A New Class of Automata Networks

Nino Boccara\textsuperscript{a,b}, Henryk Fukś\textsuperscript{a} and Servie Geurten\textsuperscript{a}

\textsuperscript{a}Department of Physics, University of Illinois, Chicago, IL 60607-7059, USA
\textsuperscript{b}DRECAM-SPEC, CE Saclay, 91191 Gif-sur-Yvette Cedex, France

Abstract

A new class of automata networks is defined. Their evolution rules are determined by a probability measure $p$ on the set of all integers $\mathbb{Z}$ and an indicator function $I_A$ on the interval $[0,1]$. It is shown that any cellular automaton rule can be represented by a (nonunique) rule formulated in terms of a pair $(p, I_A)$. This new class of automata networks contains discrete systems which are not cellular automata. For a given $p$, a metric can be defined on the space of all rules which induces a metric on the space of all cellular automata rules.

1 The Evolution Operator

Systems which consist of a large number of simple identical elements evolving in time according to simple rules often exhibit a complex behavior as a result of the cooperative effect of their components. Automata network are models of such systems. They consist\textsuperscript{[9]} of a graph with a discrete variable at each vertex which evolves in discrete time steps according to a definite rule involving the values of neighbouring vertex variables. The vertex variables may be updated sequentially or synchronously. More formally, automata networks may be defined as follows:

Let $G = (V, E)$ be a graph, where $V$ is a set of vertices and $E$ a set of edges. Each edge joins two vertices not necessarily distinct. An automata network, defined on $V$, is a triple $(G, Q, \{f_i| i \in V\})$, where $G$ is a graph on $V$, $Q$ a finite set of states and $f_i: Q^{|U_i|} \rightarrow Q$ a mapping, called the transition rule associated to vertex $i$. $U_i = \{j \in V|\{j, i\} \in E\}$ is the neighbourhood of $i$, i.e., the set of vertices connected to $i$, and $|U_i|$ denotes the number of vertices belonging to $U_i$. If $|U_i|$ is finite, the rule is local. For the automata networks considered in this paper, $|U_i|$ is infinite, however it will be shown that, if certain conditions are fulfilled, even if the number of vertices in the neighborhood of a vertex is infinite, the rule may be local. Cellular automata (CA) are particular automata...
networks in which the set of vertices $V$ is the set of all integers $\mathbb{Z}^d$, where $d$ is called the space dimensionality, and the rule is translationally invariant, local and applied synchronously.

Since the class of automata networks to be described is a generalization of CA, to fix the notations, we define CA as follows: Let $s : \mathbb{Z} \times \mathbb{N} \mapsto \{0, 1\}$ be a function that satisfies the equation

$$
(\forall i \in \mathbb{Z}) \ (\forall t \in \mathbb{N}) \ s(i, t + 1) = f(s(i - r, t),
\ s(i - r + 1, t), \ldots , s(i + r, t)),
$$

where $\mathbb{N}$ is the set of nonnegative integers and $\mathbb{Z}$ the set of all integers. Such a discrete dynamical system is a two-state one-dimensional CA. The mapping $f : \{0, 1\}^{2r+1} \rightarrow \{0, 1\}$ is the rule, and the positive integer $r$ is the radius of the rule. More general rules could be site- and/or time-dependent and/or involve different left and right radii. The function $S_t : i \mapsto s(i, t)$ is the state of the CA at time $t$. $S = \{0, 1\}^\mathbb{Z}$ is the state space. An element of the state space is also called a configuration. Since the state $S_{t+1}$ at time $t + 1$ is entirely determined by the state $S_t$ at time $t$ and the rule $f$, there exists a unique mapping $F_f : S \rightarrow S$ such that $S_{t+1} = F_f(S_t)$. $F_f$, which is the evolution operator, is also referred to as the CA rule.

CA have been widely used to model complex systems in which the local character of the rule plays an essential role [8,16,13,10,3]. Nevertheless, not all complex systems exhibit purely local interactions. Consider, for example, diffusion of innovations in a social system. For a given individual, deciding to buy, say, her first computer is based on the word of mouth from, more or less, closed neighbors as her family members, friends and colleagues as well as mass-media communications. It is clear that a CA rule cannot correctly model this type of system. The new class of automata networks which described in this paper might be useful to model such systems.

The evolution operator $F_{p,A}$ of our new class of automata networks is defined in terms of a probability measure $p$ on $\mathbb{Z}$ and an indicator function $I_A$ on $[0, 1]$. Then, we shall show that, for any CA rule $f$, we can find such an evolution operator, that is, a measure $p$ and an indicator $I_A$, that emulates the CA rule.

Let $S_t$ be the state of the system at time $t$, that is, $S_t : i \mapsto s(i, t)$, and put

$$
\sigma(i, t) = \sum_{n=-\infty}^{\infty} s(i + n, t)p(n),
$$

where $p$ is a given probability measure on $\mathbb{Z}$, that is, a nonnegative function
on the set of all integers such that
\[ \sum_{n=-\infty}^{\infty} p(n) = 1. \] (3)

For all \( i \in \mathbb{Z} \) and \( t \in \mathbb{N} \), \( \sigma(i, t) \in [0, 1] \). The state \( S_{t+1} \) of the system at time \( t + 1 \) is then determined by the function
\[ i \mapsto s(i, t + 1) = I_A(\sigma(i, t)) = I_A \left( \sum_{n=-\infty}^{\infty} s(i + n, t)p(n) \right), \] (4)
where \( I_A \) is a given indicator function on \([0, 1]\), that is, a function such that, for all \( x \in [0, 1] \),
\[ I_A(x) = \begin{cases} 1, & \text{if } x \in A \subset [0, 1], \\ 0, & \text{otherwise}. \end{cases} \] (5)

Since the state \( S_{t+1} \) at time \( t + 1 \) is entirely determined by the state \( S_t \) at time \( t \) and the measure \( p \) and the subset \( A \) of \([0, 1]\), there exists a unique mapping \( F_{p,A} : S \to S \) such that
\[ S_{t+1} = F_{p,A}(S_t). \]

\( F_{p,A} \) will be referred to as the evolution operator.

For a given probability measure \( p \) and a given indicator function \( I_A \), the operator \( F_{p,A} \) is entirely determined. That is, the word “probability measure” should not be misleading: the evolution operator \( F_{p,A} \) is deterministic. Probabilistic evolution operators could also be defined. If, for instance, we replace \( I_A \) by \( XI_A \), where \( X \) is a Bernoulli random variable taking values on \([0, 1]\), the resulting evolution operator would be probabilistic.

**Theorem 1** Let \( F_f \) be the evolution operator on \( S \) associated to a CA rule \( f \); then, there exists an evolution operator \( F_{p,A} \) such that, for any configuration \( x \in S \), \( F_{p,A}(x) = F_f(x) \).

To simplify the proof, we only consider translation-invariant symmetric CA rules, that is, rules \( f \) such that, for all \( i \in \mathbb{Z} \) and all \((2r + 1)\)-block \( B(i, r) = \{x(i-r), x(i-r+1), \ldots, x(i+r)\} \) \((r > 0)\),
\[ f(x(i-r), x(i-r+1), \ldots, x(i+r)) = f(x(i+r), x(i+r-1), \ldots, x(i-r)). \] (6)
The generalization to nonsymmetric rules is straightforward. Since \( f \) is symmetric, we shall verify that we may assume that \( p \) is even, that is, for all \( n \in \mathbb{Z} \), \( p(-n) = p(n) \). The set of configurations with prescribed values at a finite number of sites is called a cylinder set. To each \((2r+1)\)-block \( B(i, r) \) corresponds a cylinder set denoted \( C(i, r) \). Since the rule is translation-invariant, it is sufficient to consider \( i = 0 \).

For any configuration \( x : n \mapsto x(n) \) belonging to the cylinder set \( C(0, r) \), the set of all numbers \( \xi(C(0, r)) \) defined by

\[
\xi(C(0, r)) = \sum_{n=-\infty}^{\infty} x(n)p(n)
\]

belongs to the subinterval \([\xi_{\text{min}}(C(0, r)), \xi_{\text{max}}(C(0, r))]\) of \([0, 1]\) (called a \( C \)-interval in what follows) such that

\[
\xi_{\text{min}}(C(0, r)) = \sum_{n=-r}^{r} x(n)p(n) = x(0)p(0) + \sum_{n=1}^{r} (x(-n) + x(n))p(n)
\]

and

\[
\xi_{\text{max}}(C(0, r)) = \xi_{\text{min}}(C(0, r)) + 2 \sum_{n=r+1}^{\infty} p(n),
\]

where we have taken into account that \( p(-n) = p(n) \).

Since, for \( n \neq 0 \), \( \xi_{\text{min}}(C(0, r)) \) depends on \( x(-n) + x(n) \) and not on \( x(-n) \) and \( x(n) \) separately, there are only \( 2 \times 3^r \) different \( \xi_{\text{min}}(C(0, r)) \) (and as many \( C \)-intervals). It is, therefore, more convenient to label them \( \xi_{\text{min}}(\nu) \), where \( \nu \) is an integer in \( \{0, 1, \ldots, 2 \times 3^r - 1\} \) defined by

\[
\nu = x(0)3^r + \sum_{k=1}^{r} (x(-k) + x(k))3^{r-k}.
\]

The corresponding \( \xi_{\text{min}}(\nu) \) is then given by

\[
\xi_{\text{min}}(\nu) = x(0)p(0) + \sum_{k=1}^{r} (x(-k) + x(k))p(k).
\]

If we want to define \( 2^{2 \times 3^r} \) different operators \( F_{p,A} \) representing the \( 2^{2 \times 3^r} \) different symmetric CA rules \( f \) of radius \( r \), we have first to find a probability measure \( p \) such that the \( 2 \times 3^r \) \( C \)-intervals are disjoint. Then, according to the rule \( f \) to be represented, we will choose a subset \( A \) of \([0, 1]\) such that some of
these \( C \)-intervals are strictly included in \( A \) whereas the others have an empty intersection with \( A \). The only problem is, therefore, to find the conditions to be satisfied by \( p \).

The sequence \( \{\xi_{\min}(\nu) \mid \nu = 0, 1, \ldots, 2 \times 3^r - 1\} \) is totally ordered for increasing values of \( \nu \) if, and only if, the \( r + 1 \) conditions

\[
p(r - k) > 2p(r - k + 1) + 2p(r - k + 2) \cdots + 2p(r) \tag{12}
\]

\((k = 0, 1, \ldots, r)\)

are satisfied, with the convention: \( p(m) = 0 \) if \( m < 0 \).

It is easy to verify that, for \( \nu = 0, 1, \ldots, 2 \times 3^r - 1 \), the difference \( \xi_{\min}(\nu + 1) - \xi_{\min}(\nu) \) is always of the form

\[
p(r - k) - 2p(r - k + 1) - 2p(r - k + 2) \cdots - 2p(r) \tag{13}
\]

\((k = 0, 1, \ldots, r)\).

Therefore, if all these differences are greater than

\[
\xi_{\max}(C(0, r)) - \xi_{\min}(C(0, r)) = 2 \sum_{n=r+1}^{\infty} p(n),
\]

all the intervals

\[
[\xi_{\min}(\nu), \xi_{\min}(\nu) + 2 \sum_{n=r+1}^{\infty} p(n)]
\]

will be disjoint. For this to be the case, the \( r + 1 \) conditions

\[
p(r - k) > 2 \sum_{n=r-k+1}^{\infty} p(n) \quad (k = 0, 1, \ldots, r). \tag{14}
\]

should be satisfied. Since \( p \) satisfies (3), Conditions (14) are identical to Conditions (12).

2 Examples

Here are two examples of probability measures \( p \).
Example 1. If, for all $n \in \mathbb{Z}$,

$$p(n) = \tanh\left(\frac{1}{2}\lambda\right) \exp(-\lambda|n|),$$

where $\lambda > 0$, it is easy to verify that Conditions (12) are satisfied for all positive values of $r$ if $\lambda > \log 3$. That is, choosing a subset $A$ of $[0, 1]$ we can represent, in this case, any symmetric CA rule. For a given CA rule $p$ and $A$ are clearly not unique.

If we take $\lambda = 1.2$, the corresponding $C$-intervals are approximately

$$
[\xi_{\min}(0), \xi_{\max}(0)] = [0, 0.1395] \\
[\xi_{\min}(1), \xi_{\max}(1)] = [0.1618, 0.3012] \\
[\xi_{\min}(2), \xi_{\max}(2)] = [0.3235, 0.4630] \\
[\xi_{\min}(3), \xi_{\max}(3)] = [0.5370, 0.6765] \\
[\xi_{\min}(4), \xi_{\max}(4)] = [0.6988, 0.8382] \\
[\xi_{\min}(5), \xi_{\max}(5)] = [0.8606, 1].
$$

And to obtain Rule 18 [15] defined by

$$f(x_1, x_2, x_3) = \begin{cases} 1, & \text{if } (x_1, x_2, x_3) = (0, 0, 1) \text{ or } (1, 0, 0), \\ 0, & \text{otherwise}, \end{cases}
$$

we may choose $A = [0.15, 0.31]$.

Example 2. If, for all $n \in \mathbb{Z}$,

$$p(n) = \frac{(1 + |n|)^{-\alpha}}{2\zeta(\alpha) - 1},$$

where $\alpha > 1$ and $\zeta$ is the zeta function, then, for a radius equal to $r_0$, there exist a threshold value $\alpha_0$ such that Conditions (12) are satisfied for all $r \leq r_0$ if $\alpha \geq \alpha_0$ where $\alpha_0$ is the solution of the equation

$$(1 + r_0)^{-\alpha_0} - 2 \sum_{n=r_0+1}^{\infty} (1 + n)^{-\alpha_0} = 0.
$$

The table below give some approximate threshold values.

| $r_0$ | 0   | 1   | 2   | 3   | 10  | 20  | 100 |
|-------|-----|-----|-----|-----|-----|-----|-----|
| $\alpha_0$ | 2.18529 | 3.28994 | 4.39061 | 5.49023 | 13.1824 | 24.169 | 112.058 |
\( \alpha \) is approximately a linear function of \( r_0 \).

For \( \alpha = 3.5 \), the corresponding \( C \)-intervals are approximately

\[
\begin{align*}
[\xi_{\text{min}}(0), \xi_{\text{max}}(0)] &= [0, 0.0612] \\
[\xi_{\text{min}}(1), \xi_{\text{max}}(1)] &= [0.0705, 0.1317] \\
[\xi_{\text{min}}(2), \xi_{\text{max}}(2)] &= [0.1410, 0.2022] \\
[\xi_{\text{min}}(3), \xi_{\text{max}}(3)] &= [0.7978, 0.8590] \\
[\xi_{\text{min}}(4), \xi_{\text{max}}(4)] &= [0.8683, 0.9295] \\
[\xi_{\text{min}}(5), \xi_{\text{max}}(5)] &= [0.9388, 1].
\end{align*}
\]

And to obtain Rule 18, we may choose, in this case, \( A = [0.062, 0.133] \).

If some \( C \)-intervals are not disjoint but are either strictly included in \( A \) or have an empty intersection with \( A \), certain CA rules could be represented, but not all CA rules of a given radius. To represent nonsymmetric CA rules we should consider noneven functions \( p \).

An operator \( F_{p,A} \) will not, in general, represent a CA rule in the following two cases.

(i) If \( p \) is such that for a given \( r \) the \( C \)-intervals are disjoint but \( A \) has a nonempty intersection with some \( C \) intervals without, however, strictly including these intervals.

(ii) If \( p \) is such that for a given \( r \) the \( C \)-intervals are not disjoint.

In the first case, the spatio-temporal pattern will look like the pattern of a CA whose evolution is governed by a mixed rule. But, a mixed rule being local, the velocity of propagation of a local perturbation cannot be greater than \( r \), whereas in this case the value of the \( \sigma \)-variable defined by (2) depends on the whole configuration and not only on a finite block. Note that the nonlocal character of the evolution operator \( F_{p,A} \) comes essentially from the definition of the indicator function since if we consider a measure \( p \) whose moments are all defined (as in Example 1) the range of the interaction between the different sites is clearly finite whatever the precise meaning we give to this range. As a consequence of the nonlocal character of the rule in this case the velocity of propagation of a local perturbation may be time-dependent and grow to infinity.

In the second case, the spatio-temporal pattern may look like a CA pattern, but no finite radius can be defined. As an example, consider the exponential probability measure, and choose \( \lambda = 1 \). The \( C \)-intervals are not disjoint and the resulting evolution operator is not a CA rule. For two different initial configurations, the corresponding spatio-temporal patterns of such an operator are represented in Figures 1a and 1b. While Pattern 1a is reminiscent of a
class-2 CA, Pattern 1b looks more like class-3. Wolfram [15] classification is, therefore, not relevant for rules of this type (failure of Wolfram classification has already been noticed even for standard CA, as reported in [11] and [7]).

CA have been widely used to model systems, say in epidemiology [1,2,4,5] or in ecology [5], in which the local character of the interaction is an essential feature. But, there are cases in which the interactions are both local and nonlocal as for the diffusion of innovations [12,14] which can be influenced by two types of communication channels: mass media and interpersonal. The new class of discrete dynamical systems we have defined may be well adapted to model such a situation. If, for instance, we consider a probability measure $p$ such that $p(n)$ is proportional to

$$e^{-\lambda_1 |n|} + a \left( e^{-\lambda_2 |n-n_0|} + e^{-\lambda_2 |n+n_0|} \right),$$

where $\lambda_1$, $\lambda_2$ and $a$ are positive constants, and $n_0$ is a positive integer. $\lambda_1$ and $\lambda_2$ are inversely proportional to the range of the nearby and distant sources of information, $a$ measures the relative weight of the distant source compared to the nearby one and $n_0$ is the location of the distant source. A model of diffusion of innovations employing this idea is currently investigated and a detailed report will be published elsewhere.
3 Metric

For a given probability measure $p$, let $F$ be the probability distribution of the random variable $\Sigma : S \to [0, 1]$ defined by

$$\Sigma = \sum_{n = -\infty}^{\infty} X(n)p(n),$$  \hspace{1cm} (18)

where $\{X(n) \mid n \in \mathbb{Z}\}$ is a doubly infinite sequence of equally distributed Bernoulli random variables such that

$$\forall n \in \mathbb{Z} \quad P(X(n) = 0) = P(X(n) = 1) = \frac{1}{2},$$  \hspace{1cm} (19)

**Theorem 2** If $\lambda > \log(3)$, the random variable $\Sigma$ is singular.

That is, for $\lambda > \log 3$, the distribution function $F$ of $\Sigma$ is continuous but has no density. The derivative of $F$ is almost everywhere equal to zero, and $F$ increases on a Cantor-like set of zero Lebesgue measure (see Figures 2a and 2b). This implies that, if an evolution operator is selected at random by selecting a subset $A$ of $[0, 1]$ at random, then, with probability one, the corresponding rule will be a CA rule.

As for the usual triadic Cantor set, the set on which $\Sigma$ is defined can be constructed by removing step by step “forbidden intervals.”

1st step

$$\sigma \notin \left[2 \sum_{n=1}^{\infty} p(n), p(0) \right]$$

2nd step

$$\sigma \notin \left[2 \sum_{n=2}^{\infty} p(n), p(1) \right]$$

\hspace{1cm} \bigcup \left[p(1) + 2 \sum_{n=2}^{\infty} p(n), 2p(1) \right]$$

\hspace{1cm} \bigcup \left[p(0) + 2 \sum_{n=2}^{\infty} p(n), p(0) + p(1) \right]$$

\hspace{1cm} \bigcup \left[p(0) + p(1) + 2 \sum_{n=2}^{\infty} p(n), p(0) + 2p(1) \right]$$

etc. For a given radius $r$, the sum $m(r)$ of the Lebesgue measures of the removed intervals is
Fig. 2. Distribution function for a) $\lambda = 1.3$ and b) $\lambda = 2$.

$$m(r) = 2 \sum_{\nu=1}^{3^r-1} \left( \xi_{\min}(\nu) - \xi_{\max}(\nu - 1) \right)$$

$$= (p(0) + 2p(1) + \cdots + 2p(r))$$

$$+ (2 \times 3^r - 1) 2 \sum_{k=r+1}^{\infty} p(k).$$
Since

\[ \lim_{r \to \infty} (2 \times 3^r - 1) 2 \sum_{k=r+1}^{\infty} p(k) = 0. \] (21)

it follows that

\[ \lim_{r \to \infty} m(r) = 1. \] (22)

This result could be made more intuitive in the following way. If we put \( \lambda = \log b \),

\[ \Sigma = \frac{b - 1}{b + 1} \left( X(0) + \sum_{n=1}^{\infty} \frac{X(-n) + X(n)}{b^n} \right). \] (23)

Since, for all positive integers \( n \), the sum of the Bernoulli random variables \( X(-n) \) and \( X(n) \) is either equal to 0, 1 or 2, the summation over \( n \) in (23) represents a random variable whose values are expressed in base \( b \) (\( b \) is not necessarily an integer). If \( b > 3 \), the set of all possible values of \( \Sigma \) is, therefore, a Cantor-like set.

For \( b = 3 \), the random variable \( \Sigma \) is given by

\[ \Sigma = \frac{1}{2} \left( X(0) + \sum_{n=1}^{\infty} \frac{X(-n) + X(n)}{3^n} \right). \] (24)

\( X(-n) + X(n) \) being a binomial random variable such that

\[ P(X(-n) + X(n) = 0) = \frac{1}{4} \]
\[ P(X(-n) + X(n) = 1) = \frac{1}{2} \]
\[ P(X(-n) + X(n) = 2) = \frac{1}{4} \]

\( \Sigma \) is absolutely continuous but not uniformly distributed on \([0,1]\) (see Figure 3), and the set of all possible values of \( \Sigma \) has a nontrivial Hausdorff dimension \( d_H \), which, in this very particular case, can be exactly determined (see, for example, [6]). It is found that

\[ d_H = \frac{3 \log 2}{2 \log 3} = 0.946395 \ldots \] (25)
For a fixed probability measure $p$ we define the distance between two evolution operators $F_{p,A_1}$ and $F_{p,A_2}$ by

$$d(F_{p,A_1}, F_{p,A_2}) = \int_0^1 |I_{A_1}(\sigma) - I_{A_2}(\sigma)| \, dF(\sigma) = \int_{A_1 \Delta A_2} \, dF(\sigma), \quad (26)$$

where $A_1 \Delta A_2$ denotes the symmetrical difference between $A_1$ and $A_2$, that is, the set of all numbers $\sigma \in [0,1]$ that belong to $A_1$ XOR $A_2$.

Note that $d(F_{p,A_1}, F_{p,A_1}) = 0$ does not imply $A_1 = A_2$. This means that two evolution operators may be different but the corresponding dynamics will be identical. Therefore, we could, instead of considering evolution operators we could consider classes of evolution operators. Two evolution operators would belong to the same class if their distance is equal to zero. This notion of distance may be useful to study sequences of, say, cellular automaton rules and study if such sequences converge in the topology defined by this metric. This topic will be developed elsewhere.

**Theorem 3** If $\lambda > \log(3)$, the distance between two CA rules does not depend upon $\lambda$.

This result follows from the fact that the subset $A$, characterizing a CA rule, is a reunion of disjoint intervals whose end points belong to removed intervals. It can even be proved that the distance will not depend upon the form of $p$ provided that $p$ is such that Conditions (12) are satisfied for all values of $r$ less or equal than the larger radius of the rules under consideration. For example,
the distance between Rule 18 (defined above) and Rule 22 defined by

\[
f(x_1, x_2, x_3) = \begin{cases} 
1, & \text{if } (x_1, x_2, x_3) = (0, 0, 1) \text{ or } (1, 0, 0) \text{ or } (0, 1, 0), \\
0, & \text{otherwise}, 
\end{cases}
\]

is

\[
d(F_{18}, F_{22}) = \int_{\{0,1,0\}} dF(\sigma) = \frac{1}{8}.
\]  

(27)

It is equal to the probability of selecting \{0, 1, 0\} among the 8 equally probable blocks of radius one.

4 Conclusion

We have described a new class of automata networks whose evolution operators are defined in terms of a probability measure \(p\) and an indicator function \(I_A\), where \(A\) is a subset of the interval \([0, 1]\). This class contains all cellular automaton rules, and other rules which have no finite radius. For a given \(p\), we have defined a metric on the space of these new rules which induces a metric on the space of all CA rules that does not depend upon the particular \(p\). In the case of evolution operators which are not CA rules, the Wolfram classification into 4 classes does not seem to be relevant since we may obtain spatio-temporal patterns reminiscent of two different classes by changing the initial configuration. The parameters characterizing the evolution operators of these new automata networks can be modified continuously, and, as a consequence, starting from a given initial configuration, we can, say, pass continuously from a class-3 to a class-2 CA. Finally these new rules could be useful to model systems in which the interactions between the different elements are short- and long-range, like in the diffusion of innovations.

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