Maximum Entropy Principle and the Higgs Boson Mass

Alexandre Alves¹, Alex G. Dias², Roberto da Silva³

¹Universidade Federal de São Paulo, Departamento de Ciências Exatas e da Terra, Diadema - SP, Brasil,
²Universidade Federal do ABC, Centro de Ciências Naturais e Humanas, Santo André - SP, Brasil,
³Universidade Federal do Rio Grande do Sul, Departamento de Física, Porto Alegre - RS, Brasil

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A successful connection between Higgs boson decays and the Maximum Entropy Principle is presented. Based on the information theory inference approach we determine the Higgs boson mass as \( M_H = 125.04 \pm 0.25 \text{ GeV} \), a value fully compatible to the LHC measurement. This is straightforwardly obtained by taking the Higgs boson branching ratios as the target probability distributions of the inference, without any extra assumptions beyond the Standard Model. Yet, the principle can be a powerful tool in the construction of any model affecting the Higgs sector. We give, as an example, the case where the Higgs boson has an extra invisible decay channel. Our findings suggest that a system of Higgs bosons undergoing a collective decay to Standard Model particles is among the most fundamental ones where the Maximum Entropy Principle applies.

The mechanism of mass generation via spontaneous electroweak symmetry breaking (EWSB), as realized in the Standard Model (SM), is essential to understand the electroweak interactions and has been almost entirely confirmed with the discovery of the Higgs boson and the observation of its couplings to some of the heavier particles of the SM spectrum. In the form we know it, EWSB is able to generate the particle masses but cannot fix their values. An exception is the photon which is massless due the unbroken \( U(1) \) gauge symmetry of electromagnetic interactions. Several attempts have been made to predict the Higgs boson mass from extensions of the SM, but until now there is no clue whether the best guesses are meant to represent a higher level of evidence for a more fundamental theory and consequently a deeper understanding of subatomic physics.

It has been recognized that entropy can be used as an universal method for statistical inference in the information theory approach. It is universal in the sense of its wide range of applicability and it is not necessarily tied to physical interpretations, despite the Gibbs-Shannon (GS) entropy is equivalent to the usual thermodynamical entropy.

In this Letter we show that the Maximum Entropy Principle (MEP) is a powerful inference tool which provides us with the most accurate theoretical determination to date of the Higgs mass. The principle also naturally leads us to assign a theoretical probability density function (PDF) for the Higgs boson mass parameter. It does not assume any beyond the Standard Model (BSM) hypothesis nor untested physical principles, rather it does only assume the present status of physical knowledge encoded in SM.

Based on this successful mass parameter inference, we propose using MEP in order to furnish further pieces of evidence to favor or not a given theory affecting the Higgs sector. According to this proposal, any candidate model is required to maximize a physically well motivated entropy associated to the Higgs bosons decays to all available channels for a given mass. As an illustration of the usefulness of the proposed inference method, we investigate a Higgs-portal scalar dark matter model.

It has been firstly observed in Ref. [7] that the product of the SM Higgs branching ratios is a distribution with a peak close to the experimentally measured value of the Higgs boson mass. Moreover, the author also mentions a possible connection to this observation with entropy arguments in the Higgs decay process, and yielding a potential mean to constrain theoretical extensions of the SM. It was also suggested in Ref. [8] that some fundamental principle might be responsible for the maximization of Higgs branching ratio to photons, a fact that could be used for constraining new physics as well. We shall see that all those observations are consequences of MEP.

Our findings suggest further that entropy production in Higgs decays to massless quanta, namely, to photons and gluons, is maximized nearly the same point of the entropy from all channels. This places the system of a large number of Higgs bosons among the most fundamental ones where it is possible to observe MEP in action.

MEP inference on the Higgs boson mass– Maximum-entropy distributions are the best estimates that can be made from partial knowledge about the PDF of a given set of random variables. The goal of MEP inference is to determine the least-biased probability density function involved in the evolution of a given system from the computation of the GS entropy measure where the partial knowledge is coded as a list of constraints. It is universal in its scope and is known, for example, to lead to the same computation rules of the statistical mechanics, without any further assumptions beyond the usual laws of mechanics (quantum or classical).

Suppose, however, we can compute a parametric family of PDFs from first principles (as the Higgs branching ratios). In this situation, the partial knowledge reflects
all the prior information available encoded in the model (the SM, for example) from which one calculates those PDFs. The correspondence between MEP and the Maximum Likelihood Estimate (MLE) is particularly crucial for the inference of the PDF parameters [6].

We are going to show that a physically well motivated entropy measure for an ensemble of Higgs bosons naturally leads us to a MLE for the Higgs mass parameter and the assignment of a PDF for the Higgs mass as well. For that purpose, consider an ensemble of $N$ non-interacting Higgs bosons which decay to SM particles according to their respective branching ratios $BR_i$, $i = \gamma\gamma, gg, Z\gamma, qq, \ell^+\ell^-, WW^*, ZZ^*$, where $q = u, d, s, c, b, t$ and $\ell = e, \mu, \tau$. There are 14 primary SM decay modes.

The probability that a system evolves from the $N$ Higgs bosons to a bath of final state SM particles due to each of the $n_k$ particles of type $i$, $n_2$ of type 2, and so on until the $m$-th mode

$$P\left(\{n_k\}_{k=1}^m\right) \equiv \frac{N!}{n_1! \cdots n_m!} \prod_{k=1}^m (p_k)^{n_k}$$  

(1)

where $p_k(M_H(\theta)) \equiv BR_k(M_H, \theta)$ are the Higgs branching ratios as a function of the Higgs boson mass $M_H$ and the remaining SM parameters $\theta$ related to the EWSB. Moreover, $\sum_{k=1}^m n_k = N$ where $n_k$ is the number of Higgs bosons decaying to the $k$th mode.

The total entropy is now given by the sum of the multinomial distribution of $N$ Higgs bosons decaying to each possible partition of $n_1$ particles of type 1, $n_2$ of type 2, and so on until the $m$-th mode

$$S_N = \sum_{\{n\}} \prod_{i=1}^N \ln \left[ P\left(\{n_k\}_{k=1}^m\right) \right] \delta \left( N - \sum_{i=1}^m n_i \right) (-\ln(P))$$  

(2)

where $\sum_{\{n\}} \cdot \equiv \sum_{n_1=0}^N \cdots \sum_{n_m=0}^N \cdot$.

The number of possible configurations involved in the sum of Eq. (2) is huge for large $N$. Recently, an asymptotic formula up to order $1/N$ has been derived [9] and is given right below in Eq. (3)

$$S_N = \frac{1}{2} \ln \left[ (2\pi N e)^m \prod_{k=1}^m p_k \right] + \frac{1}{12N} \left( 3m - 2 - \sum_{k=1}^m \frac{1}{p_k} \right) + O\left( \frac{1}{N^2} \right)$$  

(3)

We checked the formula up to $N = 10^9$ using an importance sampling Monte Carlo technique. In practice, for $N \gtrsim 10^7$, the $O(1/N)$ term is negligible for the SM Higgs branching ratios $p_k$.

An important remark about $S_N$ for the multinomial distribution is due. Calculating the entropy of a system with elementary states grouped into macrostates, with consequent loss of information, is the essence of coarse-graining [10]. The physically motivated entropy $S_N$ gathers each type of decay into subgroups counting their occurrence numbers and nothing more. Any additional information about their identities is lost. This way, $S_N$ naturally comprises the concept of irreversibility, in obedience to the second law of thermodynamics [11].

The most important feature of the asymptotic formula of Eq. (3) is the term involving the product of probabilities $p_k$ in the thermodynamic limit $N \rightarrow \infty$. This is, up to a normalization factor, a joint PDF to observe a fraction $p_1$ of decays of type 1, and so on until a fraction $p_m$ to observe decays of type $m$. With fixed probabilities, the normalized product is the likelihood function of the parameter $M_H$. This induces the assignment of the normalized product $\prod_{k=1}^m p_k(M_H(\theta))$ to a theoretical Higgs mass PDF. Moreover, requiring that the Higgs mass parameter maximizes $S_N$ in the thermodynamic limit is equivalent to a MLE of the Higgs boson mass. In this respect, the physically motivated multinomial distribution is fundamental, once the asymptotic limit of its the entropy (Eq. [3]) equals the logarithm of the likelihood function $\prod_{k=1}^m p_k(M_H(\theta))$.

A maximum of $S_N$ (a concave function in $M_H$), in the thermodynamic limit, is a solution of the equation

$$\lim_{N \rightarrow \infty} \frac{\partial S_N}{\partial M_H} = \frac{\partial S_\infty}{\partial M_H} = 0$$  

(4)

where

$$S_\infty(M_H(\theta)) \equiv \ln \left[ \prod_{k=1}^m p_k(M_H(\theta)) \right]$$  

(5)

The mass parameter that maximizes $S_\infty(M_H(\theta))$, a solution of Eq. (4) that we call $\hat{M}_H$, is distributed according to some PDF $P_{\hat{M}_H}(\hat{M}_H)$ which is obtained after marginalizing over all $\theta$ taking into account their current experimental uncertainties. This should not be confused with the theoretical Higgs mass PDF that Eq. (3) suggests. We calculate $P_{\hat{M}_H}$ just to access the theoretical error on the inferred $\hat{M}_H$ from uncertainties on the remaining SM parameters.

We adopted the SM parameters values, $\theta_{SM}$, recommended by the Higgs Working Group (HWG) [12] and calculated the SM branching ratios with HDECAY [13] taking into account all the leading EW and QCD corrections available. We do not take double off-shell top quark decay into account as its contribution is expected to be negligible in the mass region of interest [14], so we effectively have $m = 13$ decay modes. We checked our results against those quoted in the HWG report within less than 1% for the whole mass range from 10 GeV to 1 TeV. The $P_{\hat{M}_H}$ PDF was estimated by randomly choosing $2000$ points in the SM parameter space, calculating $\hat{M}_H$ for each point, and fitting the resulting histogram to a gaussian function.
a theoretical Higgs mass PDF. What kind of distribution should we expect? Under weak conditions, which we checked that all the branching ratios fulfill, a normalized likelihood function $L$ of a parameter $\xi$ asymptotically converges to the normal distribution $N(\hat{\xi},1/\sqrt{-L''(\hat{\xi})})$ about the maximum $\hat{\xi}$ in the limit of a large number of PDFs in the product [17]. The reduced number of branching ratios in the product leads to a PDF somewhat departed from a gaussian actually. The mean and the standard deviation (s.d.) of the normalized product of the Higgs branching ratios is given by $\hat{M}_H = 125.04$ GeV and 6.13 GeV, respectively.

The nearly gaussian shape of $\prod_{i=1}^{m} BR_i(M_H)$ was observed in Ref. [7] with very similar mean and standard deviation. It was argued that the maximum of this distribution (in the sense of MLE), which is close to the experimental mass value, could be due some kind of entropy argument and that the Higgs mass is placed in a window of “maximum opportunity” for its experimental observation. In fact, an information theoretic interpretation can be offered: the log-likelihood is the sum of the information content of the Higgs decays, defined by $\sum_{k=1}^{m} -\ln(p_k)$, which is maximized precisely for the observed Higgs mass. An estimate of the Higgs mass based on the maximum of the photons branching ratio has also been presented in [8].

**MEP inferences on a new physics scenario**—We now present an example of another possible application of MEP inference. Several BSM scenarios add particles the Higgs boson might decay into. A class of interesting models are those where the Higgs boson gains an extra invisible mode, in special, those ones where dark matter (DM) couples to the Higgs field as the Higgs-portal models [18]. In Ref. [19], several experimental constraints are used to determine a viable portion of the parameters space for a real scalar dark matter $\chi$ interacting with the SM Higgs boson doublet $H$ according to $\mathcal{L}_{int} = \frac{1}{2} \chi^2 H^\dagger H$.

Of course, any new decay channel participates the total Higgs entropy and might change the Higgs boson mass that maximizes it. In this respect, imposing that $\hat{M}_H$ remains unaltered, or within certain confidence region around the measured Higgs mass, might narrow the parameters space of the model.

In Ref. [19] the DM mass $m_\chi$ is constrained to lie in a small region of 55–62 GeV, and a coupling $\lambda$ in the range $10^{-2}$–$10^{-3}$ in order to evade direct detection and relic abundance bounds. In order to place constraints on the model, we mapped the allowed region found in Ref. [19] with a 20-point grid and searched for $\hat{M}_H$ including the new DM channel $p_{14} = p_4 = BR(H \to \chi\chi)$. We found only two points leading to a MEP inference of the Higgs mass compatible with the experimental value, the remaining points were excluded at 95% CL at least.

The two points not excluded are displayed in Table [10]. As we are going to discuss next, the maximum

**Figure 1.** Theoretical and experimental Higgs mass distributions. The solid black line represents the mass distribution obtained from the maximum of the Higgs entropy marginalized over $\theta$, $P_H(M_H)$. The dashed line is a crude combination of both results shown in Fig. [1]. The solid black line ($P_H$) is the distribution of the masses that lead to a maximum of $S_\infty$ of Eq. [6], while the dashed blue line represents the combination of the most recent CMS and ATLAS measurement of the Higgs mass. The theoretical determination based on MEP inference is

$$\hat{M}_H = (125.04 \pm 0.25) \text{ GeV}$$

(6)

to be compared with the latest Higgs mass measurements from CMS [15] and ATLAS [16]

$$M_H = (125.03 \pm 0.30) \text{ GeV} \quad \text{CMS}$$
$$M_H = (125.36 \pm 0.41) \text{ GeV} \quad \text{ATLAS}$$

(7)

A crude weighted-averaged combination of both results is $M_H = (125.14 \pm 0.24) \text{ GeV}$.

In Fig. [1] we also show the evolution of the measured Higgs mass since its discovery in 2012 until its most recent determination. We observe an steady decrease in the value of the mass as the measurements get more accurate converging to the theoretical MEP determination. As far as we know, this is the first time an underlying principle is used to determine accurately the Higgs boson mass, without assuming any BSM scenario or untested physical assumptions.

As discussed above, we can interpret the joint PDF given by the normalized product of branching ratios as...
of $p_{\gamma \gamma} = BR(H \rightarrow \gamma \gamma)$, and its associated GS entropy $S_{\gamma} = -p_{\gamma \gamma} \ln(p_{\gamma \gamma})$ too, are very close to the measured mass and this might indicate that the entropy created in these decays is also maximized. Interestingly, for these two points, an almost perfect alignment with the $S_{\gamma}$ maximum is achieved. Moreover, the individual entropy of the DM channel $S_{\chi} = -p_{\chi \chi} \ln(p_{\chi \chi})$ is also maximized close to 125 GeV.

### Entropy production in Higgs decays to massless quanta

From the thermodynamic point of view, each decay of the Higgs boson increases the entropy of the bath consisting of SM particles as the energy concentrated in the Higgs boson mass is dissipated into an increased number of quanta. In this respect, photons and gluons can be considered the more efficient channels to increase entropy once they are massless particles \[20, 21\]. In Fig. 1 we show $P_{\gamma}(M_H)$, obtained exactly as $P_{\gamma}$, but maximizing the GS entropy of the photons channel decay $S_{\gamma}$. The maximum points of both PDFs are very close.

In fact, the gluonic peak is also near, close to $\sim 119$ GeV, which lead us to hypothesize that the massless channels play an important role in the explanation of why $S_{H}$ displays a maximum in first place. As a function of the Higgs mass, $S_{H}$ is small wherever the Higgs decays to photons and gluons are negligible and strongly increases around the peaks of $BR(H \rightarrow \gamma \gamma)$ and $p_{gg} = BR(H \rightarrow gg)$. The creation of massive quanta in a co-moving volume of space is less effective in increasing entropy as they organize part of the released energy in their own masses. This is, in part, why entropy is taken as the photon density when we consider the thermodynamic evolution of the universe – radiation is the best way to spread energy across the space and increase entropy.

To confirm these intuitions, we calculated $\ln \left( \prod_{k=1}^{11} p_k \right)$ taking into account all channels but the massless ones (photons and gluons) multiplying the remaining branching ratios by $1/(1 - p_{\gamma \gamma} - p_{gg})$. The maximum point of the total entropy barely changes to $\hat{M}_H = 125.02$ GeV, a negligible shift compared to the value obtained with the full product.

Withdrawing just the gluonic decay channel, consistently changing the remaining ones, not only marginally changes the point of maximum, now at $\hat{M}_H = 125.00$ GeV, but it shortens the distance to the point of maximum of the entropy created in photons decays which now is given by 124.85 GeV.

From the observations made above we conclude that: (1) an experimentally compatible inference on the Higgs mass can be made considering only tree-level Higgs couplings; (2) entropy creation from decays to massless gauge bosons, especially the photon, is maximized close to the same Higgs mass of the full collectively determined entropy; (3) if photons were the only massless fields the Higgs could decay into, the agreement of the maximal points would improve considerably.

All these findings suggest that the decay of a system of scalar particles participating an EWSB mechanism, as in SM, respects MEP concerning the scalars mass in two independent ways – collectively including all particles the Higgs couples to generate masses and individually for the massless gauge bosons related to the unbroken symmetries. Nevertheless, the interplay between massive and massless modes is necessary to accurately determine the Higgs mass.

### Discussions and perspectives

The reason why the massless modes peak around the measured Higgs mass might be due the interplay of all the other SM parameters and perhaps by the structure of its symmetry group and the form of the Higgs sector \[8\]. Different parameters, for example, lead to another Higgs mass according to the MEP constraint. Removing a channel of the product of branching ratios may perturb $M_H$. In particular, the light fermion decays, whose observation is beyond the LHC capabilities, cannot be neglected and the inference $M_H = 125.04$ GeV would change a lot if they were withdrawn.

From the point of view of the information theory, MEP inference applies to any new model affecting the Higgs boson decays apart from physical considerations. Its scope of applications is wide, for example, extended Higgs sectors, BSM scenarios predicting new SM Higgs decays and/or deviations of SM Higgs couplings, models for electroweak baryogenesis and Higgsogenesis, Higgs-inflaton proposals and new physics solutions to absolute stabilization of the vacuum. By the way, it is worthy mentioning that MEP should apply to any new Higgs boson from a BSM scenario. Requiring maximization of entropy might bring information about decay rates and its mass as we have delineated in this work.

Testing the validity of MEP in Higgs decays is also straightforward. Any new information that could be related to the Higgs sector can be checked to keep the maximum of $S_H$ compatible with the measured Higgs mass. We expect that as the experimental errors on the SM parameters get smaller, the gaussian PDFs of Fig. 1 get narrower with aligned means.

We conclude this Letter stating that the Maximum En-

| $(m_{\chi} \text{ [GeV]}, \lambda)$ | $M_{H}(S_{H}^{\text{max}})$ | $M_{H}(S_{\gamma}^{\text{max}})$ | $M_{H}(S_{\chi}^{\text{max}})$ |
|-----------------|-----------------|-----------------|-----------------|
| $(57.8, 3.7 \times 10^{-4})$ | 125.3 | 125.3 | 126.1 |
| $(58.4, 5 \times 10^{-4})$ | 125.4 | 125.4 | 126.8 |

Table I. The Higgs boson mass in GeV that maximizes the total, the photon, and the dark matter GS entropies, respectively. Two points out of 20 sampled points from Ref. [19], shown in the first column, were found to lead to a Higgs mass that maximizes the entropies. The two points have $BR(H \rightarrow \chi \chi) < 1\%$. From the observations made above we conclude that: (1) an experimentally compatible inference on the Higgs mass can be made considering only tree-level Higgs couplings; (2) entropy creation from decays to massless gauge bosons, especially the photon, is maximized close to the same Higgs mass of the full collectively determined entropy; (3) if photons were the only massless fields the Higgs could decay into, the agreement of the maximal points would improve considerably.

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tropy Principle seems to be an innate inference tool for a system composed of a large number of Higgs bosons undergoing a collective decay into SM particles. According to this interpretation, such a system can be placed amongst the most fundamental ones where MEP applies.

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