ON THE BLOWUP AND LIFESPAN OF SMOOTH SOLUTIONS TO A CLASS OF 2-D NONLINEAR WAVE EQUATIONS WITH SMALL INITIAL DATA

BY

LI JUN (Department of Mathematics and IMS, Nanjing University, Nanjing 210093, People’s Republic of China),

INGO WITT (Mathematical Institute, University of Göttingen, Bunsenstr. 3-5, D-37073 Göttingen, Germany),

AND

YIN HUICHENG (Department of Mathematics and IMS, Nanjing University, Nanjing 210093, People’s Republic of China)

Abstract. We are concerned with a class of two-dimensional nonlinear wave equations $\partial_t^2 u - \text{div}(c^2(u) \nabla u) = 0$ or $\partial_t^2 u - c(u) \text{div}(c(u) \nabla u) = 0$ with small initial data $(u(0, x), \partial_t u(0, x)) = (\varepsilon u_0(x), \varepsilon u_1(x))$, where $c(u)$ is a smooth function, $c(0) \neq 0$, $x \in \mathbb{R}^2$, $u_0(x), u_1(x) \in C_0^\infty(\mathbb{R}^2)$ depend only on $r = \sqrt{x_1^2 + x_2^2}$, and $\varepsilon > 0$ is sufficiently small. Such equations arise in a pressure-gradient model of fluid dynamics, as well as in a liquid crystal model or other variational wave equations. When $c'(0) \neq 0$ or $c'(0) = 0$, $c''(0) \neq 0$, we establish blowup and determine the lifespan of smooth solutions.

1. Introduction and main results. In this paper, we shall focus on two-dimensional nonlinear wave equations of the form

$$\begin{cases}
\partial_t^2 u - \text{div}(c^2(u) \nabla u) = 0, \\
u(0, x) = \varepsilon u_0(x), \quad \partial_t u(0, x) = \varepsilon u_1(x),
\end{cases}$$

(1.1)
where \( c(u) \) is a smooth function with \( c(0) \neq 0, \ x \in \mathbb{R}^2, \ u_0(x), \ u_1(x) \in C_0^\infty(\mathbb{R}^2) \) depend only on \( r = \sqrt{x_1^2 + x_2^2} \), and \( \varepsilon > 0 \) is sufficiently small. We will assume \( c(u) = 1 + u + O(u^2) \) or \( c(u) = 1 + u^2 + O(u^3) \) without loss of generality.

Equation (1.1) has an interesting physical background. In [1], [28], a pressure-gradient model for the positive pressure function \( P \) derived from the 2-D compressible full Euler system takes the form

\[
\partial_t \left( \frac{\partial_t P}{P} \right) - \Delta P = 0.
\]

When initial data \( (P(0,x), \partial_t P(0,x)) = (1 + \varepsilon P_0(x), \varepsilon P_1(x)) \) is given and one sets \( u(t,x) = \ln P(t,x) \), then one obtains \( \partial_t^2 u - \text{div}(e^u \nabla u) = 0 \) with initial data \( u(0,x) = \ln(1 + \varepsilon P_0(x)) \) and \( \partial_t u(0,x) = \varepsilon P_1(x)(1 + \varepsilon P_0(x))^{-1} \). This is the case of \( c(u) = \exp \left( \frac{u}{2} \right) \) in (1.1). By [7], [13], [23], [27], a 2-D liquid crystal equation, or variational wave equation, takes the form \( \partial_t^2 u - c(u) \text{div}(c(u) \nabla u) = 0 \). Especially for the nematic liquid crystal equation, one has \( c(u) = \alpha \cos^2 u + \beta \sin^2 u \) with positive constants \( \alpha \) and \( \beta \) satisfying \( \alpha \neq \beta \). In this case, \( c(u) = \alpha + (\beta - \alpha) \sin^2 u = \alpha + (\beta - \alpha)u^2 + O(u^3) \), which essentially corresponds to \( c(u) = 1 + u^2 + O(u^3) \) in (1.1).

There has been extensive and remarkable work concerning the global existence or blowup and lifespan of smooth solutions to \( n \)-dimensional \( (n \geq 2) \) nonlinear wave equations of the form

\[
\begin{cases}
\sum_{i,j=0}^n g_{ij}(u, \nabla u) \partial_{ij}^2 u = f(u, \nabla u, \nabla^2 u), \\
u(0,x) = \varepsilon u_0(x), \quad \partial_{x_0} u(0,x) = \varepsilon u_1(x),
\end{cases}
\]

(1.2)

where \( x_0 = t, \ x = (x_1, \ldots, x_n), \ \nabla = \nabla_{x_0,x}, \ g_{ij} \) and \( f \) are smooth functions of their arguments that satisfy \( g_{ij}(u, \nabla u) = c_{ij} + O(|u| + |\nabla u|) \) and \( f(u, \nabla u, \nabla^2 u) = O(|u|^2 + |\nabla u|^2 + |\nabla^2 u|^2) \), respectively, the \( c_{ij} \) are constants, and the linear operator \( \sum_{i,j=0}^n c_{ij} \partial_{ij}^2 \) is strictly hyperbolic with respect to time \( t \). For the functions \( g_{ij} \) and \( f \) independent of \( u \), for \( n \geq 4 \), it has been shown that (1.2) admits a global smooth solution (see [9], [10], [17]). For \( n = 2, 3 \), the authors of [3], [5], [12], [15] obtained the global existence if null conditions hold. Otherwise, if these null conditions do not hold, then smooth solutions blow up in finite time and their lifespan can be explicitly determined in terms of the initial data (see [2], [3], [8], [11], [14], [25], and the references therein). If the functions \( g_{ij} \) and \( f \) depend on \( u \) and the derivatives of \( u \), then problem (1.2) is much harder and there are some partial results on the global existence or blowup and lifespan of smooth solutions when \( 2 \leq n \leq 4 \) (for \( n \geq 5 \), (1.2) admits a global solution; see [18], [22]). For \( 2 \leq n \leq 4 \), lower bounds on the lifespan under some suitable restrictions were obtained in [18], [19], [22], and the references therein. We especially point out that if the equation in (1.2) has the form \( \partial_t^2 u - (1 + u) \Delta u = 0 \), then, for \( n = 3 \), the authors of [4] and [20], [21] established the global existence of smooth solution. These solutions, however, often exhibit a behavior at infinity much different from that of solutions to a linear wave equation.
In this paper, we will concentrate on the nonlinear wave equation (1.1), show the finite-time blowup of smooth solution, and give an explicit expression for the lifespan $T_\varepsilon$ as $\varepsilon \to 0$. The main result reads:

**Theorem 1.1.** Let $u_0(x)$, $u_1(x) \in C_0^\infty(\mathbb{R}^2)$ depend only on $r = \sqrt{x_1^2 + x_2^2}$. If $u_0(x) \neq 0$ or $u_1(x) \neq 0$, then problem (1.1) possesses a $C^\infty$-solution for $0 \leq t < T_\varepsilon$, where $T_\varepsilon$ stands for the lifespan of smooth solution $u(t, x)$.

(i) For $c(u) = 1 + u + O(u^2)$,

$$
\lim_{\varepsilon \to 0} \varepsilon \sqrt{T_\varepsilon} = \tau_0 \equiv -\frac{1}{2 \min_{\sigma} F_0'(\sigma)}. 
$$

(ii) For $c(u) = 1 + u^2 + O(u^3)$,

$$
\lim_{\varepsilon \to 0} \varepsilon^2 \ln T_\varepsilon = \nu_0 \equiv -\frac{1}{2 \min_{\sigma} \{F_0(\sigma)F_0'(\sigma)\}}. 
$$

Here $F_0(\sigma)$ is the Friedlander radiation field for the 2-D linear wave equation $\Box w = 0$ with initial data $(w(0, x), \partial_t w(0, x)) = (u_0(r), u_1(r))$.

Recall that $F_0(\sigma) = \frac{1}{2\pi\sqrt{2}} \int_{-\infty}^{+\infty} \frac{R(s; u_1) - R_0(s; u_0)}{\sqrt{s - \sigma}} ds$, where

$$
R(s; u_i) = \int \delta(s - \langle \omega, x \rangle) u_i(x) \, dx = \int_{-\infty}^{\infty} u_i(\sqrt{s^2 + y^2}) \, dy 
$$

is the Radon transform of $u_i(r)$ ($i = 0, 1$).

Henceforth, we shall also assume that $u_0$, $u_1$ are supported in the disk $B(0, M)$, where $M > 0$.

**Remark 1.1.** It follows from Theorem 1.1 that smooth solutions to (1.1) blow up in finite time provided that $u_0(x) \neq 0$ or $u_1(x) \neq 0$ because $F_0 \neq 0$, $F_0(M) = 0$, and $\lim_{\sigma \to \infty} F_0(\sigma) = 0$. Further properties of the function $F_0(\sigma)$ can be found in [10, Theorem 6.2.2].

**Remark 1.2.** For $c(u) = 1 + O(u^3)$, (1.1) admits a global smooth solution (see [15]).

**Remark 1.3.** If $c(u) = 1 + c_1 u + O(u^2)$ or $c(u) = 1 + c_2 u^2 + O(u^3)$ in (1.1), with $c_1 \neq 0$ and $c_2 \neq 0$, then one also has finite-time blowup of smooth solutions, and one can establish an explicit expressions for $T_\varepsilon$ as in Theorem 1.1.

**Remark 1.4.** For the Cauchy problem for the 1-D liquid crystal equation $\partial_t^2 u - c(u)\partial_x(c(u)\partial_x u) = 0$ with $(u(0, x), \partial_t u(0, x)) = (u_0(x), u_1(x))$, it has been shown in [7, Theorem 1] that if there exist positive constants $c_0 < c_1$ and an $m_0 \in \mathbb{R}$ such that $c_0 \leq c(u) \leq c_1$ for all $u \in \mathbb{R}$ and $c'(m_0) \neq 0$, then the $C^1$-solution $u(t, x)$ with special initial data $u_0(x) = m_0 + \varepsilon \phi(\frac{x}{\varepsilon})$ and $u_1(x) = -\text{sgn}(c'(m_0))c(u_0(x))u_0'(x)$ blows up in finite time; here $\phi \in C_0^1(0, 1)$ and $\varepsilon > 0$ is sufficiently small. However, for small and in general smooth initial data $(\varepsilon u_0(x), \varepsilon u_1(x))$ with compact support, which certainly do not satisfy the assumptions on the initial data made in [7, Theorem 1], there are no blowup results and precise estimates of the lifespan for the 1-D or 2-D liquid crystal equations available in the literature so far. In fact, from the proof of Theorem 1.1 and Remark 2.1(ii), it follows that smooth solutions to the 2-D liquid crystal equation
\[ \partial_t^2 u - c(u) \text{div}(c(u) \nabla u) = 0 \]
develop singularities in finite time for arbitrarily given small and smooth, spherically symmetric initial data with compact support.

**Remark 1.5.** From the proof of Theorem 1.1, we infer that \( \lim_{t \to T^-} \| \nabla u(t, \cdot) \|_{L^\infty} = \infty. \)

Note that this is different from the geometric blowup \( \lim_{t \to T^-} \| \nabla^2 u(t, \cdot) \|_{L^\infty} = \infty \) that occurs for solutions \( u \in C^2([0, T_\varepsilon] \times \mathbb{R}^2) \) to the nonlinear wave equation \( \sum_{i,j=0}^2 g_{ij}(\nabla u) \partial_{ij}^2 u = 0 \) with small initial data \( (u(0, x), \partial_t u(0, x)) = (\varepsilon u_0(x), \varepsilon u_1(x)) \) (see \cite{2, 3}). In this latter case, a local shock is formed at blowup time for the unsteady potential flow equation (see \cite{24}).

Let us comment on the proof of Theorem 1.1. To show (1.3) or (1.4), first we study the lower bound on the lifespan \( T_\varepsilon \) for problem (1.1). As in \cite[Chapter 6]{10} or \cite{4, 6}, by constructing a suitable approximate solution \( u_a \) to (1.1) and then considering the difference of the exact solution \( u \) and \( u_a \), applying the Klainerman-Sobolev inequality, and further establishing some delicate energy estimate, we obtain the desired lower bound on the lifespan \( T_\varepsilon \). On the other hand, the solution \( u \) to (1.1) is spherically symmetric for \( t < T_\varepsilon \) due to the spherical symmetry of the initial data \((u_0, u_1)\). Based on this, we can change (1.1) to a \( 2 \times 2 \) equation system in the coordinates \((t, r)\), with \( u \) still appearing in the coefficients. Thanks to the good properties of the difference of the real solution \( u \) and the approximate solution \( u_a \) before blowup time \( T_\varepsilon \), we can treat the solution \( u \) and its derivatives simultaneously to obtain a precise estimate on the upper bound of \( T_\varepsilon \). We point out that the methods in this paper are partly motivated by \cite{9} and \cite{14}, where equations of the form \( \partial_t^2 u - c^2(\partial_t u) \Delta u = 0 \) with \( c'(0) \neq 0 \) were studied, but only the estimates of the first-order derivatives of \( u \) were required.

The paper is organized as follows. In Section 2, we construct an approximate solution \( u_a \) to (1.1) in the two cases \( c(u) = 1 + u + O(u^2) \) and \( c(u) = 1 + u^2 + O(u^3) \), and we establish some related estimates. In Section 3, we obtain the lower bound on the lifespan \( T_\varepsilon \) by continuous induction, studying the nonlinear equation satisfied by \( u - u_a \). In Section 4, we change the second-order equation in (1.1) to a \( 2 \times 2 \) first-order partial differential system and further establish some delicate estimates on \( u \) and \( \nabla u \). From this, together with the blowup lemma of Hörmander \cite[Lemma 1.3.2]{10}, we obtain the upper bound on \( T_\varepsilon \) and thus complete the proof of Theorem 1.1. Some useful auxiliary lemmas and conclusions are given in the appendix.

In what follows, we will make use of the following notation:

- \( Z \) stands for one of the Klainerman vector fields in the symmetric case,
  \[ \partial_r, \partial_t, S = t \partial_t + r \partial_r, H = r \partial_t + t \partial_r. \]
- \( \partial \) represents \( \partial_r \) or \( \partial_t \).
- The norm \( \| f \|_{L^2} \) stands for \( \| f(t, \cdot) \|_{L^2(\mathbb{R}^2)} \).

2. **Construction of an approximate solution to (1.1).** In this section, we construct a suitable approximate solution \( u_a \) to (1.1). Then the lower bound on \( T_\varepsilon \) is obtained, in Section 3, by a continuous induction argument through estimating the difference of the solution \( u \) and \( u_a \). As \( c(u) \) assumes the two different forms of \( c(u) = \ldots \)
1 + u + O(u^2) and c(u) = 1 + u^2 + O(u^3), respectively, and since the constructions will be slightly different in these two cases, we divide this section into two parts.

2.1. Construction of \( u_a \) when \( c(u) = 1 + u + O(u^2) \). As in [10, Chapter 6], we introduce the slow time variable \( \tau = \varepsilon \sqrt{1 + t} \) and assume that the solution to (1.1) can be approximated by

\[
\varepsilon r^{-\frac{1}{2}} V(\tau, \sigma), \quad r > 0,
\]

where \( \sigma = r - t \).

Let \( V(\tau, \sigma) \) solve the equation

\[
\begin{aligned}
\partial^2_{\tau\sigma} V + 2V \partial^2_{\sigma} V + 2(\partial_{\sigma} V)^2 &= 0, \quad (\tau, \sigma) \in \mathbb{R}^+ \times \mathbb{R}, \\
V(0, \sigma) &= F_0(\sigma), \\
\text{supp} V &\subseteq \{(\tau, \sigma) : \sigma \leq M\};
\end{aligned}
\]

(2.1)

\( F_0(\sigma) \) was introduced in Theorem 1.1.

Recall that \( \tau_0 \equiv -\frac{1}{2 \min_{\sigma} F_0'(\sigma)} > 0 \). With regard to problem (2.1), one then has

**Lemma 2.1.** Problem (2.1) admits a \( C^\infty \)-solution \( V(\tau, \sigma) \) for \( 0 \leq \tau < \tau_0 \), and \( V(\tau, \sigma) \) blows up as \( \tau \to \tau_0^- \).

**Proof.** Set \( W = \partial_{\sigma} V \). Then it follows from (2.1) that

\[
\begin{aligned}
\partial_{\tau} W + 2V \partial_{\sigma} W + 2W^2 &= 0, \quad (\tau, \sigma) \in \mathbb{R}^+ \times \mathbb{R}, \\
W(0, \sigma) &= F_0'(\sigma),
\end{aligned}
\]

(2.2)

The characteristic curve \( \sigma = \sigma(\tau, s) \) of (2.1) emanating from \( (0, s) \) is defined by

\[
\begin{aligned}
\frac{d\sigma}{d\tau}(\tau, s) &= 2V(\tau, \sigma(\tau, s)), \\
\sigma(0, s) &= s.
\end{aligned}
\]

(2.3)

Along characteristic curves, one has

\[
\begin{aligned}
\frac{dW}{d\tau}(\tau, \sigma(\tau, s)) + 2W^2(\tau, \sigma(\tau, s)) &= 0, \\
W(0, \sigma(0, s)) &= F_0'(s),
\end{aligned}
\]

which yields, for \( \tau < \tau_0 \),

\[
W(\tau, \sigma(\tau, s)) = \frac{F_0'(s)}{1 + 2F_0'(s)\tau}.
\]

(2.4)

Note that the equation in (2.1) is equivalent to \( \partial_{\sigma}(\partial_{\sigma} V + 2V \partial_{\sigma} V) = 0 \). Together with the boundary condition for \( V \) in (2.1), this yields

\[
\begin{aligned}
\frac{dV}{d\tau}(\tau, \sigma(\tau, s)) &= 0, \\
V(0, \sigma(0, s)) &= F_0(s).
\end{aligned}
\]

This means

\[
V(\tau, \sigma(\tau, s)) = F_0(s).
\]

(2.5)
From (2.5) and (2.3), one concludes
\[ \sigma(\tau, s) = s + 2F_0(s)\tau, \]
which implies that \( \partial_s \sigma(\tau, s) = 1 + 2F'_0(s)\tau > 0 \) for \( \tau < \tau_0 \). Thus it follows from the implicit function theorem that \( s = s(\tau, \sigma) \) is a smooth function of the variables \( \tau \) and \( \sigma \). Consequently, \( V(\tau, \sigma) = F_0(s(\tau, \sigma)) \) is a smooth solution to (2.1) for \( \tau < \tau_0 \), and as \( \tau \to \tau_0^- \), the derivative \( V_\sigma(\tau, \sigma) \) blows up due to (2.4). Lemma 2.1 is proved. \( \square \)

From [10, Chapter 6], one has that \( F_0(\sigma) \in C^\infty(\mathbb{R}) \) is supported in \(( -\infty, M) \] and obeys the estimates
\[ |F_0^{(k)}(\sigma)| \leq C_k(1 + |\sigma|)^{-\frac{1}{2} - k}, \quad k \in \mathbb{N}_0. \]

From (2.7), we now derive a decay estimate for \( V(\tau, \sigma) \) in (2.1) for \( \tau < \tau_0 \) and \( \sigma \to -\infty \).

**Lemma 2.2.** For any positive constant \( b < \tau_0 \) and \( 0 \leq \tau \leq b \), the smooth solution \( V \) to (2.1) satisfies the estimates
\[ |Z^a \partial^l \sigma^m V(\tau, \sigma)| \leq C_{ab}^{lm}(1 + |\sigma|)^{-\frac{1}{2} - l - m}, \quad \alpha, l, m \in \mathbb{N}_0, \]
where \( C_{ab}^{lm} \) are positive constants depending on \( b, \alpha, l, \) and \( m \).

**Proof.** When \( \tau \leq b \), it follows from (2.6)–(2.7) that \( |s| \leq |\sigma| \leq 2|s| \) for large \( |s| \). Together with (2.4)–(2.5), this yields
\[ |V(\tau, \sigma)| \leq C_b(1 + |\sigma|)^{-\frac{1}{2}}, \quad |\partial_\sigma V(\tau, \sigma)| \leq C_b(1 + |\sigma|)^{-\frac{3}{2}}. \]

By (2.6) and (2.4), one has
\[ \partial_\sigma s(\tau, \sigma) = \frac{1}{1 + 2F'_0(s)\tau} \]
and
\[ \partial^2_\sigma V(\tau, \sigma(s, \tau), s) = \frac{F''_0(s)}{(1 + 2F'_0(s)\tau)^2} - \frac{2F'_0(s)F''_0(s)}{(1 + 2F'_0(s)\tau)^3}, \]
which yields
\[ |\partial^2_\sigma V(\tau, \sigma)| \leq C_b(1 + |\sigma|)^{-\frac{3}{2}}. \]

On the other hand, it follows from (2.1) and (2.10) that
\[ |\partial^2_\tau V(\tau, \sigma)| \leq C_b(1 + |\sigma|)^{-\frac{3}{2}} \]
and further
\[ |\partial_\tau V(\tau, \sigma)| \leq C_b(1 + |\sigma|)^{-\frac{3}{2}}. \]

Based on (2.9)–(2.11), by an inductive argument, one arrives at
\[ |Z^a \partial^l \sigma^m V(\tau, \sigma)| \leq C_{ab}^{lm}(1 + |\sigma|)^{-\frac{1}{2} - l - m}, \quad l, m \in \mathbb{N}_0. \]

Due to \( S = \sigma \partial_\sigma + \frac{\varepsilon t}{2\sqrt{1 + t}} \partial_\tau \) and \( H = -\sigma \partial_\sigma + \frac{\varepsilon r}{2\sqrt{1 + t}} \partial_\tau \) by Lemma A.1(ii), one analogously obtains
\[ |Z^a \partial^l \sigma^m V(\tau, \sigma)| \leq C_{ab}^{lm}(1 + |\sigma|)^{-\frac{1}{2} - l - m}, \quad \alpha, l, m \in \mathbb{N}_0, \]
and this completes the proof of Lemma 2.2. \( \square \)
Next we construct an approximate solution $u^I_a$ to (1.1) for $0 \leq \tau = \varepsilon \sqrt{1 + t} < \tau_0$. Let $w_0$ be the solution of the linear wave equation
\[
\begin{cases}
\partial_t^2 w_0 - \Delta w_0 = 0, \\
 w_0(0, x) = u_0(x), \\
\partial_t w_0(0, x) = u_1(x).
\end{cases}
\]
It follows from [10, Theorem 6.2.1] that, for any constants $t > 0$ and $0 < m < 1$,
\[
|Z^\alpha (w_0(t, x) - r^{-\frac{1}{2}} F_0(\sigma))| \leq C_{\alpha l} (1 + t)^{-\frac{2}{3}} (1 + |\sigma|)^{\frac{1}{2}}, \quad r \geq \ell t, \tag{2.12}
\]
\[
|\partial^k w_0(t, x)| \leq C_{km} (1 + t)^{-1-|k|}, \quad r \leq mt. \tag{2.13}
\]
Choose a $C^\infty$-function $\chi(s)$ such that $\chi(s) = 1$ for $s \leq 1$ and $\chi(s) = 0$ for $s \geq 2$. For $0 \leq \tau = \varepsilon \sqrt{1 + t} < \tau_0$, we take the approximate solution $u^I_a$ to (1.1) to be
\[
u^I_a(t, x) = \varepsilon \left( \chi(\varepsilon t) w_0(t, x) + r^{-\frac{1}{2}} (1 - \chi(\varepsilon t)) \chi(-3\varepsilon \sigma) V(\sigma, \tau) \right). \tag{2.14}
\]
By Lemma 2.2 and [10, Theorem 6.2.1], one has that, for a fixed positive constant $b < \tau_0$,
\[
|Z^\alpha u^I_a(t, x)| \leq C_{\alpha b} \varepsilon (1 + t)^{-\frac{1}{2}} (1 + |\sigma|)^{-\frac{1}{2}}, \quad \tau \leq b. \tag{2.15}
\]
Set $J^I_a = \partial_t^2 u^I_a - c^2(u^I_a) \Delta u^I_a - 2c(u^I_a)c'(u^I_a) |\nabla u^I_a|^2$.

**Lemma 2.3.** One has
\[
\int_0^{\frac{\varepsilon^2}{\varepsilon^2 - 1}} ||Z^\alpha J^I_a||_{L^2} dt \leq C_{\alpha b} \varepsilon^{\frac{3}{2}}.
\]

**Proof.** We divide the proof into three parts.

(i) $0 \leq t \leq \frac{1}{\varepsilon}$. In this case, $\chi(\varepsilon t) = 1$ and $u^I_a = \varepsilon w_0$. This yields
\[
J^I_a = \varepsilon (1 - c^2(\varepsilon w_0)) \Delta w_0 - 2\varepsilon^2 c(\varepsilon w_0)c'(\varepsilon w_0) |\nabla w_0|^2.
\]

It follows from (2.15) and a direct computation that
\[
||Z^\alpha J^I_a||_{L^2} \leq C \varepsilon^2 (1 + t)^{-\frac{1}{2}}, \quad 0 \leq t \leq \frac{1}{\varepsilon}. \tag{2.16}
\]

(ii) $\frac{1}{\varepsilon} \leq t \leq \frac{2}{\varepsilon}$. Now we rewrite $u^I_a$ as
\[
u^I_a = \varepsilon w_0(t, x) + \varepsilon (1 - \chi(\varepsilon t)) \left( r^{-\frac{1}{2}} \chi(-3\varepsilon \sigma) V(\tau, \sigma) - w_0(t, x) \right).
\]
Then
\[
J^I_a = J_1 + J_2 + J_3 + J_4, \tag{2.17}
\]
where
\[
J_1 = (1 - c^2(u^I_a)) \Delta u^I_a - 2c(u^I_a)c'(u^I_a) |\nabla u^I_a|^2,
\]
\[
J_2 = \varepsilon (\partial_t^2 - \Delta) \left\{ (1 - \chi(\varepsilon t)) r^{-\frac{1}{2}} \chi(-3\varepsilon \sigma) \left( V(\tau, \sigma) - F_0(\sigma) \right) \right\},
\]
\[
J_3 = \varepsilon (\partial_t^2 - \Delta) \left\{ \chi(-3\varepsilon \sigma) \left( r^{-\frac{1}{2}} F_0(\sigma) - w_0(t, x) \right) \right\},
\]
\[
J_4 = \varepsilon (\partial_t^2 - \Delta) \left\{ (1 - \chi(\varepsilon t)) \chi(-3\varepsilon \sigma) - 1 \right\} w_0(t, x).
\]
We treat each $J_i$ ($1 \leq i \leq 4$) in (2.17) separately.
From (2.15) one obtains
\[ \|Z^\alpha J_1\|_{L^2} \leq C_{ab} \varepsilon^2 (1 + t)^{-\frac{1}{2}}. \]  
(2.18)

Since
\[
J_2 = \varepsilon r^{-\frac{1}{2}} (\partial_t - \partial_r)(\partial_t + \partial_r) \left\{ (1 - \chi(\varepsilon t)) \chi(\varepsilon r) (V(\tau, \sigma) - F_0(\sigma)) \right\} \\
- \frac{\varepsilon}{4} r^{-\frac{5}{2}} (1 - \chi(\varepsilon t)) \chi(\varepsilon r) (V(\tau, \sigma) - F_0(\sigma)) \\
= O(\varepsilon^3) r^{-\frac{1}{2}} (1 + |\sigma|)^{-\frac{3}{2}} + O(\varepsilon^2) r^{-\frac{1}{2}} \int_0^\tau \partial^2_{r\sigma} V(s, \sigma) \, ds \\
+ O(\varepsilon^2)(1 + t)^{-\frac{1}{2}} r^{-\frac{1}{2}} (1 + |\sigma|)^{-\frac{3}{2}} + O(\varepsilon)(1 + t)^{-\frac{3}{2}}
\]
and
\[
\left| Z^\alpha \left( r^{-\frac{1}{2}} \int_0^\tau \partial^2_{s\sigma} V(s, \sigma) \, ds \right) \right| \leq C_{ab} \varepsilon (1 + t)^{\frac{1}{2}} r^{-\frac{1}{2}} (1 + |\sigma|)^{-\frac{3}{2}},
\]
one has
\[ \|Z^\alpha J_2\|_{L^2} \leq C_{ab} \varepsilon^2 (1 + t)^{-\frac{1}{2}}. \]  
(2.19)

Note that \(-\frac{2}{3\varepsilon} \leq \sigma \leq M\) holds in the support of \(J_3\), which implies \(r \geq \frac{1}{3} t\). Together with (2.12), this yields
\[
J_3 = O(\varepsilon^3)(1 + |\sigma|)^{\frac{1}{2}} (1 + t)^{-\frac{3}{2}} + O(\varepsilon^2) \partial (r^{-\frac{1}{2}} F_0(\sigma) - w_0) \\
+ O(\varepsilon)(1 + t)^{-\frac{3}{2}} (1 + |\sigma|)^{-\frac{3}{2}}.
\]

On the other hand, it follows from property (i) of Lemma A.1 that
\[
|Z^\alpha \partial (r^{-\frac{1}{2}} F_0(\sigma) - w_0(t, x))| \leq C_{ab} |Z^\alpha (r^{-\frac{1}{2}} F_0(\sigma) - w_0(t, x))| \\
\leq C_{ab} (1 + |\sigma|)^{-\frac{1}{2}} |Z^\alpha (r^{-\frac{1}{2}} F_0(\sigma) - w_0(t, x))| \\
\leq C_{ab} (1 + t)^{-\frac{3}{2}} (1 + |\sigma|)^{-\frac{3}{2}}.
\]

One then obtains
\[ \|Z^\alpha J_3\|_{L^2} \leq C_{ab} \varepsilon^2 (1 + t)^{-\frac{1}{2}}. \]  
(2.20)

Analogously, together with (2.13), one arrives at
\[ \|Z^\alpha J_4\|_{L^2} \leq C_{ab} \varepsilon^2 (1 + t)^{-2}. \]  
(2.21)

Collecting (2.18)–(2.21) yields
\[ \|Z^\alpha J_5\|_{L^2} \leq C_{ab} \varepsilon^2 (1 + t)^{-\frac{3}{2}}, \quad \frac{1}{\varepsilon} \leq t \leq \frac{2}{\varepsilon}. \]  
(2.22)
(iii) $\frac{2}{\varepsilon} \leq t \leq \frac{b^2}{\varepsilon^2} - 1$. Together with (2.1), by a direct computation one has
\begin{align*}
J_a^I &= -\varepsilon^2 r^{-\frac{1}{2}} \partial_{\tau \sigma} \hat{V} \left( \frac{1}{\sqrt{1 + t}} - r^{-\frac{1}{2}} \right) \\
&\quad - \varepsilon^2 r^{-1} \left( \partial_{\tau \sigma} \hat{V} + 2 \hat{V} \partial_{\sigma}^2 \hat{V} + 2(\partial_{\sigma} \hat{V})^2 \right) \\
&\quad + O(\varepsilon^3) r^{-\frac{1}{2}} (1 + t)^{-1} \partial \hat{V} + O(\varepsilon^2)(1 + t)^{-\frac{3}{2}} r^{-\frac{1}{2}} \partial \hat{V},
\end{align*}
(2.23)
where $\hat{V}(\tau, \sigma) = \chi(-3\varepsilon \sigma) V(\tau, \sigma)$.

It follows from (2.1) that
\begin{align*}
\varepsilon^2 r^{-1} \left( \partial_{\tau \sigma} \hat{V} + 2 \hat{V} \partial_{\sigma}^2 \hat{V} + 2(\partial_{\sigma} \hat{V})^2 \right)
&= O(\varepsilon^3) r^{-1} (1 + |\sigma|)^{-\frac{3}{2}} - \varepsilon^2 r^{-1} \chi(-3\varepsilon \sigma)(1 - \chi(-3\varepsilon \sigma)) \partial_{\tau \sigma}^2 V \quad (2.24)
\end{align*}

where we have used the fact that $\chi(-3\varepsilon \sigma)(1 - \chi(-3\varepsilon \sigma))$ is supported in the interval $[-\frac{2}{3\varepsilon^3}, \frac{1}{3\varepsilon^3}]$.

Substituting (2.24) into (2.23) yields
\begin{align*}
\| Z^\alpha J_a^I \|_{L^2} \leq C_{\alpha b} \left( \varepsilon^3 (1 + t)^{-\frac{1}{2}} + \varepsilon^2 (1 + t)^{-\frac{3}{2}} \right).
\end{align*}
(2.25)
Consequently, combining (2.16), (2.22), and (2.25), one obtains
\begin{align*}
\int_0^{\frac{b^2}{\varepsilon^2} - 1} \| Z^\alpha J_a^I \|_{L^2} \, dt \leq C_{\alpha b} \varepsilon^{\frac{3}{2}},
\end{align*}
which finishes the proof of Lemma 2.3. \qed

2.2. Construction of $u_a$ when $c(u) = 1 + u^2 + O(u^3)$. When $c(u) = 1 + u^2 + O(u^3)$, set the slow time variable to $\tau = \varepsilon^2 \ln(1 + t)$ as in [4], and assume that the solution to (1.1) can be approximated by
\begin{align*}
\varepsilon r^{-\frac{1}{2}} G(\tau, \sigma), \quad r > 0,
\end{align*}
where $\sigma = r - t$ and where $G(\tau, \sigma)$ solves the equation
\begin{align*}
\begin{cases}
\partial_{\tau \sigma}^2 G + G \partial_{\sigma}^2 G + 2G(\partial_{\sigma} G)^2 = 0, & (\tau, \sigma) \in \mathbb{R}^+ \times \mathbb{R}, \\
G(0, \sigma) = F_0(\sigma), \\
supp G \subseteq \{(\tau, \sigma): \sigma \leq M\}.
\end{cases}
\end{align*}
(2.26)

Recall that $\nu_0 \equiv -\frac{1}{2 \min_{\sigma} \{F_0(\sigma) F'_0(\sigma)\}} > 0$. With regard to problem (2.26), one has

**Lemma 2.4.** Problem (2.26) admits a $C^\infty$-solution $G(\tau, \sigma)$ for $0 \leq \tau < \nu_0$, and $G(\tau, \sigma)$ blows up as $\tau \to \nu_0^-$. **
\textbf{Proof.} Set }Q = \partial_{\sigma}G. \text{ Then it follows from (2.26) that}
\begin{align}
\begin{cases}
\partial_{\tau}Q + G^{2}\partial_{\sigma}Q + 2GQ^{2} = 0, \\
Q(0, \sigma) = F_{0}'(\sigma), \\
\text{supp} Q \subseteq \{(\tau, \sigma) : \sigma \leq M\}.
\end{cases}
\end{align}
\text{(2.27)}

The characteristic curve \(\sigma = \sigma(\tau, s)\) of (2.26) emanating from \((0, s)\) is defined by
\begin{align}
\begin{cases}
\frac{d\sigma}{d\tau}(\tau, s) = G^{2}(\tau, \sigma(\tau, s)), \\
\sigma(0, s) = s.
\end{cases}
\end{align}
\text{(2.28)}

Along characteristic curves, one has
\begin{align}
\begin{cases}
\frac{dQ}{d\tau}(\tau, \sigma(\tau, s)) + 2(GQ^{2})(\tau, \sigma(\tau, s)) = 0, \\
Q(0, \sigma(0, s)) = F_{0}'(s)
\end{cases}
\end{align}
\text{and}
\begin{align}
\begin{cases}
\frac{dG}{d\tau}(\tau, \sigma(\tau, s)) = 0, \\
G(0, \sigma(0, s)) = F_{0}(s).
\end{cases}
\end{align}

Then, for \(\tau < \nu_{0}\),
\begin{align}
\begin{cases}
G(\tau, \sigma(\tau, s)) = F_{0}(s), \\
Q(\tau, \sigma(\tau, s)) = \frac{F_{0}'(s)}{1 + 2F_{0}(s)F_{0}'(s)}.
\end{cases}
\end{align}

Together with (2.28), this yields
\begin{align}
\sigma(\tau, s) = s + F_{0}^{2}(s)\tau,
\end{align}
\text{(2.29)}

which means that, for \(\tau < \nu_{0}\),
\begin{align}
\partial_{\tau}\sigma(\tau, s) = 1 + 2F_{0}(s)F_{0}'(s)\tau > 0.
\end{align}

Therefore, it follows from the implicit function theorem that \(s = s(\tau, \sigma)\) can be taken as a smooth function of \(\tau\) and \(\sigma\). Thus \(G(\tau, \sigma) = F_{0}(s(\tau, \sigma))\) is a smooth solution of (2.26) when \(\tau < \nu_{0}\). One then completes the proof of Lemma 2.4 as the one of Lemma 2.1. \(\square\)

Parallel to Lemma 2.2, one has

\textbf{Lemma 2.5. For any positive constant }b < \nu_{0} \text{ and } 0 \leq \tau \leq b, \text{ the smooth solution }G \text{ of (2.26) satisfies the estimates}
\begin{align}
|Z^{\alpha_{l}}\partial_{\tau}^{l}\partial_{\sigma}^{m}G(\tau, \sigma)| \leq C_{\alpha l m}^{\varepsilon}(1 + |\sigma|)^{-\frac{1}{2} - l - m}, \quad \alpha, l, m \in \mathbb{N}_{0}.
\end{align}
\text{(2.30)}

\textbf{Proof.} Since the proof is similar to that of Lemma 2.2, it is omitted. \(\square\)

Next we construct an approximate solution \(u_{a}^{\varepsilon}\) to (1.1) for \(0 \leq \tau = \varepsilon^{2}\ln(1 + t) < \nu_{0}\). As in the first part, we take the approximate solution \(u_{a}^{\varepsilon}\) of (1.1) to be
\begin{align}
\begin{aligned}
u_{a}^{\varepsilon}(t, x) &= \varepsilon \left(\chi(\varepsilon t)w_{0}(t, x) + r^{-\frac{1}{2}}(1 - \chi(\varepsilon t))(1 - 3\varepsilon \sigma)G(\tau, \sigma)\right).
\end{aligned}
\end{align}
\text{(2.31)}

By Lemma 2.5 and \cite{10} Theorem 6.2.1, one has
\begin{align}
|Z^{\alpha}u_{a}^{\varepsilon}(t, x)| \leq C_{\alpha l}^{\varepsilon}(1 + t)^{-\frac{1}{2} - \frac{1}{2}}(1 + |\sigma|)^{-\frac{1}{2}}, \quad \tau \leq b < \nu_{0}.
\end{align}
\text{(2.32)}
Set \( J^{II}_a = \partial_t^2 u^{II}_a - c_2 (u^{II}_a) \Delta u^{II}_a - 2c(u^{II}_a) c'(u^{II}_a) |\nabla u^{II}_a|^2 \). Then one has

**Lemma 2.6.**

\[
\int_0^{\frac{1}{\varepsilon^2} - 1} \| Z^a J^{II}_a \|_{L^2} \, dt \leq C_{ab} \varepsilon^\frac{3}{2} |\ln \varepsilon|.
\]

**Proof.** We divide this proof procedure into two parts.

(i) \( 0 \leq t \leq \frac{2}{\varepsilon} \). As in (i) and (ii) of the proof of Lemma 2.3, one has

\[
\| Z^a J^{II}_a \|_{L^2} \leq C_{ab} \varepsilon^2 |\ln \varepsilon| (1 + t)^{-\frac{1}{2}}.
\]

(2.33)

Note that the factor \( \ln \varepsilon \) in (2.33) appears due to

\[
G(\tau, \sigma) - F_0(\sigma) = \tau \int_0^\tau \partial_s G(s, \sigma) \, ds = \varepsilon^2 \ln(1 + t) \int_0^\tau \partial_s G(s, \sigma) \, ds = O(\varepsilon^2 |\ln \varepsilon|), \quad 1 \leq t \leq \frac{2}{\varepsilon}.
\]

(ii) \( \frac{2}{\varepsilon} \leq t \leq e^{\frac{1}{\varepsilon^2}} - 1 \). It follows from a direct computation that

\[
J^{II}_a = -2\varepsilon^3 r^{-\frac{1}{2}} \partial^2_{\tau \sigma} \hat{G} \left( \frac{1}{1 + t} - \frac{1}{r} \right)
- 2\varepsilon^3 r^{-\frac{1}{2}} \left( \partial^2_{\tau \sigma} \hat{G} + \hat{G}^2 \partial^2_{\sigma} \hat{G} + 2\hat{G} (\partial_{\sigma} \hat{G})^2 \right) + O(\varepsilon^3) (1 + t)^{-\frac{1}{2}},
\]

(2.34)

where \( \hat{G}(\tau, \sigma) = \chi(-3\varepsilon \sigma) G(\tau, \sigma) \).

Note that

\[
\varepsilon^3 r^{-\frac{1}{2}} \left( \partial^2_{\tau \sigma} \hat{G} + \hat{G}^2 \partial^2_{\sigma} \hat{G} + 2\hat{G} (\partial_{\sigma} \hat{G})^2 \right) = O(\varepsilon^4) r^{-\frac{3}{2}} (1 + |\sigma|)^{-\frac{3}{2}} + \varepsilon^3 r^{-\frac{3}{2}} \chi(-3\varepsilon \sigma) (1 - \chi^2(-3\varepsilon \sigma)) \partial^2_{\tau \sigma} G = O(\varepsilon^4) (1 + t)^{-1} r^{-\frac{3}{2}} (1 + |\sigma|)^{-\frac{3}{2}}.
\]

One then obtains

\[
\| Z^a J^{II}_a \|_{L^2} \leq C_{ab} \left( \varepsilon^4 (1 + t)^{-1} + \varepsilon^3 (1 + t)^{-\frac{3}{2}} \right).
\]

(2.35)

Combining (2.35) and (2.33) yields

\[
\int_0^{\frac{1}{\varepsilon^2} - 1} \| Z^a J^{II}_a \|_{L^2} \, dt \leq C_{ab} \varepsilon^\frac{3}{2} |\ln \varepsilon|.
\]

This completes the proof of Lemma 2.6. \( \square \)

**Remark 2.1.** Consider the 2-D variational wave equation \( \partial_t^2 u - c(u) \text{div}(c(u) \nabla u) = 0 \) with initial data \((u(0,x), \partial_t u(0,x)) = (\varepsilon u_0(x), \varepsilon u_1(x))\).
where \( \varepsilon \) of smooth solution to (2.37) (see Lemma A.5).

Applying the method of characteristics, one easily proves that (2.36) admits a smooth solution only for \( 0 \leq t < \tau_0 = -\frac{1}{\min F_0'(\sigma)} \).

(ii) Let \( c(u) = 1 + u^2 + O(u^3) \). As in Section 2.2, \( u \) can then be approximated by \( \varepsilon r^{-\frac{1}{2}} G(\tau, \sigma) \), where \( \tau = \varepsilon^2 \ln(1 + t) \), \( \sigma = r - t \), and \( G(\tau, \sigma) \) solves the equation

\[
\begin{align*}
\begin{aligned}
\frac{\partial^2 G}{\partial \tau^2} + 2 \frac{\partial G}{\partial \tau} + (\partial \sigma G)^2 &= 0, \\
G(0, \sigma) &= F_0(\sigma),
\end{aligned}
\end{align*}
\]  

(2.37)

As shown in [26, Theorem 1.2], for some special class of initial data \( G(0, \sigma) \), \( C^1 \)-solutions \( G \) to (2.37) blow up in finite time. The Friedlander radiation field \( F_0(\sigma) \), however, does not meet the assumptions of [26] due to \( F_0^{(k)}(M) = 0 \) for any \( k \in \mathbb{N}_0 \), which is different from \( F_0'(M) \neq 0 \) assumed in [26]. Nonetheless, one can still obtain the finite-time blowup of smooth solution to (2.37) (see Lemma A.5).

3. The lower bound on the lifespan \( T_\varepsilon \). In this section, based on the preparations in Section 2, we establish the lower bound on the lifespan \( T_\varepsilon \) by utilizing continuous induction and the energy method.

First we deal with the case that \( c(u) = 1 + u + O(u^2) \) in (1.1).

**Lemma 3.1.** Let \( c(u) = 1 + u + O(u^2) \). Then, for sufficiently small \( \varepsilon \) and \( 0 \leq \tau = \varepsilon \sqrt{1 + t} \leq b < \tau_0 \), (1.1) admits a \( C^\infty \)-solution \( u \) which satisfies the estimate

\[
|Z^\kappa \partial (u - u_\alpha^I)| \leq C_b \varepsilon^{\frac{3}{2}} (1 + t)^{-\frac{3}{2}} (1 + |t - r|)^{-\frac{1}{2}}
\]  

(3.1)

for \( |\kappa| \leq 2 \); here \( u_\alpha^I \) was introduced in (2.14).

**Proof.** Set \( v = u - u_\alpha^I \). Then

\[
\begin{align*}
\begin{aligned}
\frac{\partial^2 v}{\partial \tau^2} - \frac{\partial^2 v}{\partial x^2} &= F, \\
v(0, x) = \partial_x v(0, x) &= 0,
\end{aligned}
\end{align*}
\]  

(3.2)

where

\[
F = -J_\alpha^I + (c^2(u) - c^2(u_\alpha^I)) \Delta u_\alpha^I + 2c(u)c'(u) |\nabla v|^2 + 4c(u)c'(u) \nabla v \cdot \nabla u_\alpha^I + 2(c(u)c'(u) - c(u_\alpha^I)c'(u_\alpha^I)) |\nabla u_\alpha^I|^2.
\]  

(3.3)

We will use continuous induction to prove (3.1). To this end, we assume that, for some \( T \leq \frac{b^2}{\varepsilon^2} - 1 \),

\[
|Z^\kappa \partial v| \leq \varepsilon (1 + t)^{-\frac{1}{2}} (1 + |t - r|)^{-\frac{1}{2}}, \quad |\kappa| \leq 2, \quad t \leq T,
\]  

(3.4)
holds and then prove that
\[|Z^\alpha \partial v| \leq \frac{1}{2} \varepsilon (1 + t)^{-\frac{1}{2}} (1 + |t - r|)^{-\frac{1}{2}}, \quad |\kappa| \leq 2, \ t \leq T. \tag{3.5}\]

Note that from (3.4) one has
\[|Z^\kappa v| \leq C \varepsilon (1 + t)^{-\frac{1}{2}} (1 + |t - r|)^{\frac{1}{2}}, \quad |\kappa| \leq 2, \ t \leq T. \tag{3.6}\]

Applying \(Z^\alpha\) to both sides of (3.2) yields, for \(|\alpha| \leq 4\),
\[(\partial_t^2 - c^2(u)\Delta)Z^\alpha v = G \equiv \sum_{|\beta| \leq |\alpha|} C_{\alpha\beta}Z^\beta F + [Z^\alpha, (c^2(u) - 1)\Delta]v + \sum_{|\beta| < |\alpha|} C'_{\alpha\beta}Z^\beta ((c^2(u) - 1)\Delta v); \tag{3.7}\]

here the commutator relation \([Z^\alpha, \partial_t^2 - \Delta] = \sum_{|\beta| < |\alpha|} C''_{\alpha\beta}Z^\beta (\partial_t^2 - \Delta)\) with suitable constants \(C_{\alpha\beta}, C'_{\alpha\beta}, C''_{\alpha\beta}\) has been made use of.

Next we derive an estimate of \(\|\partial Z^\alpha v\|_{L^2}\) from (3.7). Define the energy
\[E(t) = \frac{1}{2} \sum_{|\alpha| \leq 4} \int_{\mathbb{R}^2} (|\partial_t Z^\alpha v|^2 + c^2(u)|\nabla Z^\alpha v|^2) \, dx.\]

Multiplying both sides of (3.7) by \(\partial_t Z^\alpha v\) (\(|\alpha| \leq 4\)), integrating by parts in \(\mathbb{R}^2\), and noting that \(|\partial u| = |\partial u^I_a + \partial v| \leq C_b \varepsilon (1 + t)^{-\frac{1}{2}}\) from the construction of \(u^I_a\) and assumption (3.4), one arrives at
\[E'(t) \leq \frac{C_b \varepsilon}{\sqrt{1 + t}} E(t) + \sum_{|\alpha| \leq 4} \int_{\mathbb{R}^2} |G| \cdot |\partial_t Z^\alpha v| \, dx. \tag{3.8}\]

Moreover, due to the inductive hypothesis (3.4) and (2.15), one has
\[|Z^\kappa u| \leq C_b \varepsilon (1 + t)^{-\frac{1}{2}} (1 + |\sigma|)^{\frac{1}{2}} \leq C_b \varepsilon, \quad |\kappa| \leq 2, \ t \leq T. \tag{3.9}\]

We now treat each term in the sum \(\sum_{|\alpha| \leq 4} \int_{\mathbb{R}^2} |G| \cdot |\partial_t Z^\alpha v| \, dx\) separately.

(A) Treatment of \(\sum_{|\beta| < |\alpha|} \int_{\mathbb{R}^2} |Z^\beta ((c^2(u) - 1)\Delta v)| \cdot |\partial_t Z^\alpha v| \, dx\). It follows from (3.9) that, for \(|\beta| < |\alpha|\),
\[
\int_{\mathbb{R}^2} |Z^\beta ((c^2(u) - 1)\Delta v)| \cdot |\partial_t Z^\alpha v| \, dx
\leq C_b \sum_{|\beta_1| + |\beta_2| = |\beta|} \int_{\mathbb{R}^2} |Z^{\beta_1} u| \cdot |Z^{\beta_2} \Delta v| \cdot |\partial_t Z^\alpha v| \, dx
\leq C_b \sum_{|\beta_1| + |\beta_2| = |\beta|} \int_{\mathbb{R}^2} |Z^{\beta_1} v| \cdot |Z^{\beta_2} \Delta v| \cdot |\partial_t Z^\alpha v| \, dx
+ C_b \sum_{|\beta_1| + |\beta_2| = |\beta|} \int_{\mathbb{R}^2} |Z^{\beta_1} u^I_a| \cdot |Z^{\beta_2} \Delta v| \cdot |\partial_t Z^\alpha v| \, dx. \tag{3.10}\]
The troublesome term in (3.10) is $Z^\beta v$ since a term of the form $Z^\beta v$ might not be contained in the energy $E(t)$. However, thanks to property (i) of Lemma A.1, one has

$$|Z^{\beta_2} \Delta v| \leq \frac{2}{1 + |t - r|} \sum_{|\beta_2| = |\beta_2| + 1} |Z^{\beta_2} \partial v|.$$ 

Due to $|\beta| < |\alpha| \leq 4$, by (3.4) and Lemma A.2, the first term in the right-hand side of (3.10) can then be estimated as

$$\int_{\mathbb{R}^2} |Z^{\beta_2} v| \cdot |Z^{\beta_2} \Delta v| \cdot |\partial_t Z^\alpha v| \, dx$$

$$\leq C_b \sum_{|\beta_2| = |\beta_2| + 1} \int_{\mathbb{R}^2} \frac{1}{1 + |t - r|} |Z^{\beta_2} v| \cdot |Z^{\beta_2} \partial v| \cdot |\partial_t Z^\alpha v| \, dx$$

$$\leq \frac{C_b \varepsilon}{\sqrt{1 + t}} E(t). \tag{3.11}$$

Analogously,

$$\int_{\mathbb{R}^2} |Z^{\beta_2} v| \cdot |\partial_t Z^\alpha v| \, dx \leq \frac{C_b \varepsilon}{\sqrt{1 + t}} E(t).$$

Therefore, one obtains

$$\sum_{|\beta| < |\alpha|} \int_{\mathbb{R}^2} |Z^\beta ((c^2(u) - 1)\Delta v) - |\partial_t Z^\alpha v| \, dx \leq \frac{C_b \varepsilon}{\sqrt{1 + t}} E(t). \tag{3.12}$$

**B) Treatment of**

$$\int_{\mathbb{R}^2} \left| \left[ Z^\alpha, (c^2(u) - 1) \Delta \right] v \cdot |\partial_t Z^\alpha v| \right| \, dx.$$

For

$$\int_{\mathbb{R}^2} \left| \left[ Z^\alpha, (c^2(u) - 1) \Delta \right] v \cdot |\partial_t Z^\alpha v| \right| \, dx$$

$$\leq C_b \sum_{|\alpha_1| + |\alpha_2| = |\alpha|, |\alpha_1| \geq 1} \int_{\mathbb{R}^2} |Z^{\alpha_1} u| \cdot |Z^{\alpha_2} \Delta v| \cdot |\partial_t Z^\alpha v| \, dx$$

$$\leq C_b \left( \sum_{|\alpha_1| + |\alpha_2| = |\alpha|, |\alpha_1| \geq 1} \int_{\mathbb{R}^2} |Z^{\alpha_1} u| \cdot |Z^{\alpha_2} \Delta v| \cdot |\partial_t Z^\alpha v| \, dx \right.$$  

$$+ \sum_{|\alpha_1| + |\alpha_2| = |\alpha|, |\alpha_1| \geq 1} \int_{\mathbb{R}^2} |Z^{\alpha_1} v| \cdot |Z^{\alpha_2} \Delta v| \cdot |\partial_t Z^\alpha v| \, dx \right)$$

by the same argument as in (3.11), one has

$$\int_{\mathbb{R}^2} \left| \left[ Z^\alpha, (c^2(u) - 1) \Delta \right] v \cdot |\partial_t Z^\alpha v| \right| \, dx \leq \frac{C_b \varepsilon}{\sqrt{1 + t}} E(t). \tag{3.13}$$

Next we treat each term $\int_{\mathbb{R}^2} |Z^\beta F| \cdot |\partial_t Z^\alpha v| \, dx$, $|\beta| \leq |\alpha|$, that is included in $\sum_{|\alpha| \leq 4} \int_{\mathbb{R}^2} |G| \cdot |\partial_t Z^\alpha v| \, dx$. 
(C) Treatment of \( \int_{\mathbb{R}^2} |Z^\beta J^I_a| : |\partial_t Z^\alpha v| \, dx \). In this case, one has
\[
\int_{\mathbb{R}^2} |Z^\beta J^I_a| : |\partial_t Z^\alpha v| \, dx \leq \|Z^\beta J_a\|_{L^2} \cdot \sqrt{E(t)}. \tag{3.14}
\]

(D) Treatment of \( \int_{\mathbb{R}^2} |Z^\beta ((c^2(u) - c^2(u^I_a))\Delta u^I_a)| : |\partial_t Z^\alpha v| \, dx \). Due to (3.9) and Lemmas A.1 and A.2, a direct computation yields
\[
\int_{\mathbb{R}^2} |Z^\beta ((c^2(u) - c^2(u^I_a))\Delta u^I_a)| : |\partial_t Z^\alpha v| \, dx
\leq C_b \sum_{|\beta_1| + |\beta_2| = |\beta|} \int_{\mathbb{R}^2} |Z^{\beta_1} v| : |Z^{\beta_2} \Delta u^I_a| : |\partial_t Z^\alpha v| \, dx
\leq C_b \sum_{|\beta_1| + |\beta_2| = |\beta|} \int_{\mathbb{R}^2} \frac{1}{1 + |t - r|} |Z^{\beta_1} v| : |Z^{\beta_2} \partial v^I_a| : |\partial_t Z^\alpha v| \, dx \tag{3.15}
\leq \frac{C_b \varepsilon}{\sqrt{1 + t}} E(t).
\]

(E) Treatment of \( \int_{\mathbb{R}^2} |Z^\beta (c(u)c'(u)|\nabla v|^2) : |\partial_t Z^\alpha v| \, dx \). Similarly to (D), one has
\[
\int_{\mathbb{R}^2} |Z^\beta (c(u)c'(u)|\nabla v|^2) : |\partial_t Z^\alpha v| \, dx \leq \frac{C_b \varepsilon}{\sqrt{1 + t}} E(t). \tag{3.16}
\]

(F) Treatment of \( \int_{\mathbb{R}^2} |Z^\beta (c(u)c'(u)|\nabla v \cdot \nabla u^I_a| : |\partial_t Z^\alpha v| \, dx \). It follows from a direct computation that
\[
\int_{\mathbb{R}^2} |Z^\beta (c(u)c'(u)|\nabla v \cdot \nabla u^I_a| : |\partial_t Z^\alpha v| \, dx
\leq C_b \sum_{|\beta_1| + |\beta_2| \leq |\beta|} |Z^{\beta_1} \partial v| : |Z^{\beta_2} \partial u^I_a| : |\partial_t Z^\alpha v| \, dx \tag{3.17}
\leq \frac{C_b \varepsilon}{\sqrt{1 + t}} E(t).
\]

(G) Treatment of \( \int_{\mathbb{R}^2} |Z^\beta ((c(u)c'(u) - c(u^I_a)c'(u^I_a))|\nabla u^I_a|^2) : |\partial_t Z^\alpha v| \, dx \). This case is also similar to (D). In particular, one has
\[
\int_{\mathbb{R}^2} |Z^\beta ((c(u)c'(u) - c(u^I_a)c'(u^I_a))|\nabla u^I_a|^2) : |\partial_t Z^\alpha v| \, dx \leq \frac{C_b \varepsilon}{\sqrt{1 + t}} E(t). \tag{3.18}
\]
Substituting (3.12)–(3.18) into (3.8) yields
\[
E'(t) \leq \frac{C_b \varepsilon}{\sqrt{1 + t}} E(t) + \sum_{|\beta| \leq 4} \|Z^\beta J^I_a\|_{L^2} \sqrt{E(t)}.
\]
Thus, by Lemmas 2.3 and A.3, one obtains
\[
\|\partial Z^\alpha v\|_{L^2} \leq C_b \varepsilon^2, \quad |\alpha| \leq 4,
\]
and further

\[ \|Z^\alpha \partial v\|_{L^2} \leq C_b \varepsilon^{\frac{3}{2}}, \quad |\alpha| \leq 4. \quad (3.19) \]

By (3.19) and the Klainerman-Sobolev inequality (see [10], [16]), one has

\[ |Z^\kappa \partial v| \leq C_b \varepsilon^{\frac{3}{2}} (1 + t)^{-\frac{3}{2} (1 + |t - r|)^{-\frac{1}{2}}}, \quad |\kappa| \leq 2, \ t \leq T, \quad (3.20) \]

which means that, for small \( \varepsilon \),

\[ |Z^\kappa \partial v| \leq \frac{1}{2} \varepsilon (1 + t)^{-\frac{1}{2} (1 + |t - r|)^{-\frac{1}{2}}}, \quad |\kappa| \leq 2, \ t \leq T. \]

This completes the proofs of (3.6) and (3.1). \( \square \)

When \( c(u) = 1 + u^2 + O(u^3) \) in (1.1), then similarly to Lemma 3.1, one has

**Lemma 3.2.** Let \( c(u) = 1 + u^2 + O(u^3) \). Then, for sufficiently small \( \varepsilon \) and \( 0 \leq \tau = \varepsilon^2 \ln(1 + t) \leq b < \nu_0 \), (1.1) admits a \( C^\infty \)-solution \( u \) which satisfies the estimate

\[ |Z^\kappa \partial (u - u^{II}_a)| \leq C_b \varepsilon^{\frac{3}{2}} |\ln \varepsilon|(1 + t)^{-\frac{3}{2} (1 + |t - r|)^{-\frac{1}{2}}}, \quad (3.21) \]

for all \( |\kappa| \leq 2. \)

**Proof.** As in the proof of Lemma 3.1, we define the energy

\[ E(t) = \frac{1}{2} \sum_{|\alpha| \leq 4} \int_{\mathbb{R}^2} (|\partial_t Z^\alpha v|^2 + c^2(u)|\nabla Z^\alpha v|^2) \, dx \]

and obtain

\[ E'(t) \leq \frac{C_b \varepsilon^2}{1 + t} E(t) + \sum_{|\beta| \leq 4} C_b \|Z^\beta J^{II}_a\|_{L^2} \sqrt{E(t)}. \]

Due to Lemmas 2.6 and A.3, one then obtains (3.21) as in the proof of Lemma 3.1. \( \square \)

**Remark 3.1.** Lemma 3.1 implies that \( \lim_{\varepsilon \to 0} \varepsilon \sqrt{1 + T_\varepsilon} \geq \tau_0 \) holds for the lifespan \( T_\varepsilon \) of solutions to (1.1) in the case \( c(u) = 1 + u + O(u^2) \). Hence,

\[ \lim_{\varepsilon \to 0} \varepsilon \sqrt{T_\varepsilon} \geq \tau_0. \quad (3.22) \]

Similarly, Lemma 3.2 implies for the lifespan \( T_\varepsilon \) of (1.1) in the case \( c(u) = 1 + u^2 + O(u^3) \) that

\[ \lim_{\varepsilon \to 0} \varepsilon^2 \ln T_\varepsilon \geq \nu_0. \quad (3.23) \]

4. The upper bound on the lifespan \( T_\varepsilon \). In this section, we establish the upper bound on \( T_\varepsilon \). Some of our ideas are inspired by [9], [14], and [6]. Since, in contrast to [9], [14], \( c(u) \) in (1.1) contains the solution \( u \) and not the derivatives of \( u \), our derivation has to be more careful. Thanks to estimates of \( Z^\alpha (u - u^I_a) \) and \( Z^\alpha (u - u^{II}_a) \) with \( |\alpha| \leq 2 \) in Lemmas 3.1 and 3.2, respectively, one observes that \( |Z^\alpha (u - u^I_a)| \leq C_b \varepsilon^{\frac{3}{2}} (1 + t)^{-\frac{3}{2}} \) for \( t \leq \frac{b^2}{\varepsilon^2} - 1 \) and \( |Z^\alpha (u - u^{II}_a)| \leq C_b \varepsilon^{\frac{3}{2}} |\ln \varepsilon|(1 + t)^{-\frac{3}{2}} \) for \( t \leq \frac{b^2}{\varepsilon^2} - 1 \), respectively, near the light cone. This will play a crucial role in the analysis later on.
Set $U = r^{\frac{1}{2}}u$. Because of the spherical symmetry of $u$, (1.1) can be written as

$$
\begin{aligned}
\frac{\partial^2 U}{\partial t^2} - c^2(u)\frac{\partial^2 U}{\partial r^2} &= \frac{1}{4}r^{-\frac{3}{2}}c^2(u)u + 2r^{-\frac{1}{2}}c(u)c'(u)\left(\partial_r U - \frac{1}{2}r^{-\frac{1}{2}}u\right)^2, \\
U(0, r) &= \varepsilon r^{\frac{1}{2}}u_0, \\
\partial_t U(0, r) &= \varepsilon r^{\frac{1}{2}}u_1.
\end{aligned}
$$

(4.1)

Define the operators $L_1$ and $L_2$ by

$$
L_1 = \partial_t + c(u)\partial_r, \quad L_2 = \partial_t - c(u)\partial_r.
$$

We also set

$$
w_1 = L_2 U = (\partial_t - c(u)\partial_r)U, \quad w_2 = L_1 U = (\partial_t + c(u)\partial_r)U,
$$

which means $\partial_t U = \frac{w_2 + w_1}{2}$ and $\partial_r U = \frac{w_2 - w_1}{2c(u)}$.

For

$$
L_1 L_2 = \partial_t^2 - c^2(u)\partial_r^2 - (L_1 c(u))\partial_r, \quad L_2 L_1 = \partial_t^2 - c^2(u)\partial_r^2 + (L_2 c(u))\partial_r,
$$

one has

$$
\begin{aligned}
L_1 w_1 &= \frac{1}{2r^{\frac{1}{2}}c(u)}c'(u)w_1^2 + \frac{c'(u)}{4r^{\frac{3}{2}}c(u)}\left(\frac{3}{r^{\frac{1}{2}}}c(u)u - 2w_1\right)w_1 \\
&\quad + \frac{1}{4r^{\frac{3}{2}}}c^2(u)u + \frac{1}{2r^{\frac{5}{2}}}c(u)c'(u)u^2 - \frac{3}{4}c'(u)uw_2, \\
L_2 w_2 &= \frac{1}{2r^{\frac{1}{2}}c(u)}c'(u)w_2^2 - \frac{c'(u)}{4r^{\frac{3}{2}}c(u)}\left(\frac{3}{r^{\frac{1}{2}}}c(u)u + 2w_1\right)w_2 \\
&\quad + \frac{1}{4r^{\frac{3}{2}}}c^2(u)u + \frac{1}{2r^{\frac{5}{2}}}c(u)c'(u)u^2 + \frac{3}{4}c'(u)uw_1.
\end{aligned}
$$

(4.2)

Because $\partial_r c(u) = c'(u)\partial_r u = \frac{c'(u)}{2r^{\frac{1}{2}}c(u)}(w_2 - w_1) - \frac{1}{2r}c'(u)u$, one also has

$$
\begin{aligned}
L_1 w_1 + \omega_1 \partial_r c(u) &= \frac{1}{4r}c'(u)uw_1 + \frac{1}{4r^{\frac{3}{2}}}c^2(u)u + \frac{1}{2r^{\frac{5}{2}}}c(u)c'(u)u^2 - \frac{3}{4r}c'(u)uw_2, \\
L_2 w_2 - \omega_2 \partial_r c(u) &= -\frac{1}{4r}c'(u)uw_2 + \frac{1}{4r^{\frac{3}{2}}}c^2(u)u + \frac{1}{2r^{\frac{5}{2}}}c(u)c'(u)u^2 + \frac{3}{4r}c'(u)uw_1
\end{aligned}
$$

and

$$
\begin{aligned}
d(|\omega_1|dr - cdt) &= sgn \omega_1 (L_1 w_1 + \omega_1 \partial_r c) \, dt \wedge dr \\
&= sgn \omega_1 \left[\frac{1}{4r}c'(u)uw_1 + \frac{1}{4r^{\frac{3}{2}}}c^2(u)u + \frac{1}{2r^{\frac{5}{2}}}c(u)c'(u)u^2 - \frac{3}{4r}c'(u)uw_2\right] \, dt \wedge dr,
\end{aligned}
$$

(4.4)

$$
\begin{aligned}
d(|\omega_2|dr + cdt) &= sgn \omega_2 (L_2 w_2 - \omega_2 \partial_r c) \, dt \wedge dr \\
&= sgn \omega_2 \left[-\frac{1}{4r}c'(u)uw_2 + \frac{1}{4r^{\frac{3}{2}}}c^2(u)u + \frac{1}{2r^{\frac{5}{2}}}c(u)c'(u)u^2 + \frac{3}{4r}c'(u)uw_1\right] \, dt \wedge dr.
\end{aligned}
$$

(4.5)

We first deal with the case $c(u) = 1 + u + O(u^2)$. 
For \( c(u) = 1 + u + O(u^2) \), one knows from Section 2 that, for \( \varepsilon \sqrt{1 + T_b} = b < \tau_0 \) with \( b > 0 \) a fixed constant, (1.1) has a \( C^\infty \)-solution for \( t \leq T_b \). Choose \( \varepsilon > 0 \) so small that \( \frac{1}{\varepsilon} < \frac{b^2}{\varepsilon^2} - 1 \). Define the characteristic curve \( \Gamma^\pm_\lambda \) by \( \frac{dr}{dt} = \pm c(u(t, r)) \) and let it pass through \( (\lambda, 0) \) in the \( (r,t) \)-plane. Let \( D \) be the domain that is bounded by \( \Gamma^+_M \), \( \Gamma^+_{\rho_0 - 1} \), \( \{ t = 0 \} \), and \( \{ t = T_b \} \) (see Figure 1), where \( \rho_0 \) is chosen in such a way that \( F'_0(\rho_0) = \min_{\sigma \leq M} F'_0(\sigma) \). Obviously, \( \Gamma^+_M \) is the straight line \( r = t + M \).

One now has

**Lemma 4.1.** If \( (t,r), (t', r') \in \Gamma^-_\mu \cap D \ (\mu \in \mathbb{R}) \) and \( (t,r) \in \Gamma^+_\lambda \), \( (t', r') \in \Gamma^+_\lambda' \), where \( \lambda, \lambda' \in [\rho_0 - 1, M] \), then

\[
|t - t'| \leq C_b. \tag{4.6}
\]

**Proof.** The equation \( r = r(t) \) for \( \Gamma^+_\lambda \) is

\[
\begin{aligned}
\frac{dr(t)}{dt} &= c(u(t, r(t))) \equiv c(t), \\
r(0) &= \lambda,
\end{aligned}
\]

which yields

\[
r(t) - \lambda = \int_0^t (c(s) - 1) \, ds + t.
\]
Because
\[ |c(t) - 1| \leq C|u(t, r(t))| \leq C_b \varepsilon (1 + t)^{-\frac{1}{2}} (1 + |t - r(t)|)^{\frac{1}{2}}, \quad 0 \leq \tau = \varepsilon \sqrt{1 + t} \leq b < \tau_0, \]
one has
\[
|t + M - r(t)| \leq C_b, \quad (4.8)
\]
which means that the horizontal width between \( \Gamma_{0-1}^{+} \) and \( \Gamma_M^{+} \) in \( D \) is finite.

On the other hand, the equation \( \tilde{r} = \tilde{r}(t) \) for \( \Gamma_\mu^{+} \) is
\[
\begin{cases}
\frac{d\tilde{r}(t)}{dt} = -c(u(t, \tilde{r}(t))) \equiv -\tilde{e}(t), \\
\tilde{r}(0) = \mu.
\end{cases}
\]

Let \( \text{supp} u \subset \{(t, r): r < t + M\} \); then \( |\tilde{r}(t) + t - \mu| = 0 \) for \( t < t_* \) and \( |\tilde{r}(t) + t - \mu| \leq \int_{t_0}^{t} |\tilde{c}(s) + 1| ds \leq C_b \varepsilon \int_{t_*}^{t} (1 + s)^{-\frac{1}{2}} (1 + |\tilde{r}(s) - s|)^{\frac{1}{2}} ds \)
for \( t \geq t_* \), where \( (t_*, t_* + M) \) denotes the intersection of the curves \( \Gamma_\mu^{-} \) and \( \Gamma_M^{+} \).

Combining this with (4.8) yields
\[
|\tilde{r}(t) + t - \mu| \leq C_b \varepsilon \sqrt{1 + t} \leq C_b.
\]

Thus, if \( (t, r), (t', r') \in \Gamma_\mu^{+} \cap D \) (\( \mu \in \mathbb{R} \)) and \( (t, r) \in \Gamma_\lambda^{+}, (t', r') \in \Gamma_{\lambda'}^{+} \), one then arrives at
\[
|t - t'| \leq \frac{1}{2} \left( |t + r - \mu| + |t' + r' - \mu| + |t - r - \lambda| + |t' - r' - \lambda'| + |\lambda - \lambda'| \right) \leq C_b,
\]
which finishes the proof of Lemma 4.1. \( \square \)
For $t \leq T_b$, define
\[ A(t) = \sup_{1/\varepsilon \leq s \leq t} \int_{(s, r) \in D} |w_1(s, r)| \, dr, \]
\[ B(t) = \sup_{1/\varepsilon \leq s \leq t} s^{\frac{1}{2}} |u(s, r)|, \]
\[ C(t) = \sup_{1/\varepsilon \leq s \leq t} s |w_2(s, r)|. \]

Then one obtains

**Lemma 4.2.** There exists a constant $E > 0$ such that, for small $\varepsilon$,
\[ A\left(\frac{1}{\varepsilon}\right) \leq \frac{E \varepsilon}{2}, \quad B\left(\frac{1}{\varepsilon}\right) \leq E \varepsilon, \quad C\left(\frac{1}{\varepsilon}\right) \leq E^2 \varepsilon^2. \] (4.9)

**Proof.** Since $w_1 = r^{\frac{1}{2}} \partial_t u + c(u)r^{\frac{1}{2}} \partial_r u - \frac{1}{2} r^{-\frac{1}{2}} c(u) u$, for $t \leq \frac{b^2}{\varepsilon^2} - 1$, one has
\[ |w_1(t, r)| \leq C_b \varepsilon. \] (4.10)

Thus, it follows from (4.8) and (4.10) that
\[ \int_{(\frac{1}{\varepsilon}, r) \in D} |w_1(s, r)| \, dr \leq C_b \varepsilon. \] (4.11)

Furthermore, because $|u(t, r)| \leq C_b \varepsilon (1 + t)^{-\frac{1}{2}} (1 + |t - r|)^{\frac{1}{2}}$ for $t \leq \frac{b^2}{\varepsilon^2} - 1$ together with (4.8), one has, for $(\frac{1}{\varepsilon}, r) \in D$,
\[ \frac{1}{\varepsilon^2} \left| u \left( \frac{1}{\varepsilon}, r \right) \right| \leq C_b \varepsilon. \] (4.12)

Note that
\[ w_2(t, r) = r^{\frac{1}{2}} \left( \partial_t u + c(u) \partial_r u \right) + \frac{1}{2} r^{-\frac{1}{2}} c(u) u \]
\[ = r^{\frac{1}{2}} S + \frac{H}{t + r} u + r^{\frac{1}{2}} (c(u) - 1) \partial_r u + \frac{1}{2} r^{-\frac{1}{2}} c(u) u, \]
which implies that $|w_2(t, r)| \leq C_b \varepsilon (1 + t)^{-\frac{1}{2}}$ for $(t, r) \in D$ and $\varepsilon \sqrt{1 + t} \leq b$ in view of (3.9). Together with (4.3), this yields
\[ |L_2 w_2| \leq \frac{C_b \varepsilon^2}{1 + t}. \] (4.13)

For $w_2(t, t + M) = 0$ and (4.6), from (4.13) one then obtains
\[ C\left(\frac{1}{\varepsilon}\right) \leq C_b \varepsilon^2. \] (4.14)

Collecting (4.11), (4.12) and (4.14) completes the proof of (4.9), where $E = 8(1 + C_b)$. □

Based on Lemma 4.2, we will use continuous induction to estimate the upper bound on $T_\varepsilon$ when $c(u) = 1 + u + O(u^2)$. To this end, we assume that, for $0 \leq t \leq T' \leq T_b$,
\[ A(t) \leq E \varepsilon, \quad B(t) \leq 2 E \varepsilon, \quad C(t) \leq 3 E^2 \varepsilon^2. \] (4.15)

We now establish
Lemma 4.3. Under the hypothesis (4.15) and for $\varepsilon$ sufficiently small, one has, for $\frac{1}{\varepsilon} \leq t \leq T'$,\[ A(t) \leq \frac{2}{3} E\varepsilon, \quad B(t) \leq E\varepsilon, \quad C(t) \leq \frac{5}{2} E^2 \varepsilon^2. \] (4.16)

Proof. First we estimate $A(t)$. By (4.4) and Green's formula, one has, for $\frac{1}{\varepsilon} \leq t \leq T'$,\[ \int_{(t,r) \in D} |w_1(t,r)| \, dr \leq \int_{(1/\varepsilon,r) \in D} |w_1(1/\varepsilon,r)| \, dr + \int_{1/\varepsilon \leq s \leq t, (s,r) \in D} \left| \frac{1}{4r} c'(u)uw_1 + \frac{1}{4r^2} c^2(u)u \right| \, dsdr + \frac{1}{2r^2} c'(u)c(u)u^2 - \frac{3}{4r} c'(u)uw_2 \right\| (s,r) \, dsdr \leq \frac{1}{2} E\varepsilon + \int_{1/\varepsilon \leq s \leq t, (s,r) \in D} \left| \frac{1}{4r} c'(u)uw_1 + \frac{1}{4r^2} c^2(u)u \right| \, dsdr + \frac{1}{2r^2} c'(u)c(u)u^2 - \frac{3}{4r} c'(u)uw_2 \right\| (s,r) \, dsdr. \] (4.17)

By the inductive hypothesis (4.15), one has $|u(s,r)| \leq 2E\varepsilon s^{-\frac{3}{2}}$ for $\frac{1}{\varepsilon} \leq s \leq T'$ and $(s,r) \in D$. Note also that $c$ is near 1 for small $\varepsilon$ and $|r-s| \leq C_b$ holds for $s \geq 1/\varepsilon$. One then has $r \geq s/2$ and \[ \int_{1/\varepsilon \leq s \leq t} \left| \frac{1}{4r} c'(u)uw_1 \right| \, dsdr \leq \int_{1/\varepsilon \leq s \leq t} \left| w_1 \right| (s,r) \, dsdr \leq 2E\varepsilon \int_{1/\varepsilon}^t \frac{1}{s^{\frac{3}{2}}} ds \int_{(s,r) \in D} \left| w_1 \right| (s,r) \, dr \leq 4E\varepsilon^{\frac{3}{2}} A(t) \leq 4E^2 \varepsilon^{\frac{5}{2}}. \] (4.18)

Similarly, one has \[ \int_{1/\varepsilon \leq s \leq t} \left| \frac{1}{4r^2} c^2(u)u \right| \, dsdr \leq 2E\varepsilon \int_{1/\varepsilon \leq s \leq t} \frac{1}{s^2} dsdr \leq C_b E\varepsilon^2, \] (4.19)\[ \int_{1/\varepsilon \leq s \leq t} \left| \frac{1}{2r^2} c'(u)c(u)u^2 \right| \, dsdr \leq C_b E^2 \varepsilon^2 \int_{1/\varepsilon \leq s \leq t} \frac{1}{s^2} dsdr \leq C_b E^2 \varepsilon^7, \] (4.20)
Fig. 2

and

\[
\int\int_{\substack{1/\varepsilon \leq s \leq t \\ (s,r) \in D}} \left| \frac{3}{4r} c'(u)uw_2 \right| (s,r) \, dsdr \leq C_b E^2 \varepsilon^2 \int\int_{\substack{1/\varepsilon \leq s \leq t \\ (s,r) \in D}} \frac{1}{s^{3/2}} \, dsdr \leq C_b E^2 \varepsilon^{7/2}. \tag{4.21}
\]

Substituting (4.18)–(4.21) into (4.17) yields

\[
A(t) \leq \frac{1}{2} E \varepsilon + C_b E \varepsilon^2 + C_b E^2 \varepsilon^{7/2},
\]

which implies \(A(t) \leq \frac{2}{3} E \varepsilon\) for sufficiently small \(\varepsilon\).

Next we estimate \(B(t)\). Note that \(u\) satisfies the equation

\[
L_2 u = L_2 (r^{-\frac{1}{2}} U) = r^{-\frac{1}{2}} w_1 + \frac{1}{2r^{\frac{3}{2}}} c(u)u. \tag{4.22}
\]

Integrate (4.22) along the characteristic curve \(\Gamma_{\mu_-}^\mu\) which insects \(\Gamma_{M}^\mu\) at the point \((t', r')\). When \(t' \geq 1/\varepsilon\), denote by \(D_1\) the domain bounded by \(\Gamma_{\mu_-}^\mu\), the line \(\{t = t'\}\), and \(\Gamma_{M}^\mu\).

Let \(\Gamma_1, \Gamma_2,\) and \(\Gamma_3\) be the faces of the boundary (see Figure 2). By (4.4), one then has

\[
\int\int_{(s,r) \in D_1} \text{sgn} w_1 \left[ \frac{1}{4r} c'(u)uw_1 + \frac{1}{4r^{3/2}} c^2(u)u + \frac{1}{2r^{3/2}} c'(u)c(u)u^2 - \frac{3}{4r} c'(u)uw_2 \right] (s,r) \, dsdr
\]

\[= \left( \int_{\Gamma_1} + \int_{\Gamma_2} \right) |w_1| (dr - c \, dt), \]
which implies
\[
\int_{\Gamma_1} |w_1|(dr - c \, ds) \\
\leq \iint_{(s,r) \in D_1} \left| \frac{1}{4r} c'(u)uw_1 + \frac{1}{4r^2} c^2(u)u + \frac{1}{2r^2} c'(u)c(u)u^2 - \frac{3}{4r} c'(u)uw_2 \right| (s,r) \, dsdr \\
+ \int_{\Gamma_2} |w_1|(dr - c \, dt) \\
\leq \iint_{\frac{1}{\varepsilon} \leq s \leq t \atop (s,r) \in D} \left| \frac{1}{4r} c'(u)uw_1 + \frac{1}{4r^2} c^2(u)u + \frac{1}{2r^2} c'(u)c(u)u^2 \\
- \frac{3}{4r} c'(u)uw_2 \right| (s,r) \, dsdr + \int_{(t,r) \in D_1} |w_1(t,r)| \, dr \\
\leq \frac{1}{6} E\varepsilon + \frac{2}{3} E\varepsilon = \frac{5}{6} E\varepsilon,
\]
the last inequality resulting the estimates (4.18)–(4.21). This yields
\[
\int_{\nu}^{t} \frac{|w_1(s,\tilde{r}(s))|}{(\tilde{r}(s))^{\frac{3}{2}}} \, ds \\
\leq \int_{\nu}^{t} \frac{|w_1(s,\tilde{r}(s))|}{(\tilde{r}(t))^{\frac{3}{2}}} \, ds \leq \frac{2}{t^{\frac{3}{2}}} \int_{\nu}^{t} |w_1(s,r(s))| \, ds \\
= \frac{2}{t^{\frac{3}{2}}} \int_{\Gamma_1} \frac{|w_1(s,r)|}{\sqrt{1 + c^2}} \, ds = \frac{2}{t^{\frac{3}{2}}} \int_{\Gamma_1} \frac{|w_1|}{2c\sqrt{1 + c^2}} (dr - c \, ds) \\
\leq \frac{5}{6t^{\frac{3}{2}}} E\varepsilon.
\]

Note also that \( r(s) \geq \frac{s}{2} \geq \frac{1}{2\varepsilon} \). By (4.22), one then has
\[
|u(t,r)| \leq \int_{\nu}^{t} \frac{|w_1(s,\tilde{r}(s))|}{(\tilde{r}(s))^{\frac{3}{2}}} \, ds + \int_{\nu}^{t} \frac{|c'u(s,\tilde{r}(s))|}{2(\tilde{r}(s))^{\frac{3}{2}}} \, ds \\
\leq \frac{5}{6t^{\frac{3}{2}}} E\varepsilon + 3\varepsilon \int_{\nu}^{t} |u(s,r(s))| \, ds.
\]

By (4.6), one obtains
\[
|u(t,\tilde{r}(t))| \leq \frac{5}{6t^{\frac{3}{2}}} E\varepsilon e^{C\varepsilon} \leq E\varepsilon t^{-\frac{1}{2}}.
\]

One has \( t \leq t' + |t - t'| \leq 2/\varepsilon \) for \( t' \leq 1/\varepsilon \). Thus, \( t^{\frac{3}{2}} |u(t,r)| \leq E\varepsilon \) for \((t,r) \in D\), and \( B(t) \leq E\varepsilon \) follows.

Finally, we estimate \( C(t) \). We rewrite (4.3) as
\[
L_2w_2 = aw_2 + b,
\]
where
\[ a = \frac{1}{2r^\frac{3}{2}c(u)}c'(u)w_2 - \frac{c'(u)}{4r^\frac{3}{2}c(u)}\left(\frac{3}{r^\frac{1}{2}}c(u)u + 2w_1\right), \]
\[ b = \frac{1}{4r^\frac{3}{2}}c^2(u)u + \frac{1}{2r^\frac{3}{2}}c(u)c'(u)u^2 + \frac{3}{4}c'(u)uw_1. \]

Integrating (4.24) along \(\Gamma_\mu^+\) as above, one obtains
\[
|w_2(t, r)| \leq \int_{t'}^t |aw_2 + b|(s, \tilde{r}(s)) \, ds. \tag{4.25}
\]

Noting that \(t' \geq t - |t - t'| \geq t - C_b, |w_2(t, r)| \leq \frac{3E^2\varepsilon^2}{t}\) and using (4.6), (4.8), and (4.23), by the choice of \(E = 8(1 + C_b)\) in Lemma 4.2, one arrives at
\[
\int_{t'}^t |b(s, \tilde{r}(s))| \, ds
\leq \int_{t'}^t \frac{1}{4(\tilde{r}(s))^\frac{3}{2}}|c^2(u)u|(s, \tilde{r}(s)) \, ds + \int_{t'}^t \frac{1}{2(\tilde{r}(s))^\frac{3}{2}}|c(u)c'(u)u^2|(s, \tilde{r}(s)) \, ds
+ \int_{t'}^t \frac{3}{4(\tilde{r}(s))^\frac{3}{2}}|c'(u)uw_1|(s, \tilde{r}(s)) \, ds \tag{4.26}
\]
\[
\leq \frac{1}{3}C_bE\varepsilon^2 t^{-1} + \frac{2E^2\varepsilon^2}{t} \int_{t'}^t \frac{1}{s^\frac{3}{2}} \, ds + \frac{5E\varepsilon}{4t} \int_{t'}^t \frac{|w_1(s, \tilde{r}(s))|}{(\tilde{r}(s))^\frac{3}{2}} \, ds
\leq \frac{1}{3}E^2\varepsilon^2 t^{-1} + \frac{1}{2}E^2\varepsilon^2 t^{-1} + \frac{1}{2}E^2\varepsilon^2 t^{-1}
\leq \frac{5}{3}E^2\varepsilon^2 t^{-1},
\]
and similarly
\[
\int_{t'}^t |aw_2|(s, r(s)) \, ds \leq \frac{2E^2\varepsilon^2}{t} \int_{t'}^t |a|(s, \tilde{r}(s)) \, ds \leq \frac{C_bE^3\varepsilon^3}{t}. \tag{4.27}
\]
Substituting (4.26) and (4.27) into (4.25) yields
\[
|w_2(t, r)| \leq \int_{t'}^t |aw_2 + b|(s, \tilde{r}(s)) \, ds \leq \frac{5E^2\varepsilon^2}{2t},
\]
which shows that \(C(t) \leq \frac{5E^2\varepsilon^2}{2}.\)

We will use Lemma A.4 to estimate the upper bound on the lifespan \(T_\varepsilon\) when \(c(u) = 1 + u + O(u^2)\) in (1.1), based on Lemmas 4.2 and 4.3. More specifically, we will show that
\[
\lim_{\varepsilon \to 0} \varepsilon \sqrt{T_\varepsilon} \leq -\frac{1}{2F'(\rho_0)} = \tau_0. \tag{4.28}
\]

It follows from (4.2) that on the characteristic curve \(\Gamma_\rho^+, w_1(t, r(t))\) satisfies
\[
\frac{dw_1}{dt}(t, r(t)) = L_1w_1 = a_0(t)w_1^2 + a_1(t)w_1 + a_2(t), \tag{4.29}
\]
where

\[ a_0(t) = \left( \frac{1}{2r^2} c'(u) \right)(t, r(t)), \]

\[ a_1(t) = \left( \frac{c'(u)}{4r^2} \right)\left( \frac{3}{r^2} c(u) u - 2w_2 \right)(t, r(t)), \]

\[ a_2(t) = \left( \frac{1}{4r^2} c^2(u) + \frac{1}{2r^2} c(u) c'(u) u^2 - \frac{3}{4r} c'(u) u w_2 \right)(t, r(t)). \]

By (4.15), one has, for \( \frac{1}{\varepsilon} \leq t \leq T_b, \)

\[ |a_1| \leq \frac{5E\varepsilon}{t^2}, \quad |a_2| \leq \frac{10E\varepsilon^2}{t^2}, \]

which implies

\[ \int_{1/\varepsilon}^{T_b} |a_1| \, ds \leq 10E\varepsilon^{3/2}, \quad \int_{1/\varepsilon}^{T_b} |a_2| \, ds \leq 20E^2\varepsilon^2. \] (4.30)

This also yields

\[ K = \left( \int_{1/\varepsilon}^{T_b} |a_2(t)| \, dt \right) \exp\left( \int_{1/\varepsilon}^{T_b} |a_1(t)| \, dt \right) = O(\varepsilon^2). \] (4.31)

By the definition of \( u^I_0 \) in (2.14), one has that \( u^I_0(1/\varepsilon) = \varepsilon w_0(1/\varepsilon) \) on \( \Gamma^+_{\rho_0} \). Moreover, it follows from Lemma 3.1 that

\[ |Z^\alpha(u - u^I_0)| \leq C_{\alpha\beta}\varepsilon^{3/2}(1 + t)^{-\frac{1}{2}}(1 + |t - r|)^{1/2}. \]

On the other hand, by \[10\] Theorem 6.2.1, one has

\[ \left| \partial^\alpha Z^\beta \left( u^I_0(1/\varepsilon) - \left( \frac{\varepsilon}{r^{1/2}} F_0 \right)(r(1/\varepsilon) - 1/\varepsilon) \right) \right| \leq C_{\alpha\beta}\varepsilon^{3/2}. \]

Therefore,

\[ w_1(1/\varepsilon) = \left( r^{1/2} \partial_t u(1/\varepsilon) - \left[ c \left( \frac{u}{2r^{1/2}} + r^{1/2} \partial_r u \right) \right] (1/\varepsilon) \]

\[ = \left( r^{1/2} \partial_t u^I_0(1/\varepsilon) - \left[ c \left( \frac{u^I_0}{2r^{1/2}} + r^{1/2} \partial_r u^I_0 \right) \right] (1/\varepsilon) + O(\varepsilon^{3/2}) \]

\[ = \varepsilon F_0'(r(1/\varepsilon) - 1/\varepsilon)(-c(1/\varepsilon) - 1) + O(\varepsilon^{3/2}) \]

\[ = -2\varepsilon F_0'(r(1/\varepsilon) - 1/\varepsilon) + O(\varepsilon^{3/2}). \]

Note that one has \( |r - t| \leq C + |\rho_0| \) and \( |u(t, r)| \leq C_b\varepsilon(1 + t)^{-\frac{1}{2}} \) on \( \Gamma^+_{\rho_0} \). Hence,

\[ |r(1/\varepsilon) - 1/\varepsilon - \rho_0| \leq \int_0^{1/\varepsilon} C_b\varepsilon(1 + s)^{-\frac{1}{2}} \, ds = C_b\varepsilon \sqrt{1 + \frac{1}{\varepsilon}}. \]

We now prove (4.28). By Lemma A.4 and (4.30) and (4.31), one has

\[ \left( \int_{1/\varepsilon}^{T_b} \frac{1}{2(r(t))^{1/2}c(u)} c'(u) \, dt \right) \exp\left( - \int_{1/\varepsilon}^{T_b} |a_1(t)| \, dt \right) < (w_1(1/\varepsilon) - K)^{-1}; \]
that is,
\[
(\sqrt{T_\varepsilon} - \sqrt{1 + \frac{1}{\varepsilon}}) (1 + O(\varepsilon)) < (-2\varepsilon F_0(\rho_0) + O(\varepsilon^{3/2}))^{-1} \exp(5E\varepsilon^2).
\]
Thus,
\[
\lim_{\varepsilon \to 0} \sqrt{T_\varepsilon} \leq -\frac{1}{2F_0'(\rho_0)} = \tau_0,
\]
and (4.28) is proved.

Next we show that in the case \(c(u) = 1 + u^2 + O(u^2)\) in (1.1), the lifespan \(T_\varepsilon\) satisfies
\[
\lim_{\varepsilon \to 0} \varepsilon^2 \ln T_\varepsilon \leq -\frac{1}{2\min_\sigma \{F_0(\sigma)F_0'(\sigma)\}}.
\]

Although the proof is analogous to that of (4.28), for the readers' convenience, we provide the details.

Set \(\tilde{T}_b = \varepsilon r^2 - 1\), where \(0 < b < \nu_0\) is a fixed constant. As above, define \(\tilde{\Gamma}_\lambda^\pm\) to be the characteristic curve given by \(\frac{dr}{d\varepsilon} = \pm c(u(t, r))\) and passing through the point \((\lambda, 0)\). The domain \(\tilde{D}\) is bounded by \(\tilde{\Gamma}_\lambda^+\), \(\tilde{\Gamma}_{\rho_0 - 1}^+\), \(\{t = 0\}\), and \(\{t = \tilde{T}_b\}\), where \(\tilde{\rho}_0\) is chosen so that \(F_0(\tilde{\rho}_0)F_0'(\tilde{\rho}_0) = \min_\sigma \{F_0(\sigma)F_0'(\sigma)\}\).

Similarly to Lemma 4.1, one has

**Lemma 4.4.** If \((t, r), (t', r') \in \tilde{\Gamma}_\nu^- \cap \tilde{D} (\nu \in \mathbb{R})\) and \((t, r) \in \tilde{\Gamma}_\lambda^+, (t', r') \in \tilde{\Gamma}_{\lambda'}^+\), where \(\lambda, \lambda' \in [\tilde{\rho}_0 - 1, M]\), then
\[
|t - t'| \leq C_b. \tag{4.33}
\]

**Proof.** For \(\lambda \in [\tilde{\rho}_0 - 1, M]\), the equation \(r = r(t)\) of \(\tilde{\Gamma}_\lambda^+\) is
\[
\begin{align*}
\frac{dr(t)}{dt} &= c(u(t, r(t))) \equiv c(t),
\frac{dr}{d\varepsilon} &= \lambda.
\end{align*}
\]

Because \(|c(t) - 1| \leq C_b |u(t, r(t))| \leq C_b \varepsilon^2 (1 + t)^{-1}(1 + |r - r(t)|)\) for \(0 < \tau = \varepsilon^2 \ln(1 + t) \leq b < \nu_0\), one has
\[
|r(t) - t| \leq |\lambda| + \int_0^t |c(u(s, r(s))) - 1| ds
\]
\[
\leq m_0 + C_b \varepsilon^2 \int_0^t (1 + s)^{-1}(1 + |r(s) - s|) ds,
\]
which implies \(|r(t) - t| \leq C_b\) for \(t \leq \tilde{T}_b\). The proof of Lemma 4.4 then concludes by an argument similar to that in the proof of Lemma 4.1.

Define \(A(t), B(t),\) and \(C(t)\) as in Lemma 4.2. When \(c(u) = 1 + u^2 + O(u^3)\) in (1.1), one obtains the paralleling Lemmas 4.2 and 4.3:

**Lemma 4.5.** There exists a positive constant \(E\) such that, for small \(\varepsilon\),
\[
(i) \quad A\left(\frac{1}{\varepsilon}\right) \leq \frac{E\varepsilon}{2}, \quad B\left(\frac{1}{\varepsilon}\right) \leq E\varepsilon, \quad C\left(\frac{1}{\varepsilon}\right) \leq E^2\varepsilon^2. \tag{4.34}
\]
Moreover, it follows from Lemma 3.2 that
\[ A(t) \leq E\varepsilon, \quad B(t) \leq 2E\varepsilon, \quad C(t) \leq 3E^2\varepsilon^2. \] (4.35)

**Proof.** Since the proof is analogous to those of Lemmas 4.2 and 4.3, it is omitted. \(\square\)

Next we prove (4.32). It follows from (4.2) that along the characteristic curve \(\tilde{\Gamma}_{\tilde{\rho}_0}^+\), \(w_1(t,r(t))\) satisfies
\[
\frac{dw_1}{dt}(t,r(t)) = a_0(t)w_1^2 + a_1(t)w_1 + a_2(t),
\] (4.36)
where \(a_0(t), a_1(t),\) and \(a_2(t)\) are defined as in (4.29).

By (4.35), one has, for \(\frac{1}{\varepsilon} \leq t \leq \tilde{T}_b\),
\[ |a_1| \leq \frac{5E\varepsilon^2}{t^2}, \quad |a_2| \leq \frac{10E\varepsilon^2}{t^4}, \]
which implies
\[
\int_{1/\varepsilon}^{\tilde{T}_b} |a_1| \, ds \leq 10E\varepsilon^3, \quad \int_{1/\varepsilon}^{\tilde{T}_b} |a_2| \, ds \leq 20E^2\varepsilon^2
\] (4.37)
and
\[
K = \left( \int_{1/\varepsilon}^{\tilde{T}_b} |a_2(t)| \, dt \right) \exp \left( \int_{1/\varepsilon}^{\tilde{T}_b} |a_1(t)| \, dt \right) = O(\varepsilon^2).
\] (4.38)

By the definition of \(u_a^{II}\) in (2.31), one has that \(u_a^{II}(1/\varepsilon) = \varepsilon w_0(1/\varepsilon)\) on \(\tilde{\Gamma}_{\tilde{\rho}_0}^+\) holds true. Moreover, it follows from Lemma 3.2 that
\[
|Z^\alpha (u - u_a^{II})| \leq C_{\alpha \beta} \varepsilon^{3/2} |\ln \varepsilon|(1 + t)^{-1/2}(1 + |t - r|)^{1/2}. \] (4.39)

On the other hand, by [10] Theorem 6.2.1 one has
\[
\left| \partial^\alpha Z^\beta \left( u_a^{II}(1/\varepsilon) - \left( \frac{\varepsilon}{r^2} F_0 \right) (r(1/\varepsilon) - 1/\varepsilon) \right) \right| \leq C_{\alpha \beta} \varepsilon^{3/2}.
\]

Therefore,
\[
w_1(1/\varepsilon) = (r^{1/2} \partial_t u)(1/\varepsilon) - \left( c \left( \frac{u}{2r^{1/2}} + r^{1/2} \partial_r u \right) \right)(1/\varepsilon)
\]
\[
= (r^{1/2} \partial_t u_a^{II})(1/\varepsilon) - \left( c \left( \frac{u_a^{II}}{2r^{1/2}} + r^{1/2} \partial_r u_a^{II} \right) \right)(1/\varepsilon) + O(\varepsilon^{3/2})
\]
\[
= \varepsilon F_0'(r(1/\varepsilon) - 1/\varepsilon) \left( -c(1/\varepsilon) - 1 \right) + O(\varepsilon^{3/2} |\ln \varepsilon|)
\]
\[
= -2\varepsilon F_0'(r(1/\varepsilon) - 1/\varepsilon) + O(\varepsilon^{3/2} |\ln \varepsilon|).
\]

Note that \(|r - t| \leq C + |\tilde{\rho}_0|\) and \(|u(t,r)| \leq C_b \varepsilon(1 + t)^{-1/2}\) on \(\tilde{\Gamma}_{\tilde{\rho}_0}^+\). Hence,
\[
|r(1/\varepsilon) - 1/\varepsilon - \tilde{\rho}_0| \leq \int_0^{1/\varepsilon} C_b \varepsilon^2 (1 + s)^{-1} \, ds = C_b \varepsilon^2 |\ln \varepsilon|.
\]

Thus one has
\[
w_1 \left( \frac{1}{\varepsilon} \right) = -2\varepsilon F_0'(\tilde{\rho}_0) + O(\varepsilon^{3/2} |\ln \varepsilon|).
\] (4.40)

For later reference, we now provide properties of \(u\) when restricted to \(\tilde{\Gamma}_{\tilde{\rho}_0}^+\).
By the definition of $u_\alpha^{II}$ in (2.31), one has, for $t \geq \frac{1}{\varepsilon}$,

$$u_\alpha^{II}(t, x) = \varepsilon \left( \chi(\varepsilon t)w_0(t, x) + (1 - \chi(\varepsilon t))r^{-1/2}F_0(\tilde{\rho}_0) \right) \text{ on } \tilde{\Gamma}_{\tilde{\rho}_0}^+. \quad (4.41)$$

Moreover, it follows from \cite{10} Lemma 6.2.1 that

$$|\partial^\alpha Z^\beta(w_0(t, x) - r^{-1/2}F_0(\tilde{\rho}_0))| \leq C_{\alpha\beta}\varepsilon^{\frac{1}{2}}(1 + t)^{-1} \quad \text{on } \tilde{\Gamma}_{\tilde{\rho}_0}^+. \quad (4.42)$$

Substituting (4.42) into (4.41) yields, for $t \geq \frac{1}{\varepsilon}$,

$$u_\alpha^{II} = \varepsilon r^{-1/2}F_0(\tilde{\rho}_0) + O(\varepsilon^3)(1 + t)^{-1} \quad \text{on } \tilde{\Gamma}_{\tilde{\rho}_0}^+. \quad (4.43)$$

Relying on the preparations above, we now prove (4.32).

As $F_0(\tilde{\rho}_0)F'_0(\tilde{\rho}_0) < 0$, without loss of generality we can assume that $F_0(\tilde{\rho}_0) < 0$ and $F'_0(\tilde{\rho}_0) > 0$. One then has, for $t \geq \frac{1}{\varepsilon}$,

$$w_1 \left( \frac{1}{\varepsilon} \right) < 0 \quad \text{and} \quad a_0(t) = \frac{2u + O(u^2)}{2(r(t))^{1/2}c(u)} < 0 \quad \text{on } \tilde{\Gamma}_{\tilde{\rho}_0}^+. \quad (4.44)$$

Consider the equation for $\tilde{w}_1 = -w_1$. It follows from (4.36) that on the characteristic curve $\tilde{\Gamma}_{\tilde{\rho}_0}^+$, for $t > \frac{1}{\varepsilon}$,

$$\frac{d\tilde{w}_1}{dt}(t, r(t)) = -a_0(t)(\tilde{w}_1)^2 + a_1(t)\tilde{w}_1 - a_2(t),$$

where $\tilde{w}_1 \left( \frac{1}{\varepsilon} \right) > 0$. By Lemma A.4, one has

$$\left( -\int_{1/\varepsilon}^{T_b} \frac{1}{2(r(t))^{1/2}c(u)} c'(u) \, dt \right) \exp \left( -\int_{1/\varepsilon}^{T_b} |a_1(t)| \, dt \right) < (\tilde{w}_1(1/\varepsilon) - K)^{-1}. \quad (4.45)$$

From this, together with (4.37), (4.38), (4.40), (4.43), and $c'(u) = 2u + O(u^2)$, one arrives at

$$\varepsilon F_0(\tilde{\rho}_0)(\ln T_b - \ln(1/\varepsilon))(1 + O(\varepsilon)) < (-2\varepsilon F'_0(\tilde{\rho}_0) + O(\varepsilon^{3/2}|\ln \varepsilon|))^{-1} \exp(10E\varepsilon^2),$$

which implies

$$\lim_{\varepsilon \to 0} \varepsilon^2 \ln T_\varepsilon \leq -\frac{1}{2F_0(\tilde{\rho}_0)F'_0(\tilde{\rho}_0)} = \nu_0. \quad (4.46)$$

Consequently, (4.32) is proved.

**Proof of Theorem 1.1.** Under the assumptions of Theorem 1.1, it follows from (3.22), (3.23), (4.28), and (4.32) that the lifespan $T_\varepsilon$ satisfies

$$\lim_{\varepsilon \to 0} \varepsilon \sqrt{T_\varepsilon} = \tau_0 \quad \text{when} \quad c(u) = 1 + u + O(u^2)$$

and

$$\lim_{\varepsilon \to 0} \varepsilon^2 \ln T_\varepsilon = \nu_0 \quad \text{when} \quad c(u) = 1 + u^2 + O(u^3).$$

Thus, we have completed the proof of Theorem 1.1. \qed
Appendix A. Some useful lemmas.

**Lemma A.1.**  
(i) For \( \phi(t,r) \in C^1 \),

\[
|\partial \phi| \leq \frac{2}{1 + |t-r|} \sum_{|\beta|=1} |Z^\beta \phi|.
\]  
(A.1)

(ii) Klainerman fields have the following expressions in \((\tau,\sigma)\) coordinates:

\[
\begin{align*}
\partial_t &= -\partial_\sigma + \frac{\epsilon}{2\sqrt{1+t}} \partial_r, \\
\partial_r &= \partial_\sigma, \\
S &= \sigma \partial_\sigma + \frac{\epsilon t}{2\sqrt{1+t}} \partial_r, \\
H &= -\sigma \partial_\sigma + \frac{\epsilon r}{2\sqrt{1+t}} \partial_r.
\end{align*}
\]

**Proof.**  
(i) For \( \partial_t = \frac{tS - rH}{t^2 - r^2} \) and \( \partial_r = \frac{tH - rS}{t^2 - r^2} \),

\[
(1 + |t - r|)(|\partial_t \phi| + |\partial_r \phi|) \leq 2(|S\phi| + |H\phi| + |\partial_t \phi| + |\partial_r \phi|),
\]

and (A.1) is proved.

(ii) This follows from a direct computation. \( \square \)

**Lemma A.2.** If \( f(t,x) \in C^1(\mathbb{R}^+ \times \mathbb{R}^2) \) depends only on \((t,r)\) and \( \text{supp} f \subseteq \{(t,x): r \leq M + t\} \), then

\[
\| (1 + |t - r|)^{-1} f \|_{L^2} \leq C \| \partial_r f \|_{L^2}.
\]

**Remark A.1.** In the case of \( x \in \mathbb{R}^3 \), Lemma A.2 has been proved in [19].

**Proof.** Since \( \text{supp} f \subseteq \{r \leq M + t\} \),

\[
f(t,r) = -\int_{r}^{M+t} \partial_r f(t,s) \, ds.
\]

It follows that

\[
|f(t,r)|^2 \leq \left( \int_{r}^{M+t} |\partial_r f(t,s)|^2 (1 + |t-s|)^{1/2} \, ds \right) \int_{r}^{M+t} (1 + |t-s|)^{-1/2} \, ds
\]

\[
\leq C \left( \int_{r}^{M+t} |\partial_r f(t,s)|^2 (1 + |t-s|)^{1/2} \, ds \right) (1 + |t-r|)^{1/2}.
\]
Thus,
\[
\int_0^{M+t} (1 + |t - r|)^{-3/2} |f(t, r)|^2 r \, dr \\
\leq C \int_0^{M+t} \left( \int_r^{M+t} |\partial_r f(t, s)|^2 (1 + |t - s|)^{1/2} \, ds \right) (1 + |t - r|)^{-3/2} r \, dr \\
\leq C \int_0^{M+t} |\partial_r f(t, s)|^2 (1 + |t - s|)^{1/2} s \, ds \int_0^s (1 + |t - r|)^{-3/2} r \, dr \\
\leq C \int_0^{M+t} |\partial_r f(t, s)|^2 s \, ds,
\]
and Lemma A.2 is proved. \(\blacksquare\)

**Lemma A.3** (Generalized Gronwall inequality). Let \( f \in C^1[0, \infty) \), \( g, h \in C[0, \infty) \) be nonnegative and let
\[
\frac{df^2(t)}{dt} \leq f(t)g(t) + h(t)f^2(t).
\]
Then
\[
f(t) \leq \left( f(0) + \frac{1}{2} \int_0^t g(s) \, ds \right) \exp \left( \frac{1}{2} \int_0^t h(s) \, ds \right).
\]

**Lemma A.4** ([10, Lemma 1.3.2]). Let \( w \) be a solution in \([0, T]\) to the ordinary differential equation
\[
\frac{dw}{dt} = a_0(t)w^2 + a_1(t)w + a_2(t)
\]
with \( a_j \) continuous and \( a_0 \geq 0 \). Let
\[
K = \left( \int_0^T |a_2(t)| \, dt \right) \exp \left( \int_0^T |a_1(t)| \, dt \right).
\]
Then
\[
\left( \int_0^T a_0(t) \, dt \right) \exp \left( - \int_0^T |a_1(t)| \, dt \right) < (w(0) - K)^{-1}
\]
provided that \( w(0) > K \).

**Lemma A.5** (Blowup of smooth solution to problem (2.37)). The smooth solution to (2.37) blows up in finite time if \( F_0(\sigma) \neq 0 \).

**Proof.** Assume that (2.37) admits a global smooth solution. Due to \( (F_0')^2(M) = (F_0')^2(-\infty) = 0 \) and \( (F_0')^2 \neq 0 \), one has
\[
F_0'(\sigma)F_0''(\sigma) = \left( \frac{1}{2}(F_0'(\sigma)) \right)' < 0 \text{ on some interval } I \subset (-\infty, M).
\]
Without loss of generality, we can assume \( F_0'(\sigma) < 0 \) and \( F_0''(\sigma) > 0 \) on \( I \).
Let $\Sigma = \{(\tau, \sigma(\tau, l)) : \tau \geq 0, l \in I\}$, where $\sigma(\tau, l)$ stands for the characteristics of (2.37) emanating from the point $(l, 0)$; i.e., $\sigma(\tau, l)$ satisfies

$$\begin{cases}
    \frac{d\sigma(\tau, l)}{d\tau} = G^2(\tau, \sigma(\tau, l)), \\
    \sigma(0, l) = l.
\end{cases} \quad (A.2)$$

Set $Q(\tau, l) = (\partial_\sigma G)(\tau, \sigma(\tau, l))$ and $G(\tau, l) = G(\tau, \sigma(\tau, l))$. It follows from the equation in (2.37) that

$$Q(\tau, l) = \frac{F_0'(l)}{1 + F_0'(l) \int_0^\tau G(t, l) dt} \quad (A.3)$$

and

$$(\partial_\sigma Q)(\tau, \sigma(\tau, l)) \partial_\tau \sigma(\tau, l) = \frac{F''_0(l) - (F_0')^2(l) \int_0^\tau Q(t, l) \partial_\tau \sigma(t, l) dt}{(1 + F_0'(l) \int_0^\tau G(t, l) dt)^2}.$$

Therefore,

$$Q < 0 \text{ and } \partial_\sigma Q > 0 \text{ in } \Sigma. \quad (A.4)$$

Choose $l_i \in I$ ($i = 0, 1, 2$) such that $l_0 < l_1 < l_2$ and denote $E_j = \int_{l_i}^{l_{i+1}} (F_0')^2(l) dl$ for $j = 0, 1$. It follows from the conservation of energy for problem (2.37) and (A.4) that, for $j = 0, 1$,

$$0 < E_j = \int_{\sigma(\tau, l_i)}^{\sigma(\tau, l_{i+1})} Q^2(\tau, s) ds$$

$$\leq (-Q(\tau, l_j)) \int_{\sigma(\tau, l_i)}^{\sigma(\tau, l_{j+1})} (-Q)(\tau, s) ds = -Q(\tau, l_j)(G(\tau, l_j) - G(\tau, l_{j+1})),$$

which yields

$$G(\tau, l_j) - G(\tau, l_{j+1}) \geq -\frac{E_j}{Q(\tau, l_j)}, \quad j = 0, 1. \quad (A.5)$$

By (A.2), one has that $\sigma(\tau, l) < M$ holds for $l \in I$ and all $\tau$. Therefore,

$$\sum_{i=0}^{2} \int_0^\infty G^2(\tau, l_i) d\tau \leq 3M + \sum_{i=0}^{2} |l_i|,$$

which implies that there exists a sequence $\{\tau_k\} \subset [0, \infty)$ with $\tau_k \to \infty$ as $k \to \infty$ such that

$$G(\tau_k, l_i) \to 0 \text{ as } k \to \infty \text{ for } i = 0, 1, 2. \quad (A.6)$$

It then follows from (A.5) and (A.6) that

$$Q(\tau_k, l_j) \to -\infty \text{ as } k \to \infty \text{ for } j = 0, 1. \quad (A.7)$$

On the other hand, by (A.4), one has

$$\int_0^{\tau_k} G(t, l_0) dt > \int_0^{\tau_k} G(t, l_1) dt.$$

Together with (A.4) and (A.7), this yields as $k \to \infty$

$$-\frac{1}{F_0'(l_0)} \geq -\frac{1}{F_0'(l_1)}.$$
and $F'_0(l_1) \leq F'_0(l_0)$. The latter, however, contradicts the fact that $F'_0(l_0) < F'_0(l_1)$ holds due to $F''_0(\sigma) > 0$ in $I$ and $l_0 < l_1$.

Thus, the proof of Lemma A.5 has been completed. □

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