THE NONCOMMUTATIVE POISSON BRACKET AND THE DEFORMATION OF THE FAMILY ALGEBRAS

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ABSTRACT. A.A. Kirillov introduced the family algebras in 2000. In this paper we study the noncommutative Poisson bracket \( P \) on the classical family algebra \( \mathcal{C}_\tau(g) \). We show that \( P \) is the first-order deformation from \( \mathcal{C}_\tau(g) \) to \( \mathcal{Q}_\tau(g) \) where the later is the quantum family algebra. We will prove that the noncommutative Poisson bracket is in fact a Hochschild 2-coboundary therefore the deformation is infinitesimally trivial. In the last part of this paper we also talk about Mackey’s analogue and the quantization problem of the family algebras as in [11].

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1. INTRODUCTION

The family algebras are introduced by A. A. Kirillov in the year 2000 in [12] and [13] as follows:

Let \( g \) be a finite dimensional complex Lie algebra, \( S(g) \) and \( U(g) \) be the symmetric algebra and the universal enveloping algebra of \( g \) respectively. Let \( G \) be a connected and simply connected Lie group with \( \text{Lie}(G) = g \). \( G \) has adjoint actions ad on \( S(g) \) and \( U(g) \).

On the other hand, let \( V_\tau \) be a finite dimensional complex representation of \( g \). Then \( \tau \) gives rise to a representation of \( G \). Therefore \( G \) has a natural action on \( \text{End}_C V_\tau \):

\[
\forall A \in \text{End}_C V_\tau, g \in G, \quad g \cdot A := \tau(g) \circ A \circ \tau(g)^{-1}.
\]

As a result, \( G \) has natural diagonal actions on \( \text{End}_C V_\tau \otimes_S S(g) \) and \( \text{End}_C V_\tau \otimes_U U(g) \): for any \( g \in G \) and for any \( A_i \otimes a^i \in \text{End}_C V_\tau \otimes_S S(g) \), \( B_i \otimes b^i \in \text{End}_C V_\tau \otimes_U U(g) \),

\[
g \cdot (A_i \otimes a^i) := g \cdot A_i \otimes \text{ad}(g) a^i = \tau(g) \circ A_i \circ \tau(g)^{-1} \otimes (\text{ad}g) a^i;
\]

\[
g \cdot (B_i \otimes b^i) := g \cdot B_i \otimes \text{ad}(g) b^i = \tau(g) \circ B_i \circ \tau(g)^{-1} \otimes (\text{ad}g) b^i.
\]
Now we come to the definition of the family algebras, see [12] and [13]:

**Definition 1.1** (The family algebras). The *classical family algebra* is defined to be:

\[ C_r(g) := (\text{End}_{\mathbb{C}} V_r \otimes_{\mathbb{C}} S(g))^G. \] (1)

The *quantum family algebra* is defined to be:

\[ Q_r(g) := (\text{End}_{\mathbb{C}} V_r \otimes_{\mathbb{C}} U(g))^G. \] (2)

Kirillov proved that \( C_r(g) \) and \( Q_r(g) \) are algebras, that is, they are closed under multiplications. He ([12], [13]) and Rozhkovskaya ([19]) have found various the relation between family algebras and the representations of \( g \). On the other hand, in 2011, N. Higson relates family algebras with the admissible representations of complex semisimple Lie groups in [11].

In this paper, we study family algebras in another approach. It is well-known that we have a *Poisson bracket* on \( S(g) \) (see [14]): Let \( X_i \) be a basis of \( g \) and \( c_{ij}^k \) be the structure constant with respect to the basis \( X_i \), then for any \( a, b \in S(g) \), the Poisson bracket is defined to be

\[ \{ a, b \} := c_{ij}^k X_k \cdot \partial^i a \cdot \partial^j b. \]

Now we can define the *noncommutative Poisson bracket* on the classical family algebra:

**Definition 1.2** (The noncommutative Poisson bracket on \( C_r(g) \)). Let \( A, B \in C_r(g), A = A_i \otimes a^i, B = B_j \otimes b^j \). We define the noncommutative Poisson bracket \( P \) as follows:

\[ \{ A, B \} := A_i B_j \otimes \{ a^i, b^j \}. \] (3)

In this paper we will study the properties of the noncommutative Poisson bracket (for short, Poisson bracket) on \( C_r(g) \). The following are two important results we get:

- The Poisson bracket on \( C_r(g) \) characterizes the first-order deformations from \( C_r(g) \) to \( Q_r(g) \), just as the Poisson bracket on \( S(g) \) characterizes the first-order deformations from \( S(g) \) to \( U(g) \), see Proposition 4.6.
- In the Hochschild cochain complex of \( C_r(g) \), the Poisson bracket is a 2-coboundary. In fact we can explicitly find a Hochschild 1-cochain \( \nabla \) which maps to the Poisson bracket under the Hochschild differential, see Theorem 5.4.

It is expected that this result can help us find a quantization map \( C_r(g) \to Q_r(g) \), as proposed by Higson in [11].

This paper is organized as follows: In Section 2 we review the family algebras, in Section 3 we study the first properties of the noncommutative Poisson bracket \( P \), in Section 4 we give the relation between \( P \) and the deformation from \( C_r(g) \) to \( Q_r(g) \), in Section 5 we prove that the noncommutative Poisson bracket \( P \) is a Hochschild 2-coboundary and therefore the deformation is infinitesimally trivial. In Section 6 we talk about the quantization problem of the family algebras. In the three appendices we summarize the results on Hochschild cohomology, Gerstenhaber bracket and their relation to the deformation theory.

**Remark 1.** Although Kirillov and Higson in [12], [13] and [11] require the Lie algebra \( g \) to be semisimple and the representation \( \tau \) to be irreducible, in this paper we do not need this restriction, except for Section 6.

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2. A BRIEF INTRODUCTION TO THE FAMILY ALGEBRAS

Most of the materials in this section can be found in [12] and [13].

First of all, we use the following notation-definition

**Definition 2.1.**
\[
\begin{align*}
\widetilde{C}_r(g) &:= \text{End}_C V_r \otimes_C S(g), \\
\widetilde{Q}_r(g) &:= \text{End}_C V_r \otimes_C U(g).
\end{align*}
\]

\(\widetilde{C}_r(g)\) and \(\widetilde{Q}_r(g)\) consist of matrices with entries in \(S(g)\) and \(U(g)\), respectively. Therefore they are algebras in a natural way: for any \(A_i \otimes a^i, B_j \otimes b^j \in \widetilde{C}_r(g)\) (or \(\widetilde{Q}_r(g)\)), their product is given by the following formula:
\[
(A_i \otimes a^i) \cdot (B_j \otimes b^j) := A_iB_j \otimes a^ib^j.
\]

**Remark 2.** \(A_i\) and \(B_j\) do not commute in \(\widetilde{C}_r(g)\) and \(\widetilde{Q}_r(g)\). Moreover, in the \(\widetilde{Q}_r(g)\) case \(a^i\) and \(b^j\) do not commute either.

The following simple result will be frequently used:

**Proposition 2.1.** In both \(\widetilde{C}_r(g)\) and \(\widetilde{Q}_r(g)\), the matrix component and the \(S(g)\) component always commute. In more detail, for any \(A_i \otimes a^i, B_j \otimes b^j \in \widetilde{C}_r(g)\) (or \(\widetilde{Q}_r(g)\)), we have
\[
(A_i \otimes a^i) \cdot (B_j \otimes b^j) = A_iB_j \otimes a^ib^j
\]
\[
= (Id \otimes a^i) \cdot (A_iB_j \otimes b^j)
\]
\[
= (A_iB_j \otimes a^i) \cdot (Id \otimes b^j).
\]

Proof: It is obvious. \(\square\)

By Definition [11] we know
\[
C_r(g) = (\widetilde{C}_r(g))^G \text{ and } Q_r(g) := (\widetilde{Q}_r(g))^G.
\]

Now we show that the Lie group action can be reduced to the Lie algebra action.

**Proposition 2.2** (The criterion for classical family algebra, [12] Section 1). Let \(A_i \otimes a^i \in \widetilde{C}_r(g)\), then \(A_i \otimes a^i \in C_r(g)\) if and only if
\[
\forall X \in g, [\tau(X), A_i] \otimes a^i + A_i \otimes \{X, a^i\} = 0,
\]
or in other words,
\[
\forall X \in g, [\tau(X), A_i] \otimes a^i = A_i \otimes \{a^i, X\}.
\]

Proof: By definition [11] we know that \(A_i \otimes a^i \in C_r(g)\) if and only if:
\[
\tau(g) \circ A_i \circ \tau(g)^{-1} \otimes (\text{ad}g) a^i = A_i \otimes a^i.
\]

It is well-known that the adjoint action of \(g \otimes a^i\) on \(S(g)\) is exactly the Poisson bracket. As a result, Equation [7] and Equation [8] are infinitesimal versions of the above equation. Since \(G\) is connected and simply connected, they are equivalent. \(\square\)

Similarly we have

**Proposition 2.3** (The criterion for quantum family algebra, [12] Section 1). Let \(A_i \otimes a^i \in \widetilde{Q}_r(g)\), then \(A_i \otimes a^i \in Q_r(g)\) if and only if
\[
\forall X \in g, [\tau(X), A_i] \otimes a^i + A_i \otimes [X, a^i] = 0,
\]
or in other words,
\[
\forall X \in g, [\tau(X), A_i] \otimes a^i = A_i \otimes [a^i, X].
\]
Proof: Similar to the proof of Proposition 2.3 □

Then we can prove the following result:

**Corollary 2.4** (see also [12] and [13]). \( C_\tau(g) \) and \( Q_\tau(g) \) are subalgebras of \( \tilde{C}_\tau(g) \) and \( \tilde{Q}_\tau(g) \) respectively.

Proof: Let \( A_i \otimes a^i \) and \( B_j \otimes b^j \) be two elements in \( C_\tau(g) \). Their product
\[
(A_i \otimes a^i) \cdot (B_j \otimes b^j) = A_iB_j \otimes a^ib^j.
\]

Now \( \forall X \in g \),
\[
[\tau(X), A_iB_j] \otimes a^ib^j = [\tau(X), A_i]B_j \otimes a^ib^j + A_i[\tau(X), B_j] \otimes a^ib^j
\]
\[
= ([\tau(X), A_i] \otimes a^i) \cdot (B_j \otimes b^j) + (A_i \otimes a^i) \cdot ([\tau(X), B_j] \otimes b^j).
\]
The second equality is because of Proposition 2.3, the matrix component always commutes with the \( S(g) \) component.

Now by Proposition 2.2, we know
\[
\text{the above } = (A_i \otimes \{a^i, X\}) \cdot (B_j \otimes b^j) + (A_i \otimes a^i) \cdot B_j \otimes \{b^j, X\}
\]
\[
= A_iB_j \otimes \{a^i, X\}b^j + A_iB_j \otimes a^i\{b^j, X\} \quad \text{(Proposition 2.2)}
\]
\[
= A_iB_j \otimes \{a^ib^j, X\}.
\]

Hence we get
\[
A_iB_j \otimes a^ib^j \in C_\tau(g).
\]

In the same way we can show that if \( A_i \otimes a^i \) and \( B_j \otimes b^j \) are in \( Q_\tau(g) \), then
\[
A_iB_j \otimes a^ib^j \in Q_\tau(g). \quad \square
\]

It is not difficult to see that the family algebras are nontrivial. In fact, let \( I(g) = S(g)^g \) be the invariant subalgebra of \( S(g) \) and \( Z(g) \) be the center of \( U(g) \). We have

**Proposition 2.5** ([13]). \( I(g) \) embeds into \( C_\tau(g) \) as scalar matrices
\[
I(g) \hookrightarrow C_\tau(g)
\]
\[a \mapsto \text{Id} \otimes a. \quad (9)
\]

Similarly \( Z(g) \) embeds into \( Q_\tau(g) \) as scalar matrices too.

Proof: It is obvious that \( I(g) \) embeds into \( \tilde{C}_\tau(g) \) as scalar matrices. Now by Proposition 2.2, it is easy to see that the image is contained in \( C_\tau(g) \).

The proof for \( Z(g) \) and \( Q_\tau(g) \) is the same. □

**Example 1.** For any \( g \), when the representation \( \tau \) is the trivial representation, we see that \( I(g) = C_\tau(g) \) and \( Z(g) = Q_\tau(g) \).

**Example 2.** For \( g = \text{sl}(2, \mathbb{C}) \) and \( \{e, f, h\} \) be the standard basis of \( \text{sl}(2, \mathbb{C}) \) which satisfies the commutation relation
\[
[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f. \quad (10)
\]
Let \( \tau \) be the 2-dimensional standard representation, we can find an element \( M \in C_\tau(g) \) which is not in \( I(g) \). In fact
\[
M = \begin{pmatrix}
\frac{h}{e} & f \\
-f & \frac{h}{e}
\end{pmatrix} \quad (11)
\]
We can also find an element in \( Q_\tau(g) \) with the same expression of \( M \), see [12] and [13].

**Remark 3.** When \( \tau \) is nontrivial irreducible and \( g \) is semisimple, \( I(g) \) is not equal to \( C_\tau(g) \) and \( Z(g) \) is not equal to \( Q_\tau(g) \) either, see Corollary 5.3 below or [12].
3. THE NONCOMMUTATIVE POISSON BRACKET ON $\widehat{\mathcal{C}}_r(g)$

The noncommutative Poisson bracket on $C_r(g)$ in Definition 3.2 can be easily extended to $\widehat{C}_r(g)$:

**Definition 3.1.** Let $A, B \in \widehat{C}_r(g)$, $A = A_i \otimes a^i$, $B = B_j \otimes b^j$. We define the noncommutative Poisson bracket as follows:

$$\{A, B\} := A_i B_j \otimes \{a^i, b^j\}.$$  \hspace{1cm} (12)

We will also denote the noncommutative Poisson bracket by $P$.

**Remark 4.** P. Xu gives a similar construction in [22] Example 2.2. Nevertheless, our construction has different purpose than his.

**Remark 5.** The Poisson bracket on $\widehat{C}_r(g)$ is not anti-symmetric and does not satisfies the Leibniz rule and the Jacobi identity.

Nevertheless, J. Block and E. Getzler in 1992 give a definition of Poisson bracket on noncommutative algebra in [3] and we can prove that our noncommutative Poisson bracket $P$ satisfies the requirement of Poisson bracket in that sense:

**Definition 3.2 (3 Definition 1.1).** A Poisson bracket on a (possibly noncommutative) algebra $A$ is a Hochschild 2-cocycle $P \in Z^2(A, A)$ such that $P \circ P \in C^3(A, A)$ is a 3-coboundary. In other words

$$P \circ P \in B^3(A, A) \subset Z^3(A, A) \subset C^3(A, A).$$ \hspace{1cm} (13)

For Hochschild cohomology see Appendix A and for the definition of $P \circ P$ see Proposition B.1

**Remark 6.** In Definition 3.2, we can consider the condition $P \in Z^2(A, A)$ as a noncommutative Leibniz rule and $P \circ P \in B^3(A, A)$ as a noncommutative Jacobi identity. They together implies that $P$ can be lift to an associative product on $A$ up to order 3, see Corollary C.2.

For our algebra $\widehat{C}_r(g)$ and the Poisson bracket $P$ in Definition 3.1 first we can prove that $P$ is a 2-cocycle. We have the following proposition:

**Proposition 3.1.** For any $A, B, C \in \widehat{C}_r(g)$, we have

$$A\{B, C\} - \{AB, C\} + \{A, BC\} - \{A, B\}C = 0.$$ \hspace{1cm} (14)

In other words, we have $d_H P = 0$ where $d_H$ is the Hochschild differential. Therefore

$$P \in Z^2(\widehat{C}_r(g), \widehat{C}_r(g)).$$ \hspace{1cm} (15)

**Proof:** Let $A = A_i \otimes a^i$, $B = B_j \otimes b^j$ and $C = C_k \otimes c^k$. Remember Proposition 2.1 the matrix component and the $S(g)$ component always commute. Then by the definition of the Poisson bracket we get

$$A\{B, C\} - \{AB, C\} + \{A, BC\} - \{A, B\}C$$

$$= A_i B_j C_k \otimes (a_i b_j - a_i b_j - a_i b_j - a_i b_j) c_k.$$  \hspace{1cm} (16)

By the Leibniz rule of the (ordinary) Poisson bracket on $S(g)$ we know that

$$a_i b_j - a_i b_j - a_i b_j - a_i b_j = 0.$$

**Proposition 3.2.** $P \circ P$ is a 3-coboundary. In other words, $P \circ P \in B^3(\widehat{C}_r(g), \widehat{C}_r(g))$.

In fact, we can define a 2-cocahin $\Phi \in C^2(\widehat{C}_r(g), \widehat{C}_r(g))$ as follows: Let $A = A \otimes a$ and $B = B \otimes b$ (to simplify the notation we omit the super and sub-indices)

$$\Phi(A, B) := AB \otimes \frac{1}{2} c^s_{ijkl} X_s X_t \partial^i \partial^j a \partial^k \partial^l b$$

$$+ AB \otimes \frac{1}{3} c^s_{ijkl} X_s \partial^k \partial^l a \partial^i b + \partial^i a \partial^k \partial^j b.$$ \hspace{1cm} (17)

Then we have

$$P \circ P + d_H \Phi = 0.$$
Lemma 3.3. We can define a \( t \) is called the deformation of \( \star \) the deformation of \( T \) in Definition 1.2 is well-defined.

Proposition 3.4. That is the following proposition:

Proof: We can proof this proposition by computation using Proposition 2.2 and the definition of the \( \star \) PBW:

\[
\phi(a, b) := \frac{1}{2} c_{ij} c_{kl} X_s X_{\tau} \partial^j \partial^k a \partial^i \partial^l b + \frac{1}{3} c_{js} c_{kt} X_{\tau}(\partial^s \partial^j a \partial^t \partial^k b + \partial^t \partial^k a \partial^j \partial^s b).
\]

Then for any \( a, b, c \in S(\mathfrak{g}) \) we have

\[
\{ a, \{ b, c \} \} = \{ \{ a, b \}, c \} + (d_H \phi)(a, b, c) = 0.
\]

Assume we have Lemma 3.3, by abusing the notations we have \( \Phi = \text{Id} \otimes \phi \), then we immediately get

\[
P \circ P + d_H \Phi = 0.
\]

Proof of Lemma 3.3 We can check it by hand using Jacobi identity. Another approach involves the star-product on \( S(\mathfrak{g}) \) and the general result of deformation theory and we defer it to Proposition 4.2.

By Proposition 3.1 and 3.2 we know that the noncommutative Poisson bracket in Definition 3.1 is indeed a Poisson bracket in the sense of Definition 3.2.

Before we move on, we need to prove that the Poisson bracket indeed maps \( C_r(\mathfrak{g}) \otimes C_r(\mathfrak{g}) \) to \( C_r(\mathfrak{g}) \). That is the following proposition:

Proposition 3.4. For any \( \mathcal{A}, \mathcal{B} \in C_r(\mathfrak{g}) \), we have that \( \{ \mathcal{A}, \mathcal{B} \} \) is still in \( C_r(\mathfrak{g}) \). In other words, the noncommutative Poisson bracket in Definition 3.2 is well-defined.

Proof: We can proof this proposition by computation using Proposition 2.2 and the definition of the noncommutative Poisson bracket \( P \). In Section 5 we will give another proof using a different construction of \( P \). See Corollary 5.1.

4. The noncommutative Poisson bracket and the deformation of \( \widetilde{C}_r(\mathfrak{g}) \)

We will see in this section that the Poisson bracket plays an essential role in the deformation of \( \widetilde{C}_r(\mathfrak{g}) \).

4.1. A quick review of the deformation from \( S(\mathfrak{g}) \) to \( U(\mathfrak{g}) \) and the Poisson bracket. Before studying the deformation of \( \widetilde{C}_r(\mathfrak{g}) \), let us first review the corresponding theory of \( S(\mathfrak{g}) \) and \( U(\mathfrak{g}) \).

It is well-known that

\[
S(\mathfrak{g}) = T(\mathfrak{g})/(X \otimes Y - Y \otimes X)
\]

and

\[
U(\mathfrak{g}) = T(\mathfrak{g})/(X \otimes Y - Y \otimes X - [X, Y])
\]

where \( T(\mathfrak{g}) \) is the tensor algebra of \( \mathfrak{g} \).

Moreover, we consider the algebra

\[
U_t(\mathfrak{g}) = T(\mathfrak{g})/(X \otimes Y - Y \otimes X - t[X, Y]).
\]

For \( t \neq 0 \) all the algebras \( U_t(\mathfrak{g}) \) are isomorphic to \( U(\mathfrak{g}) \), and when \( t = 0 \), \( U_0(\mathfrak{g}) \) is isomorphic to \( S(\mathfrak{g}) \). \( t \) is called the deformation parameter.

We have the Poincaré-Birkhoff-Witt map \( \text{I}_{PBW} \) from \( S(\mathfrak{g}) \) to \( U_t(\mathfrak{g}) \) given by:

\[
\text{I}_{PBW} : \quad S(\mathfrak{g}) \longrightarrow U_t(\mathfrak{g})
\]

\[
X_1 X_2 \ldots X_k \mapsto \sum_{\sigma \in S_k} \frac{1}{k!} X_{\sigma(1)} X_{\sigma(2)} \ldots X_{\sigma(k)}.
\]

The Poincaré-Birkhoff-Witt theorem (see [15]) tells us that the above map \( \text{I}_{PBW} \) is an isomorphism between \( \mathfrak{g} \)-vector spaces.
Remark 7. The map $I_{\text{PBW}}$ is not an algebraic isomorphism unless $g$ is an abelian Lie algebra or $t = 0$.

Therefore we have the following definition

Definition 4.1. The map $I_{\text{PBW}}$ pulls back the multiplication of $U_t(g)$ to $S(g)$ and we call it the \textbf{star-product} on $S(g)$, denoted by $*_t$. For any $a, b \in S(g)$

$$a *_t b := I_{\text{PBW}}^{-1}(a \cdot I_{\text{PBW}}(b)).$$

In particular we denote $*_t$ simply as $*$. When $t = 0$ the star-product reduces to the original production on $S(g)$. Obviously $*_t$ satisfies the associativity law because the multiplication on $U_t(g)$ is associative.

Now by definition, the map $I_{\text{PBW}}$ gives an algebraic isomorphism

$$I_{\text{PBW}} : (S(g), *_t) \sim \to (U_t(g), \cdot).$$

Therefore we can identify $U_t(g)$ with $(S(g), *_t)$, especially we can identify $U(g)$ with $(S(g), *)$.

Remark 8. Our star-product $*_t$ is not exactly the same as the star-product constructed by Kontsevich in Section 8. Nevertheless, they give isomorphic algebra structures on $S(g)$.

The star-product $*_t$ depends on the deformation parameter $t$. In fact we can write the first few terms of $*_t$.

Proposition 4.1 ([10] Section 3). We can write $*_t$ as

$$a *_t b = ab + \frac{t}{2} \{a, b\} + O(t^2).$$

In other words, the Poisson bracket on $S(g)$ is exactly the \textbf{first-order deformation} from $S(g)$ to $U(g)$.

Remark 9. In fact we can find the expressing of the $t^2$ term in the star-product. According to [9] Remark 4.7, for any $a, b \in S(g)$, the $t^2$ term is

$$m_2(a, b) := \frac{1}{8} c_{ij}^k c_{kl}^s X_s \partial^i \partial^k a \partial^j \partial^l b + \frac{1}{12} c_{ks}^i c_{js}^l X_i (\partial^k \partial^j a \partial^l b + \partial^i \partial^j b \partial^k b).$$

Now we can give another proof of Lemma 3.3

Proposition 4.2 (Lemma 3.3). We can define a 2-cochain $\phi \in C^2(S(g), S(g))$ as follows: for any $a, b \in S(g)$

$$\phi(a, b) := \frac{1}{2} c_{ij}^k c_{kl}^s X_s \partial^i \partial^k a \partial^j \partial^l b + \frac{1}{3} c_{ks}^i c_{js}^l X_i (\partial^k \partial^j a \partial^l b + \partial^i \partial^j b \partial^k b).$$

Then for any $a, b, c \in S(g)$ we have

$$\{a, \{b, c\}\} - \{\{a, b\}, c\} + (d_H \phi)(a, b, c) = 0.$$  

Proof: In the framework of deformation theory (see Appendix C). Let $m = *_t$ be the star-product. Compare Proposition 4.1 Equation 28 and Equation 75 we get

$$P = 2m_1 \text{ and } \phi = 4m_2.$$  

where we denote the Poisson bracket on $S \mathfrak{g}$ by $P$ also.

Since we know from the definition that the star-product is associative, by Proposition C.2 especially Equation 82 we get

$$m_1 \circ m_1 + d_H m_2 = 0$$

hence

$$P \circ P + d_H \phi = 0$$

and this is exactly Equation 30. □

If we restrict ourselves to the invariant subalgebra $I(g) = S(g)^{\mathfrak{g}}$, then we have the following well-known result about the Poisson bracket:

$$\{a, \{b, c\}\} - \{\{a, b\}, c\} + (d_H \phi)(a, b, c) = 0.$$  

Therefore we have the following definition

Our star-product $*_{\text{PBW}}$.
Proposition 4.3 ([13]). The Poisson bracket vanishes on $I(\mathfrak{g})$. In other words, for any $a, b \in I(\mathfrak{g})$, we have

$$\{a, b\} = 0. \quad (31)$$

Proof: This is almost the definition of $I(\mathfrak{g})$. In fact

$$\{a, b\} = c_{ij}^k X_k(\partial^j a)(\partial^i b) = (c_{ij}^k X_k \partial^j a)(\partial^i b) = (\text{ad}X_j(a))(\partial^j b).$$

Since $a \in I(\mathfrak{g})$, we know that $\text{ad}X_j(a) = 0$ for any $X_j$, as a result, $\{a, b\} = 0$. □

Remark 10. Proposition 4.3 and Proposition 4.4 tell us that the first-order deformation from $I(\mathfrak{g})$ to $Z(\mathfrak{g})$ is zero.

In fact we have the much deeper Duflo’s isomorphism theorem:

Theorem 4.5 ([6], [16] Section 8, [1] and [4]). There exists an algebraic isomorphism:

$$\text{Duf} : I(\mathfrak{g}) \to Z(\mathfrak{g}) \quad (33)$$

□

Remark 11. In general, the map $\text{Duf}$ will be different from the Poincaré-Birkhoff-Witt map $I_{\text{PBW}}$ in Proposition 4.4 although they have the same domain and image.

Remark 12. Since $Z(\mathfrak{g})$ is isomorphic to $Z_t(\mathfrak{g})$ as algebras, the map $\text{Duf}$ can be easily generalized to the map $\text{Duf}_t : I(\mathfrak{g}) \to Z_t(\mathfrak{g})$ for any $t$.

4.2. The deformation from $\widetilde{C}_t(\mathfrak{g})$ to $\widetilde{Q}_t(\mathfrak{g})$ and the noncommutative Poisson bracket. Parallel to the constructions of $S(\mathfrak{g})$, let us make the following definition

Definition 4.2. We define the algebra $\widetilde{Q}_t^G(\mathfrak{g})$ as

$$\widetilde{Q}_t^G(\mathfrak{g}) := \text{End}_{\mathfrak{g}} \otimes U_t(\mathfrak{g}). \quad (34)$$

Moreover, we define

$$Q_t^G(\mathfrak{g}) := (\text{End}_{\mathfrak{g}} \otimes U_t(\mathfrak{g}))^G. \quad (35)$$

By definition, we have $\widetilde{Q}_0(\mathfrak{g}) = \widetilde{C}(\mathfrak{g})$, $Q_0(\mathfrak{g}) = C(\mathfrak{g})$ and for any $t \neq 0$ we have $\widetilde{Q}_t(\mathfrak{g}) \cong \widetilde{Q}_t(\mathfrak{g})$.

We also have the Poincaré-Birkhoff-Witt map on the family algebras:

Definition 4.3. The Poincaré-Birkhoff-Witt map $F_{\text{PBW}}$ on family algebras is defined to be $\text{Id} \otimes I_{\text{PBW}}$. In other words:

$$F_{\text{PBW}} : \widetilde{C}_t(\mathfrak{g}) \longrightarrow \widetilde{Q}_t^G(\mathfrak{g})$$

$$A_i \otimes a^i \longmapsto A_i \otimes I_{\text{PBW}}(a^i). \quad (36)$$

$F_{\text{PBW}}$ is an isomorphism between $\mathfrak{g}$-vector spaces.
As $I_{PBW}$, $F_{PBW}$ is not an algebraic isomorphism either. Nevertheless it can also pull back the product on $\widetilde{Q}^t(g)$ to $\widetilde{C}_r(g)$:

**Definition 4.4.** The star-product $\ast_t$ on $\widetilde{C}_r(g)$ is defined to the pull-back of the product on $\widetilde{Q}^t(g)$ via the map $F_{PBW}$. In other words, for any $A, B \in \widetilde{C}_r(g)$

$$A \ast_t B := F_{PBW}^{-1}(F_{PBW}(A) \cdot F_{PBW}(B)).$$  \hspace{1cm} (37)

Moreover, if we write $A = A_i \otimes a^i$ and $B = B_j \otimes b^j$, then

$$(A_i \otimes a^i) \ast_t (B_j \otimes b^j) = A_i B_j \otimes (a^i \ast_t b^j).$$  \hspace{1cm} (38)

Therefore the map $F_{PBW}$ gives an algebraic isomorphism

$$F_{PBW} : (\widetilde{C}_r(g), \ast_t) \longrightarrow (\widetilde{Q}^t(g), \cdot).$$  \hspace{1cm} (39)

Therefore we can identify $\widetilde{Q}^t(g)$ with $(\widetilde{C}_r(g), \ast_t)$, especially we can identify $\widetilde{C}_r(g)$ with $(\widetilde{C}_r(g), \ast)$.

For the star-product on $\widetilde{C}_r(g)$, we also have

**Proposition 4.6.** We can write the star-product $\ast_t$ on $\widetilde{C}_r(g)$ as

$$A \ast_t B = AB + \frac{t}{2}([A, B]) + O(t^2).$$  \hspace{1cm} (40)

In other words, the Poisson bracket on $\widetilde{C}_r(g)$ is exactly the first-order deformation from $\widetilde{C}_r(g)$ to $\widetilde{Q}^t(g)$.

Proof: This is just a combination of the definition of star-product (Definition 4.4), the definition of noncommutative Poisson bracket (Definition 3.1) and Proposition 4.1. \hfill \Box

**Remark 13.** By now, the results in this subsection exemplified the slogan "the deformation theory of an algebra $A$ is the same as that of the matrix algebra Mat$_{n \times n}(A)$." However, when restrict to the invariant subalgebras, these two become different.

If we restrict ourselves to the family algebra $C_r(g)$, i.e. the invariant subalgebra of $\widetilde{C}_r(g)$, we get the follow proposition which is similar to Proposition 4.4.

**Proposition 4.7.** The image of $C_r(g)$ under the Poincaré-Birkhoff-Witt map $F_{PBW}$ is exactly $Q^t_r(g)$, the invariant subalgebra of $\widetilde{Q}^t_r(g)$. In other words,

$$F_{PBW} : C_r(g) \rightarrow Q^t_r(g)$$  \hspace{1cm} (41)

is an isomorphism between vector spaces.

Proof: Just remember that $F_{PBW} : \widetilde{C}_r(g) \rightarrow \widetilde{Q}^t_r(g)$ is an isomorphism between $g$-vector spaces, i.e. it is compatible with the $g$-actions. \hfill \Box

Now it is natural to ask for the corresponding result of Proposition 4.4 and the Duflo’s isomorphism theorem 4.5 on family algebras.

In fact, in Theorem 5.4 of this paper we will prove that the noncommutative Poisson bracket vanishes in the Hochschild cohomology. The generalization of Duflo’s isomorphism theorem to family algebras is still an open problem, see Section 6.

5. The vanishing of the noncommutative Poisson bracket in $\text{HH}^2(C_r(g))$

5.1. The map $\nabla$. In this section we focus on the classical family algebra $C_r(g)$ and the matrix algebra $C_r(g)$. 

Definition 5.1 (The definition of $\nabla$). We define the map $\nabla : \widetilde{C}_{r}(g) \to \widetilde{C}_{r}(g)$ as follows: Fix a basis $X_k$ of $g$. Let $A_i \otimes a^i \in \widetilde{C}_{r}(g)$.

$$\nabla( A_i \otimes a^i) := A_i \tau(X_k) \otimes \partial^k (a^i).$$

(42)

Notice that $\widetilde{C}_{r}(g)$ is nothing but a matrix algebra with entries in $S(g)$. In the form of matrices,

$$\nabla( A) = \partial^k (A) \tau(X_k).$$

(43)

Hence $\nabla$ is a first-order differential operator on $\widetilde{C}_{r}(g)$.

From Equation (43) it is not difficult to see that the map $\nabla$ dose not depend on the expression of $A \in \widetilde{C}_{r}(g)$ as $A_i \otimes a^i$.

To show $\nabla$ is a well-defined map, it is now sufficient to prove the following proposition:

Proposition 5.1. The map $\nabla : \widetilde{C}_{r}(g) \to \widetilde{C}_{r}(g)$ is independent of the choice of the basis of $g$.

Proof: We need to do some computation. Let $\tilde{X}_j$ be another basis of $g$. Then

$$\tilde{X}_j = T^k_j X_k$$

where $T^k_j$ is the transition matrix. Then, let $\tilde{\partial}^j$ be the partial derivation with respect to $\tilde{X}_j$, we have

$$\tilde{\partial}^j = (T^{-1})^j_i \partial^k.$$ 

Let $\tilde{\nabla}$ be the $\nabla$ map under the basis $\tilde{X}_j$, for $A_i \otimes a^i \in \widetilde{C}_{r}(g)$, we have

$$\tilde{\nabla}(A_i \otimes a^i) = A_i \tau(\tilde{X}_j) \otimes \tilde{\partial}^j(a^i)$$

$$= A_i \tau(T^k_j X_k) \otimes (T^{-1})^j_i \partial^k(a^i).$$

The constant $(T^{-1})^j_i$ can be moved to the first component, hence

$$(T^{-1})^j_i$$

the above $= T^k_j (T^{-1})^j_i A_i \tau(X_k) \otimes \partial^k(a^i)$$

$$= \delta^k_i A_i \tau(X_k) \otimes \partial^k(a^i)$$

$$= A_i \tau(X_k) \otimes \partial^k(a^i)$$

$$= \nabla( A_i \otimes a^i)$$

So $\nabla$ is invariant under the change of basis of $g$. $\square$

The map $\nabla$ is obviously $\mathbb{C}$-linear, moreover it has the following important property:

Proposition 5.2. The image under $\nabla$ of the subalgebra $C_{r}(g)$ is contained in $C_{r}(g)$ itself.

Proof: The proof requires some careful computation.

Let $A_i \otimes a^i \in C_{r}(g)$, then

$$\nabla( A_i \otimes a^i) = A_i \tau(X_k) \otimes \partial^k (a^i).$$

By Proposition 2.2, we only need to show that, for $X_j$ which is one of the basis of $g$, we have

$$[\tau(X_j), A_i \tau(X_k)] \otimes \partial^k a^i = A_i \tau(X_k) \otimes \{ \partial^k a^i, X_j \}.$$ 

(44)

In fact,

the left hand side

$$= [\tau(X_j), A_i] \tau(X_k) \otimes \partial^k a^i + A_i [\tau(X_j), \tau(X_k)] \otimes \partial^k a^i$$

$$= \nabla([\tau(X_j), A_i] \otimes a^i) + A_i [\tau(X_j), \tau(X_k)] \otimes \partial^k a^i.$$
To make the following computation more clear, let us denote:

\[ I := \nabla (\{ \tau (X_j), A_i \} \otimes a^i), \]
\[ II := A_i [\tau (X_j), \tau (X_k)] \otimes \partial^k a^i. \]

For I, since \( A_i \otimes a^i \in C_\tau (g) \), by Proposition 2.2 we have:

\[ I = \nabla (\{ \tau (X_j), A_i \} \otimes a^i) = \nabla (A_i \otimes \{ a^i, X_j \}) \]

From the definition of the Poisson bracket on \( S(g) \), we know

\[ \{ a^i, X_j \} = c_{ij}^r X_r \partial^r a^i \partial^j X_j = c_{ij}^r X_r \partial^r a^i \delta_j^i = c_{sj}^r X_r \partial^r a^i. \]

Therefore

\[ I = \nabla (A_i \otimes c_{sj}^r X_r \partial^r a^i) \]
\[ = A_i \tau (X_i) \otimes \partial^i (c_{sj}^r X_r \partial^r a^i) \]
\[ = A_i \tau (X_i) \otimes c_{sj}^r (\partial^i (X_r) \partial^r a^i + X_r \partial^i \partial^r a^i) \]
\[ = A_i \tau (X_i) \otimes c_{sj}^r \delta_i^r \partial^s a^i + A_i \tau (X_i) \otimes c_{sj}^r X_r \partial^r \partial^s a^i \]
\[ = A_i \tau (X_i) \otimes c_{sj}^r \partial^s a^i + A_i \tau (X_i) \otimes c_{sj}^r X_r \partial^r \partial^s a^i. \]

Nevertheless, we have

\[ c_{sj}^r X_r \partial^r \partial^s a^i = \{ \partial^i a^i, X_j \} \]

As a result

\[ I = A_i \tau (X_i) \otimes \partial^i a^i + A_i \tau (X_i) \otimes \{ \partial^i a^i, X_j \} \tag{45} \]

As for II, we know

\[ II = A_i [\tau (X_j), \tau (X_k)] \otimes \partial^k a^i \]
\[ = A_i \tau (X_i) \otimes \partial^i a^i \]
\[ = A_i [\tau (X_j), \tau (X_k)] \otimes \partial^k a^i. \]

We know that \( [X_j, X_k] = c_{jk}^r X_r \) hence

\[ \tau ([X_j, X_k]) = \tau (c_{jk}^r X_r) = c_{jk}^r \tau (X_r) = -c_{kj}^r \tau (X_r). \]

As a result

\[ II = A_i \tau ([X_j, X_k]) \otimes \partial^k a^i \]
\[ = -A_i c_{kj}^r \tau (X_r) \otimes \partial^k a^i \]
\[ = -A_i \tau (X_i) \otimes c_{kj}^r \partial^k a^i \tag{46} \]

Combine Equation 45 and 46 we get

the left hand side of 44
\[ = I + II \]
\[ = A_i \tau (X_i) \otimes c_{sj}^r \partial^s a^i + A_i \tau (X_i) \otimes \{ \partial^i a^i, X_j \} - A_i \tau (X_i) \otimes c_{sj}^r \partial^k a^i \]
\[ = A_i \tau (X_i) \otimes \{ \partial^i a^i, X_j \} \]
\[ = \text{the right hand side of 44} \]

Therefore we finishes the proof. \( \square \)

Now according to Proposition 5.2, we can say that \( \nabla \) is a \( \mathbb{C} \)-linear map from \( C_\tau (g) \) to \( C_\tau (g) \). In other words, \( \nabla \) belongs to the Hochschild 1-cochain \( C^1 (C_\tau (g), C_\tau (g)) \). see Appendix A for a review of Hochschild cohomology.

Before moving on to the next section, we give a direct application of the map \( \nabla \).
Corollary 5.3 ([12] Section 1). When the Lie algebra $\mathfrak{g}$ is semisimple and $\tau$ is a nontrivial irreducible representation, the classical family algebra $C_\tau(\mathfrak{g})$ is more than $I(\mathfrak{g})$, i.e. $I(\mathfrak{g}) \subsetneq C_\tau(\mathfrak{g})$, and we also have $Z(\mathfrak{g}) \subsetneq Q_\tau(\mathfrak{g})$.

Proof: Let $\text{Cas}$ be the quadratic Casimir element in $I(\mathfrak{g})$, $\deg \text{Cas} = 2$. Then by Proposition 5.2 we know that $\nabla(\text{Cas}) \in C_\tau(\mathfrak{g})$ but $\deg \nabla(\text{Cas}) = 1$. Since $\tau$ is nontrivial we know that $\nabla(\text{Cas}) \neq 0$. On the other hand, since $\mathfrak{g}$ is semisimple, there is no nonzero degree-1 element in $I(\mathfrak{g})$, therefore $\nabla(\text{Cas}) \notin I(\mathfrak{g})$ hence $I(\mathfrak{g}) \subsetneq C_\tau(\mathfrak{g})$.

Since there is a PBW map $F_{\text{PBW}} : C_\tau(\mathfrak{g}) \rightarrow Q_\tau(\mathfrak{g})$ which maps $I(\mathfrak{g})$ to $Z(\mathfrak{g})$, we know that $Z(\mathfrak{g}) \subsetneq Q_\tau(\mathfrak{g})$. □

Remark 14. In fact, in Example 2 the element $M$ is obtained in the same way as $\nabla \text{Cas}$ in the above corollary.

Remark 15. Then map $\nabla$ is motivated by the element $M_P$ defined in Section 1 of [12]. Nevertheless in that paper $M_P$ is defined only for $P \in I(\mathfrak{g})$ and here we extend the domain to all $C_\tau(\mathfrak{g})$.

5.2. $\nabla$ and the Poisson bracket. In this subsection, we build up the relation between $\nabla$ and the Poisson bracket $P$.

First we review some notations of Hochshchild cohomology. Notice that $\nabla : C_\tau(\mathfrak{g}) \rightarrow C_\tau(\mathfrak{g})$ is a Hochshchild 1-cochain, i.e.

$$\nabla \in C^1(C_\tau(\mathfrak{g}), C_\tau(\mathfrak{g})).$$

Let

$$d_H : C^1(C_\tau(\mathfrak{g}), C_\tau(\mathfrak{g})) \rightarrow C^2(C_\tau(\mathfrak{g}), C_\tau(\mathfrak{g}))$$

be the differential map in the Hochschild complex.

Let $A, B \in C_\tau(\mathfrak{g})$. Then by the definition of $d_H$, we have

$$(d_H \nabla)(A, B) = A \nabla(B) - \nabla(AB) + \nabla(A)B. \quad (47)$$

The following theorem is the main result of this paper.

Theorem 5.4. For any $A = A_i \otimes a^i$, $B = B_j \otimes b^j \in C_\tau(\mathfrak{g})$, we have

$$\{ A, B \} = -A \nabla(B) + \nabla(AB) - \nabla(A)B. \quad (48)$$

In other words

$$P + d_H \nabla = 0 \quad (49)$$

as elements in the Hochschild 2-cochain $C^2(C_\tau(\mathfrak{g}), C_\tau(\mathfrak{g}))$. Therefore the Poisson bracket is a coboundary in $C^2(C_\tau(\mathfrak{g}), C_\tau(\mathfrak{g}))$.

Proof: First let us see what is $\nabla(AB)$:

$$\nabla(AB) = \nabla(A_i B_j \otimes a^i b^j) = A_i B_j \tau(X_k) \otimes \partial^k (a^i b^j) = A_i B_j \tau(X_k) \otimes (\partial^k a^i) b^j + A_i B_j \tau(X_k) \otimes a^i (\partial^k b^j) \quad (50)$$

To make the computation more clear, let us denote:

$$I := A_i B_j \tau(X_k) \otimes (\partial^k a^i) b^j, \quad \text{II} := A_i B_j \tau(X_k) \otimes a^i (\partial^k b^j).$$

Then

$$\nabla(AB) = I + \text{II}. \quad (51)$$

It is easy to see that $\text{II} = A \nabla(B)$. In fact

$$\text{II} = A_i B_j \tau(X_k) \otimes a^i (\partial^k b^j) = (A_i \otimes a^i) \cdot (B_j \tau(X_k) \otimes \partial^k b^j) = A \nabla(B). \quad (52)$$
Unfortunately, $I \neq (\nabla A)B$ in general. We know that
\[ I = A_i B_j \tau(X_k) \otimes (\partial^k a^i) b^j \]
and
\[
(\nabla A)B = (A_i \tau(X_k) \otimes \partial^k a^i) \cdot (B_j \otimes b^j)
= A_i \tau(X_k) B_j \otimes (\partial^k a^i) b^j.
\]
Therefore
\[
I - (\nabla A)B = (A_i B_j \tau(X_k) - A_i \tau(X_k) B_j) \otimes (\partial^k a^i) b^j
= A_i [B_j, \tau(X_k)] \otimes (\partial^k a^i) b^j.
\] We need to further simplify $A_i [B_j, \tau(X_k)] \otimes (\partial^k a^i) b^j$. In fact we have the following lemma

**Lemma 5.5.** *any $A = A_i \otimes a^i$, $B = B_j \otimes b^j \in C_\tau(g)$, we have*
\[
A_i [B_j, \tau(X_k)] \otimes (\partial^k a^i) b^j = A_i B_j \otimes \{a^i, b^j\} = \{A, B\}.
\]
Assuming Lemma 5.5, then by Equation 53 we have
\[
I = \{A, B\} + (\nabla A)B.
\] Put equations 51, 52, and 55 together, we have:
\[
\nabla(AB) - A \nabla(B) - \nabla(A)B
= I + II - A \nabla(B) - \nabla(A)B
= \{A, B\} + (\nabla A)B + A \nabla(B) - A \nabla(B) - (\nabla A)B
= \{A, B\}.
\] This proves Theorem 5.4. 

**Proof of Lemma 5.5** First by Proposition 2.1, we have
\[
A_i [B_j, \tau(X_k)] \otimes (\partial^k a^i) b^j = (A_i \otimes \partial^k a^i) \cdot ([B_j, \tau(X_k)] \otimes b^j).
\]
Since $B = B_j \otimes b^j$ is contained in $C_\tau(g)$, by Proposition 2.2, we know that
\[
(A_i \otimes \partial^k a^i) \cdot (B_j \otimes \{X_k, b^j\})
= A_i B_j \otimes \{X_k, b^j\}
= A_i B_j \otimes \{a^i, b^j\}
= \{A, B\}.
\]
This proves Lemma 5.5.

**Remark 16.** Although both the map $\nabla$ and the Poisson bracket $P$ can be defined on the larger algebra $\widetilde{C}_\tau(g)$, we do not have the relation
\[
\{A, B\} = -d_H \nabla(A, B)
\]
for any $A, B \in \widetilde{C}_\tau(g)$. In fact, in the proof of Lemma 5.5 we see that we need $B \in C_\tau(g)$.

From the view point of Proposition 3.3, we have the following

**Corollary 5.6.** *The deformation from $C_\tau(g)$ to $Q_\tau(g)$ is infinitesimally trivial.*

**Proof:** We know in Proposition 4.6 that the first order deformation $m_1$ is $\frac{1}{2} P$, therefore this corollary is just a direct consequence of Theorem 5.4.

We can also give an alternative proof of Proposition 3.4 using Theorem 5.4.

**Corollary 5.7 (Proposition 3.4).** *For any $A, B \in C_\tau(g)$, we have that $\{A, B\}$ is still in $C_\tau(g)$.***
that the first Chern class is closed in the Hochschild cochain, in other words, we can check that \( \nabla(\tilde{c}) = \nabla'(\tilde{c}) \rightarrow \tilde{C}_r(g) \) to be
\[
\nabla'(A_i \otimes a^i) := \tau(X_k)A_i \otimes \partial^k a^i.
\] (57)

Similar to Proposition 5.2, we can check that \( \nabla' \) also maps \( C_r(g) \) to \( C_r(g) \).

**Remark 17.** The difference between \( \nabla \) and \( \nabla' \) is: in the definition of \( \nabla \), the matrix \( \tau(X_k) \) is multiplied from the right; while in the definition of \( \nabla' \), the matrix \( \tau(X_k) \) is multiplied from the left.

In general \( \nabla'(A) \neq \nabla(A) \), we want to compute their difference. First we define the first Chern class on \( \tilde{C}_r(g) \) following [4] Section 1.1.

**Definition 5.2.** The first Chern class \( c_1 \) is a map \( \tilde{c}_r(g) \rightarrow \tilde{C}_r(g) \), \( c_1 := \text{tr(ad)} \). More precisely
\[
c_1 : \tilde{C}_r(g) \rightarrow \tilde{C}_r(g)
A \otimes a \rightarrow A \otimes c^{ij}_k \partial^k a.
\] (58)

It is easy to check that \( c_1 \) is \( g \)-invariant hence \( c_1 \) maps \( C_r(g) \) to \( C_r(g) \). Moreover, it is also easy to check that the first Chern class is closed in the Hochschild cochain, in other words, \( c_1 \in Z^1(\tilde{C}_r(g), \tilde{C}_r(g)) \) and \( c_1 \in Z^1(C_r(g), C_r(g)) \).

Having the first Chern class, we can express the difference between \( \nabla \) and \( \nabla' \) in \( C_r(g) \):

**Proposition 5.8.** In \( C_r(g) \) we have \( \nabla - \nabla' = -c_1 \).

Proof: For any \( A_i \otimes a^i \in C_r(g) \)
\[
\nabla(A_i \otimes a^i) - \nabla'(A_i \otimes a^i) = A_i \tau(X_k) \otimes \partial^k a^i - \tau(X_k)A_i \otimes \partial^k a^i
= [A_i, \tau(X_k)] \otimes \partial^k a^i
= \partial^k([A_i, \tau(X_k)] \otimes a^i) \quad \text{(We can move the partial derivative out)}.
\]

Since \( A_i \otimes a^i \in C_r(g) \), we have
\[
[A_i, \tau(X_k)] \otimes a^i = A_i \otimes \{X_k, a^i\}.
\]

Therefore
\[
\nabla(A_i \otimes a^i) - \nabla'(A_i \otimes a^i) = \partial^k(A_i \otimes \{X_k, a^i\})
= A_i \otimes \partial^k \{X_k, a^i\}
= A_i \otimes \partial^k (c^l_{kj}X_l \partial^j a^i)
= A_i \otimes (c^l_{kj} \partial^j a^i + c^l_{kj}X_l \partial^k \partial^j a^i)
= A_i \otimes c^l_{kj} \partial^j a^i + A_i \otimes c^l_{kj}X_l \partial^k \partial^j a^i
= -c_1(A_i \otimes a^i) + A_i \otimes c^l_{kj}X_l \partial^k \partial^j a^i.
\]

Since \( c^l_{kj} \) is anti-symmetric with respect to \( k, j \), it is easy to see that
\[
A_i \otimes c^l_{kj}X_l \partial^k \partial^j a^i = 0
\]

Hence we get
\[
\nabla(A_i \otimes a^i) - \nabla'(A_i \otimes a^i) = -c_1(A_i \otimes a^i) \quad \square
\]

**Corollary 5.9.** In \( C_r(g) \) we have \( d_H \nabla' = d_H \nabla = P \) the Poisson bracket. Therefore we can replace \( \nabla \) by \( \nabla' \) in Theorem 5.4.
Proof: We know that $d_H \nabla = P$ and $\nabla - \nabla' = -c_1$. In Definition 5.2 we also know that $c_1$ is closed, i.e. $d_H c_1 = 0$. □

Moreover, we have the following result

**Proposition 5.10.** When $g$ is a semisimple Lie algebra, we have $\nabla' = \nabla$ in $C_\tau(g)$.

Proof: We know that for semisimple Lie algebra, the adjoint representation is traceless, in other words $c_{ij} = 0$ for any $j$.

Therefore $c_1 = 0$ for semisimple $g$. □

Remark 18. I am still not clear about the significance of the result in this subsection, especially its relation with the quantization problem, see Section 6.

6. Open Problem: Mackey’s analogue and the quantization of the family algebras

In this section we restrict to the case that $g$ is complex semisimple and the representation $\tau$ to be semisimple.

In 1975 G. Mackey ([18]) studied the analogies between the representations of a semisimple Lie group $G$ and those of its Cartan motion group $G_c$. Moreover, as we have mentioned in the introduction, N. Higson find the relation between family algebras and Mackey’s analogue in [11].

Remark 19. In fact, Higson introduced the spherical Hecke algebras $R(g, \tau)$ and $R(g_c, \tau)$ respectively. These algebras have the importance that the irreducible $R(g, \tau)$ modules are 1-1 correspondent to irreducible $(g, K)$-modules of $G$ with nonzero $\tau$-isotypical component, and the similar result holds for $R(g_c, \tau)$. When $G$ is complex semisimple, Higson proved that the spherical Hecke algebras are isomorphic to the family algebras. For details see [11].

Let $h \subset g$ be the Cartan subalgebra. Higson also constructed the generalized Harish-Chandra homomorphisms:

$$\text{GHC}_\tau : R(g, \tau) \rightarrow U(h)$$

$$\text{GHC}_{\tau,c} : R(g_c, \tau) \rightarrow S(h)$$

and relates them to the admissible dual of $G_c$ with minimal $K$-type $\tau$.

The Mackey’s analogue for admissible dual of complex semisimple $G$ has the following form:

**Theorem 6.1** ([11], Section 8). Under the identification $U(h) \cong S(h)$, the two homomorphisms $\text{GHC}_\tau$ and $\text{GHC}_{\tau,c}$ has the same image.

In the end of [11], Higson proposed the problem of constructing a quantization map $Q$ between $C_\tau(g)$ and $Q_\tau(g)$ such that the following diagram commutes.

$$\begin{align*}
C_\tau(g) & \xrightarrow{Q} Q_\tau(g) \\
\downarrow \text{GHC}_{\tau,c} & \quad \downarrow \text{GHC}_\tau \\
S(h) & \cong U(h)
\end{align*}$$

(60)

Here $Q$ is a vector space isomorphism but need not to be an algebraic isomorphism.

Remark 20. According to Theorem 5.4 and Corollary 5.6, the deformation from $C_\tau(g)$ to $Q_\tau(g)$ is infinitesimally trivial, which suggests that they are very closed to each other. The quantization problem ask us to find precisely the relations between $C_\tau(g)$ and $Q_\tau(g)$. 
Remark 21. On the other hand, in the 2002 Ph. D thesis [19] Chapter 6, N. Rozhkovskaya studied the family algebras for $g = \mathfrak{sl}(2, \mathbb{C})$ and for any finite dimensional irreducible representation of $g$. In fact she gave explicitly the generators and generation relations of $C_{\tau}(g)$ and $Q_{\tau}(g)$. According to her formulas, $C_{\tau}(g)$ and $Q_{\tau}(g)$ are not isomorphic as algebras unless $\tau = \text{the trivial}$, the standard or the adjoint representations. This suggests that we cannot expect the quantization map $Q$ to be an algebraic isomorphism.

In [1] and [2], A. Alekseev, and E. Meinrenken give a new proof of Duflo’s isomorphism theorem using the quantization map of the Weil algebras.

In [20], Z. Wei introduced the covariant Weil algebras as simultaneous generalizations of Weil algebras and family algebras. It is expected that the quantization problem of family algebras can be solved in the adjoint representations. This suggests that we cannot expect the quantization map $Q$ to be an algebraic isomorphism.

In [7] or [3] Section 1. For further topics see the survey [5].

We see that $d$ or $\Delta$ Section 1. For further topics see the survey [5].

In this section we give a quick review of the Hochschild cohomology.

Let us review the theory of Hochschild cohomology, see [21] or [4] Section 2 for references.

Let $A$ be an associative $\mathbb{C}$-algebra. The associated Hochschild complex $C^*(A, A)$ is defined as follows:

$$C^n(A, A) := \text{Hom}_\mathbb{C}(A^\otimes n, A), \ n \geq 0.$$  \hfill (61)

The differential $d_H$ is defined on homogeneous elements $f \in C^n(A, A)$ by the formula

$$(d_H(f))(a_0, a_1, \ldots, a_n) := a_0 f(a_1, \ldots, a_n) + \sum_{k=1}^n (-1)^{k} f(a_0, \ldots, a_{k-1}a_k, \ldots, a_n)$$

$$+ (-1)^{n+1} f(a_0, \ldots, a_{n-1}a_n).$$  \hfill (62)

We see that $d_H f \in C^{n+1}(A, A)$. We can prove $d_H \circ d_H = 0$ therefore $C^*(A, A)$ is a cochain complex.

The Hochschild cohomology of $A$ is defined as the cohomology group of the cochain complex $C^*(A, A)$, and we denote it by $\text{HH}^*(A, A)$ or for short $\text{HH}^*(A)$:

$$\text{HH}^n(A) := H^n(C^*(A, A)).$$  \hfill (63)

Now let us look at the case $n = 2$. The following observation is easy to get:

**Proposition A.1.** Let $f \in C^2(A, A) = \text{Hom}_\mathbb{C}(A \otimes A, A)$. Then $f$ is a 2-coboundary if and only if there exists a $g \in C^1(A, A) = \text{Hom}_\mathbb{C}(A, A)$ such that for any $a, b \in A$

$$f(a, b) = ag(b) - g(ab) + g(g)b.$$  \hfill (64)

Moreover, $f$ is a 2-cocycle if and only if for any $a, b, c \in A$

$$af(b, c) - f(ab, c) + f(a, bc) - f(a,b)c = 0.$$  \hfill (65)

Proof: Direct check by definition. $\square$

**APPENDIX B. THE GERSTENHABER BRACKET ON HOCHSCHILD COCHAINS AND COHOMOLOGIES**

In this section we give a quick review of the Gerstenhaber bracket. For more details and proofs see [7] or [3] Section 1. For further topics see the survey [5].

First, we define an operation $\circ : C^k(A, A) \otimes C^l(A, A) \to C^{k+l-1}(A, A)$. Let $f_1 \in C^k(A, A)$ and $f_2 \in C^l(A, A)$,

$$(f_1 \circ f_2)(a_1, \ldots, a_{k+l-1}) :=$$

$$\sum_{i=0}^{k-1} (-1)^{(k-i-1)(l-1)} f_1(a_1, \ldots, a_i, f_2(a_{i+1}, \ldots, a_{i+l}), a_{i+l+1}, \ldots, a_{k+l-1}).$$  \hfill (66)

In particular, for 2-cochains we have
Proposition B.1. Let \( f_1, f_2 \in C^2(A, A) \), then \( f_1 \circ f_2 \in C^3(A, A) \) and is given by
\[
(f_1 \circ f_2)(a_1, a_2, a_3) = f_1(f_2(a_1, a_2), a_3) - f_1(a_1, f_2(a_2, a_3)).
\] (67)
In particular, for \( f \in C^2(A, A) \) we have
\[
(f \circ f)(a_1, a_2, a_3) = f(f(a_1, a_2), a_3) - f(a_1, f(a_2, a_3)).
\] (68)
Proof: This is just the definition. \( \square \)

The Gerstenhaber bracket is defined to be
\[
[f_1, f_2]_G := f_1 \circ f_2 - (-1)^{(k-1)(l-1)} f_2 \circ f_1.
\] (69)
The Gerstenhaber bracket is a Lie bracket. In fact we have the following

Theorem B.2. The operation \( \circ \) gives a pre-Lie algebra structure on \( C^{*-1}(A, A) \). Therefore we obtain that \( (C^{*-1}(A, A), [\cdot, \cdot]_G) \) is a graded Lie algebra.
Proof: See \[7\]. \( \square \)

Proposition B.3. Let \( f \in C^2(A, A) \), then
\[
[f, f]_G = 2f \circ f.
\] (70)
Proof: We get this directly from the definitions. \( \square \)

The Gerstenhaber bracket is compatible with the Hochschild differential \( d_1 \). In fact \( d_1 \) is inner in the Gerstenhaber bracket. More precisely, let \( \mu : A \otimes A \to A \) denote the multiplication map in \( A \). Then \( \mu \in C^2(A, A) \). We have the following

Proposition B.4. For any \( f \in C^k(A, A) \), we have
\[
d_H f = [\mu, f]_G \in C^{k+1}(A, A).
\] (71)
We also have \([\mu, \mu]_G = 0\).
Proof: Compare the definition of \( d_1 \) in Equation (62) and the definition of the Gerstenhaber bracket in Equation (66) and Equation (69). The fact that \([\mu, \mu]_G = 0\) is exactly the associativity of \( \mu \). \( \square \)

As a result, we have the following theorem:

Theorem B.5. The Gerstenhaber bracket is compatible with the Hochschild differential \( d_1 \). In other words, for any \( f_1 \in C^k(A, A) \) and \( f_2 \in C^l(A, A) \), we have
\[
d_H([f_1, f_2]_G) = [d_H f_1, f_2]_G + (-1)^{k-1}[f_1, d_H f_2]_G.
\] (72)
Therefore the Gerstenhaber bracket reduces to the Hochschild cohomology \( HH^{*-1}(A) \).
Proof: Since \( d_1 \) is an inner derivation according to Proposition B.4, Equation (72) is a consequence of the super-Jacobi identity of the graded Lie algebra \( (C^{*-1}(A, A), [\cdot, \cdot]_G) \). \( \square \)

Appendix C. \( HH^*(A) \) and the Deformations of \( A \)

The Hochschild cohomology plays an important role in the deformation theory, see \[8\] or \[4\] Section 2.
Let \( A \) be an associative \( \mathbb{C} \) algebra (in fact we can replace \( \mathbb{C} \) by any field). A deformation of the algebra structure of \( A \) means that we fix \( A \) as a \( \mathbb{C} \)-vector space and change the multiplication operation on \( A \). More precisely let \( \mathbb{C}[[t]] \) be the formal power series of \( t \) and we define
\[
A[[t]] := A \otimes_{\mathbb{C}} \mathbb{C}[[t]].
\] (73)
\( A[[t]] \) is obviously a \( \mathbb{C}[[t]] \)-module.
A deformation of the algebra structure on $A$ is given by a map
\[ m : A[[t]] \otimes A[[t]] \to A[[t]] \] (74)
where $m$ is required to be $\mathbb{C}[[t]]$-bilinear. So we only need to know the value of $m$ on $A \otimes A$.

For any $a, b \in A$, we can write $m(a, b)$ as
\[ m(a, b) = ab + \sum_{k=0}^{\infty} t^k m_k(a, b). \] (75)

We see that each $m_k$ belongs to $C^2(A, A)$.

**Remark 22.** The element $t$ is called the deformation parameter. If we evaluate at $t = 0$ we get the original multiplication on $A$. On the other hand if we evaluate at $t \neq 0$, omit the convergence problem, we get a new binary operation $A \otimes A \to A$.

Being a multiplication, $m$ needs to satisfy the associativity law.

**Theorem C.1** (Formal deformation, see [8] Chapter I.1). Let $m(a, b) = ab + \sum_{k=0}^{\infty} t^k m_k(a, b)$ as in Equation (75). Then $m$ satisfies the associativity law if and only if for each $k \geq 1$, we have
\[ d_H m_k + \frac{1}{2} \sum_{i=1}^{k-1} [m_i, m_{k-i}] = 0. \] (76)

If this holds, we say that $m$ gives a formal deformation of $A$.

**Proof:** The associativity law means that for any $a, b, c \in A$, we have
\[ m(a, m(b, c)) - m(m(a, b), c) = 0. \] (77)

Now consider $m$ as an element in $C^2(A[[t]], A[[t]])$, then Equation (77) is exactly
\[ [m, m]_G = 0. \] (78)

We write $m = \mu + \sum_{k=1}^{\infty} t^k m_k$ where $\mu$ is the original multiplication on $A$. Then because we know $[\mu, f]_G = d_H f$ and $[\mu, \mu]_G = 0$ in Proposition B.4, Equation (78) becomes the Maurer-Cartan Equation
\[ d_H (\sum_{k=1}^{\infty} t^k m_k) + \frac{1}{2} \sum_{k=1}^{\infty} t^k m_k, \sum_{k=1}^{\infty} t^k m_k]_G = 0. \] (79)

In the expansion of Equation (79) we take the $t^k$ term and get Equation (76) \(\square\)

**Corollary C.2** (Infinitesimal deformation). $m$ satisfies the associativity law mod $t^2$ if and only if $d_H m_1 = 0$, i.e. for any $a, b, c \in A$, we have
\[ am_1(b, c) - m_1 ab, c + m_1(a, bc) - m_1(a, b)c = 0. \] (80)

If this holds, we say that $m$ gives an infinitesimal deformation of $A$.

Moreover, $m$ satisfies the associativity law mod $t^3$ if and only if $d_H m_1 = 0$ together with
\[ d_H m_2 + \frac{1}{2} [m_1, m_1]_G = 0. \] (81)

The above equation is equivalent to
\[ d_H m_2 + m_1 \circ m_1 = 0 \] (82)

since in Proposition B.3 we know that $[m_1, m_1]_G = 2 m_1 \circ m_1$.

**Proof:** This is an direct corollary of Theorem C.1 \(\square\)

On the other hand, we need to study the problem that when the deformation $m$ is trivial. In other words, wether or not we can find an algebraic isomorphism
\[ \theta : (A[[t]], \mu) \to (A[[t]], m) \] (83)
where $\theta$ is $\mathbb{C}[[t]]$-linear and is given by

$$\theta(a) = a + \sum_{k=1}^{\infty} t^k \theta_k(a). \quad (84)$$

The requirement for $\theta$ is for any $a, b \in A$

$$\theta(ab) = m(\theta(a), \theta(b)). \quad (85)$$

The existence of $\theta$ is a complicated problem. First we have:

**Proposition C.3** (Infinitesimally trivial deformation). There exists a $\theta_1 \in C^1(A,A)$ such that $\theta = id + \theta_1$ satisfies Equation (85) mod $t^2$ if and only if $m_1 \in B^2(A,A)$. If this holds, we say that $m$ is an infinitesimally trivial deformation of $A$.

Proof: We expand both sides of Equation (85) and look at the $t$ term we get

$$\theta_1(ab) = \theta_1(a)b + a\theta_1(b) + m_1(a,b) \quad (86)$$

In other words

$$m_1 + d_H \theta_1 = 0. \quad (87)$$

Further discussion of the triviality of deformations involves the concept of gauge equivalence of Maurer-Cartan elements, see [16] Section 1 or [17] Chapter 13.

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