MODAL EXTENSION OF ŁUKASIEWICZ LOGIC
FOR REASONING ABOUT COALITIONAL POWER

TOMÁŠ KROUPA AND BRUNO TEHEUX

Abstract. Modal logics for reasoning about the power of coalitions capture
the notion of effectivity functions associated with game forms. The main goal
of coalition logics is to provide formal tools for modeling the dynamics of
a game frame whose states may correspond to different game forms. The two
classes of effectivity functions studied are the families of playable and truly
playable effectivity functions, respectively. In this paper we generalize the
concept of effectivity function beyond the yes/no truth scale. This enables us
to describe the situations in which the goals of players are evaluated by utility
functions on states. Then we introduce two modal extensions of Łukasiewicz
\((n+1)\)-valued logic together with many-valued neighborhood semantics in
order to encode the properties of many-valued effectivity functions associated
with game forms. As our main results we prove completeness theorems for the
two newly introduced modal logics.

1. Introduction

Modeling collective actions of agents and capturing their effectivity is among the
important research topics on the frontiers of game theory, computer science and
mathematical logic. The main efforts are concentrated on answering the following
question: what is the set of outcome states that can effectively be implemented by
a coalition of agents? A game-theoretic framework for studying collective actions
and their enforceability is based on the notion of game forms. Loosely speaking,
a game form is a pure description of a game and its rules, without regard to the
agents’ preferences. The game frames enable us to capture a more general action
model in which a game form is associated with every state of the frame and the
outcome states of the game forms are the states of the frame. From the game-
theoretic viewpoint, the game frames are extensive form games with simultaneous
moves of the players; see [17, 22].

The concept of \(\alpha\)-effectivity ([1, 20]) is one of the key approaches to characterize
the coalitional effectivity within game form models. A coalition \(C\) is \(\alpha\)-effective for
a set of outcome states \(X\) if the players in \(C\) can choose a joint strategy that enforces
the outcome in \(X\) no matter what strategies are adopted by the other players.
The previous definition gives rise to the concept of a (truly) playable effectivity
function. In his seminal paper [19], Pauly introduces Coalition Logic \(\text{CL}_N\) to
reason about \(\alpha\)-effectivity in game forms with player set \(N\). The axiomatization
of \(\text{CL}_N\) is an attempt at a characterization of the class of \(\alpha\)-effectivity functions
in a multi-modal language. Pauly also defined a neighborhood semantics with
respect to which \(\text{CL}_N\) is complete. The logic \(\text{CL}_N\) was subsequently analyzed
and extended by many authors see [2, 5]. Namely Goranko et al. [9] found a
gap in Pauly’s characterization of playable effectivity functions (see [19] Theorem

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3.2], which led to the introduction of truly playable effectivity functions. The reader is invited to consult Appendix A in which we recall all the necessary notions regarding Boolean effectivity functions and the distinction between playable and truly playable functions, respectively.

In this paper we extend the results of Pauly [19] and Goranko et al. [9] to the situations in which the effectivity of coalitions is evaluated on a finer finite scale than \( \{0, 1\} \). Our main goal is to investigate the properties of many-valued modal logics \( P_n \) and \( TP_n \) devised for reasoning about the refined notion of effectivity. To this end, we proceed as follows. In Section 2 we generalize the notion of \( \alpha \)-effectivity. Since we use finitely-valued Łukasiewicz logic, our scale is always the set

\[
L_n = \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}, \text{ where } n \text{ is a positive integer.}
\]

Thus we introduce the concept of \( L_n \)-valued effectivity function whose purpose is to capture the effectivity of coalitions in those settings where goals are evaluated by utility functions. The motivation for this generalization is illustrated with examples: we show how to model the degree of satisfaction of agents’ goals in a strategic game (Example 2.5). The properties of (true) playability of \( L_n \)-valued effectivity functions, which are crucial for understanding the relation effectivity functions to game forms, are analyzed in Section 3. In particular, we establish the characterization of truly playable \( L_n \)-valued effectivity functions (Theorem 3.9). In Section 4 we develop the tools to capture the properties of \( L_n \)-valued effectivity functions in a many-valued modal language. These developments rely not only on the recent advances in modal extensions of Łukasiewicz logic (see [9, 12]), but they also require the introduction of neighborhood semantics, which has never been considered in the modal many-valued setting before, to the best of our knowledge. The newly introduced logics \( P_n \) and \( TP_n \) axiomatize in the many-valued modal language the properties of playable and truly playable \( L_n \)-valued effectivity functions, respectively. Our main results are Theorem 4.17 and Theorem 4.28 which show that the logics \( P_n \) and \( TP_n \) are complete with respect to the corresponding classes of \( L_n \)-valued coalitional frames. The key ingredient in the proof of completeness of \( TP_n \) is the many-valued generalization of filtration technique for neighborhood models [7, Chapter 7.5].

2. Game forms and many-valued effectivity functions

In what follows, \( S \) denotes a nonempty set, \( N = \{1, \ldots, k\} \) is a finite set and, for any \( i \in N \), \( \Sigma_i \) denotes a nonempty set. Recall the following definition [1].

**Definition 2.1.** A game form is a tuple \( G = \langle N, \{\Sigma_i \mid i \in N\}, S, o \rangle \), where \( N \) is a set of players, \( \Sigma_i \) is a set of strategies for each \( i \in N \), \( S \) is a set of outcome states, and \( o : \prod_{i \in N} \Sigma_i \rightarrow S \) is an outcome function.

The game forms are not to be confused with strategic games. While a preference relation over \( S \) must be defined for each player \( i \in N \) in a strategic game [17], no such requirement exists for a game form. Below we recall the basic examples of game forms.

**Example 2.2.** (i) Let \( N = \{1, 2\} \) and \( \Sigma_1, \Sigma_2 \) be some strategy sets. Assume that the players choose their strategies simultaneously. Then we may set \( S = \Sigma_1 \times \Sigma_2 \) and define \( o \) as the identity function, which turns \( (\{1, 2\}, \{\Sigma_1, \Sigma_2\}, \Sigma_1 \times \Sigma_2, o) \) into a game form.

(ii) Suppose, on the other hand, that Player 2 makes his choice only after observing the strategic choice of Player 1. This sequential procedure is modeled by a game form such that \( \Sigma_2 \) is the set of all functions \( r : \Sigma_1 \rightarrow \Sigma_2' \), where \( \Sigma_2' \) can be viewed as the set of all possible moves that can be played by Player 2. Hence,
Appendix A). We are going to generalize the concept of effectivity function: our

Example 2.3. Let $S'$ be any nonempty set of outcome states and $\Pi(S')$ be a set of

An important example is when the outcome function coincides with some social

Example 2.5. As an example, consider the situation in which we evaluate the
capacity of groups of agents to fulfill specified goals.

Example 2.4. Let $G = (N, \{\Sigma_i | i \in N\}, S, o)$ be a game form. The $L_n$-valued
effectivity function of $G$ is the map $E_G: \mathcal{P}N \times L_n^S \to L_n$ defined by

\[
E_G(C, f) = \max_{\sigma_C} \min_{\sigma_{\overline{C}}} f(o(\sigma_C \sigma_{\overline{C}})), \quad C \in \mathcal{P}N, f \in L_n^S,
\]

where $\sigma_C$ and $\sigma_{\overline{C}}$ range in the set of all joint strategies of coalitions $C$ and $\overline{C}$, respectively.

As an example, consider the situation in which we evaluate the capacity of groups
of agents to fulfill specified goals.

Example 2.5. Let $N$ be the set of the countries of the European Union (EU) and
set $n = k$. Let $\Sigma_i$ be the set of possible policies that country $i$ can adopt at the
local (country) level for each $i \in N$. Assume that for every $i \in N$ we evaluate the
following question:

\[
\text{`Given a global strategy vector } \sigma_N, \text{ does the economy of country}
\]

\[
i \in N \text{ operate at full employment provided that } \sigma_N \text{ is applied?'}
\]

This situation can be modeled by a set of states $S = \mathcal{P}N$. The outcome state
$A \in \mathcal{P}N$ means that only the countries in $A$ realize full employment. Experts
define the outcome function $o: \prod_{i \in N} \Sigma_i \to \mathcal{P}N$ by setting $i \in o(\sigma_N)$ if and only if
the global strategy $\sigma_N$ leads to a state in which country $i \in N$ has reached full
employment.
In this setting, an example of a relevant function $f \in \mathbb{L}_n^S$ can be defined by

$$f(A) = \frac{|A|}{|A|},$$

where $|A|$ is the cardinality of $A \in S = \mathcal{P}N$. For any coalition $C$, the value $E_G(C, f)$ measures the level with which the countries in $C$ can collaborate to enforce full employment in Europe, regardless of the policies adopted by the countries in $\mathcal{C}$. As another example, consider now that $n = \text{lcm}(1, \ldots, k)$ and define $g_C: \mathcal{P}N \rightarrow \mathbb{L}_n$ by setting $g_C(A) = \frac{|A|}{n}$ for every $A \in \mathcal{P}N$ with $C \neq \emptyset$. Then the value of $E_G(C, g_C)$ can be seen as measuring how efficient is coalition $C$ in enforcing full employment in the countries of $C$ (regardless of the result of their strategies in countries in $\mathcal{C}$).

The previous example is just an instance of a more general scheme. Assume that each player in $N$ is trying to achieve his/her own specific goal and that each goal can be partially achieved with a degree of achievement quantified in the scale $\mathbb{L}_1$. Given a player set $N$ and the set of strategy spaces $\{\Sigma_i \mid i \in N\}$, we consider for each player $i \in N$ utility (payoff) function $o_i: \prod_{j \in N} \Sigma_j \rightarrow \mathbb{L}_n$. The range of each $o_i$ is included in $\mathbb{L}_n$, which means that we are effectively measuring utility only in a finite scale. This assumption is not, however, too restrictive as many game-theoretic models have been developed in this framework (cf. [13, 15]).

Example 2.6. Given a player set $N$ and the set of strategy spaces $\{\Sigma_i \mid i \in N\}$, we consider for each player $i \in N$ the utility (payoff) function $o_i: \prod_{j \in N} \Sigma_j \rightarrow \mathbb{L}_n$. The range of each $o_i$ is included in $\mathbb{L}_n$, which means that we are effectively measuring utility only in a finite scale. This assumption is not, however, too restrictive as many game-theoretic models have been developed in this framework (cf. [13, 15]). Put $o = (o_1, \ldots, o_k)$ and notice that $o$ is a mapping sending each joint strategy profile $\sigma \in \prod_{j \in N} \Sigma_j$ to an element $o(\sigma) \in \mathbb{L}_n$. The tuple $G = (N, \{\Sigma_i \mid i \in N\}, \mathbb{L}_n, o)$ then becomes a game form.

How do coalitions $C \subseteq N$ assess their utility in this scheme? For a sufficiently large $n \in \mathbb{N}$, each coalition $C$ can employ an aggregation function $f_C: \mathbb{L}_n^N \rightarrow \mathbb{L}_n$ evaluating the total utility realized by $C$. For example, the average utility $f_C^\sigma$ and the minimum utility $f_C^\sigma$ are given by

$$f_C^\sigma(a) = \frac{1}{|C|} \sum_{i \in C} a_i \quad \text{and} \quad f_C^\sigma(a) = \min\{a_i \mid i \in C\},$$

respectively, where $a = (a_1, \ldots, a_k) \in \mathbb{L}_n^N$. For every joint strategy profile $(\sigma_C, \sigma_{C'}) \in \prod_{j \in N} \Sigma_j$, the number $f_C(o(\sigma_C, \sigma_{C'})) \in \mathbb{L}_n$ is the aggregated utility resulting from the simultaneous choice of the strategic vectors $\sigma_C$ and $\sigma_{C'}$. Assume that the cooperative outcomes are evaluated by $f_C$: what is the minimum guaranteed utility degree that the coalition $C$ is able to ensure? It is simply the value of effectivity function at $(C, f_C)$, that is,

$$E_G(C, f_C) = \max_{\sigma_C} \min_{\sigma_{C'}} f_C(o(\sigma_C, \sigma_{C'})).$$

Adopting the framework of social choice correspondences employed in Example 2.3 we obtain the following kind of many-valued effectivity function.

Example 2.7. Assume now that each state $s \in S'$ is assigned a utility value $u(s) \in \mathbb{L}_n$. For instance, this value can measure the utility of outcome state $s$ with respect to a criterion that is not under the control of players (the consumption of available resources, say). Then, consider a function $A_u: \mathcal{P}S' \rightarrow \mathbb{L}_n$ aggregating the utilities $u(s)$ for $s \in T$, where $T \in \mathcal{P}S'$. Set $A_u(T) = \max\{u(s) \mid s \in T\}$, for instance. With these definitions, $A_u \in \mathbb{L}_n^S$ and for every $C \in \mathcal{P}N$, the value $E_G(C, A_u)$ is the minimum guaranteed utility (with regard to $u$) that coalition $C$ is able to ensure.

3. Playability of $\mathbb{L}_n$-valued effectivity functions

Analogously to the classical literature [16, 20] on effectivity functions, we can study the notion of effectivity in a setting independent on game forms. Let $S$ be
We say that $\chi$ we denote by $t$ is effective, homogeneous, and has liveness and safety properties. We say that $A$ is a set of outcomes and $N$ is finite player set. We always assume that $|S| \geq 2$ and $|N| \geq 2$.

**Definition 3.1.** An $L_n$-valued effectivity function is a mapping $E: \mathcal{P}N \times L_n^S \to L_n$.

Note that the $L_1$-valued effectivity functions are exactly the effectivity functions $\mathcal{P}N \times \{0, 1\}^S \to \{0, 1\}$ arising in the Boolean framework (19). Therefore we call any $L_1$-valued effectivity function a Boolean effectivity function.

Our goal is to characterize the class of $L_n$-valued effectivity functions that are associated with game forms. This characterization is related to the properties of effectivity functions listed in Definition 3.2. We use the following standard notation used in Łukasiewicz logic; see [8]. The set $L_n$ is equipped with Łukasiewicz interpretation of the implication $\to$ and the negation $\neg$ defined by

\[
\neg x = 1 - x,
\]

for every $x, y \in L_n$. Moreover, the strong disjunction $\oplus$ and the strong conjunction $\odot$ are defined by $p \oplus q = \neg p \to q$ and $p \odot q = \neg (\neg p \oplus \neg q)$, respectively, and hence they are interpreted in $L_n$ by the following associative binary operations:

\[
x \oplus y = \min(x + y, 1) \quad \text{and} \quad x \odot y = \max(x + y - 1, 0).
\]

Finally, the connectors $\land$ and $\lor$ are defined as $p \land q = (q \odot \neg p) \oplus p$ and $p \lor q = (q \odot \neg p) \oplus q$, and are interpreted on $L_n$ as the minimum and the maximum operation, respectively.

**Definition 3.2.** Let $E$ be an $L_n$-valued effectivity function. We say that $E$

1. is outcome monotonic whenever $E(C, f) \geq E(C, g)$, for every $C \in \mathcal{P}N$ and every $f, g \in L_n^S$ with $f \geq g$;
2. is $N$-maximal if $\neg E(\emptyset, \neg f) \leq E(N, f)$ for every $f \in L_n^S$;
3. is regular if it satisfies $E(C, f) \leq \neg E(C, \neg f)$ for every $C \in \mathcal{P}N$ and $f \in L_n^S$;
4. is superadditive if $E(C_1, f) \land E(C_2, g) \leq E(C_1 \cup C_2, f \land g)$ for every $C_1, C_2 \in \mathcal{P}N$ such that $C_1 \cap C_2 = \emptyset$ and every $f, g \in L_n^S$;
5. is coalition monotonic if $E(C, f) \leq E(C', f)$ for every $C \subseteq C' \in \mathcal{P}N$ and every $f \in L_n^S$;
6. is homogeneous if $E(C, f \oplus g) = E(C, f) \oplus E(C, f)$ and $E(C, f \odot f) = E(C, f) \odot E(C, f)$ for every $C \in \mathcal{P}N$ and every $f \in L_n^S$;
7. has liveness property if $E(C, 1) = 1$ for every $C \in \mathcal{P}N$;
8. has safety property if $E(C, 0) = 0$ for every $C \in \mathcal{P}N$;
9. is principal if there is $g \in L_n^S$ with $E(\emptyset, \neg) = \neg(1) = \{ f \in L_n^S \mid f \geq \bigodot_{i=1}^n g \}$.

We say that $E$ playable whenever it is outcome monotonic, $N$-maximal, superadditive, homogeneous, and has liveness and safety properties. We say that $E$ is truly playable if it is playable and principal.

If $n = 1$, then the definitions of (truly) playable Boolean effectivity function coincide with the corresponding definitions used in the Boolean setting (19); see also Appendix [A].

Note that if $E$ is an outcome monotonic and homogeneous $L_n$-valued effectivity function, then $E(C, -)^{-1}(1)$ is an MV-filter of the MV-algebra $L_n^S$ [8]. Moreover, the $L_n$-valued effectivity function $E$ is principal whenever the MV-filter $E(\emptyset, \neg)^{-1}(1)$ is principal.

It may be difficult to get some intuition about the definition of a homogeneous $L_n$-valued effectivity function in the game form framework. We refer to Remark [4.18] for an equivalent formulation of this definition. The following result illustrates that homogeneity arises naturally in the context of game forms. For every set $Y \subseteq S$, we denote by $\chi_Y$ the characteristic function of $Y$. 
Proposition 3.3. If \( G = (N, \{ \Sigma_i \mid i \in N \}, S, o) \) is a game form, then \( E_G \) is a truly
playable \( \mathbb{L}_n \)-valued effectivity function.

Proof. It follows directly from Definition 2.4 that \( E_G \) is outcome monotonic, \( N \)-maximal and has liveness and
safety properties. Homogeneity of \( E_G \) follows from the fact that the maps \( \tau_{\boxplus} : x \mapsto x \oplus x \) and \( \tau_{\odot} : x \mapsto x \odot x \) are lattice homomorphisms of \( \mathbb{L}_n \). Moreover, \( E_G(\emptyset, -)^{-1}(1) = \{ g \mid g \geq \chi_{\text{ran}(o)} \} \), which shows that \( E_G \) is
principal since \( \bigcap_{j=1}^n \chi_{\text{ran}(o)} = \chi_{\text{ran}(o)} \).

It remains to prove that \( E_G \) is superadditive. Let \( C_1, C_2 \in \mathcal{P} N \) be such that \( C_1 \cap C_2 = \emptyset \) and \( f_1, f_2 \in \mathbb{L}_n^S \). Denote by \( \sigma_{C_1}^* \) and \( \sigma_{C_2}^* \) two strategy vectors satisfying
\( E(C_1, f_1) = \min_{\sigma_{C_1}} f_1(o(\sigma_{C_1}^* \sigma_{C_2}^*)) \) and \( E(C_2, f_2) = \min_{\sigma_{C_2}} f_2(o(\sigma_{C_2}^* \sigma_{C_2}^*)) \). It
follows that \( E(C_1, f_1) \land E(C_2, f_2) = \min_{\sigma_{C_1}} \min_{\sigma_{C_2}} f_1(o(\sigma_{C_1}^* \sigma_{C_2}^*)) \land f_2(o(\sigma_{C_2}^* \sigma_{C_2}^*)) \), which is not greater than \( \min_{\sigma_{C_1} \land \sigma_{C_2}^*} (f_1 \land f_2)(o(\sigma_{C_1}^* \sigma_{C_2}^* \sigma_{C_2}^* \sigma_{C_2}^*)) \). The conclusion follows from the fact that the latter is upper bounded by \( E_G(C_1 \cup C_2, f_1 \land f_2) \). \( \square \)

As the next result shows, a Boolean effectivity function can be associated with any homogeneous \( \mathbb{L}_n \)-valued
effectivity function. The Boolean algebra \( \mathbb{L}_n^S = \{ 0, 1 \}^S \)
is called the Boolean skeleton of \( \mathbb{L}_n^S \). In other words, the Boolean skeleton of \( \mathbb{L}_n^S \) is
the powerset of \( S \) if we identify the subsets of \( S \) with their characteristic functions on \( S \). An element \( f \in \mathbb{L}_n^S \) belongs to the Boolean skeleton of \( \mathbb{L}_n^S \) if and only if \( f \oplus f = f \), and such an element is said to be idempotent. For every \( \mathbb{L}_n \)-valued
effectivity function \( E : \mathcal{P} N \times \mathbb{L}_n^S \rightarrow \mathbb{L}_n \), we denote by \( E^\sharp \) the restriction of \( E \) to
\( \mathcal{P} N \times \mathbb{L}_n^S \).

Lemma 3.4. If \( E \) is a homogeneous \( \mathbb{L}_n \)-valued effectivity function, then \( E^\sharp \) is
a Boolean effectivity function. If in addition \( E \) is playable (respectively, truly
playable), then so is \( E^\sharp \).

Proof. For any idempotent element \( f \in \mathbb{L}_n^S \) and for every \( C \in \mathcal{P} N \), we obtain
\( E(C, f) \oplus E(C, f) = E(C, f \oplus f) = E(C, f) \).

Therefore \( E^\sharp \) is a Boolean effectivity function.

If in addition \( E \) is playable (respectively, truly playable), then it satisfies conditions \( 1 \), \( 2 \), \( 4 \), \( 7 \) and \( 8 \) (respectively, conditions \( 1 \), \( 2 \), \( 4 \), \( 7 \), \( 8 \) and
\( 9 \)) of Definition 3.2. It follows that \( E^\sharp \) also satisfies the analogous Boolean conditions (see Appendix A) since they do not involve any existential quantifier over the elements of \( \mathbb{L}_n^S \). \( \square \)

In order to study the playability property, we need more technical preliminaries. To this end, put
\( \tau_{\boxplus}(x) = x \oplus x \) and \( \tau_{\odot}(x) = x \odot x \), for every \( x \in \mathbb{L}_n^S \).

Definition 3.5. Let \( i \in \{ 1, \ldots, n \} \). We denote by \( \tau_{i/n} \) a mapping \( \mathbb{L}_n \rightarrow \mathbb{L}_n \) such that
\begin{enumerate}
\item \( \tau_{i/n} \) is a finite multiple composition of the maps \( \tau_{\boxplus} \) and \( \tau_{\odot} \),
\item the following equality holds:\n\[ \tau_{i/n}(x) = \begin{cases} 0 & \text{if } x < \frac{i}{n}, \\ 1 & \text{if } x \geq \frac{i}{n}. \end{cases} \]
\end{enumerate}

A constructive proof of the existence of \( \tau_{i/n} \) appears in [15].

Any mapping \( \tau : \mathbb{L}_n \rightarrow \mathbb{L}_n \) can be composed with any \( f \in \mathbb{L}_n^S \). Thus \( \tau(f)(s) = \tau(f(s)) \) for every \( s \in S \) and \( f \in \mathbb{L}_n^S \).

Lemma 3.6. Let \( E, E' : \mathcal{P} N \times \mathbb{L}_n^S \rightarrow \mathbb{L}_n \) be homogeneous \( \mathbb{L}_n \)-valued
effectivity functions. Then \( E = E' \) if and only if \( E^\sharp = E'^\sharp \).
To prove that \( E^2 = E^\sigma \). Let \( C \in \mathcal{PN} \), \( f \in \mathbb{L}^S_n \) and \( i \in \{1, \ldots, n\} \). Since \( \tau_i/f(f) \) is idempotent and \( E \) and \( E^\sigma \) are homogeneous, we have \( E(C, f) \geq \frac{1}{n} \) if and only if \( 1 = E(C, \tau_i/f(f)) = E^\sigma(C, \tau_i/f(f)) \), which is equivalent to \( E^\sigma(C, f) \geq \frac{1}{n} \).

The following lemma is straightforward. Its statement uses the notion of Boolean effectivity function \( H_G \) associated with a game form \( G \), which is recalled in Appendix A.

**Lemma 3.7.** Let \( G \) be a game form. If \( H_G \) and \( E_G \) are the Boolean and the \( \mathbb{L}_n \)-valued effectivity function associated with \( G \), respectively, then \( H_G = E_G^\sigma \).

For any \( r \in [0,1] \), we denote by \( \lfloor r \rfloor \) the element \( \min \{ a \in \mathbb{L}_n \mid a \geq r \} \). The following lemma will turn out to be crucial for understanding the limits of expressive power of the language associated with (truly) playable effectivity functions; see Proposition 1.8.

**Lemma 3.8.** If \( H : \mathbb{P} \times \mathbb{L}^S_1 \to \mathbb{L}_1 \) is a playable Boolean effectivity function, then the function \( E : \mathbb{P} \times \mathbb{L}^S_n \to \mathbb{L}_n \) defined by

\[
E(C, f) = \max \left\{ \frac{1}{n} \in \mathbb{L}_n \mid H(C, \tau_i/f(f)) = 1 \right\},
\]

for every \( C \in \mathcal{PN} \) and \( f \in \mathbb{L}^S_n \), is a playable \( \mathbb{L}_n \)-valued effectivity function that satisfies \( E^2 = H \). If in addition \( H \) is truly playable, then so is \( E \).

**Proof.** Clearly \( E^2 = H \). It follows from outcome monotonicity of \( H \) that \( E \) is outcome monotonic. Since \( H \) satisfies liveness and safety, so does the function \( E \).

To prove that \( E \) is superadditive, assume on the contrary that there exist \( f, g \in \mathbb{L}^S_n \) and \( C, D \in \mathcal{PN} \) such that \( C \cap D = \varnothing \) and \( E(C \cup D, f \wedge g) < \frac{1}{n} \leq E(C, f) \wedge E(D, g) \) for some \( i \in \{1, \ldots, n\} \). On the one hand, it follows that \( H(C \cup D, \tau_i/f(f) \wedge \tau_i/g(g)) = H(C \cup D, \tau_i/f(g \wedge g)) = 0 \). On the other hand, we obtain \( H(C, \tau_i/f(f)) = H(D, \tau_i/g(g)) = 1 \) and by superadditivity of \( H \) we get \( H(C \cup D, \tau_i/f(f) \wedge \tau_i/g(g)) = 1 \), a contradiction.

Now, let \( f \in \mathbb{L}^S_n \). Since \( \tau_i/f(f + f) = \tau_i/f(\lfloor f \rfloor) \in \mathbb{L}^S_n \), it follows from the definition of \( E \) that \( E(C, f \circ f) \geq \frac{1}{n} \) if and only if \( H(C, \tau_i/f(\lfloor f \rfloor)) = 1 \). Using again the definition of \( E \), the latter is equivalent to \( E(C, f) \geq \frac{1}{n} \), which is the same as \( E(C, f) = E(C, f \circ f) \geq \frac{1}{n} \). We can proceed in a similar way to prove that \( E(C, f \circ f) = E(C, f) \circ E(C, f) \) for every \( f \in \mathbb{L}^S_n \).

To prove \( N \)-maximality of \( E, \) consider \( f \in \mathbb{L}^S_n \) and \( i \in \{0, \ldots, n-1\} \). It follows from the definition of \( E \) and \( N \)-maximality of \( H \) that \( E(N, f) \leq \frac{1}{n} \) if and only if \( H(\varnothing, \tau_i\neg\tau_i/f(f)) = 1 \). Since \( \tau_i\neg\tau_i/f(f) = \tau_i\neg\tau_i/f(\neg f) \), the last identity is equivalent to \( H(\varnothing, \tau_i\neg\tau_i/f(\neg f)) = 1 \) and finally to \( E(\varnothing, \neg f) \leq \frac{1}{n} \).

Assume that \( H \) is truly playable. Definition 1.8 yields existence of \( g \in \mathbb{L}^S_1 \) such that \( H(\varnothing, \neg) = \frac{1}{n} \). Since \( E \) is homogeneous, \( E(\varnothing, \neg) = \frac{1}{n} \).

The following result, which is the \( \mathbb{L}_n \)-valued generalization of [19, Theorem 3.2] and [9, Theorem 1], completes the characterization of truly playable \( \mathbb{L}_n \)-valued effectivity functions.

**Theorem 3.9.** An \( \mathbb{L}_n \)-valued effectivity function \( E : \mathcal{PN} \times \mathbb{L}^S_n \to \mathbb{L}_n \) is truly playable if and only if there is a game form \( G \) such that \( E = E_G \).

**Proof.** By Proposition 3.8, \( E_G \) is truly playable. Conversely, assume that \( E \) is truly playable. By Lemma 3.4 and Theorem 1.4, there exists a game form \( G \) such that \( H_G = E^\sigma \), where \( H_G \) is the Boolean effectivity function of \( G \). We obtain by
Lemma 3.7 that $E^2 = H_G = E^2_G$, where $E_G$ is the $L_n$-valued effectivity function associated with $G$. The conclusion $E = E_G$ follows from Lemma 3.9 \( \square \)

It is useful to introduce a weaker notion of playability.

**Definition 3.10.** An $\mathbb{L}_n$-valued effectivity function $E : \mathcal{P}N \times \mathbb{L}_n^S \to \mathbb{L}_n$ is semi-playable if $E(C, f) \leq E(C, g)$ for every $f \leq g \in \mathbb{L}_n^S$ and every coalition $C \neq N$, if $E$ has liveness and safety property for coalitions $C \neq N$, and if it satisfies superadditivity for coalitions $C_1$ and $C_2$ such that $C_1 \cap C_2 = \emptyset$ and $C_1 \cup C_2 \neq N$.

**Lemma 3.11.** If $E$ is a playable $\mathbb{L}_n$-valued effectivity function, then $E$ is coalition monotonic and regular.

**Proof.** We proceed by contradiction to prove that $E$ is regular. Assume that there are $C \in \mathcal{P}N$, $f \in \mathbb{L}_n^S$ and $i \in \{1, \ldots, n\}$ such that $\neg E(C, \neg f) < \frac{i}{n} \leq E(C, f)$. Put $j = 1 - \frac{i}{n}$. Since $E$ is homogeneous, it follows that $E(C, \tau_{ij}(f)) = 1$ and $E(C, \tau_{ij}(\neg f)) = 1$. Thus we obtain by superadditivity $E(N, \tau_{ij}(f) \land \tau_{ij}(\neg f)) = 1$, which contradicts the safety property since $\tau_{ij}(f) \land \tau_{ij}(\neg f) = 0$.

We prove that $E$ is coalition monotonic. Let $C \subseteq C' \in \mathcal{P}N$ and $f \in \mathbb{L}_n^S$. By applying superadditivity to $C_1 = C$ and $C_2 = C' \setminus C$ we obtain $E(C, f) \leq E(C', f)$, which is the desired result. \( \square \)

**Proposition 3.12.** An $\mathbb{L}_n$-valued effectivity function $E : \mathcal{P}N \times \mathbb{L}_n^S \to \mathbb{L}_n$ is playable if and only if it is semi-playable, homogeneous, regular and $N$-maximal.

**Proof.** The first implication follows from Lemma 3.11 Conversely, assume that $E$ is semi-playable, homogeneous, regular and $N$-maximal. First we prove superadditivity. Let $C \in \mathcal{P}N$ with $C \neq N$. We have to verify that $E(C, f) \land E(C, g) \leq E(C, f \land g)$. By way of contradiction, assume that there is $i \in \{1, \ldots, n\}$ such that $E(N, f \land g) < \frac{i}{n} \leq E(C, f) \land E(C, g)$. Since $E$ is homogeneous, we obtain $E(C, \tau_{ij}(f)) = 1 = E(C, \tau_{ij}(g))$, while, by $N$-maximality, $E(\emptyset, \neg \tau_{ij}(f \land g)) = 1$. It follows from superadditivity that $E(C, \tau_{ij}(f) \land \neg \tau_{ij}(f \land g)) = 1$, which is equivalent to $E(C, \tau_{ij}(f) \land \neg (\tau_{ij}(f) \land \tau_{ij}(g))) = 1$, since $\tau_{ij}(f \land g) = \tau_{ij}(f) \land \tau_{ij}(g)$. From the fact that $\tau_{ij}(f)$ and $\tau_{ij}(g)$ belong to the Boolean skeleton of $\mathbb{L}_n^S$ we deduce $E(C, \neg \tau_{ij}(g)) = 1$. We conclude that $E(C, \neg \tau_{ij}(g)) = 1$ by outcome monotonicity and finally that $E(C, \tau_{ij}(g)) = 0$ by regularity, which is the desired contradiction.

It is easy to check that liveness and safety conditions are satisfied for $C = N$. Moreover, $E$ is $N$-maximal and homogeneous by assumption. It remains to prove that $E$ is outcome monotonic. If $f \leq g \in \mathbb{L}_n^S$, we obtain successively

$$E(N, f) \leq \neg E(\emptyset, \neg f) \leq \neg E(\emptyset, \neg g) \leq E(N, g)$$

where the first inequality is obtained by regularity, the second by monotonicity and the third by $N$-maximality. \( \square \)

4. $\mathbb{L}_n$-valued modal language and semantics for effectivity functions

In this section we build a many-valued modal logic with neighborhood semantics in the spirit of [19, 9] that capture the properties of (truly) playable $\mathbb{L}_n$-valued effectivity functions.

4.1. Neighborhood semantics for playable $\mathbb{L}_n$-valued effectivity functions.

Let $\mathcal{L}$ be the language $\{\to, \neg, 1\} \cup \{[\mathcal{C}] \mid \mathcal{C} \in \mathcal{P}N\}$ where $\to, \neg, 1$ are respectively binary, unary and constant, and $[\mathcal{C}]$ is a unary modality for every $\mathcal{C} \in \mathcal{P}N$. 
The set $\mathcal{L}$ of formulas is defined inductively from the countably infinite set $\text{Prop}$ of propositional variables by the following rules:

$$\phi ::= 1 \mid p \mid \phi \rightarrow \phi \mid \neg \phi \mid [C] \phi$$

where $p \in \text{Prop}$ and $C \in \mathcal{P}N$. We use $0$ as an abbreviation of $\neg 1$. The intended reading of the formula $[C] \phi$ is

‘coalition $C$ can enforce an outcome state in which $\phi$ holds’.

We introduce a semantics for $\mathcal{L}$ which is based on a class of action models called $\mathbb{L}_n$-valued $\mathcal{L}$-frames. Such frames are $\mathbb{L}_n$-valued extensions of coalition frames introduced in [10]. The coalition frames are a very general model of interaction in which an effectivity function over the player set $N$ is associated with each outcome state in $S$. Under the assumption of true playability, this is equivalent to specifying a game form for every outcome state in $S$.

**Definition 4.1.** An $\mathbb{L}_n$-valued $\mathcal{L}$-frame is a tuple $\mathcal{F} = (S, E)$, where $S$ is a set of outcome states and $E$ is a mapping sending each outcome state $u \in S$ to an $\mathbb{L}_n$-valued effectivity function $E(u) : \mathcal{P}N \times \mathbb{L}_n \to \mathbb{L}_n$. A tuple $\mathcal{M} = (\mathcal{F}, \text{Val})$ is an $\mathbb{L}_n$-valued $\mathcal{L}$-model (based on $\mathcal{F}$) if $\mathcal{F} = (S, E)$ is an $\mathbb{L}_n$-valued $\mathcal{L}$-frame and $\text{Val} : S \times \mathcal{L} \to \mathbb{L}_n$.

For every $\mathbb{L}_n$-valued $\mathcal{L}$-model $\mathcal{M}$, the valuation map $\text{Val}$ is extended inductively to $S \times \mathcal{L}$ by setting

$$(4.1) \quad \text{Val}(u, \phi \rightarrow \psi) = \text{Val}(u, \phi) \rightarrow \text{Val}(u, \psi),$$

$$(4.2) \quad \text{Val}(u, \neg \psi) = \neg \text{Val}(u, \psi),$$

$$(4.3) \quad \text{Val}(u, 1) = 1,$$

using the Łukasiewicz interpretations $[4.1]$ and $[4.2]$ of the connectors $\neg, \rightarrow, 1$ in $\mathbb{L}_n$ and by setting

$$(4.4) \quad \text{Val}(u, [C] \phi) = E(u)(C, \text{Val}(\neg, \phi)),$$

for every $C \in \mathcal{P}N$ and every $\phi, \psi \in \mathcal{L}$. We use the standard notation and terminology. We say that a formula $\phi$ is true in $\mathcal{M} = (\mathcal{F}, \text{Val})$ and write $\mathcal{M} \models \phi$ if $\text{Val}(u, \phi) = 1$ for every $u \in S$. A formula $\phi$ is valid in an $\mathbb{L}_n$-valued $\mathcal{L}$-frame $\mathcal{F}$ if it is true in every $\mathbb{L}_n$-valued $\mathcal{L}$-model based on $\mathcal{F}$.

In order to proceed further, we need to generalize the technique of filtration [7, Chapter 7.5] for neighborhood models.

**Definition 4.2.** Let $\mathcal{M} = (S, E, \text{Val})$ be an $\mathbb{L}_n$-valued $\mathcal{L}$-model and $\Gamma$ be a set of formulas closed under subformulas and the unary operators $\tau_\oplus$ and $\tau_\oplus$. Consider the equivalence relation $\equiv_\Gamma$ defined on $S$ by

$$u \equiv_\Gamma v \quad \text{if} \quad \forall \phi \in \Gamma \ \text{Val}(u, \phi) = \text{Val}(v, \phi).$$

We denote by $|S|$ the set of equivalence classes $[u]$ for $\equiv_\Gamma$. An $\mathbb{L}_n$-valued $\mathcal{L}$-model $\mathcal{M}^* = (|S|, E^*, \text{Val}^*)$ is a $\Gamma$-filtration of $\mathcal{M}$ if the following conditions are satisfied for every $u \in S$:

1. $\text{Val}^*([u], p) = \text{Val}(u, p)$ for every $p \in \text{Prop} \cap \Gamma$,
2. $E^*(u)(C, \text{Val}([\neg, \phi])) = E^*([u])(C, [\text{Val}([\neg, \phi])])$ for every $C \in \mathcal{P}N$ and $\phi \in \Gamma$,

where the map $\text{Val}([\neg, \phi]) : |S| \to \mathbb{L}_n$ is defined by $\text{Val}([u], \phi) = \text{Val}(u, \phi)$.

**Lemma 4.3.** Let $\mathcal{M} = (S, E, \text{Val})$ be an $\mathbb{L}_n$-valued $\mathcal{L}$-model and $\Gamma$ be a set of formulas closed under subformulas and the operators $\tau_\oplus$ and $\tau_\oplus$. If $\mathcal{M}^* = (|S|, E^*, \text{Val}^*)$ is a $\Gamma$-filtration of $\mathcal{M}$, then

$$(4.5) \quad \text{Val}(u, \phi) = \text{Val}^*([u], \phi)$$

for every $\phi \in \Gamma$ and every $u \in S$. 


Proof. Note that identity (4.5) is equivalent to $\text{Val}^*(\neg, \phi) = |\text{Val}(\neg, \phi)|$. The proof is a standard induction argument on the length of $\phi \in \Gamma$. We consider only the case where $\phi = [C]\psi \in \Gamma$ for $C \in \mathcal{PN}$. By the definition of $\text{Val}^*$ and the induction hypothesis, we obtain

$$\text{Val}^*((u)|C, \text{Val}^*(-, \psi)) = E^*((u))((C, \text{Val}(-, \psi))).$$

It follows from Definition 4.2(2) that

$$E^*((u)|(C, \text{Val}(-, \psi))) = E(u)(C, \text{Val}(-, \psi)) = \text{Val}(u, [C]\psi).$$

In what follows we focus on the relations between the language $\mathcal{L}$ and the $\mathcal{L}_n$-valued $\mathcal{L}$-frames in which the effectivity functions are (truly) playable.

**Definition 4.4.** An $\mathcal{L}_n$-valued $\mathcal{L}$-frame $\mathfrak{F} = (S, E)$ is said to be (truly) playable if $E(u)$ is (truly) playable for every $u \in S$. An $\mathcal{L}_n$-valued $\mathcal{L}$-model $M$ is (truly) playable if it is based on a (truly) playable $\mathcal{L}_n$-valued $\mathcal{L}$-frame.

Our first aim is to prove that, similarly as in the Boolean case [9], there is no set of $\mathcal{L}$-formulas that can define truly playable $\mathcal{L}_n$-valued $\mathcal{L}$-frames inside the class of playable $\mathcal{L}_n$-valued $\mathcal{L}$-frames. To this end, we show that for any playable $\mathcal{L}_n$-valued $\mathcal{L}$-model $M$ and any formula $\phi$, there is a finite playable $\mathcal{L}_n$-valued $\mathcal{L}$-model $M_\phi$ such that $M \models \phi$ if and only if $M_\phi \models \phi$. We use this property and the fact that finite playable $\mathcal{L}_n$-valued $\mathcal{L}$-models are truly playable to prove Proposition 4.6. The construction of $M_\phi$ is based on a refinement of filtration for playable $\mathcal{L}_n$-valued $\mathcal{L}$-models. We proceed in two steps. The next definition constitutes the first step in this direction.

**Definition 4.5.** Using the notation of Definition 4.2, an $\mathcal{L}_n$-valued $\mathcal{L}$-model $M^* = ((S), E^*, \text{Val}^*)$ is an intermediate $\Gamma$-filtration of a playable $\mathcal{L}_n$-valued $\mathcal{L}$-model $M = (S, E, \text{Val})$ if $M^*$ is a $\Gamma$-filtration of $M$ that satisfies

\begin{align}
(4.6) & \quad E^*((u)|(C, f) = \max\{E(u)(C, \text{Val}(-, \phi)) | \phi \in \Gamma \text{ and } |\text{Val}(-, \phi)| \leq f\}, \\
(4.7) & \quad E^*((u)|(N, f) = \neg E^*((u)|(\emptyset, \neg f)),
\end{align}

for every proper coalition $C \in \mathcal{PN}$ and every $f \in \mathcal{L}_n^{[S]}$.

Observe that since we have assumed playability ($\mathcal{N}$-maximality, in particular) of $M$ in Definition 4.5, an intermediate $\Gamma$-filtration of $M$ is indeed a $\Gamma$-filtration in the sense of Definition 4.2.

For every formula $\mu$ we denote by $\text{Cl}(\mu)$ the closure of the set of subformulas of $\mu$ for the operators $\neg \phi$ and $\phi \rightarrow \psi$. The next lemma shows how an intermediate $\text{Cl}(\mu)$-filtration is an intermediate step in the construction of a playable $\text{Cl}(\mu)$-filtration of a playable $\mathcal{L}_n$-valued $\mathcal{L}$-model.

**Lemma 4.6.** Let $\mu \in \mathcal{L}$ and $M = (S, E, \text{Val})$ be a playable $\mathcal{L}_n$-valued $\mathcal{L}$-model. If $M^* = ((S), E^*, \text{Val}^*)$ is an intermediate $\text{Cl}(\mu)$-filtration of $M$, then the Boolean effectivity function $E^*((u)|) : \mathcal{PN} \times \mathcal{L}_n^{[S]} \rightarrow \mathcal{L}_n$ is playable for every $u \in S$.

**Proof.** By $n, \phi$ we denote the formula $\bigcup_{i=1}^{n} \phi$. First, observe that if $u \in S$, $f \in \mathcal{L}_n^{[S]}$ and $C \neq N$ is a coalition, then

$$E^*((u)|(C, f) = \max\{E(u)^2(C, \text{Val}(-, n, \phi)) | \phi \in \Gamma \text{ and } |\text{Val}(-, n, \phi)| \leq f\},$$

which shows that $E^*((u)|)^2$ is a Boolean effectivity function. We prove that $E^*((u)|)^2$ is regular and semi-playable (see Definition 5.10). It is straightforward to show that $E^*((u)|)^2(C, \neg)$ is monotonic and satisfies liveness and safety for every coalition $C \neq N$. Moreover, $\mathcal{N}$-maximality holds for $E^*((u)|)^2$ according to (4.7).
Let us prove superadditivity for coalitions $C, D$ such that $C \cap D = \emptyset$ and $C \cup D \neq N$. If $f, g \in \mathbb{L}^{|S|}$, then $E^*(|u|)(C, f) \wedge E^*(|u|)(D, g)$ is by definition equal to the maximum of the values $E(u)(C, \text{Val}(-, \psi)) \wedge E(u)(D, \text{Val}(-, \rho))$, where $\psi$ and $\rho$ run through the elements of $\text{Cl}(\mu)$ satisfying $|\text{Val}(-, \psi)| \leq f$ and $|\text{Val}(-, \rho)| \leq g$. By superadditivity of $E$ we have

$$E(u)(C, \text{Val}(-, \psi)) \wedge E(u)(D, \text{Val}(-, \rho)) \leq E(u)(C \cup D, \text{Val}(-, \psi \wedge \rho)),$$

for every formula $\psi$ and $\rho$. Thus it follows from the definition of $E^*$ that

$$E^*(|u|)(C, f) \wedge E^*(|u|)(D, g) \leq E^*(|u|)(C \cup D, f \wedge g),$$

which is the sought result.

It remains to prove that for every $C \in \mathcal{P}N$ and every $f \in \mathbb{L}^{|S|}$ such that $E^*(|u|)(C, f) = 1$, we have $E^*(|u|)(\overline{C}, \neg f) = 0$. By condition (4.7), we may assume that $C \neq N$. Suppose for the sake of contradiction that $E^*(|u|)(\overline{C}, \neg f) = 1$. By (4.6) it means that there are some $\psi, \rho \in \text{Cl}(\mu)$ such that $|\text{Val}(-, \psi)| \leq f$ and $|\text{Val}(-, \rho)| \leq \neg f$, and $E(u)(C, \text{Val}(-, \psi)) = E(u)(\overline{C}, \text{Val}(-, \rho)) = 1$. By superadditivity of $E$, we obtain $E(N, \text{Val}(-, \psi \wedge \rho)) = 1$ with $\psi \wedge \rho \in \text{Cl}(\mu)$ satisfying $|\text{Val}(-, \psi \wedge \rho)| \leq f \wedge \neg f$. By (4.6), Definition (4.2), $N$-maximality of $E$ and the fact that $f \in \mathbb{L}^{|S|}_1$, we obtain $E^*(|u|)(N, 0) = 1$. This is a contradiction since $E^*(|u|)(N, 0) = \neg E^*(|u|)(\emptyset, 1) = 0$ by (4.7) and liveness of $E^*$ for the empty coalition.

We combine Lemma 4.6 together with Lemma 3.5 to construct $\text{Cl}(\mu)$-filtrations that preserve playability.

**Proposition 4.7.** If $\mu \in \mathcal{L}$ and $M = (S, E, \text{Val})$ is a playable $\mathbb{L}_n$-valued $\mathcal{L}$-model, then there is a playable $\text{Cl}(\mu)$-filtration $M^* = (|S|, E^*, \text{Val}^*)$ of $M$.

**Proof.** Consider the intermediate $\text{Cl}(\mu)$-filtration $M^* = (|S|, E^*, \text{Val}^*)$ of $M$. By Lemma 4.6, the Boolean effectiveness function $E^*(|u|) : \mathcal{P}N \times \mathbb{L}^{|S|}_1 \rightarrow \mathbb{L}_1$ is playable for every $u \in S$. By Lemma 3.5, the map $E^*(|u|) : \mathcal{P}N \times \mathbb{L}^{|S|}_n \rightarrow \mathbb{L}_n$ defined by

$$E^*(|u|)(C, f) = \max \{ \frac{n}{i} \in \mathbb{L}_n \mid E^*(|u|)^i(C, \tau_{\mu}(f)) = 1 \}$$

is also playable. We prove that $M^* := (|S|, E^*, \text{Val}^*)$ is a $\text{Cl}(\mu)$-filtration of $M$. It suffices to check that $M^*$ satisfies condition (5) of Definition 4.2. Let $u \in S$, $C \in \mathcal{P}N$, $\phi \in \text{Cl}(\mu)$ and $i \in \{1, \ldots, n\}$. We obtain by definition of $E^*$ that $E^*(|u|)(C, |\text{Val}(-, \phi)|) \geq i/n$ if and only if $E^*(|u|)^i(C, |\text{Val}(-, \tau_{\mu}(\phi))|) = 1$. Since $M^*$ is an intermediate $\text{Cl}(\mu)$-filtration of $M$, the last identity is in turn equivalent to $E(u)(C, |\text{Val}(-, \phi)|) = 1$. Finally, this gives $E(u)(C, |\text{Val}(-, \phi)|) \geq i/n$.

The next result shows that the gain of expressive power induced by the many-valued nature of $\mathcal{L}$ and of its associated semantics is not enough to single out those playable $\mathcal{L}$-models that are truly playable.

**Proposition 4.8.** There is no set $\Lambda$ of $\mathcal{L}$-formulas such that a playable $\mathbb{L}_n$-valued $\mathcal{L}$-frame $\mathfrak{F}$ is truly playable if and only if every formula of $\Lambda$ is valid in $\mathfrak{F}$.

**Proof.** Assume that there exists a set $\Lambda$ of $\mathcal{L}$-formulas such that a playable $\mathbb{L}_n$-valued $\mathcal{L}$-frame $\mathfrak{F}$ is truly playable if and only if every formula of $\Lambda$ is valid in $\mathfrak{F}$.

Let $\mathfrak{F}$ be a playable $\mathbb{L}_n$-valued $\mathcal{L}$-frame which is not truly playable. The existence of such $\mathfrak{F}$ is a consequence of Lemma 3.5 applied to the effectiveness function $E$ defined in 4. Proposition 4. For every $\phi \in \Lambda$ and every model $M$ based on $\mathfrak{F}$, Proposition 4.7 provides a playable $\text{Cl}(\phi)$-filtration $M^*$. Since $M^*$ has a finite set of outcomes, it is truly playable. It follows from the definition of $\Lambda$ that $M^* \models \phi$ and from Lemma 4.3 that $M \models \phi$. We have proved that every formula of $\Lambda$ is true in every model based on $\mathfrak{F}$ and we conclude that $\mathfrak{F}$ is truly playable, which is the desired contradiction.
4.2. Ł_n-valued playable logic for finite playable Ł_n-valued Ł-frames. Proposition 4.5 says that Ł is not adequate for capturing the properties of Ł_n-valued effectivity functions associated with game forms. Indeed, this language is not even expressive enough to distinguish between the playable and the truly playable Ł_n-valued Ł-frames. Nevertheless, when the set of outcome states S is finite, every playable Ł_n-valued effectivity function is truly playable and it turns out that playability can be encoded by Ł-formulas; see our completeness result, Theorem 4.17. We start with axiomatizing the properties of playable Ł_n-valued effectivity functions.

Definition 4.9. An Ł_n-valued playable logic is a subset L of Ł which is closed under Modus Ponens, Uniform Substitution and Monotonicity (if φ → ψ ∈ L, then [C]φ → [C]ψ ∈ L for every C ∈ PN) and that contains an axiomatic base of Łukasiewicz (n + 1)-valued logic (see [10] or [8, Section 8.5]) together with the following axioms:

The axioms of Ł_n-valued playable logic

\begin{align*}
(1) \quad [C](p \circ p) & \leftrightarrow [C]p \circ [C]p, \\
(2) \quad [C](p \oplus p) & \leftrightarrow [C]p \oplus [C]p, \\
(3) \quad \neg[C]0, \\
(4) \quad ([C]p \land [C']q) & \rightarrow [C \cup C'](p \land q), \\
(5) \quad [\varnothing]p & \rightarrow \neg[N]\neg p,
\end{align*}

for every C, C' ∈ PN such that C \cap C' = \emptyset.

We denote by P_n the smallest Ł_n-valued playable logic, that is, the intersection of all the Ł_n-valued playable logics. We conform with common usage and we often write \(\vdash_{P_n} \phi\) instead of \(\phi \in P_n\).

The axioms (1)-(5) together with the Monotonicity rule reflect the properties defining playability. In Remark 4.13 at the end of this section, we give equivalent and more intuitive axioms that can replace (1)-(2) in the axiomatization of P_n.

The following lemma can be proved by a standard induction argument.

Lemma 4.10. Let M be a playable Ł_n-valued Ł-model. If \(\vdash_{P_n} \phi\), then \(M \models \phi\).

We will prove completeness of P_n with respect to the class of playable Ł_n-valued Ł-models. Our proof is based on the construction of the canonical model.

4.2.1. Construction of the canonical model. Let us denote by 3P_n the Lindenbaum-Tarski algebra of P_n, that is, the quotient of Ł under the syntactic equivalence relation \(\equiv\) defined by

\[\phi \equiv \psi \quad \text{if} \quad \vdash_{P_n} \phi \leftrightarrow \psi,\]

equipped with the operations 1, \(\neg\), \(\rightarrow\) and [C] defined as 1 := 1/ \equiv, \(\neg(\phi/ \equiv) := \neg\phi/ \equiv, \phi/ \equiv \rightarrow \psi/ \equiv := (\phi \rightarrow \psi)/ \equiv\) and [C](\phi/ \equiv) := [C]φ/ \equiv, for every C ∈ PN and every φ, ψ ∈ Ł (these definitions are meaningful because P_n is closed under Equivalence). By abuse of notation, we denote the class φ/ \equiv by φ.

Since P_n contains every tautology of Łukasiewicz n+1-valued logic, the \(\{\rightarrow, \neg, 1\}\)-reduct of 3P_n is an MV-algebra that belongs to the variety \(\mathcal{MV}_n\) generated by Ł_n.

In the Boolean setting, one of the key ingredients of the construction of the canonical model is the ultrafilter theorem that allows us to separate by an ultrafilter any two different non-top elements of a Boolean algebra B. We can rephrase this separation result using the bijective correspondence between the ultrafilters of B and the homomorphisms of B into the two-element Boolean algebra 2: for every \(a \neq b \in B \setminus \{1\}\), there is a homomorphism \(u: B \rightarrow 2\) such that \(u(a) = 1\) and \(u(b) = 0\). The variety \(\mathcal{MV}_n\) has an analogous property [8].
Lemma 4.11. Let $\mathbf{A} \in \mathcal{MV}_n$. For every $a \neq b$ in $\mathbf{A} \setminus \{1\}$, there is a $[\neg, \rightarrow, 1]$-homomorphism $u: \mathbf{A} \rightarrow \mathbb{L}_n$ such that $u(a) = 1$ and $u(b) 
eq 1$.

This separation property explains our choice of the set $W^c := \mathcal{MV}(\mathfrak{A}_{P_n}, \mathbb{L}_n)$ of $[\neg, \rightarrow, 1]$-homomorphisms from $\mathfrak{A}_{P_n}$ to $\mathbb{L}_n$ as the universe of the canonical model of $\mathbf{P}_n$.

We will use the following technique to associate an $\mathbb{L}$-valued effectivity function $E^c(u)$ with every $u \in W^c$. For each $i \in \{1, \ldots, n\}$ we will define a subset $P_i$ of $\mathcal{P}N \times W^c \times \mathbb{L}_n^{W^c}$, such that $P_1 \supseteq \cdots \supseteq P_n$. Then we will safely set $E^c(u)(C, f) := \max \left\{ \frac{i}{n} \in \mathbb{L}_n \mid (C, u, f) \in P_i \right\}$.

Definition 4.12. For every $i \in \{1, \ldots, n\}$ let $P_i$ be the subset of $\mathcal{P}N \times W^c \times \mathbb{L}_n^{W^c}$ defined by

\begin{equation}
P_i = \left\{ (C, u, f) \mid \exists \phi(u([C] \phi) \geq \frac{i}{n} \text{ and } \forall v \in W^c(v(\phi) \geq \frac{i}{n} \implies f(v) \geq \frac{i}{n}) \right\}.
\end{equation}

We use the convention $P_0 = \mathcal{P}N \times W^c \times \mathbb{L}_n^{W^c}$.

Lemma 4.13. The inclusion $P_i \subseteq P_{i-1}$ holds for each $i \in \{1, \ldots, n\}$.

Proof. Assume that $\phi$ satisfies the condition defining $P_i$ in (4.8) for $C$, $f$, $u$ and $i > 0$ and put $\rho = \tau_{f, u}(\phi)$. Then $u([C] \rho) = \tau_{f, u}(u([C] \phi)) = 1 \geq \frac{i - 1}{n}$. Moreover, if $v(\rho) \geq \frac{i - 1}{n}$, then $v(\rho) = 1$ since $\rho$ belongs to the Boolean skeleton of $\mathfrak{A}_{P_n}$. Therefore $v(\phi) \geq \frac{i}{n}$, which gives $f(v) \geq \frac{i}{n} \geq \frac{i - 1}{n}$. \hfill $\square$

Definition 4.14. The canonical model of $\mathbf{P}_n$ is the $\mathbb{L}_n$-valued $\mathcal{L}$-model $\mathcal{M} = (\mathfrak{A}, \mathcal{L}, \mathcal{V})$ with $\mathfrak{A} = (W^c, E^c)$ where $E^c(u)(C, f)$ is defined for every $u \in W^c$, every $C \in \mathcal{P}N$ and every $f \in \mathbb{L}_n^{W^c}$ by

\begin{equation}
E^c(u)(C, f) = \left\{ \begin{array}{ll}
\max \left\{ \frac{i}{n} \in \mathbb{L}_n \mid (C, u, f) \in P_i \right\} & \text{if } C \neq N, \\
\neg E^c(u)(\varnothing, \neg f) & \text{if } C = N,
\end{array} \right.
\end{equation}

and where $\mathcal{V}$ is defined by

\begin{equation}
\mathcal{V}(u, p) = u(p),
\end{equation}

for every $p \in \mathcal{P}rop$ and $u \in W^c$.

In particular, for every $(C, u, f) \in \mathcal{P}N \times W^c \times \mathbb{L}_n^{W^c}$ with $C \neq N$, it holds

\begin{equation}
E^c(u)(C, f) \geq \frac{i}{n} \quad \text{if and only if} \quad (C, u, f) \in P_i.
\end{equation}

The next proposition shows that the identity (4.10) remains true in the canonical model after replacing $p$ by any formula $\phi \in \mathcal{L}$.

Proposition 4.15 (Truth Lemma). The canonical model $(W^c, E^c, \mathcal{V})$ of $\mathbf{P}_n$ satisfies $\mathcal{V}(u, \mu) = u(\mu)$ for every $\mu \in \mathcal{L}$ and every $u \in W^c$.

Proof. We proceed by induction on the number of connectors in $\mu$. If $\mu \in \mathcal{P}rop$ or $\mu \in \{1, \neg \psi, \psi \rightarrow \rho\}$, then the result follows immediately from (4.11) and (4.3).

Let $\mu = [C] \psi$ for some $\psi \in \mathcal{L}$ and $C \in \mathcal{P}N \setminus \{N\}$. We will prove that for any $u \in W^c$ and $i \leq n$,

\begin{equation}
E^c(u)(C, \mathcal{V}(\neg, \psi)) \geq \frac{i}{n} \quad \text{if and only if} \quad u([C] \psi) \geq \frac{i}{n}.
\end{equation}

First, assume $E^c(u)(C, \mathcal{V}(\neg, \psi)) \geq \frac{i}{n}$. Then, by (4.11) and (4.3), there is $\rho \in \mathcal{L}$ such that $u([C] \rho) \geq \frac{1}{n}$ and $\mathcal{V}(v, \psi) \geq \frac{1}{n}$ for any $v \in W^c$ satisfying $v(\rho) \geq \frac{1}{n}$. By induction hypothesis, it means that for every $v \in W^c$ with $v(\rho) \geq \frac{1}{n}$, we have
\(v(\psi) \geq \frac{1}{n}\). It follows that \(v(\tau_{i/n}(\rho) \rightarrow \tau_{i/n}(\psi)) = 1\) for every \(\psi \in W^c\). This yields 
\[\vdash_{P_n} \tau_{i/n}(\rho) \rightarrow \tau_{i/n}(\psi)\] 
since the \(\{\rightarrow, \neg, 1\}\)-reduct of \(\mathcal{F}_{P_n}\) has the separation property (Lemma 4.11). As \(P_n\) is closed under Monotonicity, we obtain 
\[\vdash_{P_n} [\mathcal{C}]\tau_{i/n}(\rho) \rightarrow [\mathcal{C}]\tau_{i/n}(\psi)\].

By axioms 1 and 2 of \(P_n\) (Definition 4.9) and Uniform Substitution, this is equivalent to 
\[\vdash_{P_n} \tau_{i/n}([\mathcal{C}]\rho) \rightarrow \tau_{i/n}([\mathcal{C}]\psi)\].

Hence, if \(\psi \in W^c\) and \(v([\mathcal{C}]\rho) \geq \frac{1}{n}\), then \(v([\mathcal{C}]\psi) \geq \frac{1}{n}\). We can thus conclude that 
\(u([\mathcal{C}]\psi) \geq \frac{1}{n}\).

Conversely, let \(u([\mathcal{C}]\psi) \geq \frac{1}{n}\). We obtain \((\mathcal{C}, u, \text{Val}(-, \psi)) \in P_i\) by induction hypothesis and by considering \(\phi = \psi\) in the definition 4.11 of \(P_i\). This proves 4.12.

Finally, assume \(\mu = [N]\psi\) for some \(\psi \in \mathcal{L}\). We will prove 
\[E^c(u)(N, \text{Val}(-, \psi)) = u([N]\psi)\].

Indeed, on the one hand we obtain 
\[E^c(u)(N, \text{Val}(-, \psi)) = \neg E^c(u)(\emptyset, \text{Val}(-, \neg \psi))\]
by 4.9, which is in turn equal to \(u(\neg [\emptyset] \neg \psi)\) by induction hypothesis and the first part of this proof. Axiom 4.10 of \(P_n\) yields 
\[E^c(u)(N, \text{Val}(-, \psi)) \leq u([N]\psi)\].

To prove the converse inequality, let us assume for the sake of contradiction that there is \(i \in \{1, \ldots, n\}\) such that 
\[E^c(u)(N, \text{Val}(-, \psi)) < \frac{1}{n} = u([N]\psi)\]. Therefore 
\(u([N]\tau_{i/n}(\psi)) = 1\), while 
\[E^c(u)(\emptyset, \text{Val}(-, \neg \tau_{i/n}(\psi))) = 1\] by 4.9. It follows from this identity that \(u([\emptyset] \neg \tau_{i/n}(\psi)) = 1\) by induction hypothesis and the first part of the proof. From axiom 4.10 applied with \(C = N\) and \(C' = \emptyset\) we get 
\[u([N]\tau_{i/n}(\psi) \land \neg \tau_{i/n}(\psi)) = 1\]; however, this is in contradiction with axiom 4.10 since \(\tau_{i/n}(\psi) \land \neg \tau_{i/n}(\psi) = 0\).

4.2.2. Completeness result for \(P_n\). In order to use the canonical model for the proof of completeness of \(P_n\) with respect to the class of the playable \(L_n\)-valued \(\mathcal{L}\)-models, we need the following result.

**Lemma 4.16.** The canonical model of \(P_n\) is a playable \(L_n\)-valued \(\mathcal{L}\)-model.

**Proof.** Let \(u \in W^c\). It suffices to prove that \(E^c(u)\) is semi-playable, homogeneous, \(N\)-maximal and regular. It is easily checked that \(E^c(u)(C, \neg -)\) is monotonic for every coalition \(C \neq N\). The property of \(N\)-maximality is obtained by 4.10.

For homogeneity, let \(C \neq N \in PN\) and \(f \in L_n^c\). We will prove that for any 
\[i \in \{1, \ldots, n\},\]
\[(4.13) \quad E^c(u)(C, f \oplus f) \geq \frac{i}{n} \quad \text{if and only if} \quad E^c(u)(C, f) \oplus E^c(u)(C, f) \geq \frac{i}{n}.

First, assume \(E^c(u)(C, f) \oplus E^c(u)(C, f) \geq \frac{i}{n}\) or, equivalently, \(E^c(u)(C, f) \geq \frac{i}{2n}\). By the definition of \(E^c\), there is a formula \(\rho\) such that \(u([C]\rho) \geq \frac{1}{2n}\) and \(f(\rho) \geq \frac{i}{2n}\) for every \(v\) satisfying \(v(\rho) \geq \frac{1}{2n}\). On the one hand, by axioms 1 and 2 of \(P_n\) and considering \(\phi = \tau_{i/2n}(\rho)\), we get 
\[E^c(u)(\emptyset, \text{Val}(-, \tau_{i/2n}(\rho))) = 1 \geq \frac{i}{n}\]. On the other hand, if \(v \in W^c\) is such that \(v(\phi) \geq \frac{1}{n}\), then \(v(\phi) = 1\) since \(\phi\) is an idempotent element of \(\mathcal{F}_{P_n}\). Therefore \(v(\rho) \geq \frac{1}{2n}\). This implies \(f(\rho) \geq \frac{1}{2n}\) or, equivalently, \((f \oplus f)(\rho) \geq \frac{i}{2n}\). We conclude that \(E^c(u)(C, f \oplus f) \geq \frac{i}{n}\).

Conversely, assume \(E^c(u)(C, f) \geq \frac{i}{n}\) for some \(i > 0\). The definition of \(E^c\) yields a formula \(\rho\) such that \(u([C]\rho) \geq \frac{1}{n}\) and \(f(\rho) \geq \frac{i}{2n}\) for any \(v \in W^c\) with \(v(\rho) \geq \frac{1}{n}\). By considering \(\phi = \tau_{i/n}(\rho)\), we obtain on the one hand that 
\[u([C]\phi) = \tau_{i/n}(u([C]\rho)) = 1 \geq \frac{i}{2n}\]. On the other hand, if \(v \in W^c\) is such that
Definition 4.9) by a family of axioms, which are easier to understand. Indeed, put (2).

Remark (Completeness of $A_m$)

We can use the formulas $\varphi$ for every $v \in W^\phi$. We deduce that $\vdash_{P_n} \varphi \rightarrow 0$ and hence $\vdash_{P_n} [\varphi] \rightarrow [0]$ by Monotonicity. It follows that $\frac{1}{n} \leq u([\varphi]) \leq u([0]) = 0$, a contradiction.

To prove that $E^u(\varphi)$ has liveliness property for every $C \neq N$, it suffices to consider $\varphi = 1$ in (1.8) in order to show $(C, u, 1) \in P_1$.

We have to prove coalition monotonicity for $C_1$ and $C_2$ such that $C_1 \cup C_2 \neq C$. It is enough to prove the following: if $E^u(\varphi)(C_1, f_1) \wedge E^u(\varphi)(C_2, f_2) \geq \frac{i}{n}$ for some $i \in \{1, \ldots, n\}$, then $E^u(\varphi)(C_1 \cup C_2, f_1 \wedge f_2) \geq \frac{i}{n}$. Let $\ell \in \{1, 2\}$ and denote by $\phi_\ell$ a formula such that $u([\phi_\ell]) \geq \frac{i}{n}$ and $f_\ell(v) \geq \frac{i}{n}$ for every $v$ satisfying $\varphi(\ell) \geq \frac{i}{n}$. Thus we can consider $\varphi = \phi_1 \wedge \phi_2$ in (1.8) to get $(C_1 \cup C_2, u, f_1 \wedge f_2) \in P_1$.

It remains to check that $E^u(\varphi)$ is regular. By (1.9), it suffices to prove that it is $\mathcal{C}$-regular for every $C \neq N$. For the sake of contradiction, assume that there exists $i \in \{1, \ldots, n\}$ such that $\neg E^u(\varphi)(\mathcal{C}, \neg f) < \frac{i}{n} \leq E^u(\varphi)(C, f)$. It follows that $E^u(\varphi)(C, \tau_{/i}(f)) = 1$, while $E^u(\varphi)(\mathcal{C}, \tau_{/i}(\neg f)) = 1$ for $\frac{i}{n} = 1 - \frac{i}{n}$. By superadditivity, we obtain $E^u(\varphi)(N, \tau_{/i}(f) \wedge \tau_{/i}(\neg f)) = 1$, which is a contradiction since $\tau_{/i}(f) \wedge \tau_{/i}(\neg f)$ is the constant map $0$.

\section*{Theorem 4.17 (Completeness of $P_n$)}

For any $\varphi \in \mathcal{C}$, the following assertions are equivalent:

1. $\vdash_{P_n} \varphi$.
2. $\varphi$ is true in every playable $L_n$-valued $\mathcal{C}$-model.
3. $\varphi$ is true in every finite playable $L_n$-valued $\mathcal{C}$-model.
4. $\varphi$ is true in every truly playable $L_n$-valued $\mathcal{C}$-model.

\section*{Proof}

The implication $[\varphi] \implies [\varphi]$ is the content of Lemma 4.16. The equivalences $[\varphi] \iff [\varphi]$ and $[\varphi] \iff [\varphi]$ follow from Proposition 4.17 and Lemma 4.13. Finally, it remains to argue for the implication $[\varphi] \implies [\varphi]$. We know by Lemma 4.16 that $\mathcal{M}^c$ is a playable $L_n$-valued model. According to Proposition 4.15, $\mathcal{M}^c \models \varphi$ means that the class of $\varphi$ is equal to $1$ in $\exists_{P_n}$, or, equivalently, $\vdash_{P_n} \varphi$.

\section*{Remark 4.18}

We can use the formulas $\tau_{/i}(p)$ to replace axioms (1) and (2) of $P_n$ (Definition 4.14) by a family of axioms, which are easier to understand. Indeed, put

\begin{align*}
A & = \{ [\varphi] \cdot [p] \leftrightarrow [\varphi] \cdot [p] \cdot [\varphi] \mid \forall \in \{ \otimes, \oplus \}, C \in \mathcal{P}N \}, \\
B & = \{ [\varphi]_{/i}(p) \leftrightarrow \tau_{/i}(p) \mid i \in \{1, \ldots, n\}, C \in \mathcal{P}N \}.
\end{align*}

It follows from the definition of an $L_n$-valued playable logic that $B \subseteq P_n$. A careful analysis of the proofs of Lemma 4.13, Proposition 4.15, Lemma 4.16 and Theorem 4.17 shows that we have only used the axioms in $A$ in the form of substitutions in formulas of $B$. Denote by $P_n$ the smallest set of formulas that contains an axiomatic base of Łukasiewicz logic, the set $B$, and that is closed under Modus Ponens, Uniform Substitution and Monotonicity. It follows from the previous observation.
that for any $\phi \in \mathcal{L}$ we have $\vdash_{P_n} \phi$ if and only if $M \models \phi$ for every playable $L_n$-valued $\mathcal{L}$-model. It results that $P_n^* = P_n$.

Thus the set of axioms $A$ can be equivalently replaced by $B$. Informally speaking, the content of axioms $[1] - [2]$ of $P_n$ is rephrased below.

For any $i \leq n$, the following two assertions are equivalent:

- The truth value of the statement ‘coalition $C$ can enforce an outcome state in which $\phi$ holds’ is at least $\frac{1}{n}$.
- Coalition $C$ can enforce an outcome state in which the truth value of $\phi$ is at least $\frac{1}{n}$.

4.3. $L_n$-valued truly playable logics for truly playable $L_n$-valued $\mathcal{L}^+$-frames.

Theorem 4.17 says that $P_n$ is the logic of playable rather than truly playable effectivity functions. Moreover, by Proposition 4.8 there is no axiomatization of truly playable effectivity functions in the language $\mathcal{L}$. Thus the presented many-valued approach is a faithful generalization of the Boolean framework; see [9]. In fact the authors of [9] go beyond this limitation of the Boolean setting by adding a new connective to $\mathcal{L}$ and by enriching the neighborhood semantics with a Kripke relation. We follow this idea by designing the modal equivalent of truly playable $L_n$-valued effectivity functions.

Let $\mathcal{L}^+$ be the language $\mathcal{L} \cup \{[O]\}$ where $[O]$ is unary. The set $\mathcal{L}^+$ of formulas is defined inductively from the countably infinite set $\text{Prop}$ of propositional variables by the following rules:

\[
\phi ::= 1 \mid p \mid \phi \rightarrow \phi \mid \neg \phi \mid [C]\phi \mid [O]\phi
\]

where $p \in \text{Prop}$ and $C \in \mathcal{P}N$.

In order to interpret $\mathcal{L}^+$-formulas, we enrich the $L_n$-valued $\mathcal{L}$-frames with a binary relation.

**Definition 4.19.** A tuple $\mathfrak{F} = (S, E, R)$ is an $L_n$-valued $\mathcal{L}^+$-frame if $(S, E)$ is an $L_n$-valued $\mathcal{L}$-frame and $R \subseteq S \times S$. We say that $\mathfrak{F} = (S, E, R)$ is standard if $R = \{(u, v) \mid E(u)(\emptyset, \neg \chi(v)) = 0\}$.

A tuple $M = (S, E, R, \text{Val})$ is an $L_n$-valued $\mathcal{L}^+$-model (based on $(S, E, R)$) if $(S, E, R)$ is an $L_n$-valued $\mathcal{L}$-frame and $\text{Val} : S \times \text{Prop} \rightarrow L_n$. An $L_n$-valued $\mathcal{L}^+$-frame $\mathfrak{F} = (S, E, R)$ or an $L_n$-valued $\mathcal{L}^+$-model $M = (S, E, R, \text{Val})$ is called playable (truly playable, respectively) if $(S, E)$ is a playable (truly playable, respectively) $L_n$-valued $\mathcal{L}$-frame.

In an $L_n$-valued $\mathcal{L}^+$-model, the valuation map $\text{Val}$ is extended inductively to $S \times \mathcal{L}^+$ by using rules [1.1] - [1.3] for the connectors $\top, \neg$ and $\rightarrow$, by using rule [1.4] for the connectors $[C]$, where $C \in \mathcal{P}N$, and by putting

\[
\text{Val}(u, [O]\phi) = \min\{\text{Val}(v, \phi) \mid (u, v) \in R\}
\]

for any $\phi \in \mathcal{L}^+$ and $u \in S$.

It turns out that the class of standard truly playable $L_n$-valued $\mathcal{L}^+$-frames can be defined inside the class of standard playable $L_n$-valued $\mathcal{L}^+$-frames by an $\mathcal{L}^+$-formula. The next assertion is the $L_n$-valued generalization of [9] Proposition 14.

**Proposition 4.20.** A standard playable $L_n$-valued $\mathcal{L}^+$-frame $\mathfrak{F}$ is truly playable if and only if $[O]\phi \leftrightarrow [O]\phi$ is valid in $\mathfrak{F}$.

**Proof.** First assume that $\mathfrak{F}$ is truly playable and $\mathfrak{F}^\dagger$ is standard. It follows from Lemma 4.4 that $\mathfrak{F}^\dagger : = (S, E^\dagger, R)$ is a standard truly playable $L_1$-valued $\mathcal{L}^+$-frame. By [9] Proposition 14, for every $\phi \in \mathcal{L}^+$ the formula $[O]\phi \leftrightarrow [O]\phi$ is valid in $\mathfrak{F}^\dagger$ and we must prove that $[O]\phi \leftrightarrow [O]\phi$ is also valid in $\mathfrak{F}$.

Assume that there is $\phi \in \mathcal{L}^+$, and a model $M = (S, E, R, \text{Val})$ based on $\mathfrak{F}$ such that $\text{Val}(u, [O]\phi) < \frac{1}{n}$ for some $i \leq n$. It follows from homogeneity
of $E(u)$ that $Val(u, [\emptyset] \tau_{\mathcal{L}}(\phi)) = 0$, while $Val(u, [\mathcal{O}] \tau_{\mathcal{L}}(\phi)) = 1$. Moreover, the map $Val(-, \tau_{\mathcal{L}}(\phi))$ ranges in $\mathcal{L}_1$. Hence, any map $Val': S \times \text{Prop} \rightarrow \mathcal{L}_n$ with $Val'(w, p) = Val(w, \tau_{\mathcal{L}}(\phi))$ for every $w \in S$ defines an $\mathcal{L}_1$-valued model based on $\mathfrak{A}$ that falsifies $[\emptyset]p \leftrightarrow [\mathcal{O}]p$, which is the sought contradiction. We can derive a similar contradiction in case there is some $\phi \in \mathcal{L}_+^+$ and a model $\mathcal{M} = (S, E, R, Val)$ based on $\mathfrak{A}$ such that $Val(u, [\emptyset] \phi) \geq \frac{1}{n} > Val(u, [\mathcal{O}] \phi)$ for some $i \in \{1, \ldots, n\}$.

Conversely, let $\mathfrak{A} = (S, E, R)$ be a standard playable $\mathcal{L}_n$-valued $\mathcal{L}_+^+$-frame in which $[\emptyset] \phi \leftrightarrow [\mathcal{O}] \phi$ is valid for every $\phi \in \mathcal{L}_+^+$. Then $\mathfrak{A} = (S, E^+, R^+, Val^+)$ is a $\Gamma$-filtration of $\mathcal{M}$ if it satisfies (1) and (2) of Definition 4.21, and the following conditions:

(3) if $(u, v) \in R$, then $\langle |u|, |v| \rangle \in R^*$,
(4) if $\langle |u|, |v| \rangle \in R^*$ and $u(\phi) = 1$ for every $\phi \in \Gamma$, then $v(\phi) = 1$.

Lemma 4.22. Let $\mathcal{M} = (S, E, R, Val)$ be an $\mathcal{L}_n$-valued $\mathcal{L}_+^+$-model and $\Gamma$ be a set of formulas closed under subformulas and the unary operators $\tau_{\mathcal{L}}$ and $\tau_{\mathcal{R}}$. With the notation introduced in Definition 4.21, an $\mathcal{L}_n$-valued $\mathcal{L}_+^+$-model $\mathcal{M}^* = (|S|, E^+, R^+, Val^*)$ is a $\Gamma$-filtration of $\mathcal{M}$ if it satisfies (1) and (2) of Definition 4.21 and the following conditions:

$$Val(u, \phi) = Val^*(|u|, \phi)$$

for every $\phi \in \Gamma$.

Proof. The proof is a routine induction argument on the length of $\phi \in \Gamma$. Considering the proof of Lemma 4.21, the only case we have to discuss is $\phi = [\mathcal{O}]\psi \in \Gamma$. First, we note that by our assumption on $\mathcal{M}$ and condition (4) of Definition 4.21, we have $Val(u, [\mathcal{O}]\psi) \leq Val(v, \psi)$ for every $\langle |u|, |v| \rangle \in R^*$.

Let $u \in S$. Then the definition of $Val^*$ and the induction hypothesis yield

$$Val^*(|u|, [\mathcal{O}]\psi) = \min \{Val(v, \psi) \mid \langle |u|, |v| \rangle \in R^*\}.$$ 

The inequality $Val^*(|u|, [\mathcal{O}]\psi) \leq Val(u, [\mathcal{O}]\psi)$ holds true since $\langle |u|, |v| \rangle \in R^*$ for every $\langle u, v \rangle \in R$. The other inequality is obtained by (1.16) and (1.17).

Definition 4.23. An $\mathcal{L}_n$-valued truly playable logic is a subset $\mathcal{L}$ of $\mathcal{L}_+$ that is closed under Modus Ponens, Uniform Substitution and Monotonicity for every $\mathcal{C}$, where $C \in \mathcal{P}_N$, and such that $\mathcal{L}$ contains an axiomatic base of Łukasiewicz $(n+1)$-valued logic together with axioms (4)–(5) of logic $\mathcal{P}_n$ and the following axioms:

$$\begin{align*}
(6) & [\mathcal{O}]1 \\
(7) & [\mathcal{O}]p \leftrightarrow [\emptyset]p. \\
(8) & [\emptyset](p \rightarrow q) \rightarrow ([\emptyset]p \rightarrow [\emptyset]q).
\end{align*}$$

We denote by $\mathcal{TP}_n$ the smallest $\mathcal{L}_n$-valued truly playable logic.
Contrary to the Boolean case, it is not known if axiom $\mathbf{S}$ can be removed from the axiomatization of $\text{TP}_n$ without changing $\text{TP}_n$. Nevertheless, the following result holds true.

**Lemma 4.24.** $\text{TP}_n$ is closed under the necessitation rule for $[\emptyset]$.

**Proof.** Let $\vdash_{\text{TP}_n} \phi$. Then $\vdash_{\text{TP}_n} 1 \to \phi$ and $\vdash_{\text{TP}_n} [\emptyset]1 \to [\emptyset]\phi$ by Monotonicity, which is equivalent to $\vdash_{\text{TP}_n} [\emptyset]\phi$ by axiom $\mathbf{M}$. The conclusion follows from axiom $\mathbf{E}$ and Modus Ponens. $\square$

We prove completeness of $\text{TP}_n$ with respect to the standard truly playable $\mathcal{L}_n$-valued $\mathcal{L}^+$-models by the technique of the canonical model. We denote by $\mathfrak{T}_{\text{TP}_n}$ the Lindenbaum-Tarski algebra of $\text{TP}_n$.

**Definition 4.25.** The canonical model of $\text{TP}_n$ is the $\mathcal{L}_n$-valued $\mathcal{L}^+$-model $\mathcal{M} = (\mathfrak{S}, \text{Val}^\mathfrak{S})$ with $\mathfrak{S} = (W^c, E^c, R^c)$, where $W^c = \mathcal{M}V(\mathfrak{T}_{\text{TP}_n}, \mathcal{L}_n)$, $E^c$ and Val are as in Definition 4.14 and $R^c$ is defined by

$$R^c = \{ (u, v) \mid \forall \phi \in \text{Cl}(\mu) \ u([\emptyset]\phi) = 1 \implies v(\phi) = 1 \}.$$ 

**Proposition 4.26** (Truth Lemma). The canonical model $\mathcal{M} = (\mathfrak{S}, \text{Val}^\mathfrak{S})$ of $\text{TP}_n$ satisfies $\text{Val}^\mathfrak{S}(u, \mu) = u(\mu)$ for every $\mu \in \mathcal{L}_+^+$ and every $u \in W^c$.

**Proof.** The proof is carried out by induction on the number of connectives in $\mu$. The only case not considered in the proof of Proposition 4.15 is $\mu = [\emptyset]\phi$. However, this case was considered in the proof of [12, Proposition 5.6] or in the proof of [11, Proposition 5.5]. $\square$

It is worth noticing that Proposition 4.26 relies on the fact that the modality $[\emptyset]$ is normal.

**Proposition 4.27.** Let $\mu \in \mathcal{L}_+^+$ and let $\mathcal{M}^* = ([W^c], E^*, \text{Val}^*)$ be an intermediate $\text{Cl}(\mu)$-filtration of $(W^c, E^c, \text{Val}^\mathfrak{S})$. If

$$R^* = \{ ([u], [v]) \mid \forall \phi \in \text{Cl}(\mu) \ u([\emptyset]\phi) = 1 \implies v(\phi) = 1 \},$$

then the model $\mathcal{M}^* = ([W^c], E^*, R^*, \text{Val}^*)$ is a standard truly playable $\text{Cl}(\mu)$-filtration of the canonical model $\mathcal{M} = (W^c, E^c, R^c, \text{Val}^\mathfrak{S})$ of $\text{TP}_n$, where $E^*$ is obtained from $E^c$ as in Proposition 4.7.

**Proof.** By Proposition 4.7 we know that $\mathcal{M}$ is playable. As $\mathcal{M}$ is finite, it is truly playable. By Proposition 4.17 and by definition of $R^*$, the model $\mathcal{M}^*$ is a $\text{Cl}(\mu)$-filtration of $\mathcal{M}$ in the sense of Definition 4.24.

It remains to prove that $\mathcal{M}^*$ is standard. First, assume that $u, v \in W^c$ and $E^*([u])(\mathcal{O}, \neg \chi_{\{[v]\}}) = 0$. It follows from the definition of $E^*$ that

$$E(u)(\mathcal{O}, \text{Val}(\mathcal{O}, \phi)) = 0$$

for every $\phi \in \text{Cl}(\mu)$ such that $\text{Val}(v, \phi) = 0$. We conclude that $([u], [v]) \in R^*$ by the definition of $R^*$.

Let $E([u], [v]) \in R^*$. We will prove that $E^*([u])(\mathcal{O}, \neg \chi_{\{[v]\}}) = 0$. By way of contradiction, assume that there is $\phi \in \text{Cl}(\mu)$ such that $\text{Val}(v, \phi) = 0$ and

$$E(u)(\mathcal{O}, \text{Val}(-, \phi)) = \frac{i}{n} > 0.$$ 

Since $\chi_{\{[v]\}}$ is idempotent and $E(u)$ is homogeneous, we may assume $i = n$. It follows from Definition 4.18 and Proposition 4.26 that $u([\mathcal{O}]\phi) = 1$. By axiom $\mathbf{C}$ of $\text{TP}_n$, we deduce $u([\emptyset]\phi) = 1$. The last identity is a contradiction since $([u], [v]) \in R^n$ and $v(\phi) = 0$. $\square$

**Theorem 4.28** (Completeness of $\text{TP}_n$). For any $\phi \in \mathcal{L}_+^+$, the following assertions are equivalent:
\[ \vdash_{\text{TP}_n} \phi. \]

(2) \( \phi \) is true in every standard truly playable \( L_n \)-valued \( \mathcal{L}^+ \)-model.

(3) \( \phi \) is true in every finite standard playable \( L_n \)-valued \( \mathcal{L}^+ \)-model.

Proof. It is clear that (2) \( \implies \) (3). Moreover, (1) \( \implies \) (2) can be proved by a straightforward induction argument. To prove (2) \( \implies \) (1), we obtain by Proposition 4.27 and Proposition 4.26 that \( \phi \) is true in the canonical model of TP\(_n\), which means \( \vdash_{\text{TP}_n} \phi \).

5. Conclusions and future research

In this paper we have studied some generalizations of Pauly’s Coalition Logic in a modal extension of Łukasiewicz logic. We believe that these generalizations could prove vital for the logical treatment of problems arising in game theory and social choice. Below we list some ideas for possible applications and topics for further investigations.

(1) The gain of expressive power owing to the used many-valued modal language could be exploited to encode some properties of strategic or voting games, such as, for instance, the distribution of power among coalitions in weighted voting games.

(2) In modal extensions of \((n + 1)\)-valued Łukasiewicz logics, two types of relational structures can naturally be considered, giving rise to two types of completeness results [12]. On the one hand, there is the class of frames (structures with binary accessibility relations), while, on the other hand, there is the class of \( L_n \)-valued frames. The latter are frames in which the set of allowed truth values in a world is a prescribed subalgebra of \( L_n \) for every world of the frame. Such a prescription could also be considered in the context of \( L_n \)-valued (truly) playable logics, where the neighborhood semantics replace the relational ones. The possible aim is to obtain new completeness results with respect to this enriched semantics.

(3) We have based our generalizations of Coalition Logic on modal extensions of Łukasiewicz logic. Other families of many-valued logics could be considered as a basis for many-valued versions of Coalition Logic. For example, it would be interesting to compare expressive power between the language developed in this paper and a many-valued coalitional language based on modal extensions of Gödel logics [13].

(4) Coalition Logic is among many formal calculi developed to model the deductive aspects of games. Other systems have been considered, such as ATL [4, 3] and its epistemic extensions [23]. A natural task could be to design the many-valued versions of those calculi in order to capture wider classes of games or protocols in which errors are allowed; see [21], for instance.

Appendix A. Boolean effectivity functions and their representation

In this appendix we summarize the results about Boolean effectivity functions and their representation by game forms as investigated in [20, 19, 9]. Let \( N \) be a player set and \( S \) be a set of outcomes. Recall Definition 2.1 a game form is a tuple \( G = (N, \{ \Sigma_i \mid i \in N \}, S, o) \), where \( \Sigma_i \) is a set of strategies for each player \( i \in N \) and \( o: \prod_{i \in N} \Sigma_i \rightarrow S \) is an outcome function. The effectivity function \( H_G: \mathcal{P}N \rightarrow \mathcal{P}S \) of a game form \( G \) is defined as follows:

\[ X \in H_G(C) \text{ if there exists } \sigma_C \text{ such that for every } \sigma_C, o(\sigma_C \sigma_C) \in X. \]

In words, the condition \( X \in H_G(C) \) is true whenever the coalition \( C \) has the power to enforce an outcome to lie in \( X \). The interested reader is referred to [19] for a discussion and examples of effectivity functions in game theory and social choice.
More generally and without any reference to game forms, any mapping \( E: \mathcal{P}N \rightarrow \mathcal{P}PS \) is called an effectivity function. Note that we may equivalently view the mapping \( \mathcal{P}N \rightarrow \mathcal{P}PS \) as \( \mathcal{P}N \times \{0,1\}^S \rightarrow \{0,1\} \). The notion of truly playable effectivity function was introduced by Goranko et al. in order to fix the error in Pauly’s characterization result in Theorem A.2. Roughly speaking, the problems arise by the existence of Pauly’s playable effectivity functions that are not generated by game forms over infinite sets of outcomes. We collect the basic notions and results concerning truly playable functions. The following properties of effectivity functions are considered in Definition A.1.

**Definition A.1.** Let \( E: \mathcal{P}N \rightarrow \mathcal{P}PS \) be an effectivity function. We say that \( E \)

1. is superadditive if \( C_1 \cap C_2 = \emptyset \) and \( X \in E(C_1), Y \in E(C_2) \) imply that \( X \cap Y \in E(C_1 \cup C_2) \), for every \( C_1, C_2 \in \mathcal{P}N \);
2. is outcome monotonic if \( X \in E(C) \) and \( X \subseteq Y \) imply \( Y \in E(C) \), for every \( C \in \mathcal{P}N \);
3. is \( N \)-maximal if \( \overline{X} \notin E(\emptyset) \) implies \( X \in E(N) \);
4. has liveness property if \( \emptyset \notin E(C) \), for every \( C \in \mathcal{P}N \);
5. has safety property if \( S \in E(C) \), for every \( C \in \mathcal{P}N \).

We call \( E \) playable whenever (1)–(5) are satisfied.

We will prove that the characterization of effectivity functions generated by game forms from Theorem 1 can be obtained as a consequence of Peleg’s Theorem 3.5* in [29]. Let \( N = \{1, \ldots, k\} \) be a finite set of players with \( k \geq 2 \) and \( S \) be a (possibly infinite) set of outcomes such that \( |S| \geq 2 \). A family \( B \subseteq \mathcal{P}S \) that contains \( S \) is called a structure on \( S \). An effectivity function \( E: \mathcal{P}N \rightarrow \mathcal{P}PS \) is said to be compatible with a structure \( B \) on \( S \) if \( E(C) \subseteq B \) for every \( C \in \mathcal{P}N \), \( E \) has liveness and safety properties, \( E(\emptyset) = \{S\} \), and \( E(N) = B \setminus \{\emptyset\} \). We say that \( B \) is closed under finite intersections if \( X_1, \ldots, X_j \in B \) implies \( \bigcap_{i=1}^{j} X_i \in B \) for every \( j \in \mathbb{N} \). An effectivity function is outcome monotonic w.r.t. \( B \) when the following implication holds true: if \( X \in E(C), X \subseteq Y \) and \( Y \in B \), then \( Y \in E(C) \), for every \( C \in \mathcal{P}N \).

**Theorem A.2 ([20] Theorem 3.5*).** Let \( B \) be a structure on \( S \) closed under finite intersections and \( E: \mathcal{P}N \rightarrow \mathcal{P}PS \) be an effectivity function compatible with \( B \). The following conditions are equivalent:

1. \( E \) is superadditive and outcome monotonic w.r.t. \( B \).
2. There exists a game form \( G = (N, \{\Sigma_i \mid i \in N\}, S, o) \) with the property \( E(C) = H_G(C) \cap B \) for every \( C \in \mathcal{P}N \).

Below we repeat the condition of true playability, which was introduced by Goranko et al. in Definition 8]. Our formulation is equivalent to the original definition by Proposition 5.

**Definition A.3.** Let \( E: \mathcal{P}N \rightarrow \mathcal{P}PS \) be an effectivity function. We say that \( E \) is truly playable if it is playable and \( E(\emptyset) \) is a principal filter in \( \mathcal{P}S \).

Since every filter on a finite set is principal, the class of truly playable functions and that of playable functions coincide provided that \( S \) is finite. The following result, which was originally proved in Theorem 1, amends the gap in the proof of Pauly’s correspondence result in case of an infinite outcome space \( S \). As announced, we now show that Theorem 1 can be considered as a consequence of Peleg’s Theorem A.2.

**Theorem A.4 ([9] Theorem 1]).** Let \( E: \mathcal{P}N \rightarrow \mathcal{P}PS \) be an effectivity function. There exists a game form \( G = (N, \{\Sigma_i \mid i \in N\}, S, o) \) satisfying \( E = H_G \) if and only if \( E \) is truly playable.
Proof. As for the first implication, it is easy to see that $H_G$ is playable. Set $Z = \{ z \in S \mid z = o(\sigma_N) \text{ for some strategy profile } \sigma_N \}$. Then $Z \in H_G(\emptyset)$. Clearly, for any set of outcomes $X \subseteq S$ we have $X \in H_G(\emptyset)$ if and only if $X$ contains the “range” $Z$. This means that $H_G(\emptyset)$ is the principal filter generated by $Z$ and $H_G$ is truly playable.

In order to show the converse implication, let $E$ be truly playable and put $Z = \bigcap E(\emptyset)$. Since $E$ satisfies safety, $Z \neq \emptyset$, and since $E(\emptyset)$ is a principal filter, $E(\emptyset) = \{ X \in \mathcal{P}S \mid Z \subseteq X \}$. Consider the mapping $E' : \mathcal{P}Z \to \mathcal{P}PZ$ defined as follows:

$$X \in E'(C) \text{ if } X \in E(C), \text{ for every } X \in \mathcal{P}Z \text{ and every } C \in \mathcal{P}N.$$  

We claim that $E'$ is compatible with the structure $\mathcal{P}Z$ on $Z$. Indeed, it follows that $E'(\emptyset) = \{ Z \}$ and $E'(C) \subseteq \mathcal{P}Z \setminus \{ \emptyset \}$ for every $C \in \mathcal{P}N$. By monotonicity, $Z \in E'(C)$ for every $C \in \mathcal{P}N$. It remains to prove that $\mathcal{P}Z \setminus \{ \emptyset \} \subseteq E(N)$. Let $\emptyset \neq X \subseteq Z$. If $X = Z$, then we already know that $X \in E(N)$. Otherwise, $X$ is a nonempty proper subset of $Z$ and thus $X \notin E(\emptyset)$. We obtain by $N$-maximality of $E$ that $\overline{X} \in E(N)$ and by superadditivity that $\overline{X} \cap Z \in E(N)$. We have proved that the complement in $Z$ of any nonempty proper subset of $Z$ is in $E(N)$, which yields $\mathcal{P}Z \setminus \{ \emptyset \} \subseteq E(N)$. We can conclude that $E'(N) = \mathcal{P}Z \setminus \{ \emptyset \}$.

By Theorem $[1,2]$ there is a game form $G' = (N, \{ \Sigma_i \mid i \in N \}, Z, o)$ such that $E' = H_{G'}$. Put $G = (N, \{ \Sigma_i \mid i \in N \}, S, o)$. We will show that $E = H_G$. To this end, let $C \in \mathcal{P}N$ and $X \in E(C)$. By superadditivity, $X \cap Z \in E(C)$, therefore $X \cap Z \in E'(C) = H_{G'}(C)$. By the definition of $H_{G'}$ and $H_G$, we get $X \in H_G(C)$. For the converse inclusion $H_G \subseteq E$, assume that $X \in \mathcal{P}S$ belongs to $H_G(C)$. By the definition of $H_G$, there exists $\sigma_C$ such that $o(\sigma_C o_{\tau_N}) \in X \cap Z$ for every $\tau_N$. This means $X \cap Z \in H_G(C) = E'(C)$, which gives $X \cap Z \in E(C)$. Finally, $X \in E(C)$ follows from outcome monotonicity. \qed

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Tomáš Kroupa, Institute of Information Theory and Automation of the ASCR, Pod Vodárenskou věží 4, 182 08 Prague, Czech Republic
E-mail address: kroupa@utia.cas.cz

Bruno Teheux, University of Luxembourg, Faculté des Sciences, de la Technologie et de la Communication, 6, rue Richard Coudenhove-Kalergi L-1359 Luxembourg
E-mail address: bruno.teheux@uni.lu