SOME IDENTITIES OF THE GENERALIZED TWISTED BERNOULLI NUMBERS AND POLYNOMIALS OF HIGHER ORDER

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Abstract  The purpose of this paper is to derive some identities of the higher order generalized twisted Bernoulli numbers and polynomials attached to \( \chi \) from the properties of the \( p \)-adic invariant integral. We give some interesting identities for the power sums and the generalized twisted Bernoulli numbers and polynomials of higher order using the symmetric properties of the \( p \)-adic invariant integral.

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1. Introduction and preliminaries

Let \( p \) be a fixed prime number. Throughout this paper, the symbol \( \mathbb{Z}, \mathbb{Z}_p, \mathbb{Q}_p, \) and \( \mathbb{C}_p \) denote the ring of rational integers, the ring of \( p \)-adic integers, the field of \( p \)-adic rational numbers, and the completion of algebraic closure of \( \mathbb{Q}_p \), respectively. Let \( \mathbb{N} \) be the set of natural numbers and \( \mathbb{Z}_+ = \mathbb{N} \cup \{0\} \). Let \( \nu_p \) be the normalized exponential valuation of \( \mathbb{C}_p \) with \( |p|_p = p^{-\nu_p(p)} = p^{-1} \).

Let \( UD(\mathbb{Z}_p) \) be the space of uniformly differentiable function on \( \mathbb{Z}_p \). For \( f \in UD(\mathbb{Z}_p) \), the \( p \)-adic invariant integral on \( \mathbb{Z}_p \) is defined as

\[
I(f) = \int_{\mathbb{Z}_p} f(x) dx = \lim_{N \to \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x). \tag{1.1}
\]

(see [4-5]). From (1.1), we note that

\[
I(f_1) = I(f) + f'(0), \tag{1.2}
\]

where \( f'(0) = \left. \frac{df(x)}{dx} \right|_{x=0} \) and \( f_1(x) = f(x+1) \). For \( n \in \mathbb{N} \), let \( f_n(x) = f(x+n) \). Then we can derive the following equation from (1.2).

\[
I(f_n) = I(f) + \sum_{i=0}^{n-1} f'(i), \quad \text{(see [4-5])}. \tag{1.3}
\]

Let \( d \) be a fixed positive integer. For \( n \in \mathbb{N} \), let

\[
X = X_d = \lim_{N} \mathbb{Z}/dp^N\mathbb{Z}, \quad X_1 = \mathbb{Z}_p,
\]

\[
X^* = \bigcup_{a \in \mathbb{Z}_p, 0<p \cdot a < dp} (a + dp\mathbb{Z}_p),
\]

\[
a + dp^N\mathbb{Z}_p = \{ x \in X | x \equiv a \pmod{dp^N} \}.
\]
where $a \in \mathbb{Z}$ lies in $0 \leq a < dp^N$. It is easy to see that

$$
\int_X f(x) dx = \int_{\mathbb{Z}_p} f(x) dx, \quad \text{for} \quad f \in U D(\mathbb{Z}_p).
$$

(1.4)

The ordinary Bernoulli polynomials $B_n(x)$ are defined as

$$
\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!},
$$

and the Bernoulli numbers $B_n$ are defined as $B_n = B_n(0)$ (see [1-19]).

For $n \in \mathbb{N}$, let $T_p$ be the $p$-adic locally constant space defined by

$$
T_p = \bigcup_{n \geq 1} C_{p^n} = \lim_{n \to \infty} C_{p^n},
$$

where $C_{p^n} = \{ \omega | \omega p^n = 1 \}$ is the cyclic group of order $p^n$. It is well known that the twisted Bernoulli polynomials are defined as

$$
\frac{t}{\xi e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_{n,\xi}(x) \frac{t^n}{n!}, \quad \xi \in T_p,
$$

and the twisted Bernoulli numbers $B_{n,\xi}$ are defined as $B_{n,\xi} = B_{n,\xi}(0)$ (see [14-18]).

Let $\chi$ be the Dirichlet’s character with conductor $d \in \mathbb{N}$. Then we have

$$
\int_X \chi(x) \xi^x e^{xt} dx = \frac{\sum_{n=0}^{d-1} \chi(a) \xi^{a t} e^{at}}{\xi^{d e^t} - 1}.
$$

(1.5)

It is known that the generalized twisted Bernoulli numbers attached to $\chi$, $B_{n,\chi,\xi}$, are defined as

$$
\frac{t}{\xi e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_{n,\chi,\xi}(x) \frac{t^n}{n!}, \quad \xi \in T_p.
$$

(1.6)

The generalized twisted Bernoulli polynomials attached to $\chi$, $B_{n,\chi,\xi}(x)$, are defined as

$$
\frac{t}{\xi e^t - 1} e^{xt} = \sum_{n=0}^{d-1} \sum_{a=0}^{\infty} \chi(a) \frac{t^n}{n!}, \quad \xi \in T_p.
$$

(1.7)

(see [13], [16]). From (1.5), (1.6) and (1.7), we derive that

$$
\int_X \chi(x) \xi^x x^n dx = B_{n,\chi,\xi} \quad \text{and} \quad \int_X \chi(y) \xi^y (x+y)^n dy = B_{n,\chi,\xi}(x).
$$

(1.8)

By (1.3) and (1.4), it is easy to see that for $n \in \mathbb{N},$

$$
\int_X f(x+n) dx = \int_X f(x) dx + \sum_{i=0}^{n-1} f'(i),
$$

(1.9)

where $f'(i) = \frac{df(x)}{dx} |_{x=i}$. From (1.0), it follows that
In the next section, we will consider the extension of (1.14) related to the generalized twisted Bernoulli numbers and polynomials of higher order attached to $\chi$, $n$, $d$.

Let $k, n, d \in \mathbb{N}$. By (1.8) and (1.12), we have that

$$\int_X \chi(x) \xi^{nd+e} e^{(nd+e)t} dx = \int_X \chi(x) \xi^e e^{xt} dx$$

For $k \in \mathbb{Z}_+$, let us define the $p$-adic functional $T_{k, \chi, \xi}(n)$ as follows:

$$T_{k, \chi, \xi}(n) = \sum_{l=0}^{n} \chi(l) \xi^{lk}.$$  (1.11)

Let $k, n, d \in \mathbb{N}$. By (1.10) and (1.11), we see that

$$\int_X \chi(x) \xi^{nd+e} (nd + x) e^{xt} dx = k T_{k-1, \chi, \xi}(nd - 1).$$  (1.12)

From (1.8) and (1.12), we have that

$$\frac{\xi^{nd} B_{k, \chi, \xi}(nd) - B_{k, \chi, \xi}}{k} = T_{k-1, \chi, \xi}(nd - 1).$$  (1.13)

For $w_1, w_2, d \in \mathbb{N}$, we note that

$$\frac{d}{dx} \int_X \chi(x) \xi^{w_1 x + w_2 x^2} e^{(w_1 x + w_2 x^2)t} dx = \frac{d}{dx} \int_X \xi^{w_1 x + w_2 x^2} e^{(w_1 x + w_2 x^2)t} dx$$

In the next section, we will consider the extension of (1.14) related to the generalized twisted Bernoulli numbers and polynomials of higher order attached to $\chi$.

The generalization of twisted Bernoulli polynomials of order $k$ attached to $\chi$, $B_{n, \chi, \xi}^{(k)}(x)$, are defined as

$$\left( \frac{d}{dt} \sum_{a=0}^{d-1} \chi(a) \xi^a e^{at} \right)^k e^{xt} = \sum_{n=0}^{\infty} B_{n, \chi, \xi}^{(k)}(x) \frac{t^n}{n!}, \quad \xi \in T_p,$$  (1.15)

and $B_{n, \chi, \xi}^{(k)} = B_{n, \chi, \xi}^{(k)}(0)$ are called the generalized twisted Bernoulli numbers of order $k$ attached to $\chi$. When $k = 1$, the polynomials and numbers are called the generalized twisted Bernoulli polynomials and numbers attached to $\chi$, respectively (see [12]).

The authors of this paper have studied various identities for the Bernoulli and the Euler polynomials by the symmetric properties of the $p$-adic invariant integrals (see [6-8], [10]). T. Kim [6] established interesting identities by the symmetric properties of the $p$-adic invariant integrals and some relationships between the power sum and the Bernoulli polynomials. In [8], Kim et al. gave some identities of symmetry for the generalized Bernoulli polynomials. The twisted Bernoulli polynomials and numbers are very important in several field of mathematics and physics, and so have been studied by many authors (cf. [9-18]). Recently, Kim-Hwang [10] obtained some relations between the power sum polynomials and twisted Bernoulli polynomials.

In this paper, we extend our results to the generalized twisted Bernoulli numbers and polynomials of higher order attached to $\chi$. The purpose of this paper is to derive some identities of the generalized twisted Bernoulli numbers and polynomials attached to...
\( \chi \) from the properties of the \( p \)-adic invariant integral. In Section 2, we give interesting identities for the power sums and the generalized twisted Bernoulli numbers and polynomials of higher order using the symmetric properties for the \( p \)-adic invariant integral.

2. Some identities of the generalized twisted Bernoulli numbers and polynomials of higher order

Let \( w_1, w_2, d \in \mathbb{N} \). For \( \xi \in T_p \), we set

\[
Y(m, \chi, \xi; w_1, w_2) = \left( \frac{d \int_X \left( \prod_{i=1}^{m} \chi(x_i) \xi \left( \sum_{i=1}^{m} x_i w_1 \sum_{i=1}^{m} x_i w_1 t \right) dx_1 \cdots dx_m \right)}{\int_X \xi^{d w_1 w_2 x} e^{d w_1 w_2 x t} dx} \right)^{m} \times \left( \int_X \left( \prod_{i=1}^{m} \chi(x_i) \xi \left( \sum_{i=1}^{m} x_i w_2 \sum_{i=1}^{m} x_i w_2 t \right) dx_1 \cdots dx_m \right) \right) \tag{2.1}
\]

where

\[
\int_X f(x_1, \cdots, x_m) dx_1 \cdots dx_m = \left[ \int_X f(x_1, \cdots, x_m) dx_1 \cdots dx_m \right]^{m \text{-times}}.
\]

In (2.1), we note that \( Y(m, \chi, \xi; w_1, w_2) \) is symmetric in \( w_1, w_2 \). From (2.1), we derive that

\[
Y(m, \chi, \xi; w_1, w_2) = \left( \frac{d \int_X \left( \prod_{i=1}^{m} \chi(x_i) \xi \left( \sum_{i=1}^{m} x_i w_1 \sum_{i=1}^{m} x_i w_1 t \right) dx_1 \cdots dx_m \right)}{\int_X \xi^{d w_1 w_2 x} e^{d w_1 w_2 x t} dx} \right)^{m} \times \left( \int_X \left( \prod_{i=1}^{m} \chi(x_i) \xi \left( \sum_{i=1}^{m} x_i w_2 \sum_{i=1}^{m} x_i w_2 t \right) dx_1 \cdots dx_m \right) \right) e^{w_1 w_2 y}. \tag{2.2}
\]

From (1.10) and (1.11), it follows that

\[
\frac{dw_1 \int_X \chi(x) \xi^{x} e^{x t} dx}{\int_X \xi^{d w_1 x} e^{d w_1 x t} dx} = \sum_{i=0}^{w_1 d - 1} \chi(i) \xi^i e^i = \sum_{k=0}^{\infty} T_{k, \chi, \xi}(w_1 d - 1) \frac{x^k}{k!}. \tag{2.3}
\]

By (1.15), we also see that

\[
\xi^{d w_1 w_2 x - 1} \sum_{\alpha=0}^{d - 1} \chi(\alpha) \xi^\alpha e^{\alpha w_1 t} = \sum_{n=0}^{\infty} \lambda_{n, \chi, \xi}(w_2 x) \frac{w_1^n x^n}{n!} \tag{2.4}
\]

\[
\int_X \left( \prod_{i=1}^{m} \chi(x_i) \xi \left( \sum_{i=1}^{m} x_i w_1 \sum_{i=1}^{m} x_i w_1 t \right) dx_1 \cdots dx_m \right) \int_X \left( \prod_{i=1}^{m} \chi(x_i) \xi \left( \sum_{i=1}^{m} x_i w_2 \sum_{i=1}^{m} x_i w_2 t \right) dx_1 \cdots dx_m \right) \right) \tag{2.1}
\]

\[
eq \left( \frac{w_1 t}{\xi^{d w_1 e^{d w_1 t} - 1} - 1} \sum_{\alpha=0}^{d - 1} \chi(\alpha) \xi^\alpha e^{\alpha w_1 t} \right)^{m} \times \left( \sum_{n=0}^{\infty} \lambda_{n, \chi, \xi}(w_2 x) \frac{w_1^n x^n}{n!} \right).
\]
By (2.2), (2.3) and (2.4), we have that

\[
Y(m, \xi_2, \xi_2 | t_1, t_2) = \left( \sum_{l=0}^{\infty} B_l^{(m)} (w_2) \frac{t_1^l}{l!} \right) \left( \frac{1}{w_1} \sum_{k=0}^{\infty} T_{k, \xi_1} (w_1 d - 1) \frac{w_1^k}{k!} \right) \left( \sum_{i=0}^{\infty} B_i^{(m-1)} (w_1 y) \frac{t_2^i}{i!} \right)
\]

(2.5)

From the symmetry of \(Y(m, \xi_1, \xi_2 | t_1, t_2)\) in \(w_1\) and \(w_2\), we see that

\[
Y(m, \xi_1, \xi_2 | t_1, t_2) = \sum_{n=0}^{\infty} \left( \sum_{j=0}^{n} \binom{n}{j} w_1^j w_2^{n-j} B_{n-j, \xi_1, \xi_2} (w_2) \sum_{k=0}^{j} \binom{j}{k} T_{k, \xi_1} (w_1 d - 1) B_{j-k, \xi_1, \xi_2} (w_1 y) \right) \frac{t_1^n}{n!}
\]

(2.6)

Comparing the coefficients on the both sides of (2.5) and (2.6), we obtain an identity for the generalized twisted Bernoulli polynomials of higher order as follows.

**Theorem 1.** Let \(d, w_1, w_2 \in \mathbb{N}\). For \(n \in \mathbb{Z}_+\) and \(m \in \mathbb{N}\), we have

\[
\sum_{j=0}^{n} \binom{n}{j} w_1^j w_2^{n-j} B_{n-j, \xi_1, \xi_2} (w_2) \sum_{k=0}^{j} \binom{j}{k} T_{k, \xi_1} (w_1 d - 1) B_{j-k, \xi_1, \xi_2} (w_1 y)
\]

\[
= \sum_{n=0}^{\infty} \left( \sum_{j=0}^{n} \binom{n}{j} w_1^j w_2^{n-j} B_{n-j, \xi_1, \xi_2} (w_1 x) \sum_{k=0}^{j} \binom{j}{k} T_{k, \xi_1} (w_1 d - 1) B_{j-k, \xi_1, \xi_2} (w_1 y) \right) \frac{t_1^n}{n!}
\]

(2.7)

**Remark 1.** Taking \(m = 1\) and \(y = 0\) in (2.7) derives the following identity:

\[
\sum_{j=0}^{n} \binom{n}{j} w_1^j w_2^{n-j} B_{n-j, \xi_1, \xi_2} (w_2) T_{j, \xi_1} (w_1 d - 1)
\]

\[
= \sum_{n=0}^{\infty} \left( \sum_{j=0}^{n} \binom{n}{j} w_1^j w_2^{n-j} B_{n-j, \xi_1, \xi_2} (w_1 x) T_{j, \xi_1} (w_1 d - 1) \right) \frac{t_1^n}{n!}
\]

Moreover, if we take \(x = 0\) and \(y = 0\) in Theorem 1, then we have the following identity for the generalized twisted Bernoulli numbers of higher order.

**Corollary 2.** Let \(d, w_1, w_2 \in \mathbb{N}\). For \(n \in \mathbb{Z}_+\) and \(m \in \mathbb{N}\), we have

\[
\sum_{j=0}^{n} \binom{n}{j} w_1^j w_2^{n-j} B_{n-j, \xi_1, \xi_2} \sum_{k=0}^{j} \binom{j}{k} T_{k, \xi_1} (w_1 d - 1) B_{j-k, \xi_1, \xi_2}
\]

\[
= \sum_{n=0}^{\infty} \left( \sum_{j=0}^{n} \binom{n}{j} w_1^j w_2^{n-j} B_{n-j, \xi_1, \xi_2} T_{j, \xi_1} (w_1 d - 1) \right) \frac{t_1^n}{n!}
\]

(2.8)

We also note that taking \(m = 1\) in Corollary 2 shows the following identity:

\[
\sum_{j=0}^{n} \binom{n}{j} w_1^j w_2^{n-j} B_{n-j, \xi_1, \xi_2} T_{j, \xi_1} (w_1 d - 1)
\]

\[
= \sum_{j=0}^{n} \binom{n}{j} w_1^j w_2^{n-j} B_{n-j, \xi_1, \xi_2} T_{j, \xi_1} (w_1 d - 1).
\]

(2.8)
Now we will derive another interesting identities for the generalized twisted Bernoulli numbers and polynomials of higher order. From (1.15), (2.2) and (2.3), we can derive that

\[
Y(m, \chi, \xi | w_1, w_2) = \frac{1}{w_1} \left( \sum_{k=0}^{d-1} \binom{n}{k} w_1^{k-1} w_2^{n-k} B_{n-k, \chi, \xi}^{(m-1)}(w_1 y) \sum_{i=0}^{w_1 d-1} \chi(i) \xi^{w_1 i} B_{k, \chi, \xi}^{(m)}(w_2 x + \frac{w_2}{w_1} i) \right) t^n \quad \text{for } \frac{y}{t} \geq 0. \tag{2.10}
\]

Comparing the coefficients on the both sides of (2.9) and (2.10), we obtain the following theorem which shows the relationship between the power sums and the generalized twisted Bernoulli polynomials.

**Theorem 3.** Let \(d, w_1, w_2 \in \mathbb{N}\). For \(n \in \mathbb{Z}_+\) and \(m \in \mathbb{N}\), we have

\[
\sum_{k=0}^{n} \binom{n}{k} w_1^{k-1} w_2^{n-k} B_{n-k, \chi, \xi}^{(m-1)}(w_1 y) \sum_{i=0}^{w_1 d-1} \chi(i) \xi^{w_1 i} B_{k, \chi, \xi}^{(m)}(w_2 x + \frac{w_2}{w_1} i) = \sum_{k=0}^{n} \binom{n}{k} w_1^{k-1} w_2^{n-k} B_{n-k, \chi, \xi}^{(m-1)}(w_1 y) \sum_{i=0}^{w_2 d-1} \chi(i) \xi^{w_2 i} B_{k, \chi, \xi}^{(m)}(w_1 x + \frac{w_1}{w_2} i).
\]

**Remark 2.** Let \(m = 1\) and \(y = 0\) in Theorem 3. Then it follows that

\[
w_1^{n-1} \sum_{i=0}^{w_1 d-1} \chi(i) B_{n, \chi, \xi}^{(1)}(w_2 x + \frac{w_2}{w_1} i) = w_2^{n-1} \sum_{i=0}^{w_2 d-1} \chi(i) B_{n, \chi, \xi}^{(1)}(w_1 x + \frac{w_1}{w_2} i). \tag{2.11}
\]

Moreover, if we take \(x = 0\) and \(y = 0\) in Theorem 3, then we have the following identity for the generalized twisted Bernoulli numbers of higher order.

**Corollary 4.** Let \(d, w_1, w_2 \in \mathbb{N}\). For \(n \in \mathbb{Z}_+\) and \(m \in \mathbb{N}\), we have

\[
\sum_{k=0}^{n} \binom{n}{k} w_1^{k-1} w_2^{n-k} B_{n-k, \chi, \xi}^{(m-1)}(w_1 y) \sum_{i=0}^{d w_1 - 1} \chi(i) \xi^{w_1 i} B_{k, \chi, \xi}^{(m)}(\frac{w_2}{w_1} i) = \sum_{k=0}^{n} \binom{n}{k} w_2^{k-1} w_1^{n-k} B_{n-k, \chi, \xi}^{(m-1)}(w_2 y) \sum_{i=0}^{d w_2 - 1} \chi(i) \xi^{w_2 i} B_{k, \chi, \xi}^{(m)}(\frac{w_1}{w_2} i).
\]

If we take \(m = 1\) in Corollary 4, we derive the identity for the generalized twisted Bernoulli numbers : for \(d, w_1, w_2 \in \mathbb{N}\) and \(n \in \mathbb{Z}_+\),

\[
w_1^{n-1} \sum_{i=0}^{d w_1 - 1} \chi(i) \xi^{w_1 i} B_{n, \chi, \xi}^{(1)}(\frac{w_2}{w_1} i) = w_2^{n-1} \sum_{i=0}^{d w_2 - 1} \chi(i) \xi^{w_2 i} B_{n, \chi, \xi}^{(1)}(\frac{w_1}{w_2} i). \tag{2.12}
\]
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