Abstract. The existence of a Seshadri stratification on an embedded projective variety provides a flat degeneration of the variety to a union of projective toric varieties, called a semi-toric variety. Such a stratification is said to be normal when each irreducible component of the semi-toric variety is a normal toric variety. In this case, we show that a Gröbner basis of the defining ideal of the semi-toric variety can be lifted to define the embedded projective variety. Applications to Koszul and Gorenstein properties are discussed.

1. Introduction

Seshadri stratifications on an embedded projective variety $X \subseteq \mathbb{P}(V)$ have been introduced in [5] as a far reaching generalization of the construction in [15]. The aim is to provide a geometric framework of standard monomial theories such as Hodge algebras [12], LS-algebras [3], etc.

Such a stratification consists of certain projective subvarieties $X_p \subseteq X$ and homogeneous functions $f_p \in \text{Sym}(V^*)$ indexed by a finite set $A$. The set $A$ inherits a partially ordered set (poset) structure from the inclusion relation between the subvarieties $X_p$. These data: the collection of subvarieties $X_p$ and of homogeneous functions $f_p$, $p \in A$, and the poset structure on $A$ should satisfy the regularity and compatibility conditions in Definition 2.1.

Out of a Seshadri stratification we construct in [5] a quasi-valuation $\mathcal{V}$ on the homogeneous coordinate ring $R := \mathbb{K}[\hat{X}]$ taking values in the vector space $\mathbb{Q}^A$, where $\hat{X}$ is the affine cone of $X$. The quasi-valuation has one-dimensional leaves, hence its image in $\mathbb{Q}^A$, denoted by $\Gamma$, parametrizes a vector space basis of the homogeneous coordinate ring $R$. The set $\Gamma$, called a fan of monoids, carries fruitful structures: it is a finite union of finitely generated monoids in $\mathbb{Q}^A$, each monoid corresponds to a maximal chain in $A$. Geometrically, such a quasi-valuation provides a flat degeneration of $X$ into a union of projective toric varieties whose irreducible components arise from the monoids in $\Gamma$. Geometrically, such a quasi-valuation provides a flat degeneration of $X$ into a union of projective toric varieties whose irreducible components arise from the monoids in $\Gamma$. Such a flat family is called a semi-toric degeneration of $X$. In general, the degeneration constructed in this way is different from the degeneration in Gröbner theory using a monomial order: the ideal defining the semi-toric variety is radical. Roughly speaking, it is the deepest degeneration without introducing any nilpotent elements.

We associate in [5] a Newton-Okounkov simplicial complex to a Seshadri stratification, and introduce an integral structure on it to establish a connection between the volume of the simplicial complex and the degree of $X$ with respect to the embedding.

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In the article, toric varieties are reduced and irreducible, but not necessarily normal.
When all toric varieties appearing in the semi-toric degeneration are normal, or equivalently, all monoids in the fan of monoids $\Gamma$ are saturated, such a Seshadri stratification is called normal. From such stratifications, we are able to derive a standard monomial theory in loc.cit.

As an application, the Lakshmibai-Seshadri path model [19, 20] for a Schubert variety is recovered from the Seshadri stratification consisting of Schubert subvarieties contained in it (see [7], [8] for details).

In this article, we study certain properties and applications of normal Seshadri stratifications.

First we will show (Theorem 3.2) that for such a stratification, the subduction algorithm lifts a reduced Gröbner basis of the defining ideal of the semi-toric variety to a reduced Gröbner basis of the defining ideal of $X$ with respect to an embedding. The example of the flag variety $\text{SL}_3/B$ in $\mathbb{P}(V(\rho))$, with the Seshadri stratification given by its Schubert varieties, is discussed in Section 5. As an application, we study how to determine the Koszul property of the homogeneous coordinate ring $R$ from properties of the stratification. For this we introduce Seshadri stratifications of LS-type (Definition 2.6), and prove (Theorem 3.4): if the stratification is of LS-type and the functions $f_p$ are linear, then the algebra $R$ is Koszul. We also show that the Gorenstein property of the semi-toric variety can be lifted to $R$. As an application we show (Proposition 4.4) that the irreducible components of the semi-toric variety are not necessarily weighted projective spaces.

The Gröbner basis and the Koszul property have already been addressed for Schubert varieties in [18], and for LS-algebras in [2, 4]. Our approach in this article is different. For example, the Gröbner basis of the defining ideal of $X$ is obtained in an algorithmic way by lifting the semi-toric relations; moreover, instead of being assumptions, weaker versions of quadratic straightening relations in the definition of LS-algebras become now consequences. In our paper [6] the relation between quasi-valuations and LS-algebras is studied in yet another way: starting from an LS-algebra and defining a quasi-valuation similar to the geometric one coming from Seshadri stratifications.

This article is organized as follows. In Section 2 we give a recollection on normal Seshadri stratifications and several constructions around them. Lifting Gröbner bases from the semi-toric varieties to the original variety is discussed in Section 3, which is then used to study the Koszul property. The Gorenstein property is discussed in Section 4; it is then applied to answer the question whether all irreducible components in the semi-toric variety are weighted projective spaces. Section 5 is devoted to an explicit example, when $X$ is the flag variety $\text{SL}_3/B$, to illustrate the lifting procedure of Gröbner bases.

2. Seshadri stratifications

Throughout the paper we fix $\mathbb{K}$ to be an algebraically closed field and $V$ to be a finite dimensional vector space over $\mathbb{K}$. The vanishing set of a homogeneous function $f \in \text{Sym}(V^*)$ will be denoted by $\mathcal{H}_f := \{[v] \in \mathbb{P}(V) \mid f(v) = 0\}$. For a projective subvariety $X \subseteq \mathbb{P}(V)$, we let $\hat{X}$ denote its affine cone in $V$. 
In this section we briefly recall the definition of a Seshadri stratification on an embedded projective variety. We quickly outline the construction of associated quasi-valuations and their associated fan of monoids.

Certain special classes, such as normal Seshadri stratifications and Seshadri stratifications of LS-type will be discussed. Details can be found in [5].

2.1. Definition. Let \( X \subseteq \mathbb{P}(V) \) be an embedded projective variety, \( X_p, p \in A \), be a finite collection of projective subvarieties of \( X \) and \( f_p \in \text{Sym}(V^*) \), \( p \in A \), be homogeneous functions of positive degrees. The index set \( A \) inherits a poset structure by requiring: for \( p, q \in A \), \( p \geq q \) if \( X_p \supseteq X_q \). We assume that there exists a unique maximal element \( p_{\text{max}} \in A \) with \( X_{p_{\text{max}}} = X \).

Definition 2.1 ([5]). The collection of subvarieties \( X_p \) and functions \( f_p \) for \( p \in A \) is called a Seshadri stratification on \( X \), if the following conditions are fulfilled:

(S1) the projective subvarieties \( X_p, p \in A \), are smooth in codimension one; if \( q < p \) is a covering relation in \( A \), then \( X_q \) is a codimension one subvariety in \( X_p \);

(S2) for \( p, q \in A \) with \( q \not\leq p \), the function \( f_q \) vanishes on \( X_p \);

(S3) for \( p \in A \), it holds set-theoretically

\[
\mathcal{H}_{f_p} \cap X_p = \bigcup_{q \text{ covered by } p} X_q.
\]

The functions \( f_p \) will be called extremal functions.

It is proved in [5, Lemma 2.2] that if \( X_p \) and \( f_p \), \( p \in A \), form a Seshadri stratification on \( X \), then all maximal chains in \( A \) share the same length \( \dim X \). This allows us to define the length \( \ell(p) \) of \( p \in A \) to be the length of a (hence any) maximal chain joining \( p \) with a minimal element in \( A \). With this definition, \( \ell(p) = \dim X_p \).

The set of all maximal chains in \( A \) will be denoted by \( \mathcal{C} \).

To such a Seshadri stratification, we associate an edge-coloured directed graph \( \mathcal{G}_A \): as a graph it is the Hasse diagram of the poset \( A \); the edges, which correspond to covering relations in \( A \), point to the larger element.

For a covering relation \( p > q \) in \( A \), the affine cone \( \hat{X}_q \) is a prime divisor in \( \hat{X}_p \). According to (S1), the local ring \( \mathcal{O}_{\hat{X}_p, \hat{X}_q} \) is a discrete valuation ring (DVR). Let \( \nu_{p,q} : \mathcal{O}_{\hat{X}_p, \hat{X}_q} \to \mathbb{Z} \) be the associated discrete valuation. It extends to the field of rational functions \( \mathbb{K}(\hat{X}_p) = \text{Frac}(\mathcal{O}_{\hat{X}_p, \hat{X}_q}) \), also denoted by \( \nu_{p,q} \), by requiring

\[
\nu_{p,q} \left( \frac{f}{g} \right) := \nu_{p,q}(f) - \nu_{p,q}(g), \quad \text{for } f, g \in \mathcal{O}_{\hat{X}_p, \hat{X}_q} \setminus \{0\}.
\]

The edge \( q \to p \) in the directed graph \( \mathcal{G}_A \) is colored by the integer \( b_{p,q} := \nu_{p,q}(f_p) \), called the bond between \( p \) and \( q \). According to (S3), the bonds \( b_{p,q} \geq 1 \).

Since we will mainly work with the affine cones later in the article, it is helpful to extend the construction one step further. If \( p \in A \) is a minimal element, the affine cone \( \hat{X}_p \) is an affine line \( \mathbb{A}^1 \) hence \( 0 \in V \) is contained in \( \hat{X}_p \). We set \( \hat{A} := A \cup \{p_{-1}\} \) with \( X_{p_{-1}} := \{0\} \). The set \( \hat{A} \) is endowed with the structure of a poset by requiring \( p_{-1} \) to be the unique minimal element. This partial order is compatible with the inclusion of affine cones \( \hat{X}_p \) with \( p \in \hat{A} \).
We associate to the extended poset \( \hat{A} \) the directed graph \( \mathcal{G}_{\hat{A}} \), an edge between a minimal element \( p \) in \( A \) and \( p_{-1} \) is colored by \( b_{p,p_{-1}} \), the vanishing order of \( f_p \) at \( \hat{X}_{p_{-1}} \) = \( \{0\} \): it is nothing but the degree of \( f_p \).

### 2.2. A family of higher rank valuations.

From now on we fix a Seshadri stratification on \( X \subseteq \mathbb{P}(V) \). Let \( R_p := \mathbb{K}[\hat{X}_p] \) denote the homogeneous coordinate ring of \( X_p \) and \( \mathbb{K}(\hat{X}_p) \) the field of rational functions on \( X_p \).

Let \( N \) be the least common multiple of all bonds appearing in \( \mathcal{G}_{\hat{A}} \).

To a fixed maximal chain \( \mathcal{C} : p_{\max} = p_r > p_{r-1} > \ldots > p_1 > p_0 \) in \( A \), we associate a higher rank valuation \( \mathcal{V}_\mathcal{C} : \mathbb{K}[\hat{X}] \setminus \{0\} \to \mathbb{Q}^\mathcal{C} \) as follows.

First choose a non-zero rational function \( g_r := g \in \mathbb{K}(\hat{X}) \) and denote by \( a_r \) its vanishing order in the divisor \( \hat{X}_{p_{r-1}} \subset \hat{X}_{p_r} \). We consider the following rational function

\[
h := \frac{g_r^N}{f_{p_r}} \in \mathbb{K}(\hat{X}_{p_r}),
\]

where \( b_r := b_{p_r,p_{r-1}} \) is the bond between \( p_r \) and \( p_{r-1} \). By [5, Lemma 4.1], the restriction of \( h \) to \( \hat{X}_{p_{r-1}} \) is a well-defined non-zero rational function on \( \hat{X}_{p_{r-1}} \). Let \( g_{r-1} \) denote this rational function. This procedure can be iterated by restarting with the non-zero rational function \( g_{r-1} \) on \( \hat{X}_{p_{r-1}} \). The output is a sequence of rational functions

\[
g_\mathcal{C} := (g_r, g_{r-1}, \ldots, g_1, g_0)
\]

with \( g_k \in \mathbb{K}(\hat{X}_{p_k}) \setminus \{0\} \).

Collecting the vanishing orders together, we define a map

\[
\mathcal{V}_\mathcal{C} : \mathbb{K}[\hat{X}] \setminus \{0\} \to \mathbb{Q}^\mathcal{C},
\]

\[
g \mapsto \sum_{k=0}^{r} \frac{\nu_k(g_k)}{b_k} e_{p_k} + \frac{1}{N} \sum_{k=1}^{r-1} \frac{\nu_{k-1}(g_{k-1})}{b_{k-1}} e_{p_{k-1}} + \ldots + \frac{1}{N^r} \frac{\nu_0(g_0)}{b_0} e_{p_0},
\]

where \( \nu_k := \nu_{p_k,p_{k-1}} \) is the discrete valuation on the local ring \( \mathcal{O}_{\hat{X}_{p_k},\hat{X}_{p_{k-1}}} \), extended to the fraction field, and \( e_{p_k} \) is the coordinate function in \( \mathbb{Q}^\mathcal{C} \) corresponding to \( p_k \in \mathcal{C} \). Such a map defines a valuation [5, Proposition 6.10] having at most one-dimensional leaves [5, Theorem 6.16].

### 2.3. A higher rank quasi-valuation.

For a fixed maximal chain \( \mathcal{C} \in \mathcal{C} \), the image of the valuations \( \mathcal{V}_\mathcal{C} \) is not necessarily finitely generated. To overcome this problem we introduce a quasi-valuation by minimizing this family of valuations. We refer to [5, Section 3.1] for the definition and basic properties of quasi-valuations.

**Definition 2.2.** A linearization \( >^t \) of the partial order on \( A \) is called length preserving, if for any \( p, q \in A \) with \( \ell(p) > \ell(q) \), \( p >^t q \) holds.

We fix a length preserving linearization \( >^t \) of \( A \) and enumerate elements in \( A \) as

\[
q_M >^t q_{M-1} >^t \ldots >^t q_1 >^t q_0
\]

to identify \( \mathbb{Q}^A \) with \( \mathbb{Q}^{M+1} \) by sending

\[
a = a_M e_{q_M} + a_{M-1} e_{q_{M-1}} + \ldots + a_0 e_{q_0} \in \mathbb{Q}^A
\]
to \((a_M, a_{M-1}, \ldots, a_1, a_0)\). We will consider the lexicographic ordering on \(\mathbb{Q}^{M+1}\) defined by: for \(a, b \in \mathbb{Q}^{M+1}\), \(a > b\) if the first non-zero coordinate of \(a - b\) is positive. We will write \(a \geq b\) if either \(a = b\) or \(a > b\). The vector space \(\mathbb{Q}^A\) is then endowed with a total order which is clearly compatible with vector addition.

We define a map
\[
\mathcal{V} : \mathbb{K}[\hat{X}] \setminus \{0\} \to \mathbb{Q}^A, \quad g \mapsto \min\{\mathcal{V}_\mathcal{C}(g) \mid \mathcal{C} \in C\},
\]
where \(\mathbb{Q}^C\) is naturally embedded into \(\mathbb{Q}^A\) and the minimum is taken with respect to the total order defined above. By [5, Lemma 3.4], \(\mathcal{V}\) is a quasi-valuation.

The set \(\Gamma := \{\mathcal{V}(g) \mid g \in \mathbb{K}[\hat{X}] \setminus \{0\}\} \subseteq \mathbb{Q}^A\) be the image of the quasi-valuation. For a fixed maximal chain \(\mathcal{C} \in C\), we define a subset \(\Gamma_{\mathcal{C}} := \{a \in \Gamma \mid \text{supp} a \subseteq \mathcal{C}\}\) of \(\Gamma\) where for \(a = \sum_{p \in A} a_p e_p \in \mathbb{Q}^A\), \(\text{supp} a := \{p \in A \mid a_p \neq 0\}\).

**Theorem 2.3** ([5, Proposition 8.6, Corollary 9.1, Lemma 9.6]). The following hold:

1. The quasi-valuation \(\mathcal{V}\) takes values in \(\mathbb{Q}^A_{\geq 0}\).
2. The set \(\Gamma\) is a finite union of finitely generated monoids \(\Gamma_{\mathcal{C}}\).

The set \(\Gamma\) will be called a fan of monoids.

For a homogeneous element \(g \in R \setminus \{0\}\), we can recover its degree from its quasi-valuation [5, Corollary 7.5, Proposition 8.7]: we denote \(a := \mathcal{V}(g)\) with \(a = (a_p)_{p \in A}\), then \(\text{deg}(g) = \sum_{p \in A} \text{deg}(f_p) a_p\) ([5, Corollary 7.5]). This suggests to define the degree of \(a = \sum_{p \in A} a_p e_p \in \mathbb{Q}^A\) to be
\[
\text{deg}(a) := \sum_{p \in A} \text{deg}(f_p) a_p.
\]

### 2.4. Fan of monoids, semi-toric degenerations

We define a fan algebra \(\mathbb{K}[\Gamma]\) as the quotient of the polynomial ring \(\mathbb{K}[x_a \mid a \in \Gamma]\) by an ideal \(I(\Gamma)\) generated by the following elements: (1) \(x_a x_b - x_a + x_b\) if there exists a chain \(C \subseteq A\) containing both \(\text{supp} a\) and \(\text{supp} b\); (2) \(x_a x_b\) if there is no such a chain.

The quasi-valuation \(\mathcal{V}\) defines a filtration on \(R := \mathbb{K}[\hat{X}]\) as follows: for \(a \in \Gamma\) we define
\[
R_{\geq a} := \{g \in R \setminus \{0\} \mid \mathcal{V}(g) \geq a\} \cup \{0\}
\]
and similarly \(R_\geq a\) by replacing the inequality \(\geq\) with \(>\). By Theorem 2.3, \(R_{\geq a}\) and \(R_\geq a\) are ideals. The successive quotients \(R_{\geq a}/R_\geq a\) is one-dimensional [5, Lemma 10.2], and the associated graded algebra
\[
\text{gr}_\mathcal{V} R := \bigoplus_{a \in \Gamma} R_{\geq a}/R_\geq a
\]
is isomorphic to the algebra \(\mathbb{K}[\Gamma]\) [5, Theorem 11.1].

Geometrically, it means that there exists a flat family \(\pi : X \to \mathbb{A}^1\) with the generic fibre isomorphic to \(X\) and the special fibre \(\text{Proj}(\text{gr}_\mathcal{V} R)\) a (reduced) union of toric varieties [5, Theorem 12.2]. The projective variety \(\text{Proj}(\text{gr}_\mathcal{V} R)\) is called a semi-toric variety, and we say \(X\) admits a semi-toric degeneration to it.
2.5. **Normal Seshadri stratifications.** So far we have associated to a Seshadri stratification on $X \subseteq \mathbb{P}(V)$ a fan of monoids $\Gamma$, which is a finite union of finitely generated monoids $\Gamma_\mathcal{C}$.

**Definition 2.4.** A Seshadri stratification is called *normal* if for any maximal chain $\mathcal{C} \in \mathcal{C}$, the monoid $\Gamma_\mathcal{C}$ is saturated, that is to say, $\mathcal{L}_\mathcal{C} \cap \mathbb{Q}_{\geq 0} = \Gamma_\mathcal{C}$, where $\mathcal{L}_\mathcal{C}$ is the group generated by $\Gamma_\mathcal{C}$.

When a Seshadri stratification is normal, we can characterize a nice generating set of the fan algebra $\mathbb{K}[\Gamma]$.

A non-zero element $a \in \Gamma_\mathcal{C}$ is called *indecomposable* if there does not exist non-zero elements $a_1, a_2 \in \Gamma_\mathcal{C}$ with $\min \text{ supp } a_1 \geq \max \text{ supp } a_2$ such that $a = a_1 + a_2$.

Every element $a \in \Gamma_\mathcal{C}$ admits [5, Proposition 15.3] a decomposition into a sum $a = a_1 + \ldots + a_s$ of indecomposable elements in $\Gamma_\mathcal{C}$ satisfying $\min \text{ supp } a_i \geq \max \text{ supp } a_{i+1}$ for $i = 1, 2, \ldots, s - 1$. Such a decomposition is unique if $\Gamma_\mathcal{C}$ is saturated.

Let $G$ be the set of indecomposable elements in $\Gamma \subseteq \mathbb{Q}^A$. If the Seshadri stratification is normal, then any $a \in \Gamma$ admits a unique decomposition as above into a sum of elements in $G$. The set $G$ is not necessarily finite. In this article we will concentrate on the case when $G$ is finite.

**Definition 2.5.** A normal Seshadri stratification is called *of finite type* if $G$ is a finite set.

If this is the case, we let

$$S := \mathbb{K}[y_{u_1}, \ldots, y_{u_m}]$$

denote the polynomial ring indexed by $G$. We sometimes write $y_i := y_{u_i}$ for short.

In certain applications it is needed that the monoid $\Gamma_\mathcal{C}$ is not only saturated, but also of some special form. For this we recall the LS-lattice and the LS-monoid associated to a maximal chain.

For a maximal chain $\mathcal{C}: p_r > p_{r-1} > \ldots > p_1 > p_0$ in $A$, we abbreviate $b_k := b_{p_k,p_{k-1}}$ to be the bond between $p_k$ and $p_{k-1}$. The *LS-lattice* $\text{LS}_\mathcal{C}$ associated to $\mathcal{C}$ is defined as follows

$$\text{LS}_\mathcal{C} := \left\{ u = \begin{pmatrix} u_r \\ u_{r-1} \\ \vdots \\ u_0 \end{pmatrix} \in \mathbb{Q}_\mathcal{C} \right| \begin{array}{c} b_r u_r \in \mathbb{Z} \\ b_{r-1}(u_r + u_{r-1}) \in \mathbb{Z} \\ \vdots \\ b_1(u_r + u_{r-1} + \ldots + u_1) \in \mathbb{Z} \\ u_0 + u_1 + \ldots + u_r \in \mathbb{Z} \end{array} \right\}.$$  

The *LS-monoid* is its intersection with the positive octant:

$$\text{LS}_\mathcal{C}^+ := \text{LS}_\mathcal{C} \cap \mathbb{Q}_{\geq 0}^\mathcal{C}.$$  

Being an intersection of a lattice and an octant, the monoid $\text{LS}_\mathcal{C}^+$ is saturated.

**Definition 2.6.** A Seshadri stratification is called *of LS-type*, if for the extremal functions $f_p, p \in A$, are all of degree one, and for every maximal chain $\mathcal{C} \in \mathcal{C}$, $\Gamma_\mathcal{C} = \text{LS}_\mathcal{C}^+$.
Remark 2.7. A Seshadri stratification of LS-type is normal and of finite type (see Lemma 3.3).

For a fixed maximal chain $\mathcal{C} : p_r > p_{r-1} > \ldots > p_0$ in $\mathcal{C}$ as above, a monomial basis of the algebra generated by the monoid $\text{LS}^+_{\mathcal{C}}$ can be described in the following way as in [4]. We set $b_{r+1}$ and $b_0$ to be 1 and for $k = 0, 1, \ldots, r$, $M_k$ to be the l.c.m of $b_k$ and $b_{k+1}$. We consider the following map

$$
\iota_{\mathcal{C}} : \text{LS}^+_{\mathcal{C}} \to \mathbb{K}[x_0, x_1, \ldots, x_r],
$$

$$(u_r, u_{r-1}, \ldots, u_0) \mapsto x_0^{M_{u_0}}x_1^{M_{u_1}} \cdots x_r^{M_{u_r}}.
$$

We need to verify that for any $k = 0, 1, \ldots, r$, $M_k u_k \in \mathbb{N}$. Indeed, from $b_k (u_r + \ldots + u_k) \in \mathbb{N}$ it follows $M_k(u_r + \ldots + u_{k+1}) + M_k u_k \in \mathbb{N}$. Since $b_{k+1}$ divides $M_k$, $M_k(u_r + \ldots + u_{k+1}) \in \mathbb{N}$ and hence $M_k u_k \in \mathbb{N}$.

It is then straightforward to show as in loc.cit that the map is injective and extends to an injective $\mathbb{K}$-algebra homomorphism $\iota_{\mathcal{C}} : \mathbb{K}[(\text{LS}^+_{\mathcal{C}})] \to \mathbb{K}[x_0, x_1, \ldots, x_r]$.

3. Gröbner bases and applications

3.1. Lifting defining ideals. We assume that the Seshadri stratification is normal and we keep the notation as in the previous sections. Let $\mathcal{G} = \{u_i \mid i \in J\}$ be the set of indecomposable elements in $\Gamma \subseteq \mathbb{Q}^A$, indexed by the (possibly infinite) set $J$. For each $u_i \in \mathcal{G}$ we fix a homogeneous element $g_{u_i} \in R$ such that $\mathcal{V}(g_{u_i}) = u_i$. Again we use the abbreviation $g_i := g_{u_i}$. According to [5, Proposition 15.6], $\{g_i \mid i \in J\}$ forms a generating set of the algebra $R$. Moreover, for $i \in J$ let $\overline{g}_i$ be the class of $g_i$ in $\text{gr}_V R$. It is shown in loc.cit that $\{\overline{g}_i \mid i \in J\}$ generates $\text{gr}_V R$ as an algebra.

We consider the following commutative diagram of algebra homomorphisms:

$$
\begin{array}{ccc}
S & \xrightarrow{\psi} & R \\
\varphi \downarrow & & \downarrow g_i \\
\text{gr}_V R & & \overline{g}_i \\
\end{array}
$$

Let $I := \ker \psi$ and $I_V := \ker \varphi$ be the defining ideals of $R$ and $\text{gr}_V R$.

We recall the subduction algorithm from [5, Algorithm 15.15]. The input of the algorithm is a non-zero homogeneous element $f \in R$, and the output \( \sum c_{\underline{a}_1} \cdots \underline{a}_s g_{\underline{a}_1} \cdots g_{\underline{a}_s} \) is a linear combination of standard monomials which coincides with $f$ in $R$.

Algorithm:

1. Compute $a := \mathcal{V}(f)$.
2. Decompose $a$ into a sum of indecomposable elements $a = a_1 + \ldots + a_s$ such that $\min \text{supp} a_j \geq \max \text{supp} a_{j+1}$.
3. Compute \( \overline{f} \) and $\overline{g}_{\underline{a}_1} \cdots \overline{g}_{\underline{a}_s}$ in $\text{gr}_V R$ to find $\lambda \in \mathbb{K}^*$ such that $\overline{f} = \lambda \overline{g}_{\underline{a}_1} \cdots \overline{g}_{\underline{a}_s}$.
4. Print $\lambda g_{\underline{a}_1} \cdots g_{\underline{a}_s}$ and set $f_1 := f - \lambda g_{\underline{a}_1} \cdots g_{\underline{a}_s}$. When $f_1 \neq 0$ return to Step (1) with $f$ replaced by $f_1$.
5. Done.
We take \( r \in I_V \). To emphasize that it is a polynomial in \( y_i \), we write it as \( r(y_i) \). Let \( g := r(g_i) \in R \) be its value at \( y_i = g_i \) (i.e. its image under \( \psi \)). Applying the subduction algorithm to \( g \) returns the output \( h \in R \), which is a linear combination of standard monomials in \( R \). This allows us to write down the polynomial \( h(y_i) \in S \) such that \( h(g_i) = h \). We set
\[
\tilde{r}(y_i) := r(y_i) - h(y_i) \in S.
\]
The element \( \tilde{r}(g_i) = g - h \) is contained in \( I \). It has been shown in [5, Corollary 15.17] that the ideal \( I \) is generated by \( \{ \tilde{r}(g_i) \mid r \in I_V \} \).

3.2. Lifting Gröbner bases. In this paragraph we assume that the fixed normal Seshadri stratification is of finite type.

The ideal \( I_V \) is radical and generated by monomials and binomials. A Gröbner basis of such an ideal is not hard to describe. In this section we will lift a Gröbner basis of \( I_V \) to a Gröbner basis of \( I \). Later in Section 5, we will work out as an example a Gröbner basis of the defining ideal of the complete flag varieties \( \text{SL}_3/B \), embedded as a highest weight orbit.

We fix in this section a normal Seshadri stratification. Let \( G := \{ u_1, \ldots, u_n \} \) be the set of indecomposable elements in \( \Gamma \). Since the set \( G \), as a subset of \( \Gamma \), is totally ordered by \( >^t \), we assume without loss of generality that
\[
u_1 >^t u_2 >^t \ldots >^t u_m.
\]

To be coherent with respect to the standard convention in Gröbner theory [10], we consider the following total order \( > \) on monomials in \( S := \mathbb{K}[y] \mid a \in G \) defined by: for two monomials \( y_1^{k_1} \ldots y_m^{k_m} \) and \( y_1^{k_1'} \ldots y_m^{k_m'} \) with \( k_1, \ldots, k_m, \ell_1, \ldots, \ell_m \geq 0 \), we declare
\[
y_1^{k_1} \ldots y_m^{k_m} > y_1^{k_1'} \ldots y_m^{k_m'}
\]
if \( \deg(y_1^{k_1} \ldots y_m^{k_m}) > \deg(y_1^{k_1'} \ldots y_m^{k_m'}) \), or \( \deg(y_1^{k_1} \ldots y_m^{k_m}) = \deg(y_1^{k_1'} \ldots y_m^{k_m'}) \) and the first non-zero coordinate in the vector \( (k_1 - \ell_1, \ldots, k_m - \ell_m) \) is negative. The total order \( > \) is a monomial order.

Identifying the monomials in \( S \) with \( \mathbb{N}^G \), the above monomial order gives a monomial order on \( \mathbb{N}^G \). With this identification, the fan of monoids \( \Gamma \) can be embedded into both \( \mathbb{Q}^G \) and \( \mathbb{N}^G \). Therefore \( \Gamma \) is endowed with two monomial orders \( >^t \) and \( > \).

Lemma 3.1. For \( a, a' \in \Gamma \) with \( \deg a = \deg a' \), the following holds: if \( a >^t a' \), then \( a < a' \).

Proof. Let \( \supp a = \{ q_1, \ldots, q_s \} \) with \( q_1 >^t \ldots >^t q_s \) and \( \supp a' = \{ q'_1, \ldots, q'_{s'} \} \) with \( q'_1 >^t \ldots >^t q'_{s'} \). This allows us to write
\[
a = \sum_{i=1}^{s} \lambda_i e_{q_i} \quad \text{and} \quad a' = \sum_{i=1}^{s'} \lambda'_i e_{q'_i}
\]
as elements in \( \mathbb{Q}^A \) and
\[
a = \sum_{i=1}^{m} \mu_i e_{u_i} \quad \text{and} \quad a' = \sum_{i=1}^{m} \mu'_i e_{u_i}
\]
in \( \mathbb{N}^G \).
Theorem 3.2. Let $\lambda_1 = \lambda_1' = \ldots, q_{k-1} = q_{k-1}' = \lambda_{k-1}'$ but $q_k > q_k'$. Let $1 \leq t \leq m$ (resp. $1 \leq t' \leq m$) be minimum such that $q_k \in \text{supp} u_t$ and $\mu_t \neq 0$ (resp. $q_k' \in \text{supp} u_{t'}$ and $\mu_{t'} \neq 0$). From $q_k > t' q_k'$ it follows $t \leq t'$. When $t = t'$, $q_k$ will appear in $\text{supp} q_k'$, which is not possible because this would imply that $q_k = q_k'$. Therefore $t < t'$ and hence $a < a'$.

Proof. For a polynomial $f \in S$ (resp. an ideal $J \subseteq S$), let $\text{in}_\succ(f)$ (resp. $\text{in}_\succ(J)$) be the initial term of $f$ (resp. initial ideal of $J$). Let $\mathcal{G}_{\text{red}}(I_V, \succ)$ denote the reduced Gröbner basis of $I_V$ with respect to $\succ$.

Theorem 3.2. The set $\{ \tilde{r} \mid r \in \mathcal{G}_{\text{red}}(I_V, \succ) \}$ forms a reduced Gröbner basis of $I$ with respect to $\succ$.

Proof. In the proof we will slightly abuse the notation: for $f \in S$, we will write $\mathcal{V}(f)$ for $\mathcal{V}(\psi(f))$, the quasi-valuation of the value of $f$ at $g_i$.

We first show that the set $\{ \tilde{r} \mid r \in \mathcal{G}_{\text{red}}(I_V, \succ) \}$ forms a Gröbner basis. Let $\mathcal{G}_{\text{red}}(I_V, \succ)$ = $\{ r_1, \ldots, r_p \}$. According to [5, Theorem 11.1], $\text{gr}_\succ R$ is isomorphic to $K[\Gamma]$ as $K$-algebra, hence the ideal $I_V$ is generated by homogeneous binomials and monomials. By Buchberger algorithm [10, Chapter 2, Section 7], for each $1 \leq i \leq p$, if $r_i$ is not a monomial, then it has the form $\text{in}_\succ(r_i) - s_i$, where $s_i \notin \text{in}_\succ(I_V)$ is a monomial in $S$. In this case we have $\mathcal{V}(\text{in}_\succ(r_i)) = \mathcal{V}(s_i)$, hence $\mathcal{V}(r_i) \geq \mathcal{V}(\text{in}_\succ(r_i))$.

We claim that $\tilde{r}_i = \text{in}_\succ(r_i) + t_i$ where $1 \leq i \leq p$ and $t_i$ is a linear combination of monomials which are strictly smaller than $\text{in}_\succ(r_i)$ with respect to $\succ$. Indeed, the monomials appearing in $t_i$ are either $s_i$, or, according to the subduction algorithm, those strictly larger than $\mathcal{V}(r_i)$ with respect to $\succ_t$, hence they are strictly larger than $\mathcal{V}(\text{in}_\succ(r_i))$ with respect to $\succ_t$. Since the homogeneity is preserved in the subduction algorithm, by Lemma 3.1, all monomials appearing in $t_i$ are strictly smaller than $\text{in}_\succ(r_i)$ with respect to $\succ$.
Since \( \{r_1, \ldots, r_p\} \) is a Gröbner basis of \( I_V \) with respect to \( \succ \), we have:

\[
\operatorname{in}_\succ(I_V) = (\operatorname{in}_\succ(r_1), \ldots, \operatorname{in}_\succ(r_p)) = (\operatorname{in}_\succ(\tilde{r}_1), \ldots, \operatorname{in}_\succ(\tilde{r}_p)) \subseteq \operatorname{in}_\succ((\tilde{r}_1, \ldots, \tilde{r}_p)) \subseteq \operatorname{in}_\succ(I).
\]

As \( \operatorname{gr}_V R \) is the associated graded algebra of \( R \), the above inclusion implies \( \operatorname{in}_\succ(I_V) = \operatorname{in}_\succ(I) \). This shows that \( \{\tilde{r}_1, \ldots, \tilde{r}_p\} \) is a Gröbner basis of \( I \) with respect to the monomial order \( \succ \).

For the reducedness, it suffices to notice that monomials appearing in \( t_i \) are not contained in the initial ideal \( \operatorname{in}_\succ(I) \).

3.3. Koszul property. We apply Theorem 3.2 to study the Koszul property of the homogeneous coordinate ring \( R \). In the case of Schubert varieties, the Koszul property is sketched in [18, Remark 7.6] from a standard monomial theoretic point of view. For LS-algebras such a property is proved in [2, 4].

In this paragraph we fix a Seshadri stratification of LS-type on \( X \). We keep the notation introduced in the previous sections.

Recall that for an indecomposable element \( \underline{u} \in \mathbb{G} \), we have fixed a homogeneous element \( g_{\underline{u}} \in R \) with \( V(g_{\underline{u}}) = \underline{u} \).

**Lemma 3.3.** Assume that the Seshadri stratification is of LS-type. The degree of any indecomposable element \( \underline{u} \in \mathbb{G} \) is one, hence \( \deg(g_{\underline{u}}) = 1 \). In particular, the Seshadri stratification is of finite type.

**Proof.** Let \( \underline{u} \in \Gamma \) be an indecomposable element and let \( \mathcal{C} : p_r > p_{r-1} > \ldots > p_0 \) be a maximal chain in \( A \) such that \( \operatorname{supp} \underline{u} \subseteq \mathcal{C} \). We will look at \( \underline{u} \) as an element in \( \mathbb{Q}^\mathcal{C} \) and abbreviate its coordinate \( u_{pk} \) to be \( u_k \) for \( 0 \leq k \leq r \). Assume that \( \deg(\underline{u}) > 1 \) (the degree is defined in (1)). There exists a maximal index \( j \) such that

\[
u_r + u_{r-1} + \ldots + u_j \geq 1.
\]

We consider \( \underline{u}' \in \mathbb{Q}^\mathcal{A} \) with \( \operatorname{supp} \underline{u}' \subseteq \mathcal{C} \) defined by:

\[
u_k :\begin{cases} u_k, & \text{if } k > j; \\
1 - (u_r + \ldots + u_{j+1}), & \text{if } k = j; \\
0, & \text{if } k < j;
\end{cases}
\]

where we wrote \( u_k' := u_{pk}' \) for short.

We show that \( \underline{u}' \in \Gamma_\mathcal{C} \). By the assumption \( \Gamma_\mathcal{C} = \operatorname{LS}_\mathcal{C}^+ \), it suffices to show that for any \( 1 \leq k \leq r \), \( b_k(u_r' + \ldots + u_j') \in \mathbb{N} \). When \( k > j \), it follows from the corresponding property of \( \underline{u} \); when \( k \leq j \), it suffices to notice that \( u_r' + \ldots + u_k' = 1 \) and \( b_k \in \mathbb{N} \).

The difference \( \underline{u} - \underline{u}' \) lies in the lattice \( \operatorname{LS}_\mathcal{C} \), and by construction its coordinates are non-negative. Since the LS-monoid is saturated,

\[
\underline{u} - \underline{u}' \in \operatorname{LS}_\mathcal{C} \cap \mathbb{Q}_\geq 0^\mathcal{A} = \operatorname{LS}_\mathcal{C}^+.
\]

By comparing the degree, \( \underline{u} - \underline{u}' \neq 0 \), contradicts to the assumption that \( \underline{u} \) is indecomposable. The other statement \( \deg(g_{\underline{u}}) = 1 \) follows from [5, Corollary 7.5].

\[\blacksquare\]
As an application of the lifting of Gröbner basis, we prove the following

**Theorem 3.4.** The homogeneous coordinate ring $R := \mathbb{K}[\hat{X}]$ is a Koszul algebra.

**Proof.** The algebra $R$ is generated by $\{q_u \mid u \in \mathbb{G}\}$. We prove that $R$ admits a quadratic Gröbner basis, hence by [1, Page 654], $R$ is Koszul. According to Theorem 3.2 and the fact that the lifting preserves the degree, it suffices to show the following lemma:

**Lemma 3.5.** The fan algebra $\mathbb{K}[\Gamma]$ is generated by degree 2 elements.

**Proof.** We first define an ideal $J \subseteq \mathbb{K}[y_{u_1}, \ldots, y_{u_m}]$ generated by $J(u_i, u_j)$ for $u_i, u_j \in \mathbb{G}$ with $1 \leq i, j \leq m$. These elements $J(u_i, u_j)$ are defined as follows:

1. If $\supp u_i \cup \supp u_j$ is not contained in a maximal chain in $A$, then
   \[ J(u_i, u_j) := y_{u_i}y_{u_j}. \]

2. Otherwise $u_i + u_j \in \Gamma$ is well-defined. If $\min \supp u_i \not\geq \max \supp u_j$ and $\min \supp u_j \not\geq \max \supp u_i$, then by [5, Proposition 15.3], we can write
   \[ u_i + u_j = u_{i_1} + \ldots + u_{i_s}. \]
   Comparing the degree using Lemma 3.3, we have $s = 2$. By assumption we have $\min \supp u_{i_1} \geq \max \supp u_{i_2}$, then define
   \[ J(u_i, u_j) := y_{u_i}y_{u_j} - y_{u_{i_1}}y_{u_{i_2}}. \]

3. For the remaining cases we set $J(u_i, u_j) := 0$.

We single out a property which will be used later in the proof: in the case (2), if $u_i >^t u_j$, then from the proof of Lemma 3.3, $u_{i_1} >^t u_{i_2}$.

We consider an algebra homomorphism
\[ \varphi : \mathbb{K}[y_{u_1}, \ldots, y_{u_m}] \rightarrow \mathbb{K}[\Gamma], \quad y_u \mapsto x_u. \]
Recall that for $a_1, \ldots, a_k \in \mathbb{G}$, the monomial $x_{a_1} \cdots x_{a_k}$ is called standard if for any $i = 1, \ldots, k - 1$, $\min \supp a_i \geq \max \supp a_{i+1}$. This notion is similarly defined for monomials in $\mathbb{K}[y_{u_1}, \ldots, y_{u_m}]$. The standard monomials form a linear basis of $\mathbb{K}[\Gamma]$. This implies that the map $\varphi$ is surjective.

From the definition of the defining ideal $I(\Gamma)$ of $\mathbb{K}[\Gamma]$, $\varphi$ sends the ideal $J$ to zero. The map $\varphi$ induces a surjective algebra homomorphism $\overline{\varphi} : \mathbb{K}[y_{u_1}, \ldots, y_{u_m}] / J \rightarrow \mathbb{K}[\Gamma]$. We show that modulo the ideal $J$, we can write any non-zero monomial in $y_{u_1}, \ldots, y_{u_m}$ as a standard monomial, hence standard monomials generate $\mathbb{K}[y_{u_1}, \ldots, y_{u_m}] / J$, implying that $\overline{\varphi}$ is an isomorphism.

Indeed, we consider a non-zero monomial $y_{a_1} \cdots y_{a_s}$ where $a_1, \ldots, a_s \in \mathbb{G}$ and proceed by induction on $s$. We assume that this monomial is not standard because otherwise there is nothing to prove. When $s = 2$, we can use $J(a_1, a_2) \in J$ to write it as a standard monomial. For general $s > 2$, without loss of generality we can assume that
\[ a_1 >^t a_2 >^t \ldots >^t a_s, \]
with respect to the total order $>^t$ on $\mathbb{Q}^d$, and their supports are contained in a maximal chain $\mathcal{C}$ in $A$. There are two cases to consider:
(Case 1). If $y_{a_1} y_{a_2}$ is standard, then apply induction hypothesis to write $y_{a_1} \cdots y_{a_s}$ into a standard monomial $y_{b_1} \cdots y_{b_s}$ with $b_1, \ldots, b_s \in G$. Since $a_2$ is the largest element among $a_2, \ldots, a_s$ with respect to $>_t$, we have $\max \supp b_2 = \max \supp a_2$, hence $\min \supp a_1 \geq \max \supp b_2$ and the monomial $y_{a_1} y_{b_2} \cdots y_{b_s}$ is standard.

(Case 2). If $y_{a_1} y_{a_2}$ is not standard, we use the $s = 2$ case to write it into a standard monomial $y_{\ell_1} y_{\ell_2}$: we have furthermore $a_{\ell_1} > a_1$. If the monomial $y_{\ell_2} y_{a_3} \cdots y_{a_s}$ is standard, then we are done. Otherwise we apply the induction hypothesis to write it into a standard monomial $y_{b_2} \cdots y_{b_s}$. Denote $b_1 := a_{\ell_1}$, we obtain a monomial $y_{b_1} \cdots y_{b_s}$ with $y_{b_1} > y_{a_1}$. If $y_{b_1} y_{b_2}$ is standard then we are done, otherwise repeat the above procedure. Such a process will eventually terminate because there are only finitely many elements in $G$.

The lemma is proved.

The proof of the theorem is then complete.

Remark 3.6. One may also argue as in [2, 4]: By [5, Theorem 12.1], there exists a flat family over $A^1$ with special fibre Spec($\text{gr}_V R$) and generic fibre Spec($R$). By [17, Theorem 1], if $\text{gr}_V R$ is Koszul, so is $R$. Then one uses [16] and Lemma 3.5.

Example 3.7. Let $X(\tau) \subseteq \mathbb{P}(V(\lambda))$ be a Schubert variety in a partial flag variety $G/Q$ where $G$ is a semi-simple simply connected algebraic group, $Q$ is a parabolic subgroup in $G$ and $V(\lambda)$ is the irreducible representation of $G$ with a regular highest weight with respect to $Q$. We consider the Seshadri stratification on $X(\tau)$ defined in [7] consisting of all Schubert subvarieties in $X(\tau)$ and the extremal weight functions (see also Section 5). In loc.cit. we have proved that this Seshadri stratification is of LS-type. Theorem 3.4 implies that the homogeneous coordinate ring $\mathbb{K}[ X(\tau) ]$ is a Koszul algebra (see also [18]).

3.4. Relations to LS-algebras. We briefly discuss in this paragraph relations between Seshadri stratifications and LS-algebras [3, 6]. In [6] we have proved that given an LS algebra with some additional assumptions, then one can construct a quasi-valuation on the algebra having as values exactly the LS-paths. Here we see how from a Seshadri stratification of LS-type one can recover a “partial” LS algebra structure. For the definition of an LS-algebra we refer to the version in [4, 6]. Note that the conditions (LS1), (LS2) and (LS3) in [6] are labeled (LSA1), (LSA2) and (LSA3) in [4].

Assume that the Seshadri stratification on $X \subseteq \mathbb{P}(V)$ is of LS-type. We examine which conditions for being an LS-algebra hold on the homogeneous coordinate ring $R := \mathbb{K}[ X ]$.

Elements in the fan of monoids

$$\Gamma = \bigcup_{c \subseteq C} LS_c^+$$

are called LS-paths. By Lemma 3.3, the set of indecomposable elements $G$ coincides with the set of all degree one elements in $\Gamma$. For each $u \in G$ we fix homogeneous element $g_u \in R$ of degree one satisfying $V(g_u) = u$.

The condition (LS1) is fulfilled because by [5, Proposition 15.6], standard monomials form a linear basis of $R$. 

The condition (LS3) also holds by [5, Theorem 11.1, Proposition 15.6]. Indeed, since the associated graded algebra $\text{gr}_\mathbb{F} R$ is isomorphic to the fan algebra $\mathbb{K}[\Gamma]$, the homogeneous generators $g_u, u \in \mathbb{G}$, of $R$ can be chosen to meet the coefficient 1 condition in (LS3).

The condition (LS2) is almost satisfied by $R$. First, the degree two straightening relations in (LS2) are guaranteed by Theorem 3.2 and Lemma 3.5. But in the definition of an LS-algebra it is required that the standard monomials appearing in a straightening relation should be larger than the non-standard monomial with respect to a stronger relation $\subseteq$.

Summary: If the Seshadri stratification on $X \subseteq \mathbb{P}(V)$ is of LS-type, then the homogeneous coordinate ring $R$ of $X$ admits a structure of a “partial” LS-algebra, where “partial” means that in the definition of $\subseteq$ one replaces “for any total order refining the partial order” by “for the fixed total order $\geq^1$”. Note that $R$ is an LS-algebra when the Seshadri stratification is balanced with respect to all linearizations of the order on $A$; see [8] for details.

4. Gorenstein property

Following [2, 4], we study the Gorenstein property of $R$ from the viewpoint of Seshadri stratifications. As an application, we will show that the irreducible components of the semi-toric variety Proj($\text{gr}_\mathbb{F} R$) are not necessarily weighted projective spaces.

We assume that the collection $X_p$ and $f_p, p \in A$ defines a Seshadri stratification on the embedded projective variety $X \subseteq \mathbb{P}(V)$, and denote by $R := \mathbb{K}[\hat{X}]$ the homogeneous coordinate ring.

4.1. Gorenstein property. We start from the following

Proposition 4.1. If the fan algebra $\mathbb{K}[\Gamma]$ is Gorenstein, then $R$ is Gorenstein.

Proof. By [5, Theorem 11.1], $\mathbb{K}[\Gamma]$ is isomorphic to $\text{gr}_\mathbb{F} R$ as an algebra. The latter is the special fibre in a flat family [5, Theorem 12.1], the proposition follows from the fact that being Gorenstein is an open property.

Remark 4.2. For LS-algebras, under certain assumptions, the above proposition is proved in [2, 4].

When the poset $A$ is linearly ordered, the Gorenstein property of $R$ can be determined effectively.

Let the poset $A = \{p_0, \ldots, p_r\}$ in the Seshadri stratification be linearly ordered with $p_r > p_{r-1} > \ldots > p_0$. The bond between $p_k$ and $p_{k-1}$ will be denoted by $b_k$. Let $M_k$ be the l.c.m of $b_k$ and $b_{k+1}$ where $b_0$ and $b_{r+1}$ are set to be 1. Assume furthermore that the Seshadri stratification is of LS-type (Definition 2.6).

Theorem 4.3 ([4, Theorem 7.3]). Under the above assumptions, the algebra $R$ is Gorenstein if and only if for any $k = 0, 1, \ldots, r$,

$$b_k \left( \frac{1}{M_r} + \frac{1}{M_{r-1}} + \ldots + \frac{1}{M_k} \right) \in \mathbb{N}.$$
The proof of the theorem realizes $R$ as an invariant algebra of a finite abelian group acting on a polynomial ring. Such a group can be chosen to contain no pseudo-reflections, then the Gorenstein criterion in [22] can be applied. In the proof, to show that $R$ is indeed the invariant algebra, one makes use of the homomorphism $\iota_C$ after Definition 2.6: this is the reason why the Seshadri stratification is assumed to be of LS-type.

4.2. Weighted projective spaces. If all bonds appearing in the extended graph $G_\hat{A}$ are 1, such a Seshadri stratification is called of Hodge type [5, Section 16.1]. In this case the irreducible components appearing in the semi-toric variety are all projective spaces. It is natural to ask whether in general the irreducible components are weighted projective spaces. In this section we give a Seshadri stratification of LS-type on a toric variety, which is not a weighted projective space, such that the semi-toric variety associated to the stratification is the toric variety itself.

We consider the graded $\mathbb{C}$-algebra

$$R := \mathbb{C}[x_1, x_2, \ldots, x_6]/(x_2^2 - x_1x_3, x_5^2 - x_4x_6).$$

Let $X := \text{Proj}(R) \subseteq \mathbb{P}^5$ be the associated projective variety where the embedding comes from the canonical surjection $\mathbb{C}[x_1, x_2, \ldots, x_6] \to R.$

We consider the following subvarieties in $X$: $X_{p_3} := X,$

$$X_{p_2} := X_{p_3} \cap \{[0 : 0 : a : b : c : d] \in \mathbb{P}^5 \mid a, b, c, d \in \mathbb{C}\},$$

$$X_{p_1} := X_{p_2} \cap \{[0 : 0 : b : c : d] \in \mathbb{P}^5 \mid b, c, d \in \mathbb{C}\},$$

$$X_{p_0} := X_{p_1} \cap \{[0 : 0 : 0 : 0 : d] \in \mathbb{P}^5 \mid d \in \mathbb{C}\};$$

they are projective subvarieties by taking the reduced structure. Let

$$f_{p_3} := x_1, \quad f_{p_2} := x_3, \quad f_{p_1} := x_4, \quad f_{p_0} := x_6.$$

We leave it to the reader to verify that these data define indeed a Seshadri stratification on $X$ with the following colored Hasse graph

$$p_3 \quad 2 \quad p_2 \quad 1 \quad p_1 \quad 2 \quad p_0.$$

The index set $A$ is a linear poset.

This Seshadri stratification is of LS-type. Indeed, we need to show that

$$\Gamma = \left\{ u = \left( \begin{array}{c} u_3 \\ u_2 \\ u_1 \\ u_0 \end{array} \right) \in \mathbb{Q}^4 \left| \begin{array}{c} 2u_3 \in \mathbb{N} \\ u_3 + u_2 \in \mathbb{N} \\ 2(u_3 + u_2 + u_1) \in \mathbb{N} \\ u_3 + u_2 + u_1 + u_0 \in \mathbb{N} \end{array} \right. \right\}.$$

Since there exists only one maximal chain, the quasi-valuation $V$ is in fact a valuation. It is straightforward to verify that for a monomial $x_1^{a_1} \cdots x_6^{a_6},$

$$V(x_1^{a_1} \cdots x_6^{a_6}) = \left( \begin{array}{c} a_1 \\ a_3 \\ a_4 \\ a_6 \end{array} \right) + \frac{1}{2} \left( \begin{array}{c} a_2 \\ a_2 \\ a_5 \\ a_5 \end{array} \right).$$
The monomials
\[ \{x_1^{a_1} \cdots x_6^{a_6} \mid a_1, a_5 \in \{0, 1\}, a_2, a_3, a_4, a_6 \in \mathbb{N}\} \]
generate the ring \( R \), and they have different valuations. As a consequence, \( \Gamma \) is contained in the LS-monoid \( \text{LS}_X^+ \). To show the other inclusion, for \( u := (u_3, u_2, u_1, u_0) \in \text{LS}_X^+ \), the monomial with exponent

\[ ([u_3], 2(u_3 - [u_3]), u_2 - (u_3 - [u_3]), [u_1], 2(u_1 - [u_1]), u_0 - (u_1 - [u_1])) \]

has \( u \) as valuation, where \( [u] \) is the integral part of \( u \).

The associated graded algebra \( \text{gr}_v R \) is isomorphic to \( R \), and the flat family over \( \mathbb{A}^1 \) is trivial. So \( X \) itself appears as the irreducible component in the degenerate variety.

**Proposition 4.4.** The projective variety \( X \) is not isomorphic to a weighted projective space.

**Proof.** Since \( \dim X = 3 \), we consider the weighted projective spaces \( \mathbb{P}(a) \) with \( a = (a_0, a_1, a_2, a_3) \) where \( a_0 \leq a_1 \leq a_2 \leq a_3 \). Without loss of generality, we assume that the weights \( a \) are normalized, that is to say,

\[ \gcd(a_1, a_2, a_3) = \gcd(a_0, a_2, a_3) = \gcd(a_0, a_1, a_3) = \gcd(a_0, a_1, a_2) = 1. \]

By Theorem 4.3, the algebra \( R \) is Gorenstein. It suffices to consider those weighted projective spaces which are Gorenstein. For weighted projective spaces with normalized weights, being Gorenstein and being Fano are equivalent, hence by [11, Example 8.3.3, Exercise 8.3.2], \( \mathbb{P}(a) \) is Gorenstein if and only if

\[ a_i \mid a_0 + a_1 + a_2 + a_3 \quad \text{for} \quad i = 0, 1, 2, 3. \]

It is not hard to see that there are only 14 of them (see also [13, Table 1]) with

\[ a = (1, 1, 1, 1), \quad (1, 1, 1, 3), \quad (1, 1, 2, 2), \quad (1, 1, 2, 4), \quad (1, 1, 4, 6), \quad (1, 2, 2, 5), \quad (1, 2, 3, 6), \]

\[ (1, 2, 6, 9), \quad (1, 3, 4, 4), \quad (1, 3, 8, 12), \quad (1, 4, 5, 10), \quad (1, 6, 14, 21), \quad (2, 3, 3, 4), \quad (2, 3, 10, 15). \]

We compare the singular locus of \( X \) to \( \mathbb{P}(a) \) with the weights \( a \) from the above list. The singular locus of \( X \) is a disjoint union of two \( \mathbb{P}^1 \). To determine the singular locus of \( \mathbb{P}(a) \), we use the criterion from [14, Section 1]. For a prime number \( p \), denote

\[ \mathbb{P}_p(a) := \{ \bar{x} \in \mathbb{P}(a) \mid p \mid a_i \text{ for those } i \text{ with } x_i \neq 0 \}. \]

Then the singular locus of \( \mathbb{P}(a) \) is given by the union of all \( \mathbb{P}_p(a) \).

From this description, it is clear that only \( \mathbb{P}(2, 3, 3, 4) \) has as singular locus a disjoint union of two copies of \( \mathbb{P}^1 \).

It remains to show that \( X \) is not isomorphic to \( \mathbb{P}(2, 3, 3, 4) \). The variety \( X \) (resp. \( \mathbb{P}(2, 3, 3, 4) \)) is a toric variety with torus \( T = (\mathbb{C}^*)^4 \) (resp. \( T' = (\mathbb{C}^*)^4 \)). Since \( \mathbb{P}(2, 3, 3, 4) \) is a complete simplicial toric variety, if they were isomorphic as abstract varieties, such an isomorphism would be in the automorphism group of \( \mathbb{P}(2, 3, 3, 4) \). By [9, Corollary 4.7], such an automorphism group is a linear algebraic group with maximal torus \( T' \). Since the maximal tori are conjugate, the isomorphism between \( X \) and \( \mathbb{P}(2, 3, 3, 4) \) could be chosen to be toric. The toric variety \( X \) has 4 two-dimensional torus orbit closures which are all isomorphic. However, \( \mathbb{P}(2, 3, 3, 4) \) has \( \mathbb{P}(2, 3, 4) \cong \mathbb{P}(1, 2, 3) \) and \( \mathbb{P}(2, 3, 3) \cong \mathbb{P}(1, 1, 2) \) as two-dimensional torus orbit closures; they are not isomorphic by looking at the singularities. This contradiction completes the proof.
Remark 4.5. If $X$ admits a Seshadri stratification of LS-type, the proof of [4, Theorem 6.1] can be applied verbatim to show that the irreducible components appearing in the semi-toric variety (see Section 2.4) are quotients of a projective space by a finite abelian subgroup in a general linear group. Moreover, such a subgroup can be chosen to contain no pseudo-reflections.

5. Example

In this last section, we illustrate the lifting procedure in Theorem 3.2 in an example related to flag varieties. To avoid technical assumptions we fix in this section $\mathbb{C}$ as the base field.

5.1. Setup. Let $G$ be a simple simply connected algebraic group, $B \subseteq G$ be a fixed Borel subgroup and $T$ be the maximal torus contained in $B$. Let $\alpha_1, \ldots, \alpha_n$ be the simple roots with respect to the above choice, $\Delta_+$ be the set of positive roots and $\varpi_1, \ldots, \varpi_n$ be the fundamental weights generating the weight lattice $\Lambda$. For a positive root $\beta \in \Delta_+$, $U_{\pm \beta}$ are the root subgroups in $G$ associated to $\pm \beta$. Let $W := N_G(T)/T$ be the Weyl group of $G$. For $\tau \in W$, we denote $\Delta_\tau := \{ \gamma \in \Delta_+ \mid \tau^{-1}(\gamma) \notin \Delta_+ \}$ and $U_\tau := \prod_{\gamma \in \Delta_\tau} U_\gamma$ for any chosen ordering of elements in $\Delta_+$. Let $g$ be the Lie algebra of $G$ with the Cartan decomposition $g = n_+ \oplus h \oplus n_-$ such that $n_+ \oplus h$ is the Lie algebra of $B$. For a positive root $\beta \in \Delta_+$ we fix root vectors $X_{\pm \beta} \in n_\pm$ of weights $\pm \beta$. When $\beta = \alpha_i$ is a simple root, we abbreviate $X_{\pm \alpha_i} := X_{\pm i}$ for $1 \leq i \leq n$. For $k \in \mathbb{N}$, the $k$-th divided power of $X_{\pm i}$ is denoted by $X_{\pm i}^{(k)}$. For a dominant integral weight $\lambda \in \Lambda$, we denote $V(\lambda)$ the finite dimensional irreducible representation of $g$: it is a highest weight representation and we choose a highest weight vector $v_\lambda \in V(\lambda)$. We associate to a fixed element $\tau \in W$ with reduced decomposition $\tau = s_{i_1} \cdots s_{i_t}$ an extremal weight vector $v_\tau := X_{-i_1}^{(m_1)} \cdots X_{-i_t}^{(m_t)} v_\lambda \in V(\lambda)$ with $m_k$ the maximal natural number such that $X_{-i_k}^{(m_k)} \cdots X_{-i_t}^{(m_t)} v_\lambda \neq 0$. By Verma relations, $v_\tau$ is independent of the choice of the reduced decomposition of $\tau$. The dual vector of $v_\tau$ is denoted by $p_\tau \in V(\lambda)^*$. In the following example we will take $G = \text{SL}_3$. Let $X := \text{SL}_3/B$ be the complete flag variety embedded into $\mathbb{P}(V(\rho))$, where $\rho = \varpi_1 + \varpi_2$, as the highest weight orbit $\text{SL}_3 \cdot [v_\rho]$ through the chosen highest weight line $[v_\rho] \in \mathbb{P}(V(\rho))$. The homogeneous coordinate ring will be denoted by $R := \mathbb{C}[\tilde{X}]$, where the degree $k$ component is $V(k\rho)^*$. In this case the Weyl group $W$ is the symmetric group $\mathfrak{S}_3$. The longest element in $W$ will be denoted by $w_0$.

We consider the Seshadri stratification on $\text{SL}_3/B$ as in [7] given by the Schubert varieties $X(\tau)$ and the extremal weight functions $p_\tau$ for $\tau \in W$. 
The Hasse diagram with bonds associated to this Seshadri stratification is depicted as follows:

We choose $N := 2$ to be the l.c.m of all bonds appearing in the above diagram.

By [7, Theorem 7.1, Theorem 7.3], for any maximal chain $C \in C$, the monoid $\Gamma_C$ coincides with the LS-monoid $LS_C^+$, the Seshadri stratification is therefore normal. Moreover, the fan of monoid $\Gamma$ is independent of the choice of the linearization of the partial order on $A$. Without loss of generality, we choose the following length preserving linear extension of the Bruhat order on $W$:

$w_0 >^t s_1 s_2 >^t s_2 s_1 >^t s_1 >^t s_2 >^t \text{id}.$

With this total order one identifies $Q^W$ with $Q^6$.

The indecomposable elements in $G$ are $e_1, \ldots, e_6$ in $Q^6$ and the following two extra elements:

$$\pi_1 := \begin{pmatrix} 0, 0, \frac{1}{2}, \frac{1}{2}, 0, 0 \end{pmatrix}, \quad \pi_2 := \begin{pmatrix} 0, \frac{1}{2}, 0, 0, \frac{1}{2}, 0 \end{pmatrix}.$$ 

For each element $a \in G$, in [21] and [7, Lemma 13.3] with $\ell = 2$ we have introduced the path vector associated to $a$ and $\ell$, denoted by $p_a^\ell$. More precisely, for $\tau \in W$, the path vector associated to the coordinate function $e_\tau \in Q^W$ is the extremal functions $p_\tau$. For $\pi_1, \pi_2 \in G$, we denote the associated path vector by $p_{\pi_1}$ and $p_{\pi_2}$. By [7, Theorem 7.1], for $\tau \in W$, $\mathcal{V}(p_\tau) = e_\tau; \mathcal{V}(p_{\pi_1}) = \pi_1$ and $\mathcal{V}(p_{\pi_2}) = \pi_2$.

On the polynomial ring

$$S := [y_{w_0}, y_{s_1 s_2}, y_{s_2 s_1}, y_{\pi_1}, y_{s_1}, y_{s_2}, y_{\text{id}}].$$

we consider the following monomial order $\gg$. The generators of $S$ are enumerated with respect to $>^t$:

$$y_{w_0} >^t y_{s_1 s_2} >^t y_{s_2 s_1} >^t y_{\pi_1} >^t y_{s_1} >^t y_{s_2} >^t y_{\text{id}},$$

then the monomial order $\gg$ is the one defined in Section 3.2.

The associated graded algebra $\text{gr}_s R$ is generated by

$$\overline{p}_{\text{id}}, \overline{p}_{s_1}, \overline{p}_{s_2}, \overline{p}_{s_2 s_1}, \overline{p}_{w_0}, \overline{p}_{\pi_1}, \overline{p}_{\pi_2}$$

subject to the following relations:

$$\begin{align*}
\overline{p}_{s_2 s_1} \overline{p}_{s_1 s_2} &= 0, \quad \overline{p}_{s_2} \overline{p}_{s_1} = 0, \quad \overline{p}_{\pi_1} \overline{p}_{s_1 s_2} = 0, \quad \overline{p}_{\pi_1} \overline{p}_{s_2} = 0, \quad \overline{p}_{\pi_2} \overline{p}_{s_2 s_1} = 0, \\
\overline{p}_{\pi_2} \overline{p}_{s_1} &= 0, \quad \overline{p}_{s_2}^2 - \overline{p}_{s_2 s_1} \overline{p}_{s_1} = 0, \quad \overline{p}_{\pi_2}^2 - \overline{p}_{s_1 s_2} \overline{p}_{s_2} = 0, \quad \overline{p}_{\pi_2} \overline{p}_{\pi_1} = 0.
\end{align*}$$

They form a reduced Gröbner basis of the defining ideal of $\text{gr}_s R$ in $S$ with respect to the monomial order $\gg$. 

5.2. Birational charts. There are four maximal chains in $W$:

$$
\mathcal{C}_1 : w_0 > s_2 s_1 > s_1 > \text{id}, \quad \mathcal{C}_2 : w_0 > s_1 s_2 > s_2 > \text{id}, \\
\mathcal{C}_3 : w_0 > s_1 s_2 > s_1 > \text{id}, \quad \mathcal{C}_4 : w_0 > s_2 s_1 > s_2 > \text{id}.
$$

For $i = 1, 2, 3, 4$, we let $\mathcal{V}_{\mathcal{C}_i}$ denote the valuation associated to the maximal chain $\mathcal{C}_i$ in Section 2.2. We will introduce birational charts of $\text{SL}_3/B$ and its Schubert varieties to calculate these valuations.

We will work out $\mathcal{V}_{\mathcal{C}_1}(p_{\pi_2})$ and the method can be applied in a straightforward way to determine other valuations. We will freely use the notations in [7, Section 12, 13].

First consider the following birational chart of $\text{SL}_3/B$ introduced in [7, Lemma 3.2]: we write $\beta = \alpha_1 + \alpha_2$,

$$
(3) \quad U_{\beta} \times U_{\alpha_2} \times U_{-\alpha_1} \rightarrow \text{SL}_3/B \rightarrow \mathbb{P}(V(\rho))
$$

$$(\exp(t_1 X_\beta), \exp(t_2 X_{\alpha_2}), \exp(y X_{-\alpha_1})) \mapsto \exp(t_1 X_\beta) \exp(t_2 X_{\alpha_2}) \exp(y X_{-\alpha_1}) \cdot [v_{s_2 s_1}].$$

The vanishing order of the path vector $g := p_{\pi_2} \in V(\rho)^*$ along the Schubert variety $X(s_2 s_1)$ is the lowest degree of $y$ in the polynomial

$$
(4) \quad p_{\pi_2} (\exp(t_1 X_\beta) \exp(t_2 X_{\alpha_2}) \exp(y X_{-\alpha_1}) \cdot v_{s_2 s_1}) \in \mathbb{C}[t_1, t_2][y].
$$

To compute this polynomial we work in the tensor product of Weyl module $M(\rho) \otimes M(\rho)$ as in [7, Section 12.4, Lemma 13.3], where the embedding of $V(\rho)$ into $M(\rho) \otimes M(\rho)$ is uniquely determined by $v_\rho \mapsto m_\rho \otimes m_\rho$. The path vector $p_{\pi_2}$ is defined as the restriction of $x_{s_1 s_2} \otimes x_{s_2} \in M(\rho)^* \otimes M(\rho)^*$ to $V(\rho)^*$ (notation in [7, Section 13]). Direct computation shows that the polynomial in (4) equals to $-t_1 y$, hence the vanishing order of $p_{\pi_2}$ along $X(s_2 s_1)$ is 1.

The rational function $\frac{p_{\pi_2}^2}{p_{\pi_0}^2}$ coincides with $\frac{p_{\pi_2}^2}{p_{s_2 s_1}^2}$ on the birational chart (3) because both of them evaluate to the polynomial $t_1^2$ on the above chart. The function $g_2 := \frac{p_{\pi_2}^2}{p_{s_2 s_1}^2}$ is a rational function on $X(s_2 s_1)$.

In order to determine the vanishing order of $g_2$ along the Schubert variety $X(s_1)$, we make use of the following birational chart

$$
U_{\alpha_1} \times U_{-\alpha_2} \rightarrow X(s_2 s_1) \rightarrow \mathbb{P}(V(\rho)_{s_2 s_1}),
$$

$$(\exp(t_1 X_{\alpha_1}), \exp(y X_{-\alpha_2})) \mapsto \exp(t_1 X_{\alpha_1}) \exp(y X_{-\alpha_2}) \cdot [v_{s_1}]$$

where $V(\rho)_{s_2 s_1}$ is the Demazure module associated to $s_2 s_1 \in W$. From similar computation as above, $p_{\pi_2}^2$ (resp. $p_{s_2 s_1}^2$) evaluates to the polynomial $t_1^2 y^2$ (resp. $y^4$), hence $g_2$ has a pole of order 2 along $X(s_1)$.

Continue this computation, we obtain

$$
ge_{\mathcal{C}_1} = \left( p_{\pi_2}, \frac{p_{\pi_2}^2}{p_{s_2 s_1}^2}, \frac{p_{\text{id}}^4}{p_{s_1}^2}, p_{\text{id}}^8 \right),
$$

and the valuation is hence

$$
\mathcal{V}_{\mathcal{C}_1}(p_{\pi_2}) = \left( 1, -\frac{1}{2}, -\frac{1}{2}, 1 \right).
$$
5.3. Lift semi-toric relations. As an example, we lift the relation $p_{s_2s_1}p_{s_1s_2} = 0$ to $R$. Other relations can be dealt with similarly.

In order to determine $V(p_{s_2s_1}p_{s_1s_2})$, we need to work out the above four valuations on $p_{s_2s_1}$ and $p_{s_1s_2}$. By [5, Example 6.8], the valuation $V_{\epsilon_1}(p_{s_2s_1}) = (0, 1, 0, 0)$.

Similar computation as in the previous paragraph, one has:

$$V_{\epsilon_1}(p_{s_1s_2}) = \left(1, -\frac{1}{2}, \frac{1}{2}, 0\right).$$

Summing them up we obtain:

$$V_{\epsilon_1}(p_{s_2s_1}p_{s_1s_2}) = \left(1, \frac{1}{2}, -\frac{1}{2}, 0\right).$$

In the same way we have:

$$V_{\epsilon_2}(p_{s_2s_1}p_{s_1s_2}) = \left(1, \frac{1}{2}, \frac{1}{2}, 0\right), \quad V_{\epsilon_3}(p_{s_2s_1}p_{s_1s_2}) = \left(1, 1, -\frac{1}{2}, \frac{1}{2}\right).$$

$$V_{\epsilon_4}(p_{s_2s_1}p_{s_1s_2}) = (1, 1, -1, 1).$$

Taking the minimum with respect to the total order defined above, we obtain

$$V(p_{s_2s_1}p_{s_1s_2}) = \left(1, 0, \frac{1}{2}, -\frac{1}{2}, 0, 0\right).$$

It decomposes into indecomposable elements in $G$ as follows:

$$\left(1, 0, \frac{1}{2}, -\frac{1}{2}, 0, 0\right) = (1, 0, 0, 0, 0, 0) + \left(0, 0, \frac{1}{2}, \frac{1}{2}, 0, 0\right).$$

The standard monomial having this quasi-valuation is hence $p_{w_0}p_{\pi_1}$.

In the next step we consider the function $p_{s_2s_1}p_{s_1s_2} - p_{w_0}p_{\pi_1}$. The coefficient $-1$ is uniquely determined by the property

$$V(p_{s_2s_1}p_{s_1s_2}) < V(p_{s_2s_1}p_{s_1s_2} + \lambda p_{w_0}p_{\pi_1})$$

for $\lambda \in \mathbb{C}$, where both sides can be computed using the birational chart (3).

Along the maximal chains $\mathcal{C}_2$ and $\mathcal{C}_3$, the valuations of $p_{s_2s_1}p_{s_1s_2}$ and $p_{w_0}p_{\pi_1}$ are different. It follows:

$$V_{\epsilon_2}(p_{s_2s_1}p_{s_1s_2} - p_{w_0}p_{\pi_1}) = \left(1, \frac{1}{2}, \frac{1}{2}, 0\right),$$

$$V_{\epsilon_3}(p_{s_2s_1}p_{s_1s_2} - p_{w_0}p_{\pi_1}) = \left(1, 1, -\frac{1}{2}, \frac{1}{2}\right).$$

Along both maximal chains $\mathcal{C}_1$ and $\mathcal{C}_4$, both valuations $V_{\epsilon_1}$ and $V_{\epsilon_4}$ on $p_{s_2s_1}p_{s_1s_2} - p_{w_0}p_{\pi_1}$ have the first coordinate 2. Since this element is homogeneous of degree 2, from [5, Corollary 7.5], in both of the valuations there exist at least one negative coordinate. According to the non-negativity of the quasi-valuation [5, Proposition 8.6], neither of them can be the minimum.

As a summary, we have shown that

$$V(p_{s_2s_1}p_{s_1s_2} - p_{w_0}p_{\pi_1}) = \left(1, \frac{1}{2}, 0, 0, \frac{1}{2}, 0\right).$$
Again decompose it into indecomposable elements 
\[
\left(1, \frac{1}{2}, 0, 0, \frac{1}{2}, 0\right) = (1, 0, 0, 0, 0, 0) + \left(0, 1 \frac{1}{2}, 0, 0, \frac{1}{2}, 0\right),
\]
we obtain the next standard monomial \(p_{w_0} p_{\pi_2}\).

On the birational chart (3) we have used before, the function \(p_{s_2 s_1} p_{s_1 s_2} - p_{w_0} p_{\pi_2} - p_{w_0} p_{\pi_1}\) is zero, giving out the lifted relation
\[
p_{s_2 s_1} p_{s_1 s_2} - p_{w_0} p_{\pi_2} - p_{w_0} p_{\pi_1} = 0.
\]

By lifting all relations in (2), the reduced Gröbner basis of the defining ideal of SL\(_3/B\) in \(\mathbb{P}(V(\rho))\) with respect to \(\succ\) is given by:

\[
\begin{align*}
p_{s_1} p_{s_2} &= p_{w_0} p_{\pi_1} + p_{w_0} p_{\pi_2}, \quad p_{s_1} p_{\pi_2} = p_{s_1 s_2} p_{w_0}, \\
p_{\pi_1} p_{\pi_2} &= p_{s_2 s_1} p_{s_1} - p_{w_0} p_{w_0}, \quad p_{\pi_1} p_{\pi_2} = p_{s_2 s_1} p_{w_0}, \\
p_{s_2 s_1} p_{s_1 s_2} &= p_{w_0} p_{\pi_1} + p_{w_0} p_{\pi_2}, \quad p_{s_2 s_1} p_{s_1 s_2} = p_{s_2 s_1} p_{w_0} - p_{w_0} p_{w_0}.
\end{align*}
\]

These relations coincide with those given in [2], although the bases are defined in a different way.

**Remark 5.1.** The Seshadri stratification on \(SL_3/B \subseteq \mathbb{P}(V(\rho))\) consisting of Schubert varieties is normal and balanced (see [7, Theorem 7.3] for details on the balanced condition). This property can be used to determine a Gröbner basis of the defining ideal of a Schubert variety in \(SL_3/B\).

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