The quantum Pesin theorem: Lyapunov exponents in the classical limit

IGNACIO GOMEZ\(^1\), MARCELO LOSADA\(^1\), SEBASTIAN FORTIN\(^2\) AND MARIO CASTAGNINO\(^2\)

January 16, 2014

Abstract

A quantum version of the Pesin theorem is presented and it is related to the Lyapunov exponents of the classical limit. We apply the quantum Pesin theorem to a phenomenological Gamow type model and we conclude that its classical limit is chaotic due to the positivity of its Lyapunov exponents. Moreover, the quantum Pesin theorem provides a semiclassical method expressed in terms of quantum mean values which allows us to determine if the classical limit is chaotic or not in a simple way.

1- Instituto de Física de Rosario (IFIR-CONICET), Rosario, Argentina.
2- Instituto de Astronomía y Física del Espacio (IAFE-CONICET), Buenos Aires, Argentina.

Key words: Pesin theorem-Lyapunov exponents-Kolmogorov Sinai entropy-classical limit

1 Introduction

The presence of (classical) Lyapunov exponents in quantum systems was reported in several papers [1, 2, 3, 4, 5, 6], and the positivity of them is a necessary condition for chaos. For a quantum system with a classically chaotic Hamiltonian the decoherence formalism can be used to define quantum chaos, where purity decreases with a Lyapunov rate [7, 8]. A complete definition of classical chaos can be found in [9], where the three most important features of chaos, the Lyapunov exponents, the ergodic hierarchy and the complexity, are studied. The Brudno theorem is the link between the Kolmogorov-Sinai entropy and complexity, while the Pesin theorem is the link between the Lyapunov exponents and the Kolmogorov-Sinai entropy as defined in the ergodic hierarchy [10].

A reasonable definition of quantum systems having a chaotic classical description was given by M. Berry: “A quantum system is chaotic if its classical limit is chaotic” [15]. This “quantum caology”, as was named originally by M. Berry, was what later ended up referred as quantum chaos.

In previous works [11, 12] we studied the quantum ergodic hierarchy (QEH). It ranks the chaotic level of quantum systems according to how the quantum correlations between states and observables are canceled for large times. In [12] we used the QEH to characterize typical chaos phenomena like the exponential localization of the kicked rotator and the quantum interference destruction of the Casati-Prosen model in terms of the ergodic and mixing levels.

Moreover, the QEH is an attempt, among several theoretical and phenomenological approaches (like the WKB approximation, the random matrix theory, etc. [13, 14, 15, 16, 17]), towards a theoretical framework of quantum systems which admits a chaotic classical description assuming the Berry definition.

In this letter, we use the Berry definition and the QEH idea of ranking the quantum chaos with quantum mean values and we present a simple quantum (or better semiclassical) version of the Pesin theorem. Also, we show how this theorem relates: classical Lyapunov exponents of the classical limit to mean values of projectors which are the corresponding quantum operators of characteristic functions defined over the phase space. From the quantum Pesin theorem we obtain an associated method to determine if a quantum system has a chaotic classical limit or not.

Finally, we apply this to an example of the decoherence literature, a phenomenological Gamow type model [19, 20], and we conclude that its classical limit is chaotic.
2 The Kolmogorov-Sinai entropy

In this section, following [21], we give the general notions of the Kolmogorov-Sinai entropy. We consider a dynamical system \((\Gamma, \Sigma, \mu, \{T_t\}_{t \in J})\), where \(\Gamma\) is the phase space, \(\Sigma\) is a \(\sigma\)-algebra, \(\mu : \Sigma \to [0, 1]\) is a normalized measure and \(\{T_t\}_{t \in J}\) is a semigroup of preserving measure transformations. For instance, \(T_t\) could be the classical Liouville transformation or the corresponding classical transformation associated to the quantum Schrödinger transformation and \(J\) is usually \(\mathbb{R}\) for continuous dynamical systems, respectively. Let us divide the phase space \(\Gamma\) into a partition \(Q\) of \(n\) small cells \(A_i\), each having positive measure. We define the entropy of the partition as

\[
H(Q) = -\sum_{i=1}^{n} \mu(A_i) \log \mu(A_i). \tag{1}
\]

Then we call KS-entropy of partition \(Q\) to

\[
h_\mu(T, Q) = \lim_{n \to \infty} \frac{1}{n} H(\chi_{n=0}^{n} Q). \tag{2}
\]

From the KS-entropy \(h_\mu(T, Q)\) of the partition \(Q\), we define the KS-entropy \(h_{k\mu}\) of the dynamical system as the supreme \(h\) over all measurable initial partitions of phase space \(\Gamma\) when the initial size of the cells tends to zero, i.e.

\[
h_{k\mu}(T) = \sup_Q h_\mu(T, Q). \tag{3}
\]

Then the Pesin theorem says (23, 21, 24)

\[
h_{k\mu}(T) = \int_{\Gamma} \left[ \sum_{\sigma_i(x) > 0} \sigma_i(x) \right] d\phi(2^{N+1}) \tag{4}
\]

where \(\sigma_i(x)\) are all the positive Lyapunov exponents of the physical system.

3 The Weyl-Wigner-Moyal Mapping

In this section we review the main tools of the Wigner transformation. We will use a formalism for states and observables which has been proposed by the Brussels school (led by Ilya Prigogine) in [22].

Given a quantum system, we consider the quantum algebra of operators \(\hat{A}\). If \(\hat{f} \in \hat{A}\), then the Wigner transformation is given by (see [25, 26])

\[
symb(\hat{f}) = f(\phi) = \int (q + \Delta |q - \Delta| e^{2i\Delta \phi} d^{N+1} \Delta,
\]

where \(f(\phi)\) is a distribution function over the phase space \(\Gamma = \mathbb{R}^{2(N+1)}\). The set of all these distribution functions is the quasiclassical algebra \(\mathcal{A}_q\), which is defined by \(\mathcal{A}_q = symb(\hat{A})\). \(\mathcal{A}_q\) is not the classical algebra \(\mathcal{A}\) because in the general case, the Wigner transform is not positive for all state \(\rho\). For this reason, \(\rho(\phi) = symb(\hat{\rho})\) is called a quasi-probability distribution. More precisely, \(\rho(\phi) = symb(\hat{\rho})\) is the Wigner quasi-probability distribution of the state \(\hat{\rho}\). We can also introduce the star product (see [27]),

\[
symb(\hat{f} \ast \hat{g}) = symb(\hat{f} \ast \hat{g}) = (f \ast g)(\phi) = f(\phi) \exp \left( -\frac{i\hbar}{2} \frac{\partial}{\partial a} \frac{\partial}{\partial b} \right) g(\phi) \tag{6}
\]

where \(\omega^{ab}\) is the metric of the phase space.

The Moyal bracket is the symbol corresponding to the quantum commutator, i.e.

\[
\{f, g\}_{mb} = \frac{1}{i\hbar} (f \ast g - g \ast f) = symb \left( \frac{1}{i\hbar} [f, g] \right). \tag{7}
\]

It can be proved that (see [25])

\[
\{f \ast g\}(\phi) = f(\phi)g(\phi) + 0(\hbar),
\]

\[
\{f, g\}_{mb} = \langle f, g \rangle_{pb} + 0(\hbar^2). \tag{8}
\]

To define the inverse \(symb^{-1}\) we will use the symmetrical or Weyl ordering prescription, namely,

\[
symb^{-1}[q^i(\phi), p^j(\phi)] = \frac{1}{2} \left( q^i p^j + p^j q^i \right). \tag{9}
\]

Therefore, by means of the transformations \(symb\) and \(symb^{-1}\), we have defined an isomorphism between the quantum algebra \(\hat{A}\) and the quasiclassical algebra of distribution functions \(\mathcal{A}_q\),

\[
symb^{-1} : \mathcal{A}_q \to \hat{A}, \quad symb : \hat{A} \to \mathcal{A}_q \tag{10}
\]

The mapping so defined is the Weyl-Wigner-Moyal symbol. When \(\hbar \to 0\), then \(\mathcal{A}_q \to \hat{A}\), where \(\mathcal{A}\) is the classical algebra of observables.

The Wigner transformation for states is

\[
\rho(\phi) = symb(\hat{\rho}) = (2\pi\hbar)^{-(N+1)} symb(\text{for operators}) \hat{\rho}. \tag{11}
\]
As it is well-known, an important property of the Wigner transformation is that [25]:

\[
\langle \tilde{O}_r \rangle = \langle \tilde{\rho} \tilde{O} \rangle = \langle \text{symb}(\tilde{\rho}), \text{symb}(\tilde{O}) \rangle = \langle \rho(\phi), O(\phi) \rangle = \int d\phi 2^{(N+1)} \rho(\phi) O(\phi)
\]

(12)

where \( \langle f, g \rangle \) is the scalar product between \( f \) and \( g \). \( [\tilde{O}] = \int_0^\infty \int_0^\infty O(\omega, \omega') |\omega, \omega'\rangle d\omega d\omega' \in \mathcal{O} \subset \tilde{A} \) and \( \tilde{A} \) is the observable space whose basis is \( \{|\omega, \omega'\rangle = |\omega\rangle |\omega'\rangle\} \). The subalgebra \( \mathcal{O} \) is a mathematical way to introduce the equilibrium arrival of quantum systems. Physically, \( \mathcal{O} \) contains all the relevant information of the system to which we can have access. Moreover, \( \mathcal{O} \) can be chosen in different ways [19, 25, 29, 30, 31, 32]. In [12] the state (\( \tilde{\rho} \)) belong to the dual space \( \mathcal{O}' \subset \tilde{A}' \). This means that the definition of \( \tilde{\rho} \in \mathcal{O}' \subset \tilde{A}' \) as a functional on \( \mathcal{O} \) is equivalent to the definition of \( \text{symb}(\tilde{\rho}) \in A_q^\ast \) as a functional on \( A_q^\ast \). The equation (12) is exact (e.g. with no orders \( O(\hbar) \)), while other equations, like (3), are only approximate to an order \( \hbar/2 \). In fact, equation (12) will be the only tool we will use to obtain the Pesin-quantum theorem.

The particular case of equation (12) for the expectation value of the identity operator \( \tilde{I} \) reads

\[
\langle \tilde{I} \rangle = \langle \tilde{\rho} \tilde{I} \rangle = \langle \text{symb}(\tilde{\rho}), \text{symb}(\tilde{I}) \rangle = \langle \rho(\phi), I(\phi) \rangle = \int d\phi 2^{(N+1)} \rho(\phi) = 1
\]

(13)

Moreover, given a partition \( Q \) and an element \( A_l \), if we make \( \rho(\phi) = I_{A_l(\phi)} \), we have

\[
\langle I_{A_l(t)} \rangle = \langle \tilde{I}_{A_l(t)} \rangle = \langle \text{symb}(\tilde{I}_{A_l(t)}), \text{symb}(\tilde{I}) \rangle = \langle I_{A_l(t)}(\phi), I(\phi) \rangle = \int d\phi 2^{(N+1)} I_{A_l(t)}(\phi) = \mu(A_l(t))
\]

(14)

where we have used that \( \text{symb}(\tilde{I}_{A_l(t)}) = I_{A_l(t)} \). Therefore, \( \mu(A_l(t)) = I_{A_l(t)}(\tilde{I}) \) and it will be essential to obtain a quantum version of the Pesin theorem in the next section.

4 The Quantum Pesin Theorem

In this section we translate the classical Pesin theorem (equation (4)) to a quantum version (semi-classical) in the classical limit \( \hbar \to 0 \). Given a partition \( Q = \{A_1, A_2, ..., A_m\} \) we have the partition \( B(\hbar) = \bigcap_{n=0}^T T^{-n} A_{k_j} \) be a generic element of \( B(\hbar) \). On the other hand, \( \mu(B_0(\hbar)) = (\prod_{j=0}^n \hat{I}_{A_{k_j}(j)}(\hat{I}) \) when \( \hbar \approx 0 \) (see appendix) where \( \hat{I}_{A_{k_j}}(j) = U(j) \hat{I}_{A_{k_j}}(0) U(j)^\dagger \) is the projector \( \hat{I}_{A_{k_j}}(0) \) at \( t = j \) and \( \hat{I}_{A_{k_j}}(0) = \text{symb}^{-1}(I_{A_{k_j}(\phi)}) \) [4]. Therefore, if we replace \( \mu(B(\hbar)) \) by \( (\prod_{j=0}^n \hat{I}_{A_{k_j}(j)}(\hat{I}) \) in the eq. (2), we have

\[
\frac{h_{\mu}(T, Q)}{n} = \lim_{n \to \infty} \frac{1}{n} \sup_{B_{k_j}(\hbar)} H(B(\hbar)) = \\
\frac{1}{n} \sum_{n} \mu(B(\hbar)) \log \mu(B(\hbar)) = \\
\frac{1}{n} \sum_{n} \left( \sum_{l=1}^{R_n} \log(I_{A_{l}(\phi)}(\tilde{I})) \right) \\
= \int_\mathcal{M} \sum_{\sigma_i(x) > 0} \sigma_i(x) d\mu = \\
\sup_{Q} \left\{ - \frac{1}{n} \sum_{n} \left( \sum_{l=1}^{R_n} \log(I_{A_{l}(\phi)}(\tilde{I})) \right) \right\}
\]

(15)

where \( R_n \) is the number of elements of \( B(\hbar) \). Then from eq. (3) and (4) we have the quantum Pesin theorem

\[
\int_\mathcal{M} \sum_{\sigma_i(x) > 0} \sigma_i(x) d\mu = \\
\sup_{Q} \left\{ - \frac{1}{n} \sum_{n} \left( \sum_{l=1}^{R_n} \log(I_{A_{l}(\phi)}(\tilde{I})) \right) \right\}
\]

(16)

where \( \sigma_i(x) \) are all the positive Lyapunov exponents contained in a finite region \( \mathcal{M} \) of phase space \( \Gamma \) and \( d\mu = d\hat{\phi}^{2(N+1)} \).

The main goal of this theorem is that the Lyapunov exponents of the classical limit can be calculated from the sum of (16) which only involves quantum mean values. Therefore, we see that the quantum Pesin theorem has an associated method to obtain the Lyapunov exponents. The prescription of this method is the following:

\[1\] Where it is not necessarily \( \hat{H} = \hat{H}^\dagger \). In an irreversible process \( \hat{H} \neq \hat{H}^\dagger \) and typically \( \hat{H} \) describes an effective Hamiltonian of the system in interaction with its environment.
(1) Given a partition \( Q = \{ A_i : i = 1, \ldots, m \} \).

(2) we calculate the operators \( \hat{I}_{A_{k_j}}(j) \) where \( \hat{I}_{A_{k_j}}(0) = \text{sym}^{-1}(I_{A_{k_j}}(\phi)) \) and \( k_j = 1, \ldots, m \).

(3) Then we calculate \( \prod_{j=0}^{n} \hat{I}_{A_{k_j}}(j)\hat{I} \) for all \( n \).

(4) Now we perform the sum and the limit of \( \prod_{j=0}^{n} \hat{I}_{A_{k_j}}(j)\hat{I} \) and take the supreme over all partitions \( Q \).

(5) Therefore, from the quantum Pesin theorem we conclude that the result obtained is the sum of the Lyapunov exponents integrated over all phase space.

However, the sum of \( \prod_{j=0}^{n} \hat{I}_{A_{k_j}}(j)\hat{I} \) can be very difficult to calculate or we might only be interested in determining if the classical limit is chaotic or not. In both cases the prescription can be reduced using the lemma (see [21] pag. 364). It says that if \( \mu(B_l(-n)) \) decreases exponentially then the Kolmogorov-Sinai entropy is positive. From this and eq. (11), (2) and (3) we have the positivity of the Lyapunov exponents. Moreover, since

\[ \mu(B_l(-n)) = (\prod_{j=0}^{n} \hat{I}_{A_{k_j}}(j)\hat{I}) \]

in the classical limit \( h \to 0 \) then we have the following “weaker” prescription.

(1') Idem (1).

(2') Idem (2).

(3') Idem (3).

(4') If \( (\prod_{j=0}^{n} \hat{I}_{A_{k_j}}(j)\hat{I}) \) decreases exponentially when \( n \to \infty \), we conclude the positivity of the Lyapunov exponents of the classical limit and therefore, we have a chaotic classical limit.

In the next section we will use the prescription (1' - 4') on an example of the literature to illustrate the physical relevance of the quantum Pesin theorem.

5 Physical relevance

In this section we illustrate the quantum Pesin theorem with an example of the decoherence literature: A phenomenological Gamow type model is a single oscillator embedded in an environment composed of a large bath of noninteracting oscillators which can be considered a continuum. The degeneracies of this system become useless the application of the perturbation theory. Instead, we can apply an analytical extension of the Hamiltonian (see [20], [34], [35], [36], [37]) to obtain an non-hermitian effective Hamiltonian given by

\[ \hat{H} = \sum_{n=0}^{\infty} z_n |n\rangle \langle \tilde{n}| \]

where \( z_n = n(\omega_0 - i\gamma_0) \) are complex eigenvalues (except \( z_0 = \omega_0 \), \( n = 0, 1, 2, \ldots \)). The natural frequency of the single oscillator is \( \omega_0 \) and \( \gamma_0 \) is associated with the relaxation time \( t_R \) by \( t_R = \frac{\omega_0}{\gamma_0} \) (see [19] pag. 288). From the formula (29) (see appendix section 7.1) we have

\[ \hat{I}_{A_{k_j}}(j) = \alpha_{A_{k_j}}(0,0)\langle 0| + \sum_{n=1}^{\infty} \alpha_{A_{k_j}}(n,n)e^{-2\pi n j} |n\rangle \langle \tilde{n}| + \]

\[ + \sum_{n=1}^{\infty} \alpha_{A_{k_j}}(n,0)e^{\pi n j} e^{-2\pi nj} |n\rangle \langle \tilde{n}| + \]

\[ + \sum_{n,m>0, n\neq m} \alpha_{A_{k_j}}(n,m)e^{-2\pi (n+m) j} |n\rangle \langle \tilde{n}| \]

(18)

Then we see that for \( j \gg \frac{\omega_0}{\gamma_0} = t_R \) all the sums of \( \prod_{j=0}^{n} \hat{I}_{A_{k_j}}(j)\hat{I} \) decay very rapidly and therefore we can neglect these terms, that is

\[ \hat{I}_{A_{k_j}}(j) \simeq \alpha_{A_{k_j}}(0,0)\langle 0| \text{ for } j \gg \frac{\omega_0}{\gamma_0} = t_R \]

(19)

Thus from (19) it follows that for \( n \gg t_R \) we have

\[ \prod_{j=0}^{n} \hat{I}_{A_{k_j}}(j) \simeq \left( \prod_{j=0}^{n} \alpha_{A_{k_j}}(0,0) \right) |0\rangle \langle 0| \]

(20)

and therefore,

\[ \left( \prod_{j=0}^{n} \hat{I}_{A_{k_j}}(j)\hat{I} \right) \simeq \prod_{j=0}^{n} \alpha_{A_{k_j}}(0,0) \]

(21)

\(^2\)If \( n \gg t_R \) in particular \( \hat{I}_{A_{k_n}}(n) \simeq \alpha_{A_{k_n}}(0,0)\langle 0| \langle 0| \) is diagonal. Then, \( \prod_{j=0}^{n} \hat{I}_{A_{k_j}}(j) = \hat{I}_{A_{k_0}}(0)\hat{I}_{A_{k_1}}(1)\ldots\hat{I}_{A_{k_n}}(n) \simeq \hat{I}_{A_{k_0}}(0)\hat{I}_{A_{k_1}}(1)\ldots\hat{I}_{A_{k_n}}(n) \simeq \prod_{j=0}^{n} \alpha_{A_{k_j}}(0,0) |0\rangle \langle 0| \) is diagonal regardless of whether the projectors \( \hat{I}_{A_{k_0}}(0), \hat{I}_{A_{k_1}}(1), \ldots, \hat{I}_{A_{k_{n-1}}}(n-1) \) are diagonals or not.
On the other hand, when $j \to \infty$ we have

$$
\mu(A_{kj}(j)) = (\hat{I}_{A_{kj}(j)}|\hat{I}) = \alpha_{A_{kj}}(0,0) + \\
+ \sum_{n=1}^{\infty} \alpha_{A_{kj}}(n,n)e^{-2\pi n \hbar j} \to \alpha_{A_{kj}}(0,0)
$$

where $\mu(A_{kj}(j))$ can not tend to zero. In such case since $\{A_k\}$ is a partition the measure of the phase space $\Gamma$ will tend to zero. Therefore, $\alpha_{A_{kj}}(0,0) \neq 0$.

Moreover, since $\hat{I}_{A_{kj}(0)} = \text{symb}^{-1}(I_{A_{kj}}(\phi))$ is a projector then $\alpha_{A_{kj}}(0,0) = 0|\hat{I}_{A_{kj}}(0)|0$ is positive for all $k_j \in \{1,\ldots,m\}$ where $m$ is the number of elements of the partition $Q = \{A_1,A_2,\ldots,A_m\}$. Now since the phase space $\Gamma$ is normalized then

$$
\mu(\Gamma) = 1 = \sum_k \mu(A_k) \geq \alpha_{A_{kj}}(0,0).
$$

Then we have $0 < \alpha_{A_{kj}}(0,0) < 1$ for all $k_j$. If we define $\alpha = \min_k \{\alpha_{A_k}\}, \beta = \max_k \{\alpha_{A_k}\}$ then from (21) we have

$$
\alpha^{n+1} \leq \prod_{j=0}^{n} \alpha_{A_{kj}}(0,0) \leq \beta^{n+1} \text{ as } n \to \infty
$$

with $0 < \alpha, \beta < 1$. From the inequality (23) and the eq. (20) we conclude that $(\prod_{j=0}^{n} \hat{I}_{A_{kj}}(j)|\hat{I})$ decreases exponentially and then from the prescription (1') - (4') we conclude the positivity of the Lyapunov exponents of the classical limit of a phenomenological Gamow type model. Therefore, the classical limit of this model is chaotic.

6 Conclusions

We have obtained a semiclassical version of the Pesin theorem [10] that can allow us to express the Lyapunov exponents in terms of quantum mean values $(\prod_{j=0}^{n} \hat{I}_{A_{kj}}(j)|\hat{I})$ in the classical limit $\hbar \to 0$. Applying this relation to the phenomenological Gamow type model we have concluded the positivity of the Lyapunov exponents of its classical limit and therefore its chaotic behavior.

The exponential decay of $\mu(B_l(-n))$ was obtained as a result of the diagonalization of the operators $\hat{I}_{A_{kj}}(j)$ in the limit $j \to \infty$. As this also happens in any quantum system whose Hamiltonian has only one real eigenvalue and the others complex, we conclude that all these systems have chaotic classical limit (more precisely, their classical limit are Kolmogorov-Sinai).

We emphasize the most remarkable consequence of the quantum Pesin theorem: It provides a simple method (see the prescription (1') - (4') of section 4) to determine if the classical limit is chaotic

$$
(\prod_{j=0}^{n} \hat{I}_{A_{kj}}(j)|\hat{I}) \text{ decreases exponentially}
$$

\[\downarrow\downarrow\]

chaotic classical limit

From this relation we see the usefulness of the quantum Pesin theorem. The chaotic classical limit is simply obtained from the positivity of the Lyapunov exponents which can be determined by the exponential decreasing of the mean values.

References

[1] R. Alicki, A. Lozinski, P. Pakonski, K. Zyczkowski, J. Phys. A, 37, 5157-5172, 2004.
[2] W. Slomczynski, K. Zyczkowski, J. Math. Phys., 35, 5674-5700, 1994.
[3] W. Slomczynski, K. Zyczkowski, Phys. Rev. Lett., 9, 80, 1998.
[4] K. Zyczkowski, H. Wiedemann, W. Slomczynski, Vistas in Astronomy, 37, 153-156, 1993.
[5] F. M. Cucchietti, D. A. R. Dalvit, J. P. Paz and W. H. Zurek, Phys. Rev. Lett. 91, 210403 (2003).
[6] F. M. Cucchietti, H. M. Pastawski, and R. A. Jalabert, Phys. Rev. B 70, 035311 (2004).
[7] D. Monteoliva, J. P. Paz, Phys. Rev. Lett., 85, 3373,(2000).
[8] D. Monteoliva, J. P. Paz, Phys. Rev. E., 64, 056238,(2001).
[9] G. Bellot and J. Earman, Studies In History and Philosophy of Science 28, 147-182 (1997).
[10] J. Berkovitz, R. Frigg, F. Kronz, Stud. Hist. Phil. Mod. Phys., 37, 661-691, 2006.
[11] M. Castagnino, O. Lombardi, Phys. A, 388, 247-267, 2009.
[12] I. Gomez, M. Castagnino Physica A, 393, 112-131, 2014.
[13] H. Stockmann, Quantum Chaos - An Introduction, Cambridge Univ. Press, Cambridge, 1999.
In order to evaluate the KS entropy, we have to generate the following partition

\[ B(-n) = \bigcup_{j=0}^{n} T^{-j}Q = \{ \bigcap_{j=0}^{n} T^{-j}A_{k_j} : A_{k_j} \in Q \} \]

If \( B(-n)=\bigcap_{j=0}^{n} T^{-j}A_{k_j} \) is a generic element of \( B(-n) \) then the measure of \( B(-n) \) is given by

\[ \mu(B(-n)) = \mu(\bigcap_{j=0}^{n} T^{-j}A_{k_j}) = \int_{\bigcap_{j=0}^{n} T^{-j}A_{k_j}} d\phi = \int_{\phi} \left( \prod_{j=0}^{n} I_{A_{k_j}}(T^j \phi) \right) d\phi = \langle \bigwedge_{j=0}^{n} I_{A_{k_j}} \circ T^j, I(\phi) \rangle = \langle \text{symb} \left( \bigwedge_{j=0}^{n} I_{A_{k_j}} \circ T^j \right), \text{symb}(\hat{I}) \rangle = \langle \bigwedge_{j=0}^{n} \hat{I}_{A_{k_j}} \circ T^j, \hat{I} \rangle = \left( \prod_{j=0}^{n} \hat{I}_{A_{k_j}}(j) \right) \hat{I} \]

where we have used the following properties:

- The characteristic function of an intersection of sets is the product of the characteristic functions of each set.
- If \( T \) is bijective \( \Rightarrow I_{T^{-j}A_{k_j}}(\phi) = I_{A_{k_j}}(T^j \phi) \).
If $\hbar \to 0 \Rightarrow symb(\prod_{j} f_{j})(\phi) \simeq \prod_{j} f_{j}(\phi)$ (e.g. neglecting terms of order $O(\hbar)$). This property is the generalization of (8) for a product of $n$ functions $f_{i}$ which was used in the sixth equality of (26).

- $\hat{I}_{A_{kj}} \circ T_{j} = \hat{I}_{A_{kj}}(j) = \hat{U}(j)\hat{I}_{A_{kj}}(0)\hat{U}(j)^{\dagger}$ with $\hat{U}(j) = e^{-\frac{i}{\hbar}\hat{H}_{j}}$ the evolution operator. This property is a consequence of the Wigner transformation (5) which was used in the seventh equality of (26).

7.2 An expression for the operators $\hat{I}_{A_{kj}}$, the quantum version of the characteristic functions $I_{A_{kj}}$

We consider a Hamiltonian of the form

$$\hat{H} = \sum_{r} z_{r}|r\rangle\langle r|$$

(27)

where $z_{r} = Re(z_{r}) + iIm(z_{r})$ are complex eigenvalues (poles) and the set $\{|r\rangle\}$ are their eigenvectors. The set $\{|r\rangle\langle s|\}$ is a basis of the algebra of operators and then we have

$$\hat{I}_{A_{kj}}(0) = \sum_{r,s} \alpha_{A_{kj}}(r,s)|r\rangle\langle s|$$

(28)

Therefore,

$$\hat{I}_{A_{kj}}(j) = e^{-\frac{i}{\hbar}\hat{H}_{j}} \left( \sum_{r,s} \alpha_{A_{kj}}(r,s)|r\rangle\langle s| \right) e^{\frac{i}{\hbar}\hat{H}_{j}} =$$

$$e^{-\frac{i}{\hbar}\hat{H}_{j}} \left( \sum_{r,s} \alpha_{A_{kj}}(r,s)|r\rangle\langle s| \right) e^{\frac{i}{\hbar}\hat{H}_{j}}$$

$$= \sum_{p} \sum_{q} \alpha_{A_{kj}}(p,q) e^{-\frac{i}{\hbar}z_{p}} e^{\frac{i}{\hbar}z_{q}} |p\rangle\langle q|$$

(29)

where we have used the exponential of an operator ($e^{\hat{A}} = \sum_{k=0}^{\infty} \frac{1}{k!} \hat{A}^{k}$) and the orthogonal relations of the projectors $|p\rangle\langle p|$ and $|q\rangle\langle q|$. That is,

$$|p\rangle\langle p|^k = |p\rangle\langle p|,$$

$$|q\rangle\langle q|^k = |q\rangle\langle q|, \text{ and}$$

$$\langle \tilde{p}|p'\rangle = \langle \tilde{q}|q'\rangle = 0 \text{ if } p \neq p', q \neq q'$$

(30)