Non-parametric estimation of quadratic Hawkes processes for order book events

Antoine Fosset\textsuperscript{a,b}, Jean-Philippe Bouchaud\textsuperscript{b,c} and Michael Benzaquen \textsuperscript{a,b,c}

\textsuperscript{a}Ladhyx, UMR CNRS, 7646, Ecole polytechnique, Palaiseau Cedex, France; \textsuperscript{b}Chair of Econophysics & Complex Systems, Ecole polytechnique, Palaiseau Cedex, France; \textsuperscript{c}Capital Fund Management, Paris, France

ABSTRACT

We propose an actionable calibration procedure for general Quadratic Hawkes models of order book events (market orders, limit orders, cancellations). One of the main features of such models is to encode not only the influence of past events on future events but also, crucially, the influence of past price changes on such events. We show that the empirically calibrated quadratic kernel is well described by a diagonal contribution (that captures past realised volatility), plus a rank-one ‘Zumbach’ contribution (that captures the effect of past trends). We find that the Zumbach kernel is a power-law of time, as are all other feedback kernels. As in many previous studies, the rate of truly exogenous events is found to be a small fraction of the total event rate. These two features suggest that the system is close to a critical point – in the sense that slightly stronger feedback kernels would lead to endogenous liquidity crises.

ARTICLE HISTORY

Received 13 May 2020
Accepted 12 April 2021

KEYWORDS

Quadratic Hawkes processes; order book dynamics; Zumbach effect; endogenous liquidity crises

1. Introduction

The accumulation of empirical clues over the past few years provides mounting evidence that a large fraction of market volatility is of endogenous nature (Cutler, Poterba, and Summers 1989; Fair 2002; Joulin et al. 2008; Bouchaud 2010; Fosset, Bouchaud, and Benzaquen 2020). This obviously does not mean that significant news, such as the very recent Covid-19 crisis, do not impact financial markets, but rather that these only account for a small fraction of large price moves. The S&P500 flash crash of May 6th, 2010 (Zweig 2010; Kirilenko et al. 2017) is one example among many of an extreme event that has not been triggered by any outstanding piece of news. Furthermore, while one may argue that in some cases large drops are exogenously triggered, their amplification is often due to endogenous mechanisms (Bouchaud et al. 2018).

The behaviourally supported idea that agents tend to overreact, especially during crises, has driven the market modelling community to fall back on self-exciting processes, better known as Hawkes processes (Hawkes 1971). The latter have proven to be extremely efficient to tackle the intricate dynamics of the order flow and other self-excited effects in financial markets (Toke 2011; Filimonov and Sornette 2012; Bacry et al. 2013; Hardiman, Bercot, and Bouchaud 2013; Bacry and Muzy 2014; Bacry, Mastromatteo, and Muzy 2015; Bormetti et al. 2015; Huang, Lehalle, and Rosenbaum 2015; Rambaldi, Pennesi, and Lillo 2015; Alfonsi and Blanc 2016a, 2016b; Morariu-Patrichi and Pakkanen 2018; Rambaldi, Bacry, and Lillo 2017; Achab et al. 2018; Calcagnile et al. 2018; Wu et al. 2019; Koyama and Shinomoto 2020). Nonetheless, linear Hawkes processes are unable to account for an empirical finding essential to our eyes to model fat-tails and endogenous instabilities: the Zumbach effect (Zumbach 2010; Chicheportiche and Bouchaud 2014; Blanc, Donier, and Bouchaud 2017; Euch et al. 2020). The latter states that past price trends increase future activity, regardless of their sign. Quadratic Hawkes processes (Q-Hawkes), inspired by quadratic ARCH processes (Sentana 1995; Chicheportiche and Bouchaud 2014), were recently introduced to circumvent this issue (Blanc, Donier, and Bouchaud 2017; Dandapani, Jusselin, and
Rosenbaum 2019), and have proven key to understand fat-tails in the distribution of returns, as well as spread, volatility and liquidity dynamics (Fosset, Bouchaud, and Benzaquen 2020). The Q-Hawkes processes we shall present in this paper account for feedback of past price trends and volatility on the liquidity flow.

In our recent paper (Fosset, Bouchaud, and Benzaquen 2020) we indeed argued that price or spread jumps could be the result of endogenous feedback loops that trigger liquidity seizures, see also Dall’Amico et al. (2019). In particular, we empirically showed that Zumbach-like effects exist in order book data, i.e. past trends and volatility tend to promote future activity, and in particular cancellations that diminish liquidity and fragilise the system, possibly leading to a liquidity crisis. Combining Q-Hawkes processes with a stylized order book model (Daniels et al. 2003; Smith et al. 2003) revealed an interesting scenario with a second order phase transition between a stable regime for weak feedback and an unstable regime for strong feedback, in which liquidity crises arise with high probability. However, for such a scheme to be relevant for financial markets, the system must sit very close to the instability threshold (perhaps as the result of ‘self-organized criticality’). As an alternative scenario, we also proposed a non-linear Hawkes process which exhibits liquidity crises as occasional ‘activated’ events, separating locally stable periods of normal activity.

In the present paper, we calibrate on real market data a version of the generalised Q-Hawkes process proposed in our recent work (Fosset, Bouchaud, and Benzaquen 2020). We provide successively more parsimonious (and thus more robust and insightful) representations of our results and deliver convincing evidence for the price/liquidity feedback mechanism described above and quantify its implications. In Section 2 we briefly recall the ingredients of the model and present the non-parametric calibration procedure, inspired by the methods introduced by Bacry and Muzy (2016), Bacry, Jaisson, and Muzy (2016), and Bacry, Dayri, and Muzy (2012). We apply such calibration to order book data on the EURO STOXX and BUND futures contracts. In Section 3 we present an alternative method that needs fewer assumptions to compute the overall effect of past price moves on future liquidity flow (this method is expected to be less noisy since it involves one numerical step less). We then perform a deeper investigation into the nature of the quadratic kernel, by introducing a well-suited low rank (Zumbach-like) approximation that allows us separate the effects of trend and volatility, thereby allowing for further interpretation of our results when applied to our futures contracts. In Section 4, we focus on the liquidity flow and analyse spread time series in relation with adequate trend and volatility signals. Results appear to favour the ‘self-organised criticality’ scenario over the metastable, ‘activated’ scenario discussed above and in Fosset, Bouchaud, and Benzaquen (2020). In Section 5 we conclude.

2. Brute force calibration of a Q-Hawkes process

2.1. Definition of the model

We present a simplified version of the Generalized Quadratic Hawkes process (GQ-Hawkes) introduced in Fosset, Bouchaud, and Benzaquen (2020), where the influence of the size of the queues on event rates is neglected. Consider a 6-dimensional process \( N_t = (N_t^{C,b}, N_t^{LO,b}, N_t^{MO,b}, N_t^{MO,a}, N_t^{LO,a}, N_t^{C,a}) \) counting six types of order book events: limit orders (LO), cancellations (C), and market orders (MO), for both the bid (b) and ask (a) sides of the order book; we consider best quotes only. This is justified by the fact our present study focuses on large tick assets for which most of the liquidity lies at the best quotes, and most of the activity takes place at the best quotes (for the EURO STOXX futures contract presented below the number of events at the second best is only 9% of that at the best, and only 4.5% for the third best). Note that our analysis can be easily extended to small tick assets, by aggregating the volume and activity of a few queues near the best (as done e.g. in Dall’Amico et al. 2019). But even for small tick stocks, it is well documented that the fluctuations of the overall volume in the order book are captured by the volume at best (more precisely, the principal component of the whole order book fluctuations is a pure dilation, see e.g. Bouchaud, Mézard, and Potters 2002).

We further assume that the process \( N_t \) is coupled to the past price process \( P_{t-s} \) in the following way. Denoting \( \lambda_t \) the intensity of the the 6-dimensional process \( N_t \) we let:

\[
\lambda_t = \alpha_0 + \int_0^t \phi(t-s) \, dN_s + \int_0^t L(t-s) \, dP_s + \int_0^t \int_0^t K(t-s, t-u) \, dP_s \, dP_u, \tag{1}
\]
with $\phi$ a causal $6 \times 6$ matrix kernel, $L$ a causal 6-dimensional vector kernel, and $K$ a causal 6-dimensional vector double-kernel. One can always choose $K(u, s) = K(s, u)$ without loss of generality.

The intensity $\lambda_t$ is the sum of four different contributions, from left to right in the RHS of Equation (1), one has the base rate $\alpha_0$, taken here to be constant, the standard linear Hawkes contribution, followed by the linear and the quadratic contributions of price fluctuations. In particular, the kernel $\phi$ encodes the effect of past events on the present event rate (self-excitation); $\|\phi^j(t - s)\|$ corresponds to the average number of events of type $i$ at time $t$ induced by one event of type $j$ at time $s$. The kernel $L$ signifies the impact of signed price changes on the event rates; $\int_0^t L(t - s) dP_s$ can be interpreted as a local trend. Finally, $K$ encodes the joint impact of price changes at times $s$ and $u$ on the event rates. As we shall see below, this term can be interpreted as the effect of both volatility and square trend. As pointed out in Fosset, Bouchaud, and Benzaquen (2020) and Blanc, Donier, and Bouchaud (2017), assuming that $P_t$ is a martingale makes analytical calculations, and numerical calibration, much more congenial. Finally, assuming as we shall do hereafter that a stationary state is reached allows us to replace the lower bound of the integrals in Equation (1) by $-\infty$.

2.2. A non-parametric calibration procedure

Here we introduce a non-parametric scheme to calibrate Equation (1) to real market data, that is, estimate the unknown kernels $\phi$, $L$, $K$ and $\alpha_0$ from the observable event rates and price changes. Our method is an extension of the second moment method introduced by Bacry and Muzy (2016) and Bacry, Jaisson, and Muzy (2016), see also Chicheportiche and Bouchaud (2014).

2.2.1. Covariances and Wiener-Hopf-like equations

Before deriving the equations that will be used for the calibration, we introduce the following averages and covariances:

$$\Delta_k \, dt := \mathbb{E} \left[ (dP_t)^k \right],$$  
$$\Lambda^i \, dt := \mathbb{E} \left[ dN_t^i \right],$$  
$$\chi^{ij}_{NN}(t - s) \, dt \, ds := \text{Cov} \left( dN_t^i, dN_s^j \right) - \Lambda^j \delta_{ij} \delta(t - s) \, dt \, ds, $$  
$$\chi^{ij}_{NP}(t - s) \, dt \, ds := \text{Cov} \left( dN_t^i, dP_s \right), $$  
$$\chi^{ij}_{Np^2}(t - s) \, dt \, ds := \text{Cov} \left( dN_t^i, dP_s^2 \right), $$

$$\chi^{ij}_{Np^2}(t - s) \, dt \, ds := \text{Cov} \left( dP_t^i, dP_s^2 \right) - \Delta_4 \delta(t - s) \, ds \, dt, $$

where we have assumed for simplicity that the jumps of $P$ and $N$ are not simultaneous. Note that while price jumps can only occur if one order book event triggers them, the relative frequency of the latter is so much larger that this approximation is fully justified. Combining Equation (1) with Equations (2) yields the following set of equations for the first and second moments of the processes. Introducing the notations $\|f\| = \int_{\mathbb{R}} f(t) \, dt$ and $K_d(t) := K(t, t)$ the diagonal part of $K$, one obtains for $t, x > 0$ with $t \neq x$:

$$\Lambda^i = \alpha_0^i + \sum_k \|\phi^{ik}\| \Lambda^k + \|K_d^i\| \Delta, $$

$$\chi^{ij}_{NN}(t) = \Lambda^j \phi^{ij}(t) + \int_{\mathbb{R}^+} \left[ \sum_k \phi^{ik}(s) \chi^{kj}_{NN}(t - s) + \Lambda^i(s) \chi^{ij}_{NP}(s - t) + K_d^i(s) \chi^{ij}_{Np^2}(s - t) \right] \, ds,$$

$$+ \int_{[t, +\infty[^2} K^i(s, u) \chi^{ij}_{Np^2}(s - t, u - t) \, 1_{[s \neq u]} \, du \, ds, $$

where $\chi^{ij}_{NP}(t - s) \, dt \, ds := \text{Cov} \left( dN_t^i, dP_s \right)$,
\[ \chi^{i}_{NP}(t) = \int_{\mathbb{R}^+} \sum_{k} \phi^{ik}(s) \chi^{k}_{NP}(t-s) \, ds + L^{i}(t) \Delta_2 + K^{i}_{d}(t) \Delta_3, \quad (3c) \]

\[ \chi^{i}_{Np^2}(t) = \int_{\mathbb{R}^+} \sum_{k} \phi^{ik}(s) \chi^{k}_{Np^2}(t-s) \, ds + L^{i}(t) \Delta_3 + K^{i}_{d}(t) \Delta_4 + \int_{\mathbb{R}^+} \chi^{p^2 p^2}(t-s) K^{i}_{d}(s) \, ds, \quad (3d) \]

\[ \chi^{i}_{NPP}(t, x) = \int_{\mathbb{R}^+} \sum_{k} \phi^{ik}(s) \chi^{k}_{NPP}(t-s, x-s) \, ds + 2 \Delta^2 K^{i}(t, x). \quad (3e) \]

Note that while Equation (3) is linear in the unknowns \( \phi, L, K \) and \( \alpha_0 \), there is no analytical formula that can be used in practice to invert the system. Provided the number of events generated by price fluctuations is small compared to that generated by the linear Hawkes contribution, i.e. \( \sum_{i,k} \| \phi^{ik} \| \Lambda^{k} \gg \sum_{i} \| K^{i}_{d} \| \Delta_2 \), Equation (3b) conveniently simplifies to:

\[ \chi^{ij}_{NN}(t) = \Lambda^{j} \phi^{ij}(t) + \sum_{k} \int_{\mathbb{R}^+} \phi^{ik}(s) \chi^{kj}_{NN}(t-s) \, ds. \quad (4) \]

This approximation is relatively well supported by real data for short enough times (see below). It is essential at this stage as it allows us to decouple the estimation of the Hawkes kernel from that of \( L \) and \( K \): one can first estimate \( \phi \) from Equation (4) and then compute \( L \) and \( K \) from Equations (3c), (3d) and (3e). The base rate is finally obtained from Equation (3a). Note that while in principle an exact calibration of Equations (3) is possible, it does not perform well on real data – but see Section 3 below.

### 2.2.2. Micro-price, discretisation and calibration recipe

In Section 2 we stressed that the point process \( P_t \) needs to be a martingale for Equations (3) to be valid. Yet, it is well established that the mid-price in financial markets displays substantial mean-reversion at short timescales. To circumvent this issue we consider the volume weighted mid-price, sometimes called the micro-price, \( p_{micro}^{t} \), known to be closer to a martingale at high frequency (Stoikov 2018; Gould and Bonart 2016).\(^3\) It is defined as:

\[ p_{micro}^{t} = \frac{v^{b}_{t} b_{t} + v^{a}_{t} a_{t}}{v^{b}_{t} + v^{a}_{t}}, \quad (5) \]

where \( v^{b}_{t} \), \( v^{a}_{t} \) denote the available volume at the best bid \( b_{t} \) and ask \( a_{t} \), respectively. To enforce the martingale property we use the so-called surprise price, that we shall henceforth denote by \( P_{s} \), and which consists in subtracting to the price its (linear) statistical predictability. Mathematically speaking, this reads:

\[ dP_{t} = dP_{micro}^{t} - \int_{-\infty}^{t} \rho_{p}(t-s) \, dP_{s}^{micro}, \quad (6) \]

where \( \rho_{p}(t-s) := \text{Cor}(dP_{micro}^{t}, dP_{s}^{micro}) \) denotes the price auto-correlation function.

We also note that the intensity of order book events exhibit an intraday U-shape, very much like the well known U-shaped volatility pattern. Computing the total intensity of events \( \Lambda_{tot} = \sum_{i} \Lambda_{i} \) over 5-min bins and averaging over trading days, a U-shape is clearly visible. To avoid spurious effects related to these intraday seasonalities, we rescale time flow by this average pattern to enforce a constant rate of events in the new time variable.

In order to estimate the kernels from real order book data, one must choose a time grid \( t_{n}^{H} \) with weights \( w_{n}^{H} \) for kernel \( \phi \), such that \( \| \phi \| \approx \sum_{n} w_{n}^{H} \phi(t_{n}^{H}) w_{n}^{H} \). We decide to use quadrature points (Bacry, Jaisson, and Muzy 2016) to ensure a good approximation of the integrals with a minimal number of points. Further, given that we expect power-law kernels, see e.g. Hardiman, Bercot, and Bouchaud (2013), Bacry, Jaisson, and Muzy (2016), and Blanc, Donier, and Bouchaud (2017), we choose a linear scale at short times that switches to logarithmic at longer times. Finally, given that typical timescales are usually quite different (see below), we choose a different time grid \( t_{n} \), \( w_{n} \) for the kernels \( L \) and \( K \). See Appendix 1 for more details.
Finally, the empirical covariances are usually very noisy, so we choose to smooth them using a convenient fitting function in order to obtain better behaved kernels. Concerning the volatility covariance $\chi_{P2P2}$, it is found to behave like a power law at large times so the chosen fitting function is $4A(1 + t/B)^{-C}$. We also fit the logarithm of $\chi_{NP}(t)$, $\chi_{NP2}(t)$ by a polynomial in log $t$, and smooth the off-diagonal kernel, see Section 3.2 for details. Plots of the 'raw' kernels obtained without smoothing fits are provided in Appendix 2. Apart from being more noisy, as expected, these raw kernels are very similar to the smoothed ones.

The calibration recipe then amounts to the following steps.

- Compute the surprise price from the micro-price using Equations (5) and (6).
- Rescale time by the typical daily pattern of $\Lambda_{\text{tot}} = \sum_i \Lambda_i$.
- Estimate $\Delta_k$, $\Lambda$ and the covariances $\chi_{P2P2}$, $\chi_{NN}$, $\chi_{NP}$, $\chi_{NP2}$ and $\chi_{NPP}$ from the data using Equations (2),
- Use adequate fitting functions to smooth the empirical covariances (optional),
- Discretise and solve Equation (4) to obtain the Hawkes kernel $\Phi$,
- Discretise and solve Equations (3c), (3d), and (3e) to obtain the kernels $L$ and $K$,
- Discretise and solve Equation (3a) to obtain the base rate $\alpha_0$.

Further details on how to solve these equations in practice are provided in Appendix 1.

### 2.3. Empirical results

We now apply the calibration procedure presented above to the EURO STOXX futures contract in the period 2016/09/12–2020/02/07. For this contract, the average time between two order book events is $\tau_e \approx 0.03$ s, two orders of magnitude below the average time between two price changes $\tau_p \approx 7$ s, indicating that the range of the kernels $L$ and $K$ is likely to be greater than that of $\Phi$, and allowing one to choose discretisation time grids accordingly. We also apply the procedure to the BUND futures contract but do not show all the (redundant) results for the sake of readability; summarising results are displayed in Figure 5 and Tables 1, A1 and A2.

As specified in Section 2.2.2, we start with the calibration of the Hawkes kernel $\Phi$. The results are displayed in Figure 1 for the norms of the kernels, and in Figure A1 in the Appendix for the full time-dependence. The temporal decay of the kernels appears to be power law with exponent $\approx -1.5$, consistent with previous reports (Hardiman, Bercot, and Bouchaud 2013; Bacry, Jaisson, and Muzy 2016; Blanc, Donier, and Bouchaud 2017).

The calibration leads to a stable Hawkes process with spectral radius of $\|\Phi\|$ (computed over 1000s) found to be $\approx 0.75$ for the EURO STOXX contract and $\approx 0.74$ for the BUND (Toke 2011; Bacry, Mastromatteo, and Muzy 2015). The results show that the expected bid-ask symmetry holds with a high level of accuracy (see Fosset, Bouchaud, and Benzaquen 2020), such that one can average the kernels accordingly to improve the statistics without loss of information.

Plugging the obtained Hawkes kernels into Equations (3c), (3d) and (3e) allows us to calibrate the kernels $L$ and $K$, see Figure 2. Again the expected bid-ask symmetry properties hold rather well: while the linear kernel $L$ is anti-symmetric (the effect of the positive trend on the bid is the same as that of a negative trend on the ask), the quadratic kernel $K$ is bid-ask symmetric. We will therefore not distinguish further bid and ask events in the following. To provide an estimate of the error in our calibration procedure, we use a bootstrap method: we run the calibrations on overlapping sub-samples to compute the standard deviation of our results in a statistically significant manner. For the sake of visibility throughout the manuscript we only present the error band for one of the kernels in each plot, it being very similar in magnitude for the others.

| Table 1. Average order volumes (in shares). |
|-------------------------------------------|
| $\bar{v}_{CB}$ | $\bar{v}_{OL}$, $\bar{v}_{MO,b}$ | $\bar{v}_{MO,a}$ | $\bar{v}_{OL,a}$ | $\bar{v}_{CA}$ |
|----------------|---------------------------------|-----------------|-----------------|----------------|
| EUROSTOXX      | 10.1                            | 9.2             | 7.2             | 8.2            | 9.2            |
| BUND           | 4.5                             | 4.8             | 4.4             | 4.2            | 4.8            | 4.5 |
Figure 1. Norms of the Hawkes kernel $\|\phi\|$ for the EURO STOXX futures contract between 2016/09/12 and 2020/02/07, calibrated using Equation (4).

Figure 2. Kernels resulting from the non-parametric calibration on the EURO STOXX futures contract between 2016/09/12 and 2020/02/07. (a) Linear kernels $L$. Note that the sign is such that an up (resp. down) trend increases all the event rates at the bid (resp. ask) at short times. (b) Diagonal of quadratic kernels $K_d$. (c) Full quadratic kernels $K(t,x)$. A bootstrap error band around the MO kernels is displayed in light green. The error band is similar for other kernels.

Figure 2(c) shows that the quadratic contribution cannot be reduced to the diagonal part $K_d$ only. Indeed, the off-diagonal contribution of the kernel is non-zero and rather long-ranged. The decay of the diagonal contribution is a power law with exponent $\approx -1$. Such a decay is very slow and means that $\|K_d\|$ is
logarithmically sensitive to long timescales, for which we do not have much information since we only use data belonging to the same trading day to avoid the thorny discussion of overnight effects and how to treat them.

Finally, while the Hawkes and price feedback effects are difficult to compare as they do not operate on the same timescales, one can argue that the approximation presented at the end of Section 2.2.1 is well supported by data: considering a cut-off of 1000 s to compute the norms, one finds: $\sum_i \|K^i_d\| \Delta_2 / \sum_{i,k} \|\phi^i_k\| \Delta^k \approx 0.06$. Another useful piece of information is the global effect of the quadratic term on order book events, measured by $\sum_i \|K^i_d\| \Delta_2$, which must be compared to the total activity $\sum_i \Lambda^i$. The ratio of these two quantities is found to be 5% for the EURO STOXX and 7% for the BUND (see Table A2 for more details). Although not dominant, this feedback is clearly not negligible. Together with the standard Hawkes contribution, this means that the exogenous contribution $\alpha$ to the total activity is only 19% of the total for the EURO STOXX (17% for the BUND). Note that this fraction is expected to decreases further as the upper cut-off of the slowly decaying kernels is extended beyond 1000 s (see e.g. Hardiman, Bercot, and Bouchaud 2013).

3. A simplified framework

Here we present a framework which improves the above calibration in a threefold manner. As we shall see, (i) it allows to circumvent the approximation given in Equation (4) which, we recall, is not perfectly satisfied by real data, (ii) it helps cleaning further the noisy off-diagonal contribution of the quadratic kernel, and (iii) it gives a more relevant measure of the global effect of price fluctuations on event rates with no longer having to consider, nor calibrate, the Hawkes contribution.

3.1. Effective kernels

Using the resolvent method, see Bacry, Jaisson, and Muzy (2016) and Jaisson and Rosenbaum (2015), one can rewrite Equation (1) as:

$$\lambda_t = (1 - \|\phi\|)^{-1} \alpha_0 + \int_{-\infty}^{t} \mathcal{R}(t-s) \, dB_s + \int_{-\infty}^{t} \tilde{L}(t-s) \, dB_s + \int_{-\infty}^{t} \int_{-\infty}^{t} \tilde{K}(t-s, t-u) \, dB_s \, dB_u, \quad (7)$$

with $M$ a martingale satisfying $dB_t = dB_s - \lambda_t \, dt$, $\mathcal{R} = \sum_{n \geq 1} \phi^n$ the resolvent, $\tilde{L} = L + \mathcal{R} \ast L$ and $\tilde{K}(t,s) = K(t,s) + \int_{0}^{\infty} \mathcal{R}(u) K(t-u, s-u) \, du$. The kernels $\tilde{L}$ and $\tilde{K}$ account for the overall feedback effect of $P_t$, including all subsequent Hawkes self-excited events that are induced by price fluctuations. The remarkable property of such kernels is that they solve a much simpler set of equations:

$$\chi_{NP}^i(t) = \tilde{L}^i(t) \Delta_2 + \tilde{K}^i_d(t) \Delta_3$$

$$\chi_{NP2}^i(t) = \tilde{L}^i(t) \Delta_3 + \tilde{K}^i_d(t) \Delta_4 + \int_{\mathbb{R}} \chi_{P2P}^i(t-s) \tilde{K}^i_d(s) \, ds$$

$$\chi_{NPP}^i(t,x) = 2 \tilde{K}^i(x,t) \Delta_2^2$$

where we have again enforced that $\tilde{K}$ is symmetric. The results obtained from the inversion of Equations (8) for the EURO STOXX futures contract are displayed in Figure 3. These lead to similar, though slightly cleaner, conclusions to Figure 2. In particular, the values of $\sum_i \|\tilde{K}^i_d\| \Delta_2$ are compatible with those obtained above (taking into account the $1 - \|\phi\|$ factor, see Table A2).

3.2. The Zumbach factorisation

Here we further dissect the results of the calibration presented in the previous section, with the objective in particular of separating the contributions of trend and of volatility to the quadratic feedback. A meaningful
Figure 3. Effective kernels resulting from the simplified calibration on the EURO STOXX futures contract between 2016/09/12 and 2020/02/07. (a) Linear kernels $\bar{L}$. Note that the sign is such that an up (resp. down) trend increases all the event rates at the bid (resp. ask) at short times. (b) Diagonal of quadratic kernels $\bar{K}_d$. (c) Full quadratic kernels $\bar{K}(t,x)$. A bootstrap error band around the MO kernel is displayed in light green. The error band is similar for other kernels.

approximation for the quadratic kernel $\bar{K}$ was introduced in Blanc, Donier, and Bouchaud (2017), as the sum of a purely diagonal matrix and a rank-one contribution:

$$\bar{K}^i(t - s, t - u) := \bar{K}^i_d \psi^i(t - s) I_{[s=u]} + \bar{K}^i_1 Z^i(t - s) Z^i(t - u).$$

(9)

The first term on the right hand side of Equation (9) reflects feedback of past volatility on current order book events. Its contribution in Equation (7) can indeed by written as:

$$\left[\sigma^i(t)\right]^2 := \int_0^t \psi^i(t - s) (dP_s)^2,$$

(10)

The second term is in turn a reflection of the effect of past trends, as measured in Equation (7) by $[\mu^i(t)]^2$, where:

$$\mu^i(t) := \int_0^t Z^i(t - s) dP_s.$$

(11)

This last term is reminiscent of the so-called Zumbach effect: past trends, regardless of their sign, lead to an increase in future activity. An alternative interpretation is that $[\mu^i(t)]^2$ is a local measure of a low-frequency volatility, to be contrasted with $[\sigma^i(t)]^2$ which is a local measure of high-frequency volatility. Note that the kernels $\psi$ and $Z$ are normalised:

$$\int \psi^i(s) \, ds = \int Z^i(s)^2 \, ds = 1,$$

(12)

such that the overall strength of the volatility contribution is $\bar{K}_d$ while that of the trend contribution is $\bar{K}_1$.

While in practice such an approximation is of course not perfect, one can check that including higher rank contributions is unessential as the latter do not carry much additional signal. The rank-one kernel is obtained by minimising $\int \int (\bar{K}^i(s, u) - \bar{K}^i_1 Z^i(s) Z^i(u))^2 I_{[s\neq u]} \, ds \, du$, which consists in finding the first eigenvector of a well chosen linear map, see Srebro and Jaakkola (2003) for more details. The $\psi$ contribution is then obtained by
Figure 4. Zumbach approximation of the effective kernel $\bar{K}$ on the EURO STOXX futures contract between 2016/09/12 and 2020/02/07. (a) Zumbach kernel $Z$, (b) Volatility kernel $\psi$. Both kernels are normalised such that $\|\psi\| = \|Z\|^2 = 1$, with a cut-off in the time integrals at 1000 s. A bootstrap error band around the MO kernels is displayed in light green. The error band is similar for other kernels.

We have tested the robustness of our results by calibrating our model on different sub-periods using a bootstrap method. All calibrated kernels are quantitatively consistent through different sub-periods of our sample. We also have tested the validity of the rank one approximation by comparing the norm of the rank two component with the rank one. We find that the ratios of these quantities is 15%–20%, which justifies our rank one approximation.

4. Liquidity dynamics & crises

4.1. Quadratic feedback on liquidity

So far we have focussed on the impact of past price moves on event rates. Here we wish to go on step further and estimate the effect of past price changes on liquidity, i.e. volume weighted events. For this one needs to consider order volumes. The average volumes are given in Table 1 for the different types of orders.

Assuming bid/ask symmetry (consistent with the empirical results), Figure 5 displays the amount of shares per second that can be attributed to the quadratic effect (both volatility and Zumbach) for each event type, namely $\bar{K}_d^2 V^i \Delta_2$ and $\bar{K}_1^2 V^i \Delta_2$ where $\bar{K}_d, \bar{K}_1$ are obtained as explained in the previous section, $V^i$ are given in Table 1, and $\Delta_2$ is defined in Equation (2a).
Introducing the overall average quadratic liquidity flux as:

\[ I_K := \left( \| R^{LO} \| V^{LO} - \| R^C \| V^C - \| R^{MO} \| V^{MO} \right) \Delta_2, \]  

one consistently finds that the quadratic (price) feedback has an overall negative effect on liquidity \( I_K < 0 \), most of it associated to volatility, see Figure 5(c). In other terms, the quadratic feedback tends to decrease liquidity on average. Figure 5(b) shows that both the volatility and Zumbach terms have an average negative impact on liquidity (i.e. the green bars represent less than 50% of the total contribution). The Zumbach term is responsible for non-trivial long-range liquidity anomalies. In particular, Blanc, Donier, and Bouchaud (2017) showed that the price process resulting from a quadratic Hawkes process follows is diffusive with fat tailed stochastic diffusivity at large times, which can be attributed to the Zumbach effect, rather than its volatility counterpart (see also the discussion in Dandapani, Jusselin, and Rosenbaum 2019). In any case, we believe that the quadratic feedback of price trends on order book events is a crucial ingredient to understand liquidity crises. In the next section we provide a direct test of this hypothesis.

4.2. Spread dynamics and liquidity crises

With the aim of making contact with our previous work (Fosset, Bouchaud, and Benzaquen 2020), we now focus on the analysis of spread dynamics. Since the EUROSTOXX futures is a large tick contract (the spread is equal to one over 99% of the time and seldom higher than two), we characterise the dynamics of liquidity using an effective spread \( S^e \) which is defined as follows. Calling \( v^a_t(x) \) (resp \( v^b_t(x) \)) the ask (resp bid) volume at price level \( x \), we construct cumulative volumes as \( Q^a_t(x) = \sum_{n \leq x} v^a_t(n) \) and \( Q^b_t(x) = \sum_{n \geq x} v^b_t(n) \). We then choose the average volume at best \( V_{best} \) as a reference volume, and define:

\[ S^e_t := (Q^a_t)^{-1}(V_{best}) - (Q^b_t)^{-1}(V_{best}), \]  

where \((Q^a/Q^b)^{-1}\) denotes the inverse function of \( Q^a/Q^b \). The effective spread is a natural proxy for liquidity in the close vicinity of the midprice: when the liquidity is close to its average, the effective spread coincides with the regular spread; but when liquidity is low, it can be much larger as aggregating the volume of several queues is needed to recover the reference volume \( V_{best} \). Figure 6(a) displays the survival function of the effective spreads, revealing that \( P(S^e > s) \sim s^{-\gamma} \). This power-law tail is interesting for the following reason: the effective spread can be seen as a proxy for the size of latent price jumps, i.e. the jumps that are likely to happen if an aggressive market order hits the market. Hence, one expects the distribution of effective spread is not far from the distribution of price returns \( r \), which is well known to decay as \( P(r) \sim r^{-\gamma} \).

Let us now study the relation between effective spread, square volatility \( \sigma^2 \) and square trend \( \mu^2 \), as defined in Equations (10) and (11). Figures 6(b–d) display the correlation functions \( C_\mu(\tau) := \text{Cor}[\mu(t+\tau)^2, S^e(t)] \), \( C_\sigma(\tau) := \text{Cor}[\sigma(t+\tau)^2, S^e(t)] \) and \( C_{\mathcal{J}}(\tau) := \text{Cor}[\mathcal{J}(t+\tau), S^e(t)] \) respectively, with \( \mathcal{J} = \mu^2/\sigma^2 \). Note that a causal positive impact of past trends on future spreads should translate as a strong contribution to \( C_\mu(\tau) \).
for negative $\tau$. Interestingly, this is compatible with Figure 6(b), which confirms in a model-free fashion that the Zumbach-like coupling is important: past square trends increase future effective spread, or equivalently decrease future liquidity. While also slightly asymmetric, the volatility/spread correlation $C_{\sigma}(\tau)$ does not reveal such a level of asymmetry (see Figure 6(c)). Figure 6(d) shows an even more pronounced asymmetry when we rescale the trend by the local volatility: $T$ is a proxy of the autocorrelation of returns, independently of their amplitude. In this sense, it is a better signature of trend behaviour, as the volatility aspect of recent price changes is discarded.

5. Conclusion
In this work, we have proposed an actionable procedure to calibrate general Quadratic Hawkes models for order book events (market orders, limit orders, cancellations). One of the main features of such models is to encode not only the influence of past events on future events but also, crucially, the influence of past price changes on such events. We have shown that the empirically calibrated quadratic kernel (describing the part of the feedback that is independent of the sign of past returns) is well described by the shape postulated in Blanc, Donier, and Bouchaud (2017), Fosset, Bouchaud, and Benzaquen (2020), and Dandapani, Jusselin, and Rosenbaum (2019), namely:

- a diagonal contribution that captures past realised volatility, and
- a rank-one contribution that captures the effect of past trends.

The latter contribution can be interpreted as the microstructural origin of the Zumbach effect: past trends, independently of their sign, tend to reduce the liquidity present in the order book, and therefore increase future volatility. As we have shown in our companion paper (Fosset, Bouchaud, and Benzaquen 2020), such coupling can in fact be strong enough to destabilise the order book and lead to liquidity crises.

One of the perhaps unexpected result of our calibration is that the Zumbach kernel is found to be a power-law of time for the futures contracts studied here, and not an exponential as was found in Blanc, Donier, and Bouchaud (2017) for US stock prices. Hence, all Hawkes kernels in our study are found to be power-laws of time. Furthermore, as in many previous studies (Filimonov and Sornette 2012; Hardiman, Bercot, and Bouchaud 2013; Bacry, Mastromatteo, and Muzy 2015), the rate of truly exogenous events is found to be much smaller than the total event rate, typically 1/5 when all kernels are truncated beyond 1000 s, and probably even smaller when longer lags are taken into account, due to the slow decay of the kernels. These two features suggest that the system is close to a critical point – in the sense that slightly stronger feedback kernels would lead to instabilities. In our setting, we have shown that the effective spread (which is a measure of the (il-)liquidity of the order book) has itself a power-law tailed distribution, which we see as a precursor of the famous ‘inverse cubic’ power-law tails of the return distribution (in the present context, see e.g. Chicheportiche and Bouchaud 2014; Blanc, Donier, and Bouchaud 2017). Such a power-law is not compatible with the alternative ‘activated’ scenario proposed in Fosset, Bouchaud, and Benzaquen (2020), which would rather suggest a bimodal distribution with a hump at large effective spreads.

Hence, we favour at this stage the scenario of markets poised close to a point of instability, although the detailed mechanisms that lead to such a fine tuning are still somewhat obscure. We note that the near-criticality has also been argued to be crucial to understand the ‘rough’ nature of volatility (Jaisson and Rosenbaum 2016; Rosenbaum and Jusselin 2018; Dandapani, Jusselin, and Rosenbaum 2019). We believe that understanding these mechanisms is probably one of the most intellectually challenging (and exciting) issue for market microstructure theorists.

Notes
1. The intensity of a point process can be interpreted as follows: the probability of occurrence of an event of type $i$ during the time interval $[t, t + \delta t]$ is equal to $\lambda_i \delta t$.
2. One could account for exogenous events by including them as a time dependent $\sigma_0$, as in Rambaldi, Pennesi, and Lillo (2015) and Rambaldi, Filimonov, and Lillo (2018). We leave this extension for future investigations.
3. More refined definitions of the micro-price, even closer to a martingale, are discussed in Stoikov (2018).

4. For the EUROSTOXX, the fitting parameters are found to be $A = 1.7 \times 10^{-4} \, \text{s}^{-2}$, $B = 81 \, \text{s}$ and $C = 0.71$.

5. The slight abuse of notation here since the diagonal part of $\bar{K}(s)$ is in fact $\bar{K}_d(\psi) + \bar{K}_1(Z^2(s))$.

6. Also note that all three procedures presented in the paper give the same relative magnitude for the effects of the quadratic terms.

7. The normalisation of all kernels is computed with a time cut-off at 1000 s.

8. Note that the linear terms give no net contribution, i.e. $V^{LO}||\bar{L}^{LO}|| - V^{C}||\bar{L}^{C}|| - V^{MO}||\bar{L}^{MO}|| \approx 0$, which explains why we focus on the quadratic term. In other words, the trend has almost no linear effect on the liquidity flux at large time scales.

9. Changing the reference volume to $2V_{\text{best}}$ or $V_{\text{best}}/2$ does not change the qualitative conclusions below.

Acknowledgments

We thank Jonathan Donier, Iacopo Mastromatteo, José Moran, Mehdi Tomas, Stephen Hardiman and Mathieu Rosenbaum for fruitful discussions. This research was conducted within the Econophysics & Complex Systems Research Chair, under the aegis of the Fondation du Risque, the Fondation de l’Ecole polytechnique, the Ecole polytechnique and Capital Fund Management.

Disclosure statement

No potential conflict of interest was reported by the author(s).

ORCID

Michael Benzaquen http://orcid.org/0000-0002-9751-7625

References

Achab, Massil, Emmanuel Bacry, Jean-François Muzy, and Marco Rambaldi. 2018. "Analysis of Order Book Flows Using a Non-Parametric Estimation of the Branching Ratio Matrix." Quantitative Finance 18 (2): 199–212.

Alfonsi, Aurélien, and Pierre Blanc. 2016a. "Dynamic Optimal Execution in a Mixed-Market-Impact Hawkes Price Model." Finance and Stochastics 20 (1): 183–218.

Alfonsi, Aurélien, and Pierre Blanc. 2016b. "Extension and Calibration of a Hawkes-Based Optimal Execution Model." Market Microstructure and Liquidity 2 (2): Article ID: 1650005.

Bacry, Emmanuel, Khalil Dayri, and Jean-François Muzy. 2012. "Non-Parametric Kernel Estimation for Symmetric Hawkes Processes. Application to High Frequency Financial Data." The European Physical Journal B 85 (5): 157.

Bacry, Emmanuel, Sylvain Delattre, Marc Hoffmann, and Jean-François Muzy. 2013. "Modelling Microstructure Noise with Mutually Exciting Point Processes." Quantitative Finance 13 (1): 65–77.

Bacry, Emmanuel, Thibault Jaissie, and Jean-François Muzy. 2016. "Estimation of Slowly Decreasing Hawkes Kernels: Application to High-Frequency Order Book Dynamics." Quantitative Finance 16 (8): 1179–1201.

Bacry, Emmanuel, Iacopo Mastromatteo, and Jean-François Muzy. 2015. "Hawkes Processes in Finance." Market Microstructure and Liquidity 1 (1): Article ID: 1550005.

Bacry, Emmanuel, and Jean-François Muzy. 2014. "Hawkes Model for Price and Trades High-Frequency Dynamics." Quantitative Finance 14 (7): 1147–1166.

Bacry, Emmanuel, and Jean-François Muzy. 2016. "First- and Second-Order Statistics Characterization of Hawkes Processes and Non-Parametric Estimation." IEEE Transactions on Information Theory 62 (4): 2184–2202.

Blanc, Pierre, Jonathan Donier, and J.-P. Bouchaud. 2017. "Quadratic Hawkes Processes for Financial Prices." Quantitative Finance 17 (2): 171–188.

Bormetti, Giacomo, Lucio Maria Calcagnile, Michele Treccani, Fulvio Corsi, Stefano Marmi, and Fabrizio Lillo. 2015. "Modelling Systemic Price Cojumps with Hawkes Factor Models." Quantitative Finance 15 (7): 1137–1156.

Bouchaud, Jean-Philippe. 2010. "The Endogenous Dynamics of Markets: Price Impact and Feedback Loops." Preprint, arXiv:1009.2928.

Bouchaud, Jean-Philippe, Julius Bonart, Jonathan Donier, and Martin Gould. 2018. Trades, Quotes and Prices: Financial Markets Under the Microscope. Cambridge: Cambridge University Press.

Bouchaud, Jean-Philippe, Marc Mézard, and Marc Potters. 2002. "Statistical Properties of Stock Order Books: Empirical Results and Models." Quantitative Finance 2 (4): 251–256.

Calcagnile, Lucio Maria, Giacomo Bormetti, Michele Treccani, Stefano Marmi, and Fabrizio Lillo. 2018. "Collective Synchronization and High Frequency Systemic Instabilities in Financial Markets." Quantitative Finance 18 (2): 237–247.

Chicheportiche, Rémy, and Jean-Philippe Bouchaud. 2014. "The Fine-Structure of Volatility Feedback I: Multi-Scale Self-Reflexivity." Physica A: Statistical Mechanics and its Applications 410: 174–195.

Cutler, D. M., J. M. Poterba, and L. H. Summers. 1989. "What Moves Stock Prices?" Journal of Portfolio Management 15 (3): 4–12.

Dall’Amico, Lorenzo, Antoine Fosset, Jean-Philippe Bouchaud, and Michael Benzaquen. 2019. "How Does Latent Liquidity Get Revealed in the Limit Order Book?" Journal of Statistical Mechanics: Theory and Experiment 2019 (1): Article ID: 013404.
Appendices

Appendix 1. Estimation procedure

Here we show how to practically estimate the kernels presented in Section 2.2 from empirical data. First, we detail the empirical estimators for averages and covariances, then focus on the time grids used for estimation, and finally discuss the numerical discretisation of Equations (3).

Covariance estimators

We assume that we have a sample of events of type $i$ that happen at times $(T^i_n)_n$, with $i = P$ for the price process. Calling $T$ the total length of observation, the estimators of the average intensities read:

$$\Lambda^i \approx \frac{N^i_T}{T}$$ (A1a)
\[ \Delta_k \approx \frac{1}{T} \sum_n \left( \Delta r_n^k \right)^k. \] (A1b)

For the covariance estimators, we use a classical approach for asynchronous data. Denoting \( \Delta t \), \( \Delta x \) the time steps associated with times \( t \) and \( x \), one has:

\[ \chi_{ij}^{NN}(t) \approx \frac{1}{T \Delta t} \sum_{n,p} \mathbb{I}_{\left[ T_n - T_p \in \left[ t - \Delta t/2, t + \Delta t/2 \right] \right]} - \Delta^i \Lambda^j \] (A2a)

\[ \chi_{NP}^i(t) \approx \frac{1}{T \Delta t} \sum_{n,p} \Delta r_{TP}^p \mathbb{I}_{\left[ T_n - T_p \in \left[ t - \Delta t/2, t + \Delta t/2 \right] \right]} \] (A2b)

\[ \chi_{NP}^2(t) \approx \frac{1}{T \Delta t} \sum_{n,p} \left( \Delta r_{TP}^p \right)^2 \mathbb{I}_{\left[ T_n - T_p \in \left[ t - \Delta t/2, t + \Delta t/2 \right] \right]} - \Delta^i \Delta^j \chi_{NP}(t, x) \] (A2c)

\[ \chi_{PP}^{12}(t) \approx \frac{1}{T^2 \Delta t \Delta x} \sum_{n,p,q} \Delta r_{TP}^p \mathbb{I}_{\left[ T_n - T_p \in \left[ t - \Delta t/2, t + \Delta t/2 \right], T_n - T_q \in \left[ x - \Delta x/2, x + \Delta x/2 \right] \right]} \] (A2d)

\[ \chi_{PP}^{12}(t) \approx \frac{1}{T \Delta t} \sum_{n,p} \left( \Delta r_{TP}^p \right)^2 \mathbb{I}_{\left[ T_n - T_p \in \left[ t - \Delta t/2, t + \Delta t/2 \right] \right]} - \Delta^2 \] (A2e)

**Figure A1.** Hawkes kernels for the EURO STOXX futures contract between 2016/09/12 and 2020/02/07 (t in seconds).
Note that, as mentioned above, one can choose different time grids for the Hawkes and price contributions. Symmetry properties of the covariances enable us to estimate them only for positive times:

- $\chi_{NN}^\mu(-t) = \chi_{NN}^\mu(t)$
- $\chi_{NP}^\nu(t) = 0$ and $\chi_{NP}^\nu(t) = 0$ for $t < 0$
- $\chi_{NPP}^\nu(t,x) = 0$ for $\min(t,x) < 0$
- $\chi_{P2P}^\nu(-t) = \chi_{P2P}^\nu(t)$.

One can reasonably assume that the covariances are $\mathcal{C}^1$ except in zero.

Choice of time grids A good choice of time grid to estimate the kernels is provided in Bacry, Jaisson, and Muzy (2016). Indeed, quadrature points in log-scale are well suited to accurately account for long range behaviour in the norm of the kernels. Consistently, it is advised to have time intervals increasing at the same rate as the grid of points we use. On the other hand, taking disjoined intervals $[t - \Delta t/2, t + \Delta t/2]$ enables fast computations of the covariances. To enforce all of this, we compute the differences between the quadrature points, sort them and take the cumulative sum. This gives the disjoined time intervals suited for fast computations. Then, with linear interpolation, we obtain the final values on the quadrature points.

**Figure A2.** Raw effective kernels resulting from the calibration on the EURO STOXX futures contract between 2016/09/12 and 2020/02/07, without any smoothing procedure – compare with Figure 3. (a) Linear kernels $\bar{L}$. (b) Diagonal of quadratic kernels $\bar{K}_d$. (c) Full quadratic kernels $\bar{K}(t,x)$. A bootstrap error band around the MO kernel is displayed in light green.

**Figure A3.** Zumbach approximation of the effective kernel $\bar{K}$ on the EURO STOXX futures contract between 2016/09/12 and 2020/02/07 – without any smoothing procedure – compare with Figure 4. (a) Zumbach kernel $Z$. (b) Volatility kernel $\psi$. Both kernels are normalised such that $\|\psi\| = \|Z\| = 1$, with a cut-off in the time integrals at 1000 s. A bootstrap error band around the MO kernel is displayed in light green. The error band is similar for other kernels.
Discretisation Equations (3) can be discretised in two different ways, using properties of the covariances and time grids. To show how to approximate the integrals, we provide an example of discretisation of \( \int_{\mathbb{R}} f(s) \, ds \) for an arbitrary function \( f \) using the time grid \((t_n)\). The two possibilities are:

- The quadrature technique: \( \int_{\mathbb{R}} f(s) \, ds \approx \sum_n f(t_n)w_n \).
- The piece-wise \( \mathcal{C}^1 \) approximation: \( \int_{\mathbb{R}} f(s) \, ds \approx \sum_n \frac{t_{n+1}-t_n}{2} \left[ f(t_n^+) + f(t_{n+1}^-) \right] \), with \( f(x^+) = \lim_{y \to x^+} f(y) \) and \( f(x^-) = \lim_{y \to x^-} f(y) \).

The first approximation is very efficient to compute \( TrK \) or \( \|\phi\| \) using \((t^h_n)\) and \((w^h_n)\). The second handles very well the behaviour around zero and can be useful to solve Equation (4).

### Appendix 2. Additional plots and tables

**Table A1.** Quadratic, Zumbach and volatility contributions to the liquidity rate of events (in shares per second).

|                  | C    | LO   | MO   |
|------------------|------|------|------|
| \( VTrK \Delta_2 \) EUROSTOXX | 20.4 | 18.8 | 2.1  |
|                  | BUND | 5.7  | 6.6  | 1.7  |
| \( VK_1 \Delta_2 \) EUROSTOXX | 9.7  | 8.6  | 0.5  |
|                  | BUND | 3.5  | 4.6  | 0.7  |
| \( VK_d \Delta_2 \) EUROSTOXX | 10.7 | 10.1 | 1.6  |
|                  | BUND | 2.2  | 2.1  | 1.0  |

**Table A2.** Different ratios between the quadratic contributions, base rates and Hawkes contributions, truncated at different time scales \( t(s) \) for the EURO STOXX futures contract between 2016/09/12 and 2020/02/07. The top three entries are the most important ones.

|                  | C     | LO     | MO     |
|------------------|-------|--------|--------|
| \( \alpha^0_i / \Lambda \) | 10    | 0.25   | 0.14   | 0.29   |
|                  | 100   | 0.24   | 0.14   | 0.27   |
|                  | 1000  | 0.23   | 0.13   | 0.27   |
| \( \Delta_2 TrK' / \Lambda \) | 10    | 0.04   | 0.03   | 0.03   |
|                  | 100   | 0.05   | 0.03   | 0.04   |
|                  | 1000  | 0.06   | 0.04   | 0.05   |
| \( \Delta_2 TrK'' / \Lambda \) | 10    | 0.15   | 0.16   | 0.14   |
|                  | 100   | 0.23   | 0.23   | 0.21   |
|                  | 1000  | 0.28   | 0.28   | 0.24   |
| \( \alpha^0_i / \sum_j \|\phi_j\| \Lambda \) | 10    | 0.35   | 0.17   | 0.42   |
|                  | 100   | 0.34   | 0.17   | 0.40   |
|                  | 1000  | 0.32   | 0.16   | 0.40   |
| \( \Delta_2 K_1' / \sum_j \|\phi_j\| \Lambda \) | 10    | 0.06   | 0.05   | -0.01  |
|                  | 100   | 0.06   | 0.05   | -0.01  |
|                  | 1000  | 0.06   | 0.05   | -0.01  |
| \( \Delta_2 K_0' / \sum_j \|\phi_j\| \Lambda \) | 10    | -0.01  | -0.01  | 0.06   |
|                  | 100   | 0.00   | -0.01  | 0.08   |
|                  | 1000  | 0.02   | 0.00   | 0.08   |
| \( \Delta_2 K_1' / \Lambda \) | 10    | 0.13   | 0.13   | 0.05   |
|                  | 100   | 0.13   | 0.13   | 0.05   |
|                  | 1000  | 0.13   | 0.13   | 0.05   |
| \( \Delta_2 K_0' / \Lambda \) | 10    | 0.02   | 0.03   | 0.09   |
|                  | 100   | 0.10   | 0.11   | 0.16   |
|                  | 1000  | 0.15   | 0.15   | 0.19   |

Note: For sake of simplicity, we have here approximated \( K_1 \) as \( (1 - \|\phi\|)K_1 \) and \( K_d \) as \( (1 - \|\phi\|)K_d \).