On the time dependence of
the rate of convergence towards Hartree dynamics
for interacting Bosons

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Abstract

We consider interacting $N$-Bosons in three dimensions. It is known that the difference between the many-body Schrödinger evolution in the mean-field regime and the corresponding Hartree dynamics is of order $1/N$. We investigate the time dependence of the difference. To have sub-exponential bound, we use the results of time decay estimate for small initial data. We also refine time dependent bound for singular potential using Strichartz estimate. We consider the interaction potential $V(x)$ of type $\lambda \exp(-\mu |x|^\gamma)$ for $\lambda \in \mathbb{R}$, $\mu \geq 0$, and $0 < \gamma < 3/2$, which covers the Coulomb and Yukawa interaction.

1 Introduction and the main results

We consider a many-body particle system of $N$-Bosons with two body interaction via Coulomb type interaction or Yukawa type interaction, i.e., $V(x) = \lambda \exp(-\mu |x|^\gamma)$ with $\lambda \in \mathbb{R}$, $\mu \geq 0$, and $0 < \gamma < 3/2$. The system can be described by a complex valued function $\psi_N = \psi_N(x_1, \ldots, x_N) : (\mathbb{R}^3)^N \to \mathbb{C}$, which is called wave function. The wave function $\psi_N$ for the Bosonic system is symmetric under the permutation of variables, i.e., for each $x_i, x_j \in \mathbb{R}^3$, $1 \leq i, j \leq N$, $\psi_N(x, x_j, x_i, \ldots) = \psi_N(x, x_i, x_j, \ldots)$. Our system is governed by the following Hamiltonian:

$$H_N = \sum_{j=1}^{N} -\Delta_j + \frac{1}{N-1} \sum_{i<j} V(x_i - x_j),$$

and we call it a many-body mean-field Hamiltonian.

Now, suppose that the system is fully condensed, i.e., the initial wave function is given by

$$\psi_N^0 = \varphi^\otimes N$$

with a one-body wave function $\varphi : \mathbb{R}^3 \to \mathbb{C}$ in some appropriate function space which will be described later. We want to argue that the system is almost condensed at the time $t \geq 0$ as well, i.e.,

$$\psi_{N,t} = e^{-iH_N t} \psi_N \simeq \varphi_t^\otimes N \quad \text{for large } N$$

for some $\varphi_t : \mathbb{R}^3 \to \mathbb{C}$.

Heuristically, from the point of view of particle $x_1$, it ‘feels’ averaged potential

$$\frac{1}{N-1} \sum_{j=2}^{N} V(x_1 - x_j)$$
from other particles. Since the Hamiltonian is symmetric under the permutation of the particles, the averaged potential is the same for every particle \(x_j\). Thus, we can expect that \(\varphi_t\) evolves according to the Hartree equation

\[
\partial_t \varphi_t = -\Delta \varphi_t + (V * |\varphi_t|^2) \varphi_t
\]

(3)

with initial data \(\varphi_{t=0} = \varphi\). Non-rigorous derivation of the Hartree equation can be found in literature. (See, e.g., Section 1 of [4]).

To understand the ‘almost condensation’ of the system at the time \(t \geq 0\) in a mathematically rigorous way, we proceed as follows. First, we consider the density matrix \(\gamma_{N,t} = |\psi_{N,t}\rangle \langle \psi_{N,t}|\) associated with \(\psi_{N,t}\), which can be understood as the orthogonal projection onto \(\psi_{N,t}\). More precisely, the kernel of \(\gamma_{N,t}\) is given by

\[
\gamma_{N,t}(x;x') = \psi_{N,t}(x)\psi_{N,t}(x').
\]

The one-particle marginal density is then defined through its kernel

\[
\gamma_{N,t}^{(1)}(x) = \int d|x'\rangle \langle x'| |\psi_{N,t}\rangle \langle \psi_{N,t}| (x',x).
\]

(4)

We now focus on the trace-norm distance between the one-particle marginal density \(\gamma_{N,t}^{(1)}\) and the projection operator \(|\varphi_t\rangle \langle \varphi_t|\).

In particular, we will prove that

\[
\text{Tr} \left| \gamma_{N,t}^{(1)} - |\varphi_t\rangle \langle \varphi_t| \right| \leq \frac{C(t)}{N}
\]

(5)

and find \(C(t)\) according to the conditions on \(V\). It is known that the optimal \(N\)-dependence for the rate of convergence is of \(O(1/N)\). (See, e.g., [2, 4, 13, 14].) For the necessity of the trace-norm in (5), we refer to [20], where it is also provided an example that explains why \(L^2\)-norm is counterintuitive. Moreover, if the initial many-body state is fully factorized, for every \(t > 0\), the evolved state is never close to the state \(\varphi_{N,t}^{\otimes N}\) in the \(L^2\)-norm, except in the non interacting case. One can quote in this contest the several works aimed to find a norm-approximation of the many-body evolution, by taking into account fluctuations around the Hartree dynamics, see for example [5, 11, 12, 13, 22, 25], and the pioneering papers by Hepp and Ginibre-Velo [3, 10, 18].

Historically, Spohn [26] first proved that \(\text{Tr} \left| \gamma_{N,t}^{(1)} - |\varphi_t\rangle \langle \varphi_t| \right| \to 0\) as \(N \to \infty\) for bounded potential. It was extended by Erdős and Yau [8] to prove the same result for singular potential (including the Coulomb case) by using the BBGKY hierarchy. The rate of convergence, especially the \(N\)-dependence of the bound in (5), has been intensively studied in last ten years. First, a new method based on coherent state approach was introduced by Rodnianski and Schlein in [25] to give an explicit rate of convergence as in (5) with an \(O(1/\sqrt{N})\) bound. The proof is based on the Fock space approach that was introduced by Hepp [18] and extended by Ginibre and Velo [10]. Soon after [25], Knowles and Pickl [20] considered more singular interaction potentials and obtained similar estimates on the rate of convergence. The proofs are based on the use of projection operators, and their approach allows for a large class of possibly time-dependent external potentials. The proof in [20] is based on the use of projection operators in the \(N\)-particle space \(L^2(\mathbb{R}^{3N})\), and allows for a large class of possibly time-dependent external potentials. The \(O(1/N)\) rate of convergence, which is optimal in \(N\)-dependence, was proved by Chen, Lee, and Schlein in [13] for the Coulomb case. It was later extended in [4] to cover the case \(V \in L^2 + L^\infty\). We also remark that Hott [19] pointed out the initial condition may stay in bigger space than \(H^1(\mathbb{R}^3)\) for \(V(x) = \lambda |x|^{-\gamma}\) with \(1 < \gamma < 3/2\).

Unlike the \(N\)-dependence in the rate of the convergence, the time dependence of the bound has mostly been of order \(e^{Kt}\) (or even worse) in most of the works mentioned above, with the exception of [20] where the authors also showed that the rate of convergence can be uniform in time if the solution of the Hartree equation satisfies an integrability condition. For example, in [3], where the use of Strichartz estimate was the main strategy of the proof to generalize the interaction potential, the time dependence of the bound is of order \(e^{Kt^{3/2}}\) which grows faster than \(e^{Kt}\).

It is in general harder to obtain the better bound in terms of \(N\)-dependence in (5) for more singular interaction potential, e.g. \(\gamma > 1\). On the contrary, it is typically more difficult to prove the better bound in terms of \(t\)-dependence for slowly decaying interaction potential or long range potential, e.g. \(\gamma < 1\). (Heuristically, we can also argue that the optimal bound can be proved relatively easily, since it decays sufficiently fast.) It was also remarked by Knowles and Pickl in [20] that the time dependence can be removed for interaction potential with strong decay. Such a phenomenon is known in the Gross-Pitaevskii regime; we

\footnote{For the details of the Strichartz estimate, see [25].}
refer to the work of Chong [7], where the scattering results of the cubic nonlinear Schrödinger equation were used. For a inverse power law potential \( V(x) = \lambda |x|^{-\gamma} \) with \( 0 < \gamma < 3/2 \), however, one may not have the corresponding scattering result for the solution. Nevertheless, it is possible to use the Strichartz estimates as in [4], which can be regarded as a generalized time decay estimate in the time averaged sense. We remark that the time decay estimates of the Hartree equation has been deeply researched in many important works by Hayashi, Naumkin, and Ozawa [15][16][17]. Moreover, the existence of the modified operator of the equation was studied, e.g., by Nakanishi [23][24].

A similar approach can also be applied to many-body semi-relativistic Schrödinger equations which describes a Boson star. Lee [21] provide the optimal rate of convergence \( O(1/N) \) for Coulomb interaction. Following the approach presented in this article, it is believed that one can obtain a corresponding bound for the semi-relativistic case by exploiting the properties of the mean-field solution. We refer to the work of Cho and Ozawa [3] for more detail on the solution of the semi-relativistic Hartree equation. The time dependence of the bound in the semi-relativistic case will be discussed in a future paper.

In this article, we investigate the time dependence \( C(t) \) in (5) by using the results of time decay estimates and Strichartz estimates for \( V(x) = \lambda \exp(-\mu |x|) |x|^{-\gamma} \) for \( \lambda \in \mathbb{R}, \mu \geq 0 \) and \( 0 < \gamma < 3/2 \). More precisely, we prove that the bound in (5) is time-independent if the interaction constant is below a threshold, i.e., \( |\lambda| < \lambda_c \) for some \( \lambda_c = \lambda_c(\gamma, \mu) \). We also improve the time dependence on the bound for more singular potential with \( 1 < \gamma < 3/2 \) and \( \alpha \in [2\gamma/3, 1] \) to \( C_\alpha e^{K_{\gamma, \alpha} t} \), which was \( C e^{K_{\gamma, \alpha} t^{3/2}} \) in [4]. For the exact Coulomb interaction case with \( \gamma = 1 \), we prove a bound that is a polynomial of \( t \) whose degree is proportional to \( \lambda \), hence sublinear in \( t \) if \( \lambda \) is sufficiently small. The bounds are collected in Table 1 which describes the time dependence of the rate of convergence.

**Notational Remark.** We use \( \|f\|_p = \|f\|_{L^p(X)} \) for the standard \( L^p \) norm of \( f : X \to \mathbb{C} \). We also use \( \|f\|_{H^p} = \|f\|_{H^p(X)} \) for the standard \( H^p \) norm of \( f : X \to \mathbb{C} \). We denote \( \|J\|_{op} \) as an operator norm of an operator \( J \). In many lines of inequalities we will face constants \( C \) here and there, and note that the constant may differ line by line. The time dependent constant \( C(t) \) also can differ line by line. Sometimes we may use \( C_\alpha \) if we want to emphasize the dependence on a variable \( \alpha \). We write \( S \) to denote the Schwartz space and \( S' \) to denote the dual space of \( S \).

**Definition 1.1.** We define a generalized Sobolev space, or weighted Sobolev space, such that

\[
H^{m,s}_p = \left\{ \phi \in S' : \|\phi\|_{m,s,p} = \| (1 + |x|^2)^{s/2} (1 - \Delta)^{m/2} \phi \|_p < \infty \right\}
\]

for \( m, s \in \mathbb{R} \). We may simply write \( H^{m,s} \) to denote \( H^{2,m,s}_2 \).

Note that \( H^{0,0} = H^s \) and \( \varphi \in H^{0,k} \) implies \( \hat{\varphi} \in H^k \). Moreover, because one can think of \( |\varphi|^2 \) as a probability distribution under normalization, if \( \varphi \in H^{0,\gamma} \), one can understand that the \( \gamma \)-th moment of \( |\varphi|^2 \) is finite.

**Assumption 1.2.** We assume initial data \( \varphi \) for given \( \lambda, \gamma, \mu \) such that

1. for \( |\lambda| \leq \lambda_c \) and \( 0 < \gamma < 1 \), let \( \varphi \in H^{5,0} \cap H^{0,5} \) with \( \|\varphi\|_{H^{5,0}} + \|\varphi\|_{H^{0,5}} = 1 \),
2. for \( |\lambda| \leq \lambda_c \) and \( 1 \leq \gamma < 3/2 \), let \( \varphi \in H^{5,0} \cap H^{0,5} \) with \( \|\varphi\|_{H^{5,0}} + \|\varphi\|_{H^{0,5}} = 1 \) for \( S > 3/2 \),
3. for \( |\lambda| \leq \lambda_c \) and \( \mu > 0 \), let \( \varphi \in H^{5,0} \cap H^{0,5} \) with \( \|\varphi\|_{H^{5,0}} + \|\varphi\|_{H^{0,5}} = 1 \) for \( S > 3/2 \),
4. for \( \lambda > \lambda_c \), \( \mu = 0 \) and \( 1 \leq \gamma < 3/2 \), let \( \varphi \in H^{2,0} \cap H^{0,2} \), or
5. for \( \lambda > \lambda_c \) and \( \mu > 0 \), let \( \varphi \in H^{2,0} \cap H^{0,2} \),
6. otherwise, let \( \varphi \in H^4(\mathbb{R}^3) \).

**Theorem 1.3.** Assume that the potential \( V(x) = \lambda \exp(-\mu |x|) |x|^{-\gamma} \) with interaction constant \( \lambda \in \mathbb{R} \) and positive \( \mu \geq 0 \). Let \( \lambda_c = \lambda_c(\mu, \gamma) \) be a threshold of interaction constant. Assume that \( \varphi \) follows the Assumption 1.2 for each case. Let \( \varphi_t \) be the solution of the Hartree equation

\[
i \partial_t \varphi_t = -\Delta \varphi_t + (V * |\varphi|^2) \varphi_t
\]

with initial data \( \varphi_{t=0} = \varphi \). Let \( \psi_{N,t} = e^{-i H_{N,t} t} \psi_0 \otimes N \) and \( \gamma^{(1)}_{N,t} \) be the one-particle reduced density associated with \( \psi_{N,t} \), as defined in [4]. Then there exists a time-dependent constant \( C(t) \), depending only on \( \varphi, \lambda, \mu, \) and \( t \) such that

\[
\text{Tr} |\gamma^{(1)}_{N,t} - |\varphi_t\rangle \langle \varphi_t| | \leq \frac{C(t)}{N},
\]

(6)
Moreover, we can choose the time dependent factor $C(t)$ in \ref{4} as in the Table \ref{7} with constants $C$ and $K$ independent of $t$, arbitrary constant $\alpha \in [2\gamma/3, 1)$, and $\lambda_c = \lambda_c(\mu, \gamma)$.

| Table 1: Time dependent factor $C(t)$ of the rate of convergence |
|------------------|------------------|------------------|
| $V(x) = \lambda|x|^{-\gamma}$ | $V(x) = \lambda \exp(-\mu|x|) |x|^{-\gamma}$, $\mu > 0$ |
| $\lambda > \lambda_c$ | $Ce^{Kt}$ | $Ce^{Kt}$ |
| $|\lambda| \leq \lambda_c$ | $Ce^{Kt \gamma/\alpha}$ | $Ce^{Kt \gamma/\alpha}$ |
| $\lambda < -\lambda_c$ | $Ce^{Kt}$ | $Ce^{Kt}$ |

Remark 1.4. Note that for $V(x) = \lambda|x|^{-\gamma}$ with $\lambda < -\lambda_c$, according to \ref{4}, the exponent of $t$ was $3/2$ which is the case $\alpha = 2\gamma/3$. The current paper provides a better time growth rate.

Remark 1.5. Notice that in the case of Coulomb interaction the exponent $K$ in the bound $(1 + t)^K$ is sufficiently small, for small enough $\lambda$. In the proof, we show that $K$ is proportional to $|\lambda|$, i.e., $K = \kappa |\lambda|$ for fixed $\kappa > 0$. Because we are dealing with $|\lambda| < \lambda_c$ with small $\lambda_c$, $K = \kappa |\lambda|$ is also sufficiently small for some constant $\kappa$. Thus, even though it is written as a polynomial of $(1 + t)$, it is actually sublinear in $(1 + t)$.

Remark 1.6. In \ref{20}, the authors remarked that if $\|\varphi_t\|_{L^1}$ and $\|\varphi_t\|_{L^q}$ is integrable in $t$ over $\mathbb{R}$, then the time dependent factor is uniform in time, i.e., $C(t) < \infty$, where $V \in L^p(\mathbb{R}^3) + L^{p_2}(\mathbb{R}^3)$ and $1/2 = 1/p_i + 1/q_i$ for $i = 1, 2$. They also noted that such an integrability condition describes a scattering regime and it requires an interaction potential with strong decay. The result of the current article suggests that the strong decay of $V$, i.e., large $\gamma$, may not be enough to guarantee the scattering behavior but one actually needs to consider the size of the interaction constant $\lambda$. Intuitively, if the interaction constant is too large, the interactions between particles are hard to ignore even with strong decay. Thus, the particles cannot be asymptotically free even for large $t$, and one cannot expect the usual scattering behavior.

Remark 1.7. Theorem \ref{1.3} is valid also if one considers more general initial data $\tilde{\Psi}_N \in L^2(\mathbb{R}^{3N})$, not necessarily factorized, which exhibits condensation (in the sense of the convergence of the one-particle marginal density). More precisely, let $\Psi_N = \varphi_{0N}$ and consider $\tilde{\Psi}_N$ such that $\|\tilde{\Psi}_N - \Psi_N\|_{L^2(\mathbb{R}^{3N})} = O(1/N)$. Let $\gamma_N^{(1)}$ be the one particle reduced density of $\tilde{\Psi}_N$. Then

$$\int \text{d}x \left( \psi_{N,t} \bar{\psi}_{N,t} - \tilde{\psi}_{N,t} \bar{\psi}_{N,t} \right) = \text{Re} \int \text{d}x \left( \psi_{N,t} \bar{\psi}_{N,t} - \tilde{\psi}_{N,t} \bar{\psi}_{N,t} \right)$$

$$\leq \|\psi_{N,t} - \tilde{\psi}_{N,t}\|_{L^2(\mathbb{R}^{3N})} \|\psi_{N,t}\|_{L^2(\mathbb{R}^{3N})} + \|\tilde{\psi}_{N,t}\|_{L^2(\mathbb{R}^{3N})}$$

One may hope to have the following bound:

$$\|\tilde{\Psi}_{N,t} - \Psi_{N,t}\|_{L^2(\mathbb{R}^{3N})} \leq \frac{C(t)}{N}$$

for some $C(t)$. Suppose we have such bound. This leads us to

$$\text{Tr} \left| \gamma_N^{(1)} - |\varphi_t\rangle \langle \varphi_t| \right| \leq \text{Tr} \left| \gamma_N^{(1)} - \gamma_N^{(1)} \right| + \text{Tr} \left| \gamma_N^{(1)} - |\varphi_t\rangle \langle \varphi_t| \right| \leq \frac{C(t)}{N}.$$

We follow the approach in \ref{2, 3, 4, 25} for the proof of Theorem \ref{1.3}. In this method based on the analysis of the coherent states in the Fock space, the main obstacle is that a bound on the term $\int_0^t \text{d}s \|V(\cdot - x)|\varphi_s\|_{\infty}$ is required. For this reason, we begin by establishing the time dependence of $\int_0^t \text{d}s \|V(\cdot - x)|\varphi_s\|_{\infty}$. The estimate is based on several time decay estimates of the solution of the Hartree equation.
The rest of the paper is organized as follows: We will provide the estimates for $\int_0^t ds \| V(\cdot - x)\varphi_s \|_\infty$ in Section 2. In Section 2.2, we will provide a sketch of proof of time decay estimates for Yukawa interaction, because it is a simple adjustment of previous results [13, 17]. In Section 3, we briefly provide definitions and properties of Fock space which we are going to use. Section 4 is devoted to give proof of the main theorem. We have many useful bounds for operators in Fock space to prove the main theorem in Section 5. While the most of the materials in Sections 3 through 5 are similar to those in the previous works [3, 4, 25], we do not omit them in the current paper in order to provide a logically complete explanation of our proof.

2 Properties of solution of mean-field equation

This section is devoted to provide time dependent or time independent bounds of $\int_0^t ds \| V(\cdot - x)\varphi_s \|_\infty$ for each case appeared in Table 1.

2.1 Time decay estimate of the Hartree equation for Coulomb type interaction

This section introduces time decay estimates of the Hartree equation. We will show that $\int_0^\infty dt \| V(\cdot - x)\varphi_t \|_\infty < C$ using time decay estimates for weakly attracting Hartree equation.

**Proposition 2.1.** Suppose that $\varphi_t$ is a solution of (5). Suppose that $\lambda$ in (5) is sufficiently small, the suitable size of $\lambda$ is depending on $\gamma$, $\mu$, and $\varphi$. We assume that

1. $\varphi \in H^{5,0} \cap H^{0,5}$ for $\mu = 0$ and $0 < \gamma < 1$,
2. $\varphi \in H^{S,0} \cap H^{0,S}$ for $\mu = 0$ and $1 \leq \gamma < 3/2$ with $S > 3/2$, or
3. $\varphi \in H^{S,0} \cap H^{0,S}$ for $\mu > 0$ and $0 \leq \gamma < 3/2$ with $S > 3/2$.

Then there exists a unique global solution $\varphi_t$ of (5) such that

$$\| \varphi_t \|_\infty \leq C\lambda (1 + |t|)^{-3/2}.$$

**Proposition 2.2.** Suppose that $\varphi_t$ is a solution of (5). We assume that $\varphi \in H^{2,0} \cap H^{0,2}$ for (i) $\lambda > 0$, $\mu = 0$ and $1 < \gamma < 3/2$ or (ii) $\lambda > 0$, $\mu > 0$ and $0 < \gamma < 3/2$. Then there exists a unique global solution $\varphi_t$ of (5) such that

$$\| \varphi_t \|_\infty \leq C\lambda (1 + |t|)^{-1/2}.$$

To prove this theorem, we are going to use small data scattering theory for Hartree dynamics; Hayashi and Namukin found that:

**Lemma 2.3** (Hayashi and Namukin 98). We assume that $\varphi \in H^{S,0}(\mathbb{R}^n) \cap H^{0,S}(\mathbb{R}^n)$ and $\| \varphi \|_{S,0} + \| \varphi \|_{0,S} = \epsilon' < \epsilon$, where $\epsilon$ is sufficiently small and $n/2 < S < p = 1 + 2/n$. Then there exists a unique global solution $\varphi_t$ to the Hartree equation (5) with

$$V(x) = \lambda |x|^{-1} + \mu |x|^{-\delta}$$

for $1 < \delta < n$, such that

$$\varphi_t \in C(\mathbb{R}, H^{S,0} \cap H^{0,S})$$

and

$$\| \varphi_t \|_\infty \leq C\epsilon' (1 + |t|)^{-3/2}.$$

**Proof.** See Theorem 1.1 of [13].
Lemma 2.4 (Hayashi and Naumkin 01’). We assume that \( \varphi \in H^{5,0}(\mathbb{R}^n) \cap H^{0,5}(\mathbb{R}^n) \) and \( \|\varphi\|_{5,0} + \|\varphi\|_{0,5} = \epsilon' < \epsilon \), where \( \epsilon \) is sufficiently small. Then there exists a unique global solution \( \varphi_t \) of the Hartree equation (3), with

\[
V(x) = \lambda |x|^{-\delta}
\]

for \( 0 < \delta < 1 \), such that

\[
\|\varphi_t\|_{\infty} \leq C \epsilon'(1 + |t|)^{-3/2}.
\]

Proof. See Theorem 1.1 of [16]. If we put \( n = 3 \) and \( p = \infty \) for our discussion, we get the result.

Lemma 2.5 (Hayashi and Ozawa 87’). We assume that \( \varphi \in H^{2,0}(\mathbb{R}^n) \cap H^{0,2}(\mathbb{R}^n) \). Then there exists a unique global solution \( \varphi_t \) of the Hartree type equation (3), with

\[
V(x) = |x|^{-1}
\]

Then,

\[
\|\varphi_t\|_{\infty} \leq C (1 + |t|)^{-1/2}.
\]

Proof. See Theorem 1.1 [17].

Notice that Lemma 2.4 and 2.5 were proven under the condition of small initial initial data. We will interpret (or convert) this result into the case of generic initial data with weak interaction. The strategy is the following:

We substitute \( \varphi \) with \( \tilde{\varphi}(\epsilon'/M) \) for suitable constant \( M > 0 \). Then \( \tilde{\varphi} \) solves the partial differential equation

\[
i\partial_t \tilde{\varphi}_t = -\Delta \tilde{\varphi}_t + ((\epsilon'/M)^2 V * |\tilde{\varphi}|^2) \tilde{\varphi}_t
\]

with initial data \( \|\tilde{\varphi}\|_{\gamma,0} + \|\tilde{\varphi}\|_{0,\gamma} = M \). Now, letting \( \tilde{\lambda} := \lambda \epsilon'^2/M^2 \),

\[
(\epsilon'/M)^2 V = \frac{\lambda \epsilon'^2 e^{-\mu|x|}}{M^2 |x|^{-\gamma}} = \frac{\lambda \epsilon'^2 e^{-\mu|x|}}{M^2 |x|^{-\gamma}}.
\]

Note that \( \epsilon' = \epsilon'(\tilde{\lambda}) \) was small enough and \( M > 0 \) was arbitrarily chosen. Hence, we have new Hartree equation

\[
i\partial_t \tilde{\varphi}_t = -\Delta \tilde{\varphi}_t + (\tilde{\lambda} \frac{\epsilon'^2 e^{-\mu|x|}}{|x|^{-\gamma}}) \tilde{\varphi}_t
\]

with small interaction constant \( \tilde{\lambda} \) such that \( |\tilde{\lambda}| \leq \lambda_c = \lambda_c(\mu, \gamma, M) \). Therefore, using this ‘interpretation’, we have Proposition 2.4 and Proposition 2.2.

2.2 Time decay estimates of the Hartree equation for Yukawa type interaction

In this section, we provide decay estimates for Yukawa type interaction potential. Since the proofs will closely follow [15] and [17], we only provide the sketch of proofs. For time decay estimates, heuristically, the main difficulty stems from attractive, long-range interaction potential; if the range of the interaction is short enough, then ‘far sides’ of wave function would not interact with each other. Hence, if there is a time decay estimate for Coulomb interaction, one can also expect that there is a similar bound for Yukawa type interaction. Even though the explanation here is rather heuristic, this can be made rigorous as in the following lemmas, whose proofs are based on fixed point arguments.

Lemma 2.6. We assume that \( \varphi \in H^{S,0}(\mathbb{R}^n) \cap H^{0,S}(\mathbb{R}^n) \) and \( \|\varphi\|_{S,0} + \|\varphi\|_{0,S} = \epsilon' < \epsilon \), where \( \epsilon \) is sufficiently small and \( 3/2 < S < 5/3 \). Then there exists a unique global solution \( \varphi_t \) of the Hartree equation (3), with

\[
V(x) = \frac{\lambda e^{-\mu|x|}}{|x|^{-\gamma}}
\]

for \( \mu > 0 \) and \( 0 < \gamma < 3/2 \), such that

\[
\varphi_t \in C(\mathbb{R}, H^{S,0} \cap H^{0,S})
\]

and

\[
\|\varphi_t\|_{\infty} \leq C \epsilon'(1 + |t|)^{-3/2}.
\]
Lemma 2.9. Because the proof in [15] relies on the fact that
\[ \| |x|^{-\gamma} * |u| \|_{L^p(\mathbb{R}^n)} < \infty \]
and
\[ \| (-t^2 \Delta)^{s/2} |x|^{-\gamma} * |u| \|_{L^p(\mathbb{R}^n)} < \infty \]
for some \( t > 0, 0 < s < 1, 1 \leq \gamma < 3/2, \) and \( n \in \mathbb{Z}, \)
we have
\[ \| \exp(-\mu|x|) |x|^{-\gamma} * |u| \|_{L^p(\mathbb{R}^n)} < \infty \]
and
\[ \| (-t^2 \Delta)^{s/2} \exp(-\mu|x|) |x|^{-\gamma} * |u| \|_{L^p(\mathbb{R}^n)} \leq \| (-t^2 \Delta)^{s/2} \exp(-\mu|x|) |x|^{-\gamma} * |u| \|_{L^p(\mathbb{R}^n)} < \infty. \]

Lemma 2.7. We assume that \( \varphi \in H^{2,0}(\mathbb{R}^n) \cap H^{0,2}(\mathbb{R}^n). \) Then there exists a unique global solution \( \varphi_t \) of the Hartree type equation (3), with
\[ V(x) = \frac{\lambda e^{-\mu|x|}}{|x|^\gamma}. \]
Then,
\[ \| \varphi_t \|_{\infty} \leq C(1 + |t|)^{-1/2}. \]

Idea of proof. Noting that \( e^{-\mu|x|} |x|^{-\gamma} < C|x|^{-1} \) for some \( C = C(\mu, \gamma). \) We follow the proof of [17]. □

2.3 On the time dependence of \( \int_0^t ds \| V(x - \cdot) \varphi_s \|_{L^2} \)

We are going to prepare for Section 5.1. Proposition 2.10 below is the key lemma to improve the time dependence of the Lemma presented in Section 5.1. The proof of Proposition 2.10 is based on the following two lemmas.

Lemma 2.8 (Boundedness of \( H^1 \)-norm of \( \varphi_t \)). For the solution \( \varphi_t \) of the Hartree equation (3) for \( V \in L^2 + L^\infty, \varphi_0 \in H^1(\mathbb{R}^3) \) for the Hartree equation then there exist constant \( C \) depending only on \( \varphi_0 \) and \( V \) such that
\[ \| \varphi_t \|_{H^1(\mathbb{R}^3)} \leq C. \]

Proof. See Lemma 2.1 of [3]. □

Lemma 2.9 (Strichartz estimate for \( V \in L^2 \)). Suppose that \( V \in L^2(\mathbb{R}^3). \) Let \( \varphi_t \) be the solution of the Hartree equation (3) with initial data \( \varphi_0 = \varphi \in H^1(\mathbb{R}^3), \) then there exists a constant \( C, \) depending only on \( \| \varphi \|_{H^1} \) and \( \| V \|_{L^2}, \) such that
\[ \| \varphi_t \|_{L^2((0,T),L^\infty)} \leq C \sqrt{1+T}. \]

Proof. We closely follow [1, Theorem 2.3.3] for the proof of the lemma. The result for \( V \in L^2 + L^\infty \) is in the proof of Lemma 2.8 and here we remove terms for \( L^\infty \) part of \( V. \) From the Sobolev inequality and the Strichartz’s estimate, we have
\[ \| \varphi_t \|_{L^2((0,T),L^\infty)} \leq C\| \varphi_t \|_{L^2((0,T),W^{1,s})} \]
\[ \leq C\| \varphi_0 \|_{H^1} + C \| (V * |\varphi_t|^2) \varphi_t \|_{L^2((0,T),W^{1,s/5})}. \]
(7)

From the definition of the Sobolev norm,
\[ \| (V * |\varphi_t|^2) \varphi_t \|_{L^2((0,T),W^{1,s/5})} \]
\[ \leq C\| (V * |\varphi_t|^2) \varphi_t \|_{L^2((0,T),L^{6/5})} + C \| \nabla ((V * |\varphi_t|^2) \varphi_t) \|_{L^2((0,T),L^{6/5})}. \]
(8)
We first focus on the spatial integral; integration with respect to the time variable \( t \) will be considered later. In the first term in the right-hand side of (8), the integrand of the spatial integral is bounded by

\[
\| (V \ast |\varphi|^2) \varphi_t \|_{L^{6/5}} \leq \| V \ast |\varphi|^2 \|_{L^3} \| \varphi_t \|_{L^2} \leq \| V \|_{L^2} \| |\varphi|^2 \|_{L^{6/5}} \| \varphi_t \|_{L^2} \\
\leq \| V \|_{L^2} \| |\varphi|^2 \|_{L^{12/5}} \| \varphi_t \|_{L^2} \leq \| \varphi_t \|_{L^2}^{5/2} \| \varphi_t \|_{L^6}^{1/2} \\
\leq \| V \|_{L^2} \| \varphi_t \|_{L^2}^{5/2} \| \varphi_t \|_{\dot{H}^1}^{1/2},
\]

where we used Hölder’s inequality, Young’s inequality, and Riesz-Thorin Theorem. Similarly, we decompose the integrand of the second term in the right-hand side (8) into two parts and find that

\[
\| \nabla ((V \ast |\varphi|^2) \varphi_t) \|_{L^2((0,T),L^{6/5})} \\
\leq \| V \ast (\nabla |\varphi|^2) \varphi_t \|_{L^2((0,T),L^{6/5})} + \| (V \ast |\varphi|^2) (\nabla \varphi_t) \|_{L^2((0,T),L^{6/5})}.
\]

We again apply Hölder’s inequality, Young’s inequality, and Riesz-Thorin Theorem to get

\[
\| V \ast (\nabla |\varphi|^2) \varphi_t \|_{L^{6/5}} \leq \| V \ast (\nabla |\varphi|^2) \|_{L^1} \| \varphi_t \|_{L^2} \leq C \| V \|_{L^2} \| \nabla \varphi_t \|_{L^{6/5}} \| \varphi_t \|_{L^2} \\
\leq C \| V \|_{L^2} \| \varphi_t \|_{L^2} \| \nabla \varphi_t \|_{L^2} \| \varphi_t \|_{L^2} \leq C \| V \|_{L^2} \| \varphi_t \|_{L^3}^{3/2} \| \varphi_t \|_{\dot{H}^1}^{3/2},
\]

\[
\| (V \ast |\varphi|^2) (\nabla \varphi_t) \|_{L^{6/5}} \leq \| V \ast |\varphi|^2 \|_{L^2} \| \nabla \varphi_t \|_{L^2} \leq \| \varphi_t \|_{L^2}^{3/2} \| \varphi_t \|_{\dot{H}^1}^{3/2}.
\]

Thus, after taking \( L^2 \)-norm according to (7) with respect to the time variable \( t \), with the mass conservation \( \| \varphi_t \|_{L^2} = 1 \) and Lemma (2.8) we conclude that

\[
\| \varphi_t \|_{L^2((0,T),L^\infty)} \leq C \sqrt{1 + T}.
\]

\[\Box\]

**Proposition 2.10 (Key estimate).** Suppose that \( \varphi_s \) a solution of (3) with initial data \( \varphi \) satisfies Assumption (7.2). We have

\[
\int_0^t ds \| V(x - \cdot) \varphi_s \|_2 \leq C(t).
\]

where \( C(t) = C(\varphi_0, V, t) \) is depends only on initial data \( \varphi_0 \), interaction potential \( V \) and time \( t \), given in Table 7.

**Remark 2.11.** Strichartz estimate was used to obtain

\[
\int_0^t ds \sup_x \| V(\cdot - x) \varphi_s \|_\infty \leq C(1 + t)^{3/2}
\]

in [4]. Here we use Proposition 2.10 so that

\[
\int_0^t ds \sup_x \| V(\cdot - x) \varphi_s \|_\infty \leq C(t).
\]

**Proof of Proposition 2.10** Throughout this proof, (i) for \( |\lambda| < \lambda_c \), we use the time decay estimate to prove a sub-exponential bound in time, and (ii) for \( |\lambda| > \lambda_c \), we prove an exponential (or slightly bigger) bound in time without time decay estimate.

For Coulomb cases, we consider the following: For a fixed \( x \in \mathbb{R}^3 \), let \( B_r = \{ y \in \mathbb{R}^3 : |x - y| \leq r \} \) be the ball centered at \( x \) with radius \( r \). By Hölder inequality, the fact that \( |x - y|^{-2\gamma} < 1 \) for \( y \in B_r \), Sobolev embedding, and Lemma (2.8) we have

\[
\frac{1}{\lambda^2} \| V(x - \cdot) \varphi_s \|_2^2 = \int dy \frac{\| |\varphi_s(y)|^2 \|_{L^2}}{|x - y|^{2\gamma}} = \int_{B_Y} dy \frac{\| |\varphi_s(y)|^2 \|_{L^2}}{|x - y|^{2\gamma}} + \int_{B_{Y}^c} dy \frac{\| |\varphi_s(y)|^2 \|_{L^2}}{|x - y|^{2\gamma}}
\]

\[
\leq C \| |\varphi_s| \|_\infty \left( \int_{B_Y} dy \frac{1}{|x - y|^{2\gamma}} \right) + C \left( f(s) \right)^{-2\gamma} \| |\varphi_s| \|_2^2
\]

(11)
for a positive valued function $f(s)$ with arbitrary $s > 0$, which will be determined later.

Note that
\[
\int_{B_{f(s)}} dy \frac{1}{|x-y|^{2\gamma}} = 4\pi \int_0^{f(s)} r^{2-2\gamma} dr = \frac{4\pi}{3-2\gamma} (f(s))^{3-2\gamma}
\]

implies, by time decay estimate, that
\[
\frac{1}{\lambda^2} \|V(x-\cdot)\varphi_s\|_2^2 \leq C \|\varphi_s\|_{\infty}^2 (f(s))^{3-2\gamma} + C (f(s))^{-2\gamma}
\]
\[
\leq C(1+s)^{-3} (f(s))^{3-2\gamma} + C (f(s))^{-2\gamma}.
\]

By letting $f(s) = 1 + s$, we get
\[
\|V(x-\cdot)\varphi_s\|_2^2 \leq C(1+s)^{-2\gamma}.
\]

**Case 1.** $V(x) = \lambda |x|^{-\gamma}$ with $0 < \gamma < 1$ and $\lambda \in \mathbb{R}$.

From Hölder inequality and Hardy inequality, we get
\[
\frac{1}{\lambda^2} \|V(x-\cdot)\varphi_s\|_2^2 = \int dy |x-y|^{-2\gamma} |\varphi_s(y)|^2
\]
\[
= \int dy |x-y|^{-2\gamma} |\varphi_s(y)|^2 \cdot |\varphi_s(y)|^{2-2\gamma}
\]
\[
\leq \left( \int dy |x-y|^{-2\gamma} |\varphi_s(y)|^2 \right)^{\gamma} \left( \int dy |\varphi_s(y)|^{2-2\gamma} \right)^{1-\gamma}
\]
\[
\leq C \|\varphi_s\|_{H^1}^2 \|\varphi_s\|_{L^{2-2\gamma}}^{2-2\gamma}
\]
\[
\leq C \|\varphi_s\|_{H^1}^2 \leq C.
\]

**Case 2.** $V(x) = \lambda |x|^{-\gamma}$ with $0 < \gamma < 1$ and $|\lambda| \leq \lambda_c$.

We have from (12) that
\[
\int_0^t ds \|V(x-\cdot)\varphi_s\|_2 \leq \int_0^t ds (1+s)^{-\gamma} \leq \frac{1}{1-\gamma} (1+t)^{1-\gamma}.
\]

Then
\[
\exp \left( \int_0^t ds \|V(x-\cdot)\varphi_s\|_2 \right) \leq \exp \left( K (1+t)^{1-\gamma} \right).
\]

**Case 3.1.** $V(x) = \lambda |x|^{-\gamma}$ with $\gamma = 1$ and $|\lambda| \leq \lambda_c$.

From Kato’s inequality and (12),
\[
\int_0^t ds \|V(x-\cdot)\varphi_s\|_2 \leq \int_0^t ds 2\sqrt{\pi} |\lambda| (1+s)^{-1} \leq 2\sqrt{\pi} |\lambda| \log(1+t).
\]

Then
\[
\exp \left( \int_0^t ds \|V(x-\cdot)\varphi_s\|_2 \right) \leq (1+t)^{2\sqrt{\pi} |\lambda|}.
\]

**Case 3.2.** $V(x) = \lambda |x|^{-\gamma}$ with $1 < \gamma < 3/2$ and $|\lambda| \leq \lambda_c$.

From (12),
\[
\int_0^t ds \|V(x-\cdot)\varphi_s\|_2 \leq C \int_0^t ds (1+s)^{-\gamma} \leq \frac{C}{\gamma-1}.
\]

Thus
\[
\exp \left( \int_0^t ds \|V(x-\cdot)\varphi_s\|_2 \right) \leq C.
\]

**Case 3.3.** $V(x) = \lambda |x|^{-\gamma}$ with $0 < \gamma < 3/2$ and $\lambda \in \mathbb{R}$.
Let \( 2\gamma/3 \leq \alpha = \alpha(\gamma) < 1 \) so that

\[
\frac{1}{\lambda^2} \| V(x - \cdot) \varphi_s \|_2^2 = \int dy \frac{|\varphi_s(y)|^2}{|x - y|^{2\gamma}} = \int_{B_1} dy \frac{|\varphi_s(y)|^2}{|x - y|^{2\gamma}} + \int_{B_1^c} dy \frac{|\varphi_s(y)|^2}{|x - y|^{2\gamma}}.
\]

(13)

Note that by Hölder inequality with a pair \((\frac{3\alpha}{2\gamma}, \frac{3\alpha}{3\alpha - 2\gamma})\), we have

\[
\int_{B_1} dy \frac{|\varphi_s(y)|^2}{|x - y|^{2\gamma}} \leq \| |x - \cdot|^{-2\gamma}\|_\frac{3\alpha}{2\gamma} \| \varphi_s^2\|_\frac{3\alpha}{3\alpha - 2\gamma} \leq \left( \int_{B_1} dy |x - \cdot|^{-3\alpha} \right)^{\frac{2\alpha}{3\alpha - 2\gamma}} \| \varphi_s\|_2^{\frac{6\alpha}{3\alpha - 2\gamma}}.
\]

Since \( \alpha < 1 \), the first factor \( \left( \int_{B_1} dy |x - \cdot|^{-3\alpha} \right)^{2\gamma/3\alpha} = C_\alpha < \infty \). By Riesz–Thorin theorem

\[
\| \varphi_s \|_{\frac{6\alpha}{3\alpha - 2\gamma}} \leq \| \varphi_s \|_6^{\frac{3\alpha - 2\gamma}{\alpha}} \| \varphi_s \|_\infty^{\frac{2\gamma - 2\alpha}{\alpha}}.
\]

Thus

\[
\int_{B_1} dy \frac{|\varphi_s(y)|^2}{|x - y|^{2\gamma}} \leq C \| \varphi_s \|_\infty^{4(\gamma - \alpha)/\alpha}.
\]

Next, we bound the second term of (13) using that \( |x - y|^{-2\gamma} \leq 1 \) for \( y \in B_1^c \) so that

\[
\int_{B_1^c} dy \frac{|\varphi_s(y)|^2}{|x - y|^{2\gamma}} \leq \int_{B_1^c} dy |\varphi_s(y)|^2 \leq \| \varphi_s \|_2^2 \leq C.
\]

Hence

\[
\frac{1}{\lambda^2} \| V(x - \cdot) \varphi_s \|_2^2 \leq C \| \varphi_s \|_\infty^{4(\gamma - \alpha)/\alpha} + C.
\]

Now we have, using Hölder inequality in time and Strichartz estimate,

\[
\int_0^t ds \| V(x - \cdot) \varphi_s \|_2 \leq C \int_0^t ds \left( \| \varphi_s \|_\infty^{2(\gamma - \alpha)/\alpha} + 1 \right)
\]

\[
\leq C \left( \int_0^t ds \right)^{\frac{2\alpha - \gamma}{\alpha}} \left( \int_0^t ds \| \varphi_s \|_\infty^{2(\gamma - \alpha)/\alpha} \right) + Ct
\]

\[
\leq C(1 + t)^{\frac{2\alpha - \gamma}{\alpha} + \frac{2\gamma - 2\alpha}{\alpha}} + Ct \leq \max \{ C(1 + t)^{\gamma/\alpha}, Ce^t \}
\]

for any \( \alpha \in [2\gamma/3, 1) \).

**Case 3.4.** \( V(x) = \lambda \exp(-\mu|x|)|x|^{-\gamma} \) with \( 0 < \gamma < 3/2, \mu > 0 \), and \( \lambda \in \mathbb{R} \).

Note that then \( V \in L^2 \). By Hölder inequality and Sobolev embedding,

\[
\frac{1}{\lambda^2} \| V(x - \cdot) \varphi_s \|_2^2 = \int dy \frac{e^{-2\mu|x-y|}}{|x - y|^{2\gamma}} |\varphi_s(y)|^2
\]

\[
\leq \| \varphi_s(y) \|_\infty^2 \int dy \frac{e^{-2\mu|x-y|}}{|x - y|^{2\gamma}} \leq C \| \varphi_s \|_\infty^2.
\]

Then by Cauchy–Schwarz inequality and Lemma 2.9 (Strichartz estimate), we get

\[
\int_0^t ds \| V(x - \cdot) \varphi_s \|_2 \leq C \int_0^t ds \| \varphi_s \|_\infty \leq C \left( \int_0^t ds \right)^{1/2}
\]

\[
\left( \int_0^t ds \| \varphi_s \|_\infty^2 \right)^{1/2} \leq C(1 + t).
\]
**Case 4.** \(V(x) = \lambda \exp(-\mu|x|)|x|^{-\gamma}\) with \(\mu > 0, 0 < \gamma < 3/2,\) and \(|\lambda| \leq \lambda_c\).

Using Hölder inequality,

\[
\frac{1}{\lambda^2} \|V(x - \cdot) \varphi_s\|_2^2 = \int dy \frac{e^{-2\mu|x-y|}|\varphi_s(y)|^2}{|x-y|^{2\gamma}} \\
\leq C \|\varphi_s\|_2^2 \int dy \frac{e^{-2\mu|x-y|}}{|x-y|^{2\gamma}} \leq C(1+s)^{-3},
\]

hence we get a time independent bound

\[
\int_0^t ds \|V(x - \cdot) \varphi_s\|_2 \leq \int_0^t ds C(1+s)^{-3/2} \leq C.
\]

**Case 5.** \(V(x) = \lambda |x|^{-\gamma}\) with \(1 \leq \gamma < 3/2\) and \(\lambda > 0\).

From (11),

\[
\frac{1}{\lambda^2} \|V(x - \cdot) \varphi_s\|_2^2 \leq C \|\varphi_s\|_\infty^2 \left( f(s) \right)^{3-2\gamma} + C \left( f(s) \right)^{-2\gamma} \\
\leq C(1+s)^{-1} \left( f(s) \right)^{3-2\gamma} + C \left( f(s) \right)^{-2\gamma}.
\]

By putting \(f(s) = (1+s)^{1/3}\), we find that

\[
\|V(x - \cdot) \varphi_s\|_2^2 \leq C(1+s)^{-2\gamma/3},
\]

hence

\[
\exp\left( \int_0^t ds \|V(x - \cdot) \varphi_s\|_2 \right) \leq C \exp\left( K(1+t)^{1-2\gamma/3} \right).
\]

**Case 6.** \(V(x) = \lambda \exp(-\mu|x|)|x|^{-\gamma}\) with \(0 < \gamma < 3/2, \mu > 0,\) and \(\lambda > 0\).

Similarly to (11),

\[
\frac{1}{\lambda^2} \|V(x - \cdot) \varphi_s\|_2^2 \leq C \|\varphi_s\|_\infty^2 \left( \int_{B(f(s))} dy \frac{e^{-2\mu|x-y|}}{|x-y|^{2\gamma}} \right) + C \left( f(s) \right)^{-2\gamma} \|\varphi_s\|_2^2 \\
\leq C \|\varphi_s\|_\infty^2 + C \frac{e^{-2f(s)}}{f(s)^{2\gamma}}.
\]

Letting \(f(s) = 1+s\), we get

\[
\frac{1}{\lambda^2} \|V(x - \cdot) \varphi_s\|_2^2 \leq C(1+s)^{-1},
\]

which implies that

\[
\int_0^t ds \|V(x - \cdot) \varphi_s\|_2 \leq C \int_0^t ds(1+s)^{-1} \leq C \log(1+t).
\]

This completes the proof. \(\Box\)

**Remark 2.12.** In the table below, we summarize the cases considered in the proof of Proposition 2.10.

---

11
This section is devoted to explain Fock space formalism for studying the dynamics of the system of $N$-Bosons. We consider Bosonic Fock space as in $[3, 21, 25]$. The Bosonic Fock space is a Hilbert space defined by

$$\mathcal{F} = \bigoplus_{n \geq 0} L^2 \left( \mathbb{R}^3, dx \right)^{\otimes n} = \mathbb{C} \oplus \bigoplus_{n \geq 1} L^2_s \left( \mathbb{R}^{3n}, dx_1, \ldots, dx_n \right),$$

where $L^2_s = L^2_s(\mathbb{R}^{3n}, dx_1, \ldots, dx_n)$ is a subspace of $L^2(\mathbb{R}^{3n}, dx_1, \ldots, dx_n)$ that is the space of all functions symmetric under any permutation of $x_1, x_2, \ldots, x_n$. It is convenient to let $L^2_s(\mathbb{R}^3)^{\otimes 0} = \mathbb{C}$. An element $\psi \in \mathcal{F}$ can be understood as a sequence $\psi = \{ \psi^{(n)} \}_{n \geq 0}$ of $n$-particle wave functions $\psi^{(n)} \in L^2_s(\mathbb{R}^{3n})$ or as a vector in a countable dimensional vector space such that each $n$-th component is a function $\psi^{(n)}(\cdot) \in L^2_s(\mathbb{R}^{3n})$. The inner product on $\mathcal{F}$ is defined by

$$\langle \psi_1, \psi_2 \rangle = \sum_{n \geq 0} \langle \psi_1^{(n)}, \psi_2^{(n)} \rangle_{L^2(\mathbb{R}^{3n})} = \psi_1^{(0)} \psi_2^{(0)} + \sum_{n \geq 1} \int dx_1 \ldots dx_n \psi_1^{(n)}(x_1, \ldots, x_n) \psi_2^{(n)}(x_1, \ldots, x_n).$$

We denote $\| \psi \| = \langle \psi, \psi \rangle^{1/2}. The vector $\Omega := \{1, 0, 0, \ldots \} \in \mathcal{F}$ is called the vacuum. Note that an element $\psi \in \mathcal{F}$ is denoting a many-body quantum state which can have uncertainty of the number of particles of the quantum system. Because of that one can think of generation or annihilation of a particle. For $f \in L^2(\mathbb{R}^3)$, we define the creation operator $a^*(f)$ and the annihilation operator $a(f)$ on $\mathcal{F}$ by

$$a^*(f) \psi^{(n)}(x_1, \ldots, x_n) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} f(x_j) \psi^{(n-1)}(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$$

and

$$a(f) \psi^{(n)}(x_1, \ldots, x_n) = \sqrt{n+1} \int dx f(x) \psi^{(n+1)}(x, x_1, \ldots, x_n),$$

(15)

each of which denotes the creation or annihilation of a particle having wave function $f$. By definition, the creation operator $a^*(f)$ is the adjoint of the annihilation operator of $a(f)$, and in particular, $a^*(f)$ and $a(f)$ are not self-adjoint. We will use the self-adjoint operator $\phi(f)$ defined as

$$\phi(f) = a^*(f) + a(f).$$

Let $a^*_x$ and $a_x$ operator-valued distributions such that

$$a^*(f) = \int dx f(x) a^*_x, \quad a(f) = \int dx f(x) a_x$$

for any $f \in L^2(\mathbb{R}^3)$. For each non-negative integer $n$, we introduce the projection operator onto the $n$-particle sector of the Fock space, for $\psi = (\psi^{(0)}, \psi^{(1)}, \ldots) \in \mathcal{F}$,

$$P_n(\psi) := (0, 0, \ldots, 0, \psi^{(n)}, 0, \ldots).$$

(16)
For simplicity, with slight abuse of notation, we will use \( \psi^{(n)} \) to denote \( P_n \psi \). The will use number operator \( \mathcal{N} \) which counts the expected number of particles of a vector in \( \mathcal{F} \) and is defined by

\[
\mathcal{N} = \int dx \, a^*_x a_x. \tag{17}
\]

Note that \( \mathcal{N} \) satisfies that \( (\mathcal{N} \psi)^{(n)} = n \psi^{(n)} \). Let \( J \) be an operator defined on the one-particle sector \( L^2(\mathbb{R}^3, dx) \), then we extend this operator into Fock space by \( d\Gamma(J) \), which is called its second quantization and whose action on the \( n \)-particle sector is given by

\[
(d\Gamma(J)\psi)^{(n)} = \sum_{j=1}^n J_j \psi^{(n)}
\]

where \( J_j = 1 \otimes \ldots \otimes J \otimes \ldots \otimes 1 \) is the operator \( J \) acting on the \( j \)-th variable only. With a kernel \( J(x; y) \) of the operator \( J \), the second quantization \( d\Gamma(J) \) can be also be written as

\[
d\Gamma(J) = \int dx dy J(x; y) a^*_x a_y.
\]

The following lemma shows that the annihilation operator and the creation operator can be bounded roughly \( \mathcal{N}^{1/2} \) or \( (\mathcal{N} + 1)^{1/2} \). Moreover, it gives a bound of the second quantization operators.

**Lemma 3.1 (Lemma 2.1 in [3])**. For \( \alpha > 0 \), let \( D(\mathcal{N}^\alpha) = \{ \psi \in \mathcal{F} : \sum_{n \geq 1} n^{2\alpha} \| \psi^{(n)} \|^2_2 < \infty \} \) denote the domain of the operator \( \mathcal{N}^\alpha \). For any \( f \in L^2(\mathbb{R}^3, dx) \) and any \( \psi \in D(\mathcal{N}^{1/2}) \), we have

\[
\| a(f) \psi \|_F \leq \| f \|_2 \| \mathcal{N}^{1/2} \psi \|_F,
\]

\[
\| a^*(f) \psi \|_F \leq \| f \|_2 \| (\mathcal{N} + 1)^{1/2} \psi \|_F,
\]

\[
\| \phi(f) \psi \|_F \leq 2 \| f \|_2 \| (\mathcal{N} + 1)^{1/2} \psi \|_F.
\]

Moreover, for any bounded one-particle operator \( J \) on \( L^2(\mathbb{R}^3, dx) \) and for every \( \psi \in D(\mathcal{N}) \), we find

\[
\| d\Gamma(J)\psi \|_F \leq \| J \|_{op} \| \mathcal{N} \psi \|_F. \tag{19}
\]

To consider the problem embedded into the Fock space, we extend Hamiltonian in (1) to the Fock space by

\[
\mathcal{H}_N := \int dx \, \nabla_x a^*_x \nabla_x a_x + \frac{1}{2 \mathcal{N}} \int dx dy \, V(x - y) \, a^*_x a_y a_x. \tag{20}
\]

This definition satisfies \( (\mathcal{H}_N \psi)^{(N)} = H_N \psi^{(N)} \) for \( \psi \in \mathcal{F} \). Hence it is a generalization of (1) into the Fock space. The one-particle marginal density \( \gamma^{(1)}_\psi \) associated with \( \psi \) is

\[
\gamma^{(1)}_\psi(x; y) = \frac{1}{\langle \psi, \mathcal{N} \psi \rangle_F} \langle \psi, a^*_y a_x \psi \rangle_F. \tag{21}
\]

Note that \( \gamma^{(1)}_\psi \) is a trace class operator on \( L^2(\mathbb{R}^3) \) and \( \text{Tr} \gamma^{(1)}_\psi = 1 \). It can be easily checked that (21) is equivalent to (4).

We defined a coherent state which is an eigenvector of annihilation operator \( a(f) \) such that

\[
\psi(f) = e^{-\| f \|^2/2} \sum_{n \geq 0} \frac{(a^*_f(f))^n}{n!} \Omega = e^{-\| f \|^2/2} \sum_{n \geq 0} \frac{1}{\sqrt{n!}} f^\otimes n.
\]

For \( f \in L^2(\mathbb{R}^3) \), the Weyl operator \( W(f) \) is defined by

\[
W(f) := \exp(a^*_f(f) - a(f))
\]
and it also satisfies
\[ W(f) = e^{-\|f\|_2^2/2} \exp(a^*(f)) \exp(-a(f)), \]
which is known as the Hadamard lemma in Lie algebra. The Weyl operator is closely related to the coherent states. The coherent state can also be expressed in terms of the Weyl operator as
\[ \psi(f) = W(f) \Omega = e^{-\|f\|_2^2/2} \exp(a^*(f)) \Omega = e^{-\|f\|_2^2/2} \sum_{n \geq 0} \frac{1}{\sqrt{n!}} f^\otimes n. \tag{22} \]

We collect the useful properties of the Weyl operator and the coherent states in the following lemma.

**Lemma 3.2** (Part of Lemma 2.2 in [3]). Let \( f, g \in L^2(\mathbb{R}^3, dx) \).

1. The commutation relation between the Weyl operators is given by
\[ W(f) W(g) = W(g) W(f) e^{-2i \text{Im}(f, g)} = W(f + g) e^{-i \text{Im}(f, g)}. \]
2. The Weyl operator is unitary and satisfies that
\[ W(f)^* = W(f)^{-1} = W(-f). \]
3. The coherent states are eigenvectors of annihilation operators, i.e.,
\[ a_x \psi(f) = f(x) \psi(f) \quad \Rightarrow \quad a(g) \psi(f) = \langle g, f \rangle_{L^2} \psi(f). \]

The commutation relation between the Weyl operator and the annihilation operator (or the creation operator) is thus
\[ W^*(f) a_x W(f) = a_x + f(x) \quad \text{and} \quad W^*(f) a_x^* W(f) = a_x^* + \overline{f(x)}. \]
4. The distribution of \( N \) with respect to the coherent state \( \psi(f) \) is Poisson. In particular,
\[ \langle \psi(f) , \mathcal{N} \psi(f) \rangle_F = \|f\|_2^2, \quad \langle \psi(f) , \mathcal{N}^2 \psi(f) \rangle_F - \langle \psi(f) , \mathcal{N} \psi(f) \rangle_F^2 = \|f\|_2^2. \]

We define, for following lemmas,
\[ d_N^2 := \frac{\sqrt{N!}}{N^{N/2} e^{-N/2}}, \tag{23} \]
and note that \( C^{-1} N^{1/4} \leq d_N \leq C N^{1/4} \) for some constant \( C > 0 \) independent of \( N \), which can be easily checked by using Stirling’s formula.

**Lemma 3.3.** There exists a constant \( C > 0 \) independent of \( N \) such that, for any \( \varphi \in L^2(\mathbb{R}^3) \) with \( \|\varphi\|_2 = 1 \), we have
\[ \left\| (N + 1)^{-1/2} W^*(\sqrt{N} \varphi) \frac{(a^*(\varphi))^N}{\sqrt{N!}} \right\|_F \leq \frac{C(t)}{d_N}. \]

**Proof.** See [2] Lemma 6.3. \( \square \)

**Lemma 3.4.** Let \( P_m \) be the projection onto the \( m \)-particle sector of the Fock space \( \mathcal{F} \) for a non-negative integer \( m \). Then, for any non-negative integers \( k \leq (1/2) N^{1/3} \),
\[ \left\| P_{2k} W^*(\sqrt{N} \varphi) \frac{(a^*(\varphi))^N}{\sqrt{N!}} \right\|_F \leq \frac{2}{d_N} \]
and
\[ \left\| P_{2k+1} W^*(\sqrt{N} \varphi) \frac{(a^*(\varphi))^N}{\sqrt{N!}} \right\|_F \leq \frac{2(k + 1)^{3/2}}{d_N \sqrt{N}}. \]

**Proof.** See [21] Lemma 7.2. \( \square \)
4 Proof of Main Theorem

In this section, we prove the main result of the paper, Theorem 1.1 following the same logic given in [4].

4.1 Unitary operators and their generators

We let 
\[ \psi_t = e^{-i H_N t} \varphi^\otimes N \]
so that \( \psi_t \) is the time evolution of the factorized state \( \varphi^\otimes N \) with respect to the Hamiltonian \( H_N \). Noting the definition of \( k \)-particle marginal density \( \gamma^{(k)} \), the one-particle marginal density associated with \( \psi_t \) can be written as
\[
\gamma^{(1)}_{N,t} = \frac{\langle e^{-i H_N t} \varphi^\otimes N, a_y^* a_x e^{-i H_N t} \varphi^\otimes N \rangle}{\langle e^{-i H_N t} \varphi^\otimes N, \varphi^\otimes N \rangle} = \frac{1}{N} \left( \langle \varphi^\otimes N, e^{i H_N t} a_y^* a_x e^{-i H_N t} \varphi^\otimes N \rangle \right). 
\]

We want to argue that (24) can be approximated by the one-particle marginal density associated with the coherent states. To use the coherent state, we expand \( a_y^* a_x \) around \( N \varphi_t(y) \varphi_t(x) \). The expansion leads us to investigate
\[ W^*(\sqrt{N} \varphi_t) e^{-i H_N (t-s)} W(\sqrt{N} \varphi_s) \]
(25)
and (28) with respect to \( t \) as in [4, 21, 25], we have
\[
i \partial_t W^*(\sqrt{N} \varphi_t) e^{-i H_N (t-s)} W(\sqrt{N} \varphi_s) = \sum_{k=0}^{4} \mathcal{L}_k(t) \]
(26)
where
\[ \mathcal{L}_0(t) := \frac{N}{2} \int \int dx \int dx (V \ast |\varphi_t|^2)(x)|\varphi_t(x)|^2, \]
\[ \mathcal{L}_1(t) = 0, \]
\[ \mathcal{L}_2(t) := \int dx \nabla_x a_y^* \nabla_x a_x + \int dx \left( V \ast |\varphi_t|^2 \right) (x) a_x^* a_x \]
\[ + \int dx dy V (x-y) \varphi_t(x) \varphi_t(y) a_y^* a_x + \frac{1}{2} \int dx dy V (x-y) \left( \varphi_t(x) \varphi_t(y) a_x^* a_y + \varphi_t(x) a_x \varphi_t(y) a_y \right), \]
\[ \mathcal{L}_3(t) := \frac{1}{\sqrt{N}} \int \int dx dy V (x-y) \left( \varphi_t(y) a_y^* a_x + \varphi_t(y) a_x^* a_y \right) a_x, \]
and
\[ \mathcal{L}_4 := \frac{1}{2N} \int \int dx dy V (x-y) a_x^* a_y^* a_x a_y. \]
(27)
(28)
(29)
Because the phase factor \( \mathcal{L}_0(t) \) is just a complex-valued function, we can cancel this term by multiplying the right-hand side of (26) by a function \( e^{-i \mathcal{L}_0(t)} \) (see Section 3 of [21]). Thus, if we define the unitary operator \( U(t; s) \) by
\[ U(t; s) := e^{-i \omega(t; s) W^*(\sqrt{N} \varphi_t) e^{-i H_N (t-s)} W(\sqrt{N} \varphi_s)} \]
(15)
As explained in Section 1, we use the technique developed in [21] to prove Theorem 1.3. The proof of Theorem 1.3 consists of the following two propositions.

**Proposition 4.1.** Suppose that the assumptions in Theorem 1.3 hold. For a Hermitian operator $J$ on $L^2(\mathbb{R}^3)$, let

$$E^1_1(J) := \frac{d_N}{N} \left< W^*(\sqrt{N}\varphi) \left( (a^*(\varphi))^N \sqrt{N!} \right) \Omega, \mathcal{U}^*(t) \mathcal{U}(t) \Omega \right>_{\mathcal{F}}.$$

Then, there exist a constant $C(t)$ depending only on $\lambda$, $\varphi_0$, and $t$ such that

$$|E^1_1(J)| \leq \frac{C(t)}{N} \| J \|_{\text{op}}.$$

**Proposition 4.2.** Suppose that the assumptions in Theorem 1.3 hold. For a Hermitian operator $J$ on $L^2(\mathbb{R}^3)$, let

$$E^2_1(J) := \frac{d_N}{N} \left< W^*(\sqrt{N}\varphi) \left( (a^*(\varphi))^N \sqrt{N!} \right) \Omega, \mathcal{U}^*(t) \phi(J \varphi_t) \mathcal{U}(t) \Omega \right>_{\mathcal{F}}.$$

Then, there exist a constant $C(t)$ depending only on $\lambda$, $\varphi_0$, and $t$ such that

$$|E^2_1(J)| \leq \frac{C(t)}{N} \| J \|_{\text{op}}.$$

Proof of Propositions 4.1 and 4.2 will be given later in section 5.2. With Propositions 4.1 and 4.2, we now prove Theorem 1.3.

**Proof of Theorem 1.3** By the definition of $k$-particle density, in (24) we have

$$\gamma^{(1)}_{N,t} = \frac{1}{N} \left< \left( (a^*(\varphi))^N \sqrt{N!} \right) \Omega, e^{i\mathcal{H}_N t} a_y a_x e^{-i\mathcal{H}_N t} (a^*(\varphi))^N \sqrt{N!} \Omega \right>_{\mathcal{F}}.$$

From (14), the factorized state $\varphi \otimes N$ in $\mathcal{F}$ can be written in the following form:

$$\{0, 0, \ldots, 0, \varphi \otimes N, 0, \ldots \} = \frac{(a^*(\varphi))^N}{\sqrt{N!}} \Omega.$$  \hfill (33)
From \(16\) and \(22\), we find that
\[
\frac{(a^*(\varphi))^N}{\sqrt{N!}} \Omega = \frac{\sqrt{N!}}{N^{N/2} e^{-N/2}} P_N W(\sqrt{N} \varphi) \Omega = d_N P_N W(\sqrt{N} \varphi) \Omega.
\]

Since \([\mathcal{H}_N, N] = 0\), we also have that
\[
\gamma^{(1)}_{N,t}(x; y) = \frac{1}{N} \left\langle \frac{(a^*(\varphi))^N}{\sqrt{N!}}, e^{i\mathcal{H}_N t} a_y^* a_x e^{-i\mathcal{H}_N t} \frac{(a^*(\varphi))^N}{\sqrt{N!}} \right\rangle \Omega
\]
\[
eq \frac{d_N}{N} \left\langle \frac{(a^*(\varphi))^N}{\sqrt{N!}}, e^{i\mathcal{H}_N t} a_y^* a_x e^{-i\mathcal{H}_N t} P_N W(\sqrt{N} \varphi) \Omega \right\rangle \Omega
\]
\[
eq \frac{d_N}{N} \left\langle \frac{(a^*(\varphi))^N}{\sqrt{N!}}, P_N e^{i\mathcal{H}_N t} a_y^* a_x e^{-i\mathcal{H}_N t} W(\sqrt{N} \varphi) \Omega \right\rangle \Omega
\]
\[
eq \frac{d_N}{N} \left\langle \frac{(a^*(\varphi))^N}{\sqrt{N!}}, \gamma^{(1)}_{N,t}(x; y) \varphi_t(y) \right\rangle \Omega.
\]

Moreover, using
\[
e^{i\mathcal{H}_N t} a_x e^{-i\mathcal{H}_N t} = W(\sqrt{N} \varphi) \mathcal{U}^*(t)(a_x + \sqrt{N} \varphi_t(x)) \mathcal{U}(t) W^*(\sqrt{N} \varphi)
\]
and similar relation for the \(a_y^*\), we obtain that
\[
\gamma^{(1)}_{N,t}(x; y) = \frac{d_N}{N} \left\langle \frac{(a^*(\varphi))^N}{\sqrt{N!}}, e^{i\mathcal{H}_N t} a_y^* a_x e^{-i\mathcal{H}_N t} W(\sqrt{N} \varphi) \Omega \right\rangle \Omega
\]
\[
eq \frac{d_N}{N} \left\langle \frac{(a^*(\varphi))^N}{\sqrt{N!}}, W(\sqrt{N} \varphi) \mathcal{U}^*(t)(a_y^* + \sqrt{N} \varphi_t(y))(a_x + \sqrt{N} \varphi_t(x)) \mathcal{U}(t) \Omega \right\rangle \Omega.
\]

Hence,
\[
\gamma^{(1)}_{N,t}(x; y) - \varphi_t(y) \varphi_t(x) = \frac{d_N}{N} \left\langle \frac{(a^*(\varphi))^N}{\sqrt{N!}}, W(\sqrt{N} \varphi) \mathcal{U}^*(t) a_y^* a_x \mathcal{U}(t) \Omega \right\rangle \Omega
\]
\[
+ \varphi_t(y) \frac{d_N}{N} \left\langle \frac{(a^*(\varphi))^N}{\sqrt{N!}}, W(\sqrt{N} \varphi) \mathcal{U}^*(t) a_y^* \mathcal{U}(t) \Omega \right\rangle \Omega
\]
\[
+ \varphi_t(x) \frac{d_N}{N} \left\langle \frac{(a^*(\varphi))^N}{\sqrt{N!}}, W(\sqrt{N} \varphi) \mathcal{U}^*(t) a_y^* \mathcal{U}(t) \Omega \right\rangle \Omega.
\]

By the definition of \(E^1_{\varphi}(J)\) and \(E^2_{\varphi}(J)\) in Propositions \(4.1\) and \(4.2\) for any compact one-particle Hermitian operator \(J\) on \(L^2(\mathbb{R}^3)\), we obtain
\[
\text{Tr}(J (\gamma^{(1)}_{N,t} - |\varphi_t\rangle \langle \varphi_t|)) = \int dx dy J(x; y) \left( \gamma^{(1)}_{N,t}(y; x) - \varphi_t(y) \varphi_t(x) \right)
\]
\[
= \frac{d_N}{N} \left\langle \frac{(a^*(\varphi))^N}{\sqrt{N!}}, W(\sqrt{N} \varphi) \mathcal{U}^*(t) d\Gamma(J) \mathcal{U}(t) \Omega \right\rangle \Omega
\]
\[
+ \varphi_t(y) \frac{d_N}{N} \left\langle \frac{(a^*(\varphi))^N}{\sqrt{N!}}, W(\sqrt{N} \varphi) \mathcal{U}^*(t) \phi(J\varphi_t) \mathcal{U}(t) \Omega \right\rangle \Omega
\]
\[
= E^1_{\varphi}(J) + E^2_{\varphi}(J).
\]
Thus, Propositions 4.1 and 4.2 lead us to
\[ \left| \text{Tr} J(\gamma_{N,t}^{(1)} - |\varphi_t\rangle \langle \varphi_t|) \right| \leq C(t) \frac{\|J\|_{\text{op}}}{N}. \]

Since the space of compact operators is the dual to that of the trace class operators, and since \( \gamma_{N,t}^{(1)} \) and \( |\varphi_t\rangle \langle \varphi_t| \) are Hermitian,
\[ \text{Tr} \left| \gamma_{N,t}^{(1)} - |\varphi_t\rangle \langle \varphi_t| \right| \leq \frac{C(t)}{N} \]

which concludes the proof of Theorem 1.3.

5 Comparison of Dynamics and Proof of Propositions

5.1 Comparison of dynamics

This section follows [25]. Rodnianski and Schlein used Hardy inequality to bound the time integration of \( \sup_x \|V(x)\varphi_t\|_\infty \leq C \) in [25]. In [4], the authors used Strichartz estimate to bound the time integration of \( \sup_x \|V(x)\varphi_t\|_\infty \), i.e.,
\[ \int_0^t ds \sup_x \|V(x)\varphi_s\| \leq \left( \int_0^t ds \right)^{1/2} \sup_x \|V(x)\varphi_s\| \leq C t^{3/2}. \]

This section will bound \( \sup_x \|V(x)\varphi_t\|_\infty \) by \( C(t) \) so that we can use the Table 1. Since the structure of each proof coincides with previous results [4,25], here we just provide the lemmas without proofs, because one can easily change all the \( C e^{\kappa t} \) appeared in [25] by \( C(t) \).

**Lemma 5.1.** Suppose that the assumptions in Theorem 1.3 hold. Then, for any \( \psi \in F \) and \( j \in \mathbb{N} \), there exist a constant \( C \equiv C(j) \) such that
\[ \left\| (N + 1)^{1/2} L_3(t) \psi \right\|_F \leq \frac{C}{\sqrt{N}} \sup_x \|V(x)\varphi_t\|_2 \left\| (N + 1)^{(j+3)/2} \psi \right\|_F. \]

**Proof.** See Lemma 4.6 of [4].

**Lemma 5.2.** Suppose that the assumptions in Theorem 1.3 hold. Let \( \mathcal{U}(t;s) \) be the unitary evolution defined in (30). Then for any \( \psi \in F \) and \( j \in \mathbb{N} \), there exist constants \( C(t) \equiv C(t,j) \) such that
\[ \left\langle \mathcal{U}(t;s) \psi, N^j \mathcal{U}(t;s) \psi \right\rangle_F \leq C(t) \left\langle \psi, (N + 1)^{2j+2} \psi \right\rangle_F. \]

**Proof.** See Lemma 4.1 of [4].

**Lemma 5.3.** Suppose that the assumptions in Theorem 1.3 hold. Let \( \tilde{\mathcal{U}}(t;s) \) be the unitary evolution defined in (31). Then, for any \( \psi \in F \) and \( j \in \mathbb{N} \), there exist a constant \( C(t) \equiv C(t,j) \) such that
\[ \left\langle \tilde{\mathcal{U}}(t;s) \psi, N^j \tilde{\mathcal{U}}(t;s) \psi \right\rangle_F \leq C(t) \left\langle \psi, (N + 1)^{2j+2} \psi \right\rangle_F. \]

**Proof.** See Lemma 4.5 of [4].

The following lemma will be used in the proof of Proposition 4.2 in the following Section 5.2

**Lemma 5.4.** Suppose that the assumptions in Theorem 1.3 hold. Let \( \mathcal{U}(t;s) \) and \( \tilde{\mathcal{U}}(t;s) \) be the unitary evolution defined in (30) and (31) respectively. Then, for all \( j \in \mathbb{N} \), there exist constants \( C(t) \equiv C(t,j) \) such that, for any \( f \in L^2(\mathbb{R}^3) \),
\[ \left\| (N + 1)^{1/2} \left( \mathcal{U}^* (t) \phi(f) \mathcal{U}(t) - \tilde{\mathcal{U}}^* (t) \phi(f) \tilde{\mathcal{U}}(t) \right) \right\|_F \leq C(t) \frac{\|f\|_2}{\sqrt{N}}. \]

**Proof.** See Lemma 4.7 of [4]. To prove this lemma, we use Lemma 5.1 and 5.3 as Lemma 4.7 of [4] used Lemma 4.1 and 4.5 of [4].
5.2 Proof of Propositions 4.1 and 4.2

In this section, we prove Propositions 4.1 and 4.2 by applying the lemmas provided in Subsection 5.1.

**Proof of Proposition 4.1.** Note that

\[ E_1(t)(J) = \frac{d_N}{N} \left\langle W^* \left( \sqrt{N} \varphi \right) \frac{(a^*(\varphi))^N}{\sqrt{N!}} \Omega, \mathcal{U}(t) \Gamma(J) \mathcal{U}(t) \Omega \right\rangle \]

implies that

\[ |E_1(t)(J)| = \left| \frac{d_N}{N} \left\langle W^* \left( \sqrt{N} \varphi \right) \frac{(a^*(\varphi))^N}{\sqrt{N!}} \Omega, \mathcal{U}(t) \Gamma(J) \mathcal{U}(t) \Omega \right\rangle \right| \]

\[ \leq \frac{d_N}{N} \left\| (N+1)^{-\frac{1}{2}} W^* \left( \sqrt{N} \varphi \right) \frac{(a^*(\varphi))^N}{\sqrt{N!}} \Omega \right\|_F \]

\[ \times \left\| (N+1)^{\frac{1}{2}} \mathcal{U}(t) \Gamma(J) \mathcal{U}(t) \Omega \right\|_F. \]

Using Lemma 3.3, we have

\[ \left\| (N+1)^{-\frac{1}{2}} W^* \left( \sqrt{N} \varphi \right) \frac{(a^*(\varphi))^N}{\sqrt{N!}} \Omega \right\|_F \leq \frac{C(t)}{d_N}. \]  

By applying Lemma 5.2 and (19) several times, we get

\[ \left\| (N+1)^{\frac{1}{2}} \mathcal{U}(t) \Gamma(J) \mathcal{U}(t) \Omega \right\|_F \leq C(t) \left\| (N+1)^{\frac{1}{2}} \mathcal{U}(t) \Gamma(J) \mathcal{U}(t) \Omega \right\|_F \]

\[ \leq C(t) \| J \|_{\text{op}} \left\| (N+1)^{\frac{1}{2}} \mathcal{U}(t) \Gamma(J) \mathcal{U}(t) \Omega \right\|_F \]

\[ \leq C(t) \| J \|_{\text{op}} \left\| (N+1)^{\frac{1}{2}} \Omega \right\|_F. \]

Therefore, from (34), (35), and (36), we have the desired bound

\[ |E_1(t)(J)| \leq \frac{C(t)\| J \|_{\text{op}}}{N}. \]

\[ \square \]

For the proof of Proposition 4.2, we apply a very similar approach to the one used in the proof of Lemma 4.2 in [21]. To obtain the logical completeness, we fill the detail.

**Proof of Proposition 4.2.** Let

\[ \mathcal{R}(f) = \mathcal{U}(t)f \mathcal{U}(t) - \mathcal{U}(t)f \mathcal{U}(t). \]

According to (32), the even sector will have zero amplitude, i.e.

\[ P_{2t} \mathcal{U}(t)f \mathcal{U}(t) = 0. \]
for all \( \ell = 0, 1, \ldots \). (See Lemma 8.2 in [21] for more detail.) This gives us that

\[
|E^\ell_J| = \frac{d_N}{\sqrt{N}} \left\langle \frac{(a^\ast(\phi))^N}{\sqrt{N!}} \Omega, W^* (\sqrt{N} \varphi) \bar{U}^* (t) \phi(J, \varphi) \bar{U}(t) \Omega \right\rangle_F \\
+ \frac{d_N}{\sqrt{N}} \left\langle \frac{(a^\ast(\phi))^N}{\sqrt{N!}} \Omega, W^* (\sqrt{N} \varphi) \mathcal{R}(J, \varphi) \bar{U}(t) \Omega \right\rangle_F
\]

\[
\leq \frac{d_N}{\sqrt{N}} \left\| \sum_{\ell=1}^{\infty} (\mathcal{N} + 1)^{-\frac{\ell}{2}} P_{2\ell-1} W^* (\sqrt{N} \varphi) \frac{(a^\ast(\phi))^N}{\sqrt{N!}} \Omega \right\|_F
\times \left\| (\mathcal{N} + 1)^{\frac{\ell}{2}} \bar{U}^* (t) \phi(J, \varphi) \bar{U}(t) \Omega \right\|_F
\]

\[
+ \frac{d_N}{\sqrt{N}} \left\| (\mathcal{N} + 1)^{-\frac{\ell}{2}} W^* (\sqrt{N} \varphi) \frac{(a^\ast(\phi))^N}{\sqrt{N!}} \Omega \right\|_F
\times \left\| (\mathcal{N} + 1)^{\frac{\ell}{2}} \mathcal{R}(J, \varphi) \bar{U}(t) \Omega \right\|_F
\]

(37)

We divide the sum into two group using \( L = \frac{1}{2} N^{1/3} \), Lemma 5.3 and Lemma 5.4, such that

\[
\left\| \sum_{\ell=1}^{L} (\mathcal{N} + 1)^{-\frac{\ell}{2}} P_{2\ell-1} W^* (\sqrt{N} \varphi) \frac{(a^\ast(\phi))^N}{\sqrt{N!}} \Omega \right\|_F^2
\leq \sum_{\ell=1}^{L} \left\| (\mathcal{N} + 1)^{-\frac{\ell}{2}} P_{2\ell-1} W^* (\sqrt{N} \varphi) \frac{(a^\ast(\phi))^N}{\sqrt{N!}} \Omega \right\|_F^2
\]

\[
+ \frac{1}{L^L} \sum_{\ell=1}^{\infty} \left\| (\mathcal{N} + 1)^{-1/2} P_{2\ell-1} W^* (\sqrt{N} \varphi) \frac{(a^\ast(\phi))^N}{\sqrt{N!}} \Omega \right\|_F^2
\]

\[
\leq \left( \frac{C}{L^2 d_N^2 N} \right) + \frac{C}{N^{4/3}} \left\| (\mathcal{N} + 1)^{-1/2} W^* (\sqrt{N} \varphi) \frac{(a^\ast(\phi))^N}{\sqrt{N!}} \Omega \right\|_F \leq \frac{C(t)}{d_N^2 N}. \tag{38}
\]

Applying Lemma 5.3,

\[
\left\| (\mathcal{N} + 1)^{\frac{\ell}{2}} \bar{U}^* (t) \phi(J, \varphi) \bar{U}(t) \Omega \right\|_F \leq C(t) \left\| (\mathcal{N} + 1)^{\frac{\ell}{2}} \phi(J, \varphi) \bar{U}(t) \Omega \right\|_F
\]

\[
\leq C(t) \left\| J \varphi \right\| \left\| (\mathcal{N} + 1)^{\frac{\ell}{2}} \bar{U}(t) \Omega \right\|_F \leq C(t) \left\| J \right\| \left\| (\mathcal{N} + 1)^{\frac{\ell}{2}} \Omega \right\|_F \leq C \left\| J \right\|_{\text{op}}.
\]

For the second term of (38), we apply Lemmas 5.3 and 5.4 and put \( J \varphi \) into \( J \). Altogether, we get the desired bound

\[
\left\| (\mathcal{N} + 1)^{1/2} \mathcal{R}(f) \Omega \right\|_F \leq \frac{C(t) \left\| f \right\|_{2}}{N}.
\]

\[ \square \]

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