Lorentz-Invariant “Elements of Reality” and the Question of Joint Measurability of Commuting Observables

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ABSTRACT

It is shown that the joint measurements of some physical variables corresponding to commuting operators performed on pre- and post-selected quantum systems invariably disturb each other. The significance of this result for recent proofs of the impossibility of realistic Lorentz invariant interpretation of quantum theory (without assumption of locality) is discussed.
Recently, few authors\textsuperscript{1−4} by taking very plausible definition of “elements of reality” claimed to show that Lorentz invariant realistic interpretation of quantum mechanics is not possible. Contrary to the Einstein-Podolsky-Rosen (EPR) argument, their proof have not based on the locality assumption. In this Letter we will show that contradiction disappear when we abandon a “product rule” of elements of reality, even for elements of reality corresponding to commuting operators. We will show that the product rule has to be abandoned because joint measurements of commuting operators in the considered situations invariably disturb each other.

The plan of this Letter is as follows: We shall start with the discussion of the problem of measurement performed on a pre- and post-selected quantum system. We will show that even commuting operators can not be measured on such systems without disturbing each other. Then we shall briefly present the arguments against Lorentz invariant realistic interpretation of quantum mechanics due to Pitowsky,\textsuperscript{1−2} and Hardy,\textsuperscript{3−4} and we shall explain their usage of the “product rule”. We shall conclude with a brief discussion of the two-state vector approach in which the elements of reality are Lorentz invariant.

In every textbook of quantum mechanics we can find a condition for simultaneous measurability of variables $A$ and $B$: the corresponding operators must commute:

$$[A, B] = 0. \quad (1)$$

Commutativity of the operators $A$ and $B$ is a strong sufficient condition, in fact, for a given quantum state $|\Psi\rangle$ it is enough to require “commutativity on a state”:

$$[A, B]|\psi\rangle = 0. \quad (2)$$
Commutativity condition (2) is sufficient and necessary condition for simultaneous measurability of $A$ and $B$. If, the operators $A$ and $B$ do not commute, the measurement of one disturbs the outcome of the other. There is correspondence between unmeasurability of both variables and the fact that standard formalism of quantum theory can not associate well-defined values to two noncommuting operators.

One other way to see this property is to consider the standard measuring procedure. The interaction Hamiltonian is given by

$$H_{int} = g(t)pA,$$  

(3)

where $p$ is a canonical momentum of the measuring device; the conjugate position $q$ corresponds to the position of a pointer on the device. The time-dependent coupling $g(t)$ is constant for a short time interval corresponding to the measurement. Therefore, during the time of the measurement we obtain (in the Heisenberg picture):

$$\frac{dB}{dt} = i[H, B] = ig(t)p[A, B].$$  

(4)

Thus, it is clear why variables corresponding to commuting operators are measurable simultaneously without mutual disturbance, while measurement of non-commuting operators disturb each other.

If $A$ and $B$ commute, and if at a given moment we know that measurement of $A$ must yield $A = a$ while measurement of $B$ must yield $B = b$ one can safely claim that also the result of a measurement of $AB$ is known and equal $ab$. I repeat this well known fact because surprisingly it is not true when we consider a pre- and post-selected quantum system.
Let us spell out what is the pre- and post-selected quantum system. We consider a quantum system at time $t$. For simplicity we choose zero free Hamiltonian. At time $t_1 < t$ the system was prepared in a quantum state $|\Psi_1\rangle$, and at the time $t_2 > t$ another measurement was performed and it was found in a state $|\Psi_2\rangle$. We ask questions about possible measurements at time $t$.

Consider a measurement of a variable $A$. If either $|\Psi_1\rangle$ or $|\Psi_2\rangle$ is an eigenstate of $A$, then clearly the outcome of the measurement is well defined (it is the corresponding eigenvalue of $A$) and, measuring of commuting variable $B$ before, after, or even during the measurement of $A$ will not, in principle, disturb the measurement of $A$. However, for a pre- and post-selected quantum system it might be that the result of the measurement of $A$ is certain even if neither $|\Psi_1\rangle$ nor $|\Psi_2\rangle$ is an eigenstate of $A$. This is the case in which a measurement at any time in the time period $(t_1, t_2)$ of some variables commuting with $A$, invariably disturb the $A$-measurement.

The simplest example is the setup proposed by Bohm for analyzing the EPR argument: two separate spin-1/2 particles prepared, at the time $t_1$, in a singlet state (which has the same form in every basis)

$$|\Psi_1\rangle = \frac{1}{\sqrt{2}}(|\uparrow_1\downarrow_2\rangle - |\downarrow_1\uparrow_2\rangle).$$

At the time $t_2$ the measurements of $\sigma_{1x}$ and $\sigma_{2y}$ are performed and some certain results are obtained. If at the time $t$, $t_1 < t < t_2$, the measurement of $\sigma_{1y}$ was performed (and this is the only measurement performed between $t_1$ and $t_2$), then the outcome of the measurement is known with certainty: $\sigma_{1y}(t) = -\sigma_{2y}(t_2)$. If, instead, only a measurement of $\sigma_{2x}$ was performed at the time $t$, the result of the measurement is also certain: $\sigma_{2x}(t) = -\sigma_{1x}(t_2)$. The operators $\sigma_{1y}$ and $\sigma_{2x}$ are
obviously commute, but nevertheless, measuring $\sigma_{2x}(t)$ clearly disturb the outcome of the measurement of $\sigma_{1y}(t)$: it is not certain anymore. The EPR argument yields that $\sigma_{1x}(t)$ is known in this case (it is equal to $-\sigma_{2x}(t)$), but $\sigma_{1y}$ is uncertain.

Measuring of the product $\sigma_{1y}\sigma_{2x}$, is, in principle, different from the measurement of both $\sigma_{1y}$ and $\sigma_{2x}$ together. In our example the outcome of the measurement of the product is certain, but it does not equal to the product of the results which must come out of the measurements of $\sigma_{1y}$ and $\sigma_{2x}$ when every one of them is performed without the other. Note that we can rewrite the operator of the product as a modular sum: $\sigma_{1y}\sigma_{2x} = (\sigma_{1y} + \sigma_{2x})_{\text{mod}4} - 1$ and for such operators there is a method for instantaneous measurements which uses solely local interactions.6

To show this, and for future applications to other pre- and post-selected quantum systems let us apply the formalism pioneered by Aharonov Bergmann and Lebowitz7 for calculating probabilities of the results of measurement performed between two other measurements. If the first measurement prepared the state $|\Psi_1\rangle$, the final measurement found the state $|\Psi_2\rangle$, then the probability for a result $A = a_n$ is given by

\[
\text{prob}(A = a_n) = \frac{|\langle \Psi_2 | P_{A=a_n} | \Psi_1 \rangle|^2}{\sum_k |\langle \Psi_2 | P_{A=a_k} | \Psi_1 \rangle|^2}.
\]

where the sum is over all possible eigenvalues of $A$. The formula immediately yields probability 1 when $|\Psi_1\rangle$ or $|\Psi_2\rangle$ is an eigenstate, but it also might yield 1 when neither of the states is an eigenstate, as we proceed to show on the example presented above.

In our example the state $|\Psi_1\rangle$ is given by Eq.(5). Consider for concreteness the following results of the final measurements: $\sigma_{1x} = 1$ and $\sigma_{2y} = 1$. Then
the state $|\Psi_2\rangle = |\uparrow_1\uparrow_2\rangle$. For finding the probability of the outcome of the measurement of $\sigma_{1y}$ we have to use the projection operators $P_{[\sigma_{1y}=1]} = |\uparrow_1 y\rangle \langle \uparrow_1 y|$, $P_{[\sigma_{1y}=-1]} = |\downarrow_1 y\rangle \langle \downarrow_1 y|$. Applying all this to formula (6), we, indeed, obtain $\text{prob}[\sigma_{1y} = -1] = 1$. In the same way we obtain $\text{prob}[\sigma_{2x} = -1] = 1$. For calculation of the probabilities of the measurement of the product $\sigma_{1y}\sigma_{2x}$ we shall use the projection operators

$$P_{[\sigma_{1y}\sigma_{2x}=1]} = |\uparrow_1 y\uparrow_2 x\rangle \langle \uparrow_1 y\uparrow_2 x| + |\downarrow_1 y\downarrow_2 x\rangle \langle \downarrow_1 y\downarrow_2 x|,$$

$$P_{[\sigma_{1y}\sigma_{2x}=-1]} = |\uparrow_1 y\downarrow_2 x\rangle \langle \uparrow_1 y\downarrow_2 x| + |\downarrow_1 y\uparrow_2 x\rangle \langle \downarrow_1 y\uparrow_2 x|.$$  

The Eq.(6) yields, then, $\text{prob}[\sigma_{1y}\sigma_{2x} = 1] = 0$ contrary to the consequence of the product rule according to which $\sigma_{1y}\sigma_{2x} = 1$ with probability 1. It follows that the value of the product $\sigma_{1y}\sigma_{2x}$ is certain, but it equals to $-1$.

Let us state the main result: for pre- and post-selected quantum system it might be the case that the operators corresponding to two observables $A$ and $B$ commute $[A, B] = 0$, and the value of $A$ is well defined (the outcome of a measurement, if it is the only one to be performed, is certain), but measuring $B$ invariably disturbs the results of the measurement of $A$. Therefore, for pre- and post-selected quantum system one cannot apply frequently used “product rule” which is: if it is known with certainty that $A = a$ and $B = b$, then $AB = ab$. In fact it might be that the value of $AB$ is also known with certainty, but it does not equal to $ab$.

Let us turn now to presenting the arguments against possibility of Lorentz invariant realistic interpretation of quantum mechanics. Starting point of these arguments were the definition of elements of reality and the principle of Lorentz invariance. Contrary to the usual EPR-type arguments, no locality principle, i.e.,
the impossibility of the action at a distance, were assumed. In the discussed works there were adopted the following definitions:

(i) **Element of reality** \( \text{RED} \): “If quantum predictions dictate with certainty what the result of measuring a physical quantity at some time would be then, whether or not the prediction is actually verified, there exist an element of reality at that time corresponding to this physical quantity and having equal to the predicted measurement result.”

(ii) **The Principle of Lorentz invariance**: “If an element of reality corresponding to some Lorentz invariant physical quantity exists and has a value within space-time region \( R \) with respect to one space-like hyperplane containing \( R \), then it exists and has the same value in \( R \) with respect to any other hyperplane containing \( R \).”

In the usual EPR argument the element of reality corresponding to the outcome of a measurement fixed just by the *possibility* to infer this outcome from the results of measurements in a causally disconnected region. Contrary to this, in the present approach, which does not rely on the locality assumption, the elements of reality are fixed by actual measurements performed in a space-like separated region.

The first argument\(^1\text{--}^2\) is based on the modified Greenberger-Horne-Zeilinger\(^10\) (GHZ) setup for proving the nonexistence of local hidden variables. Three spin-1/2 particles located in the corners of a very large triangle move fast in the directions pointing out of the center of the triangle. At the time \( t_1 \) the particles are prepared in the state:

\[
|\Psi_1\rangle = |GHZ\rangle = \frac{1}{\sqrt{3}}(|\uparrow_{1z}\uparrow_{2z}\uparrow_{3z}\rangle - |\downarrow_{1z}\downarrow_{2z}\downarrow_{3z}\rangle).
\]  

At the time \( t_2 \) the the spin components in \( x \) direction are measured on all particles.
and the results $\sigma_{ix} = x_i$ are obtained. Consider now some possible measurements performed on the particles at the time $t$, $t_1 < t < t_2$. For the three observers who perform the $\sigma_{ix}$ measurements (at the rest frame time $t_2$), the measurements on the other particles (rest frame time $t$) are performed after their $\sigma_{ix}$ measurement, and they can predict (each in his Lorentz frame) the following result with certainty:

\begin{align}
\sigma_{2y}\sigma_{3y} &= x_1 \\
\sigma_{1y}\sigma_{3y} &= x_2 \\
\sigma_{1y}\sigma_{2y} &= x_3
\end{align}

Eqs.(9a-c) represent elements of reality in certain space-time regions corresponding to certain Lorentz frames. The principle of Lorentz invariance yields that these are also the elements of reality in the rest frame. Multiplying Eqs. (9b) and (9c) we obtain:

$$\sigma_{1y}^2\sigma_{3y}\sigma_{2y} = x_2x_3$$

Taking in account that $\sigma_{1y}^2$ is an identity operator for spin variables of particle 1, we conclude that $x_1 = x_2x_3$. The obtained equation, however, contradicts quantum mechanics: in GHZ state it must be that $x_1x_2x_3 = -1$.

Another example uses just two particles: electron and positron. Hardy have used two entangled setups proposed by Elitzur and Vaidman (EV) for interaction-free measurements. They proposed to place a Mach-Zehnder interferometer tuned to zero counts of one of the detectors in such a way that the particle’s trajectory of one arm of the interferometer passes through the observed region. One EV device
tests the point \( P \) with a single electron, while the other tests the same point \( P \) at the time with a single positron. If both electron and the positron come to the point \( P \) together then they annihilate, and it might happen that both devices yield that the point \( P \) is not empty. This is the case considered in Hardy’s work.\(^3\) Clifton and Niemann\(^4\) considered a variation of this idea using Stern-Gerlach devices rather than interferometers.

Consider the Lorentz frame in which the observer of the electron EV device has its result first. She infers\(^12\) that the \textit{positron} was in \( P \). In other Lorentz frame, however, the observer of the positron EV device was the first to obtain the result. He deduces that the \textit{electron} was in \( P \) at that time. The principle of Lorentz invariance yields that there are two elements of reality: the electron in \( P \) and the positron in \( P \). The product rule here is very natural: if the electron in \( P \) and the positron in \( P \) then the electron and the positron in \( P \). The latter, however, leads to contradiction since the particles in \( P \) have to annihilate and cannot be detected by the observers.

Both Hardy and Pitowsky obtain their elements of reality as \textit{predictions} of different observers, but their arguments run only when they consider “predictions” of all observers. However, there is no Lorentz observer for which all the predictions are inferences from the past toward the future: at least some of the inferences must be \textit{retrodictions}. In fact, in both cases we have a quantum system on which two complete measurement are performed in succession and the claims about the elements of reality are made for the time in between these two measurements. In the example of Pitowsky it is the preparation of \(|GHZ\rangle\) state and then measurement \( x \) components of spin for all particles, while in Hardy’s case this is preparation of the electron-positron state and then detection of electron and positron in certain
detectors. Therefore, the discussion in the beginning of this Letter is relevant for both examples.

Consider the first example. The state $|\Psi_1\rangle$ is given by Eq.(8); $|\Psi_2\rangle = |x_1, x_2, x_3\rangle$, i.e., the state with certain $x$ components of spin. The operators considered between these two states are: $\sigma_2 y \sigma_3 y$, $\sigma_1 y \sigma_3 y$, and $\sigma_1 y \sigma_2 y$.

The formalism, Eq.(6), yields (as it should be) the probability 1 for the outcomes given by Eqs.(9a-c). But it also shows that the measurements of commuting operators $\sigma_1 y \sigma_3 y$, and $\sigma_1 y \sigma_2 y$ disturb each other. Eq.(6) yields that the probability to find both results (9b) and (9c), when measured together is just $\frac{1}{2}$. Again the measurement of the product differs from the measuring both of the operators separately and the probability to find the outcome of the product measurement corresponding to the product of (9b) and (9c) is zero since the outcome have to be given by Eq.(9a).

In Hardy’s example the free Hamiltonian is not zero, it describes the interaction of the electron and the positron with beam splitters and mirrors of the interferometers as well as the annihilation of the electron and the positron in $P$. Therefore, the state $|\Psi_1\rangle$ in the formula (6) must be the initial state evolved forward in time until the time $t$, while the state $|\Psi_2\rangle$ must be obtained by evolving the final state backward in time until the time $t$. (The time $t$, is the moment when the particles might reach the point $P$.) Straightforward calculations shows that Eq.(6) reproduces Hardy’s result: in the conditions specified by Hardy’s example if one was testing: “was the electron in $P$?” – his result must be “yes”, if another observer was looking for the positron in $P$, her answer must be “yes” too, but if both of them making the measurements each one will have just the probability $\frac{1}{3}$ for answer “yes” (they will never answer “yes” together). Again, the operator considered by
Hardy is the product of the two projection operators and its measurement is not equivalent to two simultaneous inquiries one about the existence of the electron in $P$ and another about the existence of the positron in $P$. The measurement of the product is easy to implement: we have to just look for the photons created by the annihilation of the electron-positron pair. The formalism, i.e. Eq.(6), yields the probability zero (contrary to 1 obtained from the product rule).

We believe that Redhead’s definition of elements of reality is a plausible one. It does not lead to contradiction with Lorentz invariance if we do not adopt the product rule. But in the light of what was shown in the beginning of the Letter it is clear that the product rule is inconsistent with Redhead’s elements of reality. The elements of reality are obtained under the assumption that there are no measurements disturbing the inference of the values of the elements of reality. Pitowsky and Clifton state it explicitly in their works: “For our argument, we shall assume that no such intervening measurements take place.” But as we showed the measurements of the operators corresponding to the elements of reality the product of which they consider do intervene with each other. So, it is inconsistent with the way of the definition of the elements of reality to apply the product rule. If there is an element of reality that $A=a$ and there is an element of reality $B=b$, it does not follow that there is an element of reality $AB=ab$. It might be that the product $AB$ has a certain value, and therefore it is an element of reality in the Redhead’s language, but it need not to be equal $ab$.

In fact, this is what happens in all the examples considered here. In the first example we have elements of reality: “$\sigma_{1y} = -1$”, “$\sigma_{2x} = -1$”, and the product is also an element of reality, but $\sigma_{1y}\sigma_{2x} = -1$. In Pitowsky example the elements of reality are: “$\sigma_{2y}\sigma_{3y} = x_1$,” $\sigma_{1y}\sigma_{3y} = x_2$”;and the product is also known,
\[\sigma_2 \sigma_3 \sigma_1 \sigma_3 = \sigma_1 \sigma_2 = x_3.\] Nevertheless \(x_1 x_2 \neq x_3, (x_1 x_2 = -x_3).\) And, in Hardy’s example \(P_{e^-} = 1, P_{e^+} = 1,\) but \(P_{e^-} P_{e^-} = 0,\) where \(P_{e^-}, P_{e^+}\) are projection operators on the states “an electron in \(\mathcal{P}\)” and “a positron in \(\mathcal{P}\)” respectively.

In no-hidden-variables theorems where only the wave function evolving from the past towards the future is considered (i.e. there is no Lorentz frame in which some facts are retrodicted), the product rule or its generalization to any function of commuting operators is certainly valid.\(^{13}\) Its validity is based on joint measurability of commuting operators. However, in considerations in which retrodictions (at least in the eyes of some Lorentz observers) are involved, measurements of commuting operators do disturb each other, and the product rule contradicts the spirit of the definition of elements of reality.

Giving up the product rule allows us to extend the concept of elements of reality. Since we anyway consider circumstances in which retrodictions are involved, we propose to include them fully and give to it the same status as to predictions. In the examples presented here predictions were applied to the future events as well as to the causally disconnected (space-like separated) events, while retrodictions appeared only for space-like separated events. We propose to allow retrodiction to the past also. It will fit the Redhead definition of elements of reality with a minor change of “quantum predictions” to “quantum inferences”. Then, in the two spin-1/2 particles example, the observer which measures \(\sigma_{1x}(t_2) = 1\) not only infers that \(\sigma_{2x}(t) = -1\) but also that \(\sigma_{1x}(t) = 1.\) Or, in the example of Pitowsky there are also elements of reality at the time \(t,\) before the actual measurements of spin \(x\) components: \(\sigma_{1x} = x_1, \sigma_{2x} = x_2,\) and \(\sigma_{3x} = x_3.\)

According to the definition, the element of reality exists “whether or not the prediction [inference] is actually verified...” The first corresponds to the counterfac-
tual statement: if someone performed a measurement at the time $t$ (and no other measurements were performed between $t_1$ and $t_2$) then the outcome of his measurement is certain (given by the element of reality). But even the part corresponding to “no actual verification measurement” has physical meaning. Recently introduced weak measurements can test the elements reality almost without disturbing the quantum system. We need an ensemble of identically pre- and post-selected systems. Each system is practically undisturbed by the measuring interaction (which is a standard measuring procedures with very weak coupling), but also each individual measurement yields almost no information. However, collecting the results on the ensemble we can find the results of the weak measurement which is called weak value. It has been proven that in all cases there exist an element of reality i.e., the outcome value of the measurement, if measured, is certain, the weak value is equal to this value.

Taking in account acute difficulties in building Lorentz invariant realistic description of a time evolution of a quantum system using a single quantum state evolving from the past toward the future, we suggest a description of a quantum system using the two state vectors: one evolving from the measurement in the past and teh other evolving backward in time from the measurement in the future (relative to a given time). The elements of reality defined by prediction and retrodiction yield Lorentz invariant description of the history of a quantum system.
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