A $q$-Lorentz Algebra From $q$-Deformed Harmonic Oscillators

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Abstract

A mapping between the operators of the bosonic oscillator and the Lorentz rotation and boost generators is presented. The analog of this map in the $q$-deformed regime is then applied to $q$-deformed bosonic oscillators to generate a $q$-deformed Lorentz algebra, via an inverse of the standard chiral decomposition. A fundamental representation, and the co-algebra structure, are given, and the generators are reformulated into $q$-deformed rotations and boosts. Finally, a relation between the $q$-boson operators and a basis of $q$-deformed Minkowski coordinates is noted.

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I. INTRODUCTION

The existence of symmetries has proved to be a universal feature in many aspects of physics. Lie group theory, as a mathematical description of these symmetries, has also proved to be a powerful tool for prediction and unification in the last thirty years. This may suggest that consideration of possible deformations of these symmetries would not be highly profitable. However, it is important to realise that arbitrary deformations can be considered as generalisations to the standard results, and can lead to new and more powerful symmetries that were not previously apparent. Indeed deformations of physical theories have been very important in the advance of physics in the last hundred years. For example, special relativity can be considered as a deformation of Galilean relativity with deformation parameter $1/c$, while quantum mechanics can similarly be regarded as a deformation of classical mechanics with parameter $\hbar$.

There has recently been considerable interest in the deformation, via an arbitrary parameter $q$, of the Lie group structures commonly found in physics. These quantum groups are deformations of the universal enveloping algebra of the underlying Lie group and have a Hopf algebra structure. While quantum groups associated with many of the simple Lie algebras initially received most attention, deformations of the spacetime symmetry groups, i.e. the Lorentz group, have also been considered recently by a number of authors.

Ogievetsky et al. have considered a chiral decomposition of a $q$-Lorentz group into two inequivalent $SL_q(2)$ groups acting on dotted and undotted spinors. This resulted in a $q$-decomposition since the two $SL_q(2)$ groups only $q$-commuted. In this letter we consider the construction of a $q$-deformed Lorentz algebra which admits a complete decomposition into two commuting chiral sub-algebras. We base this construction on a $q$-deformation of the classical group isomorphism $SO(4) \cong SU(2) \otimes SU(2)$. Recognising that the non-compact Lorentz group $SO(3,1)$ is equivalent to its compact relation $SO(4)$, with an altered metric, we can thus consider the $q$-deformed Lorentz algebra resulting from the direct product.
of two copies of the $SU_q(2)$ (or $U_q(su(2))$) quantum group in the Drinfel’d-Jimbo basis. This algebra can then be linked back to the $q$-boson ($q$-SHO) operators by making use of the $q$-deformed Jordan-Schwinger map \[14,19\].

In Section 2 the mapping between harmonic oscillator annihilators and creators and the “classical” Lorentz rotations and boosts is considered. An analogous construction is then applied to the $q$-oscillator operators to generate a $q$-deformed Lorentz algebra in Section 3. Fundamental representations are obtained and the generators are reformulated into $q$-deformed rotations and boosts. It is also noted that there exists a mapping from $q$-boson operators to the $q$-Minkowski coordinates of Ogievetsky et al. \[3,11\].

II. CONSTRUCTION OF THE LORENTZ GROUP GENERATORS

Consider two commuting sets of SHO annihilators and creators

$$[a_i, a_j^\dagger] = \delta_{ij}, \quad \text{for } i, j \in 1, 2,$$

(1)

with number operator

$$N_i = a_i^\dagger a_i, \quad i = 1, 2. \quad (2)$$

Then we construct three generators

$$J_+ = a_1^\dagger a_2 \quad (3)$$

$$J_- = a_2^\dagger a_1 \quad (4)$$

$$J_3 = \frac{1}{2}(N_1 - N_2) \quad (5)$$

according to the Jordan-Schwinger map. These generators then satisfy the commutation relations

$$[J_3, J_\pm] = J_\pm \quad (6)$$

$$[J_+, J_-] = 2J_3, \quad (7)$$

and generate $SU(2)$. Writing $J_\pm = J_1 \pm iJ_2$ we have
\[ [J_i, J_j] = i \epsilon_{ijk} J_k, \quad (8) \]

where

\[
J_1 = \frac{1}{2}(a_1^\dagger a_2 + a_2^\dagger a_1) \quad (9)
\]
\[
J_2 = \frac{1}{2i}(a_1^\dagger a_2 - a_2^\dagger a_1) \quad (10)
\]
\[
J_3 = \frac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2) \quad (11)
\]

Choosing two commuting sets of \(SU(2)\) generators

\[
[J^m_i, J^p_j] = i \delta^{mp} \epsilon_{ijk} J^p_k, \quad (12)
\]

where \(p, m = 1, 2\), and \(i, j, k = 1, 2, 3\), we construct the generators

\[
J_i = J^1_i + J^2_i \quad \text{(rotations)} \quad (13)
\]
\[
K_i = -i(J^1_i - J^2_i) \quad \text{(boosts)} \quad (14)
\]

which satisfy

\[
[K_i, K_j] = -i \epsilon_{ijk} J_k \quad (15)
\]
\[
[J_i, J_j] = i \epsilon_{ijk} J_k \quad (16)
\]
\[
[J_i, K_j] = i \epsilon_{ijk} K_k \quad (17)
\]

These clearly generate

\[
SO(3, 1) \cong SU(2) \otimes SU(2), \quad (19)
\]

which with the Minkowski metric becomes the Lorentz group \([1]\). Thus combining the two maps we can write the Lorentz group generators in terms of the bosonic oscillator operators as follows,

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\(^1\)A change of sign in the right hand side of Eq. \([15]\) will give the generator algebra of the compact group \(SO(4)\).
\[ J_1 = \frac{1}{2} \sum_{i=1}^{2} (a_i^\dagger a_i^1 + a_i^\dagger a_i^1) \quad (20) \]
\[ J_2 = \frac{1}{2i} \sum_{i=1}^{2} (a_i^\dagger a_i^2 - a_i^\dagger a_i^2) \quad (21) \]
\[ J_3 = \frac{1}{2} \sum_{i=1}^{2} (N_i^1 - N_i^2) \quad (22) \]
\[ K_1 = -\frac{i}{2} (a_1^\dagger a_2^1 + a_2^\dagger a_1^1 - a_1^\dagger a_2^2 - a_2^\dagger a_1^2) \quad (23) \]
\[ K_2 = -\frac{1}{2} (a_1^\dagger a_2^1 - a_2^\dagger a_1^1 - a_1^\dagger a_2^2 + a_2^\dagger a_1^2) \quad (24) \]
\[ K_3 = -\frac{i}{2} (N_1^1 - N_2^1 - N_1^2 + N_2^2), \quad (25) \]

where the lower index corresponds to the individual SHO operators in each of the commuting copies of SU(2) denoted by the upper index.

The rotation and boost operators are often combined into one tensor \( \sigma_{\mu\nu} \), as follows,

\[ (\sigma_{\mu\nu}) = 2 \begin{pmatrix} 0 & \vec{K}^T \\ -\vec{K} & (J_{ij}) \end{pmatrix}, \quad (26) \]

where \( \vec{K} \) is the boost vector,

\[ \vec{K}^T = (K_1, K_2, K_3), \quad (27) \]

and \( (J_{ij}) \) is the standard \( 3 \times 3 \) rotation matrix where

\[ J_i = \frac{1}{2} \epsilon_{ijk} J_{jk}. \quad (28) \]

\( \sigma^{\mu\nu} \) gives the reducible spinorial representation of \( SO(3, 1) \) (or \( SO(4) \)) which is most naturally formulated in terms of the elements of a Dirac algebra as follows,

\[ \sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu], \quad \text{where} \]

\[ \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}. \quad (29) \]

If \( \eta^{\mu\nu} \) is replaced by the Euclidean metric \( \delta^{\mu\nu} \), as in the case of \( SO(4) \), the Dirac algebra becomes the standard Clifford algebra. Since the generators \( \sigma^{\mu\nu} \) can be related to the bosonic oscillator operators this suggests that when considered as operators we can represent
elements of the $\gamma$-matrices in terms of the bosonic annihilators and creators. i.e. in the Weyl representation, in $2 \times 2$ block form,

\begin{align}
(\gamma^1)^{01} &= -(\gamma^1)^{10} = (a_1^\dagger a_2 + a_2^\dagger a_1) \\
(\gamma^2)^{01} &= -(\gamma^2)^{10} = -i(a_1^\dagger a_2 - a_2^\dagger a_1) \\
(\gamma^3)^{01} &= -(\gamma^3)^{10} = -(a_1^\dagger a_1 - a_2^\dagger a_2)
\end{align}

Note that the standard form for $\gamma^5$ still holds and Dirac bi-spinors $\psi$ transforming under the spinorial representation of the Lorentz group, i.e. $\psi \rightarrow \psi' = \exp(-i/4\theta_{\mu\nu}\sigma^{\mu\nu})$, can still be decomposed into $SU(2)$ transforming spinors via the standard chiral projectors, $P_{\pm} = (1 \pm \gamma^5)/2$. Vectors and higher spin representations obtained via direct products of the chiral spinors, $\psi_R = (1/2, 0) = P_+ \psi$, and $\psi_L = (0, 1/2) = P_- \psi$, will also transform in the standard manner and thus the formulation in terms of the bosonic oscillator operators is entirely equivalent to the standard one.

**III. CONSTRUCTION OF THE $Q$-LORENTZ ALGEBRA**

Having introduced a mapping between the bosonic oscillator operators and the Lorentz group generators we can consider its $q$-deformation, and thereby obtain a $q$-deformation of the Lorentz algebra. Our mapping will now be

\[ q\text{-SHO} \rightarrow SU_q(2) \rightarrow SO_q(3,1). \]

The standard $q$-oscillator is defined via the deformed commutation relations \[^{[7]}\,\]

\[^{2}\text{The standard chiral decomposition of Lorentz bi-spinors produces spinors transforming under the self representation and complex conjugate self representation of } SL(2, C), \text{ with undotted and dotted indices respectively. However, since we are working for convenience with the compact } SO(4) \text{ group, a discussion involving } SU(2) \text{ spinors is more relevant.} \]
\[ [a, a^\dagger]_{q^{-1}} = q a^\dagger a - a^{-1} a^\dagger a = q N_q \]  
(35)

\[ [a, N_q] = a \]  
(36)

\[ [a^\dagger, N_q] = -a^\dagger, \]  
(37)

where \( N_q \) is the \( q \)-number operator. The substitution \( q \to q^{-1} \) also implies

\[ a a^\dagger - q a^\dagger a = q^{-N_q}. \]  
(38)

The Jordan - Schwinger map can be applied as before, giving the so-called \( q \)-Jordan-Schwinger map (see [16–19]). Taking a pair of commuting \( q \)-SHO, \( a_i, a_i^\dagger, N_{q_i}, \, i=1,2 \), we define

\[ J_+ = a^\dagger_1 a_2 \]  
(39)

\[ J_- = a_2^\dagger a_1 \]  
(40)

\[ J_3 = \frac{1}{2} (N_{q_1} - N_{q_2}). \]  
(41)

These are the generators of the quantum universal enveloping algebra of \( SU(2), U_q(su(2)) \). i.e.

\[ [J_3, J_{\pm}] = \pm J_\pm \]  
(42)

\[ [J_+, J_-] = \frac{q^{2J_3} - q^{-2J_3}}{q - q^{-1}} = [2J_3]_q, \]  
(43)

where Eq. 42 follows directly, and Eq. 43 follows from consideration of Eqs. 35 and 38.

We shall return to this formulation later in the consideration of \( q \)-rotation and \( q \)-boost generators. However, for construction of the \( q \)-Lorentz algebra in analogy to the classical chiral decomposition, a different basis for \( SU_q(2) \) is more appropriate. If we choose a basis consisting of generators

\[ J_+ = a_1^\dagger a_2 \]  
(44)

\[ J_- = a_2^\dagger a_1 \]  
(45)

\[ q^{J_3} = q^{1/2(N_{q_1} - N_{q_2})}, \]  
(46)
then the $SU_q(2)$ algebra has the form (see Appendix A):

\[ q^{J_3} J_\pm q^{-J_3} = q^{\pm 1} J_\pm \]  \hspace{1cm} (47)

\[ [J_+, J_-] = [2J_3]_q. \]  \hspace{1cm} (48)

In analogy to the classical direct product, $SO(3,1) \cong SU(2) \otimes SU(2)$, we can construct the $q$-Lorentz algebra from two copies of $SU_q(2)$. The basis will thus be \{ $J_\pm$, $q^{J_3}$, $\overline{J}_\pm$, $q^{\overline{J}_3}$ \}, where

\[ q^{J_3} J_\pm q^{-J_3} = q^{\pm 1} J_\pm \]  \hspace{1cm} (49)

\[ [J_+, J_-] = [2J_3]_q \]  \hspace{1cm} (50)

\[ q^{\overline{J}_3} \overline{J}_\pm q^{-\overline{J}_3} = q^{\pm 1} \overline{J}_\pm \]  \hspace{1cm} (51)

\[ [\overline{J}_+, J_-] = [2\overline{J}_3]_q, \]  \hspace{1cm} (52)

where barred and unbarred generators commute. The explicit commutation properties of the two copies of $SU_q(2)$ are in contradistinction to the chiral decomposition considered by Ogievetsky et al. \cite{9,10}, where the two chiral subalgebras only explicitly $q$-commute.

Since the two algebras, $SU_q(2)$ and $\overline{SU}_q(2)$, are distinct, and in line with the direct product interpretation, we note that the fundamental representation must be four dimensional. This can be explicitly obtained by embedding the $SU_q(2)$ fundamental representation (see Appendix A) into $4 \times 4$ matrices as follows,

\[ \rho(q^{J_3}) = \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix} 0_2 \]  \hspace{1cm} (53)

\[ \rho(J_+) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} 0_2 \]  \hspace{1cm} (54)

\[ \rho(J_-) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} 0_2 \]  \hspace{1cm} (55)
The barred and unbarred algebras then form distinct $2 \times 2$ blocks, as expected for the direct product decomposition. We also note that a similar representation has been obtained by Dabrowski et al. [13].

The coalgebra structure of this $q$-Lorentz algebra can be obtained by analogy with the results of Dabrowski et al. [13]. The co-product ($\Delta$), the co-unit ($\epsilon$), and the antipode ($S$) are given by

\[ \Delta(q^{J_3}) = q^{J_3} \otimes q^{J_3} \]  
\[ \Delta(J_+) = J_+ \otimes q^{-J_3} q^{J_3} + q^{J_3} q^{-J_3} \otimes J_+ \]  
\[ \Delta(J_-) = J_- \otimes q^{-J_3} q^{J_3} + q^{J_3} q^{-J_3} \otimes J_- \]  
\[ \Delta(q^{\overline{J}_3}) = q^{\overline{J}_3} \otimes q^{\overline{J}_3} \]  
\[ \Delta(\overline{J}_+) = \overline{J}_+ \otimes q^{-J_3} q^{J_3} + q^{J_3} q^{-J_3} \otimes \overline{J}_+ \]  
\[ \Delta(\overline{J}_-) = \overline{J}_- \otimes q^{-J_3} q^{J_3} + q^{J_3} q^{-J_3} \otimes \overline{J}_- \]

\[ \epsilon(q^{J_3}) = \epsilon(q^{\overline{J}_3}) = 1 \]  
\[ \epsilon(J_+) = \epsilon(\overline{J}_+) = 1 \]  
\[ \epsilon(J_-) = \epsilon(\overline{J}_-) = 1 \]
\[ S(q^{J_3}) = q^{-J_3} \]  
\[ S(J_\pm) = -q^{\pm 1} J_\pm \]  
\[ S(q^{\bar{J}_3}) = q^{-\bar{J}_3} \]  
\[ S(J_\pm) = -q^{\pm 1} \bar{J}_\pm. \]

This completes the Hopf algebra structure of the quantum Lorentz group in this formulation.

To obtain a formalism of the \( q \)-Lorentz algebra corresponding to the standard formulation in terms of rotations and boosts we return to the original Drinfel’d-Jimbo basis of \( U_q(\mathfrak{su}(2)) \). Again choosing two commuting sets of \( U_q(\mathfrak{su}(2)) \) generators, \( \{J_3, J_\pm\} \), and \( \{\bar{J}_3, \bar{J}_\pm\} \), we construct the generators

\[ J_1 = \frac{1}{2} \left( J_+ + J_- + \bar{J}_+ + \bar{J}_- \right) \]  
\[ J_2 = \frac{1}{2} \left( J_+ - J_- + \bar{J}_+ - \bar{J}_- \right) \]  
\[ J_3 = J_3 + \bar{J}_3 \]  
\[ \mathcal{K}_1 = \frac{1}{2i} \left( J_+ + J_- - \bar{J}_+ - \bar{J}_- \right) \]  
\[ \mathcal{K}_2 = -\frac{1}{2} \left( J_+ - J_- - \bar{J}_+ + \bar{J}_- \right) \]  
\[ \mathcal{K}_3 = -i \left( J_3 - \bar{J}_3 \right), \]

where we associate \( J_i, i = 1, 2, 3 \) with \( q \)-rotations, and \( \mathcal{K}_i, i = 1, 2, 3 \) with \( q \)-boosts. With this construction these \( q \) generators have exactly the same structure in terms of \( q \)-boson operators, as do the classical rotations and boosts in terms of the boson operators. i.e. Eqs. 20-25 still hold in the \( q \)-deformed regime.

In order to make contact with the analysis of Schmidke et al. \[8\] and Ogievetsky et al. \[8,10\] we can transform our formulation of the \( q \)-Lorentz generators to correspond to the Woronowicz basis of \( SU_q(2) \). In terms of the Woronowicz basis generators, the generators of \( U_q(\mathfrak{su}(2)) \) in the Drinfel’d-Jimbo basis are (see Appendix A):

\[ J_3 = \frac{1}{4 \ln q} \ln \left( 1 - (q - q^{-1} T_3) \right) \]  
\[ J_\pm = T_\pm \left( 1 - (q - q^{-1} T_3)^{1/4} q^{\mp 1/2} \right). \]
Thus the \( q \)-rotations and boosts can be written in this basis as

\[
\begin{align*}
\mathcal{J}_1 &= \frac{1}{2} \left( T^i_+ (1 - (q - q^{-1}) T^i_3)^{1/4} q^{-1/2} + T^i_- (1 - (q - q^{-1}) T^i_3)^{1/4} q^{1/2} \right) \\
\mathcal{J}_2 &= \frac{1}{2i} \left( T^i_+ (1 - (q - q^{-1}) T^i_3)^{1/4} q^{-1/2} - T^i_- (1 - (q - q^{-1}) T^i_3)^{1/4} q^{1/2} \right) \\
\mathcal{J}_3 &= \frac{1}{4 \ln q} \ln \left[ (1 - (q - q^{-1}) T^i_3) (1 - (q - q^{-1}) T^i_3)^2 \right] \\
\mathcal{K}_1 &= -\frac{i}{2} \left( (-1)^{i-1} T^i_+ (1 - (q - q^{-1}) T^i_3)^{1/4} q^{-1/2} + (-1)^{i-1} T^i_- (1 - (q - q^{-1}) T^i_3)^{1/4} q^{1/2} \right) \\
\mathcal{K}_2 &= -\frac{1}{2} \left( (-1)^{i-1} T^i_+ (1 - (q - q^{-1}) T^i_3)^{1/4} q^{-1/2} - (-1)^{i-1} T^i_- (1 - (q - q^{-1}) T^i_3)^{1/4} q^{1/2} \right) \\
\mathcal{K}_3 &= -\frac{i}{4 \ln q} \ln \frac{1 - (q - q^{-1}) T^i_3}{1 - (q - q^{-1}) T^i_3} \\
\end{align*}
\]

where \( i = 1, 2 \) is summed.

In either formulation these generators satisfy the following commutation relations:

\[
\begin{align*}
[\mathcal{J}_i, \mathcal{J}_j] &= \begin{cases} 
\frac{1}{2} i \epsilon_{ijk} [2 \mathcal{J}_k], & \text{if } k \neq 3 \\
\frac{1}{2} i \epsilon_{ijk} \{2 \mathcal{J}_k\}_q, & \text{if } k = 3
\end{cases} \\
[\mathcal{K}_i, \mathcal{K}_j] &= \begin{cases} 
-\frac{1}{2} i \epsilon_{ijk} [2 \mathcal{J}_k], & \text{if } k \neq 3 \\
-\frac{1}{2} i \epsilon_{ijk} \{2 \mathcal{J}_k\}_q, & \text{if } k = 3
\end{cases} \\
[\mathcal{J}_i, \mathcal{K}_j] &= \begin{cases} 
\frac{1}{2} i \epsilon_{ijk} [2 \mathcal{K}_k], & \text{if } k \neq 3 \\
\frac{1}{2} i \epsilon_{ijk} \{2 \mathcal{K}_k\}_q, & \text{if } k = 3
\end{cases}
\]

where

\[
\begin{align*}
\{2 \mathcal{J}_k\}_q &= \left[ \frac{1}{i} (i \mathcal{J}_k - \mathcal{K}_k) \right]_q + \left[ \frac{1}{i} (i \mathcal{J}_k + \mathcal{K}_k) \right]_q \\
\{2 \mathcal{K}_k\}_q &= -i \left( \left[ \frac{1}{i} (i \mathcal{J}_k - \mathcal{K}_k) \right]_q - \left[ \frac{1}{i} (i \mathcal{J}_k + \mathcal{K}_k) \right]_q \right)
\end{align*}
\]

and \([A]_q\) represents the normal \( q \)-integers. We see that in the limit \( q \to 1 \) the generators \( \mathcal{J}_i \) and \( \mathcal{K}_i \) reduce to the standard Lorentz rotations and boosts.

In this formalism the map between the \( q \)-SHO operators and the \( q \)-Lorentz generators is formally identical to the classical case and the \( q \)-structure is explicit only in the deformed commutation relations. This suggests that, by construction, the generators of a \( q \)-spinorial representation, \( \sigma^{\mu \nu} \), should have a matrix representation in terms of the rotation and boost.
generators equivalent to the classical case, with the $q$-structure hidden in its algebra. This immediately implies a deformed Dirac (Clifford) algebra for $\gamma$-matrices. The form of this deformation has not been determined, but we note in the next section that the form of an associated deformed Minkowski metric can be obtained in a simple manner from the $q$-boson operators.

IV. Q-MINKOWSKI SPACE

The existence and form of a $q$-deformed Minkowski spacetime have been considered in detail recently by a number of authors [8,10,20–23]. Here we shall simply consider how the deformed Minkowski space algebra of Ogievetsky et al. can be related to the algebra of the $q$-SHO.

In previous discussions [8,10] generators for $q$-Minkowski space coordinates were built out of $q$-spinor bilinears, their commutation relations being determined by the $SU_q(2)$ $q$-spinor relations. In these discussions the $q$-spinors were taken as coordinates of the non-commutative quantum plane $\{(x, y) : xy = qyx\}$, the underlying carrier space of $SU_q(2)$ [24]. However, we now show that the $q$-Minkowski generators can be constructed directly out of the $q$-boson operators. Consider the basis $\{A, B, C, D\}$, where the generators are defined as follows:

$$A = q^{-(N_{q_1} - N_{q_2})}$$  \hspace{1cm} (90)

$$B = q^{-1/2(N_{q_1} - N_{q_2})}a_1^\dagger a_2$$  \hspace{1cm} (91)

$$C = \frac{(q - q^{-1})^2}{q}a_2^\dagger a_1 q^{-1/2(N_{q_1} - N_{q_2})}$$  \hspace{1cm} (92)

$$D = \frac{(q - q^{-1})^2}{q}a_2^\dagger a_1 a_2^\dagger a_2 + q^{(N_{q_1} - N_{q_2})},$$  \hspace{1cm} (93)

where $\{a_i^\dagger, a_i, N_{q_i}\}, i = 1, 2$ are the generators of two commuting $q$-SHOs. We note that this map differs from the construction in terms of $q$-oscillators, and $SU_q(2)$ generators, considered by Kulish [20]. This operator basis generates the $q$-Minkowski algebra of Schmidke et al. [8], in the basis of Kulish [20].
\[ AC = q^2 CA \]
\[ AB = q^{-2} BA \]
\[ AD = DA \]  
(94)

\[ [B, D] = -\frac{q - q^{-1}}{q} AB \]
\[ [C, D] = \frac{q - q^{-1}}{q} CA \]
\[ [B, C] = \frac{q - q^{-1}}{q} (AD - A^2), \]

where central elements are: the \( q \)-trace,

\[
\text{tr}_q(K) = \text{Tr}_I q K = \frac{1}{q} A + q D
\]
(95)

\[
I_q = \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix}
\]
(96)

and the invariant Minkowski length,

\[
L = CB - \frac{1}{q^2} DA
\]
(97)

\[ 0 = [L, \{A, B, C, D\}]. \]
(98)

Kulish showed, via consideration of the reflection equation, that the homomorphism,

\[
K = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \rightarrow MK\tilde{M}^{-1},
\]
(99)

where \( M \) is an element of the quantum group algebra \( SL_q(2, C) \) ("dual" to our formulation of \( SU_q(2) \)), and \( M^\dag = \tilde{M}^{-1} \), leaves the defining relations (Eq.s 94) invariant. Thus this procedure does indeed give an analog of Minkowski space in the \( q \)-deformed regime. The \( q \)-Minkowski 4-vector, in \( 2 \times 2 \) formalism, can be written in terms of real Minkowski coordinates \( \{X^0, X^1, X^2, X^3\} \) as follows:

\[
X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \frac{\sqrt{q}}{2} (X^0 - X^3) & \frac{\sqrt{q}}{1+iq} (X^1 - X^2) \\ \frac{\sqrt{q}}{(1+i)q} (X^1 + X^2) & \frac{\sqrt{q}}{2q} (X^0 + X^3) \end{pmatrix}.
\]
(100)
This then allows the invariant Minkowski length to be represented in the form

$$L = (q^2 + 1)^{-1} g_{ij} X^i X^j, \quad (101)$$

where the deformed Minkowski metric is given by

$$g_{ij} = \begin{pmatrix}
0 & q^2 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & -1 & q(q - q^{-1})
\end{pmatrix}, \quad (102)$$

as was obtained by Ogievetsky et al. \[10\]. Clearly such a metric does not reduce to the standard form in the $q \to 1$ limit.

V. CONCLUDING REMARKS

In this letter we have considered various maps from SHO operators to the generators of $SU(2)$ and the Lorentz group. Consideration of equivalent maps in the $q$-deformed regime has led to a $q$-deformation of the algebra of Lorentz rotations and boosts, and its corresponding chirally decomposed form. We have also noted that $q$-deformed Minkowski 4-vectors can also be obtained via construction from the $q$-SHO. The relationship between these algebras suggests that the $q$-SHO is fundamentally related not only to the compact quantised universal enveloping algebras but also to the structure of the $q$-deformed spacetime symmetry groups. The fact that the $q$-SHO has also been found to possess a quantum group (Hopf algebra) structure [25,27] implies that it could play a central role in the classification of quantum groups.

It was noted earlier that the deformation of the rotation and boost algebra of the Lorentz group should lead to a deformation of the $\gamma$-matrix Clifford algebra associated with the spinorial representation. This was not considered here but we note in passing that such deformed Clifford algebras have been obtained [10,11] using $R$-matrix methods, and consideration of
the differential calculus on quantum spaces has allowed Song [11] to determine the form of
the Dirac equation for $q$-deformed Dirac bi-spinors.

We finally note that the analysis here has focussed on the spacetime symmetry
applications of the various mappings. If, however, we were to concentrate on the compact
direct product group $SO(4) \cong SU(2) \otimes SU(2)$ then the analysis presented here could easily
be modified to allow consideration of the $q$-deformation of internal symmetries such as chiral
symmetry, i.e. $SU_{qL}(2) \otimes SU_{qR}(2)$.

**APPENDIX A: COMPARISON OF $SU_q(2)$ GENERATORS IN VARIOUS BASES**

Quantum group theory has developed from a number of rather different starting points.
The mathematical structure has been abstracted from fields as seemingly diverse as quan-
tum inverse scattering theory, solvable two dimensional statistical models, non-commutative
geometry, not to mention knot theory. The manifestation of this structure in such diverse
fields is certainly suggestive of deeper underlying relationships. However, one problem as-
associated with the multitude of approaches to the subject is that there have been a number
of different formulations of the quantum group structure. In particular the most widely
studied $q$-deformed algebra, $SU_q(2)$ (or $U_q(su(2))$), has been formulated in a number of
different ways. It is not immediately apparent whether these different formulations are in
fact equivalent. For discussion of the $q$-deformed Lorentz algebra it is useful to have direct
mappings between the generators of $SU_q(2)$ in different formulations.

To achieve this we start with the following basis $\{qJ^3, J^\pm\}$ of $SU_q(2)$, which has the
following algebra (Basis 1):

\begin{equation}
qJ^3 J^\pm q^{-J^3} = q^{\pm 1} J^\pm \tag{A1}
\end{equation}

\begin{equation}
[J^+, J^-] = \frac{q^{2J^3} - q^{-2J^3}}{q - q^{-1}}, \tag{A2}
\end{equation}

due in a somewhat different form, to Kulish and Reshitikhin [28]. It can easily be shown
that the following $2 \times 2$ matrices constitute a fundamental representation for the generators:
\[
\rho(q^{J_3}) = \begin{pmatrix}
q^{1/2} & 0 \\
0 & q^{-1/2}
\end{pmatrix}
\]  
(A3)

\[
\rho(J_+) = \begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}
\]  
(A4)

\[
\rho(J_-) = \begin{pmatrix}
0 & 0 \\
1 & 0
\end{pmatrix}
\]  
(A5)

The standard basis for the quantum universal enveloping algebra \( U_q(su(2)) \), due to Drinfel’d and Jimbo [2–4], which we denote (Basis 2),

\[
[J_3, J_{\pm}] = \pm J_{\pm}
\]  
(A6)

\[
[J_+, J_-] = \frac{q^{2J_3} - q^{-2J_3}}{q - q^{-1}} = \frac{\sinh(2\eta J_3)}{\sinh(\eta)},
\]  
(A7)

where \( q = e^\eta \), is then a direct consequence of the algebra above [29]. Eq. A7 follows as a direct consequence of Eq. A2. Eq. A6 follows from Eq. A1 by noting that iteration of Eq. A1 implies

\[
\ln(q^{J_3}) J_{\pm} = J_{\pm} \ln(q^{\pm} q^{J_3}).
\]  
(A8)

The fundamental representation for the generators in this basis is simply given by a linear combination of Pauli matrices (e.g. [30]). i.e.

\[
\rho(J_3) = \frac{1}{2} \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}
\]  
(A9)

\[
\rho(J_+) = \begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}
\]  
(A10)

\[
\rho(J_-) = \begin{pmatrix}
0 & 0 \\
\frac{\sinh(2\eta)}{\sinh(\eta)} & 0
\end{pmatrix}
\]  
(A11)

The basis for \( SU_q(2) \) obtained by Woronowicz [3,31] via analysis of non-commutative differential calculus has also been widely studied. This basis (Basis 3) \( \{T_3, T_\pm\} \) has the algebra
\[ q^{-1}T_+T_+ - qT_-T_= T_3 \]  
\[ q^2T_3T_+ - q^{-2}T_+T_3 = (q + q^{-1})T_+ \]  
\[ q^{-2}T_3T_+ - q^2T_-T_3 = -(q + q^{-1})T_- \]

which at first sight may appear rather different to the bases of $SU_q(2)$ previously considered. However, a mapping between the generators does exist, as was noted by Rosso \[30\]. We can obtain an explicit mapping in two stages as follows. First create the new generator basis \{\tau_3, T_\pm\} via the mapping \[14\]

\[ \tau_3 = q^{-4}J_3 \]  
\[ T_\pm = q^{\pm1/2}J_\pm q^{-J_3} \].

It then follows, from the algebra of Basis 1 (Eq.\[A1\] \[A2\]), that

\[ q^{-1}T_+T_+ - qT_-T_= \frac{1 - \tau_3}{q - q^{-1}} \]  
\[ T_+\tau_3 = q^4\tau_3T_+ \]  
\[ T_-\tau_3 = q^{-4}\tau_3T_- \].

We can now obtain the algebra of the Woronowicz basis via the relation \[9\], 

\[ \tau_3 = 1 - (q - q^{-1})T_3 \].

Eq. \[A12\] follows immediately, and Eqs. \[A13\] \[A14\] follow from Eqs. \[A18\] \[A19\]. This basis has the fundamental representation

\[ \rho(T_3) = \begin{pmatrix} q^{-1} & 0 \\ 0 & -q \end{pmatrix} \]  
\[ \rho(T_+) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \]  
\[ \rho(T_-) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \].

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For the cases of the Drinfel’d-Jimbo and Woronowicz bases (2 and 3), which are of interest in the discussion of the $q$-Lorentz group, we can now give the explicit map between the generators. i.e.

\[ T_3 = \frac{1 - q^4 J_3}{q - q^{-1}} \]  \hspace{1cm} (A24)  

\[ T_\pm = J_\pm q^{-j_3 \pm 1/2}. \]  \hspace{1cm} (A25)

The inverse map is a little more complex and is given by

\[ J_3 = \frac{1}{4 \ln q} \ln \left( 1 - (q - q^{-1}) T_3 \right) \]  \hspace{1cm} (A26)  

\[ J_\pm = T_\pm \left( 1 - (q - q^{-1}) T_3 \right)^{1/4} q^{\pm 1/2}. \]  \hspace{1cm} (A27)
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