Dissipation and tunnelling at the sonic horizon of Bondi accretion

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ABSTRACT
Viscous dissipation, as a small effect about the Bondi flow, shrinks its sonic sphere. An Eulerian perturbation on the steady flow gives a wave equation. The perturbation is a high-frequency travelling wave, in which dissipation is taken iteratively. A WKB analysis shows that an acoustic wave, propagating radially against the bulk inflow, is blocked within the sonic horizon. By Cauchy’s residue analysis, the acoustic wave tunnels through the horizon with a viscosity-dependent decaying amplitude, that is scaled by the analogue Hawking temperature.

Key words: accretion, accretion discs – hydrodynamics – methods: analytical

1 INTRODUCTION

Bondi (1952) accretion is a classic textbook example of a transonic flow in astrophysics (Chakrabarti 1990; Frank et al. 2002). This compressible astrophysical flow is steady and spherically symmetric, with its fluid elements being driven radially inwards by the gravitational field of a centrally located accretor, which can be an ordinary star or a neutron star or a black hole (Peterson et al. 1980). Far away from the accretor, the conservative velocity field has a very low subsonic value (idealized to vanish at infinity), but it becomes highly supersonic as it approaches the accretor. With the flow thus being subsonic at the outer boundary and supersonic at the inner boundary, it crosses the sonic barrier at an intermediate radius (Bondi 1952; Ray & Bhattacharjee 2002). The surface of this sonic barrier, where the flow becomes transonic, is spherical. Travelling acoustic waves in the supersonic region are completely trapped within this spherical sonic surface. As a result the region bounded by the surface can be viewed as a spherically symmetric acoustic black hole (Unruh 1981, 1995; Visser 1998), and the sonic surface itself becomes a sonic horizon – the acoustic analogue of the event horizon of a general-relativistic black hole.

The specific features of Bondi (1952) accretion, namely, spherically symmetric, compressible and irrotational, suit it well as a three-dimensional transonic potential flow. Natural examples of such flows are otherwise uncommon. These reasons make the Bondi (1952) inflow, apart from being a paradigm in studies of astrophysical accretion, a physical model of interest in the study of viscosity-dependent decaying amplitude, that is scaled by the analogue Hawking temperature.

2 THE HYDRODYNAMIC EQUATIONS

The viscous compressible flow is governed by the equation of continuity (Landau & Lifshitz 1987), given as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \quad (1)$$

and by the Navier-Stokes equation (Landau & Lifshitz 1987), which, for a flow driven radially by the gravity of an accretor of mass, $M$, is expressed in full as

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} + \frac{\nabla P}{\rho} + \frac{GM}{r^2} \mathbf{r} = \frac{1}{\rho} \left[ \eta_1 \nabla^2 \mathbf{v} + \left( \frac{\eta_2}{3} + \zeta_2 \right) \nabla \left( \nabla \cdot \mathbf{v} \right) \right]. \quad (2)$$

In Eq. (2), the pressure is prescribed by the polytropic equation, $P = K \rho^\gamma$, with $1 < \gamma < 5/3$, within the isothermal and the...
adiabatic limits (Chandrasekhar 1934). With \( P \equiv P(\rho) \), both \( \rho \) and \( v \) are mathematically closed in Eqs. (1) and (2). In the latter equation, \( \eta \) and \( \zeta \) are the first (shear) and second (bulk) coefficients of viscosity, respectively, both with positive values of the same order (Landau & Lifshitz 1987). The viscous effects arise due to internal friction on the molecular scale. Its smallness suits our purpose of introducing viscosity as a feeble presence about the conservative background of the inviscid flow. Inasmuch as the molecular viscosity of a gas depends on its pressure and temperature (hence, on its density as well), the spatial variation of the molecular viscosity is less compared to that of the bulk velocity of a spherically symmetric inflow. This is certainly true for a transonic radial flow in an open astrophysical system, where radiative processes facilitate efficient cooling and maintain the system close to being isothermal. By this argument, \( \eta \) and \( \zeta \) are approximated as constants (Balbus & Hawley 1998; Ray 2003). We adopt this approximation in our present study of a viscous spherically symmetric flow. Since the bulk flow here is vorticity-free, the left hand side of the vector identity, \( \nabla \times (\nabla \times \mathbf{v}) = \nabla (\nabla \cdot \mathbf{v}) - \nabla^2 \mathbf{v} \), vanishes to give \( \nabla (\nabla \cdot \mathbf{v}) = \nabla^2 \mathbf{v} \), which we use to simplify the viscosity-dependent terms in Eq. (2). We further note that since accretion is a compressible flow, \( \nabla \cdot \mathbf{v} \neq 0 \), as it is to be seen in Eq. (1).

Tailored according to spherical symmetry (Landau & Lifshitz 1987), with \( \nu = \nu(r,t) \) and \( \rho = \rho(r,t) \), Eqs. (1) and (2) become

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\nu \rho \mathbf{v}) = 0
\]

and

\[
\frac{\partial \nu}{\partial t} + \frac{v \partial \nu}{\partial r} + \frac{1}{\rho} \frac{\partial P}{\partial r} + \frac{GM}{r^2} = \frac{1}{\rho} \left( \frac{4}{3} \frac{\partial \eta}{\partial r} + \zeta \right) \frac{\partial}{\partial r} \left( \frac{1}{r^2} \frac{\partial}{\partial r} (\nu^2) \right),
\]

respectively. The two coefficients of viscosity add up like scalars in Eq. (4), to give a total viscosity, \( \eta = (4/3) \eta + \zeta \). With Eqs. (3) and (4), the coupled dynamics of \( \rho \) and \( \nu \) in the viscous spherically symmetric inflow is formulated completely.

3 THE STEADY SONIC HORIZON

In the steady state, partial time derivatives vanish, i.e. \( \partial / \partial t \equiv 0 \), leaving behind only full spatial derivatives. As such, integrating Eq. (3) gives \( 4\pi \rho \nu r^2 = -\bar{m} \), in which \( \bar{m} \) is the matter inflow rate, with its negative sign, due to \( v < 0 \), indicating the inflow (Frank et al. 2003). These constraints, together with scaling \( v \) by the local speed of sound, \( a = \sqrt{\partial P/\partial \rho} = \sqrt{\gamma K \rho^{-\gamma-1}} \), reduce Eq. (3) to

\[
\frac{\partial \nu}{\partial t} = \frac{GM}{r^2} = \frac{2\eta}{(\gamma - 1)\rho} \frac{d}{dr} \left( \frac{\nu \rho}{a} \frac{da}{dr} \right).
\]

The viscosity-dependent term on the right hand side of Eq. (5) is positive (Ray 2003), whose physical implication is that viscosity opposes the gravity-driven inflow in spherical symmetry (Ray 2003). Contrary to this, viscosity in accretion discs effects the outward transport of angular momentum, and aids the axially symmetric inflow (Balbus & Hawley 1998; Frank et al. 2003). Substituting \( \rho \) with \( a \) in the steady continuity equation, the spatial derivatives of \( a \) are replaced in Eq. (5), to give us

\[
\left( \frac{v^2}{2} - v \frac{a^2}{\rho r^2} \right) \frac{d \nu}{dr} = \frac{2a^2}{r^2} - GM \frac{\eta}{\rho} \frac{d^2 a}{dr^2} - 2 \frac{v^2}{r^2}.
\]

When \( \eta = 0 \), Bondi (1952) accretion becomes transonic at the sonic horizon (Chakrabarti 1990; Frank et al. 2002). A smooth passage of the inflow through the sonic horizon requires both the right hand and the left hand side of Eq. (6) to vanish together, while \( d\nu/dr \neq 0 \) (Ray & Bhattacharjee 2002). The values of \( \nu \) and \( r \) at the sonic horizon are thus obtained from Eq. (6) as

\[
\nu^2 = a^2, \quad r = \frac{GM}{2\nu^2},
\]

with the subscript “\( \nu \)” indicating the critical values when \( \eta = 0 \). The boundary conditions of the flow are \( \nu \rightarrow 0 \) and \( a \rightarrow a_\infty \) (a fixed ambient limit of \( a \)), for \( r \rightarrow \infty \). We now introduce the polytropic index, \( n = (\gamma - 1)^{-1} \) (Chandrasekhar 1939). With it, and with \( \eta = 0 \), the integral of Eq. (5) is \( (v^2/2) + na^2 - (GM/r) = na^2_\infty \), from which, on applying the critical values given by Eqs. (7), we further get \( \nu^2 = 2na^2_\infty/(2n - 3) \). This fixes \( \nu \), and in terms of the outer boundary conditions. Since \( a \) is related to \( \rho \), we also get \( \rho_\infty = 2n/(2n - 3)\), with \( \rho \rightarrow \rho_\infty \) for \( r \rightarrow \infty \).

Having established the transonic conditions for \( \eta = 0 \) in Eq. (6), we now look at how they are modified with the inclusion of viscosity as a small perturbative effect about the inviscid state (Ray 2003). Molecular viscosity is sufficient for this purpose, and we scale it in terms of quantities that are determined by the microscopic molecular properties of the fluid on the large scale of the accretion radius, \( r_a \approx GM/a^2_\infty \) (Frank et al. 2002). We write \( \eta = \eta_0 \eta_r \) in which \( \eta_0 \) is dimensionless and \( \eta_r = a_\infty \rho_\infty r_a \). The latter sets the scale of \( \eta \) and the former controls its magnitude, with \( \eta_r \ll 1 \) in our study. Once again we require Eq. (6) to vanish on both sides, while \( d\nu/dr \neq 0 \). This will give two quadratic equations in \( \nu_{\infty} \) and \( r_{\infty} \), respectively, from the left hand and the right hand sides of Eq. (6). The subscript “\( \infty \)” in both now denotes critical values when \( \eta \neq 0 \). While solving the two quadratic equations, we neglect all terms of \( \eta \) that are of orders higher than linear (due to the smallness of \( \eta_r \)), and in all the \( \eta_r \)-dependent terms we iteratively bring in the conditions provided by Eqs. (7). First we get

\[
\nu_{\infty} \approx \pm a_c + \frac{\eta}{\rho \nu r_c},
\]

in which we note that the critical velocities of inflows (\( v < 0 \)) and outflows (\( v > 0 \)) have different magnitudes, as opposed to their equality in the inviscid case, seen clearly in Eqs. (6). Viscosity is responsible for breaking the symmetry. Using the scale of \( \eta_r \) and the critical values of \( \rho \) and \( r \) in the inviscid limit, we simplify Eq. (6) as \( \nu_c \approx \pm a_c + \eta_r a_c \Delta \), with \( \Delta = 2[(2n - 3)/2n]^{1/2} \).

Next we solve the quadratic equation in \( r_c \) for small \( \eta_r \), with a binomial expansion of the discriminant and a choice of its physically relevant positive sign, to get

\[
r_{\infty} \approx \frac{GM}{2a^2_c} + \frac{\nu_c}{\rho_c a^2_c}.
\]

In Eq. (7), we see that \( \nu_c = \pm a_c \), for outflows and inflows, respectively. Using this, we simplify Eq. (9) as \( r_{\infty} \approx r_c \).}

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4 A WAVE EQUATION

In the steady state, the solutions of Eqs. (4) and (5) are two coupled time-independent fields, which we write as $\rho_0(r)$ and $v_0(r)$. About these steady flow profiles, we impose small time-dependent, radial perturbations and then linearize the perturbed quantities. The prescription for the time-dependent radial perturbation is $v(r, t) = v_0(r) + v'(r, t)$ and $\rho(r, t) = \rho_0(r) + \rho'(r, t)$, with the primed quantities being perturbations about a steady background. Following an Eulerian perturbation scheme employed by Petterson et al. (1980), we define a new variable, $f(r, t) = \rho v^2$, which emerges as a constant of the motion from the steady limit of Eq. (4). This constant, $f_0 = \rho_0 v_0^2$, is the matter flow rate, within a geometrical factor of $4\pi$ (Frank et al. 2002). Applying the perturbation scheme for $v$ and $\rho$, the perturbation in $f$ is derived as

$$f' = \frac{\rho'}{\rho_0} + \frac{v'}{v_0},$$ (10)

connecting $v'$, $\rho'$ and $f'$ together. To relate only $\rho'$ and $f'$ to each other, we use the perturbation scheme in Eq. (4), resulting in

$$\frac{\partial \rho'}{\partial t} = -\frac{1}{r^2} \frac{\partial f'}{\partial r}.$$ (11)

To obtain a similar relation between $v'$ and $f'$, we combine the conditions given in Eqs. (10) and (11), to get

$$\frac{\partial v'}{\partial t} = \frac{v_0}{f_0} \left( \frac{\partial f'}{\partial t} + \frac{\partial f'}{\partial r} \right).$$ (12)

Together, Eqs. (11) and (12) form a closed set, with $\rho'$ and $v'$ expressed exclusively in terms of $f'$. Now we need an independent condition, in which we can use Eqs. (11) and (12). Just such a condition is offered by Eq. (4), with $P = K\rho^2$. We take the second-order time derivative of Eq. (4), to which we then apply Eqs. (11) and (12) and the second-order time derivative of Eq. (4).

This exercise leads us to the wave equation,

$$\frac{\partial}{\partial t} \left( h_{tt} \frac{\partial f'}{\partial t} \right) + \frac{\partial}{\partial r} \left( h_{tr} \frac{\partial f'}{\partial r} \right) + \frac{1}{r} \frac{\partial}{\partial r} \left( h_{r} \frac{\partial f'}{\partial r} \right) = \eta v_0 r \rho_0 \left[ \frac{v_0}{f_0} \frac{\partial^2 f'}{\partial r^2} + \frac{\partial f'}{\partial r} \left( \frac{1}{\rho_0} \frac{\partial f'}{\partial r} + \frac{v_0}{f_0} \frac{\partial f'}{\partial r} \right) \right] + \eta v_0 \rho_0 \left( \frac{\partial^2 f'}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{1}{v_0 f_0} \frac{\partial f'}{\partial r} \right) \right),$$ (13)

in which $h_{tt} = v_0$, $h_{tr} = v_0$, and $h_{r} = v_0(v_0^2 - a_0^2)$, with $a_0$ in $h''$ being the steady value of $a$.

We should find it instructive to examine Eq. (13) for $\eta = 0$. In this inviscid limit, going by the symmetry of the left hand side of Eq. (13), we can recast it in a compact form as $\partial_\mu (h^{\mu \nu} \partial_\nu f') = 0$, with the Greek indices running from 0 to 1, under the equivalence that 0 stands for $t$ and 1 stands for $r$. All the $h^{\mu \nu}$ in Eq. (13) are to be seen as elements of the matrix

$$h^{\mu \nu} = \begin{pmatrix} \rho_0 & v_0 \\ v_0 & v_0 v_0^2 - a_0^2 \end{pmatrix}.$$ (14)

Now, in Lorentzian geometry the d’Alembertian of a scalar field in curved space is obtained from the metric, $g_{\mu \nu}$, as

$$\Delta \varphi \equiv \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu \nu} \partial_\nu \varphi),$$ (15)

where $g^{\mu \nu}$ is the inverse of the matrix, $g_{\mu \nu}$ (Visser 1998; Barceló et al. 2011). By comparing Eq. (15) with Eq. (13) when $\eta = 0$, we identify $h^{\mu \nu} = \sqrt{-g} g^{\mu \nu}$ (Visser 1998), and see that the wave equation of $f'$ in Eq. (13) is similar to Eq. (15). The metric that is implicit in Eq. (13), is to be read from Eq. (14), and its inverse establishes an acoustic metric and an acoustic horizon, when $v_0^2 = a_0^2$ (Visser 1998). In the radial inflow of the Bondi (1952) accretion, this horizon is due to an acoustic black hole. The radius of the horizon is the critical radius, $r_c$, given in Eq. (17), which cannot be breached by any acoustic wave propagating against the bulk inflow, after having originated in the supercritical region, where $v_0^2 > a_0^2$ and $r < r_c$. Borne by the wave, the flow of information across the acoustic horizon is, therefore, only inwards. With the inclusion of viscosity ($\eta \neq 0$), however, a source term appears in Eq. (13) to compromise the symmetry of the acoustic metric. For a perturbatively small viscosity, the sharp position of the acoustic horizon is then blurred in a thin layer of uncertainty. The width of this thin layer is in proportion to the small viscosity, agreeing with what Eq. (15) implies.

5 WAVE BLOCKING AT THE SOLAR HORIZON

Without viscosity in the right hand side of Eq. (13), linear perturbations do not destabilize the steady background flow (Petterson et al. 1980). We now look at the effect that viscosity has on this condition. We treat the perturbation as a high-frequency travelling wave, whose wavelength, $\lambda$, is much less than the natural length scale in the inflow, the radius of the acoustic horizon, $r_c$. Thus specifying $\lambda \ll r_c$, we use a separable solution for the travelling wave as $f'(r, t) = \exp [i \omega(r - i \omega t)]$, with the understanding that $\omega$ is much greater than any characteristic frequency of the system. Applying the foregoing solution to Eq. (13), and carrying out some algebraic simplifications, deliver

$$\begin{align*}
& \left( v_0^2 - a_0^2 \right) \left[ \frac{d^2 s}{dr^2} - \left( \frac{ds}{dr} \right)^2 \right] + i \frac{d}{dr} \left[ v_0 \left( v_0^2 - a_0^2 \right) \frac{ds}{dr} \right] \\
& + 2 v_0 \omega \frac{ds}{dr} - 2 v_0 \frac{d\omega}{dr} - 2 \omega^2 = \eta \left( \frac{v_0}{\rho_0} \frac{ds}{dr} \right)^3 - 3 \frac{v_0}{\rho_0} \frac{ds^2}{dr} + \left[ \frac{v_0}{\rho_0} \frac{ds}{dr} - 2 v_0 \frac{d\omega}{dr} \right] + \frac{\omega}{\rho_0} + i \frac{\omega}{\rho_0} v_0^2 + \frac{d\omega^2}{dr} \\
& + \left[ \frac{v_0}{\rho_0} \frac{ds}{dr} - 2 v_0 \frac{d\omega}{dr} \right] \frac{d^2 s}{dr^2} \left( \frac{\omega}{\rho_0} - 2 i \frac{d\omega}{dr} \frac{v_0}{\rho_0} \right) \\
& + \left[ \frac{v_0}{\rho_0} \frac{ds}{dr} - 2 v_0 \frac{d\omega}{dr} \right] \frac{d^2 s}{dr^2} \left( \frac{\omega}{\rho_0} - 2 i \frac{d\omega}{dr} \frac{v_0}{\rho_0} \right) \\
& - \frac{i}{\rho_0} \frac{d}{dr} \left( \frac{v_0}{\rho_0} \frac{ds}{dr} - 2 v_0 \frac{d\omega}{dr} \right) - \frac{2 i}{r} \frac{d\rho_0}{dr} + \frac{2 i}{r} \frac{d\rho_0}{dr} - i \frac{d\rho_0}{dr} \right) - i \frac{d^2 s^2}{dr^2} \left( \frac{\omega}{\rho_0} - 2 i \frac{d\omega}{dr} \frac{v_0}{\rho_0} \right).
\end{align*}$$

(16)
\( \beta_0 \), and solve a resulting quadratic equation in \( d\alpha_0/dr \) to obtain
\[
\alpha_0 = \int \frac{\omega}{{v_0} + a_0} \, dr. \tag{17}
\]

Likewise, from the imaginary part, in which we need to use the solution of \( \alpha_0 \), we obtain \( \beta_0 = \ln \left( \sqrt{v_0 a_0} \right) \).

We now perform a self-consistency check that \( \alpha_0 \gg \beta_0 \), as a basic requirement of our \textit{WKB} analysis. First, we note \( \alpha_0 \) contains a \( \omega \) (the high frequency of the travelling wave), and in this respect is of a leading order over \( \beta_0 \), which contains \( \omega^0 \). Next, on very large scales of length, i.e. \( r \to \infty \), the asymptotic behaviour of the background velocity is \( {v_0} \to 0 \), and the corresponding speed of acoustic propagation, \( a_0 \), approaches a constant asymptotic value.

In that case, \( \alpha_0 \sim \omega r \) in Eq. (17). Moreover, on similar scales of length, going by \( {v_0} \sim f_0 r^{-2} \), we see that \( \beta_0 \sim \ln r \). Further, near the acoustic horizon, where \( |{v_0}| \approx a_0 \), for the wave that goes against the bulk inflow with the speed, \( a_0 - |{v_0}| \), there is a singularity in \( \alpha_0 \). All of these facts taken together, we see that our solution scheme is well in conformity with the \textit{WKB} prescription.

Thus far we have worked with \( \eta = 0 \) (absence of viscosity).

To know how viscous dissipation affects the travelling wave, we have to find a solution of \( s \) from Eq. (16), with \( \eta \neq 0 \). To this end, we adopt an iterative approach, exploiting the condition that \( \eta = \eta_0 \eta_0 \) has a very small value, the smallness being set by \( \eta_0 \ll 1 \). Then taking up Eq. (15) in full, we propose a solution for it as \( \eta_0 \eta_0 = s = s_0 + \delta s_0 \), with \( \delta \) being another dimensionless parameter that, like \( \eta_0 \), obeys the requirement, \( \delta \ll 1 \). Therefore, in the right hand side of Eq. (16) all terms that carry the product, \( \alpha_0 \delta \), can be safely neglected as being very small. This, in keeping with the principle of our iterative treatment, effectively means that all the surviving viscosity-related terms in the right hand side of Eq. (16) will go as \( \eta s_0 \). Further, by the \textit{WKB} analysis, we have also assured ourselves that \( \alpha_0 \gg \beta_0 \), by which we ignore all dependence on \( \beta_0 \) in the right hand side of Eq. (16), when we compare them with all the terms containing \( \alpha_0 \). With these arguments, we approximate \( s \approx \alpha_0 \) in the right hand side of Eq. (16), and see here that the most dominant \( \alpha_0 \)-dependent real term is of the quadratic degree. Preserving only this term in the right hand side of Eq. (16) and extracting only the \( \beta \)-independent real terms from the left hand side, we arrive at a quadratic equation in \( \alpha_0/dr \),
\[
\left( v_0^2 - a_0^2 \right) \frac{d\alpha_0}{dr} - 2v_0 \omega \frac{d\alpha_0}{dr} + \omega^2 - \eta \left\{ -3 \frac{v_0}{\rho_0} \frac{d\alpha_0}{dr} \frac{d^2\alpha_0}{dr^2} + 2 \frac{d\alpha_0}{dr} \left[ \frac{v_0}{r \rho_0} - \frac{d}{dr} \left( \frac{v_0}{r \rho_0} \right) \right] \right\} = 0. \tag{18}
\]

We solve Eq. (18) under the provision of \( \eta_0 \ll \left( \lambda/r_c \right)^2 \), which accords well with our requirement that \( \eta_0 \) may be arbitrarily small. A binomial approximation of terms with \( \eta_0 \) in the discriminant gives us \( \alpha = \alpha_0 + \alpha_\eta \), with the viscosity-dependent correction to \( \alpha \) being
\[
\alpha_\eta \simeq \pm \int \frac{\eta_0 |v_0|^2}{2 \rho_0 a_0 (v_0 + a_0)} \left( \frac{d}{dr} \left[ \frac{v_0}{v_0 + a_0} \right] \right) (v_0 + a_0) dr. \tag{19}
\]

Next, to take up \( \beta \), we extract all the imaginary terms from the left hand side of Eq. (16), and noting that the most dominant contribution to the imaginary terms in the right hand side is of the cubic degree in \( \alpha_0 \), we are required to solve the equation,
\[
2 \left[ v_0 \omega - (v_0^2 - a_0^2) \frac{d\beta}{dr} \right] \frac{d\beta}{dr} + \frac{d}{v_0} \frac{d}{dr} \left[ v_0 (v_0^2 - a_0^2) \frac{d\alpha_0}{dr} \right] - 2 \beta \frac{d\alpha_0}{dr} = -\frac{\eta_0}{\rho_0} \left( \frac{d\alpha_0}{dr} \right)^3. \tag{20}
\]

We observe that \( \alpha_0 \) and \( \alpha_\eta \) are both linear in \( \omega \), whereas the right hand side of Eq. (20), with \( (d\alpha_0/dr)^3 \), is cubic in \( \omega \). Therefore, the dominant correction in \( \beta \) due to viscosity, can be found with the approximation \( \alpha \simeq \alpha_0 \) in the left hand side of Eq. (20). This reasoning gives us \( \beta = \beta_0 + \beta_\eta \), in which
\[
\beta_\eta \simeq \pm \int \frac{\eta_0 |v_0|^2}{2 \rho_0 a_0 (v_0 + a_0)} dr. \tag{21}
\]

A noteworthy aspect of both Eqs. (19) and (21) is that in the former, the correction to \( \beta_0 \) is of the order of \( \omega \) (an odd order contributing to the phase), and in the latter the correction to \( \beta_0 \) is of the order of \( \omega^2 \) (an even order contributing to the amplitude). Since \( \alpha_0 \) is of the order of \( \omega \), the correction in \( \alpha_0 \) appears comparable to \( \alpha_0 \). More crucially, since \( \beta_0 \) is of the order of \( \omega^0 \), the correction in \( \beta_0 \) appears to be dominant over \( \beta_0 \) for large \( \omega \). This, however, is not really the case. We have obtained the results given by Eqs. (19) and (21) under the restriction that \( \eta_0 \ll (\lambda/r_c)^2 \). Considering the wavelength as \( \lambda(\rho) = 2\pi(v_0 + a_0)/\omega \), we immediately see that \( \eta_0 \omega \) in Eq. (19) and \( \eta_0 \omega^2 \) in Eq. (21), reduce both \( \alpha_0 \) and \( \beta_0 \) to be sub-leading to \( \alpha_0 \) and \( \beta_0 \) respectively. This is true in most of the spatial range of the flow, except in the close neighbourhood of the acoustic horizon, where \( v_0 = a_0 \). In this region, for outgoing waves against the inflow, \( \beta_0 \) will diverge, as Eq. (21) shows, while \( \beta_0 \) itself remains finite.

For the travelling wave, \( f'(r, t) = e^{-\beta_0} \exp(\sigma_0 - i\omega t) \), from which, by extracting only the amplitude, we get \( |f'(r, t)| \sim e^{-\beta_0} \). Our primary concern is the stability of waves propagating outwards against the steady Bondi (1952) transonic inflow, for which \( v_0 < 0 \). We write \( v_0 = -|v_0| \) for the Bondi (1952) inflow, and choose the lower sign in Eq. (21) for outgoing waves. These specifications give
\[
|f'(r, t)| \sim \frac{1}{\sqrt{\alpha_0 |v_0|}} \exp \left[ \int \frac{\eta_0 |v_0|^2}{2 \rho_0 a_0 (|v_0| - a_0)} dr \right]. \tag{22}
\]

In the supersonic region, bounded within the spherical sonic horizon, \( |v_0| > a_0 \). For a wave approaching the sonic horizon from the supercritical region, with \( |v_0| \to a_0 \), the integrand in Eq. (22) suffers a divergence, and \( |f'(r, t)| \to \infty \). The instability of the wave amplitude stands out very clearly in this case. In contrast, in the subsonic region just outside the sonic horizon, where \( |v_0| < a_0 \), the same integrand acquires a negative sign overall, and \( |f'(r, t)| \to 0 \). Recalling that \( f' \) is a perturbation on the steady matter inflow rate, we realize that the divergence of the perturbation just within the horizon implies an accumulation of the fluid matter, with none of it allowed to percolate outside the horizon. The sonic horizon of Bondi (1952) accretion forces a discontinuity in the outward propagation of the wave, and acts like an impenetrable barrier to block acoustic waves within itself, much like a black hole.\footnote{All of this is the exact reverse of what occurs in the two-dimensional outflow of the shallow-water hydraulic jump, where a wave in the subcritical region, propagating upstream against the steady outflow, accumulates a wall of water at the horizon (Ray & Bhattacharjee 2007), but fails to breach it, thus characterizing the horizon as a white hole (Volovik 2003; Ray & Bhattacharjee 2007; Bhattacharjee 2017). It is remarkable that a gas flow on an astrophysical scale and a liquid flow on a laboratory...}
The energy flux of the perturbation also behaves in a manner similar to its amplitude. The kinetic energy per unit volume is $E_{\text{kin}} = (1/2)(\rho_0 + \rho^2)(v_0 + v)^2$ (Petterson et al. 1980). The potential energy per unit volume, with contributions from both the gravitational energy and the internal energy (Petterson et al. 1980), is $E_{\text{pot}} = (\rho_0 + \rho^2)(GM/r) + \rho_0\varepsilon + \frac{\partial(\rho_0\varepsilon)/\partial\rho_0\rho^2}{\partial r_0^2}$, where $\rho$ is the internal energy per unit mass (Landau & Lifshitz 1987). In both of the foregoing expressions of energy, the zeroth-order terms refer to the steady flow, and the first-order terms disappear upon time-averaging. Thereafter, the time-averaged total energy in the perturbation, per unit volume of fluid, is to be obtained by summing the second-order terms in $E_{\text{kin}}$ and $E_{\text{pot}}$. All of these terms go either as $\rho^2$ or $v^2$, or as a product of $r'$ and $v'$. Making use of Eqs. (10), (11) and (17), we get $\langle r' v' \rangle \simeq \alpha_0 \langle v_0 + \rho \rangle^{-1} (f'/f_0)$ and $\langle v'/v_0 \rangle \simeq \alpha_0 \langle v_0 + \rho \rangle^{-1} (f'/f_0)$. Once we have two relations explicitly connecting $r'$ and $v'$ with $f'$, the time-averaged total energy per unit volume is $E_{\text{tot}} \sim \langle |f'(r, t)|^2 \rangle$. The energy flux of the spherical wavefront, moving with the speed, $(v_0 + \alpha_0)$, is $F = 4\pi r^2 E_{\text{tot}}(v_0 + \alpha_0)$. Clearly, for the wave travelling outwards against the Bondi (1952) inflow, very close to the sonic horizon, both $E_{\text{tot}}$ and $F$ will exhibit the same instability implied by Eq. (22).

6 RESIDUES AND TUNNELLING AT THE HORIZON

Waves propagating outwards against the steady Bondi (1952) inflow (for which $v_0 = -|v_0|$), encounter a singularity at the sonic horizon, where $|v_0| = \alpha_0$. This is obvious from the integrands in Eqs. (17), (19) and (21). In each case, circumstance of the singularity requires rendering it as a simple pole on the path of the integration, and then applying Cauchy’s residue theorem on the path. We first demonstrate this procedure in full for the simplest case, which is in Eq. (17), by considering its lower sign, as only this pertains to an outward wave against the inflow. The main contribution to the integral comes from the immediate neighbourhood of $|v_0| = \alpha_0$, which is also where $r = r_c$, as Eqs. (7) show. A Taylor expansion about the horizon, up to the first order, gives $a_0 - |v_0| \simeq (a_0 - |v_0|)\varepsilon + \frac{d(a_0 - |v_0|)}{dr_0}\varepsilon (r - r_c)$. The Taylor expansion in the neighbourhood of the horizon transforms the singularity at $|v_0| = \alpha_0$ to a simple pole at $r = r_c$. Going by what Eq. (7) suggests, the zeroth-order term in the Taylor expansion vanishes, in consequence of which, we approximate Eq. (17) as

$$\alpha_0 \simeq \int \frac{\omega}{|d(a_0 - |v_0|)/dr_0|} (r - r_c) \, dr. \quad (23)$$

The analogue surface gravity, $g_s = \alpha_0 |d(a_0 - |v_0|)/dr_0|$, at the sonic horizon (Visscher 1998), and the analogue Hawking temperature, $T_H = (h g_s)/(2\pi k_B a_s)$ (Visscher 1998). In terms of $g_s$ and $T_H$, the integral in Eq. (23), taking the residue at the pole, becomes

$$\alpha_0 \simeq \frac{2h g_s}{2h g_s} (\pm i\pi) + \mathcal{P} \{\alpha_0\} = \frac{\hbar \omega}{2k_B T_H} (\pm i\pi) + \mathcal{P} \{\alpha_0\}, \quad (24)$$

where $\mathcal{P} \{\alpha_0\}$ is the principal value of the integral. Furthermore, the negative sign in $\pm i\pi$ is due to a clockwise detour of the pole, and the positive sign is due to an anti-clockwise detour. Both are valid mathematically, but in a real physical sense, the ultimate choice of the sign is determined by the boundary condition at the pole (Dennery & Krzywicki 1996). Since $\alpha = \alpha_0 + \alpha_s$, the imaginary part of $\alpha_0$ in Eq. (24) contributes to the amplitude of $f'(r, t) = e^{-\beta} \exp(\mathrm{i} \alpha - \mathrm{i} \omega t)$. Now, the horizon is like an unyielding barrier to outgoing waves. This boundary condition at the horizon necessitates the choice of the positive sign of $\alpha$ in Eq. (24), and as such, the wave, which can only be very weak with a decaying amplitude, tunnels through the barrier.

An additional contribution to the tunnelling amplitude comes from $\alpha_s$, as given in Eq. (19), with the integral being a sum of two terms. For simplicity of notation, we write $\gamma = 2 + (4\pi/v_0)d\Omega_0/dr$. Very close to the accretor, for a freely falling inflow, $v_0 \sim r^{-1/2}$, and very far away from the accretor, for a highly subsonic inflow, $v_0 \sim r^{-2}$ (Chakrabarti 1998). In the former case, $\gamma = 0$, and in the latter, $\gamma = -6$. We expect $\gamma(r_c)$ to have an intermediate value between these two limits. Following what we did to arrive at Eq. (23), a Taylor expansion up to the first order gives $(a_0 - |v_0|)^2 \simeq (a_0 - |v_0|)^2 + 2(a_0 - |v_0|)\varepsilon (a_0 - |v_0|)/dr_0\varepsilon (r - r_c)$. Now that we explicitly account for viscosity, however small, we realize from Eq. (24) that $(a_0 - |v_0|)^2$ is a small non-vanishing quantity of the order of $\eta_0^2$. Set against this, the first-order term in the Taylor expansion is of the order of $\eta_0$. This argument allows us to neglect the zeroth-order term in the Taylor series, and write $(a_0 - |v_0|)^2 \simeq -2(a_0 - |v_0|)\varepsilon (a_0 - |v_0|)/dr_0\varepsilon (r - r_c)$. By the same token, we also approximate $(a_0 - |v_0|)^2 \simeq (a_0 - |v_0|)^2 \varepsilon (a_0 - |v_0|)/dr_0\varepsilon (r - r_c)$. These conditions, imposed about the sonic horizon (where the singularity contributes the most to the integral), approximates Eq. (19), with its lower sign, to

$$\alpha_s \simeq \int \frac{\eta_0 \gamma(r_c)}{4\rho_c r_c (|v_0| - a_0)\varepsilon (a_0 - |v_0|)/dr_0\varepsilon (r - r_c)} \, dr - \int \frac{\eta_0 \gamma(r_c)}{2\rho_c (a_0 - |v_0|)^2} \, dr. \quad (25)$$

About the sonic horizon, the kinematic viscosity, $\nu_s = \eta_0/\rho_0$. The wave number, $k(r) = \omega/(|v_0| - \alpha_0)$, is blue-shifted near the horizon, justifying the WKB approximation (Panik & Wilczek 2000). We define a frequency, $\Omega = \nu_s k_n r_c$, and a temperature, $T_H = [k_n (a_0 - |v_0|)]/2\pi k_B r_c$. With these definitions, and by taking the residue at the pole, the integral in Eq. (25) gives

$$\alpha_s \simeq \pm \frac{\hbar \Omega \gamma(r_c)}{8k_B T_H} + \frac{\hbar \Omega}{4k_B T_H} \mathcal{P} \{\alpha_0\}. \quad (26)$$

with $\mathcal{P} \{\alpha_0\}$ being the principal value of the integral. The boundary condition at the singularity will only allow an acoustic wave to tunnel through with a decaying amplitude. Consequently, we select the positive sign of $\pm i$ in Eq. (24). Likewise, not to overlook the positive sign of $\pm i$ in Eq. (24) as well, the total contribution to the amplitude of the tunnelling wave, from both Eqs. (24) and (26), is now written as $|f'_T| \sim e^{-\gamma}$, in which

$$\Gamma = \frac{\hbar \omega}{2k_B T_H} + \frac{\hbar \Omega \gamma(r_c)}{8k_B T_H} + \frac{\hbar \Omega}{4k_B T_H}. \quad (27)$$

With $|f'_T|$ determined thus, the tunnelling probability is easily found from $|f'_T|^2$. In Eq. (27), the second term is of particular interest to us. The frequency, $\Omega$, is dependent on viscosity, and it is the dissipative influence of viscosity that shrinks the sonic sphere, as we have already deduced from Eq. (9). What is more, in the tunnelling amplitude, $\hbar \Omega$ is scaled by the fluid analogue of the Hawking temperature. The combined effect of all these facts is that the second term in Eq. (27) is likely to be responsible for the phenomenon of black hole evaporation by phonon radiation in fluid analogues (Unruh 1981; Jacobson 1991; Unruh 1995). However, we also note that $T_s \ll T_H$, from which we realize that the third term in Eq. (27) will overwhelm both the terms scaled by $T_H$. 

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The residues in the singularities of $\alpha_0$ and $\alpha_0$, have contributed to the amplitude of the tunnelling wave. In reciprocation, the residue in the singularity of $\beta_0$, as shown by Eq. (21), will contribute to the phase of the tunnelling wave. This is obtained as $\beta_0 \approx [(\pm i \Omega r_0)/[(2k_0 T_{0r})]] + P[\beta_0]$, in which the tunnelling frequency, $\Omega r_0 = \nu (k r_0)^2 \gg \Omega$, and $P[\beta_0]$ is the principal value of the integral in Eq. (21). We observe that through the singularity, the mutual exchange of amplitude and phase, between $\alpha$ and $\beta$, respectively, occurs only in the presence of viscosity.

7 CONCLUDING REMARKS

We have treated viscous spherically symmetric transonic accretion as an astrophysical model to demonstrate the analogue Hawking radiation of phonons through the sonic horizon of an acoustic black hole. Viscosity plays a part in this process, as well as in shrinking the acoustic black hole. Both phenomena are consistent with each other, when we recall the possibility of the evaporation of black holes [Unruh 1981; Jacobson 1991; Unruh 1995]. Since viscosity appears to be instrumental in the Hawking radiation, which is fundamentally a quantum effect, we mention a study where viscosity in the Navier-Stokes equation was shown to be equivalent to Planck’s constant in Schrödinger’s equation (Bhattacharjee et al. 2009).

In the physical flow that we have studied here, viscosity has a very weak perturbative presence about an inviscid background. In astrophysical accretion, however, viscosity is enhanced because of turbulence in the flow, without which, for instance, the outward transport of angular momentum in accretion discs is not feasible (Balbus & Hawley 1998; Frank et al. 2002). In spherically symmetric accretion, the coupling of the mean flow and the turbulent fluctuations scales viscosity up significantly, and shifts the sonic horizon inwards (Ray & Bhattacharjee 2005), qualitatively in the same way that we have seen here. Hence, we believe that an effective “turbulent viscosity” (Ray & Bhattacharjee 2005) in transonic inflows, can make the tunnelling of acoustic waves through the sonic horizon more pronounced. That said, we should remember that turbulence is a nonlinear phenomenon, while the results of our present study have been derived through linearization.

In passing, we mention that as long as an equation of state provides a means for acoustic propagation, transonic hydrodynamic flows can produce an analogue metric and an acoustic horizon. While this appears to be a universal feature of flows that pass through a critical point (Naskar et al. 2007; Bhattacharjee 2007; Sarkar et al. 2013; Sen & Ray 2014), the symmetric form of the acoustic metric can be disrupted because of the physical circumstances of a particular fluid flow. For example, the coupling of the flow and the geometry of Schwarzschild spacetime adversely affects the acoustic horizon (Naskar et al. 2007). The same behaviour is also exhibited due to viscous dissipation in the shallow-water circular hydraulic jump (Ray & Bhattacharjee 2007). Dispersion, arising out of interactions between baryons and vector mesons in a nuclear outflow, similarly breaks the symmetry of the acoustic metric (Sarkar et al. 2013). Perturbations of nonlinear order in spherically symmetric accretion shift the acoustic horizon about its static position, although the symmetry of the acoustic metric remains intact (Sen & Ray 2014). We surmise that some of the aforementioned systems can help to detect the analogue Hawking radiation.

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REFERENCES

Axford W. I., Newman R. C., 1967, ApJ, 147, 230
Balbus S. A., Hawley J. F., 1998, Rev. Mod. Phys., 70, 1
Barceló C., Liberati S., Visser V., 2011, Liv. Rev. Relativity, 14, 3
Bhattacharjee J. K., 2017, Phys. Lett. A, 381, 733
Bhattacharjee J. K., Bhattacharya A., Das T. K., Ray A. K., 2009, MNRAS, 398, 841
Bondi H., 1952, MNRAS, 112, 195
Chakrabarti S. K., 1990, Theory of Transonic Astrophysical Flows, World Scientific, Singapore
Chandrasekhar S., 1939, An Introduction to the Study of Stellar Structure, The University of Chicago Press, Chicago
Das T. K., 2004, Class. Quantum Grav., 21, 5253
Dennery P., Krzywicki A., 1996, Mathematics for Physicists, Dover Publications, New York
Frank J., King A., Raine D., 2002, Accretion Power in Astrophysics, Cambridge University Press, Cambridge
Jacobson T., 1991, Phys. Rev. D, 44, 1731
Landau L. D., Lifshitz E. M., 1987, Course of Theoretical Physics – Fluid Mechanics, Butterworth-Heinemann, Oxford
Mach M., Malec E., 2008, Phys. Rev. D, 78, 124016
Naskar T., Chakravarty N., Bhattacharjee J. K., Ray A. K., 2007, Phys. Rev. D, 76, 123002
Parikh M. K., Wilczek F., 2000, Phys. Rev. Lett., 85, 5042
Petterson J. A., Silk J., Ostriker J. P., 1980, MNRAS, 191, 571
Ray A. K., 2003, MNRAS, 344, 1085
Ray A. K., Bhattacharjee J. K., 2002, Phys. Rev. E, 66, 066303
Ray A. K., Bhattacharjee J. K., 2005, ApJ, 627, 368
Ray A. K., Bhattacharjee J. K., 2007, Phys. Lett. A, 371, 241
Sarkar N., Basu A., Bhattacharjee J. K., Ray A. K., 2013, Phys. Rev. C, 88, 055205
Sen S., Ray A. K., 2014, Phys. Rev. D, 89, 063004
Unruh W. G., 1981, Phys. Rev. Lett., 46, 1351
Unruh W. G., 1995, Phys. Rev. D, 51, 2827
Visser M., 1998, Class. Quantum Grav., 15, 1767
Volovik G. E., 2006, J. Low. Temp. Phys., 145, 337

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