We investigate the effect of stochastic control errors on the Hamiltonian that controls a closed quantum system. Quantum information technologies require careful control for preparing a desired state used as an information resource. However, because the stochastic control errors inevitably appear in realistic situations, it is difficult to completely implement the control Hamiltonian. Under this error, the actual performance of quantum control is far away from the ideal one, and thus it is of great importance to evaluate the effect of the control errors. In this paper, we derive a lower bound of the fidelity between two closed quantum systems obeying the dynamics with and without errors. This bound reveals a reachable and unreachable set of the controlled quantum system under stochastic noises. Also, it is easily computable without considering the stochastic process and needing the full dynamics of the states. We demonstrate the actual performance of this bound via a simple control example. Furthermore, based on this result, we quantitatively evaluate the probability of obtaining the target state in the presence of control errors.

1. Introduction

Recently, there has been considerable interest in quantum information technologies including quantum computing. To realize these technologies, it is necessary to prepare a desired quantum state used as an information resource. Thus, an accurate technique for manipulating the quantum system and preparing the desired target state plays an essential role [1]. When the effect of external environments such as decoherence can be ignored,
the ideal time evolution of the quantum system is described by the Schrödinger equation. Therefore, the target state can be prepared by suitably implementing the control Hamiltonian. For example, the schemes of quantum annealing [2–5] or quantum adiabatic computation [6–10] are modelled by the Schrödinger equation.

However, it is difficult to completely implement the control Hamiltonian without any errors in realistic situations. In particular, considering experimental control set-up, analogue control errors on the parameters are inevitable. Mainly, there are two types of analogue control errors; deterministic control error and stochastic control error. Deterministic control error that deterministically effects quantum dynamics, such as a bias of the magnetic field, have been investigated so far in several studies [11–13]. Also, stochastic control errors, such as a stochastic fluctuation occurring on the control Hamiltonian at each time, is one of the critical obstacles to overcome [14–18]. In unitary dynamics, stochastic errors can be formulated as a time-varying stochastic noise, and the time evolution under it can be described by the stochastic differential equation. Although some studies have investigated the stochastic noise [16–18], none have looked at its effect in a general and rigorous manner.

This paper aims to quantitatively investigate the influence of the stochastic errors on quantum state preparation. We present a lower bound of the fidelity between the two closed quantum states obeying the dynamics with and without stochastic control errors. This bound is computable without needing the full dynamics of the states. Also, it gives a quantitative limit on the Hamiltonian control for preparing the quantum state under control errors as a function of the noise strength. Furthermore, based on this result, we quantitatively evaluate the probability of obtaining the target state in the presence of control errors.

2. Lower bound for the closed system under stochastic noise

(a) Dynamics of the closed system

We begin with the explanation of the set-up. Let \( |\psi(t)\rangle \) be the quantum state in the absence of the stochastic control errors, which obeys the following equation:

\[
\frac{d|\psi(t)\rangle}{dt} = -i\hat{H}(t)|\psi(t)\rangle \quad \text{and} \quad |\psi(0)\rangle = |\psi_0\rangle,
\]  

where \( \hat{H}(t) \) is the time-dependent Hamiltonian that controls the quantum system (we set \( \hbar = 1 \)). We assume that \( |\psi(t)\rangle \) reaches the target state \( |\psi(T)\rangle \) at final time \( t = T \).

Next, we consider another state \( |\phi(t)\rangle \) driven by \( \hat{H}(t) \) and \( \sum_k \hat{B}_k(t) \). \( \hat{B}_k(t) \) denotes a kth stochastic control error and satisfies \( \hat{B}_k^\dagger(t) = \hat{B}_k(t) \). In this case, it is known that the time evolution of \( |\phi(t)\rangle \) is described by the following stochastic differential equation [17]:

\[
\frac{d|\phi(t)\rangle}{dt} = -\left(i\hat{H}(t)dt + \sum_k \hat{B}_k(t) \circ dW_k(t)\right)|\phi(t)\rangle,
\]  

where the symbol \( \circ \) denotes the Stratonovich interpretation and \( W_k(t) \) is the Gaussian Wiener process satisfying \( dW_k^2(t) = dt \). Another expression of this stochastic process is written by

\[
\frac{d|\phi(t)\rangle}{dt} = -\left(i\hat{H}(t) + \frac{1}{2} \sum_k \hat{B}_k^2(t)\right)|\phi(t)\rangle dt - i \sum_k \hat{B}_k(t)|\phi(t)\rangle \bullet dW_k(t),
\]  

where the symbol \( \bullet \) is the Itō interpretation. By modifying the stochastic process from the Stratonovich form to the Itō form, we can take the expectation \( \mathbb{E} \), as will be seen later. Moreover, we impose the following condition on \( \hat{B}_k(t) \):

\[
\hat{B}_k^2(t) = \gamma_k^2(t)^2 \hat{I},
\]  

where \( \hat{I} \) denotes the identity operator and \( \gamma_k(t) \) is the time-dependent noise strength. For example, when \( \hat{B}_k(t) \) is constructed from the tensor product of the Pauli matrices \( \hat{B}_k(t) = \gamma_k(t) \hat{S} \otimes \hat{S} \otimes \cdots \),
this error condition (2.4) is satisfied. However, for example, when \( \hat{B}_k(t) \) corresponds to the spin angular momentum operator \( \hat{B}_k(t) = \gamma_k(t) \sum_j \hat{S}^{(j)} \) or the annihilation or creation operator, equation (2.4) is not satisfied.

Under the condition (2.4), equation (2.3) becomes

\[
d\phi(t) = -\left( i\hat{H}(t) + \frac{1}{2} \sum_k \gamma_k^2(t) \hat{I} \right) \phi(t) dt - i \sum_k \hat{B}_k(t) \phi(t) \bullet dW_k(t). \tag{2.5}
\]

Taking the expectation of both sides of equation (2.5) with respect to \( dW_k(t) \), due to \( \mathbb{E}[dW_k(t)] = 0 \), we have

\[
\frac{d\mathbb{E}[\phi(t)]}{dt} = -\left( i\hat{H}(t) + \frac{1}{2} \sum_k \gamma_k^2(t) \hat{I} \right) \mathbb{E}[\phi(t)]. \tag{2.6}
\]

Now we assume that the initial states of \( |\psi(t)\rangle \) and \( |\phi(t)\rangle \) are identical \( |\psi(0)\rangle = |\phi(0)\rangle = |\psi_0\rangle \). Then, the desired final state \( |\psi(T)\rangle \) is given by

\[
|\psi(T)\rangle = \exp_+ \left( -i \int_0^T \hat{H}(t) dt \right) |\psi_0\rangle, \tag{2.7}
\]

where \( \exp_+ \) denotes the time-ordered exponential. From this \( |\psi(T)\rangle \) and the fact that the terms \( \gamma^2(t) \hat{I} \) commute with \( \hat{H}(t) \) for all \( t \), we can write the expectation state \( \mathbb{E}[|\phi(T)\rangle] \) as follows:

\[
\mathbb{E}[|\phi(T)\rangle] = \exp \left( -\frac{1}{2} \int_0^T \sum_k \gamma_k^2(t) dt \right) |\psi(T)\rangle. \tag{2.8}
\]

This indicates that the stochastic control error makes the final obtained state \( \mathbb{E}[|\phi(T)\rangle] \) away from the ideal one exponentially. Even if \( \sum_k \gamma_k(t) \) is small, as the driving time becomes longer, the control achievement degrades. Therefore, it is important to quantify how much the preparation of a desired state can be achieved under stochastic noises.

**b) Derivation of the lower bound**

Consider the fidelity between \( |\psi(t)\rangle \) and \( |\phi(t)\rangle \)

\[
F(t) := |\langle \psi(t) | \phi(t) \rangle|^2, \tag{2.9}
\]

where \( 0 \leq F(t) \leq 1 \). \( F(t) \) is employed as a cost function in various quantum science scenarios, because it can be easily computed both analytically and numerically [1]. If and only if \( |\psi(t)\rangle = e^{i\theta} |\phi(t)\rangle \) (\( \theta \in \mathbb{R} \) is a global phase factor), \( F(t) = 1 \) is achieved. Because \( F(t) \) decreases from 1 for \( t > 0 \) under the control errors, our targeted result is a lower bound of \( F(t) \) such that \( F(t) \geq F_* \geq 0 \).

We find the time evolution of the overlap \( \langle \psi(t) | \phi(t) \rangle \)

\[
d(\langle \psi(t) | \phi(t) \rangle) = (\psi(t) | i\hat{H}(t) | \phi(t) \rangle dt - (\psi(t) | \hat{H}(t) | \phi(t) \rangle dt + \sum_k \gamma_k^2(t) \hat{I} dt + \sum_k i\hat{B}_k(t) \bullet dW_k(t) | \phi(t) \rangle
\]

\[
= -\sum_k \gamma_k^2(t) (\langle \psi(t) | \phi(t) \rangle dt - \sum_k (\langle \psi(t) | i\hat{B}_k(t) | \phi(t) \rangle \bullet dW_k(t). \tag{2.10}
\]

Taking the real part and the ensemble average of this equation,

\[
\frac{d\mathbb{E}[\Re \{\langle \psi(t) | \phi(t) \rangle\}]}{dt} = -\sum_k \gamma_k^2(t) / 2 \mathbb{E}[\Re \{\langle \psi(t) | \phi(t) \rangle\}]. \tag{2.11}
\]

Solving this equation yields

\[
\mathbb{E}[\Re \{\langle \psi(t) | \phi(t) \rangle\}] = \exp \left( -\frac{1}{2} \int_0^t \sum_k \gamma_k^2(t') dt' \right). \tag{2.12}
\]
Further, by using the inequality $\mathbb{E}[x^2] \geq \mathbb{E}[x]^2$,
\[
\mathbb{E}[\|\langle \psi(t)|\phi(t)\rangle\|] \leq \mathbb{E}[|\langle \psi(t)|\phi(t)\rangle|] \leq \sqrt{\mathbb{E}[|\langle \psi(t)|\phi(t)\rangle|^2]}
\]
\[
= \sqrt{\mathbb{E}[F(t)]}. \tag{2.13}
\]

From (2.12) to (2.13), we end up with the following lower bound
\[
\mathbb{E}[F(T)] \geq F_* := \exp \left( - \int_0^T \sum_k \|\hat{B}_k(t)\|^2 dt \right), \tag{2.14}
\]
for $T \in [0, \infty)$. This inequality is the main result of this paper. We list some notable features of $F_*$ below.

(i) For a given target state, $F_*$ gives a fundamental limit on the state preparation under stochastic control errors. Namely, $F_*$ clarifies the set that the controlled quantum system can or cannot reach, under given control errors and control time.

(ii) $F_*$ can be calculated without needing the full dynamics of the quantum system, which does not necessitate solving any equations.

(iii) If the assumption $\hat{B}^2(t) = \gamma^2(t)\hat{I}$ is not imposed, $F_*$ can be generalized as follows (see appendix A for the proof)
\[
\mathbb{E}[F(T)] \geq F_* := \exp \left( - \int_0^T \sum_k \|\hat{B}_k(t)\|^2 dt \right). \tag{2.15}
\]

However, it should be noted that the tightness of $F_*$ is drastically weaker than $F_*$, e.g. when $\hat{B}$ satisfies $\hat{B}^2 = \gamma^2 \hat{I}$, $F_* = e^{-\gamma^2 T}$ while $F'_* = e^{-N\gamma^2 T}$.

### 3. Examples

(a) One-qubit

We here examine how tight the obtained bound $F_*$ is in practice through a simple example. We consider the one-qubit system such as a two level atom consisting of the excited state $|0\rangle = [1, 0]^T$ and the ground state $|1\rangle = [0, 1]^T$. We aim to drive the state from $|\psi_0\rangle = |1\rangle = [0, 1]^T$ to $|\psi(T)\rangle = |0\rangle = [1, 0]^T$ under the following setting:
\[
\hat{H} = u(t)\hat{S}_y \quad \text{and} \quad \hat{B} = \gamma(t)\hat{S}_x, \tag{3.1}
\]
where $\hat{S}_x = |0\rangle\langle 1| + |1\rangle\langle 0|$, $\hat{S}_y = i(|1\rangle\langle 0| - |0\rangle\langle 1|)$ and $\hat{S}_z = |0\rangle\langle 0| - |1\rangle\langle 1|$ are the Pauli matrices. $\hat{H}$ and $\hat{B}$ represent the control Hamiltonian and the control error, respectively. $\hat{B}$ fluctuates the state vector around the $x$-axis with frequency $\gamma(t)$. For simplicity, we assume that $u(t)$ and $\gamma(t)$ are time-independent $u(t) = u$ and $\gamma(t) = \gamma$. Then, the lower bound $F_*$ is calculated as $F_* = \exp(-\gamma^2 \pi/(2u))$. Also, by solving the equation $d|\psi(t)\rangle/dt = -iu\hat{S}_y|\psi(t)\rangle$, we obtain the exact control time $T = \pi/(2u)$. Moreover, assuming $2u > \gamma^2$, the average fidelity $\mathbb{E}[F(T)]$ can be analytically calculated as follows:
\[
\mathbb{E}[F(T)] = \frac{1}{2} \left\{ 1 + \frac{e^{-\gamma^2 T}}{\sqrt{D}} \left( \gamma^2 \sin \left( \sqrt{D} T \right) - \sqrt{D} \cos \left( \sqrt{D} T \right) \right) \right\}, \tag{3.2}
\]
where $D = 4u^2 - \gamma^2$ (see appendix B for the proof).

Figure 1a depicts $F_*$ and $\mathbb{E}[F(T)]$ as a function of $\gamma$, in unit of $u = 1$. We find that $F_*$ is effective for small $\gamma$. Further, let us focus on the ratio of $\mathbb{E}[F(T)]$ and $F_*$. When $\gamma$ is fixed, as the energy spent for the control increases (i.e. the control time becomes faster), $F_*$ becomes tighter (figure 1b). From this result, we can say that a rapid state control for the quantum system is efficient to make the influence of the control errors small.
Next, we consider a time-dependent rotation error modelled by
\[ \hat{B}(t) = \gamma (\cos(\omega t)\hat{S}_x + \sin(\omega t)\hat{S}_y), \]
which are common in the current quantum computing experiments. \( \omega \) is the positive constant representing the rotation frequency. Under the same Hamiltonian and the initial and final states, \( F_* \) has the same expression as above and gives a slightly tighter bound for all \( \gamma \), compared to the case of \( \hat{B} = \hat{S}_x \), as illustrated in figure 1a. Thus, it can be seen that the lower bound \( F_* \) works well for such realistic time-dependent control errors.

(b) Two-qubit

Next, we study the two-qubit system under stochastic noise. Let us focus on the SWAP operation exchanging the states of qubits 1 and 2, which is of particular use in the scenario of quantum information such as quantum Fourier transform (QFT) [1]. It is realized by the SWAP gate
\[ \hat{U}_{\text{SWAP}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \]
and \( \hat{U}_{\text{SWAP}} \) is generated by the time-independent Hamiltonian
\[ \hat{H} = \frac{u}{2} (\hat{S}_x \otimes \hat{S}_x + \hat{S}_y \otimes \hat{S}_y + \hat{S}_z \otimes \hat{S}_z). \]

We assume that the initial state is \( |\psi_0\rangle = |+\rangle \otimes |0\rangle \) where \( |+\rangle = (|0\rangle + |1\rangle)/\sqrt{2} \), and the final state is given by
\[ |\psi(T)\rangle = \hat{U}_{\text{SWAP}} |\psi_0\rangle = |0\rangle \otimes |+\rangle. \]
It takes the time \( T = \pi/(2u) \) for this transformation.

Moreover, we now introduce the two types of decay processes
\[ \hat{B}_S = \gamma \hat{S}_x \otimes \hat{S}_x \]
and
\[ \hat{B}_1 = \gamma \hat{S}_x \otimes \hat{I} \quad \text{and} \quad \hat{B}_2 = \gamma \hat{I} \otimes \hat{S}_x, \]
\( F \) and \( \mathbb{E}[F(T)] \) (blue dots) for (a) global noise \( \hat{B}_g \) and (b) local one \( (\hat{B}^1_l, \hat{B}^2_l) \) when \( u = 1 \) is fixed.

\( \hat{B}_g \) is a global noise acting on both qubits spontaneously. On the other hand, \( \hat{B}^1_l \) and \( \hat{B}^2_l \) are local noises acting on the each qubits. For each noise case, the lower bound is calculated as follows:

\[
F_* = \begin{cases} 
  e^{-\pi \gamma^2/2u} & \text{(global noise)} \\
  e^{-\pi \gamma^2} & \text{(local noise).}
\end{cases}
\]

As illustrated in figure 2a, b, we find that \( F_* \) works as a more effective bound for the local noise than the global one, in particular when \( \gamma \) is large. And also, when the noise is small, \( F_* \) gives a sharper estimate compared to the one-qubit case.

(c) Atomic ensemble

As a further example, we study an atomic ensemble system composed of \( N \) identical qubits. We consider a case where the system is controlled by any Hamiltonian under a stochastic error. We specify the general form representing the global noise and local one as follows:

\[
\hat{B}_g = \gamma(t) \hat{S}_x^1 \otimes \hat{S}_x^2 \otimes \cdots \otimes \hat{S}_x^N
\]

and

\[
\hat{B}_l^1 = \gamma(t) \hat{S}_x^1 \otimes \hat{I} \otimes \cdots \otimes \hat{I}, \ldots, \hat{B}_l^N = \gamma(t) \hat{I} \otimes \cdots \otimes \hat{S}_x^N.
\]

For each noise, \( F_* \) is calculated as follows:

\[
F_* = \begin{cases} 
  e^{-\int_0^T \gamma^2(t) \, dt} & \text{(global noise)} \\
  e^{-N \int_0^T \gamma^2(t) \, dt} & \text{(local noise).}
\end{cases}
\]

From these results, \( F_* \) is more effective for the global noise than the local noise, and also we may say that the local noise acts on the system more seriously than the global noise. Therefore, if the system is subjected to the local noise, it is required for highly suppressing the magnitude of the noise.

4. Estimation for obtaining the target state

Here, we quantitatively investigate how likely it is to obtain the target state, in the presence of stochastic noise. Let us expand the two final states \( |\psi(T)\rangle \) and \( |\phi(T)\rangle \) as follows:

\[
|\psi(T)\rangle = \sum_{i=1}^n p_i |i\rangle
\]

\[
|\phi(T)\rangle = \sum_{i=1}^n q_i |i\rangle
\]
and
\[ |\phi(T)\rangle = \sum_{i=1}^{n} q_i |i\rangle, \]  
(4.2)

where \{ |n\rangle, n = 1, \ldots, m, \ldots, n \} is the measurement basis. In the following, the derivation is based on projective measurement. We assume that the target state is the mth eigenstate \(|m\rangle \) of \(|\psi(T)\rangle \) in the measurement basis and its probability amplitude is given by
\[ |p_m|^2 = 1 - \epsilon^2, \]  
(4.3)

where \(0 < \epsilon < 1\) is the acceptable value. Let us calculate the upper bound of the overlap \(|\langle \psi(T)|\phi(T)\rangle|\) as follows:
\[ |\langle \psi(T)|\phi(T)\rangle| = \left| \left( \sum_{i=1}^{n} p_i^* |i\rangle \right) \left( \sum_{i=1}^{n} q_i |i\rangle \right) \right| \leq \sum_{i=1}^{n} |p_i q_i| \]
\[ \leq \sqrt{1 - \epsilon^2} |q_m| + \sum_{i \neq m} |p_i q_i| \]
\[ \leq \sqrt{1 - \epsilon^2} |q_m| + \epsilon \sqrt{1 - |q_m|^2}, \]  
(4.4)

where the Cauchy–Schwarz inequality was used. Therefore, combining equation (2.14), the probability amplitude of the mth eigenstate under stochastic noises has the following lower bound:
\[ |q_m| \geq q_* := \sqrt{1 - \epsilon^2} \exp\left(-\frac{1}{2} \int_0^T \sum_k \gamma_k^2(t) \, dt\right) + \epsilon \sqrt{1 - \exp\left(-\int_0^T \sum_k \gamma_k^2(t) \, dt\right)} \]  
(4.5)

where \(q_*\) is always greater than or equal to zero \(q_* \geq 0\), and thus, \(q_*\) gives a meaningful information on state preparation. If we set \(\epsilon = 0\), we recover the result (2.14) as \(|q_m|^2 \geq \exp(-\int_0^T \sum_k \gamma_k^2(t) \, dt)\).

When \(\epsilon\) is fixed for a certain value, \(q_* \to \epsilon\) in the limit \(\int_0^T \sum_k \gamma_k^2(t) \, dt \to \infty\); in this case, if \(\epsilon = 0\), it is impossible to completely obtain the desired target state.

5. Conclusion

In this paper, we have presented the lower bound of the fidelity between the two controlled quantum systems in the presence and absence of stochastic control errors. The derived bound can be straightforwardly calculated and used to derive the theoretical estimation for obtaining the target state. We would like to emphasize that this bound clarifies a set that the quantum system can or cannot reach. That is to say, once the stochastic noise \(\hat{B}(t)\) that satisfies the condition \(\hat{B}_k^2(t) = \gamma_k^2(t)I\) is specified, we can immediately grasp how much the quantum state can approach the target at least. Also, we have presented another lower bound \(F_*\) without imposing this condition, but its effectiveness is much weaker than \(F_*\). Thus, the derivation of a general and useful bound without limiting the type of noise is of interest to us. Finally, aside from these results, there is the problem that there is no methodology for dealing with stochastic errors. An important remaining work is to develop a method to overcome stochastic control errors.

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A. Derivation of the lower bound for general noise

We derive the generalized lower bound $F'_s$ without imposing the assumption $\hat{B}^2_\omega(t) = \gamma^2(t)\hat{I}$. First, we present a time evolution of the operator representing the pure state $\hat{\rho}(t) = |\psi(t)\rangle\langle\psi(t)|$

$$d\hat{\rho}(t) = d|\phi(t)\rangle\langle\phi(t)| + |\phi(t)\rangle d\langle\phi(t)| + d|\phi(t)\rangle \cdot d\langle\phi(t)|$$

$$= \left( -i\hat{H}(t) dt - \frac{1}{2} \sum_k \hat{B}_k^2(t) dt - i \sum_k \hat{B}_k(t) \cdot dW_k(t) \right) |\phi(t)\rangle\langle\phi(t)|$$

$$+ |\phi(t)\rangle\langle\phi(t)| \left( i\hat{H}(t) dt - \frac{1}{2} \sum_k \hat{B}_k^2(t) dt + i \sum_k \hat{B}_k(t) \cdot dW_k(t) \right)$$

$$+ \left( -i\hat{H}(t) dt - \frac{1}{2} \sum_k \hat{B}_k^2(t) dt - i \sum_k \hat{B}_k(t) \cdot dW_k(t) \right) |\phi(t)\rangle\langle\phi(t)|$$

$$\times \left( i\hat{H}(t) dt - \frac{1}{2} \sum_k \hat{B}_k^2(t) dt + i \sum_k \hat{B}_k(t) \cdot dW_k(t) \right)$$

$$= -i[\hat{H}(t), \hat{\rho}(t)] dt + \sum_k D[\hat{B}_k(t)] \hat{\rho}(t) dt - i \sum_k [\hat{B}_k(t), \hat{\rho}(t)] \cdot dW_k(t), \quad (A\ 1)$$

where we used $d\tau^2 = dt \cdot dW(t) = dW(t) dt = 0$ and defined $D[\hat{B}]\hat{\rho} = \hat{B}\hat{\rho}\hat{B} - \hat{B}^2\hat{\rho}/2 - \hat{\rho}\hat{B}^2/2$. In the same manner, the time evolution of $\hat{\sigma}(t) = |\psi(t)\rangle\langle\psi(t)|$ is given by

$$\frac{d\hat{\sigma}(t)}{dt} = -i[\hat{H}(t), \hat{\sigma}(t)]. \quad (A\ 2)$$

Using (A 1) and (A 2), we calculate the infinitesimal change of the fidelity $F(t) = \text{Tr} [\hat{\rho}(t)\hat{\sigma}(t)]$ as follows:

$$dF(t) = \text{Tr} [d\hat{\rho}(t)\hat{\sigma}(t) + \hat{\rho}(t) d\hat{\sigma}(t)]$$

$$= \text{Tr} \left\{ \left( -i[\hat{H}(t), \hat{\rho}(t)] + \sum_k D[\hat{B}_k(t)]\hat{\rho}(t) \right) \hat{\sigma}(t) \right\} \text{dt} + \text{Tr} \{ \hat{\rho}(t)(-i[\hat{H}(t), \hat{\sigma}(t)]) \} \cdot dW_k(t)$$

$$- i \sum_k \text{Tr} ([\hat{B}_k(t), \hat{\rho}(t)]) \cdot dW_k(t)$$

$$= - \sum_k \text{Tr} [\hat{B}_k(t)\hat{\rho}(t)\hat{\sigma}(t)] \text{dt} + \sum_k \text{Tr} [\hat{B}_k(t)\hat{\rho}(t)\hat{B}_k(t)\hat{\sigma}(t)] \text{dt} - i \sum_k \text{Tr} ([\hat{B}_k(t), \hat{\rho}(t)]) \cdot dW_k(t)$$

$$\geq - \sum_k \|\hat{B}_k(t)\|_F^2 F(t) \text{dt} - i \sum_k \text{Tr} ([\hat{B}_k(t), \hat{\rho}(t)]) \cdot dW_k(t), \quad (A\ 3)$$

where we have used the inequality $\text{Tr}(\hat{A}\hat{B}) \leq \text{Tr}(\hat{A})\text{Tr}(\hat{B})$ for positive semidefinite matrices $\hat{A}$ and $\hat{B}$. Taking the ensemble average of (A 3), we thus have

$$\frac{d\mathbb{E}[F(t)]}{dt} \geq - \sum_k \|\hat{B}_k(t)\|_F^2 \mathbb{E}[F(t)]. \quad (A\ 4)$$

Then, by integrating the above from 0 to $T$, we obtain the result given in (2.14)

$$\mathbb{E}[F(T)] \geq F'_s = \exp \left( -\int_0^T \sum_k \|\hat{B}_k(t)\|_F^2 \text{dt} \right). \quad (A\ 5)$$
B. Detailed calculations of average fidelity

To obtain the exact solution of the average final fidelity $\mathbb{E}[F(T)]$, it is needed to calculate $\mathbb{E}[\hat{\rho}(T)] = \mathbb{E}[|\psi(T)\rangle \langle \psi(T)|]$, because

$$
\mathbb{E}[F(T)] = \mathbb{E}[|\langle \phi(T) | \psi(T) \rangle|^2] = \text{Tr}[\mathbb{E}[\hat{\rho}(T)] \hat{\sigma}(T)].
$$

(B 1)

For the set-up $(\hat{H}, \hat{B}) = (u \hat{S}_y, \gamma \hat{S}_x)$, by using equation (A 1), the time evolution of $\mathbb{E}[\hat{\rho}(t)]$ is described as follows:

$$
\frac{d\mathbb{E}[\hat{\rho}(t)]}{dt} = -i[u \hat{S}_y, \mathbb{E}[\hat{\rho}(t)]] + D[\gamma \hat{S}_x] \mathbb{E}[\hat{\rho}(t)].
$$

(B 2)

This equation leads to the two differential equations

$$
\frac{d\mathbb{E}[z(t)]}{dt} = -2u\mathbb{E}[x(t)] - 2\gamma^2 \mathbb{E}[z(t)]
$$

(B 3a)

and

$$
\frac{d\mathbb{E}[x(t)]}{dt} = 2u\mathbb{E}[z(t)],
$$

(B 3b)

where we defined $x(t) = \text{Tr}[\hat{\rho}(t) \hat{S}_x]$ and $z(t) = \text{Tr}[\hat{\rho}(t) \hat{S}_z]$. In order to calculate $\mathbb{E}[F(T)]$, it is necessary to obtain the information on $\mathbb{E}[z(t)]$ only, because $\mathbb{E}[F(T)] = \langle 0 | \mathbb{E}[\hat{\rho}(T)] | 0 \rangle = (1 + \mathbb{E}[z(T)])/2$. From these two equations, we have the second-order differential equation

$$
\frac{d^2\mathbb{E}[z(t)]}{dt^2} = -2\gamma^2 \frac{d\mathbb{E}[z(t)]}{dt} - 4u^2 \mathbb{E}[z(t)].
$$

(B 4)

By assuming $2u > \gamma^2$ and using the initial conditions $\mathbb{E}[z(0)] = -1$ and $\mathbb{E}[x(0)] = 0$, the solution of this equation is thus

$$
\mathbb{E}[z(T)] = \frac{e^{-\gamma^2 t}}{\sqrt{D}} \left( 2 \sin \left( \sqrt{D}t \right) - \sqrt{D} \cos \left( \sqrt{D}t \right) \right), \quad D = 4u^2 - \gamma^4.
$$

(B 5)

Therefore, we obtain the analytical solution of $\mathbb{E}[F(T)]$

$$
\mathbb{E}[F(T)] = \frac{1}{2} \left( 1 + \frac{e^{-\gamma^2 T}}{\sqrt{D}} \left( 2 \sin \left( \sqrt{D}T \right) - \sqrt{D} \cos \left( \sqrt{D}T \right) \right) \right).
$$

(B 6)

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