Magnetic flux quantum in 2D correlated states of multiparticle charged system

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Abstract
We demonstrate that in a correlated 2D systems of electrons in the presence of perpendicular magnetic field the magnetic flux quantum may not achieve its value determined for a single or a noncorrelated electron. Correlations induced by the repulsion of electrons at strong magnetic field presence impose topological-type limits on planar cyclotron orbits which cause specific homotopy of trajectories resulting in constraints of the magnetic field flux quantum value. These restrictions occur at discrete series of magnetic field values corresponding to hierarchy of 2D correlated Hall states observed experimentally in GaAs thin films and in graphene. The similar homotopy property is observed in 2D Chern topological insulators when the magnetic field is substituted by the Berry field.

1. Introduction
Magnetic flux quantum, \( \Phi_0 = \frac{\hbar}{e} \) (if include superconductors, \( \frac{\hbar}{2e} \)) is one of the basic notions in quantum physics and concerns both single particle systems as well as multiparticle systems [1]. Magnetic flux quantization has been confirmed experimentally [2, 3] and lies in the core of Abrikosov vortices in superconductors [4], Little–Parks effect [5], Josephson effect [6], integer and fractional quantum Hall effect (IQHE) [7], (FQHE) [8, 9], Aharonov–Bohm effect [10], Berry-phase quantization [11] and of many others including new concepts of long-range entanglement [12] or topological transitions supporting hybrid Majorana phases [13] in 2D electron systems in perpendicular magnetic field. Magnetic flux quantization reflects the fact that for a charge particle, an electron with the charge \( e \) or Cooper pair with \( 2e \) charge, not arbitrarily small cyclotron orbit is possible but only just one given by the flux quantum, \( \Phi_0/H \), where \( H \) is the magnetic field. The surface \( \Phi_0/H \) is well defined though a cyclotron movement loses its meaning in quantum mechanics as the operators of coordinates of cyclotron orbit center do not commute. Thus, at the magnetic field \( H \) the subject of quantization is the size of the cyclotron orbit \( D \) perpendicular to the field, \( D = \frac{\Phi_0}{2n} \). The quantization of cyclotron orbits meets with the quasiclassical approximation which allows for the identification of the reason of flux quantization in the Bohr–Sommerfeld rule applied to components of kinematic momentum perpendicular to a magnetic field.

One can expect that in the case when the homotopy of a cyclotron orbit is modified by a topological factor, the magnetic flux quantum can be also affected. In 3D and for single or noncorrelated electron in any dimensional space none topological modifications of cyclotron trajectories occur. But in 2D strongly correlated multielectron systems some restrictions concern trajectories which can, at specific conditions, limit the effective magnetic flux quantum to a multiple of \( \frac{\hbar}{2} \). Such a restriction must induce a significant effect, and indeed it is experimentally observed at FQHE (and in fractional Chern topological insulators if the magnetic field is substituted by the Berry field [14]).

The present paper concerns a derivation of the magnetic flux quantum in the case of correlated 2D electron systems in perpendicular magnetic (or Berry) field changing the homotopy of trajectories. It occurs that at sufficiently strong field corresponding to fractional, \( \frac{\hbar}{2n} \) (\( k \)-positive integer), fillings of the lowest
Landau level (LLL), the effective magnetic flux quantum is \((2k + 1)\) times larger than \(\Phi_0 = \frac{\hbar}{e}\). This topological effect arises due to the correlations of 2D electrons is closely linked with FQHE. The repulsion of electrons governed over correlation induces exclusively in 2D geometry a special rearrangement of cyclotron trajectories, which causes an increase of the effective magnetic flux quantum.

2. Magnetic field flux quantum

Let us consider a quasiclassical wave function of a single charged particle (let say of an electron with a charge \(e\)) in the form of, \(\Psi = C \exp\left(\frac{\mathbf{p} \cdot \mathbf{r}}{\hbar}\right)\), where \(\mathbf{p}\) is the kinematic momentum, \(C\) is the normalization constant. In the presence of the magnetic field, \(\mathbf{H}\), this wave function attains the form, \(\Psi = C \exp\left(\frac{\mathbf{p} - e\mathbf{A} \cdot \mathbf{r}}{\hbar}\right)\), where \(\nabla \times \mathbf{A} = \mathbf{H}\). If the contour integral is taken along a closed loop, \(\delta D\), in the configuration space of the system, then the requirement of the single-valued wave function definition causes that \(\oint_{\delta D} \mathbf{A} \cdot d\mathbf{r} = n 2\pi\), or, due to the Stokes theorem, \(\int_D \nabla \times \mathbf{A} \cdot d\mathbf{s} = \int_D \mathbf{H} \cdot d\mathbf{s} = nh/e\). The magnetic field flux quantum, for \(n = 1\), is thus given by,

\[
\Phi_0 = \int_D \mathbf{H} \cdot d\mathbf{s} = \frac{\hbar}{e},
\]

where \(D\) is an arbitrary membrane spanned on the contour \(\delta D\).

However, the above simple derivation of the magnetic field flux quantum may be modified in 2D multielectron correlated system exposed to a strong perpendicular magnetic field, which could change the homotopy of electron trajectories. It is a matter of a topology of the multiparticle configuration space of which type can be the contour \(\delta D\) and it would be different than that for a single electron. Closed loops in any metric space, \(\mathcal{A}\), are defined by the first homotopy group, \(\pi_1(\mathcal{A})\) (called often as the fundamental group). Especially interesting is this group in the case of multiparticle systems of identical indistinguishable particles, known in this case as the braid group \([15–17]\). The loops in braid groups must reflect the indistinguishability of particles (of \(N\) particles enumerated in an arbitrary but fixed manner)—they are loops in the multiparticle configuration space, \(\Omega = (\mathcal{M}^N - \Delta)/S_N\), where \(\mathcal{M}\) is a manifold at which particles are located, \(\mathcal{M}^N = M \times M \times \cdots \times M\), is \(N\)th fold normal product, \(\Delta\) is a diagonal subset of \(\mathcal{M}^N\) with at least two particle positions coinciding, subtracted in order to assure particle number conservation, and the division by the permutation group \(S_N\) expresses the indistinguishability of particles.

It has been proven \([18, 19]\) that the braid group, \(\pi_1(\Omega)\), is generated by finite number of generators, \(\sigma_i\), \(i = 1, \ldots, N - 1\), describing exchanges of \(i\)th particle position with \((i + 1)\)th one. Because of particle indistinguishability, \(\sigma_i\) are closed loops in \(\Omega\). Despite a braid has in general \(N\) strands mutually entangled, the braids \(\sigma_i\) have a structure of two entangled strands only representing \(i\)th and \((i + 1)\)th particle exchange, whereas the other particles maintain their positions and representing them strands are straight not entangled lines. Such a simple braid can be associated with a semicircle along which the \(i\)th particle traverses with respect to \((i + 1)\)th particle on the manifold \(M\) on which particles are distributed. \(\sigma^2_i\) is thus a full circle on \(M\) which is traversed by \(i\)th particle around \((i + 1)\)th one (a circle can be substituted by any deformed but homotopic loop, because braids are classes of homotopic trajectories \([15, 19]\); this loop can be contractible or not, depending on \(M\)). Any multiparticle-trajectory closed loop in the configuration space \(\Omega\) can be expressed as some braid being the group-type product of braid group generators, as usual in multiply cyclic algebraic groups. The generators fulfill the conditions depending on the manifold \(M\). If \(M = \mathbb{R}^2\) these conditions have the form \([18]\),

\[
\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad \text{for} \ 1 \leqslant i \leqslant N - 2,
\]

\[
\sigma_i \sigma_j = \sigma_j \sigma_i, \quad \text{for} \ 1 \leqslant i, j \leqslant N - 1, \ |i - j| \geqslant 2,
\]

whereas for \(M = \mathbb{R}^d, d \geqslant 3\), \(\pi_1(\Omega) = S_N\) (the permutation group), with the condition for generators, \(\sigma^2_i = e\) (\(e\) is the neutral element) \([19]\). In the latter case the loop \(\sigma^d_i\) on \(M\) is contractible, whereas for \(M = \mathbb{R}^2\), not. In the case of a compact \(M\), like a sphere or torus, the conditions for generators are more complicated due to global topology of compact manifolds \([20, 15]\).

In the result, the braid groups for 2D manifolds (or locally 2D) are infinite groups \([15, 18–20]\), whereas for arbitrary 3D (or higher dimensional) manifolds, braid groups are finite and equal to the permutation group \(S_N\) \([19]\).

The question on the magnetic field flux quantum resolves itself thus to the form of closed loops from the appropriate \(\pi_1\) group. In majority of cases, including multiparticle systems, the formula (1) holds. However, in some special case of 2D multiparticle system of interacting indistinguishable particles with confined homotopy of trajectories by correlations, the situation can change and the realization of the minimal flux quantum (1) by braid loops may be impossible.
2.1. Metrics of braids imposed by the cyclotron-type effect in correlated 2D electron system

Loops from the braid group reflect the topology of the Ω space in dependence of M manifold. The dynamic particularities including interaction between particles do not influence the homotopy classes and do not change the braid group, except, however, for a special case of 2D repulsing charged particles located on a planar oppositely charged uniform jellium and exposed to a perpendicular magnetic field or, of more general, Berry-type field [17, 21, 22, 14].

To be more specific, let us consider N electrons distributed on the planar positive jellium with surface S (in thermodynamic limit, N → ∞, but with surface density kept constant, \( \frac{N}{S} = \text{const} \)). Electrons repulse themselves and the system is stable due to the interaction with the jellium of mutually repulsing electrons. The repulsion of electrons deposited on the positive jellium causes that at \( T = 0 \) K the stable lowest energy classical distribution of electrons is uniform of triangle form Wigner lattice. Still arbitrary loops are admitted here with not confined dimension.

If, however, the perpendicular magnetic field is included, and all electrons fall into the LLL, then all trajectories must be of cyclotron type and are identical due to LL degeneracy. Planar cyclotron orbits are of finite size and all the braids must be built of pieces of cyclotron orbits, thus are also of finite size. The planar cyclotron effect (in 2D without a driving motion along the perpendicular magnetic field direction) imposes thus some metrics on braids. The braids representing generators \( \sigma_i \) must be of cyclotron orbit size as they are halves of cyclotron orbits (the latter can be associated with \( \sigma'_i \)) and, moreover, can be defined only if the cyclotron metrics agrees with the metrics of the Wigner crystal, because braids must match electron pairs (at a specific electron numeration, that the consecutive electrons are the nearest neighbors). This is possible only at some discrete values of the magnetic field. If the cyclotron orbit surface perfectly fit to the surface per single particle in Wigner lattice, then the exchanges of nearest electrons are possible and \( \sigma_i \) can be defined at such a numeration of particles that consecutive ones are nearest neighbors. In this case the multielectron system is correlated. Let us emphasize that the interaction of electrons induces the correlation because it conditions the Wigner lattice classical distribution of electrons required to the definition of the commensurability of electron separation with the cyclotron metrics of braids.

The cyclotron orbit size is defined by, \( \Phi_0 = \frac{h}{2eH} \), and the commensurability of braid metrics with the particle separation in Wigner crystal is given by the condition for the field \( H = H_0 \), at which the commensurability (figure 1(a)),

\[
\frac{\Phi_0}{H} = \frac{\hbar}{eH} = \frac{S}{N},
\]

is fulfilled (i.e., equation (3) is fulfilled only at \( H = H_0 \)). In the above equation \( \Phi_0 = \frac{\hbar}{2e} \), as in equation (1), i.e., with integration along the \( \delta D \) defined by doubled braid \( \sigma'_i \), being of the same size for each \( i \), as \( \sigma_i \) is the half of the cyclotron orbit on \( M \) and of the same size and homotopy as for a single electron. None other simpler trajectories of electrons exist in the braid group at \( H = H_0 \) and the correlations are defined here fully by \( \sigma_i, i = 1, \ldots, N - 1 \).

If the magnetic field grows, then for \( H > H_0 \) the condition (3) is no longer fulfilled and \( \sigma_i \) generators cannot be defined—cyclotron orbits are too short and cannot match the closest particles in the Wigner lattice (figure 1(c)). Therefore in the 2D system of interacting electrons upon the field \( H > H_0 \) the flux quantum formula (1) is no longer applied for braid loops. Cyclotron braid orbits \( \sigma'_i \) are too small in comparison to electron separation in Wigner lattice described with constant \( \frac{S}{N} \). In other words, the contour \( \delta D \) required to the definition of the flux \( \Phi_0 \) cannot be built of \( \sigma_i \) as the latter cannot be defined at field \( H > H_0 \).

Nevertheless, in the braid group \( \pi_1(\Omega) \) there exist other group elements which could be defined in consistency with flux quantum definition and being commensurate with the Wigner lattice. Such elements are \( \sigma^{2k+1}_i \), \( k \) positive integer, which are also exchanges of neighboring particles, but with additional \( k \) loops, \( \sigma^k_i \), in \( \Omega \). The corresponding to these braids cyclotron orbits on \( M \) with \( 2k + 1 \) loops, \( (\sigma^{2k+1}_i)^2 \), can be taken as the contour \( \delta D' \) for the definition of the flux quantum. For such a multi-loop trajectory, \( \delta D' \), the integrals, \( \int_{\delta D'} A \cdot d\mathbf{r} = (2k + 1) \int_{\delta D} A \cdot d\mathbf{r} = (2k + 1) \Phi_0 = \Phi_0 \), and the \( 2k + 1 \) times larger flux can be now considered as the flux quantum consistent with the braid group structure (in the case when \( \sigma_i \) does not exist). Note, that \( \sigma^{2k+1}_i \) are also exchanges of neighboring particles and can be used as the braid group generators instead of \( \sigma_i \). The cyclotron multi-loops orbits given by \( (\sigma^{2k+1}_i)^2 \) have a larger size, \( \frac{\Phi_0}{H} = (2k + 1) \frac{\hbar}{eH} \) (e.g., for \( k = 1 \) in figure 1(b)). The commensurability condition of these cyclotron orbits with the Wigner crystal defines the field \( H = H_0 \), at which the following commensurability condition,

\[
(2k + 1) \frac{\hbar}{eH} = \frac{S}{N}
\]

is fulfilled (i.e., equation (4) is fulfilled only if \( H = H_0 \)). The values of magnetic fields \( H_k \) and \( H_0 \), at which equations (4) and (3) are fulfilled (i.e., the fields at which the FQHE or IQHE occur) depend on the
Figure 1. (a) Single-loop cyclotron orbit (left) and the corresponding braid for the generator $\sigma_i$ of the full braid group (center) in the case when its metrics fits to the separation of electrons on the plane at $\nu = 1$; (right) the electron Wigner lattice is shown with single-loop cyclotron orbit fit to nearest neighbors separation. (b) Three-loop cyclotron orbit (left) for 3 times larger magnetic field and the corresponding braid (center) for generator $\sigma_3 i$ of the cyclotron subgroup with metrics which perfectly fits to electron separation in the Wigner lattice at $\nu = \frac{1}{3}$ (right). (c) Single-loop cyclotron orbit for the field 3 times larger than in the case of (a) at the same density of 2D electrons, $\nu = \frac{1}{3}$ (left)—the braid $\sigma_i$ cannot be defined because is too small (cannot reach even the closest electrons in the Wigner lattice) (center and right).

electron surface concentration $N_s$. In GaAs samples these fields are in the range of 10–20 T [23], whereas in graphene monolayer these fields can be strongly reduced (even to single T only) because in graphene the number of electrons in the conduction band can be changed by the application of a lateral voltage which shifts the Fermi level near the Dirac point [24].

If one expresses the fields $H_0$ and $H_k$ in terms of the Landau level degeneracy, $N_0(H) = \frac{H_0}{h/e}$, one can find that the condition (3) resolves itself to the filling ratio, $\nu = \frac{N}{N_0(H_0)} = \frac{N_0}{H_0 h/e} = 1$, whereas the condition (4), to the filling ratio, $\nu = \frac{N}{N_0(H_k)} = \frac{1}{2k+1}$. One can recognize the above filling rates as corresponding to the IQHE at $\nu = 1$ and FQHE (Laughlin states [9]) at $\nu = \frac{1}{2k+1}$, i.e., to the correlated states when the correlations are imposed by the Wigner lattice commensurability with the cyclotron orbits defined by single-loop or multi-loop cyclotron orbits consistent with the braid structure.

The new braid groups generated by $\sigma_i^{2k+1}$ do not contain $\sigma_i$ elements and are subgroups of the full braid group generated by $\sigma_i$. These subgroups we call as the cyclotron braid subgroups. The latter can be generalized including the commensurability of cyclotron orbits (single-loop and multi-loop ones) with next-nearest neighbors in the Wigner crystal and for the commensurability of each loop of a multi-loop cyclotron orbit separately with $x$th order neighbors of the Wigner crystal.

The resulted commensurability condition patterns reproduce the filling rate hierarchy perfectly coinciding with all know from experiments FQHE and IQHE hierarchy in the LLL and in higher LLs in GaAs 2D systems and in graphene monolayer and bilayer [25] as well as in the Chern topological insulators when the cyclotron effect is substituted by the Berry-type field [14] instead of the magnetic field [26].

3. Multi-loop cyclotron braids have larger size than single-loop ones

We have shown above that the magnetic field flux quantum $\Phi_0 = \frac{h}{e}$ may not be attainable in some homotopy phases of multiparticle 2D electron correlated systems in magnetic field. Though the flux $\Phi_0$ given by equation (1) still is the fundamental magnetic flux quantum for a single electron, the correlation in multiparticle state requires the commensurability of braid size with electron distribution, which may modify cyclotron trajectories limiting also an effective quantization of magnetic flux. Various patterns of
this commensurability generate effective flux quanta for multi-loop trajectories in consistence with the cyclotron braid subgroup structure. In particular, a larger flux quantum for multi-loop cyclotron braid orbits defines larger spatial dimension of such orbits. Larger orbits can reach particles too distant for single-loop orbits and cyclotron braid subgroups define then correlations of electrons instead of the full braid group. This fact is the origin of the FQHE as the correlated states. The role of electron repulsion is fundamental for these correlation, because the interaction defines the Wigner lattice in which braid correlations can be considered. FQHE was heuristically modeled by auxiliary field flux quanta attached to electrons in order to construct a hypothetical composite fermions (CFs) experienced reduced field [27]. The CFs model in an pictorial manner illustrates the fundamental braid commensurability of multi-loop cyclotron orbits with the Wigner lattice. It has been demonstrated [26] that CFs can be utilized as the model of homotopy patters but only for the commensurability of multi-loop braids with nearest neighbors in Wigner lattice [26]. This suffices to describe only the main hierarchy of FQHE in the LLL of GaAs and fails in explanation of so-called enigmatic states in the LLL of GaAs and in higher LLs of all Hall materials, in bilayer graphene and monolayer graphene also in theLLL and in the fractional Chern insulators, wherever next nearest neighbors correlations are important. All the cases beyond the reach of CF model can be described by the cyclotron braid subgroups including correlations with next-nearest neighbors in the Wigner crystal, in consistence with experiment [25, 26] (figure1).

The proof for the increase of multi-loop orbit size can be independently performed also via the Bohr–Sommerfeld rule, which links the area of 1D phase space ranged by the classical phase trajectory loop with the corresponding number of quantum states. The quasiclassical wave function in a 1D well, \( U(x) \), with turning points \( a \) and \( b \) has the form,

\[
\Psi(x) = \left\{ \begin{array}{ll}
\frac{e}{\sqrt{\pi}} \sin \frac{1}{\hbar} \int_{a}^{c} p \, dx, & \text{for } \Psi(a) = 0, \\
\frac{e}{\sqrt{\pi}} \sin \frac{1}{\hbar} \int_{b}^{c} p \, dx, & \text{for } \Psi(b) = 0,
\end{array} \right.
\]

(5)

where \( p(x) = \sqrt{2m(E - U(x))} \) (for simplicity, assuming vertical infinite borders of the well). Uniqueness of the wave function requires,

\[
2 \int_{a}^{b} p \, dx = \oint_{a}^{b} p \, dx = S_{\text{xp}} = n2\pi\hbar = nh,
\]

(6)

which is the Bohr–Sommerfeld quantization rule (\( n \) is an integer; for finite borders, \( S_{\text{xp}} = (n + \frac{1}{2})\hbar [28] \)). The above has been derived upon the condition that the trajectory is single-loop. For a different homotopy class and for a multi-loop trajectory one gets, however,

\[
2 \int_{a}^{b} p \, dx = \oint_{a}^{b} p \, dx = S_{\text{px}} = (2k + 1)n2\pi\hbar = n(2k + 1)h,
\]

(7)

for a trajectory \( (a, b) \) with additional \( k \) loops. Each loop of all \( 2k \) loops symmetrically pinned (by \( k \) loops) to both branches, ‘upper’ \( (+p) \) and ‘lower’ \( (-p) \), of the closed trajectory between \( a \) and \( b \) in the integral \( \oint p \, dx \) adds \( 2\pi \).

This is of particular importance when the Bohr–Sommerfeld rule is applied to an effective 1D phase-space \( (Y, P_{y}) \) of \( x, y \) components of the 2D kinematic momentum in the presence of a perpendicular magnetic field. The kinematic momentum components (at the Landau gauge, \( A = (0, Hx, 0) \)),

\[
P_{x} = -ih \frac{\partial}{\partial x},
\]

(8)

\[
P_{y} = -ih \frac{\partial}{\partial y} - eHx,
\]

do not commute,

\[
[P_{x}, P_{y}] = i\hbar eH.
\]

(9)

The pair of operators, \( Y = \frac{1}{i\hbar} P_{x} \) and \( P_{y} \), can be treated as operators of canonically conjugated generalized position \( Y \) and momentum \( P_{y} \), because \( [Y, P_{y}] = i\hbar \). Thus, the 1D effective phase space, \( (Y, P_{y}) \), is actually the 2D space, \( (P_{x}, P_{y}) \). The latter 2D kinematic momentum space is, on the other hand, the renormalized by the factor \( \frac{1}{i\hbar} \) and turned in plane by \( \pi/2 \) the ordinary 2D space \( (x, y) \), which is visible due to the quasiclassical formula for the Lorentz force, \( F = \frac{\partial}{\partial t} \mathbf{P} = e \frac{\partial}{\partial t} \times \mathbf{H} \), which gives \( dP_{x} = eHdy \) and \( dP_{y} = -eHdx \).

In 2D position space, trajectories \( (x, y) \) may belong to different homotopy classes if these trajectories correspond to braids in multiparticle 2D correlated state. If not, then the multiparticle state is not correlated. The loops of particles in correlated multiparticle state must be represented in braid group terms,
in particular of cyclotron braid subgroup form. For such trajectories one obtains from the generalized Bohr–Sommerfeld rule,

$$S_{Yj} = n(2k + 1)\hbar,$$

or rewritten to \((x,y)\) space,

$$S_{xy} = (2k + 1)n\hbar \frac{eH}{\epsilon},$$

which defines the quantum of the magnetic field flux \(\Phi_L\) in correlated state,

$$\Phi_L = \Delta S_{xy}H = \frac{(2k + 1)\hbar}{e},$$

\(\Delta S_{xy}\) is the change of \(S_{xy}\) in equation (11) when \(n\) is shifted by 1. Only for \(k = 0\), i.e., for the homotopy class without additional loops, the flux quantum equals to \(\Phi_L = \frac{\hbar}{e}\).

Different magnetic field flux quanta \(\Phi_L\) define different sizes of multi-loop cyclotron orbits, \(\Phi_L/H\). The IQHE corresponds to \(k = 0\) (the homotopy class of single-loop cyclotron orbits) and the cyclotron orbit size for \(k = 0\) equals to \(\Delta S_{xy} = \frac{\hbar}{2eH} = \frac{s}{N} = \frac{\hbar}{Ny}\), which gives \(\nu = \frac{N}{N_0} = 1\), \(N_0 = \frac{h\nu}{e}\) is the LL degeneracy taken here for \(H_0\)—the magnetic field for \(\nu = 1\).

The FQHE main line corresponds to \(k = 1, 2, \ldots\) (the homotopy classes with \(q = (2k + 1)\)-loop cyclotron orbits or braids with \(k\) additional loops); e.g., for \(k = 1\) (the simplest Laughlin state), the triple-loop cyclotron orbit has the size \(\Delta S_{xy} = \frac{3\hbar}{2eH}\). This orbit for \(H_1 = 3H_0\) fits to interparticle separation \(\frac{3h}{2e}\) — hence, from the commensurability condition, \(\frac{3h}{2e} = \frac{1}{N}\), one obtains, \(\nu = \frac{N}{N_0} = \frac{N}{1/N_0} = \frac{1}{3}\), as for the simplest FQHE state.

The quasiclassical method of Bohr–Sommerfeld quantization applied to many particle systems is interaction independent, i.e., it holds for arbitrarily strongly interacting multiparticle systems. The sizes of magnetic flux quanta are also interaction independent for different homotopy classes, although the existence of nonhomotopic trajectories in \((x,y)\) space is conditioned by the Coulomb interaction of 2D charged particles. In a gas system of noninteracting particles their mutual positions are arbitrary, which dismisses correlations and nontrivial homotopies. Moreover, the step in the quasiclassical Bohr–Sommerfeld formula is exact and not limited to the quasiclassical approximation only.

The Laughlin correlations expressed by the exponential, \(q = 2k + 1\), in the Jastrow polynomial manifesting themselves by the phase shift \(q\pi\) when particle interchange [9], is born by the one dimensional unitary representation (1DUR) of the cyclotron braid group generator with additional \(k\) loops [29]. Actually, for the initial 1DUR for ordinary electrons, \(\sigma_i \rightarrow e^{i\pi}\), we obtain the projective 1DUR for the cyclotron subgroup at \(q = 2k + 1\), \(\sigma_i^q \rightarrow e^{iq\pi}\), which agrees with the Laughlin phase shift \(q\pi\) [9]. The multiparticle wave function must transform according to 1DUR of the braid if the exchange of coordinates of these wave function (classical positions of particles) undergoes according to this particular braid [30–32] (note that in 2D this is not only a permutation).

### 3.1. Versatility of the degeneracy of Landau levels

The degeneracy of LLs is the property which is independent of material and interaction and results from the form of ladder operators of kinematic momentum of a 2D charged particle (electron) exposed to a perpendicular magnetic field—it is the single-particle property. At Landau gauge, \(A = [0, Hx, 0]\), \(B = \nabla \times A = [0, 0, H]\), the kinetic energy of an electron,

$$\hat{H} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \left( -i\hbar \frac{\partial}{\partial y} - eHx \right)^2,$$

and the wave function,

$$\Psi = e^{ip_y/\hbar}f(x),$$

result in the equation for \(f(x)\),

$$-\frac{\hbar^2}{2m} \frac{d^2f(x)}{dx^2} + \frac{e^2H^2}{2m} \left( x - \frac{p_y}{eH} \right)^2 f(x) = Ef(x).$$

Equation (15) is the oscillator equation with energy spectrum, \(E_n = (n + \frac{1}{2})\hbar\omega_H\), where \(\omega_H = \frac{eB}{\hbar}\), with the degeneracy due to a shift of the oscillator center, \(x_0 = \frac{p_y}{eH}\). This oscillator center must be located inside the sample \(L_x \times L_y\), i.e., \(0 < x_0 < L_x\), whereas \(p_y = \frac{2\pi\hbar x_0}{L_y}\). The condition for \(x_0\) gives,

$$0 < p_y < eBL_x,$$
thus the total number of states for a whole range of \( p_x \) equals to \( \frac{L_{\perp} l_H}{h} \), which is the degeneracy of LLs, \( N_0 \).

A filling rate of LLs is expressed by the ratio \( \nu = \frac{N}{N_0} \). In the case of Chern topological insulators without LLs, a filling rate is counted per crystalline node, i.e., \( \nu = \frac{N}{N_0} \), where \( n_0 \) is number of nodes in the lattice.

### 3.2. Magnetic flux quantum for noninteracting or noncorrelated electrons

It is interesting to outline the linkage of the magnetic flux quantization with LL structure. The degeneracy of LLs equals to \( N_0 = \frac{h c}{R} \). In the case of the electron gas, the Fermi surface is spherical and in the system none correlations exist. The electron motion in magnetic field is quasclassically governed by the Lorentz force, \( \mathbf{\dot{p}} = e \mathbf{v} \times \mathbf{B} \), thus with constant energy, \( \varepsilon = \text{const.} \) and constant \( p_z = \text{const.} \). (in 3D case), which follows from the Lorentz force formula in the following manner, \( \frac{\partial \varepsilon}{\partial t} = 0 \) for \( \mathbf{B} = \mathbf{0} \) and \( \mathbf{\dot{p}} = \mathbf{\dot{r}} \times \mathbf{B} = 0 \) (note that \( \mathbf{P} = \mathbf{p} - e \mathbf{A} \)).

Hence, for isotropic gaseous Fermi surface, the quasiclassical trajectory in the momentum space is a circle with the surface,

\[
\pi (p_x^2 + p_y^2) = \pi \left( 2m \varepsilon - p_z^2 \right),
\]

as for \( \varepsilon = \frac{p_z^2}{2m} \). Quantization of the magnetic field flux defines, on the other hand, the surface of this orbit in the kinematic momentum space as,

\[
\oint \mathbf{P}_\alpha d\mathbf{P}_\gamma = \left( n + \frac{1}{2} \right) \hbar c \mathbf{H}.
\]

Thus, from equations (17) and (18) one gets immediately,

\[
\varepsilon = \left( n + \frac{1}{2} \right) \frac{c H m}{\hbar} + \frac{p_z^2}{2m} = \hbar \omega_H \left( n + \frac{1}{2} \right) + \frac{p_z^2}{2m},
\]

which is the Landau energy spectrum in the gas (in 2D case, the free motion term in \( z \) direction, \( \frac{p_z^2}{2m} \), disappears).

One can notice, that for the Dirac-like dispersion in the graphene monolayer, with \( \varepsilon = v_F p_x \), one gets,

\[
\pi (p_x^2 + p_y^2) = \pi \frac{\varepsilon^2}{v_F^2},
\]

which with flux quantization (18) imposed, gives the Landau energy spectrum,

\[
\varepsilon = \sqrt{n + \frac{1}{2}} \frac{c h v_F}{2 l_H} \simeq \sqrt{n} \frac{h v_F}{l_H},
\]

where \( l_H = \sqrt{\frac{h}{e B}} \) is the magnetic length. The quasiclassical approximation holds for large \( n \), thus \( \sqrt{n + \frac{1}{2}} \simeq \sqrt{n} \), and equation (21) reproduces the exact LL energy formula in graphene monolayer [33].

The zero energy for \( n = 0 \) is, however, beyond the quasiclassical approach [33].

In graphene bilayer, the single-particle energy spectrum close to Dirac points again is parabolic one, which gives the linear dependence on \( n \) for LL energy, \( \varepsilon = (n + \frac{1}{2}) \hbar \omega_B \) (for large \( n \), \( n + \frac{1}{2} \simeq \sqrt{n} \)), the latter is an exact form of the dispersion, which accounts the so-called ‘accidental’ degeneracy of oscillator states with \( n = 0 \) and \( n = 1 \) in bilayer-graphene [34]).

Though the formula (18) is quasiclassical, the value of its step, \( \Delta \sigma_{p_x,p_y} = \hbar c H \), is exact. The above illustrates the universal character of the magnetic field flux quantization and proves that Landau quantization is in fact its consequence in the case of gaseous systems, as a single-particle property.

### 4. Unitary representations of cyclotron braid subgroups

The full braid group generated by \( \sigma_i \) for 2D manifold \( M \) on which particles are located has 1DURs in the form, \( \sigma_i \rightarrow e^{i \alpha}, i = 1, \ldots, N - 1, \alpha \in [0, 2\pi) \) [19]. These 1DURs do not depend on index \( i \) of generators due to relation (2). For 3D \( M \) (or higher dimensional), when the braid group is the permutation group, \( \alpha \) can be only 0 or \( \pi \) (corresponding to bosons and fermions, respectively) [15, 17]. For 2D manifolds \( \alpha \) from the entire compartment \( [0, 2\pi) \) defines anyons [17, 21, 22]. 1DUR determines the statistics of particles, in particular, fractional statistics of anyons, because the multiparticle wave function \( \Psi(r_1, \ldots, r_N) \) acquires the phase factor equal to the 1DUR of the particular braid when coordinates \( r_1, \ldots, r_N \) exchange according to this braid [30, 32].
For the cyclotron subgroup generated by $\sigma_y^{2k+1}$, the projective 1DURs have thus the form, $\sigma_y^{2k+1} \rightarrow e^{i(2k+1)\pi \gamma}$. For $\alpha = \pi$, as for original electrons, the 1DUR of cyclotron subgroup is thus $\sigma_y^{2k+1} \rightarrow e^{i(2k+1)\pi}$.

Because the multiparticle wave function of interacting 2D electrons upon the magnetic field must be a holomorphic function and homogeneous with respect to particle coordinates, the unique form of it is the Laughlin function [9] to fulfill the condition imposed by the 1DUR for particular $k$. The Laughlin function, $\Psi(z_1, \ldots, z_N) = \mathcal{A} \prod_{i=1}^{N} (z_i - z_{i+1})^{2k+1} e^{-\sum_{i=1}^{N} |z_i|^2/\nu_i}$, ($z_i$ is the complex representation of $r_i$), has a Jastrow polynomial factor of order $(2k + 1)$.

The same holds for more complicated cyclotron subgroups, when multi-loop orbits accommodate to separation of nearest or next-nearest neighboring electrons, independently for each loop. Assigning by $x_j$, $j = 1, \ldots, 2k + 1$ the next-nearest neighbor order (for $x_j = 1$, the nearest neighbors), one can write out the cyclotron commensurability condition [26],

$$\frac{HS}{N} = \frac{h}{x_1 e} \pm \cdots \pm \frac{h}{x_{2k+1} e}.$$  \hfill (22)

Assuming that $x_1 = x_2 = \cdots = x_{2k} = x$ and $x_{2k+1} = y$, and maintaining $\pm$ only before the last term ($-$ denotes the inverted direction of the loop) one can obtain the subclass of the FQHE hierarchy (22) at filling rates,

$$\nu = \frac{N}{N_0} = \frac{xy}{2ky \pm x}. \hfill (23)$$

The hierarchy (23) gives all experimentally observed FQHE features in the LLL of GaAs [26], including all those described by the CF model [27] for $x = 1$, and also all so-called enigmatic states for $x > 1$ being beyond the reach of phenomenological CF model [23]. In the LLL of monolayer graphene the hierarchy (23) is, however, too simplified and the general hierarchy (22) must be applied to match with experimental observations [35].

For the arbitrary cyclotron subgroup, we can write out the general form of the braid generators, the form of the generators corresponding to homotopy phases enumerated by the hierarchy (23) is as follows (for $\pm$ in (23), $q = 2k + 1$):

$$B_i^{\pm \sigma_y^{q}} = (\sigma_i \sigma_{i+1} \cdots \sigma_{i+x-2} \sigma_{i+x-1} \cdots \sigma_{i+2} \sigma_i) \quad \left. \begin{array}{l} \sigma_i \sigma_{i+1} \cdots \sigma_{i+x-2} \sigma_{i+x-1} \sigma_{i+1} \cdots \sigma_i \sigma_{i+2} \sigma_i \\ \sigma_i \sigma_{i+1} \cdots \sigma_{i+x-2} \sigma_{i+x-1} \sigma_{i+1} \cdots \sigma_i \sigma_{i+2} \sigma_i \end{array} \right\} \quad \left( \begin{array}{l} \sigma_i \sigma_{i+1} \cdots \sigma_{i+x-2} \sigma_{i+x-1} \sigma_{i+1} \cdots \sigma_i \sigma_{i+2} \sigma_i \\ \sigma_i \sigma_{i+1} \cdots \sigma_{i+x-2} \sigma_{i+x-1} \sigma_{i+1} \cdots \sigma_i \sigma_{i+2} \sigma_i \end{array} \right)$$  \hfill (24)

The 1DURs for the generators (24) equal to $e^{i\pi}$ (for + in the superscript of $b$) and $e^{-i\pi}$ (for -), $q = 2k + 1$, both for $\alpha = \pi$ in $\sigma_i \rightarrow e^{i\alpha}$, as needed for original old electrons.

The structure of the generators of cyclotron braid subgroups (24) together with their 1DURs determine the shape of trial wave functions for particular homotopy phases. In the LLL the wave function for interacting particles must be holomorphic function with its polynomial part uniquely defined by nodes, whereas the exponential confining envelope function, $e^{-\sum_{j=1}^{N} |z_j|^2/\nu_j}$, is the same for all arbitrary linear combinations of noninteracting $N$ particle Landau functions (thus, is the same for interacting particles). Therefore, the uniquely defined trial wave functions for the hierarchy (23) are as follows,

$$\Psi_{q}^{\pm \sigma_y^{q}}(z_1, z_2, \ldots, z_N) = \mathcal{A} \prod_{i=1}^{\text{mod } N/x} (z_i - z_{i \text{ mod } x+(j-1)x})^{q-1} \quad \left. \begin{array}{l} \prod_{j=1}^{x} (z_i - z_{i \text{ mod } x+(j-1)x})^{q-1} \\ \prod_{j=1}^{x} (z_i - z_{i \text{ mod } x+(j-1)x})^{q-1} \end{array} \right\} \prod_{i=1}^{\text{mod } N/y} (z_i - z_{i \text{ mod } y+(j-1)y})^{q-1} \quad \left. \begin{array}{l} \prod_{j=1}^{y} (z_i - z_{i \text{ mod } y+(j-1)y})^{q-1} \\ \prod_{j=1}^{y} (z_i - z_{i \text{ mod } y+(j-1)y})^{q-1} \end{array} \right\}$$  \hfill (25)

For $x = 1$ these wave functions describe homotopy phases for the CF hierarchy. For $x = y = 1$ they resolve themselves to the Laughlin functions. The similarity to multi-component Halperin functions [36] is also
Table 1. Comparison of energy values (per particle in units, $\epsilon_{\text{xc}}/e^2$) obtained by exact diagonalization (Ex. diag.)
and by quantum MMC simulation for few exemplary filling fractions with FQHE (MMC simulation for trial wave
functions acc. to equation (25) for 200 particles [26]).

| $\nu = N/N_0$ | 2/5 | 3/7 | 4/9 | 5/11 | 2/9 | 3/13 | 4/17 |
|---------------|-----|-----|-----|------|-----|------|------|
| MMC sim.      | −0.432 677 | −0.441 974 | −0.446 474 | −0.451 056 | −0.342 379 | −0.348 134 | −0.351 857 |
| Ex. diag.     | −0.432 804 | −0.442 281 | −0.447 442 | −0.450 797 | −0.342 742 | −0.348 349 | −0.351 189 |

noticeable. For comparison, in table 1 we show averaged energies of states (25) calculated by Metropolis
Monte Carlo (MMC) method [37] (for $N = 200$ particles) compared with energies obtained by the exact
diagonalization of the interaction in small models for the same filling rates. The agreement is better than for
trial wave functions in CF model constructed by the procedure of the projection from higher LLs onto the
LLL in order to remove singularities [38]. The projection procedure leading to holomorphic functions
required in the LLL is not uniquely defined and is semi-empirically accommodated to optimize the energy.
This causes, however, uncontrolled accidental symmetry perturbations of the resulting trial wave function
in comparison to the accurate symmetry kept by the functions (25) in the limit $\lambda = 1$. Such perturbations
eventually cause some discrepancy in mean energy in comparison to more accurate homotopy phase
energies (calculated acc. (25)) in consistence with exact diagonalization). The energies in table 1 are in
compliance, on the other hand, with experimentally measured activation energies of correlated states for
particular $\nu$ in 2DES GaAs. In monolayer graphene the more general hierarchy (22) is required because the
exponential term in wave functions is not as for GaAs (25) due to different form of single particle wave
function envelope in monolayer graphene [35, 33].

5. Two-particle illustration of homotopy classes in 2D related to different magnetic
flux quantization

To visualize transparently the change of magnetic flux quantization in different correlated states of 2D
electrons in magnetic field, let us consider the simplest multiparticle, two-electron 2D system
located on the surface $S$ (positive jellium with homogeneously smeared $+2e$ charge) and exposed to a
perpendicular strong magnetic field.

In the case when the jellium is present, one can apply the same approach as for many electron 2D
systems for IQHE and FQHE. For the field $H_0$ such that the degeneracy of corresponding LLs,
$L_0(H_0) = \frac{H_0 S}{2e} = 2$, two electrons completely fill the LLL, i.e., $\nu = \frac{N}{N_0} = 1$. The related homotopy phase is
defined here by the same commensurability condition as for IQHE, $\frac{N}{N_0} = \frac{\phi_0}{\Phi_0}$, which means that the
cyclotron braid $\sigma_1$ (for $N = 2$, $i = 1$ only) perfectly fits to the electron separation, as visualized in
figure 2(b). The wave function for this simplest homotopy phase at $H_0$ is

$$
\Psi_1(z_1, z_2) = \mathcal{A}(z_1 - z_2) e^{-i(|z_1|^2 + |z_2|^2)/4\Phi_0},
$$

(26)

where $\mathcal{A}$ is the normalization constant, $z_i = x_i + iy_i$ is a complex representation of the 2D position of $i$th
particle, $r_i = (x_i, y_i)$ and $\Phi_0 = \frac{\hbar}{e}$ is the magnetic length at field $H$, here taken at $H = H_0$. This function
is the Slater function for the completely filled LLL. It must be emphasized here that this Slater function, being,
on the other hand the wave function of two noninteracting particles in the LLL of gas, is simultaneously the
ground state for the interacting electrons at $\nu = 1$. It is an exceptional situation when the different systems,
with and without interaction, have the same ground state eigenfunctions corresponding, however, to
different Hamiltonians (and different energies),

$$
\hat{H}_{\text{int}} = \sum_{i=1}^{2} \left( \hat{p}_i - e A_i \right)^2 / 2m,
$$

(27)

where the vector potential $A_i = \frac{\hbar}{2}(-H y_i, H x_i)$ in the symmetrical gauge for magnetic field, and

$$
\hat{H}_{\text{int}} = \hat{H}_{\text{int}} + \frac{e^2}{4\pi \varepsilon_0 |\mathbf{r}_1 - \mathbf{r}_2|} + \hat{H}_{jj} + \hat{H}_{ij},
$$

(28)

where the jellium–jellium interaction,

$$
\hat{H}_{jj} = \frac{\rho_j^2}{2} \int_S d^2 r \int_S d^2 r' \frac{e^2}{4\pi \varepsilon_0 |\mathbf{r} - \mathbf{r}'|},
$$

(29)
Figure 2. (a) Wigner lattice for $N = 2$ electrons on the jellium of surface $S$ with homogeneously smeared charge $+2e$ separation of electrons corresponds to $S/2$ surface. (b) At magnetic field $H_0$ for which the degeneracy of LLs is 2 (i.e., $N_0(H_0) = \frac{h}{eH_0} = 2$) the single-loop cyclotron orbit of size, $\Phi_0/H_0 = \pi$, perfectly fits to electron separation, just as for IQHE and the 2-electron system is correlated with braids $\sigma_i$ at filling rate $\nu = \frac{2}{3} = 1$. (c) For $H_1 = 3H_0$ the single-loop orbit is too short to match electrons—the system cannot be correlated with single-loop orbits, or braids $\sigma_i$. (d) For $H_1 = 3H_0$ three-loop cyclotron orbit of size, $3\Phi_0/H_1 = \frac{3\pi}{3} = \pi$, again perfectly fits to particle separation and the system is correlated (FQHE) with three-loop cyclotron orbits, or braids $\sigma_3$ at filling rate $\nu = \frac{1}{3}$ (as $N_0(H_1) = 6$).

(with the charge density $\rho_0 = \frac{1}{2\pi \hbar_0}$ and the electron–jellium interaction,

$$\hat{H}_{\text{dj}} = -\rho_0 \sum_{i=1}^{2} \int_S d^2r \frac{e^2}{4\pi \varepsilon_0 |\mathbf{r} - \mathbf{r}_i|},$$

(30)

$\varepsilon_0$ and $\varepsilon$ are the dielectric constant and the material permittivity, respectively. The commensurability pattern, $S = \frac{\hbar}{\varepsilon_0}$, uniquely determines the symmetry of the wave function (in the LLL, the wave function of interacting electrons must be a holomorphic function, defined uniquely by its nodes). Here, the 1DUR of the braid group is $\sigma_i \rightarrow e^{i\alpha}$, where $\alpha = \pi$ is chosen for original fermions, which induces the polynomial part. The confining exponent, $e^{-\left(|z_1|^2+|z_2|^2\right)/4\Phi_0}$, invariant for particle exchange is the same common factor in the subspaces of the two-particle Hilbert space spanned by eigenfunctions of noninteracting particles in the LLL for any filling rate (also for $\nu = 1$). The resulted two-particle wave function has the form of the Slater function for $N = 2$ in the LLL of noninteracting particles. This function, if related to $\hat{H}_{\text{int}}$ (without interaction), is, however, none correlated state (in the gas, none correlations present, and any braid commensurability cannot be defined in the gas), whereas the same function if related to $\hat{H}_{\text{int}}$ (as uniquely determined by the braid group 1DUR for $\nu = 1$) describes the correlated state of IQHE; this is the simplest homotopy phase. Correlations are characterized here by cyclotron braid without any loop, $\sigma_i$ ($i = 1$ here), being a half piece of the single-loop cyclotron orbit ($\sigma_2$) with the size defined by the magnetic flux quantum $\Phi_0$, i.e., $\frac{\pi}{\Phi_0}$.

Thus, we see that the braid correlations are not explicitly built-into the form of the wave function. The same wave function may describe both the correlated and gaseous 2D electron systems, as shown above. The charge distributions and the averaged electron distance at $\nu = 1$ shown in figure 3 (right) are the same for interacting and noninteracting systems because the two-particle wave function has the same form in both systems. In gas any commensurability does not, however, hold (the Wigner crystal disappears in the absence of electron interaction) and does not impose any restrictions on a full braid group. In the gas, the
finite separation of electrons is caused by the fermionic ‘repulsion’ and might be called Pauli virtual crystallization, although none other homotopy class exists in the gaseous system.

The Wigner lattice is the matter of interaction (cf, figure 2(a)) and creates the platform for the cyclotron braid commensurability condition. For example, for \( H_1 = 3H_0 \) single-loop cyclotron orbits are too short to allow correlations defined by \( \sigma_i \) (cf figure 2(c)). But the three-loop cyclotron orbit of size, \( \frac{S}{2\pi} = \frac{3\pi l_H}{2} \), again perfectly fits to Wigner lattice separation, \( \frac{S}{2\pi} \). The correlated state is possible with correlations defined by the cyclotron braid \( \sigma^i \) (\( i = 1 \) here) corresponding to the three-loop cyclotron orbit, \( (\sigma^i)^2 \). The filling factor for \( H_1 \) is, \( \nu = \frac{1}{4} \) (because the degeneracy of LLs is now, \( N_0(H_1) = 6 \) and \( N \) is still 2).

The commensurability pattern for this homotopy phase is, \( \frac{S}{2\pi} = \frac{3\pi l_H}{4} \) at \( \nu = \frac{1}{4} \) (cf figure 2(d)). The Laughlin wave function for this state has the form,

\[
\Psi_{1/3}(z_1, z_2) = B(z_1 - z_2)^3 e^{-[(|z_1|^2 + |z_2|^2)/4l_H^2]},
\]

(31)

The energies for both homotopy phases shown above can be calculated directly as,

\[
\Delta E = \langle \Psi_\nu(z_1, z_2) | \hat{H}_{\text{int}} - \hat{H}_{\text{aim}} | \Psi_\nu(z_1, z_2) \rangle, \tag{32}
\]

for \( \nu = 1 \) or \( \nu = 1/3 \). From the MMC estimation we have obtained the energies \( \Delta E / N = -0.58 \left( \frac{e^2}{4\pi \epsilon \sigma j_{\text{HH}}^N} \right) \) for \( \nu = 1 \), \( \Delta E / N = -0.39 \left( \frac{e^2}{4\pi \epsilon \sigma j_{\text{HH}}^N} \right) \) for \( \nu = \frac{1}{4} \). The sign minus indicates stability, i.e., that the jellium–electron attraction energy overcomes the jellium–jellium repulsion and electron–electron repulsion energies. The energies for the considered homotopy phases are different.

For both these homotopy classes, for \( \nu = 1 \) and \( \nu = \frac{1}{3} \), one can compare the charge density distribution,

\[
\int d^2 r_2 |\Psi_\nu(r_1, r_2)|^2,
\]

(33)

and the averaged separation of electrons,

\[
\int d^2 r_1 \int d^2 r_2 |\Psi_\nu(r_1, r_2)|^2 / |r_1 - r_2| \Psi_\nu(r_1, r_2).
\]

(34)

As is shown in figure 3 for multi-loop cyclotron braids the single-particle charge distribution is pushed out from the center of the jellium plaque resulting in a larger mean separation of the electron distributions, which minimizes the electron repulsion energy. A comparison of the energies for both considered phases reveals that the reduction of electron repulsion causes an increase in the electron–jellium interaction, which prefers a more uniform charge distribution. Both homotopy phases are, however, stable with respect to the nearby quantum states, which cannot be correlated if \( \nu \) is slightly shifted out of the cyclotron braid commensurability condition.

These simple examples of the homotopy phases in the case of \( N = 2 \) can be generalized to large electron number systems (also in the thermodynamic limit, provided that the planar density remains constant).
6. Discussion and comments

We have demonstrated that in the correlated 2D electron systems at some values of the strong perpendicular magnetic field, the magnetic flux quantum, \( \Phi_0 = \frac{\hbar}{e} \), must be substituted by its multiple, \( \Phi_k = (2k + 1)\Phi_0 \), where \( k \) is the number of loops in the simplest admissible braids \( \sigma_{2k+1} \), when the braids \( \sigma_i \) are too short to match neighboring electrons uniformly distributed due to Coulomb repulsion. Elementary braids, \( \sigma_{2k+1} \), also define exchanges of indistinguishable closest particles and identify correlations in 2D system of interacting electrons. Larger values of the magnetic flux quantum in correlated planar system display the size of a multi-loop cyclotron orbits and via the cyclotron commensurability with electron distribution (Wigner lattice) result in the specific hierarchy of correlated phases, which is experimentally observed as the discrete hierarchy of FQHE and IQHE.

The effect of by correlations restricted magnetic field flux quantization is general and should be observed in all 2D electron systems in perpendicular magnetic (or Berry) field, provided that the surface density of electrons is sufficiently small (the lower density \( N \) the lower \( H_k \) and \( H_0 \) are in equations (4) and (3)) and the electrons have a sufficiently large mobility (of order of \( 10^6 \text{ cm}^2 \text{ V s}^{-1} \)). Such high mobility allows the mean free path of electrons to exceed the sample size (of micrometer order), which is required to organize the correlated state of all electrons in the system [39]. The conditions needed for the observation of IQHE and FQHE are achieved in GaAs and in graphene monolayer (in graphene the mobility is additionally increased by the Klein tunneling effect for Dirac electrons). Except of these materials the fractional and integer quantum Hall effects are observed also in other semiconductor 2DESs, e.g., in Si [40], CdTe [41], ZnO [42], CdSAsS [43] (including even diluted magnetic semiconductors with large effective gyromagnetic factor [44]) and in bilayer systems, like bilayer GaAs [45] and bilayer graphene [46, 47].

The important feature of correlated topological multiparticle states like quantum Hall states in 2D systems, is the long-range quantum entanglement of all electrons in contrary to conventional multiparticle phases with local order parameters and some symmetry broken [12]. In the latter case the quantum entanglement is also local, usually binary corresponding to the order parameter definition, like the binary Cooper pairing in superconductors with the gauge symmetry broken. This deep difference between conventional phases and topological homotopy phases in multiparticle systems is noticeable e.g., in the form of the Laughlin function [9], \( \Psi(z_1, \ldots, z_N) \sim \prod_{i<j}|z_i - z_j|^{2k+1}e^{-\sum_i |z_i|^2/4l_H^2} \) (where \( z_i \) are complex coordinates of electrons on the plane, \( l_H = \sqrt{\frac{\hbar}{eB}} \) is the magnetic length, \( k = 0, 1, 2, \ldots, \)) which is apparently nonseparable function in the tensor product of single particle Hilbert spaces. The related quantum entanglement of all electrons simultaneously is the characteristic feature for strongly correlated topological states without any local order parameter and any symmetry broken, like in 2D quantum Hall states.

The braid group approach finds application to the development of the path integration for multiparticle systems of indistinguishable identical particles as it was originally presented by Wu, Laidlaw and de Witt [17, 22]. This original method has been confined, however, to the ordinary multiparticle configuration space with the full braid group and via unitary scalar representations (1DURs) of the braid group allows for the definition of bosons, fermions and anyons. The main idea is that to any multiparticle path in the configuration space one can hitch an arbitrary loop from the braid group. As braids are nonhomotopic the whole domain of the path integral decomposes into separate subdomains of trajectories which cannot be transformed ones onto another ones by any continuous deformations without cutting. Hence, the uniform measure in the path space cannot be defined and the path integration must be performed separately over independent subdomains numerated by the braid group elements and next summed over all these segments with a unitary (because of causality) weight factors. In reference [22] it has been proved that these factors create a 1DUR of the braid group. Various 1DURs define distinct quantum statistics for the same classical particles. The progress presented in [26] consists in the observation that the braid group can change itself for 2D interacting electrons at the presence of strong magnetic (or Berry) field, because of the cyclotron commensurability constraints. The same constraints result in the limits for the attainability of quantum of magnetic field flux which appears in 2D at strong magnetic fields to be a multiple of the elementary flux quantum, \( (2k + 1)\Phi_0, \Phi_0 = \frac{\hbar}{e} \). Independently of the path integration one can utilize this result via the linkage of the braid group and the symmetry of multiparticle wave functions originally formulated by Sudarshan et al. [30–32]. According to this idea the multiparticle wave function must transform as the 1DUR of the braid if the arguments of this wave function (which can be considered as the classical positions of particles) interchange their positions according to this particular braid. In 2D the braids are not permutations and at the magnetic field presence they must be elements of some subgroups of the full braid groups which we call as the cyclotron subgroups. These subgroups are severely limited by the described in the present paper cyclotron commensurability condition closely related with the cyclotron orbit size.
including multiloop orbits and possible to be defined at only some discrete series of the magnetic field values corresponding to fractional filling rates of LLS, the hierarchy of FQHE. This can be done without invoking to the path integration. However, the latter is helpful in the theoretical derivation of the longitudinal resistivity at FQHE, $R_{xx}(\nu)$, as the function the LLL filling rate, $\nu$. This highly varying curve observed experimentally [23], cf figure 4 right panel, is not understandable upon any conventional model of FQHE. Only by the assessment of the relative values of path integrals for various $\nu$ resulting from the different number of various commensurability braid patterns varying with $\nu$, one can obtain a similar behavior of the theoretical $R_{xx}$ [26] to that from experiments. This is due to the fact that the path integral, i.e., the matrix element of the evolution operator in the position representation has a direct link to the conductivity. Hence, to a nonstationary kinetical effect like the conductance there contribute all commensurability patterns despite that the corresponding to them homotopy phases differ in energy. Remarkable, that nonzero local minima in $R_{xx}$ occur at filling fractions for which the homotopy patterns with nesting of braids to next-nearest neighbors in the Wigner lattice are possible only, whereas the correlations with nearest neighbors give zero $R_{xx}$ in agreement with experimental observations shown in figure 4. Note, that the longitudinal resistance curve is more characteristic for FQHE than the transverse Hall resistivity fractional quantization, $R_{xy} = \frac{h}{e^2 \nu}$, which is usually regarded as the hallmark of FQHE. In fact the formula for $R_{xy}$ is classical one and the FQHE consists in the selection of the hierarchy of filling rates, which is essentially quantum as demonstrated in the present paper, similarly as quantum is the specific variation of $R_{xx}$ with respect to filling rate $\nu$ [26].

The similar trajectory homotopy induced quantization holds also in the case when the magnetic field in 2D electron systems is substituted by the Berry field in the case of Chern topological insulator. The discrete hierarchy of correlated states called as fractional Chern topological insulators, encountered mostly in exact diagonalization of electron repulsion at degenerated (extremely narrow) bands in planar crystal lattices with fractional on-site fillings by electrons, repeats the general homotopy patterns known from FQHE [48, 49]. Multi-loop trajectories and corresponding braids, with size governed by multiples of the smallest Berry field flux quantum, allow for organization of correlated 2D electron states in fractional Chern topological insulators.

7. Conclusion

Topological correlations are induced by the electron interaction but are controlled by the cyclotron commensurability with electron distribution, which selects admissible states. Any such admissible state is stable because its eigenenergy is lower than that of uncorrelated states. However, for the same filling rates many various patterns of correlations are usually possible, and the energy competition between them selects the lowest one.

The crucial role for the correlation patterns play the size of the cyclotron trajectory which must be commensurate with the electron distribution. This commensurability is the necessary condition for organization of the correlated 2D state at perpendicular magnetic field, which apparently must be incompressible. The cyclotron trajectory size is given by the effective magnetic flux quantum, the lowest flux...
quantum admissible at particular correlation type. As the correlation patterns can change exclusively in a
discrete manner via the multiplication of loops in multi-loop orbits and the commensurability with nearest
or next-nearest neighboring electrons, hence the effective flux quanta change also in a discrete way,
correspondingly.

The presented discussion does not change a fundamental magnetic flux quantum, that universally
always holds for a free electron, but only identifies constraints imposed on the minimal magnetic flux by
trajectory homotopy topological conditions arisen from the homotopy correlations. In homotopy
noncorrelated phases, both out of the commensurability despite interaction, or in noninteracting systems,
the magnetic flux quantum is always of its fundamental smallest value.

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