Violation of local realism in a high-dimensional two-photon setup with non-integer spiral phase plates

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We propose a novel setup to investigate the quantum non-locality of orbital angular momentum states living in a high-dimensional Hilbert space. We incorporate non-integer spiral phase plates in spatial analyzers, enabling us to use only two detectors. The resulting setup is somewhat reminiscent of that used to measure polarization entanglement. However, the two-photon states that are produced, are not confined to a 2×2-dimensional Hilbert space, and the setup allows the probing of correlations in a high-dimensional space. For the special case of half-integer spiral phase plates, we predict a violation of the Clauser-Horne-Shimony-Holt version of the Bell inequality ($S \leq 2$), that is even stronger than achievable for two qubits ($S = 2\sqrt{2}$), namely $S = 3\frac{\sqrt{3}}{2}$.

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I. INTRODUCTION

Recently, the orbital angular momentum (OAM) of light has drawn considerable interest in the context of quantum information processing. The spatial degrees of freedom involved in OAM provide a high-dimensional alphabet to quantum information processing (i.e. quNits instead of qubits). Additionally, since OAM is associated with the topology of the electromagnetic field, the use of this observable in quantum entanglement may lead to states that are inherently robust against decoherence.

The most popular OAM analyzer when dealing with conservation, correlation and entanglement of OAM consists of a so-called fork hologram, i.e. a binary phase hologram containing a fork in its center, together with a spatial-mode detector consisting of a single-mode fiber connected to a single-photon detector; such analyzers have been used in the three-dimensional case, i.e. $N = 3$, by Vaziri et al. In that experiment, proof of entanglement of the OAM degree of freedom of two photons was given by showing that a generalized Bell inequality was violated; this scheme requires 6 detectors, namely 3 in each arm, and one has to measure $3 \times 3$ coincidence count rates to perform a measurement for a single setting of the analyzers.

In the present paper, we consider the use of spiral phase plates (SPPs) instead of phase holograms in an OAM entanglement setup, enabling us to investigate high-dimensional entanglement with only two detectors. More specifically, we will consider SPPs that impose on an optical beam a non-integer OAM expectation value per photon, in units of $\hbar$. With such devices, combined with single-mode fibers to form quantum-state analyzers, we propose to build an OAM entanglement setup that is reminiscent of the usual setup to measure polarization entanglement, where the rotational settings of the analyzers (polarizers in that case) is varied. We will show that it is possible to identify SPP analyzer settings, in the spirit of horizontal and vertical aligned polarizers, when using half-integer SPPs, allowing observation of high-dimensional entanglement ($N > 2$), in contrast to the polarization case ($N = 2$). These claims are supported by calculations; we predict highly non-classical quantum correlations ($S = 3\frac{\sqrt{3}}{2}$), i.e. stronger quantum correlation between two photons than the maximum correlation between two qubits ($S = 2\sqrt{2}$).

II. SPIRAL PHASE PLATES

A SPP, shown in Fig. 1(a), is a transparent dielectric plate with a thickness that varies as a smooth ramp, thus phase shifting an incident field linearly with the azimuthal angle $\theta$. As can be seen, the plate carries a screw discontinuity, expressed by the spiraling thickness, and an edge discontinuity, i.e. a radially oriented step with height $h_s$. The difference between the maximum and minimum phase shift is written as $2\ell \pi$, where $\ell$ is not necessarily integer; in fact, $\ell$ depends on depends on the step height $h_s$, the difference in refractive indices of the SPP and surrounding medium, and the wavelength of the incident light. Thus, a photon propagating through this plate will acquire an OAM with expectation value equal to $\ell \hbar$. Placing such a plate in the waist of Laguerre-Gaussian beam, the field in the exit plane just behind the plate will be described by

$$|r, \theta \rangle \langle \ell |l, p\rangle = u^{p}_{LG}(r, \theta) \exp{(i\ell \theta)},$$

where $|l, p\rangle$ are the Laguerre-Gaussian (LG) field states and $\hat{S}(\ell)$ is the operator representing the effect of the SPP on the input mode. Note that we have neglected the uniform phase shift that is caused by the SPP’s base with height $h_0$, since it acts as a plane-parallel plate. By adding another plane-parallel plate with the appropriate thickness, the total phase shift of these two can be made equal to an integer multiple of $2\pi$.

The function $u^{p}_{LG}(r, \theta)$ in Eq. (1) is the complex amplitude of the Laguerre-Gaussian beam in its waist plane,
LG modes and will be no space propagation. For integer values of the step index \( l \), the output will generally be in a \( \ell \)-mode. When placing a SPP in a beam in a single LG on the number of nodes along a transverse radius of the field behind the SPP, while for \( \ell \) components are required. As SPPs with non-integer \( \ell \) components are required. As SPPs with non-integer \( \ell \) can also create such high-dimensional superpositions of OAM modes, we anticipate that, when employed suitably, they can also project onto such superpositions. We therefore propose to incorporate such non-integer SPPs in analyzers for OAM states living in a high-dimensional Hilbert space.

From a topological point of view, SPPs with non-integer \( \ell \) imprint a mixed screw-edge dislocation on an incident field. The result is rotational asymmetry of the imprinted phase distribution and thus of the emerging field, which becomes visible in the far-field intensity profile (Fig. 1b)). It is the orientation of the step in the transverse plane, that we wish to exploit as an analyzer setting in a new multipartite entanglement scheme.

Since \( \ell \) shall be chosen to have a non-integer value, it is important to realize that, when an incident field passes through an SPP in combination with its complement (i.e. a SPP with the same step height and orientation, but an inverted vorticity), the beam basically passes through a plane-parallel plate that shifts the phase of the field, in an azimuthally uniform way, by \( 2\pi \ell \).

\[
\hat{S}^{\text{compl}}(\ell) \hat{S}(\ell) = \exp(\ii 2\pi \ell) \hat{I},
\]

where \( \hat{I} \) is the identity operator and where we keep the exponent since \( \exp(\ii 2\pi \ell) \neq 1 \) for non-integer \( \ell \). Since \( \hat{S}(\ell) \) is unitary, it follows that

\[
\hat{S}^{\text{compl}}(\ell) = \exp(\ii 2\pi \ell) \hat{S}^{\dagger}(\ell),
\]

where \( \hat{S}^{\dagger}(\ell) \) is the Hermitian conjugate of \( \hat{S}(\ell) \). As \( \hat{S}^{\dagger}(\ell) \) and \( \hat{S}^{\text{compl}}(\ell) \) only differ by a multiplicative phase factor we can again use a plane-parallel plate in the experiment to compensate for this phase factor. Similar to the uniform phase shift caused by the SPP’s base \( h_0 \), we will neglect this phase shift as well, as it can be trivially dealt with. Henceforth, the operator for the compensating SPP, with inverted vorticity, will be represented by the operator \( \hat{S}^{\dagger}(\ell) \).

III. PROPOSAL FOR AN EXPERIMENT

In the experiment on OAM qutrits \((N = 3)\), fork holograms were used. Those holograms can only modify the OAM expectation value by an integer number, depending on the diffraction order of the hologram. It also required the use of three analyzers in each arm of a spontaneous parametric down-conversion (SPDC) setup. In contrast, our proposed OAM entanglement experiment uses non-integer SPP state detectors and only requires a single analyzer-detector combination in each arm. This scheme is shown in Fig. 2 it has been inspired by the setup used in polarization entanglement. The SPPs that are inserted in the two arms should be chosen to obey conservation of OAM (see also section VI). Thus, if the classical pump beam does not contain any OAM, a typical choice for the signal and idler SPPs would be to have step indices \( \ell \) and \(-\ell\) respectively, where, in our case, \( \ell \) should have a non-integer value. The role of the lenses in signal and idler path, \( L_s \) and \( L_i \) respectively, will be discussed in section VI. Finally, single-mode fibers are indicated by \( F_s \) and \( F_i \).

By manipulating the transverse axes of our analyzers, namely the SPP steps, we have access to various photon states that live in a high-dimensional OAM Hilbert.
space, as mentioned earlier. We may orient these edges arbitrarily in the transverse plane, as shown in Fig. 4, thus allowing their use as angular analyzers. When combined with a single-mode fiber, the SPP with step index \(\ell\), set at an azimuthal angle \(\alpha\), projects the incident photon state out onto the OAM state with expectation value \(-\ell\) with edge angle \(\alpha_s\). As we will argue in section VI, the coincidence count rate will depend, like in polarization entanglement, only on the relative angle of the transverse axes.

We stress that, in spite of the superficial similarity between a polarizer and a non-integer SPP, they are of course very different devices; for example, while polarization corresponds to alignment, the SPP edge corresponds to orientation. In other words, with a polarizer, one can analyze the alignment of the electrical field oscillation, either horizontal or vertical, thus yielding a periodicity of \(\pi\) when rotated. With a SPP with non-integer \(\ell\) one can analyze the spatial orientation of a field, thus yielding a periodicity of \(2\pi\) when rotated (see e.g. Fig. 1(b)). Thus we expect that the coincidence count rate will have a periodicity of \(2\pi\) when one of our analyzers is rotated.

An equally important difference between the two cases is that, whereas polarization Hilbert space is two dimensional, OAM Hilbert space is infinite dimensional. As we will see, this makes rotation of SPPs fundamentally different from rotation of a polarizer. In order to address these aspects explicitly, we need a basis of the OAM Hilbert space that is suited for our purpose.

### IV. NON-INTEGER OAM STATES

Our aim is to construct a complete, orthonormal basis that contains non-integer OAM states as basis elements. To this end, we consider all states in the polar representation and separate the radial and angular parts, so that the arbitrary state \(|\vec{r}\rangle = |r,\theta\rangle\) can be written as \(|r|\hat{r}\rangle\), where we have introduced the direction (i.e. angular) ket \(|\hat{r}\rangle\). This allows us to introduce a complete basis set of angular states, the eigenstates \(|l\rangle\) of the OAM operator \(\hat{L}_z\), so that \(\hat{L}_z|l\rangle = l|l\rangle\). A Laguerre-Gaussian state can thus be separated into a radial and angular part, so that \(|l,p\rangle = |\rho_p\rangle|l\rangle\), where \(|\rho_p\rangle\) is the radial part of the state, which is of less importance in this paper. The OAM eigenstates can be transformed from integer to non-integer OAM states. By applying a unitary operator to the integer OAM basis, its completeness and orthonormality are conserved; the unitary operator that we use is the SPP operator, introduced in Eq. (1):

\[
|\hat{r}\rangle\hat{S}(\lambda)|l\rangle \equiv |\hat{r}\rangle a^{(0)}_\lambda |l\rangle = \frac{1}{\sqrt{2\pi}} \exp i(l + \lambda)\theta. \tag{5}
\]

The new basis \(\{|a^{(0)}_\lambda\rangle\}\) has its components enumerated by \(\lambda\), each with OAM equal to \((l + \lambda)\hbar\), where \(\lambda \in [0,1)\) is a constant (not to be confused with the wavelength of the light). Different values of \(\lambda\) define different bases of OAM Hilbert space, each basis being complete and orthonormal.

As an example, passing a photon in a fundamental Gaussian mode \(|0,0\rangle = |\rho_{00}\rangle|0\rangle\) through a SPP with \(\ell = \frac{3}{4}\), creates the state \(|\rho_{00}\rangle|a^{(0)}_{2/3}\rangle\); sending this latter photon through a SPP with an aligned step and \(\ell = -\frac{3}{4}\), combined with a single-mode fiber, then yields a detection probability of unity.

### V. ORIENTATION OF THE EDGE DISLOCATION

Since we intend to rotate the non-integer SPPs, it is important to also have a description of the states that arise when the SPP is rotated while the coordinate system is fixed. To gain insight, we show in Fig. 3 the phase shift imposed by a SPP as a function of the azimuthal angle. The solid line represents the imprinted phase for a SPP with its edge oriented at an angle \(\theta = 0\), while the dashed line corresponds to a SPP with edge orientation \(\theta = \alpha\). We can now generalize the definition of the operator \(\hat{S}(\ell)\) to the operator \(\hat{S}(\alpha, \ell)\), which includes the
overlap amplitude into the basis these basis states. Thus the decomposition of
where \( l \) is the integer part of the step (not to be confused
half-integer states is orthogonal to the non-rotated state. We will call the
index \( \lambda \) equal to \( \hat{\lambda} \). For this, we choose a SPP with step \( \ell \)
respectively, referring to the orientation of the edge part
of the dislocation. We find

\[ \langle \hat{\lambda} \rangle \hat{\lambda} \langle 0 \rangle = \hat{\lambda} \langle 0 \rangle \]

FIG. 4: The overlap (see Eq. (8)) between a non-integer OAM state and the identical state rotated over \( \alpha \). When \( \lambda = \frac{1}{2} \), the states are identical aside from a trivial phase shift, due to the vanishing edge dislocation. For half-integer OAM, i.e. \( \lambda = \frac{1}{2} \), the two states are generally different, leading to a parabolic dependence of their overlap; the two states are orthogonal when \( \alpha = \pi \).

to the polarizer setup, we focus from now on mainly on the case \( \lambda = \frac{1}{2} \). Note that, as Eq. (8) is independent of \( j \), any SPP with \( \ell = j + \lambda \), where \( j \) is the integer part of the step (not to be confused with the integer LG index \( l \)), and \( \lambda \in [0, 1) \), yielding the overlap amplitude

\[ A_\lambda(\alpha) = \langle a^{(l+j)}(0) | a^{(l+j)}(0) \rangle \]

\[ = \langle l | \hat{S}^\dagger (0, j + \lambda) \hat{S}(\alpha, j + \lambda) | l \rangle \]

\[ = \frac{1}{2\pi} |2\pi - \alpha + \alpha \exp(2i\pi\lambda)| \times \exp[-i(l + j + \lambda)\alpha], \]

where \( |l\rangle \) is the OAM operator eigenstate with eigenvalue \( l \). The overlap probability is then

\[ |A_\lambda(\alpha)|^2 = \left(1 - \frac{\alpha}{\pi}\right)^2 \sin^2(\lambda\pi) + \cos^2(\lambda\pi), \]

which depends neither on the integer part of the step index \( j \) nor on the OAM state \( l \). For non-zero values of \( \lambda \), this overlap function has a quadratic dependence on the orientation \( \alpha \). This function is plotted in Fig. 4 for various values of \( \lambda \). It illustrates that, when \( \lambda = 0 \), the projection does not change, as expected. For values of \( \alpha \neq 0 \), the outcome of the projection is less trivial, with \( \lambda = \frac{1}{2} \) providing an especially interesting case: when the state is rotated over \( \alpha = \pi \) by rotating the SPP, the state is orthogonal to the non-rotated state. We will call the half-integer states \( |a^{(l)}(0)\rangle \) and \( |a^{(l)}(\pi)\rangle \) ‘up’ and ‘down’, respectively, referring to the orientation of the edge part of the dislocation. In principle, this orientational label can be used for any non-integer OAM state, but for half-integer OAM states it carries an analogy to fermionic spin, as our ‘up’ and ‘down’ states are orthogonal just like up and down spin \( \frac{1}{2} \). In polarization entanglement the comparable orientational labels are known as ‘H’ for horizontal and ‘V’ for vertical polarization. To make our proposed OAM entanglement setup maximally equivalent

VI. ENTANGLEMENT OF HALF-INTEGER ORBITAL ANGULAR MOMENTUM STATES

When dealing with OAM entanglement, issues of conservation of OAM during the spontaneous parametric down-conversion process arise. This conservation is discussed in many papers \( \frac{2}{2} \frac{10}{10} \frac{11}{11} \frac{12}{12} \) and it seems that OAM is indeed conserved if two conditions are fulfilled, namely (i) the paraxial limit, and (ii) the thin-crystal limit. In practice this is usually the case and we will therefore work within these limits.

The two-photon state can be described in the spatial polar representation as \( \frac{15}{15} \)

\[ \Psi = \int d\vec{r} \mathcal{P}(\vec{r}) \langle \vec{r} | 1 \rangle \langle \vec{r}' | 2 \rangle, \]

where \( \mathcal{P}(\vec{r}) \) is the mode function of the pump beam. We restrict ourselves to a pump beam in a pure OAM mode
so that $\mathcal{P}(\hat{r})$ can be written as $\mathcal{P}(r) \exp \left( i q \theta \right) / \sqrt{2 \pi}$, where $\mathcal{P}(r)$ is the radial part of the pump mode function. We can now write the two-photon state in the half-integer SPP basis $\left\{ |\alpha \rangle \right\}$, thus yielding

$$|\Psi\rangle = \frac{1}{\sqrt{2 \pi}} \int dr \mathcal{P}(r) |r\rangle_1 |r\rangle_2$$

$$\times \sum_{m,n=-\infty}^{\infty} \left[ \int d\theta \exp \left( i q \theta \right) |a^{(n)}_{\frac{1}{2}}(0)|\hat{r}\rangle \langle a^{(m)}_{\frac{1}{2}}(0)|\hat{r}\rangle \right]$$

$$\times |a^{(n)}_{\frac{1}{2}}(0)|a^{(m)}_{\frac{1}{2}}(0)\rangle$$

$$= \frac{1}{\sqrt{2 \pi}} |R\rangle \sum_{n} |a^{(n)}_{\frac{1}{2}}(0)\rangle a^{(q-n-1)}_{\frac{1}{2}}(0)\rangle,$$

where we have defined the radial ket,

$$|R\rangle \equiv \int dr \mathcal{P}(r) |r\rangle_1 |r\rangle_2.$$

We now continue to calculate the coincidence fringe that is expected in the proposed experiment. For the pump beam, we shall assume a fundamental Gaussian that is expected in the proposed experiment. For the same appearance as Eq. (7) with $\lambda = \frac{1}{2}$. The coincidence fringe is then given by the modulus squared,

$$|B(\alpha)|^2 = |C|^2 \left( 1 - \frac{\alpha}{\pi} \right)^2,$$

which is proportional to Eq. (8): we find a parabolic coincidence fringe.

The above reasoning to obtain the coincidence fringe $|B(\alpha)|^2$ is valid for any choice of the signal SPP orientation and only depends on the relative orientation $\alpha$ of the signal and idler SPPs. Thus a coincidence measurement on entangled OAM pairs using half-integer OAM analyzers, will bring forth a coincidence fringe that is parabolic, regardless of the individual settings of the analyzers.

VII. THE CHSH VERSION OF THE BELL INEQUALITY

There have been several theoretical papers that address the generalization of the Bell inequality \cite{20} to quantify the violation of local realism of two $N$-dimensional bipartite Bell inequality, it is allowed to use finite-$N$ quNits, but imprint the infinite OAM dimensionality of the (oriented) edge on a transmitted light field. Rotation of this edge is equivalent to a partial exploration of the complete Hilbert space along a certain path, namely an iso-OAM path; due to this complexity, it is not clear how a generalized Bell inequality could be applied to our case.

However, instead of using a generalized high-dimensional bipartite Bell inequality, it is allowed to use an inequality for lower-dimensional two-particle entanglement \cite{25}. Thus we choose, as in the polarization case, the inequality introduced by Clauser, Horne, Shimony and Holt (CHSH) for a measurement where the coincidence probability is expected to be a function of only $\alpha_s - \alpha_i$, \cite{25}. When relabelling $\alpha_s$ and $\alpha_i$ as $\alpha_1$ and $\alpha_2$ (in no particular order), the CHSH inequality is given by \cite{8,27}\cite{28}\cite{E},

$$S = E(\alpha_1, \alpha_2) - E(\alpha'_1, \alpha_2) + E(\alpha_1, \alpha'_2)$$

$$+ E(\alpha'_1, \alpha'_2) \leq 2.$$

The function $E$ is specified, for the variables $x, y$, as \cite{8,28}\cite{E},

$$E(x, y) = \frac{P(x, y) + P(x^+, y^-) - P(x^+, y^+) - P(x, y^-)}{P(x, y) + P(x^+, y^+) + P(x^+, y^-) + P(x, y^-)}.$$

The notation $x^+$ (and similarly for $y^+$) is used to indicate an analyzer setting that analyses a state orthogonal to the state with setting $x$. Thus in our case, $x^+ \equiv x + \pi$ and $y^+ \equiv y + \pi$. $P(x, y)$ is the coincidence probability
function, which is equal to

\[ P(x, y) = |B(|y - x|)|^2 = C^2 \left( 1 - \frac{|y - x|}{\pi} \right)^2. \]  

(18)

As the periodicity in the present case is half that of the case of polarization entanglement, we use the standard analyzer settings for polarization entanglement multiplied by a factor of two: \( \alpha_1 = -\frac{1}{4}\pi, \alpha'_1 = \frac{1}{4}\pi, \alpha_2 = -\frac{3}{4}\pi, \alpha'_2 = 0. \)

Substitution yields a Bell parameter \( S = 3\frac{1}{2}. \) This is the key result of our paper; it indicates that in the case of entanglement of half-integer OAM states, the maximum violation of the CHSH inequality, given by Eq. (16), is stronger than the maximum that is allowed in polarization entanglement, namely \( S = 2\sqrt{2}. \) In other words, quantum non-locality of the photons in the proposed setup is stronger than the maximum achievable for two qubits. To achieve this, only two detectors are required and only one coincidence count rate is measured per analyzer setting, in contrast to the OAM quNit setup requiring \( N \) detectors and \( N^2 \) coincidence count rates per analyzer setting.

VIII. CONCLUSIONS

In this paper we have put forward a novel approach to demonstrate high-dimensional entanglement of orbital angular momentum states. The proposed setup uses analyzers that consist of non-integer SPPs and single-mode fibers, enabling detection of high-dimensional entanglement with only two detectors.

The key idea is to use the orientation of the edge dislocation in the SPPs. We specialize to the case of half-integer \( \ell, \) so that the orientation of the edge as an analyzer setting can, to a certain extent, be treated similarly as the axis of a polarizer in polarization entanglement. Instead of horizontal and vertical polarization states, we deal with ‘up’ and ‘down’ states, referring to the orientation of the edge dislocation. We analytically calculate the coincidence fringe in the entanglement setup and find it to be parabolic in shape, and periodic over \( 2\pi. \) When evaluating the well-known CHSH Bell parameter, we find \( S = 3\frac{1}{2}, \) i.e. we predict beyond-Bell pairing of two photons. To achieve this, we require only two detectors, as opposed to the standard multiport approach. This economic exploitation of the spatial degrees of freedom seems to be a consequence of the singular nature of our half-integer SPPs, which implies, in principle, infinite dimensionality.

Experimental verification of the outlined proposal is under way.

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APPENDIX: PURE EDGE DISLOCATION

In the present paper, we have seen that, for spiral phase plates with half-integer step index \( \ell, \) i.e. imprinting a mixed screw-edge dislocation, the coincidence fringe in a twin-photon experiment is parabolic as a function of the relative orientation of the two radial edges, with visibility equal to 1. This then results in a value of the CHSH-Bell parameter equal to \( S = 3\frac{1}{2}. \)

An intriguing question is whether there are other devices with optical singularities, besides the half-integer spiral phase plate, that, in a quantum experiment, will give rise to a similarly large value of the \( S \)-parameter. More precisely, can one design a twin-photon experiment that results in a value of the \( S \)-parameter larger than \( 2\sqrt{2}, \) using devices that are simpler to produce and handle than the spiral phase plate discussed so far?

As we will see below, this question can be answered positively, and the proposed device is surprisingly simple. It is a non-integer step phase plate carrying a straight edge dislocation, as shown in Fig. 5. Obviously, such a plate can be manufactured more easily than the spiral phase plates, whose production and characterisation is extensively discussed elsewhere.

FIG. 5: Similar to the spiral phase plate in Fig. 1(a), the non-integer step phase plate has its step height \( h_s \) chosen so that the phase shift due to the thick part is a non-integer multiple of \( 2\pi \) with respect to the thin part. The edge dislocation is chosen so that it goes through the centre of the field mode.

Similar to the non-integer spiral phase plate, the non-integer step phase plate contains an orientational degree of freedom, \( \alpha. \) It is possible to write a unitary operator for the plate’s action,

\[ \langle x, y | \hat{F}(\alpha, \phi) | l \rangle = \frac{\exp(i\theta)}{\sqrt{2\pi}} \times \begin{cases} \exp(i\phi), & \alpha \leq \theta < \alpha + \pi, \\ 1, & \end{cases} \]

(19)

where \( \alpha \in [0, \pi) \) represents the orientation of the step, \( \phi \) the optical phase delay resulting from the step, \( \theta \in [0, 2\pi) \) the azimuthal angle and \( | l \rangle \) the orbital angular momentum eigenstates. For the special case \( \phi = \pi, \) Eq. (19) corresponds to the Hilbert transform.
FIG. 6: The overlap (see Eq. 21) between a state with an imprinted line dislocation, and the identical state rotated over \( \alpha \). When there is no dislocation (\( \phi = 0 \)), the states are identical. For other phase differences \( \phi \), the two states are generally different; \( \phi = \frac{\pi}{2} \) and \( \phi = \pi \) result in a continuous parabolic fringe.

non-integer step phase plate obviously also transforms the pure orbital angular momentum states into high-dimensional states, due to the edge dislocation. The definition as given here can be extended for \( \alpha \in [-\pi, \pi) \), seemingly complicating matters, but without any impact on the calculations ahead.

Since the operator \( \mathcal{F} \) is unitary, we can define a new, complete basis \( \{ |b_\phi^{(l)}(\alpha)\rangle \} \), where \( \mathcal{F}(\alpha, \phi)|l\rangle = |b_\phi^{(l)}(\alpha)\rangle \).

We can write the states after rotation \( |b_\phi^{(l)}(\alpha)\rangle \) in terms of superpositions of the non-rotated states, \( |b_\phi^{(l)}(0)\rangle \), where the decomposition depends on the angle \( \alpha \). We can illustrate this by making a projection of such a state with \( \alpha = 0 \) onto the same state \( \alpha \neq 0 \). The overlap amplitude thus becomes

\[
A(\alpha) = \frac{\langle b_\phi^{(l)}(0) | b_\phi^{(l)}(\alpha) \rangle}{\frac{\alpha}{\pi} (\cos \phi - 1) + 1},
\]

which is valid for values of \( \alpha \in [-\pi, \pi) \). The overlap probability is given by

\[
|A(\alpha)|^2 = \left( \frac{\alpha}{\pi} \right)^2 (\cos \phi - 1)^2 + 2 \frac{\alpha}{\pi} (\cos \phi - 1) + 1.
\]

Figure 6 shows the intensity fringe as a function of \( \alpha \) for three different values of \( \phi \). For clarity, we show the functions over the range \( \alpha \in [0, 2\pi) \) instead of \( \alpha \in [-\pi, \pi) \), which is allowed when \( \alpha \) is periodic over \( 2\pi \).

The most interesting cases occur when \( \phi = \pi \) and \( \phi = \pi/2 \), when the fringe becomes a parabola, periodic over \( \pi \) and \( 2\pi \), respectively. The result is then, aside from the periodicity, identical to that for a half-integer spiral phase plate.

A plate with a single edge dislocation with phase difference \( \pi \) or \( \pi/2 \) can thus be used, similar to a spiral phase plate, to investigate entanglement. The setup would be identical to Fig. 2 with the spiral phase plates replaced by the edge dislocation devices discussed here. The calculation from two-photon state to coincidence fringe is the same as discussed in section 7 using the states \( |b_\phi^{(l)}(\alpha)\rangle \), or \( |b_\phi^{(l)}(\alpha)\rangle \) instead of \( |a_\phi^{(l)}(\alpha)\rangle \).

The CHSH-Bell inequality is violated maximally for these two-photon states, using the set of sixteen angles as used in polarisation entanglement for \( \phi = \pi \), and the set as used for the half-integer spiral phase plates for \( \phi = \pi/2 \). Both cases yield an CHSH-Bell parameter equal to \( S = 3 \frac{3}{4} \).

Finally, we note that binary phase plates only slightly more complex than the type discussed above and pictured in Fig. 3 can yield an even stronger violation of the CHSH-Bell inequality. When using a plate with the shape as indicated in the inset of Fig. 7 the maximum CHSH-Bell parameter will be equal to \( S = 4 \), the highest value at all possible. For such a plate, we predict a coincidence fringe with a shape as indicated in Fig. 4.

![Figure 7](image)

FIG. 7: When using a plate as indicated in the inset, the overlap (see Eq. 21) as a function of \( \alpha \) is as shown in the graph. This results in \( S = 4 \).

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