ON THE ESSENTIAL SPECTRUM OF ELLIPTIC DIFFERENTIAL OPERATORS

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ABSTRACT. Let $\mathcal{A}$ be a $C^*$-algebra of bounded uniformly continuous functions on a finite dimensional real vector space $X$ such that $\mathcal{A}$ is stable under translations and contains the continuous functions that have a limit at infinity. Denote $\mathcal{A}^\dagger$ the boundary of $X$ in the character space of $\mathcal{A}$. Then to each operator $A$ in the crossed product $\mathcal{A} \rtimes X$ one may naturally associate a family of bounded operators $A_\kappa$ on $L^2(X)$ indexed by the characters $\kappa \in \mathcal{A}^\dagger$. We show that the essential spectrum of $A$ is the union of the spectra of the operators $A_\kappa$. The applications cover very general classes of singular elliptic operators.

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1. Introduction

1.1. Elliptic algebra. Let $X = \mathbb{R}^d$ and $L^2 = L^2(X)$. We denote $\mathcal{B} = \mathcal{B}(X)$ the algebra of bounded operators on $L^2$ and $\mathcal{K} = \mathcal{K}(X)$ that of compact operators. Let $\mathcal{B}_{\text{loc}}$ be the space $\mathcal{B}$ equipped with the local norm topology defined by the family of seminorms $\|A\|_{\theta} = \|A\theta(q)\|$ where $\theta \in C_0(X)$ (continuous functions which tend to zero at infinity) and $\theta(q)$ means multiplication by $\theta$. Clearly this topology is metrizable and finer than the strong operator topology. If $\theta(x) > 0 \forall x$ then $\|\cdot\|_{\theta}$ is a norm on $\mathcal{B}$ which on bounded subsets defines the local norm topology. If $(A_s)_{s}$ is a sequence which converges in $\mathcal{B}_{\text{loc}}$ to $A$, we say that the sequence is locally norm convergent and write $u\text{-}\lim_s A_s = A$.

If $a \in X$ then $e^{iaq}$ and $e^{iap}$ are the unitary operators on $L$ that act as follows:

$$
(e^{iaq}u)(x) = e^{iax}u(x) \quad \text{and} \quad (e^{iap}u)(x) = u(x + a).
$$

(1.1)

We also use an alternative notation for the translation by $a \in X$ of a function, namely $\tau_a(\varphi)(x) = \varphi(a + x)$, and extend it to operators: $\tau_a(A) = e^{iap}Ae^{-iap}$ for $A \in \mathcal{B}$. The elliptic algebra of $X$ (the name will be justified page 4) is defined by

$$
\mathcal{E} = \{ A \in \mathcal{B} \mid \lim_{a \to 0} \|e^{iap} - 1\|A^{(e)} = 0, \lim_{a \to 0} \|e^{-iap}A - A\| = 0 \}.
$$

(1.2)

The notation $A^{(e)}$ means that the relation must hold for both $A$ and $A^*$. Clearly $\mathcal{E}$ is a $C^*$-algebra. $\mathcal{E}_{\text{loc}}$ is the set $\mathcal{E}$ equipped with the local norm topology inherited from $\mathcal{B}_{\text{loc}}$.

Our main result requires more formalism but we can state right now the simplest particular case, which does not require any $C^*$-algebra background. We denote $X^\dagger$ the set of all ultrafilters finer than the Fréchet filter on $X$. We denote $\text{Sp}(A)$ the spectrum and $\text{Sp}_{\text{ess}}(A)$ the essential spectrum of an operator $A$.

**Theorem 1.1.** If $A \in \mathcal{B}$ then the limit $u\text{-}\lim_{x \to \epsilon} \tau_x(A) = A_\epsilon$ exists $\forall \epsilon \in X^\dagger$ and

$$
\text{Sp}_{\text{ess}}(A) = \bigcup_{\epsilon \in X^\dagger} \text{Sp}(A_\epsilon).
$$

(1.3)

**Remark.** That the convergence holds locally in norm is important for the proof of the theorem. This type of convergence has been used in [19], see for example (4.24) there.

For the convenience of the reader, we recall in an appendix §5 some facts concerning filters and ultrafilters and also reformulate Theorem 1.1 in terms of sequences, as in [20, 24, 25] for example. We mention that ultrafilters have been first used in this context in [11, Th. 4.1] and then in [27]. In §3 we will see that ultrafilters play a quite natural role in the theory.

In §4.1 we will extend this result to the unbounded operators whose resolvent belongs to $\mathcal{E}$. We mention a simpler result in the self-adjoint case. Note that non-densely defined self-adjoint operators could appear as limits. For example, quite often $H_\epsilon = \infty$ where $\infty$ is the operator with $\{0\}$ as domain and $0$ as resolvent. Clearly, $H$ has purely discrete spectrum, i.e. $\text{Sp}_{\text{ess}}(H) = \emptyset$, if and only if $H_\epsilon = \infty$ for all $\epsilon$. See also §4.2.8.

**Corollary 1.2.** Let $H$ be a self-adjoint operator on $L^2$ such that $R(z) = (H - z)^{-1}$ satisfies for some, hence for all, $z$ in the resolvent set of $H$:

$$
\lim_{a \to 0} \|(e^{iap} - 1)R(z)\| = 0 \quad \text{and} \quad \lim_{a \to 0} \|(e^{iaq}, R(z))\| = 0.
$$

(1.4)

If $\epsilon \in X^\dagger$ then $u\text{-}\lim_{x \to \epsilon} \tau_x(R(z)) = R_\epsilon(z)$ exists for all $z$ in the resolvent set of $H$ and $R_\epsilon(z)$ is a self-adjoint pseudo-resolvent, hence is the resolvent of a self-adjoint operator $H_\epsilon$ acting in a closed subspace $L^2_\epsilon$ (which could be reduced to $0$) of $L^2$. We have

$$
\text{Sp}_{\text{ess}}(H) = \bigcup_{\epsilon \in X^\dagger} \text{Sp}(H_\epsilon).
$$

(1.5)

where the spectrum of $H_\epsilon$ is computed in the subspace $L^2_\epsilon$.

This gives a complete proof of Corollary 4.2 in [11]: the proof sketched on page 31 there gives (1.5) but with union replaced by the closure of the union [15, Th. 1.2].
1.2. General algebras. This subsection contains some preliminary notations and material required for the presentation of our main result Theorem 1.6.

$C^0_c(X)$ is the algebra of bounded uniformly continuous functions on $X$ and the subalgebras consisting of functions which have compact support, or tend to zero at infinity, or have a limit at infinity are denoted $C_c(X), C_0(X),$ and $C_0(X) = C_b(X) + C,$ respectively.

A complex measurable function $\varphi$ on $X$ is usually identified with the operator of multiplication by $\varphi$ on $L^2,$ but for clarity it is sometimes convenient to denote $\varphi(q)$ this operator. Then we define $\varphi(p) = F^{-1}(\varphi(q))F,$ where $F$ is the Fourier transform (formally $p = -i \nabla$). With the notation $\tau_a$ introduced before: $\tau_a(\varphi(q)) = (\tau_a(\varphi))(q)$ and $\tau_a(\varphi(p)) = \varphi(p)$.

Let $\mathcal{A}$ be a $C^*$-algebra of bounded uniformly continuous functions on $X$ such that $\mathcal{A}$ is stable under translations and contains the set of continuous functions that have a limit at infinity are denoted $\varphi \in \mathcal{A}$.

The largest algebra $\mathcal{A}$ allowed by our conditions is $C^0_c(X)$ and the corresponding crossed product coincides with the elliptic algebra $C^0_c(X) \rtimes \tau \varphi \in \mathcal{A}$.

The character space, or spectrum, of $\mathcal{A}$ is the compact space $\sigma(\mathcal{A})$ consisting of nonzero morphisms $\mathcal{A} \to \mathbb{C}$ equipped with the weak$^*$ topology inherited from the embedding $\sigma(\mathcal{A}) \subset \mathcal{A}'$ (dual of $\mathcal{A}$). Each $x \in X$ defines a character $\chi_x : \varphi \mapsto \varphi(x)$ and the map $x \mapsto \chi_x$ is a homeomorphism of $X$ onto an open dense subset of $\sigma(\mathcal{A})$.

The boundary of $X$ in $\sigma(\mathcal{A})$ is the compact set

$$\mathcal{A}^1 = \sigma(\mathcal{A}) \setminus X = \{ \xi \in \sigma(\mathcal{A}) \mid \xi(\varphi) = 0 \forall \varphi \in C_0(X) \}. \quad (1.6)$$

$\mathcal{A}$ is canonically isomorphic with the $C^*$-algebra $C(\sigma(\mathcal{A}))$: each $\varphi \in \mathcal{A}$ extends to a continuous function on $\mathcal{A}$ for which we keep the notation $\varphi.$ We just set $\varphi(\chi) = \chi(\varphi)$.

1.3. Translations by characters. The additive group $X$ naturally acts on $\sigma(\mathcal{A}).$ Indeed, $a \mapsto \tau_a$ is a strongly continuous homomorphism from $X$ into the set of automorphisms of $\mathcal{A}$ so if $\tau'_a : \mathcal{A}' \to \mathcal{A}'$ is the dual map then we get an action $a \mapsto \tau'_a$ of $X$ on the dual space $\mathcal{A}'$ of $\mathcal{A}.$ If $\chi$ is a linear form on $\mathcal{A}$ then $\tau'_a(\chi)$ is the linear form $\tau'_a(\chi) = \chi \circ \tau_a$ so if $\chi$ is a character then $\tau'_a(\chi)$ is a character. From (1.6) we see that this action of $X$ leaves invariant $\mathcal{A}$ and $\tau'_a(\chi_b) = \chi_{a+b}$ for $a, b \in X.$ When there is no ambiguity we simplify the notation and set $\tau'_a(\chi) = \chi + a$ for $a \in X$ and $\chi \in \sigma(\mathcal{A}).$

To each $\chi \in \sigma(\mathcal{A})$ we associate a morphism $\tau_\chi : \mathcal{A} \to C^0_c(X)$ uniquely determined by the condition $\chi_x \circ \tau_\chi = \chi \circ \tau_x \forall x \in X$ and we say that $\tau_\chi$ is the translation morphism associated to the character $\chi.$ More explicitly, if $\varphi \in \mathcal{A}$ then $\tau_\chi(\varphi)$ is the function

$$\tau_\chi(\varphi)(x) = \chi(\tau_x(\varphi)) \quad \forall x \in X. \quad (1.7)$$

\(^1\) In the sense that it is the smallest $C^*$-algebra which contains the resolvents of these operators.
If \( \chi = \chi_\alpha \) is the evaluation at \( a \in X \) then this is the usual translation \( \tau_\alpha(\varphi) \). If \( \chi = \kappa \in \mathcal{A}^1 \) then think of \( \tau_\kappa(\varphi) \) as the translation of \( \varphi \in \mathcal{A} \) by the point at infinity \( \kappa \). Note that
\[
\tau_\kappa(\varphi)(x) = \chi(\tau_\kappa(\varphi)) = (\tau'_\kappa(\chi))(\varphi) = (x + \chi)(\varphi) = \varphi(x + \chi).
\]
So \( \tau_\kappa(\varphi) \) is the function \( x \mapsto \varphi(x + \chi) \) which we sometimes call localization of \( \varphi \) at \( \chi \).
The translations by characters \( \chi \in \sigma(A) \) of functions \( \varphi \in \mathcal{A} \) extend to translations of operators \( A \in \mathcal{A} \) as follows: there is a unique continuous map \( \tau_\chi : \mathcal{A} \rightarrow \mathcal{E} \) such that
\[
\tau_\chi(\varphi(q)\psi(p)) = (\tau_\chi(\varphi))(q)\psi(p) \quad \forall \varphi \in \mathcal{A} \forall \psi \in \mathcal{C}_0(X).
\]
(1.8)
If \( \chi = x \in X \) then \( \tau_x(A) = e^{ixp}Ae^{-ixp} \) and if \( \chi = \kappa \in \mathcal{A}^1 \) then
\[
\tau_\kappa(A) = \lim_{x \rightarrow \kappa} e^{ixp}Ae^{-ixp}
\]
where \( x \in X \) tends to \( \kappa \) in \( \sigma(A) \) and the limit holds in the local norm topology on \( \mathcal{E} \) (Theorem 1.4). We keep the name translation morphism for the map \( \tau_\chi \) extended to the operator level.

### 1.4. Fredholm operators
We recall some facts here concerning the essential spectrum of bounded operators \([6, \S 4.3]\). Let \( \mathcal{H} \) be a Hilbert space. \( F \in \mathcal{B}([\mathcal{H}] \) is a Fredholm operator if its kernel has finite dimension and its range finite codimension (then the range is closed). Equivalently, this means that there is an operator \( G \in \mathcal{B}(\mathcal{H}) \) such that \( 1 - FG \) and \( 1 - GF \) are compact. Clearly, this can be rephrased as follows: the image of \( F \) in the quotient \( C^*-\)algebra of \( \mathcal{B}(\mathcal{H}) \) with respect to the ideal of compact operators \( K(\mathcal{H}) \) is an invertible operator (Atkinson theorem \([6, \text{Th. 4.3.7}]\)). It is this last version that we will use.

The spectrum of \( A \in \mathcal{B}(\mathcal{H}) \) is denoted \( \text{Sp}(A) \). Then the essential spectrum of \( A \) is the set \( \text{Sp}_{\text{ess}}(A) \) of complex numbers \( \lambda \) such that \( A - \lambda \) is not a Fredholm operator.

If \( F \in \mathcal{E} \) is Fredholm then there is \( G \in \mathcal{B} \) such that \( 1 = FG + K \) with \( K \) compact. Since \( \|(e^{ixp} - 1)F\| \rightarrow 0 \) and \( \|(e^{ixp} - 1)K\| \rightarrow 0 \) as \( x \rightarrow 0 \) we get \( \|(e^{ixp} - 1)\| \rightarrow 0 \), which is impossible. Thus:

**Remark 1.3.** There are no Fredholm operators in \( \mathcal{E} \). Thus \( 0 \in \text{Sp}_{\text{ess}}(A) \) if \( A \in \mathcal{E} \).

### 1.5. Main results
We recall a fact proved in \([15, \text{Th. 5.16}]\) for \( X \) an arbitrary locally compact abelian groups.

**Theorem 1.4.** For any \( A \in \mathcal{A} \) the map \( x \mapsto A_x = \tau_x(A) = e^{ixp}Ae^{-ixp} \) is norm continuous and extends to a continuous map \( \sigma(A) \ni \chi \mapsto A_\chi \in \mathcal{E}_{\text{loc}} \). We have \( A_\chi = \tau_\chi(A) \) where \( \tau_\chi \) is the translation morphism associated to \( \chi \). And
\[
\tau_\chi(A) = 0 \forall \chi \in \mathcal{A}^1 \iff A \in \mathcal{K}.
\]
(1.10)
A new proof of (1.10) will be given in Section 2.

**Remark 1.5.** The morphism \( \tau_\chi \), hence \( A_\chi \), has been defined independently of the extension by continuity procedure used in Theorem 1.4. This is important in concrete situations because the computation of \( A_\chi \) does not require a knowledge of the topology of \( \sigma(A) \).

If \( \mathcal{C} \) is a \( C^*-\)algebra and \( I \) is a set then we denote \( \prod_{i \in I} \mathcal{C} \) the \( C^*-\)algebra consisting of all bounded functions \( C : I \rightarrow \mathcal{C} \) with the natural operations and norm \( \|C\| = \sup_{i \in I} \|C(i)\| \).

It follows that the map \( \Phi(A) = (A_\chi)_{\chi \in \mathcal{A}^1} \) defines a morphism
\[
\Phi : \mathcal{A} \rightarrow \prod_{\chi \in \mathcal{A}^1} \mathcal{E}
\]
whose kernel is \( \mathcal{K} \) hence it induces an injective morphism
\[
\tilde{\Phi} : \mathcal{A} / \mathcal{K} \rightarrow \prod_{\chi \in \mathcal{A}^1} \mathcal{E}.
\]
From (1.12) we get for any normal operator $A \in \mathcal{A}$ (cf. [15, Th. 1.15])

$$\text{Sp}_{\text{ess}}(A) = \bigcup_{\kappa \in \mathcal{A}^+} \text{Sp}(A_{\kappa})$$

(1.13)

where $\bigcup$ means closure of the union. Our main result is an improvement of this relation.

**Theorem 1.6.** For any operator $A \in \mathcal{A}$ we have

$$\text{Sp}_{\text{ess}}(A) = \bigcup_{\kappa \in \mathcal{A}^+} \text{Sp}(A_{\kappa}).$$

(1.14)

The relation (1.14) is a significative improvement of (1.13): the condition of normality of $A$ is eliminated and one takes the union instead of the closure of the union. To get the present version we use the techniques of M. Lindner and M. Seidel [20] who solved a problem left open by V. Rabinovich, S. Roch, and B. Silberman, see [25] and their earlier papers (one may also find in [15, §1.4] a detailed discussion of the earlier literature). In their theory the Euclidean space $X$ is replaced by the abelian group $\mathbb{Z}^d$ and instead of $\mathcal{A}$ they work with algebras similar to $\ell_\infty(X) \rtimes X$ acting in $\ell_p$ spaces of Banach space valued functions.. Their results have been extended by J. Špakula and R. Willett [30] (see also [29]) to a general class of discrete metric spaces (without any group structure) under the condition that the space has the property $A$ in the sense of Guoliang Yu. This property also plays a fundamental rôle in [9] where Theorem 1.4 is extended to not necessarily discrete metric spaces. Špakula and Willett also use the metric sparsification property of Chen, Tessera, Wang, and Yu [3] and we follow them in this respect. Of course, in the Euclidean case this could be replaced by an ad hoc construction, as in [20], but this simplifies a lot the argument and puts things in the proper perspective. It seems clear to us that the proofs given in Section 3 work for a class of (non-abelian) groups much more general then the Euclidean spaces, e.g. locally compact groups with finite asymptotic dimension, and this together with [9, Th. 6.8] would give an analog of Theorem 1.6 for such groups. This would cover magnetic Schrödinger operators in the framework of [12, §5]. We shall treat such extensions in a later publication. For an alternative approach to these topics, see the paper [22] by V. Nistor and N. Prudhon.

In [11, 15] the space $X$ is an arbitrary locally compact abelian group. Although the proofs in Section 3 clearly extend to a more general class of non-abelian groups, we decided to consider here only the case $X = \mathbb{R}^d$ which does not require much formalism and the applications we have in mind concern only differential operators on Euclidean spaces.

**Remark 1.7.** Theorem 1.1 is the particular case of Theorem 1.6 corresponding to $\mathcal{A} = C_b^0(X)$. The procedure of going from the characters of $C_b^0(X)$ to ultrafilters is explained in §5: for any $\chi \in \mathcal{A}^+$ there is $\kappa \in X^\dagger$ such that $\chi(\varphi) = \lim_{\kappa \to \varphi} \varphi(x)$ for all $\varphi \in \mathcal{A}$.

**Remark 1.8.** The operator $A_{\kappa}$ with $\kappa \in \mathcal{A}^+$ will be called localization at $\kappa$ of $A$ and we refer generically to the operators $A_{\kappa}$ as localizations at infinity of $A$. By (1.8) and a notation introduced page 5, if $A$ is the pseudodifferential operator $\sum_i \varphi_i(q)\psi_i(p)$ then $A_{\kappa}$ is the pseudodifferential operator $\sum_i \varphi_i(q + \kappa)\psi_i(p)$. Theorem 1.6 says that the essential spectrum of $A$ is determined by its localizations at infinity. Note that this is not true in some simple and physically significative cases like the Stark Hamiltonian, cf. §4.2.7, and the constant magnetic field case. The point is that we consider only the “infinity” defined by the position observable $q$, while for other Hamiltonians one has to take into consideration the contribution of other regions at infinity in phase space.

**Remark 1.9.** If $\varphi \in C_b^0(X)$ and $\psi \in C_0(X)$ then the operator $A = \varphi(q)\psi(p)$ belongs to $\mathcal{B}$ hence it has localizations at infinity. On the other hand, if $\varphi(x) = e^{ix^2}$ and $0 \not\in \mathcal{F}\psi \in C_0^\infty(X)$ then $A \not\in \mathcal{B}$ [9, Ex. 7.2]. The importance of the uniform continuity condition can also be seen as follows: it is clear that $A$ is an integral operator with kernel of the form $k(x,y) = e^{ix^2}\theta(x - y)$ where $\theta \in C_0^\infty(X)$, so $k$ is controlled bounded and $C^\infty$ but not uniformly continuous. Moreover, the operator $A$ has no localizations at infinity. Indeed,
we have $A_x = e^{ixp}A e^{-ixp} = e^{(x^2+2qx)}A$ hence $\|A_x u\| = \|Au\| \neq 0$ if $u \neq 0$ and clearly \( w\lim_{x \to \infty} A_x u = 0 \). Thus if $u \neq 0$ then a sequence of vectors of the form $A_{x_n} u$ with $x_n \to \infty$ cannot converge strongly.

**Remark 1.10.** The character space $\mathcal{A}^\dagger$ may sometimes be reduced to a much smaller set by the following procedure. It may happen that there is $\Omega \subset \mathcal{A}^\dagger$ such that for each $\chi \in \mathcal{A}^\dagger$ the morphism $\tau_\chi$ may be factorized $\tau_\chi = \eta\tau_\omega$ with $\omega \in \Omega$ and $\eta$ a morphism. If this is the case then one has $\text{Sp}(A_\omega) = \text{Sp}(\eta(\tau_\omega(A))) \subset \text{Sp}(\tau_\omega(A)) = \text{Sp}(A_\omega)$ hence (1.14) will hold with $\mathcal{A}^\dagger$ replaced by $\Omega$. We will see an example of this mechanism in §4.3.

2. **Proof of Theorem 1.4.**

We shall give here a proof of (1.10) which requires much less formalism and is simpler than that from [15]. Clearly it suffices to consider the case $\mathcal{A} = C_0^\dagger(X)$.

In this section we abbreviate $\mathcal{E} = C_0^\dagger(X)$. We denote $\mathcal{K}$ the uniform compactification of $X$, i.e. the character space $\sigma(\mathcal{E})$ of the algebra $\mathcal{E}$, and let $\mathcal{X} = \mathcal{K} \setminus X = \mathcal{E}^\dagger$ be the boundary of $X$ in $\mathcal{K}$. Then, according to the first part of Theorem 1.4, for each $\mathcal{A} \in \mathcal{K}$ there is a continuous map $\mathcal{K} \ni \chi \mapsto A_\chi \in \mathcal{A}_{\text{loc}}$ such that $A_x = e^{ixp}A e^{-ixp}$ if $x \in X$. We will show that $\mathcal{A} \in \mathcal{K}$ if $A_\chi = 0$ for all $\chi \in \mathcal{X}$.

Let us state this in more explicit terms. Let $\theta \in C_0(X)$ be a strictly positive function. Then $\chi \mapsto A_\chi \theta(q)$ is a norm continuous function $\mathcal{K} \to \mathcal{B}$ and, since $\mathcal{K}$ is compact, it is uniformly continuous. We assume that this function is zero on the boundary of $X$ in its compactification $\mathcal{K}$. This means in fact that the restriction to $X$ of the function $\chi \mapsto A_\chi \theta(q)$ belongs to the space $C_0(X, \mathcal{B})$ of continuous functions $X \to \mathcal{B}$ which are of class $C_0$. Even more explicitly, this means that

$$x \mapsto A_x \theta(q) = e^{ixp}A e^{-ixp} \theta(q)$$

is a norm continuous function $X \to \mathcal{B}$ which tends to zero at infinity. Or

$$\lim_{x \to \infty} \|A e^{-ixp} \theta(q)\| = 0.$$

Of course, this relation will hold for any $\theta \in C_0(X)$: first we get it for $\theta$ replaced by any continuous function with compact support and then we extend it to any function in $C_0(X)$ by density and uniform boundedness of $A e^{-ixp}$. Since $e^{-ixp} \theta(q) e^{ixp} = \theta(q - x)$ we get

$$\lim_{x \to \infty} \|A \theta(q - x)\| = 0 \quad \forall \theta \in C_0(X). \quad (2.1)$$

To summarize, we have to prove the following: if $\mathcal{A} \in \mathcal{K}$, which means

$$\lim_{\alpha \to 0} \|(e^{iap} - 1)A^+(\alpha)\| = 0 \quad \text{and} \quad \lim_{\alpha \to 0} \|e^{-iaq}A e^{iaq} - A\| = 0 \quad (2.2)$$

and if (2.1) is satisfied, then $\mathcal{A} \in \mathcal{K}$. By using the Riesz-Kolmogorov characterization of compactness, and by taking into account that $\mathcal{A}$ is compact if and only if $A^*$ is compact, we see that it suffices to prove

$$\lim_{\alpha \to 0} \|A(e^{iap} - 1)\| = 0 \quad \text{and} \quad \lim_{\alpha \to 0} \|A(e^{iaq} - 1)\| = 0.$$

But the first of these relations is automatically satisfied because of (2.2) hence in fact we just have to prove that $\lim_{\alpha \to 0} \|A(e^{iaq} - 1)\| = 0$ if (2.1) and (2.2) are satisfied.

**Lemma 2.1.** Let $\phi_r$ be the characteristic function of the region $|x| > r$. Then for any $A \in \mathcal{B}$ we have

$$\lim_{a \to 0} \|A(e^{iaq} - 1)\| = 0 \iff \lim_{r \to \infty} \|A \phi_r(q)\| = 0 \quad (2.3)$$
Let $B_x(r) = \{ y \in X \mid |y - x| < r \}$ and $B_x = B_x(1)$. We denote $1_{B_x}$ the characteristic function of $B_x$ and often identify $1_{B_x}(q) = 1_{B_x}$.

**Lemma 2.2.** For any $A \in \mathcal{B}$ we have
\[
\lim_{x \to \infty} \|A1_{B_x}\| = 0 \iff \lim_{x \to \infty} \|A\theta(q - x)\| = 0 \quad \forall \theta \in C_0(X). \tag{2.4}
\]

**Proof.** If $B = B_0$ then $0 \leq 1_{B_x}(q) = 1_B(q - x) \leq \theta(q - x)$ if $\theta \in C_0(X)$ and $\theta \geq 1$ on the unit ball, hence the implication $\Rightarrow$ is obvious. Reciprocally, it suffices to show that the left hand side of (2.4) implies the right hand side if $\theta$ has compact support. Then if $Z$ is a finite set such that $\text{supp} \theta \subset \bigcup_{z \in Z} B_z$ then there is a number $C$ such that
\[
|\theta(q - x)|^2 \leq C \sum_{z \in Z} 1_{B_z}(q - x) \leq C \sum_{z \in Z} 1_{B_{z+\varepsilon}}(q)
\]

hence
\[
\|A\theta(q - x)\|^2 = \|A\theta(q - x)\|^2 A^* = C \sum_{z \in Z} \|A1_{B_{z+\varepsilon}}(q)A^*\|
\]

and the right hand side tends to zero as $x \to \infty$. \hfill $\square$

**Lemma 2.3.** There is a family of linear maps $\Phi_\varepsilon : \mathcal{E} \to \mathcal{E}$, with $\varepsilon > 0$, such that for any $A \in \mathcal{E}$ the operator $A_\varepsilon = \Phi_\varepsilon(A)$ is controlled, $\|A_\varepsilon\| \leq \|A\|$, and $\lim_{\varepsilon \to 0} A_\varepsilon = A$ in norm. Moreover, $\Phi_\varepsilon(A\varphi(q)) = \Phi_\varepsilon(A)\varphi(q)$ for all $\varphi \in C_0(X)$.

**Proof.** The next argument is based on an idea from the proof of Proposition 4.11 from [10]. To each $\xi \in L^1(X)$ real we associate a map $\Phi_\varepsilon : \mathcal{E} \to \mathcal{E}$ defined by
\[
\Phi_\varepsilon(A) = \int_X e^{-ikq}Ae^{ikq}\xi(k)dk.
\]
Clearly $\Phi_\varepsilon$ is linear and norm continuous, in fact $\|\Phi_\varepsilon(A)\| \leq \|\xi\|_1 \|A\|$. It is also clear that $\Phi_\varepsilon(A\varphi(q)) = \Phi_\varepsilon(A)\varphi(q)$ and $\Phi_\varepsilon(\varphi(q)A) = \varphi(q)\Phi_\varepsilon(A)$ if $\varphi \in L^\infty(X)$.

Let us write $A \in C^0(q)$ if the second condition in (2.2) is satisfied. Since $e^{-i\alpha q}\Phi_\varepsilon(A)e^{i\alpha q} = \Phi_\varepsilon(e^{-i\alpha q}Ae^{i\alpha q})$ we obviously have $A \in C^0(q) = \Phi_\varepsilon(A) \subseteq C^0(q)$. Then by using the relation $e^{i\alpha q}e^{-i\varepsilon q} = e^{-i\varepsilon q}e^{i\alpha q}e^{i\alpha q}$ we get
\[
(e^{i\alpha q} - 1)\Phi_\varepsilon(A) = \int_X e^{-i\varepsilon q}(e^{i\alpha q} - 1)Ae^{i\varepsilon q}\xi(k)dk
\]
\[
+ \int_X (e^{-i\varepsilon q} - 1)e^{i\varepsilon q}e^{i\alpha q}Ae^{-i\varepsilon q}\xi(k)dk
\]
so that
\[
\|(e^{i\alpha q} - 1)\Phi_\varepsilon(A)\| \leq \|(e^{i\alpha q} - 1)A\|\|\xi\|_1 + \|A\| \int_X |e^{-i\varepsilon q} - 1| |\xi(k)|dk.
\]
Note also that $\Phi_\varepsilon(A^*) = \Phi_\varepsilon(A^*)$. Thus we see that if $A$ satisfies the first condition in (2.2) then $\Phi_\varepsilon(A)$ satisfies it too. So, we proved that $\Phi_\varepsilon$ leaves $\mathcal{E}$ invariant.
Now we prove that if the Fourier transform $\hat{\xi}$ of $\xi$ has compact support then $\Phi_\xi(A)$ is a controlled operator, for any $A \in \mathcal{B}$. The next computation is slightly formal but easy to justify [10, Prop. 4.11]. If $\varphi, \psi$ are bounded continuous functions then $\varphi(q) = \int_\mathbb{R} \varphi(x) E(dx)$ where $E$ is the spectral measure of the observable $q$ and similarly for $\psi$. Then

$$\varphi(q)\Phi_\xi(A)\psi(q) = \int_\mathbb{R} \varphi(q)e^{-ikq}Ae^{ikq}\psi(q)\xi(k)dk$$

$$= \int_\mathbb{R} \varphi(x)e^{-ikx}E(dx)AE(dy)e^{iky}\psi(y)\xi(k)dk$$

$$= \int_\mathbb{R} \varphi(x)\psi(y)\xi(x-y)E(dx)AE(dy).$$

So if $\hat{\xi}(z) = 0$ for $|z| > R$ and if the distance between the supports of $\varphi$ and $\psi$ is $> R$ then $\varphi(q)\Phi_\xi(A)\psi(q) = 0$, hence $\Phi_\xi(A)$ is a controlled operator.

Let us fix a positive function $\xi$ such that $\int_\mathbb{R} \xi(k)dk = 1$ which is the Fourier transform of a function with compact support. For $\varepsilon > 0$ we set $\xi_\varepsilon(k) = e^{-\varepsilon^2 k^2}$, function whose Fourier transform is $x \mapsto \xi_\varepsilon(\varepsilon x)$ which is also of compact support. If we set $\Phi_\varepsilon = \Phi_{\xi_\varepsilon}$ then

$$\Phi_\varepsilon(A) = \int_\mathbb{R} e^{-ikq}Ae^{ikq}\xi_\varepsilon(k)dk$$

hence clearly $\lim_{\varepsilon \to 0} \Phi_\varepsilon(A) = A$ in norm if $A \in C^0(q)$.

**Proposition 2.4.** If $A \in \mathcal{E}$ then $A \in \mathcal{K}$ if and only if $\lim_{x \to \infty} \|A1_{B_x}\| = 0$.

**Proof.** If $A \in \mathcal{K}$ then $\lim_{x \to \infty} \|A1_{B_x}\| = 0$ is true if $A$ has rank 1, hence if $A$ is compact. Reciprocally, let $A \in \mathcal{E}$ such that $\lim_{x \to \infty} \|A1_{B_x}\| = 0$ and let $A_x$ be as in Lemma 2.3. Since $\|A_x - A\| \to 0$ as $\varepsilon \to 0$ it suffices to show that $A_x$ is compact. The operator $A_x$ belongs to $\mathcal{E}$ and is controlled. Moreover, if $\theta \in C_0(X)$ then $A_x\theta(q - x) = \Phi_\varepsilon(\Lambda \theta(q - x))$ hence $\|A_x\theta(q - x)\| \leq \|A \theta(q - x)\|$ by Lemma 2.3 and so $\lim_{x \to \infty} \|A_x\theta(q - x)\| = 0$ by Lemma 2.2 hence $\lim_{x \to \infty} \|A_x1_{B_x}\| = 0$ by the same lemma. Since $A_x \in \mathcal{E}$ the operators $\theta(q)A_x$ and $A_x\theta(q)$ are compact if $\theta \in C_c(X)$. Thus, due to [9, Lem. 3.8], $A_x$ is a compact operator. The quoted lemma says that $\lim_{x \to \infty} \|A_x\phi\| = 0$ holds because $\lim_{x \to \infty} \|A_x1_{B_x}\| = 0$ and $A_x$ is controlled. Then we get compactness by Lemma 2.1.

### 3. Proof of Theorem 1.6.

The algebra $\mathcal{A}$ has no unit and its unitization may be identified with the $C^*$-subalgebra $\mathcal{A}_1 = \mathcal{A} + C \subset \mathcal{B}$. Then the map $\Phi$ introduced in (1.11) extends to a unital morphism $\mathcal{A}_1 \to \prod_{\mathcal{A} \in \mathcal{A}_1} \mathcal{B}$, that we also denote $\Phi$, and the kernel of this extension is $\mathcal{K}$. If $S = A - \lambda$ with $A \in \mathcal{A}$ and $\lambda \in \mathbb{C}$ then $S_x = e^{i\varphi}S_0 e^{-i\varphi}$ and $S_x = e^{i\varphi}A e^{-i\varphi} - \lambda$ hence the map $x \mapsto S_x \in \mathcal{B}$ extends to a continuous map $\sigma(A) \ni \chi \mapsto S_\chi \in \mathcal{B}_{\text{loc}}$ and we have $S_\lambda = A_\lambda - \lambda$ for all $\chi \in \sigma(A)$. Thus the extended $\Phi$ is given by the same formula $\Phi(S) = (S_\chi)_{\mathcal{A} \in \mathcal{A}_1}$ for $S \in \mathcal{A}_1$.

Let $\mathcal{S}$ be the quotient of $S$ in $\mathcal{B}/\mathcal{K}$. Then $S$ is Fredholm if and only if $\mathcal{S}$ is invertible in $\mathcal{B}/\mathcal{K}$. If $S \in \mathcal{A}_1$ then this happens if and only if $\Phi(S)$ is invertible in $\prod_{\mathcal{A} \in \mathcal{A}_1} \mathcal{B}$, hence

$$S \in \mathcal{A}_1 \text{ is Fredholm} \iff \left\{ \begin{array}{l} S_\chi \text{ is invertible in } \mathcal{B} \forall \chi \in \mathcal{A}_1^+ \\ \text{ and } \sup_{\chi \in \mathcal{A}_1} \|S_\chi^{-1}\| < \infty. \end{array} \right. \quad (3.1)$$

On the other hand, the relation (1.14) is equivalent to

$$\mathbb{C} \setminus \text{Sp}_{\text{ess}}(A) = \bigcap_{\mathcal{A} \in \mathcal{A}_1} \left( \mathbb{C} \setminus \text{Sp}(A_\chi) \right)$$
which means that we have \( \lambda \notin \text{Sp}_{\text{res}}(A) \) if and only if \( A_{x} - \lambda \) is an invertible operator in \( \mathcal{B} \) for all \( x \in A^\dagger \). Thus, with the notation \( S = A - \lambda \), (1.14) says

\[
S \text{ is Fredholm} \iff S_{x} \text{ is invertible in } \mathcal{B} \forall x \in A^\dagger.
\]  

(3.2)

So to prove the theorem we have to show that the second condition (3.1) is automatically satisfied. To summarize, we have to prove

\[
S \in \mathcal{A}_1 \text{ and } S_{x} \text{ invertible } \forall x \in A^\dagger \implies \sup_{x \in A^\dagger} \|S_{x}^{-1}\| < \infty.
\]  

(3.3)

Following [20] we set \( \nu(T) = \inf_{\|u\| = 1} \|Tu\| \) if \( T \) is a bounded operator in a Hilbert space. If \( T \) is bijective then \( \nu(T) = \|T^{-1}\|^{-1} \). Thus the relation \( \sup_{x \in A^\dagger} \|S_{x}^{-1}\| < \infty \) is equivalent to \( \inf_{x \in A^\dagger} \nu(S_{x}) > 0 \). Hence (3.3) follows from

\[
\forall S \in \mathcal{A}_1 \exists \omega \in A^\dagger \text{ such that } \inf_{x \in A^\dagger} \nu(S_{x}) = \nu(S_{\omega}).
\]  

(3.4)

because in our case \( \hat{S} \) is invertible, so we know that \( S_{x} \) is invertible \( \forall x \in A^\dagger \). We will prove (3.4) by adapting the ideas of the papers [20, 30] to the Euclidean context.

If \( L \in \mathcal{B} \) then localized versions of \( \nu(L) \) are defined as follows. Let \( \Omega \subset X \) open and \( 0 < \theta \leq \infty \). We assume (here and later) that \( \Omega \) is not empty, so \( L^2(\Omega) \) is a closed not trivial subspace of \( L^2(X) \), and we set

\[
\nu(L|\Omega) = \inf\{ \|Lu\| \mid u \in L^2(\Omega), \|u\| = 1 \},
\]

\[
\nu_0(L|\Omega) = \inf\{ \|Lu\| \mid u \in L^2(\Omega), \text{diam}(\text{supp}u) < \theta, \|u\| = 1 \}.
\]

We have \( \nu(L|\Omega) = \nu_\infty(L|\Omega) \) because the functions with compact support are dense in \( L^2(\Omega) \). Moreover, one may easily check the relations:

\[
\nu_0'(L|\Omega') \leq \nu_\infty'(L|\Omega'') \quad \text{if } \theta' \geq \theta'', \Omega' \supset \Omega'',
\]

\[
\nu_0(\nu_{\text{ap}} L e^{-i\rho} |\Omega) = \nu_0(L|a + \Omega) \quad \forall a \in X.
\]

(3.5)

(3.6)

Let us denote \( \rho(L) \) the lower bound of the numbers \( r \) such that \( \langle u|Lv \rangle = 0 \) if the distance between the supports of \( u \) and \( v \) is \( > r \). Then \( L \) is called \textit{controlled} if \( \rho(L) < \infty \). This means that there is \( r > 0 \) such that \( \langle u|Lv \rangle = 0 \) whenever the distance between the supports of \( u \) and \( v \) is larger than \( r \) [26, pp. 67, 69].

The proofs of the next two lemmas closely follow those of Proposition 7.6 and Corollary 7.10 from [30].

**Lemma 3.1.** For any \( \varepsilon > 0, r > 0, \ell < \infty \) there is \( \theta > 0 \) such that \( \forall \Omega \subset X \) open

\[
\nu_0(L|\Omega) \leq \nu(L|\Omega) + \varepsilon \quad \text{if } L \in \mathcal{B} \text{ with } \|L\| \leq \ell \text{ and } \rho(L) \leq r.
\]

**Proof.** \( X = \mathbb{R}^d \) has the metric sparsification property, for example because it has finite asymptotic dimension [3]; see also [23, 26]. To state this in precise terms we introduce a notation and a notion. We denote \( \mathcal{M}(X) \) the set of finite positive measure on \( X \). Then let \( R > 0 \) a real number; a closed subset \( Y \subset X \) will be called \( R \)-\textit{sparse} if it can be written as a union \( Y = \bigcup_{i \in I} Y_i \) with dist\( (Y_i, Y_j) \geq R \) if \( i \neq j \) and \( \text{sup}_i \text{diam}(Y_i) < \infty \). Clearly each \( Y_i \) will also closed. Then, by taking into account [3, Prop. 3.3], we have:

\[
c < 1, \ R > 0, \ \mu \in \mathcal{M}(X) \implies \exists Y = R \text{-sparse set with } \mu(Y) \geq c \mu(X).
\]  

(3.7)

In the Definition 3.1 from [3] the set \( Y \) is only assumed Borel but it is easy to check that its closure will have the same properties. We choose \( c, R \) such that

\[
1/2 < c < 1, \ 6\ell(c^{-1} - 1)^{1/2} < \varepsilon, \ R > 2r.
\]  

(3.8)

Now let \( u \in L^2(\Omega) \) with \( \|u\| = 1 \) and \( \|Lu\| < \nu(L|\Omega) + \varepsilon/4 \). Define the measure \( \mu \) by \( \mu(A) = \int_A |u(x)|^2dx \). Finally, let \( Y \) as in (3.7) and choose any \( \theta > \text{sup}_i \text{diam}(Y_i) \).
Denote $1_Y$ the characteristic function of $Y$ and $1_Y' = 1 - 1_Y$ the characteristic function of $X \setminus Y$. If $u_Y = 1_Y u$ then

$$
\|Lu - L_{1_Y}u\|^2 \leq \|L\|^2 \|1_Y' u\|^2 = \|L\|^2 \mu(X \setminus Y) \leq \|L\|^2 (1 - c) \mu(X) = (1 - c) \|L\|^2
$$

hence $\|L_{1_Y}u\| \leq \|Lu\| + \ell(1 - c)^{1/2}$. Set $u_1 = 1_{Y_1} u$, so $L_{1_Y} = \sum_i L_{1_i}$ and the functions $L_{1_i}$ have disjoint supports, hence $\|L_{1_Y}\|^2 = \sum_i \|L_{1_i}\|^2$. We keep only the terms with $u_i \neq 0$ in this sum and write

$$
\|L_{1_Y}\|^2 = \sum_i \|L_{1_i}\|^2/\|u_i\|^2 \cdot \|u_i\|^2 \geq \inf_i \|L_{1_i}\|^2/\|u_i\|^2 \sum_j \|u_j\|^2
$$

and since we also have $\sum_j \|u_j\|^2 = \|u_{1_Y}\|^2 \geq c \|u\|^2 = c$ we get

$$
\inf_i \|L_{1_i}\|^2/\|u_i\|^2 \leq c^{-1/2} \|Lu\| + \ell(c^{-1} - 1)^{1/2} \leq \|Lu\| + \ell(c^{-1/2} - 1 + (c^{-1} - 1)^{1/2}).
$$

If $a = c^{-1}$ then from (3.8) we get $1 < a < 2$

$$
a - 1 + (a^2 - 1)^{1/2} = (a - 1)^{1/2}((a - 1)^{1/2} + (a + 1)^{1/2}) < (a - 1)^{1/2}(1 + \sqrt{3}) < 3(a - 1)^{1/2} < \varepsilon/2 \ell.
$$

Thus $\inf_i \|L_{1_i}\|^2/\|u_i\|^2 \leq \|Lu\| + \varepsilon/2$. Choose $i$ such that $\|L_{1_i}\|^2/\|u_i\|^2 \leq \|Lu\| + 3 \varepsilon/4$ and denote $v = u_i/\|u_i\| \in L^2(\Omega)$. Then $\text{supp} v \subset Y_i$ which has diameter $\varepsilon < \theta$ and we have $\|Lu\| \leq \nu(L(\Omega)) + \varepsilon$ by the choice of $u$. Hence $\nu_\theta(L(\Omega)) \leq \nu(L(\Omega)) + \varepsilon$. □

**Lemma 3.2.** If $S \in \mathcal{A}$ and $\varepsilon > 0$ then there is $\theta > 0$ such that

$$
\nu_\theta(S, \Omega) \leq \nu(S, \Omega) + \varepsilon \quad \forall \varepsilon \in \mathcal{A} \text{ and } \forall \Omega \subset X \setminus \text{open set}.
$$

**Proof.** We have $S = A - \lambda$ with $A \in \mathcal{A}$ and $\lambda \in \mathcal{C}$. The subset of $\mathcal{A}$ consisting of controlled operators is dense in $\mathcal{A}$: indeed, it contains the linear subspace generated by the operators $\varphi(\psi)\psi(p)$ with $\varphi \in A$ and $\Psi \in C_c(X)$. So there is a controlled operator $T \in \mathcal{A} + \mathcal{C}$ such that $\|S - T\| < \varepsilon$. This clearly implies $\|S_{x} \cdot T_{x}\| < \varepsilon$ for all $x \in A'$. Moreover, $\rho(T_{x}) \leq \rho(T)$ because if $r > \rho(T)$ and $\text{dist}(\text{supp}u, \text{supp}v) > r$ then for each $x \in X$ we also have $\text{dist}(\text{supp}(\psi^{x \cdot p} u), \text{supp}(\psi^{x \cdot p} v)) > r$ hence $\|u_{x}T_{x}v\| = 0$ so by passing to the limit in the direction $x$ we get $\langle u_{x}T_{x}v \rangle = 0$, so $\rho(T_{x}) \leq r$. Since we also have $\|T_{x}\| \leq \|T\| \leq \|S\| + \varepsilon$, we may use Lemma 3.1 and find $\theta > 0$ such that

$$
\nu_\theta(T_{x}\Omega) \leq \nu(T_{x}\Omega) + \varepsilon \quad \forall \varepsilon \in \mathcal{A} \text{ and } \forall \Omega \subset X.
$$

Finally, from $\|S_{x} - T_{x}\| < \varepsilon$ we obviously get $|\nu_\theta(S_{x}\Omega) - \nu_\theta(T_{x}\Omega)| \leq \varepsilon$ and similarly for the $\nu(\cdot, \Omega)$ quantities. Thus $\nu_\theta(S_{x}\Omega) \leq \nu(S_{x}\Omega) + 3 \varepsilon \forall \varepsilon \in \mathcal{A}$ and $\forall \Omega \subset X$. □

In the proof of the next lemma we use an argument from the proof of Theorem 8 in [20]. Let $B(r) = \{x \in X \mid |x| < r\}$ be the open ball of center 0 and radius $r$ in $X$. If $S \in \mathcal{B}$ then the operators $S_{\varepsilon} = e^{i \varepsilon p}S_{0} - e^{i \varepsilon p}$ are called translations of $S$.

**Lemma 3.3.** Let $S \in \mathcal{B}$ and $n > 0$ integer. For $1 \leq i \leq n$ let $\varepsilon_i, \theta_i > 0$ such that

$$
\nu_\theta(S, \Omega) < \nu(S, \Omega) + \varepsilon_i \quad \forall \varepsilon \in \mathcal{A} \text{ and } \forall \Omega \subset X \text{ open ball}.
$$

(3.9)

Then there is a translate $T$ of $S$ such that for all $1 \leq m \leq n$

$$
\nu(T, B(\sum_{1 \leq i \leq m} \theta_i + \theta_m)) < \nu(S) + \varepsilon_m + \cdots + \varepsilon_n.
$$

(3.10)

**Proof.** To simplify the writing, in this proof we set $V_{a} = e^{a p}$. We begin with two remarks which will be used in the next argument. First, due to (3.6), the estimate (3.9) is also satisfied by any translate of $S$. Then, if $v$ is a function with $\text{diam}(\text{supp}v) < \theta$ and if $a \in \text{supp}v$ then $\text{supp}V_{a}v \subset B(\theta)$ because $\text{supp}V_{a}v = \text{supp}v - a$. 

Denote \((\varepsilon'_i)\) a permutation of the numbers \((\varepsilon_i)\) (we shall specify it later on) and let \((\theta'_i)\) be the corresponding numbers defined as in (3.9). Set \(S_0 = S\) and \(\theta'_0 = \infty\). Let us prove that there are translations \(S_1, \ldots, S_n\) of \(S\) such that
\[
S_i = V_x S_i - 1 V_x^* \text{ with } |x_i| < \theta'_{i-1} \quad \text{and} \quad \nu(S_i|B(\theta'_i)) < \nu(S) + \varepsilon'_2 + \cdots + \varepsilon'_n. \quad (3.11)
\]
From (3.9) we get \(\nu(\theta_i)|S < \nu(S) + \varepsilon'_1\) hence there is \(v \in L^2(X)\) with \(|v| = 1\) and \(\text{diam}(\text{supp } v) < \theta'_1\) such that \(|S v| < \nu(S) + \varepsilon'_1\). If \(x_1 \in \text{supp } v\) and \(u_1 = V_x v\) then \(\text{supp } u_1 \subset B(\theta'_1)\) so if we set \(S_1 = V_x S V_x^*\) then
\[
\|S_1 u_1\| < \nu(S) + \varepsilon'_1 \quad \text{hence } \quad \nu(S_1|B(\theta'_1)) < \nu(S) + \varepsilon'_1.
\]
Thus \(S_1\) has been constructed. Assume that \(S_1, \ldots, S_i\) have been constructed with \(i < n\) and let us construct \(S_{i+1}\). From (3.9) and the induction assumption (3.11) we get
\[
\nu_{\theta_i+1}(S_i|B(\theta'_i)) < \nu(S_i|B(\theta'_i)) + \varepsilon'_{i+1} < \nu(S) + \varepsilon'_1 + \cdots + \varepsilon'_i.
\]
So there is unit vector \(v \in L^2(X)\) with \(\text{supp } v \subset B(\theta'_i)\) and \(\text{diam}(\text{supp } v) < \theta'_{i+1}\) such that \(|S_i v| < \nu(S) + \varepsilon'_1 + \cdots + \varepsilon'_i\). Let \(x_{i+1} \in \text{supp } v\) and let us denote \(u_{i+1} = V_{x_{i+1}} v\) and \(S_{i+1} = V_{x_{i+1}} S V_{x_{i+1}}^*\). Then \(|x_{i+1}| < \theta'_i\), \(\text{supp } u_{i+1} \subset B(\theta'_{i+1})\), and
\[
\|S_{i+1} u_{i+1}\| < \nu(S) + \varepsilon'_1 + \cdots + \varepsilon'_i \quad \text{hence } \quad \nu(S_{i+1}|B(\theta'_{i+1})) < \nu(S) + \varepsilon'_1 + \cdots + \varepsilon'_i.
\]
This proves the existence of operators \(S_1, \ldots, S_n\) verifying (3.11).

Next, starting with \(S_n = V_{x_n} S_{n-1} V_{x_n}^*\), then replacing \(S_{n-1}\) by \(V_{x_{n-1}} S_{n-2} V_{x_{n-1}}^*\), and so on, we get \(S_n = V_{x_0} S V_{x_0}^*\) with \(y_i = x_n + \cdots + x_{i+1}\). Then in (3.11) we use \(S_i = V_{y_i} S_{n-i} V_{y_i}\) and by taking into account the relation (3.6) we get
\[
\nu(S_n|B(\theta'_i) - y_i) = \nu(V_{y_i} S_{n-i} V_{y_i}|B(\theta'_i)) = \nu(S_i|B(\theta'_i)) < \nu(S) + \varepsilon'_1 + \cdots + \varepsilon'_i.
\]
Since \(|y_i| \leq |x_n| + \cdots + |x_{i+1}| < \theta'_i + \cdots + \theta'_{n-i}\) we have
\[
B(\theta'_i) - y_i \subset B(\theta'_i + |y_i|) \subset B(2\theta'_i + \cdots + \theta'_{n-1}) \subset B(2\theta'_i + \cdots + \theta'_{n-i})
\]
and hence
\[
\nu(S_n|B(2\theta'_i + \cdots + \theta'_{n-i})) < \nu(S) + \varepsilon'_1 + \cdots + \varepsilon'_i.
\]
If we take \(\varepsilon'_k = \varepsilon_{n-k+1}\) and \(\theta'_k = \theta_{n-k+1}\) we get
\[
\nu(S_n|B(2\theta_{n-i+1} + \cdots + \theta_{n-i})) < \nu(S) + \varepsilon'_1 + \cdots + \varepsilon'_{n-i+1}
\]
and hence (3.10) is satisfied with \(T = S_n\).

\[\square\]

**Lemma 3.4.** For each \(S \in \mathcal{A}\) the set \(\mathcal{S}_\kappa = \{S_\kappa \mid \kappa \in A^1\}\) is a compact stable under translations subset of \(\mathcal{B}_{loc}\).

**Proof.** As explained at the beginning of Section 3, the map \(\chi \mapsto S_\chi\) is a continuous function \(\sigma(A) \to \mathcal{B}_{loc}\). Since \(A^1\) is a compact subset of \(\sigma(A)\) it follows that the set \(S\) is compact in \(\mathcal{B}_{loc}\). To prove the invariance under translations, it suffices to note that if \(a \in X\) and \(\kappa \in A^1\) then \(\tau_a(S_\kappa) = \tau_a \tau_a(S) = \tau_{a \kappa}(S)\) and \(a + \kappa \in A^1\) (page 4). \[\square\]

**Lemma 3.5.** For each \(S \in \mathcal{A}\) there is \(\omega \in A^1\) such that \(\inf_{\chi \in A^1} \nu(S_\chi) = \nu(S_\omega)\).

**Proof.** Let \(\{\varepsilon_i\}_{i \geq 0}\) be a sequence of strictly positive numbers such that \(\sum \varepsilon_i < \infty\) and let us set \(\eta_m = \sum_{i \geq m} \varepsilon_i\). By Lemma 3.2, there is a sequence of numbers \(\theta_i\) such that
\[
\nu_{\theta_i}(S_\omeg) \leq \nu(S_\omeg) + \varepsilon_i \quad \forall \omeg \in A^1, \forall i, \text{ and } \forall \omeg \subset X \text{ open}.
\]
Then choose a sequence of points \(\omega_m \in A^1\) such that \(\lim_m \nu(S_{\omega_m}) = \inf_{\chi \in A^1} \nu(S_\chi)\). To simplify notations we set \(S_n = S_{\omega_n}\), so \(\lim_n \nu(S_n) = \inf_{\chi \in A^1} \nu(S_\chi)\). Now we apply Lemma 3.3 to the operator \(S_n\), which satisfies the preceding inequality for \(0 \leq i \leq n\). If we set \(c_m = \sum_{1 \leq i \leq m} \theta_i + \theta_m\) we see that there is a translate \(T_n\) of \(S_n\) such that
\[
\nu(T_n|B(c_m)) < \nu(S_n) + \eta_m \quad \forall 0 \leq m \leq n.
\]
Lemma 3.4 implies that the sequence \( \{ T_n \} \) has a subsequence \( \{ T_{n_k} \} \) convergent in \( \mathcal{R}_{loc} \) to some \( S_\omega \) with \( \omega \in A^1 \). We have

\[
\nu(T_{n_k})|B(\zeta_m)) < \nu(S_{n_k}) + \eta_m \quad \forall 0 \leq m \leq n_k
\]

and \( \lim_k T_{n_k}1_{B(\zeta_m)} = S_\omega 1_{B(\zeta_m)} \) in norm by the definition of the local norm topology. This implies \( \lim_k \nu(T_{n_k})|B(\zeta_m)) = \nu(S_\omega)|B(\zeta_m)) \). But \( \lim_k \nu(S_{n_k}) = \inf_{\nu \in A^1} \nu(S_\nu) \) by our initial choice, hence we obtain

\[
\nu(S_\omega) \leq \nu(S_\omega)|B(\zeta_m)) \leq \inf_{\nu \in A^1} \nu(S_\nu) + \eta_m
\]

for any \( m \). Making \( m \to \infty \) we get \( \nu(S_\omega) \leq \inf_{\nu \in A^1} \nu(S_\nu) \) which finishes the proof. \( \square \)

4. Applications.

This section is devoted to applications of Theorem 1.6 in the context of differential operators. In §4.1 we develop some tools which allow one to extend Theorem 1.6 to unbounded not necessarily self-adjoint operators. Then in §4.2 we consider singular elliptic differential operators affiliated to \( \mathcal{D} \). In [15] one may find many other examples of algebras \( \mathcal{A} \) and in each of these examples our Theorem 1.6 can be applied and gives a significant improvement of the results. For example, if \( X = \mathbb{R}^d \), the condition that \( H \) be a normal operator in Proposition 6.3 or Theorems 6.13 and 6.27 from [15] is eliminated. Note that in these three examples we have been able to show there, by ad hoc arguments, that the union is already a closed set, but the non normal case was clearly not accessible. The \( C^* \)-algebra involved in Theorem 6.27 in [15] is the “usual” \( N \)-body algebra, i.e. the smallest \( C^* \)-algebra to which the \( N \)-body Hamiltonians with 2-body interactions tending to zero at infinity are affiliated. But for the larger \( C^* \)-algebra generated by \( N \)-body Hamiltonians with asymptotically homogeneous 2-body interactions introduced in [16] both questions (closedness of the union and not normal case) remained open; they are solved in §4.3.

4.1. Unbounded operators. We extend here to non self-adjoint operators the notion of affiliation to \( C^* \)-algebras developed in [1, 5] and references therein.

For physical and technical reasons we are forced to consider non densely defined self-adjoint operators. In fact, even in the simplest physically interesting case of operators of the form \( H = p^2 + v \) on \( L^2(\mathbb{R}) \) with \( v \) a positive unbounded function some localizations at infinity are not densely defined operators, cf. §4.2.8.

There are three natural ways of viewing self-adjoint operators affiliated to a \( C^* \)-algebra \( \mathcal{A} \); as morphisms \( C_0(\mathbb{R}) \to \mathcal{A} \), as resolvents, or as usual self-adjoint operators living in closed subspaces of \( \mathcal{H} \) [1, §8.1.2]. If we think of \( H \) as the Hamiltonian of a quantum system, the morphism point of view is the most natural one, but it has not an obvious extension to non self-adjoint operators. So we shall treat the non self-adjoint case by using the resolvent approach. Under a supplementary condition which suffices in our applications we will also define an associated operator which lives in a closed subspace of \( \mathcal{H} \).

4.1.1. Closed operators. To justify later definitions, we recall some facts concerning a closed operator \( H \) acting in \( \mathcal{H} \). Let \( D(H) \) be its domain equipped with the graph topology. Its resolvent set is \( \Omega(H) = \{ z \in \mathbb{C} \mid H - z : D(H) \to \mathcal{H} \) is bijective \} and its resolvent is the map \( R : z \mapsto (H - z)^{-1} \) with domain \( \Omega(H) \). Let \( R_{za} = (H - z)^{-1}R(a) \) for \( a \in \Omega(H) \) and \( z \in \mathbb{C} \); this is a bijective map \( \mathcal{H} \to \mathcal{H} \) if \( a, z \in \Omega(H) \).

The spectrum of \( H \) is the set \( \text{Sp}(H) = \mathbb{C} \setminus \Omega(H) \) and the essential spectrum of \( H \) is defined as the essential spectrum of the continuous operator \( H : D(H) \to \mathcal{H} \), i.e. the set of numbers \( \lambda \) such that \( H - \lambda : D(H) \to \mathcal{H} \) is not Fredholm. So \( \lambda \in \text{Sp}_{ess}(H) \) means that
either $\lambda$ is an eigenvalue of infinite multiplicity of $H$ or the range of $H - \lambda$ is not of finite codimension in $H$. If $a \in \Omega(H)$ then we have the following spectral mapping theorem:

\[
\text{Sp}(H) = \{ \lambda \in \mathbb{C} \mid (\lambda - a)^{-1} \in \text{Sp}(R(a)) \}, \tag{4.1}
\]

\[
\text{Sp}_{\text{ess}}(H) = \{ \lambda \in \mathbb{C} \mid (\lambda - a)^{-1} \in \text{Sp}_{\text{ess}}(R(a)) \}. \tag{4.2}
\]

$H$ is affiliated to a $C^*$-algebra $\mathcal{C}$ of operators on $H$ if its resolvent set is not empty and $(H - z)^{-1} \in \mathcal{C}$ for some, hence all, $z \in \Omega(H)$ (cf. the text just above Theorem 4.1).

4.1.2. Resolvents. We call resolvent\(^1\) any map $R : \Omega \rightarrow B(\mathcal{H})$ with $\Omega \subset \mathbb{C}$ not empty satisfying $R(a) - R(b) = (a - b)R(a)R(b)$ for all $a, b \in \Omega$. Equivalently:

\[
(1 - (b - a)R(a))(1 - (a - b)R(b)) = 1 \quad \forall a, b \in \Omega. \tag{4.3}
\]

If we denote $R_{za} = 1 - (z - a)R(a)$ for $a \in \Omega$ and $z \in \mathbb{C}$ then the preceding condition may be written $R_{za}R_{ba} = 1 \forall a, b \in \Omega$. Equivalently, $R_{za}$ is invertible and $R_{za}^{-1} = R_{ba}$ if $a, b \in \Omega$. One may easily check that for such $a, b$ we have $R_{za}R_{ab} = R_{zb} \forall z \in \mathbb{C}$. Moreover, we clearly have $R(z) = R(a)R_{az} = R(a)R_{za}^{-1}$ if $a, z \in \Omega$.

$R$ is called maximal if there is no resolvent which extends $R$ to a strictly larger domain. Fix some $a \in \Omega$. According to [17, Th. 5.8.6], each resolvent $R$ has a unique maximal extension, the domain of this extension is the open set consisting of all $z \in \mathbb{C}$ such that $R_{za}$ is invertible, and the value of the extension at such $z$ is $R(a)R_{za}^{-1}$ (the proof is easy by the comments above). We keep the notation $R$ for the maximal extension of $R$.

Following [7, §5], we define the spectrum of the resolvent $R$ as the complement of the domain of its maximal extension, in other terms

\[
\text{Sp}(R) = \{ z \in \mathbb{C} \mid R_{za} : \mathcal{H} \rightarrow \mathcal{H} \text{ is not bijective} \}. \tag{4.4}
\]

Then the essential spectrum of $R$ is the subset of the spectrum defined by

\[
\text{Sp}_{\text{ess}}(R) = \{ z \in \mathbb{C} \mid R_{za} : \mathcal{H} \rightarrow \mathcal{H} \text{ is not Fredholm} \}. \tag{4.5}
\]

These definitions are independent of the choice of $a$: if, for example, $R_{za}$ is Fredholm then $R_{zb} = R_{za}R_{ab}$ is Fredholm too because $R_{ab}$ is invertible. Then by taking into account the expression of $R_{za}$ we get for any $a \in \Omega$

\[
\text{Sp}(R) = \{ z \in \mathbb{C} \mid (z - a)^{-1} \in \text{Sp}(R(a)) \}, \tag{4.6}
\]

\[
\text{Sp}_{\text{ess}}(R) = \{ z \mid (z - a)^{-1} \in \text{Sp}_{\text{ess}}(R(a)) \}. \tag{4.7}
\]

Clearly $R$ is a restriction of the resolvent of a closed operator $H$ as in §4.1.1 if and only if $R(z) : \mathcal{H} \rightarrow \mathcal{H}$ is injective for some, hence for all, $z \in \Omega$. Then from (4.1) and (4.2) we get $\text{Sp}(R) = \text{Sp}(H)$ and $\text{Sp}_{\text{ess}}(R) = \text{Sp}_{\text{ess}}(H)$.

Now let $\mathcal{C}$ be a $C^*$-algebra of operators on $\mathcal{H}$ and $R : \Omega \rightarrow B(\mathcal{H})$ a resolvent. We say that $R$ is affiliated to $\mathcal{C}$ if $R$ is $\mathcal{C}$-valued, i.e. $R(z) \in \mathcal{C} \forall z \in \Omega$. Note that for this it suffices that $R(a) \in \mathcal{C}$ for some $a \in \Omega$. Indeed, then $R_{za} \in \mathcal{C}$ hence its inverse $R_{az}$ also belongs to $\mathcal{C}$ and so $R(z) \in \mathcal{C}$. Notice that if $\pi : \mathcal{C} \rightarrow \mathcal{D}$ is a morphism from $\mathcal{C}$ to a $C^*$-algebra $\mathcal{D}$ then $\pi \circ R$ will be a resolvent affiliated to $\mathcal{D}$.

**Theorem 4.1.** Let $\mathcal{A}$ be as in §1.2 and let $R : \Omega \rightarrow \mathcal{A}$ be a resolvent. Then for any $z \in \Omega$ and any $\varphi \in A^1$ the limit $\lim_{\varphi \rightarrow \infty} e^{iz\varphi}R(z)e^{-iz\varphi} \equiv R_{\varphi}(z)$ exists locally in norm and defines a resolvent $R_{\varphi} : \Omega \rightarrow \mathcal{D}$. We have

\[
\text{Sp}_{\text{ess}}(R) = \bigcup_{\varphi \in A^1} \text{Sp}(R_{\varphi}). \tag{4.8}
\]

\(^1\) The usual terminology is “pseudo-resolvent” but in our context, and especially in the case of self-adjoint operators, it is unnatural to emphasize the distinction between densely and non densely defined operators.
Proof. The existence of the limit and the fact that $R_{\kappa}$ is a resolvent affiliated to $\mathcal{E}$ are consequences of Theorem 1.4. Then $\text{Sp}_{\text{ess}}(R(\alpha)) = \bigcup_{\kappa \in A} \text{Sp}(R_{\kappa}(\alpha))$ for any $\alpha \in \Omega$ by Theorem 1.6. The relation (4.8) follows from (4.6) and (4.7).

4.1.3. Regular resolvents. Our next purpose is to express Theorem 4.1 in terms of an operator $H$ such that $R(z) = (H - z)^{-1}$ in a generalized sense. As we already mentioned, even if we assume that $R$ is the resolvent of an operator as in §4.1.1, the resolvents $R_{\kappa}$ will in general not be resolvents of operators in this sense and we need this generalization to treat them.

Consider a resolvent $R : \Omega \to B(\mathcal{H})$. Then the subspace $R(z)\mathcal{H}$ is independent of $z$ and we denote $\mathcal{H}_R$ its closure. The closed subspace $\mathcal{N}_R = \ker R(z)$ is also independent of $z$. And $R$ is a restriction of the resolvent of a closed operator if and only if $\mathcal{N}_R = \{0\}$ and this operator is densely defined if and only if $\mathcal{H}_R = \mathcal{H}$. We say that $R$ is a regular resolvent if

$$\exists z_n \in \Omega \text{ with } |z_n| \to \infty \text{ such that } ||z_n R(z_n)|| \leq \text{const}. \quad (4.9)$$

If $R$ is a regular resolvent affiliated to a $C^*$-algebra $\mathcal{E}$ and if $\pi : \mathcal{E} \to \mathcal{D}$ is a morphism, then $\pi \circ R$ is regular and affiliated to $\mathcal{D}$. The following fact has been proved in [18].

**Lemma 4.2.** If $R$ is a regular resolvent then $\mathcal{H}_R \cap \mathcal{N}_R = \{0\}$ and $\mathcal{H} = \mathcal{H}_R + \mathcal{N}_R$.

In particular: if $R$ is regular and $\mathcal{N}_R = \{0\}$ then there is a closed densely defined operator $H$ in $\mathcal{H}$ such that $R(z) = (H - z)^{-1}$ for all $z \in \Omega$.

Since $R(z)\mathcal{H} \subset \mathcal{H}_R$ for any $z \in \Omega$, the restrictions $R^\mathcal{E}(z) = R(z)|_{\mathcal{H}_R}$ define a resolvent in $\mathcal{H}_R$ and the range of $R^\mathcal{E}(z)$ is dense in $\mathcal{H}_R$. If $R$ is regular then $R^\mathcal{E}$ satisfies (4.9) hence is regular, so there is a closed densely defined operator $H$ in the Hilbert space $\mathcal{H}_R$ such that $R^\mathcal{E}(z) = (H - z)^{-1}$ for all $z \in \Omega$. Obviously the domain of $H$ is $D(H) = \mathcal{H}_R$ and we may, and we shall, think of $H$ as a closed operator in $\mathcal{H}$ such that $H D(H)$ is contained in the closure $\mathcal{H}_R$ of $D(H)$ in $\mathcal{H}$. We say that $H$ is the operator associated to the regular resolvent $R$. Note that the resolvent $R$ is not completely defined by its associated operator $H$, one must also specify a closed subspace $\mathcal{N}_R$ supplementary to $\mathcal{H}_R = D(H)$.

**Example 4.3.** The operator associated to the resolvent $R = 0$ with domain $\Omega = \mathbb{C}$ is denoted $H = \infty$. We have $D(H) = \{0\}$ and $\sigma(H) = \emptyset$.

**Lemma 4.4.** Let $R$ be a regular resolvent and $H$ the operator associated to it. Denote $\text{Sp}(H)$ and $\text{Sp}_{\text{ess}}(H)$ the spectrum and the essential spectrum of $H$ considered as operator in the Hilbert space $\mathcal{H}_R$. Then $\text{Sp}(R) = \text{Sp}(H)$ and $\text{Sp}_{\text{ess}}(R) = \text{Sp}_{\text{ess}}(H)$.

**Proof.** We have already shown this in case $\mathcal{N}_R = \{0\}$, so we may assume $\mathcal{N}_R$ is not trivial. Relatively to the topological direct sum (but not orthogonal in general) decomposition $\mathcal{H} = \mathcal{H}_R \oplus \mathcal{N}_R$ we have $R(z) = R^\mathcal{E}(z) \oplus 0$. As mentioned after (4.6) and (4.7) we have $\text{Sp}(R^\mathcal{E}) = \text{Sp}(H)$ and $\text{Sp}_{\text{ess}}(R^\mathcal{E}) = \text{Sp}_{\text{ess}}(H)$. Clearly $R_{\kappa:} = R^\mathcal{E} \oplus 1$ hence $R_{\kappa:} : \mathcal{H} \to \mathcal{H}$ is bijective if and only if $R^\mathcal{E}_{\kappa:} : \mathcal{H}_R \to \mathcal{H}_R$ is bijective, so $\text{Sp}(R) = \text{Sp}(R^\mathcal{E})$. We also have $\ker(R_{\kappa:}) = \ker(R^\mathcal{E}_{\kappa:}) \oplus \{0\}$ and $R_{\kappa:} = R^\mathcal{E}_{\kappa:} \mathcal{H}_R \oplus \mathcal{N}_R$ hence $R_{\kappa:}$ is Fredholm if and only if $R^\mathcal{E}_{\kappa:}$ is Fredholm, so $\text{Sp}_{\text{ess}}(R) = \text{Sp}_{\text{ess}}(H)$.

Let $R$ be a regular resolvent affiliated to $\mathcal{E}$ and let $H$ be the operator associated to it. If $R_{\kappa}$ is as in Theorem 4.1 then $R_{\kappa}$ is a regular resolvent affiliated to $\mathcal{E}$ and we denote $H_{\kappa}$ the operator associated to it. Then the relation $\lim_{\kappa \to \infty} e^{izp} R(z)e^{-izp} = H_{\kappa}$ is an abbreviation for "$\lim_{\kappa \to \infty} e^{izp} R(z)e^{-izp} = R_{\kappa}(z)$ locally in norm for any $z \in \Omega$, $\kappa \in A^\mathcal{E}$." Then according to Theorem 4.1 and Lemma 4.4 we have

$$\text{Sp}_{\text{ess}}(H) = \bigcup_{\kappa \in A^\mathcal{E}} \text{Sp}(H_{\kappa}). \quad (4.10)$$
4.1.4. Self-adjoint resolvents. Finally, let us summarize these results in the self-adjoint case (see [1, §8.1.2] for detailed proofs). A resolvent $R$ is called self-adjoint if $\Omega$ is stable under conjugation and $R(z)^* = R(z)$ $\forall z \in \Omega$. This implies $\mathcal{N}_R = \mathcal{H}_R^\perp$. Then there is a unique morphism $\Phi : C_0(\mathbb{R}) \rightarrow B(\mathcal{H})$ such that $\Phi(r_z) = R(z)$ for all $z \in \Omega$, where $r_z(\lambda) = (\lambda - z)^{-1}$ for $\lambda \in \mathbb{R}$. The map $R \mapsto \Phi$ is a bijective correspondence between self-adjoint resolvents on $\mathcal{H}$ and morphisms $C_0(\mathbb{R}) \rightarrow B(\mathcal{H})$. A self-adjoint resolvent is clearly regular and the operator associated to it is a densely defined self-adjoint operator in the Hilbert $\mathcal{H}_R$.

Let us call observable on $\mathcal{H}$ any densely defined self-adjoint operator acting in a closed subspace of $\mathcal{H}$. The map $R \mapsto H$ is a bijective correspondence between self-adjoint resolvents on $\mathcal{H}$ and observables on $\mathcal{H}$. Thus we may identify observables $H$, self-adjoint resolvents $R$, and morphisms $\Phi$. An observable is affiliated to $\mathcal{E}$ if $R$ is affiliated to $\mathcal{E}$, or if $\Phi : C_0(\mathbb{R}) \rightarrow \mathcal{E}$.

If $H$ is an observable affiliated to $\mathcal{A}$ then $\lim_{z \to -1, \mathcal{E}} \mathbb{E}^{i\mathbb{E}p} \Phi(u) e^{-i\mathbb{E}p} = \Phi_{\mathcal{E}}(u)$ exists for any $u \in C_0(\mathbb{R})$, by the Stone-Weierstrass theorem. Now if we apply the preceding results with $\mathcal{A} = \mathcal{E}$ we get:

**Theorem 4.5.** Let $H$ be an observable on $L^2(X)$ such that for some number $z$ in its resolvent set its resolvent $R(z) = (H - z)^{-1}$ satisfies

$$
\lim_{a \to 0} \| (e^{i\mathbb{E}p} - 1) R(z) \| = 0 \text{ and } \lim_{a \to 0} \| [e^{i\mathbb{E}p}, R(z)] \| = 0. \tag{4.11}
$$

Then for any $\varphi \in X^1$ the limit $\lim_{z \to \infty} e^{i\mathbb{E}p} H e^{-i\mathbb{E}p} = H_{\mathcal{E}}$ exists and

$$
\operatorname{Sp}_{\text{ess}}(H) = \bigcup_{\varphi \in X^1} \operatorname{Sp}(H_{\mathcal{E}}). \tag{4.12}
$$

4.2. Differential operators. We now present some applications of the abstract results from §1.5 and §4.1 to differential operators emphasising those of interest in quantum mechanics. In fact all we have to do is to give examples of operators affiliated to the algebra $\mathcal{A}$ and which are of some independent interest. In the maximal case $\mathcal{A} = \mathcal{E}$ this is easy because the conditions of affiliation to $\mathcal{E}$ are very explicit and easy to check. For other $\mathcal{A}$ one may use the perturbative affiliation criteria developed in [5], see for example Theorems 2.5 and 2.8 there (these results can be extended to non self-adjoint operators in a rather obvious way, but we shall not discuss this topic here). For some $\mathcal{A}$ there are criteria of affiliation of the same nature as in the case of $\mathcal{E}$, e.g. [16, Th. 5.2].

4.2.1. Non self-adjoint operators. We consider first a class of non self-adjoint differential operators affiliated to $\mathcal{E}$. For simplicity of the presentation we consider only relatively bounded perturbations of the Laplacian, the extension to more general operators requires more formalism [15, §6.6]. More singular operators will be treated in the self-adjoint case.

We recall a general fact. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{G}$ a dense subspace equipped with a Hilbert structure such that the embedding $\mathcal{G} \hookrightarrow \mathcal{H}$ is continuous. We identify the adjoint space $\mathcal{H}^* = \mathcal{H}$ via the Riesz isomorphism and then embed $\mathcal{G} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{G}^*$ as usual. The operator $\tilde{T}$ in $\mathcal{H}$ associated to, or induced by, $T \in B(\mathcal{G}, \mathcal{G}^*)$ is the restriction of $T$ to $T^{-1}(\mathcal{H}) = \{ u \in \mathcal{G} \mid Tu \in \mathcal{H} \}$ considered as operator in $\mathcal{H}$. If $z \in \mathcal{C}$ then $\tilde{T}_{-z} = \tilde{T} - z$. One may easily check that if there is $z$ such that $T - z : \mathcal{G} \rightarrow \mathcal{G}^*$ is bijective then $\tilde{T}$ is a densely defined closed operator in $\mathcal{H}$ such that $z$ belongs to its resolvent set and $\tilde{T}_z = \tilde{T}_r$.

From now on, unless otherwise explicitly stated, we take $\mathcal{H} = L^2(X)$ with norm $\| \cdot \|$ and for real $s$ we denote $\mathcal{H}^s = \mathcal{H}^s(X)$ the usual Sobolev space with norm $\| \cdot \|_s$.

Consider an operator $V \in B(\mathcal{H}^k, \mathcal{H}^{-1})$ such that $\Re V \geq -\mu \Delta - \nu$ as forms on $\mathcal{H}^1$ for some numbers $\mu, \nu$ with $\mu < 1$; here $\Re V = (V + V^*)/2$. Then $\Delta + V : \mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$ is a
continuous operator whose adjoint is $\Delta + V^* : \mathcal{H}_1 \to \mathcal{H}_{-1}$. If $u \in \mathcal{H}_1$ and $\lambda > \nu$ then

$$\Re \langle u | (\Delta + V + \lambda)u \rangle \geq \|u\| (1 - \mu) \Delta + (\lambda - \nu) |u| \geq c \|u\|^2$$

(4.13)

for a constant $c > 0$. This implies $c \|u\|_1 \leq \|\Delta + V + \lambda\|_{-1}$ and the same estimate with $V$ replaced by $V^*$. Hence $\Delta + V + \lambda : \mathcal{H}_1 \to \mathcal{H}_{-1}$ is bijective if $\lambda > \nu$ so the operator $H$ associated to $\Delta + V$ in $\mathcal{H}$ is closed, densely defined, the half-line $\lambda > \nu$ is included in the resolvent set of $H$, and $H^*$ is the operator induced by $\Delta + V^*$ in $\mathcal{H}$. Moreover from (4.13) we get for $u \in D(H)$

$$(\lambda - \nu)\|u\|^2 \leq \Re \langle u | (\Delta + V + \lambda)u \rangle \leq \|u\| \|\Delta + V + \lambda\|_1$$

hence the resolvent of $H$ satisfies $(\lambda - \nu)\|R(\lambda)\|_1 \leq 1$ for $\lambda > \nu$, so is regular.

**Theorem 4.6.** Let $V \in B(\mathcal{H}_1, \mathcal{H}_{-1})$ such that $\lim_{\mu \to 0} \|e^{i\mu a} V\|_{\mathcal{H}_1 \to \mathcal{H}_{-1}} = 0$ and $\Re V \geq -\mu \Delta - \nu$ with $\mu < 1$. Then the operator $H$ in $\mathcal{H}$ associated to $\Delta + V$ is affiliated to $\mathcal{E}$ hence $\text{Sess}(H) = \cup_{\lambda \in \mathcal{X}} \text{Sp}(H_\lambda)$.

**Proof.** We are here in the setting of §4.1.3 and the formula for the essential spectrum has to be interpreted as in (4.10). We just have to prove that $H$ is affiliated to $\mathcal{E}$ and for this it suffices to show that for some $\lambda > \nu$

$$\lim_{a \to 0} \|e^{i\mu a} - 1\| H(\lambda) = 0 \quad \text{and} \quad \lim_{a \to 0} \|\text{R}(\lambda), e^{i\mu a}\| = 0.$$  

(4.14)

According to (1.2) we should also prove that the first condition above is satisfied with $R(\lambda)$ replaced by $R(\lambda)^*$, but this is obvious here because $R(\lambda)^*$ is the resolvent of the operator $H^*$ which involves $V^*$ and $V^*$ satisfies the same conditions as $V$. The first condition in (4.14) is satisfied because the range of $R(\lambda)$ is included in $\mathcal{H}_1$. For the second one we note that $\mathcal{H}_1$ is stable under $e^{i\mu a}$ so if we denote $\|\cdot\|_B$ the norm in $B(\mathcal{H}_1, \mathcal{H}_{-1})$ then

$$\|e^{i\mu a}, R(\lambda)\|_B = \|R(\lambda) V, e^{i\mu a} R(\lambda)\|_B \leq \|R(\lambda)\|_B^2 \cdot \|V, e^{i\mu a}\|_{\mathcal{H}_1 \to \mathcal{H}_{-1}}.$$ 

Thus $\|e^{i\mu a}, R(\lambda)\|_B \to 0$ which is more than needed.

Under stronger conditions on $V$ we have a more explicit description of the operators $H_\lambda$. We will say that a function $V : X \to \mathbb{C}$ is $\Delta$-small if $V$ is locally integrable and for each $\mu > 0$ there is $\nu$ such that $|V| \leq \mu \Delta + \nu$. Also, a symmetric operator $S : \mathcal{H}_1 \to \mathcal{H}_{-1}$ is $\Delta$-small if for each $\mu > 0$ there is $\nu$ such that $\pm S \leq \mu \Delta + \nu$.

**Theorem 4.7.** Let $V : X \to \mathbb{C}$ be $\Delta$-small. Then $\lim_{x \to x_\infty} V(x + q) = V_\infty$ exists in the strong topology in $B(\mathcal{H}_1, \mathcal{H}_{-1})$ for each $x_\infty \in X_1$, the operators $\Re V_\infty : \mathcal{H}_1 \to \mathcal{H}_{-1}$ are $\Delta$-small, and $H_\infty$ is the operator in $\mathcal{H}$ associated to $\Delta + V_\infty : \mathcal{H}_1 \to \mathcal{H}_{-1}$.

**Proof.** If $W$ is the real or imaginary part of $V$ then $W$ is $\Delta$-small hence by Proposition 6.33 from [15] the translates $W(x + q)$ of the operator $W(q) \in B(\mathcal{H}_1, \mathcal{H}_{-1})$ converge strongly to some $W_\infty \in B(\mathcal{H}_1, \mathcal{H}_{-1})$ hence the same holds for the translates $V(x + q)$. Obviously an estimate like $\pm \Re V(x + q) \leq \mu \Delta + \nu$ remains valid in the limit $x \to x_\infty$, hence $\Re V_\infty : \mathcal{H}_1 \to \mathcal{H}_{-1}$ is $\Delta$-small. If $K_\infty$ is the operator in $\mathcal{H}$ associated to $\Delta + V_\infty$, it remains to prove that $K_\infty = H_\infty$. Fix $\mu < 1$ and let $\nu$ such that $|V| \leq \mu \Delta + \nu$. Clearly, if $\lambda > \nu$ then $\lambda$ is in the resolvent set of $K_\infty$ and of $H_\infty = e^{i\mu p} H e^{i\mu p} = \Delta + V_\infty$ for any $x \in X$, where $V_\infty = V(x + q)$. Then

$$(\Delta + V_\infty + \lambda)^{-1} - (\Delta + V_\infty + \lambda)^{-1} = (\Delta + V_\infty + \lambda)^{-1} (V_\infty - V_\infty) (\Delta + V_\infty + \lambda)^{-1}$$

holds in $B(\mathcal{H}_1, \mathcal{H}_1)$, hence for $u \in \mathcal{H}_1$ we have

$$\|((\Delta + V_\infty + \lambda)^{-1} - (\Delta + V_\infty + \lambda)^{-1}) u\|_1$$

$$\leq \|\Delta + V + \lambda)^{-1}\| \|V_\infty - V_\infty\| \|V_\infty - (\Delta + V_\infty + \lambda)^{-1} u\|_1.$$
where $\| \cdot \|_B$ is the norm in $B(H^{-1}, H^1)$. The last factor above converges to zero as $x \to \infty$ hence $(\Delta + V_x + \lambda)^{-1} \to (\Delta + V_x + \lambda)^{-1}$ strongly in $B(H^{-1}, H^1)$, which clearly implies $(H_x + \lambda)^{-1} \to (K_x + \lambda)^{-1}$ strongly in $B(H)$. But $\lim_{x \to \infty} e^{ipx}R(-\lambda)e^{-ipx} = R(H_x(-\lambda))$ locally in norm, hence $R_x(-\lambda) = (K_x + \lambda)^{-1}$. Thus $H_x = K_x$.

4.2.2. Affiliation criteria. We discuss here general conditions which ensure the affiliation to $\mathcal{E}$ of self-adjoint operators [15, §4.2] and in the next paragraphs give some concrete examples. In each of these cases one may use Theorem 4.5.

Recall that a self-adjoint operator $H$ on $\mathcal{H} \equiv L^2(X)$ is affiliated to $\mathcal{E}$ if and only if for some $z \notin \text{Sp}(H)$ the operator $R(z) = (H - z)^{-1}$ satisfies

$$\lim_{a \to 0} \| (e^{iap} - 1)R(z) \| = 0 \quad \text{and} \quad \lim_{a \to 0} \| e^{-iaq}R(z)e^{iaq} - R(z) \| = 0. \quad (4.15)$$

The first factor above is equivalent to the existence of a continuous function $\phi : X \to \mathbb{R}$ with $\lim_{x \to \infty} \phi(x) = +\infty$ such that $D(H) \subset D(\phi(p))$. For example, it suffices to have $D(H) \subset H^s$ for some $s > 0$. The second condition in (4.15) is a sort of regularity condition on the dependence on $p$ of $H$. One could check it by justifying the writing

$$e^{-iap}R(z)e^{iaq} - R(z) = (e^{-iap}He^{iaq} - z)^{-1} - (H - z)^{-1}$$

$$= -(e^{-iap}He^{iaq} - z)^{-1}(e^{-iap}He^{iaq} - H)(H - z)^{-1}$$

and imposing a condition on $e^{-iap}He^{iaq} - H$. In the simplest case $H = h(p) + v(q)$ with some real function $h$ we have $e^{-iap}He^{iaq} - H = h(p + a) - h(p)$ so one is forced to impose a smoothness condition on the function $h$.

From now on in this subsection we use the same notation for a function on $X$ and for the operator of multiplication by this function in function spaces on $X$, e.g. $V \equiv V(q)$.

Let $h : X \to \mathbb{R}$ with $\lim_{k \to \infty} h(k) = +\infty$. Let $m > 0$ an integer and assume that $h$ is of class $C^m$, its derivatives of order $m$ are bounded, and $|h^{(\alpha)}(k)| \leq C(1 + |h(k)|)$ if $|\alpha| \leq m$. For example, $h$ could be a hypoelliptic polynomial, or $h(k) = \sqrt{k^2 + 1}$, etc.

Then $h(p)$ is a self-adjoint operator on $\mathcal{H}$. Denote $\mathcal{G} = D(|h(p)|^{1/2})$ its form domain equipped with the graph topology and $\mathcal{G}^*$ the space adjoint to $\mathcal{G}$. As usual $\mathcal{G} \subset \mathcal{H} \subset \mathcal{G}^*$.

Clearly $e^{iaq}G = \mathcal{G}$ and $e^{iaq}$ extends to a bounded operator on $\mathcal{G}^*$ for which we keep the notation $e^{iaq}$. Thus $(e^{iaq})_{a \in X}$ is a $C_0$-group in each of the spaces $\mathcal{G}, \mathcal{H}, \mathcal{G}^*$ and the commutator $[e^{iaq}, S]$ is a well defined element of $B(\mathcal{G}, \mathcal{G}^*)$ if $S \in B(\mathcal{G}, \mathcal{G}^*)$.

Let $W : \mathcal{G} \to \mathcal{G}^*$ be a symmetric operator satisfying $W \geq -\mu h(p) - \nu$ for some real numbers $\mu, \nu$ with $\mu < 1$. Then the sum $h(p) + W$ is a well defined operator $\mathcal{G} \to \mathcal{G}^*$ and the operator associated to it in $\mathcal{H}$ is a self-adjoint bounded from below operator that we denote $H_0$. Assume $\lim_{a \to 0} \| e^{iaq}W \|_{\mathcal{G} \to \mathcal{G}^*} = 0$. Then $H_0$ is affiliated to $\mathcal{E}$.

Now let $V : X \to \mathbb{R}$ be a locally integrable function and let $V_-$. be its negative part. Assume that there are numbers $\mu, \nu$ with $\mu < 1$ such that $V_- \leq \mu H_0 + \nu$ in form sense.

Then the self-adjoint operator $H$ associated to the form sum $H_0 + V$ is affiliated to $\mathcal{E}$.

4.2.3. Uniformly elliptic operators. We emphasize that the conditions on the perturbation $W$ considered above are such that $W$ may contain terms of the same order as $h(p)$. For example, the next fact is a consequence of the preceding statement. Recall the notations $p_{1} = -i\partial_1$ and $p_{\alpha} = p_{1}^{\alpha_1} \ldots p_{1}^{\alpha_d}$ for $\alpha \in \mathbb{N}^d$.

Let $a_{\alpha, \beta} \in L^\infty(X)$ such that $L = \sum_{|\alpha|, |\beta| \leq m} \nu^\alpha a_{\alpha, \beta} p^{\beta}$ is uniformly elliptic on $H^m$, i.e.

$$\langle u | Lu \rangle \geq \mu \| u \|_{H^m}^2 - \nu \| u \|_{L^2}^2$$

with $\mu, \nu > 0$. If $V \in L^\infty_{loc}(X)$ is positive then the self-adjoint operator $H$ associated to the form sum $L + V$ is affiliated to $\mathcal{E}$. 
4.2.4. Schrödinger operators. We consider now Schrödinger operators with singular potentials. Note that $\Delta = -\Delta^2$ is the positive Laplacian.

Let $W$ be a continuous symmetric sesquilinear form on $\mathcal{H}^1$ such that $W \geq -\mu \Delta - \nu$ with $\mu < 1$ and $\lim_{u \to 0} \|[e^{iaq}, W]\|_{\mathcal{H}^1 \to \mathcal{H}^{-1}} = 0$. Denote $H_0$ the self-adjoint operator associated to the form $\Delta + W$ on $\mathcal{H}^1$. Let $V : X \to \mathbb{R}$ locally integrable such that its negative part is form bounded with respect to $H_0$ with relative bound $< 1$. Then the self-adjoint operator $H$ associated to the form sum $H_0 + V$ is affiliated to $\mathcal{E}$.

These conditions are satisfied if $W = 0$ and $V$ is of Kato class, hence we see that closures are not needed in the Theorems 3.12 and 4.5 in the paper [19] of Y. Last and B. Simon.

4.2.5. Magnetic fields. The next example involves magnetic fields and a perturbation of the Euclidean metric. Let $L$ be the form sum

$$L = \sum_{ij} (p_i - a_i) g_{ij} (p_j - a_j) + v$$

where $g_{ij}, a_i, v$ are operators of multiplication by real functions such that:

1) $g_{ij} \in L^\infty(X)$ and the matrix $G(x) = (g_{ij}(x))$ is uniformly positive definite, i.e. there is a number $\varepsilon > 0$ such that $G(x) \geq \varepsilon$ for all $x$;

2) $\forall \mu > 0 \exists \nu > 0$ such that $\sum_j \|a_j u\|^2 + \langle u | w u \rangle \leq \mu \|u\|^2 + \nu \|u\|^2 \forall u \in \mathcal{H}^1$, where $w$ is the negative part of $v$.

Then the self-adjoint operator $H$ associated to the form sum $L$ is affiliated to $\mathcal{E}$.

4.2.6. Dirac operators. The theory extends easily to operators in $\mathcal{H} = L^2(X, E) = L^2(X) \otimes E$ where $E$ is a finite dimensional Hilbert space and covers singular Dirac operators [15, Prop. 1.11 and Cor. 4.8]. Now by $\mathcal{E}$ we understand $\mathcal{E} \otimes B(E)$.

Let $D$ be the Dirac operator acting in $\mathcal{H}$. So $D$ is a first order symmetric differential operator with constant coefficients realized as a self-adjoint operator in $\mathcal{H}$ with domain the Sobolev space $\mathcal{H}^1$. Then let $V$ be a continuous symmetric form on $\mathcal{H}^{1/2}$ such that $\pm V \leq \mu |D| + \nu$ with $\mu < 1$ and $\lim_{u \to 0} \|[e^{iaq}, V]\|_{\mathcal{H}^{1/2} \to \mathcal{H}^{-1/2}} = 0$. Then the operator in $\mathcal{H}$ associated to the sum $D + V : \mathcal{H}^{1/2} \to \mathcal{H}^{-1/2}$ is self-adjoint and affiliated to $\mathcal{E}$.

4.2.7. Stark Hamiltonian. There are physically interesting differential operators which are not affiliated to $\mathcal{E}$. Indeed, the Stark Hamiltonian $H = p^2 + q$ is a self-adjoint operator on $L^2(\mathbb{R})$ which does not satisfy any of the conditions (4.15). And we have $\text{s-lim}_{|\alpha| \to \infty} e^{iaq} H e^{-iaq} = \infty$ and $\text{s-lim}_{|\alpha| \to \infty} e^{iaq} H e^{-iaq} = \infty$, while $\text{Sp}_{\text{ess}}(H) = \mathbb{R}$.

Proof. Note first that if $\theta \in L^\infty(\mathbb{R})$ is a nonzero function then $\|(e^{iaq} - 1)\theta(q)\|$ does not tend to zero as $a \to 0$. Indeed, by a remark made just after (4.15), in the contrary case there is $\psi \in C_0(\mathbb{R})$ such that $\theta(q) = \psi(p)T$ for some bounded operator $T$. Then for any $\eta \in C_0(\mathbb{R})$ the operator $\eta(q)\theta(q) = \eta(q)\psi(p)T$ is compact, which is obviously wrong.

Let $R = (H + i)^{-1}$ and $s = p^3/3$. Since $H = e^{isq} e^{-is}$ we have $R = e^{is} (q + i)^{-1} e^{-is}$ hence $\|(e^{iaq} - 1)R\| = \| (e^{iaq} - 1)(q + i)^{-1} \|$ which does not tend to zero as $a \to 0$ by the preceding remark. Then a short computation gives

$$e^{iaq} Re^{-iaq} = e^{is}(q - 2ap + a^2 + i)^{-1} e^{-is}$$

hence

$$\|e^{iaq} Re^{-iaq} - R\| = \| (q - 2ap + a^2 + i)^{-1} - (q + i)^{-1} \|
\triangleq \| (q - 2ap + i)^{-1} - (q + i)^{-1} \| + O(a^2),$$

To show that this does not tend to zero as $a \to 0$ it suffices to use the estimate

$$\|\varphi(q) + \psi(p + aq)\| \geq \max \{ \sup |\varphi|, \sup |\psi| \}$$
which is valid and easy to prove for any $\varphi, \psi \in C_0(\mathbb{R})$. Finally, note for example that $s\lim_{|a|\to \infty} e^{iaq} H e^{-iaq} = \infty$ means $s\lim_{|a|\to \infty} e^{iaq} Re^{-iaq} = 0$ which is an exercise. □

4.2.8. Unbounded potentials. We give two one dimensional examples of Hamiltonians $H = p^2 + v$ affiliated to $\mathcal{D}(\mathbb{R})$ with potentials $v$ not relatively bounded with respect to $p^2$.

The first one is such that one of its localizations at infinity is not densely defined. For $n > 0$ integer let $v(x) = n$ if $n^2 - n < x < n^2$ and $v(x) = 0$ if $n^2 - n < x < n^2 + n$. Let $v$ be an arbitrary bounded positive function on $x < 0$. Denote $R = (H + 1)^{-1}$ and let $\Delta_+$ be the Dirichlet Laplacian on $(0, \infty)$. Then $u\lim_{n \to \infty} \tau_{n,\infty}(R) = R_+$ where $R_+ = (\Delta_+ + 1)^{-1}$ on $L^2(0, \infty)$ and $R_+ = 0$ on $L^2(-\infty, 0)$. Thus one of the localizations at infinity of $H$ is the Dirichlet Laplacian on $(0, \infty)$ considered as a non densely defined self-adjoint operator on $L^2(\mathbb{R})$ which “lives” in the subspace $L^2(0, \infty)$.

We now give a second one dimensional example where the essential spectrum has an interesting structure. Note that if $X = \mathbb{R}$ and $\kappa$ is an ultrafilter then either $[0, +\infty] \in \kappa$ or $]-\infty, 0[ \in \kappa$ hence there are two contributions to the essential spectrum of $H$, one coming from the behaviour at $+\infty$, that we denote $\text{Sp}_{\text{ess}}^+(H)$, and one coming from the behaviour at $-\infty$, that we denote $\text{Sp}_{\text{ess}}^-(H)$.

Let $H = h(p) + v(q)$, where $h : \mathbb{R} \to \mathbb{R}$ is of class $C^1$, polynomially bounded, tends to $+\infty$ if $p \to -\infty$, and $|h'(p)| \leq C(1 + |h(p)|)$. The function $v$ is real, continuous, and bounded from below. Assume that for large positive $x$ we have $v(x) = x^a \omega(x^\theta)$ with $a \geq 0$, $0 < \theta < 1$ and $\omega$ a positive continuous periodic function with period 1. Moreover, assume that $\omega$ vanishes only at the points of $\mathbb{Z}$ and that there are real numbers $\lambda, \mu > 0$ such that $\omega(t) \sim \lambda |t|^\mu$ when $t \to 0$. Then there are three possibilities:

1. If $a < \mu(1 - \theta)$ then $\text{Sp}_{\text{ess}}^+(H) = [\inf h, +\infty)$.
2. If $a = \mu(1 - \theta)$ then $\text{Sp}_{\text{ess}}^+(H) = \text{Sp}(h(P) + \lambda |\theta Q|^{\mu})$, a discrete not empty set.
3. If $a > \mu(1 - \theta)$ then $\text{Sp}_{\text{ess}}^+(H) = \emptyset$.

4.2.9. Discrete spectrum. A self-adjoint operator $H$ has purely discrete spectrum if and only if $\text{Sp}_{\text{ess}}^+(H) = \emptyset$. If $H$ is affiliated to $\mathcal{D}$, i.e. if the conditions of (1.4) are satisfied, then due to (1.5) this means $H_{\mathcal{D}} = \infty$ for any ultrafilter $\kappa$. Thus Corollary 1.2 implies:

**Proposition 4.8.** If $H$ is a self-adjoint operator affiliated to $\mathcal{D}$ then $H$ has purely discrete spectrum if and only if $w\lim_{n \to \infty} e^{iaq} R(z) e^{-iaq} = 0$ for some $z$ in the resolvent set of $H$.

The next result is a consequence of this proposition, we refer to [8] for the proof. See [28] for results of the same nature obtained by other techniques. Let $B_a$ be the ball $|x - a| < 1$.

**Corollary 4.9.** Let $H_0$ be a bounded from below self-adjoint operator with form domain equal to the Sobolev space $H^m$ for some real $m > 0$ and satisfying $\lim_{a \to 0} e^{iaq} H_0 e^{-iaq} = H_0$ in norm in $B(H^m, H^{-m})$. Let $V$ be a positive locally integrable function such that $\lim_{a \to 0} |\{ x \in B_a : |V(x) - \lambda| \} | = 0$ for each $\lambda > 0$. Then the self-adjoint operator $H$ associated to the form sum $H_0 + V$ has purely discrete spectrum.

4.3. N-body Hamiltonians. The $C^*$-algebra involved in the preceding subsection was the maximal one $\mathcal{A} = C^*_B(X)$. We will consider now an algebra $\mathcal{A} = \mathcal{R}(X)$ introduced in [16] (with a different notation) which has an interesting structure and character space so its study is worthwhile independently of the quantum mechanical applications mentioned in [16]. Theorem 1.6 will allow us to solve a problem left open in [16].

The terminology “N-body” is misleading in a certain measure, for example N has no significance, it should only suggest that the operators affiliated to the algebra $\mathcal{R}(X) \otimes X$
include the Hamiltonians describing quantum \( N \)-particle systems. But in fact the usual such Hamiltonians are affiliated to a smaller algebra studied in [5, §4] (and previous articles quoted there) with [15, Th. 6.27] as analog of Theorem 4.5: note that the \( N \)-body Hamiltonians with hard-core interactions studied in [2] are also affiliated to this algebra. But the \( N \)-body type Hamiltonians affiliated to the algebra \( \mathcal{R}(X) \times X \) are much more general since the allowed interactions are only required to be asymptotically homogeneous.

Let \( E \) be a finite dimensional real vector space \( E \). The half-line determined by a vector \( \alpha \neq 0 \) in \( E \) is the set \( \{ r \alpha \mid r > 0 \} \), the sphere at infinity \( S_E \) of \( E \) is the set of all half-lines in \( E \), and as a set the spatial compactification of \( E \) is the disjoint union \( E = E \cup S_E \). This space has a natural compact space topology [16, §3] such that \( E \to \overline{E} \) as an open dense subset hence \( \mathcal{C}(\overline{E}) \) can be identified with an algebra of continuous functions on \( E \). More precisely, by restricting functions on \( E \) to \( E \) one may realize \( \mathcal{C}(\overline{E}) \) as a translation invariant \( C^* \)-subalgebra of \( C^*_0(\overline{E}) \) which contains \( C^*_0(E) \).

The simplest example of this type is \( \mathcal{A} = C(\mathbb{R}) \) the algebra of continuous functions on \( \mathbb{R} \) that have limits at \( \pm \infty \). Then \( \mathcal{A}^1 = (\mathbb{R}, +\infty) \) consists of just two points, for any \( A \in \mathcal{A} \) the limits \( A_{\pm} = \lim_{r \to \pm \infty} e^{-irp} A e^{irp} \) exist, and \( \text{Sp}_{\text{ess}}(A) = \text{Sp}(A_{-}) \cup \text{Sp}(A_{+}) \). This is easy to prove directly, without any formalism.

The higher dimensional analog \( \mathcal{A} = C(\mathbb{R}^d) \) is less trivial. Now \( \mathcal{A}^1 = S_X \) and for any \( A \in \mathcal{A} \) and any \( a \in S_X \) the limit \( A_a = \lim_{r \to \infty} e^{-irp} A e^{irp} \) exists (and is clearly independent of the choice of \( a \)). It has been shown in [16, Cor. 4.5] that if \( A \) is a normal operator then \( \text{Sp}_{\text{ess}}(A) = \bigcup_{a \in S_X} \text{Sp}(A_a) \), in particular the union is a closed set, but the techniques used in [16] are not applicable to not normal operators. We shall give now a complete treatment of an \( N \)-body type generalization of this example.

Let \( X \) be a real finite dimensional vector space (it is not convenient to identify it with \( \mathbb{R}^d \)). If \( Y \subset X \) is a linear subspace and if we denote \( \pi_Y : X \to X/Y \) the canonical surjection, then the map \( \phi \mapsto \phi \circ \pi_Y \) gives a natural embedding \( C_0^*(X/Y) \subset C_0^*(X) \) so we may, and we shall, think of \( C_0^*(X/Y) \) as a \( C^* \)-subalgebra of \( C_0^*(X) \). On the other hand, \( C(\overline{X/Y}) \) is a \( C^* \)-subalgebra of \( C_0^*(X/Y) \), so we obtain a realization of \( C(\overline{X/Y}) \) as a \( C^* \)-subalgebra of \( C_0^*(X) \). Finally, we define the algebra \( \mathcal{A} \) of interest in this example:

\[
\mathcal{R}(X) \doteq \begin{cases} 
C^* \text{-algebra generated by the subalgebras } C(\overline{X/Y}) \\
\text{when } Y \text{ runs over the set of all subspaces of } X. 
\end{cases}
\tag{4.16}
\]

It is easy to see that \( C^*_0(X) \subset \mathcal{R}(X) \subset C_0^*(X) \) and that \( \mathcal{R}(X) \) is stable under translations, so we may take \( \mathcal{A} = \mathcal{R}(X) \) in Theorem 1.6. The corresponding \( \mathcal{A}^1 \) is the crossed product \( \mathcal{A}(X) = \mathcal{R}(X) \times X \) identified with the closed linear subspace of \( \mathcal{A}(X) \) generated by the operators \( \varphi(q)(p) \) with \( \varphi \in \mathcal{R}(X) \) and \( p \in C_0(X^*) \).

We shall keep the notation \( e^{irp} \) for the translation operator by \( a \in X \) although we do not give a meaning to the symbol \( p \). This operator acts as before: \( (e^{irp}u)(x) = u(a + x) \). To be precise, \( r \to \infty \) means \( r \to +\infty \).

**Theorem 4.10.** If \( A \in \mathcal{A}(X) \) and \( a \in S_X \) then \( u\lim_{r \to \infty} e^{irp} A e^{-irp} \equiv A_a \) exists for each \( a \in \alpha \) and is independent of \( a \). We have \( \text{Sp}_{\text{ess}}(A) = \bigcup_{a \in S_X} \text{Sp}(A_a) \).

**Corollary 4.11.** If \( H \) is a self-adjoint operator affiliated to \( \mathcal{A}(X) \) then for any \( a \in S_X \) the limit \( H_a = u\lim_{r \to \infty} e^{irp} H e^{-irp} \) exists for each \( a \in \alpha \) and is independent of \( a \). We have \( \text{Sp}_{\text{ess}}(H) = \bigcup_{a \in S_X} \text{Sp}(H_a) \).

---

1 We refer to [16] for details. Note however that the \( C^* \)-algebras \( \mathcal{A}(X) \), \( \mathcal{K}(X) \) do not depend on the choice of a Lebesgue measure on \( X \). We denote \( X^* \) the vector space dual to \( X \).
The first part of the theorem is a consequence of [16, Th. 6.18] and the corollary is an improvement of [16, Th. 6.21], where the formula for the spectrum is shown with the union replaced by the closure of the union. This question is discussed in [16] after Theorem 6.21 and Lemma 6.22 there gives a hint about the difficulty of the problem. On the other hand, the question of non self-adjoint \( A \) is not at all discussed in [16] because the relation (6.20) from [16] gives nothing for such operators.

Let us mention a recent result from [21] which is related to Corollary 4.11 but is proved using the different ideas introduced in [22]. Consider a finite set \( \mathcal{L} \) of linear subspaces of \( X \) which is stable under intersections and contains \( \{0\} \) and \( X \). Let us denote \( \mathcal{R}_L(X) \) the \( C^* \)-subalgebra of \( \mathcal{R}(X) \) generated by the \( C(X/Y) \) with \( Y \in \mathcal{L} \). Then \( \mathcal{R}_L(X) \) is stable under translations so the crossed product \( \mathcal{R}_L(X) \rtimes X \) is a well defined \( C^* \)-subalgebra of \( \mathcal{R}(X) \). Then Theorem 1.1 from [21] says that the assertion of Corollary 4.11 holds for the self-adjoint operators \( H \) affiliated to \( \mathcal{R}_L(X) \).

The rest of this subsection is devoted to the proof of the last assertion of Theorem 4.10. The main ingredient of the proof is a sufficiently explicit description of the character space \( \mathcal{R}(X) \) which is stable under intersections and contains \( \{0\} \) and \( X \).

The character space of the algebra \( C(X) \) is \( X \) and each \( \alpha \in \mathcal{S}_X \) defines a character of \( C(X) \), that we also denote by \( \alpha \), namely \( \alpha(u) = u(\alpha) = \lim_{r \to \infty} u(ra) \) if \( a \in \alpha \). In fact \( \alpha \) extends naturally to a character of \( \mathcal{R}(X) \):

\[
\alpha(u) = \lim_{r \to \infty} u(ra) \quad \forall u \in \mathcal{R}(X).
\]

Indeed, it suffices to prove that the limit above exists if \( u \in C(X/Y) \) for any subspace \( Y \). But this is clear: since \( u \equiv u \circ \pi_Y \), if \( \alpha \subset Y \) then \( u(ra) = u(0) \) and if \( \alpha \not\subset Y \) then \( u(ra) = u(r \pi_Y(a)) \) hence if we set \( \beta = \pi_Y(\alpha) \in \mathcal{S}_X/Y \) we get \( \lim_{r \to \infty} u(ra) = \beta(u) \).

Thus we obviously get a canonical embedding \( \mathcal{S}_X \subset \mathcal{R}(X) \). Now let us compute the translation morphism \( \tau_\alpha \) associated to \( \alpha \in \mathcal{S}_X \). From (1.7) we get

\[
\tau_\alpha(u)(x) = \alpha(\tau_x(u)) = \lim_{r \to \infty} \tau_x(u)(ra) = \lim_{r \to \infty} u(x + ra)
\]

hence we see that for each \( u \in \mathcal{R}(X) \) and \( x \in X \) the limit

\[
\tau_\alpha(u)(x) = \lim_{r \to \infty} u(ra + x)
\]

exists and is independent of \( \alpha \) (4.18)

which is the explicit expression for the translation morphism associated to \( \alpha \).

To better understand the action of \( \tau_\alpha \) we have to recall more of the formalism introduced in [16]. If \( Y \subset X \) is a linear subspace then the \( C^* \)-algebra \( \mathcal{R}(X/Y) \) associated to the vector space \( X/Y \) is well defined and, by using the embedding \( C^*_0(X/Y) \subset C^*_0(X) \), one may identify it with a subalgebra of \( \mathcal{R}(X) \):

\[
\mathcal{R}(X/Y) = C^* - \text{algebra generated by the subalgebras } C(X/Z) \text{ with } Z \supset Y.
\]

Let \([\alpha]\) be the one dimensional subspace generated by \( \alpha \) and set \( X/\alpha = X/[\alpha] \) to simplify the writing. Then the function \( \tau_\alpha(u) \) belongs to the subalgebra \( \mathcal{R}(X/\alpha) \) and

the map \( \tau_\alpha : \mathcal{R}(X) \to \mathcal{R}(X/\alpha) \) is a surjective morphism and a projection. (4.20)

Thus the translation morphism \( \tau_\alpha \) also has the property \( \tau^2_\alpha = \tau_\alpha \), more precisely \( \tau_\alpha(u) = u \) if and only if \( u \in \mathcal{R}(X/\alpha) \). Recall that the operation \( \tau_\alpha(u) \) on the function \( u \) should be thought as a translation by the point at infinity \( \alpha \in \mathcal{S}_X \). On the other hand, the operator \( \tau_\alpha \) of translation by some \( a \in X \) is an automorphism of \( \mathcal{R}(X) \) but never a projection.

Lemma 4.12. Let \( \kappa \) be a character in \( \mathcal{R}(X) \).

(1) There is a unique \( \alpha \in \mathcal{S}_X \) such that \( \kappa(C(X)) = \alpha \). This \( \alpha \) will be denoted \( \alpha_\kappa \).
(2) The map \( R(X)^1 \ni \kappa \mapsto \alpha_\kappa \in S_X \) is surjective.

(3) Denote \( \hat{\kappa} \in \sigma(R(X/\alpha_\kappa)) \) the restriction of \( \kappa \) to \( R(X/\alpha_\kappa) \). Then \( \kappa = \hat{\kappa} \tau_{\alpha_\kappa} \).

(4) Let \( Y \subset X \) with \( \alpha_\kappa \not\subset Y \) and \( \beta_\kappa = \pi_Y(\alpha_\kappa) \in S_{X/Y} \). Then \( \kappa|C(X/Y) = \beta_\kappa \).

(5) We have \( \tau_\kappa \tau_{\alpha_\kappa} = \tau_\kappa \) and \( \tau_\kappa = \tau_\hat{\kappa} \tau_{\alpha_\kappa} \).

**Proof.** To prove (1), note that the restriction of \( \kappa \) to \( C(\overline{X}) \) is a character of \( C(\overline{X}) \) hence it must be the evaluation at a uniquely determined point of \( X \). If this point is some \( \alpha \in X \), and since \( C_0(X) \) is an ideal in \( R(X) \), then we will have \( \kappa = \chi_\alpha \) which is not possible because \( \kappa \not\in X \). Hence the point belongs to \( S_X \). On the other hand, any point of \( S_X \) can be obtained in this way because each character of \( C(\overline{X}) \) is the restriction of a character of \( R(X) \); this proves (2). Now let us prove (3). Note first that for any \( \alpha \in S_X \) the restriction \( \tau_\alpha|C(\overline{X}) \) is just the character \( \kappa \) associated to the character \( \alpha \) of \( C(\overline{X}) \). Then \( \kappa \) and \( \hat{\kappa} \tau_\alpha \) are characters of \( R(X) \) whose restrictions to \( C(\overline{X}) \) and \( R(X/\alpha) \) are \( \alpha \) and \( \hat{\kappa} \), so are equal, hence \( \kappa = \hat{\kappa} \tau_\alpha \) as a consequence of \([16, \text{Cor. 6.8}]\). The assertion (4) follows immediately from \([16, \text{Lem. 6.7}]\). Finally, we prove (5). To simplify notations we write \( \alpha = \alpha_\kappa \) and first prove \( \tau_\kappa \tau_{\alpha} = \tau_\kappa \). The equality holds on \( R(X/\alpha) \) because \( \tau_{\alpha} \) is the identity on this subalgebra. Thus it remains to show that \( \tau_\kappa \tau_{\alpha} \) and \( \tau_\kappa \) are equal on any \( C(X/Y) \) if \( \alpha \not\subset Y \). We saw before that \( \tau_{\alpha}(u) = u(\pi_Y(\alpha)) \) if \( u \in C(X/Y) \) hence \( \tau_{\alpha}|C(X/Y) \) is the character \( \beta \) associated to the point \( \beta = \pi_Y(\alpha) \in S_{X/Y} \). Since \( \tau_{\alpha} \) is a unital morphism we see that \( \tau_{\kappa} \tau_{\alpha}|C(X/Y) = \beta \) is just the character \( \kappa \) of \( C(\overline{X}) \). to \( C(\overline{X}) \) and \( \pi_{X/\alpha} \). Hence \( \tau_{\alpha} \) is a morphism, so \( \tau_{\alpha} \tau_{\alpha} \tau_{\alpha} \tau_{\alpha} \tau_{\alpha} \) is also a morphism. This proves the assertion.

It remains to prove the relation \( \tau_{\kappa} \tau_{\alpha} = \tau_{\kappa} \tau_{\alpha} \). Here \( \tau_{\kappa} \tau_{\alpha} \) is the endomorphism of the algebra \( R(X/\alpha) \) associated to the character \( \tau_{\kappa} \) by the rule \( \tau_{\kappa}(v)(y) = \tau_{\alpha}(v)(y) \) for \( v \in R(X/\alpha) \) and \( y \in X/\alpha \). Since \( \tau_{\kappa} \) is a restriction of \( \kappa \) we get \( \tau_{\kappa}(v)(y) = \kappa(\tau_{\alpha}(v)) \). Let us denote \( \pi_{\alpha} \) the canonical surjection \( X \rightarrow X/\alpha \) and recall that we decided to identify a function \( w \) on \( X/\alpha \) with the function \( w \circ \pi_{\alpha} \) on \( X \). Then the preceding identity can be written \( \tau_{\kappa}(v)(x) = \kappa(\tau_{\alpha}(v)) \) for all \( x \in X \) (in fact \( y = \pi_{\alpha}(x) \) and \( \tau_{\alpha}(v) = \tau_{\alpha}(v) \)). If \( u \in R(X) \) then \( \tau_{\alpha}(u) \in R(X/\alpha) \) hence we obtain \( \tau_{\kappa}(\tau_{\alpha}(u))(x) = \kappa(\tau_{\alpha}(u))(x) \) for all \( x \in X \). But \( \kappa(\tau_{\alpha}(u)) = \kappa(\tau_{\alpha}(u))(x) \) so we get \( \kappa(\tau_{\alpha}(u))(x) = \kappa_{\alpha}(\tau_{\alpha}(u))(x) \) for all \( x \in X \) and \( u \in R(X) \) so that \( \tau_{\kappa} \tau_{\alpha} = \tau_{\kappa} \tau_{\alpha} \tau_{\alpha} \).

**Corollary 4.13.** The map \( \kappa \mapsto (\alpha_{\kappa}, \hat{\kappa}) \) induces a bijection of \( R(X)^1 \) onto the disjoint union of the character spaces \( \sigma(R(X/\alpha)) \), where \( \alpha \) runs over \( S_X \). The inverse map is \( (\alpha, \chi) \mapsto \chi_{\alpha} \). Thus we may identify

\[
R(X)^1 \simeq \bigoplus_{\alpha \in S_X} \sigma(R(X/\alpha)).
\]

(4.21)

From Theorem 1.6, relation (4.21), and (5) of Lemma 4.12, we obtain for any \( A \in \mathcal{B}(X) \):

\[
\text{Sp}_{\text{ess}}(A) = \bigcup_{\alpha \in R(X/\alpha)} \text{Sp}(\tau_{\alpha}(A)) = \bigcup_{\alpha \in R(X/\alpha)} \text{Sp}(\tau_{\kappa} \tau_{\alpha}(A))
\]

(4.22)

The maps \( \tau_{\kappa} : \mathcal{B}(X) \rightarrow \mathcal{B}(X/\alpha) \) are morphisms, so \( \text{Sp}(\tau_{\kappa} \tau_{\alpha}(A)) \subset \text{Sp}(\tau_{\alpha}(A)) \).

Note that we have equality here if \( \kappa \) is such that \( \hat{\kappa} \in X/\alpha_{\kappa} \). Then by using (2) of Lemma 4.12 we obtain

\[
\text{Sp}_{\text{ess}}(A) = \bigcup_{\alpha \in S_X} \text{Sp}(\tau_{\alpha}(A))
\]

(4.23)

It remains to give a convenient expression to \( \tau_{\alpha}(A) \). Consider first the translation morphism \( \tau_{\alpha} : R(X) \rightarrow R(X/\alpha) \) associated to \( \alpha \). If we embed \( R(X) \subset \mathcal{B}(X) \) as a an algebra of multiplication operators, then the definition (4.18) implies

\[
\tau_{\alpha}(u(q)) = s \lim_{r \rightarrow \infty} e^{irap}u(q)e^{-irap}.
\]

(4.24)
This clearly gives for the induced morphism \( \tau_\alpha : \mathcal{R}(X) \to \mathcal{R}(X/\alpha) \) the formula
\[
\tau_\alpha(A) = \text{slim}_{\tau \to \infty} \text{e}^{irap} A e^{-irap} \quad \forall A \in \mathcal{R}(X).
\] (4.25)

Finally, by using (4.23) and (4.25) we get the last assertion of Theorem 4.10.

5. APPENDIX

A filter on \( X \) is a set \( \mathcal{F} \) of subsets which does not contain the empty set, is stable under intersections, and satisfies \( T \supset S \in \mathcal{F} \Rightarrow T \in \mathcal{F} \). If \( f : X \to Y \) where \( Y \) is a topological space and if \( y \in Y \) then \( \lim_{x \to \infty} f(x) = y \) means that \( f^{-1}(V) \in \mathcal{F} \) for any neighborhood \( V \) of \( y \). A filter is finer than the Fréchet filter if it contains the complements of bounded subsets. Un ultrafilter is a filter which is not included in any other filter.

The character space of \( C^*_b(X) \) is called uniform compactification of \( X \) and is a quotient of the Stone-Čech compactification \( \beta X \) of \( X \), which is the character space of the \( C^*_b \)-algebra \( C_0(X) \) of bounded continuous functions on \( X \). In turn, \( \beta X \) is a quotient of the Stone-Čech compactification \( \gamma X \) of the discrete space \( X \), which is the character space of the \( C^*_b \)-algebra of all bounded functions on \( X \), and its elements are the ultrafilters on \( X \). The analogue of \( \mathcal{A}^\uparrow \) in this context is the set of ultrafilters \( \mathcal{F} \) finer than the Fréchet filter. We denote \( X^\uparrow \) the set of all such ultrafilters.

Now we restate Theorem 1.1 without using the notion of ultrafilters and thus make the connection with the results from \([24, 25, 19, 20]\) and references there. Following \([24, 25]\), we use the operator spectrum of \( A \):
\[
\sigma_{op}(A) = \{ B \in \mathcal{B} \mid B = \text{ulim}_{n \to \infty} \tau_{x_n}(A) \text{ for a sequence } \{x_n\} \text{ with } |x_n| \to \infty \}. \tag{5.1}
\]

**Theorem 5.1.** If \( A \in \mathcal{E} \) then the set \( \{ \tau_x(A) \mid x \in X \} \) is relatively compact in \( \delta_{\text{loc}} \) and \( \sigma_{op}(A) \) is a compact subset of \( \delta_{\text{loc}} \). We have
\[
\text{Sp}_{\text{loc}}(A) = \bigcup_{B \in \sigma_{op}(A)} \text{Sp}(B). \tag{5.2}
\]

**Proof.** It suffices to show that \( \sigma_{op}(A) = \{ A_\kappa \mid \kappa \in X^\uparrow \} \), cf. Theorem 1.4 and Lemma 3.4. If \( \kappa \in X^\uparrow \) then \( A_\kappa = \text{ulim}_{n \to \infty} \tau_{x_n}(A) \) by (1.9). Since \( \mathcal{B}_{\text{loc}} \) is metrizable, there is a metric \( d \) which defines its topology and then for each integer \( n > 0 \) there is \( x_n \) with \( |x_n| > n \) and \( d(\tau_{x_n}(A), A_\kappa) < 1/n \), hence \( A_\kappa \in \sigma_{op}(A) \). Reciprocally, if \( B \in \sigma_{op}(A) \) then there is a sequence \( \{x_n\} \) with \( |x_n| \to \infty \) such that \( B = \text{ulim}_{n \to \infty} \tau_{x_n}(A) \). Let \( \mathcal{F} \) be the set of subsets \( F \) of \( X \) with the property: there is \( N \) such that \( x_n \in F \) for all \( n > N \). Clearly \( \mathcal{F} \) is an ultrafilter finer than the Fréchet filter and \( B = \text{ulim}_{n \to \infty} \tau_{x_n}(A) = A_\kappa. \) \( \square \)
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