Nonexistence of certain binary and ternary linear complementary dual codes

Ken Saito

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Abstract

A linear complementary dual codes are codes whose intersections with their dual codes are trivial. In this paper, we give results on the nonexistence of some linear complementary dual codes with large minimum weights. We completely determine the largest minimum weight among all binary linear complementary dual codes for dimension 4. We also determine the largest minimum weight among all ternary linear complementary dual codes for dimensions 2 and 3.

1 Introduction

Linear complementary dual codes codes were introduced by Massey [12]. We say that such a code is LCD for brevity. LCD codes are an important class of linear codes for both theoretical and practical reasons (see [1], [2], [3], [5], [6], [7], [8], [9], [12]).

It is a fundamental problem to determine the largest minimum weight $d_q(n,k)$ among all LCD $[n,k]$ codes over $\mathbb{F}_q$ for given $n$, $k$ and $q$. It has been shown that any code over $\mathbb{F}_q$ is equivalent to an LCD code for $q \geq 4$ [6]. This motivates us to study binary and ternary LCD codes. Recently, the values $d_q(n,k)$ have been determined for $q \in \{2,3\}$ and small $k$. The values $d_2(n,1)$ were determined in [7]. The values $d_2(n,2)$ were determined in [8].

*Research Center for Pure and Applied Mathematics, Graduate School of Information Sciences, Tohoku University, Sendai 980–8579, Japan. email: kensaito@ims.is.tohoku.ac.jp.
The values $d_2(n, 3)$ were determined in [9]. The number of all inequivalent LCD $[n, k]$ codes over $\mathbb{F}_q$ were determined for $q \in \{2, 3\}$ and $k \in \{1, n-1\}$ in [2]. The aim of this paper is to determine the values $d_q(n, k)$ for $(q, k) \in \{(2, 4), (3, 2), (3, 3)\}$.

In this paper, we study LCD $[n, k]$ codes over $\mathbb{F}_q$ having large minimum weights, under the condition that $(q, k_0) \in \{(2, 3), (3, 2)\}$ and $k \geq k_0$. Then the simplex $[n, k]$ codes over $\mathbb{F}_q$ are constant weight and self-orthogonal codes, which meet the Griesmer bound [10, Theorems 1.4.8 (ii), 1.4.10 (i) and 2.7.5]. The simplex codes play an important role in our study.

The paper is organized as follows. In Section 2, we give definitions, notations and basic results for LCD codes. In Section 3, we give some constructions of LCD codes using self-orthogonal codes. In Section 4, we give results on the nonexistence of some LCD codes with large minimum weights. These results are used in Sections 5 and 6. In Section 5, we study the minimum weights of binary LCD $[n, k]$ codes. We give results on the nonexistence of some binary LCD $[n, k]$ codes with large minimum weights for $k \in \{4, 5\}$. We show that $d_2(n, 4) = \left\lfloor \frac{8n}{15} \right\rfloor$ if $n \equiv 5, 9, 13 \pmod{15}$, $\left\lfloor \frac{8n}{15} \right\rfloor - 1$ if $n \equiv 1, 2, 3, 4, 6, 7, 8, 10, 11, 12, 14 \pmod{15}$ and $\left\lfloor \frac{8n}{15} \right\rfloor - 2$ if $n \equiv 0 \pmod{15}$. We also give an observation on $d_2(n, 5)$. In Section 6, we study the minimum weights of ternary LCD $[n, k]$ codes. We give some results on the nonexistence of some ternary LCD $[n, k]$ codes with the largest minimum weights for $k \in \{2, 3, 4\}$. We show that $d_3(n, 2) = \left\lfloor \frac{3n}{7} \right\rfloor$ if $n \equiv 1, 2 \pmod{4}$, $\left\lfloor \frac{3n}{7} \right\rfloor - 1$ if $n \equiv 0, 3 \pmod{4}$, $d_3(n, 3) = \left\lfloor \frac{3n}{13} \right\rfloor$ if $n \equiv 2, 3, 4, 6, 7, 10 \pmod{13}$ and $d_3(n, 3) = \left\lfloor \frac{3n}{13} \right\rfloor - 1$ if $n \equiv 0, 1, 5, 8, 9, 11, 12 \pmod{13}$. We also give an observation on $d_3(n, 4)$.

All computer calculations in this paper were done by the program written in MAGMA [4].

2 Preliminaries

2.1 Definitions, notations and basic results

We denote the finite field of order $q$ by $\mathbb{F}_q$, where $q$ is a prime power. An $[n, k]$ code $C$ over $\mathbb{F}_q$ is a $k$-dimensional subspace of $\mathbb{F}_q^n$. The parameter $n$ is called the length of $C$. A generator matrix of an $[n, k]$ code $C$ is a $k \times n$ matrix whose rows are basis of $C$. The support of a vector $x = (x_1, \ldots, x_n) \in \mathbb{F}_q^n$ is the subset $\{i \mid x_i \neq 0\}$ of $\{1, \ldots, n\}$. The weight of a vector $x \in \mathbb{F}_q^n$ is the
cardinality of the support of \( x \). A vector of \( C \) is called a codeword of \( C \). The minimum nonzero weight of all codewords in \( C \) is called the minimum weight \( d(C) \) of \( C \). An \([n, k]\) code with minimum weight \( d \) is called an \([n, k, d]\) code.

We use the following notations throughout this paper. Let \( \mathbf{0}_n \) and \( \mathbf{1}_n \) denote the zero vector and the all-one vector of length \( n \), respectively. Let \( O_k \) denote the \( k \times k \) zero matrix. For a matrix \( A \), let \( A^T \) and \( A^{(s)} \) denote the transpose of \( A \) and the juxtaposition \((A, \ldots, A)\) of \( s \)-copies of \( A \), respectively. Let \( \mathbb{Z}_{\geq 0} \) denote the set of nonnegative integers.

The dual code \( C^\perp \) of an \([n, k]\) code \( C \) over \( \mathbb{F}_q \) is defined as \( C^\perp = \{ x \in \mathbb{F}_q^n \mid x \cdot y = 0 \text{ for all } y \in C \} \), where \( x \cdot y \) is the standard inner product. A code \( C \) is called self-orthogonal if \( C \subset C^\perp \). A code \( C \) is called a linear complementary dual (LCD for brevity) code if \( C \cap C^\perp = \{ \mathbf{0}_n \} \). The following characterization is due to Massey [12].

**Proposition 2.1.** Let \( G \) be a generator matrix of a code \( C \) over \( \mathbb{F}_q \). Then \( C \) is LCD if and only if \( GG^T \) is nonsingular.

Throughout this paper, we use the above proposition. Let \( d_q(n, k) \) denote the largest minimum weight among all LCD \([n, k]\) codes over \( \mathbb{F}_q \). Let

\[
\alpha_q(n, k) = \max \left\{ d \in \mathbb{Z}_{\geq 0} \mid n \geq \sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil \right\}
\]

for given \( n \), \( k \) and \( q \). By the Griesmer bound, any \([n, k]\) code over \( \mathbb{F}_q \) has minimum weight at most \( \alpha_q(n, k) \).

The following lemma is a straightforward generalization of [9, Lemma 2.2].

**Lemma 2.2.** Suppose that there is an LCD \([n, k, d]\) code \( C \) over \( \mathbb{F}_q \). If \( d_q(n-1, k) \leq d - 1 \), then \( d(C^\perp) \geq 2 \).

**Proof.** If \( d(C^\perp) = 1 \), then an LCD \([n-1, k, d]\) code over \( \mathbb{F}_q \) is constructed by deleting a zero column of a generator matrix of \( C \). \( \square \)

### 3 Construction of some LCD codes

In this section, we assume that \((q, k_0) \in \{(2, 3), (3, 2)\}\). For \( k \geq k_0 \), we give a construction of LCD \([n, k]\) codes over \( \mathbb{F}_q \) with large minimum weights. We
use the notation \([k]_q = (q^k - 1)/(q - 1)\) for a positive integer \(k\). We define the \(k \times [k]_q\) \(\mathbb{F}_q\)-matrices \(S_{q,k}\) as follows:

\[
S_{q,1} = (1), \\
S_{2,k} = \begin{pmatrix} S_{2,k-1} & 0_{k-1}^T \\ 0_{[k-1]_2} & 1 \end{pmatrix}, \\
S_{3,k} = \begin{pmatrix} S_{3,k-1} & 0_{k-1}^T \\ 0_{[k-1]_3} & 1 \end{pmatrix}.
\]

The matrix \(S_{q,k}\) is a generator matrix of the simplex \([k]_q, k, q^{k-1}\) code. The simplex \([[k]_q, k, q^{k-1}]\) code is a constant weight code [10, Theorem 2.7.5]. It is trivial that \(S_{q,k}\) is LCD.

Then we have

\[
\begin{align*}
\text{Lemma 3.1.} & \quad \text{Let } G_0 \text{ and } G' \text{ be generator matrices of an LCD } [n_0, k, d_0] \text{ code } C_0 \text{ and a self-orthogonal } [n', k, d'] \text{ code } C' \text{ over } \mathbb{F}_q, \text{ respectively. Then the code with generator matrix } (G_0 | G') \text{ is an LCD } [n_0 + n', k, d] \text{ code } C \text{ over } \mathbb{F}_q \text{ with } d \geq d_0 + d'.
\end{align*}
\]

\textbf{Proof.} It is trivial that \(C\) is an \([n_0 + n', k, d]\) code over \(\mathbb{F}_q\) with \(d \geq d_0 + d'\). Since \(C'\) is self-orthogonal, we have \(G'(G')^T = O_k\). Then we have

\[
(G_0 | G')(G_0 | G')^T = G_0 G_0^T + G'(G')^T = G_0 G_0^T.
\]

Since \(C_0\) is LCD, the matrix \(G_0 G_0^T\) is nonsingular. Hence, \(C\) is LCD. \(\Box\)
Proposition 3.2. Suppose that \( k \geq k_0 \). Let \( m = (m_1, \ldots, m_{[k]q}) \in \mathbb{Z}_{\geq 0}^{[k]q} \). If \( C_{q,k}(m) \) is an LCD code with minimum weight \( d_0 \), then \( C_{q,k}(m; s) \) is an LCD code of length \( n \) with minimum weight \( d \), where

\[
(n, d) = \left( \sum_{i=1}^{[k]q} m_i + [k]q \cdot s, d_0 + q^{k-1}s \right).
\]

Proof. Suppose that \( C_{q,k}(m) \) is LCD. Since \( S_{q,k}S_{q,k}^T = O_k \), we have

\[
G_{q,k}(m; s)G_{q,k}(m) = G_{q,k}(m)G_{q,k}(m)^T.
\]

Hence, \( C_{q,k}(m; s) \) is LCD. Since a code with generator matrix \( S_{q,k}(s) \) is a constant weight code, there is a codeword of weight \( d_0 + q^{k-1}s \) in \( C_{q,k}(m; s) \). Hence, the result follows by Lemma 3.1.

Remark 3.3. Recently, quaternary Hermitian LCD codes with large minimum weights have been studied by considering simplex codes in [3] and [11]. Lemma 3.1 is an \( \mathbb{F}_q \)-analogy \( (q=2, 3) \) of [3, Lemma 2.3]. Proposition 3.2 is an \( \mathbb{F}_q \)-analogy \( (q=2, 3) \) of [11, Lemmas 3.1 and 3.3].

4 Nonexistence of some LCD codes

In this section, we assume that \( (q, k_0) \in \{(2, 3), (3, 2)\} \). We give results on the nonexistence of some LCD codes with large minimum weight.

Let \( \mathcal{P} = \{1, \ldots, v\} \) and \( \mathcal{B} = \{B_1, \ldots, B_b\} \) be a collection of \( k \)-subsets of \( \mathcal{P} \). Then, a pair \( (\mathcal{P}, \mathcal{B}) \) is called a \( t-(v, k, \lambda) \) design if the number of sets \( B \in \mathcal{B} \) such that \( T \subseteq B \) is \( \lambda \) for any \( t \)-subset \( T \) of \( \mathcal{P} \). An element in \( \mathcal{P} \) and a set in \( \mathcal{B} \) are called a point and a block, respectively. A 2-design \( (\mathcal{P}, \mathcal{B}) \) is called symmetric if \( v = b \). The \( b \times v \) incidence matrix \( A = (a_{i,j}) \) of a design \( (\mathcal{P}, \mathcal{B}) \) is defined by

\[
a_{i,j} = \begin{cases} 
1 & \text{if } j \in B_i, \\
0 & \text{if } j \notin B_i.
\end{cases}
\]

Proposition 4.1. Suppose that \( k \geq k_0 \). Let \( m = (m_1, \ldots, m_{[k]q}) \in \mathbb{Z}_{\geq 0}^{[k]q} \). If an \([n, k]\) code \( C_{q,k}(m) \) over \( \mathbb{F}_q \) has minimum weight at least \( \alpha \), then

\[
q\alpha - (q - 1)n \leq m_i \leq n - \frac{q^{k-1} - 1}{(q - 1)q^{k-2}}\alpha
\]

for each \( i \in \{1, \ldots, [k]q\} \).
Proof. Let $A$ be the incidence matrix of a 2-$(v, k, \lambda)$ design $D$. Let $b$ and $r$ be the number of blocks in $D$ and the number of blocks containing a particular point in $D$, respectively. If each entry of the $n \times 1$ matrix $A(m_1, \ldots, m_v)^T$ is at least $\alpha$ and

$$\sum_{i=1}^{v} m_i = n,$$

then

$$\frac{r\alpha - \lambda n}{r - \lambda} \leq m_i \leq n - \frac{b - r}{r - \lambda} \alpha$$

for each $i \in \{1, \ldots, v\}$ [3, Lemma 3.1].

By the Assmus–Mattson theorem, the supports of all nonzero codewords in the simplex $[[k]_q, k, q^{k-1}]$ code yield a symmetric 2-$(k|q_0, q, (q-1)q^{k-2})$ design if $k \geq k_0$. By applying this design to (1), we have

$$r\alpha - \lambda n \leq q\alpha - (q-1)n,$$

$$n - \frac{b - r}{r - \lambda} \alpha = n - \frac{q^{k-1} - 1}{(q-1)q^{k-2}} \alpha.$$

Since the simplex $[[k]_q, k, q^{k-1}]$ code is a constant weight code, the weight of any nonzero codeword in $C_{q,k}(m)$ is one of the entries of $A(m_1, \ldots, m_v)^T$. Hence, $C_{q,k}(m)$ has minimum weight at least $\alpha$ if and only if each entry of $A(m_1, \ldots, m_v)^T$ is at least $\alpha$. This completes the proof. \hfill \Box

Two $[n, k]$ codes $C_1$ and $C_2$ over $\mathbb{F}_q$ are equivalent if there is an $n \times n$ monomial matrix $P$ over $\mathbb{F}_q$ such that $C_2 = \{cP \mid c \in C_1\}$. For any $[n, k]$ code $C$ over $\mathbb{F}_q$ with $d(C^\perp) \geq 2$, there is a vector $m = (m_1, \ldots, m_{[k]_q}) \in \mathbb{Z}_{\geq 0}^{[k]_q}$ such that $C$ is equivalent to the code $C_{q,k}(m)$.

Theorem 4.2. Suppose that $k \geq k_0$ and $n \equiv 0 \pmod{[k]_q}$. Then there is no LCD $[n, k; \alpha_q(n, k)]$ code over $\mathbb{F}_q$.

Proof. Write $n = [k]_qs$. Then we have $\alpha_q(n, k) = q^{k-1}s$ and $\alpha_q(n-1, k) = \alpha_q(n, k) - 1$. Suppose that there is an LCD $[n, k; \alpha_q(n, k)]$ code $C$ over $\mathbb{F}_q$. Since $d(C^\perp) \geq 2$ by Lemma 2.2, $C$ is equivalent to a code $C_{q,k}(m)$ for a vector $m = (m_1, \ldots, m_{[k]_q})$. By Proposition 4.1, we have $m_i = s$ for each $i$. Hence, $m = s1_{[k]_q}$. Since $S_{q,k}S_{q,k}^T = O_k$, we have

$$G_{q,k}(s1_{[k]_q})G_{q,k}(s1_{[k]_q})^T = S_{q,k}^{(s)}S_{q,k}^{(s)^T} = O_k.$$
Therefore, $C_{q,k}(m)$ is self-orthogonal. This is a contradiction since $C$ is LCD. Hence, the result follows. 

We are now in a position to give the following theorem. Let

$$r_q(n, k, \alpha) = q^{k-1}n - [k]_q \cdot \alpha$$

for positive integers $n$, $k$, $\alpha$ and $q$.

Theorem 4.3. Suppose that $k \geq k_0$ and $q\alpha - (q - 1)n \geq 1$.

(i) If $qr_q(n, k, \alpha) < k$, then there is no LCD $[n, k, \alpha]$ code $C$ over $\mathbb{F}_q$ with $d(C^\perp) \geq 2$.

(ii) If there is no LCD $[qr_q(n, k, \alpha), k, (q - 1)r_q(n, k, \alpha)]$ code $C_0$ over $\mathbb{F}_q$ with $d(C_0^\perp) \geq 2$, then there is no LCD $[n, k, \alpha]$ code $C$ over $\mathbb{F}_q$ with $d(C^\perp) \geq 2$.

Proof. Suppose that there is an LCD $[n, k, \alpha]$ code $C$ over $\mathbb{F}_q$ with $d(C^\perp) \geq 2$. Then $C$ is equivalent to a code $C_{q,k}(m)$ for a vector $m = (m_1, \ldots, m_{[k]_q}) \in \mathbb{Z}^{[k]_q}_{\geq 0}$. Since $d(C) = \alpha$, we have

$$q\alpha - (q - 1)n \leq m_i$$

by Proposition 4.1. Then a matrix $G_{q,k}(m)$ consists of least $q\alpha - (q - 1)n$ column vectors $h_{q,k}^{(i)}$ for each $i \in \{1, \ldots, [k]_q\}$ and we obtain a matrix $G$ of the following form:

$$G = \left( G_0 \mid S_{q,k}^{(q\alpha - (q - 1)n)} \right)$$

by permuting columns of $G_{q,k}(m)$. Then $S_{q,k}^{(q\alpha - (q - 1)n)}$ is a $k \times n'$ matrix and $G_0$ is a $k \times n_0$ matrix, where

$$n' = (q\alpha - (q - 1)n)[k]_q$$

and

$$n_0 = n - n' = q\left(q^{k-1}n - [k]_q \cdot \alpha\right) = qr_q(n, k, \alpha).$$

Since $S_{q,k}S_{q,k}^T = O_k$, $GG^T = G_0G_0^T$ and rank$(GG^T) = k$, we have

$$n_0 \geq \text{rank}(G_0) \geq \text{rank}(G_0G_0^T) = k.$$
Then the result of part (i) follows. The code $C'$ with generator matrix $S_{q,k}^{(q\alpha-(q-1)n)}$ is a self-orthogonal $[n', k, d']$ code, where and
\[
d' = (q\alpha - (q-1)n)q^{k-1}.
\]
Hence, the code $C_0$ with generator matrix $G_0$ is an LCD $[n_0, k, d_0]$ code with $\alpha \geq d_0 + d'$ by Lemma 3.1. Since $C'$ is a constant weight code, there is a codeword of weight $d_0 + d'$ in $C_{q,k}(m)$. Therefore, $\alpha = d_0 + d'$ and we have
\[
d_0 = (q-1)(q^{k-1}n - [k]_q \cdot \alpha) = (q-1)r_q(n, k; \alpha).
\]
Then the result of part (ii) follows. This completes the proof.

\[\square\]

Remark 4.4. If $q\alpha - (q-1)n \geq 1$, then we have
\[
(q-1)(n - qr_q(n, k; \alpha)) = -(q-1)n(q^k - 1) + q\alpha(q^k - 1) \\
\geq -(q-1)n(q^k - 1) + ((q-1)n + 1)(q^k - 1) \\
= q^k - 1.
\]
This means that the length of $C_0$ is less than the length of $C$ in Theorem 4.3 (ii). If we write $n = [k]_q \cdot s + t$, then the assumption $q\alpha - (q-1)n \geq 1$ of Theorem 4.3 can be written as follows:
\[
s \geq \frac{qr_q(n, k; \alpha) - t}{[k]_q} + 1.
\]
The inequality of this form is used to the propositions in Sections 5 and 6.

Remark 4.5. Proposition 4.1, Theorem 4.2 and Theorem 4.3 are $\mathbb{F}_q$-analogies ($q = 2, 3$) of Lemma 3.2, Theorem 3.3 and Theorem 3.4 in [3], respectively.

5 Binary LCD codes

In this section, we give results on the nonexistence of some binary LCD $[n, k]$ codes with large minimum weights for $k \in \{4, 5\}$.

5.1 Determination of $d_2(n, 4)$

In [2], the values $d_2(n, 4)$ are determined for $n \equiv 2, 3, 4, 5, 6, 9, 10, 13 \pmod{15}$. In this section, we determine the values $d_2(n, 4)$ for $n \equiv 0, 1, 7, 8, 11, 12, 14 \pmod{15}$. Write $n = 15s + t$, where $s \in \mathbb{Z}_{\geq 0}$ and $t \in \{0, 1, \ldots, 14\}$. We list the values $\alpha_2(n, 4)$ in Table 1.
Lemma 5.1. Suppose that $n \equiv 1, 7, 8, 11, 12, 14 \pmod{15}$. If there is a binary LCD $[n, 4, \alpha_2(n, 4)]$ code $C$, then $d(C^\perp) \geq 2$.

Proof. Suppose that $n \equiv 1 \pmod{15}$. Then $d_2(n-1, 4) \leq \alpha_2(n-1, 4) - 1 = \alpha_2(n, 4) - 1$ by Theorem 4.2 and Table 1. Suppose that $n \equiv 7, 8, 11, 12, 14 \pmod{15}$. Then $d_2(n-1, 4) \leq \alpha_2(n-1, 4) = \alpha_2(n, 4) - 1$ by Table 1. Hence, the result follows by Lemma 2.2.

Proposition 5.2. For $n \geq 4$ and $n \equiv 1, 7, 8, 11, 12, 14 \pmod{15}$, there is no binary LCD $[n, 4, \alpha_2(n, 4)]$ code.

Proof. Suppose that $t \in \{8, 12, 14\}$. Then we have $3\alpha_3(n, 3) - 2n = s$ and $r_3(n, 3, \alpha_3(n, 3)) = 4, 6, 7$ for $t = 8, 12, 14$, respectively. Suppose that $t \in \{1, 7, 11\}$. Then we have $3\alpha_3(n, 3) - 2n = s - 1$ and $r_3(n, 3, \alpha_3(n, 3)) = 8, 11, 13$ for $t = 1, 7, 11$, respectively. By [2, Table 14] and [9, Table 3], there is no binary LCD $[2r, 4, r]$ code $C_0$ with $d(C_0^\perp) \geq 2$ for $r \in \{4, 6, 7, 8, 11, 13\}$. Hence, there is no binary LCD $[n, 4, \alpha_2(n, 4)]$ code $C$ with $d(C^\perp) \geq 2$ for $n \equiv 1, 7, 8, 11, 12, 14 \pmod{15}$ by Theorem 4.3 (ii). Then the result follows by Theorem 2.2.

Lemma 5.3. Suppose that $n \equiv 0 \pmod{15}$. If there is a binary LCD $[n, 4, \alpha_2(n, 4) - 1]$ code $C$, then $d(C^\perp) \geq 2$.

Proof. We have $d_2(n-1, 4) \leq \alpha_2(n-1, 4) - 1 = \alpha_2(n, 4) - 2$ by Proposition 5.2 and Table 1. Hence, the result follows by Lemma 2.2.

Proposition 5.4. For $n \geq 4$ and $n \equiv 0 \pmod{15}$, there is no binary LCD $[n, 4, \alpha_2(n, 4) - 1]$ code.

Proof. Write $n = 15s$. Then we have $3(\alpha_3(n, 3) - 1) - 2n = s - 2$ and $r_3(n, 3, \alpha_3(n, 3) - 1) = 15$ for $t = 0$. By [2, Proposition 4], there is no binary
LCD \([30, 4, 15]\) code. Hence, there is no binary LCD \([n, 4, \alpha_2(n, 4) - 1]\) code \(C\) with \(d(C^\perp) \geq 2\) for \(n \equiv 0 \pmod{15}\) by Theorem 4.3 (ii). Then the result follows by Lemma 5.3.

Suppose that \(n \geq 4\). In [2, Theorem 1], it is shown that \(d_2(n, 4) = \lfloor \frac{8n}{15} \rfloor\) if \(n \equiv 5, 9, 13 \pmod{15}\) and \(d_2(n, 4) = \lfloor \frac{8n}{15} \rfloor - 1\) if \(n \equiv 2, 3, 4, 6, 10 \pmod{15}\). In [2, Proposition 3], it is shown that \(d_2(n, 4) \geq \lfloor \frac{8n}{15} \rfloor - 2\) if \(n \equiv 1, 7, 8, 11, 12, 14 \pmod{15}\) and \(d_2(n, 4) \leq \lfloor \frac{8n}{15} \rfloor - 1\) if \(n \equiv 1, 7, 8, 11, 12, 14 \pmod{15}\). By Proposition 5.2, we have \(d_2(n, 4) \leq \lfloor \frac{8n}{15} \rfloor - 2\) if \(n \equiv 0 \pmod{15}\). By Theorem 4.2 and Proposition 5.4, we have \(d_2(n, 4) \leq \lfloor \frac{8n}{15} \rfloor - 2\) if \(n \equiv 0 \pmod{15}\). We summarize these results as follows:

**Theorem 5.5.** For \(n \geq 4\),

\[
d_2(n, 4) = \begin{cases} 
\lfloor \frac{8n}{15} \rfloor & \text{if } n \equiv 5, 9, 13 \pmod{15}, \\
\lfloor \frac{8n}{15} \rfloor - 1 & \text{if } n \equiv 1, 2, 3, 4, 6, 7, 8, 10, 11, 12, 14 \pmod{15}, \\
\lfloor \frac{8n}{15} \rfloor - 2 & \text{if } n \equiv 0 \pmod{15}.
\end{cases}
\]

**5.2 Observation on \(d_2(n, 5)\)**

| \(n\) | \(\alpha_2(n, 5)\) | \(n\) | \(\alpha_2(n, 5)\) | \(n\) | \(\alpha_2(n, 5)\) | \(n\) | \(\alpha_2(n, 5)\) |
|---|---|---|---|---|---|---|---|
| 31s | 16s | 31s + 8 | 16s + 3 | 31s + 16 | 16s + 8 | 31s + 24 | 16s + 12 |
| 31s + 1 | 16s | 31s + 9 | 16s + 4 | 31s + 17 | 16s + 8 | 31s + 25 | 16s + 12 |
| 31s + 2 | 16s | 31s + 10 | 16s + 4 | 31s + 18 | 16s + 8 | 31s + 26 | 16s + 12 |
| 31s + 3 | 16s | 31s + 11 | 16s + 4 | 31s + 19 | 16s + 8 | 31s + 27 | 16s + 13 |
| 31s + 4 | 16s | 31s + 12 | 16s + 5 | 31s + 20 | 16s + 9 | 31s + 28 | 16s + 14 |
| 31s + 5 | 16s + 1 | 31s + 13 | 16s + 6 | 31s + 21 | 16s + 10 | 31s + 29 | 16s + 14 |
| 31s + 6 | 16s + 2 | 31s + 14 | 16s + 6 | 31s + 22 | 16s + 10 | 31s + 30 | 16s + 15 |
| 31s + 7 | 16s + 2 | 31s + 15 | 16s + 7 | 31s + 23 | 16s + 11 |    |    |

In [2], the values \(d_2(n, 5)\) are determined for \(n \geq 5\) and \(n \equiv 3, 4, 5, 7, 11, 19, 20, 22, 26 \pmod{31}\). In this section, we determine the values \(d_2(n, 4)\) for \(n \geq 5\) and \(n \equiv 24, 28, 30 \pmod{31}\). Write \(n = 31s + t\), where \(s \in \mathbb{Z}_{\geq 0}\) and \(t \in \{0, 1, \ldots, 30\}\). We list the values \(\alpha_2(n, 5)\) in Table 2.

**Lemma 5.6.** Suppose that \(n \equiv 16, 24, 28, 30 \pmod{31}\). If there is a binary LCD \([n, 5, \alpha_2(n, 5)]\) code \(C\), then \(d(C^\perp) \geq 2\).
Proof. We have \(d_2(n-1,5) \leq \alpha_2(n-1,5) = \alpha_2(n,4) - 1\) by Table 2. Hence, the result follows by Lemma 2.2.

\[\text{Proposition 5.7.} \quad \text{For} \quad n \geq 5 \quad \text{and} \quad n \equiv 16, 24, 28, 30 \pmod{31}, \quad \text{there is no binary LCD} \quad [n, 5, \alpha_2(n,5)] \quad \text{code.}\]

Proof. Suppose that \(t \in \{16, 24, 28, 30\}\). Then we have \(2\alpha_2(n,5) - n = s\) and \(r_2(n,5,\alpha_2(n,5)) = 8, 12, 14, 15\), respectively. By [2, Proposition 6 and Table 14] and [9, Table 3], there is no binary LCD \([n, 5, r_2(n,5,\alpha_2(n,5))]\) code for \(r \in \{8, 12, 14, 15\}\). Hence, there is no binary LCD \([n, 5, \alpha_2(n,5)]\) code \(C\) with \(d(C^\perp) \geq 2\) by Theorem 4.3 (ii). The result follows by Lemma 5.6.

In [2, Proposition 5], it is shown that \(d_2(n,5) \geq \left\lfloor \frac{16n}{31} \right\rfloor - 1\) if \(n \equiv 24, 28, 30 \pmod{15}\) and \(d_2(n,5) \geq \left\lfloor \frac{16n}{31} \right\rfloor - 2\) if \(n \equiv 0, 16 \pmod{15}\) for \(n \geq 5\). By Theorem 4.2 and Proposition 5.7, we have the following:

\[\text{Theorem 5.8.} \quad \text{(i)} \quad \text{If} \quad n \geq 5 \quad \text{and} \quad n \equiv 24, 28, 30 \pmod{31}, \quad \text{then} \]

\[d_2(n,5) = \left\lfloor \frac{16n}{31} \right\rfloor - 1.\]

\[\text{(ii) If} \quad n \geq 5 \quad \text{and} \quad n \equiv 0, 16 \pmod{31}, \quad \text{then} \]

\[d_2(n,5) \in \left\{ \left\lfloor \frac{16n}{31} \right\rfloor - 1, \left\lfloor \frac{16n}{31} \right\rfloor - 2 \right\}.\]

Now, we consider the largest minimum weights of binary LCD \([n, 5]\) codes for \(n \not\equiv 16, 24, 28, 30 \pmod{31}\).

\[\text{Lemma 5.9.} \quad \text{(i) Suppose that} \quad n \equiv 1, 6, 8, 9, 12, 13, 15, 17, 21, 23, 25, 27, 29 \pmod{31}. \quad \text{If there is a binary LCD} \quad [n, 5, \alpha_2(n,5)] \quad \text{code} \quad C, \quad \text{then} \quad d(C^\perp) \geq 2.\]

\[\text{(ii) Suppose that} \quad n \equiv 0 \pmod{31}. \quad \text{If there is a binary LCD} \quad [n, 5, \alpha_2(n,5) - 1] \quad \text{code} \quad C, \quad \text{then} \quad d(C^\perp) \geq 2.\]

Proof. (i) Suppose that \(n \equiv 6, 8, 9, 12, 13, 15, 17, 21, 23, 27 \pmod{31}\). Then we have \(d_2(n-1,5) \leq \alpha_2(n-1,5) = \alpha_2(n,5) - 1\) by Table 2. Suppose that \(n \equiv 1, 17, 25, 29 \pmod{31}\). Then we have \(d_2(n-1,5) \leq \alpha_2(n-1,5) - 1 = \alpha_2(n,5) - 1\) by Theorem 4.2, Table 2 and Proposition 5.7.
(ii) Suppose that $n \equiv 0 \pmod{31}$. Then we have $d_3(n-1,5) \leq \alpha_2(n-1,5) - 1 = \alpha_2(n,5) - 2$ by Table 2 and Proposition 5.7.

Hence, the result follows by Lemma 2.2. 

\begin{proof}

Proposition 5.10. Let $s \in \mathbb{Z}_{\geq 0}$, $s_0 = \frac{2r-t}{31} + 1$ and

\[ \delta_i = \begin{cases} 
0 & \text{if } i \in \{1,3\}, \\
1 & \text{if } i \in \{2,4\}. 
\end{cases} \]

Set

\[ X_1 = \left\{ (16, 1), (34, 6), (35, 8), (20, 9), (37, 12), (22, 13), (23, 15), (24, 17), (26, 21), (27, 23), (28, 25), (29, 27), (30, 29) \right\}, \]

\[ X_2 = \{ (31, 0) \}, \]

\[ X_3 = \left\{ (32, 2), (48, 3), (64, 4), (49, 5), (50, 7), (36, 10), (52, 11), (38, 14), (40, 18), (56, 19), (41, 20), (42, 22), (44, 26) \right\}, \]

\[ X_4 = \{ (39, 16) \}. \]

(i) Suppose that $i \in \{1,2\}$, $(r,t) \in X_i$ and there is no binary LCD $[2r, 5, r]$ code $C_0$ with $d(C_0^\perp) \geq 2$. Then there is no binary LCD $[31s+t, 5, \alpha_2(31s+t, 5) - \delta_i]$ code for all $s \geq s_0 + 1$.

(ii) Suppose that $i \in \{3,4\}$, $(r,t) \in X_i$ and there is no binary LCD $[2r, 5, r]$ code $C_0$ with $d(C_0^\perp) \geq 2$. Then there is no binary LCD $[31s+t, 5, \alpha_2(31s+t, 5) - \delta_i]$ code $C$ with $d(C^\perp) \geq 2$ for all $s \geq s_0 + 1$.

\end{proof}

Proof. Suppose that $(r,s_0,t) \in X_i$ for $i \in \{1,2,3,4\}$ and there is no binary LCD $[2r, 5, r]$ code $C_0$ with $d(C_0^\perp) \geq 2$. Then there is no binary LCD $[31s+t, 5, \alpha_2(31s+t, 5) - \delta_i]$ code $C$ with $d(C^\perp) \geq 2$ for all $s \geq s_0 + 1$ by Theorem 4.3 (ii). Hence, the result follows for (ii). If $i \in \{1,2\}$, there is no binary LCD $[31s+t, 5, \alpha_2(31s+t, 5) - \delta_i]$ code $C$ with $d(C^\perp) = 1$ for all $s \geq s_0 + 1$ by Lemma 5.9 (i)-(ii). Hence, the result follows for (i). This completes the proof. 

\end{proof}

6 Ternary LCD codes

In [2, Proposition 4], it was shown that $d_3(n,1) = n$ if $n \not\equiv 0 \pmod{3}$ and $d_3(n,1) = n - 1$ if $n \equiv 0 \pmod{3}$. In this section, we give results on the nonexistence of some ternary LCD $[n,k]$ codes with large minimum weights for $k \in \{2,3,4\}$. 

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Table 3: Values $\alpha_3(n, 2)$

| $n$  | $\alpha_3(n, 2)$ | $n$  | $\alpha_3(n, 2)$ |
|------|-----------------|------|-----------------|
| $4s$ | $3s$            | $4s + 2$ | $3s + 1$      |
| $4s + 1$ | $3s$        | $4s + 3$ | $3s + 2$      |

6.1 Determination of $d_3(n, 2)$

Write $n = 4s + t$, where $s \in \mathbb{Z}_{\geq 0}$ and $t \in \{0, 1, 2, 3\}$. We list the values $\alpha_3(n, 2)$ in Table 3. By [1, Table 4], there is a ternary LCD $[n, 2, d]$ code for $(n, d) \in \{(5, 3), (6, 4)\}$.

Hence, by Proposition 3.2, ternary LCD $[n, 2, \alpha_3(n, 2)]$ codes are constructed for $n \geq 5$ and $n \equiv 1, 2 \pmod{4}$.

Lemma 6.1. Suppose that $n \geq 2$ and $n \equiv 3 \pmod{4}$. If there is a ternary LCD $[n, 2, \alpha_3(n, 2)]$ code $C$, then $d(C^\perp) \geq 2$.

Proof. We have $d_3(n - 1, 2) \leq \alpha_3(n - 1, 2) = \alpha_3(n, 2) - 1$ for $n \equiv 3 \pmod{4}$ by Table 3. Hence, the result follows by Lemma 2.2.

Proposition 6.2. For $n \geq 2$ and $n \equiv 3 \pmod{4}$, there is no ternary LCD $[n, 2, \alpha_3(n, 2)]$ code.

Proof. We have $r_3(4s + 3, 2, 3s + 1) = 1$. There is no ternary LCD $[3, 2, 2]$ code [1, Table 4]. Hence, there is no ternary LCD $[n, 2, \alpha_3(n, 2)]$ code $C$ with $d(C^\perp) \geq 2$ by Theorem 4.3 (ii). Then the result follows by Lemma 6.1.

By Table 3 and Proposition 6.2, we have the following:

Theorem 6.3. For $n \geq 2$,

$$d_3(n, 2) = \begin{cases} \left\lfloor \frac{3n}{4} \right\rfloor & \text{if } n \equiv 1, 2 \pmod{4}, \\ \left\lfloor \frac{3n}{4} \right\rfloor - 1 & \text{if } n \equiv 0, 3 \pmod{4}. \end{cases}$$
Table 4: Values $\alpha_3(n,3)$

| $n$   | $\alpha_3(n,3)$ | $n$   | $\alpha_3(n,3)$ | $n$   | $\alpha_3(n,3)$ |
|-------|----------------|-------|----------------|-------|----------------|
| 13s   | 9s             | 13s + 5 | 9s + 3          | 13s + 10 | 9s + 6         |
| 13s + 1 | 9s           | 13s + 6 | 9s + 3          | 13s + 11 | 9s + 7         |
| 13s + 2 | 9s           | 13s + 7 | 9s + 4          | 13s + 12 | 9s + 8         |
| 13s + 3 | 9s + 1        | 13s + 8 | 9s + 5          |        |               |
| 13s + 4 | 9s + 2        | 13s + 9 | 9s + 6          |        |               |

Table 5: Ternary LCD $[t + 3, 3, d]$ codes $C_{3,3}^{t+3}$ for $t \in \{8, 9, 10, 11, 12\}$

| Code  | $d$ | $(m_1, m_2, m_3, m_4, m_5, m_6, m_7, m_8, m_9, m_{10})$ |
|-------|-----|------------------------------------------------------|
| $C_{3,3}^{14}$ | 6   | $(0,0,0,0,0,0,1,2,1,2,2,2)$                          |
| $C_{3,3}^{12}$ | 7   | $(0,0,0,0,0,1,1,1,2,2,1,2)$                          |
| $C_{3,3}^{13}$ | 8   | $(0,0,0,0,1,1,1,2,2,1,2)$                            |
| $C_{3,3}^{14}$ | 8   | $(0,0,0,0,0,1,2,2,1,2,2,3)$                          |
| $C_{3,3}^{15}$ | 9   | $(0,0,0,0,1,1,1,2,2,2,2,2)$                          |

6.2 Determination of $d_3(n, 3)$

Write $n = 13s + t$, where $s \in \mathbb{Z}_{\geq 0}$ and $t \in \{0, 1, \ldots, 12\}$. We list the values $\alpha_3(n,3)$ in Table 4.

It is trivial that $\mathbb{F}_3^3$ is a ternary LCD $[3, 3, 1]$ code. By [1, Proposition 5], there is a ternary LCD $[4, 3, 2]$ code. By [1, Table 4], there is a ternary LCD $[n, 3, d]$ code for $(n, d) \in \{(5, 2), (6, 3), (7, 4), (8, 4), (9, 5), (10, 6)\}$.

For $t \in \{8, 9, 10, 11, 12\}$, we found the codes $C_{3,3}^{t+3} = C_{3,3}(m)$, where the vectors $m \in \mathbb{Z}_{\geq 0}^{13}$ are listed in Table 5. Hence, by Proposition 3.2, ternary LCD $[n, 3, d]$ codes are constructed for $n \geq 16$, where

$$d = \begin{cases} 
\alpha_3(n,3) & \text{if } n \equiv 2, 3, 4, 6, 7, 10 \pmod{13}, \\
\alpha_3(n,3) - 1 & \text{if } n \equiv 0, 1, 5, 8, 9, 11, 12 \pmod{13}.
\end{cases}$$

Lemma 6.4. Suppose that $n \equiv 1, 5, 8, 9, 11, 12 \pmod{13}$. If there is a ternary LCD $[n, 3, \alpha_3(n,3)]$ code $C$, then $d(C^\perp) \geq 2$. 

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Proof. Suppose that \( n \equiv 1, 5, 8, 9, 11, 12 \pmod{13} \). Then \( d_3(n - 1, 3) \leq \alpha_3(n, 2) - 1 \) by Proposition 4.2 and Table 3. Hence, the result follows by Lemma 2.2.

\[ \]

Table 6: Sets \( L_i \)

| \( i \) | \( L_i \) | \( i \) | \( L_i \) |
|-------|-------|-------|-------|
| 1     | \( \{1, 2, 3, 5, 8, 9, 10, 11, 13\} \) | 8     | \( \{1, 3, 4, 5, 7, 10, 11, 12, 13\} \) |
| 2     | \( \{5, 6, 7, 8, 9, 10, 11, 12, 13\} \) | 9     | \( \{2, 3, 4, 5, 6, 9, 10, 11, 12\} \) |
| 3     | \( \{2, 3, 4, 7, 8, 9, 11, 12, 13\} \) | 10    | \( \{2, 3, 4, 5, 6, 7, 8, 10, 13\} \) |
| 4     | \( \{1, 2, 4, 6, 7, 9, 10, 11, 13\} \) | 11    | \( \{1, 2, 3, 6, 7, 8, 10, 11, 12\} \) |
| 5     | \( \{1, 2, 4, 5, 6, 8, 11, 12, 13\} \) | 12    | \( \{1, 2, 4, 5, 7, 8, 9, 10, 12\} \) |
| 6     | \( \{1, 3, 4, 6, 8, 9, 10, 12, 13\} \) | 13    | \( \{1, 2, 3, 5, 6, 7, 9, 12, 13\} \) |
| 7     | \( \{1, 3, 4, 5, 6, 7, 8, 9, 11\} \)  |       |       |

We describe how the values \( d_3(n, 3) \) were determined. The code \( C_{3,3}(m) \) has weight enumerator \( 1 + 2 \sum_{i=1}^{13} y^{w_i} \) for a vector \( m = (m_1, \ldots, m_{13}) \in \mathbb{Z}_{\geq 0}^{13} \), where \( w_i = \sum_{j \in L_i} m_j \) and the sets \( L_1, \ldots, L_{13} \) are listed in Table 6. Note that \( L_1, \ldots, L_{13} \) are the supports of codewords in the simplex code \( C_{3,3}(1_{13}) \). Fix positive integers \( n \) and \( \alpha \). Then we found all vectors \( m = (m_1, \ldots, m_{13}) \in \mathbb{Z}_{\geq 0}^{13} \) such that \( n = \sum_{i=1}^{13} m_i \) and \( w_i \geq \alpha \) for each \( i \in \{1, \ldots, 13\} \). For such vectors \( m \), the codes \( C_{3,3}(m) \) have minimum weight at least \( \alpha \). Then we checked whether the matrix \( G_{3,3}(m)G_{3,3}(m)^T \) is nonsingular or not. If there is no ternary LCD \( [n, 3, d] \) code for \( d = \alpha \), then we consider the case \( d = \alpha - 1 \). By using the above method, our computer calculation shows the following:

**Lemma 6.5.** There is no ternary LCD \( [n, 3, d] \) code \( C \) with \( d(C^\perp) \geq 2 \) for

\[
(n, d) \in \begin{cases} (5, 3), (8, 5), (9, 6), (11, 7), (12, 8), (14, 9), \\ (15, 10), (18, 12), (21, 14), (24, 16), (27, 18) \end{cases}.
\]

**Proposition 6.6.** For \( n \geq 3 \) and \( n \equiv 1, 5, 8, 9, 11, 12 \pmod{13} \), there is no ternary LCD \( [n, 3, \alpha_3(n, 3)] \) code.

Proof. Write \( n = 13s + t \), where \( s \in \mathbb{Z}_{\geq 0} \) and \( t \in \{0, 1, \ldots, 12\} \). Suppose that \( t = 1 \). Then we have \( 3\alpha_3(n, 3) - 2n = s - 2 \) and \( r_3(n, 3, \alpha_3(n, 3)) = 9 \). By Lemma 6.5, there is no ternary \( [14, 3, 9] \) LCD code \( C_0 \) with \( d(C_0^\perp) \geq 2 \).
Suppose that \( t \in \{5, 8, 9, 11, 12\} \). Then we have \( 3\alpha_3(n, 3) - 2n = s - 1 \) and \( r_3(n, 3, \alpha_3(n, 3)) = 6, 7, 3, 8, 4 \) for \( t = 5, 8, 9, 11, 12 \), respectively. By Lemma 6.5, there is no ternary LCD \([3r, 3, 2r] \) code \( C_0 \) with \( d(C_0^\perp) \ge 2 \) for \( r \in \{3, 4, 6, 7, 8, 9\} \). Hence, there is no ternary LCD \([n, 3, \alpha_3(n, 3)] \) code \( C \) with \( d(C^\perp) \ge 2 \) for \( t \in \{1, 5, 8, 9, 11, 12\} \) by Theorem 4.3 (ii). The result follows by Lemma 6.4.

By Table 5 and Proposition 6.6, we have the following:

\[
d_3(n, 3) = \begin{cases} \left\lfloor \frac{3n}{13} \right\rfloor & \text{if } n \equiv 2, 3, 4, 6, 7, 10 \text{ (mod 13)}, \\ \left\lfloor \frac{3n}{13} \right\rfloor - 1 & \text{if } n \equiv 0, 1, 5, 8, 9, 11, 12 \text{ (mod 13)}. \end{cases}
\]

### 6.3 Observation on \( d_3(n, 4) \)

| \( n \) | \( \alpha_3(n, 4) \) | \( n \) | \( \alpha_3(n, 4) \) | \( n \) | \( \alpha_3(n, 4) \) | \( n \) | \( \alpha_3(n, 4) \) |
|---|---|---|---|---|---|---|---|
| 40s | 27s | 40s + 10 | 27s + 6 | 40s + 20 | 27s + 12 | 40s + 30 | 27s + 19 |
| 40s + 1 | 27s | 40s + 11 | 27s + 6 | 40s + 21 | 27s + 13 | 40s + 31 | 27s + 20 |
| 40s + 2 | 27s | 40s + 12 | 27s + 7 | 40s + 22 | 27s + 14 | 40s + 32 | 27s + 21 |
| 40s + 3 | 27s | 40s + 13 | 27s + 8 | 40s + 23 | 27s + 15 | 40s + 33 | 27s + 21 |
| 40s + 4 | 27s + 1 | 40s + 14 | 27s + 9 | 40s + 24 | 27s + 15 | 40s + 34 | 27s + 22 |
| 40s + 5 | 27s + 2 | 40s + 15 | 27s + 9 | 40s + 25 | 27s + 16 | 40s + 35 | 27s + 23 |
| 40s + 6 | 27s + 3 | 40s + 16 | 27s + 9 | 40s + 26 | 27s + 17 | 40s + 36 | 27s + 24 |
| 40s + 7 | 27s + 3 | 40s + 17 | 27s + 10 | 40s + 27 | 27s + 18 | 40s + 37 | 27s + 24 |
| 40s + 8 | 27s + 4 | 40s + 18 | 27s + 11 | 40s + 28 | 27s + 18 | 40s + 38 | 27s + 25 |
| 40s + 9 | 27s + 5 | 40s + 19 | 27s + 12 | 40s + 29 | 27s + 18 | 40s + 39 | 27s + 26 |

Write \( n = 40s + t \), where \( s \in \mathbb{Z}_{\ge 0} \) and \( t \in \{0, 1, \ldots, 39\} \). We list the values \( \alpha_3(n, 4) \) in Table 7.

It is trivial that \( \mathbb{F}_3^4 \) is a ternary LCD \([4, 4, 1] \) code. By [2, Proposition 5], there is a ternary LCD \([5, 4, 2] \) code. By [1, Table 4], there is a ternary LCD \([n, 4, d] \) code for \( (n, d) \in \{(6, 2), (7, 3), (8, 4), (9, 4), (10, 5)\} \).
Hence, by Proposition 3.2, ternary LCD \([n, 4, d]\) codes are constructed for \(n \geq 40\), where

\[
d = \begin{cases} 
\alpha_3(n, 4) & \text{if } n \equiv 4, 5, 7, 8 \pmod{40}, \\
\alpha_3(n, 4) - 1 & \text{if } n \equiv 6, 9, 10 \pmod{40}.
\end{cases}
\]

By Theorem 4.3 (ii), we have the following:

**Proposition 6.8.** Let \(s \in \mathbb{Z}_{\geq 0}\) and \(s_0 = \frac{3r-t}{40}\). Suppose that

\[
(r, t) \in \begin{cases} 
(27, 1), (54, 2), (81, 3), (68, 4), (55, 5), (42, 6), (69, 7), \\
(56, 8), (43, 9), (30, 10), (57, 11), (44, 12), (31, 13), (18, 14), \\
(45, 15), (72, 16), (59, 17), (46, 18), (33, 19), (60, 20), (47, 21), \\
(34, 22), (21, 23), (48, 24), (35, 25), (22, 26), (9, 27), (36, 28), \\
(63, 29), (50, 30), (37, 31), (24, 32), (51, 33), (38, 34), (25, 35), \\
(12, 36), (39, 37), (26, 38), (13, 39)
\end{cases}.
\]

If there is no ternary LCD \([3r, 4, 2r]\) code \(C_0\) with \(d(C_0^\perp) \geq 2\), then there is no ternary LCD \([40s + t, 4, \alpha_3(40s + t, 4)]\) code \(C\) with \(d(C^\perp) \geq 2\) for all \(s \geq s_0 + 1\).

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**References**

[1] M. Araya and M. Harada, On the classification of linear complementary dual codes, Dis. Math. **342** (2019), 270–278.

[2] M. Araya and M. Harada, On the minimum weights of binary linear complementary dual codes, (submitted), arXiv:1807.03525v1.

[3] M. Araya, M. Harada and K. Saito, Quaternary Hermitian linear complementary dual codes, (submitted), arXiv:1904.07517.

[4] W. Bosma, J. Cannon and C. Playoust, The Magma algebra system I: The user language, J. Symbolic Comput. **24** (1997), 235–265.
[5] C. Carlet and S. Guilley, Complementary dual codes for countermeasures to side-channel attacks, In: E.R. Pinto et al. (eds.), Coding Theory and Applications, CIM Series in Mathematical Sciences, vol. 3, pp. 97–105, Springer, 2014.

[6] C. Carlet, S. Mesnager, C. Tang, Y. Qi and R. Pellikaan, Linear codes over $\mathbb{F}_q$ are equivalent to LCD codes for $q > 3$, IEEE Trans. Inform. Theory 64 (2018), 3010–3017.

[7] S.T. Dougherty, J.-L. Kim, B. Ozkaya, L. Sok and P. Solé, The combinatorics of LCD codes: linear programming bound and orthogonal matrices, Int. J. Inf. Coding Theory 4 (2017), 116–128.

[8] L. Galvez, J.-L. Kim, N. Lee, Y.G. Roe and B.-S. Won, Some bounds on binary LCD codes, Cryptogr. Commun. 10 (2018), 719–728.

[9] M. Harada and K. Saito, Binary linear complementary dual codes, Cryptogr. Commun. 11 (2019), 677–696.

[10] W.C. Huffman and V. Pless, Fundamentals of error-correcting codes, Cambridge University Press, Cambridge, (2003).

[11] L. Lu, R. Li, L. Guo and Q. Fu, Maximal entanglement entanglement-assisted quantum codes constructed from linear codes, Quantum Inf. Process. 14 (2015), 165–182.

[12] J.L. Massey, Linear codes with complementary duals, Dis. Math. 106/107 (1992), 337–342.