HODGE-TATE CONDITIONS FOR LANDAU-GINZBURG MODELS

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Abstract. We give a sufficient condition for a class of tame compactified Landau-Ginzburg models in the sense of Katzarkov-Kontsevich-Pantev to satisfy some versions of their conjectures. We also give examples which satisfy the condition. The relations to the quantum D-modules of Fano manifolds and the original conjectures are explained in Appendices.

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1. Introduction

Let $X$ be a smooth projective variety over $\mathbb{C}$ with a Zariski open subset $Y$. We assume that $D := X \setminus Y$ is a simple normal crossing hypersurface. Let $f : X \to \mathbb{P}^1$ be a flat projective morphism such that the restriction $w := f|_Y$ is a regular function. The pair $(Y, w)$ is called a Landau-Ginzburg model. In general, the meromorphic flat connection $(\mathcal{O}_X(D), d + df)$ has irregular singularities along $D$. Let $H^\bullet_{dR}(Y, w)$ denote the de Rham cohomology group of $(\mathcal{O}_X(D), d + df)$. It has been studied from the viewpoint of generalized Hodge theories. (See twistor $D$-modules [27], [28], irregular Hodge structures [10], [17], [34], [35], non-commutative Hodge structures [24], [25], TERP-structures [22], and so on.)

In some cases, $(Y, w)$ can be considered as a ‘mirror dual’ of a smooth projective Fano variety $F$ called a sigma model. In that case, it is predicted that some categories associated to $(Y, w)$ are equivalent to the corresponding categories associated to $F$. This prediction is called a Homological Mirror Symmetry conjecture (HMS). Some parts of HMS are proved in some cases [1], [2], [39].

From this point of view, Katzarkov-Kontsevich-Pantev [25] proposed some conjectures as conjectural consequences of HMS. As emphasized in [25], some of their conjectures can be seen as “purely algebro-geometric” conjectures on the generalized Hodge theory of $H^\bullet_{dR}(Y, w)$. Such conjectures are the main subjects of this paper.

As an introduction, we survey some versions of the conjectures in §1.1 and §1.2. (The relations to the original ones are explained in Appendix [B]) Then, we explain our main result in §1.3. In this paper, we always assume that the pole divisor $(f)_\infty$ of $f$ is reduced and the support $|(f)_\infty|$ is equal to $D$, although this assumption is more restrictive than that of [25].

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1.1. Landau-Ginzburg Hodge numbers. The cohomology group $H^k_{\text{dR}}(Y,w)$ is given by taking the hypercohomology of the complex $(\Omega^*_Y(\ast D),d+df\wedge)$. There are $\mathcal{O}_X$-coherent subsheaves $\Omega^k_j$ of $\Omega^k_X(\ast D)$ which give a subcomplex $(\Omega^*_j,d+df\wedge)$ (see [33, 11]). It is known that the inclusion $(\Omega^*_j,d+df\wedge) \hookrightarrow (\Omega^*_Y(\ast D),d+df\wedge)$ is a quasi-isomorphism (see [17] for example). The Landau-Ginzburg Hodge number $f^{p,q}(Y,w)$ is defined by

$$f^{p,q}(Y,w) := \dim H^q(X,\Omega^p_j).$$

It is proved by Esnault-Sabbah-Yu, Kontsevich, and M. Saito [17] that we have $\dim H^k(Y,w) = \sum_{p+q=k} f^{p,q}(Y,w)$, which can be considered as a consequence of $E_1$-degeneration property of the “Hodge filtration”.

Take sufficiently small holomorphic disk $\Delta$ in $\mathbb{P}^1$ centered at infinity so that $Y_b := f^{-1}(b)$ is smooth for any $b \in \Delta \setminus \{\infty\}$. It is proved in [25] that we have the following equation:

$$\dim H^k_{\text{dR}}(Y,w) = \dim H^k(Y_b),$$

where $b \in \Delta \setminus \{\infty\}$, and $H^k(Y,Y_b)$ denotes the relative cohomology with $\mathbb{C}$-coefficient. In our situation, the monodromy $T_b$ at infinity is known to be unipotent. Let $k^W$ be the monodromy weight filtration of $N_k := \log T_k$ on $H^k(Y,Y_b)$ centered at $k$ (see [22], [23]). The Landau-Ginzburg Hodge number $h^{p,q}(Y,w)$ is defined by

$$h^{p,q}(Y,w) := \dim \text{Gr}_k^W H^k(Y,Y_b), \quad (k = p + q).$$

By a HMS consideration, Katzarkov-Kontsevich-Pantev [25] conjectured:

$$(1.1) \quad f^{p,q}(Y,w) = h^{p,q}(Y,w).$$

It is easy to observe that the conjecture (1.1) does not hold if the fiber $D$ at infinity is smooth and $f^{p,q}(Y,w)$ are not zero for two different pairs $(p,q)$ and $(p',q')$ with $p + q = p' + q'$. Actually, such example is given in [26]. However, in loc. cite., there are examples of $(X,f)$ which satisfies (1.1). There remains a question when the equation (1.1) holds. The counter-example suggests that we need to impose some conditions on the degeneration property of $Y_b$ as $b \to \infty$.

1.2. Speciality. Let $(\lambda,\tau)$ be a pair of complex numbers. The dimension of the hypercohomology $\mathbb{H}^*(X;\Omega^*_Y,\lambda d + \tau df\wedge)$ is known to be independent of the choice of $(\lambda,\tau)$ ([17], [27]). Let $\mathbb{C}_\lambda, \mathbb{C}_\tau$ be complex planes with coordinate $\lambda$ and $\tau$ respectively. Put $\mathbb{P}^1_1 := \mathbb{C}_\lambda \cup \{\infty\}$ and $S := \mathbb{P}^1_1 \times \mathbb{C}_\tau$. It follows that we have a locally free $\mathbb{Z}/2\mathbb{Z}$-graded $\mathcal{O}_S(\ast(\lambda)_\infty)$-module $bH$ whose fiber at $(\lambda,\tau)$ is $\mathbb{H}^*(X;\Omega^*_Y,\lambda d + \tau df\wedge)$. The $\mathcal{O}_S(\ast(\lambda)_\infty)$-module $bH$ is equipped with a grade-preserving meromorphic flat connection:

$$b\nabla : bH \to bH \otimes_{\mathcal{O}_S} \Omega^1_S(\log \lambda \tau)((\lambda)_0),$$

where $\Omega^1_S(\log \lambda \tau)((\lambda)_0)$ denotes the $\mathcal{O}_S$-module locally generated by $\lambda^{-1} \tau^{-1} d\tau$ and $\lambda^{-2} d\lambda$.

For a smooth projective Fano variety $F$, the quantum $\mathcal{D}$-module for the quantum parameters $c_1(F) \log \tau \in H^2(F)$ gives a similar pair $(aH,a\nabla)$. These pairs are considered as one parameter variation of non-commutative Hodge structures $(\mathbb{H},\mathbb{A}\nabla) := (aH,a\nabla)|_{\tau=1}$, and $(bH,b\nabla) := (bH,b\nabla)|_{\tau=1}$. It is conjectured [25, Conjecture 3.11] that homological mirror correspondences for a pair $F / (Y,w)$ should induce an isomorphism $(aH,a\nabla) \simeq (bH,b\nabla)$ (more precisely, we need to fix more data to determine the mirror pair).

On the one hand, $(\mathbb{H},\mathbb{A}\nabla)$ has a trivial logarithmic extension to $\lambda = \infty$. On the other hand, it is a non-trivial problem to construct a logarithmic extension of $(bH,b\nabla)$ such that the induced vector bundle on $\mathbb{P}^1_1$ is trivial. The problem is called Birkhoff problem (see e.g. [33]), and the solution to the problem for $(bH,b\nabla)$ plays a key role in the construction of primitive forms [11], [32].
Katzarkov-Kontsevich-Pantev observed that the trivial solution of the Birkhoff problem for the connection \((A_H, A^\nabla)\) can be described in terms of the Deligne’s canonical extension and the weight filtration for the nilpotent part of the residue endomorphism along \(\{\lambda = \infty\}\). An extension given in a similar way is called a skewed canonical extension in \cite{25}. The skewed canonical extension can be defined for more general objects including \((B_H, B^\nabla)\). The property that the skewed canonical extension gives a solution to the Birkhoff problem is called “speciality” (see \cite{25, Definition 3.21}, or Definition \ref{def:2.13} for details).

§ 1.3. Rescaling structures and Hodge-Tate conditions. To treat the conjectures in \ref{1.1} and \ref{1.2} simultaneously, we introduce a notion of rescaling structure (See \ref{2} for details). Let \(\sigma : \mathbb{C}_0^* \times S \to S\) be a \(\mathbb{C}_0^*\) be an action of \(\mathbb{C}_0^*\) defined by \((\theta, \lambda, \tau) \mapsto (\theta \lambda, \theta \tau)\). Let \(p_2 : \mathbb{C}_0^* \times S \to S\) denote the projection. A rescaling structure is a triple \((\mathcal{H}, \nabla, \chi)\) of \(\mathbb{Z}\)-graded locally free \(\mathcal{O}_S(\ast (\lambda)_{\infty})\)-module \(\mathcal{H}\), a grade-preserving meromorphic flat connection
\[
\nabla : \mathcal{H} \to \mathcal{H} \otimes \Omega^1_\tau(\log \lambda \tau)((\lambda)_0),
\]
and an isomorphism \(\chi : p_2^{\ast} \mathcal{H} \xrightarrow{\sim} \sigma^{\ast} \mathcal{H}\) with some conditions (see Definition \ref{def:2.5}).

For a rescaling structure \((\mathcal{H}, \nabla, \chi)\), take a fiber \(V\) of \(\mathcal{H}\) at \((\lambda, \tau) = (1, 0)\). Under an assumption, we associate two filtrations \(F\) and \(W\) on \(V\), where \(F\) is called Hodge filtration and \(W\) is called weight filtration of \(\mathcal{H}\) (\ref{2.4}). We also define an abstract version of Hodge numbers \(f^{p,q}(\mathcal{H})\) and \(h^{p,q}(\mathcal{H})\).

The rescaling structure is said to satisfy the Hodge-Tate condition if these two filtration behave like a Hodge filtration and a weight filtration of a mixed Hodge structure of Hodge-Tate type in the sense of Deligne \cite{0} (see Definition \ref{2.11} for more precise). If \((\mathcal{H}, \nabla, \chi)\) satisfies Hodge-Tate condition, we have \(f^{p,q}(\mathcal{H}) = h^{p,q}(\mathcal{H})\), and we also have that \(\mathcal{H}_{\tau=1}\) is special.

In Appendix \ref{a}, we show that a “Tate twisted” version \(\mathcal{H}_F\) of \(\mathcal{H}\) comes equipped with a rescaling structure for any smooth projective Fano variety \(F\). The rescaling structure \(\mathcal{H}_F\) satisfies the Hodge-Tate condition, and we have
\[
f^{p,q}(\mathcal{H}_F) = h^{p,q}(\mathcal{H}_F) = \dim H^q(F, \Omega^{n-p}_F).
\]

For the pair \((X, f)\), we also have a version \(\mathcal{H}_f\) of \(\mathcal{H}\), which comes equipped with a rescaling structure (See \ref{3} The relation between \(\mathcal{H}_f\) and \(\mathcal{H}\) is given in Appendix \ref{b}). The main result of this paper is the following:

**Theorem 1.1** (Theorem \ref{3.25}). Let \(\mathcal{H}_f\) be the rescaling structure for \((X, f)\).

1. If \(\mathcal{H}_f\) satisfies the Hodge-Tate condition, then the equation \(\ref{1.1}\) holds and \(\mathcal{H}_{f|_{\tau=1}}\) is special.
2. The rescaling structure \(\mathcal{H}_f\) satisfies Hodge-Tate condition if and only if a mixed Hodge structure \((H^k(Y, Y_\infty; \mathbb{Q}), F, W)\) is Hodge-Tate for every \(k \in \mathbb{Z}\).

The definition of the mixed Hodge structure \((H^k(Y, Y_\infty; \mathbb{Q}), F, W)\) is given in \ref{3.4.3}. In \ref{3} we also give some examples such that \(\mathcal{H}_f\) satisfies the Hodge-Tate condition in the case where the dimension of \(X\) is two or three.

2. Rescaling structures

2.1. Holomorphic extensions and filtrations. Let \(\mathbb{C}\) denote a complex plane. Set \(\mathbb{C}^* := \mathbb{C} \setminus \{0\}\). Let \(H\) be a finitely generated locally free \(\mathcal{O}_\mathbb{C}(\ast \{0\})\)-module. Let \(V\) denote the fiber of \(H\) at \(1 \in \mathbb{C}\).
Assume that we are given an increasing filtration $G_*V = (G_mV \mid m \in \mathbb{Z})$ on $V$ such that

\begin{equation}
G_mV := \begin{cases} 
0 & (m \ll 0) \\
V & (m \gg 0).
\end{cases}
\end{equation}

We shall recall some methods to construct an extension of $H$ to an $\mathcal{O}_C$-module by using $G_*V$. Here, by an extension of $H$, we mean a locally free $\mathcal{O}_C$-submodule $L$ of $H$ such that $L \otimes \mathcal{O}_C(*\{0\}) = H$.

2.1.1. Construction using $\mathbb{C}^*$-actions. Let $m : \mathbb{C}^* \times \mathbb{C}^* \rightarrow \mathbb{C}^*$ and $\sigma : \mathbb{C}^* \times \mathbb{C} \rightarrow \mathbb{C}$ denote the multiplications. Let $p_2 : \mathbb{C}^* \times \mathbb{C} \rightarrow \mathbb{C}$ be the projection. Assume that $H$ is $\mathbb{C}^*$-equivariant with respect to $\sigma$. Namely, we have an isomorphism $\chi : p_2^*H \overset{\sim}{\rightarrow} \sigma^*H$ with the cocycle condition:

\[(m \times \mathrm{id}_C)^*\chi = (\mathrm{id}_C \times \sigma)^*\chi \circ p_2^*\chi,
\]

where $p_{23} : \mathbb{C}^* \times \mathbb{C}^* \rightarrow \mathbb{C}^* \times \mathbb{C}$ is given by $p_{23}(t_1, t_2, z) := (t_2, z)$. This case is considered in [37], for example. For any vector $v \in V$, there is a unique invariant section $\phi_v \in \Gamma(\mathbb{C}, H)$ with $\phi_v(1) = v$. There exists a unique extension $L_1$ such that $v \in G_mV$ if and only if $\phi_v \in L_1(m\{0\})$. The extension $L_1$ is isomorphic to the extension $\sum_m G_mV \otimes \mathcal{O}_C(-m\{0\})$ of $V \otimes \mathcal{O}(\{0\})$. This construction gives a one to one correspondence between the sets of increasing filtrations on $V$ with [2.1] and $\mathbb{C}^*$-equivariant holomorphic extensions of $H$.

Example 2.1. Let $V$ be a finite dimensional $\mathbb{C}$-vector space with a decomposition $V = \bigoplus_{p \in \mathbb{Z}} V_p$. Put $H := \mathcal{O}_C(*\{0\}) \otimes_{\mathbb{C}} V$. Remark that $p_2^*H \simeq \mathcal{O}_{\mathbb{C}^* \times \mathbb{C}^*}(\{\ast\} \times \{0\}) \otimes V \simeq \sigma^*H$. Define $\chi : p_2^*H \overset{\sim}{\rightarrow} \sigma^*H$ by $\chi_{|\mathcal{O}_{\mathbb{C}^* \times \mathbb{C}^*}(\{\ast\} \times \{0\}) \otimes V_h(t, z)} := (p \otimes \mathrm{id}_V)_h$. Consider $V$ as the fiber of $H$ at $1 \in \mathbb{C}$. Then the trivial extension $L_1 := \mathcal{O}_C \otimes V$ corresponds to the following filtration:

\[G_mV = \bigoplus_{-p \leq m} V_p.
\]

Indeed, for $v \in V_p$, the invariant section $\phi_v$ is given by $\phi_v(z) = z^p v \in L_1(-p\{0\})$.

2.1.2. Double complex. Let $(\mathbb{C}^* \bullet, \delta_1, \delta_2)$ be a double complex of $\mathbb{C}$-vector spaces where $\delta_1 : \mathbb{C}^{p,q} \rightarrow \mathbb{C}^{p+1,q}$ and $\delta_2 : \mathbb{C}^{p,q} \rightarrow \mathbb{C}^{p,q+1}$ are the differentials. We assume that $\mathbb{C}^{p,q} = 0$ if $p < 0$ or $q < 0$, and that the total complex $(\mathbb{C}^\bullet, \delta)$ has finite dimensional cohomology. Here, we put $C^k := \bigoplus_{p+q=k} \mathbb{C}^{p,q}$ and $\delta := \delta_1 + \delta_2$. Let $F$ be the filtration on $(\mathbb{C}^\bullet, \delta)$ given by $F_mC^\ell := \bigoplus_{p+q = \ell, -p \leq m} \mathbb{C}^{p,q}$. We also assume that the morphisms $H^k(F_m(\mathbb{C}^\bullet, \delta)) \rightarrow H^k(\mathbb{C}^\bullet, \delta)$ are injective for all $k$ and $m$.

Put $C^{p,q} := \mathcal{O}_C \otimes \mathbb{C}^{p,q}$ and $C^k := \bigoplus_{p+q = k} \mathbb{C}^{p,q}$. We have a complex $(C^\bullet, z\delta_1 + \delta_2)$. Let $L_1$ be the $k$-th cohomology group of this complex. By the assumption, $L_1$ is a finitely generated locally free $\mathcal{O}_C$-module. Put $H := L_1 \otimes \mathcal{O}_C(*\{0\})$ and consider $L_1$ as an extension of $H$. Define $\chi_p : p_2^*C^{p,q} \overset{\sim}{\rightarrow} \sigma^*C^{p,q}$ by $\chi_p(t, z) := t^p \otimes \mathrm{id}$. This induces an isomorphism $\chi : p_2^*H \overset{\sim}{\rightarrow} \sigma^*H$ with the cocycle condition.

Lemma 2.2. Consider the $k$-th cohomology $H^k(\mathbb{C}^\bullet, \delta)$ as the fiber of $H$ at $1 \in \mathbb{C}$. Then the extension $L_1$ corresponds to the following filtration:

\[G_mH^k(\mathbb{C}^\bullet, \delta) := \mathrm{Im}(H^k(F_m(\mathbb{C}^\bullet, \delta)) \rightarrow H^k(\mathbb{C}^\bullet, \delta)).
\]

Proof. Put $F_mC^k := \bigoplus_{p+q = k, -p \leq -m} \mathbb{C}^{p,q}$. It induces a filtration on the complex $(\mathbb{C}^\bullet, z\delta_1 + \delta_2)$, which is also denoted by $F$. The induced filtration on $L_1$ is also denoted by $F$. By the assumption, we have $\text{Gr}_F^k L_1 \simeq H^k(\text{Gr}_F^k(\mathbb{C}^\bullet))$. Hence it reduces to the case where there exists a $p_0 \in \mathbb{Z}$ such that $\mathbb{C}^{p,q} = 0$ for $p \neq p_0$. In this case, we have $L_1 \simeq H^{k-p_0}(\mathbb{C}^{p_0}, \delta_2) \otimes \mathcal{O}_C$, and we obtain the conclusion by Example 2.1. \qed
2.1.3. Construction using flat connections with regular singularities. Assume that \( H \) is equipped with a flat connection \( \nabla \) with a regular singularity at \( \{0\} \). We also assume that each \( G_kV \) is invariant with respect to the monodromy of \( \nabla \). This case is considered in \([24, 25, 33]\) for example. We have the flat subbundles \( G_*H \) on \( H \) such that the fiber of \( G_kH \) at 1 is \( G_kV \). For any \( t \in \mathbb{C}^* \), let \( V_t \) be the fiber of \( H \) at \( t \). Let \( G_*V_t \) denote the induced filtration on \( V_t \). Set \( I_t := \{st \mid 0 < s \leq 1\} \).

For any vector \( v \in V_t \), we have the flat section \( \psi_{v,t} \in \Gamma(I_t, H) \) with \( \psi_{v,t}(t) = v \). There exists a unique logarithmic lattice \( L_2 \) with the following property: Fix a frame of \( L_2 \) near 0, and let \( \|*\|_{L_2} \) be the Hermitian metric on \( L_2 \) near 0 so that the frame is a orthogonal with respect to \( \|*\|_{L_2} \). A vector \( v \in V_t \) is contained in \( G_mV_t \) if and only if \( \psi_{v,t} \) satisfies

\[
\|\psi_{v,t}(r \cdot t)\|_{L_2} \leq C|r|^{-m}(-\log r)^N \quad (0 < r < 1)
\]

for some positive constants \( C \) and \( N \). This construction also gives a one to one correspondence between the logarithmic extension of \( H \) and monodromy invariant filtrations on \( V \) with \((2.1)\).

2.1.4. Characterization by using the Deligne lattice. The extension \( L_2 \) can be characterized by using the Deligne lattice of \((H, \nabla)\). Let \( L' \) be the Deligne lattice of \((H, \nabla)\), which means that \( L' \) is the logarithmic at 0 and the residue with eigenvalues whose real parts are contained in \((-1,0)\). The flat subbundles \( G_mH \) extend to \( \{0\} \) and give subbundles of \( L' \). Let \( G_mL' \) denote the subbundles of \( L' \).

**Lemma 2.3** ([25 §3.3.1]). The extension \( L_2 \) is given by

\[
L_2 = \sum_{m \in \mathbb{Z}} G_mL'(-m\{0\})
\]

as a submodule of \( L'(*\{0\}) \).

**Proof.** It is enough to show that \( L_2 = L' \) if \( G_*V \) is given by \( G_{-1}V = 0 \) and \( G_0V = V \). Let \( \text{rk}L' \) be the rank of \( L' \). We have an isomorphism of logarithmic connections \((L', \nabla) \simeq (O_C^{\text{rk}L'}, \nabla')\), where \( \nabla' = d - Ut^{-1}dt \) for a matrix \( U \in \text{End}(\mathbb{C}^{\text{rk}L'}) \) with eigenvalues whose real parts are contained in \([0,1)\). Take the standard frame \( v_1, \ldots, v_{\text{rk}L'} \) of \( O_C^{\text{rk}L'} \). It induces a Hermitian metric \( \|*\|_{L'} \). For fixed \( t \in \mathbb{C}^* \), take \( \alpha \in \mathbb{C} \) with \( \exp \alpha = t \). We have the flat section \( \psi_i(r \cdot t) := \exp(\alpha \log r U)v_i(rt) \) on \( I_t \) for all \( i = 1, \ldots, \text{rk}L' \). Since the flat sections on \( I_t \) are \( \mathbb{C} \)-linear combinations of \( \psi_i \), we obtain the conclusion. \( \square \)

2.1.5. Relation between two constructions. Assume that \( H \) is \( \mathbb{C}^* \)-equivariant and equipped with a flat connection \( \nabla \). We also assume the compatibility of the action and flat connection. In other words, for all \( t \in \mathbb{C}^* \), the action of \( t \) on \( H \) is assumed to be equal to the parallel transport of \( \nabla \). Then we have the following.

**Lemma 2.4.** The connection \( \nabla \) is regular singular at \( \{0\} \). The extensions \( L_1 \) and \( L_2 \) constructed in \((2.1)\) and \((2.1.3)\) coincide to each other.

**Proof.** By the compatibility of the action and the connection, the invariant section \( \phi_v \) for \( v \in V \) is \( \nabla \)-flat. Since \( L_2 \) is generated by \( t^m\phi_v \) (\( v \in G_mV \), \( t \) is a coordinate on \( \mathbb{C} \)), it gives a logarithmic extension of \( H \). This shows that the connection \( \nabla \) is regular singular at \( \{0\} \). Fix a trivialization of \( L_1 \) around \( \{0\} \) and let \( \|*\|_{L_1} \) be the induced Hermitian metric on \( L_1 \) around \( \{0\} \). For \( v \in G_mV \), \( \phi_v \) is in \( L_1(m\{0\}) \), which implies

\[
\|\phi_v(t)\|_{L_1} \leq C|t|^{-m} \quad (t \in \mathbb{C}^*)
\]

for some positive constant \( C \). This shows the conclusion: \( L_1 = L_2 \). \( \square \)
2.2. Definition of rescaling structure. Let $C_{\lambda}$, $C_{\tau}$ be complex planes with coordinate $\lambda$ and $\tau$ respectively. Put $P_{1} := C_{\lambda} \cup \{\infty\}$ and $S := P_{1} \times C_{\tau}$. Let $\sigma : C_{\theta}^{*} \times S \to S$ be an action of $C_{\theta}^{*}$ defined by $\sigma(\theta, \lambda, \tau) := (\theta \lambda, \theta \tau)$. For a meromorphic function $h$ on a variety, $(h)_0$ and $(h)_{\infty}$ denote the zero divisor of $h$ and the pole divisor of $h$, respectively. The supports of these divisors are denoted by $|(h)_0|$ and $|(h)_{\infty}|$, respectively. Let $p_{2} : C_{\theta}^{*} \times S \to S$ be the projection. We define the notion of rescaling structure as follows.

**Definition 2.5.** A rescaling structure is a triple $(H, \nabla, \chi)$ of a $\mathbb{Z}$-graded locally free $O_{S}(\ast(\lambda)_{\infty})$-module $H$, a grade-preserving meromorphic flat connection

$$\nabla : H \to H \otimes O_{S}^{1}(\ast(\lambda)_{\infty} \cup |(\lambda \tau)|),$$

and an grade-preserving isomorphism $\chi : p_{2}^{*}H \tilde{\to} \sigma^{*}H$ with the following properties:

1. We have $\nabla_{\lambda_{\tau}}(H) \subset H$ and $\nabla_{\lambda_{\tau}}(H) \subset H$.
2. On $C_{\lambda} \times C_{\tau}^{*}$, $\chi$ is flat with respect to $p_{2}^{*}\nabla$ and $\sigma^{*}\nabla$.
3. The isomorphism $\chi$ satisfies the cocycle condition. In other words, we have

$$(m \times \text{id}_{S})^{*}\chi = (\text{id}_{C_{\tau}} \times \sigma)^{*}\chi \circ p_{23}^{*}\chi,$$

where $m : C_{\theta}^{*} \times C_{\theta}^{*} \to C_{\theta}^{*}$ denotes the multiplication and $p_{23} : C_{\theta}^{*} \times C_{\theta}^{*} \times S \to C_{\theta}^{*} \times S$ denotes the projection given by $p_{23}(\theta_{1}, \theta_{2}, (\lambda, \tau)) = (\theta_{2}, (\lambda, \tau))$.

We often abbreviate $\nabla$ and $\chi$ if there is no confusion. The $k$-th graded piece of $H$ is denoted by $H^{k}$. We assume $\sum_{k} \text{rank } H^{k} < \infty$ in this paper.

We note that we introduce the notion of rescaling structure only for convenience for the later use. Similar structures have been studied in [22, 25, 27, 34, 35], for example. Operations acting on $H$ is often assumed to preserve the grading without a mention. If $H$ and $H'$ are rescaling structures, we can naturally define the tensor product $H \otimes H'$ which is also a rescaling structure. The dual $H^{\vee}$ can also be defined canonically.

**Example 2.6.** Set $T := O_{S}(\ast(\lambda)_{\infty})v$ where $v$ is a global section, and $\deg v = 2$. The connection $\nabla$ is defined by $\nabla v := -v\lambda^{-1}d\lambda$. The isomorphism $\chi : p_{2}^{*}T \tilde{\to} \sigma^{*}T$ is given by $\chi(p_{2}^{*}v) := \theta \sigma^{*}v$. Then the tuple $T(-1) := (T, \nabla, \chi)$ is a rescaling structure. We define

$$T(-k) := \begin{cases} (T(-1)^{\otimes k} & \text{if } k \in \mathbb{Z}_{\geq 0} \\ (T(-1)^{\vee})^{\otimes -k} & \text{if } k \in \mathbb{Z}_{< 0}. \end{cases}$$

For a rescaling structure $H$, we define $H(k) := H \otimes T(k)$.

2.3. Hodge numbers and Hodge-Tate condition for rescaling structures.

2.3.1. Hodge filtrations for rescaling structures. Let us consider the restriction $H|_{\tau = 0} := H/\tau H$. It admits $C_{\theta}^{*}$ action, and hence we can apply the correspondence of [2.1.1] to get the filtration $F_{\bullet}V$ on $V := H|_{\lambda = 1, \tau = 0}$ corresponding to the lattice at $\lambda = 0$.

**Definition 2.7.** Let $(H, \nabla, \chi)$ be a rescaling structure. Then we define

$$f^{p,q}(H) := \dim \text{Gr}_{p}^{F}V^{p+q},$$

where $V^{k}$ is the $k$-th graded part of $V$. 
2.3.2. Weight filtrations for nilpotent rescaling structures. We consider the following condition on rescaling structures.

**Definition 2.8.** A rescaling structure \((\mathcal{H}, \nabla, \chi)\) is called nilpotent if the residue endomorphism \(\text{Res}_{(\tau=0)} \nabla\) on \(\mathcal{H}|_{\tau=0}\) is nilpotent.

By definition, we have the following:

**Lemma 2.9.** \(\mathcal{H}(\ast(\lambda_0))\) is the Deligne lattice of the meromorphic connection \(\mathcal{H}(\ast(\lambda \tau)|_{\lambda=1})\) along the divisor \(|(\tau)|\).

We have a nilpotent endomorphism \(N := (\text{Res}_{(\tau=0)} \nabla)|_{\lambda=1}\) on \(V\), where \(V\) is the fiber of \(\mathcal{H}\) at \((\lambda, \tau) = (1, 0)\). Let \(V^k\) be the fiber of \(\mathcal{H}^k\) at \((\lambda, \tau) = (1, 0)\). The graded piece of \(N\) on \(V^k\) is denoted by \(N_k\). Let \(W^k\) denote the weight filtration of \(N_k\) centered at \(k\), i.e., \(W^k\) is the unique filtration on \(V^k\) with the following properties:

\[
\begin{align*}
N_k (W_i) & \subset W_{i-2} & \text{for all } i \in \mathbb{Z}, \\
N_k^2 : \text{Gr}_{k+j} W^k & \homotopyequivalent \text{Gr}_{k-j} W^k & \text{for all } j \in \mathbb{Z}.
\end{align*}
\]

The induced filtration on \(V\) is simply denoted by \(W\).

**Definition 2.10.** Let \((\mathcal{H}, \nabla, \chi)\) be a nilpotent rescaling structure. We define

\[
h^{p,q}(\mathcal{H}) := \dim \text{Gr}_{2p} W^{p+q}.
\]

2.3.3. Hodge-Tate condition. In [9], a mixed \((\mathbb{Q})\)-Hodge structure \((V_Q, F, W)\) is called Hodge-Tate if the Hodge filtration \(F\) on \(V := V_Q \otimes_{\mathbb{Q}} \mathbb{C}\) and the weight filtration \(W\) satisfy the following:

\[
\begin{align*}
W_{2i+1} &= W_{2i} & \text{for all } i \in \mathbb{Z}, \\
F_{-j} \oplus W_{2j+2} & \homotopyequivalent V & \text{for all } j \in \mathbb{Z}.
\end{align*}
\]

We use the same notation in this paper. Imitating this notion, we define the following:

**Definition 2.11.** Let \((\mathcal{H}, \nabla, \chi)\) be a nilpotent rescaling structure. Let \(F\) and \(W\) be the filtrations on \(V := \mathcal{H}|_{(\lambda, \tau) = (1, 0)}\) defined in [2.3.1] and [2.3.2]. Then \((\mathcal{H}, \nabla, \chi)\) is said to satisfy the Hodge-Tate condition if \((V, F, W)\) satisfies (2.4) and (2.5). A rescaling structure is called of Hodge-Tate type if it satisfies the Hodge-Tate condition.

The following is trivial by definition.

**Lemma 2.12.** If a rescaling structure \((\mathcal{H}, \nabla, \chi)\) satisfies Hodge-Tate condition, then \(h^{p,q}(\mathcal{H}) = h^{p,q}(\mathcal{H})\) for all \(p, q\).

2.4. Hodge-Tate condition implies the speciality. Let \(H = \bigoplus_k H^k\) be a \(\mathbb{Z}\)-graded finitely generated locally free \(\mathcal{O}_{\mathbb{P}^1}(\ast \infty)\) module with a grade-preserving meromorphic flat connection \(\nabla\). We assume that \(\nabla\) has singularity at most \(\{\lambda = 0\}\) in \(\mathbb{C}_\lambda\) and \(\nabla_{\lambda \partial_\lambda}\) \((H) \subset H\). We also assume that \(\nabla\) is regular singular at infinity. Take the Deligne lattice \(U_0 H\) at \(\lambda = \infty\). Let \(N\) be the nilpotent part of \(\text{Res}_{(\lambda = \infty)} \nabla\). Define \(W^k(U_0 H^k|_{\lambda=\infty})\) as the weight filtration of \(N\) centered at \(k\). It induces a filtration \(\mathcal{W} \mathcal{U}(U_0 H|_{\lambda \neq 0})\) of \(\mathbb{Z}\)-graded logarithmic subbundles of \(U_0 H|_{\lambda \neq 0}\).

**Definition 2.13** [25 Definition 3.21]. Let \(H, \nabla, U_0 H\), and \(W^k(U_0 H|_{\lambda \neq 0})\) be as above. We define a vector bundle \(\tilde{H}\) on \(\mathbb{P}^1\) by

\[
\tilde{H}|_{\lambda \neq 0} := \text{Im} \left\{ \bigoplus_{\ell} W_{2\ell}(U_0 H) \otimes \mathcal{O}_{\mathbb{P}^1}(-\ell \cdot \infty) \to U_0 H(\ast \infty) \right\},
\]

and \(\tilde{H}|_{\lambda = \infty} := H\). We call \(\tilde{H}\) a skewed canonical extension of \(H\). The \(\mathbb{Z}\)-graded flat bundle \((H, \nabla)\) is called special if \(\tilde{H}\) is isomorphic to a trivial bundle over \(\mathbb{P}^1\).
Remark 2.14. Our definition of speciality is slightly different from that of [25]. This construction of $\hat{H}$ is the same as in [2.1.3] if we take the filtration $G_{\tau}V$ to be $G_{\ell} := W_{2\ell}$.

Proposition 2.15. Let $(\mathcal{H}, \nabla, \chi)$ be a rescaling structure of Hodge Tate type. Then $H_{1} := \mathcal{H}_{|\tau=1}$ is special.

The rest of this section is devoted to prove this proposition.

2.4.1. Regular singularity along $|(\lambda)_{\infty}|$. Let $(\mathcal{H}, \nabla, \chi)$ be a rescaling structure. Put $S^{*} := \mathbb{C}^{*} \times \mathbb{C}^{*} \subset S$. Let $\iota : S^{*} \hookrightarrow \mathbb{C}^{*} \times S$ be the embedding given by $\iota(\lambda, \tau) := (\lambda^{-1} \tau^{-1}, \lambda, \tau)$. We observe that $\iota_{\sigma} := \sigma \circ \iota$ gives $\iota_{\sigma}(\lambda, \tau) = (\tau^{-1}, \lambda^{-1})$ and $\iota_{p} := p_{2} \circ \iota$ is the inclusion $S^{*} \hookrightarrow S$. Hence we have the isomorphism

$$
\iota^{*} \chi : \iota_{p}^{*} \mathcal{H} = \mathcal{H}_{|S^{*}} \xrightarrow{\sim} \iota_{\sigma}^{*} \mathcal{H}.
$$

We also remark that $\iota_{\sigma}$ extends to the map $S \setminus \{(\lambda)_{0}\} \to S$ given by $(\lambda, \tau) \mapsto (\tau^{-1}, \lambda^{-1})$, which is denoted by $\tau_{\sigma}$.

Lemma 2.16. The meromorphic connection $(\mathcal{H}((\lambda)_{0}), \nabla)$ is regular singular along $|(\lambda)_{\infty}|$.

Proof. The isomorphism (2.6) gives a logarithmic extension $\hat{\mathcal{H}}$ of $\mathcal{H}_{|\tau \neq 0}$ along $|(\lambda)_{\infty}|$. The pull back $\tau_{\sigma}^{*} \hat{\mathcal{H}}$ is isomorphic to $\mathcal{H}_{|S \setminus \{(\lambda)_{0}\}}$. □

2.4.2. Deligne lattice. Since $\mathcal{H}((\lambda)_{0})$ is regular singular along $|(\lambda)_{\infty}| \cup |(\tau)_{0}| \subset S$, we have the Deligne lattice $U_{0}\mathcal{H}$ of $\mathcal{H}((\lambda)_{0})$ along $|(\lambda)_{\infty}| \cup |(\tau)_{0}|$. Assume that $\mathcal{H}$ is nilpotent. Then $U_{0}\mathcal{H}_{|\tau=0}$ is equal to $\mathcal{H}((\lambda)_{0})_{|\tau=0}$ by Lemma 2.19. In particular, we have $U_{0}\mathcal{H}_{|(\lambda, \tau)=(1,0)} = V$. By (2.6), we have that the residue endomorphism $N := \text{Res}_{0} \nabla$ on $U_{0}\mathcal{H}_{|\lambda=\infty}$ is nilpotent. We have the weight filtration $^{\dagger}W$ on degree $k$ part of $U_{0}\mathcal{H}_{|\lambda=\infty}$ with respect to $N$ centered at $k$. Let $W$ be the resulting filtration on $U_{0}\mathcal{H}_{|\lambda=\infty}$. Then we have logarithmic $\mathcal{O}_{S}((\lambda)_{0})$-submodules $W_{\bullet}(U_{0}\mathcal{H})$ of $\mathcal{H}((\lambda)_{0})$ which coincide with $W_{\bullet}U_{0}\mathcal{H}_{|\lambda=\infty}$ on $\lambda = \infty$. Define $\hat{\mathcal{H}}$ by

$$
\hat{\mathcal{H}}_{|\lambda \neq 0} := \text{Im} \left( \bigoplus_{\ell} W_{2\ell}(U_{0}\mathcal{H}) \otimes \mathcal{O}_{S}(-\ell(\lambda)_{\infty}) \to \mathcal{H}((\lambda)_{0})_{|\lambda \neq 0} \right),
$$

and $\hat{\mathcal{H}}_{|\lambda = 0} = \mathcal{H}$. It is easy to see that $\hat{\mathcal{H}}_{|\tau=1}$ is $\hat{\mathcal{H}}_{1}$.

Lemma 2.17. The filtration on $V$ induced by $W_{\bullet}U_{0}\mathcal{H}$ is equal to the weight filtration given in (2.3.2)

Proof. Let $T_{1}$ be the monodromy around $\{\lambda = \infty\}$ acting on $V' := \mathcal{H}_{|(\lambda, \tau)=(1,1)}$. Let $T_{2}$ be the monodromy around $\{\tau = 0\}$ acting on $V'$. By the $\mathbb{C}^{*}$-equivariance of $\mathcal{H}$ (or, by (2.6)), $N^{(1)} := \log T_{i}$ ($i = 1, 2$) coincide with each other (both of them are nilpotent). We have a trivialization $(U_{0}\mathcal{H}, \nabla) \simeq (V' \otimes \mathcal{O}_{S}((\lambda)_{0}), \nabla^{\prime})$, where $\nabla^{\prime} = d - N^{(1)} \lambda^{-1} d\lambda + N^{(2)} \tau^{-1} d\tau$. Identify $V$ and $V'$ via this isomorphism. Then the filtration induced by $W_{\bullet}U_{0}\mathcal{H}$ corresponds to the filtration induced by $N^{(1)}$, and the filtration given in (2.3.2) corresponds to the filtration induced by $N^{(2)}$. Since $N^{(1)} = N^{(2)}$, these filtrations are equal. □

2.4.3. Proof of the Proposition 2.15. Put $\hat{H}_{0} := \hat{\mathcal{H}}_{|\tau=0}$. By Lemma 2.17 and Lemma 2.19, $\hat{H}_{0}|_{\lambda \neq 0}$ is given by construction in (2.1.1) taking $G_{\ell} = W_{2\ell}$ ($\ell \in \mathbb{Z}$). Then the Hodge-Tate condition implies the triviality of $\hat{H}_{0}$. By the rigidity of triviality of vector bundles on $\mathbb{P}^{1}$, there is an open neighborhood $U$ in $\mathbb{C}_{\tau}$ such that the restriction $\hat{\mathcal{H}}_{|\tau \neq 0}$ is trivial along $\mathbb{P}^{1}_{\lambda}$. Using the $\mathbb{C}^{\ast}_{\tau}$-action, we can show that $\hat{\mathcal{H}}$ itself is trivial along $\mathbb{P}^{1}_{\lambda}$. In particular, $\hat{H}_{1}$ is trivial. □
3. Landau-Ginzburg models

In this section, we consider the following pair \((X, f)\), referred to as a Landau-Ginzburg model:

- A smooth projective variety \(X\) of dimension \(n\) over \(\mathbb{C}\).
- A flat projective morphism \(f: X \to \mathbb{P}^1\) of varieties.

We also consider \(f\) as a meromorphic function on \(X\). We assume that the pole divisor \((f)_{\infty}\) of \(f\) is reduced. The support \(|(f)_{\infty}|\) is denoted by \(D\). We also assume that \(D\) is simple normal crossing. Put \(Y := X \setminus D\). The restriction of \(f\) to \(Y\) is denoted by \(w\).

3.1. Rescaling structure for Landau-Ginzburg models

3.1.1. The Kontsevich complex. Let \(df : \Omega^k_X(\log D) \to \Omega^{k+1}_X(\log D)(D)\) be a morphism induced by the multiplication of \(df\). The inverse image of \(\Omega^{k+1}_X(\log D) \subset \Omega^{k+1}_X(\log D)(D)\) is denoted by \(\Omega^k_f\).

The multiplication \(df\) induces a morphism \(df : \Omega^k_f \to \Omega^{k+1}_f\). The exterior derivative \(d\) induces a morphism \(d : \Omega^k_f \to \Omega^{k+1}_f\).

Let \(\pi_S : S \times X \to X\) be the projection. Recall that \(S = \mathbb{P}^1 \times \mathbb{C}_\tau\). Put \(\Omega^k_{f,\lambda,\tau} := \pi_S^{-1}\Omega^k_f \otimes \lambda^{-k}\mathcal{O}_{S \times X}((\lambda)_{\infty})\). We have morphisms of sheaves \(d + \lambda^{-1}\tau df : \Omega^k_{f,\lambda,\tau} \to \Omega^{k+1}_{f,\lambda,\tau}\) where \(d\) is the relative exterior derivative, i.e., \(d = d_{S \times X/S}\). Since \((d + \lambda^{-1}\tau df)^2 = 0\), we have a complex \((\Omega^k_{f,\lambda,\tau}, d + \lambda^{-1}\tau df)\).

**Definition 3.1.** Let \(p_S : S \times X \to S\) denote the projection. For each \(k \in \mathbb{Z}\), we put

\[
\mathcal{H}^k_f := \mathbb{R}^k p_S^* (\Omega^k_{f,\lambda,\tau}, d + \lambda^{-1}\tau df). \tag{3.1}
\]

We define a \(\mathbb{Z}\)-graded \(\mathcal{O}_S((\lambda)_{\infty})\)-module by \(\mathcal{H}_f := \bigoplus_{k \in \mathbb{Z}} \mathcal{H}^k_f\).

3.1.2. The rescaling structure. Let \(\sigma : \mathbb{C}^*_\theta \times S \to S\) denote the action of \(\mathbb{C}^*_\theta\) given in [22]. Let \(\tilde{\sigma} : \mathbb{C}^*_\theta \times S \times X \to S \times X\) be the action induced by \(\sigma\) and trivial \(\mathbb{C}^*_\theta\)-action on \(X\). Let \(\tilde{p}_2 : \mathbb{C}^*_\theta \times S \times X \to S \times X\) denote the projection. We have the natural isomorphism \(\tilde{\chi}_f : \tilde{p}_2^* (\Omega^k_{f,\lambda,\tau}, d + \lambda^{-1}\tau df) \cong \tilde{\sigma}^* (\Omega^k_{f,\lambda,\tau}, d + \lambda^{-1}\tau df)\). It induces an isomorphism \(\chi_f : p_2^* \mathcal{H}_f \cong \sigma^* \mathcal{H}_f\) with the cocycle condition (Definition [2.5] (3)).

**Proposition 3.2.** The pair \((\mathcal{H}_f, \chi_f)\) comes equipped with a rescaling structure.

**Proof.** By the theorem of Esnault-Sabbah-Yu, M. Saito, and M. Kontsevich [17] (see also [25], [27]), \(\mathcal{H}_f\) is locally free over \(\mathcal{O}_S((\lambda)_{\infty})\). Moreover, [27] Theorem 3.5 (see also its consequences in [27] [3.1.8]) implies that we have a connection \(\nabla\) on each \(\mathbf{H}^k_f\) with the properties in Definition [2.5] \(\square\).

3.1.3. Hodge filtration. Since \(\mathcal{H}_f\) is a rescaling structure, \(V_f := \mathcal{H}_{f,(\lambda,\tau) = (1,0)}\) is equipped with a filtration \(F_* V_f\) (See [2.3.1]). Note that \(V_f \cong \mathbb{H}^*(X, (\Omega^*_f, d))\).

**Lemma 3.3 ([17], [27]).** Let \(F_*(\Omega^*_f, d)\) be the stupid filtration on \((\Omega^*_f, d)\), that is, we put \(F_{-p} \Omega^k_f = 0\) for \(p > k\) and \(F_{-p} \Omega^k_f = \Omega^k_f\) for \(p \leq k\). Then we have the following:

\[
F_{-p} V^k_f \cong \mathrm{Im} (\mathbb{H}^k(X, (\mathbb{F}_f, d)) \to \mathbb{H}^k(X, (\Omega^*_f, d))). \tag{3.2}
\]

**Proof.** Let \(\pi_\lambda : \mathbb{C}_\lambda \times X \to X\) be the projection. Define \(\Omega^k_{f,\lambda} := \pi_\lambda^* \Omega^k_f\). Let \(p_\lambda : \mathbb{C}_\lambda \times X \to \mathbb{C}_\lambda\) denote the projection. By the local freeness, we have an \(\mathbb{C}^*_\theta\)-equivariant isomorphism

\[
\mathcal{H}^k_f|_{\tau = 0} \cong \mathbb{R}^k p_\lambda^* (\Omega^*_{f,\lambda}, \lambda d).
\]

The isomorphism \(\chi\) on \(\mathbb{R}^k p_\lambda^* (\Omega^*_{f,\lambda}, \lambda d)\) is induced by \(\theta^p \tilde{\chi}_f|_{\tau = 0} : (\tilde{p}_2^* \Omega^p_{f,\lambda,\tau})|_{\tau = 0} \cong (\tilde{\sigma}^* \Omega^p_{f,\lambda,\tau})|_{\tau = 0}\).

Let \(\mathcal{A}_X^{p, q}\) denote the sheaf of \((p, q)\)-forms on \(X\). Let \(\partial : \mathcal{A}_X^{p, q} \to \mathcal{A}_X^{p+1, q}\) and \(\bar{\partial} : \mathcal{A}_X^{p, q} \to \mathcal{A}_X^{p+1, q}\) be the Dolbeault operators. Set \(\mathcal{A}_X^{p, q} := \Omega^p_f \otimes_{\mathcal{O}_X} \mathcal{A}_X^{q}\). Put \(\mathcal{A}_X^{p, q} := \mathcal{O}_{\mathbb{C}_\lambda \times X} \otimes_{\pi^{-1}_\lambda \mathcal{O}_{\mathbb{C}_\lambda}} \pi^{-1}_\lambda \mathcal{A}_X^{p, q}\). The
operators on $\mathcal{A}^{p,q}_{f,\lambda}$ induced by $\partial$ and $\overline{\partial}$ are denoted by the same notation. Then we obtain the double complex $(\mathcal{A}^{p,q}_{f,\lambda}, \lambda \partial, \overline{\partial})$. Let $(\mathcal{A}^{p,q}_{f,\lambda}, \lambda \partial + \overline{\partial})$ be the total complex. Remark that $\mathcal{A}^{k}_{f,\lambda} = \bigoplus_{p+q=k} \mathcal{A}^{p,q}_{f,\lambda}$. We obtain a $\mathbb{C}^*_\partial$-equivariant quasi-isomorphism
\[(\Omega^{\bullet}_{f,\lambda}, \lambda \partial) \xrightarrow{\sim} (\mathcal{A}^{\bullet}_{f,\lambda}, \lambda \partial + \overline{\partial}),\]
where the isomorphism on $\mathcal{A}^{p,q}_{f,\lambda}$ is induced by $\theta^p (\partial f)_{\tau=0}$. Hence, we have $\mathbb{C}^*_\partial$-equivariant isomorphism:
\[(3.3) \quad \mathbb{R}^k_p \mathcal{A}^{\bullet}_{\lambda, \delta} \simeq \mathcal{H}^{k-\rho}(X, \Omega^\bullet_{\partial}) \ast \mathcal{I}^{\bullet - \rho}(f, \lambda).\]

Applying Lemma 3.2 for $C^{\rho, q} := \Gamma(X, \mathcal{A}^{\rho, q}_{f, \lambda})$, $\delta_1 := \partial$, and $\delta_2 := \overline{\partial}$, the fiber of the cohomology sheaf $\mathcal{H}^{k \rho}_p (\mathcal{A}^{\bullet}_{\lambda, \delta})$ at $\lambda = 1$ has the filtration $G_\bullet$ as in Lemma 3.2. (The fact that we can apply the lemma is due to [17, Theorem 1.3.2]). Since the restriction of (3.3) to $\lambda = 1$ gives a filtered isomorphism $(V^k_f, F) \simeq (H^k(C^{\bullet - \rho}(\partial), G)$, we obtain the conclusion.

By this lemma, we have $Gr^F V^k_f = H^k(X, \Omega^\bullet_{\partial})$. Define $f^{\rho, q}(Y, w) := \dim H^q(X, \Omega^{\bullet - \rho}_{\partial})$. Then we have $f^{\rho, q}(Y, w) = f^{\rho, q}(H_f)$. In the rest of [3] we investigate $H^{\rho, q}(H_f)$, or the weight filtration of the rescaling structure.

3.2. Meromorphic connections for Landau-Ginzburg models. We set $X^{(1)} := \mathbb{C}_\tau \times X$. We also set $D^{(1)} := \mathbb{C}_\tau \times D$. Let $p_{\tau} : X^{(1)} \to \mathbb{C}_\tau$ and $\pi_{\tau} : X^{(1)} \to X$ denote the projections. We shall review some results on a meromorphic flat bundle $\mathcal{M} := \mathcal{O}(\ast D^{(1)}) v$ with $\nabla v = d(\tau f) v$ in [27], where $v$ denotes a global frame. We have
\[\mathcal{M} \simeq \left( \mathcal{O}_{X^{(1)}}(\ast D^{(1)}) + d(\tau f)v \right) ; v \mapsto 1.\]
Remark that, in our case, some of the results in [27] are simplified since we assume that $(f)_{\infty}$ is reduced and the horizontal divisor (denoted by $H$ in [27]) is empty.

3.2.1. $V$-filtration along $\tau$. Regard $\pi_{\tau}^* D_X$ as a sheaf of subalgebra in $D_{X^{(1)}}$. Let $\nabla_0 D_{X^{(1)}}$ denote the sheaf of subalgebra generated by $\pi_{\tau}^* D_X$ and $\nabla \tau$. For $\alpha = 0, 1$, we set
\[U_{\alpha} \mathcal{M} := \pi_{\tau}^* D_X \cdot \mathcal{O}_{X^{(1)}}((\alpha + 1)D^{(1)}) v \subset \mathcal{M}.\]
For $\alpha \in \mathbb{Z}_{\leq 0}$, we set $U_{\alpha} \mathcal{M} := \tau^{-\alpha} U_{0} \mathcal{M}$. For $\alpha \in \mathbb{Z}_{> 0}$, we set $U_{\alpha} \mathcal{M} := \sum_{p+q \leq \alpha} \partial_{\tau} U_{p+q} \mathcal{M}$. Then we have the following:

**Proposition 3.4 ([27] Proposition 2.3).** $U_{\alpha} \mathcal{M}$ is a $V$-filtration on $\mathcal{M}$ along $\tau$ indexed by integers with the standard order (up to shift of degree by 1). More precisely, we have the following:
- $U_{\alpha} \mathcal{M}$ are coherent $\nabla_0 D_{X^{(1)}}$-modules such that $\bigcup U_{\alpha} \mathcal{M} = \mathcal{M}$.
- We have $\tau U_{\alpha} \mathcal{M} \subset U_{\alpha+1} \mathcal{M}$ and $\partial_\tau U_{\alpha} \mathcal{M} \subset U_{\alpha+1} \mathcal{M}$.
- Define $Gr_{\alpha} U_{\alpha} \mathcal{M} := U_{\alpha} \mathcal{M} / U_{\alpha+1} \mathcal{M}$. Then $\tau \partial_\tau + \alpha$ is nilpotent on $Gr_{\alpha} U_{\alpha} \mathcal{M}$. \hfill $\square$

3.2.2. Relative de Rham complexes. We set $\Omega^k_{X^{(1)}} := \pi_{\tau}^* \Omega^k_f$. We obtain a complex $(\Omega^\bullet_{X^{(1)}} + d \tau f)$ where $d = d_{X^{(1)}} / \mathbb{C}_\tau$ is the relative exterior derivative. We have the following:

**Proposition 3.5 ([27]).** We have a quasi-isomorphism of complexes
\[(3.4) \quad (\Omega^\bullet_{X^{(1)}} + d \tau f) \xrightarrow{\sim} U_0 \mathcal{M} \otimes \Omega^\bullet_{X^{(1)} / \mathbb{C}_\tau}.\]

**Proof.** Combine [27] Proposition 2.21 and [27] Proposition 2.22 in the case $\alpha = 0$. \hfill $\square$

As a consequence, we have the following (see also the proof of [27] Corollary 2.23):
Corollary 3.6. We have the following isomorphism of logarithmic connections:
\[
\mathcal{H}_{f|\lambda=1}^k \cong \mathbb{R}^k p_{\tau*} \left( U_0 \mathcal{M} \otimes \Omega^\bullet_{X(1)/C} \right).
\]
We also have a quasi-isomorphism of complexes:
\[
(\Omega^\bullet_f, d) := (\Omega^\bullet_{\tau \cdot}, d + \tau df)|_{\tau = 0} \cong Gr^U_0 \mathcal{M} \otimes \Omega^\bullet_X,
\]
which induces \( V^k_f \cong \oplus^k (X, (\mathcal{O}_X^0 \mathcal{M} \otimes \Omega^\bullet_X)) \). The residue endomorphism on \( V^k_f \) is identified with the nilpotent endomorphism on \( \oplus^k (X, (\mathcal{O}_X^0 \mathcal{M} \otimes \Omega^\bullet_X)) \) associated with \( \varphi_0 \) on \( Gr^U_0 \mathcal{M} \otimes \Omega^\bullet_X \), where \( \varphi_0 \) denotes the endomorphism induced by \( \tau \partial_\tau \).

3.2.3. Residue endomorphisms. We shall give another description of \( \varphi_0 \) in Corollary 3.6. Consider \( \Omega^k_{X(1)}(\log \tau_0) := \mathcal{O}_{\{0\} \times X} \otimes \Omega^1_{X(1)}(\log \tau) \) as an \( \mathcal{O}_X \)-module. It naturally decomposes to the following module:
\[
\Omega^k_X \oplus [\tau^{-1}d\tau] \cdot \Omega^{k-1}_X,
\]
where \([\tau^{-1}d\tau]\) denotes the section induced by \( \tau^{-1}d\tau \).

Since \( U_\alpha \mathcal{M} \) is a \( V_0 \mathcal{D}_{X(1)} \)-module, we have \( \nabla : U_\alpha \mathcal{M} \otimes \Omega^k_{X(1)}(\log \tau) \rightarrow U_\alpha \mathcal{M} \otimes \Omega^{k+1}_{X(1)}(\log \tau) \). This induces
\[
\nabla' : Gr^U_0 \mathcal{M} \otimes \Omega^k_{X(1)}(\log \tau) \rightarrow Gr^U_0 \mathcal{M} \otimes \Omega^{k+1}_{X(1)}(\log \tau) \rightarrow Gr^U_0 \mathcal{M} \otimes \Omega^k_X.
\]

The morphisms
\[
\nabla_0 : Gr^U_0 \mathcal{M} \otimes \Omega^k_X \rightarrow Gr^U_0 \mathcal{M} \otimes \Omega^{k+1}_X \quad \text{and}
\]
\[
\nabla_0 : Gr^U_0 \mathcal{M} \otimes [\tau^{-1}d\tau] \cdot \Omega^{k-1}_X \rightarrow Gr^U_0 \mathcal{M} \otimes [\tau^{-1}d\tau] \cdot \Omega^k_X
\]

induced by \( \nabla' \) are the same as the flat connection \( \nabla_0 \) given by the \( \mathcal{D}_X \)-module structure of \( Gr^U_0 \mathcal{M} \). The morphism \( Gr^U_0 \mathcal{M} \otimes \Omega^k_X \rightarrow Gr^U_0 \mathcal{M} \otimes [\tau^{-1}d\tau] \cdot \Omega^k_X \) induced by \( \nabla' \) is given by \( m \mapsto [\tau^{-1}d\tau] \varphi_0(m) \).

We have the following exact sequence of complexes:
\[
(3.5) \quad 0 \rightarrow Gr^U_0 \mathcal{M} \otimes (\tau^{-1}d\tau) \cdot \Omega^k_X \rightarrow Gr^U_0 \mathcal{M} \otimes \Omega^1_X \rightarrow Gr^U_0 \mathcal{M} \otimes \Omega^k_X \rightarrow 0
\]

From this exact sequence, we obtain a morphism
\[
\varphi_1 : Gr^U_0 \mathcal{M} \otimes \Omega^k_X \rightarrow Gr^U_0 \mathcal{M} \otimes (\tau^{-1}d\tau) \cdot \Omega^k_X \simeq Gr^U_0 \mathcal{M} \otimes \Omega^k_X
\]
in the derived category \( D^b(\mathbb{C}_X) \) of \( \mathbb{C}_X \)-modules.

Lemma 3.7. \( \varphi_0 = \varphi_1 \).

Proof. Let \( C^\bullet(h) \) be the mapping cone of \( h \) in (3.5), that is,
\[
C^k(h) := Gr^U_0 \mathcal{M} \otimes \Omega^{k+1}_{X(1)}(\log \tau) \oplus Gr^U_0 \mathcal{M} \otimes \Omega^k_X , \quad dC^\bullet(h)(a, b) := (-\nabla' a, ha + \nabla_0 b),
\]
where \( a \in Gr^U_0 \mathcal{M} \otimes \Omega^{k+1}_{X(1)}(\log \tau) \) and \( b \in Gr^U_0 \mathcal{M} \otimes \Omega^k_X \). Then the morphism
\[
Gr^U_0 \mathcal{M} \otimes \Omega^k_X \ni \omega \mapsto [\tau^{-1}d\tau] \cdot \omega \in Gr^U_0 \mathcal{M} \otimes \Omega^{k+1}_{X(1)}(\log \tau)
\]
induces a quasi-isomorphism \( \iota_0 : Gr^U_0 \mathcal{M} \otimes \Omega^k_X \rightarrow C^\bullet(h) \). The morphism \( \varphi_1 \) is induced by a natural morphism \( \iota_1 : Gr^U_0 \mathcal{M} \otimes \Omega^k_X \rightarrow C^\bullet(h) \).

Using the identification \( \Omega^k_{X(1)}(\log \tau) = \Omega^k_X \oplus [\tau^{-1}d\tau] \cdot \Omega^{k-1}_X \), we obtain a morphism \( \Omega^k_X \rightarrow \Omega^k_{X(1)}(\log \tau) \) of \( \mathcal{O}_X \)-modules. This morphism induces \( \Psi : Gr^U_0 \mathcal{M} \otimes \Omega^k_X \rightarrow \mathcal{C}^{k-1}(h) \). For a section
Lemma 3.8. Relative cohomology groups for Landau-Ginzburg models. Let $t$ denote a coordinate on the target space of $w : Y \to \mathbb{C}$. Put $s := 1/t$ and let $\mathbb{C}_s \subset \mathbb{P}^1$ be the complex plane with coordinate $s$. Take a sufficiently small holomorphic disk $\Delta_s \subset \mathbb{C}_s$ centered at infinity so that no critical values of $f$ are contained in $\Delta_s^\circ := \Delta_s \setminus \{\infty\}$.

Set $\mathfrak{F} := X \times \Delta_s, \mathfrak{D} := D \times \Delta_s$. Let $\pi_s : \mathfrak{F} \to X$ and $p_s : \mathfrak{F} \to \Delta_s$ be the projections. Put $g := 1/f$. Set $\Gamma := \{(x, s) \in \mathfrak{F} | g(x) = s\}$. The inclusion $\mathfrak{F} \supset \mathfrak{D}$ is denoted by $i_\mathfrak{F}$ and $\mathfrak{D} \cap \Gamma$ is denoted by $\mathfrak{D}_\Gamma$.

3.3.1. De Rham complexes. For $k \in \mathbb{Z}_{\geq 0}$, we have a natural morphism

$$\phi^k : \Omega^k_X (\log \mathfrak{D} \cup \{s = 0\}) \longrightarrow i_{\mathfrak{F}}^* \Omega^k_X (\log \mathfrak{D}_\mathfrak{F}).$$

Let $E^k$ be the kernel of $\phi^k$. This gives a subcomplex $E^\bullet$ of $\Omega^\bullet_X (\log \mathfrak{D} \cup \{s = 0\})$.

**Lemma 3.8.** For each $k$, we have

$$E^k = \left(\frac{ds}{s} - \frac{dg}{g}\right) \cdot \pi_s^* \Omega^{k-1}_X (\log D) \oplus (s - g) \cdot \pi_s^* \Omega^k_X (\log D).$$

In particular, $E^k$ is a locally free $O_X$-module.

**Proof.** It is trivial that the right hand side of (3.6) is included in $E^k$. Let $s^{-1}ds \cdot \omega_1 + \omega_2$ be a section of $E^k$, where $\omega_1 \in \pi_s^* \Omega^{k-1}_X (\log D)$ and $\omega_2 \in \pi_s^* \Omega^k_X (\log D)$. Since $(s^{-1}ds - g^{-1}dg) \cdot \omega_1$ is a (local) section of $E^k$, $g^{-1}dg \cdot \omega_1 + \omega_2$ is a section of $E^k$. We observe that $g^{-1}dg \cdot \omega_1 + \omega_2$ is also a section of $\pi_s^* \Omega^k_X (\log D)$. Since $E^k \cap \pi_s^* \Omega^k_X (\log D) = (g - s)\pi_s^* \Omega^k_X (\log D)$, we obtain that $g^{-1}dg \cdot \omega_1 + \omega_2$ is a section of $(s - g)\pi_s^* \Omega^k_X (\log D)$. This implies that $s^{-1}ds \cdot \omega_1 + \omega_2$ is a section of the right hand side of (3.6). □

3.3.2. Relative de Rham complex. For $k \in \mathbb{Z}_{\geq 0}$, we have a canonical morphism

$$\phi^k : \pi_s^* \Omega^k_X (\log D) \longrightarrow i_{\mathfrak{F}}^* \Omega^k_{\mathfrak{F}/\Delta_s} (\log \mathfrak{D}_\mathfrak{F}).$$

Remark that $\pi_s^* \Omega^k_X (\log D)$ is given by

$$\pi_s^* \Omega^k_{\mathfrak{F}/\Delta_s} (\log \mathfrak{D}_\mathfrak{F}) := \frac{\Omega^k_{\mathfrak{D}_\mathfrak{F}} (\log \mathfrak{D}_\mathfrak{F})}{\Omega^k_{\mathfrak{D}/\Delta_s} (\log s)}.$$

where $pr$ denotes the composition of $i_\mathfrak{F}$ and $p_s$.

**Definition 3.9.** The kernel of the morphism $\phi^k$ is denoted by $E^k$. The induced subcomplex of $\pi_s^* \Omega^k_X (\log D)$ is denoted by $E^\bullet$.

By definition, $E^k_{|X \setminus D} \simeq ((s - g)\pi_s^* \Omega^k_X (\log D))_{|X \setminus D}$.

**Lemma 3.10.** Let $Q$ be a point in $D$. If we take a sufficiently small neighborhood $U$ of $Q$, we have the following:

$$E^k_{\mid U} = \frac{dg}{g} \cdot (\pi_s^* \Omega^{k-1}_X (\log D))_{|U} + ((s - g)\pi_s^* \Omega^{k-1}_X (\log D))_{|U},$$

where $\Omega := U \times \Delta_s$. Moreover, $E^k$ is a locally free $O_X$-module.
Proof. Since the complex $(\Omega_X^k(\log D), g^{-1}dg)$ is acyclic near $Q \in D$, we have a decomposition
\[ \pi^*_s \Omega_X^k(\log D)|_{\Delta_s} = \mathcal{F}^\ell \oplus \mathcal{G}^\ell \]
such that $g^{-1}dg : \mathcal{F}^\ell \to \mathcal{G}^\ell$ (for some $\ell \in \mathbb{Z}_{\geq 0}$) for a sufficiently small neighborhood $U$ of $Q$ (see the proof of [27] Lemma 2.29). We have $E^k|_U \Rightarrow \mathcal{G}^k$, and $E^k|_U \cap \mathcal{F}^k = (s-g)E^k$. The local freeness of $E^k$ and the equation (3.7) are obvious by this description.

By this lemma, the restriction of $E^k$ to $s = 0$ is identified with $(\Omega_f, d)$. Here, we remark that we have
\[ \Omega^k_{f|U} = g : \Omega_X^k(\log D)|_U + df \wedge \Omega^{-1}_X(\log D)|_U \]
for sufficiently small $U$ (see [25] (2.3.1), [27] Lemma 2.29 for example).

3.3.3. Gauss-Manin connection. We have a canonical epi-morphism $\varphi : E^k \to E^k$.

Lemma 3.11. $\ker \varphi = s^{-1}ds \cdot E^{k-1}$.

Proof. $\ker \varphi \supset s^{-1}ds \cdot E^{k-1}$ is trivial. Let $(s^{-1}ds - g^{-1}dg)\omega_1 + (s-g)\omega_2$ be a (local) section of $\ker \varphi$, where $\omega_1 \in \pi^*_s \Omega^k_X(\log D)$ and $\omega_2 \in \pi^*_s \Omega^{-1}_X(\log D)$. We have
\[ g^{-1}dg\omega_1 + (s-g)\omega_2 = 0. \]
Hence, we have
\[ \omega_1 = (s-g)\tau_1 + g^{-1}dg\tau_2, \quad \omega_2 = g^{-1}dg\tau_1 \]
for some $\tau_1 \in \Omega^{k-1}_X(\log D)$ and $\tau_2 \in \Omega^{k-2}_X(\log D)$. We obtain
\[ (s^{-1}ds - g^{-1}dg)\omega_1 + (s-g)\omega_2 = (s^{-1}ds - g^{-1}dg)((s-g)\tau_1 + g^{-1}dg\tau_2) + (s-g)g^{-1}dg\tau_1 = s^{-1}ds(s-g)\tau_1 + (s^{-1}ds - g^{-1}dg)g^{-1}dg\tau_2 = s^{-1}ds((s-g)\tau_1 + g^{-1}dg\tau_2). \]
This implies $\ker \varphi \subset s^{-1}ds \cdot E^{k-1}$.

By this lemma, we have the following diagram, whose rows and columns are exact:

\[
\begin{array}{ccccccc}
0 & \to & s^{-1}ds \cdot E^*[-1] & \to & E^* & \to & E^* & \to & 0 \\
0 & \to & s^{-1}ds \cdot \pi^*_s \Omega^k_X(\log D)[-1] & \to & \Omega^k_X(\log(D \cup \{s = 0\})) & \to & \pi^*_s \Omega^k_X(\log D) & \to & 0 \\
0 & \to & s^{-1}ds \cdot i_{\Gamma_s} \Omega^k_{\Gamma/\Delta_s}(\log D_{\Gamma})[-1] & \to & i_{\Gamma_s} \Omega^k_{\Gamma}(\log D_{\Gamma}) & \to & i_{\Gamma_s} \Omega^k_{\Gamma/\Delta_s}(\log D_{\Gamma}) & \to & 0 \\
0 & \to & 0 & \to & 0 & \to & 0 & \to & 0
\end{array}
\]

From this exact sequence, we obtain a morphism
\[ E^* \to s^{-1}ds \cdot E^* \]
in the derived category $D^b(C_X)$. This gives a logarithmic connection
\[(3.9) \quad \nabla^{GM} : \mathbb{R}^k P_s E^\bullet \rightarrow \mathbb{R}^k P_s E^\bullet \otimes \Omega^1_{\Delta_b}(\log s).\]

On $\Delta^X_b$, the kernel of $\nabla^{GM}$ is the local system of the relative cohomology $H^k(Y_b, b \in \Delta^X_b)$ \((b \in \Delta^X_b)\) \((23)\).

Hence we obtain a morphism $\Omega^\bullet_X(\log D) : (\Omega^\bullet_X(\log D) \otimes s^{-1} ds \otimes \Omega^{k-1}_X(\log D))$.

3.3.4. Residue endomorphisms.

Put $E^\bullet_0 := E^\bullet \otimes \mathcal{O}_{X \times \{0\}}$. The complex $E^\bullet_0$ can naturally be considered as a complex on $X$. The complex $E^\bullet_0$ is a subcomplex of the complex $\Omega^\bullet_X(\log D) \otimes s^{-1} ds \otimes \Omega^{k-1}_X(\log D)$. On $\pi_s(\Gamma)$, we have
\[E^k_0 = g \cdot \Omega^k_X(\log D) \oplus \left( \frac{ds}{s} - \frac{dg}{g} \right) \otimes \Omega^{k-1}_X(\log D).\]

On $X \setminus \pi_s(\Gamma)$, we have
\[E^k_0 = \Omega^k_X(\log D) \oplus \frac{ds}{s} \otimes \Omega^{k-1}_X(\log D).\]

From the exact sequence (3.8), we have the following exact sequence:
\[(3.10) \quad 0 \rightarrow \frac{ds}{s} \otimes (\Omega^\bullet_f, d)[{-1}] \rightarrow E^\bullet_0 \rightarrow (\Omega^\bullet_f, d) \rightarrow 0\]

From this exact sequence, we obtain a morphism
\[\varphi_2 : (\Omega^\bullet_f, d) \rightarrow \frac{ds}{s} \otimes (\Omega^\bullet_f, d) \simeq (\Omega^\bullet_f, d).\]

This induces a residue endomorphism
\[\text{Res}_{s=0}(\nabla^{GM}) : \mathbb{H}^k(X, (\Omega^\bullet_f, d)) \rightarrow \mathbb{H}^k(X, (\Omega^\bullet_f, d))\]

of $\nabla^{GM}$ along $\{s = 0\}$.

3.4. Hodge-Tate conditions for Landau-Ginzburg models.

3.4.1. Comparison of the residue endomorphisms. We shall compare the residue endomorphisms given in 3.2.2 (see also 3.2.3 and 3.3.4).

Put $\Omega^\bullet_X(\log s)_0 := \Omega^\bullet_X(\log s) \otimes \mathcal{O}_{X \times \{0\}}$. Let $[s^{-1} ds]$ denote the section of $\Omega^\bullet_X(\log s)_0$ induced by $s^{-1} ds$. The correspondence $[s^{-1} ds] \leftrightarrow [\tau^{-1} d\tau]$ gives an isomorphism $\Omega^\bullet_X(\log s)_0 \simeq \Omega^\bullet_{X^{(1)}}(\log \tau)_0$. Via this isomorphism, we identify $\Omega^\bullet_X(\log s)_0$ with $\Omega^\bullet_{X^{(1)}}(\log \tau)_0$. Similarly, we identify
\[\Omega^\bullet_X(\log(\mathcal{D} \cup \{s = 0\}))_0 := \Omega^\bullet_X((\log(\mathcal{D} \cup \{s = 0\})) \otimes \mathcal{O}_{X \times \{0\}}\]

with
\[\Omega^\bullet_{X^{(1)}}((\log(D^{(1)} \cup \{\tau = 0\}))_0 := \Omega^\bullet_{X^{(1)}}((\log(D^{(1)} \cup \{\tau = 0\})) \otimes \mathcal{O}_{\{0\} \times X}.\]

By the construction of $U_0M$, we have an inclusion
\[\Omega^k_{X^{(1)}}(\log \tau) \otimes \mathcal{O}_{X^{(1)}}(D^{(1)}) \cdot v \hookrightarrow \Omega^k_{X^{(1)}}(\log \tau) \otimes M.\]

We also have another inclusion
\[\Omega^k_{X^{(1)}}((\log(D^{(1)} \cup \{\tau = 0\})) \cdot v \hookrightarrow \Omega^k_{X^{(1)}}((\log \tau) \otimes \mathcal{O}_{X^{(1)}}(D^{(1)}) \cdot v.\]

Hence we obtain a morphism
\[\Omega^k_X((\log(\mathcal{D} \cup \{s = 0\}))_0 \rightarrow U_0M \otimes \Omega^k_{X^{(1)}}(\log \tau).\]

Since the filtration $U_\bullet M$ is indexed by $\mathbb{Z}$, we have a morphism
\[\Omega^k_X((\log(\mathcal{D} \cup \{s = 0\}))_0 \rightarrow \text{Gr}_0U_\bullet M \otimes \Omega^k_{X^{(1)}}(\log \tau)_0\]
given by \( \eta \rightarrow v \otimes \eta \), where \( v \) denotes the section of \( \mathcal{G}_0^{U} \mathcal{M} \) induced by the global section \( v \) of \( \mathcal{M} \).

By restricting this morphism to \( E_0^k \), we obtain a morphism
\[
\Phi : E_0^k \rightarrow \mathcal{G}_0^{U} \mathcal{M} \otimes \Omega_{X,1}^k (\log \tau)_0.
\]

**Lemma 3.12.** \( \Phi \) defines a morphism of complexes.

**Proof.** Firstly, we verify the lemma on \( \pi_s(\Gamma) \). Since \( f = 1/g \), we have
\[
\nabla'(v) = v \tau dg^{-1} + v \tau g^{-1} [\tau^{-1} d\tau] = v \tau g^{-1} ( [\tau^{-1} d\tau] - g^{-1} dg )
\]
in \( \mathcal{G}_0^{U} \otimes \Omega_{X,1}^k (\log \tau)_0 \). Hence, we have
\[
\nabla'(v) \cdot (- g^{-1} dg + [\tau^{-1} d\tau]) = 0
\]
in \( \mathcal{G}_0^{U} \otimes \Omega_{X,1}^k (\log \tau)_0 \). Since \( v g^{-1} dg \) and \( v \tau^{-1} d\tau \) are sections of \( U_0 \mathcal{M} \otimes \Omega_{X,1}^k (\log \tau) \), we have
\[
g \nabla'(v) = v \tau (- g^{-1} dg + [\tau^{-1} d\tau]) = 0
\]
in \( \mathcal{G}_0^{U} \otimes \Omega_{X,1}^k (\log \tau)_0 \).

Let \((g^{-1}dg - [\tau^{-1}d\tau]) \omega_1 + gw_2 \) be a section of \( E_0^k \), where \( \omega_1 \in \Omega_X^{k-1}(\log D) \) and \( \omega_2 \in \Omega_X^k(\log D) \) (see \([3.3.4]\)). We then obtain
\[
\nabla'(v \cdot (g^{-1}dg - [\tau^{-1}d\tau]) \omega_1)
= \nabla'(v) \cdot (g^{-1}dg - [\tau^{-1}d\tau]) \omega_1 + v \cdot d((g^{-1}dg - [\tau^{-1}d\tau]) \omega_1))
= v \cdot d((g^{-1}dg - [\tau^{-1}d\tau]) \omega_1))
\]
and
\[
\nabla'(v \cdot gw_2) = \nabla'(v) \cdot g \cdot w_2 + vd(g \cdot w_2) = vd(g \cdot w_2).
\]

Hence we have \( \nabla' \circ \Phi = \Phi \circ d \) on \( \pi_s(\Gamma) \).

On \( X \setminus \pi_s(\Gamma) \), \( f = 1/g \) is a holomorphic function. Hence, \( \nabla'(v) = v \tau df + v \tau f \cdot \tau^{-1} d\tau \) is a section of \( \tau U_0 \mathcal{M} \otimes \Omega_{X,1}^k (\log \tau) \). This implies \( \nabla'(v) = 0 \) on \( \mathcal{G}_0^{U} \mathcal{M} \otimes \Omega_{X,1}^k \). Then we can prove \( \nabla' \circ \Phi = \Phi \circ d \) on \( X \setminus \pi_s(\Gamma) \) similarly.

We then obtain the following.

**Theorem 3.13.** The nilpotent endomorphism \( \text{Res}_{(\tau=0)} \nabla \big|_{\lambda=1} \) on \( V_f^k \) coincide with the residue endomorphism of the Gauss-Manin connection \( \nabla_{GM} \) for the relative cohomology group.

**Proof.** By Lemma \([3.12]\) we obtain the following commutative diagram in the abelian category of complexes on \( X \):
\[
\begin{array}{ccccccccc}
0 & \xrightarrow{d} & \frac{ds \cdot (\Omega^*_f, d)[-1]}{\text{qis}} & \xrightarrow{E^*_0} & (\Omega^*_f, d) & \xrightarrow{\Phi} & (\Omega^*_f, d) & \xrightarrow{\text{qis}} & 0 \\
\downarrow{\text{qis}} & & & & & & & & \\
0 & \xrightarrow{\text{qis}} & \mathcal{G}_0^{U} \mathcal{M} \otimes ([\tau^{-1} d\tau] \cdot \Omega^*_X[-1]) & \xrightarrow{\Phi} & \mathcal{G}_0^{U} \mathcal{M} \otimes \Omega^*_X \otimes (\log \tau)_0 & \xrightarrow{\text{qis}} & \mathcal{G}_0^{U} \mathcal{M} \otimes \Omega^*_X & \xrightarrow{\text{qis}} & 0
\end{array}
\]

The rows of this diagram are the exact sequences \(([3.10]) \) and \(([3.5]) \). Left and right columns are the quasi-isomorphisms given in Corollary \([3.6]\). This diagram shows \( \varphi_1 = \varphi_2 \) in the derived category, which implies the Theorem (See Lemma \([3.7]\)).
3.4.2. **Koszul complex.** Let $W_{m} \Omega_{X}^{\ell}(\log D)$ be the weight filtration given by
\[
W_{m} \Omega_{X}^{\ell}(\log D) := \begin{cases} 
\Omega_{X}^{\ell}(\log D) & (m \geq \ell) \\
\Omega_{X}^{\ell-m} \land \Omega_{X}^{m}(\log D) & (0 \leq m < \ell) \\
0 & (m < 0).
\end{cases}
\]
Take the irreducible decomposition $D = \bigcup_{i \in I} D_{i}$. Fix an order of $I$. Note that each $D_{i}$ is a smooth hypersurface in $X$ by the assumption. Put $D(0) := X$, and $D(m) := \bigcup_{I \in \Lambda, |I|=m}(\bigcap_{i \in I} D_{i})$ for $m \in \mathbb{Z}_{>0}$. We have the isomorphism of complexes: $R\text{es}_{m} : \mathcal{V}_{m}^{W} \Omega_{X}^{\bullet}(\log D) \xrightarrow{\sim} a_{m} \omega_{D(m)[1-m]}$, where $a_{m} : D(m) \to X$ denotes the morphism induced by inclusions ([8], [20], [29]).

We recall that the morphism $R\text{es}_{m}$ is locally described as follows. Let $(U; (z_{1}, \ldots, z_{n}))$ be a local coordinate system such that $U \cap D = \bigcup_{1 \leq j \leq k} \{z_{j} = 0\}$. Assume that we have $\{i_{1} < i_{2} < \cdots < i_{k}\} \subset \Lambda$ such that $D_{i_{j}} \cap U = \{z_{j} = 0\}$. For $J = (j_{1}, \ldots, j_{m})$ with $1 \leq j_{1} < j_{2} < \cdots < j_{m} \leq k$, put $D_{J} := \{z_{j_{1}} = \cdots = z_{j_{m}} = 0\}$ and $(z^{-1} dz)_{J} := z_{j_{1}}^{-1} dz_{j_{1}} \wedge \cdots \wedge z_{j_{m}}^{-1} dz_{j_{m}}$. For $\omega \in W_{m} \Omega_{X}^{\ell}(\log D)$, we have a unique expression
\[
\omega = (z^{-1} dz)_{J} \wedge \alpha + \beta,
\]
where $\alpha \in \Omega_{X}^{\ell-m}$, $\beta \in \Omega_{X}^{m}(\log D)$ such that $\beta$ does not have the component $(z^{-1} dz)_{J}$. The residue $R\text{es}_{J} \omega$ is defined by $R\text{es}_{J} \omega := \alpha|_{D_{J}}$, and $R\text{es}_{m}$ is defined by
\[
R\text{es}_{m}(\omega) := \sum_{J \subset \{1, \ldots, k\}, |J|=m} R\text{es}_{J}(\omega).
\]

Let $\mathcal{M}_{X,D}^{\mathbb{R}}$ be the sheaf of invertible sections of $\mathcal{O}_{X}(\ast D)$. We have the morphism $\mathcal{O}_{X} \to \mathcal{M}_{X,D}^{\mathbb{R}}$ given by $h \mapsto \exp(2\pi i h)$, where $i := \sqrt{-1}$. We have the following exact sequence of $\mathbb{Z}_{X}$-modules:
\[
0 \to \mathbb{Z}_{X} \to \mathcal{O}_{X} \xrightarrow{\exp(2\pi i \cdot)} \mathcal{M}_{X,D}^{\mathbb{R}} \xrightarrow{\nu_{D(1)}} a_{1} \mathcal{M}_{D(1)}^{\mathbb{R}} \to 0,
\]
where $\mathbb{Z}_{X}$-module structure of $\mathcal{M}_{X,D}^{\mathbb{R}}$ is given by the multiplication and $\nu_{D(1)}$ denotes taking the valuation along the divisors. The induced morphism $\mathcal{O}_{X} \to \mathcal{M}_{X,D}^{\mathbb{R}} \otimes_{\mathbb{Z}} \mathbb{Q}$ is denoted by $e$. We also have the following exact sequence:
\[
0 \to \mathcal{Q}_{X} \to \mathcal{O}_{X} \xrightarrow{e} \mathcal{M}_{X,D}^{\mathbb{R}} \otimes_{\mathbb{Z}} \mathbb{Q} \to a_{1} \mathcal{Q}_{D(1)} \to 0.
\]
We shall consider the following "Koszul complex" of $e$ ([23], [29]):
\[
K_{m}^{\ell} := \text{Sym}_{\mathbb{Q}}^{m-\ell}(\mathcal{O}_{X}) \otimes_{\mathbb{Q}} \bigwedge^{\ell}(\mathcal{M}_{X,D}^{\mathbb{R}} \otimes_{\mathbb{Z}} \mathbb{Q}).
\]
We have the natural inclusion $K_{m}^{\ell} \hookrightarrow K_{m+1}^{\ell}$ by $h_{1} \cdots h_{m-\ell} \otimes y \mapsto 1 \cdot h_{1} \cdots h_{m-\ell} \otimes y$ and the differential $d : K_{m}^{\ell} \to K_{m+1}^{\ell}$ by
\[
d(h_{1} \cdots h_{m-\ell} \otimes y) := \sum_{i=1}^{m-\ell} h_{1} \cdots h_{i-1} \cdot h_{i+1} \cdots h_{m-\ell} \otimes e(h_{i}) \land y.
\]

**Lemma 3.14** ([23 Proposition 4.3.1.6], [29 Theorem 4.15]).
\[
\mathcal{H}^{q}(K_{m}^{\bullet}) \simeq \begin{cases} 
a_{q} \mathcal{Q}_{D(1)} & \text{for } q \leq p \\
0 & \text{for } q > p. \end{cases}
\]

By this lemma, the natural inclusion $K_{p}^{\bullet} \hookrightarrow K_{p+1}^{\bullet}$ is a quasi-isomorphism for $p \geq n = \dim X$. We put $K_{\infty}^{\bullet} := K_{\bullet}$ and let $W_{m} K_{\infty}^{k}$ be the image of $K_{m}^{k}$ to $K_{n}^{k}$ for $m < n$ and $W_{m} K_{\infty}^{k} := K_{\infty}^{k}$ for $m \geq n$. We obtain a filtered complex $(K_{\infty}^{\bullet}, W)$. 


Theorem 3.15 ([29] Theorem 4.15, Corollary 4.16]). The morphism $K_m^\ell \to W_m \Omega^\ell_X (\log D)$ given by
\[(3.12) \quad h_1 \cdots h_{m-\ell} \otimes y_1 \wedge \cdots \wedge y_\ell \mapsto \frac{1}{(2\pi i)^\ell} \left( \prod_{i=1}^{m-\ell} h_i \right) \cdot \frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_\ell}{y_\ell} \]
induces a filtered quasi-isomorphism $\alpha : (K^\bullet_m, W) \otimes \mathbb{C} \to (\Omega^\bullet_X (\log D), W)$, or an isomorphism in the derived category of filtered complexes $D^b (\mathbb{F} \mathbb{C}_X)$ [37] §7.1.

Corollary 3.16 ([29] Proposition-Definition 4.11, Corollary 4.17]). Let $F$ be the stupid filtration on $\Omega^\bullet_X (\log D)$. Then the tuple
\[\mathcal{H}dg(X \log D) := ((K^\bullet_\infty, W), (\Omega^\bullet_X (\log D), F, W), \alpha)\]
is isomorphic to the cohomological mixed $\mathbb{Q}$-Hodge complex $((\mathbb{R} J_\bullet \mathbb{Q} Y, \tau_\leq), (\Omega^\bullet_X (\log D), F, W), \alpha')$ on $X$ in [37] (8.1.8)]. Here, $\mathcal{J} : Y \hookrightarrow X$ is the inclusion, $\tau_\leq$ denotes the filtration by truncation functor and $\alpha' : (\mathbb{R} J_\bullet \mathbb{C} Y, \tau_\leq) \to (\Omega^\bullet_X (\log D), W)$ is an isomorphism in $D^+ (\mathbb{F} \mathbb{C}_X)$.

Proof. By Theorem 3.15 we have the following commutative diagram:
\[(3.13) \quad (\mathbb{R} J_\bullet \mathbb{C} Y, \tau_\leq) \xrightarrow{\sim} (\mathbb{R} J_\bullet \Omega^\bullet_X (\log D), \tau_\leq) \xrightarrow{\sim} (\Omega^\bullet_X (\log D), \tau_\leq) \xrightarrow{\sim} (\Omega^\bullet_X (\log D), W)\]
\[\xrightarrow{\mathbb{R} J_\bullet J^{-1} \alpha} \xrightarrow{\alpha} \xrightarrow{\alpha} (K^\bullet_\infty, \tau_\leq) \otimes \mathbb{C} \xrightarrow{\sim} (K^\bullet_\infty, \tau_\leq) \otimes \mathbb{C} \xrightarrow{\sim} (K^\bullet_\infty, W) \otimes \mathbb{C} \]
Here, the arrows $\sim$ and $\uparrow \sim$ denote filtered quasi-isomorphisms. Since the natural morphism
\[(\Omega^\bullet_X (\log D), \tau_\leq) \to (\Omega^\bullet_X (\log D), W)\]
is a filtered quasi-isomorphism, the morphism
\[(K^\bullet_\infty, \tau_\leq) \otimes \mathbb{C} \to (K^\bullet_\infty, W) \otimes \mathbb{C}\]
is also a filtered quasi-isomorphism. Remark that $\alpha'$ is defined by the first row of (3.13), and the second row comes from the following sequence:
\[(3.14) \quad (\mathbb{R} J_\bullet \mathbb{Q} Y, \tau_\leq) \xrightarrow{\sim} \mathbb{R} J_\bullet J^{-1} (K^\bullet_\infty, \tau_\leq) \xrightarrow{\sim} (K^\bullet_\infty, \tau_\leq) \xrightarrow{\sim} (K^\bullet_\infty, W)\]
It follows that (3.14) defines the isomorphism of cohomological mixed Hodge complexes. \hfill \Box

The cohomological mixed Hodge complex $\mathcal{H}dg(X \log D)$ gives a mixed $\mathbb{Q}$-Hodge structure on the cohomology groups $H^k (Y, \mathbb{Q})$, $k \in \mathbb{Z}_{\geq 0}$, which is denoted by $H^k (Y) := (H^k (Y, \mathbb{Q}), F, W)$.

3.4.3. Cohomological mixed Hodge complex. Put $\tilde{A}^{p,q} := \Omega^{p,q}_X (\log D)/W_{q-1} \Omega^{p+q}_X (\log D)$ and $\tilde{C}^{p,q} := (K^p_\infty/W_{q-1} K^{p+q}_\infty) (q)$, where $p, q \in \mathbb{Z}_{\geq 0}$ and $(q)$ denotes the Tate twist. We have the following differentials
\[
\delta' : \tilde{A}^{p,q} \to \tilde{A}^{p+1,q} : [\eta \mod W_{q-1}] \mapsto [d\eta \mod W_{q-1}],
\delta'' : \tilde{A}^{p,q} \to \tilde{A}^{p,q+1} : [\eta \mod W_{q-1}] \mapsto [g^{-1} dg \wedge \eta \mod W_q],
\delta' : \tilde{C}^{p,q} \to \tilde{C}^{p+1,q} : [x \otimes y \mod W_{q-1}] \otimes (2\pi i)^q \mapsto [dx \otimes y \mod W_{q-1}] \otimes (2\pi i)^q,
\delta'' : \tilde{C}^{p,q} \to \tilde{C}^{p,q+1} : [x \otimes y \mod W_{q-1}] \otimes (2\pi i)^q \mapsto [x \otimes g \wedge y \mod W_q] \otimes (2\pi i)^{q+1},
\]
where $\eta \in \Omega^{p,q}_X (\log D)$, $x \in \text{Sym}^k (O_X)$ for $k \geq 0$, and $y \in \Lambda^{p+q}_O (\mathcal{M}^{p,q}_{X,D} \otimes \mathbb{Q})$. The total complex of this double complexes are denoted by $s(\tilde{A}^{\bullet \bullet})$ and $s(\tilde{C}^{\bullet \bullet})$, that is, $s(\tilde{A}^{\bullet \bullet})^k := \bigoplus_{p+q=k} \tilde{A}^{p,q}$ and
\[ \delta := \delta' + \delta'' : s(\tilde{A}^{\bullet \bullet})^k \to s(\tilde{A}^{\bullet \bullet})^{k+1} \] 

is the differential. \( s(\tilde{C}^{\bullet \bullet}) \) is defined similarly. We also have the filtrations

\[
W_r \tilde{A}^{p,q} := W_{r+2q} \Omega_X^{p+q}(\log D)/W_{r-1} \Omega_X^{p+q}(\log D) \subset \tilde{A}^{p,q},
\]

\[
W_r \tilde{C}^{p,q} := (W_{r+2q} K^{p+q}_\infty/W_{r-1} K^{p+q}_\infty)(q) \subset \tilde{C}^{p,q}.
\]

It induces the following filtrations on \( s(\tilde{A}^{\bullet \bullet}) \) and \( s(\tilde{C}^{\bullet \bullet}) \):

\[
W_r s(\tilde{A}^{\bullet \bullet})^k := \bigoplus_{p+q=k} W_r \tilde{A}^{p,q},
\]

and

\[
W_r s(\tilde{C}^{\bullet \bullet})^k := \bigoplus_{p+q=k} W_r \tilde{C}^{p,q}.
\]

We define the filtration \( F \) by

\[
F_{k} s(\tilde{A}^{\bullet \bullet})^k := \bigoplus_{p+q=k} \bigoplus_{\ell \geq -\ell} A^{p,q}.
\]

Since \( \delta'' W_r \subset W_{r-1} \) for the filtrations \( W \) on \( s(\tilde{A}^{\bullet \bullet}) \) and \( s(\tilde{C}^{\bullet \bullet}) \), we obtain the isomorphisms

\[
Gr^{W}_j s(\tilde{A}^{\bullet \bullet}) \approx \bigoplus_{k \geq -j} Gr^{W}_{j+2k} \Omega_X^{\bullet \bullet}(\log D)
\]

and

\[
Gr^{W}_j s(\tilde{C}^{\bullet \bullet}) \approx \bigoplus_{k \geq -j} Gr^{W}_{j+2k} K^{\infty}_\infty(k).
\]

Then the following lemma is trivial by Theorem 3.16.

**Lemma 3.17.** The morphisms \( K^{\infty+q}(q) \to \Omega_X^{p+q}(\log D) \) given by

\[
(\eta_1 \cdot \eta_{n-p-q} \otimes \ldots y_1 \otimes \ldots y_{p+q}) \mapsto \frac{1}{(2\pi i)^p} \prod_{i=1}^{n-p-q} \frac{dy_i}{\eta_i} \wedge \ldots \frac{dy_{p+q}}{y_{p+q}}
\]

induces a filtered quasi-isomorphism \( \alpha_1 : (s(\tilde{C}^{\bullet \bullet}), W) \otimes \mathbb{C} \to (s(\tilde{A}^{\bullet \bullet}), W) \).

**Theorem 3.18** ([23] Theorem 11.22]). The tuple

\[
\psi_g^{\text{Hdg}} := ((s(C^{\bullet \bullet}), W), (s(\tilde{A}^{\bullet \bullet}), F, W), \alpha_0)
\]

is a cohomological mixed \( Q \)-Hodge complex on \( X \), which defines a mixed Hodge structure on the hypercohomology \( \mathbb{H}^k(X, \psi_g(Q_X)) \) of the nearby cycle \( \psi_g(Q_X) \).

The mixed \( Q \)-Hodge structure on the hypercohomology group \( \mathbb{H}^k(X, \psi_g(Q_X)) \) is denoted by \( H^k(Y_{\infty}) \). Define \( \vartheta_\infty : \Omega_X^{\bullet \bullet}(\log D) \to A^{p,0} \) by \( \vartheta_\infty(q) := (-1)^p [g^{-1} \eta \wedge \eta] \mod W_0 \). It induces a morphism of complexes \( \vartheta_\infty : \Omega_X^{\bullet \bullet}(\log D) \to s(\tilde{A}^{\bullet \bullet}) \). Define \( \vartheta_0 : K^{\infty}_\infty \to C^{p,0} \) by \( \vartheta_0(x \otimes y) := (-1)^p [x \otimes \eta \wedge y] \). It induces a morphism of complexes \( \vartheta_0 : K^{\infty}_\infty \to s(C^{\bullet \bullet}) \). By the construction, we have \( \alpha_0 \circ \vartheta_0 = \vartheta_\infty \circ \alpha \).

Hence, we obtain a morphism of cohomological mixed \( Q \)-Hodge complexes \( \vartheta : \text{Hdg}(X \log D) \to \psi_g^{\text{Hdg}} \) \((\text{See } \text{[19] } \S 3.3.4.2) \) for the definition of morphism of cohomological mixed Hodge complex. We have the mixed cone complex \( C(\vartheta) := ((C(\vartheta_\infty), W), (C(\vartheta_0), F, W), \alpha_0) \) \((\text{[14] } \S 3.3.4.2) \). We also have the notion of shift \((\text{[16] } \S 3.3.3.1) \) for cohomological mixed Hodge complexes.

**Proposition 3.19.** The tuple \( \Xi^{\text{Hdg}} := ((s(C^{\bullet \bullet}), W), (s(\tilde{A}^{\bullet \bullet}), W, F), \alpha_1) \) constitutes a cohomological mixed \( Q \)-Hodge complex on \( X \), which is isomorphic to \( C(\vartheta)[-1] \).

**Proof.** The shifted cone \( C(\vartheta_\infty)[-1] \) of \( \vartheta_\infty \) is given by

\[
(C(\vartheta_\infty)[-1])^k = K^{\infty}_k \oplus s(C^{\bullet \bullet})^{k-1}
\]

\[
= \tilde{C}^{k,0} \oplus \bigoplus_{p+q=k-1} C^{p,q} = \bigoplus_{p+q=k} \tilde{C}^{p,q}.
\]

The differential \( d : \bigoplus_{p+q=k} \tilde{C}^{p,q} \to \bigoplus_{p+q=k+1} \tilde{C}^{p,q} \) of \( C(\vartheta_\infty)[-1] \) is given by \( d_{|\tilde{A}^{p,q}} = -\delta \) for \( q > 0 \), and \( d_{|\tilde{A}^{p,0}} = \delta' + (-1)^{p+1} \delta'' \). The isomorphism \( h_0 : s(\tilde{C}^{\bullet \bullet}) \to C^{\bullet \bullet}(\vartheta_\infty)[-1] \) is given by
Remark that \( h_{q|C^p.0} := \text{id}_{C^p.0} \) and \( h_{q|C^p.s+1} = (-1)^{p+q} \text{id}_{C^p.s+1} \). The weight filtration on \( C(\vartheta_Q)[-1] \) is given by

\[
W^k(C^k(\vartheta_Q)[-1])^k = W^kH^k \oplus W^l_{s(\C^{\bullet \bullet})}^{k-1}
\]

This shows the compatibility of the weight filtrations. Similar argument can be applied to \( C(\vartheta_C) \). The compatibility of Hodge filtration \( F \) can easily be checked. Let \( h_C : s(\A^{\bullet \bullet}) \to C(\vartheta_C) \) be the isomorphism defined by the same way as \( h_Q \). It can also be checked that

\[
(h_Q[-1] \circ (h_Q \otimes \text{id}_C) = h_C \circ \alpha_1.
\]

Remark that \( \alpha_\sigma : C(\vartheta_Q) \otimes C \to C(\vartheta_C) \) is denoted by \( \alpha_\sigma(x, y) := (\alpha x, \alpha y) \) for \( x \in K^{k+1} \otimes \C \), \( y \in s(\C^{\bullet \bullet})^k \otimes \C \). This proves the proposition.

The mixed Hodge complex \( \Xi_{Qg}^{\text{Hdg}} \) defines a mixed Hodge structure on \( \mathbb{H}^k(X, s(\C^{\bullet \bullet})) \), which we denote by \( H^k(Y, Y_{\infty}) = (H^k(Y, Y_{\infty}; Q), F, W) \).

**Corollary 3.20** ([29]). We have the following long exact sequence of mixed Hodge structures:

\[
\cdots \to H^{k-1}(Y_{\infty}) \to H^k(Y, Y_{\infty}) \to \mathcal{H}^k(Y) \to H^k(Y_{\infty}) \to \cdots
\]

**Proof.** Apply [29, Theorem 3.22 (2)] to the cone \( C(\vartheta) \). □

**3.4.4. Monodromy weight filtration.** Let \( \nu : \tilde{\A}^{p,q} \to \tilde{\A}^{p-1,q+1} \) be the morphism given by \( \nu([\eta \mod W_{q-1}]) := [\eta \mod W_q] \). It induces a nilpotent endomorphism on \( s(\A^{\bullet \bullet}) \), which is also denoted by \( \nu \). It can easily be observed that \( \nu(W_i) \subset W_{i-2} \), and \( \nu(F_i) \subset F_{i+1} \). We also define \( \nu : \tilde{C}^{p,q} \to \tilde{C}^{p-1,q+1} \) similarly: \([x \otimes y \mod W_{q-1}] \otimes (2\pi i)^{q-1} \mapsto [x \otimes y \mod W_q] \otimes (2\pi i)^{q-1} \).

Hence we have a morphism \( \nu : H^k(Y, Y_{\infty}) \to H^k(Y, Y_{\infty})(-1) \) of mixed Hodge structures for each \( k \).

The following theorem is proved in [3.4.6].

**Theorem 3.21.** The map \( \nu \) induces isomorphisms

\[
\nu^*: \text{Gr}_{k+r}^W H^k(Y, Y_{\infty}) \xrightarrow{\sim} \text{Gr}_{k-r}^W H^k(Y, Y_{\infty})(-r),
\]

i.e., the weight filtration \( W \) on \( H^k(Y, Y_{\infty}) \) is the monodromy weight filtration of \( \nu \) centered at \( k \).

The way to prove this theorem is essentially the same as in [20, Theorem 5.2]. We remark that \( n \) in [20] corresponds to \( n - 1 \) in this paper.

**3.4.5. Monodromy weight spectral sequence.** By Proposition 3.19, we have the following.

**Corollary 3.22.** The spectral sequence for \( (\mathbb{R} \Gamma(X, s(\C^{\bullet \bullet})), W) \) whose \( E_1 \)-term is given by

\[
E_1^{-r,q+r} = \mathbb{H}^q(Y, X, \text{Gr}_r^W s(\C^{\bullet \bullet}))
\]

degenerates at \( E_2 \)-term. In other words, \( \text{Gr}_r^W \mathbb{H}^q(X, s(\C^{\bullet \bullet})) \) is the cohomology of the complex:

\[
E_1^{-r-1,q+r} \xrightarrow{d_1} E_1^{-r,q+r} \xrightarrow{d_2} E_1^{-r+1,q+r}.
\]

**Proof.** Apply [8 (8.1.9)] to the cohomological mixed \( \Q \)-Hodge complex \( \Xi_{Qg}^{\text{Hdg}} \) on \( X \). □

By Theorem 3.15, we have a quasi-isomorphism \( \text{Gr}_r^W K_\infty^\bullet \simeq a_{m,4} \mathbb{Q} D_{\{m\}}[-m](-m) \). Recall that \( \text{Gr}_j^W s(\C^{\bullet \bullet}) \simeq \bigoplus_{k \geq 0, j} \text{Gr}_{j+2k}^W K_\infty^\bullet(k) \).

Hence,

\[
E_1^{-r,q+r} = \mathbb{H}^q(Y, X, \text{Gr}_r^W s(\C^{\bullet \bullet})) \simeq \bigoplus_{k \geq 0,-r} \mathbb{H}^q(Y, X, \text{Gr}_{r+2k}^W K_\infty^\bullet(k))
\]

\[
\simeq \bigoplus_{k \geq 0,-r} H^{q-r-2k}(D(2k + r); \Q)(-r - k).
\]
Following [20], we put $K_{i,j,k} = H^{i+j-2k+n}(D(2k-i); \mathbb{Q})(i-k)$ for $k \geq 0$, $i$, and $K_{i,j,k} = 0$ otherwise. Then we have $E_1^{r,q+r} = \bigoplus_{k \in \mathbb{Z}} K_{r,q-k,n-k}^{r,q-n,k}$. We also put $E_1^{r,q+r} = E_1^{r,q+r} \otimes \mathbb{R}$, $K_{i,j,k} = K_{i,j,k}^{i,j,k} \otimes \mathbb{R}$, and $K^{i,j} = \bigoplus_k K_{i,j,k}$. The induced morphism $d_1 \otimes \text{id}_{\mathbb{R}}$ is also denoted by $d_1$.

**Proposition 3.23** (cf. [20] Lemma (2.7), Proposition (2.9)). The restriction of $d_1$ to $K^{i,j,k}$ decomposes to $d_1' : K^{i,j,k} \rightarrow K^{i+1,j+1,k}$ and $d_1'' : K^{i,j,k} \rightarrow K^{i+1,j,k+1}$. Moreover, $d_1'$ is the alternating sum of the Gysin map $\gamma_{(2k-i)}$ in [20] (1.3) times $(-1)$, and $d_1''$ is the alternating sum of restriction map $\rho_{(2k-i)}$ in [20] (1.3).

**Proof.** By the definition, $d_1 : E_1^{r,q+r} \rightarrow E_1^{r-1,q+r}$ is induced by the following short exact sequence.

$$0 \rightarrow Gr_{i-1}^W s(\overline{C}^{i,j,k}) \rightarrow W_r s(\overline{C}^{i,j,k})/W_{r-2} s(\overline{C}^{i,j,k}) \rightarrow Gr_{i}^W s(\overline{C}^{i,j,k}) \rightarrow 0.$$ 

We shall compute the complex version of $d_1$ using Dolbeault resolution, and then observe the compatibility with the rational structure.

Let $\mathcal{O}_{X,D}^n := (\Omega^n_X/(\Omega^n_X \otimes O_X / O_X))$ be sheaf of $(p,q)$-forms on $X$. Put $\mathcal{O}_{X,D}^0 := \Omega_X^0 / O_X / O_X$ and $\mathcal{O}_{X,D}^1 := \Omega_X^1 / O_X / O_X$. Let $\delta := \partial + \overline{\partial} : \mathcal{A} \rightarrow \mathcal{A}[+1]$, be the differential ($X = X, or X, D$). We have a resolution $\mathcal{A}(d)$ compatible with the filtrations. Put $\mathcal{O}_{X,D}^p := \mathcal{O}_{X,D}^{p+1}/W_{-1}^{p+1} \mathcal{O}_{X,D}^{p+1}$ and define $\delta' : \mathcal{A} \rightarrow \mathcal{A}[+1]$. Let $\delta'' : \mathcal{A} \rightarrow \mathcal{A}[+1]$ be the induced morphism. We can take the quasi-isomorphism $s(\mathcal{A}^{i,j,k}) \simeq s(\mathcal{A}^{i,j,k})$ compatible with the filtrations.

Let $d_1' \equiv [x] \in H^q(X,Gr_{r+2k}^W \Omega_X^r(\log D)) \subset H^q(X,Gr_{r+2k}^W s(A^{i,j,k}))$. Since we have the isomorphism $H^q(X,Gr_{r+2k}^W \Omega_X^r(\log D)) \simeq H^q(\Gamma(X,Gr_{r+2k}^W \Omega_X^r(\log D)))$, we can take a representative $x \in \Gamma(X,Gr_{r+2k}^W \Omega_X^r(\log D))$ with $0 = dx \in \Gamma(X,Gr_{r+2k}^W \Omega_X^r(\log D))$. Take a lift $\overline{x} \in \Gamma(X,Gr_{r+2k}^W \Omega_X^r(\log D)) = \Gamma(X,Gr_{r+2k}^W \Omega_X^r(\log D))$. We have $\delta'' \overline{x} \in \Gamma(X,\Omega_{r+2k}^r(\log D))$. Since $dx = 0$, we have $\delta'' \overline{x} = \delta' \overline{x}$. We obtain that

$$d_1'[x] = [\delta' \overline{x}] + [\delta'' \overline{x}] \in H^{q+1}(X,Gr_{r+2k}^W \Omega_X^r(\log D)) \subset H^{q+1}(X,Gr_{r+2k}^W s(A^{i,j,k})) $$

Defining $d_1''[x] := [\delta'' \overline{x}]$, we have the decomposition $d_1' = d_1'' + d_1''$. By the construction, $d_1'' : H^q(X,Gr_{r+2k}^W \Omega_X^r(\log D)) \rightarrow H^{q+1}(X,Gr_{r+2k}^W \Omega_X^r(\log D))$ is induced by the short exact sequence

$$0 \rightarrow Gr_{r+2k}^W \Omega_X^r(\log D) \rightarrow W_{r+2k}^r \Omega_X^r(\log D) \rightarrow Gr_{r+2k}^W \Omega_X^r(\log D) \rightarrow 0.$$ 

The differential $d_1'' : H^q(X,Gr_{r+2k}^W \Omega_X^r(\log D)) \rightarrow H^{q+1}(X,Gr_{r+2k}^W \Omega_X^r(\log D))$ is induced by $g^{-1}dg : Gr_{r+2k}^W \Omega_X^r(\log D) \rightarrow Gr_{r+2k}^W \Omega_X^r(\log D) \quad (m \geq 0)$.

In [20], it is shown that $\text{Res}_{r+2k-1} \circ d_1'' = \gamma^{(r+2k)} \circ \text{Res}_{r+2k}$ and $\text{Res}_{r+2k} \circ d_1'' = \rho_{(r+2k)} \circ \text{Res}_{r+2k}$ holds, where $\gamma^{(m)} : \Omega^k(D(m); \mathbb{C}) \rightarrow \Omega^{k+m}(D(m); \mathbb{C})$ denotes the (alternating sum of) Gysin map and $\rho_{(m)} : \Omega^k(D(m); \mathbb{C}) \rightarrow \Omega^{k}(D(m+1); \mathbb{C})$ denotes the (alternating sum of) restriction [20] (1.3). It is also shown that similar commutativity holds for rational cohomology ([20] (1.8),(2.9)). Hence, we obtain the conclusion. \(\square\)

The morphism $\nu : s(\overline{C}^{i,j,k}) \rightarrow s(\overline{C}^{i,j,k})(-1)$ induces morphisms $\nu : K^{i,j,k} \rightarrow K^{i+2, j,k+1}(-1)$, which is identity whenever $k \geq 0, i$. Hence, we obtain:
Lemma 3.24 ([20] Lemma (2.7), Proposition (2.9) [29] Proposition 11.34).

1. For all \( i \geq 0 \), \( \nu \) induces an isomorphism \( \nu^i : K^{-i,j} \xrightarrow{\sim} K^{i,j}(-i) \).
2. \( \text{Ker}(\nu^{i+1}) \cap K^{-i,j} = K^{-i,j,0} \).

3.4.6. Polarized Hodge-Lefschetz modules. We shall use the Guillén-Navarro Aznar’s formulation [20 §4] of the result of Saito [35] and Deligne on the Hodge-Lefschetz modules. Let \( \bigoplus_{i,j} L^{i,j} \) be a bi-graded finite dimensional \( \mathbb{R} \)-vector space. Let \( \ell_1, \ell_2 \) be endomorphisms on \( L \) such that \( \ell_1(L^{i,j}) \subset L^{i+2,j}, \ell_2(L^{i,j}) \subset L^{i,j+2} \), and \( [\ell_1, \ell_2] = 0 \). The tuple \( (\ell_1, \ell_2) \) is called Lefschetz module if \( \ell_1 : L^{-i,j} \to L^{i,j} \) are isomorphisms for all \( i > 0 \) and \( \ell_2 : L^{-i,j} \to L^{i,j} \) are isomorphisms for all \( j > 0 \). A Lefschetz module \( (L^{i,j}, \ell_1, \ell_2) \) is called Hodge-Lefschetz module if every \( L^{i,j} \) has real Hodge structure and \( \ell_1, \ell_2 \) are morphisms of real Hodge structures of some types ([19 (1.2)], or [10 Definition 7.22]).

A polarization \( \psi \) of a Hodge-Lefschetz module \( (L^{i,j}, \ell_1, \ell_2) \) is a morphism of real Hodge structures \( \psi : L^{i,j} \otimes L^{i,j} \to \mathbb{R} \) of certain type with the following properties:

\[
\begin{align*}
\text{(P1)} & \quad \psi(\ell_1 x, y) + \psi(x, \ell_2 y) = 0 & \text{for } i = 1, 2 \text{ and} \\
\text{(P2)} & \quad \psi(\gamma x, \ell_1^2 C) - \psi(\gamma x, 1) & \text{is symmetric positive definite on } L_0^{-i,-j} \cap \text{Ker}(\ell_1^{i+1}) \cap \text{Ker}(\ell_2^{j+1}).
\end{align*}
\]

Here, \( C \) denotes the Weil operator. The tuple \( (L^{i,j}, \ell_1, \ell_2, \psi) \) of Hodge-Lefschetz module and its polarization is called polarized Hodge-Lefschetz module.

A differential \( d \) on a polarized Hodge-Lefschetz module \( (L^{i,j}, \ell_1, \ell_2, \psi) \) is a morphism of real Hodge structures \( d : L^{i,j} \to L^{i,j} \) of certain type such that

\[
\begin{align*}
\text{(D1)} & \quad d(L^{i,j}) \subset L^{i+1,j+1} & \text{for } i, j \in \mathbb{Z}, \\
\text{(D2)} & \quad d^2 = 0, \\
\text{(D3)} & \quad [d, \ell_1] = 0 & \text{for } i = 1, 2, \text{ and} \\
\text{(D4)} & \quad \psi(dx, y) = \psi(x, dy).
\end{align*}
\]

The tuple \( (L^{i,j}, \ell_1, \ell_2, \psi, d) \) is called differential polarized Hodge-Lefschetz module. By definition, \( \ell_i \) defines an endomorphism on the cohomology group \( H^*(L^{i,j}, d) \) for \( i = 1, 2 \), which is denoted by the same notation. We also have a bilinear map on \( H^*(L^{i,j}, d) \), which is also denoted by \( \psi \).

Theorem 3.25 ([20] Theorem (4.5)). Let \( (L^{i,j}, \ell_1, \ell_2, \psi, d) \) be a differential polarized Hodge-Lefschetz module. Then \( (H^*(L^{i,j}, d), \ell_1, \ell_2, \psi) \) is a polarized Hodge-Lefschetz module.

Proof. Fix a Kähler form \( \omega_{\text{Kah}} \) on \( X \). Let \( [\omega_{\text{Kah}}] \in H^2(X; \mathbb{R}) \) be its cohomology class. Cup product with the restriction of the class \( [\omega_{\text{Kah}}] \) to \( H^2(D(2k+i); \mathbb{R}) \) defines mappings \( L : K^{i,j,k} \to K^{i,j+2,k} \) for all \( k \geq 0, i \). Define the linear mapping \( \psi : K^{i,j} \otimes K^{i,j} \to \mathbb{R} \) by

\[
\psi(x, y) := \begin{cases} 
(\varepsilon(i + j - n)/(2\pi i))^{2k+1} \int_{D(2k+i)} x \wedge y & \text{if } x \in K^{-i,j,k}, y \in K^{i,j,k+1} \\
0 & \text{else,}
\end{cases}
\]

where \( \varepsilon(a) := (-1)^a(a-1)/2 \).

Theorem 3.26 (cf. [20] Theorem (5.1)). The tuple \( (K^{i,j}, (2\pi i)^i \nu, L, \psi, d_1) \) is a differential polarized Hodge-Lefschetz module.

Proof. By Lemma 3.24 (\( (2\pi i)^i \nu : K^{-i,j} \to K^{i,j} \)) for \( i > 0 \). By the hard Lefschetz theorem, we also have \( L^i : K^{-i,j} \to K^{i,j} \) for \( j > 0 \). Hence, \( (K^{i,j}, (2\pi i)^i \nu, L) \) is a Hodge-Lefschetz module. Since the trace map and the cup product are the morphisms of Hodge structures, \( \psi \) is a morphism of real Hodge structures. By some direct computations as in [20 Proposition 3.5], we have \( \psi(x, y) = (-1)^n \psi(y, x) \), \( \psi(2\pi i^i \nu x, y) + \psi(x, 2\pi i^i \nu y) = 0 \), \( \psi(Lx, y) + \psi(x, Ly) = 0 \). This proves (P2). By Lemma 3.24 and the last formula in [20 (1.3)], we also have \( \psi(d_1 x, y) = \psi(x, d_1 y) \). It follows that \( \psi(d_1 x, y) = \psi(x, d_1 y) \). This proves (D4). (D1), (D2) are trivial by definition. (D3) follows from Proposition 3.23.
It remains to prove (P1). Put \( K^{-i,-j}_0 := K^{-i,-j} \cap \ker(\wp^{i+1}) \cap \ker(L^{j+1}) \). By the hard Lefschetz theorem and Lemma 3.24, \( K^{-i,-j}_0 \) is the primitive part of \( H^{n-i-j}(D(i); \mathbb{R})(-i) \). If we put \( Q(x, y) := \psi(x, ((2\pi i)^i) \wedge L^j C \gamma) \) for \( x, y \in K^{-i,-j}_0 \), we have
\[
Q(x, y) = \varepsilon(i + j - n) \int_{D(i)} ((2\pi i)^i) \wedge L^j (2\pi i)^i y
\]
Note that \( \xi := (2\pi i)^i x \) and \( \eta := (2\pi i)^i y \) are the element of the primitive part of \( H^{n-i-j}(D(i); \mathbb{R}) \). Since \( L \) is the Lefschetz operator on \( D(i) \), the map \( (\xi, \eta) \mapsto \varepsilon(i + j - n) \int_{D(i)} \xi \wedge L^j \eta \) is positive definite by the classical Hodge-Riemann bilinear relations. This implies (P1). \( \square \)

**Proof of Theorem 3.21.** By Theorem 3.25 and Theorem 3.26, the tuple
\[
(H^*(K^{\bullet, \bullet}, d), (2\pi i)^i)\nu, L, \psi
\]
is a polarized Hodge-Lefschetz module. In particular, \((2\pi i)^i : H^*(K^{\bullet, \bullet}, d_1)^{-i, j} \rightarrow H^*(K^{\bullet, \bullet}, d_1)^{i, j} \) are isomorphisms for \( i > 0 \). By Corollary 3.22, this implies the theorem. \( \square \)

### 3.4.7. Main theorem.
We firstly compare the nilpotent endomorphisms in \( 3.4.1 \) with \( \nu \) in \( 3.4.3 \). Recall that the stupid filtration on \( (\Omega^i_f, d) \) was denoted by \( F \) in Lemma 3.3.

**Proposition 3.27.** We have a filtered quasi-isomorphism \( \rho : ((\Omega^i_f, d), F) \sim (s(A^{\bullet, \bullet}), F) \), which is compatible with the nilpotent endomorphisms \( \varphi_2 \) and \( \nu \). In other words, \( \nu \circ \rho = \rho \circ \varphi_2 \) in the derived category.

**Proof.** The morphism \( \rho \) is given by the natural inclusion \( \Omega^i_f \rightarrow \Omega^i_X(\log D) = \widetilde{A}^{p,0} \). It is trivial that \( \rho \) is strictly compatible with \( F \). By \( 3.8 \), we have a short exact sequence
\[
0 \rightarrow \Omega^i_f \rightarrow \Omega^p_X(\log D) \rightarrow \Omega^p_X(\log D) \otimes \mathcal{O}_D \rightarrow 0
\]
By \( 3.8 \), we have an exact sequence
\[
0 \rightarrow \Omega^p_X(\log D) \otimes \mathcal{O}_D \xrightarrow{\theta_p} A^{p,0} \xrightarrow{\delta'} A^{p,1} \xrightarrow{\delta''} \ldots
\]
where \( \theta_p(\eta) := (-1)^p [g^{-1} dg \wedge \eta \mod W_0] \). Hence, we obtain an exact sequence
\[
0 \rightarrow \Omega^i_f \xrightarrow{\rho} \widetilde{A}^{p,0} \xrightarrow{\delta'} \widetilde{A}^{p,1} \xrightarrow{\delta''} \ldots
\]
This implies that \( \rho \) is a filtered quasi-isomorphism.

Take the shifted cone \( B^\bullet := C^\bullet(\nu)[-1] \) of \( \nu \). Define \( \varphi : E^k_0 \rightarrow B^k = s(A^{\bullet, \bullet})^k \oplus s(\widetilde{A}^{\bullet, \bullet})^{k-1} \) as the restriction of the following morphism:
\[
\Omega^k_X(\log(D \cup \{ s = 0 \}))_0 = \Omega^k_X(\log D) \oplus s^{-1} ds \Omega^{k-1}_X(\log D) \oplus \omega_1 + s^{-1} ds \omega_2 \mapsto \omega_1 \oplus \omega_2
\]
\[
\in A^{k,0} \oplus A^{k-1,0} \subset s(A^{\bullet, \bullet})^k \oplus s(\widetilde{A}^{\bullet, \bullet})^{k-1}.
\]
Then, \( \varphi \) gives a morphism of complex. Indeed, it is trivial on \( X \setminus \pi_s(\Gamma) \). On \( \pi_s(\Gamma) \), take a section \( g \omega_1 + (s^{-1} ds - g^{-1} dg) \omega_2 \) of \( E^k_0 \). Remark that \( [g \omega_1 \mod W_0] = 0 \), and \( [dg \wedge \omega_1 \mod W_0] = \ldots \).
\[
[g(g^{-1}dg \wedge \omega_1) \mod W_0] = 0. \quad \text{Then we have}
\]
\[
dg(g\omega_1) = (dg \wedge \omega_1 + gd\omega_1) \oplus 0
\]
\[
= g(d(g\omega_1)),
\]
\[
dg((s^{-1}ds - g^{-1}dg) \cdot \omega_2) = d((-g^{-1}dg\omega_2) \oplus \omega_2)
\]
\[
= (g^{-1}dg\omega_2) \oplus (-d\omega_2, [-g^{-1}dg \wedge \omega_2 \mod W_0])
\]
\[
= (g^{-1}dg\omega_2) \oplus -d\omega_2
\]
\[
= g \circ d((s^{-1}ds - g^{-1}dg)\omega_2).
\]
We obtain the following diagram:
\[
\begin{array}{ccc}
0 & \longrightarrow & s^{-1}ds \cdot (\Omega^*, d)[-1] \\
\downarrow \rho & & \downarrow \rho \\
0 & \longrightarrow & s(\tilde{A}^{\bullet \bullet})[-1] \\
\end{array}
\begin{array}{ccc}
& & \longrightarrow E_0 \longrightarrow (\Omega^*, d) \longrightarrow 0 \\
\end{array}
\begin{array}{ccc}
& & \longrightarrow B^\bullet \longrightarrow s(\tilde{A}^{\bullet \bullet}) \longrightarrow 0 \\
\end{array}
\]

The compatibility with \( \varphi_2 \) and \( \nu \) follows from this diagram. \( \square \)

Combining the Theorem 3.13, Theorem 3.21, and Proposition 3.27, we attain the following main theorem of this paper.

**Theorem 3.28.** The filtrations \( F \) and \( W \) on \( V_f^k \) are identified with the Hodge filtration and the weight filtration on \( H^k(Y, Y_\infty; \mathbb{C}) \). In particular, the rescaling structure \( \mathcal{H}_f \) is of Hodge-Tate type if and only if the mixed Hodge structures \( (H^k(Y, Y_\infty; \mathbb{Q}), F, W) \) are Hodge-Tate for all \( k \). \( \square \)

We also have the equation
\[
(3.16) \quad h^{p,q}(\mathcal{H}_f) = \dim \text{Gr}_W^{p+q}(Y, Y_\infty).
\]

The right hand side of (3.16) is denoted by \( h^{p,q}(Y, w) \) in §1. By Lemma 2.12 and Proposition 2.15 we obtain Theorem 1.1 (1). Theorem 1.1 (2) follows from Theorem 3.28 immediately.

**Remark 3.29.** A similar relation between \( V_f \) and \( H^\bullet(Y, Y_\infty) \) is obtained in [31] Theorem (4.3), Theorem (5.3)] in terms of Hodge modules. However, it is not clear whether the weight filtrations are the same as ours.

By the strictness of the morphisms of mixed Hodge structures [7 Theorem (2.3.5)], we have the following well known fact (see [29] Corollary 3.8, for example):

**Lemma 3.30.** Let \( V^i = (V^i_Q, F, W) \) \((i = 1, 2, 3)\) be mixed \( \mathbb{Q} \)-Hodge structures, where \( V^i_Q \) is the \( \mathbb{Q} \)-vector space, \( F \) is the Hodge filtration on \( V^i_Q := V^i_Q \otimes \mathbb{C} \), and \( W \) is the weight filtration for each \( i \). Assume that we have the following
\[
V^1 \longrightarrow V^2 \longrightarrow V^3
\]
be a exact sequence of mixed \( \mathbb{Q} \)-Hodge structures.

Then, for all \( k, p \in \mathbb{Z} \), the sequences
\[
\text{Gr}_F^{p} \text{Gr}_k^W V^1_c \longrightarrow \text{Gr}_F^{p} \text{Gr}_k^W V^2_c \longrightarrow \text{Gr}_F^{p} \text{Gr}_k^W V^3_c
\]
of complex vector spaces are exact. \( \square \)

Remark that a mixed \( \mathbb{Q} \)-Hodge structure \( V = (V_Q, F, W) \) is Hodge-Tate if and only if
\[
\text{Gr}_F^{p} \text{Gr}_k^W V^i_c = 0
\]
for \( p \neq q \). Then, we immediately have the following:
**Corollary 3.31.** Let $V^i$ be as in Lemma 3.30. If $V^1$ and $V^3$ are Hodge-Tate, then so is $V^2$. □

By the long exact sequence (3.15) of mixed Hodge structures, we have the following:

**Corollary 3.32.** If the mixed Hodge structures $H^k(Y)$ and $H^k(Y_\infty)$ are of Hodge-Tate type for all $k$, then $\mathcal{H}_f$ is of Hodge-Tate type. □

4. **Examples**

In this section, we shall give some examples of Landau-Ginzburg models $(X, f)$ in §3 such that the induced rescaling structures $\mathcal{H}_f$ are of Hodge-Tate type. In §3 we consider the case dim $X = 2$. In §4.2 we consider the case dim $X = 3$.

4.1. **Two dimensional examples.** We shall prove the following:

**Proposition 4.1.** Let $f : X \to \mathbb{P}^1$ be a rational elliptic surface such that $(f)_\infty$ is reduced normal crossing, and $D = |(f)_\infty|$ is a wheel of $d$ smooth rational curves for $2 \leq d \leq 9$. Then the rescaling structure $\mathcal{H}_f$ of $(X, f)$ is of Hodge-Tate type.

**Proof.** Since $X$ is a rational surface, we have $h^{p,q}(X) = 0$ for $p \neq q$. Since $D$ is a wheel of $d$ rational curves, the (co)homology of $D$ is of Hodge-Tate type (see [29] Example 5.34) for example). We have the exact sequence of mixed Hodge structures [8, (9.2.1.2)]:

$$\cdots \to H^k(X) \to H^k(Y) \to H^{k-1}(D)(-1) \to \cdots.$$  

By Corollary 3.31 it follows that $H^k(Y)$ are Hodge-Tate for all $k$. By the Clemens-Schmid exact sequence [19] (10.14), Theorem (10.16)], we have the following exact sequence of mixed Hodge structures:

$$H^k(D) \to H^k(Y_\infty) \mathcal{N} \to H^k(Y_\infty)(-1) \to H_{2-k}(D)(-2),$$

where $0 \leq k \leq 2$ and $N$ is the nilpotent endomorphism. Since $H^k(D)$ and $H_{2-k}(D)$ are Hodge-Tate, by Corollary 3.31 we have the exact sequence

$$(4.1) \quad 0 \to A_1 \to H^k(Y_\infty) \to H^k(Y_\infty)(-1) \to A_2 \to 0,$$

where $A_1$ and $A_2$ are Hodge-Tate. Let $p_k(x, y)$ be the Hodge number polynomial of $H^k(Y_\infty)$ (see [29] (II-1), Lemma 2.8, and (III-2)) for example). The exact sequence (4.1) implies that $(1-x^t)\sum p_k(x, y) = \sum a_p x^p y^p$ for some $a_p$. Hence, we have $p_k(x, y) = \sum b_p x^p y^p$ for some $b_p$. Namely, we have that $H^k(Y_\infty)$ is of mixed Hodge Tate for each $k$. By Corollary 3.32 we have the conclusion. □

By Theorem 3.28 Lemma 2.12 Proposition 2.15 3.2, and 3.10, we obtain the following:

**Corollary 4.2.** Let $(X, f)$ be as in Proposition 4.1. Then, we have $f^{p,q}(Y, w) = h^{p,q}(Y, w)$, and $\mathcal{H}_{f|_{\tau=1}}$ is special.

**Remark 4.3.** This example was studied by Auroux-Katzarkov-Orlov [1] as homological mirrors of del Pezzo surfaces. The equality of Hodge numbers $f^{p,q}(Y, w)$ and $h^{p,q}(Y, w)$ was proved by Lunts-Przyjalkowski [26] who directly computed both of the numbers (The number $f^{p,q}(Y, w)$ was also computed in Harder’s thesis [21]). Here, we gave a more conceptual proof of the equality. To the best of the author’s knowledge, the speciality of $\mathcal{H}_{f|_{\tau=1}}$ was not known.

4.2. **Three dimensional examples.** We consider toric Landau-Ginzburg models considered in Harder’s thesis [21].
4.2.1. Fano polytope. Let $M$ be a free Abelian group of rank 3. Put $M_{\mathbb{R}} := M \otimes \mathbb{R}$, and $N := \text{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$. We have the natural pairing $\langle \cdot, \cdot \rangle : M \times N \to \mathbb{Z}$. Define $N_{\mathbb{R}}$ similarly. We consider an integral polytope $P$ with the following properties:

(a) There is a finite set $\{ u_F | F \text{ is facet of } P \}$ of primitive vectors in $N$ indexed by all facets of $P$ such that

$$\begin{align*}
P &= \{ m \in M_{\mathbb{R}} | \langle m, u_F \rangle \geq -1, \text{ for all } F \}, \\
F &= \{ m \in P | \langle m, u_F \rangle = -1 \}.
\end{align*}$$

In particular, the origin $0 \in M$ is contained in the interior of $P$.

(b) For each facet $F$, the set of vertex of $F$ form a basis of $M$. In particular, $F$ is a triangle whose interior does not contain the point of $M$.

Remark 4.4. The condition (a) is called reflexivity. The condition (b) implies that the cone generated by $F$ is smooth. These cones generates a smooth fan, which defines a smooth Fano variety.

4.2.2. Toric varieties. For a face $Q$ of $P$, let $\sigma_Q$ be the cone generated by $\{ u_F | Q \subset F \}$. We remark that $\sigma_P = \{ 0 \}$ since $\{ 0 \}$ is the cone generated by empty set. Then we have a fan $\Sigma_P := \{ \sigma_Q | Q \text{ is a face of } P \}$ (see [B], Theorem 2.3.2), for example). Although this fan is not smooth in general, we have a smooth refinement $\Sigma$ of $\Sigma_P$. Since the dimension of $\Sigma_P$ is 3, the refinement is given by a triangulation of the convex hull of the set $\{ u_F | F \text{ is a facet of } P \}$. In particular, together with the condition (a), we may assume that for every primitive vector $u_\rho$ of a ray $\rho$ in $\Sigma$, we have $\min_{m \in P} \langle m, u_\rho \rangle = -1$. The toric variety corresponding to $\Sigma$ is denoted by $X_\Sigma$. It contains the algebraic torus $T_N = \text{Spec}(\mathbb{C}[M])$ as an open dense subset. Put $D_\Sigma := X_\Sigma \setminus T_N$.

4.2.3. A non-degenerate Laurent polynomial. We consider a Laurent polynomial

$$f_P(\chi) = \sum_{m \in M} c_m \chi^m \in \mathbb{C}[M],$$

where $c_m$ are complex numbers and $\chi^m$ is the monomial corresponding to $m \in M$. The polynomial $f_P$ is considered as an algebraic function on $T_N$. Since $T_N$ is an open dense subvariety of $X_\Sigma$, $f_P$ is considered as a meromorphic function on $X_\Sigma$, whose pole divisor is contained in $D_\Sigma$. We impose the following non-degenerate condition on $f_P$:

(c) The convex hull of $\{ m | c_m \neq 0 \}$ in $M_{\mathbb{R}}$ is $P$.

(d) For every face $Q \subset P$, put $f_Q(\chi) := \sum_{m \in Q} c_m \chi^m$. Then, the intersection of $(df_Q)^{-1}(0)$ and $f_Q^{-1}(0)$ in $T_N$ is empty for every $Q$.

The meaning of the non-degenerate condition considering $f_P$ as a meromorphic connection on $X_\Sigma$ is explained later.

4.2.4. Coordinate system with respect to a cone. Fix an isomorphism $M \cong \mathbb{Z}^3$; $m \mapsto (m_1, m_2, m_3)$. Let $(e_i)_{i=1}^3$ be a canonical base of $M$ via $M \cong \mathbb{Z}^3$. We have an isomorphism $\mathbb{C}[M] \cong \mathbb{C}[x_1^\pm, x_2^\pm, x_3^\pm]$ by $\chi^m \mapsto x_1^{m_1} x_2^{m_2} x_3^{m_3}$. For a maximal cone $\sigma \in \Sigma(3)$, take primitive vectors $u_\rho$ for rays $\rho$ of $\sigma$. Then the open subvariety $U_\sigma = \text{Spec}(\mathbb{C}[\sigma^\vee \cap M])$ of $X_\Sigma$ have coordinate $(y_\rho)_{\rho \in \sigma(1)}$. The relation between the two coordinates is given by $x_i = \prod_{\rho \in \sigma(1)} y_\rho^{(e_i, u_\rho)}$. The function $f_P$ considered as a meromorphic function on $U_\sigma$ is given by

$$f_P(y) = \sum_{m \in P} c_m \prod_{\rho \in \sigma(1)} y_\rho^{(m, u_\rho)}.$$
4.2.5. Pole orders along invariant divisors. For each ray $\rho \in \Sigma(1)$, we have the divisor $D_{\rho}$ invariant under the action of $T_N$. If $\rho \in \sigma(1)$, the intersection $U_{\rho} \cap D_{\rho}$ is given by $\{ y_{\rho} = 0 \}$. Let $Q_{\rho}$ be a face defined by

$$Q_{\rho} := \left\{ m \in \mathbb{P} \left| \langle m, u_{\rho} \rangle = \min_{m' \in \mathbb{P}} \langle m', u_{\rho'} \rangle = -1 \right. \right\}.$$  

Remark that $Q_{\rho} \neq \emptyset$. The equation (4.2) is written as follows:

$$f_{\rho}(y) = y_{\rho}^{-1} \left( y_{\rho} f_{Q_{\rho}}(y) + y_{\rho} \sum_{m \in \mathbb{P}, (m, u_{\rho}) \geq 0} c_m \prod_{\rho' \in \sigma(1)} y_{\rho'}^{\langle m, u_{\rho'} \rangle} \right).$$

Remark that $y_{\rho} f_{Q_{\rho}}(y)$ does not depend on $y_{\rho}$ for all ray $\rho$. The pole order along $D_{\rho}$ is one.

4.2.6. Non-degenerate condition. For a $\tau \in \sigma(2)$, take $\rho, \rho' \in \sigma(1)$ so that $\tau = \rho + \rho'$. Put $Q_{\tau} := Q_{\rho} \cap Q_{\rho'}$. We have

$$f_{\rho}(y) = y_{\rho}^{-1} y_{\rho'}^{-1} \left( y_{\rho} y_{\rho'} f_{Q_{\tau}}(y) + y_{\rho} y_{\rho'} \sum_{m \in \mathbb{P}, (m, u_{\rho}) \geq 0, (m, u_{\rho'}) \geq 0} c_m \prod_{\rho'' \in \sigma(1)} y_{\rho''}^{\langle m, u_{\rho''} \rangle} \right).$$

Note that $y_{\rho} y_{\rho'} f_{Q_{\tau}}(y)$ does not depend on $y_{\rho}$ or $y_{\rho'}$. There is also a similar description of $f_{\rho}$ for the vertex $Q_{\sigma} = \bigcap_{\rho \in \sigma(1)} Q_{\rho}$. From these descriptions, we have the following properties of the zero divisor $(f_{\rho})_0$ in $X_{\Sigma}$:

- The divisor $(f_{\rho})_0$ is a (reduced) smooth hypersurface of $X_{\Sigma}$.
- The fixed points of the action of $T_N$ is not contained in $(f_{\rho})_0$.
- The divisor $D_{\Sigma} \cup (f_{\rho})_0$ is simply normal crossing.

4.2.7. Base locus. Put $B_{\rho} := \{(f_{\rho})_0 \cap D_{\rho}\}$ for all ray $\rho \in \Sigma$.

**Lemma 4.5.** For every $\rho$, $B_{\rho}$ is isomorphic to a projective line.

**Proof.** By the non-degenerateness of $f_{\rho}$, all $B_{\rho}$ are smooth curves in $X_{\Sigma}$. Since $D_{\Sigma} \cup (f_{\rho})_0$ is normal crossing, the intersections of $B_{\rho}$ and the lower dimensional $T_N$-orbits in $D_{\rho}$ are 0-dimensional. Therefore, it is enough to show that the intersection of $(f_{\rho})_0$ and the two dimensional orbit in $D_{\rho}$ is rational.

Take a facet $F \subset \mathbb{P}$ which contains $Q_{\rho}$. By the assumption (a), (b) in \textbf{4.2.2} F is a triangle, whose vertexes $e_1, e_2, e_3$ form a $\mathbb{Z}$-basis of $M$. Using this basis, we take an isomorphism $M \simeq \mathbb{Z}^3$. Let $(x_1, x_2, x_3)$ be the corresponding coordinate as in \textbf{4.2.3}. Put

$$I := \{ i \in \{1, 2, 3\} \mid e_i \text{ is a vertex of } Q_{\rho} \}.$$  

Remark that $I \neq \emptyset$, and $f_{Q_{\rho}} = \sum_{i \in I} e_i x_i \neq 0$.

Take $\sigma \in \Sigma(2)$ so that $\rho \in \sigma(1)$. Let $\rho_1 := \rho, \rho_2, \rho_3$ be the three ray of $\sigma$. Put $y_i := y_{\rho_i}$ for $i = 1, 2, 3$. Then $g := y_1 f_{Q_{\rho}}$ is a Laurent polynomial depending only on $y_2, y_3$. We need to show that $\{(y_2, y_3) \in (\mathbb{C}^*)^2 \mid g(y_2, y_3) = 0 \}$ is rational. This space is isomorphic to the quotient space of $\{(y_1, y_2, y_3) \in (\mathbb{C}^*)^3 \mid f_{Q_{\rho}}(y_1, y_2, y_3) = 0 \}$ by the $\mathbb{C}^*$-action defined by $t \cdot (y_1, y_2, y_3) := (ty_1, ty_2, ty_3)$.

Using the coordinate $(x_1, x_2, x_3)$, the $\mathbb{C}^*$-action is given by $t \cdot (x_1, x_2, x_3) = (t^{-1} x_1, t^{-1} x_2, t^{-1} x_3)$ since $(e_i, u_{\rho_i}) = -1$. We are considering quotient space of $\{(x_1, x_2, x_3) \in (\mathbb{C}^*)^3 \mid \sum_{i \in I} e_i x_i = 0 \}$. Since the quotient of $\{(x_1, x_2, x_3) \in \mathbb{C}^3 \mid \sum_{i \in I} e_i x_i = 0 \}$ by the action defined above is a line in $\mathbb{P}^2$, we obtain the rationality. \qed
4.2.8. Blowing ups. Take an ordering $\Sigma(1) = \{\rho_1, \ldots, \rho_\ell\}$ for the set of all rays in $\Sigma$. We consider the following sequence of blowing ups:

$$X = X^{(0)} \overset{p^{(0)}}{\rightarrow} X^{(1)} \overset{p^{(1)}}{\rightarrow} \cdots \overset{p^{(j)}}{\rightarrow} X^{(j)} = X_\Sigma,$$

where $p^{(j)} : X^{(j)} \rightarrow X^{(j+1)}$ is the blowing up along the strict transform of $B_{\rho_{j+1}}$ in $X^{(j)}$. The composition $X \rightarrow X_\Sigma$ is denoted by $\pi_\Sigma$. The strict transform of $D_{\rho_{j}}$ is denoted by $D_{j}$ ($1 \leq j \leq \ell$).

**Lemma 4.6.** We have the following:

1. The divisor $D_{j}$ is given by the composition of blowing ups of $D_{\rho_{j}}$ along reduced 0-schemes.
2. The union $D := \bigcup_{j} D_{j}$ is simple normal crossing.
3. The pole divisor of $\pi_{\Sigma}^* f_{\rho}$ is reduced and the support $(\pi_{\Sigma}^* f_{\rho})_\infty$ is $D$.
4. The pull back of $f_{\rho}$ by $\pi_{\Sigma}$ gives a well defined morphism $\pi_{\Sigma}^* f_{\rho} : X \rightarrow \mathbb{P}^1$.

**Proof.** Let $\pi^{(i)} : X^{(i)} \rightarrow X^{(0)}$ be the composition $p^{(i-1)} \circ \cdots \circ p^{(0)}$ for $i = 1, 2, \ldots, \ell$. We put $\pi^{(0)} := \text{id}_{X^{(0)}}$. Let $f^{(i)}$ be the pull back of $f_{\rho}$ by $\pi^{(i)}$ for $i = 0, 1, \ldots, \ell$. Let $D^{(i)}_{j}$ (resp. $B^{(i)}_{j}$) denote the strict transform of $D_{\rho_{j}}$ (resp. $B_{\rho_{j}}$) in $X^{(i)}$ for $i, j = 1, 2, \ldots, \ell$. Put $D^{(0)}_{j} := D_{\rho_{j}}$ and $B^{(0)}_{j} := B_{\rho_{j}}$, respectively. We define $D^{(i)} := \bigcup_{j} D^{(i)}_{j}$. We shall prove the following by the induction on $i$:

1. The divisor $D^{(i)}_{j}$ is given by the composition of blowing ups of $D^{(0)}_{j}$ along reduced 0-schemes.
2. The zero divisor $(f^{(i)})_0$ is a reduced smooth hypersurface of $X^{(i)}$, and the union $(f^{(i)})_0 \cup D^{(i)}$ is simple normal crossing.
3. The pole divisor $(f^{(i)})_\infty$ is reduced and the support $(f^{(i)})_\infty$ is $D^{(i)}$.
4. The intersection $(f^{(i)})_0 \cap (f^{(i)})_\infty \cap \left(\bigcup_{j=1}^{\ell} D^{(j)}_{j}\right)$ is empty.

Remark that (1)$_0$, and (4)$_0$ are trivial. We also remark that (2)$_0$ and (3)$_0$ are shown in \textit{4.2.6}. Take $i \in \{1, 2, \ldots, \ell\}$. Assume that (1)$_{i-1}$, (2)$_{i-1}$, (3)$_{i-1}$, (4)$_{i-1}$ holds. Let $Q$ be an arbitrary point in $D^{(i-1)}_{i}$. By the assumption (2)$_{i-1}$, (3)$_{i-1}$, we have a local coordinate system $(U_Q; z_0, z_1, z_2)$ centered at $Q$ with the following properties:

$$D^{(i-1)} \cap U_Q = \bigcup_{i=1}^{k} \{z_i = 0\}, \quad D^{(i-1)} \cap U_Q = \{z_1 = 0\}, \quad f^{(i-1)}_{U_Q}(z) = z_0 \cdot \prod_{i=1}^{k} z_i^{-1},$$

where $k = 1$, or 2. We have $B^{(i-1)}_i \cap U_Q = \{z_0 = z_1 = 0\}$. Let $V_Q$ be the inverse image of $U_Q$ by $p^{(i-1)}$. Then we have

$$V_Q = \left\{(z_0, z_1, z_2), [w_0 : w_1] \in U_Q \times \mathbb{P}^1 | z_0 w_1 - z_1 w_0 = 0\right\}.$$

If $k = 2$ and $\{z_2 = 0\} = D^{(i-1)}_j$ then $j > i$ by the assumption (4)$_{i-1}$. $D^{(i)}_j \cap V_Q$ is given by the blowing up of $D^{(i-1)}_j \cap U_Q$ at the reduced point $Q$. On $V^{+}_Q := V_Q \cap \{w_0 \neq 0\}$, we have a local coordinate $(u_0, u_1, u_2)$ with $z_0 = u_0, z_1 = u_0 u_1, z_2 = u_2$, and $w_0/w_1 = u_1$. We have $f^{(i)}_{V^{+}_Q}(u) = \prod_{i=1}^{k} u_i^{-1}$. The strict transform $D^{(i)}_i \cap V^{+}_Q$ is given by $\{u_1 = 0\}$. On $V^{-}_Q := V_Q \cap \{w_1 \neq 0\}$, we have a local coordinate $(v_0, v_1, v_2)$ with $z_0 = v_0 v_1, z_1 = v_1, z_2 = v_2$, and $w_0/w_1 = v_0$. We have $f^{(i)}_{V^{-}_Q}(v) = v_0$ if $k = 1$, and $f^{(i)}_{V^{-}_Q}(v) = v_0 v_2^{-1}$ if $k = 2$. The strict transform $D^{(i)}_i \cap V^{-}_Q$ is given by $\{v_1 = 0\}$. By this description and the assumptions, we have (1)$_\ell$, (2)$_\ell$, (3)$_\ell$, (4)$_\ell$. Then, by the induction, we obtain (1)$_\ell$, (2)$_\ell$, (3)$_\ell$, (4)$_\ell$. It is easy to prove that (1)$_\ell$, (2)$_\ell$, (3)$_\ell$, (4)$_\ell$ implies the lemma. \qed
4.2.9. **Hodge-Tate condition.** We obtain the following:

**Proposition 4.7.** Let \( f : X \to \mathbb{P}^1 \) be the pull back of \( f_\nu \) by \( \pi_\nu \). Then the rescaling structure \( \mathcal{H}_f \) is of Hodge-Tate type.

**Proof.** By Lemma 4.6 the pair \((X, f)\) satisfies the condition in [38]. Since \( X \) is given by blowing ups of a toric manifold along projective lines, \( h^{p,q}(X) = 0 \) for \( p \neq q \) ([10], Theorem 7.31]). Since \( D_j \) is given by the composition of blowing ups of \( D_{\lambda j} \) along reduced 0-schemes (Lemma 4.6 (1)), and each \( D_i \cap D_j \) is isomorphic to \( \mathbb{P}^1 \), the (co)homology of \( D \) is Hodge-Tate (see [29], Example 5.34) for example). Hence, by Lemma 3.31 and the exact sequence

\[
\cdots \to H^k(X) \to H^k(Y) \to H^{k-1}(D)(-1) \to \cdots,
\]

we have that the mixed Hodge structure on \( H^k(Y) \) is Hodge-Tate for each \( k \). By Corollary 3.32 it remains to show that the limit mixed Hodge structure \( H^k(Y_\infty) \) is of Hodge-Tate type. From the Clemens-Schmid exact sequence \([19\), (10.14), Theorem (10.16)]\), we obtain the following exact sequence of mixed Hodge structures:

\[
H^k(D) \to H^k(Y_\infty) \xrightarrow{\nabla} H^k(Y_\infty)(-1) \to H_{4-k}(D)(-3),
\]

where \( 0 \leq k \leq 4 \). Since by Corollary 3.31, we have the exact sequence

\[
0 \to A_1 \to H^k(Y_\infty) \to H^k(Y_\infty)(-1) \to A_2 \to 0,
\]

where \( A_1 \) and \( A_2 \) are Hodge-Tate. Then, by the similar argument as in the proof of Proposition 4.1, \( H^k(Y_\infty) \) is also Hodge-Tate for each \( k \).

Similarly as Corollary 4.2, we have the following:

**Corollary 4.8.** Let \((X, f)\) be as in Proposition 4.7. Then we have \( f^{p,q}(Y, w) = h^{p,q}(Y, w) \). We also have that \( \mathcal{H}_{f|\nu = 1} \) is special. □

**Remark 4.9.** In [21], A. Harder computed the number \( f^{p,q}(Y, w) \) and compare it with the Hodge number of the smooth toric Fano manifold \( X_\nu \) associated to \( P \) [21], Theorem 2.3.7]. In [30], Reichelt-Sevenheck studied hypergeometric \( \mathcal{D} \)-module associated to (a family of) \( f_\nu \), and solved a kind of Birkhoff problem. The result here is a priori different from theirs since the cohomology considered here is different from the one considered in [30]. We also remark that T. Mochizuki informed that we can obtain similar but a priori different results from the viewpoint of twistor \( \mathcal{D} \)-modules.

**Appendix A. Rescaling structures for quantum \( \mathcal{D} \)-modules of Fano manifolds**

**A.1. Square roots of Tate twists.** We use the notation in [22]. Set \( T^{1/2} := \mathcal{O}_S(*)_{\infty} w \) where \( w \) is a global section with \( \deg w = 1 \). We define a connection \( \nabla \) on \( T^{1/2} \) by \( \nabla w := -(1/2) w \lambda^{-1} d\lambda \). Since \( p^*_S(T^{1/2}, \nabla) \) is not isomorphic to \( \sigma^*(T^{1/2}, \nabla) \), \( \nabla \) is not equipped with a rescaling structure. However, we have a flat isomorphism \( (T^{1/2})^{\otimes 2} \sim \sim T; w^{\otimes 2} \to v \). Hence we use the notation \( T(-1/2) := (T^{1/2}, \nabla) \). For each \( k \in \mathbb{Z} \), we define

\[
T(-k/2) := \begin{cases} 
T(-1/2)^{\otimes k} & (k \geq 0) \\
(T(-1/2)^v)^{\otimes -k} & (k < 0).
\end{cases}
\]

In the case where \( k \in 2\mathbb{Z} \), \( T(k/2) \) is identified with the rescaling structure defined in Example 2.6. For a meromorphic connection \( (\mathcal{H}, \nabla) \) as in Definition 2.5 we also define \( \mathcal{H}(k/2) := \mathcal{H} \otimes T(k/2) \).
A.2. **Tate twisted quantum D-modules.** Let \( F \) be a smooth projective Fano variety over \( \mathbb{C} \) of dimension \( n \). Put \( HH_a(F) := \bigoplus_{p-q=n} H^q(F, \Omega^p_F) \). Set \( HH_*(F) := \bigoplus_a HH_a(F) \) and identify it with \( H^*(F; \mathbb{C}) \) by the Hodge decomposition. Let \( \ast \) be the quantum cup product of \( F \) with respect to the parameter \( c_1(F) \log \tau \in H^2(F; \mathbb{C}) \), where \( c_1(F) \) is the first Chern class of the tangent bundle of \( F \). This is well defined for all \( \tau \in \mathbb{C} \). Indeed, the right hand side of

\[
(a \ast \beta)_{\tau} = \sum_{d \in H_2(F; \mathbb{Z})} (a, \beta, \gamma)_{0,3,d} \tau^{c_1(F) \cdot d}
\]

is a finite sum since \( F \) is Fano, where \( a, \beta, \gamma \in H^*(F; \mathbb{C}) \simeq HH_*(F) \), \((\cdot, \cdot)_F \) denotes the Poincaré pairing, and \((\cdot, \cdot)_{0,3,d} \) denotes genus-zero 3-points Gromov-Witten invariant of degree \( d \in H_2(F; \mathbb{Z}) \) (see [3, 4, 5], and references therein).

For any non-negative integer \( k \), we take a finite rank free \( \mathcal{O}_S(*\lambda)_{(\infty)} \)-module \( aH^k_k := HH_{k-n}(F) \otimes \mathcal{O}_S(*\lambda)_{(\infty)} \). The \( \mathbb{Z} \)-grading of \( aH^k_k \) is defined to be 0. Define \( \mu_F \in \text{End}(HH_{k-n}(F)) \) by \( \mu_{F;HF_F} := (p + q - n)/2 \cdot \text{id}_{HF_F} \). We also have an endomorphism \( c_1(F) \ast \tau \) on \( HH_{k-n}(F) \). We have the Dubrovin connection \( a\nabla \) on \( aH^k_k \) as follows ([12, 13, 14]):

\[
a\nabla := d + \frac{c_1(F) \ast \tau}{\lambda} \frac{d\tau}{\tau} + \mu_F \frac{d\lambda}{\lambda} - c_1(F) \ast \frac{d\lambda}{\lambda^2}.
\]

**Proposition A.1.** \( \mathcal{H}_F^k := aH^k_k(-k/2) \) comes equipped with a rescaling structure.

**Proof.** \( \mathcal{H}_F^k \) is identified with the free \( \mathcal{O}_S(*\lambda)_{(\infty)} \)-module \( HH_{k-n}(F) \otimes \mathcal{O}_S(*\lambda)_{(\infty)} \) with the connection:

\[
\nabla = d + \frac{c_1(F) \ast \tau}{\lambda} \frac{d\tau}{\tau} + \left( \mu_F - \frac{k}{2} \cdot \text{id} \right) \left( \frac{d\lambda}{\lambda} + \frac{d\theta}{\theta} \right) - c_1(F) \ast \frac{d\lambda}{\lambda^2}.
\]

Taking the pull back by \( \sigma : \mathbb{C}^*_\tau \times S \rightarrow S ; (\theta, \lambda, \tau) \mapsto (\theta\lambda, \theta\tau) \), we have

\[
\sigma^* \nabla = d + \frac{c_1(F) \ast \theta \tau}{\theta \lambda} \frac{d\tau}{\tau} + \left( \mu_F - \frac{k}{2} \cdot \text{id} \right) \left( \frac{d\lambda}{\lambda} + \frac{d\theta}{\theta} \right) - c_1(F) \ast \frac{d\lambda}{\lambda^2}.
\]

Put \( \mu_k := \mu_F - (k/2) \cdot \text{id} \). On \( H^q(F, \Omega^p_F) \) with \( q - p = k - n \), we have \( \mu_k = (q - k) \cdot \text{id} = (p - n) \cdot \text{id} \). Hence we have a morphism of \( \mathcal{O}_{\mathbb{C}^*_\tau \times S(*\lambda)_{(\infty)}} \)-modules:

\[
\theta^{-\mu_k} : p_2^* \mathcal{H}_F^k \xrightarrow{\sim} \sigma^* \mathcal{H}_F^k.
\]

By \( \text{(A.2)} \), we obtain

\[
c_1 \ast \tau = \theta^{\mu_k} \left( \frac{c_1(F) \ast \theta \tau}{\theta} \right) \theta^{-\mu_k},
\]

which implies that \( \theta^{-\mu_k} \) is flat with respect to the connections (see [18 §2.2] for example).

**Definition A.2.** We define a rescaling structure \( \mathcal{H}_F \) by

\[
\mathcal{H}_F := \bigoplus_{k \in \mathbb{Z}} \mathcal{H}_F^k.
\]

We call \( \mathcal{H}_F \) a Tate twisted quantum D-module of \( F \).

**Remark A.3.** The \( \mathbb{Z}/2\mathbb{Z} \)-graded flat meromorphic connection \( aH \) in the introduction (or [25]) is given by \( aH = \bigoplus_k aH^k \), where the \( \mathbb{Z}/2\mathbb{Z} \)-grading on \( aH^k \) is given by \( (k \mod 2) \).
A.3. Hodge-Tate condition and Hodge numbers. The fiber of $\mathcal{H}_k^\tau$ at $\lambda, \tau = (1, 0)$ is naturally identified with $\text{HH}_k$ of $F$. We shall describe the Hodge and weight filtrations on $\text{HH}_k$ in the sense of [23, 3.1].

As we have seen in the proof of Proposition B.1, the $\mathbb{C}^*$-action on $\mathcal{H}_k^\tau|_{\tau = 0}$ is given by $\theta^{-q-k}$ on $H^q(F, \Omega^k_F) \otimes O_{C_\lambda}$ with $q - p = k - n$. Hence the Hodge filtration on $\text{HH}_k$ is given as follows:

\begin{equation}
F_i \text{HH}_k = \bigoplus_{p-n \leq i, \; q-p = k-n} H^q(F, \Omega^k_F).
\end{equation}

We obtain $f^{p,q}(\mathcal{H}_k) = \dim H^q(F, \Omega^{n-p}_F) = h^{n-p,q}(F)$.

The residue endomorphism $N_k := \text{Res}_v \nabla$ on $\text{HH}_k$ is identified with $c_1(F)$. It follows that the monodromy weight filtration centered at $k$ is given as follows:

\begin{equation}
kW_i \text{HH}_k = \bigoplus_{p+n-i/2, \; q-p = k-n} H^q(F, \Omega^k_F).
\end{equation}

Hence, we have $h^{p,q}(\mathcal{H}_k) = h^{n-p,q}(F)$. By (A.3) and (A.4), we obtain the following:

**Proposition A.4.** The Tate twisted quantum $\mathcal{D}$-module $\mathcal{H}_k$ satisfies the Hodge-Tate condition for any smooth projective Fano variety $F$. \hfill $\square$

### Appendix B. Relation to the work of Katzarkov-Kontsevich-Pantev

B.1. Tame compactified Landau-Ginzburg model. In [25], Katzarkov-Kontsevich-Pantev considered the following:

**Definition B.1** ([25 Definition 2.4, (T)], see also [26 Definition 3]). A tame compactified Landau-Ginzburg model is a tuple $((X, f), D, \text{vol}_X)$, where

1. $X$ is a smooth projective variety and $f : X \to \mathbb{P}^1$ is a flat projective morphism.
2. $D = (\bigcup_i D_i^h) \cup (\bigcup_j D_j^f) \subset X$ is a reduced normal crossing divisor such that
   a. $D_j^f = \bigcup_i D_i^h$ is a scheme theoretic pole divisor of $f$, i.e. $(f)_\infty = D_j^f$. In particular, the pole order of $f$ along $D_j^f$ is one;
   b. each component $D_i^h$ of $D^h := \bigcup_i D_i^h$ is smooth and horizontal for $f$, i.e. $f|_{D_i^h}$ is a flat morphism;
   c. the critical locus of $f$ does not intersect $D^h$;
3. $\text{vol}_X$ is a nowhere vanishing meromorphic section of the canonical bundle $K_X$ with poles of order exactly one along each component of $D$. In other words, we have an isomorphism $\mathcal{O}_X \cong K_X(D)$.

In this paper ([33], the horizontal divisor $D^h$ is assumed to be empty, and each component $D_j^f$ is assumed to be smooth. Although we do not impose the existence of $\text{vol}_X$ in [33] all examples in [44] have $\text{vol}_X$.

B.2. Landau-Ginzburg Hodge numbers. The Hodge number $f^{p,q}(Y,w)$ in this paper corresponds to $f^{q,p}(Y,w)$ in [25, Definition 3.1]. The definition in this paper suits to the convention in the classical Hodge theory. The Hodge number $h^{p,q}(Y,w)$ in [25] is $\dim \text{Gr}^W_{H^{p+q}}(Y, Y_\infty)$ in our notation. Our definition of $h^{p,q}(Y,w)$ is $\dim \text{Gr}^W_{H^{p+q}}(Y, Y_\infty)$, which is different from their definition. As mentioned in [26], their definition seems not to be what they had in mind. The definition of $h^{p,q}(Y,w)$ in this paper corresponds to $h^{q,p}(Y,w)$ in [26, Definition 3]. In [25], they also gave a counter-example for the part of equality with the numbers $i^{p,q}(Y,w)$ in [25, Conjecture 3.6].
B.3. One parameter families. Recall that $S = \mathbb{P}^1_\lambda \times \mathbb{C}_\tau$. We also recall that $\pi_S : S \times X \to X$ and $p_S : S \times X \to S$ denote the projections. Put

$$\Omega^k_{X,S}(\ast D) := \mathcal{O}_{X \times S}(\ast \lambda_\infty) \otimes \pi_S^{-1} \Omega^k_X(\ast D).$$

Let $^bH^k$ be the $\mathcal{O}_S(\ast \lambda_\infty)$-module defined by

$$^bH^k := \mathbb{R}^k p_{S*}(\Omega^k_{X,S}(\ast D), \lambda d + \tau df \wedge).$$

Let $\nabla : \Omega^\bullet_{X,S}(\ast D) \to \Omega^\bullet_{X,S}(\ast D) \otimes p_S^*\Omega^1_S((\lambda \tau)_0)$ be the connection on $\Omega^\bullet_{X,S}(\ast D) := \bigoplus_k \Omega^k_{X,S}(\ast D)$ defined by

$$\nabla = d_S + \frac{f}{\lambda} d\tau + \xi \frac{d\lambda}{\lambda} - \tau f \frac{d\lambda}{\lambda^2},$$

where $G = -(k/2) \text{id}$ on $\Omega^k_{X,S}(\ast D)$. Then, we have $[\nabla_{\partial_\lambda}, \lambda d + \tau df \wedge] = 0$, and $[\nabla_{\partial_\lambda}, \lambda d + \tau df \wedge] = (2\lambda)^{-1}(\lambda d + \tau df \wedge)$. Let $\mathcal{A}^{p,q}_{X}$ be the sheaf of $(p,q)$-forms on $X$ and $\partial\overline{\partial}$ be the Dolbeault operators. Put $\mathcal{A}^{p,q}_{X,S,D} := \Omega^p_S(\ast D) \otimes \pi_S^{-1} \mathcal{A}^{p,q}_X$. Let $\partial : \mathcal{A}^{p,q}_{X,S,D} \to \mathcal{A}^{p+1,q}_{X,S,D}$, and $\overline{\partial} : \mathcal{A}^{p,q}_{X,S,D} \to \mathcal{A}^{p,q+1}_{X,S,D}$ be the induced operators. Put $\mathcal{A}^{p,q}_{X,S,D} := \bigoplus_{p+q} \mathcal{A}^{p,q}_{X,S,D}$ and

$$d_{tot} := \lambda \partial + \overline{\partial} + \tau f : \mathcal{A}^{p,q}_{X,S,D} \to \mathcal{A}^{p,q+1}_{X,S,D}.$$

We have a natural quasi-isomorphism

$$\iota_{\text{Dol}} : (\Omega^\bullet_{X,S}(\ast D), \lambda d + \tau df \wedge) \xrightarrow{\sim} (\mathcal{A}^\bullet_{X,S,D}, d_{tot}).$$

We also have the connection $\nabla : \mathcal{A}^\bullet_{X,S,D} \to \mathcal{A}^\bullet_{X,S,D} \otimes \Omega^1_S((\lambda \tau)_0)$ by

$$\nabla := d_S + \frac{f}{\lambda} d\tau + \mu \frac{d\lambda}{\lambda} - \tau f \frac{d\lambda}{\lambda^2},$$

where $\mu|_{\mathcal{A}^q_{X,S,D}} = 2^{-1}(q-p) \cdot \text{id}$. Then $\iota_{\text{Dol}} \circ \nabla = \nabla \circ \iota_{\text{Dol}}$ by definition. We have $[\nabla_{\partial_\lambda}, d_{tot}] = 0$, and

$$[\nabla_{\partial_\lambda}, d_{tot}] = [\partial_\lambda + \lambda^{-1} \mu f - \lambda^{-2} \tau f, \lambda \partial + \overline{\partial} + \tau f]$$

$$= \partial + (1/2) \partial + (1/2) \lambda^{-1} \overline{\partial} - (1/2) \lambda^{-1} \tau f + \lambda^{-1} \tau f$$

$$= (2\lambda)^{-1}(\lambda \partial + \overline{\partial} + \tau f + \lambda^{-1} \tau f).$$

Hence $\nabla$ gives a connection $^b\nabla^k$ on $^bH^k \simeq \mathcal{A}^k p_{S*}(\mathcal{A}^\bullet_{X,S,D}, d_{tot})$. We remark that similar discussions are given in [15] and [24].

**Lemma B.2.** For each $k \in \mathbb{Z}_{\geq 0}$, we have $(^bH^k, ^b\nabla^k)((-k/2)) \simeq \mathcal{H}^k_T$.

**Proof.** We have a natural isomorphism $(^bH^k, ^b\nabla^k)((-k/2)) \simeq (^bH^k, ^b\nabla^k - (k/2) \lambda^{-1} d\lambda)$. Then the connection $^b\nabla^k - (k/2) \lambda^{-1} d\lambda$ is induced from the following connection on $\mathcal{A}^\bullet_{X,S,D}$:

$$\nabla' := d_S + \frac{f}{\lambda} d\tau + P \frac{d\lambda}{\lambda} - \tau f \frac{d\lambda}{\lambda^2},$$

where $P|_{\mathcal{A}^q_{X,S,D}} = 2^{-1}((q-p) - (p+q)) \cdot \text{id} = -(p) \cdot \text{id}$. Remark that $[\nabla', d_{tot}] = 0$. Moreover, it is induced from the following connection on $\mathcal{A}^\bullet_{X,S}(\ast D)$:

$$\nabla' := d_S + \frac{f}{\lambda} d\tau + P \frac{d\lambda}{\lambda} - \tau f \frac{d\lambda}{\lambda^2},$$

where $P|_{\mathcal{A}^q_{X,S}(\ast D)} = -(p) \cdot \text{id}$. We also remark that $[\nabla', \lambda d + \tau df] = 0$. Then, the quasi-isomorphism $\text{iso} : (\Omega^\bullet_{X,S}(\lambda d + \lambda^{-1} \tau df) \xrightarrow{\sim} (\Omega^\bullet_{X,S}(\ast D), \lambda d + \tau df)$ on $S^* \times X = (\mathbb{C}_X^* \times \mathbb{C}^*_\tau) \times X$ defined by $\text{iso}|_{\mathcal{A}^p_{X,S}(\ast D)} = \lambda^p$ induces the conclusion naturally. \(\square\)
Remark B.3. It seems that the connection on $b\ H$ which Katzarkov-Kontsevich-Pantev had in mind in [25, (3.2.2)] was the one where $f$ is replaced by $q^f$. The dual of it (or, the connection $(b\ H, b\ \nabla)$ defined firstly in [25, §3.2.2]) is isomorphic to $\bigoplus_{k\in\mathbb{Z}}(b\ H^k, b\ \nabla^k)$.

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