WEIGHTED ISOPERIMETRIC INEQUALITIES IN CONES
AND APPLICATIONS

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ABSTRACT. This paper deals with weighted isoperimetric inequalities relative to cones of $\mathbb{R}^N$. We study the structure of measures that admit as isoperimetric sets the intersection of a cone with balls centered at the vertex of the cone. For instance, in case that the cone is the half-space $\mathbb{R}_+^N = \{x \in \mathbb{R}^N : x_N > 0\}$ and the measure is factorized, we prove that this phenomenon occurs if and only if the measure has the form $d\mu = ax^k N \exp(c|x|^2) dx$, for some $a > 0$, $k, c \geq 0$. Our results are then used to obtain isoperimetric estimates for Neumann eigenvalues of a weighted Laplace-Beltrami operator on the sphere, sharp Hardy-type inequalities for functions defined in a quarter space and, finally, via symmetrization arguments, a comparison result for a class of degenerate PDE’s.

Key words: relative isoperimetric inequalities, Neumann eigenvalues, weighted Laplace-Beltrami operator, Hardy inequalities, degenerate elliptic equations.

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1. Introduction

This paper deals with weighted relative isoperimetric inequalities in cones of $\mathbb{R}^N$. Let $\omega$ be an open subset of $\mathbb{S}^{N-1}$, the unit sphere of $\mathbb{R}^N$, and $\Omega$ the cone

\[ \Omega = \left\{ x \in \mathbb{R}^N : \frac{x}{|x|} \in \omega, x \neq 0 \right\}. \]

We consider measures of the type $d\nu = \phi(x) dx$ on $\Omega$, where $\phi$ is a positive Borel measurable function defined in $\Omega$. For any measurable set $M \subset \Omega$, we define the $\nu$-measure of $M$

\[ \nu(M) = \int_M d\nu = \int_M \phi(x) dx \]

and the $\nu$-perimeter of $M$ relative to $\Omega$

\[ P_\nu(M, \Omega) = \sup \left\{ \int_M \text{div} (v(x)\phi(x)) dx : v \in C_0^1(\Omega, \mathbb{R}^N), |v| \leq 1 \right\}. \]
We also write $P_\nu(M, \mathbb{R}^N) = P_\nu(M)$. Note that if $M$ is a smooth set, then

$$P_\nu(M, \Omega) = \int_{\partial M \cap \Omega} \phi(x) dH_{N-1}(x).$$

The isoperimetric problem reads as

$$I_\nu(m) = \inf \{ P_\nu(M, \Omega) : M \subset \Omega, \nu(M) = m \}, \quad m > 0.$$  

One says that $M$ is an isoperimetric set if $\nu(M) = m$ and $I_\nu(m) = P_\nu(M, \Omega)$.

We give necessary conditions on the function $\phi$ for having $B_R \cap \Omega$ as an isoperimetric set, in Section 2. Here and throughout the paper, $B_R$ and $B_R(x)$ denote the ball of radius $R$ centered at zero and at $x$, respectively. In Theorem 2.1 we prove that if $B_R \cap \Omega$ is an isoperimetric set for every $R > 0$, then

$$\phi = A(r) B(\Theta),$$

where $r = |x|$ and $\Theta = \frac{x}{|x|}$.

As an application of Theorem 2.1, we prove a sharp Hardy-type inequality for functions defined in $Q = \{ x_1 > 0, x_N > 0 \}$ involving a power-type weight, (see Theorem 2.6).

We are able to give an explicit expression of the density $\phi$ in some special cases. For instance, when $\Omega$ is the half space

$$\Omega = \mathbb{R}^N_+ = \{ x = (x_1, \ldots, x_N) \in \mathbb{R}^N : x_N > 0 \},$$

if $\phi$ is a smooth function with a factorized structure,

$$\phi(x) = \prod_{i=1}^N \phi_i(x),$$

and if $B_R \cap \mathbb{R}^N_+$ is an isoperimetric set, then

$$\phi(x) = ax^k_N \exp(c |x|^2),$$

for some numbers $a > 0$, $k \geq 0$ and $c \geq 0$, (see Theorem 2.8).

Section 3 is dedicated to the case $\Omega = \mathbb{R}^N$, and to the proof of the following Theorem, which is the main result of our paper.

**Theorem 1.1.** Let $\mu$ be the measure defined by

$$d\mu = x_N^k \exp(c |x|^2)dx, \quad x \in \mathbb{R}^N_+,$$

with $k, c \geq 0$, and let $M$ be a measurable subset of $\mathbb{R}^N_+$ with finite $\mu$-measure. Then

$$P_\mu(M) \geq P_\mu(M^\star),$$

where $M^\star = B_{r^\star} \cap \mathbb{R}^N_+$, with $r^\star$ such that $\mu(M) = \mu(M^\star)$.

The proof of Theorem 1.1 requires some technical effort which is due to the degeneracy of the measure on the hyperplane $\{ x_N = 0 \}$.

Note that Theorem 1.1 is imbedded in a wide bibliography related to the isoperimetric problems for “manifolds with density” (see, for instance, [10], [13], [14], [15], [17], [31], [32], [37], [39]). Further references will be given in Section 2.
It was shown in [26] that the isoperimetric set for measures of the type \( y^k dx dy \), with \( k \geq 0 \) and \((x,y) \in \mathbb{R}_+^2 \), is \( B_R \cap \mathbb{R}_+^2 \). In [11] C. Borell proved that balls centered at the origin are isoperimetric sets for measures of the type \( \exp(c |x|^2) dx \) in \( \mathbb{R}^N \) with \( c \geq 0 \) (see also [13] and [37] for this and related results).

In Section 4 we consider degenerate elliptic problems of the type

\[
\begin{aligned}
- \text{div}(A(x) \nabla u) &= x_N^k \exp(c |x|^2)f(x) & \text{in } D, \\
uf(x) &= \text{on } \Gamma_+,
\end{aligned}
\]

(1.8)

where \( D \) is a bounded open set in \( \mathbb{R}_+^N \), whose boundary is decomposed into a part \( \Gamma_0 \), lying on the hyperplane \( \{ x_N = 0 \} \) and a part \( \Gamma_+ \) contained in \( \mathbb{R}_{+}^N \). (For precise definitions, see Section 4). Assume that \( c,k \geq 0 \), \( A(x) = (a_{ij}(x))_{ij} \) is an \( N \times N \) symmetric matrix with measurable coefficients satisfying

\[
x_N^k \exp(c |x|^2) |\zeta|^2 \leq a_{ij}(x) \zeta_i \zeta_j \leq \Lambda x_N^k \exp(c |x|^2) |\zeta|^2, \quad \Lambda \geq 1,
\]

(1.9)

for almost every \( x \in D \) and for all \( \zeta \in \mathbb{R}^N \). Assume also that \( f \) belongs to the weighted Lebesgue space \( L^2(D,d\mu) \) where \( d\mu \) is the measure defined in (1.7).

The type of degeneracy in (1.9) occurs, for \( k \in \mathbb{N} \), when one looks for solutions to linear PDE’s which are symmetric with respect to a group of \((k+1)\) variables (see, e.g., [12], [26], [40] and the references therein). The case of a non-integer \( k \) has been the object of investigation, for instance, in the generalized axially symmetric potential theory (see, e.g., [44] and the subsequent works of A. Weinstein).

We obtain optimal bounds for the solution to problem (1.8) using a symmetrization technique which is due to G. Talenti (see [41] and also [3], [6], [8], [12], [26], [36]).

If \( M \) is measurable set with finite \( \mu \)-measure, and if \( f : M \to \mathbb{R} \) is a measurable function, the weighted rearrangement \( f^\star : M^\star \to [0,\infty] \) is uniquely defined by the following condition

\[
\left\{ x \in M^\star : f^\star(x) > t \right\} = \left\{ x \in M : |f(x)| > t \right\}^\star \quad \forall t \geq 0.
\]

(1.10)

This means that the super level sets of \( f^\star \) are half-balls centered at the origin, having the same \( \mu \)-measure of the corresponding super level sets of \( |f| \).

Let \( C_\mu \) denote the \( \mu \)-measure of \( B_1 \cap \mathbb{R}_+^N \). Using Theorem 1.1, we obtain the following comparison result.

**Theorem 1.2.** Let \( u \) be the weak solution to problem (1.8), and let \( w \) be the function

\[
w(x) = w^\star(x) = \frac{1}{C_\mu} \int_{|x|}^{r^\star} \left( \int_0^f f^\star(\sigma) \sigma^{N-1+k} \exp \left( c \sigma^2 \right) d\sigma \right) \rho^{-N+1-k} \exp \left( -c \rho^2 \right) d\rho,
\]

which is the weak solution to the problem

\[
\begin{aligned}
- \text{div} \left( x_N^k \exp \left( c |x|^2 \right) \nabla w \right) &= x_N^k \exp \left( c |x|^2 \right) f^\star & \text{in } D^\star, \\
w &= 0 & \text{on } \partial D^\star \cap \mathbb{R}_+^N.
\end{aligned}
\]

(1.11)

Then

\[
u^\star(x) \leq w(x) \ a.e. \ in \ D^\star,
\]

(1.12)
2. Weighted isoperimetric inequalities in a cone of $\mathbb{R}^N$

In this section we study isoperimetric problems with respect to measures, relative to cones in $\mathbb{R}^N$. Notice that such problems have been investigated for instance in [1], [4], [18], [25], [33] and [35]. Our aim is to characterize those measures for which an isoperimetric set is given by the intersection of a cone with the ball having center at the vertex of the cone.

We begin by fixing some notation that will be used throughout: $\omega_N$ is the $N$-dimensional Lebesgue measure of the unit ball in $\mathbb{R}^N$. For points $x \in \mathbb{R}^N - \{0\}$ we will often use $N$-dimensional polar coordinates $(r, \Theta)$, where $r = |x|$ and $\Theta = x|x|^{-1} \in S^{N-1}$. $\nabla_\Theta$ denotes the gradient on $S^{N-1}$.

By $S^{N-1}_+ = S^{N-1}_- \cap \mathbb{R}^N_+$ we denote the half sphere.

Consider the isoperimetric problem (1.3) where $\Omega$ is the cone defined in (1.1) and $\nu$ the measure given by (1.2).

The first result of this section says that, if the isoperimetric set of (1.3) is $B_R \cap \Omega$ for a suitable $R$, then the density of the measure $d\nu$ is a product of two functions $A$ and $B$ of the variables $r$ and $\Theta$, respectively.

Theorem 2.1. Consider Problem (1.3), with $\phi \in C^1(\Omega) \cap C(\overline{\Omega})$, $\phi(x) > 0$ for $x \in \Omega$. Suppose that $I_{\nu}(m) = P_{\nu}(B_R \cap \Omega)$ whenever $m = \nu(B_R \cap \Omega)$, for every $R > 0$. Then

\[
\phi = A(r)B(\Theta),
\]

where $A \in C^1((0, +\infty)) \cap C([0, +\infty))$, $A(r) > 0$ if $r > 0$, and $B \in C^1(\omega)$, $B(\Theta) > 0$ for $\Theta \in \omega$. Moreover, if $\phi \in C^2(\Omega)$, then

\[
\lambda(B, \omega) \geq N - 1 + r^2 \left[ \frac{(A'(r))^2}{(A(r))^2} - \frac{A''(r)}{A(r)} \right] \quad \forall r > 0,
\]

where

\[
\lambda(B, \omega) := \inf \left\{ \frac{\int_\omega |\nabla_\Theta u|^2 B d\Theta}{\int_\omega u^2 B d\Theta} : u \in C^1(\omega), \int_\omega u B d\Theta = 0, u \neq 0 \right\}.
\]

Remark 2.1. Observe that $\lambda(B, \omega)$ is the first nontrivial eigenvalue of the Neumann problem

\[
\begin{cases}
-\nabla_\Theta (B \nabla_\Theta u) = \lambda Bu & \text{in } \omega \\
\frac{\partial u}{\partial n} = 0 & \text{on } \partial \omega
\end{cases}
\]

where $u \in W^{1,2}(\omega)$, and $n$ is the exterior unit normal to $\partial \omega$. 
Proof of Theorem 2.1: Let $R > 0$. For $\varepsilon \in \mathbb{R}$ we define the following measure-preserving perturbations $G_{\varepsilon}$ from $B_R \cap \Omega$:

$$G_{\varepsilon} := \{(r, \Theta) : 0 < r < R + \varepsilon h(\Theta) + s(\varepsilon), \Theta \in \omega, \quad |\varepsilon| \leq \varepsilon_0 \}$$

where $h \in C^1(\overline{\omega})$, and $s$ is to be chosen such that $s \in C^2([-\varepsilon_0, \varepsilon_0])$, $s(0) = 0$, and $\nu(G_{\varepsilon}) = \nu(B_R)$ for $|\varepsilon| \leq \varepsilon_0$. Writing $\phi = \phi(r, \Theta)$, and $R_{\varepsilon} := R + \varepsilon h + s(\varepsilon)$, we have, for $|\varepsilon| \leq \varepsilon_0$,

$$(2.4) \quad \nu(G_{\varepsilon}) = \int_{\omega} \int_0^{R_{\varepsilon}} r^{N-1} \phi(r, \Theta) \, dr \, d\Theta = \nu(B_R)$$

and

$$(2.5) \quad P_\nu(G_{\varepsilon}, \Omega) = \int_{\omega} (R_{\varepsilon})^{N-2} \phi(R_{\varepsilon}, \Theta) \sqrt{(R_{\varepsilon})^2 + |\nabla_\Theta R_{\varepsilon}|^2} \, d\Theta \geq P_\nu(B_R \cap \Omega, \Omega).$$

Denote $s_1 := s'(0)$ and $s_2 := s''(0)$. Differentiating (2.4) gives

$$(2.6) \quad 0 = \int_{\omega} \phi(R, \Theta)(h(\Theta) + s_1) \, d\Theta,$$

and

$$(2.7) \quad 0 = \int_{\omega} ((N - 1)\phi(R, \Theta) + R\phi_r(R, \Theta))(h(\Theta) + s_1)^2 \, d\Theta + s_2 R \int_{\omega} \phi(R, \Theta) \, d\Theta.$$ 

Using (2.5) we get

$$(2.8) \quad \begin{cases}
\frac{\partial}{\partial \varepsilon} P_\nu(G_{\varepsilon}, \Omega) \bigg|_{\varepsilon = 0} = 0 \\
\frac{\partial^2}{\partial \varepsilon^2} P_\nu(G_{\varepsilon}, \Omega) \bigg|_{\varepsilon = 0} \geq 0.
\end{cases}$$

The first condition in (2.8) gives

$$(2.9) \quad \int_{\omega} ((N - 1)\phi(R, \Theta) + R\phi_r(R, \Theta))(h(\Theta) + s_1) \, d\Theta = 0.$$ 

In other words, we have that $\int_{\omega} ((N - 1)\phi + R\phi_r)v \, d\theta = 0$ for all functions $v \in C^1(\overline{\omega})$ satisfying $\int_{\omega} \phi v \, d\theta = 0$. Then the Fundamental Lemma in the Calculus of Variations tells us that there is a number $k(R) \in \mathbb{R}$ such that

$$(2.10) \quad \phi_r(R, \Theta) = k(R)\phi(R, \Theta) \quad \forall \Theta \in \omega.$$ 

Integrating this with respect to $R$ implies (2.1). Hence (2.6) and (2.7) give

$$(2.11) \quad 0 = \int_{\omega} B(\Theta)(h(\Theta) + s_1) \, d\Theta,$$

and

$$(2.12) \quad 0 = \left\{ \frac{N - 1}{R} + \frac{A'(R)}{A(R)} \right\} \cdot \int_{\omega} B(\Theta)(h(\Theta) + s_1)^2 \, d\Theta + s_2 \int_{\omega} B(\Theta) \, d\Theta.$$
Next assume that $\phi \in C^2(\Omega)$. Then, using (2.1) and the second condition in (2.8) a short computation shows that

$$0 \leq \left\{ (N-2)(N-1)R^{N-3}A(R) + 2(N-1)R^{N-2}(A'(R))^{N-1}A''(R) \right\} \\ \cdot \int_\omega B(\Theta)(h(\Theta) + s_1)^2 \, d\Theta \\ + s_2 \left\{ (N-1)R^{N-2}A(R) + R^{N-1}A'(R) \right\} \int_\omega B(\Theta) \, d\Theta \\ + R^{N-3}A(R) \int_\omega B(\Theta) |\nabla_\Theta(h(\Theta)) + s_1|^2 \, d\Theta.$$

Together with (2.12) this implies

$$0 \leq \left\{ -(N-1)R^{N-3}A(R) - R^{N-1}A'(R) + R^{N-1}A''(R) \right\} \\ \cdot \int_\omega B(\Theta)(h(\Theta) + s_1)^2 \, d\Theta \\ + R^{N-3}A(R) \int_\omega B(\Theta) |\nabla_\Theta(h(\Theta)) + s_1|^2 \, d\Theta.$$

This implies (2.2), in view of (2.11), and the definition of $\lambda(B, \omega)$.

**Remark 2.2.** The value of $\lambda(B, \omega)$ is explicitly known in some special cases. For instance (see, e.g. [38]), if $B \equiv 1$, and $\omega = S^{N-1}$, we have

$$\lambda(1, S^{N-1}) = N - 1,$$

the eigenvalue has multiplicity $N$, with corresponding eigenfunctions $u_i(x) = x_i$, $(i = 1, \ldots, N)$, so that (2.2) reads as

$$A'^2 \leq A''(r)A(r),$$

or equivalently, $A$ is log-convex, that is,

$$A(r) = e^{g(r)},$$

with a convex function $g$. It has been conjectured in [37], Conjecture 3.12, that for weights $\phi = A(r)$, with log-convex $A$, balls $B_R$, $(R > 0)$, solve the isoperimetric problem in $\mathbb{R}^N$.

After finishing this paper, S. Howe kindly informed us about his new preprint [21] where he gives some partial answers to this conjecture. He also determines the isoperimetric sets for some radial weights. Further, some numerical evidence for the validity of the log-convex conjecture is provided in [24].

It is interesting to note that Theorem 1.1, whose proof will be the object of the next section, and Theorem 2.1 imply the following result.

**Proposition 2.1.** Let $k \geq 0$, and

$$B = B_k(\Theta) = \left( \frac{x_N}{|x|} \right)^k, \quad (x \in S^{N-1}_+).$$
Then
\begin{equation}
\lambda(B_k, S^{N-1}_+) = N - 1 + k,
\end{equation}
with corresponding eigenfunctions
\begin{equation}
u_i = x_i, \quad (i = 1, \ldots, N - 1).\end{equation}
Proof: Let \( u_i \) be given by (2.17). Theorem 1.1 and Theorem 2.1 imply that (2.2) holds, with \( \omega = S^{N-1}_+ \), \( A(r) = r^k e^{cr^2} \), (c \( \geq 0 \)), and \( B(\Theta) = B_k(\Theta) \). Hence \( \lambda(B_k, S^{N-1}_+) \geq N - 1 + k - 2c r^2 \) for all \( r > 0 \), which implies that \( \lambda(B_k, S^{N-1}_+) \geq N - 1 + k \). The assertion follows from the identities
\[ \int_{S^{N-1}_+} |\nabla \Theta u_i|^2 B_k d\Theta = (N - 1 + k) \int_{S^{N-1}_+} (u_i)^2 B_k d\Theta, \] and
\[ \int_{S^{N-1}_+} u_i B_k d\Theta = 0, \quad (i = 1, \ldots, N - 1). \]

The next result gives the sharp constant in a weighted Hardy inequality with respect to the measure \( x_k^k |x|^m dx \) in the quarter space \( \{ x_1 > 0, x_N > 0 \} \) (for related results in half spaces, see e.g., [2], [5], [29], [34] and [43]).

First we introduce some notation. Let \( D \) be an open set in \( \mathbb{R}^N_+ \), and \( \nu \) a measure given by \( d\nu = \phi(x)dx \), where \( \phi \in L^\infty_{\text{loc}}(\mathbb{R}^N_+) \), and \( \phi(x) > 0 \). The weighted Hölder space \( L^2(D, d\nu) \) is the set of all measurable functions \( u : D \to \mathbb{R} \) such that \( \int_D u^2 d\nu < +\infty \), and the weighted Sobolev space \( W^{1,2}(D, d\nu) \) is the set of functions \( u \in L^2(D, d\nu) \) that possess weak partial derivatives \( u_{x_i} \in L^2(D, d\nu) \), \( (i = 1, \ldots, N) \). Norms in these spaces are given respectively by
\[ \|u\|_{L^2(D, d\nu)} := \left( \int_D u^2 d\nu \right)^{1/2}, \] and
\[ \|u\|_{W^{1,2}(D, d\nu)} := \left( \int_D (|u|^2 + |\nabla u|^2) d\nu \right)^{1/2}. \]

**Definition 2.1.** Let \( X \) be the set of all functions \( u \in C^1(\overline{D}) \) that vanish in a neighborhood of \( \partial D \setminus \{ x_N = 0 \} \). Then let \( V^{1,2}(D, d\nu) \) be the closure of \( X \) in the norm of \( W^{1,2}(D, d\nu) \).

Next, let
\begin{equation}
Q := \{ x \in \mathbb{R}^N : x_1 > 0, x_N > 0 \},
\end{equation}
and specify
\begin{equation}
d\nu := x_N^k |x|^m dx,
\end{equation}
where \( k \geq 0 \) and \( m \in \mathbb{N} \).

**Theorem 2.2.** With \( Q \) and \( \nu \) given by (2.18) and (2.19) respectively, we have
\begin{equation}
\int_Q |\nabla u|^2 d\nu \geq C(k, m) \int_Q \frac{u^2}{|x|^2} d\nu,
\end{equation}
for all $u \in V^2(Q, dv)$, where

$$C(k, m) = \left(\frac{N + m + k - 2}{2}\right)^2 + N + k - 1 = \left(\frac{N + m + k}{2}\right)^2 - m.$$  

The constant $C(k, m)$ in (2.21) is sharp, and is not attained for any nontrivial function $u$.

**Proof**: We proceed as in [34, proof of Proposition 4.1]. Extend $u$ to an odd function onto $\mathbb{R}_+^N$ by setting $u(-x_1, x_2, \ldots, x_N) := -u(x)$, $x \in Q$. Writing $u = u(r, \Theta)$, and $B_k(\Theta) = w(x) = x_k |x|^{-k}$, we have for a.e. $r > 0$,

$$\int_{\mathbb{S}^{N-1}_+} u(r, \Theta) B_k(\Theta) \, d\Theta = 0,$$

and thus by,

$$\int_{\mathbb{S}^{N-1}_+} |\nabla u(r, \Theta)|^2 B_k(\Theta) \, d\Theta \geq (N + k - 1) \int_{\mathbb{S}^{N-1}_+} [u(r, \Theta)]^2 B_k(\Theta) \, d\Theta.$$

Further, the one-dimensional Hardy inequality (see [9]) tells us that for a.e. $\Theta \in \mathbb{S}^{N-1}_+$,

$$\int_0^{+\infty} r^{N+m+k-1} [u_r(\Theta)]^2 \, dr \geq \left(\frac{N + m + k - 2}{2}\right)^2 \int_0^{+\infty} r^{N+m+k-3} [u(\Theta)]^2 \, dr.$$

Integrating (2.22) and (2.23) gives

$$\int_{\mathbb{R}_+^N} |\nabla u|^2 \, dv = \int_0^{+\infty} \int_{\mathbb{S}^{N-1}_+} \left( [u_r]^2 - r^{-2} |\nabla u|^2 \right) r^{N-1+m+k} B_k \, d\Theta \, dr$$

$$\geq \left[ \left(\frac{N + m + k - 2}{2}\right)^2 + N + k - 1 \right] \int_0^{+\infty} \int_{\mathbb{S}^{N-1}_+} u^2 r^{N+m+k-3} B_k \, d\Theta \, dr$$

$$= C(k, m) \int_{\mathbb{R}_+^N} \frac{u^2}{|x|^2} \, dv.$$

The constant $C(k, m)$ is not attained since the constant is not attained in the one-dimensional Hardy inequality. Moreover, the exactness of $C(k, m)$ follows in a standard manner by considering functions of the form $u = u_n = x_1 |x|^{-(N-m-k)/2} \psi_n(|x|)$, $(n \in \mathbb{N})$, where $\psi_n \in C_0^{\infty}((0, +\infty))$, $0 \leq \psi_n \leq 1$, $|\psi_n'| \leq 4/n$, $\psi_n(t) = 0$ for $t \in (0, (1/n)] \cup [2n, +\infty)$, and $\psi_n(t) = 1$ for $t \in [(2/n), n]$, and then passing to the limit $n \to \infty$. The details are left to the reader. \qed

Theorem 2.1 has some further consequences when the cone $\Omega$ contains the wedge

$$W_+ := \{ x = (x_1, \ldots, x_N) : x_i > 0, i = 1, \ldots, N \},$$

and if

$$\phi(x) = \prod_{i=1}^N \phi_i(x_i),$$

for some smooth functions $\phi_i$, $i = 1, \ldots, N$.

In the following, let

$$\omega_+ := W_+ \cap \mathbb{S}^{N-1}.$$
We first show

**Lemma 2.1.** Assume that \( \phi \in C^2(W_+) \) satisfies (2.2) and (2.24), where \( A, \phi_i \in C^2((0, +\infty)) \cap C([0, +\infty)), \) \( B \in C^2(\omega_+) \cap C(\mathbb{R}^+), \) \( \phi_i(x_i) > 0 \) for \( x_i > 0, \) \( i = 1, \ldots, N, \) \( A(r) > 0 \) for \( r > 0, \) and \( B(\Theta) > 0 \) for \( \Theta \in \omega_+. \) Then

\[
\phi(x) = a \prod_{i=1}^{N} x_i^{k_i} e^{c|x|^2}, \quad x \in W_+,
\]

where \( a > 0, \) \( k_i \geq 0, \) \( i = 1, \ldots, N, \) and \( c \in \mathbb{R}. \)

**Proof:** Differentiating the equation \( \log[A(r)B(\Theta)] = \log[\prod_{i=1}^{N} \phi_i(x_i)] \) with respect to \( r \) gives

\[
\frac{rA'(r)}{A(r)} = \sum_{i=1}^{N} \frac{x_i\phi'_i(x_i)}{\phi_i(x_i)}.
\]

Differentiating this with respect to \( x_i \) yields

\[
\frac{A'(r)}{rA(r)} + \frac{A''(r)}{A(r)} = \frac{(A'(r))^2}{(A(r))^2} \frac{\phi'_i(x_i)}{x_i\phi_i(x_i)} + \frac{\phi''_i(x_i)}{\phi_i(x_i)} = 4c_i \quad (i = 1, \ldots, N),
\]

for some number \( c \in \mathbb{R}. \) In other words,

\[
\frac{d}{dx_i} \left\{ \frac{x_i\phi'_i(x_i)}{\phi_i(x_i)} \right\} = 4cx_i, \quad (i = 1, \ldots, N).
\]

Integrating this and dividing by \( x_i \) give

\[
\frac{\phi'_i(x_i)}{\phi_i(x_i)} = 2cx_i + \frac{k_i}{x_i}, \quad (i = 1, \ldots, N),
\]

for some numbers \( k_i \in \mathbb{R}, \) \( (i = 1, \ldots, N). \) Then another integration leads to

\[
\log[\phi_i(x_i)] = b_i + k_i \log x_i + c(x_i)^2, \quad (b_i \in \mathbb{R}),
\]

that is,

\[
\phi_i(x_i) = a_i x_i^{k_i} e^{c(x_i)^2},
\]

where \( a_i = e^{b_i}, \) \( (i = 1, \ldots, N). \) Since \( \phi_i \in C([0, +\infty)), \) and \( \phi_i(x_i) > 0 \) for \( x_i > 0, \) we have \( a_i > 0, \)

and \( k_i \geq 0, \) \( (i = 1, \ldots, N). \) Now (2.25) follows, (with \( a = \prod_{i=1}^{N} a_i). \)

As pointed out in the Introduction, we can specify the expression of the density \( \phi \) of the measure, when the cone \( \Omega \) is \( \mathbb{R}^N_+ \) and \( \phi \) is factorized.

**Theorem 2.3.** Assume \( \Omega = \mathbb{R}^N_+ \) and consider Problem (1.3), where \( \phi \in C^1(\mathbb{R}^N_+) \cap C(\mathbb{R}^N_+), \) and satisfies (2.2), for some functions \( \phi_i \in C^2(\mathbb{R}), \phi_i(t) > 0 \) for \( t \in \mathbb{R}, \) \( (i = 1, \ldots, N - 1), \) and \( \phi_N \in C^2((0, +\infty)) \cap C([0, \infty)), \phi_N(t) > 0 \) for \( t > 0. \) Suppose that \( I_P/m = P(\mathbb{B}_R \cap \mathbb{R}^N_+, \mathbb{R}^N) \) for \( m = \nu(\mathbb{B}_R \cap \mathbb{R}^N_+). \) Then

\[
\phi(x) = ax^k e^{c|x|^2},
\]

for some numbers \( a > 0, \) \( k \geq 0 \) and \( c \geq 0. \)
Proof: By Theorem 2.1 we have \( \phi = A(r)B(\Theta) \) with smooth positive functions \( A \) and \( B \), and
\[
\lambda(B, S_+^{n-1}) \geq N - 1 + r^2 \left[ \frac{(A')^2}{A(r)^2} - \frac{A''(r)}{A(r)} \right] \quad \forall r > 0.
\]
Then, Lemma 2.1 shows that \( \phi \) satisfies (2.25). Since \( \varphi(x) > 0 \) whenever \( x_N > 0 \) and \( x_i = 0 \), for some \( i \in \{1, \ldots, N - 1\} \), it follows that we must have \( k_i = 0, (i = 1, \ldots, N - 1) \). This proves (2.26), for some numbers \( a > 0, k \geq 0 \) and \( c \in \mathbb{R} \). Hence, \( B(\Theta) = [x_N|x|^{-1}]^k \), and \( A(r) = ar^kc^{r^2} \).

Therefore (2.27) and (2.16) imply that
\[
N - 1 + k \geq N - 1 + k - 2cr^2 \quad \forall r > 0.
\]
Hence we must have \( c \geq 0 \).

We end this section by analyzing the case where the cone \( \Omega \) is \( \mathbb{R}^N \setminus \{0\} \).

**Theorem 2.4.** Assume \( \Omega = \mathbb{R}^N \setminus \{0\} \) and consider Problem (1.3), with \( \phi \in C^2(\mathbb{R}^N \setminus \{0\}) \cap C(\mathbb{R}^N) \), \( \phi(x) > 0 \) for \( x \neq 0 \), and satisfies (2.24), where \( \phi_i \in C^2(\mathbb{R} \setminus \{0\}) \cap C(\mathbb{R}) \), and \( \phi_i(t) > 0 \) for \( t \neq 0 \), \( (i = 1, \ldots, N) \). Suppose that \( I_\nu(m) = P_\nu(B_R) \) for \( m = \nu(B_R) \). Then
\[
\phi(x) = ae^{\nu|x|^2},
\]
for some numbers \( a > 0 \), and \( c \geq 0 \).

Proof: By Theorem 2.1 we have \( \phi = A(r)B(\Theta) \) with smooth positive functions \( A \) and \( B \), and
\[
\lambda(B, S^{n-1}) \geq N - 1 + r^2 \left[ \frac{(A')^2}{A(r)^2} - \frac{A''(r)}{A(r)} \right] \quad \forall r > 0.
\]
Then, Lemma 2.1 shows that \( \phi \) satisfies (2.25). Since \( \varphi(x) > 0 \) whenever \( x \neq 0 \) and \( x_i = 0 \), for some \( i \in \{1, \ldots, N\} \), it follows that \( k_i = 0, (i = 1, \ldots, N) \). This proves (2.28), for some numbers \( a > 0, k \geq 0 \), that is, \( B(\Theta) \equiv 1 \) and \( A(r) = ae^{cr^2} \). Hence, (2.29) and (2.13) imply that \( A \) is log-convex, that is, we must have \( c \geq 0 \).

### 3. A Dido’s problem

In this section we provide the proof of Theorem 1.1. As pointed out in the Introduction, we have to find the set having minimum \( \mu \) - perimeter among all the subsets of \( \mathbb{R}_+^N \) having prescribed \( \mu \) - measure, where \( \mu \) is the measure defined in (1.7). In order to face such a problem we first show a simple inequality for measures defined on the real line related to \( du \). Then the isoperimetric problem is addressed in the plane: the one-dimensional results allow to restrict the search of optimal sets to the ones which are starlike with respect to the origin. Finally Theorem 1.1 is achieved in its full generality.

#### 3.1. Dido’s problem on the real line.

Let \( \mathbb{R}_+ = (0, +\infty) \). The following isoperimetric inequality holds.

**Proposition 3.1.** Let \( \phi: \mathbb{R}_+ \to \mathbb{R}_+ \) be a nondecreasing continuous function, \( d\nu = \phi(x)dx \) and \( M \) be a measurable subset of \( \mathbb{R}_+ \) with \( \nu(M) < +\infty \). Then
\[
P_\nu(M) \geq P_\nu(S(M)),
\]
where \( S(M) \) denotes the interval \( (0, d) \), with \( d \geq 0 \) chosen such that \( \nu(M) = \nu(S(M)) \).
Proof: First assume that $M$ is of the form
\begin{equation}
M = \bigcup_{j=1}^{k} (a_j, b_j),
\end{equation}
with
\begin{equation}
0 \leq a_j < a_{j+1}, \quad a_j < b_j, \quad b_j < b_{j+1} < +\infty,
\end{equation}
for all $j \in \{1, ..., k-1\}$. By the properties of the weight function $\phi$ we have that $b_k \geq d$ and hence
\begin{equation}
P_\nu(M) = \sum_{j=1}^{k} [\phi(a_j) + \phi(b_j)] \geq \phi(0) + \phi(d) = P_\nu(S(M)).
\end{equation}
Next let $M$ be measurable and $\nu(M) < +\infty$. By the basic properties of the perimeter, there exists a sequence of sets $\{M_n\}$ of the form \([3.2]\) such that $\lim_{n \to +\infty} \nu(M \Delta M_n) = 0$ and $\lim_{n \to +\infty} P_\nu(M_n) = P_\nu(M)$. The first limit implies that also $\lim_{n \to +\infty} P_\nu(S(M_n)) = P_\nu(S(M))$, so that the assertion follows from inequality \([3.3]\). □

3.2. Dido’s problem in two dimensions. In our study of the measure $d\mu$, an important role will be played by the following isoperimetric theorem (see [13] and [37]) relative to the measure
\[d\tau = \exp(c|x|^2)dx, \quad x \in \mathbb{R}^m, \text{ with } m \geq 1 \text{ and } c \geq 0.\]

**Theorem 3.1.** If $M$ is any measurable subset of $\mathbb{R}^N$ and $M^*$ is the ball of $\mathbb{R}^N$ centered at the origin having the same $\tau$-measure of $M$, then
\begin{equation}
P_\tau(M) \geq P_\tau(M^*).
\end{equation}

We write $(x, y)$ for points in $\mathbb{R}^2$, and we consider in $\mathbb{R}_+^2$ the measure
\[d\mu = y^k \exp\left(c(x^2 + y^2)\right) dx dy,
\]
where $c \geq 0$ and $k \geq 0$. If $M$ is a measurable subset of $\mathbb{R}_+^2$, given any number $m > 0$, the isoperimetric problem on $\mathbb{R}_+^2$ reads as:
\begin{equation}
I_\mu(m) := \inf\{P_\mu(M), \text{ with } M : \mu(M) = m\}.
\end{equation}
The following result holds true.

**Theorem 3.2.** Let $m > 0$. Then $I_\mu(m)$ is attained for the half-disk $B_r \cap \mathbb{R}_+^2$, centered at zero, having $\mu$-measure $m$. Equivalently there exists $r > 0$ such that
\begin{equation}
I_\mu(m) = P_\mu(B_r \cap \mathbb{R}_+^2) = \exp\left(cr^2\right) r^{k+1} \int_{0}^{\pi} \sin^k \theta d\theta = B\left(\frac{k+1}{2}, \frac{1}{2}\right) \exp\left(cr^2\right) r^{k+1},
\end{equation}
where $B$ denotes the Beta function.

Proof: If $k = 0$, and $c = 0$ (unweighted case), the result is well-known. Further, if $c > 0$ and $k = 0$, that is, $d\mu = e^{c(x^2+y^2)} dx dy$, the result follows from Theorem 3.1. Finally, the result has been shown in the case $c = 0$ and $k > 0$ by Maderna and Salsa, [26], (see also [19]).

Therefore we may restrict ourselves to the case that both $c$ and $k$ are positive.

Our proof requires some technical effort which is mainly due to the degeneracy of the measure on the $x$-axis. The strategy is as follows: First we use symmetrization arguments in order to reduce the isoperimetric problem to sets which are starlike w.r.t. the origin (Step 1). Then we...
obtain some a-priori-estimates for a minimizing sequence (Step 2). This allows us to show that a (starlike) minimizer exists (Step 3), which is also bounded (Step 4) and smooth (Step 5). In Step 6 we evaluate the second variation of the Perimeter functional, and we show that the minimizer is a half-disk centered at the origin.

Throughout our proof, $C$ will denote a generic constant which may vary from line to line.

**Step 1: Symmetrization**

Our aim is to simplify the isoperimetric problem using Steiner symmetrization in two directions. This method has already been employed in the case $c = 0$ (see [26], and [19]).

Let $\{D_n\} \subset \mathbb{R}^2$ be a minimizing sequence for problem (3.5), i.e.

$$
\mu(D_n) = m \ \forall n \in \mathbb{N} \quad \text{and} \quad \lim_{n \to +\infty} P_\mu(D_n) = I_\mu(m),
$$

where, without loss of generality, we may assume that the sets $D_n$ are smooth.

Let $D$ be a smooth set of $\mathbb{R}^2$. We denote by $S_x(D)$ and $S_y(D)$ the Steiner symmetrization in $x$-direction, with respect to the measure $d\mu_x = e^{xy} dx$, and the Steiner symmetrization in $y$-direction, with respect to the measure $d\mu_y = e^{xy} y^k dy$, of $D$, respectively.

More precisely, $S_x(D)$ is the subset of $\mathbb{R}^2$ whose cross sections parallel to the $x$-axis are open intervals centered at the $y$-axis, and such that their $\mu_x$-lengths are equal to those of the corresponding cross sections of $D$.

The set $S_y(D)$ is defined in a similar way: its cross sections parallel to the $y$-axis are open intervals with an endpoint lying on the $x$-axis, and such that their $\mu_y$-lengths are equal to those of the corresponding cross sections of $D$.

Now consider the sequence of sets $M_n = S_y(S_x(D_n))$. By Proposition 3.11 and Theorem 3.1 we have that $P_\mu(S_y(S_x(D_n))) \leq P_\mu(D_n)$ and by Cavalieri’s principle $\mu(S_y(S_x(D_n))) = \mu(D_n)$. Therefore $\{M_n\}$ is still a minimizing sequence for (3.5). On one hand, the sets $M_n$ can lose regularity under symmetrization: the symmetrized sets are not more then locally Lipschitz continuous, in general. On the other hand, they acquire some nice geometrical property: they are all starlike with respect to the origin. Thus, introducing polar coordinates $(r, \theta)$ by $x = r \cos \theta$ and $y = r \sin \theta$, we have

$$
M_n = \{(r, \theta) : 0 < r < \rho_n(\theta), \ \theta \in (0, \pi)\}, \ \forall n \in \mathbb{N},
$$

for some functions $\rho_n(\theta) : (0, \pi) \to (0, +\infty)$. Note that, defining $\rho_n(0) := \lim_{\theta \to 0^+} \rho_n(\theta) =: \rho_n(0)$, and $\rho_n(\pi/2) := \lim_{\theta \to \pi/2^-} \rho_n(\theta)$, we have also have $\rho_n \in C([0, \pi])$. Then

(i) the functions $\rho_n(\theta)$ are locally Lipschitz in $(0, \pi/2)$;
(ii) $\rho_n(\theta) = \rho_n(\pi - \theta)$, $\forall n \in \mathbb{N}$, $\forall \theta \in (0, \pi)$;
(iii) the functions $x_n(\theta) := \rho_n(\theta) \cos \theta$ and $y_n(\theta) := \rho_n(\theta) \sin \theta$ are nonincreasing and nondecreasing, respectively, on $(0, \pi/2)$.

Hence we may assume that the minimizing sequence is of the form (3.7), with conditions (i)--(iii) in force. Under these conditions, the set $M_n$, its $\mu$-measure and $\mu$-perimeter are uniquely determined
by the function $\rho_n(\theta)$. More precisely, setting
\[ z := \sin^k \theta, \quad \theta \in [0, \pi], \]
\[ F(r) := \int_0^r e^{cr^2} t^{k+1} dt, \quad \text{and} \]
\[ G(r, p) := e^{cr^2} r^k \sqrt{r^2 + p^2}, \quad r > 0, \quad p \in \mathbb{R}, \]
we find that
\[ \mu(M_n) = \int_0^\pi F(\rho_n) zd\theta =: \mu(\rho_n), \quad \text{and} \]
\[ P_\mu(M_n) = \int_0^\pi G(\rho_n, \rho'_n) zd\theta =: P_\mu(\rho_n). \]

With this notation, the isoperimetric problem (3.5) now reads as
(3.8) Minimize $P_\mu(\rho)$ over
\[ K := \{ \rho : (0, \pi/2) \cup (\pi/2, \pi) \to (0, +\infty) : \rho \text{ satisfies (i)-(iii) and } \mu(\rho) = m \}. \]

Step 2: Some estimates
Next we will obtain some uniform estimates for the minimizing sequence $\{\rho_n\}$ of problem (3.8).

Condition (iii) implies
(3.9) \[ -\rho_n(\theta) \cot \theta \leq \rho'_n(\theta) \leq \rho_n(\theta) \tan \theta \quad \text{a.e. on } (0, \pi/2), \quad n \in \mathbb{N}. \]

Set
\[ y^0_n := \sup_{\theta \in (0, \pi/2)} y_n(\theta) = y_n(\pi/2) = \rho_n(\pi/2). \]

We claim that
(3.10) \[ \sup_{n \in \mathbb{N}} y^0_n =: y^0 < +\infty. \]

Indeed, since $\{P_\mu(\rho_n)\}$ is a bounded sequence, we obtain for every $n \in \mathbb{N},$
\[ C \geq P_\mu(\rho_n) \]
\[ = 2 \int_0^{\pi/2} e^{c(x_n^2(\theta) + y^2_n(\theta))} y_n(\theta) \sqrt{(x_n(\theta))^2 + (y_n'(\theta))^2} d\theta \]
\[ \geq 2 \int_0^{\pi/2} e^{cy^2_n(\theta)} y_n(\theta) y_n'(\theta) d\theta = 2 \int_0^{y^0_n} e^{ct^2} t^k dt, \]
and (3.10) follows.

From (3.9) and (3.10) we further deduce that for every $\theta \in (0, \pi),$
(3.11) \[ \rho_n(\theta) = \frac{y_n(\theta)}{\sin \theta} \leq \frac{y_n(\pi/2)}{\sin \theta} \leq \frac{y^0}{\sin \theta} \quad \forall n \in \mathbb{N}. \]

Conditions (3.11) and (3.9) imply that for every $\delta \in (0, \pi/4)$ there is a number $d_\delta > 0$ such that
(3.12) \[ \sup_{\theta \in (\delta, \pi/2-\delta)} \{ \rho_n(\theta), |\rho'_n(\theta)| \} \leq d_\delta. \]
Next we claim:

There exists a number \( d_1 > 0 \), such that

\[
\rho_n(\theta) \geq d_1 \quad \forall \theta \in (0, \pi), \text{ and } \forall n \in \mathbb{N}.
\]

Assume (3.13) was not true. Then the fact that \( x_n(\theta) \) and \( y_n(\theta) \) are nonincreasing, respectively nondecreasing, \( \forall n \in \mathbb{N} \), means that there is a subsequence, still labelled as \( \{\rho_n\} \), such that

\[
\lim_{n \to \infty} \rho_n(\pi/4) = 0.
\]

Set \( \delta_n := \rho_n(\pi/4)/\sqrt{2} \), and note that \( x_n(\pi/4) = y_n(\pi/4) = \delta_n \). In view of (3.11) we have that

\[
\lim_{n \to \infty} \mu(M_n \cap \{|x| < \delta_n\}) = 0.
\]

Since \( \mu(\rho_n) = m \), this implies that there is a number \( d_2 > 0 \), such that for all \( n \in \mathbb{N} \),

\[
d_2 \leq \mu(M_n \cap \{x > \delta_n\}) = -\int_0^{\pi/4} x'_n(\theta)e^{\epsilon x^2_n(\theta)} \int_0^{y_n(\theta)} e^{\epsilon t^2} t^{k+1} \, dt \, d\theta
\]

\[
\leq -d_2 \int_0^{\pi/4} x'_n(\theta)e^{\epsilon (x^2_n(\theta)+y^2_n(\theta))} y^k_n(\theta) \, d\theta.
\]

On the other hand, the sequence \( \{P_\mu(\rho_n)\} \) is bounded, so that

\[
\begin{align*}
C & \geq P_\mu(\rho_n) \\
& \geq \int_0^{\pi/4} e^{\epsilon (x^2_n(\theta)+y^2_n(\theta))} y^k_n(\theta) \sqrt{(x'_n(\theta))^2 + (y'_n(\theta))^2} \, d\theta \\
& \geq -\int_0^{\pi/4} x'_n(\theta)e^{\epsilon (x^2_n(\theta)+y^2_n(\theta))} y^k_n(\theta) \, d\theta.
\end{align*}
\]

Hence we obtain \( d_2 \leq \delta^2_n C \) for all \( n \in \mathbb{N} \), which is a contradiction.

Next we claim that there is a number \( d_3 > 0 \) such that holds for every \( \theta \in (0, \pi/2) \) and for all \( n \in \mathbb{N} \),

\[
y^k_n(\theta) \int_0^{x_n(\theta)} e^{\epsilon t^2} \, dt \leq d_3.
\]

Consider the set

\[
\tilde{M}_n(\theta) := \{(x, y) \in M_n : y \leq y_n(\theta)\}.
\]

It is easy to verify that

\[
\begin{align*}
\frac{1}{2} \left( P_\mu(M_n) - P_\mu(\tilde{M}_n(\theta)) \right) & = \int_0^{\pi/2} e^{\epsilon (x^2_n(\tau)+y^2_n(\tau))} y^k_n(\tau) \sqrt{(x'_n(\tau))^2 + (y'_n(\tau))^2} \, d\tau \\
& \quad - \int_0^{x_n(\theta)} e^{\epsilon (t^2+y^2_n(t))} y^k_n(\theta) \, dt \\
& \quad \geq \int_0^{\pi/2} \left( -x'_n(\theta) \right) e^{\epsilon (x^2_n(\tau)+y^2_n(\tau))} \left( e^{2y^2_n(\tau)} y^k_n(\tau) - e^{y^2_n(\theta)} y^k_n(\theta) \right) \, d\tau \geq 0.
\end{align*}
\]

Hence

\[
C \geq P_\mu(M_n) \geq P_\mu(\tilde{M}_n(\theta)) \geq 2y^k_n(\theta)e^{\epsilon y^2_n(\theta)} \int_0^{x_n(\theta)} e^{\epsilon t^2} \, dt,
\]
and (3.16) follows.

Below we will frequently make use of the following limit which holds for all \( \alpha > -1 \),

\[
\lim_{z \to +\infty} \int_{z^\alpha}^{e^{cz^2/2} \alpha - 1} e^{ct^2} \, dt = \frac{1}{2c}.
\]

In view of (3.18) with \( \alpha = 0 \), and (3.17), and since \( x_n(\theta) \geq d/\sqrt{2} \) for \( \theta \in (0, \pi/4) \), we obtain

\[
C \geq y_n^k(\theta)e^{c\rho_n^2(\theta)}e^{x_n^2(\theta)}/x_n(\theta), \quad \forall \theta \in (0, \pi/4).
\]

Since \( y_n(\theta) \geq (1/2)\rho_n(\theta) \) for \( \theta \in (0, \pi/4) \), and \( x_n(\theta) \leq \rho_n(\theta) \), we further deduce from (3.19),

\[
C \geq \rho_n^{k-1}(\theta)\theta^k e^{c\rho_n^2(\theta)}, \quad \forall \theta \in (0, \pi/4).
\]

Now recall (3.13), and \( \lim_{z \to +\infty} e^{cz^2/2} z^{-k-1} = +\infty \). Hence (3.20) shows that there is a number \( d_4 > 0 \) such that for all \( n \in \mathbb{N} \),

\[
\rho_n(\theta) \leq \sqrt{d_4 - \frac{2k}{c} \ln \theta}, \quad \forall \theta \in (0, \pi/4).
\]

Finally we show:

\[
\mu(M \cap \{0 < \theta < s\}) = \int_0^s \sin^k \theta \int_0^{\rho_n(\theta)} e^{ct^2} t^{k+1} dt \, d\theta \leq C \int_0^s \theta^k e^{c\rho_n^2(\theta)} \rho_n^k(\theta) \, d\theta \leq C \int_0^s \sqrt{d_4 - \frac{2k}{c} \ln \theta} d\theta \to 0, \quad as \ s \to 0.
\]

Now the claim (3.22) follows from the uniform estimate (3.23) and from the fact that for every \( s \in (0, \pi/2) \),

\[
m/2 = \mu(M_n \cap \{0 < \theta < s\}) + \mu(M_n \cap \{s < \theta < \pi/2\}).
\]

Step 3: The minimum is achieved

In this step we show that a minimizer of problem (3.8) exists.

In view of the properties (i)–(iii), (3.9), and the estimates (3.11), (3.13), (3.14) and (3.21) there exists a function \( \rho : (0, \pi/2) \cup (\pi/2, \pi) \to [0, +\infty) \) which is locally Lipschitz continuous, and a
subsequence, still denoted by \( \{ \rho_n \} \), such that
\[
\rho_n \to \rho \quad \text{uniformly on compact subsets of } (0, \pi/2),
\]
\[
\rho(\theta) = \rho(\pi - \theta) \quad \forall \theta \in (0, \pi/2),
\]
\[
-\rho(\theta) \cot \theta \leq \rho'(\theta) \leq \rho(\theta) \tan \theta \quad \text{a.e. on } (0, \pi/2),
\]
\[
\rho(\theta) \leq \frac{y^0}{\sin \theta} \quad \forall \theta \in (0, \pi/2),
\]
\[
\rho(\theta) \geq d_1 \quad \forall \theta \in (0, \pi/2),
\]
\[
\sup_{\theta \in (\delta, \pi/2 - \delta)} \{ \rho(\theta), |\rho'(\theta)| \} \leq d_\delta, \quad \forall \delta \in (0, \pi/4),
\]
\[
\rho(\theta) \leq \sqrt{d_4 - \frac{2k}{c} \ln \theta}, \quad \forall \theta \in (0, \pi/4).
\]

Note, setting \( x(\theta) := \rho(\theta) \cos \theta \), and \( y(\theta) := \rho(\theta) \sin \theta \), condition (3.26) implies that the functions \( x(\theta) \) and \( y(\theta) \) are nonincreasing, respectively nondecreasing on \((0, \pi/2)\). Further, defining \( \rho(\pi/2) := \lim_{\theta \to \pi/2} \rho(\theta) \), we see that \( \rho \in C((0, \pi)) \).

Let
\[
M := \{ (r, \theta) : 0 < r < \rho(\theta), \ \theta \in (0, \pi) \},
\]
and \( P_\mu(\rho) := P_\mu(M) \). We claim
\[
\mu(M) = m.
\]

Indeed, the estimate (3.22) shows
\[
\text{For every } \epsilon \in (0, m) \text{ there is a } \delta \in (0, \pi/2), \text{ such that } \mu(M \cap \{ \delta < \theta < \pi - \delta \}) \geq m - \epsilon.
\]

Since we also have \( \mu(M) \leq m \), (3.31) follows.

Finally, the lower semicontinuity of the perimeter shows that
\[
I_\mu(m) = \lim_{n \to \infty} P_\mu(M_n) \geq P_\mu(M).
\]

But \( \rho \in K \), therefore \( I_\mu(m) = P_\mu(M) \), and \( M \) is a minimizer.

Note that our weight function \( \phi(x, y) := y^k e^{c(x^2 + y^2)} \) is positive and \( \phi \in C^\infty(\mathbb{R}_+^2) \). Due to a regularity result of F. Morgan, \cite{30}, Corollary 3.7 and Remark 3.10, this implies that \( \partial M \cap \mathbb{R}_+^2 \) is a one-dimensional \( C^1 \)-manifold which is locally analytic. In view of the symmetry of \( M \) this implies that \( \rho \) is differentiable at \( \pi/2 \), with \( \lim_{\theta \to \pi/2} \rho'(\theta) = \rho'(\pi/2) = 0 \). Using the properties (3.25)–(3.30) this implies that \( \rho \in C^\infty((0, \pi)) \).

Then standard Calculus of Variations (see \cite{24}) shows that there is a number \( \gamma \in \mathbb{R} \) - a Lagrangian multiplier - such that
\[
-\frac{d}{d\theta} (G_\rho z) + G_r z = \gamma F' z \quad \text{on } (0, \pi).
\]

Here and in the following, the functions \( G, F \) and their derivatives are evaluated at \( (\rho, \rho') \).

**Step 4 : The minimizer is bounded**
We will argue by contradiction, that is, we assume that \( \rho \) was unbounded. Then (3.26) would imply that

\[
\lim_{t \to 0} \rho(t) = +\infty.
\]

First we claim that (3.34) further means that there exists a sequence \( t_n \to 0 \) such that

\[
\rho'(t_n) \geq \rho^3(t_n).
\]

Indeed, assume (3.35) was not true. Then there exists a number \( t_0 > 0 \) such that

\[
- \rho'(t) < \rho^3(t) \quad \text{for} \quad t \in (0, t_0).
\]

By the estimate (3.30) we can find a number \( t_1 \in (0, t_0) \) such that

\[
- \frac{2}{t_1} + (\rho(t_1))^{-2} \geq \delta_0 > 0.
\]

Integrating (3.36) gives

\[
1 - \frac{2}{t} - \frac{1}{(\rho(t))^{-2}} = \delta_0 + 2t \quad \forall t \in (0, t_1),
\]

which implies that \( \rho \) is bounded, a contradiction. Hence (3.35) follows. Note that (3.35), together with our assumption (3.34) implies that

\[
\lim_{n \to \infty} \frac{\rho(t_n)}{\rho'(t_n)} = 0.
\]

Using the Euler equation (3.33), a short calculation shows that

\[
\frac{d}{d\theta} \left( G - \rho'G_p - \gamma F \right) = \rho'G_p \frac{z'}{z}.
\]

Integrating (3.38) on the interval \( (t_n, \pi/2) \) gives

\[
\gamma \int_{t_n}^{\pi/2} e^{c(t-s)} s^{k+1} ds - e^{c(\rho(t_n))} (\rho(t_n))^{k+2} ((\rho(t_n))^{2} + (\rho'(t_n))^{2})^{-1/2}
\]

\[
= -c_1 + \int_{t_n}^{\pi/2} e^{c(t-s)} (\rho(t))^{k} (\rho'(t))^{2} ((\rho(t))^{2} + (\rho'(t))^{2})^{-1/2} \cot \theta dt,
\]

where we have put

\[
c_1 = (G - \rho'G_p - \gamma F) \bigg|_{\theta=\pi/2}.
\]

In view of (3.34), (3.35), (3.37) and (3.38) with \( \alpha = k + 1 \) we find that

\[
\lim_{n \to \infty} \frac{\gamma \int_0^{\rho(t_n)} e^{c(\rho(t_n))} s^{k+1} ds}{e^{c(\rho(t_n))^2} (\rho(t_n))^{k+2} ((\rho(t_n))^{2} + (\rho'(t_n))^{2})^{-1/2}} = +\infty.
\]
Hence the left-hand side of equation (3.39) tends to $+\infty$ as $n \to +\infty$. Using de l’Hospital’s rule, (3.31), (3.35) and (3.37), we obtain from (3.40),

\[
1 = \lim_{n \to \infty} \frac{\gamma \int_0^{\rho(t_n)} e^{c/s^2} s^{k+1} ds - e^{c/(\rho(t_n))^2} (\rho(t_n))^{k+2} ((\rho(t_n))^2 + (\rho'(t_n))^2)^{-1/2}}{-c_1 + \int_{t_n}^{\pi/2} e^{c/(\rho(t))^2} (\rho(t))^k (\rho'(t))^2 ((\rho(t))^2 + (\rho'(t))^2)^{-1/2} k \cot t dt}
\]

\[
= \lim_{n \to \infty} \frac{\gamma \int_{t_n}^{\pi/2} e^{c/(\rho(t))^2} (\rho(t))^k (\rho'(t))^2 ((\rho(t))^2 + (\rho'(t))^2)^{-1/2} k \cot t dt}{-e^{c/(\rho(t))^2} (\rho(t))^2 (\rho'(t))^2 ((\rho(t))^2 + (\rho'(t))^2)^{-1/2} k \cot t_n + \gamma \rho'(t_n) e^{c/(\rho(t))^2} (\rho(t_n))^{k+1}}
\]

\[
= \lim_{n \to \infty} \frac{\gamma \rho'(t_n) e^{c/(\rho(t))^2} (\rho(t_n))^{k+1}}{-e^{c/(\rho(t))^2} (\rho(t))^2 (\rho'(t))^2 ((\rho(t))^2 + (\rho'(t))^2)^{-1/2} k \cot t_n + \gamma \rho'(t_n) e^{c/(\rho(t))^2} (\rho(t_n))^{k+1}}
\]

\[
= \lim_{n \to \infty} \frac{\gamma \rho(t_n)}{k \cot t_n} = \lim_{n \to \infty} \frac{\gamma}{k} t_n \rho(t_n).
\]

But the last limit is zero in view of (3.30), and we have obtained a contradiction. In other words, $\rho$ is bounded on $(0, \pi)$.

Putting $\rho(0) := \lim_{t \to 0} \rho(t) =: \rho(\pi)$, we then have

(3.41) \[ \rho \in C([0, \pi]). \]

Step 5: $\rho''$ is bounded

We will first need some integrability properties of the functions

\[
G_r = e^{c \rho^2} \rho^k \left( (2c\rho + (k/\rho)) \{ \rho^2 + (\rho')^2 \}^{1/2} + \rho^{2} + (\rho')^2 \right)^{-1/2},
\]

\[
G_p = e^{c \rho^2} \rho^k \rho' \{ \rho^2 + (\rho')^2 \}^{-1/2},
\]

\[
F' = e^{c \rho^2} \rho^{k+1}.
\]

By (3.41) and (3.28), $G_p$ and $F'$ are bounded on $(0, \pi)$. Moreover, since $P_\mu(\rho) < +\infty$, we also have $G_r z \in L^1((0, \pi))$. Integrating (3.33) between 0 and $t \in (0, \pi/2)$ gives

\[
- G_p(\rho(t), \rho'(t)) z(t) = \int_0^t (\gamma F' - G_r) z d\theta
\]

\[
= \int_0^t e^{c \rho^2} \rho^k \left( \gamma \rho - (2c\rho + (k/\rho)) \{ \rho^2 + (\rho')^2 \}^{1/2} - \rho^{2} + (\rho')^2 \right)^{-1/2} \rho z d\theta
\]

(3.42) \[ \leq \gamma \int_0^t e^{c \rho^2} \rho^{k+1} z d\theta \leq C \int_0^t z d\theta \leq C t^{k+1}. \]

On the other hand, if $\rho'(t) < 0$, then (3.28) and the boundedness of $\rho$ show that

(3.43) \[ - G_p(\rho(t), \rho'(t)) z(t) = -e^{c/(\rho(t))^2} (\rho(t))^k \rho'(t) \{ \rho^2(t) + (\rho')^2 \}^{-1/2} \sin^k t
\]

\[ \geq -C \rho'(t) \{ C^2 + (\rho')^2 \}^{-1/2} t^k. \]
Furthermore, the estimate (3.26) and the boundedness of \( \rho \) imply that there is a constant \( d_5 > 0 \) such that
\[
(3.44) \quad \rho'(t) \leq d_5 t \quad \forall t \in (0, \pi/2).
\]
Now (3.43), (3.42) and (3.44) imply that
\[
(3.45) \quad \rho'(t)/t \quad \text{is bounded on } (0, \pi/2).
\]
In particular we have \( \rho \in C^1([0, \pi]) \) and \( \rho'(0) = \rho'(\pi) = 0 \).

Finally, using (3.33), a short calculation gives
\[
(3.46) \quad \gamma \rho = \frac{-\rho^2 \rho'' + \rho(\rho')^2}{(\rho^2 + (\rho')^2)^{3/2}} \quad \frac{[k/\rho] + 2c\rho(\rho')^2 + k\rho' \cot t}{(\rho^2 + (\rho')^2)^{1/2}}.
\]
By (3.28), (3.41) and (3.45) this implies that
\[
(3.47) \quad \rho'' \in L^\infty((0, \pi)).
\]

Step 6: \( M \) is a half-disk
Note first that the derivatives \( G_{rr}, G_{rp}, G_{pp} \) and \( F'' \) are bounded, in view of the properties (3.28), (3.41) and (3.45).

Since \( \rho \) is a minimizer of (3.8), the second variation of \( P_\mu \) at \( \rho \) in \( K \) is nonnegative. This means that
\[
(3.48) \quad 0 \leq \int_0^\pi \left( G_{rr}\kappa^2 + 2G_{rp}\kappa' + G_{pp}(\kappa'')^2 - \gamma F''\kappa^2 \right) z \, d\theta,
\]
for every \( \kappa \in W^{1,2}((0, \pi)) \) such that
\[
(3.49) \quad \int_0^\pi F'\kappa \, d\theta = 0.
\]

Furthermore, dividing (3.33) by \( z \) and then differentiating yields
\[
(3.50) \quad G_{rr}\rho' + G_{rp}\rho'' - \frac{d}{d\theta} \left( G_{rp}\rho' + G_{pp}\rho'' \right) - \left( G_{pr}\rho' + G_{pp}\rho'' \right) \frac{z'}{z} - G_p \left( \frac{z'}{z} \right)' = \gamma F''\rho' \quad \text{in } (0, \pi).
\]

Multiplying (3.50) by \( \rho' z \) and then integrating by parts, we obtain
\[
(3.51) \quad \int_0^\pi G_p \rho' \left( \frac{z'}{z} \right)' \, d\theta = \int_0^\pi \left( G_{rr}(\rho')^2 + 2G_{rp}\rho' + G_{pp}(\rho'')^2 - \gamma F''(\rho')^2 \right) z \, d\theta.
\]
Note that we may use (3.48) with \( \kappa = \rho' \in W^{1,\infty}((0, \pi)) \). This shows that the right-hand side of (3.51) is nonnegative. On the other hand,
\[
(3.52) \quad \int_0^\pi G_p \rho' \left( \frac{z'}{z} \right)' \, d\theta = -k \int_0^\pi e^{1/2} \rho^k \left( \rho^2 + (\rho')^2 \right)^{-1/2} (\rho')^2 \sin^{k-2} \theta \, d\theta \leq 0.
\]

Hence
\[
(3.53) \quad \int_0^\pi e^{1/2} \rho^k \left( \rho^2 + (\rho')^2 \right)^{-1/2} (\rho')^2 \sin^{k-2} \theta \, d\theta = 0,
\]
which implies that \( \rho' = 0 \) in \([0, \pi]\). This means that \( \rho \) is constant in \([0, \pi]\), and the result follows. \( \square \)
3.3. The N-dimensional case. Proof of Theorem 1.1. We proceed by induction over the dimension \( N \). Note that the result for \( N = 2 \) is Theorem \[ \text{3.2} \]
Assume that the assertion holds true for sets in \( \mathbb{R}^N \), for some \( N \geq 2 \), and more precisely, for all measures of the type
\[
\delta \mu = x_N^k \exp \{c|x|^2\} \, dx
\]
where \( k \geq 0 \) and \( c \geq 0 \).
We write \( y = (x', x_N, x_{N+1}) \) for points in \( \mathbb{R}^{N+1} \), where \( x' \in \mathbb{R}^{N-1} \), and \( x_N, x_{N+1} \in \mathbb{R} \). Let a measure \( \nu \) on
\[
\mathbb{R}^{N+1}_+ := \{y = (x', x_N, x_{N+1}) \in \mathbb{R}^{N+1} : x_{N+1} > 0\}
\]
be given by
\[
d \nu = x_{N+1}^k \exp \{c(x'^2 + x_N^2 + x_{N+1}^2)\} \, dy.
\]
We define two measures \( \nu_1 \) and \( \nu_2 \) by
\[
d \nu_1 = \exp \{c|x'|^2\} \, dx', \quad \text{and}
\]
\[
d \nu_2 = x_{N+1}^k \exp \{c(x_N^2 + x_{N+1}^2)\} \, dx_N \, dx_{N+1},
\]
and note that \( d \nu = d \nu_1 d \nu_2 \).
Let \( M \) be a subset of \( \mathbb{R}^{N+1}_+ \) having finite and positive \( \nu \)-measure.
We define 2-dimensional slices
\[
M(x') := \{(x_N, x_{N+1}) : (x', x_N, x_{N+1}) \in M\}, \quad (x' \in \mathbb{R}^{N-1}).
\]
Let \( M' := \{x' \in \mathbb{R}^{N-1} : 0 < \nu_2(M(x'))\} \), and note that \( \nu_2(M(x')) < +\infty \) for a.e. \( x' \in M' \). For all those \( x' \), let \( H(x') \) be the half disc in \( \mathbb{R}^{N}_{+} \) centered at \( (0,0) \) with \( \nu_2(M(x')) = \nu_2(H(x')) \). (For convenience, we put \( H(x') = \emptyset \) for all \( x' \in M' \) with \( \nu_2(M(x')) = +\infty \).) By Theorem \[ \text{3.2} \]
we have
\[
P_{\nu_2}(H(x')) \leq P_{\nu_2}(M(x')) \quad \text{for a.e.} \quad x' \in M'.
\]
Let
\[
H := \{y = (x', x_N, x_{N+1}) : (x_N, x_{N+1}) \in H(x'), x' \in M'\}.
\]
The product structure of the measure \( \nu \) tells us that
(i) \( \nu(M) = \nu(H) \), and
(ii) the isoperimetric property for slices, \[ \text{3.54} \], carries over to \( M \), that is,
\[
P_{\nu}(H) \leq P_{\nu}(M),
\]
(see for instance Theorem 4.2 of \[ \text{7} \]).
Note again, the slice \( H(x') = \{(x_N, x_{N+1}) : (x', x_N, x_{N+1}) \in H\} \) is a half disc \( \{(r \cos \theta, r \sin \theta) : 0 < r < R(x'), \theta \in (0, \pi)\} \), with \( 0 < R(x') < +\infty \), \( (x' \in M') \). Set
\[
K := \{(x', r) : 0 < r < R(x'), x' \in M'\},
\]
and introduce a measure \( \alpha \) on \( \mathbb{R}^{N}_{+} \) by
\[
d \alpha := a_k r^{k+1} \exp \{c(|x'|^2 + r^2)\} \, dx' \, dr,
\]
where
\[
a_k := \int_0^\pi \sin^k \theta \, d\theta = B \left( \frac{k + 1}{2}, \frac{1}{2} \right).
\]
An elementary calculation then shows that
\[ \nu(H) = \alpha(K), \]
and
\[ P_\nu(H) = P_\alpha(K). \]
Let \( B_R \) denote the open ball in \( \mathbb{R}^N \) centered at the origin, with radius \( R \), and choose \( R > 0 \) such that
\[ \alpha(B_R \cap \mathbb{R}^N_+) = \alpha(K). \]
By the induction assumption it follows that
\[ (3.56) \quad P_\alpha(B_R \cap \mathbb{R}^N_+) \leq P_\alpha(K). \]
Finally, let \( M^\star \) be the half ball in \( \mathbb{R}^{N+1}_+ \) centered at the origin, with radius \( R \),
\[ M^\star := \{ y = (x', x_N, x_{N+1}) : |x'|^2 + x_N^2 + x_{N+1}^2 < R^2, x_{N+1} > 0 \}. \]
Then
\[ \nu(M^\star) = \nu(M) \]
and
\[ P_\nu(M^\star) = P_\alpha(B_R \cap \mathbb{R}^N_+). \]
Together with (3.56) and (3.55) we find
\[ P_\nu(M^\star) \leq P_\nu(M), \]
that is, the isoperimetric property holds for \( N + 1 \) in place of \( N \) dimensions. The Theorem is proved.

\[ \square \]

4. Application to a class of degenerate elliptic equations

4.1. Notation and preliminary results. First we introduce the notion of weighted rearrangement. For an exhaustive treatment of rearrangements we refer to \[1, 16, 20, 22, 23].\n
Let the measure \( \mu \) be given by (1.7), and let \( M \) be a measurable subset of \( \mathbb{R}^N_+ \). The distribution function of a Lebesgue measurable function \( u : M \rightarrow \mathbb{R} \), with respect to \( d\mu \), is the function \( m_\mu \) defined by
\[ m_\mu(t) = \mu \left( \{ x \in M : |u(x)| > t \} \right), \forall t \geq 0. \]
The decreasing rearrangement of \( u \) is the function \( u^\star \) defined by
\[ u^\star(s) = \inf \left\{ t \geq 0 : m_\mu(t) \leq s \right\}, \forall s \in (0, \mu(M)]. \]
Let \( C_\mu \) be the \( \mu \)-measure of \( B_1 \cap \mathbb{R}^N_+ \), that is,
\[ C_\mu = \frac{1}{2} (N - 1) \omega_{N-1} B \left( \frac{k + 1}{2}, \frac{N - 1}{2} \right), \]
and let a function \( \psi(r) \) be defined by
\[ \psi(r) = \int_0^r \exp \left( ct^2 \right) t^{N+k-1} dt. \]
Let \( M^\star \) be defined as in Theorem 1.1, that is,
\[ (4.1) \quad M^\star = B_{r^\star} \cap \mathbb{R}^N_+, \]
where

\[(4.2) \quad r^\star = \psi^{-1}\left(\frac{\mu(M)}{C_\mu}\right).\]

The rearrangement \(u^\star\) of \(u\), by its definition given in (1.10), is

\[u^\star(x) = u^\star(C_\mu \psi(|x|)), \forall x \in M^\star.\]

The isoperimetric inequality in Theorem 3.1 can be also stated as follows

\[P_\mu(M) \geq I_\mu(\mu(M)),\]

where \(I_\mu(\tau)\) is the function such that

\[P_\mu(M^\star) = I_\mu(\mu(M^\star)),\]

or equivalently

\[(4.3) \quad I_\mu(\tau) = C_\mu \exp\left(c \left[\psi^{-1}\left(\frac{\tau}{C_\mu}\right)\right]^2\right) \left[\psi^{-1}\left(\frac{\tau}{C_\mu}\right)\right]^{N+k-1}.\]

The fact that half balls \(B_R \cap \mathbb{R}^N_+\) are isoperimetric for the weighted measure \(\mu\) imply a Polya-Szegö-type inequality (see [42], p. 125).

**Theorem 4.1.** Let \(D\) be an open set with finite \(\mu\)-measure, and let the space \(V^2(D, d\mu)\) be given by Definition 2.1. Then we have for every function \(u \in V^2(D, d\mu)\),

\[(4.4) \quad \int_D |\nabla u|^2 d\mu \geq \int_{D^\star} |\nabla u^\star|^2 d\mu.\]

Since rearrangements preserves the \(L^p\) norms, we have that the Rayleigh-Ritz quotient decreases under rearrangement i.e.

\[\frac{\int_D |\nabla u|^2 d\mu}{\int_D u^2 d\mu} \geq \frac{\int_{D^\star} |\nabla u^\star|^2 d\mu}{\int_{D^\star} (u^\star)^2 d\mu}, \forall u \in V^2(D, d\mu).\]

The following Poincaré type inequality states the continuous embedding of \(V^2(D, d\mu)\) in \(L^2(D, d\mu)\). It is a consequence of some one-dimensional inequalities (see [29], Theorem 2, p. 40).

**Corollary 4.1.** Let \(D\) be an open subset of \(\mathbb{R}^N_+\). Then there exists a constant \(C\), such that for every \(u \in V^2(D, d\mu)\),

\[\int_D u^2 d\mu \leq C \int_D |\nabla u|^2 d\mu.\]

**4.2. Comparison result.** Now we are in a position to obtain sharp estimates for the solution to problem (1.8). By a weak solution to such a problem we mean a function \(u\) belonging to \(V^2(D, d\mu)\) such that

\[(4.5) \quad \int_D A(x)\nabla u \nabla \chi d\mu = \int_D f \chi d\mu,\]

for every \(\chi \in C^1(\bar{D})\) such that \(\chi = 0\) on the set \(\partial D \setminus \{x_N = 0\}\).

**Proof of Theorem 1.2:** Note first that the existence of a unique solution to problems (1.8) and
We have \((1.11)\) is ensured by the Lax-Milgram Theorem. Arguing as in \([41]\) (see for instance \([12]\), p. 363),

we get

\[
1 \leq \left\{ \frac{1}{[\mu(m_u(t))]^{-2}} \int_0^{m_u(t)} f^*(\sigma) d\sigma \right\} (-m_u'(t))
\]

and

\[
u^*(s) \leq \int_s^\mu(D) \left(I_{\mu}^{-1}(l) \int_0^l f^*(\sigma) d\sigma \right) dl,
\]

Using (4.3) in (4.7) we obtain

\[
u^*(x) \leq \frac{1}{C_{\mu}^2} \int_{C_{\mu} \psi(|x|)}^\mu(D) \exp \left(-2c \left(\psi^{-1}\left(\frac{l}{C_{\mu}}\right)\right)^2 \right) \left(\psi^{-1}\left(\frac{l}{C_{\mu}}\right)\right)^{-2N-2k+2} \int_0^l f^*(\sigma) d\sigma \right) dl
\]

\[
= \frac{1}{C_{\mu}} \int_{|x|}^r \exp \left(-c\eta^2\right) \eta^{-N-k+1} \left(\int_0^{C_{\mu} \psi(\eta)} f^*(\sigma) d\sigma \right) d\eta \hspace{1cm} (\eta := \psi^{-1}\left(\frac{l}{C_{\mu}}\right))
\]

\[
= \int_{|x|}^r \exp \left(-c\eta^2\right) \eta^{-N-k+1} \left(\int_0^{C_{\mu} \psi(\xi)} f^*(\xi) \xi^{N+k-1}\exp\left(c\xi^2\right) d\xi \right) d\eta
\]

\[
= \int_{|x|}^r \exp \left(-c\eta^2\right) \eta^{-N-k+1} \left(\int_0^{C_{\mu} \psi(\xi)} f^*(\xi) \xi^{N+k-1}\exp\left(c\xi^2\right) d\xi \right) d\eta
\]

\[
= w(x).
\]

Now let us show (1.13). Arguing as in \([12]\), p. 363–364 (see also \([41]\)), we derive

\[-\frac{d}{dt} \int_{|u|>t} |\nabla u|^q d\mu \leq \left(\int_0^{m_u(t)} f^*(s) ds \right)^{q/2} (-m_u'(t))^{1-q/2}
\]

\[
\leq (I(m_u(t)))^{-q} \left(\int_0^{m_u(t)} f^*(s) ds \right)^{q} (-m_u'(t)).
\]

Integrating the last inequality between 0 and \(+\infty\), we get

\[
\int_D |\nabla u|^q d\mu = \int_0^{+\infty} \left[-\frac{d}{dt} \int_{|u|>t} |\nabla u|^q d\mu \right] dt
\]

\[
\leq \int_0^{+\infty} (I_{\mu}(m_u(t)))^{-q} \left(\int_0^{m_u(t)} f^*(\sigma) d\sigma \right)^{q} (-m_u'(t)) dt
\]

\[
\leq \int_0^{\mu(D)} (I_{\mu}(s))^{-q} \left(\int_0^{s} f^*(\sigma) d\sigma \right)^{q} ds.
\]
Now a straightforward calculation yields
\[
\int_D |\nabla u|^q \, d\mu \leq C_\mu \int_0^{\mu(D)} \exp \left( -qc \left[ \psi^{-1} \left( \frac{s}{C_\mu} \right) \right]^2 \right) \left[ \psi^{-1} \left( \frac{s}{C_\mu} \right) \right]^{-q(N+k-1)} \left( \int_0^s f^*(\sigma) \, d\sigma \right)^q \, ds
\]
\[
= C_\mu^2 \int_0^{R^*} \exp \left( -qc \eta^2 \right) \eta^{-q(N+k-1)} \left( \int_0^{C_\mu \psi(\eta)} f^*(\sigma) \, d\sigma \right)^q \, d\eta
\]
\[
= C_\mu^2 \int_0^{R^*} \left( \int_0^\eta f^*(C_\mu \psi(\rho)) \, C_\mu \exp \left( cp^2 \right) \rho^{N+k-1} \, d\rho \right)^q \exp \left( (1-q) \eta^2 \right) \eta^{(1-q)(N+k-1)} \, d\eta
\]
\[
= C_\mu^2 \int_0^{R^*} \left( \int_0^\eta f^*(\rho) \exp \left( cp^2 \right) \rho^{N+k-1} \, d\rho \right)^q \exp \left( (1-q) \eta^2 \right) \eta^{(1-q)(N+k-1)} \, d\eta
\]
\[
= \int_D |\nabla w|^q \, d\mu.
\]

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