THE CHOW RING OF RELATIVE FULTON–MACPHERSON SPACE

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Abstract. Suppose that $X$ is a nonsingular variety and $D$ is a nonsingular proper subvariety. Configuration spaces of distinct and non-distinct $n$ points in $X$ away from $D$ were constructed by the author and B. Kim in [4] by using the method of wonderful compactification. In this paper, we give an explicit presentation of Chow motives and Chow rings of these configuration spaces.

1. Introduction

Let $X$ be a complex connected nonsingular algebraic variety and let $D$ be a smooth divisor.

In [4], two generalizations of Fulton–MacPherson spaces were constructed by using the method of wonderful compactifications [5]. Two spaces are following:

(1) A compactification $X_D^{[n]}$ of the configuration space of $n$ labeled points in $X \setminus D$, i.e. “not allowing those points to meets $D$.”

(2) A compactification $X_D[n]$ of the configuration spaces of $n$ distinct labeled points in $X \setminus D$, i.e. ”not allowing those points to meet each other as well as $D$.”

The goal of this paper is to give an explicit presentation of Chow motives and Chow rings of these configuration spaces. Our main theorems are:

Theorem 1.1. The Chow ring $A^*(X_D^{[n]})$ is isomorphic to the polynomial ring $A^*(X^n)[x_S]$ modulo the ideal generated by

1. $x_S \cdot x_T$ for $S, T$ that overlap,
2. $J_{DS/X^n} \cdot x_S$ for all $S$,
3. $P_{DS/X^n}(-\Sigma_{S' \supset S} x_{S'})$ for all $S$.

Theorem 1.2. The Chow ring $A^*(X_D[n])$ is isomorphic to the polynomial ring $A^*(X^n)[x_S, y_I]$ modulo the ideal generated by

1. $y_I \cdot y_J$ for $I$ and $J$ that overlap,
2. $x_S \cdot x_T$ for $S$ and $T$ that overlap,
3. $x_S \cdot y_I$ unless $I \subset S$,
(4) $\mathcal{I}_{\Delta_I/X^n} \cdot y_I$ for all $I$,
(5) $\mathcal{I}_{DS/X^n} \cdot x_S$ for all $S$,
(6) $c_{a,b}(\sum_{a,b \in I} y_I)$ for $a, b \in \{1, \ldots, n\}$ (distinct),
(7) $P_{DS/X^n}(-\Sigma_{S' \supset S} x_{S'})$ for all $S$.

The paper is organized as follows. In section 2, we review the theory of wonderful compactification and Chow rings and motives after blow-up. In section 3, we review the construction of compactifications of $n$ points in $X \setminus D$. In section 4, we compute Chow groups and motives explicitly. In section 5, we compute Chow rings under the assumptions such that $X^n$ has the Kunneth decomposition and the embedding $D \hookrightarrow X$ is a Lefshetz embedding.

1.1. Notation.

- As in [1], for a subset $I$ of $N := \{1, 2, \ldots, n\}$, let
  $$I^+ := I \cup \{n + 1\}.$$

- Let $Y_1$ be the blowup of a nonsingular complex variety $Y_0$ along a nonsingular closed subvariety $Z$. If $V$ is an irreducible subvariety of $Y_0$, we will use $\tilde{V}$ or $V(Y_1)$ to denote
  - the total transform of $V$, if $V \subset Z$;
  - the proper transform of $V$, otherwise.
  If there is no risk to cause confusion, we will use simply $V$ to denote $\tilde{V}$. The space $Bl_{Y_1} Y_1$ will be called the iterated blowup of $Y_0$ along centers $Z, V$ (with the order).

- For a partition of $I$ of $N$, $\Delta_I$ denotes the polydiagonal associated to $I$. And consider the binary operation $I \wedge J$ on the set of all partitions satisfying
  $$\Delta_I \cap \Delta_J = \Delta_{I \wedge J}.$$ 

  We use $\Delta_{I_0}$ instead of $\Delta_I$ when $I = \{I_0, I_1, \ldots, I_l\}$ such that $|I_i| = 1$ for all $i \geq 1$.

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2. Wonderful Compactification of Arrangements of Subvarieties

In this section, we review the theory of wonderful compactification of arrangements of subvarieties. See the detail and proofs in [5], [6].
2.1. Arrangement, building set and nest.

**Definition 2.1** (of clean intersection). Let $Y$ be a nonsingular algebraic variety and let $U$ and $V$ be two smooth subvarieties of $Y$.

$U$ and $V$ intersect cleanly if $U \neq V$ and their scheme-theoretic intersection is nonsingular and the tangent bundles satisfy $T(U \cap V) = T U \cap T V$.

**Remark 2.2.** If the intersection is transversal, then it is a clean intersection.

**Definition 2.3** (of arrangement). A simple arrangement of subvarieties of $Y$ is a finite set $S = \{S_i\}$ of nonsingular closed irreducible subvarieties of $Y$ satisfying the following conditions

1. $S_i$ and $S_j$ intersect cleanly,
2. $S_i \cap S_j$ is either empty or some $S_k$’s.

**Definition 2.4** (of building set). Let $S$ be an arrangement of subvarieties of $Y$. A subset $G \subset S$ is called a building set with respect to $S$, if, for any $S \in S$, the minimal elements in $G$ which contain $S$ intersect transversally and their intersection is $S$. These minimal elements are called the $G$-factors of $S$.

**Definition 2.5** (of $G$-nest). A subset $T \subset G$ is called a $G$-nest if there is a flag of elements in $S$: $S_1 \subset S_2 \subset \cdots \subset S_k$ such that

$$T = \bigcup_{i=1}^{k} \{ A : A \text{ is a } G\text{-factor of } S_i \}.$$

2.2. Construction of $Y_G$ by a sequence of blow-ups. Let $Y$ be a nonsingular algebraic variety, $S$ be a simple arrangement of subvarieties and $G$ be a building set with respect to $S$. Order $G = \{G_1, \ldots, G_N\}$ such that $i < j$ if $G_i \subset G_j$.

We define $(Y_k, S^{(k)}, G^{(k)})$ inductively, where $Y_k$ is a blow-up of $Y_{k-1}$ along a nonsingular variety, $S^{(k)}$ is a simple arrangement of subvarieties of $Y_k$ and $G^{(k)}$ is a building set with respect to $S^{(k)}$.

**Definition/Theorem 2.6.** Assume $S$ is a simple arrangement of subvarieties of $Y$ and $G$ is a building set. Let $G$ be a minimal element in $G$ and consider $\pi : \tilde{Y} := \text{Bl}_G Y \to Y$. Denote the exceptional divisor by $E$. For any nonsingular variety $V$ in $Y$, we define $\tilde{V} \subset \text{Bl}_G Y$, the $\sim$ transform of $V$, to be the proper transform of $V$ if $V \not\subset G$, and to be $\pi^{-1}(V)$ if $V \subset G$.

For simplicity of notation, for a sequence of blow-ups, we use the same notation $\tilde{V}$ to denote the iterated one.
(1) The collection $\mathcal{S}'$ of subvarieties in $\tilde{Y}$ defined by

$$\mathcal{S}' := \{\tilde{S}\}_{S \in \mathcal{S}} \cup \{\tilde{S} \cap E\}_{\emptyset \subseteq S \subseteq G \subseteq \mathcal{S}}$$

is a simple arrangement in $\tilde{Y}$.

(2) $\mathcal{G}' := \{\tilde{G}_i\}_{G_i \in \mathcal{G}}$ is a building set with respect to $\mathcal{S}'$.

(3) Given a subset $\mathcal{T}$ of $\mathcal{G}$. Define $\mathcal{T}' := \{\tilde{A}\}_{A \in \mathcal{T}} \subseteq \mathcal{G}'$. $\mathcal{T}$ is a $G$-nest if and only if $\mathcal{T}'$ is a $G'$-nest.

Let’s go back to the construction of $Y_G$.

(1) For $k = 0$, $Y_0 = Y$, $\mathcal{S}^{(0)} = \mathcal{S}$, $\mathcal{G}^{(0)} = \mathcal{G} = \{G_1, \cdots, G_N\}$, $G_i^{(0)} = G_i$.

(2) Assume $Y_{k-1}$ is already constructed. Let $Y_k$ be the blow-up of $Y_{k-1}$ along the nonsingular subvariety $G_{k-1}^{(k-1)}$. Define $G_i^{(k)} := G_i^{(k-1)}$. Since $G_i^{(k-1)}$ for $i < k$ are all divisors, $G_k^{(k-1)}$ is minimal in $\mathcal{G}^{(k-1)}$. Thus there is a naturally induced arrangement $\mathcal{S}^{(k)}$ and a building set $\mathcal{G}^{(k)}$ by the theorem 2.6.

(3) Continue the inductive construction to $k = N$, where all elements in the building set $\mathcal{G}^{(N)}$ are divisors.

**Theorem 2.7.** Denote $Y^\circ = Y \setminus \cup_{G \in \mathcal{G}} G$. There is a natural locally closed embedding

$$Y^\circ \hookrightarrow Y \times \prod_{G \in \mathcal{G}} \text{Bl}_GY,$$

and its closure is denoted by $Y_G$ and called the wonderful compactification of $\mathcal{G}$. Then $Y_G$ is isomorphic to $Y_N$ which is constructed in the above.

The variety $Y_G$ is nonsingular. For each $G \in \mathcal{G}$, there is a nonsingular divisors $D_G \subset Y_G$ such that

(1) The union of these divisors is $Y_G \setminus Y^\circ$.

(2) Any set of these divisors meets transversally. An intersection of divisors $D_{T_1} \cap \cdots D_{T_i}$ is not empty exactly when $\{T_1, \cdots T_i\}$ form a $G$-nest.

**Theorem 2.8** (order of blow-ups). (1) Let $\mathcal{I}_i$ be the ideal sheaf of $G_i \in \mathcal{G}$. Then

$$Y_G \cong \text{Bl}_{\mathcal{I}_i \cdots \mathcal{I}_N}Y.$$

(2) If we arrange $\mathcal{G} = \{G_1, \cdots G_N\}$ in such an order that

(*) for any $1 \leq i \leq N$, the first $i$ terms $G_1, \cdots G_i$ form a building set

Then

$$Y_G \cong \text{Bl}_{G_N} \cdots \text{Bl}_{G_2} \text{Bl}_{G_1}Y.$$
2.3. **Chow group and motive of $Y_G$.** Let $Y_0 := Y, Y_0^T := \cap_{T \in T} T$ where $T$ is a $G$-nest. Define $r_T(G) := \dim(\cap_{G \subseteq T} T)$ (here we use a convention that $\cap_{G \subseteq T} T = Y$ if no $T$ strictly contains $G$). Then define $M_T := \{ \bar{\mu} \in M_T : 1 \leq \mu_G \leq r_T(G) - 1 \}$ and let $\| \bar{\mu} \| := \sum_{G \in G} \mu_G$ for $\bar{\mu} \in M_T$.

**Theorem 2.9.** We have the Chow group decomposition $$A^*(Y_G) = A^*(Y) \oplus \bigoplus_T \bigoplus_{\bar{\mu} \in M_T} A^*(\| \bar{\mu} \|) (Y_0^T)$$ where $T$ runs through all $G$-nests.

If $Y$ is complete, we also have the Chow motive decomposition $$h(Y_G) = h(Y) \oplus \bigoplus_T \bigoplus_{\bar{\mu} \in M_T} h(Y_0^T)(\| \bar{\mu} \|)$$ where $T$ runs through all $G$-nests.

2.4. **Chow ring of $Y_S$.** In this section, we will review the result of Chow rings after blow-up \[3\] and the result of Hu \[2\] concerning the Chow ring of $Y_S$.

**Definition 2.10** (of Lefschetz embedding). An embedding $U \hookrightarrow Y$ is called a Lefschetz embedding if the restriction map $A^*(Y) \rightarrow A^*(U)$ is surjective. Under this situation, let $J_{U/Y}$ be the kernel of $A^*(Y) \rightarrow A^*(U)$ and let $P_{U/Y}$ be the Chern polynomial for the normal bundle $N_{U/Y}$.

**Definition 2.11.** A Chern polynomial $P_{U/Y}(t)$ for a Lefschetz embedding $U \hookrightarrow Y$ is a polynomial $$P_{U/Y}(t) = t^d + a_1 t^{d-1} + \cdots + a_{d-1} t + a_d \in A^*(Y)[t],$$ where $d$ is the codimension of $U$ in $Y$ and $a_i \in A^i(Y)$ is a class whose restriction in $A^i(U)$ is the Chern class $c_i(N_{U/Y})$.

**Lemma 2.12.**
1. If $D$ is a divisor, then $P_{D/Y}(t) = t + D$.
2. If $V_1, V_2 \subset Y$ are subvarieties meeting transversally and their intersection is $Z$, then $$P_{Z/Y}(t) = P_{V_1/Y}(t) \cdot P_{V_2/Y}(t)$$

**Lemma 2.13.** Let $U$ and $V$ are non-singular closed subvarieties of $Y$ meeting cleanly in a non-singular closed subvariety $Z$. We also assume that both embeddings $U \hookrightarrow Y$ and $V \hookrightarrow Y$ are Lefschetz. Then all the relevant inclusions below are Lefschetz and
Lemma 2.14. Let \( \{ U_i \} \) be disjoint non-singular closed subvarieties of a smooth variety \( Y \), such that \( U_i \hookrightarrow Y \) are Lefschetz. Then the Chow ring \( A^*(\text{Bl}_i U_i Y) \) is isomorphic to the polynomial ring \( A^*(Y)[x_i] \), where \( x_i \) corresponds to the exceptional divisor \( \tilde{U}_i \), modulo the ideal generated by

1. \( x_i \cdot x_j \) for \( i \neq j \),
2. \( J_{U_i/Y} \cdot x_i \) for all \( i \),
3. \( P_{U_i/Y}(-x_i) \) for all \( i \).

Definition 2.15. A regular simple arrangement \( S \) is a simple arrangement such that for any \( S_l \subset S_i \), there is \( S_j \supset S_l \) such that \( S_l = S_i \cap S_j \).

Theorem 2.16. Let \( S \) be a regular simple arrangement of subvarieties such that all the inclusions \( S_i \subset S_j \) and \( S_i \subset Y \) are Lefschetz embedding. Then the Chow ring of \( Y_S \) is isomorphic to the polynomial ring \( A^*(Y)[x_{S_i}, \ldots, x_{S_N}] \) (where \( x_{S_i} \) corresponds to the exceptional divisor \( S_i^{(i+1)} \)) modulo the ideal generated by

1. \( x_{S_i} \cdot x_{S_j} \) for incomparable \( S_i, S_j \),
2. \( J_{S_i/Y} \cdot x_{S_i} \) for all \( i \),
3. \( P_{S_i/Y}(-\sum_{S_j \subset S_i} x_{S_j}) \) for all \( i \).

3. Construction of \( X_D^{[n]} \) and \( X_D[n] \)

Fix a nonsingular divisor \( D \) of an algebraic variety \( X \) of dimension \( m \). In this section, we review construction of a compactification of configuration spaces of \( n \) point in \( X \setminus D \), \( X_D^{[n]} \), and a compactification of configuration spaces of \( n \) distinct point in \( X \setminus D \), \( X_D[n] \). In this paper, we assume that \( D \) is a divisor but every thing will work in the case of \( D \) is a smooth subvariety after some adjustment. See the details in [4].

3.1. Construction. For a subset \( S \) of \( N := \{ 1, 2, \ldots, n \} \) define a nonsingular subvariety in \( X^n \)

\[ D_S := \{ x \in X^n \mid x_i \in D, \forall i \in S \}. \]

Let \( \mathcal{A} \) be the collection of \( D_S \) for all \( S \subset N := \{ 1, \ldots, n \} \) with \( |S| \geq 2 \). It is clear that the collection is a simple arrangement of smooth
subvarieties of $X^n$ and take a building set $\mathcal{G} = \mathcal{A}$. Then define $X_D[n]$ to be the closure of $X^n \setminus \bigcup S D_S$ in

$$X^n \times \prod_S \text{Bl}_{D_S} X^n$$

It can be constructed by a successive blowups by theorem 2.7. In particular we may order $\mathcal{G}$ as $D_{12}, D_{123}; D_{13}, D_{23};..., D_{12...n}; D_{U \cup \{n\}}$ with $|U| = n - 2$ and $U \subset N \setminus \{n\};...; D_m$ for $i = 1, ..., n - 1$ by theorem 2.8.

**Lemma 3.1.** Let $I_1$ and $J_2$ be partitions of $N$. The intersection of proper transforms of $\Delta_{I_1}$ and $\Delta_{I_2}$ is the proper transform of the intersection $\Delta_{I_1 \wedge I_2}$.

**Corollary 3.2.** For $I \subset N$ with $|I| \geq 2$, $\Delta_I(X_D[n])$ form a building set of nonsingular subvarieties of $X_D[n]$ with respect to the set of all polydiagonals.

**Definition 3.3.** Define $X_D[n]$ to be the closure of $X_D[n] \setminus \bigcup_{|I| \geq 2} \Delta_I(X_D[n])$

$$X_D[n] \times \prod_{|I| \geq 2} \text{Bl}_{\Delta_I(X_D[n])} X_D[n]$$

**Theorem 3.4.**

1. $X_D[n]$ is a nonsingular variety. There is a natural projection from $X_D[N]$ to $X_D[I]$ for any subset $I$ of $N$.
2. There is a natural $S_n$-action on $X_D[n]$.
3. The boundary is the union of divisors $\widetilde{D}_S$ with $|S| \geq 1$, and $\widetilde{\Delta}_I$ with $|I| \geq 2$ of normal crossings.
4. The intersections of boundary divisors are nonempty if and only if they are nested. Here $\{D_S, \Delta_I\}$ is nested if each pair $S_i$ and $S_k$ ($T_j$ and $T_l$) is either disjoint or one is contained in the other and each pair $S_i$ and $T_j$ is either disjoint or $T_k$ is contained in $S_i$.
5. We may take order $D_S$; $\Delta_I$ for $n \notin S, I$; and then $D_T$ with $n \in T$, then $\Delta_J$ with $n \in J$.

4. CHOW GROUPS AND MOTIVES

In this section, we will apply theorem 2.9 to $X_D[n]$ and $X_D[n]$. For simplicity, we assume that $X$ is complete.

4.1. CHOW GROUP AND MOTIVE OF $X_D[n]$. In this case, our $Y = X^n, S = \mathcal{G} = \{D_S : S \subset N \text{with} |S| \geq 2\}$ where $D_S = \{x \in X^n \mid x_i \in D, \forall i \in S\}$. We have $S = \mathcal{G}$, so a $\mathcal{G}$-nest is just a chain of elements in $S$, $T = \{D_{S_1} \subset D_{S_2} \subset \cdots \subset D_{S_k}\}$. Thus $Y_0 T = D_{S_1}$.
A chain $\mathcal{CH}$ is a chain of subset of $N$, $S_k \subsetneq \cdots \subsetneq S_2 \subsetneq S_1$, such that $S_k$ is not a singleton. Obviously, there is one-to one correspondence between a set of chains of $S$ and a set of chains of $N$. We say $\emptyset$ is also a chain. We define $\max_{\mathcal{CH}(T)} S$ as the maximal element of $\mathcal{CH}(T)$ which is strictly contained in $S$, where $\mathcal{CH}(T)$ is the chain of $N$ which corresponds to $T$. If there is no such element, then we define $\max_{\mathcal{CH}(T)} S = \emptyset$.

Now let $G = D_S$ and let’s compute $r_T(G)$:

$$r_T(G) = \dim(\bigcap_{G \subseteq T \in T} T) - \dim G$$
$$= \dim(D_{\max_{\mathcal{CH}(T)} S}) - \dim D_S$$
$$= |S| - |\max_{\mathcal{CH}(T)} S|.$$

Remark 4.1 (When $D$ is not a divisor). When $D$ is not a divisor, then we also blow up $D_{\{i\}}$. So we will not exclude the case such that $S_k$ is a singleton for $\{S_k \subsetneq \cdots \subsetneq S_2 \subsetneq S_1\}$. $r_T(G)$ will be also changed, it will be multiplied by the codimension of $D$ in $X$.

For a chain $\mathcal{CH}(\neq \emptyset)$, define

$$M_{\mathcal{CH}} := \{\overrightarrow{\mu} = \{\mu_S\}_{S \in \mathcal{CH}} : 1 \leq \mu_S \leq |S| - |\max_{\mathcal{CH}} S| - 1\}.$$

For $\mathcal{CH} = \emptyset$, define $M_{\mathcal{CH}}$ is consist of one $\overrightarrow{\mu}$ with $\|\overrightarrow{\mu}\| = 0$ and $D_{\emptyset} = X^n$.

**Theorem 4.2.** Let $X$ be a complete nonsingular variety. Then we have the Chow group and motive decompositions

$$A^*(X_D[n]) = \bigoplus_{\mathcal{CH}} \bigoplus_{\overrightarrow{\mu} \in M_{\mathcal{CH}}} A^* - \|\overrightarrow{\mu}\|(D_{S_{\mathcal{CH}}}),$$
$$h(X_D[n]) = \bigoplus_{\mathcal{CH}} \bigoplus_{\overrightarrow{\mu} \in M_{\mathcal{CH}}} h(D_{S_{\mathcal{CH}}})(\|\overrightarrow{\mu}\|),$$

where $\mathcal{CH}$ runs through all the chains of $N$ and $S_{\mathcal{CH}}$ is the maximal element in $\mathcal{CH}$.

**4.2. Chow group and motif of $X_D[n]$.** We use the same notation as [6].

(1) We call two subsets $I, J \subset N$ are overlapped if $I \cap J$ is not a nonempty proper subset of both $I$ and $J$. For a set $\mathcal{N}$ of subsets of $N$, we call $I$ is compatible with $\mathcal{N}$, denoted by $I \sim \mathcal{N}$, if $I$ does not overlap any elements of $\mathcal{N}$.

A nest $\mathcal{N}$ is a set of subset of $N$ such that any pair $I \neq J \in \mathcal{N}$ are not overlapped and contains all singletons.
For a given nest $N$, define $N^\circ := N \setminus \{\{1\}, \ldots, \{n\}\}$.

A nest $N$ naturally corresponds to a tree (which may not be connected) with each node labeled by an element of $N$. Let $c(N)$ be the number of connected components of the forest which corresponds to $N$. Denote by $c_I(N)$ the number of maximal elements of the set $\{J \in N : J \subseteq I\}$, which is called the number of sons of the node $I$.

Let $\Delta_N := \cap_{I \in N} \Delta_I(X^{|I^c|})$ in this section.

(2) For a nest $N (\neq \{\{1\}, \ldots, \{n\}\})$, define

$$M_N := \{\mu = \{\mu_I\}_{I \in N} : 1 \leq \mu_I \leq m(c_I - 1) - 1\}$$

where $m = \dim X$.

For $N = \{\{1\}, \ldots, \{n\}\}$, define $M_N = \{\mu\}$ with $\|\mu\| = 0$.

As in [6], we have

**Proposition 4.3.** We have the Chow group and motive decompositions

$$A^\ast(X_D[n]) = \bigoplus_{N} \bigoplus_{\mu \in M_N} A^{\ast-\|\mu\|}(\Delta_N),$$

$$h(X_D[n]) = \bigoplus_{N} \bigoplus_{\mu \in M_N} h(\Delta_N)(\|\mu\|),$$

where $N$ runs through all the nest of $N$.

Now we need to simplify $A^\ast(\Delta_N)$ and $h(\Delta_N)$.

**Lemma 4.4.** $D_S$ and $\Delta_I$ intersect cleanly.

*Proof.* We only need to prove that $TD_S \cap T\Delta_I \subset T(D_S \cap \Delta_I)$. An arc in $\Delta_I$ have a coordinate representative $(x_i) \in X^n$ such that $x_i = x_j$ for $i, j \in I$. For an arc in $\Delta_I$ to be an arc in $D_S$, $x_i \in D$ for all $i \in S$. Thus the arc should be an arc in $D_S \cap \Delta_I$. \qed

**Proposition 4.5.** $\Delta_I$ is isomorphic to $X^{[|I^c|]+1}$.

*Proof.* We need to know which blow ups of $D_S$ have an effect to $\Delta_I$ in a specific order of blow ups. We can assume that $I = \{1, \ldots, n\}$ by arranging the order and denote $a = |I^c|$ and $b = |I|$. We will denote $\Delta_I$ by $X^a \times \Delta(\cong X^{[I^c]|+1})$. Then we have two different kinds of $D_S$. The first one is that $S \subseteq I^c$, which we call the first kind, the second one is that $S \nsubseteq I^c$, which we call the second kind. We will change the order of blow ups so that we first blow up along $D_S$ of the first kind, and then along the second kind. More precisely, we order $D_{I^c} \times X^b, D_{I^c,1} \times X^b, \ldots, D_{I^c,1,\ldots,a} \times X^b, \ldots, D_{I^c,1,\ldots,a} \times X^b(\sum, j \in \{1, \ldots, a\})$ and then $D_{I^c} \times X^b, \ldots, D_{|S'|} \times X^b, \ldots, (|S''| > 0$ and $(|S'|, |S''|) :
non-increasing in lexicographical order \(\). This order satisfies (\(\ast\))-condition in definition/theorem 2.6, so that we can blow up in this order.

In this order of blow ups, notice that \(X^a \times \Delta\) and \(D_{S'} \times D_{S''}\) for \(S'' \subset I\) are separated when we blow up along \(D_{S'} \times D^b\). Thus we can forget the process of blow ups by \(D_{S'} \times D_{S''}\), where we only need to care about \(D_{S'} \times D^b\) for the second kind. Under the isomorphism \(X^a \times \Delta \cong X^{a+1}\), they are just \(D_{S'} \times D^b\).

□

We can also apply the same technique to polydiagonals term by term. Thus we can go further from proposition 4.3.

**Theorem 4.6.** We have the Chow group and motive decompositions

\[
A^*(X_D[n]) = \bigoplus_{N} \bigoplus_{\mu \in M_N} \bigoplus_{\chi \in M_{\text{CH}}} A^{*-\|\mu\|-\|\chi\|}(D_{S_{\text{CH}}}),
\]

\[
h(X_D[n]) = \bigoplus_{N} \bigoplus_{\mu \in M_N} \bigoplus_{\chi \in M_{\text{CH}}} h(D_{S_{\text{CH}}})(\|\mu\| + \|\chi\|),
\]

where \(N\) runs through all the nest of \(N\) and \(\text{CH}\) runs through all the chains of \(c(N)\).

5. **Chow rings**

In this section we assume that \(X\) has a cellular decomposition and \(D\) is a smooth divisor of \(X\) such that \(D \hookrightarrow X\) is a Lefshetz embedding. The reason we assume these conditions is that we need a Kunnneth decomposition and S. Keel’s formula for intersection ring of blow-up.

5.1. **Chow ring of** \(X_D^n\). Note that \(D_S \hookrightarrow D_{S'}\) for \(S \supset S'\) and \(D_S \hookrightarrow X^n\) are Lefshetz embedding.

Obviously, the arrangement \(\mathcal{A}\) is regular, so we can apply theorem 2.16.

**Theorem 5.1.** The Chow ring \(A^*(X_D^n)\) is isomorphic to the polynomial ring \(A^*(X^n)[x_S]\) modulo the ideal generated by

1. \(x_S \cdot x_T\) for \(S, T\) that overlap,
2. \(J_{D_S/X^n} \cdot x_S\) for all \(S\),
3. \(P_{D_S/X^n}(-\sum_{S' \supset S} x_{S'})\) for all \(S\).
5.2. Chow ring of $X_D[n]$. We will compute the Chow ring of $X_D[n]$ from $X_D^\natural$ by a sequence of blow ups along, which is same as \[1\],

$$\Delta_{(1,2)} \Delta_{(1,2,3)} \Delta_{(1,3)} \Delta_{(2,3)} \cdots \Delta_{(1\cdots n)} \cdots \Delta_{(n-1,n)}.$$

Let

$$Y^{[i]}_i \to \cdots \to Y^{[i]}_{k+1} \to Y^{[i]}_k \to \cdots \to Y^{[i]}_0$$

be a part of the above sequence of blow-ups along

$$\Delta_{(1\cdots i+1)} \cdots \Delta_{(1\cdots i-k-1,i+1)} \cdots \Delta_{(k-1,i,i+1)} \cdots \Delta_{(1,i+1)} \cdots \Delta_{(i,i+1)}.$$

Note $1 \leq i \leq n-1$.

We will compute Chow rings of $Y^{[i]}_k$’s inductively by using theorem 2.14.

**Lemma 5.2.** If $I'$ and $J'$ are subsets of $\{1, \cdots, i, i+1\}$ that overlap, then $\Delta_{I'}$ and $\Delta_{J'}$ are disjoint at $Y^{[i]}_k$, except, up to the order of $I'$ and $J'$, in exactly the following cases:

1. $I' = I \subset \{1, \cdots, i\}, |I| \leq i-k$, $J' = J^+$, with $J \subset I$,
2. $I' = I^+, J' = J^+$, with $I \cap J = \emptyset, |I \cup J| \leq i-k$.

**Proof.** We change the order of blow ups in the following way;

$$D_S, \Delta_I; D_{S^+}, \Delta_{I^+}; D_{S^{++}},$$

where $S, I \subset \{1, \cdots, i\}, |I^+| \leq i-k+2$ and $S^{++} \not\subset \{1, \cdots, i, i+1\}$. After blowing up along $D_{S^+}, \Delta_I$, the space is $X_D[i] \times X^{(n-i)}$. If $I', J' \subset \{1, \cdots, i\}$, then $\Delta_{I'}$ and $\Delta_{J'}$ are disjoint by theorem 3.4.

For $I' = I^+, J' = J^+$, $\Delta_{I^+}$ is a product of the graph of $p_a : \Delta_I \to X$ and $X^{n-i-1}$ where $a \in I$ and we use a convention $\Delta_a = X^n$. Same for $\Delta_{J^+}$. To have non-empty intersection, $I$ and $J$ must be nested by theorem 3.4. But we have an assumption that $I^+$ and $J^+$ overlap, so that $I$ and $J$ must be disjoint. $\Delta_{I^+}$ and $\Delta_{J^+}$ will be separted after blowing up along $\Delta_{(I, J^+)}$

Now let’s move to the case that $I' = I \subset \{1, \cdots, i\}$ and $J' = J^+$. In this case, $\Delta_I = \Delta_I \times X^{n-i}$. To have non-empty intersectin, $I$ and $J$ are nested, i.e. $J \subset I$ or $I \subset J$. But the latter case $I' \subset J'$, which contradict to the assumption of overlapping. Thus $J \subset I$. $\Delta_I$ and $\Delta_{J^+}$ will be separted after blowing up along $\Delta_{I^+}$.

Note that $D_S$ and $\Delta_I$ are intersting cleanly and its intersection is a proper subset of $\Delta_I$. \[\square\]
Lemma 5.3. For $a \in I \subset \{1, \ldots, i\}$ such that $2 \leq |I| \leq i - k$, then at $Y_k^{[i]}$,

$$\widetilde{\Delta}_I^+ = \widetilde{\Delta}_I \cap \widetilde{\Delta}_a^+.$$  

Proof. Proof is very similar to proposition 3.1.

Lemma 5.4. If $\widetilde{\Delta}_{I'}$ is a divisor in $Y_k^{[i]}$, then the inverse image $\pi^*(\widetilde{\Delta}_{I'})$ in $Y_k^{[i]}$ is the divisor $\widetilde{\Delta}_{I}$, except cases such that $I' = J \subset \{1, \ldots, i\}$ with $|J| = i - k$ and in that case

$$\pi^*(\widetilde{\Delta}_{J}) = \widetilde{\Delta}_{J} + \widetilde{\Delta}_J^+.$$  

Proof. For the case described in the statement, by lemma 5.3, the statement is true. For other cases, it is obvious that the divisor $\widetilde{\Delta}_{I'}$ does not contain any blow up center by considering the space $X_D^I \times X^{(n-i)}$.

For $a \in N$, let $p_a$ be the corresponding projection from $X^n$ to $X$, and for $a, b \in N$ (distinct), let $p_{a,b}$ be the projection from $X^n$ to $X^{a,b}$. Let $[\Delta] \in A^m(X^{a,b})$ be the class of the diagonal, where $m = \dim X$. Define a polynomial $c_{a,b}(t) \in A^*(X^n)[t]$ by

$$c_{a,b}(t) = \sum_{i=1}^{m} (-1)^i p_a^*(c_{m-i}) t^i + [\Delta_{(a,b)}]$$

where $c_{m-i}$ is the $(m-i)$-th Chern class of $X$ and $[\Delta_{(a,b)}] = p_{a,b}^*([\Delta])$.

Let’s compute Chern polynomials and Lefshetz kernels of $\Delta$’s at the stage of $Y_0^{[i]}$.

Lemma 5.5.  

1. $J_{\Delta_{I}(Y_0^{[i]})/Y_0^{[i]}} = (J_{\Delta_I/X^n}, x_S)$ where $S \supsetneq I$.

2. $P_{\Delta_{I}(Y_0^{[i]})/Y_0^{[i]}}(t) = P_{\Delta_I/X^n}(t)$.

Proof.  

1. By the proof of proposition 4.5 we know that $\widetilde{D}_S$ for $S \supsetneq I$ is disjoint from $\widetilde{\Delta}_I$, and others intersect cleanly and non-trivially. By lemma 2.13 we have the statement.

2. We know that $\Delta$ is intersecting with $\widetilde{D}_S$ cleanly including the cases disjoint by the proof of proposition 4.5. By lemma 2.13 we know that a Chern polynomial will not be changed.

Proposition 5.6.  

1. For $a \in \{1, \ldots, i\}$, let $0 \leq k \leq i - 1$, a Chern polynomial of $\widetilde{\Delta}_{a, i + 1}$ at $Y_k^{[i]}$ is

$$c_{a,i+1}(-t + \sum_{a,i+1 \in I'} D_k I').$$
(2) For $I \subset \{1, \cdots, i\}, 2 \leq |I| \leq i - k$, a Chern polynomial of $\Delta_I$ at $Y_k[i]$ is 
\[(t + D_kI) \cdot c_{a,i+1}(-t + \sum_{I' \subset I'} D_kI')\]
for any $a \in I$.
Here $D_kI$ is the divisor of $Y_k[i]$ corresponding to $\Delta_I$.

Proof. Exactly same as [1].

Proposition 5.7. Let $I' = I^+ \subseteq \{1, \cdots, i, i + 1\}$ such that $|I'| = i - k + 1$. Then the restriction $\widetilde{\Delta}_{I'} \to Y_k[i]$ is Lefschetz embedding, and its Lefschetz kernel is generated by
\[(1) \ D_kJ' \text{ for any } J' \subseteq \{1, \cdots, i, i + 1\} \text{ that overlaps with } I', \text{ except if } I \subset J' \subseteq \{1, \cdots, i, i + 1\}.
(2) \ J_{\Delta_{I'}/X^n}.
(3) \ x_S \text{ for } S \notin I'.\]

Proof. By lemma 2.13, $\widetilde{\Delta}_{I'} \to Y_k[i]$ is Lefschetz embedding.

Now let’s prove the statement for generators. By lemma 2.13, we have to show that, for $J'$ which overlap with $I'$, those exceptional cases are exactly blow up centers which intersect $\Delta_{I'}$ with non-empty intersection. The order of blow ups does not matter to the statement, so that we can change the order as we want.

First consider a case that $I' \cap J' \neq \emptyset$. We can assume $i + 1 \in I' \cap J'$ by changing numbering. In this case, by lemma 5.2 we know exactly when the intersection is non-empty or not.

Now, consider a case that $I' \cap J' = \emptyset$. We can assume that $J' = \{1, \cdots, j\}$ and $I' \subset \{j + 1, \cdots, j + i + 1\}$. Then by the inductive construction of $X_D[n]$, it is obvious they intersect.
\( J_{D_S/X^n} \cdot x_S \) for all \( S \),
\( P_{D_S/X^n}(-\Sigma_{S \supset S} x_{S'}) \) for all \( S \).

**Proof.** For \( Y_0^{[1]} \), it is just theorem 5.1. Also note that \( Y_0^{[i]} = Y_0^{[i+1]} \)
and the statement for \( Y_i^{[i]} \) will imply \( Y_0^{[i+1]} \) because condition (4b) is
vacuous when \( k = 0 \).

We only need to prove that the statement for \( Y_k^{[i]} \) will imply the one
for \( Y_{k+1}^{[i]} \). The conditions (5) to (7) are coming from blow up along \( D_S \)
and these are not new.

For (1), proof is exactly same as [1].
(1), (2), and (3) follow from proposition 5.7.

\( \square \)

Especially, we have

**Theorem 5.9.** The Chow ring \( A^*(X_D[n]) \) is isomorphic to the polynomial ring \( A^*(X^n)[x_S, y_I] \) modulo the ideal generated by

1. \( y_I \cdot y_J \) for \( I \) and \( J \) that overlap,
2. \( x_S \cdot y_T \) for \( S \) and \( T \) that overlap,
3. \( x_S \cdot y_I \) unless \( I \subset S \),
4. \( J_{\Delta_I/X^n} \cdot y_I \) for all \( I \),
5. \( J_{D_S/X^n} \cdot x_S \) for all \( S \),
6. \( c_{a,b}(\sum_{a,b \in I} y_I) \) for \( a, b \in \{1, \ldots, n\} \) (distinct),
7. \( P_{D_S/X^n}(-\Sigma_{S \supset S} x_{S'}) \) for all \( S \).

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