THE DEGREE OF THE DIVISOR OF JUMPING RATIONAL CURVES

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Abstract

For a semistable reflexive sheaf $E$ of rank $r$ and $c_1 = a$ on $\mathbb{P}^n$ and an integer $d$ such that $r \mid ad$, we give sufficient conditions so that the restriction of $E$ on a generic rational curve of degree $d$ is balanced, i.e. a twist of the trivial bundle (for instance, if $E$ has balanced restriction on a generic line, or $r = 2$ or $E$ is an exterior power of the tangent bundle). Assuming this, we give a formula for the 'virtual degree', interpreted enumeratively, of the (codimension-1) locus of rational curves of degree $d$ on which the restriction of $E$ is not balanced, generalizing a classical formula due to Barth for the degree of the divisor of jumping lines of a semistable rank-2 bundle. This amounts to computing a certain determinant line bundle associated to $E$ on a parameter space for rational curves, and is closely related to the 'quantum K-theory' of projective space.

INTRODUCTION

Let $E$ be a reflexive sheaf (or vector bundle) of rank $r$ on a projective space $\mathbb{P}^n$, $n \geq 2$. By a theorem of Grothendieck, the pullback $E_C$ of $E$ to the nonsingular model of any rational curve in $\mathbb{P}^n$ can be decomposed as

$$E_C = \bigoplus_{i=1}^{r} \mathcal{O}(k_i), \quad k_1 \geq \ldots \geq k_r,$$

where the sequence $(k_1, \ldots, k_r)$ is uniquely determined and called the splitting type of $E$ on $C$. By semi-continuity, splitting type determines a natural locally closed stratification of any parameter space for rational curves, which is an important aspect of the geometry of $E$.

For $E$ semistable of rank $r = 2$, a well-known theorem of Grauert-M"ulich [OSS] says that if $r = 2$, the splitting type of $E$ on a generic line $L \subset \mathbb{P}^n$ is either $(k, k)$ ($c_1(E)$ even) or $(k, k-1)$ ($c_1(E)$ odd). One of the most important geometric objects associated to $E$ of rank 2 is the locus of jumping lines, i.e. the locus of lines $L$ such that $E_L$ is not of the above generic form. This locus is especially important in case $c_1(E)$ is even, when it is actually a divisor. A well-known formula due originally to Barth computes the degree of the divisor of jumping lines (with a suitable scheme structure).

The purpose of this paper is to generalize Barth’s formula to the case of bundles of arbitrary rank $r$ and rational curves of any degree $d$ such that

$$r \mid dc_1(E),$$

essentially under the condition that the restriction of $E$ on a generic rational curve $C$ of degree $d$ is balanced, i.e. has splitting type $(k^r)$, so that the locus of rational
curves $C$ for which $E_C$ is unbalanced is of pure codimension 1 and may be called the divisor of jumping rational curves of degree $d$ of $E$.

Now the balancedness condition is not satisfied for all semistable sheaves $E$ even for lines. However, we will introduce below a condition (condition AB), which states that the restriction of $E$ on a generic line $L$ is 'almost balanced', i.e. has splitting type $(k^s, (k - 1)^{r - s})$ for some $k, s$, (so if $r | c_1(E)$ then $E_L$ is actually balanced). Assuming this, we will then introduce a certain transversality condition T and show that with this condition the restriction $E_C$ to a generic rational curve of any degree $d$ is almost balanced, so that under (0.1) a divisor of jumping rational curves of degree $d$ may be defined. We will also show that conditions AB and T are satisfied whenever either $r | c_1(E)$ and AB holds, or $r = 2$ or $E$ is an exterior power of the tangent bundle.

Our main result (Theorem 3.1), which assumes conditions AB and T, computes the 'degree' of the divisor of jumping rational curves, which is interpreted as the weighted number of these curves incident to a generic collection of linear spaces (assuming of course that the total codimension of the incidence conditions equals the dimension of the divisor). More precisely, our formula expresses this degree in terms of some other enumerative invariants which have been computed before, e.g. in [P], [R1],[R2],[R3].

Note that even a homogeneous bundle like the tangent bundle in general admits jumping rational curves of degree $d > n$. In fact, the divisors of jumping rational curves associated to homogeneous vector bundles are an interesting class of projectively invariant divisors on any parameter space of rational curves (be it the Chow variety considered here, or Kontsevich's space or whatever).

This note was motivated by a talk by Givental [G] on his 'quantum K-theory', still under construction. This theory seeks to compute expressions of the form

$$\chi(M_{0,0}(\mathbb{P}^n, d), ev^*(E_1 \boxtimes ... \boxtimes E_m)),$$

where $M_{0,0}(\mathbb{P}^n, d)$ is Kontsevich's space of stable $m$-pointed rational curves of degree $d$ and $ev : M_{0,m}(\mathbb{P}^n, d) \rightarrow (\mathbb{P}^n)^m$ is the evaluation map. As will emerge below, our formula essentially amounts to a computation of

$$\chi(M_{0,1}(\mathbb{P}^n, d), ev^*(E)),$$

where $M_{0,1}(\mathbb{P}^n, d)$ is the normalization of a 1-dimensional subvariety of $M_{0,1}(\mathbb{P}^n, d)$ defined by incidence to a generic collection $(A_i)$ of linear spaces. By the Riemann-Roch formula, the latter may also be identified as an intersection number in $M_{0,m}(\mathbb{P}^n, d)$; for instance in case all the $A_i$ are points, hence have trivial normal bundle, it equals

$$\chi(M_{0,m}(\mathbb{P}^n, d), ev_1^*(E), ev_2^*(h^n) \cup ... \cup ev_m^*(h^n)),$$

where $h$ is the hyperplane class, and in general the 'relative' Euler characteristic may be defined by

$$\chi(X, E, b) = \int_X ch(E) \cup td(T_X) \cup b,$$

and coincides with the Euler characteristic of the restriction of $E$ on any smooth subvariety $B$ with fundamental class $b$ and trivial normal bundle (which may or may not exist); in our case $B = (ev_2 \times ... \times ev_m)^{-1}(pt.)$ has trivial normal bundle.
Though our formula is apparently new and independent of any quantum methods, it might in principle become accessible by Givental’s methods at some point. Indeed these methods, unlike ours, might yield results when the jumping locus has codimension > 1.

The paper is organised as follows. In Sect.1 we study the Chow compactification of the family of rational space curves. It does not seem to be generally known that this compactification is well-behaved at least in codimension 1 (and perhaps in codimension 2 as well, as long as multiple components don’t appear), so we have given a self-contained treatment here. The results come out as expected. In Sect.2 we study qualitatively the restriction of a bundle on a projective space to rational curves, focusing on criteria to ensure that the restriction on a generic rational curve is balanced. Our enumerative formula is given in Sect.3. The proof is based, not surprisingly, on the Riemann-Roch formula.

1. RATIONAL CURVES

Here we review some qualitative results about families of rational curves in $\mathbb{P}^n$. See also [R1][R2][R3] and references therein for details and proofs. In what follows we fix $n \geq 2$ and denote by $\bar{V}_d$ or $\bar{V}_{d,n}$ the closure in the Chow variety of the locus of irreducible nodal (if $n = 2$) or nonsingular (if $n > 2$) rational curves of degree $d$ in $\mathbb{P}^n$, with the scheme structure as closure, i.e. the reduced structure (recall that the Chow form of a reduced 1-cycle $Z$ is just the hypersurface in $G(n-2, \mathbb{P}^n)$ consisting of all linear spaces meeting $Z$). Thus $\bar{V}_d$ is irreducible reduced of dimension $(n+1)d + n - 3$. Let $A_1, \ldots, A_k$ be a generic collection of linear subspaces of respective codimensions $a_1, \ldots, a_k$, $2 \leq a_i \leq n$ in $\mathbb{P}^n$. We denote by

$$B = B_d = B_d(a.) = B_d(A.)$$

the normalization of the locus (with reduced structure)

$$\{(C, P_1, \ldots, P_k) : C \in \bar{V}_d, P_i \in C \cap A_i, i = 1, \ldots, k\}$$

which is also the normalization of its projection to $\bar{V}_d$, i.e. the locus of degree-$d$ rational curves (and their specializations) meeting $A_1, \ldots, A_k$. We have

$$\dim B = (n+1)d + (n-3) - \sum (a_i - 1).$$

When $\dim B = 0$ we set

$$N_d(a.) = \deg(B).$$

When $n = 2$, all $a_i = 2$ so the $a_i$’s may be suppressed. For $n > 2$, $k$ is called the length of the condition-vector $(a.)$. The numbers $N_d(a.)$, first computed in general by Kontsevich and Manin, are computed in [R2],[R3] by an elementary method based on recursion on $d$ and $k$.

Now suppose $\dim B = 1$ and let

$$\pi : X \to B$$

be the normalization of the tautological family of rational curves, and $f : X \to \mathbb{P}^n$ the natural map. The following summarizes results from [R2],[R3] (proved mostly in the references therein):
Theorem 1.1. (i) $X$ is smooth.

(ii) Each fibre $C$ of $\pi$ is either
(a) a $\mathbb{P}^1$ on which $f$ is either an immersion with at most one exception which maps to a cusp ($n = 2$) or an embedding ($n > 2$); or
(b) a pair of $\mathbb{P}^1$’s meeting transversely once, on which $f$ is an immersion with nodal image ($n = 2$) or an embedding ($n > 2$); or
(c) if $n = 3$, a $\mathbb{P}^1$ on which $f$ is a degree-1 immersion such that $f(\mathbb{P}^1)$ has a unique singular point which is an ordinary node.

(iii) If $n > 2$ then $\tilde{V}_{d,n}$ is smooth along the image $\tilde{B}$ of $B$, and $\tilde{B}$ is smooth except, in case some $a_i = 2$, for ordinary nodes corresponding to curves meeting some $A_i$ of codimension 2 twice. If $n = 2$ then $\tilde{V}_{d,n}$ is smooth in codimension 1 except for a cusp along the cuspidal locus and normal crossings along the reducible locus, and $\tilde{B}$ has the singularities induced from $\tilde{V}_{d,n}$ plus ordinary nodes corresponding to curves with a node at some $A_i$, and no other singularities.

The basic idea is that one can first get a handle on what curves occur in $\tilde{B}$ by the standard technique of semistable reduction (actually, normalization is sufficient) plus dimension counting (as, e.g. in Harris’ work on the Severi problem); then doing the deformation theory for the curves which do occur is easy enough. For the convenience of the reader we will give a complete proof for $n > 2$.

Let $H_d$ denote the (scheme-theoretic) closure in the Hilbert of the family of nonsingular rational curves of degree $d$, and $H^0_d \subset H_d$ be the open subset of reduced curves with normal crossings, which is well known to be smooth (see also below). Let

$$
\pi_d : X_d \to \tilde{V}_d, \quad \pi_{H_d} : X_{H_d} \to H_d
$$

be the universal cycle (resp. universal curve). There is a natural morphism (cf. [K])

$$
c : H_d \to \tilde{V}_d
$$

which assigns to a curve $C$ its Chow form, which is the divisor on the Grassmannian $G(n-2, \mathbb{P}^n)$ consisting of subspaces meeting $C$, with multiplicities if $C$ is not generically reduced. Thus $\tilde{V}_d$ is naturally embedded in a projective space parametrizing a suitable linear system on the Grassmannian. Clearly $c$ is one to one on the subset $H^1_d \subset H_d$ consisting of reduced curves. In fact, more is true:

**Lemma 1.2.** $c$ is unramified at any reduced subscheme.

**proof.** Let $C$ be reduced and pick a nonzero $v \in T_{[C]}H_d$. Let $p \in C$ be a general point where $v_p \neq 0$, and let $L$ be a general $(n-2)$–plane through $p$, which is also a general point in some component of $c(C)$. Let $\tilde{L}$ be the lift of $L$ to $\mathbb{C}^{n+1}$, and $(\tilde{p}) \subset (\tilde{L})$ the 1-dimensional subspace lifting $p$. Then the tangent space to the Grassmannian at $L$ may be identified with

$$
\text{Hom}(\tilde{L}, \mathbb{C}^{n+1}/\tilde{L}),
$$

while the tangent space to the divisor $c(C)$ is

$$
\{ \phi \in \text{Hom}(\tilde{L}, \mathbb{C}^{n+1}/\tilde{L}) : \phi(\tilde{p}) \equiv 0 \mod T_pC \}.
$$
Choosing $L$ general through $p$, we can arrange that
\[ v_p \not\in \langle T_p C, T_p L \rangle, \]
and it follows that as $C$ moves infinitesimally according to $v$, we can move $L$ preserving incidence to $C$ and going outside of $c(C)$, so $d_{C'}(v) \neq 0$. Note that the Lemma and the proof are valid for pure-dimensional subschemes of any dimension. □

Now the basic codimension-1 dimension counting result for $\bar{V}_d$ is the following

**Proposition 1.3.** Let $W \subset \bar{V}_d$ be any codimension-1 subvariety and $[C] \in W$ a general curve. Then $C$ is either

(i) a smooth embedded $\mathbb{P}^1$; or

(ii) a pair of smooth embedded $\mathbb{P}^1$’s meeting transversely at one point; or

(iii) only if $n = 3$, an irreducible immersed rational curve with one normal crossing.

Moreover $\bar{V}_d$ is smooth at $[C]$ in each case and has tangent space $H^0((I_C/I_C^2)^*)$ in case (i) or (ii), or $H^0(N_f)$, where $f : \mathbb{P}^1 \to C$ is the normalization, in cases (i) or (iii).

**proof.** We will use the following variant of Kleiman transversality:

**Lemma 1.4.** Let $\{Z_s : s \in S\}$ be a family of $k$-cycles in $\mathbb{P}^n$ which is $PGL_n$-equivariant (i.e. $S$ is $PGL_n$-invariant and $gZ_s = Z_{gs}$), and let $U \subset \mathbb{P}^n$ be a (purely) codimension-$c$ subvariety, $c > k$. Then the locus $S_U := \{s \in S : Z_s \cap U \neq \emptyset\}$ is of codimension $c - k$ in $S$.

**proof.** Take any $s \in S$ and any $(c - k - 1)$- dimensional subvariety $Q \subset PGL_n$. Kleiman transversality says that

\[ \left( \bigcup_{g \in Q} gZ_s \right) \cap g_0 U = \emptyset \]

for general $g_0 \in PGL_n$, hence

\[ \left( \bigcup_{g \in Q} g_0^{-1} gZ_s \right) \cap U = \emptyset. \]

This easily implies that the intersection of $S_U$ with the $PGL_n$-orbit of $Z_s$ is of codimension $c - k$. Thus $S_U$ meets every orbit in codimension $c - k$, and it follows easily that $S_U$ is of codimension $c - k$. □

We now return to the proof of the Proposition. If $W$ fails to be $PGL_n$-invariant, then its general element $[C]$ is general in $\bar{V}_d$, hence smooth. Hence we may assume $W$ is $PGL_n$-invariant. By semistable reduction, there exists a family

\[ Y \to T \]
with general fibre $\mathbb{P}^1$ and special fibre

$$Y_0 = \bigcup Y_{0,i}$$

with normal crossings, and with a surjective map $h : Y_0 \to C$. Set

$$h_i = h|_{Y_{0,i}}, h_i : [Y_{0,i}] = m_i C_i \text{ or } 0, d_i = \deg(C_i), k = \# \{i : m_i > 0\}.$$  

We may assume $m_1 > 0$ and that $C_1$ is non-disconnecting (i.e. $\bigcup_{i>1} C_i$ is connected). Then from Lemma 1.4 it follows that

$$(n + 1)d + n - 4 = \dim W = \dim\{C\} \leq \dim\{C_1\} + \dim\{\bigcup_{i>1} C_i\} - n + 2$$

$$\leq (n + 1)d_1 + n - 3 + \dim\{\bigcup_{i>1} C_i\} - n + 2 \leq \ldots$$

$$\leq (n + 1) \sum d_i + k(n - 3) - (k - 1)(n - 2) = (n + 1) \sum d_i + n - 3 - (k - 1).$$  

It follows at once that $\sum d_i = d$, so that all nonzero $m_i$ are equal to 1, and that $k = 1$ or 2, and in the latter case all inequalities above are equalities.

Now suppose $k = 2$, so

$$C = C_1 \cup_p C_2.$$  

Then $(C_1, C_2)$ must be a general point in the locus of intersecting curves in $V_{d_1} \times V_{d_2}$, which has codimension $n - 2$, and it follows easily that $C_1$ and $C_2$ are smooth; moreover as the stabiliser in $PGL_n$ of a point $p \in \mathbb{P}^n$ acts transitively on $\mathbb{P}^n - p$ and on $T_p \mathbb{P}^n$, it follows easily that $C_1$ and $C_2$ meet transversely once. Now as $C$ is a locally complete intersection, the tangent space to $H_d$ at $[C]$ is given by

$$H^0(N), \ N = (I_C/I_C^2)^*.$$  

On the other hand the tangent space to the locally trivial deformations is given by

$$H^0(N'),$$

where $N'$ is the kernel of a map

$$N \to \mathbb{C}_p$$

to a skyscraper sheaf at $p$, which assigns to a deformation of $C$ the corresponding deformation of the germ $(C, p)$, which is just an ordinary node (cf. [S]). Note that $N'$ may also be identified as the kernel of the natural map

$$N_{C_1} \oplus N_{C_2} \to T_p \mathbb{P}^n/(T_p C_1 + T_p C_2),$$

$$N' = \{(v_1, v_2) \mapsto v_{1,p} - v_{2,p}\},$$

hence clearly $H^1(N') = 0$. It follows easily as in [S] that the germ of $H_d$ at $[C]$ is smooth and maps unramifiedly to the deformation space of $(C, p)$. Since the total space of the versal deformation of $(C, p)$ is just given by $xy = t$, it is smooth, and it follows that the universal curve $X_d$ is smooth along its fibre over $[C]$.  


It remains to consider the case where $C$ is irreducible of degree $d$, hence given by projecting a rational normal curve

$$C_d \subset \mathbb{P}^d$$

from a center

$$M = \mathbb{P}^{d-n-1}, \ M \cap C_d = \emptyset.$$ 

Recall that an arbitrary length-$k$ subscheme of $C_d$ spans a $\mathbb{P}^{k-1}$, so if $C$ is not embedded then for any $(C_d, M)$ such that $C = \text{proj}_M(C_d)$, $M$ must meet some $k$-secant $\mathbb{P}^{k-1}$ to $C_d$ in a $\mathbb{P}^{k-2}$, $k \geq 2$. By a straightforward dimension count, the family of pairs $(C_d, M)$ with the latter property, for any fixed $k$, is of codimension

$$1 + (k - 1)(n - 2)$$

and, under the mapping

$$(C_d, M) \mapsto \text{proj}_M(C_d)$$

maps to a subfamily of the same codimension in $\tilde{V}_d$. Since $n > 2$, this codimension could equal 1 only if $n = 3$, $k = 2$, hence the set of $C$’s which are not embedded has codimension 1 only if $n = 3$. We can see similarly that if $n = 3$ a general nonembedded $C$ has one normal crossing and corresponds to $M$ meeting the secant variety of $C_d$ in one general point.

Now for any irreducible $C$ of degree $d \geq 3$, pick general points

$$p_1, p_2, p_3 \in C$$

and transverse hyperplanes $H_i \ni p_i$, and let

$$f : \mathbb{P}^1 \to C$$

be the normalisation and set $p'_i = f^{-1}(p_i)$. Consider a space $D$ parametrising deformations $f'$ of $f$ so that

$$f'(p'_i) \in H_i.$$ 

Then clearly

$$T_f D = H^0(T')$$

where

$$T' \subset f^*(T_{\mathbb{P}^n})$$

is the (full-rank) subsheaf of vector fields tangent to $H_i$ at $p'_i$, $i = 1, 2, 3$. As

$$f^*(T_{\mathbb{P}^n})(-p'_1 - p'_2 - p'_3) \subset T',$$

clearly $H^1(T') = 0$, hence $D$ is smooth at $f$. Moreover the natural map

$$D \to \tilde{V}_d$$

$$f' \mapsto f'(\mathbb{P}^1)$$
is clearly one to one and is unramified at \( f \) by an evident variant of Lemma 1.2. Therefore \( \tilde{V}_d \) is smooth at \( [C] \) (which clearly implies that the normalization of the total space of the universal cycle, which has a smooth fibre over \([C]\), is itself smooth along this fibre). Also, it is clear that the natural map \( T' \to N_f \) induced an isomorphism

\[
H^0(T') \to H^0(N_f).
\]

This completes the proof of Proposition 1.3. \( \square \)

We can now complete the proof of Theorem 1.1. Let

\[
X_d' \to \tilde{V}_d
\]

be the normalization of the universal cycle. Then \( X_d' \) is irreducible of dimension \((n + 1)d + n - 2\) and nonsingular in codimension 1. Likewise the fibred product

\[
(X_d')^k \to \tilde{V}_d
\]

is irreducible of dimension \((n + 1)d + n - 3 + k\) and nonsingular in codimension 1 (its singularities come from singularities of \( \tilde{V}_d \) and repeated singular points of fibres). Moreover the fibres of \( (X_d')^k \) over \( \tilde{V}_d \) are obviously \( k \)-dimensional. Now consider the incidence variety

\[
I = \{(C, p_1, \ldots, p_k, A_1, \ldots, A_k) : p_i \in A_i, i = 1, \ldots, k\}
\]

\[
\subset (X_d')^k \times G(n - a_1, \mathbb{P}^n) \times \ldots \times G(n - a_k, \mathbb{P}^n).
\]

This is obviously a fibre bundle over \((X_d')^k\), hence irreducible of dimension

\[
(n + 1)d + n - 3 + k + \sum a_i(n - a_i),
\]

i.e. codimension \( \sum a_i \) in the product, and nonsingular in codimension 1. It follows that if

\[
\sum (a_i - 1) = (n + 1)d + n - 4
\]

then a general fibre \( B \) of \( I \) over \( G(n - a_1, \mathbb{P}^n) \times \ldots \times G(n - a_k, \mathbb{P}^n) \) is smooth and 1-dimensional and the image \( \overline{B} \) of \( B \) in \( \tilde{V}_d \) can be made disjoint from any given codimension-2 subvariety, hence the curves in \( \overline{B} \) are as claimed. The smoothness of \( X \) as in Theorem 1.1 can be proved similarly by considering a suitable incidence variety in

\[
(X_d')^{k+1} \times G(n - a_1, \mathbb{P}^n) \times \ldots \times G(n - a_k, \mathbb{P}^n)
\]

(with incidence conditions on the first \( k \) points). Finally, as \( B \) is smooth, clearly singularities of \( \overline{B} \) come from curves meeting some \( A_i \) more than once (or nontransversely), and the remaining assertions about these singularities can be proved by a dimension-counting argument on the rational normal curve similar to the one we did above. This completes the proof of Theorem 1.1. \( \square \)

It follows from Theorem 1.1 that we can speak about a 'general reducible boundary curve' of \( V_d \) as being a 1-nodal curve

\[
C_1 \cup_m C_2
\]

which is either embedded as such (if \( n > 2 \)) or maps to a reducible nodal curve with one distinguished separating node (if \( n = 2 \)).
Remark 1.5. Another fact which follows easily from the above discussion is that for any 
\[ b = (C, p_1, ..., p_k) \in B, \]
we can identify the tangent space \( T_b B \) with
\[ \{ v \in H^0(N) : v_{p_i} \equiv 0 \mod T_{p_i} A_i, i = 1, ..., k \} \]
where \( N \) is either \( (I_C/I_C^2)^* \) in case (a) or (b), or \( N_f \) in case (a) or (c). This implies, in the notation of [R3], Sect 2, that not only is \( B = \bigcap B_i \) but
\[ T_b B = \bigcap_i T_b B_i \]
as well, so that \( B \) is the complete transverse intersection of the \( B_i \) in \( B^+ \), a fact which was implicitly used in the computation of the genus of \( B \) in [R3].

2. JUMPING RATIONAL CURVES

Here we discuss some qualitative generalities about restriction of vector bundles from \( \mathbb{P}^n \) to rational curves. See [OSS] for details on vector bundles over projective spaces.

A vector bundle \( E_C \) of rank \( r \) on a rational curve \( C \) is said to be \textit{almost balanced} if it can be decomposed
\[
E_C \simeq s \mathcal{O}(k) \oplus (r - s) \mathcal{O}(k - 1).
\]
In this case the subsheaf \( s \mathcal{O}(k) \subseteq E_C \) is well-defined and determines a canonical 'positive' subspace \( V_C(p) \subseteq E(p) := E \otimes \mathcal{O}_p \) for any \( p \in C \).

Now let \( E \) be a semistable reflexive sheaf of rank \( r \) on \( \mathbb{P}^n \) and chern class \( c_1(E) = a \in \mathbb{Z} \). We introduce the following

\textbf{Condition AB.} \textit{The restriction} \( E_L \) \textit{of} \( E \) \textit{on a general line is almost balanced.}

By the Grauert-Mülich Theorem, condition AB is satisfied if \( E \) is semistable of rank 2. Also, this condition is obviously satisfied whenever
\[ E = \bigwedge^m T_{\mathbb{P}^n} \]
for any \( m \) (though it fails for \( E = \text{Sym}^m T_{\mathbb{P}^n}, m > 1 \)). Assuming this condition holds, we try to get a similar conclusion for rational curves of higher degree. To this end, consider the following

\textbf{Transversality condition T.} \textit{Given a general point} \( p \in \mathbb{P}^n \), \textit{an arbitrary subspace} \( W \subseteq E(p), \) \textit{and a general line} \( L \ni p \), \textit{the positive subspace} \( V_L(p) \) \textit{is transverse to} \( W \).
Lemma 2.1. Assuming condition AB, condition T is satisfied provided either

(i) $a \equiv 0 \mod r$; or
(ii) $r = 2$; or
(iii) $E = \bigwedge^m T_{P^n}$.

proof. Case (i) is obvious since then $V_L(p) = E(p)$. In case (ii), if $a$ is odd, it is well known [OSS] that by semistability of $E$ the 1-dimensional subspace $V_L(p)$ varies with $L$, which is sufficient. In case (iii) $V_L(p)$ is the space of 'multiples' of $T_L(p)$ and again the result is clear.

Thus, conditions AB and T are both satisfied whenever either $E$ has generic splitting type $(k^r)$ on lines, or has rank 2 or is an exterior power of the tangent bundle.

Now we want to study restrictions of $E$ either to general rational curves or to general reducible curves in $\bar{V}_{d,n}$. To this end we make another definition. Consider a 1-nodal curve

$$C = C_1 \cup_p C_2$$

with each $C_i \simeq \mathbb{P}^1$. A bundle $E_C$ on $C$ is said to be almost balanced if each $E_{C_i}$ is almost balanced and the induced positive subspaces

$$V_1, V_2 \subseteq E(p)$$

are in general position. It is an easy exercise that in this case we can write

$$E_C \simeq (\oplus L_i) \oplus (\oplus M_j)$$

where for some $k$, each $L_i$ (resp. $M_j$) is a line bundle of total degree $k$ (resp $k-1$). Moreover the 'positive' subsheaf $\oplus L_i \subseteq E_C$ is canonically defined and for any general point $q \in C_1$ or $C_2$ the corresponding subspace $V \subseteq E(q)$ can be identified in an evident sense with either $V_1 + V_2$ or $V_1 \cap V_2$. Note that the decomposition (2.2) implies easily that for any small deformation $(E_{C'}, C')$ of $(E_C, C)$ where $C'$ is a smooth $\mathbb{P}^1$, $E_{C'}$ is almost balanced.

Proposition 2.2. Let $E$ be a reflexive sheaf on $\mathbb{P}^n$ satisfying conditions AB and T, and let $C$ be either a general element or a general reducible boundary element of $\bar{V}_d$. Then $E_C$ is almost balanced.

proof. We use induction on $d$, the case $d = 1$ being exactly condition AB. Assuming the assertion holds for $d - 1$, specialise a general curve $C$ of degree $d$ to a general reducible $C_0 = C_1 \cup_p L$ with $L$ a line. By property T, clearly $E_{C_0}$ is almost balanced, hence so is $E_C$ for a general $C$.

For a general reducible $C = C_1 \cup_p C_2$, we know $E_{C_i}$ is almost balanced and moreover, as each $C_i$ may be viewed as a specialisation of a polygon, it follows that the positive subspaces $V_i \subseteq E(p)$ may be assumed transverse, hence $E_C$ is almost balanced. □

Example 2.3. We consider the case of the tangent bundle $E = T_{P^n}$. For any irreducible rational curve

$$C \to \mathbb{P}^n$$
of degree $d \equiv 0 \mod n$, given by a polynomial vector

$$(f_0, ..., f_n) \in \oplus H^0(\mathcal{O}_C(d)),$$

we have an exact sequence

$$0 \to \mathcal{O}_C \to \oplus \mathcal{O}_C(d) \to E_C \to 0.$$ Dualising and twisting by $d$, we see that sections of $E_C^{*}(d+k)$ correspond to syzygies of degree $k$ among the $f_i$, i.e. relations of the form

$$\sum g_i f_i = 0, \quad g_i \in H^0(\mathcal{O}_C(k)).$$

It is immediate from this that $C$ is a jumping curve iff it admits a syzygy of degree $\frac{d}{n} - 1$, while any curve admits a syzygy of degree $\frac{d}{n}$. It is also easy to see that $C$ is a jumping curve iff the ideal generated by $f_0, ..., f_n$ in the homogeneous coordinate ring of $C$ fails to contain $H^0(\mathcal{O}_C(d + \frac{d}{n} - 1))$, while this ideal never contains $H^0(\mathcal{O}_C(d + \frac{d}{n} - 2))$.

3. COUNTING JUMPING RATIONAL CURVES

We continue with the notations above and assume that $E$ satisfies conditions AB and T. Then we may define the jumping locus

$$\mathcal{J}_{d,E} \subset \bar{V}_d,$$

as a set, to be the closure of the set of irreducible reduced $C$ such that $E_C$ is not almost balanced. Now if it happens that

$$(3.1) \quad -r < ad \leq 0,$$

then it is easy to endow $\mathcal{J}_{d,E}$ with a global scheme structure: namely let

$$\Pi : \mathcal{X} \to V_d$$

be the tautological family and

$$\mathcal{F} : \mathcal{X} \to \mathbb{P}^n$$

the natural map, and let $\mathcal{J}_{d,E}$ be the Fitting subscheme

$$Fit_1(R^1\Pi_*((\mathcal{F}^*E)))$$

defined by 1st fitting ideal of $R^1\Pi_*((\mathcal{F}^*E))$. This what is done in [OSS] for $d = 1$. For our purposes however, the hypothesis (3.1) is too restrictive. Without it one can still endow $\mathcal{J}_{d,E}$, at least in the event it had codimension 1, with a scheme structure 'slice by slice', as we now proceed to do.
We now assume that
\[(3.2)\quad r \mid \text{ad}.\]
In view of Proposition 2.2 this implies that for a general \( C \in V_d \), \( E_C \) is in fact balanced. Let \( \pi : X \to B, f : X \to \mathbb{P}^n \) be as in Sect. 1, and let
\[(3.3)\quad s_i \subset X, i = 1, \ldots, k\]
be the tautological section corresponding to \( A_i \). Let \( D = D_t \) be any divisor of the form
\[(3.4)\quad D = \sum t_is_i\]
such that \( \sum t_i = \frac{ad}{r} + 1 \). Set
\[(3.5)\quad G = f^*(E)(-D)\]
Thus for a general fibre \( F_b = \pi^{-1}(b) \) we have
\[G_{F_b} = rO(-1)\]
We define \( J_{d,E,B} \) to be the part of the first Fitting scheme \( \text{Fit}_1(R^1\pi_*(G)) \) supported in the open subset \( B^0 \subset B \) corresponding to irreducible curves. It is easy to see that this is independent of the choice of twisting divisor \( D \). In particular, taking \( D = (ad + r)s_i \), our scheme structure coincides with the natural scheme structure on \( J_{d,E,B} \) which defined, at least through codimension 1 over the locus of curves in \( \overline{V}_d \) incident to \( A_i \), by virtue of the existence of a canonical section. Now set
\[(3.6)\quad J_{d,E}(a.) = c_1(J_{d,E,B}).\]
This evidently depends only on the \((a.)\), and it is this number that we will compute.

To state our formula conveniently we introduce some objects from [R2][R3]. Set
\[(3.7)\quad m_i = m_i(a.) = -s_i^2, i = 1, \ldots, k.\]
Note that if \( a_i = a_j \) then \( m_i = m_j \). In particular for \( n = 2 \) they are all equal. It is shown in [R2][R3] that these numbers can all be computed recursively in terms of data of lower degree \( d \) and (for \( n > 2 \)) lower length. For instance for \( n = 2 \) we have
\[(3.8)\quad 2m_1 = \sum_{d_1 + d_2 = d} N_{d_1}N_{d_2}d_1d_2 \left( \frac{3d - 4}{3d_1 - 2} \right).\]
Note that
\[(3.9)\quad s_i.s_j = N_d(\ldots, a_i + a_j, \ldots, \hat{a}_j, \ldots), i \neq j,\]
so this number may be considered known. Hence \( D^2 \) may be considered known. Also, let \( R_i \) be the sum of all fibre components not meeting \( s_i \). We have
\[(3.10)\quad R_i.s_j = m_i + m_j + 2s_i.s_j\]
so this number is computable as well (actually we showed in [R2] that this is computable in terms of lower-degree data, and the \( m_i \) were computed from that).

Next, set
\[
L = f^*(\mathcal{O}(1)),
\]
and note that
\[
(3.11) \quad L^2 = N_d(2, a.), \ L.s_i = 0, \ i = 1, ..., k.
\]

Also, it is easy to see as in [R3] that
\[
(3.12) \quad L.R_i = \sum d_1N_{d_1}(a_1, n_1)N_{d_2}(a_2, n_2), \ n > 2,
\]
the summation for \( n > 2 \) being over all \( d_1 + d_2 = d, n_1 + n_2 = n \) and all decompositions
\[
A. = (A_i) \prod (A_i^1) \prod (A^2)
\]
(as unordered sequences or partitions).

Recall that the geometric genus \( g = g(B) \) was computed in [P][R2] for \( n = 2 \) and in [R3] in general. Next, define for any index-set \( I \subseteq \{1, ..., k\} \) of cardinality \( |I| \)
\[
(3.13) \quad t_I = \sum_{i \in I} t_i,
\]
\[
h(a., t.) = \sum N_{d_1}N_{d_2} \left( \frac{3d-1}{3d_1-1} \right)^{\left( r(t_I - ad_1 - r) \right)}, \ n = 2
\]
\[
(3.14) \quad h(a., t.) = \sum N_{d_1}(a_i : i \in I)N_{d_2}(a_i : i \notin I)(rt_I - ad_1 - r), \ n > 2
\]
the sum being extended over all \( d_1 + d_2 = d \) and all index-sets \( I \) such that \( t_I > \left\lceil \frac{ad_1}{r} \right\rceil \)
(and, for \( n = 2 \), also \( |I| \leq 3d_1 - 1 \)). Now we can finally state our formula.

**Theorem 3.1.** Notations as above, assume \( E \) satisfies conditions \( AB \) and \( T \) and set \( b = c_2(E) \in \mathbb{Z} \). Then the weighted number of jumping rational curves meeting \( A_1, ..., A_k \) is given by
\[
(3.15) \quad J_{d,E}(a.) =
\]
\[
\frac{r(g-1)}{2}((a^2 - 2b)L^2 + rD^2) + \frac{1}{2}((-r)(2g - 2 - m_1)
\]
\[
+ aR_1.L + 2rs_1.D - \sum_{i=2}^{k} t_iR_1.s_i] - h(t., a.)
\]
proof. We apply the Riemann-Roch formula in Grothendieck’s form ([F],15.2) (though the Hirzebruch form would have worked too) to the vector bundle \( G \) and the map \( \pi : X \to B \). Clearly \( \pi_*(G) = 0 \) while \( R^1\pi_*(G) \) is a torsion sheaf supported firstly on those \( b \in B \) corresponding to irreducible jumping curves \( C \), where it has length \( h^1(G_{Fb}) \), and secondly on those \( b \) corresponding to reducible fibres \( F_b = C_1 \cup_p C_2 \) such that \( h^1(G_{Fb}) \neq 0 \) where its length is again equal to this \( h^1 \). By property T, \( E \) is almost balanced on all reducible fibres \( F_b \). This implies easily that \( F_b \) has at most a unique component, say \( C_1 \) of degree \( d_1 \), such that \( h^1(G_{C_1}) \neq 0 \), and in this case
\[
\begin{align*}
h^0(G_{C_1}) &= 0 \\
h^1(G_{C_2}(-p)) &= 0
\end{align*}
\]
hence
\[
h^1(G_{Fb}) = h^1(G_{C_1}).
\]
Now we have
\[
E_{C_1} \simeq s\mathcal{O}(j) \oplus (r - s)\mathcal{O}(j - 1)
\]
with \( j = \lceil ad_1 \rceil \), and it is immediate from this that \( h^1(G_{C_1}) \neq 0 \) only if \( t_I > j \), in which case
\[
h^1(G_{Fb}) = h^1(G_{C_1}) = rt_I - ad_1 - r.
\]
It follows that the total \( h^1 \) coming from reducible fibres equals \( h(t.,a.) \), so one side of GRR yields \(-J_{d,E}(a.) - h(t.,a.)\).

Now the other side of GRR generally equals
\[
(3.16) \quad (r1_X + c_1(G) + \frac{1}{2}(c_1^2 - 2c_2)(G))(1_X - \frac{1}{2}K_X + \chi(\mathcal{O}_X)[pt])_2
\]
where \([pt]\) is a point and 2 denotes degree-2 part. Clearly
\[
\chi(\mathcal{O}_X) = 1 - g.
\]
Next, the canonical class \( K_X \) was computed in [R2][R3] as
\[
K_X = -2s_1 + (2g - 2 - m_i)F + R_i
\]
for any \( i \) (we take \( i = 1 \)), where \( F \) is a fibre. Given this, the computation of (3.16) is routine, yielding the formula (3.15). \( \square \)

Example 3.2. Take \( n = 2, d = 4 \) and let \( E \) be the tangent bundle of \( \mathbb{P}^2 \), with Chern classes \( a = 3, b = 3 \). It is known that in this case \( N_4 = 620, g = 725 \). It is easy to compute that
\[
m_1 = 284, R_1.L = 5220.
\]
We take \( D = 7s_1 \) and compute that in this case the \( h^1 \) contribution from the reducible curves is
\[
h(t,a.) = 9180,
\]
therefore finally

\[ J_{4,T_{x_2}} = 7944. \]

It is easy to see that on an irreducible rational quartic the tangent bundle cannot have \( O(4) \) as a direct summand so the jumping quartics all have splitting type \((7,5)\). It is also easy to see that a conic-pair \( C_1 \cup_p C_2 \) with \( C_1, C_2 \) irreducible is jumping iff \( C_1 \) and \( C_2 \) are tangent at \( p \).

**Example 3.3.** Again for \( n = 2 \), let \( E \) be a semistable bundle with \( c_1 = 0, c_2 = m - 1 \) corresponding to \( m \) general points in \( \mathbb{P}^2 \) (cf. [OSS]). Then we compute

\[ J_{d,E} = (m - 1)N_d. \]

For \( m = 2 \), \( J_{d,E} \) is just a hyperplane section of \( V_d \) corresponding to a certain point in \( \mathbb{P}^2 \). For \( m = 3 \), we get an interesting class of quadric sections of \( V_d \).

**Remark 3.4.** Note that Theorem 3.1 may be applied to the restriction of \( E \) to a general linear subspace \( A_0 \subset \mathbb{P}^n \). Hence the Theorem generalizes immediately to the case where one of the incidence conditions on the rational curve becomes containment in \( A_0 \).
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