COMPACTIFICATIONS OF $M_{0,n}$ ASSOCIATED WITH ALEXANDER SELF DUAL COMPLEXES: CHOW RING, $\psi$-CLASSES, AND INTERSECTION NUMBERS

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Abstract. An Alexander self-dual complex gives rise to a compactification of $M_{0,n}$, called ASD compactification, which is a smooth algebraic variety. ASD compactifications include (but are not exhausted by) the polygon spaces, or the moduli spaces of flexible polygons. We present an explicit description of the Chow rings of ASD compactifications. We study the analogs of Kontsevich’s tautological bundles, compute their Chern classes, compute top intersections of the Chern classes, and derive a recursion for the intersection numbers.

Alexander self dual complex, modular compactification, tautological ring, Chern class, Chow ring

1. Introduction

The moduli space of $n$-punctured rational curves $M_{0,n}$ and its various compactifications is a classical object, bringing together algebraic geometry, combinatorics, and topological robotics. Recently, D.I. Smyth[1] classified all modular compactifications of $M_{0,n}$. We make use of an interplay between different compactifications, and:

- describe the classification in terms of (what we call) preASD simplicial complexes;
- describe the Chow rings of the compactifications arising from Alexander self-dual complexes (ASD compactifications);
- compute for ASD compactifications the associated Kontsevich’s $\psi$-classes, their top monomials, and give a recurrence relation for the top monomials.

Oversimplifying, the main approach is as follows. Some (but not all) compactifications are the well-studied polygon spaces, that is, moduli spaces of flexible polygons. A polygon space corresponds to a threshold Alexander self-dual complex. Its cohomology ring (which equals the Chow ring) is known due to J.-C. Hausmann and A. Knutson[2], and A. Klyachko[10]. The paper[3] gives a computation-friendly presentation of the ring. Due to Smyth[1], all the modular compactifications correspond to preASD complexes, that is, to those complexes that are contained in an ASD complex. A removal of a facet of a preASD complex amounts to a blow up of the associated compactification. Each ASD compactification is achievable from a threshold ASD compactification by a sequence of blow ups and blow downs. Since the changes in the Chow ring are controllable, one can start with a polygon space, and then (by elementary steps) reach any of the ASD compactifications and describe its Chow ring (Theorem 26).

M. Kontsevich’s $\psi$-classes[4] arise here in a standard way. Their computation of is a mere modification of the Chern number count for the tangent bundle over $S^2$ (a classical exercise in a topology course). The recursion (Theorem 36) and the top monomial counts (Theorem 37) follow.

It is worthy mentioning that a disguised compactification by simple games, i.e., ASD complexes, is discussed from a combinatorial viewpoint in[5].

Now let us give a very brief overview of moduli compactifications of $M_{0,n}$. A compactification by a smooth variety is very desirable since it makes intersection theory applicable. We also expect that (1) a compactification is modular, that is, itself is the moduli space of some curves and marked points lying on it, and (2) the complement of $M_{0,n}$ (the “boundary”) is a divisor.

The space $M_{0,n}$ is viewed as the configuration space of $n$ distinct marked points (“particles”) living in the complex projective plane. The space $M_{0,n}$ is non-compact due to forbidden collisions of the marked

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points. Therefore, each compactification should suggest an answer to the question: what happens when two (or more) marked points tend to each other? There exist two possible choices: either one allows some (not too many!) points to coincide, either one applies a blow up. It is important that the blow ups amount to adding points that correspond to $n$-punctured nodal curves of arithmetic genus zero.

A compactification obtained by blow ups only is the celebrated Deligne–Mumford compactification. If one avoids blow ups and allows (some carefully chosen) collections of points to coincide, one gets an ASD-compactification; among them are the polygon spaces. Diverse combinations of these two options (in certain cases one allows points to collide, in other cases one applies a blow up) are also possible; the complete classification is due to [1].

Now let us be more precise and look at the compactifications in more detail.

1.1. Deligne–Mumford compactification.

**Definition 1.** [6] Let $B$ be a scheme. A family of rational nodal curves with $n$ marked points over $B$ is

- a flat proper morphism $\pi : C \to B$ whose geometric fibers $E_*$ are nodal connected curves of arithmetic genus zero, and
- a set of sections $(s_1, \ldots, s_n)$ that do not intersect nodal points of geometric fibers.

In this language, the sections correspond to marked points. The above condition means that a nodal point of a curve may not be marked.

A family $(\pi : C \to B; s_1, \ldots, s_n)$ is stable if the divisor $K_C + s_1 + \cdots + s_n$ is $\pi$-relatively ample.

Let us rephrase this condition: a family $(\pi : C \to B; s_1, \ldots, s_n)$ is stable if each irreducible component of each geometric fiber has at least three special points (nodal points and points of the sections $s_i$).

**Theorem 2.** [6] (1) There exists a smooth and proper over $\mathbb{Z}$ stack $\overline{\mathcal{M}}_{0,n}$, representing the moduli functor of stable rational curves. Corresponding moduli scheme is a smooth projective variety over $\mathbb{Z}$.

(2) The compactification equals the moduli space for $n$-punctured stable curves of arithmetic genus zero with $n$ marked points. A stable curve is a curve of arithmetic genus zero with at worst nodal singularities and finite automorphism group. This means that (i) every irreducible component has at least three marked or nodal points, and (ii) no marked point is a nodal point.

The Deligne-Mumford compactification has a natural stratification by stable trees with $n$ leaves. A stable tree with $n$ leaves is a tree with exactly $n$ leaves enumerated by elements of $[n] = \{1, \ldots, n\}$ such that each vertex is at least trivalent.

Here and in the sequel, we use the following notation: by vertices of a tree we mean all the vertices (in the usual graph-theoretical sense) excluding the leaves. A bold edge is an edge connecting two vertices (see Figure 1).

The initial space $\mathcal{M}_{0,n}$ is a stratum corresponding to the one-vertex tree. Two-vertex trees (Fig 1(b)) are in a bijection with bipartitions of the set $[n]$: $T \cup T^c = [n]$ s.t. $|T|, |T^c| > 1$. We denote the closure of the corresponding stratum by $D_T$. The latter are important since the (Poincaré duals of) closures of the strata generate the Chow ring $A^*(\overline{\mathcal{M}}_{0,n})$:

**Theorem 3.** [6, Theorem 1] The Chow ring $A^*(\overline{\mathcal{M}}_{0,n})$ is isomorphic to the polynomial ring

$$Z[D_T : T \subset [n]; |T| > 1, |T^c| > 1]$$

factorized by the relations:

1. $D_T = D_T^c$;
2. $D_T D_S = 0$ unless $S \subset T$ or $T \subset S$ or $S \subset T^c$ or $T \subset S^c$;
3. For any distinct elements $i, j, k, l \in [n]$:

$$\sum_{i,j \in T; k,l \notin T} D_T = \sum_{i,k \in T; j,l \notin T} D_T = \sum_{i,l \in T; j,k \notin T} D_T$$
1.2. Weighted compactifications. The next breakthrough step was done by B. Hassett in [8].

Define a weight data as an element \( A = (a_1, \ldots, a_n) \in \mathbb{R}^n \) such that
\begin{itemize}
  \item \( 0 < a_i \leq 1 \) for any \( i \in [n] \),
  \item \( a_1 + \cdots + a_n > 2 \).
\end{itemize}

**Definition 4.** Let \( B \) be a scheme. A family of nodal curves with \( n \) marked points \( (\pi : C \to B; s_1, \ldots, s_n) \) is \( A \)-stable if
\begin{enumerate}
  \item \( K_\pi + a_1 s_1 + \cdots + a_n s_n \) is \( \pi \)-relatively ample,
  \item whenever the sections \( \{ s_i \}_{i \in I} \) intersect for some \( I \subset [n] \), one has \( \sum_{i \in I} a_i < 1 \).
\end{enumerate}

The first condition can be rephrased as: each irreducible component of any geometric fiber has at least three distinct special points.

**Theorem 5.** [8, Theorem 2.1] For any weight data \( A \) there exist a connected Deligne–Mumford stack \( \overline{M}_{0,A} \) smooth and proper over \( \mathbb{Z} \), representing the moduli functor of \( A \)-stable rational curves. The corresponding moduli scheme is a smooth projective variety over \( \mathbb{Z} \).

The Deligne–Mumford compactification arises as a special case for the weight data \((1, \ldots, 1)\).

It is natural to ask: how much does a weighted compactification \( \overline{M}_{0,A} \) depend on \( A \)? Pursuing this question, let us consider the space of parameters:
\[
A_n = \left\{ A \in \mathbb{R}^n : 0 < a_i \leq 1, \sum_i a_i > 2 \right\} \subset \mathbb{R}^n.
\]

The hyperplanes \( \sum_{i \in I} a_i = 1, I \subset [n], |I| \geq 2 \), (called walls) cut the polytope \( A_n \) into chambers. The Hassett compactification depends on a chamber only [8, Proposition 5.1].

Combinatorial stratification of the space \( \overline{M}_{0,A} \) looks similarly to that of the Deligne–Mumford’s with the only difference — some of the marked points now can coincide [9].

More precisely, a weighted tree \( (\gamma, I) \) is an ordered \( k \)-partition \( I_1 \sqcup \cdots \sqcup I_k = [n] \) and a tree \( \gamma \) with \( k \) ordered leaves marked by elements of the partition such that \( \sum_{j \in I_m} a_j \leq 1 \) for any \( m \), and \( (2) \) for each vertex, the number of emanating bold edges plus the total weight is greater than 2. Open strata are enumerated by weighted trees: the stratum of the space \( \overline{M}_{0,A} \) corresponding to a weighted tree \( (\gamma, I) \) consists of curves whose irreducible components form the tree \( \gamma \) and collisions of sections form the partition \( I \). Closure of this stratum is denoted by \( D_{(\gamma, I)} \).
1.3. Polygon spaces as compactifications of $\mathcal{M}_{0,n}$. Assume that an $n$-tuple of positive real numbers $L = (l_1, \ldots, l_n)$ is fixed. We associate with it a flexible polygon, that is, $n$ rigid bars of lengths $l_i$ connected in a cyclic chain by revolving joints. A configuration of $L$ is an $n$-tuple of points $(q_1, \ldots, q_n)$, $q_i \in \mathbb{R}^3$, with $|q_iq_{i+1}| = l_i$, $|q_0q_1| = l_n$.

The following two definitions for the polygon space, or the moduli space of the flexible polygon are equivalent:

**Definition 6.**

I. The moduli space $M_L$ is a set of all configurations of $L$ modulo orientation preserving isometries of $\mathbb{R}^3$.

II. Alternatively, the space $M_L$ equals the quotient of the space

$$\left\{ (u_1, \ldots, u_n) \in (\mathbb{S}^2)^n : \sum_{i=1}^{n} l_iu_i = 0 \right\}$$

by the diagonal action of the group $SO_3(\mathbb{R})$.

The second definition shows that the space $M_L$ does not depend on the ordering of $\{l_1, \ldots, l_n\}$; however, it does depend on the values of $l_i$.

Let us consider the parameter space

$$\left\{ (l_1, \ldots, l_n) \in \mathbb{R}_{>0}^n : l_i < \sum_{j \neq i} l_j \text{ for } i = 1, \ldots, n \right\}.$$ 

This space is cut into open chambers by walls. The latter are hyperplanes with defining equations

$$\sum_{i \in I} l_i = \sum_{j \notin I} l_j.$$

The diffeomorphic type of $M_L$ depends only on the chamber containing $L$. For a point $L$ lying strictly inside some chamber, the space $M_L$ is a smooth $(2n - 6)$-dimensional algebraic variety $[10]$. In this case we say that the length vector is generic.

**Definition 7.** For a generic length vector $L$, we call a subset $J \subset [n]$ long if

$$\sum_{i \in J} l_i > \sum_{i \notin J} l_i.$$ 

Otherwise, $J$ is called short. The set of all short sets we denote by $SHORT(L)$.

Each subset of a short set is also short, therefore $SHORT(L)$ is a (threshold Alexander self-dual) simplicial complex. Rephrasing the above, the diffeomorphic type of $M_L$ is defined by the simplicial complex $SHORT(L)$.

2. **ASD and preASD compactifications**

2.1. ASD and preASD simplicial complexes. Simplicial complexes provide a necessary combinatorial framework for the description of the category of smooth modular compactifications of $\mathcal{M}_{0,n}$.

A simplicial complex (a complex, for short) $K$ is a subset of $2^{[n]}$ with the hereditary property: $A \subset B \in K$ implies $A \in K$. Elements of $K$ are called faces of the complex. Elements of $2^{[n]} \setminus K$ are called non-faces. The maximal (by inclusion) faces are called facets.

We assume that the set of 0-faces (the set of vertices) of a complex is $[n]$. The complex $2^{[n]}$ is denoted by $\Delta_{n-1}$. Its $k$-skeleton is denoted by $\Delta^k_{n-1}$. In particular, $\Delta^{n-2}_{n-1}$ is the boundary complex of the simplex $\Delta_{n-1}$.

**Definition 8.** For a complex $K \subset 2^{[n]}$, its Alexander dual is the simplicial complex

$$K^\circ := \{ A \subset [n] : A^c \notin K \} = \{ A^c : A \in 2^{[n]} \setminus K \}.$$
Here and in the sequel, $A^c = [n] \setminus A$ is the complement of $A$. A complex $K$ is *Alexander self-dual* (an ASD complex) if $K^c = K$. A pre Alexander self-dual (a pre ASD) complex is a complex contained in some ASD complex.

In other words, ASD complexes (pre ASD complexes, respectively) are characterized by the condition: for any partition $[n] = A \sqcup B$, exactly one (at most one, respectively) of $A$, $B$ is a face.

Some ASD complexes are *threshold complexes*: they equal $\text{SHORT}(\mathcal{L})$ for some generic weight vectors $\mathcal{L}$ (Section 1.3). It is known that threshold ASD complexes exhaust all ASD complexes for $n \leq 5$. However, for bigger $n$ this is no longer true. Moreover, for $n \to \infty$ the percentage of threshold ASD complexes tends to zero.

To produce new examples of ASD complexes, we use *flips*:

**Definition 9.** For an ASD complex $K$ and a facet $A \in K$ we build a new ASD complex

$$\text{flip}_A(K) := (K \setminus A) \cup A^c.$$ 

It is easy to see that

**Proposition 10.** (1) [5] Inverse of a flip is also some flip. (2) [5] Any two ASD complexes are connected by a sequence of flips. (3) For any ASD complex $K$ there exists a threshold ASD complex $K'$ that can be obtained from $K$ by a sequence of flips with some $A_i \subset [n]$ such that $|A_i| > 2, |A_i^c| > 2$.

**Proof.** We prove (3). It is sufficient to show that for any ASD complex, there exists a threshold ASD complex with the same collection of 2-element non-faces. For this, let us observe that any two non-faces of an ASD complex necessarily intersect. Therefore, all possible collections of 2-element non-faces of an ASD complex (up to renumbering) are:

(1) empty set;
(2) $(12), (23), (31)$;
(3) $(12), (13), \ldots, (1k)$.

It is easy to find appropriate threshold ASD complexes for all these cases. $\square$

ASD complexes appear in the game theory literature as “simple games with constant sum” (see [11]). One imagines $n$ players and all possible ways of partitioning them into two teams. The teams compete, and a team *looses* if it belongs to $K$. In the language of flexible polygons, a short set is a loosing team.

**Contraction, or freezing operation.** Given an ASD complex $K$, let us build a new ASD complex $K_{(ij)}$ with $n - 1$ vertices $\{1, \ldots, \hat{i}, \ldots, \hat{j}, \ldots, n, (i, j)\}$ by contracting the edge $(i, j) \in K$, or freezing $i$ and $j$ together.

The formal definition is: for $A \subset \{1, \ldots, \hat{i}, \ldots, \hat{j}, \ldots, n\}$, $A \in K_{(ij)}$ iff $A \in K$, and $A \cup \{(i, j)\} \in K_{(ij)}$ iff $A \cup \{i, j\} \in K$.

Contraction $K_I$ of any other face $I \subset K$ is defined analogously.
Informally, in the language of simple game, contraction of an edge means making one player out of two.
In the language of flexible polygons, “freezing” means producing one new edge out of two old ones (the lengths sum up).

2.2. Smooth extremal assignment compactifications. Now we review the results of [1] and [12], and indicate a relation with preASD complexes.

For a scheme $B$, consider the space $U_{0,n}(B)$ of all flat, proper, finitely-presented morphisms $\pi : \mathcal{C} \to B$ with $n$ sections $\{s_i\}_{i \in [n]}$, and connected, reduced, one-dimensional geometric fibers of genus zero. Denote by $\mathcal{V}_{0,n}$ the irreducible component of $U_{0,n}$ that contains $\mathcal{M}_{0,n}$.

**Definition 11.** A modular compactification of $\mathcal{M}_{0,n}$ is an open substack $\mathcal{X} \subset \mathcal{V}_{0,n}$ that is proper over $\mathcal{Z}$. A modular compactification is stable if every geometric point $(\pi : \mathcal{C} \to B; s_1, \ldots, s_n)$ is stable. We call a modular compactification smooth if it is a smooth algebraic variety.
Figure 2. Contraction of \{1, 2\} in a simplicial complex

Definition 12. A smooth extremal assignment \(Z\) over \(\overline{M}_{0,n}\) is an assignment to each stable tree with \(n\) leaves a subset of its vertices

\[\gamma \mapsto Z(\gamma) \subset \text{Vert}(\gamma)\]

such that:

1. for any tree \(\gamma\), the assignment is a proper subset of vertices: \(Z(\gamma) \subsetneq \text{Vert}(\gamma)\),
2. for any contraction \(\gamma \sim \tau\) with \(\{v_i\}_{i \in I} \subset \text{Vert}(\gamma)\) contracted to \(v \in \text{Vert}(\tau)\), we have \(v_i \in Z(\gamma)\) for all \(i \in I\) if and only if \(v \in Z(\tau)\),
3. for any tree \(\gamma\) and \(v \in Z(\gamma)\) there exists a two-vertex tree \(\gamma'\) and \(v' \in Z(\gamma')\) such that \(\gamma' \sim \gamma\) and \(v' \sim v\).

Definition 13. Assume that \(Z\) is a smooth extremal assignment. A curve \((\pi : C \to B; s_1, \ldots, s_n)\) is \(Z\)-stable if it can be obtained from some Deligne–Mumford stable curve \((\pi' : C' \to B'; s'_1, \ldots, s'_n)\) by (maximal) blowing down irreducible components of the curve \(C'\) corresponding to the vertices from the set \(Z(\gamma(C'))\).

A smooth assignment is completely defined by its value on two-vertex stable trees with \(n\) leaves. The latter bijectively correspond to unordered partitions \(A \sqcup A^c = [n]\) with \(|A|, |A^c| > 1\): sets \(A\) and \(A^c\) are affixed to two vertices of the tree. The first condition of Definition 13 implies that no more than one of \(A\) and \(A^c\) is “assigned”. One concludes that preASD complexes are in bijection with smooth assignments.

All possible modular compactifications of \(M_{0,n}\) are parametrized by smooth extremal assignments:

Theorem 14. [1, Theorems 1.9 & 1.21] and [12, Theorem 1.3]

- For any smooth extremal assignment \(Z\) of \(M_{0,n}\), or equivalently, for any preASD complex \(K\), there exists a stack \(\overline{M}_{0,Z} = \overline{M}_{0,K} \subset \mathcal{V}_{0,n}\) parameterizing all \(Z\)-stable rational curves.
- For any smooth modular compactification \(\mathcal{X} \subset \mathcal{V}_{0,n}\), there exist a smooth extremal assignment \(Z\) (a preASD complex \(K\)) such that \(\mathcal{X} = \overline{M}_{0,Z} = \overline{M}_{0,K}\).

There are two different ways to look at a moduli space. In the present paper we look at the moduli space as at a smooth algebraic variety equipped with \(n\) sections (fine moduli space). The other way is to look at it as at a smooth algebraic variety (coarse moduli space). Different preASD complexes give rise to different fine moduli spaces. However, two different complexes can yield isomorphic coarse moduli spaces.

Indeed, consider two preASD complexes \(K\) and \(K \cup \{ij\}\) (we abbreviate the latter as \(K + (ij)\)), assuming that \(\{ij\} \notin K\). The corresponding algebraic varieties \(\overline{M}_{0,K}\) and \(\overline{M}_{0,K+(ij)}\) are isomorphic. A
vivid explanation is: to let a couple of marked points to collide is the same as to add a nodal curve with these two points sitting alone on an irreducible component. Indeed, this irreducible component would have exactly three special points, and \( \text{PSL}_2 \) acts transitively on triples.

**Theorem 15.**\(^\text{[13, Statements 7.6–7.10]} \) The set of smooth modular compactifications of \( \mathcal{M}_{0,n} \) is in a bijection with objects of the pre\( \text{ASD} \)\(_n \) \( / \sim \), where \( K \sim L \) whenever \( K \setminus L \) and \( L \setminus K \) consist of two-element sets only.

**Example 16.** Pre\( \text{ASD} \) complexes and corresponding compactifications.

1. the 0-skeleton \( \Delta^0_{n-1} = [n] \) of the simplex \( \Delta_{n-1} \) corresponds to the Deligne–Mumford compactification;
2. the complex \( \mathcal{P}_n := \text{pt} \sqcup \Delta_{n-2}^{n-3} \) (disjoint union of a point and the boundary of a simplex \( \Delta_{n-2} \)) is \( \text{ASD} \). It corresponds to the Hassett weights \((1, \varepsilon, \ldots, \varepsilon)\); this compactification is isomorphic to \( \mathbb{P}^{n-3} \);
3. the Losev–Manin compactification \( \overline{\mathcal{M}}_{0,n}^{LM} \)\(^\text{[13]} \) corresponds to the weights \((1,1,\varepsilon,\ldots,\varepsilon)\) and to the complex \( \text{pt}_1 \sqcup \text{pt}_2 \sqcup \Delta_{n-3} \);
4. the space \((\mathbb{P}^1)^{n-3} \) corresponds to weights \((1,1,1,\varepsilon,\ldots,\varepsilon)\), and to the complex \( \text{pt}_1 \sqcup \text{pt}_2 \sqcup \text{pt}_3 \sqcup \Delta_{n-4} \).

2.3. \( \text{ASD} \) compactifications via stable point configurations. \( \text{ASD} \) compactifications can be explained in a self-contained way, without referring to \( \text{[1]} \).

Fix an \( \text{ASD} \) complex \( K \) and consider configurations of \( n \) (not necessarily all distinct) points \( p_1, \ldots, p_n \) in the projective line. A configuration is called \( \text{stable} \) if the index set of each collection of coinciding points belongs to \( K \). That is, whenever \( p_{i_1} = \ldots = p_{i_k} \), we have \( \{i_1, \ldots, i_k\} \in K \).

Denote by \( \text{STABLE}(K) \) the space of stable configurations in the complex projective line. The group \( \text{PSL}_2(\mathbb{C}) \) acts naturally on this space. Set

\[
\overline{\mathcal{M}}_{0,K} := \text{STABLE}(K)/\text{PSL}_2(\mathbb{C}).
\]

If \( K \) is a threshold complex, that is, arises from some flexible polygon \( \mathcal{L} \), then the space \( \overline{\mathcal{M}}_{0,K} \) is isomorphic to the polygon space \( M_{\mathcal{L}} \)\(^{[10]} \).

Although the next theorem fits in a broader context of \( \text{[1]} \), we give here its elementary proof.

**Theorem 17.** The space \( \overline{\mathcal{M}}_{0,K} \) is a compact smooth variety with a natural complex structure.

**Proof.** **Smoothness.** For a distinct triple of indices \( i, j, k \in [n] \), denote by \( U_{i,j,k} \) the subset of \( \overline{\mathcal{M}}_{0,K} \) defined by \( p_i \neq p_j, p_j \neq p_k, \) and \( p_i \neq p_k \). For each of \( U_{i,j,k} \), we get rid of the action of the group \( \text{PSL}_2(\mathbb{C}) \), setting

\[
U_{i,j,k} = \{(p_1, \ldots, p_n) \in \overline{\mathcal{M}}_{0,K} : p_i = 0, p_j = 1, \text{ and } p_k = \infty\}.
\]

Clearly, each of the charts \( U_{i,j,k} \) is an open smooth manifold. Since all the \( U_{i,j,k} \) cover \( \overline{\mathcal{M}}_{0,K} \), smoothness is proven.

**Compactness.** Let us show that each sequence of \( n \)-tuples has a converging subsequence.

Assume the contrary. Without loss of generality, we may assume that the sequence \((p_1^i = 0, p_2^i = 1, p_3^i = \infty, p_4^i, \ldots, p_n^i)_{i=1}^\infty \) has no converging subsequence. We may assume that for some set \( I \notin K \), all \( p_j^i \) with \( j \in I \) converge to a common point. We say that we have a **collapsing long set** \( I \). This notion depends on the choice of a chart. We may assume that our collapsing long set has the minimal cardinality among all long sets that can collapse without a limit (that is, violate compactness) for this complex \( K \). We may assume that \( I = \{3, 4, 5, \ldots, k\} \).

This long set can contain at most one of the points \( p_1, p_2, p_3 \). We consider the case when it contains \( p_3 \); other cases are treated similarly.

That is, all the points \( p_4^i, \ldots, p_k^i \) tend to \( \infty \). Denote by \( C_i \) the circle with the minimal radius embracing the points \( p_3^i = \infty, p_4^i, p_5^i, \ldots, p_k^i \). The circle contains at least two points of \( p_4^i, \ldots, p_k^i, p_3 = \infty \). Apply a transform \( \phi_i \in \text{PSL}_2(\mathbb{C}) \) which turns the radius of \( C_i \) to 1, and keeps at least two of the points...
Proposition 18. 

Elements we denote by the same symbols as the perfect cycles. In this new chart the cardinality of the collapsing long set gets smaller. A contradiction to the minimality assumption. \hfill \Box

A natural question is: what if one takes a simplicial complex (not a self-dual one), and cooks the analogous quotient space. Some heuristics are: if the complex contains simultaneously some set \( A \) and its complement \( [n] \setminus A \), we have a stable tuple with a non-trivial stabilizer in \( \text{PSL}_2(\mathbb{C}) \), so the factor has a natural nontrivial orbifold structure. If a simplicial complex is smaller than some ASD complex \( K' \), and therefore, we get a proper open subset of \( \overline{\mathcal{M}}_{0,K'} \), that is, we lose compactness.

2.4. Perfect cycles. Assume that we have an ASD complex \( K \) and the associated compactification \( \overline{\mathcal{M}}_{0,K} \). Let \( K_I \) be the contraction of some face \( I \subseteq K \). Since the variety \( \overline{\mathcal{M}}_{0,K_I} \) naturally embeds in \( \overline{\mathcal{M}}_{0,K} \), the contraction procedure gives rise to a number of subvarieties of \( \overline{\mathcal{M}}_{0,K} \). These varieties (1) "lie on the boundary" \(^1\) and (2) generate the Chow ring (Theorem 26). Let us look at them in more detail.

An elementary perfect cycle \((ij) = (ij)_K \subset \overline{\mathcal{M}}_{0,K}\) is defined as

\[
(ij) = (ij)_K = \{ (p_1, \ldots, p_n) \in \overline{\mathcal{M}}_{0,K} : p_i = p_j \}.
\]

Let \([n] = A_1 \sqcup \ldots \sqcup A_k\) be an unordered partition of \([n]\). A perfect cycle associated to the partition \((A_1) \cdot \ldots \cdot (A_k) = (A_1)_K \cdot \ldots \cdot (A_k)_K = \{(p_1, \ldots, p_n) \in \overline{\mathcal{M}}_{0,K} : i, j \in A_m \Rightarrow p_i = p_j \}.

Each perfect cycle is isomorphic to \( \overline{\mathcal{M}}_{0,K'} \) for some complex \( K' \) obtained from \( K \) by a series of contractions.

Singletons play no role, so we omit all one-element sets \( A_i \) from our notation. Consequently, all the perfect cycles are labeled by partitions of some subset of \([n]\) such that all the \( A_i \) have at least two elements.

Note that for arbitrary \( A \in K \), the complex \( K_A \) might be ill-defined. This happens if \( A \notin K \). In this case the associated perfect cycle \((A)\) is empty.

For each perfect cycle there is an associated Poincaré dual element in the cohomology ring. These dual elements we denote by the same symbols as the perfect cycles.

The following rules allow to compute the cup-product of perfect cycles:

**Proposition 19.**

1. Let \( A \) and \( B \) be disjoint subsets of \([n]\). Then
   a. \((A) \sim (B) = (A) \cdot (B)\).
   b. \((A_i) \sim (B_i) = (ABi)\).
2. For \( A \notin K \), we have \((A) = 0\). If one of \( A_k \) is a non-face of \( K \), then \((A_1) \cdot \ldots \cdot (A_k) = 0\).
3. The four-term relation: \((ij) + (kl) = (jk) + (il)\) holds for any distinct \( i, j, k, l \).

**Proof.** In the cases (1) and (2) we have a transversal intersection of holomorphically embedded complex varieties. The item (3) will be proven in Theorem 26. \hfill \Box

Examples:

\[
(123) \cdot (345) = (12345); \quad (12) \cdot (34) \cdot (23) = (1234).
\]

A more sophisticated computation:

\[
(12) \cdot (12) = (12) \cdot ((13) + (24) - (34)) = (123) + (124) - (12) \cdot (34).
\]

**Proposition 19.** A cup product of perfect cycles is a perfect cycle.

**Proof.** Clearly, each perfect cycle is a product of elementary ones. Let us prove that the product of two perfect cycles is an integer linear combination of perfect cycles. We may assume that the second factor is an elementary perfect cycle, say, \((12)\). Let the first factor be \((A_1) \cdot (A_2) \cdot (A_3) \cdot \ldots \cdot (A_k)\).

We need the following case analysis:

\(^1\)That is do not intersect the initial space \( \mathcal{M}_{0,n} \).
flips and blow ups. Let $K$ be an ASD complex, and let $A \subset [n]$ be its facet.

Lemma 21. The perfect cycle $(A)$ is isomorphic to $\overline{M}_{0,|A|+1} \cong \mathbb{P}^{|A|-2}$.

Proof. Contraction of $A$ gives the complex $pt \cup \Delta_{|A|} = \mathcal{P}_{|A^c|+1}$ from the Example 16 (2).

Lemma 22. For an ASD complex $K$ and its facet $A$, there are two blow up morphisms

$$\pi_A : \overline{M}_{0,K\setminus A} \to \overline{M}_{0,K} \text{ and } \pi_{A^c} : \overline{M}_{0,K\setminus A} \to \overline{M}_{0,\text{flip}_{A^c}(K)}.$$ 

The centers of these blow ups are the perfect cycles $(A)$ and $(A^c)$ respectively. The exceptional divisors are equal: $D_A = D_{A^c}$. Both are isomorphic to $\overline{M}_{0,\mathcal{P}_{|A|+1}} \times \overline{M}_{0,\mathcal{P}_{|A^c|+1}} \cong \mathbb{P}^{|A|} \times \mathbb{P}^{|A^c|}$. The maps $\pi_A|D_A$ and $\pi_{A^c}|D_{A^c}$ are projections to the first and the second components respectively.

The proof literally repeats [8, Corollary 3.5]: $K$–stable but not $K_A(K^c)$–stable curves have two connected components. The marked points with indices from the set $A$ lie on one of the irreducible components, and marked points with indices from the set $A^c$ lie on the other.

Corollary 23. For an ASD complex $K$ and its facet $A$, the algebraic varieties $\overline{M}_{0,K}$ and $\overline{M}_{0,K\setminus A}$ are $\text{HI}$–schemes, i.e., the canonical map from the Chow ring to the cohomology ring is an isomorphism.

Proof. This follows from Lemma 22 and Theorem 11.

3. Chow rings of ASD compactifications

As it was already mentioned, many examples of ASD compactifications are polygon spaces, that is, come from a threshold ASD complex. Their Chow rings were computed in [2]. A more relevant to the present paper presentation of the ring is given in [8]. We recall it below.

Definition 24. Let $A^*_\text{univ} = A^*_{\text{univ},n}$ be the ring

$$\mathbb{Z}[I] : I \subset [n], 2 \leq |I| \leq n - 2$$

factorized by relations:

1. “The four-term relations”: $(ij) + (kl) - (ik) - (jl) = 0$ for any $i, j, k, l \in [n]$.
2. “The multiplication rule”: $(Ik) \cdot (Jk) = (IJk)$ for any disjoint $I, J \subset [n]$ not containing element $k$.

There is a natural graded ring homomorphism from $A^*_{\text{univ}}$ to the Chow ring of an ASD-compactification that sends each of the generators $(I)$ to the corresponding perfect cycle.
Theorem 25. The Chow ring (it equals the cohomology ring) of a polygon space equals the ring $A^*_{\text{univ}}$ factorized by

$$ (I) = 0 \quad \text{whenever } I \text{ is a long set.} $$

The following generalization of Theorem 25 is the first main result of the paper:

**Theorem 26.** For an ASD complex $L$, the Chow ring $A^*_{L} := A^*(\mathcal{M}_{0,L})$ of the moduli space $\mathcal{M}_{0,L}$ is isomorphic to the quotient $A^*_{\text{univ}}$ by the ideal $I_L := \langle (I) : I \notin L \rangle$.

The idea of the proof is: the claim is true for threshold ASD complexes (i.e., for polygon spaces), and each ASD complex is achievable from a threshold ASD complex by a sequence of flips. Therefore it is sufficient to look at a unique flip. Let us consider an ASD complex $K^+B$ where $B \notin K$ is a facet in $K^+B$. Set $A := [n] \setminus B$, and consider the ASD complex $K^+A = \text{flip}_B(K^+B)$.

We are going to prove that if the claim of the theorem holds true for $K^+B$, then it also holds for $K^+A$.

By Lemma 22 the space $\mathcal{M}_{0,K}$ is the blow up of $\mathcal{M}_{0,K+B}$ along the subvariety $(B)$ and the blow up of $\mathcal{M}_{0,K+A}$ along the subvariety $(A)$. The diagram of the blow ups looks as follows:

```
(B) ← g_B ↓ i_B D → g_A ↑ j_A (A)
\mathcal{M}_{0,K+B} ← π_B ↓ \mathcal{M}_{0,K} → π_A ↑ \mathcal{M}_{0,K+A}
```

The induced diagram of Chow rings is:

```
A^*_{(B)} = A^*_{P_{[A]+1}} \xrightarrow{g^*_B} A^*_{P_{[B]+1}} × A^*_{P_{[B]+1}} \xleftarrow{i^*_B} A^*_{(A)} = A^*_{P_{[B]+1}}
```

Let $A^*_{K+A,comb}$ be the quotient of $A^*_{\text{univ}}$ by the ideal $I_{K+A}$. We have a natural graded ring homomorphism

$$ \alpha = \alpha_{K+A} : A^*_{K+A,comb} \rightarrow A^*_{K+A} =: A^*_{K+A,alg}, $$

where the map $\alpha$ sends each symbol $(I)$ to the associated perfect cycle.

A remark on notation: as a general rule, all objects related to $A^*_{K+A,comb}$ we mark with a subscript “comb”, and objects related to $A^*_{K+A,alg}$ we mark with “alg”.

We shall show that $\alpha$ is an isomorphism. The outline of the proof is:

1. The ring $A^*_{K+A,alg}$ is generated by the first graded component. (The ring $A^*_{K+A,comb}$ is also generated by the first graded component; this is clear by construction.)
2. The restriction of $\alpha$ to the first graded components is a group isomorphism. Therefore, $\alpha$ is surjective.
3. The map $\alpha$ is injective.

**Lemma 27.** The ring $A^*_{K+A,alg}$ is generated by the group $A^1_{K+A,alg}$.

**Proof.** By Theorem 38

$$ A^*_K \cong \frac{A^*_{K+A,alg}[T]}{(f_A(T), T \cdot \ker(i^*_A))}. $$

Observe that:
The zero graded components of $A^*_{K+A,alg}, A^*_{K+A,comb}$ equals $\mathbb{Z}$.
The map $\pi_A : A^*_{K+A,alg} \to A^*_K$ is a homomorphism of graded rings. Moreover, the variable $T$
stands for the additive inverse of the class of the exceptional divisor $D$. And so, $T$ a degree one
homogeneous element.

Since $i^*_A$ is the multiplication by the cycle $(A)$, the kernel ker$(i^*_A)$ equals the annihilator Ann$(A)_{alg}$
in the ring $A^*_{K+A,alg}$. Since the space $M_{0,K+A}$ is an HIl-scheme, the degree of the ideal Ann$(A)_{alg}$
is strictly positive.

The polynomial $f_A(T)$ is a homogeneous element whose degree equals the degree deg$_T(f_A(T))$.
Besides, its coefficients are generated by elements from the first graded component since they all
belong to the ring $\alpha(A^*_{K+A,comb})$.

Denote by $(A^1_{K+A,alg})$ the subalgebra of $A^*_{K+A,alg}$ generated by the first graded component.

First observe that the restriction of the map $A^*_{K+A,alg}[T] \to A^*_K$ to the first graded components is
injective.

Assuming that the lemma is not true, consider a homogeneous element $r$ of the ring $A^*_{K+A,alg}$ with
minimal degree among all not belonging to $(A^1_{K+A,alg})$. There exist elements $b_i \in (A^1_{K+A,alg})$
 such that $b_p \cdot T^p + \cdots + b_1 \cdot T + b_0 = r$ in the ring $A^*_{K+A,alg}[T]$. The elements $b_i$ are necessarily homogeneous.

Equivalently, $b_p \cdot T^p + \cdots + b_1 \cdot T + b_0 - r$ belongs to the ideal $(f_A(T), T \cdot \text{Ann}(A_{alg}))$. Therefore
$b_p \cdot T^p + \cdots + b_1 \cdot T + b_0 - r = x \cdot f_A(T) + y \cdot T \cdot i$ with some $x, y \in R[T]$ and $i \in \text{Ann}(A_{alg})$.

Setting $T = 0$, we get $b_0 - r = x_0 \cdot f_0$. If the element $x_0$ belongs to $(A^1_{K+A,alg})$, then we are done.
Assume the contrary. Then from the minimality assumption we get the following inequalities: $\deg(b_0 - r) =
\deg(x_0 \cdot f_0) > \deg(x_0) \geq \deg(r)$. A contradiction.

\textbf{Lemma 28.} For any ASD complex $L$ the groups $A^1_{L,comb}$ and $A^1_{L,alg}$ are isomorphic. The isomorphism is induced by the homomorphism $\alpha_L$.

The proof analyses how do these groups change under flips.

We know that the claim is true for threshold complexes. Due to Lemma 10 we may consider flips only
with $n - 2 > |A| > 2$. Again, we suppose that the claim is true for the complex $K + A$ and will prove for the
complex $K + A$ with $A \sqcup B = [n]$. Under such flips $A^1_{comb}$ does not change. The group $A^1$ does not change neither. This becomes clear with the following two short exact sequences (see Theorem 39(e)):

\[ 0 \to A_{n-4}([\overline{M}_{0,|A|+1}]) \to A_{n-4}([\overline{M}_{0,|A|+1} \times \overline{M}_{0,|B|+1}] \oplus A_{n-4}([\overline{M}_{0,K+B}]) \to A_{n-4}([\overline{M}_{0,K}]) \to 0, \]

\[ 0 \to A_{n-4}([\overline{M}_{0,|B|+1}]) \to A_{n-4}([\overline{M}_{0,|A|+1} \times \overline{M}_{0,|B|+1}] \oplus A_{n-4}([\overline{M}_{0,K+A}]) \to A_{n-4}([\overline{M}_{0,K}]) \to 0. \]

Now we know that $\alpha : A^*_{K+A,comb} \to A^*_{K+A,alg}$ is surjective.

\textbf{Proposition 29.} Let $\Gamma$ be a graph $\text{Vert}(\Gamma) = [n]$ which equals a tree with one extra edge. Assume that
the unique cycle in $\Gamma$ has the odd length. Then the set of perfect cycles $\{(ij)\}$ corresponding to the edges
of $\Gamma$ is a basis of the (free abelian) group $A^1_{univ}$.

\textbf{Proof.} Any element of the group $A^1_{univ}$ by definition has a form $\sum_{ij} a_{ij} \cdot (ij)$ with the sum ranges over
all edges of the complete graph on the set $[n]$. The four-term relation can be viewed as an alternating
relation for a four-edge cycle. One concludes that analogous alternating relation holds for each cycle of
even length. Example: $(ij) - (jk) + (kl) - (lm) + (mp) - (pi) = 0$. Such a cycle may have repeating
vertices. Therefore, if a graph has an even cycle, the perfect cycles associated to its edges are dependant.

It remains to observe that the graph $\Gamma$ is a maximal graph without even cycles. \hfill $\square$

By Theorem 38 the Chow rings of the compactifications corresponding to complexes $K, K + A,$ and
$K + B$ are related in the following way:

\[ A^*_K \cong \frac{A^*_{K+A,alg}[T]}{(f_A(T), T \cdot \text{ker}(i_A^*))} \cong \frac{A^*_ {K+B}[S]}{(f_B(S), S \cdot \text{ker}(i_B^*))}. \]

Now we need an explicit description of the polynomials $f_A$ and $f_B$. 

Assuming that $A = \{x, x_2, \ldots, x_a\}$ and $B = \{y, y_2, \ldots, y_b\}$, where $|A| = a$ and $|B| = b$, take the generators

1. $\{(xy) : (xy_i), i \in \{2, \ldots, b\} ; (x_jy) : j \in \{2, \ldots, a\} ; (yy_2)\}$ for $A_{K+B}^*$, and

2. $\{(xy) : (xy_i), i \in \{2, \ldots, b\} ; (x_jy) : j \in \{2, \ldots, a\} ; (xx_2)\}$ for $A_{K+A,comb}^*$.

Denote by $A$ the subring of the Chow rings $A_{K+A,comb}^*$ and $A_{K+B}^*$ generated by the elements $\{(xy) : (xy_i), i \in \{2, \ldots, b-1\} ; (x_jy) : j \in \{2, \ldots, a-1\}\}$. Then $A_{K+A,comb}^*$ is isomorphic to $A[I]/F_B(I)$ where $I := (xx_2)$ and $F_B(I)$ is an incarnation of the expression $(B) = (yy_2) \cdots (yy_b) = 0$ via the generators. Analogously, $A_{K+B,comb}^* \cong A[J]/F_A(J)$ with $V := (yy_2)$.

The cycles $(A)$ and $(B)$ equal to the complete intersection of divisors $(xx_2), (xx_3), \ldots, (xx_{a-1})$ and $(yy_2), (yy_3), \ldots, (yy_{b-1})$ respectively. So the Chern polynomials are:

\[
f_A(T) = (T + (xx_2)) \cdots (T + (xx_{a-1}))\]

and

\[
f_B(S) = (S + (yy_2)) \cdots (S + (yy_{b-1})).\]

Moreover, the new variables $T$ and $S$ correspond to one and the same exceptional divisor $D_A = D_B$.

Relation between polynomials $f_A$ and $F_A$ are clarified in the following lemma.

**Lemma 30.** The Chow class of the image of a divisor $(ab)_{K+A}, a, b \in [n]$ under the morphism $\pi_{K+A}$ equals

1. $(ab)_K$ for $a \in A, b \in B$, or vice versa;
2. $\text{bl}_{(ab)_{K+A}}$ for $\{a, b\} \subset B$;
3. $\text{bl}_{(A)}((ab)_{K+A}) + D_A$ for $\{a, b\} \subset A$.

Proof. In case (1), the cycle $(ab)_{K+A}$ does not intersect $(A)_{K+A}$. It is by definition $\text{bl}_{(ab)_{K+A}}((ab)_{K+A})$. Then (1) and (2) follow directly from Theorem 10 (2) by dimension counts.

The claim (3) also follows from the blow-up formula Theorem 10

\[
\pi_{K+A}(ab) = \text{bl}_{(ab)_{K+A}} + j_{A,\pi} (g_A^*[(ab) \cap (A)] \cdot s(N_{D_A \overline{M}_{0,K}}))_{n-4},
\]

where $N_{D_A \overline{M}_{0,K}}$ is a normal bundle and $s( )$ is a total Segre class. This follows from the equalities by the functoriality of the total Chern and Segre classes and the equality $s(U) \cdot c(U) = 1$. Namely, we have

\[
s((ab) \cap (A)) = [(ab) \cap (A)] \cdot s(N_{(ab)_{K+A}}((ab)_{K+A}) = [(ab) \cap (A)] \cdot s((N_{(A) \overline{M}_{0,K}})).
\]

Finally, we note that $g_A^*[(ab) \cap (A)] = D_A$.

\[
f_A(T) = \pi_A^*((xx_2) \cdots (xx_{a-1})) = \pi_A^*(A) \text{ and } f_B(S) = \pi_B^*(B).\]
Lemma 31.

1. The ideal $\text{Ann}(A)_{\text{comb}}$ is generated by its first graded component.
2. More precisely, the generators of the ideal $\text{Ann}(A)_{\text{comb}}$ are
   (a) the elements of type $(ab)$ with $a \in A, b \in B$, and
   (b) the elements of type
   $$(a_1a_2) - (b_1b_2),$$
   where $a_1, a_2 \in A; b_1, b_2 \in B = A^c$.
3. The annihilators $\text{Ann}(A)_{\text{comb}}$ and $\text{Ann}(B)$ are canonically isomorphic.

Proof. First observe that the kernel $\ker(i_A^*)$ equal the annihilator of the cycle $(A)$. Set $\kappa$ be the ideal
generated by $\ker(i_A^*) \cap A^1$. Without loss of generality, we may assume that $A = \{1, 2, \ldots, m\}$. Observe that:

1. If $I \subset [n]$ has a nonempty intersection with both $A$ and $A^c$, then $(A)(I) = 0$. In this case, $(I)$ can be expressed as $(ab)(I')$, where $a \in A, b \in A^c$.
2. (i) If $I \subset A$, then $(A) \sim (I) = (A)(m+1 \ldots m + |I|)$.
   (ii) In this case, the element $(I) - (m + 1, \ldots, m + |I|)$ is in $\kappa$.
   Let us demonstrate this by giving an example with $A = \{1, 2, 3, 4\}$, $(I) = (12)$:
   $$(123) \sim (124) = (12) \sim (15) + (16) - (56) = 0 + 0 - (1234)(56).$$
   We conclude that $(12) - (56) \in \kappa$.
   Let us show that $(123) - (567) \in \kappa$. Indeed, $(123) - (567) = (12) \sim (23) - (567) \in \kappa \equiv (56) \sim (23) - (567) \in \kappa \equiv (56) \sim (23) - (67) \in \kappa$. Since $(23) - (67) \in \kappa$, the claim is proven.
3. If $I \subset A^c$, then $(A) \sim (I) = (A)(m + 1, \ldots, m + |I|)$. The element $(I) - (m + 1, \ldots, m + |I|)$ is in $\kappa$.
   This follows from (1) and (2).

Now let us prove the lemma. Assume $x \sim (A) = 0$. Let $x = \sum a_i(I_i) \ldots (I_{k_i})$.

We may assume that $x$ is a homogeneous element. Modulo $\kappa$, each summand $(I_1) \ldots (I_{k_i})$ can be reduced to some $(m + 1 \ldots m + r_1)(m + r_2 + 1, \ldots, m + r_2 \ldots m + r_2 \ldots m + r_k + 1, \ldots, m')$. Modulo $\kappa$, $(m + 1 \ldots m + r_1)(m + r_2 + 1 \ldots m + r_2 \ldots m + r_k + 1, \ldots, m')$ can be reduced to a one-bracket element $(m + 1 \ldots m + r_1 m + r_1 + 1 \ldots m + r_2 \ldots m + r_k + 1, \ldots, m')$.

Indeed, for two brackets we have:
$$(m + 1 \ldots m + r_1)(m + r_2 + 1, \ldots, m + r_2) \equiv (m + 1 \ldots m + r_1)(m + r_2 + 1, \ldots, m + r_2 - 1) \equiv (m + 1 \ldots m + r_2 - 1) \pmod{\kappa}.$$ For a bigger number of brackets, the statement follows by induction.

We conclude that a homogeneous $x \in \ker(i_A^*)$ modulo $\kappa$ reduces to some $a(m + 1 \ldots m + m')$, where $a \in \mathbb{Z}$. Then $a = 0$. Indeed, $(A)(m + 1 \ldots m + m') \neq 0$ since by Lemma [20] $(A)(m + 1 \ldots m + m')(m + m' \ldots n) \neq 0$. □

Remark 32. Via the four-term relation any element from b) can be expressed as a linear combination of elements from a). So only a)–elements are sufficient to generate the annihilators. Actually,
$$A \cong C \oplus \text{Ann}(A)_{\text{comb}}.$$ We arrive at the following commutative diagram of graded rings:

$$\begin{array}{c}
\text{Ann}(A)_{\text{comb}} \\
\downarrow \uparrow
\\
\text{Ann}(B) \\
\downarrow \uparrow
\\
\text{Ann}(A)_{\text{comb}}
\end{array}$$

$$\begin{array}{c}
\text{A}^*_K \cong \overline{\text{A}}^*_K / \text{Ann}(B) \cong \overline{\text{A}}^*_K / \text{Ann}(A)_{\text{comb}}
\\
\overline{\text{A}}^*_K := \text{A}^*_{K=A+\text{comb}}[T] / f_A(T) \cong \text{A}^*_{K+B}[S] / f_B(S)
\\
\text{A}^*_{K+B} \cong \text{A}[J] / F_A(J)
\\
\text{A}^*_{K+A, \text{comb}} \cong \text{A}[I] / F_B(I)
\end{array}$$
Therefore, the following diagram commutes:

\[
\begin{array}{ccc}
0 & \to & \text{Ann}(A)_{\text{comb}} \\
\downarrow & & \downarrow \cong \\
0 & \to & \text{Ann}(A)_{\text{alg}} \\
\end{array}
\]

\[
\begin{array}{ccc}
\mathbb{A}^*_K[\mathcal{M}_{0,K}] & \to & \mathbb{A}^*_K[\mathcal{M}_{0,K}] \\
\alpha & & \cong \\
\mathbb{A}^*_K[\mathcal{M}_{0,K}] & \to & \mathbb{A}^*_K[\mathcal{M}_{0,K}]
\end{array}
\]

Here \( R[g] \) denotes the extension of a ring \( R \) by a polynomial \( g(t) \). All three vertical maps are induced by the map \( \alpha \); the last vertical map is an isomorphism since both rings are isomorphic to \( \mathbb{A}^*_K \).

The ideals \( \text{Ann}(A)_{\text{comb}} \) and \( \text{Ann}(A)_{\text{alg}} \) coincide, so the homomorphism

\[
\alpha : \mathbb{A}^*_K[\mathcal{M}_{0,K}] \to \mathbb{A}^*_K[\mathcal{M}_{0,K}]
\]

is injective, and the theorem is proven. \( \square \)

4. Poincaré polynomials of ASD compactifications

**Theorem 33.** Poincaré polynomial \( P_q(\mathcal{M}_{0,L}) \) for an ASD complex \( L \) equals

\[
P_q(\mathcal{M}_{0,L}) = \frac{1}{q(q-1)} \left( (1+q)^{n-1} - \sum_{l \in L} q^{|l|} \right).
\]

**Proof.** This theorem is proven by Klyachko [10, Theorem 2.2.4] for polygon spaces, that is, for compactifications coming from a threshold ASD complex. Assume that \( K + A \) is a threshold ASD complex. For the blow up of the space \( \mathcal{M}_{0,K+B} \) along the subvariety \( (B) \) we have an exact sequence of Chow groups

\[
0 \to \mathbb{A}_p(\mathcal{M}_{0,P_{A/B} \cup 1}) \to \mathbb{A}_p(\mathcal{M}_{0,P_{A/B} \cup 1} \times \mathcal{M}_{0,P_{A/B} \cup 1}) \oplus \mathbb{A}_p(\mathcal{M}_{0,K+B}) \to \mathbb{A}_p(\mathcal{M}_{0,K}) \to 0,
\]

where, as before, \( A \cup B = [n] \) and \( p \) is a natural number. We get the equality

\[
P_q(K) = P_q(K + B) + P_q(P_{A/B}) \cdot P_q(P_{A/B}) - P_q(P_{A/B}).
\]

Also, we have the recurrent relations for the Poincaré polynomials:

\[
P_q(K + B) = P_q(K + A) + P_q(P_{A/B}) - P_q(P_{A/B}) =
\]

\[
\frac{1}{q(q-1)} \left( (1+q)^{n-1} - \sum_{l \in K+A} q^{|l|} + q^{|A|} - q - q^{|B|} + q \right) =
\]

\[
\frac{1}{q(q-1)} \left( (1+q)^{n-1} - \sum_{l \in K} q^{|l|} - q^{|A|} + q^{|A|} - q - q^{|B|} + q \right) =
\]

\[
\frac{1}{q(q-1)} \left( (1+q)^{n-1} - \sum_{l \in K+B} q^{|l|} \right).
\]

We have used the following: if \( U \) is a facet of an ASD complex, then there is an isomorphism \( \mathcal{M}_{0,P_{U/B} \cup 1} \cong \mathbb{P}|U|^{-2} \) (see Example [16(2)]); the Poincaré polynomial of the projective space \( \mathbb{P}|U|^{-2} \) equals \( \frac{q^{|U|-2}}{q(q-1)} \).

5. The tautological line bundles over \( \mathcal{M}_{0,K} \) and the \( \psi \)-classes

The *tautological line bundles* \( L_i, \ i = 1, \ldots, n \) were introduced by M. Kontsevich [4] for the Deligne-Mumford compactification. The first Chern classes of \( L_i \) are called the \( \psi \)-classes.

We now mimic Kontsevich’s original definition for ASD compactifications. Let us fix an ASD complex \( K \) and the corresponding compactification \( \mathcal{M}_{0,K} \).

**Definition 34.** The line bundle \( E_i = E_i(L) \) is the complex line bundle over the space \( \mathcal{M}_{0,K} \) whose fiber over a point \( (u_1, \ldots, u_n) \in (\mathbb{P}^1)^n \) is the tangent line\(^2\) to the projective line \( \mathbb{P}^1 \) at the point \( u_i \). The first Chern class of \( E_i \) is called the \( \psi \)-class and is denoted by \( \psi_i \).

\(^2\)In the original Kontsevich’s definition, the fiber over a point is the cotangent line, whereas we have the tangent line. This replacement does not create much difference.
Proposition 35. \(1\) For any \(i \neq j \neq k \in [n]\) we have
\[\psi_i = (ij) + (ik) - (jk).\]

\(2\) The four-term relation holds true:
\[(ij) + (kl) = (ik) + (jl)\] for any distinct \(i, j, k, l \in [n]\).

Proof. \(1\) Take a stable configuration \((x_1, \ldots, x_n) \in \overline{M}_{0,K}\). Take the circle passing through \(x_i, x_j,\) and \(x_k\). It is oriented by the order \(ijk\). Take the vector lying in the tangent complex line to \(x_i\) which is tangent to the circle and points in the direction of \(x_j\). It gives rise to a section of \(E_i\) which is defined correctly whenever the points \(x_i, x_j,\) and \(x_k\) are distinct. Therefore, \(\psi_i = A(ij) + B(ik) + C(jk)\) for some integer \(A, B, C\). Detailed analysis specifies their values.

Now \(2\) follows since the Chern class \(\psi_i\) does not depend on the choice of \(j\) and \(k\). \(\square\)

Let us denote by \([d_1, \ldots, d_n]_K\) the intersection number \(\langle \psi_1^{d_1} \cdots \psi_n^{d_n} \rangle_K = \psi_1^{d_1} \cdots \psi_n^{d_n}\) related to the ASD complex \(K\).

Theorem 36. Let \(\overline{M}_{0,K}\) be an ASD compactification. A recursion for the intersection numbers is
\[|d_1, \ldots, d_n|_K = |d_1|, \ldots, d_i + d_j - 1, \ldots, \hat{d}_j, \ldots, d_n|_{K(ij)} + |d_1, \ldots, d_i + d_k - 1, \ldots, \hat{d}_k, \ldots, d_n|_{K(ik)},\]
where \(i, j, k \in [n]\) are distinct.

Remind that \(K_{(ij)}\) denotes the complex \(K\) with \(i\) and \(j\) frozen together. Might happen that \(K_{(ij)}\) is ill-defined, that is, \((ij) \notin K\). Then we set the corresponding summand to be zero.

Proof. By Proposition \(35\)
\[\langle \psi_1^{d_1} \cdots \psi_n^{d_n} \rangle_K = \langle \psi_1^{d_1} \cdots \psi_n^{d_n} \rangle_K - \langle (1i) + (1j) - (ij) \rangle.\]
It remains to observe that \(\langle \psi_1^{d_1} \cdots \psi_n^{d_n} \rangle_K - \langle ab \rangle\) equals the \(\langle \psi_1^{d_1} \cdots \psi_n^{d_n} \rangle_{K(a,b)}\). \(\square\)

Theorem 37. Let \(\overline{M}_{0,K}\) be an ASD compactification. Any top monomial in \(\psi\)-classes modulo renumbering has a form
\[\psi_1^{d_1} \cdots \psi_m^{d_m}\]
with \(\sum_{q=1}^m d_q = n - 3\) and \(d_q \neq 0\) for \(q = 1, \ldots, m\). Its value equals the signed number of partitions \(|n - 2| = I \cup J\)
with \(m + 1 \in I\) and \(I, J \subset K\). Each partition is counted with the sign
\[(-1)^N \cdot \varepsilon,\]
where
\[N = |J| + \sum_{q \in J, q \leq m} d_q, \quad \varepsilon = \begin{cases} 1, & \text{if } J \cup \{n\} \in K, \text{and } J \cup \{n - 1\} \in K; \\ -1, & \text{if } I \cup \{n\} \in K, \text{and } I \cup \{n - 1\} \in K; \\ 0, & \text{otherwise.} \end{cases}\]

Proof goes by induction.

Although the base is trivial, let us look at it. The smallest \(n\) which makes sense is \(n = 4\). There exist two ASD complexes with four vertices, both are threshold. So there exist two types of fine moduli compactifications, both correspond to the configuration spaces of some flexible four-gon. The top monomials are the first powers of the \(\psi\)-classes.

\(1\) For \(l_1 = 1; l_2 = 1; l_3 = 1; l_4 = 0, 1\) we have \(\psi_1 = \psi_2 = \psi_3 = 0, \text{ and } \psi_4 = 2\).

Let us prove that the theorem holds for the monomial \(\psi_1\). There are two partitions of \([n - 2] = [2]\):

(a) \(J = \{1\}, I = \{2\}\). Here \(\varepsilon = 0\), so this partition contributes 0.

(b) \(J = \emptyset, I = \{1, 2\}\). Here \(I \notin K\), so this partition also contributes 0.
Theorem 38. [7, Appendix. Theorem 1] The Chow ring
Denote by \( E \) is a regular embedding, and \( \tilde{1} \) COMPACTIFICATIONS OF \( M \)
where \( P \) This isomorphism is induced by \( \tilde{\psi} \).

Theorem 39. \( \tilde{\psi} \) divisor

For the induction step, let us use the recursion. We shall show that for any partition \([n - 2] = I \cup J\), its contribution to the left hand side and the right hand side of the recursion are equal.

This is done through a case analysis. We present here three cases; the rest are analogous.

(1) Assume that \( i, j, k \in I \), and \( (I, J) \) contributes 1 to the left hand side count. Then

\[ (d_1, \ldots, d_i + d_j - 1, \ldots, \hat{d_j}, \ldots, d_n)K_{(ij)} \]

contributes 1 to the right hand side.

Indeed, neither \( N \), nor \( \varepsilon \) changes when we pass from \( K \) to \( K_{(ij)} \).

\[ \vartheta \]

(2) Assume that \( i \in I, j, k \in J \), and \( (I, J) \) contributes 1 to the left hand side count. Then

\[ (d_1, \ldots, d_i + d_j - 1, \ldots, \hat{d_j}, \ldots, d_n)K_{(ij)} \] contributes 0 to the right hand side.

\[ \epsilon \]

Indeed, \( N \) turns to \( N - 1 \), whereas \( \varepsilon \) stays the same.

(3) Assume that \( i \in J, j, k \in I \), and \( (I, J) \) contributes 1 to the left hand side count. Then

\[ (d_1, \ldots, d_i + d_j - 1, \ldots, \hat{d_j}, \ldots, d_n)K_{(ij)} \] contributes 0.

\[ \delta \]

\[ (d_1, \ldots, d_i + d_k - 1, \ldots, \hat{d_k}, \ldots, d_n)K_{(ik)} \] contributes 0, and

\[ \epsilon \]

\[ (d_1, \ldots, d_i + d_j, \ldots, \hat{d_j}, \ldots, d_k, \ldots, d_n)K_{(jk)} \] contributes 1, since \( N \) turns to \( N - 1 \), and \( \varepsilon \) stays the same.

This theorem was proven for polygon spaces (that is, for threshold ASD complexes) in [14].

6. Appendix. Chow rings and blow ups

Assume we have a diagram of a blow up \( Y := \bl_X(Y) \). Here \( X \) and \( Y \) are smooth varieties, \( \iota : X \hookrightarrow Y \) is a regular embedding, and \( \tilde{X} \) is the exceptional divisor. In this case, \( \iota^* : A^*(Y) \rightarrow A^*(X) \) is surjective.

\[
\begin{array}{ccc}
\tilde{X} & \xleftarrow{\theta} & \tilde{Y} \\
\downarrow{\tau} & & \downarrow{\pi} \\
X & \xrightarrow{\iota} & Y
\end{array}
\]

Denote by \( E \) the relative normal bundle

\[ E := \tau^*N_XY/N_{\tilde{X}}\tilde{Y}. \]

Theorem 38. [2, Appendix. Theorem 1] The Chow ring \( A^*(\tilde{Y}) \) is isomorphic to

\[
\frac{A^*(Y)[T]}{(P(T), T \cdot \ker \iota^*)},
\]

where \( P(T) \in A^*(Y)[T] \) is the pullback from \( A^*(X)[T] \) of Chern polynomial of the normal bundle \( N_XY \). This isomorphism is induced by \( \pi^* : A^*(Y)[T] \rightarrow A^*(\tilde{Y}) \) which sends \(-T\) to the class of the exceptional divisor \( \tilde{X} \).

Theorem 39. [13, Proposition 6.7] Let \( k \in \mathbb{N} \).

a) (Key formula) For all \( x \in A_k(X) \)

\[ \pi^*\iota_*(x) = \theta_*(c_{d-1}(E) \cap \tau^*x) \]
e) There are split exact sequences

\[ 0 \to A_kX \xrightarrow{v} A_k\tilde{X} \oplus A_kY \xrightarrow{\eta} A_k\tilde{Y} \to 0 \]

with \( v(x) = (c_d-1(E) \cap \tau^*x, -\iota_*x) \), and \( \eta(\tilde{x},y) = \theta_*\tilde{x} + \pi_*y \). A left inverse for \( v \) is given by \( (\tilde{x},y) \mapsto \tau_*(\tilde{x}) \).

**Theorem 40.** [15, Theorem 6.7, Corollary 6.7.2]

1. (Blow-up Formula) Let \( V \) be a \( k \)-dimensional subvariety of \( Y \), and let \( \tilde{V} \subset \tilde{Y} \) be the proper transform of \( V \), i.e., the blow-up of \( V \) along \( V \cap X \). Then

\[ \pi^*[V] = [\tilde{V}] + j_*\{c(E) \cap \tau^*s(V \cap X, V)\}_k \text{ in } A_k\tilde{Y}. \]

2. If \( \dim V \cap X \leq k - d \), then \( \pi^*[V] = [\tilde{V}] \).

An algebraic variety \( Z \) is a HI–scheme if the canonical map \( \text{cl} : A^*(Z) \to H^*(Z, \mathbb{Z}) \) is an isomorphism.

**Theorem 41.** [7, Appendix. Theorem 2]

- If \( X, \tilde{X}, \) and \( Y \) are HI, then so is \( \tilde{Y} \).
- If \( X, \tilde{X}, \) and \( Y \) are HI, then so is \( \tilde{Y} \).

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[14] J. Agapito, L. Godinho, *Intersection numbers of polygon spaces*, Trans. Amer. Math. Soc. 361 (2009), 4969–4997.

[15] Fulton, William. *Intersection Theory*, Springer-Verlag New York, 1998.