Toric \textit{AF}-algebras and faithful representation of the mapping class groups

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Abstract

There exists a covariant non-injective functor from the space of generic Riemann surfaces to the so-called toric \textit{AF}-algebras; such a functor maps isomorphic Riemann surfaces to the stably isomorphic toric \textit{AF}-algebras. We use the functor to construct a faithful representation of the mapping class group of surface of genus \(g > 1\) into the matrix group \(GL_{6g-6}(\mathbb{Z})\).

Key words and phrases: Riemann surfaces, \textit{AF}-algebras

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1 Introduction

A. The mapping class group. The mapping class group has been introduced in the 1920-ies by M. Dehn \cite{Dehn}. Such a group, \(\text{Mod} (X)\), is defined as the group of isotopy classes of the orientation-preserving diffeomorphisms of a two-sided closed surface \(X\) of genus \(g \geq 1\). The group is known to be prominent in algebraic geometry \cite{Arakelov}, topology \cite{Farrell} and dynamics \cite{Katok}. When \(X\) is a torus, the \(\text{Mod} (X)\) is isomorphic to the group \(SL_2(\mathbb{Z})\). (The \(SL_2(\mathbb{Z})\) is called a modular group, hence our notation for the mapping class groups.) A little is known about the representations of \(\text{Mod} (X)\) beyond the case \(g = 1\). Recall, that the group is called \textit{linear}, if there exists a faithful representation into the matrix group \(GL_m(R)\), where \(R\) is a commutative ring. The braid

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groups are known to be linear [3]. Using a modification of the argument for the braid groups, it is possible to prove, that $\text{Mod}(X)$ is linear in the case $g = 2$ [4]. Whether the mapping class group is linear for $g \geq 3$, is an open problem, known as a Harvey conjecture [8], p.267.

B. The toric $AF$-algebras. Denote by $T_S(g)$ the Teichmüller space of genus $g \geq 1$ with a distinguished point $S$. Let $q \in H^0(S, \Omega^2)$ be a holomorphic quadratic differential on the Riemann surface $S$, such that all zeroes of $q$ (if any) are simple. By $\tilde{S}$ we mean a double cover of $S$ ramified over the zeroes of $q$ and by $H_1^{\text{odd}}(\tilde{S})$ the odd part of the integral homology of $\tilde{S}$ relative to the zeroes. Note that $H_1^{\text{odd}}(\tilde{S}) \cong \mathbb{Z}/n$, where $n = 6g - 6$ if $g \geq 2$ and $n = 2$ if $g = 1$. The fundamental result of Hubbard and Masur [9] implies, that $T_S(g) \cong \text{Hom}(H_1^{\text{odd}}(\tilde{S}); \mathbb{R}) - \{0\}$, where 0 is the zero homomorphism. Finally, denote by $\lambda = (\lambda_1, \ldots, \lambda_n)$ the image of a basis of $H_1^{\text{odd}}(\tilde{S})$ in the real line $\mathbb{R}$, such that $\lambda_1 \neq 0$. Note that such an option always exists, since the zero homomorphism is excluded. We let $\theta = (\theta_1, \ldots, \theta_{n-1})$, where $\theta_i = \lambda_{i-1}/\lambda_1$. Recall that, up to a scalar multiple, the vector $(1, \theta) \in \mathbb{R}^n$ is the limit of a generically convergent Jacobi-Perron continued fraction [2]:

$$
\begin{pmatrix}
1 \\
\theta
\end{pmatrix} = \lim_{k \to \infty} \begin{pmatrix} 0 & 1 \\
I & b_1 \\
\vdots & \ddots & \ddots & \ddots \\
I & b_k \\
0 & 1 & 0 \\
I & 1 & 1 \\
0 & 0 & \mathbb{1}
\end{pmatrix},
$$

where $b_i = (b_{1i}^{(i)}, \ldots, b_{n-1}^{(i)})^T$ is a vector of the non-negative integers, $I$ the unit matrix and $\mathbb{1} = (0, \ldots, 0, 1)^T$. We introduce an $AF$-algebra, $A_\theta$, via the Bratteli diagram [6], shown in Fig.1. (The numbers $b_i^{(j)}$ of the diagram indicate the multiplicity of edges of the graph.) Let us call $A_\theta$ a toric $AF$-algebra.

C. The Teichmüller functor. Let $R, R' \in T_S(g)$ be a pair of isomorphic Riemann surfaces; let $A_\theta$ and $A_{\theta'}$ be the corresponding toric $AF$-algebras. We look for an answer to the following elementary question: How are the algebras $A_\theta$ and $A_{\theta'}$ related to each other? Recall, that the stable isomorphism between the $C^*$-algebras is a fundamental equivalence in noncommutative geometry; the $C^*$-algebras $A$ and $A'$ are said to be stably isomorphic, whenever $A \otimes \mathcal{K}$ is isomorphic to $A' \otimes \mathcal{K}$, where $\mathcal{K}$ is the $C^*$-algebra of compact operators. Denote by $V$ the maximal subset of $T_S(g)$, such that for each Riemann surface $R \in V$, there exists a convergent Jacobi-Perron continued fraction. Let $F$ be the map which sends the Riemann surfaces into the toric $AF$-algebras according to the formula $R \mapsto A_\theta$; we shall call $F$ the Teichmüller functor. Let $W$ be the image of $V$ under $F$; the following lemma relates the algebras $A_\theta$ and $A_{\theta'}$. 

2
Figure 1: The toric AF-algebra $A_g$ of genus $g = 2$.

Lemma 1 ([11]) The set $V$ is a generic subset of $T_S(g)$ and the map $F$ has the following properties: (i) $V \cong W \times (0, \infty)$ is a trivial fiber bundle, whose projection map $p : V \to W$ coincides with $F$; (ii) $F : V \to W$ is a covariant functor, which maps isomorphic Riemann surfaces $R, R' \in V$ to stably isomorphic toric AF-algebras $A_g, A_g' \in W$.

D. The result. Recall that $Mod (X)$ acts on $T_S(g)$ by isomorphisms of the Riemann surfaces; the action is properly discontinuous and free for a finite index subgroup of $Mod (X)$ [7]. Lemma 1 extends the action to the toric AF-algebras, where $Mod (X)$ acts by the stable isomorphisms; the latter fact is remarkable: it is known, that the stable isomorphism group of a non-stationary toric AF-algebra admits a faithful representation into the matrix group $GL_n(\mathbb{Z})$ [8]. This elementary observation implies the Harvey conjecture.

Theorem 1 For every surface $X$ of genus $g \geq 2$, there exists a faithful representation $\rho : Mod (X) \to GL_{6g-6}(\mathbb{Z})$.

The structure of the article is as follows. The notation and facts used for the proof of theorem 1 are introduced in Section 2. Theorem 1 is proved in Section 3.
2 Preliminaries

2.1 AF-algebras

A. The $C^*$-algebras. By a $C^*$-algebra one understands the Banach algebra with an involution. Namely, a $C^*$-algebra $A$ is an algebra over the complex numbers $\mathbb{C}$ with a norm $a \mapsto ||a||$ and an involution $a \mapsto a^*$, $a \in A$, such that $A$ is complete with the respect to the norm, and such that $||ab|| \leq ||a|| \cdot ||b||$ and $||a^*a|| = ||a||^2$ for every $a, b \in A$. If $A$ is commutative, then the Gelfand theorem says that $A$ is isometrically $\ast$-isomorphic to the $C^*$-algebra $C_0(X)$ of the continuous complex-valued functions on a locally compact Hausdorff space $X$. For otherwise, the algebra $A$ represents a noncommutative topological space.

B. The stable isomorphisms of $C^*$-algebras. Let $A$ be a $C^*$-algebra deemed as a noncommutative topological space. One can ask, when two such topological spaces $A, A'$ are homeomorphic? To answer the question, let us recall the topological $K$-theory. If $X$ is a (commutative) topological space, denote by $V_c(X)$ an abelian monoid consisting of the isomorphism classes of the complex vector bundles over $X$ endowed with the Whitney sum. The abelian monoid $V_c(X)$ can be made to an abelian group, $K(X)$, using the Grothendieck completion. The covariant functor $F : X \to K(X)$ is known to map the homeomorphic topological spaces $X, X'$ to the isomorphic abelian groups $K(X), K(X')$. Let $A, A'$ be the $C^*$-algebras. If one wishes to define a homeomorphism between the noncommutative topological spaces $A$ and $A'$, it will suffice to define an isomorphism between the abelian monoids $V_c(A)$ and $V_c(A')$ as suggested by the topological $K$-theory. The role of the complex vector bundle of the degree $n$ over the $C^*$-algebra $A$ is played by a $C^*$-algebra $M_n(A) = A \otimes M_n$, i.e. the matrix algebra with the entries in $A$. The abelian monoid $V_c(A) = \cup_{n=1}^{\infty} M_n(A)$ replaces the monoid $V_c(X)$ of the topological $K$-theory. Therefore, the noncommutative topological spaces $A, A'$ are homeomorphic, if the abelian monoids $V_c(A) \cong V_c(A')$ are isomorphic. The latter equivalence is called a stable isomorphism of the $C^*$-algebras $A$ and $A'$ and is formally written as $A \otimes \mathcal{K} \cong A' \otimes \mathcal{K}$, where $\mathcal{K} = \cup_{n=1}^{\infty} M_n$ is the $C^*$-algebra of compact operators. Roughly speaking, the stable isomorphism between the $C^*$-algebras means that they are homeomorphic as the noncommutative topological spaces.

C. The $AF$-algebras. An $AF$-algebra (approximately finite $C^*$-algebra) is
defined to be the norm closure of an ascending sequence of the finite dimensional 
$C^*$-algebras $M_n$'s, where $M_n$ is the $C^*$-algebra of the $n \times n$ matrices with the entries in $\mathbb{C}$. Here the index $n = (n_1, \ldots, n_k)$ represents a semi-simple matrix algebra $M_n = M_{n_1} \oplus \ldots \oplus M_{n_k}$. The ascending sequence mentioned above can be written as $M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} \ldots$, where $M_i$ are the finite dimensional $C^*$-algebras and $\varphi_i$ the homomorphisms between such algebras. The set-theoretic limit $A = \lim M_n$ has a natural algebraic structure given by the formula $a_m + b_k \rightarrow a + b$; here $a_m \rightarrow a, b_k \rightarrow b$ for the sequences $a_m \in M_m, b_k \in M_k$. The homomorphisms $\varphi_i$ can be arranged into a graph as follows. Let $M_i = M_{i_1} \oplus \ldots \oplus M_{i_k}$ and $M_{i'} = M_{i_1'} \oplus \ldots \oplus M_{i_k'}$ be the semi-simple $C^*$-algebras and $\varphi_i : M_i \rightarrow M_{i'}$ the homomorphism. One has the two sets of vertices $V_{i_1}, \ldots, V_{i_k}$ and $V_{i_1'}, \ldots, V_{i_k'}$, joined by the $a_{rs}$ edges, whenever the summand $M_{i_r}$ contains $a_{rs}$ copies of the summand $M_{i_s}$ under the embedding $\varphi_i$. As $i$ varies, one obtains an infinite graph called a Bratteli diagram of the $AF$-algebra. The Bratteli diagram defines a unique $AF$-algebra.

D. The stationary $AF$-algebras. If the homomorphisms $\varphi_1 = \varphi_2 = \ldots = Const$ in the definition of the $AF$-algebra $A$, the $AF$-algebra $A$ is called stationary. The Bratteli diagram of a stationary $AF$-algebra looks like a periodic graph with the incidence matrix $A = (a_{rs})$ repeated over and over again. Since matrix $A$ is a non-negative integer matrix, one can take a power of $A$ to obtain a strictly positive integer matrix – which we always assume to be the case. The stationary $AF$-algebra has a non-trivial group of the automorphisms [6], Ch.6.

2.2 The Jacobi-Perron continued fraction

A. The regular continued fractions. Let $a_1, a_2 \in \mathbb{N}$ such that $a_2 \leq a_1$. Recall that the greatest common divisor of $a_1$ and $a_2$, $GCD(a_1, a_2)$, can be determined from the Euclidean algorithm:

$$
\begin{aligned}
    a_1 &= a_2 b_1 + r_3 \\
    a_2 &= r_3 b_2 + r_4 \\
    r_3 &= r_4 b_3 + r_5 \\
    &\vdots \\
    r_{k-3} &= r_{k-2} b_{k-1} + r_{k-1} \\
    r_{k-2} &= r_{k-1} b_k,
\end{aligned}
$$

where $b_i \in \mathbb{N}$ and $GCD(a_1, a_2) = r_{k-1}$. The Euclidean algorithm can be
written as the regular continued fraction
\[
\theta = \frac{a_1}{a_2} = b_1 + \frac{1}{b_2 + \frac{1}{b_3 + \cdots + \frac{1}{b_k}}},
\]
If \(a_1\) and \(a_2\) are non-commensurable, in the sense that \(\theta \in \mathbb{R} - \mathbb{Q}\), then the Euclidean algorithm never stops and \(\theta = [b_1, b_2, \ldots]\). Note that the regular continued fraction can be written in the matrix form:
\[
\begin{pmatrix}
1 \\
\theta_1 \\
\vdots \\
\theta_{n-1}
\end{pmatrix} = \lim_{k \to \infty}
\begin{pmatrix}
0 & 1 \\
1 & b_1^{(1)} \\
0 & \vdots \\
0 & 1 \\
0 & \vdots \\
0 & 1
\end{pmatrix}
\cdots
\begin{pmatrix}
0 & 1 \\
1 & b_1^{(k)} \\
0 & \vdots \\
0 & 1 \\
0 & \vdots \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
0 \\
1 \\
\vdots \\
1
\end{pmatrix},
\]

B. The Jacobi-Perron continued fractions. The Jacobi-Perron algorithm and connected (multidimensional) continued fraction generalizes the Euclidean algorithm to the case \(\text{GCD}(a_1, \ldots, a_n)\) when \(n \geq 2\). Specifically, let \(\lambda = (\lambda_1, \ldots, \lambda_n), \lambda_i \in \mathbb{R} - \mathbb{Q}\) and \(\theta_{i-1} = \frac{\lambda_i}{\lambda_1}\) with \(1 \leq i \leq n\). The continued fraction
\[
\begin{pmatrix}
1 \\
\theta_1 \\
\vdots \\
\theta_{n-1}
\end{pmatrix} = \lim_{k \to \infty}
\begin{pmatrix}
0 & 0 & \ldots & 0 & 1 \\
1 & 0 & \ldots & 0 & b_1^{(1)} \\
0 & \vdots & \vdots & \vdots & \vdots \\
0 & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & b_{n-1}^{(1)}
\end{pmatrix}
\cdots
\begin{pmatrix}
0 & 0 & \ldots & 0 & 1 \\
1 & 0 & \ldots & 0 & b_1^{(k)} \\
0 & \vdots & \vdots & \vdots & \vdots \\
0 & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & b_{n-1}^{(k)}
\end{pmatrix}
\begin{pmatrix}
0 \\
1 \\
\vdots \\
1
\end{pmatrix},
\]
where \(b_j^{(j)} \in \mathbb{N} \cup \{0\}\), is called the *Jacobi-Perron algorithm* (JPA). Unlike the regular continued fraction algorithm, the JPA may diverge for certain vectors \(\lambda \in \mathbb{R}^n\). However, for points of a generic subset of \(\mathbb{R}^n\), the JPA converges. The convergence of the JPA algorithm can be characterized in terms of the measured foliations. Let \(\mathcal{F} \in \Phi_X\) be a measured foliation on the surface \(X\) of genus \(g \geq 1\). Recall that \(\mathcal{F}\) is called uniquely ergodic if every invariant measure of \(\mathcal{F}\) is a multiple of the Lebesgue measure. By the Masur-Veech theorem, there exists a generic subset \(V \subset \Phi_X\) such that each \(\mathcal{F} \in V\) is uniquely ergodic [10], [14]. We let \(\lambda = (\lambda_1, \ldots, \lambda_n)\) be the vector with coordinates \(\lambda_i = \mu(\gamma_i), \) where \(\gamma_i \in H^\text{odd}_1(S)\); by an abuse of notation, we shall say that \(\lambda \in V\). In view of duality between the measured foliations and the interval exchange transformations [10], the JPA converges if and only if \(\lambda \in V \subset \mathbb{R}^n\) [1].

6
3 Proof of theorem 1

As before, let $W$ denote the set of toric $AF$-algebras of genus $g \geq 2$. Let $G$ be a finitely generated group and $G \times W \to W$ be an action of $G$ on $W$ by the stable isomorphisms of toric $AF$-algebras; in other words, $\gamma(\mathbb{A}_\theta) \otimes \mathcal{K} \cong \mathbb{A}_\theta \otimes \mathcal{K}$ for all $\gamma \in G$ and all $\mathbb{A}_\theta \in W$. The following preparatory lemma will be important.

**Lemma 2** For each $\mathbb{A}_\theta \in W$, there exists a representation $\rho_{\mathbb{A}_\theta} : G \to GL_{6g-6}(\mathbb{Z})$.

*Proof.* The proof of lemma is based on the following well known criterion of the stable isomorphism for the $AF$-algebras: a pair of such algebras $\mathbb{A}_\theta, \mathbb{A}_{\theta'}$ are stably isomorphic if and only if their Bratteli diagrams coincide, except (possibly) a finite part of the diagram, see [6], Theorem 2.3. (Note, that the order isomorphism between the dimension groups mentioned in the original text, can be reformulated in the language of the Bratteli diagrams as stated.)

Let $G$ be a finitely generated group on the generators $\{\gamma_1, \ldots, \gamma_m\}$ and $\mathbb{A}_\theta \in W$. Since $G$ acts on the toric $AF$-algebra $\mathbb{A}_\theta$ by stable isomorphisms, the toric $AF$-algebras $\mathbb{A}_{\theta_1} := \gamma_1(\mathbb{A}_\theta), \ldots, \mathbb{A}_{\theta_m} := \gamma_m(\mathbb{A}_\theta)$ are stably isomorphic to $\mathbb{A}_\theta$; moreover, by transitivity, they are also pairwise stably isomorphic. Therefore, the Bratteli diagrams of $\mathbb{A}_{\theta_1}, \ldots, \mathbb{A}_{\theta_m}$ coincide everywhere except, possibly, some finite parts. We shall denote by $\mathbb{A}_{\theta_{\text{max}}} \in W$ a toric $AF$-algebra, whose Bratteli diagram is the maximal common part of the Bratteli diagrams of $\mathbb{A}_{\theta_i}$ for $1 \leq i \leq m$; such a choice is unique and defined correctly because the set $\{\mathbb{A}_{\theta_i}\}$ is a finite set. By the definition of a toric $AF$-algebra, the vectors $\theta_i = (1, \theta_1^{(i)}, \ldots, \theta_{6g-7}^{(i)})$ are related to the vector $\theta_{\text{max}} = (1, \theta_1^{(\text{max})}, \ldots, \theta_{6g-7}^{(\text{max})})$ by the formula:

$\begin{pmatrix}
\theta_1^{(i)} \\
\vdots \\
\theta_{6g-7}^{(i)}
\end{pmatrix} = \begin{pmatrix}
1 & 0 & \ldots & 0 & 1 \\
1 & 0 & \ldots & 0 & b_1^{(1)(i)} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & b_{6g-7}^{(1)(i)}
\end{pmatrix} A_i \begin{pmatrix}
1 & 0 & \ldots & 0 & 1 \\
1 & 0 & \ldots & 0 & b_1^{(k)(i)} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & b_{6g-7}^{(k)(i)}
\end{pmatrix} \begin{pmatrix}
1 \\
\theta_1^{(\text{max})} \\
\vdots \\
\theta_{6g-7}^{(\text{max})}
\end{pmatrix}$

The above expression can be written in the matrix form $\theta_i = A_i \theta_{\text{max}}$, where $A_i \in GL_{6g-6}(\mathbb{Z})$. Thus, one gets a matrix representation of the generator $\gamma_i$, given by the formula $\rho_{\mathbb{A}_\theta}(\gamma_i) := A_i$. The map $\rho_{\mathbb{A}_\theta} : G \to GL_{6g-6}(\mathbb{Z})$ extends to the rest of the group $G$ via its values on the generators; namely, for every
$g \in G$ one sets $\rho_{k_1}(g) = A_1^{k_1} \ldots A_m^{k_m}$, whenever $g = \gamma_1^{k_1} \ldots \gamma_m^{k_m}$. It is verified (by induction), that the map $\rho_{k_1}: G \to GL_{6g-6}(\mathbb{Z})$ is a homomorphism, since $\rho_{k_1}(g_1g_2) = \rho_{k_1}(g_1)\rho_{k_2}(g_2)$ for $g_1, g_2 \in G$. Lemma 2 follows. □

Let $W_{aper} \subset W$ be a set consisting of the toric AF-algebras, whose Bratteli diagrams do not contain periodic (infinitely repeated) blocks; these are known as non-stationary toric AF-algebras and they are generic in the set $W$ endowed with the natural topology. We call the action of $G$ on the toric AF-algebra $\kappa_0 \in W$ free, if $\gamma(\kappa_0) = \kappa_0$. Lemma 3 follows.

**Lemma 3** Let $\kappa_0 \in W_{aper}$ and $G$ be free on the $\kappa_0$. Then $\rho_{k_1}$ is a faithful representation.

**Proof.** Since the action of $G$ is free, to prove that $\rho_{k_1}$ is faithful, it remains to show, that in the formula $\theta_i = A_i \theta_{\text{max}}$, it holds $A_i = I$, if and only if, $\theta_i = \theta_{\text{max}}$, where $I$ is the unit matrix. Indeed, it is immediate that $A_i = I$ implies $\theta_i = \theta_{\text{max}}$. Suppose now that $\theta_i = \theta_{\text{max}}$ and, let to the contrary, $A_i \neq I$. One gets $\theta_i = A_i \theta_{\text{max}} = \theta_{\text{max}}$. Such an equation has a non-trivial solution, if and only if, the vector $\theta_{\text{max}}$ has a periodic Jacobi-Perron fraction; the period of such a fraction is given by the matrix $A_i$. This is impossible, since it has been assumed, that $\kappa_{\theta_{\text{max}}} \in W_{aper}$. The contradiction finishes the proof of lemma 3. □

Let $G = \text{Mod}(X)$, where $X$ is a surface of genus $g \geq 2$. The group $G$ is finitely generated [5]; it acts on the Teichmüller space $T(g)$ by isomorphisms of the Riemann surfaces. Moreover, the action of $G$ is free on a generic set, $U \subset T(g)$, consisting of the Riemann surfaces with the trivial group of the automorphisms. Let $F: V \to W$ be the Teichmüller functor between the Riemann surfaces and toric AF-algebras (lemma 1); the following is true.

**Lemma 4** The pre-image $F^{-1}(W_{aper})$ is a generic set in the space $T(g)$.

**Proof.** Note, that the set of stationary toric AF-algebras is a countable set. The functor $F$ is a surjective map, which is continuous with respect to the natural topology on the sets $V$ and $W$. Therefore, the pre-image of the complement of a countable set is a generic set. □

To finish the proof, consider the set $U \cap F^{-1}(W_{aper})$; this set is a non-empty set, since it is the intersection of two generic subsets of $T(g)$. Let $R$ be a point (a Riemann surface) in the above set. In view of lemma 3 group $G$ acts on the toric AF-algebra $\kappa_0 = F(R)$ by the stable isomorphisms. By
the construction, the action is free and \( a_\theta \in W_{aper} \). In view of lemma 3, one gets a faithful representation \( \rho = \rho_{a_\theta} \) of the group \( G = \text{Mod} (X) \) into the matrix group \( GL_{6g-6}(\mathbb{Z}) \). Theorem follows. □

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