On Dorfman connections of a Courant algebroid

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Dedicated to the memory of our colleague, Professor Kirill Mackenzie (1951-2020).

Abstract

We extend the Courant-Dorfman algebra of a Courant algebroid \( E \) to an algebra of differential operators on tensor products of \( E \) with values in tensor bundles of a vector bundle \( B \), predual of \( E \). Starting with a Dorfman connection of \( E \) on \( B \), this algebra provides the natural space in which the Cartan calculus and the curvatures of induced connections live. We conclude with examples of Dorfman connections arising from well known cases of Courant algebroids.

Keywords: Courant algebroid, Courant-Dorfman algebra, standard cohomology, differential operator, symbol, Dorfman connection

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1 Introduction

Since their introduction by Liu, Weinstein and Xu [29], Courant algebroids have enjoyed much attention due to their strong relation to higher structures [46, 5] and generalized geometry [6, 28], along to their applications in string theory [13, 21, 22] and T-duality [41, 42], among others. Roughly speaking, a Courant algebroid is a vector bundle $E \to M$ endowed with a non skew-symmetric bracket on $\Gamma(E)$, a bundle map $\rho : E \to TM$ and a fibrewise nondegenerate symmetric bilinear form on $E$, satisfying certain axioms similar to those of a Lie algebroid. Another important reason for receiving wide attention is their close relationship to Dirac structures. The latter were introduced in [11] to establish a unified framework for the study of presymplectic structures, Poisson structures and foliations, and provide a geometric approach to the study of constrained mechanical systems. A Dirac structure is a Lagrangian subbundle $L$ of a Courant algebroid $E$ that is closed under the non skew-symmetric bracket of $\Gamma(E)$. A pair $(E, L)$ is called Manin pair. In the formulation of graded geometry, a Courant algebroid is realized as a $NQ$-symplectic manifold of degree 2, which is something that allows one to study the differential geometry of such objects in a more concise way. For example, it has been proved [38, 40] that the standard cohomology of the standard complex $(\mathcal{C}, d)$ of the Courant-Dorfman algebra $\mathcal{C}$ of a Courant algebroid $E$ is isomorphic with the de Rham cohomology of the corresponding symplectic graded manifold.

Dorfman connections were initially introduced for dull algebroids in [20] by Jotz Lean in order to define, in the context of Courant algebroids, a structure that has similar properties to linear connections of Lie algebroids on vector bundles. This is used to study linear splittings of the standard Courant algebroid $TE \oplus T^*E$ over a vector bundle $E \to M$ and to establish the existence of a one to one correspondence of linear splittings of $TE \oplus T^*E$ with a special kind of Dorfman connections on $E \oplus T^*M$. This result generalizes the well known result of Dieudonné [14] that a linear $TM$-connection on a vector bundle $E$ defines a splitting of $TE$ in horizontal and vertical subbundles. The fact that the standard Courant algebroid over a vector bundle is a double vector bundle, places the new notion in a very interesting and rich geometric environment.

An important example of Dorfman connection is the Bott connection of a Dirac structure $L$ of a Courant algebroid $E$ on the vector bundle $E/L$ (see [20, Proposition 3.7] or Proposition 3.19 below). This connection is flat and this fact was one of our motivating points for looking at these connections closer. Our ultimate goal is to use such connections to define a notion of Atiyah class for Manin pairs suitable in the context of generalized geometry, Lie bialgebroids, etc, [4]. The study of this problem led us to the need to better understand Dorfman connection theory, and in particular Dorfman connections of a Courant algebroid $E$ on a predual vector bundle $B$. The results of this work, some of which are discussed in [4], are presented in this paper.

To be more precise, we modify the definition of Dorfman connection given in [20], replacing the dull algebroid by a Courant algebroid $E \to M$. The nonlinearity of such connections
requires a new environment for the study of the basic objects related to it and their properties. For example, because of nonlinearity, the curvature is not tensorial. Instead, it is a differential operator of order 2 on the first argument and of order 1 on the second argument. We thus construct an algebra of differential operators on spaces of tensor product of $E$, of different order on each entry, that contains Roytenberg’s Courant-Dorfman algebra \[39\]. The differential of the latter extends to this algebra of differential operators and defines a new complex and the respective cohomology related to Courant algebroid $E$. The usual operators of interior products and Lie derivatives forming a Cartan calculus for the Roytenberg’s Courant-Dorfman algebra, extend to this algebra of differential operators and they are shown to verify the corresponding identities. The covariant derivative of such a connection is then described as an extension of this differential taking values to some appropriate tensor bundle of a predual bundle $B \to M$ of $E$ (see Definition 3.1). Furthermore, the curvature is realized naturally as an object in this algebra of differential operators with values in $\text{End}(B)$, for which the Bianchi identity holds. In between, we show that (i) such a connection always exists (Proposition 3.4), (ii) the set of Dorfman connections carries affine structure (Proposition 3.5), and (iii) each Dorfman connection includes a linear connection of the predual bundle $B$ on the Courant algebroid $E$ (Proposition 3.6). We close the paper with section 3.5 providing detailed examples of Dorfman connections in settings that are well studied in the Courant algebroid literature.

In forthcoming papers we aim to develop our study on the Atiyah class of a Manin pair, to investigate the structure of the new algebra of differential operators associated to a Courant algebroid and its relation to the graded symplectic manifold corresponding to the Courant algebroid. Other points of interest are a comparison of the new cohomology of $E$ to other cohomologies of $E$ like the naive cohomology presented in \[43, 17\], and the notion of torsion of a Dorfman connection together with applications of a Dorfman connection to others theories.

Recently there has been interest towards what is called Courant algebroid connections, which are linear connections of a Courant algebroid to some vector bundle first introduced by Alekseev and Xu \[1\]. Cueca and Mehta \[12\] used the version of the standard cochain complex of the algebraic definition of a Courant algebroid defined by Keller and Waldmann in \[24\] to develop a theory of linear Courant algebroid connections in a way that mirrors the classical theory of connections. Even though Roytenberg’s complex as a starting point of this paper and Keller-Waldmann’s complex (starting point of \[12\] ) are isomorphic in the smooth category, our approaches are different and developed in parallel (see the first announcements of \[12\] and \[4\]). The difference is that (i) we use Dorfman connections whose behavior is completely different from the one of a linear Courant algebroid connection and (ii) we take into account the role of the contraction operator $i_f$ of degree $-2$ in the Cartan calculus.

Another approach based on the graded geometry for the case of exact Courant algebroids, is in \[2\].

The outline of the paper is the following. Section 2 introduces the basic setup for Courant algebroids, cohomology of the Courant-Dorfman algebra of a Courant algebroid, and cohomology of the Courant-Dorfman algebra of differential operators of a Courant algebroid. Section 3 is dedicated to Dorfman connections and the related notions of dual Dorfman connection, curvature of a Dorfman connection, ect. Along the way we provide concrete examples of the constructions.

**Notation:** Let $M$ be a smooth $n$-dimensional manifold, $TM$ and $T^*M$ its tangent and cotangent bundle, respectively, and $C^\infty(M, \mathbb{R})$ the space of smooth functions on $M$. For each $p \in \mathbb{N}$, we denote by $\mathfrak{X}^p$ and $\Omega^p$ the spaces of smooth sections of $\bigwedge^p TM$ and $\bigwedge^p T^*M$, respectively. By convention, for $p < 0$, we set $\mathfrak{X}^p = \Omega^p = \{0\}$, and $\mathfrak{X}^0 = \Omega^0 = C^\infty(M, \mathbb{R})$. Taking into account the skew-symmetry, we have $\mathfrak{X}^p = \Omega^p = \{0\}$, for $p > n$. Also, set
$\mathcal{X} = \oplus_{p \in \mathbb{Z}} \mathcal{X}^p$ and $\Omega = \oplus_{p \in \mathbb{Z}} \Omega^p$. Finally, for an arbitrary vector bundle $E \to M$, the space of smooth sections of $E$ is written as $\Gamma(E)$.

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## 2 Courant algebroids and their cohomologies

### 2.1 Courant algebroids

In [29] Liu, Weinstein and Xu introduced the notion of a *Courant algebroid* in order to generalize the notion of the Drinfel’d double of a Lie bialgebra to the notion of the *double of a Lie bialgebroid* $(A, A^*)$ defined by Mackenzie and Xu in [30]. This structure consists of a vector bundle $E \to M$ together with a skewsymmetric bracket $[\cdot, \cdot]$ on the space $\Gamma(E)$ whose "Jacobi anomaly" has an explicit expression in terms of a bundle map $E \to TM$ and a field of nondegenerate symmetric bilinear forms on $E$. It leads furthermore to a Courant algebroid structure on $E = A \oplus A^*$. If $E = TM \oplus T^*M$, the skewsymmetric bracket on $\Gamma(E)$ is given, for sections $(X, \zeta), (Y, \eta) \in \Gamma(TM \oplus T^*M)$, by the formula

$$[(X, \zeta), (Y, \eta)] = ([X, Y], \mathcal{L}_X \eta - \mathcal{L}_Y \zeta - \frac{1}{2}d(i_X \eta - i_Y \zeta)).$$

In his thesis [36], Roytenberg reformulated the notion of Courant algebroid introducing a non-skewsymmetric bracket $\langle \cdot, \cdot \rangle$ on $\Gamma(E)$ satisfying certain axioms and proved the equivalence of the two definitions. The bracket $[\cdot, \cdot]$ is derived by adding a symmetric part to the initial skewsymmetric bracket which is, in an appropriate sense, a coboundary. When $E = TM \oplus T^*M$, the new bracket is written as

$$[X + \zeta, Y + \eta] = [X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \zeta, \quad (1)$$

for $X + \zeta, Y + \eta \in \Gamma(TM \oplus T^*M)$. This coincides with the expression of the bracket considered by Dorfman in the context of complexes over Lie algebras, in order to characterize Dirac structures [16]. The non-skewsymmetric bracket $\langle \cdot, \cdot \rangle$ on $\Gamma(E)$ is named *Courant-Dorfman bracket*. For more details about the history of Courant algebroids, one may consult the paper of Kosmann-Schwarzbach [27]. After the remarks of Uchino [45], a Courant algebroid is defined as follows.

**Definition 2.1** A Courant algebroid over a smooth manifold $M$ is a constant rank vector bundle $E$ over $M$ equipped with: (i) a fibrewise nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ on the bundle, (ii) a $\mathbb{R}$-bilinear bracket $[\cdot, \cdot]$ on $\Gamma(E)$, and (iii) a smooth vector bundle map $\rho : E \to TM$, called the anchor map\(^1\), with the following properties:

1. The bracket $[\cdot, \cdot]$ satisfies the Jacobi identity in Leibniz form

$$[e_1, [e_2, e_3]] = [[e_1, e_2], e_3] + [e_1, [e_2, e_3]], \quad \text{for any } e_1, e_2, e_3 \in \Gamma(E). \quad (2)$$

2. The structures $\langle \cdot, \cdot \rangle$ and $[\cdot, \cdot]$ on $E$ are compatible in the sense that, for $e_1, e_2, e_3 \in \Gamma(E)$,

$$\rho(e_1)\langle e_2, e_3 \rangle = \langle [e_1, e_2], e_3 \rangle + \langle e_2, [e_1, e_3] \rangle. \quad (3)$$

\(^1\)We note also by $\rho$ the induced map by $\rho : E \to TM$ on the spaces of smooth sections, from $\Gamma(E)$ to $\Gamma(TM)$.
3. For \( e_1, e_2 \in \Gamma(E) \),
\[
[e_1, e_2] + [e_2, e_1] = d_E(e_1, e_2),
\]
where \( d_E : C^\infty(M, \mathbb{R}) \to \Gamma(E) \) is the map defined, for \( f \in C^\infty(M, \mathbb{R}) \) and \( e \in \Gamma(E) \), by
\[
\langle dEf, e \rangle = \langle e, dEf \rangle = \rho(e)(f).
\]

**Identifying the dual vector bundle** \( E^* \) **with** \( E \) **via** \( \langle \cdot, \cdot \rangle \), **we write** \( d_E = \rho^* \circ d. \)

From the above axioms, we get [45]:

4. The anchor map \( \rho : (\Gamma(E), \langle \cdot, \cdot \rangle) \to (\Gamma(TM), [\cdot, \cdot]) \) is a homomorphism, i.e.
\[
\rho([e_1, e_2]) = [\rho(e_1), \rho(e_2)].
\]

5. The right Leibniz identity is satisfied:
\[
[e_1, fe_2] = f[e_1, e_2] + \rho(e_1)(f)e_2.
\]

Furthermore, the left Leibniz identity takes the form
\[
[e_1, e_2] = f[e_1, e_2] - \rho(e_2)(f)e_1 + \langle e_1, e_2 \rangle d_E f.
\]

For \( e \in \Gamma(E) \) and \( \alpha \in \Omega^1 \), with a simple calculation one also derives the identities:

6. \( \rho \circ \rho^* = 0 \),
7. \( [e, \rho^*(\alpha)] = \rho^*(\mathcal{L}_{\rho(e)}\alpha) \),
8. \( [\rho^*(\alpha), e] = -\rho^*(i_{\rho(e)}d\alpha) \),

where \( \mathcal{L} \) and \( i \) denote, respectively, the classical Lie derivative and the contraction of differential forms with vector fields.

Finally, the initial skewsymmetric bracket \( [\cdot, \cdot] \) and the non-skewsymmetric bracket \( \langle \cdot, \cdot \rangle \) on \( \Gamma(E) \) are related by the formula
\[
[e_1, e_2] = [e_1, e_2] + \frac{1}{2}d_E(e_1, e_2).
\]

Below, we give some classical examples of Courant algebroids.

**Example 2.2 (Courant algebroid over a point)** A Courant algebroid over a point, i.e. \( M = \{p\} \), is just a quadratic Lie algebra \( g \), that is a Lie algebra endowed with a nondegenerate symmetric bilinear form \( \langle \cdot, \cdot \rangle \) invariant under the adjoint representation, namely, \( \langle \text{ad}_u v, w \rangle + \langle v, \text{ad}_u w \rangle = 0 \), for all \( u, v, w \in g \).

**Example 2.3 (Standard Courant algebroid)** Consider the vector bundle \( E = TM \oplus T^*M \) over \( M \) equipped with: (i) the nondegenerate symmetric fibrewise bilinear form \( \langle \cdot, \cdot \rangle \) given, at each point \( x \in M \) and for all \( (X, \zeta), (Y, \eta) \in T_xM \oplus T^*_xM \), by
\[
\langle (X, \zeta), (Y, \eta) \rangle = \langle \eta, X \rangle + \langle \zeta, Y \rangle,
\]
(ii) the vector bundle map \( \rho : TM \oplus T^*M \to TM \) projecting on the first summand, (iii) the Courant–Dorfman bracket (1) on the space \( \Gamma(E) \) of smooth sections of \( E \), and (iv) the map \( d_E : C^\infty(M, \mathbb{R}) \to \Gamma(TM \oplus T^*M) \) defined by \( d_Ef = (0, df) \). The above data define a Courant algebroid structure on \( TM \oplus T^*M \) which is called standard.

\(^2\)The precise definition of \( d_E \) is \( d_E = (g^*)^{-1} \circ \rho^* \circ d \), where \( g^* : E \to E^* \) is the vector bundle map defined by \( \langle \cdot, \cdot \rangle \).
Example 2.4 (The double of a Lie bialgebroid) Let \((\mathcal{A}, [\cdot, \cdot]_A, (\mathcal{A}^*, [\cdot, \cdot]_{\mathcal{A}^*}, a_*)\) be a Lie bialgebroid over a smooth manifold \(M\). This is a pair of Lie algebroids in duality verifying, for all \(X, Y \in \Gamma(A)\), the compatibility condition

\[ d_\ast [X, Y]_A = [d_\ast X, Y]_A + [X, d_\ast Y]_A, \]

where \(d_\ast\) denotes the differential operator defined on \(\Gamma(\mathcal{A})\) by the Lie algebroid structure of \(\mathcal{A}^*\) \([30, 31, 25]\). The vector bundle \(E = A \oplus \mathcal{A}^*\) has a Courant algebroid structure defined by (i) the natural nondegenerate bilinear form

\[ \langle X + \zeta, Y + \eta \rangle = \langle \zeta, Y \rangle + \langle \eta, X \rangle, \]

\(X + \zeta, Y + \eta \in \Gamma(E)\),

(ii) the anchor map \(\rho = a + a_\ast\), (iii) the operator \(d_E = d_\ast + d\), where \(d : C^\infty(M, \mathbb{R}) \to \Gamma(\mathcal{A}^*)\) is the usual differential operator associated to Lie algebroid structure of \(\mathcal{A}\), and (iv) the bracket

\[ [X + \zeta, Y + \eta] = ([X, Y]_A + \mathcal{L}_\ast \zeta Y - i_{\eta} d_\ast X) + ([\zeta, \eta]_{\mathcal{A}^*} + \mathcal{L}_X \eta - i_Y d\zeta). \]

In the case where \((\mathcal{A}, \mathcal{A}^*)\) is a Lie-quasi, a quasi-Lie, or a proto-bialgebroid, the vector bundle \(E = A \oplus \mathcal{A}^*\) has again a Courant algebroid structure, \([37, 26]\).

2.2 Cohomologies of a Courant algebroid

In \([39]\) Roytenberg defined and studied the notion of Courant-Dorfman algebra which is an algebraic analogue of Courant algebroids. The relation is analogous to that of Lie-Rinehart algebras to Lie algebroids and Poisson algebras to Poisson manifolds \([19]\).

A Courant-Dorfman algebra consists of a commutative algebra \(\mathcal{R}\), an \(\mathcal{R}\)-module \(\mathcal{E}\) equipped with a pseudo-metric \(\langle \cdot, \cdot \rangle\), an \(\mathcal{E}\)-valued derivation \(\partial\) of \(\mathcal{R}\) and a Courant-Dorfman bracket \([\cdot, \cdot]\) satisfying compatibility conditions generalizing those defining a Courant algebroid. Given a Courant-Dorfman algebra \((\mathcal{E}, \mathcal{R}, [\cdot, \cdot], \langle \cdot, \cdot \rangle, \partial)\) (or just \((\mathcal{E}, \mathcal{R})\)), a certain graded commutative \(\mathcal{R}\)-algebra \(\mathcal{C}(\mathcal{E}, \mathcal{R})\) endowed with a differential \(d\) is then defined and the resulting cochain complex \((\mathcal{C}(\mathcal{E}, \mathcal{R}), d)\) is called the standard complex of \((\mathcal{E}, \mathcal{R})\). It is an analogue of the de Rham complex of a Lie-Rinehart algebra for a Courant-Dorfman algebra. For our goals, we first recall the notions of universal enveloping and convolution algebra, and we present the structure of the Courant-Dorfman algebra of a Courant algebroid \((\mathcal{E}, [\cdot, \cdot], \langle \cdot, \cdot \rangle, \rho)\) following \([39]\).

2.2.1 Universal enveloping and convolution algebra

Let \(\mathbb{K}\) be a commutative ring containing \(\frac{1}{2}\), \(V\) and \(W\) two \(\mathbb{K}\)-modules and \(\langle \cdot, \cdot \rangle : V \otimes V \to W\) a symmetric bilinear form. Consider the graded \(\mathbb{K}\)-module \(L = V[1] \oplus W[2]\) and define the non-trivial brackets to be \(-\langle \cdot, \cdot \rangle\). Then \(L\) becomes a graded Lie algebra over \(\mathbb{K}\). Let \(J\) be the homogeneous ideal of the tensor algebra \(T(L)\) that is generated by elements of the form

\[ v_1 \otimes v_2 + v_2 \otimes v_1 + (v_1, v_2), \quad v \otimes w - w \otimes v, \quad w_1 \otimes w_2 - w_2 \otimes w_1. \]

By definition, the universal enveloping algebra of \(L\) is \(U(L) = T(L)/J\) and \(U(L)\) carries a natural filtration: Let \(S(W) = \bigoplus_{k \geq 0} S^k(W)\) be the symmetric algebra of \(W\) and define, for \(p \geq 0\),

\[ U(L)_{-p} = \bigoplus_{k=0}^{[\frac{p}{2}]} (V^\otimes (p-2k) \otimes S^kW) / R, \]
where $R$ is the submodule generated by elements of the form

$$v_1 \otimes \ldots \otimes v_i \otimes v_{i+1} \otimes \ldots \otimes v_{p-2k} \otimes w_1 \ldots w_k + v_1 \otimes \ldots \otimes v_{i+1} \otimes v_i \otimes \ldots \otimes v_{p-2k} \otimes w_1 \ldots w_k + v_1 \otimes \ldots \otimes \hat{v}_i \otimes \hat{v}_{i+1} \otimes \ldots \otimes v_{p-2k} \otimes (v_i, v_{i+1})w_1 \ldots w_k,$$

with $i = 1, \ldots, p - 2k - 1$ and $k = 0, \ldots, \left[\frac{p}{2}\right]$. The associated graded of the filtration $\cdots \subset U_{-p+1} \subset U_{-p} \subset U_{-p-1} \subset \cdots$ is $\text{gr} U(L) = \bigoplus_p U(L)_{-p}/U(L)_{-p+1}$. Since $U(L)$ is a (graded cocommutative) $\mathbb{K}$-coalgebra and $R$ is a $\mathbb{K}$-algebra, the space $\mathcal{A} = \mathcal{A}(V, W; R) = \text{Hom}_\mathbb{K}(U(L), R)$ is an associative algebra equipped with the convolution product and is called the 

convolution algebra of $U(L)$. Since $U(L)$ is non-positively graded one gets that $\mathcal{A}$ is non-negatively graded. Each element of $\mathcal{A}^p = \text{Hom}_\mathbb{K}(U(L)_p, R)$ is determined by $([\frac{p}{2}] + 1)$-tuples

$$\omega = (\omega_0, \omega_1, \ldots, \omega_{\left[\frac{p}{2}\right]})$$

of homomorphisms

$$\omega_k : V^\otimes p - 2k \otimes W^\otimes k \rightarrow R.$$

By construction, any $\omega_k$ is symmetric in the $W$-arguments and satisfies

$$\omega_k(v_1, v_2, \ldots, v_i, v_{i+1}, \ldots) + \omega_k(v_1, v_{i+1}, v_i, v_{i+2}, \ldots) = -\omega_{k+1}(v_1, \hat{v}_i, \hat{v}_{i+1}, \ldots, v_i, v_{i+1}, \ldots),$$

for $v_i, v_j \in V$ and $i = 1, \ldots, p - 2k$. Equivalently, each $\omega_k$ defines a map

$$\omega_k : V^\otimes p - 2k \rightarrow \text{Hom}_\mathbb{K}(S^k W, R).$$

Similarly, since $S(W[2])$ is a coalgebra (concentrated in even non-positive degrees), the space $\text{Hom}_\mathbb{K}(S(W[2]), R)$ is an algebra with multiplication given, for $H \in \text{Hom}_\mathbb{K}(S^p(W[2]), R)$, $K \in \text{Hom}_\mathbb{K}(S^q(W[2]), R)$, by

$$(H \cdot K)(w_1, \ldots, w_{p+q}) = \sum_{\tau \in \text{sh}(p, q)} H(w_{\tau(1)}, \ldots, w_{\tau(p)})K(w_{\tau(p+1)}, \ldots, w_{\tau(p+q)}).$$

Here and henceforth, $\text{sh}(p, q)$ is the set of $(p, q)$-shuffle permutations of $1, \ldots, p + q$, i.e., of permutations $\tau$ such that $\tau(1) < \ldots < \tau(p)$ and $\tau(p + 1) < \ldots < \tau(p + q)$. This leads to the following formula for the multiplication in $\mathcal{A}$:

$$(\omega \cdot \eta)_k(v_1, \ldots, v_{p+q-2k}) = \sum_{i + j = k} \sum_{\sigma \in \text{sh}(p-2i, q-2j)} (-1)^{\sigma} \omega_i(v_{\sigma(1)}, \ldots, v_{\sigma(p-2i)})\eta_j(v_{\sigma(p-2i+1)}, \ldots, v_{\sigma(p+q-2k)}),$$

where $(-1)^{\sigma}$ is the signature of $\sigma$ and the multiplication in each summand takes place in $\text{Hom}_\mathbb{K}(S(W[2]), R)$. In particular, for $k = 0$,

$$(\omega \cdot \eta)_0(v_1, \ldots, v_{p+q}) = \sum_{\sigma \in \text{sh}(p, q)} (-1)^{\sigma} \omega_0(v_{\sigma(1)}, \ldots, v_{\sigma(p)})\eta_0(v_{\sigma(p+1)}, \ldots, v_{\sigma(p+q)}),$$

where the multiplication in each summand takes place in $R$. 

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2.2.2 The cohomology of the Courant-Dorfman algebra of a Courant algebroid

We now define the standard complex of a Courant-Dorfman algebra associated to a Courant algebroid \((E, [\cdot, \cdot], \langle \cdot, \cdot \rangle, \rho)\). Let \(\mathcal{R} = C^\infty(M, \mathbb{R})\), \(\mathcal{E} = \Gamma(E^*) \cong \Gamma(E)\), \(\partial = d_E\), and \(\Omega^1 = \Gamma(T^*M)\). We thus have a metric \(\mathcal{R}\)-module \((\mathcal{E}, \langle \cdot, \cdot \rangle)\) and a symmetric bilinear form \(\langle \cdot, \cdot \rangle : \mathcal{E} \times \mathcal{E} \to \Omega^1\) determined by \(\langle \cdot, \cdot \rangle = \tilde{d} \langle \cdot, \cdot \rangle\). The graded \(\mathcal{R}\)-module \(L = \mathcal{E}[1] \oplus \Omega^1[2]\) is a graded Lie algebra over \(\mathcal{R}\) with the nontrivial brackets determined by \(-\langle \cdot, \cdot \rangle\). The universal enveloping \(\mathcal{R}(L)\) and corresponding convolution algebra \(\mathcal{A} = \mathcal{A}(\mathcal{E}, \Omega^1; \mathcal{R}) = \text{Hom}_\mathbb{R}(\mathcal{U}(L), \mathcal{R})\) are also defined as above. Set \(C^0 = \mathcal{R}\) and, for each \(p > 0\), let \(C^p \subset \mathcal{A}^p\) be the submodule consisting of elements \(\tilde{\omega} = (\tilde{\omega}_0, \tilde{\omega}_1, \ldots, \tilde{\omega}_{[p]}\) such that each

\[\tilde{\omega}_k : \mathcal{E}^{p-2k} \otimes \Omega^1_k \to \mathcal{R}\]

satisfies the following two additional conditions:

1. \(\tilde{\omega}_k : \mathcal{E}^{p-2k} \to \text{Hom}_\mathbb{R}(S^k \Omega^1, \mathcal{R})\) takes values in \(\text{Hom}_\mathbb{R}(S^k \Omega^1, \mathcal{R}) \subset \text{Hom}_\mathbb{R}(S^k \Omega^1, \mathcal{R})\), where \(S^k \Omega^1\) is the \(\mathcal{R}\)-module of the \(k\)-symmetric power of the \(\mathcal{R}\)-module \(\Omega^1\) and \(\text{Hom}_\mathbb{R}(S^k \Omega^1, \mathcal{R})\) is the space of \(\mathcal{R}\)-linear maps \(S^k \Omega^1 \to \mathcal{R}\).

2. \(\tilde{\omega}_k : \mathcal{E}^{p-2k} \to \text{Hom}_\mathbb{R}(S^k \Omega^1, \mathcal{R})\) is \(\mathcal{R}\)-linear in the \((p - 2k)\)-th argument of \(\mathcal{E}^{p-2k}\).

It is shown in [39], by induction and using (4), that, for all \(1 \leq i < p - 2k\) and \(f \in C^\infty(M, \mathbb{R})\),

\[
\tilde{\omega}_k(e_1, \ldots, fe_i, \ldots, \ldots) = f\tilde{\omega}_k(e_1, \ldots, e_i, \ldots, \ldots)
+ \sum_{j=1}^{p-2k-i} (-1)^j \langle e_i, e_{i+j} \rangle \tilde{\omega}_{k+1}(e_1, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, df, \ldots).
\]

From the above one concludes that each term of the sequence \(\tilde{\omega} = (\tilde{\omega}_0, \tilde{\omega}_1, \ldots, \tilde{\omega}_{[p]}\) \(\in C^p\) is a first-order differential operator in the first \(p - 2k - 1\) arguments (see Definition 2.11) and \(\mathcal{R}\)-linear in the \((p - 2k)\)-th argument.

It is easy to see that the space \(\text{Hom}_\mathbb{R}(S^k \Omega^1, \mathcal{R})\) is identified with the \(\mathcal{R}\)-module \(\mathcal{X}^k\) of symmetric \(k\)-derivations of \(\mathcal{R}\). In other words this is the space of symmetric functions on \(\mathcal{R}^{\otimes k}\) with values in \(\mathcal{R}\) which are derivations in each argument. Hence, the image \(\tilde{\omega}_k(e_1, \ldots, e_{p-2k})\) of \((e_1, \ldots, e_{p-2k}) \in \mathcal{E}^{p-2k}\) can be viewed as either a symmetric \(k\)-derivation of \(\mathcal{R}\) whose value on \(f_1, \ldots, f_k \in \mathcal{R}\) will be denoted by

\[\omega_k(e_1, \ldots, e_{p-2k}; f_1, \ldots, f_k),\]

or as a symmetric \(\mathcal{R}\)-multilinear function on \(S^k \Omega^1\) whose value on a \(k\)-tuple \((\alpha_1, \ldots, \alpha_k)\) of elements of \(\Omega^1\) will be denoted by

\[\tilde{\omega}_k(e_1, \ldots, e_{p-2k}; \alpha_1, \ldots, \alpha_k).\]

It is then obvious that

\[\tilde{\omega}_k(e_1, \ldots, e_{p-2k}; df_1, \ldots, df_k) = \omega_k(e_1, \ldots, e_{p-2k}; f_1, \ldots, f_k)\]

and so in the following we will interchange between the two realizations of elements of \(C^p\) without other notice.

**Definition 2.5** The graded subalgebra \(\mathcal{C} = \mathcal{C}(\mathcal{E}, \mathcal{R}) = \langle \mathcal{C}^p \rangle_{p \geq 0}\) of \(\mathcal{A}(\mathcal{E}, \Omega^1; \mathcal{R})\) is called the Courant-Dorfman algebra of the Courant algebroid \((E, [\cdot, \cdot], \langle \cdot, \cdot \rangle, \rho)\).
Define the map
\[ d : C^\bullet \rightarrow C^{\bullet+1} \] (7)
by setting, for all \( \omega = (\omega_0, \omega_1, \ldots, \omega_{\frac{p}{2}}) \in C^p, \ p \geq 0, \)
\[ d\omega = ((d\omega)_0, (d\omega)_1, \ldots, (d\omega)_{\frac{p+1}{2}}) \in C^{p+1}, \]
where, for any \( k = 0, \ldots, \left[ \frac{p+1}{2} \right], \)
\[ (d\omega)_k(e_1, \ldots, e_{p+1-2k}; f_1, \ldots, f_k) = \]
\[ \sum_{\mu=1}^{k} \omega_{k-1}(d\rho_\mu, e_1, \ldots, e_{p+1-2k}; f_1, \ldots, f_k) \]
\[ + \sum_{i=1}^{p+1-2k} (-1)^{i-1} \langle e_i, d\rho_\mu(\omega_{k-1}(e_1, \ldots, e_{p+1-2k}; f_1, \ldots, f_k)) \rangle \]
\[ + \sum_{i<j} (-1)^j \omega_k(e_1, \ldots, \hat{e_i}, \ldots, \hat{e_j}, \ldots, e_{p+1-2k}; f_1, \ldots, f_k). \] (8)

On the other hand, \( \overline{d\omega} = ((\overline{d\omega})_0, (\overline{d\omega})_1, \ldots, (\overline{d\omega})_{\frac{p+1}{2}}) \in C^{p+1}, \ p \geq 0, \) is given \([39, \text{Corollary 4.9}]\) for any \( \alpha_1, \ldots, \alpha_k \in \Omega^1, \) by the formula
\[ (\overline{d\omega})_k(e_1, \ldots, e_{p+1-2k}; \alpha_1, \ldots, \alpha_k) = \]
\[ \sum_{\mu=1}^{k} \overline{\omega}_{k-1}(\rho_\mu^*(\alpha_\mu), e_1, \ldots, e_{p+1-2k}; \alpha_1, \ldots, \alpha_k) \]
\[ + \sum_{i=1}^{p+1-2k} (-1)^{i-1} \rho(e_i)(\overline{\omega}_k(e_1, \ldots, \hat{e_i}, \ldots, e_{p+1-2k}; \alpha_1, \ldots, \alpha_k)) \]
\[ + \sum_{i<j} (-1)^j \overline{\omega}_k(e_1, \ldots, \hat{e_i}, \ldots, \hat{e_j}, \ldots, e_{p+1-2k}; \alpha_1, \ldots, \alpha_k). \]

Lemma 2.6 Let \( \omega = (\omega_0, \ldots, \omega_{\frac{p}{2}}) \in C^p \) and \( \eta = (\eta_0, \ldots, \eta_{\frac{p}{2}}) \in C^q. \) The map (7) satisfies the Leibniz identity
\[ d(\omega \cdot \eta) = d\omega \cdot \eta + (-1)^p \omega \cdot d\eta. \]

Proof. A straightforward computation shows that
\[ d(\omega \cdot \eta) = ((d(\omega \cdot \eta))_0, \ldots, (d(\omega \cdot \eta))_{\frac{p+q+1}{2}}), \]
where, for any \( k = 0, \ldots, \left[ \frac{p+q+1}{2} \right], \)
\[ (d(\omega \cdot \eta))_k = (d\omega \cdot \eta)_k + (-1)^p (\omega \cdot d\eta)_k. \]

Proposition 2.7 ([39]) The operator \( d \) is a derivation of degree \( +1 \) of \( C(\mathcal{E}, \mathcal{R}) \) and squares to zero.
The complex \((\mathcal{C}(\mathcal{E}, \mathcal{R}), d)\) is named \textit{standard complex} of \((\mathcal{E}, \mathcal{R})\) and its \(p\)-th cohomology group is denoted by \(H^p(\mathcal{E}, \mathcal{R})\).

Define next two inner products in \(\mathcal{C}\). For \(\alpha \in \Omega^1\), consider the operator \(i_\alpha : \mathcal{C} \to \mathcal{C}[-2]\) defined, for \(\bar{\omega} = (\bar{\omega}_0, \ldots, \bar{\omega}[\frac{p}{2}]) \in \mathcal{C}^p\), by
\[
(i_\alpha \bar{\omega})_k = ((i_\alpha \bar{\omega})_0, \ldots, ((i_\alpha \bar{\omega})_{k-1})),
\]
where
\[
(i_\alpha \bar{\omega})_k(e_1, \ldots, e_{p-2-2k}; \alpha_1, \ldots, \alpha_k) = \bar{\omega}_{k+1}(e_1, \ldots, e_{p-2(k+1)}; \alpha, \alpha_1, \ldots, \alpha_k).
\]
For \(f \in \mathcal{R}\), define similarly the operator \(i_f\) so that
\[
(i_f \omega)_k = (i_f \omega)_0, \ldots, (i_f \omega)_{k-1})
\]
where
\[
(i_f \omega)_k(e_1, \ldots, e_{p-2-2k}; \phi_1, \ldots, \phi_k) = \omega_k(e_1, \ldots, e_{p-2(k-1)}; \phi_1, \ldots, \phi_k).
\]

\textbf{Lemma 2.8} Let \(f \in \mathcal{R}\), \(e \in \mathcal{E}\), \(\omega = (\omega_0, \ldots, \omega[\frac{p}{2}]) \in \mathcal{C}^p\) and \(\eta = (\eta_0, \ldots, \eta[\frac{q}{2}]) \in \mathcal{C}^q\). The operators \(i_f, i_e\) are derivations of degree \(-2\) and \(-1\) respectively, satisfying the Leibniz rules
\[
i_f(\omega \cdot \eta) = (i_f \omega) \cdot \eta + \omega \cdot (i_f \eta),
\]
\[
i_e(\omega \cdot \eta) = (i_e \omega) \cdot \eta + (-1)^p \omega \cdot (i_e \eta).
\]

\textbf{Proof.} Use formulæ \((6), (10), (9)\) and \((12)\) to show that as an element of \(\mathcal{C}\), it is \(i_f(\omega \cdot \eta) = ((i_f(\omega \cdot \eta))_0, \ldots, ((i_f(\omega \cdot \eta))_{\frac{p+q-2}{2}})).\) A straightforward computation shows that
\[
(i_f(\omega \cdot \eta))_k = ((i_f \omega) \cdot \eta)_k + (\omega \cdot (i_f \eta))_k,
\]
for \(k = 0, \ldots, [\frac{p+q-2}{2}]\). Similarly, it is \(i_e(\omega \cdot \eta) = ((i_e \omega \cdot \eta))_0, \ldots, ((i_e \omega \cdot \eta))_{\frac{p+q-2}{2}}),\) where
\[
(i_e(\omega \cdot \eta))_k = ((i_e \omega) \cdot \eta)_k + (-1)^p (\omega \cdot (i_e \eta))_k.
\]

Recall that the \(\mathbb{K}\)-module \(L' = \mathcal{E}[1] \oplus \mathcal{R}[2]\) is a graded Lie algebra over \(\mathbb{K}\) with the non-trivial brackets given by \(-\langle \cdot, \cdot \rangle\). Let \(\langle \cdot, \cdot \rangle\) be the graded commutator on the space of graded endomorphisms of \(\mathcal{C}\). If \(P\) and \(Q\) are two graded endomorphisms of degree \(p\) and \(q\), respectively, then the graded endomorphism
\[
\{P, Q\} = P \circ Q - (-1)^{pq} Q \circ P
\]
is of degree \(p + q\). Thus, there is a graded Lie algebra representation \(i : L' \to \text{End}(\mathcal{C})\) of \(L'\) in the space \(\text{End}(\mathcal{C})\) of endomorphisms of \(\mathcal{C}\) defined by the assignments \(\mathcal{E} \ni e \mapsto i_e\) and \(\mathcal{R} \ni f \mapsto i_f\). By construction, the commutation relations are
\[
\{i_e, i_f\} = -\{i_f, i_e\} = 0 \quad \text{and} \quad \{i_f, i_g\} = -\{i_g, i_f\} = 0.
\]
Commuting the inner products \(i_e\) and \(i_f\) with the derivation \(d\), we define the corresponding Lie derivatives:
\[
\mathcal{L}_e = \{i_e, d\} = i_e \circ d + d \circ i_e \quad \text{and} \quad \mathcal{L}_f = \{i_f, d\} = i_f \circ d - d \circ i_f.
\]
Lemma 2.9 The following Cartan’s commutation relations hold:

1. \( \mathcal{L}_f = i_{d_E f} \)

2. \( \{ \mathcal{L}_f, i_e \} = \mathcal{L}_f \circ i_e + i_e \circ \mathcal{L}_f = i_{d_E f} \circ i_e + i_e \circ i_{d_E f} = i_{-d_E f, e} \) \( \Rightarrow \)

3. \( \{ \mathcal{L}_e, i_f \} = \mathcal{L}_e \circ i_f - i_f \circ \mathcal{L}_e = -\{ \mathcal{L}_f, i_e \} - \{ \{ i_f, i_e \} \}, d \) \( \Rightarrow \)

4. \( \{ \mathcal{L}_{e_1}, i_{e_2} \} = i_{[e_1, e_2]} \)

5. \( \{ \mathcal{L}_f, \mathcal{L}_g \} = 0 \)

6. \( \{ \mathcal{L}_e, \mathcal{L}_f \} = \{ \mathcal{L}_e, i_{d_E f} \} = i_{[e, d_E f]} = i_{d_E (e, d_E f)} = \mathcal{L}_{(e, d_E f)} \)

7. \( \{ \mathcal{L}_f, \mathcal{L}_e \} = -\{ \mathcal{L}_e, \mathcal{L}_f \} = -i_{d_E (e, d_E f)} = -\mathcal{L}_{(e, d_E f)} \)

8. \( \{ \mathcal{L}_{e_1}, \mathcal{L}_{e_2} \} = \mathcal{L}_{[e_1, e_2]} \)

Proof. Direct computation.

Comment 2.10 There is an alternative description of the complex \((\mathcal{C}, d)\) of the Courant-Dorfman algebra \(\mathcal{C}\) of a Courant algebroid \((E, [], \langle \cdot, \cdot \rangle, \rho)\) that uses graded geometry. More precisely there is a \(1-1\) correspondence between vector bundles equipped with pseudo-Euclidean forms and symplectic \(N\)-manifolds of degree 2. In particular, Courant algebroids \(E\) are in \(1-1\) correspondence with symplectic \(NQ\)-manifolds \((M, \Omega)\) of degree 2; \(\Omega\) is the symplectic form, and the data \([[], \langle \cdot, \cdot \rangle, \rho]\) of \(E\) are encoded in a cubic function \(\Theta\) of degree 3 satisfying the structure equation

\[ \{\Theta, \Theta\} = 0. \]

In the latter, the bracket \([\cdot, \cdot]\) is the Poisson bracket of degree \(-2\) defined by \(\Omega\) on the graded algebra \(\mathcal{O}\) of polynomial functions. The operator \(D = \{\Theta, \cdot\}\) defines a differential on \(\mathcal{O}\) and it can be shown that the standard complex of the Courant algebroid is isomorphic to the Poisson dg algebra \((\mathcal{O}, D)\). On the other hand, there is a natural Poisson structure on \(\mathcal{C}\) for which \((\mathcal{C}, d) \simeq (\mathcal{O}, D)\) as Poisson dg algebras. Details and proofs can be found in \([38, 37, 40, 39]\).

2.2.3 The cohomology of the Courant-Dorfman algebra of differential operators of a Courant algebroid

For the rest of our work, we introduce the following type of differential operators. Let \(E_i \to M\), \(i = 1, \ldots, k\), and \(B \to M\) be smooth vector bundles of constant rank over a smooth manifold \(M\), and \(\Gamma(E_i), i = 1, \ldots, k\), \(\Gamma(B)\) the corresponding spaces of smooth sections viewed as \(\mathcal{R}\)-modules.

Definition 2.11 We say that an operator \(D \in \text{Hom}_\mathcal{K}(\Gamma(E_1 \otimes \ldots \otimes E_k), \Gamma(B))\) is a differential operator of order \(s\) in the \(i\)-argument, \(i = 1, \ldots, k\), if its \(i\)-symbol \(\sigma_i(D)(f) \in \text{Hom}_\mathcal{K}(\Gamma(E_1 \otimes \ldots \otimes E_i \otimes \ldots \otimes E_k), \Gamma(B))\) given, for any \((e_1, \ldots, e_i, \ldots, e_k) \in \Gamma(E_1 \otimes \ldots \otimes E_i \otimes \ldots \otimes E_k)\) and \(f \in \mathcal{R}\), by

\[ \sigma_i(D)(f)(e_1, \ldots, e_i, \ldots, e_k) := D(e_1, \ldots, fe_i, \ldots, e_k) - fD(e_1, \ldots, e_i, \ldots, e_k), \]

is a \((s - 1)\)-order differential operator on the \(i\)-argument of \(\Gamma(E_1 \otimes \ldots \otimes E_k)\).
For each $p > 0$ and $m \in \mathbb{N}^*$, consider the submodule $\mathcal{D}_{m,m-1}^p$ of $\mathcal{A}^p$ consisting of elements $\tilde{\omega} = (\tilde{\omega}_0, \tilde{\omega}_1, \ldots, \tilde{\omega}_p)$ such that each

$$\tilde{\omega}_k : \mathcal{E}^{p-2k} \otimes S^k \Omega^1 \to \mathcal{R}$$

satisfies the following two additional conditions:

- **Condition 1**: $\tilde{\omega}_k : \mathcal{E}^{p-2k} \to \text{Hom}_{\mathcal{E}}(S^k \Omega^1, \mathcal{R})$ takes values in $\text{Diff}_m(S^k \Omega^1, \mathcal{R}) \subset \text{Hom}_{\mathcal{E}}(S^k \Omega^1, \mathcal{R})$, where $S^k \Omega^1$ is the $\mathcal{R}$-module of the $k$-symmetric power of the $\mathcal{R}$-module $\Omega^1$ and

  $$\text{Diff}_m(S^k \Omega^1, \mathcal{R}) = \{ D \in \text{Hom}_{\mathcal{E}}(S^k \Omega^1, \mathcal{R}) / D \text{ is a differential operator of order at most } m \text{ in each entry} \}.$$ 

- **Condition 2**: $\tilde{\omega}_k : \mathcal{E}^{p-2k} \to \text{Diff}_m(S^k \Omega^1, \mathcal{R})$ is a differential operator of order at most $m$ on the first $p - 2k - 1$ arguments and of order at most $m - 1$ on the $(p - 2k)$-th argument.

Note that for $p = 1$, $\tilde{\omega} = (\tilde{\omega}_0)$ and the unique argument of the map $\tilde{\omega}_0 : \mathcal{E} \to \mathcal{R}$ is considered as first argument. In particular, the spaces $\mathcal{D}_m^p$ of differential operators on $\mathcal{E}$ of order at most $m$, $m \in \mathbb{N}^*$, fit in the series

$$\mathcal{C}^1 = \mathcal{D}_0^1 \subset \mathcal{D}_1^1 \subset \ldots \subset \mathcal{D}_m^1 \subset \ldots,$$

and, for any $p > 1$ and $m \in \mathbb{N}^*$,

$$\mathcal{C}^p = \mathcal{D}_{1,0}^p \subset \ldots \subset \mathcal{D}_{m,m-1}^p \subset \ldots.$$ 

For any $(e_1,\ldots,e_{p-2k}) \in \mathcal{E}^{p-2k}$, $\tilde{\omega}_k(e_1,\ldots,e_{p-2k})$ can be viewed as a symmetric map on $\Omega^{1\otimes k}$ acting as a differential operator of order at most $m$ on each entry; its value on a $k$-tuple $(\alpha_1,\alpha_2,\ldots,\alpha_k)$ is denoted by

$$\tilde{\omega}_k(e_1,\ldots,e_{p-2k};\alpha_1,\ldots,\alpha_k).$$

We adopt the convention presented in the previous paragraph and write for any $k$-tuple of type $(df_1,df_2,\ldots,df_k)$ with $f_1,f_2,\ldots,f_k \in \mathcal{R}$,

$$\tilde{\omega}_k(e_1,\ldots,e_{p-2k};df_1,\ldots,df_k) = \omega_k(e_1,\ldots,e_{p-2k};f_1,\ldots,f_k).$$

In the following, we switch between the two realizations of elements of $\mathcal{D}_{m,m-1}^p$ without other notice.

Set $\mathcal{D} = \mathcal{D}(\mathcal{E},\mathcal{R}) = (\mathcal{D}_{m,m-1}^p)_{p \geq 0, m \in \mathbb{N}^*}$ and for any $\tilde{\omega} = (\tilde{\omega}_0,\ldots,\tilde{\omega}_p) \in \mathcal{D}_{m,m-1}^p$ let

- $s_i(\tilde{\omega}_k)$ be the symbol of $\tilde{\omega}_k : \mathcal{E}^{p-2k} \otimes S^k \Omega^1 \to \mathcal{R}$ with respect to the $i$-th position, $i = 1,\ldots,k$, of $\Omega^1$-arguments. For any $\alpha_1,\ldots,\alpha_k \in \Omega^1$ and $f \in \mathcal{R}$,

  $$s_i(\tilde{\omega}_k)(f)\alpha_1,\ldots,\alpha_i,\ldots,\alpha_k) = \tilde{\omega}_k(f)\alpha_1,\ldots,\alpha_i,\ldots,\alpha_k) - f\tilde{\omega}_k(\alpha_1,\ldots,\alpha_i,\ldots,\alpha_k).$$

- $\sigma_i(\tilde{\omega}_k)$ be the symbol of $\tilde{\omega}_k : \mathcal{E}^{p-2k} \otimes S^k \Omega^1 \to \mathcal{R}$ with respect to the $i$-th position, $i = 1,\ldots,p - 2k$, of $\mathcal{E}$-arguments. For any $(e_1,\ldots,e_{p-2k}) \in \mathcal{E}^{p-2k}$ and $f \in \mathcal{R}$,

  $$\sigma_i(\tilde{\omega}_k)(f)e_1,\ldots,e_{i-1},e_i,\ldots,e_{p-2k};\ldots) = \tilde{\omega}_k(e_1,\ldots,e_i,\ldots,e_{p-2k};\ldots) - f\tilde{\omega}_k(e_1,\ldots,e_i,\ldots,e_{p-2k};\ldots).$$
Proposition 2.12 The space \( \mathcal{D} \) is a graded subalgebra of \((A, \cdot)\). More precisely, for any \( \bar{w} \in \mathcal{D}^{p}_{m,m-1} \) and \( \bar{\eta} \in \mathcal{D}^{q}_{n,n-1} \), the differential operator \( \bar{w} \cdot \bar{\eta} \) is an element in \( \mathcal{D}^{p+q}_{\max\{m,n\}, \max\{m,n\}-1} \).

**Proof.** We must show that \( \mathcal{D} \) is closed under multiplication in \( A \) defined by (5) and (6). Let \( \bar{w} \in \mathcal{D}^{p}_{m,m-1} \) and \( \bar{\eta} \in \mathcal{D}^{q}_{n,n-1} \). Clearly, \( \bar{w} \cdot \bar{\eta} \) is a \((p+q)\)-form. We will prove that \( \bar{w} \cdot \bar{\eta} \) verifies conditions 1 and 2 defining \( \mathcal{D} \). Combining the formulas (5) and (6), we obtain that, for any \((e_1, \ldots, e_{p+q-2k}) \in \mathcal{E}^{p+q-2k} \) and \((\alpha_1, \ldots, \alpha_k) \in S^k \Omega^1 \),

\[
(\bar{w} \cdot \bar{\eta})_k(e_1, \ldots, e_{p+q-2k}; \alpha_1, \ldots, \alpha_k) = \sum_{r + t = k, \, \varrho \in sh(p-2r, q-2t)} (-1)^{l|\varrho|} \bar{w}_r(e_{\varrho(1)}, \ldots, e_{\varrho(p-2r)}; \alpha_\tau(1), \ldots, \alpha_\tau(r)) \cdot \bar{\eta}_t(e_{\varrho(p-2r+1)}, \ldots, e_{\varrho(p+q-2k)}; \alpha_{\tau(r+1)}, \ldots, \alpha_{\tau(r+t)}).
\]

**Condition 1:** Fixing \((e_1, \ldots, e_{p+q-2k}) \in \mathcal{E}^{p+q-2k}, (\alpha_1, \ldots, \alpha_k) \in S^k \Omega^1\), \((r, t) \) with \( r + t = k, \) and \( \varrho \in sh(p-2r, q-2t) \), observe that \( (\bar{w} \cdot \bar{\eta})_k(e_1, \ldots, e_{p+q-2k}; \alpha_1, \ldots, \alpha_k) \) is a sum over shuffle permutations \( \tau \in sh(r, t) \). Hence, the argument in the \( i \)-position of \( \Omega^1 \)-arguments of \( (\bar{w} \cdot \bar{\eta})_k \) occurs either as argument in \( \Omega^1 \)-arguments of \( \bar{w}_r \) or in \( \Omega^1 \)-arguments of \( \bar{\eta}_t \). A simple calculation shows that the symbol \( s_i((\bar{w} \cdot \bar{\eta})_k)(f)(\ldots; \alpha_1, \ldots, \alpha_i, \ldots, \alpha_k) \) is a sum of terms of the form

\[
s_i(\bar{w}_r)(f)(\ldots; \alpha_\tau(1), \ldots, \alpha_\tau(r)) \bar{\eta}_t(f)(\ldots; \alpha_{\tau(r+1)}, \ldots, \alpha_{\tau(r+t)}),
\]

where \( l = \tau(i) \), if \( 1 \leq \tau(i) \leq r \), and \( l' = \tau(i) \), if \( r + 1 \leq \tau(i) \leq r + t \), for \( \tau \in sh(r, t) \). The symbol \( s_i((\bar{w} \cdot \bar{\eta})_k)(f)(\ldots; \alpha_1, \ldots, \alpha_i, \ldots, \alpha_k) \) is thus a differential operator of order at most \( \max\{m - 1, n - 1\} \) in the \( \Omega^1 \)-arguments and \( (\bar{w} \cdot \bar{\eta})_k \) is a differential operator of order at most \( \max\{m - 1, n - 1\} + 1 \) in the \( \Omega^1 \)-arguments.

**Condition 2:** Similarly, fixing \((e_1, \ldots, e_{p+q-2k}) \in \mathcal{E}^{p+q-2k}, (\alpha_1, \ldots, \alpha_k) \in S^k \Omega^1\), \((r, t) \) with \( r + t = k, \) and \( \tau \in sh(r, t) \), we have that \( (\bar{w} \cdot \bar{\eta})_k(e_1, \ldots, e_{p+q-2k}; \alpha_1, \ldots, \alpha_k) \) is a sum over shuffle permutations \( \varrho \in sh(p-2r, q-2t) \). Hence, the \( i \)-argument in \( \mathcal{E} \)-entries of \( (\bar{w} \cdot \bar{\eta})_k \) occurs either as argument in \( \mathcal{E} \)-entries of \( \bar{w}_r \) or in \( \mathcal{E} \)-entries of \( \bar{\eta}_t \). Moreover, the last term of \((\bar{w} \cdot \bar{\eta})_k \) arises either as the last term of \( \bar{w}_r \) or the last term of \( \bar{\eta}_t \). A simple calculation shows again that the symbol \( s_i((\bar{w} \cdot \bar{\eta})_k)(f)(e_1, \ldots, e_i, \ldots, e_{p+q-2k}; \ldots) \) is a sum of terms of the form

\[
(-1)^{|\varrho|} s_i(\bar{w}_r)(f)(e_{\varrho(1)}, \ldots, e_{\varrho(p-2r)} \bar{\eta}_t(e_{\varrho(p-2r+1)}, \ldots, e_{\varrho(p+q-2k)}),
\]

where \( l = \varrho(i) \), if \( 1 \leq \varrho(i) \leq p - 2r \), and \( l' = \varrho(i) \), if \( p - 2r + 1 \leq \varrho(i) \leq p + q - 2k \), for \( \varrho \in sh(p-2r, q-2t) \). Hence the symbol \( s_i((\bar{w} \cdot \bar{\eta})_k)(g)(e_1, \ldots, e_i, \ldots, e_{p+q-2k}; \ldots) \) is a differential operator of order at most \( \max\{m - 1, n - 1\} \). Thus, \((\bar{w} \cdot \bar{\eta})_k \) is a differential operator of order at most \( \max\{m - 1, n - 1\} + 1 \) in the first \( p + q - 2k - 1 \) \( \mathcal{E} \)-entries. The order of the symbol \( \sigma_{p+q-2k}(\bar{w} \cdot \bar{\eta})_k \) is \( \max\{m - 2, n - 2\} \), so the order of the operator \((\bar{w} \cdot \bar{\eta})_k \) in the \((p + q - 2k)\)-entry is \( \max\{m - 2, n - 2\} + 1 \) in \( \max\{m, n\} - 1 \). ♦

**Definition 2.13** The graded subalgebra \( \mathcal{D} = \mathcal{D}(\mathcal{E}, \mathcal{R}) = (\mathcal{D}_{m,m-1}^{p})_{p \geq 0, m \in \mathbb{N}} \) of \( A \) is called the Courant-Dorfman algebra of differential operators of the Courant algebroid \((\mathcal{E}, [], \langle \cdot, \cdot \rangle, \rho)\).
We extend the map (7) to a map, also denoted by $d$,
\[ d : \mathfrak{D}_{m,m-1} \to \mathfrak{D}_{m+1,m}, \]
defined by (8). One can directly check that

**Proposition 2.14** It is a derivation of degree $+1$ of $\mathcal{D}(\mathcal{E}, \mathcal{R})$ which squares to zero.

The complex $(\mathfrak{D}, d)$ will be called as **complex of differential operators of** $(\mathcal{E}, \mathcal{R})$. Its $p$-cohomology group will be denoted by $\mathfrak{D}^p(\mathcal{E}, \mathcal{R})$.

For any $f \in \mathcal{R}$ and $e \in \mathcal{E}$, the operators $i_f : C \to C[-2]$ and $i_e : C \to C[-1]$ given respectively by (10) and (11), are extended in a similar way to operators $i_f : \mathfrak{D} \to \mathfrak{D}[-2]$ and $i_e : \mathfrak{D} \to \mathfrak{D}[-1]$, respectively. More precisely, they preserve the order of differential operators, that is
\[ i_f : \mathfrak{D}^p_{m,m-1} \to \mathfrak{D}^{p-2}_{m,m-1} \quad \text{and} \quad i_e : \mathfrak{D}^p_{m,m-1} \to \mathfrak{D}^{p-1}_{m,m-1}. \]  (19)
The commutator $\{\cdot, \cdot\}$ on the space of graded endomorphisms of $\mathfrak{D}$ is similarly defined, and the relations (16) - (18) together with Cartan’s formulas of Lemma 2.9 hold as well.

## 3 Dorfman connections

The notion of **Dorfman connection** introduced in [20] by Jotz Lean is a new concept adapted to Courant algebroid theory and plays a role similar to the one that linear connections play for tangent bundles and Lie algebroids. In the following we present a modified definition of this concept where the role of the **dull algebroid** is given to a Courant algebroid.

### 3.1 Predual vector bundle

We first define the notion of **predual vector bundle** $B$ of a Courant algebroid $E$ and discuss the properties of two vector subbundles of $B$ and $E$ with particular significance.

**Definition 3.1** Let $(E, [[\cdot, \cdot], \langle \cdot, \cdot \rangle, \rho)$ be a Courant algebroid over $M$, $B \to M$ a vector bundle of constant rank, $\langle \cdot, \cdot \rangle : E \times_M B \to \mathbb{R}$ a fiberwise bilinear pairing, and $d_B : C^\infty(M, \mathbb{R}) \to \Gamma(B)$ a map defined, for any $e \in \Gamma(E)$ and $f \in C^\infty(M, \mathbb{R})$, by
\[ \langle e, d_B f \rangle = \rho(e)(f). \]  (20)
The triple $(B, d_B, \langle \cdot, \cdot \rangle)$ is called predual of $E$, and $E$ and $B$ are said to be paired by $\langle \cdot, \cdot \rangle$.

**Remark 3.2** From definition (20) of $d_B$, it is obvious that determining the section $d_B f$ of $B$ involves the partial derivatives of $f$. Therefore, we can write $d_B$ as a composition
\[ d_B = \alpha \circ d, \]
where $\alpha : T^*M \to B$ is a smooth vector bundle map.

The fiberwise pairing $\langle \cdot, \cdot \rangle : E \times_M B \to \mathbb{R}$ gives rise to the $C^\infty(M, \mathbb{R})$-linear maps
\[ \langle e, \cdot \rangle : \Gamma(B) \to C^\infty(M, \mathbb{R}) \quad \text{and} \quad \langle \cdot, b \rangle : \Gamma(E) \to C^\infty(M, \mathbb{R}). \]
The first can be viewed as an element of $\Gamma(B^*)$ and the second as an element of $\Gamma(E^*)$. Denote by $K$ the subbundle $\ker E$ of $B$ over $M$, called the **kernel of** $E$ in $B$ **with respect to** $\langle \cdot, \cdot \rangle$, whose fiber at a point $x \in M$ is the space
\[ K_x = \ker E_x = \{ b \in B_x / \langle e, b \rangle = 0, \quad \text{for all } e \in E_x \}. \]
Denote also by $F$ the subbundle $\ker B$ of $E$ over $M$, called the kernel of $B$ in $E$ with respect to $\langle \cdot, \cdot \rangle$, whose fiber at a point $x \in M$ is the space

$$F_x = \ker B_x = \{ e \in E_x / \langle e, b \rangle = 0, \text{ for all } b \in B_x \}. $$

By definition of $F$, it is $\langle e, d_B f \rangle = 0$, for any $e \in \Gamma(F), f \in C^\infty(M, \mathbb{R})$. Since

$$\langle e, d_B f \rangle = 0 (20) \iff \rho(e)(f) = 0,$$

one has that any $f \in C^\infty(M, \mathbb{R})$ is first integral of any vector field $\rho(e), e \in \Gamma(F)$. This is possible, only if $\rho(e) = 0 \iff e \in \Gamma(\ker \rho)$. Thus $F$ can also be determined as the subbundle of $\ker \rho \subseteq E$ whose fiber at each point $x \in M$ is the space

$$F_x = \ker B_x = \{ e \in \ker \rho_x \subseteq E_x / \langle e, b \rangle = 0, \text{ for all } b \in B_x \}. $$

Assuming that the pairing $\langle \cdot, \cdot \rangle$ is of constant rank on $M$, the bundles $K$ and $F$ are also of constant rank on $M$. We now make some observations for the cases $\text{rank} K \geq 0$ and $\text{rank} F \geq 0$.

### 3.1.1 The case $\text{rank} K \geq 0$

Suppose that $\text{rank} K \geq 0$ and write $B \cong B/K \oplus K$. Consider the pairing $\langle \cdot, \cdot \rangle'$ of $E$ with $B/K$ defined by $\langle \cdot, \cdot \rangle$ so that, for $e \in \Gamma(E)$ and $[b] \in \Gamma(B/K), [b]$ being the class of $b$ in $B/K$,

$$\langle e, [b] \rangle' = \langle e, b \rangle. \quad (21)$$

Obviously $\langle e, [b] \rangle' = 0 \iff \langle e, b \rangle = 0$, for any $e \in \Gamma(E)$, if and only if $b \in \Gamma(K)$. The latter is equivalent to $[b] = 0$ and so $\langle e, \cdot \rangle'$ has trivial kernel. Consequently, $B/K \cong E^*$ and, composing with the isomorphism $E^* \cong E$, one has

$$B \cong E^* \oplus K \cong E \oplus K.$$

Hence in this case, $\text{rank} B \geq \text{rank} E$. The class $[b]$ of a section $b \in \Gamma(B)$, can be considered as an element of $\Gamma(E^*) \cong \Gamma(E)$ which will be denoted by $e_{[b]}$. In fact, this is the representative of $[b]$ in $\Gamma(E^*) \cong \Gamma(E)$. Thus, for any $e \in \Gamma(E)$, by the definition of $e_{[b]}$ and by the identification of $E$ with $E^*$, we have, respectively,

$$e_{[b]}(e) = \langle e, [b] \rangle' = \langle e, b \rangle \quad \text{and} \quad e_{[b]}(e) = \langle e_{[b]}, e \rangle = \langle e, e_{[b]} \rangle,$$

and so

$$\langle e, [b] \rangle' = \langle e, e_{[b]} \rangle = \langle e_{[b]}, e \rangle. \quad (22)$$

The above allow us to write any $b \in \Gamma(B)$ in the form

$$b = e_{[b]} + k,$$

where $k$ is the part of $b$ that is a section of $K$.

Let $b, b' \in \Gamma(B), f \in C^\infty(M, \mathbb{R})$ and let $b = e_{[b]} + k, b' = e_{[b']} + k'$, and $b + b' = e_{[b+b']} + k''$. Obviously, $e_{[b+b']} = e_{[b]} + e_{[b']}$, and $k'' = k + k'$. In the same way one gets that $[f b] = [f e_{[b]} + k] = [f e_{[b]} + f k]$. Thus we have

$$e_{[b+b']} = e_{[b]} + e_{[b']} \quad \text{and} \quad e_{[f b]} = f e_{[b]} \quad (23)$$

Next, we discuss the relationship between $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle'$. For $e \in \Gamma(E)$ and $b = e_{[b]} + k \in \Gamma(B)$, it is

$$\langle e, b \rangle = \langle e, e_{[b]} + k \rangle = \langle e, [b] \rangle' \iff \langle e, e_{[b]} \rangle = \langle e, e_{[b]} \rangle,$$

where
i.e. the restriction $\langle \cdot, \cdot \rangle_{\Gamma(E) \times \Gamma(E^*)}$ of $\langle \cdot, \cdot \rangle$ on $\Gamma(E) \times \Gamma(E^*) \cong \Gamma(E) \times \Gamma(E)$ recovers the bilinear nondegenerate symmetric form $\langle \cdot, \cdot \rangle$. This can be reformulated as follows: Let $(e_1, \ldots, e_r, k_1, \ldots, k_{s-r})$ be an appropriate local frame of smooth sections of $E$ whose first $r$ elements constitute a local frame of smooth sections of $E$ and the last $s-r = \text{rank}K$ a local frame of smooth sections of $K$. Let also $G$ denote the $r \times r$ nondegenerate matrix of $\langle \cdot, \cdot \rangle$ in $(e_1, \ldots, e_r)$. Denoting by $P$ the $s \times r$ matrix associated to $\langle \cdot, \cdot \rangle$, one then has

$$P = \begin{pmatrix} G & 0_{(s-r) \times r} \end{pmatrix}.$$ Finally, we make some comments about the sections $d_B f \in \Gamma(B)$. Consider the operator $\alpha$ introduced in Remark 3.2 as a smooth section of $\Gamma(B \otimes TM)$, and let $A$ be the $s \times n$ matrix associated to $\alpha$ as a bundle map. Similarly, let $\rho$ be the $n \times r$ matrix associated to anchor map $\rho$, viewed as smooth section of $TM \otimes E^*$. Relation (20) is then equivalent to $A^T P = \rho$.

Writting $A$ in block form as $A = \begin{pmatrix} A_{1 \times n}^T & A_{2 \times n}^T \end{pmatrix}$ and using the relation $A^T P = \rho$ one gets that $A = \begin{pmatrix} G^{-1} \rho \\ A_{2}^T \end{pmatrix}$. Thus,

$$d_B f = \begin{pmatrix} d_E f \\ A^2 df \end{pmatrix}$$
i.e., $e_{d_B f} = d_E f$ and $d_B f = d_E f + k$, while $k = A^2 df \in \Gamma(K)$. Since the part $k = A^2 df$ of $d_B f$ does not play any role in definition (20), assume without loss of generality that $A^2 = 0$. If $f$ is constant along the leaves of the integrable distribution $\text{Im} \rho \subseteq TM$, we have

$$\langle e, d_E f \rangle = \rho(e)(f) = 0 \Leftrightarrow d_E f = 0,$$
because $\langle \cdot, \cdot \rangle$ is non degenerate. It is then $e_{d_B f} = 0$ and $d_B f = 0$. Summarizing,

$$d_B f = \begin{cases} 0, & \text{if } f \text{ is constant along } \text{Im} \rho, \\
 d_E f, & \text{if } f \text{ is not constant along } \text{Im} \rho. \end{cases} \quad (24)$$

### 3.1.2 The case $\text{rank} F \geq 0$

Suppose $\text{rank} F \geq 0$. With an argument similar to the previous subsection, it is $E/F \cong B^*$. Hence, $E \cong B^* \oplus F$, and so $\text{rank} E \geq \text{rank} B$. For any $e \in \Gamma(E)$, let $[e]$ be its class in $E/F \cong B^*$ and $b^*_{[e]}$ its representative in $B^*$. Then,

$$e = b^*_{[e]} + f, \quad f \in \Gamma(F).$$
The pairing $\langle \cdot, \cdot \rangle$ induces a nondegenerate pairing between $B^*$ and $B$ denoted by $\langle \cdot, \cdot \rangle''$. For any $b^* \in \Gamma(B^*)$, there is an $e \in \Gamma(E)$ such that $b^*_{[e]} = b^*$. So,

$$\langle b^*, b \rangle'' = \langle b^*_{[e]}, b \rangle'' = \langle b_{[e]}^* + f, b \rangle = \langle e, b \rangle.$$ On the other hand, $E^* \cong B \oplus F^*$ and $(E/F)^* \cong F^0 \Leftrightarrow B \cong F^0$, where $F^0$ denotes the annihilator of $F \subset E$ in $E^*$. It is easy to see that the pairing $\langle \cdot, \cdot \rangle$ between $E$ and $E^*$, defined by the symmetric bilinear form on $E$, induces a nondegenerate pairing between $B$ and $B^*$ and between $F$ and $F^*$. Identifying $B$ with $B^*$ and $F$ with $F^*$ we get that the pairings between $B$ and $B^*$ induced by $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle''$ coincide. In conclusion, write

$$E \cong B \oplus F. \quad (25)$$
Consider now an appropriate local frame of smooth sections \((b_1, \ldots, b_s, f_{s+1}, \ldots, f_r)\) of \(E\) adapted to the splitting \((25)\). Let again \(G, P\) and \(\rho\) be the matrices corresponding to \((\cdot, \cdot), (\cdot, \cdot)\) and the anchor map respectively. In block form, they are
\[
G = \begin{pmatrix} G^1 & 0 \\ 0 & G^2 \end{pmatrix}, \quad P = \begin{pmatrix} G^1 & 0 \end{pmatrix} \quad \text{and} \quad \rho = \begin{pmatrix} R & 0 \end{pmatrix}.
\]
Hence, \(A^TP = \rho \iff A = (G^1)^{-1}R^T\) and
\[
d_Bg = d_Eg, \quad \text{for any } g \in C^\infty(M, \mathbb{R}). \tag{26}
\]
Of course, if \(g\) is constant along the integral manifolds of \(\text{Im}\rho\), then \(d_Eg = 0\), and so \(d_Bg = 0\).³

The above remarks explain the term \textit{predual} of \(E\) for the triple \((B, d_B, (\cdot, \cdot))\).

### 3.2 Dorfman connections

In this subsection we present and use a modified definition of the notion of \textit{Dorfman connection}. The difference is that we consider a Courant algebroid \((E, [[\cdot, \cdot], (\cdot, \cdot), \rho]_Q)\) [20, Definition 3.3].⁴ We then prove that the set \(\mathcal{E}(E, B)\) of \(E\)-Dorfman connections on a predual vector bundle \((B, d_B, (\cdot, \cdot))\) of \(E\) is non-empty (Proposition 3.4) and that it is endowed with an affine structure (Proposition 3.5). Finally, in Proposition 3.6, we prove that any \(E\)-Dorfman connection on a predual vector bundle \(B\) defines a \(B\)-linear connection on \(E\).

**Definition 3.3** Let \((E, [[\cdot, \cdot], (\cdot, \cdot), \rho])\) be a Courant algebroid over a smooth manifold \(M\) and \((B, d_B, (\cdot, \cdot))\) a predual of \(E\). An \(E\)-Dorfman connection on \(B\) is an \(\mathbb{R}\)-bilinear map
\[
\nabla : \Gamma(E) \times \Gamma(B) \to \Gamma(B)
\]
such that, for all \(e \in \Gamma(E), b \in \Gamma(B), \) and \(f \in C^\infty(M, \mathbb{R}),\) the following three properties hold:

1. \(\nabla_fe b = f \nabla_e b + (e, b) dB f,\)
2. \(\nabla_e (fb) = f \nabla_e b + \rho(e)(f)b,\)
3. \(\nabla_e (dBf) = dB(\mathcal{L}_\rho(e))f.\)

The spaces of sections \(\Gamma(E \otimes B) \cong \Gamma(E) \otimes \Gamma(B),\) and \(\Gamma(B)\) being \(C^\infty(M, \mathbb{R})\)-bimodules, the first two conditions of Definition 3.3 mean that a Dorfman connection \(\nabla\) is a first-order differential operator in the bimodules \(E \otimes B\) and \(B\) in the sense of [15]. The last condition means that the space \(\Gamma(\text{Im}d_B)\) of smooth sections of \(\text{Im}d_B\) is invariant under the map \(\nabla_e,\) for any \(e \in \Gamma(E).\)

³Note that \(d_Eg\) are sections of \(\ker \rho\) (because of property 6 of the Courant algebroid structure), for all \(g \in C^\infty(M, \mathbb{R}).\) However, these are not, in general, sections of \(F \subset \ker \rho.\) Since \(B \cong F^0\) and \(\rho(B) = \rho(E)\) as subbundles of \(TM,\) if \(d_Eg \in \Gamma(F),\) then \((d_Eg)b = 0 \iff \rho(b)(g) = 0,\) for any \(b \in \Gamma(B).\) This is equivalent to saying that \(g\) is constant along the leaves of the integrable distribution \(\text{Im}\rho.\) In this case though, \(d_Eg = 0,\) which is not always true.

⁴A \textit{dull algebroid} is a vector bundle \(Q\) over a smooth manifold \(M\) endowed with an anchor map \(\rho_Q : Q \to TM\) and a bracket \([\cdot, \cdot]_Q\) on \(\Gamma(Q)\) such that, for all \(q_1, q_2 \in \Gamma(Q)\) and \(f_1, f_2 \in C^\infty(M, \mathbb{R}),\) \(\rho_Q[q_1, q_2]_Q = [\rho_Q(q_1), \rho_Q(q_2)]\) and satisfying the Leibniz identity in both terms:
\[
[f_1q_1, f_2q_2]_Q = f_1f_2[q_1, q_2]_Q + f_1\rho_Q(q_1)(f_2)q_2 - f_2\rho_Q(q_2)(f_1)q_1.
\]
Another way to read the third condition of Definition 3.3, is the following. It can be easily checked that the Lie derivative $\mathcal{L} : \Gamma(TM) \times \Gamma(T^*M) \to \Gamma(T^*M)$ verifies the three conditions of Definition 3.3. Thus $\mathcal{L}$ can be considered as a $TM$-Dorfman connection on $T^*M$, in the sense of [20, Definition 3.3], since the Lie algebroid $(TM, [\cdot, \cdot])$ is a special case of dull algebroid. Writting $d_B = \alpha \circ d$ (see Remark 3.2), we have
\[
\nabla_e(d_B f) = d_B(\mathcal{L}_e(f)) \leftrightarrow \nabla_e\alpha(df) = \alpha(\mathcal{L}_{\rho(e)}(df)),
\]
meaning that the following diagram is commutative:

$$
\begin{array}{c}
\begin{align*}
T^*M \supset \text{Im}d & \xrightarrow{\alpha} B \\
\mathcal{L}_{\rho(e)} & \downarrow \nabla_e \\
T^*M \supset \text{Im}d & \xrightarrow{\alpha} B.
\end{align*}
\end{array}
$$

Proposition 3.4 Let $(E, B)$ be as in Definition 3.3. The set $\mathcal{E}(E, B)$ of $E$-Dorfman connections on $B$ is non-empty.

Proof. We use the notation introduced in Remark 3.2. Let $(U, x^1, \ldots, x^n)$ be a local coordinate system of $M$, $(e_1, \ldots, e_r)$ a local frame of smooth sections of $E$ over $U$ with $(e^1, \ldots, e^s)$ its dual frame of local smooth sections of $E^*$, $r = \text{rank}E$, and $(b_1, \ldots, b_s)$ a local frame of smooth sections of $B$ over $U$ with $(b^1, \ldots, b^s)$ its dual frame of local sections of $B^*$, $s = \text{rank}B$. With respect to these choices, the local expressions of the operators $\alpha$, $\langle \cdot, \cdot \rangle$ and $\rho$ are, respectively,
\[
\alpha = \alpha^{ij} b_i \otimes \frac{\partial}{\partial x^j}, \quad \langle \cdot, \cdot \rangle = p_{ij} b^i \otimes e^j, \quad \rho = \rho^i_j \frac{\partial}{\partial x^i} \otimes e^j.
\]
Their coefficients are smooth functions on $U$. With the same notation for their associated matrices, $A = (\alpha^{ij})$, $P = (p_{ij})$, and $\rho = (\rho^i_j)$, we have that (20) is equivalent to $A^T P = \rho \iff P^T A = \rho^T$.

Let $\pi : E \to M$, $\tau : B \to M$ be the projection of $E, B$, respectively, on $M$. Consider the $\mathbb{R}$-bilinear map $\nabla^0 : \Gamma(E) \times \Gamma(B) \to \Gamma(B)$ defined, for any $e \in \pi^{-1}(U) \subset \Gamma(E)$, and $b = f^1 b_1 + f^2 b_2 + \ldots + f^s b_s \in \tau^{-1}(U) \subset \Gamma(B)$, by
\[
\nabla_{e^0} b = \nabla^0_e (f^1 b_1 + \ldots + f^s b_s) := (\mathcal{L}_e(f^1)) b_1 + \ldots + (\mathcal{L}_e(f^s)) b_s + \sum_{i=1}^s f^i d_B(P_i(e)), \quad (27)
\]
where $P_i(e)$ is the $i$-component function of the local section $\langle e, \cdot \rangle$ of $B^*$. One can easily check that (27) satisfies the first and second condition of Definition 3.3, but not the third. The latter is satisfied under the following strong condition on $\alpha$: For any $k = 1, \ldots, s$,
\[
\alpha^{il} \frac{\partial^2 \alpha^{kl}}{\partial x^l} = \alpha^{kt} \frac{\partial^2 \alpha^{il}}{\partial x^t}.
\]
For this reason, we search for a vector bundle map $C : \Gamma(E \otimes B) \to \Gamma(B)$, i.e. for an element of $C$ of $\Gamma(\text{Hom}_{\mathbb{R}}(E, B \otimes B^*)) \cong \Gamma(B \otimes B^* \otimes E^*) \cong \mathbb{C}^1 \otimes \Gamma(\text{End}(B))$, such that the map $\nabla : \Gamma(E) \times \Gamma(B) \to \Gamma(B)$, with
\[
\nabla = \nabla^0 + C, \quad (28)
\]
is a Dorfman connection. We now show that it is always possible to find such a map $C$; In the local coordinates of $M$ and the local frames of $E$ and $B$ considered above, $C$ is written
Therefore, it is always possible to find, locally, a vector bundle map \( C(e_k, \cdot) : \Gamma(B) \to \Gamma(B) \), and \( N_k = (N^q_k) \) with \( N^q_k = (\alpha^q \partial \partial x^i - \alpha^q \partial \partial x^j) p_{tk} \), equation (29) is written, in matrix form, as

\[
A^T C_k - N_k = 0 \Leftrightarrow A^T C_k = N_k. \tag{30}
\]

Hence, \( \nabla = \nabla^0 + C \) verifies the third condition of Definition 3.3 if and only if, for any \( k = 1, \ldots, r \), there exists a matrix \( C_k \) satisfying (30).

- If \( \text{rank} A = s \), i.e. \( \alpha : T^*M \to B \) is a surjective vector bundle map, the matrix \( A^T \) has a left inverse matrix \( A^T_{-1} \), thus (30) has a unique solution \( C_k = A^T_{-1} N_k \), for any \( k = 1, \ldots, r \).

- If \( \text{rank} A = m < s \), we have more than one ways to define \( C_k \), for any \( k = 1, \ldots, r \). Without loss of generality, suppose that the first \( m \) columns of \( A^T = \begin{pmatrix} A^1_{n \times m} & A^2_{n \times (s-m)} \end{pmatrix} \) are linearly independent at each point \( x \in U \) and so the block \( A^1 \) has a left inverse matrix \((A^1)^{-1}_L\). One can then solve the equation (30). Writing \( C_k = \begin{pmatrix} C^1_{k,n \times s} \\ C^2_{k(s-m) \times s} \end{pmatrix} \),

we have

\[
A^T C_k = N_k \Leftrightarrow \begin{pmatrix} A^1_{n \times m} & A^2_{n \times (s-m)} \end{pmatrix} \begin{pmatrix} C^1_{k,n \times s} \\ C^2_{k(s-m) \times s} \end{pmatrix} = N_k
\]

\[
\Leftrightarrow A^1 C^1_k + A^2 C^2_k = N_k
\]

\[
\Leftrightarrow C^1_k = (A^1)^{-1}_L(N_k - A^2 C^2_k).
\]

For any choice of the block \( C^2_k \) of \( C_k \), we find the block \( C^1_k \) by the last equation and thus the matrix \( C_k, k = 1, \ldots, r \).

Therefore, it is always possible to find, locally, a vector bundle map \( C : \Gamma(E|_U) \times \Gamma(B|_U) \to \Gamma(B|_U) \) such that \( \nabla = \nabla^0 + C \) defines on \( U \) an \( E \)-Dorfman connection on \( B \).

Choose an open cover \((U_i)_{i \in I}\) of \( M \) such that, for any \( i \in I \), \( E|_{U_i} \) and \( B|_{U_i} \) are trivial bundles, and pick a smooth partition of unity \((\psi_j)_{j \in J}\) subordinate to this cover. By the previous discussion, each bundle \( B|_{U_i} \) admits at least one \( E|_{U_i} \)-Dorfman connection \( \nabla^i \) of type (28). Denote such a connection by \( \nabla^i \) and let

\[
\nabla = \sum_{i,j} \psi_j \nabla^i.
\]

Clearly, \( \nabla \) is an \( E \)-Dorfman connection on \( B \) and so \( \mathfrak{C}(E, B) \) is non empty. \( \diamond \)

**Proposition 3.5** The set \( \mathfrak{C}(E, B) \) carries a natural affine structure with corresponding vector space \( \mathcal{C}^1 \oplus \Gamma((B/\text{Im}d_B)^* \otimes B) \).
**Proof.** Let $\nabla^0$, $\nabla^1$ be two $E$-Dorfman connections on $B$. Then, for any $g \in C^\infty(M, \mathbb{R})$, the affine combination

$$\nabla := (1-g)\nabla^0 + g\nabla^1$$

is an $E$-Dorfman connection on $B$. Thus, $\mathcal{C}(E, B)$ is an affine space.

Let $\nabla$ be an $E$-Dorfman connection on $B$ and $S$ an element of $\Gamma(\text{Hom}_E(E, B \otimes B^*)) \cong \Gamma(B \otimes B^* \otimes E^*) \cong C^1 \otimes \Gamma(\text{End}(B))$. It is easy to check that $\nabla + S$ is an $E$-Dorfman connection on $B$ if and only if $\text{Im}d_B \subseteq \ker S(e, \cdot)$, for any $e \in \Gamma(E)$. On the other hand, if $\nabla$ and $\nabla'$ are two $E$-Dorfman connections on $B$, by the first condition of Definition 3.3 we get that, for any $f \in C^\infty(M, \mathbb{R})$, $e \in \Gamma(E)$ and $b \in \Gamma(B)$,

$$\nabla_f e - \nabla'_f e = f(\nabla_e - \nabla'_e)b.$$

This shows that $\nabla - \nabla'$ is a $C^\infty(M, \mathbb{R})$-linear homomorphism from $\Gamma(E)$ to $\Gamma(B)$. Using the Leibniz rule and the third condition of Definition 3.3 for $\nabla$ and $\nabla'$ we obtain, respectively, that

$$\nabla_e(fb) - \nabla'_e(fb) = f(\nabla_e - \nabla'_e)b \quad \text{and} \quad \nabla_e df - \nabla'_e df = 0.$$

This shows that $\nabla - \nabla'$ is a $C^\infty(M, \mathbb{R})$-linear endomorphism of $\Gamma(B)$ vanishing identically on $\text{Im}d_B$ and so can be viewed as a section of the vector bundle $(B/\text{Im}d_B)^* \otimes B$. As a result, $\nabla - \nabla'$ is an element of $C^1 \otimes \Gamma((B/\text{Im}d_B)^* \otimes B)$, which means that $\mathcal{C}(E, B)$ is an affine space modeled on $C^1 \otimes \Gamma((B/\text{Im}d_B)^* \otimes B)$. ♦

**Proposition 3.6** Let $(E, B)$ be as in Definition 3.3. Any $E$-Dorfman connection $\nabla$ on $B$ defines a $B$-linear connection $D$ on $E$.

**Proof.** Consider the cases $B \cong E \oplus K$ and $E \cong B \oplus F$ separately.

If $B \cong E \oplus K$, extend the vector bundle map $\rho : E \to TM$ to a vector bundle map $a : B \to TM$ setting $a(b) = \rho(e_{[b]})$. One can then easily prove that the map $D : \Gamma(B) \times \Gamma(E) \to \Gamma(E)$ defined by

$$D_b e = e_{[\nabla e b]} - [e, e_{[b]}] \quad (31)$$

is a $B$-linear connection on $E$.

If $B \cong E/F$, let $a : B \to TM$, with $a = \rho|_B$. One then checks that the map $D : \Gamma(B) \times \Gamma(E) \to \Gamma(E)$ defined by

$$D_b e = \nabla e b - [e, b] \quad (32)$$

is a $B$-linear connection on $E$.

By (31) and (32), the compatibility condition (3) between $\langle \cdot, \cdot \rangle$ and $[\cdot, \cdot]$ can be rewritten, in both cases, as

$$\rho(e)\langle e', b \rangle = \langle [e, e'], b \rangle - \langle D_b e, e' \rangle + \langle e', \nabla_e b \rangle, \quad (33)$$

for any $e, e' \in \Gamma(E), b \in \Gamma(B)$. The expression above can be considered as a compatibility condition between the Courant algebroid structure of $E$ and the $E$-Dorfman connection on $B$.

**Remark 3.7** The compatibility condition (33) is slightly different from the one given in [20, Proposition 3.4] between the bracket of a dull algebroid and a Dorfman connection. The difference lies in the fact that a Courant bracket does not satisfy, like a dull bracket, the Leibniz identity in both arguments.

---

\[^5\]Let $A \to M$ be a vector bundle over a smooth manifold $M$ endowed with an anchor map $a : A \to TM$. According to [7], an $A$-linear connection on a fiber bundle $E \to M$ is a $\mathbb{R}$-bilinear operator $D : \Gamma(A) \times \Gamma(E) \to \Gamma(E)$ such that, for any $s \in \Gamma(A), e \in \Gamma(E)$ and $f \in C^\infty(M, \mathbb{R})$,

$$D_s e = f D_s e \quad \text{and} \quad D_s(f e) = f D_s e + a(s)(f)e.$$
3.3 Curvature of Dorfman connections

Let $\mathcal{D}^p_{m,m-1}(\mathcal{E};B)$ be the spaces of $p$-cochains with values in $\Gamma(B)$, $m \in \mathbb{N}^*$, and $\mathcal{D}(\mathcal{E};B) = (\mathcal{D}^p_{m,m-1}(\mathcal{E};B))_{p \geq 0, m \in \mathbb{N}^*}$. Each element $H$ of $\mathcal{D}^p_{m,m-1}(\mathcal{E};B)$ is determined by a $([\frac{p}{2}] + 1)$-tuple $H = (H_0, H_1, \ldots, H_{[\frac{p}{2}]})$ of homomorphisms

$$H_k : \mathcal{E}^\otimes S^k \to \Gamma(B)$$

such that

1. $H_k : \mathcal{E}^\otimes S^k \to \text{Hom}_R(S^k \Omega^1, \Gamma(B))$ takes values in the space $\text{Diff}^m(S^k \Omega^1, \Gamma(B)) \subset \text{Hom}_R(S^k \Omega^1, \Gamma(B))$, where $S^k \Omega^1$ is the $\mathcal{R}$-module of the $k$-symmetric power of the $\mathcal{R}$-module $\Omega^1$ and

$$\text{Diff}^m(S^k \Omega^1, \Gamma(B)) = \{D \in \text{Hom}_R(S^k \Omega^1, \Gamma(B)) / D \text{ is a differential operator on } S^k \Omega^1 \text{ of order at most } m \text{ in each entry with values in } \Gamma(B)\}.$$

2. $H_k : \mathcal{E}^\otimes S^k \to \text{Diff}^m(S^k \Omega^1, \Gamma(B))$ is a differential operator of order at most $m$ on the first $p - 2k - 1$ arguments and of order at most $m - 1$ on the $(p - 2k)$-th argument.

Note that in the case where $p = 1$, the elements of the corresponding spaces, noted by $\mathcal{D}^m_{m-1}(\mathcal{E};B)$, are of type $H = (H_0)$ and the map $H_0 : \mathcal{E} \to \Gamma(B)$ is viewed as a differential operator of order at most $m$ on $\mathcal{E}$ with values in $\Gamma(B)$.

Clearly, the space $\mathcal{D}^p_{m,m-1}(\mathcal{E};B)$ is identified with $\mathcal{D}^p_{m,m-1} \otimes \Gamma(B)$ and $\mathcal{D}(\mathcal{E};B) \cong \mathcal{D} \otimes \Gamma(B)$, since $\mathcal{E} = \Gamma(E^*) \cong \Gamma(E)$ as vector bundle over $M$ is a finitely generated and projective module over $\mathcal{R}$ (see Serre-Swan Theorem in [44]).

**Definition 3.8** An $E$-Dorfman connection $\nabla$ on $B$ defines a map

$$d^\nabla : \Gamma(B) \to \mathcal{D}^1_B(\mathcal{E};B),$$

called Dorfman covariant derivation, satisfying, for any $f \in C^\infty(M, \mathbb{R}), b \in \Gamma(B)$, the Leibniz rule

$$d^\nabla(fb) = df \otimes b + fd^\nabla b.$$ 

We extend the multiplication $\cdot$ in $\mathcal{D}$ (as subalgebra of $\mathcal{A}$, see formula (6)) to a multiplication, also denoted by $\cdot$, between elements of $\mathcal{D}$ and of $\mathcal{D} \otimes \Gamma(B)$, setting, for any $\omega \in \mathcal{D}^p_{m,m-1}$ and $\eta \otimes b = (\eta_0, \eta_1, \ldots, \eta_{[\frac{p}{2}]} \otimes b) \in \mathcal{D}^q_{n,n-1} \otimes \Gamma(B)$,

$$\omega \cdot (\eta \otimes b) := (\omega \cdot \eta) \otimes b.$$ 

As in the theory of linear connections, one can prove that the Dorfman covariant derivation $d^\nabla$ extends uniquely to an operator of degree $+1$, denoted also by $d^\nabla$,

$$d^\nabla : \mathcal{D}^p_{m,m-1} \otimes \Gamma(B) \to \mathcal{D}^{p+1}_{m+1,m} \otimes \Gamma(B),$$ 

satisfying the Leibniz rule

$$d^\nabla(\omega \otimes b) = d\omega \otimes b + (-1)^p \omega \cdot d^\nabla b,$$

for all $\omega \in \mathcal{D}^p_{m,m-1}, b \in \Gamma(B)$. Taking into account the formula (8) as well as the second axiom of Definition 3.3, we have that, for any $H = (H_0, H_1, \ldots, H_{[\frac{p}{2}]}) \in \mathcal{D}^p_{m,m-1}(\mathcal{E};B) \cong \mathcal{D}^p_{m,m-1}(\mathcal{E};B)$.
\[ \mathcal{D}^p_{m,m-1} \otimes \Gamma(B), \] its covariant derivative \( d^\nabla H \in \mathcal{D}^{p+1}_{m+1,m} \otimes \Gamma(B) \) is given by the \( ([\frac{p+1}{2}]+1)-\)tuple \( ((d^\nabla H)_0, (d^\nabla H)_1, \ldots, (d^\nabla H)_{\frac{p+1}{2}}) \) with

\[
(d^\nabla H)_k(e_1, \ldots, e_{p+1-2k}; f_1, \ldots, f_k) = \\
\sum_{\mu=1}^k H_{k-1}(d_E f_\mu, e_1, \ldots, e_{p+1-2k}; f_1, \ldots, \hat{f}_\mu, \ldots, f_k) \\
+ \sum_{i=1}^{p+1-2k} (-1)^{i-1} \nabla e_i(H_k(e_1, \ldots, \hat{e}_i, \ldots, e_{p+1-2k}; f_1, \ldots, f_k)) \\
+ \sum_{i<j} (-1)^j H_k(e_1, \ldots, \hat{e}_i, \ldots, \hat{e}_j, [e_i, e_j], e_{j+1}, \ldots, e_{p+1-2k}; f_1, \ldots, f_k).
\]

For each \( e_i \in \mathcal{E}, \nabla e_i \) acts as a derivation on \( H_k(e_1, \ldots, \hat{e}_i, \ldots, e_{p+1-2k}; f_1, \ldots, f_k) \) and it is a first order differential operator with respect to \( e_i \). Furthermore, \([e_i, e_j]\) is a first order differential operator on the first item. Hence, the order of the differential operators of the tuple \( ((d^\nabla H)_0, (d^\nabla H)_1, \ldots, (d^\nabla H)_{\frac{p+1}{2}}) \) is increased by 1 in each term with respect to the order of the operators of \( H = (H_0, H_1, \ldots, H_{\frac{p}{2}}) \) on the same term.

Elements of the space \( (d^\nabla)^m(C^s(\mathcal{E}; B)), m + s = p \), can be thought of as elements of \( \mathcal{D}^p_{m,m-1}(\mathcal{E}; B) \). One simply applies successively \( m \) times the Dorfman covariant derivative \( d^\nabla \) on \( C^s(\mathcal{E}; B) \cong C^s \otimes \Gamma(B) \). \(^6\) For \( \omega \otimes b \in C^s \otimes \Gamma(B) \), it is \( (d^\nabla)^m(\omega \otimes b) = ((d^\nabla)^m(\omega \otimes b))_0, \ldots, ((d^\nabla)^m(\omega \otimes b))_{\frac{m}{2}} \) with

\[
((d^\nabla)^m(\omega \otimes b))_k : \mathcal{E} \otimes_{p-2k} S^k \Omega^1 \to \Gamma(B).
\]

By induction on \( m \in \mathbb{N}^* \), one can show that for any \( k = 0, \ldots, \frac{m}{2} \),

\[
((d^\nabla)^m(\omega \otimes b))_k : \mathcal{E} \otimes_{p-2k} \to \text{Diff}_{m-2k}(S^k \Omega^1, \Gamma(B))
\]

is a differential operator of order \( m - 2k \) on the first \( p - 2k - 1 \) arguments and of \( (m - 2k - 1) \) order on the last argument.

Moreover, we may also prove by induction on \( m \in \mathbb{N} \) that

- if \( m = 2r, r \in \mathbb{N} \), \( (d^\nabla)^2(\omega \otimes b) = \omega \cdot (d^\nabla)^2 b \);

- if \( m = 2r + 1, r \in \mathbb{N} \), \( (d^\nabla)^{2r+1}(\omega \otimes b) = d\omega \cdot (d^\nabla)^{2r} b + (-1)^{r} \omega \cdot (d^\nabla)^{2r+1} b \).

As a result, the elements of the spaces \( (d^\nabla)^m(C^s(\mathcal{E}; B)), m + s = p \), are either of type \( \omega \cdot (d^\nabla)^m b \) or of type \( d\omega \cdot (d^\nabla)^{m-1} b \), with \( \omega \in C^s \).

Extend the operators \( (19) \) to operators on \( \mathcal{D}^*_s, m - 1 \otimes \Gamma(B) \) by

\[
i_f : \mathcal{D}^p_{m,m-1} \otimes \Gamma(B) \to \mathcal{D}^{p-2}_{m,m-1} \otimes \Gamma(B) \\
\omega \otimes b \mapsto (i_f \omega) \otimes b
\]

and

\[
i_e : \mathcal{D}^p_{m,m-1} \otimes \Gamma(B) \to \mathcal{D}^{p-1}_{m,m-1} \otimes \Gamma(B) \\
\omega \otimes b \mapsto (i_e \omega) \otimes b.
\]

\(^6\)Due to the Serre-Swan Theorem for finitely generated and projective modules over \( \mathcal{R} \) as \( \mathcal{E} \).

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More explicitly, for an element of type \( \omega \cdot (d\nabla)^m b \in \mathcal{D}^p_{m,m-1}(\mathcal{E};B) \), with \( \omega \in C^s \) and \( m+s = p \),

\[
i_f(\omega \cdot (d\nabla)^m b) = (i_f \omega) \cdot (d\nabla)^m b + \omega \cdot (i_f (d\nabla)^m b)
\]  

(38)

and

\[
i_e(\omega \cdot (d\nabla)^m b) = (i_e \omega) \cdot (d\nabla)^m b + (-1)^p \omega \cdot (i_e (d\nabla)^m b).
\]  

(39)

The commutator \((15)\) naturally extends to the space of graded endomorphisms of \( \mathcal{D}(\mathcal{E};B) \). This way we obtain the operators

\[
\nabla_e = \{i_e,d\nabla\} \equiv i_e \circ d\nabla + d\nabla \circ i_e,
\]  

(40)

\[
\mathcal{L}^\nabla_f = \{i_f,d\nabla\} \equiv i_f \circ d\nabla - d\nabla \circ i_f,
\]  

(41)

satisfying the identities

\[
\{i_{e_1},i_{e_2}\} = i_{e_1} \circ i_{e_2} + i_{e_2} \circ i_{e_1} = i_{-\langle e_1,e_2\rangle},
\]

\[
\{\nabla_{e_1},\nabla_{e_2}\} = \nabla_{e_1} \circ \nabla_{e_2} - \nabla_{e_2} \circ \nabla_{e_1} = i_{[e_1,e_2]}.
\]  

(42)

**Proposition 3.9** The following identities hold for the elements of \( \mathcal{D}^p_{m,m-1}(\mathcal{E};B) \equiv \mathcal{D}^p_{m,m-1} \otimes \Gamma(B) \):

\[
\nabla_e(\omega \otimes b) = (\mathcal{L}^\nabla_e \omega) \otimes b + \omega \cdot \nabla_e b \quad \text{and} \quad \mathcal{L}^\nabla_f(\omega \otimes b) = (i_d \mathcal{E} f \omega) \otimes b.
\]  

(43)

**Proof.** Let \( \omega \otimes b \in \mathcal{D}^p_{m,m-1} \otimes \Gamma(B) \). Then

\[
\nabla_e(\omega \otimes b) \overset{(40)}{=} (i_e \circ d\nabla)(\omega \otimes b) + (d\nabla \circ i_e)(\omega \otimes b)
\]

\[
\overset{(35)}{=} i_e(d\omega \otimes b + (-1)^p \omega \cdot d\nabla b) + d(i_e \omega) \otimes b + (-1)^{p-1}(i_e \omega) \cdot d\nabla b
\]

\[
\overset{(41)}{=} (i_e d\omega + d i_e \omega) \otimes b + (-1)^p (i_e \omega) \cdot d\nabla b + (-1)^{2p} \omega \otimes (i_e d\nabla b)
\]

\[
+ (-1)^{p-1}(i_e \omega) \cdot d\nabla b
\]

\[
\overset{(18)}{=} \mathcal{L} \omega \otimes b + \omega \cdot \nabla_e b.
\]

Also,

\[
\mathcal{L}^\nabla_f(\omega \otimes b) \overset{(41)}{=} (i_f \circ d\nabla - d\nabla \circ i_f)(\omega \otimes b)
\]

\[
\overset{(35)}{=} i_f(d\omega \otimes b + (-1)^p \omega \cdot d\nabla b) - d\nabla ((i_f \omega) \otimes b)
\]

\[
\overset{(35),(36),(38)}{=} (i_f d\omega) \otimes b + (-1)^p (i_f \omega) \cdot d\nabla b - d(i_f \omega) \otimes b - (-1)^{p-2}(i_f \omega) \cdot d\nabla b
\]

\[
\overset{(18)}{=} \mathcal{L} f \omega \otimes b
\]

\[
= (i_d \mathcal{E} f \omega) \otimes b.
\]

\[\blacktriangleleft\]

**Lemma 3.10** The map

\[
(d\nabla)^2 : \mathcal{D}^p_{m,m-1} \otimes \Gamma(B) \to \mathcal{D}^{p+2}_{m+2,m+1} \otimes \Gamma(B)
\]

is \( C^\infty(M,\mathbb{R}) \)-linear on the sections of \( B \).
Proof. For $\omega \otimes b \in \mathcal{D}^0_{m,m-1} \otimes \Gamma(B)$ and $f \in \mathcal{R}$,

\[
(d\nabla)^2(\omega \otimes (fb)) = \omega \cdot (d\nabla)^2(fb) = \omega \cdot d\nabla (d_E f \otimes b + fd\nabla b) = \omega \cdot (d(d_E f) \otimes b - d_E f \cdot d\nabla b + d_E f \cdot d\nabla b + f(d\nabla)^2 b) = \omega \cdot (f(d\nabla)^2 b).
\]

It follows that the map $(d\nabla)^2 : \Gamma(B) \to (d\nabla)^2(\Gamma(B)) \subset \mathcal{D}^2_{2,1}(\mathcal{E}; B)$ is an $\text{End}(B)$-valued element of $\mathcal{D}^2_{2,1}(\mathcal{E}; \mathcal{R})$. We identify it with an element $R^\nabla = (R^\nabla_0, R^\nabla_1)$ of $\mathcal{D}^2_{2,1}(\mathcal{E}; \mathcal{R}) \otimes \Gamma(\text{End}(B))$.

Definition 3.11 For any $E$-Dorfman connection $\nabla$ on a smooth vector bundle $B \to M$ with a predual structure of $E$, the element $(d\nabla)^2 \in \mathcal{D}^2_{2,1}(\mathcal{E}; \mathcal{R}) \otimes \Gamma(\text{End}(B))$ is called the curvature of $\nabla$. An $E$-Dorfman connection $\nabla$ on $B$ whose curvature $(d\nabla)^2$ is identically zero is called flat or $E$-Dorfman action on $B$. In this case, $B$ is also called $E$-Dorfman module.

If $\nabla$ is flat, $d\nabla$ is a differential on $\mathcal{D}(\mathcal{E}; B)$ and we denote by $\mathcal{H}^p(\mathcal{E}; B)$ the $p$-cohomology group of the cochain complex $(\mathcal{D}(\mathcal{E}; B), d\nabla)$.

Proposition 3.12 The curvature $(d\nabla)^2 : \Gamma(B) \to \mathcal{D}^2_{2,1}(\mathcal{E}; \mathcal{R}) \otimes \Gamma(B)$ of an $E$-Dorfman connection $\nabla$ on a predual vector bundle $B$ of $E$ satisfies the following identities:

\[
i_{e_2} \circ i_{e_1} ((d\nabla)^2 b) = \nabla_{e_1} \nabla_{e_2} b - \nabla_{e_2} \nabla_{e_1} b - \nabla_{[e_1,e_2]} b,
\]

\[
i_f((d\nabla)^2 b) = \nabla_{d_E f} b.
\]

Furthermore, the restriction of $(d\nabla)^2$ on $\text{Im} d_E$ vanishes.

Proof. Let $b \in \Gamma(B)$, $e_1, e_2 \in \Gamma(E)$, and $f \in \mathcal{R} \cong \mathcal{C}^\infty(M, \mathbb{R})$. Then one has

\[
i_{e_2} \circ i_{e_1} ((d\nabla)^2 b) = i_{e_2} \circ (i_{e_1} \circ d\nabla)(d\nabla b)
\]

\[
\overset{(40)}{=} i_{e_2} \circ (\nabla_{e_1} - d\nabla \circ i_{e_1})(d\nabla b)
\]

\[
= (i_{e_2} \circ \nabla_{e_1})(d\nabla b) - (i_{e_2} \circ d\nabla \circ i_{e_1})(d\nabla b)
\]

\[
\overset{(42)}{=} (\nabla_{e_1} \circ i_{e_2} - i_{[e_1,e_2]})(d\nabla b) - (\nabla_{e_2} - d\nabla \circ i_{e_2})(d\nabla b)
\]

\[
= (\nabla_{e_1} \nabla_{e_2} - \nabla_{e_2} \nabla_{e_1} - \nabla_{[e_1,e_2]})(b),
\]

which is the well known formula of curvature. We also write $i_{e_2} \circ i_{e_1} ((d\nabla)^2 b) = R^\nabla_0 (e_1, e_2) b$, and so

\[
R^\nabla_0 (e_1, e_2) b = \nabla_{e_1} \nabla_{e_2} b - \nabla_{e_2} \nabla_{e_1} b - \nabla_{[e_1,e_2]} b.
\]

(44)

Since $i_f$ is of degree $-2$, it is

\[
i_f((d\nabla)^2 b) = (i_f \circ d\nabla)(d\nabla b) \overset{(41)}{=} (\mathcal{L}^\nabla_f + d\nabla \circ i_f)(d\nabla b)
\]

\[
= \mathcal{L}^\nabla_f (d\nabla b) \overset{(43)}{=} i_{d_E f} (d\nabla b) = \nabla_{d_E f} b,
\]

and so

\[
R^\nabla_1 (f) b = \nabla_{d_E f} b.
\]

(45)

Clearly $R^\nabla_0 (e_1, e_2) d_B g = 0$ and $R^\nabla_1 (f) d_B g = 0$, for any $g \in \mathcal{C}^\infty(M, \mathbb{R})$. Thus, $R^\nabla = (R^\nabla_0, R^\nabla_1)$ is an element of $\mathcal{D}^2_{2,1}(\mathcal{E}; \mathcal{R}) \otimes \Gamma((B/\text{Im} d_B)^* \otimes B)$.

One can justify the claim that $(d\nabla)^2(b)$ is an element of $\mathcal{D}^2_{2,1}(\mathcal{E}, \mathcal{R}) \otimes \Gamma(B)$, by computing the symbols of $R^\nabla_0$ and $R^\nabla_1$. After a straightforward calculation, we get that, for any $e_1, e_2 \in \Gamma(E)$ and $f \in \mathcal{C}^\infty(M, \mathbb{R})$:
1. The symbol $\sigma_1(R_0^\nabla)(f)$ of $R_0^\nabla$ is a 1-order differential operator in the first argument since

$$
\sigma_1(R_0^\nabla)(f)(e_1, e_2) = R_0^\nabla(f e_1, e_2)b - f R_0^\nabla(e_1, e_2)b
$$

$$
= (\langle [e_2, e_1], b \rangle + \langle e_1, \nabla e_2 b \rangle - \rho(e_2)\langle e_1, b \rangle) dB f
$$

$$
- \langle d_E\langle e_1, e_2 \rangle, b \rangle dB f - \nabla\langle e_1, e_2 \rangle dB f
$$

$$
= \langle e_1, D_b e_2 \rangle dB f - \langle d_E\langle e_1, e_2 \rangle, b \rangle dB f - \nabla\langle e_1, e_2 \rangle dB f.
$$

2. The symbol $\sigma_2(R_0^\nabla)(f)$ of $R_0^\nabla$ is a 0-order (i.e. $C^\infty(M, \mathbb{R})$-linear) differential operator in the second argument since

$$
\sigma_2(R_0^\nabla)(f)(e_1, e_2) = R_0^\nabla(e_1, f e_2)b - f R_0^\nabla(e_1, e_2)b
$$

$$
= (\rho(e_1)\langle e_2, b \rangle - \langle [e_1, e_2], b \rangle - \langle e_2, \nabla e_1 b \rangle) dB f
$$

$$
= -\langle D_b e_1, e_2 \rangle dB f.
$$

3.4 Induced connections and Bianchi identity

Definition 3.13 The dual $E$-Dorfman connection $\nabla^*$ of an $E$-Dorfman connection $\nabla$ on a predual vector bundle $B$ of $E$ is the map

$$
\nabla^*: \Gamma(E) \times \Gamma(B^*) \to \Gamma(B^*),
$$

such that

$$
\rho(e)\langle b^*, b \rangle = \langle \nabla^*_e b^*, b \rangle + \langle b^*, \nabla_e b \rangle,
$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $\Gamma(B^*)$ and $\Gamma(B)$.

By the properties of $\nabla$ (see Definition 3.3) one has that

1. $\nabla^*_e b^* f = f \nabla^*_e b^* f - \langle b^*, d_B f \rangle \langle e, \cdot \rangle$, and

2. $\nabla^*_e (f b^*) = f \nabla^*_e b^* + \rho(e)(f)b^*$.

The curvature $R^{\nabla^*} = (R_0^{\nabla^*}, R_1^{\nabla^*})$ of $\nabla^*$ is then defined by the relations

$$
\langle R_0^{\nabla^*}(e_1, e_2)b^*, b \rangle + \langle b^*, R_0^\nabla(e_1, e_2)b \rangle = 0 \quad \text{and} \quad \langle R_1^{\nabla^*}(f)b^*, b \rangle + \langle b^*, R_1^\nabla(f)b \rangle = 0. \quad (46)
$$

As in the classical case, $\nabla$ induces an $E$-Dorfman connection on any tensor bundle constructed from $B$. In particular, the pair $(\nabla^*, \nabla)$ induces an $E$-Dorfman connection $\nabla$ on $B^* \otimes B \cong \text{End}(B)$ by

$$
\nabla_e (b^* \otimes b) = \nabla^*_e b^* \otimes b + b^* \otimes \nabla_e b.
$$

It is then a simple calculation to check that the connection $\nabla$ has the following properties:

1. $\nabla_{fe} (b^* \otimes b) = f \nabla_e (b^* \otimes b) - \langle b^*, d_B f \rangle \langle e, \cdot \rangle \otimes b + \langle e, b \rangle b^* \otimes d_B f$,

2. $\nabla_e f (b^* \otimes b) = f \nabla_e (b^* \otimes b) + \rho(e)(f) (b^* \otimes b)$,

3. $\nabla_e (b^* \otimes d_B f) = \nabla^*_e b^* \otimes d_B f + b^* \otimes d_B (L_{\rho(e)} f)$.

Note that the space $\Gamma(B^* \otimes \text{Im}d_B) \subset \Gamma(B^* \otimes B)$ is invariant by $\nabla_e$.

Proposition 3.14 Let $e \in \Gamma(E)$ and $\tau \in \Gamma(\text{End}(B))$. Then $\nabla$ satisfies

$$
\nabla_e \tau = [\nabla_e, \tau] = \nabla_e \circ \tau - \tau \circ \nabla_e. \quad (47)
$$
Denote by $\mathcal{D}^p_{m,m-1}(\mathcal{E}; \text{End}(B))$ the spaces of $\text{End}(B)$-valued $p$-cochains, $m \in \mathbb{N}^*$, and set $\mathcal{D}(\mathcal{E}; \text{End}(B)) = (\mathcal{D}^p_{m,m-1}(\mathcal{E}; \text{End}(B)))_{p \geq 0, m \in \mathbb{N}^*}$. Each element $\Phi$ of $\mathcal{D}^p_{m,m-1}(\mathcal{E}; \text{End}(B))$ is determined by $([\frac{k}{2}] + 1)$-tuples $\Phi = (\Phi_0, \Phi_1, \ldots, \Phi_{[k]})$ of homomorphisms

$$\Phi_k : \mathfrak{c}^{p-2k} \otimes S^k \Omega^1 \to \Gamma(\text{End}(B))$$

categorized by similar conditions to those of operators $\hat{\omega}_k \in \mathcal{D}^p_{m,m-1}$ and $H_k \in \mathcal{D}^p_{m,m-1}(\mathcal{E}, B)$ (see, respectively, sections 2.2.3 and 3.3). The space $\mathcal{D}^p_{m,m-1}(\mathcal{E}; \text{End}(B))$ is identified with $\mathcal{D}^p_{m,m-1} \otimes \Gamma(\text{End}(B))$ and $\mathcal{D}(\mathcal{E}; \text{End}(B)) \cong \mathcal{D} \otimes \Gamma(\text{End}(B))$.

Note that for $p = 1$, it is $\Phi = (\Phi_0)$ and the unique argument of the map $\Phi_0 : \mathcal{E} \to \Gamma(\text{End}(B))$ is considered as first argument. Hence, we denote by $\mathcal{D}^1_m(\mathcal{E}; \text{End}(B))$ spaces of differential operators on $\mathcal{E}$ of order at most $m$, $m \in \mathbb{N}^*$, with values in $\text{End}(B)$.

The $E$-Dorfman connection $\tilde{\nabla}$ on $\text{End}(B)$ defines a covariant derivation operator

$$d\tilde{\nabla} : \Gamma(\text{End}(B)) \to \mathcal{D}^1 \otimes \Gamma(\text{End}(B))$$

such that, for any $f \in C^\infty(M, \mathbb{R})$ and $\tau \in \Gamma(\text{End}(B))$, it is

$$d\tilde{\nabla}(f \tau) = d_E f \otimes \tau + f d\tilde{\nabla} \tau.$$

This extends uniquely to an operator of degree +1, denoted also $d\tilde{\nabla}$, on the space of $\text{End}(B)$-valued $p$-cochains $\mathcal{D}^p_{m,m-1} \otimes \Gamma(\text{End}(B))$:

$$d\tilde{\nabla} : \mathcal{D}^p_{m,m-1} \otimes \Gamma(\text{End}(B)) \to \mathcal{D}^{p+1}_{m+1,m} \otimes \Gamma(\text{End}(B)).$$

More precisely, the image of an element $\Phi = (\Phi_0, \Phi_1, \ldots, \Phi_{[k]}) \in \mathcal{D}^p_{m,m-1} \otimes \Gamma(\text{End}(B))$, is

$$d\tilde{\nabla} \Phi = ((d\tilde{\nabla} \Phi)_0, (d\tilde{\nabla} \Phi)_1, \ldots, (d\tilde{\nabla} \Phi)[2k+1]) \in \mathcal{D}^{p+1}_{m+1,m} \otimes \Gamma(\text{End}(B))$$

where

$$(d\tilde{\nabla} \Phi)_k(e_1, \ldots, e_{p+1-2k}; f_1, \ldots, f_k) = \sum_{\mu=1}^k \Phi_{k-1}(d_E f_{\mu}, e_1, \ldots, e_{p+1-2k}; f_1, \ldots, f_{\mu}, f_k) + \sum_{i=1}^{p+1-2k} (-1)^{i-1} \tilde{\nabla} e_i (\Phi_k(e_1, \ldots, e_i, \ldots, e_{p+1-2k}; f_1, \ldots, f_k)) + \sum_{i<j} (-1)^j \Phi_k(e_1, \ldots, e_i, \ldots, e_j, \ldots, [e_i, e_j], e_{j+1}, \ldots, e_{p+1-2k}; f_1, \ldots, f_k).$$

(48)

Proposition 3.15 (Bianchi identity) Let $(E, B, \nabla, R^\nabla)$ as above. The Bianchi identity

$$d\tilde{\nabla}(R^\nabla) = 0$$

holds.

Proof. Consider the curvature $R^\nabla = (R^\nabla_0, R^\nabla_1) \in \mathcal{D}_{2,1}^2(\mathcal{E}; \text{End}(B))$. Its image through $d\tilde{\nabla}$ is $d\tilde{\nabla}(R^\nabla) = ((d\tilde{\nabla}(R^\nabla)_0), (d\tilde{\nabla}(R^\nabla)_1)) \in \mathcal{D}_{3,2}^3(\mathcal{E}; \text{End}(B))$ and for any $e_1, e_2, e_3 \in \Gamma(E)$ it is

$$(d\tilde{\nabla}(R^\nabla))_{0}(e_1, e_2, e_3) = \tilde{\nabla} e_1 (R^\nabla_0(e_2, e_3)) - \tilde{\nabla} e_2 (R^\nabla_0(e_1, e_3)) + \tilde{\nabla} e_3 (R^\nabla_0(e_1, e_2)) - R^\nabla_0([e_1, e_2], e_3) - R^\nabla_0(e_2, [e_1, e_3]) + R^\nabla_0(e_1, [e_2, e_3])$$

(48)

$$= \nabla e_1 \circ R^\nabla_0(e_2, e_3) - R^\nabla_0(e_2, e_3) \circ \nabla e_1 - \nabla e_2 \circ R^\nabla_0(e_1, e_3) + R^\nabla_0(e_1, e_2) \circ \nabla e_3 - R^\nabla_0([e_1, e_2], e_3) - R^\nabla_0(e_2, [e_1, e_3]) + R^\nabla_0(e_1, [e_2, e_3])$$

$$= 0.$$
For the last equation use the curvature expression (44) and the fact that the bracket $[\cdot, \cdot]$ verifies the Jacobi identity (2). Similarly, for $(d\tilde{\nabla})(R\nabla))_1$ one gets

$$(d\tilde{\nabla})(R\nabla))_1(e; f) \overset{(48)}{=} R^\nabla_0(d_E f, e) + \tilde{\nabla}_e(R^\nabla_1(f))$$

$$(d\tilde{\nabla})(R\nabla))_1(e; f) \overset{(47)}{=} R^\nabla_0(d_E f, e) + \nabla_e \circ R^\nabla_1(f) - R^\nabla_1(f) \circ \nabla_e$$

$$(d\tilde{\nabla})(R\nabla))_1(e; f) \overset{(44),(45)}{=} \nabla_{d_E f} \nabla e - \nabla e \nabla_{d_E f} - \nabla_{[d_E f, e]} + \nabla e \nabla_{d_E f} - \nabla_{d_E f} \nabla e$$

$$= 0. \quad \text{(50)}$$

With respect to the extension $\tilde{\nabla}$ one can prove that its curvature $R\tilde{\nabla} = (R^\nabla_0, R^\nabla_1)$ is given, for any $e_1, e_2 \in \Gamma(E), f \in C^\infty(M, \mathbb{R})$, and $\tau \in \Gamma(\text{End}(B))$, by

$$R^\nabla_0(e_1, e_2) \tau = R^\nabla_0(e_1, e_2) \circ \tau - \tau \circ R^\nabla_0(e_1, e_2) \quad \text{and} \quad R^\nabla_1(f) \tau = R^\nabla_1(f) \circ \tau - \tau \circ R^\nabla_1(f). \quad \text{(51)}$$

**Remark 3.16** From (46) and (51), respectively, it follows that if $\nabla$ is a flat $E$-Dorfman connection on $B$, then $\nabla^*$ is also a flat $E$-Dorfman connection on $B^*$ and $\tilde{\nabla}$ is a flat $E$-Dorfman connection on $B^* \oplus B$.

### 3.5 Examples of Dorfman connections

**Example 3.17** This example is inspired by [20, Example 4.2]. Consider the standard Courant algebroid $E = TM \oplus T^*M$ (Example 2.3) and a linear $TM$-connection $\triangle$ on $TM$.\(^7\) Let $\triangle^*$ be its dual connection on $T^*M$. The map

$$\nabla : \Gamma(E) \times \Gamma(E) \to \Gamma(E),$$

defined, for any $(X, \zeta), (Y, \eta) \in \Gamma(E)$, by

$$\nabla_{(X, \zeta)}(Y, \eta) = (\triangle_X Y, \mathcal{L}_X \eta + \langle \triangle^* \zeta, Y \rangle),$$

defines an $E$-Dorfman connection on $(E, d_E, \langle \cdot, \cdot \rangle)$. Its dual $E$-Dorfman connection $\nabla^*$ on $E^* = T^*M \oplus TM$ is given, for any $(X, \zeta) \in \Gamma(E)$ and $(\eta, Y) \in \Gamma(E^*)$, by

$$\nabla^*_{(X, \zeta)}(\eta, Y) = (\triangle^* \eta - \triangle_X \zeta, \triangle_X Y + \langle \triangle^* \zeta, -\mathcal{L}_X \eta, Y \rangle).$$

**Examples 3.18 (Regular Courant algebroids)** Let $(E, [\cdot, \cdot], \langle \cdot, \cdot \rangle, \rho)$ be a regular Courant algebroid, i.e. $F := \rho(E) \subseteq TM$ is an integrable distribution of constant rank on the base manifold $M$ and so defines a regular foliation of $M$. Then, $\ker \rho$ and its orthogonal (ker $\rho$)⊥, with respect to the metric $\langle \cdot, \cdot \rangle$, are constant rank smooth subbundles of $E$. It can be checked that $G = \ker \rho/(\ker \rho)\bot$ is a bundle of quadratic Lie algebras over $M$ and, as it was proved in [9], $E$ is isomorphic to $F^* \oplus G \oplus F$. In this case, $d_E f = d_F f + 0 + 0$, where $d_F : C^\infty(M, \mathbb{R}) \to \Gamma(F^*)$ denotes the leafwise de Rham differential. In the following, we construct two examples of Dorfman connections in the framework of regular Courant algebroids.

**Example 1.** Consider the vector bundle of constant rank $B = F \oplus F^*$ endowed with the natural predual structure $(\langle \cdot, \cdot \rangle, d_B)$ of $E$. More precisely, $\langle \cdot, \cdot \rangle : E \times_M B \to \mathbb{R}$ is given, for any $\zeta + r + X \in \Gamma(E)$ and $Y + \eta \in \Gamma(B)$, by

$$\langle \zeta + r + X, Y + \eta \rangle = \langle \eta, X \rangle + \langle \zeta, Y \rangle$$

For a vector bundle $A \to M$, a linear $TM$-connection $\triangle$ on $A$ is an $\mathbb{R}$-bilinear map $\triangle : \Gamma(TM) \times \Gamma(A) \to \Gamma(A)$ such that: (i) $\triangle_X a = f \triangle_X a$, (ii) $\triangle_X fa = f \triangle_X a + X(f)a$. It is called also Koszul connection and it always exists [31, p. 185].
and \( df = 0 + dF \). Let \( pr_{F^*} : F \oplus F^* \to F^* \) be the projection onto the second summand and \( \mathcal{Q} : \Gamma(F) \times \Gamma(\mathcal{G}) \to \Gamma(F^*) \) be a \( C^\infty(M, \mathbb{R}) \)-bilinear map defined by the Courant algebroid structure on \( F^* \oplus \mathcal{G} \oplus F \), [9, Lemma 2.1]. Choose a classical \( F \)-connection \( \Delta \) on \( F \) (there always exists one) and denote by \( \Delta^* \) its dual connection on \( F^* \). One can then check directly that the map
\[
\nabla: \Gamma(F^* \oplus \mathcal{G} \oplus F) \times \Gamma(F \oplus F^*) \to \Gamma(F \oplus F^*)
\]
defined, for \( \zeta + r + X \in \Gamma(F^* \oplus \mathcal{G} \oplus F) \) and \( Y + \eta \in \Gamma(F \oplus F^*) \), by
\[
\nabla_{\zeta + r + X}(Y + \eta) = ([X, Y] + \Delta X) + pr_{F^*}([L_X \eta - i(Y)d\zeta] + \Delta Y \zeta + \mathcal{Q}(Y, r))
\]
is a \( F^* \oplus \mathcal{G} \oplus F \)-Dorfman connection on \( F \oplus F^* \).

**Example 2.** According to Proposition 4.12 in [18], the bundle of a quadratic Lie algebra \( \mathcal{G} \), endowed with the induced Courant algebroid structure from the one of \( F^* \oplus \mathcal{G} \oplus F \), is a Courant algebroid. More precisely, if \( \iota : \mathcal{G} \to F^* \oplus \mathcal{G} \oplus F \) is the injection of \( \mathcal{G} \) into \( F^* \oplus \mathcal{G} \oplus F \) and \( pr_\mathcal{G} : F^* \oplus \mathcal{G} \oplus F \to \mathcal{G} \) the projection of \( F^* \oplus \mathcal{G} \oplus F \) on the second summand, the Dorfman bracket on \( \Gamma(\mathcal{G}) \) is given, for any \( r_1, r_2 \in \Gamma(\mathcal{G}) \), by \([r_1, r_2]\mathcal{G} = pr_\mathcal{G}(\iota(r_1), \iota(r_2))\), the anchor map by \( \rho_\mathcal{G} = \rho \circ \iota = 0 \), and the inner product by \( \langle r_1, r_2 \rangle_\mathcal{G} = \langle \iota(r_1), \iota(r_2) \rangle \). Consider the map
\[
\nabla : \Gamma(\mathcal{G}) \times \Gamma(F^* \oplus \mathcal{G} \oplus F) \to \Gamma(F^* \oplus \mathcal{G} \oplus F)
\]
defined, for any \( r \in \Gamma(\mathcal{G}) \) and \( \xi + s + X \in \Gamma(F^* \oplus \mathcal{G} \oplus F) \), by
\[
\nabla_r(\xi + s + X) = [\iota(r), \xi + s + X] + 2Q(X, r) - \iota_{Xr}, \tag{52}
\]
where \( Q \) is the map mentioned in the previous example and \( \iota : \Gamma(F) \times \Gamma(\mathcal{G}) \to \Gamma(\mathcal{G}) \) is the \( F \)-linear connection on \( \mathcal{G} \) provided by the Courant algebroid structure on \( F^* \oplus \mathcal{G} \oplus F \), [9]. Then, (52) yields a \( \mathcal{G} \)-Dorfman connection on \( F^* \oplus \mathcal{G} \oplus F \).

**Proposition 3.19** ([20]) Let \((E, L)\) be a Manin pair and \( L^0 \) the annihilator of \( L \) in \( E^* \).

1. The quotient \( E/L \) is a \( L \)-Dorfman module with respect to the Dorfman action \( \nabla^L : \Gamma(L) \times \Gamma(E/L) \to \Gamma(E/L) \) defined, for any \( l \in \Gamma(L) \) and \( \bar{e} \in \Gamma(E/L) \), by
\[
\nabla^L_l \bar{e} = [l, \bar{e}],
\]
where \( \bar{e} \) denotes the class of \( e \in \Gamma(E) \) in \( \Gamma(E/L) \).

2. The space \( (E/L)^* \cong L^0 \) is also a \( L \)-Dorfman module relative to the dual \( L \)-Dorfman connection \( (\nabla^L)^* \).

**Proof.** For the first point, note that, since \( L \) is Lagrangian, \( E/L \) is isomorphic to the dual bundle \( L^* \) of \( L \) and the symmetric bilinear form of the Courant algebroid structure on \( E \) induces a nondegenerate pairing \( \langle \cdot, \cdot \rangle : L \times M E/L \to \mathbb{R} \). Further equip the vector bundle \( E/L \to M \) with the map \( d_{E/L} : C^\infty(M, \mathbb{R}) \to \Gamma(E/L) \) given by \( d_{E/L}(f) = d_{E}f \). Then \( E/L \) is a predual bundle of \( L \) in the sense of Definition 3.1. It is easy to check that \( \nabla^L \) satisfies the axioms of a Dorfman connection. As an element of \( \Gamma(\wedge^2 L^*) \otimes \Gamma(\text{End}(E/L)) \), the curvature \( R_{\nabla^L} \) vanishes identically on the sections of \( L \). For \( l_1, l_2 \in \Gamma(L), \bar{e} \in \Gamma(E/L) \), we have
\[
R_{\nabla^L}(l_1, l_2)\bar{e} = \nabla^L_{l_1} \nabla^L_{l_2} \bar{e} - \nabla^L_{l_2} \nabla^L_{l_1} \bar{e} - \nabla^L_{[l_1, l_2]} \bar{e}
\]
\[
= [l_1, [l_2, \bar{e}]] - [l_2, [l_1, \bar{e}]] - [[l_1, l_2], \bar{e}]
\]
\[
\leq 0.
\]
For the second point, let \((\nabla^L)^*: \Gamma(L) \times \Gamma((E/L)^*) \to \Gamma((E/L)^*)\) be the dual \(L\)-Dorfman connection of \(\nabla^L\) and \(R(\nabla^L)^*\) be its curvature. By (46) it follows that \((E/L)^*\) is an \(L\)-Dorfman module.

Remarks 3.20

1. In [20] it was noted that the Dorfman connection \(\nabla^L\) of the last Proposition is analogous with the Bott connection defined by an involutive subbundle of \(TM\). For this reason, it is named Bott-Dorfman connection associated to \(L\).

2. Since \(L\) is a Lagrangian subbundle, hence \(E/L \cong L^*\) and, because \(\nabla^L\) is flat, therefore \((d\nabla^L)^2 = 0\), where the covariant derivation \(d\nabla^L\) is the Lie algebroid differential of \(L\) on \(\Gamma(L^*)\), [31, 32].

Example 3.21 (Courant algebroid related to port-Hamiltonian systems) Port–Hamiltonian systems are a generalization of Hamiltonian systems that aim to describe the dynamics of a Hamiltonian system in interaction with control units, energy dissipating or energy storing units (ports), [47]. The state space of such a system is modeled by a manifold \(M\) endowed with a Dirac structure \(L\) in a Courant algebroid, [34]. More specifically, start with the standard Courant algebroid \(TM \oplus T^*M\), a vector bundle \(V\) over \(M\) endowed with a flat \(TM\)-connection \(\Delta\), and its dual bundle \(V^*\) equipped with the dual connection \(\Delta^*\). The sections \((\lambda_{\text{out}}, \lambda_{\text{in}})\) of \(V \oplus V^*\) model the output and input of the port. The vector bundle \(E = TM \oplus T^*M \oplus V \oplus V^*\) equipped with the projection \(\rho: E \to TM\) as anchor map, the symmetric nondegenerate bilinear form

\[
\langle (X, \zeta, \lambda_{\text{out}}, \lambda_{\text{in}}), (Y, \eta, \mu_{\text{out}}, \mu_{\text{in}}) \rangle = \langle \eta, X \rangle + \langle \zeta, Y \rangle + \langle \mu_{\text{in}}, \lambda_{\text{out}} \rangle + \langle \lambda_{\text{in}}, \mu_{\text{out}} \rangle,
\]

and the bracket

\[
\begin{align*}
\llbracket (X, \zeta, \lambda_{\text{out}}, \lambda_{\text{in}}), (Y, \eta, \mu_{\text{out}}, \mu_{\text{in}}) \rrbracket = & \\
& ([X, Y], \mathcal{L}_X \eta - i(Y)d\zeta + \langle \Delta^* \lambda_{\text{in}}, \mu_{\text{out}} \rangle + \langle \mu_{\text{in}}, \Delta \lambda_{\text{out}} \rangle, \\
& \Delta_X \mu_{\text{out}} = \Delta_Y \lambda_{\text{out}}, \Delta_X \lambda_{\text{in}} - \Delta_Y \mu_{\text{in}})
\end{align*}
\]

is a Courant algebroid over \(M\). The dynamics of the system are determined by a Hamiltonian \(H\) via the Hamiltonian condition \((\dot{x}, dH, \lambda_{\text{out}}, \lambda_{\text{in}}) \in \Gamma(L)\).

Consider the vector bundle \(B = T^*M \oplus V\). Clearly, the natural coupling of \(T^*M\) with \(TM\) and of \(V\) with \(V^*\), gives a coupling between \(E\) and \(B\) which, with the map \(d_B: C^\infty(M, \mathbb{R}) \to \Gamma(B), d_B f = (df, 0)\), define a predual structure of \(E\) on \(B\). Then, the map \(\nabla: \Gamma(E) \times \Gamma(B) \to \Gamma(B)\) defined, for any \(e = (X, \zeta, \lambda_{\text{out}}, \lambda_{\text{in}}) \in \Gamma(E)\) and \(b = (\eta, \mu_{\text{out}}) \in \Gamma(B)\), by

\[
\nabla_e b = (\mathcal{L}_X \eta + \langle \Delta^* \lambda_{\text{in}}, \mu_{\text{out}} \rangle, \Delta_X \mu_{\text{out}}),
\]

establishes an \(E\)-Dorfman connection on \(B\). It is the projection on \(\Gamma(B)\) of the restriction of \([\cdot, \cdot]\) on \(\Gamma(E) \times \Gamma(B)\).

We arrive at a similar result, if we consider \(B' = T^*M \oplus V^*\), which evidently has a predual structure of \(E\), and take \(\nabla': \Gamma(E) \times \Gamma(B') \to \Gamma(B')\) to be the projection on \(\Gamma(B')\) of the restriction of \([\cdot, \cdot]\) on \(\Gamma(E) \times \Gamma(B')\). More precisely, for any \(e = (X, \zeta, \lambda_{\text{out}}, \lambda_{\text{in}}) \in \Gamma(E)\) and \(b' = (\eta, \mu_{\text{in}}) \in \Gamma(B')\),

\[
\nabla'_e b' = (\mathcal{L}_X \eta + \langle \mu_{\text{in}}, \Delta \lambda_{\text{out}} \rangle, \Delta_X \mu_{\text{in}}).
\]
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