NONCOMMUTATIVE DIFFERENTIAL FORMS AND QUANTIZATION OF THE ODD SYMPLECTIC CATEGORY

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Abstract

There is a simple and natural quantization of differential forms on odd Poisson supermanifolds, given by the relation \([f, dg] = \{f, g\}\) for any two functions \(f\) and \(g\). We notice that this non-commutative differential algebra has a geometrical realization as a convolution algebra of the symplectic groupoid integrating the Poisson manifold.

This quantization is just a part of a quantization of the odd symplectic category (where objects are odd symplectic supermanifolds and morphisms are Lagrangian relations) in terms of \(\mathbb{Z}_2\)-graded chain complexes. It is a straightforward consequence of the theory of BV operator acting on semidensities, due to H. Khudaverdian.

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1 Introduction

There is a well-known analogy (formulated by A. Weinstein [3]) between symplectic manifolds and vector spaces:

| symplectic manifold | vector space |
|---------------------|-------------|
| Lagrangian submanifold | vector |
| product             | tensor product |
| opposite sympl. form | dual space |
| Lagrangian relation  | linear map  |

It is very fruitful, though it is just an analogy, i.e. there is no functor from the symplectic category to the category of vector spaces.

However, there is such a quantization functor, if we substitute symplectic manifolds with odd symplectic supermanifolds, and vector spaces with \(\mathbb{Z}_2\)-graded chain complexes, i.e. \(\mathbb{Z}_2\)-graded vector spaces \(V = V_0 \oplus V_1\) with a differential \(D: V_0 \to V_1, V_1 \to V_0, D^2 = 0\). Namely, to an odd symplectic supermanifold \(Y\) it associates the vector space of semidensities on \(Y\), with \(D\) the BV operator (due to H. Khudaverdian). Before giving the details, let us look at the result of quantization of odd symplectic groupoids.

2 Differential forms on odd Poisson supermanifolds

Let \(X\) be a supermanifold and \(\Omega(X)\) the differential graded algebra of differential forms on \(X\). By a filtered deformation of \(\Omega(X)\) we mean a differential filtered algebra \(A\) (i.e. a differential algebra \(A\) with an increasing filtration \(F^0 \subset F^1 \subset \ldots \subset A\) such that \(dF^i \subset F^{i+1}\)) with an
isomorphism between $\Omega(X)$ and the differential graded algebra $GrA$ associated to $A$ (where $(GrA)^i = F^i/F^{i-1}$). We have the following simple theorem:

**Theorem 1.** There is a natural bijection between (isomorphism classes of) filtered deformations of $\Omega(X)$ and odd Poisson structures on $X$. It is given by the commutation relation

$$[f, dg] = \{f, g\},$$

where $f$ and $g$ are any functions on $X$, $[,]$ is the supercommutator in the deformed algebra and $\{,\}$ is the odd Poisson structure.

**Proof.** Notice that $F^0 = C^\infty(X)$. Since $\Omega(X)$ is generated (as an algebra) by functions on $X$ and by their differentials, the same is true for $A$ (as is easily proved by induction). Finally, for any $f, g \in F^0$ we have $[f, dg] \in F^0$ (because $\Omega(X)$ is graded commutative, i.e. there $[f, dg] = 0$). The algebra $A$ is thus known (up to canonical isomorphism) if we know the function $\{f, g\} := [f, dg]$ for every two functions $f$ and $g$. It is straightforward to verify that $\{f, g\}$ has to be an odd Poisson structure. The converse (i.e. given an odd Poisson structure, the formula (1) gives a filtered deformation of $\Omega(X)$) can also be easily verified directly, but we give a more conceptual proof of this fact, using quantization of odd symplectic groupoids, later.

If $\pi$ is an odd Poisson structure on $X$, we denote the corresponding filtered deformation of $\Omega(X)$ by $\Omega_\pi(X)$. Since our construction is natural, for any Poisson map $X_1 \to X_2$ we have a pullback map $\Omega_{\pi_2}(X_2) \to \Omega_{\pi_1}(X_1)$ preserving all the structure; hence, e.g., if $X$ is a Poisson Lie group, $\Omega_{\pi}(X)$ is a (differential filtered) Hopf algebra.

Let us give some examples of odd Poisson manifolds and of their algebras $\Omega_{\pi}(X)$. If $X = \Pi T^*M$ with its canonical odd symplectic form, $\Omega_{\pi}(X)$ is the algebra of differential operators acting on differential forms on $M$. The operator corresponding to a function on $X$, i.e. to a multivector field $m$ on $M$, is $i_m$. The differential is the supercommutator with the de Rham $d$.

As another example, if $g$ is a Lie algebra, take $X = \Pi g^*$ with its Kirillov-Kostant Poisson structure. Geometrically, $\Omega_{\pi}(X)$ is the convolution algebra of deRham currents on the group $G$, supported at $1 \in G$. More algebraically, it is the crossed product of $A g$ with $\bigwedge g$; the differential maps identically $g \subset \bigwedge g$ to $g \subset A g$.

These two examples were special cases, of $X = \Pi A^*$, where $A \to M$ is a Lie algebroid. Lie theory is a source of other interesting odd Poisson structures on graded supermanifolds. Recall these important observations of A. Vaintrob [2]: A Lie algebroid structure on $A$ is equivalent to a degree $-1$ odd Poisson structure $\pi$ on $A^*[1]$ (this is the one we just mentioned) and also to a degree $1$ odd vector field $Q$ on $A[1]$ with $Q^2 = 0$. A bialgebroid structure on $A$ is both $\pi$ and $Q$ on $A^*[1]$, such that $L_Q\pi = 0$. They can be combined to a single Poisson structure $\tilde{\pi} = \pi + Q\partial_t$ on $A^*[1] \times \mathbb{R}[2]$ (here $t$ is the coordinate on $\mathbb{R}[2]$; recall that an odd Poisson structure on $X$ is an odd quadratic function $\pi$ on $T^*X$ such that $\{\pi, \pi\} = 0$; $Q\partial_t$ is understood as a product of two linear functions on $T^*(\Pi A^* \times \mathbb{R})$).

A *quasi-bialgebroid* structure on $A$ is a graded principal $\mathbb{R}[2]$-bundle $X \to A^*[1]$ with a $\mathbb{R}[2]$-invariant odd Poisson structure $\tilde{\pi}$ of degree $-1$. Choosing a trivialization $X = A^*[1] \times \mathbb{R}[2]$ we get a decomposition $\tilde{\pi} = \pi + Q\partial_t + \phi$ or $\tilde{\pi}$, where $\pi$ is a degree $-1$ odd Poisson structure on $A^*[1]$, $Q$ a degree $1$ odd vector field on $A^*[1]$ and $\phi$ a degree $3$ odd function on $A^*[1]$ ($\pi$, $Q$ and $\phi$ satisfy the obvious equations coming from $\{\pi, \pi\} = 0$). It is not clear to me what role the differential algebra $\Omega_{\pi}(X)$ may play e.g. in the problem of quantization of quasi-bialgebroids.
3 Khudaverdian’s BV operator

In this section we recall several theorems of H. Khudaverdian [1]. Let $Y$ be an odd symplectic manifold and let $x^i, \xi_i$ be local Darboux coordinates (i.e. $\omega = dx^i \wedge d\xi_i$). Let

$$\Delta = \frac{\partial^2}{\partial x^i \partial \xi_i},$$

understood as a differential operator from semidensities to semidensities. Then

**Theorem 2.**
1. $\Delta$ is odd and $\Delta^2 = 0$
2. $\Delta$ is formally selfadjoint
3. $\mathcal{L}_{X_f} = [\Delta, f]$, where $X_f$ is the Hamiltonian vector field generated by $f$ and $[\cdot, \cdot]$ the supercommutator
4. $\Delta$ is independent of the choice of Darboux coordinates

**Proof.** This is just a sketch: 1. and 2. are evident and 3. can be directly computed. From 1. and 3. we get that $[\mathcal{L}_{X_f}, \Delta] = 0$, i.e. $\Delta$ is invariant under Hamiltonian diffeomorphisms, hence we get 4.

Notice that Hamiltonian diffeomorphisms act trivially of the cohomology of $\Delta$, since by 3. they are homotopies. If $Y = \Pi T^*M$ then one can identify differential forms on $M$ with semidensities on $Y$ using Fourier transform along the fibres of $\Pi T M$; then $\Delta$ becomes $d$ and 3. contains as a special case Cartan formula.

If $\alpha$ and $\beta$ are semidensities on $Y$, we set

$$(\alpha, \beta) = \int_Y \alpha \beta$$

(provided the integral is well defined). Since $\Delta$ is formally selfadjoint, if $\alpha$ is $\Delta$-closed and $\beta$ is $\Delta$-exact then $(\alpha, \beta) = 0$, i.e. $(\cdot, \cdot)$ is well defined on cohomology classes (provided the appropriate care is taken for the finiteness of the integral).

Let $L \subset Y$ be a Lagrangian submanifold. We define a $\delta$-like generalized semidensity $\delta_L$ supported on $L$: in local Darboux coordinates $x^i, \xi_i$, with $L$ given by the equations $\xi_i = 0$, we set $\delta_L = \prod_i \delta(\xi_i)$.

**Theorem 3.** $\delta_L$ is $\Delta$-closed and independent of the choice of coordinates

**Proof.** Closedness is obvious and independence can be computed.

Since Hamiltonian flows generate homotopies on semidensities, we know that $\delta_L$ and $\delta_L'$ lie in the same cohomology class whenever $L$ and $L'$ can be connected by a Hamiltonian diffeomorphism. Hence, if $\alpha$ is closed (and appropriate integrals are finite) then $(\delta_L, \alpha) = (\delta_L', \alpha)$; this is the geometrical basis for BV quantization.
4 Quantization functor

**Definition 1.** The objects of the quantum odd symplectic category (QOSC) are odd symplectic manifolds and Hom($Y_1, Y_2$) is the $\mathbb{Z}_2$-graded complex of generalized semidensities on $\bar{Y}_1 \times Y_2$ ($\bar{Y}_1$ denotes $Y_1$ with the opposite symplectic form); composition of morphisms is given by integration.

Notice that the composition is not always defined, hence we don’t really have a category. This is the same problem as with the symplectic category; we’ll keep it in mind and ignore it. Composition is $\Delta$-equivariant since $\Delta$ is formally selfadjoint. The quantization of an odd symplectic manifold $Y$ can be defined as $\text{Hom}(pt, Y)$, i.e. as the space of generalized semidensities on $Y$.

**Definition 2.** The quantization functor from the odd symplectic category to QOSC is the identity on objects and it maps any Lagrangian relation $L$ to $\delta L$.

Of course, we have to prove that composition of $L$’s corresponds to composition of $\delta L$’s. This is quite simple. If two relations $L_1$ and $L_2$ can be composed (i.e. they satisfy the transversality condition) and their composition is $L$, the composition of $\delta L_1$ with $\delta L_2$ is a $\delta$-like distribution supported at $L$, i.e. a multiple of $\delta_L$. Since both $\delta L_1$ and $\delta L_2$ are closed, so must be their composition, i.e. it is a constant multiple of $\delta_L$. Finally, we have to verify that the constant is 1. It is enough to do it at a single point. For example, we can locally deform $L_1$ and $L_2$ so that they are both given by setting some of chosen Darboux coordinates to 0. For such $L$’s the constant is clearly 1, therefore it is 1 also in the undeformed parts of $L$’s, and so it is 1 for the original $L$’s as well.

5 Differential forms on odd Poisson supermanifolds II

Now the following dream is fulfilled: whenever we have an algebraic structure in the odd symplectic world, we get a corresponding structure in the world of $\mathbb{Z}_2$-graded chain complexes. As the most obvious example, let us start with an odd Poisson manifold $X$ and let $Y$ be its (local) odd symplectic groupoid; the quantization of $Y$ will be a differential algebra.

The quantization of $Y$ is the space of generalized semidensities on $Y$. We choose a reasonable subspace so that the product in the algebra (given by composition with $\delta_L \in \text{Hom}(Y \times Y, Y)$, where $L$ is the graph of the product in the groupoid $Y$) is well defined. A cheap choice is to take smooth multiples of $\delta_X$ and of its derivatives (here $X \subset Y$ is the Lagrangian submanifold of units of the groupoid $Y$). In this way we get a differential algebra with an increasing filtration (given by the degree of the distributions). As one easily sees, it is the algebra $\Omega_\pi(X)$.

References

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