Master Formula Approach to Chiral Symmetry Breaking: $\pi\pi$-scattering

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Abstract

The master formula approach to chiral symmetry breaking is used to analyze the phase shifts for $\pi\pi$ scattering in the elastic region. The results are in excellent agreement with the data for the phase shifts up to the $K\bar{K}$ threshold. Our analysis shows that the $\pi\pi$ data near threshold in the scalar channel favors a large quark condensate in the vacuum, i.e. \( \langle \bar{q}q \rangle \sim -(240 \text{ MeV})^3 \).
1. Introduction

Chiral symmetry offers an important framework for discussing hadronic processes at low energy [1]. In the past, current algebra using the PCAC hypothesis [2] and chiral perturbation theory using a one loop expansion [3] have been applied to describe processes involving pions. Both methods are limited to the threshold region because the former is an expansion around the soft pion point and the latter is an expansion around the chirally symmetric point. These points are unphysical in nature. Proposals to remediate these shortcomings based on dispersion techniques [4], while well motivated from the point of view of unitarity, do not help in understanding quantitatively the role of chiral symmetry beyond threshold.

Recently, two of us have proposed a general framework for discussing the role and consequences of chiral symmetry for processes involving pions at threshold and beyond [5]. Our approach relies on a set of exact identities for on-shell pions, all of which following from a master equation for the extended S matrix. This equation, which has the structure of a reduction formula, captures the essence of chiral symmetry and generalizes the current algebra approach on-shell. Scattering amplitudes involving pions, nucleons and photons follow by reduction in terms of correlation functions, some of which are measurable. Our approach enforces chiral symmetry and unitarity to all orders in the pion momentum, and yields exact threshold theorems that are not necessarily contrived around the soft pion point or the chiral point.

The purpose of this paper is to discuss in detail the \( \pi \pi \) scattering amplitude as derived from the master formula in the elastic region and compare it to the available data. The amplitude is rewritten in terms of the scalar and vector pion form factors, as well as vacuum correlators. The occurrence of these terms is required by chiral symmetry and unitarity. The advantage of this approach is that not only can these quantities be estimated in the same fashion as chiral perturbation theory with fewer parameters, but they can also be approximated by well accepted forms suggested by experiment without fine tuning of the parameters or the theory. This can then be used to assess our understanding of the low energy physics without being hampered by the need to use a perturbation series or worry about unitarity.

Given the importance of \( \pi \pi \) scattering in hadronic physics, there are currently a variety
of theoretical approaches to $\pi\pi$ scattering most of which allow for a good deal of phenomenological inspiration [3, 4, 8]. Since our purpose is to assess the constraints brought about by chiral symmetry alone, we will only compare our predictions with the data and one loop chiral perturbation theory [3, 4, 10]. For clarity, we will also display the one loop analysis following from the master equation by power counting in $1/f_\pi$ and minimality [3]. The latter involves only two parameters in the isospin symmetric limit whereas chiral perturbation theory requires four. Our results show that the master formula approach is able to reproduce the phase shifts quite well up to the $K\bar{K}$ threshold using educated inputs for the vacuum correlation functions.

The formulas used throughout this paper are presented in the next section. Section 3 applies these formulas to the one loop expansion in order to analyze $\pi\pi$ phase shifts. In section 4, the relevant form factors are modeled in the simplest form consistent with data and used with the master formula approach to fit phase shifts far above the threshold region. Comparison with chiral perturbation theory ($\chi$PT) results is made throughout. Our conclusions are summarized in section 5.

2. The Calculation

The master equations of the new formalism give a relation between the scattering amplitude and various correlation functions. Taking the case of $\pi\pi$ scattering, this relationship is quite simple. Specifically [3]

$$i \mathcal{T} (\pi^a \pi^b \to \pi^c \pi^d) = i \mathcal{T}_{\text{tree}} + i \mathcal{T}_{\text{vector}} + i \mathcal{T}_{\text{scalar}} + i \mathcal{T}_{\text{rest}}$$

$$= i \delta^{ab} \delta^{cd} A(s, t, u) + i \delta^{ac} \delta^{bd} A(t, u, s) + i \delta^{ad} \delta^{bc} A(u, s, t)$$

using the Mandelstam variables $s$, $t$, and $u$. In terms of the pion vector form factor $F_V$, the pion scalar form factor $F_S$, and the respective vector-vector and scalar-scalar correlation functions in the vacuum, $\Pi_V$ and $\Pi_S$, the general result may be written for the first three terms of equation (2), in the form

$$A_{\text{tree}}(s, t, u) = \frac{s - m_\pi^2}{f_\pi^2},$$

$^1$We take $m_\pi = 140$ MeV and $f_\pi = 93$ MeV.
\[ A_{\text{vector}}(s, t, u) = \frac{(s - u)}{f_\pi^2} \left( F_V(t) - 1 - \frac{t}{4f_\pi^2} \Pi_V(t) \right) + \frac{(s - t)}{f_\pi^2} \left( F_V(u) - 1 - \frac{u}{4f_\pi^2} \Pi_V(u) \right), \]

\[ A_{\text{scalar}}(s, t, u) = \frac{2m_\pi^2}{f_\pi^2} \left[ -f_\pi F_S(s) - 1 + \frac{\langle \hat{\sigma} \rangle}{2f_\pi} - \frac{m_\pi^2}{2} \Pi_S(s) \right] \]

where

\[ < \pi^a(p_2) | \bar{q} \gamma^\alpha \gamma^b q(0) | \pi^c(p_1) > = i \epsilon^{abc}(p_1 + p_2)_\alpha F_V((p_1 - p_2)^2), \]

\[ \int d^4x e^{iq \cdot x} < 0| T^* \bar{q} \gamma^\alpha \gamma^b q(0) | 0 > = -i \delta^{ab}(-g_{\alpha\beta}q^2 + q_\alpha q_\beta) \Pi_V(q^2), \]

\[ < \pi^a(p_2) | - \frac{\hat{m}}{m_\pi^2 f_\pi} \bar{q} q(0) | \pi^b(p_1) > = \delta^{ab} F_S((p_1 - p_2)^2), \]

and

\[ \frac{\hat{m}^2}{m_\pi^4 f_\pi^2} \int d^4x e^{iq \cdot x} < 0| T^* \bar{q} q(x) \bar{q} q(0) | 0 > \text{conn} = -i \Pi_S(q^2). \]

The vacuum expectation value \( \langle \hat{\sigma} \rangle \) was defined in [5] to be

\[ \langle \hat{\sigma} \rangle = -f_\pi - \frac{\hat{m}}{f_\pi m_\pi^2} \langle \bar{q} q \rangle \]

with \( \hat{m} = 7 \text{ MeV} \) being the average of the \( SU(2) \) quark masses. Taking a reasonable value for the quark condensate \( \langle \bar{q} q \rangle = -(240 \text{MeV})^3 \) gives \( \langle \hat{\sigma} \rangle \simeq -40 \text{ MeV} \). We recall that \( \langle \hat{\sigma} \rangle = 0 \) corresponds to the Gell-Mann-Oakes-Renner (GOR) relation [11]. A discussion of the overall dependence of our result on \( \langle \hat{\sigma} \rangle \) is given at the end of section 4.

The tree amplitude was first calculated by Weinberg [1] and must appear in any low energy theory. Also appearing is the vector and scalar expressions showing in a model independent way the importance of the \( \rho \) contribution [13] and the \( \sigma \) state consisting of the two scattered pions [17]. The contribution from the “rest” is a correlation function of four one-pion reduced axial vectors [5]. Since this is not a measured quantity, its one loop form is quoted. Specifically,

\[ A_{\text{1-loop}}^{\text{rest}}(s, t, u) = \frac{(s - 2m_\pi^2)^2}{2f_\pi^4} (\hat{c}_1 + \mathcal{J}(s)) + \frac{(t - 2m_\pi^2)^2}{4f_\pi^4} (\hat{c}_1 + \mathcal{J}(t)) + \frac{(u - 2m_\pi^2)^2}{4f_\pi^4} (\hat{c}_1 + \mathcal{J}(u)) + \mathcal{O}(\frac{1}{f_\pi^6}) \]  

(2)
with
\[
\mathcal{J}(q^2) = \frac{1}{8\pi^2} \left( 1 - \sqrt{1 - \frac{4m^2}{q^2}} \arccoth \sqrt{1 - \frac{4m^2}{q^2}} \right)
\]
representing the finite part of the pion-propagated loop.

Equation (2) is an exact result. However, there is no known way at present to fundamentally calculate the correlation functions or form factors of this result. Instead, either a loop expansion or experimental fit must be used. For both methods, the different isospin contributions can be extracted from the pion scattering amplitude in the usual manner

\[
T^0(s, t) = 3A(s, t, u) + A(t, u, s) + A(u, s, t)
\]
\[
T^1(s, t) = A(t, u, s) - A(u, s, t)
\]
\[
T^2(s, t) = A(t, u, s) + A(u, s, t)
\]
\[
T^I(s, t) = 32\pi \sqrt{\frac{s}{s - 4m^2}} \sum_{l=0}^{\infty} (2l + 1) P_l(\cos \theta) e^{i\delta^I_l(s)} \sin \delta^I_l(s).
\]
Here \(\theta\) is scattering angle in the center of mass frame. Since elastic scattering will always be considered, no part of the amplitude will be lost to another channel and so the phase shifts will always be real.

To obtain the phase shift, \(\delta^I_l\), an integration of the respective \(T^I\) over the Legendre polynomial with index \(l\) must be performed. This leads to a unitarity condition on the scattering amplitude
\[
f^I_l(s) = \sqrt{1 - \frac{4m^2}{s}} \int_{-1}^{1} \frac{d(\cos \theta)}{64\pi} T^I(s, \cos \theta) P_l(\cos \theta)
\]
\[
\text{Im} f^I_l = \left| f^I_l \right|^2
\]
which serves as a consistency check on the perturbative expansion employed.

Two of the various ways to express the phase shift are
\[
\delta^I_l(s) = \arccot \left( \frac{\text{Re} f^I_l(s)}{\text{Im} f^I_l(s)} \right) = \frac{1}{2} \arcsin \left( 2 \text{Re} f^I_l(s) \right)
\]
The first expression is the definition for the phase shift and is therefore preferred. However, it is sometimes better to use the second expression as will be expounded upon below. The
equivalence of these two expressions for the phase shift may also serve as a measure of unitarity.

3. Loop expansion

Calculation of the amplitude in loops is really an expansion in powers of \( s/(4\pi f_\pi)^2 \). This expansion has relevant corrections only for \( \sqrt{s} \geq 0.5 \) GeV. The amplitude is purely real at tree level and only at one loop is there any contribution to the imaginary part. It is therefore easy to see that such an expansion does not preserve unitarity (eq. (4)). For the same reason, the arcsin rather than the arccotangent is a better approximation to the true phase shift at one loop since two terms of the expansion are known for the real part as opposed to only one term for the imaginary part. Therefore the arcsin will be used for the graphs of this section.

It is not important to quote the one loop form of the scattering amplitude since it is worked out in \([5]\) except to note that the function \( J(q^2) \) given by equation (3) appears in all the form factors. In the \( t \) and \( u \) channels, the argument of \( J \) will have a \( \cos \theta \) dependence. The integral over the Legendre polynomials then must be integrated numerically.

As explained in \([5]\), only two constants appear in the amplitude to one loop: \( c_1 \) and \( \hat{c}_1 \). There it was shown that \( c_1 = \Pi_V(0) = 2f_\pi^2F'_V(0) = 0.031 \pm 0.001 \) to one loop using the measured value for the vector charge radius \( \langle r^2 \rangle_V = 6F'_V(0) = 0.42 \pm 0.02 \) fm\(^2 \) \([14]\). In \([5]\), various scattering lengths were calculated and compared with experiment to give seven determinations of \( \hat{c}_1 \). The weighted average of these values is \( 16\pi^2\hat{c}_1 = 2.46 \pm 0.55 \). Table 1 shows how the scattering lengths and range parameters in the master formula approach using the middle value of \( \hat{c}_1 \) compare with experiment, the tree calculation, and \( \chi PT \) (using arcsin for all theoretical values). The phase shifts \( \delta_0^0, \delta_1^1, \) and \( \delta_2^0 \) are plotted as the solid line in figures 1, 2, and 3. All curves in these figures cannot be extended beyond 45 degrees because the one loop result keeps growing like \( q^2 \ln q^2 \) pushing the argument of the arcsin above one.

In all three graphs, both results fit the data well up to about 450 MeV. The subsequent deviation is most likely an artifact of the loop expansion, but a two loop calculation is needed to clear up the ambiguities \([7]\). The shaded region for the master equations in

\(^2\)We used the central values of \( \bar{t}_1 = -0.6, \bar{t}_2 = 6.3, \bar{t}_3 = 2.9, \) and \( \bar{t}_4 = 4.3 \) as quoted in \([3]\).
\( \delta^0 \) shows the variation due to the uncertainty in \( \hat{c}_1 \). This shows that up to the point of deviation, the lack of accuracy of \( \hat{c}_1 \) does not matter much. Therefore the middle value for \( \hat{c}_1 \) was used for all other calculations in this paper. The master formula approach is closer to the experimental value for \( \delta^1 \) and \( \delta^2 \) even though it has half the number of tunable parameters compared to \( \chi \)PT. For \( \delta^0 \) the master formula result is too low at threshold, but the overall shape leads more towards the higher energy values before the \( q^2 \) dependence dominates. Since the one loop results in the master formula follow from a \( 1/f_\pi \) expansion and minimality \([5]\), the present results show the overall consistency of the approach. Minimality is at the origin of KSFR \([12]\).

Taking the arccotangent form for the phase shift to one loop allows the result to be in the full range of 0 to 180 degrees. It was argued above that the imaginary part will be too small since only the first term of the perturbation series is present to one loop. This would result in a lower phase shift than the full result of the theory should give. Indeed this turns out to be the case as seen by the dashed line in figures 4, 5, and 6. This is especially apparent for \( \delta^1 \) which shows no sign of a \( \rho \) resonance from this analysis for either the master formula or \( \chi \)PT since it is impossible to see resonances from a strict perturbative expansion due to violation of unitarity.

4. \( \rho \) and \( \sigma \) saturation

The one loop result employed above works well near threshold but, as was seen, cannot model even the simplest resonances. For this reason an alternative approach would be preferred. The master formula gives a general result which is not dependent on the loop expansion and so allows such an alternative. Modeling the form factors and correlation functions that appear in eq. \((2)\) with reasonable forms suggested by experiment should lead to an accurate result without much fine tuning of the parameters or of the theory.

The only problem is that direct experimental measurements do not exist for some of the correlation functions needed. At least for the vector channel it is well accepted that the features are dominated by the \( \rho \) and a good parameterization is \([13, 14]\)

\[
F_V(q^2) = \frac{m^2_\rho + \gamma q^2}{m^2_\rho - q^2 - i m_\rho \Gamma_\rho(q^2)}
\]

with \( m_\rho = 770 \) MeV and a momentum dependent width
\[
\Gamma_\rho(q^2) = 149 \left(\frac{m_\rho}{\sqrt{q^2}}\right) \left(\frac{q^2 - 4m_\pi^2}{m_\rho^2 - 4m_\pi^2}\right)^{3/2} \theta(q^2 - 4m_\pi^2) \text{ MeV.} \quad (7)
\]

For vector dominance \(\gamma\) is zero. Fitting the curve to the data for \(F_V\) requires \(\gamma = 0.3\) and is plotted in figure 7. This corresponds to \(\langle r^2 \rangle_V = 0.51 \text{ fm}^2\) which is reasonably close to the experimental value quoted above.

The vector-vector correlation function, \(\Pi_V(q^2)\), can be assumed to be dominated by the \(\rho\) resonance as well. Therefore the imaginary part may be modeled by a delta function for the \(\rho\) and a theta function so that the perturbative QCD result is recovered at large energies

\[
\text{Im}\Pi_V(s) = f_\rho^2 \delta(s - m_\rho^2) + \frac{1}{4\pi} \theta(s - s_0)
\]

with \(s_0 \simeq 1.2 \text{ GeV}\) as a reasonable value taken for the continuum threshold \([13]\). The real part may be obtained by a once subtracted dispersion relation. The same can be done for the scalar-scalar correlation function \(\Pi_S(q^2)\) with a twice subtracted dispersion relation due to the form of the perturbative QCD result. Smearing out the delta function by giving a width to the resonances, yields

\[
\Pi_V(s) = \frac{f_\rho^2}{m_\rho^2 - s - im_\rho\Gamma_\rho} - \frac{1}{4\pi^2} \ln\left(1 - \frac{s}{s_0}\right) + \Pi_V^{\text{pert.}}(0)
\]

\[
f_\pi^2 \Pi_S(s) = \frac{f_1^2}{m_1^2 - s - im_1\Gamma_1} - \frac{3\hat{m}^2}{4\pi^2 m_\pi^4} s \ln\left(1 - \frac{s}{s_0}\right) + f_\pi^2 \left(\Pi_S^{\text{pert.}}(0) + s\Pi_S^{\text{pert.}}'(0)\right).
\]

Additional resonances can be added to the scalar channel if needed by adding further terms similar to the first term in the equation for \(\Pi_S\). Vacuum dominance in the vector channel requires \(\Pi_V^{\text{pert.}}(0) = 0\), however this requirement can be relaxed for scalar particles and used to enable a better fit to data. Therefore let \(\alpha \equiv f_\pi^2 \Pi_S^{\text{pert.}}(0)\) and \(\beta \equiv f_\pi^4 \Pi_S^{\text{pert.}}'(0)\) which are both dimensionless. The widths are zero for spacelike momenta and are taken to be eq. \([7]\) for the vector width and

\[
\Gamma_1(q^2) = \Gamma_1 \left(\frac{1 - 4m_\pi^2/q^2}{1 - 4m_\pi^2/m_1^2}\right)^{1/2}
\]

for the scalar.
Expanding the resonance forms in $s$ and matching coefficients with those of the one loop result [5] in the vector channel gives $f_\rho = (140 \pm 4) \text{ MeV}$ in agreement with the experimental value of 144 MeV. Furthermore, if one resonance is assumed in the scalar channel, matching threshold parameters for the scalar channel gives $\alpha = -2 \times 10^{-2}$, $\beta = -5 \times 10^{-4}$, and $f_1 = 125 \text{ MeV}$. These values were used for the fits to the phase shifts and are close to those used in [18] (which in our notation would be: $m_1 = 540 \text{ MeV}$, $f_1 = 90 \text{ MeV}$, and $s_0 = 1.1 \text{ GeV}$).

Finally considering the scalar form factor, $F_S(q^2)$, there is not much to hint at its form. At tree level [5] $F_S(0) = -1/f_\pi$, so a reasonable form assuming the same one resonance as used for $\Pi_S$ is

$$F_S(s) = -\frac{1}{f_\pi m_1^2} \frac{h_1 m_1^2}{s - i m_1 \Gamma_1}.$$  

Using the empirical determination for the scalar charge radius [3] $6F'_S(0)/F_S(0) = 0.7 \pm 0.2$ fm$^2$ [3] and taking $h_1 = 1$ gives $m_1 = 580 \text{ MeV}$. Fitting to the data for $\delta_0^0$ requires the only free parameter, $\Gamma_1$, to be 175 MeV. Figure 7 shows that $|F_V|$ and $\Pi_V$ both compare well with the data [16]. The imaginary part of $F_S$ normalized by $F_S(0)$ is comparable to a dispersive analysis [7] shown by the dashed line.

Even though there is no definite resonance below the $f_0(980)$, many have interpreted the large enhancement of the $\delta_0^0$ data in the region of $\sqrt{s} = 500 - 800 \text{ MeV}$ (see figure 4) to be due to a $\sigma$ “particle” with mass around 500 MeV and width $\approx 700 \text{ MeV}$ [15]. This scenario is consistent with our value for $m_1$ but our $\Gamma_1$ is much smaller. It should be noted that in the master formula approach a large contribution from the $t$-channel $\rho$ resonance is also seen. This is in agreement with conjectures set forth by other authors [8] although it is not sufficient to alone fit the data in this region. Figure 4 shows that $(m_1, \Gamma_1) = (580, 175) \text{ MeV}$ reproduces the data quite well up to about 650 MeV. Above this the fit rises slower than the data.

Enhancement of the $\delta_0^0$ resonance at higher energies and also a leveling of the sharp peak in $\Pi_S$ (figure 7) can be done through the addition of further resonances. It was found that including both of the next two $0^+(0^{++})$ resonances: $f_0(980)$ and $f_0(1300)$ worked well. The fit to the data is shown by the solid line in figure 8 and the form factors in figure 9. Values for the constants employed are shown in table 2. The result strongly
depends on the values of the $h_i$'s and only weakly depends on the $f_i$'s. Since the first resonance at 560 MeV fit the data very well, its constants were retained. For such a simple model of the resonances the fit is quite good up to the $K\bar{K}$ threshold. Here the strange particles’ contribution and the role of the “rest” term, two problems which may be related, need to be taken into account and will be studied elsewhere. This fit shows in $\Pi_S$ (see figure 9) that even a narrow resonance around 580 MeV can be obscured by wide neighboring resonances. Further analysis is required.

The results are very good in the vector channel as well, giving the $\rho$ resonance as expected in figure 5. This is primarily due to three reasons: there is a large amount of data available in this channel and so the model used for the form factors is phenomenologically based, there is only one resonance of importance in the energy region of interest, and the “rest” does not contribute much to one loop showing this is probably the case in general. In addition the scalar part of the amplitude has only a small contribution and hence the one resonance or the three resonance scalar fit discussed above give identical results. For definiteness, one scalar resonance was used.

The “rest” term is important to the $\delta_{0}^{2}$ result. In fact it is the only term which significantly contributes to the imaginary part of the scattering amplitude. This term was not taken into account in the resonance forms and so the phase shift is nearly always $0^\circ$ as seen by the solid line in figure 6. However, using the one loop expression for the “rest” is justified up to about 0.5 GeV as seen in the last section. Using the one resonance fit and including the one loop “rest” agrees very well with data and is also plotted in figure 6 as the dashed line. The imaginary part of the expanded “rest” dominates at high momenta due to the $q^2$ dependence and hence the result begins to deviate around 450 MeV. Also figure 6 includes the arccotangent form of the $\chi$PT result given by the dotted-dashed line.

To see the threshold behavior, the combination $\delta_{0}^{0} - \delta_{1}^{1}$ is plotted for the master formula approach using one resonance and for both the master formula approach and $\chi$PT in the one loop approximation in figure 10. All three give comparable results near threshold where there is data. This combination is no longer as important a parameter to plot since detailed information on both phase shifts independently is now available.

As discussed above, the value of $\langle \hat{\sigma} \rangle = -40$ MeV was used in all calculations. However, the GOR relation gives $\langle \hat{\sigma} \rangle = 0$. Furthermore, if the quark condensate $\langle \bar{q}q \rangle$ is zero then
\langle \hat{\sigma} \rangle = -93 \text{ MeV}. The dependence of the master formula solutions on the value of \langle \hat{\sigma} \rangle can be seen best in the scalar channel which contains the largest contribution. Only the real part of the scattering amplitude is affected. Figure 11 shows the dependence of \delta_0^0 on these three values of \langle \hat{\sigma} \rangle for the one loop master formula calculation. This graph shows that \langle \hat{\sigma} \rangle = -93 \text{ MeV}, corresponding to a zero quark condensate, has the wrong shape at threshold. The other two values for \langle \hat{\sigma} \rangle are comparable with the data. Figure 12 shows how the combination \delta_0^0 - \delta_1^1 changes with the three different values of \langle \hat{\sigma} \rangle. Here the \(-40 \text{ MeV} \) value seems to fit best.

As was pointed out by [20], the difference \delta_0^0 - \delta_0^2 at the \(K^0\) mass (498 MeV) is a direct measurement of CP violation. The most recent data [21] gives 29.2 \(\pm\) 3\(^\circ\) from \(\pi p \rightarrow \pi \pi \Delta\). A more direct measurement from \(K \rightarrow \pi \pi\) decay is not available, although analyses based on chiral perturbation theory have been carried out [22]. A weighted average of the data prior to 1975 gives [23] 41.4\(^\circ\) \(\pm\) 8.1\(^\circ\). Using the master formula resonance result (with one scalar resonance) gives 29.8\(^\circ\). Use of the one resonance result with the one loop “rest” added is not reliable at this energy but gives 53.0\(^\circ\). The one loop expansion in general can only be used in the arccotangent form since the arcsin form of \delta_0^0 does not extend to the \(K^0\) mass. This gives 37.2\(^\circ\). In comparison, the arccotangent form for \chiPT gives 48.3\(^\circ\) (to be specific, [9] quote 45\(^\circ\) \(\pm\) 6\(^\circ\) where they use the arccotangent with the real part only).

5. Conclusions

The master formula approach to \(\pi \pi\) scattering offers powerful constraints on the scattering amplitude solely on the basis of chiral symmetry and unitarity. As a loop expansion, it fits the threshold region as well as chiral perturbation theory with half as many parameters to fix, and is compatible with KSFR. As a resonance fitting of the form factors and the vacuum correlation functions, it is in excellent agreement with the data up to the \(K \bar{K}\) threshold. We stress that the various form factors and correlation functions are amenable to fundamental estimates in QCD or empirical measurements. With a minimum number of assumptions, this approach provides a testing ground for an empirical understanding of scalar and vector correlators from the \(\pi \pi\) data. Also, the \(\pi \pi\) data near threshold in the scalar channel, although not very accurate, suggests that the vacuum supports a large quark condensate, and points in favor of the GOR relation. Better measurements at
threshold will settle these important theoretical issues.

The present results also show that chiral symmetry constraints go far beyond the threshold region. In particular, we have found that in the vector channel the $\rho$ is dominant, while in the scalar channel the presence of a low-lying though not necessarily broad resonance is unavoidable. Since our approach shows clearly the interplay between correlations in the vacuum and measured scattering amplitudes, it is well suited for lattice estimations. Improvements can even be made to the master formula approach by an expansion to $SU(3)$ and inclusion of kinematical nucleons. These additions should allow a much better understanding of the inelastic region and will be carried out elsewhere [19].

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Figure Captions

- Fig.1. The $\delta_0^0$ phase shift using the arcsin for the one loop expansion. The shaded region is the master formula result taking into account the uncertainty in $\hat{c}_1$. The central value of $16\pi^2\hat{c}_1 = 2.46$ is given by the middle solid line. The tree result is shown by the dashed line and the dotted-dashed line is the $\chi$PT result. Data is from [24] below 400 MeV and [25] otherwise.

- Fig.2. The $\delta_1^1$ phase shift using the arcsin for the one loop expansion. The solid line is the master formula results, dotted line is the tree result, and the dotted-dashed line is the $\chi$PT result. Data is from [24, 25].

- Fig.3. The $\delta_2^0$ phase shift using the arcsin for the master formula (solid line), tree (dashed line), and $\chi$PT (dotted-dashed line). Data is from [28].

- Fig.4. $\delta_0^0$ using the arccotangent. Here the solid line is the one resonance fit of the master equations for one resonance, the dotted-dashed line is $\chi$PT to one loop, and the dashed line is the master formula result to one loop. Data is from [24, 25].

- Fig.5. $\delta_1^1$ using the arccotangent for the master formula with one resonance (solid line), master formula expanded to one loop (dashed line), and $\chi$PT (dotted-dashed line). Data is from [24, 25].

- Fig.6. $\delta_2^0$ using the arccotangent for the master formula approach using one resonance (solid line) and the one loop expanded “rest” along with the one resonance (dotted line). The $\chi$PT result using arccotangent (dashed line) is also shown. Data is from [28].

- Fig.7. The form factors for the one resonance fit of the master formula approach. Data for $F_V$ and $\Pi_V$ is from [10] (see [27] for details) and the estimate for $F_S$ from a dispersive analysis [7].

- Fig.8. Fitting the $\delta_0^0$ to three resonances in the master equation approach. Also plotted are the master formula to one loop (dashed line) and $\chi$PT (dotted-dashed line). Data is from [24, 25].
• Fig.9. The form factors for the three resonance fit to the master formula (solid line). Other data is as in figure 7.

• Fig.10. $\delta_0^0 - \delta_1^1$ with the one resonance fit for the master formula result (solid line), the one loop master formula result (dashed line), and $\chi$PT (dotted-dashed line). Data is from [24].

• Fig.11. The $\delta_0^0$ phase shift for the one loop expansion of the master formula using $\langle \hat{\sigma} \rangle = 0$ (solid line), $-40$ MeV (dotted line), and $-93$ MeV (dashed line). These values correspond to $\langle \bar{q}q \rangle = -(290 \text{ MeV})^3$, $-(240 \text{ MeV})^3$, and 0 respectively. Data is from [24, 25].

• Fig.12. $\delta_0^0 - \delta_1^1$ for the master formula one resonance fit using $\langle \hat{\sigma} \rangle = -93$ MeV (solid line), $-40$ MeV (dashed line), and 0 (dotted-dashed line). These values correspond to $\langle \bar{q}q \rangle = 0$, $-(240 \text{ MeV})^3$, and $-(290 \text{ MeV})^3$ respectively. Data is from [24].
Table 1: Comparison of the experimental scattering lengths and range parameters to those predicted by the tree, $\chi$PT, and one loop master formula calculations. For the master formula, the values $c_1 = 0.031$ and $16\pi^2\hat{c}_1 = 2.46$ were used. Data is from \cite{26} and $\chi$PT results are from \cite{3, 9}.

|                | Experiment | Tree | $\chi$PT | Master |
|----------------|------------|------|----------|--------|
| $a_0^0$        | 0.26 ± 0.05 | 0.16 | 0.20     | 0.20   |
| $m_\pi^2 b_0^0$| 0.25 ± 0.03 | 0.18 | 0.26     | 0.24   |
| $m_\pi^2 a_1^1$| 0.038 ± 0.002 | 0.030 | 0.033 | 0.038 |
| $a_0^2$        | −0.019 ± 0.021 | −0.045 | −0.041 | −0.041 |
| $m_\pi^2 t_0^3$| −0.082 ± 0.008 | −0.089 | −0.065 | −0.068 |

Table 2: The parameters used for the three resonance fit of the master formal approach. All constants are in MeV except for $\alpha$, $\beta$, $\gamma$, and the $h_i$'s which are dimensionless.

| $m_1$ | 580 | $f_1$ | 125 | $\Gamma_1$ | 175 |
|-------|-----|-------|-----|-----------|-----|
| $m_2$ | 980 | $f_2$ | 300 | $\Gamma_2$ | 400 |
| $m_3$ | 1300 | $f_3$ | 300 | $\Gamma_3$ | 400 |
| $\hat{m}$ | 7 | $h_1$ | 1 | $\langle \hat{\sigma} \rangle$ | −40 |
| $\sqrt{s_0}$ | 1200 | $h_2$ | 13 | $\alpha, \gamma$ | −0.17, 0.3 |
| $f_\rho$ | 140 | $h_3$ | −11 | $\beta$ | $−1.6 \times 10^{-3}$ |
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Figure 1

\[ \delta_0^0 \]

using arcsin

Legend:
- MF 1 loop
- Tree
- \( \chi PT \)
Figure 2

$\delta_1^1$

using arccsin

- MF 1 loop
- Tree
- $\chi$PT

$\sqrt{s}$ (MeV)

degrees

$\delta_1^1$ using arccsin

Figure 2
Figure 3

\[ \delta_0^2 \]

using \textit{arcsin}

- MF 1 loop
- Tree
- \(\chi\text{PT}\)
Figure 5

$\delta_1^1$

using $\text{arccot}$

- --- MF 1 res
- --- MF 1 loop
- --- $\chi$PT
Figure 6

$\delta_0^2$

using $\text{arccot}$

- MF 1 res
- MF 1 res + rest
- $\chi$PT
Figure 7
Figure 8

\[ \delta_0 \]

*using arccot*

- **MF 3 res**
- **MF 1 loop**
- **χPT**

\[ \sqrt{s} \text{ (MeV)} \]

\[ \text{degrees} \]
Figure 9

The diagrams show the behavior of various quantities as a function of $\sqrt{s}$ (MeV), where $s$ is the center-of-mass energy squared. The left graphs depict the real and imaginary parts of $F_\pi^0 F_S(0)$, while the right graphs illustrate the real and imaginary parts of $\Pi_v$ and $f^2_\pi \Pi_S$, respectively. The data points are accompanied by error bars, indicating the uncertainty in the measurements.
Figure 10

\[ \delta_0^0 - \delta_1^1 \]

- MF 1 res
- MF 1 loop
- \( \chi PT \)
Figure 11

\[ \delta_0^0 \]

using \textit{arcsin}

\begin{itemize}
  \item \( \langle \sigma \rangle = 0 \)
  \item \( \langle \sigma \rangle = -40 \text{ MeV} \)
  \item \( \langle \sigma \rangle = -93 \text{ MeV} \)
\end{itemize}
Figure 12

\[ \delta_0^0 - \delta_1^1 \]

- Solid line: \( \langle \sigma \rangle = -93 \text{ MeV} \)
- Dashed line: \( \langle \sigma \rangle = -40 \text{ MeV} \)
- Dotted line: \( \langle \sigma \rangle = 0 \)