The spaces of curvature tensors for holonomy algebras of Lorentzian manifolds

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Abstract

The holonomy algebra \( g \) of an indecomposable Lorentzian \((n+2)\)-dimensional manifold \( M \) is a weakly-irreducible subalgebra of the Lorentzian algebra \( \mathfrak{so}_{1,n+1} \). L. Berard Bergery and A. Ikemakhen divided weakly-irreducible not irreducible subalgebras into 4 types and associated with each such subalgebra \( g \) a subalgebra \( \mathfrak{h} \subset \mathfrak{so}_n \) of the orthogonal Lie algebra. We give a description of the spaces \( \mathcal{R}(g) \) of the curvature tensors for algebras of each type in terms of the space \( \mathcal{P}(\mathfrak{h}) \) of \( \mathfrak{h} \)-valued 1-forms on \( \mathbb{R}^n \) that satisfy the Bianchi identity and reduce the classification of the holonomy algebras of Lorentzian manifolds to the classification of irreducible subalgebras \( \mathfrak{h} \) of \( \mathfrak{so}(n) \) with \( L(\mathcal{P}(\mathfrak{h})) = \mathfrak{h} \). We prove that for \( n \leq 9 \) any such subalgebra \( \mathfrak{h} \) is the holonomy algebra of a Riemannian manifold. This gives a classification of the holonomy algebras for Lorentzian manifolds \( M \) of dimension \( \leq 11 \).

Keywords: Lorentzian manifold, holonomy algebra, curvature tensor
Mathematical subject codes: 53c29, 53c50

Introduction

The connected irreducible holonomy groups of pseudo-Riemannian manifolds have been classified by M. Berger, see [5]. The classification problem for not irreducible holonomy groups is still open. The main difficulty is that the holonomy group can preserve an isotropic subspace of the tangent space.

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We tackle the classification problem for the holonomy algebras of Lorentzian manifolds, i.e. the Lie algebras of the holonomy groups. There are some partial results in this direction (see [4, 7, 9, 10]). In [3] the classification of holonomy algebras for 4-dimensional Lorentzian manifolds was given.

Wu’s theorem (see [14]) reduces the classification problem for holonomy algebras to the classification of weakly-irreducible holonomy algebras (i.e. algebras that preserve no nondegenerate proper subspace of the tangent space).

If a holonomy algebra is irreducible, then it is weakly-irreducible. The Berger list ([5]) of irreducible holonomy algebras of pseudo-Riemannian manifolds shows that the only irreducible holonomy algebra of Lorentzian manifolds is \( \mathfrak{so}(1, m - 1) \), see also [12].

We study weakly-irreducible holonomy algebras that are not irreducible.

Let \((V, \eta)\) be an \((n + 2)\)-dimensional Minkowski vector space, where \(\eta\) is a metric of signature \((1, n + 1)\). Using \(\eta\) we identify the space \(V\) with the dual space \(V^*\). Then the Lorentzian algebra \(\mathfrak{so}(V)\) is identified with the space \(V \wedge V\) of bivectors. Denote by \(\mathfrak{so}(V)_{\mathbb{R}p}\) the subalgebra of \(\mathfrak{so}(V)\) that preserves an isotropic line \(\mathbb{R}p\), where \(p \in V\). Denote by \(E\) a vector subspace \(E \subset V\) such that \((\mathbb{R}p)^{\perp \eta} = \mathbb{R}p \oplus E\). The vector space \(E\) is an Euclidean space with respect to the inner product \(-\eta|_E\). Denote by \(q\) an isotropic vector \(q \in V\) such that \(\eta(q, E) = 0\) and \(\eta(p, q) = 1\).

We have
\[
\mathfrak{so}(V)_{\mathbb{R}p} = \mathbb{R}p \wedge q + p \wedge E + \mathfrak{so}(E).
\]

Any weakly-irreducible and not irreducible subalgebra of \(\mathfrak{so}(V)\) is conjugated to a subalgebra \(\mathfrak{g}\) of \(\mathfrak{so}(V)_{\mathbb{R}p}\). We denote by \(\mathfrak{h}^0\) the projection of such subalgebra \(\mathfrak{g}\) to \(\mathfrak{so}(E)\) with respect to the above decomposition and call \(\mathfrak{h}^0\) the orthogonal part of the Lie algebra \(\mathfrak{g}\).

Conversely, for any subalgebra \(\mathfrak{h} \subset \mathfrak{so}(E)\) we construct two Lie algebras
\[
\mathfrak{g}_1 = \mathbb{R}p \wedge q + \mathfrak{h} + p \wedge E
\]
and
\[
\mathfrak{g}_2 = \mathfrak{h} + p \wedge E
\]
with the orthogonal part \(\mathfrak{h}\). Moreover, if the center \(\mathfrak{z}(\mathfrak{h})\) of \(\mathfrak{h}\) is non-trivial, then any non-zero linear map
\[
\varphi : \mathfrak{z}(\mathfrak{h}) \to \mathbb{R}
\]
defines the Lie algebra
\[
\mathfrak{g}_3^{\mathfrak{h}, \varphi} = p \wedge E + \{A + \varphi(A)p \wedge q : A \in \mathfrak{h}\}
\]
with the orthogonal part \(\mathfrak{h}\). Here \(\varphi\) is considered as the linear map \(\varphi : \mathfrak{h} = \mathfrak{z}(\mathfrak{h}) \oplus \mathfrak{h}' \to \mathbb{R}\) that vanishes on the commutant \(\mathfrak{h}'\) of \(\mathfrak{h}\).

Suppose moreover that the subalgebra \(\mathfrak{h} \subset \mathfrak{so}(E)\) acts trivially on a subspace \(E_0 \neq \{0\}\), such that we can consider \(\mathfrak{h}\) as a subalgebra of \(\mathfrak{so}(E_1)\), where \(E = E_0 \oplus E_1\) is the orthogonal
decomposition. Then any surjective linear map

$$\psi : \mathfrak{g}(\mathfrak{h}) \to E_0$$

extended to $$\mathfrak{h}$$ by $$\psi(\mathfrak{h}') = 0$$ defines the Lie algebra

$$\mathfrak{g}_4^{\mathfrak{h},\psi} = p \wedge E_1 + \{ A + p \wedge \psi(A) : A \in \mathfrak{h} \}$$

with the orthogonal part $$\mathfrak{h}$$. We call Lie algebras $$\mathfrak{g}_1^\mathfrak{h}, \mathfrak{g}_2^\mathfrak{h}, \mathfrak{g}_3^{\mathfrak{h},\varphi}$$ and $$\mathfrak{g}_4^{\mathfrak{h},\psi}$$ with the orthogonal part $$\mathfrak{h} \subset \mathfrak{so}(E)$$ the algebras of type 1, 2, 3 and 4 respectively.

These Lie algebras were considered by L. Berard Bergery and A. Ikemakhen [4], who proved that the Lie algebras of the form $$\mathfrak{g}_1^\mathfrak{h}, \mathfrak{g}_2^\mathfrak{h}, \mathfrak{g}_3^{\mathfrak{h},\varphi}$$ and $$\mathfrak{g}_4^{\mathfrak{h},\psi}$$ exhaust all weakly-irreducible subalgebras of $$\mathfrak{so}(V)_{\mathbb{R}^p}$$ (theorem 1). The other result is that the orthogonal part of the holonomy algebra of a Lorentzian manifold satisfies a Borel-Lichnerowicz-type decomposition property (theorem 2).

Remark. Note that the Lie algebra $$\mathfrak{so}(V)_{\mathbb{R}^p} = \mathbb{R}^p \wedge q + p \wedge E + \mathfrak{so}(E)$$ is isomorphic to the tangent Lie algebra for the Lie group Sim$$E$$ of similarity transformations of $$E$$, the elements $$\lambda p \wedge q$$ and $$p \wedge u$$ correspond to the homothetic transformation $$v \mapsto \lambda v$$ and to the shift $$v \mapsto v + u$$ respectively, here $$\lambda \in \mathbb{R}$$ and $$u, v \in E$$. In another paper we will give a geometrical interpretation to the result of L. Berard Bergery and A. Ikemakhen.

Let $$\mathfrak{g} \subset \mathfrak{so}(V)_{\mathbb{R}^p}$$ be a subalgebra. Recall that the space of curvature tensors of type $$\mathfrak{g}$$ is defined as the space $$\mathcal{R}(\mathfrak{g})$$ of $$\mathfrak{g}$$-valued 2-forms on $$V$$ that satisfy the Bianchi identity. We denote by

$$L(\mathcal{R}(\mathfrak{g})) = \text{span}(\{ R(u \wedge v) : R \in R(\mathfrak{g}), u, v \in V \})$$

the vector subspace of $$\mathfrak{g}$$ spanned by curvature operators from $$\mathcal{R}(\mathfrak{g})$$. If $$\mathfrak{g}$$ is the holonomy algebra of an indecomposable Lorentzian manifold, then

$$L(\mathcal{R}(\mathfrak{g})) = \mathfrak{g}.$$ (\ast)\)

A weakly-irreducible subalgebra $$\mathfrak{g} \subset \mathfrak{so}(V)_{\mathbb{R}^p}$$ that satisfies (\ast) is called a Berger algebra.

In this paper we give a description of the spaces $$\mathcal{R}(\mathfrak{g})$$ of curvature tensors for weakly-irreducible subalgebras $$\mathfrak{g} \subset \mathfrak{so}(V)_{\mathbb{R}^p}$$ of each type in terms of the orthogonal part $$\mathfrak{h}^\mathfrak{g} \subset \mathfrak{so}(E)$$ and we reduce the classification of Berger algebras to the classification of irreducible subalgebras $$\mathfrak{h} \subset \mathfrak{so}(E)$$ that satisfy some conditions (weak-Berger algebras).

More precisely, for any subalgebra $$\mathfrak{h} \subset \mathfrak{so}(E)$$ we define the space

$$\mathcal{P}(\mathfrak{h}) = \{ P \in \text{Hom}(E, \mathfrak{h}) : \eta(P(u)v, w) + \eta(P(v)w, u) + \eta(P(w)u, v) = 0 \text{ for all } u, v, w \in E \}$$

of $$\mathfrak{h}$$-valued 1-forms on $$E$$ that satisfy the Bianchi identity and denote by

$$L(\mathcal{P}(\mathfrak{h})) = \text{span}(\{ P(u) : P \in \mathcal{P}(\mathfrak{h}), u \in E \})$$
the vector subspace of \( h \) spanned by tensors \( P \in \mathcal{P}(h) \). We call \( \mathcal{P}(h) \) the space of weak-curvature tensors of type \( h \).

A subalgebra \( h \subset \mathfrak{so}(E) \) is called a weak-Berger algebra if \( L(\mathcal{P}(h)) = h \).

We give a description of the spaces of curvature tensors \( \mathcal{R}(\mathfrak{g}) \) for algebras of each type associated with a given orthogonal part \( h \subset \mathfrak{so}(E) \) in terms of the space \( \mathcal{P}(h) \) of weak-curvature tensors (theorem 3).

Corollary 1 shows that a weakly-irreducible subalgebra \( \mathfrak{g} \subset \mathfrak{so}(V)_{\mathbb{R}^p} \) is a Berger algebra iff \( h^0 \) is a weak-Berger algebra.

Note that the direct sum \( h_1 \oplus h_2 \subset \mathfrak{so}(E_1) \oplus \mathfrak{so}(E_2) \) of two weak-Berger algebras \( h_i \subset \mathfrak{so}(E_i) \), \( i = 1, 2 \) is a weak-Berger algebra.

In part (I) of theorem 4 we prove that if a subalgebra \( h \subset \mathfrak{so}(E) \) is a weak-Berger algebra, then there exists an orthogonal decomposition \( E = E_0 \oplus E_1 \oplus \cdots \oplus E_r \) and the corresponding decomposition into the direct sum of ideals \( h = \{0\} \oplus h_1 \oplus h_2 \oplus \cdots \oplus h_r \) such that \( h_i(E_j) = 0 \) if \( i \neq j \), \( h_i \subset \mathfrak{so}(E_i) \), and \( h_i \) acts irreducibly on \( E_i \). This result is stronger than theorem 2, without this it was necessary to suppose that a Berger algebra satisfies the conclusion of theorem 2. My attention to this statement was taken by A.J. Di Scala.

Part (II) of theorem 4 states that if \( h \subset \mathfrak{so}(E) \) is the holonomy algebra of a Riemannian manifold, then it is a weak-Berger algebra.

Using theory of representation of compact Lie algebras, we prove the converse statement in the case when \( \dim E \leq 9 \). Theorem 5 states that if \( \dim V \leq 11 \), then a weakly-irreducible subalgebra \( \mathfrak{g} \subset \mathfrak{so}(V)_{\mathbb{R}^p} \) is a Berger algebra iff the algebra \( h^0 \) is the holonomy algebra of a Riemannian manifold. This gives a classification of Berger algebras for Lorentzian manifolds of dimension \( \leq 11 \), which can be stated in the following way.

Let \( n_0, n_1, \ldots, n_r \) be positive integers such that \( 2 \leq n_1 \leq \cdots \leq n_r \) and \( n_0 + n_1 + \cdots + n_r = n \). Let \( h_i \subset \mathfrak{so}_{n_i} \) be the holonomy algebra of an irreducible Riemannian manifold \( (i = 1, \ldots, r) \). The Lie algebras of the form \( h = h_1 \oplus \cdots \oplus h_r \) exhaust all weak-Berger subalgebras of \( \mathfrak{so}_n \). The Lie algebras of the form \( \mathfrak{g}_1^h, \mathfrak{g}_2^h, \mathfrak{g}_3^{h,\varphi} \) and \( \mathfrak{g}_4^{h,\psi} \) (if \( \mathfrak{g}_3^{h,\varphi} \) and \( \mathfrak{g}_4^{h,\psi} \) exist) exhaust all Berger algebras for Lorentzian manifolds of dimension \( n + 2 \). Note that for each \( n > 1 \) there exists infinite number of weakly-irreducible Berger subalgebras of \( \mathfrak{so}(V)_{\mathbb{R}^p} \).

The full list of irreducible holonomy algebras of Riemannian manifolds of dimension \( \leq 9 \) is
given in the table below. In the table \( \otimes \) stands for the tensor product of representations; \( \otimes \) stands for the highest irreducible component of the corresponding product.

| \( n \) | Irreducible weak-Berger subalgebras of \( \mathfrak{so}_n \) |
|---|---|
| \( n = 1 \) | \( \mathfrak{so}_2 \) |
| \( n = 2 \) | \( \mathfrak{so}_3 \) |
| \( n = 3 \) | \( \mathfrak{so}_4, \mathfrak{su}_2, \mathfrak{u}_2 \) |
| \( n = 4 \) | \( \mathfrak{so}_5, \mathfrak{su}_2 \otimes \mathfrak{su}_3 \) |
| \( n = 5 \) | \( \mathfrak{so}_6, \mathfrak{su}_3, \mathfrak{u}_3 \) |
| \( n = 6 \) | \( \mathfrak{so}_7, \mathfrak{g}_2 \) |
| \( n = 7 \) | \( \mathfrak{so}_8, \mathfrak{su}_4, \mathfrak{u}_4, \mathfrak{sp}_2, \mathfrak{sp}_2 \otimes \mathfrak{sp}_1, \mathfrak{so}_3 \otimes \mathfrak{so}_3, \mathfrak{su}_2 \otimes \mathfrak{su}_3 \) |
| \( n = 8 \) | \( \mathfrak{so}_9, \mathfrak{so}_3 \otimes \mathfrak{so}_3 \) |

Recall that the holonomy group of an indecomposable Lorentzian manifold can be not closed. In [4] it was shown that the connected Lie subgroups of \( SO_{1,n+1} \) corresponding to Lie algebras of type 1 and 2 are closed; the connected Lie subgroup of \( SO_{1,n+1} \) corresponding to a Lie algebra of type 3 (resp. 4) is closed if and only if the connected subgroup of \( \mathfrak{so}_n \) corresponding to the subalgebra \( \ker \varphi \subset \mathfrak{z}(\mathfrak{h}) \) (resp. \( \ker \psi \subset \mathfrak{z}(\mathfrak{h}) \)) is closed. We give a criteria for Lie groups corresponding to Lie algebras of type 3 and 4 to be closed in terms of the Lie algebras \( \ker \varphi \) and \( \ker \psi \).

Let \( \mathfrak{h} \subset \mathfrak{so}_n \) be a weak-Berger algebra such that \( \mathfrak{z}(\mathfrak{h}) \neq \{0\} \). We have \( \mathfrak{h} = \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_r \), where \( \mathfrak{h}_i \) are irreducible weak-Berger algebras. We see that \( \mathfrak{z}(\mathfrak{h}) = \mathfrak{z}(\mathfrak{h}_1) \oplus \cdots \oplus \mathfrak{z}(\mathfrak{h}_r) \) and \( \dim \mathfrak{z}(\mathfrak{h}_i) = 0 \) or 1, \( i = 1, \ldots, r \). Hence we can identify \( \mathfrak{z}(\mathfrak{h}) \) with \( \mathbb{R}^m \), where \( m = \dim \mathfrak{z}(\mathfrak{h}) \).

We prove that the connected Lie group corresponding to a Lie algebra of type 3 (resp. 4) is closed if and only if there exists a basis \( v_1, \ldots, v_l \) (\( l = \dim \ker \varphi \) or \( \dim \ker \psi \)) of the vector space \( \ker \varphi \) (resp. \( \ker \psi \)) such that the coordinates of the vector \( v_i \) with respect to the canonical basis of \( \mathbb{R}^m \) are integer for \( i = 1, \ldots, l \).

**Remark.** Recently Thomas Leistner proved that a subalgebra \( \mathfrak{h} \subset \mathfrak{so}(E) \) is a weak-Berger algebra iff \( \mathfrak{h} \) is the holonomy algebra of a Riemannian manifold, see [9] [10] [11].

From this result and corollary 1 it follows that a weakly-irreducible subalgebra \( \mathfrak{g} \subset \mathfrak{so}(V)_{\mathbb{R}^p} \) is a Berger algebra iff \( \mathfrak{h}^\vartheta \) is the holonomy algebra of a Riemannian manifold.

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1 Preliminaries

Let \((V, \eta)\) be a Minkowski space of dimension \(n + 2\), where \(\eta\) is a metric on \(V\) of signature \((1, n + 1)\). We fix a basis \(p, e_1, \ldots, e_n, q\) of \(V\) such that the Gram matrix of \(\eta\) has the form

\[
\begin{pmatrix}
0 & 0 & 1 \\
0 & -E_n & 0 \\
1 & 0 & 0
\end{pmatrix},
\]

where \(E_n\) is the \(n\)-dimensional identity matrix.

Let \(E \subset V\) be the vector subspace spanned by \(e_1, \ldots, e_n\). We will consider \(E\) as an Euclidean space with the metric \(-\eta|_E\).

Denote by \(\mathfrak{so}(V)\) the Lie algebra of all \(\eta\)-skew symmetric endomorphisms of \(V\) and by \(\mathfrak{so}(V)_{\mathbb{R}p}\) the subalgebra of \(\mathfrak{so}(V)\) that preserves the line \(\mathbb{R}p\).

The algebra \(\mathfrak{so}(V)_{\mathbb{R}p}\) can be identified with the following matrix algebra:

\[
\mathfrak{so}(V)_{\mathbb{R}p} = \left\{ \begin{pmatrix}
a & X^t & 0 \\
0 & A & X \\
0 & 0 & -a
\end{pmatrix} : a \in \mathbb{R}, X \in \mathbb{R}^n, A \in \mathfrak{so}(n) \right\}.
\]

We identify the dual vector space \(V^*\) with \(V\) using \(\eta\). Hence we can identify \(\text{End} V = V \otimes V^*\) with \(V \otimes V\). In particular, we identify \(\mathfrak{so}(V)\) with \(V \wedge V = \text{span}\{u \wedge v = u \otimes v - v \otimes u : u, v \in V\}\).

Similarly, we identify \(\mathfrak{so}(E)\) with \(E \wedge E\) and consider \(\mathfrak{so}(E)\) as a subspace of \(\mathfrak{so}(V)\) that acts trivially on \(\mathbb{R}p \oplus \mathbb{R}q\).

For \(a \in \mathbb{R}\), the endomorphism \(ap \wedge q\) has the matrix \(\begin{pmatrix} a & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -a \end{pmatrix} \in \mathfrak{so}(V)_{\mathbb{R}p}\); for \(X \in E\),

\[
\begin{pmatrix}
0 & X^t & 0 \\
0 & 0 & X \\
0 & 0 & 0
\end{pmatrix} \in \mathfrak{so}(V)_{\mathbb{R}p}.
\]

Thus we see that

\[
\mathfrak{so}(V)_{\mathbb{R}p} = E \wedge E + p \wedge E + \mathbb{R}p \wedge q \text{ is a direct sum of the subalgebras.}
\]

**Definition 1.** A subalgebra \(\mathfrak{g} \subset \mathfrak{so}(V)\) is called irreducible if it preserves no proper subspace of \(V\); \(\mathfrak{g}\) is called weakly-irreducible if it preserves no nondegenerate proper subspace of \(V\).

Obviously, if \(\mathfrak{g} \subset \mathfrak{so}(V)\) is irreducible, then it is weakly-irreducible. If \(\mathfrak{g} \subset \mathfrak{so}(V)\) preserves a degenerate proper subspace \(U \subset V\), then it preserves the isotropic line \(U \cap U^\perp\); any such algebra is conjugated to a subalgebra of \(\mathfrak{so}(V)_{\mathbb{R}p}\).

**Definition 2.** Let \(W\) be a vector space and \(\mathfrak{f} \subset \mathfrak{gl}(W)\) a subalgebra. Put

\[
\mathcal{R}(\mathfrak{f}) = \{ R \in \text{Hom}(W \wedge W, \mathfrak{f}) : R(u \wedge v)w + R(v \wedge w)u + R(w \wedge u)v = 0 \text{ for all } u, v, w \in W \}.
\]

The set \(\mathcal{R}(\mathfrak{f})\) is called the space of curvature tensors of type \(\mathfrak{f}\). Denote by \(L(\mathcal{R}(\mathfrak{f}))\) the vector subspace of \(\mathfrak{f}\) spanned by \(R(u \wedge v)\) for all \(R \in \mathcal{R}(\mathfrak{f})\), \(u, v \in W\),

\[
L(\mathcal{R}(\mathfrak{f})) = \text{span}\{R(u \wedge v) : R \in \mathcal{R}(\mathfrak{f}), u, v \in W\}.
\]
Let $\mathfrak{g} \subset \mathfrak{so}(V)$ be a subalgebra. Recall that a curvature tensor $R \in \mathcal{R}(\mathfrak{g})$ satisfies the following property

$$\eta(R(u \wedge v)z, w) = \eta(R(z \wedge w)u, v)$$

for all $u, v, z, w \in V$. \hfill (1)

Let $(M, g)$ be a Lorentzian manifold of dimension $n+2$ and $\mathfrak{g}$ the holonomy algebra (that is the Lie algebra of the holonomy group) at a point $x$. By Wu’s theorem (see [14]) $(M, g)$ is locally indecomposable, i.e. is not locally a product of two pseudo-Riemannian manifolds if and only if the holonomy algebra $\mathfrak{g}$ is weakly-irreducible. If the holonomy algebra $\mathfrak{g}$ is irreducible, then $\mathfrak{g} = \mathfrak{so}(1, n+1)$. So we may assume that it is reducible and weakly-irreducible. Then it preserves an isotropic line $\ell \subset T_xM$. We can identify the tangent space $T_xM$ with $V$ such that $\ell$ corresponds to the line $\mathbb{R}p$. Then $\mathfrak{g}$ is identified with weakly-irreducible subalgebra of $\mathfrak{so}(V)_{\mathbb{R}p}$.

We need the following

**Proposition 1.** Let $\mathfrak{g}$ be the holonomy algebra of a Lorentzian manifold. Then

$$L(\mathcal{R}(\mathfrak{g})) = \mathfrak{g}.$$ 

**Proof.** The inclusion $L(\mathcal{R}(\mathfrak{g})) \subset \mathfrak{g}$ is obvious.

Let $R$ be the curvature tensor of $(M, g)$. Theorem of Ambrose and Singer ([2]) states that the vector space $\mathfrak{g}$ is generated by all endomorphisms

$$(\tau(\lambda))^{-1} \circ R_{\lambda(b)}(\tau(\lambda)(X), \tau(\lambda)(Y)) \circ \tau(\lambda) : T_xM \to T_xM,$$

where $\lambda : [a, b] \to M$ is a piecewise smooth curve in $M$ such that $\lambda(a) = x$, $\tau(\lambda)$ is the parallel transport along $\lambda$, and $X, Y \in T_{\lambda(a)}M$. Obviously, the above transformations are curvature tensors of type $\mathfrak{g}$, hence, $\mathfrak{g} \subset L(\mathcal{R}(\mathfrak{g}))$. Thus, $\mathfrak{g} = L(\mathcal{R}(\mathfrak{g}))$. \hfill $\square$

**Definition 3.** A weakly-irreducible subalgebra $\mathfrak{g} \subset \mathfrak{so}(V)_{\mathbb{R}p}$ is called a Berger algebra if $L(\mathcal{R}(\mathfrak{g})) = \mathfrak{g}$.

**Definition 4.** Let $\mathfrak{h} \subset \mathfrak{so}(E)$ be a subalgebra. Put

$$\mathcal{P}(\mathfrak{h}) = \{P \in \text{Hom}(E, \mathfrak{h}) : \eta(P(u)v, w) + \eta(P(v)w, u) + \eta(P(w)u, v) = 0 \text{ for all } u, v, w \in E\}.$$ 

We call $\mathcal{P}(\mathfrak{h})$ the space of weak-curvature tensors of type $\mathfrak{h}$. A subalgebra $\mathfrak{h} \subset \mathfrak{so}(E)$ is called a weak-Berger algebra if $L(\mathcal{P}(\mathfrak{h})) = \mathfrak{h}$, where

$$L(\mathcal{P}(\mathfrak{h})) = \text{span}(\{P(u) : P \in \mathcal{P}(\mathfrak{h}), u \in E\})$$

is the vector subspace of $\mathfrak{h}$ spanned by $P(u)$ for all $P \in \mathcal{P}(\mathfrak{h})$ and $u \in E$.

Let $\mathfrak{h} \subset \mathfrak{so}(E)$ be a subalgebra. Since $\mathfrak{h}$ is a compact Lie algebra, we have $\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{z}(\mathfrak{h})$ (the direct sum of ideals), where $\mathfrak{h}'$ is the commutant of $\mathfrak{h}$ and $\mathfrak{z}(\mathfrak{h})$ is the center of $\mathfrak{h}$. 

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Consider a weakly-irreducible subalgebra $g \subset \mathfrak{so}(V)_{\mathbb{R}_p}$. Let $h^g$ be the projection of $g$ to $\mathfrak{so}(E)$ with respect to the decomposition $\mathfrak{so}(V)_{\mathbb{R}_p} = \mathfrak{so}(E) + p \wedge E + \mathbb{R}p \wedge q$.

**Definition 5.** The Lie algebra $h^g$ is called the orthogonal part of $g$.

Conversely, with any subalgebra $h \subset \mathfrak{so}(E)$ we associate two Lie algebras

$$g^h_1 = \mathbb{R}p \wedge q + h + p \wedge E$$

and

$$g^h_2 = h + p \wedge E.$$

Moreover, suppose $\mathfrak{z}(h) \neq \{0\}$. Let

$$\varphi : \mathfrak{z}(h) \to \mathbb{R}$$

be a non-zero linear map. Extend $\varphi$ to the linear map $\varphi : h \to \mathbb{R}$ by putting $\varphi|_h = 0$. Then

$$g^{h,\varphi}_3 = p \wedge E + \{A + \varphi(A)p \wedge q : A \in h\}$$

is a Lie algebra with the orthogonal part $h$.

Suppose moreover that we have an orthogonal decomposition $E = E_0 \oplus E_1$ such that $E_0 \neq \{0\}$, $h \subset \mathfrak{so}(E_1)$, and $\dim \mathfrak{z}(h) \geq \dim E_0$. Let

$$\psi : \mathfrak{z}(h) \to E_0$$

be a surjective linear map. As above, we extend $\psi$ to a linear map $\psi : h \to E_0$ by putting $\psi|_h = 0$. Then

$$g^{h,\psi}_4 = p \wedge E_1 + \{A + \psi(A) : A \in h\}$$

is a Lie algebra with the orthogonal part $h$.

We call the Lie algebras $g^{h}_1$, $g^{h}_2$, $g^{h,\varphi}_3$ and $g^{h,\psi}_4$ the Lie algebras of type 1, 2, 3 and 4 respectively.

These Lie algebras were considered by L. Berard Bergery and A. Ikemakhen, who proved the following fundamental results (see [4]).

**Theorem 1.** Let $h \subset \mathfrak{so}(E)$ be a subalgebra. Then (if $g^{h,\varphi}_3$ and $g^{h,\psi}_4$ exist) the subalgebras $g^{h}_1$, $g^{h}_2$, $g^{h,\varphi}_3$, $g^{h,\psi}_4 \subset \mathfrak{so}(V)_{\mathbb{R}_p}$ are weakly-irreducible. Moreover, Lie algebras of the form $g^{h}_1$, $g^{h}_2$, $g^{h,\varphi}_3$ and $g^{h,\psi}_4$ exhaust all weakly-irreducible subalgebras of $\mathfrak{so}(V)_{\mathbb{R}_p}$.

**Theorem 2.** Let $g$ be the holonomy algebra of a Lorentzian manifold. Then there exists an orthogonal decomposition $E = E_0 \oplus E_1 \oplus \cdots \oplus E_r$ and the corresponding decomposition into the direct sum of ideals $h^g = \{0\} \oplus h_1 \oplus h_2 \oplus \cdots \oplus h_r$ such that $h_i(E_j) = 0$ if $i \neq j$, $h_i \subset \mathfrak{so}(E_i)$, and $h_i$ acts irreducibly on $E_i$.

The metric $\eta$ on $V$ induces the metrics on $V \otimes V$ and $V \wedge V$. Denote those metrics by $\eta \otimes \eta$ and $\eta \wedge \eta$ respectively.
Let $\theta : V \to V$ be an endomorphism, $u, v \in V$. Then $\eta(\theta(u), v) = \eta \otimes \eta(\theta, u \otimes v)$. In particular, for $\theta \in V \wedge V$ we have

$$
\eta(\theta(u), v) = 1/2 \eta \wedge \eta(\theta, u \wedge v).
$$

(2)

Let $R \in \mathcal{R}(g)$. Combining (1) and (2), we see that $\eta \wedge \eta(R(u \wedge v), z \wedge w) = \eta \wedge \eta(R(z \wedge w), u \wedge v)$ for all $u, v, z, w \in V$. This means that the linear map $R : V \wedge V \to g \subset V \wedge V$ is $\eta \wedge \eta$-symmetric.

Let $(E_1, \mu_1)$ and $(E_2, \mu_2)$ be two Euclidean spaces. Let $f : E_1 \to E_2$ be a linear map. Denote by $f^* : E_2 \to E_1$ the dual linear map for $f$. We identify the symmetric square $S^2(E)$ of $E$ with the space of all $\eta$-symmetric endomorphisms of $E$.

\section{Main results}

Let $\mathfrak{h} \subset \mathfrak{so}(E)$ be a subalgebra. We will define some sets of endomorphisms, in theorem 3 we will see that those sets consist of the curvature tensors for appropriate algebras.

For any $\lambda \in \mathbb{R}$, $L \in \text{Hom}(E, \mathbb{R})$, $T \in S^2(E)$ and $P \in \mathcal{P}(\mathfrak{h})$ we define the endomorphisms

$$
R^\lambda \in \text{Hom}(V \wedge V, \mathfrak{g}_1^b),
R^L \in \text{Hom}(V \wedge V, \mathfrak{g}_1^b),
R^T \in \text{Hom}(V \wedge V, p \wedge E)
$$

and

$$
R^P \in \text{Hom}(V \wedge V, \mathfrak{g}_2^b)
$$

by conditions

$$
R^\lambda(p \wedge q) = \lambda p \wedge q, \quad R^\lambda|_{p \wedge E + q \wedge E \wedge E} = 0,
$$

$$
R^L(q \wedge \cdot) = L(\cdot)p \wedge q, \quad R^L(p \wedge q) = p \wedge L^*(1), \quad R^L|_{p \wedge E + p \wedge E \wedge E} = 0,
$$

$$
R^T(q \wedge \cdot) = p \wedge T(\cdot), \quad R^T|_{p \wedge q + p \wedge E \wedge E} = 0
$$

and

$$
R^P(q \wedge \cdot) = P(\cdot), \quad R^P|_{E \wedge E} = -1/2 p \wedge p^*, \quad R^P|_{E \wedge q + p \wedge E} = 0,
$$

and define by

$$
\mathcal{R}(\mathbb{R}, \mathbb{R}), \quad \mathcal{R}(E, \mathbb{R}), \quad \mathcal{R}(E, E)
$$

and

$$
\mathcal{R}(E, \mathfrak{h}),
$$

respectively the vector spaces of all such endomorphisms.

We have the isomorphisms $\mathcal{R}(\mathbb{R}, \mathbb{R}) \simeq \mathbb{R}$, $\mathcal{R}(E, \mathbb{R}) \simeq E$, $\mathcal{R}(E, E) \simeq S^2(E)$ and $\mathcal{R}(E, \mathfrak{h}) \simeq \mathcal{P}(\mathfrak{h})$.

Moreover, if a Lie algebra $\mathfrak{g}_3^{b, \varphi}$ exists, then for any $P \in \mathcal{P}(\mathfrak{h})$ we define the endomorphism

$$
R^P \in \text{Hom}(V \wedge V, \mathfrak{g}_3^{b, \varphi})
$$

by conditions

$$
R^P(q \wedge \cdot) = P(\cdot) + \varphi(P(\cdot))p \wedge q, \quad R^P|_{E \wedge E} = -1/2 p \wedge p^*,
$$

$$
R^P(p \wedge q) = -1/2 p \wedge p^*(\varphi^*(1)), \quad R^P|_{p \wedge E} = 0
$$

and denote by $\mathcal{R}(E, \mathfrak{h}, \varphi)$ the vector space of all such endomorphisms.
If a Lie algebra \( g_i^{h,\psi} \) exists, then for any \( P \in \mathcal{P}(h) \) we define the endomorphism \( R^P \in \text{Hom}(V \wedge V, g_i^{h,\psi}) \) by conditions

\[
R^P(q \wedge u_1) = P(u_1) + p \wedge \psi(P(u_1)) \text{ for all } u_1 \in E_1, \quad R^P|_{E_1} = -1/2p \wedge P^*,
\]

\[
R^P(q \wedge u_0) = -1/2p \wedge P^*(\psi^*(u_0)) \text{ for all } u_0 \in E_0, \quad R^P|_{E_0} = 0
\]

and denote by \( \mathcal{R}(E_1, h, \psi) \) vector space of all such endomorphisms.

We have the isomorphisms \( \mathcal{R}(E, h, \varphi) \cong \mathcal{P}(h) \) and \( \mathcal{R}(E_1, h, \psi) \cong \mathcal{P}(h) \).

Let \( g \subset \mathfrak{so}(V)_{\mathbb{R}_p} \) be a weakly-irreducible subalgebra, \( h^0 \) be the orthogonal part of \( g \) and \( h \subset h^0 \) be a subalgebra. Suppose \( R \in \mathcal{R}(h) \). Extend the linear map \( R : E \wedge E \rightarrow h \) to the linear map \( R : V \wedge V \rightarrow h \) by putting \( R^P|_{\mathbb{R}_p \wedge q + p \wedge E + q \wedge E} = 0 \). It is obvious that \( R \in \mathcal{R}(g) \). We can write \( \mathcal{R}(h) \subset \mathcal{R}(g) \).

**Theorem 3.** Let \( h \subset \mathfrak{so}(E) \) be a subalgebra. Then we have

(I) \( \mathcal{R}(g_1^{h}) = \mathcal{R}(g_2^{h}) \oplus \mathcal{R}(E, \mathbb{R}) \oplus \mathcal{R}(\mathbb{R}, \mathbb{R}) \);

(II) \( \mathcal{R}(g_2^{h}) = \mathcal{R}(h) \oplus \mathcal{R}(E, h) \oplus \mathcal{R}(p \wedge E) \) and \( \mathcal{R}(p \wedge E) = \mathcal{R}(E, E) \);

(III) if a Lie algebra \( g_3^{h,\varphi} \) exists, then

\[
\mathcal{R}(g_3^{h,\varphi}) = \mathcal{R}(\ker \varphi) \oplus \mathcal{R}(E, h, \varphi) \oplus \mathcal{R}(p \wedge E);
\]

(IV) if a Lie algebra \( g_4^{h,\psi} \) exists, then

\[
\mathcal{R}(g_4^{h,\psi}) = \mathcal{R}(\ker \psi) \oplus \mathcal{R}(E_1, h, \psi) \oplus \mathcal{R}(p \wedge E_1).
\]

**Remark.** It is known (see [3]) that the holonomy algebra of a weakly irreducible, non-irreducible locally symmetric Lorentzian manifold equals \( p \wedge E = g_2^{(0)} \) but this algebra can also be the holonomy algebra of a nonlocally symmetric Lorentzian manifold (see [4]).

**Corollary 1.** Let \( g \subset \mathfrak{so}(V)_{\mathbb{R}_p} \) be a weakly-irreducible subalgebra. Then \( g \) is a Berger algebra if and only if \( h^0 \) is a weak-Berger algebra.

**Corollary 2.** Let \( g \subset \mathfrak{so}(V)_{\mathbb{R}_p} \) be a weakly-irreducible subalgebra such that \( h^0 \) is the holonomy algebra of a Riemannian manifold. Then \( g \) is a Berger algebra.

**Theorem 4.** Let \( h \subset \mathfrak{so}(E) \) be a weak-Berger algebra. Then

(I) There exists an orthogonal decomposition \( E = E_0 \oplus E_1 \oplus \cdots \oplus E_r \) and the corresponding decomposition into the direct sum of ideals \( h = \{0\} \oplus h_1 \oplus h_2 \oplus \cdots \oplus h_r \) such that \( h_i(E_j) = 0 \) if \( i \neq j \), \( h_i \subset \mathfrak{so}(E_i) \), and \( h_i \) acts irreducibly on \( E_i \).

(II) We have a decomposition

\[
\mathcal{P}(h) = \mathcal{P}(h_1) \oplus \cdots \oplus \mathcal{P}(h_r).
\]

In particular, \( h_i \) is a weak-Berger algebra for \( i = 1, ..., k \).

Corollary 1 and theorem 4 reduce the classification problem for the weakly-irreducible, non-irreducible holonomy algebras of Lorentzian manifolds to the classification of irreducible weak-Berger algebras.

In section 4 we will obtain the following theorem.
Theorem 5. Let \( \dim V \leq 11 \), let \( \mathfrak{g} \subset \mathfrak{so}(V)_{\mathbb{R}p} \) be a weakly-irreducible subalgebra. Then \( \mathfrak{g} \) is a Berger algebra if and only if \( \mathfrak{h}^0 \) is the holonomy algebra of a Riemannian manifold.

3 Proof of the results

3.1 Proof of part (II) of theorem 3

Let \( R \in \mathcal{R}(\mathfrak{g}^0) \). Above we saw that \( R : V \wedge V \to \mathfrak{g}^0 \subset V \wedge V \) is a \( \eta \wedge \eta \)-symmetric linear map.

It is clear that \( V \wedge V = E \wedge E + p \wedge E + q \wedge E + \mathbb{R}p \wedge q \).

**Lemma 1.** \( R|_{(p \wedge E + p \wedge q)} = 0 \).

**Proof.** By (1), we have \( \eta(R(p \wedge v)z, w) = \eta(R(z \wedge w)p, v) \) for all \( v, z, w \in V \). Since \( R(w \wedge z) \in \mathfrak{h} \), we obtain \( R(w \wedge z)p = 0 \) and \( \eta(R(p \wedge v)z, w) = 0 \). Since \( \eta \) is nondegenerate, we have \( R(p \wedge v)z = 0 \). Thus, \( R(p \wedge v) = 0 \). \( \square \)

We see that \( R \) is a linear map from \( E \wedge E + q \wedge E \) to \( \mathfrak{h} + p \wedge E \).

Let \( \Gamma \subset V \wedge V \) be one of the subspaces \( E \wedge E, p \wedge E, q \wedge E \) and \( \mathbb{R}p \wedge q \). Denote by \( p_{\Gamma} \) the projection of \( V \wedge V \) onto \( \Gamma \) with respect to the decomposition \( V \wedge V = E \wedge E + p \wedge E + q \wedge E + \mathbb{R}p \wedge q \).

Denote by \( \mathfrak{h}^\perp \) the orthogonal complement of \( \mathfrak{h} \) in \( E \wedge E \), \( \mathfrak{h}^\perp = \{ \xi \in E \wedge E : \eta(\eta(\xi, \mathfrak{h})) = \{0\} \} \).

Denote by \( p_{\mathfrak{h}} \) the projection of \( V \wedge V \) onto \( \mathfrak{h} \) with respect to the decomposition \( V \wedge V = \mathfrak{h} + \mathfrak{h}^\perp + p \wedge E + q \wedge E + \mathbb{R}p \wedge q \).

We associate with \( R \) the following linear maps:

\[
R_h = p_{E \wedge E} \circ R|_{E \wedge E} : E \wedge E \to \mathfrak{h},
\]

\[
R^h_E = p_{p \wedge E} \circ R|_{E \wedge E} : E \wedge E \to p \wedge E,
\]

\[
R^E_h = p_{E \wedge E} \circ R|_{q \wedge E} : q \wedge E \to \mathfrak{h},
\]

\[
R_E = p_{p \wedge E} \circ R|_{q \wedge E} : p \wedge E \to q \wedge E
\]

and extend these maps to \( V \wedge V \) mapping the natural complement to zero. Then \( R = R_h + R^h_E + R^E_h + R_E \).

**Lemma 2.** \( R^h_E|_{\mathfrak{h}^\perp} = 0; R_h|_{\mathfrak{h}^\perp} = 0 \).

**Proof.** Let \( \theta \in \mathfrak{h}^\perp \) and \( \xi \in \mathfrak{h} \). Since the linear map \( R \) is \( \eta \wedge \eta \)-symmetric, we have \( \eta \wedge \eta(R(\theta), \xi) = \eta \wedge \eta(R(\xi), \theta) \). Hence, \( \eta \wedge \eta(R_h(\theta) + R^h_E(\theta), \xi) = \eta \wedge \eta(R_h(\xi) + R^h_E(\xi), \theta) \). Since \( (p \wedge E) \perp (E \wedge E) \), \( \theta \perp R_h(\xi) \), and \( R_h(\xi) \in \mathfrak{h} \), we obtain \( \eta \wedge \eta(R_h(\theta), \xi) = 0 \). Since the restriction of the form \( \eta \wedge \eta \) to \( E \wedge E \) is nondegenerate, we have \( R_h(\theta) = 0 \).

Similarly, suppose \( \xi \in q \wedge E \); then \( R(\xi) = R^h_E(\xi) + R_E(\xi) \). Since \( (q \wedge E) \perp (E \wedge E) \) and the restriction of the form \( \eta \wedge \eta \) to \( p \wedge E + q \wedge E \) is nondegenerate, we see that \( R^h_E(\theta) = 0 \). \( \square \)

We define the linear maps \( Q : \mathfrak{h} \to E, T : E \to E \) and \( P : E \to E \) by conditions

\[
R^h_E(u \wedge v) = -1/2(p \wedge Q(u \wedge v)),
\]

\[
(3)\]
\[ R_E(q \wedge u) = p \wedge T(u) \]  \hfill (4)

and

\[ P(u) = R^E_b(q \wedge u) \]  \hfill (5)

for all \( u, v \in E \).

**Lemma 3.** \( P^* = Q, T^* = T \).

Proof. By (1) we have \( \eta(R(u \wedge v)q, w) = \eta(R(q \wedge w)u, v) \) for all \( u, v, w \in E \). Hence, \( \eta(R^b_E(u \wedge v)q, w) = \eta(R^E_b(q \wedge w)u, v) \). Using (3) and (4), we get

\[ -1/2 \eta(p \wedge Q(u \wedge v))q, w) = \eta(P(w)u, v). \]

Hence, \(-1/2 \eta(Q(u \wedge v), w) = \eta(P(w)u, v)\). Identity (2) implies

\[ -\eta(Q(u \wedge v), w) = \eta(\eta(P(w), u \wedge v). \]

We have proved the first part of the lemma. The second part follows from the equality \( \eta(R(q \wedge u)q, v) = \eta(R(q \wedge v)q, u) \) for all \( u, v \in E \). \( \Box \)

For the tensor \( R \) we must check the Bianchi identity \( R(u \wedge v)w + R(v \wedge w)u + R(w \wedge u)v = 0 \) for all \( u, v, w \in V \). It is sufficient to check the Bianchi identity only for the basis vectors. If two of the vectors \( u, v, w \) are equal or one of the vectors \( u, v, w \) equals \( p \), the identity holds trivially. Thus it is sufficient to check the Bianchi identity in the two cases: \( u, v, w \in E; u, v \in E, w = q \). We do this in the following lemma.

**Lemma 4.** \( R_b \in R(h), P \in P(h) \).

Proof. Let us write the Bianchi identity for \( u, v, w \in E; R(u \wedge v)w + R(v \wedge w)u + R(w \wedge u)v = 0 \). From the equalities like \( R(u \wedge v) = R_b(u, v) + R^b_E(u, v) \) and (3) it follows that \( R_b(u \wedge v)w + R_b(v \wedge w)u + R_b(w \wedge u)v = -1/2(p \wedge Q(u \wedge v))w - 1/2(p \wedge Q(v \wedge w))u - 1/2(p \wedge Q(w \wedge u))v = 0. \)

Since \((p \wedge Q(u \wedge v))w = -\eta(Q(u \wedge v), w)p \in \mathbb{R}p \) and \( R_b(u \wedge v)w \in E \), we obtain

\[ R_b(u \wedge v)w + R_b(v \wedge w)u + R_b(w \wedge u)v = 0 \]  \hfill (6)

and

\[ \eta(Q(u \wedge v), w) + \eta(Q(v \wedge w), u) + \eta(Q(w \wedge u), v) = 0. \]  \hfill (7)

Identity (6) shows that \( R_b \in R(h) \).

Now write the Bianchi identity for \( u, v \in E \) and \( q; R(u \wedge v)q + R(v \wedge q)u + R(q \wedge u)v = 0 \). Hence, \( R^b_E(u \wedge v)q + R^b_E(v \wedge q)u + R^E_b(q \wedge u)v = 0 \). Combining this with (3) and (5), we obtain

\[ -1/2(p \wedge Q(u \wedge v))q - P(v)u + P(u)v = 0. \]

Hence, \( 1/2Q(u \wedge v)q + P(v)u - P(u)v = 0 \). Combining this with (7) and using the fact that for \( z \in E \) the endomorphism \( P(z) \) is \( \eta \)-skew symmetric, we obtain

\[ \eta(P(u)v, w) + \eta(P(v)w, u) + \eta(P(w)u, v) = 0 \text{ for all } u, v, w \in E. \]  \hfill (8)
Identity (4) implies $P \in \mathcal{P}(\mathfrak{h})$.

We put $R^p = R^h_E + R^E_h$, $R^T = R_E$. Then $R^p \in \mathcal{R}(E, \mathfrak{h})$, $R^T \in \mathcal{R}(p \wedge E)$, and $R = R_h + R^p + R^T$. We have proved that $\mathcal{R}(\mathfrak{g}^h_2) \subset \mathcal{R}(\mathfrak{h}) \oplus \mathcal{R}(E, \mathfrak{h}) \oplus \mathcal{R}(p \wedge E)$. Now we check the inverse inclusion.

Suppose $T \in S^2(E)$, i.e. $T : E \to E$ is a linear map such that $T^* = T$. Put $R^T(q \wedge u) = p \wedge T(u), R^T(p \wedge q) = R^T(p \wedge u) = R^T(u \wedge v) = 0$ for all $u, v \in E$. Then $R^T \in \mathcal{R}(p \wedge E) \subset \mathcal{R}(\mathfrak{g}^h_2)$.

Let $P \in \mathcal{P}(\mathfrak{h})$ and $R^p \in \mathcal{R}(E, \mathfrak{h})$. We must check the Bianchi identity for the tensor $R^p$.

For $u, v, w \in E$ the identity follows from (4).

Suppose $u, v \in E$; then $R^p(u \wedge v)q + R^p(v \wedge q)u + R^p(q \wedge u)v = -1/2(p \wedge P^*(u \wedge v))q - P^*(u \wedge v) + P(u)v = -1/2P^*(u \wedge v) - P(v)u + P(u)v$.

Suppose $w \in E$; then $\eta(-1/2P^*(u \wedge v) - P(v)u + P(u)v, w) = \eta(-1/2P^*(u \wedge v), w) - \eta(P(u)v, w) + \eta(P(v)u, w) = -1/2\eta(P^*(u \wedge v), w) + \eta(P(v)w, u) + \eta(P(u)v, w) = -1/2\eta(P^*(u \wedge v), w) - \eta(P(w)u, v) = -1/2(\eta(P^*(u \wedge v), w)) + \eta \wedge \eta(P(w), u \wedge v) = 0$. Above we used the fact that $P(v) \in \mathfrak{h} \subset \mathfrak{so}(E)$, (2) and (1). Since the restriction of $\eta$ to $E$ is nondegenerate, we obtain $R^p(u \wedge v)q + R^p(v \wedge q)u + R^p(q \wedge u)v = 0$.

Now we have $\mathcal{R}(\mathfrak{h}) \oplus \mathcal{R}(E, \mathfrak{h}) \oplus \mathcal{R}(p \wedge E) \subset \mathcal{R}(\mathfrak{g}^h_2)$. Thus, $\mathcal{R}(\mathfrak{g}^h_2) = \mathcal{R}(\mathfrak{h}) \oplus \mathcal{R}(E, \mathfrak{h}) \oplus \mathcal{R}(p \wedge E)$.

### 3.2 Proof of part (I) of theorem 3

Let $R \in \mathcal{R}(\mathfrak{g}^h_2)$. Similarly to lemma 1, we can prove that $R|_{p \wedge E} = 0$. Hence $R$ is a linear map from $E \wedge E + q \wedge E + \mathbb{R}p \wedge q$ to $\mathfrak{h} + p \wedge E + \mathbb{R}p \wedge q$.

We can define the linear maps $R^h, R^h_E, R^E_h$ and $R_E$ as in section 3.1. It is easily shown that the map $R_2 = R_h + R^h_E + R^E_h + R_E$ is a curvature tensor of type the Lie algebra $\mathfrak{g}^h_2$.

We define the following linear maps:

$$R^h = p_{\mathbb{R}p \wedge q} \circ R|_{E \wedge E} : E \wedge E \to \mathbb{R}p \wedge q,$$

$$R^E = p_{E \wedge E} \circ R|_{E \wedge E} : \mathbb{R}p \wedge q \to E \wedge E,$$

$$R^p = p_{p \wedge E} \circ R|_{p \wedge q} : \mathbb{R}p \wedge q \to p \wedge E,$$

$$R^q = p_{q \wedge E} \circ R|_{q \wedge E} : q \wedge E \to \mathbb{R}p \wedge q,$$

and extend these maps to $V \wedge V$ sending the natural complement to zero.

We have $R = R_2 + R^h + R^h_E + R^E_h + R_E$.

**Lemma 5.** $R^h = 0, R^E = 0$.

Proof. Let as write the Bianchi identity for vectors $u, v \in E$ and $p$; $R(u \wedge v)p + R(v, p)u + R(p \wedge u)v = 0$. Since $R(v \wedge p) = R(p \wedge u) = 0$ and $R(u \wedge v)p = R^h(u \wedge v)p$, we see that $R^h(u \wedge v) = 0$. 

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Using (1), we get \( \eta(R(p \land q)u, v) = \eta(R(u \land v)p, q) = 0. \) Since \( u \) and \( v \) are arbitrary and the restriction of \( \eta \) to \( E \) is not degenerate, we obtain \( R^\mathbb{R}_h = R(p \land q)|_E = 0. \) \( \square \)

Define the linear maps \( L : E \to \mathbb{R} \) and \( K : \mathbb{R} \to E \) by conditions

\[
R^E_{\mathbb{R}}(q \land u) = L(u)p \land q \text{ for all } u \in E,
\]

\[
R^\mathbb{R}_E(a p \land q) = p \land K(a) \text{ for all } a \in \mathbb{R}.
\]

**Lemma 6.** \( K = L^*. \)

The proof is similar to the proof of lemma 3. \( \square \)

Let \( \lambda \) be a real number such that \( R_\mathbb{R}(p \land q) = \lambda p \land q. \) Put \( R_L = R^E_{\mathbb{R}} + R^\mathbb{R}_E \) and \( R^\lambda = R_\mathbb{R}. \) We see that \( R_L \in \mathcal{R}(E, \mathbb{R}), \) \( R^\lambda \in \mathcal{R}(\mathbb{R}, \mathbb{R}), \) and \( R = R_2 + R_L + R^\lambda \in \mathcal{R}(g^h_3) \oplus \mathcal{R}(E, \mathbb{R}) \oplus \mathcal{R}(\mathbb{R}, \mathbb{R}). \) Thus we have \( \mathcal{R}(g^h_1) \subset \mathcal{R}(g^h_3) \oplus \mathcal{R}(E, \mathbb{R}) \oplus \mathcal{R}(\mathbb{R}, \mathbb{R}). \) The inverse inclusion is obvious.

### 3.3 Proof of part (III) of theorem 3

We have \( g^h_2 \subset g^{h, \varphi}_3 \subset g^h_3. \) Suppose \( R \in \mathcal{R}(g^{h, \varphi}_3), \) then we have \( R \in \mathcal{R}(g^h_1). \) From section 3.2 it follows that \( R = R_2 + R^E_{\mathbb{R}} + R^\mathbb{R}_E + R_L, \) where \( R_2 = R_1 + R^h_1 + R^E_{\mathbb{R}} + R_E \in \mathcal{R}(g^h_3). \)

We claim that \( R^\mathbb{R}_L = 0. \) Indeed, there exists a \( \lambda \in \mathbb{R} \) such that \( R_\mathbb{R}(p \land q) = \lambda p \land q; \) we have \( R(p \land q) = R_\mathbb{R}(p \land q) + R^E_{\mathbb{R}}(p \land q) = \lambda p \land q + R^\mathbb{R}_E(p \land q). \) Since \( R^\mathbb{R}_E(p \land q) \in p \land E \subset g^{h, \varphi}_3 \) and \( \mathbb{R} p \land q \cap g^{h, \varphi}_3 = \{0\}, \) we see that \( \lambda = 0. \)

For any \( u \in E \) we have \( R(q \land u) = R^E_{\mathbb{R}}(q \land u) + R^E_{\mathbb{R}}(q \land u) \in \mathbb{R} p \land q. \) Since \( R(q \land u) \in g^{h, \varphi}_3, \) we see that \( R^E_{\mathbb{R}}(q \land u) = \varphi(R^E_{\mathbb{R}}(q \land u) p \land q. \) Hence, \( R^E_{\mathbb{R}} = p \land q \varphi \circ R^E_{\mathbb{R}}. \) Using (3) and (6), we get \( L(u) = \varphi(P(u)). \) Hence for any \( a \in \mathbb{R} \) we have \( L^*(a) = P^*(\varphi^*(a)). \) Using this, (10) and lemma 6, we obtain \( R^E_E(a p \land q) = p \land L^*(a) = p \land P^*(\varphi^*(a)). \) Using (3), we obtain \( R^E_E(a p \land q) = R^h_p \circ \varphi^*(a) \) for all \( a \in \mathbb{R}. \)

Suppose \( u, v \in E; \) then we have \( R(u \land v) = R_h(u \land v) + R^h_{E}(u \land v) \in g^{h, \varphi}_3. \) Hence, \( R_h(u \land v) \in \ker \varphi. \) We see that \( R_h \in \mathcal{R}(\ker \varphi). \) Put \( R_{\ker \varphi} = R_h. \)

Put \( R^P = R^h + R^h + R^E_{\mathbb{R}} + R^E_E. \) We see that \( R^P \in \mathcal{R}(E, \mathbb{H}, \varphi). \) Thus, \( R = R_{\ker \varphi} + R^P + R_E \in \mathcal{R}(\ker \varphi) \oplus \mathcal{R}(E, H, \varphi) \oplus \mathcal{R}(\mathbb{R} \land E). \)

Conversely, let \( R = R_{\ker \varphi} + R^P + R_E \in \mathcal{R}(\ker \varphi) \oplus \mathcal{R}(E, H, \varphi) \oplus \mathcal{R}(\mathbb{R} \land E). \) From section 3.2 it follows that \( R \in \mathcal{R}(g^h_1). \) Since for any \( u, v \in V \) we have \( R(u \land v) \in g^{h, \varphi}_3, \) we see that \( R \in \mathcal{R}(g^{h, \varphi}_3). \)

### 3.4 Proof of part (IV) of theorem 3

By definition, \( g^2 = h + p \land E_1 \) is a Lie algebra of type 2. We have \( g^2 \subset g^{h, \psi}_4 \subset g^h_2. \) Suppose that \( R \in \mathcal{R}(g^{h, \psi}_4), \) then we have \( R \in \mathcal{R}(g^h_2). \) From section 3.1 it follows that \( R = R_h + R^h + R^E_E + R_E. \) There exists a \( P \in \mathcal{P}(h) \) such that \( R^E_{\mathbb{R}}(q \land u) = P(u) \) for all \( u \in E. \) Let \( u_1, v_1 \in E_1, u_0 \in E_0. \) We
have \( \eta(P(u_1)v_1, u_0) + \eta(P(v_1)u_0, u_1) + \eta(P(u_0)u_1, v_1) = 0 \). Since \( h(E_0) = \{0\} \) and \( h(E_1) \subset E_1 \), we see that \( \eta(P(u_0)u_1, v_1) = 0 \) for all \( u_1, v_1 \in E_1, u_0 \in E_0 \). Hence, \( P(E_0) = \{0\}, P^*(h) \subset E_1 \).

We can write \( R^E_h = R^E_{E_1} \) and \( R^h_E = R^h_{E_1} \).

We consider the following linear maps:

\[
R^E_{E_1} = p_{p \land E_1} \circ R^E_{|q \land E_1} : q \land E_1 \to p \land E_1,
\]

\[
R^E_{E_0} = p_{p \land E_0} \circ R^E_{|q \land E_0} : q \land E_0 \to p \land E_1,
\]

\[
R^E_{E_0} = p_{p \land E_0} \circ R^E_{|q \land E_1} : q \land E_1 \to p \land E_0,
\]

\[
R_{E_0} = p_{p \land E_0} \circ R^E_{|q \land E_0} : q \land E_0 \to p \land E_0
\]

and extend these maps to \( V \lor V \) mapping the complementary subspace to zero. Obviously, \( R_E = R_{E_1} + R^E_{E_0} + R^E_{E_1} + R_{E_0} \).

We claim that \( R_{E_0} = 0 \). Indeed, for \( u_0 \in E_0 \) we have \( R(q \land u_0) = R_E(q, u_0) = R^E_{E_0}(q \land u_0) + R_{E_0}(q \land u_0) \in \mathfrak{g}_4^{h, \psi} \); since \( R^E_{E_1}(q \land u_0) \in \mathfrak{g}_4^{h, \psi}, R_{E_0}(q \land u_0) \in \mathfrak{g}_4^{h, \psi} \) and \( p \land E_0 \cap \mathfrak{g}_4^{h, \psi} = \{0\} \), we obtain \( R_{E_0}(q \land u_0) = 0 \), hence, \( R_{E_0} = 0 \).

As in section 3.3, we can prove that \( R_h \in \mathcal{R}(\ker \psi), R^E_{E_1} = p \land \psi \circ R^E_{E_1}, \) and \( R^E_{E_0}(q \land u_0) = R^E_{E_1} \circ \psi^*(u_0) \) for all \( u_0 \in E_0 \).

Put \( R^P = R^E_{h_1} + R^E_{E_1} + R^E_{E_0} + R^E_{E_1} \). Thus we have \( R = R_{\ker \psi} + R^P + R_{E_1} \in \mathcal{R}(\ker \psi) \oplus \mathcal{R}(E_1, h, \psi) \oplus \mathcal{R}(p \land E_1) \).

Conversely, let \( R = R_{\ker \psi} + R^P + R_{E_1} \in \mathcal{R}(\ker \psi) \oplus \mathcal{R}(E_1, h, \psi) \oplus \mathcal{R}(p \land E_1) \). From section 3.1 it follows that \( R \in \mathcal{R}(\mathfrak{g}_4^h) \). Since for any \( u, v \in V \) we have \( R(u \land v) \in \mathfrak{g}_4^h \), we see that \( R \in \mathcal{R}(\mathfrak{g}_4^h) \).

This concludes the proof of theorem 3. □

### 3.5 Proof of theorem 4

(I) Suppose \( h \) preserves a proper subspace \( E_1 \subset E \), then \( h \) preserves the orthogonal complement \( E_1^\perp \) to \( E_1 \) in \( E \). Put \( E_2 = E_1^\perp \). We have \( E = E_1 \oplus E_2, \ h(E_1) \subset E_1, \) and \( h(E_2) \subset E_2 \). Put \( h_1 = \{ \xi \in h : \xi(E_2) = \{0\} \} \) and \( h_2 = \{ \xi \in h : \xi(E_1) = \{0\} \} \). Obviously, \( h_1 \) and \( h_2 \) are ideals in \( h \) and \( h_1 \cap h_2 = \{0\} \).

Let \( P \in \mathcal{P}(h) \). Let \( u_1, v_1 \in E_1, u_2, v_2 \in E_2 \). We have \( \eta(P(u_1)v_1, u_2) + \eta(P(v_1)u_2, u_1) + \eta(P(u_2)u_1, v_1) = 0 \). Since \( h(E_1) \subset E_1 \) and \( h(E_2) \subset E_2 \), we have \( \eta(P(u_2)u_1, v_1) = 0 \) for all \( u_1, v_1 \in E_1, u_2 \in E_2 \). We see that \( P(E_1) \subset h_1 \) and \( P(E_2) \subset h_2 \). Hence, \( L(\mathcal{P}(h)) \subset h_1 \oplus h_2 \).

Combining this with the equality \( L(\mathcal{P}(h)) = h \), we obtain \( h = h_1 \oplus h_2 \).

(II) Let \( P \in \mathcal{P}(h) \). As above, we can prove that \( P(E_1) \subset h_1 \) and \( P(E_2) \subset h_2 \). By definition, put \( P_1 = P|_{E_1} \), \( P_2 = P|_{E_2} \). It is clearly that \( P_1 \in \mathcal{P}(h_1), P_2 \in \mathcal{P}(h_2), \) and \( P = P_1 + P_2 \).

Conversely, for any \( P_1 \in \mathcal{P}(h_1) \) and \( P_2 \in \mathcal{P}(h_2) \) we have \( P = P_1 + P_2 \in \mathcal{P}(h) \).

The proof of the theorem follows easily by induction over the number of the summands. □
3.6 Proof of corollaries

Let \( h \subset \mathfrak{so}(E) \) be a subalgebra and \( R \in \mathcal{R}(h) \). We claim that for any \( z \in E \) the tensor \( P \) defined by \( P(\cdot) = R(\cdot \wedge z) \) belongs to \( \mathcal{P}(h) \). Indeed, we have \( R(u \wedge v)w + R(v \wedge w)u + R(w \wedge u)v = 0 \) for all \( u, v, w \in E \). Multiplying both sides innerly by \( z \in E \), we obtain \( \eta(R(u \wedge v)w, z) + \eta(R(v \wedge w)u, z) + \eta(R(w \wedge u)v, z) = 0 \) for all \( u, v, w, z \in E \). Using (1), we get \( \eta(R(w \wedge z)u, v) + \eta(R(u \wedge z)v, w) + \eta(R(v \wedge z)w, u) = 0 \).

From theorem 3 and the claim it follows that for any weakly-irreducible algebra \( g \) we have \( p_{\mathfrak{so}(E)}(L(\mathcal{R}(g))) = L(\mathcal{P}(h^g)) \).

Corollary 1 follows from the following obvious facts: \( L(\mathcal{R}(p \wedge E)) = p \wedge E \) and \( L(\mathcal{R}(\mathbb{R}, \mathbb{R})) = \mathbb{R}p \wedge q \).

Corollary 2 follows from corollary 1 and the above claim. \( \square \)

4 Examples

Above we have reduced the classification problem for Berger algebras of Lorentzian manifolds to the classification of irreducible weak-Berger algebras.

Suppose we have the full list of irreducible weak-Berger algebras. Corollary 1 and theorem 4 imply that the full list of Berger algebras of Lorentzian manifolds can be obtained in the following way.

For each Euclidean space \( E \) we must consider all orthogonal decompositions \( E = E_0 \oplus E_1 \oplus \cdots \oplus E_r \) such that \( 2 \leq \dim E_1 \leq \cdots \leq \dim E_r \), and for each Euclidean space \( E_i \) all the irreducible weak-Berger algebras \( h_i \subset \mathfrak{so}(E_i) \). From theorem 4 it follows that the algebras \( h = h_1 \oplus \cdots \oplus h_r \) exhaust all weak-Berger algebras. Corollary 1 implies that the Lie algebras \( \mathfrak{g}^1, \mathfrak{g}^2, \mathfrak{g}^{1,\varphi} \) and \( \mathfrak{g}_4^{1,\psi} \) (if \( \mathfrak{g}^{1,\varphi} \) and \( \mathfrak{g}_4^{1,\psi} \) exist) exhaust all Berger algebras.

Below we list all the irreducible subalgebras \( h \subset \mathfrak{so}_n \) for \( n \leq 9 \) and state the result of computing of the spaces \( \mathcal{P}(h) \) for algebras that are not the holonomy algebras of Riemannian manifolds.

Since \( h \subset \mathfrak{so}(E) \), the Lie algebra \( h \) is compact. Hence, \( h = h' \oplus \mathfrak{z}(h) \), where \( h' \) is a compact semisimple ideal, \( \mathfrak{z}(h) \) is an Abelian ideal. Since the subalgebra \( h \subset \mathfrak{so}(E) \) is irreducible, by Schur lemma the center \( \mathfrak{z}(h) \) has dimension 0 or 1.

It is known that if a subalgebra \( h \subset \mathfrak{so}(E) \) is irreducible, the subalgebra \( h' \subset \mathfrak{so}(E) \) is irreducible too (see [13]). Let \( h \subset \mathfrak{so}(E) \) be a semisimple irreducible subalgebra. Denote by \( \mathfrak{z}_{\mathfrak{so}(E)}(h) \) the centralizer of \( h \) in \( \mathfrak{so}(E) \). If \( \mathfrak{z}_{\mathfrak{so}(E)}(h) \neq \{0\} \), then for each one-dimensional subspace \( t \subset \mathfrak{z}_{\mathfrak{so}(E)}(h) \) the Lie algebra \( h \oplus t \) is a compact non-semisimple irreducible subalgebra of \( \mathfrak{so}(E) \). Hence it is sufficient to get the list of all semisimple irreducible subalgebras of \( \mathfrak{so}(E) \).

The classification of irreducible representations of compact semisimple Lie algebras is well known, see for example [13]. Any irreducible real representation \( \pi : h \rightarrow \mathfrak{gl}(E) \) of a real
semisimple Lie algebra \( \mathfrak{f} \) can be obtained in the following way.

Suppose we have a complex irreducible representation \( \rho : \mathfrak{f} \to \mathfrak{gl}(U) \) of a complex semisimple Lie algebra \( \mathfrak{f} \). Let \( \mathfrak{h} \subset \mathfrak{f} \) be a compact real form of \( \mathfrak{f} \) (\( \mathfrak{h} \) is unique up to conjugation).

There are the following three cases:

1) The representation \( \rho \) is self-dual and orthogonal (i.e. \( \rho \sim \rho^* \), where \( \rho^* : \mathfrak{f} \to \mathfrak{gl}(U^*) \) is the dual representation for \( \rho \) and \( \rho \) admits an invariant not degenerate symmetric bilinear form)

2) The representation \( \rho \) is self-dual and symplectic (i.e. \( \rho \) admits an invariant not degenerate skew symmetric bilinear form)

3) The representation \( \rho \) is not self-dual.

The first condition holds if and only if the representation \( \rho \) admits a real form \( J \), i.e. \( J : U \to U \) is a \( \mathbb{R} \)-linear map such that \( J(iu) = -iJ(u) \) for all \( u \in U \), \( J^2 = \text{id} \), and \( J\rho(\xi) = \rho(\xi)J \) for all \( \xi \in \mathfrak{f} \). In this case the real representation \( \pi = \rho|_{\mathfrak{h}} \) in the realificated vector space \( U^{\mathbb{R}} \) preserves the space \( U^J = \{ u \in U : J(u) = u \} \) and acts irreducibly on \( U^J \). We get an irreducible real representation \( \pi : \mathfrak{h} \to \mathfrak{gl}(U^J) \).

In the cases 2) and 3) the real representation \( \pi = \rho|_{\mathfrak{h}} : \mathfrak{h} \to \mathfrak{gl}(U^{\mathbb{R}}) \) is irreducible. In the case 2) we have \( \pi(\mathfrak{h}) \subset \mathfrak{sp}_m \), in the case 3) we have \( \pi(\mathfrak{h}) \subset \mathfrak{su}_n \), where \( 2n = 4m = \dim_{\mathbb{C}} U \).

Let \( \pi : \mathfrak{h} \to \mathfrak{gl}(E) \) be an irreducible real representation of a compact Lie algebra \( \mathfrak{h} \) in a real vector space \( E \). Since \( \mathfrak{h} \) is compact, we see that \( \pi \) admits an invariant symmetric positively definite bilinear form. This form is unique up to non-zero real factor. The linear space \( E \) is an Euclidean space with respect to this form and we can write \( \pi(\mathfrak{h}) \subset \mathfrak{so}(E) \).

Any irreducible complex representation of a complex semisimple Lie algebra \( \mathfrak{f} \) is uniquely defined (up to equivalence of representations) by its highest weight \( \Lambda \). The highest weight \( \Lambda \) can be given by the labels \( \Lambda_1, ..., \Lambda_l \) on the Dynkin diagram of the Lie algebra \( \mathfrak{f} \) (\( l = \text{rk}(\mathfrak{f}) \)). There exists a criteria for a complex representation to be orthogonal, symplectic or self-dual in terms of the highest weight.

In the table 1 we list all irreducible subalgebras \( \pi(\mathfrak{h}) \subset \mathfrak{so}_n \) for \( n \leq 9 \).

The second column of the table contains the irreducible holonomy algebras of Riemannian manifolds. The third column of the table contains algebras that are not the holonomy algebras of Riemannian manifolds.

Let \( \mathfrak{h} \) be a compact semisimple Lie algebra. We denote by \( \pi^K_{\Lambda_1, ..., \Lambda_l} : \mathfrak{h} \to \mathbb{R}^n \) the real irreducible representation that is obtained as above from the complex representation \( \rho_{\Lambda_1, ..., \Lambda_l} : \mathfrak{h}(\mathbb{C}) \to \mathfrak{gl}(U) \) with the highest weight \( \Lambda \), here \( \Lambda_1, ..., \Lambda_l \) are the labels of \( \Lambda \) and \( K = \mathbb{R}, \mathbb{C} \) or \( \mathbb{H} \) if the representation \( \rho_{\Lambda_1, ..., \Lambda_l} \) is orthogonal, not self-dual or symplectic respectively. If the representation \( \rho_{\Lambda_1, ..., \Lambda_l} \) is orthogonal, then \( n = \dim_{\mathbb{C}} U \), otherwise we have \( n = 2 \dim_{\mathbb{C}} U \). We denote by \( \mathfrak{t} \) the 1-dimensional Lie algebra \( \mathbb{R} \).
Table 1. Irreducible subalgebras of $\mathfrak{so}_n$

| $n$ | irreducible holonomy algebras of $n$-dimensional Riemannian manifolds | other irreducible subalgebras of $\mathfrak{so}_n$ |
|-----|------------------------------------------------|------------------------------------------------|
| $n = 1$ | | |
| $n = 2$ | $\mathfrak{so}_2$ | |
| $n = 3$ | $\pi_2^R(\mathfrak{so}_3)$ | |
| $n = 4$ | $\pi_{1,1}^R(\mathfrak{so}_3 + \mathfrak{so}_3), \pi_1^C(\mathfrak{su}_2), \pi_1^C(\mathfrak{su}_2) \oplus t$ | |
| $n = 5$ | $\pi_{1,0}^R(\mathfrak{so}_5), \pi_1^R(\mathfrak{so}_3)$ | |
| $n = 6$ | $\pi_{1,0}^R(\mathfrak{so}_6), \pi_1^C(\mathfrak{su}_3), \pi_1^C(\mathfrak{su}_3) \oplus t$ | |
| $n = 7$ | $\pi_{1,0,0}^R(\mathfrak{so}_9), \pi_{1,2}^R(\mathfrak{so}_3 \oplus \mathfrak{so}_3)$ | $\pi_6^R(\mathfrak{so}_3)$ |
| $n = 8$ | $\pi_{1,0,0,0}^R(\mathfrak{so}_8), \pi_1^C(\mathfrak{su}_4), \pi_1^C(\mathfrak{su}_4) \oplus t, \pi_{1,0}^R(\mathfrak{sp}_2), \pi_3^C(\mathfrak{so}_3), \pi_3^C(\mathfrak{so}_3) \oplus t$ | $\pi_{1,0}^H(\mathfrak{sp}_2) \oplus t$ |
| $n = 9$ | $\pi_{1,0,0,0,0}^R(\mathfrak{so}_9)$ | $\pi_8^R(\mathfrak{so}_3)$ |

Now we must verify the equality $L(\mathcal{P}(\mathfrak{h})) = \mathfrak{h}$ for algebras from the third column of the table.

Let $\mathfrak{h} \subset \mathfrak{so}(E)$ be a subalgebra. We claim that $L(\mathcal{P}(\mathfrak{h}))$ is an ideal in $\mathfrak{h}$. Indeed, let $P \in \mathcal{P}(\mathfrak{h})$ and $\xi \in \mathfrak{h}$; put $P_\xi(u) = -\xi \circ P(u) + P(u) \circ \xi + P(\xi u)$ for all $u \in E$. It can be easily checked that $P_\xi \in \mathcal{P}(\mathfrak{h})$. We see that $[P(u), \xi] = P_\xi(u) - P(\xi u)$ for all $u \in E, \xi \in \mathfrak{h}$.

Suppose $\mathfrak{h} \subset \mathfrak{so}(E)$ is an irreducible subalgebra. We have $\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{j}(\mathfrak{h})$ and $\dim \mathfrak{j}(\mathfrak{h}) = 0$ or 1. Since $\mathfrak{h}'$ is semisimple, we have $\mathfrak{h}' = \mathfrak{h}_1 \oplus \ldots \oplus \mathfrak{h}_r$, where $\mathfrak{h}_i \subset \mathfrak{h}'$ are simple ideals. Any ideal $\tilde{\mathfrak{h}} \subset \mathfrak{h}$ is a sum of some of the ideals $\mathfrak{h}_1, \ldots, \mathfrak{h}_r$ and, may be, $\mathfrak{j}(\mathfrak{h})$. From the above claim and the obvious equality $\mathcal{P}(\mathfrak{h}) = \mathcal{P}(L(\mathcal{P}(\mathfrak{h})))$ it follows that $L(\mathcal{P}(\mathfrak{h})) = \mathfrak{h}$ if and only if $\mathcal{P}(\mathfrak{h}) \neq \{0\}$ and there is no proper ideal $\tilde{\mathfrak{h}} \subset \mathfrak{h}$ such that $\mathcal{P}(\mathfrak{h}) = \mathcal{P}(\tilde{\mathfrak{h}})$. In particular, if $\mathfrak{h}$ is simple, then $L(\mathcal{P}(\mathfrak{h})) = \mathfrak{h}$ if and only if $\mathcal{P}(\mathfrak{h}) \neq \{0\}$.

Let $\mathfrak{h} \subset \mathfrak{so}(E)$ be a subalgebra and $N = \dim \mathfrak{h}$. Denote by $C^1, \ldots, C^N$ a basis of the vector space $\mathfrak{h}$. Let $P \in \text{Hom}(E, \mathfrak{h})$ and $P_{\alpha i} (\alpha = 1, \ldots, N; i = 1, \ldots, n)$ be real numbers such that $P(e_i) = \sum_{\alpha=1}^{N} P_{\alpha i} C^\alpha$ for $i = 1, \ldots, n$. We have $P \in \mathcal{P}(\mathfrak{h})$ if and only if

$$
\sum_{\alpha=1}^{N} (P_{\alpha i} C^\alpha_{jk} + P_{\alpha j} C^\alpha_{ki} + P_{\alpha k} C^\alpha_{ij}) = 0
$$

for all $i, j, k$ such that $1 \leq i < j < k \leq n$ ($\{C^\alpha_{ij} \}_{i,j=1}^{n}$ is the matrix of the endomorphism $C^\alpha$).

Thus the space $\mathcal{P}(\mathfrak{h})$ can be found as the solution of the system of $n(n-1)(n-2)/6$ equations in $nN$ unknowns. We used computer program Mathematica 4.1 to solve such systems for algebras from the third column of table 1.

The result is $\mathcal{P}(\pi_2^R(\mathfrak{so}_3)) = \{0\}$, $\mathcal{P}(\pi_3^C(\mathfrak{so}_3)) = \{0\}$, $\mathcal{P}(\pi_3^C(\mathfrak{so}_3) \oplus t) = \{0\}$, $\dim(\mathcal{P}(\pi_{1,0}^H(\mathfrak{sp}_2) \oplus t)) = 40$, $\mathcal{P}(\pi_6^R(\mathfrak{so}_3)) = \{0\}$. We also have $\dim(\mathcal{P}(\pi_{1,0}^H(\mathfrak{sp}_2))) = 40$. 
Since $P(\pi_{1,0}^H(sp_2)) = P(\pi_{1,0}^H(sp_2) \oplus t)$, we see that $L(P(\pi_{1,0}^H(sp_2) \oplus t)) = \pi_{1,0}^H(sp_2)$. Hence the algebra $\pi_{1,0}^H(sp_2) \oplus t$ is not a weak-Berger algebra.

In table 2 we list all irreducible weak-Berger algebras $\mathfrak{h} \subset \mathfrak{so}_n$ ($n \leq 9$). This list coincides with the list of the irreducible holonomy algebras of Riemannian manifolds. We use the standard notation. In the table $\otimes$ stands for the tensor product of representations; $\otimes$ stands for the highest irreducible component of the corresponding product.

**Table 2. Irreducible weak-Berger algebras**

| n   | Irreducible weak-Berger subalgebras of $\mathfrak{so}_n$ |
|-----|-------------------------------------------------------|
| n=1 |                                                        |
| n=2 | $\mathfrak{so}_2$                                     |
| n=3 | $\mathfrak{so}_3$                                     |
| n=4 | $\mathfrak{so}_4, \mathfrak{su}_2, \mathfrak{u}_2$   |
| n=5 | $\mathfrak{so}_5, \mathfrak{su}_3 \oplus \mathfrak{so}_3$ |
| n=6 | $\mathfrak{so}_6, \mathfrak{su}_3, \mathfrak{u}_3$   |
| n=7 | $\mathfrak{so}_7, \mathfrak{g}_2$                    |
| n=8 | $\mathfrak{so}_8, \mathfrak{su}_4, \mathfrak{u}_4, sp_2, sp_2 \otimes sp_1, \mathfrak{su}_3 \otimes \mathfrak{so}_3, \mathfrak{su}_3 \otimes \mathfrak{so}_3$ |
| n=9 | $\mathfrak{so}_9, \mathfrak{so}_3 \otimes \mathfrak{so}_3$ |

Recall that the holonomy group of an indecomposable Lorentzian manifold can be not closed. In [4] it was shown that the connected Lie subgroups of $SO_{1,n+1}$ corresponding to Lie algebras of type 1 and 2 are closed; the connected Lie subgroup of $SO_{1,n+1}$ corresponding to a Lie algebra of type 3 (resp. 4) is closed if and only if the connected Lie subgroup of $SO_n$ corresponding to the subalgebra $\ker \varphi \subset \mathfrak{z}(\mathfrak{h})$ (resp. $\ker \psi \subset \mathfrak{z}(\mathfrak{h})$) is closed. We give a criteria for Lie groups corresponding to Lie algebras of type 3 and 4 to be closed in terms of the Lie algebras $\ker \varphi$ and $\ker \psi$.

Let $\mathfrak{h} \subset \mathfrak{so}_n$ be a weak-Berger algebra such that $\mathfrak{z}(\mathfrak{h}) \neq \{0\}$. Denote by $T$ the connected Lie subgroup of $SO_n$ corresponding to the Lie subalgebra $\mathfrak{z}(\mathfrak{h}) \subset \mathfrak{so}_n$. Since $\mathfrak{h}$ is a sum of irreducible weak-Berger algebras, we see that the Lie subgroup $T$ is closed. Hence $T$ is compact and $T$ is isomorphic to the torus of dimension $k = \dim \mathfrak{z}(\mathfrak{h})$. Thus, $T = S_1 \times \cdots \times S_k$, where $S_i$ is a closed Lie subgroup of $T$ isomorphic to the unit circle $S^1$ ($i = 1, \ldots, k$). Denote by $u_1, \ldots, u_k$ the tangent vectors to $S_1, \ldots, S_k$ respectively corresponding to a unit tangent vector at the unity element of the circle. The vectors $u_1, \ldots, u_k$ form a basis of $\mathfrak{z}(\mathfrak{h})$.

Let $\mathfrak{i} \subset \mathfrak{z}(\mathfrak{h})$ be a subalgebra and $\tilde{T} \subset T$ the corresponding connected Lie subgroup. We claim that the subgroup $\tilde{T}$ is closed if and only if there exists a basis $v_1, \ldots, v_l$ ($l = \dim \mathfrak{i}$) of the vector space $\mathfrak{i}$ such that the coordinates of the vector $v_i$ with respect to the basis $u_1, \ldots, u_k$ are integer for all $i = 1, \ldots, l$.

For $l = 1$ this statement was proved in [13].
Let \( l > 1 \). Suppose that there exists a basis of \( \tilde{\mathfrak{t}} \) as above. Denote by \( \tilde{S}_1, \ldots, \tilde{S}_l \) the connected Lie subgroups of \( \tilde{T} \) corresponding to the subalgebras \( \mathbb{R}v_1, \ldots, \mathbb{R}v_l \subset \mathfrak{z}(\mathfrak{h}) \). The Lie subgroups \( \tilde{S}_1, \ldots, \tilde{S}_l \) are closed. Hence the Lie groups \( \tilde{S}_1, \ldots, \tilde{S}_l \) are compact and isomorphic to the unit circle. Denote these isomorphisms by \( f_1, \ldots, f_l \). Put \( T^l = S^1 \times \cdots \times S^1 \). Define a map \( f : T^l \to T \) by putting \( f(x_1, \ldots, x_l) = f_1(x_1) \cdot \cdots \cdot f_l(x_l) \), where \( x_i \in S^1 \). We have \( f(T^l) = \tilde{T} \), hence \( \tilde{T} \) is closed in \( T \). The inverse statement is obvious.

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