SHARP EIGENVALUE BOUNDS ON QUANTUM STAR GRAPHS

SEMRA DEMIREL–FRANK

Abstract. We prove that the optimal constant in the Lieb–Thirring inequality on a star graph with $N$ edges coincides with that on $\mathbb{R}$ if $N$ is even. For odd $N$ we show that this property holds when restricting to radial potentials and we prove an almost optimal bound for general potentials.

1. Introduction

Recently there has been a lot of activity in a mathematical understanding of quantum graphs, which appear as idealized models of linear, network-shaped structures in mesoscopic physics. A large literature on the subject has arisen and we refer, for instance, to the bibliography given in [15] and the textbook [2]. In particular, in the papers [3,4,6,7] bounds we derived on the discrete eigenvalues of Schrödinger operators on metric graphs. In the present paper we will be interested in optimal constants in such bounds for one of the simplest classes of metric graphs, namely star graphs. By $\Gamma_N$ we denote $N$ half-lines $[0, \infty)$ with their endpoints 0 identified. Thus, $\Gamma_N$ is a graph with a single vertex and $N$ edges.

We consider the Schrödinger operator

$$H = -\frac{d^2}{dx^2} + V$$

in $L_2(\Gamma_N)$ with a potential $V : \Gamma_N \to \mathbb{R}$. It is well-known that, if $V_- \in L_p(\Gamma_N)$ for some $p \geq 1$ and $V_+ \in L_{loc}^1(\Gamma_N)$, then the Schrödinger operator can be defined as a self-adjoint operator in $L_2(\Gamma_N)$ via the lower semi-bounded and closed quadratic form

$$h[\psi] := \int_{\Gamma_N} (|\psi'|^2 + V|\psi|^2) \, dx, \quad \psi \in H^1(\Gamma_N) \cap L_2(\Gamma_N, V_+ \, dx).$$

By definition, a function $\psi$ on $\Gamma_N$ belongs to the Sobolev space $H^1(\Gamma_N)$ if its $N$ restrictions $\psi_1, \ldots, \psi_N$ to the edges of $\Gamma_N$ belong to $H^1(0, \infty)$ and if their values at the vertex coincide. This definition of the Schrödinger operator via quadratic forms gives rise, in a generalized sense, to the so-called Kirchhoff boundary conditions at the vertex,

$$\sum_{j=1}^N \psi'_j(0+) = 0.$$

Moreover, the condition $V_- \in L_p(\Gamma_N)$ with $p < \infty$ guarantees that the negative spectrum of the Schrödinger operator consists of discrete eigenvalues of finite multiplicities.
As usual, we write $\text{Tr} H^\gamma$ for the sum of the $\gamma$-th power of the absolute values of the negative eigenvalues of $H$.

One can prove [4] that for any $\gamma \geq 1/2$ there is a constant $L_{\gamma,N}$ such that

$$\text{Tr} H^\gamma \leq L_{\gamma,N} \int_{\Gamma_N} V^{\gamma+1/2} \, dx.$$  \hspace{1cm} (1)

In the following, we will denote by $L_{\gamma,N}$ the optimal (that is, smallest possible) value of the constant in (1). We are interested in characterizing this value and, in particular, in relating it to $L_{\gamma,2} =: L_\gamma$ for $\Gamma_2 = \mathbb{R}$, that is, the optimal constant in the inequality

$$\text{Tr} \left( -\frac{d^2}{dx^2} + V \right)^\gamma \leq L_\gamma \int_{\mathbb{R}} V^{\gamma+1/2} \, dx.$$  \hspace{1cm} (2)

Finding the optimal constant in (2) is a famous open problem due to Lieb and Thirring [11]. What is currently known is that

$$L_{1/2} = 1/4 \quad \text{and} \quad L_\gamma = (4\pi)^{-1/2} \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+3/2)} \quad \text{if } \gamma \geq 3/2;$$  \hspace{1cm} (3)

see [8, 11] and also [9, 10] for a review and results in higher dimensions.

By taking a compactly supported almost-optimal potential for (2) and transplanting it very far out on a single edge of $\Gamma_N$ it is easy to see that

$$L_{\gamma,N} \geq L_\gamma \quad \text{for all } \gamma \geq 1/2 \text{ and all } N \in \mathbb{N}.$$  \hspace{1cm} (4)

Thus, in the following we will be interested in upper bounds on $L_{\gamma,N}$.

In [3] we have shown that

$$L_{\gamma,N} = L_\gamma \quad \text{for all } \gamma \geq 2 \text{ and all } N \in \mathbb{N}.$$  \hspace{1cm} (5)

In fact, this equality is valid for a large number of graphs, but, remarkably, not for all graphs; see [3] for an explicit counterexample. As far as we know, there are no optimal results on Lieb-Thirring constants on quantum graphs apart from (5). We emphasize that the proof in [3] proceeds by showing $L_{\gamma,N} \leq (4\pi)^{-1/2} \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+3/2)}$ directly, without comparing $L_{\gamma,N}$ to $L_\gamma$.

In this paper we shall do exactly the latter, namely, we find a comparison method to relate $L_{\gamma,N}$ to $L_\gamma$, without needing to know the explicit value of $L_\gamma$. This allows us to settle the problem completely for even $N$ as well as, under a symmetry assumption, for odd $N$. The following two theorems are our main results.

**Theorem 1.** Let $\gamma \geq 1/2$. If $N$ is even, then

$$L_{\gamma,N} = L_\gamma$$

and, if $N$ is odd, then

$$L_{\gamma,N} \leq \frac{N+1}{N} L_\gamma,$$

where $L_{\gamma,N}$ and $L_\gamma$ are the optimal constants in (1) and (2), respectively.
Remarks. (1) For even \(N\), this theorem together with (3) yields explicitly the optimal constant for \(\gamma = 1/2\) and \(\gamma \geq 3/2\). This improves our earlier bound from [3] for \(\gamma \geq 2\). We emphasize that none of the methods used to prove (3) seem to generalize in an obvious way to \(\Gamma_N\).

(2) A variant of our proof shows that if \(L_{\gamma,N_0} = L_{\gamma}\) for some odd \(N_0\), then \(L_{\gamma,N} = L_{\gamma}\) for all \(N \geq N_0\); see Proposition 4.

(3) For \(N = 1\), our bound states \(L_{\gamma,1} \leq 2L_{\gamma}\). The proof of Lemma 3 shows that this bound is optimal as long as the optimal potential for \(L_{\gamma}\) has a single bound state. This holds, in particular, for \(\gamma = 1/2\).

(4) For odd \(N \geq 3\) our bound uses the bound \(L_{\gamma,1} \leq 2L_{\gamma}\) for \(N = 1\). If the latter bound can be improved for some (large) \(\gamma\), then also our bounds for arbitrary odd \(N \geq 3\) improve automatically.

We call a function \(V\) on \(\Gamma_N\) radial if the value of \(V(x)\) depends only on the distance of \(x\) from the vertex of \(\Gamma_N\). Let us denote by \(L^{(\text{rad})}_{\gamma,N}\) the optimal constant in (1) when restricted to radial functions \(V\).

**Theorem 2.** Let \(\gamma \geq 1/2\). For any \(N \geq 2\),

\[
L^{(\text{rad})}_{\gamma,N} = L_{\gamma},
\]

where \(L^{(\text{rad})}_{\gamma,N}\) is the optimal constant in the radial version of (1) and \(L_{\gamma}\) is the optimal constant in (2).

We will prove Theorem 1 in Section 2 and Theorem 2 in Section 3.

**Acknowledgement.** The author is grateful to T. Weidl for drawing my attention to Lieb–Thirring inequalities on quantum graphs and helpful comments.

## 2. Proof of Theorem 1

We begin with the proof of Theorem 1 for \(N = 1\). This is the following bound on the eigenvalues of a half-line Schrödinger operator with Neumann boundary conditions. More precisely, this operator is defined via the quadratic form \(\int_{\mathbb{R}^+}(|\psi'|^2 + V|\psi|^2)\,dx\) in \(L^2(\mathbb{R}^+)\) with form domain \(H^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+,V_+\,dx)\).

**Lemma 3.** Let \(H^{(\text{Neu})} = -\frac{d^2}{dx^2} + V\) in \(L^2(\mathbb{R}^+)\) with Neumann boundary conditions. Then, for all \(\gamma \geq 1/2\),

\[
\text{Tr} \left( H^{(\text{Neu})} \right)^\gamma_+ \leq 2L_\gamma \int_{\mathbb{R}^+} V^{\gamma+1/2} dx.
\]

**Proof.** We extend \(V\) to a symmetric function \(\tilde{V}\) on \(\mathbb{R}\) and obtain, by the same arguments as in the proof of Theorem 2 below,

\[
\text{Tr} \left( H^{(\text{Neu})} \right)^\gamma_+ + \text{Tr} \left( H^{(\text{Dir})} \right)^\gamma_+ = \text{Tr} \left( H^\mathbb{R} \right)^\gamma_+,
\]
where $H^{(\text{Dir})}$ is the same as $H^{(\text{Neu})}$ but with Dirichlet boundary conditions and $H^{\mathbb{R}}$ is the operator $-\frac{d^2}{dx^2} + \tilde{V}$ in $L_2(\mathbb{R})$. The claimed bound follows from the inequalities $\text{Tr} (H^{(\text{Dir})}\gamma) \geq 0$ and (2), that is,

$$\text{Tr} (H^{\mathbb{R}}\gamma) \leq L_\gamma \int_{\mathbb{R}} \tilde{V}^{\gamma+1/2} \, dx = 2L_\gamma \int_{\mathbb{R}^+} \tilde{V}^{\gamma+1/2} \, dx. \quad \square$$

We now turn to star graphs $\Gamma_N$ with $N \geq 3$. Lower bounds on the eigenvalues can be obtained by decoupling the edges. If we would decouple all the edges, we would end up with $N$ half-line Schrödinger operators with Neumann boundary conditions. Applying Lemma 3 to each of these operators we would obtain the bound $L_{\gamma,N} \leq 2L_\gamma$, which is not optimal. The idea in the following proof is to apply a more subtle decoupling.

**Proof of Theorem 4.** Case $N$ even. We write $N = 2n$ and consider the quadratic form $h^{(\text{cut})}[\psi]$, given by the same expression as $h[\psi]$, but with form domain

$$\{ \psi \in L_2(\Gamma_N) : \forall \, 1 \leq j \leq N : \psi_j \in H^1(\mathbb{R}^+) \text{ and } \forall \, 1 \leq j \leq n : \psi_j(0) = \psi_{j+N}(0)\}. \quad (6)$$

In other words, we decompose $\Gamma_N$ into $n$ copies of $\mathbb{R}$, namely, $e_1 \cup e_{n+1}, \ldots, e_n \cup e_N$, where $e_1, \ldots, e_N$ are the edges of $\Gamma_N$. Since the form domain of $h^{(\text{cut})}$ contains that of $h$, the corresponding operator $H^{(\text{cut})}$ satisfies $H^{(\text{cut})} \leq H$ in the sense of quadratic forms, and therefore

$$\text{Tr} H^{\gamma} \leq \text{Tr}(H^{(\text{cut})}\gamma) \quad (6)$$

for any $\gamma$. Since for the operator $H^{(\text{cut})}$ each edge is only connected to one other edge, we have

$$H^{(\text{cut})} \sim \bigoplus_{i=1}^{n} H_i,$$

where $H_i$ is the Schrödinger operator in $L_2(\mathbb{R})$ with potential $\tilde{V}_i$ given for $t > 0$ by

$$\tilde{V}_i(t) = V_i(t), \quad \tilde{V}_i(-t) = V_{n+i}(t).$$

(Here, $V_i$ and $V_{n+i}$ denote the restrictions of $V$ to the $i$-th and $n+i$-th edge.) Thus,

$$\text{Tr}(H^{(\text{cut})}\gamma) = \sum_{i=1}^{n} \text{Tr}(H_i\gamma). \quad (7)$$

Finally, if $\gamma \geq 1/2$, we can use the Lieb–Thirring inequality (2) to bound

$$\text{Tr}(H_i\gamma) \leq L_\gamma \int_{\mathbb{R}} (\tilde{V}_i)^{\gamma+1/2} \, dt. \quad (8)$$

Combining (6), (7) and (8) we obtain for $\gamma \geq 1/2$

$$\text{Tr} H^{\gamma} \leq L_\gamma \sum_{i=1}^{n} \int_{\mathbb{R}} (\tilde{V}_i)^{\gamma+1/2} \, dt = L_\gamma \int_{\Gamma_N} V^{\gamma+1/2} \, dx,$$

as claimed.
Case $N$ odd. We shall show that for $\gamma \geq 1/2$,
\begin{equation}
\text{Tr } H_\gamma^\gamma \leq L_\gamma \int_{\Gamma_N} V_{\gamma + 1/2}^\gamma \, dx + L_\gamma \int_{\mathbb{R}_+} (V_N)^{\gamma + 1/2} \, dt. \tag{9}
\end{equation}

After relabelling the edges this yields
\begin{equation}
\text{Tr } H_\gamma^\gamma \leq L_\gamma \int_{\Gamma_N} V_{\gamma + 1/2}^\gamma \, dx + L_\gamma \int_{\mathbb{R}_+} (V_i)^{\gamma + 1/2} \, dt
\end{equation}
for any $i = 1, \ldots, N$, and summing this inequality over $i$, we obtain
\begin{equation}
N \text{Tr } H_\gamma^\gamma \leq (N + 1) L_\gamma \int_{\Gamma_N} V_{\gamma + 1/2}^\gamma \, dx,
\end{equation}
which is the claimed inequality.

Thus, it remains to prove (9). This time we define a quadratic form $h^{(\text{cut})}$ by the same expression as $h[\psi]$ but with form domain
\[ \{ \psi \in L_2(\Gamma_N) : \forall 1 \leq j \leq N : \psi_j \in H^1(\mathbb{R}_+) \text{ and } \psi_1(0) = \ldots = \psi_{N-1}(0) \} . \]
As before, we have (6). Since the $N$-th edge is disconnected from the rest of the edges, we have
\begin{equation}
H^{(\text{cut})} \sim \tilde{H} \oplus \tilde{H}_N,
\end{equation}
where $\tilde{H}$ is the operator in $L_2(\Gamma_{N-1})$, which is obtained by ignoring the $N$-th edge, and $\tilde{H}_N$ is the Schrödinger operator in $L_2(\mathbb{R}_+)$ with potential $V_N$ and a Neumann boundary condition. Thus,
\begin{equation}
\text{Tr}(H^{(\text{cut})}_\gamma) = \text{Tr } \tilde{H}_\gamma^\gamma + \text{Tr}(\tilde{H}_N)^\gamma. \tag{10}
\end{equation}
Since $N - 1$ is even, we have according to Step 1
\begin{equation}
\text{Tr } \tilde{H}_\gamma^\gamma \leq L_\gamma \sum_{i=1}^{N-1} \int_{\mathbb{R}_+} (V_i)^{\gamma + 1/2} \, dt. \tag{11}
\end{equation}
On the other hand, by Lemma 3
\begin{equation}
\text{Tr}(\tilde{H}_N)^\gamma \leq 2L_\gamma \int_{\mathbb{R}_+} (V_N)^{\gamma + 1/2} \, dt. \tag{12}
\end{equation}
The claimed inequality (9) now follows from (6), (10), (11) and (12). This concludes the proof of the theorem. \hfill \Box

A refinement of the previous proof yields

**Proposition 4.** Let $\gamma \geq 1/2$. If $N_0 < N$ are both odd, then
\begin{equation}
L_{\gamma,N} \leq (N - N_0)/N L_\gamma + (N_0/N)L_{\gamma,N_0}.
\end{equation}
In particular, if $L_{\gamma,N_0} = L_\gamma$ for some odd $N_0 \in \mathbb{N}$, then $L_{\gamma,N} = L_\gamma$ for all $N \geq N_0$.

Note that the bound in Theorem 1 follows by taking $N_0 = 1$ and using $L_{\gamma,1} \leq 2L_\gamma$ according to Lemma 3.
Proof. We argue as in the odd \( N \) case of Theorem 1 and decouple \( \Gamma_N \) into two star graphs \( \Gamma_{N_0} \) and \( \Gamma_{N-N_0} \). For \( \Gamma_{N_0} \) we use the bound with \( L_{\gamma,N_0} \) and for \( \Gamma_{N-N_0} \) we use the bound with \( L_\gamma \) (since \( N-N_0 \) is even). Finally, we sum over all possible choices of \( N_0 \) edges, as in the equations after (9). We omit the details. \( \square \)

3. Proof of Theorem 2

We now turn our attention to radial potentials \( V \) on \( \Gamma_N \) and show that the constant \( L^{(\text{rad})}_{\gamma,N} \) coincides with the optimal one-dimensional constant \( L_\gamma \). This holds both for even and odd \( N \).

The symmetry of \( \Gamma_N \) allows one to construct an orthogonal decomposition of the space \( L_2(\Gamma_N) \) which reduces the Kirchhoff Laplacian. If, in addition, \( V \) is radial, it also reduces the operator \( H \). The study of the spectrum of \( H \) is then reduced to the study of the spectrum of the orthogonal components in the decomposition, where each component can be identified with a Schrödinger operator acting in the space \( L_2(\mathbb{R}^+) \).

In [7,12,13] a decomposition of the \( L_2 \) space was given for so-called regular, rooted metric trees. In what follows, we reformulate the decomposition of \( L_2(\Gamma_N) \) for our purposes. We denote by \( \mathcal{H}^{(0)} \) the closed subspace of \( L_2(\Gamma_N) \) of all radial functions on \( \Gamma_N \), i.e.,

\[
\mathcal{H}^{(0)} := \{ \psi \in L_2(\Gamma_N) : \forall r \geq 0 : \psi_1(r) = \psi_2(r) = \ldots = \psi_N(r) \},
\]

where \( \psi_j := \psi|_{e_j} \). Any radial function \( \psi \) on \( \Gamma_N \) can be identified with the function \( s := R\psi \) on the half-line \([0, \infty)\) such that \( \psi(x) = s(|x|) \) for each \( x \in \Gamma_N \), and

\[
\int_{\Gamma_N} |\psi(x)|^2 \, dx = N \int_0^\infty |s(x)|^2 \, dx, \quad \psi \in \mathcal{H}^{(0)}, s = R\psi.
\]

Thus, the operator \( \sqrt{NR} \) defines an isometry of the subspace \( \mathcal{H}^{(0)} \) onto the space \( L_2(\mathbb{R}^+) \).

To state the orthogonal decomposition of \( L_2(\Gamma_N) \) we define for \( 1 \leq \ell \leq N-1 \), the following orthogonal subspaces,

\[
\mathcal{H}^{(\ell)} := \{ \psi \in L_2(\Gamma_N) : \forall j = 1, \ldots, N, \forall r \geq 0 : \psi_{j+1}(r) = e^{i2\pi(\ell/N)}\psi_j(r) \}.
\]

(Here, we write \( \psi_{N+1} = \psi_1 \).) Clearly, as for \( \ell = 0 \) there are isometries from \( \mathcal{H}^{(\ell)} \) onto \( L_2(\mathbb{R}^+) \).

Lemma 5. The subspaces \( \mathcal{H}^{(\ell)} \), \( \ell = 0, \ldots, N-1 \), are mutually orthogonal and

\[
L_2(\Gamma_N) = \bigoplus_{\ell=0}^{N-1} \mathcal{H}^{(\ell)}.
\] (13)

Proof. First, we show that \( L_2(\Gamma_N) = \text{span} \{ \mathcal{H}^{(\ell)} : \ell \} \), i.e., for every function \( \psi \in L_2(\Gamma_N) \) there are functions \( \psi^{(\ell)} \in \mathcal{H}^{(\ell)} \) such that \( \psi = \sum_{\ell=0}^{N-1} \psi^{(\ell)} \). (Note that for \( N = 2 \) this corresponds to the fact that every function on the real line is given as a sum of even and odd functions.)
We can write \( \psi = \sum_{\ell=0}^{N-1} \psi^{(\ell)} \), where the functions \( \psi^{(\ell)} \) are defined via their restrictions \( \psi^{(\ell)}_k \) to the \( k \)-th edge, \( k = 1, \ldots, N \), by

\[
\psi^{(0)}_k(t) = \frac{1}{N} \sum_{j=1}^{N} \psi_j(t)
\]

and, for \( \ell = 1, \ldots, N-1 \),

\[
\psi^{(\ell)}_k = \frac{1}{N} \left( \psi_k(t) + \sum_{j \neq k} e^{i2\pi\ell/N} \psi_j(t) \right).
\]

The identity \( \psi = \sum_{\ell=0}^{N-1} \psi^{(\ell)} \) follows from the fact that

\[
\sum_{\ell=0}^{N-1} e^{i2\pi\ell/N} = \sum_{\ell=0}^{N-1} (e^{i2\pi/|N|})^\ell = \left(\frac{e^{i2\pi/N}}{e^{i2\pi/N}} \right)^N - 1 = 0.
\]

Moreover, it is easy to verify that \( \psi^{(\ell)} \in \mathcal{H}^{(\ell)} \).

To prove the lemma, it remains to show that the spaces \( \mathcal{H}^{(\ell)} \), \( 0 \leq \ell \leq N-1 \), are mutually orthogonal. For \( \psi^{(\ell)} \in \mathcal{H}^{(\ell)} \) and \( \psi^{(m)} \in \mathcal{H}^{(m)} \) with \( \ell \neq m \) consider

\[
\int_{\Gamma} \overline{\psi^{(\ell)}} \psi^{(m)} \, dx = \sum_{j=1}^{N} \int_{\mathbb{R}^+} \overline{\psi_j^{(\ell)}} \psi_j^{(m)} \, dt = \sum_{j=1}^{N} \int_{\mathbb{R}^+} e^{2i\pi\ell(j-1)/N} \psi_1^{(\ell)} e^{-2i\pi m(j-1)/N} \overline{\psi_1^{(m)}} \, dt
\]

\[
= \int_{\mathbb{R}^+} \psi_1^{(\ell)} \overline{\psi_1^{(m)}} \, dt \sum_{j=1}^{N} (e^{2i\pi\ell(j-m)/N})^{j-1}.
\]

The right-hand side equals zero since

\[
\sum_{j=1}^{N} (e^{2i\pi\ell(j-m)/N})^{j-1} = \sum_{j=0}^{N-1} (e^{2i\pi\ell(j-m)/N})^j = \left(\frac{e^{i2\pi\ell(m-j)/N}}{e^{i2\pi\ell(m-j)/N}} \right)^N - 1 = 0.
\]

Hence, the spaces \( \mathcal{H}^{(\ell)} \), \( 0 \leq \ell \leq N-1 \), are mutually orthogonal, as claimed.

A function in \( \mathcal{H}^{(\ell)} \) is completely determined by its restriction to one of the edges. We now characterize the \( H^1(\Gamma_N) \) property of a function in \( \mathcal{H}^{(\ell)} \) in terms of its restrictions. Clearly, a function in \( \mathcal{H}^{(0)} \) belongs to \( H^1(\Gamma_N) \) iff its restrictions belong to \( H^1(\mathbb{R}^+) \).

On the other hand, a function \( \psi \in \mathcal{H}^{(\ell)} \) with \( \ell = 1, \ldots, N-1 \) belongs to \( H^1(\Gamma_N) \) iff its restrictions belong to \( H^{1,0}(\mathbb{R}^+) = \{ \psi \in H^1(\mathbb{R}^+) : \psi(0) = 0 \} \). The crucial point here is the Dirichlet boundary condition at the origin. Moreover, we have

\[
\int_{\Gamma_N} \left| \psi' \right|^2 \, dx = \sum_{\ell=0}^{N-1} \int_{\Gamma_N} |(\psi^{(\ell)})'|^2 \, dx,
\]

where \( \psi^{(\ell)} \) denotes the projection of \( \psi \) onto \( \mathcal{H}^{(\ell)} \).

We conclude that the subspaces \( \mathcal{H}^{(\ell)} \) reduce the Schrödinger operator \( H \) and that the operators \( H|_{\mathcal{H}^{(\ell)}} \) are unitarily equivalent to operators \( H^{(\ell)} \) in \( L^2(\mathbb{R}^+) \). These operators act as \( -\frac{d^2}{dx^2} + V(x) \) and have Neumann (if \( \ell = 0 \)) and Dirichlet (if \( \ell = 1, \ldots, N-1 \))
boundary conditions. Here we identify the radial function $V$ on $\Gamma_N$ in a natural way with a function on $\mathbb{R}_+$. (More precisely, the operators $H^{(\ell)}$ are defined via the quadratic form $\int_{\mathbb{R}_+} (|\psi'|^2 + V|\psi|^2) \, dx$ with form domain $H^1(\mathbb{R}_+)$ for $\ell = 0$ and $H^{1,0}(\mathbb{R}_+)$ for $\ell = 1, \ldots, N - 1$.) Clearly, the operators $H^{(\ell)}$ for $\ell = 1, \ldots, N - 1$ coincide.

To summarize, the operator $H^\gamma$ in $L^2(\Gamma_N)$ is unitary equivalent to the orthogonal sum of the operators $H^{(\ell)}$ on $L^2(\mathbb{R}_+)$,

$$H \sim \bigoplus_{\ell=0}^{N-1} H^{(\ell)},$$

and therefore its eigenvalues, counting multiplicities, are given by the union of the eigenvalues of $H^{(\ell)}$, counting multiplicities. Then, for any $\gamma$,

$$\text{Tr } H^\gamma = \text{Tr } (H^{(0)})^\gamma + (N - 1) \text{Tr } (H^{(1)})^\gamma.$$  \hfill (15)

Consider now the Schrödinger operator

$$\tilde{H} = -\frac{d^2}{dx^2} + \tilde{V} \quad \text{in } L^2(\mathbb{R}),$$

where the potential $\tilde{V}$ is the symmetric extension of the potential $V|_{\epsilon_j}$ to the whole-line. The unitary equivalence $\text{(14)}$ with $N = 2$ implies that $\tilde{H} \sim H^{(0)} \oplus H^{(1)}$. Reinserting this into $\text{(14)}$ we find

$$H \sim \tilde{H} \oplus \bigoplus_{\ell=2}^{N-1} H^{(\ell)},$$

and hence

$$\text{Tr } H^\gamma = \text{Tr } (\tilde{H})^\gamma + (N - 2) \text{Tr } (H^{(1)})^\gamma.$$  \hfill (16)

This is the key identity in the radial case.

According to the Lieb–Thirring inequality $\text{(2)}$, for the first trace on the right side of $\text{(16)}$ and $\gamma \geq 1/2$ we have

$$\text{Tr } (\tilde{H})^\gamma \leq L_{\gamma} \int_{\mathbb{R}} (\tilde{V})_{\gamma+1/2}^\gamma dx = 2L_{\gamma} \int_{\mathbb{R}_+} V_\gamma^{\gamma+1/2} dx.$$

On the other hand, by the variational principle, inequality $\text{(2)}$ remains also true for the eigenvalues of Dirichlet half-line operators, and therefore for the second trace on the right side of $\text{(16)}$ we have

$$\text{Tr } (H^{(1)})_{\gamma} \leq L_{\gamma} \int_{\mathbb{R}_+} V_\gamma^{\gamma+1/2} dx.$$

Thus, the right side of $\text{(16)}$ is bounded from above by

$$NL_{\gamma} \int_{\mathbb{R}_+} V_\gamma^{\gamma+1/2} dx = L_{\gamma} \int_{\Gamma_N} V_\gamma^{\gamma+1/2} dx,$$

which proves the bound $L^{(\text{rad})}_{\gamma,N} \leq L_{\gamma}$ claimed in Theorem 2.
Conversely, for any $\varepsilon > 0$ there is a compactly supported $V$ on $\mathbb{R}$ such that
\[
\text{Tr} \left( -\frac{d^2}{dx^2} + V \right)_-^\gamma \geq (1 - \varepsilon)L_\gamma \int_{\mathbb{R}} V_-^\gamma + 1/2 dx.
\]  
(17)

We denote by $V_a(x) = V(x - a)$ the translate of this potential and choose $a$ so large that the support of $V_a$ is contained in $\mathbb{R}_+$. We use $V_a$ as a radial potential on $\Gamma_N$ and denote the corresponding operator by $H_a$ and its parts on $\mathcal{H}^{(0)}$ and $\mathcal{H}^{(1)}$ by $H_a^{(0)}$ and $H_a^{(1)}$, respectively. It is easy to see that as $a \to \infty$,
\[
\frac{\text{Tr} \left( H_a^{(0)} \right)_-^\gamma}{\text{Tr} \left( -\frac{d^2}{dx^2} + V_a \right)_-^\gamma} \to 1 \quad \text{and} \quad \frac{\text{Tr} \left( H_a^{(1)} \right)_-^\gamma}{\text{Tr} \left( -\frac{d^2}{dx^2} + V_a \right)_-^\gamma} \to 1.
\]

On the other hand, by translation invariance, \( \text{Tr} \left( -\frac{d^2}{dx^2} + V_a \right)_-^\gamma = \text{Tr} \left( -\frac{d^2}{dx^2} + V \right)_-^\gamma \) and $\int_{\Gamma_N} (V_a)_-^\gamma + 1/2 dx = N \int_{\mathbb{R}} V_-^\gamma + 1/2 dx$. Therefore (15) and (17) yield
\[
\liminf_{a \to \infty} \frac{\text{Tr}(H_a)_-^\gamma}{\int_{\Gamma_N} (V_a)_-^\gamma + 1/2 dx} \geq (1 - \varepsilon)L_\gamma.
\]

This proves $L^{\text{rad}}_{\gamma,N} \geq (1 - \varepsilon)L_\gamma$ and, since $\varepsilon > 0$ is arbitrary, we obtain $L^{\text{rad}}_{\gamma,N} \geq L_\gamma$. This concludes the proof of the theorem.

**References**

[1] Gregory Berkolaiko, Robert Carlson, Stephen A. Fulling, and Peter Kuchment, editors. *Quantum Graphs and Their Applications*, volume 415 of *Contemporary Mathematics*, Providence, RI, 2006. American Mathematical Society.

[2] Gregory Berkolaiko and Peter Kuchment. *Introduction to quantum graphs*, volume 186 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2013.

[3] Semra Demirel and Evans M. Harell II. On semiclassical and universal inequalities for eigenvalues of quantum graphs. *Rev. Math. Phys.*, 22:305–329, 2010.

[4] Tomas Ekholm, Rupert L. Frank, and Hynek Kovářík. Eigenvalue estimates for Schrödinger operators on metric trees. *Adv. Math.*, 226(6):5165–5197, 2011.

[5] Pavel Exner, Jonathan P. Keating, Peter Kuchment, Toshikazu Sunada, and Alexander Teplyaev, editors. *Analysis on graphs and its applications*, volume 77 of *Proceedings of Symposia in Pure Mathematics*. American Mathematical Society, Providence, RI, 2008. Papers from the program held in Cambridge, January 8–June 29, 2007.

[6] Pavel Exner, Ari Laptev, and Muhammad Usman. On some sharp spectral inequalities for Schrödinger operators on semiaxis. *Comm. Math. Phys.*, 326(2):531–541, 2014.

[7] Rupert L. Frank and Hynek Kovářík. Heat kernels of metric trees and applications. *SIAM J. Math. Anal.*, 45(3):1027–1046, 2013.

[8] Dirk Hundertmark, Elliott H. Lieb, and Lawrence E. Thomas. A sharp bound for an eigenvalue moment of the one-dimensional Schrödinger operator. *Adv. Theor. Math. Phys.*, 2(4):719–731, 1998.

[9] Norman E. Hurt. *Mathematical physics of quantum wires and devices*, volume 506 of *Mathematics and its Applications*. Kluwer Academic Publishers, Dordrecht, 2000. From spectral resonances to Anderson localization.
[10] Ari Laptev and Timo Weidl. Sharp Lieb-Thirring inequalities in high dimensions. *Acta Math.*, 184(1):87–111, 2000.

[11] Elliott H. Lieb and Walter Thirring. Inequalities for the moments of the eigenvalues of the Schrödinger Hamiltonian and their relation to Sobolev inequalities. *Studies in Mathematical Physics, Princeton University Press, Princeton, NJ*, pages 269–303, 1976.

[12] K. Naimark and M. Solomyak. Eigenvalue estimates for the weighted Laplacian on metric trees. *Proc. London Math. Soc. (3)*, 80(3):690–724, 2000.

[13] Michael Solomyak. On the spectrum of the Laplacian on regular metric trees. *Waves Random Media*, 14(1):S155–S171, 2004. Special section on quantum graphs.

Semra Demirel–Frank, Mathematics 253-37, Caltech, Pasadena, CA 91125, USA

E-mail address: sdemirel@caltech.edu