ASYMPTOTIC RESULTS FOR ROUGH CONTINUOUS-STATE BRANCHING PROCESSES

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In this paper we provide explicit representations of Laplace transforms of extinction time and total progeny of rough continuous-state branching processes introduced in [7]. Also, we show that their tail distributions are much fatter than those of Feller branching diffusions.

1. Introduction. A rough continuous-state branching process (rough CB-process) was introduced in [7] in the generalization of the classical second Ray-Knight theorem to a spectrally positive stable process \( \{\xi(t) : t \geq 0\} \) with index \( 1 + \alpha \in (1, 2) \) and Laplace exponent

\[
\Phi(\lambda) = b\lambda + c\lambda^{1+\alpha} = b\lambda + \int_0^\infty (e^{-\lambda y} - 1 + \lambda y)\nu_\alpha(dy),
\]

where \( b \geq 0, c > 0 \) and the Lévy measure \( \nu_\alpha(dy) \) is given by

\[
\nu_\alpha(dy) := \frac{c\alpha}{2} (1-\alpha) y^{\alpha-2}dy.
\]

It is described as the unique weak solution to the stochastic Volterra equation

\[
X_\zeta(t) = \zeta - b \int_0^t \frac{(t-s)^{\alpha-1}}{c\Gamma(\alpha)} X_\zeta(s)ds + \int_0^t \int_0^\infty \int_0^\infty X_\zeta(s) \int_{(t-s-y)}^{t-s} \frac{r^{\alpha-1}}{c\Gamma(\alpha)} d\tilde{N}_\alpha(ds, dy, dz),
\]

where \( \tilde{N}_\alpha(ds, dy, dz) \) is a compensated Poisson random measure on \( (0, \infty) \times \mathbb{R}_+^2 \) with intensity \( ds d\nu_\alpha(dy)dz \).

The rough CB-process \( X_\zeta \) is locally Hölder-continuous of any order strictly less than \( \alpha/2 \) and will fall into the trap 0 in finite time, i.e. its extinction time

\[
\tau_{X_\zeta} := \inf\{t \geq 0 : X_\zeta(t) = 0\}
\]

is finite almost surely and \( X_\zeta(\tau_{X_\zeta} + t) = 0 \) for any \( t \geq 0 \). Thus its total progeny defined by

\[
T_{X_\zeta} := \int_0^\infty X_\zeta(t)dt = \int_0^{\tau_{X_\zeta}} X_\zeta(t)dt
\]

is also finite almost surely. In this work, we mainly explore the asymptotic behavior of tail distributions of \( \tau_{X_\zeta} \) and \( T_{X_\zeta} \).

To understand the laws of \( \tau_{X_\zeta} \) and \( T_{X_\zeta} \), we first study the characteristic functionals of \( X_\zeta \). From Theorem 1.3 in [7], there exist two continuous functions \( \{v_t(\lambda) : t, \lambda \geq 0\} \) and \( \{V_T(\lambda) : \lambda \geq 0\} \) such that

\[
\mathbb{E}\left[\exp\{-\lambda X_\zeta(t)\}\right] = \exp\{-\zeta \cdot v_t(\lambda)\} \quad \text{and} \quad \mathbb{E}\left[\exp\{-\lambda T_{X_\zeta}\}\right] = \exp\{-\zeta \cdot V_T(\lambda)\}.
\]

Enlightened by the method developed in the proof of Theorem 1.1 in [7], we first provide explicit representations of \( v_t(\lambda) \) and \( V_T(\lambda) \) in terms of functionals of scale function and inverse Laplace exponent of \( \zeta \). Our main idea is characterizing the limits of characteristic functionals of local time processes of rescaled compounded Poisson processes which converge weakly to the stable process \( \xi \). Noting that the

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function \( v_t(\lambda) \) increases to a finite limit \( \bar{v}_t \) as \( \lambda \to \infty \) and the function \( \bar{v}_t \) decreases to 0 as \( t \to \infty \), we have
\[
P\{\tau_{X_\zeta} > t\} = P\{X_\zeta(t) > 0\} = 1 - e^{-\zeta \cdot \bar{v}_t},
\]
as \( t \to \infty \). Applying Karamata-Tauberian theorem together with our observation that both \( v_t(\lambda) \) and \( V_T(\lambda) \) are regularly varying at infinity, we obtain that both \( \tau_\zeta \) and \( T_\zeta \) have heavy-tailed distributions. In precise, for both \( b = 0 \) and \( b > 0 \) we have as \( t \to \infty \),
\[
(1.7) \quad P\{\tau_{X_\zeta} > t\} \sim Ct^{-\alpha}.
\]
Differently, as \( x \to \infty \) we have
\[
(1.8) \quad P\{T_{X_\zeta} > x\} \sim C \cdot x^{-\alpha/(1+\alpha)} \quad \text{when } b = 0 \quad \text{and} \quad P\{T_{X_\zeta} > x\} \sim C \cdot x^{-\alpha} \quad \text{when } b > 0.
\]
These are in sharp contrast to the asymptotic behavior of Feller branching diffusion defined as the unique solution to
\[
(1.9) \quad Y_\zeta(t) = \zeta - \int_0^t bY_\zeta(s)ds + \int_0^t \sqrt{2cY_\zeta(s)}dB(s),
\]
where \( \{B(t): t \geq 0\} \) is a standard Brownian motion. In detail, denote by \( \tau_{Y_\zeta} \) and \( T_{Y_\zeta} \) its extinction time and total progeny respectively. Theorem 3.4 in [5, p.17] together with a simply calculation shows that
\[
(1.10) \quad P\{\tau_{Y_\zeta} > t\} \sim Ct^{-1} \quad \text{when } b = 0 \quad \text{and} \quad P\{\tau_{Y_\zeta} > t\} \sim Ce^{-bt} \quad \text{when } b > 0.
\]
Moreover, from Corollary 10.9 in [4, p.280] we also have
\[
(1.11) \quad P\{T_{Y_\zeta} > x\} \sim C \cdot x^{-1/2} \quad \text{when } b = 0 \quad \text{and} \quad P\{T_{Y_\zeta} > x\} \leq Ce^{-\frac{4b}{\pi^2}x} \quad \text{when } b > 0.
\]

Organization of this paper: In Section 2, we first give a criticality for a rough CB-process and then provide the asymptotic results for its extinction time and total progeny. The proofs are given in Section 3.

2. Main results. Define the Mittag-Leffler function \( E_{\alpha,\alpha} \) on \( \mathbb{R}_+ \) by
\[
E_{\alpha,\alpha}(x) := \sum_{n=0}^\infty \frac{x^n}{\Gamma(\alpha(n+1))}.
\]
It is locally Hölder-continuous with index \( \alpha \); see [2] for a precise definition of it and a survey of some of its properties, e.g. for any \( a \geq 0 \) we have the well-known Laplace transform
\[
(2.1) \quad \int_0^\infty e^{-\lambda x}ax^{\alpha-1}E_{\alpha,\alpha}(-a \cdot x^\alpha)dx = \frac{a}{a + \lambda^\alpha}, \quad \lambda \geq 0.
\]
Let \( \{W(x): x \in \mathbb{R}\} \) be the scale function of \( \xi \), which is identically zero on \( x \in (-\infty, 0) \) and characterized on \( [0, \infty) \) as a strictly increasing function whose Laplace transform is given by
\[
(2.2) \quad \int_0^\infty e^{-\lambda x}W(x)dx = \frac{1}{\Phi(\lambda)}, \quad \lambda > 0.
\]
The scale function \( W \) is infinitely differentiable on \( (0, \infty) \) with derivative denoted as \( W' \). Applying integration by parts to (2.2) and the using (2.1), we see that the derivative function \( W' \) is of the form:
\[
W'(x) = c^{-1}x^{\alpha-1} \cdot E_{\alpha,\alpha}(-b/c \cdot x^\alpha), \quad x \geq 0.
\]
Specially, when \( b = 0 \) we have \( E_{\alpha,\alpha}(0) = \Gamma(\alpha) \) and

\[
(2.3) \quad W'(x) = \frac{x^{\alpha-1}}{c\Gamma(\alpha)}, \quad W(x) = \frac{x^\alpha}{c\Gamma(1+\alpha)}.
\]

When \( b > 0 \), the function \( bW' \) turns to be *Mittag-Leffler probability density function* and hence \( bW(t) \to 1 \) as \( t \to \infty \).

From Theorem 1.4 in [7], the stochastic Volterra equation (1.3) is equivalent to

\[
(2.4) \quad X_\zeta(t) = \zeta(1 - bW(t)) + \int_0^t \int_0^\infty W(s) (W(t - s) - W(t - s - y)) \tilde{N}_\alpha(ds,dy,dz).
\]

Taking expectations on both sides of the equation above, we can get the following result.

**Lemma 2.1.** For \( t > 0 \), we have \( \mathbb{E}[X_\zeta(t)] = \zeta(1 - bW(t)) \), which is identical to \( \zeta \) when \( b = 0 \) and decreases to 0 as \( t \to \infty \) when \( b > 0 \).

We say the rough CB-process \( X_\zeta \) is *critical* or *subcritical* according as \( b = 0 \) or \( b > 0 \), which is consistent to the criticality for Feller branching diffusions; see Definition 10.4 in [4, p.277].

2.1. *Extinction time.* We now provide an explicit representation for the Laplace exponent \( \nu_t(\lambda) \) in term of a functional of \( W \) and then explore the tail distribution of extinction time \( \tau_{X_\zeta} \).

**Theorem 2.2.** For \( \lambda, t \geq 0 \), the Laplace exponent \( \nu_t(\lambda) \) is of the form

\[

\nu_t(\lambda) = \frac{c}{\Gamma(1-\alpha)} \int_0^\infty \lambda \cdot \nabla_x W(t) \cdot \frac{\alpha dx}{\lambda W(t) + 1} \cdot x^{\alpha+1},
\]

and as \( \lambda \to \infty \) it increases strictly to

\[

\bar{\nu}_t := \frac{c}{\Gamma(1-\alpha)} \int_0^\infty \frac{\nabla_x W(t) \cdot \alpha dx}{W(t)} \cdot x^{\alpha+1} < \infty.
\]

**Corollary 2.3.** The function \( \bar{\nu}_t \) decreases to 0 as \( t \to \infty \). In precise,

1. when \( b = 0 \), we have

\[

\mathbb{P}\{\tau_{X_\zeta} \geq t\} \sim \zeta \cdot \bar{\nu}_t = \zeta \cdot c\Gamma(1+\alpha) \cdot t^{-\alpha};
\]

2. when \( b > 0 \), we have

\[

\mathbb{P}\{\tau_{X_\zeta} \geq t\} \sim \zeta \cdot \bar{\nu}_t \sim \frac{\zeta \cdot c}{\Gamma(1-\alpha)} t^{-\alpha}.
\]

**Remark 2.4.** For \( \lambda \geq 0 \), from Theorem 3.4 in [5, p.17] we have when \( b = 0 \),

\[

\mathbb{E}[\exp\{-\lambda Y_\zeta(t)\}] = \exp\left\{-\zeta \cdot \frac{\lambda}{1+c\lambda t}\right\}
\]

and hence as \( t \to \infty \),

\[

\mathbb{P}\{\tau_{Y_\zeta} > t\} = 1 - \exp\left\{-\frac{\zeta}{ct}\right\} \sim \frac{\zeta}{ct}.
\]

When \( b > 0 \), from Example 2.3 in [5, p.14] we have

\[

\mathbb{E}[\exp\{-\lambda Y_\zeta(t)\}] = \exp\left\{-\zeta \cdot \frac{\lambda e^{-bt}}{1+c\lambda \cdot b^{-1}(1-e^{-bt})}\right\}
\]

and hence as \( t \to \infty \),

\[

\mathbb{P}\{\tau_{Y_\zeta} > t\} = 1 - \exp\left\{-\zeta \cdot \frac{e^{-bt}}{c \cdot b^{-1}(1-e^{-bt})}\right\} \sim \frac{b}{c} e^{-bt}.
\]
2.2. Total progeny. It is obvious that the function $\Phi$ is infinitely differentiable and strictly increasing on $(0, \infty)$ with $\Phi(0) = 0$ and $\Phi(\lambda) \to \infty$ as $\lambda \to \infty$. This allows us to define its inverse $\Psi$ on $[0, \infty)$ with $\Psi(0) = 0$.

**Theorem 2.5.** For $\lambda \geq 0$ we have $V_T(\lambda) = c \cdot |\Psi(\lambda)|^\alpha$.

**Corollary 2.6.** The function $V_T(\lambda)$ increases strictly to infinity as $\lambda \to \infty$. In precise, (1) when $b = 0$, we have $V_T(\lambda) = c^{1/(1+\alpha)} \cdot \lambda^{\alpha/(1+\alpha)}$ and hence as $x \to \infty$,

$$\mathbb{P}\{T_{X_y} > x\} \sim \frac{\zeta \cdot c^{1/(1+\alpha)}}{\Gamma(1/(1+\alpha))} \cdot x^{-\alpha/(1+\alpha)};$$

(2) when $b > 0$, we have $V_T(\lambda) \sim \zeta \cdot c \cdot (\lambda/b)^\alpha$ as $\lambda \to 0+$ and hence as $x \to \infty$,

$$\mathbb{P}\{T_{X_y} > x\} \sim \frac{\zeta \cdot c}{b^\alpha \Gamma(1-\alpha)} \cdot x^{-\alpha}.$$

**Remark 2.7.** From Corollary 10.9 in [4, p.280], for $\lambda \geq -\frac{b^2}{4c}$ we have

$$E\{\exp\{-\lambda T_{Y_\zeta}\}\} = \exp\left\{-\zeta \cdot \frac{\sqrt{b^2 + 4c\lambda} - b}{2c}\right\}. \quad (2.5)$$

When $b = 0$, we have $E\{\exp\{-\lambda T_{Y_\zeta}\}\} \sim 1 - \zeta \cdot \sqrt{1/c}$ as $\lambda \to 0+$. Applying Karamata-Tauberian theorem; see Theorem 8.1.6 in [1, p.333], we have as $x \to \infty$,

$$\mathbb{P}\{T_{Y_\zeta} > x\} \sim \zeta \cdot \frac{x^{-1/2}}{\sqrt{c}(1/2)}.$$

When $b > 0$, by Chebyshev's inequality we have

$$\mathbb{P}\{T_{Y_\zeta} > x\} = \mathbb{P}\{e^{\frac{b^2}{4c} T_{Y_\zeta}} > e^{\frac{b^2}{4c} x}\} \leq e^{-\frac{b^2}{4c} x} E[e^{\frac{b^2}{4c} T_{Y_\zeta}}] \leq \exp \left\{ \frac{b\zeta}{2c} - \frac{b^2}{4c} x \right\}.$$

3. Proofs.

3.1. Compound Poisson Processes. Let $\Lambda$ be a Pareto II distribution on $\mathbb{R}_+$ with location 0 and shape $\alpha + 1$, i.e.

$$\Lambda(dx) = (\alpha + 1)(1 + x)^{-\alpha - 2}dx.$$

For $n \geq 1$, let $\{\xi^{(n)}(t) : t \geq 0\}$ be a compound Poisson process with drift $-1$, arrival rate $\gamma_n > 0$, jump size distribution $\Lambda$ and initial state $\xi^{(n)}(0)$ distributed as

$$\Lambda^*(dx) := \alpha \bar{\Lambda}(x)dx = \alpha(1 + x)^{-1-x}\alpha dx. \quad (3.1)$$

Here $\bar{\Lambda}(x) := \Lambda(x, \infty)$ is the tail distribution of $\Lambda$. It is a spectrally positive Lévy process with Laplace exponent

$$\phi^{(n)}(\lambda) := \lambda + \int_0^\infty (e^{-\lambda x} - 1) \gamma_n \Lambda(dx), \quad \lambda \geq 0.$$

We are interested in the case $E[\xi^{(n)}(1)] = \gamma_n/\alpha - 1 \leq 0$ in which the function $\phi^{(n)}$ increases strictly to infinity.

Let $c_0 = (c/\Gamma(1-\alpha))^{-1\alpha}$. We define the first passage time of $\xi^{(n)}$ in $(-\infty, 0]$ by

$$\tau^{(n)}_0 := \inf\{t \geq 0 : \xi_t^{(n)} \leq 0\}. \quad (3.2)$$
Let \( \{L_{\xi(n)}(y,t): y \in \mathbb{R}, t \geq 0\} \) be the local time process of \( \xi(n) \) with

\[
L_{\xi(n)}(y,t) := \# \{s \in (0,t]: \xi(n)(s) = y\}.
\]

We have \( L_{\xi(n)}(y,\infty) = \infty \) a.s. if and only if \( \mathbb{E}[\xi(n)(1)] = 0 \); equivalently, if and only if \( \gamma_n = \alpha \). Let \( \{Z_k^{(n)}(t): t \geq 0\}_{k \geq 1} \) be a sequence of i.i.d. copies of local time process \( \{L_{\xi(n)}(t,\tau_0^{(n)}): t \geq 0\} \). We now consider under the following condition, the weak convergences of rescaled processes \( \{\xi_0^{(n)}(t): t \geq 0\} \) and \( \{S^{(n)}(t): t \geq 0\} \) with

\[
(3.3) \quad \xi_0^{(n)}(t) := \frac{c_0}{n}\xi(n)(nt/c_0) \quad \text{and} \quad S^{(n)}(t) := \frac{1}{c_0n^{\alpha}} \sum_{k=1}^{[c_0n^{\alpha}]} Z_k^{(n)}(nt/c_0).
\]

**CONDITION 3.1.** Assume that \( n^{\alpha}(1 - \gamma_n/\alpha) \rightarrow b/c_0 \) as \( n \rightarrow \infty \).

It is easy to see that the Lévy process \( \xi(n) \) has Laplace exponent \( n^{1+\alpha}\Phi(n)(c_0\lambda/n) \) for \( \lambda \geq 0 \). Under Condition 3.1, a routine computation shows that \( n^{1+\alpha}\Phi(n)(c_0\lambda/n) \rightarrow \Phi(\lambda) \) as \( n \rightarrow \infty \). By Corollary 4.3 in [3, p.440], we have \( \xi_0^{(n)} \rightarrow \xi \) weakly in \( D([0,\infty); \mathbb{R}) \). Denote by \( \{W^{(n)}(x): x \in \mathbb{R}\} \) the weak solution \( X_\xi \) for \( (1.3) \) in \( D([0,\infty); \mathbb{R}_+) \) as \( n \rightarrow \infty \).

The next lemma follows directly from Lemma 2.3 and Proof of Theorem 1.1 in [7].

**LEMMA 3.2.** Under Condition 3.1, for any \( \zeta > 0 \) we have \( S^{(n)}(t) \) converges weakly to the unique weak solution \( X_\zeta \) to \( (1.3) \) in \( D([0,\infty); \mathbb{R}_+) \) as \( n \rightarrow \infty \).

**3.2. Proofs for Theorem 2.2 and Corollary 2.3.** From Lemma 3.2, it is necessary to consider the distribution of \( Z_1^{(n)} \) at first. For \( x > 0 \), denote by \( \mathbb{P}_x \) and \( \mathbb{E}_x \) the probability law and expectation of \( \xi^{(n)} \) conditioned on \( \xi(n)(0) = x \).

**PROPOSITION 3.3.** For \( n \geq 1 \) and \( x, t > 0 \) with \( x \neq t \), we have

\[
(3.5) \quad \mathbb{E}_t[\exp\{-\lambda Z_1^{(n)}(t)\}] = \frac{W^{(n)}(0)}{W^{(n)}(t)} \frac{1 - e^{-\lambda}}{1 - e^{-\lambda}(1 - \frac{W^{(n)}(0)}{W^{(n)}(t)})},
\]

\[
(3.6) \quad \mathbb{E}_x[\exp\{-\lambda Z_1^{(n)}(t)\}] = 1 - \frac{\nabla_x W^{(n)}(t)}{W^{(n)}(t)} \frac{1 - e^{-\lambda}}{1 - e^{-\lambda}(1 - \frac{W^{(n)}(0)}{W^{(n)}(t)})}.
\]

**PROOF.** For \( t > 0 \), let \( \tau^+_t := \inf\{s \geq 0: \xi^{(n)}(s) > t\} \) and \( T_t := \inf\{s \geq 0: \xi^{(n)}(s) = t\} \). Since \( \xi^{(n)} \) has neither diffusion nor nonnegative jumps, we have \( \tau^+_t < T_t \) a.s. Thus

\[
(3.7) \quad \mathbb{E}_x[\exp\{-\lambda Z_1^{(n)}(t)\}] = \mathbb{P}_x\{\tau_0^{(n)} < \tau^+_t\} + \mathbb{E}_x[\exp\{-\lambda Z_1^{(n)}(t)\}; \tau_0^{(n)} > \tau^+_t]\]

By the strong Markov property of \( \xi^{(n)} \), we have

\[
(3.8) \quad \mathbb{E}_x[\exp\{-\lambda Z_1^{(n)}(t)\}; \tau_0^{(n)} > \tau^+_t] = e^{-\lambda}\mathbb{E}_t[\exp\{-\lambda Z_1^{(n)}(t)\}] \mathbb{P}_x\{\tau^+_t < \tau_0^{(n)}\}.
\]

Solving (3.7)-(3.8) with \( x = t \), we have

\[
\mathbb{E}_t[\exp\{-\lambda Z_1^{(n)}(t)\}] = \frac{\mathbb{P}_t\{\tau^-_t < \tau^+_t\}}{1 - e^{-\lambda}\mathbb{P}_t\{\tau^+_t < \tau^-_t\}}.
\]
Plugging this back into (3.7)-(3.8), for \( x \neq t \) we have
\[
\mathbb{E}_x \left[ \exp \left\{ -\lambda Z_1^{(n)}(t) \right\} \right] = \mathbb{P}_x \{ \tau_0^- < \tau_t^+ \} + \mathbb{P}_x \{ \tau_t^+ < \tau_0^- \} \frac{e^{\lambda P_t \{ \tau_0^- < \tau_t^+ \}}}{1 - e^{\lambda P_t \{ \tau_t^+ < \tau_0^- \}}}.
\]

From Theorem 8.1(iii) in [4, p.215], we have \( \mathbb{P}_t \{ \tau_0^- < \tau_t^+ \} = W^{(n)}(0)/W^{(n)}(t) \) and \( \mathbb{P}_x \{ \tau_0^- < \tau_t^+ \} = W^{(n)}(t-x)/W^{(n)}(t) \). With a simple calculation we can get the desired result. \( \square \)

**Proof for Theorem 2.2.** From Proposition 3.3, we have
\[
\mathbb{E} \left[ \exp \left\{ -\lambda Z_1^{(n)}(t) \right\} \right] = 1 - \frac{1 - e^{-\lambda}}{1 - e^{-\lambda} (1 - W^{(n)}(0)/W^{(n)}(t))} \int_0^\infty \frac{\nabla_x W^{(n)}(t)}{W^{(n)}(t)} \Lambda^*(dx)
\]

By the mutual independence among \( \{ Z_k^{(n)} \}_{n \geq 1} \),
\[
\mathbb{E}_x \left[ \exp \left\{ -\lambda S_\zeta^{(n)}(t) \right\} \right] = \left( \mathbb{E} \left[ \exp \left\{ - \frac{\lambda}{c_0 n^\alpha} Z_1^{(n)}(nt/c_0) \right\} \right] \right)^{[n^\alpha c_0 \zeta]}
\]

where
\[
v_t^{(n)}(\lambda) = -c_0 n^\alpha \cdot \log \left( 1 - \frac{1 - e^{-\frac{\lambda}{c_0 n^\alpha}}}{1 - e^{-\frac{\lambda}{c_0 n^\alpha}} (1 - W^{(n)}(0)/W^{(n)}(nt/c_0))} \right) \int_0^\infty \frac{\nabla_x W^{(n)}(nt/c_0)}{W^{(n)}(nt/c_0)} \Lambda^*(dx).
\]

From Lemma 3.2 we have \( v_t^{(n)}(\lambda) \to v_t(\lambda) \) as \( n \to \infty \). Noting that
\[
v_t^{(n)}(\lambda) \sim \frac{c_0 n^\alpha}{c_0 n^\alpha [1 - e^{-\frac{\lambda}{c_0 n^\alpha}} (1 - W^{(n)}(0)/W^{(n)}(nt/c_0))]} \int_0^\infty \frac{\nabla_x W^{(n)}(nt/c_0)}{n^{-\alpha} W^{(n)}(nt/c_0)/c_0} \Lambda^*(dx).
\]

From (3.4) and the fact that \( W^{(n)}(0) = 1 \), we have as \( n \to \infty \)
\[
c_0 n^\alpha \left[ 1 - e^{-\frac{\lambda}{c_0 n^\alpha}} (1 - W^{(n)}(0)/W^{(n)}(nt/c_0)) \right] \to \lambda + 1/W(t).
\]

and
\[
\int_0^\infty \nabla_x W^{(n)}(nt/c_0) \Lambda^*(dx) = \int_0^\infty \nabla_{nx} W^{(n)}(nt/c_0) \Lambda^*(n \cdot dx) \to c_0^{1+\alpha} \int_0^\infty \nabla_x W(t) \frac{\alpha dx}{x^{\alpha+1}}.
\]

Putting all estimates above together, we can immediately get the desire result. \( \square \)

**Proof for Corollary 2.3.** Since \( X_\zeta \) falls into the trap 0 in finite time, we have \( \bar{\nu}_t \to 0 \) as \( t \to \infty \). Moreover, we have \( \mathbb{P} \{ \bar{\tau}_{X_\zeta} > t \} = 1 - \exp \{ -\zeta \cdot \bar{\nu}_t \} \sim \zeta \cdot \bar{\nu}_t \) as \( t \to \infty \). Changing the order of integration, we have
\[
\int_0^\infty \nabla_x W(t) \frac{\alpha dx}{x^{\alpha+1}} = \int_0^\infty \int_0^x W'(t-s) ds \frac{\alpha dx}{x^{\alpha+1}}
\]
\[
= \int_0^\infty W'(t-x) dx \int_x^\infty \frac{\alpha ds}{s^{\alpha+1}} = \int_0^t W'(t-x) x^{-\alpha} dx.
\]

When \( b = 0 \), from (2.3) we have
\[
\int_0^\infty \nabla_x W(t) \frac{\alpha dx}{x^{\alpha+1}} = \int_0^t \frac{(t-x)^{\alpha-1} x^{-\alpha} dx}{c \Gamma(\alpha)} = \frac{\Gamma(1 - \alpha)}{c}.
\]
and

\[ \mathbb{P}\{\tau_{X_\zeta} > t\} \sim \zeta/W(t) = W(t) = \zeta \cdot c \Gamma(1 + \alpha) \cdot t^{-\alpha}. \]

When \( b > 0 \), from (2.4) in [7] we have as \( t \to \infty \),

\[ W'(t) \sim \frac{c\alpha \cdot t^{-\alpha - 1}}{b^2 \Gamma(1 - \alpha)} = o(t^{-\alpha}). \]

Applying Proposition 3.11 in [6] to (3.9), we have as \( t \to \infty \),

\[ \int_0^\infty \nabla_x W(t) \frac{c dx}{x^{\alpha + 1}} \sim W(t) \cdot t^{-\alpha} \]

and hence \( \mathbb{P}\{\tau_{X_\zeta} > t\} \sim \zeta \cdot c_0 \cdot t^{-\alpha}. \)

\[
3.3. \text{Proofs for Theorem 2.5 and Corollary 2.6.}\] From Lemma 3.2, we have

\[ \int_0^\infty S_\zeta^{(n)}(t) dt \to T_{X_\zeta} \]

in distribution. From (3.3),

\[ \int_0^\infty S_\zeta^{(n)}(t) dt = \sum_{k=1}^{[c_0 \alpha n \zeta]} \frac{1}{n^{1+\alpha}} \int_0^\infty Z_k^{(n)}(t) dt. \]

Applying the occupation density formula, we immediately have

\[ \int_0^\infty Z_1^{(n)}(t) dt = \int_0^\infty L_{\xi^{(n)}}(t, \tau_0^{(n)}) dt = \tau_0^{(n)}. \]

The next proposition can be found in [4, p.212].

**Proposition 3.4.** Let \( \psi^{(n)} \) be the inverse of \( \Phi^{(n)} \). For \( \lambda \geq 0 \), we have

\[ \mathbb{E}_x \left[ \exp\{-\lambda \tau_0^{(n)}\} \right] = \exp\{-x \psi^{(n)}(\lambda)\}. \]

**Proof for Theorem 2.5.** By the mutual independence among \( \{Z_k^{(n)}\}_{k \geq 1} \) and Proposition 3.4, for \( \lambda \geq 0 \) we have

\[
\mathbb{E} \left[ \exp \left\{ - \lambda \int_0^\infty S_\zeta^{(n)}(t) dt \right\} \right] = \left( \mathbb{E} \left[ \exp \left\{ - \frac{\lambda}{n^{1+\alpha}} \int_0^\infty Z_k^{(n)}(t) dt \right\} \right] \right)^{[c_0 n^\alpha \zeta]}
\]

\[ = \exp \left\{ - \frac{c_0 n^\alpha \zeta}{c_0 n^\alpha} V_T^{(n)}(\lambda) \right\}, \]

where

\[ V_T^{(n)}(\lambda) = c_0 n^\alpha \log \left( 1 - \int_0^\infty \left( 1 - e^{-x \psi^{(n)}(\lambda/n^{1+\alpha})} \right) \lambda^*(dx) \right). \]

It is obvious that \( V_T^{(n)}(\lambda) \to V_T(\lambda) \) as \( n \to \infty \). Since \( n^{1+\alpha} \Phi^{(n)}(c_0 \lambda/n) \to \Phi(\lambda) \) as \( n \to \infty \), we have

\[ n \psi^{(n)}(\lambda/n^{1+\alpha}) \to c_0 \psi(\lambda). \]

Thus by the dominated convergence theorem,

\[
V_T^{(n)}(\lambda) \sim c_0 n^\alpha \int_0^\infty \left( 1 - e^{-x \psi^{(n)}(\lambda/n^{1+\alpha})} \right) \lambda^*(dx) = c_0 \int_0^\infty \left( 1 - e^{-x \cdot n \psi^{(n)}(\lambda/n^{1+\alpha})} \right) \frac{an^{\alpha+1} dx}{(1 + nx)^{1+\alpha}}.
\]
\[
\sim c_0 \int_0^\infty \left(1 - e^{-x \cdot c_0 \Psi(\lambda)}\right) \frac{\alpha dx}{x^{1+\alpha}}.
\]

A simple calculation induces that \( V_T(\lambda) = \Gamma(1 - \alpha) c_0^{1+\alpha} |\Psi(\lambda)|^{\alpha} = c \cdot |\Psi(\lambda)|^{\alpha}. \)

**Proof for Corollary 2.6.** When \( b = 0 \), we have \( \Psi(\lambda) = (\lambda/c)^{1/(1+\alpha)} \) and \( E\left[ \exp\{-\lambda T_{X_\zeta}\}\right] = \exp\{-\zeta \cdot c^{1/(1+\alpha)} \cdot \lambda^{\alpha}/(1+\alpha)\} \).

By Karamata-Tauberian theorem, we have as \( x \to \infty, \)
\[
P\{T_{X_\zeta} > x\} \sim \frac{\zeta \cdot c^{1/(1+\alpha)}}{\Gamma(1/(1+\alpha))} \cdot x^{-\alpha/(1+\alpha)}.
\]

When \( b > 0 \), we have \( \Psi(\lambda) \sim \lambda/b \) as \( \lambda \to 0^+ \) and hence
\[
1 - E\left[ \exp\{-\lambda T_{X_\zeta}\}\right] \sim \frac{\zeta \cdot c}{b^{\alpha}} \cdot \lambda^{\alpha}.
\]

Applying Karamata-Tauberian theorem again, we have as \( x \to \infty, \)
\[
P\{T_{X_\zeta} > x\} \sim \frac{\zeta \cdot c}{b^{\alpha} \Gamma(1 - \alpha)} \cdot x^{-\alpha}.
\]

\[\square\]

**REFERENCES**

[1] N. H. Bingham, C. M. Goldie, and J. L. Teugels, *Regular Variation*, vol. 27, Cambridge University Press, 1987.
[2] M. A. M. Haubold, H. J. and R. K. Saxena, *Mittag-Leffler functions and their applications*, J. Appl. Math., (2011), p. Article ID 298628.
[3] J. Jacod and A. N. Shiryaev, *Limit Theorems for Stochastic Processes*, Springer, Berlin, 2003.
[4] A. E. Kyprianou, *Introductory Lectures on Fluctuations of Lévy Processes with Applications*, Springer, Berlin, 2006.
[5] Z. Li, *Continuous-State Branching Processes with Immigration*, From Probability to Finance, (2019).
[6] W. Xu, Asymptotic results for heavy-tailed Lévy processes and their exponential functionals, To appear in Bernoulli. arXiv:1912.04795, (2021).
[7] , *A Ray-Knight theorem for spectrally positive stable processes*, arXiv:2105.02349, (2021).