The dynamo effect in decaying helical turbulence

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We consider decaying hydromagnetic turbulence with initial kinetic helicity in an electrically conducting fluid and show that a weak nonhelical magnetic field eventually becomes fully helical. Already before this happens, the magnetic field undergoes inverse cascading with the magnetic energy decaying approximately like $t^{-0.5}$, which is even slower than in the fully helical case. In this parameter range, the product of magnetic energy and correlation length to the power 1.2 is approximately constant. Our result has applications to a wide range of experimental dynamos and astrophysical time-dependent plasmas, including primordial turbulence in the early universe.

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In electrically conducting fluids such as plasmas and liquid metals, steady helical turbulence is known to lead to an efficient conversion of kinetic energy into magnetic energy—a process referred to as a dynamo. Dynamos with swirling (helical) motions can be excited at relatively small magnetic Reynolds numbers, i.e., at moderate turbulent velocities and length scales, as well as moderate electric conductivities. This is why many dynamo experiments have employed helical flows both in the constrained (nonturbulent) flows of the experiments performed in Riga and Karlsruhe, as well as the unconstrained von Kármán flows in the experiments in Cadarache. Many other experiments are currently being worked upon. Their success is limited by the power that can be delivered by the propellers or pumps. A more economic type of dynamo experiment is driven by the flow that results inside a spinning torus of liquid sodium after abruptly breaking it. This leads to turbulence from the screw-like diverters inside the torus. Theoretical studies of laminar screw dynamos have been performed, but the evolution of hydromagnetic turbulence is usually parameterized.

The problem of magnetic field evolution in decaying helical turbulent flows in conducting media is far more general. Neutron stars, for example, have convective turbulence during the first minute after their formation. The early universe is another example of turbulence driven by expanding bubbles after a first-order phase transition. Transient turbulence is also being generated as a consequence of merging galaxy clusters. Even accretion discs may provide an example of decaying turbulence when the magnetorotational instability is not excited during certain phases. Dynamo effects are suspected to occur over durations of microseconds in inertial fusion confinement plasmas. In all these cases, the magnetic field evolution depends on the initial field strength, making the interpretation in terms of a dynamo effect difficult. In this Letter, we demonstrate that in decaying helical turbulent flows, an initially nonhelical seed magnetic field becomes eventually fully helical. It develops inverse cascade-type behavior already before this happens. To what extent this can be modeled in terms of advanced dynamo theory remains open, although potentially suitable tools such as two- and three-scale dynamo theories have been developed. Previous decay simulations were always performed with strong initial magnetic fields. Only recently, the need for studying the evolution of hydromagnetic turbulence in kinetically dominated systems been emphasized.

In stationary isotropic turbulence with finite kinetic helicity, $\langle \mathbf{\omega} \cdot \mathbf{u} \rangle$, where $\mathbf{\omega} = \nabla \times \mathbf{u}$ is the vorticity and $\mathbf{u}$ is the turbulent velocity, a statistically averaged mean magnetic field $\mathbf{B}$ obeys

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left[ \alpha_{\text{dyn}} \mathbf{B} - (\mathbf{\eta} + \mathbf{\eta}_m) \mathbf{J} \right],$$

where $\alpha_{\text{dyn}} \approx -\tau (\mathbf{\omega} \cdot \mathbf{u}) / 3$ is the $\alpha$ effect, $\mathbf{\eta} \approx \tau \langle \mathbf{u}^2 \rangle / 3$ is the turbulent magnetic diffusivity, $\mathbf{\eta}_m$ is the microphysical magnetic diffusivity, and $\mathbf{J} = \nabla \times \mathbf{B} / \mu_0$ is the mean magnetic current density.

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current density with $\mu_0$ being the vacuum permeability. If the coefficients are constant and the domain is periodic, the solutions are eigenfunctions of the curl operator with eigenvalue $k$, and $\mathbf{T}$ is proportional to $\exp(i\mathbf{k} \cdot \mathbf{x} + \gamma t)$. The solutions obey $\gamma = |\alpha_{\text{dyn}}| - \eta_T k^2$ with $\eta_T = \eta + \eta_B$ and can grow exponentially if $C \equiv |\alpha_{\text{dyn}}|/\eta_T k_1 > 1$, where $k_1 = 2\pi/L$ is the smallest wave number that fits into the cubic domain of size $L^3$.

We define the fractional helicity $\epsilon_i$ such that $\langle \omega \cdot u \rangle = \epsilon_i k_i(u^2)$, where $k_i$ is the wave number of the energy-carrying eddies. Thus, $C = \epsilon_i k_i/k_1$, where $\epsilon = 1 + 3 R^{-1}_m$ with $R_m = u_{\text{rms}}/\eta k_1$ being the magnetic Reynolds number, $\tau = (u_{\text{rms}}/k_1)^{-1}$ is the turnover time, and $u_{\text{rms}} = \langle u^2 \rangle^{1/2}$ is the rms velocity [32]. The effective wave number of the large-scale field is not normally at $k = k_1$, but at $k_m$ with $k_1 \leq k_m \leq k_1/2$.

In decaying hydrodynamic turbulence, we have $u_{\text{rms}} \propto t^{-p}$ with exponent $p = 10/7$ if the Loitsiansky integral [29] is conserved, or $p = 6/5$ if the Saffman integral [30] is conserved. In these time-dependent cases, we have $B = B_0^2 \exp[-t/(\alpha_1 + \alpha_2)]$, where $\gamma(t) = (\epsilon_1 - \omega k_m)/k_1 u_{\text{rms}}k_m/3$ with $\epsilon_1 = \epsilon_i(t)$, $k_i = k_1(t)$, $k_m(t) \geq k_1$, and $\epsilon = \epsilon(t)$ now all being time-dependent functions.

With these expectations in mind, let us now turn to three-dimensional turbulence simulations. We solve the compressible hydrodynamic equations for an isothermal gas with constant sound speed in a periodic domain of size $L$, so $k_1 = 2\pi/L$; see [31] for details. We define the initial velocity in Fourier space as

$$u_i(k) = \left[ P_{ij} + i\alpha_K \epsilon_i j \right](k) \frac{k_i}{k} \frac{u_{0} k_0^{-3/2} g_y(k/k_0)^{\alpha/2-1}}{[1 + (k/k_0)^{2(\alpha+5/3)}]^{1/4}}$$

where $P_{ij} = \delta_{ij} - k_i k_j/k^2$ is the projection operator, $g(k)$ is the Fourier transform of a spatially $\delta$-correlated vector field in three dimensions with Gaussian fluctuations, and $k_0$ is the initial wavenumber of the energy-carrying eddies. We choose $k_0/k_1 = 60$ and $\alpha = 4$ for a causally generated solenoidal field [32]. The fractional helicity is controlled by the parameter $\alpha_K$ and given by $\epsilon_1 = 2\alpha_K/(1 + \alpha_K^2)$. For the initial magnetic field, we take the same spectrum, but with $\alpha_M$ instead of $\alpha_K$ and smaller amplitude $B_0$ instead of $u_0$. The velocity is initially fully helical ($\alpha_K = 1$) and solenoidal. We consider initial $\mathbf{B}(k)$ with $\alpha_M = 0$, 1, and 1. 

Viscosity $\nu$ and magnetic diffusivity $\eta$ are usually very small in physical systems of interest. This is generally difficult to simulate, especially at early times if we fix $\nu$ and $\eta$ to be so small. However, a self-similar evolution is possible by allowing $\nu$ and $\eta$ to be time-dependent (after some time $t_0$, given below) with $\nu(t) = \nu_0 \max(t, t_0)^r$, $\eta(t) = \eta_0 \max(t, t_0)^r$, where $r = (1 - \alpha)/(3 + \alpha)$ [33], which gives $r \approx -3/7$ for $\alpha = 4$. We adopt this choice here as a practical matter.

For our numerical simulations, we use the PENCIL Code [https://github.com/pencil-code], a public MHD code that particularly well suited for simulating turbulence. In all cases we use 1152$^3$ meshpoints, which is large enough to ensure that the inverse-cascade effects are well reproduced [34].

We compute kinetic and magnetic energy spectra, $E_K(k, t)$ and $E_M(k, t)$, respectively. They are normalized such that $\int E_i(k, t) \, dk = \xi_i$ for $i = K$ or $M$, where $E_K = \rho_0 u_{\text{rms}}^2/2$ and $E_M = B_{\text{rms}}^2/2\mu_0$ are the kinetic and magnetic mean energy densities. Time is given in units of the initial turnover time, $\tau_0 = \tau(0)$, where $\tau(t) = \xi_K/u_{\text{rms}}$ and $\xi_i(t) = \int k^{-1} E_i(k, t) \, dk/E_i(t)$. We have chosen $t_0/\tau_0 = 0.1$. Our runs are given in Table I, where $v_{\text{AL}} = B_0/\rho_0^{1/2}$ has been introduced and the end time of the run $t_e$ is given.

In Fig. 1 we plot $\xi_K(t)$ and $\xi_M(t)$ for Runs A-E. $\xi_M$ is found to increase at first, reaches a maximum at $t/t_0 \approx 10$, and then approaches a late-time decay law approximately proportional to $t^{-p}$ with $p \approx 0.5$. We see that kinetic energy is transferred to magnetic energy, which does eventually exceed $\xi_K$. In particular, with $\alpha_M = 1$ (Run B) the decay is slower than for $\alpha_M = 0$ (Run C), while for $\alpha_M = -1$ it is faster. The reason for $\alpha = 4$.

![FIG. 1: Evolution of $\xi_K$ (blue) and $\xi_M$ (red) for $\alpha_M = 0$ (solid), $\alpha_M = 1$ (dashed), and $\alpha_M = -1$ (dotted) for $v_{\text{AL}}/u_0 = 0.1$ (Runs A–C), as well as $0.01$ (dashed, Run D) and $0.001$ (dotted, Run E).](image-url)
magnetic helicity that is a conserved quantity, and it can only change through resistive effects and at small scales. To understand how magnetic helicity gets produced, we show in Fig. 4 magnetic and kinetic helicity spectra, $H_M(k,t)$ and $H_K(k,t)$, respectively. They obey the realizability condition, $k^{-1}|H_K(k,t)| \leq 2E_K(k,t)$ and $|kH_M(k,t)| \leq 2E_M(k,t)$, respectively, and are normalized such that $\int H_K dk = \langle \omega \cdot u \rangle$ and $\int H_M dk = \langle A \cdot B \rangle$, where $A$ is the magnetic vector potential with $B = \nabla \times A$.

We see that, at early times, a bihelical magnetic helicity spectrum is produced, where positive and negative contributions are present simultaneously, though separated in $k$ space, just like in driven turbulence [21, 30]. Thus, there remains a near-cancellation of the net magnetic helicity until the magnetic helicity spectrum saturates at $k \approx k_0$. When that happens, magnetic helicity at large scales continues to increase only slowly such that at small scales magnetic helicity continues to dissipate resistively; see Fig. 1. Eventually, the magnetic helicity spectrum has at all wave numbers a negative sign. Since Run C starts with $\sigma_M = -1$, there is more efficient transfer of kinetic energy to magnetic energy, which is why we see a stronger growth in Fig. 1. The total magnetic energy decays then subject to resistive decay in the presence of magnetic helicity.
FIG. 5: Evolution of $\langle B^2 \rangle_\xi M$ for Re = $8 \times 10^4$ (black solid, Run A), $4 \times 10^4$ (red dotted, Run F), and $2 \times 10^4$ (blue dashed, Run G). In fully helical turbulence, we expect $\langle B^2 \rangle_\xi M \rightarrow \text{const}$, but here $\langle B^2 \rangle_\xi M^{1+\beta} \approx \text{const}$ with $\beta = 0.2$.

In Fig. 6 we plot the evolution of $\langle \omega \cdot u \rangle$ (blue), $\langle \omega \cdot u - J \cdot B / \rho_0 \rangle$ (red) and $\langle A \cdot B \rangle$ (green) for Re = $8 \times 10^4$ (Run A).

FIG. 6: Evolution of $\langle \omega \cdot u \rangle$ (blue), $\langle \omega \cdot u - J \cdot B / \rho_0 \rangle$ (red) and $\langle A \cdot B \rangle$ (green) for Re = $8 \times 10^4$ (Run A).

FIG. 7: Normalized instantaneous growth rate, $2\gamma \tau$ for Runs A, D, and E.

In Fig. 5 we plot the evolution of $\langle B^2 \rangle_\xi M$ for different Reynolds numbers (Runs A, F, and G). We see that the magnetic helicity produced depends on the magnetic Reynolds number. For comparison, we also plot $\langle B^2 \rangle_\xi M^{1.2}$ for Run A.

To make contact with the mean-field interpretation, we show in Fig. 6 that $\langle \omega \cdot u \rangle$ dies out while $-\langle J \cdot B \rangle / \rho_0 \approx \text{const}$. It is this combination that replaces the $\alpha$ effect in the nonlinear regime [26, 27]. We also plot $\langle A \cdot B \rangle$ and see that it is still increasing at the end of the run. This explains why $p$ and $q$ are still different from $2/3$.

Finally, we determine $2\gamma(t) = d\ln E_M / dt$ and find that $2\gamma \tau \approx 0.25$ at early times during the time interval $0.3 \leq t / \tau_0 \leq 1$; see Fig. 7. This suggests that $k_m \approx 0.25 k_1$, which is closer to $k_1/2$ (i.e., the wave number of the fastest growing mode) than to $k_1$. A predictive theory that describes also the late time decline of $\gamma$ would need to account for the change of $k_m(t)$.

Our work has demonstrated for the first time that the decay of turbulence with kinetic helicity leads to a non-conventional intermediate decay law with $p \approx q \approx 0.5$. Qualitatively, our results are easily explained. At early times, a bihelical magnetic helicity spectrum develops and it grows until it reaches equipartition at the wave number where the spectrum peaks. At later times, the magnetic helicity at small scales gets dissipated resistivity and the two parts of the magnetic helicity spectrum have the same sign at all $k$. After that time, the spectrum shows an inverse cascade during which $\langle B^2 \rangle_\xi M^{1+\beta} \approx \text{const}$. These new insights affect our understanding of all cases of decaying turbulence with kinetic helicity in electrically conducting media, such as plasma and liquid metal experiments, specifically the braked torus experiment, neutron stars, galaxy clusters, inertial fusion confinement plasmas, and the early universe.

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