A New Left Truncated Gumbel Distribution: Properties and Estimation

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Abstract

Gumbel probability distribution is an important probability distribution that has wide applications in various fields such as engineering, physics, hydrology, geology, etc. The study aims to generate a new truncated distribution by truncate the probability density function of the first type Gumbel distribution from the left side to obtain the Truncated Gumbel distribution from the left side within the period (0, ∞). As we work in this paper on the derivation of the new truncated distribution formula and its characteristics. As well as discussing different methods of estimating its parameters, represented by the maximum likelihood method and Percentiles Estimation Method, then making a simulation of the comparison between methods of estimation.
1. Introduction

Gumbel distribution is one of the extreme distributions, also known as type I extreme distribution, which has wide applications in various fields, including engineering, insurance, economics, risk management, environmental science, telecommunications, failure time analysis, and more[1]. In fact, despite the expansion of the application of traditional distributions in various fields, the complexity in the behavior of real data for recent phenomena represented by the high degree of skewness and kurtosis or the limited appearance of data values on the right side without the left and vice versa, prompted researchers to build probability models for new distributions From classical distributions. So that these distributions are more flexible in describing the behavior of the areal data of the phenomena[2, 3].

The research aims to suggest a probability model for a new distribution for the period \((0, \infty)\) from the Gumbel distribution using the method of truncating the probability distributions from the left. The new distribution is appropriate for the random variables whose values are within the period \((0, \infty)\), including failure times for equipment and machines or survival times.

Several kinds of researches and studies have appeared that dealt with the use of the truncating method in proposing new probability distributions, including: (Khams et al.) [4] in (2013) dealt with a study in which they presented the use of the truncating method on two probability models, namely (the natural logarithm distribution and the combined Weibull distribution), and the estimation of the features of the two amputated probabilistic models using the maximum likelihood method and then compared them through a mse. In the same year, (Zaninetti)[5] proposed a probability model for a new distribution by truncating the gamma distribution from the left and right sides, then he worked on applying it to real data and comparing it with the lognormal distribution and some other distributions using some criteria, including the AIC. In (2015), (Crénin)[6] derived the truncated probability distributions of the Weibull distribution with two parameters and three parameters and studied the moments of these truncated distributions. (Mohamed et al.)[7] in (2019) suggested a continuous distribution within the period \([0, 1]\) represented by the truncated Fréchet - Pareto distribution, as the probability density function of the distribution was derived, then studying its properties represented by the moment, the characteristic function, expectation, variance, Median and coefficients of skewness and kurtosis, as well as finding the Shannon entropy. In (2020) (Hassan et al.)[8] published a paper, which included a proposal for a new distribution represented by the \([0, 1]\) truncated Lumax distribution, then deriving the new distribution family and studying its properties: probability density function, moments, ordered statistics, and Shannon's entropy function, in addition to that, A set of distributions from the \([0, 1]\) truncated Lumax family, including the uniform - truncated lumax, a distribution of Fréchet- truncated lumax, and then parameterized estimation using the maximum likelihood method.

2. Gumbel distribution (GU)

The mathematical formula for the probability density and distribution functions of the Gumbel distribution is known by equations (1) and (2) respectively.[9]

\[
f(t) = \frac{1}{b} e^{-\frac{(t-a)}{b}} e^{-e^{-\frac{(t-a)}{b}}}; \quad -\infty \leq T \leq \infty 
\]  \( \text{... (1)} \)
\[ F(t) = e^{-\frac{\left(t-a\right)}{b}} ; -\infty \leq T \leq \infty \] \hspace{1cm} (2)

Since \((a)\) is the location parameter and \((b)\) is the scale parameter. \(-\infty < a < \infty, 0 < b < \infty\)

And when \(a = 0\) and \(b = 1\), the Gumbel distribution is called the standard Gumbel distribution, and its probability density and distribution functions are abbreviated by equations (3) and (4).[1]

\[ f(t) = e^{-e^{\left(t-a\right)}} ; -\infty \leq T \leq \infty \] \hspace{1cm} (3)

\[ F(t) = e^{-e^{\left(-t\right)}} ; -\infty \leq T \leq \infty \] \hspace{1cm} (4)

Figures 1. and 2. below illustrate a plotting of the probability density function and the distribution function of the Gumbel distribution at different values of the two parameters \(a\) and \(b\).

![Graph showing probability density function and distribution function of Gumbel distribution with different values of a and b.](image)

Figure 1. Plot the probability density function for the Gumbel distribution with different values for the \(a\) and \(b\) parameters.
3. Derivation of the left truncated Gumbel distribution (LTG)$^u$

Suppose that $T$ is a random variable defined within the interval $(-\infty, \infty)$ with the probability density function and the distribution function, $f(t)$ and $F(t)$ respectively. In order to derive the probability density function $f^\ast(t)$ of the random variable $T$ defined on the interval $[c, \infty)$, we truncate the probability density function of the random variable $T$ from the left and find its mathematical formula by the equation (5).

$$f^\ast(t) = [F(\infty) - F(c)]^{-1} f(t) = [1 - F(c)]^{-1} f(t) \quad ; c \leq T \leq \infty$$

Since $(c)$ is a real constant defined within the sample space $\Omega$.

Using equation (3), (4), and (5) we work to find the probability density function for the left truncated Gumbel distribution for the period $(0, \infty)$ and as in equation (6).

$$f^\ast(t) = [1 - e^{-e^\frac{a}{b}}]^{-1} \frac{1}{b} e^{\frac{-(t-a)}{b} - \frac{(t-a)}{b}} ; T \geq 0$$

Below is a check whether the function $f^\ast(t)$ of the left truncated Gumbel distribution, which we obtained in equation (7), is a probability or not, and as follows:-

$$f^\ast(t) = \int_0^\infty [1 - e^{-e^\frac{a}{b}}]^{-1} \frac{1}{b} e^{\frac{-(t-a)}{b} - \frac{(t-a)}{b}} dt$$

$$= [1 - e^{-e^\frac{a}{b}}]^{-1} \left[ e^{\frac{-(t-a)}{b}} \right]_0^\infty$$

$$= [1 - e^{-e^\frac{a}{b}}]^{-1} \left[ e^{-\infty} - e^{-e^\frac{a}{b}} \right]$$

$$= [1 - e^{-e^\frac{a}{b}}]^{-1} \left[ e^{-\infty} - e^{-e^\frac{a}{b}} \right]$$

$$= 1$$

From the result of equation (7), we conclude that the function $f^\ast(t)$, which represents the function of the left truncated Gumbel distribution, is a probability density function.

After defining the probability density function for the left truncated Gumbel distribution, we now work to derive the cumulative distribution function and the reliability function of the LTG$^u$ distribution with the following eq.(8) and (9):
\[ F^*(t) = \int_0^t \left[ 1 - e^{-e^{\frac{a}{b}}} \right]^{-1} \frac{1}{b} e^{-e^{-\frac{(x-a)}{b}}} \frac{(x-a)}{b} dx \]

\[ = \left[ 1 - e^{-e^{\frac{a}{b}}} \right]^{-1} \int_0^t \frac{1}{b} e^{-e^{-\frac{(x-a)}{b}}} \frac{(x-a)}{b} dx \]

\[ = \left[ e^{-e^{-\frac{(x-a)}{b}}} - e^{-e^{\frac{a}{b}}} \right] \left[ 1 - e^{-e^{\frac{a}{b}}} \right]^{-1} \]

... (8)

\[ R^*(t) = 1 - \left[ 1 - e^{-e^{\frac{a}{b}}} \right]^{-1} \left[ e^{-e^{-\frac{(t-a)}{b}}} - e^{-e^{\frac{a}{b}}} \right] \]

... (9)

Figures 3 and 4 show the plotting of the probability density function and the distribution function of the left truncated Gumbel distribution at different values for the two parameters (a) and (b).

Figure 3. plotting the density function of the left truncated Gumbel distribution with different values for the (a) and (b) parameters.

Figure 4. plotting the cumulative distribution of the left truncated Gumbel distribution with different values for the (a) and (b) parameters.
4. Properties

4.1 Noncentral moments

Equation (10) defines the rth noncentral moment of the LTGE distribution

\[
E(T^r) = \int_0^\infty t^r f(t) \, dt
\]

\[
= \int_0^\infty t^r \frac{1}{b} [1 - e^{-\frac{t}{b}}]^{-1} e^{-\frac{(t-a)}{b}} \frac{(t-a)}{b} \, dt
\]

... (10)

Suppose that

\[y = e^{-\frac{(t-a)}{b}}\]

By taking the natural logarithm of the two sides and simplifying, we get

\[\ln y = -\frac{(t-a)}{b}\]
\[t = (a - blny)\]
\[dt = \frac{-b}{y} \, dy\]

if \(t = 0 \Rightarrow y = e^a\); \(t = \infty \Rightarrow y = 0\)

Now by substituting in equation (10) we find that

\[
E(T^r) = \int_{e^a}^{\infty} (a - blny)^r \frac{1}{b} [1 - e^{-\frac{a}{b}}]^{-1} e^{-\frac{b}{y}} y \left(\frac{-b}{y}\right) \, dy
\]

\[
= \frac{1}{[1 - e^{-\frac{a}{b}}]} \int_{0}^{\frac{a}{b}} (a - blny)^r e^{-y} \, dy
\]

... (11)

\[
= \frac{1}{[1 - e^{-\frac{a}{b}}]} \sum_{k=0}^{r} C_k^r (a)^r (-b)^k \int_{0}^{\frac{a}{b}} (\ln y)^k e^{-y} \, dy
\]

\[
= \frac{1}{[1 - e^{-\frac{a}{b}}]} \sum_{k=0}^{r} C_k^r (a)^r (-b)^k \gamma^k
\]

... (12)

As: \(\gamma^k = \int_{0}^{\frac{a}{b}} (\ln y)^k e^{-y} \, dy\)

We then note that equation (12) represents the noncentral moment of wrath of the LTG distribution. To find the noncentral moments of the first and second order, we substitute \(r = 1\) and \(r = 2\) into equation (12), respectively. Equations (13),(14) and (15) summarize the results of the first and second noncentral moments.
\[ E(T) = \frac{1}{1 - e^{-eb}} \int_0^a (a - b \ln y) \ e^{-y} \, dy \]
\[ = \frac{1}{1 - e^{-eb}} \left( a \int_0^a e^{-y} \, dy - b \int_0^a \ln e^{-y} \, dy \right) \]
\[ = \frac{1}{1 - e^{-eb}} \left( a \left[ -e^{-y} \right]_0^a - by \right) \]
\[ = \frac{1}{1 - e^{-eb}} \left( a \left[ 1 - e^{-eb} \right] - by \right) \]
\[ = a - \frac{by}{1 - e^{-eb}} \]
\[ \ldots (13) \]

\[ E(T^2) = \frac{1}{1 - e^{-eb}} \int_0^a (a - b \ln y)^2 \ e^{-y} \, dy \]
\[ = \frac{1}{1 - e^{-eb}} \left( a^2 \int_0^a e^{-y} \, dy - 2ab \int_0^a \ln e^{-y} \, dy \right. \]
\[ + b^2 \int_0^a (\ln y)^2 e^{-y} \, dy \]  
\[ = \frac{1}{1 - e^{-eb}} \left( a^2 \left[ 1 - e^{-eb} \right] - 2ab \gamma + b^2 \gamma^2 \right) \]
\[ \ldots (14) \]

\[ E(T^2) = \frac{1}{1 - e^{-eb}} \left( a^2 \left[ 1 - e^{-eb} \right] - 2ab \gamma + b^2 \gamma^2 \right) \]
\[ \ldots (15) \]

As:
\[ \gamma = \int_0^a \ln y e^{-y} \, dy \; \gamma^2 = \int_0^a (\ln y)^2 e^{-y} \, dy \]

**4.2 Central moments**

\[ E(t - \mu)^r = \int_0^\infty (t - \mu)^r f(t) \, dt \]
\[ = \int_0^\infty (t - \mu)^r \frac{1}{b} \left[ 1 - e^{-eb} \right]^{-1} e^{-e^{-y} - \frac{(t-a)}{b}} \frac{(t-a)}{b} \, dt \]  
\[ \ldots (16) \]

Let that
\[ y = e^{-\frac{(t-a)}{b}} \]
By performing the math and simplifying, we get:

\[
\ln y = -\frac{(t - a)}{b}
\]

\[
t = (a - b\ln y)
\]

\[
dt = \frac{-b}{y} \ dy
\]

if \( t = 0 \Rightarrow y = e^{\frac{a}{b}} \); \( t = \infty \Rightarrow y = 0 \)

Now by substituting in equation (16) we obtain the central moment of order \( r \), the formula of which is shown in equation (17) below:

\[
E(t - \mu)^r = \int_{\frac{a}{e^b}}^{\frac{a}{e^b}} (a - b\ln y - \mu)^r \frac{1}{b} \left[1 - e^{-e^b} \right]^{-1} e^{-y} y \left(\frac{b}{y}\right) dy
\]

\[
= \frac{1}{[1 - e^{-e^b}]} \int_{\frac{a}{e^b}}^{\frac{a}{e^b}} (a - \mu - b\ln y)^r e^{-y} dy
\]

\[
= \frac{1}{[1 - e^{-e^b}]} \int_{\frac{a}{e^b}}^{\frac{a}{e^b}} \sum_{k=0}^{r} C_k^r (a - \mu)^{r-k} (-b\ln y)^k e^{-y} dy
\]

\[
= \frac{1}{[1 - e^{-e^b}]} \sum_{k=0}^{r} C_k^r (a - \mu)^{r-k} (-b)^k y^k \gamma_k
\]

\[
... (17)
\]

To get the variance, we substitute \( r = 2 \) and \( \mu = E(T) \) into equation (17) as follows:

\[
\sigma^2 = \frac{1}{[1 - e^{-e^b}]} \sum_{k=0}^{2} C_k^r (a - \mu)^{r-k} (-b)^k \gamma_k
\]

\[
= \frac{1}{[1 - e^{-e^b}]} \left[ C_0^2 (a - \mu)^2 \gamma^0 + C_1^2 (a - \mu) (-b) \gamma + C_2^2 (-b)^2 \gamma^2 \right] \quad ... (18)
\]

4.3 skewness coefficient

The steps for deriving the mathematical formula for the skewness coefficient are shown in the following equation (19):

\[
C_S = \frac{E(t - \mu)^3}{\sigma^3}
\]
4.4 kortisies coefficient

As for the mathematical definition of the kortisies coefficient, the equation (20) summarizes it.

\[
C.K = \frac{E(t - \mu)}{\sigma^4} - 3
\]

\[
= \left[ \frac{1}{1 - e^{-e^{\gamma}}} \right] \left[ C_0^4 (a - \mu)^4 \gamma^6 + C_1^4 (a - \mu)^3 (-b) \gamma + C_2^4 (a - \mu) (-b)^2 \gamma^4 + C_3^4 (a - \mu)^2 (-b)^3 \gamma^2 + C_4^4 \gamma^3 + C_5^4 (-b)^4 \gamma^4 \right]^{\frac{1}{2}} ... (19)
\]

4.5 Mod

To find the mode, we work on the derivation of the probability density function of the left truncated Gumbel distribution on the left, and then equating it to zero, as shown by equation (23).

\[
f^*(t) = \left[ 1 - e^{-e^{\gamma}} \right]^{-1} \frac{1}{b} e^{-e^{-\frac{(t-a)}{b}} - \frac{(t-a)}{b}} \left( -e^{-\frac{(t-a)}{b}} - e^{\gamma} \right)
\]

\[
\left[ 1 - e^{-e^{\gamma}} \right]^{-1} \frac{1}{b} e^{-e^{-\frac{(t-a)}{b}} - \frac{(t-a)}{b}} \left( e^{-\frac{(t-a)}{b}} - 1 \right) = 0 \]

\[
\left[ 1 - e^{-e^{\gamma}} \right]^{-1} \frac{1}{b} e^{-e^{-\frac{(t-a)}{b}} - \frac{(t-a)}{b}} \left( e^{-\frac{(t-a)}{b}} - 1 \right) = 0 \] ...

From the above result, we notice that the equation (21) is non-linear and cannot be solved by classical analytical methods, so its solution will be using iterative numerical methods.

4.6 Median

It is possible to define the mathematical formula for the median of the variable T with the equation (22) as follows:

\[
F^*(t) = \frac{e^{-e^{-\frac{(t-a)}{b}} - e^{-e^{\gamma}}}}{1 - e^{-e^{\gamma}}} = 0.5
\]

\[
t = Med = a - b \ln(\ln(0.5 + 0.5 e^{-e^{\gamma}})^{-1}) \] ...

(22)
5. Estimation

5.1 Maximum likelihood method

The equations (23) and (24) below represent the maximum likelihood function of the left truncated Gumbel distribution.

\[ L = \left[ 1 - e^{-e^{\frac{a}{b}}} \right]^{-n} \left( \frac{1}{b} \right)^n e^{n} \sum \frac{e^{\frac{- (t-a)}{b}}}{b} \] … (23)

\[ \ln L = -n \ln \left( 1 - e^{\frac{a}{b}} \right) - n \ln b - \sum \frac{(t-a)}{b} - \sum \frac{e^{-\frac{(t-a)}{b}}}{b} \] … (24)

To find the estimate for the parameter \( a \), we differentiate the equation (24) with respect to \( a \) and set it equal to zero, as shown in the equation (25).

\[ \frac{\partial \ln L}{\partial a} = -ne^{\frac{a}{b}} + \frac{a}{b} - \frac{n}{b} \sum \frac{e^{-\frac{(t-a)}{b}}}{b} = 0 \] … (25)

We repeat the steps for estimating parameter \( a \) above to find the estimate for parameter \( b \). As it appears in the equation (28).

\[ \frac{\partial \ln L}{\partial b} = \frac{nae^{\frac{a}{b}}}{b^2 \left( 1 - e^{\frac{a}{b}} \right)} - \frac{n}{b} + \frac{\sum(t-a)}{b^2} - \frac{1}{b^2} \sum(t-a)e^{\frac{-(t-a)}{b}} = 0 \] … (26)

We note from equations (25) and (26) that it cannot be solved using the usual analytical methods, and therefore it is necessary to use iterative numerical methods to obtain estimates of parameters \( a, b \) including the Fsolve method.

5.2 Percentiles Estimation Method

This method was proposed by Kao and it is a statistical method used to estimate unknown parameters by comparing sample points with corresponding theoretical points. This method has been widely used with distributions that have a closed quantile function. Below is an estimation of the parameters of the amputated Campbell distribution from the left using the method of segmental estimations [10,11].

Assuming that \( p_i \) represents the nonparametric estimate of the cumulative distribution function and substituting in equation (8) and simplifying, we obtain:

\[ p_i = \frac{e^{-\frac{(t-a)}{b}} - e^{-\frac{a}{b}}}{\left[ 1 - e^{-\frac{a}{b}} \right]} \]

\[ p_i \left[ 1 - e^{-\frac{a}{b}} \right] = \left[ e^{-\frac{(t-a)}{b}} - e^{-\frac{a}{b}} \right] \]
\[ p_l - p_l e^{-\frac{a}{b}} + e^{-\frac{a}{b}} e^{-\frac{(t-a)}{b}} = 0 \]

\[ p_l + (1 - p_l) e^{-\frac{a}{b}} e^{-\frac{(t-a)}{b}} = 0 \]

... (27)

So that \( p_l = \frac{i}{n+1} ; i = 1, 2, \ldots, n \)

Square both sides and take the sum of both sides of the equation (27)

\[ \sum_{i=1}^{n} \left[ p_l + (1 - p_l) e^{-\frac{a}{b}} e^{-\frac{(t-a)}{b}} \right]^2 = 0 \]

... (28)

To obtain an estimate of the parameters using Percentiles Estimation Method, we work on deriving equation (28) for the two parameters (a) and (b) as follows:

\[ \frac{\partial}{\partial a} \sum_{i=1}^{n} \left[ p_l + (1 - p_l) e^{-\frac{a}{b}} e^{-\frac{(t-a)}{b}} \right] = 0 \]

\[ \sum_{i=1}^{n} \left[ p_l + (1 - p_l) e^{-\frac{a}{b}} e^{-\frac{(t-a)}{b}} \right] \left[ -(1 - p_l) e^{-\frac{a}{b}} \left( \frac{1}{b} \right) \right] e^{-\frac{(t-a)}{b}} \left( \frac{1}{b} \right) = 0 \]

\[ \sum_{i=1}^{n} \left[ p_l + (1 - p_l) e^{-\frac{a}{b}} e^{-\frac{(t-a)}{b}} \right] \left[ -(1 - p_l) e^{-\frac{a}{b}} + e^{-\frac{(t-a)}{b}} \left( \frac{t-a}{b} \right) \right] = 0 \]

... (29)

\[ \frac{\partial}{\partial b} \sum_{i=1}^{n} \left[ p_l + (1 - p_l) e^{-\frac{a}{b}} e^{-\frac{(t-a)}{b}} \right] = 0 \]

\[ \sum_{i=1}^{n} \left[ p_l + (1 - p_l) e^{-\frac{a}{b}} e^{-\frac{(t-a)}{b}} \right] \left[ (1 - p_l) e^{-\frac{a}{b}} \left( \frac{a}{b^2} \right) + e^{-\frac{(t-a)}{b}} \left( \frac{t-a}{b^2} \right) \right] = 0 \]

\[ \sum_{i=1}^{n} \left[ p_l + (1 - p_l) e^{-\frac{a}{b}} e^{-\frac{(t-a)}{b}} \right] \left[ a(1 - p_l) e^{-\frac{a}{b} + \frac{a}{b^2}} + e^{-\frac{(t-a)}{b}} \left( \frac{t-a}{b} \right) \right] = 0 \]

... (30)

We note that equations (29) and (30) are non-linear, and therefore it is not possible to solve them by normal analytical methods to obtain an estimate of the two parameters a and b. Thus, one of the numerical methods is used to solve nonlinear equations by more than two unknowns, including the Fsolve method, which is an algorithm in the Matlab program.
6. Simulation

The simulation experiment was carried out by adopting four sample sizes (150, 100, 50, 25) and three default values for the two parameters of the left-truncated Gumbel distribution, and the experiment was repeated 1000 times. As for the results of the experiment, they are summarized in tables 1, 2, 3 and 4 below.

Table 2. shows the results of the comparison between the Mle and Pc estimation methods for the left-truncated Gumbell distribution model using the mean square error (Mse) standard for sample sizes (150, 100, 50, 25) and for the default values cases for the two distribution parameters (a, b). We note that the best estimation method was the Mle method, for all sample sizes, and for all cases of default values for the two parameters of the left-truncated Gumbell distribution. We also note:

- The first case (a = 1, b = 2)

It was found that the least mean squares of error were (0.00035) for estimating the Mle method at sample size (n = 150) and the estimated values for parameters (a,b) were -1.0967 and 2.0996 respectively.

- The second case (a = 2, b = 3)

It was found that the least mean squares of error were (0.00034) for estimating the Mle method at a sample size of (n = 150) and the estimated values for the two parameters (a,b) were (-6.9350) and (3.5664) respectively.

- The third case (a = 3, b = 4)

It was found that the lowest mean squares of error was (0.000097) for estimating the Mle method at a sample size of (n = 150), and the estimated values for the two parameters (a,b) were (-1.1466) and (4.2615) respectively.

Table 1. Estimation of the mean squares of error for the left-truncated Gumbell distribution model and for the MLE and PC estimation methods for a 1000 repeat experiment

| Cases | n   | Parameter estimation | MSE   | Best |
|-------|-----|----------------------|-------|------|
|       |     | mle | pc |       |       |       |       |       |
|       |     | a   | b  | a    | b    | Mle  | Pc   |       |
| a=1,b=2 | 25  | -6.6042 | 2.3337 | 0.2627 | 3.3546 | 0.0017 | 0.0111 | Mle |
|       | 50  | -4.6417 | 2.2662 | 0.3174 | 3.4135 | 0.00095 | 0.0235 | Mle |
|       | 100 | -1.5640 | 2.1185 | 0.4085 | 3.5216 | 0.000507 | 0.0202 | Mle |
|       | 150 | -1.0967 | 2.0996 | 0.4555 | 3.5835 | 0.00035 | 0.0211 | Mle |
| a=2,b=3 | 25  | -12.6443 | 3.8975 | -0.7090 | 3.1119 | 0.000853 | 0.0102 | Mle |
As for Table 2, the results of the comparison between the two estimation methods for the two parameters of the left-truncated Gumbell distribution at different sample sizes (150,100,50,25) and the first case of the default values for the two parameters (a = 1, b = 2).

• At size n = 25

Pc method was shown to be the best for estimating parameter (a) with mean squares of error (Mse = 1.14), while Mle method was best for estimating parameter (b) with mean squares of error (Mse = 0.55).

• At size n = 50

Pc method was shown to be the best for estimating parameter (a) with mean squares of error (Mse = 1.01), while Mle method was best for estimating parameter (b) with mean squares of error (Mse = 0.37).

• At size n = 100

The Pc method was shown to be the best for estimating parameter (a) with mean squares of error (Mse = 0.79), while the Mle method was the best for estimating parameter (b) with mean squares of error (Mse = 0.18).

• At size n = 150

Pc method was shown to be the best for estimating parameter (a) with mean squares of error (Mse = 0.65), while Mle method was best for estimating parameter (b) with mean squares of error (Mse = 0.14).

Table 2. Mean squares of error Mse for feature estimates for the two estimation methods Mle and Pc at sample sizes (150,100,50,25) and for initial values (a = 1, b = 2) for a 1000 repeat experiment

| sample size | parameters | mle   | pc    | Best |
|-------------|------------|-------|-------|------|
| 50          | -9.0676    | 3.7219| -0.7948| 3.0013 | Mle  |
| 100         | -7.9188    | 3.6276| -0.8349| 2.9856 | Mle  |
| 150         | -6.9350    | 3.5664| -0.8647| 2.9743 | Mle  |
| 25          | -13.3636   | 5.1472| -0.9122| 3.0918 | Mle  |
| 50          | -8.1550    | 4.7449| -0.9509| 3.0398 | Mle  |
| 100         | -3.7119    | 4.4175| -0.9832| 2.9924 | Mle  |
| 150         | -1.1466    | 4.2615| -0.9926| 2.9696 | Mle  |
| a=3,b=4     |            |       |       |       |      |
| 25          | -13.3636   | 5.1472| -0.9122| 3.0918 | Mle  |
| 50          | -8.1550    | 4.7449| -0.9509| 3.0398 | Mle  |
| 100         | -3.7119    | 4.4175| -0.9832| 2.9924 | Mle  |
| 150         | -1.1466    | 4.2615| -0.9926| 2.9696 | Mle  |
Table 3. shows the results of a comparison between the two estimation methods for the two parameters of the left-truncated Gumbell distribution at different sample sizes (150, 100, 50, 25) and the second case for the default values for the two parameters (a = 2, b = 3).

• At size n = 25

Pc method was found to be the best for estimating the two parameters (a, b) with mean squares of error (Mse = 15.42) and (Mse = 2.44) respectively.

• At size n = 50

It was found that the Pc method is the best for estimating the two parameters (a) and (b) with mean squares of error (Mse = 7.76) and (Mse = 1.21) respectively.

• At size n = 100

The Pc method was found to be the best for estimating the two parameters (a) and (b) with mean squares of error (Mse = 8.15) and (Mse = 0.70) respectively.

• At size n = 150

The Pc method was found to be the best for estimating the two parameters (a) and (b) with mean squares of error (Mse = 8.26) and (Mse = 0.42) respectively.
Table 3. The average squares of error Mse for feature estimates for the Mle and Pc estimation methods at sample sizes (150,100,50,25) and for initial values (a = 2, b = 3) for a 1000 repeat experiment

| Sample size | Parameters | mle  | pc  | Best |
|-------------|------------|------|-----|------|
| 25          | a          | 461.61 | 7.76 | pc   |
|             | b          | 1.68  | 2.05 | mle  |
| 50          | a          | 337.07 | 8.01 | pc   |
|             | b          | 1.25  | 1.21 | pc   |
| 100         | a          | 307.29 | 8.15 | pc   |
|             | b          | 1.009 | 0.70 | pc   |
| 150         | a          | 275.38 | 8.26 | pc   |
|             | b          | 0.91  | 0.42 | pc   |

Finally, Table 4 shows the results of the comparison between the two estimation methods for the two parameters of the left-truncated Gumbell distribution at different sample sizes (150,100,50,25) and the third case for the default values for the two parameters (a = 3, b = 4).

- At size n = 25
  Pc method was found to be the best for estimating the two parameters (a) and (b) with mean squares of error (Mse = 15.42) and (Mse = 2.44) respectively.

- At size n = 50
  The Pc method was found to be the best for estimating the two parameters (a) and (b) with mean squares of error (Mse = 15.66) and (Mse = 1.73) respectively.

- At size n = 100
  The Pc method was found to be the best for estimating the parameter (a) with mean squares of error (Mse = 15.87), while the Mle method was the best for estimating parameter (b) with mean squares of error (Mse = 0.8997).

- At size n = 150
  The Pc method was shown to be the best for estimating the parameter (a) with mean squares of error (Mse = 15.94), while the Mle method was the best for estimating parameter (b) with mean squares of error (Mse = 0.58).

Table 4. The average squares of error Mse for feature estimates for the Mle and Pc estimation methods at sample sizes (150,100,50,25) and for the initial values (a = 3, b = 4) for a 1000 repeat experiment
### 7. Conclusion

This paper reached a set of conclusions, which we summarize as follows:

- The proposed left-truncated Gumbel distribution is appropriate for random variables within \((0, \infty)\). And the formulation of the functions of probability density and cumulative distribution and the reliability function, as well as the study of its properties represented by the mean, variance, central moments, noncentral moments, kurtosis, skewness and median.

- Through the simulation results, it was found that the best method to estimate the general model of left-truncated Gumbel distribution was the method of Mle because it achieved the lowest standard mean squared error at all sample sizes and for all cases of default values for parameters \((a)\) and \((b)\).

- It was found from the results of the estimation of the two parameters \((a, b)\) that Pc estimation for parameter \((a)\) was the closest to the default value of the parameter \((a)\), while the estimation of the Mle method for parameter \((b)\) was the closest to the default value of parameter \((b)\) and for all sample sizes and for all cases of default values.

- Through the results of the estimation of the two parameters, we find that the change in the value of the estimator of the location parameter \((a)\) and for the two methods of estimation is relatively large, and the reason for this is due to the method of truncation in the distribution. It is inferred that the truncation of the distribution affects the location parameter of the distribution more than the scale parameter.

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