The Burkholder-Davis-Gundy Inequality for Enhanced Martingales

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Abstract

Multi-dimensional continuous local martingales, enhanced with their stochastic area process, give rise to geometric rough paths with a.s. finite homogenous $p$-variation, $p > 2$. Here we go one step further and establish quantitative bounds of the $p$-variation norm in the form of a BDG inequality. Our proofs are based on old ideas by Lépingle. We also discuss geodesic and piecewise linear approximations.

1 Introduction

The theory of rough paths provides a new and robust way to drive differential equations by multi-dimensional stochastic processes in a deterministic way. In most cases, this is achieved by taking into account a certain stochastic area process and by establishing fine regularity properties of the resulting enhanced process. The object of study in this paper is a $d$-dimensional continuous local martingale $M$ null at $0$ for which the area is defined by iterated stochastic integration; the area process $A_t$ is simply the anti-symmetric part of the iterated Stratonovich integral,

$$M^2_t \equiv \int_0^t \int_0^s dM_r \otimes cdM_s \in \mathbb{R}^d \otimes \mathbb{R}^d.$$  

Note that the symmetric part of $M^2_t$ is given by $\frac{1}{2} M_t \otimes M_t$ and hence redundant if one knows $M^1_t \equiv M_t$. It follows that the enhanced process $M \equiv (1, M^1, M^2) \in \mathbb{R}^d \oplus \mathbb{R}^d \otimes \mathbb{R}^d$ lives in submanifold, namely in $G^2 \left( \mathbb{R}^d \right) \equiv \exp \left( \mathbb{R}^d \oplus so(d) \right)$, where $\exp : (x, a) \mapsto (1, x, a + \frac{1}{2} x \otimes x)$. The space $\mathbb{R}^d \oplus so(d)$ carries a Lie algebra structure and induces a (Lie-)group structure on $G^2 \left( \mathbb{R}^d \right)$. The interest in this algebraic exercise is that the resulting product operation on $G^2 \left( \mathbb{R}^d \right)$ is exactly what one needs to patch together "iterated integral increments" over adjacent intervals. $G^2 \left( \mathbb{R}^d \right)$ is also a metric (in fact, Polish) space under the Carnot-Caratheodory metric $d$. Intuitively, the distance of two points under this metric is the length of the shortest path in $\mathbb{R}^d$ which wipes out a prescribed
area. When \( d = 2 \), geodesics are seen to be parts of circles. \( G^2(\mathbb{R}^d) \) carries a dilation induced by \( (x, a) \mapsto (\lambda x, \lambda^2 a) \) for real \( \lambda \). In fact, the CC-metric is induced by a sub-additive norm, homogenous w.r.t. dilation. Since all continuous homogenous norms are Lipschitz equivalent, computations are often carried out w.r.t. \( ||(x, a)|| = |x| + |a|^{1/2} \). We refer to [6, 7] for background on rough paths, [2] contains a more detailed discussion of the relevant geometry and algebra. The notion of a (weak) geometric \( p \)-rough path [3] becomes quite elegant: by definition, one requires that the \( G^2(\mathbb{R}^d) \)-valued path \( M \) has finite \( p \)-variation

\[
\|M\|_{p\text{-var};[0,T]} = \left( \sup_{0\leq t_1 \leq \ldots \leq t_n \leq T} \sum d\left( M_{t_i}, M_{t_{i+1}} \right)^p \right)^{1/p} < \infty.
\]

Is is known [1] that this holds for a.e. \( M = M(\omega) \) when \( p > 2 \). The first main topic of this paper is to establish quantitative bounds of the \( p \)-variation norm in the form of a two-sided BDG inequality: for any moderate function \( F \) such as \( x \mapsto x^r \) for \( r > 0 \),

\[
\mathbb{E}\left( F\left( \|M\|_{p\text{-var};[0,T]} \right) \right) \sim \mathbb{E}\left( F\left( |\langle M \rangle_T|^{1/2} \right) \right).
\]

The algebraic and geometric preparations made above prove crucial to recycle many of the arguments given in Lépingle’s seminal paper [5] from 1976. Secondly, we discuss approximations and show \( L^q \)-convergence (at least for \( q > 1 \)) of lifted piecewise linear approximations of a continuous \( L^q \)-martingale w.r.t. homogenous \( p \)-variation topology.

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## 2 Preliminaries

We write \( \mathcal{M}_{0,\text{loc}}([0,\infty),\mathbb{R}^d) \) or \( \mathcal{M}_{0,\text{loc}}(\mathbb{R}^d) \) for the class of \( \mathbb{R}^d \)-valued continuous local martingales \( M : [0, \infty) \to \mathbb{R}^d \) null at 0. The bracket process \( \langle M \rangle : [0, \infty) \to \mathbb{R}^d \) is defined component-wise, the \( i \)th component is given by the usual bracket \( \langle M^i \rangle = \langle M^i, M^i \rangle \).

The area-process \( A : [0, \infty) \to so(d) \) is defined by Itô- or Stratonovich stochastic integration. As the matrix \( \langle M^i, M^j \rangle \) is symmetric both lead to the same area,

\[
A_{t}^{i,j} = \frac{1}{2} \left( \int_{0}^{t} M^i dM^j - \int_{0}^{t} M^j dM^i \right) = \frac{1}{2} \left( \int_{0}^{t} M^i \circ dM^j - \int_{0}^{t} M^j \circ dM^i \right).
\]

We note that the area-process is a vector-valued continuous martingale. By disregarding a null-set we can and will assume that \( M \) and \( A \) are continuous.
**Definition 1** Set $S_2(M) := M := \exp(M + A)$ so that $M \in C([0, \infty), G^2(\mathbb{R}^d))$. The resulting class of enhanced (continuous, local) martingales is denoted by $M^c_{0,\text{loc}}(G^2(\mathbb{R}^d))$. We refer to the operation $S_2: M \mapsto M$ as lift.

The lift is compatible with the stopping and time-changes.

**Lemma 2** (i) Let $\tau$ be a stopping time. Then $M^{\tau} = S_2(M^{\tau})$. (ii) Let $\phi$ be a time-change, that is, a family $\phi_s$, $s \geq 0$, of stopping times such that the maps $s \mapsto \phi_s$ are a.s. increasing and right-continuous. Assume that $M$ is constant on each interval $[\phi_{s-}, \phi_s]$. Then $M \circ \phi$ is a continuous local martingale and $M \circ \phi = S_2(M \circ \phi)$.

**Proof.** Stopped processes are special cases of time-changed processes (take $\phi_t = t \land \tau$) so it suffices to show the second statement. To this end, recall the compatibility of a time change $\phi$ and stochastic integration w.r.t. a continuous local martingale, constant on each interval $[\phi_{s-}, \phi_s]$, Proposition V.1.5. (ii) of [8]. The lift is a special case of stochastic integration.

The lift is also compatible with respect to scaling and concatenation of (local martingale) paths.

**Lemma 3** (i) If $\delta_c : G^2(\mathbb{R}^d) \to G^2(\mathbb{R}^d)$ is the dilation operator given by $\delta_c \exp(x + a) = \exp(cx + c^2a)$ then $\delta_c \cdot M = S_2(cM)$. (ii) We have $S_2(M)_{0,t} = S_2(M)_{0,s} \otimes S_2(M)_{s,t}$, $0 \leq s \leq t < \infty$.

**Proof.** (i) follows is trivial consequence of linearity of stochastic integrals. (ii) is true whenever a first order calculus underlies the lift. It now suffices to note that $S_2$ is (equivalently) defined as Stratonovich lift,

$$S_2(M)_t = \exp(M_t + A_t) = 1 + M_t + \int_0^t M_s \otimes dM_s.$$

**Definition 4** $F: \mathbb{R}^+ \to \mathbb{R}^+$ is moderate if (i) $F$ is continuous and increasing, (ii) $F(x) = 0$ if and only if $x = 0$ and (iii) for some (and then for every) $\alpha > 1$,

$$\sup_{x > 0} \frac{F(\alpha x)}{F(x)} < \infty.$$

The following result is found, for instance, in [9, p93].

**Theorem 5 (Burkholder-Davis-Gundy)** Let $F$ be a moderate function, $M \in M^c_{0,\text{loc}}(\mathbb{R})$. Then there exists a constant $C = C(F, d, \cdot|\cdot)$ so that

$$C^{-1}E\left(F\left(|(M)_{\infty}|^{1/2}\right)\right) \leq E\left(F\left(\sup_{s \geq 0} |M_s|\right)\right) \leq CE\left(F\left(|(M)_{\infty}|^{1/2}\right)\right).$$

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We collect a few properties of moderate functions.

**Lemma 6**

(i) \( x \mapsto F(x) \moderate \iff x \mapsto F \left( \frac{x}{2} \right) \moderate. \)

(ii) Given \( c, A, B > 0 : c^{-1}A \leq B \leq cA \implies \exists C = C(c, F) : C^{-1}F(A) \leq F(B) \leq CF(A). \)

(iii) \( \exists C : \forall x, y > 0 : F(x + y) \leq C[F(x) + F(y)]. \)

**Proof.** (i),(ii) are left to the reader. Ad (iii): W.l.o.g. \( x < y \), then \( F(x + y) \leq F(2y) \leq CF(y) \) by moderate growth of \( F \).

**Corollary 7** Let \( F \) be a moderate function, \( M \in \mathcal{M}_{0,loc}^c(\mathbb{R}^d) \) and \( |\cdot| \) a norm on \( \mathbb{R}^d \). Then there exists a constant \( C = C(F, d, |\cdot|) \) so that

\[
C^{-1}E\left(F\left(|\langle M \rangle_\infty^{1/2}\right)\right) \leq E\left(F\left(\sup_{s \geq 0} |M_s|\right)\right) \leq CE\left(F\left(|\langle M \rangle_\infty^{1/2}\right)\right).
\]

**Proof.** When \( |a| = \max \{|a^1|, \ldots, |a^d|\} \) this is a simple consequence of BDG for \( \mathcal{M}_{0,loc}^c(\mathbb{R}) \), applied componentwise. The lemma above shows that one can switch to Lipschitz equivalent norms.

From Lépingle [5], \( \sup_{s \geq 0} |M_s| \) above can be replaced by the \( p \)-variation norm\(^1\). Noting that the \( p \)-variation of a discrete-time martingale \( (Y_n) \) is naturally defined as

\[
|Y|_{p-\text{var}} \equiv \left[ \sup_{(n_k)_k} \sum_k |Y_{n_k+1} - Y_{n_k}|^p \right]^{1/p},
\]

the following lemma is best viewed as a BDG-type upper bound for discrete-time martingales.

**Lemma 8** Let \( F \) be moderate. If \( 1 < q < p \leq 2 \) or \( 1 = q = p \) then there exists a constant \( c \) such that for all, possibly \( \mathbb{R}^d \)-valued, discrete-time martingales \( (Y_n : n \in \mathbb{Z}^+) \)

\[
\mathbb{E}\left(F\left(|Y|_{p-\text{var}}\right)\right) \leq c\mathbb{E}\left[F\left(\sum_n |Y_{n+1} - Y_n|^q \right)^{1/q}\right].
\]

**Proof.** For \( d = 1 \) we can use Proposition 2.b in [5] with the remark that a discrete-time martingale can be viewed as a particular case of a continuous-time martingale with purely discontinuous sample paths. As above, the extension to \( d > 1 \) does not pose any difficulty.

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\(^1\)In the next section, we will see a more general version of this.
3 BDG on the group

Lemma 9 (Good λ inequality, [9, p.94]) Let $X,Y$ be nonnegative random variables, and suppose there exists $\beta > 1$ such that for all $\lambda > 0, \delta > 0$,

$$P(X > \beta \lambda, Y < \delta \lambda) \leq \psi(\delta) P(X > \lambda)$$

where $\psi(\delta) \searrow 0$ when $\delta \searrow 0$. There, for each moderate function $F$, there exists a constant $C$ depending only on $\beta, \psi, F$ such that

$$E(F(X)) \leq C E(F(Y)).$$

Proposition 10 Let $|\cdot|, \|\cdot\|$ continuous homogeneous norm on $\mathbb{R}^d, G^2(\mathbb{R}^d)$ respectively. Then there exists a constant $A = A(d, |\cdot|, \|\cdot\|)$ such that

$$\forall M \in \mathcal{M}_0^{c,0}(G^2(\mathbb{R}^d)) \forall \lambda > 0 : P\left(\sup_{u,v \geq 0} \|M_{u,v}\| \geq \lambda \right) \leq A \frac{E(\langle|\langle M\rangle\rangle\rangle_\infty)}{\lambda^2}.$$

Proof. We note that $\sup_{u,v \geq 0} \|M_{u,v}\| \leq 2 \sup_{t \geq 0} \|M_t\|$. By equivalence of homogeneous norm,

$$\|M_t\|^2 \leq C \left(|M_t|^2 + |A_t|\right).$$

From BDG, $E\left(\sup_{u \geq 0} |M_u|^2\right) \leq C E(\langle|\langle M\rangle\rangle\rangle_\infty)$. Note that $u \mapsto |\langle M\rangle_u| = \sum_{i=1}^d \langle M_i\rangle_u$ is increasing (in fact, there is no loss in generality in assuming that $|\cdot|$ on $\mathbb{R}^d$ is given by $|a| = \sum |a^i| ...$). Then, using BDG again,

$$E\left(\sup_{u \geq 0} |A_u|\right) \leq C E\left(\left|\int_0^{\langle M\rangle_\infty} |M_u|^2 d|\langle M\rangle|_u\right|^{1/2}\right)$$

$$\leq C E\left(\sup_{u \geq 0} |M_u| |\langle M\rangle|_\infty^{1/2}\right)$$

$$\leq C \frac{E \sup_{u \geq 0} |M_u|^2}{\sqrt{E(|\langle M\rangle|_\infty)}}$$

$$\leq C E\langle|\langle M\rangle\rangle\rangle_\infty.$$

An application of Chebyshev’s inequality finishes the proof. ■

Theorem 11 Let $F$ be a moderate function, $M \in \mathcal{M}_0^{c,0}(G^2(\mathbb{R}^d))$, and $|\cdot|, \|\cdot\|$ continuous homogeneous norm on $\mathbb{R}^d, G^2(\mathbb{R}^d)$ respectively. Then there exists a constant $C = C(F,d,|\cdot|,\|\cdot\|)$ so that

$$C^{-1} E\left(F\left(|\langle M\rangle|_\infty^{1/2}\right)\right) \leq E\left(F\left(\sup_{s,t \geq 0} \|M_{s,t}\|\right)\right) \leq C E\left(F\left(|\langle M\rangle|_\infty^{1/2}\right)\right).$$
Proof. The lower bound comes from
\[ \|M_{s,t}\| \geq |M_{s,t}| \]
the monotonicity of $F$ and the classical BDG lower bound. We prove the upper-bound: we fix $\lambda, \delta > 0$ and $\beta > 1$, and we define the stopping times
\[
S_1 = \inf \left\{ t > 0, \sup_{u,v \in [0,t]} \|M_{u,v}\| > \beta \lambda \right\},
\]
\[
S_2 = \inf \left\{ t > 0, \sup_{u,v \in [0,t]} \|M_{u,v}\| > \lambda \right\},
\]
\[
S_3 = \inf \left\{ t > 0, |\langle M \rangle_t|^{1/2} > \delta \lambda \right\},
\]
with the convention that the infimum of the empty set is $\infty$. Define the local martingale $N_t = M_{S_3 \wedge S_2, (t+S_2) \wedge S_3}$ noting that $N_t \equiv 0$ on \{ $S_2 = \infty$ \}. It is easy to see that
\[
\sup_{u,v \in [0,S_3]} \|M_{u,v}\| \leq \sup_{u,v \in [0,S_3 \wedge S_2]} \|M_{u,v}\| + \sup_{u,v \geq 0} \|N_{u,v}\|. \tag{1}
\]
By definition of the relevant stopping times,
\[
\mathbb{P} \left( \sup_{u,v \geq 0} \|M_{u,v}\| > \beta \lambda, |\langle M \rangle_\infty|^{1/2} \leq \delta \lambda \right) = \mathbb{P} (S_1 < \infty, S_3 = \infty).
\]
On the event \{ $S_1 < \infty, S_3 = \infty$ \} one has
\[
\sup_{u,v \in [0,S_3]} \|M_{u,v}\| > \beta \lambda
\]
and, since $S_2 \leq S_1$, one also has $\sup_{u,v \in [0,S_3 \wedge S_2]} \|M_{u,v}\| = \lambda$. Hence, on \{ $S_1 < \infty, S_3 = \infty$ \},
\[
\sup_{u,v \geq 0} \|N_{u,v}\| \geq \sup_{u,v \in [0,S_3]} \|M_{u,v}\| - \sup_{u,v \in [0,S_3 \wedge S_2]} \|M_{u,v}\| \geq (\beta - 1) \lambda.
\]
Therefore, using Proposition 10,
\[
\mathbb{P} \left( \sup_{u,v \geq 0} \|M_{u,v}\| > \beta \lambda, |\langle M \rangle_\infty|^{1/2} \leq \delta \lambda \right) \leq \mathbb{P} \left( \sup_{u,v \geq 0} \|N_{u,v}\| \geq (\beta - 1) \lambda \right) \leq \frac{A}{(\beta - 1)^2 \lambda^2} \mathbb{E} (|\langle N \rangle_\infty|).
\]
From the definition of $N$, for every $t \in [0, \infty]$,
\[
\langle N \rangle_t = \langle M \rangle_{S_3 \wedge S_2, (t+S_2) \wedge S_3}.
\]
On \( \{ S_2 = \infty \} \) we have \( \langle N \rangle_\infty = 0 \) while on \( \{ S_2 < \infty \} \) we have, from definition of \( S_3 \),
\[
|\langle N \rangle_\infty| = |\langle M \rangle_{S_3 \wedge S_2, S_3} - \langle M \rangle_{S_3} - \langle M \rangle_{S_3 \wedge S_2}| \leq 2 |\langle M \rangle_{S_3}| = 2 \delta^2 \lambda^2.
\]
It follows that
\[
E (|\langle N \rangle_\infty|) \leq 2 \delta^2 \lambda^2 \mathbb{P} (S_2 < \infty) = 2 \delta^2 \lambda^2 \mathbb{P} \left( \sup_{u, v \geq 0} \| M_{u, v} \| > \lambda \right)
\]
and we have the estimate
\[
\mathbb{P} \left( \sup_{u, v \geq 0} \| M_{u, v} \| > \beta \lambda, |\langle M \rangle_\infty| \leq \delta \lambda \right) \leq \frac{2 A \delta^2}{(\beta - 1)^2} \mathbb{P} \left( \sup_{u, v \geq 0} \| M_{u, v} \| > \lambda \right).
\]
An application of the good \( \lambda \)-inequality finishes the proof.

4 Path regularity and \( p \)-variation BDG

Let \( p > 2 \). From [1] it is known that for every \( M \in \mathcal{M}_{0, \text{loc}}^c \left( G^2 (\mathbb{R}^d) \right) \) and every \( T > 0 \)
\[
\| M \|_{p\text{-var}; [0, T]} < \infty \text{ a.s.} \quad (2)
\]
Here, we go one step further and provided quantitative bounds for the \( p \)-variation of the enhanced martingale in terms of \( \langle M \rangle_T \). En passant, we give a simplified proof of (2).

**Proposition 12** Let \( M \in \mathcal{M}_{0, \text{loc}}^c \left( G^2 (\mathbb{R}^d) \right) \). Then, for every \( T > 0 \),
\[
\| M \|_{p\text{-var}; [0, T]} < \infty \text{ a.s.}
\]
**Proof.** There exists a sequence of stopping times \( \tau_n \to \infty \) a.s. such that \( M^{\tau_n} \) and \( \langle M^{\tau_n} \rangle \) are bounded (for instance, \( \tau_n = \inf \{ t : |M_t| > n \text{ or } |\langle M \rangle_t| > n \} \) will do.) Since
\[
\mathbb{P} \left( \| M \|_{p\text{-var}; [0, T]} \neq \| M \|_{p\text{-var}; [0, T \wedge \tau_n]} \right) \leq \mathbb{P} (\tau_n < T) \to 0 \text{ as } n \to \infty
\]
it suffices to consider the lift of a bounded continuous martingale with bounded quadratic variation. We can work with the \( l^1 \)-norm on \( \mathbb{R}^d \), \( |a| = \sum_{i=1}^d |a_i| \). The time change \( \phi (t) := \inf \{ s : |\langle M \rangle_s| > t \} \) may have jumps but continuity of \( |\langle M \rangle| \) ensures that \( |\langle M \rangle_{\phi(t)}| = t \). From definition of \( \phi \) and the BDG inequality on the group, both \( |\langle M \rangle| \) and \( M \) are constant on the intervals \( [\phi_{t-}, \phi_t] \). It follows that \( X_t = M_{\phi(t)} \) defines a continuous\(^2\) path from \( [0, |\langle M \rangle_T|] \) to \( G^2 (\mathbb{R}^d) \) and it is easy to see that
\[
\| X \|_{p\text{-var}; [0, |\langle M \rangle_T|]} = \| M \|_{p\text{-var}; [0, T]}.
\]
\(^2\)From Lemma 2, \( X = S_2 (M \circ \phi) \), the lift of a continuous local martingale. In particular, this is another way to see continuity of \( X \).
As argued in the beginning of the proof, we may assume that \(|\langle M \rangle_T| \leq R\) for some deterministic \(R\) large enough. Therefore,

\[
P\left(\|M\|_{p\text{-var},[0,T]} > K\right) = \mathbb{P}\left(\|X\|_{p\text{-var},[0,|\langle M \rangle_T|]} , |\langle M \rangle_T| \leq R\right) \leq \mathbb{P}\left(\|X\|_{p\text{-var},[0,R]} > K\right),
\]

(3)

We go on to show that \(X\) is in fact Hölder continuous. For \(0 \leq s \leq t \leq R\), we can use the BDG inequality on the group, theorem 11, to obtain

\[
\mathbb{E}\left(\|X_{s,t}\|^{2q}\right) = \mathbb{E}\left(\|M_{\phi(s),\phi(t)}\|^{2q}\right) \leq C_q \mathbb{E}\left(|\langle M \rangle_{\phi(t)} - \langle M \rangle_{\phi(s)}|^q\right).
\]

Observe that

\[
\left|\langle M \rangle_{\phi(t)} - \langle M \rangle_{\phi(s)}\right| = \sum_i \left(\langle M^i \rangle_{\phi(t)} - \langle M^i \rangle_{\phi(s)}\right) = \left|\langle M^i \rangle_{\phi(t)}\right| - \left|\langle M^i \rangle_{\phi(s)}\right| = t - s.
\]

Thus, for all \(q < \infty\) there exists a constant \(C_q\) s.t.

\[
\mathbb{E}\left(\|X_{s,t}\|^{2q}\right) \leq C_q |t - s|^q.
\]

Knowing that \(X\) is continuous, we can apply GRR\(^3\) for paths in \((G^2(\mathbb{R}^d),d)\) to see that \(\|X\|_{1/p\text{-Hölder},[0,R]} \in L^q\) for all \(q \in [1, \infty)\) and

\[
P\left(\|X\|_{p\text{-var},[0,R]} > K\right) \leq \frac{\mathbb{E}\left(\|X\|_{p\text{-var},[0,R]}\right)}{K} \leq \frac{\mathbb{E}\left(\|X\|_{1/p\text{-Hölder},[0,R]}\right)}{K}
\]

tends to zero as \(K \to \infty\). Together with (3) we see that \(\|M\|_{p\text{-var},[0,T]} < \infty\) with probability 1 as claimed.

We are now going to prove a \(p\)-variation version of BDG. For \(\mathbb{R}\)-valued martingales this result is due to Lépingle, [5]. With the preparations made, our proof follows the same lines.

**Lemma 13** There exists a constant \(A\) such that for all continuous local martingales \(M\), for all \(\lambda > 0\),

\[
P\left(\|M\|_{p\text{-var},[0,\infty]} > \lambda\right) \leq \frac{A \mathbb{E}\left(|\langle M \rangle_\infty|\right)}{\lambda^p}.
\]

**Proof.** If suffices to prove the statement when \(\lambda = 1\) (the general case follows by considering \(M/\lambda\) with lift \(\delta_{1/\lambda} M\)). The statement then reduces to

\[
\exists A : \forall M : P\left[\|M\|_{p\text{-var},[0,\infty]} > 1\right] \leq A \mathbb{E}\left(|\langle M \rangle_\infty|\right).
\]

\(^3\)There is no modification of \(X\) needed.
Assume this is false. Then for every $A$, and in particular for $A(k) \equiv k^2$, there exists $M \equiv M^{(k)}$ with lift $\tilde{M}^{(k)}$ s.t. the condition is violated,

$$
k^2 \mathbb{E} \left[ \left\langle \langle M^{(k)} \rangle_\infty \right\rangle \right] < \mathbb{P} \left[ \left\| M^{(k)} \right\|_{\text{p-var};[0,\infty)} > 1 \right] \leq 1.
$$

Set $u_k = \mathbb{P} \left[ \left\| M^{(k)} \right\|_{\text{p-var};[0,\infty)} > 1 \right]$, $n_k = \lfloor 1/u_k + 1 \rfloor \in \mathbb{N}$ and note that $1 \leq n_ku_k \leq 2$. Take $n_k$ copies of each $M^{(k)}$ and get a sequence of martingales of form

$$
\left( \tilde{M} \right) \equiv \left( M^{(1)}, \ldots, M^{(1)}; M^{(2)}, \ldots, M^{(2)}; M^{(3)}, \ldots \right).
$$

Then

$$
n_kk^2 \mathbb{E} \left[ \left\langle \langle M^{(k)} \rangle_\infty \right\rangle \right] \leq n_k \mathbb{P} \left[ \left\| M^{(k)} \right\|_{\text{p-var};[0,\infty)} > 1 \right] = n_ku_k \leq 2.
$$

and

$$
\sum_k \mathbb{P} \left[ \left\| \tilde{M}^{(k)} \right\|_{\text{p-var};[0,\infty)} > 1 \right] = \sum_k n_ku_k = +\infty
$$

while

$$
\sum_k \mathbb{E} \left[ \left\langle \tilde{M}^{(k)} \right\rangle_\infty \right] = \sum_k n_k \mathbb{E} \left[ \left\langle M^{(k)} \right\rangle_\infty \right] \leq \sum_k \frac{2}{k^2} < \infty.
$$

Thus, if the claimed statement is false, there exists a sequence of martingales, we now revert to write $M^{(k)}$, $\tilde{M}^{(k)}$ instead of $\tilde{M}^{(k)}$, $\tilde{M}^{(k)}$ respectively, each defined on some filtered probability space $(\Omega^k, (\mathcal{F}_t^k), \mathbb{P}^k)$ with the two properties

$$
\sum_k \mathbb{P}^k \left[ \left\| M^{(k)} \right\|_{\text{p-var};[0,\infty)} > 1 \right] = +\infty \quad \text{and} \quad \sum_k \mathbb{P}^k \left[ \left\langle M^{(k)} \right\rangle_\infty \right] < \infty.
$$

Define the probability space $\Omega = \bigotimes_{k=1}^{\infty} \Omega^k$, the probability $\mathbb{P} = \bigotimes_{k=1}^{\infty} \mathbb{P}^k$, and the filtration $(\mathcal{F}_t)$ on $\Omega$ given by

$$
\mathcal{F}_t = \left( \bigotimes_{i=1}^{k-1} \mathcal{F}_i^\infty \right) \otimes \mathcal{F}_t^k \otimes \left( \bigotimes_{j=k+1}^{\infty} \mathcal{F}_0^j \right)
$$

for $k-1 \leq t < k$.

where $g(u) = 1/u - 1$ maps $[0, 1] \to [0, \infty]$. Then, a continous martingale on $(\Omega, (\mathcal{F}_t), \mathbb{P})$ is defined by concatenation,

$$
M_t = \sum_{i=1}^{k-1} M^{(i)}_\infty + M^{(k)}_{g(k-t)} \quad \text{for} \quad k-1 \leq t < k.
$$

which implies

$$
M_t = \left( \bigotimes_{i=1}^{k-1} M^{(i)}_\infty \right) \otimes M^{(k)}_{g(k-t)}.
$$
We also observe that, again for \( k - 1 \leq t < k \),
\[
\langle M \rangle_t = \sum_{i=1}^{k-1} \langle M(i) \rangle_{\infty} + \langle M(k) \rangle_{g(k-t)}.
\]
In particular, \( \langle M \rangle_{\infty} = \sum_{k} \langle M(k) \rangle_{\infty} \) and, using the second property of the martingale sequence, \( \mathbb{E}(\langle M \rangle_{\infty}) < \infty \). Define the events
\[
A_k = \left\{ \|M\|_{p\text{-var};[k-1,k]} > 1 \right\}.
\]
Then, using the first property of the martingale sequence,
\[
\sum_{k} \mathbb{P}(A_k) = \sum_{k} \mathbb{P}^k \left( \|M^k\|_{p\text{-var};[0,\infty)} > 1 \right) = \infty.
\]
Since the events \( \{A_k : k \geq 1\} \) are independent, the Borel-Cantelli lemma implies that \( \mathbb{P}(A_k \text{ i.o.}) = 1 \). Thus, almost surely, for all \( K > 0 \) there exists a finite number of increasing times \( t_0, \ldots, t_n \in [0, \infty) \) so that
\[
\sum_{i=1}^{n} \|M_{t_{i-1},t_{i}}\| > K
\]
and \( \|M\|_{p\text{-var};[0,\infty)} \) must be equal to \( +\infty \) with probability one. We now define a martingale \( N \) by time-change, namely via \( f(t) = t/(1-t) \) for \( 0 \leq t < 1 \) and \( f(t) = \infty \) for \( t \geq 1 \),
\[
N : t \mapsto M_{f(t)}.
\]
Note that \( \mathbb{E}(\langle M \rangle_{\infty}) < \infty \) so that \( M \) can be extended to a (continuous) martingale indexed by \([0, \infty]\) and \( N \) is indeed a continuous martingale with lift \( N \).
Since lifts interchange with time changes, \( \|N\|_{p\text{-var};[0,1]} = \|M\|_{p\text{-var};[0,\infty)} = +\infty \)
with probability one. But this contradicts to \( p \)-variation regularity result above.

The very same argument that was used in the proof of Theorem 11 now leads to the following BDG inequality for enhanced continuous local martingales w.r.t. homogeneous \( p \)-variation norm.

**Theorem 14** Let \( F \) be a moderate function, \( M \in \mathcal{M}^c_{0,\text{loc}} \left( G^2(\mathbb{R}^d) \right) \), and \( \|\cdot\|, \|\cdot\|_{|\cdot|} \) continuous homogenous norm on \( \mathbb{R}^d, G^2(\mathbb{R}^d) \) respectively and \( p > 2 \). Then there exists a constant \( C = C(p, F, d, |\cdot|, \|\cdot\|) \) so that
\[
C^{-1} \mathbb{E} \left( F \left( \langle M \rangle_{\infty}^{1/2} \right) \right) \leq \mathbb{E} \left( F \left( \|M\|_{p\text{-var};[0,\infty)} \right) \right) \leq C \mathbb{E} \left( F \left( \langle M \rangle_{\infty}^{1/2} \right) \right).
\]

**Remark 15** When \( p \in (2, 3) \) and \( N \in \{3, 4, \ldots\} \), \( M \) lifts uniquely to a \( G^N(\mathbb{R}^d) \)-valued path with finite homogeneous \( p \)-variation regularity, denoted by \( S_N(M) \), which is identified with the first \( N \) iterated Stratonovich integrals of \( M \). A basic theorem of Lyons asserts that
\[
\|S_N(M)\|_{p\text{-var}} \leq C(N) \|M\|_{p\text{-var}}
\]
and BDG inequalities for the \( p \)-variation of this step-\( N \) lift are an immediate corollary of Theorem 14.
5 Approximations

We now only consider (lifted) local martingales on \([0, T]\), defined or identified with local martingales stopped at \(T > 0\).

5.1 Geodesic approximations

The \(p\)-variation norm of geodesics approximations is uniformly controlled by the original \(p\)-variation norm. Therefore
\[
\mathbb{E} \left( F \left( \sup_D \left\| M^{[D]} \right\|_{p\text{-var};[0,T]} \right) \right) \leq C \mathbb{E} \left( F \left( \left| \left| M \right| \right|_{T}^{1/2} \right) \right)
\]
where \(M^{[D]}\) denotes the geodesics approximation to \(M\) based on some dissection \(D\) of \([0,T]\). Note that this is stronger than
\[
\sup_D \mathbb{E} \left( F \left( \left\| M^{[D]} \right\|_{p\text{-var};[0,T]} \right) \right) \leq C \mathbb{E} \left( F \left( \left| \left| M \right| \right|_{T}^{1/2} \right) \right)
\]
which is what we are going to show for piecewise linear approximations.

5.2 Piecewise linear approximations

Let \(D = (t_i)\) be a subdivision of \([0,T]\). Given \(x \in C \left( [0,T], \mathbb{R}^d \right)\) we define \(x^D\) to be the piecewise linear approximation of \(x\) which coincides with \(x\) on \(D\). Since \(x^D\) is of bounded variation, it admits a canonical lift to a \(G^2(\mathbb{R}^d)\)-valued path, denoted by \(x^D\). This notation applies path-by-path to \(M \in M^2_{\text{loc}}(\mathbb{R}^d)\), we write \(M^D = M^D(\omega)\) for the lifted piecewise linear approximation to \(M(\omega)\). The next lemma involves no probability.

Lemma 16 Set \(x^D = S_2(x^D)\) where \(x^D\) is linear between the points of \(D\). Then there exists a constant \(C = C = C(d, |\cdot|, \|\cdot\|)\) such that
\[
\|x^D\|_{p\text{-var};[0,T]} \leq C \left( \max_{(s_k) \subset D} \sum_k \|x^D_{s_k,s_{k+1}}\|^p \right)^{1/p} + C |x|_{p\text{-var};[0,T]}.
\]

Proof. First we note that
\[
\|x^D_{s,t}\|^p \leq 3^{p-1} \left[ \|x^D_{s,s,D}\|^p + \|x^D_{s,D,t,D}\|^p + \|x^D_{t,D,t}\|^p \right].
\]
Now let \((u_k)\) be a dissection of \([0,T]\), unrelated to \(D\). Recall that \(u_D\) resp. \(u_{D,k}\) refers to the right- resp. left-neighbours of \(u\) in \(D\).
\[
\sum_k \|x^D_{u_k,u_{k+1}}\|^p \leq 3^{p-1} \sum_k \|x^D_{u_k,u_{k+1},D}\|^p + 3^{p-1} \sum_k \left[ \|x^D_{u,k,u_D}\|^p + \|x^D_{u_{k+1},D,u_k}\|^p \right]
\]
\[
\leq 3^{p-1} \left( \max_{(s_k) \subset D} \sum_k \|x^D_{s_k,s_{k+1}}\|^p \right) + 3^{p-1} \|x^D\|_{p\text{-var};[0,T]}^p + C 3^{p-1} |x|_{p\text{-var};[0,T]}.
\]
Theorem 17. Let $F$ be a moderate function, $M \in \mathcal{M}^{c}_{0,\text{loc}}(G^{2}(\mathbb{R}^{d}))$, and $|\cdot|$, $\|\cdot\|$ continuous homogeneous norm on $\mathbb{R}^{d},G^{2}(\mathbb{R}^{d})$ respectively. Then there exists a constant $C = C(p,F,d,|\cdot|,\|\cdot\|)$ so that for all dissections $D$ of $[0,T]$,

$$E \left( F \left( \|M^{D}\|_{p\text{-var};[0,T]} \right) \right) \leq C E \left( F \left( |\langle M \rangle_{T}^{1/2} | \right) \right).$$

Proof. From Lemma 16, $\|M^{D}\|_{p\text{-var};[0,T]}$ is bounded by $\|M\|_{p\text{-var};[0,T]}$ plus

$$C \left( \max_{(s_{k}) \subset D} \sum_{k} \|M_{s_{k},s_{k+1}}^{D}\|^{p} \right)^{1/p} \leq C \left( \max_{(s_{k}) \subset D} \sum_{k} \|M_{s_{k},s_{k+1}}\|^{p} \right)^{1/p}$$

$$+ C \left( \sum_{k} d \left( M_{s_{k},s_{k+1}}, M_{s_{k},s_{k+1}}^{D} \right) \right)^{1/p}.$$ 

Trivially, $\|M\|_{p\text{-var};[0,T]} \leq \|M^{D}\|_{p\text{-var};[0,T]}$ and with a new constant $C$,

$$\|M^{D}\|_{p\text{-var};[0,T]} \leq C \|M\|_{p\text{-var};[0,T]} + C \left( \sum_{k} d \left( M_{s_{k},s_{k+1}}, M_{s_{k},s_{k+1}}^{D} \right) \right)^{1/p}.$$ 

For fixed $k$, there are $i < j$ so that $s_{k} = t_{i}$ and $s_{k+1} = t_{j}$. Then

$$M_{s_{k},s_{k+1}} = \bigotimes_{l=i}^{j-1} \exp \left( M_{t_{i},t_{l+1}} + A_{t_{l},t_{l+1}} \right)$$

$$M_{s_{k},s_{k+1}}^{D} = \bigotimes_{l=i}^{j-1} \exp \left( M_{t_{i},t_{l+1}} \right).$$

Hence, $d \left( M_{s_{k},s_{k+1}}, M_{s_{k},s_{k+1}}^{D} \right)$ equals

$$\left\| M_{s_{k},s_{k+1}}^{-1} \otimes M_{s_{k},s_{k+1}}^{D} \right\| = \left\| \exp \left( \sum_{l=i}^{j-1} A_{t_{l},t_{l+1}} \right) \right\| \leq C \left( \sum_{l=i}^{j-1} A_{t_{l},t_{l+1}} \right)^{1/2}. \quad (4)$$

The key idea is to introduce the (vector-valued) discrete-time martingale

$$Y_{j} = \sum_{l=0}^{j-1} A_{t_{i},t_{l+1}} \in so(d).$$

From (4) and equivalence of homogeneous norms we have

$$\max_{(s_{k}) \subset D} \sum_{k} d \left( M_{s_{k},s_{k+1}}, M_{s_{k},s_{k+1}}^{D} \right)^{p} \leq C \max_{\{i_{1},\ldots,i_{n}\} \subset \{1,\ldots,\#D\}} \sum_{k} \left| Y_{t_{k+1}} - Y_{t_{k}} \right|^{p/2},$$

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which leads to
\[
\|M^D\|_{p\text{-var}} \leq C \|M\|_{p\text{-var}} + C \sqrt{\max_{\{i_1,\ldots,i_n\} \subset \{1,\ldots,#D\}} \sum_k |Y_{k+1} - Y_k|^{p/2}}^{2/p}
\]
\[
= C \|M\|_{p\text{-var}} + C \sqrt{|Y|_{p/2\text{-var}}}
\]

Using basic properties of moderate functions we have
\[
E\left[F\left(\|M^D\|_{p\text{-var}}\right)\right] \leq CE\left[F\left(\|M\|_{p\text{-var}}\right)\right] + CE\left[F\left(\sqrt{|Y|_{p/2\text{-var}}}\right)\right]
\]
\[
= C E\left[F\left(\|M\|_{p\text{-var}}\right)\right] + CE\left[F \circ \sqrt{\left(\sum_l |A_{l,t,l+1}|^{p'/2}\right)^{2/p'}}\right]
\]
\[
\leq C E\left[F \left(\sum_l \|M_{l,t,l+1}\|^{p'/2}\right)^{1/p'}\right]
\]
\[
\leq CE\left[F \left(\|M\|_{p'/\text{-var};[0,T]}\right)\right].
\]

Combing the last two estimates and using Theorem 14 (with \(p' = 1 + p/2 > 2\) and \(p\) respectively) gives
\[
E\left[F\left(\|M^D\|_{p\text{-var};[0,T]}\right)\right] \leq CE\left[F\left(\|M\|_{p\text{-var};[0,T]}\right)\right] + CE\left[F \left(\|M\|_{p'/\text{-var};[0,T]}\right)\right]
\]
\[
\leq 2CE\left[F \left(|M|_{T}^{1/2}\right)\right].
\]

**Remark 18** We don’t expect a lower BDG bound uniformly over all dissections \(D\) of \([0,T]\). For instance,
\[
C^{-1}E \left(F \left(|M|_{T}^{1/2}\right)\right) \leq E \left(F \left(|M^D|_{\infty;[0,T]}\right)\right)
\]
can’t hold since \(D = \{0,T\}\) implies \(M^D_{\infty;[0,T]} = |M_T|\) and for \(F(x) = x\) we would control
\[
E \left(|M|_{\infty;[0,T]}\right) \sim E \left(|M|_{T}^{1/2}\right)
\]
in terms of \(E(|M_T|)\) which is Doob’s \(L^q\) maximal inequality with \(q = 1\). But, as is well known, one needs \(q > 1\) for Doob’s \(L^q\)-inequality to hold true.
Let us now bound the supremum distance between \( M \) and \( M^D \):

**Lemma 19** Assume that \( M \) is a martingale such that

\[
|M|_{\infty,[0,T]} \in L^q(\Omega) \text{ for some } q \geq 1. \tag{5}
\]

If \( D^n \) is a sequence of subdivisions whose time steps tends to 0 when \( n \) tends to \( \infty \), then \( d_{\infty,[0,T]}(M, M^{D_n}) \) converges to 0 in \( L^q \).

**Remark 20** If \( q > 1 \), Doob’s maximal inequality implies that (5) holds for any \( L^q \)-martingale.

**Proof of Lemma 19.** As in the proof of Theorem 17, equation (4) more specifically, we have that when \( t = t_i \in D \)

\[
d(M_t, M^D_t) \leq C \left| \sum_{k=0}^{i-1} A_{t_k,t_{k+1}} \right|^1_2.
\]

Next, consider \( t \in [t_i, t_{i+1}] \) for some \( i \). The path \( M^D \) restricted to \( [t_i, t_{i+1}] \) is a straight line with no area, hence

\[
M^D_{t_{i+1}, t_i} = \exp \left( \frac{t-s}{t_{i+1} - t_i} M_{t_i, t_{i+1}} \right).
\]

and

\[
d(M_t, M^D_t) = d(M_t \otimes M^D_{t_i, t}, M^D_t \otimes M^D_{t_i, t})
\]

\[
= \left\| (M^D_{t_i, t})^{-1} \otimes (M^D_{t})^{-1} \otimes M_{t_i, t} \right\|
\]

\[
\leq \left\| (M^D_{t_i, t}) \right\| + \left\| (M^D_{t})^{-1} \otimes M_{t_i, t} \right\| + \left\| M_{t_i, t} \right\|
\]

\[
\leq 2 \sup_{u,v \in [t_i, t_{i+1}]} \left\| M_{u,v} \right\| + C \max_{i,j} \sum_{l=i}^{j-1} A_{t_l,t_{l+1}}^{1/2}.
\]

For the \( L^q \) convergence, because \( M \) is almost surely continuous (in fact, uniformly continuous on the compact \([0,T] \))

\[
\max_{i=0,\ldots,\#D-1} \sup_{s,t \in [t^n_i, t^n_{i+1}]} \left\| M_{t_i, t_{i+1}} \right\| \to 0 \text{ a.s.}
\]

Hence, by dominated convergence,

\[
\lim_{|D_n| \to 0} \mathbb{E} \left( \max_{i=0,\ldots,\#D-1} \sup_{s,t \in [t^n_i, t^n_{i+1}]} \left\| M_{t_i, t_{i+1}} \right\|^q \right) = 0.
\]
With $Y$ defined as in the proof of Theorem 17, 

\[
\max_{i,j} \left| \sum_{l=i}^{j-1} A_{l,t_{l+1}} \right|^{1/2} \leq C \left[ \left( \max_{\{i_1, \ldots, i_n\} \subset \{1, \ldots, \#D\}} \sum_{k} |Y_{k+1} - Y_{k}|^{p/2} \right)^{2/p} \right]^{1/2}
\]

the computation given therein with $F(x) = x^q$ shows

\[
E \left( \max_{i,j} \left| \sum_{l=i}^{j-1} A_{l,t_{l+1}} \right|^{q/2} \right) \leq C \mathbb{E} \left[ F \circ \left( \max_{\{i_1, \ldots, i_n\} \subset \{1, \ldots, \#D\}} \sum_{k} |Y_{k+1} - Y_{k}|^{p/2} \right)^{2/p} \right]
\]

\[
\leq C \mathbb{E} \left[ F \left( \left( \sum_{l:t_{l} \in D_n} \|M_{l,t_{l+1}}\|^q \right)^{1/q} \right) \right]
\]

\[
= \mathbb{E} \left( \left( \sum_{l:t_{l} \in D_n} \|M_{l,t_{l+1}}\|^q \right) \right)
\]

But this last expression tends to zero, combining the bounded convergence theorem with a.s. convergence

\[
\lim_{n \to \infty} \sum_{l:t_{l} \in D_n} \|M_{l,t_{l+1}}\|^q = 0.
\]

Indeed, this follows from $M \in C^{0,q\text{-var}}$ since $q > 2$ and using the usual squeezing argument. To show $L^q$ convergence with respect to $d_{\infty} = d_{\infty;[0,T]}$, we also write $\|\cdot\|_\infty = \|\cdot\|_{\infty;[0,T]}$ here, recall that

\[
d_{\infty} \left( M, M^D \right) \leq \sup_{t \in [0,T]} d \left( M_t, M^D_t \right) + c \|M\|_\infty \sup_{t \in [0,T]} d \left( M_t, M^D_t \right)^{1/2}.
\]

We just showed that $\sup_{t \in [0,T]} d \left( M_t, M^D_t \right) \to 0$ in $L^q$. Then

\[
E \left( \left( \|M\|_\infty \sup_{t \in [0,T]} d \left( M_t, M^D_t \right) \right)^{q/2} \right)
\]

\[
\leq \left( E \left( \|M\|_\infty^q \right) \right)^{1/2} \left( E \left( \left( \sup_{t \in [0,T]} d \left( M_t, M^D_t \right) \right)^q \right) \right)^{1/2}.
\]

(Note that by the our BDG inequalities

\[
E \left( \left( \|M\|_{\infty;[0,T]} \right)^q \right) \leq C E \left( |M|_{T}^q \right) \leq C E \left( \left( \|M\|_{\infty;[0,T]} \right)^q \right)
\]

and the last expression is finite by assumption.)

\[\blacksquare\]
**Theorem 21** Let $M$ be as in Lemma 19. Then, $d_{p\text{-var};[0,T]}(M^D, M)$ converges to 0 in $L^q$. If $M$ is a local martingale, then convergence holds in probability.

**Proof.** The result for the local martingale will hold if the first result holds, by a localisation argument that we leave to the reader. We already saw that $L^q$-convergence holds w.r.t. $d_{\infty} = d_{\infty;[0,T]}$. To go further, writing $d_{p\text{-var}} = d_{p\text{-var};[0,T]}$, we use the interpolation formula

$$d_{p\text{-var}}(M, M^D) \leq C d_{\infty}(M, M^D)^{1-\frac{p'}{p}} \left( \|M\|_{p\text{-var}}^{\frac{q}{p'}} + \|M^D\|_{p\text{-var}}^{\frac{q}{p'}} \right), \quad 2 < p' < p.$$  

Hence,

$$E\left( |d_{p\text{-var}}(M^D, M)|^q \right) \leq CE\left( \left( \|M\|_{p\text{-var}}^{\frac{q}{p'}} + \|M^D\|_{p\text{-var}}^{\frac{q}{p'}} \right) d_{\infty}(M, M^D)^{q\left(1-\frac{p'}{p}\right)} \right)$$

Using Hölder with conjugate exponents $1/(p'/p)$ and $1/(1-p'/p)$ gives

$$E\left( |d_{p\text{-var}}(M^D, M)|^q \right) \leq CE\left( \|M\|_{p\text{-var}}^{\frac{q}{p'}} + \|M^D\|_{p\text{-var}}^{\frac{q}{p'}} \right)^{p'/p} \left[ E\left( d_{\infty}(M, M^D)^{q} \right) \right]^{1-p'/p}.$$  

But now it suffices to remark, using our BDG estimates, that

$$E\left( \|M\|_{p\text{-var};[0,T]}^{q} \right), E\left( \|M^D\|_{p\text{-var};[0,T]}^{q} \right) \leq CE\left( \|\langle M \rangle_T\|_{\infty;[0,T]}^{q/2} \right) \leq CE\left( \|M\|_{\infty;[0,T]}^{q} \right)$$

and the last term is finite by assumption. $\blacksquare$

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