(CO)HOMOLOGY OF SOME CYCLIC LINEAR CYCLE SETS

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Abstract. For each member \( A \) of a family of linear cycle sets whose underlying abelian group is cyclic of order a power of a prime number, we compute all the central extensions of \( A \) by an arbitrary abelian group.

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Introduction

A cycle set, as defined in [21], is a set \( A \) endowed with a binary operation \( \cdot \), such that the left translations \( a \mapsto b \cdot a \) are bijective and the identities

\[(a \cdot b) \cdot (a \cdot c) = (b \cdot a) \cdot (b \cdot c)\]

are satisfied. In [21] it was proved that non-degenerate cycle sets (i.e., with invertible squaring map \( a \mapsto a^2 \)) are in bijective correspondence with non-degenerate involutive set-theoretic solutions of the Yang-Baxter equation, whose study was started by Etingof, Schodler, and Soloviev in [10]. These solutions are connected with many domains of algebra: Garside structures, Hopf-Galois theory, affine torsors, Artin-Schelter regular rings, groups of I-type, left symmetric algebras, etcetera (see, for instance [4, 5, 7–9, 11–16, 18, 22, 23]). A linear cycle set is a cycle set \((A, \cdot)\) endowed with an abelian group operation + satisfying the identities

\[a \cdot (b + c) = a \cdot b + a \cdot c \quad \text{and} \quad (a + b) \cdot c = (a \cdot b) \cdot (a \cdot c).\]

The interest in these structures is due to the fact that they are equivalent to brace structures, and so they are strongly related with the non-degenerated involutive set-theoretic solutions of the Yang-Baxter equations. For instance, the structure group of a non-degenerate solution [10] is a brace in a natural way.

Motivated by the problem of the classify braces, in [2] the authors point out the importance of to develop a extensions theory of braces (or equivalently, of linear cycle sets). This was made out by Bachiller in [1], using the language of braces; by Ben David and Ginosar in [3], using the language of bijective 1-cocycles (other avatar of linear cycle sets); and by Lebed and Vendramin in [19], using the language of linear cycle sets. In the last approach the authors introduce a cohomology theory \( H^*_{\mathbb{N}}(A, \Gamma) \), in order to classify the central extensions of a linear cycle set \( A = (A, \cdot) \) by an abelian group \( \Gamma \). This cohomology is defined by using a explicit cochain complex \( (C^*_\mathbb{N}(A, \Gamma), \partial^*) \). This allows to use homological methods in order to studied these extensions. To be something more precise, when \( A \) is a group \((A, +)\) endowed with the trivial linear cycle set structure \( a \cdot b := b \), one can use resolutions, satellite functors, simplicial methods, etcetera, to make calculations and to obtain theoretical results about \( H^*_\mathbb{N}(A, \Gamma) \); and it is reasonable to expect that, under right circumstances, these calculations and results can be extended to more general
types linear cycle sets. For example, this occurs if the necessary hypotheses to apply the Perturbation Lemma are satisfied.

Let $p \in \mathbb{N}$ be a prime number and let $\nu, \eta \in \mathbb{N}$ be such that $0 < \nu, \eta \leq 2\nu$. Let $u := p^\nu$, $v := p^\eta$, $t := p^{\nu-\eta}$, $u' := p^{2\nu-\eta}$ and let $A$ be the linear cycle set $(\mathbb{Z}/v\mathbb{Z}; \cdot)$, where $i, j := (1 - u_1)$. Let $\Gamma$ be an additive abelian group. The main result of this paper are the following:

**Theorem A.** Assume that $u = v$. For each $\gamma, \gamma_1 \in \Gamma$ such that $v\gamma_1 = 0$, let $\tilde{\xi}_1, \tilde{\xi}_2 : \mathbb{Z} \times \mathbb{Z} \to \Gamma$ be the maps defined by

\[
\tilde{\xi}_1(t_1, t_2) := \xi_1^1([g^{t_1} \otimes g^{t_2}]) \quad \text{and} \quad \tilde{\xi}_2(t_1, t_2) := \xi_2^2(g^{t_1} \otimes g^{t_2}),
\]

where $\xi_1^1$ and $\xi_2^2$ are as above of Proposition 3.15. The following facts hold:

1. $\Gamma \times \mathbb{Z}$ is a linear cycle set via

\[
(c, i) + (c', i') := (c + c' + \xi_1^1(i, i'), i + i') \quad \text{and} \quad (c, i) \cdot (c', i') := (c' + \xi_2^2(0, i'), i \cdot i').
\]

Following [19] we denoted this linear cycle set by $\Gamma \oplus_{\xi_1^1, \xi_2^2} \mathbb{Z} \times \mathbb{Z}$. Moreover

\[
0 \longrightarrow \Gamma \longrightarrow \Gamma \oplus_{\xi_1^1, \xi_2^2} \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0,
\]

where $\iota$ and $\pi$ are the evident maps, is a central extension of $(\mathbb{Z}/v\mathbb{Z}; \cdot)$ by $\Gamma$ in the sense of [19, Definition 5.5].

2. The extension associated with $(\tilde{\xi}_1^1, \tilde{\xi}_2^2)$ is equivalent if and only if $\gamma_1 = \gamma_1$ and $v\gamma' = v\gamma$; and each central extension of $(\mathbb{Z}/v\mathbb{Z}; \cdot)$ by $\Gamma$, is equivalent to one of these.

**Theorem B.** Assume that $2 < u < v$. For each $\gamma, \gamma_1 \in \Gamma$ such that $v\gamma_1 = u\gamma$, let $\tilde{\xi}_1, \tilde{\xi}_2 : \mathbb{Z} \times \mathbb{Z} \to \Gamma$ be the maps defined by

\[
\tilde{\xi}_1(t_1, t_2) := \xi_1^1([g^{t_1} \otimes g^{t_2}]) \quad \text{and} \quad \tilde{\xi}_2(t_1, t_2) := \xi_2^2(g^{t_1} \otimes g^{t_2}),
\]

where $\xi_1^1$ and $\xi_2^2$ are as above of Proposition 3.17, $0 \leq i < u$ and $0 \leq j < t$. The following facts hold:

1. $\Gamma \times \mathbb{Z}$ is a linear cycle set via

\[
(c, i) + (c', i') := (c + c' + \xi_1^1(i, i'), i + i') \quad \text{and} \quad (c, i) \cdot (c', i') := (c' + \xi_2^2(i, i'), i \cdot i').
\]

Following [19] we denoted this linear cycle set by $\Gamma \oplus_{\xi_1^1, \xi_2^2} \mathbb{Z} \times \mathbb{Z}$. Moreover

\[
0 \longrightarrow \Gamma \longrightarrow \Gamma \oplus_{\xi_1^1, \xi_2^2} \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0,
\]

where $\iota$ and $\pi$ are the evident maps, is a central extension of $(\mathbb{Z}/v\mathbb{Z}; \cdot)$ by $\Gamma$ in the sense of [19, Definition 5.5].

2. The extension associated with $(\tilde{\xi}_1^1, \tilde{\xi}_2^2)$ is equivalent if and only if $\gamma_1 - \gamma_1^1 \in u\Gamma$ and $f(\gamma_1 - \gamma_1^1) = \gamma - \gamma'$; and each central extension of $(\mathbb{Z}/v\mathbb{Z}; \cdot)$ by $\Gamma$, is equivalent to one of these.

**Theorem C.** Assume that $u = 2$ and $v = 4$. For each $\gamma, \gamma_1, \gamma_1' \in \Gamma$ such that $4\gamma_1 = 2\gamma$ and $2\gamma_2 = 0$, let $\tilde{\xi}_1, \tilde{\xi}_2 : \mathbb{Z} \times \mathbb{Z} \to \Gamma$ be the maps defined by

\[
\tilde{\xi}_1(t_1, t_2) := \xi_1^1([g^{t_1} \otimes g^{t_2}]) \quad \text{and} \quad \tilde{\xi}_2(t_1, t_2) := \xi_2^2(g^{t_1} \otimes g^{t_2}),
\]

where $\xi_1^1$ and $\xi_2^2$ are as above of Proposition 3.19, $0 \leq i, j < 2$. The following facts hold:

1. $\Gamma \times \mathbb{Z}$ is a linear cycle set via

\[
(c, i) + (c', i') := (c + c' + \xi_1^1(i, i'), i + i') \quad \text{and} \quad (c, i) \cdot (c', i') := (c' + \xi_2^2(i, i'), i \cdot i').
\]

Following [19] we denoted this linear cycle set by $\Gamma \oplus_{\xi_1^1, \xi_2^2} \mathbb{Z} \times \mathbb{Z}$. Moreover

\[
0 \longrightarrow \Gamma \longrightarrow \Gamma \oplus_{\xi_1^1, \xi_2^2} \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0,
\]

where $\iota$ and $\pi$ are the evident maps, is a central extension of $(\mathbb{Z}/v\mathbb{Z}; \cdot)$ by $\Gamma$ in the sense of [19, Definition 5.5].

2. The extension associated with $(\tilde{\xi}_1^1, \tilde{\xi}_2^2)$ and $(\tilde{\xi}_1^1, \tilde{\xi}_2^2)$ are equivalent if and only if $\gamma_1 - \gamma_1 \in 2\Gamma$, $\gamma - \gamma = 2(\gamma_1 - \gamma_1)$ and $\gamma_1 = \gamma_1$; and each central extension of $(\mathbb{Z}/v\mathbb{Z}; \cdot)$ by $\Gamma$, is equivalent to one of these.
In order to prove these results we first compute the normalized full linear cycle set cohomology $H^*_\mathcal{N}(\mathcal{A}, \Gamma)$. This is done in Theorems 3.14, 3.16 and 3.18.

1 Preliminaries

In this paper we work in the category of abelian groups, all the maps are $\mathbb{Z}$-linear, $\otimes$ means $\otimes_{\mathbb{Z}}$ and $\text{Hom}$ means $\text{Hom}_{\mathbb{Z}}$.

1.1 Group homology

Let $G$ be a group, let $D := \mathbb{Z}[G]$ and let $\mathcal{D} := D/\mathbb{Z}1$. We call $S_n(G)$ the simplicial complex of right $D$-modules with objects $S_n(G) := D^{\otimes n+1}$, face maps $\mu_i : S_n(G) \rightarrow S_{n-1}(G)$ and degeneracy maps $\epsilon_i : S_n(G) \rightarrow S_{n+1}(G) (i = 0, \ldots, n)$, defined by:

$$
\mu_i(x_1 \otimes \cdots \otimes x_{n+1}) := \begin{cases} 
  x_2 \otimes \cdots \otimes x_{n+1} & \text{if } i = 0, \\
  x_1 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_{n+1} & \text{if } 0 < i \leq n,
\end{cases}
$$

$$
\epsilon_i(x_1 \otimes \cdots \otimes x_{n+1}) := x_1 \otimes \cdots \otimes x_i \otimes 1 \otimes x_{i+1} \otimes \cdots \otimes x_{n+1}.
$$

The chain complex associated with $S_2(G)$ is the bar resolution $(D^{\otimes n+1}, b'_n)$ of the trivial right $D$-module $\mathbb{Z}$, and the chain complex $(\mathcal{D}^{\otimes n}, D^\otimes, b'_n)$, obtained dividing $(D^{\otimes n+1}, b'_n)$ by the subcomplex generated by images of the degeneracy maps is the bar normalized resolution of $\mathcal{D}$.

Let $\Upsilon$ be the family of all the epimorphism of right $D$-modules which split as morphisms of abelian groups. We say that a right $D$-module $X$ is $\Upsilon$-relative projective if for each $f : Y_1 \rightarrow Y_2$ in $\Upsilon$ and each right $D$-module map $g : X \rightarrow Y_2$, there exists a right $D$-module map $h : X \rightarrow Y_1$ such that $g = fh$. It is well known that a right $D$-module $X$ is $\Upsilon$-relative projective if and only if there exists an abelian group $X'$ such that $X$ is a direct sum of $X' \otimes D$. A complex of right $D$-modules $(X_*, d_*)$ is a $\Upsilon$-relative projective resolution of $\mathcal{D}$ if each $X_n$ is $\Upsilon$-relative projective, and there exists a right $D$-module morphism $\pi : X_0 \rightarrow \mathcal{D}$ such that

$$
\mathcal{D} \leftarrow \pi X_0 \leftarrow d_1 \mathcal{X} \leftarrow d_2 \mathcal{X} \leftarrow d_3 \mathcal{X} \leftarrow d_4 \mathcal{X} \leftarrow d_5 \mathcal{X} \leftarrow \cdots,
$$

is contractile as complex of abelian groups. The complex $(\mathcal{D}^{\otimes n} \otimes D, b'_n)$ is an $\Upsilon$-relative projective resolution of $\mathcal{D}$. Let $\pi : D \rightarrow \mathcal{D}$ be the augmentation map. A contracting homotopy of

$$
\mathcal{D} \leftarrow \pi D \leftarrow b'_n \mathcal{D} \otimes D \leftarrow b'_2 \mathcal{D} \otimes D \leftarrow b'_3 \mathcal{D} \otimes D \leftarrow b'_4 \mathcal{D} \otimes D \leftarrow \cdots,
$$

as a complex of abelian groups, is the degree 1 map $\xi_n$, given by $\xi_n(x) := (-1)^{n+1}x \otimes 1$ for $x \in D^{\otimes n} \otimes D$. Using relative projective resolutions, a theory of relative derived functors can be developed, which is similar to the standard one (see [20]). Thus, we can define the group homology of $G$ with coefficients in a left $D$-module $M$ as $\text{Tor}_{\mathcal{D}}(D, M)$, and use relative derived functors like $\text{Hom}_{\mathcal{D}}$. Consequently, the group homology $H_\Upsilon(G, M)$, of $G$ with coefficients in $M$, is the homology of $(D \otimes \mathcal{D}^{\otimes *}, b'_n) \otimes_D M$. There are canonical identifications $\Gamma_n : \mathcal{D}^{\otimes n} \otimes M \rightarrow (\mathcal{D}^{\otimes n} \otimes D) \otimes_D M$, given by $\Gamma_n(x \otimes m) := (x \otimes 1) \otimes_D m$. Using them we obtain that $(\mathcal{D}^{\otimes n} \otimes D, b'_n) \otimes_D M \simeq (D \otimes \mathcal{D}^{\otimes n} \otimes M, b_n)$, where

$$
b_n(x_1 \otimes \cdots \otimes x_n \otimes m) := x_2 \otimes \cdots \otimes x_n m + \sum_{i=1}^{n-1} (-1)^i x_1 \otimes \cdots \otimes x_{i-1} x_i x_{i+1} \otimes \cdots \otimes x_n m + (-1)^n x_1 \otimes \cdots \otimes x_{n-1} \otimes x_n m.
$$

We call $(\mathcal{D}^{\otimes n} \otimes M, b_n)$ the canonical normalized complex of $G$ with coefficients in $M$. Since the right $D$-modules $D^{\otimes n+1}$ and $\mathcal{D}^{\otimes n} \otimes D$ are projective, the group homology can be defined using the usual functor $\text{Tor}$. The main purpose of our comment on relative derived functors above is to make subsection 2.1 and the reference [17] more understandable.
1.2 Linear cycle sets

A linear cycle set $\mathcal{A} := (A; \cdot)$ is an abelian additive group $A$, endowed with a binary operation $\cdot$ such that the left translations $a \mapsto a' \cdot a$ are permutations of $A$ and the following conditions are fulfilled

$$
(a \cdot a') \cdot (a' \cdot a') = (a' \cdot a) \cdot (a' \cdot a'), \quad [1.1]
$$

$$
a \cdot (a' + a'') = a \cdot a' + a \cdot a'', \quad [1.2]
$$

$$
(a + a') \cdot a'' = (a \cdot a') \cdot (a \cdot a''), \quad [1.3]
$$

We will use multiplicative notation. Let $G_A := \{X^a : a \in A\}$, endowed with the group structure given by $X^a X^{a'} := X^{a+a'}$. We set $X^a \cdot X^{a'} := X^a X^{a'}$ and $X^a \cdot a' := a \cdot a'$.

1.2.1 The normalized full linear cycle set (co)homology

In [19, Section 4] the authors introduce theories of (co)homology, $H^*_N(A, \Gamma)$ and $H^*_N(A, \Gamma)$, that we recall now. For each $s \geq 1$, we let $sh(D^\otimes s)$ denote the subgroup of $D^\otimes s$ generated by the shuffles

$$
\sum_{\sigma \in sh_{s-1}} sg(\sigma) d_{s-1(1)} \otimes \cdots \otimes d_{s-1(s)},
$$

taken for all $1 \leq t < s$ and $d_k \in D$. Here $sh_{s-1}$ is the subset of all the permutations $\sigma$ of $s$ elements satisfying $\sigma(1) < \cdots < \sigma(l)$ and $\sigma(l+1) < \cdots < \sigma(s)$. For each $r \geq 0$ and $s \geq 1$, let $\hat{C}^N_r(A, \mathcal{Z}) := D^\otimes s \otimes \mathcal{M}(s)$, where $\mathcal{M}(s) := D^\otimes s_{sh_{s-1}}$. Given $g_1, \ldots, g_s \in G_A$, we let $[g_1 \otimes \cdots \otimes g_s]$ denote the class of $g_1 \otimes \cdots \otimes g_s$ in $\mathcal{M}(s)$. Consider the double complex $(\hat{C}^N_r(A, \mathcal{Z}), \partial^{rb}_r, \partial^s_r)$, where

$$
\partial^{rb}_r(g_1 \otimes \cdots \otimes g_r \otimes [g_{r+1} \cdots \otimes g_{r+s}]) := g_1 \cdot g_2 \cdots g_r \otimes [g_1 \cdot g_2 \cdots g_r \cdot g_{r+1} \cdots g_{r+s}]
$$

$$
+ \sum_{j=1}^{r-1} (-1)^j g_1 \otimes \cdots \otimes g_{j-1} \otimes g_j g_{j+1} \otimes g_{j+2} \otimes g_r \otimes [g_{r+1} \cdots \otimes g_{r+s}]
$$

$$
+ (-1)^r g_1 \otimes \cdots \otimes g_r \otimes [g_{r+1} \cdots \otimes g_{r+s}]
$$

and

$$
\partial^s_r(g_1 \otimes \cdots \otimes g_r \otimes [g_{r+1} \cdots \otimes g_{r+s}]) := (-1)^{r+1} g_1 \otimes \cdots \otimes g_r \otimes [g_{r+1} \cdots \otimes g_{r+s}]
$$

$$
+ \sum_{j=r+1}^{r+s-1} (-1)^j g_1 \otimes \cdots \otimes g_{j-1} \otimes g_j g_{j+1} \otimes g_{j+2} \cdots \otimes g_{r+s}
$$

$$
+ (-1)^{r+s+1} g_1 \otimes \cdots \otimes g_r \otimes [g_{r+1} \cdots \otimes g_{r+s-1}].
$$

Recall that the total complex of $(\hat{C}^N_r(A, \mathcal{Z}), \partial^{rb}_r, \partial^s_r)$ is the chain complex $(\hat{C}^N_r(A, \mathcal{Z}), \partial_r)$, where

$$
\hat{C}^N_r(A, \mathcal{Z}) := \bigoplus_{r+s=n} \hat{C}^N_{r+s}(A, \mathcal{Z}) \quad \text{and} \quad \partial_r [\hat{C}^N_{r+s}(A, \mathcal{Z})] := \partial^{rb}_r + \partial^s_r.
$$

Let $\Gamma$ be an abelian additive group. The normalized full homology groups and the normalized full cohomology groups of $A$ with coefficients in $\Gamma$ are the homology groups of $\hat{C}^N_*(A, \Gamma)$ := $\Gamma \otimes (\hat{C}^N_*(A, \mathcal{Z}), \partial_*)$ and the cohomology groups of $\hat{C}^N_*(A, \Gamma)$ := Hom$(\hat{C}^N_*(A, \mathcal{Z}), \partial_*)$, respectively. We let $\hat{H}^N_*(A, \Gamma)$ and $\hat{H}^*_N(A, \Gamma)$ denote the full normalized homology and the full normalized cohomology, of $A$ with coefficients in $\Gamma$.

Remark 1.1. The complex $\hat{C}^N_*(A, \Gamma)$ is not the complex $(C_*(A, \Gamma), \partial_*)$ introduced in [19, Definition 4.2], but they are isomorphic via the maps $\Xi_r : \hat{C}^N_{r+s}(A, \Gamma) \rightarrow C^N_{r+s}(A, \Gamma)$, given by

$$
\Xi_r(X^{a_1} \otimes \cdots \otimes X^{a_r} \otimes [X^{a_{r+1}} \otimes \cdots \otimes X^{a_{r+s}}]) := (a_1, \ldots, a_s, a_{s+1}, \ldots, a_{r+s}).
$$

Similarly, $\hat{C}^N_*(A, \Gamma) \simeq (C^N_*(A, \Gamma), \partial^*)$, and so $\hat{H}^N_*(A, \Gamma) = H^N_*(A, \Gamma)$ and $\hat{H}^*_N(A, \Gamma) = H^*_N(A, \Gamma)$.

1.3 The perturbation lemma

Next, we recall the perturbation lemma. We present the version given in [6].

A special deformation retract

$$
(X_s, d_s) \xrightarrow{p_s} (C_s, \partial_s) \quad C_s \xrightarrow{h_{s+1}} C_{s+1}, \quad [1.4]
$$

consists of the following:
(1) Chain complexes \((X, d), (C, \partial)\) and morphisms \(\iota, \rho\) between them, such that \(p_1 = \text{id}\).

(2) A homotopy \(h\) from \(\iota \rho\) to \id, such that \(h_1 = 0, \rho h = 0\) and \(hh = 0\).

A perturbation of \([1.4]\) is a map \(\delta_*: C_* \to C_{* - 1}\) such that \((\partial + \delta)^2 = 0\). We call it small if \(\delta d\) is invertible. In this case we write \(A := (\id - \delta d)^{-1} \delta\) and we consider the diagram

\[
\begin{array}{c}
\text{(1.5)}
\end{array}
\]

where \(d^1 := d + pA i, \ i^1 := i + h A i, \ p^1 := p + p A h\) and \(h^1 := h + h A h\).

In all the cases considered in this paper the morphism \(\delta h\) is locally nilpotent (in other words, for all \(x \in C_*\) there exists \(n \in \mathbb{N}\) such that \((\delta h)^n(x) = 0\). Consequently, \((\id - \delta h)^{-1} = \sum_{n=0}^{\infty} (\delta h)^n\).

**Theorem 1.2** ([6]). If \(\delta\) is a small perturbation of \([1.4]\), then the diagram \([1.5]\) is a special deformation retract.

**Proposition 1.3.** Consider morphisms of double complexes

\[
\begin{array}{c}
(X, d^k_\ast, d^r_\ast) \xrightarrow{p_* \delta_{\ast, r} \partial_{\ast, r}} (C, \partial_{\ast, r}),
\end{array}
\]

such that \(p_* 1_* = \text{id}\). Assume that in each row \(s\) we have a special deformation retract

\[
\begin{array}{c}
(X, d^k_\ast, d^r_\ast) \xrightarrow{p_* \delta_{\ast, r} \partial_{\ast, r}} (C, \partial_{\ast, r}),
\end{array}
\]

deeded with a small perturbation \(\delta_{\ast, r}: C_* \to C_{* - 1, 5}\). Let \(A_{\ast, r} := (\id - \delta_{\ast, r} h_{\ast, r})^{-1} \delta_{\ast, r}\) and consider the diagram

\[
\begin{array}{c}
(X, d^k_\ast, d^r_\ast) \xrightarrow{p_* \delta_{\ast, r} \partial_{\ast, r}} (C, \partial_{\ast, r}),
\end{array}
\]

where \(d^1 := d + p \partial A t, \ i^1 := i + h A t, \ p^1 := p + p A h\) and \(h^1 := h + h A h\). The following facts hold:

1. The maps \(i_* \) and \(p_* \) are morphisms of double complexes such that \(p_* i_* = \text{id}\).

2. For each row \(s\), the map \(h_{s+1, 5} = \text{id}\) from \(i_* p_* \) to \(i_* p_*\).

**Proof.** Let \((X, d^k_\ast, d^r_\ast)\) and \((C, \partial_{\ast, r})\) be the total chain complexes of \((X, d^k_\ast, d^r_\ast)\) and \((C, \partial_{\ast, r}), \) respectively. We have an homotopy equivalence data

\[
\begin{array}{c}
(X, d^k_\ast, d^r_\ast) \xrightarrow{p_* \delta_{\ast, r} \partial_{\ast, r}} (C, \partial_{\ast, r}),
\end{array}
\]

where \(i_* \) and \(p_* \) are given by \(i_* := \bigoplus_{r+\neq s} \partial_{r+\neq s} \) and \(p^1 := \bigoplus_{r+\neq s} p_{r+\neq s}\). Consider the small perturbation \(\delta^s_*: C_* \to C_{* - 1, 5}\), given by \(\delta^s_* := \bigoplus_{r+\neq s} \delta^s_{r+\neq s}\). The result follows immediately by applying the perturbation lemma to this case.

\[
\text{□}
\]

2 A complex for the group homology of cyclic groups

Let \(C_v\) be a cyclic group of order \(v \in \mathbb{N}\) and let \(D := \mathbb{Z}[C_v]\). In this section we construct a chain complex suitable for our purposes, giving the group homology of \(C_v\) with coefficients in an abelian group \(M\), considered as a left \(D\)-module via the trivial action. This complex is the complex \((X, d^k_\ast, d^r_\ast)\) in a special deformation retract as in \([1.4]\) in which \((C, \partial_{\ast, r})\) is the normalized bar complex of \(C_v\) with coefficients in \(M\). It is natural to try to use the minimal resolution of \(C_v\) in order to construct \((X, d^k_\ast, d^r_\ast)\), but this does not work because, in this case, the perturbation is not small. So we are forced to use a more involved complex.

2.1 A resolution for a cyclic group

Let \(v, u, t \in \mathbb{N}\) such that \(u > 1\) and \(ut = v\). Consider the cycle groups \(C_v := \langle q \rangle, C_u := \langle x \rangle\) and \(C_t := \langle y \rangle\) of order \(v, u\) and \(t\), respectively. The group \(C_v\) is isomorphic to the crossed product \(C_u \times z C_t\), in which \(C_t\) acts trivially on \(C_u\) and \(z\) is the cocycle given by

\[
\zeta(y^j, y^j') := \begin{cases} 1 & \text{if } j + j' < t, \\ x & \text{otherwise,} \end{cases}
\]
where $0 \leq j, j' < t$. We recall that $C_u \rtimes C_t = \{ x^i w_y : 0 \leq i < u \}$ endowed with the multiplication map
\[ x^i w_y, x^j w_{y'} = x^{i+j'}((y')^i) w_{y+j'} \quad \text{where} \quad 0 \leq j, j' < t. \]

The map $f : C_u \rtimes C_t \rightarrow C_u$, defined by $f(x^i w_y) = g^{i+j}$, where $0 \leq j < t$, is an group isomorphism.

Let $E := \mathbb{Z}[C_u \rtimes C_t]$. For all $\alpha, \beta \geq 0$, let $Y_\beta := \mathbb{Z}[C_t]$ and $X_{0, \beta} := E$. The groups $X_{0, \beta}$ and $Y_\beta$ are right $E$-modules via the right regular action and the groups $Y_\beta$ are right $E$-modules via $y^i x^j w_y := y^{i+j}$. Consider the diagram of right $E$-modules and right $E$-module maps
\[ \cdots \xrightarrow{0} X_2 \xleftarrow{\partial_1} X_1 \xleftarrow{\partial_0} X_0 \xrightarrow{0} \cdots \]
where $v_\beta(w_1) := 1$ and
\[ \partial_{2 \beta-1}(1) := y - 1, \quad \partial_{2 \beta}(1) := \sum_{l=0}^{t-1} y^l, \quad d_{2 \alpha-1, \beta}(w_1) := x w_1 - w_1 \quad \text{and} \quad d_{2 \alpha, \beta}(w_1) := \sum_{l=0}^{u-1} x^i w_y. \]

Clearly, the columns and the rows of this diagram are chain complexes.

**Proposition 2.1.** Each one of the rows of the above diagram is contractable as a complex of abelian groups. A contracting homotopy $\sigma_{0, \beta}^0 : Y_\beta \rightarrow X_{0, \beta}$ and $\sigma_{\alpha+1, \beta}^0 : X_{0, \beta} \rightarrow X_{\alpha+1, \beta}$ for $\alpha \geq 0$, of the $\beta$-th row, is given by
\[ \sigma_{0, \beta}^0(y') := w_y, \quad \sigma_{2 \alpha-1, \beta}(x^i w_y) := \sum_{l=0}^{t-1} x^l w_y, \quad \text{and} \quad \sigma_{2 \alpha, \beta}(x^i w_y) := \delta_{i, u-1} w_y \]

where $0 \leq i < u$ and $\delta_{i, u-1}$ is the delta of Kronecker.

**Proof.** We must check that
\[ v_\beta \sigma_{0, \beta}^0 = \text{id}_{Y_\beta}, \quad \sigma_{0, \beta}^0 v_\beta + d_{1 \beta}^0 \sigma_{1 \beta}^0 = \text{id}_{X_{0, \beta}} \quad \text{and} \quad \sigma_{\alpha, \beta}^0 d_{\alpha, \beta}^0 + d_{\alpha+1, \beta}^0 \sigma_{\alpha+1, \beta}^0 = \text{id}_{X_{\alpha, \beta}}. \]  \[\text{[2.1]}\]

A direct computation shows that
\[ v_\beta \sigma_{0, \beta}^0(y') = v_\beta(w_y) = y', \quad \sigma_{0, \beta}^0 v_\beta(x^i w_y) = \sigma_{0, \beta}^0(y') = w_y, \]
\[ d_{2 \alpha+1, \beta}^0 \sigma_{2 \alpha+1, \beta}^0(x^i w_y) = \sum_{l=0}^{t-1} d_{2 \alpha+1, \beta}^0(x^l w_y) = x^i w_y - w_y, \]
\[ d_{2 \alpha, \beta}^0 \sigma_{2 \alpha, \beta}^0(x^i w_y) = \delta_{i, u-1} d_{2 \alpha, \beta}^0(w_y) = \delta_{i, u-1} \sum_{l=0}^{t-1} x^l w_y \]
\[ \sigma_{2 \alpha+1, \beta}^0 d_{2 \alpha+1, \beta}^0(x^i w_y) = \sigma_{2 \alpha+1, \beta}^0(x^{i+1} w_y) = \begin{cases} x^i w_y & \text{if } 0 \leq i < u - 1, \\ - \sum_{l=0}^{i} x^l w_y & \text{if } i = u - 1, \end{cases} \]
\[ \sigma_{2 \alpha, \beta}^0 d_{2 \alpha, \beta}^0(x^i w_y) = \sum_{l=0}^{u-1} \sigma_{2 \alpha, \beta}^0(x^{i+l} w_y) = w_y. \]

Equalities [2.1] follows immediately from these facts. \( \square \)

**Proposition 2.2.** Consider $\mathbb{Z}$ as a right $E$-module via the trivial action. The complex of right $E$-modules
\[ \mathbb{Z} \xrightarrow{0} Y_0 \xleftarrow{\partial_1} Y_1 \xleftarrow{\partial_2} Y_2 \xleftarrow{\partial_3} Y_3 \xleftarrow{\partial_4} Y_4 \xleftarrow{\partial_5} Y_5 \xleftarrow{\partial_6} \cdots, \]
where \( \pi \) is the right \( E \)-module morphism given by \( \pi(w_1) := 1 \), is contractible as a complex of abelian groups. A contracting homotopy \( \sigma_0^{-1} : E \to Y_0 \) and \( \sigma_{\beta+1}^{-1} : Y_\beta \to Y_{\beta+1} \) for \( \beta \geq 0 \), is given by

\[
\sigma_0^{-1}(1) := w_1, \quad \sigma_{2\beta}^{-1}(y^l) := \delta_{l-1,j} \quad \text{and} \quad \sigma_{2\beta+1}^{-1}(y^l) := \sum_{i=0}^{j-1} y^i
\]

where \( 0 \leq j < t \).

**Proof.** A direct computation shows that

\[
\pi \sigma_0^{-1}(1) = \pi(1) = 1,
\]

\[
\sigma_0^{-1}(y^l) = \sigma_0^{-1}(1) = 1,
\]

\[
\partial_{2\beta+1} \sigma_{2\beta+1}^{-1}(y^l) = \sum_{i=0}^{j-1} \partial_{2\beta+1}(y^l) = y^l - 1,
\]

\[
\partial_{2\beta} \sigma_{2\beta}^{-1}(y^l) = \sigma_{2\beta} \delta_{l-1,j} = \delta_{l-1,j} \sum_{i=0}^{j-1} y^l,
\]

\[
\sigma_{2\beta+1}^{-1} \partial_{2\beta+1}(y^l) = \sigma_{2\beta+1}(y^{l+1} - y^l) = \begin{cases} y^l & \text{if } j < t - 1, \\ -\sum_{i=0}^{t-2} y^i & \text{if } j = t - 1 \end{cases}
\]

\[
\sigma_{2\beta}^{-1} \partial_{2\beta}(y^l) = \sum_{i=0}^{t-1} \sigma_{2\beta}^{-1}(y^i) = 1.
\]

The result follows easily from these facts. \( \square \)

For \( \alpha \geq 0 \) and \( 1 \leq l \leq \beta \), we define right \( E \)-module maps \( d_{\alpha\beta}^l : X_{\alpha\beta} \to X_{\alpha+l-1,\beta-l} \), recursively by:

\[
d^l(w_1) := \begin{cases} -\sigma^0 \partial \nu(w_1) & \text{if } l = 1 \text{ and } \alpha = 0, \\ -\sigma^0 d^l d^0(w_1) & \text{if } l = 1 \text{ and } \alpha > 0, \\ -\sum_{j=1}^{l-1} \sigma^0 d^{l-j} d^l(w_1) & \text{if } l < 1 \text{ and } \alpha = 0, \\ -\sum_{j=0}^{l-1} \sigma^0 d^{l-j} d^l(w_1) & \text{if } l < 1 \text{ and } \alpha > 0. \end{cases}
\]

**Theorem 2.3.** Let \( \Upsilon \) be the family of all the epimorphism of right \( E \)-modules which split as morphisms of abelian groups. The chain complex

\[
\mathbb{Z} \leftarrow X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow X_3 \leftarrow X_4 \leftarrow \cdots,
\]

where \( \pi_E \) is the augmentation map, \( X_n := \bigoplus_{\alpha + \beta = n} X_{\alpha\beta} \) and \( d_n \) is the right \( E \)-module map defined by

\[
d_n(x) := \begin{cases} \sum_{i=1}^{n} d_{i0}(x) & \text{if } x \in X_{0n}, \\ \sum_{i=0}^{n} d_{i,n-\alpha}(x) & \text{if } x \in X_{\alpha,n-\alpha} \text{ with } \alpha > 0, \end{cases}
\]

is a \( \Upsilon \)-relative projective resolution of \( E \).

**Proof.** This is an immediate consequence of [17, Corollary A2]. \( \square \)

**Remark 2.4.** In the previous definition and in the rest of this work we identify each \( X_{rs} \) with its image inside \( X_{r+s} \).

In order to carry out our computations we also need to give an explicit \( \mathbb{Z} \)-linear contracting homotopy of this resolution. For this we define morphisms of abelian groups

\[
\sigma_{l,\beta-1}^i : Y_\beta \to X_{l,\beta-1} \quad \text{and} \quad \sigma_{\alpha+l+1,\beta-1}^i : X_{\alpha\beta} \to X_{\alpha+l+1,\beta-1},
\]

recursively by \( \sigma_{\alpha+l+1,\beta-1}^i := -\sum_{i=0}^{l-1} \sigma^0 d^{l-i} \sigma^i \) \((0 < l \leq \beta \text{ and } \alpha \geq -1)\).
Proposition 2.5. A contracting homotopy \( \sigma_0: E \to X_0 \) and \( \sigma_{n+1}: X_n \to X_{n+1} \) \((n \geq 0)\), of the resolution introduced in Theorem 2.3, is given by \( \sigma_0(x) := \sigma_{00}^{-1}(x) \) and
\[
\sigma_{n+1}(x) := \begin{cases} 
- \sum_{l=0}^{n+1} \sigma_{l,n-l+1}^{-1} v_{n}(x) + \sum_{l=0}^{n} \sigma_{l+1,n-l}(x) & \text{if } x \in X_{0n}, \\
\sum_{l=0}^{n} \sigma_{l+n-1,n-l}(x) & \text{if } x \in X_{n,n} \text{ with } \alpha > 0.
\end{cases}
\]

Proof. This is a direct consequence of [17, Corollary A2]. \( \Box \)

The following theorem gives a closed expression of the homomorphisms \( d_{\alpha \beta} \) that appear in the relative projective resolution of \( E \), obtained above.

Theorem 2.6. The maps \( d^l \) vanish for \( l > 2 \). Moreover
\[
d^1_{0,2\beta-1}(w_1) = (-1)^\alpha (w_1 - w_y), \quad d^2_{0,\beta}(w_1) = -w_1,
\]
\[
d^1_{0,2\beta}(w_1) = (-1)^{\alpha+1} \sum_{h=0}^{t-1} w_{y^h}, \quad d^2_{2\alpha+1,\beta}(w_1) = 0.
\]

Proof. We sketch the proof. We first prove the formula for \( d^1_{\alpha \beta} \) by induction on \( \alpha \). By equality [2.2],
\[
d^1_{0,2\beta+1}(w_1) = -\sigma^0_{0,2\beta} \partial_{2\beta+1} v_{2\beta+1}(w_1) = w_1 - w_y
\]
and
\[
d^1_{0,2\beta}(w_1) = -\sigma^0_{0,2\beta} \partial_{2\beta} v_{2\beta}(w_1) = -\sum_{h=0}^{t-1} w_{y^h},
\]
which proves the case \( \alpha = 0 \). Assume the formula is true for \( \alpha \). Then
\[
d^1_{0,1,2\beta+1}(w_1) = -\sigma^0_{0,1,2\beta} d^1_{0,2\beta+1} d^0_{0,1,2\beta+1}(w_1) = (-1)^{\alpha+1} (w_1 - w_y)
\]
and
\[
d^1_{0,1,2\beta}(w_1) = -\sigma^0_{0,1,2\beta} d^1_{0,2\beta} d^0_{0,1,2\beta}(w_1) = (-1)^{\alpha} \sum_{h=0}^{t-1} w_{y^h},
\]
as desired. We next prove the formula for \( d^2_{\alpha \beta} \). For \( \alpha = 0 \), we have
\[
d^2_{0,\beta}(w_1) = -\sigma^0_{1,\beta-2} d^1_{0,\beta-1} d^1_{0,\beta}(w_1) = -w_1.
\]
Assume the formula is true for \( \alpha \). Then
\[
d^2_{0,1,\beta}(w_1) = -\sigma^0_{0,2,\beta-2} (d^2_{\alpha \beta} d^0_{0,1,\beta} + d^1_{0,1,\beta-1} d^1_{0,1,\beta})(w_1) = \begin{cases} 
0 & \text{if } \alpha \text{ is even}, \\
-w_1 & \text{if } \alpha \text{ is odd},
\end{cases}
\]
as desired. Finally, since
\[
\sigma^0_{0,2,\beta-3} d^1_{0,1,\beta-1} d^0_{0,\beta}(w_1) = \sigma^0_{0,2,\beta-3} d^1_{0,1,\beta-1} d^1_{0,\beta}(w_1) = \sigma^0_{0,2,\beta-4} d^2_{0,1,\beta-2} d^2_{0,\beta}(w_1) = 0,
\]
from equality [2.2] it follows that \( d^l = 0 \) for \( l > 2 \). \( \Box \)

Proposition 2.7. The homotopy \( \nu \) found in Proposition 2.5 satisfies
\[
\sigma_{n+1}(x) = -\sigma^{-1}_{0,n+1} v_n(x) + \sum_{l=0}^{n} \sigma_{l+1,n-l}(x) \quad \text{for all } x \in X_0n.
\]

Proof. By the definitions of \( \nu \), \( v \) and \( \sigma^{-1} \), it suffices to prove that
\[
\sigma^{l}_{1,2\beta-1}(1) = 0 \quad \text{and} \quad \sigma^{l}_{1,2\beta+1-l}(y^k) = 0 \quad \text{for all } l \geq 1 \text{ and } 0 \leq k < t-1. \tag{2.3}
\]
By the definition of \( \sigma^l \) and Theorem 2.6, for this it sufficient to consider the cases \( l = 1 \) and \( l = 2 \). We have
\[
\sigma^{1}_{1,2\beta-1}(1) = -\sigma^0_{1,2\beta-1} d^1_{0,2\beta} \sigma^0_{0,2\beta}(1) = 0 \quad \text{and} \quad \sigma^{1}_{1,2\beta}(y^k) = -\sigma^0_{1,2\beta} d^1_{0,2\beta} \sigma^0_{0,2\beta}(y^k) = 0.
\]
Therefore,
\[
\sigma^{2}_{2,2\beta-2}(1) = -\sigma^0_{2,2\beta-2} d^2_{0,2\beta} \sigma^0_{0,2\beta}(1) = 0 \quad \text{and} \quad \sigma^{2}_{2,2\beta-1}(y^k) = -\sigma^0_{2,2\beta-1} d^2_{0,2\beta-1} \sigma^0_{0,2\beta-1}(y^k) = 0,
\]
which finishes the proof. \( \Box \)
Remark 2.8. Let $0 \leq j < t$. A direct computation shows that

$$
\sigma^0 \sigma^{-1} v(x^i w_y) = \begin{cases} 
\sum_{i=0}^{j-1} w_y & \text{if } x^i w_y \in X_{0,2\beta}, \\
\delta_{i-1,j} w_1 & \text{if } x^i w_y \in X_{0,2\beta+1}.
\end{cases}
$$

**Proposition 2.9.** Let $0 \leq i < u$ and $0 \leq j < t$. For all $\alpha \geq 0$ and $\beta \geq 1$, we have

$$
\sigma^1_{\alpha+2,2\beta}(x^i w_y) = (-1)^{\alpha+1} \delta_{-1,i} \delta_{i-1,j} w_1 \quad \text{and} \quad \sigma^1_{\alpha+2,2\beta-1}(x^i w_y) = (-1)^{\alpha+1} \delta_{-1,i} \sum_{i=0}^{j-1} w_y.
$$

Moreover, $\sigma^1_{\alpha+l+1,\beta-1} = 0$ for all $l \geq 2$, $\alpha \geq 0$ and $\beta \geq 1$.

**Proof.** We sketch the proof. By the definition of $\sigma^1$ above Proposition 2.5, we have

$$
\sigma^1_{\alpha+2,2\beta}(x^i w_y) = -\sigma^0_{\alpha+2,2\beta+1} d^1_{\alpha+2,2\beta+1}(x^i w_y) = (-1)^{\alpha+1} \delta_{-1,i} \delta_{i-1,j} w_1
$$

and

$$
\sigma^1_{\alpha+2,2\beta-1}(x^i w_y) = -\sigma^0_{\alpha+2,2\beta-1} d^1_{\alpha+2,2\beta-1}(x^i w_y) = (-1)^{\alpha+1} \delta_{-1,i} \sum_{i=0}^{j-1} w_y,
$$

which proves the statement for $\sigma^1$. Our next purpose is to prove that $\sigma^2_{\alpha+3,3\beta-2} = 0$. We assert that $\sigma^2_{\alpha+3,3\beta-2} d^2_{\alpha+1,\beta} = 0$. In fact, if $\alpha$ is even this follows from the fact that $d^2_{\alpha+1,\beta} = 0$, while if $\alpha$ is odd, then the assertion is also true, because

$$
\sigma^2_{\alpha+3,3\beta-2} = -\sigma^0_{\alpha+3,3\beta-2} d^2_{\alpha+1,\beta} = 0,
$$

as desired. Since, moreover $d^3 = 0$, in order to prove that $\sigma^3_{\alpha+4,3\beta-3} = 0$ it suffices to check that the equality $\sigma^2_{\alpha+4,3\beta-3} d^2_{\alpha+2,\beta-1} = 0$. If $\alpha$ is odd this follows from the fact that $d^2_{\alpha+2,\beta-1} = 0$, while if $\alpha$ is odd, then a direct computation proves that we also have $\sigma^2_{\alpha+4,3\beta-3} d^2_{\alpha+2,\beta-1} = 0$. The proof that $\sigma^1_{\alpha+l+1,\beta-1} = 0$ for $l \geq 4$, follows easily by induction. \(\square\)

### 2.1.1 Comparison with the normalized bar resolution

Let $(\mathcal{E}^{\otimes n} \otimes E, b'_n)$ be the normalized bar resolution of $\mathbb{Z}$ as a trivial right $E$-module. It is easy to see that there exist unique morphisms of right $E$-module chain complexes

$$
\phi_* : (X_*, d_*) \longrightarrow (\mathcal{E}^{\otimes n} \otimes E, b'_n) \quad \text{and} \quad \varphi_* : (\mathcal{E}^{\otimes n} \otimes E, b'_n) \longrightarrow (X_*, d_*),
$$

such that

- $\phi_0 = \varphi_0 = \text{id}_E$,
- $\varphi_{n+1}(x \otimes w_1) = \sigma_{n+1} \varphi_n b'_{n+1}(x \otimes w_1)$ for all $n \geq 0$ and $x \in \mathcal{E}^{\otimes n+1}$,
- the restriction of $\phi_n$ to $X_{n+1}$ satisfies $\phi_{n+1}(w_1) = \xi_{n+1} \phi_n d_{n+1}(w_1)$, where $\xi_{n+1}$ is as in subsection 1.1.

**Proposition 2.10.** $\varphi_*, \phi_* = \text{id}$ and $\phi_* \varphi_*$ is homotopically equivalent to the identity map. A homotopy is the one degree map $\omega_{n+1} : \phi_* \varphi_* \rightarrow \text{id}$, recursively defined by

$$
\omega_1 = 0 \quad \text{and} \quad \omega_{n+1}(y) := \xi_{n+1} (\phi_n \varphi_n - \text{id} - \omega_n b'_n)(y) \quad \text{for } n \geq 0 \quad \text{and} \quad y \in \mathcal{E}^{\otimes n} \otimes \mathbb{Z} w_1. \tag{2.4}
$$

Moreover, $\varphi_* \omega_1 = 0$, $\omega_{n+1} \phi_* = 0$ and $\omega_{n+1} \omega_* = 0$.

**Proof.** We prove the first two assertions by induction. Clearly $\varphi_0 \phi_0 = \text{id}$. Assume that $\varphi_n \phi_n = \text{id}$. Since the image of $\xi_{n+1}$ is included in $\mathcal{E}^{\otimes n+1} \otimes \mathbb{Z} w_1$, we have

$$
\varphi_{n+1} \phi_{n+1}(y) = \sigma_{n+1} \varphi_n b'_{n+1} \xi_{n+1} \phi_n d_{n+1}(y) = \sigma_{n+1} \varphi_n \phi_n d_{n+1}(y) = \sigma_{n+1} \phi_n d_{n+1}(y) = y - d_{n+2} \sigma_{n+2}(y),
$$

for $y \in X_{n+1} \otimes \mathbb{Z} w_1$. So, to conclude that $\varphi_{n+1} \phi_{n+1} = \text{id}$ it suffices to check that $\sigma_{n+2}(w_1) = 0$, which follows easily from Remark 2.8 and Propositions 2.7 and 2.9. Next we prove the second assertion. Clearly...
\(\phi \varphi - \text{id} = 0 = b_1 \omega_1\). Let \(U_n := \phi_n \varphi_n - \text{id}\) and \(T_n := U_n - \omega_n b_n\). Assuming that \(b_n \omega_n + \omega_{n-1} b_{n-1} = U_{n-1}\), we get that
\[
b_{n+1} \omega_{n+1}(y) + \omega_n b_n(y) = b_{n+1} \xi_{n+1} T_n(y) + \omega_n b_n(y)
\]
\[
= T_n(y) - \xi_n b_n T_n(y) + \omega_n b_n(y)
\]
\[
= U_n(y) - \xi_n b_n U_n(y) + \xi_n \omega_n b_n(y)
\]
\[
= U_n(y) - \xi_n U_{n-1} b_n(y) + \xi_n \omega_n b_n(y)
\]
\[
= U_n(y) - \xi_n U_{n-1} b_n(y) + \xi_n U_{n-1} b_n(y) - \xi_n \omega_n - 1 b_n b_n(y)
\]
\[
= U_n(y),
\]
for \(y \in E \otimes^n \mathbb{Z} w_1\), where the first equality holds by identity [2.4]; the second one, since \(\xi\) is a contracting homotopy; the third one, by the definition of \(T_n\); the fourth one, since \(U_n\) is a morphism; and the fifth one, by the assumption.

It remains to prove the last assertions. We check the last equality assuming that \(\omega_{n+1} \phi_n = 0\) and \(\varphi_n \omega_n = 0\), and let the other ones, which are easier, to the reader. It is evident that \(\omega_2 \omega_1 = 0\). Assume that \(n \geq 1\) and \(\omega_{n+1} \omega_n = 0\) and let \(y \in E \otimes^n \mathbb{Z} w_1\). Since
\[
\omega_{n+1} b_{n+1} \omega_{n+1}(y) = \omega_{n+1}(\phi_n \varphi_n(y) - y - \omega_n b_n(y) = -\omega_{n+1}(y),
\]
we have
\[
\omega_{n+2} \omega_{n+1}(y) = \xi_{n+2} (\phi_n \varphi_n - \text{id} - \omega_{n+1} b_{n+1}) \omega_{n+1}(y) = -\xi_{n+2} (\omega_{n+1} + \omega_{n+1} b_{n+1} \omega_{n+1})(y) = 0,
\]
as desired. \(\square\)

For each \(\alpha, \beta, n \in \mathbb{N}\) such that \(\alpha + \beta = n\), we let \(\varphi_n^{\alpha \beta} : E \otimes^n \mathbb{Z} \to X_{\alpha \beta}\) denote the unique map such that \(\varphi_n = \sum_{\alpha + \beta = n} \varphi_n^{\alpha \beta}\).

**Remark 2.11.** A direct computation using Proposition 2.7, Remark 2.8 and the definitions of \(\phi_n\) and \(\varphi_n\), shows that
\[
\phi_1(w_1) = w_y \otimes w_1
\]
\[
\phi_1(w_1) = -xw_1 \otimes w_1
\]
\[
\phi_2(w_1) = - \sum_{h=1}^{t-1} w_y \otimes w_y^h \otimes w_1
\]
\[
\phi_2(w_1) = w_y \otimes xw_1 \otimes w_1 - xw_1 \otimes w_y \otimes w_1
\]
\[
\phi_2(w_1) = - \sum_{h=1}^{u-1} xw_1 \otimes x^h w_1 \otimes w_1
\]
\[
\varphi_1^0 (x^i w_y \otimes w_1) = \sum_{h=0}^{j-1} w_y^h
\]
and
\[
\varphi_1^0 (x^i w_y \otimes w_1) = - \sum_{h=0}^{i-1} x^h w_y^i,
\]
where \(0 \leq i < u\) and \(0 \leq j < t\).

**Remark 2.12.** A direct computation shows that
\[
\omega_2 (x^j w_y \otimes w_1) = \sum_{h=0}^{i-1} xw_1 \otimes x^h w_y^j \otimes w_1 + \sum_{h=1}^{j-1} w_y \otimes w_y^h \otimes w_1,
\]
where \(0 \leq i < u\) and \(0 \leq j < t\).

### 2.2 A complex for the homology of cyclic groups

Let \(v, u, t, C_u \cong \mathbb{Z} C_t\) and \(E\) be as in Section 2.1. Let \(D := \mathbb{Z}[C_u]\) and let \(\mathbb{D} := D/\mathbb{Z} 1\). Recall that the map \(f : C_u \cong \mathbb{Z} C_t \to C_v\), defined by \(f(x^i w_y) := g^{i+j}\), where \(0 \leq j < t\), is a group isomorphism. Here we will obtain a chain complex giving the group homology of \(C_v\) with coefficients in a commutative group.
we will use the following result with φ := \(d_\omega\) and ω := \(2.10\). For each \(\omega\), we have

\[d_{\omega+1,\beta} := 0, \quad d_{\omega,\beta+1} := 0, \quad d_{\omega,\beta} := -1^{\omega+1}t, \quad d_{\omega+1,\beta+1} := 0.\]

By the definition of the maps \(d_{2\omega-1,\beta}\) and \(d_{2\omega,\beta}\), given above Proposition 2.1, and by Theorem 2.6, tensoring \(M\) over \(D\) with \((X_+, d_+)\) and using the identifications \(\theta_{\omega} : M_{\omega} \to X_{\omega} \otimes D M\), given by \(\theta_{\omega}(m) := w_1 \otimes m\), we obtain the chain complex

\[\mathcal{X}_0(M) \xleftarrow{d_1} \mathcal{X}_1(M) \xleftarrow{d_2} \mathcal{X}_2(M) \xleftarrow{d_3} \mathcal{X}_3(M) \xleftarrow{d_4} \mathcal{X}_4(M) \xleftarrow{d_5} \mathcal{X}_5(M) \xleftarrow{d_6} \ldots,\]

where \(\mathcal{X}_n(M) := \bigoplus_{\alpha + \beta = n} M_{\alpha\beta}\) and \(d_n\) is the morphism of abelian groups defined by

\[d_n(m) := \begin{cases} \sum_{l=1}^{\min(n,2)} d_{n-\alpha}(m) & \text{if } m \in M_{0n}, \\ \sum_{l=0}^{\min(n-\alpha,2)} d_{n-\alpha}(m) & \text{if } m \in M_{n0} \text{ with } \alpha > 0. \end{cases}\]

Let \((\mathcal{D}^\otimes \otimes M, b_\ast)\) be the canonical normalized complex of \(C_v\) with coefficients in \(M\). Recall that there is a canonical identification \((\mathcal{D}^\otimes \otimes M, b_\ast) \simeq (\mathcal{D}^\otimes \otimes D, b') \otimes D M\). Let

\[\phi_\ast : (\mathcal{X}_\ast, \partial_\ast) \longrightarrow (\mathcal{D}^\otimes \otimes M, b_\ast) \quad \text{and} \quad \varphi_\ast : (\mathcal{D}^\otimes \otimes M, b_\ast) \longrightarrow (\mathcal{X}_\ast, \partial_\ast)\]

be the morphisms of chain complexes induced by \(\phi_\ast\) and \(\varphi_\ast\), respectively. By definition \(\phi_0 = \varphi_0 = \text{id}_M\). Moreover, by Proposition 2.10 we know that \(\varphi_\ast \phi_\ast = \text{id}\) and \(\phi_\ast \varphi_\ast\) is homotopy equivalent to the identity map. More precisely, a homotopy \(\varphi_\ast, \phi_\ast\) is given by

\[\begin{pmatrix} \varphi_{n+1} : \mathcal{D}^\otimes \otimes M \to \mathcal{D}^\otimes \otimes M \\ \phi_{n+1} : \mathcal{D}^\otimes \otimes M \to \mathcal{D}^\otimes \otimes M \end{pmatrix}_{n \geq 0},\]

induced by \((\omega_{n+1} : \mathcal{E}^\otimes \otimes E \to \mathcal{E}^\otimes \otimes E)_{n \geq 0}\). By Proposition 2.10 we also know that \(\varphi_1 = 0\), \(\varphi_\ast \varphi_\ast = 0\), and \(\varphi_\ast \phi_\ast \circ \phi_\ast \circ \varphi_\ast = 0\).

For each \(\alpha, \beta, n \in \mathbb{N}_0\) such that \(\alpha + \beta = n\), we let \(\varphi_{n+1} : \mathcal{D}^\otimes \otimes M \to M_{\alpha\beta}\) denote the unique map such that \(\varphi_n = \sum_{\alpha + \beta = n} \varphi_{n+1}\). In Section 3 we will use the following result with \(M := \mathcal{D}^\otimes \otimes \text{sh}(\mathcal{D}^\otimes)\).

**Proposition 2.13.** The following assertions hold:

1. For each \(\alpha, \beta \geq 0\), there exists \(x_{\alpha\beta} \in \mathcal{D}_{\alpha+\beta}^\otimes\) such that \(\varphi_{\alpha+\beta}(m) = x_{\alpha\beta} \otimes m\) for all \(m \in M_{\alpha\beta}\).
2. For each \(\alpha, \beta \geq 0\), there exists a map \(\varphi_{\alpha+\beta} : \mathcal{D}_{\alpha+\beta}^\otimes \to \mathbb{Z}\) such that

\[\varphi_{\alpha+\beta}(g^{i_1} \otimes \cdots \otimes g^{i_{\alpha+\beta}} \otimes m) = \varphi_{\alpha+\beta}(g^{i_1} \otimes \cdots \otimes g^{i_{\alpha+\beta}} m) \quad \text{for all } m \in M.\]
3. For each \(n \geq 0\), there exists a map \(\varphi_{n+1} : \mathcal{D}^\otimes \otimes M \to \mathcal{D}^\otimes \otimes M\) such that

\[\varphi_{n+1}(g^{i_1} \otimes \cdots \otimes g^n \otimes m) = \varphi_{n+1}(g^{i_1} \otimes \cdots \otimes g^n m) \quad \text{for all } m \in M.\]

**Proof.** All the assertions follow from the fact that the left and right actions of \(D\) on \(M\) are trivial. \(\square\)

**Remark 2.14.** By Remark 2.11, we have

\[\varphi_1(m) = g \otimes m \quad \text{on } M_{01}, \quad \varphi_2(m) = -\sum_{i=1}^{t-1} g \otimes g^i \otimes m \quad \text{on } M_{02}, \quad \varphi_1(g^{i+1}) \otimes m = jm, \quad \varphi_1(g^{i+1}) \otimes m = -m,\]

where \(0 \leq i < u\) and \(0 \leq j < t\).
Remark 2.15. By Remark 2.12, we have
\[ \mathcal{W}_2(g^{t+j} \otimes m) = \sum_{i=0}^{s-1} g^i \otimes g^{t+j} \otimes m + \sum_{i=1}^{s-j} g \otimes g^j \otimes m, \]
where \( 0 \leq i < u \) and \( 0 \leq j < t \).

3 Full linear cycle set cohomology of cyclic cycle sets

Let \( p \in \mathbb{N} \) be a prime number and let \( \nu, \eta \in \mathbb{N} \) be such that \( 0 < \nu \leq \eta \leq 2\nu \). Let \( v = p^\nu \), \( u = p^\nu \), \( t = p^{\nu} \) and \( u' = p^{\nu-\eta} \). Note that \( u' = u \) and \( u = v \). Consider the linear cycle set \( \mathcal{A} := (\mathbb{Z}/u\mathbb{Z}; \cdot) \), where \( r \cdot j := (1 - ut)j \). Note that the set of invariants of \( \mathcal{A} \) is formed by the multiples of \( t \) and that it has \( u \) elements. In this section we will compute the cohomologies \( H^1_{\partial}(\mathcal{A}, \Gamma) \) and \( H^1_{\phi}(\mathcal{A}, \Gamma) \) of \( \mathcal{A} \) with coefficients in an arbitrary abelian group \( \Gamma \). Then, using the last result we prove Theorems A, B and C of the introduction. Let \( \mathcal{C}_v := (g) \) be the multiplicative cyclic group of order \( v \), endowed with the binary operation \( g^i \cdot g^j := g^{i+j} \). Let \( D := \mathbb{Z}[C_v] \) and \( \mathcal{D} := D/\mathbb{Z}1 \). Let \( \text{sh}(\mathcal{D}^{1}) \) be as in Subsection 1.2.1. For each \( r \geq 0 \) and \( s \geq 1 \), let \( \mathcal{M}(s) := \mathcal{D}^s / \text{sh}(\mathcal{D}^{s}) \) and let \( \overline{\mathcal{M}} := \mathcal{M}(s) \) where \( \mathcal{M}(s) \) as is in Subsection 2.2. Thus \( \overline{\mathcal{M}}_{rs} := \mathcal{D}_{rs} / \text{sh}(\mathcal{D}) \), where \( \mathcal{D}_{rs} := \mathcal{D}_{r}(\mathcal{M}(s)) \) and each \( \mathcal{D}_{rs} \) is a copy of \( \mathcal{M}(s) \). Consider the double complex \( (\overline{\mathcal{M}}_{rs}, \overline{\partial}_{rs}, \overline{\delta}_{rs}) \), where the \( s \)-row \( (\overline{\mathcal{M}}_{rs}, \overline{\partial}_{rs}) \) is the complex \( (\overline{\mathcal{M}}(s), \overline{\partial}_{rs}) \), introduced in [2.6], and \( \overline{\delta}_{rs} := \overline{\delta}_{r:1+rs} := \overline{\delta}_{r:1+rs} \), in which \( \overline{\delta}_{r:1+rs} := \overline{\delta}_{r:1+rs} \) is the map defined by
\[ \overline{\delta}_{r:1+rs}(g^{i_1} \cdot \ldots \cdot g^{i_s}) := (-1)^{r+1}[g^{i_1} \cdot \ldots \cdot g^{i_s}]
+ \sum_{j=1}^{s-1} (-1)^{r+j+1}[g^{i_1} \cdot \ldots \cdot g^{i_j} \cdot g^{i_{j+1}} \cdot \ldots \cdot g^{i_s}]
+ (-1)^{r+s+1}[g^{i_1} \cdot \ldots \cdot g^{i_s}], \]
where \( g^{i_1} \cdot \ldots \cdot g^{i_s} \), etcetera, are as in Subsection 1.2.1. Let \( \mathcal{A}_{tr} \) be the group \( \mathbb{Z}/u\mathbb{Z} \), endowed with the trivial structure of linear cycle set. For each \( s \geq 1 \), let
\[ \overline{\phi}_{rs} : (\overline{\mathcal{M}}_{rs}, \overline{\partial}_{rs}) \longrightarrow (\overline{\mathcal{C}}^N_{rs}(\mathcal{A}_{tr}), \partial^N_{rs}) \quad \text{and} \quad \overline{\psi}_{rs} : (\overline{\mathcal{C}}^N_{rs}(\mathcal{A}_{tr}), \partial^N_{rs}) \longrightarrow (\overline{\mathcal{M}}_{rs}, \overline{\partial}_{rs}) \]
be the maps \( \overline{\phi}_{rs} \) and \( \overline{\psi}_{rs} \) introduced in [2.8], with \( M \) replaced by \( \mathcal{M}(s) \). By items (1) and (2) of Proposition 2.13, in the diagram
\[ (\overline{\mathcal{M}}_{rs}, \overline{\partial}_{rs}) \xrightarrow{\overline{\phi}_{rs}} (\overline{\mathcal{C}}^N_{rs}(\mathcal{A}_{tr}), \partial^N_{rs}) \xrightarrow{\overline{\psi}_{rs}} (\overline{\mathcal{M}}_{rs}, \overline{\partial}_{rs}), \]
the maps \( \overline{\phi}_{rs} \) and \( \overline{\psi}_{rs} \) are morphisms of double complexes. Moreover, we know that \( \overline{\psi}_{rs} \overline{\phi}_{rs} = \text{id} \), and that in each row \( s \), we have a special deformation retract
\[ (\overline{\mathcal{M}}_{rs}, \overline{\partial}_{rs}) \xrightarrow{\overline{\phi}_{rs}} (\overline{\mathcal{C}}^N_{rs}(\mathcal{A}_{tr}), \partial^N_{rs}) \xrightarrow{\overline{\psi}_{rs}} (\overline{\mathcal{M}}_{rs}, \overline{\partial}_{rs}), \]
where \( (\overline{\mathcal{M}}_{rs+1, s})_{n \geq 0} \) is the family of maps \( \overline{\mathcal{M}}_{rs+1, s} \), introduce in [2.9], with \( M \) replaced by \( \mathcal{M}(s) \). For each \( s \in \mathbb{N} \), we have a perturbation \( \delta^N_{rs} : \mathcal{D}^N_{s} \otimes \mathcal{M}(s) \longrightarrow \mathcal{D}^{N+1}_{s} \otimes \mathcal{M}(s) \), where
\[ \delta^N_{rs}(g^{i_1} \otimes g^{i_2}) = g^{i_1} \cdot g^{i_2} - g^{i_2} \cdot g^{i_1} = g^{i_2} \cdot g^{i_3} - g^{i_3} \cdot g^{i_2} = g^{i_1} \cdot g^{i_3} - g^{i_3} \cdot g^{i_1}, \]
\[ \delta^N_{rs}(g^{i_1} \otimes g^{i_2} \otimes g^{i_3}) = g^{i_1} \cdot g^{i_2} \otimes g^{i_3} - g^{i_3} \otimes g^{i_2} \otimes g^{i_1} = g^{i_1} \cdot g^{i_2} \otimes g^{i_3} - g^{i_3} \otimes g^{i_2} \otimes g^{i_1} = g^{i_1} \cdot g^{i_2} \otimes g^{i_3} - g^{i_3} \otimes g^{i_2} \otimes g^{i_1}, \]
\[ \delta^N_{rs} = 0 \quad \text{for} \quad ns \notin \{11, 21, 12\}. \]

In order to carry out our computations we are going to apply Proposition 1.3 to this data. For this, first must prove that \( \delta^N_{rs} \) is small. Since \( \mathcal{M}_{rs} = \mathcal{M}_{rs+1} = 0 \), the unique non-trivial point is that \( \delta^N_{21} \mathcal{M}_{21} \) is nilpotent. But, by Remark 2.15 and the fact that \( g^i \cdot g^j = g^{i+j} \) and \( g \cdot g^j = g^{(1-u)j} = g^{(u-u')j} \), we have
\[ \delta^N_{21} \mathcal{M}_{21}(g^{i+j} \otimes g^{i}) = \sum_{l=1}^{j-1} g^l \cdot g^j \otimes g^{i} - \sum_{l=1}^{i-1} g^l \otimes g^{i} = \sum_{l=1}^{j-1} g^{(u-u')l + 1} \otimes g^{(1-u)i} - \sum_{l=1}^{i-1} g^l \otimes g^{i}, \]
where \( 0 \leq j < t \). Using this it is easy to see that \( \delta^N_{21} \mathcal{M}_{21}^{t-1} = 0 \).
Remark 3.1. The chain double complex \((\mathcal{C}_s^N(A_\alpha, Z), \partial_{rs}^+, \partial_{rs}^-)\), obtained by applying the perturbation [3.4] to \((\mathcal{C}_s^N(A, Z), \partial_{rs}^+, \partial_{rs}^-)\), only coincides with \((\mathcal{C}_s^N(A_\alpha, Z), \partial_{rs}^+, \partial_{rs}^-)\) for \(*_s = 01, *_s = 11, *_s = 21, *_s = 02, *_s = 12\) and \(*_s = 03\). Thus, the chain double complex \(\mathcal{X}(\mathcal{A}) := (\mathcal{X}_{*s}, \delta_{*s}^+, \delta_{*s}^-)\), obtained by applying Proposition 1.3 to the above data, will be useful only to compute the full (co)homology of \(\mathcal{A}\) in degrees 1 and 2. Note that \(\delta_{*s}^+ = \mathcal{D}_{*s}\).

For each \(\alpha, \beta, r \in \mathbb{N}_0\) and \(s \in \mathbb{N}\) such that \(\alpha + \beta = r\), we let \(\mathcal{M}_{rs}^\alpha : \mathcal{D}_{rs}^{\alpha} \otimes \mathcal{M}(s) \to \mathcal{M}(s)_{\alpha \beta}\) denote the unique map such that \(\mathcal{M}_{rs} = \sum_{\alpha + \beta = r} \mathcal{M}_{rs}^\beta\). Clearly \(\mathcal{M}_{rs}^\beta\) is the map \(\mathcal{M}_{rs}^\beta\) introduced above Proposition 2.13, with \(\mathcal{M}\) replaced by \(\mathcal{M}(s)\). A direct computation using equality [3.5] and Remark 2.14 shows that

\[
\mathcal{M}_{11}^{01} \mathcal{M}_{21}^{01} (g^{t_1} \otimes g^{u_1}) = \sum_{l=1}^{j-1} l(g^{t_1} g^{u_1} - g^{u_1}) = \left(\frac{j}{2}\right) g^{t_1} (g^{u_1} - 1)
\]

and

\[
\mathcal{M}_{11}^{10} \mathcal{M}_{21}^{10} (g^{t_1} \otimes g^{u_1}) = - \sum_{l=1}^{j-1} (u - u') (g^{t_1} g^{u_1} - 1) = u' \left(\frac{j}{2} - \left(\frac{j}{2} - 1\right) t\right) g^{t_1} (g^{u_1} - 1),
\]

where \(0 \leq j \leq t\). An inductive argument using these equalities, [3.5] and the fact that \(g^{u_2} = 1\) shows that,

\[
\mathcal{M}_{11}^{01} \mathcal{M}_{21}^{10} (g^{t_1} \otimes g^{u_1}) = \left(\frac{j}{s+1}\right) g^{t_1} (g^{u_1} - 1)^{s+1}
\]

and

\[
\mathcal{M}_{11}^{10} \mathcal{M}_{21}^{10} (g^{t_1} \otimes g^{u_1}) = u' \left(\frac{j}{s+1} - \left(\frac{j-1}{s}\right) t\right) g^{t_1} (g^{u_1} - 1)^{s-1},
\]

for all \(0 \leq j \leq t\) and \(s \in \mathbb{N}\). Let \(\mathcal{X}(\mathcal{A})_T\) be the subcomplex of \(\mathcal{X}(\mathcal{A})\). Recall that \(\mathcal{M}_{rs} = \bigoplus_{\alpha, \beta \geq 0} \mathcal{M}(s)_{\alpha \beta}\).

Theorem 3.2. The chain double complex \(\mathcal{X}(\mathcal{A})_T\) is a partial total complex of the diagram
where \( \tilde{d}_{\ast \ast}^0 = \overline{d}_{\ast \ast} \) (see formula [3.1]), and the other not zero maps are given by:

\[
\begin{align*}
\tilde{d}_{201}^0 (g') &:= ug', \\
\tilde{d}_{111}^1 (g') &:= g'(1 - g^{-uw}), \\
\tilde{d}_{012}^0 (g'^{1} \otimes g'^{2}) &:= [g'^{1} \otimes g'^{2}] - [g'^{1} \otimes g'^{2}], \\
\tilde{d}_{011}^1 (g') &:= g'(g^{-uw} - 1), \\
\tilde{d}_{221}^1 (g') &:= g'(1 - g^{-uw}), \\
\tilde{d}_{221}^2 (g') &:= -g' \sum_{s=0}^{t-1} g^{-uw}. 
\end{align*}
\]

Proof. In order to prove this theorem we will apply Proposition 1.3 to the data consisting of diagrams [3.2] and [3.3]. We begin by computing the first row of \( \mathcal{X}(A) \). Since \( \overline{\sigma}_{11} = 0 \) and \( \overline{\sigma}_{01} = \text{id}_{\mathcal{M}(1)} \), we know that \( \tilde{d}_{11} = \tilde{d}_{11}^0 + \delta_{11}^0 \overline{\sigma}_{11} \). Moreover, by Remark 2.14 and the definition of \( \delta_{11}^0 \), we have

\[
\delta_{11}^0 \overline{\sigma}_{11} (g') = 0 \quad \text{on} \quad \overline{\mathcal{M}}(1)_{10} \quad \text{and} \quad \delta_{11}^0 \overline{\sigma}_{11} (g' \otimes g') = g'(1 - g^{-uw}) - g' \quad \text{on} \quad \overline{\mathcal{M}}(1)_{01}.
\]

Since \( \tilde{d}_{11}^0 = 0 \) (by equalities [2.5] and [2.7]), this implies that

\[
\tilde{d}_{11}^0 (g') = 0 \quad \text{on} \quad \overline{\mathcal{M}}(1)_{10} \quad \text{and} \quad \tilde{d}_{11}^1 (g') = g' - 1 = \tilde{d}_{011}^1 (g') \quad \text{on} \quad \overline{\mathcal{M}}(1)_{01}.
\]

We next compute \( \tilde{d}_{21}^2 \). By Remark 2.14 and the definition of \( \delta_{21}^2 \), we have

\[
\begin{align*}
\varphi_{11}^0 \delta_{21}^2 \varphi_{21}^0 (g') &= 0 & \text{on} \quad \overline{\mathcal{M}}(1)_{20}, & [3.8] \\
\varphi_{11}^0 \delta_{21}^2 \varphi_{21}^0 (g') &= g'(1 - g^{-uw}) & \text{on} \quad \overline{\mathcal{M}}(1)_{11}, & [3.9] \\
\varphi_{11}^0 \delta_{21}^2 \varphi_{21}^0 (g') &= \sum_{i=1}^{t-1} g'^{i} \otimes g'^{i} - \sum_{i=1}^{t-1} g'^{(s+u')i+} \otimes g'^{-uw} & \text{on} \quad \overline{\mathcal{M}}(1)_{22}. & [3.10]
\end{align*}
\]

Hence, again by Remark 2.14,

\[
\begin{align*}
\varphi_{11}^0 \delta_{21}^2 \varphi_{21}^0 (g') &= 0 & \text{on} \quad \overline{\mathcal{M}}(1)_{11}, & [3.11] \\
\varphi_{11}^0 \delta_{21}^2 \varphi_{21}^0 (g') &= g'(1 - g^{-uw}) & \text{on} \quad \overline{\mathcal{M}}(1)_{11}, & [3.12] \\
\varphi_{11}^0 \delta_{21}^2 \varphi_{21}^0 (g') &= \sum_{i=1}^{t-1} g'^{i} \otimes g'^{i} - \sum_{i=1}^{t-1} g'^{(s+u')i+} \otimes g'^{-uw} & \text{on} \quad \overline{\mathcal{M}}(1)_{22}. & [3.13]
\end{align*}
\]

By equality [3.8] we know that \( \tilde{d}_{21}^2 (g') = \tilde{d}_{21}^2 (g') \quad \text{on} \quad \overline{\mathcal{M}}(1)_{20}. \) Consequently, by equalities [2.5] and [2.7], we have \( \tilde{d}_{21}^2 (g') = \tilde{d}_{201}^0 (g') \quad \text{on} \quad \overline{\mathcal{M}}(1)_{20}. \) Moreover, by equalities [3.6], [3.7] and [3.9],

\[
\varphi_{11}^0 (\delta_{21}^2 \varphi_{21}^2)^s \delta_{21}^2 \varphi_{21}^2 (g') = 0 \quad \text{on} \quad \overline{\mathcal{M}}(1)_{11} \quad \text{and} \quad \overline{\mathcal{M}}(1)_{11} \quad \text{for all} \quad s \geq 1,
\]

while, by equalities [2.5] and [2.7], we know that \( \tilde{d}_{21}^2 = 0 \quad \text{on} \quad \overline{\mathcal{M}}(1)_{11}. \) Hence, by equalities [3.11] and [3.12],

\[
\tilde{d}_{21}^2 (g') = \varphi_{11}^0 \delta_{21}^2 \varphi_{21}^0 (g') + \varphi_{11}^0 \delta_{21}^2 \varphi_{21}^0 (g') = \tilde{d}_{011}^1 (g') \quad \text{on} \quad \overline{\mathcal{M}}(1)_{11}.
\]

We now compute \( \tilde{d}_{21}^2 \) on \( \overline{\mathcal{M}}(1)_{12} \). Equations [3.6] and [3.7] implies that

\[
\begin{align*}
\varphi_{11}^0 (\delta_{21}^2 \varphi_{21}^2)^s \sum_{i=1}^{t-1} g'^{i} \otimes g'^{i} = \sum_{i=1}^{t-1} g'^{i} (g'^{-uw} - 1)^s = \left( \frac{t}{s+2} \right) g'^{(s+2)} (g'^{-uw} - 1)^s, \\
\varphi_{11}^0 (\delta_{21}^2 \varphi_{21}^2)^s \sum_{i=1}^{t-1} g'^{(s+u')i+} \otimes g'^{-uw} = \left( \frac{t}{s+2} \right) g'^{(s+2)} (g'^{-uw} - 1)^s, \\
\varphi_{11}^0 (\delta_{21}^2 \varphi_{21}^2)^s \sum_{i=1}^{t-1} u' \left( \left( \frac{t}{s+2} \right) g'^{(s+2)} (g'^{-uw} - 1)^s \right), \\
\varphi_{11}^0 (\delta_{21}^2 \varphi_{21}^2)^s \sum_{i=1}^{t-1} u' \left( \left( \frac{t}{s+2} \right) g'^{(s+2)} (g'^{-uw} - 1)^s \right) = -u'(s+1) \left( \frac{t}{s+2} \right) g'^{(s+2)} (g'^{-uw} - 1)^s,
\end{align*}
\]

where \( \varphi_{11}^0 \delta_{21}^2 \varphi_{21}^0 (g') = 0 \quad \text{on} \quad \overline{\mathcal{M}}(1)_{12} \).
for all \( s \geq 1 \) (in the computation of last equality we had used that \( g^{u^s} = 1 \)). So, by equality [3.10],

\[
\varphi_1^{11}(\delta_{21}^h \omega_{21})^s \delta_{21} \phi_{21}(g') = - \left( \frac{t}{s+2} \right) g^t (g^{-u^s} - 1)^{s+1}
\]

on \( \overline{M}(1)_{02} \). \[ \text{[3.15]} \]

\[
\varphi_{11}^{0}(\delta_{21}^h \omega_{21})^s \delta_{21} \phi_{21}(g') = u'(s+1) \left( \frac{t}{s+2} \right) g^t g^{-u^s} (g^{-u^s} - 1)^s
\]

on \( \overline{M}(1)_{02} \). \[ \text{[3.16]} \]

for all \( s \geq 1 \). Thus, by [3.13] and [3.15],

\[
\sum_{s=0}^{t-2} \varphi_1^{11}(\delta_{21}^h \omega_{21})^s \delta_{21} \phi_{21}(g') = - \sum_{s=1}^{t-2} \left( \frac{t}{s+2} \right) g^t (g^{-u^s} - 1)^s = -g^t \left( 1 - t + \sum_{s=1}^{t-2} g^{-su^s} \right)
\]

on \( \overline{M}(1)_{02} \). (The last equality follows by induction on \( t \) using that \( \left( \frac{t}{s+1} \right) = \left( \frac{t-1}{s+1} \right) + \left( \frac{t}{s+1} \right) \). A similar argument using [3.14] and [3.16], shows that, on \( \overline{M}(1)_{02} \) we have

\[
\sum_{s=0}^{t-2} \varphi_{11}^{0}(\delta_{21}^h \omega_{21})^s \delta_{21} \phi_{21}(g') = \sum_{s=0}^{t-2} u'(s+1) \left( \frac{t}{s+2} \right) g^t g^{-u^s} (g^{-u^s} - 1)^s = u'g^t \left( \sum_{s=0}^{t-2} (s+1)g^{-u^s(s+1)} \right)
\]

Combining this with [2.5] and [2.7], we obtain that, on \( \overline{M}(1)_{02} \),

\[
\partial_{21}^0(g') = \partial_{21}(g') - g^t \left( 1 - t + \sum_{s=1}^{t-2} g^{-su^s} \right) + u'g^t \left( \sum_{s=0}^{t-2} (s+1)g^{-u^s(s+1)} \right) = \partial_{02}^{11}(g') + \partial_{02}^{12}(g').
\]

We now compute the second row of \( \mathcal{X}(A)_{1^r} \). Since \( \overline{\omega}_{12} = 0 \) and \( \overline{\varphi}_{02} = \text{id}_{\overline{M}(2)} \), we know that \( \partial_{12}^0 = \partial_{12}^1 + \partial_{12}^2 \varphi_{12} \). Moreover, by Remark 2.14 and the definition of \( \delta_{12}^h \),

\[
\delta_{12} \phi_{12}([g^{1^s} \otimes g^{s^2}]) = 0 \quad \text{on} \quad \overline{M}(2)_{10} \quad \text{and} \quad \delta_{12}^h \phi_{12}([g^{1^s} \otimes g^{s^2}]) = [g^{1^s-1^s_1} \otimes g^{1^s_1-1^s_2}] - [g^{1^s} \otimes g^{s^2}] \quad \text{on} \quad \overline{M}(2)_{01}.
\]

Therefore, by equalities [2.5] and [2.7],

\[
\partial_{12}^1([g^{1^s} \otimes g^{s^2}]) = 0 \quad \text{on} \quad \overline{M}(2)_{10} \quad \text{and} \quad \partial_{12}^2([g^{1^s} \otimes g^{s^2}]) = \partial_{02}^{12}([g^{1^s} \otimes g^{s^2}]) \quad \text{on} \quad \overline{M}(2)_{01},
\]

as desired. \[ \square \]

Remark 3.3. Let \( (\widetilde{C}_s^N(A, \mathbb{Z}), \partial_s) \) be as in subsection 1.2.1, let \( \text{Tot}(\mathcal{X}(A)) \) be the total complex of \( \mathcal{X}(A) \) and let \( \tilde{\phi}_s : (\widetilde{C}_s^N(A, \mathbb{Z}), \partial_s^h, \partial_s^i) \longrightarrow (\mathcal{X}(A)) \) be the map obtained by applying Proposition 1.3 to the special deformation retracts \( \mathcal{X}_s \), endowed with the perturbations \( \tilde{\varphi}_s \) given in [3.4]. By that proposition, the map \( \tilde{\varphi}_s : (\widetilde{C}_s^N(A, \mathbb{Z}), \partial_s) \longrightarrow \text{Tot}(\mathcal{X}(A)), \) induced by \( \tilde{\phi}_s \), is an homotopy equivalence. Since \( \overline{\omega}_{11} = 0 \) and \( \overline{\varphi}_{12} = 0 \), we have \( \overline{\varphi}_{01} = \overline{\varphi}_{02} = \text{id}_{\overline{M}(1)} \) and \( \overline{\varphi}_{02} = \overline{\varphi}_{02} = \text{id}_{\overline{M}(2)} \). On the other hand, by Remark 2.14 and equalities [3.6] and [3.7], for each \( 0 \leq i < u \) and \( 0 \leq j < t \), we have

\[
\tilde{\varphi}_{11}^{10}(g^{1^i} \otimes g^{s^1}) = \tilde{\varphi}_{11}^{11}(g^{1^i} \otimes g^{s^1}) + \tilde{\varphi}_{11}^{10}(g^{1^j} \otimes g^{s^1}),
\]

where \( \tilde{\varphi}_{11}^{01} : \overline{M} \otimes \overline{M}(1) \longrightarrow \overline{M}(1)_{10} \) and \( \tilde{\varphi}_{11}^{10} : \overline{M} \otimes \overline{M}(1) \longrightarrow \overline{M}(1)_{10} \) are the maps given by

\[
\tilde{\varphi}_{11}^{01}(g^{1^i} \otimes g^{s^1}) := \sum_{s=0}^{j-1} \left( \frac{j}{s+1} \right) g^{s^1} (g^{-u^1} - 1)^s
\]

and

\[
\tilde{\varphi}_{11}^{10}(g^{1^i} \otimes g^{s^1}) := -ig^{s^1} + \sum_{s=0}^{j-1} u' \left( \frac{j}{s+1} - \frac{j-1}{s} \right) t \ g^{s^1} g^{-u^1} (g^{-u^1} - 1)^{s-1}.
\]

Proposition 3.4. For all \( 0 \leq i < u \) and \( 0 \leq j < t \), the following identities hold:

\[
\tilde{\varphi}_{11}^{01}(g^{1^j} \otimes g^{s^1}) = g^{s^1} \sum_{l=0}^{j-1} g^{-ul^1} \quad \text{and} \quad \tilde{\varphi}_{11}^{10}(g^{1^j} \otimes g^{s^1}) = -ig^{s^1} + u'g^{s^1} \sum_{l=1}^{j-1} (j - l - t)g^{-ul^1}.
\]

Proof. The first equality holds by the last equality in [3.17]. By this and [3.19] in order to prove the second one it suffices to show that

\[
\sum_{s=1}^{j-1} \left( \frac{j}{s+1} \right) g^{s^1} (g^{-u^1} - 1)^{s-1} = \sum_{l=1}^{j-1} (j - l - t)g^{-ul^1}.
\]

But this follows by induction on \( j \) using that \( \left( \frac{j}{s+1} \right) = \left( \frac{j-1}{s} \right) + \left( \frac{j-1}{s+1} \right). \] \[ \square \]
3.1 Computing the full linear cycle set cohomology

Here we will use freely the notations introduced in subsection 1.2.1. Let \( A \) be as at the beginning of this section and let \( \Gamma \) be an additive abelian group. In this subsection we compute \( H^2_N(A, \Gamma) \). Moreover, we obtain a family of 2-cocycles of \( (C^*_N(A, \Gamma), \partial^s) \) that applies surjectively on \( H^2_N(A, \Gamma) \), and we determine when two of these cocycles are cohomologous. By [19, Theorem 5.8] this classify the central extensions of \( A \) by \( \Gamma \). We use this fact in order to prove Theorems A, B and C.

For each \( l \in \mathbb{N} \) and \( \alpha, \beta \in \mathbb{N}_0 \) we define \( \Gamma(l)^{\alpha \beta} := \text{Hom}(M(l)_{\alpha \beta}, \Gamma) \). There are obvious identifications

\[
\Gamma(1)^{\alpha \beta} = \left\{ \sum_{0 \leq i < v} \gamma_i g^i : \gamma_i \in \Gamma \text{ for all } i \text{ and } \gamma_0 = 0 \right\}
\]

and

\[
\Gamma(2)^{\alpha \beta} = \left\{ \sum_{0 \leq i, j < v} \gamma_{ij} g^i \otimes g^j : \gamma_{ij} \in \Gamma \text{ for all } i, j, \gamma_{ij} = \gamma_{ji} \text{ and } \gamma_{0j} = 0 \text{ for all } j \right\}.
\]

Let \( \mathcal{X}(A, \Gamma) := (\check{X}^{**}, \check{d}^{**}_i, \check{d}^{**}_{ij}) \) be the cochain double complex obtained by applying the functor \( \text{Hom}(-, \Gamma) \) to \( \mathcal{X}(A)_T \). In Proposition 3.6 and Theorems 3.14, 3.16 and 3.18, we are going to calculate \( H^1_N(A, \Gamma) \) and \( H^2_N(A, \Gamma) \). By Remarks 1.1 and 3.1 in order to carry out this task we must compute \( H^1(\mathcal{X}(A, \Gamma)) \) and \( H^2(\mathcal{X}(A, \Gamma)) \). For this we will use strongly that \( \mathcal{X}(A, \Gamma) \) is a partial total complex of the diagram \( D(A, \Gamma)^T := \text{Hom}(D(A)_T, \Gamma) \). In particular

\[
\check{X}^{01} = \Gamma(1)^{00}, \quad \check{X}^{02} = \Gamma(2)^{00} \quad \text{and} \quad \check{X}^{11} = \Gamma(1)^{01} \oplus \Gamma(1)^{10}.
\]

We also want to obtain 2-cocycles of \( \check{C}^*_N(A, \Gamma) \) that represent all the elements of \( H^2_N(A, \Gamma) \). These cocycles are obtained by applying \( \text{Hom}(\hat{\varphi}_2, \Gamma) \) to 2-cocycles of \( \mathcal{X}(A, \Gamma) \), where

\[
\hat{\varphi}_2 : \check{M}(2) \oplus (\check{D} \otimes \check{M}(1)) \rightarrow \check{X}^{02} \oplus \check{X}^{11} = \check{M}(2)^{00} \oplus \check{M}(1)^{01} \oplus \check{M}(1)^{10},
\]

is as in Remark 3.3. Thus \( \hat{\varphi}_2 \) is given by the matrix

\[
[\hat{\varphi}_2] := \begin{pmatrix}
\text{id}_{\check{M}(2)} & 0 \\
0 & \varphi_{11}^{01} \\
0 & \varphi_{11}^{11}
\end{pmatrix}.
\]

From now on we set \( \check{d}^{ab}_{ij} = \text{Hom}(\check{d}^{a}_{ij}, \Gamma) \) and \( \check{d}^{ab}_{hi} = \text{Hom}(\check{d}^{a}_{hi}, \Gamma) \).

**Remark 3.5.** Applying the functor \( \Gamma \otimes - \) to \( \mathcal{X}(A)_T \) we obtain a chain double complex that gives \( H^1_N(A, \Gamma) \) and \( H^2_N(A, \Gamma) \). But we are not interested in the computation of these homology groups in this paper.

Here and subsequently, we adopt the convention that \( \gamma_{ij} := \gamma_{i'j'} \), where \( i' \) and \( j' \) are the remainder of the integer division of \( i \) by \( v \) and \( j \) by \( v \), respectively.

**Proposition 3.6.** We have \( H^1_N(A, \Gamma) = \hat{H}^1_N(A, \Gamma) = \Gamma_u \).

**Proof.** By Remarks 1.1 and 3.1 we know that \( H^1_N(A, \Gamma) = \hat{H}^1_N(A, \Gamma) = H^1(\mathcal{X}(A, \Gamma)) \). By definition

\[
\partial^{02}_v \left( \sum \gamma_i g^i \right) = \sum_{i,j} \gamma_{ij} g^j \otimes g^i \quad \text{and} \quad \partial^{11}_{hi} \left( \sum \gamma_i g^i \right) = \sum_i (\gamma_{i-1} - \gamma_i) g^i.
\]

From the first equality we get

\[
\ker(\partial^{02}_v) = \left\{ \sum \gamma_i g^i : \gamma_i = r i \gamma_1 \text{ and } \gamma_v = 0 \right\}.
\]

Consequently, \(-u\gamma_1 = (1-u)\gamma_1 - \gamma_1 = \gamma_1 - u \gamma_1 \), and so

\[
H^1(\mathcal{X}(A, \Gamma)) = \ker(\partial^{02}_v) \cap \ker(\partial^{11}_{hi}) = \left\{ \sum \gamma_i g^i : \gamma_i = r i \gamma_1 \text{ and } u \gamma_1 = 0 \right\},
\]

which is clearly isomorphic to \( \Gamma_u \). \( \square \)

Our next purpose is to compute \( \hat{H}^2_N(A, \Gamma) \).

**Lemma 3.7.** \( \sum \gamma_{ij} g^i \otimes g^j \in \ker(\partial^{03}_v) \) if and only if \( \gamma_{ij} = \sum_{k=j}^{i+j-1} \gamma_{ik} - \sum_{k=1}^{i-1} \gamma_{1k} \) for \( 1 \leq i, j < v \).
Proof. Assume that $\sum g_i \otimes g^j \in \ker(\mathcal{A}_0^{\text{003}})$. Then, for all $a \leq b$, we have

$$0 = \mathcal{A}_0^{\text{003}} \left( \sum g_i \otimes g^j \right) \left( g \otimes g^a \otimes g^b \right) = -\gamma_{ab} + \gamma_{a+1,b} - \gamma_{1,a+b} + \gamma_{1a}.$$ 

Thus, $\gamma_{a+1,b} = \gamma_{ab} + \gamma_{1,a+b} - \gamma_{1a}$. An inductive argument using this fact proves that the statement is true when $a \leq b$. For $a > b$, we have

$$\gamma_{ab} = \gamma_{ba} = \sum_{a \leq k < c} \gamma_{1k} - \sum_{1 \leq k < c} \gamma_{1k} = \gamma_{1c} - \gamma_{a+b+c} - \gamma_{ab} = 0$$

for all $0 \leq a, b, c < v$.

Conversely, assume that $\gamma_{ij} = \sum_{k=j}^{i+j-1} \gamma_{1k} - \sum_{k=1}^{i-1} \gamma_{1k}$ for $1 \leq i, j < v$. We must show that $\gamma_{bc} = \gamma_{a+b+c} - \gamma_{ab} = 0$ for all $0 \leq a, b, c < v$.

But this follows easily using that $\gamma_{ij} = \sum_{k=j}^{i+j-1} \gamma_{1k} - \sum_{k=1}^{i-1} \gamma_{1k}$ for all $i, j \in \mathbb{N}$. \hfill $\Box$

Remark 3.8. Lemma 3.7 implies that each $\sum g_i \otimes g^j \in \ker(\mathcal{A}_0^{\text{003}})$ is uniquely determined by $\gamma_{11}, \ldots, \gamma_{1,v-1}$. For example, for each $1 \leq b < v$, the element $f_b(\gamma) := \sum_{\gamma, b} \Lambda(\gamma, b) \sigma^b$ where

$$\Lambda(\gamma, b) := \begin{cases} \gamma & \text{if } i \leq b \text{ and } b - i < j \leq b, \\ -\gamma & \text{if } i > b \text{ and } b < j \leq v - i + b, \\ 0 & \text{otherwise}, \end{cases}$$

is the unique $\sum g_i \otimes g^j \in \ker(\mathcal{A}_0^{\text{003}})$ with $\gamma_{1b} = \gamma$ and $\gamma_{1j} = 0$ for $j \neq b$. Note that

$$\sum_{b=1}^{v-1} f_b(\gamma_{1b}) \quad \text{for each } \sum g_i \otimes g^j \in \ker(\mathcal{A}_0^{\text{003}}).$$

Thus, for $f_b(\gamma) : 1 \leq b < v$ and $\gamma \in \Gamma$, generate $\ker(\mathcal{A}_0^{\text{003}})$.

Remark 3.9. A direct computation shows that

$$\mathcal{A}_0^{\text{003}}(\sigma g^i) = -\sum_{j=1}^{v-1} \gamma(g^i \otimes g^j + g^i \otimes g^j) - 2 \gamma g^i \otimes g^j + \sum_{a,b=1}^{v-1} \gamma g^a \otimes g^b \quad (a + b \equiv 1 \mod v).$$

Thus, by Remark 3.8,

$$\mathcal{A}_0^{\text{003}}(\sigma g^i) = \begin{cases} -2f_1(\gamma) - f_2(\gamma) - \cdots - f_{v-1}(\gamma) & \text{if } i = 1, \\ f_{i-1}(\gamma) - f_i(\gamma) & \text{if } i \neq 1. \end{cases}$$

Consequently, since the $f_i(\gamma)$’s generate $\ker(\mathcal{A}_0^{\text{003}})$, we have $\ker(\mathcal{A}_0^{\text{003}}) / \text{Im}(\mathcal{A}_0^{\text{003}}) = \Gamma / uv\Gamma$.

Lemma 3.10. Let $\gamma \in \Gamma$. If $u = v$, then $\mathcal{A}_0^{\text{112}}(f_i(\gamma)) = 0$. Otherwise

$$\mathcal{A}_0^{\text{112}}(f_i(\gamma)) = -\gamma g \otimes g - \sum_{k=1}^{u'} \gamma g \otimes g^{k+1} - \sum_{k=u'+1}^{2u'-1} \gamma g \otimes g^{k+1} - \sum_{h=2}^{t+1} \sum_{k=hu'}^{t+1} \gamma g \otimes g^{k+h}$$

$$+ \sum_{i=2}^{u-1} \sum_{j=1}^{v-1} (\Lambda(\gamma, 1)_{(1-u')_i,(1-u')_j} - \Lambda(\gamma, 1)_{ij}) g^i \otimes g^j \quad (3.22)$$

Proof. By definition

$$\mathcal{A}_0^{\text{112}}(f_i(\gamma)) = \sum (\Lambda(\gamma, 1)_{(1-u')_i,(1-u')_j} - \Lambda(\gamma, 1)_{ij}) g^i \otimes g^j.$$ 

We will use [3.21] in order to compute $\Lambda(\gamma, 1)_{(1-u')_i,(1-u')_j}$. In order to carry out this task, for each $0 < j < v$ we need to find $k$ such that $0 \leq kv + (1 - u')_j < v$. But this happens if and only if $(k-1)v < j(u-1) < kv$, and it is evident that such a $k$ there exists and it is unique. Moreover, $1 \leq k < u$ and $j(u-1) \neq kv$. In fact, if $k \geq u$, then $kv + (1 - u')_j > uv + (1 - u)v$ while if $j(u-1) = kv$, then $v \mid j$, because $\gcd(u-1, v) = 1$. By equality [3.21], for all $j, k, u$ such that $0 < j < v$ and $0 < k < (1 - u')_j < v$, we have

$$\Lambda(\gamma, 1)_{e-u+1,kv+j(1-u)} = \begin{cases} \gamma & \text{if } u = v \text{ and } 0 < kv + j(1 - u') \leq 1, \\ -\gamma & \text{if } u < v \text{ and } 1 < kv + j(1 - u') \leq u, \\ 0 & \text{otherwise}. \end{cases}$$
Consequently, $\Lambda(\gamma, 1)_{v-u+1, kv+j(1-u)} = \gamma$ if and only if $u = v$ and $k = j = 1$. From this and Remark 3.8 it follows easily that if $u = v$, then $\hat{a}^{012}_{hi}(f_1(\gamma)) = 0$. Assume now that $u < v$. We next purpose is to determine when $\Lambda(\gamma, 1)_{v-u+1, kv+j(1-u)} = -\gamma$. Note that
\[
1 < kv + j(1 - u) \leq u \Leftrightarrow kv - u \leq j(u - 1) < kv - 1 \Leftrightarrow kt + \frac{kt - u}{u - 1} \leq j < kt + \frac{kt - u}{u - 1} + 1.
\]
Thus,
\[
\Lambda(\gamma, 1)_{v-u+1, kv+j(1-u)} = -\gamma \quad \text{if and only if} \quad j = kt + \left\lfloor \frac{kt - u}{u - 1} \right\rfloor.
\]
Write $k = hu' + l$, where $h = 0$ and $1 \leq l \leq u'$, or $h = 1$ and $1 \leq l < u'$, or $1 < h < t$ and $0 \leq l < u'$. Then
\[
\frac{kt - u}{u - 1} = \frac{(hu' + l)t - u}{u - 1} = \frac{hu + lt - u}{u - 1} = \frac{(h - 1)(u - 1) - h + lt - h - 1}{u - 1} = h - 1 + \frac{lt + h - 1}{u - 1}.
\]
We claim that $\left\lfloor \frac{kt - u}{u - 1} \right\rfloor = h$. To check this we must prove that $0 < lt + h - 1 < l$. Assume first that $h = 0$. Then $lt - 1 > 0$, because $t > 1$ and $l > 0$; while $lt - 1 < u't - 1 = u - 1$, because $0 < l < u'$. Assume now that $1 \leq h < t$. Then $lt + h - 1 > 0$, because $l > 0$ or $h > 1$; while $lt + h - 1 \leq (u' - 1)t + h - 1 = u - t + h - 1 < u - 1$. Summarizing, for all $j, k$, such that $0 < j < v$ and $0 < kv + (1 - u)j < v$, we have
\[
\Lambda(\gamma, 1)_{v-u+1, kv+j(1-u)} = -\gamma \quad \text{if and only if} \quad j = \begin{cases} \frac{kt - u}{u - 1} & \text{if } 1 \leq k \leq u', \\ kt + 1 & \text{if } u' + 1 \leq k \leq 2u', \\ kt + h & \text{if } h u' \leq k < (h + 1)u' \end{cases} \quad \text{with } 1 < h < t.
\]
Using this fact it is easy to see that equality [3.22] holds. \hfill \Box

Remark 3.11. Let $B, Z \subseteq \Gamma(2)^{00} \oplus \Gamma(1)^{01} \oplus \Gamma(1)^{10}$ be the 2-coboundaries and the 2-cocycles of $\mathcal{X}(A, \Gamma)$ respectively, and let $Z' \subseteq Z$ be the subgroup of cocycles $z = (\sum_{\gamma} \gamma_{i} g_{i} \otimes g', \sum_{\gamma} \gamma_{i} g_{i}', \sum_{\gamma} \gamma_{i} g_{i}'')$, such that $\sum_{\gamma} \gamma_{i} g_{i} \otimes g' = f_{1}(\gamma)$ for some $\gamma \in \Gamma$. By Remark 3.9 we know that $H^{2}_{\ast}(\mathcal{X}(A, \Gamma)) = Z'/B \cap Z'$. Moreover, since $f_{1}(\gamma) \in \ker(\hat{a}^{00}_{v})$, a triple $z = (f_{1}(\gamma), \sum_{\gamma} \gamma_{i} g_{i}, \sum_{\gamma} \gamma_{i} g_{i}')$ is in $Z'$ if and only if
\begin{align*}
(1) & \quad \hat{a}^{00}_{hi}(\sum_{\gamma} \gamma_{i} g_{i}') = 0, \quad \hat{a}^{011}_{hi}(\sum_{\gamma} \gamma_{i} g_{i}) = 0 \quad \text{and} \quad \hat{a}^{012}_{hi}(\sum_{\gamma} \gamma_{i} g_{i}') = 0, \\
(2) & \quad \hat{a}^{012}_{hi}(\sum_{\gamma} \gamma_{i} g_{i}') = -\hat{a}^{021}_{hi}(\sum_{\gamma} \gamma_{i} g_{i}), \\
(3) & \quad \hat{a}^{012}_{hi}(\sum_{\gamma} \gamma_{i} g_{i}') = -\hat{a}^{012}_{hi}(f_{1}(\gamma)).
\end{align*}
Clearly the first condition is satisfied if and only if $u \gamma_{i}' = 0$ for all $i$, $\gamma_{i}' = \gamma_{i+1}'$ for all $i$, and $\gamma_{i+1}' = \gamma_{i}' + \gamma_{i+1}'$ for all $i$. But this happens if and only if
\[
\gamma_{i}' = \gamma_{i}' = 0.
\]
On the other hand, item (3) says that
\[
\sum_{\gamma} (\gamma_{i+1} - \gamma_{i+1}) g_{i} \otimes g' = \hat{a}^{012}_{v}(\sum_{\gamma} \gamma_{i} g_{i}') = -\hat{a}^{012}_{hi}(f_{1}(\gamma)),
\]
which, by Lemma 3.10, implies that
\begin{enumerate}
\item[(4)] If $t = 1$ (or, equivalently, $u = v$), then $\gamma_{i} = \gamma_{v}$. Moreover $v \gamma_{i} = \gamma_{v} = 0$.
\item[(5)] If $1 < t = u$ (or, equivalently, $u' = 1$), then
\[
\gamma_{kt+i} = \begin{cases} (kt + l) \gamma_{1} - (k + 1) \gamma & \text{if } k = 0 \text{ and } 2 \leq l \leq t, \\
(kt + l) \gamma_{1} - (k + 1) \gamma & \text{if } k = 1 \text{ and } 1 \leq l \leq t + 1, \\
(kt + l) \gamma_{1} - (k + 1) \gamma & \text{if } u' < k < 2u' - 2 \text{ and } 2 \leq l \leq t + 1, \\
(kt + l) \gamma_{1} - (k + 1) \gamma & \text{if } 2 \leq h < t, \text{ } h u' - 1 < h \leq t + h, \\
(kt + l) \gamma_{1} - (k + 1) \gamma & \text{if } 2 \leq h < t, \text{ } h u' - 1 \leq (h + 1) u' - 2 \text{ and } h < l \leq t + h, \\
\end{cases}
\]
and $v \gamma_{1} - u \gamma = \gamma_{v} = 0$.
\end{enumerate}
\begin{enumerate}
\item[(6)] If $1 < t < u$ (or equivalently, $1 < u' < u$), then
\[
\gamma_{kt+i} = \begin{cases} (kt + l) \gamma_{1} - (k + 1) \gamma & \text{if } k = 0 \text{ and } 2 \leq l \leq t, \\
(kt + l) \gamma_{1} - (k + 1) \gamma & \text{if } 1 \leq k \leq u' \text{ and } 1 \leq l \leq t + 1, \\
(kt + l) \gamma_{1} - (k + 1) \gamma & \text{if } 2 \leq h < t, \text{ } h u' - 1 < h \leq t + h, \\
(kt + l) \gamma_{1} - (k + 1) \gamma & \text{if } 2 \leq h < t, \text{ } h u' - 1 \leq (h + 1) u' - 2 \text{ and } h < l \leq t + h, \\
\end{cases}
\]
and $v \gamma_{1} - u \gamma = \gamma_{v} = 0$ (note that, if $u' = 2$, then the fourth line in [3.25] is empty; while, if $t = 2$, then the last two lines are empty).
Conversely under these conditions, item (3) holds. Summarizing, items (1) and (3) are satisfied if and only if equality [3.23] and conditions (4), (5) or (6) are fulfilled, depending on the case. Finally, by the definition of \( \partial_{h_2}^{\Omega_1} \) and equality [3.23],

\[
\partial_{h_2}^{\Omega_1} \left( \sum \gamma_i g_i \right) = \sum \left( -\gamma_i' + u \sum_{s=1}^{t-1} s \gamma_{i-s} \right) g_i = \sum \left( u' \left( \frac{t}{2} \right) - 1 \right) \gamma_i' g_i = \begin{cases} \sum \frac{u-2}{2} \gamma_i' g_i & \text{if } t \neq 2, \\
0 & \text{if } t = 2. \end{cases} \tag{3.26}
\]

**Lemma 3.12.** Let \( \sum \gamma_i g_i \in \Gamma(1)^{01} \) and \( \gamma \in \Gamma \). If \( 1 < u < v = u^2 \) and condition (5) holds, or \( 1 < u < v < u^2 \) and condition (6) holds, then

\[
\partial_{h_1}^{\Omega_1} \left( \sum \gamma_i g_i \right) = -\sum_{j=0}^{\eta-\nu-1} \sum_{\{x : v(x) = j\}} \left( u + u' p^j \left( \frac{t(j)}{2} \right) \right) \left( t(j) \gamma - \gamma \right) g_i - \sum_{\{v(i)\}} v(t(j) \gamma) g_i, \tag{3.27}
\]

where \( v(i) := \max \{ l \geq 0 : p^l \mid i \} \) and \( t(j) := t/p^j \).

**Proof.** Assume first that \( v < u^2 \). By Remark 3.11 we know that \( v(\gamma) = 0 \) and equality [3.25] is satisfied. A direct computation shows that this equality can be written as

\[
(kt + l) \gamma_1 - (k + 1) \gamma \quad \text{if } k = 0 \text{ and } 2 \leq l < t, \\
(k + l) \gamma_1 - k \gamma \quad \text{if } 1 \leq k \leq u' \text{ and } l = 0, \\
(k + l) \gamma_1 - (k + 1) \gamma \quad \text{if } 1 \leq k \leq u' \text{ and } 1 \leq l < t, \\
(k + l) \gamma_1 - k \gamma \quad \text{if } u' < k < 2u' \text{ and } 0 \leq l < 1, \\
(k + l) \gamma_1 - (k + 1) \gamma \quad \text{if } u' < k < 2u' \text{ and } 2 \leq l < t, \\
(k + l) \gamma_1 - k \gamma \quad \text{if } 2 \leq h < t, hh' \leq k < (k + 1)u' \text{ and } 0 \leq l < h, \\
(k + l) \gamma_1 - (k + 1) \gamma \quad \text{if } 2 \leq h < t, hh' \leq k < (k + 1)u' \text{ and } h < l < t.
\]

Write \( t := p^{v(i)} \gamma_0 \). Clearly

\[
\sum_{s=0}^{t-1} \gamma_{1-su} = \sum_{s=0}^{t-1} \gamma_{1-sup^{v(i)} \gamma} = \sum_{s=0}^{t-1} \gamma_{1-sup^{v(i)}} = p^{v(i)} \gamma \sum_{s=0}^{t(v(i)} \gamma_{1-sup^{v(i)}}.
\]

Consequently,

\[
\partial_{h_1}^{\Omega_1} \left( \sum \gamma_i g_i \right) = -\sum_{i} \left( \sum_{s=0}^{t-1} \gamma_{1-su} \right) g_i = \sum_{j=0}^{\eta-\nu-1} \sum_{\{x : v(x) = j\}} p^j \sum_{s=0}^{t(j)-1} \gamma_{1-sup^j} g_i + \sum_{\{v(i)\}} t(j) \gamma_i g_i.
\]

By equality [3.28], if \( t \mid j \), then \( t(j) \gamma_i = t(j) \gamma_i - t \gamma_i \). So, in order to finish the proof of equality [3.27], we only must check that

\[
\sum_{s=0}^{t(j)-1} \gamma_{1-sup^j} = \left( \frac{t}{p^j} + u' p^j \left( \frac{t(j)}{2} \right) \right) \left( t(j) \gamma - \gamma \right) \quad \text{for all } i \text{ such that } v(i) = j. \tag{3.29}
\]

We divided the proof of this in five cases. In the first four we use equality [3.28] and that \( u = tu' \).

1) If \( i = 1 \), then \( j := v(i) = 0 \), and so

\[
\sum_{s=0}^{t(j)-1} \gamma_{1-sup^j} = \sum_{s=0}^{t-1} (1+su) \gamma_1 - \left( u' + 1 + \sum_{s=2}^{t-1} su \right) \gamma = \left( 1 + u' \left( \frac{t}{2} \right) \right) \left( t(j) \gamma - \gamma \right).
\]

2) If \( 1 < i < t \), then

\[
\sum_{s=0}^{t(j)-1} \gamma_{1-sup^j} = \sum_{s=0}^{t(j)-1} (t+su)^{v(i)} \gamma_1 - \left( \sum_{s=0}^{t-1} (su'p^j + 1) + \sum_{s=0}^{t(j)-1} su'p^j \right) \gamma = \left( \frac{1}{p^j} + u' p^j \left( \frac{t(j)}{2} \right) \right) \left( t(j) \gamma - \gamma \right).
\]

3) If \( t < i < u \), then \( i = tq + \bar{i} \) with \( 0 < q < u' \) and \( 0 < \bar{i} < t \), which implies that

\[
\sum_{s=0}^{t(j)-1} \gamma_{1-sup^j} = \sum_{s=0}^{t(j)-1} \gamma_{1+(su'p^j+q)t} = \sum_{s=0}^{t(j)-1} (t+su'p^j) \gamma_1 - \left( \sum_{s=0}^{t-1} (su'p^j + q + 1) + \sum_{s=0}^{t(j)-1} (su'p^j + q) \right) \gamma.
\]

Thus

\[
\sum_{s=0}^{p-1} \gamma_{1-sup^j} = \left( \frac{t}{p^j} + u' p^j \left( \frac{t(j)}{2} \right) \right) \gamma_1 - \left( \frac{t}{p^j} + t(j)q + u' p^j \left( \frac{t(j)}{2} \right) \right) \gamma = \left( \frac{t}{p^j} + u' p^j \left( \frac{t(j)}{2} \right) \right) \left( t(j) \gamma - \gamma \right).
\]
4) If \( \zeta u < \varphi < (\zeta + 1)u \), where \( 0 < \zeta < \nu \), then \( \varphi = \zeta q + \varphi \) with \( \zeta \nu' \leq q < (\zeta + 1)u \) and \( 0 < \varphi < t \). So,

\[
\sum_{s=0}^{t(j)-1} \gamma_{i + \text{sup} p's} = \sum_{s=0}^{t(j)-1} \gamma_{i + (\text{sup} p')q + t} = \sum_{s=0}^{t(j)-1} \left( i + \nu' + t(j)q + u'p\right) \gamma_{i - \text{sup} p'} u - 1
\]

\[
= \left( \left[ \frac{i}{p'} \right] - 1 \right) + \sum_{s=0}^{t(j)-1} \left( i + \nu' + t(j)q + u'p\right) \gamma_{i - \text{sup} p'} u - 1
\]

Since \( \left[ \frac{i}{p'} \right] = \frac{p'}{p} \), this implies that

\[
\sum_{s=0}^{t(j)-1} \gamma_{i + \text{sup} p's} = \left( i + \nu' + t(j)q + u'p\right) \gamma_{i - \text{sup} p'} u - 1
\]

\[
(\gamma_{i - \text{sup} p'} u - 1) \gamma_{i - \text{sup} p'} u - 1
\]

\[
(t(j) - 1) + \sum_{s=0}^{t(j)-1} \left( i + \nu' + t(j)q + u'p\right) \gamma_{i - \text{sup} p'} u - 1
\]

5) If \( p^j u < \varphi \), then by the previous cases we have

\[
\sum_{s=0}^{\nu'} \gamma_{i + \text{sup} p's} = \sum_{s=0}^{\nu'} \gamma_{i + \text{sup} p's} = \left( \left[ \frac{i}{p'} \right] + \nu' + t(j)q + u'p\right) \gamma_{i - \text{sup} p'} u - 1
\]

\[
(\gamma_{i - \text{sup} p'} u - 1) \gamma_{i - \text{sup} p'} u - 1
\]

\[
(t(j) - 1) + \sum_{s=0}^{\nu'} \left( i + \nu' + t(j)q + u'p\right) \gamma_{i - \text{sup} p'} u - 1
\]

where \( 1 \leq \varphi \leq p^j u \) is the remainder of the integer division of \( \varphi \) by \( p^j u \) (the last equality follows from the fact that \( v\gamma_2 - v\gamma_1 = 0 \)).

Assume now that \( v = u^2 \). By Remark 3.11 we know that \( v\gamma_1 - v\gamma_2 = 0 \) and equality [3.24] is satisfied. A direct computation shows that this equality can be written as

\[
(kt + l)\gamma_1 - (k + 1)\gamma_0
\]

\[
\gamma_{kt+1} = \begin{cases}
(kt + l)\gamma_1 - (k + 1)\gamma & \text{if } k = 0 \text{ and } 2 \leq l < t, \\
(kt + l)\gamma_1 - k\gamma & \text{if } k = 1 \text{ and } l = 0, \\
(kt + l)\gamma_1 - (k + 1)\gamma & \text{if } k = 1 \text{ and } 1 \leq l < t, \\
(kt + l)\gamma_1 - k\gamma & \text{if } 1 \leq k < t \text{ and } 0 \leq l \leq k, \\
(kt + l)\gamma_1 - k\gamma & \text{if } 1 < k < t \text{ and } k \leq l < t.
\end{cases}
\]

[3.30]

In the case \( v = u^2 \) the proof of equality [3.27] follows the same pattern than in the case \( v < u^2 \), but using equality [3.30] instead of [3.28]. We leave the details to the reader.

\[\square\]

Lemma 3.13. Let \( \sum \gamma_i g_i \in \Gamma((1)^{l_1}) \), \( \sum \gamma_i g_i \in \Gamma((1)^{l_1}) \) and \( \gamma \in \Gamma \). Assume the hypothesis of Lemma 3.12 holds and that \( u > 2 \). Then equality [3.23] and condition (2) are satisfied if and only if \( \gamma_i = -i(t\gamma_1 - \gamma) \) for all \( t \).

Proof. By Lemma 3.12 we have

\[
\tilde{\gamma}_1 \tilde{\gamma}_1 \left( \sum \gamma_i g_i \right) = -\sum_{j=0}^{\eta-\nu-1} \sum_{i=1}^{\nu-1} \left( t + u'p^2 \right) \left( t(j) \gamma_1 - \gamma \right) g_i - \sum_{i=1}^{\nu-1} i(t\gamma_1 - \gamma) g_i.
\]

[3.31]

Assume first that \( p \) is odd. Then \( u \left| u'p^2 \left( t(j) \gamma_1 - \gamma \right) \right| \) for all \( 0 \leq j < \eta - \nu \), and thus \( u'p^2 \left( t(j) \gamma_1 - \gamma \right) = 0 \), since \( u(t\gamma_1 - \gamma) = v\gamma_1 - v\gamma = 0 \). Consequently, by equalities [3.26] and [3.31], condition (2) holds if and only if \( \gamma_i = -i(t\gamma_1 - \gamma) \) for all \( i \) (note that these \( \gamma_i \)'s satisfy condition [3.23]). Assume now that \( p = 2 \) and \( \nu > 1 \). Since \( 2 \parallel t \), we have \( u'p^2 = 1 \equiv 1 \pmod{u} \). Consequently, if condition (2) is true, then

\[
\left( \frac{u}{2} - 1 \right) \gamma_1 = \left( 1 + u' \left( \frac{t}{2} \right) \right) \left( t(j) \gamma_1 - \gamma \right) = \left( 1 + \frac{u}{2} \right) \left( t(j) \gamma_1 - \gamma \right).
\]

[3.32]

Since \( u\gamma_1 = u(t\gamma_1 - \gamma) = 0 \), this implies that \( -2\gamma_1 = 2(t\gamma_1 - \gamma) \), and so \( -\frac{u}{2} \psi = \frac{\psi}{u}(t\gamma_1 - \gamma) \), because \( 4 \parallel u \). Adding this equality to [3.32], we obtain that \( \gamma_1 = (1 + u)(t\gamma_1 - \gamma) = t\gamma_1 - \gamma \). By condition [3.23] this implies that \( \gamma_i = -i(t\gamma_1 - \gamma) \) for all \( i \) (note that these \( \gamma_i \)'s satisfy condition [3.23]). Conversely assume that \( \gamma_i = -i(t\gamma_1 - \gamma) \) for all \( i \). By equalities [3.26] and [3.31], in order to prove that condition (2) is satisfied, we must check that

\[
\left( \frac{u}{2} - 1 \right) \gamma_i = \left( 1 + u' \left( \frac{t(j) \gamma_1 - \gamma}{2} \right) \right) \left( t(j) \gamma_1 - \gamma \right),
\]

[3.33]

for all \( i \) such that \( j := v(i) \in \{0, \ldots, \eta - \nu - 1\} \). If \( j > 0 \), then

\[
u'p^2 \left( t(j) \gamma_1 - \gamma \right) = u'p^2 \left( t(j) \gamma_1 - \gamma \right) = 2^{j-1}u(t(j) - 1)(t\gamma_1 - \gamma) = 0,
\]

and so, since \( 2 \parallel t \), we have

\[
\left( \frac{u}{2} - 1 \right) \gamma_i = \left( 1 + \frac{u}{2} \right) i(t\gamma_1 - \gamma) = \left( 1 + u' \left( \frac{t(j) \gamma_1 - \gamma}{2} \right) \right) \left( t(j) \gamma_1 - \gamma \right).
\]
as desired. Assume then that \( j = 0 \). Hence

\[
u'2^2j \left( f(j) \right)_2 = u' \left( f \right)_2 = \frac{u}{2}(t - 1) \equiv -\frac{u}{2} \pmod{u}
\]

where the last equality holds since \( 2 \mid t \). Since, moreover \( \frac{u}{2}(t - 1)\gamma_1 = 0 \), we have \( \frac{u}{2}\gamma_1' = \frac{u}{2}\gamma_1 = \frac{u}{2}\gamma_1' \), and so

\[
\left( 1 + u'2^2j \left( f(j) \right)_2 \right) (t\gamma_1 - \gamma) = \left( 1 - \frac{u}{2} \right) (t\gamma_1 - \gamma) = -\gamma + \frac{u}{2}\gamma_1' = \left( -1 + \frac{u}{2} \right) \gamma_1,
\]

which finishes the proof. \( \square \)

Let \( A \) be as at the beginning of this Section, let \( \Gamma \) be an additive abelian group and let \( B \) and \( Z \) be the groups of 2-coboundaries and 2-cocycles of \( X(A, \Gamma) \), respectively. Recall that \( \Gamma_r := \{ \gamma \in \Gamma : r\gamma = 0 \} \), for each natural number \( r \).

In the following result we set \( \gamma(\gamma_1, \gamma) := (f_1(\gamma), \sum v\gamma_1 g^1, -\sum v\gamma_1 g^1) \), where \( \gamma, \gamma_1 \in \Gamma \).

**Theorem 3.14.** If \( u = v \), then

\[
H^2_b(A, \Gamma) = \tilde{H}^2_b(A, \Gamma) = H^2(X(A, \Gamma)) \approx \Gamma_v \oplus \frac{\Gamma}{v\Gamma}. \quad [3.34]
\]

Moreover, the set \( \overline{Z} := \{(z(\gamma_1, \gamma)) : \gamma \in \Gamma \text{ and } \gamma_1 \in \Gamma_v \} \), is a subgroup of \( Z \), that applies surjectively on \( H^2(X(A, \Gamma)) \) and \( B \cap \overline{Z} = \{(f_1(\gamma), 0, 0) : \gamma \in \Gamma \} \). Finally the map \( \Theta : \overline{Z} \to \Gamma \oplus \Gamma_v \), defined by \( \Theta(z(\gamma_1, \gamma)) := (\gamma_1, \gamma) \), is an isomorphism that induces the isomorphism in [3.34].

**Proof.** By Remarks 1.1 and 3.1 we have \( H^2_b(A, \Gamma) = \tilde{H}^2_b(A, \Gamma) = H^2(X(A, \Gamma)) \). Let \( Z' \) be as in Remark 3.11. Thus

\[
Z' = \left\{ \left( f_1(\gamma), \sum \gamma_1 g^1, \sum \gamma_1' g^1' : v\gamma_1 = u\gamma_1' = 0 \text{ and condition } (2) \text{ is satisfied} \right) \right\}.
\]

Since \( t = 1 \), by the definition of \( \tilde{d}_v^{02} \) and equality [3.26], condition \( (2) \) holds if and only if \( \gamma_1 = \gamma_1' \). Hence \( Z' = \overline{Z} \). Clearly the map \( \Theta : \overline{Z} \to \Gamma \oplus \Gamma_v \), defined by \( \Theta(z(\gamma_1, \gamma)) := (\gamma_1, \gamma) \), is an isomorphism. We now compute

\[
B \cap \overline{Z} = \left\{ \left( \tilde{d}_v^{02}(x), d_v^{01}(x), 0 \right) : x \in \Gamma(1)^{00} \right\} \text{ and } d_v^{02}(x) = f_1(\gamma) \text{ for some } \gamma \in \Gamma.
\]

Write \( x = \sum \gamma''_i g^1 \). By the definition of \( \tilde{d}_v^{02} \), we have

\[
\tilde{d}_v^{02}(x) = \sum_{j=1}^{v-2} (\gamma''_{j+1} - \gamma''_j) g \otimes g^1 - (\gamma''_1 + \gamma''_{v-1}) g \otimes g^1 \oplus \sum_{i=1}^{v-1} \sum_{j=1}^{i-1} (\gamma''_{i+j} - \gamma''_j - \gamma''_j) g^1 \otimes g^1.
\]

So, by Remark 3.8, we get that \( \tilde{d}_v^{02}(x) = f_1(\gamma) \) if and only if \( \gamma''_{i+j} = \gamma''_j + \gamma''_{j+1} \) for \( 1 \leq i < v-1 \), and \( \gamma''_{i,j} = -\gamma''_j \) (or, equivalently, if and only if \( \gamma = -v\gamma_1'' \) and \( \gamma''_i = -(v-i)\gamma_1''_i \) for \( 1 \leq i < v \)). Moreover, since \( u = v \), we have \( d_v^{01}(x) = 0 \). Consequently \( B \cap \overline{Z} = \{(f_1(-v\gamma_1''), 0, 0) : \gamma'' \in \Gamma \} \). Thus the map \( \Theta \) induces an isomorphism \( H^2(X(A, \Gamma)) \approx \Gamma_v \oplus \frac{\Gamma}{v\Gamma} \). \( \square \)

Assume that we are under the hypothesis of Theorem 3.14. For each \( \gamma \in \Gamma \) and \( \gamma_1 \in \Gamma_v \), let \( \xi_1^2 : \overline{M}(2) \to \Gamma \) and \( \xi_1^2 : \overline{M}(1) \to \Gamma \) be the maps defined by

\[
\xi_1^2([g^{i1} \otimes g^{i2}]) := \begin{cases} 
\gamma & \text{if } t_1 = t_2 = 1, \\
-\gamma & \text{if } t_1, t_2 \geq 2 \text{ and } t_1 + t_2 \leq v + 1, \text{ and } \\
0 & \text{otherwise},
\end{cases}
\]

\[
\xi_1^2(\gamma_1) := x_{1,2} \gamma_1.
\]

**Proposition 3.15.** The map \( (\xi_1, \xi_2) : \overline{M}(2) \otimes (\overline{M}(2) \otimes \overline{M}(1)) \to \Gamma \) is a 2-cocycle of \( \tilde{C}_N(A, \Gamma) \). Moreover, each 2-cocycle of \( \tilde{C}_N(A, \Gamma) \) is cohomologous to a \( (\xi_1^2, \xi_1^2) \) and 2 two 2-cocycles \( (\xi_1^2, \xi_2^2) \) and \( (\xi_1, \xi_2^2) \) are cohomologous if and only if \( \gamma_1 = \gamma_1' \) and \( v\gamma' = v\gamma \).

**Proof.** By Remark 3.3, Theorem 3.14 and the discussion above Remark 3.5, it suffices to check that

\[
(\xi_1 \xi_1) = z(\gamma_1, \gamma) [\phi_2],
\]

where \( z(\gamma_1, \gamma) \) is as in the statement of Theorem 3.14 and \([\phi_2]\) is as in [3.20]. But this follows by that theorem, equality [3.21] and Proposition 3.4 with \( t = 1 \). \( \square \)

**Proof of Theorem A.** This follows from Remark 1.1, Proposition 3.15 and [19, Theorem 5.8]. \( \square \)
In the following result for each $\gamma, \gamma_1 \in \Gamma$, we set $z(\gamma_1, \gamma) := (f_1(\gamma), \sum \gamma_1 g^i, -\sum i(t_{\gamma_1} - \gamma)g^i)$, where the $\gamma_i$’s with $i \geq 2$ are as in [3.25] if $v < u^2$, and the $\gamma_i$’s with $i \geq 2$ are as in [3.24] if $v = u^2$.

**Theorem 3.16.** If $2 < u < v \leq u^2$, then

$$H_2^N(\mathcal{A}, \Gamma) = \hat{H}_2^N(\mathcal{A}, \Gamma) = H^2(\mathcal{X}(\mathcal{A}, \Gamma)) \simeq \frac{\Gamma}{u\Gamma} \oplus \Gamma_u.$$  \[3.35\]

Moreover the set $\mathcal{Z} := \{z(\gamma_1, \gamma) : v_{\gamma_1} = u\gamma\}$ is a subgroup of $Z$ that applies surjectively on $H^2(\mathcal{X}(\mathcal{A}, \Gamma))$ and $B \cap \mathcal{Z} = \{z(wu, v_\gamma) : \gamma \in \Gamma\}$. Finally the map $\Theta : \mathcal{Z} \to \Gamma \oplus \Gamma_u$, defined by $\Theta(z(\gamma_1, \gamma)) := (\gamma_1, t_{\gamma_1} - \gamma)$ is an isomorphism that induces the isomorphism in [3.35].

**Proof.** Assume first that $v < u^2$. By Remarks 1.1 and 3.1 we have $H_2^N(\mathcal{A}, \Gamma) = \hat{H}_2^N(\mathcal{A}, \Gamma) = H^2(\mathcal{X}(\mathcal{A}, \Gamma))$. Let $Z'$ be as in Remark 3.11. We have

$$Z' = \left\{\left(f_1(\gamma), \sum \gamma_1 g^i, \sum \gamma_1' g^i\right) : \gamma_1' = v_1, w_{\gamma_1} = 0, v_{\gamma_1} = u\gamma \text{ and condition (2) and equality [3.25] hold}\right\}.$$

By Lemma 3.13 we have $Z' = \mathcal{Z}$. Clearly, the map $\Theta : \mathcal{Z} \to \Gamma \oplus \Gamma_u$, given by $\Theta(z(\gamma_1, \gamma)) := (\gamma_1, t_{\gamma_1} - \gamma)$, is an isomorphism. We now compute

$$B \cap \mathcal{Z} = \left\{\left(d_{\mathcal{Z}_{02}}(x), d_{\mathcal{Z}_{11}}(x), 0\right) : x \in \Gamma(1)^{00} \text{ and } d_{\mathcal{Z}_{02}}(x) = f_1(\gamma) \text{ for some } \gamma \in \Gamma\right\}.$$

Write $x = \sum \gamma_i g^i$. By the definition of $d_{\mathcal{Z}_{02}}$, we have

$$d_{\mathcal{Z}_{02}}(x) = \sum_{j=1}^{v-2}(\gamma_j - 1 - \gamma_j' - 2\gamma_j')g^j \otimes g^i - (\gamma_i + \gamma_{i-1})g^j \otimes g^{i-1} \sum_{j=1}^{v-1} \sum_{i=2}^{v-1} (\gamma_{i+1} - \gamma_j - \gamma_j')g^j \otimes g^i.$$

So, by Remark 3.8, we get that $d_{\mathcal{Z}_{02}}(x) = f_1(\gamma)$ if and only if $\gamma_j' - 2\gamma_j' = \gamma_i$, $\gamma_j'' = \gamma_j' + \gamma_i'$ for $1 < i < v - 1$, and $\gamma_{v-1}' = -\gamma_j'(i)$ (or, equivalently, if and only if $\gamma = -\gamma_j'$ and $\gamma'' = i - v\gamma_j' - 1$ for $1 < i < v$). Hence,

$$d_{\mathcal{Z}_{11}}(x) = \sum_{j=1}^{v-1} \sum_{i=2}^{v} (\gamma_i - \gamma_i')g^j = -u\gamma_i'g^j + \sum_{i=2}^{v} (\gamma_i - \gamma_i')g^j.$$

Thus, $B \cap \mathcal{Z} = \{z(u\gamma, v_\gamma) : \gamma \in \Gamma\}$, and so the map $\Theta$ induces an isomorphism $H^2(\mathcal{X}(\mathcal{A}, \Gamma)) \simeq \frac{\Gamma}{u\Gamma} \oplus \Gamma_u$.

The case $v = u^2$ follows in the same way. The unique difference is that, in the characterization of $Z'$ we must use equality [3.24] instead of [3.25].

Assume that we are under the hypothesis of Theorem 3.16. For each $\gamma, \gamma_1 \in \Gamma$ such that $v_{\gamma_1} = u\gamma$, let $\xi^1_\gamma : \overline{\mathcal{M}}(2) \to \Gamma$ and $\xi^2_{\gamma_1, \gamma} : \overline{\mathcal{D}} \otimes \overline{\mathcal{M}}(1) \to \Gamma$ be the maps defined by

$$\xi^1_\gamma([g^1 \otimes g^{i+1}]) := \begin{cases} \gamma & \text{if } i_1 = i_2 = 1, \\
-\gamma & \text{if } i_1, i_2 \geq 2 \text{ and } i_1 + i_2 \leq v + 1, \\
0 & \text{otherwise,} \end{cases}$$

and

$$\xi^2_{\gamma_1, \gamma}([g^{i+1} \otimes g^{i+1}]) := i_1 \left(1 - u\left(\frac{j}{2}\right)\right)(t_{\gamma_1} - \gamma) + \sum_{i=0}^{j} \gamma_{i} - u\gamma_{i},$$

where $0 \leq i < u$, $0 \leq j < t$ and the $\gamma_i$’s are as in equality [3.25] if $v = u^2$ and they are in equality [3.25] if $v < u^2$ (take into account that if $r < 0$ or $r \geq v$, then to apply equalities [3.24] and [3.25], in order to compute explicitly the map $\gamma_r$ in function of $\gamma_1$ and $\gamma$, it is necessary to replace $r$ by the remainder of the integer division of $r$ by $v$).

**Proposition 3.17.** The map $\left(\xi^1_\gamma, \xi^2_{\gamma_1, \gamma}\right) : \overline{\mathcal{M}}(2) \oplus (\overline{\mathcal{D}} \otimes \overline{\mathcal{M}}(1)) \to \Gamma$ is a 2-cocycle of $\hat{C}^N(\mathcal{A}, \Gamma)$. Moreover each 2-cocycle of $\hat{C}^N(\mathcal{A}, \Gamma)$ is cohomologous to a $(\xi^1, \xi^2_{\gamma_1, \gamma})$ and two 2-cocycles $(\xi^1_1, \xi^2_{\gamma_1, \gamma})$ and $(\xi^1_2, \xi^2_{\gamma_1, \gamma})$ are cohomologous if and only if $\gamma_1 - \gamma_1' \in u\Gamma$ and $t_{\gamma_1 - \gamma_1'} = \gamma - \gamma'$.\[3.20\]

**Proof.** By Remark 3.3 and Theorem 3.16 it suffices to check that \(\left(\xi^1_1, \xi^2_{\gamma_1, \gamma}\right) = z(\gamma_1, \gamma)[\hat{\alpha}_2]\), where $z(\gamma_1, \gamma)$ is as in the statement of Theorem 3.16 and $[\hat{\alpha}_2]$ is as in [3.20]. But this follows by that theorem, equality [3.21], the fact that $u(t_{\gamma_1 - \gamma}) = 0$, and Proposition 3.4.

**Proof of Theorem B.** This follows from Remark 1.1, Proposition 3.17 and [19, Theorem 5.8]. \[3.4\]

In the following result we set $z(\gamma_1, \gamma, \gamma') := (f_1(\gamma), \sum \gamma_1 g^i, \sum \gamma_1' g^i)$, where $\gamma, \gamma_1, \gamma_1' \in \Gamma$, the $\gamma_i$’s with $i \geq 2$ are as in [3.24] and $\gamma_i' = t_{\gamma_1'}$ for $i \geq 2$.\[3.4\]
Theorem 3.18. If \( v = u^2 = 4 \), then
\[
H^2_N(A, \Gamma) = \hat{H}^2_N(A, \Gamma) = H^2(\mathcal{X}(A, \Gamma)) \cong \frac{\Gamma}{2\Gamma} \oplus \Gamma_2 \oplus \Gamma_2. \tag{3.36}
\]
Moreover, the subgroup \( Z := \{ z(\gamma_1, \gamma_1) : 2\gamma_1 = 0 \text{ and } 4\gamma_1 = 2\gamma \} \) of \( Z \) that surjectively on \( H^2(\mathcal{X}(A, \Gamma)) \), \( \bar{Z} \cap B = \{ z(2\gamma, 0, 4\gamma) : \gamma \in \Gamma \} \) and the map \( \Theta : Z \to \Gamma \oplus \Gamma_2 \oplus \Gamma_2 \), defined by \( \Theta(z(\gamma_1, \gamma_1)) := (\gamma_1, 2\gamma_1 + \gamma, \gamma_1) \), is an isomorphism that induces the isomorphism in \( [3.36] \).

Proof. By Remarks 1.1 and 3.1 we have \( H^2_N(A, \Gamma) = \hat{H}^2_N(A, \Gamma) = H^2(\mathcal{X}(A, \Gamma)) \). Let \( Z' \) be as in Remark 3.11. Then
\[
Z' = \left\{ \left( f_1(\gamma), \sum \gamma_i g_i', \sum \gamma_i g_i'' \right) : \gamma_i' = r_i \gamma_1, \gamma_i'' = r_i \gamma_1, w_1 = 0, w_1 = u \gamma \text{ and condition (2) and equality [3.24] hold} \right\}.
\]

By the fact that \( 4\gamma_1 = 2\gamma = 0 \) and equalities [3.26] and [3.27], we have
\[
d_{2N}^\Gamma \left( \sum \gamma_i g_i' \right) = -2(2\gamma_1 - \gamma)g - 2(2\gamma_1 - \gamma)g^2 - 4(2\gamma_1 - \gamma)g^3 = 0 = d_{2N}^\Gamma \left( \sum \gamma_i g_i'' \right),
\]
which shows in particular that condition (2) is fulfilled. Hence \( Z' = Z \). Clearly the map \( \Theta : Z \to \Gamma \oplus \Gamma_2 \oplus \Gamma_2 \), defined by \( \Theta(z(\gamma_1, \gamma_1)) := (\gamma_1, 2\gamma_1 + \gamma, \gamma_1) \), is an isomorphism. We now compute the group \( \bar{Z} \cap B \). Let \( x := \gamma_1' g + \gamma_2' g^2 + \gamma_3' g^3 \in \Gamma(1)^{090} \). Arguing as above we get that \( d_{2N}^\Gamma(x) = f_1(\gamma) \) if only if \( \gamma_2'' = -2\gamma_1', \gamma_2'' = -\gamma_1' \) and \( \gamma = 4\gamma_1' \). Hence,
\[
d_{h,N}^\Gamma(\gamma_1 g + \gamma_2' g^2 + \gamma_3' g^3) = (\gamma_1 - \gamma_1'')g + (\gamma_2'' - \gamma_1'')g^2 + (\gamma_3'' - \gamma_1'')g^3 = -2\gamma_1''g + 2\gamma_1''g^3,
\]
and so \( Z \cap B = \{ z(2\gamma, 0, 4\gamma) : \gamma \in \Gamma \} \). Thus, \( \Theta \) induces an isomorphism \( H^2(\mathcal{X}(A, \Gamma)) \cong \frac{\Gamma}{2\Gamma} \oplus \Gamma_2 \oplus \Gamma_2 \). \( \square \)

Assume that we are under the hypothesis of Theorem 3.18. For each \( \gamma_1' \in \Gamma_2 \) and \( \gamma_1 \in \Gamma \) such that \( 4\gamma_1 = 2\gamma \), let \( \xi_1 : M(2) \to \Gamma \) be as above of Proposition 3.17 and \( \xi_{2N}^{\gamma_1, \gamma_1} : D \otimes M(1) \to \Gamma \) be the map defined by
\[
\xi_{2N}^{\gamma_1, \gamma_1}(\gamma) \otimes \gamma_{2N}^{\gamma_1, \gamma_1} = \sum_{i=0}^{3} \gamma_{i} \otimes \gamma_{2N}^{\gamma_1, \gamma_1} = \begin{cases}
0 & \text{if } i = 0 \text{ or } i = j = 0, \\
n\gamma_1 & \text{if } i = 1, \text{ } j = 0 \text{ and } i \in \{1, 3\}, \\
0 & \text{if } i = 1, \text{ } j = 0 \text{ and } i \neq 2, \\
\gamma_1 - \gamma_1' & \text{if } j = 1 \text{ and } i = 1, \\
2\gamma_1 - \gamma & \text{if } j = 1 \text{ and } i = 2, \\
-\gamma_1 - \gamma_1' & \text{if } j = 1 \text{ and } i = 3,
\end{cases}
\]
where \( i, j \in \{0, 1\} \).

Proposition 3.19. The map \( (\xi^1, \xi_{2N}^{\gamma_1, \gamma_1}, \gamma) : M(2) \oplus (D \otimes M(1)) \to \Gamma \) is a 2-cocycle of \( C_N^\Gamma(A, \Gamma) \). Moreover each 2-cocycle of \( C_N^\Gamma(A, \Gamma) \) is cohomologous to a \( (\xi^1, \xi_{2N}^{\gamma_1, \gamma_1}, \gamma) \) and two 2-cocycles \( (\xi^1, \xi_{2N}^{\gamma_1, \gamma_1}, \gamma) \) and \( (\xi^2, \xi_{2N}^{\gamma_1, \gamma_1}, \gamma) \) are cohomologous if and only if \( \gamma_1 - \gamma_1 \in 2\Gamma, \gamma - \gamma = 2(\gamma_1 - \gamma_1) \) and \( \gamma_1 = \gamma_1 \).

Proof. Mimic the proof of Proposition 3.17. \( \square \)

Proof of Theorem C. This follows from Remark 1.1, Proposition 3.19 and [19, Theorem 5.8]. \( \square \)

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