TOPOLOGY OF THE ISO-SPECTRAL REAL MANIFOLDS ASSOCIATED WITH THE GENERALIZED TODA LATTICES ON SEMISIMPLE LIE ALGEBRAS

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Abstract. This paper concerns the topology of isospectral real manifolds of certain Jacobi elements associated with real split semisimple Lie algebras. The manifolds are related to the compactified level sets of the generalized (nonperiodic) Toda lattice equations defined on the semisimple Lie algebras. We then give a cellular decomposition and the associated chain complex of the manifold by introducing colored Dynkin diagrams which parametrize the cells in the decomposition. We also discuss the Morse chain complex of the manifold.

1. The generalized Toda lattice equations

Let \( g \) denote a real split semisimple Lie algebra of rank \( l \). We fix a split Cartan subalgebra \( h \) with root system \( \Delta \), real root vectors \( e_{\alpha_i} \) associated with simple roots \( \{ \alpha_i : i = 1, \ldots, l \} = \Pi \). We also denote \( \{ h_{\alpha_i}, e_{\pm \alpha_i} \} \) the Cartan-Chevalley basis of \( g \) which satisfies the relations,

\[
[h_{\alpha_i}, h_{\alpha_j}] = 0, \quad [h_{\alpha_i}, e_{\pm \alpha_j}] = \pm C_{j,i} e_{\pm \alpha_j}, \quad [e_{\alpha_i}, e_{-\alpha_j}] = \delta_{i,j} h_{\alpha_j},
\]

(1.1)

where the \( l \times l \) matrix \( (C_{i,j}) \) is the Cartan matrix corresponding to \( g \), and \( C_{i,j} = \langle \alpha_i, h_{\alpha_j} \rangle = \langle \alpha_i, h_{\alpha_j} \rangle \).

Then the generalized Toda lattice equation related to real split semisimple Lie algebra is defined by the following system of 2nd order differential equations for the real variables \( \{ f_j(t) : j = 1, \ldots, l \} \),

\[
\frac{d^2 f_i}{dt^2} = \epsilon_i \exp(-\langle \alpha_i, f \rangle),
\]

(1.2)

where \( f = \sum_{j=1}^{l} f_j(t) h_{\alpha_j} \in h \) and \( \epsilon_i \in \{ \pm 1 \} \).

Remark 1.1. The case with \( g = \mathfrak{sl}(l+1, \mathbb{R}) \) corresponds to the indefinite Toda lattice introduced in [7]. The main feature of the indefinite Toda equation having at least one of \( \epsilon_i \) being \( -1 \) is that the solution blows up to infinity in finite time [7]. Having introduced the signs, the group corresponding to the Toda lattice is a real split Lie group \( \tilde{G} \) with Lie algebra \( g \). For example, in the case of \( g = \mathfrak{sl}(n, \mathbb{R}) \), if \( n \) is odd, \( \tilde{G} = SL(n, \mathbb{R}) \), and if \( n \) is even, \( \tilde{G} = Ad(SL(n, \mathbb{R})^\pm) \).

Remark 1.2. If we consider the complex Toda equation, \( \epsilon_i \) in (1.2) can be absorbed in \( f_i \in \mathbb{C} \), so that the present study deals with the disconnected Cartan subgroup, where the generalized Toda lattice defines a flow in each connected component.

Remark 1.3. The original Toda lattice in [12] is obtained as the case with all \( \epsilon_i = 1 \) where the position of the \( i \)-th particle is given by \( q_i = f_i - f_{i+1} \) for \( i = 1, \ldots, l \),

\[
\frac{d^2 q_i}{dt^2} = \exp(q_{i-1} - q_i) - \exp(q_i - q_{i+1}),
\]

(1.3)

where \( f_{l+1} = 0 \) and \( f_0 = f_{l+2} = -\infty \) indicating \( q_0 = -\infty \) and \( q_{l+1} = \infty \).

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1.1. Lax formulation: isospectral manifold $\mathcal{Z}(\gamma)_{\mathbb{R}}$. The system (1.2) can be written in a Lax equation which describes an iso-spectral deformation of a Jacobi element of $\mathfrak{g}$. Define the set of real functions $\{(a_i(t), b_i(t)) : i = 1, \ldots, l\}$, 

$$a_i(t) = \frac{d}{dt} f_i(t), \quad b_i(t) = \epsilon_i \exp(-\langle \alpha_i, f \rangle).$$

Then the Toda equation (1.2) can be written in the Lax form [4, 9],

$$\frac{dX}{dt} = [P, X]$$

where the Lax pair $(X, P)$ are defined by

$$\begin{align*}
X(t) &= \sum_{i=1}^{l} a_i(t) h_{\alpha_i} + \sum_{i=1}^{l} (b_i(t)e_{-\alpha_i} + e_{\alpha_i}) \\
P(t) &= -\sum_{i=1}^{l} b_i(t) e_{-\alpha_i}
\end{align*}$$

The Lax form (1.3) represents an isospectral deformation of the Jacobi element $X$.

We denote the disconnected manifold given by the set of the elements in the form $X$ of $\mathfrak{g}$,

$$\mathcal{Z}_{\mathbb{R}} = \left\{ X = x + \sum_{i=1}^{l} (e_{\alpha_i} + b_i e_{-\alpha_i}) \in \mathfrak{g} : x \in \mathfrak{h}, \ b_i \in \mathbb{R}^* \right\} = \bigcup_{\epsilon \in \mathcal{E}} \mathcal{Z}_\epsilon,$$

where $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$, and the connected component $\mathcal{Z}_\epsilon$ is given by

$$\mathcal{Z}_\epsilon = \{ X \in \mathcal{Z}_{\mathbb{R}} : \epsilon = (\epsilon_1, \ldots, \epsilon_l) \in \mathcal{E}, \ \text{sign}(b_i) = \epsilon_i \}. \quad (1.8)$$

Here $\mathcal{E}$ is the set of all the signs $\epsilon$, so the set $\mathcal{Z}_{\mathbb{R}}$ is the disjoint union of the $2^l$ connected components.

A real isospectral leaf in $\mathcal{Z}_{\mathbb{R}}$ is defined by the level sets of the Chevalley invariants, denoted as $(I_1, \ldots, I_l)$, which are the polynomials of the variables $(a_i, b_i)$. The invariants then define a differentiable map,

$$\mathcal{I} : \mathcal{Z}_{\mathbb{R}} \longrightarrow \mathbb{R}^l$$

$$X \mapsto \gamma = (I_1, \ldots, I_l) \quad (1.9)$$

The real isospectral leaf $\mathcal{Z}(\gamma)_{\mathbb{R}}$ is then given by

$$\mathcal{Z}(\gamma)_{\mathbb{R}} = \mathcal{I}^{-1}(\gamma) \cap \mathcal{Z}_{\mathbb{R}},$$

Our main purpose in this paper is to give a detailed structure of the compactified manifold $\hat{\mathcal{Z}}(\gamma)_{\mathbb{R}}$ from the viewpoint of the Lie group theory.

**Remark 1.4.** The Chevalley invariants provide $l$-involutive integrals for the generalized Toda lattice equation, so that this proves the integrability of the equation in the Liouville-Arnold sense.

**Remark 1.5.** The construction of the compactified manifold $\hat{\mathcal{Z}}(\gamma)_{\mathbb{R}}$ for the case of $\mathfrak{sl}(l+1, \mathbb{R})$ was given in [3] based on the explicit solution structure in terms of the $\tau$-functions, which provide a local coordinate system for the manifold. By tracing the solution orbit of the indefinite Toda equation, the disconnected components in $\mathcal{Z}(\gamma)_{\mathbb{R}}$ are all glued together to make a smooth compact manifold. The result is maybe well explained by Figure [1] for the case of $A_2 \cong \mathfrak{sl}(3, \mathbb{R})$. In the figure, the Toda orbits are shown as the dotted lines, and each region labeled by the same signs in $(\epsilon_1, \epsilon_2)$ with $\epsilon_i \in \{\pm\}$ are glued together through the boundary (the wavy-lines) of the hexagon. At a point of the boundary the Toda orbit blows up in finite time, but the orbit can be uniquely traced to the one in the next region (marked by the same letter $A, B$ or $C$). Also the flows on the solid lines show the solutions of the subsystems (i.e. either $b_1 = 0$ or $b_2 = 0$). Then the compactified manifold $\hat{\mathcal{Z}}(\gamma)_{\mathbb{R}}$ by adding the blow-up points (the wavy-lines) and the
flows of the subsystems (the solid-lines) to \( Z(\gamma)_R \) in this case is shown to be isomorphic to the connected sum of two Klein bottles, that is, the integral homology \( H_k(Z(\gamma)_R, \mathbb{Z}) \) is given by \( H_0 = \mathbb{Z}, H_1 = \mathbb{Z}^3 \oplus \mathbb{Z}_2, \) and \( H_2 = 0. \) In the case of \( \mathfrak{sl}(n, \mathbb{R}) \) for \( n \geq 3, \) \( Z(\gamma)_R \) is shown to be nonorientable and the symmetry group is the semi-direct product of \((\mathbb{Z}_2)^n\) and the Weyl group \( W = S_n, \) the permutation group. One should compare this with the result of Tomei [13] where the compact manifolds are associated with the definite (original) Toda lattice equation and the compactification is done by adding only the subsystems. (Also see [3] for some topological aspects of the manifolds which are identified as permutohedrons.)

1.2. Leznov-Saveliev formulation: Cartan subgroup \( H_R. \) In the zero curvature formulation in [10], the generalized Toda lattice equation (1.2) with the sign \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_l) \) can be expressed as an orbit on the connected component \( H_\varepsilon \) of the Cartan subgroup \( H_R, \)

\[
H_R = \bigcup_{\varepsilon \in \mathcal{E}} H_\varepsilon
\]

(1.11)

where \( H_{(1,\ldots,1)} := H = \exp h, \) the connected component with the identity. Thus the set \( H_R \) consists of \( 2^l \) connected components. Let \( g_\varepsilon \) be an element of \( H_\varepsilon \) given by

\[
g_\varepsilon = h_\varepsilon \exp f
\]

(1.12)

which can be also considered as a map from \( Z_R \) to \( H_R. \) Here the element \( h_\varepsilon \in H_\varepsilon \) satisfies \( \chi_{\alpha_i}(h_\varepsilon) = \varepsilon_i \) with the group character \( \chi_{\phi} \) determined by a root \( \phi \in \Delta, \) and each connected component of \( H_R \) can be written as \( H_\varepsilon = h_\varepsilon H. \) Then the Toda lattice (1.2) is written as an evolution of \( g_\varepsilon(t), \)

\[
\frac{d}{dt} g_\varepsilon^{-1} \frac{d}{dt} g_\varepsilon = \left[ g_\varepsilon^{-1} e_+ g_\varepsilon, e_- \right],
\]

(1.13)

where \( e_\pm \) are fixed elements in the simple root spaces \( \mathfrak{g}_\pm \) so that all the elements in \( \mathfrak{g}_\pm \) can be generated by \( e_\pm, \) i.e. \( \mathfrak{g}_\pm = \{ \text{Ad}_h(e_\pm) : h \in H \}. \) In particular, we take

\[
e_\pm = \sum_{i=1}^{n} e_{\pm \alpha_i}.
\]

(1.14)

With the group character \( \chi_{\alpha_i}, \) the solution \( b_i(t) \) of the Toda lattice is given by

\[
b_i(t) = [\chi_{\alpha_i}(g_\varepsilon)]^{-1} = \chi_{-\alpha_i}(g_\varepsilon).
\]

(1.15)
Remark 1.6. With the fundamental weights $\omega_i$ defined as $\langle \omega_i, h_{a_j} \rangle = \delta_{ij}$, i.e. $\alpha_i = \sum_{j=1}^l C_{i,j} \omega_j$, we can write the solution

$$b_i = \prod_{j=1}^l \left[ \chi_{\omega_j}(g_e) \right]^{-C_{i,j}},$$

which is the well-known $\tau$-function representation of the solution with $\tau_i(t) := \chi_{\omega_i}(g_e)$.

Remark 1.7. In the compactification of the disconnected Cartan subgroup $H_R$, we need to add pieces corresponding to the blow-ups ($|b_i| = \infty$) and the subsystems ($b_i = 0$). The subsystems are determined by the subset $A = \{ \alpha_i \in \Pi : b_i \neq 0 \}$, and the corresponding Cartan subgroup, denoted by $H^A_R$, may be defined as

$$H^A_R = \bigcup_{\epsilon \in E^A} h_\epsilon H^A,$$

where the set $E^A \subset E$ and the Cartan subgroup $H^A$ are defined by

$$E^A = \{ (\epsilon_1, \ldots, \epsilon_l) \in E : \epsilon_i = 1 \text{ if } \alpha_i \in A \}$$

$$H^A = \exp \mathfrak{h}^A \quad \text{with} \quad \mathfrak{h}^A = \text{Span}_R \{ h_{\alpha_i} \in \mathfrak{h} : \alpha_i \notin A \}.$$

Then the subsystems are also expressed as the same form of (1.13) with $g^A_e \in H^A$,

$$g^A_e = h_\epsilon \exp \left( \sum_{\alpha_i \notin A} f_i(t) h_{\alpha_i} \right).$$

The corresponding Lax pair $(X^A, P^A)$ is given by

$$X^A = \sum_{\alpha_i \in \Pi} a_i h_{\alpha_i} + \sum_{\alpha_i \notin A} b_i e_{-\alpha_i} + e_+,$$

$$P^A = -\sum_{\alpha_i \notin A} b_i e_{-\alpha_i},$$

which is just the Lax pair (1.6) with $b_i = 0$ for $\alpha_i \in A$. Note here that $a_i(t) = \text{constant if } b_i = 0$. We also consider that the dimension of the manifold $H^A_R$ is $l - |A|$, the number of parameters $f_i$.

Remark 1.8. The compactification of the isospectral manifold $Z(\gamma)_R$ for a fixed $\gamma \in \mathbb{R}^l$ can be obtained by sending it to the flag manifold $\tilde{G}/B_+$ with the Borel subgroup $B_+$ of $\tilde{G}$, so that the compactified manifold $\tilde{Z}(\gamma)_R$ is a toric variety $\mathbb{H}_R x B_+$ with a generic element $x \in \tilde{G}$. Then we can show:

Theorem 1.1. The isospectral manifold $\tilde{Z}(\gamma)_R$ is a smooth compact manifold diffeomorphic to $\tilde{H}_R$.

The complex version of this theorem is given in [3], and the proof of the present case is essentially given in the same manner (the detail of the proof is given in [3]).

In the following two sections, we will describe the structure of $\tilde{H}_R$ using the Weyl action on the manifold. This is a brief summary of a preprint [4], and the proofs of the results (Proposition 2.1, Theorem 3.1 and 3.2) can be found there. Then in the section 4, we will present the Morse theory to compute the integral homology of the manifold $\tilde{H}_R$.

2. The structure of $\tilde{H}_R$ as the union of the subsystems

As was shown in the previous section, the set $H_R$ can be parametrized by the group characters $\chi_{\alpha_i}$, that is, $H_R = \cup_{\epsilon \in E} H_\epsilon$ with $H_\epsilon = h_\epsilon H$,

$$H_\epsilon = \{ h \in H_R : \text{sign}(\chi_{\alpha_i}(h)) = \chi_{\alpha_i}(h_\epsilon) = \epsilon_i, \text{ for } i = 1, \ldots, l \}.$$ (2.1)

Note that each $H_\epsilon$ is diffeomorphic to $\mathbb{R}^l$. 
Since the Weyl group acts on $H_{\mathbb{R}}$, one can partition $H_{\mathbb{R}}$ into the $|W|$ convex cones of the Weyl chambers. We denote the cone in the antidominant chamber as

$$H_{\mathbb{R}} = \bigcup_{\epsilon \in \mathcal{E}} H_{\epsilon}^-$$

where the connected component $H_{\epsilon}^-$ is defined by

$$H_{\epsilon}^- = \{ h \in H_{\mathbb{R}} : |\chi_{-\alpha_i}(h)| \leq 1, \ sign(\chi_{\alpha_i}(h)) = \epsilon_i \}$$

The boundaries of the chamber $H_{\epsilon}^-$ corresponding to $\chi_{\alpha_i}(h) = 1$ and $\chi_{\alpha_i}(h) = -1$ are called the positive and negative $\alpha_i$-walls, and especially the positive $\alpha_i$-wall gives the hyperplane of the Weyl-reflection with respect to the root $\alpha_i$. Then the connected component $H_{\epsilon}^-$ of $H_{\epsilon}^-$ is expressed as the union of $W$-translations of $H_{\epsilon}^-$, i.e.

$$H_{\epsilon}^- = \bigcup_{w \in W} w \left( H_{\epsilon(w)}^- \right).$$

Here the $W$-action on $H_{\epsilon}^-$ is obtained through the action on the group characters,

$$s_{\alpha_i}(\chi_{\alpha_j}) = \chi_{\alpha_i} \chi_{\alpha_j}^{-C_{j,i}},$$

from which we also have the $W$-action on the set $\mathcal{E}$ as $s_{\alpha_i} : \epsilon_j \mapsto \epsilon_j'$ with $\epsilon_j = \chi_{\alpha_j}(h_{\epsilon})$, $\epsilon_j' = \epsilon_j - C_{j,i}$,

$$w_{\epsilon_j}(\epsilon_j') = \epsilon_j,$$

where we have used the Weyl-reflection, $s_{\alpha_i} \alpha_j = \alpha_j - C_{j,i} \alpha_i$. Thus the element $h_{\epsilon}$ is $W$-translated to $h_{\epsilon'}$ with the sign $\epsilon' = (\epsilon_1', \ldots, \epsilon_l')$ given by (2.6), which we denote $\epsilon' = w(\epsilon)$ with $w = s_{\alpha_i}$-Weyl reflection. Thus, with the decomposition (2.4), we can consider only the antidominant chamber $H_{\epsilon}^-$, and obtain the whole $H_{\mathbb{R}}$ by the $W$-translates. This is also true for the compactified manifold $\hat{H}_{\mathbb{R}}$.

Let us first make the closure of $H_{\epsilon}^-$ by adding the pieces corresponding to the subsystems having the lower dimensions $l - |A|$ where $A \subset \Pi$ determines the subsystem (see Remark 1.7). We let

$$H_{\epsilon}^{A,-} = \left\{ h \in H_{\epsilon}^- : |\chi_{-\alpha_i}(h)| \leq 1, \ \epsilon = (\epsilon_{i_1}, \ldots, \epsilon_{i_m}) \text{ for } \alpha_{i_j} \notin A \right\}$$

Then the closure of the set $H_{\epsilon}^-$ can be obtained by

$$\overline{H_{\epsilon}^-} = \bigcup_{A \subset \Pi} H_{\epsilon}^{A,-},$$

and the compactified manifold $\hat{H}_{\mathbb{R}}$ is given by the $W$-translates of (2.8), i.e.

$$\hat{H}_{\mathbb{R}} = \bigcup_{\epsilon \in \mathcal{E}} \bigcup_{w \in W} w \left( \overline{H_{\epsilon(w)}^-} \right).$$

We summarize the result as:

**Proposition 2.1.** The closed set $\overline{H_{\epsilon}^-}$ is isomorphic to the box $\{(t_1, \ldots, t_l) \in \mathbb{R}^l : -1 \leq t_j \leq 1\}$, and the manifold $\hat{H}_{\mathbb{R}}$ is compact and has an action of the Weyl group $W$.

3. Topology of $\hat{H}_{\mathbb{R}}$

3.1. Colored Dynkin diagrams. We here give a cellular decomposition and construct the associated chain complex of the compactified manifold $\hat{H}_{\mathbb{R}}$. We first introduce the set of colored Dynkin diagrams to parametrize the cells in the decomposition. A colored Dynkin diagram is simply a Dynkin diagram in which some of the vertices have been colored either red ($R$) or blue ($B$). For examples, in the case of $A_2 \cong \mathfrak{sl}(3, \mathbb{R})$, we have $\alpha_R - \alpha, \alpha_B - \alpha_R$, etc. Thus a colored Dynkin diagram $D$ corresponds to a pair $(S, \eta)$ with $S \subset \Pi$ and $\eta : S \to \{ \pm 1 \}$, where $\eta(\alpha_i) = -1$ if $\alpha_i$ is colored $R$, and $\eta(\alpha_i) = 1$ if $\alpha_i$ is colored $B$. We denote the set of colored Dynkin diagrams as

$$\mathcal{D}(S) = \{ D = (S, \eta) : S \subset \Pi, \ \eta(\alpha) \in \{ \pm 1 \} \text{ for } \alpha \in S \}.$$
Let $W_S$ be the group generated by the simple reflections corresponding to the roots in $S$. We then define the $W_S$-action on the set $\mathbb{D}(S)$ as follows: For any $\alpha_i \in S$, $s_{\alpha_i}D = D'$ is a new colored Dynkin diagram having the colors corresponding to the sign change $\epsilon_j' = \epsilon_j \epsilon_i^{-c_{j,i}}$ in (2.6) with the identification that $R$ if the sign is $-1$, and $B$ if it is $+1$. For example, in the case of $A_2$, we have $s_{\alpha_1}(\alpha_R - \alpha_B) = \alpha_R - \alpha_R, s_{\alpha_1}(\alpha_B - \alpha_B) = \alpha_B - \alpha_R$.

The $W_S$-action induces the $W$-translates on the set $\mathbb{D}(S)$ as $W \times W_S \mathbb{D}(S)$. The elements of this set are given by pairs $(w, D)$, and among the elements we have an equivalence relation $\sim$, that is, $(w, D) \sim (w, xD)$ for any $x \in W_S$. The equivalent relation gives a bijective correspondence between $W \times W_S \mathbb{D}(S)$ and $\mathbb{D}(S) \times W/W_S$. We then define the set $\mathbb{D}^k$ as

$$\mathbb{D}^k := \{(D, [w]_{\Pi - S}) : D \in \mathbb{D}(S), [w]_{\Pi - S} \in W/W_S, |S| = k\},$$

which parametrizes all the connected components of the Cartan subgroups of the form $H^H_{\mathbb{R}}$ corresponding to the subsystems defined in Remark 1.7. In this parametrization, $\mathbb{D}^k$ corresponds explicitly to the dual of the set $H^H_{\mathbb{R}}$, so that the parametrized cell has the codimension $k$, and $\dim H^H_{\mathbb{R}} = k$. Thus all the cells in $H_{\mathbb{R}}$ can be parametrized by the sets $\mathbb{D}^k$, and we have:

**Theorem 3.1.** The collection of the sets $\mathbb{D}^k$ defined (3.2) gives a cell decomposition of the compact manifold $\hat{H}_{\mathbb{R}}$.

**Remark 3.1** There is a more convenient cell decomposition of $\hat{H}_{\mathbb{R}}$ for the purpose of calculating homology explicitly. The only change is that the $l$ dimensional cell becomes the union of all the $l$-cells together with all the (internal) boundaries corresponding to colored Dynkin diagrams where all the colored vertices are colored $B$. This is the set,

$$\hat{H}_{\mathbb{R}} \setminus \bigcup_{\eta(\alpha_i) = -1 \text{ for some } \alpha_i \in S} \eta^S (S, \eta, [w]_{\Pi - S}).$$

This set can be seen to be homeomorphic to $\mathbb{R}^I$. With this cell decomposition there is exactly one $l$-cell, and the other lower dimensional cells correspond to colored Dynkin diagrams $D$, in which at least one vertex of $D$ has been colored $R$.

**Example 3.1.** In the case of $A_2$, we have:

1. For $k = 2$, i.e. $S = \Pi$, we have 4 vertices (0-cell) parametrized by the elements of $\mathbb{D}^2$, which correspond to the 4 connected components of $H_{\mathbb{R}}$ as dual cells,

$$(\circ_B - \circ_B, [e]), (\circ_B - \circ_R, [e]), (\circ_R - \circ_B, [e]), (\circ_R - \circ_R, [e]),$$

where $[e] = [e]_0 = [w]_0$ for any $w \in W$.

2. For $k = 1$, if $S = \{\alpha_1\}$, we have 6 1-cells parametrized by,

$$(\circ_B - \circ, [w]_{\{\alpha_2\}}), (\circ_R - \circ, [w]_{\{\alpha_2\}}),$$

where $W/W_S = \{e, s_{\alpha_2}, s_{\alpha_1}, s_{\alpha_2}\}$, and if $S = \{\alpha_2\}$, we have also 6 1-cells,

$$(\circ - \circ_B, [w]_{\{\alpha_1\}}), (\circ - \circ_R, [w]_{\{\alpha_1\}}),$$

where $W/W_S = \{e, s_{\alpha_1}, s_{\alpha_2} s_{\alpha_1}\}$. Those colored Dynkin diagrams with $w = e$ correspond to the 4 walls (2-positive and 2-negative walls) of the antidominant chamber $H^H_{\mathbb{R}}$, which is isomorphic to a square.

3. For $k = 0$, i.e. $S = \emptyset$, we have 6 $|W|$ 2-cells of the convex cones corresponding to the Weyl chambers parametrized by

$$(\circ - \circ, w)$$

for $w \in W$. Those are dual to the 6-vertices corresponding to $H^H_{\mathbb{R}}$. As mentioned in Remark 3.1, we have a simpler cell decomposition. Namely, the union of all colored Dynkin diagrams having no $R$-colored vertices forms the unique 2-cell which is the set of internal points of the hexagon homeomorphic to $\mathbb{R}^I$, and all other cells consists of the boundary of the hexagon.
Figure 3.1 illustrates the example. One should note that the full parametrization of the cells are obtained by the $W$-translates of $(D, [e]_{|n-S})$ corresponding to the subsystems in the antidominant chamber $\hat{H}_{\mathbb{R}}^{\Pi-S,-}$ for all the choices of $S \subset \Pi$.

3.2. **Boundary maps and chain complex.** With these parametrizations of the cells, we now define the boundary maps on the colored Dynkin diagrams. Let $(j, c)$ be a pair of integers with $j = 1, \ldots, m$ and $c = 1, 2$. Then we define the $(j, c)$-boundary, denoted as $\partial_{j,c}D$, of a colored Dynkin diagram $D$ with the set of uncolored vertices $\{\alpha_{i_j} : 1 \leq i_1, \ldots, i_m \leq l\}$ as a new colored Dynkin diagram by coloring the $i_j$-th vertex with $R$ if $c = 1$ and with $B$ if $c = 2$. Recall that a colored Dynkin diagram $D$ corresponds to a pair $(S, \eta)$ with $S \subset \Pi$ and $\eta : S \to \{\pm 1\}$. Thus the boundary operator $\partial_{j,c}$ determines a new pair $(S, \eta')$ with $\eta' : S \cup \{\alpha_{i_j}\} \to \{\mp 1\}$. Giving an orientation on the boundary, we define the map $\tilde{\partial}_{j,c} := (-1)^{i_j+c+1}\partial_{j,c}$ which gives a map on the $\mathbb{Z}$-modules,

$$\tilde{\partial}_{j,c} : \mathbb{Z}[D(S)] \to \mathbb{Z}[D(S \cup \{\alpha_{i_j}\})].$$

Let us now consider the $\mathbb{Z}$-modules of the full set of colored Dynkin diagrams, $\mathbb{D}^k = \mathbb{D}(S) \times W/WS||S|=k$. We denote the module as

$$\mathcal{M}(S) = \mathbb{D}[W] \otimes_{\mathbb{Z}[WS]} \mathbb{Z}[\mathbb{D}(S)],$$

so that the $k$-chain is given by the direct sum of all these modules over all sets with $|S| = l - k$,

$$\mathcal{M}_k = \bigoplus_{|S|=l-k} \mathcal{M}(S).$$

The boundary map $\partial_k : \mathcal{M}_k \to \mathcal{M}_{k-1}$ can be defined by

$$\partial_k (D, [w]_{|n-S}) = \sum_{1 \leq j \leq k \atop c_1, c_2} \left( \tilde{\partial}_{j,c}D, [w]_{|n-S \cup \{\alpha_{i_j}\}} \right),$$

where $\Pi - S = \{\alpha_{i_j} : 1 \leq j \leq k\}$. The condition for the boundary map, $\partial_k \circ \partial_{k+1} = 0$, is then easily verified, and we have:

**Theorem 3.2.** The map $\partial_k$ of (3.4) defines a chain complex $\mathcal{M}_*$,

$$0 \to \mathcal{M}_l \xrightarrow{\partial_1} \mathcal{M}_{l-1} \xrightarrow{\partial_2} \cdots \xrightarrow{\partial_2} \mathcal{M}_1 \xrightarrow{\partial_1} \mathcal{M}_0 \to 0,$$
where $\mathcal{M}_k$ is defined by (3.3).

Since we have an explicit representation of the $k$-chains as (3.5) with (3.4), the integral homology $H_k(\hat{H}_R; \mathbb{Z}) = \text{Ker} \partial_k / \text{Im} \partial_{k+1}$ can be also computed. However an explicit formula may be too complicated.

4. Morse theory and Homology

4.1. Morse theory. The generalized Toda equation can be expressed as a gradient flow on the adjoint orbit of $\mathfrak{g}[1]$. Here we consider a Morse decomposition of the manifold $\hat{H}_R$ based on the gradient structure of the Toda flow. Each critical point of the Toda vector field can be parametrized by a unique element of the Weyl group $W$. Then we define the unstable and stable Weyl subgroups for $a \in W$ as

$$W^u(a) = W_{\Pi^u_a}, \quad \Pi^u_a := \{ \alpha_i \in \Pi : \ell(a s_{\alpha_i}) > \ell(a) \}$$

$$W^s(a) = W_{\Pi^s_a}, \quad \Pi^s_a := \{ \alpha_i \in \Pi : \ell(a s_{\alpha_i}) < \ell(a) \}$$

(4.1)

where $\ell(a)$ denotes the length of $a$. We will use the same notation for the unstable and stable manifolds generated by the Toda vector field corresponding to the critical point $a$. Thus, depending on the context, $W^u(a)$, $W^s(a)$ denotes either a subgroup of $W$ or a submanifold of $\hat{H}_R$. We also introduce labels in the Dynkin diagram to characterize the critical point by assigning "0" in the $i$-th place in the diagram if $s_{\alpha_i} \in W^s(a)$, and "*" if $s_{\alpha_i} \in W^u(a)$. For example, in the case of $\mathfrak{g} = \mathfrak{sl}(6 : \mathbb{R})$, the element [2143] $\in W$ is labeled as (0 * 0 * *), where [2143] := $s_{\alpha_2} s_{\alpha_1} s_{\alpha_4} s_{\alpha_3}$. The $W^u([2143])$ is then the subgroup generated by $\{s_{\alpha_2}, s_{\alpha_1}, s_{\alpha_4}\}$ and is diffeomorphic to $\mathbb{R}^3$. In terms of handle body, the critical point [2143] is identified as the product $D^2 \times D^3$ where $D^n$ is the $n$-dimensional disc.

With this identification, we have the Morse decomposition of the manifold $\hat{H}_R$,

$$\hat{H}_R = \bigcup_{a \in W} W^u(a).$$

(4.2)

The index of the critical point $a \in W$ is defined as

$$\text{Ind}(a) := \text{dim} W^u(a) = |\Pi^u_a|,$$

(4.3)

which is also given by the number of *’s in the labeled Dynkin diagram, e.g. for $\mathfrak{sl}(6 : \mathbb{R})$, $\text{Ind}([2143]) = 3$.

The Toda flow defines a (directed) graph which provides 1-dimensional connections among the critical points corresponding to the 1-dimensional flow of a $\mathfrak{sl}(2, \mathbb{R})$ subsystem. We call the graph Toda graph and is defined by

**Definition 4.1:** Toda graph. A directed graph is called Toda graph if each vertex defined by $\langle a \rangle = a^{-1} \langle e \rangle$ with $a \in W$ has the connections to another vertex $\langle b_i \rangle$ by

$$b_i = a s_{\alpha_i}, \quad \text{for } i = 1, \ldots, l.$$

The direction in the connection between two vertices $a$ and $b$ is defined by

$$a \rightarrow b, \quad \text{if } \ell(a) < \ell(b).$$

In order to construct a Morse complex, a vector field on the manifold must satisfy the Morse-Smale condition, that is, the intersection between (the manifolds) $W^u(a)$ and $W^s(b)$ for the critical points $a$ and $b$ must be transversal. However the corresponding intersections in the case of the Toda lattice are, in general, NOT transversal. We have:

**Definition 4.2:** Transversal connection (algebraic version). A connection $a \rightarrow b$ is transversal if

1. $|\Pi^u_a \cap \Pi^s_b| = \text{Ind}(a) - \text{Ind}(b)$,
2. \((W^u(a), W^s(b)) = W\),
3. \(|aW^u(a) \cap bW^s(b)| = |W_{\Pi^u \cap \Pi^s}|\).

This definition is motivated by:

**Theorem 4.1.** For the Toda lattice vector field, each CLOSURE, \(\overline{W^u}(a), (\overline{W^s}(a))\) \(a \in W\) of the unstable (stable) manifold respectively, is smooth. Moreover each smooth manifold \(\overline{W^u}(a)\) \((\overline{W^s}(a))\) is orientable and produces a cycle if and only if the subgroup \(W^u(a)\) \((W^s(a))\) is abelian. A connection \(a \rightarrow b\) is transversal (Definition (4.2)) if and only if \(W^u(a), W^s(b)\) intersect transversally. The intersection is diffeomorphic to a circle.

The proof of the theorem can be obtained from the methods developed in [2], and the detail will be given elsewhere.

We call a graph with vertices given by \(W\) and oriented edges \(a \rightarrow b\) satisfying the (algebraic) transversality conditions above with \(\text{Ind}(a) = \text{Ind}(b) + 1\), a Morse-Smale graph, if in addition, i) there is a perturbation of the Toda lattice which is Morse-Smale and has the same set of critical points \((W)\), ii) \(a \rightarrow b\) only if the manifolds \(W^u(a)\) and \(W^s(b)\) for this new vector field intersect transversally. A Morse-Smale vector field can be obtained by a small smooth perturbation of the Toda lattice as in [1]. We have confirmed that the conditions 1 through 3 in Definition 4.2 are sufficient to determine uniquely a Morse-Smale graph in the cases of \(\mathfrak{g} \cong A_l\) up to \(l = 3\). However this may not be true in general.

We now define a boundary map on the chain \(C_s\) of the cells of unstable Weyl groups \(W^u(a)\), i.e.

\[
C_s = \bigoplus_{k=0}^{l} C_k, \quad C_k = \sum_{\text{Ind}(a) = k} \mathbb{Z}(a).
\] (4.4)

where \(\langle a \rangle\) is the cell corresponding to \(W^u(a)\). The chain \(C_k\) is the set of all cells \(\langle a \rangle\) with the labeled Dynkin diagram having \(k\) number of \(*\)'s. The boundary map \(\partial_k : C_k \rightarrow C_{k-1}\) is then defined by

\[
\partial_k : \langle a \rangle \mapsto \partial_k \langle a \rangle = \sum_{\text{Ind}(b) = k-1} [a; b] \langle b \rangle,
\] (4.5)

where all the connections \(a \rightarrow b\) are edges in the Morse-Smale graph, and the incidence number \([a; b]\) is given by

\[
[a; b] = \left(1 + (-1)^{\sigma[a; b]}\right) (-1)^{\ell(a^{-1}b) + i}
\] (4.6)

with

\[
\sigma[a; b] = \left| \{ j : \epsilon_j \rightarrow \epsilon'_j < 0, \alpha_j \in \Pi^u_a \} \right|,
\]

where the index \(i\) is given by \(\{s_{\alpha_i}\} = \Pi^u_a \cap \Pi^s_b\); i.e. the Dynkin diagram corresponding to \(\langle b \rangle\) has 0 in the \(i\)-th place in addition to the 0’s in \(\langle a \rangle\). The sign change \(\epsilon_i \rightarrow \epsilon'_i\) under the connection \(a \rightarrow b\) is defined as follows:

\[
a^{-1}b \cdot (\epsilon_1, \ldots, \epsilon_i) = (\epsilon'_1, \ldots, \epsilon'_i)
\] (4.7)

where the initial signs \(\epsilon_i\)'s are taken as

\[
\epsilon_j = \left\{ \begin{array}{ll} + & \text{if } \alpha_j \in \Pi^s_a \cup \Pi^u_b, \\ - & \text{if } \alpha_j \in \Pi^s_a \cap \Pi^u_b \end{array} \right.
\]

and \(\epsilon'_j\) is defined as \(2\epsilon_j\). With this definition, we can determine the change of orientations of the hypersurfaces for \(b_i > 0\) and \(b_i < 0\) parallel to the surface given by \(b_i = 0\) under the action of \(x = a^{-1}b\).
4.2. Example of $A_l = \mathfrak{sl}(l+1; \mathbb{R})$. Let us first introduce the following elements of $W = S_{l+1}$, the symmetry group of order $l + 1$;

$$s_{ij} := s_{\alpha_i} \cdots s_{\alpha_j} = [i \cdots j], \quad (4.8)$$

where the numbers $i \cdots j$ denotes the consecutive numbers between $i$ and $j$ for $1 \leq i, j \leq l$. Then for example some (but not all) of the connections from the top cell $\langle e \rangle = (\ast \cdots \ast)$ to the cells labeled $(\ast \cdots *0 \ast \cdots *)$ with $0$ in the $j$-th place are expressed by $s_{ij} : \langle e \rangle \rightarrow \langle s_{ij} \rangle$ with

$$\langle s_{ij} \rangle := s_{ij}^{-1} \cdot \langle e \rangle = s_{ji} \cdot \langle e \rangle, \quad \text{for} \quad 1 \leq i, j \leq l. \quad (4.9)$$

Note that the cell $\langle s_{ij} \rangle$ is isomorphic to $A_{j-1} \times A_{l-j}$ as an unstable manifold $W^u(s_{ij})$ with index $l - 1$ generated by the Toda flows. In particular, all the cells of $A_{l-1}$-type are given by $\langle s_{i1} \rangle$ and $\langle s_{il} \rangle$ for $i = 1, \ldots, l$. We call the $k$-cells of $A_k$-type the principal part (of $k$-cells), and those of the $A_1 \times \cdots \times A_n$ with $j_1 + \cdots + j_n = k, n > 1$ the whisker part. The set of all the (principal) $k$-cells of $A_k$-type is denoted as $A_k$. For example, the boundary of a principal $k$-cell labeled by $(0 \cdots 0 * \cdots * 0 \cdots 0)$ with $l - k$ zeros are written in the sum of the principal and the whisker parts of $(k - 1)$-cells. We then define a boundary map $\partial$ on the principal $k$-cells into the projection of the boundary map $\partial$ on the principal parts of $(k - 1)$-cells:

$$\partial_k : A_k \rightarrow A_{k-1}. \quad (4.10)$$

For the boundary of the top cell, we have

$$\partial_l \langle e \rangle = \sum_{i=1}^{l} \left( [e; s_{i1}] \langle s_{i1} \rangle + [e; s_{il}] \langle s_{il} \rangle \right), \quad (4.11)$$

where the incidence numbers are computed as

$$[e; s_{i1}] = [e; s_{l-i+1,l}] = 2(-1)^{i+1} (1 - \delta_{il}).$$

Thus the principal part of the boundary of the top cell consists of $2(l - 1)$ cells of $A_{l-1}$-type, and the cells $\langle s_{i1} \rangle$ and $\langle s_{il} \rangle$ are not in the part of the boundary. We also note that $\langle s_{i1} \rangle$ and $\langle s_{il} \rangle$ are only cells of $A_{l-1}$-type separated from the others and invariant under the subgroup generated by $W^u(s_{i1})$ for $\langle s_{i1} \rangle$ and $W^u(s_{il})$ for $\langle s_{il} \rangle$. Then one can identify the cells which are not included in any parts of the boundaries of $A_k$-type as in the following Proposition,

**Proposition 4.1.** All the cells which are free from the boundaries of cells with higher indices are generated by the following commutative diagram starting from $\langle a_{0,0} \rangle := \langle e \rangle$,

$$\begin{align*}
\langle a_{i,j} \rangle & \xrightarrow{s_{i-j,i+1}} \langle a_{i,j+1} \rangle & \langle a_{i+1,j} \rangle & \xrightarrow{s_{i+1,l-j}} \langle a_{i+1,j+1} \rangle & \langle a_{i,j+1} \rangle & \xrightarrow{s_{i-j+2}} \\
\downarrow & & \downarrow & & \downarrow & \\
\langle a_{i,j} \rangle & & \langle a_{i+1,j} \rangle & & \langle a_{i+1,j+1} \rangle & \\
\langle a_{i,j} \rangle & \xrightarrow{s_{i-j,i+1}} \langle a_{i,j+1} \rangle & \langle a_{i+1,j} \rangle & \xrightarrow{s_{i-j+2}} \langle a_{i+1,j+1} \rangle & \langle a_{i,j+1} \rangle & \xrightarrow{s_{i-j+2}}
\end{align*} \quad (4.12)$$

where $\langle a_{i,j} \rangle$ represents a unique cell labeled with $\langle 0 \cdots 0 * \cdots * 0 \cdots 0 \rangle \in A_{l-(i+j)}$.

**Proof.** The braid relation $[i \cdot i+1 \cdot i] = [i+1 \cdot i \cdot i+1]$ shows the commutativity of the diagram. One can also show in a similar way as in (4.11) that the principal part of the boundary of the $\langle a_{i,j} \rangle$ consists of $2(l - (i + j) - 1)$ cells and is given by

$$\partial_{l-(i+j)} \langle a_{i,j} \rangle = 2 \sum_{k=1}^{l-(i+j)} (-1)^{k+1} \left( 1 - \delta_{k,l-(i+j)} \right) \left( \langle b_{i+1,i+k} \rangle + \langle b_{l-j,l-j-k+1} \rangle \right), \quad (4.13)$$

where $\langle b_{i,j} \rangle = s_{i,j} \cdot \langle a_{i,j} \rangle$, and the cells $\langle b_{i+1,l-j} \rangle = \langle a_{i+1,j} \rangle$ and $\langle b_{l-j,i+1} \rangle = \langle a_{i,j+1} \rangle$ do not appear. □

The cells defined in Proposition $[\square]$ give the seed elements of the ”principal graph” defined as the graph on the sets $A_* := \bigoplus_{k=1}^{l} A_k$ where the connections indicate the nonzero incidence
numbers. Thus in the principal graph there are \( \frac{l(l + 1)}{2} \) disconnected subgraphs, each of which has a cell \( \langle a_{i,j} \rangle \) as the highest dimensional cell (the seed cell) in the subgraph with \( \text{dim}(a_{i,j}) = l - (i + j) \). Then one can show that the pair \((\mathcal{A}_*, \partial_*)\) forms a subchain complex, that is, the boundary map satisfies

\[
\partial_k \circ \partial_{k+1} = 0.
\]

Figure 4.1 illustrates the example of \( \mathcal{A}_3 = \mathfrak{sl}(4, \mathbb{R}) \). As in Fig.4.1, one can identify the cells in \( \mathcal{A}_k \) as \((k - 1)\)-dimensional cells in the graph consisting of \( l(l + 1)/2 \) number of disconnected hypercubes. In each hypercube of dimension \( k - 1 \), the top cell is represented by \( \langle a_{i,j} \rangle \) with \( k = l - (i + j) \), and the vertices represent \( \mathcal{A}_1 \)-cycles. For the case of \( \mathcal{A}_3 \), we identify the seed cell \( \langle e \rangle \) as the face (square), the cells in \( \mathcal{A}_2 \) as the edges, and those in \( \mathcal{A}_1 \) as the vertices. Then counting the numbers of those cells, we obtain

**Theorem 4.2.** The generating function (Poincaré polynomial) \( P(\mathcal{A}_*; q) \) of the number of cells \( |\mathcal{A}_k| \) is given by

\[
P(\mathcal{A}_*; q) = \sum_{k=1}^{l} |\mathcal{A}_k| q^{k-1} = \sum_{n=1}^{l} n(q + 2)^{l-n}.
\]

**Proof.** It is easy to see that the number of \( k \)-dimensional cells \( a_k \) in the \( n \)-dimensional hypercube is given by \( \binom{n}{k} 2^{n-k} \) so that we have

\[
\sum_{k=0}^{n} |a_k| q^k = (q + 2)^n.
\]

From Proposition 4.1, we have \( n \) number of the \((l - n)\) dimensional hypercubes in the principal graph. This asserts the Theorem. \( \square \)

As a corollary of Theorem 4.2, we obtain

**Corollary 4.1.** The Betti number of \( H_1(\hat{\mathcal{H}}_\mathbb{R}, \mathbb{Z}) \) is given by

\[
b_1(\hat{\mathcal{H}}_\mathbb{R}) := \text{rank}(H_1) = P(\mathcal{A}_*; -1) = \frac{l(l + 1)}{2}.
\]

**Proof.** Total number of \( \mathcal{A}_1 \)-cycles is given by the number of vertices in the graph, i.e.

\[
|Z_1| = \sum_{n=0}^{l-1} 2^n(l - n) = 2^{l+1} - (l + 2).
\]

From the graph, we can also find the number of boundaries, that is, in each graph of \( n \)-dimensional hypercubes there are \( 2^n - 1 \) boundaries, and we have \( l - n \) disconnected graph in this dimension. Then we have

\[
|B_1| = \sum_{n=1}^{l-1} (l - n) \times (2^n - 1),
\]

and obtain the Betti number \( b_1 = |Z_1| - |B_1| \) as stated. \( \square \)

Although we have a complete characterization of the cells in terms of colored Dynkin diagrams (Section 3), we have not obtained explicitly higher homology. It is however natural to consider...
the following conjecture on the Betti numbers $b_k$ as the alternative sums of the numbers of whiskers:

$$b_k = \begin{cases} 
\sum_{n=k}^{l-k+1} \left| A_n^{(k)} \right| (-1)^{n-k}, & \text{for } 1 \leq k \leq \frac{l+1}{2}, \\
0, & \text{for } k > \frac{l+1}{2}, 
\end{cases}$$

where $|A_n^{(k)}|$ is the number of whiskers defined by

$$|A_n^{(k)}| := \sum_{n_1 + \cdots + n_k = n} \left| A_{n_1} \times \cdots \times A_{n_k} \right|.$$ 

where $|A_n^{(k)}|$ is the number of whiskers defined by

$$|A_n^{(k)}| := \sum_{n_1 + \cdots + n_k = n} \left| A_{n_1} \times \cdots \times A_{n_k} \right|.$$ 

Note here that all the $k$-cycles are given by the products of $A_1$-cycles, i.e. $|Z_k| = |A_k^{(k)}|$. The conjecture is confirmed for the cases of $g \cong A_l$ up to $l = 3$.

5. Final remark

In this paper, we have studied the topology of the isospectral manifolds associated with the compactified level variety of the generalized Toda (Kostant-Toda) lattices on real split semisimple Lie algebras. The details of the decomposition based on the colored Dynkin diagrams can be found in our recent paper [2], and the proofs of the results stated in the sections 2 and 3 can be also found in this paper.

As a final remark, we would like to mention a possible extension of the present study for the full Kostant-Toda lattices which are recently shown to be integrable in [6]. Our methods may then shed some light on the structure of the real full flag manifold.

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