First-Order Theory of Probabilistic Independence and Single-Letter Characterizations of Capacity Regions

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Abstract

We consider the first-order theory of random variables with the probabilistic independence relation, which concerns statements consisting of random variables, the probabilistic independence symbol, logical operators, and existential and universal quantifiers. Although probabilistic independence is the only non-logical relation included, this theory is surprisingly expressive, and is able to interpret the true first-order arithmetic over natural numbers (and hence is undecidable). We also construct a single-letter characterization of the capacity region for a general class of multiuser coding settings (including broadcast channel, interference channel and relay channel), using a first-order formula. We then introduce the linear entropy hierarchy to classify single-letter characterizations according to their complexity.

I. INTRODUCTION

In this paper, we study the first-order theory of random variables with the probabilistic independence relation. We first review some fragments of the theory studied in the literature. The probabilistic independence implication problem studied by Geiger, Paz, and Pearl \[1\] and Matúš \[2\] concerns the problem of deciding whether a list of probabilistic independence statements among random variables implies another probabilistic independence statement, e.g., deciding whether \(X \perp Y \wedge XY \perp Z \Rightarrow X \perp YZ\), where \(\perp\) denotes probabilistic independence, and juxtaposition \(XY\) denotes the joint random variable of \(X\) and \(Y\). It was shown in \[1\] that probabilistic independence implication is finitely axiomatizable (the previous example is one of the axioms), and hence is algorithmically decidable.

The conditional independence implication problem \[3\], \[4\], \[5\], \[6\] generalizes the probabilistic independence implication problem by considering probabilistic conditional independence. Pearl and Paz \[6\] introduced the semi-graphoid axioms, which was proved to be incomplete by Studený \[7\]. As shown in \[8\], no finite axiomization of probabilistic conditional independence is possible. Nevertheless, the semi-graphoid axioms are complete for saturated conditional independence statements \[9\], \[10\].

It is unknown whether the conditional independence implication problem is decidable \[11\], though some variants of this problem have been proved to be decidable or undecidable. If the cardinalities of all random variables are bounded, then it was shown by Niepert \[12\] that the problem is decidable (also see \[13\]). However, if only the cardinalities of some random variables are bounded, then it was proved by Li \[14\] that the problem is undecidable. Khamis, Kolaitis, Ngo and Suciu \[11\] showed that the general conditional independence implication problem is at most in \(\Pi^0_1\) in the arithmetical hierarchy.

Linear information inequalities \[15\] concern linear inequalities among entropy terms on the random variables. Pippenger \[16\] raised the question whether the axiom \(I(X;Y|Z) = H(XZ) + H(YZ) - H(XYZ) - H(Z) \geq 0\) is sufficient to characterize every true linear information inequality. This was answered by Zhang and Yeung \[17\], \[18\] in the negative, who showed the existence of non-Shannon-type inequalities not implied by the axiom. More non-Shannon-type inequalities were given in \[19\], \[20\], \[21\], \[22\], \[23\]. Linear information inequalities are closely related to the problem of finding the capacity region in network coding \[24\], \[25\], \[26\]. General logical combinations of linear inequalities (using \(\land, \lor, \neg\)) are investigated in \[11\]. It is unknown whether the verification of conditional linear information inequalities is decidable \[27\], \[28\], \[11\], though if the problem is extended to allow affine inequalities, then it was shown in \[14\] that the problem is undecidable.

While all aforementioned problems are not existential (they are purely universal statements in the form \(\forall X^n.P(X^n)\), where \(P\) is a predicate, and all random variables \(X^n = (X_1, \ldots, X_n)\) are universally quantified, i.e., they are in the \(\forall^*\)-fragment of the first-order theory of random variables), existential results on random variables (concerning predicates on \(X^n\) in the form \(\exists U^m.P(X^n, U^m)\)) are widely used in information theory. For example, in network information theory \[29\], capacity regions are often expressed as statements concerning the existence of some auxiliary random variables. Some examples of useful existential formulae include the double Markov property \[30\] (\(X \perp Y|Z\) denotes conditional independence)

\[
X \perp Z|Y \wedge Y \perp Z|X
\Rightarrow \exists U.(U \perp U|X \wedge U \perp U|Y \wedge XY \perp Z|U),
\]

\[\]
the copy lemma [18], [23], and the functional representation lemma [29] (also see [31]). The existential theory of random variables with information inequalities has been studied systematically in [32].

There are examples of rate regions and bounds in network information theory expressed in a nested “for all, there exists” form (i.e., a predicate on $X^n$ in the form $\forall Y^n.P(X^n, Y^n) \rightarrow \exists U^n.Q(X^n, Y^n, U^n)$), e.g. the outer bound for multiterminal source coding in [33], the upper bound on key agreement in [34], the outer bound for the broadcast problem in [35, Thm 1], and the auxiliary receivers in [36]. In these examples, there are both existentially and universally quantified auxiliary random variables. They can be considered as formulae in (the $\forall^n \exists^n$-fragment of) the first-order theory of random variables, with a larger depth of quantifier alternation compared to existential formulae.

Although the aforementioned special cases have been studied extensively, to the best of the author’s knowledge, there has not been a systematic treatment on the general first-order theory of random variables with arbitrary depth of quantifier alternation (refer to the related work section). In this paper, we investigate the first-order theory of random variables with probabilistic independence (FOTPI), which concerns formulae consisting of variables (which represents random variables), the probabilistic independence symbol $\perp \perp$, logical operators ($\land$, $\lor$, $\neg$) and existential and universal quantifiers ($\exists$, $\forall$). Even though probabilistic independence is the only non-logical relation included, we can use it to define other concepts in probability such as conditional independence, functional dependency, uniformity, cardinality and entropy. Therefore, most of the aforementioned problems can be considered as fragments of FOTPI. Since cardinality bounds can be defined using first-order formulae, as a corollary of [14], FOTPI is undecidable. Also, we show that FOTPI can interpret the true first-order arithmetic over natural numbers.

Furthermore, we prove that for any setting within a general class of multiuser coding settings (including broadcast channel [37], interference channel [38], relay channel [39], and finite-state Markov channel [40, 41]), the capacity region has a single-letter characterization that can be stated as a formula in FOTPI (here “single-letter” means the number of random variables in the formula is fixed). Whether this can be regarded as a solution to the open problem of finding single-letter characterizations of the capacity regions of broadcast channel and interference channel depends on the definition of “single-letter characterization”. While there is no generally accepted definition of what constitutes a single-letter characterization [44], it can be argued that if “for all, there exists” statements (e.g. [33], [42], [44], [45], [46]) are considered single-letter, then there is no reason to exclude statements with a larger depth of quantifier alternation.

The single-letter characterization given in this paper is very complex (its depth of quantifier alternation is 17), which is against the purpose of single-letter characterizations. Perhaps, instead of asking for any single-letter characterization of the capacity region of the broadcast/interference/relay channel, a more precise question is to find the simplest single-letter characterization, which will hopefully provide more insight to the optimal coding schemes. We propose a classification of first-order formulae, called the linear entropy hierarchy, according to their depth of quantifier alternation. We can then rigorously state the open problems of finding single-letter characterizations of capacity regions of the aforementioned channels in the lowest possible level in the linear entropy hierarchy.

This paper is organized as follows. In Section II we define some relations over random variables in FOTPI. In Section III we show that FOTPI can interpret the first-order arithmetic. In Section IV we investigate definability in FOTPI. In Section V we study a representation of events in FOTPI. In Section VI we study a representation of random sequences in FOTPI. In Section VII we present the main result, which is a single-letter characterization of the capacity region for a general class of multiuser coding settings. In Section VIII we propose the linear entropy hierarchy as a classification of first-order formulae according to their complexity. In Section IX we study the extension to continuous random variables.

A. Related Work

A computer program called PSITIP is described in [32], which is capable of expressing and verifying some first-order statements on random variables, though the paper [32] is focused only on existential and implication problems.

Regarding inner bounds of capacity regions of multiuser coding settings, several general inner bounds were studied in [45], [46], [47], [48]. These bounds can be regarded as purely existential formulae in FOTPI. The author is unaware of any general result on outer bounds of multiuser coding settings, though Gallager’s strategy [49], a standard method for proving outer bounds, can be applied automatically by the PSITIP software [32] to discover outer bounds for general multiuser settings. The resultant bounds are also purely existential.

Khamis, Kolaitis, Ngo and Suciu [11] studied several classes of statements about conditional independence and linear inequalities on entropy (which are all purely universal statements), and gave upper-bounds on their hardness in the arithmetical

We remark that this paper does not take an axiomatic approach. Theorems are not derived from a finite set of axioms of the first-order theory of probabilistic independence, but rather from the underlying model of random variables in a probability space (i.e., the axioms of the theory are taken to be the set of true first-order sentences about random variables). Since FOTPI can interpret the true first-order arithmetic, by Gödel’s first incompleteness theorem, FOTPI is not recursively axiomatizable.

This is perhaps surprising, considering that the capacity of the finite-state Markov channel is uncomputable [41], [42]. This is used in [44] to show that the capacity of the finite-state Markov channel does not admit a single-letter characterization in a certain form.

Note that even purely existential predicates (which include most existing single-letter formulae for capacity regions) can be undecidable without cardinality bounds. See [14] and Proposition [14]. Therefore, simplicity of a formula does not necessarily imply ease of computation.
hierarchy, which is, loosely speaking, characterized by the depth of quantifier alternation in a first-order formula on natural numbers. In other words, [11] attempted to express purely universal first-order formulae on random variables as first-order formulae on natural numbers with the lowest depth of quantifier alternation (general first-order formulae on random variables are not studied in [11]). In comparison, this paper tries to express capacity regions as first-order formulae on random variables with the lowest depth of quantifier alternation.

Another undecidability result in information theory is the capacity of the finite-state Markov channel [41], [42]. Using this fact, Agarwal [43] showed that the capacity of the finite-state Markov channel does not admit a single-letter characterization expressible as a conjunction of linear inequalities in the form \( R \leq \sum \alpha_i I(U_{A_i}; U_{B_i} | U_{C_i}) \) (\( U_{A_i} = \{ U_a \}_{a \in A_i}, A_i, B_i, C_i \subseteq [n] \)) and polynomial constraints on the joint probability mass function, where the alphabets of all auxiliary random variables are fixed. This definition of single-letter characterization shares some similarities to the first existential level of the linear entropy hierarchy in this paper (refer to Section VIII and Remark 16 for the similarities and differences).

We also remark that the graphoid and separoid axioms are expressed using first-order languages in the work by Córdoba-Sánchez, Bielza, and Larrañaga [50], though [50] has not studied the expressive power of a language with only probabilistic independence.

The notion of normalized entropy vectors was introduced in [51], which used it to give a characterization of the capacity region of a communication network, which is single-letter in a certain sense. However, it appears that representing a discrete memoryless multiuser channel in the setting in [51] is not entirely straightforward.

We write \( N_+ := \{ 1, 2, \ldots \} \), \( N_0 := \{ 0, 1, 2, \ldots \} \), \( \{a..b\} := [a, b] \cap \mathbb{Z} \), \( [n] := \{ 1..n \} \). The uniform distribution over the set \( S \) is denoted as \( \text{Unif}(S) \). The Bernoulli distribution is denoted as \( \text{Bern}(a) \) (which is 1 with probability \( a \), 0 with probability \( 1-a \)).

II. RELATIONS OVER RANDOM VARIABLES

For simplicity, all random variables are assumed to be discrete unless otherwise stated (the case for general random variables is discussed in Section X). Since we only consider discrete random variables, we may assume that they are all defined in the standard probability space \((\{0, 1\}, \mathcal{F}, P)\) (where \( P \) is the Lebesgue measure, and \( \mathcal{F} \) is the \( \sigma \)-algebra of Lebesgue measurable subsets of \([0, 1]\)). Therefore, in this paper, the set of all random variables is taken to be the set of measurable functions from \([0, 1]\) to \( N_+ \). Let this set be \( \mathcal{M} \). Since the labelling of the random variable does not matter in the probabilistic independence relation, we may also regard a random variable as a finite or countably-generated \( \sigma \)-subalgebra of \( \mathcal{F} \), though we will use random variables instead of \( \sigma \)-subalgebras in this paper for notational simplicity.

The FOTPI is the first-order theory of \((\mathcal{M}, \perp)\), where \( \perp \) stands for probabilistic independence between two random variables. This theory consists of all first-order sentences \( \psi \) where \( \mathcal{M} \models \psi \) (i.e., it is the complete theory \( \text{Th}(\mathcal{M}, \perp) \) of the system of random variables \( \mathcal{M} \)).

We write \( X \leq^\psi Y \) for the condition that \( X \) is (almost surely) a function of \( Y \) (i.e., there exists a function \( f \) such that \( X = f(Y) \) almost surely). This can be expressed using independence as

\[
X \leq^\psi Y := \forall U. (U \perp \perp Y \rightarrow U \perp \perp X).
\]

It is clear that \( X \leq^\psi Y \) implies \( \forall U : (U \perp \perp Y \rightarrow U \perp \perp X) \). For the other direction, if \( X \leq^\psi Y \) does not hold, then there exists \( x_o, y_o \) such that \( 0 < p_{X|Y}(x_o|y_o) < 1 \). Let \( U \in \{ 0, 1 \} \) be a random variable such that

\[
p_U|_{X,Y}(1|x, y) = \begin{cases} 
1 & \text{if } y = y_o, x = x_o \\
0 & \text{if } y = y_o, x \neq x_o \\
p_{X|Y}(x_o|y_o) & \text{if } y \neq y_o.
\end{cases}
\]

It is clear that \( U \perp \perp Y \) and \( U \perp \perp X \).

We write \( X \equiv^\psi Y \) for the condition that \( X \) is informationally equivalent to \( Y \) (i.e., there exists an injective function \( f \) such that \( X = f(Y) \) almost surely). This can be expressed as

\[
X \equiv^\psi Y := X \leq^\psi Y \wedge Y \leq^\psi X.
\]

\[4\] It appears that the channel constraint in [51] is for one-shot communication. While one may apply [51] on the infinite product channel, and use the cardinality of the output to normalize the rate against the number of channel uses, such approach would not be considered single-letter in the usual sense.
We use juxtaposition $XY$ to denote the joint random variable of $X$ and $Y$ (while we assume random variables take values over natural numbers, we may apply any bijection to pairs of values of $X,Y$ to the natural numbers, since any two bijections are equivalent under $\equiv$). Joint random variable can be characterized by the lattice join operation \[55\]:

\[
Z \equiv XY \\
\iff X \leq Z \land Y \leq Z \\
\land \forall U. \left( (X \leq U \land Y \leq U) \rightarrow Z \leq U \right).
\] (2)

Therefore, it is not necessary to include joint random variable in the language of FOTPI, though we would still use the notation $XY$ for the $Z$ satisfying the above formula for notational simplicity.

Mutual independence among random variables $X_1, \ldots, X_n$ can be expressed as

\[
X_1 \perp \cdots \perp X_n \\
\iff \bigwedge_{i=2}^{n} \left( X_i \perp (X_1 \cdots X_{i-1}) \right).
\]

Write $X \perp Y | Z$ for the condition that $X,Y$ are conditionally independent given $Z$. Using the functional representation lemma \[29\], we can express conditional independence as

\[
X \perp Y | Z \\
\iff \exists U. U \perp XZ \land Y \leq ZU.
\] (3)

We also define

\[
X \equiv Y := X \leq Y \land Y \leq X, \\
X \neq Y := \neg(X \equiv Y), \\
X \prec Y := X \leq Y \land \neg(Y \leq X).
\]

III. Representation of Integers

In this section, we describe a representation of integers as random variables. We show that the first-order theory of arithmetic over nonnegative integers is interpretable in the first-order theory of probabilistic independence.

A. Uniformity

To test whether $X$ is uniformly distributed over its support, we use the result in \[17\] that if $X,Y,Z$ are discrete random variables such that any one of them is a function of the other two, and they are pairwise independent, then they are all uniformly distributed over their supports, which have the same size. Using the notations in \[14\], the condition that $X$ is uniformly distributed over its support can be expressed as

\[
\text{unif}(X) := \exists Y,Z. \text{triple}(X, Y, Z),
\]

where

\[
\text{triple}(X, Y, Z) := X \leq YZ \land Y \leq XZ \land Z \leq XY \\
\land X \perp Y \land X \perp Z \land Y \perp Z.
\]

B. Cardinality

We write $|X|$ for the support of $X$, and $|X|$ for the cardinality of $X$. To test whether $X$ is (at most) a binary random variable (i.e., $|X| \leq 2$), note that any random variable with strictly less information than $X$ must be degenerate, and hence the condition that $X$ is (at most) a binary random variable can be expressed as

\[
\text{card}_{\leq 2}(X) := \forall U. \left( U \prec X \rightarrow U \equiv \emptyset \right).
\]

By $U \equiv \emptyset$, we mean that $U$ is informationally equivalent to the constant random variable. This can be expressed without introducing a new constant $\emptyset$ by $U \equiv \emptyset \iff U \perp U.$
If $X$ has cardinality at most $n$, then any random variable with strictly less information than $X$ has cardinality at most $n - 1$. Therefore, the condition that $|X| \leq n$ ($n \geq 2$) can be defined recursively as

$$
\text{card}_{\leq n}(X) := \forall U (U \prec X \rightarrow \text{card}_{\leq n-1}(U)),
$$

$$
\text{card}_{\leq 1}(X) := (X = \emptyset).
$$

We can then define

$$
\text{card}_{= n}(X) := \text{card}_{\leq n}(X) \land \neg \text{card}_{\leq n-1}(X),
$$

$$
\text{card}_{\geq n}(X) := \neg \text{card}_{\leq n-1}(X).
$$

### C. Relations over Integers

Given the tests for uniformity and cardinality, a natural way to represent a positive integer $k$ as a random variable is to represent it as a uniformly distributed random variable with cardinality $k$. In this section, we express several relations over positive integers using first-order formulae. Note that some of these formulae have appeared in [14]

- **(Equality)** The formula for checking $|X| = |Y|$ for uniform $X, Y$ is given by [14]:
  
  $$
  \text{ueq}(X, Y) := \exists U_1, U_2, U_3.
  $$
  
  $$
  \text{triple}(X, U_1, U_2) \land \text{triple}(Y, U_1, U_3).
  $$

  We also define

  $$
  \text{ueq}_{= n}(X) := \text{unif}(X) \land \text{card}_{= n}(X)
  $$
  
  to check for equality against constants.

- **(Multiplication)** The formula for checking $|X| \cdot |Y| = |Z|$ for uniform $X, Y$ is given by [14]:

  $$
  \text{uprod}(X, Y, Z) := \exists X, \tilde{Y}. (\text{ueq}(X, \tilde{X}) \land \text{ueq}(Y, \tilde{Y}) \land \tilde{X} \perp \tilde{Y} \land \tilde{X}\tilde{Y} \perp Z).
  $$

- **(Comparison)** The formula for checking $|X| \leq |Y|$ for uniform $X, Y$ is given by [14] (with slight modification):

  $$
  \text{ule}(X, Y) := \exists G, \tilde{Y}. (\text{uprod}(X, Y, G) \land \text{ueq}(Y, \tilde{Y}) \land G \leq Y\tilde{Y}).
  $$

  We briefly repeat the reason given in [14] here. Note that $\text{uprod}(X, Y, G) \land \text{ueq}(Y, \tilde{Y}) \land G \leq Y\tilde{Y}$ implies $|X||Y| = |G| \leq |Y|^2$, which implies $|X| \leq |Y|$. For the other direction, assume $X = \{0, \ldots, |X| - 1\}$, $Y = \{0, \ldots, |Y| - 1\}$, $|X| \leq |Y|$. Take $G = (Y, \tilde{X})$, where $\tilde{X} \sim \text{Unif}(X)$ is independent of $Y$. Take $\tilde{Y} = X + Y \mod |Y|$. It is clear that $G \leq Y\tilde{Y}$. Therefore, the formula for strict inequality $|X| < |Y|$ is

  $$
  \text{ult}(X, Y) := \neg \text{ule}(Y, X).
  $$

  Also define $\text{uge}$ and $\text{ugt}$ similarly. We define

  $$
  \text{ule}_{= n}(X) := \exists U. \text{ueq}_{= n}(U) \land \text{ule}(X, U),
  $$

  and similar for $\text{ult}_{= n}$, $\text{uge}_{= n}$, $\text{ugt}_{= n}$.

- **(Divisibility)** The condition that $|Y|$ is divisible by $|X|$ for uniform $X, Y$ can be expressed as

  $$
  \text{udiv}(X, Y) := \exists U. \text{uprod}(X, U, Y).
  $$

  The condition that $|X|$ is a prime number is

  $$
  \text{uprime}(X) := \neg \exists U, V. (U \neq \emptyset \land V \neq \emptyset \land \text{uprod}(U, V, X)).
  $$

- **(Successor)** The condition that $|Y| = |X| + 1$ for uniform $X, Y$ can be expressed as

  $$
  \text{usucc}(X, Y) := \text{ult}(X, Y) \land \forall U. (\text{ult}(X, U) \rightarrow \text{ule}(Y, U)).
  $$
D. Interpreting First-order Arithmetic

In order to interpret the first-order theory of arithmetic, it is left to define addition. Given a discrete random variable \(X\), we call \(Y\) a single-mass indicator of \(X\) if there exists \(x\) such that \(P(X = x) > 0\) and \(Y = 1_{\{x\}}(X)\) (i.e., \(Y\) is the indicator function of \(X = x\)). This can be characterized by the first-order formula

\[
\text{smi}(X,Y) := (X \lor Y \lor 0)
\]

\[
\lor (Y < X \land \text{card}_{=2}(Y))
\]

\[
\land \forall U.(U < X \land \text{card}_{=4}(U))
\]

\[
\rightarrow \neg \exists V. (\text{card}_{=2}(V) \land U < Y V).
\]

(6)

To check this, note that if \(Y = 1_{\{x\}}(X)\) and \(U < X \land \text{card}_{=4}(U)\), then there are at least 3 possible values of \(U\) given \(Y = 0\), and there does not exist \(V\) with \(\text{card}_{=2}(V) \land U < Y V\). For the other direction, assume \(Y \in \{0,1\}\) is binary but is not a single-mass indicator of \(X\). We can therefore construct \(U < X\) which takes two possible values given \(Y = 0\) or \(Y = 1\), and there exists \(V\) with \(\text{card}_{=2}(V) \land U < Y V\) (which indicates which of the two values \(U\) takes).

Consider

\[
\text{frac}(X,Y,Z,U) := (\text{ueq}(U) \land \text{uprod}(X,U,Z) \land \text{uprod}(Y,U,Z))
\]

\[
\lor \exists \tilde{X}, \tilde{Y}. (\text{smi}(X, \tilde{X}) \land \text{smi}(Y, \tilde{Y}) \land \text{unif}(Z))
\]

\[
\land \text{card}_{=2}(U) \land \neg \text{unif}(U)
\]

\[
\land U < Z \land \tilde{X} \perp \tilde{Y} \perp U \land Z < \tilde{X} \tilde{Y} U
\]

\[
\land \forall V. (\text{smi}(Z, V) \rightarrow \text{smi}(\tilde{X} U, V) \lor \text{smi}(\tilde{Y} U, V)).
\]

(7)

We will show that \(\text{frac}(X,Y,Z,U)\) holds if and only if \(X,Y,Z\) are uniform, \(|Z| = |\mathcal{X}| + |\mathcal{Y}|\), and \(U \sim \text{Bern}(|\mathcal{X}|/(|\mathcal{X}|+|\mathcal{Y}|))\). Note that the first case \(\text{ueq}(U) \land \text{uprod}(X,U,Z) \land \text{uprod}(Y,U,Z)\) checks for \(|\mathcal{X}| = |\mathcal{X}| + |\mathcal{Y}|\), \(|\mathcal{X}| \neq |\mathcal{Y}|\). To check the second case, note that if \(|Z| = |\mathcal{X}| + |\mathcal{Y}|\) (assume \(X = |\mathcal{X}|\) and the same for \(Y,Z\)), then we can let \(U \sim \text{Bern}(|\mathcal{X}|/(|\mathcal{X}|+|\mathcal{Y}|))\), \(X \sim \text{Unif}(|\mathcal{X}|, \mathcal{Y} \sim \text{Unif}(|\mathcal{Y}|), X \perp \mathcal{Y} \perp U\) and \(Z = X\) if \(U = 1\). It is straightforward to check \(Z \sim \text{Unif}(|\mathcal{X}|+|\mathcal{Y}|), Z < X \mathcal{Y} U\) and \(V. (\text{smi}(Z, V) \rightarrow \text{smi}(\tilde{X} U, V) \lor \text{smi}(\tilde{Y} U, V)).

For the other direction, assume \(\text{frac}(X,Y,Z,U)\) holds and \(|\mathcal{X}| \neq |\mathcal{Y}|\). Let \(U \sim \text{Bern}(\theta), \theta \in (0,1)\setminus \{1/2\}\). Note that if \(\text{smi}(Z,V)\), then \(V \sim \text{Bern}(1/|Z|)\). Also, \(\text{smi}(\tilde{X} U, V) \lor \text{smi}(\tilde{Y} U, V)\) holds only if \(V \sim \text{Bern}(\phi)\) where \(\phi = \theta/|\mathcal{X}|, \theta/|\mathcal{Y}|, (1-\theta)/|\mathcal{X}|, \text{or } (1-\theta)/|\mathcal{Y}|.\) Since \(\theta/|\mathcal{X}| \neq (1-\theta)/|\mathcal{Y}|, \) we either have \(1/|Z| = \theta/|\mathcal{X}| = (1-\theta)/|\mathcal{Y}|\). Then \(1/|Z| = 1 - (1-\theta)/|\mathcal{Y}|\) (the other cases are similar by symmetry). The first case gives \(\theta = |\mathcal{X}|/(|\mathcal{X}|+|\mathcal{Y}|), |Z| = |\mathcal{X}|+|\mathcal{Y}|\). For the second case, it implies \(|Z| \geq 2\), and hence \(|\mathcal{Y}| = 1\). Since \(\theta/|\mathcal{X}| = 1- (1-\theta)/|\mathcal{Y}|, \) we have \(|\mathcal{X}| = 1\). Since \(Z \leq X \tilde{Y} U\), we have \(|Z| = 2\), and \(U \sim \text{Bern}(1/2)\), giving a contradiction. For the third case, it implies \(|Z| \geq 2\), \(|\mathcal{X}| = |\mathcal{Y}| = 1\). Since \(1-\theta/|\mathcal{X}| = 1- (1-\theta)/|\mathcal{Y}|, \) we have \(\theta = 1/2\), giving a contradiction. Hence, only the first case is possible, and \(|Z| = |\mathcal{X}| + |\mathcal{Y}|\).

We can therefore define addition as follows. The formula for checking \(|Z| = |\mathcal{X}| + |\mathcal{Y}|\) for uniform \(X,Y,Z\) is given by

\[
\text{usum}(X,Y,Z) := \exists U. \text{frac}(X,Y,Z,U).
\]

We now show that true arithmetic \([\mathbb{N}]\) (i.e., the theory \(\text{Th}(\mathbb{N}_0,+,-,\cdot,\prec)\) containing all true first-order sentences over nonnegative integers with addition, multiplication and comparison) is interpretable in the first-order theory of probabilistic independence.

**Theorem 1.** True arithmetic is interpretable in the first-order theory of probabilistic independence.

**Proof:** We represent \(a \in \mathbb{N}_+\) by a uniform random variable with cardinality \(a\). Note that true arithmetic concerns \(\mathbb{N}_0\) instead of \(\mathbb{N}_+\), so we need a special representation for 0. We represent 0 by a random variable with distribution \(\text{Bern}(1/3)\) (up to relabeling). This distribution can be checked by observing that \(X \sim \text{Bern}(1/3)\) (up to relabeling) if and only if

\[
is0(X) := \exists U. (\text{ueq}(U) \land \emptyset < X < U).
\]

The following formula checks whether \(X\) is the representation of an integer in \(\mathbb{N}_0:\)

\[
isnat(X) := \text{is0}(X) \lor \text{unif}(X).
\]

(8)

It is straightforward to modify the definitions of usum and uprod to accommodate this special value of 0.
As a result, by Tarski’s undefinability theorem [57], [56], FOTPI is not arithmetically definable.

IV. DEFINABLE DISTRIBUTIONS

In this section, we investigate the concept of definability in FOTPI.

**Definition 2** (Definability). We use the following definitions of definability:

- (Definability of distributions) We call a probability mass function \( p \) definable in FOTPI if there exists a first-order formula \( P(X) \) such that \( P(X) \) holds if and only if \( X \) follows the distribution \( p \) up to relabeling (i.e., there exists an injective function \( f \) such that \( f(X) \sim p \)). We call a set \( S \subseteq \mathcal{M}^n \) definable in FOTPI if there exists a first-order formula \( P(X_1, \ldots, X_n) \) which holds if and only if \( (X_1, \ldots, X_n) \in S \) (note that a relation is a special case of a set for \( n = 2 \)).

- (Bernoulli definability over reals) We call a real number \( \theta \in [0, 1/2] \) Bernoulli-definable in FOTPI if the probability mass function of the Bernoulli distribution \( \text{Bern}(\theta) \) is definable in FOTPI. We call a set \( S \subseteq [0, 1/2]^n \) Bernoulli-definable in FOTPI if there exists a first-order formula \( P(X_1, \ldots, X_n) \) which holds if and only if \( X_i \sim \text{Bern}(\theta_i) \) (up to relabelling), \( \theta_i \in [0, 1/2] \), and \((\theta_1, \ldots, \theta_n) \in S\). For \( S \subseteq [0, 1/2]^n \), we call a function \( f : S \rightarrow [0, 1/2] \) Bernoulli-definable in FOTPI if its graph \( \{((\theta_1, \ldots, \theta_n), f(\theta_1, \ldots, \theta_n)) : (\theta_1, \ldots, \theta_n) \in S\} \subseteq [0, 1/2]^n \) is Bernoulli-definable in FOTPI.

- (Uniform definability over natural numbers) We call a set \( S \subseteq \mathbb{N}_+^n \) uniform-definable in FOTPI if there exists a first-order formula \( P(X_1, \ldots, X_n) \) which holds if and only if \( X_i \sim \text{Unif}[k_i] \) (up to relabelling) and \((k_1, \ldots, k_n) \in S\). For \( S \subseteq \mathbb{N}_+^n \), we call a function \( f : S \rightarrow \mathbb{N}_+ \) uniform-definable in FOTPI if its graph \( \{((\theta_1, \ldots, \theta_n), f(\theta_1, \ldots, \theta_n)) : (\theta_1, \ldots, \theta_n) \in S\} \subseteq \mathbb{N}_+^{n+1} \) is uniform-definable in FOTPI.

Since
\[
\text{qeq}(X, Y, B) := \exists Z. \text{frac}(X, Y, Z, B) \tag{9}
\]
holds if and only if \( X, Y \) are uniform, and \( B \sim \text{Bern}([|X|/(|X| + |Y|)]) \), we know that all rational numbers in \([0, 1/2] \) are Bernoulli-definable. We then show some Bernoulli-definable relations.

**Lemma 3.** The relations “\( \leq \)”, “\( < \)” and “\( = \)” over \([0, 1/2] \) are Bernoulli-definable in FOTPI, i.e., there is a first-order formula \( \text{ble}(X, Y) \) which holds if and only if \( X \sim \text{Bern}(\theta) \) and \( Y \sim \text{Bern}(\phi) \) (up to relabelling), where \( 0 \leq \theta \leq \phi \leq 1/2 \), and there are first-order formulas \( \text{blt}(X, Y) \), \( \text{beq}(X, Y) \) which hold if and only if \( \theta < \phi \) and \( \theta = \phi \) respectively.

**Proof:** Consider
\[
\text{qlt}(X, Y, B) := \exists X. \text{frac}(X, Y, B)
\]
\[
\lor \left( \text{card}_{=2}(B) \land \exists C, D. (\text{qeq}(X, Y, C) \land \text{ueq}_{12}(D) \wedge \text{smi}(BCD, C) \land \text{smi}(BCD, D) \land \neg \text{smi}(BCD, B)) \right).
\]

We will show that if \( X, Y \) are uniform random variables with \(|X| = a, |Y| = b \), \( a < b \), then \( \text{qlt}(X, Y, B) \) holds if and only if \( B \sim \text{Bern}(\theta) \) (up to relabelling) for some \( a/(a+b) < \theta \leq 1/2 \). For the “\( \leq \)” direction, if \( \theta < 1/2 \) (the case \( \theta = 1/2 \) is clear), take
\[
(B, C, D) = \begin{cases}
(1, 1, 0) & \text{with prob. } a/(a+b) \\
(1, 0, 0) & \text{with prob. } \theta - a/(a+b) \\
(0, 0, 0) & \text{with prob. } 1/2 - \theta \\
(0, 1, 0) & \text{with prob. } 1/2.
\end{cases}
\]

For the “only if” direction, assume \( B, C, D \in \{0, 1\} \). Assume the single-mass indicator in \( \text{smi}(BCD, C) \) is \( 1_{(1)}(C) \), and the indicator in \( \text{smi}(BCD, D) \) is \( 1_{(1)}(D) \). We must have \( P(C = 1) = a/(a+b) \) (if \( P(C = 1) = b/(a+b) > 1/2 \), then \( 1_{(1)}(C) \) cannot be a single-mass indicator of \( BCD \)). Consider the distribution of \( B \). Since we cannot break the split the masses \( P(C = 1) = a/(a+b) \) and \( P(D = 1) = 1/2 \) among different values of \( B \), we either assign them to the same or to different values of \( B \). The former case is impossible due to \( \neg \text{smi}(BCD, B) \). Therefore the masses \( a/(a+b) \) and \( 1/2 \) are assigned to different values of \( B \), giving \( a/(a+b) \leq \gamma \leq 1/2 \). Note that \( \gamma = a/(a+b) \) and \( \gamma = 1/2 \) are impossible due to \( \neg \text{smi}(BCD, B) \).

We also define
\[
\text{qle}(X, Y, B) := \text{qlt}(X, Y, B) \lor \text{qeq}(X, Y, B). \tag{10}
\]
Using the fact that for \( \theta_1, \theta_2 \geq 0 \), we have \( \theta_1 \leq \theta_2 \Leftrightarrow \forall a, b \in \mathbb{N}_+. a/b < \theta_1 \rightarrow a/b < \theta_2 \), we can define \( \text{ble}(B, C) \), \( \text{blt}(B, C) \) and \( \text{beq}(B, C) \) by
\[
\text{ble}(B, C) := \text{card}_{\leq 2}(B) \land \text{card}_{\leq 2}(C) \\
\land \forall X, Y. (\text{ult}(X, Y) \land \text{qlt}(X, Y, B) \rightarrow \text{qlt}(X, Y, C)),
\]
\[
\text{blt}(B, C) := \neg \text{ble}(C, B),
\]
\[
\text{beq}(B, C) := \text{ble}(B, C) \land \text{ble}(C, B).
\]

We can use this to show that any arithmetically definable number \([59], [55]\) in \([0, 1/2]\) (i.e., a real number \( \theta \in [0, 1/2] \)) such that the set \( \{(a, b) \in \mathbb{N}_+^3 : a/b \leq \theta\} \) is definable using a formula in first-order arithmetic) is Bernoulli-definable.

**Theorem 4.** Any arithmetically definable number in \([0, 1/2]\) is Bernoulli-definable in FOTPI.

**Proof:** Let \( \theta \in [0, 1/2] \) be an arithmetically definable number. By Theorem 1, we can find a first-order formula (in the theory of probabilistic independence) \( \psi(X, Y) \) which holds if and only if \( X, Y \) are uniform and \( |X|/|Y| \leq \theta \). We can check whether \( X \sim \text{Bern}(\theta) \) by checking
\[
\forall X, Y. (\psi(X, Y) \leftrightarrow \forall C. (\text{qe}(X, Y, C) \rightarrow \text{ble}(C, B))).
\]

We call a function \( f : \mathbb{N}_+ \rightarrow \mathbb{R}_{>0} \) arithmetically definable if the set \( \{(x, a, b) \in \mathbb{N}_+^3 : a/b \leq f(x)\} \) is definable using a formula in first-order arithmetic. We show that any arithmetically definable probability mass function is definable in FOTPI.

**Theorem 5.** For any probability mass function \( p \) over \( \mathbb{N}_+ \), if the function \( p : \mathbb{N}_+ \rightarrow [0, 1] \) is arithmetically definable, then it is definable in FOTPI.

**Proof:** Let \( X \leq Y \) where there are at least 3 possible values of \( Y \) given any \( X = x \). We say that two single-mass indicators \( B = 1_{\{y\}}(Y), C = 1_{\{y\}}(Y) \) correspond to the same value of \( X \) if the value of \( X \) given \( Y = b \) is the same as the value of \( X \) given \( Y = c \). This can be checked by
\[
\text{smis}(X, Y, B, C) := X \leq Y \land \text{smi}(Y, B) \land \text{smi}(Y, C) \land (B \equiv C \lor
\neg \exists U. (\text{card}_{\leq 2}(U) \land BC \leq X U)).
\]

To check this, note that if \( B \neq C \) correspond to the same value of \( X \), since there are at least 3 possible values of \( Y \) corresponding to that value of \( X \), and \( B, C \) correspond to 2 of them, it is impossible to have \( \text{card}_{\leq 2}(U) \land BC \leq X U \). For the other direction, if \( B \neq C \) correspond to different values of \( X \), we can take \( U = \max\{B, C\} \), and have \( BC \leq X U \). We also define the formula for checking whether \( B, C \) correspond to different values of \( X \):
\[
\text{smid}(X, Y, B, C) := X \leq Y \land \text{smi}(Y, B) \land \text{smi}(Y, C) \land \neg \text{smis}(X, Y, B, C).
\]

In order to check whether a random variable \( A \) follows \( p \), we assign labels in \( \{3, 4, \ldots\} \) to values of \( A \). Assume \( A \) takes values over \( \{3, 4, \ldots\} \). We call a random variable \( L \) a label of \( A \) if the conditional distribution of \( L \) given \( A = a \) is uniform among \( a \) different values, and these values are different for different \( a \). This can be checked by (up to relabelling)
\[
\text{label}_{\{l\}}(A, L) := A \leq L \\
\land \forall B. (\text{smi}(L, B) \rightarrow \exists U. (\text{UGE}_{\{l\}}(U) \land U \perp A) \\
\land \forall C. (\text{smis}(A, L, B, C) \rightarrow \text{smi}(AU, C)) \\
\land \forall D. (\text{smid}(A, L, B, D) \rightarrow \neg \exists V. (\text{eq}(U, V) \land V \perp A \land \text{smi}(AV, D))).
\]

Assume \( L \) is a label of \( A \). Consider \( B \) where \( \text{smi}(L, B) \) holds, and assume \( B = 1_{\{l\}}(L) \). Let the value of \( A \) conditional on \( L = l \) be \( a \). Then \( l \) is one of the \( a \geq 3 \) values of \( L \) corresponding to \( A = a \). The line \( \{11\} \) holds since we can have \( U \sim \text{Unif}[a] \), and any single-mass indicator \( C \) of \( L \) corresponding to \( A = a \) (there are \( a \) such \( C \)'s) are single-mass indicators of \( AU \) (since \( U \) divides the mass \( A = a \) into \( a \) equal pieces). For \( \{12\} \), if \( D \) is a single-mass indicator of \( L \) corresponding to a value of \( A \) other than \( a \) (let it be \( \tilde{a} \)), then \( \text{P}(D = 1) = \text{P}(A = \tilde{a})/\tilde{a} \neq \text{P}(A = \tilde{a})/a \), and it is impossible to have
and we can check for $\text{ueq}(U, V) \land V \perp \! \! \! \! \! \! \perp A \land \text{smi}(AV, D)$. For the other direction, using similar arguments, we can deduce that if $\text{label}_3(A, L)$ holds, then conditional on any $A = a$, $L$ is uniformly distributed in a set $S_a$ (of size that equals the size of $U$ in the definition of $\text{label}_3(A, L)$), and the sizes of $S_a$ are distinct (due to (12)), and hence we can assign the labels $a = |S_a|$ to the values of $A$.

Given $A$ with label $L$, we call $B$ a divided mass of the value $A = |U|$ if $B = 1_{|U|}(L)$ is a single-mass indicator of $L$, and we have $A = |U|$ given $L = l$. Note that this implies $P(B = 1) = P(A = |U|)/|U|$. This can be checked by

$$\text{divmass}_3(A, L, U, B) := \exists \tilde{U}. (\text{label}_3(A, L) \land \text{smi}(L, B) \land \text{ueq}(U, \tilde{U})$$

$$\land \text{uge}_3(U) \land \tilde{U} \perp \! \! \! \! \! \! \perp A \land \text{smi}(A\tilde{U}, B)).$$

(13)

Let $p$ be an arithmetically definable probability mass function over $\{3, 4, \ldots\}$ (we can use the domain $\{3, 4, \ldots\}$ instead of $\mathbb{N}_+$ by shifting). Let $\tilde{p} : \{3, 4, \ldots\} \rightarrow [0, 1/3]$ be defined as $\tilde{p}(a) := p(a)/a$ (which is also arithmetically definable). By Theorem 1 we can find a first-order formula (in the theory of probabilistic independence) $\psi(W, X, Y)$ which holds if and only if $W, X, Y$ are uniform, $|W| \geq 3$ and $|X|/(|X| + |Y|) \leq \tilde{p}(|W|)$. To show that $\tilde{p}$ is definable in $\mathcal{L}$, we can check whether $A \sim p$ (up to relabelling) by

$$\exists L. (\text{label}_3(A, L) \land \forall B, U. \left( \begin{array}{c}
\text{divmass}_3(A, L, U, B) \\
\rightarrow \forall X, Y, C. (\text{ult}(X, Y) \land \text{ueq}(X, Y, C) \rightarrow (\text{ble}(C, B) \leftrightarrow \psi(U, X, Y)))
\end{array} \right)).$$

By $\text{divmass}_3$, the $B$ (assume $B \sim \text{Bern}(\theta)$, $\theta \leq 1/2$) and $U$ in the above definition satisfies $P(A = |U|) = \theta|U|$. The second line of the definition states that for any uniform $X, Y$ with $|X| < |Y|$ and $C \sim \text{Bern}(|X|/(|X| + |Y|))$, we have $|X|/(|X| + |Y|) \leq \theta = P(A = |U|)/|U|$ if and only if $\psi(U, X, Y) \leftrightarrow |X|/(|X| + |Y|) \leq \tilde{p}(|U|) = p(|U|)/|U|$. \hfill $\blacksquare$

We say that $X, Y$ have the same distribution up to relabelling, written as $X \equiv Y$, if there exists an injective function $f$ such that $f(X)$ has the same distribution as $Y$. This relation is also definable in FOTPI.

**Proposition 6.** The following relations over random variables are definable in FOTPI:

1. The “same distribution up to relabelling” relation $X \equiv Y$.
2. Comparison of cardinality: $|X| = |Y|$ and $|X| \leq |Y|$.

**Proof:** Using similar arguments as in Theorem 5, we can check whether $A_1 \equiv A_2$ by

$$\exists L_1, L_2. (\text{label}_3(A_1, L_1) \land \text{label}_3(A_2, L_2)$$

$$\land \forall B, U. (\exists B_1, U_1. (\text{beq}(B, B_1) \land \text{ueq}(U, U_1) \land \text{divmass}_3(A_1, L_1, U_1, B_1)))$$

$$\leftrightarrow (\exists B_2, U_2. (\text{beq}(B, B_2) \land \text{ueq}(U, U_2) \land \text{divmass}_3(A_2, L_2, U_2, B_2))))).$$

Intuitively, this means that there is a labelling of $A_1, A_2$ such that for any $B \sim \text{Bern}(\theta)$ and uniform $U$, we have $P(A_1 = |U|) = \theta|U|$ if and only if $P(A_2 = |U|) = \theta|U|$, which clearly implies $A_1$ has the same distribution as $A_2$.

We can check whether $V$ is uniform and $|A| + 2 \leq |V|$ by

$$\text{cardleu}_2(A, V) := \exists L. (\text{label}_3(A, L)$$

$$\land \forall B, U. (\text{divmass}_3(A, L, U, B) \rightarrow \text{ule}(U, V))).$$

The reason is that the smallest possible $\max A$ among labellings of $A$ using the set of values $\{3, 4, \ldots\}$ is $|A| + 2$. We can then check for $|A_1| \leq |A_2|$ by

$$\forall V. (\text{cardleu}_2(A_2, V) \rightarrow \text{cardleu}_2(A_1, V)),$$

and we can check for $|A_1| = |A_2|$ by

$$\forall V. (\text{cardleu}_2(A_1, V) \leftrightarrow \text{cardleu}_2(A_2, V)).$$

$\blacksquare$
V. REPRESENTATION OF EVENTS

In this section, we discuss a representation of events. While the event $E$ can be represented by the indicator random variable $C = 1\{E\}$, there is an ambiguity since $C$ can also be the representation of the complement $E^c$ (we do not concern the labelling of $C$).

Instead, we represent an event $E$ with $P(E) < 1$ as a random variable $D$ where $D \sim \text{Unif}[k]$ conditional on $E$, where $k \geq 2$ satisfies $P(E)/k < P(E^c)$, and $D = 0$ if $E$ does not occur ($E$ can be recovered by taking the complement of the largest mass of $D$). If $P(E) = 1$, it is represented by any $D \sim \text{Unif}[k]$ where $k \geq 2$. Note that there are only two cases where $D$ is uniform: $P(E) = 1$ (which can be checked by $\text{uge}_2(D)$) and $P(E) = 0$ (which can be checked by $D \equiv 0$. Technically, $D$ represents $E$ only up to a difference of a set of measure 0, though measure 0 sets do not affect the truth value of formulae concerning probabilistic independence.

We can check whether $C$ is the indicator function of the event represented by $D$ using

$$\text{ind}(D, C) := (\text{unif}(D) \wedge C \equiv 0) \lor (\text{card}_{\geq 2}(C) \wedge \text{smi}(D, C) \wedge \exists U, V. (\text{uge}_2(U) \wedge \text{uge}_2(V) \wedge U \perp V \perp C \wedge |D| \leq CU \lor |D| = |U| + 1 \wedge \forall F, G.(\text{smi}(DV, F) \lor \neg \text{smi}(CV, F) \wedge \text{smi}(DV, G) \wedge \text{smi}(CV, G)) \lor \text{blt}(F, G)))$$

Note that $|D| = |U| + 1$ can be checked using Proposition 6 and Theorem 1. To check the above formula, note that $\text{card}_{\geq 2}(C), \text{smi}(D, C), D \leq CU$ and $|D| = |U| + 1$ ensures that $C, D$ is in the form $C \in \{0, 1\}, D = 0$ if $C = 0$, and $D \setminus \{C = 1\} \sim \text{Unif}[k]$ (up to relabelling). In the last two lines, note that $DV$ divides each mass of $D$ into two equal halves, $F$ is a single mass indicator of $DV$ where $D \neq 0$ (and hence $P(F = 1) = P(C = 1)/(2k)$), and $G$ is a single mass indicator of $DV$ where $D = 0$ (and hence $P(G = 1) = P(C = 0)/2$). The condition $\text{blt}(F, G)$ means $P(C = 1)/(2k) < P(C = 0)/2$, which is the condition needed for $D$ to be the representation of the event $C = 1$. Note that $V$ is needed since $\text{blt}$ is defined only for Bernoulli random variables with parameters in $[0, 1/2]$.

We can check whether $D$ is the representation of some event by

$$\text{i ev}(D) := \exists C. \text{ind}(D, C).$$

To check whether the event represented by $D_1$ is the complement of the event represented by $D_2$ (up to a difference of measure 0):

$$\text{compl}(D_1, D_2) := \exists C. (\text{ind}(D_1, C) \wedge \text{ind}(D_2, C) \wedge \neg \text{smi}(D_1D_2, C)) \wedge (\text{uge}_2(D_2) \rightarrow D_1 \equiv \emptyset) \lor (\text{uge}_2(D_1) \rightarrow D_2 \equiv \emptyset).$$

To check whether the event represented by $D_1$ is the same as the event represented by $D_2$ (up to a difference of measure 0):

$$\text{eveq}(D_1, D_2) := \exists C. (\text{ind}(D_1, C) \wedge \text{ind}(D_2, C)) \wedge \neg \text{compl}(D_1, D_2).$$

To check whether the event represented by $D_1$ is a subset of the event represented by $D_2$ (up to a difference of measure 0):

$$\text{subset}(D_1, D_2) := \text{i ev}(D_1) \land \text{i ev}(D_2) \land (D_1 \equiv \emptyset \lor \text{uge}_2(D_2) \lor \neg \text{unif}(D_1) \land \neg \text{unif}(D_2) \land \exists C_2. (\text{ind}(D_2, C_2) \wedge \forall D_1. (\text{ev eq}(D_1, D_1) \rightarrow \text{smi}(D_2D_1, C_2))))).$$

The reason is that if the nondegenerate event represented by $D_1$ (let it be $E_1$) is a subset of the nondegenerate event represented by $D_2$ (let it be $E_2$), then any representation $D_1$ of $E_1$ will be constant conditional on $E_2^c$, and hence $C_2 = 1\{E_2\}$ is a single mass indicator of $D_1$. For the other direction, if $P(E_1 \setminus E_2) > 0$, then we can have $D_1|E_1 \sim \text{Unif}[k]$ for $k$ large enough so that $P(E_1)/k < P(E_1 \setminus E_2)$, and hence $D_2$ is not constant conditional on $E_1 \setminus E_2$, and hence is not constant conditional on $E_2^c$. This means $\text{smi}(D_2D_1, C_2)$ cannot hold.
We can also take the union of a collection of events. Let $P$ be a first-order formula. To check whether $D$ is the representation of the union of all events $E$ with a representation satisfying $P$:

$$\text{union}_P(D) := \forall D_2, \left( P(D_2) \land \text{isev}(D_2) \rightarrow \text{subset}(D_2, D) \right) \land \forall \tilde{D}, \left( \text{isev}(\tilde{D}) \land \forall D_2, \left( P(D_2) \land \text{isev}(D_2) \rightarrow \text{subset}(D_2, \tilde{D}) \right) \rightarrow \text{subset}(\tilde{D}, \tilde{D}) \right).$$

(15)

Technically, since a representation only identifies the event up to a difference of measure 0, an uncountable union may not be well-defined (in the equivalence classes of events mod 0). Instead of the ordinary union of sets, the above definition actually describes the essential union of measurable sets [59 Def. 2], which is always measurable. Nevertheless, since this paper concerns discrete settings, we can regard essential union as ordinary union. We also define

$$\text{union}(D_1, \ldots, D_n, \tilde{D}) := \text{union}_D \cdot \text{ev}(D, D_1)(\tilde{D}).$$

(16)

Define inter for intersection similarly.

We can also check whether the event represented by $D_1$ is disjoint of the event represented by $D_2$ (up to a difference of measure 0):

$$\text{disjoint}(D_1, D_2) := \exists \tilde{D}_1, (\text{compl}(D_1, \tilde{D}_1) \land \text{subset}(D_2, \tilde{D}_1)).$$

To check whether the event represented by $D_1$ is independent of the event represented by $D_2$:

$$\text{indep}(D_1, D_2) := \exists \tilde{D}_2, (\text{eveq}(D_1, \tilde{D}_1) \land \text{eveq}(D_2, \tilde{D}_2) \land \tilde{D}_1 \perp \tilde{D}_2).$$

To check whether $P(E_1) \leq P(E_2)$, where $E_i$ is represented by $D_i$:

$$\text{prle}(D_1, D_2) := \exists \tilde{D}_1, (D_1 \subset \tilde{D}_1 \land \text{subset}(\tilde{D}_1, D_2)).$$

(17)

Also define

$$\text{preq}(D_1, D_2) := \exists \tilde{D}_2, (D_1 \subset \tilde{D}_2 \land \text{eveq}(\tilde{D}_1, D_2)).$$

(18)

Note that this allows us to perform addition and multiplication on probability of events. For example, to check whether $P(E_1) = P(E_2)P(E_3) + P(E_4)$:

$$\exists \tilde{D}_1, \ldots, \tilde{D}_4, \tilde{D}_{23}, \left( \bigwedge_{i=1}^{4} \text{preq}(D_i, \tilde{D}_i) \land \text{indep}(\tilde{D}_2, \tilde{D}_3) \land \text{inter}(\tilde{D}_2, \tilde{D}_3, \tilde{D}_{23}) \land \text{disjoint}(\tilde{D}_{23}, \tilde{D}_4) \land \text{union}(\tilde{D}_{23}, \tilde{D}_4, \tilde{D}_1) \right).$$

(19)

Given $A$ with label $L$ [12], we can check whether $P(A = |U| > 0$, and $D$ is the representation of the event $A = |U|$ (where $U$ is a uniform random variable) by

$$\text{labelevne}_3(A, L, U, D) := \exists C, \tilde{U}, \left( \text{ind}(D, C) \land \text{label}_3(A, L) \land \text{smi}(A, C) \land \text{ueq}(U, \tilde{U}) \right) \land \text{ueq}_3(U) \land \tilde{U} \perp A \land \text{∀B.} (\text{divmass}_3(A, L, U, B) \rightarrow \text{smi}(A\tilde{U}, B) \land \text{smi}(C\tilde{U}, B) \land \exists \tilde{D}. (\text{eveq}(D, \tilde{D}) \land B \subseteq \tilde{D})).$$

To show the validity of this formula, we let $C = 1_{\{a\}}(A)$. Note that if $|A| \geq 3$ (otherwise the above formula is obviously valid), smi$(A\tilde{U}, B) \land \text{smi}(C\tilde{U}, B)$ implies that $B = 1$ (assuming $B = 1_{\{|U|\}}(L)$) only if $C = 1$. Hence smi$(C\tilde{U}, B)$ implies $P(B = 1) = P(A = a)/|U|$, and the mass $A = a$ is divided into $|U|$ equal pieces by $L$, and hence $a = |U|$ by the definition of label$_3(A, L)$. It is left to check that $D$ is the representation of the event $A = |U|$ (instead of $A \neq |U|$). This is ensured by $\exists \tilde{D}. (\text{eveq}(D, \tilde{D}) \land B \subseteq \tilde{D})$. If $D$ represents $A \neq |U|$, then $\tilde{D}$ is constant given $A = |U|$, so $B$ cannot be a function of $\tilde{D}$ since $P(B = 1) = P(A = |U|)/|U|$. If $D$ represents $A = |U|$, we can take $\tilde{D}|\{A = |U|\} \sim \text{Unif}[k|U|]$ (for large enough $k$ so $\tilde{D}$ satisfies the definition of a representation of an event) such that $L$ is a function of $D$ conditional on $A = |U|$.
Note that labelevne$_3$ is false if $P(A = |\mathcal{U}|) = 0$. To check whether $D$ is the representation of the event $A = |\mathcal{U}|$ (which is empty if $P(A = |\mathcal{U}|) = 0$), we use

$$\text{labelev}_3(A, L, U, D)$$

$$:= \text{labelev}_3(A, L, U, D)$$

$$\vee (D \not\in \emptyset \land \neg\exists D_2, \text{labelev}_3(A, L, U, D_2)).$$

The condition that $X$ has the same distribution as $Z$ is written as $X \overset{d}{=} Z$. We say that $Y \mid X$ follows the conditional distribution of $W \mid Z$, written as $Y \mid X \overset{d}{=} W \mid Z$, if $P_X(x) > 0$ implies $p_Z(x) > 0$ and $P_Y(y \mid x) = P_W(y \mid x)$ for any $y$. Note that since FOTPI does not concern the labelling of a random variable, we have to use another random variable $L_X$ (the label) to specify the values of $X$, as described in \([12]\). The statements $X \overset{d}{=} Z$ and $Y \mid X \overset{d}{=} W \mid Z$ are not valid statements in FOTPI without the labels.

Proposition 7. The following conditions are definable in FOTPI:

- The condition $X \overset{d}{=} Z$, where $L_X, L_Z$ are the labels of $X, Z$ respectively \([12]\).
- The condition $Y \mid X \overset{d}{=} W \mid Z$, where $L_X, L_Y, L_Z, L_W$ are the labels of $X, Y, Z, W$ respectively \([12]\).
- The condition $Y \overset{d}{=} W \mid Z$.

Proof: Note that $X \overset{d}{=} Z$ can be checked by

$$\text{deq}(X, L_X, Z, L_Z)$$

$$:= \text{labelev}_3(X, L_X) \land \text{labelev}_3(Z, L_Z) \land \forall U, D_X, D_Z, \ldots$$

$$\text{labelev}_3(X, L_X, U, D_X) \land \text{labelev}_3(Z, L_Z, U, D_Z)$$

$$\rightarrow P(D_X) = P(D_Z),$$

where $P(D_X) = P(D_Z)$ is checked by \([13]\). Also, $Y \mid X \overset{d}{=} W \mid Z$ can be checked by

$$\text{cdeq}(X, L_X, Y, L_Y, Z, L_Z, W, L_W)$$

$$:= \text{labelev}_3(X, L_X) \land \text{labelev}_3(Y, L_Y) \land \text{labelev}_3(Z, L_Z) \land \text{labelev}_3(W, L_W)$$

$$\land \forall U, V, D_X, D_Y, D_Z, D_W, \ldots$$

$$D_X \not\in \emptyset \land \text{labelev}_3(X, L_X, U, D_X) \land \text{labelev}_3(Z, L_Z, U, D_Z)$$

$$\land \text{labelev}_3(Y, L_Y, V, D_Y) \land \text{labelev}_3(W, L_W, V, D_W)$$

$$\rightarrow D_Z \not\in \emptyset \land P(D_X)P(D_Z \cap D_W) = P(D_Z)P(D_X \cap D_Y),$$

where $D_Z \cap D_W$ denotes the representation of the intersection of the events represented by $D_Z$ and $D_W$. $P(D_X)$ denotes the probability of the event represented by $D_X$, and $P(D_X)P(D_Z \cap D_W) = P(D_Z)P(D_X \cap D_Y)$ can be expressed in the same way as \([19]\). Note that $Y \underset{X}{\overset{d}{\sim}} W \mid Z$ if and only if $p_X(x) > 0 \rightarrow p_Z(x) > 0 \land p_X(x)p_Z, W(x, y) = p_Z(x)p_X, Y(x, y)$ for all $x, y$.

For $Y \overset{d}{=} W \mid Z$, it can be checked by

$$\text{cdeqr}(X, Y, Z, W)$$

$$:= \exists L_X, L_Y, Z, L_Z, L_W, \text{cdeq}(X, L_X, Y, L_Y, Z, L_Z, W, L_W).$$

VI. Representation of Random Sequences

In the remainder of this paper, we use the following notations within first-order formulae:

- Random variables are denoted by uppercase letters (except $E$).

- $\mathbb{N}_0$-valued variables are denoted by lowercase (English or Greek) letters. Any $\mathbb{N}_0$-valued variable $\alpha$ is understood as a random variable obtained using the representation in Theorem \([11]\) restricted to satisfy $\text{isnat}(\alpha)$ \([8]\), and hence can be
expressed in FOTPI. Addition, comparison and multiplication can be performed on \( \mathbb{N}_0 \)-valued variables due to Theorem 1.

- Event variables are denoted by \( E \) (or with subscripts). Any event variable \( E \) is understood as a random variable obtained using the representation in Section V restricted to satisfy labelev(\( E \)) \( \text{[14]} \), and hence can be expressed in FOTPI. Set operations and relations such as \( E_1 \cup E_2, E_1 \cap E_2, E_1 \sim E_2 \) (i.e., \( E_1 \leftrightarrow E_2 \) holds almost surely), \( E_1 \subseteq E_2 \) (i.e., \( E_1 \rightarrow E_2 \) holds almost surely) and \( E_1 \perp \perp E_2 \) can be defined (see Section V). For the union of a collection of events \( E \) satisfying \( P(E) \) \( \text{[15]} \), we use the notation \( \bigcup_{E : P(E)} E \).

We have defined labelev\(_3\)(\( A, L \)) \( \text{[12]} \) to check whether \( A \) can be labelled using values in \( \{3, 4, \ldots\} \), and \( L|A \sim \text{Unif}[A] \). For the purpose of simplicity, we will shift the values of \( A \) to \( \{0, 1, \ldots\} \) (i.e., \( L|A \sim \text{Unif}[A + 3] \)). Given \( A \in \{0, 1, \ldots\} \) with label \( L \), we can check whether \( E \) is (the representation of) the event \( A = a, a \in \mathbb{N}_0 \) (recall that \( a \) is understood to be obtained using the representation in Theorem 1) by

\[
\text{labelev}_0(A, L, a, E) := \text{labelev}_3(A, L, a + 3, E).
\]

For notational simplicity, we will denote the \( E \) satisfying the above formula by

\[
\{ A \circ L = a \}. \quad \text{(20)}
\]

The notation \( A \circ L \) intuitively means the random variable \( A \) with labels given by \( L \) (note that \( A \) itself, as for any random variable in FOTPI, does not inherently have well-defined values, since formulae in FOTPI do not concern the labelling of random variables).

We can also obtain the event where the random variable \( A \) (with label \( L \)) equals the random variable \( B \) (with label \( M \)) by

\[
\bigcup_{E : \exists a. E \equiv \{A \circ L = a\} \cap \{B \circ M = a\}} E,
\]

which is defined using \( \text{[15]} \). The above event is denoted as

\[
\{ A \circ L = B \circ M \}. \quad \text{(21)}
\]

If the above event occurs with probability 1 (recall that we can check whether \( P(E) = 1 \) by \( \text{ueq}_2(E) \)), we simply denote this condition as

\[
A \circ L \equiv B \circ M.
\]

The conditional distribution relation (Proposition \( \text{[7]} \)) is denoted as

\[
Y \circ L_Y |X \circ L_X \sim W \circ L_W |Z \circ L_Z. \quad \text{(22)}
\]

When we check independence or conditional independence for labelled random variables, e.g.,

\[
A \circ L \perp \perp B \circ M, \quad \text{(23)}
\]

we ignore the labels (the above line means \( A \perp \perp B \)).

In this section, we use the notation \( X^n = (X_1, \ldots, X_n) \) to denote a sequence of random variables. The main challenge of expressing a coding setting in FOTPI is that coding settings are often defined asymptotically, where \( n \), the length of the random sequences, tends to infinity. However, the number of random variables in a formula in FOTPI is fixed. Therefore, we have to design a method to extract \( X_i \), given \( X^n \) as a single random variable. We utilize the classical Gödel encoding \( \text{[60], [61], [62]} \), which we briefly recall below:

**Definition 8.** The Gödel beta function \( \text{[61]} \) is defined as

\[
\text{beta}(b, c, i) := b \mod (c(i + 1) + 1),
\]

where \( a \mod b \) denotes the remainder when \( a \) is divided by \( b \). It satisfies the property that for any sequence \( a_0, a_1, \ldots, a_n \in \mathbb{N}_0 \), there exists \( b, c \in \mathbb{N}_0 \) such that \( \text{beta}(b, c, i) = a_i \) for \( i \in [0..n] \). The Cantor pairing function (which is a bijection from \( \mathbb{N}_0 \times \mathbb{N}_0 \) to \( \mathbb{N}_0 \)) is defined as

\[
\text{pair}(b, c) := \frac{(b + c)(b + c + 1)}{2} + c.
\]

For a sequence \( a_1, a_2, \ldots, a_n \in \mathbb{N}_0 \), its Gödel encoding is defined as

\[
\text{enc}(\{a_i\}_{i \in [n]}) := \text{pair}(b, c),
\]
where \(b, c \in \mathbb{N}_0\) satisfy \(\text{beta}(b, c, 0) = n\) and \(\text{beta}(b, c, i) = a_i\) for \(i \in [n]\) (if multiple \((b, c)\) satisfies the requirements, take the smallest \(\text{pair}(b, c)\)). The decoding predicate \(\text{dec}(r, i, a)\) is defined so that \(\text{dec}(\text{enc}(\{a_i\}_{i \in [n]}), i, a)\) is true if and only if \(i \in [n]\) and \(a_i = a\). It can be defined by

\[
\begin{align*}
\text{dec}(r, i, a) := & \exists b, c. (\text{pair}(b, c) = r) \\
& \land 1 \leq i \leq \text{beta}(b, c, 0) \land \text{beta}(b, c, i) = a_i.
\end{align*}
\]

\(\text{dec}(r, i, a) := \text{dec}(r, i, a) \land \forall r'. (\forall i', a'. (\text{dec}(r, i', a') \leftrightarrow \text{dec}(r', i', a'))) \rightarrow r' \geq r).

Note that the definition of \(\text{dec}\) enforces the minimality condition in the definition of \(\text{enc}\) (\(r\) is the smallest among all \(r'\) which gives the same decoded values for all \(i\)). Due to Theorem \(\text{dec}\) can be defined in FOTPI (though \(\text{enc}\) cannot since FOTPI does not natively support quantifying over sequences).

Assume \(X_1, \ldots, X_n \in \mathbb{N}_0\). Let \(\bar{X} := \text{enc}(X^n)\) be the Gödel encoding of \(X^n\) (which is a random integer), and \(L|\bar{X} \sim \text{Unif}[\bar{X} + 3]\) is the label of \(\bar{X}\). We can check whether \(\bar{X} = \text{enc}(X^n)\) for some \(X^n\) by the following formula in FOTPI:

\[
iseq(\bar{X}, \bar{L}, n) := \text{label}(3(\bar{X}, \bar{L}) \land \\
\forall l. (\{\bar{X}[\bar{L}] = l\} \neq \emptyset \\
\rightarrow (\forall i, (1 \leq i \leq n \rightarrow \exists x. \text{dec}(l, i, x))) \\
\land (\forall i, x. (i > n \rightarrow \neg \text{dec}(l, i, x)))).
\]

where \(\{\bar{X}[\bar{L}] = l\}\) is defined in (27). Intuitively, this means if \(P(\bar{X} = l) > 0\), then we can decode the \(i\)-th entry of \(l\) for \(1 \leq i \leq n\), and we cannot decode its \(i\)-th entry for \(i > n\).

We now define a formula to check whether \(\bar{X}\) is the \(i\)-th component (i.e., \(X_i\)) of \(\bar{X}\). Let \(L|X \sim \text{Unif}[X + 3]\) be the labelling of \(X\). This can be checked by

\[
\text{entry}(\bar{X}, \bar{L}, n, X, L, i)
:= \isq(\bar{X}, \bar{L}, n) \land \text{label}(3(X, L) \land 1 \leq i \leq n \\
\land \forall x, l. (\text{dec}(l, i, x) \rightarrow \{\bar{X} o \bar{L} = l\} \subseteq \{X o L = x\}).
\]

This means that if \(\text{dec}(l, i, x)\), then \(\bar{X} = l\) implies \(X = x\).

We can also obtain subsequences of \(\bar{X} = \text{enc}(X^n)\). To check whether \(\bar{Y}\) with label \(\bar{M}\) satisfies \(\bar{Y} = \text{enc}(X_i, \ldots, X_j)\), we use the formula

\[
\begin{align*}
\text{subseq}(\bar{X}, \bar{L}, n, \bar{Y}, \bar{M}, i, j) := & 1 \leq i \leq j \leq n \land \isq(\bar{X}, \bar{L}, n) \land \isq(\bar{Y}, \bar{M}, j - i + 1) \\
\land & \forall k. (i \leq k \leq j \rightarrow \\
\forall & X, L. (\text{entry}(\bar{X}, \bar{L}, n, X, L, k) \\
\leftrightarrow & \text{entry}(\bar{Y}, \bar{M}, j - i + 1, X, L, k - i + 1))).
\end{align*}
\]

For notational simplicity, in the remainder of this paper, we use the notation

\[
X \circ L = (X \circ \bar{L})_i
\]

to denote \(\text{entry}(\bar{X}, \bar{L}, X, L, i)\), and

\[
\bar{Y} \circ \bar{M} = (\bar{X} \circ \bar{L})_i\to
\]
to denote \(\text{subseq}(\bar{X}, \bar{L}, \bar{Y}, \bar{M}, i, j)\). Given \(X_1, \ldots, X_n\) with labels \(L_1, \ldots, L_n\) for a fixed (non-variable) \(n\), we write

\[
\bar{X} \circ \bar{L} = (X_1 \circ L_1, \ldots, X_n \circ L_n)
\]

(24) if

\[
isq(\bar{X}, \bar{L}, n) \land \bigwedge_{i=1}^{n} \text{entry}(\bar{X}, \bar{L}, n, X_i, L_i, i),
\]

i.e., \(\bar{X} = \text{enc}(X^n)\) with label \(\bar{L}\).
To check whether $X = \text{enc}(X^n)$ (with label $\bar{L}$) where $X_1, \ldots, X_n$ are i.i.d. with the same distribution as $X$ (with label $L$),
\[
\text{iid}(\bar{X}, \bar{L}, n, X, L) := \text{isseq}(\bar{X}, \bar{L}, n) \land \forall i, X', L', \bar{Y}, \bar{M}. (X' \circ L' = (\bar{X} \circ \bar{L})_i \land \bar{Y} \circ \bar{M} = (\bar{X} \circ \bar{L})_{i-1} \\
\rightarrow X' \circ L' \overset{d}{=} X \circ L \land X' \perp \bar{Y}),
\]
where $X' \circ L' \overset{d}{=} X \circ L$ is defined in (22) (we can let the conditioned random variables be degenerate). The above formula checks that $X_i \overset{d}{=} X$ is independent of $X_1, \ldots, X_{i-1}$.

We now characterize the entropy $H(X)$. By the source coding theorem, $H(X)$ is the smallest $R$ such that for any $R' > R$ and $\epsilon > 0$, there exists $n$, $W$, $Y^n$ with $Y^n \leq W \leq X^n$, $|W| \leq 2^{nR'}$, and $P(X^n \neq Y^n) \leq \epsilon$, where $X_1, \ldots, X_n$ are i.i.d. with the same distribution as $X$. Therefore, we can check whether $H(X) \leq a/b$ for $a, b \in \mathbb{N}_0$, $b > 0$ by
\[
\text{hle}(X, a, b) := \forall a', b'. (a'b > ab' \rightarrow \\
\forall E. (\text{isev}(E) \land E \neq \emptyset \rightarrow \exists \bar{X}, \bar{L}, \bar{Y}, \bar{M}, L, W, n. (\text{iid}(\bar{X}, \bar{L}, n, X, L) \land \text{isseq}(\bar{Y}, \bar{M}, n) \\
\land \bar{Y} \overset{b}{\leq} W \overset{a}{\leq} \bar{X} \land |W|^b \leq 2^{na'} \\
\land \text{prl}(\{X \circ L \neq \bar{Y} \circ \bar{M}, E\}))),
\]
where $|W|^b \leq 2^{na'}$ can be defined since exponentiation is definable using a first-order formula over natural numbers (alternatively, we can construct an i.i.d. sequence $\tilde{W}$ by $\text{iid}(W, L_W, b', W, L_{\tilde{W}})$, which has cardinality $|W|^b$, and use Proposition 6 to check $|W|^b \leq 2^{na'}$), $\{X \circ L \neq \bar{Y} \circ \bar{M}\}$ represents the event $X^n \neq Y^n$ (this notation is defined in (21)), and $\text{prl}(\{X \circ L \neq \bar{Y} \circ \bar{M}, E\})$ checks that $P(X^n \neq Y^n) \leq P(E)$. The above formula means that for any $a', b'$ such that $a'b > ab$, for any $E$ with $P(E) > 0$, there exists $n$, $W$, $Y^n$ with $Y^n \leq W \overset{a}{\leq} X^n$, $|W| \leq 2^{na'/b'}$, and $P(X^n \neq Y^n) \leq P(E)$, where $X_1, \ldots, X_n$ are i.i.d. with the same distribution as $X$.

Using (23), we can check inequalities among entropy of random variables. For example, $H(X) \leq 2H(Y)$ can be checked by
\[
\forall a, b. (\text{hle}(Y, a, 2b) \rightarrow \text{hle}(X, a, b)).
\]

VII. SINGLE-LETTER CHARACTERIZATION OF CAPACITY REGIONS

In this section, we study a general network which encompasses the discrete memoryless networks in [63], [64], [65], the finite-state Markov channel [40], and the Markov network [45].

Definition 9 (Joint source-channel Markov network). We define a network with $k$ terminals as follows. Consider the source distribution $p_{Y_1, \ldots, Y_k}$, the channel $p_{Y_1, \ldots, Y_k, S'|X_1, \ldots, X_k, S'}$, input alphabet $X_i$ (where $X_i \in X_i$), initial state distribution $p_S$, and decoding requirement $p_{Z_1, \ldots, Z_k|W_1, \ldots, W_k}$, where the random variables take values in $\mathbb{N}_0$. At the beginning of the communication scheme, the source $W_{1,1}, \ldots, W_{1,n}$ is given to terminal $i$, where $(W_{1,i}, \ldots, W_{k,i}) \sim p_{W_1, \ldots, W_k}$ i.i.d. across $i \in [n]$. Let $S_{1}, X_{1,i}, Y_{1,i}$ be the channel state, the channel input given by terminal $i$, and the channel output observed by terminal $i$ at time $t \in [n]$ respectively. Let $S_1 \sim p_S$ independent of $\{W_{1,i}\}_{i,t}$. At time $t \in [n]$, terminal $i$ outputs $X_{1,i} \in X_i$ as a (possibly stochastic) mapping of $W_{1,1}, \ldots, W_{1,n}, X_{1,1}, \ldots, X_{1,t-1}$ and $Y_{1,1}, \ldots, Y_{1,t-1}$ for $i \in [k]$, and then $Y_{1,1}, \ldots, Y_{k,t}, S_{t+1}$ are generated given $X_{1,1}, \ldots, X_{k,t}, S_t$ following $p_{Y_1, \ldots, Y_k, S'|X_1, \ldots, X_k, S'}$. At the end, terminal $i$ outputs $\hat{Z}_{1,1}, \ldots, \hat{Z}_{1,n}$ as a (possibly stochastic) mapping of $W_{1,1}, \ldots, W_{1,n}, X_{1,1}, \ldots, X_{1,n}$ and $Y_{1,1}, \ldots, Y_{1,n}$. The probability of error is defined as the total variation distance
\[
P_e := d_{TV}(\{W_{1,1}, \ldots, W_{k,t}, \hat{Z}_{1,1}, \ldots, \hat{Z}_{t,t}\}_{t \in [n]}, (p_{W_1, \ldots, W_k, p_{Z_1, \ldots, Z_k|W_1, \ldots, W_k})^n),
\]
i.e., this is a strong coordination problem [66] where $W_{1,1}, \ldots, W_{k,t}, \hat{Z}_{1,1}, \ldots, \hat{Z}_{t,t}$ should approximately follow $p_{W_1, \ldots, W_k, p_{Z_1, \ldots, Z_k|W_1, \ldots, W_k}}$. We say that this network admits a communication scheme if for any $\epsilon > 0$, there exists $n$ and a communication scheme (encoding and decoding functions) such that $P_e \leq \epsilon$.

Note that the joint source-channel Markov network encompasses channel coding settings by letting $M_1, \ldots, M_l$ to be independent (they are the messages), $W_i = \{M_j\}_{j \in E_i}$, where $E_i \subseteq [k]$ is the set of messages that terminal $k$ can access, and $Z_i = \{M_j\}_{j \in D_i}$, where $D_i \subseteq [k]$ is the set of messages that terminal $k$ intends to decode. In this case, $P_e$ is the probability that any terminal makes an error in decoding the intended messages. The sources $\{W_{1,1}\}$ can be regarded as messages by the source-channel separation theorem. For example, to represent the broadcast channel $p_{Y_2, Y_3|X_1}$, let $W_1 = (M_1, M_2)$, $Z_2 = M_1$,...
We will show that the capacity region of the joint source-channel Markov network can be characterized in FOTPI.

**Theorem 10.** Fix \( k \geq 1 \). There exists a first-order formula

\[
Q_k(W_1, \ldots, W_k, X_1, \ldots, X_k, Y_1, \ldots, Y_k, Z_1, \ldots, Z_k, S, L_S, S', L_{S'})
\]

such that for any joint source-channel Markov network \((p_{W_1}, \ldots, p_{W_k}, p_{Y_1}, \ldots, p_{Y_k}, S'|X_1, \ldots, X_k, S, p_S, p_{Z_1}, \ldots, p_{Z_k})|_{W_1, \ldots, W_k}\) and any \( p_{X_1, \ldots, X_k} \) which assigns positive probability for each \((x_1, \ldots, x_k) \in X_1 \times \cdots \times X_k\), the network admits a communication scheme if and only if the formula \( Q_k \) holds for

\[
(W_1, \ldots, W_k, Z_1, \ldots, Z_k) \sim p_{W_1, \ldots, W_k} p_{Z_1, \ldots, Z_k}|_{W_1, \ldots, W_k},
\]

and \( L_S, L_{S'} \) are labels \( \{2\} \) of \( S, S' \) respectively (i.e., \( \text{label}_3(S, L_S) \) and \( \text{label}_3(S', L_{S'}) \) hold).

**Proof:** The main idea of the proof is to state the multi-letter operational definition of the joint source-channel Markov network using a first-order formula. Since the operational definition only consists of distribution constraints, conditional independence constraints, and a bound on the total variation distance, all of which can be expressed in first-order formulae, the proof is a rather straightforward rewriting of these constraints in the language of first-order formulae.

Let \( W_{[k]} := \text{enc}(W_1, \ldots, W_k) \) (with label \( L_{W_{[k]}} \)) and define \( X_{[k]}, Y_{[k]}, Z_{[k]} \) similarly. This can be checked by

\[
\text{isseq}(W_{[k]}, L_{W_{[k]}}, k) \land \bigwedge_{i=1}^k \exists L. W_i \circ L \equiv (W_{[k]} \circ L_{W_{[k]}})_i,
\]

and similar for \( X_{[k]}, Y_{[k]}, Z_{[k]} \). Note that \( k \) is fixed and is not a variable, so the above formula is valid.

We write \( W_t^n := (W_{i,1}, \ldots, W_{i,n}) \). Let \( W_{[k],t} := \text{enc}(W_{1,t}, \ldots, W_{k,t}) \) and \( W_{[k],n} := \text{enc}(W_{[k],1}, \ldots, W_{[k],n}) \) (with label \( L_{W_{[k]}} \)). Define \( \tilde{X}_{[k]}, \tilde{Y}_{[k]}, \tilde{Z}_{[k]} \) similarly. Note that \( \{Z_{[k],t}\}_{t \in [n]} \) (used to define \( \tilde{Z}_{[k]} \)) are auxiliary random variables that do not appear in the operational setting (they are not the same as \( \tilde{Z}_{t,x} \)), which will be used later in the bound on \( P_e \). We check that they are nested sequences by

\[
\text{isseq}(W_{[k]}, L_{W_{[k]}}, n)
\]

\[\land \forall t, \tilde{W}, L_{\tilde{W}}, (\tilde{W} \circ L_{\tilde{W}} = (W_{[k]})^o_t \rightarrow \text{isseq}(W', L_{W', k}),)\]

and similar for \( \tilde{X}_{[k]}, \tilde{Y}_{[k]}, \tilde{Z}_{[k]} \), where we write \( W_{[k]})^o := W_{[k]} \circ L_{W_{[k]}} \) for brevity (when the label corresponding to the random variable is clear from the context). We write \( (W_{[k]})^o_t := ((W_{[k]})^o)_t \) (which corresponds to \( W_{[k],t} \)).

First we enforce that \( (W_{1,t}, \ldots, W_{k,t}, Z_{1,t}, \ldots, Z_{k,t}) \sim p_{W_{1,t}, \ldots, W_{k,t}} p_{Z_{1,t}, \ldots, Z_{k,t}} \) i.i.d. across \( t \in [n] \) by

\[
\forall t. (1 \leq t \leq n \rightarrow (W_{[k]})^o_t \overset{d}{=} W_{[k]} \circ (\tilde{Z}_{[k]})^o_t ((W_{[k]})^o_1 \cdots (W_{[k]})^o_t) \sim Z_{[k]} \circ W_{[k]} \circ \tilde{Z}_{[k]}^o_t \substack{\cup
\quad ((W_{[k]})^o_1 \cdots (W_{[k]})^o_{t-1}) (\tilde{Z}_{[k]})^o_{t-1}}),
\]

where \( (W_{[k]})^o_t \overset{d}{=} W_{[k]} \circ \tilde{Z}_{[k]}^o_t \) means

\[
\exists W', L'. (W' \circ L' = (W_{[k]} \circ L_{W_{[k]}})^o_t \land W' \circ L' \overset{d}{=} W_{[k]} \circ L_{W_{[k]}},)
\]

\[\text{and refer to } (25) \text{ for the meaning of } (W_{[k]})^o_t ((\tilde{Z}_{[k]})^o_t) \cup ((W_{[k]})^o_1 \cdots (W_{[k]})^o_{t-1}) (\tilde{Z}_{[k]})^o_{t-1} \text{. Note that } (25) \text{ checks that}
\]

\[
(W_{1,t}, \ldots, W_{k,t}) \overset{d}{=} (W_1, \ldots, W_k),
\]

\[(Z_{1,t}, \ldots, Z_{k,t}) ((W_{1,t}, \ldots, W_{k,t}) \sim (Z_1, \ldots, Z_k)) (W_1, \ldots, W_k),
\]

\[\text{Technically, the capacity region is often defined as the closure of the set of achievable } (R_1, R_2). \] We can represent the closure operation in FOTPI by declaring the rate pair \((H(M_1), H(M_2))\) to be in the capacity region if for any \( M_1, M_2 \) such that \( H(M_1) < H(M_1), H(M_2) < H(M_2) \) (which can be expressed using (16), the network admits a communication scheme for the messages \( M_1, M_2 \). For the purpose of simplicity, we ignore the closure operation.

\[\text{We require labels only for the present state } S \text{ and the next state } S', \text{ but not for } W_t, X_t, Y_t, Z_t, \text{ since the channel } p_{Y_1, \ldots, Y_k}|_{S'} |_{X_1, \ldots, X_k, S} \text{ is essentially unchanged under relabelling of } W_t, X_t, Y_t, Z_t, \text{ but not under relabelling of } S \text{ while keeping } S' \text{ unmodified. How the values of } S \text{ correspond to the values of } S' \text{ is essential.}\]
where the \( \cdots \) in (21) by

By the coupling definition of total variation distance, we can enforce

\[
\text{prle}\{(\tilde{Z}_k)^o = (\tilde{Z}_k)^o \}, \quad E).
\]

Finally, since the network admits a communication scheme if and only if \( P_e \leq P(E) \) is possible for any \( P(E) > 0 \), the final formula \( Q_k \) is

\[
\forall E. (\text{isev}(E) \land E \neq \emptyset) \rightarrow \\
\exists n, W_k, L_{W_k}, X_k, L_{X_k}, Y_k, L_{Y_k}, Z_k, L_{Z_k}, \\
\tilde{W}_k, L_{\tilde{W}_k}, \tilde{X}_k, L_{\tilde{X}_k}, \tilde{Y}_k, L_{\tilde{Y}_k}, \tilde{Z}_k, L_{\tilde{Z}_k}, \tilde{Z}_k, L_{\tilde{Z}_k}.
\]

where the “\( \cdots \)” in the last line is the conjunction of (26)–(33).
VIII. Definition of Single-Letter Characterization

Theorem 10 shows that the capacity regions of a large class of multiuser settings can be expressed as first-order formulae. Arguably, this single-letter characterization is against the spirit of single-letter characterizations in network information theory, considering its complexity. Theorem 10 suggests that general first-order formulae are perhaps too powerful, and focusing on first-order “single-letter” formulae is not a meaningful restriction.

As there was no generally accepted definition of single-letter characterization, Körner [44] raised the question of finding a logical theory on single-letter characterizations. After this, to the best of the author’s knowledge, the only attempt in providing a definition of single-letter characterization is [43], which studies single-letter characterizations in the form of a conjunction of linear inequalities on mutual information terms, where the alphabets of all auxiliary random variables are fixed, which might be a little restrictive (refer to Remark 16). In this section, we propose some possible definitions that are (in a sense) more general than [43], yet are more restrictive than general first-order formulae. This may allow open problems regarding single-letter characterizations of capacity regions to be stated in a rigorous manner.

We first define the probabilistic independence hierarchy in a similar manner as the arithmetical hierarchy [58] and the Lévy hierarchy [67].

Definition 11 (The probabilistic independence hierarchy). A first-order formula is in the set $\Delta^\Pi_0 = \Sigma^\Pi_0 = \Pi^\Pi_0$ if and only if it is logically equivalent to a quantifier-free (i.e., all variables are free) formula. We define $\Delta^\Pi_i, \Sigma^\Pi_i, \Pi^\Pi_i$ recursively. A first-order formula is in the set $\Sigma^\Pi_i$ if and only if it is logically equivalent to a formula in the form $\exists U_1, \ldots, U_k, P(X_1, \ldots, X_n, U_1, \ldots, U_k)$, where $P$ is a formula in $\Pi^\Pi_i$. A first-order formula is in the set $\Pi^\Pi_i$ if and only if it is logically equivalent to a formula in the form $\forall U_1, \ldots, U_k, P(X_1, \ldots, X_n, U_1, \ldots, U_k)$, where $P$ is a formula in $\Sigma^\Pi_i$. We then define $\Delta^\Pi_i = \Sigma^\Pi_i \cap \Pi^\Pi_i$.

As a direct corollary of [14], $\Sigma^\Pi_3$ and $\Pi^\Pi_3$ are undecidable fragments of FOTPI. Finding the lowest level in the probabilistic independence hierarchy which is undecidable is left for future studies.

Proposition 12. The problem of deciding the truth value of a $\Sigma^\Pi_3$ formula without free variables is undecidable. The same is also true for $\Pi^\Pi_3$.

Proof: Note that $X \leq Y$ (1) is $\Pi^\Pi_1$, $Z \leq XY$ (2) (which can be extended to $Z \leq X_1 \cdots X_n$) is $\Pi^\Pi_2$, $X \perp Y | Z$ (3) is $\Sigma^\Pi_3$, and $\text{card}_{=2}(X)$ (5) is $\Pi^\Pi_3$. The undecidability of $\Sigma^\Pi_3$ follows from [14 Cor. 3], which states that the problem of deciding whether there exists random variables $X_1, \ldots, X_n$ satisfying some conditional independence constraints where $X_1$ is nondegenerate binary is undecidable. The undecidability of $\Pi^\Pi_3$ is due to the fact that the negation of a $\Sigma^\Pi_3$ formula is a $\Pi^\Pi_3$ formula.

In FOTPI, the atomic formulae are probabilistic independence statements. In network information theory, rate regions are more often stated using linear inequalities on entropy terms. We can also define another hierarchy with linear inequalities as atomic formulae.

Definition 13 (The linear entropy hierarchy). A first-order formula is in the set $\Delta^H_{\text{atom}}$ if and only if it is either in the form $\alpha^T h(X_1, \ldots, X_n) \geq 0$ for some $\alpha \in \mathbb{Q}^{2^n-1}$ (where $h(X_1, \ldots, X_n) \in \mathbb{R}^{2^n-1}$ is the entropic vector [12] of $(X_1, \ldots, X_n)$, or in the form $Y | X_1, \ldots, X_n \sim Y | X_1, \ldots, X_n$ (see Proposition 7) (4). A first-order formula is in the set $\Delta^H_0 = \Sigma^H_0 = \Pi^H_0$ if and only if it is logically equivalent to a composition of $\Delta^H_{\text{atom}}$ formulae using logical conjunction, disjunction and negation. We define $\Delta^H_i, \Sigma^H_i, \Pi^H_i$ in a similar manner as $\Delta^\Pi_i, \Sigma^\Pi_i, \Pi^\Pi_i$.

Note that $\Sigma^H_i$ shares some similarities with [43] (refer to Remark 16). Since entropy and the $\sim$ relation can be defined in FOTPI, the linear entropy hierarchy is in FOTPI as well. Similar to Proposition [12], $\Sigma^H_2$ and $\Pi^H_2$ are undecidable fragments of FOTPI.

Proposition 14. The problem of deciding the truth value of a $\Sigma^H_2$ formula without free variables is undecidable. The same is also true for $\Pi^H_2$. Also, the problem of deciding the truth value of $P(X)$, where $P$ is a $\Sigma^H_1$ formula and $X \sim \text{Bern}(1/2)$, is undecidable. The same is also true for $\Pi^H_1$.

Proof: Note that $\text{card}_{=2}(X)$ (5) is $\Pi^H_1$. The result follows from [14 Cor. 3], which states that the problem of deciding whether there exists $X_1, \ldots, X_n$ satisfying some conditional independence constraints where $X_1$ is nondegenerate binary (also holds for the constraint $X_1 \sim \text{Bern}(1/2)$) is undecidable.

Theorem 10 states that for a general class of multiuser coding settings, the capacity region can be stated as a first-order formula, and hence is in the linear entropy hierarchy.

This is needed for channel coding problems to allow changing the input distribution. For example, the capacity region of the point-to-point channel $p(y|x)$ is $\exists X, Y \cdot T(X, Y) \leq I(X; Y)$ (where the rate is $R = H(M)$). This is also useful for outer bounds involving coupling, e.g. [68].
Proposition 15. The formula for the capacity region of the joint source-channel Markov network in Theorem 10 is in $\Delta_{18}^H$.

Proof: Note that $X \perp Y$ can be expressed as $H(XY) - H(X) - H(Y) \geq 0$, so Theorem 10 is still valid when the atomic formulae are linear inequalities on entropy terms. Technically, using linear inequalities would restrict attention to random variables with finite entropy, though it can be checked that the construction in Theorem 10 does not require any random variable with infinite entropy (as long as the input random variables have finite entropy).

By tracing the construction in the proof of Theorem 10 and counting the depth of quantifier alternation, we find out that the formula is in $\Pi_1^H \subseteq \Delta_{18}^H$.

Using the linear entropy hierarchy, we propose some possible definitions of single-letter characterization, in decreasing order of generality.

1) Any first-order formula in $\bigcup_{i \geq 0} \Delta_i^H$. By Theorem 10, the capacity regions of a large class of multiuser settings can be expressed as first-order formulae.

2) A first-order formula in $\Delta_i^H$, $\Sigma_i^H$, or $\Pi_i^H$ for a fixed $i \geq 2$. For example, we may restrict attention to $\Pi_2^H$ to allow only existential formulae and “for all, there exists” formulae (e.g. [33], [34], [35], [36]).

3) A first-order formula in $\Sigma_1^H$, i.e., an existential formula. The majority of capacity regions and bounds in network information theory [29] are $\Sigma_1^H$ formulae, where all auxiliary random variables are existentially quantified. However, restricting to $\Sigma_1^H$ does not give any provable benefit on the ease of computation of the region, since Proposition 14 shows that such formula is undecidable even for the simple input distribution $X \sim \text{Bern}(1/2)$. Therefore, restricting to $\Sigma_1^H$ formulae is merely a simplicity concern rather than a computability concern.

4) A first-order formula in $\Sigma_1^H$, together with cardinality bounds for all existentially-quantified random variables. Cardinality bounds on auxiliary random variables are given for many capacity regions in network information theory [29], which might allow more efficient computation. Technically, we should restrict the cardinality bounds to be computable functions of the cardinalities of the (non-auxiliary) random variables in the coding setting. Considering that logarithm is involved in the definition of entropy, whether the cardinality bounds can make the regions computable depends on the solution to Tarski’s exponential function problem [69] (also see [28], [11]).

5) A first-order formula in $\Delta_0^H = \Sigma_0^H = \Pi_0^H$. This would disallow auxiliary random variables, and hence is probably too restrictive. Regions without auxiliary random variables such as the Slepian-Wolf region [70] are in $\Delta_0^H$.

Perhaps, instead of setting a fixed limit on which levels in the linear entropy hierarchy counts as single-letter characterizations, we can find the lowest level that can express the capacity region of a given setting. Theorem 10 implies that such level always exists. Given the linear entropy hierarchy, we can state open problems in network information theory, e.g. for the broadcast channel, as follows.
Open problem. What is the lowest level in the linear entropy hierarchy $\Delta_i^H, \Sigma_i^H, \Pi_i^H$ such that the capacity region of the broadcast channel $p_{Y,Z|X}$ can be expressed as a first-order formula (with free variables $X,Y,Z,W_1,W_2$, where $W_i$ represents the rate $R_i$ via $R_i = H(W_i)$) in that level?

Theorem [10] implies that the capacity region is expressible in $\Delta_{i=8}^H$ (and hence in $\Sigma_{i=8}^H$ and $\Pi_{i=8}^H$). The remaining question is to find the lowest possible level. If the conjecture that the 3-ary auxiliary Marton’s inner bound [74], [75], [76] is optimal is correct, then the lowest level would be $\Sigma_1^H$. Some outer bounds for the broadcast channel are the UV outer bound [77], [78] ($\Sigma_1^H$ formula, suboptimal) and the J version of the UV outer bound [56] ($\Pi_2^H$ formula, optimality unknown).

We can also raise the same question for interference channel [33]. The Han-Kobayashi inner bound [79] ($\Sigma_1^H$ formula) was shown to be suboptimal by Nair, Xia and Yazdanpanah [80]. To the best of the author’s knowledge, there is no candidate single-letter characterization of the capacity region that is conjectured to be optimal. Some outer bounds for the interference channel are given in [51] ($\Sigma_1^H$ formula), [52] ($\Sigma_1^H$ formula), [53] ($\Sigma_1^H$ formula), and [56] ($\Pi_2^H$ formula). Perhaps the reason we are unable to give a $\Sigma_1^H$ candidate for the capacity region is that the actual lowest possible level is $\Pi_2^H$ (or higher).

Remark 16. In [43], two slightly different definitions of single-letter characterizations are given (specialized to the case where there is only one rate $R$):

- [43] eqn (1): A formula which is the conjunction of inequalities in the form $\beta R + \sum_i \alpha_i I(U_{A_i};U_{B_i}|U_{C_i}) \leq 0$ (where $U_{A_i} = \{U_a\}_{a \in A_i}, A_i, B_i, C_i \subseteq [n], \beta, \alpha_i \in \mathbb{R}$ are computable; note that $\beta$ can be nonpositive) and polynomial constraints on the joint probability mass function.
- [43] eqn (3): A formula which is the conjunction of inequalities in the form $\beta R + \sum_i \alpha_i I(U_{A_i};U_{B_i}|U_{C_i}) \leq 0$ and polynomial constraints on the joint probability mass function.

The proof of the non-existence of single-letter characterization for the Markov channel in [43] is performed on the second definition, which is more restrictive than the first definition. While [43] shows that the optimal rate in the second definition can be computed up to arbitrary precision, whether this also holds for the first definition is not entirely clear (it may depend on the solution to Tarski’s exponential function problem [69]).

The first definition [43] eqn (1) is closer to $\Sigma_1^H$ in this paper. Still there are some important differences. In [43], the alphabets of all random variables are fixed, and general polynomial constraints on the joint probability mass function are allowed. These are not the case for $\Sigma_1^H$. The fixed alphabet assumption is crucial to the proof of the non-existence of single-letter characterization in [43] (since fixing the alphabet transforms the problem into an existential problem on real numbers, which are the entries of the joint probability mass function). On the other hand, unlike [43], $\Sigma_1^H$ allows logical negation and disjunction, making the set of rate regions expressible in $\Sigma_1^H$ closed under union. It is unclear whether the Markov network in [43], or the joint source-channel Markov network in this paper, has a capacity region expressible in $\Sigma_1^H$.

IX. Extension to Continuous Random Variables

In the previous sections, we assumed all random variables are discrete. In this section, we consider an extension to general random variables (in the standard probability space). It is unclear a priori whether this extension will increase or decrease the interpretability strength. For example, the true first-order theory of natural numbers is undecidable, whereas the true first-order theory of real numbers is decidable [69], [84]. Hence, it is possible that a theory in a continuous setting is not stronger than its discrete counterpart. Nevertheless, we will show that the condition that a random variable is discrete can be defined using a first-order formula, and hence the first-order theory of general random variables is at least as expressive as that of discrete random variables.

Previously, we assumed all random variables are defined on the same standard probability space $([0,1], \mathcal{F}, P)$. This is possible since the standard probability space is rich enough to allow defining any new discrete random variable in addition to an existing finite collection of discrete random variables. This is no longer possible for continuous random variables, since one random variable (e.g. the random variable defined by the identity map) may already exhaust the randomness in the space, and it is impossible to define any nondegenerate random variable independent of it. Therefore, we have to assign slightly different semantics to logical symbols, in order to allow extension of the probability space when new random variables are introduced.

We define the first-order theory of general random variables. Assume all random variables take value in $[0,1]$. The predicate $\psi(X_1,\ldots,X_n)$ should be interpreted as a predicate on the joint probability distribution $P_{X_1,\ldots,X_n}$ over $[0,1]^n$, i.e., $\psi(X_1,\ldots,X_n)$ is a shorthand for $(P_{X_1,\ldots,X_n})$. For a first-order predicate $\psi(X_1,\ldots,X_n,Y)$ on $P_{X_1,\ldots,X_n,Y}$, the formula $\exists Y.\psi(X_1,\ldots,X_n,Y)$ (which is a predicate on $P_{X_1,\ldots,X_n}$) means $\exists P_{X_1,\ldots,X_n,Y}.((P_{X_1,\ldots,X_n,Y})_{X_1,\ldots,X_n} = P_{X_1,\ldots,X_n} \land \psi(P_{X_1,\ldots,X_n,Y}))$, that is, there exists joint distribution $P_{X_1,\ldots,X_n,Y}$ over $[0,1]^{n+1}$ such that its $(X_1,\ldots,X_n)$-marginal $(P_{X_1,\ldots,X_n,Y})_{X_1,\ldots,X_n}$ is $P_{X_1,\ldots,X_n}$, and $\psi(P_{X_1,\ldots,X_n,Y})$ holds. We define $\forall Y.\psi(X_1,\ldots,X_n,Y)$ similarly. For logical conjunction

$$\psi_1(X_1,\ldots,X_l,Y_1,\ldots,Y_m) \land \psi_2(X_1,\ldots,X_l,Z_1,\ldots,Z_n),$$
it is interpreted as
\[
\psi_1((P_{X_1},...,X_1,Y_1,...,Y_m)_{x_1,...,x_1,y_1,...,y_m})
\land \psi_2((P_{X_1},...,X_1,Y_1,...,Y_m)_{x_1,...,x_1,z_1,...,z_n})_{x_1,...,x_1,z_1,...,z_n},
\]
which is a predicate on \(P_{X_1},...,X_1,Y_1,...,Y_m,z_1,...,z_n\). Define logical disjunction similarly. We can therefore define any first-order formula recursively (the base cases are \(\exists Y.\psi(Y)\) and \(\forall Y.\psi(Y)\) without any free variable, which are interpreted as \(\exists Y_1.\psi(Y_1)\) and \(\forall Y_1.\psi(Y_1)\) respectively).

We now check that some of the formulae in the previous sections still hold in the first-order theory of general random variables. The condition \(U \ni 0\) (i.e., \(U\) is almost surely a constant, or \(P_U\) is a degenerate distribution) can be checked using the same formula \(U \ni 0 \iff U \not\perp \perp U\). For general random variables, \(X \perp Y\) means there exists a measurable function \(f: [0,1] \to [0,1]\) such that \(X = f(Y)\) with probability 1. We can check that (1) is still valid for general random variables. It is straightforward to check that \(X \perp Y \implies \forall U. (U \not\perp \perp Y \to X \not\perp U.\).

Finally, we define the formula for discrete random variables. Since any probability distribution can be decomposed into a mixture of a discrete distribution and an atomless distribution, checking whether \(P\) is atomless (i.e., \(P\) is a discrete random variable) is equivalent to checking if it does not contain an atomless component, i.e., a measurable set \(S \subseteq [0,1]\) such that \(P(X \in S) > 0\) and \(P(X = x) = 0\) for any \(x \in S\). This can be checked by the formula
\[
\text{atomless}(X) := \neg \exists Y.\text{smi}(X, Y).
\]

To check this, we first show that if there exists \(V, W\) satisfying the above constraints, then \(X\) has an atomless component. Since \(\text{card}_{\leq n}(V\land W)\), we may assume \((V, W) \in \{(0, 0), (1, 0), (0, 1)\}\). Let \(S\) be the set of values of \(X\) corresponding to \((V, W) = (0, 0)\). Assume the contrary that \(S\) is not an atomless component. Then there exists \(x \in S\) such that \(P(X = x) > 0\). Let \(U = 1\{X = x\}\). It is straightforward to check that \(U\) satisfies the constraints in (34), leading to a contradiction.

We then show that if \(X\) has an atomless component, then there exists \(V, W\) satisfying (34). Let \(S \subseteq [0,1]\) such that \(P(X \in S) > 0\) and \(P(X = x) = 0\) for any \(x \in S\). Divide \(S\) into three disjoint sets \(S_0, S_1, S_2\) with positive probabilities. Let \(S_0 := S_0, S_0 := S_1, S_1 := [0,1]\), \(S_0 \cup S_1\). Let \((V, W) = (0, 0)\) if \(X \in S_0\), \((V, W) = (1, 0)\) if \(X \in S_1\), \((V, W) = (1, 0)\) if \(X \in S_2\). Assume the contrary that there exists \(U\) satisfying the constraints in (34). Since \(\text{smi}(X, U)\), we assume \(U = 1\{x_0\}(X), P(X = x_0) > 0\). Since \(\text{smi}(VWU, U)\), \(U\) is degenerate conditional on \((V, W) = (1, 0)\), and hence \(x_0 \notin S_{1,0}\) (note that \(S_{1,0} \neq \{x_0\}\) since \(S_2 \subseteq S_{1,0}\) is atomless). Since \(\text{smi}(VWU, U)\), \(U\) is degenerate conditional on \((V, W) = (0, 1)\), and hence \(x_0 \notin S_{0,1}\). Hence we have \(x_0 \in S_{0,0} \subseteq S\), leading to a contradiction.
As a result, the first-order theory of general random variables is at least as expressive as the first-order theory of discrete random variables (if we can impose discrete(X) on each variable X in the theory of general random variables, it reduces to the theory of discrete random variables). Hence, the first-order theory of general random variables is also algorithmically undecidable.

We remark that it is unclear whether Theorem 10 holds for channels with continuous input or output alphabet. The above argument only shows that the capacity of discrete channels can still be characterized using a first-order formula with general random variables by the construction in Theorem 10.

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