Infinitely many non-isotopic real symplectic forms on $S^2 \times S^2$

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Abstract
Let $(S^2, \omega)$ be a symplectic sphere, and let $\tau: S^2 \rightarrow S^2$ be an anti-symplectic involution of $(S^2, \omega)$. We consider the product $(S^2, \omega) \times (S^2, \omega)$ endowed with the anti-symplectic involution $\tau \times \tau$, and study the space of monotone anti-invariant symplectic forms on this four-manifold. We show that this space is disconnected. In addition, during the course of the proof, we produce a diffeomorphism of $\text{Gr}(2,4)$ which induces the identity map on all homology and homotopy groups, but which is not homotopic to the identity.

1. Introduction

Is there a closed four-manifold $X$ and a cohomology class $\xi \in H^2(X; \mathbb{R})$ such that the space $\Omega_\xi$ of symplectic forms of class $\xi$ is connected? This uniqueness problem up to isotopy for cohomologous symplectic forms is completely open in dimension four, though disconnected examples are known in higher dimensions, see [5, Problem 2 and Theorem 9.7.4]. This short note concerns the following modified version of this problem.

A real symplectic 4-manifold† is a triple $(X, \sigma, \omega)$, where $\sigma$ is an involution which is anti-symplectic $\sigma^* \omega = -\omega$. Pick a class $\xi \in H^2(X; \mathbb{R})$ such that $\sigma^* \xi = -\xi$ and let $\mathbb{R} \Omega_\xi$ denote the space of those symplectic forms on $X$ which are $\sigma$-anti-invariant and are in the class $\xi$. A natural question to ask is: Are there any examples of disconnected spaces $\mathbb{R} \Omega_\xi$?

Let us introduce a couple of simple examples of real symplectic manifolds. Consider the real ruled quadric $X_1$ defined as

$$X_1 := \{ x \in \mathbb{C}^3 \mid x_0^2 + x_1^2 = x_2^2 + x_3^2 \}.$$ (1.1)

The set of real points of $X_1$ is a 2-torus in $\mathbb{R}^3$ which is doubly ruled by real projective lines. Being a smooth projective variety, the surface $X_1$ inherits a Kähler form $\omega$ from the ambient space $(\mathbb{C}^3, \Omega_{st})$. Here $\Omega_{st}$ stands for the Fubini–Study 2-form. After rescaling $\omega$, we assume it is monotone meaning that

$$[\omega] = -K_{X_1} \in H^2(X_1; \mathbb{Z}),$$

where $K_{X_1}$ is the canonical class of $X_1$. The complex conjugation $\sigma: \mathbb{C}^3 \rightarrow \mathbb{C}^3$, $\sigma(x_i) = \bar{x}_i$, is an anti-symplectic involution with respect to $\Omega_{st}$. If a projective hypersurface is cut out by a real polynomial, then it is preserved by $\sigma$ and hence itself carries an anti-symplectic involution, namely $(\sigma|_{X_1})^* \omega = -\omega$. In this anti-invariant setting, we answer to the question of uniqueness in the negative proving
Theorem 1. The space of monotone anti-invariant symplectic forms on $X_1$ has infinitely many connected components.

A weaker statement applies to the quadric $X_2$ defined as

$$X_2 := \{ x \in \mathbb{CP}^3 \mid x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0 \}.$$  \hspace{1cm} (1.2)

In this case, we have:

Theorem 2. The space of monotone anti-invariant symplectic forms on $X_2$ has at least two connected components.

To prove both theorems, concrete representatives of different connected components of $\mathbb{R}\Omega_{(-K_{X_1})}$ (for abbreviation, we let $\mathbb{R}\Omega_K$ stand for $\mathbb{R}\Omega_{(-K_{X_1})}$) will be presented; these forms will be related by a diffeomorphism and so belong to the same connected component of the moduli space of $\sigma$-anti-invariant forms. It is not uninteresting to compare Theorems 1 and 2 with the recent results of Kharlamov and Shevchishin (see [3]), who study real symplectic 4-manifolds up to the equivalence relation generated by deformations and diffeomorphisms.

In particular, [3, Theorem 1.1] states that if a real rational symplectic 4-manifold $(X, \sigma, \omega)$ is $\sigma$-minimal, then it is a real Kähler surface. We do not discuss the notion of $\sigma$-minimality but note that a minimal surface (for example, $S^2 \times S^2$) is also $\sigma$-minimal.

Although there are countably many non-isomorphic complex structures on $S^2 \times S^2$, we stick to the one coming from the product $\mathbb{CP}^1 \times \mathbb{CP}^1$, for it is the only complex structure which admits a Kähler form in the anti-canonical class. A classical result is that there are exactly four different types of anti-holomorphic involutions on $\mathbb{CP}^1 \times \mathbb{CP}^1$ (see [4, Lemma 1.16]), in this note we only discuss the types $Q^{2,2}$ and $Q^{4,0}$. Given an underlying real algebraic surface $(X, \sigma)$ and a class $\xi \in H^2(X; \mathbb{R})$, the space of $\sigma$-anti-invariant Kähler forms in class $\xi$ is convex. It then follows from Moser’s argument that, up to isotopy, there exists exactly one Kähler form in a given cohomology class. The classical Moser’s trick considers a family of symplectic forms $\omega_t$ such that the cohomology class of $\omega_t$ is constant and provides a family of diffeomorphisms $\varphi_t$ such that $\varphi_t^* \omega_t = \omega_0$. The argument is easily adapted to the case of anti-invariant forms: if the family $\omega_t$ consists of $\sigma$-anti-invariant forms, then the derived isotopy $\varphi_t$ commute with $\sigma$.

Given a pair of rational symplectic 4-manifolds $(X, \sigma, \omega)$ and $(X, \sigma', \omega')$ with $X$ diffeomorphic to $S^2 \times S^2$ and with both $\omega$ and $\omega'$ being monotone. Then $(X, \sigma, \omega)$ is diffeomorphic to $(X, \sigma', \omega')$ if and only if $\sigma$ is diffeomorphic to $\sigma'$. In other words, the natural mapping

$$\text{Diff}_K(X, \sigma) \to \mathbb{R}\Omega_K, \quad f \to f_* \omega$$

is surjective. Here $\text{Diff}_K(X, \sigma)$ stands for the subgroup of those diffeomorphisms which preserve $K_X \in H^2(X; \mathbb{Z})$ and are $\sigma$-equivariant. Therefore, the moduli space $\mathbb{R}\Omega_K / \text{Diff}(X, \sigma)$ consists of a single point, yet, according to our claim, the space $\mathbb{R}\Omega_K$ itself may have infinitely many connected components.

Another problem worth considering in this anti-invariant setting is to construct a pair of symplectic forms which are not deformation equivalent. Two $(\sigma$-anti-invariant) symplectic forms $\omega$ and $\omega'$ are said to be deformation equivalent if there exists a $(\sigma$-equivariant) diffeomorphism $\varphi$ such that $\varphi_* \omega'$ and $\omega$ are connected by a path of $(\sigma$-anti-invariant) symplectic forms. In general, without taking in account any involutions, inequivalent symplectic forms in dimension 4 have been obtained by McMullen and Taubes [6] and later by Smith [9] and Vidussi [10]. It is not immediately clear how to construct similar examples in the presence of an anti-holomorphic involution. Note, however, that according to [3], one cannot produce such examples in the realm of real rational 4-manifolds.
2. Proof of Theorem 1

We now describe $X_1$ in a way that visibly exhibits its complex structure. There is a projective transformation sending equation (1.1) to

$$y_0 y_1 = y_2 y_3,$$

and transforming $\sigma: x_i \to \bar{x}_i$ into $\sigma: (y_0, y_1, y_2, y_3) \to (\bar{y}_1, \bar{y}_0, \bar{y}_3, \bar{y}_2)$. The rational functions

$$z = \frac{y_2}{y_0}, \quad w = \frac{y_3}{y_0}$$

define a biholomorphism from $X_1$ onto $\mathbb{CP}^1(z) \times \mathbb{CP}^1(w)$. In the inhomogeneous coordinates $(z, w)$, the map $\sigma$ takes the form

$$\sigma(z, w) = (\bar{z}^{-1}, \bar{w}^{-1}).$$

Finally, the form $\omega$ on $X_1$ splits into a product form $\omega_{\mathbb{CP}^1} \oplus \omega_{\mathbb{CP}^1}$.

We now introduce an invariant which is capable to distinguish between some connected components of the space $\mathbb{R}\Omega_K$ of anti-invariant monotone ($\xi = -K_X$) symplectic forms on $X_1$. To understand this invariant, it is the easiest to start with the product Kähler form $\omega$. We let $L$ denote the fixed point set of $\sigma$, which is the product of the two copies of $\mathbb{RP}^1$ defined, respectively, by $|z| = 1$ and $|w| = 1$. Pick a point $p$ on $L$ and observe that there is but one smooth complex sphere passing through $p$ for each of the generators $H_2(X_1; \mathbb{Z}) \cong \mathbb{Z}(A) \oplus \mathbb{Z}(B)$. Denote these spheres by $C_A$ and $C_B$, respectively. Note that the curves

$$\gamma_A = C_A \cap L, \quad \gamma_B = C_B \cap L$$

are transversally intersecting simple closed curves in $L$, which form a basis for $H_1(L; \mathbb{Z})$. Our invariant associates to $\omega \in \mathbb{R}\Omega_\xi$ the class $[\gamma_A] \in H_1(X; \mathbb{Z})$. There is no natural choice for orientation of $\gamma_A$, so we orient it somehow. To see $\gamma_A$ is indeed an invariant for connected components of $\mathbb{R}\Omega_K$, we use the following observation of Gromov:

**Theorem 3 [1, 2.4.A1].** Let $(X, \omega)$ be $S^2 \times S^2$ endowed with a product monotone form. Then, every $\omega$-compatible almost-complex structure $J$ defines two transversal fibrations of $X$ into $J$-holomorphic spheres and these fibrations continuously (even smoothly) depend on $J$.

Therefore, for every $\omega$-compatible almost-complex structure $J$, there is but one $J$-holomorphic sphere $C_A$ in class $A$ passing through $p$.

We let $J_\omega$ denote the space of $\omega$-compatible almost-complex structures, where one can find the subspace of $\sigma$-anti-invariant ($\sigma_\ast \circ J = -J \circ \sigma_\ast$) structures, denoted by $\mathbb{R}J_\omega$. We have already seen that $C_A$ intersects $L$ by a simple closed curve in the integrable case, but we wish to see this for an arbitrary $J \in \mathbb{R}J_\omega$. Since $\sigma$ is anti-holomorphic, it must send the curve $C_A$ to another $J$-curve in class $A$. Since both $C_A$ and $\sigma(C_A)$ pass through the point $p$, they must coincide. As such, the restriction of $\sigma$: $C_A \to C_A$ is an anti-holomorphic involution of $C_A$ that has a fixed point; it has, therefore, a fixed smooth circle, which exhausts the fixed points. We conclude that for every $J \in \mathbb{R}J_\omega$ the sphere $C_A$ in class $A$ intersects $L$ by a closed simple curve $\gamma_A$. The class $[\gamma_A] \in H_1(L; \mathbb{Z})$ does not depend on $J$, for the space $\mathbb{R}J_\omega$ is connected (see Lemma 1.) Nor it depends on the isotopy class of $\omega$, as long as our isotopy is $\sigma$-equivariant: given two forms at the same connected component of $\mathbb{R}\Omega_K$, we use Moser’s trick to obtain a family of $\sigma$-equivariant diffeomorphisms between them, thus identifying the corresponding spaces of almost-complex structures, spaces of holomorphic spheres, etc.

**Lemma 1.** Let $(X, \sigma, \omega)$ be a real symplectic manifold, and let $\mathbb{R}J_\omega$ be the space of $\omega$-compatible almost-complex structures which are anti-invariant under the anti-symplectic
involution. The space \( \mathbb{R}J_\omega \) is non-empty and connected, and in fact it is contractible by [11, Proposition 1.1].

Proof. Let \( \mathcal{R}_\sigma \) be the space of Riemannian metrics on \( X \) which are invariant under the anti-symplectic involution. Clearly, the space \( \mathcal{R}_\sigma \) is convex and hence, contractible. There is a natural embedding \( i : \mathbb{R}J_\omega \to \mathcal{R}_\sigma \) defined by

\[
J \mapsto \omega(\cdot, J\cdot) \quad \text{for} \quad J \in \mathbb{R}J_\omega.
\]

We will prove that \( \mathbb{R}J_\omega \) is a retract of \( \mathcal{R}_\sigma \). Since \( \mathcal{R}_\sigma \) is connected, this would imply that \( \mathbb{R}J_\omega \) is connected too. For every \( g \in \mathcal{R}_\sigma \), there is a unique field of endomorphisms \( A_g \) of \( T_X \) such that

\[
\omega(\cdot, \cdot) = g(A_g\cdot, \cdot).
\]

Since \( \omega \) is \( \sigma \)-anti-invariant and \( g \) is \( \sigma \)-invariant, it follows that \( A_g \) and \( \sigma \) anti-commute, that is,

\[
A_g \circ \sigma = -\sigma \circ A_g. \tag{2.2}
\]

Furthermore, as \( \omega \) is skew-symmetric, so is \( A_g \), that is, \( A_g^t = -A_g \), where \( A_g \) is the adjoint for \( A_g \) with respect to \( g \). Set

\[
J_g := (-A_g^2)^{-1/2}A_g, \tag{2.3}
\]

where the square root in the right-hand side of (2.3) is well defined because \( (-A_g^2) \) is \( g \)-self-adjoint and positive-definite. In fact, self-adjoint positive operators have a unique self-adjoint positive square root, and this is the root we pick for \( (-A_g^2) \). Also, since \( (-A_g^2) \) and \( \sigma \) commute, it follows that

\[
(-A_g^2)^{1/2} \circ \sigma = \sigma \circ (-A_g^2)^{1/2} \quad \text{and} \quad (-A_g^2)^{-1/2} \circ \sigma = \sigma \circ (-A_g^2)^{-1/2}. \tag{2.4}
\]

Here we have used that \( \sigma \) is an involution. Using (2.2) and (2.4), we see that \( J_g \) and \( \sigma \) anti-commute. Moreover, \( J_g \) satisfies

\[
J_g^2 = -\text{id}. \tag{2.5}
\]

To prove (2.5), it suffices to check that

\[
(-A_g^2)^{-1/2} \circ A_g = A_g \circ (-A_g^2)^{-1/2}. \tag{2.6}
\]

One should pass to the complexification of \( T_X \) to see this. Also, using (2.6), it is straightforward to check that \( J_g \) is \( \omega \)-compatible. We conclude that \( J_g \in \mathbb{R}J_\omega \). Define \( u : \mathcal{R}_\sigma \to \mathbb{R}J_\omega \) as

\[
u(g) := J_g.
\]

It is easy to see that \( u \circ i = \text{id} \), so \( u \) is a retraction. \( \Box \)

To prove our theorem, we shall construct a sequence of forms \( \omega_k \in \mathbb{R}\Omega^2_K \) whose invariants \( [\gamma_k] \in H_1(L; \mathbb{Z}) \) are pairwise distinct. To this end, we find a diffeomorphism \( f : X_1 \to X_1 \) such that:

1. we wish \( f \) to satisfy \( f \circ \sigma = \sigma \circ f \), so that \( f_* \omega \) would be anti-invariant. The condition also implies that \( f \) keeps \( L \) invariant.
2. The restriction of \( f \) to \( L \) is smoothly isotopic to \( (a \text{ power of}) \) the Dehn twist along the curve \( \gamma_B \), so, in particular, we would have \( f_*[\gamma_A] = [\gamma_A] + m[\gamma_B] \neq [\gamma_A] \) in \( H_1(L; \mathbb{Z}) \cong \mathbb{Z}^2 \) for some \( m \in \mathbb{Z} \).
3. And, finally, we wish \( f : H_2(X_1; \mathbb{Z}) \to H_2(X_1; \mathbb{Z}) \) to be the identity isomorphism. This is necessary for using the invariant \( [\gamma_A] \), as it was defined with the class \( A \).
Once \( f \) is found, it can be used to obtain infinitely many non-isotopic forms \( \omega_k \) by setting \( \omega_k := f^k \omega \). This finishes the proof.

To construct \( f \), we first do it on the subset
\[
\mathbb{CP}^1(z) \times K_\varepsilon \subset X, \quad K_\varepsilon = \{ w \in \mathbb{CP}^1(w) \mid 1 - \varepsilon < |w| < 1 + \varepsilon \}
\]
with the formula
\[
f(z, w) := (z e^{\text{Arg}(w)}, w) \quad \text{for some} \quad m \in \mathbb{Z},
\]
which can be seen as a mapping from \( K_\varepsilon \) to \( \text{Diff}_+(\mathbb{CP}^1(z)) \), the group of orientation-preserving diffeomorphisms of \( \mathbb{CP}^1(z) \). We ask whether this mapping can be extended to a mapping defined over the whole sphere \( \mathbb{CP}^1(w) \). When \( m \) is even, the answer is yes since: \( K_\varepsilon \) is homotopy equivalent to \( S^1 \) and \( \pi_1(\text{Diff}_+(S^2)) = \mathbb{Z}_2 \). We thus constructed \( f \) as an orientation-preserving fiberwise diffeomorphism of the fibration \( \mathbb{CP}^1(z) \times \mathbb{CP}^1(w) \to \mathbb{CP}^1(w) \).

Whether \( f \) is smoothly isotopic to the identity? If we do not impose the condition \( \sigma \circ f = f \circ \sigma \) on the isotopy, then the answer is yes since: \( \pi_2(\text{Diff}_+(S^2)) = 0 \). Therefore, all the forms \( f^k \omega \) are in the same connected component of \( \Omega_K \) but not of \( \mathbb{R}\Omega_K \).

3. **Proof of Theorem 2**

We employ a similar model for \( X_2 \). Namely, \( X_2 \) is again the product \( \mathbb{CP}^1(z) \times \mathbb{CP}^1(w) \), but now \( \sigma \) is given by
\[
\sigma(z, w) = (-z^{-1}, -w^{-1}). \tag{3.1}
\]
Just like in (2.1), the action of \( \sigma \) is componentwise, but in contrast to (2.1), the new involution has no fixed points. Using Cartesian coordinates \((x, y)\) on \( S^2 \times S^2 \subset \mathbb{R}^3 \times \mathbb{R}^3 \), we describe \( \sigma \) as
\[
\sigma(x, y) = (-x, -y).
\]
Consider the diffeomorphism \( f : X_2 \to X_2 \) given by
\[
f(x, y) := (-x + 2(x, y) y, y).
\]
Here \((x, y)\) stands for the Euclidean inner product of \( x \) and \( y \). This map does nothing on the second sphere and does the reflection of the first sphere with respect to the axis passing through the antipodal points \( y \) and \(-y\). As \( f \) and \( \sigma \) commute, we obtain a descendant self-mapping \( g \) of \( Z := X_2/\sigma \) and the commutative diagram
\[
\begin{array}{ccc}
X_2 & \xrightarrow{f} & X_2 \\
\downarrow p & & \downarrow p \\
Z & \xrightarrow{g} & Z,
\end{array}
\tag{3.2}
\]
where \( p : X \to Z \) is the covering map, which identifies \((x, y)\) with \((-x, -y)\). Geometrically, \( Z \cong \text{Gr}(2, 4) \), the Grassmannian of two-planes in \( \mathbb{R}^4 \), and \( X_2 \cong S^2 \times S^2 \) is the corresponding Grassmannian of oriented planes.

It is interesting to look at the algebraic properties \( g \). Taking \( \pi_k \) and \( H_k \) of (3.2), we obtain commutative diagrams of abelian groups. Since \( f \) is homotopic to the identity (not equivariantly!), we have
\[
f_* = \text{id} : \pi_k(X_2) \to \pi_k(X_2) \quad \text{and} \quad f_* = \text{id} : H_k(X_2) \to H_k(X_2) \quad \text{for all} \quad k. \tag{3.3}
\]
Since \( p : X_2 \to Z \) is a connected covering, we also have isomorphisms
\[
p_* : \pi_k(X_2) \to \pi_k(Z) \quad \text{for} \quad k \neq 1. \tag{3.4}
\]
From (3.3) and (3.4), we obtain

\[ g_∗ = \text{id}: \pi_k(Z) \to \pi_k(Z) \quad \text{for all } k \neq 1. \]  \hspace{1cm} (3.5)

To prove that \( g_∗ = \text{id} \) for \( k = 1 \), observe that \( \pi_1(Z) = \mathbb{Z}_2 \) and that the only automorphism of \( \mathbb{Z}_2 \) is the identity.

Recall that

\[
\begin{align*}
H_0(Z; \mathbb{Z}_2) & = \mathbb{Z}_2, \quad H_1(Z; \mathbb{Z}_2) = \mathbb{Z}_2, \quad H_2(Z; \mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2, \\
H_3(Z; \mathbb{Z}_2) & = \mathbb{Z}_2, \quad H_4(Z; \mathbb{Z}_2) = \mathbb{Z}_2.
\end{align*}
\]  \hspace{1cm} (3.6)

Here the first and the last equality follow from the connectivity of \( Z \). The group \( H_1 \) is easy to recover since we know that \( \pi_1(Z) = \mathbb{Z}_2 \), whereas the group \( H_2 \) is recovered via Poincaré duality. What is left to compute is \( H_2 \). Recall that the Schubert cell decomposition of \( \text{Gr}(2, 4) \) consists of one 0-cell, one 1-cell, two 2-cells, one 3-cell, and of a single 4-cell. Thus, the dimension of \( H_2 \) is not greater than 2. To show that \( H_2 \) is exactly \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \), we explicitly describe two non-homologous cycles in \( Z \).

Let us introduce the diagonal sphere \( \Delta := \{(x, y) \in S^2 \times S^2 | x = y\} \). The sphere \( \Delta \) is invariant with respect to \( \sigma \), as \( \sigma(x, x) = (-x, -x) \). Let \( Q \) denote \( p(\Delta) \), which is an embedded \( \mathbb{R}P^2 \) in \( Z \). The map

\[ s: X_2 \to \Delta, \quad s(x, y) := (x, x) \]

fits into the diagram

\[
\begin{array}{ccc}
X_2 & \longrightarrow & \Delta \\
\downarrow \sigma & & \downarrow \sigma \\
X_2 & \longrightarrow & \Delta,
\end{array}
\]

and hence induces a map \( Z \to Q \) which we denote by the same letter \( s \). Note that \( s: Z \to Q \)

is a fiber bundle over \( Q \). Let \( F \) be any fiber of \( s \). We claim that the classes \([F], [Q] \in H_2(Z; \mathbb{Z}_2)\) are non-zero and are not equal to each other. Indeed, since \( Q \), being a section, intersects \( F \) at exactly one point, we have \([F] \cdot [Q] = 1\).

This implies that both \( F \) and \( Q \) are not homologically trivial. They are also not homologous to each other, as for if they were that would imply the equality \([F] \cdot [Q] = [F]^2\). However, the cycle \( F \), being a fiber, must have self-intersection number 0.

We claim that \( g_∗ = \text{id}: H_k(Z; \mathbb{Z}_2) \to H_k(Z; \mathbb{Z}_2) \) for all \( k \). This is obvious for \( k \neq 2 \) for dimension reasons. To prove that for \( k = 2 \), observe that \( g \) and \( s \) commute and that \( g \) keeps \( Q \) fixed. Hence, we must have \( g_∗[F] = [F], g_∗[Q] = [Q] \).

A slightly more subtle computation reveals:

\[
\begin{align*}
H_0(Z; \mathbb{Z}) & = \mathbb{Z}, \quad H_1(Z; \mathbb{Z}) = \mathbb{Z}_2, \quad H_2(Z; \mathbb{Z}) = \mathbb{Z}_2, \quad H_3(Z; \mathbb{Z}) = 0, \quad H_4(Z; \mathbb{Z}) = \mathbb{Z},
\end{align*}
\]  \hspace{1cm} (3.7)

with \( H_0(Z; \mathbb{Z}) \) generated by \([F]\). Clearly, the action of \( g \) on \( H_k(Z; \mathbb{Z}) \) is also trivial for all \( k \).

We conclude that \( g \) induces the identity morphisms on all homology and homotopy groups. Nevertheless, we claim that \( g: Z \to Z \) is not homotopic to the identity. To see this, we need to introduce the mapping torus

\[ T_g := Z \times [0, 1]/\{(z, 1) \sim (g(z), 0)\}. \]

We will show now that the Stiefel–Whitney class \( w_3(T_g) \) does not vanish; by contrast, the mapping torus \( T_\text{id} \) has vanishing \( w_3 \). Therefore, \( g \) is not homotopic to the identity.
LEMMA 2. \( w_3(T_g) \neq 0 \).

Proof. Remember that \( g|_Q = \text{id} \). Therefore, there is a natural embedding
\[
Q \times S^1 = Q \times [0,1]/\{(q,1) \sim (g(q),0)\}
\]
into \( T_g \).

We refer to this embedded copy of \( Q \times S^1 \) as \( \mathcal{R} \). It suffices to prove that the restriction of \( w_3(T_g) \) on \( \mathcal{R} \) does not vanish. Let us look at the restriction of the tangent bundle to \( T_g \) on \( \mathcal{R} \). It splits as
\[
N_{\mathcal{R}/T_g} \oplus T_{\mathcal{R}},
\]
where \( N_{\mathcal{R}/T_g} \) is the normal bundle to \( \mathcal{R} \) in \( T_g \) and \( T_{\mathcal{R}} \) is the tangent bundle to \( \mathcal{R} \). As \( \mathcal{R} = Q \times S^1 \), we have
\[
T_{\mathcal{R}} = T_Q \oplus \mathbb{R},
\]
where \( T_Q \) is the pull-back of the tangent bundle to \( Q \) under the projection \( \mathcal{R} \to Q \), and \( \mathbb{R} \) is a trivial line bundle.

Now, we will prove the formula
\[
N_{\mathcal{R}/T_g} = N_{Q/Z} \otimes L_{S^1}.
\]
(3.9)

Here \( L_{S^1} \) is the non-trivial (non-orientable) line bundle over \( S^1 \), and \( N_{Q/Z} \) is the normal bundle to \( Q \) in \( Z \). In the right-hand side of the formula, both factors are interpreted as the pull-backs with respect to the projections \( \mathcal{R} \to Q \) and \( \mathcal{R} \to S^1 \).

Observe that, although \( g|_Q = \text{id} \), the derivative \( dg: N_{Q/Z} \to N_{Q/Z} \) is not the identity but
\[
dg = -\text{id}: N_{Q/Z} \to N_{Q/Z}.
\]

Now (3.9) follows from a general fact. Suppose we have a topological space \( X \) with a vector bundle \( I: E \to X \), and that we form a twisted bundle \( I^\times \) over \( X \times S^1 \) with the total space
\[
E \times [0,1]/\{(e,x,1) \sim (-e,x,0)\},
\]
where a triple \((e,x,1)\) consists of \( x \in X \times \{1\} \) and \( e \in I^{-1}(x) \), and likewise for \((e,x,0)\). Then it is easy to see that
\[
I^\times = I \otimes L_{S^1},
\]
and as \( N_{\mathcal{R}/T_g} = N_{Q/Z}^\times \), formula (3.9) follows.

Now, we will establish an isomorphism
\[
N_{Q/Z} \cong T_Q.
\]
(3.10)

This is again a general fact. Suppose we are given a manifold \( Y \) and a free involution \( \tau: Y \to Y \). Let us consider the diagonal \( \Delta = \{(y_1,y_2) \in Y \times Y \mid y_1 = y_2\} \). Set
\[
Z := Y \times Y/\sim, \quad Q := \Delta/\sim, \quad \text{where } (y_1,y_2) \sim (\tau(y_1),\tau(y_2)).
\]
There is a canonical way (there are two, choose either) to identify \( T_\Delta \) with the normal bundle to \( \Delta \) in \( Y \times Y \). This identification is equivariant with respect to the involution \( \tau \times \tau: Y \times Y \to Y \times Y \), and hence gives rise to an identification between \( T_Q \) and \( N_{Q/Z} \).

Combining (3.8) with (3.9) and (3.10) yields
\[
N_{\mathcal{R}/T_g} \oplus T_{\mathcal{R}} \cong T_Q \oplus \mathbb{R} \oplus (T_Q \otimes L_{S^1}).
\]
(3.11)

We wish to compute the class \( w_3 \) of this bundle. To this end, we apply the Whitney sum formula to obtain
\[
w_3(T_Q \oplus (T_Q \otimes L_{S^1})) = w_1(Q)w_2(T_Q \otimes L_{S^1}) + w_2(Q)w_1(T_Q \otimes L_{S^1}),
\]
(3.12)
where \( w_i(Q) = w_i(T_Q) \) are the Stiefel–Whitney classes \( Q \) (more precisely, their pull-backs to \( \mathcal{R} \) under the projection \( \mathcal{R} \to Q \)), and \( w_i(T_Q \otimes L_{S^1}) \) are the classes of the bundle \( T_Q \) twisted by \( L_{S^1} \). In computing \( w_3 \), we have ignored the \( \mathbb{R} \) component of (3.11) because characteristic classes are invariant under taking the sum with a trivial bundle.

In general, if \( L \) is a line bundle and \( E \) is an arbitrary vector bundle, then there are closed expressions for both the first and the top classes of \( E \otimes I \) in terms of \( w_i(E) \) and \( w_i(L) \). They are as follows:

\[
\begin{align*}
  w_n(E \otimes L) &= w_n(E) + w_{n-1}(E)w_1(L) + w_{n-2}(E)w_1(L)^2 + \cdots, \\
  w_1(E \otimes L) &= w_1(E) + nw_1(L),
\end{align*}
\]

where \( n = \text{rk } E \). The proof of both formulas is a simple application of the splitting principle and is omitted. Since \( \text{rk } T_Q = 2 \), we get

\[
  w_3(T_Q \otimes L) = w_2(Q) + w_1(Q)w_1(L_{S^1}), \quad w_1(T_Q \otimes L) = w_1(Q).
\]

Since \( \dim Q = 2 \), we have \( w_1(Q)w_2(Q) = 0 \). Substituting (3.14) into (3.12), we obtain \( w_1(Q)^2w_1(L_{S^1}) \). Recall that \( Q \cong \mathbb{RP}^2 \) and \( \mathcal{R} \cong Q \times S^1 \). The class \( w_1(\mathbb{RP}^2)^2 \) generates \( H^2(\mathbb{RP}^2; \mathbb{Z}_2) \), while the class \( w_1(L_{S^1}) \) generates \( H^1(S^1; \mathbb{Z}_2) \). Clearly, their product does not vanish.

**Lemma 3.** \( w_3(T_\text{id}) = 0 \).

**Proof.** Since \( T_\text{id} \cong Z \times S^1 \), the Whitney formula yields

\[
  w_3(T_\text{id}) = w_3(Z).
\]

One of the definitions of \( w_3 \) says

\[
  w_3(Z) \cdot [Z] = \chi(Z) \mod 2.
\]

Since there is a double covering \( S^2 \times S^2 \to Z \), it follows that

\[
  \chi(Z) = \frac{\chi(S^2 \times S^2)}{2} = 2.
\]

**Remark.** Suppose that \( X \) is a simply-connected 4-manifold. Then a classical result of Quinn [7] states that a homeomorphism \( f: X \to X \) is homotopic to the identity if and only if it induces the identity morphism on homology. The theorem’s generalization to the non-simply-connected case is unknown, not even for the simplest case of the group \( \mathbb{Z}_2 \). Suppose that \( Z \) is a closed 4-manifold with \( \pi_1(Z) \cong \mathbb{Z}_2 \). Of course, for a homeomorphism \( g: Z \to Z \) to be homotopic to the identity, it is first necessary to have \( g \) acting identically on all \( \pi_k \) and \( H_k \). However, this is not sufficient. For example, a simple obstruction is that any self-homeomorphism of \( Z \) that is homotopic to the identity must have a lift (to the universal cover) that is also homotopic to the identity, but even that in general is not sufficient, as the example above shows.

Thus, although \( f \) is isotopic to the identity as a diffeomorphism of \( X_2 \), it is not isotopic to the identity within \( \text{Diff}(X_2, \sigma) \).

**Lemma 4.** Suppose that \( h: (X_2, \sigma, \omega) \to (X_2, \sigma, \omega) \) is a symplectomorphism such that \( h \circ \sigma = \sigma \circ h \) and such that \( h_* = \text{id} : H_2(X_2; \mathbb{Z}) \to H_2(X_2; \mathbb{Z}) \). Then \( h \) is isotopic to the identity within \( \text{Diff}(X_2, \sigma) \).

**Proof.** Let \( J_0 \) be the underlying complex structure of the Kähler surface \( X_2 \). Since \( h \) is a symplectomorphism and \( J_0 \in \mathbb{R}J_\omega \), it follows that \( h_*J_0 \in \mathbb{R}J_\omega \). Since \( \mathbb{R}J_\omega \) is connected, there is a path \( J(t) \in \mathbb{R}J_\omega, t \in [0, 1] \) such that \( J(0) = J_0 \) and \( J(1) = h_*J_0 \). Recall that every
structure \( J(t) \in \mathbb{R} J_\omega \) gives rise to a pair \((\mathcal{F}_A^t, \mathcal{F}_B^t)\) of transversal fibrations of \( X_2 \) into \( J(t) \)-holomorphic spheres. Since \( \sigma \) is anti-holomorphic for each \( J(t) \), these fibrations are invariant with respect to \( \sigma \) in the following sense: if \( C \) is a fiber of \( \mathcal{F}_A^t \), then so is \( \sigma(C) \), and likewise for \( \mathcal{F}_B^t \). Following [1, 2.4.A1], one constructs a path of diffeomorphisms \( \alpha(t) \in \text{Diff}(X_2, \sigma) \) with \( \alpha(0) = \text{id} \) such that \( \alpha(t) \) sends \((\mathcal{F}_A^0, \mathcal{F}_B^0)\) to \((\mathcal{F}_A^t, \mathcal{F}_B^t)\). Composing \( h \) with \( \alpha(t) \), we can assume henceforth that \( h \) preserves \((\mathcal{F}_A^0, \mathcal{F}_B^0)\). This is a very restrictive condition implying that \( h \) is componentwise in the sense that

\[
h = h_1 \times h_2 : S^2 \times S^2 \to S^2 \times S^2,
\]

where both diffeomorphisms \( h_1 : S^2 \to S^2 \) must be orientation-preserving and must also commute with the antipodal map \( \tau : S^2 \to S^2, \ \tau(x) = -x \). Here the condition \( h_* = \text{id} \) has been used twice, once to ensure \( h \) does not interchange the fibrations \( \mathcal{F}_A^0 \) and \( \mathcal{F}_B^0 \), and once to ensure that \( h \) preserves the natural complex orientation of their fibers. What is left is to show that the group

\[
\text{Diff}(S^2, \tau) \times \text{Diff}(S^2, \tau), \quad (3.15)
\]

is connected. Here \( \text{Diff}(S^2, \tau) \) stands for the subgroup of those diffeomorphisms of \( S^2 \) which are orientation-preserving and \( \tau \)-equivariant.

A more general statement is that \( \text{Diff}(S^2, \tau) \) is homotopy equivalent to \( SO(3) \). Note that every element of \( \text{Diff}(S^2, \tau) \) induces a self-diffeomorphism of \( \mathbb{R}P^2 \cong S^2/\tau \), and we have a surjective homomorphism

\[
\text{Diff}(S^2, \tau) \to \text{Diff}(\mathbb{R}P^2), \quad (3.16)
\]

which is not injective, since \( \text{id} : \mathbb{R}P^2 \to \mathbb{R}P^2 \) has exactly two lifts, the trivial lift \( \text{id} \) and the non-trivial lift \( \tau \). However, if we replace \( \text{Diff}(S^2, \tau) \) by \( \text{Diff}_+(S^2, \tau) \), then \( (3.16) \) becomes an isomorphism. One can prove that \( \text{Diff}(\mathbb{R}P^2) \) is homotopy equivalent to \( SO(3) \) by following the steps of the proof of Smale’s theorem (see [8]), as in [2, Appendix A]. A shorter proof is as follows. It is a classical result that the 2-sphere \( S^2 \) admits a unique complex structure. That is, the group of diffeomorphisms of \( S^2 \) acts transitively on the space of complex structures on \( S^2 \). From here, it is easy to show that \( \text{Diff}_+(S^2, \tau) \) acts transitively on the collection of \( \tau \)-anti-invariant complex structures on \( S^2 \), and that we have a fibration

\[
\text{Aut}(\mathbb{C}P^1, \tau) \to \text{Diff}_+(S^2, \tau) \to J_\tau,
\]

where \( \text{Aut}(\mathbb{C}P^1, \tau) \) is the subgroup of those biholomorphisms of \( \mathbb{C}P^2 \) which are \( \tau \)-equivariant, and \( J_\tau \) is a connected component (there are two of them!) of the space of \( \tau \)-anti-invariant complex structures on \( S^2 \). The space \( J_\tau \) is contractible, while \( \text{Aut}(\mathbb{C}P^1, \tau) \), being a Lie group, contracts on its maximal compact subgroup, which is \( SO(3) \). Hence, the group \( \text{Diff}_+(S^2, \tau) \) itself contracts onto \( SO(3) \).

We are now in a position to show \( \omega \) and \( f_\omega \) are at different components of \( \mathbb{R} \Omega_{\nu K} \). Suppose they are not. Then we could join them with a path of monotone anti-invariant forms. By Moser’s trick, that would imply that \( f \) is isotopic to a symplectomorphism within \( \text{Diff}(X_2, \sigma) \), and that, by the previous lemma, would imply that \( f \) is isotopic to the identity within \( \text{Diff}(X_2, \sigma) \). This is a contradiction.

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