CALABI-YAU EXTENSIONS AND LOCALIZATIONS OF KOSZUL REGULAR ALGEBRAS

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Abstract. Let $A$ be a Koszul Artin-Schelter regular algebra, $\theta$ an algebra automorphism and $\sigma$ an algebra homomorphism from $A$ to $M_{2 \times 2}(A)$. We compute the Nakayama automorphisms of a skew polynomial extension $A[t; \theta]$ and of a trimmed double Ore extension $A_P[y_1, y_2; \sigma]$ (introduced in [ZZ08]) respectively. This leads to the characterization of the Calabi-Yau property of $A[t; \theta]$, $A_P[y_1, y_2; \sigma]$ and the skew Laurent extension $A[t^\pm; \theta]$.

Introduction

Let $A$ be a Koszul AS-regular algebra with a Nakayama automorphism $\nu_A$ in the sense of [BZ08]. In [HVZ13] the authors proved that the skew polynomial extension $A[t; \nu]$ is Calabi-Yau. Similar results have been also proved in [GYZ14, RRZ13, GK13]. In order to classify Koszul Artin-Schelter regular algebra of dimension 4, in [ZZ08] Zhang and Zhang introduced the notion of a double Ore extension $A_P[y_1, y_2; \sigma]$, where $A$ is an algebra and $\sigma$ is an algebra homomorphism from $A$ to $M_{2 \times 2}(A)$ (see Section 3 for the precise definition). The relation between a double Ore extension and an iterated Ore extension was discussed in [CLM11]. In this note, we first study the Calabi-Yau property of a double extension $A_P[y_1, y_2; \sigma]$ of a Koszul AS-regular algebra $A$. We then apply the method to study the Calabi-Yau property of a skew Laurent extension $A[t^\pm; \theta]$. Our first main result reads as follows (Theorem 2.13).

**Theorem 1.** A double Ore extension $A_P[y_1, y_2; \sigma]$ of a Koszul Artin-Schelter regular algebra $A$ is Calabi-Yau if and only if $\det_{\sigma} = \nu_A$ and a homological determinant type condition is satisfied. Here, $\det_{\sigma}$ is an algebra automorphism induced by $\sigma$.

It follows from Farinati’s result on the Van den Bergh duality under the noncommutative localization [F05] that the Calabi-Yau property is stable under localization. Here, by computing the Nakayama automorphism we characterize the Calabi-Yau property of the skew Laurent extension which is the localization of skew polynomial extension with respect to the Ore set $\{t^i, i \in \mathbb{N}\}$. The second main result is the following (Theorem 3.1 and Theorem 3.3):

**Theorem 2.** (1) The skew Laurent extension $A[t^\pm; \theta]$ of $A$ is Calabi-Yau if and only if there exists an integer $n$ such that $\theta^n = \nu_A$ and the homological determinant.
hdet(θ) of θ equals 1.

(2) Given two automorphisms τ and ξ of A, A[y_1^{±1}, y_2^{±1}; τ, ξ] is Calabi-Yau if and only if there exists two integers m, n such that τ^mξ^n = ν_A and hdet(τ) = hdet(ξ) = 1.

The paper is organized as follows. We recall some preliminary results mainly on the relation of the Nakayama automorphism for Koszul Artin-Schelter regular algebras and their Yoneda Ext algebras in Section 1. The Nakayama automorphism and the Calabi-Yau property of a trimmed double Ore extension are discussed in Section 2. In Section 3 we consider the skew Laurent extension A[t^{±1}; θ]. Throughout, k is a field and all algebras are k-algebras; unadorned ⊗ means ⊗_k and Hom means Hom_k; * always denotes the dual over k.

1. Preliminaries

An N-graded algebra A = ∐_{i≥0} A_i is called connected if A_0 = k. By a graded algebra we mean a locally finite graded algebra generated in degree 1. A module means a left (graded) module. Shifting of a graded module is denoted by ( ). For a module M over A, φM stands for a twisted module by an algebra automorphism φ, where the action is defined by a˙m := φ(a)m. Similarly, M^φ and 1_M^φ denote the twisted right module and the twisted bimodule respectively.

Let V be a finite-dimensional vector space, and T_k(V) the tensor algebra with the usual grading. A connected graded algebra A = T_k(V)/⟨R⟩ is called a quadratic algebra if R is a subspace of V ⊗_k V. The homogeneous dual of A is then defined as A! := T_k(V^*)/⟨R⊥⟩, where R⊥ is the orthogonal complement of R in (V^*)⊗_k V.

Definition 1.1. A quadratic algebra A is called Koszul if the trivial A-module A/k admits a projective resolution

\[ \cdots \rightarrow P_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow A/k \rightarrow 0 \]

such that P_n is generated in degree n for all n ≥ 0.

For more details about Koszul algebras and the Koszul duality, we refer to [Sm96].

Now, recall the definitions of an AS-regular algebra, a Nakayama automorphism and a Calabi-Yau algebra.

Definition 1.2. A connected graded algebra A is called Artin-Schelter (AS, for short) Gorenstein of dimension d with parameter l for some integers d and l, if

(i) inj. dim(A_A) = inj. dim(A_A) = d; and

(ii) Ext^i_A(k, A) ≅ Ext^i_A^φ(k, A) ≅ \begin{cases} 0, & i ≠ d, \\ k(l), & i = d. \end{cases}

If, furthermore, A has a finite global dimension, then A is called AS-regular.

Definition 1.3. [G06, BZ08] Let A be a graded algebra and ν an automorphism. A is called twisted Calabi-Yau of dimension d if

(i) A is homologically smooth, i.e., A, as an A^φ-module, has a finitely generated projective resolution of finite length.
Now, there are two notions of Nakayama automorphisms: one for twisted Calabi-Yau algebras and the other for Frobenius algebras. We use Koszul duality, see Proposition 1.4. For this, we need the following preparation.

The automorphism $\nu$ is called the Nakayama automorphism of $A$. If, in addition, $A^r$ is isomorphic to $A$ as $A^r$-modules, or equivalently, $\nu$ is inner, then $A$ is called Calabi-Yau of dimension $d$. Ungraded Calabi-Yau algebras are defined similarly but without degree shift.

Let $E$ be a Frobenius algebra. By definition, there is an isomorphism $\varphi : E \cong E^*$ as right modules. Equivalently, there is a nondegenerate bilinear form, often called Frobenius pair, $\langle -, - \rangle : E \times E \to k$ such that $\langle ab, c \rangle = \langle a, bc \rangle$ for all $a, b, c \in E$ (where the bilinear form is defined by $\langle a, b \rangle := \varphi(b)(a)$). By the nondegeneracy of the bilinear form, there exists an automorphism $\mu$, unique up to an inner automorphism, such that $\langle a, b \rangle = \langle \mu(b), a \rangle$ for all $a, b \in E$. Thus, $\varphi$ becomes an isomorphism of $E$-bimodules $^*E \cong E^*$. Classically, the automorphism $\mu$ is called the Nakayama automorphism for $E$. For more detail, see [Sm96].

Now, there are two notions of Nakayama automorphisms: one for twisted Calabi-Yau algebras and the other for Frobenius algebras. We use $\nu$ for the former and $\mu$ for the latter if there is no confusion. In fact, the notion of a Nakayama automorphism in [BZ08] can be defined for algebras with finite injective dimension, and it coincides with the classical Nakayama automorphism of a Frobenius algebra. But in this paper, we focus ourselves on twisted Calabi-Yau algebras (or equivalently, AS-regular algebras in the graded case). It is well known that a connected graded algebra $A$ is AS-regular if and only if its Yoneda Ext algebra is Frobenius. In this case, the two notions of Nakayama automorphisms will coincide in the sense of the Koszul duality, see Proposition 1.4. For this, we need the following preparation.

Let $A = T_k(V)/\langle R \rangle$ be a Koszul algebra. Then its Yoneda Ext algebra $E(A)$ is isomorphic to $T_k(V^*)/\langle R^+ \rangle$, see [Sm96]. Let $\theta$ be a graded automorphism of $A$, and define $\theta^* : V^* \to V^*$ by $\theta(f)(x) = f(\theta(x))$ for each $f \in V^*$ and $x \in V$. It is easy to see that $\theta^*$ induces a graded automorphism of $E(A)$ because $\theta$ is assumed to preserve the relation of $A$. We still use the notation $\theta^*$ for this algebra automorphism. Suppose that $\{e_1, e_2, \cdots, e_n\}$ is a $k$-basis of $V$ and $\{e_1^*, e_2^*, \cdots, e_n^*\}$ the corresponding dual basis of $V^*$. If $\theta(e_i) = \sum_j c_{ij} e_j$ for $c_{ij} \in k$ ($1 \leq i, j \leq n$), then we have:

$$\theta^*(e_i^*) = \sum_j c_{ji} e_j^*.$$  

(1.1)

**Proposition 1.4.** [VdB97] Let $A$ be a Koszul AS-regular algebra of dimension $d$. Then, the Nakayama automorphism $\nu$ of $A$ is equal to $\epsilon^{d+1} \mu^*$, where $\mu$ is the Nakayama automorphism of the Frobenius algebra $E(A)$ and $\epsilon$ is the automorphism of $A$ defined by $a \mapsto (-1)^{\deg a} a$, for any homogeneous element $a \in A$.

The homological determinant plays an important role in this paper. Roughly speaking, for an AS-Gorenstein algebra $A$, the homological determinant, $\text{hdet}$, is a homomorphism from the graded automorphism group $\text{GrAut}(A)$ of $A$ to the multiplicative group $k^\times \setminus \{0\}$ generalizing the usual determinant of a matrix [JZ00]. For the precise definition and its application, we refer to [JZ00] [RRZ13]. Here, we recall a
characterization of the homological determinant of an automorphism of a Koszul algebra.

**Proposition 1.5.** [WZ11 Proposition 1.11] Suppose that $A$ is a Koszul AS-regular algebra with global dimension $d$. If $\sigma$ is a graded automorphism of $A$, then $\sigma^*(u) = (\text{hdet}\sigma)u$ for any $u \in \text{Ext}_A^d(k, k)$.

Finally, we recall the definition and some basic properties of a double Ore extension from [ZZ08, ZZ09].

**Definition 1.6.** Let $A$ be a subalgebra of a $k$-algebra $B$. Then:

1. $B$ is called a right double Ore extension of $A$ if:
   - (i) $B$ is generated by $A$ and two new variables $y_1$ and $y_2$;
   - (ii) $y_1$ and $y_2$ satisfy the relation
     \[ y_2y_1 = py_1y_2 + qy_1^2 + \tau_1y_1 + \tau_2y_2 + \tau_0 \]
     for some $p, q \in k$ and $\tau_1, \tau_2, \tau_0 \in A$;
   - (iii) $B$ is a free left $A$-module with basis $\{y_1^iy_2^j : i, j \geq 0\}$;
   - (iv) $y_1A + y_2A + A \subseteq Ay_1 + Ay_2 + A$.

2. $B$ is called a left double Ore extension of $A$ if:
   - (i) $B$ is generated by $A$ and two new variables $y_1$ and $y_2$;
   - (ii) $y_1$ and $y_2$ satisfy the relation
     \[ y_1y_2 = p'y_2y_1 + q'y_2^2 + \tau'_1y_1 + \tau'_2y_2 + \tau'_0 \]
     for some $p', q' \in k$ and $\tau'_1, \tau'_2, \tau'_0 \in A$;
   - (iii) $B$ is a free right $A$-module with basis $\{y_1^iy_2^j : i, j \geq 0\}$;
   - (iv) $Ay_1 + Ay_2 \subseteq y_1A + y_2A + A$.

3. $B$ is called a double Ore extension of $A$ if it is a left and right double Ore extension of $A$ with the same generators $\{y_1, y_2\}$.

Condition (1).\(iv\) in the above definition is equivalent to the existence of two maps:

\[ \sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} : A \to M_{2 \times 2}(A) \quad \text{and} \quad \delta = \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} : A \to M_{4 \times 2}(A) \]

subject to

\[ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} a = \sigma(a) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \delta(a) \]

for all $a \in A$. In case $B$ is a right double Ore extension of $A$, we will write $B = A_P[y_1, y_2; \sigma, \delta, \tau]$, where $P = \{p, q\} \subset k$, $\tau = \{\tau_0, \tau_1, \tau_2\} \subset A$, and $\sigma, \delta$ as above. Similar to the Ore extension, $\sigma$ is a homomorphism of algebras and $\delta$ is a $\sigma$-derivation, that is, $\delta$ is $k$-linear and satisfies $\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$, for all $a, b \in A$. If $\delta$ is the zero map and $\tau = \{0\}$, then the right double extension is denoted $A_P[y_1, y_2; \sigma]$, called a trimmed right double Ore extension. Dually, Condition (2).\(iv\) in the above definition is equivalent to the existence of two maps

\[ \phi = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} : A \to M_{2 \times 2}(A) \quad \text{and} \quad \delta' = \begin{pmatrix} \delta'_1 & \delta'_2 \end{pmatrix} : A \to M_{2 \times 1}(A) \]
satisfying
\[(1.3) \quad a (y_1 \ y_2) = (y_1 \ y_2) \phi(a) + \delta'(a)\]
for all \(a \in A\). For a double Ore extension, the relation of equations (1.2) and (1.3) will be explained in the following.

**Definition 1.7.** Let \(\sigma : A \to M_{2 \times 2}(A)\) be an algebra homomorphism. We say that \(\sigma\) is invertible if there is an algebra homomorphism \(\phi = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} : A \to M_{2 \times 2}(A)\) satisfying:
\[
(\phi_{11} \ \phi_{12}) \cdot (\sigma_{11} \ \sigma_{21}) = (\sigma_{11} \ \sigma_{21}) \cdot (\phi_{11} \ \phi_{12}) = \begin{pmatrix} \text{Id}_A & 0 \\ 0 & \text{Id}_A \end{pmatrix},
\]
where \(\cdot\) is the multiplication of the matrix algebra \(M_{2 \times 2}(\text{End}_k(A))\). The multiplication of \(\text{End}_{\mathbb{C}Z}(A)\) is the composition of \(k\)-linear maps. The map \(\phi\) is called the inverse of \(\sigma\).

**Lemma 1.8.** [ZZ08] Let \(B = A_P[y_1, y_2; \sigma, \delta, \tau]\) be a right double Ore extension of \(A\).

1. If \(B\) is a double Ore extension, then \(\sigma\) is invertible.
2. Suppose that both \(A\) and \(B\) are connected graded algebras. If \(p \neq 0\) and \(\sigma\) is invertible, then \(B\) is a double Ore extension.

In order to study the regularity of double Ore extensions, Zhang and Zhang introduced an invariant of \(\sigma\), called the (right) determinant of \(\sigma\), which is similar to the quantum determinant of the 2 \(\times\) 2-matrix. As we will see, it will play an important role in the description of the Nakayama automorphism of a double Ore extension.

Let \(B = A_P[y_1, y_2; \sigma, \delta, \tau]\) be a right double Ore extension of \(A\). The right determinant of \(\sigma\) is defined to be the map:
\[
\text{det}_r \sigma : a \mapsto -q\sigma_{12}(\sigma_{11}(a)) + \sigma_{22}(\sigma_{11}(a)) - p\sigma_{12}(\sigma_{21}(a))
\]
for each \(a \in A\). If \(\sigma\) is invertible with the inverse \(\phi\), then the left determinant of \(\phi\) is defined by:
\[
\text{det}_l \phi := -q\phi_{11} \circ \phi_{21} + \phi_{11} \circ \phi_{22} - p\phi_{12} \circ \phi_{21}.
\]

The following properties of the determinant of \(\sigma\) were given in [ZZ08].

**Proposition 1.9.** [ZZ08] Let \(B = A_P[y_1, y_2; \sigma, \delta, \tau]\) be a right double Ore extension of \(A\), and \(\sigma\) be invertible with inverse \(\phi\). Then,

1. \(\text{det}_r \sigma\) is an algebra endomorphism of \(A\);
2. if \(p \neq 0\), then
\[
\text{det}_r \sigma = \frac{q}{p} \sigma_{11} \circ \sigma_{12} + \sigma_{11} \circ \sigma_{22} - \frac{1}{p} \sigma_{21} \circ \sigma_{12},
\]
\[
\text{det}_l \phi = \frac{q}{p} \phi_{21} \circ \phi_{11} + \phi_{22} \circ \phi_{11} - \frac{1}{p} \phi_{21} \circ \phi_{12};
\]
3. \(\text{det}_r \sigma\) is invertible with inverse \(\text{det}_l \phi\).

Double Ore extensions share lots of ring-theoretic and homological properties with the original algebras [ZZ08]. We list the following one which we will use later.
Lemma 1.10. [ZZ08, Theorem 0.2] Let $A$ be an AS-regular algebra. If $B$ is a connected graded double Ore extension of $A$, then $B$ is AS-regular and $\text{gldim } B = \text{gldim } A + 2$.

2. Double Ore Extension

In this section, we study the Calabi-Yau property of a double Ore extension. First of all, we discuss the Koszul property of a double Ore extensions.

Proposition 2.1. Let $A$ be a Koszul algebra and $B = A[y_1, y_2; \sigma]$ a trimmed right double Ore extension of $A$. Then, $B$ is a Koszul algebra.

Proof. Suppose that $M$ is a $B \otimes_A P$-module and $\varphi$ is an automorphism of $A$. Recall that $M^{\varphi}$ is the twisted bimodule on the $k$-space $M$ with bimodule structure:

$$b \cdot m \cdot a = bm \varphi(a)$$

for all $m \in M, b \in B$ and $a \in A$. On the direct sum $M \oplus M$, there is another right $A$-module structure defined by using $\sigma$ as follows:

$$(s, t) \circ a = (s, t) \left( \begin{array}{cc} \sigma_{11}(a) & \sigma_{12}(a) \\ \sigma_{21}(a) & \sigma_{22}(a) \end{array} \right) = (s\sigma_{11}(a) + t\sigma_{12}(a), s\sigma_{21}(a) + t\sigma_{22}(a))$$

for all $s, t \in M$ and $a \in A$. Since $\sigma$ is an algebra homomorphism, $M \oplus M$ is also a $B \otimes_A P$-module with the original left $B$-action and the right $A$-action defined by $\circ$. Denote by $(M \oplus M)^{\sigma}$ this $B \otimes_A P$-module. By [ZZ08, Theorem 2.2], there is an exact sequence of $B \otimes_A P$-modules

$$(2.1) \quad 0 \to B^{\det, \sigma} \xrightarrow{\partial} (B \oplus B)^{\sigma} \xrightarrow{f} B \otimes_A P_0 \to A \to 0,$$

where, $f$ maps $(s, t)$ to $sy_1 + ty_2$, $g$ sends $r$ to $(r(qy_1 - y_2), rpy_1)$ and the last term $A$ is identified with $B/(y_1, y_2)$. Moreover, $(2.1)$ is a linear resolution of $A$ as a left $B$-module if we assume that both $y_1$ and $y_2$ are of degree 1.

Now by assumption, $A_k$ admits a projective resolution:

$$(2.2) \quad \cdots \to P_{n} \to P_{n-1} \to \cdots \to P_1 \to P_0 \to A_k \to 0$$

with $P_n$ generated in degree $n$ for each $n \geq 0$. We consider the third quadrant bicomplex:
Since each term in the sequence (2.1) is projective as a right \( A \)-module, all the rows of the bicomplex are exact except at the \((-1)\)-th column. Thus, the homology along the rows yields a single nonzero column, that is,

\[
\cdots \rightarrow 0 \rightarrow B^\text{det, } \sigma \otimes_A k \rightarrow (B \oplus B)^\sigma \otimes_A k \rightarrow B \otimes_A k \rightarrow 0.
\]

Moreover, the sequence (2.1) is a split exact sequence. Therefore, the homology of (2.3) is an algebra homomorphism. Then, (2.3) is a \( \sigma \)-linear map, denoted \( \sigma^* \). Extend \( \sigma^* \) to an algebra homomorphism \( \sigma^* : T_k(V^*) \rightarrow M_{2 \times 2}(T_k(V^*)) \) by letting:

\[
\sigma^*(xy) := \sigma^*(x) \sigma^*(y)
\]

for each \( x, y \in T_k(V^*) \). Furthermore, it is easy to see that \( \sigma^* \) induces an algebra homomorphism from \( A^1 \) to \( M_{2 \times 2}(A^1) \) because \( \sigma : A \rightarrow M_{2 \times 2}(A) \) is assumed to be an algebra homomorphism. We still use the notation \( \sigma^* \) for this algebra homomorphism if no confusion occurs. Moreover, similar to the relation between the automorphism group of a Koszul algebra and the one of its Koszul dual, we have the following easy property.

**Lemma 2.2.** \( \sigma \) is invertible with inverse \( \phi \) if and only if \( \sigma^* \) is invertible with inverse \( \phi^* \).

In the rest of this section, \( A = T_k(V)/(R) \) is a Koszul algebra. Let \( \sigma : A \rightarrow M_{2 \times 2}(A) \) be an algebra homomorphism. Then,

\[
\begin{pmatrix}
\sigma_{11} & \sigma_{12} \\
\sigma_{21} & \sigma_{22}
\end{pmatrix} : V^* \rightarrow M_{2 \times 2}(V^*)
\]

defines a \( k \)-linear map, denoted \( \sigma^* \). Extend \( \sigma^* \) to an algebra homomorphism \( \sigma^* : T_k(V^*) \rightarrow M_{2 \times 2}(T_k(V^*)) \) by letting:

\[
\sigma^*(xy) := \sigma^*(x) \sigma^*(y)
\]

for each \( x, y \in T_k(V^*) \). Furthermore, it is easy to see that \( \sigma^* \) induces an algebra homomorphism from \( A^1 \) to \( M_{2 \times 2}(A^1) \) because \( \sigma : A \rightarrow M_{2 \times 2}(A) \) is assumed to be an algebra homomorphism. We still use the notation \( \sigma^* \) for this algebra homomorphism if no confusion occurs. Moreover, similar to the relation between the automorphism group of a Koszul algebra and the one of its Koszul dual, we have the following easy property.

**Lemma 2.2.** \( \sigma \) is invertible with inverse \( \phi \) if and only if \( \sigma^* \) is invertible with inverse \( \phi^* \).

In the rest of this section, \( A = T_k(V)/(R) \) is a Koszul AS-regular algebra of global dimension \( d \) with the Nakayama automorphism \( \nu \), and \( B = A_P[y_1, y_2; \sigma] \) is a trimmed double Ore extension of \( A \). Let \( \{e_1, e_2, \cdots, e_n\} \) is a \( k \)-basis of \( V \) and \( \{e_1^*, e_2^*, \cdots, e_n^*\} \) the corresponding dual basis of \( V^* \). Let \( \delta \) be a base element of the 1-dimensional space \( A_d \). Since \( A^1 \) is a Frobenius algebra, we may pick a basis \( \{\eta_1, \eta_2, \cdots, \eta_n\} \) of \( A_{d-1} \) such that \( \eta_i e_i^* = \delta_{ij} \). Then \( \eta_i e_i^* = \lambda_{ij} \delta \) for some \( \lambda_{ij} \in k \). It follows from the construction of the Frobenius pair that the Nakayama automorphism \( \mu_{A^1} \) of \( A^1 \) is given by:

\[
(2.4) \quad \mu_{A^1}(e_i^*) = \sum \lambda_{ji} e_j^*.
\]

We can assume that \( B = T_k(V \oplus ky_1 \oplus ky_2)/(R_B) \). The generating relations in \( B \) are of following three types:

1. the relations defining \( A \);
2. \( y_2 y_1 - p y_1 y_2 - q y_1^2 \);
3. \( \{y_j e_i - \sigma j_1(e_i)y_j - \sigma j_2(e_i)y_2; j = 1, 2, i = 1, \cdots, n\} \).

In terms of Definition 1.6 and Definition 1.7, the relation (3) in the above is equivalent to

\[
(3') \quad \{e_i y_j - y_1 \phi_{1j}(e_i) - y_2 \phi_{2j}(e_i); j = 1, 2, i = 1, \cdots, n\}.
\]
Obviously, \( \{e_1^*, e_2^*, \ldots, e_n^*, y_1^*, y_2^*\} \) is a basis for \( B^l_1 \). Now we can characterize the dual algebra of \( B \).

**Lemma 2.4.** The relations for \( B^l \) consist of

1. the relations for \( A^l \);
2. the relations for \( C^l \);
3. \( \{y_j^* e_i^* + \phi_{i1}^j(e_i^*)y_1^* + \phi_{i2}^j(e_i^*)y_2^*; j = 1, 2, i = 1, \ldots, n\} \), where \( \phi \) is the inverse of \( \sigma \).

**Proof.** According to the description of the generating relations of \( B \), all of the relations listed above in (1), (2) and (3) belong to \((R_B)^+\).

On the other hand, it suffices to show that every element \( f = \sum_i k_i e_i^* y_1^* + l_i e_i^* y_2^* + m_i y_1^* e_i^* + n_i y_2^* e_i^* \in (R_B)^+ \) can be written as

\[
f = \sum_i a_i (y_1^* e_i^* + \phi_{i1}^1(e_i^*)y_1^* + \phi_{i2}^1(e_i^*)y_2^*) + b_i (y_2^* e_i^* + \phi_{i1}^2(e_i^*)y_1^* + \phi_{i2}^2(e_i^*)y_2^*),
\]

for \( a_i, b_i \in k \). Firstly, we have

\[
k_i = \sum_j m_j e_j^* (\phi_{i1}(e_i)) + n_j e_j^* (\phi_{i2}(e_i))
\]

and

\[
l_i = \sum_j m_j e_j^* (\phi_{i2}(e_i)) + n_j e_j^* (\phi_{i1}(e_i))
\]

for any \( i \). Further,

\[
\sum_i e_j^*(\phi_{i1}(e_i))e_i^* = \phi_{i1}^1(e_j^*)
\]

by the definition of \( \phi_{i1}^1 \). Hence, we have

\[
f = \sum_j m_j \phi_{i1}^1(e_j^*)y_1^* + n_j \phi_{i2}^1(e_j^*)y_2^*
+ \sum_j m_j \phi_{i2}^2(e_j^*)y_2^* + n_j \phi_{i1}^2(e_j^*)y_1^*
+ \sum_i m_i y_1^* e_i^* + n_i y_2^* e_i^*
= \sum_i m_i (y_1^* e_i^* + \phi_{i1}^1(e_i^*)y_1^* + \phi_{i2}^2(e_i^*)y_2^*)
+ n_i (y_2^* e_i^* + \phi_{i1}^2(e_i^*)y_1^* + \phi_{i2}^1(e_i^*)y_2^*),
\]

which completes the proof. \( \square \)

**Proposition 2.5.** Suppose that \( A \) is a Koszul algebra and \( B = A_P[y_1, y_2; \sigma] \) is a trimmed right double Ore extension of \( A \). Then,

1. the map \( A^l \to B^l \) is injective;

For convenience, we list the following well-known property of the algebra \( C := k(y_1, y_2)/(y_2 y_1 - pqy_1 y_2 - qy_1^2)(p \neq 0) \).

**Proposition 2.3.** The algebra \( C \) is Koszul AS-regular of dimension 2. Its Koszul dual \( C^l \) is \( k(y_1^*, y_2^*)/(y_1^*)^2 + qy_1^* y_2^*,y_1^* y_2^* + py_2^* y_1^*,(y_2^*)^2) \).
\[ (2) \ B^i = A^i \oplus A^i y_1^i \oplus A^i y_2^i \oplus A^i y_1^i y_2^i = A^i \oplus y_1^i A^i \oplus y_2^i A^i \oplus y_1^i y_2^i A^i. \] Moreover, \( B^i \) is a free right (and left) \( A^i \)-module with a basis \( \{ 1, y_1^i, y_2^i, y_1^i y_2^i \} \).

**Proof.** As a matter of fact, the proof is similar to [LSV96 Proposition 2.5]. So we just give a sketch of it. First of all, since \( B \) is a free left \( A \)-module with basis \( \{ y_1^i y_2^i; i, j \geq 0 \} \) by definition, the Hilbert series of \( B \) is equal to the Hilbert series of \( A \otimes k[y_1, y_2] \), i.e.,
\[
H_B(t) = \frac{H_A(t)}{(1-t)^2}.
\]

It is well known that there is a functional equation on Hilbert series \( H_C(t) H_{C'}(-t) = 1 \) for a Koszul algebra \( C \). Since both \( A \) and \( B \) are Koszul algebras, so we have \( H_B(t) = (1+t)^2 H_A(t) \). Thus, Statement (1) follows from the following equivalent conditions whose proofs are similar to those of [LSV96 Theorem 2.6]:

(i) \( H_B(t) = (1 + t)^2 H_A(t) \);
(ii) \( A^i \to B^i \) is injective;
(iii) \( A^3 \to B_3 \) is injective.

Moreover, there is a surjective algebra homomorphism \( A^i \prod k(Y_1, Y_2) \to B^i \) from the coproduct of \( A^i \) and the free algebra \( k(Y_1, Y_2) \), which sends \( Y_i \) to \( y_i^* \) for \( i = 1, 2 \). By Lemma [2.2], the kernel of this map is the ideal generated by
\[
\{ Y_i^j + q Y_2 Y_1, Y_1 Y_2 + p Y_2 Y_1, Y_2^j \} \cup \{ Y_j e_i^j + \phi_{ij}^1(e_i^*) Y_1 + \phi_{ij}^2(e_i^*) Y_2; j = 1, 2, i = 1, \ldots, n \}.
\]

Now consider the algebra
\[
E := (A^i \prod k(Y_1, Y_2))/(Y_j e_i^j + \phi_{ij}^1(e_i^*) Y_1 + \phi_{ij}^2(e_i^*) Y_2; j = 1, 2, i = 1, \ldots, n).
\]

It is not hard to see that \( E \) is a free right (also left) \( A^i \)-module with the same basis as the free algebra \( k(Y_1, Y_2) \). Then, Statement (2) will follows from the claim: the following three elements
\[
Y_1^2 + q Y_2 Y_1, \quad Y_1 Y_2 + p Y_2 Y_1, \quad Y_2^2
\]
are normal elements in the algebra \( E \). To see this, it suffices to note that \( \phi_{ij} \)'s, determined by Definition [1.6], (iv), satisfy the relations dual to R3.1-R3.3 in [ZZ08]. \( \square \)

In order to state and prove the main result of this section, we need more notation. Assume that \( (\phi_{ij}^{kl})_{n \times n} \) is the matrix of the restriction of the \( k \)-linear map \( \phi_{ij} \) to \( V \), i.e.,
\[
\phi_{ij}(e_l) = \sum_k \phi_{ij}^{lk} e_k
\]
for each \( l \). Both \( \sigma^* \) and \( \phi^* \) are algebra homomorphisms from \( A^i \) to \( M_{2 \times 2}(A^i) \). Since \( \delta \) is a basis element of the highest nonzero component \( A^i_d \) of \( A^i \), we assume:
\[
(2.5) \quad \sigma^*(\delta) = \begin{pmatrix} W \delta & X \delta \\ Y \delta & Z \delta \end{pmatrix}, \quad \phi^*(\delta) = \begin{pmatrix} W' \delta & X' \delta \\ Y' \delta & Z' \delta \end{pmatrix}
\]
for some \( W, X, Y, Z, W', X', Y', Z' \in k \). Thus we have the following obvious property from Lemma [2.2]

**Lemma 2.6.** \[
\begin{pmatrix} W & X \\ Y & Z \end{pmatrix} \begin{pmatrix} W' & X' \\ Y' & Z' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]
In order to compute the Nakayama automorphism of $B^1$, we prove the following:

**Lemma 2.7.**

1. $\varepsilon := \delta y_1^* y_2^*$ is a basis element of the 1-dimensional space $B_{d+2}^1$.
2. $\{\eta y_1^* y_2^*, \eta y_1^* y_2^*, \ldots, \eta y_1^* y_2^*, \delta y_1^*, \delta y_2^*\}$ forms a basis of $B_{d+1}^1$.
3. For any $1 \leq i, j \leq n$ and $m = 1, 2$, the following equations hold:

\[
\begin{align*}
\epsilon_i^* \eta_j y_1^* y_2^* &= \delta_{ij} \varepsilon, \\
\epsilon_i^* \delta y_m^* &= (a) 0, \\
\delta y_1^* \delta y_2^* &= (c_1)(-1)^dW' \varepsilon, \\
\delta y_2^* \delta y_2^* &= (c_2)(-1)^dY' \varepsilon, \\
\delta y_1^* y_1^* &= (f_1)-\frac{1}{p} \varepsilon, \\
\delta y_1^* y_2^* &= (f_2)\varepsilon, \\
\eta_m \eta_j y_1^* y_2^* &= (b) 0,
\end{align*}
\]

**Proof.** (1) is obvious by Proposition 2.4 (2). The equation (a) follows the definition. Since $A^1 \rightarrow B^1$ is injective, equations (c) and (d) hold naturally. Equations (g) and (h) follow from the relations (2) and (3) of Lemma 2.4. Now for (b), by relation (3) of Lemma 2.4 and Proposition 2.3, we have

\[
y_1^* y_2^* \varepsilon_j = - y_1^* (\phi_{21}(e_j^*)) y_1^* + \phi_{22}(e_j^*) y_2^*
\]

\[
= - \sum_k (\phi_{21}^k y_1^* e_k^* y_1^* + \phi_{22}^k y_1^* e_k^* y_2^*)
\]

\[
= \sum_k (\phi_{21}^k \phi_{11}^k (e_k^*) y_1^* y_1^* + \phi_{22}^k \phi_{12}^k (e_k^*) y_2^* y_1^* + \phi_{22}^k \phi_{11}^k (e_k^*) y_1^* y_2^*)
\]

\[
= \sum_{k,l} (\phi_{21}^k \phi_{11}^l (e_l^*) y_1^* y_1^* + \phi_{22}^k \phi_{12}^l (e_l^*) y_2^* y_1^* + \phi_{22}^k \phi_{11}^l (e_l^*) y_1^* y_2^*)
\]

Hence, the equation (b) will follow from the assumption. Next we prove the remaining equations. For a fixed $j$, suppose that $\eta_j = \sum_m \lambda_m e_{m_1}^* e_{m_2}^* \cdots e_{m_{d-1}}^*$, where $\lambda_m \in k$, then:

\[
\begin{align*}
\begin{pmatrix} y_1^* \\
\eta_j \end{pmatrix} &= \begin{pmatrix} y_1^* \\
\eta_j \end{pmatrix} \epsilon_j^* \eta_j \\
&= \sum_m \lambda_m \begin{pmatrix} y_1^* \\
\eta_j \end{pmatrix} e_j^* e_{m_1}^* e_{m_2}^* \cdots e_{m_{d-1}}^* \\
&= - \sum_m \lambda_m \phi^*(e_j^*) \begin{pmatrix} y_1^* \\
\eta_j \end{pmatrix} e_{m_1}^* e_{m_2}^* \cdots e_{m_{d-1}}^*.
\end{align*}
\]
Finally, we show the claim (2). Suppose that:

Then, the other equations follow.

Hence, by the definition of \( \phi^* \), we obtain:

\[
\begin{pmatrix}
y_1^* \\
y_2^*
\end{pmatrix}
\delta = (-1)^d \begin{pmatrix}
\phi^*_{11}(\delta) & \phi^*_{12}(\delta) \\
\phi^*_{21}(\delta) & \phi^*_{22}(\delta)
\end{pmatrix}
\begin{pmatrix}
y_1^* \\
y_2^*
\end{pmatrix}
= (-1)^d \begin{pmatrix}
W^1\delta y_1^* + X^1\delta y_2^* \\
Y^1\delta y_1^* + Z^1\delta y_2^*
\end{pmatrix}.
\]

Then, the other equations follow.

Finally, we show the claim (2). Suppose that:

\[
a_1 \eta_1 y_1^* y_2^* + \cdots + a_n \eta_n y_1^* y_2^* + b_1 \delta y_1^* + b_2 \delta y_2^* = 0
\]

for some coefficients, \( a_1, \cdots, a_n, b_1, b_2 \in \mathbb{k} \). By the construction of the Frobenius pair of \( B^! \) and the equations (a), (c), (e), (f) and (g), we get \( n + 2 \) linear equations with indeterminates \( a_1, \cdots, a_n, b_1, b_2 \). From the fact that \( \{ \eta_1, \eta_2, \cdots, \eta_n \} \) is a basis of \( A_{d-1} \) and Lemma 2.6, it follows that this system of linear equations has only the zero solution. That is, the vectors \( \eta_1 y_1^* y_2^*, \eta_2 y_1^* y_2^*, \cdots, \eta_n y_1^* y_2^*, \delta y_1^* \) and \( \delta y_2^* \) are linear independent. On the other hand, by the identity \( H_B^!(t) = (1 + t)^2 H_A^!(t) \), we have \( \dim B_{d+1} = 2 \dim A_d + \dim A_{d-1} = n + 2 \). Hence, the vectors form a basis. \( \square \)

**Proposition 2.8.** The restriction of the Nakayama automorphism \( \mu_B^! \) to \( A^! \) equals \( \mu_A^!(\det_t \phi)^* \).

**Proof.** It follows from the equations (a), (b), (c) and (d) in Lemma 2.7. \( \square \)

We need the following technical result although the proof is obvious.

**Lemma 2.9.** Let \( E = \mathbb{k} \oplus E_1 \oplus \cdots \oplus E_d \) be a graded Frobenius algebra. Suppose that \( \{ \alpha_1, \alpha_2 \} \) and \( \{ \beta_1, \beta_2 \} \) are bases of \( E_1 \) and \( E_{d-1} \) respectively. If

\[
\begin{align*}
\langle \alpha_1, \beta_1 \rangle &= a, & \langle \beta_1, \alpha_1 \rangle &= e, \\
\langle \alpha_1, \beta_2 \rangle &= b, & \langle \beta_2, \alpha_1 \rangle &= f, \\
\langle \alpha_2, \beta_1 \rangle &= c, & \langle \beta_1, \alpha_2 \rangle &= g, \\
\langle \alpha_2, \beta_2 \rangle &= d, & \langle \beta_2, \alpha_2 \rangle &= h,
\end{align*}
\]

then \( \langle \alpha_2, \beta_1 \rangle = c \) and \( \langle \beta_2, \alpha_1 \rangle = f \).
then the Nakayama automorphism of $E$ is given by:

$$
\mu(\alpha_1) = \frac{de - cf}{ad - bc}\alpha_1 + \frac{af - be}{ad - bc}\alpha_2,
$$

$$
\mu(\alpha_2) = \frac{dg - ch}{ad - bc}\alpha_1 + \frac{ah - bg}{ad - bc}\alpha_2.
$$

Proof. One just needs to observe that the Frobenius pair $\langle -,- \rangle$ is a nondegenerate bilinear form and therefore $ad - bc \neq 0$. □

Proposition 2.10. 

$$
\mu_B(y_1^*) = (-1)^d \left( qY' + \frac{q}{p}X + \frac{1}{p}W \right) y_1^* + \left( -qW' - \frac{q}{p}X' \right) y_2^*,
$$

$$
\mu_B(y_2^*) = (-1)^d \left( \frac{pY'}{W'Z' - X'Y'} y_1^* + \frac{pW'}{W'Z' - X'Y'} y_2^* \right).
$$

Combining Proposition 2.10 with Lemma 2.6, we obtain:

Corollary 2.11. 

$$
\mu_B(y_1^*) = (-1)^{d+1} \left( (qX + \frac{q}{p}X + \frac{1}{p}W) y_1^* + (qZ + \frac{q}{p}Z + \frac{1}{p}Y) y_2^* \right),
$$

$$
\mu_B(y_2^*) = (-1)^{d+1} (pX y_1^* + pZ y_2^*).
$$

Proposition 2.12. The restriction of the Nakayama automorphism $\nu_B$ of $B$ to $A$ equals $(\det, \sigma)^{-1} \nu_A$, and

$$
\nu_B(y_1) = (qX + \frac{q}{p}X + \frac{1}{p}W) y_1 + pX y_2,
$$

$$
\nu_B(y_2) = (qZ + \frac{q}{p}Z + \frac{1}{p}Y) y_1 + pZ y_2.
$$

Proof. The conclusion follows from Proposition 2.8, Corollary 2.11 and Proposition 1.4. □

Now we are ready to state the main result of this section.

Theorem 2.13. Suppose that $A$ is a Koszul AS-regular algebra. Let $B = A[y_1, y_2; \sigma]$ be a trimmed double Ore extension of $A$. Then $B$ is Calabi-Yau if and only if the following two conditions are satisfied:

1. $\det, \sigma = \nu_A$;
2. $W = p, X = 0, Y = -(1 + \frac{1}{p})q$ and $Z = \frac{1}{p}$.

Remark 2.14. According to the assumption on constants $W, X, Y$ and $Z$ (see equation (2.5)) and the characterizations of the homological determinant (see Proposition 1.5), the condition (2) of Theorem 2.13 is called a homological determinant type condition.

Remark 2.15. For a Koszul AS-regular algebra $A$, there exists a unique skew polynomial extension $B$ such that $B$ is Calabi-Yau by the result in [HVZ13, GYZ14, LWW12, RRZ13, GK13]. Here, we consider the existence and the uniqueness of a Calabi-Yau double Ore extension of a Koszul AS-regular algebra.
(1) Consider the trimmed double Ore extension $B = A_P[y_1, y_2; \sigma]$, with $P = (1, 0)$ and $\sigma = \begin{pmatrix} \nu_A & 0 \\ 0 & \text{id} \end{pmatrix}$. Then $B$ is Calabi-Yau. But it is easy to see that $B$ is an iterated Ore extension of $A$ (see [ZZ09, Proposition 3.6] or its proof). Hence, we ask if there exists a nontrivial double Ore extension $B$ (not iterated one) such that $B$ is Calabi-Yau? The answer is negative from the following example. Let $A = k[x_1, x_2]/(x_2x_1 - x_1x_2 - x_1^2)$ be the Jordan plane. Then, there is only one nontrivial double Ore extension by the classification in [ZZ09], namely, the type $H := A_P[y_1, y_2; \sigma]$ with $P = (-1, 0)$ and $\sigma$ given by the matrix $\begin{pmatrix} 0 & h & 0 \\ 0 & h & 0 \\ h & 0 & 0 \\ hf & h & 0 \end{pmatrix}$, with $h \neq \in k$ and $f \in k$. Moreover, $\det(\sigma) = \nu_A$ if and only if $h^2 = 1$ and $f = 1$. Now, $\sigma^*(\delta) = \begin{pmatrix} -\delta & 0 \\ 0 & -\delta \end{pmatrix}$.

Therefore, there is no Calabi-Yau algebra of the type $H$ by Theorem 2.13.

(2) For the uniqueness, let $A = k[x_1, x_2]/(x_2x_1 + x_1x_2)$ be the quantum plane and $B := A_P[y_1, y_2; \sigma]$ with $P = (-1, 0)$, where $\sigma$ is given by the matrix $\begin{pmatrix} 0 & g & f \\ g & 0 & f \\ f & 0 & g \end{pmatrix}$, with $f, g \in k$ and $f^2 \neq g^2$. So $B$ is of type $\mathbb{N}$ in the classification of [ZZ09]. In this case, $\det(\sigma) = \nu_A$ if and only if $f^2 - g^2 = -1$. If such a condition is satisfied, then $\sigma^*(\delta) = \begin{pmatrix} -\delta & 0 \\ 0 & -\delta \end{pmatrix}$. Hence, $B$ is Calabi-Yau if and only if $f^2 - g^2 = -1$ by Theorem 2.13. Therefore, the double Ore extension, which is Calabi-Yau, of a Koszul AS-regular algebra may not be unique if it exists.

**Remark 2.16.** By the example in Remark 2.15 (1), the condition (1) and condition (2) in Theorem 2.13 are independent.

To end this section, we return to discuss the Nakayama automorphism and the Calabi-Yau property of the skew polynomial extension. For a twist Calabi-Yau algebra $A$ with the Nakayama automorphism $\nu_A$, it was proved in [LWW12] that the Nakayama automorphism of its Ore extension $D = A[t; \theta, \delta]$ is given by

$$\nu_D(x) = \begin{cases} \theta^{-1} \circ \nu_A(x), & x \in A; \\ ax + b, & x = t, \end{cases}$$

for some $a, b \in A$ with $a$ invertible. It was also remarked that if $\delta = 0$, then $\nu_D(t) = at$. Now if we restrict to Koszul algebras, we can describe the Nakayama automorphism more explicitly as follows.

**Proposition 2.17.** Suppose that $A$ is a Koszul AS-regular algebra with Nakayama automorphism $\nu_A$, $\theta$ is a graded algebra automorphism of $A$ and $D = A[t; \theta]$. The
Nakayama automorphism $\nu_D$ of $D$ is given by:

$$\nu_D(x) = \begin{cases} 
\theta^{-1} \circ \nu_A(x), & x \in A \\
(hdet \, \theta)x, & x = t.
\end{cases}$$

Proof. The proof is similar to the one of Proposition 2.12. □

Note that the homological determinant of the Nakayama automorphism of a Koszul algebra is equal to 1 [RRZ13]. Thus, we arrive at the following result which was proved in [HVZ13, GYZ14, LWW12, RRZ13, GK13]:

**Theorem 2.18.** Suppose that $A$ is a Koszul AS-regular algebra with Nakayama automorphism $\nu_A$, and $\theta$ is a graded algebra automorphism of $A$ and $D = A[t; \theta]$. Then, $D$ is Calabi-Yau if and only if $\theta^n = \nu_A$.

3. **Skew Laurent Extensions**

In this section, we consider the Calabi-Yau property of the Ore localization of $A[t; \theta]$. For a skew polynomial extension $A[t; \theta]$ of an algebra $A$, the multiplicatively closed set $\{t^i; i \in \mathbb{N}\}$ is an Ore set. The localization of $D$ with respect to this Ore set is just the skew Laurent polynomial extension $A[t^{\pm 1}; \theta]$. In case $A$ is a Koszul AS-regular algebra, the Nakayama automorphism $\nu$ of $D$ extends to an automorphism $\tilde{\nu}$ of $A[t^{\pm 1}; \theta]$ by $\tilde{\nu}(t^{-1}) = \frac{1}{hdet \, \theta} t^{-1}$ and $\tilde{\nu}(x) = \nu(x)$ for $x \in D$. In fact, $\tilde{\nu}$ is the Nakayama automorphism of $A[t^{\pm 1}; \theta]$ [F05].

**Theorem 3.1.** Suppose that $A$ is a Koszul AS-regular algebra with Nakayama automorphism $\nu_A$, $\theta$ is a graded algebra automorphism of $A$. Then, $A[t^{\pm 1}; \theta]$ is graded Calabi-Yau if and only if there exists an integer $n$ such that $\theta^n = \nu_A$ and the homological determinant of $\theta$ equals 1.

Proof. First of all, it is easy to see that the only invertible elements in $k[t^{\pm 1}]$ are monomials. Suppose that $A[t^{\pm 1}; \theta]$ is Calabi-Yau. Then, it’s Nakayama automorphism $\tilde{\nu}$ is inner, i.e., there exists an integer $n \in \mathbb{Z}$ such that $\tilde{\nu}(x) = t^n x t^{-n}$ for each $x \in A[t^{\pm 1}; \theta]$. In particular, $\tilde{\nu}(t) = t$. Therefore, $hdet(\theta) = 1$ by Proposition 2.17. If $n$ is nonnegative, then for each $x \in A$ we have

$$\tilde{\nu}(x) = \theta^{-1} \nu_A(x) = t^n x t^{-n}$$

$$= t^n (t^{-1} \theta(x) t) t^{-n}$$

$$= t^{n-1} \theta(x) t^{1-n}$$

$$= \cdots = \theta^n(x).$$

Hence, $\nu_A(x) = \theta^{n+1}(x)$. Similarly, the claim also holds for the case when $n$ is a negative integer.

Conversely, if $\theta^n = \nu_A$ for some integer $n$ and the homological determinant of $\sigma$ equals 1, then $\tilde{\nu}(t) = t$. Next, for each $x \in A$, we have

$$\tilde{\nu}(x) = \theta^{-1} \nu_A(x) = \theta^{n-1}(x).$$

But in $A[t^{\pm 1}; \theta]$, $\theta(x) = t x t^{-1}$. That is, both $\theta$ and its inverse are inner. Therefore, $\tilde{\nu}$ is an inner automorphism. The proof is completed. □
Example 3.2. Let $A = \mathbb{k}\langle x, y \rangle/(yx - xy - x^2)$ be the Jordan plane. It is a twisted Calabi-Yau algebra of dimension 2, with the Nakayama automorphism $\nu$ given by $\nu(x) = x$ and $\nu(y) = 2x + y$. Then, $A[t; \theta]$ is Calabi-Yau if and only if $\theta = \nu$ by Proposition 2.18. It is not hard to see that each graded automorphism $\nu$ has the form $\theta(x) = ax$ and $\theta(y) = bx + ay$ for some $a, b \in \mathbb{k}$. By Proposition 3.4, the homological determinant of $\theta$ equals $a^2$. Thus, $A[t^{\pm 1}; \theta]$ is Calabi-Yau if and only if $\theta$ is either given by

$$\begin{align*}
\theta(x) &= x \\
\theta(y) &= \pm \frac{a}{n}x + y
\end{align*}$$

for some nonzero integer $n$, or given by

$$\begin{align*}
\theta(x) &= -x \\
\theta(y) &= \pm \frac{a}{n}x - y
\end{align*}$$

for some even integer $n$.

Finally, we are going to study the localization or the quotient ring of the double Ore extension $B$ with respect to the Ore set generated by new generators. However, we can only do this in some special case as follows.

Proposition 3.3. Let $B = Ap[y_1, y_2; \sigma]$ be a trimmed double Ore extension with $P = (p, 0)$ and $\sigma = \begin{pmatrix} \tau & 0 \\ 0 & \xi \end{pmatrix}$. Then,

1. $\tau$ and $\xi$ are automorphisms of the algebra $A$. Moreover, they commute with each other.
2. The multiplicatively closed set $S := \{ay_1^{n_1}y_2^{n_2}; a \in k, n_1, n_2 \in \mathbb{Z}_{\geq 0}\}$ is an Ore set.
3. The quotient ring $B_S$ of $B$ with respect to $S$ exists and $B_S = Ap[y_1^{\pm 1}, y_2^{\pm 1}; \sigma]$.

Proof. Part (1) follows from Proposition 1.9. The rest is easy.

Theorem 3.4. Suppose that $A$ is a Koszul AS-regular algebra with Nakayama automorphism $\nu_A$, and $B = Ap[y_1, y_2; \sigma]$ is a trimmed double Ore extension with $P = (p, 0)$ and $\sigma = \begin{pmatrix} \tau & 0 \\ 0 & \xi \end{pmatrix}$. Then, $B_S$ is Calabi-Yau if and only if there exist two integers $m, n$ such that the following conditions are satisfied:

1. $\tau^m \theta^n = \nu_A$;
2. $\text{hdet}(\tau) = p^{n+1}$ and $\text{hdet}(\xi) = \frac{1}{p^{m+1}}$.

Proof. The proof is similar to the one of Theorem 5.1. So we omit it.

In general, we may consider an iterated skew polynomial extension of algebra $A$: $R = A[y_1, \ldots, y_m; \theta_1, \ldots, \theta_m]$, where $\theta_1, \ldots, \theta_m$ are commutative graded automorphisms of $A$. Note that $y_iy_j = y_jy_i$ for all $i, j$, and that $y_ia = \theta_i(a)y_i$ for all $i$ and all $a \in A$. The quotient ring $R_S$ of $R$ with respect to the multiplicatively closed set $S := \{y_1^{n_1} \cdots y_m^{n_m}; n_1, \ldots, n_m \in \mathbb{Z}_{\geq 0}\}$ exists and is isomorphic to the iterated skew Laurent ring $A[y_1^{\pm 1}, \ldots, y_m^{\pm 1}; \theta_1, \ldots, \theta_m]$. For detail, see the book [GW04].
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have the following criterion for an iterated skew polynomial extension of a Koszul AS-regular algebra to be Calabi-Yau.

**Theorem 3.5.** Suppose that \( A \) is a Koszul AS-regular algebra with Nakayama automorphism \( \nu_A \), \( R = A[y_1, \ldots, y_m; \theta_1, \ldots, \theta_m] \) and \( S := \{ y_1^{n_1} \cdots y_m^{n_m} : n_1, \ldots, n_m \in \mathbb{Z}_{\geq 0} \} \). Then,

1. the Nakayama automorphism \( \nu_R \) of \( R \) is
   \[
   \nu_R(x) = \begin{cases} 
   (\theta_1 \circ \cdots \circ \theta_m)^{-1} \circ \nu_A(x), & x \in A \\
   \text{lhdet}(\theta_i) y_i, & x = y_i, 1 \leq i \leq m;
   \end{cases}
   \]
2. \( R \) is Calabi-Yau if and only if \( \theta_1 \circ \cdots \circ \theta_m = \nu_A \) and \( \text{lhdet}(\theta_i) = 1 \) for all \( i \);
3. \( RS \) is Calabi-Yau if and only if
   - (i) \( \text{lhdet}(\theta_i) = 1 \) for all \( i \), and
   - (ii) there exists a series of integers \( k_1, \ldots, k_m \) such that \( \theta_1^{k_1} \cdots \theta_m^{k_m} = \nu_A \).

Note that a typical example of such a Calabi-Yau \( RS \) is the smash product of a Koszul AS-regular algebra with a free abelian group algebra. For example, some algebras in the classification of Calabi-Yau pointed Hopf algebras of finite Cartan type in [YZ11].

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