Dirac returns: non-Abelian statistics of vortices with Dirac fermions

Shigeiho Yasui, Kazunori Itakura, and Muneto Nitta

KEK Theory Center, Institute of Particle and Nuclear Studies, High Energy Accelerator Research Organization (KEK), 1-1 Oho, Tsukuba, Ibaraki 305-0801, Japan
Department of Physics, and Research and Education Center for Natural Sciences, Keio University, 4-1-1 Hiyoshi, Yokohama, Kanagawa 223-8521, Japan

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Topological superconductors classified as type D admit zero-energy Majorana fermions inside vortex cores, and consequently the exchange statistics of vortices becomes non-Abelian, giving a promising example of non-Abelian anyons. On the other hand, types C and DIII admit zero-energy Dirac fermions inside vortex cores. It has been long believed that an essential condition for the realization of non-Abelian statistics is non-locality of Dirac fermions made of two Majorana fermions trapped inside two well-separated vortices as in the case of type D. Contrary to this conventional wisdom, however, we show that vortices with local Dirac fermions also obey non-Abelian statistics.

One of remarkable achievements in condensed matter physics in these years is the classification of topological insulators and superconductors [1, 2]. In particular, among them, the type D superconductors exhibit a unique property that zero-energy Majorana fermions [3, 4] are trapped inside a vortex core [5] or appear at the edge of superconductors. Examples of materials for type D are given by chiral $p$-wave superconductors/superfluids such as Sr$_2$RuO$_4$ and the A-phase of $^3$He. One of important aspects of the zero-energy Majorana fermions is that exchange statistics of vortices is non-Abelian [6, 7] so that a set of them can be regarded as non-Abelian anyons. They are expected to be quite useful for a fault-tolerant quantum computing [8]. The idea for topological quantum computation utilizing non-Abelian anyons on a topological state of matter has been developed [9]. This is why it is one of hot topics to experimentally realize Majorana fermions.

In the first theoretical discovery of non-Abelian statistics of vortices with Majorana fermions (we call them the Majorana vortices) [10], the key ingredient was an unusual way to define Dirac fermions that are necessary for the construction of a Hilbert space of many vortices. With a single Majorana fermion at each vortex core, one has to compose a Dirac fermion from two Majorana fermions located at spatially well-separated vortices. Since then, it has been long believed that the essential property that leads to non-Abelian statistics is the non-locality of Dirac fermions. This is one of the reasons why other types of topological insulators/superconductors have not attracted more attentions than the type D insulators/superconductors as candidates of non-Abelian anyons.

On the other hand, the type C and DIII insulators/superconductors admit Dirac fermions inside vortex cores [11]. We call such vortices the Dirac vortices. There are several systems allowing Dirac fermions such as cores of integer (singular) vortices in $^3$He A-phase [12], and dislocation lines in topological insulators [13]. Dirac fermions already exist locally in these cases, and therefore they have been believed for a long time to admit no non-Abelian statistics. Contrary to conventional wisdom, however, we show in the present paper that they actually do give non-Abelian statistics, and consequently it implies that Majorana fermions are not necessary anymore for constructing non-Abelian statistics.

We treat a set of $n$ vortices, each of which allows for a single zero-energy Dirac fermion at its core, and consider an exchange of two vortices in such a system. An operation $T_k$ is defined as the anti-clockwise exchange of the $k$-th and $(k+1)$-th vortices. Then, $\{T_k|k=1, \cdots, n-1\}$ are in general the generators of the braid group. They satisfy the braid relations

\[
\begin{align*}
T_k T_{k+1} T_k &= T_{k+1} T_k, \\
T_{k+1} T_k &= T_k T_{k+1} \quad \text{for } |k - (k+1)| = 1.
\end{align*}
\]

We assume that exchange of two vortices is performed as an adiabatic process, and that quantum states of vortices are essentially determined by the zero-energy modes. Let us define an (annihilation) operator $\hat{\psi}_k$ for the zero-mode Dirac fermion at the $k$-th vortex. It satisfies the following algebra:

\[
\{\hat{\psi}_k, \hat{\psi}_l^\dagger\} = \delta_{kl}, \quad \{\hat{\psi}_k, \hat{\psi}_l\} = \{\hat{\psi}_k^\dagger, \hat{\psi}_l^\dagger\} = 0.
\]

Then, the exchange of two vortices is expressed in terms of the change of operators $\hat{\psi}_k$ under the operation $T_k$. Note that the wave function of the zero modes obtains an additional phase, a minus sign, when the zero-mode fermion turns around a vortex, implying that the zero-mode wave function is a double-valued function. In order to regard this wave function as a single-valued function, we introduce a cut so that the zero-mode wave function changes its sign when it crosses the cut. Under the operation $T_k$, we find that the $k$-th vortex crosses the cut of the $(k+1)$-th vortex. The operation $T_k$ acts on the Dirac zero-mode operators $\hat{\psi}_k$ and $\hat{\psi}_{k+1}$ for the $k$-th and $(k+1)$-th vortices, respectively, as

\[
T_k : \begin{cases} 
\hat{\psi}_k &\rightarrow \hat{\psi}_{k+1} \\
\hat{\psi}_{k+1} &\rightarrow -\hat{\psi}_k
\end{cases}
\]
with keeping the others \( \hat{\psi}_\ell \) (\( \ell \neq k, k + 1 \)) unchanged.\(^1\)

We confirm that the above transformation satisfies the braid relations (i) and (ii). We should emphasize that, because of the minus sign in Eq. (1), we obtain the relation \( T_k^{4} = 1 \), i.e., four successive exchanges are equivalent to the identity.\(^2\) This may be contrasted with the exchange of two conventional particles, in which two successive exchanges are equivalent to the identity.\(^3\) This fact by itself suggests that the operation \( T_k \) induces a non-trivial statistics, as discussed below.

Now, let us find the representation of the operation \( T_k \) following the procedure discussed in Ref. [7]. First of all, we note that \( T_k \) defined in Eq. (1) can be represented with respect to the Dirac fermion operator \( \hat{\psi}_\ell \). The corresponding operator \( \hat{\tau}_k \) reads

\[
\hat{\tau}_k = 1 + \hat{\psi}_{k+1}^{\dagger} \hat{\psi}_k + \hat{\psi}_{k+1}^{\dagger} \hat{\psi}_k - \hat{\psi}_{k+1} \hat{\psi}_k + 2 \hat{\psi}_{k+1}^{\dagger} \hat{\psi}_k + 2 \hat{\psi}_{k+1} \hat{\psi}_k \hat{\psi}_{k+1}^{\dagger} \hat{\psi}_k. \tag{2}
\]

We confirm that \( \hat{\tau}_k \hat{\psi}_\ell \hat{\tau}_k^{-1} (\ell = 1, \ldots, n) \) reproduces the transformation \( \hat{T}_k \). We also note that \( \hat{\tau}_k \) satisfies the braid relations: (i) \( \hat{\tau}_k \hat{\tau}_{k'} = \hat{\tau}_{k'} \hat{\tau}_k \) for \( |k - \ell| = 1 \) and (ii) \( \hat{\tau}_k \hat{\tau}_{k'} = \hat{\tau}_{k'} \hat{\tau}_k \) for \( |k - \ell| > 1 \). Moreover, we have \( \hat{\tau}_{k}^{4} = 1 \). Therefore, \( \hat{\tau}_k \) gives a representation of the braid group with a condition \( \hat{\tau}_{k}^{4} = 1 \). In general, \( \hat{\tau}_k \)'s are non-Abelian; \( \hat{\tau}_k \hat{\tau}_{k+1} \neq \hat{\tau}_{k+1} \hat{\tau}_k \), which will be explicitly seen in the matrix representation of \( \hat{\tau}_k \) in the Hilbert space.\(^5\)

Next, we note that, with the operators \( \hat{\psi}_\ell \) (\( \ell = 1, \ldots, n \)) for the zero-mode Dirac fermions, it is straightforward to construct the Hilbert space of the multiple vortex systems. The Fock vacuum \( |0\rangle \) is defined by \( \hat{\psi}_\ell |0\rangle = 0 \) for all \( \ell \), and one can construct the Hilbert space by multiplying \( \hat{\psi}_\ell \)'s on the vacuum. Since the fermion number \( f \) is conserved under the transformation \( \hat{\tau}_k \) (the fermion number operator \( \hat{f} = \sum_{\ell=1}^{n} \hat{\psi}_\ell \hat{\psi}_\ell^{\dagger} \)) commutes with \( \hat{\tau}_k \) for any \( k \), the total Hilbert space \( \mathbb{H}^{(n)} \) for \( n \) vortices is decomposed into a direct sum of the sectors \( \mathbb{H}^{(n,f)} \) with the fermion number \( f \) (\( 0 \leq f \leq n \)); \( \mathbb{H}^{(n)} = \bigoplus_{f=0}^{n} \mathbb{H}^{(n,f)} \). By successive multiplications of \( \hat{\psi}_\ell^{\dagger} \) on \( |0\rangle \), we obtain states with the fermion number \( f \). We use the following notation:

\[
|0 \cdots 0_{1} \cdots 0_{f} \cdots 0_{j} \cdots 0 \rangle = \hat{\psi}_{1}^{\dagger} \cdots \hat{\psi}_{i}^{\dagger} \cdots \hat{\psi}_{j}^{\dagger} |0\rangle, \tag{3}
\]

and the others are 0. Notice that the matrix representation is allowed for both even and odd number of vortices.\(^6\)

Then, the total matrix is given as \( \hat{\tau}_k^{(n)} = \bigoplus_{f=0}^{n} \hat{\tau}_k^{(n,f)} \). We will see below that this basic structure essentially determines the non-Abelian statistics of the representation.

Let us see concretely how to construct the Hilbert space \( \mathbb{H}^{(n)} \) and how the non-Abelian structure emerges in the matrix expression of \( \hat{\tau}_k^{(n,f)} \) (\( 0 \leq f \leq n \)) by using simpler cases with \( n = 2, 3, 4 \) vortices. In the case of \( n = 2 \), we have only one operation \( \hat{T}_1 \). The Hilbert space \( \mathbb{H}^{(2)} \) can be decomposed into a direct sum of three sectors \( \mathbb{H}^{(2,0)} = \{|00\rangle\}, \mathbb{H}^{(2,1)} = \{|10\rangle, |01\rangle\} \) and \( \mathbb{H}^{(2,2)} = \{|11\rangle\} \). Then, we obtain the matrix representation of \( \hat{\tau}_1 \) as (see Eq. (4))

\[
\tau_1^{(2,0)} = 1, \quad \tau_1^{(2,1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_1^{(2,2)} = 1. \tag{5}
\]

Here, \( \tau_1^{(2,0)} \) and \( \tau_1^{(2,2)} \) for the empty/fully-occupied sectors \( \mathbb{H}^{(2,0)} \) and \( \mathbb{H}^{(2,2)} \) are identities, while \( \tau_1^{(2,1)} \) for \( \mathbb{H}^{(2,1)} \) contains off-diagonal components. Even though the matrix \( \tau_1^{(2,1)} \) is non-diagonal, we have only one such matrix, and thus can simply diagonalize it to find an Abelian statistics. Indeed, \( \tau_1^{(2,1)} \) can be diagonalized by a unitary transformation, and we find that it has eigenvalues \( \pm 1 \). Therefore the exchange statistics is an Abelian anyon statistics in the sector \( \mathbb{H}^{(2,1)} \).

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1 The transformation \( \hat{T}_k \) is consistent with the Bogoliubov-de Gennes equation which is a microscopic equation for the fermions in the presence of vortices.

2 Two successive exchanges yield \( T_k^2 = -1 \).

3 If there is no minus sign in Eq. (1), we obtain \( T_k^2 = 1 \) which implies the Bose-Einstein/Fermi-Dirac statistics or para-statistics. Thus, the existence of the minus sign in Eq. (1) is essential.

4 Recall that, in the case of the Majorana vortices, four-times multiplication of an exchange operator is projectively equivalent to the identity \( \hat{e} \).

5 In contrast, \( \hat{\tau}_k \)'s are Abelian when \( \hat{\tau}_k \)'s are just complex numbers \( e^{i\theta_k} \), with real \( \theta_k \)'s (\( \theta = 0 \) for bosons, \( \theta = \pi \) for fermions, and \( \theta \neq 0, \pi \) for anyons).

6 It should be noticed that it makes sense to discuss the exchange of occupied \( \cdots 1 \) and empty \( \cdots 0 \) states which corresponds to the matrix elements, such as \( \cdots 10 \cdots \hat{\tau}_k \cdots 01 \cdots \). This is because we are considering the exchange of two vortices, and it is possible to have the situation where the vortex does not contain a Dirac zero-mode fermion in it.
Truly non-Abelian statistics emerges when the number of Dirac vortices is larger than or equal to 3. In the case of \( n = 3 \), the Hilbert space \( \mathbb{H}^{(3,0)} \) can be given as a direct sum of \( \mathbb{H}^{(3,0)} = \{000\}, \mathbb{H}^{(3,1)} = \{010, 011, 001\}, \mathbb{H}^{(3,2)} = \{110, 011, 101\} \) and \( \mathbb{H}^{(3,3)} = \{111\} \). We thus have the matrix representation of \( \tau_k \) as

\[
\begin{align*}
\tau^{(3,0)}_1 &= \tau^{(3,0)}_2 = 1, \\
\tau^{(3,1)}_1 &= \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tau^{(3,1)}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \\
\tau^{(3,2)}_1 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \tau^{(3,2)}_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\
\tau^{(3,3)}_1 &= \tau^{(3,3)}_2 = 1. \quad (6)
\end{align*}
\]

The empty/fully-occupied sectors \( \mathbb{H}^{(3,0)} \) and \( \mathbb{H}^{(3,3)} \) are again trivial. For the sectors \( \mathbb{H}^{(3,1)} \) and \( \mathbb{H}^{(3,2)} \), the matrices show the basic structure \( \mathbb{H} \). This time, it is not possible to simultaneously diagonalize all the matrices. Indeed, we find \( \tau^{(n,f)}_{(n,f)} \) and \( \tau^{(n,f)}_{(n,f)} \) are non-commutative; \( \tau^{(n,f)}_{(n,f)} \neq \tau^{(n,f)}_{(n,f)} \) with \( n = 3 \) and \( f = 1, 2 \). Therefore the exchange statistics is non-Abelian in the sectors \( \mathbb{H}^{(3,1)} \) and \( \mathbb{H}^{(3,2)} \).

We again note that, with the Dirac fermions, the Hilbert space is constructed for both \textit{even} and \textit{odd} numbers of vortices. This is in contrast with the vortices with Majorana fermions \( \mathbb{H} \), in which case the Hilbert space can be constructed from only \textit{even} numbers of vortices because a single Dirac fermion is constructed from two Majorana fermions belonging to different vortices.

Finally, let us also show the results for the case of \( n = 4 \). We will see the basic structure \( \mathbb{H} \) and thus non-Abelian statistics again when the fermion number \( f = 1, 2, 3 \). The Hilbert space \( \mathbb{H}^{(4)} \) can be given as a direct sum of five sectors: \( \mathbb{H}^{(4,0)} = \{0000\}, \mathbb{H}^{(4,1)} = \{1000, 0100, 0010, 0001\}, \mathbb{H}^{(4,2)} = \{1100, 1010, 0110, 0101, 0011\}, \mathbb{H}^{(4,3)} = \{1110, 1101, 1011, 0111\} \), and \( \mathbb{H}^{(4,4)} = \{1111\} \). The matrix representation of \( \tau_k \) is:

\[
\begin{align*}
\tau^{(4,0)}_1 &= \tau^{(4,0)}_2 = \tau^{(4,0)}_3 = 1, \\
\tau^{(4,1)}_1 &= \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \tau^{(4,1)}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\
\tau^{(4,1)}_3 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \tau^{(4,1)}_4 = \tau^{(4,1)}_5 = 1. \quad (7)
\end{align*}
\]

in the sectors \( \mathbb{H}^{(4,0)} \) and \( \mathbb{H}^{(4,1)} \),

\[
\begin{align*}
\tau^{(4,2)}_1 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \tau^{(4,2)}_2 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \\
\tau^{(4,2)}_3 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \\
\tau^{(4,2)}_4 &= \tau^{(4,2)}_5 = 1, \quad (8)
\end{align*}
\]

in the sector \( \mathbb{H}^{(4,2)} \), and lastly

\[
\begin{align*}
\tau^{(4,3)}_1 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \tau^{(4,3)}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\
\tau^{(4,3)}_3 &= \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\
\tau^{(4,4)}_1 &= \tau^{(4,4)}_2 = \tau^{(4,4)}_3 = 1, \quad (9)
\end{align*}
\]

in the sectors \( \mathbb{H}^{(4,3)} \) and \( \mathbb{H}^{(4,4)} \). As in the previous cases, the empty/fully-occupied sectors \( \mathbb{H}^{(4,0)} \) and \( \mathbb{H}^{(4,4)} \) are trivial. For the sectors \( \mathbb{H}^{(4,1)}, \mathbb{H}^{(4,2)} \) and \( \mathbb{H}^{(4,3)} \), we find \( \tau^{(n,f)}_{(n,f)} \) and \( \tau^{(n,f)}_{(n,f)} \) non-commutative; \( \tau^{(n,f)}_{(n,f)} \neq \tau^{(n,f)}_{(n,f)} \) with \( n = 4 \) and \( f = 1, 2, 3 \). Then the exchange statistics is again non-Abelian in the sectors \( \mathbb{H}^{(4,1)}, \mathbb{H}^{(4,2)} \) and \( \mathbb{H}^{(4,3)} \). In this way, we can construct the matrix representation of the exchange operators \( \tau_k \) for arbitrary number of Dirac vortices.

Our procedure to find the matrix representation for the exchange of two Dirac vortices is very similar to the one adopted by Ivanov \cite{ivanov} for the exchange of two Majorana vortices. However, let us clarify that these two are quite different in several aspects and emphasize that the complexity appearing in the manipulation of the Majorana vortices is not an essential requisite to realize the non-Abelian statistics. We here point out two aspects: locality of the Dirac fermions and conservation of the Dirac fermion number under the exchange of two vortices. While these two hold for the Dirac vortices as explicitly demonstrated in the present paper, they are violated in the Majorana vortices.
Concerning the locality of the Dirac fermions, we recall that the Dirac fermions employed to construct the Hilbert space for the Majorana vortices are nonlocal. More precisely, the Dirac fermion operator \( \psi^M_k = \frac{1}{\sqrt{2}} (\gamma^{2k} + i \gamma^{2k+1}) \) is made of two Majorana operators \( \gamma^{2k} \) and \( \gamma^{2k+1} \) in spatially separated \( 2k \)-th and \( (2k + 1) \)-th vortices. Namely, \( 2n \) Majorana vortices generate \( n \) nonlocal Dirac fermions. In contrast, since we started with the Dirac vortices themselves, they are by definition local, i.e., a Dirac fermion is located at the core of a single vortex.

In the Majorana vortices, non-Abelian statistics appears when one exchanges two ‘unpaired’ Majorana vortices, each of which is used to define different Dirac fermions. Then, the exchange of two unpaired Majorana vortices induces non-conservation of the fermion number \( \xi \). For example, exchange in the zero Dirac fermion sector generates a state with two Dirac fermions. Mathematically, one finds that the fermion number operator is (is not) conserved under the exchange of two Dirac fermions. In contrast, since we started with the Dirac vortices, they are by definition local, i.e., a Dirac fermion is located at the core of a single vortex.

We find interestingly enough that the Dirac operator in Eq. (2) can be decomposed into two Majorana operators \( \tilde{\psi}^M_k \) and \( \tilde{\psi}^M_{k+1} \) as \( \hat{\psi}_k = \frac{1}{\sqrt{2}} (\tilde{\psi}^M_k + i \tilde{\psi}^M_{k+1}) \), up to an overall phase. (Notice that this expression is consistent with the exchange rule \( \xi \), because each Majorana operator \( \tilde{\psi}^M_k \) changes the sign like Eq. (1) under the exchange of vortices.) Then, the operator (2) can be rewritten as \( \tilde{\psi} = \prod_{k=1,2} \sqrt{2} (1 + \tilde{\psi}^M_k \tilde{\psi}^M_{k+1}) \). This suggests that if we start with two Majorana fermions localized in a single vortex, then the Hilbert space of the Majorana vortices can be spanned by local Dirac fermions, and the fermion number is conserved under the exchange of vortices.

However, if we have three, or in general, odd numbers of Majorana fermions in a single vortex, we cannot construct Dirac fermions only by using the Majorana fermions in a single vortex, and thus necessarily encounter non-locality of Dirac fermions and non-conservation of fermion number. In fact, this was explicitly demonstrated when the number of Majorana fermions is three and they have SO(3) symmetry [12]. The resulting non-Abelian statistics was found to be given by a tensor product of the matrices in a single Majorana case and a Coxeter group. An example of physical systems to realize it is a color superconductor [12].

Hence, we observe that non-Abelian statistics can be realized by using either even or odd number of Majorana fermions. When the fermion number is even (odd), one uses local (nonlocal) Dirac fermions, and the fermion number operator is (is not) conserved under the exchange of two vortices. Therefore, we emphasize that both the non-locality of the Dirac fermion and the non-conservation of the fermion number are not essential requisite for the non-Abelian statistics, but the minus sign in the exchange of vortices as in Eq. (1) is rather important.

One interesting question is if two Majorana vortices behave similarly to one Dirac vortex, and moreover if exchange of two sets of two Majorana vortices show non-Abelian statistics. However, the exchange of two Dirac fermions is not equivalent to exchange of two sets of two Majorana vortices, see Fig. 1. Consider two Dirac fermion operators in a system of (even number \( M \)) of the 2\( M \)-th Majorana fermions, and the (2\( M \)+1)-th Majorana fermions (Goldstino) for spontaneously broken supersymmetry (4 supercharges). Prasad-Sommerfield vortices spontaneously break and the Nambu-Goldstone charges are conserved. Therefore the fermion trapped in the vortex is a color superconductor [14]. A color superconductor is a system to realize it is a color superconductor [14].

![FIG. 1: The exchanges of (a) two sets of two Majorana vortices and (b) two Dirac vortices, which yield the statistics given by Eq. (10) and Eq. (1).](image)

Thus, there is no minus sign, unlike Eq. (1). See Fig. 1. (a). The matrices for exchange operators \( \tilde{\xi}_k \) in this case can be obtained by ignoring the minus signs in our matrices. It satisfies \( \tilde{\xi}_k^2 = 1 \) giving at most parafermion statistics, see footnote [14].

Finally we make a comment on supersymmetric gauge theories as a possibility of Dirac vortices. Bogomol’nyi-Prasad-Sommerfield vortices spontaneously break and preserve a half of supersymmetry in supersymmetric gauge theories. Consequently Nambu-Goldstone fermions (Goldstino) for spontaneously broken supersymmetry is trapped in the core of the vortex [14]. When the theory possesses \( N = 2 \) supersymmetry (eight supercharges) \( N = 1 \) supersymmetry (four supercharges) is preserved. Therefore the fermion trapped in the vortex [14].

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[1]: Reference
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is a Dirac fermion because it belongs to a chiral multiplet. Therefore vortices in supersymmetric theories may give non-Abelian statistics studied in this paper. It remains as a future problem to study this case.

In summary, we have explicitly constructed the exchange statistics of vortices with Dirac fermions, which is relevant to the type C and DIII topological insulators/superconductors and some other systems, and have found that it is non-Abelian, contrary to conventional wisdom. It is expected that the existence of local Dirac fermions will provide us an interesting approach to observe the non-Abelian statistics in laboratories, helping us to utilize quantum computations.

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