THE MODULI SPACE OF THREE DIMENSIONAL LIE ALGEBRAS

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Abstract. In this paper, we consider the versal deformations of three dimensional Lie algebras. We classify Lie algebras and study their deformations by using linear algebra techniques to study the cohomology. We will focus on how the deformations fasten the space of all such structures together. This space is known as the moduli space. We will give a geometric description of this space, derived from deformation theory, in order to illustrate general features of Lie algebras’ moduli spaces.

1. Introduction

Here we establish the basic language in which we will express the Lie Algebras and their cohomology. Recall that the exterior algebra $\bigwedge V$ of an ungraded vector space $V$ has a natural $\mathbb{Z}$-grading, with $\deg(v_1 \wedge \cdots \wedge v_n) = n$ and that there is a corresponding $\mathbb{Z}$-graded coalgebra structure with comultiplication $\Delta$ given by

$$\Delta(v_1 \wedge \cdots \wedge v_n) = \sum_{k=1}^{n-1} \sum_{\sigma \in \text{Sh}(k,n-k)} (-1)^{\sigma} v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(k)} \otimes v_{\sigma(k+1)} \wedge \cdots \wedge v_{\sigma(n)}.$$ 

Here $\text{Sh}(k,n-k)$ represents the unshuffles of type $(k,n-k)$, that is, the permutations which are increasing on $1, \ldots, k$ and $k+1, \ldots, n$ and $(-1)^{\sigma}$ represents the sign of the permutation $\sigma$. A linear map $\varphi \in \text{Hom}(\bigwedge^k V, V)$ has degree $k-1$, and extends uniquely to a coderivation by the rule

$$\varphi(v_1 \wedge \cdots \wedge v_n) = \sum_{\sigma \in \text{Sh}(k,n-k)} (-1)^{\sigma} \varphi(v_{\sigma(1)} \wedge \cdots v_{\sigma(k)}) \wedge v_{\sigma(k+1)} \wedge \cdots \wedge v_{\sigma(n)}.$$ 

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Recall that a coderivation of $\bigwedge V$ is a linear map $\varphi : \bigwedge V \rightarrow \bigwedge V$ satisfying $\Delta \varphi = (\varphi \otimes 1 + 1 \otimes \varphi) \Delta$. We can view $L_n = \text{Hom}(\bigwedge^n V, V)$ as a subspace of the space $L = \text{Coder}(\bigwedge V)$. Moreover $L = \prod_{n=1}^{\infty} L_n$ in a natural way. The space of coderivations of $V$ has a natural structure of a $\mathbb{Z}$-graded Lie algebra. If $\varphi, \psi \in L_k$ and $\psi \in L_l$ then $[\varphi, \psi] \in L_{k+l-1}$ is given by $[\varphi, \psi] = \varphi \psi - (-1)^{\deg \varphi \deg \psi} \psi \varphi$. More explicitly, we compute

$$(\varphi \psi)(v_1 \wedge \cdots \wedge v_{k+l-1}) =$$

$$\sum_{\sigma \in \text{Sh}(k,l-1)} (-1)^{\sigma} \varphi(\psi(v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(l)})) \wedge v_{\sigma(i+1)} \wedge \cdots \wedge v_{\sigma(k+l-1)},$$

which determines $\varphi \psi$ as an element of $L_{k+l-1}$.

If $V = \langle f_i, i = 1 \ldots N \rangle$, and $I = (i_1, \ldots, i_k)$ is an increasing multi-index, i.e., $1 \leq i_1 < \cdots < i_k \leq N$, then $\varphi^I_k \in L_k$ is defined by $\phi^I_k(f_j) = \delta^I_j f_k$, where $f_j = f_{j_1} \wedge \cdots \wedge f_{j_k}$. Then we have

$$L_k = \langle \varphi^I_j | 1 \leq i_1 \leq \cdots \leq i_k \leq N, 1 \leq j \leq N \rangle, \quad \dim(L_k) = \binom{n}{k}.$$
Any coderivation in $L_k$ can be expressed in the form

$$\varphi = a^i_j \varphi^S_{i,j,k},$$

using the Einstein summation convention, so that we can represent $\varphi$ but the $N \times \binom{N}{k}$ matrix $A = a^i_j$. More generally, $\varphi : L_n \to L_{m-k+1}$ is responsible by the $\binom{N}{m-k+1} \times \binom{N}{m}$ matrix $B = b^i_j$, where

$$\varphi(f_{S(j,m)}) = b^i_j f_{S(i,m-k+1)}.$$ 

In particular when $m = 2k - 1$ the matrix $AB$ represents $\varphi^2$. Thus $\varphi^2 = 0$ is equivalent to the matrix equation $AB = 0$. Moreover, if $d \in L_2$ is a codifferential, then the Jacobi relation is equivalent to $AB = 0$ where $d = a^i_j \varphi^S_{i,j,2}$ and the coefficients $b^i_j$ are determined as follows.

It is easy to see that

$$\varphi_{(u,v)}(f_{(r,s,t)}) =$$

$$\left[\delta^r_s \delta^t_u \left(\delta^k_l \delta^i_t - \delta^k_i \delta^t_l\right) - \delta^u_s \delta^t_i \left(\delta^k_l \delta^i_s - \delta^k_s \delta^i_l\right) + \delta^u_t \delta^s_i \left(\delta^k_i \delta^l_s - \delta^k_s \delta^l_i\right) + \delta^u_t \delta^s_i \left(\delta^k_l \delta^i_s - \delta^k_s \delta^i_l\right)\right] f_{(k,l)}.$$ 

Suppose that

$$S(j,3) = (r,s,t)$$
$$S(i,2) = (k,l).$$

Then

$$b^i_j = a^w_{u,v} \left[\delta^r_u \delta^t_s \left(\delta^k_l \delta^i_t - \delta^k_i \delta^t_l\right) - \delta^u_s \delta^t_i \left(\delta^k_l \delta^i_s - \delta^k_s \delta^i_l\right) + \delta^u_t \delta^s_i \left(\delta^k_i \delta^l_s - \delta^k_s \delta^l_i\right) + \delta^u_t \delta^s_i \left(\delta^k_l \delta^i_s - \delta^k_s \delta^i_l\right)\right].$$

The above formula is easily implemented on a computer and determines the coefficients of $B$ linearly in terms of the coefficients of $A$. Since $AB$ is an $N \times \binom{N}{3}$ matrix, this means the Jacobi relation is given by $N \binom{N}{3}$ homogenous quadratic equations in the $N \binom{N}{3}$ coefficients of $a^i_j$.

An invertible linear map $g : V \to V$ extends to a coalgebra automorphism of $\wedge V$ by $g(v_1 \wedge \cdots \wedge v_n) = g(v_1) \wedge \cdots \wedge g(v_n)$. Two Lie algebra structures are isomorphic, or in other words, they are associated codifferentials $d, d'$ are equivalent if there is an invertible linear map $g : V \to V$ such that $g^*(d) = g^{-1} \circ d \circ g = d'$. The set of equivalence classes of codifferentials on $V$ is called the moduli space of Lie algebra structures on $V$.

Let $G = g^j_i$ where $g(f_j) = g^j_i f_i$ represents $g : V \to V$ as an $N \times N$ matrix. Define the $\binom{N}{2} \times \binom{N}{2}$ matrix $Q = q_j^i$ by $g(f_{S(j,2)}) = q_j^i f_{S(i,2)}$ so that $Q$ represents $g : \wedge^2 V \to \wedge^2 V$. Suppose that

$$S(j,2) = (u,v)$$
$$S(i,2) = (k,l).$$
Then
\[ q^i_j = g^k_i g^l_j - g^l_i g^k_j. \]

If \( A, A' \) represent the codifferential \( d, d' \) respectively, the condition \( g^*(d) = d' \) is represented by the matrix equation \( A = G^{-1}A Q \). It is easier to solve the equation \( GA' = AQ \) as long as the solution matrix \( G \) satisfies \( \det G \neq 0 \).

2. THREE DIMENSIONAL LIE ALGEBRAS

Let \( V = \langle f_1, f_2, f_3 \rangle \) be a three dimensional vector space with Lie algebra structure determined by the codifferential \( d \) which is represented by the \( 3 \times 3 \) matrix \( A = (a^i_j) \). Then if
\[
B = \begin{bmatrix}
-a_{1,2} - a_{2,3} \\
a_{1,1} - a_{3,3} \\
a_{2,1} + a_{3,2}
\end{bmatrix},
\]
the Jacobi identity is equivalent to the matrix equation \( AB = 0 \).

Recall that the derived subalgebra of a Lie algebra is the image of \( d : S^2(V) \to V \). Since \( A \) is a matrix representing \( d_1 \) it follows that the rank of \( A \) is the dimension of the derived subalgebra.

We first consider the case where the derived subalgebra has dimension three. Then the matrix \( A \) of \( d \) is invertible, so it must be the case that \( B = 0 \). In [6] it is shown that any \( n \) dimensional Lie algebra structure such that the bracket of any two elements is a linear combination of those elements has an abelian ideal of dimension \( n - 1 \). We shall see later that whenever there is an ideal of dimension \( n - 1 \) the rank of the associated matrix can never be larger than \( n - 1 \). Thus by choosing an appropriate basis we may assume that \( d(f_1 f_2) = f_3 \). Taking into account the fact that \( B = 0 \) we see that the matrix \( A \) has to be of the form
\[
A = \begin{bmatrix}
0 & x & y \\
0 & z & -x \\
1 & 0 & 0
\end{bmatrix}
\]
where \( x^2 + yz \neq 0 \).

It is easy to see that \( d \) is equivalent to a codifferential whose matrix is of the form \( A' = \begin{bmatrix} 0 & 0 & y \\ 0 & \beta & 0 \\ 1 & 0 & 0 \end{bmatrix} \). In fact, if \( z \neq 0 \), then the linear automorphism \( g \) whose matrix is \( G = \begin{bmatrix} 0 & 0 & -z \\ 0 & 1 & x \\ 1 & 0 & 0 \end{bmatrix} \) yields \( A' = G^{-1}A Q \), where \( \beta = x^2 + yz, \eta = z \); if \( y \neq 0 \) then \( G = \begin{bmatrix} 0 & 0 & -z \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \) gives \( \beta = y, \eta = -x^2 - yz \); while if both \( y \) and \( z \) vanish then \( G = \begin{bmatrix} -1/2 & -1/2 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \) gives \( \beta = \eta = x \).
It follows that any codifferential whose matrix is invertible is equivalent to one whose matrix is of the form \( A = \begin{bmatrix} 0 & 0 & \mu \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \). The automorphism determined by \( G = \text{Diag}(r, s, rs) \) preserves the form of \( A \) with \( \lambda \rightarrow \lambda' = r^2\lambda, \mu \rightarrow \mu' = s^2\mu \), which is enough to see that there is only one equivalence class of codifferentials with invertible matrix over \( \mathbb{C} \).

Moreover \( G = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \) preserves the form as well, with \( \lambda \rightarrow \lambda' = -\lambda, \mu \rightarrow \mu' = -\mu \), which shows that there are at most two nonequivalent such codifferentials over \( \mathbb{R} \). Of course, these two codifferentials correspond to the two nonequivalent compact real forms of the simple complex Lie algebra \( \mathfrak{sl}(2, \mathbb{C}) \).

Now suppose that the derived algebra of a Lie algebra on a vector space \( V \) of dimension \( n \) has dimension smaller than \( n \). Then there is an ideal \( V' \) in \( V \) of dimension \( n-1 \). This gives rise to the exact sequence

\[
0 \rightarrow V' \rightarrow V \rightarrow \mathbb{C} \rightarrow 0,
\]

where \( \mathbb{C} \) is the trivial lie algebra. Let \( d' \) be the induced codifferential on \( V' \), given by \( d'(u, v) = d(u, v) \) for \( u, v \in V' \), with matrix \( A' \) in terms of the basis \( \{f_1, \ldots, f_{n-1}\} \) of \( V' \). Let \( \{f_n\} \) be a basis of the complementary subspace to \( V' \) and define \( \rho : V' \rightarrow V' \) by \( \rho(u) = d(u, f_n) \). Then \( \rho \) is a derivation of \( d' \). If \( R = (r^i_j) \) is the matrix of \( \rho \), given by \( \rho(f_j) = r^i_j f_i \), then the matrix of \( A \) is just \( A = [A' \, R] \).

If \( f'_n = f_n + v \), where \( v \in V' \), then the induced derivation \( \rho'(u) = d(u, f'_n) \) determines an equivalent coderivation to \( d \). Moreover, \( \rho'(u) = \rho(u) + \text{ad}_v(u) \), where \( \text{ad}_v(u) = d(u, v) \) is the inner derivation of \( V' \) determined by \( v \). Thus we can always replace \( R \) by the matrix \( R' \) given by \( \rho \), and we see that the extensions of \( V' \) by \( \mathbb{C} \) are determined by the space of outer derivations \( \text{Out}(d') \) of the induced Lie algebra structure on \( V' \). If we denote the space of inner derivations of \( V' \) by \( \text{ad}(d') \), then \( \text{Out}(d') = H^1(d')/\text{ad}(d') \).

Now let us specialize this to the three dimensional case of Lie algebras. We use the fact that there are, up to equivalence, only two Lie algebra structures on a two dimensional vector space.

**Case 1.** The nonabelian Lie algebra, given by \( d(f_1, f_2) = f_1 \).

**Case 2.** The abelian Lie algebra, given by \( d(f_1, f_2) = 0 \).

Let us study the possible forms for the matrix \( R \) in case 1. We have

\[
\begin{align*}
r^1_1 f_1 + r^2_2 f_2 &= \rho(f_1) = \rho(d(f_1, f_2)) = d(\rho(f_1), f_2) + d(f_1, \rho(f_2)) \\
&= d(r^1_1 f_1 + r^2_2 f_2, f_2) + d(f_1, r^1_2 f_1 + r^2_2 f_2) = r^1_1 f_1 + r^2_2 f_1
\end{align*}
\]

It follows that \( r^2_1 = r^2_2 = 0 \). Consider the inner derivations

\[
\rho_i(u) = d(u, f_i).
\]
It is easy to see that $\rho = -r_1^1 \rho_1 + r_2^1 \rho_2$. Thus every derivation is inner, and we may assume that $R = 0$. But then $\{f_1, f_2\}$ span an ideal on which $d$ acts as the zero matrix. Thus, the nonabelian case reduces to the abelian one.

Now assume that $d' = 0$. Then $d$ is represented by the matrix $A = \begin{bmatrix} 0 & R \\ 0 & 0 \end{bmatrix}$, where $R$ is any $2 \times 2$ matrix. However, if $g'$ is any linear automorphism of $V'$, given by a matrix $G$, and $g$ is the linear automorphism of $V$ given by the matrix $R = \begin{bmatrix} G & 0 \\ 0 & 0 \end{bmatrix}$, then the $d$ is represented by the matrix $A = \begin{bmatrix} 0 & R' \\ 0 & 0 \end{bmatrix}$, where $R' = (G')^{-1}RG'$. Thus similar matrices $R$ determine equivalent codifferentials. Moreover, multiplication of $R$ by any nonzero constant $\lambda$ also determines an equivalent codifferential, corresponding to the linear automorphism given by the matrix $G = \text{Diag}(1, 1, \lambda)$.

As a consequence, we immediately reduce to the following possibilities.

$d(\lambda : \mu)$: given by the matrix $R = \begin{bmatrix} \lambda & 1 \\ 0 & \mu \end{bmatrix}$. Note that $R$ is similar to the matrix $\begin{bmatrix} \mu & 1 \\ 0 & \lambda \end{bmatrix}$, so that $d(\mu : \lambda) \sim d(\lambda : \mu)$. Moreover, if we consider the linear automorphism determined by the diagonal matrix $G = \text{Diag}(t, 1, t)$, then $R$ is replaced by the matrix $\begin{bmatrix} t\lambda & 1 \\ 0 & t\mu \end{bmatrix}$, so that $d(t\lambda : t\mu) \sim d(\lambda : \mu)$. As a consequence, we can view $(\lambda : \mu) \in \mathbb{P}^1$. The similarity of matrices with the same eigenvalues determines an action of $\Sigma_2$ on $\mathbb{P}^1$, so the set of codifferentials of this type are parameterized by the orbifold $\mathbb{P}^1/\Sigma_2$.

$d_2$: given by the matrix $R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

d_1$: given by the matrix $R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Together with the codifferential $d_3$ representing the Lie algebra structure $\mathfrak{s}l_2(\mathbb{C})$ these codifferentials represent a complete classification of all three dimensional Lie algebras. The codifferential $d_1$ represents the Heisenberg algebra $\mathfrak{n}_3(\mathbb{C})$, $d_2$ represents the solvable Lie algebra $\mathfrak{r}_{3,1}(\mathbb{C})$, $d(0 : 1)$ represents the Lie algebra $\mathfrak{r}_2(\mathbb{C}) \oplus \mathbb{C}$, $d(1 : 1)$ represents the solvable Lie algebra $\mathfrak{r}_3(\mathbb{C})$, while $d(\lambda : \mu)$ represents the solvable Lie algebra $\mathfrak{r}_{3,\mu/\lambda}(\mathbb{C})$ when $\lambda \neq \mu$. Note that our alignment of algebras into families differs slightly from the classical alignment. If we interchange $d_2$ with $d(1 : 1)$ then the alignment would agree, but our reasons for not making this change will become apparent.

3. Miniversal Deformations of Lie Algebras

The classical notion of an infinitesimal deformation of a Lie algebra structure on a vector space $V$ given by a codifferential $d \in L_2$ is a
coderivation

\[ d_t = d + t\psi, \]

where \( \psi \in L_2 \) which satisfies the Jacobi identity infinitesimally; i.e., \([d_t, d_t] = 0 \ mod t^2\). The condition is equivalent to \( D(\psi) = 0 \), where \( D : L \to L \) is the coboundary operator determined by \( d \), given by \( D(\psi) = [d, \psi] \). The fact that \( D^2 = 0 \), so that \( D \) is a differential on \( L \), follows from the Jacobi identity for \( d \), and the homology of this differential is the cohomology \( H(d) \) (with coefficients in the adjoint representation). Let \( A = \mathbb{C}[t]/(t^2) \). Then \( A \) is a local algebra, with maximal ideal \( m = (t) \). Then we can consider \( d_t \) as being an element of \( L \otimes A \), which may be thought of as the coderivations of \( \bigwedge(V) \) with coefficients in \( A \). In this sense \([d_t, d_t] = 0\), in terms of the natural bracket on \( L \otimes A \).

One can generalize this construction by allowing \( A \) to be any local (commutative) algebra over \( \mathbb{C} \), and say that \( d_A \in L \otimes A \) is a deformation of \( d \) with local base \( A \) if \( \epsilon^*(d_A) = d \), where \( \epsilon : A \to \mathbb{C} \) is the augmentation homomorphism, and \( \epsilon_* : L \otimes A \to L \otimes \mathbb{C} = L \) is the induced map.

If \( m^2 = 0 \), then we call \( A \) an infinitesimal algebra, and say that \( d_A \) is an infinitesimal deformation. This notion of infinitesimal deformation generalizes the classical notion. An infinitesimal deformation \( d_A \) with base \( A \) is universal if whenever \( d_B \) is any infinitesimal deformation with base \( B \), then there is a unique homomorphism \( f : A \to B \) such that \( f_* (d_A) \sim d_B \), where \( \sim \) means infinitesimal equivalence. Two deformations \( d_B \) and \( d_B' \) with the same base \( B \) are said to be infinitesimally equivalent if there is an infinitesimal automorphism \( g_B \in Aut(V) \otimes B \), satisfying \( g_B^*(d_B) = d_B' \). We say that \( g_B \) is an infinitesimal automorphism if \( \epsilon_*(g_B) = 1_V \).

If \( \dim(H^2(d)) = n \), then the deformation \( d^{\inf} \), with base \( A = \mathbb{C}[t^1, \ldots, t^n]/(t^1, \ldots, t^n)^2 \), given by

\[ d^{\inf} = d + \delta_i t^i, \]

where \( \{\delta_i\} \) is a prebasis of \( H^2(d) \) is a universal infinitesimal deformation.

The classical notion of a formal deformation is given by a power series

\[ d_t = d + \psi_i t^i, \]

where \( \psi_i \in L_2 \) and \([d_t, d_t] = 0\). We may consider \( d_t \in L \otimes \mathfrak{a} \), where \( \mathfrak{a} = \mathbb{C}[[t]] \) is the ring of formal power series in the variable \( t \), and \( \otimes \) means the formal completion of the tensor product \( L \otimes \mathfrak{a} = \lim_{k \to \infty} L \otimes A/m^k \).
that requirement for universal deformation with the exception that we drop the uniqueness where

definition of deformation $H_d$ that miniversal deformations exist whenever $\dim(H^2(d)) < \infty$. To see how to construct the miniversal deformation, we proceed as follows.

Let $d^1 = d + \delta$, where $\delta = \delta_i t^i$, for some prebasis $\{\delta_i\}$ of $H^2(d)$, the $t^i$ are parameters, and let $m = (t^i)$ be the ideal in $C[[t^i]]$ generated by the $t^i$. Denote $B = D(L_1)$ and $P_i$ be a preimage of $D(L_i)$. Note that

$[d^1, d^1] = [\delta, \delta] \in H^2 \otimes m^2.$

Thus

$[d^1, d^1] = \alpha_i r^i_2 + \beta_2,$

where $\{\alpha_i\}$ is a prebasis of $H^3$, $r^i_2 \in m^2$ and $\beta_2 \in B \otimes m^2$. Then $\beta_2 = -2D(\gamma_2)$ for some $\gamma_2 \in P_2 \otimes m^2$, and we define $d^2 = d^1 + \gamma_2$. Let $R_2 = (r^i_2)$ be the ideal generated by the $r^i_2$. Then

$[d^2, d^2] = [d^1, d^1] + 2[d^1, \gamma_2] + [\gamma_2, \gamma_2]$

$= \alpha_i r^i_2 + \beta_2 + 2[d, \gamma_2] + 2[d^1 \cdot d, \gamma_2] + [\gamma_2, \gamma_2]$

$= \alpha_i r^i_2 + 2[d^1 \cdot d, \gamma_2] + [\gamma_2, \gamma_2]$

$\in H^3 \otimes R_2 + L_3 \otimes m^3 + L_3 \otimes m^4.$

Moreover,

$D[d^2, d^2] = -2[d^1 \cdot d, D\gamma_2] + 2[D\gamma_2, \gamma_2]$

$= [d^1 \cdot d, \beta_2] + 2[D\gamma_2, \gamma_2]$

$= [d^1, \beta_2] + 2[D\gamma_2, \gamma_2]$

$= [d^1, [d^1, d^1]] - [d^1, \alpha_i r^i_2] + 2[D\gamma_2, \gamma_2]$

$= -[d^1 \cdot d, \alpha_i r^i_2] + 2[D\gamma_2, \gamma_2]$

$\in H^3 \otimes mR_2 + L_3 \otimes m^4 \subseteq L_3 \otimes (mR_2 + m^4)$

Thus

$[d^2, d^2] = \alpha_i r^i_3 + \beta_3 + \tau_3,$
where \( r_3^i - r_2^i \in m^3 \), \( \beta_3 \in B \otimes m^3 \) and \( \tau_3 \in P_3 \otimes (mR^2 + m^4) \). If \( R_3 \) is the ideal generated by \( r_3^i \), then \( mR^2 \subseteq R^3 + m^4 \), so \( \tau_3 \in P_3 \otimes (R^3 + m^4) \). Now suppose inductively that we have been able to construct \( d^n \) satisfying

\[
[d^n, d^n] = \alpha_n r_{n+1}^i + \beta_{n+1} + \tau_{n+1},
\]

where \( r_{n+1}^i \in m^{n+1} \), \( \beta_{n+1} \in B \otimes m^{n+1} \), and \( \tau_{n+1} \in P \otimes (R_{n+1} + m^{n+2}) \), where \( R_{n+1} = (r_{n+1}^i) \). Then \( 2\beta_{n+1} = -D\gamma_{n+1} \), for some \( \gamma_{n+1} \in P \otimes m^{n+1} \), and if we define \( d^{n+1} = d^n + \gamma_{n+1} \), we have

\[
[d^{n+1}, d^{n+1}] = [d^n, d^n] + 2[d^n, \gamma_{n+1}] + [\gamma_{n+1}, \gamma_{n+1}]
= \alpha_n r_{n+1}^i + \tau_{n+1} + 2[d^n - d, \gamma_{n+1}] + [\gamma_{n+1}, \gamma_{n+1}]
\in H^3 \otimes R_{n+1} + P \otimes (R_{n+1} + m^{n+2}) + L_3 \otimes m^{n+2}
\]

Note that \( L_3 = H^3 \oplus B \oplus P \), and thus the coboundary part in the above lies strictly in \( B \otimes m^{n+2} \). Now

\[
D[d^{n+1}, d^{n+1}] = [d^{n+1}, d^{n+1}] + 2[D(d^n), \gamma_{n+1}] - 2[d^n - d, D\gamma_{n+1}]
+ 2[D\gamma_{n+1}, \gamma_{n+1}]
= [d^n, \tau_{n+1}] + [d - d^n, \tau_{n+1}] + 2[D(d^n), \gamma_{n+1}] + [d^n, \beta_{n+1}]
+ 2[D\gamma_{n+1}, \gamma_{n+1}]
= -[d^n, \alpha_n r_{n+1}^i] + [d - d^n, \tau_{n+1}] + 2[D(d^n), \gamma_{n+1}]
+ 2[D\gamma_{n+1}, \gamma_{n+1}]
\in H^3 \otimes mR_{n+1} + L \otimes (mR_{n+1} + m^{n+3}) + L \otimes m^{n+3}
\subseteq L_3 \otimes (mR_{n+1} + m^{n+3}).
\]

Consequently, we can express

\[
[d^{n+1}, d^{n+1}] = \alpha_n r_{n+2}^i + \beta_{n+2} + \tau_{n+2},
\]

where \( r_{n+2}^i - r_{n+1}^i \in m^{n+2} \), \( \beta_{n+2} \in B \otimes m^{n+2} \) and \( \tau_{n+2} \in P_3 \otimes (mR_{n+1} + m^{n+3}) \). Thus we can continue the construction indefinitely, and we obtain finally a deformation \( d^{\text{inf}} = d + \sum_{i=2}^{\infty} \gamma_i \), which satisfies

\[
[d^{\text{inf}}, d^{\text{inf}}] = \alpha_i r_{i}^i,
\]

where \( r_i^i = \lim_{n \to \infty} r_n^i \) give the relations on the base of the miniversal deformation.

The process described above is not very efficient in constructing the miniversal deformation, because it potentially requires an infinite number of steps, although, in practice, it often terminates after a finite number of steps. In some recent papers [3, 4, 7, 5, 6, 1], the authors have shown how to construct a miniversal deformation \( d^\infty \) as follows.

\[
d^\infty = d + \delta_i t^i + \gamma_i x^i,
\]
where \( \{ \delta_i \} \) is a prebasis of \( H^2(d) \), and \( \{ \gamma_i \} \) is a prebasis of the 3-coboundaries \( D(L_2) \), \( t^i \) are the parameters which appear in the base \( \mathcal{A} \), and \( x^i \) are formal power series in the parameters \( t^i \) which are found as follows. To determine the coefficients \( x^i \), we compute
\[
[d^\infty, d^\infty] = \alpha_i r^i + \beta_i s^i + \tau_i y^i,
\]
where \( \{ \alpha_i \} \) is a prebasis of \( H^3(d) \), \( \{ \beta_i \} \) is a basis of the 3-coboundaries \( D(L_2) \), and \( \{ \tau_i \} \) is a prebasis of the 4-coboundaries \( D(L_3) \). Thus, taken together, the \( \alpha_i, \beta_i, \tau_i \) give a basis of \( L_3 \). By the construction of the miniversal deformation given in [3], it follows that \( s^i = 0 \) for all \( i \), \( r^i \) are formal power series in the parameters \( t^i \), and \( y^i = 0 \mod (r^1, \ldots) \). Moreover, the equations \( s^i = 0 \) can be solved to obtain the expressions for \( x^i \) as formal power series in the parameters \( t^i \). Actually, the \( x^i \) can always be expressed as rational functions of the parameters.

The construction above can be implemented on a computer, and we have constructed Maple worksheets that carry out this implementation for an arbitrary Lie algebra of any dimension. Using these programs we have constructed miniversal deformations of all three dimensional Lie algebras, which we give below. We will also give prebases for the cohomology for each of these examples.

### 4. Calculation of the Miniversal Deformations

| Type          | Codiff   | \( H^1 \) | \( H^2 \) | \( H^3 \) |
|---------------|----------|-----------|-----------|-----------|
| \( d_1 = n_3 \) | \( \psi_1^{23} \) | 4         | 5         | 2         |
| \( d_2 = r_{3,1}(\mathbb{C}) \) | \( \psi_1^{13} + \psi_2^{23} \) | 3         | 3         | 0         |
| \( d(1 : 1) = r_3(\mathbb{C}) \) | \( \psi_1^{13} + \psi_1^{23} + \psi_2^{23} \) | 1         | 1         | 0         |
| \( d(\lambda : \mu) = r_{3,\mu/\lambda}(\mathbb{C}) \) | \( \psi_1^{13} \lambda + \psi_1^{23} + \psi_2^{23} \mu \) | 1         | 1         | 0         |
| \( d(1 : 0) = r_2(\mathbb{C}) \oplus \mathbb{C} \) | \( \psi_1^{13} + \psi_1^{23} \) | 2         | 1         | 0         |
| \( d(1 : -1) = r_{3,-1}(\mathbb{C}) \) | \( \psi_1^{13} + \psi_1^{23} - \psi_2^{23} \) | 1         | 2         | 1         |
| \( d_3 = sl_2(\mathbb{C}) \) | \( \psi_1^{13} + \psi_1^{23} + \psi_2^{23} \) | 0         | 0         | 0         |

Table 1. Cohomology of Three Dimensional Algebras

Table 1 gives the cohomology for each of the types of 3-dimensional Lie algebras. Note that for Lie algebras of type \( d(\lambda : \mu) \) there are two special values of the parameters. For \( d(1 : 0) \) the only variation is in \( H^1 \), which plays no role in the miniversal deformation. For \( d(1 : -1) \) \( H^2 \) and \( H^3 \) are not the same as for generic elements of the family.
$d(\lambda : \mu)$, and this difference plays an important role in understanding the moduli space.

4.1. **The codifferential $d_3$.** From Table 1 we see that $H^2(d_3) = 0$, so there are no deformations of this codifferential. This is not surprising, as this codifferential corresponds to the simple Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. As we shall see, some of the other codifferentials can deform into $d_3$.

4.2. **The codifferential $d(\lambda : \mu), \mu \neq -\lambda$.** For generic values of $(\lambda : \mu)$ we have

\begin{align*}
H^1 &= \langle \varphi_1^1 + \varphi_2^2 \rangle \\
H^2 &= \langle \psi_2^{13} \rangle.
\end{align*}

The universal infinitesimal deformation is

\[ d^1 = \psi_1^{13} \lambda + \psi_1^{23} \mu + \psi_2^{13} t. \]

Because $[d^1, d^1] = 0$, the universal infinitesimal deformation is universal. Since $H^3 = 0$ there can be no relations on the base, which is therefore $\mathcal{A} = \mathbb{C}[[t]]$. The matrix of the miniversal deformation is

\[ A = \begin{bmatrix}
0 & \lambda & 1 \\
0 & t_1 & \mu \\
0 & 0 & 0
\end{bmatrix}. \]

This matrix is just the matrix of $d(\alpha : \beta)$ where

\[ \{\alpha, \beta\} = \frac{\lambda + \mu \pm \sqrt{(\lambda - \mu)^2 + 4t}}{2}. \]

Thus deformations of $d(\lambda : \mu)$ simply move along the family. We see that there are no jump deformations. The codifferential $d(1 : 1)$ has no special behavior in this context so it is more natural to include it in the family than $d_2$, which as we will see later has a more complicated deformation picture.

4.3. **The codifferential $d(1 : 0)$.** The only special thing about $d(1 : 0)$ is that $H^1$ is 2-dimensional. We have

\[ H^1 = \langle \varphi_1^1 + \varphi_2^2 \rangle. \]

The significance of this is that there are more outer derivations of $d(1 : 0)$, which plays a role in considering extensions of a Lie algebra by the algebra corresponding to $d(1 : 0)$. The deformation picture is generic so deformations of $d(1 : 0)$ simply move along the family. This justifies our unconventional inclusion of this element in the one parameter family of Lie algebras.
4.4. **The codifferential** $d(1 : -1)$. We have

\[
H^1 = \langle 2\phi_1^1 + \phi_2^2 \rangle \\
H^2 = \langle \psi_{13}^1, \psi_{12}^3 \rangle \\
H^3 = \langle \varphi_{3}^{123} \rangle
\]

The universal infinitesimal deformation is

\[
d^1 = \psi_{13}^1(1 + t^1) + \psi_{1}^{23} - \psi_{2}^{23} + \psi_{3}^{12} t^2.
\]

This deformation also coincides with the miniversal deformation $d^\infty$, but in this case, we do have one relation on the base

\[
t_1 t^2 = 0,
\]

so the base of the miniversal deformation is $A = \mathbb{C}[[t^1, t^2]]/(t^1 t^2)$. This relation follows from the bracket calculation

\[
[d^\infty, d^\infty] = -2\varphi_{3}^{123} t^1 t^2.
\]

In order to study the deformations, we have to take into account the relation. Thus, in the matrix

\[
A = \begin{bmatrix}
0 & 1 + t^1 & 1 \\
0 & 0 & -1 \\
t^2 & 0 & 0
\end{bmatrix}
\]

of the miniversal deformation, we must either have $t^1 = 0$, or $t^2 = 0$. This means that although the tangent space $H^2$ is 2 dimensional, the actual deformations only occur along two curves.

Along the curve $t^1 = 0$, $d^\text{inf} \sim d_3$. In fact, if we let $g$ be the automorphism of $\mathbb{C}^3$ whose matrix $G$ is given by

\[
G = \begin{bmatrix}
0 & -1/2 \frac{t^2-1}{t^2} & -1/2 \frac{1+t^2}{t^2} \\
0 & 1 & 1 \\
1 & 0 & 0
\end{bmatrix},
\]

then $d_3 = g^*(d^\infty)$. Thus we have a jump deformation from $d(1 : -1)$ to $d_3$ along this curve.

Along the curve $t^2 = 0$, we have $d^\infty = d(1 + t^1, -1)$, which gives a deformation along the family.

Thus the picture for $d(1 : -1)$ is more interesting.
4.5. The codifferential $d_2$. We have
\[ H^1 = \langle \varphi_2, \varphi_1^2, \varphi_2^3 \rangle \]
\[ H^2 = \langle \psi_{13}^1, \psi_{13}^2, \psi_1^{23} \rangle. \]
The universal infinitesimal deformation is
\[ d_1 = \psi_1^{13}(1 + t^4) + \psi_2^{23} + \psi_1^{13}t^2 + \psi_1^{23}t^3. \]
Because $[d_1, d_1] = 0$, the universal infinitesimal deformation is miniversal, and there are no relations on the base $\mathcal{A} = \mathbb{C}[[t^1, t^2, t^3]].$ Along the curve $t^2 = t^1 = 0$, we have $g^*(d^\infty) = d(1 : 1)$ where $g$ is determined by the matrix
\[ G = \begin{bmatrix} t_3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \]
This means that there is a jump deformation from $d_2$ to $d(1 : 1)$. Otherwise, we have that $d^\infty \sim d(\alpha : \beta)$ where
\[ \{\alpha, \beta\} = \frac{2 + t^1 \pm \sqrt{(t^1)^2 - 4t^2t^3}}{2}, \]
as long as $t^1 \neq -2$. Thus we see that locally (for small values of the parameters $t^i$) $d_2$ deforms into elements of the family $d(\lambda : \mu)$ which are “near” to $d(1 : 1)$. Thus $d_2$ deforms as if it were the element $d(1 : 1)$, but is distinguishable from it in terms of deformation behaviour because it has a jump deformation, and in addition, it has a larger parameter space of deformations.

4.6. The codifferential $d_1$. We have
\[ H^1 = \langle \varphi_3^2, \varphi_2^3, \varphi_1^1 + \varphi_2^2, \varphi_2^3 - \varphi_3^3 \rangle \]
\[ H^2 = \langle \psi_{11}^{12}, \psi_3^{12}, \psi_{13}^{13}, \psi_2^{13}, \psi_3^{13} - \psi_2^{12} \rangle \]
\[ H^3 = \langle \psi_{123}^{123}, \varphi_3^{123} \rangle. \]
The universal infinitesimal deformation is
\[ d_1 = \psi_1^{23} + \psi_1^{12}t^1 + \psi_3^{12}t^2 + \psi_1^{13}t^3 + \psi_2^{13}t^4 + (\psi_3^{13} - \psi_2^{12})t^5. \]
In this case, the miniversal deformation $d^\infty$ is equal to the infinitesimal deformation, and we compute
\[ [d^\infty, d^\infty] = 2\varphi_2^{123}(t^1t^4 + t^3t^5) + 2\varphi_3^{123}(t^1t^5 - t^2t^3), \]
which means that there are two relations on the base $\mathcal{A}$ of the versal deformation. We have $\mathcal{A} = \mathbb{C}[[t^1, t^2, t^3, t^4, t^5]]/(t^1t^4 + t^3t^5, t^1t^5 - t^2t^3)$.
The matrix of the versal deformation is
\[
A = \begin{bmatrix}
t_1 & t_3 & 1 \\
-t_5 & t_4 & 0 \\
t_2 & t_5 & 0
\end{bmatrix}.
\]
However, due to the relations, not every such \( A \) is actually the matrix of a 3 dimensional Lie algebra. We have to solve the relations and then consider the resulting matrices, in order to classify the deformations. It is easy to obtain the solutions using Maple. We have three solutions, given by
\[
\begin{align*}
(1) & \quad t^1 = t^3 = 0, \\
(2) & \quad t^3 = t^4 = t^5 = 0, \\
(3) & \quad t^5 = -\frac{t^4 t^2}{t_1}, t^2 = -\frac{t^1 (t^2)^2}{t_1^2}.
\end{align*}
\]
Thus the solutions give rise to a 2-dimensional and 2 3-dimensional pieces.

For the first solution, along the surface \( t^2 t^4 + (t^5)^2 = 0 \), we have \( d^\infty \sim d(1 : -1) \), and otherwise \( d^\infty \sim d_3 \), which means that there is a three parameter family of jump deformations to \( d_3 \) and a two parameter family of jump deformations to \( d(1 : -1) \).

For the second solution, along the curve \( t^1 = \alpha + \beta, t^2 = \alpha \beta \), we have \( d^\infty \sim d(\alpha : \beta) \), which means that there is a jump deformation from \( d^\infty \) to \( d(\alpha : \beta) \) for every value of \( (\alpha : \beta) \).

For the third solution, along the surface \( t^3 = \alpha + \beta, t^4 = -\alpha \beta \), we have \( d^\infty \sim d(\alpha : \beta) \), which again gives a family of jump deformations.

As a consequence, we see that there are jump deformations from \( d_1 \) to every codifferential in the moduli space except \( d_2 \). We also see that although the tangent space is 5 dimensional at the point \( d_1 \), deformations of this codifferential actually live along lower dimensional varieties, which illustrates a common situation in a moduli space of Lie algebras.

5. Conclusions

The moduli space of 3-dimensional Lie algebras has a natural stratification by orbifolds, three of which are just points, with the remaining piece given by \( \mathbb{P}^1/\Sigma_2 \). The maps between these pieces are given by jump deformations. In the case of \( d_1 \), there are jump deformations to all the points in the \( \mathbb{P}^1/\Sigma_2 \) stratum. The orbifold points play a special role in the moduli space, either because they have extra deformations, or because there are extra deformations to them. We illustrate the moduli space of 3 dimensional Lie algebras by the following picture.
In [6], the moduli space of 4-dimensional Lie algebras was studied. Although that picture is more complex, the main features, such as the stratification by orbifolds, with jump deformations connecting the strata, already occur in the three dimensional picture.

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