The capability and certain functors of some nilpotent lie algebras of class two

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ABSTRACT
Recently, the authors obtained the Schur multiplier, the non-abelian tensor square and the non-abelian exterior square of \( d \)-generator generalized Heisenberg Lie algebras of rank 1. Here, we intend to obtain the same results for \( d \)-generator generalized Heisenberg Lie algebras of rank \( t \) when \( \frac{1}{2}d(d - 1) - 3 \leq t \leq \frac{1}{2}d(d - 1) - 1 \). Then, as a result, we give similar consequences for a nilpotent Lie algebra \( L \) of class two when \( \dim(L/Z(L)) = d \), \( \dim L^2 = t \) such that \( \frac{1}{2}d(d - 1) - 3 \leq t \leq \frac{1}{2}d(d - 1) - 1 \).

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1. Motivation and preliminary results

The Schur multiplier of a group \( G \) comes from the work of Schur in 1904. It is well known that for a group \( G \) with a free representation \( F/R \), we have

\[
\mathcal{M}(G) \cong (R \cap F')/[R, F].
\]

There are many papers devoted to obtain the structure of the Schur multiplier of groups, when the structure of groups is in hand. For instance, the Schur multiplier of abelian groups, extra-special \( p \)-groups, and generalized extra-special \( p \)-groups are obtained in [1,2]. Rai [3] succeeded in obtaining the Schur multiplier of special \( p \)-groups minimality generated by \( d \geq 3 \) elements with derived subgroup of order \( p^{\frac{1}{2}d(d-1)} \). It is well known that the results of the class of finite \( p \)-groups have analogues on the class of finite-dimensional nilpotent Lie algebras. This motivation allows us to write this paper. Let \( L \) be a Lie algebra and \( L \cong F/R \) for a free Lie algebra \( F \). It is well known that the Schur multiplier \( \mathcal{M}(L) \) of \( L \) is equal to \( (R \cap F^2)/[R, F] \). Recall that a Lie algebra \( L \) is called capable if and only if \( L \cong E/Z(E) \) for some Lie algebra \( E \). By the results of [4–9], the Schur multiplier and the capability of nilpotent Lie algebras with derived subalgebra of dimension \( t \) such that...
Clearly, if \( \Theta \in \mathfrak{g} \) is bounded. If \( \mathfrak{g} \) is exact. Moreover, if any two elements of the set \( \mathfrak{g} \) are obtained. Recall that a Lie algebra \( H \) is called a generalized Heisenberg Lie algebra of rank \( n \) if \( H^2 = Z(H) \) and \( \dim H^2 = n \). The capability and the structures of the Schur multiplier, the non-abelian tensor square, and the exterior square of \( d \)-generator generalized Heisenberg Lie algebras of rank \( \frac{1}{2}d(d-1) \) when \( d \geq 3 \) are given in [10]. The motivation of this paper is to obtain the Schur multiplier, the non-abelian tensor square and the exterior square of \( d \)-generator generalized Heisenberg Lie algebras of rank \( t \) such that \( \frac{1}{2}d(d-1) - 3 \leq t \leq \frac{1}{2}d(d-1) - 1 \). Moreover, we give the same results for a nilpotent Lie algebra \( L \) of class two with \( \dim(L/Z(L)) = d \) and \( \dim L^2 = t \) such that \( \frac{1}{2}d(d-1) - 3 \leq t \leq \frac{1}{2}d(d-1) - 1 \). Finally, we show that all of \( d \)-generator generalized Heisenberg Lie algebras of rank \( \frac{1}{2}d(d-1) - 1 \) and \( \frac{1}{2}d(d-1) - 2 \) are capable and also we show that all of nilpotent Lie algebras of class two with central factor of dimension \( d \) and derived subalgebra of dimension \( t \) such that \( \frac{1}{2}d(d-1) - 2 \leq t \leq \frac{1}{2}d(d-1) - 1 \) are capable. Throughout the paper \( \mathcal{A}(n) \) is used to denote an abelian Lie algebra of dimension \( n \). For a given Lie algebra \( L \) and \( x \in L \), let \( Y = L/(L^2 + Z(L)) \) and \( \tilde{x} = x + L^2 + Z(L) \). Then

**Theorem 1.1 ([11, Proposition 2.3]):** Let \( L \) be a finite-dimensional nilpotent Lie algebra of class two. Then the map

\[
\Psi_2 : Y \otimes Y \otimes Y \to L^2 \otimes L/L^2
\]

given by \( \bar{x} \otimes \bar{y} \otimes \bar{z} \mapsto ((x,y) \otimes z + L^2) + ((z,x) \otimes y + L^2) + ([y,z] \otimes x + L^2) \) is a homomorphism. Moreover, if any two elements of the set \( \{x,y,z\} \) are linearly dependent, then \( \Psi_2(\bar{x} \otimes \bar{y} \otimes \bar{z}) \) is identity.

Similar to the result of Blackburn for the group theory case (see [12]), we begin with the following result for Lie algebras. Let \( L \) be a finite-dimensional nilpotent Lie algebra of class two with a free presentation \( 0 \rightarrow R \rightarrow F \xrightarrow{\pi} L \rightarrow 0 \) for a free Lie algebra \( F \) and an ideal \( R \) in \( F \). If \( x \in L \), then \( \bar{x} \in F \) denotes a fixed pre-image of \( x \) under \( \pi \). As was shown in [13, Proposition 4.1], the following map is a homomorphism:

\[
\beta : L^2 \otimes L/L^2 \rightarrow \mathcal{M}(L) = [\bar{x}, \bar{z}] + [R, F],
\]

\[
x \otimes (z + L^2) \mapsto [\bar{x}, \bar{z}] + [R, F].
\]

\[
\text{(1)}
\]

Clearly, \( \text{Im}(\beta) = F^3 + [R, F]/[R, F] \).

**Lemma 1.2 ([11, Theorem 2.1] and [13, Proposition 4.1]):** For a given finite-dimensional nilpotent Lie algebra \( L \) of class two, the sequence

\[
0 \rightarrow \ker \beta \rightarrow L^2 \otimes L/L^2 \xrightarrow{\beta} \mathcal{M}(L) \rightarrow \mathcal{M}(L/L^2) \rightarrow L^2 \rightarrow 0
\]

is exact. Moreover,

\[
K = \langle [x,y] \otimes z + L^2 + [z,x] \otimes y + L^2 + [y,z] \otimes x + L^2 | x, y, z \in L \rangle \subseteq \ker \beta.
\]

Recall that a pair of Lie algebras \((E, M)\) is said to be a defining pair for a Lie algebra \( L \) if \( M \subseteq Z(E) \cap E^2 \) and \( E/M \cong L \). If \( L \) is finite-dimensional, then the dimension of \( E \) is bounded. If \((E, M)\) is a defining pair for \( L \), then a \( E \) of maximal dimension is called
a cover for \( L \). Moreover, from [6,14] in this case \( M \cong \mathcal{M}(L) \). From [6], all covers of a finite-dimensional Lie algebra \( L \) are isomorphic.

Let \( L \) be a finite-dimensional nilpotent Lie algebra of class two such that

\[
M^* = (L/L^2 \wedge L/L^2) \oplus ((L^2 \otimes L/L^2)/K) \quad \text{and} \quad N \cong ((L^2 \otimes L/L^2)/K),
\]

where \( K \) defined in Lemma 1.2. Then \( M^*/N \cong (L/L^2 \wedge L/L^2) \). It is clear that the map \( \rho : L/L^2 \wedge L/L^2 \to L^2 \) given by \( x + L^2 \wedge y + L^2 \mapsto [x, y] \) is an epimorphism. Let \( M \) be a subalgebra of \( M^* \) containing \( N \) for which \( M/N \cong \ker \rho \). Since \( M^* \) is abelian, we have \( M = N \oplus \ker \rho \). Clearly,

\[
dim M/N = \dim (L/L^2 \wedge L/L^2) - \dim L^2.
\]

In the following result, we will construct a Lie algebra \( L^* \) containing \( M^* \) such that \( M^* \subseteq Z(L^*) \), \( L^*/M^* \cong L \) and \( M^* \cap (L^*)^2 = M \). Let \( 0 \to R \to F \xrightarrow{\pi} L \to 0 \) be a free presentation for \( L^* \) for a free Lie algebra \( F \) and an ideal \( R \) in \( F \). Assume that \( M^* \cong S/R \) for some ideal \( S \) of \( F \). Then \( L^*/M^* \cong F/S \) and \( M^* \cap (L^*)^2 \cong (F^2 \cap S) + R/R \cong (F^2 \cap S)/(F^2 \cap R) \). Since \( M^* \) is a central ideal, we get \([S, F] \subseteq R \). Now, using the proof of [13, Proposition 4.1], there is the following epimorphism

\[
\alpha : \mathcal{M}(L^*/M^*) = (F^2 \cap S)/[S,F] \to M^* \cap (L^*)^2 \cong (F^2 \cap S)/(F^2 \cap R)
\]

\[
f + [S,F] \mapsto f + (F^2 \cap R). \tag{2}
\]

With the above notations and assumptions, we have

**Theorem 1.3:** \( \mathcal{M}(L) \cong M \) and \( \ker \beta = K \).

**Proof:** Using Lemma 1.2, we have

\[
dim \mathcal{M}(L) = \dim \mathcal{M}(L/L^2) - \dim L^2 + \dim (F^3 + [R,F]/[R,F])
\]

\[
= \dim \mathcal{M}(L/L^2) - \dim L^2 + \dim ((L^2 \otimes L/L^2)/\ker \beta)
\]

\[
\leq \dim \mathcal{M}(L/L^2) - \dim L^2 + \dim ((L^2 \otimes L/L^2)/K).
\]

Since \( \dim \mathcal{M}(L/L^2) = \dim (L/L^2 \wedge L/L^2) \), we have

\[
dim \mathcal{M}(L) \leq \dim (L/L^2 \wedge L/L^2) - \dim L^2 + \dim ((L^2 \otimes L/L^2)/K)
\]

\[
= \dim (M/N) + \dim N = \dim M.
\]

Hence, \( \dim \mathcal{M}(L) \leq \dim M \). To prove \( \mathcal{M}(L) \cong M \), it is sufficient to construct a Lie algebra \( L^* \) containing \( M^* \) such that \( M^* \subseteq Z(L^*) \), \( L^*/M^* \cong L \) and \( M^* \cap (L^*)^2 = M \), since in this case by using (2), we can consider the epimorphism \( \alpha : \mathcal{M}(L) \to M \). Now since \( \dim \mathcal{M}(L) \leq \dim M \), \( \alpha \) is an isomorphism. Hence, \( \mathcal{M}(L) \cong M \). In particular, since

\[
dim \mathcal{M}(L) = \dim (L/L^2 \wedge L/L^2) - \dim L^2 + \dim ((L^2 \otimes L/L^2)/\ker \beta) \leq
\]

\[
\dim (L/L^2 \wedge L/L^2) - \dim L^2 + \dim ((L^2 \otimes L/L^2)/K)
\]

\[
= \dim M/N + \dim N = \dim M = \dim \mathcal{M}(L)
\]
and $K \subseteq \ker \beta$, we have $\dim \left( (L^2 \otimes L/L^2) / K \right) = \dim \left( (L^2 \otimes L/L^2) / \ker \beta \right)$ and so $K = \ker \beta$, as required.

We are going to construct a Lie algebra $L^*$ such that $L^*/M^* \cong L$. Choose a basis $\{x_1 + L^2, \ldots, x_m + L^2\}$ for $L/L^2$ and a basis $\{y_1, \ldots, y_r\}$ for $L^2$. Since $L$ is of class two, $L$ has the following presentation:

$$L = \langle x_i, y_s \mid [x_i, x_j] = \sum_{d=1}^{r} \alpha_{dij}y_d, [y_s, x_i] = [y_s', y_s] = 0, \alpha_{dij} \in \mathbb{F}, 1 \leq i, j \leq m, 1 \leq s, s' \leq r \rangle.$$  

Suppose that $a_{si} = (y_s \otimes (x_i + L^2)) + K$ and $e_{ij} = x_i + L^2 \land x_j + L^2 \in L/L^2 \land L/L^2$ for all $i, j, s$ such that $1 \leq s \leq r$ and $1 \leq i, j \leq m$. Then $N = \langle a_{si} | 1 \leq s \leq r, 1 \leq i \leq m \rangle$ and $M^* = \langle e_{ij} | 1 \leq i, j \leq m \rangle \oplus N$. Let $\bar{x}_i$ and $\bar{y}_s$ be pre-images of elements $x_i$ and $y_s$ in $L$. Assume that $L^*$ is a Lie algebra generated by $M^* \cup \{\bar{x}_i, \bar{y}_s | 1 \leq s \leq r, 1 \leq i \leq m \}$ with the following relations:

$$[\bar{x}_i, \bar{x}_j] = \sum_{d=1}^{r} \alpha_{dij} \bar{y}_d + e_{ij}, [\bar{y}_s, \bar{x}_i] = a_{si},$$

$$[e_{ij}, \bar{x}_t] = [e_{ij}, \bar{y}_t] = [a_{si}, \bar{x}_t] = [a_{si}, \bar{y}_t] = [\bar{y}_t, \bar{y}_t] = 0$$

for all $i, j, s, t, i', t'$ such that $1 \leq s, t, t' \leq r$ and $1 \leq i, j, i' \leq m$. One can easily check that $(L^*)^2 = N \oplus (\sum_{d=1}^{r} \alpha_{dij} \bar{y}_d + e_{ij} | 1 \leq i, j \leq m \rangle, M^* \subseteq Z(L^*), \text{ and } L^*/M^* \cong L$. In particular, $\bar{x}_i + M^*$ and $\bar{y}_s + M^*$ correspond to $x_i$ and $y_s$, respectively. Since $N \subseteq M^*$, we have $M^* \cap (L^*)^2 = N \oplus (M^* \cap (\sum_{d=1}^{r} \alpha_{dij} \bar{y}_d + e_{ij} | 1 \leq i, j \leq m \rangle)$. We claim that $M^* \cap (\sum_{d=1}^{r} \alpha_{dij} \bar{y}_d + e_{ij} | 1 \leq i, j \leq m \rangle) = \ker \rho$. If $\sum_{1 \leq i, j \leq m} \omega_{ij} e_{ij}$ is in $\ker \rho$, where $\omega_{ij}$ is a scalar, then

$$\sum_{1 \leq i, j \leq m} \sum_{d=1}^{r} \omega_{ij} \alpha_{dij} \bar{y}_d = 0.$$  

By considering the natural isomorphism $\sigma : L^*/M^* \rightarrow L$ given by $\bar{x}_i \mapsto x_i$ and $\bar{y}_s \mapsto y_s$ for all $i, s$ such that $1 \leq s \leq r$ and $1 \leq i \leq m$, we have $\sigma(\sum_{1 \leq i, j \leq m} \sum_{d=1}^{r} \omega_{ij} \alpha_{dij} \bar{y}_d + M^*) = 0$ and so $\sum_{1 \leq i, j \leq m} \sum_{d=1}^{r} \omega_{ij} \alpha_{dij} \bar{y}_d \in M^*$. By the definition of $M^*$, we conclude that $\sum_{1 \leq i, j \leq m} \sum_{d=1}^{r} \omega_{ij} \alpha_{dij} \bar{y}_d = 0$. It follows that

$$\sum_{1 \leq i, j \leq m} \omega_{ij} e_{ij} + \sum_{1 \leq i, j \leq m} \sum_{d=1}^{r} \omega_{ij} \alpha_{dij} \bar{y}_d = \sum_{1 \leq i, j \leq m} \omega_{ij} e_{ij} \in M^* \cap (\sum_{d=1}^{r} \alpha_{dij} \bar{y}_d + e_{ij} | 1 \leq i, j \leq m \rangle).$$
Hence \( \ker \rho \subseteq M^* \cap \left( \sum_{d=1}^{r} \alpha_{dij}y_d + e_{ij} \mid 1 \leq i, j \leq m \right) \). To prove the opposite inclusion, let \( b \) be an element of \( M^* \). If \( b \in \left( \sum_{d=1}^{r} \alpha_{dij}y_d + e_{ij} \mid 1 \leq i, j \leq m \right) \), then

\[
b = \sum_{1 \leq ij \leq m} \omega_{ij}e_{ij} + \sum_{1 \leq ij \leq m} \sum_{d=1}^{r} \omega_{ij}\alpha_{dij}y_d,
\]

where \( \omega_{ij} \) is a scalar. Since \( b \in M^* \), we have

\[
\sum_{1 \leq ij \leq m} \sum_{d=1}^{r} \omega_{ij}\alpha_{dij}y_d = 0
\]

and so \( \sum_{1 \leq ij \leq m} \omega_{ij}\alpha_{dij}y_d = 0 \) for all \( r \) such that \( 1 \leq d \leq r \). Since \( (\overline{y}_1, \ldots, \overline{y}_r) \cong L^2 \), we have \( \overline{y}_d \) is not trivial for all \( r \) such that \( 1 \leq d \leq r \). It follows that \( \sum_{1 \leq ij \leq m} \omega_{ij}\alpha_{dij} = 0 \). Therefore

\[
b = \sum_{1 \leq ij \leq m} \omega_{ij}e_{ij} \text{ and } \sum_{1 \leq ij \leq m} \sum_{d=1}^{r} \omega_{ij}\alpha_{dij} = 0.
\]

Since

\[
\rho \left( \sum_{1 \leq ij \leq m} \omega_{ij}e_{ij} \right) = \rho \left( \sum_{1 \leq ij \leq m} \omega_{ij}(x_i + L^2 \wedge x_j + L^2) \right)
\]

\[
\sum_{1 \leq ij \leq m} \omega_{ij}[x_i, x_j] = \sum_{1 \leq ij \leq m} \sum_{d=1}^{r} \omega_{ij}\alpha_{dij}y_d = 0,
\]

we get

\[
\sum_{1 \leq ij \leq m} \omega_{ij}e_{ij} = \sum_{1 \leq ij \leq m} \omega_{ij}(x_i + L^2 \wedge x_j + L^2) \in \ker \rho \quad \text{and so}
\]

\[
M^* \cap \left( \sum_{d=1}^{r} \alpha_{dij}y_d + e_{ij} \mid 1 \leq i, j \leq m \right) \subseteq \ker \rho.
\]

It follows that \( M^* \cap \left( \sum_{d=1}^{r} \alpha_{dij}y_d + e_{ij} \mid 1 \leq i, j \leq m \right) = \ker \rho \). Hence,

\[
M^* \cap (L^*)^2 = N \oplus \ker \rho.
\]

On the other hand, \( M = N \oplus \ker \rho \). We conclude that \( M = M^* \cap (L^*)^2 \). The proof is completed. \( \blacksquare \)

2. Main results

This section is devoted to obtain the Schur multiplier, the tensor square and the non-abelian exterior square of all \( d \)-generator generalized Heisenberg Lie algebras of rank \( t \) when \( \frac{1}{2}d(d-1) - 3 \leq t \leq \frac{1}{2}d(d-1) - 1 \). Moreover, we show that such Lie algebras of rank \( \frac{1}{2}d(d-1) - 1 \) and \( \frac{1}{2}d(d-1) - 2 \) are capable. In the final part of the section, these results are extended for nilpotent Lie algebras of class two with the central factor
of dimension \(d\) and derived subalgebra of dimension \(k\) such that \(\frac{1}{2}d(d-1) - 3 \leq k \leq \frac{1}{2}d(d-1) - 1\).

In the following, we compute the dimension of the image of the homomorphism \(\Psi_2\) defined in Theorem 1.1 for all \(d\)-generator generalized Heisenberg Lie algebras of rank \(t\) such that \(\frac{1}{2}d(d-1) - 3 \leq t \leq \frac{1}{2}d(d-1) - 1\).

**Proposition 2.1:** Let \(L\) be a \(d\)-generator generalized Heisenberg Lie algebra of rank \(m\) and \(d \geq 3\). Then

(i) if \(m = \frac{1}{2}d(d - 1) - 1\) or \(m = \frac{1}{2}d(d - 1) - 2\), then

\[
B = \{\Psi_2(x_n + L^2 \otimes x_q + L^2 \otimes x_k + L^2) | 1 \leq n < q < k \leq d\}
\]

is a basis of \(\text{Im} \Psi_2\) and \(\dim \text{Im} \Psi_2 = \frac{1}{6}d(d - 1)(d - 2)\).

(ii) If \(m = \frac{1}{2}d(d - 1) - 3\), then \(\dim \text{Im} \Psi_2 = \frac{1}{6}d(d - 1)(d - 2) - 1\) or \(\dim \text{Im} \Psi_2 = \frac{1}{6}d(d - 1)(d - 2)\).

**Proof:**

(i) Let \(\{x_1, x_2, \ldots, x_d\}\) be a minimal generating set of \(L\). Clearly, \(L^2 = \langle [x_n, x_q] | 1 \leq n < q \leq d \rangle\). Since \(\dim L^2 = \frac{1}{2}d(d - 1) - 1\), we have

\[
[x_i, x_j] = \sum_{1 \leq n < q \leq d, (n, q) \neq (i, j)} \beta_{nq}[x_n, x_q],
\]

where \(\beta_{nq}\) is a scalar for some \((i, j)\) such that \(1 \leq i < j \leq d\). Let \(x = x + L^2\). We know that \(B_1 = \{[x_r, x_i] | 1 \leq r < t \leq d, (r, t) \neq (i, j)\}\) generates \(L^2\) and \(L/L^2\) is generated by \([x_1, \ldots, x_d]\). Since \(L_2 \cong \bigoplus_{1 \leq r < t \leq d, (r, t) \neq (i, j)} \langle [x_r, x_t] \rangle\), \(B_1\) is also linearly independent. Now, since

\[
L^2 \otimes L/L^2 \cong \bigoplus_{1 \leq k \leq d} \bigoplus_{1 \leq r < t \leq d, (r, t) \neq (i, j)} \langle [x_r, x_t] \otimes x_k + L^2 \rangle,
\]

the set \(A = \{[x_r, x_i] \otimes x_k | 1 \leq k \leq d, 1 \leq r < t \leq d, (r, t) \neq (i, j)\}\) is a basis of \(L^2 \otimes L/L^2\). We claim that \(\Psi_2(\overline{x_i} \otimes \overline{x_j} \otimes \overline{x_k})\) is non-trivial for all \(i, j, k\) such that \(1 \leq i < j < k \leq d\). Assume on the contrary that \(\Psi_2(\overline{x_i} \otimes \overline{x_j} \otimes \overline{x_k}) = 0\). Then \([x_i, x_j] \otimes \overline{x_k} + [x_j, x_k] \otimes \overline{x_i} + [x_k, x_i] \otimes \overline{x_j} = 0\) and so \([x_k, x_i] \otimes \overline{x_j} = -[x_i, x_j] \otimes \overline{x_k} = [x_j, x_k] \otimes \overline{x_i}\), which is a contradiction, since \([x_k, x_i] \otimes \overline{x_j} \in A\). Put \(B = \{\Psi_2(\overline{x_n} \otimes \overline{x_q} \otimes \overline{x_k}) | 1 \leq n < q < k \leq d\}\).

Suppose that \(D = \{\Psi_2(\overline{x_i} \otimes \overline{x_j} \otimes \overline{x_k}) | 1 \leq k \leq d, k \neq j, k \neq i\}\) and \(C = \{\Psi_2(\overline{x_n} \otimes \overline{x_q} \otimes \overline{x_k}) | 1 \leq n < q < k \leq d, (n, q) \neq (i, j)\}\). Obviously, \(C \cap D = \emptyset\) and \(B = C \cup D\). We claim that \(B\) is a basis of \(\text{Im} \Psi_2\). Clearly, \(B\) generates \(\text{Im} \Psi_2\). Now, we show that \(B\) is linearly independent. It is enough to see that \(C\) and \(D\) are linearly independent.

First, let \(\sum_{1 \leq n < q < k \leq d, (n, q) \neq (i, j)} \alpha_{nqk} \Psi_2(\overline{x_n} \otimes \overline{x_q} \otimes \overline{x_k}) = 0\), where \(\alpha_{nqk}\) is a scalar. Then

\[
\sum_{1 \leq n < q < k \leq d, (n, q) \neq (i, j)} \alpha_{nqk} ([x_n, x_q] \otimes \overline{x_k} + [x_q, x_k] \otimes \overline{x_n} + [x_k, x_n] \otimes \overline{x_q}) = 0.
\]

For all \(n, q, k\) such
that \(1 \leq n < q < k \leq d\) and \((n, q) \neq (i, j)\), we have \([x_q, x_k] \otimes \overline{x_n}, [x_q, x_k] \otimes \overline{x_n},\) and \([x_k, x_n] \otimes \overline{x_q} \in A\). Since \(A\) is linearly independent, we have \(\alpha_{nqk} = 0\) for all \(n, q, k\) such that \(1 \leq n < q < k \leq d\) and \((n, q) \neq (i, j)\). It implies that \(C\) is linearly independent.

Now, we show that \(D\) is linearly independent. Let \(\sum_{k=1}^{d} \lambda_{ijk} \Psi_2(\overline{x_i} \otimes \overline{x_j} \otimes \overline{x_k}) = 0\), where \(\lambda_{ijk}\) is a scalar. Since \([x_i, x_j] = \sum_{1 \leq n < q \leq d \atop (n, q) \neq (i, j)} \beta_{nq}[x_n, x_q]\), where \(\beta_{nq}\) is a scalar, we have

\[
\sum_{k=1}^{d} \lambda_{ijk} ([x_i, x_j] \otimes \overline{x_k} + [x_j, x_k] \otimes \overline{x_i} + [x_k, x_i] \otimes \overline{x_j})
\]

\[
= \left( \sum_{k=1}^{d} \lambda_{ijk} \sum_{1 \leq n < q \leq d \atop (n, q) \neq (i, j)} \beta_{nq} ([x_n, x_q] \otimes \overline{x_k}) \right)
\]

\[
+ \sum_{k=1}^{d} \lambda_{ijk} ([x_j, x_k] \otimes \overline{x_i} + [x_k, x_i] \otimes \overline{x_j}) = 0.
\]

For all \(n, q, k\) such that \(1 \leq n < q \leq d, 1 \leq k \leq d\) and \((n, q) \neq (i, j)\), we have \([x_j, x_k] \otimes \overline{x_i}, [x_k, x_i] \otimes \overline{x_j}\) and \([x_n, x_q] \otimes \overline{x_k} \in A\). Since \(A\) is linearly independent, we have \(\lambda_{ijk} = \beta_{nq} = 0\) for all \(n, q, k\) such that \(1 \leq n < q \leq d, 1 \leq k \leq d\) and \((n, q) \neq (i, j)\). It implies that \(D\) is linearly independent. Thus \(B\) is a basis of \(\text{Im} \Psi_2\) and so \(\dim \text{Im} \Psi_2 = \frac{1}{6} d(d - 1)(d - 2)\). Similarly, we can see that the result holds for every \(d\)-generator generalized Heisenberg Lie algebra of rank \(\frac{1}{2} d(d - 1) - 2\).

(ii) Let \(\{x_1, x_2, \ldots, x_d\}\) be a minimal generating set of \(L\). Clearly, \(L^2 = \langle [x_n, x_q] \rangle 1 \leq n < q \leq d\). Since \(\dim L^2 = \frac{1}{2} d(d - 1) - 3\), we have

\[
[x_i, x_j] = \sum_{1 \leq n < q \leq d \atop (n, q) \neq ((i, j), (t, s), (t', s'))} \beta_{nq}[x_n, x_q],
\]

\[
[x_t, x_s] = \sum_{1 \leq n < q \leq d \atop (n, q) \neq ((i, j), (t, s), (t', s'))} \beta_{nq}[x_n, x_q],
\]

and \([x_{t'}, x_{s'}] = \sum_{1 \leq n < q \leq d \atop (n, q) \neq ((i, j), (t, s), (t', s'))} \beta_{nq}[x_{n2}, x_{q2}],\) where \(\beta_{nq}, \beta_{nq1}\) and \(\beta_{nq2}\) are scalars for some \((i, j), (t, s), (t', s')\) such that \(1 \leq t' < s' \leq d, 1 \leq t < s \leq d\) and \(1 \leq i < j \leq d\). If \(\Psi_2(\overline{x_i} \otimes \overline{x_j} \otimes \overline{x_k})\) is trivial for some \(i, j, k\) such that \(1 \leq i < j < k \leq d\), then \([x_i, x_j] = [x_j, x_k] = [x_k, x_i] = 0\). Otherwise, \(\Psi_2(\overline{x_i} \otimes \overline{x_j} \otimes \overline{x_k})\) is non-trivial. By a similar way used in part (i), we have \(\dim \text{Im} \Psi_2 = \frac{1}{6} d(d - 1)(d - 2) - 1\) or \(\dim \text{Im} \Psi_2 = \frac{1}{6} d(d - 1)(d - 2)\). The proof is completed.

We are ready to determine the structure of the Schur multiplier of a \(d\)-generator generalized Heisenberg Lie algebra of rank \(m\) such that \(\frac{1}{2} d(d - 1) - 3 \leq m \leq \frac{1}{2} d(d - 1) - 1\).
Theorem 2.2: Let $L$ be a $d$-generator generalized Heisenberg Lie algebra of rank $m$ and $d \geq 3$. Then

(i) If $m = \frac{1}{2}d(d - 1) - 1$, then $\mathcal{M}(L) \cong A(\frac{1}{3}d(d - 1)(d + 1) - d + 1)$.

(ii) If $m = \frac{1}{2}d(d - 1) - 2$, then $\mathcal{M}(L) \cong A(\frac{1}{3}d(d - 1)(d + 1) - 2d + 2)$.

(iii) If $m = \frac{1}{2}d(d - 1) - 3$, then $\mathcal{M}(L) \cong A(\frac{1}{3}d(d - 1)(d + 1) - 3d + 3)$ or $\mathcal{M}(L) \cong A(\frac{1}{3}d(d - 1)(d + 1) - 3d + 2)$.

Proof: (i) Theorem 1.1, Lemma 1.2 and Theorem 1.3 imply that

$$\dim \mathcal{M}(L) = \dim(L^2 \otimes L/L^2) - \dim \text{Im}\Psi_2 + \dim(L/L^2 \wedge L/L^2) - \dim L^2.$$ 

By [15, Lemma 2.6] and Proposition 2.1(i), we have $\dim \mathcal{M}(L/L^2) = \frac{1}{2}d(d - 1)$ and $\dim \text{Im}\Psi_2 = \frac{1}{6}d(d - 1)(d - 2)$. Now $\dim(L^2 \otimes L/L^2) = \frac{1}{2}d^2(d - 1) - d$. Thus,

$$\dim \mathcal{M}(L) = \frac{1}{2}d^2(d - 1) - d - \frac{1}{6}d(d - 1)(d - 2) + \frac{1}{2}d(d - 1) - \frac{1}{2}d(d - 1) + 1$$

$$= \frac{1}{2}d^2(d - 1) - \frac{1}{6}d(d - 1)(d - 2) - d + 1$$

$$= \frac{1}{3}d(d - 1)(d + 1) - d + 1.$$ 

Therefore $\dim \mathcal{M}(L) = \frac{1}{3}d(d - 1)(d + 1) - d + 1$. It completes the proof.

(ii) Theorem 1.1, Lemma 1.2 and Theorem 1.3 imply that

$$\dim \mathcal{M}(L) = \dim(L^2 \otimes L/L^2) - \dim \text{Im}\Psi_2 + \dim(L/L^2 \wedge L/L^2) - \dim L^2.$$ 

Using Proposition 2.1(i) and [15, Lemma 2.6], we have

$$\dim \text{Im}\Psi_2 = \frac{1}{6}d(d - 1)(d - 2) \text{ and } \dim \mathcal{M}(L/L^2) = \frac{1}{2}d(d - 1).$$

Now, $\dim(L^2 \otimes L/L^2) = \frac{1}{2}d^2(d - 1) - 2d$. Therefore

$$\dim \mathcal{M}(L) = \frac{1}{2}d^2(d - 1) - 2d + \frac{1}{2}d(d - 1) - \frac{1}{6}d(d - 1)(d - 2) - \frac{1}{2}d(d - 1) + 2$$

$$= \frac{1}{2}d^2(d - 1) - \frac{1}{6}d(d - 1)(d - 2) - 2d + 2 =$$

$$= \frac{1}{3}d(d - 1)(d + 1) - 2d + 2.$$ 

Hence $\dim \mathcal{M}(L) = \frac{1}{3}d(d - 1)(d + 1) - 2d + 2$. It completes the proof.

(iii) Similarly, the result holds for every $d$-generator generalized Heisenberg Lie algebra of rank $\frac{1}{2}d(d - 1) - 3$. 

\[ \blacksquare \]
Now we show that the converse of Theorem 2.2(i) and (ii) is also true.

**Theorem 2.3:** Let \( L \) be a \( d \)-generator generalized Heisenberg Lie algebra of rank \( m \). Then

(i) \( m = \frac{1}{2}d(d - 1) - 1 \) if and only if \( \dim \mathcal{M}(L) = \frac{1}{3}d(d - 1)(d + 1) - d + 1 \).

(ii) \( m = \frac{1}{2}d(d - 1) - 2 \) if and only if \( \dim \mathcal{M}(L) = \frac{1}{3}d(d - 1)(d + 1) - 2d + 2 \).

**Proof:** Let \( \dim \mathcal{M}(L) = \frac{1}{2}d(d - 1)(d + 1) - d + 1 \). By a similar way used in the proof of Theorem 2.2, we can see that if \( \dim L^2 < \frac{1}{2}d(d - 1) - 1 \), then \( \dim \mathcal{M}(L) < \frac{1}{3}d(d - 1)(d + 1) - d + 1 \). Thus, \( \dim L^2 = \frac{1}{2}d(d - 1) - 1 \). The converse holds by Theorem 2.2. Let \( \dim \mathcal{M}(L) = \frac{1}{3}d(d - 1)(d + 1) - 2d + 2 \). By a similar way used in the proof of Theorem 2.2, we can see that if \( \dim L^2 < \frac{1}{2}d(d - 1) - 2 \), then \( \dim \mathcal{M}(L) < \frac{1}{3}d(d - 1)(d + 1) - 2d + 2 \). Thus \( \dim L^2 = \frac{1}{2}d(d - 1) - 2 \). The converse holds by Theorem 2.2. □

We are in a position to determine the capability of all \( d \)-generator generalized Heisenberg Lie algebras of rank \( \frac{1}{2}d(d - 1) - 1 \) or \( \frac{1}{2}d(d - 1) - 2 \).

**Theorem 2.4:** Let \( L \) be a \( d \)-generator generalized Heisenberg Lie algebra of rank \( \frac{1}{2}d(d - 1) - 1 \) or \( \frac{1}{2}d(d - 1) - 2 \). Then \( L \) is capable.

**Proof:** Theorem 2.2 implies that \( \dim \mathcal{M}(L) > \dim \mathcal{M}(L/K) \) for every one-dimensional central ideal \( K \) of \( L \) and so the homomorphism \( \mathcal{M}(L) \to \mathcal{M}(L/K) \) is not monomorphism. Thus the result follows from [13, Corollary 4.6]. □

From [6], a Lie algebra \( S \) is called stem if \( Z(S) \subseteq S^2 \). The following lemma is useful to prove Theorem 2.6.

**Lemma 2.5:** Let \( L \) be a \( d \)-generator generalized Heisenberg Lie algebra of rank \( m \) such that \( \frac{1}{2}d(d - 1) - 3 \leq m \leq \frac{1}{2}d(d - 1) - 1 \) and \( L^* \) be a cover for \( L \). Then \( L^* \) is a stem nilpotent Lie algebra of class 3.

**Proof:** Let \( L^* \) be a cover Lie algebra of \( L \). Then there exists an ideal \( B \) of \( L^* \) such that \( B \cong \mathcal{M}(L), B \subseteq Z(L^*) \cap (L^*)^2 \), and \( L^*/B \cong L \). Since \( Z(L^*/B) = (L^*/B)^2 = (L^*)^2/B \), we have \( Z(L^*) \subseteq (L^*)^2 \). We claim that \( L^* \) is nilpotent of class 3. Clearly \( L^* \) is not abelian since \( L \) is non-abelian. Assume on the contrary that \( L^* \) is nilpotent of class 2. Since \( B \subseteq (L^*)^2 = Z(L^*) \), we have

\[
d = \dim L/Z(L) = \dim L/L^2 = \dim \left( (L^*/B)/(L^*/B)^2 \right) = \dim \left( L^*/(L^*)^2 \right) = \dim \left( L^*/Z(L^*) \right).
\]

By [6, Lemma 14], we have \( \dim (L^*)^2 \leq \frac{1}{2}d(d - 1) \). Now we have

\[
\dim \mathcal{M}(L) = \dim B \leq \dim (L^*)^2 \leq \frac{1}{2}d(d - 1).
\]

By Theorem 2.2, we have a contradiction. Thus \( L^* \) is nilpotent of class 3. The result holds. □
We are ready to describe a cover Lie algebra $L^*$ of a $d$-generator generalized Heisenberg Lie algebra $L$ of rank $m$ when $\frac{1}{2}d(d - 1) - 3 \leq m \leq \frac{1}{2}d(d - 1) - 1$.

**Theorem 2.6:** Let $L$ be a $d$-generator generalized Heisenberg Lie algebra of rank $m$. Then

(i) if $m = \frac{1}{2}d(d - 1) - 1$, then $L^*$ is the cover of $L$ if and only if $L^*$ is a stem nilpotent Lie algebra of class 3 with a central ideal $B$ such that $L^*/B \cong L$ and $B = (L^*)^3 \cong \mathcal{M}(L)$ or $B = (L^*)^3 \oplus A(1) \cong \mathcal{M}(L)$.

(ii) If $m = \frac{1}{2}d(d - 1) - 2$, then $L^*$ is the cover of $L$ if and only if $L^*$ is a stem nilpotent Lie algebra of class 3 with a central ideal $B$ such that $L^*/B \cong L$ and $B = (L^*)^3 \cong \mathcal{M}(L)$; or $B = (L^*)^3 \oplus A(1) \cong \mathcal{M}(L)$; or $B = (L^*)^3 \oplus A(2) \cong \mathcal{M}(L)$.

(iii) If $m = \frac{1}{2}d(d - 1) - 3$, then $L^*$ is the cover of $L$ if and only if $L^*$ is a stem nilpotent Lie algebra of class 3 with a central ideal $B$ such that $L^*/B \cong L$ and $B = (L^*)^3 \cong \mathcal{M}(L)$; or $B = (L^*)^3 \oplus A(1) \cong \mathcal{M}(L)$; or $B = (L^*)^3 \oplus A(3) \cong \mathcal{M}(L)$.

**Proof:** (i) Let $L^*$ be a cover Lie algebra of $L$. Then there exists an ideal $B$ of $L^*$ such that $B \cong \mathcal{M}(L)$, $B \subseteq Z(L^*) \cap (L^*)^2$, and $L^*/B \cong L$. Since $L^*/(L^*)^3$ is of class two and $\dim \left( (L^*/(L^*)^3)/(L^*/(L^*)^3)^2 \right) = d$, we have $\dim((L^*/(L^*)^3) \leq \frac{1}{2}d(d - 1)$, by [6, Lemma 14]. Using Lemma 2.5, $L^*$ is a stem nilpotent Lie algebra of class 3 and so $(L^*)^3 \subseteq B$ and $\dim(L^*)^2 - \dim B = \dim L^2$. Therefore

\[
\frac{1}{2}d(d - 1) - 1 + \dim B - \dim(L^*)^3 = \dim L^2 + \dim B - \dim(L^*)^3 = \dim(L^*)^2 - \dim(L^*)^3 = \dim((L^*)^2/(L^*)^3) \leq \frac{1}{2}d(d - 1).
\]

Hence $\frac{1}{2}d(d - 1) - 1 + \dim B - \dim(L^*)^3 \leq \frac{1}{2}d(d - 1)$ and so

\[
0 \leq \dim B - \dim(L^*)^3 \leq 1.
\]

It follows that $B \cong (L^*)^3 \oplus A(1)$ or $B = (L^*)^3$. The proof is completed.

(ii) Let $L^*$ be a cover Lie algebra of $L$. Then there exists an ideal $B$ of $L^*$ such that $B \cong \mathcal{M}(L)$, $B \subseteq Z(L^*) \cap (L^*)^2$, and $L^*/B \cong L$. Since $L^*/(L^*)^3$ is of class two and $\dim \left( (L^*/(L^*)^3)/(L^*/(L^*)^3)^2 \right) = d$, we have

\[
\dim((L^*/(L^*)^3) \leq \frac{1}{2}d(d - 1)
\]

by [6, Lemma 14]. Using Lemma 2.5, $L^*$ is a stem nilpotent Lie algebra of class 3 and so $(L^*)^3 \subseteq B$ and $\dim(L^*)^2 - \dim B = \dim L^2$ and so $(L^*)^3 \subseteq B$ and $\dim(L^*)^2 - \dim B = \dim L^2$. Therefore

\[
\frac{1}{2}d(d - 1) - 2 + \dim B - \dim(L^*)^3 = \dim L^2 + \dim B - \dim(L^*)^3 = \dim(L^*)^2 - \dim(L^*)^3
\]
\[ \dim((L^*)^2/(L^*)) \leq \frac{1}{2}d(d-1). \]

Hence \( \frac{1}{2}d(d-1) - 2 + \dim B - \dim(L^*)^3 \leq \frac{1}{2}d(d-1) \) and so

\[ 0 \leq \dim B - \dim(L^*)^3 \leq 2. \]

Thus \( B \cong (L^*)^3 \oplus A(2), B \cong (L^*)^3 \oplus A(1), \) or \( B = (L^*)^3. \) The proof is completed.

(iii) Similarly, the result holds for every \( d \)-generator generalized Heisenberg Lie algebra of rank \( \frac{1}{2}d(d-1) - 3 \).

\[ \blacksquare \]

From [16], \( L \wedge L, L \otimes L, \) and \( J_2(L) \) are used to denote the exterior square, the non-abelian tensor square and the kernel of the commutator map \( \kappa : L \otimes L \rightarrow L^2 \) of a Lie algebra \( L \), respectively. The authors assume that the reader is familiar with these concepts. The following results give the structures of the non-abelian tensor square and the exterior square of a \( d \)-generator generalized Heisenberg Lie algebra of rank \( m \) such that \( \frac{1}{2}d(d-1) - 3 \leq m \leq \frac{1}{2}d(d-1) - 1 \).

**Theorem 2.7:** Let \( L \) be a \( d \)-generator generalized Heisenberg Lie algebra of rank \( m \). Then

(i) If \( m = \frac{1}{2}d(d-1) - 1 \), then

\[
L \wedge L \cong A(\frac{1}{6}(d-1)(2d^2 + 5d - 6) - 1),
\]

\[
L \otimes L \cong A(\frac{1}{3}d(d^2 + 3d - 4)),
\]

\[
J_2(L) \cong A((\frac{1}{6}d(2d^2 + 3d - 5) + 1).
\]

(ii) If \( m = \frac{1}{2}d(d-1) - 2 \), then

\[
L \wedge L \cong A(\frac{1}{6}(d-1)(2d^2 + 5d - 12) - 2),
\]

\[
L \otimes L \cong A(\frac{1}{3}d(d^2 + 3d - 7)),
\]

\[
J_2(L) \cong A((\frac{1}{6}d(2d^2 + 3d - 11) + 2).
\]

**Proof:**

(i) By [16, Theorem 35(iii)] and Theorem 2.2, we have

\[
L \wedge L \cong \mathcal{M}(L) \oplus L^2 \cong A(\frac{1}{6}(d-1)(2d^2 + 5d - 6) - 1)).
\]

Using [17, Lemmas 2.2 and 2.3], we have \( L/L^2 \square L/L^2 \cong A(\frac{1}{2}d(d+1)). \) Now [17, Theorem 2.5] shows that

\[
L \otimes L \cong (L \wedge L) \oplus (L/L^2 \square L/L^2)
\]
\[ \cong A\left(\frac{1}{6}(d-1)(2d^2 + 5d - 6) - 1\right) \oplus A\left(\frac{1}{2}d(d + 1)\right) \]
\[ \cong A\left(\frac{1}{3}d(d^2 + 3d - 4)\right). \]

Now [17, Corollary 2.6] implies that \( J_2(L) \cong A((\frac{1}{6}d(2d^2 + 3d - 5) + 1) \). The proof is completed.

(ii) By a similar way used in the proof of part (i), we can obtain the result.

\[ \text{Theorem 2.8: Let } L \text{ be a } d \text{-generator generalized Heisenberg Lie algebra of rank } \frac{1}{2}d(d - 1) - 3. \text{ Then} \]

(i) if \( \mathcal{M}(L) \cong A\left(\frac{1}{3}d(d - 1)(d + 1) - 3d + 3\right) \), then
\[ L \wedge L \cong A\left(\frac{1}{6}(d-1)(2d^2 + 5d - 18) - 3\right), \]
\[ L \otimes L \cong A\left(\frac{1}{3}d(d^2 + 3d - 10)\right), \]
\[ J_2(L) \cong A\left(\frac{1}{6}(d+1)(2d^2 + 3d - 20) + 3\right). \]

(ii) If \( \mathcal{M}(L) \cong A\left(\frac{1}{3}d(d - 1)(d + 1) - 3d + 2\right) \), then
\[ L \wedge L \cong A\left(\frac{1}{6}(d-1)(2d^2 + 5d - 18) - 4\right), \]
\[ L \otimes L \cong A\left(\frac{1}{3}d(d^2 + 3d - 10) - 1\right), \]
\[ J_2(L) \cong A\left(\frac{1}{6}(d+1)(2d^2 + 3d - 20) + 2\right). \]

\[ \text{Proof: The result is obtained by a similar way used in the proof of Theorem 2.7(i).} \]

The following result gives the structures of the non-abelian tensor square, the exterior square, and the Schur multiplier of a nilpotent Lie algebra \( L \) of class two such that \( \dim(L/Z(L)) = d \) and \( \frac{1}{2}d(d - 1) - 3 \leq \dim L^2 \leq \frac{1}{2}d(d - 1) - 1 \).

\[ \text{Theorem 2.9: Let } L \text{ be an } n \text{-dimensional Lie algebra of class two such that } \dim(L/Z(L)) = d. \text{ Then} \]

(i) if \( \dim L^2 = \frac{1}{2}d(d - 1) - 1 \), then
\[ \mathcal{M}(L) \cong A\left(\frac{1}{3}d(d - 1)(d + 1) + \frac{1}{2}(t - 1)(t + 2d) + 1\right). \]
\[ L \wedge L \cong A\left(\frac{1}{6}(d-1)(2d^2 + 5d - 6) + \frac{1}{2}t(t - 1) + dt - 1\right). \]
\[ L \otimes L \cong A\left(\frac{1}{3}d(d^2 + 3d - 4) + \frac{1}{2}(2t^2 + 4dt + d^2 + d)\right). \]
\[ J_2(L) \cong A\left(\frac{1}{3}d(d-1)(d+1) + \frac{1}{2}(2t^2 + 4dt + d^2 - d) + 1\right). \]

(ii) If \( \text{dim} L^2 = \frac{1}{2}d(d-1) - 2 \), then

\[
\mathcal{M}(L) \cong A\left(\frac{1}{3}d(d-1)(d+1) + \frac{1}{2}(t-1)(t+2d) - d + 2\right). \\
L \wedge L \cong A\left(\frac{1}{6}(d-1)(2d^2 + 5d - 12) + \frac{1}{2}t(t-1) + dt - 2\right). \\
L \otimes L \cong A\left(\frac{1}{3}d(d^2 + 3d - 7) + \frac{1}{2}(2t^2 + 4dt + d^2 + d)\right). \\
J_2(L) \cong A\left(\frac{1}{3}d(d-1)(d+1) + \frac{1}{2}(2t^2 + 4dt + d^2 - 3d) + 2\right).
\]

(iii) If \( \text{dim} L^2 = \frac{1}{2}d(d-1) - 3 \), then

\[
\mathcal{M}(L) \cong A\left(\frac{1}{3}d(d-1)(d+1) + \frac{1}{2}(t-1)(t+2d) - 2d + 3\right). \\
L \wedge L \cong A\left(\frac{1}{6}(d-1)(2d^2 + 5d - 18) + \frac{1}{2}t(t-1) + dt - 3\right). \\
L \otimes L \cong A\left(\frac{1}{3}d(d^2 + 3d - 10) + \frac{1}{2}(2t^2 + 4dt + d^2 + d)\right). \\
J_2(L) \cong A\left(\frac{1}{3}d(d-1)(d+1) + \frac{1}{2}(2t^2 + 4dt + d^2 - 5d) + 3\right).
\]

or

\[
\mathcal{M}(L) \cong A\left(\frac{1}{3}d(d-1)(d+1) + \frac{1}{2}(t-1)(t+2d) - 2d + 2\right). \\
L \wedge L \cong A\left(\frac{1}{6}(d-1)(2d^2 + 5d - 18) + \frac{1}{2}t(t-1) + dt - 4\right). \\
L \otimes L \cong A\left(\frac{1}{3}d(d^2 + 3d - 10) + \frac{1}{2}(2t^2 + 4dt + d^2 + d) - 1\right). \\
J_2(L) \cong A\left(\frac{1}{3}d(d-1)(d+1) + \frac{1}{2}(2t^2 + 4dt + d^2 - 5d) + 2\right).
\]

**Proof:**

(i) By [18, Proposition 2.2], we have \( L \cong H \oplus A(t) \) such that \( H \) be a \( d \)-generator generalized Heisenberg Lie algebra of rank \( \frac{1}{2}d(d-1) - 1 \) and \( A(t) \) is an abelian Lie algebra. Using Theorem 2.7 and [16, Proposition 10], we have

\[
L \wedge L \cong (H \wedge H) \oplus (A(t) \wedge A(t)) \oplus (H/H^2 \otimes A(t)) \\
\cong A\left(\frac{1}{6}(d-1)(2d^2 + 5d - 10) - 1\right) \oplus A\left(\frac{1}{2}t(t-1)\right) \oplus A(dt) \\
\cong A\left(\frac{1}{6}(d-1)(2d^2 + 5d - 6) + \frac{1}{2}t(t-1) + dt - 1\right).
\]

Hence \( \mathcal{M}(L) \cong A\left(\frac{1}{3}d(d-1)(d+1) + \frac{1}{2}(t-1)(t+2d) - d + 1\right) \), by [16, Theorem 35(iii)]. Also [17, Theorem 2.5 and Corollary 2.6] imply that

\[
L \otimes L \cong A\left(\frac{1}{3}d(d^2 + 3d - 4) + \frac{1}{2}(2t^2 + 4dt + d^2 + d)\right),
\]
\[ J_2(L) \cong A(\frac{1}{3}d(d - 1)(d + 1) + \frac{1}{2}(2t^2 + 4dt + d^2 - d) + 1), \]
as required.

(ii),(iii) By a similar technique used in the proof of part (i), we can obtain the result. ■

In the following, we determine the capability of a nilpotent Lie algebra \( L \) of class two such that \( \dim(L/Z(L)) = d \) and \( \dim L^2 = \frac{1}{2}d(d - 1) - 1 \) or \( \dim L^2 = \frac{1}{2}d(d - 1) - 2 \).

**Corollary 2.10:** Let \( L \) be an \( n \)-dimensional Lie algebra of class two such that \( \dim(L/Z(L)) = d \) and \( \dim L^2 = \frac{1}{2}d(d - 1) - 1 \) or \( \dim L^2 = \frac{1}{2}d(d - 1) - 2 \). Then \( L \) is capable.

**Proof:** The result follows from [18, Proposition 2.2] and Theorem 2.4. ■

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