Non-metricities, torsion and fermions

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April 2018

Abstract

I investigate the general extension of Einstein’s gravity by considering the third rank non-metricity tensor and the torsion tensor. The minimal coupling to Dirac fields faces an ambiguity coming from a severe arbitrariness of the Fock-Ivanenko coefficients. This arbitrariness is fed in part by the covariant derivative of Dirac matrices, which is not completely determined as well. It is remarkable that this feature is not exclusive to the non-metricity case: it happens also for gravity with torsion alone. Nevertheless, theory in vacuum is well defined and non-trivial, where torsion is the source of non-metricity or vice-versa. I point also to the existence of two independent non-metricities.

PACS 04.20.-q; 04.50.Kd.

1 Introduction

Besides the usual Riemannian geometric degrees of freedom, non-Riemannian counterparts such as torsion [1] and non-metricity [2, 3] have been extensively investigated in literature as gravitation components which would be relevant in high-energy regimes. Torsion and non-metricity are respectively related mathematically to the antisymmetric part of the affine connection

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and the non-vanishing covariant derivative of metric tensor, and the study of their physical relevance is going on until the present days. Torsion has been considered in a greater number of works (see, for example, the reviews [4, 5]), but the interest on non-metricity have been increased in last years.

Like torsion, non-metricity can be studied in different ways according to which component of non-metricity is chosen to be considered. For example, one can consider only a scalar degree of freedom of non-metricity. In such theories, the spacetime is called “Weyl integrable spacetime” (see, for example, Refs. [6, 7, 8]). One can consider instead the vector degrees of freedom, which brings us the Weyl spacetime, where the covariant derivative of metric tensor is proportional to the metric, \( \nabla_\mu g_{\alpha\beta} = \phi_\mu g_{\alpha\beta} \), and the vector \( \phi_\mu \) describes all the non-metricity. If this quantity can be written as a scalar derivative, \( \phi_\mu = \partial_\mu \Omega \), then the spacetime is Weyl integrable (previous case). For this approach, see, for instance, Refs. [2, 9, 10, 11, 12]. In the most general case, the gravity theories are called sometimes by Metric-Affine Gravity (MAG), and the spacetime is more general than the Weyl spacetime. I can cite some works in this approach in Refs. [13, 14, 15, 16, 17, 18]. In each one of these three different approaches, you can find many papers (from which I cited just a few) investigating several topics, such as Dirac fermions in curved space and consequent experimental bounds, as well as cosmological effects of non-metricity like singularity avoidance or accelerated expansion. In particular, paper in Ref. [16] obtains experimental bounds for non-metricity from results already obtained for Lorentz violating theories. One finds also an increasing number of works exploring non-metricity as an equivalent to General Relativity, called Symmetric Teleparallel Gravity [19] (see also the review [20]).

In the present work, I consider the most general extension of Einstein-Hilbert action, which includes torsion and general non-metricity. Firstly, in order to find how fermions couple minimally to non-metricity, one has to consider the issue of covariant derivative with respect to both diffeomorphism

\[1\] There is no scalar irreducible component in the decomposition of non-metricity tensor, such that I mean scalar degree of freedom as the unique degree of freedom coming from the vector component.
and Lorentz transformation, the last one understood as operating in the tangent space of each point in the manifold. Because of the independence between spacetime and the tangent space, it seems unnatural the identification of Minkowskian non-metricity, $\tilde{\nabla}_\mu \eta_{ab}$, to the usual non-metricity $\nabla_\mu g_{\alpha\beta}$. In fact, this identification lead not to any contradiction, that is why nobody has spoken about two different non-metricities. Some authors study the usual non-metricity, and another authors study the Minkowskian one. I argue that it is more natural to accept both non-metricities, mutually independent. This issue is extremely relevant for investigating how non-metricity couples to matter.

But to study the coupling to matter, one has also to understand the structure of the spinorial covariant derivative, i.e., the Fock-Ivanenko coefficients. Then, in Section 2, after setting up notations and basic features of theory, I consider gravity with torsion and the usual non-metricity in vacuum. It turns out that, in vacuum, non-metricity is the source of torsion (or vice-versa), confirming the result obtained by Ponomarov and Obukhov. Thus, in principle, the detection of non-Riemannian structure in this context can face the difficulty of saying which one exists: torsion or non-metricity. In order to avoid this problem of arbitrary torsion (or non-metricity), one has to consider another action.

In Section 3, the covariant derivative of Dirac matrices are calculated and restricted maximally although it remains some arbitrariness. In the next section, I express the Fock-Ivanenko coefficients in terms of the covariant derivative of Dirac matrices for the most simple choice dropping the arbitrariness but not all of them, and the standard form of the Fock-Ivanenko coefficients (present in other papers) are reproduced. In Section 4, I calculate the Fock-Ivanenko coefficients with all explicit arbitrariness, and draw my conclusions in Section 5 by observing that the same arbitrariness

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The necessity of a precise definition of how the covariant derivative with Greek index acts upon objects with Latin indices is attenuated by observing that one can adopt $\tilde{\nabla}_\mu = e^a_\mu \nabla_a$. Notice that the operator $\nabla_\mu$ is actually defined by acting on both types of indices. Indeed, even when one considers only Greek indices, no one can ignore that, for example, $\tilde{\nabla}_\mu g_{\alpha\beta} = \tilde{\nabla}_\mu (e^a_\alpha e_{a\beta})$, such that the complete definition of $\tilde{\nabla}_\mu$ is always hidden in the equations without Latin indices.
is also present in theory with only torsion.

2 Formulation of theory

We use two kinds of labels in specifying the components of a tensor: a Latin letter ($a, b, ...$) and a Greek letter ($\mu, \nu, ...$). Latin letters indicate components of an object which is invariant under Poincaré transformations; and Greek letters indicate components of an object which is invariant under general coordinate transformations.

Let us denote the known objects (like connection and covariant derivative) with tilde in the presence of torsion and non-metricity, and without tilde for the case where there is only torsion. Thus, the covariant derivative of metric is written as

$$\tilde{\nabla}_\mu g_{\alpha\beta} = \partial_\mu g_{\alpha\beta} - \tilde{\Gamma}^\rho_{\alpha\mu} g_{\rho\beta} - \tilde{\Gamma}^\rho_{\beta\mu} g_{\alpha\rho} = Q_{\mu\alpha\beta}. \quad (1)$$

Of course, one has, at the same time, $\nabla_\mu g_{\alpha\beta} = 0$. In fact, equation (1) is the definition of non-metricity tensor $Q_{\mu\alpha\beta}$ (notice that $Q_{\mu\alpha\beta} = Q_{\mu\beta\alpha}$).

Similarly, the covariant derivative of the Minkowski metric tensor has the form

$$\tilde{\nabla}_\mu \eta_{ab} = \partial_\mu \eta_{ab} + \tilde{\omega}^c_{a\mu} \eta_{cb} + \tilde{\omega}^c_{b\mu} \eta_{ac} = Q_{\mu ab}, \quad (2)$$

where, obviously, $\partial_\mu \eta_{ab} = 0$ (we put it in the equation just for the sake of completeness). The spin connection here, $\tilde{\omega}_{ab\mu}$, differs from the usual spin connection with torsion and metricity, $\omega_{ab\mu}$, in respect to the known antisymmetry in the last case, $\omega_{ab\mu} = -\omega_{ba\mu}$. By equation (2), the non-metricity object $Q_{\mu ab}$ defines precisely the symmetric part of $\tilde{\omega}_{ab\mu}$:

$$\tilde{\omega}_{(ab)\mu} = \frac{1}{2} (\tilde{\omega}_{ab\mu} + \tilde{\omega}_{ba\mu}) = Q_{\mu ab}. \quad (3)$$

So far we have defined two kinds of non-metricity, $Q_{\mu\alpha\beta}$ and $Q_{\mu ab}$, without indicating any relation between them. One can always set what the

\footnote{The covariant derivative of a spinor in the presence of non-metricity is a complicated subject, admitting different approaches. See, for instance, Refs. [22, 23, 24].}
notation suggests, which is:

\[ Q_{\mu\alpha\beta} = e^a_\alpha e^b_\beta Q_{\mu ab} . \]  

(4)

It should be stressed that the above relation is NOT necessary. It can, of course, be put by hand, but no one is obliged to set this relation, which is an interesting logic issue: observe that the equation (4) does not follow from definitions (1) and (2), but is only suggested by notation. For the sake of comparison, let us mention that the inverse happens with the standard definition of these notations themselves. We define, for example, the objects \( A_\alpha \) and \( A_a \) such that

\[ A_\alpha = e^a_\alpha A_a . \]  

(5)

In this case, equation (5) comes unseparated from the definitions of \( A_\alpha \) and \( A_a \) (actually, the above equation is the definition of \( A_\alpha \) or \( A_a \) in terms of the definition of the other one). In the case of non-metricities, the objects \( Q_{\mu\alpha\beta} \) and \( Q_{\mu ab} \) was not defined by (4) at all! One can always adopt (4) for gaining simplicity, but we would like to point out, for the first time, the possibility to deal with two independent kinds of non-metricity.

2.1 Calculation of \( \tilde{\nabla}_\mu e^a_\alpha \)

As all the objects with a Latin and a Greek label, the covariant derivative of \( e^a_\alpha \) shall necessarily have the corresponding connection for the Latin label, the spin connection, and the one for Greek label, the affine connection:

\[ \tilde{\nabla}_\mu e^a_\alpha = \partial_\mu e^a_\alpha - \tilde{\omega}^a_{\alpha \mu} e^c_\alpha - \tilde{\Gamma}^\rho_{\alpha \mu} e^a_\rho . \]  

(6)

From equation (1), one can write the connection with non-metricity and torsion in terms of the connection with only torsion, as follows:

\[ \tilde{\Gamma}^\rho_{\alpha \mu} = \Gamma^\rho_{\alpha \mu} + N^\rho_{\alpha \mu} , \]

(7)

where \( N^\rho_{\alpha \mu} = (Q^\rho_{\alpha \mu} - Q^\rho_\alpha \mu - Q^\mu_\rho \alpha) / 2 \) and, as usual, \( \Gamma^\rho_{\alpha \mu} = \{ \alpha \mu \} + K^\rho_{\alpha \mu} \), where \( \{ \alpha \mu \} \) is the Christoffel symbol and \( K^\rho_{\alpha \mu} \) is the contortion tensor.\footnote{The torsion tensor is defined by \( T^\lambda_{\alpha \beta} = \tilde{\Gamma}^\lambda_{\alpha \beta} - \tilde{\Gamma}^\lambda_{\beta \alpha} \).}
\( K^\rho_{\alpha\mu} = (T^\rho_{\alpha\mu} - T_\alpha^\rho_{\mu} - T_\mu^\rho_{\alpha})/2. \) Taking equation (3) into account (we mean \( \tilde{\omega}_{ab\mu} = Q_{ab\mu}/2 + \omega_{ab\mu} \)), we finally achieve, substituting the above equation into equation (6), and also using \( \nabla_\mu e^\alpha_a = 0, \)

\[
\tilde{\nabla}_\mu e^\alpha_a = -\frac{1}{2} Q_{\alpha b}^a e^b_\alpha - N^{\rho}_{\alpha\mu} e^\alpha_\rho.
\] (8)

From the above formula, one can calculate also another derivatives, \( \tilde{\nabla}_\mu e^{ab}_a, \tilde{\nabla}_\mu e_{ab} \) and \( \tilde{\nabla}_\mu e_{ab}^\alpha, \) which must be done carefully, keeping in mind that, for example, \( \tilde{\nabla}_\mu e_{ab}^\alpha \neq \eta^{ab} \tilde{\nabla}_\mu e_{ab} \neq g_\alpha^\beta \tilde{\nabla}_\mu e_{ab}^\beta. \)

2.2 Non-trivial vacuum solutions

One should of course pay attention to \( \tilde{\nabla}_\mu \gamma^a, \) as it is directly related to quantities like the spinor covariant derivative, \( \tilde{\nabla}_\mu \psi, \) required to formulate interaction between matter and geometric variables. Let us consider, for now, the vacuum solutions (only geometric quantities without matter). For this purpose, the most simple action is

\[
S = -\frac{1}{8\pi G} \int \sqrt{-g} d^4x \tilde{R},
\] (9)

where \( \tilde{R} \) is the curvature scalar obtained by index contractions of the total curvature tensor

\[
\tilde{R}^\rho_{\lambda\mu\nu} = \tilde{\Gamma}^\rho_{\lambda\nu,\mu} + \tilde{\Gamma}^\rho_{\tau\mu} \tilde{\Gamma}^\tau_{\lambda\nu} - (\mu \leftrightarrow \nu),
\] (10)

and the index after the dot means (ordinary) partial derivative introducing the same index. The connections above are the total connections, given by (7). Substituting these connections, we get

\[
\tilde{R}^\rho_{\lambda\mu\nu} = \tilde{\Gamma}^\rho_{\lambda\nu,\mu} + M^\rho_{\lambda\|\mu} - M^\rho_{\mu\|\nu} + M^\rho_{\tau\mu} M^\tau_{\lambda\nu} - M^\rho_{\tau\nu} M^\tau_{\lambda\mu}
\] (11)

where \( \tilde{R}^\rho_{\lambda\mu\nu} \) is the Riemannian curvature, also \( M^\rho_{\mu\nu} = K^\rho_{\mu\nu} + N^\rho_{\mu\nu} \) and the double bar means Riemannian covariant derivative (constructed with Riemannian connection).

The equations of motion for torsion and non-metricity can be achieved (neglecting the surface terms) by variation of action with respect to contortion, \( K^\gamma_{\alpha\beta}, \) and the tensor \( N^\gamma_{\alpha\beta}, \) yielding, respectively,

\[
T^\beta_{\ [\alpha\gamma]} + Q_{[\alpha\gamma]}^\beta + \delta^\beta_{ [\alpha} q_{\gamma]} - \delta^\beta_{ [\alpha} Q_{\gamma]} - 2\delta^\beta_{ [\alpha} T_{\gamma]} = 0,
\] (12)
and
\[ Q_\gamma^{\alpha\beta} + T^{(\alpha\beta)}_\gamma + \frac{1}{2} \delta_\gamma^{(\alpha} q^{\beta)} - \delta_\gamma^{(\alpha} Q^{\beta)} - \delta_\gamma^{(\alpha} T^{\beta)} + \frac{1}{2} g^{\alpha\beta} (2T_\gamma - q_\gamma) = 0, \tag{13} \]
where we use the notation for symmetrization and anti-symmetrization such that
\[ a^{(\mu\nu)} = (a^{\mu\nu} + a^{\nu\mu})/2 \text{ and } a^{[\mu\nu]} = (a^{\mu\nu} - a^{\nu\mu})/2, \]
and the traces \( Q_\alpha, q_\alpha \) and \( T_\alpha \) are defined as
\[ T_\alpha = T^\rho_{\alpha\rho}, \quad Q_\alpha = Q^\rho_{\alpha\rho} \quad \text{and} \quad q_\alpha = Q_\alpha^\rho. \tag{14} \]
By taking traces of equations (12) and (13), we arrive at the result
\[ q_\alpha = 4Q_\alpha \quad \text{and} \quad T_\alpha = \frac{3}{2} Q_\alpha, \tag{15} \]
which can be inserted back in (12) and (13), yielding, after some algebraic manipulations,
\[ Q_{\mu\alpha\beta} = g_{\alpha\beta} Q_\mu \tag{16} \]
\[ T^\alpha_{\mu\nu} = \frac{1}{2} \left( \delta^\alpha_\mu Q_\nu - \delta^\alpha_\nu Q_\mu \right). \tag{17} \]
Thus, the vacuum solution with torsion and non-metricity has non-trivial torsion and non-metricity (although not dynamical, obviously), both expressed in terms of the same 4-vector \( Q_\alpha \) (all other degrees of freedom vanishes). Consider, for example, the Einstein-Cartan action together with minimally coupled Dirac fields. In that case, torsion is non-trivial and is algebraically related to fermions. The axial current is the source of torsion (its pseudo-trace). Similar feature happens in our solution: torsion is the source of non-metricity, or vice-versa: non-metricity is the source of torsion.

Of course the situation can change dramatically if we include fermions, which can give rise to other degrees of freedom. Let us investigate then how Dirac fields couple minimally with geometry.

3 Calculation of \( \tilde{\nabla}_\mu \gamma^\alpha \)

The Fock-Ivanenko coefficients are the four matrices \( \tilde{\Gamma}_\mu \), defined by
\[ \tilde{\nabla}_\mu \psi = \partial_\mu \psi + i \tilde{\Gamma}_\mu \psi, \quad \text{and} \quad \tilde{\nabla}_\mu \bar{\psi} = \partial_\mu \bar{\psi} - i \bar{\psi} \tilde{\Gamma}_\mu. \tag{18} \]
The above equations allow us to write the covariant derivative of the matrix \( \psi \bar{\psi} \) as \( \tilde{\nabla}_\mu (\psi \bar{\psi}) = \partial_\mu (\psi \bar{\psi}) + i[\tilde{\Gamma}_\mu , \psi \bar{\psi}] \), and from this we conclude that one should include the commutator \( i[\tilde{\Gamma}_\mu , M] \) in the expression for the covariant derivative of the matrix \( M \). Thus,

\[
\tilde{\nabla}_\mu \gamma^a = \partial_\mu \gamma^a - \tilde{\omega}^a b_\mu \gamma^b + i[\tilde{\Gamma}_\mu , \gamma^a],
\]

where \( \partial_\mu \gamma^a \) was written just for completeness (it vanishes). At the same time, one can suppose that the quantity \( \tilde{\nabla}_\mu \gamma^a \) can be written in terms of combinations of Dirac matrices \( \gamma^a \), the commutators \( \sigma_{bc} = i[\gamma^b , \gamma^c]/4 \) and also the 4x4 identity \( \hat{1} \) (\( \gamma_5 \) and \( \gamma_5 \gamma_b \) can be disregarded because of the parity symmetry of \( \tilde{\nabla}_\mu \gamma^a \)). So, expansion in this basis yields

\[
\tilde{\nabla}_\mu \gamma^a = \tilde{A}_\mu^a \hat{1} + \tilde{B}_\mu^{ab} \gamma^b + \tilde{C}_\mu^{abc} \sigma_{bc},
\]

where \( \tilde{A}_\mu^a, \tilde{B}_\mu^{ab} \) and \( \tilde{C}_\mu^{abc} \) are tensor components, with \( \tilde{C}_\mu^{abc} = -\tilde{C}_\mu^{acb} \). Substituting equation (20) into \( \tilde{\nabla}_\mu (\gamma_a \gamma^a) = 0 \), we get, after some algebra,

\[
2\tilde{A}_\mu^a \gamma^a + (2\tilde{B}_\mu^{ab} + Q_\mu^{ab}) \gamma^a \gamma^b + \tilde{C}_\mu^{abc} (\gamma_a \sigma_{bc} + \sigma_{bc} \gamma^a) = 0.
\]

This implies

\[
\tilde{A}_\mu^a = 0, \quad \eta_{ab} \tilde{B}_\mu^{ab} = -\eta_{ab} Q_\mu^{ab} \quad \text{and} \quad \tilde{C}_\mu^{[abc]} = 0,
\]

where \( \tilde{C}_\mu^{[abc]} \) is the normalized totally antisymmetric combination of \( \tilde{C}_\mu^{abc} \), and we used the identity \( \gamma_a \sigma_{bc} + \sigma_{bc} \gamma^a = \epsilon_{abcd} \gamma^5 \gamma^d \).

4 Relation between spin connection and the Fock-Ivanenko coefficients

Let us make the simplest choice, restricting by hand the quantities \( \tilde{B}_\mu^{ab} \) and \( \tilde{C}_\mu^{abc} \), according to

\[
\tilde{B}_\mu^{ab} = -Q_\mu^{ab} \quad \text{and} \quad \tilde{C}_\mu^{abc} = 0.
\]

\[\text{This equation, } \text{(19)}, \text{can also be achieved independently by the requirement that } \tilde{\nabla}_\mu (\bar{\psi} \gamma^a \psi) \text{ behaves as a genuine 4-vector in tangent space: } \nabla_\mu (\bar{\psi} \gamma^a \psi) = \partial_\mu (\bar{\psi} \gamma^a \psi) - \tilde{\omega}^a b_\mu (\bar{\psi} \gamma^b \psi).\]
Observe that the above equations are more restrictive than equations (21). Thus, in this particular case, we have

\[ \tilde{\nabla}_\mu \gamma^a = -Q_\mu^{\ ab} \gamma_b. \]  

(23)

Now we shall substitute the above expression in the equation (19), in order to find a way to write the Fock-Ivanenko coefficients, \( \tilde{\Gamma}_\mu \), in terms of the spin connection, \( \tilde{\omega}^{ab}_\mu \), and the non-metricity, \( Q_\mu^{\ ab} \) (see in Ref. [2] a very similar approach for Weyl geometry).

By using equation (23) in (19) we get then

\[ -Q_\mu^{\ ab} \gamma_b = -\tilde{\omega}^{a\ b}_\mu \gamma^b + i[\tilde{\Gamma}_\mu, \gamma^a]. \]  

(24)

In order to solve this equation for \( \tilde{\Gamma}_\mu \), consider the expansion of the unknown Fock-Ivanenko coefficients \( \tilde{\Gamma}_\mu \) in the basis \( \{\mathbf{1}, \gamma^c, \sigma_{cd}\} \):

\[ \tilde{\Gamma}_\mu = \tilde{D}_\mu \mathbf{1} + \tilde{E}_\mu^a \gamma_a + \tilde{F}_\mu^{ab} \sigma_{ab}, \]  

(25)

where \( \tilde{F}_\mu^{ab} = -\tilde{F}_\mu^{ba} \). It is very important that these tensor components are real numbers, because, together with equations (18), it guarantees that \( \bar{\psi} \psi \) is a scalar: \( \tilde{\nabla}_\mu (\bar{\psi} \psi) = \partial_\mu (\bar{\psi} \psi) \). Substituting the above expansion into equation (24), one can write, after straightforward algebra,

\[ (-Q_\mu^{\ ab} + \tilde{\omega}^{a\ b}_\mu - 2\tilde{F}_\mu^{ab}) \gamma_b - 4\tilde{E}_\mu^a \eta^{ac} \sigma_{bc} = 0, \]  

(26)

where we have used the identity \( [\sigma_{bc}, \gamma_a] = i\gamma_b \eta_{ac} - i\gamma_c \eta_{ab} \). With respect to Latin indices, remember that \( \tilde{F}_\mu^{ab} \) is antisymmetric and \( Q_\mu^{\ ab} \) is symmetric, so we conclude

\[ \tilde{F}_\mu^{ab} = \frac{1}{2} \tilde{\omega}^{[ab]}_\mu \quad \text{and} \quad \tilde{E}_\mu^a = 0, \]  

(27)

Notice that equation (14) is also proven (independently) from (26). It is worth mentioning that \( \tilde{D}_\mu \) does not need to be zero. Then (25) reads

\[ \tilde{\Gamma}_\mu = \frac{1}{2} \tilde{\omega}^{[ab]}_\mu \sigma_{ab} + \tilde{D}_\mu \mathbf{1}. \]  

(28)

It is very interesting that the literature on this subject point to the presence of the arbitrary 4-vector \( \tilde{D}_\mu \). The first term, proportional to \( \sigma_{ab} \), is the usual

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6See, for instance, the works in Refs. [2, 9, 13, 14, 18].
term that appears in the covariant derivative of a spinor under the minimal coupling prescription. Since all 4-vectors from the geometric content refer to traces of torsion and non-metricity, it is natural to consider these traces (or its combinations) as good candidates for the quantity \( \tilde{D}_\mu \).

5 The arbitrariness of the Fock-Ivanenko coefficients and \( \tilde{\nabla}_\mu \gamma^a \)

If we did not make the very restrictive choices \( [22] \), equation \( [23] \) would have to be rewritten as

\[
\tilde{\nabla}_\mu \gamma^a = \tilde{B}_\mu^{ab} \gamma_b + \tilde{C}_\mu^{abc} \sigma_{bc},
\]

where \( \tilde{B}_\mu^{ab} \) and \( \tilde{C}_\mu^{abc} \) satisfy the necessary conditions \( [21] \). This means that the covariant derivative of \( \gamma^a \) is actually much more arbitrary throughout a second rank tensor \( \tilde{B}_\mu^{ab} \) with arbitraries anti-symmetric and traceless parts and a third rank tensor satisfying \( \tilde{C}_\mu^{[abc]} = 0 \).

Now, substituting \( [29] \) into \( [19] \) together with the expansion \( [25] \), one is able to find the following result:

\[
\tilde{\Gamma}_\mu = \tilde{D}_\mu \hat{1} + \frac{1}{6} \tilde{C}^{ba}_{\ b} \gamma_a + \frac{1}{2} \left( \tilde{B}_\mu^{[ab]} + \tilde{\omega}^{[ab]}_{\ \mu} \right) \sigma_{ab}.
\]

In this equation, the arbitrariness of \( \tilde{\Gamma}_\mu \) (except \( \tilde{D}_\mu \)) comes from the arbitrariness of \( \tilde{\nabla}_\mu \gamma^a \). This arbitrariness can not be eliminated or reduced by some additional condition besides \( \tilde{\nabla}_\mu (\bar{\psi} \gamma^a \psi) = \partial_\mu (\bar{\psi} \gamma^a \psi) - \tilde{\omega}^{[a}_{\ b\mu} (\bar{\psi} \gamma^b \psi) \).

For example, if one considers

\[
\tilde{\nabla}_\mu (\bar{\psi} \gamma^a \gamma^b \psi) = \partial_\mu (\bar{\psi} \gamma^a \gamma^b \psi) - \tilde{\omega}^{a}_{\ c\mu} (\bar{\psi} \gamma^c \gamma^b \psi) - \tilde{\omega}^{b}_{\ c\mu} (\bar{\psi} \gamma^a \gamma^c \psi),
\]

no more and no less than equations \( [21] \) would be derived likewise.

6 Conclusions: the case with metricity

The notations we have used are very convenient for investigating the case with torsion alone. All quantities with tilde are defined in the spacetime
with non-metricity. For the case with metricity and non-zero torsion, the quantities is written without tilde, such that all equations will be essentially the same, with the obvious feature $Q_{\mu}^{ab} = 0$. Equation (30), for example, would read

$$\Gamma_{\mu} = D_{\mu} \hat{1} + \frac{1}{6} C^{ba}_{\ b} \gamma_a + \frac{1}{2} \left( B_{\mu}^{[ab]} + \omega^{ab}_{\ \mu} \right) \sigma_{ab}.$$

(32)

Here, there are also those arbitrariness found in the non-metricity case. It is interesting to observe that the alleged reason by which there should be a 4-vector $\tilde{D}_{\mu}$ in equation (28) works perfectly well in saying that this 4-vector is also present in the case with just torsion. The other quantities, $C^{ba}_{\ b}$ and $B_{\mu}^{[ab]}$, can also be present in the expression of $\Gamma_{\mu}$ without contradiction, just as it happens in the non-metricity case, $\tilde{\Gamma}_{\mu}$. I haven’t seen any consistency condition that can rule out those arbitrariness.

**Acknowledgments**

G.B.P. is grateful to CNPq and FAPEMIG for partial support. The author acknowledges Prof. I. Shapiro (UFJF) and Mr. A. F. Andrade (IFMG) for fruitful discussions in the beginning of this work.

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