ON CORE AND BAR-CORE PARTITIONS

JEAN-BAPTISTE GRAMAIN AND RISHI NATH

Abstract. If s and t are relatively prime J. Olsson proved in [7] that the s-core of a t-core partition is again a t-core partition, and that the s-bar-core of a t-bar-core partition is again a t-bar-core partition. Here generalized results are proved for partitions and bar-partitions when the restriction that s and t be relatively prime is removed.

1. Introduction

The basic facts about partitions, hooks and blocks can be found in [3, Chapter 2] or [6, Chapter 1]. We recall a few key definitions here. A partition λ of n is defined as a non-increasing sequence of nonnegative integers \((\lambda_1, \lambda_2, \cdots)\) that sum to n. A partition is represented graphically by its Young diagram \([\lambda]\), which consists of the set of nodes \(\{(i, j) \mid (i, j) \in \mathbb{N}^2, j \leq \lambda_i\}\). The node \((i, j)\) is in the \(i\)th row and \(j\)th column of \([\lambda]\). The rows of \([\lambda]\) are labelled from top to bottom, while its columns are labelled from left to right.

To each node \((i, j)\) in \([\lambda]\) we associate the hook \(h_{ij}\) of \(\lambda\), which consists of the node \((i, j)\) itself, together with all the nodes \(\{(i, k) \mid j < k\}\) in \([\lambda]\) (i.e. in the same row as and to the right of \((i, j)\)) and all the nodes \(\{(\ell, j) \mid i < \ell\}\) (i.e. in the same column as and below \((i, j)\)). The length of \(h_{ij}\) is the total number of nodes contained in the hook. For any integer \(\ell \geq 1\), we call \(\ell\)-hook a hook of length \(\ell\), and \((\ell)\)-hook a hook of length divisible by \(\ell\). The information about the \((\ell)\)-hooks in \(\lambda\) is encoded in the \(\ell\)-quotient \(q_\ell(\lambda) = (\lambda_0, \ldots, \lambda_{\ell-1})\) of \(\lambda\). The \(\lambda_i\)'s are partitions whose sizes sum to the number \(w\) of \((\ell)\)-hooks in \(\lambda\) (called the \(\ell\)-weight of \(\lambda\)).

The removal of an \(\ell\)-hook \(h\) in \(\lambda\) is obtained by removing the \(\ell\) nodes of \([\lambda]\) in \(h\), and migrating the disconnected nodes in \([\lambda]\) up and to the left. The result is a partition of \(n - \ell\) denoted by \(\lambda \setminus h\). By removing all the \((\ell)\)-hooks in \(\lambda\), one obtains the \(\ell\)-core \(\gamma_\ell(\lambda)\) of \(\lambda\). The partition \(\gamma_\ell(\lambda)\) contains no \((\ell)\)-hooks, and is uniquely determined by \(\ell\) (i.e. doesn’t depend on the order in which we remove the \(\ell\)-hooks in \(\lambda\)). The partition \(\lambda\) is entirely determined by its \(\ell\)-core and \(\ell\)-quotient.

It is well-known that the irreducible complex characters of the symmetric group \(\mathfrak{S}_n\) are labelled by the partitions of \(n\). If \(p\) is a prime, then the distribution of irreducible characters of \(\mathfrak{S}_n\) into \(p\)-blocks has a combinatorial description known as the Nakayama Conjecture: two characters \(\chi_\lambda, \chi_\mu \in \text{Irr}(\mathfrak{S}_n)\) belong to the same \(p\)-block if and only if \(\lambda\) and \(\mu\) have the same \(p\)-core (see [3, Theorem 6.1.21]). Hence we define, for each integer \(\ell \geq 1\), an \(\ell\)-block of partitions of \(n\) to be the set of all partitions of \(n\) having a common \(\ell\)-core.

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We now recall the analogous notions and results for bar-partitions, which can be found in [5, Chapter 1]. A bar-partition is a partition \( \lambda \) comprised of distinct parts. To each bar-partition we associate a shifted Young diagram \( S(\lambda) \) obtained by shifting the \( i \)th row of the usual Young diagram \( (i-1) \) positions to the right. The \( j \)-th node in the \( i \)-th row will be called the \((i,j)\)-node. To each node \((i,j)\) in \( S(\lambda) \), one can associate a bar and bar-length. For any odd integer \( \ell \), a bar-partition \( \lambda \) is entirely determined by its \( \ell \)-core \( \gamma_{\ell}(\lambda) \) and its \( \ell \)-quotient \( \bar{\gamma}_{\ell}(\lambda) \). The bar-core \( \gamma_{\ell}(\lambda) \) is obtained by removing from \( \lambda \) all the bars of length divisible by \( \ell \) (called \((\ell)\)-bars). The bar-quotient of \( \lambda \) is of the form \( \bar{\gamma}_{\ell}(\lambda) = (\lambda_0, \lambda_1, \ldots, \lambda_{(\ell-1)/2}) \), where \( \lambda_0 \) is a bar-partition, \( \lambda_1, \ldots, \lambda_{(\ell-1)/2} \) are partitions, and the sizes of the \( \lambda_i \)'s sum to the number of \((\ell)\)-bars in \( \lambda \) (called \( \ell \)-weight of \( \lambda \)).

It is well-known that the bar-partitions of \( n \) label the faithful irreducible complex characters of the 2-fold covering group \( \bar{S}_n \) of \( S_n \). These correspond to irreducible projective representations of \( \bar{S}_n \), and are known as spin-characters. If \( p \) is an odd prime, then the distribution of spin-characters of \( \bar{S}_n \) of positive defect into \( p \)-blocks has a combinatorial description known as the Morris Conjecture: two \( \bar{s} \)-bar-partitions of \( n \) correspond to irreducible \( S_n \)-modules if and only if the bar-partitions labelling them have the same \( \bar{p} \)-core (see [6, Theorem 13.1]).

In analogy with this, we define, for each odd integer \( \ell \geq 1 \), an \( \ell \)-block of partitions of \( n \) to be the set of all bar-partitions of \( n \) having a common given \( \ell \)-core.

2. Some new results on cores and bar-cores

In this section, we generalize to arbitrary integers \( s \) and \( t \) the results on cores and bar-cores proved by J. B. Olsson in [7] when \( s \) and \( t \) are coprime. Note that Olsson’s result ([7, Theorem 1]) was interpreted by M. Fayers through alcove geometry and actions of the affine symmetric group (see [2]). It was also used by F. Garvan and A. Berkovich to bound the number of distinct values their partition statistic (the GBG-rank) can take on a \( \ell \)-core (mod \( s \)) (see [1, Theorem 1.2]).

We keep the notation as in Section [1].

**Theorem 2.1.** For any two positive integers \( s \) and \( t \), the \( s \)-core of a \( t \)-core partition is again a \( t \)-core partition.

**Remark 2.2.** This result was proved by J. B. Olsson in [7], under the extra hypothesis that \( s \) and \( t \) are relatively prime. R. Nath then gave in [5] a proof of the result in general. We give here another proof which, unlike the one given by Nath, uses Olsson’s result, and provides the framework for the proof for bar-partitions.

**Proof.** Consider a \( t \)-core partition \( \lambda \). Let \( g = \gcd(s,t) \), and write \( s_0 = s/g \) and \( t_0 = t/g \). It’s a well-known fact (see e.g. [6, Theorem 3.3]) that there is a canonical bijection \( \varphi \) between the set of hooks of length divisible by \( g \) in \( \lambda \) and the set of hooks in \( q_g(\lambda) = (\lambda_0, \ldots, \lambda_{g-1}) \) (i.e. hooks in each of the \( \lambda_i \)'s). For each positive integer \( k \) and hook \( h \) of length \( kg \) in \( \lambda \), the hook \( \varphi(h) \) has length \( k \). Furthermore, we have \( q_g(\lambda \setminus h) = q_g(\lambda) \setminus \varphi(h) \).

In particular, since \( \lambda \) is an \( t \)-core, and since \( t = t_0g \), we see that \( q_g(\lambda) \) contains no \( t_0 \)-hook, so that each \( \lambda_i \) is an \( t_0 \)-core.

Now, the \( s \)-hooks in \( \lambda \) are in bijection with the \( s_0 \)-hooks in \( q_g(\lambda) \). When we remove them all, we obtain that the \( s \)-core \( \gamma_s(\lambda) \) has \( g \)-core \( \gamma_g(\gamma_s(\lambda)) = \gamma_g(\lambda) \) and \( g \)-quotient \( q_g(\gamma_s(\lambda)) = (\gamma_{s_0}(\lambda_0), \ldots, \gamma_{s_0}(\lambda_{g-1})) \). But, since \( s_0 \) and \( t_0 \) are coprime, the \( s_0 \)-core of each \( t_0 \)-core \( \lambda_i \) is again a \( t_0 \)-core ([7, Theorem 1]). This shows that
$q_g(\gamma_s(\lambda))$ has no $t_0$-hook, which in turn implies that $\gamma_s(\lambda)$ contains no $t$-hook, whence is an $t$-core.

As we mentioned in Section 1 when $p$ is a prime, the study of $p$-cores is linked to that of the $p$-modular representation theory of the symmetric group $\mathfrak{S}_n$ (as they label the $p$-blocks of irreducible characters). When $\ell \geq 2$ is an arbitrary integer, it turns out that it is still possible to describe an $\ell$-modular representation theory of $\mathfrak{S}_n$ (see [3]). The theory of $\ell$-blocks obtained in this way is in fact related to the ordinary representation theory of an Iwahori-Hecke algebra of type $\mathfrak{S}_n$, when specialized at an $\ell$-root of unity. Külshammer, Olsson and Robinson proved in [3] the following analogue of the Nakayama Conjecture: two characters $\chi_\lambda, \chi_\mu \in \text{Irr}(\mathfrak{S}_n)$ belong to the same $\ell$-block if and only if $\lambda$ and $\mu$ have the same $\ell$-core.

It is therefore legitimate to study $\ell$-cores and $\ell$-blocks of partitions. In particular, we obtain from Theorem 2.1 a generalization of [7] Corollary 3]. We call principal $\ell$-block of $n$ the $\ell$-block of partitions of $n$ which contains the partition $(n)$ (i.e. the set of partitions labelling the characters of the principal $\ell$-block of $\mathfrak{S}_n$).

**Corollary 2.3.** Let $r, s$ and $t$ be any positive integers such that $s > r > t$, and let $n = as + r$ for some $a \in \mathbb{Z}_{\geq 0}$. Then the principal $s$-block of $n$ contains no $t$-core.

**Proof.** Suppose the partition $\lambda$ of $n$ is a $t$-core. The $s$-core $\gamma$ of $\lambda$, which is obtained by removing $s$-hooks, must therefore be a partition of some $m$ which differs from $n$ by a multiple of $s$, i.e. $m = bs + r$ for some $b$ such that $a > b \geq 0$. By Theorem 2.1, $\gamma$ is also a $t$-core. Now, if $\lambda$ was in the principal $s$-block of $n$, then its $s$-core would be the same as that of the cycle $(n)$, hence also a cycle. We would thus have $\gamma = (m)$. But since $m \geq r \geq t$, the cycle $(m)$ contains a $t$-hook, hence cannot be a $t$-core.\[\square\]

In terms of blocks of characters, this means that, if $s$, $t$ and $n$ are as above, then there is no trivial block inclusion of a $t$-block in the principal $s$-block of $\mathfrak{S}_n$ (see [8]).

We now prove the analogue results for bar-cores, which was proved by Olsson when $s$ and $t$ are odd and coprime ([7] Theorem 4]).

**Theorem 2.4.** For any two odd positive integers $s$ and $t$, the $s$-core of an $\bar{t}$-core partition is again a $\bar{t}$-core partition.

**Proof.** Take any $\bar{t}$-core $\lambda$. Let $g = \text{gcd}(s, t)$, and write $s_0 = s/g$ and $t_0 = t/g$. There is a canonical bijection $\varphi$ between the set of bars of length divisible by $g$ in $\lambda$ and the set of bars in its $\bar{g}$-quotient $\bar{q}_g(\lambda) = (\lambda_0, \lambda_1, \ldots, \lambda_{(g-1)/2})$, where a bar in $\bar{q}_g(\lambda)$ is either a bar in the bar-partition $\lambda_0$ or a hook in one of the partitions $\lambda_1, \ldots, \lambda_{(g-1)/2}$ (see [3] Theorem 4.3]). For each positive integer $k$ and bar $b$ of length $kb$ in $\lambda$, the bar $\varphi(b)$ has length $k$. Furthermore, we have $\bar{q}_g(\lambda \setminus b) = \bar{q}_g(\lambda) \setminus \varphi(b)$.

The same argument as in the proof of Theorem 2.1 thus proves that $\lambda_0$ is an $\bar{t}_0$-core, that each $\lambda_i$ $(1 \leq i \leq (g-1)/2)$ is an $t_0$-core, and that the $s$-core $\bar{\gamma}_s(\lambda)$ of $\lambda$ has $\bar{g}$-quotient $\bar{q}_g(\bar{\gamma}_s(\lambda)) = (\bar{\gamma}_{s_0}(\lambda_0), \bar{\gamma}_{s_0}(\lambda_1), \ldots, \bar{\gamma}_{s_0}(\lambda_{(g-1)/2}))$. And, since $s_0$ and $t_0$ are coprime, the $s_0$-core of each $t_0$-core $\lambda_i$ $(1 \leq i \leq (g-1)/2)$ is again a $t_0$-core ([7] Theorem 1]), and the $s_0$-core of the $t_0$-core $\lambda_0$ is again a $t_0$-core ([7] Theorem 2.2]).
Theorem 4]). This shows that the \( \bar{g} \)-quotient of \( \bar{\gamma}_s(\lambda) \) contains no \( t_0 \)-bar, which finally implies that \( \bar{\gamma}_s(\lambda) \) contains no \( t \)-bar, whence is an \( \bar{t} \)-core.

\[ \square \]

In analogy with the partition case, we call principal \( \bar{\ell} \)-block of bar-partitions of \( n \) (for \( \ell \) odd) the \( \bar{\ell} \)-block containing the bar-partition \( (n) \). Then the same argument as for the proof of Corollary 2.3 yields

**Corollary 2.5.** Let \( r, s \) and \( t \) be any positive integers such that \( s \) and \( t \) are odd and \( s > r \geq t \), and let \( n = as + r \) for some \( a \in \mathbb{Z}_{\geq 0} \). Then the principal \( \bar{s} \)-block of \( n \) contains no \( \bar{t} \)-core.

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