Antiphase versus in-phase synchronization of coupled pendulum clocks and metronomes

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In 1665, Huygens observed that two pendulum clocks hanging from the same board became synchronized in antiphase after hundreds of swings. On the other hand, modern experiments with metronomes placed on a movable platform show that they tend to synchronize in phase, not antiphase. Here, using a simple model of coupled clocks and metronomes, we calculate the regimes where antiphase and in-phase synchronization are stable. Unusual features of our approach include its treatment of the escapement mechanism, a small-angle approximation up to cubic order, and a three-time scale asymptotic analysis.

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There is still no fully satisfactory explanation for the “sympathy of clocks” that Huygens discovered more than 350 years ago. Here we explore the roles played by the escapement mechanism and a pendulum’s amplitude-dependent frequency, an otherwise well-known effect whose importance in this context has been surprisingly overlooked. We show it explains why coupled pendulum clocks typically synchronize in antiphase but coupled metronomes synchronize in phase. Given the historical significance of Huygens’s work and the pervasiveness of synchronization in nature and technology, we hope our work will bring the nonlinear science community closer to solving the ancient riddle of the sympathy of clocks.

I. INTRODUCTION

Synchronization occurs in diverse physical, biological, and chemical systems, from the coordinated beating of heart cells to the coherent voltage oscillations of Josephson junction arrays.1–4 Historically, the study of synchronization began with Huygens’s discovery of the “sympathy of clocks,” an effect he described as “marvelous”5,6. While confined to his room with an ailment, Huygens noticed that two of his pendulum clocks were synchronized. Suspecting that they must be coupled somehow, perhaps through vibrations in their common support, Huygens did a series of experiments to test the idea.

In one experiment, he attached two clocks to a board suspended on the backs of two chairs (Fig. 1) and noticed, to his amazement, that no matter how he started the clocks, within about thirty minutes their pendulums always synchronized in antiphase, repeatedly swinging toward each other and then apart.

Modern-day versions of a similar effect with metronomes have attracted millions of views on YouTube. In these experiments, following the work of Pantaleone6, anywhere from two to 32 metronomes are placed on a light platform that is free to move sideways, typically by rolling on empty soda cans or other light cylinders (Fig. 2). As with Huygens’s clocks, synchronization gradually occurs after several minutes. But in this case, the mode of synchronization is in phase rather than antiphase.

In this article, we revisit these problems in hopes of shedding new light on the conditions that favor one form of synchronization over another. Notable features of our approach are: (1) the attention given to modeling the escapement mechanism, (2) a small-angle approximation expanded past the linear term, and (3) a scaling of the model equations that disentangles different physical effects through a two- and a three-time scale asymptotic analysis. While each of these ingredi-
ents can be found in the literature, this is the first time that all three have been considered simultaneously. Our results reveal the importance of a nonlinear effect—the dependence of a pendulum’s frequency on its amplitude—that has been neglected in previous studies. In our model, this nonlinear effect selects for antiphase synchronization, in-phase synchronization, or the bistability of both.

II. THE ESCAPEMENT MECHANISM

We begin by describing the mechanics of the escape mechanism. The left panel of Fig. 3 shows the components of a so-called deadbeat anchor escapement in clocks (the escapements for metronomes work similarly, except their energy source is a spring that unwinds instead of a weight that descends). The main components are the axle, the escapement wheel, and the weight. The escapement wheel is a gear with teeth. The axle extends in the direction perpendicular to the page and goes through the center of the escapement wheel. The escapement wheel and the axle rotate together. The weight provides energy to the system; it hangs from a cord wound around the axle, and as it descends it applies a torque to the axle to turn the axle-escapement wheel system in the clockwise direction.

The right panel of Fig. 3 shows a pendulum rigidly attached to an anchor. The sides of the anchor are known as pallets. The pendulum, anchor, and pallets all oscillate together about their common pivot, as shown in Fig. 4, which in turn causes the teeth of the wheel to interact with the pallets. Whenever a tooth strikes a pallet, it does so without recoil; this is where the “dead” in “deadbeat” comes from. Moreover, a tooth in contact with a pallet slides along the pallet face without applying torque to the system. Torque is applied only when a tooth reaches the end of a pallet. Note that the right and left ends of the pallets are differently shaped, as shown in the right panel of Fig. 3; this shape difference is crucial to obtain the desired clock dynamics (but, for visual clarity, those shape differences are suppressed in Fig. 4).

To see how energy is transferred from the escapement to the pendulum, consider four key moments in a swing cycle (Fig. 4). At time $t_1$, the pendulum is swinging counterclockwise, and the green tooth (located near 1 o’clock on the escapement wheel) is contacting the end of the right pallet, thereby applying a force on it perpendicular to the pallet’s end (this is where the end shape of matters). Because the force points in the direction of the blue arrow shown in Fig. 4, the anchor-pendulum system experiences an impulse that increases its kinetic energy.

Once the green tooth is no longer in contact with the right pallet, the escapement wheel accelerates counterclockwise due to the torque caused by the weight. Then the escapement wheel stops abruptly when the pink tooth (located near 11 o’clock on the wheel) meets the left pallet. Meanwhile, the anchor-pendulum system continues turning counterclockwise. At time $t_2$ in Fig. 4 the pendulum makes its largest angle with the vertical. While the pink tooth is in contact with the pallet face, the tooth applies a force that points toward the pivot of the anchor-pendulum system (because the pallet face is a circular arc at a constant radial distance from the pivot). Hence this force does not apply any torque to the anchor-pendulum system with respect to the pivot, and so the dynamics of the anchor-pendulum system is not affected when the tooth is in contact with the pallet face.

Similar events occur in the next half of the cycle, with times $t_3$ and $t_4$ playing the parts of $t_1$ and $t_2$. Energy is pumped into the pendulum at time $t_3$, and only then.

The self-sustained oscillations of the pendulum continue until the cord that holds the weight is no longer wound around the axle of the escapement wheel. The periodic input of energy that the anchor-pendulum system receives from the escapement wheel-weight system makes up for the energy lost due to damping.
III. MODEL OF TWO COUPLED CLOCKS

In the cartoon shown in Fig. 2 both \( \hat{\theta}_1 \) and \( \hat{\theta}_2 \) are functions of time \( t \). We use primes to denote derivatives with respect to \( t \). The position of the center of mass of the platform is denoted \( \bar{x} \), where \( e \) is the constant unit dimensionless vector parallel to the platform in the counterclockwise direction when below the pivots.

To model the action of the escapement on the pendulum \( i \) (\( 1 \leq i \leq 2 \)), we assume there is a constant impulse \( J \) and a critical angle \( \theta_c \) such that pendulum \( i \) receives a positive impulse \( J \) whenever \( \hat{\theta}_i = \hat{\theta}_c \) and \( \hat{\theta}'_i > 0 \). Such an impulse occurs at time \( \bar{t}_1 \) in Fig. 4. Similarly, a negative impulse \( -J \) is received whenever \( \hat{\theta}_i = -\hat{\theta}_c \) and \( \hat{\theta}'_i < 0 \) (as at time \( \bar{t}_3 \) in Fig. 4). Let \( \{ \bar{t}_i \} \) be the set of times when pendulum \( i \) receives a positive impulse, and let \( \{ \bar{T}_i \} \) be the set of times when it receives a negative impulse. We define

\[
\bar{f}_1(t) = \sum_{\bar{t}_i \in \bar{T}_i} J \delta(t - \bar{t}_i) - \sum_{\bar{t}_i \in \bar{T}_i} J \delta(t - \bar{t}_i)
\]

and

\[
\bar{f}_2(t) = \sum_{\bar{t}_i \in \bar{T}_i} J \delta(t - \bar{t}_i) - \sum_{\bar{t}_i \in \bar{T}_i} J \delta(t - \bar{t}_i),
\]

where \( \delta \) is the delta function.

To complete the model, let \( m \) be the mass of each pendulum; \( M \) is the combined mass of both metronomes or clocks, including their pendulums, and the platform; \( L \) is the length of each pendulum, namely the distance from the pivot to the center of mass of the pendulum; \( \nu \) is a damping constant; and \( g \) is the acceleration due to gravity. For simplicity, we neglect the mass of the escapement wheels. Then, Newton’s second law yields

\[
ml\bar{\theta}_1'' = -mg \sin \bar{\theta}_1 - \nu l \bar{\theta}_1' - mx \cos \bar{\theta}_1 + \bar{f}_1
\]

\[
ml\bar{\theta}_2'' = -mg \sin \bar{\theta}_2 - \nu l \bar{\theta}_2' - mx \cos \bar{\theta}_2 + \bar{f}_2
\]

\[
\frac{M}{ml} \bar{x}'' = -\bar{x}'' \cos \bar{\theta}_1 + \bar{\theta}_1'^2 \sin \bar{\theta}_1 - \bar{x}'' \cos \bar{\theta}_2 + \bar{\theta}_2'^2 \sin \bar{\theta}_2.
\]

The first and second equations are obtained by taking torques about the pivots of the corresponding pendulum and dividing by \( L \). The third equation reflects the assumption that the wheels have no mass, which implies that the center of mass of the whole system experiences no acceleration in the direction parallel to the platform.

IV. CHARACTERISTIC SCALES AND NONDIMENSIONALIZATION

Since synchronization takes place after hundreds of swings, the relevant physics occurs at two different time scales. We introduce a small parameter \( \varepsilon \ll 1 \) that encodes the separation of these time scales. We will choose the rest of the dimensionless parameters and variables so that the different physical effects take place on either the time scale of a single swing of a pendulum, or a much longer time scale given by \( 1/\varepsilon \) times the pendulum’s period.

Specifically, we scale the variables and parameters as follows. The natural choice for the dimensionless time is

\[
t = \bar{t} \sqrt{g/L}
\]

so that the periods of the pendulums are \( O(1) \) in \( t \). An \( O(1) \) phase adjustment of the pendulums, due to inertial forcing from the motion of the platform, occurs in long times of \( O(M/m) \) in \( t \); thus, we want \( M/m = O(1/\varepsilon) \), or equivalently, the mass ratio

\[
m/M = O(\varepsilon).
\]

This choice leads us to introduce a dimensionless parameter

\[
b = m/(\varepsilon M),
\]

assumed to be \( O(1) \). Physically, \( b \) quantifies how strongly the pendulums’ motion affects the platform’s motion, and vice versa. Indirectly, \( b \) also controls how much one pendulum couples to the other.

To scale the angle variables, note first that the \( \bar{\theta}_i \) are of the order \( \bar{\theta}_c \). To make the nonlinear equations of motion as tractable as possible, we want to use a small-angle approximation, but we also want to retain the leading effects of nonlinear terms. With these ideas in mind, note that \( \sin \bar{\theta}_i \approx \bar{\theta}_i + \varepsilon (\bar{\theta}_i^3) \), so the leading nonlinear effects take place in times of \( O(1/\bar{\theta}_c^3) \) in \( t \). Thus, we want \( 1/\bar{\theta}_c^3 = O(1/\varepsilon) \), which motivates the following scaling:

\[
\bar{\theta}_c = \bar{\theta}_c/\sqrt{\varepsilon}, \quad \bar{\theta}_c = \bar{\theta}_c/\sqrt{\varepsilon},
\]

where the dimensionless parameters \( r \) and \( \theta_0 \) are \( O(1) \).

To scale the remaining quantities in the model, we estimate that the position of the center of mass \( \bar{x} \) satisfies \( \bar{x} = O(L\bar{\theta}_m/M) \). Since \( \bar{\theta}_i = O(\sqrt{\varepsilon}) \) and \( m/M = O(\varepsilon) \), we introduce

\[
x = \bar{x}/(L\varepsilon \sqrt{\varepsilon}),
\]

so that \( x \) is \( O(1) \). The damping due to friction takes place in times of \( O((m/\nu \sqrt{g/L})^2) \) in \( t \). Since we want this quantity to be \( O(1/\varepsilon) \), we introduce the \( O(1) \) dimensionless parameter

\[
v = (\nu/\varepsilon)^2 \sqrt{\varepsilon}/g.
\]

The impulse \( J \) causes an increase in the amplitude of oscillations of \( O(J/(m\sqrt{gLr})) \) in the dimensionless variables \( \theta_i \). We want this quantity to be \( O(\varepsilon) \) so the cumulative effects of the impulses take place in times of \( O(1/\varepsilon) \) in \( t \). Therefore we define

\[
J = \varepsilon^{-3/2} J/(m\sqrt{gLr}),
\]

and assume it to be \( O(1) \).

Next, we nondimensionalize the governing equations. Let \( \{ \bar{T}_i \} \) and \( \{ \bar{T}_i \} \) be the set of dimensionless times \( t \) when pendulum \( i \) receives a positive or negative impulse, respectively. We define

\[
f_i(t) = \sum_{\bar{t}_x \in \bar{t}_x} \delta(t - \bar{t}_x) - \sum_{\bar{t}_x \in \bar{T}_x} \delta(t - \bar{t}_x)
\]

(1)
and
\[ f_2(t) = \sum_{i \in F_1} \delta(t - t_i) - \sum_{i \in F_2} \delta(t - t_i). \]  
(2)

Then, neglecting terms of \( O(\varepsilon^2) \) in the first two equations and terms of \( O(\varepsilon) \) in the third equation, we find that the equations of motion become
\[ \ddot{\theta}_1 + \theta_1 = \varepsilon^2 \theta_1^3 - \varepsilon \nu \dot{\theta}_1 + \varepsilon J f_1 - \varepsilon \ddot{x} \]
\[ \ddot{\theta}_2 + \theta_2 = \varepsilon^2 \theta_2^3 - \varepsilon \nu \dot{\theta}_2 + \varepsilon J f_2 - \varepsilon \ddot{x} \]
\[ \ddot{x} = -b (\dot{\theta}_1 + \dot{\theta}_2), \]
where dots denote derivatives with respect to \( t \).

The upshot is that our choice of scaling has converted the first two equations into undamped linear oscillators perturbed by various forces of size \( O(\varepsilon) \). Although these forces are small, their effects accumulate on a long time scale of \( O(1/\varepsilon) \).

The cubic terms multiplied by \( \varepsilon r \) turn out to be especially important. As we will see in subsequent sections, the size of \( r \) determines whether the coupled system will ultimately synchronize in phase or in antiphase, or whether both of those states are locally stable. From a physical standpoint, varying \( r \) corresponds to varying the critical angle \( \bar{\theta} \) and the impulse \( J \) in the same proportion while keeping all the other dimensional parameters constant.

V. NUMERICAL SIMULATIONS

We simulated the system of equations (3) for several values of the parameters and initial conditions. In all our simulations, the pendulums synchronized either in antiphase or in phase.

As a first example, consider Fig. 5. In this simulation we started the two pendulums nearly in phase (top panel) and they continued to swing in that mode for a while. But after very long times, on the order of hundreds of swing cycles, the pendulums settled into a state of antiphase synchronization (bottom panel), much like what one sees in experiments on coupled metronomes. Note that the same parameter values were used as in Fig. 5, except for the value of \( r \). As we will see in the next section, this parameter \( r \) plays a decisive role in determining whether the long-term mode of synchronization is in phase or antiphase.

VI. ASYMPTOTIC ANALYSIS

As discussed in previous sections, we are assuming the mass ratio \( m/M \) is of the same order as a small parameter \( \varepsilon \ll 1 \). With suitable scaling of the other physical parameters, the dynamics then take place on two time scales, one of \( O(1) \) and the other of \( O(1/\varepsilon) \) in \( t \). Specifically, the pendulum angles \( \theta_1, \theta_2 \) and the dimensionless location \( x \) of the system’s center of mass are oscillatory variables with periods of \( O(1) \) in \( t \), but their amplitude and phase differences change by \( O(1) \) on time scales of \( O(1/\varepsilon) \) in \( t \). Thus, we make the following ansatz:
\[ \theta_i(t) \sim \theta_{i0}(t, \tau) + \varepsilon \theta_{i1}(t, \tau) + \cdots, \quad i = 1, 2 \]
\[ x(t) \sim x_0(t, \tau) + \varepsilon x_1(t, \tau) + \cdots \]
(4)
where \( \sim \) means asymptotic approximation in the parameter regime \( \varepsilon \ll 1 \), and \( \tau = \varepsilon t \) is a slow time variable. Each \( \theta_j \) and
\( x_t \) are functions of \( t \) and \( \tau \), and these functions are periodic in their first argument \( t \).

We carry out a standard two-time scale analysis. Namely, we plug the ansatz (4) into the system of equations (3), then replace

\[
\frac{d}{dt} \text{ by } \frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial \tau}
\]

in that system (this is because of the form of the ansatz (4)), and finally collect terms having like powers of \( \varepsilon \).

From the terms that contain the power \( \varepsilon^0 \) in the expansion of (3), we obtain

\[
\frac{\partial^2 \theta_{10}}{\partial \tau^2}(t, \tau) + \theta_{10}(t, \tau) = 0
\]

\[
\frac{\partial^2 \theta_{20}}{\partial \tau^2}(t, \tau) + \theta_{20}(t, \tau) = 0
\]

\[
\frac{\partial^2 x_0}{\partial \tau^2}(t, \tau) + b \left( \frac{\partial^2 \theta_{10}}{\partial t^2}(t, \tau) + \frac{\partial^2 \theta_{20}}{\partial t^2}(t, \tau) \right) = 0.
\]

The general solution of these equations (recalling that \( x_0 \) is periodic in \( t \)) is

\[
\theta_{10}(t, \tau) = A_1(\tau) \sin(t + \varphi_1(\tau))
\]

\[
\theta_{20}(t, \tau) = A_2(\tau) \sin(t + \varphi_2(\tau))
\]

and

\( x_0 = -b(\theta_{10} + \theta_{20}) \).

As usual, differential equations for the evolution of the slow variables \( A_1, A_2, \varphi_1, \varphi_2 \) will be obtained at the next order of \( \varepsilon \). But before we proceed to that order, we need to deal with an unusual feature of our model system (3): it contains delta-function forcing terms due to the repeated impulses provided by the escape mechanism. Now that we have an asymptotic approximation for the fast oscillations of the pendulums, we can find the times when the escape acts; by solving for these times and inserting them into the delta functions, we get the following asymptotic approximations for the impulsive forcing terms \( f_1 \) and \( f_2 \) in Eqs. (1) and (2):

\[
f_1(t) \sim f_{10}(t, \tau) \quad \text{and} \quad f_2(t) \sim f_{20}(t, \tau),
\]

where

\[
f_{10}(t, \tau) = \sum_{n \in \mathbb{Z}} \delta \left( t - \arcsin \left( \frac{\theta_c}{A_1(\tau)} + \varphi_1(\tau) + 2n\pi \right) \right)
\]

\[- \sum_{n \in \mathbb{Z}} \delta \left( t - \arcsin \left( \frac{\theta_c}{A_1(\tau)} + \varphi_1(\tau) + (2n + 1)\pi \right) \right)
\]

and

\[
f_{20}(t, \tau) = \sum_{n \in \mathbb{Z}} \delta \left( t - \arcsin \left( \frac{\theta_c}{A_2(\tau)} + \varphi_2(\tau) + 2n\pi \right) \right)
\]

\[- \sum_{n \in \mathbb{Z}} \delta \left( t - \arcsin \left( \frac{\theta_c}{A_2(\tau)} + \varphi_2(\tau) + (2n + 1)\pi \right) \right).
\]

Now proceeding to the \( O(\varepsilon^1) \) terms in the expansion of the system (3), we find that its first two equations give

\[
\frac{\partial^2 \theta_1}{\partial t^2} + \theta_1 = \frac{r}{6} \theta_0 - \nu \frac{\partial \theta_0}{\partial t} + J f_1 - \frac{\partial^2 x_0}{\partial t^2} - 2 \frac{\partial^2 \theta_0}{\partial \tau^2}
\]

\[
\frac{\partial^2 \theta_2}{\partial t^2} + \theta_2 = \frac{r}{6} \theta_0 - \nu \frac{\partial \theta_0}{\partial t} + J f_2 - \frac{\partial^2 x_0}{\partial t^2} - 2 \frac{\partial^2 \theta_0}{\partial \tau^2},
\]

where we have not explicitly displayed that the arguments of each variable are \((t, \tau)\).

Next, to derive the slow time equations for \( A_1, A_2, \varphi_1, \varphi_2 \), recall an elementary fact from the solvability theory of differential equations: Let \( h(t) \) be a 2\( \pi \)-periodic function of \( t \). Let \( \varphi \) be any fixed real number. The equation \( \dot{\theta} + \theta = h \) has a 2\( \pi \)-periodic solution \( \theta \) if and only if \( \int_0^{2\pi} h(t) \sin(t + \varphi) dt = 0 \) and \( \int_0^{2\pi} h(t) \cos(t + \varphi) dt = 0 \). This fact is usually stated with \( \varphi = 0 \), but in our analysis it will be convenient to use \( \varphi = \varphi_1 \) and \( \varphi = \varphi_2 \).

We can now go back to Equations (9), recall that \( \theta_{ij} \) are 2\( \pi \)-periodic in \( t \), and use the fact stated in the last paragraph to conclude that

\[
\int_0^{2\pi} \left( \frac{r}{6} \theta_0 - \nu \frac{\partial \theta_0}{\partial t} + J f_1 - \frac{\partial^2 x_0}{\partial t^2} - 2 \frac{\partial^2 \theta_0}{\partial \tau^2} \right) \sin(t + \varphi_1) dt = 0,
\]

\[
\int_0^{2\pi} \left( \frac{r}{6} \theta_0 - \nu \frac{\partial \theta_0}{\partial t} + J f_2 - \frac{\partial^2 x_0}{\partial t^2} - 2 \frac{\partial^2 \theta_0}{\partial \tau^2} \right) \cos(t + \varphi_1) dt = 0,
\]

and

\[
\int_0^{2\pi} \left( \frac{r}{6} \theta_0 - \nu \frac{\partial \theta_0}{\partial t} + J f_1 - \frac{\partial^2 x_0}{\partial t^2} - 2 \frac{\partial^2 \theta_0}{\partial \tau^2} \right) \sin(t + \varphi_2) dt = 0,
\]

\[
\int_0^{2\pi} \left( \frac{r}{6} \theta_0 - \nu \frac{\partial \theta_0}{\partial t} + J f_2 - \frac{\partial^2 x_0}{\partial t^2} - 2 \frac{\partial^2 \theta_0}{\partial \tau^2} \right) \cos(t + \varphi_2) dt = 0.
\]

By computing these four integrals (and omitting the algebraic details, which are long but straightforward), we obtain the following slow time equations:

\[
\frac{dA_1}{dt} = -\frac{\nu}{2} A_1 + \sqrt{1 - \frac{\theta_c^2}{A_1^2}} + \frac{b}{2} \sin(\varphi_1 - \varphi_2) A_2 (10)
\]

\[
A_1 \frac{d\varphi_1}{dt} = b A_2 - \theta_c A_1 \frac{\theta_c}{A_1} + b \cos(\varphi_1 - \varphi_2) A_2 - \frac{r}{16} A_3 (11)
\]

\[
\frac{dA_2}{dt} = -\frac{\nu}{2} A_2 + \sqrt{1 - \frac{\theta_c^2}{A_2^2}} + \frac{b}{2} \sin(\varphi_2 - \varphi_1) A_1 (12)
\]

\[
A_2 \frac{d\varphi_2}{dt} = b A_2 - \theta_c A_2 \frac{\theta_c}{A_2} + b \cos(\varphi_2 - \varphi_1) A_1 - \frac{r}{16} A_3 (13)
\]

This system holds for \( A_1(\tau) > \theta_c \) and \( A_2(\tau) > \theta_c \), meaning that the pendulums’ swings are large enough to engage the escape mechanism at all times.
Since we our goal is to identify whether the system evolves to antiphase or in-phase synchronization or no synchronization at all, the variable of interest to us is the phase difference \( \Psi = \phi_1 - \phi_2 \).

Dividing Eq. (11) by \( A_1 \), dividing Eq. (13) by \( A_2 \), and subtracting the results, we obtain

\[
\frac{d\Psi}{d\tau} = \frac{\theta}{\pi} \left( A_2^2 - A_1^2 \right) + \frac{r}{16} \left( A_2^2 - A_1^2 \right) + \frac{b}{2} \left( \frac{A_2}{A_1} - \frac{A_1}{A_2} \right) \cos \Psi.
\]

We rewrite Equations (10) and (12) as

\[
\frac{dA_1}{d\tau} = -\nu A_1 + \sqrt{1 - \frac{\theta^2}{A_1^2}} + \frac{b}{2} A_2 \sin \Psi
\]

\[
\frac{dA_2}{d\tau} = -\nu A_2 + \sqrt{1 - \frac{\theta^2}{A_2^2}} - \frac{b}{2} A_1 \sin \Psi.
\]

From now on we will focus on the analysis of the system (14), (15) and (16).

**VII. STABILITY ANALYSIS OF IN-PHASE AND ANTIPHASE SYNCHRONIZATION**

The system (14), (15) and (16) has four fixed points. More precisely, for \( \sigma = 1 \) and \( \sigma = -1 \), we define

\[
\alpha = \left( \frac{\pi \theta \nu}{J} \right)^2, \quad \Lambda^{(\sigma)} = \sqrt{2} \frac{J}{\pi \nu} \sqrt{1 + \sigma \sqrt{1 - \alpha}}.
\]

One fixed point is \( A_1 = A_2 = A_c^{(-1)} \) and \( \Psi = 0 \). A second fixed point is \( A_1 = A_2 = A_c^{(+1)} \) and \( \Psi = \pi \). Both of these fixed points turn out to be unstable, so we ignore them in what follows. To ease the notation, let \( A_c = A_c^{(-1)} \). A third fixed point is \( A_1 = A_2 = A_c \) and \( \Psi = 0 \), which corresponds to in-phase synchronization. The fourth fixed point is \( A_1 = A_2 = A_c \) and \( \Psi = \pi \). It represents antiphase synchronization.

The eigenvalues of the Jacobian matrix associated with the system (14), (15), and (16) can be found explicitly at these synchronized states, and thereby provide information about their stability. The calculations involved are tedious but standard, so we omit the details and summarize the results. Let

\[
r_1 = \frac{\nu \beta^2 \theta^2}{4\sqrt{\alpha}} - \beta b
\]

\[
r_2 = \frac{\nu \beta^2 \theta^2}{4\sqrt{\alpha}} + \beta b
\]

where

\[
\beta = \frac{4\alpha}{(1 + \sqrt{1 - \alpha}) \theta^2}.
\]

We find that in-phase synchronization is unstable for \( r < r_1 \) and stable for \( r > r_1 \), whereas antiphase synchronization is unstable for \( r > r_2 \) and stable for \( r < r_2 \).

Figure 7 shows a bifurcation diagram in the parameters \( r \) and \( b \). The values of the other parameters are: \( \theta_c = \sqrt{1/2} \), \( \nu = 1 \), and \( J = \pi \). The straight lines are \( r_1 \) and \( r_2 \) as indicated in the text. The other curves were generated with the numerical bifurcation program MATCONT.28

![Bifurcation diagram](image-url)

**FIG. 7. Bifurcation diagram in the parameters \( b \) and \( r \), when the other parameters are fixed at the values \( \theta_c = \sqrt{1/2} \), \( \nu = 1 \), and \( J = \pi \). The straight lines are \( r_1 \) and \( r_2 \). Other curves were generated with the numerical bifurcation program MATCONT.**

Figure 7 shows a bifurcation diagram in the parameters \( r \) and \( b \). The values of the other parameters are: \( \theta_c = \sqrt{1/2} \), \( \nu = 1 \), and \( J = \pi \). The straight lines are \( r_1 \) and \( r_2 \). The other curves in the diagram were generated with the numerical bifurcation program MATCONT.

Fix \( b \) to be small, say \( b = 0.1 \). Then for small values of \( r \), only antiphase oscillations are stable; for intermediate values of \( r \), both forms of synchronization are stable; and for large values of \( r \), only in-phase oscillations are stable.

To interpret these results physically, recall that \( r \) is a dimensionless measure of the pendulum’s nonlinearity, which can become important when the oscillations are small but not too small. Indeed, \( r \) arose when we scaled the size of the critical angle at which the escapement engages and impulses are imparted. Our analysis shows that the nonlinear effects captured by \( r \) are not negligible perturbations; they completely change the picture. We would not see a transition from antiphase to in-phase synchronization without them.

We have also seen that \( r \) reflects the dependence of a pendulum’s frequency on its amplitude, an effect that becomes increasingly important at large amplitudes. In short, as the amplitudes increase, antiphase synchronization loses stability in favor of in-phase synchronization. This finding may shed some light on why metronomes tend to synchronize in phase: they have a larger critical angle and typically swing at much larger amplitudes than the pendulums in pendulum clocks.

For larger fixed values of the coupling constant \( b \), and for larger values of \( r \), Fig. 7 shows a more complicated scenario. In particular, for \( r \) increasing from small values, the stable fixed points corresponding to antiphase oscillations branch into two stable equilibria with a phase difference \( \Psi \) that is nearly, but not exactly, equal to \( \pi \); meanwhile, the exactly antiphase oscillations become unstable. For slightly larger val-
ues of \( r \), the two stable, nearly antiphase oscillations bifurcate into two limit cycles in a Hopf bifurcation. Finally, at even larger values of \( r \), the limit cycles lose their stability and only in-phase synchrony is stable. We were able to find the nearly antiphase equilibria and the stable limit cycles in numerical simulations of the original nondimensional equations.

VIII. THREE-TIME SCALE ANALYSIS

In the parameter regime \( b \ll 1 \), the coupling effect from the motion of the platform becomes so weak that it takes place on a super-long time scale of \( t = O(e^{-b^{-1}}) \). This extra separation of time scales allow us to simplify the system of equations (14), (15) and (16) even further as explained next.

If we were to set \( b = 0 \) in the equations (15) and (16) for the amplitudes, we would find that both \( A_1 \) and \( A_2 \) would approach \( A_c \) as the time \( \tau \) increases. This suggests the ansatz

\[
A_1(\tau) \sim A_c + ba_1(\tau, b\tau) + \ldots
\]

\[
A_2(\tau) \sim A_c + ba_2(\tau, b\tau) + \ldots
\]  

(17)

in the parameter regime \( b \ll 1 \), where \( a_1 \) and \( a_2 \) are functions of two variables that we call \( \tau \) and \( s \), i.e., \( a_1 = a_1(\tau, s) \) and \( a_2 = a_2(\tau, s) \), where \( s = b\tau \).

If we were to plug the ansatz of Equations (17) into the equation (14) for the phase difference \( \psi \), we would obtain that the right hand side of that equation is of \( O(b) \). This observation suggests the following ansatz for \( \psi \):

\[
\psi \sim \psi_0(b\tau) + b\psi_1(\tau, b\tau) + \ldots,
\]  

(18)

where \( \psi_0 \) is a function of only one variable, \( \psi_0 = \psi_0(s) \) but \( \psi_1 \) is a function of two variables, \( \psi_1 = \psi_1(\tau, s) \).

We again carry out a two time scale analysis. More precisely, in Eqs. (14), (15) and (16) we replace \( \frac{d}{d\tau} \) by \( \frac{\partial}{\partial \tau} + b \frac{\partial}{\partial s} \),

(19)

plug the ansatz (17) and (18), and expand in powers of \( b \). We find that at first order in \( b \), Eqs. (15) and (16) reduce to:

\[
\frac{da_1}{d\tau} = \left( -\frac{v}{2} + \frac{J\theta^2}{\pi A_c^3} \frac{1}{\sqrt{1 - \theta^2 / \lambda^2}} \right) a_1 + \frac{A_c}{2} \sin \psi_0
\]  

(20)

\[
\frac{da_2}{d\tau} = \left( \frac{v}{2} + \frac{J\theta^2}{\pi A_c^3} \frac{1}{\sqrt{1 - \theta^2 / \lambda^2}} \right) a_2 - \frac{A_c}{2} \sin \psi_0
\]  

(21)

Simple but tedious algebra (that we do not present here) reveals that the quantity in large parentheses above is negative. This result implies that both \( a_1 \) and \( a_2 \) tend to constants on the time scale of \( \tau \). Since we are interested in much longer time scales (on the super-slow time scale of \( \tau/b \)) we arrive at the following result for the corrections terms to the amplitudes:

\[
a_1 \sim -\left( -\frac{v}{2} + \frac{J\theta^2}{\pi A_c^3} \frac{1}{\sqrt{1 - \theta^2 / \lambda^2}} \right)^{-1} \frac{A_c}{2} \sin \psi_0
\]  

(22)

\[
a_2 \sim \left( \frac{v}{2} + \frac{J\theta^2}{\pi A_c^3} \frac{1}{\sqrt{1 - \theta^2 / \lambda^2}} \right)^{-1} \frac{A_c}{2} \sin \psi_0.
\]  

(23)

We now plug the ansatz (17) and (18) with \( a_1 \) and \( a_2 \) given by the above formula into Eq. (13) to conclude, after simple arguments and more algebra not presented here, that \( \psi_0 \) evolves according to

\[
\frac{d\psi_0}{ds} = \gamma \left( \frac{v\beta^2\theta^2}{4\sqrt{\alpha}} - r \right) \sin \psi_0,
\]  

(24)

where the constant prefactor \( \gamma \) is given by

\[
\gamma = \frac{1 + \sqrt{1 - \alpha^2}}{2\alpha} \left( 2 - 2\alpha + 2\sqrt{1 - \alpha} - \alpha \right),
\]

Equation (24) shows that the antiphase mode \( \psi_0 = \pi \) is stable for \( 0 \leq r < \frac{v\beta^2\theta^2}{4\sqrt{\alpha}} \) and the in-phase mode \( \psi_0 = 0 \) is stable for \( r > \frac{v\beta^2\theta^2}{4\sqrt{\alpha}} \). This finding agrees with our previous analysis. In fact, \( \frac{v\beta^2\theta^2}{4\sqrt{\alpha}} \) is the intercept where the straight-line stability boundaries meet the \( r \)-axis in Fig. 7.

IX. DISCUSSION

We have modeled the behavior of two coupled pendulums with deadbeat escapement mechanisms driving their motion. In our analysis of this system, we focused on a parameter regime that is both physically realistic and analytically tractable: a weak-coupling regime in which the ratio of a pendulum’s mass to the mass of the entire system is assumed to be small. In this regime, phase adjustments of the pendulums due to inertial forcing from the platform occur over long times relative to the period of the pendulums. By scaling other physical parameters appropriately, we were able to use a multiple time scale analysis to study “the sympathy of clocks” in a way that appears simpler than most in the existing literature. It allows us to delineate regions in the mass ratio and escapement impulse parameter space where only in-phase synchronization is stable, or where only antiphase synchronization is stable, or where both are stable.

Let us try to situate our work relative to the enormous body of earlier work on Huygens’s clocks and related problems about coupled metronomes. One of the unusual features of our approach is that we model the escapement mechanism by using discrete impulses in the form of \( \delta \)-functions. Other approaches have used different discontinuous functions to model the escapement[2,10,14,15,31] or continuous functions such as a
van der Pol term or some other continuous function which gives self-excitation in the system. Importantly, our impulses provide a “boost” to the pendulum before it reaches the apex of its swing rather than to push it back in the opposite direction. We believe that this model more faithfully reflects how deadbeat escapements actually work.

Further, when making the small-angle approximation, we expand \( \sin \theta \) past the linear term to include the cubic term. It is much more common to either take only the first order approximation to \( \sin \) so that the analysis is more straightforward, or to avoid a small-angle approximation altogether although this choice can cause the analysis to become unwieldy. We find that including the cubic term is crucial to the dynamics of our model; we would not see a transition from antiphase to in-phase without it.

The asymptotic analyses that we have presented are similar in some respects to others in the literature. However, while previous analyses have predominantly used the mass ratio as the small parameter, we consider both the mass ratio and a new small parameter related to the critical angle at which the escapement mechanism engages. As a result, we can clearly tease out the separate roles of the mass ratio, the size of the critical angle, and the size of the impulse. The latter two are the quantities (in least in our model) that select which mode of synchronization is favored: only in-phase, only antiphase, or the bistability of both.

Regions of bistability have been found in earlier analytical studies. Bistability can also occur in reality; although we are perhaps more accustomed to antiphase synchronization of clocks and in-phase synchronization of metronomes, experimental studies have demonstrated that both kinds of devices can display bistability in certain circumstances.

One of our main results is that the slight dependence of a pendulum’s frequency on its amplitude can play an outsized role in the long-term dynamics of coupled clocks and metronomes. Although well known for individual pendulums, this effect has not been emphasized in previous analyses of these coupled systems. Indeed, we suspect that the dynamics of Huygens’s clocks have resisted a complete analysis for more than 350 years, precisely because small effects like this can play such a pivotal role.

These cautionary words underscore the necessity of finding a model that captures the essential physics but remains tractable. In this spirit, we have focused on one small piece of the puzzle: the modeling of the escapement mechanism. But much remains to be done, especially in the modeling of the platform, where damping and restoring forces may also be qualitatively important.

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DATA AVAILABILITY

The data that support the findings of this study are almost all available within the article. Any data that are not available can be found from the corresponding author upon reasonable request.

1. A. T. Winfree, The Geometry of Biological Time (Springer-Verlag, 1980).
2. A. Pikovsky, M. Rosenblum, and J. Kurths, Synchronization: A Universal Concept in Nonlinear Sciences (Cambridge University Press, 2001).
3. S. H. Strogatz, Sync (Hyperion, 2003).
4. I. Blokhman, Synchronization in Science and Technology (American Society of Mechanical Engineers Press, 1988).
5. J. G. Yoder, Unrolling Time: Christiaan Huygens and the Mathematicalization of Nature (Cambridge University Press, 2004).
6. J. Pantaleone, “Synchronization of metronomes,” Am J Phys 70, 992–1000 (2002).
7. M. Bennett, M. F. Schatz, H. Rockwood, and K. Wiesenfeld, “Huygens’s clocks,” Proc R Soc A 458, 563–579 (2002).
8. M. Kumon, R. Washizaki, J. Sato, R. Mizumoto, and Z. Iwai, “Controlled synchronization of two 1-DOF coupled oscillators,” in Proceedings of the 15th IFAC World Congress, Barcelona (2002) pp. 3–10.
9. M. Senator, “Synchronization of two coupled escapement-driven pendulum clocks,” J Sound Vib 291, 566–603 (2006).
10. A. L. Fradkov and B. Andrievsky, “Synchronization and phase relations in the motion of two-pendulum system,” International Journal of Non-Linear Mechanics 42, 895–901 (2007).
11. V. V. Kuznetsov, G. A. Leonov, H. Nijmeijer, and A. Y. Pogromsky, “Synchronization of two metronomes,” in PSYCO (2007) pp. 49–52.
12. K. Czolczynski, P. Perlikowski, A. Stefanski, and T. Kapitaniak, “Clustering of Huygens’ clocks,” Prog Theo Phys 122, 1027–1033 (2009).
13. H. Ulrichs, A. Mann, and U. Parltitz, “Synchronization and chaotic dynamics of coupled mechanical metronomes,” Chaos 19, 043120 (2009).
14. R. Dilão, “Antiphase and in-phase synchronization of nonlinear oscillators: The Huygens’s clocks system,” Chaos 19, 023118 (2009).
15. K. Czolczynski, P. Perlikowski, A. Stefanski, and T. Kapitaniak, “Why two clocks synchronize: Energy balance of the synchronized clocks,” Chaos 21, 023129 (2011).
16. V. Jovanovic and S. Koshkin, “Synchronization of Huygens’ clocks and the Poincaré method,” J Sound Vib 331, 2887–2900 (2012).
17. Y. Wu, N. Wang, L. Li, and J. Xiao, “Anti-phase synchronization of two coupled mechanical metronomes,” Chaos 22, 023146 (2012).
18. K. Czolczynski, P. Perlikowski, A. Stefanski, and T. Kapitaniak, “Synchronization of the self-excited pendula suspended on the vertically displacing beam,” Communications in Nonlinear Science and Numerical Simulation 18, 386–400 (2013).
19. P. R. Ramirez, R. H. Fey, and H. Nijmeijer, “Synchronization of weakly nonlinear oscillators with Huygens’ coupling,” Chaos 23, 033118 (2013).
20. H. M. Oliveira and L. V. Melo, “Huygens synchronization of two clocks,” Scientific reports 5, 11548 (2015).
21. J. P. Ramirez, L. A. Olvera, H. Nijmeijer, and J. Alvarez, “The symmetry of two pendulum clocks: beyond Huygens’ observations,” Scientific reports 6, 23580 (2016).
22. A. B. Willms, P. M. Kitanov, and W. F. Langford, “Huygens’ clocks revisited,” Royal Society Open Science 4, 170777 (2017).
23. M. Kapitaniak, K. Czolczynski, P. Perlikowski, A. Stefanski, and T. Kapitaniak, “Synchronization of clocks,” Physics Reports 517, 1–69 (2012).
24. A. M. Lepschy, G. Mian, and U. Vario, “Feedback control in ancient water clocks using impulsive differential equations and poincare maps,” International Journal of Control 76, 1685–1698 (2003).
25. A. Rowlings, The Science of Clocks and Watches (Caldwell Industries, Luling, TX USA, 1944).
26. A. Dhooge, W. Govaerts, Y. A. Kuznetsov, H. G. E. Meijer, and B. Sautois, “New features of the software matcont for bifurcation analysis of dynamical
systems,” Mathematical and Computer Modelling of Dynamical Systems 14, 147–175 (2008).