Instantaneous Bethe-Salpeter Equation and Its Exact Solution

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We present an approach to solve a Bethe-Salpeter (BS) equation exactly without any approximation if the kernel of the BS equation is instantaneous, and take positronium as an example to illustrate the general features of the solutions. As a middle stage, a set of coupled and self-consistent integration equations for a few scalar functions can be equivalently derived from the BS equation always, which are solvable accurately. For positronium, precise corrections to those of the Schrödinger equation in order $v$ (relative velocity) in eigenfunctions, in order $v^2$ in eigenvalues, and the possible mixing, such as that between $S$ ($P$) and $D$ ($F$) components in $J^{PC} = 1^{-+}$ ($J^{PC} = 2^{++}$) states as well, are determined quantitatively. Moreover, we also point out that there is a problematic step in the classical derivation which was proposed first by E.E. Salpeter. Finally, we emphasize that for the effective theories (such as NRQED and NRQCD etc) we should pay great attention on the corrections indicated by the exact solutions.

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Bethe-Salpeter (BS) equation [1] is a very good tool to treat various bound state systems. For a fermion-antifermion binding system, the BS equation is written as follows,

$$(p_1 - m_1)\chi_P(q)(p_2 + m_2) = i \int \frac{d^4k}{(2\pi)^4} V(P, q, k)\chi_P(k),$$

(1)

where $\chi_P(q)$ is the BS wave function, $P$ is the total momentum, $q$ is relative momentum, and $V(P, q, k)$ is the so-called BS kernel between the electron and positron in the bound state, $p_1$, $p_2$ are the momenta of the constituent electron 1 and positron 2, respectively. The total momentum $P$ and the relative momentum $q$ are related to the momenta of the two quarks as follows: $p_1 = \alpha_1 P + q$, $\alpha_1 = \frac{m_1}{m_1 + m_2}$, $p_2 = \alpha_2 P - q$, $\alpha_2 = \frac{m_2}{m_1 + m_2}$.

If the kernel of the four-dimensional BS equation, $V(P, q, k)$, in $\vec{P} = 0$ frame (C.M.S) of the concerned bound state, has the behavior: $V(P, q, k)|_{\vec{P}=0} = \frac{1}{(q-k)^0}$, the BS equation is called as ‘instantaneous one’, and may be derived to a Schrödinger equation accordingly, that was firstly realized by E.E. Salpeter [2].

In Coulomb gauge, the transverse photon exchange is considered as higher order, so at the lowest order the BS equation for positronium has the kernel $V(P, q, k)|_{\text{positronium}} = \gamma^0 V_{\gamma^0} \gamma^0 = -\gamma^0 \frac{4m_1}{(q-k)^0} \gamma^0$ only, that is instantaneous, so we take positronium as an example to pursue exact solutions of the BS equation. It certainly is interesting that the relativistic corrections, including the possible mixing, such as that between $S$ and $D$ components in $J^{PC} = 1^{-+}$ state and that between $P$ and $F$ components in $J^{PC} = 2^{++}$ state etc, will be fully fixed by the instantaneous BS equation. Here we report that as the case of positronium, an instantaneous BS equation can be really solved without any approximation, and outline the approach briefly. We derive the instantaneous BS equation (the BS wave function in 4 spin structure) into a set of coupled and self-consistent integration equations for its components (scalar functions) without any approximation, and solve them numerically. The accuracy of the solutions can be contral at all. The results are discussed finally. While the details are put in Refs. 3, 4.

The approach which we show here mainly is to follow E.E. Salpeter derivation but without any approximation. As done in Ref. 2 and in book 3, the instantaneous BS wave function $\varphi_P(\vec{q})$ is introduce:

$$\varphi_P(q^0, \vec{q}) = \frac{i}{2\pi} \frac{\chi_P(q^0, q)}{q^0},$$

(2)

then the BS equation Eq(1) is re-written

$$\chi_P(q^0, \vec{q}) = S^{(1)}_f(p^0_1)\eta(\vec{q})S^{(2)}_f(-p^0_2),$$

(3)

where $S^{(1)}_f(p^0_1)$ and $S^{(2)}_f(-p^0_2)$ are the propagators of the fermion and anti-fermion respectively and the integrated ‘BS-nut’ $\eta(\vec{q}) = \int \frac{d^4q}{(2\pi)^4} V(\vec{q}, \vec{k})\varphi_P(\vec{k})$. For general applications, we keep $m_1 \neq m_2$ at this moment, although final application in this paper is to positronium $m_1 = m_2$. The propagators can be decomposed as:

$$-iJfS^{(f)}_j(Jpf) = \frac{\Lambda^+_j(q)}{2\eta_{-\omega_i} + \omega_i - i\epsilon} + \frac{\Lambda^-_j(q)}{2\eta_{-\omega_i} + \omega_i + i\epsilon},$$

(4)

with $\omega_i = \sqrt{m_i^2 + q^2}$, $\Lambda^+_j(q) = \frac{1}{2\omega_1} \left[ \gamma^0 \omega_i \pm J(m_i + \gamma \cdot \vec{q}) \right]$, where $J = 1$ for the quark $(i = 1)$ and $J = -1$ for

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with $\omega_{1,2} = \sqrt{m_1^2,2 + q^2}$. When $m_1 = m_2$ i.e. $\omega_1 = \omega_2$, we may obtain Eq. (10) from Eq. (12) unless $M = 0$ in the extremely relativistic cases, thus in the cases $m_1 = m_2$, equations Eq. (12) are equivalent to Eq. (10). Whereas when $m_1 \neq m_2$ i.e. $\omega_1 \neq \omega_2$, the equations Eq. (12) have the ‘trivial solutions’ just as Eq. (10), but also have ‘non-trivial solutions’. Practically when solving equations, such as Eq. (11), one would not check the projections $\Lambda{i}^{(q)}_{i}\gamma^0 \circ \gamma^0 \Lambda{2}^i_{2}(q)$, $\Lambda{i}^{(q)}_{i}\gamma^0 \circ \gamma^0 \Lambda{2}^i_{2}(q)$ and $\Lambda{i}^{(q)}_{i}\gamma^0 \circ \gamma^0 \Lambda{2}^i_{2}(q)$ on the solutions precisely, and generally non-trivial solutions are chosen. Especially, when further approximations are made, certain misleading may occur i.e. the obtained solutions may not satisfy the equations Eq. (10), even in the cases $m_1 = m_2$. Therefore, we conclude that Eq. (12) are not fully equivalent to Eq. (10).

We will present the different consequences due to Eq. (10) and Eq. (12) more precisely and solve them respectively in Ref. 8.

In the literature the authors of Refs. 2, 3 etc made an additional assumption on the spin structure of the BS wave function e.g. the spin structure of the wave functions for $S$-wave positronium has the formulation (in C.M.S. $\vec{P} = 0$):

$$\varphi_{0-}(lS_0)(\vec{q}) = \left[\gamma_0 + 1\right] \gamma f(\vec{q}) (13)$$

for $JPC = 0^+ -(lS_0)$; and

$$\varphi_{1-}(lS_1,\lambda)(\vec{q}) = \left[\gamma_0 + 1\right] \Lambda f(\vec{q}) (14)$$

for $JPC = 1^- -(3S_1)$. Here $\Lambda \equiv \epsilon_1^0 \gamma^\alpha \epsilon_1^0 \gamma^\beta - \epsilon_1^0 \gamma^\Delta$. With the spin structure finally the bound state problem turned to solve a Schrödinger equation accordingly by taking the so-called positive-energy equation Eq. (8) only (the rests are ignored).

To solve the instantaneous BS equation Eq. (8) for positronium without any approximation, even in the case $m_1 = m_2$ we ‘return’ to solve the Eqs. (8) only. In fact, the approach here is also applicable in the cases $m_1 \neq m_2$.

In $\vec{P} = 0$ frame (C.M.S.), the most general formulation of the BS wave function for the bound state $JPC = 0^- (1S_0)$ may be written as the follows:

$$\varphi_{1S_0}(\vec{q}) = \gamma^0 \gamma^5 \varphi_1(|q|) + \gamma^5 \varphi_2(|q|) + \sqrt{\frac{4\pi}{3}} \frac{|q|}{2m} E\varphi_3(|q|) \gamma^0 + \sqrt{\frac{4\pi}{3}} \frac{|q|}{2m} E\varphi_4(|q|) (15)$$

where $E \equiv \left[Y_{l-1}\gamma^+ + Y_{l1}\gamma^- - Y_{10}\gamma^3\right]$, $\gamma^+ \equiv -\frac{Y_{1+} + Y_{1-}}{\sqrt{2}}$, $\gamma^- \equiv \frac{Y_{1+} - Y_{1-}}{\sqrt{2}}$ and $Y_{lm} \equiv Y_{lm}(\theta_q, \phi_q)$ are spherical harmonics. To apply the equation Eq. (10) to Eq. (15), we obtain the ‘constraints’ for the components of the wave function precisely:

$$\varphi_1(|q|) = -\frac{1}{2} \varphi_4(|q|) \quad \varphi_3(|q|) = 0 \quad (16)$$
The expansion coefficients $C_{i,n\ell}^{(j)}$ for the $J^{PC} = 0^{-+}$ states
are calculated numerically by expanding the components of the wave function into Legendre polynomials. The first three eigenvalues are given by $E^{(1)} = -6.8026952534$, $E^{(2)} = -1.70069524809$, and $E^{(3)} = -0.75586715480$, and the corresponding eigenfunctions

$$f_i^{(j)}(\vec{q}) = \sum_{n\ell} C_{i,n\ell}^{(j)} R_{n\ell}(\vec{q}) ,$$

where $|\vec{q}|$ denotes the magnitude of the momentum. The coefficients $C_{i,n\ell}^{(j)}$ for $n \leq 4S$ are given in Table I.

For $J^{PC} = 0^{-+}$, the wave function can be simplified to

$$\varphi_{1S_0}(\vec{q}) = \left( \frac{2\sqrt{6}}{5} C_\lambda + A_\lambda \right) f_1 + \left( -2\sqrt{6} D_\lambda \varphi_1^\gamma + A_\varphi \gamma \right) f_2 + \left( -2\sqrt{6} C_\varphi + B_\varphi \right) f_3 + \left( \sqrt{6} D_\varphi \varphi_1^\gamma - B_\varphi \gamma \right) f_4 ,$$

where $A_\lambda$ is the same as in Eq. (14), and $B_\lambda \equiv \sqrt{6} \varphi_1^\gamma Y_{2-2} - \sqrt{3} \left( \varphi_1^\gamma + \varphi_1^\lambda \right) Y_{2-1} - \sqrt{6} \left( \varphi_1^\gamma - \varphi_1^\lambda \right) Y_{2-2}$, $C_\lambda \equiv \frac{\sqrt{5}}{2} \left( \varphi_1^\gamma - \varphi_1^\lambda \right)$, and $D_\lambda \equiv \frac{\sqrt{5}}{2} \left( \varphi_1^\gamma - \varphi_1^\lambda \right) Y_{1-1} + \left( \varphi_1^\gamma - \varphi_1^\lambda \right) Y_{1-2}$.

The final results for the eigenvalues are

$$E^{(1)} = -6.8026952534; \quad E^{(2)} = -1.70069524809; \quad E^{(3)} = -0.75586715480.$$
\[(M - 2m)f_4 = \frac{1}{3} \frac{2m^2 + 4v^2}{m} f_3 - 2mf_4 - \frac{4\sqrt{2}}{3\sqrt{5}} \frac{m^2}{m} f_1 \]
\[-\frac{2}{3} \int \frac{d^3 k}{m^2} \left[ \left( \frac{m}{\pi} \right)^2 Q_2 + 2 \frac{\sqrt{2}}{3\sqrt{5}} \frac{m}{m} Q_1 \right] f_3 \]
\[-\frac{2\sqrt{2}}{3\sqrt{5}} \frac{m}{m} Q_1 f_1. \quad (22)\]

Four coupled equations for four independent components \(f_{1,2,3,4}\) is the request of the \(S-D\) wave mixing and a self-consistent problem is reached.

We solve the Eqs.\( (22)\) numerically with the same method as adopted in the case for \(0^-\), and obtain the results:

\[E^{(1)}(3S_1) = -6.8027275; \quad E^{(2)}(3S_1) = -1.707008; \quad E^{(3)}(3S_1) = -0.7558690; \quad E^{(4)}(3D_1) = -0.7558729; \]
\[\cdots \quad (23)\]

for eigenvalues, while the table of the coefficients \(C_{i,n_l}\) for the eigenfunctions is too big to present here. Instead, we present them here only up to the order \(v\) accuracy as in the case for \(0^-\).

For \(J^{PC} = 1^- (3S_1)\) states we can simplify: \(f_1(\vec{q}) \simeq -f_2(\vec{q}) \equiv f = R_{n_l=0}(\vec{q})\) and \(f_3(\vec{q}) \simeq f_4(\vec{q}) \simeq 0\), i.e.

\[\varphi_{3S_1,\lambda}(\vec{q}) \simeq \left[ \left( \frac{2\sqrt{2}}{3} m \right) C_{\lambda} + A_{\lambda} \right] - \left( -2 \frac{2\sqrt{2}}{3} m D_{\lambda} \gamma^5 + A_{\lambda} \gamma^0 \right) \right] f. \quad (24)\]

For \(J^{PC} = 1^- (3D_1)\) states we can simplify: \(f_1(\vec{q}) \simeq f_2(\vec{q}) \simeq 0\) and \(f_3(\vec{q}) \simeq f_4(\vec{q}) \equiv f = R_{n_l=2}(\vec{q})\), i.e.

\[\varphi_{3D_1,\lambda}(\vec{q}) \simeq \left[ \left( -2 \sqrt{2} \frac{m \vec{q}}{3} m \right) C_{\lambda} + B_{\lambda} \right] + \left( \sqrt{2} \frac{m \vec{q}}{3} m D_{\lambda} \gamma^5 - B_{\lambda} \gamma^0 \right) \right] f. \quad (25)\]

The accurate coefficients \(C^{(j)}\) for \(J^{PC} = 1^- \) can be found in our achieved paper \[4\].

We may see that the exact solutions obtained here for the wave function contain the order \(v = \frac{\vec{q}}{m}\) corrections to the approximate ones obtained by E.E. Salpeter approaches (described as above), namely, the terms in Eqs.\( [23, 24]\) explicitly proportional to \(v\). The corrections will cause the normalization for the wave function a change in order \(v^2\) (the detail in \[6\]). The most interesting fact is the exact solutions for \(1^-\) states present the possible \(S - D\) wave mixing. To compare the eigenvalues Eq.\( (18)\) for \(0^-\) and Eq.\( (23)\) for \(1^-\) with those of the Schrödinger solution (the degenerate ones without any splitting), we may see the ‘hyperfine’ splitting for spin singlet and triplet states which is in order of \(\alpha^2 \) (\(v \equiv \alpha\)), and there is a remarkable splitting between \(n \{^3S_1\}\) and \(n \{^3D_1\}(n \geq 3)\) for \(1^-\) states \(E^{(3)}(3S_1) \geq E^{(3)}_3 > E^{(4)}(3D_1)\) and \(\Delta E \simeq \alpha^2\) due to the fact that the BS equation Eq.\( (3)\) is relativistic.

Note that the approach we present here may be applicable for all the instantaneous BS equations and can obtain the exact solutions, while the features of the exact solutions for positronium described above may be different for an instantaneous BS equation with a different kernel from that of Coulomb interaction.

Based on a specific problem of positronium, we have learnt that the wave functions contain quite great relativistic corrections thus for effective theories, such as NRQED \[10\] (and NRQCD \[11\]), the relativistic corrections from the wave functions should be considered carefully if the relevant calculations declare the relativistic effects are taken into account. For NRQED (NRQCD) etc, the wave function effects may be involved either at matching the underlined theory to the effective one or into the non-pertubative matrix elements. Moreover the problems, such as gauge invariance for the matrix elements and \(E1\) (electrical dipole) radiative transitions etc, are sensitive to the effects, the exact solutions are crucial \[6\].

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