ℓ–State Solutions of Multiparameter Exponential-type Potentials

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Abstract.
In the present work, bound state solutions for a class of multiparameter exponential-type potential are obtained in the frame of the Greene and Aldrich approximation for the centrifugal term. The proposal is general and their usefulness is exemplified with the treatment of the Eckart, Manning-Rosen, Hulthén and Deng Fan potentials that are obtained straightforwardly without resorting to specialized methods of solution for each specific potential, as usually is done. Furthermore, the proposal admits other approximations for the centrifugal term indicating an improvement to procedures developed with the same objective. So, our proposal can be considered as an unified treatment of the ℓ-state solutions for exponential-type potentials and can be used to find new solvable potentials.

Keywords: Bound states, centrifugal term, point canonical transformation, Schrödinger equation.

1. Introduction
The search of analytical solutions of the Schrödinger equation for ℓ ≠ 0, of exponential-type potentials, has been of great interest as evidenced by the large number of works devoted to this subject; see for example, Ikhdair [1] and references therein. Their importance comes from the fact that this kind of potentials has been proposed as models of diatomic molecules whose analytical solutions contain all the necessary information of the quantum system under study. So, each specific potential has been studied by means of different approaches to the centrifugal term and by using various methods of solution of DE such as the Nikiforov-Uvarov [2], asymptotic iteration [3], SUSY-QM [4], path integral approach [5], numerical [6] and many others. In this paper, we propose an unified treatment to the study of bound state solutions of exponential-type potentials by considering the Greene and Aldrich approximation to the centrifugal term [7]. To that, the foundation of our proposal, given in Section 2, leads to an effective potential \( V_{\text{eff}}(r) \) containing the term \( 1/r^2 \) suitable for different approximations to the centrifugal term as shown in section 3. Also, in that section, the usefulness of the proposed \( V_{\text{eff}}(r) \) is exemplified by using the approach of Greene and Aldrich [7] with the aim to find the bound state solutions of already
known exponential potentials, as particular cases.

2. Multiparameter exponential-type potential

Recently, by using a canonical transformation method applied to the hypergeometric DE, we have obtained the exactly-solvable multiparameter exponential-type potential given by \[8\]

\[
V(r) = \frac{qA\exp(-r/k)}{1-q\exp(-r/k)} + \frac{qB\exp(-r/k)}{(1-q\exp(-r/k))^2} + \frac{q^2C\exp(-2r/k)}{(1-q\exp(-r/k))^2},
\]

for the Schrödinger equation

\[-\frac{d^2\psi_n(r)}{dr^2} + V(r)\psi_n(r) = E_n\psi_n(r),\]

with eigenvalues

\[E_n = -\frac{1}{4k^2} \left( \frac{n^2 + (h+1)(2n+1)/2 + k^2(A+B)}{n + (h+1)/2} \right)^2\]

and eigenfunctions

\[
\psi_n(r) = (q\exp(-r/k))^{(c-1)/2}(1-q\exp(-r/k))^{(h+1)/2} 2F_1(-n, b; c; q\exp(-r/k)).
\]

where \(a, b\) and \(c\) are given from the hypergeometric DE, the \(A, B\) and \(C\) parameters are related by means of

\[A + B = \frac{2ab - c(a + b + 1 - c)}{2k^2}, \quad A - C = \frac{(c - 1)^2 - (a - b)^2}{4k^2},\]

and \(h = (1 + 4k^2(B + C))^{1/2}\). Evidently, the choice of the \(A, B\) and \(C\) parameters leads to specific exponential-type potentials, as particular cases of Eq.(1), without using specific methods of solution for each one of potentials considered. This feature of our proposal, is used in the following section to give a unified treatment of \(\ell\)–state solutions of exponential-type potentials.

3. \(\ell\)–state solutions of exponential-type potentials

Let us consider the variable change

\[B = B + \frac{\ell(\ell + 1)}{k^2} \quad \ell = 0, 1, 2, ..,\]

and \(q = 1\), in such a way that \(V(r)\) of Eq. (1) becomes

\[
V(r) = \frac{A\exp(-r/k)}{1-\exp(-r/k)} + \frac{B\exp(-r/k)}{(1-\exp(-r/k))^2} + \frac{C\exp(-2r/k)}{(1-\exp(-r/k))^2} + \frac{\ell(\ell + 1)}{k^2(1-\exp(-r/k))^2}.
\]

This form of \(V(r)\) suggest the approximation for the centrifugal term given by Green and Aldrich \[7\]

\[
\frac{1}{r^2} \approx \frac{\exp(-r/k)}{k^2(1-\exp(-r/k))^2},
\]

where their \(\vartheta = 1/k\). Consequently, one have

\[
V(r) = V_{eff} = \frac{A\exp(-r/k)}{1-\exp(-r/k)} + \frac{B\exp(-r/k)}{(1-\exp(-r/k))^2} + \frac{C\exp(-2r/k)}{(1-\exp(-r/k))^2} + \frac{\ell(\ell + 1)}{r^2},
\]
and, according to Eq.(3), in this case the eigenvalues are given by

$$E_{n,\ell} = -\frac{1}{4k^2} \left( \frac{n^2 + (h_\ell + 1)(2n + 1)/2 + k^2(A + B) + \ell(\ell + 1)}{n + (h_\ell + 1)/2} \right)^2$$  \(\text{(10)}\)

where \(h_\ell = \sqrt{1 + 4\ell(\ell + 1) + 4k^2(B + C)}\). In relation with the eigenfunctions, these are

$$\psi_{n,\ell}(r) = (\exp(-r/k))^{(\ell+1)/2}(1 - \exp(-r/k))^{(h_\ell+1)/2} \, _2F_1(n, b_\ell; c_\ell; \exp(-r/k)).$$  \(\text{(11)}\)

where \(b_\ell\) and \(c_\ell\) are respectively

$$b_\ell = \frac{(h + 1)(h + n) - 2k^2(A + B) - 2\ell(\ell + 1)}{2n + h + 1}, \quad c_\ell = \frac{-2n(n + h) - 2k^2(A + B) - 2\ell(\ell + 1)}{2n + h + 1}.$$  \(\text{(12)}\)

To show the usefulness of our proposal, we are going to consider different options for the \(A, B\) and \(C\) parameters with the aim to find the \(\ell\)–bound state solutions of Schrödinger equation for specific exponential-type potentials as will see next.

a). Hulthén potential

In this case, the choice \(B = C = 0, A = -Ze^2\delta\) and \(k = \delta^{-1}\), where \(\delta\) is the screening parameter, let us to rewrite the effective potential of Eq.(10) as

$$V_{\text{eff}} = \frac{-Ze^2\delta \exp(-\delta r)}{1 - \exp(-\delta r)} + \frac{\ell(\ell + 1)}{r^2}$$  \(\text{(13)}\)

having, from Eq.(11) and Eq.(13), the energy spectra

$$E_{n,\ell} = -\delta^2 \left( \frac{n + \ell + 1}{2} - \frac{Ze^2}{2\delta(n + \ell + 1)} \right)^2$$  \(\text{(14)}\)

and wavefunctions

$$\psi_{n,\ell}(r) = (\exp(-\delta r))^{(\ell+1)/2}(1 - \exp(-\delta r))^{\ell+1} \, _2F_1(-n, b_\ell; c_\ell; \exp(-\delta r))$$  \(\text{(15)}\)

in agreement with Ikhdair [9] when using natural units. It should be noted that in this case

$$b_\ell = \frac{(\ell + 1)(n + 2\ell + 1) + Ze^2/\delta - \ell(\ell + 1)}{n + \ell + 1}, \quad c_\ell = \frac{n(2\ell - 1) + Ze^2/\delta - \ell(\ell + 1)}{n + \ell + 1}.$$  \(\text{(16)}\)

b). Manning-Rosen potential.

Similarly to the above case, let us consider now \(B = 0, A = -V_0/b^2, C = \frac{\alpha(\alpha - 1)}{\ell^2}\) with \(k = b\). Thus, the corresponding effective potential of Eq.(10) becomes

$$V_{\text{eff}} = \frac{1}{b^2} \left( \frac{\alpha(\alpha - 1) \exp(-2r/b)}{1 - \exp(-2r/b)} - V_0 \exp(-r/b) \right) + \frac{\ell(\ell + 1)}{r^2},$$  \(\text{(17)}\)

that is identified with the Manning-Rose potential with centrifugal term, with energy spectra

$$E_{n,\ell} = -\frac{1}{4b^2} \left( \frac{n^2 + (2n + 1)(h_\ell + 1)/2 + \ell(\ell + 1) - V_0}{n + (h_\ell + \ell)/2} \right)^2$$  \(\text{(18)}\)
where \( h_\ell = \sqrt{1 + 4\ell(\ell + 1) + 4\alpha(\alpha - 1)} \) and eigenfunctions

\[
\psi_{n,\ell}(r) = \frac{(\exp(-\delta r/b))^{(c_\ell - 1)/2}(1 - \exp(-\delta r/b))^{(h_\ell + 1)/2}}{2F_1(-n, b_\ell; c_\ell; \exp(-\delta r/b))},
\]

where

\[
b_\ell = \frac{(h_\ell + 1)(n + h_\ell) + 2bV_0 - \ell(\ell + 1)}{2n + h_\ell + 1}, \quad c_\ell = \frac{-2n(h_\ell + n) + 2bV_0 - 2\ell(\ell + 1)}{2n + h_\ell + 1}.
\]

It should be noticed, that eigenvalues given in Eq.(22) coincides with those of Qiang et al [10] after the identification of their parameter \( \alpha' \) with our \( h_\ell \) by means of \( \alpha' = (h_\ell + 1)/2 \) as well as their \( \exp(1/\beta) = 1 \) to be in agreement with the same approximation of the centrifugal term.

c). Eckart potential.

Similarly to the above cases, the choice of \( A = -\alpha, B = \beta, C = 0 \) and \( k = a \) in Eq.(10), leads to the Eckart potential with centrifugal term

\[
V_{\text{eff}} = -\frac{\alpha \exp(-r/a)}{1 - \exp(-r/a)} + \frac{\beta \exp(-r/a) + \ell(\ell + 1)}{r^2}.
\]

So, using Eq.(11), in this case the energy spectra is given by

\[
E_{n,\ell} = \frac{1}{4a^2} \left( \frac{n^2 + (2n + 1)(h_\ell + 1)/2 + a^2(\beta - \alpha) + \ell(\ell + 1)}{n + (h_\ell + 1)/2} \right)^2
\]

that can be written as

\[
E_{n,\ell} = \frac{1}{4a^2} \left( n + \frac{h_\ell + 1}{2} - \frac{a^2\alpha}{n + (h_\ell + 1)/2} \right)^2,
\]

where \( h_\ell = \sqrt{1 + 4\ell(\ell + 1) + 4\alpha^2\beta} \), in agreement with Taskin and Kocak [11] when using \( \lambda = \epsilon = 1 \). Finally from Eq.(13), the wavefunctions are given now by

\[
\psi_{n,\ell}(r) = \frac{(\exp(-r/a))^{(c_\ell - 1)/2}(1 - \exp(-r/a))^{(h_\ell + 1)/2}}{2F_1(-n, b_\ell; c_\ell; \exp(-r/a))},
\]

where

\[
b_\ell = \frac{(h_\ell + 1)(n + h_\ell) - 2a^2(\beta - \alpha) - 2\ell(\ell + 1)}{2n + h_\ell + 1}, \quad c_\ell = \frac{-2n(h_\ell + n) - 2a^2(\beta - \alpha) - 2\ell(\ell + 1)}{2n + h_\ell + 1}.
\]

d). Deng Fan potential.

This case can be achieved with the selection of \( A = -2bD, B = 0, C = Db^2 \) and \( k = 1/\alpha \), leads to the potential

\[
V_{\text{eff}} = \frac{-2bD \exp(-\alpha r)}{1 - \exp(-\alpha r)} + \frac{Db^2 \exp(-2\alpha r)}{(1 - \exp(-\alpha r))^2} + \frac{\ell(\ell + 1)}{r^2}.
\]

that is identified with the shifted Morse Potential or Deng Fan potential with centrifugal term [12]. In consequence, the corresponding Schrödinger equation has the eigenvalues

\[
E_{n,\ell} = -\frac{\alpha^2}{4} \left( n + \frac{h_\ell + 1}{2} - \frac{bD(b + 2)}{\alpha^2(n + (h_\ell + 1)/2)} \right)^2,
\]
where $h_\ell = \sqrt{1 + 4\ell(\ell + 1) + D(2b/\alpha)^2}$, accordingly with Hamzavi et al. [12] when using their $\delta_\ell = (h_\ell + 1)/2$ and $d_0 = 0$. Also, the wavefunctions are

$$\psi_{n,\ell}(r) = (\exp(-\alpha r))^{(c_\ell - 1)/2}(1 - \exp(-\alpha r))^{(h_\ell + 1)/2} _2F_1(-n, b_\ell; c_\ell; \exp(-\alpha r)),$$

where

$$b_\ell = \frac{(h_\ell + 1)(n + h_\ell) + (4bD)/\alpha^2 - 2\ell(\ell + 1)}{2n + h_\ell + 1}, \quad c_\ell = \frac{-2n(h_\ell + n) + (4bD)/\alpha^2 - 2\ell(\ell + 1)}{2n + h_\ell + 1}.$$

(29)

By space restrictions, other particular cases derived from our proposal will be given elsewhere along with other approximations for the centrifugal term.

**Concluding remarks**

This work has been devoted to give a unified treatment of the $\ell-$ state solutions of exponential-type potentials. Our proposal is based on a class of multiparameter exponential-type potential obtained from a canonical transformation method applied to the hypergeometric DE. The effective potential that we are proposing, accepts different approximations for the centrifugal term. However, we have only considered the Greene and Aldrich approximation and the usefulness of the proposal has been exemplified by selecting the parameters $A$, $B$ and $C$ in a such a way that one leads to the Hulthen, Eckart, Manning-Rosen and Deng Fan potentials as particular cases. That is, an important feature of our work is that instead to solve the Schrödinger equation for an specific exponential potential, by resorting specialized solution methods of DE, each single case is directly obtained from our proposal by selecting properly the involved parameters that is an advantage when compared with procedures developed with the same purpose.

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