D-Branes Probes of $G_2$ Holonomy Manifolds

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We describe how mirror symmetry of three-dimensional $\mathcal{N} = 1$ supersymmetric gauge theories can be used to determine the theory on the world-volume of a D2-brane probe of manifolds with $G_2$-holonomy. This is a much shortened companion paper to [1].

Introduction

The concept of the D-brane probe provides the link between gauge theory and geometry. It has proven to be one of the most important ideas among recent developments in string theory, shedding new light on strong coupling gauge theory dualities, providing a physical interpretation of previously obscure algebraic-geometric constructions, and resulting in examples of the AdS/CFT correspondence with increasingly realistic gauge and matter content.

The purpose of this letter is to describe some results extending this enterprise to D2-brane probes of non-compact manifolds of $G_2$ holonomy. These spaces are of interest for both theoretical and phenomenological reasons [2, 3, 4, 5, 6]. For orbifold backgrounds, and their partial resolutions, there are well known procedures for determining the gauge theory living on the probe world-volume [7, 8]. (See [9] for examples of $G_2$ holonomy orbifolds). However, recent attention has focused on asymptotically conical $G_2$ manifolds for which the orbifold prescription is not relevant. Here we suggest a different approach to understand aspects of the probe theory, using mirror symmetry of three dimensional gauge theories [10].

Recall that mirror symmetry is a duality between a pair of three-dimensional field theories which, among other things, interchanges the Coulomb and Higgs branches. The basic observation is that mirror symmetry on an M2-brane probe has a simple interpretation as an “M-theory flip” [11].

\[
\begin{array}{cccc}
\text{M-theory on } X & \text{IIA with} & \text{D6-branes} & \text{IIA on } X \\
\downarrow \text{Coulomb branch} & \text{mirror symmetry} & \uparrow \text{Higgs branch} \\
& \text{Higgs branch} &
\end{array}
\]

In order to explain this duality, let us consider a membrane probe on a (singular) space $X$ with $G_2$ holonomy. We can perform a reduction from M-theory to IIA string theory in two different ways. First, we can reduce on a circle transverse to both the manifold $X$, and to the membrane. This takes us to IIA theory on $X$. The space $X$ is reproduced as a Higgs branch in the D2-brane world-volume theory, in a way reminiscent to the hyperKähler quotient construction. By analogy, we call it the $G_2$ quotient construction.

On the other hand, we can reduce on an $S^1$ contained within $X$. There are many ways to choose the $S^1$, which lead to different IIA backgrounds with D6-branes and/or Ramond-Ramond fluxes. An illustrative example was described in [1]. One might ask: “What is the natural choice of the M-theory circle?” One simple choice is to require $X/U(1) \cong \mathbb{R}^6$. If such a quotient exists it gives, after reduction to type IIA theory, a configuration of D6-branes in a (topologically) flat space-time. The positions of the D6-branes correspond to the fixed points of the circle action. In contrast to the previous reduction, the geometry of $X$ is entirely encoded in the configuration of D6-branes, rather than in the geometry of space-time [12, 13]. The M-theory geometry $X$ is then re-constructed as the Coulomb branch of the world-volume theory on the D2-brane. When the space $X$ develops a conical singularity, the configuration of D6-branes also becomes singular. In particular, in some cases of interest it degenerates into a collection of flat D6-branes intersecting at the suitable angles [14]. For such models, it becomes a simple exercise to derive the three-dimensional gauge theory on the D2-brane probe. Then, using mirror symmetry, one can obtain the theory on the probe of $X$.

The simplest illustration of this method was given in [11], where a single D2-brane probe of $N$ parallel D6-branes was used to re-derive the original $\mathcal{N} = 4$ mirror pairs of Intriligator and Seiberg [15]. Note, however, that we here we run the logic of [11] in reverse: we use mirror symmetry, derived through independent techniques, to derive the gauge theory on the probe of $X$.

This letter contains only the barest details of our method. Many further results and examples, including the derivation of the mirror pairs and the interesting subtleties involved, as well as applications to $SU(3)$, $Sp(2)$ and $Spin(7)$ holonomy manifolds can be found in [1].

Mirror Symmetry

Let us firstly describe the mirror pairs that will be our tool in understanding the probe theories. Our mirror pairs have $\mathcal{N} = 1$ supersymmetry, and are derived in [1]...
from deforming \( \mathcal{N} = 4 \) mirror pairs using both field theory and string theory techniques. Of course with such little supersymmetry (\( \mathcal{N} = 1 \) means 2 supercharges) we have little control over the strong coupling dynamics and must be wary of any conjectured duality. Our only savior is parity symmetry which may be used to prohibit the lifting of certain vacuum moduli spaces \([3, 4]\). We hope that the success of our mirror pairs in describing manifolds of \( G_2 \) holonomy goes some way towards convincing the reader of their utility.

The mirror pairs preserve only \( \mathcal{N} = 1 \) supersymmetry, and are given by

\[
\begin{align*}
\text{Theory A} &: \quad U(1)^r \text{ with } k \text{ scalars and } N \text{ hypers} \\
\text{Theory B} &: \quad U(1)^{N-r} \text{ with } (3N-k) \text{ scalars and } N \text{ hypermultiplets}
\end{align*}
\]

The abelian vector multiplets contain only a photon and a Majorana spinor, while the scalar multiplets, which we shall denote as \( \Phi \), contain a single real scalar and a Majorana fermion. In contrast, the hypermultiplets fill out representations of the \( \mathcal{N} = 4 \) algebra: they each contain four Majorana fermions and two complex scalars, \( q \) and \( \bar{q} \). We write the superfield as a doublet, \( W = (Q, Q^\dagger)^T \). The chiral multiplets \( Q \) and \( \bar{Q} \) carry conjugate charges under the gauge group. For Theory A, we denote the charge of the hypermultiplets as \( R^c_i \), while for Theory B it is \( \bar{R}^c_i; i = 1, \ldots, N; a = 1, \ldots, r; p = 1, \ldots, N-r \). Each of these matrices are assumed to be of maximal rank, and mirror symmetry requires

\[
\sum_{i=1}^{N} R^c_i \bar{R}^c_i = 0 \quad \forall \, a, p \quad (1)
\]

In \( \mathcal{N} = 1 \) theories, there are no holomorphic luxuries and interactions are determined in terms of a real superpotential, \( f \). For Theory A, this superpotential has the cubic form associated with the \( \mathcal{N} = 4 \) theories, and is determined by a triplet of \( k \times N \) matrices \( T_c, c = 1, 2, 3 \)

\[
f = \sum_{i=1}^{N} \sum_{c=1}^{3} \sum_{\alpha=1}^{k} W^i_c \tau^\alpha W_i \cdot T^\alpha_{c,3} \Phi_{\alpha} \quad (2)
\]

where \( \alpha = 1, \ldots, k \) and \( \tau^\alpha \) are the three Pauli matrices. A similar coupling exists for Theory B, now with the triplet of \( (3N-k) \times N \) coupling matrices \( T_c \), satisfying

\[
\sum_{i=1}^{N} \sum_{c=1}^{3} T^\alpha_{c,i} \hat{T}^\rho_{c,i} = 0 \quad \forall \, \alpha, \rho \quad (3)
\]

Further details of these theories, together with the methods used to derive them, can be found in \([1]\). Here let us restrict ourselves to a few comments. The Coulomb branch of Theory A has dimension \( (N + r) \), which coincides with the dimension of the Higgs branch of Theory B. (The converse also holds). Mass and FI parameters may be added to both theories, partially lifting some branches of vacua, and the mirror map for these deformations is known.

The \( G_2 \) Quotient Construction

Let us now apply the mirror pairs described above to a D2-brane probe of a D6-brane background. We take \( i = 1, \ldots, N \), flat D6-branes, each of which has spatial world-volume direction,

\[
D6_i : \quad 123[47][58][59][69][79]
\]

The D6-branes lie on a special lagrangian locus if each rotation is contained in \( SU(3) \) or, more simply, if

\[
\theta_1^i \pm \theta_2^i \pm \theta_3^i = 0 \quad \text{mod } 2\pi \quad \forall \, i \quad (4)
\]

ensuring that \( \mathcal{N} = 1 \) supersymmetry (4 supercharges) is preserved on their common world-volume. (For non-generic angles, more supersymmetry may be preserved. We will assume this is not the case).

As described in the introduction, we probe this configuration with a D2-brane lying in the \( x^1 - x^2 \) plane. This breaks supersymmetry by a further half, resulting in a \( d = 2 + 1 \) dimensional world-volume theory with \( \mathcal{N} = 1 \) supersymmetry (2 supercharges). For the singular case of intersecting, flat D6-branes, the theory on the D2-brane probe is simple to write down. The 2-2 strings give rise to the usual gauge field and seven scalars. Of these, there is one free \( \mathcal{N} = 1 \) scalar multiplet parameterizing motion in the \( x^3 \) direction common to all D6-branes. Further fields arise from the 2-6 strings. These give rise to \( N \) hypermultiplets. Thus, we have the interacting \( \mathcal{N} = 1 \) supersymmetric theory on the probe.

**Theory A:** \( U(1) \) with 6 scalar multiplets and \( N \) hypermultiplets

where each hypermultiplet has charge +1 under the gauge field. The couplings of the hypermultiplets to the scalar multiplets are determined by the geometry of the D6-branes: each hypermultiplet couples minimally to the three scalar fields orthogonal to the corresponding D6-brane. If we define the scalar fields \( \phi_\alpha = x^{\alpha + 3}, \alpha = 1, \ldots, 6 \), then the superpotential is of the form (2) with the couplings determined by the D6-brane orientations,

\[
T^\alpha_{c,i} = -\sin \theta^i_c \phi_{\alpha} \delta_{c,a} + \cos \theta^i_c \phi_{\alpha} \delta_{c,a-3} \quad c = 1, 2, 3 \quad (5)
\]

From the IIA space-time picture, we are lead to the natural conjecture that the Coulomb branch of this theory, parameterized by the six real scalars \( \phi_\alpha \), together with the dual photon \( \sigma \), is a seven dimensional manifold \( X \) that admits a metric of \( G_2 \) holonomy. However, this description of \( X \) in terms of Coulomb branch variables is not overly useful. In particular, the isometries of \( X \) are lost in the reduction to IIA, and are only expected to be recovered as isometries of the Coulomb branch in the strong coupling limit. It would be desirable to have an algebraic description of \( X \), in which the symmetries are manifest. This is exactly what the mirror theory provides for us.
Since Theory A is in the class of theories discussed above, we may simply write down the mirror theory whose Higgs branch is conjectured to give the $G_2$ manifold $X$.

**Theory B:** $U(1)^{N-1}$ with $3(N-2)$ scalar and $N$ hypermultiplets

The gauge couplings are determined by the $A_{N-1}$ quiver diagram: i.e the $i^{th}$ gauge group acts on the $i^{th}$ hypermultiplet with charge $+1$, and the $(i+1)^{th}$ hypermultiplet with charge $-1$. All other hypermultiplets are neutral. The Yukawa terms are of the form $\bar{\phi}_i \phi_j$, with the triplet of coupling matrices $T$ determined by $\hat{T}_{i,j}$. The Higgs branch of this theory is parameterized by $w_i = (q_i, \tilde{q}_i)^T$, the $N$ doublets of complex scalars in the hypermultiplets. These are constrained by the $(3(N-2))$ D-terms, modulo $(N-1)U(1)$ gauge quotients,

$$\sum_{i,c} T_{c,i}^\rho w_i^c \tau^c w_i = 0 \quad \rho = 1, \ldots, 3(N-2) \quad (6)$$

This quotient construction yields a conical manifold, which is expected to admit a metric of $G_2$ holonomy. In some cases the conical singularity may be (partially) resolved by adding constants to the right-hand side of (6). This blows up two-cycles and, in the IIA picture, corresponds to translating the $D6$-branes in the $x^4 - x^9$ directions. Note that when the Yukawa matrices $T$ fall into suitable $SU(2)$ triplets, the above method coincides with the toric hyperKähler quotient construction, supplemented by a further quotient by a tri-holomorphic isometry to yield a manifold of dimension seven. This is the construction discussed by Acharya and Witten [5]. However, in general, the charges in (6) differ.

The above theory may also be considered as a $\mathcal{N} = (1,1)$ supersymmetric linear sigma model in $d = 1+1$ dimensions, cf. [10]. However, in the absence of something akin to Yau’s theorem, we cannot be sure that the Ricci flat metric to which the theory flows has $G_2$ holonomy.

**An Example**

Let us now examine the $G_2$-quotient construction applied to a specific example. Our choice for consideration is the $G_2$ manifold $X$ given in the cone over the flag manifold $SU(3)/U(1)^2$ [11]. This example was also discussed in detail by Atiyah and Witten [1]. They show that, with a suitable choice of M-theory circle, $X$ can be reduced to three, flat, intersecting D6-branes in type IIA string theory. The symmetry of $X$ (to be discussed below), together with the special lagrangian condition [12] determines the angles of these three branes to be $\theta_1 = 0$, $\theta_2 = 2\pi/3$ and $\theta_3 = 4\pi/3$ for each $c$. The configuration is drawn in Figure [1].

In order to make the symmetries of the configuration manifest, we define two triplets of scalars, $\vec{\phi}_1 = (x^7, x^8, x^9)^T$ and $\vec{\phi}_2 = (x^4, x^5, x^6)^T$ in terms of which the orientation of the $i^{th}$ D6-brane can be described by the set of linear equations.

$$D6_1 : \quad \rho_1 = 0$$
$$D6_2 : \quad \frac{1}{2} \phi_1 + \sqrt{3} \phi_2 = 0$$
$$D6_3 : \quad -\frac{1}{2} \phi_1 + \sqrt{3} \phi_2 = 0$$

The original $G_2$ holonomy manifold $X$ enjoyed an $SU(3)$ continuous isometry. It’s not surprising that, upon taking the quotient to IIA string theory, this isometry is partially lost. In fact, the D6-brane background has only a $SU(2)$ symmetry, under which each $\phi_a$ transforms as a triplet. The M-theory circle itself provides one further, hidden, $U(1)$ action. We therefore conclude that the reduction to IIA string theory has broken the isometry group to $SU(3) \rightarrow SU(2) \times U(1)$. Now, let us introduce a probe D2-brane in this background, and look at the $\mathcal{N} = 1$ gauge theory on its world-volume.

**Theory A:** $U(1)$ with 6 scalars and 3 hypermultiplets

As described above, the 6 scalar multiplets combine into two triplets whose interactions with the hypermultiplets are of the form [11], where the interaction matrices are determined by [12]. The Coulomb branch of this theory is parameterized by the six scalars, together with the dual photon. It has the $SU(2) \times U(1)$ symmetry group, which is expected to be enhanced to the full $SU(3)$ only in the strong coupling, infra-red limit.

Using the results described earlier, the mirror theory is the $\mathcal{N} = 1$ gauge theory with matter content.

**Theory B:** $U(1)^2$ with 3 scalar and 3 hypermultiplets

The charges of the three hypermultiplets under the $U(1)^2$ gauge group are $(+1, -1, 0)$ and $(0, +1, -1)$. The three scalars couple through the usual superpotential [11], with interactions determined by [12] and [13] to be $T_{c,i}^\rho = \delta_{c,i}^\rho$ for all $i$. Let us examine the Higgs branch of this theory. The superpotential provides 3 real constraints on the 12 real scalar fields contained in the hypermultiplets. After dividing by the gauge group, we are left with a Higgs branch of dimension 7, as required. The constraints are,

$$\sum_{i=1}^{3} |q_i|^2 - |\tilde{q}_i|^2 = 0, \quad \sum_{i=1}^{3} \tilde{q}_i q_i = 0 \quad (7)$$

![FIG. 1: Intersection of special Lagrangian D6-branes dual to M-theory on G2 holonomy cone over SU(3)/U(1)^2 (a), and its non-singular deformation (b).](image)
Firstly notice that this space has a manifest $SU(3)$ symmetry, thus recovering the full isometry group of $X$. It is not difficult to further show that the space is indeed isomorphic to the cone over $SU(3)$, ensuring that the full Higgs branch is the flag manifold $SU(3)/U(1)^2$.

There is a single normalizable deformation of this space, which yields a smooth $G_2$ manifold:

$$X \cong \mathbb{R}^3 \times \mathbb{C}P^2$$  \hspace{1cm} (8)

In the D6-brane picture, the singularity is resolved by deforming the singular locus of flat, intersecting D6-branes into a smooth special Lagrangian curve $L \subset \mathbb{C}^3$:

$$L \cong \mathbb{R} \times S^2 \cup \mathbb{R}^3$$  \hspace{1cm} (9)

In the present case this deformation involves only two out of the three D6-branes. To see this more explicitly, let us choose the first and second D6-branes, which deform to lie on the special lagrangian curve:

$$\vec{\phi}_1 \cdot \vec{\phi}_2 = -|\vec{\phi}_1||\vec{\phi}_2|, \quad |\vec{\phi}_1|^2 - |\vec{\phi}_2|^2 = \rho$$  \hspace{1cm} (10)

This curve has a remarkable property: it creates a hole through which the remaining D6-brane can pass, see Figure 1. Therefore, it suffices to deform only two of the three D6-branes in order to completely remove the conical singularity. Of course, one has three different ways to pick a pair of D6-branes, leading to three different resolutions of the space, meeting at a singular point. This is precisely the picture suggested in [1].

It is natural to ask how the probe theory responds to such a deformation. From the perspective of Theory A, one can show that there is essentially a unique deformation consistent with all the symmetries of the model; it is a Yukawa term coupling a pair of hypermultiplets. Moreover, the locus of zeroes of the fermion mass matrix has the same topology as the locus 1. For more details, see [1].

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[1] S. Gukov and D. Tong, “D-brane Probes of Exceptional Holonomy Manifolds, and Dynamics of $N = 1$ Three-Dimensional Gauge Theories”, hep-th/0201018.
[2] B. Acharya, “On Realising $N=1$ Super Yang-Mills in M theory”, hep-th/0011085.
[3] M. Atiyah, J. Maldacena and C. Vafa, “An M-theory Flop as a Large $N$ Duality”, hep-th/0011256.
[4] M. Atiyah and E. Witten, “M-Theory Dynamics On A Manifold Of $G_2$ Holonomy”, hep-th/0107177.
[5] B. Acharya and E. Witten, “Chiral Fermions from Manifolds of $G_2$ Holonomy”, hep-th/0109152.
[6] E. Witten, “Deconstruction, $G_2$ Holonomy and Doublet-Triplet Splitting,” hep-ph/0201018.
[7] M. R. Douglas and G. Moore, “D-branes, Quivers, and $ALE$ Instantons”, hep-th/9603167.
[8] C. Johnson and R. Myers, “Aspects of IIB Theory on $ALE$ Spaces”, Phys. Rev. D55 (1997) 6382.
[9] M. R. Douglas, B. R. Greene and D. R. Morrison, “Orbifold resolution by D-branes”, Nucl. Phys. B506 (1997) 84; D. Morrison and M.R. Plesser, “Non-Spherical Horizons, F’, Adv. Theor. Math. Phys. 3 (1999) 1.
[10] G. Ferretti, P. Salomonson, D. Tsimpis, “D-brane probes on $G_2$ Orbifolds”, hep-th/0111050.
[11] M. Porrati, A. Zaffaroni, “M-Theory Origin of Mirror Symmetry in Three Dimensional Gauge Theories”, Nucl.Phys. B490 (1997) 10.
[12] S. Gukov and J. Sparks, “M theory on Spin(7) manifolds.”, hep-th/0109025.
[13] I. Affleck, J. Harvey and E. Witten, “Instantons and (Super)symmetry Breaking in (2+1) Dimensions”, Nucl. Phys. B206 (1982) 413.
[14] M. Gremm and E. Katz, “Mirror symmetry for $N=1$ QED in three dimensions”, JHEP 0002 (2000) 008.
[15] M. Berkooz, M. R. Douglas, R.G. Leigh, “Branes Intersecting at Angles”, Nucl. Phys. B480 (1996) 265.
[16] M. Aganagic and C. Vafa, “$G_2$ Manifolds, Mirror Symmetry and Geometric Engineering”, hep-th/0110171.
[17] R. Bryant, S. Salamon, "On the Construction of some Complete Metrics with Exceptional Holonomy", Duke Math. J. 58 (1989) 829.
[18] G. W. Gibbons, D. N. Page, C. N. Pope, “Einstein Metrics on $S^3$, $\mathbb{R}^3$ and $\mathbb{R}^4$ Bundles” Commun.Math.Phys. 127 (1990) 529-553.