Entropic uncertainty relations from equiangular tight frames

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Finite tight frames are interesting in various respects, including potential applications in quantum information science. Indeed, each complex tight frame leads to a non-orthogonal resolution of the identity in the Hilbert space. In a certain sense, equiangular tight frames are very similar to the maximal sets that provide symmetric informationally complete measurements. Hence, applications of equiangular tight frames in quantum physics deserve more attention than they have obtained. We derive entropic uncertainty relations for a quantum measurement built of the states of an equiangular tight frame. First, the corresponding index of coincidence is estimated from above, whence desired results follow. State-dependent and state-independent formulations are both presented. We also discuss applications of the corresponding measurements to detect entanglement and steerability.

Keywords: equiangular tight frames, Rényi entropy, Tsallis entropy, steering inequalities

I. INTRODUCTION

Studies of discrete structures in finite-dimensional Hilbert spaces has a long history [1, 2]. Such structures are interesting not only in their own rights but also due to potential application in quantum physics. Emerging technologies of quantum information processing gave a new stimulus to investigate finite sets of states with special properties. Mutually unbiased bases (MUBs) are one of the most known examples of this kind [3]. Another remarkable paradigm of discrete structures is given by symmetric informationally complete measurements [4]. Quantum and unitary designs are now considered as a powerful tool in quantum information science [5–8]. On the other hand, the question of building such structures are often difficult to resolve [2]. For instance, the maximal number of MUBs remains unknown even for $d = 6$, i.e., for the smallest dimension that is not a prime power.

Finite tight frames are a natural generalization of orthonormal bases [9]. Each of such frames can be applied to build a positive operator-valued measure. Positive operator-valued measures (POVMs) are an indispensable tool in quantum information science [10]. Tight frames have also found use in signal processing and coding [11]. Equiangular tight frames (ETFs) can be emphasized due to the same overlap between any two frame vectors. In a certain sense, this idea is similar to that is used to define a mutually unbiased basis. Maximal sets of complex equiangular vectors provide symmetric informationally complete POVMs [9]. The existence of such POVMs for arbitrary dimensions is still an open question, though much many exact constructions have been found [4, 12]. This gives a rigid reason to study arbitrary ETFs in more detail and use them together with the maximal ones.

In this paper, we consider entropic uncertainty relations for a quantum measurement built of the states of an equiangular tight frame. Potential applications in quantum information science will be mentioned as well. The paper is organized as follows. In Section II the required material on complex tight frames is briefly recalled. Section III is devoted to estimate from above the corresponding index of coincidence, whence many results will be obtained. The proposed approach is a natural development of the treatment of [13]. In Section IV we formulate entropic uncertainty relations for quantum measurements assigned to ETFs. To quantify the amount of related uncertainties, the Rényi and Tsallis entropies are both used. In Section V the derived relations are discussed in application to entanglement detection and steering inequalities. In Section VI we conclude the paper with a summary of the results.

II. PRELIMINARIES

In this section, we recall some material concerning equiangular tight frames. The authors of [14, 15] discussed the existence of such frames in both real and complex forms. In the following, all the frames are assumed to be complex. Let $\mathcal{H}_d$ be $d$-dimensional Hilbert space. A set of $n \geq d$ unit vectors $\mathcal{F} = \{ |\phi_j\rangle \}$ is called a frame if there exist strictly positive numbers $S_0 < S_1 < \infty$ such that

$$S_0 \leq \sum_{j=1}^{n} |\langle \phi_j | \psi \rangle|^2 \leq S_1$$

(1)

for all unit $|\psi\rangle \in \mathcal{H}_d$. The numbers $S_0$ and $S_1$ are the minimal and maximal eigenvalues of the frame operator

$$S = \sum_{j=1}^{n} |\phi_j\rangle \langle \phi_j| .$$

(2)
The special case $S_0 = S_1 = n/d$ gives a tight frame. Then the frame operator is scalar with the eigenvalue $S = n/d$ of multiplicity $d$. Parseval tight frames obtained with $S = 1$ are equivalent to orthonormal bases. A special kind of tight frames is known as equiangular ones. The tight frame $\mathcal{F}$ is called equiangular, when there exist $c > 0$ such that

$$|\langle \phi_i | \phi_j \rangle|^2 = c$$

for each pair $i \neq j$. By calculations, for an equiangular tight frame we have

$$S = nc + 1 - c = \frac{n}{d}, \quad c = \frac{S - 1}{n - 1} = \frac{n - d}{(n - 1)d}.$$  \hspace{1cm} (4)

If there exists an ETF with $n$ elements in $d$ dimensions, then $n \leq d^2$ and also exists an ETF with $n$ elements in $n - d$ dimensions $^{14,15}$. The second fact deserves to be considered in more detail. Let us put the $d \times n$ matrix

$$\Phi = \left( |\phi_1\rangle \cdots |\phi_n\rangle \right),$$

where the frame states stand as columns. Since $(d/n) \Phi \Phi^\dagger = \mathbb{1}_d$ for any ETF, suitably rescaled rows of $\Phi$ form an orthonormal set. One can convert $(d/n) \Phi \Phi^\dagger$ into a unitary $n \times n$ matrix by adding $n - d$ rows that are mutually orthogonal as well. Collecting these rows into $(n - d) \times n$ matrix and normalizing its columns gives other ETF $^{15}$. The latter contains $n$ vectors in $n - d$ dimensions. This way is very close to that is typically used to build a Naimark extension of rank-one POVM with $n$ elements. It is more important here that the given approach is constructive. In the least case $n = d^2$, we see from (3) that

$$|\langle \phi_i | \phi_j \rangle|^2 = \frac{1}{d + 1} \quad (i \neq j).$$

Here, we deal with a symmetric informationally complete measurement (SIC-POVM). As was already mentioned, we have a way to generate new ETFs from already given ones. It is not the case for SIC-POVMs, even though existing lists of ETFs are sufficiently rare $^{13}$. In more detail, symmetric informationally complete measurements are discussed in $^{16}$. The authors of $^{17}$ introduced the concept of mutually unbiased frames as a general framework for studying unbiasedness.

The states of an ETF originate the resolution $\mathcal{E} = \{ E_j \}$ of the identity, namely

$$\frac{d}{n} S = \sum_{j=1}^{n} E_j = \mathbb{1}_d.$$ \hspace{1cm} (6)

Here, the POVM elements are expressed as

$$E_j = \frac{d}{n} |\phi_j\rangle \langle \phi_j|.$$ \hspace{1cm} (7)

When the pre-measurement state is described by density matrix $\rho$ with $\text{tr} (\rho) = 1$, the probability of $j$-th outcome is equal to

$$p_j(\mathcal{E}; \rho) = \frac{d}{n} \langle \phi_j | \rho | \phi_j \rangle.$$  \hspace{1cm} (8)

Developing some ideas of $^{13}$, we aim to derive entropic uncertainty relations for POVMs assigned to ETFs. The composed technique is somehow similar to that was used to obtain uncertainty relations for mutually unbiased bases $^{18}$. Features of the mentioned discrete structures allow one to impose certain restrictions on generated probabilities. The authors of $^{19}$ introduced the notion of general symmetric informationally complete measurements. These measurements are not necessarily described by elements of rank one. Similarly, the set of $d + 1$ mutually unbiased measurements can be built for arbitrary $d$, when rank-one projectors are not required $^{20}$. Entropic uncertainty relations were obtained for mutually unbiased measurements $^{21}$ and general SIC-POVMs $^{22}$. However, excessive costs may be necessary to implement measurements with elements that are not of rank one. Such costs may be not compatible with a current tendency to use shallow quantum circuits. Thus, ETF-based measurements are still of interest as having rank-one elements. Due to the results of $^{23}$, we can often restrict a consideration to POVMs with elements of rank one. For such measurements, there is a Naimark extension with only $n - d$ extra dimensions (see, e.g., the remarks given right after (5)). In more detail, the question of building a Naimark extension is addressed in section 9-6 of $^{24}$. 
III. ON THE INDEX OF COINCIDENCE

In this section, we shall estimate the index of coincidence defined as

\[ I(\mathcal{E}; \rho) = \sum_{j=1}^{n} p_j (\mathcal{E}; \rho)^2. \]  

(9)

To each \( |\psi \rangle \in \mathcal{H}_d \), we assign \( |\psi^* \rangle \in \mathcal{H}_d \) as the ket with the conjugate components, whence \( \langle \varphi^* | \psi^* \rangle = \langle \varphi | \psi \rangle^* = \langle \psi | \varphi \rangle \). Let us begin with an auxiliary result.

**Proposition 1** Let \( n \) unit kets \( |\phi_j \rangle \) form an equiangular tight frame in \( \mathcal{H}_d \), and let

\[ |\Psi_0 \rangle = \frac{1}{\sqrt{nS}} \sum_{j=1}^{n} |\phi_j \rangle \otimes |\phi_j^* \rangle, \]

\[ |\Psi_k \rangle = \frac{1}{\sqrt{n - nc}} \sum_{j=1}^{n} \omega^{k(j-1)} |\phi_j \rangle \otimes |\phi_j^* \rangle, \]

(10)

(11)

where \( k = 1, \ldots, n - 1 \) and \( \omega \) is a primitive \( n \)-th root of unity. Then the vectors (10)–(11) form an orthonormal set in the space \( \mathcal{H}_d \otimes \mathcal{H}_d \).

**Proof.** First, we aim to show that the vectors (10)–(11) are mutually orthogonal. Up to a factor, the inner product \( \langle \Psi_0 | \Psi_k \rangle \) with \( k \neq 0 \) is represented as

\[ \sum_{i=1}^{n} \sum_{j=1}^{n} \omega^{k(j-1)} |\langle \phi_i | \phi_j \rangle|^2 = \sum_{j=1}^{n} \omega^{k(j-1)} + c \sum_{i,j=1}^{n} \omega^{(j-1)} \]

(12)

where we used (3). As \( \omega \) is a primitive root of unity, one has \( \sum_{j=1}^{n} \omega^{k(j-1)} = 0 \) for \( k = 1, \ldots, n - 1 \). We assign this zero sum by the factor \( c \), whence the right-hand side of (12) becomes

\[ c \sum_{i=1}^{n} \sum_{j=1}^{n} \omega^{k(j-1)} = nc \sum_{j=1}^{n} \omega^{k(j-1)} = 0. \]

(13)

Up to a common factor, we express \( \langle \Psi_q | \Psi_k \rangle \) with \( k, q \neq 0 \) as

\[ \sum_{i=1}^{n} \sum_{j=1}^{n} \omega^{-q(i-1)} \omega^{k(j-1)} |\langle \phi_i | \phi_j \rangle|^2 = \sum_{j=1}^{n} \omega^{(k-q)(j-1)} + c \sum_{i,j=1}^{n} \omega^{k(j-1)-q(i-1)}. \]

(14)

For \( q \neq k \), the first sum in the right-hand side of (14) is zero. Multiplying it by \( c \), one reduces (14) to the form

\[ c \sum_{i=1}^{n} \omega^{-q(i-1)} \sum_{j=1}^{n} \omega^{k(j-1)} = 0. \]

(15)

To check the normalization, we note that

\[ \sum_{i=1}^{n} \sum_{j=1}^{n} |\langle \phi_i | \phi_j \rangle|^2 = n + (n^2 - n)c = nS. \]

(16)

The inverse square root of (10) stands right before the sum in the right-hand side of (10). Substituting \( q = k \) in (14) finally leads to

\[ n + c \sum_{i=1}^{n} \sum_{j=1}^{n} \omega^{k(j-1)} - nc = n - nc, \]

whence the normalization factor of (11) follows. ■

The statement of Proposition 1 generalizes one of the results of [13]. Namely, for a SIC-POVM we have \( n = d^2 \), so that the vectors (10)–(11) form an orthonormal basis. To formulate entropic uncertainty relations for an ETF-based measurement, we proceed to estimating the index of coincidence (9) from above. Let us extend one of the derivations presented in [13] for a SIC-POVM.
Proposition 2 Let $n$ unit kets $|\phi_j\rangle$ form an ETF in $\mathcal{H}_d$, and let POVM $\mathcal{E}$ be assigned to this frame. For the given density matrix $\rho$, it holds that

$$I(\mathcal{E}; \rho) \leq \frac{Sc + (1 - c) \text{tr}(\rho^2)}{S^2},$$

(18)

where $S$ and $c$ obey [4].

**Proof.** Let us mention the two corollaries of the completeness relation for the POVM $\mathcal{E}$. For arbitrary operator $K$ on $\mathcal{H}_d$ and ket $|\psi\rangle$, one has

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \langle \phi_i | K | \phi_j \rangle \langle \phi_j | \phi_i \rangle = \frac{n^2}{d^2} \text{tr}(K),$$

(19)

$$\sum_{j=1}^{n} | \phi_j \rangle \langle \phi_j | \psi \rangle = \frac{n}{d} | \psi \rangle.$$

(20)

They follow by substituting the resolution of the identity into $\text{tr}(\mathbb{1}_d K \mathbb{1}_d) = \text{tr}(K)$ and $\mathbb{1}_d |\psi\rangle = |\psi\rangle$, respectively. Let us represent $\rho \otimes \mathbb{1}_d |\Psi_0\rangle$ in the form

$$\rho \otimes \mathbb{1}_d |\Psi_0\rangle = \sum_{k=0}^{n-1} a_k |\Psi_k\rangle + |\Theta\rangle,$$

where $\langle \Theta | \Psi_k \rangle = 0$. It follows from (10) and (19) that

$$a_0 = \langle \Psi_0 | \rho \otimes \mathbb{1}_d |\Psi_0\rangle = \frac{1}{nS} \sum_{i=1}^{n} \sum_{j=1}^{n} \langle \phi_i | \rho | \phi_j \rangle \langle \phi_j | \phi_i \rangle = \frac{n}{Sd^2} = \frac{1}{d}.$$  

(21)

For $k \neq 0$, we write the coefficient

$$a_k = \langle \Psi_k | \rho \otimes \mathbb{1}_d |\Psi_0\rangle = \frac{1}{n\sqrt{S - Sc}} \sum_{i=1}^{n} \sum_{j=1}^{n} \omega^{-k(i-1)} \langle \phi_i | \rho | \phi_j \rangle \langle \phi_j | \phi_i \rangle = \frac{S^2}{n\sqrt{S - Sc}} \sum_{i=1}^{n} \omega^{-k(i-1)} p_i .$$

(22)

Similarly to (21), one also calculates

$$\langle \Psi_0 | (\rho \otimes \mathbb{1}_d)^2 |\Psi_0\rangle = \frac{1}{nS^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \langle \phi_i | \rho^2 | \phi_j \rangle \langle \phi_j | \phi_i \rangle = \frac{\text{tr}(\rho^2)}{d} \geq \frac{1}{d^2} + \sum_{k=1}^{n-1} a_k^2 a_k .$$

(23)

Due to (22) and $\sum_{k=1}^{n-1} \omega^{-k(i-j)} = n \delta_{ij} - 1$, we obtain

$$\sum_{k=1}^{n-1} a_k^2 a_k = \frac{S^3}{n^2(1-c)} \sum_{i=1}^{n} \sum_{j=1}^{n} p_i p_j \sum_{k=1}^{n-1} \omega^{k(i-j)} = \frac{n^2 I(\mathcal{E}; \rho) - n}{d^2(1-c)} .$$

(24)

Combining (23) with (24) leads to

$$n^2 I(\mathcal{E}; \rho) \leq n - d(1-c) + d^2(1-c) \text{tr}(\rho^2) = ncd + d^2(1-c) \text{tr}(\rho^2),$$

(25)

since $n - d(1-c) = ncd$ in line with [4]. Dividing (25) by $n^2$ gives (18) due to $S = n/d$. ■

There are two cases, when the inequality (18) is saturated. For the maximally mixed state $\rho_s = \mathbb{1}_d/d$, we have $p_j(\mathcal{E}; \rho_s) = 1/n$ for all $j$ and, herewith, $I(\mathcal{E}; \rho_s) = 1/n$. The right-hand side of (18) reads as

$$\frac{Scd + 1 - c}{S^2d} = \frac{nc + S - nc}{Sd^2} = \frac{1}{n},$$

(26)

due to $1 - c = S - nc$ and $Sd = n$. Another case is obtained with any pure state taken from the set $\mathcal{F} = \{|\phi_j\rangle\}$. Then the index of coincidence is calculated as

$$\left(\frac{d}{n}\right)^2 + (n-1) \left(\frac{cd}{n}\right)^2 = \frac{1 + (n-1)c^2}{S^2} = \frac{1 + Sc - c}{S^2} = \frac{d^2 - 2d + n}{n^2 - n},$$

(27)
where we used \((n - 1)\alpha = S - 1\). For a pure state, the right-hand side of (18) coincides with (27) due to \(\text{tr}(\rho^2) = 1\).

Let us recall the case of SIC-POVMs. For \(n = d^2\), \(S = d\) and \(c = (d + 1)^{-1}\), the result (18) is rewritten as

\[
I(E; \rho) = \frac{1 + \text{tr}(\rho^2)}{d(d + 1)}.
\]  

(28)

The probabilities are posed as \(p_j(E; \rho) = d^{-1}\langle \phi_j \vert \rho \vert \phi_j \rangle\). The equality holds, since the set of vectors (10)–(11) is complete in this case. As was mentioned, the index of coincidence (28) was calculated in [13]. The inequality (18) reflects a non-trivial inner structure of the kets that form an equiangular tight frame. Hence, entropic uncertainty relations for the corresponding POVM immediately follow.

**IV. ENTROPIC UNCERTAINTY RELATIONS**

To express uncertainty relations, we will use the Rényi and Tsallis entropies. For \(\alpha > 0 \neq 1\), the Rényi \(\alpha\)-entropy is expressed as [25]

\[
R_\alpha(p) := \frac{1}{1 - \alpha} \ln \left( \sum_j p_j^\alpha \right).
\]  

(29)

In the limit \(\alpha \to 1\), this quantity reduces to the Shannon entropy \(H_1(p) = -\sum_j p_j \ln p_j\). The limit \(\alpha \to \infty\) leads to the min-entropy

\[
R_\infty(p) = -\ln \left( \max_j p_j \right).
\]  

(30)

For \(\alpha > 0 \neq 1\), the Tsallis \(\alpha\)-entropy is defined by [26]

\[
H_\alpha(p) := \frac{1}{1 - \alpha} \left( \sum_j p_j^\alpha - 1 \right) = -\sum_j p_j^\alpha \ln_\alpha(p_j).
\]  

(31)

Here, the \(\alpha\)-logarithm of positive \(\xi\) is defined as

\[
\ln_\alpha(\xi) := \begin{cases} 
\frac{\xi^{1 - \alpha} - 1}{1 - \alpha}, & \text{for } 0 < \alpha \neq 1, \\
\ln \xi, & \text{for } \alpha = 1.
\end{cases}
\]  

(32)

By \(R_\alpha(E; \rho)\) and \(H_\alpha(E; \rho)\), we respectively mean the entropies (29) and (31) computed with the probabilities \(p_j\). By construction, the Tsallis is concave with respect to probability distributions. The Rényi \(\alpha\)-entropy is certainly concave for \(0 < \alpha \leq 1\) [27]. For \(\alpha > 1\), the answer on the question about its concavity depends on dimensionality of probability distributions [28].

Entropic uncertainty relations for a POVM with elements (7) follow from the inequality (18). For the given index of coincidence, the maximal probability can be estimated from above as described in [13]. Using (18), we then have

\[
\max_j p_j(E; \rho) \leq \frac{1}{n} \left( 1 + \sqrt{n - 1} \sqrt{nI(E; \rho)} - 1 \right) \leq \frac{1}{nS} \left( S + \sqrt{n - 1} \sqrt{nSc + n(1 - c) \text{tr}(\rho^2) - S^2} \right) = \frac{1}{nS} \left( S + \sqrt{(n - 1)(1 - c)} \sqrt{n \text{tr}(\rho^2) - S} \right).
\]  

(33)

(34)

This inequality is saturated for the maximally mixed state \(\rho_* = \mathbb{1}_d/d\). Indeed, we clearly have \(n \text{tr}(\rho^2) = S\). It is not so obvious that it is also saturated for any of the frame states. Of course, the state-independent inequality

\[
\max_j p_j(E; \rho) \leq \frac{d}{n}
\]  

(35)

directly follows from [17]. In effect, the right-hand side of (34) reduces to \(d/n\) for any of the frame states. Substituting \(\rho = \vert \phi_i \rangle \langle \phi_i \vert\), we see from (27) that

\[
nI(E; \rho) = \frac{d^2}{n} \left( 1 + \frac{(n - d)^2}{(n - 1)d^2} \right) = \frac{(n - 1)d^2 + n^2 - 2nd + d^2}{n(n - 1)} = \frac{n - 1 + (d - 1)^2}{n - 1},
\]  

(36)
whence $\sqrt{n-1} \sqrt{n I(\mathcal{E}; \rho)} - 1 = d - 1$. The right-hand side of (33) then reads as $d/n$. As was mentioned above, the inequality (18) is saturated in the considered case. Hence, the right-hand side of (33) coincides with (34) as well. In the simplest case of orthonormal bases, when $n = d$ and $c = 0$, the inequality (34) is saturated for all pure states. Thus, we have arrived at uncertainty relations of the Landau–Pollak type [29].

Let us proceed to uncertainty relations in terms of Rényi entropies. Due to (30) with (34), the min-entropy uncertainty relation reads as

$$R_\infty(\mathcal{E}; \rho) \geq \ln(nS) - \ln\left( S + \sqrt{(n-1)(1-c)} \sqrt{n \text{tr}(\rho^2)} - S \right),$$

where $S$ and $c$ are defined by (4). In the case $n = d^2$, when a SIC-POVM is dealt with, the inequality (37) reduces to one of the results of [13]. Substituting $\alpha = 2$ into (29) gives the collision entropy $R_2(\mathcal{E}; \rho) = - \ln I(\mathcal{E}; \rho)$. Combining the latter with (34), we obtain

$$R_2(\mathcal{E}; \rho) \geq 2 \ln S - \ln(Sc + (1 - c) \text{tr}(\rho^2)) \geq \ln\left( \frac{n^2 - n}{d^2 - 2d + n} \right).$$

The state-independent inequality (39) is obtained by combining $\text{tr}(\rho^2) \leq 1$ with decreasing of the function $\xi \mapsto - \ln \xi$. We already mentioned two particular cases, in which the inequalities (37) and (38) are both saturated. The same holds for the relations (37) and (38). As was shown in [21], the Rényi $\alpha$-entropy of order $\alpha \in [2, \infty]$ satisfies

$$R_\alpha(p) \geq \frac{\alpha - 2}{\alpha - 1} R_\infty(p) + \frac{1}{\alpha - 1} R_2(p).$$

Combining the above inequalities, we have arrived at a conclusion.

**Proposition 3** Let $n$ unit kets $|\phi_j\rangle$ form an ETF in $\mathcal{H}_d$, and let POVM $\mathcal{E}$ be assigned to this frame by (7). For the given density matrix $\rho$ and $\alpha \in [2, \infty)$, it holds that

$$R_\alpha(\mathcal{E}; \rho) \geq \frac{\alpha \ln S + (\alpha - 2) \ln n - \ln (Sc + (1 - c) \text{tr}(\rho^2))}{\alpha - 1} - \frac{\alpha - 2}{\alpha - 1} \ln\left( S + \sqrt{(n-1)(1-c)} \sqrt{n \text{tr}(\rho^2)} - S \right),$$

$$\geq \ln n - \frac{(\alpha - 2) \ln d}{\alpha - 1} + \frac{1}{\alpha - 1} \ln\left( \frac{n - 1}{d^2 - 2d + n} \right).$$

The statement of Proposition 3 gives a Rényi-entropy uncertainty relation for the POVM assigned to a ETF. This relation reflects features of ETFs via (18) in combination with some properties of Rényi entropies. Entropic uncertainty relations are a tool for deriving several kinds of criteria used in quantum information processing. Such criteria are of interest, since ETFs are easier to construct than SIC-POVMs. In more detail, applications of the derived relations will be discussed in the next section. To derive uncertainty relations in terms of Tsallis entropies, we adopt the corresponding method of [13]. The following statement takes place.

**Proposition 4** Let $n$ unit kets $|\phi_j\rangle$ form an ETF in $\mathcal{H}_d$, and let POVM $\mathcal{E}$ be assigned to this frame by (7). For the given density matrix $\rho$ and $\alpha \in [0, 2]$, it holds that

$$H_\alpha(\mathcal{E}; \rho) \geq \ln_\alpha\left( \frac{S^2}{Sc + (1 - c) \text{tr}(\rho^2)} \right).$$

The state-independent inequality is posed as

$$H_\alpha(\mathcal{E}; \rho) \geq \ln_\alpha\left( \frac{n^2 - n}{d^2 - 2d + n} \right).$$

**Proof.** For $0 < \alpha \leq 2$, one writes

$$H_\alpha(p) \geq \ln_\alpha\left( \frac{1}{I(p)} \right).$$

The latter holds by applying Jensen’s inequality to the function $\xi \mapsto \ln_\alpha(1/\xi)$ [13]. Since this function decreases, combining (18) with (35) implies (33). The result (44) immediately follows due to $\text{tr}(\rho^2) \leq 1$. ■
The probability \( p \) where the binary entropy \( h \) satisfies distribution (46) satisfies 
\[
\text{distribution (50) is majorized by any of two local probability distributions [34]. Suppose that the POVM }
\]
\[
\text{For product states, one generates the convolution of two distributions assigned to local measurements. Hence, the }
\]
each separable state \( \rho \) assigned to an ETF according to (7). Separability conditions follow from state-independent uncertainty relations. For 
\[
\text{A bipartite mixed state is called separable, when its density matrix can be represented as a convex combination of }
\]
product states [32, 33]. To derive separability conditions, we recall the scheme proposed in [34]. Let 
\[
\text{Mutually unbiased bases give an important tool to detect entanglement [31]. The use of SIC-POVMs in entanglement }
\]
detection was briefly mentioned in [13]. This task can also be approached with measurements assigned to ETFs. 
\[
\text{The latter immediately follows from (43) and concavity of the Tsallis entropy. The state-independent formulation }
\]
\[
\text{The statement of Proposition 4 gives Tsallis-entropy uncertainty relations for the measurement assigned to a ETF. }
\]
\[
\text{It is important for applications to have a restriction valid for all states. The state-independent relation (44) will }
\]
\[
\text{be employed to formulate entanglement criteria and steering inequalities. Another reason to use Tsallis-entropy }
\]
\[
\text{The state-independent relation (44) will }
\]
\[
\text{The latter immediately follows from (43) and (47).}
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\[
\text{For a density matrix } \rho \text{ For a density matrix } \rho \text{ of the bipartite system, the (} j, k \text{)-probability reads as }
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V. SOME APPLICATIONS

Let us consider applications of the derived relations. By \( A \) and \( B \), we denote subsystems of a bipartite system. Mutually unbiased bases give an important tool to detect entanglement [31]. The use of SIC-POVMs in entanglement detection was briefly mentioned in [13]. This task can also be approached with measurements assigned to ETFs. A bipartite mixed state is called separable, when its density matrix can be represented as a convex combination of product states [32, 33]. To derive separability conditions, we recall the scheme proposed in [34]. Let \( \mathcal{N}_A = \{ N_{A_j} \} \) and \( \mathcal{N}_B = \{ N_{B_k} \} \) be two \( n \)-outcome POVMs in \( \mathcal{H}_A \) and \( \mathcal{H}_B \), respectively. It is convenient for a time to index elements by numbers from 0 up to \( n - 1 \). To the given POVMs, we assign the measurement \( \mathcal{M}(\mathcal{N}_A, \mathcal{N}_B) \) with \( n \) elements 
\[
M_k := \sum_{j=0}^{n-1} N_{A_j} \otimes N_{B_k j},
\]
where the symbol \( \otimes \) means the modular subtraction. It follows from (49) that (50)
\[
p(\mathcal{M}; \rho_A \otimes \rho_B) = p(\mathcal{N}_A; \rho_A) * p(\mathcal{N}_B; \rho_B).
\]
For product states, one generates the convolution of two distributions assigned to local measurements. Hence, the distribution (51) is majorized by any of two local probability distributions (34). Suppose that the POVM \( \mathcal{N}_A \) is assigned to an ETF according to (7). Separability conditions follow from state-independent uncertainty relations. For each separable state \( \rho_{AB} \) and \( \alpha \in (0, 2] \), it holds that 
\[
H_{\alpha}(\mathcal{M}; \rho_{AB}) \geq \ln_{\alpha} \left( \frac{n^2 - n}{d^2 - 2d + n} \right). 
\]
The latter immediately follows from (13) and concavity of the Tsallis entropy. The state-independent formulation (35) also leads to a separability criterion. For each separable state \( \rho_{AB} \), it holds that 
\[
\max_j p_j(\mathcal{M}; \rho_{AB}) \leq \frac{d}{n}.
\]
There is another to use ETFs for entanglement detection. Let us consider a bipartite system of two \( d \)-dimensional subsystems. To the given ETF, we assign the POVM \( \mathcal{N}_{AB} \) with elements 
\[
\frac{d^2}{n^2} |\phi_j\rangle \langle \phi_j| \otimes |\phi_k^*\rangle \langle \phi_k^*|.
\]
For a density matrix \( \rho_{AB} \) of the bipartite system, the (\( j, k \))-probability reads as 
\[
\frac{d^2}{n^2} \langle \phi_j \phi_k^* | \rho_{AB} | \phi_j \phi_k^* \rangle.
\]
Summing these probabilities over all $j = k$, we obtain the correlation measure

$$G(N_{AB}; \rho_{AB}) = \frac{d^2}{n^2} \sum_{j=1}^{n} \langle \phi_j \phi_j^* \rho_{AB} | \phi_j \phi_j^* \rangle .$$

(54)

For the case of SIC-POVMs, the quantity (54) reduces to that was introduced in [13]. It is also similar to the mutual predictability proposed in [31]. For each product state $\rho_A \otimes \rho_B$, the probability (54) is a product of two local probabilities. Using (18) and the Cauchy–Schwarz inequality, we then obtain

$$G(N_{AB}; \rho_A \otimes \rho_B) \leq \frac{1}{S^2} \sqrt{Sc + (1 - c) \operatorname{tr}(\rho_A^2)} \sqrt{Sc + (1 - c) \operatorname{tr}(\rho_B^2)} \leq \frac{d^2 - 2d + n}{n^2 - n} .$$

(55)

This inequality holds for all separable states. Its violation implies that the tested state is entangled. Using some orthonormal basis $\{\nu\} \in \mathcal{H}_d$, we write a maximally entangled state

$$|\Phi_+\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |\nu\rangle \otimes |\nu\rangle .$$

It is immediate to get $\langle \phi_j \phi_j^* | \Phi_+\rangle = 1/\sqrt{d}$, whence the $(j, j)$-probability is equal to $d/n^2$ and

$$G(N_{AB}; |\Phi_+\rangle \langle \Phi_+|) = \frac{d}{n} .$$

(56)

To estimate a region of detectability, we divide the right-hand side of (55) by (56). The ratio is calculated as

$$\frac{d - 2 + n/d}{n - 1} < 1 ,$$

whenever $n > d$. In this way, the most efficient scheme takes place for $n = d^2$, when the used ETF is maximal with reaching a SIC-POVM. Nevertheless, we can also utilize other ETF with sufficiently large number of vectors. Moreover, the use of several ETFs provides a collection of separability conditions. In general, this direction deserves further investigations.

It is well known that entropic uncertainty relations immediately lead to steering inequalities. The phenomenon of steering initially noticed by Schrödinger was formally posed in [35]. For a general discussion of this important issue, see the review [36] and references therein. Steering inequalities in terms of the standard entropic functions were considered in [37]. The authors of [38] focused on steering inequalities formulated with the use of Rényi entropies. Steering criteria also follow from uncertainty relations in terms of Tsallis entropies [39]. Properties of the Tsallis conditional entropy are considered in [40]. To obtain steering inequalities, the authors of [39] use the joint convexity and pseudoadditivity of the Tsallis relative entropy [41]. It must be stressed that generalized entropic functions do not succeed all the standard properties. For entropic functions of the Tsallis type, some questions of such a kind were addressed in [42, 43]. For this reason, we further refrain from a consideration of generalized conditional entropies. The desired steering inequalities can directly be formulated in terms of the corresponding probabilities [39].

Let us consider the protocol in a simplified form sufficient for our purposes. Sharing with Bob a bipartite state $\rho_{AB}$, Alice performs on her subsystem a POVM-measurement $\mathcal{E}_A$. The actual state of Bob’s subsystem is herewith conditioned on Alice’s result. This state is further subjected to Bob’s measurement $\mathcal{E}_B$. The parties repeat this procedure many times. Using the generated probabilities and classical side information from Alice, Bob recovers the joint probabilities. If for all possible measurements one can express the joint probability distribution as a convex combination

$$\sum_{\lambda} \lambda \mathcal{P}_A^{(\lambda)} \otimes \mathcal{P}_B^{(\lambda)} ,$$

(57)

then the system is called unsteerable [34]. Writing $q_i = \sum_{j=1}^{n} p_{ij}$ and combining (44) with the steering inequality of [39], we have

$$\frac{1}{\alpha - 1} \left( 1 - \sum_{i,j=1}^{n} p_{ij} q_i^{-\alpha} \right) \geq \ln_{\alpha} \left( \frac{n^2 - n}{d^2 - 2d + n} \right) ,$$

(58)

where $\alpha \in (0, 2]$. This form of the criterion is straightforward to evaluate [32]. Its violation implies steerability of the considered state. Thus, we can utilize ETF-based measurements to detect quantum steerability. Of course, the same
statement holds for SIC-POVMs. On the other hand, equiangular tight frames are easier to construct. In addition, several existing ETFs lead to a collection of steering criteria. Uncertainty relations in terms of Rényi entropies also produce steering criteria as explained in [38]. The papers [44, 45] presented Rényi-entropy uncertainty relations and respective steering inequalities for POVMs assigned to quantum designs. A similar treatment can be developed on the base of [12]. We refrain from presenting the details here.

VI. CONCLUSIONS

We have derived entropic uncertainty relations for quantum measurements on the base of equiangular tight frames. Such measurements are interesting not only in their own rights, but also due to potential applications in quantum information science. The results presented above are aimed to support this motivation. In the context of quantum information, the maximal ETFs are mainly considered as leading to informationally complete measurements. At the same time, there is no general strategy to construct SIC-POVMs for arbitrary dimensions. Equiangular tight frame without requirement of the maximal number of elements is easier to build. Indeed, there is a constructive approach to generate new ETFs from existing ones. Explicit examples of SIC-POVMs can also be used in this way [1]. Even if already known ETFs will sometimes occur, our abilities are enlarged here.

The presented relations have justified the claim that ETFs deserve wider application in quantum information than they have obtained. Each of equiangular tight frames directly leads to a POVM-measurement. The inner structure of ETFs is such that certain restrictions on generated probabilities follow. In particular, the corresponding index of coincidence and the maximal probability are bounded from above. Hence, entropic uncertainty relations and other corollaries were formulated. The elaborated framework is a natural extension of the results obtained for SIC-POVMs in the paper [13]. It is shown that ETF-based measurements can be used to detect entanglement and steerability. In this regard, one merely utilizes the treatment developed for other measurements. The above results are presented with a hope to stimulate further use of ETFs as a quantum information tool.

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