater than their gravitational force $F_G = G m_p m_e / r^2$, i.e.,

$$\ln_{10} \frac{F_e}{F_G} = \frac{e^2}{4\pi \varepsilon_0 G m_p m_e} \approx 10^{40},$$

where $m_e, e, m_p, \varepsilon_0$ and $G$ are the mass and charge of electron, the proton mass, the vacuum permittivity and gravitational constant, respectively. On the other side, the ratio of the age of
the universe and the time for light to traverse an electron is also nearly of the same size, i.e.,

\[ LN_2 = \frac{t}{e^2 / 4\pi \epsilon_0 m_e c^3} \approx 10^{40}. \tag{2} \]

Dirac then suggested that the above two quantities are equal. As a result of such a relationship, some of the fundamental constants cannot remain constant for ever since LN\(_2\) varies with the age of the universe. According to Dirac’s hypothesis, atomic parameters cannot change with time and thus \( G \) should change inversely with time, i.e., \( G \propto t^{-1} \) \([9,10]\), see also \([11]\) for recent reviews. Since the advent of this idea, it has led to interesting implications within theoretical physics and has attracted a great deal of attention during the past decades \([12–31]\). Moreover, it has even interesting power to justify baryogenesis \([31]\), the current and primary accelerated universes \([32]\) and can support the de Sitter spacetime \([27,28]\).

In Newtonian gravity, one is allowed to write an explicit time variation of \( G \) without the need of satisfying any further constraint. However, the situation is different in GR as there are further constraints to be satisfied. Consider the Einstein field equation \( G_{\mu \nu} = 8\pi G T_{\mu \nu} \) with the assumption of \( G = G(t) \) and \( c \equiv 1 \). If one takes the covariant divergence of this equation, the left-hand side vanishes as a result of Bianchi identity. Then, if the ordinary energy–momentum conservation law (OCL) is assumed to hold, i.e., \( T_{\mu \nu}^{; \mu} = 0 \), one finds that \( G \) must be a constant with respect to spacetime coordinates, i.e., \( \partial G / \partial x^\mu = 0 \) always. In this respect, GR does not allow for any variation in the gravitational coupling \( G \) owing to the fact that the Einstein tensor is divergence-free and the divergence of energy–momentum tensor is also zero. Hence, in the light of Dirac’s proposal, some modifications of GR field equation are essential. This is because, if we simply let \( G \) to be a variable, then the OCL is violated \([33,34]\). In this respect, investigating the effects of a varying \( G \) can be performed only through modified field equations along with modified conservation laws. From these arguments, one may intuitively imagine that a varying \( G \) could contribute as a new degree of freedom within the OCL. As in GR, \( G \) denotes mutual relation between geometry and matter fields; hence, variations of \( G \) together with the violation of OCL may be considered as a signal for the idea that another relation between geometry and matter fields may exist that connects their changes to each other. However, there are modifications of GR with a varying \( G \) that respect the OCL such as Brans–Dicke theory, in which the dynamical scalar field \( \phi \) can be considered as the origin of gravitational coupling and thus it varies as \( G \propto \frac{1}{\phi} \) \([20,35–38]\).

OCL, as one of the cornerstones of GR \([39]\), is not respected in all modified gravity theories; for example, it is broken in the non-minimal curvature matter coupling theories \([40–49]\). Rastall gravity is a pioneering theory in this area \([47]\) in accordance with various observations \([50–62]\), and its corresponding cosmology avoids the age and entropy problems arisen in the framework of the standard cosmology \([63]\). In fact, this theory can even provide a better platform for describing the matter dominated era compared to the Einstein theory \([52]\). A generalized form of this theory allows us to relate the current and primary accelerated universe to the ability of the spacetime to couple with the energy–momentum sources, filling the background, and in fact introduces this coupling as a candidate for dark energy and inflaton field \([48]\).

In addition to inflationary models powered by employing varying \( G \) theories \([32]\), there are also other models to describe inflation without considering an inflaton field \([48,64–66]\). In Ref. \([48]\), it has been shown that while the existence of an interaction between the geometry and matter fields may model the primary and current inflationary eras, it does not necessarily lead to the breakdown of OCL. In fact, if geometry has the ability of being non-minimally coupled with the matter fields, then this ability may support the primary inflationary era and
the current accelerated phase [48]. To obtain these results, authors focus on the special case of $T_{\mu\nu} = 0$ and find out the form of non-minimal coupling in each cosmic era.

The study of various non-minimal couplings can at least make us familiar with their consequences and properties, which may finally lead to a better understanding of spacetime that helps us provide better predictions about its behavior and nature. In GRT, cosmological scenarios [48,64,67,68] imply the power of non-minimal coupling in $i$) providing both singular and non-singular universes, $ii$) describing the particle production process, $iii$) avoiding the coincidence problem, and $iv$) playing the role of dark energy (unlike the original Rastall theory [48,69]). In this regard, thermodynamically it has also been shown that the confusion in defining energy and some of its outcomes, which may lead to the OCL generalization (or equivalently, the breakdown of OCL), could make the cosmos dark [70,71].

Since in Rastall gravity, the gravitational coupling is a constant, but differs from those of GR and Newtonian gravity (NG) [47,72], Rastall theory (and indeed a mutual non-minimal coupling between the geometry and matter fields in the Rastall way) cannot provide a theoretical basis for the probable variations of $G$ during the cosmic evolution. These points will be reopened in more detail in the next section.

Motivated by the above arguments, it is reasonable to $i$) examine the ability of non-minimal coupling between geometry and matter fields in producing a non-constant $G$ and also $ii$) study the results of a non-constant $G$ in the framework of Rastall theory. The latter is tried to be answered by some authors in Ref. [73], by combining Rastall and Brans–Dicke theories with each other. In the present study, the changes in $G$ are not originated by the Rastall theory, meaning that the first part is still unsolved and debatable. We therefore focus on GRT to describe the compatibility of a non-minimal coupling with Dirac’s idea on evolution of $G$. Indeed, we are eager to show that, at least phenomenologically, a non-minimal coupling may itself change $G$ and play the role of dark energy.

The present work is then arranged as follows: In Sects. 2 and 3, a brief review on the Rastall theory and its generalization [48] has been provided, and some of their predictions about the variations of $G$ are addressed. Section 4 includes our survey on the possibility of explaining a well-known Dirac cosmological model, previously introduced by other authors, within the framework of GRT. To show the ability of non-minimal coupling in satisfying Dirac hypothesis and describing the cosmic evolution, simultaneously, a new model is also introduced in Sect. 5. Section 6 is devoted to concluding remarks. Here, we use $c = \hbar = 1$ units.

## 2 Rastall theory and a model for varying $G$

Originally, P. Rastall argued that the OCL may not be valid in a curved spacetime leading to [47]

\[ T_{\mu\nu} \neq 0, \]

in the non-flat spacetimes. From the mathematical point of view, $T_{\mu\nu}$ is a ranked one tensor field written as $T_{\mu\nu} = Q_{\mu\nu}$ where $Q$ is an unknown scalar function found out from other parts of physics, mathematics and observations [47]. Since $Q$ is a scalar and Rastall hypothesis admits the violation of OCL in a curved spacetime (where Ricci scalar is not always zero), therefore Ricci scalar, $R$, can be considered as a suitable suggestion for $Q$, and thus, [47]

\[ T_{\mu\nu} = \lambda R_{\mu\nu}, \]

where $\lambda$ is a constant.
where $\lambda'$ is called the Rastall constant parameter. Using the Bianchi identity, it is easy to get
\[ G_{\mu\nu} + \kappa' \lambda' g_{\mu\nu} R = \kappa' T_{\mu\nu}, \] (5)
which $\kappa'$ is a constant [47] called the Rastall gravitational coupling constant. Applying the Newtonian limit on this result, we obtain [47]
\[ \frac{\kappa'}{4\kappa'\lambda' - 1} \left( 3\kappa'\lambda' - \frac{1}{2} \right) = \kappa_G, \] (6)
where $\kappa_G \equiv 4\pi G$. Hence, since $\kappa'$ and $\lambda'$ are constants, $G$ should also be a constant as well (the current value of $G$, namely $G_0$, is proper option leading to $\kappa_G \equiv \kappa_G = 4\pi G_0$). We therefore conclude that since the left-hand side of (6) is a constant then a mutual non-minimal interaction between the geometry and matter fields within the framework of original version of Rastall gravity does not support the varying $G$ theories. Equation (6) also reveals that the Rastall gravitational coupling constant ($\kappa'$) differs from that of GR ($2\kappa_G = 8\pi G$), and only if $\lambda' = 0$ then they will be equal.

It is also useful to note that one may use Eq. (5) in order to introduce the generalized energy–momentum tensor $\Theta^{\mu\nu} = T^{\mu\nu} - (\kappa'\lambda')/(4\kappa'\lambda' - 1)T g^{\mu\nu}$ which finally leads to the GR counterpart form of the Rastall field equations, given as $G_{\mu\nu} = \kappa' \Theta_{\mu\nu}$. In this manner, although the obtained field equations are similar to those of GR, their solutions for $T_{\mu\nu}$ differ in general from those of GR [71,74], a result confirmed by various observational data, see, e.g., [50,59,74] and references therein).

One can also generalize the Rastall theory by considering $\lambda' \rightarrow \lambda$, where $\lambda$ is a varying parameter. Therefore, Eq. (4) is extended as follows [48]
\[ T^{\mu\nu}_\lambda = (\lambda R)^{\mu\nu}, \] (7)
which finally leads to
\[ G_{\mu\nu} + \kappa \lambda g_{\mu\nu} R = \kappa T_{\mu\nu}, \] (8)
where $\kappa$ is again a constant but $\lambda$ can change over time. Using the trace of Eq. (8), one can also rewrite this equation as
\[ G_{\mu\nu} + \tau T g_{\mu\nu} = \kappa T_{\mu\nu}, \] (9)
in which
\[ \tau = \frac{\kappa^2 \lambda}{4\kappa \lambda - 1}. \] (10)
Now since $\kappa$ is constant, the covariant derivative of Eq. (9) leads to
\[ \tau^{;\nu} T + \tau T^{;\nu} = \kappa T^{;\nu}_{\mu}, \] (11)
meaning that even if OCL is respected and until $\tau \neq constant$ (or equally, $\lambda \neq constant$), the non-minimal coupling affects the evolution of the energy–momentum source and vice versa [48]. Therefore, unlike the Rastall theory, OCL can be met in this framework even in the presence of non-minimal coupling. In this regard, it is shown that, in the framework of Eq. (8), even if OCL is met, the accelerated universe can be explained under the shadow of $\lambda$ without resorting to a dark energy source [48].

Now, considering the Newtonian limit (ignoring the pressure of $T_{\mu\nu}$ and utilizing relation $R_{00} = \nabla^2 \phi$, in which $\phi$ denotes the Newtonian potential [75]), one can easily find
\[ \frac{\kappa}{4\kappa \lambda - 1} \left( 3\kappa \lambda - \frac{1}{2} \right) = \kappa_G. \] (12)
Due to the similarity of Eqs. (8) and (5), one could expect that the Newtonian limit of field equations (8) is obtainable by replacing $\kappa'$ and $\lambda'$ with $\kappa$ and $\lambda$, respectively, in Eq. (6). Equation (12) also indicates that $G$ (or equally $\kappa_G$) does not necessarily remain constant in this theory. Therefore, this generalization of Rastall theory provides a basis for theories including a varying $G$ [1, 2, 13–17, 19–24, 27–31]. In fact, this equation tells that a non-minimal coupling between the geometry and matter fields can make $G$ variable [76], meaning that such coupling can be considered as a theoretical basis for varying $G$ theories.

3 Newtonian limit, a model for running $G$, and the value of $\kappa$

Now, using Eq. (12), and following Ref. [48], in which $\kappa \lambda \equiv \beta = [4 + \theta (1 + z)^3]^{-1}$, where $\theta$ is an unknown constant and $z$ denotes the redshift, one can obtain

$$
\kappa_G = \frac{\kappa}{2} \left[ 1 - \frac{2}{\theta (1 + z)^3} \right],
$$

finally leading to

$$
\kappa_G = \frac{\kappa}{2} \left[ 1 - \frac{2}{\theta} \right] \equiv \kappa G_0,
$$

(14)

and

$$
\kappa G = \frac{\kappa}{2},
$$

(15)

for $z \to 0$ and $z \to \infty$, respectively. Based on Ref. [48], whenever $0 < \theta \leq 1/2$ (leading to $\beta > 0$), the current accelerated universe is explainable in the presence of OCL, and without considering a dark energy-like source. Moreover, expression $\beta = [4 + \theta (1 + z)^3]^{-1}$ is present in both of the matter-dominated era (MDE) and the current-accelerated universe [48]. Hence, Eq. (15) can be considered as the value of $G$ at the beginning of MDE, whereas the value of $\kappa$ is obtainable by using Eq. (14)

$$
\kappa = \frac{8 \pi G_0}{1 - \frac{2}{\theta}},
$$

(16)

combined with Eq. (15) to see that $\kappa$, and thus $\kappa G$ are negative at the beginning of MDE. Therefore, in the model proposed in Ref. [48], which still respects OCL in the framework of (8), $G$ is not always positive during the cosmic evolution. Negative values of $\kappa$ provide a setting for baryonic matters to support traversable wormholes in the Rastall framework [72]. Moreover, in the framework of GRT, it has been shown that negative values of $\kappa$ could have their own effects on matter perturbations and formation of structures in large-scale universe [77]. In this regard, overdense and underdense regions in the universe could form periodically so that both large-scale structures and voids could form as the universe evolves from MDE to the present time. Also, emergence of structures in a class of alternative theories of gravity has been reported in [78], where the authors considered a non-minimally coupled scalar field in addition to an induced negative gravitational constant and studied structure formation with repulsive gravitation on the large scale. In the framework of general scalar tensor theories, a cosmological mechanism has been proposed in which it is possible for $G$ to change sign from a positive branch (attracting) to a negative branch (repulsive gravity) and vice versa [79]. It is also worth mentioning that negative values of $G$ have previously been reported in some other approaches studying the variations of $G$ [20, 27, 28].
Beside the effects of repulsive gravity (represented by a universal negative coupling) on the evolution of perturbations and formation of structures, the study of possible consequences of $\kappa < 0$ on the stability of the model is of particular importance. In this regard, from the viewpoint of perturbative analysis, the existence of a repulsive gravity phase in the evolution of the universe could lead to growing models with respect to scalar perturbations producing then, large inhomogeneities. Hence, a repulsive phase may destroy homogeneity, and in this sense it may be unstable [80]. In [81], it has been discussed that a transition from positive gravitational coupling $G$ to negative one results in an instability, in such a way that small deviations from isotropy and homogeneity within the gravitational field will grow unboundedly, leading to a true cosmological singularity at the boundary between gravity and anti-gravity. Also, investigating classical stability of the model through dynamical system approach is of long-standing interest and significance. Work along this line has been carried out for a class of GRT models [82], where the authors have shown that the eventual fate of the universe ends in late time attractors, which are classically stable. However, investigating these issues for the present model needs a deeper analysis with more scrutiny, and future studies will be reported elsewhere. Finally, we note that, since $\dot{G}$ does not decrease with time for $0 < \theta \leq 1/2$ ($\dot{G} > 0$ in this manner), this model does not respect the Dirac’s hypothesis, claiming that $G$ should decrease as a function of time [1,2,13,15]. Hence, more comprehensive non-minimal couplings are needed to provide settings for Dirac hypothesis and also to model the cosmic evolution without considering a mysterious fluid (dark energy), simultaneously.

3.1 Another possibility

In Ref. [67], choosing $\lambda = (1 + d_0 H)/[3\kappa(w + 1)]$, in which $w \equiv p/\rho$ (where $p$ and $\rho$ denote the pressure and energy density of the cosmic fluid, respectively), it has been shown that non-singular cosmic evolution is obtainable in GRT. In this case, $d_0$ is a free parameter, and some outcomes of this proposal in various cosmic eras have also been studied in Ref. [67]. Accepting this proposal along with considering the unit $\kappa = 8\pi G_0$ and also assuming $G(H_0) = G_0$ (which helps us in finding $d_0$), one easily reaches

$$G(H) = G_0 \frac{3(1 - w)H_0 - 6H}{(1 - 3w)H_0 - 4H},$$

where $H_0$ is the current value of $H$ and use has been made of Eq. (12).

4 Dirac cosmological model

As in the present model there is no evolution equation for the variation of $G$, which is promoted as a dynamical field, one then has to impose a suitable ansatz on the behavior of this parameter. Based on Dirac hypothesis, $G$ should decrease with time, i.e, $G \propto t^{-1}$ [9,10]. In general, one may consider $\dot{G} = G_0 f$, in which $f$ is a decreasing function of time [1,2,13,15,24]), in order to preserve Dirac hypothesis. Now, combining Eq. (12) with $\kappa = 8\pi G_0 \alpha$ [52], along with Eqs. (7) and (8) for a flat FLRW universe, one finds

$$\gamma \equiv \lambda \kappa = \frac{f - \alpha}{4f - 6\alpha},$$

$$3 \int (\rho + p) \frac{da}{a} = \frac{1}{2\alpha} \left[ (f - 3\alpha)\rho - 3(f - \alpha)p \right].$$

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Now, let us consider a flat FRW universe filled by a pressureless fluid with energy density \( \rho \) when \( H \) other values of \( \alpha \) are unknown constants to be evaluated later, and \( k \) by only considering the baryonic content of the universe, meaning that the source for current density parameter \( \zeta \) weak compared with those of \( \zeta \) and \( \kappa \). Defining \( \Omega_1 = \frac{\zeta}{\Omega} \), the corresponding Friedmann equations read

\[
q = -1 - \frac{\dot{H}}{H^2} = -1 + \frac{3\alpha(\rho + p)}{\rho(3\alpha - f) + 3(f - \alpha)p},
\]

whenever a fluid with energy density \( \rho \) and pressure \( p \) fills the background. We note that \( \gamma \) is a varying parameter and, \( q \) and \( \alpha \) denote deceleration parameter and scale factor, respectively, and we also have assumed \( 8\pi G_0 = 1 \).

The case with \( f = a^{-n} \) leads to a decreasing function of time whenever \( n > 0 \) \( [23, 29] \). In this manner, assuming \( w \equiv p/\rho = 0 \), together with using Eq. (18), one easily finds \( q = (3\alpha - 1)^{-1} \), and \( \rho = \rho_0 a^n(1 - 3\alpha a)^{(n+2)/n} \), where \( \rho_0 \) is the integration constant. These results indicate that, at limit \( a \rightarrow 1 \), the obtained pressureless fluid can accelerate the universe expansion with \( q \leq -1/2 \) for \(-1/3 \leq \alpha < 1/3 \). Consequently, the non-minimal coupling \( \gamma = [(1 + z)^n - \alpha]/(4(1 + z)^n - 6\alpha) \) allows \( G \) to vary as \( G = G_0(1 + z)^n \) \( [23] \), where we used the \( 1 + z = 1/\alpha \) relation. It is also easy to see that the universe described by this model has begun from a primary inflationary phase \( (q = -1) \) corresponding to the \( a \rightarrow 0 \) point. In fact, in this limit, we also have \( \gamma = 1/4 \), a value that supports an inflationary phase for even an empty universe \( [48] \).

Now, let us consider two more comprehensive cases, i.e., \( i \) \( p = k \rho^{1+1/m} \), where \( m \) and \( k \) are unknown constants to be evaluated later, and \( ii \) \( p = \sigma \rho/(a - bp) - c \rho^2 \) in which \( \sigma \), \( a \), \( b \) and \( c \) are unknown coefficients. In this manner, as it is obvious from Fig. 1, a proper behavior is obtainable for the cosmos. Here, \( w \equiv p/\rho \) denotes the equation of state of cosmic fluids. Depending on the values of unknown parameters, the universe can also experience a transition at \( z_t \), which can even take values smaller than 1. Clearly, both fluids behave as dark energy sources, and the corresponding non-minimal coupling cannot be considered as a dark energy source.

5 A new proposal for \( \lambda \) parameter

Now, let us consider a flat FRW universe filled by a pressureless fluid with energy density \( \rho \) when \( \lambda R = \zeta H^n \) in which \( \zeta \) and \( n \) are unknown constants. In this manner, the \( \lambda \) parameter takes the form

\[
\lambda = \frac{\zeta}{R} = \frac{\zeta}{6} \frac{H^n}{\dot{H} + 2H^2},
\]

whence, the corresponding Friedmann equations read

\[
H^2 - \frac{\kappa \zeta}{3} H^n = \frac{\kappa}{3},
\]

\[
H^2 + \frac{2}{3} \dot{H} - \frac{\kappa \zeta}{3} H^n = 0.
\]

Defining \( \Omega = 8\pi G \rho/3H^2 \), while \( \Omega_0 \) denotes its current value \( [84] \), the evolution of \( q \) and \( G/G_0 \) is plotted in Fig. 2. For the employed parameters, transition redshift \((z_t)\) lies within the range of \( 0.4 \leq z_t \leq 0.88 \). The sensitivity of diagrams to the values of \( \Omega_0 \) and \( H_0 \) is so weak compared with those of \( \zeta \) and \( n \) and \( \kappa \). Indeed, although we only consider a baryonic source for current density parameter \( \Omega_0 = 0.049 \) \( [84] \), and \( H_0 = 67.66 \) \( [86] \), the obtained behaviors are also achievable for other candidates of \( \Omega \) (such as dark matter) and also the other values of \( H_0 \), reported in the literature. Hence, suitable behavior of \( q \) is obtainable by only considering the baryonic content of the universe, meaning that the \( \zeta H^n \) term may
Fig. 1 The evolution of $q$ and state parameter $w$ versus $z$ for $H(z = 0) = 67$ [83]. Upper panels are provided for the case (i) and the lower ones are depicted for case (ii) discussed in Sect. 4. The model parameters used to draw the curves of $w$ are the same as those of $q$ diagrams.
Fig. 2 The evolution of $q$ and $G/G_0$ assuming $w = 0$, for the case discussed in Sect. 5. The diagrams for $G/G_0$ are plotted using the same model parameters as of $q$ diagrams.

play the role of the unknown parts (dark components) of cosmos. Dirac hypothesis is also respected during the cosmic evolution. Remarkably, $G$ will take negative values in future, meaning that gravity will become repulsive which speeds the universe expansion rate up more i.e., $q$ decreases. All these happen under the shadow of the existence of non-minimal coupling $\lambda$ which varies during the evolution of the universe. In Fig. 3, $H(z)$ [85] and the distance modulus [87] are plotted for the $\Lambda$CDM model and also our model.

The negative value of $G$ is the direct result of the assumed $\lambda$, and changes in the values of model parameters do not affect this result. There are also other works that predict negative values for $G$ [20,27,28]. Theoretically, our model shows that a non-minimal coupling between geometry and matter fields can accelerate the universe expansion and has an ability to satisfy Dirac hypothesis.

6 Concluding remarks

After addressing some properties of previously cosmological models introduced in the framework of GRT [48,67], the implications of GRT on obtaining varying $G$ has been studied through considering the Newtonian limit of the field equations. Thereinafter, following a proposal of Dirac hypothesis introduced in [23,29], the required non-minimal coupling needed to support Dirac model was also obtained. Our results show that the dark sectors of cosmos can be unified into one cosmic fluid, which behaves as a pressureless fluid in high redshift limit and also accelerates the universe in line with the current observations (Fig. 1). We also proposed a non-minimal coupling (Sect. 5), which can play the role of dark side of cosmos satisfying Dirac hypothesis. Indeed, the present study addresses a deep connection between
non-minimal coupling (between the matter fields and geometry) and the idea of variable $G$. This translates into saying that one may find the footprints of non-minimal coupling between the matter fields and geometry by having the observationally confirmed profile of $G$ and conversely.

Although relying on the Rastall hypothesis on relation between the changes in spacetime curvature and violation of OCL, we only focused on the implications of the violation of OCL in cosmology and its connection with Dirac hypothesis, the OCL violation can also be allowed due to the quantum considerations such as uncertainty principle, and in the framework of unimodular gravity producing significant cosmological outcomes [64]. Indeed, even in the framework of GR and thanks to the Bianchi identity, OCL is violated as the result of the existence of a non-constant $G$. In summary, it was our goal to address $i$) probable connection between Dirac hypothesis and non-minimal couplings, and simultaneously, $ii$) the ability of such couplings in being responsible for the unknown parts (dark sides) of cosmos. Therefore, such couplings need to be further studied from both of the theoretical and observational viewpoints. Finally, we would like to mention that, though Rastall gravity and its generalizations provide interesting results, cosmological models based on this theory need to be accurately tested by observations. In the present model, we tried to explore theoretical consequences of a varying $G$ cosmology based on GRT and also briefly examined observational aspects of the theory. However, a full observational treatment of the present model, e.g., in light of [88], needs to be done, and work along this line can be considered as an interesting subject for future studies and developments.

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