Low Dimensional Models for the Nonlinear Dynamics of Transport Barrier Oscillations in Tokamak Edge Plasmas

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Abstract. Transport barriers at the plasma edge are key elements of high confinement regimes in fusion devices. In typical configurations, such barriers are not stable but exhibit quasi-periodic relaxation oscillations. In this work, one-dimensional and zero-dimensional models for such oscillations are presented that give insight into the underlying mechanisms which are found to be intrinsically nonlinear. The models are systematically derived from three-dimensional turbulence simulations based on a fluid description of the plasma. In these simulations, a transport barrier is generated by an imposed ExB shear flow. This barrier exhibits quasi-periodic relaxation oscillations even if the ExB shear flow is frozen. The models presented here are therefore different from those based on turbulent shear flow generation. They allow to isolate and illustrate a different mechanism for barrier oscillations. Furthermore, these models reproduce regimes characterized by a decrease of the oscillation frequency with the ExB flow shear.

1. Introduction
High confinement regimes in thermonuclear fusion devices are characterized by the appearance of a transport barrier at the plasma edge [1, 2, 3]. Such barriers are regions where the turbulent flux of particles and energy is reduced by a sheared rotation of the plasma. As a consequence of the local flux balance, the density, temperature and pressure profiles steepen in these regions. In the most promising operational regime of future fusion reactors, the transport barrier is not stable but exhibits quasi-periodic relaxation oscillations. A relaxation is characterized by an increase of turbulent transport through the barrier and a decrease of the pressure inside the barrier. Currently, transport barrier relaxations are modeled by phenomenologically constructed dynamical equations for the amplitudes of relevant modes [5, 6, 7].

Here, we propose one dimensional (1D) and zero dimensional (OD) models for transport barrier relaxations, derived from three dimensional (3D) turbulence simulations based on a fluid description of the plasma. A transport barrier is generated via an imposed $E \times B$ shear flow. These simulations are based on a system of normalized reduced resistive MHD equations for the
electrostatic potential $\phi$ and pressure $p$ [8],

$$\partial_t \nabla^2 \phi + \{\phi, \nabla^2 \phi\} = -\nabla_\parallel^2 \phi - Gp + \nu \nabla_\perp^4 \phi$$

$$\partial_t p + \{\phi, p\} = \delta_c G \phi + \chi_\parallel \nabla_\parallel^2 p + \chi_\perp \nabla_\perp^2 p + S(r)$$

This system describes the so-called resistive ballooning mode (RBM) turbulence of a tokamak edge plasma. Eq. (1) corresponds to the charge balance in the drift approximation involving the divergence of the polarization current, the parallel current, and the diamagnetic current, and viscosity ($\mu$), respectively. Eq. (2) is derived from energy conservation, where $\chi_\parallel$ and $\chi_\perp$ represent, respectively, parallel and perpendicular collisional heat diffusivities, $S(r)$ is an energy source, and $\delta_c$ is essentially the ratio between the pressure gradient length and the major radius of the torus. The curvature operator $G$ arises from the compressibility of diamagnetic current and $E \times B$ drift. The parallel current is evaluated using a simplified electrostatic Ohm’s law, $j_\parallel \sim -\nabla_\parallel \phi$. Magnetic flux surfaces are represented by a set of concentric circular tori, where the coordinates $(r, \theta, \phi)$ correspond to the minor radius, and the poloidal and toroidal angles. The Poisson bracket is $\{\phi, .\} = r^{-1}(\partial_r \phi \partial_\theta - \partial_\theta \phi \partial_r)$, the curvature operator is $G = \sin \theta \partial_r + \cos \theta r^{-1} \partial_\theta$.

Typical simulations of RBM turbulence in the presence of an imposed ExB shear flow show the generation of a transport barrier and the appearance of quasi-periodic relaxation oscillations of that barrier, even in the case of a frozen ExB shear flow [9]. The relaxations are found to be governed by the transitory growth of a particular mode, characterized by fixed poloidal and toroidal wavenumbers, and localized radially at the center of the transport barrier. The importance of this mode allows for the construction of a 1D model for the radial and temporal dynamics of this mode, coupled to the evolution of the pressure profile.

2. One dimensional model

For this purpose, the pressure is decomposed into a mean profile $\bar{p}(r, t)$ and a perturbation $\delta p = \tilde{p}(r, t)e^{i(m\theta - n\phi)}$ localized at the barrier center ($r = r_0$), i.e. $\nabla_\parallel^2 \delta p \sim -(r - r_0)^2 \delta p$. For simplification, a cylindrical curvature operator $G$ is assumed, $G \rightarrow r^{-1} \partial_\theta$, and the number

![Figure 1](attachment:image.png)

**Figure 1.** Time evolution of pressure fluctuations $(\int \tilde{p}^2 \, dx)^{1/2}$ with and without shear flow (left). Time evolution of the pressure gradient $|\partial_r \tilde{p}|/(\Gamma_{tot}/\chi_\perp)$ and the turbulent flux $2\gamma_0 |\tilde{p}|^2/\Gamma_{tot}$ at the barrier center (right). (Here, $\Gamma_{tot}$ is the total energy flux determined by the source $S$.)
of fields is reduced by assuming a linear relation between potential and pressure fluctuations, \( \hat{\phi} = ik_\theta / (\gamma_0 k_\perp^2) \hat{p} \) with \( k_\theta = m/r_0 \) and \( k_\perp \) representing the poloidal and perpendicular wave numbers. Here \( \gamma_0 \) is the linear growth rate in the presence of a mean pressure gradient \( \kappa \) and in absence of dissipation and \( E \times B \) shear flow. The poloidal shear flow is assumed to have the form \( \bar{u}_\theta = \partial_r \bar{\phi} = \omega_E (r - r_0) \). The evolutions equations for the pressure become,

\[
\begin{align*}
\partial_t \bar{p} &= -2\gamma_0 \partial_x |\hat{p}|^2 + \chi_\perp \partial_x^2 \bar{p} + S, \\
\partial_t \bar{\phi} &= \gamma_0 (-\partial_x \bar{p} - \kappa_0) \hat{p} - i\omega_E x \bar{p} - \chi_{\parallel}^2 x^2 \bar{p} + \chi_\perp \partial_x^2 \bar{p},
\end{align*}
\]

where \( x = r - r_0, \omega_E' = k_\theta \omega_E, \chi_{\parallel}' = k_\theta^2 \chi_\parallel, \) and \( \kappa_0 = k_\theta^2 \chi_\perp / \gamma_0 \). In the absence of shear flow \((\omega_E = 0)\), the system evolves to a stationary state. With increasing \( \omega_E > 0 \), the system first shows regular oscillations (Fig. 1a) and then reproduces relaxation oscillations (Fig. 1b). The mechanism for relaxation oscillations is governed by the existence of a time delay of the order \( \tau = \left(\frac{1}{4} \chi_\perp \omega_E'\right)^{-1/3} \) for the stabilization of fluctuations by the shear flow [9]. This essentially non linear mechanism reveals that the role of the velocity shear is different from a modification of the linear instability threshold. Indeed, if the coupling term with the shear flow is replaced by a shift of the instability threshold, no relaxation oscillations are observed except if the instability term is further modified (e.g. by a Heaviside function [7]).

3. Zero dimensional model

We now construct systematically a system of amplitude equations (0D model) for the evolution of the amplitudes of the relevant radial structures. Thus system reproduces main features of the transport barrier oscillations observed in the previous 3D and 1D models.

The relevant radial structures in the nonlinear dynamics of transport barrier oscillations are determined by applying a proper orthogonal decomposition (POD) method [10] to the spatio-temporal signal obtained from the 1D model. The data are converted to an \( M \times N \) matrix \( P_{ij} \) in which columns correspond to time series. With the POD, the matrix is decomposed into a

![Figure 2. Weights \( W_n \) of the modes obtained with the POD, as a function of the mode number \( n \).](image-url)
set of modes $A_n, P_n$ which are orthogonal in space and time,

$$P^j_i = P(x_i, t_j) = \sum_n W_n A_n(t) P_n(x)$$

These modes are ordered by decreasing weight $W_n$ (Fig. 4). The steepness of the weight distribution suggests that the dynamics of the system can be described using the first modes only (Galerkin approximation),

$$\bar{p}(x, t) = -\kappa x + a_0(t)p_0(x), \quad \tilde{p}(x, t) = a_R(t)p_R(x) + ia_I(t)p_I(x).$$

The modes $p_0, p_R$ and $p_I$ correspond to the mean pressure and the real and imaginary part of the main perturbation, respectively. They can be approximated by the following expressions,

$$p_0(x) = \beta x \exp\left(-2ax^2\right),$$

$$p_R(x) = \alpha \cos(bx) \exp\left(-ax^2\right),$$

$$p_I(x) = -\zeta \alpha \sin(bx) \exp\left(-ax^2\right),$$

where $p_R$ and $p_I$ are orthogonal, and the normalization constants $\alpha, \beta$ and $\zeta$ are such that

$$\int_{-\infty}^{\infty} p_0^2 \, dx = \int_{-\infty}^{\infty} p_R^2 \, dx = \int_{-\infty}^{\infty} p_I^2 \, dx = 1.$$

Note that $p_1(x) = p_R(x) + (i/\zeta) p_I(x)$ with

$$a = \sqrt{\chi_\parallel k_\theta \chi_\perp / 2}, \quad b = \frac{\omega E}{2\sqrt{\chi_\parallel \chi_\perp}}$$

is the most unstable eigenmode of the linear equation obtained from Eq. (4) for $\partial_x \bar{p} = -\kappa = \text{const}$. However, as will get obvious in the following, the dynamical equations for one single

Figure 3. Variation of the coefficients $\Omega$ (left) and $\zeta, \delta, \delta'$ (right) with the flow shear (characterized by the parameter $\gamma_E/\gamma_s$).
amplitude $a_1$ of this eigenmode (i.e. $a_1 = a_R = \zeta a_I$) coupled to the dynamics of $a_0$ do not reproduce oscillating states.

The projection of the 1D model equations (3), (4) with the Galerkin approximation (6), (7) onto the modes (8), (9), (10) leads to the time evolution equations for the amplitudes $a_0$, $a_R$, and $a_I$,

\begin{align*}
    \frac{d}{dt}a_0 &= -3\gamma_s a_0 + 2\delta a_R^2 + 2\delta' a_I^2, \\
    \frac{d}{dt}a_R &= (\Gamma - \delta a_0) a_R + \Omega \left(\frac{a_R}{\zeta} - a_I\right), \\
    \frac{d}{dt}a_I &= (\Gamma - \delta' a_0) a_I - \Omega (\zeta a_I - a_R),
\end{align*}

with the coefficients

$$
\Gamma = \gamma_0 (\kappa - \kappa_0) - \gamma_s - \gamma_E,
$$

$$
\Omega = 2\gamma_E \left[ \exp\left(\frac{\gamma_E}{\gamma_s}\right) + 1 \right]^{-1},
$$

$$
\zeta = \tanh^{-1/2}\left(\frac{\gamma_E}{2\gamma_s}\right),
$$

$$
\delta = \delta_0 \left[ \exp\left(\frac{\gamma_E}{2\gamma_s}\right) + 1 + \frac{\gamma_E}{\gamma_s} \right] \cosh^{-1}\left(\frac{\gamma_E}{2\gamma_s}\right),
$$

$$
\delta' = \delta_0 \left[ \exp\left(\frac{\gamma_E}{2\gamma_s}\right) - 1 - \frac{\gamma_E}{\gamma_s} \right] \sinh^{-1}\left(\frac{\gamma_E}{2\gamma_s}\right),
$$

and

$$
\gamma_E = \frac{\omega_E^2}{4\chi}, \quad \gamma_s = \sqrt{\chi_\parallel \chi_\perp k_\theta}, \quad \delta_0^2 = \frac{\gamma_0^2}{\sqrt{\pi}} \sqrt{2\chi_\perp \gamma_s^3}. 
$$

The variation of the coefficients with the flow shear is shown in Fig. 3. The system (11-13) reproduces oscillations (Fig. 4) due to the independent evolution of both, $a_R$ and $a_I$. If $a_R = \zeta a_I$

![Figure 4](image_url)
as for a linear eigenmode, oscillations would not appear. For low values of $\gamma_E/\gamma_s$, the oscillation frequency is found to increase with flow shear, and for high values of $\gamma_E/\gamma_s$, the oscillation frequency decreases with flow shear (Fig. 4). This reflects the behavior of the coupling coefficient $\Omega$ (see Fig. 3).

4. Conclusions

In conclusion, we have derived 1D and OD models based on 3D turbulence simulations that reproduce oscillations of transport barriers. These models reveal a new mechanism for such oscillations which is different from turbulent shear flow generation [11]. It is based on a time delay for stabilization by the shear flow. The role of the latter is therefore different from a shift of the linear instability threshold. Both models reproduce regimes in which the oscillation frequency decreases with the ExB flow shear. In the OD model, such behavior gets obvious from analytic expressions for the coefficients in the dynamical equations. In combination with studies concerning the frequency dependence on heating power for a fixed ExB flow shear, the model allows to reproduce regimes characterized by a decreasing frequency with heating power (if the ExB shear increases fast enough with heating power) [9]. This property is reminiscent of so-called type III edge localized mode (ELM) dynamics in tokamak edge transport barriers [12].

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