Bernstein Functions and Radial Limits of Prescribed Mean Curvature Surfaces

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Abstract
The radial limits at a point \( y \) of the boundary of the domain \( \Omega \subset \mathbb{R}^2 \) of a bounded variational solution \( f \) of Dirichlet or contact angle boundary value problems for a prescribed mean curvature equation are studied with an emphasis on the effects of assumptions about the curvatures of the boundary \( \partial \Omega \) on each side of the point \( y \). For example, at a nonconvex corner \( y \), we previously proved that all nontangential radial limits of \( f \) at \( y \) exist; here we provide sufficient conditions for the tangential radial limits to exist, even when the Dirichlet data \( \phi \in L^\infty(\partial \Omega) \) has no one-sided limits at \( y \) or the contact angle \( \gamma \in L^\infty(\partial \Omega ; [0, \pi]) \) is not bounded away from 0 or \( \pi \). We also provide a complement to a 1976 Theorem by Leon Simon on least area surfaces.

1 Introduction
Let \( \Omega \) be a locally Lipschitz domain in \( \mathbb{R}^2 \) and define \( Nf = \nabla \cdot Tf = \text{div} (Tf) \), where \( f \in C^2(\Omega) \) and \( Tf = \frac{\nabla f}{\sqrt{1 + |\nabla f|^2}} \). Consider the Dirichlet problem

\[
Nf = H(\cdot, f(\cdot)) \quad \text{in} \quad \Omega
\]

\[
f = \phi \quad \text{on} \quad \partial \Omega
\]

and the contact angle problem

\[
Nf = H(\cdot, f(\cdot)) \quad \text{in} \quad \Omega
\]

\[
Tf \cdot \nu = \cos \gamma \quad \text{on} \quad \partial \Omega,
\]
where \( \phi : \partial \Omega \to \mathbb{R} \), \( \gamma : \partial \Omega \to [0, \pi] \), and \( H : \Omega \times \mathbb{R} \to \mathbb{R} \) are prescribed functions, \( H(x, t) \) is nondecreasing in \( t \) for each \( x \in \Omega \) (cf. \( \mathbb{6} \)) and \( \nu \) is the exterior unit normal to \( \partial \Omega \).

For a smooth domain, some type of boundary curvature condition (which depends on \( H \)) must be satisfied in order to guarantee that a classical solution of (1)-(2) exists for each \( \phi \in C^{0}(\partial \Omega) \); when \( H \equiv 0 \), this curvature condition is that \( \partial \Omega \) must have nonnegative curvature (with respect to the interior normal direction of \( \Omega \)) at each point (e.g. \( \mathbb{17} \)). However, Leon Simon (\( \mathbb{30} \)) has shown that if \( \Gamma_{0} \subset \partial \Omega \) is smooth (i.e. \( C^{4} \)), \( H \equiv 0 \), \( \phi \in C^{0,1}(\partial \Omega) \), the curvature \( \Lambda \) of \( \partial \Omega \) is negative on \( \Gamma_{0} \) and \( \Gamma \) is a compact subset of \( \Gamma_{0} \), then the variational solution \( z = f(x) \), \( x \in \Omega \), extends to \( \Omega \cup \Gamma \) as a Hölder continuous function with Lipschitz continuous trace, even though \( f \) may not equal \( \phi \) on \( \Gamma \); Simon’s result holds for least area hypersurfaces in \( \mathbb{R}^{n} \), \( n \geq 2 \) when the mean curvature of \( \partial \Omega \) has a negative upper bound on \( \Gamma \subset \partial \Omega \) (see also \( \mathbb{1} \mathbb{27} \)).

One can look at this in a different way. In the case \( H \equiv 0 \), the requirement that \( \Lambda(p) < 0 \) at a point \( p \in \partial \Omega \) implies that \( Nf = 0 \) has a (continuous) Bernstein function \( \psi \) at \( p \) for \( \Omega \) (see Definition \( \mathbb{1} \) and Definition \( \mathbb{3} \)). In \( \mathbb{8} \), Bernstein functions for the minimal surface equation in \( \mathbb{R}^{2} \) are constructed for \( C^{2,\alpha} \) domains \( \Omega \subset \mathbb{R}^{2} \) whose curvature \( \Lambda \) (with respect to \(-\nu\)) vanishes at a finite number of points and satisfies \( \Lambda \leq 0 \) on a segment of \( \partial \Omega \). Using these Bernstein functions, we will prove the following generalization of (3)-(4) when \( n = 2 \).

**Corollary 1.** Let \( \Omega \) be a domain in \( \mathbb{R}^{2} \), \( \Gamma \) is a \( C^{2,\lambda} \) open subset of \( \partial \Omega \) and the curvature \( \Lambda \) (with respect to \(-\nu\)) of \( \Gamma \) is nonpositive and vanishes at only a finite number of points of \( \Gamma \), for some \( \lambda \in (0, 1) \). Suppose \( \phi \in L^{\infty}(\partial \Omega) \), \( y \in \Gamma \), either \( f \) is symmetric with respect to a line through \( y \) or \( \phi \) is continuous at \( y \), and \( f \in BV(\Omega) \) minimizes

\[
J(u) = \int_{\Omega} \sqrt{1 + |Du|^{2}} \, dx + \int_{\partial \Omega} |u - \phi| \, ds
\]

for \( u \in BV(\Omega) \). Then \( f \in C^{0}(\Omega \cup \{y\}) \). If \( \phi \in C^{0}(\Gamma) \), then \( f \in C^{0}(\Omega \cup \Gamma) \).

**Example 1.** Let \( \Omega = \{(x, y) \in \mathbb{R}^{2} : 1 < (x + 1)^{2} + y^{2} < \cosh^{2}(1) \} \) and \( \phi(x, y) = \sin \left( \frac{\pi}{x + y} \right) \) for \((x, y) \neq (0, 0)\) (see Figure \( \mathbb{1} \) for a rough illustration of the graph of \( \phi \)). Set \( \mathcal{O} = (0, 0) \). Let \( f \in C^{2}(\Omega) \) minimize (3) over \( BV(\Omega) \) (i.e. \( f \) is the variational solution of (1)-(2) with \( H \equiv 0 \)). Then Corollary \( \mathbb{2} \) (with \( y = O \)) implies \( f \in C^{0}(\overline{\Omega}) \), even though \( \phi \) has no limit at \( O \).

Variational solutions of (3)-(4) will exist in some sense (e.g. §7.3 of \( \mathbb{12} \)) but they need not be finitely valued (e.g. the discussion of extremal curves in Chapter 6 of \( \mathbb{12} \)), bounded (e.g. \( \mathbb{12} \), Corollary 5.5) or continuous at each point of the boundary (e.g. \( \mathbb{18} \)). Variational solutions of (1)-(2) will be bounded if \( \phi \in L^{\infty}(\Omega) \) but need not be continuous at each point of the boundary. Many authors (e.g. \( \mathbb{7} \mathbb{9} \mathbb{13} \mathbb{22} \mathbb{28} \mathbb{30} \mathbb{31} \)) have investigated the boundary behavior at corners of variational solutions of (1)-(2) and a number of authors (e.g. \( \mathbb{4} \mathbb{10} \mathbb{12} \mathbb{11} \mathbb{15} \mathbb{18} \mathbb{22} \mathbb{24} \mathbb{25} \mathbb{29} \)) have done so for variational solutions of (3)-(4).
Let $Q$ be the operator on $C^2(\Omega)$ given by

$$Qf(x) \overset{\text{def}}{=} Nf(x) - 2H(x, f(x)), \quad x \in \Omega,$$

where $H : \Omega \times \mathbb{R} \to \mathbb{R}$ is prescribed and $H(x, t)$ is weakly increasing in $t$ for each $x \in \Omega$. Let $\nu$ be the exterior unit normal to $\partial \Omega$, defined almost everywhere on $\partial \Omega$. We assume that for almost every $y \in \partial \Omega$, there is a continuous extension $\hat{\nu}$ of $\nu$ to a neighborhood of $y$.

For each point $y \in \partial \Omega$, polar coordinates relative to $y$ are denoted by $r_y$ and $\theta_y$. We shall assume that for each $y \in \partial \Omega$, there exists a $\delta > 0$ such that $\partial \Omega \cap B_\delta(y) \setminus \{y\}$ consists of two (open) arcs $\partial_1^y \Omega$ and $\partial_2^y \Omega$, whose tangent rays approach the rays $L_y^1 : \theta_y = \alpha(y)$ and $L_y^2 : \theta_y = \beta(y)$ respectively, as the point $y$ is approached, with $\alpha(y) < \beta(y) < \alpha(y) + 2\pi$, in the sense that the tangent cone to $\Omega$ at $y$ is $\{\alpha(y) \leq \theta_y \leq \beta(y), 0 \leq r_y < \infty\}$. (In particular, $\{\alpha(y) < \theta_y < \beta(y), 0 < r_y < \epsilon(\theta_y)\}$ is a subset of $\Omega$ for some $\epsilon \in C^0((\alpha(y), \beta(y)))$, $\epsilon(\cdot) > 0$, and $\{\beta(y) < \theta_y < \alpha(y) + 2\pi, 0 < r_y < \epsilon(\theta_y)\} \cap \Omega = \emptyset$ for some $\epsilon \in C^0((\beta(y), \alpha(y) + \pi))$, $\epsilon(\cdot) > 0$.) When $\beta(y) - \alpha(y) < \pi$, $\partial \Omega$ is said to have a **convex corner** at $y$ and when $\beta(y) - \alpha(y) > \pi$, $\partial \Omega$ is said to have a **nonconvex corner** at $y$. The radial limit of $f$ at $y = (y_1, y_2) \in \partial \Omega$ in the direction $\omega(\theta) = (\cos \theta, \sin \theta)$, $\theta \in (\alpha(y), \beta(y))$, is

$$Rf(\theta, y) \overset{\text{def}}{=} \lim_{r \to 0} f(y_1 + r \cos(\theta), y_2 + r \sin(\theta)).$$

$Rf(\alpha(y), y)$ will be defined as the limit at $y$ of the trace of $f$ restricted to $\partial_1^y \Omega$ and $Rf(\beta(y), y)$ as the limit at $y$ of the trace of $f$ restricted to $\partial_2^y \Omega$. Notice that if $f$ is a generalized (e.g. variational or Perron) solution of (1)-(2), $f$ need not equal $\phi$ on portions of $\partial \Omega$ and the tangential radial limits $Rf(\alpha(y), y)$ and $Rf(\beta(y), y)$ may, for example, differ from $\phi(y)$ when $\phi$ is continuous at $y$. 

2 Radial Limit Theorems

We shall investigate the existence and behavior of the radial limits of non-parametric prescribed mean curvature surfaces at corners of the domain, including “smooth corners” (e.g. Corollary 1). In particular, we shall use Bernstein functions to investigate the behavior of variational solutions of (1)-(2) or (3)-(4) at points of $\partial \Omega$.
Definition 1. Given a domain $\Omega$ as above, a **upper Bernstein pair** $(U^+, \psi^+)$ for a curve $\Gamma \subset \partial \Omega$ and a function $H$ is a $C^1$ domain $U^+$ and a function $\psi^+ \in C^2(U^+) \cap C^0 \left( \overline{U^+} \right)$ such that $\Gamma \subset \partial U^+$, $\nu$ is the exterior unit normal to $\partial U^+$ at each point of $\Gamma$ (i.e. $U^+$ and $\Omega$ lie on the same side of $\Gamma$; see Figure 3), $Q\psi^+ \leq 0$ in $U^+$, and $T\psi^+ \cdot \nu = 1$ almost everywhere on an open subset of $\partial U^+$ containing $\Gamma$ in the same sense as in [3]; that is, for almost every $y \in \Gamma$,

$$\lim_{U^+ \ni x \to y} \frac{\nabla \psi^+(x) \cdot \nu(x)}{\sqrt{1 + |\nabla \psi^+(x)|^2}} = 1. \quad (8)$$

Definition 2. Given a domain $\Omega$ as above, a **lower Bernstein pair** $(U^-, \psi^-)$ for a curve $\Gamma \subset \partial \Omega$ and a function $H$ is a $C^1$ domain $U^-$ and a function $\psi^- \in C^2(U^-) \cap C^0 \left( \overline{U^-} \right)$ such that $\Gamma \subset \partial U^-$, $\nu$ is the exterior unit normal to $\partial U^-$ at each point of $\Gamma$ (i.e. $U^-$ and $\Omega$ lie on the same side of $\Gamma$), $Q\psi^- \geq 0$ in $U^-$, and $T\psi^- \cdot \nu = -1$ almost everywhere on an open subset of $\partial U^-$ containing $\Gamma$ in the same sense as in [3].

In the following theorem, we consider a domain with a nonconvex corner $y$ and prove that the radial limits of $f$ at $y$ exist and behave as in [7, 20, 21].

In [20], $\Omega$ was required to be locally convex at points of $\partial U$ and $\partial \Omega$ and, in [7, 21], the curvatures of $\partial_1^y \Omega$ and $\partial_2^y \Omega$ were required to have an appropriate positive lower bound when these curves were smooth. In [9], no such curvature requirement was imposed but only nontangential radial limits were shown to exist. This theorem strengthens Theorem 1 of [9] when the curvatures of $\partial_1^y \Omega$ and $\partial_2^y \Omega$ imply Bernstein functions exist (see [4]).

**Theorem 1.** Let $f \in C^2(\Omega) \cap L^\infty(\Omega)$ satisfy $Qf = 0$ in $\Omega$ and let $H^* \in L^\infty(\mathbb{R}^2)$ satisfy $H^*(x) = H(x, f(x))$ for $x \in \Omega$. Suppose that $y \in \partial \Omega$, $\beta(y) - \alpha(y) > \pi$, and there exist $\delta > 0$ and upper and lower Bernstein pairs $(U_1^\pm, \psi_1^\pm)$ and $(U_2^\pm, \psi_2^\pm)$ for $(\Gamma_1, H^*)$ and $(\Gamma_2, H^*)$ respectively, where $\Gamma_1 = B_\delta(y) \cap \partial_1^y \Omega$ and $\Gamma_2 = B_\delta(y) \cap \partial_2^y \Omega$. Then the limits

$$\lim_{\Gamma_1 \ni x \to y} f(x) = z_1 \quad \text{and} \quad \lim_{\Gamma_2 \ni x \to y} f(x) = z_2 \quad (9)$$

exist, the radial limit $Rf(\theta, y)$ exists for each $\theta \in [\alpha(y), \beta(y)]$, $Rf (\alpha(y), y) = z_1$, $Rf (\beta(y), y) = z_2$, and $Rf(\cdot, y)$ is a continuous function on $[\alpha(y), \beta(y)]$ which behaves in one of the following ways:

(i) $Rf(\cdot, y) = z_1$ is a constant function and $f$ is continuous at $y$.

(ii) There exist $\alpha_1$ and $\alpha_2$ so that $\alpha(y) \leq \alpha_1 < \alpha_2 \leq \beta(y)$, $Rf = z_1$ on $[\alpha(y), \alpha_1]$, $Rf = z_2$ on $[\alpha_2, \beta(y)]$ and $Rf$ is strictly increasing (if $z_1 < z_2$) or strictly decreasing (if $z_1 > z_2$) on $[\alpha_1, \alpha_2]$.

(iii) There exist $\alpha_0, \alpha_1, \alpha_2$ so that $\alpha(y) \leq \alpha_0 < \alpha_1 < \alpha_2 < \alpha_2 \leq \beta(y)$, $\alpha_R = \alpha_L + \pi$, and $Rf$ is constant on $[\alpha(y), \alpha_1], [\alpha_L, \alpha_R]$, and $[\alpha_2, \beta(y)]$ and either strictly increasing on $[\alpha_1, \alpha_2]$ and strictly decreasing on $[\alpha, \alpha_2]$ or strictly decreasing on $[\alpha_1, \alpha_L]$ and strictly increasing on $[\alpha_L, \alpha_2]$. 


Let expected. Corollary 1 follows from this theorem and an additional argument.

**Theorem 2.** Let \( f \in C^2(\Omega) \cap L^\infty(\Omega) \) satisfy \( Qf = 0 \) in \( \Omega \) and let \( H^* \in L^\infty(\mathbb{R}^2) \) satisfy \( H^*(x) = H(x, f(x)) \) for \( x \in \Omega \). Suppose that \( y \in \partial \Omega \), \( \beta(y) - \alpha(y) = \pi \), and there exist \( \delta > 0 \) and upper and lower Bernstein pairs \( (U^\pm, \psi^\pm) \) for \( (\Gamma, H^*) \), where \( \Gamma = B_\delta(y) \cap \partial \Omega \). Then the limits

\[
\lim_{\gamma_1 \to x, y} f(x) = z_1 \quad \text{and} \quad \lim_{\gamma_2 \to x, y} f(x) = z_2
\]

exist, \( Rf(\theta, y) \) exists for each \( \theta \in [\alpha(y), \beta(y)] \), \( Rf(\cdot, y) \in C^0([\alpha(y), \beta(y)]) \), \( Rf(\alpha(y), y) = z_1 \), \( Rf(\beta(y), y) = z_2 \), and \( Rf(\cdot, y) \) behaves as in (i) or (ii) of Theorem 2.

In the third theorem, we consider a domain with a convex corner \( y \) and prove that the radial limits of \( f \) at \( y \) exist and behave as expected. This theorem strengthens Theorem 2 of [9].

**Theorem 3.** Let \( f \in C^2(\Omega) \cap L^\infty(\Omega) \) satisfy \( Qf = 0 \) in \( \Omega \) and let \( H^* \in L^\infty(\mathbb{R}^2) \) satisfy \( H^*(x) = H(x, f(x)) \) for \( x \in \Omega \). Suppose that \( y \in \partial \Omega \) and there exist \( \delta > 0 \) and upper and lower Bernstein pairs \( (U^\pm, \psi^\pm) \) for \( (\Gamma_2, H^*) \), where \( \Gamma_2 = B_\delta(y) \cap \partial \Omega \). Suppose further that \( z_1 = \lim_{\gamma_1 \to x, y} f(x) \) exists, where \( \Gamma_1 = B_\delta(y) \cap \partial \Omega \). Then

\[
\lim_{\gamma_2 \to x, y} f(x) = z_2
\]

exists, \( Rf(\theta, y) \) exists for each \( \theta \in [\alpha(y), \beta(y)] \), \( Rf(\cdot, y) \in C^0([\alpha(y), \beta(y)]) \), \( Rf(\alpha(y), y) = z_1 \), \( Rf(\beta(y), y) = z_2 \), and \( Rf(\cdot, y) \) behaves as in (i), (ii) or (iii) of Theorem 2.

In the fourth theorem, we generalize Theorem 2 of [10].

**Theorem 4.** Let \( f \in C^2(\Omega) \) satisfy \( Qf = 0 \) in \( \Omega \) and let \( H^* \in L^\infty(\mathbb{R}^2) \) satisfy \( H^*(x) = H(x, f(x)) \) for \( x \in \Omega \). Suppose that \( y \in \partial \Omega \), \( \beta(y) - \alpha(y) < \pi \), and there exist \( \delta > 0 \) and upper and lower Bernstein pairs \( (U^\pm, \psi^\pm) \) for \( (\Gamma_2, H^*) \), where \( \Gamma_2 = B_\delta(y) \cap \partial \Omega \). Suppose further that \( f \in C^1(\Omega \cup \partial \Omega \cup \partial \Omega \cup \partial \Omega) \), \( T \in C^1(\Omega) \), \( \nu(x) = \cos(\gamma(x)) \) for \( x \in \partial \Omega \), and

\[
\lim_{\partial \Omega \ni x \to y} \gamma(x) = \gamma_2.
\]
Suppose also that there exist \( \lambda_1, \lambda_2 \in [0, \pi] \) with \( 0 < \lambda_2 - \lambda_1 < 2 (\beta(y) - \alpha(y)) \) such that \( \lambda_1 \leq \gamma(x) \leq \lambda_2 \) for \( x \in \partial_2^c \Omega \) and \( \pi - 2\alpha - \lambda_1 < \gamma_2 < \pi + 2\alpha - \lambda_2 \). Then the conclusions of Theorem 1 hold.

In the fifth theorem, we generalize Theorem 1 of [25] at the cost of extra boundary assumptions; Theorem 1 of [5] also generalizes the Lancaster-Siegel theorem but only obtains nontangential radial limits while here the existence of all radial limits is established while not requiring the contact angle to be bounded away from zero or \( \pi \).

**Theorem 5.** Let \( f \in C^2(\Omega) \cap L^\infty(\Omega) \) satisfy (1) and (2) almost everywhere on \( \partial \Omega \). Let \( H^* \in L^\infty(\mathbb{R}^2) \) satisfy \( H^*(x) = H(x, f(x)) \) for \( x \in \Omega \). Let \( y \in \partial \Omega \) and suppose there exist \( \delta > 0 \) and upper and lower Bernstein pairs \( (U^\pm_1, \psi^\pm_1) \) and \( (U^\pm_2, \psi^\pm_2) \) for \( (\Gamma_1, H^*) \) and \( (\Gamma_2, H^*) \) respectively, where \( \Gamma_1 = B_\delta(y) \cap \partial_1^c \Omega \) and \( \Gamma_2 = B_\delta(y) \cap \partial_2^c \Omega \). If \( \beta(y) - \alpha(y) \leq \pi \), assume there exist constants \( \gamma^+_\pm, \gamma^-_\pm, 0 \leq \gamma^+_\pm, \gamma^-_\pm \leq \pi \), satisfying

\[
\pi - (\beta(y) - \alpha(y)) < \gamma^+_\pm + \gamma^-_\pm \\
\leq \pi^+_\pm + \pi^-_\pm < \pi + \beta(y) - \alpha(y)
\]
such that \( \gamma^+_\pm \leq \gamma^+_\pm(s) \leq \pi^+_\pm \) for all \( s \in (0, s_0) \), for some \( s_0 > 0 \). Then the conclusions of Theorem 1 hold.

**Example 2.** Let \( \Omega = \{(r \cos \theta, r \sin \theta) : 0 < r < 1, -\alpha < \theta < \alpha \} \) with \( \alpha > \frac{\pi}{2} \). (see Figure 3(a)). Let \( \phi(x, y) = \sin \left( \frac{\pi}{x+y^2} \right) \) for \( (x, y) \neq (0, 0) \) (see Figure 3(b) for a rough illustration of the graph of \( \phi \)). Let \( f \) satisfy (1) in \( \Omega \) with \( H \equiv 0 \) and \( f = \phi \) on \( \partial \Omega \setminus \{O\} \). Then [22] shows that \( Rf(\theta) \) exists when \( |\theta| < \alpha \). Since \( \Omega \) is locally convex at each point of \( \partial \Omega \setminus \{O\} \), we see that \( f \in C^0(\Omega \setminus \{O\}) \) and \( f = \phi \) on \( \partial \Omega \setminus \{O\} \). Since \( \phi \) has no limit at \( O \), \( Rf(\pm \alpha) \) do not exist; however \( \lim_{\theta \to -\alpha} Rf(\theta) \) and \( \lim_{\theta \to \alpha} Rf(\theta) \) both exist (e.g. from the behavior of \( Rf(\theta) \) established in [21, 22]) and, by symmetry, are equal.

Suppose we replace \( \Omega \) with a slightly larger (and still symmetric) domain \( \Omega_1 \), \( \Omega \subset \Omega_1 \subset B_1(O) \), such that \( \partial \Omega_1 \cap B_1(O) \) has negative curvature (with respect to the exterior normal to \( \Omega_1 \)) and \( \partial \Omega_1 \) and \( \partial \Omega_1 \) are tangent at \( O \) (see Figure 3(c) for an illustration of \( \Omega_1 \)). Let \( f_1 \in C^\infty(\Omega) \) minimize (1) over \( BV(\Omega_1) \), so that \( f_1 \) is the variational solution of (1) in \( \Omega_1 \) with \( H \equiv 0 \). Then Theorem 1 implies \( Rf_1(\theta) \) exists when \( |\theta| \leq \alpha \) and symmetry implies \( Rf_1(-\alpha) = Rf_1(\alpha) \). One wonders, for example, about the relationship between \( Rf_1(\alpha) \) and \( \lim_{\theta \to \alpha} Rf(\theta) \).

### 3 Proofs

**Remark 1.** The proofs of these Theorems are similar to those in [22] (and [25]). One difference is that the results in [22, 25] were only concerned with nontangential radial limits at one point, \( O \), and so restricting the solution ("f") to a subdomain which is tangent to the domain \( \Omega \) at \( O \) and therefore assuming \( f \in C^0(\Omega \setminus \{O\}) \)
caused no difficulties. Since we wish to show that tangential radial limits also exist and describe the behavior of \( f \) on \( \partial \Omega \), we cannot make such simplifying assumptions and so we have to modify the proofs in [5, 9].

**Proof of Theorem 1** We may assume \( \Omega \) is a bounded domain. Set \( S_0 = \{ (x, f(x)) : x \in \Omega \} \). From the calculation on page 170 of [25], we see that the area of \( S \) is finite; let \( M_0 \) denote this area. For \( \delta \in (0, 1) \), set

\[
p(\delta) = \sqrt{\frac{8\pi M_0}{\ln \left( \frac{1}{\delta} \right)}}.
\]

Let \( E = \{ (u, v) : u^2 + v^2 < 1 \} \). As in [7, 25], there is a parametric description of the surface \( S \),

\[
Y(u, v) = (a(u, v), b(u, v), c(u, v)) \in C^2(E : \mathbb{R}^3),
\]

which has the following properties:

- \((a)\) \( Y \) is a diffeomorphism of \( E \) onto \( S_0 \).
- \((b)\) Set \( G(u, v) = (a(u, v), b(u, v)) \), \( (u, v) \in E \). Then \( G \in C^0(E : \mathbb{R}^2) \).
- \((c)\) Set \( \sigma(y) = G^{-1}(\partial \Omega \setminus \{ y \}) \); then \( \sigma(y) \) is a connected arc of \( \partial \Omega \) and \( Y \) maps \( \sigma(y) \) onto \( \partial \Omega \setminus \{ y \} \). We may assume the endpoints of \( \sigma(y) \) are \( o_1(y) \) and \( o_2(y) \). (Note that \( o_1(y) \) and \( o_2(y) \) are not assumed to be distinct.)
- \((d)\) \( Y \) is conformal on \( E \): \( Y_u \cdot Y_v = 0 \), \( Y_u \cdot Y_\nu = Y_v \cdot Y_\nu \) on \( E \).
- \((e)\) \( \Delta Y := Y_{uu} + Y_{vv} = H^* (Y) Y_u \times Y_v \) on \( E \).

Notice that for each \( C \in \mathbb{R} \), \( Q(\psi_j^+ + C) = Q(\psi_j^-) \leq 0 \) on \( \Omega \cap U_j^+ \) and \( Q(\psi_j^- + C) = Q(\psi_j^-) \geq 0 \) on \( \Omega \cap U_j^- \), \( j = 1, 2 \), and so

\[
N(\psi_j^+ + C)(x) \leq 2H(x, f(x)) = Nf(x) \quad \text{for} \quad x \in \Omega \cap U_j^+, \quad j = 1, 2 \tag{11}
\]

and

\[
N(\psi_j^- + C)(x) \geq 2H(x, f(x)) = Nf(x) \quad \text{for} \quad x \in \Omega \cap U_j^-, \quad j = 1, 2. \tag{12}
\]

Let \( q \) denote a modulus of continuity for \( \psi_1^\pm \) and \( \psi_2^\pm \).

Let \( \zeta(y) = \partial E \setminus \sigma(y) \): then \( G(\zeta(y)) = \{ y \} \) and \( o_1(y) \) and \( o_2(y) \) are the endpoints of \( \zeta(y) \). There exists a \( \delta_1 > 0 \) such that if \( w \in E \) and \( \text{dist} (w, \zeta(y)) \leq 2\delta_1 \), then \( G(w) \in (U_1^+ \cup U_2^+) \cap (U_1^- \cup U_2^-) \). Now \( T(\psi_j^\pm) \cdot \nu = \pm 1 \) (in the sense of

\[
\begin{align*}
\text{Figure 3:} & \quad (a) \ \Omega \quad (b) \ \text{The graph of } \phi \ \text{over } \partial \Omega \quad (c) \ \Omega_1
\end{align*}
\]
\[ \text{Claim: } Y \text{ is uniformly continuous on } V^* \text{ and so extends to a continuous function on } \overline{V^*}. \]

**Pf:** Let \( \epsilon > 0 \). Choose \( \delta \in \left( 0, (\delta^*)^2 \right) \) such that \( p(\delta) + 2q(p(\delta)) < \epsilon \). Let \( w_1, w_2 \in V^* \) with \( |w_1 - w_2| < \delta \); then \( G(w_1), G(w_2) \in (U_1^+ \cup U_2^+) \cap (U_1^- \cup U_2^-) \). Set \( C_\epsilon(w) = \{ u \in E : |u - w| = \epsilon \} \) and \( B_\rho(w) = \{ u \in E : |u - w| < \rho \} \). From the Courant-Lebesgue Lemma (e.g. Lemma 3.1 in [2]), we see that there exists \( \rho = \rho(\delta) \in \left( \delta, \sqrt{3} \right) \) such that the arclength \( l_\rho(w_1) \) of \( Y(C_\rho(w_1)) \) is less than \( p(\delta) \). Notice that \( w_2 \in B_\rho(w_1) \).

Let \( k(\delta)(w_1) = \inf_{u \in C_\rho(\delta)(w_1)} c(u) = \inf_{x \in G(C_\rho(\delta)(w_1))} f(x) \)

and \( m(\delta)(w_1) = \sup_{u \in C_\rho(\delta)(w_1)} c(u) = \sup_{x \in G(C_\rho(\delta)(w_1))} f(x) \).

Then \( m(\delta)(w_1) - k(\delta)(w_1) \leq l_\rho < p(\delta) \).

Fix \( x_0 \in C'_\rho(\delta)(w_1) \). For \( j = 1, 2 \), set

\[ C_j^+ = \inf_{x \in U_j^+ \cap C'_\rho(\delta)(w_1)} \psi_j^+(x) \quad \text{and} \quad C_j^- = \sup_{x \in U_j^- \cap C'_\rho(\delta)(w_1)} \psi_j^-(x). \]

Then \( \psi_j^+ - C_j^+ \geq 0 \) on \( U_j^+ \cap C'_\rho(\delta)(w_1) \) and \( \psi_j^- - C_j^- \leq 0 \) on \( U_j^- \cap C'_\rho(\delta)(w_1) \).

Therefore, for \( j, l \in \{1, 2\} \) and \( x \in U_j^+ \cap U_l^- \cap C'_\rho(\delta)(w_1) \), we have

\[ k(\delta)(w_1) + (\psi_j^-(x) - C_j^-) \leq f(x) \leq m(\delta)(w_1) + (\psi_j^+(x) - C_j^+) \].

For \( j = 1, 2 \), set

\[ b_j^+(x) = m(\delta)(w_1) + (\psi_j^+(x) - C_j^+) \quad \text{for } x \in U_j^+ \cap \overline{G(B_\rho(\delta)(w_1))} \]

and

\[ b_j^-(x) = k(\delta)(w_1) + (\psi_j^-(x) - C_j^-) \quad \text{for } x \in U_j^- \cap \overline{G(B_\rho(\delta)(w_1))}. \]

Now \( \rho(\delta) < \sqrt{3} < \delta^* \leq \delta_2 \); notice that if \( w \in B_\rho(\delta)(w_1) \), then \( |w - w_1| < \delta_2 \)

and \( |G(w) - y| < 2p(\delta_2) \) and thus if \( x \in G(B_\rho(\delta)(w_1)) \cap \partial U_j^\pm \), then \( x \in Y_j^\pm \).
From (11), (12), the facts that $b_i^- \leq f$ on $U_i^- \cap C_i^0(w_1)$ and $f \leq b_i^+$ on $U_i^+ \cap C_i^0(w_1)$ for $j, l = 1, 2$, and the general comparison principle (Theorem 5.1, [12]), we have (see Figure 4)

$$b_i^- \leq f \text{ on } U_i^- \cap G(B_{\rho(\delta)}(w_1)) \quad \text{for } l = 1, 2 \quad (13)$$

and

$$f \leq b_j^+ \text{ on } U_j^+ \cap G(B_{\rho(\delta)}(w_1)) \quad \text{for } j = 1, 2. \quad (14)$$

Since the diameter of $G(B_{\rho(\delta)}(w_1)) \leq p(\delta)$, we have $|\psi_j^+(x) - C_j^+| \leq q(p(\delta))$

![Diagram](image)

Figure 4: General comparison principle applied on $U_2^\pm$ (left) and $U_1^\pm$ (right)

for $x \in U_j^\pm \cap G(B_{\rho(\delta)}(w_1))$. Thus, whenever $x_1, x_2 \in G(B_{\rho(\delta)}(w_1))$, at least one of the cases (a) $x_1, x_2 \in U_1^+ \cap U_1^-$, (b) $x_1, x_2 \in U_2^+ \cap U_2^-$, (c) $x_1 \in U_1^+$ and $x_2 \in U_2^-$ or (d) $x_1 \in U_2^+$ and $x_2 \in U_1^-$ holds. Since $c(w) = f(G(w))$, $G(w_1) \in U_j^\pm \cap U_j^-$ for some $i = 1, 2$ and $j = 1, 2$, and $G(w_2) \in U_l^+ \cap U_n^-$ for some $l = 1, 2$ and $n = 1, 2$, we have

$$b_j^- (G(w_1)) - b_l^- (G(w_2)) \leq c(w_1) - c(w_2) \leq b_l^+ (G(w_1)) - b_n^- (G(w_2))$$

or

$$-[m(\delta)(w_1) - k(\delta)(w_1) + (\psi_i^+(G(w_2)) - C_i^+)] - (\psi_j^- (G(w_1)) + C_j^-] \leq c(w_1) - c(w_2) \leq [m(\delta)(w_1) - k(\delta)(w_1) + (\psi_i^+(G(w_1)) - C_i^+] - (\psi_n^- (G(w_2)) + C_n^-].$$

Since $|\psi_j^+(G(w)) - C_j^+| \leq q(p(\delta))$ for $w \in B_{\rho(\delta)}(w_1) \cap U_j^\pm$, we have

$$|c(w_1) - c(w_2)| \leq p(\delta) + 2q(p(\delta)) < \epsilon.$$

Thus $c$ is uniformly continuous on $V^+$ and, since $G \in C^2(V : \mathbb{R}^2)$, we see that $Y$ is uniformly continuous on $V^*$. Therefore $Y$ extends to a continuous function, still denote $Y$, on $V^*$. \qed

Notice that

$$\lim_{t \in \mathbb{R}^3 \to y} f(x) = \lim_{t \in \mathbb{E} \to o_1(y)} c(w) = c(o_1(y))$$
As at the end of Step 1 of the proof of Theorem 1 of [25], we define $G$ here conformal (or anticonformal) map from $\supseteq\Omega$. Suppose (9) holds. Now we need to consider two cases:

(A) $o_1(y) = o_2(y)$.

(B) $o_1(y) \neq o_2(y)$.

These correspond to Cases 5 and 3 respectively in Step 1 of the proof of Theorem 1 of [25].

Case (A): Suppose $o_1(y) = o_2(y)$; set $o = o_1(y) = o_2(y)$. Then $f$ extends to a function in $C^0(\Omega \cup \{y\})$ and case (i) of Theorem holds.

Pf: Notice that $G$ is a bijection of $E \cup \{o\}$ and $\Omega \cup \{y\}$. Thus we may define $f = c \circ G^{-1}$, so $f(G(w)) = c(w)$ for $w \in E \cup \{o\}$; this extends $f$ to a function defined on $\Omega \cup \{y\}$. Let $\delta_i$ be a decreasing sequence of positive numbers converging to zero and consider the sequence of open sets $\{G(B_{\delta_i}((o)))\}$ in $\Omega$, where $\rho(i) = \rho(\delta_i(o))$. Now $y \notin G(C_{\rho(i)}(o))$ and so there exist $\sigma_i > 0$ such that $P(i) = \{x \in \Omega : |x - y| < \sigma_i\}$ is either a conformal or an indirectly conformal (or anticonformal) map from $\overline{B}$ onto $\overline{E}$ such that $g(1,0) = o_1(y)$, $g(-1,0) = o_2(y)$ and $g(u,0) \in o_1(y) o_2(y)$ for each $u \in [-1,1]$, where $ab$ denotes the (appropriate) choice of arc in $\partial E$ with $a$ and $b$ as endpoints.

Notice that $K(u,0) = y$ for $u \in [-1,1]$. Set $x = a \circ g$, $y = b \circ g$ and $z = c \circ g$, so that $X(u,v) = (x(u,v), y(u,v), z(u,v))$ for $(u,v) \in B$. Now, from Step 2 of the proof of Theorem 1 of [25], we define

$$X \in C^0 \left( \overline{B} : \mathbb{R}^3 \right) \cap C^{1,\epsilon} \left( B \cup \{u,0 : -1 < u < 1\} : \mathbb{R}^3 \right)$$

for some $\epsilon \in (0,1)$ and $X(u,0) = (y, z(u,0))$ cannot be constant on any non-degenerate interval in $[-1,1]$. Define $\Theta(u) = \arg(x_v(u,0) + iy_v(u,0))$. From equation (12) of [25], we see that

$$\alpha_1 = \lim_{u \uparrow 1} \Theta(u) \quad \text{and} \quad \alpha_2 = \lim_{u \downarrow 1} \Theta(u);$$

here $\alpha_1 < \alpha_2$. As in Steps 2-5 of the proof of Theorem 1 of [25], we see that $Rf(\theta)$ exists when $\theta \in (\alpha_1, \alpha_2)$,
$G^{-1}(L(\alpha_1)) \cap \partial E = \{o_2(y)\} \cup K^{-1}(L(\alpha_1)) \cap \partial B = \{(-1,0)\}$ when $\alpha_1 > \alpha(y)$
where $L(\theta) = \{y + (r \cos(\theta), r \sin(\theta)) : \Omega : 0 < r < \delta^*\}$, and one of the following cases holds:

(a) $Rf$ is strictly increasing or strictly decreasing on $(\alpha_1, \alpha_2)$.
(b) There exist $\alpha_L, \alpha_R$ so that $\alpha_1 < \alpha_L < \alpha_R < \alpha_2$, $\alpha_R = \alpha_L + \pi$, and $Rf$ is constant on $[\alpha_L, \alpha_R]$ and either increasing on $(\alpha_1, \alpha_L]$ and decreasing on $[\alpha_R, \alpha_2)$ or decreasing on $(\alpha_1, \alpha_L]$ and increasing on $[\alpha_R, \alpha_2)$.

We may argue as in Case A to see that $f$ is uniformly continuous on

$$\Omega^+ = \{y + (r \cos(\theta), r \sin(\theta)) : \Omega : 0 < r < \delta, \alpha_2 \leq \theta < \beta(y) + \epsilon\}$$

and $f$ is uniformly continuous on

$$\Omega^- = \{y + (r \cos(\theta), r \sin(\theta)) : \Omega : 0 < r < \delta, \alpha(y) - \epsilon < \theta \leq \alpha_1\}$$

for some small $\epsilon > 0$ and $\delta > 0$, since $G$ is a bijection of $E \cup \{o_1(y)\}$ and $\Omega \cup \{y\}$ and a bijection of $E \cup \{o_2(y)\}$ and $\Omega \cup \{y\}$. (Also see [5, 10].) Theorem 1 then follows, as in [9], from Steps 2-5 of the proof of Theorem 1 of [23] (replacing Step 3 with [9]).

Proof of Theorem 2

The proof of this theorem is essentially the same as that of Theorem 1.

Proof of Corollary 1

From pp.1064-5 in [3], we see that there exist upper and lower Bernstein pairs $(U^+, \psi^\pm)$ for $(\Gamma, H^*)$. From Theorem 2 we see that the radial limits $Rf(\theta, y)$ exist for each $\theta \in [\alpha(y), \beta(y)]$. (Since $\beta(y) - \alpha(y) = \pi$, case (iii) of Theorem 1 cannot occur.) Set $z_1 = Rf(\alpha(y), y)$, $z_2 = Rf(\beta(y), y)$ and $z_3 = \phi(y)$. If $z_1 = z_2$, then case (i) of Theorem 1 holds. (If $f$ is symmetric with respect to a line through $y$, then $z_1 = z_2$ and we are done.)

Suppose otherwise that $z_1 \neq z_2$; we may assume that $z_1 < z_3$ and $z_1 < z_2$. Then there exist $\alpha_1, \alpha_2 \in [\alpha(y), \beta(y)]$ with $\alpha_1 < \alpha_2$ such that

$$Rf(\theta, y) \begin{cases} 
\text{constant(= z_1)} & \text{for } \alpha(y) \leq \theta \leq \alpha_1 \\
\text{strictly increasing} & \text{for } \alpha_1 < \theta < \alpha_2 \\
\text{constant(= z_2)} & \text{for } \alpha_2 \leq \theta \leq \beta(y).
\end{cases}$$

From Theorem 2 we see that $Rf(\theta, y)$ exists for each $y \in \Gamma$ and $\theta \in [\alpha(y), \beta(y)]$ and $f$ is continuous on $\Omega \cup \Gamma \setminus \Upsilon$ for some countable subset $\Upsilon$ of $\Gamma$. Let $z_0 \in (z_1, \min\{z_2, z_3\})$ and $\theta_0 \in (\alpha_1, \alpha_2)$ satisfy $Rf(\theta_0, y) = z_0$. Let $C_0 \subset \Omega$ be the $z_0$-level curve of $f$ which has $y$ and a point $y_0 \in \partial \Omega \setminus \{y\}$ as endpoints. Let $y_1 \in \partial \Omega \cap \Gamma \setminus \Upsilon$ and $y_2 \in C_0$ such that the (open) line segment $L$ joining $y_1$ and $y_2$ is entirely contained in $\Omega$. Let $M = \inf_L f$, $\Pi$ be the plane containing $(y, z_0)$ and $L \times \{M\}$, and let $h$ be the affine function on $\mathbb{R}^2$ whose graph is $\Pi$. Let $\Omega_0$ be the component of $\Omega \setminus (C_0 \cup L)$ whose closure contains $B_\delta(y) \cap \partial \Omega$ for some $\delta > 0$. Then there is a curve $C \subset \Omega_0$ on which $f = h$ whose endpoints are $y_1$ and $y_2$ for some $y_3 \in \partial \Omega \setminus y_1$ and $y_2$, such that $h > f$ in $\Omega_1$, where $\Omega_1 \subset \Omega_0$ is the open set bounded by $C$ and the portion of $\partial \Omega \setminus \{y\}$ between $y$ and
Notice that $h < f$ in $L \cup C_0$. (In Figure 3 on the left, $\{(x, h(x)) : x \in C\}$ is in red, $L$ is in dark blue, $C_0$ is in yellow, and the light blue region is a portion of $\partial^1_\Omega \times \mathbb{R}$, and, on the right, $\Omega_0$ is in light green and $\partial^2_\Omega$ is in magenta.) Now let $g \in C^2(\Omega)$ be defined by $g = f$ on $\Omega \setminus \Omega_1$ and $g = h$ on $\Omega_1$ and observe that $J(g) < J(f)$, which contradicts the fact that $f$ minimizes $J$. Thus it must be the case that $z_1 = z_2$, case (i) of Theorem 1 holds and $f$ is continuous at $y$. 

**Remark 2.** Corollary 1 can be generalized to minimizers of $J(u) = \int_\Omega \sqrt{1 + |Du|^2} dx + \int_\Omega \int_c u(x) H(x, t) \, dt \, dx + \int_{\partial \Omega} |u - \phi| ds$ for $u \in BV(\Omega)$ and the conclusion remains the same; here $c$ is a reference height (e.g. $c = 0$). In the proof of Corollary 1, the only change is a replacement of the plane $\Pi$ with an appropriate surface (e.g. a portion of a sphere) over a subdomain like $\Omega_1$ such that the test function $g$ satisfies $J(g) < J(f)$.

**Proof of Example 1.** By Corollary 1, $f$ is continuous on $\Omega \cup \{(0, 0)\}$. Clearly $f$ is continuous at $(x, y)$ when $(x + 1)^2 + y^2 = \cosh^2(1)$. By [30], $f$ is continuous at $(x, y)$ when $(x + 1)^2 + y^2 = 1$ and $(x, y) \neq (0, 0)$. The parametrization [10] of the graph of $f$ (restricted to $\Omega \setminus \{(x, 0) : x < 0\}$) satisfies $Y \in C^0(E)$. Notice that $\zeta((0, 0)) = \{0\}$ (since $\beta((0, 0)) - \alpha((0, 0)) = \pi$ and $z_1 = z_2$ for some $o \in \partial E$). Suppose $G$ in (a2) is not one-to-one. Then there exists a nondegenerate arc $\zeta \subset \partial E$ such that $G(\zeta) = \{y_1\}$ for some $y_1 \in \partial \Omega$ and therefore $f$ is not continuous at $y_1$, which is a contradiction. Thus $f = g \circ G^{-1}$ and so $f \in C^0(\Omega)$. (The continuity of $G^{-1}$ follows, for example, from Lemma 3.1 in [2].) 

**Proof of Theorem 3.** The proof of Theorem 2 of [9] uses unduloids as Bernstein functions (i.e. comparison surfaces) on subdomains of $\Omega$ (see Figure 7 of [9]). The proof of Theorem 3 is essentially the same, using the Bernstein pairs $(U^\pm, \psi^\pm)$ rather than unduloids, staying on $\partial^2_\Omega$ rather than on an arc of a circle inside $\Omega$, and arguing as in the proof of Theorem 1.

**Proof of Theorem 4.** The proof of Theorem 2 of [10] uses portions of tori as Bernstein functions (i.e. comparison surfaces) on subdomains of $\Omega$ (see Figure 7 of [10]). The proof of Theorem 4 is essentially the same, using the Bernstein pairs $(U^\pm, \psi^\pm)$ rather than tori, staying on $\partial^2_\Omega$ rather than on an arc of a circle inside $\Omega$, and arguing as in the proof of Theorem 1.
4 Bernstein Functions

The value of Theorems 1-5 is dependent on the existence of Bernstein functions. The results of [8] provide Bernstein pairs for minimal surfaces.

*Proposition 1.* Let \( a < b, \lambda \in (0, 1), \psi \in C^{2,\lambda}([a,b]) \) and \( \Gamma = \{(x,\psi(x)) \in \mathbb{R}^2 : x \in [a,b] \} \) such that \( \psi'(x) < 0 \) for \( x \in [a,b], \psi''(x) < 0 \) for \( x \in [a,b] \setminus J \), there exist \( C_1 > 0 \) and \( \epsilon_1 > 0 \) such that if \( \bar{x} \in J \) and \( |x - \bar{x}| < \epsilon_1 \), then \( \psi''(x) \leq -C_1|x - \bar{x}|^{\lambda} \), and \( t\psi(x_1) + (1-t)\psi(x_2) < \psi(tx_1 + (1-t)x_2) \) for each \( t \in (0,1) \) and \( x_1, x_2 \in [a,b] \) with \( x_1 \neq x_2 \), where \( J \) is a finite subset of \( (a,b) \). Then there exists an open set \( U \subset \mathbb{R}^2 \) with \( \Gamma \subset \partial U \) and a function \( h \in C^2(U) \cap C^0(\overline{U}) \) such that \( \partial U \) is a closed, \( C^{2,\lambda} \) curve, \( \Gamma \) lies below \( U \) in \( \mathbb{R}^2 \) (i.e. the exterior unit normal \( \nu = (\nu_1(x), \nu_2(x)) \) to \( \partial U \) satisfies \( \nu_2(x) < 0 \) for \( a \leq x \leq b \), \( \text{Nh} = 0 \) in \( U \) and [8] holds for each \( y \in \Gamma \), where \( \hat{\nu} \) is a continuous extension of \( \nu \) to a neighborhood of \( \Gamma \).

*Proof:* We may assume that \( a, b > 0 \). There exists \( c > b \) and \( k \in C^{2,\lambda}([-c,c]) \) with \( k(-x) = k(x) \) for \( x \in [0,c] \) such that \( k(x) = -\psi(x) \) for \( x \in [a,b], k''(x) > 0 \) for \( x \in [-c,c] \setminus J \), where \( J \) is a finite set, \( k''(0) > 0 \), and the set

\[
K = \{(x,k(x)) \in \mathbb{R}^2 : x \in [-c,c]\}
\]

is strictly concave (i.e. \( tk(x_1) + (1-t)k(x_2) > k(tx_1 + (1-t)x_2) \) for each \( t \in (0,1) \) and \( x_1, x_2 \in [-c,c] \) with \( x_1 \neq x_2 \)). From [8] (pp.1063-5), we can construct a domain \( \Omega(K,l) \) such that \( K \subset \partial \Omega(K,l) \) and \( \Omega(K,l) \) lies below \( K \) (i.e. the outward unit normal to \( \Omega(K,l) \) at \( (x,k(x)) \) is \( \nu(x) = \frac{-(k''(x),1)}{\sqrt{1+(k''(x))^2}} \); see Figure 4 of [8]) and a function \( F^+ \in C^2(\Omega(K,l)) \cap C^0(\overline{\Omega(K,l)}) \) such that

\[
\mu(x) \equiv \frac{\langle \nabla F^+(x), -1 \rangle}{\sqrt{1+|\nabla F^+(x)|^2}}, \quad x \in \Omega(K,l),
\]

extends continuously to a function on \( \Omega(K,l) \cup K \) and \( \mu(x,k(x)) \cdot \nu(x) = 1 \) for \( x \in [-c,c] \). Now let \( V \) be an open subset of \( \Omega \) with \( C^{2,\lambda} \) boundary such that \( \{(x,\psi(x)) : x \in [a,b] \} \subset \partial V \) and \( \nu \) \( \partial V \) and \( \partial \Omega(K,l) \setminus K = \emptyset \) and then let \( U = \{(x,-y) : (x,y) \in V \} \) and \( h(x,y) = F^+(x,-y) \) for \( (x,y) \in \overline{U} \).

*Remark 3.* Let \( \Omega \subset \mathbb{R}^2 \) be an open set, \( \Gamma \subset \partial \Omega \) be a \( C^{2,\lambda} \) curve and \( y \in \Gamma \) be a point at which we wish to have upper and lower Bernstein pairs for \( H \equiv 0 \). Let \( \Sigma \subset \Gamma \) be the intersection of \( \partial \Omega \) with a neighborhood of \( y \) and suppose there is
a rigid motion $\zeta : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\zeta(\Sigma)$ and $\zeta(\Omega)$ satisfy the hypotheses of Proposition 1. Then $\zeta^{-1}(U), h \circ \zeta$ will be an upper Bernstein pair for $\Sigma$ and $H \equiv 0$ and $\zeta^{-1}(U), -h \circ \zeta$ will be a lower Bernstein pair for $\Sigma$ and $H \equiv 0$.

When $H(x, z)$ is independent of $z$, the existence of (bounded) Bernstein functions is tied to boundary curvature conditions; in Theorem 3.1 of [15] (and Theorem 6.6 of [12]), we see that

**Proposition 2.** Suppose $\Omega$ is a $C^2$ domain in $\mathbb{R}^2$ such that

$$|\int \int_A H(x)dx| < \int |D\chi_A| \quad \text{for all } A \subset \Omega, \ A \neq \emptyset, \Omega$$

and $\int \int_B H(x)dx = \int |D\chi_B|$: that is, $\Omega$ is an extremal domain. Let $y \in \partial\Omega$ and suppose

$$\Lambda(y) < 2H(y),$$

where $\Lambda(y)$ is the (signed) curvature of $\partial\Omega$ at $y$ with respect to the interior normal direction. Then the (unique up to vertical translations) solution $g$ of

$Ng(x) = H(x)$ for $x \in \Omega$ is bounded and continuous in $W = \Omega \cap B_\epsilon(y)$,

$T_g$ extends continuously to a function on $W$ and $T_g(x) = \nu(x)$ for each $x \in B_\epsilon(y) \cap \partial\Omega$ for some $\epsilon > 0$, where $\nu$ is the exterior unit normal to $\Omega$.

Using Proposition 2 and a similar procedure to that in the proof of Proposition 1 we can obtain Bernstein pairs near $y$ when $\partial\Omega \cap B_\epsilon(y)$ is a subset of the boundary of an extremal domain $W$ for some $\epsilon > 0$ such that $\Omega$ and $W$ are on the same side of $\partial\Omega \cap B_\epsilon(y)$ and the boundary curvature condition $\Lambda_W(y) < 2|H(y)|$ is satisfied. In the same manner, we can obtain Bernstein pairs near $y$, illustrated in Figure 2 by the sets $U^\pm$ and $U^\pm_2$, when $\partial^1\Omega \cap B_\epsilon(y)$ and $\partial^2\Omega \cap B_\epsilon(y)$ are subsets of the boundaries of extremal domains $W_1$ and $W_2$ for some $\epsilon > 0$, $\Omega$ and $W_j$ are on the same side of $\partial^j\Omega \cap B_\epsilon(y)$ for $j = 1, 2$, $\Lambda_{W_j}(y) < 2|H(y)|$ and $\Lambda_{W_2}(y) < 2|H(y)|$, where $\Lambda_{W_j}(y)$ denotes (signed) curvature of $\partial W_j$.

**Remark 4.** In Proposition 2 the sets $A$ are Caccioppoli sets; that is, Borel sets such that the distributional (first) derivatives of the characteristic function $\chi_A$ of $A$ are Radon measures. The notation $A \neq \emptyset, \Omega$ means that neither $A$ nor $\Omega \setminus A$ has (two-dimensional) measure zero and the notation $\int |D\chi_A|$ means the total variation of $\chi_A \in BV(\Omega)$ (e.g. §6.3 of [12]). Determining when hypothesis (15) is satisfied can be difficult; Giusti includes an Appendix in [15] which discusses the case of constant $H$.

We may use §14.4 of [14] (also see Corollary 14.13) to obtain Bernstein functions in a neighborhood $U$ of a point $y \in \Gamma$ when $\Gamma \subset \partial\Omega$ is a $C^2$ curve satisfying $\Lambda(x) < 2|H(x)|$ for $x \in \Gamma \cap U$ and $H \in C^0(\overline{U} \cap \Omega)$ is either non-negative or non-negative in $U \cap \Omega$.

**Lemma 1.** Suppose $\Omega$ is a $C^{2, \lambda}$ domain in $\mathbb{R}^2$ for some $\lambda \in (0, 1)$. Let $y \in \partial\Omega$ and $\Lambda(y)$ denote the (signed) curvature of $\partial\Omega$ at $y$ with respect to the interior...
normal direction (i.e. $-\nu$). Suppose $\Lambda(y) < 2|H(y)|$ and $H \in C^0(\overline{U} \cap \Gamma)$ is either non-positive or non-negative in $U \cap \Omega$, where $U$ is some neighborhood of $y$. Then there exist $\delta > 0$ and upper and lower Bernstein pairs $(U^\pm, \psi^\pm)$ for $(\Gamma, H)$, where $\Gamma = B_\delta(y) \cap \partial \Omega$.

**Proof:** There exists $\delta_1 > 0$ such that $B_{\delta_1}(y) \subset U$ and $\Lambda(x) < 2|H(x)|$ for each $x \in \partial \Omega \cap B_{\delta_1}(y)$. There exists a $\delta_2 \in (0, \delta_1/2)$ such that

$$\Lambda_0 \equiv \sup\{\Lambda(x) : x \in \partial \Omega \cap B_{\delta_2}(y)\} < \inf\{2|H(x)| : x \in \partial \Omega \cap B_{\delta_2}(y)\} \equiv 2H_0.$$ 

If $\Lambda_0 > 0$, set $R = \frac{1}{\Lambda_0}$; otherwise let $R$ be a small positive number. Now let $W$ be a $C^{2,\lambda}$ domain in $\mathbb{R}^2$ such that $\partial \Omega \cap B_{\delta_1}(y) \subset \partial W$, $\Omega$ and $W$ lie on the same side of $\partial \Omega \cap B_{\delta_1}(y)$ and $W$ satisfies an interior sphere condition of radius $R$ at each point of $\partial \Omega \cap B_{\delta_2}(y)$. Continuously extend $H$ outside $U$ to $W$ in such a manner that $H$ is either non-positive or non-negative in $W$. From inequality (14.73) of [14], there exists $L > 0$ such that

$$u(x) - u_0(x) \leq L \quad \text{for} \quad x \in \partial W \cap B_{\delta_2}(y),$$

where $u$ is any solution of (1) in $W$ and $u_0(x) = \sup\{u(t) : t \in \partial W \setminus B_R(x)\}$. We may assume $2\delta_2 < R$ and set $u^* = \sup\{u(t) : t \in \partial W \setminus B_{R-\delta_2}(y)\}$. Then $u_0(x) \leq u^*$ for each $x \in \partial W \cap B_{\delta_2}(y)$ and $u \leq L + u^*$ on $\partial \Omega \cap B_{\delta_2}(y)$. Now let $\phi \in C^\infty(\partial W)$ such that $\phi = 0$ on $\partial W \setminus B_{R-\delta_2}(y)$ and $\phi > L$ on $\partial W \cap B_{\delta_2}(y)$ and let $h \in C^2(W)$ be the solution of (1)-(2) in $W$ with Dirichlet data $\phi$. (Just as [14] ignores in Theorem 14.11 the question of whether $u = \phi$ on $\partial \Omega \setminus B_R(y)$, we may assume that $W$ satisfies curvature conditions (i.e. $\Lambda_W \geq 2|H|$) on $\partial W \setminus B_{R-\delta_2}(y)$ so that $h = \phi$ on $\partial \Omega \setminus B_R(y)$ and so $h^* = 0$.) It then follows (e.g. [15]) that $h \in C^0(\overline{W})$ and

$$\frac{\partial h}{\partial \nu} = +\infty \quad \text{on} \quad B_{\delta_2}(y) \cap \partial \Omega.$$ 

Thus $h$ is an upper Bernstein function. The existence of a lower Bernstein function is similar. \hfill $\Box$

**Remark 5.** In a similar manner, given $y \in \partial \Omega$ we can establish the existence of upper and lower Bernstein pairs for the intersections of $\partial^1_w \Omega$ and $\partial^2_w \Omega$ with a neighborhood of $y$ when these sets are each subsets of the boundaries of smooth (i.e. $C^{2,\lambda}$) domains $W_1$ and $W_2$ which satisfy appropriate boundary curvature conditions at $y$. (For capillary surfaces in positive gravity (and prescribed mean curvature surfaces with $\frac{\partial H}{\partial \nu}(x, z) \geq \kappa > 0$), one can examine Theorem 2 of [13].)

5 **Curvature Conditions on $\partial^1_w \Omega$ and $\partial^2_w \Omega$**

In [9], the existence of nontangential radial limits of bounded, nonparametric prescribed mean curvature surfaces at nonconvex corners was proven; in Theorem [11] we showed that all radial limits of such surfaces at nonconvex corners
exist when Bernstein functions exist. On the other hand, [22] and Theorem 3 of [25] provide examples in which no radial limit exists at a point $y$ of $\partial \Omega$ at which the boundary of $\Omega$ is smooth. In this section, we shall focus on the three intervals ($\alpha$, $\beta$, $\gamma$) of radial limits when $H$ satisfies \( H_\alpha \leq H_\beta \leq H_\gamma \).

Suppose \( H_\alpha \) or \( H_\beta \) or \( H_\gamma \) exist when Bernstein functions exist. On the other hand, [22] and Theorem 3 of [9] provide examples in which no radial limit exists at a point $y$ of $\partial \Omega$ at which the boundary of $\Omega$ is smooth. In this section, we shall focus on the three intervals ($\alpha$, $\beta$, $\gamma$) of radial limits when $H(x, t)$ is weakly increasing in $t$ for each $x \in \Omega$, provided that we include in (b) all of the cases in which $Rf(\theta, y)$ exists for $\theta$ in one of the three intervals $\{\alpha(y), \beta(y), \gamma\}$.

**Lemma 2.** Let $f \in C^2(\Omega) \cap L^\infty(\Omega)$ satisfy $Qf = 0$ in $\Omega$ and let $H^* \in L^\infty(\mathbb{R}^2)$ satisfy $H^*(x) = H(x, f(x))$ for $x \in \Omega$. Let $y \in \partial \Omega$ and suppose there exists a $\theta_1 \in [\alpha(y), \beta(y)]$ such that $Rf(\theta_1, y)$ exists. Then $Rf(\theta, y)$ exists for each $\theta \in (\alpha(y), \beta(y))$, $Rf(\cdot, y) \in C^0((\alpha(y), \beta(y)))$ and $Rf(\cdot, y)$ behaves as in Theorem 1 of [9].

Suppose, in addition, that there exist $\delta > 0$ and upper and lower Bernstein pairs $\left(U_1^+, \psi_1^+\right)$ and $\left(U_2^+, \psi_2^+\right)$ for $(\Gamma_1, H^*)$ and $(\Gamma_2, H^*)$ respectively, where $\Gamma_1 = B_\delta(y) \cap \partial_1^Y \Omega$ and $\Gamma_2 = B_\delta(y) \cap \partial_2^Y \Omega$. Then the conclusions of Theorem 2 hold.

**Proof:** The first part follows from Theorem 2 of [9]. The second part follows from Theorem 3.

Now suppose $f \in C^2(\Omega)$ satisfies $Nf(x) = H(x)$ for $x \in \Omega$ and $y \in \partial \Omega$ satisfies $\beta(y) - \alpha(y) \leq \pi$. Under what conditions do types of behavior (a), (b) or (c) occur?

**Lemma 3.** Suppose $\Omega, y$ and $H$ are as above and $\Lambda(x) \geq 2|H(x)|$ for almost all $x \in B_\epsilon(y) \cap \partial_1^Y \Omega \cup \partial_2^Y \Omega$, for some $\epsilon > 0$. Then there exists $\phi \in L^\infty(\partial \Omega)$ such that the solution $f \in C^2(\Omega) \cap L^\infty(\Omega)$ of $Nf = H$ in $\Omega$ and $f = \phi$ almost everywhere on $\partial \Omega$ has no radial limits at $y$.

**Proof:** This follows from Theorem 16.9 of [14] and the “gliding hump” argument in [22].

**Lemma 4.** Suppose $\Omega, y$ and $H$ are as above and $\Lambda(x) < 2|H(x)|$ for almost all $x \in B_\epsilon(y) \cap \partial_1^Y \Omega \cup \partial_2^Y \Omega$, for some $\epsilon > 0$. Then there exist $\delta > 0$ and upper and lower Bernstein pairs $\left(U_j^+, \psi_j^+\right)$ for $(\Gamma_j, H)$, where $\Gamma_j = B_\delta(y) \cap \partial_j^Y \Omega$, for $j = 1, 2$, and the conclusions of Theorems 2 hold when their other hypotheses are satisfied.

**Proof:** This follows from Remark 5.
Theorem 6. Suppose $\Omega$ is a $C^{2,\lambda}$ domain in $\mathbb{R}^2$ and $f \in C^2(\Omega) \cap L^\infty(\Omega)$ is a variational (i.e., BV) solution of (1)-(2) for some $\phi \in L^\infty(\Omega)$ and $\lambda \in (0,1)$. Let $y \in \partial \Omega$ and let $\Lambda(y)$ denote the (signed) curvature of $\partial \Omega$ at $y$ with respect to the interior normal direction (i.e., $-\nu$).

(i) Suppose $\Lambda(y) < 2|H(y)|$. Then the conclusions of Theorem 2 hold.

(ii) Suppose $\Lambda(y) > 2|H(y)|$. Then the conclusions of Theorem 2 hold if $\phi$ restricted to $\partial^j \Omega$ has a limit $z_j$ at $y$ for $j = 1, 2$, while for certain $\phi \in L^\infty(\Omega)$, $Rf(\cdot, y)$ does not exist for any $\theta \in [\alpha(y), \beta(y)]$.

Proof: The first part follows from Lemma 4. The second part follows from Theorem 16.9 of [14], [21] (see also [7, 25]) and Lemma 3. \hfill \Box

Remark 6. One can state a theorem similar to Theorem 6 when $f \in C^2(\Omega)$ is a variational solution of (1)-(2) for some $\phi \in L^\infty(\Omega)$ and $y \in \partial \Omega$ satisfies $\beta(y) - \alpha(y) < \pi$, $\partial^1 \Omega$ and $\partial^2 \Omega$ are smooth, and boundary curvature conditions apply on $\partial^j \Omega$, $j = 1, 2$. If, for example, $\Lambda(x) \geq 2|H(x)|$ for $x \in \partial^1 \Omega$ near $y$, $\Lambda(x) < 2|H(x)|$ for $x \in \partial^2 \Omega$ near $y$, and $\phi$ restricted to $\partial^1 \Omega$ has a limit $z_1$ at $y$, then the conclusions of Theorem 2 hold (e.g., Theorem 2 of [23]).

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