Perturbative Renormalization of Improved Lattice Operators*

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We derive bases of improved operators for all bilinear quark currents up to spin two (including the operators measuring the first moment of DIS Structure Functions), and compute their one-loop renormalization constants for arbitrary coefficients of the improvement terms. We have thus control over $O(\alpha)$ corrections, and for a suitable choice of improvement coefficients we are only left with errors of $O(\alpha^2)$.

1. INTRODUCTION

In this talk we extend previous calculations of the renormalization constants of quark bilinear operators using improved (Sheikholeslami-Wohlert) fermions. Though the calculations are performed in one-loop perturbation theory, we quote results for arbitrary coefficients of the improved action and operators. If they are properly determined (for instance by imposing Ward identities), then only $O(\alpha^2)$ errors are left.

2. QUARK PROPAGATOR

Let us first consider the quark propagator

$$S^{-1} = i \not{p} + m + arp^2/2 - \Sigma^{\text{latt}},$$

where the bare mass is given by $ma = \frac{1}{2\pi} - 4 - \frac{g^2}{16\pi^2} C_F \Sigma_0$. We write the self-energy as

$$\Sigma^{\text{latt}} = \frac{g^2}{16\pi^2} C_F (i \not{\Sigma}_1^{\text{latt}} + m \Sigma_2^{\text{latt}} + arp^2 \Sigma_3^{\text{latt}} + arm \not{p} \Sigma_4^{\text{latt}} + arm^2 \Sigma_5^{\text{latt}} + O(\alpha^2)).$$

In the covariant gauge we obtain (for $r = 1$)

\begin{align*}
\Sigma_0^{\text{latt}} &= -51.4347 + 13.7331 c_{sw} + 5.7151 c_{sw}^2
\Sigma_1^{\text{latt}} &= 15.6444 - 2.2489 c_{sw} - 1.3973 c_{sw}^2 + \gamma_1 (1 - \eta) \frac{1}{2} L(ap) + \eta
\Sigma_2^{\text{latt}} &= 9.0680 - 9.9868 c_{sw} - 0.0169 c_{sw}^2 + (\gamma_2 - \gamma_1 \eta) \frac{1}{2} L(ap) + 2\eta
\Sigma_3^{\text{latt}} &= 7.0670 + 0.4857 c_{sw} - 0.0817 c_{sw}^2 + 0.0719 \eta + (-1 + 3c_{sw} - 2\eta) \frac{1}{2} L(ap)
\Sigma_4^{\text{latt}} &= -6.2029 - 1.4850 c_{sw} + 1.2860 c_{sw}^2 - 0.1437 \eta + (-5 - 3c_{sw} + 2\eta) \frac{1}{2} L(ap)
\Sigma_5^{\text{latt}} &= -13.4623 + 16.9857 c_{sw} + 1.5234 c_{sw}^2 - 2.0719 \eta + (-10 + 6c_{sw} + \eta) \frac{1}{2} L(ap).
\end{align*}

From this we derive the renormalization constant

$$Z_m = 1 - \frac{g^2}{16\pi^2} C_F (6 \log(\alpha \mu) - 12.952 - 7.738 c_{sw} + 1.380 c_{sw}^2),$$

and the critical $\kappa$

$$\kappa_c(g) = \frac{1}{2} \left( 4 - \frac{g^2}{16\pi^2} C_F \Sigma_0 \right)^{-1}.$$  \hspace{1cm} (1)
In Eq. (1) one has the choice of using $c_{sw} = 1$ or the actual value used in true simulations. In Fig. 1 we compare the various choices with the data, and we find that the latter choice, combined with tadpole improvement, agrees best. This justifies our procedure.

$$b_2 = 1 + \frac{g^2}{16\pi^2}C_F(2\Sigma_1 - \Sigma_2 - 2\Sigma_3 - \Sigma_4)$$

$$m_R = mZ_m(1 - b_marm)$$

$$= m(1 + \frac{g^2}{16\pi^2}C_F(\Sigma_1 - \Sigma_2))(1 - \frac{arm}{2} - \frac{arm}{2}) = m(1 - b_marm)$$

$$= \frac{g^2}{2}C_F(2\Sigma_1 - \Sigma_2 - 2\Sigma_3 - 2\Sigma_4 + 2\Sigma_5).$$

We see that $\lambda$ can be a constant only if $c_{sw} = 1$:

$$2\Sigma_1 - 2\Sigma_3 = 17.1548 - 5.4691c_{sw} - 2.6311c_{sw}^2$$

$$+ 1.8562\eta + 6(1 - c_{sw})\frac{1}{2}L(ap);$$

for every other value of $c_{sw}$ there are $a\log(ap)$ contributions to $S(p)$. The same happens for the parameters $b_2$ and $b_m$: both of them involve $a\log(ap)$ terms, that at $c_{sw} = 1$ cancel out.

Two possible expressions that remove $O(a)$ effects from the quark propagator are:

$$S_{imp}(p) \equiv (1 + b_2arm)(S(p) - ar\lambda);$$

$$S^{-1}(p) = (1 + b_2arm)S_{imp}^{-1}(p)$$

$$- ar\lambda S_{imp}^{-1}(p)S_{imp}^{-1}(p).$$

The improved propagators in these two definitions differ by terms of $O(a^2)$, but both are free of $O(a)$ effects. Eq. (1) is a non-linear equation which has to be solved iteratively, but it seems a better definition in practice than the first one. This is because Eq. (1) seems to have larger $O(a^2)$ effects, thus using Eq. (3) we can reach higher momenta.

3. BASES FOR IMPROVED OPERATORS

Some fundamental bases necessary to achieve full $O(a)$ improvement for point operators are:

$$(\bar{\psi}\gamma_5\psi)^{imp} = (1 + a b m)\bar{\psi}\gamma_5\psi - \frac{1}{2}ac_1\bar{\psi}\overset{\leftrightarrow}{D}\psi$$

$$(\bar{\psi}\gamma_\mu\gamma_5\psi)^{imp} = (1 + a b m)\bar{\psi}\gamma_\mu\gamma_5\psi + \frac{1}{2}ac_1\partial_\mu(\bar{\psi}\gamma_\mu\gamma_5\psi)$$

$$(\bar{\psi}\gamma_\mu\psi)^{imp} = (1 + a b m)\bar{\psi}\gamma_\mu\psi - \frac{1}{2}ac_1\bar{\psi}\overset{\leftrightarrow}{D}_\mu \psi$$

$$+ \frac{1}{2}ac_2\partial_\lambda(\bar{\psi}\sigma_{\mu\lambda}\psi)$$
\[
(\bar{\psi}_\mu \gamma_5 \psi)^{\text{imp}} = (1 + a b m) \bar{\psi}_\mu \gamma_5 \psi \\
-\frac{1}{2} a c_1 \bar{\psi}_\mu \gamma_5 \gamma_\mu \gamma_5 D_\lambda \psi + \frac{1}{2} a c_2 \partial_\mu (\bar{\psi}_\mu \gamma_5 \psi),
\]

where \( \hat{D}_\mu = \hat{D}_\mu - \hat{D}_\mu \). For the tensor operator see Ref. [1]. The normalizations are chosen such that \( c_i = 1 + O(g^2) \) (in this way, for \( g = 0 \) we realize tree-level improvement).

Some possible bases for the improvement of the one-link DIS operators \( O_{\mu \nu} = \bar{\psi} \gamma_\mu \gamma_5 \hat{D}_\nu \psi \) and the polarized \( O_{\mu \nu}^{5} = \bar{\psi} \gamma_\mu \gamma_5 \hat{D}_\nu \psi \) are:

\[
O_{\mu \nu}^{\text{imp}} = (1 + a b m) \bar{\psi}_\mu \gamma_5 \hat{D}_\nu \psi
\]

\[
= -a c_1 g \bar{\psi}_\mu \gamma_5 F_{\mu \lambda}^{\text{clover}} \gamma_\lambda \psi - \frac{1}{4} a c_2 \bar{\psi}_\mu \{ D_\mu, \hat{D}_\nu \} \psi \\
+ \frac{1}{2} a i c_3 \partial_\mu (\bar{\psi}_\mu \gamma_5 \hat{D}_\nu \psi);
\]

\[
O_{\mu \nu}^{5, \text{imp}} = (1 + a b m) \bar{\psi}_\mu \gamma_5 \gamma_\mu \gamma_5 \hat{D}_\nu \psi
\]

\[
= -a c_1 g \bar{\psi}_\mu \gamma_5 F_{\mu \lambda}^{\text{clover}} \gamma_\lambda \psi \\
- \frac{1}{4} a c_2 \bar{\psi}_\mu \gamma_5 \gamma_\mu \gamma_5 \{ D_\mu, \hat{D}_\nu \} \psi \\
+ \frac{1}{2} a i c_3 \partial_\mu (\bar{\psi}_\mu \gamma_5 \hat{D}_\nu \psi).
\]

The relation \( [\hat{D}_\mu, \hat{D}_\nu]^{\text{last}} = 4 i g F_{\mu \nu}^{\text{clover}} + O(a^2) \) is useful to derive other bases.

If all these operators are inserted into forward matrix elements the surface terms \( \partial_\mu (\bar{\psi}_\mu \gamma_5 \psi) \) vanish due to momentum conservation. Using the equations of motion it is also possible to further reduce the number of improvement coefficients.

### 4. POINT OPERATORS

We present the general \( Z \) renormalization factors with the improvement coefficients \( c_{sw} \) (for the action) and \( c_i \) (for the operators) kept general. These factors will be essential when a determination of the improvement coefficients is done [1].

Writing (to order \( g^2 \)) \( \langle O \rangle_{p^2} = O^{(1)} + a O^{(2)} \), the calculation of the amputated matrix elements for \( A_5 = \bar{\psi}_5 \psi \) and \( A_\mu = \bar{\psi}_\mu \gamma_5 \psi \) gives

\[
A_5^{(1)} = \frac{g^2}{16\pi^2} C_F \gamma_5 (0.5750 + 3.4333 c_{sw}^2)
\]

\[
-2\eta + (-4 + \eta) L(\eta)
\]

\[
A_5^{(2)} = 0
\]

\[
A_\mu^{(1)} = \frac{g^2}{16\pi^2} C_F (\gamma_\mu \gamma_5 (0.1519 - \eta - 19.3723 c_1) \nonumber \\
+ 2.4967 c_{sw} + 10.3167 c_1 c_{sw} - 0.8541 c_{sw}^2 \\
- 0.8846 c_1 c_{sw}^2 - (1 - \eta) L(\eta) - 2(1 - \eta) \frac{\hat{p}_\rho \hat{p}_\mu}{p^2})
\]

\[
A_\mu^{(2)} = \frac{i g^2}{16\pi^2} C_F (\frac{1}{2} (\hat{p}_\rho \gamma_5 + \gamma_\mu \gamma_5 \hat{p}_\rho) \cdot (1.5323 \\
+ 0.7322 c_1 - 1.7118 c_{sw} + 0.5430 c_1 c_{sw} \\
+ 0.1302 c_{sw}^2 + 0.0537 c_1 c_{sw}^2 - \eta (0.8563 \\
- 4.0583 c_1) + (1 - \eta)(1 + c_1) L(\eta)).
\]

Similar results hold for the other bilinears [1]. In general the \( O(a) \) terms do not contribute to the tree-level structure in the massless case.

To extract the \( Z \) factors we project forward matrix elements onto their tree-level structure: \( \langle q(p) | O(\mu) | q(p) \rangle = Z_O Z_\psi^{-1} \langle q(p) | O(a) | q(p) \rangle \), and \( \langle q(p) | O(\mu) | q(p) \rangle |_{p^2 = \mu^2} = \langle q(p) | O(a) | q(p) \rangle |_{p^2 = \mu^2} \), where \( Z_\psi \) is the wave function renormalization factor. The one-loop renormalization factor \( Z_O \) for a lattice operator \( O \) can be cast into the form

\[
Z_O(a \mu, g) = 1 - \frac{g^2}{16\pi^2} C_F (\gamma_\phi \log(\alpha_\mu) + B_O),
\]

where \( \gamma_\phi \) is its anomalous dimension, and \( B_O \) the finite part of \( Z_O \).

The Wilson coefficients are usually computed in the \( \overline{\text{MS}} \) scheme; to convert the finite parts to this scheme one can use the relations \( B_\phi^{\overline{\text{MS}}} = B_\phi - B_\phi^{\text{cont}} \) and \( B_\phi^{\overline{\text{MS}}} = B_\phi^{\overline{\text{MS}}} + \frac{\gamma_E}{2}(\gamma_E - \log(4\pi)) \), where \( \gamma_\phi \) and \( B_\phi^{\text{cont}} \), the finite contributions to the continuum renormalization factors \( Z_O^{\text{cont}} \), are:

| \( O \) | \( \gamma_\phi \) | \( B_\phi^{\text{cont}} \) |
|---|---|---|
| 1, \( \gamma_5 \) | -6 | 5 + \frac{\gamma_E}{2}(\gamma_E - \log(4\pi)) - \eta |
| \( \gamma_\mu \), \( \gamma_\mu \gamma_5 \) | 0 | \gamma_\phi |
| \( \gamma_\mu \gamma_5 \) | 2 | -1 + \frac{3\gamma_E}{2}(\gamma_E - \log(4\pi)) + \eta |

The results in the \( \overline{\text{MS}} \) scheme for all point operators are then as follows:
\[ B_{1}^{\text{MS}} = 12.9524 - 19.1718 c_1 + 7.7379 c_{\text{sw}} \]
\[ + 13.8007 c_1 c_{\text{sw}} - 1.3804 c_{\text{sw}}^2 - 3.5383 c_1 c_{\text{sw}}^2 \]
\[ B_{\gamma_{\nu}}^{\text{MS}} = 22.5954 - 2.2487 c_{\text{sw}} + 2.0360 c_{\text{sw}}^2 \]
\[ B_{\gamma_{\mu},\gamma_{5}}^{\text{MS}} = 20.6178 - 9.7864 c_1 - 4.7456 c_{\text{sw}} \]
\[ + 3.4164 c_1 c_{\text{sw}} - 0.5432 c_{\text{sw}}^2 + 0.8846 c_1 c_{\text{sw}}^2 \]
\[ B_{\sigma_{\mu\nu},\gamma_{5}}^{\text{MS}} = 15.7963 - 19.3723 c_1 + 0.2478 c_{\text{sw}} \]
\[ + 10.3167 c_1 c_{\text{sw}} - 2.2514 c_{\text{sw}}^2 - 0.8846 c_1 c_{\text{sw}}^2 \]
\[ B_{\sigma_{\mu\nu},\gamma_{5}}^{\text{MS}} = 17.0181 - 16.2438 c_1 - 3.9133 c_{\text{sw}} \]
\[ + 6.8553 c_1 c_{\text{sw}} - 1.9723 c_{\text{sw}}^2 + 0.5897 c_1 c_{\text{sw}}^2. \]

With the appropriate values of \( c_{\text{sw}} \) and \( c_i \), all these operators are then fully \( O(\alpha) \) improved.

5. ONE-LINK OPERATORS

The many-link operators are essential in the OPE expansion of current-current correlators occurring in Structure Functions computations. Using Eq. (4) with \( b = 0 \) and \( c_3 = 0 \) for the unpolarized case, in one-loop perturbation theory for the representation \( \tau_3^{(3)} (O_{\{44\}} - 1/3(O_{\{11\}} + O_{\{22\}} + O_{\{33\}}) \) we have:

\[ O_{\text{imp}}^{\tau_3^{(3)}}: B_O = -1.8826 - 3.9698 c_{\text{sw}} \]
\[ - 1.0398 c_{\text{sw}}^2 + c_1 (5.9970 - 3.2685 c_{\text{sw}}) \]
\[ - c_2 (6.6733 - 4.5371 c_{\text{sw}} - 0.4462 c_{\text{sw}}^2). \]

For the other representation, \( \tau_3^{(6)} (O_{\{14\}}) \), we find full agreement with the \( O(\alpha) \) improved result given in Ref. [4] for \( c_{\text{sw}} = 1 \) and \( c_1 = c_2 = 1 \).

For the polarized case, considering Eq. (3), the result for the representation \( \tau_4^{(6)} (O_{\{14\}}) \) is

\[ O_{\text{imp}}^{5,\tau_4^{(6)}}: B_O = -4.0988 - 1.3593 c_{\text{sw}} - 1.8926 c_{\text{sw}}^2 \]
\[ - c_2 (27.5719 - 16.1193 c_{\text{sw}} + 0.7570 c_{\text{sw}}^2). \]

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