Extreme fractal structures in chaotic mechanical systems: riddled basins of attraction

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Abstract. Chaotic dynamical systems with certain phase space symmetries may exhibit riddled basins of attraction, which can be viewed as extreme fractal structures in the sense that, regardless of how small is the uncertainty in the determination of an initial condition, we cannot decrease the fraction of such points that are certain to converge to a given attractor. We investigate a mechanical system exhibiting riddled basins of attraction: a particle under a two-dimensional potential with friction and time-periodic forcing. The verification of riddling is made by checking its mathematical requirements through computation of finite-time Lyapunov exponents as well as by scaling laws describing the fine structure of basin filaments densely intertwined in phase space.

1. Introduction

The connection between chaos and fractal geometry is a cornerstone of nonlinear dynamics. Fractal sets appear quite often in nonlinear dynamics due to the stretch-and-fold construction leading to chaotic attractors [1]. Fractal basin boundaries have been observed in many dissipative dynamical systems with multiple attractors, leading to final-state sensitivity [2]. By the latter we mean that, even if a considerable improvement is made in the accuracy with which an initial condition is known, little is gained in diminishing the fraction of uncertain initial conditions, i.e. the relative number of initial conditions which we do not know a priori the final state they asymptote to [3].

Non-attracting fractal sets are also important for nonlinear dynamics, for they are the underlying structure of chaotic orbits. In dissipative dynamical systems, such sets are exemplified by the strange saddles, which are formed by the intersection of the stable and unstable manifolds of unstable hyperbolic fixed points, containing a dense orbit. In conservative dynamical systems, such non-attracting fractal sets are responsible for many phenomena like fractal escape boundaries [4], Wada basins [5], and chaotic scattering [6].

One epitome of the connection between chaos and fractals is the horseshoe set discovered by Smale in the 60’s [7]. A horseshoe is a non-attracting invariant Cantor-like fractal set with (i) a countable set of periodic orbits; (ii) an uncountable set of bounded non-periodic orbits; (iii) a dense orbit [1]. The strange saddle, for example, being formed by an infinite number of transversal intersection of invariant manifolds, is a fractal horseshoe-like set. If an orbit starts off
but near the strange saddle, it will wander erratically so as to closely approach a large number of unstable orbits embedded in the strange saddle, forming the basis of transient chaos [8].

An extreme form of fractality is represented by the so-called riddled sets. Such sets are so intertwined that they virtually defy any attempt to improve the final-state predictability. In other words, while for fractal basins a great reduction of the initial state uncertainty is necessary to yield a better predictability, for riddled basins no improvement at all would be achieved by means of this procedure. This is so because of the rather stringent mathematical requirements for riddled sets, particularly the existence of phase-space symmetries leading to invariant manifolds in which chaotic dynamics takes place, and with conspicuous transversal stability properties [9].

In spite of these mathematical requirements, it turns out that riddled sets have been observed and many dynamical systems and even in experiments. Basin riddling was named and described by Alexander et al. [10], who also presented a variety of illustrative examples chosen from discrete maps. Independently, however, riddling has been also noticed by Pikowsky and Grassberger [11]. Riddled basins have been described in mechanical systems, like coupled elastic arches [12], and were generally found to occur in spatially extended dynamical systems, like coupled oscillators (where the invariant set contains a chaotic synchronized state) [13, 14] and coupled circuits [15, 16]. Other applications include ecological population models [17, 18], learning dynamical systems [19], chemical reactions of the Belouzov-Zhabotinsky type [20], and in models of interdependent open economies [21].

In this paper we are to revisit an example first studied by Ott and co-workers [22, 23, 24], namely of a particle under a two-dimensional potential function, subjected to both friction and time-periodic forcing. In References [22, 23, 24] there was considered the case where the invariant set contains a chaotic attractor of the type exhibited by the forced Duffing equation, obtaining riddled basins of attraction for wide parameter intervals. On the other hand, the Duffing equation can be viewed as an approximation of the behavior of a nonlinear pendulum, so we have studied the full driven pendulum in order to verify that it also able to exhibit riddling.

The existence of riddling can be verified in two basic ways: the first is to check that all mathematical conditions for riddling are fulfilled by the system. This direct approach can be pursued by investigating the behavior of Lyapunov exponents along the directions transversal to the invariant set in which the chaotic attractor lies. The second approach handles scaling laws that relate the basin structures with the properties of finite-time Lyapunov exponents, and is based on a stochastic model developed by Ott et al. [24]. We have verified the presence of riddled basins using both approaches.

This paper is organized as follows: in Section II we outline the basic concepts of riddled basins of attraction, focusing on the computation of finite-time Lyapunov exponents to be used further. Section III presents the mechanical system to be investigated whereas Section IV contains the our results on riddled basins, using a characterization via two scaling laws. Our Conclusions are left to the final Section.

2. Riddled basins of attraction

The most common example of a riddled set is a riddled basin of attraction, which may appear in chaotic systems having certain symmetries and quite general topological and metric properties. A dynamical system may have a chaotic attractor $A$ whose basin of attraction is riddled with “holes” (in a measure-theoretical sense) belonging to the basin of another (non necessarily chaotic) attractor $B$. In other words, riddled basins of attraction never exhibit a disk in phase space, because every point in the basin of attraction of attractor $A$ has pieces of the basin of attraction of attractor $B$ arbitrarily nearby [9, 10].

Riddling affects dramatically our ability of predicting what attractor the trajectory originating from a given initial condition asymptotes to. Let $p$ be an arbitrary point belonging to the basin of a chaotic attractor $A$. If the basin of $A$ is riddled by the basin of the other
attractor $B$, then a small ball of radius $\epsilon$ centered at $p$ has a nonzero fraction of its volume belonging to the basin of $B$, irrespective of how small the radius $\epsilon$ might be [Fig. 1]. Hence, if we regard this $\epsilon$-ball as an uncertainty neighborhood related to the (numerical or experimental) determination of the initial condition, the resulting trajectory always has a positive probability of falling into the basin of the other attractor. In other words, the probability of escaping from the basin of attractor $A$ is nonzero for every uncertainty $\epsilon$. Consequently, riddling implies an extreme form of final-state sensitivity.

In fractal basins of attraction, the uncertain fraction of initial conditions scales with the uncertainty radius as $f(\epsilon) \sim \epsilon^\alpha$, where $0 < \alpha < 1$ is the uncertainty exponent. If $D$ is the dimension of the phase space and $D_0$ is the box-counting dimension of the fractal basin boundary, then it can be proved that $\alpha = D - D_H$ [3]. From this point of view, a riddled basin is associated with an exponent $\alpha = 0$ (within numerical accuracy), meaning that the uncertain fraction is always 100%, regardless of the value of $\epsilon$.

In order to express the mathematical requirements for riddling, we first give the definition of attractor to be used throughout this work. Let $H$ be the phase space in which the dynamical system $F$ is defined (it is a discrete-time map, but continuous-time flows can be also described by $F$, if Poincaré sections are taken). A closed subset $A \in H$ is an attractor of $F$ if it satisfies the following conditions:

- $A$ has a basin of attraction, denoted $\beta(A)$, of positive Lebesgue measure (volume) in the phase space $H$.
- $A$ is a compact set with a dense orbit. In the Milnor definition of attractor, the basin of attraction does not need to include the whole neighborhood of the attractor [25].

If the basin of attraction of $A$ has positive Lebesgue measure, $A$ is a weak Milnor attractor [26]. This measure-theoretical definition implies the following set of conditions under which riddled basins occur in a dynamical system [23]

1. There is an invariant subspace $M \in H$ whose dimension $d_M$ is less than that of the phase space $d_H$.
2. The dynamics on the invariant subspace $M$ has a chaotic attractor $A$;
3. There is another attractor $B$, chaotic or not, and not belonging to the invariant subspace $M$.
4. The attractor $A$ is transversely stable in the phase space $H$, i.e., for typical orbits on the attractor the Lyapunov exponents for infinitesimal perturbations along the directions transversal to the invariant subspace $M$ are all negative.
5. A set of unstable periodic orbits embedded in the chaotic attractor $A$ becomes transversely unstable. As a consequence, at least one of the Lyapunov exponents along directions transverse to $M$, although negative for almost any orbit of $A$, experiences positive finite-time fluctuations.

Condition 1 is a consequence of the system having some symmetry which enables it to display an invariant subspace $M$, in the sense that, once an initial condition is exactly placed on $M$, the resulting trajectory cannot escape from $M$ for further times. To have riddling, it is necessary to have a dense set of points with zero Lebesgue measure in the attractor lying in the invariant subspace which are transversely unstable, thus it is necessary that this attractor be chaotic (condition 2). The existence of another attractor (condition 3) is necessary for the basin of an attractor to be riddled with holes belonging to the basin of this second attractor.

1 The uncertainty involved in numerical simulations is related to the truncation of numbers, whereas in experiments is due to the accuracy of the measurements.
If the transverse Lyapunov exponents of typical orbits lying in the invariant subspace $\mathcal{M}$ are all negative (condition 4), then $\mathcal{A}$ is an attractor in the weak Milnor sense, and its basin has positive Lebesgue measure. Condition 5 states that, while the invariant subspace $\mathcal{M}$ is still transversely stable, there will be trajectories on the attractor $\mathcal{A}$ that are transversely unstable. Condition 4 can be quantitatively checked by computing the maximal Lyapunov exponent along a transversal direction to $\mathcal{M}$. Verifying condition 5, on the other hand, would require the computation of finite-time Lyapunov exponents.

The finite-time Lyapunov exponents for a $d_H$-dimensional map $F$ are defined as follows. Let $n$ be a positive integer and $DF^n(x_0)$ be the Jacobian matrix of the $n$-times iterated map, with entries evaluated at an initial condition $x_0 \in \mathcal{A}$. Supposing that the singular values of $DF^n(x_0)$ are ordered: $\xi_1(x_0, n) \geq \xi_2(x_0, n) \geq \ldots \geq \xi_{d_H}(x_0, n)$, the $k$-th time-$n$ Lyapunov exponent for the point $x_0$ is defined as

$$\tilde{\lambda}_k(x_0; n) = \frac{1}{n} \ln ||DF^n(x_0) \cdot v_k||,$$

where $k = 1, 2, \ldots, d_H$ and $v_k$ is the singular vector related to $\xi_k(x_0, n)$.

The infinite time limit of the above expression is the usual Lyapunov exponent $\lambda_k = \lim_{n \to \infty} \tilde{\lambda}_k(x_0, n)$. Although the time-$n$ exponent $\tilde{\lambda}_k(x_0, n)$ generally takes on a different value, depending on the chosen point, its infinite time limit takes on the same value for almost all $x_0$ with respect to the natural ergodic measure of the invariant set $\mathcal{A}$.

If the attractor in $\mathcal{A}$ is chaotic (or hyper-chaotic, in general), there are $d_M$ infinite-time Lyapunov exponents related to directions in the invariant subspace $\mathcal{M}$, of which many can be positive; and $d_H - d_M$ Lyapunov exponents along directions transverse to $\mathcal{M}$. For the sake of determining the transversal stability of $\mathcal{M}$ it suffices to consider the largest one, $\lambda_\perp$. Hence, condition 4 implies that $\lambda_\perp$ takes on a negative value:

$$\lambda_\perp = \lim_{n \to \infty} \tilde{\lambda}_\perp(x_0, n) < 0.$$

The possible existence of an infinite number of transversely unstable orbits embedded in a transversely stable attractor implies fluctuations of the finite-time largest transversal exponent, $\tilde{\lambda}_\perp(x_0, n)$. Consider the probability distribution $P(\tilde{\lambda}_\perp(x_0, n))$, from which we can obtain the
average value of this exponent (assuming proper normalization):

\[ \langle \tilde{\lambda}_\perp(n) \rangle = \int_{-\infty}^{+\infty} \tilde{\lambda}_\perp(n) P(\tilde{\lambda}_\perp(n)) d\tilde{\lambda}_\perp(n), \] (3)

When \( n \) is large enough this probability distribution function is assumed to be Gaussian [27]:

\[ P(\tilde{\lambda}_\perp(x_0, n)) \approx \sqrt{\frac{n G''(\lambda_\perp)}{2\pi}} \exp \left[ -\frac{n G''(\lambda_\perp)}{2} (\tilde{\lambda}_\perp(n) - \lambda_\perp)^2 \right], \] (4)

where \( G \) is a function with the following convexity properties:

\[ G(\lambda_\perp) = G'(\lambda_\perp) = 0, \quad G''(\lambda_\perp) > 0. \]

Substituting (4) into (3) there results

\[ \langle \tilde{\lambda}_\perp(x_0, n) \rangle = \lambda_\perp. \] (5)

The standard deviation in the Gaussian approximation approaches zero with \( n^{-1/2} \), when \( n \gg 1 \), we can write the corresponding variance as

\[ \sigma^2 = \langle (\tilde{\lambda}_\perp(n) - < \lambda_\perp >)^2 \rangle = \frac{2D}{n}, \] (6)

such that the diffusion coefficient \( D = 1/2G''(\lambda_\perp) \) is independent of \( n \).

In terms of the distribution of finite-time transversal exponents, condition 5 for riddling implies that there is a positive fraction of positive values of \( \tilde{\lambda}_\perp(n) \) for initial conditions \( x_0 \) randomly chosen in the attractor \( A \), i.e.,

\[ \phi(n) = \int_0^\infty P(\tilde{\lambda}_\perp(n)) d\tilde{\lambda}_\perp(n) > 0. \] (7)

3. A mechanical system with riddled basins

Let us consider a point particle of unit mass under the influence of a two-dimensional potential \( V(x, y) \), viscous friction, and a time-periodic external forcing, whose equation of motion is

\[ \ddot{r} + b \dot{r} + \nabla V(r) = A \sin(\omega t) \hat{x}, \] (8)

where the potential well is given as [Fig. 2]

\[ V(x, y) = -a \cos x + (x + \chi) y^2 - y^4, \] (9)

with \( b, \chi, A \) and \( \omega \) being, respectively, the dissipation coefficient, a characteristic scale length, and the amplitude and frequency of the driving force.

The phase space \( H \) of this system is five-dimensional, with coordinates \( x, p_x, y, p_y, \) and \( \theta \equiv \omega t \mod 2\pi \), in terms of which the equation of motion can be written as the autonomous vector field

\[ \begin{align*}
\dot{x} &= p_x, \\ \dot{p}_x &= -bp_x - a \sin x - y^2 + A \sin \theta, \\ \dot{y} &= p_y, \\ \dot{p}_y &= -bp_y - 2y(x + \chi) - 4y^3, \\ \dot{\theta} &= \omega,
\end{align*} \] (10-14)
Figure 2. Potential function (9), with $a = 1.0$ and $\chi = 2.5$.

Figure 3. Chaotic attractor of the driven damped pendulum in the invariant plane $y = p_y = 0$ in the phase space, with $a = 1.0$, $b = 0.22$, $A = 2.7$, $\chi = 2.5$, and $\omega = 1.0$. 
Due to the symmetry $V(x, y) = V(x, -y)$ the particle dynamics is invariant with respect to the $y \rightarrow -y$ and $p_y \rightarrow -p_y$ transformations in $H$. Thus the constraints $y = 0$ and $p_y = 0$ define a three-dimensional invariant subspace $M \subset H$ in such a way that a trajectory with an initial condition $(x(0), y(0) = 0, p_x(0), p_y(0) = 0, \theta(0))$ belonging to $M$ will remain there forever (condition 1).

Restricted to this invariant subspace, the system state is described by the coordinates $(x, p_x, \theta)$, whose dynamics is governed by the equations of the forced damped pendulum [28]

\begin{align*}
\dot{x} &= p_x, \quad (15) \\
\dot{p}_x &= -bp_x - a \sin x + A \sin \theta, \quad (16) \\
\dot{\theta} &= \omega. \quad (17)
\end{align*}

which has, for conveniently chosen values of its parameters, like $a = 1.0$, $b = 0.22$, $\chi = 2.5$, $A = 2.7$, and $\omega = 1.0$, a chaotic attractor $A$ in the invariant subspace $M$, depicted in Fig. 3, and which fulfills condition 2 for riddling. From now on we will fix all parameters but the characteristic length $\chi$, which will be our variable parameter (however, for all considered values of $\xi$ we have obtained chaotic attractors in the invariant subspace). There is another attractor at infinity $B = (|y| = \infty, |p_y| = \infty)$ for the same set of parameters (condition 3).

We integrated Eqs. (10)-(14) by using an adaptive stepsize fourth-order Runge-Kutta scheme. The dynamics of this time-periodic flow can be studied by means of a stroboscopic, or time-$T$ map: we consider the values of the phase space variables at discrete time instants satisfying $\theta = 0$, mod $2\pi$, i.e. we sample the values of $(x(t_n), p_x(t_n); y(t_n), p_y(t_n))$ where $t_n = 2n\pi/\omega$ for $n = 0, 1, 2, \ldots$. For ease of visualization we consider the two-dimensional projection of these variables in the plane $(x, y)$.

A graphical representation of the basins is obtained by considering a unit square in this projection and sparkle randomly $10^5$ over this region [Fig. 4(a)]. Each of these initial condition was integrated for $10^5$ iterations. If the resulting trajectory was found to asymptote to the attractor $A$ the corresponding initial condition was marked with a black pixel, otherwise the trajectory was supposed to go to the attractor at infinity and no dot was marked. We used $|y| > 20$ as a criterion to determine whether or not the trajectory is going to infinity (varying this number does not affect significantly our results, though). The black and white and regions in Fig. 4(a) are thus graphical approximations of the basins of $A$ and $B$, respectively. Both regions appear to be densely interwoven, which is confirmed by magnifications suggesting that

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Fig4}
\caption{(a) The white and black dots represent, respectively, the basins of a chaotic attractor in the invariant subspace at $y = p_y = 0$ and the attractor at infinity. The parameters were the same as in the previous figure. (b) Magnification of a small region chosen from (a).}
\end{figure}
regions containing black dots also contain white dots [Fig. 4(b)]. As we will see later on, this will prove to be true for any region containing black dots, irrespective of how small it might be. We remark, however, that the basin of infinity is not riddled, since magnifications of a number of regions containing black points do not present blank areas.

Although the basins appear to be intertwined, a quantitative characterization of riddling needs a deeper analysis, where the verification of the remaining conditions for riddling demands the computation of Lyapunov exponents in the directions transversal to $M$. There are two such exponents, the largest one being denoted as $\lambda_\perp$, and which is computed by taking a variation of Eqs. (12) and (13) and setting $y = p_y = 0$, what results in

$$\dot{\delta y} = \delta p_y, \quad \dot{\delta p_y} = -b \delta p_y - 2 [x(t) + \chi] \delta y - 4(\delta y)^3,$$

where $x(t)$ is computed from a trajectory in the invariant subspace $M$, obtained by solving Eqs. (15)-(17), and which can be regarded as a driving term for Eq. (19).

The largest finite-time transversal exponent, computed for the set of initial conditions $x_0 \equiv \{x(0), y(0) = 0, v_x(0), v_y(0) = 0, \theta(0)\}$ is then given by

$$\tilde{\lambda}_\perp(x_0, t) = \frac{1}{t} \ln \left[ \frac{\delta(t)}{\delta(0)} \right], \quad (20)$$

where

$$\delta(t) = \sqrt{[\delta y(t)]^2 + [\delta v_y(t)]^2} \quad (21)$$

is the magnitude of a transversal displacement to the invariant subspace.

The infinite-time limit of Eq. (20), $\lambda_\perp$, determines the transversal stability of the Duffing attractor belonging to $M$. Figure 5 depicts its variation with the system parameter $\chi$, showing that $\lambda_\perp$ takes on negative values for $\chi > \chi_c \approx 2.315$ (condition 4). Moreover, if the basin of the

![Figure 5. Transversal Lyapunov exponent as a function of the control parameter $\chi$. The remaining parameters are the same as in the previous figures.](image-url)
Figure 6. Probability distribution functions for the time-$n$ transversal Lyapunov exponents for different values of $t$ and $\chi = 2.5 < \chi_c$. The remaining parameters are the same as in the previous figures.

Chaotic attractor $A$ is riddled by the basin of the attractor at infinity, condition 5 implies that there are negative as well as positive fluctuations of $\tilde{\lambda}_\perp(t)$, in such a way that there is a nonzero fraction of positive finite-time Lyapunov exponents for $\chi > \chi_c$.

We have verified this condition by computing the probability distribution function $P(\tilde{\lambda}_\perp(x_0, t))$ related to transversal time-$t$ Lyapunov exponent. Numerical approximations of this PDF are depicted in Figure 6 for some values of $t$, when $\chi$ takes on a value higher than $\chi_c$, thus in a regime where condition 4 for riddling holds. The PDF becomes increasingly more akin of a Gaussian function as the time $t$ increases from 20 to 50, as already expected on general bases (large deviation statistics [29]). In all cases the probability distribution function includes negative as well as positive values of $\tilde{\lambda}_\perp(x_0, t)$, hence there is a nonzero positive fraction $\phi(t)$, thus verifying condition 5 for riddling. Figure 7 shows the variation of the positive fraction with the control parameter $\chi$.

4. Scaling laws for riddled basins

4.1. Fine structure of basin filaments

The onset of riddling has been shown to occur through a bifurcation suffered by some low-period unstable periodic orbit embedded in the chaotic orbit $A$ lying on the invariant manifold $M$ [30]. Exactly at the bifurcation, this orbit (which is already unstable along the invariant manifold, as it is embedded in a chaotic set) loses transversal stability and becomes a repeller along some direction orthogonal to $M$. Due to the nonlinearity of the transversal dynamics there develops, just after the bifurcation takes place, a narrow tongue anchored at the unstable orbit, as schematically shown by Fig. 8. If an initial condition starts within this narrow tongue, even if it is very close to the invariant set $M$, the resulting trajectory will asymptote to the infinity.

Once this bifurcation occurs for an unstable orbit in $M$, the same formation of tongues occurs for every preimage of this orbit, yielding a dense set of tongue-like sets anchored at the corresponding preimages [30]. These sets belong to the basin of the attractor at infinity $B$. 
Figure 7. Positive fraction of time-50 transversal Lyapunov exponent as a function of $\chi$.

Figure 8. Schematic figure showing the structure of riddled basins near the invariant subspace that contains a chaotic attractor.
Figure 9. Fraction of length of a line at $y = \ell$ belonging to the basin of infinity, as a function of $\ell$. Numerical results, least squares fit and the theoretical prediction given by Eq. (23) are given, respectively, by filled circles, thick dashed line, and thin dashed line.

The characteristic feature of riddling is that those tongues have widths that tend to zero as we approach the invariant manifold containing the chaotic attractor $A$. Hence the basin of $A$ always contains pieces of basins of $B$, regardless the distance to the manifold $M$, so forming a fine structure of basin filaments.

This fine structure can be quantitatively characterized by the following numerical experiment [23, 24]: let us consider the invariant manifold at $y = 0$ and take $y$ as the transversal distance to the manifold. We draw a horizontal line at $y = \ell$, and evaluate the fraction $V_\ell$ of its length that belongs to the basin of the chaotic attractor $A$. If the basin of $A$ is riddled with tongues belonging to the basin of infinity, it follows that for any distance $\ell$, no matter how small, there is always a nonzero value of $V_\ell$. Since the tongues are anchored at the invariant subspace $y = 0$, this fraction tends to zero as $\ell \to 0$ (in the limit it is a Lebesgue measure zero set). Accordingly, the fraction of length belonging to the basin of the infinity is $P_* = 1 - V_\ell$, and it is expected to scale with $\ell$ as a power law

$$P_*(\ell) \sim \ell^\eta,$$

where $\eta > 0$ is a characteristic scaling exponent.

Figure 9 shows (filled circles) the results of such numerical experiment for the same parameter values used in plotting the basins shown in Fig. 4. The dashed line represents a least squares fit with exponent $\eta = 0.72 \pm 0.02$, where the errors were estimated from the expression $[P_*(1 - P_*)/N]^{1/2}$, following Ref. [24]. Ott and co-workers have developed a stochastic model for the time evolution of the transversal finite-time Lyapunov exponents [24]. Although they are correlated through the deterministic chaotic process governing the system dynamics [in our case the equations (10)-(14)], if the correlation time is short enough we can ignore such correlations and treat the finite-time Lyapunov exponents $\tilde{\lambda}_\perp(x_0, n)$ as statistically independent innovations. This approximation is supposed to be valid if the system is near the point where the invariant manifold $M$ loses transversal stability as a whole.

The assumption of statistically independent values of $\tilde{\lambda}_\perp(x_0, n)$ enables us to model the
stochastic dynamics of finite-time exponents as a biased random walk described by a Fokker-Planck equation. The stationary solution of the latter, with reflecting boundary conditions at $y = 0$, predicts a scaling law for $P_*$ of the same form as (22), where the exponent is given by the following expression [23, 24]:

$$\eta = \frac{|\lambda_\perp|}{D},$$

(23)

where $\lambda_\perp$ is the infinite-time limit of the transversal Lyapunov exponent, and $D$ is the diffusion coefficient obtained from the Fokker-Planck equation through the variance of the statistically independent fluctuations. The thin dashed line in Fig. 9 represents the result of the theoretical model based on this stochastic dynamics assumption, giving a prediction of $\eta = 0.60$ for the scaling exponent. As we can see, the agreement is fairly good, specially when larger values of $\ell$ are considered.

4.2 Uncertainty exponent

Another quantitative characterization of riddled basins is based on the uncertainty exponent described in the Introduction, and which gives a measure of the extreme final-state sensitivity due to riddling. Consider again the line $y = \ell$ drawn in the phase space portrait as described earlier, and choose randomly an initial condition $x(0) = x_0$ on that line. Now choose randomly another initial condition $x'(0)$ with uniform probability within an interval of length $2\epsilon$ and centered at $x_0$ [Fig. 10]. If both points belong to different basins, they can be referred to as $\epsilon$-uncertain [2, 3].

The fraction of $\epsilon$-uncertain points, or uncertain fraction, which will be denoted by $< p >$, is the probability of making a mistake when attempting to predict to which basin the initial condition $(x_0, y_0 = \ell)$ belongs, given a measurement uncertainty $\epsilon$. This probability is expected,
on general grounds, to scale with the latter as another power law of the form
\[ <p> \sim \epsilon^\varphi, \] (24)
where \( \varphi \geq 0 \) depends on both \( x \) and \( \ell \). The stochastic model of Ott et al. gives the following expression for this exponent in terms of the infinite-time Lyapunov exponents
\[ \varphi = \frac{\lambda_2^2}{4D\lambda_\parallel}. \] (25)

Figure 11 presents our results for this second numerical experiment, for \( <p> \) versus \( \epsilon \) for a given \( \ell \). The dashed line is a least squares fit of the numerical data, yielding an exponent \( \varphi = 0.280 \pm 0.005 \), where the errors were estimated from the formula \( [<p>(1-<p>)/N]^{1/2} \), as for the previous scaling law. This value compares reasonably with the prediction of the stochastic model (25), which gives \( \varphi = 0.10 \), although the agreement here is not so good as for the first scaling law. Both values of this exponent are near zero, showing that indeed the fractality here is extreme. The smallness of \( \varphi \) is ultimately due to its quadratic dependence on \( \lambda_\perp \) which is itself very small near the transition to a riddled basin. This explains only partially, however, the disagreement between the numerical and theoretical values for the scaling exponent.

The most important factor is that the theoretical model assumes a random walk behavior for the finite-time transversal Lyapunov exponents, thus neglecting any correlation between them, what it is naturally a crude approximation, except very near the blowout bifurcation point (which, in our model, is at \( \chi_c \)). Outside this rather particular case, we cannot expect but a qualitative agreement between model and numerics.

Just to give a quantitative example of how difficult is to predict the final state of the mechanical system here studied. Suppose we are to improve the accuracy with which the initial conditions are found. A great effort is then hypothetically devoted to improve this accuracy by \( N \) orders of magnitude. This means to decrease the uncertainty by about the same factor. Due to the smallness of the uncertainty exponent, the uncertain fraction is expected to decrease a mere with respect to its previous value, whatever it was estimated. Hence no significant improvement in final state predictability was gained even though a massive effort was done to decrease uncertainties. This plagues every physical system exhibiting riddling.

5. Conclusions

Riddled basins of attraction are typical in systems of physical interest. In this paper we consider a particle under a two-dimensional potential function with friction and time-periodic forcing. Thanks to a symmetry of the potential function, the system presents an invariant subspace in which there is a chaotic attractor similar to that of a driven damped pendulum. The riddled basins appear if the transversal dynamics to this subspace present a maximal negative Lyapunov exponent but also transversely unstable periodic orbits embedded in the chaotic attractor.

These two apparently contradictory requirements are possible once the natural measure of a chaotic attractor is supported by an infinite number of unstable periodic orbits. The closed set of transversely stable orbits (embedded in the chaotic attractor) which makes the Lyapunov exponent to be negative coexists with an open set of transversely unstable orbits. The latter can be detected through positive fluctuations of the finite-time Lyapunov exponent (maximal along the transversal direction).

We have used two numerical diagnostics for characterization of riddling: (i) the existence of the scaling law for the fraction of the basin of the second attractor (at infinity) when it riddles the basin of the chaotic attractor lying in the invariant subspace; (ii) the computation of the finite-time Lyapunov exponents along the transversal directions in order to verify the existence
Figure 11. Fraction of uncertain initial condition as a function of the uncertainty radius. Numerical results, least squares fit and the theoretical prediction given by Eq. (23) are given, respectively, by filled circles, thick dashed line, and thin dashed line.

of positive values. In particular, we have verified that the scaling laws agree reasonably well with a theoretical model developed by Ott et al. [24] using a biased random walk with reflecting barrier.

Both diagnostics, however, have their shortcomings. The mere existence of a scaling law for the fraction of length of some line drawn transversely to the basin filaments does not necessarily imply riddling. The reason is that any map with an unstable fixed point belonging to an invariant subspace could display the same scaling if there are trajectories forming separatrices. The latter divide the local unstable subspace of this point into different regions according to whether points are asymptotic to an attractor inside or outside the invariant subspace. On the other hand, the mere existence of positive fluctuations of the finite-time transversal Lyapunov exponents cannot warrant the existence of riddled basins, since it may also occur in other contexts, like the proximity of a near-tangency (“glitch”) of the stable and unstable subspaces of a periodic saddle orbit. We conclude that the characterization of riddling, in practice, relies on the combination of a number of numerical diagnostics as well as some knowledge of the periodic orbit structure of the system.

The agreement between our numerical results and the model of Ott et al. is already expected to be partial, since the stochastic model implies that the finite-time Lyapunov exponents are statistically independent variables, which obviously they are not, since the deterministic chaotic dynamics unavoidably introduces correlations which, however, decay with time. If this decay time is short enough the use of stochastic models is acceptable as an approximation. Moreover, we observed that the stochastic approximation is better if the contributions of positive and negative fluctuations are of comparable importance, a situation more likely to occur near the blowout point, where the chaotic attractor as a whole loses transversal stability.
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