Study of quark distribution amplitudes of $1S$ and $2S$ heavy quarkonium states

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Abstract

In this paper, the quark distribution amplitudes of $1S$ and $2S$ heavy quarkonium states are studied in terms of Gaussian-type wave functions. The transverse momenta $p_\perp$ integrals of the formulae for the decay constant are performed analytically. Then the quark distribution amplitudes are obtained. In addition, the $\xi$-moments are also calculated. After fixing the relevant parameters appearing in the quark distribution amplitude, the curves of the quark distribution amplitude for $1S$ and $2S$ heavy quarkonium states are plotted. Finally, the numerical results of this approach are compared with the other theoretical predictions.

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I. INTRODUCTION

Since the discoveries of $J/\psi$ and $\Upsilon$ more than thirty years ago, a great deal of information on heavy quarkonium levels and their transitions has been accumulated \[1\]. On one hand, the known levels of charmonium and bottomonium include $1S$, $1P$, $1D$, $2S$, $2P$, $2D$, $3S$, and $4S$ states, with the labels $S$, $P$, $D$ corresponding to relative orbital angular momentum $L = 0, 1, 2$ between quark and antiquark. On the other hand, the numerous transitions between heavy quarkonium states are classified into strong and radiative decays, which shed light on aspects of quantum chromodynamics (QCD) in both the perturbative and the non-perturbative regimes (for a recent review see \[2\]). Recently the more precise $\psi'$, $\eta_c'$, and $\eta_b$ mass measurements have been reported \[3, 4, 5, 6\], and the errors of their relevant decay widths have decreased \[1\]. A thorough understanding of their properties, such as their quark distribution amplitudes, which are the universal non-perturbative objects, will be of great benefit when analyzing the hard exclusive processes with heavy quarkonium production.

It has been known that heavy quarkonium is relevant for a non-relativistic treatment \[7\]. Although non-relativistic QCD (NRQCD) is a powerful theoretical tool for separating the high energy modes from low energy contributions, in most cases the calculation of low energy hadronic matrix elements has relied on model-dependent non-perturbative methods. Therefore, various methods have been employed in heavy quarkonium physics, such as lattice QCD, quark-potential model, etc. \[2\]. The light-front quark model offers many insights into the internal structures of the bound states. In this study, heavy quarkonium is explored within a quark model on the light front. Light-front QCD is a promising analytic method for solving the non-perturbative problems of hadron physics \[8\], and may be the only possible method by which the low energy quark model and the high energy parton model can be reconciled. For hard processes with a large momentum transfer, light-front QCD reduces to perturbative QCD (pQCD) which factorizes the physical quantity into a convolution of the hard scattering kernel and the quark distribution amplitudes.

The basic ingredient in light-front QCD is the relativistic hadron wave function. It generalizes the distribution amplitudes by including the transverse momentum distributions, and it contains all the information of a hadron from its constituents. The hadronic quantities are represented by the overlap of wave functions and can be derived in principle. The light-front wave function is manifestly a Lorentz invariant as it is expressed in terms of the
internal momentum fraction variables which are independent of the total hadron momentum. Moreover, the fully relativistic treatment of quark spins and the center-of-mass motion can be carried out using the so-called Melosh rotation \[9, 10, 11\]. This treatment has been successfully applied to calculate many phenomenologically important meson decay constants and hadronic form factors \[12, 13, 14, 15, 16, 17\]. In addition, the covariant light-front approach \[13, 16\] has also been applied to ground-state s-wave mesons, which include \(1^1S_0\) pseudoscalar mesons \(\eta_c, \eta_b\) and \(1^3S_1\) vector mesons \(J/\psi, \Upsilon\) (see Ref. \[18\]).

As mentioned above, the quark distribution amplitude or light-cone wave function (LCWF) absorbs the non-perturbative dynamics and is the key ingredient of any hard exclusive process with hadron production. In the literature, there are many theoretical studies \[19, 20, 21, 22, 23, 24, 25, 26\] of this issue. The main purpose of this study is the calculation of the quark distribution amplitudes of pseudoscalar and vector heavy quarkonium states by integrating the transverse momenta of momentum distribution amplitudes within the light-front approach. As to the so-called \(\xi\)-moments which parameterize the quark distribution amplitudes, they are also calculated analytically.

The remainder of this paper is organized as follows. Section II comprises brief reviews of the light-front framework and the light-front analysis for the decay constants of pseudoscalar \((P)\) and vector \((V)\) mesons; the processes \(V \to P\gamma\) are given. In Section III, the quark distribution amplitudes and the \(\xi\)-moments are calculated. In Section IV, the numerical results and discussions are presented. Finally, the conclusions are given in Section V.

II. FORMALISM OF COVARIANT LIGHT-FRONT APPROACH

In heavy quarkonium, the valence quarks have equal masses, \(m_1 = m_2 = m\), with \(m\) the mass of heavy quark \(c\) or \(b\). Thus, the formulae in this section lead to simplifications for the quarkonium system.

The momentum of a particle is given in terms of the light-front component by \(k = (k^-, k^+, k_\perp)\) where \(k^\pm = k^0 \pm k^3\) and \(k_\perp = (k^1, k^2)\), and the light-front vector is written as \(\tilde{k} = (k^+, k_\perp)\). The longitudinal component \(k^+\) is restricted to positive values, i.e., \(k^+ > 0\) for the massive particle. In this way, the physical vacuum of light-front QCD is trivial except for the zero longitudinal momentum modes (zero-mode). A meson with total momentum \(P\) and two constituents, quark and anti-quark whose momenta are \(p_1\) and \(p_2\), respectively
will be studied. In order to describe the internal motion of the constituents, it is crucial to introduce the intrinsic variables \((x, p_\perp)\) through
\[
\begin{align*}
p_1^+ &= xP^+, & p_{1\perp} &= xP_\perp + p_\perp; \\
p_2^+ &= (1 - x)P^+, & p_{2\perp} &= (1 - x)P_\perp - p_\perp,
\end{align*}
\]
where \(x\) is the light-front momentum fraction. The invariant mass \(M_0\) of the constituents and the relative momentum in \(z\) direction \(p_z\) can be written as
\[
M_0^2 = \frac{p_{\perp}^2 + m^2}{x(1 - x)}, \quad p_z = \left( x - \frac{1}{2} \right) M_0.
\]
The invariant mass \(M_0\) of \(q\bar{q}\) is generally different from the mass \(M\) of meson which satisfies \(M^2 = P^2\). This is due to the fact that the meson, quark and anti-quark cannot be on-shell simultaneously. The momenta \(p_{\perp}\) and \(p_z\) constitute a momentum vector \(\vec{p} = (p_{\perp}, p_z)\), which represents the relative momenta in the transverse and \(z\) directions, respectively. The energy of the quark and antiquark \(e_1 = e_2 \equiv e\) can be obtained from their relative momenta,
\[
e = \sqrt{m^2 + p_{\perp}^2 + p_z^2}.
\]
It is straightforward to find that
\[
x = \frac{e - p_z}{2e}, \quad e = \frac{M_0}{2}.
\]
As shown in Ref. [27], one can pass to the light-front approach by integrating the \(p^-\) component of the internal momentum in the covariant Feynman momentum loop integrals. The Feynman rules for the meson-quark-anti-quark vertices were then needed to calculate the amplitudes which related to the decay constant and M1 transition. In the following formulations, we follow the notation in [16]. The vertices \(\Gamma_M\) for the incoming meson \(M\) are given as
\[
\begin{align*}
H_P \gamma_5 & \quad \text{for } P, \\
iH_V \left[ \gamma_\mu - \frac{1}{W_V} (p_1 - p_2)_\mu \right] & \quad \text{for } V.
\end{align*}
\]
The Feynman rules which are derived from quantum field theory and Eq. (3) can be used to write down the relevant matrix elements. The integration of the \(p^-\) component will force the quark or anti-quark to be on its mass shell. The specific form of the covariant vertex
function for the on-shell (anti)quark can be determined by comparing it to the conventional
vertex function, which can be written as

\[
| M(P_s^{2S+1} L_J, J_z) \rangle = \int \{d^3p_1\} \{d^3p_2\} \frac{2(2\pi)^3}{2} \delta^3(\tilde{P} - p_1 - \tilde{p}_2) \times \sum_{\lambda_1, \lambda_2} \Psi^{SS_z}(\tilde{p}_1, \tilde{p}_2, \lambda_1, \lambda_2) \ | q_1(p_1, \lambda_1) \bar{q}_2(p_2, \lambda_2) \rangle. \tag{6}
\]

The momentum-space wave function \( \Psi^{SS_z} \) can be expressed as

\[
\Psi^{SS_z}(\tilde{p}_1, \tilde{p}_2, \lambda_1, \lambda_2) = R_{\lambda_1 \lambda_2}^{SS_z}(x, p_\perp) \phi(x, p_\perp), \tag{7}
\]

where \( \phi(x, p_\perp) \) is the light-front momentum distribution amplitude for the s-wave meson
and can be chosen to be normalizable, i.e., it satisfies

\[
\int \frac{dxd^2p_\perp}{2(2\pi)^3} | \phi(x, p_\perp) |^2 = 1, \tag{8}
\]

and \( R_{\lambda_1 \lambda_2}^{SS_z} \) constructs a state of definite spin \( (S, S_z) \) out of light-front helicity \((\lambda_1, \lambda_2)\) eigenstates. In practice, it is more convenient to use the covariant form for \( R_{\lambda_1 \lambda_2}^{SS_z} \): \(12, 13\):

\[
R_{\lambda_1 \lambda_2}^{SS_z}(x, k_\perp) = \frac{1}{\sqrt{2M_0}} \bar{u}(p_1, \lambda_1) \Gamma v(p_2, \lambda_2), \tag{9}
\]

where

\[
\Gamma = \gamma_5 \quad \text{(pseudoscalar, } S = 0), \tag{10}
\]

\[
\Gamma = - \sphericalangle(S_z) + \frac{\epsilon \cdot (p_1 - p_2)}{M_0 + 2m} \quad \text{(vector, } S = 1). \tag{11}
\]

All details are shown in Appendix A of Ref. \cite{16}. The function \( H_{P,V} \) and the parameter \( W_V \)
are reduced to \( h_{P,V} \) and \( w_V \), respectively, and they are written by

\[
h_P = h_V = (M^2 - M_0^2) \sqrt{\frac{x(1-x)}{N_c}} \frac{1}{\sqrt{2M_0}} \phi(x, p_\perp),
\]

\[
w_V = M_0 + 2m. \tag{12}
\]

In principle, \( \phi(x_2, p_\perp) \) is obtained by solving the light-front QCD bound-state equation
\( H_{LF} | \Psi \rangle = M | \Psi \rangle \), which is the familiar Schrödinger equation in ordinary quantum mechanics
and \( H_{LF} \) is the light-front Hamiltonian. However, except in some simple cases, the full
solution has remained a challenge. There are several popular phenomenological light-front
momentum distribution amplitudes that have been employed to describe various hadronic
structures in the literature. A widely used one is the Gaussian-type chosen here for the 1S heavy quarkonium state \([18]\):

\[
\phi^{1S}(x, p_\perp) = 4 \left( \frac{\pi}{\beta^2} \right)^{3/4} \sqrt{\frac{dp_z}{dx}} \exp \left( -\frac{p_z^2 + p_\perp^2}{2\beta^2} \right).
\]

(13)

In addition, the momentum distribution amplitude for the 2S heavy quarkonium state is considered. Eqs. (8) and (13) can be rewritten as

\[
\int \frac{d^3\vec{p}}{2(2\pi)^3} |\phi(p)|^2 = 1,
\]

(14)

\[
\phi^{1S}(p) = 4 \left( \frac{\pi}{\beta^2} \right)^{3/4} \exp \left( -\frac{p^2}{2\beta^2} \right),
\]

(15)

where \( p = |\vec{p}| \). The Fourier transition, or the conjugate coordinate wave function of Eq. (15) is the ground-state solution of the harmonic-oscillator (HO) problem:

\[
\varphi^{1S}(r) = \left( \frac{\beta^2}{\pi} \right)^{3/4} \exp \left( -\frac{\beta^2 r^2}{2} \right).
\]

(16)

Then, for consistency, the relevant excited-state solutions

\[
\varphi^{1P}_m(r) = \sqrt{\frac{8}{3}} \beta^{3/2} \frac{3\pi^{1/4}}{\pi} \beta r \exp \left( -\frac{\beta^2 r^2}{2} \right) Y_{1m}(\theta, \varphi),
\]

(17)

\[
\varphi^{2S}(r) = \sqrt{\frac{1}{6}} \left( \frac{\beta^2}{\pi} \right)^{3/4} \left( \frac{3}{4} - \frac{\beta^2 r^2}{2} \right) \exp \left( -\frac{\beta^2 r^2}{2} \right),
\]

(18)

\( (Y_{1m}'s \text{ are the spherical harmonics}) \), are applied to the excited meson states. This suggestion has been given by the authors of Ref. [28]. Therefore, the Fourier transforms of Eqs. (17) and (18) can be rewritten as

\[
\phi^{1P}_m(x, p_\perp) = 4\sqrt{2} \left( \frac{\pi}{\beta^2} \right)^{3/4} \sqrt{\frac{dp_z}{dx}} \frac{p_m}{\beta} \exp \left( -\frac{p_z^2 + p_\perp^2}{2\beta^2} \right),
\]

(19)

\[
\phi^{2S}(x, p_\perp) = 4\sqrt{\frac{8}{3}} \left( \frac{\pi}{\beta^2} \right)^{3/4} \sqrt{\frac{dp_z}{dx}} \left( \frac{p_z^2 + p_\perp^2}{2\beta^2} - \frac{3}{4} \right) \exp \left( -\frac{p_z^2 + p_\perp^2}{2\beta^2} \right),
\]

(20)

\( (p_{m=\pm 1} = p_{\perp 1} \pm ip_{\perp 2})/\sqrt{2}, \text{ and } p_{m=0} = p_z) \) and can be treated as the momentum distribution amplitudes of the 1P and 2S meson states, respectively. In fact, Eq. (19) has been used for the \( p \)-wave mesons in Ref. [16]. This paper further analyzes the momentum distribution amplitudes as shown in Eqs. (13) and (20).

The formulae of the decay constants \( f_{P,V} \) and M1 transition form factor \( V(0) \) are needed in the latter analysis. However, their derivations have been done in Refs. [16, 18]. The definitions and formulae for them are provided here.
A. Decay constant $f_{P,V}$

The decay constants of mesons $f_{P,V}$ are defined by the matrix elements for $P$ and $V$ mesons

$$
\langle 0|A_{\mu}|P(P)\rangle = if_{P}P_{\mu},
\langle 0|V_{\mu}|V(P,\epsilon)\rangle = M_{V}f_{V}\epsilon_{\mu},
$$

(21)

(22)

where $P_{\mu}$ is the momentum of meson and $\epsilon_{\mu}$ is the polarization vector of $V$ meson. The formula for $f_{P}$ is

$$
f_{P} = \frac{N_{c}}{4\pi^{3}} \int dx d^{2}p_{\perp} \frac{h_{P}}{x(1-x)(M^{2} - M_{0}^{2})} m
= \frac{\sqrt{2}N_{c}}{8\pi^{3}} \int dx d^{2}p_{\perp} \frac{m}{\sqrt{m^{2} + p_{\perp}^{2}}} \phi_{P}(x,p_{\perp}),
$$

(23)

where $N_{c} = 3$ is the color number and $m$ denotes the quark mass. As to the formula for $f_{V}$, we considered the case with the transverse polarization $\epsilon(\pm) = \left( \frac{2}{P_{\perp}} \epsilon_{\perp} \cdot P_{\perp}, 0, \epsilon_{\perp} \right), \quad \epsilon_{\perp} = \mp \frac{1}{\sqrt{2}} (1, \pm i).

(24)

and obtain

$$
f_{V} = \frac{N_{c}}{4\pi^{3}} \int dx d^{2}p_{\perp} \frac{h_{V}}{x_{1}x_{2}(M^{2} - M_{0}^{2})} \left[ xM_{0}^{2} - p_{\perp}^{2} \right]
= \frac{\sqrt{2}N_{c}}{8\pi^{3}M} \int dx d^{2}p_{\perp} \frac{1}{\sqrt{m^{2} + p_{\perp}^{2}}} \left[ \frac{M_{0}^{2}}{2} - p_{\perp}^{2} + \frac{2m}{w_{V}} p_{\perp}^{2} \right] \phi_{V}(x,p_{\perp}).
$$

(25)

This expression can be shown to be in agreement with Eq. (2.22) of Ref. [16]. Since the momentum distribution function is even in $p_{z}$, a quality defined in Eq. [2], it follows that

$$
\int d^{2}p_{\perp} \frac{1}{\sqrt{m^{2} + p_{z}^{2}}} \left( x - \frac{1}{2} \right) M_{0} = 0.
$$

(26)

Therefore,

$$
\int d^{2}p_{\perp} \frac{1}{\sqrt{m^{2} + p_{z}^{2}}} \left( xM_{0}^{2} \right) = \int d^{2}p_{\perp} \frac{\phi_{V}(x,p_{\perp})}{\sqrt{m^{2} + p_{\perp}^{2}}} \frac{M_{0}^{2}}{2}.
$$

(27)

B. Vector current form factor $V(q^{2})$

For the transition $V \rightarrow P\gamma$, a more general process $V \rightarrow P\gamma^{*}$ where the final photon is off-shell is considered. The $V \rightarrow P\gamma^{*}$ transition is parameterized in term of a vector current.
form factor $V(q^2)$ by

$$\Gamma_\mu = ie\varepsilon_{\mu\nu\alpha\beta}q^\alpha P^\beta V(q^2),$$

where $\Gamma_\mu$ is the amplitude of the $V \to P\gamma^*$ process. $P$ ($\epsilon$) is the momentum (polarization vector) of the initial vector meson, $P'$ denotes the momentum of the final pseudoscalar meson, and the momentum transfer $q = P - P'$. The formula for the form factor $V(q^2)$ is

$$V(q^2) = \frac{e_q}{8\pi^2} \int dxd^2p_\perp \frac{\phi_V(x,p_\perp)}{M_0} \left[ \frac{\phi_P(x,p'_\perp)}{(1 - x)M'_0} + \frac{\phi_P(x,p''_\perp)}{xM''_0} \right] \times \left[ m + \frac{2}{w_V} \left( p'_\perp + \frac{(p'_\perp \cdot q_\perp)^2}{q^2} \right) \right],$$

where $p'_\perp = p_\perp - (1 - x)q_\perp$, $p''_\perp = p_\perp + xq_\perp$, $M'_0 = (m^2 + p'_\perp^2)/(1 - x)$, and $M''_0 = (m^2 + p''_\perp^2)/x(1 - x)$. The rate for $V \to P\gamma$ is

$$\Gamma(V \to P\gamma) = \frac{\alpha}{3} \frac{(M_V^2 - M_P^2)^3}{8M_V^3} |V(0)|^2.$$  

III. ANALYSIS OF MOMENTUM DISTRIBUTION AMPLITUDE

In this section, the momentum distribution amplitudes of pseudoscalar and vector heavy quarkonium states are analyzed. The forms of quark distribution amplitude and $\xi$-moments can be derived from them.

A. Quark distribution amplitude $\Phi_P(\xi)$

The quark distribution amplitude of the pseudoscalar heavy quarkonium state can be defined as follows:

$$\langle 0 | \bar{c}(z)\gamma^\alpha \gamma_5 [z, -z] c(-z) | P \rangle = if_P P^\alpha \int_{-1}^{1} d\xi e^{i(Pz)\xi} \Phi(\xi, \mu),$$

where $\xi = 2x - 1$ and $\mu$ is an energy scale which separates the perturbative and non-perturbative regimes. The factor $[z, -z]$ is defined as

$$[z, -z] = P\exp[iq \int_{-z}^{z} dx^\mu A_\mu(x)],$$

which makes the matrix element Eq. (31) gauge invariant. The quark distribution amplitude $\Phi(\xi, \mu)$ is normalized as

$$\int_{-1}^{1} d\xi \Phi(\xi, \mu) = 1,$$
and it can be expanded in Gegenbauer polynomials \( C_n^{3/2}(\xi) \) as

\[
\Phi(\xi, \mu) = \Phi_{as}(\xi) \left[ 1 + \sum_{n=1}^{\infty} a_n(\mu) C_n^{3/2}(\xi) \right],
\]

(34)

where \( \Phi_{as}(\xi) = 3(1 - \xi^2)/4 \) is the asymptotic quark distribution amplitude and \( a_n(\mu) \) the Gegenbauer moments which describe to what degree the quark distribution amplitude deviates from the asymptotic one. \( C_n^{3/2}(\xi) \)'s have the orthogonality integrals

\[
\int_{-1}^{1} (1 - \xi^2) C_m^{3/2}(\xi) C_n^{3/2}(\xi) d\xi = \frac{2(n+1)(n+2)}{2n+3} \delta_{mn}.
\]

(35)

Then \( a_n \) can be obtained by using the above orthogonality integrals as

\[
a_n(\mu) = \frac{2(2n+3)}{3(n+1)(n+2)} \int_{-1}^{1} C_n^{3/2}(\xi) \Phi(\xi, \mu) d\xi.
\]

(36)

An alternative approach to parameterize quark distribution amplitude is to calculate the so-called \( \xi \)-moments,

\[
\langle \xi^n \rangle_\mu = \int_{-1}^{1} d\xi \xi^n \Phi(\xi, \mu).
\]

(37)

It is easy to find the relations between the Gegenbauer moments and the \( \xi \)-moments. Here we list them up to \( n = 6 \)

\[
a_2 = -\frac{7}{12} [1 - 5\langle \xi^2 \rangle],
\]

\[
a_4 = \frac{11}{24} [1 - 14\langle \xi^2 \rangle + 21\langle \xi^4 \rangle],
\]

\[
a_6 = -\frac{35}{448} [5 - 135\langle \xi^2 \rangle + 495\langle \xi^4 \rangle - 429\langle \xi^6 \rangle].
\]

(38)

All the \( n \)-odd moments are vanishing because this paper only focuses on the \( s \)-wave momentum distribution amplitudes, which are \( \xi \)-even functions.

There are some similar procedures by which the quark distribution amplitude can be obtained from the equal time wave function. Within the framework of this study, the decay constant Eq. (23) can be rewritten as

\[
1 = \int_{0}^{1} dx \left[ \frac{\sqrt{2N_c}}{8\pi^3 f_P} \int d^2p_\perp \frac{m}{\sqrt{m^2 + p_\perp^2}} \Phi_P(x, p_\perp) \right]
\]

(39)

and we defined LCWF \( \hat{\Phi}_P(x, \mu) \) as

\[
\hat{\Phi}_P(x, \mu) = \frac{\sqrt{2N_c}}{8\pi^3 f_P} \int_{p_\perp^2 < \mu^2} d^2p_\perp \frac{m}{\sqrt{m^2 + p_\perp^2}} \Phi_P(x, p_\perp)
\]

(40)
where \( \hat{\Phi}_P(x, \mu) = 2\Phi_P(\xi, \mu) \). If the momentum distribution amplitudes Eqs. (13) and (20) are applied as the non-perturbative inputs, the suppression of the Gaussian function allows us to do the \( p_\perp \) integrals up to infinity with no loss of accuracy. Thus, the results can be obtained as follows:

\[
\Phi^{1S}_P(\xi) = \sqrt{\frac{3}{8}} \left( \frac{2}{\pi} \right)^{5/4} \frac{m}{f^{1S}_P} \, e^d \, \Gamma \left[ \frac{3}{4}, \frac{d}{1 - \xi^2} \right],
\]

\[
\Phi^{2S}_P(\xi) = \left( \frac{2}{\pi} \right)^{5/4} \frac{m}{f^{2S}_P} \, e^d \left\{ \Gamma \left[ \frac{7}{4}, \frac{d}{1 - \xi^2} \right] - \left( \frac{3}{4} + d \right) \Gamma \left[ \frac{3}{4}, \frac{d}{1 - \xi^2} \right] \right\},
\]

where \( \Gamma[r, y] \) is the incomplete Gamma function

\[
\Gamma[r, y] = \int_y^\infty t^{r-1} e^{-t} dt,
\]

and \( d = m^2/2\beta^2 \). The incomplete gamma function may be expressed quite elegantly in terms of the confluent hypergeometric function

\[
\Gamma[r, y] = \Gamma[r] - r^{-1}y^r \times \, _1F_1(r; r + 1; -y),
\]

where

\[
_1F_1(r_1, r_2, \ldots, r_i; r'_1, r'_2, \ldots, r'_j; y) = \sum_{n=0}^{\infty} \frac{(r_1)_n(r_2)_n \cdots (r_i)_n y^n}{(r'_1)_n(r'_2)_n \cdots (r'_j)_n n!},
\]

and \( (r)_n = (r+n-1)!/(r-1)! \) is the Pochhammer symbol. After fixing the parameters \( m \) and \( \beta \), the behaviors of the quark distribution amplitudes, Eqs. (41) and (42), are determined and compared with those of other theoretical groups in Section IV. It is worth to mention that Eqs. (41) and (42) have incorrect asymptotic behavior because the Gaussian-type wave functions are primarily only suitable for the low energy region. At energy scales accessible at current experiments, a quark distribution amplitude can be far from the asymptotic form. The worth of the asymptotic form is that it provides a criterion. In other words, one can use the Gegenbauer moments (or \( \xi \)-moments) to describe to what degree his quark distribution amplitude deviates from the asymptotic one.

In fact, the \( \xi \) integrals of \( \Phi^{(1/2)S}_P(\xi) \) can also be analytically performed, and the decay constants \( f^{(1/2)S}_P \) can be expressed as

\[
f^{1S}_P = \sqrt{\frac{3}{2}} \left( \frac{2}{\pi} \right)^{5/4} m \, e^d \left\{ \Gamma \left[ \frac{3}{4} \right] \, _1F_1 \left( -\frac{1}{2}; \frac{1}{4}; -d \right) - \frac{3d^{3/4} \Gamma \left[ -\frac{3}{4} \right]^2}{8\sqrt{2\pi}} \, _1F_1 \left( \frac{1}{4}; \frac{7}{4}; -d \right) \right\},
\]

\[
f^{2S}_P = 3 \left( \frac{2}{\pi} \right)^{5/4} m \, e^d \left\{ -2d \, \Gamma \left[ \frac{3}{4} \right] \, _1F_1 \left( -\frac{1}{2}; \frac{5}{4}; -d \right) + \frac{3d^{3/4} \Gamma \left[ -\frac{3}{4} \right]^2}{16\sqrt{2\pi}} \, _1F_1 \left( -\frac{3}{4}; \frac{3}{4}; -d \right) \right\}.
\]
It is evident that \( f_P^{\text{1(2)S}} \) is only dependent on the values of \( m \) and \( m/\beta \). Eqs. (46) and (47) can be expanded in terms of \( d \) as

\[
\begin{align*}
    f_P^{\text{IS}} &\sim c_1 \left( 1 + 3d + \frac{21}{10}d^2 + \frac{77}{90}d^3 + \ldots \right) + c'_1 d^{3/4} \left( 1 + \frac{6}{7}d + \frac{30}{77}d^2 + \frac{4}{33}d^3 + \ldots \right), \\
    f_P^{\text{2S}} &\sim -c_2 d \left( 1 + \frac{7}{5}d + \frac{77}{90}d^2 + \frac{77}{234}d^3 \ldots \right) + c'_2 d^{3/4} \left( 1 + 2d + \frac{10}{7}d^2 + \frac{20}{33}d^3 + \ldots \right),
\end{align*}
\]

where all \( c^{(i)} \) are positive constants. However, for a typical value of \( d \), all the series in Eqs. (48) and (49) converge very slowly. In addition, the \( \xi \)-moments can be expressed analytically as

\[
\begin{align*}
    \langle \xi^2 \rangle_P^{\text{IS}} &= A_P^{\text{IS}} \left\{ \frac{2}{3} \Gamma \left[ \frac{3}{4} \right] F_1 \left( -\frac{3}{2}, \frac{1}{4}, -d \right) - \frac{d^{3/4} \Gamma \left[ -\frac{3}{4} \right]}{2\sqrt{2\pi}} F_1 \left( -\frac{3}{4}, \frac{7}{4}, -d \right) \right\}, \\
    \langle \xi^4 \rangle_P^{\text{IS}} &= A_P^{\text{IS}} \left\{ \frac{2}{5} \Gamma \left[ \frac{3}{4} \right] F_1 \left( -\frac{5}{2}, \frac{1}{4}, -d \right) - \frac{3d^{3/4} \Gamma \left[ -\frac{3}{4} \right]^2}{7\sqrt{2\pi}} F_1 \left( -\frac{7}{4}, \frac{7}{4}, -d \right) \right\}, \\
    \langle \xi^6 \rangle_P^{\text{IS}} &= A_P^{\text{IS}} \left\{ \frac{2}{7} \Gamma \left[ \frac{3}{4} \right] F_1 \left( -\frac{7}{2}, \frac{1}{4}, -d \right) - \frac{30d^{3/4} \Gamma \left[ -\frac{3}{4} \right]^3}{77\sqrt{2\pi}} F_1 \left( -\frac{11}{4}, \frac{7}{4}, -d \right) \right\},
\end{align*}
\]

and

\[
\begin{align*}
    \langle \xi^2 \rangle_P^{\text{2S}} &= A_P^{\text{2S}} \left\{ \frac{7d}{6} \Gamma \left[ -\frac{1}{4} \right] F_1 \left( -\frac{3}{2}, \frac{5}{4}, -d \right) + \frac{3d^{3/4} \Gamma \left[ -\frac{3}{4} \right]^2}{8\sqrt{2\pi}} F_1 \left( -\frac{7}{4}, \frac{3}{4}, -d \right) \right\}, \\
    \langle \xi^4 \rangle_P^{\text{2S}} &= A_P^{\text{2S}} \left\{ \frac{11d}{10} \Gamma \left[ -\frac{1}{4} \right] F_1 \left( -\frac{5}{2}, \frac{5}{4}, -d \right) + \frac{9d^{3/4} \Gamma \left[ -\frac{3}{4} \right]^2}{28\sqrt{2\pi}} F_1 \left( -\frac{11}{4}, \frac{3}{4}, -d \right) \right\}, \\
    \langle \xi^6 \rangle_P^{\text{2S}} &= A_P^{\text{2S}} \left\{ \frac{15d}{14} \Gamma \left[ -\frac{1}{4} \right] F_1 \left( -\frac{7}{2}, \frac{5}{4}, -d \right) + \frac{90d^{3/4} \Gamma \left[ -\frac{3}{4} \right]^3}{308\sqrt{2\pi}} F_1 \left( -\frac{15}{4}, \frac{3}{4}, -d \right) \right\},
\end{align*}
\]

where

\[
A_P^{\text{IS}} = \sqrt{\frac{3}{8}} \left( \frac{2}{\pi} \right)^{5/4} \frac{m}{f_P^{\text{IS}}} e^d, \quad A_P^{\text{2S}} = \left( \frac{2}{\pi} \right)^{5/4} \frac{m}{f_P^{\text{2S}}} e^d. \tag{52}
\]

The derivations of Eqs. (50) and (51) have used the formula

\[
1F_1(a; b; c) = \frac{b - 1}{c} \left[ 1F_1(a; b - 1; c) - 1F_1(a - 1; b - 1; c) \right], \tag{53}
\]

which is easily checked from the definition of the confluent hypergeometric function Eq. (45).
B. Quark distribution amplitude $\Phi_V(\xi)$

Similar to the case of $\Phi_P(\xi)$, Eq. (25) can be rewritten and $\hat{\Phi}_V(x)$ defined as

$$
\hat{\Phi}_V(x) = \sqrt{\frac{2N_c}{8\pi^3 M f_V}} \int d^2p_\perp \frac{\phi_V(x, p_\perp)}{\sqrt{m^2 + p_\perp^2}} \left[ \frac{M_0^2}{2} - p_\perp^2 + \frac{2m_p^2}{w_\perp} \right].
$$

(54)

However, the $p_\perp$ integrals in Eq. (54) cannot be analytically performed. The crux is the third term in the square bracket, that is, the term proportional to $1/w_\perp$. This term may be rewritten and expanded as

$$
\frac{1}{M_0 + 2m} = \frac{1}{4m (1 + \frac{M_0 - 2m}{4m})} \approx \frac{1}{4m} \left[ 1 - \left( \frac{M_0 - 2m}{4m} \right) + \left( \frac{M_0 - 2m}{4m} \right)^2 - \ldots \right],
$$

(55)

because, in the non-relativistic limit, $M_0 \rightarrow 2m$. Then, we defined the approximate quark distribution amplitudes as

$$
\hat{\Phi}_V'(x) = \sqrt{\frac{2N_c}{8\pi^3 M f_V}} \int d^2p_\perp \frac{\phi_V(x, p_\perp)}{\sqrt{m^2 + p_\perp^2}} \left[ \frac{M_0^2}{2} - p_\perp^2 + \frac{p_\perp^2}{2} \right],
$$

(56)

$$
\hat{\Phi}_V''(x) = \sqrt{\frac{2N_c}{8\pi^3 M f_V}} \int d^2p_\perp \frac{\phi_V(x, p_\perp)}{\sqrt{m^2 + p_\perp^2}} \left[ \frac{M_0^2}{2} - p_\perp^2 + \frac{p_\perp^2}{2} \left( 1 - \frac{M_0 - 2m}{4m} \right) \right].
$$

(57)

After applying the momentum distribution function in Eq. (13) and fitting the parameters, we found that not only the center values of the parameters $\beta$ of Eqs. (56) and (57) were both in the error bar of $\beta_{J/\psi}$ (see Table I), but also that their curves were almost the same. In addition, a similar situation also existed in the case of the $2S$ states. Therefore, we have only shown the results from the form $\hat{\Phi}_V'(x)$. By performing the $p_\perp$ integrals, it can be obtained that

$$
\Phi_V^{1S}(\xi) = \sqrt{\frac{3}{8}} \left( \frac{2}{\pi} \right)^{5/4} \frac{w^2 e^d}{M f_V^{1S}} \left\{ d\Gamma \left[ \frac{3}{4}, \frac{d}{1 - \xi^2} \right] + (3 + \xi^2)\Gamma \left[ \frac{7}{4}, \frac{d}{1 - \xi^2} \right] \right\},
$$

(58)

$$
\hat{\Phi}_V^{2S}(\xi) = \left( \frac{2}{\pi} \right)^{5/4} \frac{w^2 e^d}{M f_V^{2S}} \left\{ d\Gamma \left[ \frac{7}{4}, \frac{d}{1 - \xi^2} \right] + (3 + \xi^2)\Gamma \left[ \frac{11}{4}, \frac{d}{1 - \xi^2} \right] \right\} - \left( \frac{3}{4} + d \right) \left\{ d\Gamma \left[ \frac{3}{4}, \frac{d}{1 - \xi^2} \right] + (3 + \xi^2)\Gamma \left[ \frac{7}{4}, \frac{d}{1 - \xi^2} \right] \right\},
$$

(59)

The above derivations are based on the formula:

$$
\Gamma[r, y] - (r - 1)\Gamma[r - 1, y] = y(\Gamma[r - 1, y] - (r - 2)\Gamma[r - 2, y]),
$$

(60)
which is easily checked from the definition of incomplete Gamma function, Eq. (43). The \( \xi \)-moments of \( \Phi' (\xi) \) can also be obtained analytically

\[
\langle \xi^2 \rangle_{V}^{1S} = A_{V}^{1S} \left\{ \frac{1}{5} \Gamma \left[ \frac{3}{4} \right] \left[ 10 \, 1 \, F_1 \left( -\frac{3}{2}; -\frac{3}{4}; -d \right) - 1 \, F_1 \left( -\frac{5}{2}; -\frac{3}{4}; -d \right) \right] + \frac{2d^{7/4} \Gamma \left[ -\frac{3}{4} \right]^2}{7\sqrt{2\pi}} \left[ 3 \, 1 \, F_1 \left( \frac{1}{4} ; \frac{11}{4} ; -d \right) - 1 \, F_1 \left( \frac{3}{4} ; \frac{11}{4} ; -d \right) \right] \right\},
\]

\[
\langle \xi^4 \rangle_{V}^{1S} = A_{V}^{1S} \left\{ \frac{3}{35} \Gamma \left[ \frac{3}{4} \right] \left[ 14 \, 1 \, F_1 \left( -\frac{5}{2}; -\frac{3}{4}; -d \right) - 1 \, F_1 \left( -\frac{7}{2}; -\frac{3}{4}; -d \right) \right] + \frac{3d^{7/4} \Gamma \left[ -\frac{7}{4} \right]^2}{4\sqrt{2\pi}} \left[ 7 \, 1 \, F_1 \left( -\frac{3}{4} ; \frac{11}{4} ; -d \right) - 1 \, F_1 \left( -\frac{7}{4} ; \frac{11}{4} ; -d \right) \right] \right\},
\]

\[
\langle \xi^6 \rangle_{V}^{1S} = A_{V}^{1S} \left\{ \frac{1}{21} \Gamma \left[ \frac{3}{4} \right] \left[ 18 \, 1 \, F_1 \left( -\frac{7}{2}; -\frac{3}{4}; -d \right) - 1 \, F_1 \left( -\frac{9}{2}; -\frac{3}{4}; -d \right) \right] + \frac{165d^{7/4} \Gamma \left[ -\frac{11}{4} \right]^2}{32\sqrt{2\pi}} \left[ 11 \, 1 \, F_1 \left( -\frac{7}{4} ; \frac{11}{4} ; -d \right) - 1 \, F_1 \left( -\frac{11}{4} ; \frac{11}{4} ; -d \right) \right] \right\},
\]

and

\[
\langle \xi^2 \rangle_{V}^{2S} = A_{V}^{2S} \left\{ \frac{1}{5} \Gamma \left[ \frac{3}{4} \right] \left[ 15 \, 1 \, F_1 \left( -\frac{3}{2}; -\frac{3}{4}; -d \right) + (d - 6) \, 1 \, F_1 \left( -\frac{5}{2}; -\frac{3}{4}; -d \right) \right] + \frac{d^{7/4} \Gamma \left[ -\frac{3}{4} \right]^2}{7\sqrt{2\pi}} \left[ 9 \, 1 \, F_1 \left( \frac{1}{4} ; \frac{11}{4} ; -d \right) + 2(d - 6) \, 1 \, F_1 \left( \frac{3}{4} ; \frac{11}{4} ; -d \right) \right] \right\},
\]

\[
\langle \xi^4 \rangle_{V}^{2S} = A_{V}^{2S} \left\{ \frac{3}{35} \Gamma \left[ \frac{3}{4} \right] \left[ 35 \, 1 \, F_1 \left( -\frac{5}{2}; -\frac{3}{4}; -d \right) + (d - 22) \, 1 \, F_1 \left( -\frac{7}{2}; -\frac{3}{4}; -d \right) \right] + \frac{3d^{7/4} \Gamma \left[ -\frac{7}{4} \right]^2}{8\sqrt{2\pi}} \left[ 35 \, 1 \, F_1 \left( -\frac{3}{4} ; \frac{11}{4} ; -d \right) + 2(d - 22) \, 1 \, F_1 \left( -\frac{7}{4} ; \frac{11}{4} ; -d \right) \right] \right\},
\]

\[
\langle \xi^6 \rangle_{V}^{2S} = A_{V}^{2S} \left\{ \frac{1}{21} \Gamma \left[ \frac{3}{4} \right] \left[ 63 \, 1 \, F_1 \left( -\frac{7}{2}; -\frac{3}{4}; -d \right) + (d - 46) \, 1 \, F_1 \left( -\frac{9}{2}; -\frac{3}{4}; -d \right) \right] + \frac{165d^{7/4} \Gamma \left[ -\frac{11}{4} \right]^2}{64\sqrt{2\pi}} \left[ 77 \, 1 \, F_1 \left( -\frac{7}{4} ; \frac{11}{4} ; -d \right) + 2(d - 46) \, 1 \, F_1 \left( -\frac{11}{4} ; \frac{11}{4} ; -d \right) \right] \right\},
\]

where

\[
A_{V}^{1S} = \sqrt{\frac{3}{8}} \left( \frac{2}{\pi} \right)^{5/4} \frac{w^2 e^d}{M f_{V}^{1S}}, \quad A_{V}^{2S} = \left( \frac{2}{\pi} \right)^{5/4} \frac{w^2 e^d}{M f_{V}^{2S}}.
\]

IV. NUMERICAL RESULTS AND DISCUSSIONS

In this work, the numerical results were calculated for the quark distribution amplitudes \( \Phi(\xi) \) and \( \xi \)-moments for the pseudoscalar and vector heavy quarkonium. First, it was nec-
ecessary to determine the parameters appearing in the momentum distribution functions. In total, there are five for charmonium, $m_c$, $\beta_{\eta_c}$, $\beta_{J/\psi}$, $\beta_{\psi'}$, and five ones for bottomonium: $m_b$, $\beta_{\eta_b}$, $\beta_{\eta'}$, $\beta_{\Upsilon}$, $\beta_{\Upsilon'}$. The decay constants of vector heavy quarkonium states were determined first. The decay constant $f_V$ is related to the electromagnetic decay of vector meson $V \rightarrow e^+e^-$ by

$$\Gamma(V \rightarrow e^+e^-) = \frac{4\pi \alpha^2}{3 M_V} c_V f_V^2,$$

where $c_V$ is the square of the electric charge of heavy quark. From the data [1], we obtained the values

$$f_{J/\psi} = 416 \pm 7 \text{ MeV}, \quad f_{\psi'} = 298 \pm 8 \text{ MeV},$$

$$f_{\Upsilon} = 715 \pm 5 \text{ MeV}, \quad f_{\Upsilon'} = 497 \pm 4 \text{ MeV}.$$  

(65) (66)

For the decay constants of $\eta_c$, the two decay modes $B \rightarrow K\eta_c$ and $B \rightarrow KJ/\psi$ are considered using the following relation

$$\frac{\Gamma(B \rightarrow K\eta_c)}{\Gamma(B \rightarrow KJ/\psi)} = \left(\frac{f_{\eta_c}}{f_{J/\psi}}\right)^2 \frac{C_{\eta_c}}{C_{J/\psi}} \lambda_{BK\eta_c}^2 \left(1 - \frac{M_K^2}{M_B^2} \frac{f_+(M_{\eta_c}^2)}{f_+(M_{J/\psi}^2)} + \frac{M_{\eta_c}^2}{M_B^2} \frac{f_-(M_{\eta_c}^2)}{f_+(M_{J/\psi}^2)}\right),$$

(67)

where $C_{\eta_c}$ and $C_{J/\psi}$ are related to the Wilson coefficients and can be determined to have the value $|C_{\eta_c}/C_{J/\psi}| = 0.89 \pm 0.02$ [33],

$$\lambda_{abc} = \left[\left(1 - \frac{M_b^2}{M_a^2} - \frac{M_c^2}{M_a^2}\right)^2 - 4 \frac{M_b^2 M_c^2}{M_a^2 M_a^2}\right]^{1/2},$$

(68)

and the form factors $f_\pm(q^2)$ are defined by the Lorentz decomposition of the matrix element

$$\langle K(p)|\bar{s}\gamma_\mu b|B(p+q)\rangle = f_+(q^2)(2p+q)_\mu + f_-(q^2)q_\mu.$$  

(69)

By the results of Ref. [16] one calculated the form factors $f_\pm(q^2)$ within the covariant light-front quark model. Combining the above with the experimental values, we can obtain

$$f_{\eta_c} = 421 \pm 38 \text{ MeV}.$$  

(70)

This result leads to the ratio

$$\left(\frac{f_{\eta_c}}{f_{J/\psi}}\right)^2 = 1.02 \pm 0.22,$$  

(71)
which is consistent with the Van Royen-Weisskopf formula \[34\],

$$
\left( \frac{f_{\eta_c}}{f_{J/\psi}} \right)^2 = \frac{M_{J/\psi}}{M_{\eta_c}} \left| \frac{\psi_{\eta_c}(0)}{\psi_{J/\psi}(0)} \right|^2 \simeq 1.04,
$$

when the approximation \( \psi_{\eta_c}(0) \simeq \psi_{J/\psi}(0) \) is used. For the decay constant \( f_{\eta_c'} \), however, a similar estimation of Eq. \( 67 \) has a large uncertainty because the error of \( Br(B \to K\eta_c') \) was greater than 50% \[1\]. Given the consistency between Eq. \( 71 \) and \( 72 \), the approximation \( \psi_{\eta_c'}(0) \simeq \psi_{\psi'}(0) \) was used, and the corresponding Van Royen-Weisskopf formula,

$$
\left( \frac{f_{\eta_c'}}{f_{\psi'}} \right)^2 \simeq \frac{M_{\psi'}}{M_{\eta_c'}} = 1.01,
$$

is applied and the decay constant \( f_{\eta_c'} = 300 \pm 8 \text{MeV} \) is obtained. Furthermore, an additional constraint is needed. The new data \( Br(J/\psi \to \eta_c \gamma) = (1.98 \pm 0.31)\% \) \[35\] are chosen to calculate the form factor \( V(0) \) in Eq. \( 30 \). For the charm sector, there were one assumption (Eq. \( 73 \)) and four constraints (the decay constants \( 65, \), \( 70 \), and the new data \( Br(J/\psi \to \eta_c \gamma) \)) were used to fix the five parameters appearing in the momentum distribution amplitudes. The results are listed in Table I. As to the decay constants of \( \eta_b \)

and \( \eta_b' \), a similar approximation was assumed

$$
\left( \frac{f_{\eta_b}}{f_{\Upsilon'}} \right)^2 \simeq \left( \frac{f_{\eta_b'}}{f_{\Upsilon'}} \right)^2 \simeq \frac{M_{\Upsilon'}}{M_{\eta_b'}} = 1.01,
$$

and the decay constants \( f_{\eta_b} = 718 \pm 5 \text{MeV}, f_{\eta_b'} = 499 \pm 5 \text{MeV} \) obtained. Finally we used \[16\] \( m_b = 4.64 \text{ GeV} \) and fixed the \( \beta \) parameters for the bottom sector in Table II.

After fixing all parameters, Eqs. \( 41, 42, 58 \), and \( 59 \) are used to plot the quark distribution amplitudes \( \Phi_{P,V}^{(1)S}(\xi) \) and \( \Phi_{P,V}^{(2)S}(\xi) \) for the charm sector, as shown in Fig. 1 and Fig. 2, respectively. As is seen, the momentum fraction \( x \) of the pseudoscalar charmonium

| parameter | \( m_c \) | \( \beta_{\eta_c} \) | \( \beta_{\eta_c'} \) | \( \beta_{J/\psi} \) | \( \beta_{\psi'} \) |
|-----------|-----------|----------------|----------------|----------------|----------------|
| value     | 1.56      | 0.820±0.078     | 0.665±0.028     | 0.613 ± 0.006  | 0.477 ± 0.008  |
| parameter | \( \beta'_{J/\psi} \) | \( \beta''_{J/\psi} \) | \( \beta'_{\psi'} \) | \( \beta''_{\psi'} \) |
| value     | 0.611 ± 0.06 | 0.613 ± 0.06 | 0.474 ± 0.08 | 0.477 ± 0.08 |

\[16\] m_b = 4.64 GeV and fixed the \( \beta \) parameters for the bottom sector in Table II.
TABLE II: Parameters $m_b$ and $\beta$’s for bottomonium states (in units of GeV). $\beta'$ and $\beta''$ are the parameters appearing in equations (56) and (57).

| parameter | $m_b$ | $\beta_{m_b}$ | $\beta'_{m_b}$ | $\beta_Y$ | $\beta'_{Y}$ | $\beta''_{Y}$ |
|-----------|------|--------------|--------------|---------|-----------|------------|
| value     | 4.64 | 1.47 ± 0.01  | 1.04 ± 0.01  | 1.30 ± 0.01 | 0.926 ± 0.005 |

FIG. 1: Quark distribution amplitudes $\Phi_\eta c(\xi)$ (solid lines) and $\Phi'_{J/\psi}(\xi)$ (dashed lines) of this work.

had a slightly wider distribution than that of the vector charmonium. Next, the quark distribution amplitudes $\Phi_\eta c(\xi)$ and $\Phi'_{J/\psi}(\xi)$ of this work were compared with those of Ref. [19, 20] in Fig. 3. For the latter, the equation [19, 20]

$$\Phi(\xi, \mu = 1.2 \text{GeV}) = c(\alpha, \beta, \gamma)(1 - \xi^2)(\alpha + \gamma \xi^2) \exp\left(-\frac{\beta}{1 - \xi^2}\right), \quad (75)$$

was used to plot the curves. $\Phi^{1S}(\xi, \mu)$ and $\Phi^{2S}(\xi, \mu)$ corresponded to $(\alpha = 1, \beta = 3.8, \gamma = 0)$ [19] and $(\alpha = 0.027, \beta = 2.49, \gamma = 1)$ [20], respectively. In Fig. 3, the major difference between the two curves of the 1S states was that the curve of Ref. [19], on which $\xi$ is peaked around zero, was sharper than that of this work. This meant that the momentum fraction $x$ in the function used in Ref. [19] was more centered on 1/2 than in the Gaussian-type wave function. As to the curves of the 2S state, we found the locations of extreme value by
FIG. 2: Quark distribution amplitudes $\Phi_{\eta_c}(\xi)$ (solid lines) and $\Phi_{\psi'}(\xi)$ (dashed lines) of this work.

FIG. 3: Quark distribution amplitudes $\Phi_{\eta_c}(\xi)$ and $\Phi_{\eta_c'}(\xi)$ of this work (solid lines) and that of Ref. [19, 20] (dashed lines).

differentiating both Eqs. (42) and (75) over $\xi$. The results were

$$\xi = 0, \pm \sqrt{\frac{3}{3 + 4d}},$$

$$\xi = 0, \pm \frac{1}{2\sqrt{\gamma}} \sqrt{3\gamma + \beta \gamma - \alpha - (\alpha^2 + 2\alpha \gamma + 6\alpha \beta \gamma + \gamma^2 + 6\beta \gamma^2 + \beta^2 \gamma^2)^{1/2}},$$

respectively. The values are $\xi = 0, \pm 0.463$ for this work and $\xi = 0, \pm 0.421$ for Ref. [20]. For the bottom sector, the quark distribution amplitudes $\Phi_{P,V}^{(1S)}(\xi)$ and $\Phi_{P,V}^{(2S)}(\xi)$ are plotted in Figs. 4 and 5. These figures show that the $x$-distribution of the pseudoscalar bottomonium
FIG. 4: Quark distribution amplitudes $\Phi_{\eta_b}(\xi)$ (solid lines) and $\Phi'_{\Upsilon}(\xi)$ (dashed lines) of this work.

FIG. 5: Quark distribution amplitudes $\Phi_{\eta_b'}(\xi)$ (solid lines) and $\Phi'_{\Upsilon'}(\xi)$ (dashed lines) of this work.

was almost the same as for the vector bottomonium. The reason is that since the differences between the pseudoscalar and the vector heavy quarkonium states arise from $1/m_Q$ corrections, the larger the $m_Q$, the smaller the difference between them. For comparison, the $x$-distributions of the charmonium and bottomonium states are plotted together in Fig. 6.

In addition, the differences between various quark distribution amplitudes can also be shown by $\xi$-moments. We calculated the $\xi$-moments Eqs. (50), (51), (61), and (62). Our results and those of other theoretical groups are listed in Tables III and IV. In these tables, Refs. [19, 20, 21] used the QCD sum rules. The authors of Ref. [22] calculated in the frame-
work of the Buchmuller-Tye-potential model and found that their results are in agreement with experiments, which included the leptonic widths and hyperfine splittings. The authors of [23] made calculations for the Cornell potential. The predictions of light-cone sum rules are shown in Ref. [24]. The authors of Ref. [25] not only numerically calculated the 1S charmonium states in conventional LFQM, but they also used the variational approach to fix the mass of the charm quark. The authors of Ref. [26] constructed quark distribution amplitudes by assuming the very non-relativistic limit, where $\Phi(x) = \delta(x - 1/2)$, and then redistributing the parton momenta by relativistic gluon exchange. Depending on the assumed value of $\alpha_s$ at the heavy quark scale, they found the $\xi^n$ moments of $\eta_c$ in Table III. For reasonable values of $\alpha_s$ they obtained qualitatively similar values as in [19] and the curves of them are slightly narrower than in our numerical analysis. As to the $\xi$-moments for the bottomonium states, they were evaluated and are listed in Table V.

V. CONCLUSIONS

This study performed the transverse momenta $p_\perp$ integrals of formulae for the decay constants $f_{P,V}$ of 1S and 2S heavy quarkonium states and then obtained their quark distribution amplitudes $\Phi_{P,V}(\xi)$. In addition, the $\xi$-moments $\langle \xi^{2,4,6}\rangle_{P,V}$ were also obtained by integrating out $\xi$ in $\Phi_{P,V}(\xi)$. For each heavy quarkonium state, the five parameters $m_Q, \beta_P, \beta_P, \beta_V,$
TABLE III: The $\xi$-moments for the 1S charmonium states. ($^\dagger$ $\langle \xi^n \rangle_{\eta_c} = \langle \xi^n \rangle_{J/\psi}$)

| moment | $\langle \xi^2 \rangle_{\eta_c}$ | $\langle \xi^2 \rangle_{J/\psi}$ | $\langle \xi^4 \rangle_{\eta_c}$ | $\langle \xi^4 \rangle_{J/\psi}$ | $\langle \xi^6 \rangle_{\eta_c}$ | $\langle \xi^6 \rangle_{J/\psi}$ |
|--------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|
| this work | $0.117^{+0.010}_{-0.011}$ | $0.0966^{+0.0013}_{-0.0013}$ | $0.0307^{+0.0052}_{-0.0049}$ | $0.0215^{+0.0005}_{-0.0005}$ | $0.0109^{+0.0026}_{-0.0023}$ | $0.00657^{+0.00023}_{-0.00023}$ |
| [19, 21] | $0.070^{+0.007}_{-0.007}$ | $0.073^{+0.007}_{-0.007}$ | $0.012^{+0.002}_{-0.002}$ | $0.012^{+0.002}_{-0.002}$ | $0.0032^{+0.0009}_{-0.0009}$ | $0.0033^{+0.0007}_{-0.0007}$ |
| [22] | 0.086 | 0.020 | | | | 0.0066 |
| [23] | 0.084 | 0.019 | | | | 0.0066 |
| [24]† | 0.13 | 0.040 | | | | 0.018 |
| [25] | $0.084^{+0.004}_{-0.007}$ | $0.082^{+0.004}_{-0.006}$ | $0.017^{+0.001}_{-0.003}$ | $0.016^{+0.002}_{-0.002}$ | $0.0047^{+0.0006}_{-0.0010}$ | $0.0046^{+0.0005}_{-0.0010}$ |
| [26] | 0.067 | 0.011 | | | | 0.004 |

TABLE IV: The $\xi$-moments for the 2S charmonium states. ($^\dagger$ $\langle \xi^n \rangle_{\eta_c} = \langle \xi^n \rangle_{\psi'}$)

| moment | $\langle \xi^2 \rangle_{\eta_c}$ | $\langle \xi^2 \rangle_{\psi'}$ | $\langle \xi^4 \rangle_{\eta_c}$ | $\langle \xi^4 \rangle_{\psi'}$ | $\langle \xi^6 \rangle_{\eta_c}$ | $\langle \xi^6 \rangle_{\psi'}$ |
|--------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|
| this work | $0.192^{+0.008}_{-0.008}$ | $0.136^{+0.006}_{-0.006}$ | $0.0600^{+0.0048}_{-0.0044}$ | $0.0326^{+0.0021}_{-0.0020}$ | $0.0229^{+0.0026}_{-0.0023}$ | $0.00950^{+0.00078}_{-0.00074}$ |
| [20]† | $0.18^{+0.05}_{-0.07}$ | | $0.051^{+0.031}_{-0.031}$ | | $0.017^{+0.016}_{-0.014}$ | |
| [22] | 0.16 | 0.042 | | | | 0.015 |

TABLE V: The $\xi$-moments for the 1S and 2S bottomonium states.

| moment | $\langle \xi^2 \rangle_{\eta_b}$ | $\langle \xi^2 \rangle_{\Upsilon}$ | $\langle \xi^4 \rangle_{\eta_b}$ | $\langle \xi^4 \rangle_{\Upsilon}$ | $\langle \xi^6 \rangle_{\eta_b}$ | $\langle \xi^6 \rangle_{\Upsilon}$ |
|--------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|
| this work | $0.0643^{+0.0005}_{-0.0005}$ | $0.0598^{+0.0004}_{-0.0004}$ | $0.0103^{+0.0001}_{-0.0001}$ | $0.00894^{+0.00011}_{-0.00011}$ | $0.00237^{+0.00005}_{-0.00005}$ | $0.00192^{+0.00003}_{-0.00003}$ |
| this work | $0.0844^{+0.0010}_{-0.0010}$ | $0.0729^{+0.0007}_{-0.0007}$ | $0.0128^{+0.0003}_{-0.0003}$ | $0.00984^{+0.00018}_{-0.00018}$ | $0.00255^{+0.00009}_{-0.00008}$ | $0.00176^{+0.00005}_{-0.00004}$ |

$\beta_{\psi'}$ which appeared in the momentum distribution amplitude were determined. This study first extracted the decay constants $f_{V,V'}$ from the experimental data of the leptonic decay $Br(V \rightarrow e^+e^-)$, and it then used the Van Royen-Weisskopf formula to obtain the decay constants $f_{P,P'}$. These decay constants were used as constraints to fix the above parameters. Then, the curves of the quark distribution amplitudes for the 1S and 2S heavy quarkonium states were plotted by the fixed parameters. It was found that, for the 1S charmonium state, the momentum fraction $x$ in the function used by Ref. [19] was more centered on 1/2 than the one in the Gaussian-type wave function. In addition, the $x$-distribution of the
pseudoscalar bottomonium was almost the same as that of the vector bottomonium. The reason for this was that the differences between these heavy quarkonium states, which arise from $1/m_Q$ corrections, become small when $m_Q$ is large. Finally, the numerical results of the $\xi$-moments were calculated and compared with the experimental data and other theoretical predictions.

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