$w_\infty$ Algebras, Conformal Mechanics, and Black Holes

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Abstract

We discuss BPS solitons in gauged $\mathcal{N} = 2$, $D = 4$ supergravity. The solitons represent extremal black holes interpolating between different vacua of anti-de Sitter spaces. The isometry superalgebras are determined and the motion of a superparticle in the extremal black hole background is studied and confronted with superconformal mechanics. We show that the Virasoro symmetry of conformal mechanics, which describes the dynamics of the superparticle near the horizon of the extremal black hole under consideration, extends to a symmetry under the $w_\infty$ algebra of area-preserving diffeomorphisms. We find that a Virasoro subalgebra of $w_\infty$ can be associated to the Virasoro algebra of the asymptotic symmetries of $AdS_2$. In this way spacetime diffeomorphisms of $AdS_2$ translate into diffeomorphisms in phase space: our system offers an explicit realization of the $AdS_2/CFT_1$ correspondence. Using the dimensionally reduced action, the central charge is computed. Finally, we also present generalizations of superconformal mechanics which are invariant under $\mathcal{N} = 1$ and $\mathcal{N} = 2$ superextensions of $w_\infty$.

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1 Introduction

Solitonic objects play an important role in string theory. In particular, the study of certain p-branes in supergravity theories, which interpolate between Minkowski space at infinity, and products of anti-de Sitter spaces and compact Einstein manifolds near the horizon, led to the $AdS/CFT$ correspondence [1, 2, 3]. In the present paper, we extend the discussion of interpolating solitons to the case of gauged supergravities. In particular we concentrate on the gauged $\mathcal{N} = 2$, $D = 4$ theory [4, 5]. A salient feature of the solitons in the gauged theory, which distinguishes them from corresponding objects in the ungauged case, is that their near-horizon limit involves not only products of $AdS$ spaces and positive curvature manifolds (like the $AdS_2 \times S^2$ Bertotti-Robinson solution, which arises as the near-horizon limit of the extremal Reissner-Nordström black hole), but also spacetimes like $AdS_2 \times H^2$ ($H^2$ denoting the two-dimensional hyperbolic space).

We study the motion of a superparticle in the near-horizon black hole background and analyze the associated conformal mechanics model. The isometries of the theory give rise to two copies of the Virasoro algebra, in the angular sector and in the radial-time sector of the superparticle action respectively. Focusing on the latter we show that the corresponding Hamiltonian can be written in the most general scale-invariant form, namely

$$H = \frac{p^2}{2f(u)},$$

(1.1)

where $q$, $p$ are the canonical conjugate variables and $f$ is an arbitrary function of $u = pq$. The Virasoro symmetry, which exists for any system with scale-invariant Hamiltonian of one dynamical variable [1], is only a subalgebra of a larger symmetry algebra, namely the algebra of certain volume-preserving diffeomorphisms. It is interesting to observe that the same symmetry algebra was found in string theories with two-dimensional target space [7, 8, 9, 10]. In particular we discuss a $w_\infty$ subalgebra of area-preserving diffeomorphisms [1]. This symmetry occurs due to the fact that one can find a canonical transformation which reduces the scale-invariant Hamiltonian of one dynamical variable to that of a free particle. Then the generators of $w_\infty$ act as symplectic diffeomorphisms preserving the two-form $\Omega = dp \wedge dq - dH \wedge dt$. We find that the algebra admits one central extension (since the first Betti number of the phase space is $b_1 = 1$). For this general class of models we show that in a natural way a Virasoro subalgebra of $w_\infty$ can be associated to the Virasoro algebra of the asymptotic symmetries of $AdS_2$ [15]. In this way the quantum-mechanical system explicitly realizes the asymptotic $AdS_2$ symmetries. The connection between gravity on $AdS_2$ and conformal field theory in 0+1 dimensions can be regarded as an example of the $AdS_2/CFT_1$ correspondence [1, 14, 15, 16, 17]. On the gravity (bulk) side, the Virasoro algebra is generated by spacetime diffeomorphisms preserving the asymptotic form of the metric. In the theory living on the boundary of $AdS_2$, these Virasoro generators translate into generators of diffeomorphisms in the particle phase space. For all black holes whose near-horizon metric contains an $AdS_2$ factor, a central charge appears in this Virasoro

\footnote{This algebra has been encountered previously in the theory of supermembranes [11], and has also been considered in the context of two-dimensional black holes and the matrix model [12, 13].}
Finally the above mentioned canonical transformation allows us to construct models of superconformal mechanics exhibiting the symmetries of $\mathcal{N} = 1$ and $\mathcal{N} = 2$ superextensions of $w_\infty$.

The remainder of this paper is organized as follows: In section 2 we present the model and the BPS interpolating soliton solutions. We analyze their near-horizon limit and their supersymmetry properties. Furthermore, the isometry superalgebras of the soliton and its near-horizon limit are derived. In section 3 the motion of a particle near the horizon of the solitonic black hole is studied and compared with conformal mechanics. The symmetries of conformal mechanics are studied in detail in section 4. In section 5 we show that the asymptotic symmetries of the bulk theory living on $AdS_2$ are in direct correspondence with the symmetries of the conformal theory living on the boundary. The supersymmetric extensions are presented in section 6. Finally our results are summarized and discussed.

2 Interpolating solitons in $\mathcal{N} = 2$, $D = 4$ gauged Supergravity

Let us first briefly review the gauged version of $\mathcal{N} = 2$, $D = 4$ supergravity [4, 5]. In this theory, the rigid $SO(2)$ symmetry rotating the two independent Majorana supersymmetries present in the ungauged theory, is made local by the introduction of a minimal gauge coupling between the graviphoton and the gravitinos. Local supersymmetry then requires a negative cosmological constant and a gravitino mass term. The theory has four bosonic and four fermionic degrees of freedom; it describes a graviton $V_\mu^a$, two Majorana gravitinos $\psi_\mu^I$ ($I = 1, 2$), which we combine into a single complex spinor $\psi_\mu = \psi_1^\mu + i\psi_2^\mu$, and a Maxwell gauge field $A_\mu$, minimally coupled to the gravitinos, with coupling constant $g$. The bosonic part of the Lagrangian is [4, 5]

$$ V^{-1}\mathcal{L} = -\frac{1}{4}R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{3}{2}g^2, $$

(2.1)

where $R$ is the scalar curvature, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the gauge field strength and the cosmological constant is $\Lambda = 3g^2$. We look for solutions of the field equations from (2.1), with a product metric

$$ ds^2 = f(r)dt^2 - f(r)^{-1}dr^2 - K^2d\Omega^2, $$

(2.2)

where $K$ is constant, and $d\Omega^2$ is a metric of curvature $k = 0, \pm 1$ on a two-dimensional manifold $\Sigma$. For the gauge field $A$ we make the ansatz

$$ A = \begin{cases} 
\frac{q_e}{K^2}r \, dt + q_m \cos \theta \, d\phi & k = 1 \\
\frac{q_e}{K^2}r \, dt + q_m \Theta \, d\phi & k = 0 \\
\frac{q_e}{K^2}r \, dt + q_m \cosh \theta \, d\phi & k = -1,
\end{cases} $$

(2.3)

$q_e$ and $q_m$ denoting the electric and magnetic charges respectively. One easily checks that (2.3) solves the gauge field equations of motion, $\partial_\mu (VF^{\mu\nu}) = 0$. The equations of motion
for the metric are
\[
\frac{d^2 f(r)}{dr^2} = 2(Q^2 + \Lambda),
\]
\[
K^2(Q^2 - \Lambda) = k,
\]
where we have defined
\[
Q^2 \equiv \frac{q_e^2 + q_m^2}{K^4}.
\]
From (2.4) we have
\[
f(r) = (Q^2 + \Lambda)r^2.
\]
In the following we are interested in bosonic backgrounds preserving some amount of supersymmetry, which means that the gravitino variation must vanish,
\[
\delta \psi_\mu = \hat{\nabla}_\mu \epsilon = 0.
\]
Here \(\epsilon\) is an infinitesimal Dirac spinor, and \(\hat{\nabla}_\mu\) is given by
\[
\hat{\nabla}_\mu = D_\mu + \frac{i}{2}g_\mu - \frac{1}{2}F_{\alpha\beta}\sigma^{\alpha\beta}\gamma_\mu,
\]
with the Lorentz and gauge covariant derivative
\[
D_\mu = \partial_\mu + \frac{1}{2}\omega_\mu^{\alpha\beta}\sigma_{\alpha\beta} + igA_\mu.
\]
We use standard conventions, \(\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}\), \(\sigma_{\alpha\beta} = 1/4[\gamma_\alpha, \gamma_\beta]\). The integrability conditions for (2.8) lead to the solution
\[
q_e = 0, \quad q_m = \pm gK^2.
\]
Inserting (2.11) into (2.5) yields for the curvature \(k\) of the two-manifold \(\Sigma\)
\[
k = -2g^2K^2 < 0,
\]
Thus \(\Sigma\) must be diffeomorphic to the hyperbolic space \(H^2\) or to a quotient thereof. Without loss of generality setting \(k = -1\) we obtain
\[
K^2 = \frac{1}{2g^2}, \quad q_m = \pm \frac{1}{2g}.
\]
Introducing the dimensionless coordinates \(\tau = gt, \rho = gr\), the metric becomes
\[
ds^2 = \frac{1}{2g^2}(8\rho^2 d\tau^2 - \frac{d\rho^2}{2\rho^2} - d\theta^2 - \sinh^2 \theta d\phi^2).
\]
The \((\tau, \rho)\)-part of the metric is just the line element of \(AdS_2\) in horospherical coordinates.
A short comment on the supersymmetry conditions on the charges is in order. Usually one has electromagnetic duality invariance, i.e., electric and magnetic charges enter into the BPS conditions in a symmetric way. In our case however, due to the minimal coupling of the graviphoton to the gravitino, this duality invariance is broken. The bosonic sector remains duality invariant upon gauging, but the Killing spinor equation does not, since the gauge potential \( A_\mu \) appears in (2.10). A similar nonsymmetric appearance of electric and magnetic charges in the BPS conditions was found for black holes in diverse gauged supergravity theories [18, 19, 20, 21].

Let us now turn back to the question of how many supersymmetries are preserved by our solution. In general, the integrability conditions are necessary, but not sufficient to guarantee the existence of Killing spinors. Solving explicitly the Killing spinor equations, we find the solution

\[
\epsilon(\tau, \rho) = \left[ \sqrt{\rho}(1 - \gamma_1 i) + (-4\tau \sqrt{\rho}\gamma_0 i + \frac{1}{\sqrt{\rho}})(1 + \gamma_1 i) \right] P_\pm \epsilon_0,
\]

where \( \epsilon_0 \) denotes a constant spinor and

\[
P_\pm \equiv \frac{1}{2}(1 \pm i\gamma_2\gamma_3)
\]

is a projection operator. The \( \pm \) sign corresponds to the sign of the magnetic charge \( q_m = \pm 1/(2g) \). We choose the + sign for definiteness. The appearance of the projector \( P_+ \) explicitly shows that half of the \( \mathcal{N} = 2 \) supersymmetries are broken; the dimension of the solution space is reduced from four to two (complex) dimensions.

In order to determine the residual symmetry superalgebra of the above supergravity configurations we make use of a technique described in [22, 23] (cf. also [25]). It is based on the fact that, up to purely bosonic factors, the isometry superalgebra is determined by the Killing spinors, just as the bosonic symmetry algebra is determined by the Killing vectors. To see this, one first observes that, given two Killing spinors \( \epsilon \) and \( \epsilon' \), the bilinear \( \bar{\epsilon}\gamma^\mu\epsilon'\partial_\mu \) is a Killing vector. In [22] it was shown that

\[
\{Q_F(\epsilon), Q_F(\epsilon')\} = Q_B(\bar{\epsilon}\gamma^\mu\epsilon'\partial_\mu)
\]

for the corresponding charges. This means that the determination of the linear combination \( \bar{\epsilon}\gamma^\mu\epsilon'\partial_\mu \) of Killing vectors is equivalent to the determination of the linear combination of bosonic charges appearing in the anticommutator of any pair of fermionic charges.

We define

\[
\eta_\pm = \frac{1}{2}(1 \pm i\gamma_1)P_+\epsilon_0.
\]

Here \( \frac{1}{2}(1 \pm i\gamma_1) \) is an additional projector that commutes with \( P_+ \). Using the \( \eta_\pm \), we can write for the Killing spinors in (2.13)

\[
\epsilon(\tau, \rho) = 2\sqrt{\rho}\eta_- + 2(-4\tau \sqrt{\rho}\gamma_0 i + \frac{1}{\sqrt{\rho}})\eta_+ = \epsilon_- + \epsilon_+.
\]
Now the Killing vectors $\xi_{++}, \xi_{+−}, \xi_{−−}$ can be expressed in terms of these Killing spinors as $\xi_{++} = \epsilon_{+} \gamma^\mu \epsilon_{+} \partial_\mu$, etc. In this way we obtain

$$\xi_{++} = 32 \bar{g}(\bar{\eta}_{+} \gamma_0 \eta_{+}) \ell_{−}, \quad \xi_{+−} = −8ig(\bar{\eta}_{+} \eta_{−})\ell_{0}, \quad \xi_{−−} = 2g(\bar{\eta}_{−} \gamma_0 \eta_{−})\ell_{+},$$

(2.20)

where

$$\ell_{−} = (\tau^2 + 1)\partial_r - 2\tau\rho \partial_\rho,$$

$$\ell_{+} = \partial_r,$$

$$\ell_{0} = -\tau \partial_r + \rho \partial_\rho,$$

(2.21)

are the Killing vectors of $AdS_2$, satisfying the so$(2, 1)$ algebra

$$[\ell_{+}, \ell_{−}] = -2\ell_{0},$$

$$[\ell_{0}, \ell_{±}] = \pm \ell_{±}.$$ (2.22)

We see that the anticommutator of two supercharges contains only the so$(2, 1)$ generators of $AdS_2$; the additional bosonic so$(2, 1)$ symmetries of hyperbolic space $H^2$ are not obtained by the above method. This means that the isometry superalgebra of this $AdS_2 \times H^2$ background is given by a direct sum of an appropriate superextension of so$(2, 1) \cong su(1, 1)$, and a bosonic part so$(2, 1) \cong su(1, 1)$. In our case, we thus obtain osp$(2|2) \oplus$ so$(2, 1) \cong su(1, 1|1) \oplus su(1, 1)$ for the residual superalgebra.

Finally we want to exhibit a BPS solitonic object which interpolates between the above $N = 1$ supersymmetric $AdS_2 \times H^2$ solution and the maximally supersymmetric $AdS_4$ spacetime. In fact such a soliton has been found in [18]: it represents an extremal black hole with metric

$$ds^2 = \left(rg - \frac{1}{2gr}\right)^2 dt^2 - \left(rg - \frac{1}{2gr}\right)^{-2} dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2),$$

(2.23)

and gauge field

$$A = q_m \cosh \theta d\phi, \quad q_m = \frac{1}{2g}.$$ (2.24)

The spacetime with metric (2.23) has an event horizon at $r = r_+ = 1/(g\sqrt{2})$. Introducing the new coordinates $\rho = g(r - r_+), \tau = gt$, one verifies that the near-horizon limit of (2.23) is indeed the metric in (2.14). On the other hand, for large $r$, (2.23) gives

$$ds^2 = (-1 + r^2 g^2)dt^2 - (-1 + r^2 g^2)^{-1} dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2),$$

(2.25)

which is simply $AdS_4$ seen by an accelerated observer [26]. The Killing spinors for the configuration (2.23), (2.24) have been determined in [18]. The Killing spinors for the configuration (2.23), (2.24) have been determined in [18].

$$\epsilon(r) = \left(rg - \frac{1}{2gr}\right)^{\frac{1}{2}}(1 - i\gamma_1)P_+ \epsilon_0.$$ (2.26)
Here $\epsilon_0$ is subject to a double projection, which reduces the complex dimension of the solution space from four to one. Near the horizon, where the black hole metric approaches the one in (2.14), we have a supersymmetry enhancement resulting in a doubling of the Killing spinors. Using the same technique as above, one finds for the black hole a residual superalgebra $s(2) \oplus su(1, 1)$, where $s(2)$ denotes the superalgebra introduced by Witten to formulate supersymmetric quantum mechanics [27].

3 Particle motion near the horizon

In this section we study the motion of a particle with mass $m$ and magnetic charge $q$ in the near-horizon regime of the extremal BPS black holes discussed above and find that it is governed by a model of conformal mechanics [28, 29].

We consider the $AdS_2 \times H^2$ solution of gauged $N^\prime = 2$, $D = 4$ supergravity: the metric is given in (2.14), and the gauge field is $A = q_m \cosh \theta d\phi$, with magnetic charge $q_m = 1/(2g)$. Defining new coordinates

\[
\begin{align*}
\zeta &= \frac{2 \tau}{g}, \\
\xi &= \frac{1}{2g \cosh \theta + \sinh \theta \cos \phi}, \\
x &= \frac{1}{g \sqrt{\rho}}, \\
z &= \frac{1}{g \sqrt{\cosh \theta + \sinh \theta \cos \phi}},
\end{align*}
\]  

we obtain

\[
ds^2 = \frac{d\zeta^2}{g^4 \zeta^4} - \frac{dx^2}{g^2 x^2} - 2 \left( \frac{d\xi^2}{g^4 \zeta^4} + \frac{dz^2}{g^2 z^2} \right),
\]

\[
\tilde{A} = \frac{1}{g^2 z^2} d\xi.
\]

We use a Hamiltonian formalism and define

\[
\mathcal{H} = g^{\mu\nu}(\Pi_\mu - qA_\mu)(\Pi_\nu - qA_\nu),
\]

where $\Pi_\mu$ denote generalized momenta and $g_{\mu\nu}$ is the metric. For our configuration this leads to

\[
\mathcal{H} = g^4 x^4 \Pi_\zeta^2 - g^2 x^2 \Pi_x^2 - \frac{g^2}{2} z^2 \Pi_z^2 - \frac{g^4}{2} z^4 \left( \Pi_\xi - \frac{q}{g^2 z^2} \right)^2.
\]

The Hamilton equations are

\[
\begin{align*}
\dot{\zeta} &= 2g^4 x^4 \Pi_\zeta, \\
\dot{x} &= -2g^2 x^2 \Pi_x, \\
\dot{z} &= -g^2 z^2 \Pi_z, \\
\dot{\xi} &= -g^4 z^4 \left( \Pi_\xi - \frac{q}{g^2 z^2} \right),
\end{align*}
\]

\[\text{For further aspects of the connection between black holes (0-branes) and conformal mechanics cf. [30, 31, 32, 33, 34, 35].}

\[\tilde{A}\text{ is equal to } A\text{ up to a gauge transformation, so in the following we will omit the tilde.}\]
\[ \dot{\Pi}_\zeta = 0, \]
\[ \dot{\Pi}_x = -4g^4x^3\Pi_\zeta^2 + 2g^2x^2, \]
\[ \dot{\Pi}_z = g^2\Pi_\zeta^2 + 2g^2z\Pi_\zeta(g^2z^2\Pi_\xi - q), \]
\[ \dot{\Pi}_\xi = 0, \]  

(3.6)

where the dot denotes the derivative with respect to an affine parameter \(\lambda\). Since the coordinates \(\zeta\) and \(\xi\) are cyclic, the associated conjugate momenta \(\Pi_\zeta\) and \(\Pi_\xi\) are conserved. Another constant of motion is given by

\[ \frac{g^2}{2}z^2\Pi_z^2 + \frac{g^4z^4}{2}\left(\Pi_\xi - \frac{q}{g^2z^2}\right)^2 = \frac{c^2}{2}. \]  

(3.7)

Moreover we have the on-shell relation \(H = m^2\) where \(m\) is the mass of the particle. Thus we can identify \(H = \Pi_\zeta\) with the Hamiltonian of the magnetic particle \((m, q)\),

\[ H = \frac{1}{g^2x^2}\sqrt{g^2x^2\Pi_\zeta^2 + \frac{g^2}{2}z^2\Pi_z^2 + \frac{g^4z^4}{2}\left(\Pi_\xi - \frac{q}{g^2z^2}\right)^2 + m^2}. \]  

(3.8)

In particular, if we impose (3.7), \(H\) becomes the Hamiltonian for a particle in one dimension, being \(x\) and \(\Pi_\zeta\) the conjugate variables. Defining

\[ u = x\Pi_\zeta, \quad p = \Pi_\zeta, \]  

(3.9)

and

\[ f(u) = \frac{g^2u^2}{2\sqrt{g^2u^2 + \frac{c^2}{2} + m^2}}, \]  

(3.10)

the reduced Hamiltonian becomes

\[ H = \frac{p^2}{2f(u)}. \]  

(3.11)

Thus we have obtained a conformal theory in \(0 + 1\) dimensions. The generators of the conformal algebra \(so(2, 1)\) are

\[ D = \frac{1}{2}u, \quad H = \frac{p^2}{2f(u)}, \quad K = \frac{1}{2}x^2f(u), \]  

(3.12)

where \(D\) is the generator of dilatations and \(K\) the generator of proper conformal transformations. They satisfy the Poisson bracket algebra

\[ [D, H]_{PB} = H, \quad [D, K]_{PB} = -K, \quad [H, K]_{PB} = 2D. \]  

(3.13)
A copy of this algebra is obtained by considering the isometries on the hyperbolic plane $H^2$. In fact from (3.7) we have

$$\Pi_\xi = \frac{\Pi^2}{2\psi(v)},$$

(3.14)

where

$$v = z\Pi_z, \quad \psi(v) = 2g^2v^2q + \frac{\sqrt{c^2 - g^2v^2}}{q^2 - c^2 + g^2v^2}.$$  

(3.15)

Correspondingly we introduce

$$\tilde{D} = \frac{1}{2}v, \quad \tilde{H} = \Pi_\xi = \frac{\Pi^2}{2\psi(v)}, \quad \tilde{K} = \frac{1}{2}z^2\psi(v),$$

(3.16)

which again satisfy the $so(2,1)$ algebra

$$[\tilde{D}, \tilde{H}]_{PB} = \tilde{H}, \quad [\tilde{D}, \tilde{K}]_{PB} = -\tilde{K}, \quad [\tilde{H}, \tilde{K}]_{PB} = 2\tilde{D}.$$  

(3.17)

The generators in (3.12) and the ones in (3.16) commute and give rise to the $so(2,1) \times so(2,1)$ algebra. In the same way as in [6], this symmetry can be extended to two copies of the Virasoro algebra, where the generators are given by

$$L_n = -\frac{i}{2}t^{1+n}p^{1-n}f^n,$$

$$\tilde{L}_n = -\frac{i}{2}z_z^{1+n}\Pi_z^{1-n}\psi^n.$$  

(3.18)

Now we go back to the Hamiltonian (3.11) and discuss its symmetries in detail; in particular we will see where the above Virasoro symmetry comes from.

4 Symmetries of conformal mechanics

The Hamiltonian in (3.11) describes the most general scale-invariant system of one dynamical variable $q$ and canonical conjugate momentum $p$, $f$ denoting an arbitrary function of $u = qp$.

For example, Hamiltonians of this type where shown to govern the dynamics of a particle in the near-horizon region of an extremal Reissner-Nordström black hole [28]. Also, the conformal mechanics model of De Alfaro, Fubini and Furlan [36], with Hamiltonian

$$H = \frac{p^2}{2} + \frac{g}{2q^2},$$

(4.1)

where $g$ is a dimensionless coupling constant, can be recovered from (3.11) by setting

$$f(u) = \frac{1}{1 + gu^{-2}}.$$  

(4.2)
In general one can show [3] that such theories exhibit a Virasoro symmetry with generators

\[ L_n = -\frac{i}{2} q^{1+n} p^{1-n} f^n, \]  

obeying

\[ [L_n, L_m]_{PB} = -i (m - n) L_{m+n}. \]  

In particular the \( \text{so}(2,1) \) subalgebra (the conformal algebra in 0+1 dimensions) is generated by

\[ iL_{-1} = H, \quad iL_0 = D, \quad iL_1 = K, \]  

with \( H, D \) and \( K \) as in (3.12). Let us now examine in more detail why this Virasoro symmetry arises, and how it generalizes to \( w_\infty \). We consider the action of a particle in 0+1 dimensions,

\[ S = \int dt (p \dot{q} - H(p, q)) = \int \alpha, \]  

\( \alpha \) being the one-form

\[ \alpha = p \, dq - H \, dt. \]  

A symmetry transformation for the action in (4.6) must leave invariant the two-form \( \Omega = d\alpha = dp \wedge dq - dH \wedge dt \) [9]. In order to determine these symmetries explicitly, first we perform the canonical transformation

\[ q' = q \sqrt{f}, \quad p' = \frac{p}{\sqrt{f}}. \]  

In terms of the new variables the Hamiltonian becomes

\[ \tilde{H} = H = \frac{p'^2}{2}, \]  

showing that classically the system is equivalent to a free particle. Setting

\[ y = p', \quad x = q' - p't, \]  

the two-form \( \Omega \) reduces to

\[ \Omega = dy \wedge dx. \]  

Therefore the symmetries of the system are the diffeomorphisms which preserve the symplectic two-form (1.11). They are generated by vector fields (cf. [8])

\[ \xi = \xi^a \partial_a = \Omega^{ab} (\partial_b \Lambda(y, x) + \omega_b) \partial_a + h(y, x, t) \partial_t \]

\[ = \frac{\partial \Lambda}{\partial y} \partial_x - \frac{\partial \Lambda}{\partial x} \partial_y + \omega_y \partial_x - \omega_x \partial_y + h \partial_t, \]  

where \( \Lambda(y, x) \) and \( h(y, x, t) \) are arbitrary functions, and \( \omega \in H^1(M, \mathbb{R}) \), \( M \) denoting the \( (x,y) \) phase space. This symmetry algebra is isomorphic to that of the matrix model with the standard inverted harmonic oscillator Hamiltonian [7, 8, 9]. In order to establish
a correspondence with the symmetries arising in string theories with two-dimensional target space \([7, 8, 9]\), one should instead consider only time-independent functions \(h\), i.e. \(h = h(y, x)\). In this case, the transformations generated by (4.12) preserve \(\Omega\) as well as the volume form

\[
\Theta = dy \wedge dx \wedge dt. \tag{4.13}
\]

For \(h = 0\), the vector fields (4.12) generate the algebra \(w_\infty\) of area-preserving (symplectic) diffeomorphisms \([37]\) on which we will concentrate in the following\(^4\). Explicitly one can choose as a basis of generators

\[
v^l_m = y^{l+1}x^{l+m+1} = \left(\frac{p}{\sqrt{f}}\right)^{l+1}(q\sqrt{f} - \frac{p}{\sqrt{f}}t)^{l+m+1}, \tag{4.14}
\]

where \(l = 0, 1, \ldots\) and \(m \in \mathbb{Z}\). They satisfy the \(w_\infty\) algebra

\[
[v^l_m, v^{l'}_{m'}]_{PB} = [m(l' + 1) - m'(l + 1)]v^{l+l'}_{m+m'}. \tag{4.15}
\]

The Virasoro algebra in (4.4) is the one generated by the \(v_{2n}^n\).

One easily checks that

\[
\frac{dv^l_m}{dt} = \frac{\partial v^l_m}{\partial t} + [v^l_m, H] = 0, \tag{4.16}
\]

so the \(v^l_m\) are constants of motion (conserved charges). The two fundamental integrals of motion are \(y = p/\sqrt{f}\) and \(x = q\sqrt{f} - pt/\sqrt{f}\).

Now, whenever

\[
f(u) \xrightarrow{u \to 0} \left(\alpha + \frac{\beta}{u^2}\right)^{-1} \tag{4.17}
\]

the spectrum of the Hamiltonian in (3.11) is continuous and bounded from below (\(E > 0\)), the ground state at \(E = 0\) being non-normalizable.\(^5\) A way to cure this infrared problem is to study the time evolution of the system by means of a compact operator \((4.18)\) (which has a discrete set of normalizable eigenfunctions)

\[
R = \frac{1}{2}\left(aH + \frac{1}{a}K\right), \tag{4.18}
\]

where \(a\) denotes an infrared cutoff. To be more specific we choose

\[
K = q^2f/2. \tag{4.19}
\]

Though scale and translational invariance are broken by this procedure, the new model admits a symmetry algebra isomorphic to \(so(2, 1)\) \([37]\). In fact the new system is again

\(^4\)Note that, in order for \(w_\infty\) to admit central extensions, the first cohomology group \(H^1(M)\) (\(M\) denoting \((x, y)\) phase space) must be non-trivial. There exist \(b_1\) independent central extensions, where \(b_1\) is the first Betti number of \(M\) \([37]\).

\(^5\)We observe that the functions \(f(u)\) in (3.10) and in (4.2) as well as the one of the model in \([28]\) all satisfy the asymptotic behaviour in (4.17).
invariant under the algebra $w_\infty$. Let us consider $R$, with $H$ given by (3.11) and $K$ as in (4.19). Now we have

$$\Omega = dp \wedge dq - dR \wedge dt, \quad (4.20)$$

which, by means of the (canonical) transformation

$$\bar{p} = \frac{1}{\sqrt{2}} e^{-\frac{a}{2} (\frac{ap}{\sqrt{f}} + iq \sqrt{f})},$$
$$\bar{q} = \frac{i}{a \sqrt{2}} e^{\frac{a}{2} (\frac{ap}{\sqrt{f}} - iq \sqrt{f})} \quad (4.21)$$

can be recast into the form

$$\Omega = d\bar{p} \wedge d\bar{q}. \quad (4.22)$$

Similarly to the previous case, we take as basis functions

$$v_m^l = \bar{p}^{l+1} \bar{q}^{m+1}, \quad (4.23)$$

which satisfy (4.15). As above, the $v_m^l$ are conserved, the fundamental integrals of motion being $\bar{p}$ and $\bar{q}$. A Virasoro subalgebra is spanned by

$$L_m = \frac{1}{4a} e^{int} (\frac{ap}{\sqrt{f}} + iq \sqrt{f})^{1-m} (\frac{ap}{\sqrt{f}} - iq \sqrt{f})^{1+m}. \quad (4.24)$$

In contrast to the generators in (4.3) we have now $L_{-m} = L^\dagger$ and $L_0 = R$, which means that $L_0$ plays the role of the Hamiltonian.

Let us finally discuss the question of central charges. Since the Virasoro symmetry of this model stems from the larger symmetry algebra of area-preserving diffeomorphisms, central extensions might be possible. As already observed, in order to have central extensions, the first Betti number $b_1$ of the manifold on which these diffeomorphisms act, must be nonzero. Now we show that this is indeed the case. To this end we go back to the Hamiltonian $\tilde{R}$ in (4.18) with $H$ and $K$ given in (3.11) and (4.19) respectively. First of all we note that the $w_\infty$ generators (4.23) are singular at $\bar{q} = 0$ for $m$ sufficiently negative. In order to define smooth diffeomorphisms we perform a canonical transformation

$$\varphi = 2 \arctan \frac{qf}{ap}, \quad p_\varphi = \frac{ap^2}{4f} + \frac{q^2 f}{4a}. \quad (4.25)$$

This yields the transformed Hamiltonian

$$\tilde{R} = R = p_\varphi, \quad (4.26)$$

so classically the system is equivalent to a relativistic particle on the circle. (Note that $0 \leq \varphi \leq 2\pi$, in order to have $q \geq 0$). The two-form $\Omega$ becomes

$$\Omega = dp_\varphi \wedge d\varphi', \quad (4.27)$$

where $\varphi' = \varphi - t$. Thus, for constant $t$, the diffeomorphisms preserving $\Omega$ are the area-preserving diffeomorphisms $SDiff(S^1 \times \mathbb{R})$ on the cylinder. This manifold has $b_1 = 1$, therefore $SDiff(S^1 \times \mathbb{R})$ admits exactly one central extension [37].
5 Conformal mechanics and bulk asymptotic symmetries

In this section we show that the Virasoro generators (4.24) of the infrared-regularized model with Hamiltonian \( R \) given by (4.18) can be associated in a natural way to the generators of the asymptotic symmetries of \( AdS_2 \) [15].

Let us first consider the \( AdS_2 \) metric with coordinates as in (3.2)

\[
 ds^2 = \frac{1}{g^4 q^4} dt^2 - \frac{1}{g^2 q^2} dq^2. \tag{5.1}
\]

Then the asymptotic symmetries of \( AdS_2 \) determined in [15] become

\[
 l_m = e^{imt} \left\{ \left[ 1 - \frac{m^2 g^2 q^4}{8} + o(q^8) \right] \partial_t + \left[ \frac{im}{2} q + o(q^2) \right] \partial_q \right\}. \tag{5.2}
\]

They generate the diffeomorphisms which preserve the asymptotic form of the \( AdS_2 \) metric. In [15], the generators (5.2) have been determined by imposing certain boundary conditions on the metric. These boundary conditions must be weak enough in order to allow for a larger symmetry algebra than \( so(2,1) \), but strong enough to ensure the possibility to define the associated conserved charges [15]. It is easy to show that the bulk generators (5.2) satisfy the Virasoro algebra

\[
 [l_m, l_k] = -i(m-k)l_{m+k}. \tag{5.3}
\]

They can be rewritten as

\[
 l_m = e^{imt} \left\{ -(m^2 - 1)l_0 + \frac{m}{2} (m+1)e^{-it} l_1 + \frac{m}{2} (m-1)e^{it} l_{-1} \right\}. \tag{5.4}
\]

In order to establish the correspondence with the symmetries of our conformal mechanics model we associate the generators \( l_0, l_{\pm 1} \) with the phase-space functions given in (4.24)

\[
 l_0 \rightarrow L_0 = R, \quad l_{\pm 1} \rightarrow L_{\pm 1} = e^{\pm it} \left\{ R - \frac{q^2 f}{2a} \mp \frac{i}{2} qp \right\}. \tag{5.5}
\]

In this way, using (5.3) in (5.4), we find

\[
 l_m \rightarrow \tilde{L}_m = e^{imt} \left\{ \frac{a p^2}{4f} + \frac{q^2 f}{4a} - \frac{m^2}{2a} q^2 f - \frac{im}{2} qp \right\}. \tag{5.6}
\]

At this point it is trivial to check that, when expanded in a Laurent series in \( q \) near the boundary \( q = 0 \), the \( \tilde{L}_m \) in (5.6) obtained through the above correspondence and the \( L_m \)

\[\text{Note that the boundary is at } q = 0.\]
of the conformal mechanics given in (4.24) do agree up to the order $q^6$ inclusive. The IR-regularized model realizes explicitly the asymptotic symmetries of $AdS_2$.

The central extension of the algebra is easily obtained through a comparison with the work in ref. [15]. There the asymptotic symmetries of $AdS_2$ were realized canonically in the Hamiltonian formulation of the Jackiw-Teitelboim (JT) model. In this way, the authors of [15] found a central charge, which is expressed in terms of the dilaton field evaluated at the black hole horizon. In our case, in order to identify the dilaton we proceed as follows: we consider static solutions of the form

$$ds^2 = h_{ij}(r,t)dx^idx^j - \eta^2(r,t)\sigma_{IJ}dx^I dx^J,$$

$$A = A_t(r)dt + A_\phi(\theta)d\phi,$$  

(5.7)

where $i, j = t, r$; $I, J = \theta, \phi$, and $\sigma_{IJ}$ denotes the metric on a two-dimensional space of constant curvature $k$. From Gauss theorem we have

$$F_{tr} = q_e \sqrt{-h} \frac{1}{\eta^2}, \quad F_{\theta\phi} = q_m \sqrt{\sigma},$$

(5.8)

where

$$\sqrt{\sigma} = \sqrt{\det \sigma_{IJ}} = \begin{cases} \sin \theta & k = 1, \\ \theta & k = 0, \\ \sinh \theta & k = -1. \end{cases}$$

(5.9)

The ansatz for the metric and the electromagnetic field strength can be used in the Lagrangian (2.1) of $\mathcal{N} = 2$ gauged supergravity. The corresponding equations of motion give rise to an effective two-dimensional dilatonic theory of gravity, being $h_{ij}$ the metric and $\eta$ the dilaton field. The action becomes

$$S = \Omega \int dx dt \sqrt{-h} \left[ -\frac{1}{2} (\nabla \eta)^2 - \frac{1}{4} \eta^2 R + V(\eta) \right],$$

(5.10)

where the dilaton potential is given by

$$V(\eta) = k \frac{q_e^2 + q_m^2}{2} + \frac{3g^2 \eta^2}{2},$$

(5.11)

and $\Omega$ denotes the volume of the reduction space.7

We consider now generic black holes with metric

$$ds^2 = W(r)dt^2 - W(r)^{-1} dr^2 - r^2 d\Omega^2,$$

(5.12)

where

$$W(r) = k - \frac{2m}{r} + g^2 r^2 + \frac{q_e^2 + q_m^2}{r^2}. $$

(5.13)

---

7In the case of noncompact spaces, one should consider a quotient thereof, i.e. a Riemann surface of genus $n \geq 1$. 
Near the horizon \( r = r_+ \), where \( W(r) = 0 \), the metric contains an \( AdS_2 \) factor if the black hole is extremal. This leads to the requirement

\[
W(r_+) = 2V(r_+),
\]

where \( V(r) \) is the dilaton potential \((5.11)\). This means that at the horizon, the dilaton potential \((5.11)\) vanishes. Near the horizon, we expand

\[
\eta^2(r) = (r_+ + \Phi)^2 \to r_+^2 + 2r_+\Phi, \quad \Phi \to 0,
\]

\( \Phi = 0 \) denoting the horizon position. We further have

\[
(\nabla \eta)^2 \to \left( 1 - \frac{2\Phi}{r_+} \right) (\nabla \Phi)^2,
\]

\[
V(\eta) \to V(\Phi) \equiv V'(r_+\Phi).
\]

Putting the dilaton on-shell \((\Phi = r - r_+)\), neglecting a Gauss-Bonnet term, and defining

\[
\bar{\Phi} = \frac{1}{2} r_+ \Phi,
\]

we arrive at the JT model with action

\[
S = \Omega \int dx \, dt \sqrt{-h} \bar{\Phi}(-R + 2\lambda^2),
\]

where

\[
\lambda^2 = \frac{1}{2} W''(r_+) = \frac{V'(r_+)}{r_+}.
\]

At this point we can use the results of ref. \[15\] to obtain that for all the various superconformal particle models the central charge is given by

\[
c = 24\Omega \sqrt{\frac{r_+^3}{V'(r_+)}}.
\]

In particular, for the interpolating soliton \((2.23), (2.24)\), one gets

\[
c = \frac{12\Omega}{g^2\sqrt{2}}.
\]

We emphasize that the results presented so far are quite general: they are valid for any conformal quantum mechanics describing the motion of a particle in a black hole configuration that in the near-horizon region is given by \( AdS_2 \times \Sigma_2 \) where \( \Sigma_2 \) is a two-dimensional manifold with constant curvature.
6 Superextensions

We now ask whether one can find generalizations of conformal mechanics which are invariant under superextensions of the $w_\infty$ algebra. Superconformal mechanics has first been studied in [38, 39], where the $su(1,1|1) \cong osp(2|2)$ generalization of the DFF model was constructed. The authors of [28] studied the motion of a superparticle in the $AdS_2 \times S^2$ background of an extreme RN black hole in the near-horizon limit. There, an $osp(1|2)$ superextension of the DFF model was considered, which, however, was only possible for $g = 0$ in (4.1), if one wants the standard supersymmetry to be linearly realized (cf. also [24]). This is not surprising, as for $g \neq 0$ there is no classical solution of zero energy, so there can be no ground state annihilated by the supercharge $Q$ [28]. Below we shall construct models of $\mathcal{N} = 1$ or $\mathcal{N} = 2$ superconformal mechanics in which both the standard supersymmetry and the conformal supersymmetry are nonlinearly realized. These models will turn out to admit not only the $osp(1|2)$ or $osp(2|2)$ symmetry algebras, but the entire $\mathcal{N} = 1$ or $\mathcal{N} = 2$ superextensions of $w_\infty$.

Let us first consider the $\mathcal{N} = 1$ case. We start from the bosonic Hamiltonian (3.11), which, after the canonical transformation (4.8) reduces to (4.9), i.e. to the Hamiltonian of a free particle. For a free particle, however, an $\mathcal{N} = 1$ superextension can easily be found, the action is given by

$$S = \int dt \left( \frac{1}{2} \dot{q}^2 + \frac{i}{2} \dot{\psi} \psi \right),$$

(6.1)

where $\psi$ is an anticommuting world line field. Performing now the inverse of the canonical transformation (4.8), the action (6.1) goes over in

$$S = \int dt \left( p \dot{q} - \frac{p^2}{2f} + \frac{i}{2} \dot{\psi} \psi \right)$$

(6.2)

plus a surface term, which we can drop. We are thus led to propose (6.2) as $\mathcal{N} = 1$ superparticle action, which generalizes the DFF model for $g \neq 0$. In the following we will show that the symmetry algebra admitted by (6.2) is indeed the $\mathcal{N} = 1$ superextension of $w_\infty$. Similar to the bosonic case, (6.2) can be written as

$$S = \int \alpha,$$

(6.3)

where

$$\alpha = p dq + \frac{i}{2} \psi \psi dt - H dt,$$

(6.4)

so the two-form $\Omega = d\alpha$ considered in section 3 reads

$$\Omega = dp \wedge dq - dH \wedge dt + \frac{i}{2} d\psi \wedge d\psi.$$

(6.5)

Using (4.8) and (4.10), one finds for the super two-form

$$\Omega = dy \wedge dx + \frac{i}{2} d\psi \wedge d\psi.$$  

(6.6)
It is now easy to identify the vector fields which preserve $\Omega$; they are of the form

$$\xi = \Omega^{AB} \partial_B \Lambda \partial_A + h \partial_t = \frac{\partial \Lambda}{\partial y} \partial_x - \frac{\partial \Lambda}{\partial x} \partial_y - i \frac{\partial \Lambda}{\partial \psi} \partial_\psi + h \partial_t, \quad (6.7)$$

where $\Lambda = \Lambda(y, x, \psi)$ and $h = h(y, x, \psi, t)$ are arbitrary superfunctions. As a basis on the supermanifold parametrized by the coordinates $y, x, \psi$, we can take

$$v^l_m = y^{l-1} x^{l+m-1} = \left( \frac{p}{\sqrt{f}} \right)^{l-1} (q \sqrt{f} - \frac{p}{\sqrt{f}} t)^{l+m-1},$$
$$G^l_r = y^{l-1} x^{l+r-1} \psi = \left( \frac{p}{\sqrt{f}} \right)^{l-1} (q \sqrt{f} - \frac{p}{\sqrt{f}} t)^{l+r-1} \psi. \quad (6.8)$$

Under the super Poisson bracket \[41, 42\]

$$[f, g] = \frac{\partial f}{\partial q} \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial g}{\partial q} + 2 \frac{\partial f}{\partial \psi} \frac{\partial g}{\partial \psi}, \quad (6.9)$$

the basis functions (6.8) generate the $\mathcal{N} = 1$ superextension of $w_\infty$ \[40\], which reads

$$[v^k_m, v^l_n] = [m(l-1) - n(k-1)] v^{k+l-2}_{m+n},$$
$$[G^k_r, G^l_s] = 2 v^{k+l-1}_{r+s},$$
$$[v^k_m, G^l_r] = [m(l-1) - r(k-1)] G^{k+l-2}_{m+r}. \quad (6.10)$$

This is an algebra of symplectic super-diffeomorphisms \[41\]. Note that the $v^l_m$ and the $G^l_r$ are conserved (super)charges. An $\mathcal{N} = 1$ super-Virasoro subalgebra is generated by

$$L_n = -i 2^{-n} v^{2-n} = -i 2 \left( \frac{p}{\sqrt{f}} \right)^{1-n} (q \sqrt{f} - \frac{p}{\sqrt{f}} t)^{1+n},$$
$$G_r = -i \sqrt{2} G^{3/2-r}_{2r} = -i \sqrt{2} \left( \frac{p}{\sqrt{f}} \right)^{1/2-r} (q \sqrt{f} - \frac{p}{\sqrt{f}} t)^{1/2+r} \psi, \quad (6.11)$$

where $n \in \mathbb{Z}$ and $r \in \mathbb{Z} + \frac{1}{2}$ in the Neveu-Schwarz sector and $r \in \mathbb{Z}$ in the Ramond sector. The $L_n$ and $G_r$ satisfy

$$[L_m, L_n] = -i(m-n) L_{m+n},$$
$$[G_r, G_s] = -2 i L_{r+s},$$
$$[L_m, G_r] = -i(\frac{m}{2} - r) G_{m+r}. \quad (6.12)$$

This super-Virasoro algebra generalizes the bosonic part found in \[6\]. It contains an $osp(1|2)$ subalgebra, whose generators read

$$H = i L_{-1}, \quad D = i L_0, \quad K = i L_1, \quad Q = i G_{-1/2}, \quad S = i G_{1/2}. \quad (6.14)$$

\[8\]We omitted possible harmonic one-forms $\omega$. 

16
We now show that the supersymmetries are nonlinearly realized. The infinitesimal variation of a superfunction $F(y, x, ψ, t)$ under a symplectic superdiffeomorphism generated by $ξ$ in (6.7), is given by
\[ δξF = LξF = ξ^A ∂A F, \] (6.15)
$Lξ$ denoting the Lie derivative along $ξ$. Taking $Λ = ϵQ$ and $h = 0$ in (6.7), where $ϵ$ is constant and anticommuting, and $Q$ was defined in (6.14), one gets
\[ δϵQq = ϵψ \sqrt{2} f(1 - pqf'(u)) \frac{2f}{2f}, \] (6.16)
which, for the DFF model (4.1), reduces to $δϵQq = \sqrt{2} ϵψ$, and
\[ δϵQψ = -i p ϵ. \] (6.17)
Similar relations hold for $Λ = ϵS$, i.e. for the variations under conformal supersymmetry transformations. (6.16) and (6.17) show that the supersymmetries are indeed nonlinearly realized.

One can also generalize the model considered above to $N = 2$. In this case, the action reads ($α, β = +, -$)
\[ S = \int dt (pq - \frac{p^2}{2f} + \frac{i}{2} ψαδαβ ψβ). \] (6.18)
For the super two-form $Ω$ we get
\[ Ω = dp ∧ dq - dH ∧ dt + i δαβdψα ∧ dψβ, \]
\[ = dy ∧ dx + i δαβdψα ∧ dψβ \] (6.19)
by means of (4.8) and (4.10). It is preserved by the vector fields
\[ ξ = \frac{∂Λ}{∂y} ∂x - \frac{∂Λ}{∂x} ∂y - i \frac{∂Λ}{∂ψα} ∂ψα + h∂t, \] (6.20)
with the superfunctions $Λ = Λ(y, x, ψα)$ and $h = h(y, x, ψα, t)$. In super phase space we take the basis
\[ v_l^i = y_l^{-i}x_l^{i+1} = \left(\frac{p}{\sqrt{f}}\right)^{l-1} (q\sqrt{f} - \frac{p}{\sqrt{f}} t)^{l+m-1}, \]
\[ G^l_r = y_l^{-1}x_l^{l+r-1}ψ^± = \left(\frac{p}{\sqrt{f}}\right)^{l-1} (q\sqrt{f} - \frac{p}{\sqrt{f}} t)^{l+r-1}ψ^±, \] (6.21)
\[ J_l^m = \frac{1}{2} y_l^{-2}x_l^{l+m-2}ψ^+ψ^- = \frac{1}{2} \left(\frac{p}{\sqrt{f}}\right)^l (q\sqrt{f} - \frac{p}{\sqrt{f}} t)^{l+m}ψ^+ψ^- . \]
We define the graded Poisson bracket \[42\]
\[
[f, g] = \frac{\partial f}{\partial q} \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial g}{\partial q} - 2(-1)^\text{deg} f \left( \frac{\partial f}{\partial \psi^+} \frac{\partial g}{\partial \psi^-} + \frac{\partial f}{\partial \psi^-} \frac{\partial g}{\partial \psi^+} \right),
\]
(6.22)
where \(\text{deg} f\) is the grading of \(f\). Using (6.22), our basis functions generate the \(\mathcal{N} = 2\) superextension of \(w_{\infty} [12]\),

\[
[l, v^k] = [m(l - 1) - n(k - 1)]v^{k+l-2},
\]
\[
[l, G^s] = 2[l^{k+l-1} - 2r(l - 1) - s(k - 1)]J^{k+l-1},
\]
\[
[l, J_l] = [m(l - 2) - n(k - 1)]J^{k+l-2},
\]
\[
[l, G^s] = \pm G^{k+l-2}.
\]

It contains an \(\mathcal{N} = 2\) super-Virasoro subalgebra with generators

\[
L_n = -\frac{i}{2} l_{2^n} = -\frac{i}{2} \left( \frac{p}{\sqrt{f}} \right)^{1-n} (q\sqrt{f} - \frac{p}{\sqrt{f}} t)^{1+n},
\]
\[
G^\pm_r = -\frac{i}{\sqrt{2}} G^r_{2^r} = -\frac{i}{\sqrt{2}} \left( \frac{p}{\sqrt{f}} \right)^{1/2-r} (q\sqrt{f} - \frac{p}{\sqrt{f}} t)^{1/2+r} \psi^\pm,
\]
\[
J_n = -i J_{2^n} = -\frac{i}{2} \left( \frac{p}{\sqrt{f}} \right)^{-n} (q\sqrt{f} - \frac{p}{\sqrt{f}} t)^{n} \psi^+ \psi^-,
\]
(6.24)

satisfying

\[
[L_m, L_n] = -i(m - n)L_{m+n},
\]
\[
[G^r, G^s] = -2iL_{r+s} - i(s - r)J_{r+s},
\]
\[
[L_m, G^\pm] = -i(m - r)G^\pm_{m+r},
\]
\[
[L_m, J_n] = i n J_{m+n},
\]
\[
[J_m, G^\pm] = \pm i G^\pm_{m+r}.
\]
(6.25)

Like above, we have \(n \in \mathbb{Z}\) and \(r \in \mathbb{Z} + \frac{1}{2}\) in the Neveu-Schwarz sector and \(r \in \mathbb{Z}\) in the Ramond sector. The \(osp(2|2) \cong su(1, 1|1)\) subalgebra is spanned by

\[
H = i L_{-1}, \quad D = i L_0, \quad K = i L_1, \quad Q^\pm = i G^\pm_{-1/2}, \quad S^\pm = i G^\pm_{1/2}, \quad i J_0.
\]
(6.26)

### 7 Conclusions

The initial aim of our work was the study of the (super)particle dynamics in a BPS black hole background solution of \(\mathcal{N} = 2\) gauged supergravity in \(D = 4\) dimensions. We have found that such a system is described by a quantum mechanical model whose symmetry
group is generated by a couple of conformal algebras, one in the angular sector and one in the time-radial sector. The angular conformal symmetry is new, and occurs due to the hyperbolic event horizon geometry. Restricting ourselves to a fixed value of the angular so(2, 1) Casimir, we showed that the particle Hamiltonian reduces to a universal scale invariant form, and that its symmetries extend to the algebra $w_\infty$ of area-preserving diffeomorphisms, acting in phase space. Naively, the conformal Virasoro subalgebra of $w_\infty$ has vanishing central charge \[1\]. On the other hand, we showed that a one-to-one correspondence can be established between this Virasoro algebra and the asymptotic symmetries of $AdS_2$. In addition in ref. \[15\] it had been proven that these asymptotic symmetries exhibit a nonvanishing central charge. The crucial observation to resolve this puzzle was the fact that the Virasoro algebra of the conformal mechanics model appears as a subalgebra of $w_\infty$ acting on a manifold with $b_1 = 1$, i.e. with exactly one possible central extension. The central charge was then computed by dimensional reduction of the bulk action, yielding a Jackiw-Teitelboim model, and comparison with the work in ref. \[15\]. Thus our system offers an explicit realization of the $AdS_2/CFT_1$ correspondence.

**Acknowledgements**

The part of this work due to D. K. has been partially supported by a research grant within the Common Special Academic Program III of the Federal Republic of Germany and its Federal States, mediated by the DAAD. The authors would like to thank G. Berrino, M. Cadoni, M. M. Caldarelli, V. Moretti and L. Vanzo for helpful discussions.
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20
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