Derivation of a generalized Schrödinger equation for dark matter halos from the theory of scale relativity

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Using Nottale’s theory of scale relativity, we derive a generalized Schrödinger equation applying to dark matter halos. This equation involves a logarithmic nonlinearity associated with an effective temperature and a source of dissipation. Fundamentally, this wave equation arises from the nondifferentiability of the trajectories of the dark matter particles whose origin may be due to ordinary quantum mechanics, classical ergodic (or almost ergodic) chaos, or to the fractal nature of spacetime at the cosmic scale. The generalized Schrödinger equation involves a coefficient $D$, possibly different from $\hbar/2m$ (where $\hbar$ is the Planck constant and $m$ the mass of the particles), whose value for dark matter halos is $D = 1.02 \times 10^{23} \text{ m}^2/\text{s}$. This model is similar to the Bose-Einstein condensate dark matter model except that it does not require the dark matter particle to be ultralight. It can accommodate any type of particles provided that they have nondifferential trajectories. We suggest that the cold dark matter crisis may be solved by the fractal (nondifferential) structure of spacetime at the cosmic scale, or by the chaotic motion of the particles on a very long timescale, instead of ordinary quantum mechanics. The equilibrium states of the generalized Schrödinger equation correspond to configurations with a core-halo structure. The quantumlike potential generates a solitonic core that solves the cusp problem of the classical cold dark matter model. The logarithmic nonlinearity accounts for the presence of an isothermal halo that leads to flat rotation curves (it also accounts for the isothermal core of large dark matter halos). The damping term ensures that the system relaxes towards an equilibrium state. This property is guaranteed by an $H$-theorem satisfied by a Boltzmann-like free energy functional. In our approach, the temperature and the friction arise from a single formalism. They correspond to the real and imaginary parts of the complex friction coefficient present in the scale covariant equation of dynamics that is at the basis of Nottale’s theory of scale relativity. They may be the manifestation of a cosmic aether or the consequence of a process of violent relaxation and gravitational cooling on a coarse-grained scale.

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I. INTRODUCTION

The nature of dark matter (DM) is still unknown and remains one of the most important open problems of modern cosmology. The existence of DM has been predicted by Zwicky in 1933 to account for the missing mass of the galaxies in the Coma cluster inferred from the virial theorem [1]. The robust indication of DM came later from the measurement of the rotation curves of spiral galaxies [2–5] that revealed that they were flat instead of declining with the distance like in the case of planetary systems that are dominated by a central mass (Kepler’s law). The existence of DM has been confirmed by independent observations of gravitational lensing [3], hot gas in clusters [7], and the anisotropies of the cosmic microwave background (CMB) [8]. Very recently, astronomers reported that Dragonfly 44, an ultra diffuse galaxy with the mass of the Milky Way but with no discernable stars may be made almost entirely of DM [9].

Observation of the large-scale structure (LSS) of the Universe and the CMB are consistent with the cold dark matter (CDM) model in which DM is modeled as a pressureless gas described by the Euler-Poisson equations or as a collisionless system described by the Vlasov-Poisson equations. The most studied candidate particles for DM are WIMPs (weakly interacting massive particles) with a mass in the GeV-TeV range. These particles are the lightest supersymmetric partners predicted by models of supersymmetry (SUSY) [10]. The CDM model works remarkably well at large (cosmological) scales and is consistent with ever improving measurements of the CMB from WMAP and Planck missions [11, 12]. It is able to account for the formation of structures, with the small objects forming first and merging over time to form larger objects (hierarchical clustering). This leads to a “cosmic web” made of virialized halos connected by filaments delimiting empty voids, in very good agreement with observations. However, the CDM model experiences serious difficulties at small (galactic) scales. In particular, being pressureless, numerical simulations of CDM lead to cuspy density profiles [13], with a density diverging as $r^{-1}$ for $r \to 0$, while observations favor cored density profiles with a finite density at the center [14]. This is the “cusp-core” problem [15]. Other related problems are known as the “missing satellites” problem [16, 17] and the “too big to fail” [20] problem. The expression “small-scale crisis of CDM” has been coined.1

1 Some researchers remain unconvinced that there is a real problem at the center of the galaxies. For example, the cusp-core problem could be an effect of asphericity and misalignment of the halos. We refer to [21] for a detailed discussion of this issue and additional references.
One possibility to solve the CDM crisis is to invoke the feedback of baryons that can transform cusps into cores [22,24]. Another possibility is to consider warm dark matter (WDM) where the dispersion of the particles is responsible for a pressure force that can halt gravitational collapse and prevent the formation of cusps [25]. Finally, an interesting suggestion is to invoke quantum mechanics. Indeed, DM could be made of elementary particles that have not been detected yet and whose quantum nature may solve the small-scale problems of the CDM model.

For example, the DM particle could be a fermion, like a sterile neutrino with a mass in the keV range, satisfying the Fermi-Dirac statistics (a sterile neutrino is also the most plausible candidate for WDM). In that case, gravitational collapse leading to cuspy halos is prevented by the quantum pressure arising from the Pauli exclusion principle, or by the thermal pressure. The resulting configurations have a core-halo structure with a core made of a “fermion ball” at $T = 0$ and an isothermal halo with a density profile decaying as $\rho \sim r^{-2}$ at large distances. An isothermal, or almost isothermal, halo, leads to flat rotation curves. This core-halo structure with a degenerate core and an isothermal halo may also be justified by the process of collisionless violent relaxation (for classical or quantum particles) leading to the Lynden-Bell statistics [20] that is similar to the Fermi-Dirac statistics. This violent collisionless relaxation, establishing an out-of-equilibrium Fermi-Dirac-like distribution on a very short timescale, may be more relevant than a collisional relaxation establishing a thermal Fermi-Dirac distribution on a much longer timescale, possibly larger than the age of the Universe [27,29]. This fermionic scenario, where the Fermi-Dirac distribution arises either from quantum mechanics or from violent collisionless relaxation, has been studied by several authors [27,60].

Alternatively, the DM particle could be a boson like an axion. At low temperatures, bosons are expected to form Bose-Einstein condensates (BECs). In that case, DM is described by a scalar field (SF) that can be identified with the wave function $\psi$ of the condensate. The evolution of this wave function is governed by the Schrödinger-Poisson equations. In the BEC scenario, gravitational collapse is prevented by the repulsive quantum potential accounting for the Heisenberg uncertainty principle. The resulting structure has a core-halo structure. Quantum mechanics is important in the core which is similar to a soliton (a steady state of the Schrödinger-Poisson equations). On the other hand, the halo (quanta-terferences) behaves essentially as CDM and has a density profile close to the isothermal profile ($\rho \sim r^{-2}$) or close to the NFW and Burkert profiles ($\rho \sim r^{-3}$). This core-halo structure may result from a process of gravitational cooling [61] that is similar to the process of violent relaxation [26]. Gravitational cooling may be at work during hierarchical clustering where DM halos merge to form bigger halos. This bosonic model has received different names such as wave DM, fuzzy DM (FDM), BECDM, $\psi$DM, or SFDM [23,147] (see the introduction of [92] for a short historic of this model). The relevance of this model has been demonstrated by the simulations of Schive et al. [110,111]. They showed that the BECDM model behaves as CDM at large (cosmological) scales but that differences appear at small (galactic) scales where the wave nature of the particles manifests itself. This may solve the CDM crisis. For quantum mechanics to be important at the scale of DM halos, the mass of the boson must be extremely small, of the order of $10^{-22} \text{eV/c}^2$ (this is the condition required for the de Broglie wavelength of the boson to be of the order of the size of DM halos). The standard QCD axion with a mass $m = 10^{-4} \text{eV/c}^2$ essentially behaves as CDM and does not solve the CDM crisis. However, ultralight axions with a mass up to $m \sim \hbar/\text{H}_0/c^2 \sim 10^{-35} \text{eV/c}^2$ (where $\text{H}_0$ is the Hubble constant) arise in string theory and are not excluded by particle physics. This point has been recently emphasized by Hui et al. [137] who stressed the viability of the FDM model. Nevertheless, the existence of ultralight bosons remains an hypothesis that is not confirmed yet. It may therefore be interesting to develop alternative models of DM that do not require such small particle masses while exhibiting features similar to the FDM model.

In this paper, we approach the problem of DM from the viewpoint of Nottale’s theory of scale relativity [148] relying on a fractal spacetime. Nottale has shown, quite generally, that when a particle has a nondifferentiable trajectory, its evolution is described by a Schrödinger-like equation. This equation involves a coefficient $D$ with the dimension of a diffusion coefficient, sometimes called the fractal fluctuation parameter, whose value depends upon the system under consideration (i.e. the origin of the nondifferentiability).

Nottale first argued that the fractal structure of spacetime manifests itself at the “microscale” which is the realm of ordinary quantum mechanics. He used his formalism to derive the Schrödinger equation for a quantum particle from Newton’s fundamental equation of dynamics by invoking a principle of scale covariance. In order to reproduce the results of quantum mechanics, the coefficient $D$ which appears in his Schrödinger-like equation must be equal to $D = \hbar/2m$, where $\hbar$ is the Planck constant and $m$ is the mass of the particle.

Then, Nottale proposed that the fractal structure of

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2 This $\rho \sim r^{-2}$ profile is similar to the numerical Navarro-Frenk-White (NFW) [13] profile and to the observational Burkert [14] profile both decaying as $\rho \sim r^{-3}$ at large distances. The difference in slope between the isothermal profile ($\rho \sim r^{-2}$) and the NFW and Burkert profiles ($\rho \sim r^{-3}$) may be due to nonideal effects: incomplete relaxation, tidal effects, stochastic forcing...

3 The theory of Nottale is related to, but distinct from, Nelson’s stochastic interpretation of quantum mechanics [129].
spacetime also manifests itself at the “macroscale” which is the realm of astrophysics and cosmology. He writes: “In this new approach, space becomes not only curved but also fractal beyond some characteristic scale relative to the system under consideration. The induced effects on motion (in standard space) of the internal fractal structures of geodesics (in scale space), are to transform classical mechanics into a quantum-like mechanics, i.e. Newton’s fundamental equation of dynamics into a Schrödinger-like equation.” At the macroscale, the non-differentiability of the trajectories of the particles could arise either from the chaoticity of the motion of the classical particles on a very long timescale that is larger than their predictability horizon (ergodic or almost ergodic chaos), or from the intrinsic fractal structure of spacetime itself above a certain astrophysical scale. In other words, spacetime can become fractal beyond some temporal and/or space transition. This leads to a new quantum mechanics operating now at the cosmic scale.

Nottale tried to find evidence of his theory in some astrophysical observations. He first applied his theory to the solar system based on the fact that the planets have a chaotic dynamics on a long timescale. In his approach, the solar system may be viewed as a gigantic atom described by a Schrödinger-like equation with an attractive 1/r potential. This leads to a quantization of the solar system similar to the quantization of the hydrogen atom. The difference with ordinary quantum mechanics is that the coefficient D that appears in the Schrödinger-like equation of Nottale has a value different from $\hbar/2m$. From this Schrödinger-like equation, he could obtain a quantization of the semi-major axes and eccentricities of planetary orbits. The predicted law of distance is not a Titius-Bode-like power law ($a_n = a + bc^n$) but a more constrained and statistically significant quadratic law of the form $a_n = a_0n^2$ (in these expressions $a_n$ is the semi-major axis and $n > 0$ is an integer quantumlike number). Interestingly, this law gives a much better fit to the planetary distribution than the empirical Titius-Bode law.

Nottale applied his formalism to other astrophysical objects such as extra-solar planetary systems, star-forming regions, binary stars, high-velocity clouds, planetary nebulae, galactic centers, galaxies, our Local Group of galaxies, clusters of galaxies, and the large scale structures of the Universe. In all these examples, gravity acts as an external potential that is not affected by the structures that it contributes to form.

At last, Nottale proposed to apply his formalism to DM halos with, again, the argument that the DM particles have nondifferentiable trajectories due to chaos or due to the fractal structure of spacetime at astrophysical scales. In the case of DM halos, the gravitational potential is produced by the system itself in a self-consistent manner. As a result, Nottale obtained a Schrödinger-Poisson-like equation similar to the Schrödinger-Poisson equation that arises in the BEC/SF model of DM. Again, the main difference is that the coefficient D in Nottale’s theory may be different from $\hbar/2m$.

This suggestion is very interesting because it could give a novel interpretation to the Schrödinger-Poisson equation applying to DM halos, different from its interpretation stemming from ordinary quantum mechanics. However, some arguments given by Nottale are incorrect. For example, Nottale argues that there is no need of DM to explain the flat rotation curves of spiral galaxies. He writes: “The quantumlike potential $V_Q$ is at the origin of the various dynamical and lensing effects usually attributed to unseen additional masses.”

Based on the results obtained by solving the Schrödinger-Poisson equation in the context of the BEC/SF model of DM, we know that the effect of the quantum potential is different from what is suggested by Nottale. Still, it plays a fundamental role in the physics of DM halos since it can solve the cusp problem. This leads to the first potentially important result of this paper. We propose that the cusp problem may be solved by the quantumlike potential arising in the theory of Nottale from the nondifferentiability of the trajectories of the DM particles. This nondifferentiability may be due to (i) the quantum nature of the particles if they are ultralight (as in the usual interpretation of the FDM model), (ii) their chaotic (fractal) dynamics that manifests itself on a very long timescale, or (iii) the intrinsic fractal structure of spacetime above a certain scale. If suggestions (ii) and/or (iii) are correct, that would mean that the cored density profile of DM halos is a manifestation of the fractal nature of spacetime at astrophysical scales. That would lead to a revolution of the concepts of space and time since one would have to take into account the fractal nature of spacetime in the theories of physics when dealing with the large-scale structures of the Universe or considering very long timescales of evolution.

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4 Historically, the idea to describe the solar system by a Schrödinger equation with a Newtonian gravitational potential dates back to Jehle [150] in 1938.

5 He also applied his approach to the morphogenesis of planetary nebulae (and even flowers!), obtaining particular solutions of the Schrödinger equation that spectacularly resemble to these objects. It is however difficult to say whether this impressive resemblance is just a coincidence or if this agreement is deeper than apparent at first sight.

6 Nottale did not point out this analogy probably because he did not know the literature on this subject.

7 If the nondifferentiability of spacetime and the Schrödinger-like equation are due to an effect of chaos in classical mechanics, we may wonder why cores are not formed in N-body simulations. A possibility is that the simulations (being necessarily based on approximations) are not fully reliable over long times to account for subtle effects of chaos. Inversely, assuming that the N-body simulations are fully reliable would imply that possibility (ii) has to be rejected. This would leave us, in the framework of our approach, with possibility (i) involving ordinary quantum mechanics or with possibility (iii) involving interesting new physics.
That would also have important implications for the nature of the DM particle. Indeed, if the cored density profile of DM halos is due to the quantization of the Universe at the macroscale instead of being due to ordinary quantum mechanics, the observation of cored density profiles should be independent of the mass of the DM particle in agreement with the equivalence principle. Indeed, observations of DM halos only determine the coefficient $D$ in the Schrödinger-like equation (see below), and this value may be different from $\hbar/2m$. That would enlarge the possibility of particles composing DM. Ultralight axions with a mass of the order of $10^{-22}$ eV/c$^2$ are not required anymore. Cores, instead of cusps, could be obtained with more massive bosons such as the QCD axion or even with classical particles such as WIMPs provided that the quantization of the DM halos (quantum potential) comes from the intrinsic fractal nature of spacetime or from the chaoticity of the trajectories of the particles.

In this paper, we complement Nottale’s theory of scale relativity by considering a more general situation where the particles are submitted to dissipative effects in addition to Newton’s law. The origin of this dissipation may be due to (i) the interaction of the system with an external environment (a real one or an hypothetical aether), (ii) the complex evolution of the system itself (gravitational cooling or violent relaxation) that leads to an effective dissipation on the coarse-grained scale, or (iii) the intrinsic nature of the fractal spacetime. We derive from the theory of scale relativity a generalized Schrödinger equation that involves a logarithmic nonlinearity associated with an effective temperature and a source of dissipation. In our approach, the temperature and the friction arise from a single formalism. They correspond to the real and imaginary parts of the complex friction coefficient present in the scale covariant equation of dynamics that is at the basis of Nottale’s theory of scale relativity. When applied to DM halos, our generalized Schrödinger equation has interesting properties. Its equilibrium states have a core-halo structure. The quantum-like potential accounts for the soliton-core of DM halos solving the cusp problem. The logarithmic nonlinearity accounts for their isothermal halo leading to flat rotation curves (it also accounts for the isothermal core of large DM halos). The damping term ensures that the system relaxes towards this equilibrium state. This is guaranteed by an $H$-theorem satisfied by a Boltzmann-like free energy functional. We use observations of DM halos to determine the coefficient $D$ arising in the Schrödinger-like equation and find $D = 1.02 \times 10^{23}$ m$^2$/s.

II. DERIVATION OF A GENERALIZED SCHRODINGER EQUATION

A. Basic tools of scale relativity

When a particle has a nondifferentiable trajectory $\mathbf{r}(t, dt)$, the derivative $d\mathbf{r}/dt$ is not defined and one has to introduce two velocities $\mathbf{u}_+(\mathbf{r}(t), t)$ and $\mathbf{u}_-(\mathbf{r}(t), t)$ defined from $t - dt$ to $t$ for $\mathbf{u}_+$ and from $t$ to $t + dt$ for $\mathbf{u}_-$. The two-valuedness of the velocity vector is due to the irreversibility in the reflection $dt \leftrightarrow -dt$ (time symmetry breaking). The elementary displacement $d\mathbf{r}_\pm$ for both processes is the sum of a differential part $d\mathbf{r}_\pm = \mathbf{u}_\pm dt$ and a non-differentiable part which is a scale-dependent fractal fluctuation $db_\pm$. This fractal fluctuation is described by a stochastic variable which, by definition, is of zero mean $\langle db_\pm \rangle = 0$. We shall assume that spacetime has a fractal dimension $D_F = 2$ as in ordinary quantum mechanics \[148\]. More general models could be constructed if the fractal dimension is different from $D_F = 2$ \[148\] but, in this paper, we restrict ourselves to the simplest case. Therefore, we write

$$d\mathbf{r}_\pm = \mathbf{u}_\pm dt + db_\pm,$$  \hspace{1cm} (1)

with

$$\langle db_\pm \rangle = 0, \quad \langle db_\pm db_{\pm j} \rangle = \pm 2D\delta_{ij}dt,$$  \hspace{1cm} (2)

where the indices $i, j$ denote the coordinates of space and $D$ is a a sort of “diffusion coefficient” measuring the covariance of the noise.\footnote{In Eq. (2), we consider that $dt > 0$ for the $(+)$ process and $dt < 0$ for the $(-)$ process so that $\pm 2D\delta_{ij}dt$ is always positive.} In other words, $D$ characterizes the amplitude of the fractal fluctuations. Following Nottale, we introduce two classical derivative operators $d_+/dt$ and $d_-/dt$ which yield the twin classical velocities when they are applied to the position vector $\mathbf{r}$, namely

$$\frac{d_+ \mathbf{r}}{dt} = \mathbf{u}_+ , \quad \frac{d_- \mathbf{r}}{dt} = \mathbf{u}_-.$$  \hspace{1cm} (3)

It proves convenient in the formalism to replace the twin velocities $(\mathbf{u}_+, \mathbf{u}_-)$ by the couple $(\mathbf{u}, \mathbf{u}_Q)$ where

$$\mathbf{u} = \frac{\mathbf{u}_+ + \mathbf{u}_-}{2}, \quad \mathbf{u}_Q = \frac{\mathbf{u}_+ - \mathbf{u}_-}{2}.$$  \hspace{1cm} (4)

With these two velocities, we can form a complex velocity

$$\mathbf{U} = \mathbf{u} - i\mathbf{u}_Q.$$  \hspace{1cm} (5)

The real part $\mathbf{u}$ can be identified with the classical velocity and the imaginary part $\mathbf{u}_Q$ is a manifestation of the nondifferentiability of spacetime. It will be called the quantum velocity. For a differentiable trajectory $\mathbf{u}_+ = \mathbf{u}_- = \mathbf{u}$ and $\mathbf{u}_Q = \mathbf{0}$. As we shall see, the complex velocity $\mathbf{U}$ leads to the Schrödinger equation. Therefore, the origin of complex numbers in the Schrödinger equation (the wave function $\psi$ and the complex number $i$) can be intrinsically attributed to the two-valuedness character of the velocity \[148\].

Following Nottale, we define a complex derivative operator from the classical (differential) parts as

$$\frac{D}{Dt} = \frac{d_+ + d_-}{2dt} - i \frac{d_+ - d_-}{2dt},$$  \hspace{1cm} (6)
in terms of which
\[ \frac{D \mathbf{r}}{Dt} = \mathbf{U}. \] (7)

The total derivative with respect to time of a function \( f(\mathbf{r}(t), t) \) of fractal dimension \( D_F = 2 \) writes
\[ \frac{df}{dt} = \frac{\partial f}{\partial t} + \nabla f \cdot \frac{d\mathbf{r}}{dt} + \frac{1}{2} \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j} \frac{dx_i}{dt} \frac{dx_j}{dt}. \] (8)

Using Eq. (2), we find that the classical (differentiable) part of this expression is
\[ \frac{d f}{dt} = \frac{\partial f}{\partial t} + \mathbf{u} \cdot \nabla f \pm D \Delta f. \] (9)

Substituting Eq. (9) into Eq. (6), we obtain the expression of the complex time derivative operator \( 148 \):
\[ \frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{U} \cdot \nabla - iD \Delta. \] (10)

### B. Application to self-gravitating systems

We now apply this formalism to a system of \( N \) nonrelativistic particles of mass \( m \) in gravitational interaction. If their trajectories are differentiable, each particle has an equation of motion given by Newton’s law
\[ \frac{d \mathbf{u}}{dt} = -\nabla \Phi, \] (11)
where \( \mathbf{F} = -\nabla \Phi \) is the gravitational force by unit of mass exerted on the particle. We note that the mass \( m \) of the particles does not appear in this equation in virtue of the equivalence principle. If we make a mean-field approximation, valid for \( N \rightarrow +\infty \) with \( m \sim 1/N \), the gravitational potential \( \Phi(\mathbf{r}, t) \) can be identified with the self-consistent potential produced by the system as whole \( \Phi(\mathbf{r}, t) = -G \int f(\mathbf{r}', t) / |\mathbf{r} - \mathbf{r}'| \, d\mathbf{r}' \) through the Poisson equation
\[ \Delta \Phi = 4\pi G \rho, \] (12)
where \( \rho(\mathbf{r}, t) \) denotes the density of particles. In that case, the evolution of the distribution function \( f(\mathbf{r}, \mathbf{v}, t) \) in phase space is governed by the Vlasov equation which describes the collisionless evolution of the system \( 152 \). If we take finite-\( N \) effects into account (gravitational encounters), we obtain at the order \( 1/N \) the inhomogeneous Lenard-Balescu equation \( 153, 154 \) which describes the collisional evolution of the system on a secular timescale. This equation can be derived rigorously from the \( N \)-body equations of motion in a systematic expansion in powers of \( 1/N \).

In this paper, we make a mean field approximation \( (N \rightarrow +\infty) \) but we consider the possibility that the trajectories of the particles are nondifferentiable for one of the reasons given in the Introduction (ordinary quantum mechanics, chaos, fractal nature of spacetime...). We use the fundamental postulate of Nottale’s theory of scale relativity according to which the equations of quantum mechanics (nondifferentiable trajectories) can be obtained from the equations of classical mechanics (differentiable trajectories) by replacing the standard velocity \( \mathbf{u} \) by the complex velocity \( \mathbf{U} \) and the standard time derivative \( d/dt \) by the complex time derivative \( D/Dt \). In other words, \( D/Dt \) plays the role of a “covariant derivative operator” in terms of which the fundamental equations of physics keep the same form in the classical (differentiable) and quantum (nondifferentiable) regimes.\(^9\) This is similar to the principle of covariance in Einstein’s theory of relativity according to which the form of the equations of physics should be conserved under all transformations of coordinates.

### C. Complex friction force

In order to be general, we assume that the particles are submitted to a friction force in addition to self-gravity. Introducing a source of dissipation in the fundamental equation of dynamics is the next level of complexity after the pure Newton law \( 11 \). The naive idea is to introduce a linear complex friction force of the form \( -\gamma \mathbf{U} \), where \( \gamma \) is a complex friction coefficient. However, it proves necessary to consider only the real part of the friction force in order to obtain a generalized Schrödinger equation that conserves the normalization condition locally (see Appendix F of \( 155 \)). Therefore, we write the scale covariant equation of dynamics under the form
\[ \frac{D \mathbf{U}}{Dt} = -\nabla \Phi - \text{Re}(\gamma \mathbf{U}). \] (13)

Using the expression (10) of the complex time derivative operator, the foregoing equation can be rewritten as
\[ \frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U} \cdot \nabla) \mathbf{U} = iD \Delta \mathbf{U} - \nabla \Phi - \text{Re}(\gamma \mathbf{U}). \] (14)

This equation is similar to the damped viscous Burgers equation of fluid mechanics, except that in the present case the velocity field \( \mathbf{U}(\mathbf{r}, t) \) is complex and the viscosity \( \nu = iD \) is imaginary.

We now assume that the flow is potential so that the complex velocity can be written as the gradient of a function, \( \mathbf{U} = \nabla \Sigma \), where \( \Sigma \) is a complex potential or a complex action.\(^{10}\) As a consequence, the flow is irrotational: \( \nabla \times \mathbf{U} = \mathbf{0} \). Using the well-known identities of fluid mechanics \( (\mathbf{U} \cdot \nabla) \mathbf{U} = \nabla(\mathbf{U}^2/2) - \mathbf{U} \times (\nabla \times \mathbf{U}) \)

\(^9\) In the present context, the term “quantum” should be taken in a very general sense, valid either at the microscale (ordinary quantum mechanics) or at the macroscale (new quantum mechanics).

\(^{10}\) See \( 148 \) for a justification of this assumption from Lagrangian mechanics.
and \( \Delta U = \nabla (\nabla \cdot U) - \nabla \times (\nabla \times U) \) which reduce to \( (U \cdot \nabla) U = \nabla (U^2/2) \) and \( \Delta U = \nabla (\nabla \cdot U) \) for an irrotational flow, and using the identity \( \nabla \cdot U = \Delta \Sigma \), we find that Eq. (14) is equivalent to
\[
\frac{\partial \Sigma}{\partial t} + \frac{(\nabla \Sigma)^2}{2} - iD\Delta \Sigma + \Phi + V(t) + \text{Re}(\gamma \Sigma) = 0, \tag{15}
\]
where \( V(t) \) is a “constant” of integration depending on time. Equation (15) can be viewed as a quantum Hamilton-Jacobi equation for a complex action, or as a Bernoulli equation for a complex potential.

We define the wave function \( \psi(r, t) \) through the complex Cole-Hopf transformation
\[
\Sigma = -2iD \ln \psi. \tag{16}
\]
Written under the form
\[
\psi = e^{i\Sigma/2D}, \tag{17}
\]
this relation is similar to the WKB transformation in quantum mechanics. Substituting Eq. (16) into Eq. (15), and using the identity
\[
\Delta (\ln \psi) = \frac{\Delta \psi}{\psi} - \frac{1}{\psi^2} (\nabla \psi)^2, \tag{18}
\]
we obtain the generalized Schrödinger equation
\[
iD \frac{\partial \psi}{\partial t} = -D^2 \Delta \psi + \frac{1}{2} \Phi \psi + \frac{1}{2} V(t) \psi + D \text{Im}(\gamma \ln \psi) \psi. \tag{19}
\]
Therefore, by performing the complex Cole-Hopf transformation (16), we find that the complex (damped) viscous Burgers equation (14) is formally equivalent to the (generalized) Schrödinger equation (19) in the same sense that, by performing the usual Cole-Hopf transformation, the viscous Burgers equation is equivalent to the diffusion equation in ordinary hydrodynamics. As a result, quantum mechanics may be interpreted as a generalized hydrodynamics involving a complex velocity field and an imaginary viscosity. This interpretation takes a clear meaning in the context of Nottale’s theory of scale relativity.

As will be demonstrated below, the density \( \rho \) is proportional to \( |\psi|^2 \). For commodity, we choose to normalize the wave function such that \( \int |\psi|^2 \, dr = M \), where \( M \) is the total mass of the system. This implies that
\[
\rho = |\psi|^2. \tag{20}
\]
As a result, the generalized Schrödinger equation (19) must be coupled self-consistently to the Poisson equation
\[
\Delta \Phi = 4\pi G|\psi|^2. \tag{21}
\]
Dividing Eq. (19) by \( \psi \), taking the Laplacian, and using the Poisson equation (21), we can eliminate the gravitational potential and obtain the single differential equation
\[
iD \frac{\partial \ln \psi}{\partial t} = -D^2 \Delta \left( \frac{\Delta \psi}{\psi} \right) + 2\pi G|\psi|^2 + D \text{Im}(\gamma \Delta \ln \psi). \tag{22}
\]

### D. Recovery of ordinary quantum mechanics

Before going further, let us make the connection with ordinary quantum mechanics. In the absence of dissipation \( (\gamma = V = 0) \), the wave equation (19) reduces to the Schrödinger-like equation
\[
iD \frac{\partial \psi}{\partial t} = -D^2 \Delta \psi + \frac{1}{2} \Phi \psi. \tag{23}
\]
This equation coincides with the ordinary Schrödinger equation of quantum mechanics
\[
i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + m \Phi \psi \tag{24}
\]
provided that we make the identification
\[
D = \frac{\hbar}{2m}, \tag{25}
\]
where \( \hbar \) is the Planck constant and \( m \) the mass of the particles. Therefore, in the context of ordinary quantum mechanics, the coefficient \( D \) is inversely proportional to the mass of the particles.

In the gravitational case considered here, the Schrödinger-like equation (23) satisfies the equivalence principle since it does not depend on the inertial mass \( m \) of the particles. This is consistent with the fundamental equation of dynamics (11) from which it is deduced. We note, by contrast, that the ordinary Schrödinger equation (24) breaks the equivalence principle since it explicitly depends on the inertial mass of the particles (14). This suggests that the Schrödinger-like equation (23) with a diffusion coefficient \( D \) may be more relevant to describe astrophysical systems like DM halos than the ordinary Schrödinger equation (24).

**Remark:** For a free particle \( (\Phi = 0) \) the Schrödinger-like equation (23) reduces to
\[
\frac{\partial \psi}{\partial t} = iD \Delta \psi. \tag{26}
\]
Under this form, the Schrödinger-like equation is similar to a diffusion equation with an imaginary diffusion coefficient \( iD \). This strengthens the equivalence between the Schrödinger equation and the complex Burgers equation
\[
\frac{\partial U}{\partial t} + (U \cdot \nabla) U = iD \Delta U \tag{27}
\]
through the complex Cole-Hopf transformation (16).

### E. Fluctuation-dissipation theorem

We now come back to the generalized Schrödinger equation (19) with \( \gamma \neq 0 \) including dissipation. Writing \( \gamma = \gamma_R + i\gamma_I \), where \( \gamma_R \) is the classical friction coefficient
and $\gamma_I$ is the quantum friction coefficient, and using the identity

$$\text{Im}(\gamma \ln \psi) = \gamma_I \ln |\psi| - \frac{1}{2} \gamma_R \ln \left( \frac{\psi}{\psi^*} \right), \quad (28)$$

we can rewrite Eq. (19) as

$$iD \frac{\partial \psi}{\partial t} = -D^2 \Delta \psi + \frac{1}{2} \Phi \psi + \frac{1}{2} V(t) \psi + D \gamma_I \ln |\psi| \psi - i \gamma_R \ln \left( \frac{\psi}{\psi^*} \right) \psi. \quad (29)$$

Introducing the notations

$$\gamma_R = \xi, \quad \gamma_I = \frac{k_B T}{D m}, \quad (30)$$

the generalized Schrödinger equation (19) takes the form

$$iD \frac{\partial \psi}{\partial t} = -D^2 \Delta \psi + \frac{1}{2} \Phi \psi + \frac{1}{2} V(t) \psi + \frac{k_B T}{m} \ln |\psi| \psi - \frac{1}{2} i \gamma D \ln \left( \frac{\psi}{\psi^*} \right) \psi. \quad (31)$$

Using the hydrodynamic representation of the generalized Schrödinger equation (see below), we can interpret $\xi$ as an ordinary friction coefficient and $T$ as an effective temperature ($k_B$ is Boltzmann’s constant). It is convenient to choose the function $V(t)$ so that the average value of the friction term proportional to $\xi$ is equal to zero. This gives

$$V(t) = iD \left\langle \ln \left( \frac{\psi}{\psi^*} \right) \right\rangle, \quad (32)$$

where $\left\langle X \right\rangle = (1/M) \int \rho X \, d\mathbf{r}$. We finally obtain the generalized Schrödinger equation\(^1\)

$$iD \frac{\partial \psi}{\partial t} = -D^2 \Delta \psi + \frac{1}{2} \Phi \psi + \frac{k_B T}{m} \ln |\psi| \psi - \frac{1}{2} i \gamma D \ln \left( \frac{\psi}{\psi^*} \right) \psi. \quad (33)$$

This equation has to be coupled to the Poisson equation\(^2\). They can be combined into a single differential equation

$$iD \frac{\partial \Delta \psi}{\partial t} = -D^2 \Delta \left( \frac{\Delta \psi}{\psi} \right) + 2\pi G |\psi|^2 + \frac{k_B T}{m} \Delta \ln |\psi| - \frac{1}{2} i \xi D \Delta \ln \left( \frac{\psi}{\psi^*} \right). \quad (34)$$

It is interesting to note that the complex nature of the friction coefficient

$$\gamma = \xi + i \frac{k_B T}{D m} \quad (35)$$

leads to a generalized Schrödinger equation exhibiting simultaneously a friction term (as expected) and an effective temperature term (unexpected). They correspond to the real and imaginary parts of $\gamma$. This may be viewed as a new form of fluctuation-dissipation theorem. In this respect, we note that the relation

$$D = \frac{k_B T}{m \gamma_I} \quad (36)$$

is similar to the Einstein relation of Brownian motion\(^3\). It is important to stress, however, that $T$ does not represent the true thermodynamic temperature which is here assumed to be equal to zero (see Appendix C). It could be interpreted as the temperature of an hypothetical Dirac-like aether\(^4\) (it may represent the temperature of the vacuum if it has fluctuations), or be an intrinsic property of the fractal spacetime itself. Similarly, the friction coefficient $\xi$ may characterize the friction of the system with the aether or be an intrinsic property of the fractal spacetime. Another possibility is that the effective temperature $T$ and the friction $\xi$ heuristically parametrize the process of violent relaxation and gravitational cooling experienced by the system on a coarse-grained scale.

Remark: In relation to the equivalence principle discussed in Sec. II D a comment may be in order. It seems that the mass $m$ of the particles has appeared in Eq. (33). However, its occurrence is artificial because it arises from the notation of Eq. (30) that involves the effective temperature $T$. In fact, only the ratio $k_B T/m$ matters and this ratio is independent of the mass (in other words, $m$ is an effective mass that needs not coincide with the DM particle mass). By anticipating a result that will be obtained below, we could have written $v_\infty^2/2$ instead of $k_B T/m$ so that the mass $m$ does not appear anymore in the equations.\(^5\) In this way, the equivalence principle is respected. However, in order to develop an analogy with thermodynamics (see below), we shall work in terms of an effective temperature $T$ and an effective mass $m$.

F. The Madelung transformation

Using the Madelung transformation\(^6\), the generalized Schrödinger equation (33) can be written in the

\(^1\) It is possible to generalize this equation further by taking the self-interaction of the particles into account (see Appendices A and B).

\(^2\) This notation makes sense since the coefficient in front of $\ln |\psi|\psi$ in Eq. (31) has the dimension of a velocity square. As we shall see, the velocity $v_\infty$ corresponds to the constant circular velocity of the spiral galaxies.
form of real hydrodynamic equations. To that purpose, we write the wave function as
\[
\psi(\mathbf{r}, t) = \sqrt{\rho(\mathbf{r}, t)} e^{i\sigma(\mathbf{r}, t)/2D},
\]
where \(\rho\) is the density and \(\sigma\) is a real potential or a real action. They can be expressed in terms of the wave function as
\[
\rho = |\psi|^2, \quad \sigma = -iD \ln \left( \frac{\psi}{\psi^*} \right).
\]
We note that the effective temperature term in the generalized Schrödinger equation \([33]\) can be written as \((k_B T/2m) \ln \rho \psi^2\) and the dissipative term as \((1/2)\xi(\sigma - \langle \sigma \rangle) \psi^2\). Following Madelung, we introduce the real potential velocity field \(u = \nabla \sigma\). The flow defined in this way is irrotational since \(\nabla \times u = 0\). Substituting Eq. \([37]\) into the generalized Schrödinger equation \([33]\) and separating the real and imaginary parts, we find that the generalized Schrödinger equation is equivalent to the hydrodynamic equations
\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \quad \frac{\partial \sigma}{\partial t} + \frac{(\nabla \sigma)^2}{2} + \Phi + V_Q + \frac{k_B T}{m} \ln \rho + \frac{\xi \sigma - \langle \sigma \rangle}{\rho} = 0,
\]
where
\[
V_Q = -2D \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}}.
\]
It involves a pressure force with an effective isothermal equation of state
\[
P = \rho \frac{k_B T}{m},
\]
a gravitational force, a quantum force, and a damping force. Using the continuity equation \([39]\), the quantum damped isothermal Euler equation \([42]\) can be written as
\[
\frac{\partial}{\partial t} (\rho \mathbf{u}) + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla P - \nabla \Phi - \nabla V_Q - \xi \rho \mathbf{u}.
\]
When the quantum potential is neglected (Thomas-Fermi approximation), we recover the classical damped isothermal Euler equation. For dissipationless systems (\(\xi = 0\)), we recover the quantum and classical isothermal Euler equations.

**G. Connection with the equations of Brownian theory**

In the overdamped limit \(\xi \rightarrow +\infty\), we can formally neglect the inertial term in Eq. \([42]\) so that
\[
\xi \mathbf{u} \simeq -\frac{1}{\rho} \nabla P - \nabla \Phi - \nabla V_Q.
\]
Substituting this relation into the continuity equation \([39]\), we obtain the quantum Smoluchowski equation \([40]\):
\[
\xi \frac{\partial \rho}{\partial t} = \nabla \cdot (\nabla P + \rho \nabla \Phi + \rho \nabla V_Q).
\]
When the quantum potential is neglected, it reduces to the classical Smoluchowski equation
\[
\xi \frac{\partial \rho}{\partial t} = \nabla \cdot \left( \frac{k_B T}{m} \nabla \rho + \rho \nabla \Phi \right)
\]
that was introduced in the context of Brownian motion \([160]\). The diffusion coefficient satisfies the standard Einstein relation \([159]\):
\[
D = \frac{k_B T}{\xi m}.
\]
On the other hand, if we neglect the advection term \(\nabla (\rho \mathbf{u} \otimes \mathbf{u})\) in Eq. \([44]\), but retain the term \(\partial (\rho \mathbf{u})/\partial t\), and combine the resulting equation with the continuity equation \([39]\), we obtain the quantum telegraph equation
\[
\frac{\partial^2 \rho}{\partial t^2} + \xi \frac{\partial \rho}{\partial t} = \nabla \cdot (\nabla P + \rho \nabla \Phi + \rho \nabla V_Q)
\]
which can be seen as a generalization of the quantum Smoluchowski equation \([46]\) taking inertial (or memory) effects into account. When the quantum potential is neglected, we recover the classical telegraph equation.

It is interesting to recover the equations of Brownian theory, with a completely different interpretation, from the generalized Schrödinger equation \([33]\) in a strong friction limit. In this sense, our approach makes a connection between quantum mechanics and Brownian motion. However, we emphasize that (besides the presence of the quantum potential) this analogy is essentially formal. For example, the diffusion term in the Smoluchowski equation for Brownian particles arises from stochastic processes (it is due to a random force or a noise in the Langevin \([161]\) equations of Brownian motion) while the diffusion term in the Smoluchowski equation derived from the generalized Schrödinger equation \([33]\) arises from a logarithmic nonlinearity stemming from the imaginary part of the complex friction coefficient in the covariant equation of dynamics \([13]\). One is left speculating if this complex friction force is equivalent to a stochastic force. In any case, at a formal level, the generalized Schrödinger equation \([33]\), which is equivalent to the covariant equation of dynamics \([13]\), unifies quantum mechanics and Brownian motion.
Remark: The dynamics and thermodynamics of self-gravitating Brownian particles described by the Smoluchowski-Poisson equations [12] and [47] has been studied in [162–167]. If the strong friction limit \( \xi \to +\infty \) is relevant, this work could be applied to the generalized Schrödinger-Poisson equations [21] and [43].

H. Justification of the Born interpretation

In the theory of scale relativity, the fundamental object of interest is the complex velocity \( \mathbf{U} \) of the fractal geodesics and the complex hydrodynamic equation [14] of these geodesics from which the (generalized) Schrödinger equation [43] can be derived. Now, we expect the fluid of geodesics to be more concentrated at some places and less at others as does a real fluid. To find the probability density of presence of the paths we can remark that Eqs. (14) and (33) are equivalent to the real hydrodynamic equations (39)-(44). In that context, \( \mathbf{u} = \nabla \sigma \) is not an ad hoc definition (unlike in Madelung’s original work) but it corresponds to the classical velocity (the real part of \( \mathbf{U} \)). On the other hand, in the theory of scale relativity, the quantum potential is a manifestation of the geometry of spacetime, namely, of its non-differentiability and fractality, in similarity with the Newtonian potential being a manifestation of the curvature of spacetime in Einstein’s theory of general relativity [148]. Then, Eqs. (39)-(44) describe a fluid of fractal geodesics in a non-differentiable spacetime. They have therefore a clear physical interpretation. As a result \( \rho(r,t) = |\psi|^2 \) represents the density of the geodesic fluid, and the probability density for the “quantum particle” to be found at a given position must be proportional to \( |\psi|^2 \). This is how the Born postulate can be naturally justified in the theory of scale relativity [148].

III. APPLICATION TO DM HALOS

We now apply the generalized Schrödinger equation [43] to the context of DM halos and show qualitatively how it can account for their main properties. A more quantitative study, and a comparison with observations, will be the subject of a specific paper [168].

A. Quantum hydrostatic equilibrium

Considering a solution of the generalized Schrödinger equation (33) of the form \( \psi(r,t) = \phi(r)e^{-i\epsilon t}/2D \) where \( \phi(r) \) and \( \epsilon \) (energy) are real, we obtain the time-independent generalized Schrödinger equation

\[
-2D^2 \Delta \phi + \Phi \phi + \frac{k_B T}{m} (\ln \rho) \phi = \epsilon \phi, \tag{50}
\]

which is a nonlinear eigenvalue equation. Dividing Eq. [50] by \( \phi \) and using the fact that \( \phi(r) = \sqrt{\rho(r)} \), or directly substituting \( \sigma = -\epsilon t \) into Eq. [40], we obtain the equilibrium Hamilton-Jacobi (or Bernoulli) equation

\[
-2D^2 \Delta \sqrt{\rho} + \Phi + \frac{k_B T}{m} \ln \rho = \epsilon. \tag{51}
\]

Taking the gradient of Eq. [51], we obtain the condition of quantum hydrostatic equilibrium

\[
\rho \nabla V_{\mathcal{Q}} + \nabla P + \rho \nabla \Phi = 0. \tag{52}
\]

This equation corresponds also to the equilibrium state \( \rho = 0 \) of the quantum Euler equation [12]. It describes the balance between the gravitational attraction, the quantum pressure, and the effective thermal pressure. It must be coupled self-consistently to the Poisson equation (12). Equation [51] can be rewritten as

\[
\rho = Ae^{-\frac{m\epsilon}{k_B T}(\Phi + V_{\mathcal{Q}})} \quad \text{with} \quad A = e^{m\epsilon/k_B T}, \tag{53}
\]

which can be interpreted as a quantum Boltzmann distribution. This is actually a differential equation since the quantum potential \( V_{\mathcal{Q}} \) involves derivatives of the density.

B. H-theorem

Introducing the Boltzmann free energy \( F_B = E - TS_B \) where \( E = \Theta_c + \Theta Q + W \) is the energy (including the classical kinetic energy \( \Theta_c = (1/2) \int \rho u^2 \, dr \), the quantum kinetic energy \( \Theta_Q = \int \rho V_{\mathcal{Q}} \, dr \) and the gravitational energy \( W = (1/2) \int \rho \Phi \, dr \) and \( S_B = -k_B \int (\rho/m)(\ln \rho - 1) \, dr \) is the Boltzmann entropy, we can show that the generalized Schrödinger equation (33) satisfies an H-theorem [141]:

\[
\dot{F}_B = -\xi \int \rho u^2 \, dr \leq 0. \tag{54}
\]

When \( \xi = 0 \), the generalized Schrödinger equation (33) conserves the energy \( \dot{F}_B = 0 \). When \( \xi > 0 \), the free energy decreases monotonically \( \dot{F}_B \leq 0 \). On the other hand, \( \dot{F}_B = 0 \) if, and only if, \( \mathbf{u} = 0 \), leading to the condition of quantum hydrostatic equilibrium (52). Therefore, the dissipative term ensures that the system relaxes towards an equilibrium state for \( t \to +\infty \). In this sense, it can account for the complicated processes of violent relaxation and gravitational cooling. The equilibrium state minimizes the Boltzmann free energy \( F_B \) at fixed mass \( M \) (see footnote 13). Writing the variational principle as

\[
\text{This result assumes that} \ F_B \text{is bounded from below. For isothermal self-gravitating systems this is not the case. There is no minimum of free energy at fixed mass because the system can always lose free energy by evaporating except if it is artificially confined within a box [169]. However, evaporation is a slow process. In practice, the system relaxes towards a quasi-equilibrium state that slowly evolves in time because of evaporation.}
\[ \delta F - \alpha \delta M = 0, \] where \( \alpha \) is a Lagrange multiplier (chemical potential) taking into account the conservation of mass, we recover Eq. (51) with \( \alpha = \epsilon \). Therefore, the eigenenergy \( \epsilon \) represents the chemical potential \( \alpha \).

Remark: We note that the conservative (\( \xi = 0 \)) generalized Schrödinger equation \([53]\) is reversible while the dissipative (\( \xi > 0 \)) generalized Schrödinger equation \([53]\) is irreversible. As discussed previously, irreversibility may be the manifestation of a cosmic aether or the consequence of a process of violent relaxation and gravitational cooling on a coarse-grained scale (a sort of nonlinear Landau damping like for the Vlasov-Poisson equations \([170]\)).

C. General differential equation

Combining the equation of hydrostatic equilibrium \([52]\) and the Poisson equation \([12]\), and using the effective isothermal equation of state \([43]\), we obtain a differential equation

\[ 2D^2 \Delta \left( \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right) - \frac{k_B T}{m} \Delta \ln \rho = 4\pi G \rho \] (55)

that determines the density profile of the DM halos. This profile has a core-halo structure that is studied in detail in a separate paper \([168]\) where Eq. (55) is solved numerically. We describe below how Eq. (55) simplifies in the core and in the halo respectively. Then, we give a preliminary discussion of its general solution and show how it can account for the main properties of DM halos.

1. Solitonic core

In the core, thermal effects are negligible and the condition of hydrostatic equilibrium \([52]\) reduces to

\[ \nabla V_Q + \nabla \Phi = 0. \] (56)

It describes the balance between the gravitational attraction and the quantum pressure arising from the Heisenberg uncertainty principle. The differential equation (55) becomes

\[ 2D^2 \Delta \left( \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right) = 4\pi G \rho. \] (57)

This equation has been solved numerically in \([64, 93, 110, 111, 125, 171, 173]\). Its solution is called a soliton because it corresponds to the static state of the ordinary Schrödinger-Poisson equation. This profile presents a core in which the central density is finite. As a result, it can solve the cusp problem of CDM. The exact mass-radius relation is given by \([64, 93, 171]\):

\[ R_{99} = 39.6 \frac{D^2}{GM}, \] (58)

where \( R_{99} \) is the radius of the configuration containing 99% of the mass. The density profile extends to infinity. It has an approximately Gaussian shape \([92, 93]\). Another fit of this profile has been given in \([110, 111]\).

2. Isothermal halo

In the halo, quantum effects are negligible and the condition of hydrostatic equilibrium \([52]\) reduces to

\[ \nabla P + \rho \nabla \Phi = 0. \] (59)

It describes the balance between the gravitational attraction and the effective thermal pressure. Using Eq. (43), it can be integrated into

\[ \rho = A e^{-m \Phi / k_B T}, \] (60)

which can be interpreted as Boltzmann’s law in a gravitational mean field potential. This equation must be coupled self-consistently to the Poisson equation \([12]\) leading to the Boltzmann-Poisson equation

\[ \Delta \Phi = 4\pi G A e^{-m \Phi / k_B T}. \] (61)

This equation arises in the statistical mechanics of self-gravitating systems \([169]\) but it has been derived here from rather different arguments. Equation (61) is equivalent to the differential equation

\[ \Delta \ln \rho + \frac{4\pi G m}{k_B T} \rho = 0 \] (62)

obtained from Eq. (55) by neglecting the quantum potential. It is easy to show that the asymptotic behavior of the density distribution is given by \([152]\):

\[ \rho(r) \sim \frac{k_B T}{2\pi G m r^2}. \] (63)

The mass contained within a sphere of radius \( r \) behaves at large distances as \( M(r) = \int_0^r \rho(r') 4\pi r'^2 dr' \sim 2k_B T r / G m \). Therefore, the (effective) isothermal density profile leads to flat rotation curves since \([152]\):

\[ v_c^2(r) = \frac{GM(r)}{r} \to \frac{2k_B T}{m} \text{ for } r \to +\infty, \] (64)

where \( v_c(r) \) is the circular velocity at distance \( r \). This result is consistent with the observations that show that the circular velocity of spiral galaxies tends to a constant \( v_\infty \) at large distances instead of declining according to Kepler’s law. We find that \( v_\infty = (2k_B T / m)^{1/2} \). We stress again that the effective temperature \( T \) does not

\[ \int_0^\infty \rho(r) 4\pi r'^2 dr' \sim 2k_B T r / G m. \]
From general thermodynamical arguments, we expect the system to reach an isothermal distribution. However, in practice, nonideal effects may prevent its relaxation towards thermodynamical equilibrium. This is relatively obvious in the present context since a self-gravitating isothermal system has an infinite mass \[15\]. Therefore, its atmosphere cannot be exactly isothermal. We note in this respect that the exponent \( \alpha = -3 \) (NFW/Burkert) of the observed density profile \( \rho \sim r^{-\alpha} \) at large distances is the closest exponent to \( \alpha = 2 \) (isothermal) that yields a halo with a (marginally) finite mass.

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3. Core-halo structure

In this section, we give a preliminary discussion of the differential equation \([55]\) governing the structure of DM halos in the present model. If we define

\[
\rho = \rho_0 e^{-\psi}, \quad \xi = \left( \frac{4\pi G \rho_0 m}{k_B T} \right)^{1/2} r, \tag{65}
\]

\[
\chi = \frac{D m}{k_B T} \sqrt{\pi G \rho_0}, \tag{66}
\]

where \( \rho_0 \) is the central density and \( \chi \) is a dimensionless concentration parameter, we find that Eq. \([55]\) takes the form of a quantum Emden equation

\[
4\chi^2 \Delta \left( \frac{\Delta e^{-\psi/2}}{e^{-\psi/2}} \right) + \Delta \psi = e^{-\psi}. \tag{67}
\]

The classical Emden equation is recovered for \( \chi = 0 \) \[174\]. The differential equation \([67]\) can be solved numerically. The density profiles and the circular velocity profiles are plotted in Figs. \[1\] and \[2\] for different values of \( \chi \). The density profile presents a core-halo structure which is more or less pronounced depending on the value of the concentration parameter \( \chi \). The velocity profile shows a dip which is due to the presence of the solitonic core. These profiles are qualitatively similar to the core-halo profiles obtained by Schive et al. \[110, 111\] by solving the Schrödinger-Poisson equations numerically but they are not exactly

FIG. 1: Normalized density profiles for different values of the concentration parameter \( \chi \) (specifically \( \chi = 0, 0.1, 1, 5, 10, 20, 100 \)). They present a core-halo structure with a solitonic core and an isothermal halo. The purely isothermal halo corresponds to \( \chi = 0 \) and the purely solitonic profile corresponds to \( \chi \to +\infty \).

FIG. 2: Same as Fig. 1 for the circular velocity profiles. They display a dip due to the presence of the solitonic core.
the same. In particular, in the present model, the halo is isothermal instead of being given by the NFW profile. It will be of interest in future works to compare these profiles with real DM halos. The present paper is just a first step in that direction. A more detailed study will be performed in a separate paper [168].

D. The fundamental parameters of the generalized Schrödinger equation

1. The coefficient \( \mathcal{D} \)

As discussed previously, the equilibrium states of the generalized Schrödinger equation (33) have a core-halo structure that is in qualitative agreement with the structure of DM halos [110] [111]. The quantumlike potential accounts for their solitonic core (solving the cusp problem) and the logarithmic nonlinearity accounts for their isothermal halo (leading to flat rotation curves). The extension of the atmosphere, as compared to that of the core, depends on the size of the halos through the concentration parameter \( \chi \). Large DM halos such as the Medium Spiral \((R \sim 1 \text{kpc} \text{ and } M \sim 10^{11} M_\odot)\) have a core-halo structure with a small core and a large atmosphere.\(^\text{16}\) By contrast, small DM halos such as dwarf spheroidal galaxies (dSph) like Fornax \((R \sim 1 \text{kpc} \text{ and } M \sim 10^8 M_\odot)\) are very compact and do not have an atmosphere. They are purely solitonic, corresponding to the ground state of the Schrödinger-Poisson equations. Therefore, their mass-radius relation is given by Eq. (58). We shall determine the coefficient \( \mathcal{D} \) arising in the generalized Schrödinger equation (33) by considering a typical dwarf halo of radius \( R = 1 \text{kpc} \) and mass \( M = 10^8 M_\odot \) (Fornax). Assuming that this halo represents the ground state of the Schrödinger-Poisson equation (pure soliton) we find from Eq. (58) that

\[
\mathcal{D} = 1.02 \times 10^{23} \text{ m}^2/\text{s}. \tag{68}
\]

If we assume that the nondifferentiability of spacetime is due to ordinary quantum mechanics, using Eq. (25) we find that the mass of the bosons must be ultralight, of the order of \( m = 2.92 \times 10^{-22} \text{eV}/\text{c}^2 \) (in comparison, for the electron of mass \( m = 511 \text{keV}/\text{c}^2 \), the quantum diffusion coefficient \( \mathcal{D} = h/2m = 5.79 \times 10^{-5} \text{m}^2/\text{s} \). However, the message conveyed in this paper is that the nondifferentiability of spacetime may arise from reasons different from ordinary quantum mechanics (as explained in the Introduction). In that case, the mass of the DM particle is not constrained (only \( \mathcal{D} \) is fixed), allowing for a larger class of particles including WIMPs, the QCD axion etc. The price to pay with this new approach is that we have to take into account a new physical ingredient in the equations of the problem, namely the fractal structure of spacetime at the cosmic scale. This leads to a complete reconsideration of the notion of space and time. This possibility is not ruled out by observations since it is not possible to determine if the Schrödinger equation that applies to DM halos is justified by the presence of ultralight particles (ordinary quantum mechanics) of by the fractal structure of spacetime.

2. The coefficients \( v_\infty^2/2 = k_B T/m \) and \( \xi \)

To determine the other coefficients \( k_B T/m \) and \( \xi \) that appear in the generalized Schrödinger equation (33) we proceed as follows. It is an observational evidence that the surface density of DM halos is independent of their size and has a universal value \( \Sigma_0 = \rho_0 r_h = 141 M_\odot/\text{pc}^2 \) [175] where \( \rho_0 \) is the central density and \( r_h \) is the halo radius where the central density is divided by 4. If we approximate large DM halos by an isothermal sphere, one can show that \( M_h = 1.76 \Sigma_0 r_h^3, v_h^2 = 1.76 G \Sigma_0 r_h \) and \( k_B T/m = 0.954 G \Sigma_0 r_h \) [108]. Therefore, the halo mass scales with size as \( M_h \propto r_h^2 \), the circular velocity at the halo radius scales as \( v_h \propto r_h^{1/2} \), and the temperature scales as \( k_B T/m \propto r_h \). The dynamical time \( t_D = 1/(G \rho_0)^{1/2} = (r_h/G \Sigma_0)^{1/2} \) scales as \( t_D \propto r_h^{1/2} \). For a halo of mass \( M_h = 10^{11} M_\odot \) (Medium Spiral), we find \( r_h = 2.01 \times 10^4 \text{pc}, \rho_0 = 7.02 \times 10^{-3} M_\odot/\text{pc}^2, (k_B T/m)^{1/2} = 108 \text{km/s}, \) and \( t_D = 178 \text{Myrs} \) (we also have \( v_h = (GM_h/r_h)^{1/2} = 146 \text{km/s} \) and \( v_\infty = 153 \text{km/s} \). Let us write the friction coefficient as \( \xi = \alpha \) where \( \alpha \) is a dimensionless parameter. The friction coefficient is of the order of the inverse relaxation time \( (\xi \sim 1/t_R) \). For self-gravitating systems evolving as a result of two-body encounters, relaxation (thermalization) occurs on a very long timescale, of the order \( t_R \sim (N/\ln N)t_D \) (Chandrasekhar’s time) [152], much larger than the dynamical time. In the situation that we consider here, an effective thermalization may take place on a much shorter timescale, of the order of the dynamical time \( t_D \) (as in the process of violent relaxation of collisionless self-gravitating systems [29]), or even shorter. In order to take into account all the possibilities, we leave \( \alpha \) arbitrary. As a result, the coefficients \( k_B T/m = v_\infty^2/2 \) and \( \xi \) of the generalized Schrödinger equation (33) depend on the size of DM halos according to\(^\text{17}\)

\[
\frac{v_\infty^2}{2} = k_B T/m = 0.954 G \Sigma_0 r_h, \quad \xi = \alpha \left( \frac{G \Sigma_0}{r_h} \right)^{1/2}. \tag{69}
\]

\(^\text{16}\) Actually, in large DM halos, the solitonic core is almost imperceptible. In that case, the presence of a core (instead of a cusp) is due to the effective thermal pressure rather than the quantum potential.

\(^\text{17}\) The expression of \( T \) is valid for relatively large halos. Close to the ground state \( T \rightarrow 0 \).
The scale dependence of $k_BT/m$ and $\xi$ is not a problem if the generalized Schrödinger equation (33) arises from the fractal structure of spacetime at the cosmological scale. In that case, its coefficient $D$, $\xi$ and $k_BT/m$ are intrinsic properties of the spacetime and they can change with the size of the halos. The scale dependence of $k_BT/m$ and $\xi$ is also expected if the effective temperature and the friction result from a process of violent relaxation and gravitational cooling. In that case, they will change from halo to halo depending on the efficiency of the relaxation process.

**Remark:** If we assume that $\gamma_R \sim \gamma_I$ (since these coefficients have a common origin) and use Eq. (30), we get $D \sim D_0 \xi/m$. Using Eq. (69), we obtain $\alpha \sim 0.95(G\Sigma_0\alpha_0^3)^{1/2}/D$. For a halo of mass $M_h = 10^{11} M_\odot$ and radius $r_h = 2.01 \times 10^4$ pc (Medium Spiral), we find $\alpha = \xi \sim 640 > 1$. This suggests that large DM halos are in the strong friction limit, allowing us to use the results of (162,167) valid for the Smoluchowski-Poisson equations. However, the assumption that $\gamma_R \sim \gamma_I$ remains to be established on more solid grounds. We give an argument in its favor in Appendix D by generalizing to the case of dissipative systems the original method of quantization introduced by Schrödinger [176].

**IV. CONCLUSION**

In this paper, we have proposed to describe DM halos by the generalized Schrödinger equation (33) obtained from the theory of scale relativity [148]. We have suggested that the origin of this equation is not due to ordinary quantum mechanics, as in the standard BECDM model [62,137], but to the fractal structure of spacetime (nondifferentiability) that manifests itself (i) above a certain length scale (ii) and/or for sufficiently long times. In the first case, this fractal structure is regarded as an intrinsic property of spacetime at the astrophysical or cosmological macro-scale. In the second case, it arises from the chaoticity of the trajectories of the particles. As a result, the coefficient $D$ that arises in the generalized Schrödinger equation (34) may be different from $h/2m$ allowing the mass of the DM particle to be much larger than the value $\sim 10^{-22}$ eV/c$^2$, corresponding to ultralight axions, required by the BECDM model. Indeed, the observations of dwarf DM halos (like Fornax) only determine the value of the coefficient $D = 1.02 \times 10^{23}$ m$^2$/s, not the mass $m$ of the DM particle. This is a consequence of the equivalence principle. We have proposed that the value of the coefficient $D$ is universal at the cosmological scale. It may, however, take a different value at the astrophysical and planetary scales [143]. By contrast, even at the cosmological scale, the friction parameter $\xi$ and the effective temperature $T$ are scale-dependent. They behave with the halos radius $r_h$ as $k_BT/m = 0.954(G\Sigma_0/r_h)$ and $\xi = 640 (G\Sigma_0/r_h)^{1/2}$, where $\Sigma_0 = 141 M_\odot/pc^2$ is the universal surface density of DM halos.

When applied to DM halos, the generalized Schrödinger equation (33) has attractive properties. It leads to equilibrium configurations with a core-halo structure similar to the structure of DM halos observed in the Universe or in numerical simulations [110,111]. The quantum potential accounts for their solitonic core, the logarithmic nonlinearity accounts for their isothermal core and the friction term guarantees that the system relaxes towards these core-halo structures (equilibrium states). This damping can account for the process of violent relaxation [26] and gravitational cooling [61] on a coarse-grained scale. An interest of the present model is that there is no free (arbitrary) parameter. The coefficients $D$, $k_BT/m$ and $\xi$ of the generalized Schrödinger equation (33) are determined by Eqs. (68) and (69). Therefore, for a given halo mass $M_h$ (or radius $r_h$), one can predict the DM halo profile by numerically solving the general differential equation (65). The results will be presented in detail in a forthcoming paper [168]. Preliminary results are given in [141] and in Figs. 1 and 2 of the present paper.

Apart from the reinterpretation of the coefficient $D$, our approach is similar to the BECDM model in which all the bosonic particles are in the same quantum state described by the condensate wave function $\psi$. However, the difference of interpretation is crucial because, if correct, it would lead to a complete reconsideration of the nature of space and time. In particular, following Nottale [148], we suggest that spacetime could become fractal (nondifferentiable) at the cosmic scales leading to a new form of quantum mechanics. We stress that we do not reject the BECDM model based on ordinary quantum mechanics. This is at this day the most plausible scenario. The purpose of this paper is just to propose an alternative scenario that has similar properties and can therefore solve the CDM crisis. It could become particularly interesting if one finds that the mass of the DM particle is different from what is predicted by the BECDM model. Indeed, if some arguments exclude ultralight

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18 We have treated the diffusion coefficient $D$ as a universal constant given by Eq. (68). The diffusion coefficient $D$ could also depend on the scale but that would add an indetermination in our model that is not necessary (this assumption of universality should be relaxed only if one finds that it is in conflict with observations). Of course, the value of $D$ given by Eq. (68) only applies to cosmological scales. It is very likely that $D$ changes at smaller scales (galactic, stellar, planetary...). In this respect, we note that the order of magnitude of $D$ found by Nottale [148] in the solar system is $D \sim 5 \times 10^{14}$ m$^2$/s.

19 According to Nottale’s theory [138], a classical system having a chaotic behavior may be described by a Schrödinger-like equation on a very long timescale.

20 In addition, our scenario has more flexibility than ordinary quan-
particles, the existence of DM cores (instead of cusps) could be a manifestation of the fractal (nondifferentiability) structure of spacetime at the galactic scale. To be very general, and embrace all possibilities, we could consider a generalized Schrödinger equation with a coefficient of the form $\mathcal{D} = \hbar/2m + \mathcal{D}_{\text{cosmo}}$ where the first term is the contribution of standard quantum mechanics (BEC-CDM model) and the second term corresponds to the intrinsic fractal structure of spacetime at the cosmic scale.

Another interest of our formalism is to obtain general equations that, in a sense, unify quantum mechanics and Brownian theory (even if this unification is formal or effective). Indeed, in the strong friction limit $\xi \to +\infty$, the generalized Schrödinger equation becomes equivalent to the quantum Smoluchowski equation that is similar to the one introduced in Brownian theory (up to an additional quantum potential). Self-gravitating Brownian particles described by the Smoluchowski-Poisson equations have been studied theoretically in [162–167]. Because of this formal analogy, if the strong friction limit is relevant to the case of DM (see the Remark at the end of Sec. III D), these theoretical results on the dynamics and thermodynamics of self-gravitating Brownian particles could find applications in the physics of DM halos with a new interpretation. This possibility will be considered in future works.

Finally, we would like to contrast the evolution of a collisionless self-gravitating system evolving in a differentiable spacetime with the evolution of a collisionless self-gravitating system evolving in a nondifferentiable (fractal) spacetime. In the first case, the system is described by the Vlasov-Poisson equations. These equations have a very complicated dynamics associated with phase mixing and nonlinear Landau damping. As a result, they develop intermingled filaments at smaller and smaller scales and a coarse-grained description becomes necessary to smooth out this intricate filamentation. The system is expected to relax towards a quasistationary state on the coarse-grained scale on a very short timescale of the order of the dynamical time. This is called violent relaxation. Lynden-Bell developed a statistical theory of this process and derived an equilibrium distribution function describing this quasistationary state.

A sort of exclusion principle similar to the Pauli exclusion principle in quantum mechanics arises in the theory of Lynden-Bell due to the incompressibility of the flow in phase space and the conservation of the distribution function (on the fine-grained scale) by the Vlasov equation. As a result, the Lynden-Bell distribution function is similar to the Fermi-Dirac distribution function, suggesting that the process of violent relaxation is similar in some respects to the relaxation of fermionic particles. A kinetic equation for the coarse-grained distribution function has been proposed in [177, 179]. It can be viewed as a generalized Landau or Fokker-Planck equation taking into into account the Lynden-Bell exclusion principle. We can then derive hydrodynamic equations which have the form of damped Euler equations including a pressure term with a fermionic-like equation of state. These hydrodynamic equations do not involve a quantum potential because the analogy between the Lynden-Bell theory and quantum mechanics is purely effective.

On the other hand, we have argued in this paper that a collisionless self-gravitating system evolving in a nondifferentiable spacetime (it could also be a classical system having a chaotic motion on a very long timescale) is described by the generalized Schrödinger equation. The corresponding hydrodynamic equations have the form of damped Euler equations including a quantum-like potential (arising from the non-differentiability of spacetime) and a temperature term. However, there is no Lynden-Bell (Fermi-Dirac) pressure term in these equations. In order to reconcile the two descriptions (i.e. to recover, in the differentiable limit, the hydrodynamic equations of [177] associated with the Lynden-Bell theory) we derive in Appendix B a generalized Schrödinger equation with a nonlinearity accounting for Lynden-Bell’s degeneracy pressure. This may be seen as a refinement of the generalized Schrödinger equation taking into account the Lynden-Bell exclusion principle of violent relaxation or, alternatively, a generalization of the Lynden-Bell theory in a nondifferentiable spacetime.

Let us briefly summarize the main ideas of this paper.

\[14\]

\[22\] The Lynden-Bell theory accounts for an exclusion principle arising from the Vlasov equation which plays a role similar to the Pauli exclusion principle in quantum mechanics. This is why the Lynden-Bell distribution is similar to the Fermi-Dirac distribution, and why the hydrodynamic equations of [177] involve a Fermi-Dirac-like pressure term. However, the analogy with quantum mechanics stops here because the Lynden-Bell theory is not based on a Schrödinger equation. As a result, there is no quantum potential (corresponding to the kinetic term, accounting for the Heisenberg uncertainty principle, in the Schrödinger equation). There has been some attempts to describe the Vlasov-Poisson equations on a coarse-grained scale in terms of an effective Schrödinger equation but this is essentially an heuristic procedure aimed at smoothing out the small scales and avoiding numerical instabilities (see the discussion in [24]).
Appendix A: Short-range interactions

In this Appendix, we generalize the Schrödinger equation [33] by taking the self-interaction of the particles into account.

1. Mean-field Schrödinger equation

Let us assume that the particles have a short-range interaction described by the binary potential \( u_{SR}(r-r') \). If we make a mean field approximation, we find that the potential in which a particle moves is given by

\[
\Phi_{SR}(r, t) = \int u_{SR}(r-r') \rho(r', t) dr'.
\]  
(A1)

This short-range potential has to be added in Eq. (33) to the gravitational potential corresponding to long-range interactions. The generalized Schrödinger equation taking into account both short-range and long-range interactions in the mean field approximation is

\[
iD \frac{\partial \psi}{\partial t} = -D^2 \Delta \psi + \frac{1}{2} \left( \Phi + \Phi_{SR} \right) \psi + \frac{k_B T}{m} \ln |\psi| \psi
\]

\[
- \frac{1}{2} i \xi D \left[ \ln \left( \frac{\psi}{\psi^*} \right) - \left\langle \ln \left( \frac{\psi}{\psi^*} \right) \right\rangle \right] \psi,
\]

where

\[
\Phi(r, t) = -G \int \frac{\rho(r', t)}{|r-r'|} dr'.
\]  
(A3)

is the gravitational potential determined by the Poisson equation (12) and \( \Phi_{SR}(r, t) \) is the short-range potential given by Eq. (A1).

2. Gross-Pitaevskii-like equation

If we consider a pair contact interaction \( u_{SR} = g \delta(r-r') \) with strength \( g \) as described by Dirac’s \( \delta \)-function, we find that

\[
\Phi_{SR}(r, t) = g \rho(r, t).
\]  
(A4)

Using Eq. (20), the foregoing equation can be rewritten as \( \Phi_{SR} = g |\psi|^2 \). Substituting this relation into Eq. (A2), we obtain a generalized Schrödinger equation of the form

\[
iD \frac{\partial \psi}{\partial t} = -D^2 \Delta \psi + \frac{1}{2} \Phi \psi + \frac{g |\psi|^2}{m} \ln |\psi| \psi
\]

\[
- \frac{1}{2} i \xi D \left[ \ln \left( \frac{\psi}{\psi^*} \right) - \left\langle \ln \left( \frac{\psi}{\psi^*} \right) \right\rangle \right] \psi.
\]  
(A5)

It includes a cubic nonlinearity like in the Gross-Pitaevskii equation [181] [183]. This is a particular case of the generalized GP equation studied in [141]. The corresponding hydrodynamic equations have the form of Eqs. (39)-(42) with an equation of state

\[
P = \frac{k_B T}{m} + \frac{1}{2} g |\psi|^2.
\]  
(A6)

Their equilibrium state describes DM halos with a polytropic core (soliton) of index \( \gamma = 2 \) and an isothermal atmosphere. The polytropic equation of state introduces an internal energy \( U = (1/2) g \rho^2 dr \) in the expression of the free energy (see Sec. [113] and Ref. [141]). We note that \( U = (1/2) \int \rho \Phi_{SR} dr \) where \( \Phi_{SR} \) given by Eq. (A4).

Remark: in the quantum model where the bosons have a self-interaction, Eq. (A5) takes the form

\[
i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + mF \psi + 4\pi a_s h^2 m^2 |\psi|^2 \psi
\]

\[+ 2k_B T \ln |\psi| \psi - \hbar \frac{1}{2} \xi \left[ \ln \left( \frac{\psi}{\psi^*} \right) - \left\langle \ln \left( \frac{\psi}{\psi^*} \right) \right\rangle \right] \psi,
\]  
(A7)

where \( a_s \) is the s-scattering length of the bosons and we have used \( g = 4\pi a_s h^2 /m^3 [186] \). This corresponds to the model of BECDM proposed in [131]. The transition between the region dominated by the quantum potential and the region dominated by (effective) thermal effects corresponds to the (effective) de Broglie length

\[
\lambda_{dB} \sim \frac{\hbar}{\sqrt{mk_B T}}.
\]  
(A8)

The quantum potential dominates for \( r \ll \lambda_{dB} \) and the effective thermal effects dominate for \( r \gg \lambda_{dB} \). The transition between the region dominated by the quantum pressure (due to the repulsive self-interaction of the bosons) and the region dominated by the (effective) thermal pressure corresponds to a density

\[
\rho_B \sim \frac{m^2 k_B T}{a_s h^2}.
\]  
(A9)

The quantum pressure dominates for \( \rho \gg \rho_B \) and the (effective) thermal pressure dominates for \( \rho \ll \rho_B \).
3. Cahn-Hilliard-like equation

The potential of Eq. (A1) corresponds to the dominant term in an expansion of the short-range potential of interaction (A1) in powers of the range of the interaction. Let us derive a generalized Schrödinger equation taking into account the next order term in this expansion. Setting $q = r - r$ and writing the short-range potential of interaction as

$$\Phi_{SR}(r, t) = \int u_{SR}(q)\rho(r + q, t) dq,$$  \hspace{1cm} (A10)

we can Taylor expand $\rho(r + q, t)$ to second order in $q$ to obtain

$$\rho(r + q, t) = \rho(r, t) + \sum_i \frac{\partial \rho}{\partial x_i} q_i + \frac{1}{2} \sum_{i,j} \frac{\partial^2 \rho}{\partial x_i \partial x_j} q_i q_j.$$  \hspace{1cm} (A11)

Substituting this expansion into Eq. (A10), we get

$$\Phi_{SR}(r, t) = g\rho(r, t) + \chi \Delta \rho(r, t)$$  \hspace{1cm} (A12)

with $g = 4\pi \int_0^{+\infty} u_{SR}(q) q^2 dq$ and $\chi = \frac{\chi}{\rho} \int_0^{+\infty} u_{SR}(q) q^4 dq$. Note that $l = (\chi/g)^{1/2}$ has the dimension of a length corresponding to the range of the interaction. Using Eq. (A12), Eq. (A12) can be rewritten as $\Phi_{SR} = g|\psi|^2 + \chi \Delta |\psi|^2$. Substituting this relation into Eq. (A12), we obtain a generalized Schrödinger equation of the form

$$i\hbar \frac{\partial \psi}{\partial t} = -\hbar^2 \Delta \psi + \frac{1}{2} \Phi \psi + \frac{1}{2} g |\psi|^2 \psi + \chi \Delta |\psi|^2 \psi + 
\frac{k_B T}{m} \ln |\psi| \psi - \frac{1}{2} \xi D \left[ \ln \left( \frac{\psi}{\psi^*} \right) - \left( \frac{\psi}{\psi^*} \right) \right] \psi.$$  \hspace{1cm} (A13)

It includes a cubic nonlinearity like in the Gross-Pitaevskii equation and a Laplacian term like in the Cahn-Hilliard equation (this analogy will be developed in a separate paper). The cubic term introduces an internal energy $U = (1/2)g f \rho^2 dr$, and the Laplacian term introduces a square gradient energy $W_\chi = -(1/2) \chi f (\nabla \rho)^2 dr$, in the expression of the free energy (see Sec. 11B and Ref. 14). We note that $U = (1/2) \int \rho \Phi_{SR} dr$ where $\Phi_{SR}$ given by Eq. (A12).

Appendix B: Violent relaxation in a nondifferentiable spacetime

In a differentiable spacetime, a collisionless self-gravitating system is described by the Vlasov-Poisson equations. The Vlasov-Poisson equations experience a process of violent relaxation [20]. They are expected to relax, on a coarse-grained scale, towards the Lynden-Bell distribution:

$$\mathcal{J}(r, v) = \frac{\eta_0}{1 + e^{\eta_0 (v^2/2 + f(r) - \mu)} / T_{eff}},$$  \hspace{1cm} (B1)

where $\eta_0$ is the maximum value of the fine-grained distribution function and $T_{eff}$ is an effective temperature (note that $T_{eff}$ has not the dimension of a temperature but $T_{eff}/\eta_0$ has the dimension of a velocity dispersion). The Lynden-Bell distribution is similar to the Fermi-Dirac distribution. In the nondegenerate limit $\mathcal{J} \ll \eta_0$, it takes a form similar to the Maxwell-Boltzmann distribution. The evolution of the coarse-grained distribution function $\mathcal{J}(r, v, t)$ can be described by a generalized Landau or Fokker-Planck equation [177–179] taking into account the Lynden-Bell exclusion principle $\mathcal{J}(r, v, t) \leq \eta_0$ which is similar to the Pauli exclusion principle in quantum mechanics (but with another interpretation). From this kinetic equation, one can derive generalized hydrodynamic equations [177] that incorporate a pressure force with a Lynden-Bell equation of state (similar to the Fermi-Dirac equation of state in quantum mechanics) and a linear friction force. In the nondegenerate limit, the Lynden-Bell equation of state reduces to the isothermal equation of state, but with temperature proportional to mass [20]. Indeed, the mass $m$ of the particles should not occur in a collisionless theory based on the Vlasov equation. Therefore, in Lynden-Bell’s theory, the isothermal equation of state writes

$$P = \rho \frac{T_{eff}}{\bar{\eta}_0}.$$  \hspace{1cm} (B2)

The hydrodynamic equations derived in [177] are similar to Eqs. (39)-(44), except for the presence of the quantum potential. In our approach, this term arises from the nondifferentiability of spacetime. The generalized Schrödinger equation (A24), which is equivalent to Eqs. (39)-(44), may therefore describe the process of violent relaxation in a nondifferentiable spacetime (or on a very long timescale when chaotic effects come into play). To improve this description, one needs to take into account the Lynden-Bell exclusion principle that is specific to the theory of violent relaxation. Since the hydrodynamic equations (39)-(44) already contains a thermal pressure (it can be adapted to the theory of violent relaxation by replacing $k_B T/m$ by $T_{eff}/\bar{\eta}_0$), we just have to add the contribution of the degeneracy pressure as explained below.

23 In practice, the quasistationary state reached by the system as a result of violent relaxation deviates from the Lynden-Bell distribution because of incomplete relaxation, [20] [177–179].

24 When coupled to the Poisson equation, the Lynden-Bell distribution function yields a cluster with an infinite mass because it does not take into account the escape of high energy particles. An improved model with a finite mass, which can be derived from the generalized Landau equation, is provided by the fermionic King model [28] [29] [178].
In the completely degenerate limit (corresponding to a system at $T_{\text{eff}} = 0$), the Lynden-Bell distribution function is given by
\[ f(r, v) = \eta_0 H(v_{\text{LB}}(r) - v), \tag{B3} \]
where $H(\cdot)$ is the Heaviside step function and $v_{\text{LB}}(r)$ is the Lynden-Bell velocity (similar to the Fermi velocity in quantum mechanics). The density and the pressure are then given by
\[ \rho = \int f \, dv = \int_0^{v_{\text{LB}}} \eta_0 4\pi v^2 \, dv = \frac{4\pi}{3} \eta_0 v_{\text{LB}}^3, \tag{B4} \]
\[ P = \frac{1}{3} \int f \, dv = \frac{1}{3} \int_0^{v_{\text{LB}}} \eta_0 v^3 4\pi v^2 \, dv = \frac{4\pi}{15} \eta_0 v_{\text{LB}}^5, \tag{B5} \]
leading to the equation of state
\[ P = \frac{1}{5} \left( \frac{3}{4\pi \eta_0} \right)^{2/3} \rho^{5/3}. \tag{B6} \]
This is a polytropic equation of state of index $\gamma = 5/3$ ($n = 3/2$) like in the theory of white dwarf stars \cite{174}. It is easy to determine the new term in the generalized Schrödinger equation that leads to an equation of state of that form \cite{141}. We find
\[ i \frac{\partial \psi}{\partial t} = -\Delta \psi + \frac{1}{2} m \Phi \psi + \frac{1}{4} \left( \frac{3}{4\pi \eta_0} \right)^{2/3} |\psi|^{1/3} \psi \]
\[ + \frac{T_{\text{eff}}}{\eta_0} \ln |\psi| \psi - \frac{1}{2} \xi D \left[ \ln \left( \frac{\psi}{\psi^*} \right) - \left\langle \ln \left( \frac{\psi}{\psi^*} \right) \right\rangle \right] \psi. \tag{B7} \]
This generalized Schrödinger equation includes a power-law nonlinearity $|\psi|^{1/3} \psi$ that generalizes the one arising in the Gross-Pitaevskii equation. The corresponding hydrodynamic equations have the form of Eqs. \((39)-(42)\) with an equation of state
\[ P = \rho \frac{T_{\text{eff}}}{\eta_0} + \frac{1}{5} \left( \frac{3}{4\pi \eta_0} \right)^{2/3} \rho^{5/3}. \tag{B8} \]
Their equilibrium state describes DM halos with a solitonic core due to the quantumlike potential (arising from the nondifferentiability of spacetime), a polytropic core (similar to a fermion ball) of index $n = 3/2$ due to Lynden-Bell’s type of degeneracy, and an isothermal atmosphere due to violent relaxation or being a manifestation of the temperature of an aether (it is also possible to take into account the self-interaction of the particles as in Appendix A). The polytropic equation of state introduces an internal energy $U = (3/10) \left(3/4\pi \eta_0\right)^{2/3} \int \rho^{5/3} \, dr$ in the expression of the free energy (see Sec. III B and Ref. 111). This expression can be directly obtained from the relation $U = \frac{1}{2} \int \frac{f \, dv^2}{\eta_0} = \frac{3}{2} \int P \, dv$, where $P$ is the “quantum” pressure at $T_{\text{eff}} = 0$ given by Eq. \((B6)\). In other words, the internal energy $U$ corresponds to the kinetic energy of “microscopic” motions.

The hydrodynamic equations \((39)-(42)\) corresponding to the generalized Schrödinger equation \((B7)\) are similar to the hydrodynamic equations obtained in \cite{177} except that they include a quantum potential arising from the nondifferentiability of spacetime.\textsuperscript{26} As a result, they can be viewed as a generalization of the equations of violent relaxation \cite{177} in a fractal (nondifferentiable) spacetime. Alternatively, Eq. \((B7)\) can be viewed as a refinement of the generalized Schrödinger equation \((B3)\) taking into account the specificities of violent relaxation (Lynden-Bell’s exclusion principle).

In summary, the hydrodynamic equations \((39)-(42)\) with the equation of state \((B8)\) and with the quantum potential describe the process of violent relaxation in a nondifferentiable spacetime while the hydrodynamic equations \((39)-(42)\) with the equation of state \((B5)\) but without the quantum potential describe the process of violent relaxation in a differentiable spacetime.

Remark: In the case of BECDM where the particles are bosons and where the nondifferentiability of spacetime is due to quantum mechanics, using Eq. \((25)\), we find that Eq. \((B7)\) becomes
\[ i \hbar \frac{\partial \psi}{\partial t} = -\hbar^2 2m \Delta \psi + m \Phi \psi + \frac{m^2}{2} \left( \frac{3}{4\pi \eta_0} \right)^{2/3} |\psi|^{1/3} \psi \]
\[ + \frac{2m}{\eta_0} T_{\text{eff}} \ln |\psi| \psi - i \frac{\hbar}{2} \xi D \left[ \ln \left( \frac{\psi}{\psi^*} \right) - \left\langle \ln \left( \frac{\psi}{\psi^*} \right) \right\rangle \right] \psi. \tag{B9} \]
This equation generalizes the BECDM model by taking into account the process of violent relaxation. This description may also be valid if the particles are fermions at statistical equilibrium. In that case, the isothermal equation of state corresponds to the true thermodynamic temperature, the polytropic equation of state takes into account the Pauli exclusion principle, and the quantum potential takes into account the Heisenberg uncertainty principle. This leads to a generalized Schrödinger equa-

\textsuperscript{26} In the present approach, the Lynden-Bell equation of state $P_{\text{LB}}(\rho)$ appearing in the hydrodynamic equations of \cite{177} is replaced by the simpler equation of state \cite{158}. It is easy to determine the nonlinear term $h|\psi|^{2}$ in the generalized Schrödinger equation that exactly reproduces the Lynden-Bell equation of state $P_{\text{LB}}(\rho)$. It corresponds to the associated enthalpy $h_{\text{LB}}(\rho) = \int \left\\left[ P_{\text{LB}}(\rho'/\rho) \right\\right] \, d\rho'$ \cite{141}. However, since it does not have an analytical expression, we only consider here the simpler equation of state \cite{158}.
tion of the form \([141]\).\(^{27}\)

\[
\frac{i\hbar}{\partial t} \psi = -\frac{\hbar^2}{2m} \Delta \psi + m\Phi \psi + \frac{1}{2} \left( \frac{3}{8\pi} \right)^{2/3} \frac{(2\pi\hbar)^2}{m^{5/3}} |\psi|^{4/3} \psi \\
+ 2k_B T \ln |\psi| \psi - \frac{\hbar^2}{2} \left[ \ln \left( \frac{\psi}{\bar{\psi}} \right) - \left\langle \ln \left( \frac{\psi}{\bar{\psi}} \right) \right\rangle \right] \psi.
\] (B10)

The other relevant equations can be obtained from the previous ones by using the relation \(\rho_0 = 2m^4/(2\pi\hbar)^3\) \([174]\). The transition between the region dominated by the quantum pressure (due to the Pauli exclusion principle) and the region dominated by the thermal pressure corresponds to a density

\[
\rho_F \sim \frac{m^{5/2}(k_B T)^{3/2}}{\hbar^3}.
\] (B11)

The quantum pressure dominates for \(\rho \gg \rho_F\) and the thermal pressure dominates for \(\rho \ll \rho_F\).

**Appendix C: Effective temperature**

We have seen that large DM halos have an isothermal, or almost isothermal, atmosphere which is responsible for the flat, or almost flat, rotation curves of galaxies. The temperature \(T\) is related to the circular velocity at infinity \(v_\infty\) by the relation

\[
\frac{k_B T}{m} = \frac{v_\infty^2}{2}.
\] (C1)

For the Medium Spiral, \(v_\infty \sim 153\,\text{km/s}\). If we assume that \(T\) is the true thermodynamic temperature, then \(m\) represents the mass of the DM particle and Eq. (C1) determines the temperature of the DM particle.

If we assume that DM halos are self-gravitating BECs, then the boson mass must be of the order of \(m = 2.92 \times 10^{-22} \text{eV}/\text{c}^2\) in order to account for the mass and size of ultracompact dwarf halos at \(T = 0\) such as Fornax (see Sec. III D 1 and Appendix D of \([139]\)). In that case, we find from Eq. (C1) that the temperature of large halos such as the Medium Spiral is \(T \sim 4.41 \times 10^{-25} \text{K}\).\(^{28}\) Such a small temperature may not be physical.\(^{29}\) This strongly suggests that \(T\) is not the true thermodynamic temperature.\(^{30}\) It may rather represent an effective temperature. We have proposed two possible interpretations of this effective temperature:

(i) We have suggested that the quantum-like aspects of DM halos are not due to quantum mechanics but to the fractal structure of spacetime at the cosmic scale. In that case, DM halos are described by a generalized Schrödinger-like equation

\[
i\mathcal{D} \frac{\partial \psi}{\partial t} = -\mathcal{D}^2 \Delta \psi + \frac{1}{2} \Phi \psi + \frac{1}{2} V(t) \psi \\
+ \frac{\hbar^2}{2} \ln |\psi| \psi - \frac{1}{2} i\mathcal{D} \ln \left( \frac{\psi}{\bar{\psi}} \right) \psi,
\] (C2)

where neither the mass of the DM particle nor the temperature appear explicitly. The flat rotation curves of galaxies is due to the term \((v_\infty^2/2) \ln |\psi| \psi\) in the generalized Schrödinger-like equation (C2), where \(v_\infty\) is a coefficient of this equation. It could be interpreted as a sort of fundamental constant of physics, except that it depends on the scale as discussed in Sec. III D 2. In this interpretation, there is no ultra-small mass \(m\) nor ultra-small temperature \(T\) since such quantities do not explicitly appear in the equations. One can always define a temperature \(T\) by the relation \(v_\infty^2/2 = k_B T/m\) (where \(m\) is some mass scale) in order to develop a thermodynamical analogy, but this temperature is purely effective since only the ratio \(k_B T/m\) has a physical meaning. It could be interpreted as the temperature of the aether but the process of thermalization would be completely different than in thermodynamics.

(ii) We have suggested that the envelope of DM halos arises from a process of collisionless violent relaxation like in the theory of Lynden-Bell \([26]\). Such a process tends to establish an isothermal distribution justifying Eq. (C1). However, since this process is based on

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\(^{27}\) It may also be relevant to include the Dirac-Slater exchange correction arising from the identity of the fermions as in Sec. 5.2 of \([111]\).

\(^{28}\) Bosons with a repulsive self-interaction may have a much larger mass than noninteracting bosons, up to \(m = 1.10 \times 10^{-3} \text{eV}/\text{c}^2\) (see Appendix D of \([139]\)), leading to a temperature \(T \sim 1.66 \times 10^{-6} \text{K}\) (we have \(T \sim 1 \text{K}\) for \(m \sim 662 \text{eV}/\text{c}^2\)).

\(^{29}\) Actually, we can take the point of view that we can argue that the temperature of the aether \(T \sim 4.41 \times 10^{-25} \text{K}\) is physical but it is so small that it is undetectable in earth experiments.

\(^{30}\) However, if the mass of the particle is extraordinarily small, such as \(m = 2.92 \times 10^{-22} \text{eV}/\text{c}^2\), the ratio \(k_B T/m\) becomes large and can have observable consequences such as on the rotation curves of the galaxies in astrophysics. Similarly, the friction coefficient \(\xi\) with the aether is undetectable in earth experiments (the friction time \(\xi^{-1} \sim 1 \text{Myr}\) is extremely long) but it becomes important on astrophysical and cosmological timescales. We can also make the following remark. In the present point of view, the temperature of the aether is \(T \sim 4.41 \times 10^{-25} \text{K}\) at the scale \(\sim 10 \text{kpc}\) corresponding to the Medium Spiral while \(T = 0\) at the scale \(\sim 1 \text{kpc}\) corresponding to the ground state (Fornax) of the BECDM model (see Sec. III D 1). At smaller scales, \(T\) could become negative in order to recover the results of Bielinski-Birula & Mycielski \([191]\) (gaussons) at the microscale. This is discussed in more detail in Sec. 7 of \([111]\).
Vlasov equation, the mass of the DM particle should not appear in the equations. In other words, the temperature $T$ must be proportional to mass $m$. In Lynden-Bell’s theory, $k_B T/m$ is replaced by $T_{\text{eff}}/\eta_0$ where $T_{\text{eff}}$ is an effective temperature (see Appendix B). Therefore $v_\infty^2 = T_{\text{eff}}/\eta_0$. Again, in this interpretation, there is no ultra-small mass $m$ nor ultra-small temperature $T$ since such quantities do not explicitly appear in the equations.

**Remark:** If we assume that DM halos are made of fermions, like a sterile neutrino, then the fermion mass must be of the order of $m = 170$ eV/$c^2$ in order to account for the mass and size of ultracompact dwarf halos at $T = 0$ such as Fornax (see Appendix D of [139]). In that case, we find from Eq. (C1) that the temperature of large halos such as the Medium Spiral is $T \sim 0.257$ K. This temperature is physical (and there is no condensation temperature in the Fermi-Dirac statistics) suggesting that, if DM is made of fermions, $T$ may represent the true thermodynamic temperature.

**Appendix D: Generalized Einstein relation**

In this Appendix, we derive the time-independent generalized Schrödinger equation [50] from the method of quantization introduced by Schrödinger in his first paper on quantum mechanics [176]. In this paper, he derived the fundamental eigenvalue equation of quantum mechanics from a variational principle based on the classical Hamilton-Jacobi equation (see Appendix F of [139] for a short account of his approach). We use the same approach but take into account frictional effects.

The classical Hamilton-Jacobi equation with friction is

$$\frac{\partial \sigma}{\partial t} + \frac{(\nabla \sigma)^2}{2} + \Phi + \xi \sigma = 0, \quad \text{(D1)}$$

where $\sigma$ is the classical action which is related to the classical velocity by $u = \nabla \sigma$. Introducing the energy $\epsilon = -\partial \sigma/\partial t$, we get

$$\epsilon = \frac{(\nabla \sigma)^2}{2} + \Phi + \xi \sigma. \quad \text{(D2)}$$

Following Schrödinger’s approach, we introduce a real wave function $\phi(r)$ through the substitution

$$\sigma = 2D \ln \phi. \quad \text{(D3)}$$

Equation (D2) is then rewritten in terms of $\phi$ as

$$(\nabla \phi)^2 - \frac{1}{2D^2} (\epsilon - \Phi) \phi^2 + \frac{\xi}{D} (\ln \phi) \phi^2 = 0. \quad \text{(D4)}$$

Following again Schrödinger’s approach, we introduce the functional

$$J = \int \left\{ (\nabla ^2 \phi)^2 - \frac{1}{2D^2} (\epsilon - \Phi) \phi^2 + \frac{\xi}{D} (\ln \phi) \phi^2 \right\} \, dr \quad \text{(D5)}$$

and consider its minimization with respect to variations on $\phi$. The stationarity condition $\delta J = 0$ gives

$$-2D^2 \Delta \phi + \Phi \phi + 2\xi D (\ln \phi) \phi = \epsilon \phi, \quad \text{(D6)}$$

where we have redefined the energy as $\tilde{\epsilon} = \epsilon - \xi D$ for convenience. Comparing this equation with the time-independent generalized Schrödinger equation (50), we obtain the relation

$$D = \frac{k_B T}{m \xi}. \quad \text{(D7)}$$

In the case of quantum mechanics, using Eq. (25), it takes the form

$$\frac{\hbar}{2m} = \frac{k_B T}{m \xi} \quad \text{or} \quad \frac{\hbar}{2} = \frac{k_B T}{\xi}. \quad \text{(D8)}$$

This can be viewed as a sort of generalized Einstein relation expressing a form of fluctuation-dissipation theorem. This relation can also be obtained from the formalism of scale relativity by assuming that $\gamma_R = \gamma_I$ (see Appendix G of [151]). Therefore, in a sense, the variational approach of Schrödinger applied to the present situation suggests that

$$\gamma_R = \gamma_I. \quad \text{(D9)}$$

We do not claim, however, that this equality should always be valid.

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