Paper Moebius Bands with T Patterns

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Abstract

This paper gives another proof of the key lemma in my recent paper which solves the optimal paper Moebius band conjecture of Halpern and Weaver, namely Lemma T. The proof here is longer but it offers more geometric intuition about what is going on.

1 Introduction

My recent paper [S1], which solves the Optimal Paper Moebius Band Conjecture, supercedes the original content of this paper. [S1] has a discussion of this conjecture and many references.

An embedded paper Moebius band of aspect ratio $\lambda$ is a smooth isometric embedding $I : M_\lambda \to \mathbb{R}^3$, where $M_\lambda$ is the flat Mobius band

$$M_\lambda = ([0,1] \times [0,\lambda]) / \sim, \quad (x,0) \sim (1-x,\lambda)$$  \hspace{1cm} (1)

An isometric mapping is a map whose differential is an isometry. The map is an embedding if it is injective. In [S1] I prove that a smooth embedded paper Moebius band has aspect ratio greater than $\sqrt{3}$. This result is sharp, and the solution of the Halpern-Weaver conjecture.

Being a ruled surface, a smooth paper Moebius band has a foliation by straight line segments which we call bends. We say that a $T$-pattern in a paper Moebius band is a collection of 2 co-planar bends such that the lines extending them are perpendicular. The key result in [S1] is

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Lemma 1.1 (T) An embedded paper Moebius band has a $T$ pattern.

This result also holds in the immersed case but I didn’t state it that way in [S1] because I didn’t care about the immersed case.

A weaker version of this lemma also appears in [S2]. I eventually found a very short proof of Lemma T and that is what appears in the final version of [S1]. In this paper I prove Lemma T in a different category, that of immersed piecewise linear Moebius bands, then deduce the smooth version by an approximation argument.

2 Nice Polygonal Moebius Bands

2.1 Polygonal Moebius Bands

We represent $M_\lambda$ as the quotient of a bilaterally symmetric trapezoid $\tau$, as shown in Figure 2.1. A transverse triangle in $M_\lambda$ is one having 1 edge $\partial M_\lambda$ and 2 edges with their vertices in $\partial M_\lambda$ as shown in Figure 2.1. We call the edge in $\partial M_\lambda$ the ridge. We define a pre-bend of a transverse triangle to be a segment joining the ridge to the opposite vertex.

![Figure 2.1: Transverse triangulation and pre-bend foliation](image)

A transverse triangulation of $M_\lambda$ is a partition of $\tau$ into transverse triangles. Each transverse triangle has a foliation by pre-bends, and these piece together to give the pre-bend foliation of $\tau$. Say that polygonal Moebius band is a continuous map $I : M_\lambda \to \mathbb{R}^3$ that is piecewise affine with respect to some transverse triangulation of $M_\lambda$. We always cut open along a pre-bend to get the kind of trapezoid representation shown in Figure 2.1. The map $I$ should be injective on each transverse triangle.

We define the bends to the images of the pre-bends under $I$. We define $T$-patterns as in the smooth case. More generally, we call two bends partners if the lines through the origin parallel to these bends are perpendicular.
2.2 The Space of Partners

We identify the pre-bends of $M_\lambda$ with the circle $\mathbb{R}/\lambda\mathbb{Z}$ as follows: We map each pre-bend to its intersection with the centerline of $M_\lambda$, and this is a copy of $\mathbb{R}/\Lambda\mathbb{Z}$. We call two points $r, s \in \mathbb{R}/\lambda\mathbb{Z}$ partners if the bends $I(\beta_r)$ and $I(\beta_s)$ are partners. We let $\Omega \subset (\mathbb{R}/\lambda\mathbb{Z})^2$ be the subset of partner points. We call $W$ nice if $\Omega$ is a piecewise smooth 1-manifold – i.e. a finite disjoint union of piecewise smooth embedded loops.

Lemma 2.1 Let $M$ be a polygonal Moebius band. We can find a linear transformation $\phi$ as close as we like to the identity so that $\phi(M)$ is nice.

We will prove Lemma 2.1 through a series of smaller lemmas. Say that an anchored line in $\mathbb{R}^3$ is a line through the origin. Say that an anchored plane is a plane in $\mathbb{R}^3$ through the origin. Let $\Pi_1$ and $\Pi_2$ be anchored planes.

Lemma 2.2 Suppose that $\Pi_1$ and $\Pi_2$ are not perpendicular. The set of perpendicular anchored lines $(L_1, L_2)$ with $L_j \in \Pi_j$ for $j = 1, 2$ is diffeomorphic to a circle.

Proof: For each anchored line $L_1 \in \Pi_1$ the line $L_2 = L_1^\perp \cap \Pi_2$ is the unique choice anchored line in $\Pi_2$ which is perpendicular to $L_1$. The line $L_2$ is a smooth function of $L_1$. So, the map $(L_1, L_2) \to L_1$ gives a diffeomorphism between the space of interest to us and a circle. ♠

A sector of the plane $\Pi_j$ is a set linearly equivalent to the union of the $(++)$ and $(-\cdot)\cdot$ quadrants in $\mathbb{R}^2$. Let $\Sigma_j \subset \Pi_j$ be a sector. The boundary $\partial \Sigma_j$ is a union of two anchored lines crossing at the origin.

Lemma 2.3 Suppose $\Pi_1$ and $\Pi_2$ are not perpendicular and no line of $\partial \Sigma_1$ is perpendicular to a line of $\partial \Sigma_2$. Then the set of perpendicular pairs of anchored lines $(L_1, L_2)$ with $L_j \in \Sigma_j$ for $j = 1, 2$ is either empty or diffeomorphic to a closed line segment. If $(L_1, L_2)$ corresponds to an endpoint then exactly one of these lines lies in the boundary of its sector.

Proof: Let $\Pi_1$ and $\Pi_2$ be anchored planes. Let $S^1$ denote the set of perpendicular pairs as in Lemma 2.2. Let $X \subset S^1$ denote the set of those pairs with $L_j \in \Sigma_j$. Let $\pi_1$ and $\pi_2$ be the two diffeomorphisms from Lemma 2.2.
The set of anchored lines in $\Sigma_j$ is a line segment and hence so is its inverse image $X_j \subset S^1$ under $\pi_j$. We have $X = X_1 \cap X_2$.

Suppose $X$ is nonempty. Then some $p \in X$ corresponds to a pair of lines $(L_1, L_2)$ with at most one $L_j \in \partial \Sigma_j$. But then we can perturb $p$ slightly, in at least one direction, so that the corresponding pair of lines remains in $\Sigma_1 \times \Sigma_2$. This shows that $X_1 \cap X_2$, if nonempty, contains more than one point. But then the only possibility, given that both $X_1$ and $X_2$ are segments, is that their intersection is also a segment.

If neither $L_1$ nor $L_2$ lies in the boundary of its sector then we can perturb in both directions. This implies that $X$ contains the point corresponding to $(L_1, L_2)$ in its (relative) interior. Hence the endpoints of $X$ correspond to pairs with at least one line in the boundary of a sector. Both lines cannot be in the sector boundary because a sector boundary line of one sector is not perpendicular to a sector boundary line of the other sector. ♠

**Proof of Lemma 2.1** Say that an image triangle of $M$ is the image under $I$ of one of the triangles in the transverse triangulation. Each image triangle $\mu$ defines a sector. The anchored plane containing the sector is parallel to the one containing $\mu$. The boundary of the sector is the union of the two anchored lines parallel to the apex-incident edges of $\mu$.

Now we consider an affine adjustment using a linear map $\phi$. Since we just need to destroy finitely many perpendicularity relations we can take $\phi$ as close as we like to the identity such that every pair of sectors associated to $\phi \circ I$ satisfies the hypotheses of Lemma 2.3. We also call the new polygonal Moebius band $M$ and we show that it is nice.

Let $\Omega$ be the partner set. The space $(\mathbb{R}/\lambda \mathbb{Z})^2$ is tiled by special rectangles corresponding to pairs of transverse triangles. By Lemma 2.3 any nontrivial intersection of $\Omega$ with a special rectangle is a special segment with endpoints in the relative interiors of edges of $\partial R$. Any two special segments have disjoint interiors because their interiors lie in different rectangle interiors. Let $s_1$ be any special segment, contained in a special rectangle $R_1$. Let $v$ be an endpoint of $s_1$. Let $R_2$ be the special rectangle adjacent to $R_1$ and sharing the edge containing $v$. Since $\Omega \cap R_2$ is nonempty, this intersection is another special segment $s_2$ which also contains $v$. In this manner, $s_1$ continues across $v$ to a unique special segment $s_2$.

These properties, disjoint interiors and continuance across vertices, show that $\Omega$ is an embedded piecewise smooth 1-manifold. ♠
3 Existence of T Patterns

3.1 An Odd Homology Class

Let $I : M_{\lambda} \rightarrow M$ be a nice polygonal Moebius band. Let $\Omega$ be the partner set for $M$. By hypothesis, $\Omega$ is a piecewise smooth 1-manifold, a subset of the open cylinder $\Upsilon$, which we get by removing the diagonal from $\mathbb{R}/\lambda \mathbb{Z}^2$. We call $\Omega$ odd if $\Omega$ represents the nontrivial element of the homology group $H_1(\Upsilon; \mathbb{Z}/2) = \mathbb{Z}/2$.

Lemma 3.1 $\Omega$ is odd.

Proof: We let $\Upsilon$ be the compactification of $\Upsilon$ obtained by adding 2 boundary components. The point $(a, b)$ lies near one boundary component if $b$ lies just ahead of $a$ in the cyclic order coming from $\mathbb{R}/\lambda \mathbb{Z}$. The point $(a, b)$ lies near the other boundary component if $b$ lies just behind of $a$ in the cyclic order coming from $\mathbb{R}/\lambda \mathbb{Z}$. We get a path $\gamma$ which runs from one boundary component of $\Upsilon$ to the other by holding $a$ fixed and varying $b$ all the way around from ahead of $a$ to just behind $a$. Let $\gamma$ be such a path. If we pick $a$ generically then $\gamma$ intersects $\Omega$ transversely. In particular, $\gamma$ intersects $\Omega$ a finite number of times.

We give an orientation to the pre-bend $\beta_a$ corresponding to $a$. This gives an orientation to the bend $I(\beta_a)$. We attempt to give a continuous orientation to the bends $I(\beta_b)$, knowing that this is impossible because we are on a Moebius band. But we can almost do this. When $b$ is just ahead of $a$ we orient $I(\beta_b)$ so that it points almost in the same direction as $I(\beta_a)$. After we have gone all the way along $\gamma$ until $b$ is just behind $a$, the bend $I(\beta_b)$ points almost in the opposite direction as $I(\beta_a)$. This means that the bends are partners an odd number of times along the path. Hence $\gamma$ intersects $\Omega$ an odd number of times.

If $\omega$ is a component of $\Omega$ then $\gamma$ intersects $\omega$ an even number or an odd number of times, depending respectively on whether $\omega$ is trivial or nontrivial in $H_1(\Upsilon; \mathbb{Z}/2)$. Hence $\Omega$ has an odd number of homologically nontrivial components. Hence $\Omega$ is odd. ♠
3.2 The Main Argument

Let $M$ be the nice polygonal Moebius band. Let $(x, y) \in \Upsilon$. There is a unique minimal path $x_t \in R/\lambda Z$ such that $x_0 = x$ and $x_1 = y$ and $x_t$ is locally increasing with respect to the cyclic order on $R/\lambda Z$. Let $u_t$ be the bend associated to $x_t$. We pick an orientation on $u_0$ and then extend it continuously to an orientation on $u_1 = v$. Let $\overrightarrow{u}$ and $\overrightarrow{v}$ be the vectors parallel to the orientations of $u$ and $v$. We write $\overrightarrow{u} \rightsquigarrow \overrightarrow{v}$. Note that $-\overrightarrow{u} \rightsquigarrow -\overrightarrow{v}$ and $-\overrightarrow{v} \rightsquigarrow -\overrightarrow{u}$.

Let $m_u$ and $m_v$ be the midpoints of the segments $u$ and $v$. Define

$$h(x, y) = (m_u - m_v) \cdot (\overrightarrow{u} \times \overrightarrow{v}). \tag{2}$$

This vector is well defined: If switch the orientation of $u$ then the orientation of $v$ also switches and the cross product does not. Note that

$$h(y, x) = (m_v - m_u) \cdot (\overrightarrow{v} \times (-\overrightarrow{u})) = (m_v - m_u) \cdot (\overrightarrow{u} \times \overrightarrow{v}) = -h(x, y). \tag{3}$$

Consider $\Sigma : \Upsilon \to \Upsilon$ given by $\Sigma(a, b) = (b, a)$. This map is an involution and an isomorphism on $H_1(\Upsilon; \mathbb{Z}/2)$. By construction $\Sigma$ permutes the components of $\Omega$. Since $\Omega$ is odd, there must be some component $\omega$ of $\Omega$ such that $\Sigma(\omega) = \omega$. By Equation (3) and continuity, $h$ vanishes at some point of $\omega$. Such a point corresponds to a $T$-pattern.

Smooth Case: Let $I : M_\lambda \to M$ be a smooth paper Moebius band. Take a finite list $\beta_1, ..., \beta_n$ of pre-bends in $M_\lambda$, with $\beta_1$ being the first bend and $\beta_n$ being the last. Call this a mesh. These pre-bends divide $M_\lambda$ into thin trapezoids. We add diagonals to get a transverse triangulation. In this way we can approximate a smooth paper Moebius band as closely as we like with a polygonal one. We can further perturb to make the approximation nice. We can then take a limit of $T$-patterns on the nice polygonal approximations and get one on the smooth paper Moebius band.

4 References

[S1] R. E. Schwartz, *The Optimal Paper Moebius Band*, preprint, 2023

[S2] R. E. Schwartz, *An Improved Bound on the Optimal Paper Moebius Band*, Geometriae Dedicata, 2021.