A conjecture of Herzog and Conca on counting of paths

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Abstract

A formula concerning counting of paths was conjectured by Herzog and Conca few years ago. Recently, Krattenthaler and Prohaska gave an affirmative answer to this conjecture. In this paper we generalize this formula.

1 Introduction

Let $X$ be the set $\{(i, j) \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ of the plane with the partial order given by $(i, j) \leq (i', j')$ if $i \geq i'$ and $j \leq j'$. Let $P, Q \in X$ with $P \geq Q$; a path from $P$ to $Q$ is a maximal chain in $X$ with end points $P$ and $Q$. A corner of a path $C$ is an element $(i, j) \in C$ for which $(i - 1, j)$ and $(i, j - 1)$ belong to $C$ as well. We use $w(P, Q)$ for the number of different paths from $P$ to $Q$ and $w_k(P, Q)$ for the number of different paths with $k$ corners from $P$ to $Q$. Therefore $w(P, Q) = \sum_{k \geq 0} w_k(P, Q)$.

Let $P_i, Q_i, i = 1, \ldots, r$ be points of $X$. A subset $W \subseteq X$ is called an $r$-tuple of non-intersecting paths form $P_i$ to $Q_i$ ($i = 1, \ldots, r$) if $W = C_1 \cup C_2 \cup \cdots \cup C_r$ where each $C_i$ is a path from $P_i$ to $Q_i$, and where $C_i \cap C_j = \emptyset$ if $i \neq j$.

The number of corners $c(W)$ of $W$ is the sum of the number of corners of the $C_i$. We use $w_k(P, Q)$ for the number of the families of non-intersecting paths form $P_i$ to $Q_i$ ($i = 1, \ldots, r$) with exactly $k$ corners, and $W(P, Q)$ for...

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the polynomial \((t) \sum_{k \geq 0} w_k(P, Q)t^k\). The work of Krattenthaler [2] and Kulkarni [5] showed the following.

**Theorem 1.1** Let \(X = \{(i, j) \mid 1 \leq i \leq m, 1 \leq j \leq n\}\) with the partial order giving as the above. Let \(P_i = (a_i, \mathbf{n})\), \(1 = a_1 < \cdots < a_r \leq m\), and \(Q_i = (m, b_i)\), \(1 = b_1 < \cdots < b_r \leq n\). Then

\[
W(P, Q) = \det(\sum_{k \geq 0} \binom{m-a_i+i-j}{k} \binom{n-b_j+j-i}{k+j-i} t^k)_{i,j=1,\ldots,r}.
\]

When \(a_i = b_i = i\), the above formula is equivalent to the following one obtained by Conca and Herzog [1],

\[
W(P, Q) = t^{-\binom{r}{2}} \det(\sum_{k \geq 0} \binom{m-i}{k} \binom{n-j}{k} t^k)_{i,j=1,\ldots,r}.
\]

Since \(\binom{m-i}{k} \binom{n-j}{k}\) is the number of paths from \((i, n)\) to \((m, j)\) with exactly \(k\) corners, this formula suggests that if \(P_i = (i, n)\) and \(Q_i = (m, i)\) are points of a rectangular region \(X\), then

\[
W(P, Q) = t^{-\binom{r}{2}} \det[W(P_i, Q_j)]_{i,j=1,\ldots,r}.
\]

(1)

According to this formula, Conca and Herzog made the following conjecture.

**Conjecture:** Let \(Y\) be a one-sided ladder-shaped region of the plane (see Figure 1 below) with the partial order defined as the above. Let \(P_i = (i, n)\) and \(Q_i = (m, i)\) be points of \(Y\). Then

\[
W_Y(P, Q) = t^{-\binom{r}{2}} \det[W_Y(P_i, Q_j)]_{i,j=1,\ldots,r}.
\]

Recently, Krattenthaler and Prohaska [4] gave an affirmative answer to the
Conjecture by using the notion two-rowed arrays introduced in \[3\].

Our main concern in this paper is trying to give a self-contained proof to the result obtained by Krattenthaler and Prohaska. As a consequence, we are able to generalize their result as follows.

**Theorem 1.2** Let $Y$ be a one-sided ladder-shaped region of the plane with the partial order given in the beginning. Let $P_i = (i, n)$ and $Q = (m, b_i)$, $i = 1, \ldots, r$ be points of $Y$. Let $W(n, m; b_1, \ldots, b_r)$ be the polynomial $\sum_{k \geq 0} w_k(P, Q)t^k$. Let $\tilde{W}(n, m; b_1, \ldots, b_r) = \det[W(P_i, Q)]_{i,j=1,\ldots,r}$. Then

$$t^{-\binom{r}{2}}\tilde{W}(n, m; b_1, \ldots, b_r) = \sum_{\underline{c} \in S_r} A(\underline{b}; \underline{c})(1 - t)^{\sum(b_i - c_i)}W(n, m; c_1, \ldots, c_r),$$

where $\underline{b} = (b_1, \ldots, b_r)$ and $\underline{c} = (c_1, \ldots, c_r)$. In particular, if $b_i = i \ \forall i$, then

$$t^{-\binom{r}{2}}\tilde{W}(n, m; 1, \ldots, r) = W(n, m; 1, \ldots, r).$$

Here $S_r = \{(b_1, b_2, \ldots, b_r) \in \mathbb{N}^r \mid b_1 < b_2 < \cdots < b_r\}$ and $A(\underline{b}; \underline{c})$ is defined in section 3.

## 2 Some fundamental lemmas

Let $Y$ be a one-sided ladder-shaped region of the plane with the partial order given by $(i, j) \leq (i', j')$ if $i \geq i'$ and $j \leq j'$. Let $P_i = (i, s)$ and $Q_i = (l, b_i)$, $i = 1, \ldots, r$ be points of $Y$. We use $W(s, l; b_1, \ldots, b_r)$ for the polynomial $\sum_{k \geq 0} w_k(P, Q)t^k$ and $\tilde{W}(s, l; b_1, \ldots, b_r)$ for $\det[W(P_i, Q)]_{i,j=1,\ldots,r}$. In order to obtain some useful properties of $W$ and $\tilde{W}$, we introduce the following notations.

Let $l$ be a positive integer and

$$S_l = \{(b_1, b_2, \ldots, b_l) \in \mathbb{N}^l \mid b_1 < b_2 < \cdots < b_l\}.$$  

We define a partial order on $S_l$ so that for any $(b_1, \ldots, b_l), (c_1, \ldots, c_l) \in S_l$, $(b_1, \ldots, b_l) \leq (c_1, \ldots, c_l)$ if $b_i \leq c_i \ \forall i$. Moreover, for any two lattice points $(b_1, \ldots, b_l), (c_1, \ldots, c_l) \in S_l$, we define $\left[ \begin{array}{c} c_1 \cdots c_l \\ b_1 \cdots b_l \end{array} \right]$ and $\left[ \begin{array}{c} c_1 \cdots c_l \\ b_1 \cdots b_l \end{array} \right]$ as
follows. Let \( b, c \in \mathbb{N} \). We use the symbol \( \left[ \begin{array}{c} c \\ b \end{array} \right] \) for \( t, 1, 0 \) if \( c > b, c = b \) and \( c < b \), where \( t \) is a variable. Furthermore, for any points \((b_1, \ldots, b_r), (c_1, \ldots, c_r) \in S_r\),
\[
\left[ \begin{array}{ccc} c_1 & \cdots & c_r \\ b_1 & \cdots & b_r \end{array} \right] = 0
\]
if \( c_i \geq b_{i+1} \) for some \( i < r \) and
\[
\left[ \begin{array}{ccc} c_1 & \cdots & c_r \\ b_1 & \cdots & b_r \end{array} \right] = \left[ \begin{array}{c} c_1 \\ b_1 \end{array} \right] \cdots \left[ \begin{array}{c} c_r \\ b_r \end{array} \right]
\]
if \( c_i < b_{i+1} \) for every \( i \). Finally,
\[
\left\{ \begin{array}{ccc} c_1 & \cdots & c_r \\ b_1 & \cdots & b_r \end{array} \right\} = \sum_{\sigma \in S_X} \text{sign}(\sigma) \left[ \begin{array}{c} \sigma(c_1) \\ b_1 \end{array} \right] \cdots \left[ \begin{array}{c} \sigma(c_r) \\ b_r \end{array} \right],
\]
where \( X = \{c_1, \ldots, c_r\} \) and \( S_X \) is the permutation group on \( X \).

**Lemma 2.1** Let \( Y \) be a one-sided ladder-shaped region of the plane with the partial order given in the beginning. Suppose that \( m = r \) (Then \( Y \) is a rectangular region). Let \( P_i = (i, n) \) and \( Q_i = (r, b_i), i = 1, \ldots, r \). Then
\[
t^{-(\frac{1}{2})} \tilde{W}(n, r; b_1, \ldots, b_r) = w(P, Q).
\]

**Proof.** Notice that
\[
W(P_i, Q_j) = \sum_{k \geq 0} \binom{r-i}{k} \binom{n-b_j}{k} t^k = \sum_{k \geq 0} \binom{r-i}{k} \binom{n-b_j}{k} t^k + \binom{n-b_j}{r-i} t^{r-i},
\]
therefore, the \( i \)-th row of \( [W(P_i, Q_j)]_{i,j=1,\ldots,r} \) is
\[
< \sum_{k \geq 0} \binom{r-i}{k} \binom{n-b_1}{k} t^k, \ldots, \sum_{k \geq 0} \binom{r-i}{k} \binom{n-b_r}{k} t^k > + < (n-b_1) t^{r-i}, \ldots, (n-b_r) t^{r-i} >.
\]
Since
\[
< \sum_{k \geq 0} \binom{r-i}{k} \binom{n-b_1}{k} t^k, \ldots, \sum_{k \geq 0} \binom{r-i}{k} \binom{n-b_r}{k} t^k >
\]
is a linear combination of the last \( r - i \) rows of \([W(P_j, Q_j)]_{i,j=1,\ldots,r}\), one can use elementary row operations to obtain that

\[
\tilde{W}(n, r; b_1, \ldots, b_r) = \det \begin{pmatrix}
W(P_1, Q_1) & \cdots & W(P_1, Q_r) \\
\vdots & \ddots & \vdots \\
1 + (n - b_1)t & \cdots & 1 + (n - b_r)t \\
1 & \cdots & 1 \\

W(P_1, Q_1) & \cdots & W(P_1, Q_r) \\
\vdots & \ddots & \vdots \\
(n - b_1)t & \cdots & (n - b_r)t \\
1 & \cdots & 1 \\
\end{pmatrix}
\]

\[
= \cdots
\]

\[
= \det \begin{pmatrix}
W(P_1, Q_1) & \cdots & W(P_1, Q_r) \\
\vdots & \ddots & \vdots \\
\sum_{k \geq 0} \binom{r-i}{k} \frac{1}{k!} (n - b_1)^{k} t^{k-i} & \cdots & \sum_{k \geq 0} \binom{r-i}{k} \frac{1}{k!} (n - b_r)^{k} t^{k-i} \\
\vdots & \ddots & \vdots \\
1 & \cdots & 1 \\
\end{pmatrix}
\]

\[
= \cdots
\]

\[
= \det \begin{pmatrix}
\binom{n-b_1}{r-i} t^{r-i} & \cdots & \binom{n-b_r}{r-i} t^{r-i} \\
\vdots & \ddots & \vdots \\
1 & \cdots & 1 \\
\end{pmatrix}
\]

\[
= ct^{(r)}
\]

for some constant \( c \). However, \( \tilde{W}(n, r; b_1, \ldots, b_r)(1) = w(P, Q) \), therefore the assertion follows. \( \Box \)

**Corollary 2.2** Let \( Y \) be a one-sided ladder-shaped region of the plane with the partial order given in the beginning. Suppose that \( m = r \) (Then \( Y \) is a rectangular region). Let \( P_i = (i, n) \) and \( Q_i = (r, b_i) \), \( i = 1, \ldots, r \). Then

\[
t^{-\binom{r}{2}} \tilde{W}(n, r; b_1, \ldots, b_r) = t^{-\binom{r-1}{2}} \sum_{c_i = b_1}^{b_2-1} \cdots \sum_{c_{r-1} = b_{r-1}}^{b_r-1} \tilde{W}(b_r - 1, r-1; c_1, \ldots, c_{r-1}).
\]
Proof. Let \( P'_i = (i, b_r - 1) \) and \( S_{c_i} = (r - 1, c_i) \). Then by Lemma 2.1, it is enough to show that
\[
\begin{align*}
w(P, Q) &= \sum_{c_i = b_1}^{b_2 - 1} \cdots \sum_{c_{r-1} = b_{r-1}}^{b_r - 1} w(P', S_{c_i}),
\end{align*}
\]
where \( P' = \{P'_1, \ldots, P'_{r-1}\} \) and \( S_{c_i} = \{S_{c_1}, \ldots, S_{c_{r-1}}\} \). However, this follows from two facts. One is
\[
w(P, Q) = w(P', Q'),
\]
where \( Q' = \{Q_1, \ldots, Q_{r-1}\} \). The other is
\[
w(P', Q') = \sum_{c_i = b_1}^{b_2 - 1} w(P', S_{c_1}, Q_2, \ldots, Q_{r-1})
\]
\[= \cdots \]
\[= \sum_{c_i = b_1}^{b_2 - 1} \cdots \sum_{c_{r-1} = b_{r-1}}^{b_r - 1} w(P', S_{c_i}),
\]
by Lemma 3.2. \( \Box \)

Lemma 2.3 Let \( Y \) be a one-sided ladder-shaped region of the plane with the partial order given in the beginning. Then the following hold:

(i)
\[
\tilde{W}(n, m; b_1, \ldots, b_r) = \sum_{d \in S_r, (m, d_r) \in Y} \{ d_1 \ldots d_r \} \tilde{W}(n, m-1; d_1, \ldots, d_r).
\]

(ii)
\[
W(n, m; b_1, \ldots, b_r) = \sum_{d \in S_r, (m, d_r) \in Y} [ d_1 \ldots d_r ] W(n, m-1; d_1, \ldots, d_r).
\]

Proof. (i). Let \( S_{d_j} = (m - 1, d_j) \). Since
\[
W(P_i, Q_j) = \sum_{d_j \in N_i, (m, d_j) \in Y} [ d_j ] W(P_i, S_{d_j})
\]
by Lemma 3.1, 
\[
\tilde{W}(n, m; b_1, \ldots, b_r) = \det[W(P, Q_i)]_{i,j=1,\ldots,r}
\]
\[
= \sum_{d_1 \in \mathbb{N}, (m, d_1) \in Y} \cdots \sum_{d_r \in \mathbb{N}, (m, d_r) \in Y} \det\{[d_j]_b W(P, S_{d_j})\}_{i,j=1,\ldots,r}
\]
\[
= \sum_{d_1 \in \mathbb{N}, (m, d_1) \in Y} \cdots \sum_{d_r \in \mathbb{N}, (m, d_r) \in Y} [d_1]_{b_1} \cdots [d_r]_{b_r} \det[W(P, S_{d_j})]_{i,j=1,\ldots,r}
\]
\[
= \sum_{d \in S_{d_j}, (m, d_r) \in Y} \sum_{\sigma} \sigma(d_1) [\sigma(d_r)_{b_r}] \cdots \cdots \cdots [\sigma(d_r)_{b_r}] \det[W(P, S_{\sigma(d_j)})]_{i,j=1,\ldots,r}
\]
\[
= \sum_{d \in S_{d_j}, (m, d_r) \in Y} \sum_{\sigma} \text{sign}(\sigma) [\sigma(d_1)_{b_1}] \cdots [\sigma(d_r)_{b_r}] \det[W(P, S_{d_j})]_{i,j=1,\ldots,r}
\]
\[
= \sum_{d \in S_{d_j}, (m, d_r) \in Y} \{d_1, \ldots, d_r\} \tilde{W}(n, m - 1; d_1, \ldots, d_r).
\]

(ii). Let \( S_{d_j} = (m - 1, d_j) \). By using Lemma 3.2] repeatedly, one can obtain that
\[
W(n, m; b_1, \ldots, b_r) = \sum_{d_1 = b_1}^{b_2 - 1} \cdots \sum_{d_r = b_r}^b W(n, m - 1; d_1, \ldots, d_r),
\]
where \( b \) is the maximal integer for which \((m, b) \in Y \). Thus (ii) follows from the definition of \([d_j]_b \). \(\square\)

3 Properties of \( A(b, c) \)

Let \( l \geq 2 \) be an integer and
\[
S_l = \{(b_1, b_2, \ldots, b_l) \in \mathbb{N}^l \mid b_1 < b_2 < \cdots < b_l\}
\]
with the partial order given in section 2.

For any two lattice points \((b_1, \ldots, b_l), (c_1, \ldots, c_l) \in S_l\), we define \( A(b_1, \ldots, b_l; c_1, \ldots, c_l) \) as follows.

Assume \( l = 2 \). Then \( A(b_1, b_2; c_1, c_2) = 0 \) if \( b_2 \neq c_2 \) or \( c_1 < b_1 \), and \( A(b_1, b_2; c_1, c_2) = 1 \) if \( b_2 = c_2 \) and \( b_1 \leq c_1 \).
Assume that $l > 2$. If $b_l \neq c_l$, then $A(b_1, \ldots, b_l; c_1, \ldots, c_l) = 0$. If $b_l = c_l$, then
\[ A(b_1, \ldots, b_l; c_1, \ldots, c_l) = \sum_{d' \leq d'' \in S_{l-1}} A(b_1, \ldots, b_{l-1}; c'_1, \ldots, c'_{l-1}). \]

**Theorem 3.1** Let $l \geq 2$ be a positive integer and $(b_1, \ldots, b_l), (c_1, \ldots, c_l) \in S_l$. Let
\[
B_1(b_1, \ldots, b_l; c_1, \ldots, c_l) = \sum_{d \in S_l} A(d_1, \ldots, d_l; c_1, \ldots, c_l) \left\{ \begin{array}{ccc} d_1 & \cdots & d_l \\ b_1 & \cdots & b_l \end{array} \right\} (1-t)^{\sum (c_i - d_i)},
\]
\[
B_2(b_1, \ldots, b_l; c_1, \ldots, c_l) = \sum_{d \in S_l} A(b_1, \ldots, b_l; d_1, \ldots, d_l) \left[ \begin{array}{ccc} c_1 & \cdots & c_l \\ d_1 & \cdots & d_l \end{array} \right] (1-t)^{\sum (d_i - b_i)}
\]
and
\[
B_3(b_1, \ldots, b_l; c_1, \ldots, c_l) = \sum_{d_1 = b_1}^{b_2-1} \cdots \sum_{d_{l-1} = b_{l-1}}^{b_l-1} A(d_1, \ldots, d_{l-1}; c_1, \ldots, c_{l-1}) (1-t)^{\sum_{i=1}^{l-1} (c_i - d_i)}
\]
if $l \geq 3$ and $b_l = c_l$. Then
\[ B_1(b_1, \ldots, b_l; c_1, \ldots, c_l) = B_2(b_1, \ldots, b_l; c_1, \ldots, c_l) \]
and
\[ B_1(b_1, \ldots, b_l; c_1, \ldots, c_l) = B_3(b_1, \ldots, b_l; c_1, \ldots, c_l) \]
if $l \geq 3$ and $b_l = c_l$.

**Remark 3.2** (i). $B_2(b_1, \ldots, b_l; c_1, \ldots, c_l) = 0$ if $c_{l-1} \geq b_l$.
(ii). $B_3(b_1, \ldots, b_l; c_1, \ldots, c_l) = B_3(b_1, \ldots, b_{l-1}, b; c_1, \ldots, c_{l-1}, b)$ for every $b > \max\{b_{l-1}, c_{l-1}\}$.

To show Theorem 3.1, we need several lemmas. The first one is easy to obtain, we left the proof to the reader.

**Lemma 3.3** Let $b \leq e \leq c$ be positive integers. Then
\[
\sum_{d=e}^{c} \left[ \begin{array}{c} c \\ d \end{array} \right] (1-t)^{e-d} = (1-t)^{e-b}
\]
and
\[
\sum_{d=b}^{e} \binom{d}{b}(1-t)^{c-d} = (1-t)^{c-e}.
\]

**Lemma 3.4** Let \( l \geq 2 \) be a positive integer. Let \((b_1, \ldots, b_l)\) and \((c_1, \ldots, c_l)\) be two points in \( S_l \). Then
\[
\sum_{b_i \leq d_i \leq S_l} \binom{c_i}{d_i} (1-t)^{d_i-b_i} = \sum_{c_i \leq d_i \leq S_l} \binom{d_i}{b_i} (1-t)^{c_i-d_i}.
\]

**Proof.** We may assume that \((b_1, \ldots, b_l) \leq (c_1, \ldots, c_l)\). By Lemma 3.3,
\[
\sum_{b_i \leq d_i \leq S_l} \binom{c_i}{d_i} (1-t)^{d_i-b_i} = \prod_{i=2}^{l} \{ \sum_{d_i=max\{b_i,c_{i-1}+1\}}^{c_i} \binom{c_i}{d_i} (1-t)^{d_i-b_i} \}
\]
\[
= \prod_{i=2}^{l} (1-t)^{max\{0,c_{i-1}-b_i+1\}}.
\]

Also, by Lemma 3.3,
\[
\sum_{c_i \leq d_i \leq S_l} \binom{d_i}{b_i} (1-t)^{c_i-d_i} = \prod_{i=1}^{l-1} \{ \sum_{d_i=min\{c_i,b_i+1\}}^{c_i} \binom{d_i}{b_i} (1-t)^{c_i-d_i} \}\{ \sum_{d_i=b_i}^{c_i} \binom{d_i}{b_i} (1-t)^{c_i-d_i} \}
\]
\[
= \prod_{i=1}^{l-1} (1-t)^{max\{0,c_i-b_i+1\}}.
\]

This shows the Lemma.  \( \square \)

**Lemma 3.5** Let \((b_1, b_2), (c_1, c_2) \in S_2\). If \( c_1 \leq b_2 - 1 \) and \( b_1 \leq c_1 \), then
\[
B_1(b_1, b_2; c_1, c_2) = B_2(b_1, b_2; c_1, c_2) = \binom{c_2}{b_2}.
\]

If \( c_1 < b_1 \) or \( c_1 \geq b_2 \), then
\[
B_1(b_1, b_2; c_1, c_2) = B_2(b_1, b_2; c_1, c_2) = 0.
\]
Proof. It is clear that if \( c_1 < b_1 \) then \( B_1(b_1, b_2; c_1, c_2) = B_2(b_1, b_2; c_1, c_2) = 0 \), so that we may assume \( b_1 \leq c_1 \). If \( c_1 \leq b_2 - 1 \), then by Lemma 3.3

\[
B_1(b_1, b_2; c_1, c_2) = \sum_{d_1=b_1}^{c_1} A(d_1, c_2; c_1, c_2) \left\{ \frac{d_1}{b_1} \frac{c_2}{b_2} \right\} (1-t)^{c_1-d_1} = \sum_{d_1=b_1}^{c_1} \left\{ \frac{c_2}{b_2} \right\} (1-t)^{c_1-d_1}
= \left\{ \frac{c_2}{b_2} \right\}
\]

and

\[
B_2(b_1, b_2; c_1, c_2) = \sum_{d_1=b_1}^{c_1} A(b_1, b_2; d_1, b_2) \left\{ \frac{c_1}{d_1} \frac{c_2}{b_2} \right\} (1-t)^{d_1-b_1} = \sum_{d_1=b_1}^{c_1} \left\{ \frac{c_1}{d_1} \frac{c_2}{b_2} \right\} (1-t)^{d_1-b_1}
= \left\{ \frac{c_1}{d_1} \frac{c_2}{b_2} \right\}.
\]

On the other hand, if \( c_1 \geq b_2 \) then

\[
B_1(b_1, b_2; c_1, c_2) = \sum_{d_1=b_1}^{c_1} A(d_1, c_2; c_1, c_2) \left\{ \frac{d_1}{b_1} \frac{c_2}{b_2} \right\} (1-t)^{c_1-d_1} = \sum_{d_1=b_1}^{b_2-1} \left\{ \frac{d_1}{b_1} \frac{c_2}{b_2} \right\} (1-t)^{c_1-d_1} + \left\{ \frac{b_2}{b_1} \frac{c_2}{b_2} \right\} (1-t)^{c_1-b_2}
= \sum_{d_1=b_1}^{b_2-1} \left\{ \frac{d_1}{b_1} \right\} t(1-t)^{c_1-d_1} + (t^2-t)(1-t)^{c_1-b_2}
= 0,
\]

and

\[
B_2(b_1, b_2; c_1, c_2) = \sum_{d_1=b_1}^{b_2-1} A(b_1, b_2; d_1, b_2) \left\{ \frac{c_1}{d_1} \frac{c_2}{b_2} \right\} (1-t)^{d_1-b_1} = 0
\]

as \( \left\{ \frac{c_1}{d_1} \frac{c_2}{b_2} \right\} = 0 \). This proves the lemma. \( \square \)

Lemma 3.6 Let \((b_1, \ldots, b_l) \in S_l \) with \( l \geq 3 \). Let \( b \in \mathbb{N} \) such that \( b > b_l \). For every \( i \leq l \), let \( \sigma_i : \{b_1, \ldots, b_l\} \longrightarrow \{b_1, \ldots, b_i, \ldots, b_l, b\} \) be the bijective map
such that $\sigma_i(b_j) = b_j$ if $j < i$, $\sigma_i(b_j) = b_{j+1}$ if $i \leq j \leq l - 1$ and $\sigma_i(b_l) = b_i$. Let $f(x_1, \ldots, x_{l-1}) \in \mathbb{Q}(t)[x_1, \ldots, x_{l-1}]$. Then

$$\sum_{i=1}^{l} (-1)^i \{ \sum_{d_1=\sigma_i(b_1)}^{\sigma_i(b_2)-1} \cdots \sum_{d_{l-1}=\sigma_i(b_{l-1})}^{\sigma_i(b_l)-1} f(d_1, \ldots, d_{l-1}) \}$$

$$= (-1)^l \sum_{d_1=b_1}^{b_2-1} \cdots \sum_{d_{l-1}=b_{l-1}}^{b_l-1} \sum_{j=1}^{l-1} (-1)^j \{ \sum_{d_1=\sigma_i(b_1)}^{\sigma_i(b_2)-1} \cdots \sum_{d_{l-1}=\sigma_i(b_{l-1})}^{\sigma_i(b_l)-1} f(d_1, \ldots, d_{l-1}) \}$$

Proof. We proceed by induction on $l$. The case $l = 3$ is easy to check. We may assume that $l > 3$. Then by induction

$$\sum_{i=1}^{l} (-1)^i \{ \sum_{d_1=\sigma_i(b_1)}^{\sigma_i(b_2)-1} \cdots \sum_{d_{l-1}=\sigma_i(b_{l-1})}^{\sigma_i(b_l)-1} f(d_1, \ldots, d_{l-1}) \}$$

$$= (-1)^l \sum_{d_1=b_1}^{b_2-1} \cdots \sum_{d_{l-1}=b_{l-1}}^{b_l-1} \sum_{j=1}^{l-1} (-1)^j \{ \sum_{d_1=\sigma_i(b_1)}^{\sigma_i(b_2)-1} \cdots \sum_{d_{l-1}=\sigma_i(b_{l-1})}^{\sigma_i(b_l)-1} f(d_1, \ldots, d_{l-1}) \}$$

$$+ (-1)^l \sum_{d_1=b_1}^{b_2-1} \cdots \sum_{d_{l-1}=b_{l-1}}^{b_l-1} \sum_{j=1}^{l-1} (-1)^j \{ \sum_{d_1=\sigma_i(b_1)}^{\sigma_i(b_2)-1} \cdots \sum_{d_{l-1}=\sigma_i(b_{l-1})}^{\sigma_i(b_l)-1} f(d_1, \ldots, d_{l-1}) \}$$

$$= (-1)^l \sum_{d_1=b_1}^{b_2-1} \cdots \sum_{d_{l-1}=b_{l-1}}^{b_l-1} f(d_1, \ldots, d_{l-1})$$



\[\square\]

Proof of Theorem 3.1 We prove the theorem by induction on $l$. If $l = 2$, then it is the content of Lemma 3.3. Therefore we may assume that $l \geq 3$. Without loss of generality, we may further assume that $(b_1, \ldots, b_l) \leq (c_1, \ldots, c_l)$. Let

$$D(b_1, \ldots, b_{l-1}; c_1, \ldots, c_{l-1}) = \sum_{d\leq \sum d \in S_{l-1}} B_1(b_1, \ldots, b_{l-1}; d_1, \ldots, d_{l-1})(1-t)^{\sum(c_i-d_i)}.$$
We will show in the following that

\[ D(b_1, \ldots, b_{t-1}; c_1, \ldots, c_{t-1}) = B_i(b_1, \ldots, b_{t-1}, b; c_1, \ldots, c_{t-1}, b) \]  

for \( i = 1, 2, 3 \) and for every \( b > \max\{b_{t-1}, c_{t-1}\} \).

Observe first that

\[ B_1(b_1, \ldots, b_{t-1}, b; c_1, \ldots, c_{t-1}, b) \]

\[ = \sum_{d \in S_{t-1}} A(d_1, \ldots, d_{t-1}, b; c_1, \ldots, c_{t-1}, b) \{ \begin{array}{ccc} c_1 & \cdots & b_{t-1} \\ b_1 & \cdots & b_{t-1} \end{array} \} (1 - t)^{\sum_{i=1}^{t-1} (c_i - d_i)} \]

\[ = \sum_{d \in S_{t-1}} \sum_{d' \leq d} A(d_1, \ldots, d_{t-1}; d'_{t-1}) \{ \begin{array}{ccc} c_1 & \cdots & b_{t-1} \\ b_1 & \cdots & b_{t-1} \end{array} \} (1 - t)^{\sum_{i=1}^{t-1} (d_i - d'i)} \]

\[ = \sum_{d' \leq d} B_1(b_1, \ldots, b_{t-1}; d'_{t-1}) (1 - t)^{\sum_{i=1}^{t-1} (c_i - d'i)} \]

Moreover, by induction and Lemma 3.4

\[ B_2(b_1, \ldots, b_{t-1}, b; c_1, \ldots, c_{t-1}, b) \]

\[ = \sum_{d \in S_{t-1}} A(b_1, \ldots, b_{t-1}, b; d_1, \ldots, d_{t-1}, b) \{ \begin{array}{ccc} c_1 & \cdots & c_{t-1} \\ d_1 & \cdots & d_{t-1} \\ b_1 & \cdots & b_{t-1} \end{array} \} (1 - t)^{\sum_{i=1}^{t-1} (d_i - b_i)} \]

\[ = \sum_{d \in S_{t-1}} \sum_{d' \leq d} A(b_1, \ldots, b_{t-1}; d'_{t-1}) \{ \begin{array}{ccc} c_1 & \cdots & c_{t-1} \\ d_1 & \cdots & d_{t-1} \\ b_1 & \cdots & b_{t-1} \end{array} \} (1 - t)^{\sum_{i=1}^{t-1} (d_i - b_i)} \]

\[ = \sum_{d' \leq d} B_2(b_1, \ldots, b_{t-1}; d'_{t-1}) (1 - t)^{\sum_{i=1}^{t-1} (c_i - d_i)} \]

\[ = D(b_1, \ldots, b_{t-1}; c_1, \ldots, c_{t-1}) \]
Finally, by induction and Lemma 3.3

\[ D(b_1, \ldots, b_{l-1}; c_1, \ldots, c_{l-1}) = \sum_{d \leq c \leq \sum_{i=1}^{l-1} (c_i - d_i)} B_2(b_1, \ldots, b_{l-1}; d_1, \ldots, d_{l-1})(1 - t)^{\sum_{i=1}^{l-1} (c_i - d_i)} \]

\[ = \sum_{d \leq c} \sum_{d' \in S_{l-1}} A(b_1, \ldots, b_{l-1}; d_1', \ldots, d_{l-1}')(1 - t)^{\sum_{i=1}^{l-1} (d_i' - b_i)}(1 - t)^{\sum_{i=1}^{l-1} (c_i - d_i)} \]

\[ = \sum_{d \leq c} \sum_{d' \in S_{l-2}} A(b_1, \ldots, b_{l-1}; d_1', \ldots, d_{l-2}', b_{l-1}')(1 - t)^{\sum_{i=1}^{l-1} (d_i' - b_i)}(1 - t)^{\sum_{i=1}^{l-1} (c_i - d_i)} \]

\[ = \sum_{d \leq c} \sum_{d_{l-2} \leq b_{l-1} < d'} A(b_1, \ldots, b_{l-1}; d_1', \ldots, d_{l-2}', b_{l-1})(1 - t)^{\sum_{i=1}^{l-1} (d_i' - b_i)}(1 - t)^{\sum_{i=1}^{l-2} (c_i - d_i)} \]

If \( l = 3 \), then by Lemma 3.3

\[ \sum_{d \leq c} B_2(b_1, \ldots, b_{l-1}; d_1, \ldots, d_{l-2}, b_{l-1})(1 - t)^{\sum_{i=1}^{l-2} (c_i - d_i)} \]

\[ = \sum_{d \leq c} B_2(b_1, b_2; d_1, b_2)(1 - t)^{c_1 - d_1} \]

\[ = \sum_{d \leq c \text{ min}(c_1, b_2 - 1)} B_2(b_1, b_2; d_1, b_2)(1 - t)^{c_1 - d_1} \]

\[ = \sum_{d_1 = b_1 \text{ min}(c_1, b_2 - 1)} (1 - t)^{c_1 - d_1} \]

\[ = \sum_{d_1 = b_1} A(d_1, c_2; c_1, c_2)(1 - t)^{c_1 - b_1} \]

\[ = \sum_{d_1 = b_1} \sum_{d_2 = b_2} A(d_1, d_2; c_1, c_2)(1 - t)^{c_1 + c_2 - d_1 - d_2} \]

\[ = B_3(b_1, b_2; b_1, c_1, c_2, b) \]

\[ = B_3(b_1, \ldots, b_{l-1}, b; c_1, \ldots, c_{l-1}, b) \]
If \( l > 3 \), then by induction

\[
\sum_{d \leq L} B_2(b_1, \ldots, b_{l-1}; d_1, \ldots, d_{l-2}, b_{l-1})(1 - t)\sum_{i=1}^{l-2}(c_i - d_i)
\]

\[
= \sum_{d \leq L} B_3(b_1, \ldots, b_{l-1}; d_1, \ldots, d_{l-2}, b_{l-1})(1 - t)\sum_{i=1}^{l-2}(c_i - d_i)
\]

\[
= \sum_{d \leq L} \sum_{b_2-1}^{b_{l-1}-1} \sum_{d_1 = b_1}^{b_{l-2}-1} \sum_{d_2 = b_2-2}^{b_{l-2}-1} A(d_1', \ldots, d_{l-2}; d_1, \ldots, d_{l-2})(1 - t)\sum_{i=1}^{l-2}(d_i - d_i')(1 - t)\sum_{i=1}^{l-1}(c_i - d_i)
\]

\[
= \sum_{d \leq L} \sum_{b_2-1}^{b_{l-1}-1} \sum_{d_1 = b_1}^{b_{l-2}-1} \sum_{d_2 = b_2-2}^{b_{l-2}-1} A(d_1', \ldots, d_{l-2}; d_1, \ldots, d_{l-2})(1 - t)\sum_{i=1}^{l-2}(c_i - d_i')
\]

\[
= \sum_{d \leq L} \sum_{b_2-1}^{b_{l-1}-1} \sum_{d_1 = b_1}^{b_{l-2}-1} \sum_{d_2 = b_2-2}^{b_{l-2}-1} \sum_{b_3-1}^{b_{l-3}-1} A(d_1', \ldots, d_{l-3}; d_1, \ldots, d_{l-3}, c_1, \ldots, c_{l-1})(1 - t)\sum_{i=1}^{l-1}(c_i - d_i')
\]

\[
= B_3(b_1, \ldots, b_{l-1}, b, c_1, \ldots, c_{l-1}, b).
\]

This completes the proof of (3).

We now assume that \( b_l < c_l \). Then

\[
B_1(b_1, \ldots, b_l; c_1, \ldots, c_l)
\]

\[
= \sum_{d \in S_1} A(d_1, \ldots, d_l; c_1, \ldots, c_l)\{\begin{array}{ccc}d_1 & \cdots & d_l \\ b_1 & \cdots & b_l \end{array}\} (1 - t)\sum_{i=1}^{l}(c_i - d_i)
\]

\[
= \sum_{d \in S_{l-1}} A(d_1, \ldots, d_{l-1}; c_l; c_1, \ldots, c_l)\{\begin{array}{ccc}d_1 & \cdots & d_{l-1} \\ b_1 & \cdots & b_{l-1} \end{array}\} (1 - t)\sum_{i=1}^{l-1}(c_i - d_i).
\]

For every \( i \leq l \), let \( \sigma_i : \{b_1, \ldots, b_{l-1}\} \rightarrow \{\hat{b}_1, \ldots, \hat{b}_l\} \) be the bijection such that \( \sigma_i(b_j) = b_j \) if \( j < i \) and \( \sigma_i(b_j) = b_{j+1} \) if \( j \geq i \). (Note that \( \sigma_l \) is the identity.) Then

\[
\{\begin{array}{ccc}d_1 & \cdots & d_{l-1} \\ b_1 & \cdots & b_{l-1} \end{array}\}
\]

\[
= \sum_{i=1}^{l} t(-1)^{l+i}\{\begin{array}{ccc}d_1 & \cdots & d_{l-1} \\ \sigma_i(b_1) & \cdots & \sigma_i(b_{l-1}) \end{array}\}
\]

\[
= \sum_{i=1}^{l} t(-1)^{l+i}\{\begin{array}{ccc}d_1 & \cdots & d_{l-1} \\ \sigma_i(b_1) & \cdots & \sigma_i(b_{l-1}) \end{array}\}.
\]
Therefore,

\[ B_1(b_1, \ldots, b_l; c_1, \ldots, c_l) = t \sum_{i=1}^{l} (-1)^{l+i} \sum_{d \in S_{i-1}} A(d_1, \ldots, d_{l-1}, b_l) \left\{ \begin{array}{ccc} d_1 & \cdots & d_{l-1} \\ \sigma_i(b_1) & \cdots & \sigma_i(b_{l-1}) \end{array} \right\} c_l \right\} (1 - t) \Sigma_{i=1}^{l-1} (c_i - d_i) \]

\[ = t \sum_{i=1}^{l} (-1)^{l+i} B_1(\sigma_i(b_1), \ldots, \sigma_i(b_{l-1}), c_l; c_1, \ldots, c_l). \]

We have two cases need to discuss:

**Case 1:** \( c_{l-1} < b_l \).

In this case, \( B_1(\sigma_i(b_1), \ldots, \sigma_i(b_{l-1}), c_l; c_1, \ldots, c_l) = 0 \) for \( i < l \), as \( \sigma_i(b_{l-1}) = b_l > c_{l-1} \) if \( i < l \). Therefore

\[ B_1(b_1, \ldots, b_l; c_1, \ldots, c_l) = tB_1(b_1, \ldots, b_{l-1}, c_l; c_1, \ldots, c_l). \]

On the other hand,

\[ B_2(b_1, \ldots, b_l; c_1, \ldots, c_l) \]

\[ = \sum_{d \in S_{l-1}} A(b_1, \ldots, b_l; d_1, \ldots, d_{l-1}, b_l) \left\{ \begin{array}{ccc} c_1 & \cdots & c_{l-1} \\ d_1 & \cdots & d_{l-1} \end{array} \right\} b_l \right\} (1 - t) \Sigma_{i=1}^{l-1} (d_i - b_i) \]

\[ = t \sum_{d \in S_{l-1}} A(b_1, \ldots, b_l; d_1, \ldots, d_{l-1}, b_l) \left\{ \begin{array}{ccc} c_1 & \cdots & c_{l-1} \\ d_1 & \cdots & d_{l-1} \end{array} \right\} b_l \right\} (1 - t) \Sigma_{i=1}^{l-1} (d_i - b_i) \]

\[ = t \sum_{d \in S_{l-1}} A(b_1, \ldots, b_l; c_l, d_1, \ldots, d_{l-1}, c_l) \left\{ \begin{array}{ccc} c_1 & \cdots & c_{l-1} \\ d_1 & \cdots & d_{l-1} \end{array} \right\} (1 - t) \Sigma_{i=1}^{l-1} (d_i - b_i) \]

\[ = t \sum_{d \in S_{l-1}} A(b_1, \ldots, b_l; c_l, d_1, \ldots, d_{l-1}, c_l) \left\{ \begin{array}{ccc} c_1 & \cdots & c_l \\ d_1 & \cdots & d_{l-1} \end{array} \right\} (1 - t) \Sigma_{i=1}^{l-1} (d_i - b_l) \]

\[ = tB_2(b_1, \ldots, b_l; c_l; c_1, \ldots, c_l). \]

Now, we can conclude from (4) that \( B_1(b_1, \ldots, b_l; c_1, \ldots, c_l) = B_2(b_1, \ldots, b_l; c_1, \ldots, c_l) \) in the case.

**Case 2:** \( b_l \leq c_{l-1} \).

In this case, \( B_2(b_1, \ldots, b_l; c_1, \ldots, c_l) = 0 \) by Remark 3.2. Thus it remains to show that \( B_1(b_1, \ldots, b_l; c_1, \ldots, c_l) = 0 \). Let \( \tilde{\sigma}_i : \{b_1, \ldots, b_l\} \rightarrow \{b_1, \ldots, b_l, c_l\} \) be the extension of \( \sigma_i \) such that \( \tilde{\sigma}_i(b_j) = \sigma_i(b_j) \) for \( j \leq l - 1 \) and \( \tilde{\sigma}_i(b_l) = c_l \);
then by Lemma [5, 2] and the fact that $c_{i-1} \geq b_i$,

\[ B_1(b_1, \ldots, b_l; c_1, \ldots, c_l) \]

\[ = t \sum_{i=1}^{l} (-1)^{l+i} B_1(\sigma_1(b_1), \ldots, \sigma_i(b_{i-1}), c_i; c_1, \ldots, c_l) \]

\[ = t \sum_{i=1}^{l} (-1)^{l+i} B_3(\sigma_1(b_1), \ldots, \sigma_i(b_{i-1}), c_i; c_1, \ldots, c_l) \]

\[ = t \sum_{i=1}^{l} (-1)^{l+i} \sum_{d_i=\sigma_i(b_i)}^{\tilde{\sigma}_i(b_i)-1} \sum_{d_{i-1}=\sigma_{i-1}(b_{i-1})}^{\tilde{\sigma}_{i-1}(b_{i-1})} A(d_1, \ldots, d_{i-1}; c_1, \ldots, c_{l-1})(1-t)^{\sum(c_i-d_i)} \]

\[ = t \sum_{d_1=b_1} \sum_{d_{i-1}=b_{i-1}} A(d_1, \ldots, d_{l-1}; c_1, \ldots, c_{l-1})(1-t)^{\sum(c_i-d_i)} \]

\[ = 0. \]

The proof of the theorem is now complete.

## 4 Main theory

The way to prove our main theorem is to use induction on $r$. Therefore we shall first discuss the case $r = 2$.

**Proposition 4.1** Let $Y$ be a one-sided ladder-shaped region of the plane with the partial order given in the beginning. Let $P_i = (i, n)$ and $Q = (m, b_i)$, $i = 1, 2$ be points of $Y$. Let $W(n, m; b_1, b_2)$ be the polynomial $\sum_{k \geq 0} w_k(P, Q)t^k$ and let $\tilde{W}(n, m; b_1, b_2) = \det[W(P_i, Q_j)]_{i,j=1,2}$. Then

\[ t^{-1}\tilde{W}(n, m; b_1, b_2) = \sum_{(c_1, c_2) \in S_2} A(b_1, b_2; c_1, c_2)(1-t)^{c_1+c_2-b_1-b_2}W(n, m; c_1, c_2). \]

In particular, if $b_1 = 1$ and $b_2 = 2$, then

\[ t^{-1}\tilde{W}(n, m; 1, 2) = W(n, m; 1, 2). \]

**Proof.** We prove by induction on $m$. If $m = 2$, then

\[ \sum_{(c_1, c_2) \in S_2} A(b_1, b_2; c_1, c_2)(1-t)^{c_1+c_2-b_1-b_2}W(n, 2; c_1, c_2) \]

\[ = \sum_{c_1=b_1}^{b_2-1} (1-t)^{c_1-b_1}[b_2 - c_1 - (b_2 - c_1 - 1)(1-t)] \]

\[ = b_2 - b_1 \]

\[ = t^{-1}\tilde{W}(n, 2; b_1, b_2). \]
Assume that \( m > 2 \). Then by Lemma 2.3 and induction

\[
W(n, m; b_1, b_2) = t^{-1} \sum_{(c_1, c_2) \in S_2, (m, c_2) \in Y} \{ \frac{c_1}{b_1} \frac{c_2}{b_2} \} W(n, m - 1; c_1, c_2)
\]

Furthermore, by Lemma 2.3

\[
\sum_{(c_1, c_2) \in S_2} A(b_1, b_2; c_1, c_2)(1 - t)^{c_1 + c_2 - b_1 - b_2} W(n, m; c_1, c_2)
\]

Since \( B_1(b_1, b_2; d_1, d_2) = B_2(b_1, b_2; d_1, d_2) \), the assertion follows.

**Theorem 4.2** Let \( Y \) be a one-sided ladder-shaped region of the plane (see Figure 1) with the partial order given in the beginning. Let \( P_i = (i, n) \) and \( Q = (m, b_i) \), \( i = 1, \ldots, r \) be points of \( Y \). Let \( W(n, m; b_1, \ldots, b_r) \) be the polynomial \( \sum_{k \geq 0} w_k(P, Q)t^k \) and let \( \tilde{W}(n, m; b_1, \ldots, b_r) = \det[W(P_i, Q_j)]_{i,j=1,\ldots,r} \). Then

\[
t^{-\langle \tilde{\omega} \rangle} \tilde{W}(n, m; b_1, \ldots, b_r) = \sum_{\xi \in S_r} A(b, \xi)(1 - t)^{\sum(c_i - b_i)} W(n, m; c_1, \ldots, c_r),
\]

where \( b = (b_1, \ldots, b_r) \) and \( \xi = (c_1, \ldots, c_r) \). In particular, if \( b_i = i \ \forall i \), then

\[
t^{-\langle \tilde{\omega} \rangle} \tilde{W}(n, m; 1, \ldots, r) = W(n, m; 1, \ldots, r).
\]
Proof. We prove the theorem by induction on $r$ and $m$. If $r = 2$, then it is the content of Proposition \[1.1\]. Therefore we may assume that $r > 2$.

Assume for the moment that $m = r$. Notice that by Lemma \[2.3\]

$$
\sum_{c \in S_r} A(b, c)(1-t)\sum_{i} (c_i - b_i) W(n, m; c_1, \ldots, c_r)
= \sum_{c \in S_{r-1}, c_r = b_r} A(b, c)(1-t)\sum_{i} (c_i - b_i) W(b_r - 1, r; c_1, \ldots, c_{r-1})
= \sum_{c \in S_{r-1}, c_r = b_r} A(b, c)(1-t)\sum_{i} (c_i - b_i) \{ \sum_{d \in S_{r-1}, d_r < b_r} [d_1 \cdots d_{r-1}] W(b_r - 1, r - 1; d) \}
$$

$$
= \sum_{d \in S_{r-1}, d_r < b_r} \{ \sum_{c \in S_{r-1}, c_r = b_r} A(d, c)[d_1 \cdots d_{r-1}] (1-t)\sum_{i} (c_i - b_i) \} W(b_r - 1, r - 1; d)
$$

$$
= \sum_{d \in S_{r-1}, d_r < b_r} B_2(b, \ldots, b_r; d_1, \ldots, d_{r-1}, b_r).
$$

Furthermore, by Corollary \[2.2\] and induction,

$$
\left(t^{-1}\right)_2 \bar{W}(n, m; b_1, \ldots, b_r)
= t^{-1} \sum_{c_1 = b_1}^{b_1 - 1} \cdots \sum_{c_{r-1} = b_{r-1}}^{b_{r-1} - 1} \bar{W}(b_r - 1, r - 1; c_1, \ldots, c_{r-1})
$$

$$
= \sum_{c_1 = b_1}^{b_1 - 1} \cdots \sum_{c_{r-1} = b_{r-1}}^{b_{r-1} - 1} \sum_{d \in S_{r-1}} A(c, d)(1-t)\sum_{i} (d_i - c_i) W(b_r - 1, r - 1; d_1, \ldots, d_{r-1})
$$

$$
= \sum_{d \in S_{r-1}, d_r < b_r} \{ \sum_{c_1 = b_1}^{b_1 - 1} \cdots \sum_{c_{r-1} = b_{r-1}}^{b_{r-1} - 1} A(d, c)(1-t)\sum_{i} (d_i - c_i) \} W(b_r - 1, r - 1; d_1, \ldots, d_{r-1})
$$

$$
= \sum_{d \in S_{r-1}, d_r < b_r} B_3(b, \ldots, b_r; d_1, \ldots, d_{r-1}, b_r).
$$
Thus the theorem holds for \( m = r \) by Theorem 3.1. We now assume that \( m > r \). By Lemma 2.3 and induction,

\[
\sum_{\mathcal{c} \in S_r, (m, \mathcal{c}) \in \mathcal{Y}} A(\mathcal{c}; \mathcal{d}) (1 - t)^{\sum (d_i - c_i)} W(n, m - 1; \mathcal{d}_1, \ldots, \mathcal{d}_r)
\]

Also, by Lemma 2.3

\[
\sum_{\mathcal{c} \in S_r} A(\mathcal{c}; \mathcal{b}) (1 - t)^{\sum (c_i - b_i)} W(n, m; \mathcal{c}_1, \ldots, \mathcal{c}_r)
\]

Since \( B_1(\mathcal{b}; \mathcal{d}) = B_2(\mathcal{b}; \mathcal{d}) \), the assertion follows. \( \square \)

References

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