ESTIMATES OF 1D RESONANCES IN TERMS OF POTENTIALS

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Abstract. We discuss resonances for Schrödinger operators with compactly supported potentials on the line and the half-line. We estimate the sum of the negative power of all resonances and eigenvalues in terms of the norm of the potential and the diameter of its support. The proof is based on harmonic analysis and Carleson measures arguments.

Dedicated to Lennart Carleson, on the occasion of his 85th birthday

1. Introduction and main results

In this paper we will present a global estimate of resonances in terms of the potential for Schrödinger operators

$$H = H_0 + q,$$

where $H_0$ is one of the following:

Case 1: $-d^2/dx^2$ in $L^2(\mathbb{R})$.

Case 2: $-d^2/dx^2$ in $L^2(\mathbb{R}_+)$, with $f(0) = 0$ boundary conditions.

Case 3: $-d^2/dx^2$ in $L^2(\mathbb{R}_+)$, with $f'(0) = 0$ boundary conditions.

We assume that $q$ is real, integrable and has a compact support. It is well known that the spectrum of $H$ consists of an absolutely continuous part $[0, \infty)$ and a finite number of simple negative eigenvalues $E_1 < \cdots < E_m < 0$, see well-known papers [5], [23] and the book [21] about inverse scattering. The Schrödinger equation

$$-f'' + q(x)f = k^2 f, \quad k \in \mathbb{C} \setminus \{0\},$$

has unique solutions $\psi_\pm(x, k)$ such that $\psi_+(x, k) = e^{ikx}$ for large positive $x$ and $\psi_-(x, k) = e^{-ikx}$ for large negative $x$. Outside the support of $q$ any solutions of (1.1) have to be combinations of $e^{\pm ikx}$. The functions $\psi_\pm(x, \cdot), \psi'_\pm(x, \cdot)$ for all $x \in \mathbb{R}$ are entire. We define the Wronskian $w$ for Case 1 by

$$w(k) = \{\psi_-(\cdot, k), \psi_+(\cdot, k)\},$$

where $\{f, g\} = fg' - f'g$. In Case 2 the Jost function is defined as $\psi_+(0, \cdot)$ and in Case 3 the Jost function is defined as $\psi'_+(0, \cdot)$. Let $F$ be one of the functions $w, \psi_+(0, k)$ or $\psi'_+(0, k)$. Recall that the function $F$ is entire. One has exactly $m$ simple zeros $k_1 = i|E_1|^{1/2}, \ldots, k_m = i|E_m|^{1/2}$ in the upper half-plane $\mathbb{C}_+$ and for $q \neq 0$ an infinite number of zeros $(k_n)_{n+1}^\infty$ in the lower half-plane $\overline{\mathbb{C}}_-$ labeled by

$$0 \leq |k_{m+1}| \leq |k_{m+2}| \leq \cdots \quad \text{where} \quad 0 < |k_{m+2}|,$$

see [9], [25], [28] and [16]. Here it is possible that $k_{m+1} = 0$ for some potential, but $0 < |k_{m+2}|$ for any potential. By definition, a zero $k_n \in \overline{\mathbb{C}}_-$ of $F$ is called a resonance of $H$. The multiplicity of the resonance is the multiplicity of the corresponding zero of $F$. Of course, the energies are given by $k^2$, but since $k$ is the natural parameter, we will abuse the terminology.

Date: May 2, 2014.

Key words and phrases. Resonances, Lieb-Thirring inequality.
There are only few estimates of resonances. We denote the number of zeros of function $f$ having modulus $\leq r$ by $N(r, f)$, each zero being counted according to its multiplicity. Firstly, Zworski [28] determines the asymptotics of the counting function for resonances:

$$N(r, w) = \frac{2r}{\pi}(\gamma + o(1)) \quad \text{as} \quad r \to \infty,$$

(1.3)

where $[0, \gamma]$ is the convex hull of the support of $q$.

Secondly, let $M_r$ denote the number of resonances and eigenvalues in the half-plane $\{\text{Im } k > t\}, t < 0$. Then there is a constant $C_q$ (see Theorem 3.11 in [8]) such that the following estimate holds true:

$$N_t \leq C_q \left(1 + \int \int_{R^2} e^{4|t||x-y||q(x)||q(x)|} dx \, dy\right),$$

(1.4)

where $C_q$ is some constant depending on $\|q\|$, but not on $t$. Unfortunately, (1.4) is not sharp, since $C_q$ is unknown.

Thirdly, for the Case 2 (at $\gamma = 1$) there are simple estimates from [14]

$$|k_n|e^{-2|\text{Im } k_n|} \leq \|q\|e^{\|q\|}, \quad \text{where} \quad \|q\| = \int_R |q(x)| \, dx,$$

(1.5)

for any $k_n \in \mathbb{C}$. Note that this estimate yields the well-known logarithmic curve for forbidden domain. If, in addition, $q'$ is integrable, then (1.5) will be sharper, and so on.

Define the constant $Q$ by

$$Q = \max\{\|q\|, \|q\|_1\}, \quad \|q\| = \int_R |q(t)| \, dt, \quad \|q\|_1 = \int_R |tq(t)| \, dt.$$  

(1.6)

We present theorem about new estimates of counting functions.

**Theorem 1.1.** Let $H = H_0 + q$, where $q$ is integrable and has a compact support. In Case 2, let supp $q \subset [0, \gamma]$ but in no smaller interval. In Cases 2 and 3, let $\gamma = \sup(\text{supp}(q))$. Let $r > 0$ and $r_1 = r + \frac{1}{2}$. Then the following estimates hold true:

$$N(r, w) \leq \frac{1}{\log 2} \frac{4r_1\gamma}{\pi} + \log(1 + 4r_1) + \frac{9Q}{1 + 4r_1};$$

(1.7)

$$\frac{N(r, \psi_+(0, \cdot)) + N(r, \psi'_+(0, \cdot))}{2} \leq \frac{1}{\log 2} \frac{4r_1\gamma}{\pi} + \log(1 + 4r_1) + \frac{9\|q\|\max\{1, \gamma\}}{1 + 4r_1}. $$

(1.8)

The proof is based on the Jensen formula and standard estimates of the fundamental solutions. The RHS in (1.7) has asymptotics $\frac{2}{\log 2} \frac{4r_1\gamma}{\pi} + o(1))$ as $r \to \infty$. If we compare this asymptotics with (1.3), then we obtain the coefficient $\frac{2}{\log 2}$. It means that the estimate (1.7) is sufficiently sharp.

We present our main result.

**Theorem 1.2.** Let $H = H_0 + q$, where $q$ is integrable and has a compact support. In Case 1, let supp $q \subset [0, \gamma]$ but in no smaller interval. In Cases 2 and 3, let $\gamma = \sup(\text{supp}(q))$. Then for any $p > 1$ the following estimates hold true:

$$\sum_{\pm \text{Im } k_n < 0} \frac{1}{|k_n - 2i|^p} \leq CY_p \left(1 + \frac{\gamma}{\pi} + Q\right),$$

(1.9)
where $C \leq 2^5$ is an absolute constant, $Y_p = \sqrt{\frac{\pi}{\Gamma(\frac{p}{2})}}$ and

$$Q = \begin{cases} \max\{\|q\|, \|q\|_1\} & \text{Case 1} \\ 2\|q\| \max\{1, \gamma\}, & \text{Case 2, 3}. \end{cases} \quad (1.10)$$

**Remark.**
1) The function $Y_p, p > 1$ is strongly monotonic and convex on $(1, \infty)$, since

$$Y_p' < 0, Y_p'' > 0,$$

and satisfies (see more in Lemma 2.4)

$$Y_2 = \pi, \quad Y_p = \left\{ \begin{array}{ll} \frac{1}{p} (\sqrt{2\pi} + O(1/p)) & \text{as } p \to \infty \\ \frac{1}{p-1} (2 + o(1)) & \text{as } p \to 1 \end{array} \right. \quad (1.11)$$

Thus we can control the RHS of (1.9) at $p \to 1$ and for large $p \to \infty$. Note we take $p > 1$, since the asymptotics (1.3) implies the simple fact

$$\sum_{k_n \neq 0} \frac{1}{|k_n|} = \infty,$$

see p. 17 in [18].

2) The RHS of (1.9) depends on 3 crucial parameters: $p > 1$, the diameter of the support of the potential and the magnitude $\|q\|$ of the potential. In Cases 1 and 3 we can not remove 1 in the RHS of (1.9). In Case 2 probably the number 1 should be absent in the RHS of (1.9). In order to explain this we need to add that at $q = 0$ there is a resonance in the Cases 1 and 3, but there is no a resonance in the Case 2.

3) In Case 1 the proof of (1.9) is based on analysis of the function $w$. We use harmonic analysis and the Carleson Theorem (Theorems 1.56 and 2.3.9 in [10]) about Carleson measure. The proof for Cases 2 and 3 is a simple corollary of Case 1.

4) $C$ is the constant from Carleson’s Theorem ([1], [2] and see Theorems 1.56 and 2.3.9, [10]), see also (2.19).

5) In fact, the estimates (1.7), (1.9) give a new global property of resonance stability.

6) It is well-known that the Jost function is a Fredholm determinant, see [12]. In Case 1 we consider the Fredholm determinant $D(k) = \det(I + q(H_0 - k^2)^{-1}, k \in \mathbb{C}_+$. The function $D$ is analytic in the upper half-plane (see e.g. [9]) and satisfies the well-known identity $D(k) = \frac{w(k)}{2ik}$ for all $k$. Thus Theorem 1.2 describes the zeros of the determinants also.

Resonances for the multidimensional case were studied by Melrose, Sjöstrand, and Zworski and other, see [24], [29], [26]) and references therein. We discuss the one dimensional case. A lot of papers is devoted to resonances for the 1D Schrödinger operator, see Froese [9], Simon [25], Zworski [28], Korotyaev [17] and references therein. Different properties of resonances were determined in [8], [25], [28], and [14], [16], [17]. Korotyaev solved the inverse problem for resonances for the Schrödinger operator with a compactly supported potential on the real line [16] and the half-line [14]: (i) the characterization of $S$-matrix in terms of resonances, (ii) a recovering of the potential from the resonances, (iii) the potential is uniquely determined by the resonances (about uniqueness see also [30], [3]).

The ”local resonance” stability problem was considered in [15] for Case 2. Roughly speaking, if $(k_n)_1^\infty$ is a sequence of eigenvalues and resonances of the Schrödinger operator with some compactly supported potential $q$ and $$\sum_{n \geq 1} n^{2\varepsilon}|k_n - k_n^*|^2 < \infty$$ for some sequence $(k_n^*)_1^\infty$ and $\varepsilon > 1$, then $(k_n^*_n)_1^\infty$ is a sequence of eigenvalues and resonances of a Schrödinger operator for some unique real compactly supported potential $q^*$.

Finally we note that some stability results (quite different type, so-called stability for finite data) were obtained by Marletta-Shterenberg-Weikard [22].

A lot of papers are devoted to estimates of eigenvalues in terms of the potential. We recall Lieb-Thirring type inequalities (1.9) from [6] given by:
Let $V$ be a non-negative, unbounded potential, such that the Schrödinger operator $-\Delta + V$ in $L^2(\mathbb{R}^d)$ has an unbounded sequence of discrete eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots$. Then

$$\sum_{n \geq 1} |\lambda_n|^{-\frac{p}{2}} \leq C_p(d) (4p)^{-\frac{p}{2}} \int_{\mathbb{R}^d} V(x)\frac{dx}{x^d}, \quad \text{where } C_p(d) = \frac{\Gamma\left(\frac{p-d}{2}\right)}{\Gamma\left(\frac{d}{2}\right)},$$

(1.12)

for all $p > d$. The inequality is derived from inequalities by Golden [4] and Thompson [27]. It is interesting that (1.9) and (1.12) (at $d = 1$) have the same constant $C_p(1) = Y_p$.

In our case (1.9) the number of resonances is infinite and we consider also the power $p > 1$. Furthermore, some resonances and eigenvalues can be close to zero even for small potentials. For this reason, we sum $|k_n - 2i|^{-p}$ and $|k_n + 2i|^{-p}$. Thus, roughly speaking, (1.9) is a Lieb-Thirring type inequality for the resonances [20].

2. Proof

2.1. Estimates for entire functions. An entire function $f(z)$ is said to be of exponential type if there is a constant $\beta$ such that $|f(z)| \leq \text{const} \cdot e^{\beta|z|}$ everywhere. The infimum of the set of $\beta$ for which such inequality holds is called the type of $f$.

**Definition.** Let $\mathcal{E}_\gamma, \gamma > 0$ denote the set of exponential type functions $f$, which satisfy

$$|f(k)| \geq 2|k| \quad \forall \ k \in \mathbb{R},$$

(2.1)

$$|f(k) - 2ik + f_0| \leq \frac{Q^2}{|k|^2} e^{\gamma(|\text{Im} k| + \text{Im} k) + |\phi'|}, \quad \forall \ k \in \mathbb{C},$$

(2.2)

where the constants $Q = Q(f), f_0$ depend on $f$ and $|k|_1 = \max\{1, |k|\}$.

In the proof of Theorem 1.2 we will need some properties of zeros of $f \in \mathcal{E}_\gamma$ in terms of the Carleson measure. Recall that a positive Borel measure $M$ defined in $\mathbb{C}_-$ is called a Carleson measure if there is a constant $C_M$ such that for all $(r, t) \in \mathbb{R}_+ \times \mathbb{R}$

$$M(D_-(t, r)) \leq C_M r, \quad \text{where } D_-(t, r) \equiv \{z \in \mathbb{C}_- : |z - t| < r\},$$

(2.3)

here $C_M$ is the Carleson constant independent of $(t, r)$.

For an entire function $f$ with zeroes $k_n, n \geq 1$ we define an associated measure by

$$d\Omega(k, f) = \sum_{\text{Im} k_n < 0} \delta(k - k_n + i)du dv, \quad k = u + iv \in \mathbb{C}_-.$$
iii) Let \( k_n, n \geq 1 \) be all zeros of \( f \). Then for each \( p > 1 \) the following estimate holds true:

\[
\sum_{\pm \Im k_n \leq 0} \frac{1}{|k_n - 2t|^p} \leq CY_p C(f),
\]

where \( C \leq 2^p \) is the absolute constant and \( Y_p \) is given by

\[
Y_p = \int_{\mathbb{R}} (1 + x^2)^{-\frac{p}{2}} dx = \sqrt{\pi} \frac{\Gamma\left(\frac{p}{2}\right)}{\Gamma\left(\frac{p-1}{2}\right)} \quad \forall \ p > 1.
\]

**Proof.**

i) Recall the Jensen formula (see p. 2 in [13]) for an entire function \( F \):

\[
\log |F(0)| + \int_0^r \frac{N(t,F)}{t} \, dt = \frac{1}{2\pi} \int_0^{2\pi} \log |F(re^{i\phi})| \, d\phi,
\]

Rewrite \( f \) in the form

\[
f = 2ik - f_0 + f_+, \quad \text{where} \quad |f_+(k)| \leq \frac{Q^2}{|k|} e^{\left(\frac{Q}{|k|}\right)} e^{2\gamma v}, \quad v_-(k) = \frac{\Im k - \Im k}{2},
\]

for all \( k \in \mathbb{C} \). Consider the case \( t \in \mathbb{R} \setminus \left[-\frac{1}{2}, \frac{1}{2}\right] \). Let \( z = t + k, k = re^{i\phi} \). Then (2.2) and (2.10) imply

\[
|f(t+k)| = |2it + 2ik - f_0 + f_+(z)| \leq 2|t| \left(1 + \frac{r}{|t|}\right) \left(1 + \frac{|f_0| + |f_+(z)|}{2(|t| + r)}\right)
\]

\[
\leq 2|t| \left(1 + \frac{r}{|t|}\right) e^{2\gamma v} \left(1 + \frac{|f_0| + Q^2 |z|^{-1}}{2(|t| + r)}\right)
\]

\[
\leq 2|t| \left(1 + \frac{r}{|t|}\right) e^{2\gamma v} \left(1 + \frac{|f_0|}{2(|t| + r)}\right) \left(1 + \frac{Q^2}{2(|t| + r)}|z|_1\right).
\]

Let \( N(s) = N(s, f(t+\cdot)) \). Substituting this into (2.9) we obtain for the function \( F(k) = f(t+k), k = re^{i\phi}, z = t + re^{i\phi} \):

\[
\log |f(t)| + \int_0^r \frac{N(s)}{s} ds = \frac{1}{2\pi} \int_0^{2\pi} \log |f(z)| \, d\phi
\]

\[
\leq \log 2|t| + \log \left(1 + \frac{r}{|t|}\right) + \log \left(1 + \frac{|f_0|}{2(|t| + r)}\right) + \frac{1}{2\pi} \int_0^{2\pi} 2\gamma v_- \, d\phi + X(t,r),
\]

where \( X(t,r) = \frac{1}{2\pi} \int_0^{2\pi} \frac{Q}{|z|_1} + \log \left(1 + \frac{Q^2}{2(|t| + r)|z|_1}\right) \, d\phi \),

where \( 2v_- = r(|\sin \phi| - |\sin \phi|) \) and the simple integration yields

\[
\frac{1}{2\pi} \int_0^{2\pi} 2\gamma v_- \, d\phi = \frac{r\gamma}{2\pi} \int_0^{2\pi} \left(|\sin \phi| - |\sin \phi|\right) \, d\phi = \frac{2\gamma r}{\pi}.
\]

Substituting the estimate \( \frac{|f(t)|}{2|t|} \geq 1 \) into (2.12) together with the simple one

\[
\int_0^r \frac{N(s)}{s} ds \geq N(r/2) \int_{r/2}^r \frac{ds}{s} = N(r/2) \log 2,
\]

we obtain

\[
N(r/2) \log 2 \leq \frac{2r\gamma}{\pi} + \log \left(1 + \frac{r}{|t|}\right) + \frac{|f_0|}{2(|t| + r)} + X(t,r).
\]
Let \( t = \frac{1}{r} \) and \( r_1 = r + \frac{1}{2} \) for any \( r > 0 \). Then
\[
\{ |k| < r \} \subset \{ |k - \frac{1}{2}| < r_1 \}, \quad |z| = |(1/2) + k| \geq \frac{1}{2} \left( 1 + |k| - (1/2) \right) = \frac{1}{4} (1 + 2|k|)
\]
and (2.13) give
\[
\mathcal{N}(r, f) \leq \mathcal{N}(r_1, f(\frac{1}{2} + \cdot)) \leq \frac{4r_1\gamma}{\pi} + \log(1 + 4r_1) + \frac{|f_0|}{(1 + 4r_1)} + \mathcal{X}(t, r),
\]
where \( \mathcal{X}(t, r) \leq \frac{1}{2\pi} \int_0^{2\pi} \left[ \frac{4Q}{(1 + 4r_1)} + \log \left( 1 + \frac{4Q^2}{(1 + 4r_1)^2} \right) \right] d\phi \leq \frac{8Q}{(1 + 4r_1)}.
\]
which yields (2.5).

ii) Let \( r \leq 1, t \in \mathbb{R} \). Then by the construction of \( \Omega(\cdot, f) \), we obtain \( \Omega(D_-(t, r), f) = 0 \).

In order to show (2.6) for the case \( r > 1, t \in \mathbb{R} \), we need to consider two cases:

Firstly, let \( r > 1, t \in \mathbb{R} \setminus [-\frac{1}{2}, \frac{1}{2}] \). Then due to (2.14) and (2.12), the measure \( \Omega(\cdot, f) \) satisfies
\[
\Omega(D_-(t, r), f) \leq \mathcal{N}(r, f(t + \cdot)) \leq \frac{1}{\log 2} \left( \frac{4r\gamma}{\pi} + 4r + |f_0| + Q + \log(1 + Q)^2 \right) \leq C_1 r, \tag{2.16}
\]
where
\[
C_1 = \frac{4}{\log 2} \left( \frac{\gamma}{\pi} + 1 + \frac{|f_0| + 3Q}{4} \right).
\]

Secondly, let \( r > 1, t \in [0, \frac{1}{2}] \). The proof for the case \( t \in [-\frac{1}{2}, 0] \) is similar. For two disks \( D(t, r) \) and \( D(\frac{1}{2}, r_1), r_1 = r + \frac{1}{2} \) we have
\[
D(t, r) \cap \{ \text{Im} k \leq -1 \} \subset D(\frac{1}{2}, r_1) \cap \{ \text{Im} k \leq -1 \}, \quad r_1 = r + \frac{1}{2} \leq \frac{3r}{2}.
\]

Then due to (2.5), the measure \( \Omega(\cdot, f) \) satisfies
\[
\Omega(D_-(t, r), f) \leq \Omega(D_-(1/2, r_1), f) \leq \mathcal{N}(r_1, f(\frac{1}{2} + \cdot)) \leq C_1 r_1 \leq \frac{3}{2} C_1 r. \tag{2.18}
\]

Thus \( \Omega(\cdot, f) \) is a Carleson measure with the Carleson constant \( C(f) = \frac{3}{2} C_1 \), where \( C_1 \) is given by (2.16).

iii) Consider the case \( \text{Im} k_n \leq 0 \). The proof for the case \( \text{Im} k_n > 0 \) is similar, even simpler. In order to show (2.7) we recall the Carleson result (see p. 63, Theorem 3.9, [10]):

Let \( F \) be analytic on \( \mathbb{C}_- \). For \( 0 < p < \infty \) we say \( F \in \mathcal{H}_p = \mathcal{H}_p(\mathbb{C}_-) \) if
\[
\sup_{y < 0} \int_{\mathbb{R}} |F(x + iy)|^p dx = \| F \|_{\mathcal{H}_p}^p < \infty.
\]

Note that the definition of the Hardy space \( \mathcal{H}_p \) involve all \( y < 0 \), instead of small only value of \( y \), like say, \( y \in (-1, 0) \). We define the Hardy space \( \mathcal{H}_p \) for the case \( \mathbb{C}_- \), since below we work with functions on \( \mathbb{C}_- \).

If \( M \) is a Carleson measure and satisfies (2.5), then the following estimate holds:
\[
\int_{\mathbb{C}_-} |F|^p dM \leq CC_M \| F \|_{\mathcal{H}_p}^p \quad \forall \ F \in \mathcal{H}_p, \ p \in (0, \infty), \tag{2.19}
\]
where \( C_M \) is the so-called Carleson constant from (2.5) and \( C \leq 2^5 \) is an absolute constant.
In order to prove \((2.7)\) we take the functions \(F(k) = \frac{1}{k-1}\). Then estimates \((2.19), (2.6)\) yield
\[
\sum_{\text{Im} k_n \leq 0} \frac{1}{|k_n - 2i|^p} = \int_{\mathbb{C}_-} |F(\lambda)|^p d\Omega \leq CC(f) \|F\|_{\mathcal{H}^p}^p, \quad p \in (1, \infty), \quad (2.20)
\]

\(C\) is an absolute constant and \(C(f)\) is defined in \((2.6)\). Here we have the simple identity
\[
\|F\|_{\mathcal{H}^p}^p = \int_{\mathbb{R}} \frac{dt}{t - i} = \int_{\mathbb{R}} \frac{dt}{(t^2 + 1)^{\frac{p}{2}}} = Y_p. \quad (2.21)
\]

The function \(Y\) is studied in Lemma \(2.4\). Combine \((2.20)\) and \((2.21)\) we obtain \((2.7)\). □

### 2.2. Estimates of resonances

We consider Case 1 and without loss of generality let \(supp q \subset [0, \gamma]\). The Schrödinger equation
\[
-f'' + q(x)f = k^2 f, \quad k \in \mathbb{C} \setminus \{0\}, \quad (2.22)
\]
has unique solutions \(\psi_{\pm}(x, k)\) such that \(\psi_{\pm}(x, k) = e^{\pm ikx}, \quad x > \gamma\) and \(\psi_{-}(x, k) = e^{-ikx}, \quad x \leq 0\). Outside of the support of \(q\) any solutions of \((2.22)\) have to be linear combinations of \(e^{\pm ikx}\). Functions \(\psi_{\pm}(x, \cdot), \psi'_{\pm}(x, \cdot), x \in \mathbb{R}\) are entire. We define the functions \(a, w, s\) by
\[
w(k) = 2ika(k) = \{\psi_{-}(\cdot, k), \psi_{+}(\cdot, k)\}, \quad s(k) = \{\psi_{+}(\cdot, k), \psi_{-}(\cdot, -k)\}, \quad (2.23)
\]
and \(\{f, g\} = fg' - f'g\) denotes the Wronskian. The functions \(w(k), s(k)\) are entire and the following asymptotic estimates hold true:
\[
a(k) = 1 + O(k^{-1}) \quad \text{as} \quad |k| \to \infty, \quad k \in \mathbb{C}_+. \quad (2.24)
\]
The scattering matrix for the operators \(H, H_0 = -\frac{d^2}{dx^2}\) is given by
\[
S(k) = \begin{pmatrix} a(k)^{-1} & r_-(k) \\ r_+(k) & a(k)^{-1} \end{pmatrix}, \quad r_{\pm} = \frac{s(\pm k)}{w(k)}, \quad k \in \mathbb{R}, \quad (2.25)
\]
where \(1/a\) is the transmission coefficient and \(r_{\pm}\) are the reflection coefficients. The matrix \(S(k), k \in \mathbb{R}\) is unitary, which yields
\[
|w(k)|^2 = 4k^2 + |s(k)|^2, \quad \forall \quad k \in \mathbb{R}. \quad (2.26)
\]
The solution \(\psi_{+}\) of \((2.22)\) satisfies the following equation
\[
\psi_{+}(x, k) = e^{ikx} - \int_x^\gamma \frac{\sin[k(x-t)]}{k} q(t) \psi_{+}(t, k) dt \quad \forall \quad (x, k) \in [0, \gamma] \times \mathbb{C}. \quad (2.27)
\]
It is well known that equation \((2.27)\) has a unique solution. Due to \((2.27)\) the function \(y(x, k) = e^{-ikx} \psi_{+}(x, k)\) satisfies the integral equation
\[
y(x, k) = 1 + \int_x^\gamma G(t - x, k)q(t)y(t, k) dt \quad G(t, k) = \frac{\sin kt}{k}e^{ikt}, \quad (2.28)
\]
for all \((x, k) \in [0, \gamma] \times \mathbb{C}\). We have the standard iterations
\[
y(x, k) = 1 + \sum_{n \geq 1} y_n(x, k), \quad y_n(x, k) = \int_x^\gamma G(t - x, k)q(t)y_{n-1}(t, k) dt, \quad y_0 = 1. \quad (2.29)
\]
We need some properties of the functions introduced above.
Lemma 2.2. Let \( q \in L^1(\mathbb{R}) \) and \( \text{supp} \, q \subset [0, \gamma] \). Then the functions \( w, s \) and \( \psi_+(x, \cdot), \psi'_+(x, \cdot), x \in \mathbb{R} \) are entire and we have
\[
|y_n(x, k)| \leq \frac{h^n}{n!} e^{2(\gamma-x)v_-}, \quad v_- = \begin{cases} 0, & k \in \mathbb{C}_+ \\ \Im k, & k \in \mathbb{C}_- \end{cases}
\tag{2.30}
\]
for any \( n \geq 1, (x, k) \in [0, \gamma] \times \mathbb{C} \), where
\[
h = \frac{Q}{|k|_1}, \quad Q = \max\{|q|, |q||1\}, \quad |k|_1 = \max\{1, |k|\};
\tag{2.31}
\]
where \( |q| = \int_0^\gamma |q(t)|dt \) and \( |q||1 = \int_0^\gamma t|q(t)|dt \), and
\[
|y(x, k)| \leq e^{2(\gamma-x)v_- + h}, \quad |y(x, k) - 1| \leq he^{2(\gamma-x)v_- + h},
\tag{2.32}
\]
Moreover, \( w \in \mathcal{E}_\gamma \) and satisfies for any \( k \in \mathbb{C} \):
\[
w(k) = i2k - q_0 + w_s(k), \quad w_s(k) = -\int_0^\gamma q(t)(y(t, k) - 1)dt,
\tag{2.33}
\]
where \( q_0 = \int_0^\gamma q(t)dt \).

**Proof.** The function \( G(t, k) = \frac{\sin kt}{k} e^{ikt} \) satisfy
\[
|G(t, k)| \leq \frac{e^{2v_- t}}{|k|_1} \begin{cases} 1 & \text{if } |k| \geq 1 \\ t & \text{if } |k| < 1 \end{cases}, \quad t \geq 0, \quad k \in \mathbb{C},
\tag{2.35}
\]
Consider the first case in (2.35): \(|k| \geq 1\), the proof for the second case \(|k| < 1\) is similar. Substituting the estimate in (2.35) for \(|k| \geq 1\) into the identity
\[
y_n(x, k) = \int_{x = t_0< t_1< t_2< \ldots< t_n} \left( \prod_{1 \leq j \leq n} G(t_j - t_{j-1}, k)q(t_j) \right)dt_1dt_2\ldots dt_n,
\]
we obtain
\[
|y_n(x, k)| \leq \frac{1}{|k|_1^n} \int_{x = t_0< t_1< t_2< \ldots< t_n} \left( \prod_{1 \leq j \leq n} e^{2v_-(t_j - t_{j-1})}|q(t_j)| \right)dt_1dt_2\ldots dt_n
\]
\[
= \frac{e^{-2v_- x}}{|k|_1^n} \int_{x = t_0< t_1< t_2< \ldots< t_n} \left( \prod_{1 \leq j \leq n} |q(t_j)| \right) e^{2v_- t_0} dt_1dt_2\ldots dt_n
\tag{2.36}
\]
\[
\leq \frac{e^{2(\gamma-x)v_-}}{|k|_1^n} \int_{x = t_0< t_1< t_2< \ldots< t_n} |q(t_1)q(t_2)\ldots q(t_n)|dt_1dt_2\ldots dt_n \leq e^{2(\gamma-x)v_-} \frac{|q||n}{n! |k|_1^n}.
\]
This shows that for each \( x \geq 0 \) the series (2.29) converges uniformly on any bounded subset of \( \mathbb{C} \). Each term of this series is an entire function. Hence the sum is an entire function. Summing the majorants we obtain estimates (2.32). Thus the functions \( w, s \) and \( \psi_+(x, \cdot), \psi'_+(x, \cdot), x \in \mathbb{R} \) are entire.
The Schrödinger equation \( \tilde{\psi} \) such that

\[
\psi_+(0, k) = ik - \int_0^\gamma \cos[kt] q(t) \psi_+(t, k) dt,
\]

\[
\tilde{\psi}_+(0, k) = ik + i \int_0^\gamma \sin[kt] q(t) \psi_+(t, k) dt,
\]

which yields

\[
w(k) = \psi_- (0, k) \psi_+^\prime (0, k) - \psi_+ (0, k) \psi_- (0, k) = i k \psi_+ (0, k) + \psi_+^\prime (0, k)
\]

\[
= 2ik - q_0 + w_+(k), \quad w_+(k) = - \int_0^\gamma q(t) (y(t, k) - 1) dt.
\]

Due to (2.32) we have the estimate

\[
|w_+(k)| \leq \int_0^\gamma |q(t)| h e^{2(\gamma - t)\nu + h} dt \leq h \|q\| e^{2\gamma \nu + h},
\]

which is (2.34). Together with (2.26) this implies that \( w \in \mathcal{E}_\gamma \).

The identity (2.34) and estimate (2.34) give

\[
w(k) = i 2k - q_0 + w_+(k), \quad |w_+(k)| \leq \frac{Q^2}{|k|} \frac{Q - 2}{|k|} e^{\frac{Q - 2}{|k|}}, \quad \forall k \in \mathbb{C},
\]

where \( q_0 = \int_0^\gamma q(t) dt \), \( Q = \max\{|q|, |q|_1\} \).

We show the relations between the Cases 1, 2 and 3. In order to show this we use a standard trick. For \( q \in L^1(\mathbb{R}_+) \) with \( \text{supp} \ q \subset [0, \gamma] \) we define the Schrödinger operator \( \tilde{H} \) acting in \( L^2(\mathbb{R}) \) with an even compactly supported potential \( \tilde{q} \) given by

\[
\tilde{H} = - \frac{d^2}{dx^2} + \tilde{q}, \quad \tilde{q}(x) = q(|x|), \ x \in \mathbb{R}.
\]

The Schrödinger equation \( -f'' + \tilde{q}(x) f = k^2 f, \quad k \in \mathbb{C} \setminus \{0\} \), has unique solutions \( \tilde{\psi}_\pm (x, k) \) such that

\[
\tilde{\psi}_+ (x, k) = e^{ikx}, \quad x \geq \gamma \quad \text{and} \quad \tilde{\psi}_- (x, k) = e^{-ikx}, \quad x \leq -\gamma.
\]

Note that the symmetry of the potential \( \tilde{q} \) yields

\[
\tilde{\psi}_+ (x, k) = \tilde{\psi}_+ (-x, k), \quad \forall x \in [0, \gamma].
\]

This implies that the Wronskian \( \tilde{w}(k) \) for the potential \( \tilde{q} \) satisfies

\[
\tilde{w}(k) = \{ \tilde{\psi}_+ (x, k), \tilde{\psi}_- (x, k) \} \big|_{x=0} = 2 \psi_+ (0, k) \psi_+^\prime (0, k).
\]

where \( \psi_+ (0, k) \) and \( \psi_+^\prime (0, k) \) are the Jost functions for the Cases 2 and 3 respectively with the potential \( q \). This identity is very useful. For example, if we have estimate (1.9) for the Case 1, then (2.42) gives the estimate (1.9) for the Case 2 and 3.

We obtain the estimate of the Wronskian \( \tilde{w}(k) \). We need to define the potential \( q_1 = \tilde{q}(x - \gamma) \), which has the support \( \text{supp} \ q_1 \subset [0, 2\gamma] \). Using Lemma (2.22) and (2.33), (2.34) we obtain

\[
\tilde{w}(k) = i 2k - 2q_0 + w_+(k), \quad |w_+(k)| \leq \|q_1\| \frac{Q^2}{|k|} e^{2\gamma \nu + h}.
\]
where the corresponding constant $\bar{Q}$ is given by
\[
\bar{Q} = \max\{\|q_1\|, \|q_1\|_1\} = 2\|q\| \max\{1, \gamma\},
\] (2.44)
since
\[
\|q_1\| = \int_0^{2\gamma} |q_1(x)|\,dx = 2\|q\|, \quad \|q_1\|_1 = \int_0^{2\gamma} x|q(x - \gamma)|\,dx = \int_{-\gamma}^{\gamma} (t + \gamma)|\tilde{q}(t)|\,dt = 2\gamma\|q\|.
\]

We describe few facts about the resonances in the disk $\{|k| < r\}$.

**Proposition 2.3.** Let $q \in L^1(\mathbb{R})$ and $\text{supp} \, q \subset [0, \gamma]$ for some $\gamma > 0$.

i) Let $r > 0$ and let $q$ satisfy
\[
\|q\|(1 + h e^{2\gamma r + h}) < 2r, \quad \text{where} \quad h = \frac{\max\{\|q\|, \|q\|_1\}}{\max\{1, r\}}.
\] (2.45)
Then the function $w$ has only one simple zero in the disk $\{|k| < r\}$.

ii) Let $1 \leq r \leq \frac{1}{2\gamma}$ and let $2\|q\| \leq r$. Then the function $w$ has only one simple zero in the disk $\{|k| < r\}$.

**Proof.** i) Let $w_0 = i2k$ and $|k| = r$. Then the estimates (2.33), (2.34) imply
\[
|w(k) - w_0(k)| \leq \|q\|(1 + h e^{2\gamma r + h}) = |w_0(k)| C_0, \quad C_0 = \frac{\|q\|}{2r} (1 + he^{2\gamma r + h}).
\] (2.46)
Hence, if $C_0 < 1$, then by Rouche’s theorem, $w(k)$ has only one simple zero, as $w_0 = 2ik$ in the disk $\{|k| < r\}$.

ii) If we assume that $f(h) = h + h^2 e^{1+h} < 2$, then (2.45) holds true.

We have that the function $f$ is increasing. We get the estimates $f(\frac{1}{2}) < 2$, since $h = \frac{\|q\|}{r} \leq \frac{1}{2}$, which yield that $w$ has only one simple zero in the disk $\{|k| < r\}$.

**Remark.** 1) This proposition shows that for any $\gamma, r > 0$ and sufficiently small $\|q\|$ the function $w$ has only one simple zero in the disk $\{|k| < r\}$.

2) The number of zeros of $w$ in the disk with large $r$ depends on the diameter $\gamma$. But it is very interesting to determine the asymptotics of $N(r, w)$ as $r \to \infty$. This problem remains open for long period.

3) We need to say that (1.7) holds also true for the Case 2 and 3. Then (1.9) for Case 1 yields (1.9) for Cases 2 and 3.

**Proof of Theorem 1.1.** Lemma 2.2 gives that the function $w \in \mathcal{E}_\gamma$.

We apply the estimate (2.5) to the function $w$. Thus due to the estimate (2.40) and the simple fact $|q_0| \leq Q$, we obtain (1.7).

We show (1.8). The identity (2.42) gives $N(r, \psi_+(0, \cdot)) + N(r, \psi'_+(0, \cdot)) = N(r, \tilde{w})$, where $\tilde{w}$ is the Wronskian for the potential $\tilde{q}$ given by (2.41) and $\tilde{w}$ satisfies (2.43), (2.44). Then these arguments and the estimate (1.7) implies (1.8).

**Proof of Theorem 1.2.** We consider Case 1 and let $\text{supp} \, q \subset [0, \gamma]$. By Lemma 2.2 the function $w$ belongs to $\mathcal{E}_\gamma$ with $Q = \max\{\|q\|, \|q\|_1\}$ and satisfies (2.40). Thus the estimate (2.7), (2.6) from Theorem 2.1 give (1.9) for Case 1.

Consider Cases 2 and 3. Then we have the identity $\tilde{w}(k) = 2\psi_+(0, k)\psi'_+(0, k)$, where $\tilde{w}$ is the Wronskian for the potential $\tilde{q}$ given by (2.41) (see (2.42)). Then (1.9) for $\tilde{q}$ (Case 1) together with (2.44) implies (1.9) for Cases 2 and 3.
Lemma 2.4. The function \( Y_p = \int_\mathbb{R} (1 + x^2)^{-\frac{p}{2}} dx, p > 1 \) is strongly monotonic and convex on \((1, \infty)\), since \( Y'_p < 0, Y''_p > 0 \), and satisfies (1.11) and the identity (2.8), i.e., \( Y_p = \sqrt{\pi \Gamma(\frac{p}{2})}/\Gamma(\frac{p}{2}) \) for all \( p > 1 \).

Proof. We rewrite \( Y_p \) in the form \( Y_p = \int_\mathbb{R} e^{-p f(x)} dx \), where \( f(x) = \frac{1}{2} \log(1 + x^2) \). This yields \( Y_p' = -\int_\mathbb{R} f(x)e^{-p f(x)} dx < 0 \) and \( Y_p'' = \int_\mathbb{R} f'(x)e^{-p f(x)} dx > 0 \). We rewrite \( Y_p \) in another form:

\[
Y_p = 2 \int_0^\infty (1 + x^2)^{-\frac{p}{2}} dx = \int_0^\infty t^{-\frac{p}{2}}(1 + t)^{-\frac{p}{2}} dt = \frac{\Gamma(z)}{\Gamma(z + \frac{p}{2})}, \quad z = \frac{p - 1}{2},
\]

where \( \Gamma \) is the Gamma function, see identities (2), (5) in Section 1.5 in [7].

Asymptotics \( Y_p = \frac{\Gamma(z)}{\Gamma(z + \frac{p}{2})} = z^{-\frac{1}{2}}(1 + O(\frac{1}{z})) \) as \( z \to \infty \), see (5) in Sect. 1.18 in [7] give (1.11) as \( p \to \infty \). Recall that \( \Gamma \) is meromorphic in \( \mathbb{C} \) with simple poles \( 0, -1, -2, \ldots \). Then the identities \( \Gamma(1 + z) = z \Gamma(z) \) and \( \Gamma(1) = 1, \Gamma(1/2) = \sqrt{\pi} \) yield (1.11) as \( p \to 1 \):

\[
Y_p = \sqrt{\pi} \frac{\Gamma(z)}{\Gamma(z + \frac{p}{2})} = \frac{\sqrt{\pi} \Gamma(1 + z)}{z \Gamma(z + \frac{p}{2})} = \frac{1 + O(z)}{z} \quad \text{as} \quad z = \frac{p - 1}{2} \to 0.
\]

Acknowledgments. I am grateful to Michael Loss, Atlanta, Grigori Rozenblum, Göttingen, and Sergey Morozov, Munich, for useful comments about Lieb-Thirring inequalities and Elliot Lieb, Princeton, for the reference [6]. Moreover, I am also grateful to Alexei Alexandrov (St. Petersburg) for stimulating discussions about the Carleson Theorem and about the absolute constant \( C \) in (2.19).

This work was supported by the Ministry of education and science of the Russian Federation, state contract 14.740.11.0581 and the RFFI grant ”Spectral and asymptotic methods for studying of the differential operators” No 11-01-00458.

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