MAXIMAL $L^p$-REGULARITY FOR PERTURBED EVOLUTION EQUATIONS IN BANACH SPACES

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ABSTRACT. The main purpose of this paper is to investigate the concept of maximal $L^p$-regularity for perturbed evolution equations in Banach spaces. We mainly consider three classes of perturbations: Miyadera-Voigt perturbations, Desch-Schappacher perturbations, and more general Staffans-Weiss perturbations. We introduce conditions for which the maximal $L^p$-regularity can be preserved under these kind of perturbations. We give examples for a boundary perturbed heat equation in $L^r$-spaces and a perturbed boundary integro-differential equation. We mention that our results mainly extend those in the works: [P. C. Kunstmann and L. Weis, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 30 (2001), 415-435] and [B.H. Haak, M. Haase, P.C. Kunstmann, Adv. Differential Equations 11 (2006), no. 2, 201-240].

1. INTRODUCTION

In this paper we investigate the maximal $L^p$–regularity of evolution equations of the type

$$
\begin{cases}
\dot{z}(t) = A_m z(t) + P z(t) + f(t), & t \geq 0, \\
z(0) = 0, \\
G z(t) = K z(t), & t \geq 0,
\end{cases}
$$

(1.1)

where $A_m : Z \subset X \rightarrow X$ is a linear closed operator in a Banach space $X$ with domain $D(A_m) = Z$, $P : Z \rightarrow X$ is an additive linear perturbation of $A_m$, $G, K : Z \rightarrow U$ are linear boundary operators ($U$ is another Banach space) and $f \in L^p(\mathbb{R}^+, X)$ with $p \geq 1$ is a real number. Actually, we assume that $A := A_m$ with domain $D(A) = \ker(G)$ is a generator of a strongly continuous semigroup $T := (T(t))_{t \geq 0}$ on $X$.

The concept of maximal regularity has been the subject of several works for many years, e.g. [8] [12] [9] [10], and the monograph [11]. The main purpose of these works is to give sufficient conditions on the operator $A$ so as the problem (1.1) with $P \equiv 0$ and $K \equiv 0$, which can be written as

$$
\dot{z} = Az + f, \ z(0) = 0,
$$

(1.2)

has a maximal $L^p$–regularity. A necessary condition for the maximal regularity of the evolution equation (1.2) is that $A$ is a generator of an analytic semigroup. This condition

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is also sufficient only if the work space \( X \) is a Hilbert space, see [10]. In the case of UMD Banach space \( X \), Weis [34] introduced necessary and sufficient conditions in terms of \( \mathcal{R} \)-boundedness of the operator \( A \) for the maximal \( L^p \)-regularity of (1.2).

Let us first consider the problem (1.1) with \( K \equiv 0 \), which can be regarded as the following evolution equation

\[
\dot{z} = (A + P)z + f, \quad z(0) = 0.
\] (1.3)

The maximal \( L^p \)-regularity for (1.3) has been studied by many authors, e.g. [3, 12, 18, 24, 27]. In [24], the authors proved that the problem (1.3) has the maximal \( L^p \)-regularity if \( X \) is a UMD space, the problem (1.2) has a maximal \( L^p \)-regularity and the perturbation \( P \) is small, i.e. for every \( \delta > 0 \), there exists \( c_\delta > 0 \) such that \( \|Px\| \leq \delta \|Ax\| + c_\delta \|x\| \), for all \( x \in D(A) \). The proof is mainly based on the fact that \( \mathcal{R} \)-sectoriality is preserved under \( A \)-small perturbations and the UMD property of the space \( X \). In Section 4 we generalize this result to the case of general Banach spaces. In fact, we will assume that the restriction of \( P \) on \( D(A) \) is a \( p \)-admissible observation operator for \( A \) (see Section 2 for the definition), hence \( (A + P, D(A)) \) is a generator of a strongly continuous semigroup on the Banach space \( X \), see [15]. In Theorem 4.8 we will show that if the problem (1.2) has a maximal \( L^p \)-regularity then it is also for its perturbed problem (1.3). To compare this result with the result proved in [24], we first prove in Lemma 4.10 that the fact that \( A \) generates an analytic semigroup on a Banach space \( X \) and \( P \) is \( p \)-admissible observation operator for \( A \), implies that for \( \beta > \frac{1}{p} \) the operator \( P(-A)^{-\beta} \) has a bounded extension to \( X \). As \( (-A)^\beta \) is a \( A \)-small perturbation and \( Px = P(-A)^{-\beta}(-A)^\beta x \) then \( P \) is a \( A \)-small perturbation, and then the result in [24] holds also in Banach spaces.

Return now to our initial boundary problem (1.1). This later can be reformulated as

\[
\dot{z} = (A + P)z + f, \quad z(0) = 0,
\] (1.4)

where \( A : D(A) \subset X \to X \) is the linear operator defined by

\[
A := A_m, \quad D(A) = \{ x \in Z : Gx = Kx \}.
\]

In addition to our assumption at the beginning of this section, we also suppose that \( G : Z \to U \) is surjective. Let then \( D_\lambda \in \mathcal{L}(U, Z) \) (\( \lambda \in \rho(A) \)) be the Dirichlet operator associated with \( A_m \) and \( G \), see Section 3. In order to state our main results on well-posedness and maximal \( L^p \)-regularity of the problem (1.4), we select \( B := (\lambda I - A_{-1})D_\lambda \in \mathcal{L}(U, X_{-1}) \) for \( \lambda \in \rho(A) \), where \( X_{-1} \) is the extrapolation space associated to \( A \) and \( X \), and \( A_{-1} : X \to X_{-1} \) is the extension of \( A \) to \( X \), which is a generator of a strongly continuous semigroup on \( X_{-1} \). We assume that \( B \) is a \( p \)-admissible control operator for \( A \), see the next section for the definition and notation. If \( K \) is bounded, i.e. \( K \in \mathcal{L}(X, U) \) then it is known that the operator \( A \) coincides with the part of the operator \( A_{-1} + BK \) in \( X \), which generates a strongly continuous semigroup on \( X \) (see e.g. [14, 17, 30]). In this case, we prove (see Theorem 4.13) that if the problem (1.2) has the maximal \( L^p \)-regularity, then the evolution equation

\[
\dot{z} = Az + f, \quad z(0) = 0
\] (1.5)
has also the same property. In addition if we assume that \((A, B, P|_{D(A)})\) generates a regular linear system on \(X, U, X\), then \(P\) is still \(p\)-admissible observation operator for \(A\) and then the problem \((1.4)\) is well-posed and has the maximal \(L^p\)-regularity, see Theorem 3.4 and Theorem 4.15. Let us now assume that the boundary operator \(K\) is unbounded \(K : Z \subsetneq X \to U\). This situation is quite difficult which needs additional assumption to treat the well-posedness and maximal \(L^p\)-regularity. According to [17], if we assume that \((A, B, K|_{D(A)})\) is regular on \(X, U, U\) with \(I_U : U \to U\) as an admissible feedback, then the problem \((1.5)\) is well-posed on the Banach space \(X\). Moreover, if the problem \((1.2)\) has a maximal \(L^p\)-regularity and \(\|\lambda D_\lambda\| \leq \kappa\) for any \(\Re \lambda > \lambda_0\), where \(\lambda_0 \in \mathbb{R}\) and \(\kappa > 0\) are constants, then the problem \((1.5)\) has also the maximal \(L^p\)-regularity on a non reflexive Banach space \(X\), see Theorem 4.17. On the other hand, we assume that \((A, B, P|_{D(A)})\) generates a regular linear system on \(X, U, X\). Then, in Theorem 3.4, we prove that the problem \((1.4)\) is well-posed. Corollary 4.22 shows that the problem \((1.4)\) has the maximal \(L^p\)-regularity. If \(X\) is a UMD space then we use \(\mathcal{R}\)-boundedness to prove the maximal \(L^p\)-regularity for the evolution equation \((1.4)\), see Theorem 4.20 and Corollary 4.22. We mention that in [18], the authors proved perturbation theorems for sectoriality and \(\mathcal{R}\)-sectoriality in general Banach spaces. They gives conditions on intermediate spaces \(Z\) and \(W\) such that, for an operator \(S : Z \to W\) of small norm, the operator \(A + S\) is sectorial (resp. \(\mathcal{R}\)-sectorial) provided \(A\) is sectorial (resp. \(\mathcal{R}\)-sectorial). Their results are obtained by factorizing \(S = BC\). As \(\mathcal{R}\)-sectoriality implies maximal regularity in UMD spaces, these theorems yield to maximal regularity perturbation only in UMD spaces.

In Section 5, we have used product spaces and Bergman spaces to reformulate boundary perturbed intego-differential equations as our abstract boundary evolution equation \((1.1)\). This allows us to translate the results on well-posedness and maximal \(L^p\)-regularity obtained for the problem \((1.1)\) to intego-differential equations.

In the next section, we first recall the necessary material about feedback theory of infinite dimensional linear systems. We then use this theory to prove the well-posedness of the evolution equation \((1.1)\) in Section 3. Our main results on maximal \(L^p\)-regularity for the problem \((1.1)\) are gathered in Section 4. The last section is devoted to apply the obtained results to perturbed intego-differential equations.

**Notation.** Hereafter \(p, q \in [1, \infty]\) and \(T > 0\) are real numbers such that \(\frac{1}{p} + \frac{1}{q} = 1\). If \(X\) is a Banach space, we denote by \(L^p([0, T]; X)\) the space of all \(X\)-valued Bochner integrable functions. For any \(\theta \in (0, \pi)\), \(\Sigma_\theta\) is the following sector:

\[
\Sigma_\theta := \{z \in \mathbb{C}^*; |\arg z| < \theta\}.
\]

For any \(\alpha \in \mathbb{R}\), the right half-plane is defined by

\[
\mathbb{C}_\alpha := \{z \in \mathbb{C}; \Re z > \alpha\}.
\]

Given a semigroup \(T := (T(t))_{t \geq 0}\) generated by an operator \(A : D(A) \subset X \to X\), we will always denote by \(\omega_0(T)\) (or \(\omega_0(A)\)) the growth bound of this semigroup. The resolvent set of \(A\) is denoted by \(\rho(A)\). Preferably, we denote the resolvent operator of \(A\) by \(R(\lambda, A) := (\lambda - A)^{-1}\) for any \(\lambda \in \rho(A)\), where the notation \(\lambda - A\) means \(\lambda I - A\).
2. Feedback theory of infinite dimensional linear systems

In this section, we gather definitions and results from feedback theory of infinite dimensional linear systems mainly developed in the references [30, 31, 32, 37]. We also give some new development of this theory. Hereafter, $X$ and $U$ are Banach spaces and $p \in [1, \infty[$.

It is known (see e.g. [30, 31]) that partial differential equations with boundary control and point observation can be reformulated as the following distributed linear system

\[
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t), \quad t \geq 0 \\
y(t) &= K x(t), \\
x(0) &= x_0,
\end{aligned}
\]

(2.1)

where $A : D(A) \subset Z \subset X \to X$ is the generator of a strongly continuous semigroup $T := (T(t))_{t \geq 0}$ on $X$ with $Z$ is a Banach space continuously and densely embedded in $X$, $B \in \mathcal{L}(U, X^{-1})$ is a control operator such that

$R(\lambda, A_{-1})B \in \mathcal{L}(U, Z)$, $\lambda \in \rho(A)$,

and $K \in \mathcal{L}(Z, U)$ is an observation operator. Here $X_{-1}$ is the completion of $X$ with respect to the norm $\|R(\lambda, A)\|$.

We recall that we can extend $T$ to another strongly continuous semigroup $T_{-1} := (T_{-1}(t))_{t \geq 0}$ on $X_{-1}$ with generator $A_{-1} : X \to X_{-1}$, the extension of $A$ to $X$ (see e.g. [13, chap.2]). The mild solution of the system (2.1) is given by:

\[
x(t) = T(t)x_0 + \int_0^t T_{-1}(t-s)Bu(s)ds \quad x_0 \in X,
\]

(2.2)

where the integral is taken in $X_{-1}$. Formally, the well-posedness of the system (2.1) means that the state satisfies $x(t) \in X$ for any $t \geq 0$, the observation function $y$ is extended to a locally $p$-integrable function $y \in L^p_{loc}([0, \infty), U)$ satisfying the following property: for any $\tau > 0$, there exists a constant $c_\tau > 0$ such that

$\|y\|_{L^p([0,\tau], U)} \leq c_\tau \left(\|x_0\| + \|u\|_{L^p([0,\tau], U)}\right),$

for any initial state $x_0 \in X$ and any control function $u \in L^p_{loc}([0, \infty), U)$. In order to mathematically explain this concept, let us define

$C := K|_{D(A)} \in \mathcal{L}(D(A), U)$.

We also need the following definition.

**Definition 2.1.** (i) $B \in \mathcal{L}(U, X_{-1})$ is called $p$-admissible control operator for $A$, if there exists $t_0 > 0$ such that:

$\Phi_{t_0} u := \int_0^{t_0} T_{-1}(t-s)Bu(s)ds \in X$

for any $u \in L^p_{loc}([0, \infty), U)$. We also say that $(A, B)$ is $p$-admissible.
(ii) $C \in \mathcal{L}(D(A), Y)$ is called $p$-admissible observation operator for $A$, if there exist $\alpha > 0$ and $\kappa := \kappa_\alpha > 0$ such that:

$$\int_0^\alpha \| C \mathbb{T}(t)x \|^p_Y dt \leq \kappa^p \| x \|^p,$$

(2.3)

for all $x \in D(A)$. We also say that $(C, A)$ is $p$-admissible.

Let us now describe some consequences of this definition. If $B$ is $p$-admissible control operator for $A$, then by the closed graph theorem one can see that for any $t \geq 0$,

$$\Phi_t \in \mathcal{L}(L^p([0, t], U), X).$$

This implies that the state of the system (2.1) satisfies $x(t) = \mathbb{T}(t)x_0 + \Phi_tu \in X$ for any $t \geq 0$, $x_0 \in X$ and $u \in L^p_{\text{loc}}([0, \infty), U)$. According to [36], for all $0 < \tau_1 \leq \tau_2$,

$$\| \Phi_{\tau_1} \| \leq \| \Phi_{\tau_2} \|. \quad (2.4)$$

Now if $C$ is $p$-admissible observation operator for $A$, then due to (2.3), the map $\Psi_\infty : D(A) \to L^p_{\text{loc}}([0, \infty), U)$ defined by $\Psi_\infty x := CT(\cdot)x$, can be extended to a bounded operator $\Psi_\infty : X \to L^p_{\text{loc}}([0, \infty); U)$. For any $x \in X$ and $t \geq 0$, we define the family $\Psi_t x := \Psi_\infty x$ on $[0, t]$. Then for all $t \geq 0$,

$$\Psi_t \in \mathcal{L}(X, L^p_{\text{loc}}([0, \infty), U)).$$

On the other hand, let us consider the linear operator

$$D(C_A) := \left\{ x \in X : \lim_{s \to +\infty} s\text{CR}(s, A)x \text{ exists in } U \right\},$$

$$C_A x := \lim_{s \to +\infty} s\text{CR}(s, A)x.$$  

Clearly, $D(A) \subset D(C_A)$ and $C_A = C$ on $D(A)$. This shows that $C_A$ is in fact an extension of $C$, called the Yosida extension of $C$ w.r.t. $A$. We note that if $C$ is $p$-admissible for $A$, then $\mathbb{T}(t)X \subset D(C_A)$ and

$$(\Psi_\infty x)(t) = C_A \mathbb{T}(t)x,$$

for any $x \in X$ and a.e. $t > 0$.

In the sequel, we assume that $B$ and $C$ are $p$-admissible for $A$ and set

$$W^{2,p}_{0,\text{loc}}([0, \infty), U) := \left\{ u \in W^{2,p}_{\text{loc}}([0, \infty), U) : u(0) = 0 \right\}.$$

This space is dense in $L^p_{\text{loc}}([0, \infty), U)$. Remark that for any $u \in W^{2,p}_{0,\text{loc}}([0, \infty), U)$, $t \geq 0$ and by assuming $0 \in \rho(A)$ (without loss of generality) and using an integration by parts, we have

$$\Phi_t u = R(0, A_{-1})Bu(t) - R(0, A)\Phi_t \dot{u} \in Z.$$

On the other hand, using the fact that $KR(0, A_{-1})B \in \mathcal{L}(U)$, $CR(0, A) \in \mathcal{L}(X, U)$ and (2.3), the application $(t \mapsto K\Phi_t u) \in L^p_{\text{loc}}([0, \infty), U)$ for any $u \in W^{2,p}_{0,\text{loc}}([0, \infty), U)$. Thus we have defined an application

$$F_\infty : W^{2,p}_{0,\text{loc}}([0, \infty), U) \to L^p_{\text{loc}}([0, \infty), U), \quad u \mapsto F_\infty u = K\Phi u. \quad (2.5)$$
**Definition 2.2.** Let $B$ and $C$ be $p$-admissible control and observation operators for $A$, respectively. We say that the triple $(A, B, C)$ generates a well-posed system $\Sigma$ on $X, U, U$, if the operator $F_\infty$ defined by (2.5) satisfies the following property: For any $\alpha > 0$ there exists a constant $\vartheta_\alpha > 0$ such that for all $u \in W^{2,p}_{0,loc}([0, \infty), U)$,

$$
\|F_\infty u\|_{L^p([0, \alpha], U)} \leq \vartheta_\alpha \|u\|_{L^p([0, \alpha], U)}. \tag{2.6}
$$

The operator $F_\infty$ is called the extended input-output operator of $\Sigma$.

If $(A, B, C)$ generates a well-posed system $\Sigma$ on $X, U, U$, then we have two folds: first the state of (2.1) satisfies $x(t) \in X$ for all $t \geq 0$, and second $F_\infty$ have an extension $F_\infty \in \mathcal{L}(L^p_{loc}([0, \infty), U))$, due to (2.6). Observe that the observation function $y$ verifies

$$
y(\cdot) := y(\cdot; x_0, u) = CT(\cdot)x_0 + F_\infty u = (\Psi_\infty F_\infty)(x_0, u), \tag{2.7}
$$

for all $x_0 \in D(A)$ and $u \in W^{2,p}_{0,loc}([0, \infty), U)$. By density of $D(A) \times W^{2,p}_{0,loc}([0, \infty), U)$ in $X \times L^p_{loc}([0, \infty), U)$, the function $y$ is extended to a function $y \in L^p_{loc}([0, \infty), U)$ such that

$$
y = \Psi_\infty x_0 + F_\infty u, \quad \forall (x_0, u) \in X \times L^p_{loc}([0, \infty), U).$$

We now turn out to give a representation of the observation function $y$ in terms of the observation operator $C$ and the state $x(\cdot)$. To that purpose Weiss [37] introduced the following subclass of well-posed linear systems.

**Definition 2.3.** Let $(A, B, C)$ generates a well-posed system $\Sigma$ on $X, U, U$ with extended input-output operator $F_\infty$. This system is called regular (with feedthrough $D = 0$) if :

$$
\lim_{\tau \to 0} \frac{1}{\tau} \int_0^\tau (F_\infty u_{z_0})(s) ds = 0
$$

with $u_{z_0}(s) = z_0$ for all $s \geq 0$, is a constant control function.

According to Weiss [37] [38], if $(A, B, C)$ generates a regular system $\Sigma$ on $X, U, U$, then the state and the observation function of the linear system (2.1) satisfy

$$
x(t) \in D(C_A) \quad \text{and} \quad y(t) = C_A x(t), \tag{2.8}
$$

for any initial state $x(0) = x_0 \in X$, any control function $u \in L^p([0, \infty), U)$ and a.e. $t \geq 0$.

**Definition 2.4.** Let a triple $(A, B, C)$ generates a well-posed system $\Sigma$ on $X, U, U$ with extended input-output operator $F_\infty$. Define

$$
F_\tau u := F_\infty u \quad \text{on} \quad [0, \tau].
$$

The identity operator $I_{U} : U \to U$ is called an admissible feedback for $\Sigma$ if the operator $I - F_\tau : L^p([0, t_0], U) \to L^p([0, t_0], U)$ admits a (uniformly) bounded inverse for some $t_0 > 0$ (hence all $t_0 > 0$).

A consequence of Definition 2.4 is that the feedback law $u = y(\cdot; x_0, u)$ has a sense. In fact, due to (2.7) this is equivalent to $(I - F_\tau)u = \Psi_\tau x_0$ on $[0, \tau]$. As $I - F_\tau$ is invertible
in $L^p([0,\tau], U)$, then the equation $u = y(\cdot, x_0, u)$ has a unique solution and this solution $u \in L^p([0,\tau], U)$ is given also by
\[ u(t) = C_\Lambda x(t), \quad \text{a.e. } t \geq 0, \]
due to (2.8). Using (2.2), the state $x(\cdot)$ satisfies the following variation of constants formula
\[ x(t) = T(t)x_0 + \int_0^t T_{-1}(t - s)BC_\Lambda x(s)ds \]
for any $x_0 \in X$ and any $t \geq 0$. Now we set
\[ T^{cl}(t)x_0 := x(t), \quad t \geq 0. \]
Then by using the definition of $C_0$–semigroups one can see that $(T^{cl}(t))_{t \geq 0}$ is a $C_0$–semigroup on $X$. More precisely, we have the following perturbation theorem due to Weiss [37] in Hilbert spaces and to Staffans [31, Chap.7] in Banach spaces.

**Theorem 2.5.** Let $(A, B, C)$ generates a regular linear system $\Sigma$ with the identity operator $I_U : U \rightarrow U$ an admissible feedback operator. Then the operator
\[ A^{cl} := A_{-1} + BC_\Lambda \]
\[ D(A^{cl}) := \{x \in D(C_\Lambda); (A_{-1} + BC_\Lambda)x \in X\} \]
generates a $C_0$-semigroup $(T^{cl}(t))_{t \geq 0}$ on $X$ such that range$(T^{cl}(t)) \subset D(C_\Lambda)$ for a.e. $t > 0$, and for any $\alpha > 0$, there exists $c_\alpha > 0$ such that for all $x_0 \in X$,
\[ \|C_\Lambda T^{cl}(\cdot)x_0\|_{L^p(0,\alpha, U)} \leq c_\alpha \|x_0\|. \] (2.10)
Moreover, this semigroup satisfies
\[ T^{cl}(t)x_0 = T(t)x_0 + \int_0^t T_{-1}(t - s)BC_\Lambda T^{cl}(s)x_0ds \quad x_0 \in X, \quad t \geq 0. \] (2.11)
In addition $(A^{cl}, B, C_\Lambda)$ generates a regular system $\Sigma^{cl}$.

**Definition 2.6.** Let $(A, B, C)$ generates a regular linear system on $X, U, U$ with the identity operator $I_U : U \rightarrow U$ as an admissible feedback. The operator
\[ \mathbb{P}^{sw} := BC : D(C_\Lambda) \subset X \rightarrow X_{-1} \]
is called the **Staffans-Weiss perturbation** of $A$.

It is not difficult to see that if one of the operators $B$ or $C$ is bounded (i.e. $B \in \mathcal{L}(U, X)$ or $C \in \mathcal{L}(X, U)$) and the other is $p$-admissible then the triple $(A, B, C)$ generates a regular linear system on $X, U, U$ with the identity operator $I_U : U \rightarrow U$ as an admissible feedback.

As application of the Staffans-Weiss theorem (Theorem 2.5), we distinct two subclasses of perturbations as follows:
Remark 2.7. (i) We take $B \in \mathcal{L}(X,U)$ and $C \in \mathcal{L}(D(A),U)$ a $p$-admissible observation operator for $A$. According to Theorem 2.5, the operator $A^d := A + BC$ with domain $D(A^d) = D(A)$ is a generator of a strongly continuous semigroup $T^d := (T^d(t))_{t \geq 0}$ on $X$ such that $T^d(t)X \subset D(C_A)$ for a.e. $t > 0$, the estimate (2.10) holds, and

$$T^d(t)x = T(t)x + \int_0^t T(t-s)BC_A T^d(s)ds, \quad t \geq 0, \ x \in X. \quad (2.12)$$

On the other hand, it is shown in [15], that the semigroup $T^d$ satisfy also the following formula

$$T^d(t)x = T(t)x + \int_0^t T^d(t-s)BC_A T(s)ds, \quad t \geq 0, \ x \in X. \quad (2.13)$$

Using Hölder inequality on can see that there exists $\alpha_0 > 0$ and $\gamma \in (0,1)$ such that

$$\int_0^{\alpha_0} \|BC T(t)x\| \leq \gamma \|x\|$$

for all $x \in D(A)$. The following operator

$$P_{mv} := BC : D(A) \to X$$

is a Miyadera-Voigt perturbation for $A$; (see e.g. [13, p.195]).

(ii) We take $C \in \mathcal{L}(X,U)$ and $B \in \mathcal{L}(U,X_{-1})$ a $p$-admissible control operator for $A$. Then the part of the operator $A_{-1} + BC$ in $X$ generates a strongly continuous semigroup on $X$ satisfying all properties of Theorem 2.5. In this case the operator

$$P_{ds} := BC : X \to X_{-1}$$

is called Desch-Schappacher perturbation for $A$ (see e.g. [13, p.182]).

3. WELL-POSEDNESS OF PERTURBED BOUNDARY VALUE PROBLEMS

The object of this section is to investigate the well-posedness of the perturbed boundary value problem defined by (1.1). We first rewrite (1.1) as non-homogeneous perturbed Cauchy problem of the form (1.4). Then the well-posedness of (1.1) can be obtained if for example the operator

$$\mathcal{A} := A_m, \quad D(\mathcal{A}) = \{x \in Z : Gx = Kx\} \quad (3.1)$$

generates a strongly continuous semigroup on $X$ and that $P$ is a $p$-admissible observation operator for $A$ (see Remark 2.7 (i)). Recently, the authors of [17] introduced conditions on $A_m, G$ and $K$ for which $\mathcal{A}$ is a generator. To be more precise, assume that

**(H1)** $G : Z \to U$ is onto, and

**(H2)** the operator defined by $A := A_m|_{\ker(G)}$ and $D(A) := \ker(G)$, generates a $C_0$-semigroup $(\mathbb{T}(t))_{t \geq 0}$. 


According to Greiner [14], these conditions imply that for any \( \lambda \in \rho(A) \) the restriction of \( G \) to \( \ker(\lambda - A_m) \) is invertible. We then define

\[
\mathbb{D}_\lambda := (G_{|\ker(\lambda - A_m)})^{-1} \in \mathcal{L}(U, X), \quad \lambda \in \rho(A).
\]

This operator is called the \textit{Dirichlet operator}. Define the operators :

\[
B := (\lambda - A_{-1})\mathbb{D}_\lambda \in \mathcal{L}(U, X_{-1}), \\
C := Ki \in \mathcal{L}(D(A), U),
\]

where \( i \) is the canonical injection from \( D(A) \) to \( Z \). In the rest of this paper, \( C_\Lambda \) denotes the Yosida extension of \( C \) with respect to \( A \). It is shown in [17, lem.3.6] that if \( A, B, C \) as above and if \((A, B, C)\) generates a regular linear system \( \Sigma \) on \( X, U, U \), then we have

\[
Z \subset D(C_\Lambda) \quad \text{and} \quad (C_\Lambda)|_Z = K.
\]  

(3.3)

If \( H \) is the transfer function of \( \Sigma \) and \( \alpha > \omega_0(A) \) then

\[
H(\lambda) = C_\Lambda R(\lambda, A_{-1})B = C_\Lambda \mathbb{D}_\lambda = K \mathbb{D}_\lambda,
\]

for any \( \lambda \in \mathbb{C} \) with \( \text{Re}\lambda > \alpha \). Moreover, we have

\[
\lim_{s \to +\infty} H(s) = 0.
\]

(3.5)

We have the following perturbation theorem (see [17] for the proof).

**Theorem 3.1.** Let assumptions \((H1)\) and \((H2)\) be satisfied and let \( B \) and \( C \) be the operators defined in (3.2). Assume that \((A, B, C)\) generates a regular linear system on \( X, U, U \) with the identity operator \( I_U : U \to U \) as an admissible feedback. The following assertions hold:

(i) The operator \((A, D(A))\) defined by (3.1) coincides with the following operator

\[
A^d := A_{-1} + BC_\Lambda, \quad D(A^d) = \{ x \in D(C_\Lambda) : (A_{-1} + BC_\Lambda)x \in X \}.
\]

(ii) The operator \((A, D(A))\) generates a strongly continuous semigroup \((T^{cl}(t))_{t \geq 0}\) on \( X \) as in Theorem 2.5.

(iii) For any \( \lambda \in \rho(A) \) we have

\[
\lambda \in \rho(A) \iff 1 \in \rho(D_\lambda K) \iff 1 \in \rho(K D_\lambda).
\]

(iv) Finally for \( \lambda \in \rho(A) \cap \rho(A) \):

\[
R(\lambda, A) = (I - D_\lambda K)^{-1} R(\lambda, A).
\]

Under the assumptions of Theorem 3.1, the mild solution of the problem (1.5) is given by

\[
z(t) = T^{cl}(t)x + \int_0^t T^{cl}(t-s)f(s)ds,
\]

for any \( t \geq 0, \ x \in X \) and \( f \in L^p(\mathbb{R}^+, X) \). Before giving another useful expression of \( z \) in term of the semigroup \( T \), we need the following very useful result proved in [15, prop.3.3].
Lemma 3.2. let \((S(t))_{t \geq 0}\) be a strongly continuous semigroup on \(X\) with generator \((G, D(G))\). Let \(Y \in \mathcal{L}(D(G), X)\) be a \(p\)-admissible observation operator for \(G\). Denote by \(Y_A\) the Yosida extension of \(Y\) with respect to \(G\). Then

\[
(S * f)(t) := \int_0^t S(t - s)f(s)ds \in D(Y_A), \quad \text{a.e. } t \geq 0,
\]

\[
\|Y_A(S * f)\|_{L^p([0,\alpha], X)} \leq c(\alpha)\|f\|_{L^p([0,\alpha], X)},
\]

for \(\alpha > 0\), \(f \in L^p_{\text{loc}}([0,\infty), X)\) and a constant \(c(\alpha)\) independent of \(f\) such that \(c(\alpha) \to 0\) as \(\alpha \to 0\).

Proposition 3.3. Let assumptions of Theorem 3.1 be satisfied. Let \(\alpha > 0\) and \(f \in L^p(\mathbb{R}^+, X)\). The non-homogenous Cauchy problem (1.5) is well-posed and its mild solution satisfies for any initial condition \(x \in X\),

\[
z(t) \in D(C_\Lambda) \quad \text{a.e. } t > 0,
\]

\[
\|C_\Lambda z(\cdot)\|_{L^p([0,\alpha], X)} \leq c_\alpha \left(\|x\| + \|f\|_{L^p([0,\alpha], X)}\right),
\]

\[
z(t) = T(t)x + \int_0^t T(t-s)BC_\Lambda z(s)ds + \int_0^t T(t-s)f(s)ds, \quad t \geq 0,
\]

where \(c_\alpha > 0\) is a constant independent of \(f\).

Proof. Let, by Theorem 3.1, \(T^c\) the semigroup generated by \(A\) and let \(z : [0, +\infty) \to X\) be the mild solution of the problem (1.5) given by (3.6). According to Theorem 2.5 we know that \(C_\Lambda\) is an admissible observation operator for \(A\). We denote by \(C_{\Lambda,A}\) the Yosida extension of \(C_\Lambda\) with respect to \(A\). Then \(D(C_{\Lambda,A}) \subset D(C_\Lambda)\) and \(C_{\Lambda,A} = C_\Lambda\) on \(D(C_{\Lambda,A})\). In fact, let \(x \in D(C_{\Lambda,A})\) and \(s > 0\) sufficiently large. Then by first taking Laplace transform on both sides of (2.11) and second applying \(sC_\Lambda\), we obtain

\[
sC_\Lambda R(s, A)x = sCR(s, A) + H(s)C_\Lambda sR(s, A)x,
\]

(3.8)

where we have used (3.4). Remark that

\[
\|H(s)C_\Lambda sR(s, A)x\| \leq \|H(s)\| \left(\|C_\Lambda sR(s, A)x - C_{\Lambda,A}x\| + \|C_{\Lambda,A}x\|\right)
\]

Hence, by (3.5) and the fact that \(x \in D(C_{\Lambda,A})\), we obtain

\[
\lim_{s \to +\infty} H(s)C_\Lambda sR(s, A)x = 0.
\]

Now from (3.8), we deduce that \(x \in D(C_\Lambda)\) and \(C_{\Lambda,A}x = C_\Lambda x\). Let \(x \in X\), \(\alpha > 0\) and \(f \in L^p([0,\alpha], X)\). The fact that \(C_\Lambda\) is \(p\)-admissible for \(A\), then by using (3.6) and Lemma 3.2 we obtain \(z(t) \in D(C_{\Lambda,A})\) for a.e. \(t > 0\). This shows that \(z(t) \in D(C_\Lambda)\) and \(C_{\Lambda,A}z(t) = C_{\Lambda,A}z(t)\) for a.e. \(t > 0\). The estimation in (3.7) follows immediately from (2.10) and Lemma 3.2. Let us prove the last property in (3.7). By density there exists \((f_n)_n \subset C([0,\alpha], D(A))\) such that \(f_n \to f\) in \(L^p([0,\alpha], X)\) as \(n \to \infty\). We set

\[
z_n(t) = T^c(t)x + \int_0^t T^c(t-s)f_n(s)ds, \quad t \geq 0.
\]

(3.9)
Using Hölder inequality, it is clear that \( \|z_n(t) - z(t)\| \to 0 \) as \( n \to \infty \). Now let us prove that \( z_n \) satisfies the third assertion in (3.7). In fact, the estimate in (3.7) implies that

\[
\|C_A z_n(\cdot) - C_A z(\cdot)\|_{L^p([0,\alpha],X)} \leq c_\alpha \|f_n - f\|_{L^p([0,\alpha],X)} \to 0.
\]

On the other hand, using the expression of the semigroup \( T^{cl} \) given in (2.11), change of variable and Fubini theorem we obtain

\[
z_n(t) = T^{cl}(t)x + \int_0^t T(t-s)f_n(s)ds + \int_0^t T_{-1}(t-\tau)B \int_0^\tau C_A T^{cl}(\tau - s)f_n(s)ds d\tau
\]

\[
= T(t)x + \int_0^t T(t-s)f_n(s)ds + \int_0^t T_{-1}(t-\tau)B \left( C_A T^{cl}(\tau)x + \int_0^\tau C_A T^{cl}(\tau-s)f_n(s)ds \right) d\tau.
\]

(3.10)

For simplicity we assume that \( 0 \in \rho(A) \). We then have

\[
C_A \int_0^\tau T^{cl}(\tau-s)f_n(s)ds = C_A(-A)^{-1}(-A) \int_0^\tau T^{cl}(\tau-s)f_n(s)ds
\]

\[
= C_A(-A)^{-1} \int_0^\tau T^{cl}(\tau-s)(-A)f_n(s)ds
\]

\[
= \int_0^\tau C_A(-A)^{-1}T^{cl}(\tau-s)(-A)f_n(s)ds
\]

\[
= \int_0^\tau C_A T^{cl}(\tau-s)f_n(s)ds.
\]

Now replacing this in (3.10), and using (3.9), we have

\[
z_n(t) = T(t)x + \int_0^t T_{-1}(t-s)BC_A z(s)ds + \int_0^t T(t-s)f_n(s)ds, \quad t \geq 0.
\]

Put

\[
\varphi(t) = T(t)x + \int_0^t T_{-1}(t-s)BC_A z(s)ds + \int_0^t T(t-s)f(s)ds, \quad t \geq 0.
\]

Then for any \( t \in [0,\alpha] \), we have

\[
\|z_n(t) - \varphi(t)\| \leq \gamma_\alpha \left( \|C_A z_n(\cdot) - C_A z(\cdot)\|_{L^p([0,\alpha],X)} + \|f_n - f\|_{L^p([0,\alpha],X)} \right),
\]

de to the admissibility of \( B \) for \( A \) and Hölder inequality. This shows that \( \|z_n(t) - \varphi(t)\| \to 0 \) as \( n \to \infty \), and hence \( z = \varphi \).

Now we can state the main result of this section.

**Theorem 3.4.** Let assumptions of Theorem 3.1 be satisfied. In addition, let \( P : Z \to X \) such that \((A,B,\mathbb{P})\) generates a regular linear system on \( X,U,X \), where \( \mathbb{P} = P_{|D(A)} \). The following assertions hold:

(i) The operator \( P \in \mathcal{L}(D(A),X) \) is a \( p \)-admissible observation operator for \( A \), hence the operator \((A + P,D(A))\) generates a strongly continuous semigroup on \( X \).
(ii) The boundary problem \([\text{1.1}]\) is well-posed and has a mild solution \(z : [0, +\infty) \to X\) satisfying:

\[
z(t) \in D(P_\Lambda) \quad \text{a.e. } t \geq 0, \\
z(t) = T^d(t)x + \int_0^t T^d(t - s) (P_\Lambda z(s) + f(s)) \, ds
\]

for any \(t \geq 0\), initial condition \(x \in X\) and \(f \in L^p([0, \infty), X)\), where \(P_\Lambda\) is the Yosida extension of \(P\) w.r.t. \(A\).

**Proof.** (i) We first remark from \([\text{3.3}]\) that \(Z \subset D(\mathbb{P}_{0, \Lambda})\) and \(P = \mathbb{P}_{0, \Lambda}\) on \(Z\), where \(\mathbb{P}_{0, \Lambda}\) denotes the Yosida extension of \(\mathbb{P}\) w.r.t. \(A\). Let \(x \in D(A)\) and \(\alpha > 0\). The facts that \((A, B, \mathbb{P})\) is regular and \([\text{2.10}]\), we have

\[
\int_0^t T^{-1}(t - s)BC_\Lambda T^d(s)x \in D(\mathbb{P}_{0, \Lambda}) \quad \text{a.e. } t \geq 0, \quad \text{and}
\]

\[
\left\| \mathbb{P}_{0, \Lambda} \int_0^t T^{-1}(t - s)BC_\Lambda T^d(s)x \right\|_{L^p([0, \alpha], X)} \leq \beta_\alpha \|x\|,
\]

where \(\beta_\alpha > 0\) is a constant. On the other hand, by \([\text{2.11}]\), we have

\[
PT^d(t)x = \mathbb{P}_{0, \Lambda}T^d(t)x = \mathbb{P}_{0, \Lambda}T(t)x + \mathbb{P}_{0, \Lambda} \int_0^t T^{-1}(t - s)BC_\Lambda T^d(s)x.
\]

Hence the \(p\)-admissibility of \(P\) for \(A\) follows by \([\text{3.11}]\) and the \(p\)-admissibility of \(\mathbb{P}\) for \(A\). Thus, according to Remark \([\text{2.7}]\) (i), the operator \((A + P, D(A))\) generates a strongly continuous semigroup on \(X\). The assertion (ii) follows from \([\text{15, thm.5.1}]\). \(\square\)

4. Perturbation Theorems for Maximal Regularity

4.1. Maximal regularity. Let \(G : D(G) \subset X \to X\) be the generator of a strongly continuous semigroup \(S := (S(t))_{t \geq 0}\) on a Banach space \(X\). Consider the following non-homogeneous abstract Cauchy problem:

\[
\begin{aligned}
\dot{z}(t) &= Gz(t) + f(t), \quad 0 < t \leq T \\
z(0) &= 0,
\end{aligned}
\]

where \(f : [0, T] \to X\) a measurable function.

**Definition 4.1.** We say that the operator \(G\) (or the problem \([\text{4.1}]\)) has the maximal \(L^p\)-regularity on the interval \([0, T]\), and we write \(G \in \mathcal{MR}_p(0, T; X)\), if for all \(f \in L^p([0, T], X)\), there exists a unique \(z \in W^{1,p}([0, T], X) \cap L^p([0, T], D(G))\) which verifies \([\text{4.1}]\).

By "maximal" we mean that the applications \(f, Gz\) and \(z\) have the same regularity. Due to the closed graph theorem, if \(G \in \mathcal{MR}_p(0, T; X)\) then

\[
\|\dot{z}\|_{L^p([0, T], X)} + \|z\|_{L^p([0, T], X)} + \|Gz\|_{L^p([0, T], X)} \leq C\|f\|_{L^p([0, T], X)},
\]

where \(C\) is a constant. \(\□\)
for a constant $C > 0$ independent of $f$.

It is known that a necessary condition for the maximal $L^p$-regularity is that $G$ generates an analytic semigroup. According to De Simon [10] this condition is also sufficient if $X$ is a Hilbert space. On the other hand, it is shown in [12] that if $p \in [1, \infty)$ then $G \in \mathcal{MR}_p(0, T; X)$ for all $q \in [1, \infty]$. Moreover if $G \in \mathcal{MR}_p(0, T; X)$ for one $T > 0$, then $G \in \mathcal{MR}_p(0, T'; X)$ for all $T' > 0$. Hence we simply write $G \in \mathcal{MR}(0, T; X)$.

Remark 4.2. (i) Let $\mathcal{C}([0, T]; D(G))$ be the space of all continuous functions from $[0, T]$ to $D(G)$, which is a dense space of $L^p([0, T]; X)$. It is known (see [33] (2.a) or [23] 1.5)) that $G$ has maximal $L^p$-regularity on $[0, T]$ if and only if $(S(t))_{t \geq 0}$ is analytic and the operator $\mathcal{R}$ defined by

$$\mathcal{R}(f)(t) := G \int_0^t S(t - s)f(s)ds \quad f \in \mathcal{C}([0, T], D(G)), \quad (4.3)$$

extends to a bounded operator on $L^p([0, T]; X)$. As we will see in our main results, this characterization is very useful if one works in general Banach spaces.

(ii) It is known (see [12]) that if $G \in \mathcal{MR}(0, T; X)$ then for every $\lambda \in \mathbb{C}$, $G + \lambda \in \mathcal{MR}(0, T; X)$, hence without lost of generality, we will assume throughout this paper that our generators satisfy $\omega_0(G) < 0$.

In order to recall another characterization of maximal regularity, we need some definitions.

Definition 4.3. We say that a Banach space $X$ is a UMD-space if for some (hence all) $p \in (1, \infty)$, $\mathcal{H} \in \mathcal{L}(L^p(\mathbb{R}, X))$ where

$$(\mathcal{H}f)(t) = \lim_{\epsilon \to 0} \frac{1}{\pi} \int_{|t|<\epsilon} \frac{f(t - s)}{s}ds, \quad t \in \mathbb{R}, \quad f \in \mathcal{S}(\mathbb{R}, X),$$

where $\mathcal{S}(\mathbb{R}, X)$ is the Schwartz space.

Classical UMD-spaces are Hilbert spaces and $L^p$-spaces, where $p \in (1, \infty)$. It is to be noted that every UMD-space is a reflexive space (see [2]).

Definition 4.4. A set $\tau \subset \mathcal{L}(X, Y)$ is called $\mathcal{R}$-bounded if there is a constant $C > 0$ such that for all $n \in \mathbb{N}$, $T_1, ..., T_n \in \tau$, $x_1, ..., x_n \in X$,

$$\int_0^1 \| \sum_{j=1}^n r_j(s)T_jx_j \|_Y ds \leq C \int_0^1 \| \sum_{j=1}^n r_j(s)x_j \|_X ds$$

where $(r_j)_{j \geq 1}$ is a sequence of independent $\{-1; 1\}$-valued random variables on $[0, 1]$(e.g. Rademacher variables).

Remark 4.5. Here we give examples of $\mathcal{R}$-bounded sets. We let $A$ be the generator of a bounded analytic semigroup on a Banach space $X$, $B : U \to X_{-1}$ and $C : D(A) \subset X \to U$ are linear bounded operators, where $U$ is another (boundary) Banach space. We assume that $(A, B, C)$ generates a regular linear system on $X, U, U$ with transfer function

$$H(\lambda) := C_AR(\lambda, A_{-1})B \in \mathcal{L}(U), \quad \lambda \in \rho(A),$$

13
where $C_A$ is the Yosida extension of $C$ with respect to $A$, see Section 2. It is shown in [19, p.513] that the set \{$H(is) : s \neq 0\}$ is $\mathcal{R}$-bounded.

The following result is due to Weis [34]

**Theorem 4.6.** Let $\mathbb{G}$ be the generator of a bounded analytic semigroup in a UMD-space $X$. Then $\mathbb{G}$ has maximal $L^p$-regularity for some (all) $p \in (1, \infty)$ if and only if the set \{$sR(is, \mathbb{G}) : s \neq 0\}$ is $\mathcal{R}$-bounded.

The following remark will be useful in the last section

**Remark 4.7.** Let $X, Z, U$ be a Banach spaces such that $Z \subset X$ with dense and continuous embedding, $A_m : Z \to X$ be a closed differential operator and $G : Z \to U$ be a linear surjective operator. We assume that the following operator

$$A := A_m, \quad D(A) := \ker(G)$$

generates a strongly continuous semigroup $T := (T(t))_{t \geq 0}$ on $X$. Let $\mathbb{D}_\lambda$ the Dirichlet operator associated with $A$ and $G$ (see Section 3). Moreover, we assume that the following operator

$$B := (\lambda - A_{-1})\mathbb{D}_\lambda \in \mathcal{L}(U, X_{-1})$$

is a $p$-admissible control operator for $A$. In addition, we assume that $A$ has the maximal $L^p$-regularity on $X$.

Let us first show that the operator $(-A)^\theta$, for some $\theta \in (0, \frac{1}{p})$, coincides with its Yosida extension with respect to $A$, that is:

$$D((-A)^\theta) = D((-A)^\theta) \quad \text{and} \quad (-A)^\theta \mathbb{D}_\lambda = (-A)^\theta$$

Since it is clear that $D((-A)^\theta) \subset D((-A)^\theta)$, it remains only to prove that $D((-A)^\theta) \subset D((-A)^\theta)$. Then let $x \in D((-A)^\theta)$ and set $y := (-A)^\theta x$ and $y_\lambda = (-A)^\theta \lambda R(\lambda, A)x$ for $\lambda$ sufficiently large. Clearly we have $y_\lambda \to y$ and $(-A)^\theta y_\lambda = \lambda R(\lambda, A)x \to x$ as $\lambda \to +\infty$. Then by closeness of $(-A)^{-\theta}$, $x = (-A)^{-\theta} y$. Thus $x \in D((-A)^\theta)$. This ends proof.

Finally, let us show that the triple $(A, B, (-A)^\theta)$ generates a regular system. In fact, we first prove that $\text{range}(\mathbb{D}_\mu) \subset D((-A)^\theta)$ (which is equivalent to the regularity of the system generated by $(A, B, (-A)^\theta)$ for $\mu$ sufficiently large. We know that if $B$ is $p$-admissible then $\text{range}(\mathbb{D}_\mu) \subset F^{A}_{\frac{1}{p}}$. Since $F^{A}_{\frac{1}{p}} \subset D((-A)^\theta)$ (see [13]), we have $\text{range}(\mathbb{D}_\mu) \subset D((-A)^\theta)$ and the closed graph theorem asserts that $(-A)^\theta \mathbb{D}_\mu \in \mathcal{L}(U, X)$. By virtue of analyticity of the semigroup generated by $A$, $((-A)^\theta, A)$ are $p$-admissible. To show the well-posedness of the system generated by $(A, B, (-A)^\theta)$ we have only to show that the operator $\mathbb{F}_\infty$ defined by:

$$(\mathbb{F}_\infty u)(t) := (-A)^\theta \Phi_t u, \quad u \in W^{2,p}_{0,\text{loc}}([0, \infty), U)$$
is well defined and extends to a bounded operator on $L^p_{loc}([0, \infty), U)$. In fact, by integration by parts and assuming that $0 \in \rho(A)$ we have

$$\Phi_t u = \mathbb{D}_0 u(t) - \int_0^t \mathcal{T}(t-s)\mathbb{D}_0 u'(s)ds.$$  

This shows that $\Phi_t u \in D((-A)^\theta)$ since $\int_0^t T(t-s)\mathbb{D}_0 u'(s)ds \in D(A)$ for a.e $t \geq 0$. Now we show the boundedness of $F \in \text{F}_\infty$. We have

$$(\text{F}_\infty u)(t) = (-A)^\theta \int_0^t (-A)^{1-\theta} \mathcal{T}(t-s)(-A)^\theta \mathbb{D}_0 u(s)ds$$

$$= -A \int_0^t \mathcal{T}(t-s)(-A)^\theta \mathbb{D}_0 u(s)ds$$

$$= - (\mathcal{R}(-A)^\theta \mathbb{D}_0 u(t))$$

Maximal regularity of $A$ shows the boundedness of $\text{F}_\infty$. This finishes the proof.

4.2. Perturbations that are $p$-admissible observation operators. In this part, we investigate maximal $L^p$-regularity for the problem (1.1) in the case $K = 0$. This is equivalent to study a such property for the evolution equation (1.3). As we have seen in the introductory section, we continue to assume that $P : Z \subset X \to X$ and $A := A_m$ with domain $D(A) = \ker(G)$ is the generator of strongly continuous semigroup $\mathcal{T} := (\mathcal{T}(t))_{t \geq 0}$ on $X$. We define $\mathbb{P} := Pt$ with $t : D(A) \to X$ is a continuous injection. So that $\mathbb{P} \in \mathcal{L}(D(A), X)$. We recall from Remark 2.7 (i) that if $\mathbb{P}$ is a $p$-admissible observation operator for $A$, then the following operator $A^p := A + \mathbb{P} = (A + P)t$ with domain $D(A^p) := D(A)$ is the generator of a strongly continuous semigroup $\mathcal{T}^p := (\mathcal{T}^p(t))_{t \geq 0}$ on $X$ such that $\mathcal{T}^p(t)X \subset D(\mathbb{P}_A)$ for a.e. $t > 0$, and

$$\mathcal{T}^p(t)x = \mathcal{T}(t)x + \int_0^t \mathcal{T}(t-s)\mathbb{P}_A \mathcal{T}^p(s)xds,$$

for all $x \in X$ and $t \geq 0$, where $\mathbb{P}_A$ is the Yosida extension of $\mathbb{P}$ with respect to $A$. On the other hand, as shown in [15] for any $f \in L^p([0, T])$ with $T > 0$, the mild solution of the evolution equation (1.3) satisfies $z(s) \in D(\mathbb{P}_A)$ for a.e. $s \geq 0$,

$$z(t) = \int_0^t \mathcal{T}^p(t-s)f(s)ds$$

$$= \int_0^t \mathcal{T}(t-s)(\mathbb{P}_A z(s) + f(s))ds. \quad (4.4)$$

for any $t \geq 0$. In addition if we denote by $\mathbb{P}_{A,A^p}$ the Yosida extension of $\mathbb{P}$ with respect to $A^p$, then $\mathbb{P}_{A,A^p} = \mathbb{P}_A$ on $D(\mathbb{P}_A)$. So by using (4.4) and Lemma 3.2 there exists a constant $c_T > 0$ independent of $f$ such that

$$\|\mathbb{P}_A z(\cdot)\|_{L^p([0,T],X)} \leq c_T \|f\|_{L^p([0,T],X)}. \quad (4.5)$$

We now state the main result of this paragraph.
**Theorem 4.8.** Let \( X \) be a Banach space, \( p \in \{1, \infty\} \) and \( P \) a \( p \)-admissible observation operator for \( A \). If \( A \in \mathcal{MR}(0, T; X) \) then \( A^P \in \mathcal{MR}(0, T; X) \).

**Proof.** Assume that \( A \in \mathcal{MR}(0, T; X) \), so that \( A \) generates an analytic semigroup on \( X \). This shows that there exists \( \omega \in \mathbb{R} \) such that \( \mathcal{C}_\omega := \{ \lambda \in \mathbb{C}, \Re \lambda > \omega \} \subset \rho(A) \) and for every \( \lambda \in \mathbb{C}_\omega \) we have:

\[
\| R(\lambda, A) \| \leq \frac{M}{|\lambda - \omega|}.
\]

On the other hand, for \( \lambda \in \rho(A) \),

\[
\lambda - A^P = (I - PR(\lambda, A))R(\lambda, A).
\]

By the admissibility of \( P \) for \( A \), there exists \( \tilde{M} > 0 \) such that

\[
\| PR(\lambda, A) \| \leq \frac{\tilde{M}}{(\Re \lambda - \omega)^{2q}} \quad \Re \lambda > \omega.
\]

Now for \( \Re \lambda > \omega + 2^q \tilde{M}^q := \alpha_0 \) we have

\[
\| PR(\lambda, A) \| \leq \frac{1}{2}
\]

thus \((I - PR(\lambda, A))\) is invertible and \( \| (I - PR(\lambda, A))^{-1} \| \leq 2 \). Then

\[
R(\lambda, A^P) = R(\lambda, A)(I - PR(\lambda, A))^{-1}.
\]

Finally, for \( \lambda \in \mathcal{C}_{\alpha_0} \) we have

\[
\| R(\lambda, A^P) \| \leq \frac{2M}{|\lambda - \omega|}.
\]

This implies, by [29, Thm.12.13]: that \((T^P(t))_{t \geq 0}\) is analytic. We now define, for any \( f \in \mathcal{C}([0, T], D(A)) \),

\[
(\mathcal{R} f)(t) := A \int_0^t \mathbb{T}(t - s)f(s)ds, \quad (\mathcal{R}^P f)(t) := A^P \int_0^t \mathbb{T}^P(t - s)f(s)ds, \quad t \in [0, T].
\]

Due to (4.4), we obtain

\[
\mathcal{R}^P f = \mathcal{R}(\mathbb{P}_A z(\cdot) + f) + \mathbb{P} \int_0^t \mathbb{T}(t - s) \left( \mathbb{P}_A z(s) + f(s) \right) ds.
\]

Using Remark 4.2 (i), the estimate (4.3) and Lemma 3.2 there exists a constant \( \tilde{c}_T > 0 \) independent of \( f \) such that

\[
\| \mathcal{R}^P f \|_{L^p([0, T], X)} \leq \tilde{c}_T \| f \|_{L^p([0, T], X)}.
\]

This ends the proof, due to Remark 4.2. \( \square \)

**Remark 4.9.** (1) In the proof of Theorem 4.8, we have proved that for \( p \)-admissible observation operators \( P \) for \( A \), the operator \( A \) generates an analytic semigroup on a Banach space \( X \) if and only if it is so for the operator \( A^P \). Hence if \( X \) is a Hilbert space, the maximal \( L^p \)-regularity of \( A^P \) is automatically guaranteed by [10].
(2) As explained in Remark 2.7 (i), $p$-admissible observation operators are also Miyadera-Voigt perturbations operators for $A$. We mention that the authors of [24, Cor.4] have obtained a result on maximal $L^p$-regularity under Miyadera-Voigt perturbations, where it is assumed that the state space $X$ is reflexive (or UMD) and the perturbation $P$ is a closed and densely defined operator and satisfies a very special Miyadera-Voigt condition. In our Theorem 4.8, $X$ is supposed to be a general Banach space and the perturbation $P$ is not closed and then with even minimum conditions we have obtained the maximal $L^p$-regularity for $AP$.

In the sequel we will also compare our result Theorem 4.8 with a result in [24, Thm.1] about small perturbations. To that purpose we need the following lemma.

Let $C \in \mathcal{L}(D(A), Y)$ for a Banach space $Y$. Define on $D(A)$ the operator

$$
\mathbb{J}x = \frac{1}{2i\pi} \int_\Gamma (-\mu)^{-\beta} CR(\mu, A)x d\mu,
$$

where $\Gamma := \Gamma(\psi, \epsilon) = \Gamma_1(\psi, \epsilon) \cup \Gamma_2(\psi, \epsilon) \cup \Gamma_3(\psi, \epsilon)$ denotes the upwards oriented path defined by

$$
\begin{align*}
\Gamma_1(\psi, \epsilon) &= \{ \lambda \in \mathbb{C} : |\lambda| \geq \epsilon, \arg \lambda = -\psi \}, \\
\Gamma_2(\psi, \epsilon) &= \{ \lambda \in \mathbb{C} : |\lambda| = \epsilon, |\arg \lambda| > \psi \}, \\
\Gamma_3(\psi, \epsilon) &= \{ \lambda \in \mathbb{C} : |\lambda| \geq \epsilon, \arg \lambda = \psi \},
\end{align*}
$$

for $\psi \in (\frac{\pi}{2}, \pi)$.

**Lemma 4.10.** Assume that $A$ is a sectorial operator and $\beta > \frac{1}{p}$ and let $C \in \mathcal{L}(D(A), Y)$ such that

$$
\|CR(\mu, A)\| \leq \frac{M}{(\text{Re} \mu)^{\frac{1}{q}}},
$$

for some constant $M > 0$ and $\mu$ in some half plane. Then for every $x \in D(A)$, the integral

$$
\frac{1}{2i\pi} \int_\Gamma (-\mu)^{-\beta} CR(\mu, A)x d\mu,
$$

exists as a Bochner integral, the operator $\mathbb{J}$ can be extended to a bounded operator from $X$ to $Y$. Moreover, for all $x \in D(A)$, $\mathbb{J}x = C(-A)^{-\beta}x$.

**Proof.** Fix $x \in D(A)$ and define $g(\mu) = (-\mu)^{-\beta} CR(\mu, A)x$, $\mu \in \Gamma$. Then we get

$$
CR(\mu, A)x = \frac{Cx}{\mu} + \frac{CR(\mu, A)Ax}{\mu}.
$$

Cauchy’s Theorem applied to the half plane yields that

$$
\int_\Gamma (-\mu)^{-(\beta+1)} d\mu = 0.
$$

It follows by (4.7) that

$$
\frac{1}{2i\pi} \int_\Gamma (-\mu)^{-\beta} CR(\mu, A)x d\mu = \frac{1}{2i\pi} \int_\Gamma (-\mu)^{-(\beta+1)} CR(\mu, A)Ax d\mu,
$$

$$
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$$
which exists as a Bochner integral due to \( (4.6) \). Let \( \Gamma_n := \Gamma_n(\psi, \epsilon) = \Gamma^1_n(\psi, \epsilon) \cup \Gamma^2(\psi, \epsilon) \cup \Gamma^3(\psi, \epsilon) \) denotes the upwards oriented path defined by

\[
\Gamma^1_n(\psi, \epsilon) = \{ \lambda \in \mathbb{C} : n \geq |\lambda| \geq \epsilon, \text{arg}\lambda = -\psi \}
\]

\[
\Gamma^3(\psi, \epsilon) = \{ \lambda \in \mathbb{C} : n \geq |\lambda| \geq \epsilon, \text{arg}\lambda = \psi \}
\]

It follows from Cauchy’s Theorem that

\[
\frac{1}{2i\pi} \int_{\Gamma} (-\mu)^{-\beta} CR(\mu, A) x d\mu = \lim_{n \to \infty} \frac{1}{2i\pi} \int_{\Gamma_n} (-\mu)^{-\beta} CR(\mu, A) x d\mu. \tag{4.8}
\]

Let \( C_n := C_n(\psi) \) and \( \Gamma^2 := \Gamma^2(\psi, \epsilon) \) be the upwards oriented curve defined by

\[
C_n := \{ \lambda \in \mathbb{C} : |\lambda| = n, |\text{arg}\lambda| \leq \psi \}
\]

\[
\Gamma^2(\psi, \epsilon) := \{ \lambda \in \mathbb{C} : |\lambda| = \epsilon, |\text{arg}\lambda| \leq \psi \}.
\]

Using again Cauchy’s Theorem we get

\[
\frac{1}{2i\pi} \int_{\Gamma_n} (-\mu)^{-\beta} CR(\mu, A) x dz = \frac{1}{2i\pi} \int_{C_n} (-\mu)^{-\beta} CR(\mu, A) x dz + \frac{1}{2i\pi} \int_{\Gamma^2} (-\mu)^{-\beta} CR(\mu, A) x d\mu
\]

Now, we are going to estimate the integrals over \( \Gamma^2 \), \( \Gamma^3 \) and \( C_n \). We start with the integral over \( C_n \). Then we obtain

\[
\frac{1}{2i\pi} \int_{C_n} (-\mu)^{-\beta} CR(\mu, A) x d\mu = \frac{1}{2\pi} \int_{-\psi}^{\psi} (ne^{i(\theta-\pi)})^{-\beta} CR(ne^{i\theta}, A) ne^{i\theta} x d\theta.
\]

The fact that \( C \) satisfies \((4.6)\), we obtain

\[
\left\| \frac{1}{2i\pi} \int_{C_n} (-\mu)^{-\beta} CR(\mu, A) x d\mu \right\| \leq \frac{1}{2\pi} \int_{-\psi}^{\psi} \| (ne^{i(\theta-\pi)})^{-\beta} CR(ne^{i\theta}, A) ne^{i\theta} x \| d\theta
\]

\[
\leq \frac{M}{2\pi n^{-\beta+\frac{1}{2}}} \int_{-\psi}^{\psi} \frac{d\theta}{\cos(\theta)^{\frac{1}{2}}} \| x \|.
\]

The same estimate holds for the integral over \( \Gamma^3 \). Finally for the integral over \( \Gamma^2 \), we have

\[
\left\| \frac{1}{2i\pi} \int_{\Gamma^2} (-\mu)^{-\beta} CR(\mu, A) x d\mu \right\| \leq \frac{1}{2\pi} \int_{\Gamma^2} \| (-\mu)^{-\beta+1} \| CR(\mu, A) \| x \| d\mu
\]

\[
\leq \frac{M}{\epsilon^{\frac{1}{2}}} \| x \|.
\]

since \( \Gamma^2 \) is a compact set and \( \mu \to CR(\mu, A) \) is analytic on \( \Gamma^2 \). Now, putting everything together, we find that there is a constant \( \kappa \) not depending on \( n \) such that

\[
\left\| \frac{1}{2i\pi} \int_{C_n} (-\mu)^{-\beta} CR(\mu, A) x d\mu \right\| \leq \kappa \| x \|
\]

From \((4.8)\) we obtain

\[\| J x \|_Y \leq \kappa \| x \|, \text{ for all } x \in D(A)\]
Therefore, $\mathcal{J}$ extends to a bounded linear operator on $X$.

Next we will show that for $x \in D(A)$ we have $\mathcal{J}x = C(-A)^{-\beta}x$. This is equivalent to show that the operator $C$ and the integral $\int_{\Gamma}(-\mu)^{-\beta}R(\mu, A)xd\mu$ commute. Since

$$(-A)^{-\beta}x = \frac{1}{2i\pi} \int_{\Gamma}(-\mu)^{-\beta}R(\mu, A)xd\mu,$$

$A$ commutes with $\int_{\Gamma}(-\mu)^{-\beta}R(\mu, A)xd\mu$ for every $x \in D(A)$. Now let us show that $\int_{\Gamma_n}(-z)^{-\beta}R(\mu, A)xd\mu$ converges in $D(A)$. The closedness of $A$ yields

$$A \int_{\Gamma_n}(-\mu)^{-\beta}R(\mu, A)xd\mu = \int_{\Gamma_n}(-\mu)^{-\beta}R(\mu, A)Axd\mu,$$

thus the integral $\int_{\Gamma_n}(-\mu)^{-\beta}R(\mu, A)xd\mu$ converges in $D(A)$. As $C$ is continuous on $D(A)$, we obtain

$$C(-A)^{-\beta}x = \lim_{n \to \infty} \frac{1}{2i\pi} C \int_{\Gamma_n}(-\mu)^{-\beta}R(\mu, A)xd\mu$$

$$= \lim_{n \to \infty} \frac{1}{2i\pi} \int_{\Gamma_n}(-\mu)^{-\beta}CR(\mu, A)xd\mu$$

$$= \mathcal{J}x.$$

for all $x \in D(A)$. This ends the proof. \hfill \Box

**Remark 4.11.** If $P \in \mathcal{L}(D(A), X)$ is $p$-admissible then it verifies the estimate (4.6). If in addition $A$ is sectorial, then for $\beta > \frac{1}{p}$ the operator $\mathbb{P}(-A)^{-\beta}$ has a bounded extension to $X$, due to Lemma 4.10. On the other hand, for any $x \in D(A)$ one can write

$$Px = \mathbb{P}(-A)^{-\beta}(-A)^{\beta}x.$$  

This implies that there exists a constant $c > 0$ such that

$$\|Px\| \leq c\|(-A)^{\beta}x\|.$$ 

As $(-A)^{\beta}$ is a small perturbation for $A$, then $P$ is so. Now by applying [24, Thm.1], the operator $A^p$ is sectorial as well. But if $A$ has the maximal $L^p$-regularity, the result of [24, Thm.1] confirms that $A^p$ has also the maximal $L^p$-regularity only if the state space $X$ is a UMD space. However Theorem 4.8 shows that the maximal $L^p$-regularity is preserved for $A^p$ even if we work in a general Banach space. This confirms that the $p$-admissibility for the perturbation operator is a very powerful tool to prove maximal $L^p$-regularity in Banach spaces.

### 4.3. Desch-Schappacher perturbation

In this section we will discuss maximal $L^p$-regularity of the perturbed boundary problem (1.1) (or equivalently (1.4)) under conditions (H1) and (H2) as in Section 3 and when the boundary perturbation $K$ satisfies the condition

- **(H3)** $K : X \to U$ is linear bounded (i.e. $K \in \mathcal{L}(X, U)$).

On the other hand, let $B$ as in (3.2). We shall also consider the following assumption

- **(H4)** $B$ is a $p$-admissible control operator for $A$. 

We first study the maximal $L^p$-regularity for the evolution equation (1.5), where the operator $(\mathcal{A}, D(\mathcal{A}))$ is defined by (3.1). As $K$ is bounded then, under the above conditions, the triple operator $(A, B, K)$ generates a regular linear system on $X, U, U$ with $I_U : U \to U$ as admissible feedback. By Theorem 3.1 the operator $\mathcal{A}$ generates a strongly continuous semigroup $\mathcal{T}_{cl} := (\mathcal{T}_{cl}(t))_{t \geq 0}$ on $X$ and then the unique mild solution (1.5) is given by
\[
\int_{0}^{t} \mathcal{T}_{cl}(t - s)f(s)ds,
\]
for any $t \geq 0$ and $f \in L^p([0, T], X)$ with $T > 0$. According to Proposition 3.3, this mild solution satisfies also
\[
\int_{0}^{t} (\mathcal{T}_{-1}(t - s)BKz(s)ds + \int_{0}^{t} \mathcal{T}(t - s)f(s)ds, \quad t \geq 0.
\]

**Remark 4.12.** Let us assume that $T$ is an analytic semigroup on $X$ and $B$ satisfies the condition
\[
\|R(\lambda, A_{-1})B\| \leq \frac{\kappa}{(\text{Re}\lambda - \omega)\frac{1}{p}}, \quad \text{Re}\lambda > \omega,
\]
for any $\omega > \omega_0(A)$ and a constant $\kappa > 0$. Then without assuming the condition (H4), one can use the same argument as in [24 Thm.8] (in the case $\alpha = 0$) and the fact that $\mathcal{A}$ coincides with the part of the operator $A_{-1} + BK$ on $X$ to prove that $\mathcal{A}$ generates an analytic semigroup on $X$. In the absence of analyticity of $T$ one cannot prove this generation result. Observe that our condition (H4) implies the estimate (4.11), see e.g. [32 chap.3]. With conditions (H1) to (H4) we have showed that $\mathcal{A}$ is a generator on $X$ without assuming any analyticity of $T$, see Remark 2.7 (ii).

**Theorem 4.13.** Assume that $X$ is a Banach state space and that conditions (H1) to (H4) are verified. If $A \in MR(0, T; X)$ then $A \in MR(0, T; X)$ (and hence the evolution equation (1.5) has the maximal $L^p$-regularity).

**Proof.** Let us first show that the semigroup $\mathcal{T}^{cl}$ generated by $\mathcal{A}$ is analytic. The condition $A \in MR(0, T; X)$ implies that the semigroup $\mathcal{T}$ is analytic. Hence, by [29 Thm. 12.31], we can find $\omega > \omega_0(A)$ and a constant $c > 0$ such that
\[
\|R(\lambda, A)\| \leq \frac{c}{|\lambda - \omega|},
\]
for and $\lambda \in \mathbb{C}$ such that $\text{Re}\lambda > \omega$. Now due to (4.11), for $\text{Re}\lambda > \omega + (2\kappa\|K\|)^p =: \tilde{\omega}$, we have
\[
1 \in \rho(R(\lambda, A_{-1})BK) \quad \text{and} \quad \|(I - R(\lambda, A_{-1})BK)^{-1}\| \leq 2.
\]
According to Theorem 3.1, we have \{ $\lambda \in \mathbb{C} : \text{Re}\lambda > \tilde{\omega}$ $\} \subset \rho(\mathcal{A})$ and
\[
\|R(\lambda, \mathcal{A})\| \leq \frac{2c}{|\lambda - \tilde{\omega}|},
\]

for some constant $\Re \lambda > \tilde{\omega}$, due to (4.12). This shows that $(\mathcal{T}^d(t))_{t \geq 0}$ is analytic, by [29 Thm 12.31]. Now define the following linear operators

$$\mathcal{R}^d f(t) := \mathcal{A} \int_0^t \mathcal{T}^d(t-s)f(s)ds \quad \text{and} \quad (\mathcal{R}f)(t) := \mathcal{A} \int_0^t \mathcal{T}(t-s)f(s)ds,$$

for almost every $t \in [0, T]$ and all $f \in C([0, T], D(A))$. Combining (4.9), (3.7) together with (3.1), for almost every $t \in [0, T]$ and all $f \in C([0, T], D(A))$ we obtain

$$\mathcal{R}^d f = \mathcal{A} m \int_0^t \mathcal{T}_1(t-s)BKz(s)ds + (\mathcal{R}f)(t). \quad (4.13)$$

On the other hand, taking in to account that the function $z$ is the solution of the evolution equation (1.5), using an integration by parts and the fact that range$(\mathcal{D}_\mu) \subset \ker (\mu - \mathcal{A}_m)$ for any $\mu \in \rho(A)$ we have

$$\mathcal{A} m \int_0^t \mathcal{T}_1(t-s)BKz(s)ds = \mu (\mathcal{R}(Kz(\cdot))(t) + \mathcal{A} m \int_0^t \mathcal{T}_1(t-s)(-\mathcal{A}_1)\mathcal{D}_\mu Kz(s)ds$$

$$= \mu (\mathcal{R}(Kz(\cdot))(t) + \mathcal{A} \int_0^t \mathcal{T}(t-s)\mathcal{D}_\mu K(Az(s) + f(s))ds,$$

for almost every $t \geq 0$, and all $f \in C([0, T], D(A))$. Now the identity (4.13) becomes

$$(\mathcal{R}^d f)(t) = (\mathcal{R}g_\mu(t) + (\mathcal{R}(\mathcal{D}_\mu K\mathcal{R}^d f))(t) \quad (4.14)$$

for almost every $t \geq 0$, all $\mu \in \rho(A)$ and $f \in C([0, T], D(A))$, where

$$g_\mu := (I + \mathcal{D}_\mu K)f + \mu Kz(\cdot).$$

By assumption there exists $c_T > 0$ such that

$$\|\mathcal{R}\varphi\|_{L^p([0,T],X)} \leq c_T \|\varphi\|_{L^p([0,T],X)}, \quad \varphi \in L^p([0,T],X).$$

Let $\omega > \max\{\omega_0(A), \omega_0(A)\}$ and choose and fix $\mu > \omega + (2c_T \kappa \|K\|)^p$, where the constant $\kappa > 0$ is given in (4.11). Then we have

$$\|\mathcal{D}_\mu K\| \leq \frac{1}{2c_T}. \quad (4.15)$$

Now using (4.15), (4.9) and Hölder inequality, we obtain

$$\|g_\mu\|_{L^p([0,T],X)} \leq \left(1 + \frac{1}{2c_T} + T|\mu|\|K\|e^{\omega(T)}\right) \|f\|_{L^p([0,T],X)} := \tilde{c}_T \|f\|_{L^p([0,T],X)}. \quad (4.16)$$

Now we define the operator

$$(\mathcal{F}_T g)(t) = \mathcal{R}(\mathcal{D}_\mu Kg)(t),$$

for any $t \geq 0$ and measurable functions $g : [0, T] \to X$. Using (4.14), we obtain

$$(I - \mathcal{F}_T)\mathcal{R}^d f = \mathcal{R}g_\mu. \quad (4.17)$$

Remark that the restriction of $I - \mathcal{F}_T$ to $L^p([0,T],X)$ is invertible, since by (4.15), we have

$$\|\mathcal{F}_T g\|_{L^p([0,T],X)} \leq \frac{1}{2} \|g\|_{L^p([0,T],X)}.
Now as $Rg_\mu \in L^p([0,T], X)$ then by (4.17), we have $R^d f \in L^p([0,T], X)$ and
\[ R^d f = (I - T^\mu)^{-1} Rg_\mu. \]
Finally, using (4.16), we obtain
\[ \|R^{cl}_f\|_{L^p([0,T], X)} \leq 2 c T \tilde{c} \|f\|_{L^p([0,T], X)}, \quad f \in C([0,T], D(A)). \]
The required result now follows by density.

\[ \square \]

Remark 4.14. In [24, Rem.11], the authors showed that if $A$ has a maximal $L^p$-regularity on a UMD space $X$ and a perturbation $P : X \to X_{-1}$ satisfies $\|(-A_{-1})^{-1} P\| \leq \eta$ with $\eta$ small in some sense (see condition (7) in [24]), then the part of $A_{-1} + P$ on $X$ has also the maximal $L^p$–regularity on $X$. The UMD property is an essential condition in [24] due to a Weis' perturbation theorem [34]. In our case, $X$ is a general Banach space (not necessarily UMD). However, instead of the above condition on $P$ we have assumed that the operator $P = BK$ is $p$–admissible control operator for $A$ (which is the case when $B$ is so). This condition together with (4.11) easily imply the condition (7) in [24].

We now state the result giving the maximal $L^p$-regularity for the systems (1.1) (or equivalently for the equation (1.4)) in the case when $K \in \mathcal{L}(X,U)$. This is equivalent to the

**Theorem 4.15.** Let $X, Z, U$ be Banach spaces such that $Z \subset X$ (with continuous and dense embedding ), $p \in (1, \infty)$ and consider the evolution equation (1.1) with bounded boundary perturbation operator $K \in \mathcal{L}(X,U)$. Assume that the conditions (H1) to (H4) are satisfied. Moreover, we assume that the triple $(A, B, \mathbb{P})$ generates a regular linear system on $X, U, X$ where $\mathbb{P}$ is the restriction of $P$ to $D(A)$. Then the operator $(A + P, D(A))$ is the generator of a strongly continuous semigroup on $X$. Moreover, if $A \in \mathcal{M}(0, T; X)$ then $A + P \in \mathcal{M}(0, T; X)$ (and hence the evolution equation (1.1) has the maximal $L^p$-regularity).

**Proof.** The fact that $(A + P, D(A))$ is a generator on $X$ is already proved in Theorem 3.4. Now if $A \in \mathcal{M}(0, T; X)$, then $A \in \mathcal{M}(0, T; X)$, by Theorem 4.13. On the other hand, Theorem 3.2 shows that $P$ is $p$-admissible observation operator for $A$. So, thanks to Theorem 4.8 we also have $A + P \in \mathcal{M}(0, T; X)$.

4.4. **Staffans-Weiss perturbation.** In this part, we study maximal $L^p$–regularity for the boundary perturbed equation (1.1) in the general case when the boundary perturbation $K$ is unbounded. We then assume, as in the previous part of this paper, that (H1) and (H2) are satisfied. In addition we suppose the following condition

(H3)' $K : Z \to U$ is linear bounded (i.e. $K \in \mathcal{L}(Z, U)$).

On the other hand, let $B$ and $C$ as in (3.2). We shall also consider the following assumption

(H4)' the triple $(A, B, C)$ generates a regular linear system on $X, U, U$ with $I_U : U \to U$ as an admissible feedback operator.

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Theorem 4.16. Let $X, Z, U$ be Banach spaces such that $Z \subset X$ (with continuous and dense embedding) and let conditions $(H_1), (H_2), (H_3)'$ and $(H_4)'$ be satisfied. Then the operator $(A, D(A))$ defined by (3.1) generates a strongly continuous semigroup which is analytic whenever the semigroup generated by $A$ is.

Proof. According to Theorem 3.1 (i) $A$ is a generator of a strongly continuous semigroup $T^{cl} := (T^{cl}(t))_{t \geq 0}$ on $X$. Now assume that $A$ generates an analytic semigroup $\mathbb{T}$ on $X$. Then there exist constants $\beta \in \mathbb{R}$ and $M_1 > 0$ such that

$$C_{\beta} \subset \rho(A) \quad \text{and} \quad \|R(\lambda, A)\| \leq \frac{M_1}{|\lambda - \beta|}, \quad \lambda \in C_{\beta} \quad (4.18)$$

On the other hand, let us prove that the admissibility of $B$ and $C$ for $A$, imply that $M_2 := \sup_{z \in C_0} |z|^{\frac{1}{q}} \|CR(z, A)\| < +\infty$ and $M_3 := \sup_{z \in C_0} |z|^{\frac{1}{p}} \|R(z, A^{-1})B\| < +\infty$, (4.19)

where $\frac{1}{p} + \frac{1}{q} = 1$. In fact, we will give a slight modification of the proof given in [7, lem.1.6]. Since $A$ generates a bounded analytic semigroup there exist $\omega \in ]\pi / 2, \pi[$ such that $\sigma(A) \subset \mathbb{C} \setminus \Sigma_\omega$. Let $\gamma \in ]\frac{\pi}{2}, \omega[$ and $\Gamma$ the path defined by

$$\Gamma = \{re^{\pm i\gamma}, r > 0\}$$

We can see easily that $\frac{|\text{Re} z|}{|z|} = \sin \gamma$. By virtue of the resolvent equation and using the analyticity of the semigroup, we obtain:

$$|z|^{\frac{1}{q}} \|CR(z, A)\| \leq M_\Gamma \sup_{s \in C_0} (\text{Re}(s))^{\frac{1}{q}} \|CR(s, A)\|, z \in \Gamma$$

for some constant $M_\Gamma > 0$. Since $C$ is $p$-admissible for $A$, we have

$$\sup_{s \in C_0} (\text{Re}(s))^{\frac{1}{q}} \|CR(s, A)\| < +\infty$$

and thus:

$$\sup_{z \in \Gamma} |z|^{\frac{1}{q}} \|CR(z, A)\| < +\infty$$

The application

$$\varphi : \Sigma_\omega \to \mathcal{L}(X, Y), \quad z \mapsto z^{\frac{1}{q}} CR(z, A)$$

is analytic, then for all $z \in C_0$ we have

$$z^{\frac{1}{q}} CR(z, A) = \frac{1}{2\pi i} \int_{\Gamma} \frac{u^{\frac{1}{q}} CR(u, A)}{u - z} du$$

Since $\varphi$ is bounded on $\Gamma$, then it is bounded on $C_0$ and this is what we want. The other estimation is obtained by the same arguments. Let $\omega_1 := \max\{\omega_0(A), \omega_0(A)\}$. From Theorem 3.1 (ii) we know that for any $\lambda \in C_{\omega_1}$, we have

$$R(\lambda, A) = (I - D_\lambda K)^{-1} R(\lambda, A)$$

$$= R(\lambda, A) + R(\lambda, A^{-1})B(I_U - C_\lambda D_\lambda)^{-1} CR(\lambda, A). \quad (4.20)$$
On the other hand, \((I_U - C\Lambda D\lambda)^{-1} = I + \mathcal{H}^{cl}(\lambda)\), where \(\mathcal{H}^{cl}\) is the transfer function of the (closed-loop) regular linear system generated by \((A, B, C\Lambda)\). Hence there exists \(\alpha > \omega_0(A)\) such that

\[
\nu := \sup_{C\alpha} \|(I_U - C\Lambda D\lambda)^{-1}\| < \infty. \tag{4.21}
\]

Now let \(\omega_2 := \max\{0, \alpha, \beta, \omega_1\}\). Then by using (4.18), (4.19), (4.20) and (4.21), we obtain

\[
\|R(\lambda, A)\| \leq \frac{\tilde{M}}{|\lambda - \omega_2|}
\]

for all \(\lambda \in \mathbb{C}_{\omega_2}\), where \(\tilde{M} = M_1 + \nu M_2 M_3\).

The following result shows the maximal regularity of the perturbed boundary value problem \((1.1)\) in the case of \(P \equiv 0\).

**Theorem 4.17.** Let \(X, Z, U\) be Banach spaces such that \(Z \subset X\) (with continuous and dense embedding) and let conditions \((H1)\), \((H2)\), \((H3)'\) and \((H4)'\) be satisfied and let \(p \in (1, \infty)\). Let the operator \((A, D(A))\) defined by (3.1). Assume additionally that there exists \(\lambda_0 \in \mathbb{R}\) such that

\[
\kappa_0 := \sup_{\Re\lambda > \lambda_0} \|\lambda D\lambda\| < +\infty \tag{4.22}
\]

If \(A \in \mathcal{M}\mathcal{R}(0, T; X)\) then \(A \in \mathcal{M}\mathcal{R}(0, T; X)\).

**Proof.** As \(A \in \mathcal{M}\mathcal{R}(0, T; X)\), there exists \(\mathcal{R} \in \mathcal{L}(L^p([0, T], X))\) such that

\[
(\mathcal{R}f)(t) = A \int_0^t \mathbb{T}(t-s)f(s)ds,
\]

for all \(f \in L^p([0, T], X)\) and a.e \(t \geq 0\). By Theorem 4.16, \(A\) generates an analytic semigroup \(\mathbb{T}^{cl}\) on \(X\). We then can define the following operator

\[
(\mathcal{R}^{cl}f)(t) := A \int_0^t \mathbb{T}^{cl}(t-s)f(s)ds \quad f \in C([0, T]; D(A)).
\]

Our objective is to show that the operator \(\mathcal{R}^{cl}\) admits a bounded extension on \(L^p([0, T]; X)\). On the other hand, we define the Yosida approximation operators of \(A\) by

\[
\mathcal{A}_n := nAR(n, A) = n^2 R(n, A_n) - nI,
\]

for any \(n \in \mathbb{N}\) such that \(n > \omega_0(A)\). From (4.20) and for any sufficiently integer \(n\), one can write

\[
\mathcal{A}_n = nAR(n, A) + n^2 D_n(I - C\Lambda D_n)^{-1}CR(n, A). \tag{4.23}
\]

We also set

\[
(\mathcal{R}^{cl}_n f)(t) := \mathcal{A}_n \int_0^t \mathbb{T}^{cl}(t-s)f(s)ds,
\]

for all \(t \geq 0\).
for \( f \in C([0, T]; D(A)) \) and \( n \in \mathbb{N} \) such that \( n > \omega_0(A) \). We have (see \[13\]),

\[
(\mathcal{R}^d f)(t) = \lim_{n \to \infty} (\mathcal{R}^d_n f)(t),
\]

for every \( t \in [0, T] \). Using Proposition \[3.3\] we have

\[
(\mathcal{R}^d_n f)(t) = A_n \left( \int_0^t T(t-s)f(s)ds + \int_0^t T_{-1}(t-s)C_Az(s)de \right),
\]

where

\[
z(t) = \int_0^t T^d(t-s)f(s)ds, \quad t \geq 0.
\]

Using (4.23), for large \( n \),

\[
(\mathcal{R}^d_n f)(t) = nAR(n, A) \int_0^t T(t-s)f(s)ds + n^2D_n(I - C_A\mathbb{D}_n)^{-1}CR(n, A) \int_0^t T(t-s)f(s)ds + nAR(n, A) \int_0^t T_{-1}(t-s)BC_Az(s)ds + n^2D_n(I - C_A\mathbb{D}_n)^{-1}CR(n, A) \int_0^t T_{-1}(t-s)BC_Az(s)ds := I^1_n(t) + I^2_n(t) + I^3_n(t) + I^4_n(t).
\]

We have

\[
\int_0^T \| I^1_n(t) \|^p dt = \| \mathcal{R}(nR(n, A)f) \|^p_{L^p([0, T], X)} \leq c_T \| f \|^p_{L^p([0, T], X)},
\]

for a constant \( c_T > 0 \) independent of \( f \). On the other hand,

\[
\int_0^T \| I^2_n(t) \|^p dt = \int_0^T \| n\mathbb{D}_n(I - C_A\mathbb{D}_n)^{-1}CnR(n, A) \int_0^t T(t-s)f(s)ds \|^p dt \leq (\kappa \nu)^p \int_0^T \| C \int_0^t T(t-s)nR(n, A)f(s)ds \|^p dt \leq (\kappa \nu \gamma_T)^p \| f \|^p_{L^p} := c_{T,1} \| f \|^p_{L^p},
\]

due to (4.21), (4.22) and Lemma \[3.2\] where \( c_{T,1} \) is a constant independent of \( f \). We estimate \( I^3_n(t) \) by

\[
\int_0^T \| I^3_n(t) \|^p dt = \int_0^T \| A \int_0^t T(t-s)n\mathbb{D}_nC_Az(s)ds \|^p dt = \| \mathcal{R}(n\mathbb{D}_nC_Az(s)) \|^p_{L^p([0, T], X)} \leq c_T \nu_0^p \| C_Az(s) \|^p_{L^p([0, T], U)} \leq c_{T,2} \| f \|^p_{L^p},
\]
due to (4.22) and Proposition 3.3 where $c_{T,2} > 0$ is a constant independent of $f$. Similarly,

$$
\int_0^T \| I_n^A(t) \|^p dt = \int_0^T \| nD_n(I - C_\Lambda D_n)^{-1} C \int_0^t T(t - s)R(t, A)BC_\Lambda z(s) ds \|^p dt
$$

\[\leq (\nu \kappa_0)^p \int_0^T \| C \int_0^t T(t - s)R(t, A)BC_\Lambda z(s) ds \|^p dt\]

\[\leq c_{T,3} \| f \|_{L^p},\]

by (4.21), (4.22), Lemma 3.2 and Proposition 3.3, where $c = 0$ is a constant independent of $f$. Finally one can conclude that

$$
\int_0^T \| (\mathcal{R}_n^A f)(t) \|^p dt \leq \tilde{C}_p \| f \|_{L^p},
$$

for some constant $\tilde{C}_p > 0$ depending on $p$ and independent of $f$. Since $\| (\mathcal{R}_n^A f)(t) \|^p \to \| (\mathcal{R}^A f)(t) \|^p$ for all $t \in [0, T]$, we conclude by Fatou’s lemma that

$$
\int_0^T \| (\mathcal{R}_n^A f)(t) \|^p dt \leq \lim \inf \int_0^T \| (\mathcal{R}_n^A f)(t) \|^p dt
$$

\[\leq \tilde{C}_p \| f \|_{L^p}.

Thus $\mathcal{R}_n^A$ can be extended to a bounded operator on $L^p([0, T]; X)$.

\begin{remark}
Here we will show that the result of Theorem 4.17 holds only in non reflexive Banach spaces. To that purpose, we define, for $\alpha \in (0, 1)$, the Favard space of order $\alpha$ associated to $A$ by

$$
F_\alpha^A := \left\{ x \in X : \sup_{t > 0} \left\| \frac{T(t)x - x}{t^\alpha} \right\| < \infty \right\}
$$

with norm

$$
\| x \|_{F_\alpha^A} := \sup_{t > 0} \left\| \frac{T(t)x - x}{t^\alpha} \right\|.
$$

Now the assumption $\sup_{R \lambda > \lambda_0} \left\| \lambda R(\lambda, A)B \right\| < +\infty$ in the previous theorem implies that $\text{range}(B) \subset F_1^{A-1}$ (see [25, Remark 10]) and it is important to remark that the control operator $B$ is strictly unbounded, i.e. $\text{range}(B) \cap X = \{0\}$, since it comes from the boundary (see [30]). These facts force us to work in non-reflexive Banach spaces, because if $X$ is reflexive, it is well known that $F_1^{A^{-1}} = X$, thus $\text{range}(B) \subset X$ and this cannot be true.

Now we state the previous theorem in the case of $X$ being a GT-space (e.g. if $X = L^1$ or $X = C(K)$, see for instance [21]), which is a non-reflexive space.

\begin{corollary}
Let $X, Z, U$ be Banach spaces such that $Z \subset X$ (with continuous and dense embedding) and let conditions (H1), (H2), (H3)', and (H4)' be satisfied such that either $X$ or $X^*$ is a GT-space. Let the operator $(\mathcal{A}, D(\mathcal{A}))$ be defined by (3.1). Assume additionally that there exist $\lambda_0 \in \mathbb{R}$ such that

$$
\kappa_0 := \sup_{R \lambda > \lambda_0} \| \lambda D_\lambda \| < +\infty
$$

\end{corollary}
If $A$ is an $H^\infty$-sectorial operator on $X$ with $\omega_H(A) < \frac{\pi}{2}$ then $A \in \mathcal{MR}(0,T;X)$.

Proof. According to Theorem 7.5 in [21], if $A$ is an $H^\infty$-sectorial operator on $X$ with $\omega_H(A) < \frac{\pi}{2}$ then $A$ has maximal $L^p$-regularity for all $1 < p < \infty$, which implies by Theorem 4.17 that $A^{cl} \in \mathcal{MR}(0,T;X)$. □

The next theorem present a perturbation result on UMD-spaces.

**Theorem 4.20.** Let conditions (H1), (H2), (H3)' and (H4)' be satisfied with $X,U$ be UMD-spaces and $A$ generates a bounded analytic semigroup. Assume that there exists $\omega > \max\{\omega_0(A);\omega_0(A)\}$ such that the sets $\{s\frac{1}{2}R(\omega + is, A^{-1})B; s \neq 0\}$ and $\{s\frac{1}{2}C\rho(\omega + is, A); s \neq 0\}$ are $\mathcal{R}$-bounded. If $A \in \mathcal{MR}(0,T;X)$ then $A \in \mathcal{MR}(0,T;X)$.

Proof. Assume that $A \in \mathcal{MR}(0,T;X)$ and let $\omega > \max\{\omega_0(A);\omega_0(A)\}$ such that the sets $\{s\frac{1}{2}R(\omega + is, A^{-1})B; s \neq 0\}$ and $\{s\frac{1}{2}C\rho(\omega + is, A); s \neq 0\}$ are $\mathcal{R}$-bounded. Denote $A^\omega := -\omega + A$ and $A^\omega := -\omega + A$ with domains $D(A^\omega) = D(A)$ and $D(A^\omega) = D(A)$, respectively. These operators are generators of analytic semigroups on $X$. We first observe that $A^\omega \in \mathcal{MR}(0,T;X)$. To prove our theorem it suffice to show that $A^\omega \in \mathcal{MR}(0,T;X)$. Clearly, $\omega_0(A^\omega) = \omega_0(A) - \omega < 0$ and $\omega_0(A^\omega) = \omega_0(A) - \omega < 0$, so that $i\mathbb{R}\{0\} \subset \rho(A^\omega) \cap \rho(A^\omega)$. It is not difficult to show that $(A^\omega, B, C)$ is also a regular linear system on $X, U, U$ with the identity operator $I_U : U \to U$ as an admissible feedback. Now according to Theorem 2.5 the following operator

$$A^{cl,\omega} := A^{-1} + BC_\lambda, \quad D(A^{cl,\omega}) = \{x \in D(C_\lambda) : (A^{-1} + BC_\lambda)x \in X\}.$$  

As $A^{-1} = A^{-1} - \omega$, then $D(A^{cl,\omega}) = D(A)$, and $A^{cl,\omega} = A^\omega$ due to Theorem 3.1 (i). As in (4.20) we have

$$sR(is, A^\omega) = sR(is, A^\omega) + s\frac{1}{2}R(is, A^{-1})B(I - H^\omega(is))^{-1}s\frac{1}{2}C\rho(is, A^\omega)$$

$$= sR(is, A^\omega) + s\frac{1}{2}R(\omega + is, A^{-1})B(I - H^\omega(is))^{-1}s\frac{1}{2}C\rho(\omega + is, A),$$  

where $H^\omega(\lambda) = C\lambda R(\lambda, A^{-1})B$, $\lambda \in \rho(A)$, is the transfer function of the regular linear system generated by $(A^\omega, B, C)$. Using the assumptions, the equation (4.24) and Theorem 4.6 it suffice to show that the set $\{(I - H^\omega(is))^{-1} : s \neq 0\}$ is $\mathcal{R}$-bounded. In fact, by Theorem 3.1 (i) and the condition (H4)' the triple operator $(A^\omega, B, C_\lambda)$ generates a regular linear system with transfer function

$$H^{cl,\omega}(is) = (I - H^\omega(is))^{-1}H^\omega(is), \quad s \neq 0,$$

which implies that

$$(I - H^\omega(is))^{-1} = I_U + H^{cl,\omega}(is), \quad s \neq 0.$$  

According to Remark 4.6 the set $\{H^{cl,\omega}(is) : s \neq 0\}$ is $\mathcal{R}$-bounded. Hence $\{(I - H^\omega(is))^{-1} : s \neq 0\}$ is $\mathcal{R}$-bounded. This ends the proof. □

**Proposition 4.21.** Let conditions (H1), (H2), (H3)' and (H4)' be satisfied with $X,U$ be UMD-spaces and $A$ generates a bounded analytic semigroup. Assume that there exist
constants $\omega > \max\{\omega_0(A), \omega_0(A)\}$ and $\alpha \in \left(\frac{1}{p}, 1\right)$ such that the set $\{s^{\alpha}R(\omega + is, A_{-1})B; s \neq 0\}$ is $\mathcal{R}$-bounded. If $A \in \mathcal{M}(0, T; X)$ then $A \in \mathcal{M}(0, T; X)$.

Proof. Let the operators $A^\omega$ and $\mathcal{A}^\omega$ as in the proof of Theorem 4.20. Let $s \in \mathbb{R}\setminus\{0\}$ and $\alpha \in \left(\frac{1}{q}, 1\right)$. According to Theorem 4.6, it suffices to show that the set $\{sR(is, A^\omega) : s \neq 0\}$ is $\mathcal{R}$-bounded. In fact, as in [1, 24], we obtain

$$sR(is, A^\omega) = sR(is, A^\omega) + s^{\alpha}R(is, A_{-1}^\omega)B(I - H^\omega(is))^{-1}s^{1-\alpha}CR(is, A^\omega).$$

By the proof of Theorem 4.20, we know that the set $\{(I - H^\omega(is))^{-1} : s \neq 0\}$ is $\mathcal{R}$-bounded. Now as by assumption the set $\{s^{\alpha}R(is, A_{-1}^\omega)B; s > 0\}$ is $\mathcal{R}$-bounded, it suffices to show that the set $\{s^{1-\alpha}CR(is, A^\omega); s > 0\}$ is $\mathcal{R}$-bounded. We have

$$s^{1-\alpha}CR(is, A^\omega) = s^{1-\alpha}C(-A^\omega)^{-\alpha}(-A^\omega)^{\alpha}R(is, A^\omega) = C(-A^\omega)^{-\alpha}[-(-A^\omega)^{\alpha}(is - A^\omega)^{-\alpha}]\left[s^{1-\alpha}(is - A^\omega)^{\alpha-1}\right].$$

By [21] Lemma 10, the sets $\{(-A^\omega)^{\alpha}(is - A^\omega)^{-\alpha}; s > 0\}$ and $\{s^{1-\alpha}(is - A^\omega)^{\alpha-1}; s > 0\}$ are $\mathcal{R}$-bounded. On the other hand, by Lemma 4.10 the operator $C(-A^\omega)^{-\alpha}$ has a bounded extension to $X$. Hence the set $\{s^{1-\alpha}CR(is, A^\omega); s > 0\}$ is $\mathcal{R}$-bounded. This ends the proof.

We end this section by the following result given the maximal $L^p$-regularity for the evolution equation (1.1) (or equivalently (1.4)).

**Corollary 4.22.** Let conditions (H1), (H2), (H3)' and (H4)' be satisfied on Banach spaces $X, U$ and $A \in \mathcal{M}(0, T; X)$. Moreover, we assume that $(A, B, \mathbb{P})$ generates a regular linear system on $X, U, X$ with $\mathbb{P} \in \mathcal{L}(D(A), X)$ is the restriction of $P : Z \to X$ on $D(A)$. Then the operator $\mathcal{A} + P \in \mathcal{M}(0, T; X)$ (or equivalently the evolution equation (1.1) has maximal $L^p$-regularity) if one of the following conditions hold:

(i) $X$ is a non reflexive Banach space and there exists $\lambda_0 \in \mathbb{R}$ such that

$$\sup_{\text{Re}\lambda > \lambda_0} \|\lambda \mathcal{D}\lambda\| < \infty.$$  

(ii) $X$ and $U$ are UMD spaces, the sets $\{s^{\frac{1}{2}}R(is, A_{-1})B; s \neq 0\}$ and $\{s^{\frac{1}{2}}CR(is, A); s \neq 0\}$ are $\mathcal{R}$-bounded.

(iii) $X$ and $U$ are UMD-spaces, and $A$ generates a bounded analytic semigroup and there exist constants $\omega > \max\{\omega_0(A), \omega_0(A)\}$ and $\alpha \in \left(\frac{1}{p}, 1\right)$ such that the set $\{s^{\alpha}R(\omega + is, A_{-1})B; s \neq 0\}$ is $\mathcal{R}$-bounded.

Proof. First of all the operator $P$ is $p$-admissible observation operator for $\mathcal{A}$, due to Theorem 3.4. Now the assertion (i) follows by combining Theorem 4.8 and Theorem 4.17. The assertion (ii) follows by combining Theorem 4.8 and Theorem 4.20. Finally, the assertion (iii) follows by combining Theorem 4.8 and Proposition 4.21.\]
5. Applications

The object of this section is to apply our obtained results to solve the problem of maximal \( L^p \)-regularity for integro-differential equations and boundary integro-differential equations. We then extend some results in [4].

5.1. Maximal regularity for free-boundary integro-differential equations. Let \( X_0, U_0, Z_0 \) be Banach spaces such that \( Z_0 \subset X_0 \) with continuous and dense embedding and let \( q \in (1; \infty) \). We consider the following problem:

\[
\begin{aligned}
\dot{\varrho}(t) &= A_m \varrho(t) + \int_0^t a(t-s) F \varrho(s) ds + f(t), \quad t \geq 0, \\
G \varrho(t) &= 0, \quad t \geq 0, \\
\varrho(0) &= 0,
\end{aligned}
\]

where \( A_m : Z_0 \to X_0 \) is a closed linear differential operator, \( F : Z_0 \to X_0 \) a linear operator, \( G : Z_0 \to U_0 \) is a (trace) linear boundary operator and a certain measurable function \( f : [0, \infty) \to X_0 \).

The main purpose of this subsection is to apply our abstract results on maximal regularity developed in Section 4 for the integro-differential equation (5.1). We first assume that

(A1) \( A_0 := A_m \) with domain \( D(A_0) := \ker(G) \) is a generator of a \( C_0 \)–semigroup \( T_0 := (T_0(t))_{t \geq 0} \) on \( X_0 \).

On the other hand, we denote

\( F_0 := F \iota, \quad \iota : D(A_0) \to X_0 \quad \text{is the continuous injection}. \)

We now introduce the Banach product space

\( X := X_0 \times L^q(\mathbb{R}^+, X_0) \) with norm \( \| (x, f) \| := \| x \| + \| f \|_q \).

On the other hand, we consider the matrix operator

\( \mathfrak{A} := \begin{pmatrix} A_0 & \delta_0 \\ \Upsilon & \frac{d}{ds} \end{pmatrix}, \quad D(\mathfrak{A}) = D(A_0) \times D(\frac{d}{ds}), \)

where \( \Upsilon x = a(\cdot)Fx \) for \( x \in D(A_0) \). Moreover, we define the function \( \zeta : [0, +\infty) \to X \) by

\( \zeta(t) = \begin{pmatrix} f(t) \\ 0 \end{pmatrix}, \quad t \geq 0. \)

Now to solve the integro-differential equation (5.1), we just have to solve the equation

\[
\begin{aligned}
\dot{z}(t) &= \mathfrak{A} z(t) + \zeta(t), \quad t \geq 0, \\
z(0) &= \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\end{aligned}
\]

and then the solution \( \varrho \) of (5.1) is none other than the first component of the solution \( z \) (see [13]).

Let define the left shift semigroup on \( L^q(\mathbb{R}^+, X_0) \) by

\( (S(t)f)(s) = f(t + s), \quad t, s \geq 0. \)
We recall that the left shift semigroup $S := (S(t))_{t \geq 0}$ is not analytic on $L^q(\mathbb{R}^+, X_0)$, so that the evolution equation associated with this semigroup hasn’t the maximal $L^p$-regularity on $L^q(\mathbb{R}^+, X_0)$. Hence one cannot expect the maximal regularity for the problem (5.4) on the space $X$ as well even if we assume that $A_0$ has the maximal $L^p$-regularity on $X_0$. To overcome this obstacle we shall work in a small space of $X$ with respect to the second component. That is one looks for subspaces of $L^q(\mathbb{R}^+, X_0)$ in which the left shift semigroup $S := (S(t))_{t \geq 0}$ is analytic. This observation has already been used in [4], [5] to prove analyticity and maximal regularity of a particular class of Volterra equations.

Indeed, let $h : \mathbb{R}^+ \to \mathbb{R}^+$ be an admissible function, that is $h$ is an increasing and convex function such that $h(0) = 0$. On the other hand, define the sector $\Sigma_h := \{\sigma + i\tau \in \mathbb{C}; x > 0 | y| < h(\sigma)\}$.

For $q \in (1; \infty)$, we define the Bergman space of holomorphic $L^q$-integrable functions by:

$$B^q_{h,X_0} := B^q_0(\Sigma_h; X_0) := \left\{ f : \Sigma_h \to X_0 \text{ holomorphic} ; \int_{\Sigma_h} \|f(\tau + i\sigma)\|^q_{X_0} d\tau d\sigma < \infty \right\},$$

with

$$\|f\|_{B^q_{h,X_0}} := \left( \int_{\Sigma_h} \|f(\tau + i\sigma)\|^q_{X_0} d\tau d\sigma \right)^{\frac{1}{q}}.$$

The following result is a slight modification of the one proved in [3, lem.4.3].

**Lemma 5.1.** Let $s > 1$, $p > 1$ and $h$ be an admissible function such that:

$$\int_0^1 h(\sigma)^{1-s} d\sigma < +\infty.$$  

Then for every $q = \frac{ps}{s-1}$ and every $R > 0$ there exists $C > 0$ such that:

$$\int_0^R \|f(t)\|^p_{X_0} dt \leq C \|f\|^p_{B^q_{h,X_0}},$$

holds for every $f \in B^q_{h,X_0}$.

**Proof.** The proof of this lemma is similar to the proof of Lemma 4.3 in [4]. We estimate $\|f(r)\|^p$ using Cauchy’s formula and we follow the same steps. 

Hereafter, we will take $p > 1$, $h(\sigma) = \tan(\theta)\sigma$, $s \in (1,2)$ and $q = \frac{ps}{s-1}$. It is not hard to see that $h$ is an admissible function and $\Sigma_h$ is none other than $\Sigma_\theta$.

In the rest of this paragraph, instead of the space $X$, we will work on the following subspace

$$X^q := X_0 \times B^q_{h,X_0}, \quad \|(f)\|_{X^q} := \|x\|_{X_0} + \|f\|_{B^q_{h,X_0}}.$$  

On the space $B^q_{h,X_0}$, we define the complex derivative $\frac{d}{dz}$ with its natural domain:

$$D \left( \frac{d}{dz} \right) := \left\{ f \in B^q_{h,X_0} ; f' \in B^q_{h,X_0} \right\}.$$
It is shown in [4] that the operator \((\frac{d}{dz}, D(\frac{d}{dz}))\) generates an analytic semigroup of translation on the Bergman space \(B^q_{h,X_0}\) and has the maximal \(L^p\)-regularity on \(L^q(\mathbb{R}^+, X_0)\) whenever \(X_0\) is an UMD space.

Let us now discuss the maximal regularity for the matrix operator \(\mathfrak{A}\) with domain \(D(\mathfrak{A}) = D(\mathfrak{A}_0) \times D(\frac{d}{dz})\) on the space \(X^q\). We first split the operator \(\mathfrak{A}\) as \(\mathfrak{A} := A + P\), where

\[
A := \begin{pmatrix}
\mathfrak{A}_0 & 0 \\
0 & \frac{d}{dz}
\end{pmatrix}; \quad D(A) = D(\mathfrak{A}_0) \times D(\frac{d}{dz}),
\]

and

\[
P := \begin{pmatrix}
0 & \delta_0 \\
\Upsilon_0 & 0
\end{pmatrix}; \quad D(P) = D(\mathfrak{A}_0) \times D(\frac{d}{dz}).
\]

where \(\Upsilon_0 := \Upsilon|_{D(A_0)}\). Clearly, the operator \(A\) generates on \(X^q\) the following strongly continuous semigroup

\[
T(t) = \begin{pmatrix}
T_0(t) & 0 \\
0 & S(t)
\end{pmatrix}.
\]

The next result introduce conditions for which the operator \(P\) becomes an admissible observation operator for \(A\).

**Lemma 5.2.** Assume that \(a(\cdot) \in B^q_{h,C}\) and \(F_0 \in \mathcal{L}(D(\mathfrak{A}_0), X_0)\) is a \(p\)-admissible observation operator for \(\mathfrak{A}_0\). Then \(\Upsilon_0 \in \mathcal{L}(D(\mathfrak{A}_0), B^q_{h,X_0})\) is a \(p\)-admissible operator for \(\mathfrak{A}_0\) as well. In particular the operator \(P \in \mathcal{L}(D(A), X^q)\) is a \(p\)-admissible observation operator for \(A\).

**Proof.** Since \(F_0\) is \(p\)-admissible observation operator for \(\mathfrak{A}_0\), for \(\alpha > 0\), there exists \(\gamma > 0\) such that

\[
\int_0^\alpha \|T_0(t)x\|_{X_0}^p dt \leq \gamma^p \|x\|_{X_0}^p,
\]

for any \(x \in D(\mathfrak{A}_0)\). On the other hand, we have

\[
\int_0^\alpha \|\Upsilon_0 T_0(t)x\|_{B^q_{h,X_0}}^p dt = \int_0^\alpha \left( \int_{\Sigma_h} \|a(\tau + i\sigma)|F_0 T_0(t)x|_{X_0}^q d\tau d\sigma \right)^{\frac{p}{q}} dt \\
= \int_0^\alpha \left( \int_{\Sigma_h} |a(\tau + i\sigma)|d\tau d\sigma \right)^{\frac{p}{q}} \|F_0 T_0(t)x\|_{X_0}^p dt \\
\leq \left( \gamma \|a\|_{B^q_{h,C}} \right)^p \|x\|_{X_0}^p,
\]

for any \(x \in D(\mathfrak{A}_0)\). This shows that \(\Upsilon_0\) is \(p\)-admissible observation for \(\mathfrak{A}_0\). On the other hand, Lemma 5.1 implies that the operator \(\delta_0 : D(\frac{d}{dz}) \subset B^q_{h,X_0} \to X\) is \(p\)-admissible for \(\frac{d}{dz}\). Using the above expression of the semigroup \((T(t))_{t \geq 0}\) it is clear that \(P\) is a \(p\)-admissible observation operator for \(A\). \(\square\)

Now we state the main result of this section:
Theorem 5.3. Let $X_0$ be a UMD space, $s > 1$, and $h$ be an admissible function satisfying

$$\int_0^1 h(\sigma)^{1-s} d\sigma < +\infty.$$ 

Let $p > 1$ and set $q = \frac{ps}{s-1}$. Assume that $a(\cdot) \in B_{h,\mathbb{C}}^q$ and $F_0 \in \mathcal{L}(D(A_0), X_0)$ is a $p$-admissible observation operator for $A_0$. If $A_0$ has maximal $L^p$-regularity on $X_0$, then $A$ has the maximal $L^p$-regularity on $X^q$.

Proof. In [5], the author showed that $(\frac{d}{dt}, D(\frac{d}{dt}))$ has the maximal $L^p$-regularity on $B_{h,X_0}^q$. By assumptions, $A_0$ has maximal $L^p$-regularity on $X_0$, then it is easy to see that $A$ has maximal $L^p$-regularity on $X^q$. Since $P$ is $p$-admissible for $A$ and $A = A + P$, Theorem 4.8 guaranties that $\mathfrak{A}$ has the maximal $L^p$-regularity on $X^q$. □

Remark 5.4. With the notation in [5, Thm.3.3], we have $B(\cdot) = a(\cdot)F$, but with the concept of the admissibility of the operator $F$ and according to Lemma 4.10 and Remark 4.11, $B(\cdot)$ becomes small with respect to $A_0$ in the sense of (i) in [5, Thm.3.3]. Thus the result of [5] can be obtained for this class of Volterra equation. Further, the author in [5], proved that condition (ii) of [5, Thm.3.3] is also sufficient to obtain the required result. But its not clear for us how the author in [5, Thm.3.3] can use [23, Corollary 12] to obtain the result. However, under the condition (ii), the author in [5, Thm.3.3] can get the desired result by observing only that (ii) implies easily (i). Our result of Maximal $L^p$-regularity of free-boundary Volterra equation (5.1) is not very strong, since the author in [5] shows it for a class of the operator $F$ that are not necessarily admissible. However, we show that the next example cannot be examined by the work [5, Thm.3.3].

5.2. Boundary perturbation of an integro-differential equation: We are still working under the setting of the previous section and we are now studying the problem:

$$\begin{cases}
\dot{\varrho}(t) = A_m \varrho(t) + \int_0^t a(t-s)F \varrho(s) ds + f(t), & t \geq 0 \\
G \varrho(t) = K \varrho(t), & t \geq 0 \\
\varrho(0) = 0
\end{cases}$$

(5.5)

where $A_m : Z_0 \to X_0$ is a closed linear operator, $G, K : Z_0 \to U_0$ and $F : Z_0 \to X_0$ are linear operators. Moreover, we introduce the operator

$$\mathfrak{A} := A_m, \quad D(\mathfrak{A}) := \{ x \in Z_0 : Gx = Kx \}.$$

The problem (5.5) can now be written as

$$\begin{cases}
\dot{\varrho}(t) = \mathfrak{A} \varrho(t) + \int_0^t a(t-s)F \varrho(s) ds + f(t), & t \geq 0, \\
\varrho(0) = 0
\end{cases}$$

(5.6)

This integro-differential equation is similar to that investigated in the previous subsection. We then use the same notation of Bergman space and product spaces. On $X = X_0 \times B_{h,X_0}^q$, 32
let us define the matrix operator
\[ \mathcal{G} = \begin{pmatrix} A & \delta_0 \\ \Upsilon & \frac{d}{dz} \end{pmatrix}, \quad D(\mathcal{G}) := D(A) \times D\left(\frac{d}{dz}\right), \]
where \( \Upsilon x = a(\cdot)Fx \) for \( x \in X_0 \). As discussed in the previous subsection the maximal \( L^p \)-regularity of the integro-differential equation \((5.6)\) is reduced to look for conditions for which the operator \( \mathcal{G} \) is a generator on \( X^q \) and has the maximal \( L^p \)-regularity on \( X^q \).

We introduce the following assumptions:
- \((A1)\) \( G : Z_0 \to U_0 \) is surjective
- \((A2)\) \( A_0 := A_m \) with domain \( D(A_0) := \ker(G) \) is a generator of a \( C_0 \)-semigroup on \( X_0 \).

As discussed in Section 2, the assumptions \((A1)\) and \((A2)\) imply that the Dirichlet operator
\[ D_\lambda := \left( G_{|_{\ker(\lambda - A_m)}} \right)^{-1} \in \mathcal{L}(U_0, X_0), \quad \lambda \in \rho(A_0), \]
exists. Define then
\[ B_0 := (\lambda - A_{0,-1})D_\lambda \in \mathcal{L}(U_0, X_{0,-1}), \quad \lambda \in \rho(A_0), \]
\[ K_0 := K_{| D(A_0) } \in \mathcal{L}(D(A_0), U_0), \]
\[ F_0 := F_{| D(A_0) } \in \mathcal{L}(D(A_0), X_0). \]

We also need the following hypotheses
- \((A3)\) the triple operator \((A_0, B_0, K_0)\) generates a regular linear system on \( X_0, U_0, U_0 \) with the identity operator \( I_{U_0} : U_0 \to U_0 \) as an admissible feedback.
- \((A4)\) the triple operator \((A_0, B_0, F_0)\) generates a regular linear system on \( X_0, U_0, X_0 \).

The following result shows the generation property of the operator \((\mathcal{G}, D(\mathcal{G}))\).

**Proposition 5.5.** Let assumptions \((A1)\) to \((A4)\) be satisfied. Then the operator \((\mathcal{G}, D(\mathcal{G}))\) generates a strongly continuous semigroup on \( X^q \).

**Proof.** Assumptions \((A1)\) to \((A3)\) show that the operator \((A_0, D(A_0))\) generates a strongly continuous semigroup on \( X_0 \), see Theorem 3.1. On the other hand, if in addition we consider the assumption \((A4)\), then as in the proof of Theorem 3.3 (i), one can see that the operator \( F \in \mathcal{L}(D(A), X_0) \) is a \( p \)-admissible observation of \( A \). Hence the rest of the proof follows exactly in the same way as in the previous subsection. \hfill \Box

The main result of this subsection is the following.

**Theorem 5.6.** Let \( X_0, U_0 \) be UMD spaces, \( s > 1 \) a real number, and \( \lambda \) be an admissible function satisfying
\[ \int_0^1 h(x)^{1-s} dx < +\infty. \]
Let \( p > 1 \) and set \( q = \frac{np}{n-1} \). Suppose that \( a(\cdot) \in B^p_\infty \), assumptions \((A1)\) to \((A4)\) be satisfied, \( A_0 \) generates a bounded analytic semigroup and there exists \( \omega > \max\{\omega(A_0); \omega(A)\} \) such that the sets \( \{s^{\frac{1}{p}} R(\omega + is, A_{0,-1})B_0; s \neq 0\} \) and \( \{s^{\frac{1}{p}} K_0 R(\omega + is, A_0); s \neq 0\} \) are \( R \)-bounded with \( \frac{1}{p} + \frac{1}{p'} = 1 \). If \( A_0 \in \mathcal{M}(0, T; X_0) \) then \( \mathcal{G} \in \mathcal{M}(0, T; X^q) \).
Proof. According to Theorem 4.20, we have $A \in \mathcal{MR}(0, T; X_0)$. On the other hand as we have mentioned in the proof of Proposition 5.5 the operator $F$ is a $p$-admissible observation operator for $A$. We then follow the same technique as in the proof of Theorem 5.3 to conclude that $G \in \mathcal{MR}(0, T; X^q)$. □

Remark 5.7. When perturbing boundary conditions of the integro-differential Volterra equation (5.1), even if the operator $F$ is a small perturbation of $A_0$ which in turns implies that $B(\cdot) = a(\cdot)F$ satisfies the condition (i) of [5, Thm 3.3] with respect to $A_0$, $B$ do not necessarily satisfies the condition (i) [5, Thm 3.3] with respect to $A$. Hence, the result of [5, Thm 3.3] is not applicable. It is to be noted that if we assume that $A_0$ generates an analytic semigroup on $X_0$ and $F$ is $p$-admissible observation operator for $A_0$ (in particular it is small perturbation for $A_0$, due to Remark 4.11), then $F$ is $p$-admissible for $A$ (of course under the conditions (A3) and (A4)). So that $F$ can be considered as a small perturbation for $A$ due to Remark 4.11. Even with this one cannot use [5, Thm 3.3] to conclude that the integro-differential Volterra equation (5.6) has the maximal $L^p$-regularity, because for Barta [5] it is not clear that $A$ has the maximal $L^p$-regularity on $X$.

5.3. Example of a IDE with perturbed boundary conditions: Let $r, q \in (1, \infty)$ such that $\frac{1}{r} + \frac{1}{q} = 1$. In this section, we first deal with the following PDE:

$$
\begin{align*}
\frac{\partial}{\partial t} w(t, s) &= \frac{\partial^2}{\partial s^2} w(t, s) + f(t), \quad t \in [0, T], \ s \in (0, 1) \\
\frac{\partial}{\partial s} w(t, 1) &= w(t, 1) - w(t, 0), \\
\frac{\partial}{\partial s} w(t, 0) &= 0 \\
w(0, s) &= 0,
\end{align*}
$$

(5.7)

Now let $X_0 = L^r(0, 1)$ and $A_m f := f''$ with domain $Z = \{w \in W^{2,r}(0, 1), w'(0) = 0\}$. We define on $Z$ the operator:

$$
G : Z \to \mathbb{R} \\
f \mapsto f'(0).
$$

We also define the unbounded operator $K : Z \to \mathbb{R}$ by $K f := f(1) - f(0)$. One can see that under this setting, the problem (5.7) can be transformed to:

$$
\begin{align*}
\dot{x}(t) &= A_m x(t), \quad 0 \leq t \leq T, \ x(0) = 0 \\
G x(t) &= K x(t), \quad 0 \leq t \leq T.
\end{align*}
$$

We define the operator:

$$
A_0 = A_m, \quad D(A_0) = Ker G = \{w \in W^{2,r}(0, 1), w'(0) = 0 \text{ and } w'(1) = 0\}.
$$

It is well known that this operator has maximal regularity. We set:

$$
\mathbb{D}_\lambda := (G_{|\ker(\lambda - A_m)})
$$

and

$$
\mathbb{B}_0 := (\lambda - A_{0,-1})\mathbb{D}_\lambda
$$
and
\[ K_0 := K_{D(\mathbb{A}_0)} \]

Now the main result is the following theorem

**Theorem 5.8.** The operator \( \mathbb{A} \) defined by :
\[ \mathbb{A} f = f'', \quad D(\mathbb{A}) := \{ w \in W^{2,r}(0,1), w'(0) = 0 \text{ and } w'(1) = w(1) - w(0) \}, \]
has maximal \( L^p \)-regularity for every \( p > 1 \) and the problem \( (5.7) \) has unique solution \( w \in W^{1,p}(0,T;X) \cap L^p(0,T;D(\mathbb{A})) \) such that:
\[ \| w \|_{L^p([0,T];X)} + \| w' \|_{L^p([0,T];X)} + \| w'' \|_{L^p([0,T];X)} \leq C \| f \|_{L^p([0,T];X)}, \]
for some constant \( C > 0 \).

**Proof.** We will show that conditions of Theorem 4.20 are satisfied. According to [1], Theorem 2.4, to show that the triple \((\mathbb{A}_0, \mathbb{B}_0, \mathbb{K}_0)\) generates a regular linear system, all we have to show is that there exist \( \beta \in [0,1] \) and \( \gamma \in (0,1] \) such that :

(i) \( \text{range}(D(\lambda)) \subset F^{\mathbb{A}_0}_{1-\beta} \) (where \( F^{\mathbb{A}_0}_\alpha \) is the Favard space of \( \mathbb{A}_0 \) of order \( \alpha \))

(ii) \( D((\lambda - \mathbb{A}_0)\gamma) \subset Z \)

(iii) \( \beta + \gamma < 1 \).

Let prove these three propositions:

(i) By simple calculation, one can see that there exist \( \lambda_0 > 0 \) such that
\[ \sup_{\lambda > \lambda_0} \| \lambda^{\frac{r+1}{2r}} D(\lambda) \| < +\infty. \]

This fact implies (see [1], Lemma A.1) that \( \text{range}(D(\lambda)) \subset F^{\mathbb{A}_0}_{\frac{r+1}{2r}} \) for some \( \lambda \in \rho(\mathbb{A}_0) \). We take \( \beta = 1 - \frac{r+1}{2r} = \frac{r-1}{2r} \).

(ii) According to [1] Lemma A.2], we have to show that for \( \alpha \in (0,1) \) and every \( \rho > \rho_0 > 0 \) we have
\[ |K_0 f| \leq M(\rho^\alpha \| f \|_r + \rho^{\alpha-1} \| f'' \|_r), \quad f \in D(\mathbb{A}_0) \]
for some constant \( M > 0 \). For \( f \in D(\mathbb{A}_0) \) we know that
\[ f(1) = f(0) + \int_0^1 f'(s)ds, \]
then we can easily show that :
\[ |f(1) - f(0)| \leq \| f' \|_r. \]

By [13] Example III.2.2], we have for \( \epsilon > 0 \)
\[ \| f' \|_r \leq \frac{9}{\epsilon} \| f \|_r + \epsilon \| f'' \|_r. \]

By taking \( \rho = \epsilon^{-3}, \alpha = \frac{1}{3} \) and \( \gamma \in (\frac{1}{3}, \frac{1}{r}) \) (we can always take \( 1 < r < 3 \)), we show the assertion.
Clearly we have $\beta + \gamma < 1$

Hence, by [1, Theorem 2.4], the operators $\mathbb{B}_0$ and $\mathbb{K}_0$ are $p$-admissible for $\frac{2r}{r+1} < p < \frac{1}{\gamma}$ and the triple $(\mathbb{A}_0, \mathbb{B}_0, \mathbb{K}_0)$ generates a regular system and the identity is an admissible feedback. For instance we can take $p = 2$.

To show that the sets $\{s^{\frac{1}{2}}\mathbb{K}_0 R(is, \mathbb{A}_0); s \neq 0\}$ and $\{s^{\frac{1}{2}} R(is, \mathbb{A}_{0,-1}) \mathbb{B}_0; s \neq 0\}$ are $\mathcal{R}$-bounded, it is sufficient to show that $\mathbb{K}_0$ and $\mathbb{B}_0$ are $l$-admissible ($l$-admissibility is more general than admissibility, see [19] for definitions), namely, it is sufficient to show that the sets $\{s^{\frac{1}{2}}\mathbb{K}_0 R(s + 1, \mathbb{A}_0); s > 0\}$ and $\{s^{\frac{1}{2}} R(s + 1, \mathbb{A}_{0,-1}) \mathbb{B}_0; s > 0\}$ are $\mathcal{R}$-bounded (see [19, page 514]).

In order to prove that $\{s^{\frac{1}{2}}\mathbb{K}_0 R(s + 1, \mathbb{A}_0); s > 0\}$ is $\mathcal{R}$-bounded, we follow the example in [19, page 528] and the technique used there to show the $\mathcal{R}$-boundedness of the above set, namely it suffices to find a space $\tilde{Z}_0$ such that $D(\mathbb{A}_0) \subset \tilde{Z}_0 \subset X_0$ and $\mathbb{K}_0$ is bounded in $\| \cdot \|_{\tilde{Z}_0 \to \mathbb{R}}$ and the set $\{s^{\frac{1}{2}} R(s + 1, \mathbb{A}_0); s > 0\}$ is $\mathcal{R}$-bounded in $\mathcal{L}(\tilde{Z}_0, X)$, this holds, by the same example, for $\tilde{Z}_0 = W^{1,p}(0,1)$ since $\mathbb{K}_0$ is bounded in $\mathcal{L}(\tilde{Z}_0, \mathbb{R})$.

Now to show the $\mathcal{R}$-boundedness of $\{s^{\frac{1}{2}} R(s + 1, \mathbb{A}_{0,-1}) \mathbb{B}_0; s > 0\}$ we follow the same example, we show that the set $\{s^{\frac{1}{2}} \mathbb{B}_0^* R(s + 1, \mathbb{A}_0^*); s > 0\}$ is $\mathcal{R}$-bounded. Let us first determine $\mathbb{B}_0^*$, we proceed again as in [19], multiplying the first equation in (5.7) with a fixed $v \in C^\infty([0,1])$ and integrating by parts we obtain:

$$<w'(t, \cdot), v >_{(0,1)} = < w(t, \cdot), v'' >_{(0,1)} + Gw(t, \cdot)\delta_1 v,$$

this means that $\mathbb{B}_0^* = \delta_1$. By the same argument used to show that $\{s^{\frac{1}{2}}\mathbb{K}_0 R(s + 1, \mathbb{A}_0); s > 0\}$ is $\mathcal{R}$-bounded, we show that $\{s^{\frac{1}{2}} \mathbb{B}_0^* R(s + 1, \mathbb{A}_0^*); s > 0\}$ is $\mathcal{R}$-bounded.

All hypothesis of Theorem 4.20 are satisfied, and hence the operator $\mathbb{A}$ defined above has the maximal $L^2$-regularity, thus maximal $L^p$-regularity for every $p > 1$ and then problem (5.7) has unique solution $w \in W^{1,p}([0,T]; X_0) \cap L^p([0,T]; D(\mathbb{A}))$ such that:

$$\left\| \frac{\partial}{\partial t} w(\cdot, \cdot) \right\|_{L^p([0,T]; X_0)} + \left\| w(\cdot, \cdot) \right\|_{L^p([0,T]; X)} + \left\| \frac{\partial^2}{\partial s^2} w(\cdot, \cdot) \right\|_{L^p([0,T]; X_0)} \leq C \| f \|_{L^p([0,T]; X_0)},$$

for some constant $C > 0$.

Now, we can consider the following partial integro-differential equation (PIDE) governed by the heat equation

$$\begin{align*}
\frac{\partial}{\partial t} w(t,s) &= \frac{\partial^2}{\partial s^2} w(t,s) + \int_0^t a(t - \tau) Fu(\tau, s) d\tau + f(t), \quad t \in [0,T], \ s \in (0,1), \\
\frac{\partial}{\partial s} w(t,1) &= w(t,1) - w(t,0), \\
\frac{\partial}{\partial s} w(t,0) &= 0, \\
w(0,s) &= 0,
\end{align*}$$

(5.8)

where $F : Z_0 \to X_0$ is a linear operator and $a \in B^q_{h,C}$ (see notation above). We will use the same notation as in the beginning of this subsection. We then assume that $\mathbb{A}_0$ has the
maximal $L^p$-regularity. By applying Theorem 5.6 we can easily check maximal regularity of (5.8) if the triple $(A_0, B_0, F|_{D(A_0)})$ generates a regular linear system. Let $F = (-A_0)^\theta$ for some $\theta \in (0, \frac{1}{p})$. By Remark 4.7 the triple $(A_0, B_0, (-A_0)^\theta|_{D(A_0)})$ generates a regular linear system. Now, all assumptions of Theorem 5.6 are verified, the equation (5.8) has the maximal $L^p$-regularity.

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