A Direct Algorithm to Compute the Topological Euler Characteristic and Chern-Schwartz-MacPherson Class of Projective Complete Intersection Varieties

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Abstract

Let $V$ be a possibly singular scheme-theoretic complete intersection subscheme of $\mathbb{P}^n$ over an algebraically closed field of characteristic zero. Using a recent result of Fullwood ("On Milnor classes via invariants of singular subschemes", Journal of Singularities) we develop an algorithm to compute the Chern-Schwartz-MacPherson class and Euler characteristic of $V$. This algorithm complements existing algorithms by providing performance improvements in the computation of the Chern-Schwartz-MacPherson class and Euler characteristic for certain types of complete intersection subschemes of $\mathbb{P}^n$. 
1 Introduction

Beginning with Euler’s Polyhedral Formula (circa 1750) the Euler characteristic has developed into an important invariant for the study of topology and geometry in a wide variety of settings. In addition to providing a mechanism to enable the classification of orientable surfaces, the Euler characteristic is an important component in many results in geometry. More recently several authors have noted applications of the Euler characteristic of projective varieties to problems in statistics and physics. Specifically the Euler characteristic is used for problems of maximum likelihood estimation in algebraic statistics by Huh in [15] as well as in string theory by Aluffi and Esole in [6] and by Collinucci, Denef, and Esole in [8].

Let $V$ be a subscheme of a projective space $\mathbb{P}^n$ (over an algebraically closed field of characteristic zero $k$). One of the first computational approaches to calculate the Euler characteristic of $V$, $\chi(V)$, was to do so by computing Hodge numbers and using the fact that the Euler characteristic is an alternating sum of Hodge numbers. This approach is implemented in Macaulay2 [12] as the function euler, where the Hodge numbers are found by computing the ranks of the appropriate cohomology rings. This approach, however, has significant drawbacks in both applicability and performance. Specifically, this method is only applicable for smooth subschemes and the computation of Hodge numbers is computationally expensive.

Alternatively, one may obtain the Euler characteristic of $V$ directly from the Chern-Schwartz-MacPherson class of $V$, $c_{SM}(V)$. In particular, when we consider $c_{SM}(V)$ as an element of the Chow ring of $\mathbb{P}^n$, $A^*(\mathbb{P}^n)$, we have that $\chi(V)$ is equal to the zero dimensional component of $c_{SM}(V)$. This is the method we shall use to obtain the Euler characteristic. This technique has been used by several authors (e.g. [2], [16], [14]) to construct different algorithms which are capable of calculating Euler characteristics of complex projective varieties. These previous methods will be discussed below.

In addition to containing the Euler characteristic, $c_{SM}$ classes are an important invariant in algebraic geometry, providing a generalization of the Chern class to singular schemes. While there are several other generalizations of the Chern class to singular schemes (i.e. the Chern-Fulton and Chern-Fulton-Johnson classes, see [3] for a discussion of these), the $c_{SM}$ class is the only
generalization which preserves the relation between Chern classes and the Euler characteristic. Additionally the $c_{SM}$ class has unique functorial properties (see Def. 2.1) and relationships to other common invariants. The $c_{SM}$ class has also found direct applications to problems from string theory in physics, see for example Aluffi and Esole [5].

Consider the hypersurface $V(f) \subset \mathbb{P}^n$ defined by the homogeneous polynomial $f$. All previous methods to compute $c_{SM}(V(f))$ employ Theorem 2.1 of Aluffi [2], which may be expressed as

$$c_{SM}(V(f)) = (1 + h)^{n+1} - \sum_{j=0}^{n} g_j(-h)^j(1 + h)^{n-j} \text{ in } A^*(\mathbb{P}^n) \cong \mathbb{Z}[h]/(h^{n+1}).$$

(1)

The differences between the methods lay in how the $g_j$’s are understood and computed. The first algorithm to compute $c_{SM}(V(f))$ was that of Aluffi [2]. To compute the $g_j$’s this algorithm requires the computation of the blowup of $\mathbb{P}^n$ along the singularity subscheme of $V(f)$ (that is the scheme defined by the partial derivatives of $f$). Hence the cost of computing the $c_{SM}$ class of a hypersurface using the method of Aluffi is that of computing the Resalgebra of the ideal defining the singularity subscheme of the hypersurface. This can be a quite expensive operation, making this algorithm impractical for many examples.

Another algorithm to compute the $c_{SM}$ class of a hypersurface was given by Jost in [16]. This method makes use of Fulton’s residual intersection theorem (Theorem 9.2 of Fulton [11]) which allows Jost to consider the $g_j$’s in (1) as the degrees of Fulton’s residual scheme. Jost also shows that in the context of $c_{SM}$ (and Segre) class computations these residual schemes can be computed by finding a particular saturation. Hence the computation of the saturation to find the residual scheme and the computation of its degree are the main costs of Jost’s algorithm. The algorithm of Jost is probabilistic and yields the correct result for a choice of objects lying in an open dense Zariski set of the corresponding parameter space, see Jost [16] or Eklund, Jost, and Peterson [9].

In [14], the author of this note considers the $g_j$’s as the projective degrees of a rational map defined by the partial derivatives of $f$ and gives a method to compute these projective degrees by finding the degree of a certain zero dimensional ideal (see Theorem 2.3 below). The method given in [14] to
compute the projective degrees is probabilistic and yields the correct result for a choice of objects lying in an open dense Zariski set of the corresponding parameter space. This method is implemented in [14] using both Gröbner bases methods and polynomial homotopy continuation (via Bertini [7] and PHCpack [20]); it provides a performance improvement over previous methods in many cases. A detailed comparison of these methods can be found in [14].

For $V$ a possibly singular subscheme of $\mathbb{P}^n$ all these methods require the use of the inclusion-exclusion property of $c_{SM}$ classes when $V$ has codimension higher than one. Specifically for $V_1, V_2$ subschemes of $\mathbb{P}^n$ the inclusion-exclusion property for $c_{SM}$ classes states

$$c_{SM}(V_1 \cap V_2) = c_{SM}(V_1) + c_{SM}(V_2) - c_{SM}(V_1 \cup V_2).$$

From this we may directly deduce the following.

**Proposition 1.1.** Let $V$ be a subscheme of $\mathbb{P}^n$. Write the polynomials defining $V$ as $F = (f_1, \ldots, f_m)$ and let $F_{\{S\}} = \prod_{i \in S} f_i$ for $S \subset \{1, \ldots, m\}$. Then,

$$c_{SM}(V) = \sum_{S \subset \{1, \ldots, m\}} (-1)^{|S|+1} c_{SM}(V(F_{\{S\}})),$$

where $|S|$ denotes the cardinality of the integer set $S$.

While the use of this property allows for the computation of $c_{SM}(V)$ for $V$ of any codimension, it requires exponentially many $c_{SM}$ computations relative to the number of generators of $I$. Additionally some of the schemes considered while performing inclusion-exclusion may have significantly higher degree than the original scheme $V$.

Below we discuss an algorithm that will allow for the direct computation of the $c_{SM}$ classes of arbitrary, possibly singular, globally complete intersection subschemes of $\mathbb{P}^n$ defined by a homogeneous polynomial ideal $I = (f_0, \ldots, f_m)$ where the scheme defined by $(f_0, \ldots, f_{m-1})$ is smooth (allowing for a possible rearrangement of the generators of $I$). We also give an extension of this method to all globally complete intersection subschemes of $\mathbb{P}^n$ via a form of the inclusion-exclusion property of $c_{SM}$ classes which considered only the generators of $I$ which define a singular subscheme of $\mathbb{P}^n$. This
new method can be implemented symbolically using Gröbner bases methods or numerically using polynomial homotopy continuation via a package such as Bertini [7]. We see that this new method complements existing methods for computing $c_{SM}$ classes by providing performance improvements, particularly when the input ideal has relatively few generators which define singular schemes.

In Section 2 we review several important definitions which will be used throughout this note. In particular we define several different characteristics classes including the Segre and Chern-Shwartz-MacPherson classes and explore more closely the relationships between the $c_{SM}$ class and the Euler characteristic using a recent result of Aluffi [4].

In Section 3 we give a new expression for the $c_{SM}$ class of a complete intersection subscheme $V(f_0, \ldots, f_m)$ of $\mathbb{P}^n$ such that $V(f_0, \ldots, f_{m-1})$ is smooth in Theorem 3.3. This result is based on an expression for the Milnor class of a scheme of this type due to Fullwood [10]. This expression allows us to state an algorithm to compute the $c_{SM}(V)$ for a complete intersection $V$ in $\mathbb{P}^n$. This new algorithm offers performance improvements over the standard inclusion-exclusion method when only a few of the generators of the ideal defining the scheme $V$ are singular. We give some running time results for this method in Table 3.1 and Table 3.2.

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2 Background

In this section we review the definitions of the characteristic classes that will be needed to describe the algorithm presented in Section 3. In particular we give the definition of the Chern-Shwartz-MacPherson class in Definition 2.1 and discuss its relationship with the Euler characteristic. We also define the
Segre class in Eq. (3), and the Chern-Fulton-Johnson class in Eq. (6).

The algorithm given in Section 3 will rely on an expression due to Aluffi [2] for the Segre class in terms of the projective degrees of a rational map. We give this relation in Proposition 2.4 and give the definition of the projective degrees of a rational map in Eq. (8). In Theorem 2.3 we give a result of the author’s [14] which provides a means to compute the projective degrees using a computer algebra system.

All characteristics classes considered here will be understood to be elements of some Chow ring. We will express the Chow ring of a \( n \)-dimensional non-singular variety \( M \) as \( A^*(M) = \bigoplus_{n=0}^\infty A^n(M) \), where \( A^\ell(M) \) is the Chow group of \( M \) having codimension \( \ell \) in \( M \), that is \( A^\ell(M) \) is the group of codimension \( \ell \)-cycles modulo rational equivalence. Where convenient we will also write \( A_j(M) \) for the Chow group of dimension \( j \), that is the group of \( j \)-cycles modulo rational equivalence. All computations of characteristic classes will take place in the Chow ring of \( \mathbb{P}^n \), \( A^*(\mathbb{P}^n) \). Recall that \( A^*(\mathbb{P}^n) \cong \mathbb{Z}[h]/(h^{n+1}) \) where \( h = c_1(\mathcal{O}_{\mathbb{P}^n}(1)) \) is the rational equivalence class of a hyperplane in \( \mathbb{P}^n \) (\( c_1 \) denotes the first Chern class), so that a hypersurface \( W \) of degree \( d \) will be represented by \([W] = d \cdot h \) in \( A^*(\mathbb{P}^n) \). For more details see Fulton [11].

Given \( V \) a proper closed subscheme of a variety \( W \), the Segre class of \( V \) in \( W \) may be expressed as

\[
s(V, W) = \sum_{j \geq 1} (-1)^{j-1} \eta_*(\tilde{V}^j), \tag{3}\]

where \( \tilde{V} \) is the exceptional divisor of the blow-up of \( W \) along \( V \), \( \eta : \tilde{V} \to V \) is the projection and the class \( \tilde{V}^k \) is the \( k \)-th self intersection of \( \tilde{V} \). For a more detailed description, see Fulton [11, §4.2.2]. We also note that in all cases considered here we will have \( W = \mathbb{P}^n \), allowing us to use the more concrete expression for the Segre class given in Proposition 2.4.

For a smooth scheme \( X \) let \( T_X \) denote the tangent bundle to \( X \). For a vector bundle \( E \) on \( X \) let \( c(E) \) denote the total Chern class of \( E \), see Fulton [11, §3.2]. We will write \( c(X) = c(T_X) \cap [X] \) for the total Chern class of \( X \) in the Chow ring of \( X \), \( A^*(X) \). As a consequence of the Hirzebruch-Riemann-Roch theorem, we have that the degree of the zero dimensional component of the total Chern class of a smooth projective variety is equal to the Euler
characteristic, that is
\[ \int c(T_X) \cap [X] = \chi(X). \] (4)

Here \( \int \alpha \) denotes the degree of the zero dimensional component of the class \( \alpha \in A^*(X) \), i.e. the degree of the part of \( \alpha \) in the dimension zero Chow group \( A_0(X) \). Note that we will frequently abuse notation and, given a scheme \( V \) in \( \mathbb{P}^n \) we will write \( c(V), s(V, \mathbb{P}^n) \) and \( c_{SM}(V) \) for the pushforwards to \( \mathbb{P}^n \) of each characteristic class, i.e. we will consider the various characteristic classes as their pushforwards in \( A^*(\mathbb{P}^n) \) rather than in \( A^*(V) \).

There exist several different generalizations of the total Chern class to singular schemes and all of these notions agree with \( c(T_V) \cap [V] \) for nonsingular \( V \). The Chern-Swartz-Macpherson class is, however, unique in the sense that it is the only generalization which satisfies a property analogous to (4) for any \( V \), i.e.
\[ \int c_{SM}(V) = \chi(V). \] (5)

A recent result of Aluffi [4], which we illustrate in Example 2.2, shows that the \( c_{SM} \) class has a even stronger relation to the Euler characteristic in the case of projective varieties.

We now briefly review the definition of the \( c_{SM} \) class, given in the manner of MacPherson [17]. For a scheme \( V \), denote by \( \mathcal{C}(V) \) the abelian group of finite linear combinations \( \sum_W m_W 1_W \), with the \( W \) being (closed) subvarieties of \( V \), and \( m_W \in \mathbb{Z} \); \( 1_W \) denotes the function that is 1 in \( W \), and 0 outside of \( W \). We refer to elements \( f \in \mathcal{C}(V) \) as constructible functions and write \( \mathcal{C}(V) \) for the group of constructible functions on \( V \). \( \mathcal{C} \) can be turned into a functor by letting \( \mathcal{C} \) map a scheme \( V \) to the group of constructible functions on \( V \) and map a proper morphism \( f : V_1 \to V_2 \) to
\[ \mathcal{C}(f)(1_W)(p) = \chi(f^{-1}(p) \cap W), \quad W \subset V_1, \ p \in V_2 \text{ a closed point}. \]

The Chow group functor \( A_* \) is also a functor from algebraic varieties to Abelian groups. The \( c_{SM} \) class may be realized as a natural transformation between these two functors.

**Definition 2.1.** The Chern-Schwartz-MacPherson class is the unique natural transformation between the constructible function functor and the Chow
group functor, that is $c_{SM} : \mathcal{C} \to \mathcal{A}_*$ is the unique natural transformation satisfying:

- **(Normalization)** $c_{SM}(1_V) = c(TV) \cap [V]$ for $V$ non-singular and complete.
- **(Naturality)** $f_*(c_{SM}(\phi)) = c_{SM}(C(f)(\phi))$, for $f : X \to Y$ a proper transform of projective varieties, $\phi$ a constructible function on $X$.

For a scheme $V$ let $V_{\text{red}}$ denote the support of $V$. The notation $c_{SM}(V)$ is taken to mean $c_{SM}(1_V)$ and hence, since $1_V = 1_{V_{\text{red}}}$, we have $c_{SM}(V) = c_{SM}(V_{\text{red}})$.

When $V$ is a subscheme of $\mathbb{P}^n$ the class $c_{SM}(V)$ can, in a sense, be thought of as a more refined version of the Euler characteristic since it in fact contains the Euler characteristics of $V$ and those of general linear sections of $V$ for each codimension. Specifically, if $\text{dim}(V) = m$, starting from $c_{SM}(V)$ we may directly obtain the list of invariants

$$\chi(V), \chi(V \cap L_1), \chi(V \cap L_1 \cap L_2), \ldots, \chi(V \cap L_1 \cap \ldots \cap L_m)$$

where $L_1, \ldots, L_m$ are general hyperplanes. Conversely from the list of Euler characteristics above we could obtain $c_{SM}(V)$, i.e. there exists an involution between the Euler characteristics of general linear sections and the $c_{SM}$ class in this setting. This relationship is given explicitly in Theorem 1.1 of Aluffi [4]; we give an example of this below.

**Example 2.2.** Consider $V = V(x_0x_3 - x_1x_2)$ in $\mathbb{P}^3 = \text{Proj}(k[x_0, \ldots, x_3])$ which is the variety defined by image of the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$. We may compute $c_{SM}(V) = 4h^3 + 4h^2 + 2h$ and obtain the Euler characteristics of the general linear sections using an involution formula given by Aluffi in [4], specifically:

- **First consider the polynomial** $p(t) = 4 + 4t + 2t^2 \in \mathbb{Z}[t]/(t^4)$ given by the coefficients of the $c_{SM}$ class above.

- **Next apply Aluffi’s involution**

$$p(t) \mapsto I(p) := \frac{t \cdot p(-t - 1) + p(0)}{t + 1} = 2t^2 - 2t + 4.$$
This gives $\chi(V) = 4$, $\chi(V \cap L_1) = 2$, and $\chi(V \cap L_1 \cap L_2) = 2$.

We will also make use of another generalization of the total Chern class to singular schemes called the Chern-Fulton-Johnson class and denoted $c_{FJ}$. For simplicity we will give the definition of $c_{FJ}$ only for the case where $X$ is a closed locally complete intersection subscheme of a smooth ambient variety $M$, since this will be sufficient for our purposes in this note. For a complete definition and an excellent discussion of the Chern-Fulton-Johnson classes and other related notions see Aluffi [3]. Let $X$ be a closed locally complete intersection subscheme of a smooth ambient variety $M$ and let $T_M$ denote the tangent bundle of $M$, define

$$c_{FJ}(X) = c(T_M) \cap s(X, M).$$  \hfill (6)

Also note that since we assume that $X$ is a locally complete intersection (meaning there exists a regular embedding $i : X \to M$) then by Proposition 4.1 of Fulton [11] we have

$$c_{FJ}(X) = c(T_M) \cap s(X, M) = c(T_M) \cap (c(N_X M)^{-1} \cap [X]).$$

Here $N_X M$ is the normal bundle to $X$ in $M$ (that is the vector bundle with sheaf of sections $(\mathcal{I}/\mathcal{I}^2)$ where $\mathcal{I}$ is the ideal sheaf of $X$). Finally, let $V$ be a subscheme of $M$; we define the Milnor class of $V$ as 

$$\mathcal{M}(V) = (-1)^{\text{codim}(V)}(c_{FJ}(V) - c_{SM}(V)).$$ \hfill (7)

Note that other sign conventions may be used in definition of the Milnor class, we use the sign convention used by [10], see Fullwood [10] or Aluffi [3] for more details.

All algorithms considered in this note will make use of the so-called projective degrees of a rational map to compute characteristics classes. We recall the definition of projective degrees below. Consider a rational map $\phi : \mathbb{P}^m \dashrightarrow \mathbb{P}^m$. In the manner of Harris (Example 19.4 of [13]) we may define the projective degrees of the rational map $\phi$ as a list of integers $(g_0, \ldots, g_n)$ where

$$g_i = \text{card} \left( \phi^{-1}(\mathbb{P}^{m-i}) \cap \mathbb{P}^i \right).$$ \hfill (8)

Here $\mathbb{P}^{m-i} \subset \mathbb{P}^m$ and $\mathbb{P}^i \subset \mathbb{P}^m$ are general hyperplanes of dimension $m - i$ and $i$ respectively and card is the cardinality of a zero dimensional set. Note
that points in $(\phi^{-1}(P^m) \cap P^i)$ will have multiplicity one (this follows from the Bertini theorem of Sommese and Wampler [18, §A.8.7]).

To compute the projective degrees $g_i$ we may apply Theorem 2.3 below. This computation is probabilistic and yields the correct result for a choice of objects lying in an open dense Zariski set of the corresponding parameter space.

**Theorem 2.3 (Theorem 3.1 of [14]).** Let $I = (f_0, \ldots, f_m)$ be a homogeneous ideal in $k[x_0, \ldots, x_n]$ defining an $r$-dimensional scheme $V = V(I)$, and assume, without loss of generality that all the polynomials $f_i$ generating $I$ have the same degree. The projective degrees $(g_0, \ldots, g_n)$ of $\phi: \mathbb{P}^n \to \mathbb{P}^m$,

$$\phi : p \mapsto (f_0(p) : \cdots : f_m(p)),$$

are given by

$$g_i = \dim_k (k[x_0, \ldots, x_n, T]/(P_1 + \cdots + P_i + L_1 + \cdots + L_n - i + L_A + S)).$$

Here $P_\ell, L_\ell, L_A$ and $S$ are ideals in $k[x_0, \ldots, x_n, T]$ with

$$P_\ell = \left( \sum_{j=0}^m \lambda_{\ell,j} f_j \right), \quad \lambda_{\ell,j} \text{ a general scalar in } k, \quad \ell = 1, \ldots, n,$$

$$S = \left( 1 - T \cdot \sum_{j=0}^m \vartheta_j f_j \right), \quad \vartheta_j \text{ a general scalar in } k,$$

$$L_\ell = \left( \sum_{j=0}^n \mu_{\ell,j} x_j \right), \quad \mu_{\ell,j} \text{ a general scalar in } k, \quad \ell = 1, \ldots, n,$$

$$L_A = \left( 1 - \sum_{j=0}^n \nu_j x_j \right), \quad \nu_j \text{ a general scalar in } k.$$

Additionally $g_0 = 1$.

Finally we give an expression due to Aluffi [2] for the Segre class of a projective scheme in terms of the projective degrees defined above. The expression in (12) combined with Theorem 2.3 will allow us to compute Segre classes of projective schemes. For more details see [14].
Proposition 2.4 (Proposition 3.1 of [2]). Let \( I = (f_0, \ldots, f_m) \subset k[x_0, \ldots, x_n] \) be a homogeneous ideal defining a scheme \( Y \subset \mathbb{P}^n \) and let \( h = c_1(\mathcal{O}_{\mathbb{P}^n}(1)) \) be the class of a hyperplane in \( A^*(\mathbb{P}^n) \). Since \( I \) is homogeneous we may assume that the degree of \( f_i \) is \( d \) for all \( i \). Let \( \phi : \mathbb{P}^n \to \mathbb{P}^m \) be the rational map specified by \( p \mapsto (f_0(p) : \cdots : f_m(p)) \), let \((g_0, \ldots, g_n)\) be the projective degrees of \( \phi \). Then we have:

\[
s(Y, \mathbb{P}^n) = 1 - c(\mathcal{O}(dh))^{-1} \cap \left( \sum_{i=0}^{n} \frac{g_i h^i}{c(\mathcal{O}(dh))^{i+1}} \right) \tag{11}
\]

\[
= 1 - \sum_{i=0}^{n} \frac{g_i h^i}{(1 + dh)^{i+1}} \in A^*(\mathbb{P}^n) \cong \mathbb{Z}[h]/(h^{n+1}). \tag{12}
\]

3 The Algorithm to Compute the \( c_{sm} \) Class of a Projective Complete Intersection

In this section we describe our new algorithm to compute the \( c_{SM} \) class (and hence the Euler characteristic) of a complete intersection subscheme of \( \mathbb{P}^n \) over an algebraically closed field of characteristic zero.

Let \( V = V(f_0, \ldots, f_m) \) be a complete intersection subscheme of \( \mathbb{P}^n \) such that the scheme \( V(f_0, \ldots, f_{m-1}) \) is non-singular (allowing for a possible reordering of the generators) and let \( J \) be the ideal generated by the \((m+1) \times (m+1)\) minors of the Jacobian matrix of partial derivatives of \( f_0, \ldots, f_m \). The primary result needed for the algorithms described below is given in Theorem 3.3 which gives a formula for \( c_{SM}(V) \) in terms of the Segre class of \( s(Y, \mathbb{P}^n) \) where \( Y = V(J) \cap V \) is the singularity subscheme of \( V \). This Segre class can then be computed using Eq. (12) and a method to compute the projective degrees such as Theorem 2.3. Theorem 3.3 follows from Theorem 1.1 of Fullwood [10]. We summarize this method in Algorithm 1.

In Proposition 3.4 and Corollary 3.5 we extend the result of Theorem 3.3 to any (global) complete intersection subscheme of \( \mathbb{P}^n \) with a type of inclusion-exclusion which considers only the singular generators of the ideal. Hence the
number of required Segre class computations is exponential in the number of singular generators. At worst, if all generators define singular schemes, this reduces to inclusion-exclusion as in Proposition 1.1. We present this generalized version of Algorithm 1 in Algorithm 2 below.

In Section 3.2 we compare the running time of Algorithm 2 described below to other algorithms to compute \( c_{SM} \) classes for complete intersection varieties in \( \mathbb{P}^n \). We see that for the cases considered the new algorithm does indeed provide a performance improvement. While the new method to compute \( c_{SM} \) classes is not applicable in all cases it does seem to complement existing methods by providing an efficient approach for a certain subset of problems, particularly those where the ideal defining a complete intersection \( V \) has only a few generators which define a singular scheme.

## 3.1 The Main Result

Let \( M \) be a smooth algebraic variety and let \( V \) be a subscheme of \( M \). From the definition of the Milnor class in Eq. (7) we have the following formula for the class \( c_{SM}(V) \) in \( A^\ast(M) \):

\[
c_{SM}(V) = c_{FJ}(V) - (-1)^{\text{codim}(V)}\mathcal{M}(V).
\]

We now define several notations of Aluffi \([1, \S 1.4]\) for operations in the Chow ring. Let \( \alpha = \sum_{i\geq 0} \alpha^{(i)} \) be a cycle class in \( A^\ast(M) \) with \( \alpha^{(i)} \) denoting the piece of \( \alpha \) of codimension \( i \) in \( A^i(M) \), that is \( \alpha^{(i)} \in A^i(M) \). Also let \( \mathcal{L} \) be some line bundle on \( M \). Define the following notations,

\[
\alpha^\vee = \sum_{i\geq 0} (-1)^i \alpha^{(i)}, \quad \text{and} \quad \alpha \otimes_M \mathcal{L} = \sum_{i\geq 0} \frac{\alpha^{(i)}}{c(\mathcal{L})^i}.
\]

In \([10, \S 1.1]\), Fullwood gives a new formula for the Milnor class of a subscheme \( V \subset M \) which is a global complete intersection of any codimension with an additional assumption on the structure of \( V \).

**Theorem 3.1** (Theorem 1.1 of Fullwood \([10]\)). Let \( M \) be a smooth algebraic variety over an algebraically closed field of characteristic zero. Let \( V \) be a possibly singular global complete intersection corresponding to the zero
scheme of a vector bundle \( E \rightarrow M \). Let \( j = \text{rk}(E) \). Additionally assume that \( V = M_1 \cap \cdots \cap M_j \) for some hypersurfaces \( M_1, \ldots, M_j \) and assume that, for some ordering of the hypersurfaces, \( M_1 \cap \cdots \cap M_{j-1} \) is smooth. Let \( \mathcal{L} \rightarrow M \) denote the line bundle associated to the divisor \( M_j \) and let \( Y \) denote the singularity subscheme of \( V \). Then we have

\[
\mathcal{M}(V) = \frac{c(T_M)}{c(E)} \cap (c(E^\vee \otimes \mathcal{L}) \cap (s(Y, M)^\vee \otimes_M \mathcal{L})). \tag{15}
\]

Note that if \( V \) is non-singular we will have that \( \mathcal{M}(V) = 0 \).

**Remark 3.2.** We also note that if \( V = V(I) \) is a non-singular subscheme of \( \mathbb{P}^n \) (even if it is not a complete intersection) we may simply write the following in \( A^*(\mathbb{P}^n) \cong \mathbb{Z}[h]/(h^{n+1}) \):

\[
c_{SM}(V) = c(T_{\mathbb{P}^n}) \cap s(V, \mathbb{P}^n) = (1 + h)^{n+1} s(V, \mathbb{P}^n). \tag{16}
\]

Hence we need compute only the Segre class \( s(V, \mathbb{P}^n) \); this can be done directly by calculating the projective degrees of the rational map specified by the ideal \( I \) using Theorem 2.3 and then applying the result of Proposition 2.4 to obtain the Segre class. Thus, in particular, inclusion-exclusion is not required in the smooth case. See Fulton [11, §4.2.6] or Aluffi [3] for more details.

Combining the relation (13), the result of Fullwood [10] given in (15), and the expression for the \( c_{FJ} \) class of a locally complete intersection of Suwa [19] we obtain Theorem 3.3. This result combined with Proposition 3.4 will allow us to devise a more efficient algorithm to compute \( c_{SM} \) classes of possibly singular complete intersection varieties.

**Theorem 3.3.** Let \( k \) be an algebraically closed field of characteristic zero and let \( I = (f_0, \ldots, f_m) \) be a homogeneous ideal in \( k[x_0, \ldots, x_n] \). Assume that \( V = V(I) \) is a complete intersection subscheme of \( \mathbb{P}^n \) and let \( Y \) be the singularity subscheme of \( V \). Let \( \deg(f_i) = d_i \), and further assume that \( V(f_0, \ldots, f_{m-1}) \) is smooth scheme theoretically. Let

\[
A^*(\mathbb{P}^n) \cong \mathbb{Z}[h]/(h^{n+1})
\]

denote the Chow ring of \( \mathbb{P}^n \) where \( h = c_1(\mathcal{O}_{\mathbb{P}^n}(1)) \) is the hyperplane class in
\[ \mathbb{P}^n. \text{ Then we have the following relation in } A^*(\mathbb{P}^n): \]

\[
c_{SM}(V) = (1 + h)^{n+1} \cdot \prod_{i=0}^{m} \frac{d_i h}{1 + d_i h} - \]

\[
\frac{(-1)^m (1 + h)^{n+1}}{\prod_{i=0}^{m} (1 + d_i h)} \left( \sum_{p=0}^{m} h^p \sum_{i=0}^{p} (m - i) (-1)^i d_m^{p-i} \cdot \tilde{c}_i \right) \cdot \left( \sum_{i=0}^{n} (-1)^i s_i h^i \right),
\]

where we write

\[ \prod_{i=0}^{m} (1 + d_i h) = \sum_{i=0}^{m} \tilde{c}_i h^i, \quad \text{and} \quad s(Y, \mathbb{P}^n) = \sum_{i=0}^{n} s_i h^i. \]

**Proof.** First consider the result of Eq. (15), taking \( M = \mathbb{P}^n. \) Since \( V \) is a complete intersection it may be defined as the zero scheme of a rank \( m + 1 \) vector bundle \( E. \) Let \( \mathcal{L} \to \mathbb{P}^n \) be the line bundle associated to \( V(f_m). \) Then we have that \( \mathcal{L} = \mathcal{O}(d_m h), \ c(E) = \prod_{i=0}^{m} (1 + d_i h) \) and \( c(T_{\mathbb{P}^n}) = (1 + h)^{n+1}. \) Combining this with (15) we have

\[
\mathcal{M}(V) = \frac{c(T_{\mathbb{P}^n})}{c(E)} \cap (c(E^\vee \otimes \mathcal{L}) \cap (s(Y, \mathbb{P}^n)^\vee \otimes_{\mathbb{P}^n} \mathcal{L}))
\]

\[ = \frac{(1 + h)^{n+1}}{\prod_{i=0}^{m} (1 + d_i h)} \left( \sum_{p=0}^{m} \sum_{i=0}^{p} (m - i) c_i(E^\vee)c_1(L)^{p-i} \cap (s(Y, \mathbb{P}^n)^\vee \otimes_{\mathbb{P}^n} \mathcal{O}(d_m h)) \right) \]

Let

\[ c(E) = \prod_{i=0}^{m} (1 + d_i h) = \sum_{i=0}^{m} \tilde{c}_i h^i, \quad \text{and} \quad s(Y, \mathbb{P}^n) = \sum_{i=0}^{n} s_i h^i, \]

using (14) we may expand the expression \((s(Y, \mathbb{P}^n)^\vee \otimes_{\mathbb{P}^n} \mathcal{O}(d_m h))\) as,

\[
\left( \sum_{i=0}^{n} s_i h^i \right)^\vee \otimes_{\mathbb{P}^n} \mathcal{O}(d_m h) = \left( \sum_{i=0}^{n} (-1)^i s_i h^i \right) \otimes_{\mathbb{P}^n} \mathcal{O}(d_m h)
\]

\[ = \sum_{i=0}^{n} \left( -1 \right)^i s_i h^i \mathcal{O}(d_m h) \]

\[ = \sum_{i=0}^{n} \frac{(-1)^i s_i h^i}{(1 + d_m h)^i}. \]
We may now write,

$$\mathcal{M}(V) = \frac{(1 + h)^{n+1}}{\prod_{i=0}^{m}(1 + d_i h)} \left( \sum_{p=0}^{m} h^p \sum_{i=0}^{p} \frac{(m - i)}{(p - i)} (-1)^i d_m^{p-i} \cdot \tilde{c}_i \right) \cdot \left( \sum_{i=0}^{n} \frac{(-1)^i s_i h^i}{(1 + d_i h)^i} \right).$$

Since $V$ is a complete intersection in $\mathbb{P}^n$ from Suwa [19] we have

$$c_{F,J}(V) = (1 + h)^{n+1} \cdot \prod_{i=0}^{m} \frac{d_i h}{1 + d_i h},$$

and applying the relation $c_{SM}(V) = c_{F,J}(V) - (-1)^m \mathcal{M}(V)$ gives the desired result.

Hence we may conclude that the computation of $c_{SM}$ classes in the case of the theorem above requires only the computation of $s(Y, \mathbb{P}^n)$ (where $Y$ is the singularity subscheme of $V$), which can be accomplished by means of the projective degree calculation of Theorem 2.3 for the rational map specified by the ideal corresponding to $Y$ and an application of the formula (12).

The singularity subscheme $Y$ of $V$ as given above will be $Y = V(J) \cap V$ where $J$ is the ideal in $k[x_0, \ldots, x_n]$ generated by the $(m + 1) \times (m + 1)$ minors of the $(m + 1) \times (n + 1)$ Jacobian matrix of partial derivatives, i.e. the matrix $a_{i,j} = \left( \frac{\partial f_i}{\partial x_j} \right)$ for $i = 0, \ldots, m$, $j = 0, \ldots, n$ (here we index the first row and column of the Jacobian matrix by 0). In practice we will use the ideal $(I + J) : (x_0, \ldots, x_n)\infty$ as the ideal of the singularity subscheme $Y$.

Since the only unknown in the expression of Theorem 3.3 is the Segre class $s(Y, \mathbb{P}^n)$ we may obtain an Algorithm to compute $c_{SM}$ classes (in the setting of the theorem) by combining Theorem 3.3 with the method to compute Segre classes using the projective degree of a rational map given by the author in [14]. We summarize this below.

Let $J = (w_0, \ldots, w_m) \subset R = k[x_0, \ldots, x_n]$ be a homogeneous ideal defining a scheme $Y \subset \mathbb{P}^n$ and let $h = c_1(\mathcal{O}_{\mathbb{P}^n}(1))$ be the class of a hyperplane in $A_*(\mathbb{P}^n) = \mathbb{Z}[h]/(h^{n+1})$. Since $J$ is homogeneous we may assume that the deg$(w_i) = d$ for all $i$. Also let $(g_0, \ldots, g_n)$ be the projective degrees of the map $\phi : \mathbb{P}^n \rightarrow \mathbb{P}^m$,

$$\phi : p \mapsto (w_0(p) : \cdots : w_m(p)).$$
To compute the projective degrees $g_i$ in the case where $\phi$ is specified by a homogeneous ideal we may apply Theorem 2.3. Once we have obtained the projective degrees then we may apply Proposition 2.4 to obtain the Segre class $s(Y, \mathbb{P}^n)$.

To extend the result of Theorem 3.3 to any complete intersection subscheme of $\mathbb{P}^n$ we will use Proposition 3.4 below. For a scheme $V = V(I) \subset \mathbb{P}^n$ this proposition describes a type of inclusion-exclusion for $c_{SM}$ class which considers only the generators of $I$ which define singular subschemes. If the majority of generators of $I$ define a non-singular subscheme of $\mathbb{P}^n$ this result combined with Theorem 3.3 can offer a speed advantage in comparison to methods which use only inclusion-exclusion.

Proposition 3.4. Let $Z \subset \mathbb{P}^n$ be smooth (scheme-theoretically) and let $X_1 = V(f_1), X_2 = V(f_2)$ be singular hypersurfaces in $\mathbb{P}^n$. If $V = Z \cap X_1 \cap X_2$, then we have

$$c_{SM}(V) = c_{SM}(Z \cap X_1) + c_{SM}(Z \cap X_2) - c_{SM}(Z \cap (X_1 \cup X_2)),$$

(17)

here $X_1 \cup X_2$ is the scheme generated by $f_1 \cdot f_2$. Additionally, when $V$ is a complete intersection each of the terms in (17) can be computed using Theorem 3.3.

Proof. This result follows directly from the inclusion-exclusion property of the $c_{SM}$ class, see (2).

Corollary 3.5. Let $V = Z \cap V(f_1) \cdots \cap V(f_r)$ be a subscheme of $\mathbb{P}^n$, with the subscheme $Z$ being non-singular. Write the polynomials defining $W = V(f_1) \cdots \cap V(f_r)$ as $F = (f_1, \ldots, f_r)$ and let $F_{\{S\}} = \prod_{i \in S} f_i$ for $S \subset \{1, \ldots, r\}$. Then,

$$c_{SM}(Z \cap W) = \sum_{S \subset \{1, \ldots, r\}} (-1)^{|S|+1} c_{SM}(Z \cap V(F_{\{S\}}))$$

where $|S|$ denotes the cardinality of the integer set $S$. The expressions $c_{SM}(W \cap V(F_{\{S\}}))$ can be computed using Theorem 3.3 when $V$ is a complete intersection.

This result allows us to extend the application of Theorem 3.3 to complete intersections $V = V(I) \subset \mathbb{P}^n$ where several of the generators of the ideal $I$
define a singular scheme. At worst, when all of the generators are singular, this will reduce to inclusion-exclusion. However if only a few of the generators are singular this could offer a significant computational speed boost by lowering the degrees considerably.

In Algorithm 1 we summarize the algorithm to compute $c_{SM}$ classes for projective varieties $V$ satisfying the assumptions of Theorem 3.3. In Algorithm 2 we give an algorithm which is applicable for any subscheme $V$ of $\mathbb{P}^n$ defined by a homogeneous ideal. This algorithm takes advantage of the result of Corollary 3.5 combined with Theorem 3.3 when $V$ is a complete intersection. If $V$ is smooth the result of Remark 3.2 is used. If $V$ is neither smooth nor a complete intersection then inclusion-exclusion is used.

**Input:** A homogeneous ideal $I = (f_0, \ldots, f_m)$ in $k[x_0, \ldots, x_n]$ defining a complete intersection scheme $V = V(I) \subset \mathbb{P}^n$ such that $V(f_0, \ldots, f_{m-1})$ is smooth (scheme theoretically).

**Output:** $c_{SM}(V)$ in $A^*(\mathbb{P}^n) \cong \mathbb{Z}[h]/(h^{n+1})$ and/or $\chi(V)$.

- Find the singularity subscheme $Y = V(J)$, of $X$
  - Set $K$ equal to the $(m+1) \times (m+1)$ minors of the Jacobian matrix of $I$, that is the matrix with entries $a_{i,j} = \left( \frac{df_i}{dx_j} \right)$ for $i = 0, \ldots, m$, $j = 0, \ldots, n$.
  - $J = (K + I) : (x_0, \ldots, x_n)^\infty$.
  - $Y = V(J)$.

- Apply Theorem 2.3 with the rational map defined by the ideal $J$ to compute the projective degrees $g_0, \ldots, g_n$.

- Compute $s(Y, \mathbb{P}^n)$ by using Eq. (12) and the projective degrees $g_0, \ldots, g_n$ computed above.

- Apply Theorem 3.3 to obtain $c_{SM}(V)$.
Input: a homogeneous ideal \( I = (f_0, \ldots, f_m) \) in \( k[x_0, \ldots, x_n] \) defining a scheme \( V = V(I) \subset \mathbb{P}^n \).

Output: \( c_{SM}(V) \) in \( \Lambda^*(\mathbb{P}^n) \cong \mathbb{Z}[h]/(h^{n+1}) \) and/or \( \chi(V) \).

- if \( V \) is non-singular (i.e. if the singularity subscheme \( Y \) of \( V \) is empty):
  - if \( \text{codim}(V) = m + 1 \) (i.e. \( V \) is a complete intersection):
    - \( c_{SM}(V) = (1 + h)^{m+1} \prod_{i=0}^m \frac{d_i}{1 + d_i} \).
    - Return \( c_{SM}(V) \) and/or \( \chi(V) \).
  - Compute the projective degrees \( (g_0, \ldots, g_n) \) of the rational map defined by the ideal \( I \) using Theorem 2.3.
  - Compute \( s(V, \mathbb{P}^n) \) by using Eq. (12) and the projective degrees \( (g_0, \ldots, g_n) \) obtained above.
  - Compute \( c_{SM}(V) = (1 + h)^{n+1} s(V, \mathbb{P}^n) \).
  - Return \( c_{SM}(V) \) and/or return \( \chi(V) \).

- else if \( \text{codim}(V) = m + 1 \) (i.e. \( V \) is a complete intersection):
  - for \( j = 1, \ldots, m \) and for each subset \( f_{\ell_0}, \ldots, f_{\ell_{m-j}} \) of \( f_1, \ldots, f_m \) containing \( m+1-j \) elements:
    - if \( V(f_{\ell_0}, \ldots, f_{\ell_{m-j}}) \) is non-singular:
      - Let \( Z = V(f_{\ell_0}, \ldots, f_{\ell_{m-j}}) \).
      - Let \( F \) be the set \( f_{\ell_{m-j+1}}, \ldots, f_{\ell_m} \) and let \( F_S = \prod_{i \in S} f_i \) for \( S \subset \{\ell_{m-j+1}, \ldots, \ell_m\} \).
      - Apply Corollary 3.5 to obtain
        \[
        c_{SM}(V) = \sum_{S \subset \{\ell_{m-j+1}, \ldots, \ell_m\}} (-1)^{|S|+1} c_{SM}(Z \cap V(F_S))
        \]
      - and compute each \( c_{SM} \) class in the summation using Theorem 3.3 as presented in Algorithm 1.
    - Return \( c_{SM}(V) \) and/or \( \chi(V) \).

- else: Perform the full inclusion-exclusion algorithm to obtain \( c_{SM}(V) \) as described in [14]; i.e. use Proposition 1.1, the expression for the \( c_{SM} \) class of a hypersurface in Eq. (1) and a method to compute projective degrees such as Theorem 2.3.

Algorithm 2: An algorithm to compute \( c_{SM}(V) \) for \( V = V(I) \) any subscheme of \( \mathbb{P}^n \). This algorithm takes advantage of the result of Corollary 3.5 combined with Theorem 3.3 when \( V \) is a complete intersection. If \( V \) is smooth the result of Remark 3.2 is used.
3.2 Running Time Comparison

| INPUT | CSM (Aluffi [2]) | CSM (Jost [16]) | csm\(_{\text{dir}}\) (Th. 3.3) | csm\(_{\text{J.E}}\) ([14]) |
|-------|-----------------|-----------------|-----------------------------|-----------------------------|
| \(V_1 \subset \mathbb{P}^7\) | - | - [ ] | 0.3s (0.2s) [4.8s] | - (116.5s) [ ] |
| \(V_2 \subset \mathbb{P}^4\) | - | 1.7s [-] | 0.3s (0.1s) [1.3s] | 1.2s (1.2s) [44.1s] |
| \(V_3 \subset \mathbb{P}^6\) | - | 27.7s [-] | 7.2s (2.2s) [-] | 33.2s (53.2s) [-] |
| \(V_4 \subset \mathbb{P}^5\) | - | - [-] | 4.6s (0.7s) [5.5s] | - (-) [-] |
| \(V_5 \subset \mathbb{P}^6\) | - | - [-] | 19.9s (7.9s) [24.9s] | - (-) [-] |

Table 3.1: Run times (over \(\mathbb{Q}\)) of different algorithms for computing \(c_{SM}(V)\) and \(\chi(V)\) for \(V\) a complete intersection subscheme of \(\mathbb{P}^n\). The timings in [ ] are those of numeric implementations using Bertini [7]. The timings in ( ) are from an implementation of the result of Proposition 2.3 which uses a saturation rather than computing the degree of the zero dimensional ideal to find the projective degree.

The main computational cost of Algorithm 1 is the computation of the projective degrees \(g_0, \ldots, g_n\). This can be accomplished in a number of different ways. The method we will use for this computation consists of finding the degree of the zero dimensional ideal described in Theorem 2.3. This can be accomplished symbolically using Gröbner bases calculations, or numerically using homotopy continuation via a package such as PHCpack [21] or Bertini [7]. The symbolic methods are in general much faster.

In Table 3.1 and Table 3.2 we give the running times of the algorithm discussed here in comparison to several other algorithms which use inclusion-exclusion to compute the \(c_{SM}\) class and Euler characteristic. All methods shown in the tables are implemented in Macaulay2 [12], the numeric implementations use Bertini [7]. All test computations were performed on a computer with an Intel i5-450M processor and 4 GB of RAM.
In the tables in this section we take

\[ V_1 = V(21x_0^2 + 5x_1^2 - 24x_2^2 + 13x_3^2 + 8x_4^2 - 106x_5^2 + 2x_6^2 + 14x_7^2, x_1x_5 - x_2x_4), \]

\[ V_2 = V(3x_0^2 + 19x_1^2 + 8x_2^2 + 12x_3^2 + 13x_4, 34x_0 + 5x_1 + 19x_2 + 127x_3 - 15x_4, 27x_0^2 - x_4^2), \]

\[ V_3 = V(3x_0^2 + 19x_1^2 + 8x_2^2 + 12x_3^2 + 9x_4^2 + 3x_5^2 + 25x_6^2, x_3x_3 - x_3x_3^3), \]

\[ V_4 = V(5x_0^2 + 9x_1^2 + 79x_2^2 + 2x_3^2 + 35x_4 + 73x_5^2, 23x_0 + 9x_1 + 7x_2 + 2x_3 + 4x_4 + 32x_5, x_2x_3 - x_3x_5x_4), \]

\[ V_5 = V(3x_0^2 + 17x_1^2 - 47x_2^2 + 3x_3^2 + 38x_4^2 - 727x_5^2 + 12x_6^2, x_0x_6 - x_2, 43x_6^2 + 52x_6x_1 + 94x_1^2 + 5x_0x_2 + 13x_1x_2 + x_2^2 + x_0x_3 + 4x_1x_3 + 98x_2x_3 + x_3^2 + x_0x_4 + 74x_1x_4 + 13x_2x_4 + 71x_3x_4 + 23x_3^2 + 12x_0x_5 + 2x_1x_5 + x_2x_5 + 65x_3x_5 + 92x_4x_5 + 27x_5^2 + 5x_3x_6 + 103x_1x_6 + 38x_2x_6 + x_3x_6 + 6x_4x_6 + 2x_5x_6 + 95x_6^2). \]

\[ V_6 \] is a smooth variety of degree eight and codimension three in \( \mathbb{P}^{10} \) defined by three random quadratic forms. \( V_7 \) is a variety of degree eight and codimension three in \( \mathbb{P}^{10} \) defined by two random quadratic forms and one random degree two polynomial which defines a singular scheme.

\[ V_8 = V(-2x_0^3 + 24x_1^3 + x_2^3 + x_3^3 - 7x_4^3, -9x_0^2 + 43x_1^2 + x_2^2 - 98x_3^2 - 73x_4^2, x_1x_4 - x_0x_4, x_1x_0) \]

\[ V_9 = V(-3x_0^3 + 4x_1^3 + x_3^3 - 7x_4^3, -15x_5, -31x_0 + 14x_1 - 9x_2 + 17x_3 - 7x_4 - 15x_5, (x_1 - x_5)x_4, x_3x_0). \]

For \( V_1 \subset \mathbb{P}^7 \) we have \( \text{deg}(V_1) = 4 \) and \( \text{codim}(V_1) = 2 \), for \( V_2 \subset \mathbb{P}^4 \) we have \( \text{deg}(V_2) = 4 \) and \( \text{codim}(V_2) = 3 \), for \( V_3 \subset \mathbb{P}^6 \) we have \( \text{deg}(V_3) = 6 \) and \( \text{codim}(V_3) = 2 \), for \( V_4 \subset \mathbb{P}^5 \) we have \( \text{deg}(V_4) = 2 \) and \( \text{codim}(V_4) = 3 \), and for \( V_5 \subset \mathbb{P}^6 \) we have \( \text{deg}(V_5) = 8 \) and \( \text{codim}(V_5) = 3 \). The variety \( V_8 \) has dimension zero in \( \mathbb{P}^4 \) and \( \text{deg}(V_8) = 24 \). The variety \( V_9 \) has dimension one in \( \mathbb{P}^5 \) and \( \text{deg}(V_9) = 12 \).

The method CSM (Aluffi [2]) is the implementation of Aluffi described in [2], this implementation uses inclusion-exclusion and considers the projective degrees as the multi-degree of the blowup of \( \mathbb{P}^n \) along the subscheme defined by the partial derivatives for each hypersurface considered in the inclusion-exclusion. The method CSM (Jost [16]) is the algorithm of Jost which computes the projective degrees by finding the degrees of residual
sets via saturation, this method also uses inclusion-exclusion. The method csm\_dir (Th. 3.3) is the method of Algorithm 2. The method csm\_LE ([14]) is the method described by the author in [14], this method uses inclusion-exclusion combined with (1) and uses the result of Theorem 2.3 to compute the projective degrees.

In Table 3.1 computations are performed over \( \mathbb{Q} \). In Table 3.2 computations are performed over \( \mathbb{GF}(32749) \). While the \( c_{SM} \) class is only defined over fields of characteristic zero doing the computations over \( \mathbb{GF}(32749) \) yields the same \( c_{SM} \) classes found by working over \( \mathbb{Q} \) for all examples considered here. Previous papers on computing \( c_{SM} \) classes such as Aluffi [2], Jost [16] and the author of this note [14] have also performed test computations over a finite field.

For the smooth variety \( V_6 \) the computation of \( c_{SM}(V_6) \) by Algorithm 1 or Algorithm 2 calculates the singularity subscheme \( Y \) of \( V_6 \) first, but since \( V_6 \) is smooth then \( s(Y, \mathbb{P}^n) = 0 \) is obtained immediately after \( Y \) is computed without the need to calculate the projective degrees. Hence in this case very nearly all of the time is spent computing the singularity subscheme \( Y \). Similarly, for the variety \( V_7 \) the computation of \( c_{SM}(V_7) \) using Algorithm 2 spends the majority of the computation time finding the singularity subscheme of \( V_2 \) (approximatively 90\% of the 59.5s average runtime).

For the varieties \( V_8 \) and \( V_9 \) the result of Theorem 3.3 is not directly applicable and hence the method csm\_dir (Th. 3.3), which is our implementation of Algorithm 2, must apply Corollary 3.5. We see that for the case of the variety \( V_8 \) Algorithm 2 still provides a marked advantage in comparison to inclusion-exclusion only. However for \( V_9 \) there is little practical difference between using Algorithm 2 and using an algorithm which does only inclusion-exclusion such as csm\_LE ([14]).

Overall in Tables 3.1 and 3.2 we see that, for the types of examples for which the result of Theorem 3.3 is applicable it offers a performance increase over the algorithms which use inclusion-exclusion. Additionally we see that the symbolic implementations tend to be faster than the numeric implementations, even when the symbolic versions run over \( \mathbb{Q} \), and we also see that we can expect a further speed-up using the symbolic implementations when they are run over a finite field.
Table 3.2: Run times of different algorithms for computing $c_{SM}(V)$ and $\chi(V)$ for $V$ a complete intersection subscheme of $\mathbb{P}^n$. We use - to denote computations that were stopped after ten minutes (600 s). All computations are performed over the finite field $\mathbb{F}(32749)$.

From the results in the tables we can conclude that Algorithm 1 provides a significant performance improvement for the computation of $c_{SM}(V)$ when $V = V(f_0, \ldots, f_m)$ is a complete intersection subscheme of $\mathbb{P}^n$ such that $V(f_1, \ldots, f_{m-1})$ is smooth. The performance gain offered by Algorithm 2 when one must remove several of the generators of $I = (f_0, \ldots, f_m)$ to obtain a smooth scheme is less clear, in some cases it seems to offer a performance improvement however in some cases the cost of computing several singularity subschemes and their Segre classes is too great for us to see any benefit in using Algorithm 2 over pure inclusion-exclusion.

In any case Algorithm 1 and Algorithm 2 complement other methods to compute $c_{SM}$ classes and Euler characteristics by offering an effective way to significantly improve performance for a certain class of examples. Additionally it seems likely that, with some minor heuristic adjustments to the criterion one uses to decide whether to use the specialized inclusion-exclusion of Corollary 3.5 or the usual inclusion-exclusion of Proposition 1.1, the method of Algorithm 2 would be able to offer marked improvement in many cases, and in worst cases to perform similarly to an algorithm using only inclusion-exclusion.
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