In- and out-states of scalar particles confined between two capacitor plates

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Abstract

In the present article, using a non-commutative integration method of linear differential equations, we, considering the Klein-Gordon equation with the $L$-constant electric field with large $L$ and using the light cone variables, find new complete sets of its exact solutions. These solutions can be related by integral transformations to previously known solutions that were found in Phys. Rev. D. \textbf{93}, 045033(2016). Then, using the general theory developed in Phys. Rev. D. \textbf{93}, 045002 (2016), we construct (in terms of the new solutions) the so-called in- and out-states of scalar particles confined between two capacitor plates.

1 Introduction

A particle production from a vacuum by strong electric-like external backgrounds (the Schwinger effect \cite{1}, or the effect of the vacuum instability) is one of the most interesting effect in quantum field theory (QFT) that attracts attention already for a long time. The effect can be observable if the external fields are sufficiently strong, e.g. the magnitude of an electric field should be comparable with the Schwinger critical field $E_c = m^2 e^3 / e \hbar \simeq 10^{16}$V/cm. Nevertheless, recent progress in laser physics allows one to hope that an experimental observation of the effect can be possible in the near future, see Refs. \cite{2} for a review. Moreover, electron-hole pair creation from the vacuum becomes also observable in laboratory conditions in the graphene and similar nanostructures, see, e.g. Refs. \cite{3}. Depending on the structure of such external backgrounds, different approaches have been proposed for calculating the effect, a list of relevant publications can be found in Refs. \cite{4, 5}. Calculating quantum effects in strong external backgrounds must be nonperturbative with respect to the interaction with strong backgrounds. A general formulation of QED with time-dependent external fields (so-called $t$-potential steps) was developed in Refs. \cite{6}. It can be also seen that in some situations in graphene and similar nanostructures the vacuum instability effects caused by strong (with respect of massless fermions) electric fields are of significant interest; see, e.g., Refs. \cite{5, 7, 8, 9, 10, 11, 12} and references therein. At the same time, in these...
cases electric fields can be considered as time-independent weakly inhomogeneous $x$-electric potential steps (electric fields of constant direction that are concentrated in restricted space areas) that can be approximated by a linear potential. Approaches for treating quantum effects in the explicitly time-dependent external fields are not directly applicable to the $x$-electric potential steps. In the recent work \cite{13} a consistent nonperturbative formulation of QED with critical $x$-electric potential steps, strong enough to violate the vacuum stability, was constructed. There a nonperturbative calculation technique for different quantum processes such as scattering, reflection, and electron-positron pair creation was developed. This technique essentially uses special sets of exact solutions of the Dirac and Klein Gordon equation with the corresponding external field of $x$-electric potential steps. The cases when such solutions can be found explicitly (analytically) are called exactly solvable cases. This technique was effectively used to describe particle creation effect in the Sauter field of the form $E(x) = E \cosh^{-2}(x/L_S)$, in a constant electric field between two capacitor plates separated by a distance $L$ (the so-called $L$-constant electric field), and in exponential time-independent electric steps, where the corresponding exact solutions were available, see Refs. \cite{13,14,15}. These exactly solvable models allowed one to develop a new approximate calculation method to treat nonperturbatively the vacuum instability in arbitrary weakly-inhomogeneous $x$-electric potential steps \cite{16}. Note, that the corresponding limiting case of a constant uniform electric field has many similarities with the case of the de Sitter background, see, e.g., Refs. \cite{17,18} and references therein. Thus, a study of the vacuum instability in the presence of the $L$-constant electric field with large $L \to \infty$ may be quite important for some applications. Only critical step with a potential difference $\Delta U > 2m$ ($m$ is the electron mass) can produce electron-positron pairs, moreover, pairs are born only with quantum numbers in a finite range, in the so-called Klein zone.

As a matter of fact, non-perturbative calculation techniques are related to the possibility of constructing exact solutions of the corresponding relativistic Dirac or Klein Gordon equations, in particular, solutions that have special asymptotics. Constructing of such solutions is a rather difficult task. Sometimes, to solve it, an adequate choice of variables in the corresponding equations is useful. In particular, in the work \cite{19}, see \cite{20} as well, considering the Dirac or Klein Gordon equations with a constant uniform field given by time-dependent potential and choosing the variables of the light cone, the above solutions in a special representation were found. With these solutions, it was possible to find explicitly all kinds of the corresponding QED singular functions in the Fock-Schwinger proper time representation. In the present article, using a non-commutative integration method of linear differential equations, we, considering the Klein-Gordon equation with the $L$-constant electric field with large $L$ and using the light cone variables, find new complete sets of its nonstationary exact solutions. These solutions can be related by integral transformations to previously known stationary solutions that were found in Ref. \cite{14}. Then, using the general theory developed in Ref. \cite{13}, we construct (in terms of the new nonstationary solutions) the so-called in- and out-states of scalar particles confined between two capacitor plates.

### 2 In- and out-solutions

Here we construct in- and out-solutions of the Klein-Gordon equation with an external constant electric field, which is the so-called $L$-constant electric field and belongs to the class of $x$-potential steps. The equation has the form:

\[
(P^\mu P_\mu - m) \psi(X) = 0, \quad P_\mu = i\partial_\mu - qA_\mu(X),
\]

\[
P^\mu = \eta^{\mu\nu} P_\nu, \quad \eta^{\mu\nu} = \text{diag}(1,-1,\ldots,-1), \quad d = D + 1,
\]

where $A_\mu(X)$ are corresponding electromagnetic potentials, $m$ is the particle mass and $q = -e, e > 0$ is its charge. For generality, we consider the problem in $d$-dimensional
with respect to the Klein-Gordon inner product on the momenta \( \hat{p} \). Note that for two solutions with different quantum numbers as can be easily calculated, electric field given by a linear potential, \( \hat{E} \). In fact, in this limit we may approximate the constant field by a constant uniform electric field. In fact, in this limit we may approximate the \( L \)-constant field by a constant uniform electric field given by a linear potential, \( A_0(X) = -Ex, \ A_k(X) = 0, \ x = X^1, \ E > 0. \) Let us consider stationary solutions of the Klein-Gordon equation, having the following form:

\[
\psi_n(X) = \varphi_n(t,x) \varphi_{p\perp}(r_\perp), \quad \varphi_{p\perp}(r_\perp) = (2\pi)^{-(d-2)/2} \exp(i p_\perp r_\perp), \quad X = (t,x,r_\perp),
\]

\[
\varphi_n(t,x) = \exp(-i p_\parallel t) \varphi_n(x), \quad n = (p_0, p_\perp),
\]

\[
r_\perp = (X^2, \ldots, X^D), \quad p_\perp = (p^2, \ldots, p^D), \quad \hat{p}_x = -i \partial_x.
\]

These solutions are quantum states of spinless particles with given energy \( p_0 \) and momenta \( p_\perp \) perpendicular to the \( x \)-direction. The functions \( \varphi_n(x) \) obey the second-order differential equation

\[
\left\{ \hat{p}_x^2 - [p_0 - U(x)]^2 + p_\perp^2 + m^2 \right\} \varphi_n(x) = 0, \quad U(x) = -eA_0(x).
\]

We would like to construct two complete sets of solutions of form (4), we denote them as \( \zeta \psi_n(X) \) and \( \zeta' \psi_n(X) \), \( \zeta = \pm \) in what follows, with special left and right asymptotics,

\[
\hat{p}_x \zeta \psi_n(X) = p^L \zeta \psi_n(X), \quad x \to -\infty,
\]

\[
\hat{p}_x \zeta' \psi_n(X) = p^R \zeta' \psi_n(X), \quad x \to +\infty.
\]

The solutions \( \zeta \psi_n(X) \) and \( \zeta' \psi_n(X) \) asymptotically describe particles with given real momenta \( p^{L/R} \) along the \( x \) direction. The corresponding functions \( \varphi_n(x) \) are denoted by \( \zeta \varphi_n(x) \) and \( \zeta' \varphi_n(x) \) respectively. These functions have the asymptotics:

\[
\zeta \varphi_n(x) = \zeta' \exp \left[ i \frac{|p^L|}{2} x \right], \quad x \to -\infty,
\]

\[
\zeta' \varphi_n(x) = \zeta' \exp \left[ i \frac{|p^R|}{2} x \right], \quad x \to +\infty.
\]

Solutions \( \zeta \psi_n(X) \) and \( \zeta' \psi_n(X) \) are subjected to the following orthonormality conditions with respect to the Klein-Gordon inner product on the \( x = \text{const} \) hyperplane:

\[
(\zeta \psi_n, \zeta' \psi_{n'})_x = (\zeta \psi_n, \zeta' \psi_{n'})_x = \zeta \delta_{\zeta,\zeta'} \delta_{n,n'},
\]

\[
(\psi, \psi')_x = i \int \psi^* (X) \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right) \psi' (X) \, dt dr_{\perp}.
\]

Note that for two solutions with different quantum numbers \( n \), the inner product \( (\psi, \psi')_x \) can be easily calculated,

\[
(\psi_n, \psi_{n'})_x = I \delta_{n,n'}, \quad \delta_{n,n'} = 2\pi \delta (p_0 - p_0') \delta (p_\perp - p_\perp'),
\]

\[
I = \varphi_n^* (x) \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right) \varphi_{n'} (x).
\]
Solutions $\zeta \psi_n (X)$ and $\zeta' \psi_n (X)$ can be decomposed through each other as follows:

\[
\zeta \psi_n (X) = + \psi_n (X) g \left( \frac{\xi}{\xi} \right) - \psi_n (X) g \left( - \frac{\xi}{\xi} \right),
\]

\[
\zeta' \psi_n (X) = - \psi_n (X) g \left( - \frac{\xi}{\xi} \right) + \psi_n (X) g \left( + \frac{\xi}{\xi} \right),
\]

where the expansion coefficients are defined by the equations

\[
\left( \zeta \psi_n, \zeta' \psi_n \right)_x = g \left( \frac{\xi}{\xi} \right) \delta_{n,n'}, \quad g \left( \frac{\xi}{\xi} \right) = g \left( \frac{\xi}{\xi} \right)^*.
\]

Equation (5) can be written in the following form

\[
\left[ \frac{d^2}{d\xi^2} + \xi^2 - \lambda \right] \varphi_n (x) = 0, \quad \xi = \frac{eE x - p_0}{\sqrt{eE}}, \quad \lambda = \frac{n^2}{eE}.
\]

Its general solution can be written in terms of an appropriate pair of linearly independent Weber parabolic cylinder functions (WPCFs), either $D_\rho([1 + i]\xi)$ and $D_{-\rho}([1 + i]\xi)$, or $D_\rho([1 - i]\xi)$ and $D_{-\rho}([1 - i]\xi)$, where $\rho = -i\lambda/2 - 1/2$.

Using asymptotic expansions of WPCFs, one can construct the functions $\zeta \varphi_n (x)$ and $\zeta' \varphi_n (x)$,

\[
+ \varphi_n (x) = + ND_{-\rho}(-1 + i)\xi \sim e^{-ix^2/2}, \quad \xi \to -\infty, \quad p^L = -\xi \sqrt{eE};
\]

\[
- \varphi_n (x) = - ND_\rho((1 - i)\xi) \sim e^{ix^2/2}, \quad \xi \to -\infty, \quad p^L = \xi \sqrt{eE};
\]

\[
+ \varphi_n (x) = + ND_\rho((1 - i)\xi) \sim e^{ix^2/2}, \quad \xi \to \infty, \quad p^R = \xi \sqrt{eE};
\]

\[
- \varphi_n (x) = - ND_{-\rho}(-1 + i)\xi \sim e^{-ix^2/2}, \quad \xi \to \infty, \quad p^R = -\xi \sqrt{eE};
\]

\[
\zeta N = \zeta CY_0, \quad \zeta' N = \zeta' CY_0, \quad Y_0 = (2\pi)^{-1/2}, \quad \zeta C = \zeta C = (2eE)^{-1/2} e^{-\xi^2}.
\]

Their in- and out-classifications are related with signs of the asymptotic momenta $p^L$ and $p^R$ (see, Refs. [14]). Namely,

$+ \psi_n$, $+ \psi_n$ are in - states, and $- \psi_n$, $- \psi_n$ are out - states.

It is useful to construct two different complete sets of solutions of the Klein-Gordon equation (1) that are not stationary states and have the following form:

\[
\psi_\sigma (X) = \varphi_\sigma (t, x) \varphi_{p_\perp} (r_\perp),
\]

where $\sigma$ is a set of quantum numbers, which will be defined below. In this case the function $\varphi_\sigma (t, x)$ satisfies the equation:

\[
\left\{ \hat{p}_x^2 - \left[ \hat{p}_0 - U (x) \right]^2 + \mathbf{p}_\perp^2 + m^2 \right\} \varphi_\sigma (t, x) = 0, \quad \hat{p}_0 = i\partial_t.
\]

This equation admits integrals of motion in the class of linear differential operators of the first order, which are:

\[
\hat{Y}_0 = -i e, \quad \hat{Y}_1 = \partial_t, \quad \hat{Y}_2 = \partial_x + ieEt, \quad \hat{Y}_3 = x\partial_t + t\partial_x + \frac{ieE}{2} (t^2 + x^2).
\]

The operators $\hat{Y}_a, a = 0, 1, 2, 3$ form a four-dimensional Lie algebra $\mathfrak{g}$ with nonzero commutation relations

\[
[\hat{Y}_1, \hat{Y}_2] = -E \hat{Y}_0, \quad [\hat{Y}_1, \hat{Y}_3] = \hat{Y}_2, \quad [\hat{Y}_2, \hat{Y}_3] = \hat{Y}_1.
\]

Equation (10) can be considered as an equation for the eigenfunctions of the Casimir operator $K(-i\hat{Y}) = -2E\hat{Y}_0\hat{Y}_3 + \hat{Y}_1^2 - \hat{Y}_2^2$,

\[
-K(-i\hat{Y})\varphi_\sigma (t, x) = (\mathbf{p}_\perp^2 + m^2) \varphi_\sigma (t, x).
\]
At this stage, we follow a non-commutative integration method of linear differential equations \[21\ 22\ 23\], which allows us to construct a complete set of solutions based on a symmetry of the equation. We define an irreducible representation of the Lie algebra in the space of functions of the variable \( \tilde{p} \in (-\infty, +\infty) \) by the help of the operators \( \ell_a(\tilde{p}, \partial_{\tilde{p}}, j) \),

\[
\ell_0(\tilde{p}, \partial_{\tilde{p}}, j) = i\epsilon, \quad \ell_1(\tilde{p}, \partial_{\tilde{p}}, j) = -eE\partial_{\tilde{p}} + \frac{i}{2} \tilde{p},
\]

\[
\ell_2(\tilde{p}, \partial_{\tilde{p}}, j) = eE\partial_{\tilde{p}} + \frac{i}{2} \tilde{p}, \quad \ell_3(\tilde{p}, \partial_{\tilde{p}}, j) = -\tilde{p}\partial_{\tilde{p}} + ij - \frac{1}{2}, \quad j > 0,
\]

where \( j \) parameterizes the non-degenerate adjoint orbits of a Lie algebra \( g \). The following relations hold true:

\[
[\ell_1, \ell_2] = -E \ell_0, \quad [\ell_1, \ell_3] = \ell_2, \quad [\ell_2, \ell_3] = \ell_1, \\
K(-i\ell(\tilde{p}, \partial_{\tilde{p}}, j)) = (2eE)j.
\]

Integrating the equations

\[
\left[ \hat{Y}_a + \ell_a(\tilde{p}, \partial_{\tilde{p}}, j) \right] \varphi_\sigma (t, x) = 0
\]

together with equation (11), we fix \( j = -\lambda/2 \) and derive a set of solutions which is characterized by quantum numbers \( \sigma = (\tilde{p}, p_\pm) \),

\[
\pm \varphi_\sigma(t, x) = \pm C_\sigma \exp \left( \frac{\pm i}{2} \left( \frac{1}{\sqrt{2eE}} - i^2 \right) \right) \left( \frac{\ln \left( \frac{\pm iE}{\sqrt{\pi eE}} \right)}{\frac{i}{2} \pi x_+} \right),
\]

\[
\pi_+ = \tilde{p} + eEx_+, \quad x_+ = t \pm x.
\]

The parameter \( \tilde{p} \) is an eigenvalue of the symmetry operator \( i(\hat{Y}_1 + \hat{Y}_2) \):

\[
i(\hat{Y}_1 + \hat{Y}_2) \pm \varphi_\sigma(t, x) = \tilde{p} \pm \varphi_\sigma(t, x).
\]

One can interpret the quantum numbers \( \sigma \) from the point of view of the orbit method: the parameter \( \lambda = (m^2 + \tilde{p}^2)/eE \) describes the Casimir operator \( K(-i\hat{Y}) \) spectrum and parameterizes the non-degenerate orbits of the co-adjoint representation of the local Lie group \( \exp L \) (in our case, the orbits are hyperbolic paraboloids), and the region of variation of the parameter \( \tilde{p} \) is a Lagrangian submanifold to these orbits.

In order to classify solutions (11) that are stationary states, eigenfunctions for the operator \( \hat{p}_0 \).

We represent solutions of both equation (5) and

\[
\hat{p}_0 \varphi^{(\pm)}_\sigma(t, x) = p_0 \varphi^{(\pm)}(t, x)
\]

in the following form

\[
\varphi^{(\pm)}_n(t, x) = (2\pi eE)^{-1/2} \int_{-\infty}^{+\infty} M^*(p_0, \tilde{p}) \varphi^{(\pm)}_\sigma(t, x) dp.
\]

Taking into account condition (11) one gets an equation for the function \( M(p_0, \tilde{p}) \):

\[
-i\ell_1(\tilde{p}, \partial_{\tilde{p}}, j) M(p_0, \tilde{p}) = p_0 M(p_0, \tilde{p}).
\]

We choose its particular solution

\[
M(p_0, \tilde{p}) = \exp \left( \frac{i}{4eE} \left( \tilde{p}^2 - 4\tilde{p}p_0 \right) \right),
\]

(14)
which satisfies the orthogonality relation
\[
\int_{-\infty}^{+\infty} M^*(p_0, \bar{p}) M(p_0, \bar{p}') \, dp_0 = 2\pi e E \, \delta(\bar{p} - \bar{p}').
\]  
(15)

The inverse to (13) transform reads:
\[
\pm \varphi_{\sigma}(t, x) = (2\pi e E)^{-1/2} \int_{-\infty}^{+\infty} M(p_0, \bar{p}) \varphi_{n}^{(\pm)}(t, x) \, dp_0.
\]  
(16)

Thus, we have defined direct (13) and inverse (16) integral transformation with kernel (14) that converts solutions (12) to solutions that are eigenfunctions for the operator \( \hat{p}_0 \).

Applying one of the integral transformations to solutions (12), we get:
\[
\varphi_{n}^{(\pm)}(t, x) = (2\pi e E)^{-1/2} \int_{-\infty}^{+\infty} M^*(p_0, \bar{p}) \pm \varphi_{\sigma}(t, x) \, d\bar{p}.
\]

Then, comparing Eq. (17) with Eqs. (7)–(8), we obtain the following correspondence
\[
\varphi_{n}^{(+)}(t, x) \sim +N e^{-ip_0 t} D_p [+(1 - i)\xi] = +\varphi_{n}(x) e^{-ip_0 t},
\]
\[
\varphi_{n}^{(-)}(t, x) \sim -N e^{-ip_0 t} D_p [-+(1 - i)\xi] = -\varphi_{n}(x) e^{-ip_0 t}.
\]  
(18)

Transformation (16) allows one to derive orthonormality relations on the hyperplane \( x = \text{const} \) for scalar particles constructing with the help of functions \( \pm \varphi_{\sigma}(t, x) \),
\[
\left( \pm \psi_{\sigma}, \pm \psi_{\sigma'} \right)_x = \pm \delta_{\sigma, \sigma'},
\]  
(19)

where
\[
\pm \psi_{\sigma}(X) = \pm \varphi_{\sigma}(t, x) \varphi_{P_{\pm}}(r_{\pm}),
\]  
(20)

and determine the normalizing factors \( \pm C_{\sigma} \),
\[
\pm C_{\sigma} = \frac{1}{\sqrt{4\pi e E}} e^{\pi \lambda/4}.
\]

Thus, we obtain:
\[
(-\psi_{\sigma}, + \psi_{\sigma'})_x = g (-| +) \delta_{\sigma, \sigma'}, \quad g (-| +) = ie^{\pi \lambda/2}.
\]

We introduce now the following notation:
\[
\varphi_{n}^{(\pm)}(t, x) = \pm \varphi_{n}(x) \exp (-ip_0 t),
\]
\[
\pm \psi_{n}(X) = \varphi_{n}^{(\pm)}(t, x) \varphi_{P_{\pm}}(r_{\pm}).
\]

It follows from (13) and (16):
\[
\pm \psi_{\sigma}(X) = (2\pi e E)^{-1/2} \int_{-\infty}^{+\infty} M(p_0, \bar{p}) \pm \psi_{n}(X) \, dp_0,
\]  
(21)
\[
\pm \psi_{n}(X) = (2\pi e E)^{-1/2} \int_{-\infty}^{+\infty} M^*(p_0, \bar{p}) \pm \psi_{\sigma}(X) \, d\bar{p}.
\]

Let us consider another type of solutions,
\[
\begin{align*}
\varphi_{\sigma}(t, x) &= \theta (-\pi -) \varphi_{\sigma}(t, x),
\varphi_{\sigma}(t, x) &= \theta (+\pi -) \varphi_{\sigma}(t, x).
\end{align*}
\]  
(22)
The corresponding integral transformation is:

\[ \varphi_n(t, x) = (2\pi eE)^{-1/2} \int_{-\infty}^{+\infty} M^*(p_0, \bar{p}) \theta (-\pi) + \varphi_\sigma(t, x) \, dp \]

\[ = -C_\sigma \sqrt{2\pi} (-1 + i)^\rho e^{-1-i\pi} \Gamma(\rho + 1) \epsilon^{\frac{i\pi}{2}} e^{-\pi i \rho} \psi_0(t, x) \, dp \]

\[ \varphi_n(t, x) = (2\pi eE)^{-1/2} \int_{-\infty}^{+\infty} M^*(p_0, \bar{p}) \theta (\pi) - \varphi_\sigma(t, x) \, dp \]

\[ = -C_\sigma \sqrt{2\pi} (-1 + i)^\rho e^{-1-i\pi} \Gamma(\rho + 1) \epsilon^{\frac{i\pi}{2}} e^{-\pi i \rho} \psi_0(t, x) \, dp \]

such that

\[ \varphi_n(t, x) \sim \varphi_\sigma(t, x) e^{-\pi i \rho t}, \quad -\varphi_n(t, x) \sim \varphi_\sigma(t, x) e^{-\pi i \rho t}. \]

From Eq. (22) it follows

\[ (\psi_\sigma, \bar{\psi}_\sigma') = 0, \quad (\psi_\sigma', \bar{\psi}_\sigma) = 0. \]

From (23) and (25) follows the integral transformations

\[ \bar{\psi}_\sigma(X) = (2\pi eE)^{-1/2} \int_{-\infty}^{+\infty} M(p_0, \bar{p}) \varphi_n(X) dp_0, \]

\[ \bar{\psi}_n(X) = (2\pi eE)^{-1/2} \int_{-\infty}^{+\infty} M^*(p_0, \bar{p}) \psi_\sigma(X) d\bar{p}. \]

There exist useful relations between solutions \( \psi_\sigma(X) \) and \( \bar{\psi}_\sigma(X) \). Each of them is complete for a given \( \sigma \) and can be decomposed through each other as follows:

\[ \bar{\psi}_\sigma(X) = \bar{\psi}_\sigma(X) g (\bar{\psi}_\sigma(X) g (\bar{\psi}_\sigma(X) g (\bar{\psi}_\sigma(X)), \]

\[ \bar{\psi}_\sigma(X) = \bar{\psi}_\sigma(X) g (\bar{\psi}_\sigma(X) g (\bar{\psi}_\sigma(X) g (\bar{\psi}_\sigma(X)), \]

Equations

\[ \left( \begin{array}{cc} \psi_\sigma, & \bar{\psi}_\sigma \end{array} \right) \begin{pmatrix} \psi_\sigma \bar{\psi}_\sigma \end{pmatrix} = g \begin{pmatrix} \psi_\sigma \bar{\psi}_\sigma \end{pmatrix} = g \left( \begin{array}{cc} \psi_\sigma & \bar{\psi}_\sigma \end{array} \right) \]

allow us to calculate coefficients \( g (\psi_\sigma \bar{\psi}_\sigma) \).

We note that the relations (24) are similar to relations (25) that were established for the solutions \( \psi_\sigma(X) \) and \( \bar{\psi}_\sigma(X) \) (in this case the coefficients \( g \) do not depend on \( p_0 \) and \( \bar{p} \)). From (18) and (24) it follows

\[ +\psi_\sigma, +\bar{\psi}_\sigma \text{ are in – states, and } -\psi_\sigma, -\bar{\psi}_\sigma \text{ are out – states.} \]

From Eq. (22) it follows

\[ +\psi_\sigma(X) = 0, \quad \pi_+ > 0, \]

\[ -\psi_\sigma(X) = 0, \quad \pi_- < 0. \]
Then, taking into account equations (27) and (28), we get:

\[
\begin{align*}
\psi_\sigma(X) = g \left( \begin{array}{c}
\pi_+ \\
\pi_-
\end{array} \right)^{-1} \left[ \psi_\sigma(X) \left( \begin{array}{c}
\pi_+ \\
\pi_-
\end{array} \right) + \psi_\sigma(X) \right] &= 0, \quad \pi_+ > 0, \\
\psi_\sigma(X) = g \left( \begin{array}{c}
\pi_- \\
\pi_+
\end{array} \right)^{-1} \left[ \psi_\sigma(X) \left( \begin{array}{c}
\pi_- \\
\pi_+
\end{array} \right) + \psi_\sigma(X) \right] &= 0, \quad \pi_- < 0.
\end{align*}
\]

(29)

Since the coefficient \( g \left( \begin{array}{c}
\pi_+ \\
\pi_-
\end{array} \right)^{-1} \) is not zero for all \( \sigma \), the equations (29) imply a direct connection between the solutions \( \zeta \psi_\sigma(X) \) and \( \zeta \psi_\sigma(X) \) normalized on the hyperplane \( x = \text{const} \).

Thus, using the noncommutative integration method for equation (10) we obtained in- and out-states of scalar particles in terms of new solutions (20) and (25), which are non-stationary and are determined by a set of quantum numbers \( \sigma \). Solutions \( \left\{ + \psi_\sigma, + \psi_\sigma \right\} \) describe in-solutions, and solutions \( \left\{ - \psi_\sigma, - \psi_\sigma \right\} \) describe out-states. Using integral transformations (21) and (26) the solutions \( \zeta \psi_\sigma(X) \) and \( \zeta \psi_\sigma(X) \) are related to the well-known stationary solutions \( \zeta \psi_n(X) \) (see Ref. [14]).

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