Abstract. We extend Lang's conjectures to the setting of intermediate hyperbolicity and prove two new results motivated by these conjectures. More precisely, we first extend the notion of algebraic hyperbolicity (originally introduced by Demailly) to the setting of intermediate hyperbolicity and show that this property holds if the appropriate exterior power of the cotangent bundle is ample. Then, we prove that this intermediate algebraic hyperbolicity implies the finiteness of the group of birational automorphisms and of the set of surjective maps from a given projective variety. Our work answers the algebraic analogue of a question of Kobayashi on analytic hyperbolicity.

1. Introduction

This paper is concerned with Lang's conjectures on hyperbolic varieties. Lang's conjectures relate different notions of hyperbolicity for $X$ (see [Lan86] and [Javb, §12] for a summary of his conjectures) from complex analysis to algebraic geometry and number theory. The aim of this paper is to prove several results motivated by these conjectures, and to extend Lang's conjectures on hyperbolic varieties to the more general setting of “intermediate hyperbolicity” (see Section 2). The latter is hinted at in Lang's original paper, but not worked out anywhere in the literature in full. In fact, in [Lan86, page 162] Lang explicitly says that “I omit the whole area of intermediate hyperbolicity”. In this paper, we present his conjectures in the intermediate setting, and prove several new results going in their direction.

Let us briefly explain the notion of “intermediate hyperbolicity” from a complex-analytic perspective. Recall that a complex space $X$ is Kobayashi hyperbolic if and only if the Kobayashi pseudometric $d_X$ on $X$ is a metric (see [Kob98]). In the hope of understanding the property of being Kobayashi hyperbolic for a complex space, Eisenmann [Eis70] extended Kobayashi's definition and introduced the notion of $p$-analytic hyperbolicity for all $1 \leq p \leq \dim(X)$ (see [Dem97, Definition 1.3.(ii)] for a precise definition).

We note that 1-analytic hyperbolicity and Kobayashi hyperbolicity coincide for compact complex-analytic spaces. Furthermore, a complex space $X$ of dimension $d = \dim(X)$ is $d$-analytically hyperbolic (as defined below) if and only if it is “measure-hyperbolic” (as defined in Kobayashi’s book [Kob98]). Moreover, if $X$ is $p$-analytically hyperbolic, then it is $(p + 1)$-analytically hyperbolic, so that we have a string of implications

$$X \text{ is } 1\text{-analytically hyperbolic} \implies \ldots \implies X \text{ is } (\dim X)\text{-analytically hyperbolic}.$$ 

In particular, the notions of $p$-analytic hyperbolicity interpolate between the notions of Kobayashi hyperbolicity and measure hyperbolicity. This is why we sometimes refer to the notion of $p$-hyperbolicity as being an “intermediate hyperbolicity”.

Kobayashi-Ochiai proved that a smooth projective variety $X$ of general type (i.e., $\omega_X$ is big) is $\dim(X)$-analytically hyperbolic, and Demailly generalized their result to the setting of intermediate hyperbolicity by replacing the condition that $\omega_X$ is big by a different positivity condition on $\Lambda^p\Omega^1_X$. We state here a simpler version of his result for the sake of convenience.

Theorem 1.1 (Demailly). Let $X$ be a smooth projective variety over $\mathbb{C}$. If $\bigwedge^p \Omega^1_X$ is ample, then $X$ is $p$-analytically hyperbolic.

Motivated by Lang’s conjectures, the first aim of this paper is to prove algebraic analogues of Demailly’s theorem (see Theorem 1.4 for a precise statement). Our second aim is to investigate finiteness properties of varieties which are hyperbolic in some intermediate sense (see Theorem 1.8).

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1.1. An algebraic analogue of Demailly’s analytic theorem on intermediate hyperbolicity. We introduce a notion of intermediate hyperbolicity inspired by Demailly’s notion of algebraic hyperbolicity (see [Dem97] (see also [Java1] [Java2] [JK20] [JX] [Rou19]).

**Definition 1.2** \((p\text{-algebraically hyperbolicity modulo } \Delta)\). Let \(X\) be a projective variety over \(k\), let \(\Delta\) be a closed subset of \(X\), and let \(p\) be a positive integer. We say that \(X\) is \(p\text{-algebraically hyperbolic over } k \text{ modulo } \Delta\) if, for every ample line bundle \(L\) on \(X\), there is a real number \(\alpha = \alpha(X, \Delta, L)\) such that, for every smooth projective \(p\text{-dimensional variety } Y \text{ over } k\) with ample canonical bundle \(\omega_Y\) and every rational map \(f : Y \dashrightarrow X\) which is generically finite onto its image and satisfies \(f(Y) \not\subset \Delta\), the following inequalities are satisfied for every \(0 \leq \ell \leq p\):

\[
(f^*L)_{\ell} \cdot (K_Y)^{p-\ell} \leq \alpha \cdot K_Y^p.
\]

Following Lang’s terminology, we will say that \(X\) is pseudo-\(p\text{-algebraically hyperbolic over } k\) if there is a proper closed subset \(\Delta \subset X\) such that \(X\) is \(p\text{-algebraically hyperbolic modulo } \Delta\) over \(k\). Also, we will say that \(X\) is \(p\text{-algebraically hyperbolic over } k\) if \(X\) is \(p\text{-algebraically hyperbolic modulo } \Delta\) the empty subset over \(k\).

Note that a projective variety \(X\) is 1-algebraically hyperbolic modulo the emptyset if and only if it is algebraically hyperbolic in Demailly’s sense [JK20]. More generally, a projective variety \(X\) is pseudo-1-algebraically hyperbolic if and only if it is pseudo-algebraically hyperbolic [Java1] §9.

Note that we restrict to varieties \(Y\) with ample canonical bundle. This is analogous to the fact that one may test the algebraic hyperbolicity of a projective variety on maps from curves of genus at least two (see the introduction of [JK20] for a detailed explanation).

Our first result gives an algebraic analogue of Demailly’s theorem (Theorem 1.1). To state it, we recall the definition of ampleness modulo a closed subset. We refer the reader to Lazarsfeld’s books for a definition of the augmented base locus [Laz04a] [Laz04b].

**Definition 1.3.** Let \(X\) be a projective variety, let \(\Delta \subset X\) be a closed subset, let \(E\) be a vector bundle on \(X\), and let \(p : P(E^\vee) \to X\) be the natural projection. Then \(E\) is ample modulo \(\Delta\) if the augmented base locus \(B^+(O_{P(E^\vee)}(1))\) is included in \(p^{-1}(\Delta)\).

In other words, \(E\) is ample modulo \(\Delta\), if and only if for any ample line bundle \(A\) on \(P(E^\vee)\), there is an integer \(m \geq 1\) such that the base locus of \(O_{P(E^\vee)}(m) \otimes A^{-1}\) is included in \(p^{-1}(\Delta)\).

Our first main result now reads as follows.

**Theorem 1.4.** Let \(X\) be a smooth projective variety over \(k\), and let \(\Delta \subset X\) be a proper closed subset. If \(\bigwedge^p \Omega^1_X\) is ample modulo \(\Delta\), then \(X\) is \(p\text{-algebraically hyperbolic modulo } \Delta\).

This algebraic analogue of Lazarsfeld’s analytic result fits in well with Lang’s “pseudoﬁed” notions of hyperbolicity (see [Lan80]). In fact, Lang deﬁnes a variety to be pseudo-hyperbolic if it is hyperbolic modulo some proper closed subset. He conjectured that pseudo-hyperbolicity (in any sense of the word “hyperbolic”) coincides with being of general type (see [Lan80] page 161 or the more recent [Java1] §12]). Demailly actually also proved a “pseudoﬁed” version of his theorem (Theorem 1.1). We omit a precise formulation of his result here.

The algebraic hyperbolicity of a projective variety implies that moduli spaces of maps from varieties are “bounded” (i.e., have only ﬁnitely many connected components). We refer to [JK20] for precise statements. The following deﬁnition extends the notion of boundedness introduced in [JK20] (see also [Java2] [JX]).

**Deﬁnition 1.5** \((p\text{-algebraically bounded modulo } \Delta)\). Let \(X\) be a projective variety over \(k\), let \(\Delta\) be a closed subscheme of \(X\), and let \(p\) be a positive integer. We say that \(X\) is \(p\text{-algebraically bounded modulo } \Delta\) if, for every ample line bundle \(L\) on \(X\) and every normal projective variety \(Y\) with \(\dim Y = p\), there is a real number \(C > 0\) such that, for every rational map \(f : Y \dashrightarrow X\) which is generically ﬁnite onto its image and satisﬁes \(f(Y) \not\subset \Delta\), the coefﬁcients of the Hilbert polynomial of the closure of the graph of \(f\) in \(X \times Y\) are bounded by \(C\).

**Remark 1.6.** Keeping the notations of the previous deﬁnition, and denoting \(\text{Graph}(f)\) the (closure of) graph of \(f\), we recall that the Hilbert polynomial of \(f\) is obtained by ﬁxing not only \(L\) an ample line bundle
on $X$, but also $A$ an ample line bundle on $Y$, and by computing the Hilbert polynomial of $\text{Graph}(f)$ with respect to the ample line bundle

$$A \otimes L_{|\text{Graph}},$$

which amounts to computing the Hilbert polynomial of $Y$ with respect to the ample line bundle

$$f^*L \otimes A.$$  

Note that it is straightforward to check that the boundedness condition is independent of the ample line bundle $A$ we fix on $Y$, i.e. if it holds for an ample line bundle $A$, it holds for any other ample line bundle on $Y$.

**Remark 1.7.** Voisin introduced an algebraic analogue of (analytic) measure-hyperbolicity; see [Vois03, Definition 2.20]. Indeed, she defines a variety $X$ to be algebraically measure-hyperbolic if, for every ample line bundle $L$ on $X$, there exists a constant $A > 0$ such that, for any covering family of curves $\pi : C \to B$ (with generic fiber $C$ of genus $g$) and every dominant map $\phi : C \to X$ non-constant on the fibers, one has $2g - 2 \geq A \cdot \deg \phi_* C L$. Conjecturally, if $X$ is a smooth projective variety, then $X$ is $(\dim X)$-algebraically hyperbolic if and only if it is algebraically measure-hyperbolic (in Voisin’s sense). For example, by [Vois03, Lemma 2.19], a variety of general type is algebraically measure-hyperbolic, and in this paper we prove the similar (a priori different) statement that a variety of general type is $(\dim X)$-algebraically hyperbolic; see Theorem 5.5.

Following Lang’s terminology as before, we will say that $X$ is pseudo-$p$-algebraically bounded over $k$ if there is a proper closed subset $\Delta \subset X$ such that $X$ is $p$-algebraically bounded modulo $\Delta$ over $k$. Also, we say that $X$ is $p$-algebraically hyperbolic modulo $\Delta$ over $k$ if it is $p$-algebraically bounded modulo the empty subset over $k$. With this terminology at hand, a projective variety is 1-algebraically bounded over $k$ if and only if it is bounded over $k$ [La01, §10].

The relation between algebraic hyperbolicity and boundedness in the context of intermediate hyperbolicity reads as follows.

**Proposition 1.8.** Let $X$ be a smooth projective variety over $k$, and let $\Delta \subset X$ be a proper closed subset of $X$. If $X$ is $p$-algebraically hyperbolic modulo $\Delta$ over $k$, then $X$ is $p$-algebraically bounded modulo $\Delta$ over $k$.

Note that Proposition [La8] generalizes the “trivial” implication that an algebraically hyperbolic projective variety (in Demailly’s sense) is 1-bounded (in the sense of [JK20, Definition 4.1]) to the more general setting of intermediate hyperbolicity.

The notion of boundedness is a priori weaker than the notion of being algebraically hyperbolic. However, it is conjecturally equivalent (see Section 2) and turns out to be strong enough to force certain finiteness properties, as we show now.

1.2. **Kobayashi-Ochiai’s finiteness theorem.** For our final result, our starting point is Kobayashi-Ochiai’s finiteness theorem for varieties of general type. Namely, if $X$ is a projective variety of general type over $\mathbb{C}$ and $Y$ is a projective integral variety, then the set of dominant rational maps $Y \dasharrow X$ is finite. As Lang’s “intermediate” conjectures predict that a pseudo-$p$-algebraically bounded projective variety is of general type, our following result is in accordance with Lang’s conjectures.

**Theorem 1.9.** Let $p \geq 1$ be an integer and let $X$ be a projective integral pseudo-$p$-algebraically bounded variety over $\mathbb{C}$. Then the group of birational automorphisms $\text{Bir}_k(X)$ is finite and for every projective integral variety $Y$ over $\mathbb{C}$, the set $\text{Sur}_k(Y, X)$ of surjective morphisms $Y \to X$ is finite.

We can use Theorem 1.4 and Theorem 1.9 to reprove part of Kobayashi-Ochiai’s finiteness theorem. Indeed, if $X$ is a projective variety of general type, then $X$ is $(\dim X)$-algebraically hyperbolic by Theorem 1.4 (since asking for $\bigwedge^{\dim(X)} \Omega_X$ to be ample modulo some proper closed subset $\Delta$ is equivalent to $X$ being of general type). In particular, by Proposition 1.8 and Theorem 1.9 it follows that $\text{Bir}_k(X)$ is finite and $\text{Sur}_k(Y, X)$ is finite for every projective integral variety $Y$. (Note that we do not obtain the finiteness of the set of dominant rational maps from $Y$ to $X$.)

Theorem 1.9 provides an “intermediate” version of the finiteness of $\text{Aut}_k(X)$ and $\text{Bir}_k(X)$ proven for a pseudo-bounded projective variety $X$ in [JK20, JX]. Also, it generalizes Matsumura’s finiteness theorem...
that a variety of general type has only finitely many automorphisms (by Theorem \[1.4\] and Proposition \[1.8\], and provides algebraic analogues of arithmetic finiteness results proven in \[Java\].

Kobayashi asked about analytic analogues of Matsumura’s finiteness theorem for Aut(X). In fact, in \[Kob93\] Kobayashi asked whether a dim(X)-analytically hyperbolic projective variety has only finitely many automorphisms. The answer is expected to be positive, as such a variety is expected to be of general type. For recent progress on Kobayashi’s problem we refer the reader to \[Etc\]. Our result (Theorem \[1.9\]) can be interpreted as providing a positive answer to the algebraic analogue of Kobayashi’s question.

1.3. Outline of paper. In Section \[2\] we state conjectures relating several intermediate notions of hyperbolicity. We take the opportunity to introduce several new arithmetic notions of intermediate hyperbolicity which will be the topic of future works. In Section \[3\] we prove our finiteness result (Theorem \[1.9\]) for pseudo-p-algebraically bounded varieties. In Section \[4\] we show that pseudo-p-algebraic hyperbolicity implies pseudo-p-algebraic boundedness, and thereby prove Proposition \[1.8\]. In the final section, we prove our criterion for pseudo-p-algebraic hyperbolicity (Theorem \[1.4\]).

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Conventions. Throughout this paper, we will let \(k\) denote an algebraically closed field of characteristic zero. A variety over \(k\) is a finite type separated scheme over \(k\).

2. Lang’s intermediate algebraic and arithmetic conjectures

In this section, we pursue the intermediate \(p\)-hyperbolicity analogues of Lang’s conjectures in the algebraic and arithmetic setting (thereby leaving out the complex-analytic analogues which are discussed in \[Etc\]). We will build on Lang’s original conjectures \[Lang86\] and the extensions of his conjectures summarized in \[Java\] \[section 12\].

Throughout this section, let \(k\) be an algebraically closed field of characteristic zero. Given a proper scheme \(X\) over \(k\), we refer to \[Java\] \[section 7\] for the definition of pseudo-Mordellicity, to \[Java\] \[section 9\] for the definition of pseudo-algebraic hyperbolicity, and to \[Java\] \[section 10\] for the definition of pseudo-boundedness. The notions of \(p\)-algebraic hyperbolicity and \(p\)-algebraic boundedness are defined in the introduction (see Definition \[1.2\] and Definition \[1.3\]). To state the general conjecture for varieties of general type, we will need one additional definition. To state this definition, we refer to \[Java\] \[section 3\] for the notion of a model.

Definition 2.1 (\(p\)-Mordellicity modulo \(\Delta\)). Let \(p \geq 0\) be an integer, let \(X\) be a proper variety over \(k\), and let \(\Delta \subset X\) be a closed subset. Then, we say that \(X\) is \(p\)-Mordellic over \(k\) modulo \(\Delta\) if, for every finitely generated subfield \(K \subset k\), every model \(\mathcal{X}\) for \(X\) over \(K\), and every \(p\)-dimensional smooth projective geometrically connected variety \(Y\) over \(K\), the set of rational maps \(f : Y \to X\) which are generically finite onto their image \(f(Y)\) and satisfy \(f(Y) \not\subset \Delta\) is finite.

Definition 2.2. Let \(p \geq 0\). A proper variety \(X\) over \(k\) is \(p\)-Mordellic over \(k\) if \(X\) is \(p\)-Mordellic over \(k\) modulo the empty subset.

Definition 2.3. Let \(p \geq 0\). A proper variety \(X\) over \(k\) is pseudo-\(p\)-Mordellic over \(k\) if there is a proper closed subset \(\Delta \subset X\) such that \(X\) is \(p\)-Mordellic over \(k\) modulo \(\Delta\).

Note that \(X\) is Mordellic over \(k\) (as defined in \[Java\] \[section 3\]) if and only if \(X\) is 0-Mordellic modulo the empty subset over \(k\). Indeed, a 0-dimensional smooth projective geometrically connected variety \(Y\) over \(K\) is isomorphic to \(\text{Spec } K\), so that the set of morphisms \(Y \to X\) equals the set of \(K\)-rational points of \(X\).

Note that, if a proper variety \(X\) over \(k\) is pseudo-\(p\)-Mordellic, then \(p \leq \dim X\). Also, the fact that we allow for \(p = 0\) in the above definition is an artifact of the arithmetic setting; finiteness of “points” is a reasonable property to impose (and study) over finitely generated fields of characteristic zero.

Let \(X\) be a projective variety over \(k\), and let \(\Delta \subset X\) be a closed subset. Assume that, for every algebraically closed field extension \(L/k\) of finite transcendence degree, the variety \(X_L\) is Mordellic modulo \(\Delta_L\) over \(L\).
Then, for every $1 \leq p \leq \dim X$, the variety $X$ is $p$-Mordellic modulo $\Delta$. Indeed, the set of rational maps $f : Y \to X$ with $f(Y) \not\subseteq \Delta$ equals the set of $K(Y)$-rational points of $X \setminus \Delta$, and the latter is finite by the Mordelicity assumption on the varieties $X_L$.

For varieties of general type, we expect all notions (including the intermediate ones) of pseudo-hyperbolicity to coincide. The following conjecture provides a precise statement.

**Conjecture 2.4** (Lang’s intermediate pseudo-conjectures). Let $X$ be a projective variety over $k$, and let $p \geq 1$ be an integer. Then the following are equivalent.

1. The variety $X$ is of general type.
2. There is a proper closed subset $\Delta \subseteq X$ such that, every subvariety $Y \subset X$ of dimension at least $p$ with $Y \not\subseteq \Delta$ is of general type.
3. The variety $X$ is pseudo-Mordellic over $k$.
4. The variety $X$ is pseudo-$p$-Mordellic over $k$.
5. The variety $X$ is pseudo-algebraically hyperbolic over $k$.
6. The variety $X$ is pseudo-$p$-algebraically hyperbolic over $k$.
7. The variety $X$ is pseudo-bounded over $k$.
8. The variety $X$ is pseudo-$p$-algebraically bounded over $k$.
9. The variety $X$ is $\dim(X)$-Mordellic over $k$.
10. The variety $X$ is $\dim(X)$-algebraically-hyperbolic over $k$.
11. The variety $X$ is $\dim(X)$-algebraically-bounded over $k$.

Note that (1) is independent of $p$, so that part of this conjecture is redundant. For example, (5) is equivalent to (6) with $p = 1$. Nonetheless, we chose to present the conjecture in this way to facilitate discussing known results in the following remark.

**Remark 2.5** (What do we know about Conjecture 2.4?). The following statements hold.

1. Obviously, (2) $\implies$ (1).
2. If $\dim X = 1$, then Conjecture 2.4 holds by Faltings’s proof of Mordell’s conjecture and the classical finiteness theorem of De Franchis-Severi for Riemann surfaces. More generally, if $X$ is a closed subvariety of an abelian variety, then conjecture 2.4 holds by Faltings’s proof of Mordell-Lang and the work of Ueno, Bloch-Ochiai-Kawamata [Kaw80] and Yamanoi [Yam15] on closed subvarieties of abelian varieties.
3. If $\dim X = 2$, then (3) $\implies$ (2), (5) $\implies$ (2), and (7) $\implies$ (2). This is explained in [Javb].
4. By [BJK], we have that (5) $\implies$ (7).
5. By this paper we prove (6) $\implies$ (8) (Proposition 1.8), and thus (10) $\implies$ (11). We also show that (1) $\implies$ (10) (and thus (1) $\implies$ (11)). See Proposition 1.8 and Corollary 5.5.
6. Assuming $k$ is uncountable, we show that (8) $\implies$ (11); see Lemma 3.8.
7. We show that, if $A^p \otimes \Omega^1_{X/k}$ is ample modulo some proper closed subset, then $X$ satisfies (1), (2), (6), (8), (10), and (11) (see Theorem 1.4, Lemma 3.8 and Proposition 1.8).
8. We show that, assuming $X$ satisfies (10) or (11), then $\text{Bir}_k(X)$ is finite and $\text{Sur}_k(Y,X)$ is finite for every $Y$ (see Theorem 1.9).

Part of the above conjecture already appears in Lang’s original paper. For example, the equivalence of (1), and (3) is stated explicitly in [Lan86]. Moreover, the equivalence of (1), (5), and (7) is implicit in Lang’s original conjectures (see [Javb §12]). However, the other conjectured equivalences are new.

To conclude this section, we push Lang’s conjectures further, and state the general conjecture for “intermediate” exceptional loci.

**Conjecture 2.6** (Lang’s intermediate conjectures for exceptional loci). Let $X$ be a projective integral variety over $k$, let $\Delta \subset X$ be a closed subset, and let $p$ be a positive integer. Then the following statements are equivalent.

1. Every subvariety $Y$ of $X$ of dimension at least $p$ with $Y \not\subseteq \Delta$ is of general type.
2. The variety $X$ is Mordellic modulo $\Delta$ over $k$.
3. The variety $X$ is $p$-Mordellic modulo $\Delta$ over $k$.
4. The variety $X$ is $p$-algebraically hyperbolic modulo $\Delta$ over $k$. 
(5) The variety $X$ is $p$-algebraically bounded modulo $\Delta$ over $k$.

Note that Conjecture 2.6 (for all $X$ and $p$) implies Conjecture 2.4 (for all $X$ and $p$).

Remark 2.7 (What do we know about this conjecture?). By Proposition 1.8, we have that (3) $\Longrightarrow$ (4). We also show that, if $\Lambda \Omega_X$ is ample modulo $\Delta$, then (1), (3), and (4) hold; see Theorems 1.4 and 5.3.

3. Finiteness results for intermediate pseudo-hyperbolic varieties

We will prove the finiteness results for birational automorphisms and surjective morphisms first for pseudo-$p$-algebraically bounded varieties (Definition 1.3). We then deduce the results for pseudo-$p$-algebraically hyperbolic varieties (Definition 1.2) from the fact that such varieties are pseudo-$p$-algebraically bounded. The intuition behind this line of reasoning is that "hyperbolicity" entails "boundedness".

Crucial to our proofs below is the (obvious) non-uniruledness of a pseudo-$p$-hyperbolic variety. We record this observation in the following lemma.

Lemma 3.1. Let $X$ be a projective variety over $k$, and let $1 \leq p \leq \dim X$ be an integer. If $X$ is pseudo-$p$-algebraically bounded or pseudo-$p$-algebraically hyperbolic over $k$, then $X$ is non-uniruled.

If $X$ and $Y$ are projective varieties over $k$, we let $\text{Hom}_k(Y, X)$ be the scheme parametrizing morphisms from $Y$ to $X$ over $k$. If $\text{Hilb}_k(Y \times X)$ is the Hilbert scheme of $Y \times X$, then the morphism $\text{Hom}_k(Y, X) \to \text{Hilb}_k(Y \times X)$ mapping a morphism $f : Y \to X$ to its graph $\Gamma_f \subset Y \times X$ is an open immersion [Nit03, Theorem 6.6].

We let $\text{Sur}(Y, X)$ be the open and closed subscheme of $\text{Hom}_k(Y, X)$ parametrizing surjective morphisms from $Y$ to $X$ over $k$. We start by showing that $(\dim X)$-algebraically bounded projective schemes have the property that the schemes $\text{Sur}_k(Y, X)$ are of finite type. That is, if $Y$ is a projective integral scheme over $k$ and we fix ample line bundles on $X$ and $Y$, then the set of Hilbert polynomials associated to the set of surjective morphisms $Y \to X$ (and the choices of ample line bundles on $X$ and $Y$) is finite. One also says that the Hilbert polynomial of a surjective morphism $Y \to X$ is bounded. (Although the Hilbert polynomials themselves certainly depend on the choice of ample line bundle on $X$ and $Y$, their "boundedness" does not.)

Before stating and proving our results, note that the scheme $\text{Sur}_k(Y, X)$ is of finite type over $k$ if and only if it has finitely many connected components. That is, $\text{Sur}_k(Y, X)$ is of finite type over $k$ if and only if, for every ample line bundle $L_X$ on $X$ and ample line bundle $L_Y$ on $Y$, there is an integer $n \geq 1$ and polynomials $\Phi_1, \ldots, \Phi_n$ in $\mathbb{Q}[t]$ such that, for every surjective morphism $f : Y \to X$ the Hilbert polynomial of $f$ with respect to the ample line bundle $L_X \boxtimes L_Y$ on $X \times Y$ lies in the finite set $\{\Phi_1, \ldots, \Phi_n\}$.

Lemma 3.2. Let $X$ be a projective $(\dim X)$-algebraically bounded integral variety over $k$. Then, for every projective integral variety $Y$ over $k$, the scheme $\text{Sur}_k(Y, X)$ is of finite type over $k$.

Proof. If $\dim X \leq 2$, then $X$ is of general type [LX §3], in which case the statement of the lemma is well-known. Thus, we may and do assume that $\dim X \geq 3$.

To prove the lemma, we now proceed by induction on $\dim Y$. If $\dim Y = \dim X$, then every surjective morphism $Y \to X$ is generically finite, so that the statement follows from the definition. Indeed, the definition implies in this case that the Hilbert polynomials of the set of generically finite rational (dominant) maps $Y \dashrightarrow X$ are finite.

Now, let us suppose that $\dim Y > \dim X$. Fix an ample line bundle $L$ on $X$. To prove the boundedness of $\text{Sur}_k(Y, X)$, replacing $Y$ by a resolution of singularities if necessary, we may and do assume that $Y$ is smooth.

Assume that $\text{Sur}_k(Y, X)$ is not of finite type. Then, as the connected components of $\text{Sur}_k(Y, X)$ are of finite type and indexed by Hilbert polynomials (computed with respect to $L$), there is a sequence $f_i : Y \to X$ of surjective morphisms with pairwise distinct Hilbert polynomials. Let $H \subset Y$ be a smooth ample divisor in $Y$. Note that the restrictions $f_i|_H : H \to X$ are still surjective. Indeed, for every $t \in H$, the fibre $f_i^{-1}\{t\}$ of $f_i : Y \to X$ over $t$ is of dimension $\geq 0$ and therefore intersects $H$. Then, by the induction hypothesis, the set of Hilbert polynomials associated to the morphisms $f_i|_H$ (with respect to the fixed ample line bundle $L$ on $X$) is finite. The key observation is that, in this situation, the set of numerical equivalence classes of the line bundles $(f_i|_H)^*L$ on $H$ must be finite. Indeed, observe first that the leading coefficient of the Hilbert polynomial of $(f_i|_H)^*L$ is equal to $(f_i^*L|_H + A|_H)^{\dim(H)} \geq (f_i^*L + A|_H \cdot A|_H^{\dim(H)-1})^\bullet$, where the inequality
follows from the fact that $(f_i^* L)|_H$ is nef (as it is the restriction of the pull-back of an ample). But since the set of Hilbert polynomials associated to the morphisms $f_i|_H$ is finite, we deduce that there exists a constant $\kappa$ such that, for any $i$, the following inequality is satisfied:

$$((f_i^* L + A)|_H \cdot A^\dim(H)^{-1} \leq \kappa.$$ 

Now, as $A$ is ample and $f_i^* L$ is globally generated, the line bundle $(f_i^* L + A)|_H$ is ample, hence base-point free and big, so that it follows from [JK20, Lemma 9.1] that the numerical equivalence classes of $(f_i^* L + A)|_H$ are finite. We conclude that the numerical equivalence classes of $(f_i^* L + A)|_H$ are finite. We confirm that the numerical equivalence classes of $(f_i^* L + A)|_H$ are finite as well.

As $\dim Y > 2$, it follows from the Lefschetz hyperplane theorem that $\text{NS}(Y) \to \text{NS}(H)$ is injective. In particular, as the set of numerical equivalence classes of the line bundles $(f_i|_H)^* L$ on $H$ is finite, it follows that the set of numerical equivalence classes of the line bundles $f_i^* L$ on $Y$ is finite. Now, as it is shown in [HKP06, Theorem 9.3], the Hilbert polynomial of a morphism $f : Y \to X$ is determined by the numerical equivalence class of $f^* L$ in $\text{NS}(Y)$. Indeed, the Hilbert polynomial of $f$ (with respect to $L$ and a fixed ample line bundle $A$ on $Y$) is given by $\chi(f^* L^\otimes n \otimes A^\otimes n)$. By Riemann-Roch, this number only depends on $\chi(f^* L^\otimes n \otimes A^\otimes n)$ which, in turn, only depends on the numerical equivalence class of $f^* L$ (and $A$) by basic properties of the Chern character. Therefore, we see that the set of Hilbert polynomials associated to the morphisms $f_i$ is finite which contradicts our assumption. We conclude that $\text{Sur}_k(Y, X)$ is of finite type, as required.

To prove the rigidity of surjective morphisms, we will appeal to a theorem of Hwang-Kebekus-Peternell [HKP06]. Their result relates the infinitesimal deformation space of a surjective morphism $Y \to X$ to the infinitesimal automorphisms of a suitable cover of $X$. For this reason, we investigate first the discreteness of $\text{Aut}_{X/k}$, where $\text{Aut}_{X/k}$ denotes the locally finite type group scheme of automorphisms of $X$ over $k$. Interestingly, to prove the rigidity of $\text{Aut}_{X/k}$, we will appeal to the boundedness of $\text{Sur}_k(Y, X)$ (for every $Y$) proved above.

Lemma 3.3. Let $X$ be a projective $\dim(X)$-algebraically bounded variety over $k$. Then $\text{Aut}_{X/k}$ is zero-dimensional.

Proof. Since $X$ is non-uniruled (Lemma 3.1), the connected component $A := \text{Aut}_{X/k}^0$ of the identity of $\text{Aut}_{X/k}$ is an abelian variety over $k$. For $a$ in $A$ and $x$ in $X$, we let $a \cdot x$ denote the action of $A$ on $X$. Consider the sequence of surjective morphisms $f_n : A \times X \to X$ given by

$$f_n(a, x) = (na) \cdot x.$$ 

Since the degree of the (finite étale) morphism $[n] : A \times X \to A \times X$ equals $n^{2 \dim A}$ and thus increases with $n$, we see that the Hilbert polynomials of the morphisms $f_n : A \times X \to X$ are pairwise distinct. In particular, the scheme $\text{Sur}_k(A \times X, X)$ is not of finite type over $k$. However, as $X$ is $\dim(X)$-algebraically bounded, this contradicts Lemma 3.2. 

We record the basic fact that a finite surjective cover of a $p$-algebraically bounded projective variety is a $p$-algebraically bounded projective variety in the following lemma.

Lemma 3.4. Let $Z \to X$ be a finite surjective morphism of projective varieties over $k$. If $X$ is $p$-algebraically bounded over $k$, then $Z$ is $p$-algebraically bounded over $k$.

Proof. This is a straightforward consequence of the definitions (cf. [JK20 Proposition 5.2.(2)]). 

We now prove the desired finiteness of surjective morphisms $Y \to X$, assuming $X$ is $\dim(X)$-algebraically bounded.

Theorem 3.5. If $X$ is a $\dim(X)$-algebraically bounded projective integral variety and $Y$ is a projective integral variety over $k$, then $\text{Sur}_k(Y, X)$ is finite.

Proof. By Lemma 3.2, the scheme $\text{Sur}_k(Y, X)$ is of finite type over $k$. Thus, it suffices to show that $\text{Sur}_k(Y, X)$ is zero-dimensional. To do so, let $f : Y \to X$ be a surjective morphism. As $X$ is non-uniruled (Lemma 3.1), by the theorem of Hwang-Kebekus-Peternell [HKP06], there is a finite surjective morphism $Z \to X$ and a morphism $Y \to Z$ such that $f : Y \to X$ factors as

$$Y \to Z \to X,$$
and $\text{Aut}^0_{Z/k}$ surjects onto the connected component of $f$ in $\text{Sur}_k(Y,X)$. Now, as $Z \to X$ is finite and $X$ is $(\dim X)$-algebraically bounded over $k$, by Lemma 3.3, it follows that $Z$ is $(\dim X)$-algebraically bounded over $k$. In particular, as $\dim X = \dim Z$ and $Z$ is $\dim X$-algebraically bounded, it follows from Lemma 3.3 that $\text{Aut}^0_{Z/k}$ is trivial. As $\text{Aut}^0_{Z/k}$ surjects onto the connected component of $f$ in $\text{Sur}_k(Y,X)$, it follows that the latter is trivial. We conclude that $\text{Sur}_k(Y,X)$ is zero-dimensional, as required. \hfill \Box

3.1. Birational selfmaps of $(\dim(X))$-algebraically bounded varieties. Let $X$ be a projective integral scheme over $k$. We define $\text{Bir}_{X/k}$ to be the subscheme of the Hilbert scheme $\text{Hilb}_k(X \times X)$ parametrizing, roughly speaking, closed subschemes $Z \subset X \times X$ such that $Z$ is (geometrically) integral and both projections $Z \to X$ are birational (see [Han87, Definition 1.8] for a more precise formulation). Note that the (abstract) group $\text{Bir}_k(X)$ of birational selfmaps $X \to X$ is in bijection with the set of $k$-points of the $k$-scheme $\text{Bir}_{X/k}$.

Hanamura shows that the scheme $\text{Bir}_{X/k}$ can be endowed with the structure of a group scheme structure, assuming that $X$ is a (terminal) minimal model (see [Han87, Definition 1.8]). We will use a slight extension of his result.

In fact, in [PS14, §4] Prokhorov and Shramov show that a proper non-uniruled integral variety over $k$ has a pseudo-minimal model. Hanamura’s main result on the scheme $\text{Bir}_{X/k}$ for minimal models $X$ is easily seen to extend to pseudo-minimal models by following his proof closely. Indeed, Hanamura’s proof relies on the fact that on a minimal model every pseudo-automorphism is an automorphism. This property also holds for pseudo-minimal models by [PS14, Corollary 4.7]. In particular, Hanamura’s work gives the following statement (see [Han87, §3]).

**Theorem 3.6 (Hanamura).** If $X$ is a pseudo-minimal model over $k$, then the scheme $\text{Bir}_{X/k}$ can be endowed with the structure of a group scheme over $k$ such that $\text{Bir}_{X/k}^0$ is isomorphic to $\text{Aut}_{X/k}^0$.

We use Hanamura’s structure result to prove the following finiteness result.

**Proposition 3.7.** Let $X$ be a projective integral variety over $k$. If $X$ is $(\dim(X))$-algebraically bounded, then $\text{Bir}_k(X)$ is finite.

**Proof.** Write $d := \dim(X)$. Note that, as $X$ is $d$-algebraically bounded over $k$, we have that $X$ is non-uniruled over $k$ (Lemma 3.1). Therefore, by the work of Prokhorov-Shramov [PS14, §4], the projective variety $X$ has a pseudo-minimal model. Let $Y$ be a pseudo-minimal model for $X$. Note that $\text{Bir}_k(X) = \text{Bir}_k(Y)$ (since $X$ and $Y$ are birational). Now, by Theorem 3.6, the scheme $\text{Bir}_{Y/k}$ can be endowed with the structure of a group scheme in such a way that $\text{Bir}_{Y/k}^0 = \text{Aut}_{Y/k}^0$. Now, since $X$ is $d$-algebraically bounded over $k$ and $Y$ is birational to $X$, it follows that $Y$ is $d$-algebraically bounded over $k$. In particular, $\text{Aut}_{Y/k}$ is trivial (Lemma 3.8), and thus Hanamura’s group scheme $\text{Bir}_{Y/k}$ is zero-dimensional. Finally, to conclude that $\text{Bir}_k(X)$ is finite, it suffices to show that $\text{Bir}_{Y/k}$ is of finite type over $k$. This boundedness statement is a straightforward consequence of the definition of $\text{Bir}_{Y/k}$ and the fact that $Y$ is $d$-algebraically bounded over $k$. \hfill \Box

3.2. Finiteness results for intermediate boundedness. In the previous subsection we achieved the finiteness of the set $\text{Sur}_k(Y,X)$ of surjective morphisms $Y \to X$, assuming $X$ is $(\dim(X))$-algebraically bounded. We now show the analogous statement for $p$-algebraic boundedness for $1 \leq p \leq \dim X$ an integer. To do so, we will show that pseudo-$p$-algebraic boundedness implies pseudo-$(p+1)$-algebraic boundedness, and then appeal to Theorem 3.6.

**Lemma 3.8.** Assume $k$ is uncountable. Let $X$ be a projective integral variety over $k$, let $\Delta \subset X$ be a closed subset, and let $1 \leq p < \dim(X)$. If $X$ is $p$-algebraically bounded modulo $\Delta$, then $X$ is $(p+1)$-algebraically bounded modulo $\Delta$.

**Proof.** Suppose that $X$ is not $(p+1)$-algebraically bounded modulo $\Delta$ over $k$. Let $Y$ be a smooth projective connected variety over $k$ of dimension $p + 1$. Let $f_i : Y \to X$ be a sequence of rational maps which are generically finite onto their image and satisfy $f_i(Y) \not\subset \Delta$. Assume that the $f_i$ have pairwise distinct Hilbert polynomials (recall that we define the Hilbert polynomial of a rational map $Y \to X$ to be the Hilbert polynomial of the closure of its graph in $Y \times X$ with respect to some fixed ample line bundle).

For every $i = 1, \ldots$, there is a proper closed subset $Y_i \subset Y$ such that $f_i$ is defined on $Y \setminus Y_i$. Since $k$ is uncountable, we have that $\cup_i Y_i(k) \neq Y(k)$. Let $Q$ be a point in $Y(k) \setminus \cup_i Y_i(k)$. Then, for every $i = 1, \ldots$, the morphism $f_i$ is defined at $Q$. Similarly, for $i = 1, \ldots$, let $\Delta_i := f_i^{-1}(\Delta)$, and note that $\Delta_i \subset Y$ is a proper closed subset. Then, as $k$ is uncountable, there is a point $P \in Y(k)$ in the complement of $\cup_i \Delta_i$. 

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Let $H$ be a very ample divisor of $Y$ containing $P$ and $Q$. Then, for every $i = 1, \ldots, s$, the rational map $f_{i} : Y \dashrightarrow X$ is well-defined, generically finite onto its image and satisfies $f_{i}(H) \not\subseteq \Delta$. As the rational maps $f_{i}$ have pairwise distinct Hilbert polynomials (cf. [15JK, 15K2]), we conclude that $X$ is not $p$-algebraically bounded modulo $\Delta$. This concludes the proof. \hfill \Box

**Corollary 3.9.** Assume $k$ is uncountable. Let $X$ be a pseudo-$p$-algebraically bounded projective variety over $k$. Then $\text{Bir}_{k}(X)$ is finite and, for every projective integral variety $Y$ over $k$, the set $\text{Sur}_{\Delta}(Y, X)$ is finite.

**Proof.** By Lemma 3.8 we may assume that $p = \dim(X)$. The result then follows from Theorem 3.7 and Proposition 3.7. \hfill \Box

**Proof of Theorem 1.3.** As $\mathcal{C}$ is uncountable, this clearly follows from Corollary 3.9. \hfill \Box

### 4. From intermediate algebraic hyperbolicity to intermediate boundedness

Now that we have proven our finiteness results for $p$-algebraically bounded varieties, we prove the analogous results for $p$-algebraically hyperbolic varieties by establishing that $p$-algebraic hyperbolicity implies $p$-algebraic boundedness. (Note that this proves that (8) implies (11) in Conjecture 2.4 and that (3) implies (4) in Conjecture 2.6.)

**Proof of Proposition 1.8.** Let $X$ be a smooth projective variety over $k$, and let $\Delta \subset X$ be a proper closed subset of $X$. Let $L$ be an ample line bundle on $X$. Assume that $X$ is $p$-algebraically hyperbolic modulo $\Delta$ over $k$. To do so, let $Y$ be a $p$-dimensional smooth projective variety over $k$. Let $f : Y \dashrightarrow X$ be a rational map which is generically finite onto its image and satisfies $f(Y) \not\subseteq \Delta$. Let $\tilde{\psi} : \tilde{Y} \rightarrow Y$ be a finite surjective morphism such that $K_{\tilde{Y}}$ is ample (note that $\tilde{\psi}$ does not depend on $f$), and let $g : \tilde{Y} \rightarrow Y \dashrightarrow X$ be the composed rational map. Since $g$ is generically finite onto its image and $g(\tilde{Y}) \not\subseteq \Delta$, the fact that $X$ is $p$-algebraically hyperbolic modulo $\Delta$ implies that there exists a real number $\alpha = \alpha(X, \Delta, L)$ such that, for all $0 \leq k \leq p$, the inequality

$$(g^{*}L)^{k} \cdot (K_{\tilde{Y}})^{p-k} \leq \alpha \cdot K_{\tilde{Y}}^{p}$$

holds. Note that $K_{\tilde{Y}} = \psi^{*}K_{Y} + R$, where $R$ is the ramification divisor of $\psi$.

The goal is to show that we can bound the coefficients of the Hilbert polynomial of $\text{Graph}(f) \subset Y \times X$ of $f$, computed with respect to (the restriction to $\text{Graph}(f)$) of any fixed ample line bundle on $Y \times Y$, independently of $f$. Letting $A$ be a very ample line bundle on $Y$, we fix the ample line bundle $A \boxtimes L$ on $Y \times X$. Observe that computing the Hilbert polynomial of $\text{Graph}(f)$ with respect to the ample line bundle $A \boxtimes L|_{\text{Graph}(f)}$ is equivalent to computing the Hilbert polynomial of $Y$ with respect to the ample line bundle $f^{*}L + A$.

By a result due to Kollár–Matsusaka in [KM89], to obtain the sought result, it suffices to bound the intersection numbers $(f^{*}L + A)^{p}$ and $(f^{*}L + A)^{p-1} \cdot K_{Y}$ independently of $f$. For the first intersection number, it is enough to bound independently of $f$

$$f^{*}L^{k} \cdot A^{p-k}$$

for all $1 \leq k \leq p$. To do so, since $\psi$ is surjective, note that the projection formula implies that

$$f^{*}L^{k} \cdot A^{p-k} = \deg(\psi)((g^{*}L)^{k} \cdot (\psi^{*}A)^{p-k})$$

Fix $m \in \mathbb{N}_{\geq 1}$ such that $B := mK_{\tilde{Y}} - \psi^{*}A$ is very ample (note that it is independent of $f$), and observe that

$$(g^{*}L)^{k} \cdot (\psi^{*}A)^{p-k} = (g^{*}L)^{k} \cdot (\psi^{*}A)^{p-k-1} \cdot \psi^{*}A$$

$$\leq (g^{*}L)^{k} \cdot (\psi^{*}A)^{p-k-1} \cdot mK_{\tilde{Y}}$$

$$= ((g^{*}L)^{k} \cdot (\psi^{*}A)^{p-k-2} \cdot K_{\tilde{Y}} \cdot \psi^{*}A)$$

$$\leq (g^{*}L)^{k} \cdot (\psi^{*}A)^{p-k-2} (mK_{\tilde{Y}})^{2} \leq \ldots \leq (g^{*}L)^{k} \cdot (mK_{\tilde{Y}})^{p-k},$$

so that we obtain

$$f^{*}L^{k} \cdot A^{p-k} \leq (\deg(\psi)m^{p-k})K_{\tilde{Y}}^{p}.$$
As for the second intersection number \((f^* L + A)^{p-1} \cdot K_Y\), it suffices to bound independently of \(f\)
\[(f^* L)^k \cdot A^{p-k-1} \cdot K_Y\]
for all \(1 \leq k \leq p - 1\). Note that
\[(f^* L)^k \cdot A^{p-k-1} \cdot K_Y = \deg(\psi)((g^* L)^k \cdot (\psi^* A)^{p-k-1} \cdot (K_{\tilde{Y}} - R))\]
Since \(R\) is effective, by arguing as before, we obtain that
\[(f^* L)^k \cdot A^{p-k-1} \cdot K_Y \leq (\deg(\psi) m^{p-k-1} \alpha) K_Y^p.\]
Thus, the coefficients of the Hilbert polynomial of the graph of \(f\) in \(Y \times X\) are bounded by a real number depending only on \(X, L, \Delta,\) and \(Y\). We conclude that \(X\) is \(p\)-algebraically bounded modulo \(\Delta\). \(\square\)

Combining the results of this section, we obtain the following finiteness statement.

**Theorem 4.1.** Assume \(k\) is uncountable, let \(p \geq 1\) be an integer, and let \(X\) be a projective pseudo-\(p\)-algebraically hyperbolic variety over \(k\). Then \(\text{Bir}_k(X)\) is finite and, for every projective integral variety \(Y\) over \(k\), the set \(\text{Surk}(Y, X)\) is finite.

**Proof.** Since \(X\) is pseudo-\(p\)-algebraically hyperbolic, it is pseudo-\(p\)-algebraically bounded (Proposition 1.8). In particular, the result follows from Corollary 3.9. \(\square\)

5. From positivity modulo \(\Delta\) to intermediate boundedness

In this section we prove Theorem 4.4. In our proof, we will use the following application of Bertini’s Theorem.

**Lemma 5.1.** Let \(\psi : \tilde{Y} \to Y\) be a proper birational surjective morphism between smooth projective connected varieties over \(k\) of dimension \(p\), and let \(Z \subset Y\) be the image of its exceptional locus \(\text{Exc}(\psi)\). Let \(L\) be a very ample line bundle on \(Y\), and let \(D\) be an effective divisor on \(\tilde{Y}\). Then, there exist \(F_1, \ldots, F_{p-1} \in |L|\) such that, for every \(1 \leq k \leq (p-1)\), the subvariety
\[V_k = F_1 \cap \ldots \cap F_k\]
is of dimension \((p-k) \geq 1\) and the following statements hold:

1. There is a point \(x_0\) in \(V_k \setminus Z\) such that \(V_k \setminus \{x_0\}\) is smooth and \(\psi^{-1}(x_0) \notin D\).

2. Define \(Z_0 = Z, V_0 = Y\) and \(\psi_0 = \psi\). For \(1 \leq k \leq p-1\), we inductively define \(H_k = \psi^{-1}(V_k \setminus Z_{k-1})\).

3. Then, the morphism \(\psi_k : H_k \to V_k\) induced by \(\psi\) upon restriction to \(H_k\) is surjective and birational.

4. For \(k \leq (p-2)\), the scheme \(V_k\) is normal, and
\[Z_k := \psi_k(\text{Exc}(\psi_k)).\]
is of codimension \(\geq 2\) in \(V_k\).

5. The variety \(V_k\) contains no irreducible components of \(Z_{k-1}\), as well as
\[
\text{codim}_{V_k}(V_k \cap Z) \geq 2.
\]

**Proof.** Denote \(L_0 = |L|\), pick \(x_0 \in X \setminus Z\) such that \(\psi^{-1}(x_0) \notin D\), and consider the linear system
\[L_1 := \{s \in |L| \mid s(x_0) = 0\}\]
which is an hyperplane in \(L_0\) whose base locus \(\text{Bs}(L_1)\) consists of the single point \(\{x_0\}\). Indeed, let \(x_1 \neq x_0\), and suppose that \(x_1\) is in \(\text{Bs}(L_1)\). Take then \(s \in |L|\) such that \(|L| = \text{Vect}(s) \oplus |L_1|\). As \(\text{Bs}(|L|) = \emptyset\), we must have that \(s(x_1) \neq 0\), but this implies that \(|L|\) does not separate the points \(x_0\) and \(x_1\), contradicting the fact that \(L\) is very ample.

We apply Bertini’s theorem a first time to obtain that, for a general \(s_1 \in L_1\), we have that \(F_1 := (s_1 = 0)\) is smooth away from \(\{x_0\}\). In particular, \(F_1\) is normal as it is of dimension \((p-1) \geq 2\) and smooth away from a point. Since \(\text{Bs}(L_1)\) consists of a single point not lying in \(Z_0 = Z\), we deduce that for a general \(s_1 \in L_1\),
the subvariety $F_1$ will not contain any irreducible component of $Z = Z_0$. In particular, as $\text{codim}_Y(Z) \geq 2$, we deduce that $\text{codim}_{F_1}(F_1 \cap Z) \geq 2$ as well. Define then
\[ H_1 = \psi^{-1}(F_1 \setminus Z_0), \]
and observe that since $F_1$ is irreducible, so is $F_1 \setminus Z_0$, as well as $\psi^{-1}(F_1 \setminus Z_0)$ and its closure. Accordingly, $H_1$ is irreducible, and so is $\psi(H_1) \supset F_1$, so that necessarily $\psi(H_1) = F_1$. Therefore
\[ \psi_1 : H_1 \to F_1 \]
is well defined, surjective, and obviously birational. Since $F_1$ is normal, $Z_1 := \psi_1(\text{Exc}(\psi_1))$ is of codimension 2 in $F_1$.

Consider now the linear system on $F_1$
\[ \mathcal{L}_2 := \{ s_{|F_1} : s \in \mathcal{L}_1 \}, \]
whose base locus remains equal to $\{ x_0 \}$. We apply once again Bertini’s theorem to obtain that, for a general $s_2 \in \mathcal{L}_1$, the scheme $(s_2)_{|F_1}$ is smooth away from $\{ x_0 \}$. In particular, defining $F_2 = (s_2 = 0)$, we deduce that
\[ V_2 = F_1 \cap F_2 \]
is smooth away from $\{ x_0 \}$, and thus irreducible. As before, we can always suppose that $V_2$ does not contain any irreducible component of $Z_1$, nor $Z \cap F_1 = Z_0 \cap F_1$ is irreducible, so that in particular
\[ \text{codim}_{V_2}(V_2 \cap Z) \geq 2. \]

Define
\[ H_2 = \psi^{-1}_1(V_2 \setminus Z_1). \]
By the same argument as before, we have that $\psi_1(H_2) = V_2$, and
\[ \psi_2 : H_2 \to V_2 \]
is then a birational surjective morphism.

We repeat this construction until $k = (p - 2)$. For $k = (p - 1)$, we lose the fact that $V_{p-1}$ is normal, and even irreducible. However, in this case, we can pick $F_{p-1}$ so that $V_{p-1} = F_{p-1} \cap V_{p-2}$ avoids completely $Z$ (which is possible since the base locus $\text{Bs} \mathcal{L}_{p-1}$ is equal to the single point $x_0$ which does not belong to $Z$), and therefore $\psi$ realizes an isomorphism between $H_{p-1} = \psi^{-1}(V_{p-1})$ and $V_{p-1}$.

In our proof below, we will also use basic properties of volumes of big divisors. We first recall the basic definition which can be found in Laz04a.

**Definition 5.2.** Let $L$ be a line bundle on an integral projective variety $X$ of dimension $n$ over $k$. The volume of $L$ is defined to be the non-negative real number
\[ \text{Vol}(L) = \limsup_{m \to \infty} \frac{h^0(X, L^\otimes m)}{m^n}, \]
where $h^0(X, L^\otimes m) = \dim H^0(X, L^\otimes m)$.

This notion extends to divisors (via the correspondence $D \mapsto \mathcal{O}_X(D)$), and is numerical, i.e. if $L = \mathcal{O}_X(D)$ for a divisor $D$, the volume depends only on the numerical equivalence class of $D$. Furthermore, the volume is a birational invariant, i.e. if $\nu : X' \to X$ is a birational map between two irreducible projective varieties, and $D$ is a divisor on $X$, then $\text{Vol}(D) = \text{Vol}(\nu^*D)$.

Note that the volume increases in effective directions. Moreover, by asymptotic Riemann-Roch, if $L$ is nef, the volume of $L$ is its top intersection number $L^n = L \cdot \cdots \cdot L$. Finally, the volume is non-zero if and only if the divisor is big.

We now show that, if $\Lambda^p \Omega_X$ is ample modulo $\Delta$, then $X$ is $p$-algebraically hyperbolic modulo $\Delta$.

**Proof of Theorem 1.3.** Let $X$ be a smooth projective connected variety over $k$, and let $\Delta$ be a proper closed subscheme of $X$ such that $\bigwedge^p \Omega_X^1$ is ample modulo $\Delta$ (see Definition 1.3). We are going to show that $X$ is $p$-algebraically hyperbolic modulo $\Delta$ (see Definition 1.2).
Let \( Y \) be a smooth projective connected \( p \)-dimensional variety with \( K_Y \) very ample. Let \( f : Y \rightarrow X \) be a rational map which is generically finite onto its image and such that \( f(Y) \not\subset \Delta \). Let \( \psi : \tilde{Y} \rightarrow Y \) be a proper birational surjective morphism with \( \tilde{Y} \) a smooth projective connected variety such that the composed map

\[
h : \tilde{Y} \rightarrow Y \rightarrow X
\]
is a morphism. Let \( A \) be a fixed ample line bundle on \( X \). To prove the theorem, we need to show that there exists a constant \( \alpha = \alpha(X, \Delta, A) \) such that, for every \( 0 \leq k \leq p \), the following inequality

\[
(f^*A)^k \cdot K_Y^{p-k} \leq \alpha K_Y^p
\]
holds. Since \( K_Y \) is ample, this inequality obviously holds for \( k = 0 \), as long as \( \alpha \geq 1 \).

Let \( \pi_X : \mathbb{P}(\Lambda^p T_X) \rightarrow X \) be the natural projection onto \( X \), and let \( \mathcal{O}_{\mathbb{P}(\Lambda^p T_X)}(1) \) be the dual of the tautological line bundle on \( \mathbb{P}(\Lambda^p T_X) \). As \( h \) is non-degenerate and \( \tilde{Y} \) is irreducible, the morphism \( h \) induces via its differential a rational map

\[
h : \tilde{Y} \simeq \mathbb{P}(\Lambda^p T_{\tilde{Y}}) \dashrightarrow \mathbb{P}(\Lambda^p T_X), \quad (y, [e_1 \wedge \ldots \wedge e_p]) \mapsto (h(y), [dh_y(e_1) \wedge \ldots \wedge dh_y(e_p)]).
\]

Observe that \( \tilde{h} \) is well defined outside a closed subset \( F \) of codimension at least two. Indeed, writing \( g \) is from the indeterminacy locus of \( \tilde{h} \) comes from the indeterminacy locus of a rational map

\[
g : V \subset \mathbb{C}^k \dashrightarrow \mathbb{P}^M,
\]
with \( M \in \mathbb{N}_{\geq 1} \) and, in this situation, it is of codimension \( \geq 2 \) by factoriality of \( \mathcal{O}_V \). Now, as \( F \) is of codimension at least two, it follows that

\[
\text{Pic}(\tilde{Y}) \simeq \text{Pic}(\tilde{Y} \setminus F).
\]

In particular, the pull-back by \( \tilde{h} \) of a line bundle on \( \mathbb{P}(\Lambda^p T_X) \) is well-defined.

As \( \Lambda^p \Omega_X^1 \) is ample modulo \( \Delta \), there exists an integer \( m > 0 \) such that

\[
\text{Bs}(\mathcal{O}_{\mathbb{P}(\Lambda^p T_X)}(m) \otimes \pi_X^{-1} A^{-1}) \subset \pi_{\mathbb{P}(\Lambda^p T_X)}^{-1}(\Delta).
\]

Note that \( m \) only depends on \( X \), \( A \), and \( \Delta \). Moreover, since \( h(\tilde{Y}) \not\subset \Delta \) (by assumption), the pull-back

\[
\tilde{h}^* (\mathcal{O}_{\mathbb{P}(\Lambda^p T_X)}(m) \otimes \pi_X^{-1} A^{-1}) = \tilde{h}^* \mathcal{O}_{\mathbb{P}(\Lambda^p T_X)}(m) \otimes h^* A^{-1}
\]
remains effective.

There is a natural non-trivial morphism of line bundles \( \omega_{\tilde{Y}} = \Lambda^p T_{\tilde{Y}} \rightarrow \tilde{h}^* \mathcal{O}_{\mathbb{P}(\Lambda^p T_X)}(-1) \) given by

\[
(y, v_1 \wedge \ldots \wedge v_p) \mapsto (y, dh_y(v_1) \wedge \ldots \wedge dh_y(v_p)).
\]

In particular, since

\[
\text{Hom}(\Lambda^p T_{\tilde{Y}}, \tilde{h}^* \mathcal{O}_{\mathbb{P}(\Lambda^p T_X)}(-1)) \simeq H^0(\tilde{Y}, \Lambda^p T_{\tilde{Y}} \otimes \tilde{h}^* \mathcal{O}_{\mathbb{P}(\Lambda^p T_X)}(-1)),
\]

it follows that the divisor \( K_{\tilde{Y}} \otimes \tilde{h}^* \mathcal{O}_{\mathbb{P}(\Lambda^p T_X)}(-1) \) is effective.

Define \( E := h^* A^{-1} \otimes K_{\tilde{Y}}^m \). Note that \( E \) is an effective divisor, as it can be written as the sum of two effective divisors. Write

\[
(5.1) \quad h^* A + E = mK_{\tilde{Y}}.
\]

Let \( R \subset \tilde{Y} \) be the exceptional divisor of the birational morphism \( \psi \), so that \( \text{Supp}(R) = \text{Exc}(\psi) \). Let \( Z = \psi(\text{Exc}(\psi)) \subset Y \) be the image of the exceptional locus under \( \psi \) and note that \( Z \) is of codimension \( \geq 2 \).

Take now \( r \in \mathbb{N}_{\geq 1} \) such that \( rK_Y \) is very ample. With this notation at hand, we apply Lemma 5.1.

By applying Lemma 5.1 to \( L = rK_Y \) and the effective divisor \( E \), we can find \( F_1, \ldots, F_{p-1} \in |rK_Y| \) satisfying all the conditions of the lemma.

Let \( 1 \leq k \leq (p - 1) \) and observe that \( H_k \) is not included in \( E \) (since \( \psi^{-1}(x_0) \notin E \) by construction), so that \( E_{|H_k} \) remains an effective divisor. As \( \psi \) is surjective and birational, using the projection formula, one gets that

\[
(f^*A)^k K_Y^{p-k} = \frac{1}{p^{-k}} (f^*A)^k (rK_Y)^{p-k} = \frac{1}{p^{-k}} (h^* A)^k (\psi^* (rK_Y))^{p-k}.
\]
Now, a key observation is that the intersection number \((h^* A)^k (\psi^* (rK_Y))^p^{-k}\) can be computed as follows
\[(h^* A)^k (\psi^* (rK_Y))^p^{-k} = (h^* A|_{H_{p-k}})^k.
\]
Indeed, note that it immediately follows from Lemma 5.1 that
\[\psi^* F_1 = H_1.\]
In particular, we deduce that
\[(h^* A)^k (\psi^* (rK_Y))^p^{-k} = (h^* A|_{H_1})^k (\psi^* (rK_Y))^p^{-k-1}.\]
Then, applying Lemma 5.1 again, we see that
\[(\psi^* F_2)|_{H_1} = H_2,\]
so that
\[(h^* A)^k (\psi^* (rK_Y))^p^{-k-1} = (h^* A|_{H_2})^k (\psi^* (rK_Y))^p^{-k-2}.\]
By repeating this, we end up obtaining the desired equality
\[(h^* A)^k (\psi^* (rK_Y))^p^{-k} = (h^* A|_{H_{p-k}})^k.\]
As \(A\) is an ample line bundle and \(h\) is a generically finite morphism, the pull-back \(h^* A\) is nef on \(\overline{Y}\), and so is its restriction \(h^* A|_{H_{p-k}}\). Therefore, by asymptotic Riemann-Roch, we obtain the equality
\[(h^* A|_{H_{p-k}})^k = \text{Vol}(h^* A|_{H_{p-k}}).\]
Since \(E_1|_{H_{p-k}}\) is effective and volume increases in effective direction, restricting the equation 5.1 to \(H_{p-k}\), we obtain the following inequality
\[(f^* A)^k K_{\overline{Y}}^{-1} = \frac{1}{p^{-k}} (h^* A|_{H_{p-k}})^k = \frac{1}{p^{-k}} \text{Vol} (h^* A|_{H_{p-k}})\]
\[\leq \frac{1}{p^{-k}} \text{Vol} ((mK_{\overline{Y}})|_{H_{p-k}}).\]
In order to finish the proof of the theorem, we now show that
\[\text{Vol} ((mK_{\overline{Y}})|_{H_{p-k}}) = \text{Vol} ((mK_{\overline{Y}})|_{V_{p-k}}).\]
To do so, by the very definition of the volume, it suffices to show that for all \(t \in \mathbb{N}_{\geq 1}\), we have the equality
\[H^0(H_{p-k}, \mathcal{O}_{H_{p-k}} (K_{\overline{Y}}^{\otimes t})) = H^0(V_{p-k}, \mathcal{O}_{V_{p-k}} (K_{\overline{Y}}^{\otimes t})).\]
Recall that we have the following relation between \(K_{\overline{Y}}\) and \(K_Y\)
\[K_{\overline{Y}} = \psi^* K_Y + R.\]
Since \(H_{p-k}\) is not included in \(\text{Exc}(\psi) = \text{Supp}(R)\), the divisor \(R|_{H_{p-k}}\) is effective. In particular, by taking a non-zero section \(s_0 \in H^0(X, \mathcal{O}_X(R))\), there is a natural injective map
\[i: H^0(V_{p-k}, \mathcal{O}_{V_{p-k}} (K_{\overline{Y}}^{\otimes t})) \rightarrow H^0(H_{p-k}, \mathcal{O}_{H_{p-k}} (K_{\overline{Y}}^{\otimes t}));
\]
It follows that
\[h^0(V_{p-k}, \mathcal{O}_{V_{p-k}} (K_{\overline{Y}}^{\otimes t})) \leq h^0(H_{p-k}, \mathcal{O}_{H_{p-k}} (K_{\overline{Y}}^{\otimes t})).\]
To obtain the converse inequality, we consider the following map
\[j: H^0(H_{p-k}, \mathcal{O}_{H_{p-k}} (K_{\overline{Y}}^{\otimes t})) \rightarrow H^0(V_{p-k}, \mathcal{O}_{V_{p-k}} \setminus Z, \mathcal{O}_{V_{p-k}} \setminus Z (K_{\overline{Y}}^{\otimes t}));
\]
this map is well defined as \(s_0\) is nowhere zero outside of \(\text{Exc}(\psi)\), and \(\psi^{-1}(V_{p-k} \setminus Z) \subset H_{p-k} \setminus \text{Exc}(\psi)\). Furthermore, it is obviously injective. Now, as \(Z \cap V_{p-k}\) is of codimension at least 2 in \(V_{p-k}\) by construction, the sections of \(H^0(V_{p-k} \setminus Z, \mathcal{O}_{V_{p-k}} \setminus Z (K_{\overline{Y}}^{\otimes t}))\) actually extend to sections of \(H^0(V_{p-k}, \mathcal{O}_{V_{p-k}} (K_{\overline{Y}}^{\otimes t})),\) so that
\[H^0(V_{p-k}, \mathcal{O}_{V_{p-k}} \setminus Z (K_{\overline{Y}}^{\otimes t})) \simeq H^0(V_{p-k}, \mathcal{O}_{V_{p-k}} (K_{\overline{Y}}^{\otimes t})).\]
This implies the desired inequality \(h^0(H_{p-k}, \mathcal{O}_{H_{p-k}} (K_{\overline{Y}}^{\otimes t})) \leq h^0(V_{p-k}, \mathcal{O}_{V_{p-k}} (K_{\overline{Y}}^{\otimes t})).\)
We have therefore proved that, for $1 \leq k \leq p$, the inequality
\[
(f^* A)^k K_Y^{p-k} \leq \frac{1}{r_{p-k}} \text{Vol}((mK_Y)|_{V_{p-k}})
\]
holds. However, as $mK_Y$ is ample (and thus nef), we have
\[
\text{Vol}((mK_Y)|_{V_{p-k}}) = ((mK_Y)|_{V_{p-k}})^k.
\]
Then, by the very construction of $V_{p-k}$, we obtain the equality
\[
\text{Vol}((mK_Y)|_{V_{p-k}}) = m^{k}r^{p-k}K_Y^n.
\]
We now conclude that, for every $1 \leq k \leq p$, the inequality
\[
(f^* A)^k K_Y^{p-k} \leq m^k \text{Vol}(K_Y)
\]
holds. This concludes the proof that $Y$ is $p$-algebraically hyperbolic modulo $\Delta$, as $m$ only depends on $X, \Delta$ and $A$. 

As an immediate corollary of Theorem 5.3 and Proposition 1.8 we obtain the following:

**Theorem 5.3.** Let $X$ be a smooth projective connected variety over $k$, let $\Delta$ be a proper closed subscheme of $X$, and let $1 \leq p \leq \dim X$. If $\mathcal{K}_X^p$ is ample modulo $\Delta$, then $X$ is $p$-algebraically bounded modulo $\Delta$.

Theorems 1.4 and 5.3 should be interpreted as results on “intermediate hyperbolicity”; the special case with $p = 1$ gives the following result for “hyperbolic varieties”.

**Corollary 5.4.** Let $X$ be a smooth projective connected variety over $k$. If $\mathcal{K}_X^1$ is ample modulo $\Delta$, then $X$ is algebraically hyperbolic modulo $\Delta$.

Note that, when $\Delta = \emptyset$, Corollary 5.4 is proven by Lazarsfeld [Laz04b, p.37]. If $\Delta = \emptyset$, it can also be deduced from work of Kobayashi and Demailly [Dem97]. Indeed, a smooth projective variety $X$ with $\mathcal{K}_X^1$ ample is Kobayashi hyperbolic [Kob98, Theorem 3.6.21], and thus algebraically hyperbolic [Dem97, Theorem 2.1].

The special case of Theorems 1.4 and 5.3 with $p = \dim(X)$ gives the following result for varieties of general type.

**Corollary 5.5.** If $X$ is a smooth projective variety of general type over $k$, then $X$ is $\dim(X)$-algebraically hyperbolic over $k$ and $\dim(X)$-algebraically bounded over $k$.

Corollary 5.5 does not appear explicitly in the literature. Also, note that in view of Theorem 3.5 Corollary 5.5 gives a new proof of the fact that if $X$ is a smooth projective variety of general type over $k$, then for every integral variety $Y$ over $k$, the set of surjective maps $\text{Sur}_k(Y, X)$ is finite.

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