Implication of the overlap representation for modelling generalized parton distributions

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Abstract

Based on a field theoretically inspired model of light-cone wave functions, we derive valence-like generalized parton distributions and their double distributions from the wave function overlap in the parton number conserved $s$-channel. The parton number changing contributions in the $t$-channel are restored from duality. In our construction constraints of positivity and polynomiality are simultaneously satisfied and it also implies a model dependent relation between generalized parton distributions and transverse momentum dependent parton distribution functions. The model predicts that the $t$-behavior of resulting hadronic amplitudes depends on the Bjorken variable $x_{Bj}$. We also propose an improved ansatz for double distributions that embeds this property.

Keywords: generalized parton distributions, overlap representation, duality, spectator model

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Since the discovery that the proton is a composed system, enormous amount of efforts have been made in order to reveal its contents and understand the dynamics of its constituents. Various frameworks and models have been simultaneously proposed; often based on the quantum mechanical concept of the proton wave function. Factorization theorems, derived in the framework of perturbative Quantum Chromodynamics (QCD), are the basis to relate non-perturbative parton distribution functions (PDFs), distribution amplitudes, and generalized parton distributions (GPDs) with experimental observables. It is obvious that these non-perturbative quantities are somehow related to the proton wave function. However, quantifying this relation starts with the conceptional difficulty of defining the bound state problem in the relativistic quantum field theory, and it ends up with the difficulty in matching the evaluated (generalized) parton distributions with those that are defined implicitly in the perturbative factorization approach.

The idea to write down the proton wave function in terms of the partonic degrees of freedom was spelled out in the early days [1, 2, 3]. Thinking of the proton as a bunch of partons that move nearl y on the light-cone, e.g., specified by \( n^\mu = (1, 0, 0, -1) \), allows more easily to establish the desired link, see Refs. [4, 5] and references therein. This leads to the concept of light-cone (LC) wave functions \( \psi_{n}^{\uparrow, \downarrow} \). They are the probability amplitudes for their corresponding \( n \)-parton states \( |n, \mathbf{p}_{i}^{+}, \mathbf{p}_{\perp i}, \lambda_{i} \rangle \), which build up the proton state with LC helicity \( S = \{+1/2(\uparrow), -1/2(\downarrow)\} \):

\[
|P, S = \{\uparrow, \downarrow\} \rangle = \sum_{n} \int dX_{i} d^{2}k_{\perp i} \psi_{n}^{\uparrow, \downarrow} \left( X_{i}, \mathbf{k}_{\perp i}, \lambda_{i} \right) \prod_{j=1}^{n} \frac{1}{\sqrt{X_{j}}} |n, X_{i} P^{+}, X_{i} P_{\perp} + \mathbf{k}_{\perp i}, \lambda_{i} \rangle ,
\]

where we used a shorthand for the \( n \)-parton phase space:

\[
[dX d^{2}k]_{n} = \prod_{i=1}^{n} dX_{i} d^{2}k_{\perp i} \left( \frac{1}{16\pi^{3}} \right)^{n} \delta^{(2)} \left( 1 - \sum_{j=1}^{n} X_{j} \right) \delta^{(2)} \left( \sum_{j=1}^{n} k_{\perp i} \right) .
\]

The LC wave functions depend on the longitudinal momentum fractions \( X_{i} = k_{i}^{+}/P^{+} \) (the plus component of a four-vector \( V^{\mu} \) is \( V^{+} = V^{0} + V^{3} = n \cdot V \)), the transverse momenta \( \mathbf{k}_{\perp i} \), and the LC helicities \( \lambda_{i} \). They are determined from the eigenvalue problem for the LC Hamiltonian \( P^{−} \):

\[
P^{−}|P, S\rangle = \frac{M^{2}}{P^{+}}|P, S\rangle , \quad \text{with} \quad P^{−} = P^{0} - P^{3} , \quad P^{+} = P^{0} + P^{3} , \quad \mathbf{P}_{\perp} = 0 ,
\]

which can be derived from the QCD Lagrangian. However, in practice the QCD dynamics is not well understood and it remains very challenging to develop this concept to a stage at which it can be used for quantitative evaluations of physical observables or parton distributions [5]. It is common to pin down LC wave functions by comparing their resulting model predictions with experimental observations. Certainly, so far this concept is extensively elaborated, it connects different non-perturbative quantities, like PDFs, transverse momentum dependent parton distribution functions (TMDs), distribution amplitudes, form factors, and GPDs to a more universal object.
Our study is mainly devoted to GPDs, which are accessible from hard-exclusive reactions, e.g., electroproduction of mesons and photon. They arise from the non-diagonal overlap of LC wave functions, and therefore contain a maximum of information about the struck quark wave function, compared to other non-perturbative quantities. For instance, they are reducible to elastic form factors and PDFs. Field theoretically they are defined as off-diagonal matrix elements of two field operators that live on the light cone \([6, 7, 8]\). Describing the initial and final proton states, with given momenta \(P_1\) and \(P_2 = P_1 - \Delta\) and LC helicities \(S_1\) and \(S_2\), respectively, in terms of the LC wave functions (1), one can straightforwardly derive wave function overlap representations of GPDs for the partonic s-channel exchange \([9, 10, 11]\). In this partonic process the number of partons is conserved and the momentum fraction \(x\) of the struck quark is larger than the skewness parameter \(\eta = (P_1^+ - P_2^+)/\left(\sqrt{P_1^+ + P_2^+}\right) > 0\) (up to a minus sign we use the variable conventions of \([6]\)). In this outer region of \(x\), the GPDs \(H\) and \(E\) (Ji’s conventions \([12]\)) read

\[
\left(H - \frac{n^2}{1 - n^2}\right)E(x) = \eta, \eta, t = \frac{2 - \zeta}{2\sqrt{1 - \zeta}} \sum_n \sum_{\lambda_i} \sqrt{1 - \zeta^{-2^n}} \int dX d^2k_{1, n} \delta(X - X_1)\psi_n^{\ast}(X_i', k_{1, i}', \lambda_i)\psi_n(X_i, k_{1, i}, \lambda_i),
\]

\[
\frac{\Delta_1 - i\Delta_2}{2M}E(x) = \eta, \eta, t = \sqrt{1 - \zeta} \sum_n \sum_{\lambda_i} \sqrt{1 - \zeta^{-2^n}} \int dX d^2k_{1, n} \delta(X - X_1)\psi_n^{\ast}(X_i', k_{1, i}', \lambda_i)\psi_n(X_i, k_{1, i}, \lambda_i),
\]

where \(\zeta = 2\eta/(1 + \eta), X = (x + \eta)/(1 + \eta)\), and the momenta of the outgoing partons are

\[
X_1' = \frac{X_1 - \zeta}{1 - \zeta}, \quad k_{1, i}' = k_{1, i} - \frac{1 - X_1}{1 - \zeta} \Delta_{1, i} \quad \text{for the struck quark},
\]

\[
X_i' = \frac{X_i}{1 - \zeta}, \quad k_{1, i}' = k_{1, i} + \frac{X_i}{1 - \zeta} \Delta_{1, i} \quad \text{for the spectators} \quad i = 2, \ldots, n.
\]

Anti-quark GPDs are analogously defined with a negative momentum fraction \(x \leq -\eta\). The central region, i.e., \(-\eta \leq x \leq \eta\), arises from the t-channel process in which the parton number changes from \(n + 2\) to \(n\) \([10, 11]\). Viewing this as a mesonic-like t-channel exchange makes contact to Regge phenomenology \([13, 14, 15]\). Note that positivity constraints, in its most general form \([16]\), should be satisfied in the overlap representations \([17]\), if they are not spoiled by subtraction procedures. Indeed, this can be easily shown for a two-body LC wave function, as used below.

Let us also remind of the constraints of Lorentz covariance for (quark) GPD form factors. They are not invariant under general Lorentz transformations, however, they are built by a series of local twist-two operator matrix elements, belonging to irreducible representations, that are labelled by the spin \(J \geq 1\). It turns out that Mellin moments of GPDs \(H + E\) and \(E\) with the weight \(x^{J-1}\) are polynomials in \(\eta\) of the order \(J - 1\) and \(J\), respectively. Time reversal invariance combined
with hermiticity requires that these polynomials are even [12]. These properties are manifestly implemented in the double distribution (DD) representation of GPDs [6, 7]:

$$\left\{ \frac{H + E}{E} \right\} (x, \eta, t) = \int_0^1 dy \int_{-1+y}^{1-y} dz \left\{ \frac{1}{1 - x} \right\} \delta(x - y - \eta z) \left\{ \frac{h + e}{e} \right\} (y, z, t), \quad (8)$$

where the DDs $e$ and $h$ are symmetric in $z$. The DD representation for $E$ is not uniquely defined [18, 19]; Eq. (8) shows the one which naturally occurs in our model studies, see below Eq. (20).

As explained above, the partonic interpretation of GPDs separates the support in the central ($|x| \leq \eta$) and outer ($\eta \leq |x|$) regions. Residual Lorentz covariance, ensuring the polynomiality of moments, requires that both regions are tied to each other and that the functional form of GPD is constrained, e.g., in the outer region the GPD $E$ is given by the integral representation:

$$E(x \geq \eta, \eta, t) = (1 - x) \int_{\frac{1+x}{1-x}} dy \frac{e}{\eta} e(y, (x - y)/\eta, t). \quad (9)$$

We consider both regions as dual to each other, i.e., knowing a GPD in one region allows to restore it in the other. Hence, a GPD can be entirely evaluated from the parton number conserved s-channel overlap of LC wave functions. A constructive, however, unwieldy method for the restoration of the central region is based on Mellin moments and its inverse transformation [14].

In the following we utilize two-particle LC wave functions. They serve us to describe the proton contents by a constituent quark and scalar diquark, where the latter plays the role of an collective spectator. Numerous investigations in this spirit, even much more advanced ones with specific emphases, e.g., Refs. [20, 21, 9, 22, 23, 24, 25, 26], have been made in the past. The new aspect in our study is that we take care of Lorentz constraints for the LC wave functions, cf. Refs. [27, 28]. This allows us to evaluate\(^1\) DDs from the parton number conserving overlap representations (4) and (5). We outline the generic features of such model, illuminate their $t$-dependence, point out its restrictions, and overcome them by hand, yielding improved DD and GPD ansatze.

ii. The functional form of LC wave functions is dictated by the underlying Lorentz symmetry, i.e., the longitudinal and transversal variables are tied to each other in a certain but not apparent manner. Hence, writing down LC wave functions by hand usually results in a violation of the GPD polynomiality property. Note that this failure can not be fixed by taking into account the particle number changing processes. The guidance for an appropriate model comes from a perturbative calculation in lowest order [30]. We employ the Yukawa theory and have the LC wave functions for four helicities,

$$\psi_{+1/2}^\dagger(X, k_\perp) = \psi_{-1/2}^\dagger(X, k_\perp) = \left( M + \frac{m}{X} \right) \varphi(x, k_\perp),$$

$$\psi_{+1/2}^\dagger(X, k_\perp) = \frac{k_1 - ik_2}{X} \varphi(X, k_\perp), \quad \psi_{-1/2}^\dagger(X, k_\perp) = -\frac{k_1 + ik_2}{X} \varphi(X, k_\perp), \quad (11)$$

\(^1\)Of course, this task is straightforwardly done in a Lorentz covariant formalism [29].
in terms of a scalar function \( \varphi(X, k_\perp) \). This scalar function arises from the spectator propagator in a triangle Feynman diagram [30, 31] and so the underlying Lorentz symmetry is respected. We generalize \( \varphi(X, k_\perp) \) by an adjustment of its power behavior \( p \):

\[
\varphi(X, k_\perp) = \frac{g M^{2p}}{\sqrt{1 - X}} X^{-p} \left( M^2 - \frac{k_\perp^2 + m^2}{X} - \frac{k_\perp^2 + \lambda^2}{1 - X} \right)^{-p-1},
\]

where \( M, \lambda \) and \( m \) are the proton, spectator, and quark masses, respectively. Note that the factor \( X^{-p} \) takes care of the proper Lorentz behavior and that the Yukawa theory result is for \( p = 0 \).

The GPDs are now evaluated in the outer region from the overlap representations (4) and (5),

\[
\left( H - \frac{\eta^2}{1 - \eta^2} E \right) (x \geq \eta, \eta, t) = \frac{2 - \zeta}{2\sqrt{1 - \zeta}} \int \frac{d^2k_\perp}{16\pi^3} \left[ \psi_{+1/2}^\dagger(X', k'\perp) \psi_{+1/2}(X, k\perp) + \psi_{-1/2}^\dagger(X', k'\perp) \psi_{-1/2}(X, k\perp) \right],
\]

for the two body wave functions (10)–(12). For \( p > 0 \) the \( k_\perp \) integrals are finite, while for \( p = 0 \) the representation (13) suffers from an ultraviolet divergence. We find for the GPD \( E \):

\[
E(x \geq \eta, \eta, t) = \frac{g^2}{(4\pi)^2} \frac{2\Gamma(2p + 1)}{\Gamma(p + 1)^2} \int_0^1 d\alpha \frac{[(1 - X)(1 - X')]}{(1 - \alpha^2)^{p+1}} \left[ (\alpha \overline{\alpha})^p \left( \frac{m}{M} + X - \alpha(1 - X') \right) \right],
\]

where \( t_{\text{min}} = -\zeta^2 M^2/(1 - \zeta), \overline{\alpha} = 1 - \alpha \), and the mass terms are collected in

\[
f(X|\overline{\alpha}) = \overline{\alpha} \left( (1 - X) \frac{m^2}{M^2} + X \frac{\lambda^2}{M^2} - X(1 - X) \right).
\]

The result for \( H \) has a similar structure and will not be displayed for shortness.

Since our model respects the underlying Lorentz symmetry, there must be now a possibility to transform the overlap result (15) into the form of the DD representation (9). In fact, this can be simply achieved by a linear variable transformation of the integration parameter

\[
\alpha = \frac{1}{2} \frac{1 - \eta}{1 - x} \left( 1 - y + \frac{x - y}{\eta} \right)
\]

and removing the residual skewness dependence by using \( x = y + \eta z \). We arrive at

\[
E'(x \geq \eta, \eta, t) = (1 - x) \int_{\frac{\zeta - \eta}{1 - \eta}}^{\frac{\zeta + \eta}{1 - \eta}} \frac{dy}{\eta} e \left( y, \frac{x - y}{\eta}, t \right),
\]

where the DD is given by

\[
e(y, z, t) = N \frac{\left( \frac{m}{M} + y \right) ((1 - y)^2 - z^2)^p}{\left( (1 - y) \frac{m^2}{M^2} + y \frac{\lambda^2}{M^2} - y(1 - y) - ((1 - y)^2 - z^2) \right)^{2p+1}}.
\]
Here we absorbed several factors, including $g^2$, in the normalization constant $N$. From the DD (19) and Eq. (8), we find $E$ for the central region, arising from parton number changing processes:

$$E(-\eta \leq x \leq \eta, \eta, t) = (1 - x) \int_0^{\frac{\eta + \eta}{2}} \frac{dy}{\eta} e\left(\frac{y, x - y}{\eta}, t\right).$$  \tag{20}

The evaluation of the GPD $H$ in terms of the DD $h$ goes along the same line. The combination $(H + E)(x, \eta, t)$ can be written in the form of the DD representation (8) with

$$\left(1 - \frac{2y}{M^2}\right) \left(1 - \frac{2y}{M^2}\right) $$

\[= \frac{1}{4p} \left[\frac{2m^2 + y + y^2}{M^2} - y(1 - y) - \frac{(1 - y)^2 - z^2}{t^2 M^2}\right]^{2p} + \frac{N}{2 \left[\frac{2m^2 + y + y^2}{M^2} - y(1 - y) - \frac{(1 - y)^2 - z^2}{t^2 M^2}\right]^{2p+1}}. \tag{21}\]

Finally, the overall constant $N$ is fixed by the normalization condition:

$$\int_{-1}^{1} dx H(x, \eta, t = 0) = \int_{0}^{1} dy \int_{-1+y}^{1-y} dz (h + e)(y, z, t = 0) = 1. \tag{22}\]

iii. We employ now our simple-minded model to evaluate form factors, TMDs, PDFs, and GPDs as net contributions of valence-like quarks in the proton, i.e., we deal with the isospin combination

$$H(x, \eta, t) = H_{u_{\text{val}}}(x, \eta, t) - H_{d_{\text{val}}}(x, \eta, t), \quad E(x, \eta, t) = E_{u_{\text{val}}}(x, \eta, t) - E_{d_{\text{val}}}(x, \eta, t). \tag{23}\]

We have only three model parameters, i.e., power behavior ($p$) of the LC wave function, spectator ($\lambda$) and quark ($m$) masses. Our goal in doing so is to explore this model and find its restrictions.

We start with the electromagnetic form factors, which are obtained from the Drell-Yan-West formula or, equivalently, by integrating out the momentum fraction dependence in GPDs:

$$F_1(t) = \int_{-1}^{1} dx H(x, \eta, t), \quad F_2(t) = \int_{-1}^{1} dx E(x, \eta, t). \tag{24}\]

The Dirac form factor is normalized by $F_1(t = 0) = 1$, cf. Eq. (22). The asymptotic drop off for large $-t$ is estimated to be $1/(-t)^2$, according to field theoretically inspired [2] and phenomenological [3] model counting rules and the perturbative analysis [32]. This suggest to set $p = 1$. We use the charge radius squared $R^2 = 6dF_1(t)/dt|_{t=0}$ and the anomalous magnetic moment $\kappa = F_2(t = 0)$ of the proton to pin down the remaining two parameters. The following plausible mass values yield an agreement with experimental measurements, cf. Ref. [20]:

$$p = 1, \quad \lambda = 0.75 \text{ GeV}, \quad m = 0.45 \text{ GeV} \Rightarrow R = 0.76 \text{ fm}, \quad \kappa = 1.78. \tag{25}\]

The form factor $F_1(t)$ fairly agrees with experimental data as shown in Fig. 1(a) and behaves
for $-t \lesssim 100 \text{GeV}^2$ as $1/(-t)^2$, however, in the large $-t$ asymptotics it drops faster. We would conclude, in agreement with aforementioned counting rules, that $F_2(t)$ decreases as $1/(-t)^3$. This follows from counting the powers of $\Delta^2_\perp$ in Eq. (14), arising from $k_\perp$-dependence of wave functions (10-12), and the vanishing of the overlap integral for $E$ if spherical symmetry is restored in the asymptotic $\Delta^2_\perp \to \infty$. However, we observe that the limit $\Delta^2_\perp \to \infty$ can not be taken before the $k_\perp$ integration, since divergences appear and render the integral to be infinite. This behavior shows up also in the DD (19), where the limit $-t \to \infty$ causes end-point singularities at $z = \pm(1-y)$. Hence, we effectively find a $F_2(t) \sim 1/(-t)^2$ behavior and thus naive power counting fails. We remind that the assumptions for the counting rules were carefully spelled out [2, 3, 32, 33, 35, 36].

End-point singularities arise also in the perturbative evaluation of $F_2$ and their regularization yields a logarithmical modification of the $1/(-t)^3$ scaling [37, 38]. The experimental measurements indicate a $F_2(t)/F_1(t) \propto 1/\sqrt{-t}$ scaling [34, 35, 36], which might be also interpreted as a logarithmical modified $F_2(t)/F_1(t) \propto 1/(-t)$ scaling. Interestingly, the anomalous break down of the naive power counting in our scalar diquark model leads to a $F_2(t)/F_1(t) \propto \text{const.}$ scaling. Applying the recipe of end-point regularization to our result, i.e., imposing the constraint $0.33 \leq 1-y-|z|$, we easily can fit the experimental data as demonstrated in Fig. 1(b). This exercise should not be considered as a serious attempt to explain experimental data, rather as a demonstration showing that fitting can be done easily. One might wonder whatever this failure of the scalar diquark model is related to the oversimplified spin coupling and could be cured by inclusion of an axial-vector diquark, or it might simply reflect a wrong implementation for the large $k_\perp^2$ behavior of the LC wave functions.

Our model also predicts TMDs and their $k_\perp$-integrated PDFs outcome. Note that the TMD concept has numerous fundamental issues, see, e.g., Ref. [39]. We might define here a unpolarized valence-like TMD in terms of the LC wave functions (10-12) overlap, where $\zeta = 0$ and $t = 0$:

$$q(x, k_\perp) = \frac{g^2 M^{4p} [(xM + m)^2 + k_\perp^2] (1-x)^{2p+1}}{[k_\perp^2 + (1-x)m^2 + x\lambda^2 - x(1-x)M^2]^{2p+2}}. \tag{26}$$
At large $k_t^2$ they fall off with $1/(k_t^2)^{2p+1}$ and are suppressed in the large $x$ region by $(1 - x)^{2p+1}$. Our model is similar in spirit to the spectator model utilized in Ref. [40] for the evaluation of $k_\perp$-(un)integrated parton densities and fragmentation functions. There the fermionic propagators are replaced by quark-diquark form factors with a cut-off mass $\Lambda$, while in our case they are taken to be on-shell. Replacing $m$ by $\Lambda$, we find that Eq. (26) is identical with the result (80) in Ref. [40].

The valence-like PDF is obtained from (26) and is related to the GPD $H$:

$$q(x) \equiv H(x, \eta = 0, t = 0) = \frac{\pi g^2 M^{4p+2}}{2p(2p+1)} \frac{(2p-x+1)m^2}{M^2} + \frac{4pxm}{M^2} + x\lambda^2 + x(2px + x - 1)}{(x\lambda^2 + (1 - x)(m^2 - M^2x))^{2p+1}}(1 - x)^{2p+1}. \quad (27)$$

For $p = 1$ we obtain the generic behavior of parton densities at large $x$. Note that the form factor $F_1(t)$ falls at large $-t$ with $(-t)^{-p-1}$. Hence, setting $p = n - 1$ we confirm the known counting rules. The momentum fraction carried by the valence quark combination $u - d$,

$$\langle x \rangle = \int_0^1 dx \ x \ q(x), \ q \equiv q_{\text{val}} - q_{\text{dual}}, \quad (28)$$

is $\langle x \rangle \approx 0.26$ with our parameter specification (25). It nearly agrees with the value in the Glück, Reya, and Vogt parameterization [41], given at a low input scale $\mu_0^2 = 0.4 \text{ GeV}^2$ (to perturbative next-to-leading order accuracy): $\langle x \rangle_{\text{val}} - \langle x \rangle_{\text{dual}} \approx 0.24$. This amazing coincidence of momentum fractions should be considered rather as an accident. For $x$ going to zero, the PDF (27) approaches a constant. This behavior we consider as an unrealistic feature of the diquark model. The small $x$-region, i.e., the high-energy limit might be understood in the Regge picture as an exchange of mesons in the $t$-channel which leads to the expectation that the valence-like parton densities behave as $x^{-\alpha(0)}$, where $\alpha(0)$ is the (effective) intercept of the meson trajectory. From the $s$-channel view, which we take, the true small $x$-behavior arises by summing up all Fock state components.

We come now to the GPDs. First, we comment on our duality assumption between the partonic $s$-channel and mesonic like $t$-channel exchange. We could have added to the central region a term

$$D(x/\eta, t) = \theta(|x| \leq |\eta|) \ d \left( \frac{x}{\eta}, t \right), \quad \text{with} \quad d(1, t) = d(-1, t) = 0 \quad (29)$$

that entirely lives in the central region, vanishes at the cross over point and is anti-symmetric in $x$. An explicit GPD evaluation from the parton number changing processes confirms that such a term is absent and so the underlying field theoretical model respects duality. However, such a $D$-term is needed to cure the common DD representation for the GPD $E$ or $H$ [42]:

$$\begin{cases}
H \\ E
\end{cases} (x, \eta, t) = \int_0^1 dy \int_{-1+y}^{1-y} dz \ \delta(x - y - \eta z) \ \left\{ h^\text{Rad}_{\eta, e} \right\} (y, z, t) \pm D(x/\eta, t). \quad (30)$$

Our findings suggest that both $e^{\text{Rad}}(y, z, t)$ and $d(x/\eta, t)$ have a cross talk. Indeed, they are two different projections of our DD (19); the $D$-term is extracted from $\eta \rightarrow \infty$ with fixed $x/\eta$ [43]:

$$d(x, t) = x \int_0^{1-|x|} \ dy \ \frac{N \ (m^2_M + y) \ ((1 - y)^2 - x^2)^p}{[(1 - y)m^2_M + y(1 - y) - (1 - y)^2 - x^2)^{p+1}} \quad (31)$$
Interestingly, the coefficients in its Gegenbauer expansion [44] are rather small and sign alternating:

\[ d(x, t = 0) \approx (1 - x^2) \left[ 0.345 C_1^{3/2}(x) - 0.163 C_3^{3/2}(x) + 0.026 C_5^{3/2}(x) + \cdots \right]. \] (32)

The first coefficient is in astonishing agreement with the estimate of the chiral soliton model [45], given as 0.33 at an intrinsic scale of \( \mu_0^2 \sim 0.36 \text{GeV}^2 \). Note that the flavor singlet quark \( D \)-term in the chiral soliton model is predicted to be negative and large in magnitude [44, 46, 45].

In confronting GPDs with experimental data, one usually factorizes\(^2\) the \( t \)-dependence in the DD ansatz [47, 48, 44] (VGG refers to the popular code of Vanderhaeghen, Guichon, and Guidal):

\[ h_{VGG}(y, z, t) = F_1(t) \frac{q(y)}{1 - y} \Pi_{VGG}^{VGG}(z/(1 - y)), \quad \Pi_{VGG}^{VGG}(z) = \frac{\Gamma \left( b + \frac{3}{2} \right)}{\sqrt{\pi} \Gamma(b + 1)} (1 - z^2)^b. \] (33)

The even profile function \( \Pi_{VGG}^{VGG}(z) \) is chosen to be convex and normalized to \( \int_{-1}^{1} dz \Pi_{VGG}^{VGG}(z) = 1 \).

Let us compare the spectator model (8,19,21) with the \( t \)-factorized VGG ansatz (30,33). The latter we make from the PDF (27) and \( D \)-term (32), multiplied with the form factor (24). For the profile function we take \( \Pi_{VGG}^{VGG}(z) = (3/4)(1 - z^2) \), which also appears in our DD (21) with \( p = 1 \) at \( t = 0 \), cf. Ref. [47], and employ in both models the parameters (25). In Fig. 2 we show the shape of GPDs versus \( x \) and \( \eta \), where the momentum transfer squared is set to \( t = t_{\text{min}} - 0.25 \text{GeV}^2 \).

The differences between the spectator model (a) and the \( t \)-factorized (b) GPDs are clearly visible at larger values of \( \eta \), where the later ones are broader in their \( x \) distribution and have a smaller value [see also below Fig. 3(c)] on the trajectory \( x = \eta \), compared to the former ones. The important difference between the two models is in their \( t \)-dependence. In Fig. (3) we display in panel (a) the \( t \)-dependence of the ratio \( H(\eta, \eta, t)/H(\eta, \eta, 0) \) on the trajectory \( x = \eta \) for various values of \( \eta \). Obviously, the \( t \)-dependence is flattering out for larger values of \( \eta \), while in the small \( \eta \) region it even becomes steeper, compared to the \( t \)-factorized GPD ansatz (dashed line). Analogously,
we find for the ratio \( \int_0^1 dx x^n H(x,0,t) / \int_0^1 dx x^n H(x,0,0) \) of Mellin moments at \( \eta = 0 \) that the \( t \)-dependence becomes flatter with increasing spin \( n \) as shown in panel (b). Such a behavior was also seen in lattice calculations, see Ref. [49] and references therein.

As in the case of PDFs, the small \( x \)-behavior of GPDs in a spectator model should be considered as unrealistic. Since at \( x = \eta \) the momentum fraction \( X' \) vanishes in the overlap representation (13,14), we realize that on this trajectory realistic GPDs can be obtained only if the small \( X \) behavior of the wave functions is understood. This also means that one has to sum up all Fock state components, see discussion in Refs. [9, 28]. Note that already the evolution to leading order in the flavor singlet sector knows about the small \( x \) behavior, however, in the non-singlet sector one has to perform a resummation of \( t \)-channel exchanges, perhaps, along the line as suggested in Ref. [50]. For larger values of \( x \) on \( x = \eta \) one is forced to understand at the same time the large and small \( X \) behavior of the wave functions and their interference.

Having this warning in mind, we have now a look at the resulting amplitude, which appears

Figure 3: The \( t \)-dependence of \( H(x,\eta,t)/H(x,\eta,t=0) \) at \( x = \eta \) in the \( t \)-factorized ansatz (30,33) (thin dotted) and the spectator model within \( \eta = \{0.1(\text{dotted}),0.3(\text{dashed}),0.5(\text{dashdotted}),0.7(\text{solid})\} \) (a) and for the Mellin moments \( n = \{0(\text{dotted}),1(\text{dashed}),2(\text{dashdotted}),3(\text{solid})\} \) at \( \eta = 0 \) in the spectator model (b). Imaginary (c) and real (d) part of the amplitude (34) versus \( \xi \), arising from the spectator model (solid) and factorized ansatz (dashed), where the dotted curve shows the real part without \( D \)-term.
in the hard exclusive photon or $\rho^0$ electroproduction to leading order of perturbative QCD:

$$\mathcal{H}(\xi, t) = \int_{-1}^{1} dx \left[ \frac{1}{\xi - x - i\epsilon} - \frac{1}{\xi + x - i\epsilon} \right] H(x, \eta = \xi, t).$$

(34)

The imaginary and real part of the amplitude versus $\xi$ are shown in Fig. 3 (c) and (d), respectively, where we set $t = t_{\text{min}} - 0.25$. Note that the imaginary part is given by $\pi H(x = \xi, \xi, t)$ and provides us the GPD shape on the trajectory $x = \xi$. The differences between the spectator (solid curve) and $t$-factorized (dashed curve) models are obvious. Both the imaginary and real part for the spectator model is (much) more enhanced at large $\xi$. For $\xi \to 1$ the imaginary part (3(c), solid) behaves as $(1 - \xi)$, while for the $t$-factorized GPD ansatz an additional suppression factor appear: $(1 - \xi)(-t)^{-2} \lesssim (1 - \xi)(-t_{\text{min}})^{-2} \sim (1 - \xi)^3$. On the other hand, we observe that at smaller values of $\xi$ the imaginary part of the spectator model is getting smaller. We stress, however, that small $t$-physics is not implemented in the model. The sign change of the real part in panel (d), somewhere in the valence quark region, is a feature that is also observed in other valence-like GPD ansatze. The position of the zero, however, is floating and strongly depends on the chosen ansatz. At smaller values of $\xi$ the real parts of both models approach to each other. The dotted curve shows the $t$-factorized ansatz without $D$-term, leading only to a slight constant shift.

Concerning the results of our spectator model, we conclude that the $t$-dependence of cross sections for deeply electroproduction of mesons in the large $x_{\text{Bj}}$ region is getting flatter and that the cross section could be (much) larger than the prediction from a $t$-factorized GPD ansatz, cf. Figs. 3 (c) and (d). This is consistent with preliminary measurements of the $e^- p(P_1) \to e^- p(P_2) \rho^{(0)}$ cross section, where the $t$-slope decreases if larger values of $x_{\text{Bj}}$ and $Q^2$ are approached [51]. We find, for instance, that the slope for the amplitude square $|\mathcal{H}(\xi, t)|^2$, parameterized as $e^{bt}$, decreases from $b \approx 3.5 \text{ GeV}^{-2}$ at $x_{\text{Bj}} = 0.2$ and $Q^2 = 2 \text{ GeV}^2$ to $b \approx 1.5 \text{ GeV}^{-2}$ at $x_{\text{Bj}} = 0.6$ and $Q^2 = 5 \text{ GeV}^2$.

iv. In this paper we derived from the overlap representation of LC wave functions the valence-like GPDs and their relatives: proton form factors, TMDs, and PDFs. We generalized the LC wave functions from a scalar diquark spectator model in such a way that the non-manifest Lorentz behavior of the LC wave functions is respected. This is the key to obtain the DDs from the overlap representation in the partonic $s$-channel and then to restore the full GPD support.

Our model fairly describes the Dirac form factor $F_1(t)$ and the proton anomalous magnetic moment comes out correctly by a natural choice for the constituent quark and diquark masses. However, the model fails in the description of the $t$-dependence for the Pauli form factor $F_2(t)$. Namely, its naive power counting for the large $t$-behavior is spoiled by end-point singularities that appears at the branch point $-t = \infty$. Unpolarized valence-like TMDs and PDFs, also obtained in Ref. [40], have the expected generic behavior at large momentum fraction $x$ and the average value $\langle x \rangle$ fairly agrees with the result of Ref. [41], given at a low input scale.
The GPD models satisfy by construction the positivity and polynomiality constraints. The found DD representations (8) naturally completes the polynomiality condition and avoids so a ‘misleading’ $D$-term phenomenology. Another important characteristic property of the model is that the $t$-dependence of GPDs, resulting from the DDs (21) and (19), is washed out at large $x$. This behavior is rooted in the fact that the variables $t$ and $x$ are intrinsically correlated because the transverse and longitudinal degrees of freedom in the wave functions are tied by hidden Lorentz symmetry. This flattening of $t$-dependence also appears on the trajectory $x = \eta$ and therefore it can be confronted with experimental measurements. Note that the $t$-behavior of GPDs is only tested in lattice calculations for $\eta = 0$. From the overlap representation it is clear that the $t$-dependence of GPDs and the $k_\perp$-dependence of TMDs are closely related to each other, since both arise from the $k_\perp$-dependence of wave functions, a recent discussion is given in Ref. [52].

It is in the nature of a spectator model that it fails to describe the small momentum fraction behavior, which arises from the summation over all partonic Fock state components. Hence, there is a potential problem for the predicting power of such models for GPDs on the trajectory $x = \eta$, accessible in experiments. Here a GPD arises from the interference of the LC wave function at the extreme limit of vanishing momentum fraction with that of a non-vanishing momentum fraction, controlled by the skewness parameter. Therefore, even the limit $x = \eta \to 1$ is rather intricate [53].

It remains a challenging task to construct GPDs that satisfy all theoretical constraints and are flexible enough for a ‘global’ fit of experimental data. For the time being, we might suggest to adopt the $t$-dependence in the ansatz for the DD and improve its failure at small $y$, i.e, small $x$ for the resulting GPD, by hand. Guided by Eq. (19), we would suggest for $e$, e.g., the ansatz:

$$e(y, z, t) = \frac{q_E(y, t)}{(1 - y)} \frac{N(b, P, \alpha) \left[ (1 - y) \frac{m^2}{M^2} + y \frac{\lambda^2}{M^2} - y(1 - y) \right]^P \left[ (1 - y)^2 - z^2 \right]^b}{\left[ (1 - y) \frac{m^2}{M^2} + y \frac{\lambda^2}{M^2} - y(1 - y) - ((1 - y)^2 - z^2) \frac{t}{M^2} \right]^P \left(1 - y \right)^{2b+1}}. \quad (35)$$

The Regge improved PDF analog, generically parameterized by its large and small $x$ behavior

$$q_E(x, t) = \kappa \frac{(1 - \alpha(0)) \Gamma(2 - \alpha(t) + \beta)}{\Gamma(2 - \alpha(t)) \Gamma(1 + \beta)} x^{-\alpha(t)}(1 - x)^\beta, \quad \int_0^1 dx q_E(x, t = 0) = \kappa, \quad (36)$$

where $q_E(x, t)$ is normalized at $t = 0$ to the anomalous magnetic moment $\kappa$. The normalization

$$N(b, P, \alpha) = \frac{\Gamma \left( b + \frac{3}{2} \right) \Gamma(2 - \alpha(0) + \beta) \Gamma(2 - 2P - \alpha(t) + \beta)}{\sqrt{\pi} \Gamma(b + 1) \Gamma(2 - 2P - \alpha(0) + \beta) \Gamma(2 - \alpha(t) + \beta)} \quad (37)$$

is introduced in such a way that the $t$-dependence in the form factor roughly factorizes as $1/(1 - \alpha(t))$, resulting from the Regge behavior, times an impact form factor, behaving as $(-t)^{-P}$ for $t \to -\infty$. The parameter $b$ adjusts the power behavior of $E(\xi, \xi, t) \sim (1 - \xi)^{1+b}$ at large $\xi$ and fixed $t$. The parameters should be fixed from fitting of observables; their generic values read in
acCORDANCE WITH REGGE PHENOMENOLOGY, COUNTING RULES [2, 3, 32, 53], AND THE SPECTATOR MODEL:

\[ \alpha(t) \sim 0.5 + 0.9t \text{GeV}^{-2}, \quad \beta \sim 5, \quad P \sim 2, \quad b \sim 1, \quad \lambda \sim 0.8 \text{GeV}, \quad m \sim 0.4 \text{GeV}. \]

the model features, we spelled out here in momentum fraction representation, are also implemented in the Mellin space GPD ansatz [14, 15]. We emphasize that the ansatz (35) is still not optimal, since a flexible adjustment of the resulting magnitude for the amplitude at given \( t \), in particular at small \( \xi \), is not incorporated and positivity constraints are no more automatically satisfied. A more detailed discussion of improved GPD ansatze should be given somewhere else.

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