WELL-POSEDNESS FOR THE CLASSICAL STEFAN PROBLEM AND THE ZERO SURFACE TENSION LIMIT

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Abstract. We develop a framework for a unified treatment of well-posedness for the Stefan problem with or without surface tension. In the absence of surface tension, we establish well-posedness in Sobolev spaces for the classical Stefan problem. We introduce a new velocity variable which extends the velocity of the moving free-boundary into the interior domain. The equation satisfied by this velocity is used for the analysis in place of the heat equation satisfied by the temperature. Solutions to the classical Stefan problem are then constructed as the limit of solutions to a carefully chosen sequence of approximations to the velocity equation, in which the moving free-boundary is regularized and the boundary condition is modified in a such a way as to preserve the basic nonlinear structure of the original problem. With our methodology, we simultaneously find the required stability condition for well-posedness and obtain new estimates for the regularity of the moving free-boundary. Finally, we prove that solutions of the Stefan problem with positive surface tension $\sigma$ converge to solutions of the classical Stefan problem as $\sigma \to 0$.

1. Introduction

1.1. The problem formulation. We consider the local well-posedness and boundary regularity of solutions to the classical one-phase Stefan problem, describing the evolving phase boundary of a freezing liquid. We also establish the limit of zero surface tension.

The temperature $p(t,x)$ of a liquid inside of a time-dependent domain $\Omega(t)$ and an a priori unknown moving boundary $\Gamma(t)$ satisfies the following system of equations:

\begin{align*}
& p_t - \Delta p = 0 \quad \text{in} \; \Omega(t), \tag{1.1a} \\
& \partial_n p = V_{\Gamma(t)} \quad \text{on} \; \Gamma(t), \tag{1.1b} \\
& p = \sigma \kappa_{\Gamma} \quad \text{on} \; \Gamma(t), \tag{1.1c} \\
& p(0,\cdot) = p_0, \; \Gamma(0) = \Gamma_0. \tag{1.1d}
\end{align*}

The domain $\Omega(t)$ is an evolving open subset of $\mathbb{R}^d$ with $d \geq 2$. The set $\Gamma(t)$ denotes the moving boundary (which may be a connected subset of $\partial \Omega(t)$ if a part of the boundary of $\Omega(t)$ is fixed). See Figure 1.

Figure 1. The one-phase Stefan problem. Displayed on the left side of the figure is the reference domain $\Omega$ and reference boundary $\Gamma$. The time-dependent domain $\Omega(t)$ and the moving free-boundary $\Gamma(t)$ is shown on the right side of the figure.
Equation (1.1a) expresses the fact that heat heat diffuses in the bulk $\Omega(t)$, while the boundary condition (1.1b) states that the heat flux across the boundary governs the boundary evolution; that is, $\partial_t p = \nabla p \cdot n$ is the normal derivative of $p$ on $\Gamma(t)$ where $n$ stands for the outward pointing unit normal, and $V_{\Gamma(t)}$ denotes the speed or the normal velocity of the hypersurface $\Gamma(t)$. In the case that $\sigma = 0$, (1.1c) is termed the classical Stefan condition and problem (1.1) is called the classical Stefan problem. In this case, freezing of the liquid occurs at a constant temperature $p = 0$. On the other hand, if $\sigma > 0$ in (1.1c) then the boundary condition is called the Gibbs-Thomson correction to the classical Stefan condition, and the system (1.1) is then termed the Stefan problem with surface tension, whereby $\sigma > 0$ is a given coefficient of surface tension and $\kappa_{\Gamma(t)}$ stands for the mean curvature of the moving boundary $\Gamma(t)$. Finally, we equip the problem with suitable initial conditions (1.1d): $p_0 : \Omega(0) \to \mathbb{R}$ and $\Gamma_0$ are the prescribed initial temperature and boundary, respectively.

Problem (1.1) is an example of free-boundary partial differential equation which requires the initial data to satisfy a stability condition in order to ensure well-posedness in Sobolev spaces; specifically, we shall require that

$$-\partial_n p_0 > 0 \quad \text{on } \Gamma(0),$$

which, by analogy to fluid dynamics, we shall refer to as Taylor sign condition or the Rayleigh-Taylor sign condition. Below, we will explain how this Taylor sign condition naturally appears from our analysis.

1.2. The reference domain $\Omega$ and the initial domain $\Omega_0$. We will begin the analysis with motion in $\mathbb{R}^2$, and then describe the minor modifications needed to study motion in $\mathbb{R}^3$. To simplify our presentation, we will parameterize our initial free-boundary $\Gamma_0$ as a graph over the one-dimensional torus $\mathbb{T}^1$ which we identify with $[0, 2\pi]$; we define

$$\Gamma_0 = \{ x \in \mathbb{T}^1 \times \mathbb{R}, \ x = (x', h_0(x')) \}, \quad h_0 \in H^4(\mathbb{T}^1).$$

Without loss of generality we shall further assume that $\Gamma_0$ is a small perturbation of the manifold $\mathbb{T}^1 \times \{ x^2 = 0 \}$ in the sense that

$$\|h_0\|_{H^4(\mathbb{T}^1)} \leq \epsilon_0 \ll 1,$$  \hspace{1cm} (1.3)

for some sufficiently small $\epsilon_0$. In Appendix A, we shall explain how to remove the assumption (1.3). The only reason for making this smallness assumption is that (1.3) and (1.2) allow us to to use one global Cartesian coordinate system (rather than a collection of local coordinate charts). This is ideal for describing new identities that provide very natural estimates for the second-fundamental form of the evolving free-boundary $\Gamma(t)$. All of our results apply to general domains; however, in a general setting, we must employ a finite covering of $\Omega$ by local coordinate charts, together with a partition-of-unity subordinate to that cover. In particular, the Stefan problem localizes to each chart and effectively reduces to the analysis on

$$\Omega = \mathbb{T}^1 \times (0, 1).$$

Again, we emphasize that the assumption (1.3) is not essential to our proof, and in Appendix A, we explain how to treat general $H^4$ initial geometries.

We define the initial domain

$$\Omega_0 = \{ (x_1, x_2) \in \mathbb{T}^1 \times \mathbb{R} \mid h_0(x^1) < x^2 < 1 \},$$

while the reference domain $\Omega = \mathbb{T}^1 \times (0, 1)$. The set

$$\Gamma = \mathbb{T}^1 \times \{ x^2 = 0 \}$$

is the reference boundary on which our parameterization $(x', h(t, x'))$ will be defined. The top boundary $\partial \Omega_{\text{top}} = \mathbb{T}^1 \times \{ x^2 = 1 \}$ is fixed in time, and

$$\partial_n p = 0 \quad \text{on } \partial \Omega_{\text{top}},$$  \hspace{1cm} (1.4)
1.3. Notation. For any \( s \geq 0 \) and given functions \( f : \Omega \to \mathbb{R}, \varphi : \Gamma \to \mathbb{R} \) we set
\[
\|f\|_s := \|f\|_{H^s(\Omega)}; \quad |\varphi|_s := \|\varphi\|_{H^s(\Gamma)}.
\]
In particular, when \( s \) is not an integer, the corresponding fractional Sobolev space is defined by interpolation in a standard way. If \( f : [0,T] \times \Omega \to \mathbb{R}, \varphi : [0,T] \times \Gamma \to \mathbb{R} \) are given time-dependent functions, then for any \( 1 \leq p \leq \infty \) we set
\[
\|f\|_{L^p(0,T;L^s(\Omega))} := \|f_x\|_{L^p(0,T;L^s(\Omega))}; \quad |\varphi|_{L^p(0,T;L^s(\Gamma))} := \|\varphi\|_{L^p(0,T;L^s(\Gamma))}.
\]
If \( i = 1,2 \) then \( f_{,i} := \partial_{x^i} f \) is the partial derivative of \( f \) with respect to \( x^i \) coordinate. Similarly, \( f_{,ij} := \partial_{x^i} \partial_{x^j} f \) and so on. When differentiating with respect to the time variable \( t \), we set \( f_t = f_{,t} = \partial_t f \).

For horizontal derivatives, we write
\[
\bar{\partial} f := f_{,1}, \quad \bar{\partial}^k f := \bar{\partial}^{k-1} f.
\]
We use \( C \) to denote a universal constant that may vary from line to line. In numerous estimates the sign \( \lesssim \) is used; by definition, \( X \lesssim Y \) if and only if there exists a universal constant \( C \) such that \( X \leq CY \). We use \( P \) to denote a generic real polynomial with positive coefficients that can similarly vary from line to line.

1.4. Fixing the domain. In order to obtain a priori estimates, and to facilitate the construction of solutions, we transform the Stefan problem to an equivalent problem on a fixed domain. To this end, we shall view \( \Gamma(t) \) as a graph over \( \mathbb{T}^3 \) given by the height function \( h(t,\cdot) : \Gamma \to \mathbb{R} \)
\[
\Gamma(t) := \Psi(t,\Gamma).
\]
In other words, the moving surface \( \Gamma(t) \) is parameterized as the graph of a signed height function \( h(t,x) \), so that \( \Gamma(t) = \{ x \in \mathbb{T}^3 \times \mathbb{R} \mid x = (x',h(x',t)) \} \). With this parameterization, the outward unit normal \( n(t,x') \) to \( \Gamma(t) \) at the point \( (x',h(x',t)) \) is given by
\[
n(t,x') = \frac{(\partial h,-1)}{\sqrt{1 + |\partial h|^2}}. \tag{1.5}
\]
Assuming that \( h(t,\cdot) \) is sufficiently regular and remains a graph, we can define a diffeomorphism \( \Psi(t,\cdot) : \Omega \to \Omega(t) \) as an harmonic extension of the boundary diffeomorphism \( (x',h) \), by solving the elliptic equation
\[
\Delta \Psi(t,\cdot) = 0 \quad \text{in} \; \Omega, \tag{1.6a}
\]
\[
\Psi(t,x',0) = (x',h(t,x')) \quad x' \in \Gamma, \tag{1.6b}
\]
\[
\Psi(t,\cdot) = \text{Id} \quad \text{on} \; \partial \Omega_{\text{top}}, \tag{1.6c}
\]
where \( \text{Id} \) denotes the identity map. The mapping \( \Psi(t,\cdot) \) is indeed a diffeomorphism; note that the map \( \Phi := \Psi - \text{Id} \) solves the problem
\[
\Delta \Phi(t,\cdot) = 0 \quad \text{in} \; \Omega, \tag{1.7a}
\]
\[
\Phi(t,x',0) = (0,h(t,x')) \quad x' \in \Gamma, \tag{1.7b}
\]
\[
\Phi(t,\cdot) = 0 \quad \text{on} \; \partial \Omega_{\text{top}}, \tag{1.7c}
\]
so that by elliptic estimates, we may conclude that \( \|\Psi(t,\cdot) - \text{Id}\|_{H^{s+0.5}(\Omega)} \lesssim \|h(t,\cdot)\|_{H^{s+0.5}(\Gamma)} \lesssim \epsilon_0 \) using the assumption (1.3), and the continuity of the map \( t \mapsto h(t,\cdot) \) in \( H^4(\Gamma) \) which will be proved below. By the inverse function theorem we infer that \( \Psi(t,\cdot) \) is a diffeomorphism.

As a consequence of (1.6),
\[
\|\Psi\|_{H^s(\Omega)} \leq C \|\Psi\|_{H^{s-0.5}(\Gamma)}, \tag{1.8}
\]
and thus \( \Psi(t,\cdot) \) gains a half-derivative of regularity in \( \Omega \) with respect to the height function \( h(t,\cdot) \) on \( \Gamma \).
1.5. **Reference unit normal, unit tangent, line element, and the Jacobian.** We let

\[ N = (0, -1), \quad T = (1, 0) \]

denote the outward pointing unit normal and tangent vectors to \( \Gamma = \mathbb{T}^3 \times \{ x_2 = 0 \} \), respectively. The time-dependent unit normal \( n(t, \cdot) \) and tangent \( \tau(t, \cdot) \) vectors to \( \Gamma(t) \) are given by

\[ n = Jg^{-1}A^TN, \quad \tau = Jg^{-1}A^TT, \quad (1.9) \]

where

\[ J(t, x) := \det \nabla \Psi(t, x), \quad x \in \Omega, \quad (1.10) \]
denotes the Jacobian determinant of \( \nabla \Psi \), and

\[ g(t, x) := \sqrt{1 + (\partial h(t, x))^2}, \quad x \in \mathbb{T}^1, \quad (1.11) \]

where \( g^2 dx \) is the line element associated with the metric induced on \( \Gamma \). Together with (1.9) we obtain the relationship

\[ A^2_s := J^{-1}(\partial h, -1), \quad |A^2_s| = J^{-1}g. \quad (1.12) \]

The vector \( A^2_s(t, \cdot) \) will play an important role in the derivation of energy identities as it is parallel to \( n(t, \cdot) \).

1.5.1. **The change of variables.** On the reference domain \( \Omega \), we set

\[ q := p \circ \Psi, \quad A := [\nabla \Psi]^{-1}, \quad w := \Psi_t, \quad v := -\nabla p \circ \Psi. \quad (1.13) \]

Note that, by the chain-rule, the relation \( v = -\nabla p \circ \Psi \) can be written as

\[ v^i + A^k_i q_{,k} = 0 \quad \text{in} \quad \Omega. \quad (1.14) \]

We also express \( p_t \circ \Psi \) in terms of \( q, v, w \). Again, by the chain rule, \( p_t = q_t \circ \Psi^{-1} + \nabla q \circ \Psi^{-1} \cdot \Psi_t^{-1} \). Since \( \Psi_t^{-1} = -A \circ \Psi^{-1} \Psi_t \circ \Psi^{-1} \) and \( w = \Psi_t \), using (1.14) we obtain that

\[ p_t \circ \Psi = q_t - q_{,k} A^k_i w^r = q_t + v \cdot w. \]

The transformed Laplacian \( \Delta_q q := \Delta p \circ \Psi \) is defined as

\[ \Delta_q q = A^j_i (A^k_i q_{,k})_{,j}, \quad (1.15) \]

and we define

\[ \nabla_q q := A^k_i q_{,k} = \nabla p \circ \Psi. \quad (1.16) \]

**Remark 1** (Differentiation rules). When differentiating the matrix \( A := [\nabla \Psi]^{-1} \), for a given \( i, k \in \{1, 2\} \),

\[ \partial_s A^k_i = -A^k_i w^r_{,s} A^r_i; \quad \bar{\partial} A^k_i = -A^k_i \bar{\partial} \Psi_{r,s} A^r_i. \]

In particular, a simple application of the above identities, together with the product rule, show that for any given \( a, b \in \mathbb{N} \):

\[
\bar{\partial}^m \partial_t^l A^k_i = -A^k_i \bar{\partial}^m \partial_t^l \Psi_{r,s} A^r_i + \{ \bar{\partial}^m \partial_t^l, A^k_i \}; \quad \{ \bar{\partial}^m \partial_t^l, A^k_i \} := \sum_{l + l' \geq 1} a_{l, l'} \partial^l \bar{\partial}^l (A^k_i A^l_i) \bar{\partial}^{m-l} \partial_t^{n-l'} \Psi_{r,s},
\]

where the term \( \{ \cdot, \cdot \} \) is the commutator error. Here the constants \( a_{l, l'} \) are some universal constants, depending only on \( m, n, l \) and \( l' \) (where \( 0 \leq l \leq m, 0 \leq l' \leq n \)).
1.5.2. Classical Stefan problem in the new variables. Using the family of diffeomorphisms \( \Psi(t, \cdot) \), the classical Stefan problem (i.e. problem (1.1) with \( \sigma = 0 \)) on the fixed reference domain \( \Omega \) is given by

\[
\begin{align*}
q_t - \Delta q &= -v \cdot w \quad \text{in } \Omega \times (0, T], \quad (1.18a) \\
v_t + A_k^k q_{,k} &= 0 \quad \text{in } \Omega \times (0, T], \quad (1.18b) \\
q &= 0 \quad \text{on } \Gamma \times [0, T], \quad (1.18c) \\
\Delta \Psi &= 0 \quad \text{on } \Omega \times [0, T], \quad (1.18d) \\
\Psi &= \text{Id} + hN \quad \text{on } \Gamma \times [0, T], \quad (1.18e) \\
\Psi_t \cdot n(t) &= -v \cdot n(t) \quad \text{on } \partial \Omega_{\text{top}} \times [0, T], \quad (1.18f) \\
v \cdot N &= 0 \quad \text{on } \partial \Omega_{\text{top}} \times [0, T], \quad (1.18g) \\
\Psi(0, \cdot) &= \Psi_0 \quad q(0, \cdot) = q_0 = p_0 \circ \Psi_0, \quad (1.18i)
\end{align*}
\]

where \( \Delta q \) is defined in (1.15) and \( N = (0, 1) \) is the outward-pointing unit normal to \( \partial \Omega_{\text{top}} \). Problem (1.18) is a reformulation of problem (1.1). Condition (1.18g) is equivalent to the evolution equation for the height function \( h(t, \cdot) \) which is given by

\[
h_t(t, x) = -g(t, x) \nabla \Psi q(t, x) \cdot n(t, x), \quad x \in \mathbb{T}^1,
\]

where the quantity \( q \) is defined in (1.11) and \( \nabla \Psi q \) is defined in (1.16). The time-evolution of the map \( \Psi(t, \cdot) \) is, in turn, coupled to the evolution of \( q(t, \cdot) \) via (1.18a).

1.5.3. The higher-order energy function \( E(t) \). We define the higher-order energy function as

\[
E(t) = E(q, h)(t) := \sum_{a+b \leq 4} \left\| \partial^a \partial_t^b v \right\|_{L^2 \Omega}^2 + \sum_{a+2b \leq 3} \left\| \partial^a \partial_t^b v \right\|_{C^0 \Omega}^2
\]

\[
+ \sum_{b=0}^2 \left\| \partial_t^b q \right\|_{C^0 \Omega}^2 + \sum_{b=0}^2 \left\| \partial_t^b q \right\|_{L^2 \Omega}^2 + \sum_{b=0}^2 \left\| \partial_t^b h \right\|_{C^0 \Omega}^2 + \sum_{b=0}^1 \left\| \partial_t^b h \right\|_{L^2 \Omega}^2
\]

\[
(1.20)
\]

where the time integrals in the \( L^2 \)-norms above are over the time-interval \([0, t]\). We will show that \( E(t) \) remains bounded on \([0, T]\).

1.5.4. The Taylor sign condition. In order to obtain a locally well-posed problem for arbitrarily large initial data, we must impose the Taylor sign condition on the initial data as follows:

\[
-\partial_n p_0 > 0 \quad \text{on } \Gamma(0). \quad (1.21)
\]

Expressed in terms of \( q(0, \cdot), (1.21) \) is written as

\[
q_{0,2} |_{t=0} > 0 \quad \text{on } \Gamma. \quad (1.22)
\]

The condition (1.22) ensures that

\[
\inf_{x', \in \Gamma} q_{0,2}(t, x', 0) > 0, \quad t \in [0, T] \quad (1.23)
\]

if \( T > 0 \) is taken sufficiently small. As mentioned in Section 1.1, we shall refer to (1.22) as the Taylor sign condition in analogy to the terminology used in the well-posedness theory in fluid mechanics [46, 44]. The Taylor sign condition will provide positivity of the natural energy functional.

Remark 2. Note that \( q_{0,2} = gJ^{-1} v \cdot n \) on \( \Gamma \), with \( v \) defined by (1.13). By (2.12), we conclude that \( h_t = J q_{0,2} \) at time \( t = 0 \). Since the Jacobian remains positive on a short interval of time the Taylor sign condition (1.21) shows that \( h_t(0, x) < 0 \) for all \( x \in \mathbb{T}^1 \). Thus, the domain \( \Omega(t) \) expands on a short interval of time.
1.5.5. Compatibility conditions. To ensure that the solution is continuously differentiable with respect to $t$, at $t=0$, we must impose compatibility conditions on the initial data. In particular, restricting (1.18a) to $\Gamma$ and evaluating at time $t=0$, for $H^4$ initial data, we find that

\begin{align}
q_0 &= 0 \quad \text{on } \Gamma, \quad \tag{1.24a} \\
\Delta_{\Psi_0} q_0 + J_0^{-2} g_0^2(q_0))^2 &= 0 \quad \text{on } \Gamma. \quad \tag{1.24b}
\end{align}

In the derivation of (1.24b) we have crucially used (1.9) and the identity

\begin{align}
v(0,\cdot) \cdot w(0,\cdot)|_t &= -A \Psi_0, q_0, k \Psi_0, (0,\cdot) = -q_0, A \Psi_0, N, k \Psi_0, (0,\cdot) \\
&= J_0^{-1} g_0^{-2} q_0, 2 \Psi_0, (0,\cdot) \cdot n_0 = J_0^{-1} g_0^{-2} q_0, 2 v(0,\cdot) \cdot n_0 \\
&= J_0^{-2} g_0^{-2} (q_0)^2.
\end{align}

Here $J_0, g_0$ are the initial values of $J, g$ defined in (1.10), (1.11) respectively. Conditions (1.24) are satisfied for a large class of functions. Consider simply a function $q_0$ of the Taylor sign condition

\begin{align}
\text{where we have used (1.12). It is easily checked that for such } q_0, \\
\text{and therefore the condition (1.24b) is satisfied. Note that the assumption } \alpha > 0 \text{ ensures the validity of the Taylor sign condition } (q_0)_{,2} > 0.
\end{align}

Since we imposed the homogeneous Neumann condition (1.18h) on the top boundary $\partial \Omega_{\text{top}}$, we impose the compatibility condition

\begin{align}
(q_0)_{,2} |_{t=0} &= 0 \quad \text{on } \partial \Omega_{\text{top}}. \quad \tag{1.26}
\end{align}

By employing a partition-of-unity of $\Omega$, we can now easily construct a $q_0 \in H^4(\Omega)$ such that the compatibility conditions (1.24) and (1.26) are simultaneously satisfied.

**Remark 3.** The quadratic function $q_0$ defined in (1.25) satisfies the compatibility conditions. This is one of many possible constructions of initial data satisfying the corresponding regularity and compatibility conditions.

1.6. Local-in-time well-posedness for the classical Stefan problem. We define

\begin{align}
S(t) := \{(q, h) : E(q, h)(t) < \infty\}. \quad \tag{1.27}
\end{align}

Our first result is a well-posedness statement for the classical Stefan problem.

**Theorem 1.1** (Well-posedness of the classical Stefan problem). *Given initial conditions* $(q_0, h_0) \in S(0)$ *with* $q_0$ *satisfying the Taylor sign condition (1.22) and the compatibility conditions (1.24)–(1.26), the problem (1.18) is locally-in-time well-posed, i.e. there is a $T > 0$ such that and a unique solution* $(q, h)$ *on the time interval* $[0, T]$ *with initial data* $(q_0, h_0)$, *such that*

\begin{align}
\sup_{t \in [0, T]} E(q, h) \leq 2 E(q_0, h_0).
\end{align}

**Remark 4.** The definition of our higher-order energy function $E$ restricted to time $t = 0$ requires an explanation of time-derivates of $q$ and $h$ at $t = 0$. Specifically, the values $q_{t} |_{t=0}$, $q_{tt} |_{t=0}$, $h_{t} |_{t=0}$ and $h_{tt} |_{t=0}$ are defined via space-derivatives using equations (1.18a) and (1.18g).
1.7. The vanishing surface tension limit. Our second main result establishes the vanishing surface tension limit. Denoting by $\mathcal{H}$ the mean curvature of the free-boundary, in the $\Psi$-parametrization, the boundary condition (1.18c) is replaced with

$$q = \mathcal{H} = \sigma \frac{\bar{\partial}^2 \Psi \cdot n}{|\bar{\partial}^2 \Psi|^2} = -\sigma \frac{\bar{\partial}^2 h}{(1 + |\bar{\partial} h|^2)^{3/2}}; \quad (1.28)$$

then, the problem (1.18), with the boundary condition (1.28) replacing (1.18c), is the Stefan problem with surface tension formulated in harmonic coordinates. The high-order energy function adapted for the presence of surface tension is given by

$$\mathcal{E}^\sigma = \mathcal{E}^\sigma(q, h) = \mathcal{E}(q, h) + \sigma \sum_{b=0}^2 |\bar{\partial}^b h|^2_{L^2_H H^{-2b-2}} + \sigma^2 \sum_{b=0}^2 |\partial^b h_{t}|^2_{L^2_H H^{-2b}} + \sigma^2 \sum_{b=0}^2 |\partial^b \dot{h}_{t}|^2_{L^2_H H^{-2b}}, \quad \sigma > 0. \quad (1.29)$$

1.7.1. Compatibility conditions for the Stefan problem with surface tension. To ensure the spatial continuity of the temperature function $q$ and its first derivative $q_t$ at time $t=0$, we must impose two sets of compatibility conditions. The first condition is

$$q_0 = \sigma \mathcal{H}_0 = -\sigma g_0^{-3} \bar{\partial}^2 h_0 \quad \text{on} \quad \Gamma, \quad (1.30)$$

where $\mathcal{H}_0$ denotes the mean curvature of the initial free surface $\Gamma_0$, and $g_0 = \sqrt{1 + (\bar{\partial} h_0)^2}$. To obtain the second compatibility condition, we note that $q_t \big|_{t=0} = -\sigma \partial_t \mathcal{H} \big|_{t=0}$. From the boundary condition (1.18g) we can evaluate $h_t$ at time $t=0$ as

$$h_t \big|_{t=0} = -g_0 \nabla \psi_0 \cdot n_0, \quad (1.31)$$

where the subscript 0 refers to the initial values of the quantities $g, \Psi, q,$ and $n$ defined above. Therefore,

$$\partial_t \mathcal{H} \big|_{t=0} = -g_0^{-3} \bar{\partial}^2 h_t \big|_{t=0} + 3g_0^{-5} \bar{\partial}^2 h_0 \partial_t h_t \big|_{t=0} \bar{\partial} h_0 = g_0^{-3} \bar{\partial}^2 (g_0 \nabla \psi_0 \cdot n_0) + 3g_0g_0^{-2} \partial (g_0 \nabla \psi_0 \cdot n_0) \bar{\partial} h_0, \quad (1.32)$$

where we have used (1.30) and (1.31) in the last line. After restricting (1.18a) to $\Gamma$ at time $t=0$ and using (1.32), we find that

$$-\sigma \left( g_0^{-3} \bar{\partial}^2 (g_0 \nabla \psi_0 \cdot n_0) + 3g_0g_0^{-2} \partial (g_0 \nabla \psi_0 \cdot n_0) \bar{\partial} h_0 \right) - \Delta \psi_0 g_0 = -g_0 (\nabla \psi_0 \cdot n_0) (A_0)^2 g_{0,k}. \quad \text{In particular, the right-hand side can be separated into the } \sigma \text{-dependent and } \sigma \text{-independent contributions, so that}$$

$$-g_0 (\nabla \psi_0 \cdot n_0) A_0^2 g_{0,k} = J_0^{-2} g_0^2 (q_0)^{2} + \sigma \bar{\partial} \mathcal{H}_0 (g_0 (\nabla \psi_0 \cdot n_0) (A_0)^2 + g_0 (|A_0|^2 \cdot n_0) A_0^2 (q_0)^{2}).$$

Combining the two previous identities, we find the second compatibility condition to be

$$\Delta \psi_0 g_0 + J_0^{-2} g_0^2 (q_0)^{2} = \sigma \mathcal{C}(q_0, h_0), \quad (1.33)$$

where

$$\mathcal{C}(q_0, h_0) := - \left[ g_0^{-3} \bar{\partial}^2 (g_0 \nabla \psi_0 \cdot n_0) + 3g_0g_0^{-2} \partial (g_0 \nabla \psi_0 \cdot n_0) \bar{\partial} h_0 \right]$$

$$- \partial \mathcal{H}_0 \left[ g_0 (\nabla \psi_0 \cdot n_0) (A_0)^2 + g_0 (|A_0|^2 \cdot n_0) (A_0)^2 (q_0)^{2} \right]. \quad (1.34)$$

1.7.2. Initial data satisfying compatibility conditions. When $\Psi = \text{Id}$ (and therefore $h_0(x) = 0$, $g_0 = J_0 = 1$) the compatibility conditions (1.30) and (1.33)–(1.36) simplify significantly and take the form

$$q_0 = 0 \quad \text{and} \quad (q_0)^{2} + (q_0)^{2} = \sigma (q_0)^{2} \quad \text{on} \quad \Gamma \quad (1.35)$$

$$(q_0)^{2} + (q_0)^{2} \in C^0(\Gamma). \quad (1.36)$$

It is easy to check that the function $q_0$ constructed in Section 1.5.5 satisfies (1.35)–(1.36). For general $h$ satisfying $|h|^4 \ll 1$ we can construct the initial temperature $q_0$ satisfying (1.30) and (1.33)–(1.36) by perturbative methods, using for instance the implicit function theorem.
1.7.3. Well-prepared initial data. To obtain the vanishing surface tension limit, we need to define a suitable class of initial data \((q_0^\sigma, h_0^\sigma), \sigma \geq 0\).

**Definition 1** (Well-prepared data). A family of initial data \((q_0^\sigma, h_0^\sigma)_{\sigma \geq 0}\) such that \(\mathcal{E}(q_0^\sigma, h_0^\sigma) < \infty\) is well-prepared if it satisfies 1) compatibility conditions (1.30), (1.33)–(1.36) associated to the Stefan problem with surface tension, 2) the Taylor sign condition (1.22), and 3) \(\mathcal{E}(q_0^\sigma - q_0, h_0^\sigma - h_0) \to 0\) as \(\sigma \to 0\).

We now demonstrate that the class of well-prepared initial data is non-empty. Let us assume for simplicity that \(\Psi_0 = \text{Id}\) and therefore the initial hypersurface \(\Gamma_0\) is flat. For \(\sigma \geq 0\) we have \(h^\sigma(x') = 0, x' \in \mathbb{T}^1\). Let \(b: \mathbb{T}^1 \to \mathbb{R}\) be a given smooth function and \(\alpha > 0\) a given positive real number. Consider a function \(q_0^\sigma\) independent of \(x_1\) in the slab \(T_\epsilon = \mathbb{T}^1 \times [0, \epsilon]\) for some \(0 < \epsilon < 1\) and of the form

\[
q_0^\sigma(x_1, x_2) = -\alpha^2 \frac{x_2^2}{2} + \alpha x_2 + \sigma b(x_1)x_2^3, \quad (x_1, x_2) \in T_\epsilon.
\]

It is straightforward to check that conditions (1.35)–(1.36) are both satisfied with this choice of \(q_0^\sigma\). Moreover, The Taylor sign condition holds since \(q_0^\sigma = \alpha > 0\) for any \(\sigma \geq 0\) and the convergence requirement 3) in Definition 1 is clearly satisfied. Outside the slab \(T_\epsilon\) we can extend the function \(q_0^\sigma\) smoothly so that the Neumann boundary condition \(\partial_N q_0^\sigma\) is satisfied on \(\partial \Omega_{\text{top}}\).

1.7.4. The vanishing surface tension limit. For a given \(T > 0\) let

\[
C^1_x C^0_\tau \cap C^0_\tau C^2_x := \left\{ (q, h) : q \in C^1([0, T]; C^0(\Omega)) \cap C^0([0, T]; C^2(\Omega)), \right. \\
\left. h \in C^1([0, T]; C^0(\Gamma)) \cap C^0([0, T]; C^2(\Gamma)) \right\}
\]

with the associated norm:

\[
\|(q, h)\|_{C^1_x C^0_\tau \cap C^0_\tau C^2_x} = \max_{t \in [0, T], x \in \Omega} (|q(t, x)| + |\partial_q q(t, x)| + |\nabla q(t, x)| + |\nabla^2 q(t, x)|) + \max_{t \in [0, T], x' \in \Gamma} (|h(t, x')| + |\partial_h h(t, x')| + |\partial h(t, x')| + |\partial^2 h(t, x')|).
\]

**Theorem 1.2** (The limit of zero surface tension). Let \((q_0^\sigma, h_0^\sigma)_{\sigma \geq 0}\) be a sequence of well-prepared initial conditions in the sense of Definition 1 such that

\(\mathcal{E}(q_0^\sigma - q_0, h_0^\sigma - h_0) \to 0\) as \(\sigma \to 0\).

Let \((q^\sigma(t, \cdot), h^\sigma(t, \cdot))_{\sigma \geq 0}\) denote the corresponding sequence of solutions to the Stefan problem with surface tension, such that \((q^\sigma(0, \cdot), h^\sigma(0, \cdot)) = (q_0^\sigma, h_0^\sigma)\). Then, there exists a \(\sigma\)-independent time \(T > 0\) and a constant \(C\) depending only on \((q_0, h_0)\) such that

\[
\mathcal{E}^\sigma(q^\sigma, h^\sigma)(T) \leq C, \quad \sigma \geq 0.
\]

for all \(\sigma \geq 0\).

Furthermore, the sequence \((q^\sigma, h^\sigma)\) converges in the \(C^1_x C^0_\tau \cap C^0_\tau C^2_x\)-norm to the unique solution \((q, h)\) of the classical Stefan problem (1.18) with \(\sigma = 0\) and the initial data \((q(0), h(0)) = (q_0, h_0)\).

1.8. Prior results and a motivation for the current treatment. There is a large literature on the classical one-phase Stefan problem. For a comprehensive overview, we refer the reader to MEIRMANOV [40] and VISINTIN [49]. The first weak solutions were defined by KAMENOMOSTSKAYA [35], LADYZHENSKAYA, SOLONNIKOV and URALCEVA [38]. These weak solutions were analyzed by FRIEDMAN, KINDERLEHRER [24], [25], [26], CAFFARELLI, EVANS [6], [7], wherein the regularity of weak solutions was established. Since the problem satisfies a maximum principle, it is ideally suited to the so-called viscosity solutions approach. Existence and regularity of viscosity solutions was established by ATHANASIOPOULOS, CAFFARELLI, and SALSA in [4], [5]. Existence of viscosity solutions in the one-phase case was proven by KIM [36] and in the two-phase case by KIM AND POŽAR [37]. A local-in time regularity theorem was proven in [13] which in particular shows that initially Lipschitz free-boundaries become \(C^1\) over a possibly smaller spatial region. For an exhaustive overview and introduction to the regularity theory of such solutions we refer the reader to CAFFARELLI AND SALSA [8], see also more recent results [13, 12].
Local existence of classical solutions for the classical Stefan problem was shown by MIRMANOV (see [40] and references therein) and HANZAWA [34]. In the first approach, the author regularizes the problem by adding artificial viscosity to (1.1b) and fixes the moving domain by switching to so-called von Mises variables. The obtained solutions however, lose derivatives with respect to the assumed regularity on the initial data. Similarly, in [34] the author uses Nash-Moser iteration to obtain a local-in-time solution, however again with a significant derivative loss with respect to the initial data. A local existence result for the one-phase n-dimensional Stefan problem is proved in [28], where the required regularity class for the temperature function is $W^{2,1}_p$ with $p > n + 2$. For the two-phase Stefan problem a local existence result is presented in [41] in the framework of $L^p$ maximal regularity, where the corresponding functional spaces of Sobolev-type require $p > n + 3$, where $n$ is the dimension of the ambient space.

In related work, local and global existence for the one-phase and two-phase Muskat problems has been established in [15, 16, 14, 10]. For the local and global well-posedness of the one-phase Hele-Shaw problem and optimal decay rates of the solutions, see [9] and the references therein.

As to the Stefan problem with surface tension (also known as the Stefan problem with Gibbs-Thomson correction), a global weak existence theory (without uniqueness) is given in [1, 39, 45]. In [27] the authors consider the Stefan problem with small surface tension i.e. $\sigma \ll 1$ whereby (1.1c) is replaced by $v = \sigma \kappa$. Local existence of classical solutions is studied in [43]. In [23] the authors prove a local existence and uniqueness result for classical solutions under a smallness assumption on the initial datum close to flat hypersurfaces. Global existence close to flat hyper-surfaces is proved in [30] and close to stationary spheres for the two-phase problem in [29] and later in [42].

With the Gibbs-Thomson correction, problem (1.1) can account for phenomena such as the phase nucleation, undercooling (superheating) and it is also used in modeling crystal growth [49]. It is a small-scale model as opposed to the macro-scale classical Stefan problem. In this sense, there is a fundamental importance in rigorously understanding the link between the two models. As explained in [49], [48], one can associate a free energy to the Stefan problem with surface tension defined by

$$F_\sigma(\tilde{\rho}, \tilde{\Gamma}) = \int_\Omega \tilde{\rho} dx + \sigma |\tilde{\Gamma}|,$$

where $\tilde{\rho}$, $\tilde{\Gamma}$ are time-independent. Then in the the sense of $\Gamma$-convergence of De Giorgi [22], the free energy $F_\sigma(\tilde{\rho}, \tilde{\Gamma})$ converges to the free energy for the classical Stefan problem, see [49]. This is, however, a completely time-independent consideration and does not address the vanishing $\sigma$-limit of time-dependent solutions to the full non-linear problem (1.1). In the context of the water wave problem, the vanishing surface tension limit in two and three dimensions has been studied in [2, 3]; for the full Euler equations, see [17].

Turning our attention to the Stefan problem, we can observe that there are two parallel developments in the existence theory for weak solutions briefly mentioned above. The first one applies to the classical Stefan problem and it is motivated by the validity of maximum principle; suitable notions of weak and viscosity solutions have been established [4, 5, 38, 26, 6]. The second development refers to the problem with surface tension, wherein the weak solution existence results are in BV-type spaces, and rely upon the gradient-flow structure of the problem. From the point of view of the vanishing surface tension, it is natural to ask whether the two concepts are compatible in any rigorous mathematical manner. The answer is inconclusive due to a lack compactness. While the control of solutions constructed in [39, 1] is strong enough to pass to some limit as $\sigma \to 0$, it is too weak to guarantee a sharp interface in the limit. In other words, it is not clear how to preclude the formation of so-called mushy regions [49].

We develop a new energy method for the Stefan problem with and without surface tension and prove the vanishing surface tension limit. The well-posedness is established in $H^k$ Sobolev spaces using a combination of energy estimates for tangential derivatives and elliptic-type estimates for added parabolic-type regularity. Our framework is motivated by the analysis of the free-surface incompressible Euler equations of COUTAND and SHKOLLER [19, 20].

Precise statements of our results are given in Theorems 1.1 and 1.2. The estimates that we use are nonlinear in nature and they fundamentally exploit the intricate energy structure of the problem. In particular, no derivative loss occurs with respect to the regularity of the initial data. This framework
is particularly convenient, as it allows us to rigorously establish the vanishing surface tension limit locally-in-time, as formulated in Theorem 1.2. In this way, we link two fundamental models of phase transitions that are valid on different spatial scales, thus answering the open question explained above. In forthcoming work, we shall extend our results to the two-phase Stefan problem, providing the analog of Theorem 1.1 [33], while the question of global-in-time stability of steady states using this functional-analytic framework has been addressed in [31, 32].

1.9. Methodology and outline of the paper. There are three main ingredients in our approach to the Stefan problem. First, we replace the study of the heat equation for temperature, with the equation for a velocity field \( u(t,x) \) which satisfies the equation \( u + \nabla p = 0 \). Second, we introduce the so-called Arbitrary Lagrange-Eulerian (ALE) variables, in which we introduce a family of diffeomorphisms \( \Psi(t,\cdot):\Omega \rightarrow \Omega(t) \) which fix the moving domain. With respect to this change-of-variables, we define, respectively, the new velocity and temperature fields \( v = u \circ \Psi \) and \( q = p \circ \Psi \); in these variables, the velocity equation becomes \( v + \nabla p \circ \Psi = 0 \). This equation contains the geometry of the evolving free-boundary, and by the use of energy estimates for tangential derivatives, we are able to naturally estimate the second-fundamental form as

\[
\int_{\Gamma} q,\xi |\tilde{\partial}^k \Psi \cdot A_{\Omega}^2 \xi|^2 \, dx \approx \int_{\Gamma} q,\xi |\tilde{\partial}^k h|^2 \, dx,
\]

for \( k \) some positive integer. In the original Eulerian framework (1.1), the energy dissipation law is given by

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega(t)} p(t,x)^2 \, dx + \int_{\Omega(t)} |\nabla p(t,x)|^2 \, dx = 0.
\]

This basic energy law is to weak to control the regularity of the evolving free-boundary. Observe that our higher-order control of the free-boundary given by (1.39) naturally produces the stability condition; in particular, the Taylor sign condition (1.22) arises as coefficient to the second-fundamental form, and it sign determines either the control or growth of the curvature and its derivatives via a Gronwall-type inequality.

A further subtlety consists in the discovery of another coercive energy term which is defined on the whole domain \( \Omega \) (phase), displayed in the fourth line of (2.24). It contains terms of the general form

\[
\| \tilde{\partial}^a \partial^b \xi + \tilde{\partial}^b \partial^a \xi \|_{L^2_{t}L^2_x}^2 \text{ and } \| \tilde{\partial}^a \partial^b \Psi \cdot v \|_{L^2_{t}L^2_x}^2
\]

for \( a,b \) as in (2.24). They are intrinsically linked to the problem and contain information about the regularity of the divergence of the velocity \( v \). Taking \( a = 0 \) and \( b = 1 \), the first term above becomes the norm of the ALE-divergence of \( v \), as it is easily seen from (1.18a):

\[
\| \nabla_p v \|_{L^2_{t}L^2_x} = \| \text{div} \xi v \|_{L^2_{t}L^2_x}.
\]

The gauge condition (1.6) allows us to get optimal Sobolev regularity for \( \Psi \) and hence for the temperature function \( q \). This allows us to prove that the energy \( \mathcal{E} \) defined in (2.16) is in fact bounded by the coercive quadratic form (the “natural energy”) \( \mathcal{F} \) (2.24) dictated by the Stefan problem.

Condition (1.21) is the exact equivalent of the Taylor sign condition, necessary for well-posedness of free-surface incompressible Euler equations without surface tension [19] or the water wave problem [50]. If the initial temperature \( q_0 \) is nonnegative, it is implied by the Hopf’s lemma, at least over a short period of time. In a short time regime, we prove a uniform lower bound on \( \lambda \) (cf. (1.23)), thus enabling us to close the estimates.

In many free-boundary problems, constructing the solution is in general a challenging problem despite the (possible) availability of good a-priori estimates. Our main technical idea to make the construction as straightforward as possible, is to regularize the problem via horizontal convolution by layers as introduced in [19] in the study of well-posedness of the incompressible Euler equation on a moving domain. In addition to that, we also regularize the Stefan condition \( p = 0 \) on \( \Gamma(t) \) by modifying it into a Robin-type condition. If \( \kappa \geq 0 \) is a suitable regularization parameter, to each \( \kappa \) we shall associate an energy functional \( \mathcal{E}_\kappa \) which will be shown to satisfy the following energy inequality:

\[
\mathcal{E}_\kappa(t) \leq C\mathcal{E}_\kappa(0) + C(t + \sqrt{t})P(\sqrt{\mathcal{E}_\kappa})
\]
where \( P \) is some with the leading order cubic contribution. Such a polynomial inequality, through a continuity argument leads to uniform-in-\( \kappa \) time of existence \([0, T]\) and the bound
\[
\mathcal{E}_\kappa(t) \leq 2C\mathcal{E}_\kappa(0).
\]
Passing to the limit as \( \kappa \to 0 \), we recover the solution of the Stefan problem (1.18). Our regularization is intrinsic to the problem and it does not rely on formulating a sequence of iterated linear problems.

The second part of this work focuses on the problem of the vanishing surface tension limit. Once the well-posedness framework of Theorem 1.1 is set-up, the idea is rather straightforward. Namely, at the level of energy, the presence of surface tension simply augments the high-order energy functional by a \( \sigma \)-dependent contribution coming from the boundary \( \Gamma \), so to obtain (1.29). The goal is to prove a uniform-in-\( \sigma \) upper bound on \( \mathcal{E}^\sigma \) on a \( \sigma \)-independent time interval \([0, T]\). This is made possible by one fundamental property of \( \mathcal{E}^\sigma \): it distinguishes between two boundary energy contributions of general forms
\[
|\sqrt{q_2} \partial^a \partial^b h|_0^2 \quad \text{and} \quad |\partial^a \partial^b h|_0^2
\]
for suitable \( a, b \in \mathbb{N}_0 \). Since the error terms are at least of cubic order, we can afford to estimate all lower order terms in terms of the \( \sigma \)-independent energy term, while the two terms with highest number of derivatives get bounded via the \( \sigma \)-dependent energy contribution. With uniform estimates in hand, we can pass to the limit as \( \sigma \to 0 \).

The plan of the paper is as follows. In Section 2.1 we introduce the \( \kappa \)-regularized problem and the associated high-order energy \( \mathcal{E}_\kappa \). We then state the energy identities (Lemma 2.2), prove that \( \mathcal{E}_\kappa \) is controlled by the natural energy \( \mathcal{F}_\kappa \) (Proposition 2.5) and finally prove Lemma 2.2. In Section 2.7, we provide the energy estimates for the error terms. Passage to the vanishing surface tension limit is explained in Section 3. In Section 4 we explain how to extend our results to the three dimensional setting.

2. Local well-posedness for the classical Stefan problem

2.1. A nonlinear regularization of the Stefan problem: the \( \kappa \)-problem. We regularize the problem by using the horizontal convolution by layers, introduced in [19] in the study of well-posedness of the incompressible Euler equation on a moving domain.

**Definition 2** (Horizontal convolution-by-layers). Let \( \rho_\kappa \) be a \( C^\infty(\mathbb{R}) \)-bump function supported in a ball of radius \( \kappa \) defined through: \( \rho_\kappa(x) := \frac{1}{\kappa} \rho(\frac{x}{\kappa}) \), where
\[
\rho(x) = \begin{cases} 
  c_* e^{-1/(1-|x|^2)} , & |x| < 1, \\
  0 & |x| \geq 1
\end{cases}
\]
and constant \( c_* \) is such that \( \int_\mathbb{R} \rho(x') dx' = 1 \). For any given \( g: \Omega \to \mathbb{R} \) we define the horizontal convolution by layers of \( g \) via
\[
\Lambda_\kappa g(x^1, x^2) := \int_\Gamma g(x^1, x^2) \rho_\kappa(x^1 - x') dx'.
\]

We also define the standard 2-D sequence of mollifiers: \( \eta_\kappa(x) = \kappa^{-2} \eta(\kappa x) \) where \( \eta(x) = c_* e^{-1/(1-|x|^2)} \) for \( |x| < 1 \) and \( \eta(x) = 0 \) for \( |x| \geq 1 \), and \( c_* \) is chosen so that \( \int_{\mathbb{R}^2} \eta(x) dx = 0 \). To formulate the regularized problem, we introduce the following quantities:
\[
\Psi_\kappa(t, x') = (x', h_\kappa(t, x')) \quad \text{with} \quad h_\kappa(t, x') := \Lambda_\kappa \Lambda_\kappa h(t, x'), \quad x' \in \Gamma
\]
and we define \( \Psi_\kappa \) on \( \Omega \) as a harmonic extension of its boundary value on \( \Gamma \) as in (1.6). Analogously to (1.8) the following trace estimate is true:
\[
\| \partial^a \Psi_\kappa \|_{H^s(\Omega)} \lesssim \| \partial^a \Psi_\kappa \|_{H^{s-0.5}(\Gamma)}, \quad s > 0.5, \quad a \in \mathbb{N}.
\]
We also denote \( J_\kappa := \det \nabla \Psi_\kappa \). Furthermore,
\[
\gamma := [\nabla \Psi_\kappa]^{-1}; \quad \eta := \partial_t \Psi_\kappa.
\]
In analogy to (1.12), we introduce
\[
\gamma_* := \left( \frac{\partial h_\kappa, -1}{J_\kappa} \right); \quad \eta := J_\kappa^{-1} \gamma.
\]

(2.3)
For $\kappa>0$, we now define a nonlinear regularization of the Stefan problem, which we call the $\kappa$-problem (1.18), in which the coefficients are smoothed by use of the horizontal convolution operator $\Lambda_\kappa$. On a time interval $[0,T_\kappa]$, the $\kappa$-problem is given as

$$
\begin{align*}
q_t - \Delta \psi_\kappa q &= -v \cdot \nabla w + \alpha & \text{in } [0,T_\kappa] \times \Omega, \\
v^j + \Lambda_\kappa^j q_{i,j} &= 0 & \text{in } [0,T_\kappa] \times \Omega,
\end{align*}
$$

(2.4a)

$$
q = -\kappa^2 v \cdot \nabla_\alpha^2 + \kappa^2 \beta(t,x') & \text{on } [0,T_\kappa] \times \Gamma, \\
\Psi_t \cdot n_\kappa &= -v \cdot n_\kappa & \text{on } [0,T_\kappa] \times \Gamma, \\
v \cdot N &= 0 & \text{on } [0,T_\kappa] \times \partial \Omega_{\text{top}},
\end{align*}
$$

(2.4b)

$$
\Psi(0,\cdot) = \Psi_0 \quad q(0,\cdot) = \eta_0^\kappa, 
$$

(2.4c)

where

$$
\begin{align*}
\Delta \psi_\kappa (t,\cdot) &= 0 & \text{in } \Omega, \\
\psi_\kappa (t,x',0) &= (x', h_\kappa(t,x')) & x' \in \Gamma, \\
\psi_\kappa (t,\cdot) &= \text{Id} & \text{on } \partial \Omega_{\text{top}},
\end{align*}
$$

(2.5a)

$$
\Delta \psi_\kappa q := \Lambda_\kappa^j (\Lambda_\kappa^k q_{i,k})_{ij},
$$

(2.5b)

and the time-independent forcing function $\alpha(x)$ is given by

$$
\alpha = J_0^{-2} g_0^2[q_{0,2}]^2 - J_0[0,\cdot)^{-2} g_0(0,\cdot)^2[q_{0,2}]^2.
$$

(2.6)

Here

$$
g_\kappa(t,x) := \sqrt{1 + (\partial h_\kappa(t,x))^2},
$$

and $\beta(t,x')$ is defined as

$$
\beta(t,x') := \sum_{k=0}^2 t^k (v \cdot \nabla_\alpha^2)_{i=0}.
$$

(2.7)

Note that we use the subscript and superscript $\kappa$ on dependent variables in which there is explicit use of the horizontal convolution operator $\Lambda_\kappa$; of course, all of the $q$, $h$, and $\Psi$ all implicitly depend on $\kappa$ as well, but for notational convenience, we do not indicate this implicit dependence on $\kappa$.

The presence of the horizontal mollification operator $\Lambda_\kappa$ in the approximate $\kappa$-problem changes the compatibility conditions on the initial data. The addition of the forcing functions $\alpha(x)$ and $\beta(t,x')$ ensure that the compatibility conditions (1.24) are modified to be

$$
\begin{align*}
Q^\kappa_0 &= 0 & \text{on } \Gamma, \\
\Delta \psi_\kappa Q^\kappa_0 &= -J_0^{-2} g_0^2[q_{0,2}]^2 & \text{on } \Gamma,
\end{align*}
$$

(2.8a)

where $\Psi_0^\kappa = \psi_\kappa(0,\cdot)$. The approximated initial temperature function $Q^\kappa_0$ is then defined as the solution of the fourth-order elliptic equation

$$
\begin{align*}
\Delta \psi_\kappa Q^\kappa_0 &= \eta_\kappa \ast E(\Delta \psi_\kappa \Delta \psi_\kappa q_0) & \text{in } \Omega, \\
Q^\kappa_0 &= 0 & \text{on } \Gamma,
\end{align*}
$$

(2.9a)

where $E$ continuously maps $H^k(\Omega)$ to $H^k(\mathbb{R}^2)$ for all $k \geq 0$. The fourth-order elliptic equation (2.9) can be written as a system of second-order equations given by

$$
\begin{align*}
\Delta \psi_\kappa Q^\kappa_0 &= R^\kappa_0 & \text{in } \Omega, \\
\Delta \psi_\kappa R^\kappa_0 &= \eta_\kappa \ast E(\Delta \psi_\kappa \Delta \psi_\kappa q_0) & \text{in } \Omega, \\
Q^\kappa_0 &= 0 & \text{on } \Gamma, \\
R^\kappa_0 &= -J_0^{-2} g_0^2[q_{0,2}]^2 & \text{on } \Gamma,
\end{align*}
$$

(2.10a)

According to the basic elliptic regularity theorem with Sobolev class coefficients, Theorem 3.6 in [11], we obtain estimates for $R^\kappa_0$ and then $Q^\kappa_0$ which show that

$$
\|Q^\kappa_0\|_4^2 \leq C E(g_0,h_0),
$$

(2.10b)

(2.10c)

(2.10d)
We then define the corresponding $Z$.

Proof.

where $H$

form:

commutation of the horizontal convolution operator appearing in the terms of the following schematic

energy estimates at the level of highest-in-time differentiated problem. The problem arises from the

Remark 6.

a simple existence theory for the

layers together with carefully chosen artificial viscosity terms. This approximation scheme provides

in

In analogy to (2.12), equation (2.4d) can be reformulated as an evolution equation for $h$, given by

$$(Q_\kappa^0)_2 \bigg|_{t=0} > 0 \quad \text{for sufficiently small } \kappa > 0. \quad (2.11)$$

In equation (2.4d), $n_\kappa$ denotes the outer unit normal with respect to the regularized surface $\Gamma_\kappa$, i.e. in the coordinate representation

$$n_\kappa = \frac{\partial h_\kappa}{\sqrt{1 + |\partial h_\kappa|^2}} = g_\kappa^{-1}(\partial h_\kappa, -1).$$

Note that the corresponding unit tangent to $\Gamma_\kappa$ is given via

$$\tau_\kappa = \frac{\partial \Psi_\kappa}{|\partial \Psi_\kappa|} = \frac{(1, \partial h_\kappa)}{\sqrt{1 + |\partial h_\kappa|^2}} = g_\kappa^{-1}(1, \partial h_\kappa).$$

In analogy to (2.12), equation (2.4d) can be reformulated as an evolution equation for $h$, given by

$$h_\kappa(t, x) = g_\kappa(t, x) v \cdot n_\kappa(t, x), \quad x \in \mathbb{T}^1. \quad (2.12)$$

Remark 5 (The regularization (2.4c)). The approximate $\kappa$-problem uses horizontal convolution by

layers together with carefully chosen artificial viscosity terms. This approximation scheme provides

a simple existence theory for the $\kappa$-problem while maintaining the nonlinear energy structure.

Remark 6. We introduce the regularization (2.4c) to circumvent a technical difficulty of closing the

energy estimates at the level of highest-in-time differentiated problem. The problem arises from the

commutation of the horizontal convolution operator appearing in the terms of the following schematic

form:

$$\int_\Gamma \Lambda_\kappa A_\kappa \cdot \Psi_{tt} T dx',$$

where $T$ is a lower order term. Of course, when performing a-priori estimates (i.e. assuming that

the solutions to the original problem are smooth enough to justify all the integrations by parts), such

an issue does not arise.

2.1.1. Solutions to the $\kappa$-problem.

Theorem 2.1. Let $\kappa > 0$ be fixed. Let $(Q_\kappa^0, h_0) \in H^4(\Omega) \times H^4(\Gamma)$ be given initial data satisfying the compatibility conditions (2.8). Then there is a time $T_\kappa$ depending on $\kappa$, such that there exists a unique solution $(q, h) = (q(\kappa), h(\kappa))$ to (2.4) on the time interval $[0, T_\kappa]$. The solution satisfies

$$\sum_{a=0}^2 \left( |\partial_t^a q|_{C^0_\Omega H^{4-a} + |\partial_t^a q|_{L^2_\Omega H^{4-a}} + |q_{tt}|_{L^2_\Omega (H^1)^2} + |\partial_t^{a+1} h|_{L^2_\Omega H^4 - 2a} \right) + \sum_{a=0}^1 |\partial_t^{a+1} h|_{C^0_\Omega H^2 - 2a} < \infty, \quad (2.13)$$

where $H^1(\Omega)^2$ denotes the dual space of $H^1(\Omega)$.

Proof. We briefly sketch the proof. For $T_\kappa$ fixed (and taken sufficiently small) and for $K > 0$, we define the closed set

$$Z_K := \left\{ h: [0, T] \times \Gamma \to \mathbb{R}, |\partial_t^a h|_{C^0(0, T_\kappa), H^4 - 2a} \in L^2([0, T_\kappa], H^5 - 2a) \right\}, \quad a = 0, 1, 2,$$

$$\sum_{a=0}^2 \left( |\partial_t^a h|_{C^0_\Omega H^{4-a}} + |\partial_t^a h|_{L^2_\Omega H^{4-a}} \right) \leq K, \quad h_0 \text{ and } Q_0^\kappa \text{ satisfy compatibility conditions (2.8)} \right\}. \quad (2.14)$$

Given $h \in Z_K$, we define $h_\kappa = \Lambda_\kappa^2 h$, and then we define its harmonic extension $\Psi_\kappa$ by solving (2.5).

We then define the corresponding $A, \Psi_\kappa$, and $J_\kappa$, and consider the weak formulation of the parabolic problem (2.4a)-(2.4c): for all test functions $\phi \in H^1(\Omega)$ and a.e. $t \in [0, T]$,

$$\langle q_t, J_\kappa, \phi \rangle + \int_\Omega q_k A_\kappa^k A_\kappa^j \phi_{,j} J_\kappa dx + \frac{1}{\kappa^2 \kappa^2} \int_\Gamma q_\phi dx_1 = \int_\Omega q_k A_\kappa^k \Psi_\kappa^j \phi dx + \int_\Omega \alpha \phi J_\kappa dx + \int_\Gamma \beta \phi dx_1, \quad (2.15)$$
together with the initial condition
\[ q(0, x) = Q_0^\kappa(x). \]

Since \( A, \hat{u}, \) and \( J_\kappa \) are in \( C^\infty \), and since \( A^\kappa \) \( A^\kappa_0 \geq \lambda \) for \( \lambda > 0 \), and the compatibility conditions are satisfied, standard parabolic theory provides the existence of a unique solution on a short time-interval \( [0, T_\kappa] \) with the desired regularity properties. In particular it is a standard argument to establish existence of a unique solution in \( q \in L^2(0, T_\kappa; H^5(\Omega)) \) which satisfies the estimate (2.13).

Using a Galerkin scheme on (2.15), we obtain unique solutions in \( L^2(0, T_\kappa; H^1(\Omega)) \) for \( q, q_t, \) and \( q_{tt} \) and also find that \( q_{ttt} \in L^2(0, T_\kappa; H^1(\Omega')) \), where \( H^1(\Omega') \) denotes the dual space of \( H^1(\Omega) \). Standard parabolic regularity theory, as in [47], shows that \( q \in L^2(0, T_\kappa; H^5(\Omega)) \) and that \( q_t \in L^2(0, T_\kappa; H^3(\Omega)) \).

With this solution \( q \), we define the associated velocity field \( v \) using (2.4b). We then update the height function \( h \) as
\[ \Phi(h)(t) := h_0 + \int_0^t g_\kappa(t) \cdot n_\kappa(\tau) \, d\tau, \quad t \in [0, T]. \]
Choosing \( T_\kappa \) sufficiently small, it can be shown that \( \Phi \) maps \( Z_\kappa \) into itself, and that \( \Phi \) is a contraction map. The fixed-point of \( \Phi \) is then a solution to the \( \kappa \)-problem (2.4). □

**Remark 7.** *A priori*, the time of existence \( T_\kappa \) may converge to 0 as \( \kappa \to 0 \). By obtaining \( \kappa \)-dependent bounds on solutions to (2.4), we prove that, in fact, the time of existence is independent of \( \kappa \) and given by \( T > 0 \).

### 2.2. The higher-order energy function compatible with the \( \kappa \to 0 \) asymptotics

The asymptotically consistent higher-order energy function associated to our sequence of regularized \( \kappa \)-problems is given by
\[
\mathcal{E}_\kappa = \mathcal{E}_\kappa(q, h) := \sum_{a+2b \leq 4} \| \partial^a \partial^b_t q \|_{L^2_x L^2_t}^2 + \sum_{a+2b \leq 3} \| \partial^a \partial^b_t h_1 \|_{C^0_t L^2_x}^2 + \kappa^2 \sum_{a+2b \leq 3} \| \partial^a \partial^b_t h_1 \|_{C^0_t L^2_x}^2
+ \sum_{b=0}^2 \| \partial^b_t q \|_{C^0_t H^3_x}^2 + \sum_{b=0}^2 \| \partial^b_t q \|_{L^2_t H^5_x}^2 - 2b
+ \sum_{b=0}^2 \| \partial^b_t h_1 \|_{L^2_t H^5_x}^2 - 2b + \sum_{b=0}^2 |\partial_\kappa \Lambda_\kappa h|_{L^2_t H^5_x}^2 - 2b.
\] (2.16)

As a consequence of Theorem 2.1 the map \( t \mapsto \mathcal{E}_\kappa(t) \) is continuous on \([0, T_\kappa]\).

### 2.3. Bounds on lower-order norms

Let
\[
\mathcal{A}_\kappa(t) = \sum_{a+2b \leq 2} \| \partial^a \partial^b_t q \|_{L^2_x L^2_t}^2 + \sum_{a+2b \leq 1} \| \partial^a \partial^b_t h_1 \|_{L^2_t L^2_x}^2
+ \kappa^2 \sum_{a+2b \leq 2} \| \partial^a \partial^b_t h_1 \|_{L^2_x L^2_t}^2 + \kappa^2 \sum_{a+2b \leq 1} \| \partial^a \partial^b_t h_1 \|_{L^2_t L^2_x}^2
+ \sum_{b=0}^1 \| \partial^b_t q \|_{L^2_t H^5_x}^2 - 2b + \sum_{b=0}^1 \| \partial^b_t q \|_{L^2_t H^3_x}^3 - 2b + \sum_{b=0}^1 \| \partial^b_t \Lambda_\kappa h \|_{L^2_t H^3_x}^3 - 2b + |\partial_\kappa \Lambda_\kappa h|_{L^2_t H^3_x}^3 - 2b.
\]

We then assume that
\[
\mathcal{A}_\kappa(t) \leq \mathcal{E}_\kappa(0) + 1, \quad t \in [0, T_\kappa].
\] (2.17)

By the fundamental theorem of calculus it is easy to see that
\[
\mathcal{A}_\kappa(t) \leq \mathcal{A}_\kappa(0) + \sup_{0 \leq s \leq t} \mathcal{E}_\kappa(s) \leq \mathcal{E}_\kappa(0) + t \sup_{0 \leq s \leq t} \mathcal{E}_\kappa(s).
\]

In Section 2.8 we will prove an a priori bound for \( \mathcal{E}_\kappa \) independent of \( \kappa \) and show that the time of existence \( T \) is independent of \( \kappa \). The bound (2.17) will then be justified a posteriori using the
fundamental theorem of calculus, smallness of \( T_\kappa \), and the definition of \( E_\kappa \). By choosing \( T_\kappa \) possibly smaller we assume that for certain \( \delta > 0 \)
\[
\min_{x' \in \Gamma} q_\kappa (t, x') > \delta \text{ and } |\partial h_\kappa (t, \cdot)|^2_{\infty} \leq 1/2, \quad t \in [0, T_\kappa],
\]
where \((q, h)\) is the solution of the \( \kappa \)-problem (2.4). The first inequality is true by continuity-in-time of the energy \( E_\kappa \) and the Taylor sign condition (2.11). The second inequality follows from the from the continuity-in-time and smallness of \(|\partial h_0|_{\infty} (1.3)\).

2.4. The energy identities. In this section we collect the high-order energy identities in two lemmas stated below. We use the notation \( T \) for those error terms which in an easy straightforward way are seen to satisfy the energy bound of the form:
\[
\int_0^t |T(s)| \; ds \lesssim t P(E_\kappa);
\]
this bound will then always follow from the standard \( L^\infty - L^2 - L^2 \) type estimates. Here and in the rest of the paper \( P(\cdot) \) stands for a generic polynomial satisfying \( P(0) = 0 \).

**Lemma 2.2.** Assume that \((q, h)\) is a solution to the regularized Stefan problem (2.4) given by Theorem 2.1. Then the following identities hold:

(i)
\[
\int \bar{\partial}^4 v \; \bar{\partial}^4 q + \frac{1}{2} \frac{d}{dt} \int \bar{\partial}^4 q \; \bar{\partial}^4 \kappa \; \bar{\partial}^4 v + \frac{1}{2} \int (\bar{\partial}^4 q + \bar{\partial}^4 \kappa \cdot v)^2
\]
\[
+ \kappa^2 \int J^{-1} \bar{\partial}^4 h \; |^2 = \int R_1 + \int R_2 + T;
\]
(ii)
\[
\int \bar{\partial}^2 \bar{\partial} h \; |^2 = \int R_3 + \int R_4 + T;
\]
\[
\int \bar{\partial}^2 \bar{\partial} h \; |^2 = \int R_5 + \int R_6 + T,
\]
where \( R_i, \; i = 1, \ldots, 6, \) are error terms given below respectively by (2.41), (2.42), (2.43), (2.44), (2.45) and (2.46).

We postpone the proof of Lemma 2.2 to Section 2.6.
2.5. Equivalence of the higher-order norm $\mathcal{E}_k$ and the natural energy function $\mathcal{F}_k$. By summing the left-hand sides of the identities (2.19)–(2.23) from Lemma 2.2, the natural coercive quadratic form $\mathcal{F}_k$ that arises as the energy takes the form

$$
\mathcal{F}_k = \sum_{a+2b \leq 4} \| \partial^a \partial^b_t v \|^2_{L_t^2 L^a_x^b} + \frac{1}{2} \sum_{a+2b \leq 3} \| \partial^a \partial^b_t v \|^2_{L_t^\infty L^a_x^b} + \frac{\kappa^2}{2} \sum_{a+2b \leq 4} \| J_k^{-1/2} \partial^a \partial^b_t A_k h_t \|^2_{L_t^\infty L^a_x^b} + \frac{1}{2} \sum_{a+2b \leq 4} \| J_k^{-1/2} \partial^a \partial^b_t A_k h_t \|^2_{L_t^\infty L^a_x^b} + \sum_{a+2b \leq 3} \| \sqrt{q_t} \ J_k^{-1/2} \partial^a \partial^b_t \Lambda_k h_t \|^2_{L_t^L L^a_x^b} + \sum_{a+2b \leq 3} \| \sqrt{q_t} \ J_k^{-1/2} \partial^a \partial^b_t \Lambda_k h_t \|^2_{L_t^L L^a_x^b} + \sum_{a+2b \leq 3} \| \sqrt{q_t} \ J_k^{-1/2} \partial^a \partial^b_t \Lambda_k h_t \|^2_{L_t^L L^a_x^b} + \sum_{a+2b \leq 3} \| \sqrt{q_t} \ J_k^{-1/2} \partial^a \partial^b_t \Lambda_k h_t \|^2_{L_t^L L^a_x^b} + \sum_{a+2b \leq 3} \| \sqrt{q_t} \ J_k^{-1/2} \partial^a \partial^b_t \Lambda_k h_t \|^2_{L_t^L L^a_x^b}
$$

(2.24)

The mathematical reason for imposing the Taylor sign condition (1.23) now becomes apparent. In order for the second line in the definition of $\mathcal{F}_k$ (2.24) above to make sense we must have

$$
\min_{x' \in \Gamma(t)} (q(x', t), \nu_x) > 0,
$$

as it was assumed in (1.23) for the (unregularized) classical Stefan problem. In order to perform the estimates in the next section, it is crucial to show that the energy $\mathcal{E}_k$ is bounded by $\mathcal{F}_k$. To prove this statement we first establish the following temperature estimate.

**Lemma 2.3.** Let $(q, h)$ be a solution of the regularized problem (2.4) given by Theorem 2.1. Assume that the a priori assumption (2.17) holds on $[0, T_k]$. Then

(i)

$$
\sum_{a=0}^{2} \| \partial^a_t q \|^2_{L_t^\infty H^{a-2}_x} \lesssim \mathcal{F}_k \text{ on } [0, T_k].
$$

(2.25)

(ii)

$$
\| q \|^2_{L_t^2 H^2_x} + \sum_{a=1}^{2} \| \partial^a_t q \|^2_{L_t^\infty H^{a-2}_x} \lesssim \mathcal{F}_k \text{ on } [0, T_k].
$$

(2.26)

**Proof.** We use elliptic regularity theory and the a priori assumption (2.17) to show that $\| q \|^2_{L_t^\infty H^2_x} \lesssim \mathcal{F}_k$ since $\Delta \psi = q + v \cdot \nu + \alpha$, $\| q \|_{L_t^\infty H^1_x} \lesssim \mathcal{F}_k$, and $\| \alpha \|_{L_t^\infty L^2_x} \lesssim \mathcal{F}_k(0) \lesssim \mathcal{F}_k$. Differentiating (2.4a) with respect to $x^j$ ($j = 1, 2$), we obtain that $\Delta \psi q_{,j} = (\mathcal{A}_1^m \mathcal{A}_1^n)_{,j} q_{,mn} + (q_t + \nu \cdot \nu)_t + \alpha$. Furthermore, since $q_{,j} = \Psi_{,j} \cdot v$ we have that

$$
\| q_{,j} \|_{L_t^\infty H^1_x} \lesssim \| \Psi_{,j} \|_{L_t^\infty L^2_x} \| v \|_{L_t^{\infty} L^2_x} + \| \Psi_{,j} \|_{L_t^\infty L^2_x} \| v_t \|_{L_t^{\infty} L^2_x} \lesssim \| h_t \|_{L_t^\infty H^2_x} \| v \|_{L_t^{\infty} L^2_x} + \| \nabla \psi \|_{L_t^\infty H^1_x} \| v_t \|_{L_t^{\infty} L^2_x} \lesssim \mathcal{F}_k,
$$

where we have used the trace bound (2.2) and the a priori assumption (2.17). Note that

$$
\| (v \cdot \nu)_{,j} \|_{L_t^{\infty} H^1_x} \lesssim \| v \|_{L_t^{\infty} H^1_x} \| \nu \|_{L_t^{\infty} L^2_x} + \| v \|_{L_t^{\infty} L^2_x} \| w_{,j} \|_{L_t^{\infty} L^2_x} \lesssim \| q \|_{L_t^{\infty} H^2_x} \| h_{,j} \|_{L_t^{\infty} H^2_x} + \| q \| \| h_{,j} \|_{L_t^{\infty} H^2_x} \lesssim \mathcal{F}_k,
$$

where we have used the bound (2.17) in the last estimate. It is easy to see that $\| (\mathcal{A}_1^m \mathcal{A}_1^n)_{,j} q_{,mn} \|_{L_t^\infty L^2_x} \lesssim \mathcal{F}_k(1 + P(A_k)) \lesssim \mathcal{F}_k$, where $P$ stands for a generic polynomial. Finally, $\| \nabla \psi \|_{L_t^\infty L^2_x} \lesssim \mathcal{F}_k$. Thus, by the elliptic theory again, we conclude

$$
\| q \|^2_{L_t^\infty H^2_x} \lesssim \mathcal{F}_k.
$$

Differentiating (2.4a) with respect to $t$, we obtain $\Delta \psi q_t = -(\mathcal{A}_1^m \mathcal{A}_1^n)_{,j} q_{,mn} + v_{tt} + (v \cdot \nu)_t$ (since $\alpha$ is independent of $t$). Again, using $\| v_{tt} \|^2_{L_t^{\infty} L^2_x} \lesssim \mathcal{F}_k$, the previous estimates and the bound $\| h_{,j} \|_{L_t^{\infty} H^2_x} \lesssim \mathcal{F}_k$, elliptic regularity implies $\| q_t \|^2_{L_t^{\infty} H^2_x} \lesssim \mathcal{F}_k$. Furthermore $\| q_t \|^2_{L_t^{\infty} L^2_x} \lesssim \| q_t + \nu v_t \cdot v \|_{L_t^{\infty} L^2_x} + \| w_{tt} \cdot v \|^2_{L_t^{\infty} L^2_x} \lesssim \mathcal{F}_k$. The last equality follows from the third line on the definition (2.16) of $\mathcal{F}_k$ and a
simple bound on the $L_2^\infty$-norm of $v$, which follows from Sobolev embedding. Finally, choose any $j,k \in \{1,2\}$. Applying $\partial_{x_j} \partial_{x_k}$ to (2.4a), we arrive at the elliptic equation

$$\Delta \Psi_{\kappa} q_{j,k} = - (A_{m_i}^m A_i^m)_{j} q_{m,n} k - (A_{m_i}^m A_i^n)_{j,k} q_{m,n} + (q_t + v \cdot w)_{j,k} + \alpha_{j,k}.$$  

By the estimates already derived above, (2.16), (2.6), and (2.17), the right-hand side is bounded by $F_\kappa (1 + P(A_\kappa)) \lesssim F_\kappa$ in $L_2^\infty$. Thus, by elliptic regularity, we finally conclude $\|q\|_{L_2^\infty} \lesssim F_\kappa$, concluding the proof of (2.25).

To prove (2.26) we start with the easiest case $b = 2$. For $j = 1, 2$, we have

$$q_{j,tt} = (\Psi_{\kappa,j} \cdot v)_{tt} = \Psi_{\kappa,j} \cdot v + 2 \Psi_{\kappa,j} \cdot v_t + \Psi_{\kappa,j} \cdot v_{tt}.$$  

From the above we easily infer that

$$\int_0^t \| \nabla q_{tt} \|_0^2 \, d\tau \lesssim \int_0^t \left( \| \nabla \Psi_{\kappa,tt} \|_0^2 \| v \|_\infty^2 + \| \Psi_{\kappa,tt} \|_\infty^2 \| v_t \|_0^2 + \| \Psi_{\kappa,j} \|_\infty^2 \| v_{tt} \|_0^2 \right) \, d\tau.$$  

We thereby used the trace estimate (2.2) to obtain

$$\| \nabla \Psi_{\kappa,tt} \|_0^2 \lesssim \| h_{\kappa,tt} \|_0 \lesssim F_\kappa,$$  

where we have used the definition (2.24) in the last bound above. On the other hand, using the a priori bound (2.17) and the Sobolev embedding we conclude that $\|v\|_\infty \lesssim 1 \lesssim \|q\|_2 \leq F_\kappa(0) + 1 \lesssim 1$. The remaining two terms on the right-hand side of (2.27) are estimated in a similar fashion. If $b = 1$ we apply the same ideas using (2.25), (2.17), and the Sobolev embeddings. To prove $\|q\|_1^2 L_2^\infty L_2^\infty \lesssim F_\kappa$ we need to use an interpolation estimate. The strategy consists of estimating $\|q\|_1^2 L_2^\infty L_2^\infty$ and $\|q\|_1^2 L_2^\infty L_2^\infty$ separately and then interpolating between the two estimates. The reader may consult [31] for the details.

Remark 8. The regularity of $q \in L_2^1 H_\sigma^{1,5}$ can in fact be improved.

Lemma 2.4 (Optimal regularity for $\Psi_\kappa$ and $q$). Suppose that the pair $(q,h)$ is a solution of the $\kappa$-problem (2.4) given by Theorem 2.1, and that the basic assumption (2.17) holds on $[0,T_\kappa)$. Then

$$\int_0^{T_\kappa} (\| \Psi_{\kappa} \|_0^2 + \| q \|_0^2) \, dt \leq CE_\kappa(t).$$

Proof. Step 1. We will first prove that $\int_0^{T_\kappa} \| \Psi_{\kappa} \|_0^2 \, dt \leq CE(t)$. Since $q = 0$ on $\Gamma$ it follows that

$$v(x,t) \cdot \tau_\kappa(x,t) = 0 \text{ on } \Gamma,$$  

where we recall that $\tau_\kappa$ is the unit tangent vector to $\Psi_\kappa(t,\Gamma)$. Applying the horizontal derivative $\partial$ to (2.28), and using the fact that $\partial \tau_\kappa = g_{\kappa}^{-1} \partial^2 \Psi_\kappa \cdot n_\kappa \cdot n_\kappa$ and that $\partial^2 \Psi_\kappa \cdot n_\kappa = - g_{\kappa}^{-1} \partial^2 h_\kappa$, we find that

$$\partial^2 h_\kappa = g_{\kappa}^{-1} \partial^2 v \cdot \tau_\kappa.$$  

The dominator in (2.32) is strictly positive for $T_\kappa$ small enough by the Taylor sign condition (2.11). For any $W : \Omega \to \mathbb{R}^2$ we define

$$\text{curl}_\Psi W = \varepsilon_{ji} A_j^i W_i,$$  

where $\varepsilon_{21} = - \varepsilon_{12} = 1$, $\varepsilon_{11} = \varepsilon_{22} = 0$. By the tangential trace inequality (see [11]),

$$\| \partial^3 v \cdot \tau_\kappa \|_{H^{-\frac{1}{2}}(\Gamma)} \lesssim \| \text{curl}_\Psi \partial^3 v \|_{L_2^\infty(\Omega)} + \| \partial^4 v \|_{L_2^2(\Omega)}.$$  

We observe that

$$\text{curl}_\Psi \partial^3 v = \partial^3 (\text{curl}_\Psi v) - \varepsilon_{ji} \sum_{m=1}^3 c_m \partial^m A_j^i \partial^{3-m} v_{ij}$$

$$= - \varepsilon_{ji} \sum_{m=1}^3 c_m \partial^3 A_j^i \partial^{3-m} v_{ij}.$$  

where we have used the identity $\text{curl}_\Psi v = (\text{curl} \nabla p) \circ \Psi = 0$. Using the Cauchy-Schwarz inequality and the definition of $E_\kappa$ we obtain

$$\| \text{curl}_\Psi \partial^3 v \|_0 \lesssim \sqrt{E_\kappa}.$$
From (2.31) and the definition (2.16) of $E_\kappa$ we obtain
\[
\int_0^t \left[ \partial^4 v \cdot \tau_\kappa \right]^2 \frac{1}{H^{\kappa - \frac{1}{2}}(\Gamma)} d\sigma \lesssim E_\kappa(t), \quad 0 \leq t \leq T_\kappa.
\] (2.32)

Using (2.32) and (2.31) it follows easily that $\int_0^1 |h_\kappa|^2 d\sigma \lesssim E_\kappa(t), \quad 0 \leq t \leq T_\kappa$. Recalling that $\Psi_\kappa$ is the harmonic extension of $(x', h_\kappa(x'))$, $x' \in \Gamma$ the optimal trace inequality (1.8) implies that $\int_0^1 \|\Psi_\kappa\|^2_3 d\sigma \leq CE_\kappa(t)$ for any $t \in [0, T_\kappa]$.

**Step 2.** The fact that $\int_0^{T_\kappa} \|q\|_3^2 dt \leq CE(t)$ follows from Step 1, and the elliptic regularity result in Theorem 3.6 in [11].

As a consequence of Lemmas 2.3 and 2.4 we obtain the following key bound between the norm $E_\kappa$ and the energy $F_\kappa$.

**Proposition 2.5** (Norm-energy equivalence). Let $(q, h)$ be a solution of the $\kappa$-problem (2.4) given by Theorem 2.1. Assume that the a priori assumption (2.17) holds on $[0, T_\kappa]$. Then $E_\kappa \lesssim F_\kappa$ on $[0, T_\kappa]$.

**Proof.** Due to the Taylor sign condition (2.18), the boundary integrals in $E_\kappa$ and $F_\kappa$ satisfy
\[
\sum_{b=0}^2 |\Lambda_h h|^2_{L^\infty_t L^{2b+2}} + \sum_{b=0}^1 |\Lambda_k h|^2_{L^\infty_t L^{2b}} \lesssim \sum_{a+2b \leq 4} \sqrt{-q} \partial^a \bar{\partial}^b \Lambda h |_{L^\infty_t L^2} + \sum_{a+2b \leq 3} \sqrt{-q} \partial^a \bar{\partial}^b \Lambda h |_{L^\infty_t L^2}.
\]

The remaining estimates now follow directly from Lemmas 2.3 and 2.4. \qed

**2.6. Proof of Lemma 2.2.** Proof of part (i) of Lemma 2.2. Applying the tangential differential operator $\bar{\partial}^4$ to the equation (2.4b), multiplying it by $\bar{\partial}^4 v^i$ and integrating over $\Omega$, we obtain
\[
(\bar{\partial}^4 v^i + \bar{\partial}^4 A^k_i q_{,k} + A^k_i \bar{\partial}^4 q_{,k} + \bar{\partial}^4 v^i)_{L^2} = \sum_{l=1}^3 c_l (\bar{\partial}^4 A^k_i \bar{\partial}^4 q_{,k} + \bar{\partial}^4 v^i)_{L^2},
\] (2.33)

where $c_l = \left( \begin{array}{c} 1 \\ 1 \end{array} \right)$. Recalling (1.17), we write
\[
\bar{\partial}^4 A^k_i = -A^k_i \bar{\partial}^4 \Psi_{\kappa,n} A^k_i + \{\bar{\partial}^4 A^k_i\},
\] (2.34)

where $\{\bar{\partial}^4 A^k_i\}$ stands for the lower order commutator defined in (1.17). With this identity, we obtain
\[
(\bar{\partial}^4 A^k_i q_{,k} + \bar{\partial}^4 v^i)_{L^2(\Omega)} = -\left( A^k_i \bar{\partial}^4 \Psi_{\kappa,n} A^k_i q_{,k} + \bar{\partial}^4 v^i \right)_{L^2(\Omega)} - \left( \{\bar{\partial}^4 A^k_i\} q_{,k} + \bar{\partial}^4 v^i \right)_{L^2(\Omega)}
\] (2.35)
\[
= -\int_{\Omega} q_{,k} A^k_i \bar{\partial}^4 \Psi_{\kappa,n} A^k_i \bar{\partial}^4 v^i + \int_{\Omega} A^k_i \bar{\partial}^4 \Psi_{\kappa,n} q_{,k} \bar{\partial}^4 v^i + T
\] (2.35)
\[
= -\int_{\Omega} q_{,k} A^k_i \bar{\partial}^4 \Psi_{\kappa,n} A^k_i \bar{\partial}^4 v^i - \int_{\Omega} A^k_i \bar{\partial}^4 \Psi_{\kappa,n} v^i \bar{\partial}^4 v^i + T.
\]

where we have used $(A^k_i)_{,s} = 0$ and the identity $v^r = -A^k_i q_{,k}$ to write the last line more concisely. Furthermore, integrating by parts with respect to $x^k$
\[
(\bar{\partial}^4 A^k_i q_{,k} + \bar{\partial}^4 v^i)_{L^2} = \int_{\Omega} q_{,k} \bar{\partial}^4 q A^k_i \bar{\partial}^4 v^i + \int_{\Gamma} A^k_i \bar{\partial}^4 q \bar{\partial}^4 v^i + N^k - \int_{\Omega} A^k_i \bar{\partial}^4 q \bar{\partial}^4 v^i_{,k}.
\] (2.36)

Note that the boundary contribution coming from the fixed boundary $\partial \Omega_{top}$ vanishes due to the boundary condition (2.4e), which further reduces to $v^2 = 0$ on $\partial \Omega_{top}$. Summing (2.35) and (2.36), we obtain
\[
(\bar{\partial}^4 A^k_i q_{,k} + \bar{\partial}^4 q_{,k} + \bar{\partial}^4 v^i)_{L^2(\Omega)} = -\int_{\Gamma} q_{,k} A^k_i \bar{\partial}^4 \Psi_{\kappa,n} A^k_i \bar{\partial}^4 v^i + \int_{\Gamma} A^k_i \bar{\partial}^4 q \bar{\partial}^4 v^i_{,k} + N^k
\] (2.37)
The first three terms on the right-hand side of (2.37) will be the source of positive definite quadratic contributions to the energy. To extract the quadratic coercive contribution from the first integral on the right-hand side of (2.37), we simplify it to

\[ - \int_{\Gamma} q_k \bar{A}_k^4 \bar{\partial}^4 \Psi_{\kappa} \bar{A}_\kappa^2 \bar{\partial}^4 v^i N^* = \int_{\Gamma} q_{k1} \bar{A}_k^2 \bar{\partial}^4 \Psi_{\kappa} \bar{A}_\kappa^2 \bar{\partial}^4 v^i + \int_{\Gamma} q_{k1} \bar{A}_k^2 \bar{\partial}^4 \Psi_{\kappa} \bar{A}_\kappa^2 \bar{\partial}^4 v^i \]

\[ = \int_{\Gamma} q_{k2} \bar{\partial}^4 \Psi_{\kappa} \bar{A}_\kappa^2 \bar{\partial}^4 v \cdot \bar{A}_\kappa^2 + \int_{\Gamma} q_{k1} \bar{\partial}^4 \Psi_{\kappa} \bar{A}_\kappa^2 \bar{\partial}^4 v \cdot \bar{A}_\kappa^2. \]

We rewrite the expression \( \bar{\partial}^4 v \cdot \bar{A}_\kappa^2 \) and thereby use the boundary condition (2.4d):

\[ \bar{\partial}^4 v \cdot \bar{A}_\kappa^2 = \bar{\partial}^4 w \cdot \bar{A}_\kappa^2 + \bar{\partial}^4 (v + w) \cdot \bar{A}_\kappa^2 \]

\[ = \bar{\partial}^4 w \cdot \bar{A}_\kappa^2 + \bar{\partial}^4 ((v + w) \cdot \bar{A}_\kappa^2) - \sum_{l=1}^{4} a_l \bar{\partial}^4 l (v + w) \cdot \bar{\partial}^4 \bar{A}_\kappa^2 \]

\[ = \bar{\partial}^4 w \cdot \bar{A}_\kappa^2 - \sum_{l=1}^{4} a_l \bar{\partial}^4 l (v + w) \cdot \bar{\partial}^4 \bar{A}_\kappa^2. \]

Due to the above identity and recalling \( \Psi_{\kappa} = \Lambda_{\kappa} \Lambda_{\kappa} \Psi \), we obtain

\[ \int_{\Gamma} q_{k2} \bar{\partial}^4 \Psi_{\kappa} \cdot n_{\kappa} \bar{\partial}^4 v \cdot \bar{A}_\kappa^2 = \int_{\Gamma} q_{k2} \bar{\partial}^4 \Lambda_{\kappa} \Lambda_{\kappa} \Psi \cdot \bar{A}_\kappa^2 \bar{\partial}^4 w \cdot \bar{A}_\kappa^2 - \sum_{l=1}^{4} a_l \int_{\Gamma} q_{k2} \bar{\partial}^4 \Psi_{\kappa} \cdot \bar{A}_\kappa^2 \bar{\partial}^4 l (v + w) \cdot \bar{\partial}^4 \bar{A}_\kappa^2. \]

The first term on the right-hand side of (2.38) is rewritten in the following way

\[ \int_{\Gamma} q_{k2} \bar{\partial}^4 \Lambda_{\kappa} \Lambda_{\kappa} \Psi \cdot \bar{A}_\kappa^2 \bar{\partial}^4 w \cdot \bar{A}_\kappa^2 = \int_{\Gamma} q_{k2} \bar{\partial}^4 \Lambda_{\kappa} \Lambda_{\kappa} \Psi \cdot \bar{A}_\kappa^2 \bar{\partial}^4 t \cdot \bar{A}_\kappa^2 \]

\[ + \int_{\Gamma} \bar{\partial}^4 \Lambda_{\kappa} \Lambda_{\kappa} \Psi \left[ \Lambda_{\kappa} \left( q_{k2} \left( \bar{\partial}^4 \Psi_{\kappa} \cdot \bar{A}_\kappa^2 \right) + q_{k2} \left( \bar{\partial}^4 \Lambda_{\kappa} \Lambda_{\kappa} \Psi \cdot \bar{A}_\kappa^2 \right) \right) \right]
\]

\[ = \frac{1}{2} \partial_t \int_{\Gamma} q_{k2} \left| \bar{\partial}^4 \Lambda_{\kappa} \Lambda_{\kappa} \Psi \cdot \bar{A}_\kappa^2 \right|^2 \right| - \frac{1}{2} \int_{\Gamma} q_{k2} \left| \bar{\partial}^4 \Lambda_{\kappa} \Lambda_{\kappa} \Psi \cdot \bar{A}_\kappa^2 \right|^2 - \int_{\Gamma} q_{k2} \bar{\partial}^4 \Lambda_{\kappa} \Lambda_{\kappa} \Psi \cdot \bar{A}_\kappa^2 \bar{\partial}^4 t \cdot \bar{A}_\kappa^2,
\]

\[ + \int_{\Gamma} \lambda_{\kappa} \bar{\partial}^4 \Lambda_{\kappa} \Lambda_{\kappa} \Psi \left[ \Lambda_{\kappa} \left( q_{k2} \left( \bar{\partial}^4 \Psi_{\kappa} \cdot \bar{A}_\kappa^2 \right) + q_{k2} \left( \bar{\partial}^4 \Lambda_{\kappa} \Lambda_{\kappa} \Psi \cdot \bar{A}_\kappa^2 \right) \right) \right]
\]

\[ = \frac{1}{2} \partial_t \int_{\Gamma} q_{k2} \left| \bar{\partial}^4 \Lambda_{\kappa} \Lambda_{\kappa} \Psi \cdot \bar{A}_\kappa^2 \right|^2 + \int_{\Gamma} \lambda_{\kappa} \bar{\partial}^4 \Psi_{\kappa} \left[ \Lambda_{\kappa} \left( q_{k2} \left( \bar{\partial}^4 \Psi_{\kappa} \cdot \bar{A}_\kappa^2 \right) + q_{k2} \left( \bar{\partial}^4 \Lambda_{\kappa} \Lambda_{\kappa} \Psi \cdot \bar{A}_\kappa^2 \right) \right) \right] + T. \]

The second term on the right-hand side of (2.37) turns into

\[ \int_{\Gamma} A_k^4 \bar{A}_k^4 \bar{\partial}^4 v^i \cdot N^k = - \int_{\Gamma} \bar{\partial}^4 q \bar{\partial}^4 v \cdot \bar{A}_k^2 \]

\[ = \kappa^2 \int_{\Gamma} \bar{\partial}^4 h_i (\bar{\partial}^4 (v \cdot \bar{A}_k^2) - \sum_{l=0}^{3} a_l \bar{\partial}^4 l (v \bar{\partial}^4 (-l) \bar{A}_k^2)) + \kappa^2 \int_{\Gamma} \beta (t,x') \bar{\partial}^4 v \cdot \bar{A}_k^2 \]

\[ = \kappa^2 \int_{\Gamma} \bar{\partial}^4 h_i (\bar{\partial}^4 (v \cdot \bar{A}_k^2) - \sum_{l=0}^{3} a_l \bar{\partial}^4 l (v \bar{\partial}^4 (-l) \bar{A}_k^2) + \kappa^2 \int_{\Gamma} \bar{\partial}^4 t (t,x') \bar{\partial}^4 v \cdot \bar{A}_k^2. \]

where we have used the boundary condition (2.4c) in the second equality above (recall \( v \cdot \bar{A}_k^2 = w \cdot \bar{a}_k^2 = h_k \)). As to the third term on the right-hand side of (2.37), note that

\[ \bar{A}_k^4 \bar{\partial}^4 v^i \cdot n_k = \bar{\partial}^4 (\bar{A}_k^4 v^i \cdot n_k) - \sum_{l=1}^{4} q_{l1} \bar{A}_k^4 \bar{\partial}^4 (-l) v^i \cdot n_k = -\bar{\partial}^4 (q_i + v \cdot \bar{a}_k^2) - \sum_{l=1}^{4} q_{l1} \bar{A}_k^4 \bar{\partial}^4 (-l) v^i \cdot n_k, \]
where $\mathcal{A}_t^k v_{i,k} = -\text{div}_x v = -(q_t + v \cdot \nabla v) + \alpha$ by the parabolic equation (2.4a). Thus

$$- \int_\Omega \nabla q \cdot \nabla \Psi \cdot v = \int_\Omega (q_t + \Psi_{xt} \cdot v - \alpha) (\nabla q + \nabla^2 \Psi \cdot v) + \sum_{l=1}^4 c_l \int_\Omega \nabla^3 q \cdot \nabla^2 \Psi \cdot v$$

$$= \frac{1}{2} \partial_t \int_\Omega (\nabla^4 q + \nabla^4 \Psi \cdot v) + \int_\Omega \sum_{l=1}^4 d_l \nabla^4 \Psi_{xt} \cdot \nabla v - \nabla^4 \Psi \cdot v_t (\nabla^4 q + \nabla^4 \Psi \cdot v)$$

$$= \frac{1}{2} \partial_t R_1 + \sum_{l=1}^4 c_l \int_\Omega \nabla^3 q \cdot \nabla^3 v + \int_\Omega \partial_t \alpha (\nabla^4 q + \nabla^4 \Psi \cdot v).$$

Combining (2.37), (2.38), (2.39) and (2.40) we obtain the identity (2.19) with the error terms $R_1$ and $R_2$ given by:

$$R_1 := \sum_{l=1}^3 c_l \partial_l \mathcal{A}_t^k \partial_l q, \partial_l v - \left( \sum_{l=1}^3 c_l \partial_l \mathcal{A}_t^k \partial_l q, \partial_l v \right) (\nabla^4 q + \nabla^4 \Psi \cdot v)$$

$$+ \partial_t \alpha (\nabla^4 q + \nabla^4 \Psi \cdot v);$$

$$R_2 := -A_\kappa \partial^4 q \left[ \prod_{\kappa \neq \Lambda_\kappa} \left( \frac{\bar{A}_t}{A_t} \right) \mathcal{A}_t^2 \right] - (q_{2,2} - (\partial^4 \Psi_t \cdot \mathcal{A}_t^2) | A_t^2 |) + \sum_{l=0}^3 q_{l,2} \partial^4 \Psi \cdot A_t^2 \partial^4 v$$

$$- \left( \sum_{l=0}^3 q_{l,2} \partial^4 \Psi \cdot A_t^2 \partial^4 v \right) (\nabla^4 q + \nabla^4 \Psi \cdot v).$$

Applying the tangential differential operator $\partial_t \partial_t$ to the equation (2.4b), multiplying it by $\partial_t v$ and integrating over $\Omega$, we obtain in a completely analogous fashion identity (2.20) claimed in Lemma 2.2 with error terms $R_3$ and $R_4$ given by:

$$R_3 := \sum_{1 \leq m \leq \alpha} \sum_{n \leq \alpha} c_{mn} \bar{d}_m \partial^m \partial^n \partial^4 v_{i,k}$$

$$+ \sum_{0 \leq m \leq \alpha} d_{mn} \partial^m \partial^n \partial^4 v_{i,k},$$

$$R_4 := -A_\kappa \partial^4 q \left[ \prod_{\kappa \neq \Lambda_\kappa} \left( \frac{\bar{A}_t}{A_t} \right) \mathcal{A}_t^2 \right]$$

$$+ \sum_{l \geq 1} q_{l,2} \partial^4 \partial_t v \cdot A_t^2 \partial^4 v \cdot A_t^2 + \mathcal{A}_t^2 \partial^4 v \partial^4 v \cdot A_t^2 - \kappa^2 \partial^4 v \partial^4 v \cdot A_t^2.$$
implying
\[
\frac{1}{2} \partial_t \int_{\Omega} |\bar{\partial}^3 v|_2^2 + (\bar{\partial}^3 \partial_t A^k \cdot q_{,k} + A^k_i \bar{\partial}^3 \partial_t q_{,k}, \bar{\partial}^3 v)_L^2 = \sum_{1 \leq i \leq 3, k + \ell = 1} c_{l, k, \ell, k}(\bar{\partial}^i \partial_t A^k \bar{\partial}^i \partial_t q_{,k}, \bar{\partial}^3 v)_L^2.
\] (2.47)

Recalling (1.17), we write
\[
\bar{\partial}^3 \partial_t A^k_i = -A^k_i \bar{\partial}^3 w \cdot r A^s_i + (\bar{\partial}^3 \partial_t A^k_i).
\] (2.48)

Using this decomposition we have
\[
(\bar{\partial}^3 \partial_t A^k_i q_{,k} + A^k_i \bar{\partial}^3 \partial_t q_{,k}, \bar{\partial}^3 v)_L^2 = - \int_\Omega A^k_i \bar{\partial}^3 w \cdot r A^s_i q_{,k} \bar{\partial}^3 v + \int_\Omega A^k_i \bar{\partial}^3 \partial_t q_{,k} \bar{\partial}^3 v + \int_\Gamma \bar{\partial}^3 \partial_t q_{,k} \bar{\partial}^3 v + \mathcal{T},
\] (2.49)
where the commutator term has been absorbed in the error \(\mathcal{T}.\) Integrating by parts with respect to \(s\) and \(k\) in the first two integrals on the right-hand side above respectively, we obtain analogously to the proof of Lemma 2.2:
\[
- \int_\Omega A^k_i \bar{\partial}^3 w \cdot r A^s_i q_{,k} \bar{\partial}^3 v + \int_\Omega A^k_i \bar{\partial}^3 \partial_t q_{,k} \bar{\partial}^3 v = \int_\Gamma q_{,2} \bar{\partial}^3 w \cdot A^2_i \bar{\partial}^3 v \cdot A^2_i + \int_\Gamma q_{,1} A^k_i \bar{\partial}^3 w \cdot \bar{\partial}^3 v \cdot A^2_i + \int_\Gamma \sum_{l=1}^3 d_l \bar{\partial}^3 w \cdot \bar{\partial}^3 v + \sum_{l=1}^3 \bar{\partial}^3 \partial_t A^k_i \bar{\partial}^3 v \cdot \bar{\partial}^3 w \cdot v + \mathcal{T}.
\] (2.50)

Note further that the first term on the right-hand side above can be, similarly to (2.39), further written as
\[
\int_\Gamma q_{,2} \bar{\partial}^3 w \cdot A^2_i \bar{\partial}^3 v \cdot A^2_i = \int_\Gamma q_{,2} |\bar{\partial}^3 \Lambda_\kappa w \cdot A^2_i|^2 + \int_\Gamma \Lambda_\kappa \bar{\partial}^3 w \cdot [\Lambda_\kappa q_{,2} (\bar{\partial}^3 w \cdot A^2_i) A^2_i] - q_{,2} (\bar{\partial}^3 \Lambda_\kappa w \cdot A^2_i) A^2_i].
\]

The second term on the right-hand side of (2.50) reads, using the boundary condition (2.4c)
\[
\int_\Gamma \bar{\partial}^3 \partial_t h_t \bar{\partial}^3 v \cdot A^2_i = \kappa^2 \int_\Gamma \bar{\partial}^3 \partial_t h_t (\bar{\partial}^3 (v \cdot A^2_i)) - \sum_{l=0}^2 c_l \bar{\partial}^3 v \cdot \bar{\partial}^3 \Lambda_\kappa A^2_i + \kappa^2 \int_\Gamma \bar{\partial}^3 \partial_t A^3 \cdot A^2_i + \mathcal{T},
\]
\[
= \kappa^2 \frac{d}{dt} \int_\Gamma J_\kappa^{-1} |\bar{\partial}^3 h_t|^2 - \kappa^2 \sum_{l=0}^2 c_l \int_\Gamma \bar{\partial}^3 \partial_t h_t (\bar{\partial}^3 (v \cdot A^2_i)) + \kappa^2 \int_\Gamma \bar{\partial}^3 \partial_t A^3 \cdot A^2_i + \mathcal{T},
\]
where the error term \(\mathcal{T}\) denotes the lower order terms containing the time derivative of \(J_\kappa.\) We also used the regularized boundary condition (2.4c) in the first equality above. Combining (2.47)–(2.50) and the last identity we obtain the identity (2.22) with error terms \(S_1\) and \(S_2\) given by
\[
S_1 := \sum_{1 \leq i \leq 3, k + \ell = 1} c_{l, k, \ell, k}(\bar{\partial}^i \partial_t A^k \bar{\partial}^i \partial_t q_{,k}) - \sum_{l=0}^3 d_l \bar{\partial}^3 \Lambda_\kappa w \cdot \bar{\partial}^3 v + \sum_{l=1}^3 \bar{\partial}^3 \partial_t A^k_i \bar{\partial}^3 v \cdot \bar{\partial}^3 w \cdot v; \tag{2.51}
\]
\[
S_2 := -\Lambda_\kappa \bar{\partial}^3 w \cdot [\Lambda_\kappa q_{,2} (\bar{\partial}^3 w \cdot A^2_i) A^2_i] - q_{,2} (\bar{\partial}^3 \Lambda_\kappa w \cdot A^2_i) A^2_i] + \sum_{l=1}^3 \bar{\partial}^3 \partial_t A^k_i \bar{\partial}^3 v \cdot \bar{\partial}^3 \Lambda_\kappa A^2_i \bar{\partial}^3 w \cdot A^2_i - q_{,2} (\bar{\partial}^3 \Lambda_\kappa w \cdot A^2_i) A^2_i] + \kappa^2 \sum_{l=0}^2 c_l \bar{\partial}^3 \partial_t h_t (\bar{\partial}^3 (v \cdot A^2_i)) - \kappa^2 \bar{\partial}^3 \bar{\partial}^3 v \cdot \bar{\partial}^3 w \cdot A^2_i. \tag{2.52}
\]
Applying the tangential operator $\bar{\partial}\bar{\partial}^2$ to the equation (2.4b), multiplying by $\bar{\partial}v^i$ and integrating over $\Omega$ we obtain the identity (2.23) in an analogous way, with error terms $S_3$ and $S_4$ given by

\begin{align}
S_3 := (v_i \cdot \bar{\partial}p_{\kappa tt} \bar{A}^i_k - \{ \bar{\partial}^2 (\bar{A}^i_k) \} \cdot q_k) \bar{\partial}v_i + \sum_{1 \leq m+n \leq 2} c_{mn} \bar{\partial}^m \bar{\partial}^n \bar{A}^i_k \bar{\partial}^{1-a} \bar{\partial}^{2-b} q_k \bar{\partial}v_i^t \\
+ \left( \sum_{1 \leq m+n \leq 2} d_{mn} \bar{\partial}^m \bar{\partial}^n \bar{A}^i_k \bar{\partial}^{1-a} \bar{\partial}^{2-b} v^i_{ts} - (\bar{\Psi}_{tt} v_t + \bar{\Psi}_{tt} v_t + \bar{\Psi}_{tt} \bar{\partial}v_t) \right) (q_k + \bar{\partial}\Psi_{\kappa tt} v) \right); \tag{2.53}
\end{align}

\begin{align}
S_4 := & q_2 \bar{\partial}w_i \cdot \bar{A}^i_k \left[ \left( \bar{\partial}(v + \bar{w}) \cdot \bar{A}^i_k \right) + \left( \bar{w}_t + v_t \right) \cdot \bar{\partial} \bar{A}^i_k \right] \\
& - \bar{\partial} \bar{\partial} \Lambda_w \cdot \bar{A}^i_k \left[ \left( \Lambda_w \left( - q_2 \right) \bar{\partial} \bar{\partial} \Lambda_w \cdot \bar{A}^i_k \right) + q_2 \bar{\partial} \bar{\partial} \Lambda_w \right] \\
& - q_1 \bar{A}^i_k \bar{\partial} \bar{\partial} \bar{w} \cdot \bar{\partial} \Lambda_i v \cdot \bar{A}^i_k + \kappa^2 \bar{\partial} \bar{\partial} \Lambda_i h_i \left( \bar{\partial} \bar{\partial} (v + \bar{w}) \cdot \bar{A}^i_k \right) - \kappa^2 \bar{\partial} \bar{\partial} \Lambda_i \bar{\partial} \bar{\partial} \Lambda_i v \cdot \bar{A}^i_k. \tag{2.54}
\end{align}

2.7. **Nonlinear energy estimates.** The following proposition states the desired energy bound for the classical Stefan problem (with $\sigma = 0$), will subsequently lead to a uniform-in-$\kappa$ time of existence for our family solutions to the regularized $\kappa$-problems (2.4).

**Proposition 2.6 (Main energy inequality).** There exists a constant $C$ independent of $\kappa$ and a generic polynomial function $P$ such that for any $t \in [0,T^*]$ we have the following bound:

$$ E_N(t) \leq C E_N(0) + C(t + \sqrt{t}) P(E_N). $$

The proof of the proposition proceeds by systematically estimating error terms in the energy identities from Section 2.4. We shall implicitly use the a priori bound (2.17) freely throughout the proof without explicitly making a reference to it.

**Step 1.** Estimates for $\int_0^t \int_\Omega R_1$ defined by (2.41). We start by estimating the integral $\sum_{l=1}^3 \int_0^t \int_\Omega \bar{\partial} A^i_k \bar{\partial}^{1-i} q_k \bar{\partial}^{4} v^i$ (the first term appearing in (2.41)). If $l = 1$, we have

$$ \left| \int_0^t \int_\Omega \bar{\partial} A^i_k \bar{\partial}^{1-i} q_k \bar{\partial}^{4} v^i \right| \leq \| \bar{\partial} A^i_k \|_{L^\infty L^2} \int_0^t \| \bar{\partial} q_k \|_{L^2} \| \bar{\partial}^{4} v^i \|_{L^2} $$

$$ \leq \| \bar{\partial} \bar{\partial} \Lambda_w \|_{H^{1.5}} \| \bar{\partial} q_k \|_{L^1 L^2} \| \bar{\partial}^{4} v^i \|_{L^2 L^2} \leq \sqrt{t} \| \bar{\partial} A^i_k \|_{L^\infty L^2} \| \bar{\partial} q_k \|_{L^2 L^2} \| \bar{\partial}^{4} v^i \|_{L^2 L^2} \leq \sqrt{t} P(E_N). $$

For $l = 2,3$ we have

$$ \left| \int_0^t \int_\Omega \bar{\partial} A^i_k \bar{\partial}^{1-i} q_k \bar{\partial}^{4} v^i \right| \leq \| \bar{\partial} A^i_k \|_{L^\infty L^2} \| \bar{\partial}^{1-i} q_k \|_{H^{1.5}} \| \bar{\partial}^{4} v^i \|_{L^2 L^2} $$

$$ \leq \sqrt{t} \| \bar{\partial} A^i_k \|_{L^\infty L^2} \| q \|_{H^{1.5}} \| \bar{\partial}^{4} v^i \|_{L^2 L^2} \leq \sqrt{t} P(E_N). $$

We proceed to estimate the integral $\sum_{l=1}^3 \int_0^t \int_\Omega \bar{\partial} A^i_k \bar{\partial}^{1-i} v^i_{ts} \left( \bar{\partial} q + \bar{\partial} \Lambda_w \right)$ (the second term appearing in (2.41)). Only cases $l = 1$ and $l = 4$ deserve special attention, while the cases $l = 2$ and $l = 3$ are estimated by a routine application of the Cauchy-Schwarz inequality and the Sobolev embedding. When $l = 1$, we can use Lemma B.2 to conclude that

$$ \left| \int_0^t \int_\Omega \bar{\partial} A^i_k \bar{\partial}^{4} v^i_{ts} \left( \bar{\partial} q + \bar{\partial} \Lambda_w \right) \right| \leq \int_0^t \| \bar{\partial} A^i_k \|_{L^\infty L^2} \int_0^t \| v^i_{ts} \|_{L^2} \| q \|_{L^{1.5}} + \| \bar{\partial} \Lambda_w \|_{H^{1.5}} \int_0^t \| \bar{\partial} A^i_k \|_{H^{1.5}} \int_0^t \| q \|_{L^{1.5}} $$

$$ + \| \bar{\partial} \Lambda_w \|_{H^{1.5}} \| q \|_{L^2 L^{1.5}} \| \bar{\partial} A^i_k \|_{H^{1.5}} \int_0^t \| q \|_{L^2 L^{1.5}} \leq \sqrt{t} \| \bar{\partial} A^i_k \|_{L^\infty L^2} \| q \|_{L^2 L^{1.5}} + \sqrt{t} \| \bar{\partial} A^i_k \|_{L^\infty L^2} \| \bar{\partial} \Lambda_w \|_{H^{1.5}} \int_0^t \| q \|_{L^{1.5}} + \| \bar{\partial} \Lambda_w \|_{H^{1.5}} \| q \|_{L^2 L^{1.5}} \| \bar{\partial} A^i_k \|_{H^{1.5}} \int_0^t \| q \|_{L^2 L^{1.5}} $$

$$ + \sqrt{t} \| \bar{\partial} A^i_k \|_{L^\infty L^2} \| q \|_{L^2 L^{1.5}} + \sqrt{t} \| \bar{\partial} A^i_k \|_{L^\infty L^2} \| \bar{\partial} \Lambda_w \|_{H^{1.5}} \int_0^t \| q \|_{L^2 L^{1.5}} + \| \bar{\partial} \Lambda_w \|_{H^{1.5}} \| q \|_{L^2 L^{1.5}} \| \bar{\partial} A^i_k \|_{H^{1.5}} \int_0^t \| q \|_{L^2 L^{1.5}} \leq \sqrt{t} P(E_N). $$
As for the case \( l = 4 \), we use Lemma B.2 again and obtain
\[
\left| \int_0^t \int_\Omega \bar{\partial}^4 \Psi_{\kappa, \cdot} \bar{v} \cdot (\bar{\partial}^4 q + \bar{\partial}^4 \Psi_{\kappa, \cdot} v) \right| \leq \int_0^t \int_\Omega \| \nabla^2 \Psi_{\kappa} \|_{L^2_{\kappa} H^2_\kappa} \| v \|_{W^{0, \infty}} \| \bar{\partial}^4 q + \bar{\partial}^4 \Psi_{\kappa, \cdot} v \|_{L^2_{\kappa} H^2_\kappa}.
\]
\[
\leq \int_0^t \| \nabla^2 \Psi_{\kappa} \|_{L^\infty_{\kappa} H^2_\kappa} \| q \|_{L^\infty_{\kappa} H^2_\kappa} \int_0^t \| |q|_{L^\infty_{\kappa} H^2_\kappa} \| \nabla^2 \Psi_{\kappa} \|_{L^\infty_{\kappa} H^2_\kappa} \leq \sqrt{t} \tau P(\mathcal{E}_\kappa).
\]

The next error term to estimate is \( \sum_{l=1}^4 \int_0^t \int_\Omega \bar{\partial}^{1-l} w \cdot \bar{\partial}^l v (\bar{\partial}^4 q + \bar{\partial}^4 \Psi_{\kappa, \cdot} v) \) (the third term appearing in (2.41)). If \( l = 4 \), we estimate
\[
\left| \int_0^t \int_\Omega \bar{\partial}^4 \Psi_{\kappa, \cdot} \bar{v} \cdot (\bar{\partial}^4 q + \bar{\partial}^4 \Psi_{\kappa, \cdot} v) \right| \leq \sqrt{t} \| \nabla^2 \Psi_{\kappa} \|_{L^\infty_{\kappa} H^2_\kappa} \| \bar{\partial}^4 q + \bar{\partial}^4 \Psi_{\kappa, \cdot} v \|_{L^\infty_{\kappa} L^2_\kappa} \leq \sqrt{t} \tau P(\mathcal{E}_\kappa);
\]
and analogously for \( l = 3 \)
\[
\left| \int_0^t \int_\Omega \bar{\partial}^3 \Psi_{\kappa, \cdot} \bar{v} \cdot (\bar{\partial}^4 q + \bar{\partial}^4 \Psi_{\kappa, \cdot} v) \right| \leq \sqrt{t} \| \nabla^2 \Psi_{\kappa} \|_{L^\infty_{\kappa} H^2_\kappa} \| \bar{\partial}^4 q + \bar{\partial}^4 \Psi_{\kappa, \cdot} v \|_{L^\infty_{\kappa} L^2_\kappa} \leq \sqrt{t} \tau P(\mathcal{E}_\kappa).
\]

For \( l = 1, 2 \), we have
\[
\left| \int_0^t \int_\Omega \bar{\partial}^l \Psi_{\kappa, \cdot} \bar{v} \cdot (\bar{\partial}^4 q + \bar{\partial}^4 \Psi_{\kappa, \cdot} v) \right| \leq \sqrt{t} \| \nabla^2 \Psi_{\kappa} \|_{L^\infty_{\kappa} H^2_\kappa} \| \bar{\partial}^4 q + \bar{\partial}^4 \Psi_{\kappa, \cdot} v \|_{L^\infty_{\kappa} L^2_\kappa} \| v \|_{L^2_{\kappa} H^2_\kappa} \leq \sqrt{t} \tau P(\mathcal{E}_\kappa).
\]

The next-to-last term on the right-hand side of (2.41) is estimated as follows:
\[
\left| \bar{\partial}^3 \Psi_{\kappa, \cdot} \bar{v} \cdot (\bar{\partial}^4 q + \bar{\partial}^4 \Psi_{\kappa, \cdot} v) \right| \leq \sqrt{t} \| \nabla^2 \Psi_{\kappa} \|_{L^\infty_{\kappa} L^2_\kappa} \| \bar{\partial}^4 q + \bar{\partial}^4 \Psi_{\kappa, \cdot} v \|_{L^\infty_{\kappa} L^2_\kappa} \| v \|_{L^2_{\kappa} H^2_\kappa} \leq \sqrt{t} \tau P(\mathcal{E}_\kappa).
\]

Finally, to bound \( \int_0^t \int_\Omega \bar{\partial}^4 \alpha (\bar{\partial}^4 q + \bar{\partial}^4 \Psi_{\kappa, \cdot} v) \) (the last term appearing in (2.41)) we integrate by parts and use the Cauchy-Schwarz inequality to obtain
\[
\left| \int_0^t \int_\Omega \bar{\partial}^4 \alpha (\bar{\partial}^4 q + \bar{\partial}^4 \Psi_{\kappa, \cdot} v) \right| \leq \| \bar{\partial}^3 \alpha \|_{L^\infty_{\kappa} L^2_\kappa} \sqrt{t} \| \bar{\partial}^4 q + \bar{\partial}^4 \Psi_{\kappa, \cdot} v \|_{L^\infty_{\kappa} L^2_\kappa} \leq \sqrt{t} \tau P(\mathcal{E}_\kappa).
\]

We used Lemma 2.4 and a priori bound (2.17).

Estimates for the error term \( \int_0^t \int \mathcal{R}_2 \text{ defined by (2.42)} \). For any \( i, j \in \{1, 2\} \) set \( F = qA^2_j \partial \Psi^j_i \), \( G = \bar{\partial}^3 \Psi^i_1 \) and apply Lemma B.1 to conclude
\[
\int_0^t \left| \Lambda_{\kappa} \left[ qA^2_j \partial \Psi^j_i A^2_i \right] - qA^2_j \partial \Lambda_{\kappa} \Psi^j_i A^2_i \right|^2 \leq \int_0^t \left| qA^2_j \partial \Psi^j_i A^2_i \right|^2_{W^{1, \infty}(\Gamma)} \leq P(\mathcal{E}_\kappa),
\]
where we estimate \( |\Lambda_{\kappa} w|_{L^2_{\kappa} H^2_\kappa} \) using (2.12):
\[
|h|_{L^2_{\kappa} H^2_\kappa} \leq \int_0^t \| \bar{\partial}^3 (\sqrt{1 + |\partial \kappa|^2}) |v \cdot \kappa A^2_i | \|^2 \leq P(\mathcal{E}_\kappa) \int_0^t |q|_{L^2}^2 \leq P(\mathcal{E}_\kappa). \tag{2.56}
\]

Note that we bounded \( |v|_3 \) by relating it to its norm over \( \Omega \) via the trace estimate
\[
|v|_{L^2_{\kappa} H^2_\kappa} \leq \| v \|_{L^2_{\kappa} H^2_\kappa} \leq P(\mathcal{E}_\kappa) \int_0^t |q|_{L^2}^2 \leq P(\mathcal{E}_\kappa). \tag{2.57}
\]
Thus,
\[
\left| \int_0^t \int_{\Gamma} \Lambda_\kappa \partial^4 \Psi \left[ \Lambda_\kappa \left( q_{,2} \left( \partial^4 \Psi_{\kappa} \cdot A^2_\kappa \right) A^2_\kappa \right) - q_{,2} \left( \partial^4 \Lambda_\kappa \Psi_{\kappa} \cdot A^2_\kappa \right) A^2_\kappa \right] \right|
\leq P(\mathcal{E}_\kappa)^{1/2} \left( \int_0^t \int_{\Gamma} |\partial^4 \Psi_{\kappa}|^2 \right)^{1/2} \leq t P(\mathcal{E}_\kappa).
\]

Finally, we treat the last term on the right-hand side of (2.42). For \( 1 \leq l \leq 2 \), we have
\[
\int_0^t \int_{\Gamma} q_{,2} \partial^4 \Psi_{\kappa} \cdot A^2_\kappa \partial^4 \tau(v + \gamma w) \cdot \partial^2 \Psi_{\kappa} \leq |q_{,2} \partial^4 \Psi_{\kappa} \cdot A^2_\kappa| L^2 L^2 \int_0^t |\partial^4 \Psi_{\kappa} \cdot A^2_\kappa| |\partial^4 \tau(v + \gamma w)|_0 
\lesssim |q_{,2} | L_\infty H^1 | \partial^4 \Psi_{\kappa} \cdot A^2_\kappa| L^2 L^2 \sqrt{\|v\| L^2 H^2 + \|w\| L^2 H^2} \lesssim \sqrt{t} P(\mathcal{E}_\kappa),
\]
where estimates (2.56) and (2.57) were used in the last inequality. If \( l = 3 \), we apply a similar estimate, bounding the term \( \partial^4 \Psi_{\kappa} \cdot A^2_\kappa \) in \( L^2 \)-norm and \( \partial^4 \tau(v + \gamma w) = \partial(v + \gamma w) \) via \( L^\infty \) norm and Sobolev embedding leading to:
\[
\left| \int_0^t \int_{\Gamma} q_{,2} \partial^4 \Psi_{\kappa} \cdot A^2_\kappa \partial(v + \gamma w) \cdot \partial^3 \Psi_{\kappa} \right| \lesssim \sqrt{t} P(\mathcal{E}_\kappa)
\]
Case \( l = 4 \) is the trickiest error term as four derivatives fall on \( A^2_\kappa \), thus creating a term that at highest order contains five derivatives of \( \Psi \), which is more than the number of derivatives allowed by our energy \( \mathcal{E}_\kappa \). However, we have the following identity:
\[
\int_0^t \int_{\Gamma} q_{,2} \partial^4 \Psi_{\kappa} \cdot A^2_\kappa \partial(v + \gamma w) \cdot \partial^4 \Psi_{\kappa} = \frac{1}{2} \int_{\Gamma} d_{,2} \frac{\epsilon \cdot \tau}{|\tau_\kappa|} \left( \partial^4 \Psi_{\kappa} \cdot A^2_\kappa \right)^2 + \int_{\Gamma} q_{,2} \frac{\epsilon \cdot \tau}{|\tau_\kappa|} \partial^4 \Psi_{\kappa} \cdot A^2_\kappa \partial^4 \Psi_{\kappa} \cdot A^2_\kappa + \int_{\Gamma} q_{,2} \partial^4 \Psi_{\kappa} \cdot A^2_\kappa E,
\]
where \( E \) is the lower order error term given by
\[
E = \frac{\epsilon \cdot \tau}{|\tau_\kappa|} \sum_{i=1}^{3} \sum_{i=1}^{3} \left( \xi_i \cdot \tau \cdot \partial^4 \tau \cdot A^2_\kappa \right) + \sum_{i=1}^{3} \sum_{i=1}^{3} \left( \xi_i \cdot \tau \cdot \partial^4 \tau \cdot A^2_\kappa \right).
\]
To prove (2.58) we first note that
\[
v + \gamma w = (v + \gamma w) \cdot n_\kappa n_\kappa + (v + \gamma w) \cdot \tau_\kappa \tau_\kappa = (v + \gamma w) \cdot \tau_\kappa \tau_\kappa
\]
where we have used the boundary condition (2.4d). Therefore, we have the equality
\[
(v + \gamma w) \cdot \partial^4 A^2_\kappa = (v + \gamma w) \cdot \tau_\kappa \tau_\kappa \cdot \partial^4 A^2_\kappa = \epsilon \cdot \tau \cdot \partial^4 \tau \cdot A^2_\kappa
\]
where we first used the identity \( v \cdot \tau_\kappa = 0 \) and in the last line we used the product rule expansion of the the identity \( 0 = \partial^4 (\tau_\kappa \cdot A^2_\kappa) \) with \( \epsilon_i \) the corresponding binomial coefficients. Since \( \tau_\kappa = \frac{\partial \Psi_{\kappa}}{|\partial \Psi_{\kappa}|} \), we have
\[
\partial^4 \tau_\kappa \cdot A^2_\kappa = \frac{\partial^5 \Psi_{\kappa}}{|\partial \Psi_{\kappa}|} \cdot A^2_\kappa + \sum_{i=1}^{3} \sum_{i=1}^{3} \left( \xi_i \cdot \tau \cdot \partial^4 \tau \cdot A^2_\kappa \right) = \frac{\partial^5 \Psi_{\kappa}}{|\partial \Psi_{\kappa}|} \cdot A^2_\kappa + \sum_{i=1}^{3} \sum_{i=1}^{3} \left( \xi_i \cdot \tau \cdot \partial^4 \tau \cdot A^2_\kappa \right),
\]
where we simply used the product rule to expand \( \partial^4 (\frac{\partial \Psi_{\kappa}}{|\partial \Psi_{\kappa}|}) \) and the orthogonality of \( \partial \Psi_{\kappa} \) and \( A^2_\kappa \) in the last line. Combining the previous two identities, we may write
\[
(v + \gamma w) \cdot \partial^4 A^2_\kappa = \frac{\partial^5 \Psi_{\kappa}}{|\partial \Psi_{\kappa}|} \cdot A^2_\kappa \cdot \tau_\kappa + E,
\]
where the error term \( E \) is given by (2.59). We thus obtain
\[
\int_{\Gamma} q_{,2} \partial^4 \Psi_{\kappa} \cdot A^2_\kappa (v + \gamma w) \cdot \partial^4 A^2_\kappa = - \int_{\Gamma} q_{,2} \frac{\epsilon \cdot \tau}{|\tau_\kappa|} \partial^4 \Psi_{\kappa} \cdot A^2_\kappa \partial^5 \Psi_{\kappa} \cdot A^2_\kappa + \int_{\Gamma} q_{,2} \partial^4 \Psi_{\kappa} \cdot A^2_\kappa E.
\]
Note that first integral on the right-hand side has a symmetry allowing us to extract a full tangential derivative at the level of highest order terms:

\[
- \int_{\Gamma} q_{\cdot 2} \frac{\partial \nabla \tau_5}{\partial \tau_6} \partial \Psi \cdot A^2 \partial \Psi \cdot A^2 \\
= - \frac{1}{2} \int_{\Gamma} q_{\cdot 2} \frac{\partial \nabla \tau_5}{\partial \tau_6} [\partial (\partial \Psi \cdot A^2)^2] + \int_{\Gamma} q_{\cdot 2} \frac{\partial \nabla \tau_5}{\partial \tau_6} \partial \Psi \cdot A^2 \partial \Psi \cdot A^2 \\
= \frac{1}{2} \int_{\Gamma} (q_{\cdot 2} \frac{\partial \nabla \tau_5}{\partial \tau_6}) [\partial \Psi \cdot A^2] + \int_{\Gamma} q_{\cdot 2} \frac{\partial \nabla \tau_5}{\partial \tau_6} \partial \Psi \cdot A^2 \partial \Psi \cdot A^2,
\]

where we have used integration by parts in the second equation. Finally, summing the previous two identities we arrive at (2.58).

Note that \( \Psi \) enters the right-hand side of the above identity at most with 4 derivatives. By standard \( L^\infty - L^2 - L^2 \) type estimates and identity (2.58), we finally arrive at

\[
\left| \int_0^t \int_{\Gamma} q_{\cdot 2} \partial \Psi \cdot A^2 (v \cdot \nabla \omega) \cdot \partial \Psi \cdot A^2 \right| \lesssim \sqrt{t} \mathcal{P}(\mathcal{E}_\kappa). \tag{2.60}
\]

Before we estimate the third term on the right-hand side of (2.42), we first rewrite:

\[
\partial \Psi \cdot A^2 = -\partial \Psi (v \cdot A^2) - \sum_{i=0}^3 a_i \partial \Psi \cdot A^2 \cdot \partial \Psi \cdot A^2 = -J^{-1} \partial \Psi \cdot A^2 \cdot \partial \Psi \cdot A^2,
\]

where \( a_i, l = 0, \ldots, 3 \) are the corresponding binomial coefficients. As a consequence, we have

\[
\left| \int_0^t \int_{\Gamma} q_{\cdot 2} \partial \Psi \cdot A^2 (v \cdot \nabla \omega) \cdot \partial \Psi \cdot A^2 \right| \leq \left| \int_0^t \int_{\Gamma} (v \cdot \nabla \omega + \beta)_{,1} \partial \Psi \cdot A^2 \cdot \partial \Psi \cdot A^2 \right| \\
+ \kappa^2 \sum_{i=0}^3 a_i \left| \int_0^t \int_{\Gamma} (v \cdot \nabla \omega + \beta)_{,1} \partial \Psi \cdot A^2 \cdot \partial \Psi \cdot A^2 \right| \tag{2.61}
\]

The first term on the right-hand side above is easily bounded as follows:

\[
\kappa^2 \left| \int_0^t \int_{\Gamma} (v \cdot \nabla \omega + \beta)_{,1} \partial \Psi \cdot A^2 \cdot \partial \Psi \cdot A^2 \right| \leq \sqrt{\kappa} \left| (v \cdot \nabla \omega + \beta)_{,1} A^2 \cdot \partial \Psi \cdot A^2 \right| \lesssim \sqrt{\kappa} \mathcal{P}(\mathcal{E}_\kappa).
\]

The second term on the right-hand side of (2.61) is a sum, and the hardest summand to bound is created when \( l = 0 \). In this case, roughly speaking we bound \( \partial \Psi \cdot A^2 \) by \( \kappa^{-1} |\nabla \Psi| \) trading one tangential derivative on \( \partial \Psi \cdot A^2 \) for a bound on \( \nabla \Psi \) in \( H^1 \), at the expense of a factor of \( \kappa^{-1} \). Using this observation we obtain

\[
\kappa^2 \left| \int_0^t \int_{\Gamma} (v \cdot \nabla \omega + \beta)_{,1} \partial \Psi \cdot A^2 \cdot \partial \Psi \cdot A^2 \right| \lesssim \kappa^2 \sqrt{\kappa} \left| (v \cdot \nabla \omega + \beta)_{,1} A^2 \cdot \partial \Psi \cdot A^2 \right| \lesssim \sqrt{\kappa} \mathcal{P}(\mathcal{E}_\kappa).
\]

The next-to-last term on the right-hand side of (2.42) is again a sum and the hardest term to estimate is created again when \( l = 0 \). We use the same idea as in the previous estimate to obtain

\[
\kappa^2 \left| \int_0^t \int_{\Gamma} \partial \Psi \cdot A^2 \cdot \partial \Psi \cdot A^2 \right| \lesssim \sqrt{\kappa} \left| \partial \Psi \cdot A^2 \right| \lesssim \sqrt{\kappa} \mathcal{P}(\mathcal{E}_\kappa). 
\]

Note that we exploited the presence of the \( \kappa \)-dependent energy term in our energy \( \mathcal{E}_\kappa \), using the bound \( \kappa |\partial \Psi \cdot A^2| \lesssim \sqrt{\kappa} \mathcal{P}(\mathcal{E}_\kappa) \). In analogous manner, we conclude

\[
\left| \int_0^t \int_{\Gamma} \partial \Psi \cdot A^2 \cdot \partial \Psi \cdot A^2 \right| \lesssim \sqrt{\kappa} \left| \partial \Psi \cdot A^2 \right| \lesssim \sqrt{\kappa} \mathcal{P}(\mathcal{E}_\kappa),
\]

where we note that the commutator term, i.e. the first term on the right-hand side of (2.46) deserves special attention. Due to the absence of spatial derivatives in the term \( \Psi_{tt} \) in

\[
- \partial \Psi_{tt} + \partial \Psi_{tt} \left[ \Lambda \left[ q_{\cdot 2} \left( \Psi_{tt} \cdot A^2 \right) \right] + q_{\cdot 2} \left( \partial \Psi \cdot A^2 \right) \right]
\]


where, in order to bound the second error term, we start by applying Lemma B.1 to deal with the commutator term. If \( \kappa \) is small enough, then we gain one power of \( \kappa \) in the second line above from the commutator estimate and then absorb it into the energy contribution \( \kappa |\partial_t u|_{L^2 L^2} \). The last term on the right-hand side of (2.42) contains the \( \beta \)-contribution from the regularized Dirichlet condition (2.4c). It is easily estimated using the Cauchy-Schwarz inequality by a term of the form \( C m_0 + C \kappa^2 |\partial_t u|_{L^2 L^2}^2 \) which in turn is smaller than a constant multiple of \( t m_0 + t \mathcal{E}_\kappa \). Here \( m_0 \) is a constant, which depends only on the initial data.

**Estimates for \( \int_{\Omega} S_1 \) and \( \int_{\Gamma} S_2 \).** In the second term on the right-hand side of (2.51), the hardest terms to estimate correspond to the cases \( (i, k) = (2, 1) \) and \( (i, k) = (2, 1) \). If \( (i, k) = (2, 1) \), then

\[
\left| \int_0^t \int_\Omega \partial_t A^i_k \partial^3 w \partial_t q \partial^3 v \right| \leq \left| \int_0^t \int_\Omega (\partial_t A^i_k \partial^3 v) \partial_t q \partial^3 v \right| \leq \int_0^t \left\| \partial_t A^i_k \partial^3 v \right\|_{L^2 L^2} \left\| \partial_t q \partial^3 v \right\|_{L^2 L^2} \leq \sqrt{t} \int_\Omega \left\| \partial_t A^i_k \partial^3 v \right\|_{L^2 L^2} \left\| \partial_t q \partial^3 v \right\|_{L^2 L^2} \leq \sqrt{t} \mathcal{P}(\mathcal{E}_\kappa).
\]

The second error term is rather straightforward: for any \( l = 2, 3 \)

\[
\left| \int_0^t \int_\Omega \partial_t A^i_k \partial^3 w \partial^3 q \partial^3 v \right| \leq \left| \int_0^t \int_\Omega \partial_t A^i_k \partial^3 w \partial^3 q \partial^3 v \right| \leq \left| \int_0^t \int_\Omega \partial_t A^i_k \partial^3 q \partial^3 w \partial^3 v \right| \leq \sqrt{t} \left\| \partial_t A^i_k \partial^3 q \partial^3 w \partial^3 v \right\|_{L^2 L^2} \left\| \partial^3 q \partial^3 w \partial^3 v \right\|_{L^2 L^2} \leq \sqrt{t} \mathcal{P}(\mathcal{E}_\kappa).
\]

If \( l = 1 \), then

\[
\left| \int_0^t \int_\Omega \partial_t A^i_k \partial^3 w \partial^3 q \partial^3 v \right| \leq \left| \int_0^t \int_\Omega \partial_t A^i_k \partial^3 w \partial^3 q \partial^3 v \right| \leq \left| \int_0^t \int_\Omega \partial_t A^i_k \partial^3 q \partial^3 w \partial^3 v \right| \leq \sqrt{t} \mathcal{P}(\mathcal{E}_\kappa).
\]

Similar analysis yields:

\[
\sum_{l=1}^3 \left| \int_0^t \int_\Omega \partial_t A^i_k \partial^3 q \partial^3 w \partial^3 v \right| \leq \sqrt{t} \mathcal{P}(\mathcal{E}_\kappa).
\]

As for the error term (2.52), we start by applying Lemma B.1 to deal with the commutator term. For any \( i, j \in \{1, 2\} \) set \( F = q_2 A^i_k A^k_j, \ G = \partial^3 w \) and apply Lemma B.1 to obtain

\[
\left| \int_0^t \int_\Omega \partial_t A^i_k \partial^3 q \partial^3 w \right| \leq t \sup_{0 \leq s \leq t} \left| q_2 A^i_k \partial^3 w \right| \left| \partial^3 q \partial^3 w \right| \leq t \mathcal{P}(\mathcal{E}_\kappa),
\]

where, in order to bound \( |w|_{L^2 L^2} \), we use the equation (2.4d) analogously to the bound (2.56). Upon using the Cauchy-Schwarz inequality we get

\[
\left| \int_0^t \int_\Omega \partial_t A^i_k \partial^3 w \partial^3 q \partial^3 w \right| \leq \sqrt{t} \mathcal{P}(\mathcal{E}_\kappa).
\]
As for the second term on the right-hand side of (2.52) we obtain
\[
\left| \int_0^t \int_\Gamma q \partial_t (v \cdot \delta^3 \mathcal{A}_\kappa)^2 \right| \leq \int_0^t |q|_{\infty} (|v|_{L^2}^2 + |w|_{L^2}^2) |\partial_t^3 \Lambda_{\kappa,w}|_0 \leq \|q\|_{L^p_t H^2} \|v\|_{L^p_t H^2} + |w|_{L^2} \|\partial \Psi - \text{Id}\|_3 \int_0^t |\partial_t^3 \Lambda_{\kappa,w}|_0, \lesssim \sqrt{t} P(\mathcal{E}).
\]

where, the term \(|w|_{L^p_t H^2}^2\) is bounded by \(P(\mathcal{E}_\kappa)\) for the same reason as in (2.63). The last term on the right-hand side of (2.52) is a sum, and the hardest term to bound is created when \(l=0\). We must integrate by parts with respect to the time variable, to obtain
\[
\kappa^2 \int_0^t \int_{\Gamma} \partial_t^3 h_t^i v \cdot \partial_t^3 \mathcal{A}_\kappa = \kappa^2 \int \partial_t^3 h_t^i v \cdot \partial_t^3 \mathcal{A}_\kappa \mid_0^t - \kappa^2 \int_0^t \int_{\Gamma} \partial_t^3 h_t^i v \cdot \partial_t^3 \mathcal{A}_\kappa - \kappa^2 \int_0^t \int_{\Gamma} \partial_t^3 h_t v \cdot \partial_t^3 \mathcal{A}_\kappa.
\]

Now observe that
\[
\kappa^2 \int_0^t \int_{\Gamma} \partial_t^3 h_t v \cdot \partial_t^3 \mathcal{A}_\kappa \mid_0^t \leq \kappa^2 m_0 \kappa |\partial_t h_t|_{L^\infty_t L^2_x} \|v\|_{L^\infty_t L^2_x} \lceil \int_0^t \partial_t \partial_t^3 \mathcal{A}_\kappa \rceil_{L^\infty_t L^2_x} \lesssim \sqrt{\mathcal{E}_\kappa} \sqrt{\mathcal{E}_\kappa} |\partial_t h_t|_{L^2_t L^2_x} \lesssim \sqrt{t} P(\mathcal{E}_\kappa),
\]

where \(m_0\) depends only on the initial conditions. As for the remaining three terms on the right-hand side of (2.64), they are straightforward to bound using the standard energy estimates. We arrive at
\[
\kappa^2 \int_0^t \int_{\Gamma} \partial_t^3 h_t v \cdot \partial_t^3 \mathcal{A}_\kappa \leq \kappa^2 m_0 + \sqrt{t(1 + \kappa)} P(\mathcal{E}_\kappa).
\]

In analogous manner we conclude
\[
\left| \int_0^t \int_{\Omega} S_3 \, dx \right| + \left| \int_0^t \int_{\Gamma} S_4 \, dx \right| \leq (t + \sqrt{t}) P(\mathcal{E}_\kappa).
\]

2.8. Proof of Theorem 1.1. The polynomial inequality (2.55) replaces the typically used Gronwall inequality. Since the constants appearing in (2.55) are independent of \(\kappa\) a standard continuity argument (see for instance Section 9 of [18]) yields the existence of a \(\kappa\)-independent time \(T\) such that
\[
\mathcal{E}_\kappa(t) \leq C \mathcal{E}_\kappa(0) \leq C \mathcal{E}(0) + 1
\]
for \(\kappa\) small enough.

Since \(\mathcal{E}(t) \leq \mathcal{E}_\kappa(t), \ t \in [0,T] \) (recall the definitions (1.20) of \(\mathcal{E}\) and (2.16) of \(\mathcal{E}_\kappa\)), we obtain the uniform bound
\[
\mathcal{E}(q^\kappa, h^\kappa) \leq C \mathcal{E}(0) + 1,
\]
where \((q^\kappa, h^\kappa)_\kappa\) is a family of solutions to the \(\kappa\)-regularized problem (2.4), \(0 \leq \kappa \leq 1\). Note that the assumptions (2.18) remain valid (on a possibly smaller) time interval \([0,T]\), as both \(|\partial_t h_t|_{L^\infty_t L^2_x}\) and \(\delta\) are easily controlled by the energy \(\mathcal{E}\). By the fundamental theorem of calculus, it is clear that on a possibly smaller time interval \([0,T]\) we have
\[
\sup_{0 \leq t \leq T} \mathcal{A}(t) \leq \mathcal{E}_\kappa(0) + T \mathcal{E}_\kappa(t) \leq \mathcal{E}_\kappa(0) + \frac{1}{2},
\]
thus justifying a posteriori the a priori assumption (2.17). Thus, passing to the weak limit as \(\kappa \to 0\) we obtain a solution on the time interval \([0,T]\) which belongs to the space \(\mathcal{S}(T)\) defined in (1.27). Since \(\mathcal{S}(T)\) embeds compactly into \(C^1_t C^0_x \cap C^0_t C^2_x\) the solution is also classical.

Uniqueness. We only present a brief sketch of the uniqueness argument. A simple application of the energy method also implies uniqueness of the solution. Assume that \((\bar{q}, \bar{h})\) also solves (2.4) with the corresponding \(\bar{\Psi}, \bar{v}, \bar{w}\). Then the pair \((r, \rho) := (\bar{q} - \dot{q}, \bar{h} - \dot{h})\) satisfies the following system of
equations:
\[ r_t - A^k_i (A^k_i r_{,k})_{,j} = (\Delta_{\Psi} - \Delta_{\tilde{q}})q - (v - \tilde{v}) \cdot w + \tilde{v}(w - \tilde{w}) \quad \text{in} \; \Omega; \]  
\[ (v - \tilde{v})^j + A^k_i r_{,k} + \tilde{q}_k (A^k_i - \tilde{A}^k_i) = 0 \quad \text{in} \; \Omega; \]  
\[ r = 0 \quad \text{on} \; \Gamma; \]  
\[ \rho_t = -r_{,2} \quad \text{on} \; \Gamma; \]  
\[ \partial_n r = 0 \quad \text{on} \; \partial \Omega_{\text{top}}. \]

Furthermore, initially \((r(0,x),\rho(0,x')) = (0,0)\). Applying \(\bar{\partial}\) to the identity (2.65b), multiplying by \((\bar{\partial}(v - \tilde{v}))^t\) and integrating over \(\Omega\), we derive the first identity in analogy to the proof of Lemma 2.2. Similarly, applying \(\partial_t\) to (2.65b), multiplying by \((v - \tilde{v})^t\) and integrating, we obtain the second energy identity. The natural quadratic form that emerges is equivalent to
\[ E := \|\bar{\partial}(v - \tilde{v})\|_{L^2}^2 + \|v - \tilde{v}\|_{L^2}^2 + \|r\|_{L^2}^2 + \|\rho\|_{L^2}^2 + |\rho_t|_{L^2}^2. \]

Furthermore, we have an a-priori control of the high-order derivatives of the two solutions, i.e. for some \(M > 0\): \(E(q,h) + E(\tilde{q},\tilde{h}) < M\). From here, we can easily prove the polynomial bound
\[ E(t) \leq tP(E(t)), \]
which in particular, uses the fact that the initial values for \(\rho\) and \(r\) are 0. We infer that \(E = 0\) and hence the uniqueness follows.

**Continuity in time.** Since \(q \in L^2_t H^2_x\) and \(q_t \in L^2_t H^3_x\), it follows that \(q \in C^0_t H^4_x\); similarly, since \(q_{tt} \in L^2_t H^3_x\), then \(q_t \in C^0_t H^2_x\). Passing to the limit as \(k \to 0\) in (2.32),
\[ \partial^2 h = \frac{g^2}{v \cdot n} \frac{\partial v \cdot \tau}{v \cdot n}, \]
where \(v \cdot n > 0\) by the Taylor sign condition. By passing to the limit as \(k \to 0\) in Lemma 2.4, we have that \(\Psi \in L^2_t H^2_x\), and we also have that \(\Psi \in L^2_t H^{3,5}_x\), from which it follows that \(\Psi \in C^0_t H^4_x\). Since \(q \in C^0_t H^4_x\) and , and since \(v = -\nabla_{\Psi} q \in L^2_t H^2_x\), it follows that \(v \in L^2_t H^2_x \cap C^0_t H^3_x\); hence, \(\partial v \cdot \tau \in L^2_t H^{2,5}(\Gamma)\).

Then, since \(g\) and \(n\) are in \(L^\infty \cap H^{3/2}(\Gamma)\), and \(v \cdot n \in L^\infty H^{2,5}(\Gamma)\), we see from (2.66) that
\[ h \in L^2_t H^{4,5}(\Gamma). \]

Since \(h_t = gv \cdot n\) on \(\Gamma\), we then have that
\[ h_t \in L^2_t H^{3,5}(\Gamma), \]
from which it follows that
\[ h \in C^0_t H^{4}(\Gamma). \]

Since \(h_t = gv \cdot n\) on \(\Gamma\), and since \(g\) and \(n\) are in \(C^0_t H^{3}(\Gamma)\) and \(v \in C^0_t H^{2,5}(\Gamma)\), then \(h_t \in C^0_t H^{2,5}(\Gamma)\).

Using that \(h_{tt} = \partial_t [gv \cdot n] = 0\) on \(\Gamma\) and the fact that \(v_t \in C^0_t H^{0.5}(\Gamma)\), we also have that \(h_{tt} \in C^0_t H^{0.5}(\Gamma)\).

It remains to show that \(q_{tt} \in C^0_t L^2_x\). From (1.18a),
\[ q_{tt} = (\Delta q)_t - (v \cdot w)_t. \]

Given the regularity already established for \(q, q_t, \Psi,\) and \(\Psi_t\), we need to establish the regularity for \(w_t = \Psi_{tt}\). Since \(h_{tt} \in C^0_t H^{0.5}(\Gamma)\), then \(\Psi_{tt} \in C^0_t H^1(\Omega)\), and we find that \(q_{tt} \in C^0_t L^2(\Omega)\).

3. The vanishing surface tension limit

Local-in-time existence for the Stefan problem with surface tension has been studied in a variety of papers; see, for example, [30, 29, 43, 23]. For any \((q^0, h^0) \in H^4(\Omega) \times H^{5,5}(\Omega)\) there exists a local-in-time classical solution \((q, h)\) to the Stefan problem with surface tension in the harmonic gauge:
respectively, wherein we drop the
WELL-POSEDNESS FOR THE CLASSICAL STEFAN PROBLEM AND THE ZERO SURFACE TENSION LIMIT
\[ q_t - \Delta \Psi q = -v \cdot w \quad \text{in } \Omega \times (0,T), \quad (3.1a) \]
\[ v^t + A^t_2 q_h = 0 \quad \text{in } \Omega \times (0,T), \quad (3.1b) \]
\[ q = -\sigma \frac{\partial^2 h}{(1 + |\partial h|^2)^2} \quad \text{on } \Gamma \times [0,T], \quad (3.1c) \]
\[ \Delta \Psi = 0 \quad \text{on } \Omega \times [0,T], \quad (3.1d) \]
\[ \Psi = \text{Id} + h N \quad \text{on } \Omega \times [0,T], \quad (3.1e) \]
\[ \Psi = \text{Id} \quad \text{on } \partial \Omega_{\text{top}} \times [0,T], \quad (3.1f) \]
\[ \Psi_t \cdot n(t) = -v \cdot n(t) \quad \text{on } \Gamma \times (0,T), \quad (3.1g) \]
\[ v \cdot N = 0 \quad \text{on } \partial \Omega_{\text{top}} \times (0,T), \quad (3.1h) \]
\[ \Psi(0, \cdot) = \Psi_0, \quad q(0, \cdot) = q_0^\sigma = p_0 \circ \Psi_0. \quad (3.1i) \]

With \( \sigma > 0 \), we can prove the following energy identities in the same way as Lemma 2.2.

**Lemma 3.1.** Let \((q, h)\) be a local-in-time solution to (3.1) defined on the time interval \([0, T_\sigma]\). Then we have the following energy identity:

\[
F^\sigma(q, \Psi)(t) = \int_0^t \int_\Omega \{ \mathcal{R}_1 + \mathcal{R}_3 + \mathcal{R}_5 + S_1 + S_3 \} + \int_0^t \int_\Gamma \{ \mathcal{R}_4 + \mathcal{R}_6 + S_2 + S_4 \} + \int_0^t \int_\Gamma \{ \mathcal{R}_7' + \mathcal{R}_6' + S_2' + S_4' \},
\]

where

\[
F^\sigma := F + \frac{\sigma}{2} \sum_{a + 2b \leq 4} \left| \partial \Psi \right|^{-3/2} J^{-1/2} \partial^{a+1} \partial^h (\partial^h)^2 \left| L^2_{\sigma} \right|^2 + \sigma \sum_{a + 2b \leq 3} \left| \partial \Psi \right|^{-3/2} J^{-1/2} \partial^{a+1} (\partial^h h)^2 \left| L^2_{\sigma} \right|^2 \quad (3.2)
\]

with the energy \( F \) and error terms \( \mathcal{R}_i, i = 1, \ldots, 6, S_i, i = 1, \ldots, 4 \) given by (2.24) and Lemma 2.2 respectively, wherein we drop the \( \kappa \)-dependent terms. Furthermore,

\[
\mathcal{R}_2 := -\sigma \left( \frac{\partial^2 h}{\partial \Psi^2} \right) \frac{\partial^4 \Psi}{\partial x_1 \partial x_2} A^2_1 v \cdot A^2_2
\]

\[
+ \frac{\sigma}{2} \left| \partial^3 h \right|^2 \partial_t (|\partial \Psi|^{-3} J^{-1}) - \sigma \partial^3 h \partial_t (\partial^h t) \left| \partial \Psi \right|^3 - \partial^h \partial_t \left( \partial^h \partial_t \right) \left| \partial \Psi \right|^3
\]

\[
+ \sigma \sum_{i=1}^4 a_i \partial^i (w + v) \cdot \partial^2 \Psi A^2_1 \partial^2 (\partial^h \partial_t) \left| \partial \Psi \right|^3 \quad (3.3)
\]

\[
\mathcal{R}_4 := -\sigma \left( \frac{\partial^2 h}{\partial \Psi^2} \right) \frac{\partial^2 \partial_t \Psi}{\partial x_1 \partial x_2} A^2_1 \partial^2 \partial_{t} v \cdot A^2_2
\]

\[
+ \frac{\sigma}{2} \left| \partial^3 \partial_t h \right|^2 \partial_t (|\partial \Psi|^{-3} J^{-1}) - \sigma \partial^3 \partial_t h \partial_t (\partial^h t) \left| \partial \Psi \right|^3 - \partial^h \partial_t \left( \partial^h \partial_t \right) \left| \partial \Psi \right|^3
\]

\[
+ \sigma \sum_{i=1}^4 a_i \partial^i \partial^2 \partial_t (\partial^3 \partial_t) (w + v) \cdot \partial^2 \partial_t (w + v) \partial^2 \partial_t \left( \partial^h \partial_t \right) A^2_2 \quad (3.4)
\]

\[
\mathcal{R}_7 := \sigma \left( \frac{\partial^2 h}{\partial \Psi^2} \right) \frac{\partial^2 \partial_t \Psi}{\partial x_1 \partial x_2} A^2_1 \partial^3 \partial_{tt} v \cdot A^2_2 + \frac{\sigma}{2} \left| \partial^3 \partial_{tt} h \right|^2 (|\partial \Psi|^{-3} J^{-1}) - \sigma \partial^3 \partial_{tt} h \partial_t (|\partial \Psi|^{-3})
\]

\[
+ \sigma w_1 \partial_t \left( \partial^3 \partial_{tt} h \right) \left| \partial \Psi \right|^3 - \sigma A^2_1 \partial^2 \partial_{tt} \left| \partial \Psi \right|^3 + \sigma \sum_{i=0}^3 a_i \partial^i \partial_{tt} \left( \partial^3 \partial_{tt} h \right) \left| \partial \Psi \right|^3 \quad (3.5)
\]

\[
S_2 := \sigma \sum_{i=1}^3 a_i \partial^i (w + v) \cdot \partial^2 \Psi A^2_1 \partial^3 \partial_{tt} v \partial^2 \partial_t \left( \partial^h \partial_t \right) \left| \partial \Psi \right|^3
\]

\[
+ \sigma \sum_{a + b \leq 3} a \partial^a \Psi A^2_1 \partial^b \partial^3 \partial_{tt} h \partial^3 \partial_{tt} \left( \partial^h \partial_t \right) \left| \partial \Psi \right|^3 \quad (3.6)
\]
The derivation of the energy function $E$ in analogy to Proposition 2.6. Most importantly, we establish a nonlinear polynomial inequality leads to an additional trace of the term $\bar{\partial}^4 q$ on the boundary $\Gamma$. Since $q = \sigma H$ on $\Gamma$ an integration by parts with respect to $x_k$ in the integral

$$\int_{\Omega} A^4_k \bar{\partial}^4 q_k \bar{\partial}^4 v$$

leads to an additional $\sigma$-dependent energy term in (3.2).

3.1. **Nonlinear energy estimates.** In the following proposition we prove the basic energy estimate in analogy to Proposition 2.6. Most importantly, we establish a nonlinear polynomial inequality for the energy $\mathcal{E}_\sigma$ with $\sigma$-independent coefficients. As a consequence, we show that under the assumptions of Theorem 1.2 the time interval $T_\sigma$ is independent of $\sigma$.

**Proposition 3.2.** Let $(q_0^\sigma, v_0^\sigma)_{\sigma \geq 0}$ be a given family of well-prepared initial conditions in the sense of Definition 1. There exists a constant $C$ independent of $\sigma$ and a universal polynomial $P$ such that for any $t \in [0, T_\sigma]$ the following bound holds

$$\mathcal{E}_\sigma(t) \leq C \mathcal{E}_\sigma(0) + C(t + \sqrt{t}) P(\mathcal{E}_\sigma).$$

In particular, there exists a time $T > 0$ independent of $\sigma$, a constant $C_* > 0$ and the solution $(q^\sigma, \Psi^\sigma)$ to the Stefan problem with surface tension defined on $[0, T]$ satisfying the bound

$$\mathcal{E}_\sigma(q^\sigma, \Psi^\sigma)(t) \leq C_*, \quad 0 \leq \sigma \leq 1, \quad t \in [0, T].$$

**Proof.** In comparison to the estimates for the classical Stefan problem carried over in Section 2.7 the only new error terms to estimate are the terms $R_2^\sigma$, $R_4^\sigma$, $S_2^\sigma$, $S_4^\sigma$ given in the statement of Lemma 3.1.

Estimating $\int_0^t \int_\Gamma R_2^\sigma$ defined by (3.3). We start by bounding the first term on the right-hand side of (3.3).

$$\sigma \left| \int_0^t \int_\Gamma \partial^2 q \left( \frac{\bar{\partial} h}{\bar{\partial} \Psi} \right) \bar{\partial}^4 \Psi \cdot A^4_1 \bar{\partial}^4 v \cdot A^4_2 \right| \lesssim \int_0^t P(|\sqrt{\sigma} \partial h|_4) |\sqrt{\sigma} \Psi|_5 |v|_3$$

$$\lesssim P(|\sqrt{\sigma} \partial h|_{L^\infty H^3}) |\sqrt{\sigma} \Psi|_{L^\infty H^3} |v|_{L^2 H^2} \lesssim \sqrt{t} P(\mathcal{E}_\sigma \partial h).$$

The second and the third term on the right-hand side of (3.3) are estimated analogously and rely on the standard $L^\infty - L^2 - L^2$ estimates. As for the fourth term on the right-hand side of (3.3), note that due to (2.3)

$$\sigma \left| \int_0^t \int_\Gamma J^{-1} \bar{\partial}^4 h \left( \bar{\partial}^4 \left( \frac{\bar{\partial}^2 h}{|\bar{\partial} \Psi|^3} \right) - \bar{\partial}^6 h \right) \right| \lesssim \int_0^t \sqrt{\sigma} \partial^3 h \left| v \sqrt{\sigma} \bar{\partial}^4 \left( \frac{\bar{\partial}^2 h}{|\bar{\partial} \Psi|^3} \right) - \bar{\partial}^6 h \right|$$

$$\lesssim \sqrt{t} P(\mathcal{E}_\sigma),$$

where the last estimate follows in the standard way: terms with less derivatives are bounded in the $L^\infty$-norm and then by the Sobolev embedding theorem. In the last term on the right-hand side of (3.3), the hardest case to deal with is $l = 4$. Note that

$$\sigma \int_0^t \int_\Gamma (v + w) \cdot \bar{\partial}^4 A^4_1 \bar{\partial}^4 \left( \frac{\bar{\partial}^2 h}{|\bar{\partial} \Psi|^3} \right) = \sigma \int_0^t \int_\Gamma (v + w) \cdot \bar{\partial}^4 A^4_2 \left( \partial^6 h |\partial \Psi|^{-3} \right)$$

$$+ \sigma \int_0^t \int_\Gamma (v + w) \cdot \bar{\partial}^4 A^4_1 \sum_{l=1}^4 a_l \bar{\partial}^{6-l} h \bar{\partial}^l \left( |\partial \Psi|^{-2} \right) =: I + II.$$
The more challenging term to estimate is term $I$. Since $A_2^* = J^{-1}(\partial h, -1)$ we have the identity

$$I = \sigma \int_0^t \int_\Gamma (v+w) \cdot \partial^4 (J^{-1}(\partial h, -1)) (\partial^6 h |\partial \Psi|^{-3})$$

$$= \sigma \int_0^t \int_\Gamma J^{-1}(v+w) \cdot (\partial^6 h, 0) (\partial^6 h |\partial \Psi|^{-3}) + \sigma \sum_{m=1}^3 c_m \int_0^t \int_\Gamma \partial^m (J^{-1}) (v+w) \cdot \partial^{4-m} (\partial h, -1) (\partial^6 h |\partial \Psi|^{-3})$$

$$= I_A + I_B$$

for some universal constants $c_m \in \mathbb{R}$. Note that when $m = 4$ term $(v+w) \cdot \partial^{4-m} (\partial h, -1)$ vanishes since $(\partial h, -1)$ is parallel to $\vec{n}$ and $v \cdot \vec{n} = -w \cdot \vec{n}$ by (3.1g).

$$|I_A| = \sigma \int_0^t \int_\Gamma J^{-1}(v-w^1) \partial^3 h \partial^6 h |\partial \Psi|^{-3} = \frac{1}{2} \sigma \int_0^t \int_\Gamma J^{-1}(\partial^5 h)^2 (v^1 - w^1) |\partial \Psi|^{-3}$$

$$= \frac{1}{2} \sigma \int_0^t \int_\Gamma |\partial^3 h|^2 \partial (J^{-1}(v^1 - w^1) |\partial \Psi|^{-3}) \lesssim tP(\mathcal{E}^\sigma),$$

where we have used the parametric representation of $\Psi$ in terms of $h$ and integrated by parts. The last inequality is rather standard and follows by estimating $\partial((v^1 - w^1) |\partial \Psi|^{-3})$ in $L^\infty$ norm and further via Sobolev inequality, where we also use $\sigma |\partial^5 h|_{L^\infty L^2} \lesssim \mathcal{E}^\sigma$. Terms $I_B$ and $II$ are easily estimated via the standard energy $L^\infty - L^2 - L^4$ bounds and Sobolev imbedding, and the same applies to the remaining cases $l = 1, 2, 3$. When estimating the fourth term on the right-hand side of (3.3) first integrate by parts so to remove one $\partial$-derivative from $\partial^6 \Psi$ term and then apply the standard energy estimates.

**Estimating $\int_0^t \int_\Gamma R_2^* \, d\Gamma$ defined by (3.4).** The estimates are completely analogous to the ones for $R_1^*$.

**Estimating $\int_0^t \int_\Gamma R_6^* \, d\Gamma$ defined by (3.5).** The first term on the right-hand side of (3.5) is estimated analogously to the first term on the right-hand side of (3.3). Note that

$$\frac{\sigma}{2} \int_0^t \int_\Gamma |\partial h_{tt}|^2 (|\partial \Psi|^{-3} J^{-1})_t \lesssim (|\partial \Psi|^{-3} J^{-1})_t |\partial h_{tt}| \lesssim t|\partial h_{tt}| |\partial \Psi|^{-3} \lesssim tP(\mathcal{E}^\sigma),$$

where we use Sobolev inequality and the definition of $\mathcal{E}^\sigma$ to infer

$$|\partial h_{tt}| \lesssim |\partial h_{tt}| \lesssim \int_\Gamma (-q, 2) |\partial^2 h| \lesssim \mathcal{E}^\sigma.$$  (3.9)

Space-time integrals of the third and fourth term on the right-hand side of (3.5) are bounded in the usual way by $P(\mathcal{E}^\sigma)$. To bound the last term on the right-hand side of (3.5) we distinguish the cases $l = 0$ and $l = 1$. If $l = 1$, by Leibniz rule expand

$$\partial_{tt} \partial^2 h |\partial \Psi| = \partial^2 h_{tt} |\partial \Psi|^{-1} + 2 \partial^2 h_t (|\partial \Psi|^{-1})_t + \partial^2 h (|\partial \Psi|^{-1})_{tt}.$$  

For the first two terms above integrate by parts to move one $\partial$ derivative away from $\partial^2 h_{tt}$ and $\partial^2 h_t$. Then use the standard $L^\infty - L^2 - L^4$ type estimates as well as the bound $|\partial v_t|_{L^\infty L^2} \lesssim \|q_t\|_{L^\infty L^2}$ to get the desired estimate. For the third term on right-hand side above we have

$$\sigma \int_0^t \int_\Gamma (\partial^2 h (|\partial \Psi|^{-1})_t) \partial_t (w+v) \cdot \partial_t A_2^* \lesssim |\partial^2 h|_{L^\infty} \sqrt{\sigma} (|\partial \Psi|^{-1})_t |\partial h_t (w+v) A_2^* |_{L^\infty L^2} \lesssim \sqrt{\sigma} P(\mathcal{E}^\sigma).$$
The following theorem holds:

Estimating $S^2_q$ and $S^1_q$ defined by (3.6) and (3.7) respectively. The estimates are straightforward and follow the same principle: terms with least amount of derivatives are bounded via Sobolev embedding by the $\sigma$-independent energy $E(\sigma^*, h^*)$.

Summing up the above estimates we prove the first inequality in the proposition. The existence of a $\sigma$-independent time $T$ follows from the standard continuity argument and the fact that constant $C$ in (3.8) is $\sigma$-independent. Since $E^\sigma(0) \to E(0)$ as $\sigma \to 0$ due to our assumption on initial data, the last statement of the proposition follows.

\textbf{Proof of Theorem 1.2.} Recall the definition (1.38) of $(q, h) \in C_0^\infty \cap C^0 C^2_{x}$. Assume that $(q^\sigma, h^\sigma) - (q^0, h^0)$ does not converge to 0 as $\sigma \to 0$. Then there exists an $\epsilon > 0$ and a subsequence $(\sigma_n)_{n \in \mathbb{N}}$, $\sigma_n \to 0$ as $n \to \infty$, such that

$$
\|(q^\sigma_n, h^\sigma_n) - (q^0, h^0)\|_{C_0^\infty \cap C^0 C^2_{x}} \geq \epsilon \quad \forall n \in \mathbb{N}.
$$

Since $E(q^\sigma_n, h^\sigma_n) \leq C$, there exists a subsequence of $(q^\sigma_n, h^\sigma_n)_n$ (without loss of generality indexed again by $(\sigma_n)$) and $(\bar{q}, \bar{h}) \in S$ such that

$$(q^\sigma_n, h^\sigma_n) \to (\bar{q}, \bar{h}), \quad \text{weakly in } S,$$

where we recall that $S$ is defined in (1.27). Note that the injection operator $I : S \to C_0^\infty \cap C^0 C^2_{x}$ is compact. Hence $(q^\sigma_n, h^\sigma_n) \to (\bar{q}, \bar{h})$ in $C_0^\infty \cap C^0 C^2_{x}$ where $(\bar{q}, \bar{h})_{t=0} = (q_0, h_0)$ due to the property 3) in the Definition 1 of the well-prepared initial data. Since $\sigma_n \to 0$ as $n \to \infty$, $(\bar{q}, \bar{h})$ solves the classical Stefan problem with those initial conditions. From the uniqueness statement of Theorem 1.1, we conclude that $(\bar{q}, \bar{h}) = (q, h)$. Thus $(q^\sigma_n, h^\sigma_n) \to (q, h)$ in $C_0^\infty \cap C^0 C^2_{x}$ contradicting (3.10).

### 4. The three-dimensional case

In this section, we briefly sketch how to adapt the analysis of the previous sections to prove theorems analogous to Theorem 1.1 and 1.2 in the three-dimensional setting. We assume now that $\Omega(t)$ is an evolving phase inside the reference domain

$$
\Omega := \mathbb{T}^2 \times (0, 1),
$$

where $\mathbb{T}^2$ is the 2-torus. Initially at $t = 0$ the moving boundary

$$
\Gamma_0 = \mathbb{T}^2 \times \{x^3 = h_0(x)\}
$$

is parametrized as a graph over $\Gamma = \mathbb{T}^2 \times \{x^3 = 0\}$ by the height function $h_0$. The top boundary $\partial \Omega_{\text{top}} = \mathbb{T}^2 \times \{x^3 = 1\}$ is fixed and the temperature $T$ satisfies the homogeneous Neumann boundary condition on $\partial \Omega$ just like in (1.4). We parametrize boundary as a graph over $\Gamma$ with the height function $h(t, x', x')$, where $x' := (x^1, x^2)$. Using the harmonic coordinates we can change of variables as in (1.13) to obtain a fixed boundary problem given by (1.18). The associated energy is given by

$$
E^{3D}(t) = E^{3D}(q, h)(t) := \sum_{|\alpha| + 2b \leq 5} \|\nabla^\alpha \partial_t^b q\|_{L^2}^2 + \frac{1}{2} \sum_{|\alpha| + 2b \leq 4} \|\nabla^\alpha \partial_t^b v\|_{L^2}^2
$$

$$
+ \frac{1}{2} \sum_{|\alpha| + 2b \leq 5} |\nabla^\alpha \partial_t^b h|_{L^2}^2 + \sum_{|\alpha| + 2b \leq 4} |\nabla^\alpha \partial_t^b \Psi|_{L^2}^2
$$

$$
+ \frac{1}{2} \sum_{|\alpha| + 2b \leq 5} \|\nabla^\alpha \partial_t^b q + \nabla^\alpha \partial_t^b \Psi \cdot v\|_{L^2}^2 + \sum_{|\alpha| + 2b \leq 4} \|\nabla^\alpha \partial_t^b q_t + \nabla^\alpha \partial_t^b \Psi_t \cdot v\|_{L^2}^2.
$$

In the above definition, $\alpha = (\alpha_1, \alpha_2)$ is a multi-index of order $|\alpha| = \alpha_1 + \alpha_2$, whereby $\alpha_1, \alpha_2$ are non-negative integers. Symbol $\nabla$ refers to differentiation in tangential directions, i.e. $\nabla^\alpha := \partial_{x^1}^\alpha \partial_{x^2}^\alpha$. The three-dimensional Taylor sign condition for a function $q$ reads:

$$
\min_{x' \in \Gamma}(q, x', 0) > 0.
$$

The following theorem holds:
Theorem 4.1. Let the initial conditions \((q_0, h_0)\) be such that \(E^{3D}(q_0, h_0) < \infty\) and let \(q_0\) satisfy the Taylor sign condition (4.2). Then the three-dimensional one-phase classical Stefan problem is locally-in-time well-posed, i.e. there is a \(T > 0\) such that there exists a unique solution \((q, h)\) with the initial data \((q_0, h_0)\) on the time interval \([0, T]\). In addition it satisfies the bound:

\[
E^{3D}(q, h) \leq 2E^{3D}(q_0, h_0).
\]

Furthermore, let \((q_0^n, \Psi_0^n)_{n \geq 0}\) be a given family of well-prepared initial conditions in the sense of Definition 1. Assume that it satisfies the Taylor sign condition (4.2) and the corresponding compatibility conditions. By \((q^n, h^n)_{n \geq 0}\) we denote the associated family of solutions to the problem (1.18). There exists a \(\sigma\)-independent time \(T > 0\) and a constant \(C\) depending only on \((q_0, h_0)\) such that

\[
E^{3D, \sigma}(q^n, h^n)(T) \leq C \quad \sigma \geq 0.
\]

for all \(\sigma \geq 0\). As a consequence, sequence \((q^n, h^n)\) converges to the unique solution \((q, h)\) of the classical Stefan problem (1.18) with \(\sigma = 0\) in \(C^1_0 C^0 \cap C^0 \cap C^2\) norm.

Remark 10. Note that the definition of \(E^{3D}\) contains time derivatives. Thus, to make sense out of the assumption \(E^{3D}(q_0, h_0) < \infty\), we express the time derivatives \(\partial_t q_0\) and \(\partial_t h_0\) in terms of the spatial derivatives as explained in Remark 4.

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Appendix A. Modifications of our analysis for a more general initial domain

In this section we explain how to construct a smooth reference interface for a general graph \(\Gamma_0 = \{x \mid x = (x, h_0(x))\} \subset T^1 \times [0, 1]\), where the size of \(|h|_{4,5}\) is not necessarily small. For any \(\varepsilon > 0\) we define

\[
h_0^\varepsilon(x) = \int_{T^1} h(y) \rho_\varepsilon(x - y), \quad x \in T^1, \quad \Gamma_0^\varepsilon = \{x \mid x = (x, h_0^\varepsilon(x))\} \subset T^1 \times [0, 1],
\]

and set

\[
\Omega_0^\varepsilon := \{(x, y) \in T^1 \times [0, 1] \mid x \in T^1, \ h_0^\varepsilon(x) < y < 1\}.
\]

Here \(\rho_\varepsilon\) is the standard mollifier defined in Definition 2 and the domain \(\Omega_0^\varepsilon\) will be our reference domain. Clearly \(h_0^\varepsilon \in C^\infty(\Gamma)\) and for \(\varepsilon\) sufficiently small we can parametrize the evolving surface \(\Gamma(t)\) as a graph over \(\Gamma_0^\varepsilon\) using the outward-pointing unit normal vector field \(N^\varepsilon\) of \(\Gamma_0^\varepsilon\):

\[
\Gamma(t) = \{x \mid x = (x, h_0^\varepsilon(x)) + h(t,x) N^\varepsilon(x)\}, \quad N^\varepsilon(x) = \frac{(\partial h_0^\varepsilon, -1)}{\sqrt{1 + |\partial h_0^\varepsilon|^2}}.
\]

Note that \(|h_\varepsilon - h_0|_{4,5} \to 0\) as \(\varepsilon \to 0\). The construction of the harmonic diffeomorphic extension \(\Psi : \Omega_0^\varepsilon \to \Omega(t)\) of the boundary data

\[
\Psi(t, x, h_0^\varepsilon(x)) = (x, h_0^\varepsilon(x)) + h(t, x) N^\varepsilon(x), \quad \Psi(t, x, 1) = (x, 1)
\]

is a simple consequence of the existence theory for the Dirichlet boundary value problems for systems of elliptic partial differential equations, since for small \(\varepsilon\) and small times \(t \geq 0\) we have

\[
|\Psi - \text{Id}|_{4,5} \lesssim \varepsilon \ll 1.
\]

Using the argument in (1.7) the trace estimate (1.8) is true. Fixing an \(\varepsilon > 0\) sufficiently small we drop the \(\varepsilon\)-notation and refer to the reference curve \(\Gamma_0^\varepsilon\) as \(\Gamma\), the reference domain \(\Omega_0^\varepsilon\) as \(\Omega\), the reference unit normal \(N^\varepsilon\) as \(N\), and the reference height \(h_0^\varepsilon\) as \(\bar{h}\). In the harmonic gauge, the Stefan
problem takes nearly the same form (1.18):

\[
q_t - A^k_i(A^k_i q_k)_{,i} = -v \cdot w \quad \text{in} \quad \Omega, \quad \text{(A.1a)}
\]

\[
v^i + A^k_i q_{,k} = 0 \quad \text{in} \quad \Omega, \quad \text{(A.1b)}
\]

\[
q = 0 \quad \text{on} \quad \Gamma, \quad \text{(A.1c)}
\]

\[
\Psi_t \cdot n(t) = -v \cdot n(t) \quad \text{on} \quad \Gamma, \quad \text{(A.1d)}
\]

\[
v \cdot N = 0 \quad \text{on} \quad \partial \Omega_{\text{top}}, \quad \text{(A.1e)}
\]

\[
q(0, \cdot) = q_0 = p_0 \circ \Psi; \quad \Psi(0, \cdot) = \Psi_0., \quad \text{(A.1f)}
\]

where

\[
\|\Psi_0 - \text{Id}\|_{H^5} \lesssim \varepsilon
\]

and the local coordinate realization of the unit normal \(n(t,x)\) takes the more general form:

\[
n(t,x) = \frac{(1-hH)\sqrt{1+(\partial h)^2} N - \partial h T}{\sqrt{(1+(\partial h)^2)(1-hH)^2 + (\partial h)^2}}, \quad x \in \mathbb{T}^1,
\]

where

\[
H = -\frac{\partial^2 h}{(1+(\partial h)^2)^{3/2}}, \quad T = \frac{(1, \partial h)}{\sqrt{1+(\partial h)^2}}
\]

stand for the mean curvature and the unit tangent to the reference surface \(\Gamma\) respectively. The proof of Theorem 1.1 applies to (A.1) in an analogous manner, it is simply more technical. The main technical novelty is that the tangential vector-fields to the reference surface \(\Gamma\) are not given by \(\tilde{\partial} = \partial_x\), as \(\Gamma\) may have a nontrivial curvature in general. Therefore, in the neighborhood of \(\Gamma\) for any \(C^1\) function \(f: \Omega \rightarrow \mathbb{R}\) we define the tangential derivative

\[
\tilde{\partial} f = \nabla f \cdot T,
\]

where \(T\) is a local extension of the unit tangent vector field \(T\) into the domain \(\Omega\). Choosing a smooth cut-off function \(\mu: \Omega \rightarrow [0,1]\) defined to be 1 in a neighborhood of \(\Gamma\) and 0 in a neighborhood of \(\partial \Omega_{\text{top}}\), we can replace the operator \(\tilde{\partial}\) in Lemma 2.2 by the operator

\[
\mu \tilde{\partial} + (1-\mu) \partial_i, \quad i = 1,2.
\]

The ensuing energy identities, energy estimates, and the proof of Theorem 1.1 follow in an analogous way.

**Appendix B. Auxiliary lemmas**

We collect some auxiliary estimates in this section that have been used in the proof of the energy estimates. The following commutator estimate is used in the proof of Proposition 2.6.

**Lemma B.1** (Lemma 5.1 in [20]). For \(F \in W^{1,\infty}(\Gamma)\) and \(G, \tilde{\partial} G \in L^2(\Gamma)\), there is a generic constant \(C\) independent of \(\kappa\) such that

\[
|\Lambda_{\kappa}(F \tilde{\partial} G) - f \Lambda_{\kappa}\tilde{\partial} G| \leq C|F|_{W^{1,\infty}(\Gamma)}|G|_0,
\]

where \(W^{1,\infty}(\Gamma)\) denotes the Sobolev space of functions \(h \in L^\infty(\Gamma)\) with weak derivative \(\partial h \in L^\infty(\Gamma)\).

Similarly, the following bound is used in estimating some top-order terms in the energy estimates.

**Lemma B.2** (Lemma 8.5 in [20]). Let \(H^{\frac{1}{2}}(\Omega)\) denote the dual space of \(H^{-\frac{1}{2}}(\Omega)\). Then there exists a positive constant \(C > 0\) such that

\[
\|\tilde{\partial} F\|_{H^{\frac{1}{2}}(\Omega)^\prime} \leq C\|F\|_{H^{-\frac{1}{2}}(\Omega)}.
\]

Proof. The proof is a simple consequence of an interpolation estimate between \(L^2(\Omega)\) and \(H^1(\Omega)^\prime\) spaces. The details are given in [20]. □
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