ON THE EXPANSION OF CERTAIN VECTOR-VALUED CHARACTERS
OF $U_q(gl_n)$ WITH RESPECT TO THE GELFAND-TSETLIN BASIS

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Abstract. Macdonald polynomials are an important class of symmetric functions, with
connections to many different fields. Etingof and Kirillov showed an intimate connec-
tion between these functions and representation theory: they proved that Macdonald
polynomials arise as (suitably normalized) vector-valued characters of irreducible repre-
sentations of quantum groups. In this paper, we provide a branching rule for these
characters. The coefficients are expressed in terms of skew Macdonald polynomials with
plethystic substitutions. We use our branching rule to give an expansion of the charac-
ters with respect to the Gelfand-Tsetlin basis. Finally, we study in detail the
$q=0$ case,
where the coefficients factor nicely, and have an interpretation in terms of certain $p$-adic
counts.

1. Introduction

Macdonald polynomials were originally discovered in the 1980s [10, 9], and have found
a variety of uses in mathematics, appearing in a number of disparate fields (mathematical
physics, combinatorics, representation theory and number theory, among others). These
polynomials have the key property of being invariant under all permutations of their $n$
variables. They are indexed by partitions $\lambda$ with length at most $n$, and form an orthogonal basis
for the ring of symmetric polynomials with coefficients in $\mathbb{C}(q,t)$ with respect to a certain
density function. The existence of such polynomials was proved by exhibiting particular
difference operators which have these polynomials as their eigenfunctions. Macdonald poly-
nomials contain many important families as particular degenerations of the parameters $q$
and $t$. In particular, the ubiquitous Schur functions are obtained by setting $q=t$; crucially,
these are characters of irreducible representations of $GL_n$. Hall-Littlewood polynomials are
recovered in the limit $q=0$, and these have interpretations as zonal spherical functions on
$p$-adic groups. Some other important subfamilies are the monomial, elementary, and power
sum symmetric functions.

Given the various connections to representation theory, one might ask whether Macdonald
polynomials arise as characters of certain irreducible representations. Etingof and Kirillov
discovered such a realization in [2], where they demonstrate that Macdonald polynomi-
als are ratios of vector-valued characters of representations of the quantum group $U_q(gl_n)$.
Recall that the finite-dimensional, irreducible representations $V_\lambda$ of $U_q(gl_n)$ are indexed by
$\lambda \in P_+^{(n)} = \{(\lambda_1, \ldots, \lambda_n) : \lambda_i - \lambda_{i+1} \in \mathbb{Z}_+\}$. Note that elements of $P_+^{(n)}$
can be written as $(a,a,\ldots,a)+\bar{\lambda}$, where $a \in \mathbb{C}$ and $\bar{\lambda}$ is a standard partition of length $n$. Let $k \in \mathbb{N}$ be fixed,
then they show the existence of an intertwining operator (unique up to scaling):

$$\phi^{(k)}_{\lambda} : V_{\lambda+(k-1)\rho} \to V_{\lambda+(k-1)\rho} \otimes U,$$

where $U \simeq V_{(k-1),\ldots,(n-1,-1,-1,\ldots,-1)}$ and $\rho = (\frac{n-1}{2}, \frac{n-3}{2}, \ldots, \frac{1-n}{2})$. Note that $U$ has the special
property that all weight subspaces are one-dimensional. Fix the normalization of $\phi^{(k)}_{\lambda}$
so that $v_{\lambda+(k-1)\rho} \to v_{\lambda+(k-1)\rho} \otimes u_0 + \cdots$, where $v_{\lambda+(k-1)\rho}$ is a fixed non-zero highest weight
vector for $V_{\lambda+(k-1)\rho}$ and $u_0$ is a fixed non-zero vector in the (one-dimensional) weight zero
subspace of $U$. Consider the corresponding trace function of this operator:

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Inspired by Kashiwara’s theory of crystal bases

**Definition 1.2.** Let $V_{\lambda}$ be a crystal basis for $G$ of shape $\lambda \in P^{(n)}$. Let $\Phi(\lambda)$ be the polynomials whose coefficient in the expansion of $\Phi(\mu)$ is well-known rational functions appearing in symmetric function theory, specialized to skew Macdonald polynomials, and $d$ denotes the interlacing relation:

$$\lambda_1^{(i)} \cdot \cdots \cdot \lambda_n^{(i)} \in \mathbb{Z}_+.$$ 

We visualize $\lambda$ as an array consisting of $n$ rows with the parts of $\lambda$ in the first row, parts of $\lambda$ in the second row, etc. There is a canonical basis of $V_{\lambda}$ which is indexed by GT$(\lambda)$, the Gelfand-Tsetlin patterns of shape $\lambda$.

Our aim is to compute the expansion of the trace function $\Phi^k(\mu)$ in the Gelfand-Tsetlin basis for $V_{\lambda+(k-1)\rho_n}$. We would like to index these patterns in a uniform way with respect to the parameter $k \in \mathbb{N}$. Conveniently, there is a canonical way of doing this for the Gelfand-Tsetlin patterns whose coefficient in the expansion of $\Phi(\mu)$ is non-zero:

**Definition 1.2.** Let $\lambda \in P^{(n)}$, and let $\lambda = \mu(0) \supset \mu(1) \supset \cdots \supset \mu(n-1)$ be such that:

1. $\mu^{(i)} \in P^{(n-i)}_+.$
2. $\mu_j^{(i)} - \mu_j^{(i+1)} \in \mathbb{Z}_+, 1 \leq j \leq n-i-1.$
3. $\mu_j^{(i)} - \mu_{j-1}^{(i+1)} \leq k-1, 2 \leq j \leq n-i.$

Define $\overline{\mu} = (\overline{\mu}^{(i)}) = \mu(i) + (k-1)\rho_{n-i} + (k-1)\cdot(i/2, \ldots, i/2)$, then $(\overline{\mu}^{(i)} \geq \overline{\mu}^{(i+1)} \geq \cdots \geq \overline{\mu}^{(n-1)})$ is a Gelfand-Tsetlin pattern of shape $\lambda + (k-1)\rho_n$.

We can now give our formula for the coefficients in the expansion of $\Phi^k(\lambda)$ in the Gelfand-Tsetlin basis, along with a new branching formula for the functions $\Phi(x; q, t)$. This will be expressed in terms of functions $\psi_q/\delta$, which are plethystic substitutions of skew Macdonald polynomials, and $d_n$, which is the norm with respect to a particular inner product, see the Background section for more details.
Theorem 1.3. For $\mu \subset \lambda$ with $\lambda \in \mathcal{P}_+^{(n)}$, $\mu \in \mathcal{P}_+^{(n-1)}$, define

$$c_{\lambda,\mu}(q,t) = d_{\mu}(q^2, t^2) \sum_{\beta \in \mathcal{P}_+^{(n-1)}} \psi_{\mu/\beta}(q^2, t^2) \Omega_{\beta/\mu}(q^2, t^2).$$

Then the trace functions $\bar{\Phi}(x; q, t)$ satisfy the branching rule:

$$\bar{\Phi}_{\lambda}(x; q, t) = \sum_{\mu \in \mathcal{P}_+^{(n-1)}} c_{\lambda,\mu}(q,t) \cdot \bar{\Phi}_{\mu}^{(n-1)}(x; q, t) \cdot x_{\alpha(\lambda,\mu)},$$

where $\rho(\lambda, \mu) = \sum_{i=1}^{n-1} \lambda_i - \mu_i$.

Moreover, with respect to the Gelfand-Tsetlin basis of $V_{\lambda+(k-1)\rho}$, the diagonal coefficient of the intertwining operator $\phi^{(k)}$ corresponding to the Gelfand-Tsetlin pattern $\Lambda = (\lambda^{(0)} \geq \cdots \geq \lambda^{(n-1)})$ in $GL(\lambda + (k-1)\rho)$ is equal to

$$c_{\lambda}(q, q^k) = \begin{cases} \prod_{1 \leq i < n-1} c_{\pi^{(i-1)}, \pi^{(i)}}(q, q^k), & \exists (\mu^{(0)} \geq \cdots \geq \mu^{(n-1)}) s.t. \lambda^{(i)} = \pi^{(i)} \\ 0, & \text{otherwise} \end{cases}$$

Kashiwara’s crystal bases [4] allow one to interpret finite-dimensional representations of $U_q(\mathfrak{sl}_n)$ in the “crystal limit” $q \to 0$. Remarkably, there is a rich combinatorial structure in the crystal limit. Since the Gelfand-Tsetlin basis yields a crystal basis for $U_q(\mathfrak{sl}_n)$, it seems natural to consider the limit as $q \to 0$ of the coefficients in Theorem 1.3. A priori this limit need not even exist, but in fact we are able to obtain a simple closed formula, in a factorized form:

Theorem 1.4. Let $\mu \subset \lambda$ with $\lambda \in \mathcal{P}_+^{(n)}$, $\mu \in \mathcal{P}_+^{(n-1)}$, then we have

$$\lim_{q \to 0} c_{\lambda,\mu}(q,t) = \frac{b_\lambda(t^2)}{b_\mu(t^2)} (1 - t^2) s_{\lambda/\mu}(t^2) = t^2 \sum_{j \in \mathbb{N}} (\lambda^{(j)} - \mu^{(j)}) \prod_{j \geq 1} \left( \frac{\lambda_j - \mu_{j+1}}{\lambda_j - \lambda_{j+1}} \right) t^2.$$

Here the coefficients $s_{\lambda/\mu}(t)$ are those studied in [12, 7] in the context of Pieri rules. Note that when $t = p^{-1}$ for an odd prime $p$,

$$s_{\lambda/\mu}(t) = t^{n(\lambda) - n(\mu)} a_{\lambda}(\mu; p),$$

where $a_{\lambda}(\mu; p)$ is the number of subgroups of type $\mu$ in a finite abelian $p$-group of type $\lambda$.

For $S = (\mu^{(0)} \supset \mu^{(1)} \supset \cdots \supset \mu^{(n-1)})$ with $\mu^{(i)} \in \mathcal{P}_+^{(n-i)}$, we define the coefficient

$$s_{S}(t) = s_{\mu^{(0)}/\mu^{(1)}}(t) s_{\mu^{(1)}/\mu^{(2)}}(t) \cdots s_{\mu^{(n-2)}/\mu^{(n-1)}}(t).$$

Note that when $t = p^{-1}$, $s_{S}(t)$ (is up to a power of $t$) the number of nested chains of subgroups with types specified by the sequence $S$. We also let

$$wt(S) = (\rho(\mu^{(n-1)}, 0), \rho(\mu^{(n-2)}, \mu^{(n-1)}), \ldots, \rho(\mu^{(1)}, \mu^{(2)}), \rho(\mu^{(0)}, \mu^{(1)})).$$

Theorem 1.5. Let $\lambda \in \mathcal{P}_+^{(n)}$. Then

$$\lim_{q \to 0} \bar{\Phi}_{\lambda}(x; q, t) = \frac{(1 - t^2)^n}{b_\lambda(t^2)} \sum_{S = (\lambda^{(0)} \supset \mu^{(1)} \supset \cdots \supset \mu^{(n-1)})} s_{S}(t^2) x^{wt(S)}.$$

Note that the coefficients for the Gelfand-Tsetlin basis of $V_{\lambda+(k-1)\rho}$ are obtained by specializing $t = q^k$ in (2), and hence we are only able to obtain the $t = 0$ specialization of (3) in the crystal limit. As mentioned above, there is a representation theoretic realization of $\Phi(x; q, t)$, with $t$ algebraically independent from $q$, as the trace function of an intertwiner between infinite-dimensional modules over $C(t) \otimes U_q(\mathfrak{gl}_n)$. There is an analogue of the Gelfand-Tsetlin basis for these modules, and we can obtain (3) for general $t$ as the $q \to 0$
limit of the coefficients in the expansion of \( \Phi(x; q, t) \) with respect to this basis (see Section 5 within the paper for more details about this). Unfortunately these modules do not fit into Kashiwara’s framework, and so we have not been able to find a direct interpretation of (3) in terms of crystal bases. However, the simple combinatorial structure of our formula in the limit \( q \to 0 \) does seem to suggest a possible connection, and we leave it as an open question to describe this connection more precisely.

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2. **Background on symmetric function theory**

Recall that \( \lambda = (\lambda_1, \ldots, \lambda_n) \in (\mathbb{Z}_+)^n \) is a partition if \( \lambda_i \geq \lambda_{i+1} \). There is a partial order on partitions defined by \( \lambda > \mu \) if and only if \( \sum \lambda_i = \sum \mu_i \) and for some \( k < n \) we have \( \lambda_i = \mu_i \) for all \( i \leq k \) and \( \lambda_{k+1} > \mu_{k+1} \). We will work with polynomials of \( n \) variables, i.e., over \( \mathbb{C}[x_1, \ldots, x_n] \). For \( \lambda \in \mathbb{Z}^n \), we let \( x^\lambda = x_1^{\lambda_1} \cdots x_n^{\lambda_n} \).

We fix \( k \in \mathbb{N} \), and set \( t = q^k \). Let \( \rho = (n-1 \over 2, n-3 \over 2, \ldots, 1- n \over 2) \) be half the sum of the positive roots; we will also write \( \rho_n \) when it is not clear from context. Note that

\[
\rho_n - \rho_{n-1} = \left( \frac{n-1}{2}, \frac{n-3}{2}, \ldots, \frac{1-n}{2} \right) - \left( \frac{n-2}{2}, \frac{n-4}{2}, \ldots, \frac{2-n}{2} \right) = \left( 1 - \frac{1}{2}, \ldots, 1 - \frac{n}{2} \right).
\]

We now define a number of different coefficients arising from symmetric function theory, see [10]; we also review some relevant results from the literature.

**Definition 2.1.** Define the functions \( g(\gamma; q^2, t^2) \) for a partition \( \gamma \) by

\[
g(\gamma; q^2, t^2) = t^{2|\gamma|} \frac{(q^2; q^2)_{\gamma_1} \cdots (q^2; q^2)_{\gamma_n}}{(q^2; q^2)^{\gamma_1} \cdots (q^2; q^2)^{\gamma_{n-1}}}.
\]

**Definition 2.2.** Let \( c^\delta_{\gamma \mu}(q, t) \) be the coefficients in the following Pieri rule:

\[
m^{(n)}_{\gamma}(x)P^{(n)}_{\mu}(x; q, t) = \sum_{\delta} c^\delta_{\gamma \mu}(q, t)P^{(n)}_{\delta}(x; q, t).
\]

We note that the coefficients \( c^\delta_{\gamma \mu} \) can be determined via the change of basis coefficients \( \{m^{(n)}_{\gamma}(x)\} \rightarrow \{P^{(n)}_{\mu}(x; q, t)\} \) in conjunction with the Pieri coefficients that express the product \( \{P^{(n)}_{\gamma}(x; q, t)\} \rightarrow \{P^{(n)}_{\mu}(x; q, t)\} \) in the Macdonald polynomial basis.

**Definition 2.3.** Let \( \Omega_{\beta/\mu}(q^2, q^{2k}) \) be the coefficient on \( P^{(n-1)}_{\beta}(x; q^2, q^{2k}) \) in the expansion of

\[
P^{(n-1)}_{\mu}(x; q^2, q^{2k}) \prod_{i=1}^{n-1} \frac{(q^2 x_i; q^2)^\infty}{(q^{2k} x_i; q^2)^\infty}
\]

in the basis \( \{P^{(n-1)}_{\beta}(x; q^2, q^{2k})\} \).

We now recall the branching rule for Macdonald polynomials.

**Theorem 2.4.**

\[
P^{(n)}_{\lambda}(x; q, t) = \sum_{\mu \geq \lambda} x^{(\lambda-\mu)}_{\lambda/\mu} \psi_{\lambda/\mu}(q, t)P^{(n-1)}_{\mu}(x; q, t)
\]

**Proof.** See (1.7) of [8] for example.

**Remark.** There is a product formula for the coefficients \( \psi_{\lambda/\mu}(q, t) \) appearing above ([10] p. 342)

\[
\psi_{\lambda/\mu}(q, t) = \prod_{1 \leq i \leq j \leq \ell(\mu)} \frac{f(q^{\mu_i-\mu_j+1}t^{i-j})f(q^{\lambda_i-\lambda_j+1}t^{i-j})}{f(q^{\lambda_i-\mu_j}t^{i-j})f(q^{\mu_i-\lambda_j+1}t^{i-j})},
\]
where \( f(a) = (at)_\infty / (aq)_\infty \) with \( (a)_\infty = \prod_{i \geq 0} (1 - aq^i) \).

**Proposition 2.5.** We have

\[
\lim_{q \to 0} \psi_{\lambda/\mu}(q, t) = \prod_{\{j: \lambda'_j = \mu'_j \text{ and } \lambda'_{j+1} = \mu'_{j+1} \}} (1 - t^{m_j(\mu)})
\]

if \( \lambda/\mu \) is a horizontal strip, and zero otherwise.

**Proof.** This follows from the branching rule for Hall-Littlewood polynomials (see for example [10] p228 (5.5'), (5.14')).

**□**

**Definition 2.6.** Let

\[
\phi_{\lambda/\mu}(t) = \prod_{\{j: \lambda'_j = \mu'_j + 1 \text{ and } \lambda'_{j+1} = \mu'_{j+1} \}} (1 - t^{m_j(\lambda)}),
\]

if \( \lambda/\mu \) is a horizontal strip, and zero otherwise.

Note that these coefficients are the \( q \to 0 \) limiting case of \( \phi_{\lambda/\mu}(q, t) \) which also arise as branching coefficients.

**Remark.** The functions \( \phi_{\lambda/\beta}(q, t), \Omega_{\beta/\mu}(q, t) \) have interpretations in terms of skew Macdonald polynomials (in parameters \( q, t \)) with plethystic substitutions. In particular, we have \( \phi_{\lambda/\beta}(q, t) = Q_{\lambda/\beta}(1) \) and \( \Omega_{\beta/\mu}(q, t) = Q_{\beta/\mu}(\frac{q}{1-t}) = t^{\beta/\mu} Q_{\beta/\mu}(\frac{q-t}{1+t}) \) (see for example [12]), and both these quantities have nice factorized forms.

We will write \( \psi_{\lambda/\mu}(t), g(y; t^2) \), etc. to denote the limit \( q \to 0 \) of these functions.

**Definition 2.7.** [5, 7] For any skew shape \( \lambda/\mu \), define the coefficients

\[
sk_{\lambda/\mu}(t) = t^{\sum_j (\lambda'_j - \mu'_j)} \prod_{j \geq 1} \left( \frac{\lambda'_j - \mu'_{j+1}}{m_j(\mu)} \right).
\]

**Theorem 2.8.** [5, 7] For a partition \( \lambda \) and \( r \geq 0 \), we have

\[
P^{(n)}(x; t) s^{(n)}(x) = \sum_{\lambda \vdash n} \sk_{\lambda/\lambda}(t) P^{(n)}_{\lambda}(x; t),
\]

with the sum over partitions \( \lambda \subset \lambda^+ \) for which \( |\lambda + /\lambda| = r \).

We now recall two inner products that will appear throughout the paper. We let \( \langle \cdot, \cdot \rangle_n \) denote the Macdonald inner product (defined via integration over the \( n \)-torus). In particular,

\[
\langle P^{(n)}_{\lambda}(x; q, t), P^{(n)}_{\mu}(x; q, t) \rangle_n = \int_{T_n} P^{(n)}_{\lambda}(x; q, t) P^{(n)}_{\mu}(x^{-1}; q, t) \Delta_{S}(x; q, t) dT
\]

\[
= \delta_{\lambda, \mu} \frac{1}{d_S(q, t)},
\]

where an explicit formula for \( d_S(q, t) \) can be found in [10]. Also let \( Q^{(n)}_{\mu}(x; q, t) = b_{\mu}(q, t) P^{(n)}_{\mu}(x; q, t) \) be scalar multiples of the Macdonald polynomials, and recall the other inner product \( \langle \cdot, \cdot \rangle' \) which satisfies

\[
\langle P^{(n)}_{\lambda}(x; q, t), Q^{(n)}_{\mu}(x; q, t) \rangle' = \delta_{\lambda, \mu}
\]

(so that \( \langle P^{(n)}_{\lambda}(x; q, t), P^{(n)}_{\lambda}(x; q, t) \rangle' = \frac{1}{b_{\lambda}(q, t)} \)). Note that this inner product is independent of \( n \), and we have \( \lim_{n \to \infty} \langle \cdot, \cdot \rangle_n = \langle \cdot, \cdot \rangle' \). We have

\[
b_{\lambda}(t) = \prod_{i \geq 1} \phi_{m_i(\lambda)}(t),
\]

where \( m_i(\lambda) \) denotes the number of times \( i \) occurs as a part of \( \lambda \) and

\[
\phi_i(t) = (1 - t)(1 - t^2) \cdots (1 - t^r).
\]
We also have
\[ d_\lambda(t) = \frac{1}{(1-t)^n} \prod_{i \geq 0} \phi_{m_i(\lambda)}(t), \]
so that if \( l(\lambda) = n \), \( b_\lambda(t)(1-t)^n = d_\lambda(t) \).

We recall the following fact relating the branching coefficients \( \phi_{\lambda/\beta} \) and \( \psi_{\lambda/\beta} \) [10].

**Proposition 2.9.** We have
\[ \phi_{\lambda/\beta}(q,t)/b_\lambda(q,t) = \psi_{\lambda/\beta}(q,t)/b_\beta(q,t). \]

Note that, using (7) and (8), the coefficients \( c^\delta_{\lambda \mu} \) and \( sk_{\lambda+\lambda} \) may be defined in terms of inner products. We have
\[
c^\delta_{\lambda \mu}(q,t) = \langle m^{(n)}_\gamma(x)P^{(n)}_\mu(x; q, t), Q^{(n)}_\delta(x; q, t) \rangle' = d_\delta(q,t)\langle m^{(n)}_\gamma(x; q, t), P^{(n)}_\delta(x; q, t) \rangle
\]
and similarly
\[
\langle P^{(n)}(x; q, t)s^{(n)}_{\tau}(x), Q^{(n)}_{\lambda+}(x; t) \rangle' = sk_{\lambda+\lambda}(t).
\]

3. The Gelfand-Tsetlin basis expansion

In this section, we fix \( k \in \mathbb{N} \) and set \( t = q^k \). We will prove Theorem 1.3 of the introduction. Namely, we will expand the trace function \( \Phi^{(n)}_\lambda(x) \) in the Gelfand-Tsetlin basis and compute the diagonal coefficients \( c_\lambda(q, t) \). We will use the multiplicity-one decomposition of \( V_{\lambda+(k-1)\rho} \) as a \( \mathcal{U}_q(gl_{n-1}) \)-module, and iterate, in order to do this.

Etingof and Kirillov [2] provide the following closed form for the trace function at \( \lambda = 0 \):

**Proposition 3.1.**
\[
\Phi^{(n)}_0(x) = \prod_{i=1}^{k-1} (x^{\alpha_i/2} - q^{2i}x^{-\alpha_i/2}) = x^{(k-1)\rho} \prod_{i=1}^{k-1} n \prod_{n \geq i > m \geq 1} (1 - q^{2i}/x_m).
\]

**Proposition 3.2.**
\[
\frac{\phi^{(n)}_0(x)}{\phi^{(n-1)}_0(x)} = x^{(k-1)(\rho_n - \rho_{n-1})} \prod_{i=1}^{k-1} \prod_{n \geq i > m \geq 1} (1 - q^{2i}/x_m).
\]

**Proof.** By Proposition 3.1, we have
\[
\frac{\phi^{(n)}_0(x)}{\phi^{(n-1)}_0(x)} = x^{(k-1)(\rho_n - \rho_{n-1})} \prod_{i=1}^{k-1} \prod_{j=1}^{n-1} (1 - q^{2i}/x_{n_j}).
\]

Now note that, for fixed \( 1 \leq j \leq n-1 \),
\[
\prod_{i=1}^{k-1} (1 - q^{2i}/x_{n_j}) = \frac{(x_{n_j}/x_j; q^2)_k}{(x_{n_j}/x_j; q^2)_1} = \frac{(x_{n_j}/x_j; q^2)_\infty}{(q^{2k}x_{n_j}/x_j; q^2)_\infty} \cdot \frac{(q^2x_{n_j}/x_j; q^2)_\infty}{(x_{n_j}/x_j; q^2)_\infty} = \frac{(q^2x_{n_j}/x_j; q^2)_\infty}{(q^{2k}x_{n_j}/x_j; q^2)_\infty}.
\]

Now, putting \( t = q^k \) and using the \( q \)-binomial theorem,
\[
\frac{(q^2x_{n_j}/x_j; q^2)_\infty}{(t^2x_{n_j}/x_j; q^2)_\infty} = \sum_{m=0}^{\infty} \frac{(t^2q^2;q^2)^m}{(q^2;q^2)^m}(t^2x_{n_j}/x_j)^m.
\]

Taking the product over all \( 1 \leq j \leq n-1 \) and multiplying by \( x^{(k-1)(\rho_n - \rho_{n-1})} \) gives the result.

\[\square\]
Lemma 3.3. Let $\lambda$ be fixed with $l(\lambda) = n$. Then the map
\[ \mu \to \bar{\mu} = \mu + (k - 1) \left( \rho_{n-1} + \left( \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2} \right) \right) = \mu + (k - 1) \rho_n |_{n-1} \]
is a bijection:
- $\mu \subset \lambda$, such that $\lambda_{j+1} - \mu_j \leq k - 1$ for all $j$
- $\bar{\mu} \leq \lambda + (k - 1) \rho_n$, such that $\bar{\mu} - (k - 1) \rho_n |_{n-1} \in \mathcal{P}_+$

Proof. Follows from the definition of the interlacing condition and Equation 6.

Proposition 3.4. The following branching rule for trace functions holds:
\[ \Phi^{(n)}_\lambda (x; q, q^k) = (x_1 \cdots x_{n-1})^{k-1} \sum_{\mu \subset \lambda} x^{(\lambda, \mu)} a_{\lambda, \mu}(q) \Phi^{(n-1)}_{\mu}(x; q, q^k) \]
for some coefficients $a_{\lambda, \mu}(q)$.

Proof. One first notes the multiplicity-free decomposition of $V_{\lambda+(k-1)\rho_n}$ as a module over $U_q(gl_{n-1})$:
\[ V_{\lambda+(k-1)\rho_n} |_{U_q(gl_{n-1})} = \oplus_{\bar{\mu} \leq \lambda+(k-1)\rho_n} V_{\bar{\mu}}. \]
Thus, we have
\[ \Phi_\lambda(x; q, q^k) = \text{Tr}(\phi_\lambda \cdot x^h) = \sum_{\bar{\mu} \leq \lambda+(k-1)\rho_n} \text{Tr}(\phi_\lambda \cdot x^h |_{V_{\bar{\mu}}}) \]
\[ = \sum_{\bar{\mu} \leq \lambda+(k-1)\rho_n} \text{Tr} \left( (\text{Proj}_{V_{\bar{\mu}}} \otimes \text{Id}) \circ \phi_\lambda \circ x^h |_{V_{\bar{\mu}}} \right), \]
since trace only takes into account diagonal coefficients.

We have
\[ \phi_\lambda |_{V_{\bar{\mu}}}: V_{\bar{\mu}} \to V_{\lambda+(k-1)\rho_n} \otimes U \simeq \oplus_{\alpha \leq \lambda+(k-1)\rho_n} V_\alpha \otimes U, \]
thus
\[ (\text{Proj}_{V_{\bar{\mu}}} \otimes \text{Id}) \circ \phi_\lambda |_{V_{\bar{\mu}}}: V_{\bar{\mu}} \to V_{\bar{\mu}} \otimes U \]
is an intertwining operator. By [2], this implies that
\[ (\text{Proj}_{V_{\bar{\mu}}} \otimes \text{Id}) \circ \phi_\lambda |_{V_{\bar{\mu}}} = \begin{cases} \hat{a}_{\lambda, \bar{\mu}}(q) \cdot \phi_{\bar{\mu}-(k-1)\rho_{n-1}}, & \text{if } \bar{\mu} - (k - 1) \rho_{n-1} \in \mathcal{P}_+ \\ 0, & \text{else} \end{cases} \]
for some coefficients $\hat{a}_{\lambda, \bar{\mu}}(q)$. Thus, we have,
\[ \Phi_\lambda(x; q, q^k) = \sum_{\bar{\mu} \leq \lambda+(k-1)\rho_n} \hat{a}_{\lambda, \bar{\mu}}(q) \cdot x^{(\lambda, \bar{\mu})} \cdot \Phi_{\bar{\mu}-(k-1)\rho_{n-1}}(x; q, q^k). \]
Finally we reparametrize by setting $\mu = \bar{\mu} - (k - 1) \rho_{n-1} - (k - 1) \left( \frac{1}{2} \right)^{n-1}$ and defining $a_{\lambda, \mu}(q) = \hat{a}_{\lambda, \bar{\mu}}(q)$ with the condition that $a_{\lambda, \mu} = 0$ if $\lambda_{j+1} - \mu_j \leq k - 1$ does not hold for all $j$. The result now follows by the previous Lemma.

By iterating the branching rule of the previous proposition and recalling that the Gelfand-Tsetlin basis is also obtained by iterating the multiplicity-free decomposition, we obtain the following result.

Proposition 3.5. We have the following formula for $\Phi^{(n)}_\lambda (x; q, q^k)$ as a sum over Gelfand-Tsetlin patterns $\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(n-1)}$ with $\lambda^{(0)} = \lambda + (k - 1) \rho_n$:
\[ \Phi^{(n)}_\lambda (x; q, q^k) = \sum_{\Lambda \in \text{GT}(\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(n-1)})} \prod_{\lambda = \lambda^{(0)}, \ldots, \lambda^{(n-1)}} a_{\lambda^{(i-1)}, \lambda^{(i)}}(q) x^{\text{wt}(\lambda)} \]
We will show that the coefficients $a_{\lambda, \mu}(q)$ are equal to $c_{\lambda, \mu}(q, q^k)$ defined in the introduction. We will prove this through a series of propositions. Recall the definitions of the functions $g(\cdot, \cdot), \psi_{\lambda, \mu}(\cdot, \cdot), c_{\lambda, \mu}(\cdot, \cdot)$ in the introduction.

**Lemma 3.6.** For any $m \in \mathbb{C}$, the branching coefficients $a_{\lambda, \mu}(q)$ satisfy the shift invariance:

$$a_{\lambda + mn^\mu + m n^{-1}}(q) = a_{\lambda, \mu}(q).$$

**Proof.** The intertwining operator $\phi^{(k)}$, as well as the multiplicity one decomposition used in the proof of Proposition 3.4, is determined by the $U_q(\mathfrak{sl}_n)$-module structure. Indeed, $U_q(\mathfrak{sl}_n)$ differs only from $U_q(\mathfrak{sl}_k)$ by the addition of the central element $q^{\nu_1 + \cdots + \nu_n}$. The result then follows easily from the observation that, as $U_q(\mathfrak{sl}_n)$-modules, $V_{\lambda + mn^\mu}$ is isomorphic to $V_{\lambda}$ for any partition $\lambda$ and any $m \in \mathbb{C}$.

**Remark.** By the previous Lemma, to compute $a_{\lambda, \mu}(q)$ for $\lambda \in P_+(n), \mu \in P_+(n-1)$, we may assume that $\lambda, \mu$ are partitions with $l(\lambda) = n, l(\mu) = n - 1$. We will make this assumption implicitly throughout the paper.

**Proposition 3.7.** The branching coefficients satisfy the following formula:

$$a_{\lambda, \mu}(q) = \sum_{\beta \subseteq \lambda, l(\beta) \leq n - 1} g(\gamma; q^2, q^{2k}) \psi_{\lambda/\beta}(q^2, q^{2k}) c_{\gamma, (k-1)n-1, \beta}(q^2, q^{2k}).$$

**Proof.** Combining Theorem 1.1 with Proposition 3.4 gives the following:

$$\frac{\phi_0^{(n)}(x; q, q^k)}{\phi_0^{(n-1)}(x; q, q^k)} P_{\lambda}^{(n)}(x; q^2, q^{2k}) = \sum_{\mu \subseteq \lambda} x_n^{l(\lambda)} a_{\lambda, \mu}(q) P_{\mu}^{(n-1)}(x; q^2, q^{2k}).$$

We then use Theorem 2.4 to rewrite this as

$$\frac{\phi_0^{(n)}(x; q, q^k)}{\phi_0^{(n-1)}(x; q, q^k)} = \sum_{\mu \subseteq \lambda} x_n^{l(\mu)} a_{\lambda, \mu}(q) P_{\mu}^{(n-1)}(x; q^2, q^{2k})$$

Now note from Proposition 3.2, we have

$$\frac{\phi_0^{(n)}(x; q, q^k)}{\phi_0^{(n-1)}(x; q, q^k)} = x^{(k-1)(\rho_n - \rho_{n-1})} \times$$

$$\times \sum_{\gamma \in \mathbb{Z}_{\geq 0}^n} q^{2k(\sum \gamma_i)} (q^{2k}; q^{2k})_{\gamma} \psi_{\lambda/\beta}(q^2, q^{2k}) \gamma_{n-1} x_n^{(\sum \gamma_i)} m_{\gamma}(x_1, \ldots, x_{n-1}),$$

where the sum is over $\gamma = (\gamma_1, \ldots, \gamma_{n-1})$ a partition. Now recall that $(k-1)(\rho_n - \rho_{n-1}) = (k-1)(\frac{1}{2}, \ldots, \frac{1}{2}, \frac{1-n}{2}) \in \mathbb{Z}^n$, so we have

$$\frac{\phi_0^{(n)}(x; q, q^k)}{\phi_0^{(n-1)}(x; q, q^k)} = \sum_{\gamma \in \mathbb{Z}_{\geq 0}^n} x_n^{(k-1)(\frac{1-n}{2}) + |\gamma|} g(\gamma; q^2, q^{2k}) m_{\gamma}(x_1, \ldots, x_{n-1}),$$

where we have used Definition (2.1).

We use the previous equation, along with (9), and multiply both sides by the monomial $(x_1 \cdots x_{n-1})^{(k-1)(\frac{1}{2} - \frac{1}{2})}$ to obtain the equation:
Proposition 3.8. Let $\mu \subset \lambda$ with $(\mu) \leq n - 1$. Then we have
\[
a_{\lambda, \mu}(q) = \sum_{\substack{\beta \leq \lambda, l(\beta) \leq n - 1 \\gamma \in \mathbb{P}^{n-1} \\text{a partition} \\mu \leq \lambda, l(\mu) \leq n - 1}} g(\gamma; q^2, q^{2k}) \psi_{\lambda/\beta}(q^2, q^{2k}) d_{\beta}(q^2, q^{2k}) c_{\gamma, \mu}(q^2, q^{2k}).
\]

Proof. Using standard facts about integration over $\mathbb{T}_n$, we have
\[
c_{\gamma, \mu}^{\mu+(k-1)n-1-\beta}(q^2, q^{2k}) = \langle m_{\gamma+(k-1)n-1}(x)P_{\beta}(x; q^2, q^{2k}), P_{\mu+(k-1)n-1}(x; q^2, q^{2k}) \rangle = \langle m_{\gamma}(x)P_{\beta}(x^2, q^{2k}), P_{\mu}(x; q^2, q^{2k}) \rangle = \frac{d_{\mu+(k-1)n-1}(q^2, q^{2k})}{d_{\beta}(q^2, q^{2k})} d_{\gamma, \mu}(q^2, q^{2k}),
\]
where we have used $d_{\mu+(k-1)n-1}(q^2, q^{2k}) = d_{\mu}(q^2, q^{2k})$, which follows from the definition of $\langle \cdot, \cdot \rangle$. Combining this with the previous theorem gives the result.}

We are now prepared to provide a proof of Theorem 1.3, mentioned in the introduction to this paper. The proof relies on the previous propositions proved in this section.

Proof of Theorem 1.3. By the previous proposition, we have
\[
a_{\lambda, \mu}(q) = d_{\mu}(q^2, q^{2k}) \sum_{\substack{\beta \leq \lambda, l(\beta) \leq n - 1 \\gamma \in \mathbb{P}^{n-1} \\text{a partition} \\mu \leq \lambda, l(\mu) \leq n - 1}} g(\gamma; q^2, q^{2k}) \psi_{\lambda/\beta}(q^2, q^{2k}) \frac{1}{d_{\beta}(q^2, q^{2k})} c_{\gamma, \mu}(q^2, q^{2k}).
\]
Now, note that for fixed $\beta$, we have

\[
\sum_{l(\gamma) \leq n-1} g(\gamma; q^2, q^{2k}) c^\beta_{\gamma, \mu}(q^2, q^{2k})

= \sum_{l(\gamma) \leq n-1} q^{2k|\gamma|} \frac{(q^{-2k} q^2; q^2)_{\gamma_1} \cdots (q^{-2k} q^2; q^2)_{\gamma_{n-1}}}{(q^2; q^2)_{\gamma_1} \cdots (q^2; q^2)_{\gamma_{n-1}}} c^\beta_{\gamma, \mu}(q^2, q^{2k})

= \sum_{l(\gamma) \leq n-1} q^{2k|\gamma|} \frac{(q^{-2k} q^2; q^2)_{\gamma_1} \cdots (q^{-2k} q^2; q^2)_{\gamma_{n-1}}}{(q^2; q^2)_{\gamma_1} \cdots (q^2; q^2)_{\gamma_{n-1}}} (m^{(n-1)}_\gamma(x) P^{(n-1)}_\mu(x; q^2, q^{2k}), Q^{(n-1)}_\beta(x; q^2, q^{2k}))'.
\]

Now recall the following identity [10, p314]:

\[
g_n(x; q^2, t^2) = \sum_{|\mu|=n} \frac{(t^2; q^2)_\mu}{(q^2; q^2)_\mu} m_\mu(x).
\]

Using this, we may write the previous equation as

\[
\sum_{l(\gamma) \leq n-1} g(\gamma; q^2, q^{2k}) c^\beta_{\gamma, \mu}(q^2, q^{2k})

= \left\langle \left( \sum_{r \geq 0} q^{2kr} g_r^{(n-1)}(x; q^2, q^{-2k} q^3) \right) P^{(n-1)}_\mu(x; q^2, q^{2k}), Q^{(n-1)}_\beta(x; q^2, q^{2k}) \right\rangle'.
\]

Also note [10, p311] that we have the generating function identity (for arbitrary $q, t, y$):

\[
\sum_{n \geq 0} g_n(x; q, t) y^n = \prod_{i \geq 1} \frac{(tx y; q)_\infty}{(x y; q)_\infty};
\]

thus we have

\[
\sum_{l(\gamma) \leq n-1} g(\gamma; q^2, q^{2k}) c^\beta_{\gamma, \mu}(q^2, q^{2k})

= \left\langle \prod_{i \geq 1} \frac{(q^2 x_i; q^2)_\infty}{(q^{2k} x_i; q^2)_\infty} P^{(n-1)}_\mu(x; q^2, q^{2k}), Q^{(n-1)}_\beta(x; q^2, q^{2k}) \right\rangle'.
\]

Combining this with the original sum yields

\[
a_{\lambda, \mu}(q) = d_\mu(q^2, q^{2k}) \times
\]

\[
\times \sum_{\beta \leq \lambda} \psi_{\lambda/\beta}(q^2, q^{2k}) \left\langle \prod_{i \geq 1} \frac{(q^2 x_i; q^2)_\infty}{(q^{2k} x_i; q^2)_\infty} P^{(n-1)}_\mu(x; q^2, q^{2k}), Q^{(n-1)}_\beta(x; q^2, q^{2k}) \right\rangle'.
\]

\[
= d_\mu(q^2, q^{2k}) \sum_{\beta \leq \lambda} \psi_{\lambda/\beta}(q^2, q^{2k}) \Omega_{\beta/\mu}(q^2, q^{2k})
\]

by definition of $\Omega_{\beta/\mu}(q^2, q^{2k})$. But this is exactly equal to $c_{\lambda, \mu}(q, q^k)$ as defined in (2). □

**Remarks.** The coefficients $c_{\lambda, \mu}(q, t)$ do not appear to factor nicely at the $q$-level, due to the restriction on length in the sum. For example, for $\lambda = (2, 1)$ and $\mu = (1)$ one obtains $(1-t)(1-q^2)(1-q-q^2+t)/(1-qt)$, and the term $(1-q-q^2+t)$ cannot be expressed as a product of $(1-q t^2)$.
4. The $q \to 0$ Limit

We will look at the $q \to 0$ limit of the coefficients $c_{\lambda,\mu}(q,t)$. We find that the formula has a nice product form, in terms of certain $p$-adic counts. The simplification of these coefficients at $q = 0$ may be related to the crystal basis structure of the Gelfand-Tsetlin basis, although we have not investigated a direct link.

We note that the parameters $q,t$ are linked in the finite-dimensional case since $t = q^{k}$ and $k \in \mathbb{N}$, so we cannot take $q \to 0$ without having $t \to 0$ as well. In this section, we will simply treat $t$ as a formal variable and in Section 5, we will relate it to the representation theory by using Verma modules.

The goal of this section is to prove Theorems 1.4 and 1.5 mentioned in the introduction.

**Definition 4.1.** We let $c_{\lambda,\mu}(t)$ denote $\lim_{q \to 0} c_{\lambda,\mu}(q,t)$.

**Theorem 4.2.** Let $\lambda$ be a partition of length $n$, and $\mu \subset \lambda, \mu \in \mathcal{P}_+$. Then retaining the notation of the previous sections, we have

$$c_{\lambda,\mu}(t) = \frac{b_{\mu}(t^2)}{b_{\lambda}(t^2)} \sum_{\beta \leq \lambda \atop l(\beta) \leq n-1} \phi_{\lambda/\beta}(t^2) t^{2|\beta/\mu|} s_{k_{\beta/\mu}}(t^2).$$

**Proof.** We use Theorem 1.3, the functions there admit the limit $q \to 0$. We also use that

$$\Omega_{\beta/\mu}(q,t) = Q_{\beta/\mu} \left( \frac{t-q}{1-t} \right) = t^{|\beta/\mu|} Q_{\beta/\mu} \left( \frac{1-q}{1-t} \right)$$

and

$$\lim_{q \to 0} Q_{\beta/\mu} \left( \frac{1-q}{1-t} \right) = s_{k_{\beta/\mu}}(t),$$

where the skew Macdonald polynomials are taken with respect to the parameters $(q,t)$. This gives the following

$$c_{\lambda,\mu}(t) = d_{\mu}(t^2) \sum_{\beta \leq \lambda \atop l(\beta) \leq n-1} \frac{\psi_{\lambda/\beta}(t^2)}{d_{\beta}(t^2)} t^{2|\beta/\mu|} s_{k_{\beta/\mu}}(t^2).$$

Finally, one notes that

$$d_{\beta}(t^2) = b_{\beta}(t^2) \quad \text{and} \quad d_{\mu}(t^2) = b_{\mu}(t^2),$$

and by the $q \to 0$ limit of Proposition 2.9 we have

$$\phi_{\lambda/\beta}(t)/b_{\lambda}(t) = \psi_{\lambda/\beta}(t)/b_{\beta}(t);$$

using this in the previous equation gives the result.

□

Our next goal is to obtain a nice factorized product form for $c_{\lambda,\mu}(t)$. We use a $q \to 0$ specialization of Rains’ $q$-Pfaff-Saalschütz formula:

**Theorem 4.3** ([11, Corollary 4.9]). Let $\mu \subset \lambda$ be partitions, then for arbitrary parameters $a,b,c$ we have the following identity:

$$\sum_{\beta} \frac{(a)_{\beta}}{(c)_{\beta}} Q_{\lambda/\beta} \left( \frac{a-b}{1-t} \right) Q_{\beta/\mu} \left( \frac{b-c}{1-t} \right) = \frac{(a)_{\mu}(b)_{\lambda}}{(b)_{\mu}(c)_{\lambda}} Q_{\lambda/\mu} \left( \frac{a-c}{1-t} \right).$$

**Proposition 4.4.** Let $\lambda$ be a partition of length $n$ and let $\mu \leq \lambda$ with $l(\mu) = n-1$. Then

(1) $$\sum_{\beta \leq \lambda} \phi_{\lambda/\beta}(t^2) t^{2|\beta/\mu|} s_{k_{\beta/\mu}}(t^2) = s_{k_{\lambda/\mu}}(t^2)$$

(2) $$\sum_{\beta \leq \lambda \atop l(\beta) \leq n-1} \phi_{\lambda/\beta}(t^2) t^{2|\beta/\mu|} s_{k_{\beta/\mu}}(t^2) = (1-t^2) \cdot s_{k_{\lambda/\mu}}(t^2)$$
Proof. Take \(a = q, b = qt, c = q^2\) in in Theorem 4.3:
\[
\sum_{\beta} \frac{(q)_\beta}{(q^2)_\beta} Q_{\lambda/\beta} \left(\frac{q - qt}{1 - t}\right) Q_{\beta/\mu} \left(\frac{qt - q^2}{1 - t}\right) = \frac{(q)_\mu(qt)_\lambda}{(qt)_\mu(q^2)_\lambda} Q_{\lambda/\mu} \left(\frac{q - q^2}{1 - t}\right).
\]

Using the relation \(Q_{\lambda/\mu} \left(\frac{aq - bq}{1 - t}\right) = q^{1/\mu} Q_{\lambda/\mu} \left(\frac{a - b}{1 - t}\right)\), we have
\[
\sum_{\beta} \frac{(q)_\beta}{(q^2)_\beta} Q_{\lambda/\beta} \left(\frac{t - q}{1 - t}\right) = \frac{(q)_\mu(qt)_\lambda}{(qt)_\mu(q^2)_\lambda} Q_{\lambda/\mu} \left(\frac{1 - q}{1 - t}\right).
\]
(1) then follows by taking the limit \(q \to 0\).

To prove (2), it suffices by (1) to show that
\[
\sum_{\beta \leq \lambda, l(\beta) = n} \phi_{\lambda/\beta}(t^2) t^{2(\beta/\mu)} sk_{\beta/\mu}(t^2) = t^2 \cdot sk_{\lambda/\mu}(t^2).
\]

Reindex the sum by replacing \(\beta = \beta' + 1^n\). Then we have
\[
\sum_{\beta' \leq \lambda - 1^n} \phi_{\lambda/(\beta' + 1^n)}(t^2) t^{2(\beta'/\mu)} t^{2n} sk_{\beta'/\mu}(t^2)
\]
We have the following two identities:
\[
\phi_{\lambda/(\beta' + 1^n)}(t^2) = \phi_{(\lambda - 1^n)/\beta'}(t^2)
\]
\[
sk_{(\beta' + 1^n)/\mu}(t^2) = sk_{\beta'/\mu}(t^2),
\]
which can be seen by using the explicit formulas in Section 2. It follows that
\[
\sum_{\beta' \leq (\lambda - 1^n)} \phi_{(\lambda - 1^n)/\beta'}(t^2) t^{2(\beta'/\mu - 1^{n-1})} t^2 sk_{\beta'/\mu}(t^2) = t^2 \cdot sk_{(\lambda - 1^n)/(\mu - 1^{n-1})}(t^2).
\]
Applying the identity above again completes the proof. \(\square\)

We now provide a proof of Theorem 1.4, mentioned in the introduction. The proof relies on the previous results of this section.

Proof of Theorem 1.4. By Proposition 4.4, we have
\[
c_{\lambda, \mu}(t) = \frac{b_\mu(t^2)}{b_\lambda(t^2)} \sum_{\beta \leq \lambda, l(\beta) = n - 1} \phi_{\lambda/\beta}(t^2) t^{2(\beta/\mu)} sk_{\beta/\mu}(t^2) = \frac{b_\mu(t^2)}{b_\lambda(t^2)} (1 - t^2) sk_{\lambda/\mu}(t^2),
\]
which gives the first equality. By the definitions of \(b_\lambda(t), sk_{\lambda/\mu}(t)\), this is equal to
\[
\prod_{i \geq 1} \phi_{m_i(\mu)}(t^2) (1 - t^2) t^2 \sum_{i} \left(\frac{\lambda_i - \mu_i'}{\mu_i'} \right) \prod_{j \geq 1} \left(\frac{\lambda_j - \mu_j'}{\mu_j'} \right) t^2
\]
\[
= (1 - t^2) t^2 \sum_{i} \left(\frac{\lambda_i - \mu_i'}{\mu_i'} \right) \prod_{j \geq 1} \phi_{\lambda_j - \mu_j'}(t^2) \frac{\phi_{\lambda_j' - \mu_j'}(t^2)}{\phi_{\lambda_j' - \mu_j'}(t^2)}
\]
\[
= (1 - t^2) t^2 \sum_{i} \left(\frac{\lambda_i - \mu_i'}{\mu_i'} \right) \prod_{j \geq 1} \phi_{\lambda_j - \mu_j'}(t^2) \phi_{\lambda_j' - \mu_j'}(t^2) = t^2 \sum_{i} \left(\frac{\lambda_i - \mu_i'}{\mu_i'} \right) \prod_{j \geq 1} \left(\frac{\lambda_j - \mu_j'}{\mu_j'} \right),
\]
where we have used \(m_i(\mu) = \mu_i' - \mu_i'_{i+1}\) and \(\lambda_i'_{i+1} = \mu_i'\) (because \(l(\lambda) = n\) and \(l(\mu) = n - 1\). \(\square\)
We recall that, as mentioned in the introduction, there is a $p$-adic interpretation for coefficients $sk_{\lambda/\mu}(t)$ and thus for $c_{\lambda,\mu}(t)$. More precisely,

$$sk_{\lambda/\mu}(t) = t^{\nu(\lambda)-\nu(\mu)}a_{\lambda}(\mu; t^{-1});$$

where $a_{\lambda}(\mu; p)$ is the number of subgroups of type $\mu$ in a finite abelian $p$-group of type $\lambda$, see [12] for example, and the references therein.

**Proof of Theorem 1.5.** Recall that for $S = (\mu^{(0)} \supset \mu^{(1)} \supset \cdots \supset \mu^{(n-1)})$ with $\mu^{(i)} \in \mathcal{P}_{+}^{(n-i)}$, we defined the coefficient $sk_{S}(t)$ as a product of $sk_{\mu^{(i-1)}/\mu^{(i)}}(t)$ in (4). By Definition 1.2, one can associate to $S$ a Gelfand-Tsetlin array $\Lambda$. Thus, using Theorem 1.4, we have

$$\lim_{q \to 0} c_{\lambda}(q, t) = \frac{(1 - t^{2})^{n}}{b_{\lambda}(t^{2})} sk_{S}(t^{2}).$$

Using this along with Theorem 1.3 gives the result. □

Note that when $t = p^{-1}$ for $p$ an odd prime, the coefficients appearing in both Theorems 1.4 and 1.5 are explicit $p$-adic counts.

**Corollary 4.5.** Let $\lambda$ be a partition. We have the following formula for the Hall-Littlewood polynomial:

$$P_{\lambda}(x_{1}, \ldots, x_{n}; t^{2}) = \frac{1}{b_{\lambda}(t^{2})} \sum_{S = (\lambda = \mu^{(0)} \supset \mu^{(1)} \supset \cdots \supset \mu^{(n-1)})} \frac{sk_{S}(t^{2})x^{\omega(S)}}{\sum_{S' = (0 = \mu^{(0)} \supset \mu^{(1)} \supset \cdots \supset \mu^{(n-1)})} sk_{S'}(t^{2})x^{\omega(S')}}.$$

**Proof.** Follows from Theorem 1.1 along with Theorem 1.5. □

5. VERMA MODULES AND ALGEBRAICALLY INDEPENDENT $t$

We have computed the expansion of the Macdonald vector-valued characters $\Phi_{\lambda}^{(k)}(x; q)$ with respect to the Gelfand-Tsetlin basis of $V_{\lambda+(k-1)\rho}$. These are expressed in terms of rational functions in $q, t$ which appear naturally in symmetric function theory, specialized to $t = q^{k}$. In the previous section we showed that, for algebraically independent $t$, these coefficients admit a simple limit as $q \to 0$, which is related to natural quantities appearing in $p$-adic representation theory. Note however that in the representation theoretic realization of $\Phi_{\lambda}^{(k)}$, we have $t = q^{k}$, and hence we can only obtain the $t = 0$ specialization of our formula in the $q \to 0$ limit.

In [2], Etingof and Kirillov showed that one can extend $\Phi_{\lambda}^{(k)}$ to algebraically independent $t$ by replacing the finite-dimensional irreducible module $V_{\lambda+(k-1)\rho}$ by a suitable infinite dimensional irreducible Verma module. In this section we outline their construction, which allows us to obtain a representation theoretic realization of our formula for algebraically independent $t$.

Consider the algebra $\mathbb{C}(t) \otimes U_{q}(\mathfrak{gl}_{n})$, i.e. the quantum group $U_{q}(\mathfrak{gl}_{n})$ where the coefficient field is expanded to $\mathbb{C}(t) \otimes \mathbb{C}(q)$ (note this can be identified with the subalgebra of $\mathbb{C}(q, t)$ spanned by products of the form $r_{1}(q) \cdot r_{2}(t)$ for rational functions $r_{1}, r_{2}$). We have the following analogues of the finite-dimensional modules $V_{\lambda+(k-1)\rho}$:

**Definition 5.1.** For $\lambda \in \mathcal{P}^{(n)}$, the module $M_{\lambda,t}$ over $\mathbb{C}(t) \otimes U_{q}(\mathfrak{gl}_{n})$ is uniquely defined by the following conditions:

1. There is a highest weight vector $m_{\lambda} \in M_{\lambda,t}$ satisfying:

$$e_{i} \cdot m_{\lambda} = 0, \quad (1 \leq i \leq n-1)$$

$$q^{\rho_{i}^{+}} \cdot m_{\lambda} = t^{2\rho_{i}} \cdot q^{2\langle \lambda_{i} - \rho_{i} \rangle} \cdot m_{\lambda}, \quad (1 \leq i \leq n)$$
Moreover, for the relationship between $W$ and the Verma module $M$ of weight $\lambda + (k - 1)\rho$ (see e.g. [2]). Let us attempt to clarify the relationship between $M_{\lambda,t}$ and the finite-dimensional modules $V_{\lambda+(k-1)\rho}$.

Firstly, for $k \in \mathbb{N}$ there is a quotient mapping $\mathbb{C}(t) \otimes U_q(\mathfrak{gl}_n) \to U_p(\mathfrak{gl}_n)$ sending $t \to q^k$. Moreover, for $k \in \mathbb{N}$ then there is a $\mathbb{C}(q)$-linear map $\alpha_k : M_{\lambda,t} \to V_{\lambda+(k-1)\rho}$ which is compatible with the module structures in the sense that the following diagram commutes:

\[
\begin{array}{ccc}
(C(t) \otimes U_q(\mathfrak{gl}_n)) \otimes M_{\lambda,t} & \xrightarrow{g \otimes v \mapsto g \cdot v} & M_{\lambda,t} \\
(t \mapsto q^k) \otimes \alpha_k \downarrow & & \alpha_k \\
U_q(\mathfrak{gl}_n) \otimes V_{\lambda+(k-1)\rho} & \xrightarrow{g \otimes v \mapsto g \cdot v} & V_{\lambda+(k-1)\rho}
\end{array}
\]

(We take the convention here that $V_{\lambda+(k-1)\rho} = \{0\}$ if $\lambda + (k - 1)\rho \notin P_+$, which can occur for only finitely many $k$). The kernels of $\alpha_k$ form a decreasing sequence of subspaces of $M_{\lambda,t}$, and

\[
\bigcap_{k \geq 0} \ker \alpha_k = \{0\}.
\]

The existence of $\alpha_k$ satisfying the above conditions determines the module $M_{\lambda,t}$ uniquely.

The analogue of the finite-dimensional module $U \simeq V(k-1;n-1,\ldots,1)$ as is follows:

**Definition 5.2.** The module $W_t$ over $\mathbb{C}(t) \otimes U_q(\mathfrak{gl}_n)$ is the degree zero subspace of Laurent polynomials

\[W_t = \{ p(x) \in \mathbb{C}(t) \otimes \mathbb{C}(q)[x_1^{\pm 1}x_2^{\pm 1}\cdots x_n^{\pm 1}] \mid \deg p = 0 \},\]

with the following action of the generators of $U_q(\mathfrak{gl}_n)$:

\[
e_i(p(x)) = x_i \cdot \frac{\partial}{\partial x_i} p(x)
\]

\[
e_i(p(x)) = \frac{x_i}{x_{i+1}} \cdot \frac{(tq^{-1})p(x_1,\ldots,qx_{i+1},\ldots,x_n) - (t^{-1}q)p(x_1,\ldots,q^{-1}x_{i+1},\ldots,x_n)}{(q-q^{-1})}
\]

\[
f_i(p(x)) = \frac{x_{i+1}}{x_i} \cdot \frac{(tq^{-1})p(x_1,\ldots,qx_i,\ldots,x_n) - (t^{-1}q)p(x_1,\ldots,q^{-1}x_i,\ldots,x_n)}{(q-q^{-1})}
\]

This is isomorphic to the module denoted $W_k$ in [2], and is irreducible over $\mathbb{C}(t) \otimes U_q(\mathfrak{gl}_n)$. If $k \in \mathbb{N}$ is a fixed integer, we can quotient $W_t$ by the relation $t = q^k$ to obtain an infinite-dimensional module over $U_q(\mathfrak{gl}_n)$. This is no longer irreducible, and the subspace spanned by $p(x)$ with $(x_1 \ldots x_n)^{(k-1)}$. $p(x) \in \mathbb{C}(q)[x_1,\ldots,x_n]$ is identified with the module $U$.

It is shown in [2] that there is a unique intertwining operator $\tilde{\phi} : M_{\lambda,t} \to M_{\lambda,t} \otimes W_t$ if and only if $\lambda \in P_+$. Moreover, for $k \in \mathbb{N}$ the intertwining operators $\tilde{\phi}$ and $\phi^{(k)}$ are compatible in the sense that the following diagram commutes:

\[
\begin{array}{ccc}
M_{\lambda,t} & \xrightarrow{\tilde{\phi}} & M_{\lambda,t} \otimes W_t \\
\alpha_k \downarrow & & \alpha_k \otimes (\text{Proj}_{C}(t \mapsto q^k)) \\
V_{\lambda+(k-1)\rho} & \xrightarrow{\phi^{(k)}} & V_{\lambda+(k-1)\rho} \otimes U
\end{array}
\]

The weight-zero subspace of $W_t$ is one dimensional, which allows us to define the trace function $\Phi(x;q,t) \in \mathbb{C}(q,t)[[x_1,\ldots,x_n]]$ of $\tilde{\phi}$. The compatibility (10) implies the relation $\Phi(x,q,q^k) = \Phi^{(k)}(x)$ mentioned in Theorem 1.1.

The analogue of the Gelfand-Tsetlin basis for $M_{\lambda,t}$ is obtained by iterating the multiplicity one decomposition over $U(\mathfrak{gl}_{n-1})$, as in the finite-dimensional case.
Proposition 5.3. We have the following restriction rule for $M_{\lambda,t}$ as a module over $\mathbb{C}(t) \otimes U_q(\mathfrak{gl}_{n-1}) \subset \mathbb{C}(t) \otimes U_q(\mathfrak{gl}_n)$:

$$\left(\mathbb{C}(t) \otimes U_q(\mathfrak{gl}_{n-1})\right) M_{\lambda,t} \simeq \bigoplus_{\mu \in \mathcal{P}(n-1)} M_{\mu-(\frac{1}{2})^{(n-1)},t}(\frac{1}{t})^{\mathcal{P}(n-1)} \subset \mathbb{C}(t) \otimes U_q(\mathfrak{gl}_n).$$

By iterating the restriction rule above we obtain a basis for $M_{\lambda,t}$ which is indexed by chains $\lambda = \mu^{(0)} \supset \mu^{(1)} \supset \cdots \supset \mu^{(n-1)}$ with $\mu^{(i)} \in \mathcal{P}^{(n-1)}$. We refer to this as the Gelfand-Tsetlin basis for $M_{\lambda,t}$.

Proof. This is well known to experts. It can be proved using the maps $\alpha_k$ and the decomposition of $V_{\lambda+(k-1)\rho}$ over $U_q(\mathfrak{gl}_{n-1})$. □

Theorem 5.4. With respect to the Gelfand-Tsetlin basis of $M_{\lambda,t}$, the diagonal coefficient of the intertwining operator $\tilde{\phi}$ corresponding to the chain $\lambda = \mu^{(0)} \supset \mu^{(1)} \supset \cdots \supset \mu^{(n-1)}$ is equal to:

$$\prod_{1 \leq i \leq n-1} c_{\mu^{(i-1)} \mu^{(i)}}(q,t), \quad \mu^{(i)} \in \mathcal{P}_+ \text{ for } 0 \leq i \leq n-1.$$ 

Otherwise $0$.

Proof. Using the compatibility (10), and Theorem 1.3, one can see that the formula holds for $t = q^k$ when $k \in \mathbb{N}$ is sufficiently large. Since the coefficient is a rational function of $q,t$ this determines it uniquely, and the result follows. □

REFERENCES

[1] V. G. Drinfeld, Quantum groups, Proc. Int. Congr. Math., Berkely, 1986, pp 798-820.
[2] P. I. Etingof and A. A. Kirillov, Jr., Macdonald’s polynomials and representations of quantum groups, Math. Res. Let. 1 (1994), 279-294.
[3] M. A. Jimbo, A $q$-difference analogue of $Ug$ and the Yang-Baxter equation, Lett. Math. Phys. 10 (1985), 62-69.
[4] M. Kashiwara, On crystal bases, In Representations of groups (Banff, AB, 1994), volume 16 of CMS Conf. Proc., pages 155-197. Amer. Math. Soc., Providence, RI, 1995.
[5] A. N. Kirillov, New combinatorial formula for modified Hall-Littlewood polynomials, in q-Series from a Contemporary Perspective, pp. 283-333, Contemp. Math. Vol. 254, AMS, Providence, RI, 2000.
[6] T. H. Koornwinder, Askey–Wilson polynomials for root systems of type $BC$, in Hypergeometric functions on domains of positivity, Jack polynomials, and applications (Tampa, FL, 1991), vol. 138 of Contemp. Math., Amer. Math. Soc., Providence, RI, 1992, 189–204.
[7] M. Konvalinka and A. Lauve, Skew Pieri rules for Hall-Littlewood functions, DMTCS proc. AR, 2012, 459-470.
[8] A. Lascoux and S. O. Warnaar, Branching rules for symmetric Macdonald polynomials and $\mathfrak{sl}_n$ basic hypergeometric series, Advances in Applied Mathematics 46 (2011), 424-456.
[9] I. G. Macdonald, Spherical functions on a group of $p$-adic type, Ramanujan Institute, Centre for Advanced Study in Mathematics,University of Madras, Madras, 1971. Publications of the Ramanujan Institute, No. 2.
[10] I. G. Macdonald, Symmetric functions and Hall polynomials, Oxford Mathematical Monographs, Oxford University Press, New York, second ed., 1995.
[11] E. M. Rains, $BC_n$-symmetric polynomials, Transform. Groups 10 (2005), 63-132.
[12] S. O. Warnaar, Remarks on the paper “Skew Pieri rules for Hall-Littlewood functions” by Konvalinka and Lauve, Journal of Algebraic Combinatorics 38 (2013), 519-526.