Fractional calculus ties the microscopic and macroscopic scales of complex network dynamics

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Abstract
A two-state, master equation-based decision-making model has been shown to generate phase transitions, to be topologically complex, and to manifest temporal complexity through an inverse power-law probability distribution function in the switching times between the two critical states of consensus. These properties are entailed by the fundamental assumption that the network elements in the decision-making model imperfectly imitate one another. The process of subordination establishes that a single network element can be described by a fractional master equation whose analytic solution yields the observed inverse power-law probability distribution obtained by numerical integration of the two-state master equation to a high degree of accuracy.

1. Introduction
The extraordinary advancements made in the physical sciences at the beginning of the twentieth century originated from the effort to understand the simplest—that is, the most fundamental—elements of physical phenomena. The separation of matter and light into its basic constituents—electrons, protons, atoms, and molecules in the first case and photons and phonons in the second—enabled scientists to explain the puzzling properties of solid materials, such as their sound transmission, electrical properties light emission, and reflection characteristics. The discipline of statistical mechanics demonstrated that the behavior of systems composed of millions of individual particles can be captured with simple laws involving only their average properties [21]. At that time it seemed as if the Aristotelian approach of reductionism, where complex phenomena consisted of nothing more than the sum of essential elements, was the prescription underlying the correct method of scientific discovery.

The French mathematician Poincaré was probably the first to rigorously demonstrate the failure of this attractive but overly optimistic method by extending Newton’s law of universal gravitation to a system consisting of three celestial bodies [23]. His mathematical treatment of the three-body problem demonstrated that, unlike the two-body problem of the Earth and Sun, a planet’s orbit in the three-body system need not be periodic. Poincaré proved that the long-held belief that planetary motion could be built up from the superposition of simple cycles was false. This unanticipated behavior emerges from nonlinear dynamics and has come to be known as chaos theory.

Critical phase transitions in magnetic materials are perplexing phenomena that violates the principle of superposition. If criticality were truly nothing more than collective behavior resulting from the superposition of some basic building blocks, then why does the change in an external parameter such as temperature induce such a dramatic shift in its macroscopic behavior?

More recently, the identification of emergent phenomena across multiple disciplines—from the swarming of insects [35], the schooling of fish [16] and the flocking of birds [6] observed in animal groups by naturalists; to the spatiotemporal activity of the brain [2, 7, 10] observed by neurophysiologists; to the collective and...
cooperative behavior observed in social groups studied by psychologists and sociologists—all demonstrate
collective behavior reminiscent of particle dynamics near the critical phase transitions studied by physicists [26].
Each of these disciplines has demonstrated the need to investigate the dynamics of complex networks across
scales in order to develop a deeper understanding of how large-scale behavior emerges from microscale
dynamics, and the sensitivity of the observed behavior to those dynamics.

Of particular interest to us here are the biological fields, in which we observe a need for a system-wide
approach [24]. Recent discoveries in biology were propelled by successes in molecular biology and genetics that
have made available genomic blueprints of numerous organisms, which are complemented by extensive
experimental data describing cell functions. At the same time, however, the realization came that biological
function emerges out of the interaction of numerous molecular components, making the detailed knowledge of
specific components at any level of organization insufficient to capture macroscopic functionality. There is
probably no better example of this limitation than the study of neurological systems, whose goal is to
understand, predict, and ultimately modify brain function for therapeutic purposes. Ongoing initiatives of the
Human Connectome Project [13], the Human Brain Project [14], and the Allen Brain Atlas [15] illustrate the
fact that the system–wide approach and integration of data from different spatial and temporal scales has become
the norm in modern scientific disciplines. The ultimate realization of the system approach becomes evident in
modern medicine, where patients are treated by teams of specialists from many disciplines. This comes from a
realization that separate organs in human body form a system of physiologic networks [39], where the failure of
one element affects the physiological function of others.

Despite experimental developments, the ability of science to make theoretical predictions of the behavior of
complex networks is still in its infancy. The adoption of methods from nonequilibrium statistical physics has
demonstrated limitations, resulting from the fact that living systems, in contrast to inert physical materials, are
extremely heterogeneous, nongeneric, highly specialized, and operate far from an equilibrium state [8]. Herein,
we demonstrate that fractional calculus, which for a very long time was a niche branch of mathematics, might
very well be able to span the gap between the inert materials of physics and the living networks of biology.

Although it was developed alongside classical calculus, fractional differential equations have only recently
been shown to be a convenient way to describe the dynamics of complex phenomena characterized by long-term
memory and spatial heterogeneity [18, 22, 32]. Fractional differential equations were demonstrated to capture
the time evolution of fractal processes, such as in anomalous diffusion, viscoelasticity, and turbulent fluid flow,
as reviewed by West and Grigolini [33]. In spite of the success of the mathematical descriptions of such
processes, there has been a lack of identification and interpretation of the mechanisms that entail fractional
dynamic equations in the context of complex networks. Herein we provide an explanation for one source of a
fractional differential equation that describes the dynamics of a complex network using a fractional master
equation.

Here the utility of fractional calculus is demonstrated by capturing the dynamics of the individual elements
of a complex network from the information quantifying that network’s global behavior. The phase transitions
observed in complex social and physiologic networks suggest the wisdom of using a generic model from the Ising
universality class to characterize network dynamics. Using such a model, it is possible to demonstrate that the
individual trajectory response to the collective motion of the network is described by a linear fractional
differential equation. The solution to this fractional equation is obtained through a subordination procedure
without the necessity of linearizing the underlying dynamics—that is, the solution retains the influence of the
nonlinear network dynamics on the individual. Moreover, the solutions to the fractional equations of motion
suggest a new direction for designing control mechanisms for complex networks.

In section 2 we consider the dynamics of a complex network described by a two-state master equation. The
decision-making model (DMM), defined by the two-state master equation, undergoes phase transitions at a
critical value of the control parameter [29]. It is understood that at criticality, the short-range local interactions
between the two-state elements generate long-range global correlations, thereby producing effective long-range
interactions. Consequently, at criticality there is global cooperation among the network elements.

An individual disconnected from the network is assumed to choose randomly between two states with an
exponential distribution of decision times and a given average decision time. When coupled to the other
individuals in the network, the global distribution for the time intervals between decisions is determined to be an
inverse power law [34]. In this latter case, the power–law index of the survival probability is a measure of the
complexity of the underlying dynamics determining whether that process is nonstationary and nonergodic
[28, 29]. In section 3 the DMM network dynamics is incorporated into that of an individual element through a
process known as subordination. To formalize the subordination process, we introduce the concept of
subjective time to distinguish between clock time, which that determines the activities of the network, and
operational or subjective time, which determines the activities of the individual.

The subordination process results in the two-state master equation of the DMM being replaced by a
fractional master equation for the individual whose solution is shown to be a Mittag–Leffler function in
2. Decision-making model (DMM)

The DMM realized in a complex network represents the dynamics of the probability for an individual to be in either of the two states: yes or no, up or down, on or off. The model is based on the cooperative interaction of $N$ elements, each of which is described by the two-state master equation \[28, 29\]

\[
\frac{d}{dt}p_1^{(i)} = -g_{12}^{(i)}p_1^{(i)} + g_{21}^{(i)}p_2^{(i)},
\]

(1)

\[
\frac{d}{dt}p_2^{(i)} = -g_{21}^{(i)}p_2^{(i)} + g_{12}^{(i)}p_1^{(i)}.
\]

(2)

The quantity $p_j^{(i)}(t)$ is the probability of the element $i = 1, \ldots, N$ in the network being in the state $j = 1, 2$ at time $t$ and the probability is normalized at all times such that

\[
p_1^{(i)}(t) + p_2^{(i)}(t) = 1.
\]

(3)

The network dynamics are determined by the choice of the functional form of the transition rates in the two-state master equation (equations (1) and (2)). Each probability, $p_j^{(i)}(t)$, is influenced by the states occupied by all the elements of the network linked to element $i$ as determined by the transition rates

\[
g_{12}^{(i)}(t) = g_0 \exp \left[ K \left\{ \frac{N_2^{(i)}(t) - N_1^{(i)}(t)}{N^{(i)}} \right\} \right],
\]

(4)

\[
g_{21}^{(i)}(t) = g_0 \exp \left[ K \left\{ \frac{N_1^{(i)}(t) - N_2^{(i)}(t)}{N^{(i)}} \right\} \right].
\]

(5)

The symbol $N^{(i)}$ denotes the total number of elements linked to the $i$th element and $N_s^{(i)}(t)$ is the number of those elements in the state $s = 1, 2$ at time $t$. Of course $N^{(i)} = N_1^{(i)} + N_2^{(i)}$ at all times. The parameter $K$ is the control parameter that determines the strength of the interaction between elements of the network. In the case where each element in the network is coupled to all the other elements, we have all-to-all (ATA) coupling, such that $N^{(i)} = N$, and the time dependence of the total number of elements in states $s = 1, 2$ $N_s^{(i)}(t) = N_s(t)$ implies that the transition rates become erratic functions of time \[28\].

2.1. When every element is interconnected

In the ATA coupling case, when the total number of elements within the network becomes infinite ($N \rightarrow \infty$), the fluctuation frequencies collapse into probabilities according to the law of large numbers. In physics this replacement goes by the name of the mean-field approximation, in which case the transition rates in the master equations (1) and (2) are written as

\[
g_{12}(t) = g_0 \exp \left[ -K \left\{ p_1(t) - p_2(t) \right\} \right]
\]

(6)

\[
g_{21}(t) = g_0 \exp \left[ -K \left\{ p_2(t) - p_1(t) \right\} \right]
\]

(7)

The formal manipulation of the master equation even in this simplified case is made a little easier if we introduce a new variable defined as the difference in the probabilities

\[
\Pi(t) \equiv p_1(t) - p_2(t).
\]

(8)

Subtracting equation (2) from equation (1) yields, after some algebra, the highly nonlinear rate equation for the difference variable

\[
\frac{d}{dt}\Pi = -\left( g_{12} + g_{21} \right)\Pi + (g_{21} - g_{12})
\]

(9)

where the nonlinearity enters through the transition rate dependence on the difference variable

\[
g_{12} = g_0 \exp \left[ -K\Pi \right]
\]

(10)
\[ g_{21} = g_0 \exp [K \Pi] \]  

in the mean-field approximation. By inserting equations (10) and (11) into equation (9), we obtain

\[ \frac{d}{dt} \Pi = \frac{\partial V}{\partial \Pi} \]  

and the network dynamics are determined by the potential function \( V(\Pi) \), which is a symmetric double-well potential with the explicit form

\[ V(\Pi) = \frac{2g_0}{K} \left[ \Pi \sinh K \Pi - \frac{K + 1}{K} \cosh K \Pi \right] \]  

Note that the network is not described by a Hamiltonian, and yet the global dynamics is indeed described by an effective Hamiltonian: the double-well potential given by equation (13) and depicted in figure 1.

The cooperative behavior of the infinitely large ATA coupled network described by equation (12) is that of an overdamped particle hopping from one potential minimum to the other, whose position is \( \Pi \) within the potential equation (13). For \( K < 1 \), half of the nodes are in one state and half are in the other because there is only a single broad minimum in the potential. At the critical value of the control parameter, \( K = K_c = 1.0 \), a bifurcation occurs and the potential develops two wells separated by a barrier, as discussed by Turalska et al [28]. The height of the barrier increases with the value of the control parameter.

It is now convenient to define the stochastic global variable

\[ \xi(t) = \frac{N_s(t) - N_f(t)}{N} = \frac{1}{N} \sum_{i=1}^{N} s_i(t), \]  

where \( s_i(t) \) is the state of element \( i \) at time \( t \). The fluctuations of the global variable characterize the entire network, capturing the cooperation between units at any moment of time. It is interesting that at the critical value of the control parameter, the ATA version of the DMM undergoes a phase transition. Note that the amplitude of \( \xi(t) \) depends on the value of the control parameter, \( K \). When \( K = 0 \), all elements in the network are independent Poisson processes; therefore an average taken at any moment of time over all of them yields zero. Once the value of the coupling becomes nonzero, \( K > 0 \) and single elements are less and less independent, resulting in nonzero averages. The quantity, \( K_c \), is the critical value of the control parameter \( K \), at which point a phase transition to a global majority state occurs.

In numerical calculations we use the time average, \( \xi_{\text{eq}} = \langle |\xi(t)| \rangle \), as a measure of this global majority. More precisely, after an initial \( 10^6 \) time steps, the average is taken over the same number of the consecutive time steps of the model. In figure 2, the thin line indicates the ATA phase transition as measured by \( \xi_{\text{eq}} \). The other phase transitions indicated are for the Ising model (dashed line) and the DMM model on a two-dimensional lattice, as discussed in section 2.2 and elsewhere [29].

Real networks are not ATA coupled since interactions typically have a finite range and elements are spatially separated. Thus, the ATA approximation may be valid for small networks, but it certainly breaks down for large systems. Moreover, real-world networks have finite numbers of elements. It is therefore useful to examine how strongly the mean-field solutions are violated when we relax these constraints. The stability condition can be violated in at least two different ways. The first way is by reducing the number of elements, \( N \), to a finite value.

![Figure 1. Left panel: the equilibrium mean field for different values of the control parameter \( K \). A bifurcation occurs at the critical point, \( K = K_c = 1.00 \). Right panel: potential function determined by equation (13), with barriers for \( K \) subcritical (dashed line, \( K = 0.20 \)), critical (solid line, \( K = 1.00 \)), and supercritical (dotted line, \( K = 1.80 \)).](image-url)
The second way is by restricting the number of links so the network no longer has ATA coupling. In real-world networks both sources of equilibrium disruption are expected to occur. For the time being, we retain the ATA coupling within the networks and consider the number of elements, \( N \), to be finite. In this latter case, we can no longer make the mean-field approximation, and the dynamic picture stemming from the above master equation is radically changed.

If the number of elements is still very large, but finite, we consider the mean-field approximation to be nearly valid, and we replace the average equation (14) with the stochastic quantity

\[ \xi(t) = \Pi(t) + f(t) \]

where \( f(t) \) is a small amplitude random fluctuation. After inserting equation (15) into (12) and performing some straight-forward algebra, we obtain the stochastic differential equation [4, 12] to the lowest order in the strength of the fluctuations:

\[ \frac{d\xi(t)}{dt} = -\frac{\partial V(\xi(t))}{\partial \xi} + \epsilon(t). \]

The additive fluctuations, \( \epsilon(t) \), have amplitudes that are computationally determined to be on the order of \( 1/\sqrt{N} \).

Note that the double-well potential in the mean-field approximation persists in the present description even though we have relaxed the mean-field approximation to a finite number of network elements. The random fluctuations resulting from the finite size of the network induce transitions between the two states of the potential well. Consequently, for a network with a finite but large number of elements, the phase synchronization of equation (12) is not stable and the stochastic Langevin equation (16) determines the dynamics of the network. Furthermore, the fluctuations can drive the particle from one well of the potential to the other when its amplitude is sufficient to traverse the barrier separating the wells. However, here the fluctuations arise from the finite number of elements in the network, rather than from nonexistent thermal excitations, and are consequently suppressed as the network size increases.

Although equation (16) is written in the continuous-time representation, in practice the numerical calculations of the DMM correspond to the adoption of a finite integration time step, \( \Delta t = 1 \). Note that the stochastic rate equation (16) replaces equation (12) in the case of a finite \( N \), and that equation (12) is recovered in the ideal case, \( N = \infty \). We incorporate the ATA coupling condition with a finite number of elements by numerically integrating the master equation for each element in the network and then calculating the number of elements in each of the two states. In figure 3, the fluctuating global variable, \( \xi(t) \), is depicted as a function of time under differing conditions. Note that with increasing \( N \) the fluctuation, \( \xi(t) \), become, more distinctly dichotomous-like, with an increasingly sharp transition from the ‘up’ to the ‘down’ state. This pattern corresponds to the entire network keeping a decision for a longer and longer time as the size of the network increases. The condition of a decision lasting forever is reached in the ideal case, \( N = \infty \). The global variable fluctuates between the two minima of the double-well potential as described by equation (16) for \( K = 1.05 > K_C \). The single element follows the fluctuations of the global variable, switching back and forth.

![Figure 2](image-url) The phase diagram for the global variable \( \xi(t) \). The thin solid line and the dashed line are the theoretical predictions for the fully connected and the two-dimensional regular network, respectively. In both cases, \( N = \infty \) and the latter case is the Onsager theoretical prediction [20] for a two-dimensional regular lattice. The dots correspond to the global states observed for the DMM on a two-dimensional regular lattice, \( N = 100 \times 100 \) nodes and \( g_0 = 0.01 \). Periodic boundary conditions were applied in the DMM calculations.
from the condition where the upper state is preferred statistically to that where the lower state is preferred statistically. The complete properties of the DMM on an ATA network are explored by Turalska et al [28, 29].

2.2. Nearest neighbor coupling

In this section, we consider the topology of a simple two-dimensional lattice and we confine the coupling between elements to its four nearest neighbors, thereby setting $N = 4$ in the transition rates of the two-state master equation. Similar to the ATA case, the fluctuations of the global variable $\xi(t)$ show pronounced transition as a function of the coupling parameter, $K$. As seen in figure 4(b), the global variable shifts from a configuration dominated by randomness to an organized state once the control parameter is increased above the critical value, $K_C$. For values of the control parameter $K$ corresponding to the disorganized phase $K < K_C$, single elements of the lattice are only weakly influenced by the decisions of their neighbors. Thus, the fluctuations of the global order parameter $\xi(t)$ are characterized by a small amplitude and very fast oscillations about the zero-axis. For $K > K_C$, the interaction between individuals gives rise to a majority or a consensus state, during which a significant number of agents adopts the same opinion at the same time.

To characterize the changes in temporal properties of the microscopic and macroscopic variables, we evaluate the survival probability function, $\Psi(\tau)$, of time intervals, $\tau$, between consecutive events defined as changes of the state or crossing of the zero-axis for the single element and the global variable, respectively. These calculations unveil modest deviation of the survival probability for a single individual from the exponential form $\Psi(\tau) = \exp(-g_0 \tau)$. The strict exponential is indicated in figure 4(c) by the black dashed curve. The single-particle survival probabilities do not look too different; the blue curve with the subcritical value $K = 1.5 < K_C$ is very close to the exponential. The remaining single-particle curves, whether critical $K = K_C \approx 1.70$ or supercritical $K > K_C$, appear to be very nearly exponential on this graph. Thus we conclude that when the global behavior is undergoing a phase transition, the presence of the lattice exerts only very subtle influence over the behavior of single individuals. The difference in the behavior of the individual from that in the non-interacting state is that she tends to be more reluctant to change her mind.

However, the collective dynamics of cooperating units leaves a much stronger signature in the temporal properties of the macroscopic variable. In the subcritical regime, $\Psi(\tau)$ has an exponential form, which reflects the largely independent behavior of single units from their neighbors. In the supercritical region $K > K_C$, the abrupt transitions between consecutive majority intervals that result from the lattice having finite size are responsible for an exponential shoulder present in $\Psi(\tau)$. At the critical point $K \approx K_C$, the survival probability has the distinct shape of an inverse power-law distribution, demonstrating the uniqueness of the dynamics at criticality.

Recent observations of neuronal activity during sleep, under anesthesia, and in vitro [36], as well as analysis of sleep recordings from humans [37] and other animals [38], demonstrate behavior typical of systems undergoing phase transitions or manifesting self-organized criticality. In a fashion similar to the units of the DMM, the collective action of neurons fluctuates between so-called up and down states, or wake and sleep states in the case of sleep studies. However, contrary to the temporal signature of the DMM, where both up and down states and positive and negative majority states are characterized by an inverse power-law distribution, the
dynamics of neuronal activity fluctuates between two behaviors: an inverse power-law statistic of up states and wake periods and an exponential distribution for the down and sleep states. The inhibitory and excitatory feedback loops present in the neuronal networks, together with a clear bias toward the sleep state during prolonged sleep, are the leading factors responsible for the difference in the dynamics of sleep and wake periods. The symmetry present in the DMM originates from the symmetric form of interactions and lack of preference toward one state or the other. This property would be affected by modifications to the model, such as the introduction of a skewed potential.

The deviation of the individual survival probability from the exponential form in figure 4(c) appears to be modest when compared with the dramatically greater deviation of the survival probability of the global variable from the exponential, as depicted in figure 4(d). The average network behavior differs markedly as the control parameter increases from the subcritical through the supercritical regions. However, the influence of the global variable on the behavior of the individual does not appear to induce a significant change. However, for the individual, the change is a subtle yet profound difference and is a direct result of the imitation mechanism, which is the echo response hypothesis [34]. So if the individual survival probability is not exponential, what is it? To answer this question, we turn our attention to describing an alternate construction of the dynamics of the individual elements.

3. Subordination and fractional dynamics: two views of time

In this section we demonstrate the equivalence between a fractional trajectory that solves a Caputo fractional differential equation and the ensemble average trajectory that results from a subordination process. We consider...
only Caputo-type fractional derivatives here, in part because they incorporate the initial conditions of \( P(t), \Psi(t) \), and \( \xi(t) \). However, an approach using Riemann–Liouville fractional derivatives would be equivalent as long as the initial conditions are properly specified. We begin the discussion with a derivation of the fractional derivative from a subordination argument.

With the index suppressed, the master equation for a single isolated individual is, according to the numerical simulation given by

\[
\phi(n\Delta \tau) - \phi([n - 1]\Delta \tau) = -g_0\phi([n - 1]\Delta \tau)\Delta \tau, \quad (17)
\]

whose discrete solution is

\[
\phi(n) = \left(1 - g_0\Delta \tau\right)^n\phi(0). \quad (18)
\]

Here, \( \phi(n) \) is the difference variable for a single individual chosen from the network at random, and as \( n \to \infty \) and \( \Delta \tau \to 0 \), such that clock time is \( t = n\Delta \tau \), we have the apparently trivial result

\[
\phi(t) = e^{-g_0\tau}\phi(0). \quad (19)
\]

Subordination implies the existence of two different notions of time [25, 27]. One is the operational time, \( \tau \), which is the internal time of a single individual, with an element generating the ordinary dynamics of a nonfractional system. The other notion is experimental time, \( t \), which is the time as measured by the clock of an external observer. Typically, in the operational time frame, the temporal behavior of an element is regular and evolves exactly according to the ticks of a clock, leading to the exponential. Therefore, it is assumed that the trajectory of a network’s element in operational time is well defined and given by \( \phi(t) \), which is the solution given by equation (19).

It is perhaps worthwhile to point out that this notion of two different times was introduced into psychology in the mid-nineteenth century, and it led to the general Weber–Fechner law [9]. This law has been further developed in a contemporary setting to explain the observation of 1/f noise [17] in cognition by discriminating between subjective and objective times, which are operational and experimental time, respectively.

In operational time, an element’s behavior appears ordinary, but to an experimenter observing the elements from outside the network, their temporal behavior appears erratic, evolving in time and then abruptly freezing in different states for extended time periods. Because of the random nature of the experimental time evolution of the elements, the subordination process involves an ensemble average over many realizations of the element’s dynamics, each evolving according to its own internal clock, independent of one another. Making an ensemble average over a large number of realizations results in a smooth average trajectory, which is equivalent to the fractional trajectory.

To find the average behavior, we move from the operational time solution to the experimental time solution by adopting the subordination interpretation. We assume that the experimental time lies in the interval \( (n - 1)\Delta \tau \leq t \leq n\Delta \tau \), and we obtain

\[
\langle \phi(t) \rangle = \sum_{n=1}^{\infty} \int_0^t \Psi(t - t')\psi(t')\phi(n)\,dt'. \quad (20)
\]

It is evident that the trajectory resulting from the subordination process inherently involves an ensemble average. Here we see that equation (20) replaces the solution to the single-element two-state master equation of the DMM.

Note that \( \psi(t)\,dt \) is the probability that \( n \) events have occurred, the last one in the time interval \( (t, t + dt) \). The function \( \Psi(t) \) denotes the probability that no event occurs up to time \( t \); it is given empirically by numerical calculation in figure 4(d) and mathematically by equation (27). The occurrence of an event corresponds to activating a decision with \((1 - g_0\Delta \tau)\), so that by activating \( n \), such events transform the initial condition, \( \phi(0) \), into the product, \((1 - g_0\Delta \tau)^n\phi(0) \). This form of the equation is kept from time, \( t' \), at which time the last event occurs, up to time \( t \), with the time interval \( t - t' \) being characterized by no event occurring. Of course, the expression equation (20) takes into account that the number of possible events may range from the no-event case to that of infinitely many events. The conditions necessary for this result to occur are discussed by Svenkenson et al [27]. To interpret the physical meaning of equation (20), consider each tick of the internal clock, \( \tau \), of an element measured in experimental time as an event. Since the observation is made in experimental time, the time intervals between events are random. We assume that the waiting times between consecutive events are identically distributed, independent, random variables. The integral in equation (20) is then built up according to renewal theory. After the \( n \)th event, the individual changes from state \( \phi(n) \) to \( \phi(n + 1) \), where it remains until the action of the next event. The sum over \( n \) takes into account the possibility that any number of events could have occurred prior to an observation at experimental time \( t \). The events occur randomly with a waiting-time probability density function (pdf) \( \psi(t) \) and survival probability \( \Psi(t) \). The waiting-time pdf is
related to the survival probability through

$$\psi(t) = -\frac{d\Psi(t)}{dt}. \quad (21)$$

Taking advantage of the renewal nature of the events, the waiting-time pdf for the $n$th event in a sequence is connected to the previous event by

$$\psi_n(t) = \int_0^t \psi_{n-1}(t')\psi(t-t')dt'. \quad (22)$$

At this point it is useful to introduce Laplace variables in our discussion. The Laplace transform of a function, $f(t)$, is denoted by

$$\hat{f}(s) \equiv \int_0^\infty \exp^{-st} f(t)dt. \quad (23)$$

To find an analytical expression for the behavior in experimental time, it is convenient to study the Laplace transform of equation (20)

$$\hat{\phi}(s) = \hat{\Psi}(s) \sum_{n=0}^{\infty} \left(1 - g_0\Delta \tau\right)^n \left[\hat{\psi}(s)\right]^n \phi(0). \quad (24)$$

When we assume the intervals between successive transitions are independent of one another, it is a renewal process. Consequently, the waiting-time pdf for $n$ transitions is the product of $n$ single-transition pdfs:

$$\hat{\psi}_n(s) = \left[\hat{\psi}(s)\right]^n \quad (25)$$

which was used to collapse the convolution of equation (22) to the power-law form in equation (24).

Consequently the time, $t$, is derived from a waiting-time pdf given by that of the network as a whole:

$$\psi(t) = \frac{\mu - 1}{T+t}T^{\mu-1} \quad (26)$$

and the survival probability is empirically determined from figure 4(d) to be

$$\Psi(t) = \int_t^\infty \psi(t')dt' = \left(\frac{T}{T+t}\right)^{\mu-1}. \quad (27)$$

The Laplace transform of the survival probability in terms of that for the waiting-time pdf is

$$\hat{\Psi}(s) = \frac{1}{s+g_0\Delta \tau \hat{\phi}(s)} \phi(0). \quad (28)$$

Inserting these last two expressions into equation (24) and evaluating the sum yields

$$\left\{ \hat{\phi}(s) \right\} = \frac{1}{s+g_0\Delta \tau \hat{\phi}(s)} \phi(0) \quad (29)$$

whose inverse Laplace transform yields:

$$\frac{d\left\{ \phi(t) \right\}}{dt} = -g_0\Delta \tau \int \phi(t-t')\left\{ \phi(t') \right\}dt' \quad (30)$$

which is a generalized master equation.

3.1. Fractional Langevin equation

The function $\Phi(t)$ in equation (30) is a memory kernel containing the information on how the other elements in the network influence the dynamics of the individual element under study. The Laplace transform of the memory kernel is

$$\hat{\Phi}(s) = \frac{s\hat{\phi}(s)}{1-s\hat{\psi}(s)} \quad (31)$$

and a complete discussion of its properties is now given in textbooks [33]. Equation (31) is the Laplace transform of the Montroll-Weiss memory kernel obtained using their continuous-time random walk theory.

Previous analysis, including the DMM calculations, has shown that the global waiting-time pdf is an inverse power-law distribution, as seen in figure 4(d). The asymptotic behavior of an individual in time is determined by considering the waiting-time pdf given by equation (26) as $s \to 0$:
\( \psi \Gamma \mu \lambda \approx - \mu \ll \mu - \iota \) so that equation (32) reduces to

\[
\hat{\phi}(s) = \frac{1}{s + \lambda^{-\mu-1}z^{-\mu}} \phi(0)
\]

and the rate parameter has the value

\[
\lambda^{-\mu-1} = \frac{\varepsilon_0 \tau}{\Gamma(2 - \mu)T^{\mu-1}}.
\]

We now assume that the exact equation for the individual dynamics has both an average and a fluctuating part, just as in the mean-field treatment of the double-well potential. Consequently, in terms of the Laplace variables we have the stochastic equation

\[
\hat{\phi}(s) = \frac{1}{s + \lambda^{-\mu-1}z^{-\mu}} \phi(0) + \frac{1}{s + \lambda^{-\mu-1}z^{-\mu}} \varepsilon (s)
\]

which has the inverse Laplace transform [32]

\[
\partial_{\varepsilon}^{-1}[\phi(t)] = -\lambda^{-\mu-1}\phi(t) + \varepsilon(t).
\]

Equation (36) is a stochastic fractional master equation or fractional Langevin equation in which the stochastic properties of \( \varepsilon(t) \) are determined by the fluctuations resulting from the dynamics of the finite-size DMM network. The fractional derivative in this equation is of the Caputo form and has the Laplace transform

\[
\mathcal{L} \{ \partial_{\varepsilon}^{-1}[\phi(t)]; s \} = s^\alpha \hat{\phi}(s) - s^{\alpha-1} \phi(0).
\]

It is completely equivalent to that determined using the Riemann–Liouville form of the fractional derivative.

The solution to the fractional Langevin equation is given by

\[
\phi(t) = \phi(0) E_{\mu-1} \left( - (\lambda t)^{\mu-1} \right) + \int_0^t (t - t')^{\mu-2} E_{\mu-1,\mu-1} \left( - (\lambda [t - t'])^{\mu-1} \right) \varepsilon (t') \, dt'
\]

where the homogeneous solution to the fractional equation is the Mittag–Leffler function

\[
E_{\theta} (z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1 + k \theta)}; \theta > 0
\]

and the kernel of the integral is in terms of the Mittag–Leffler function of the second kind

\[
E_{\theta,\eta} (z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\eta + k \theta)}.
\]

The dynamics of the individual is determined by the exact Laplace transform equation (28). However, it is notoriously difficult to obtain analytic expressions by direct inversion of the resulting equations due to the complexity of the exact form of the Laplace transform memory kernel. Consequently, the strategy is to consider the asymptotic forms of the solution, which was done by examining the solutions to the fractional Langevin equation given by equation (39). We find that the properties of the fluctuations change as the control parameter varies between the subcritical, critical, and supercritical regions.

### 3.2. Solution domains

In the subcritical, critical, and supercritical regions of DMM dynamics, the single elements used in the evaluation of the probability difference \( \langle \psi(t) \rangle \) were selected at random among all nodes of the lattice. The calculations were done on a 100 × 100 node two-dimensional lattice, with nearest-neighbor interactions. The time-dependent average solution calculated over an ensemble of randomly chosen individuals is depicted in figure 5, where the average is taken over \( 10^4 \) independent realizations of the dynamics. The analytic solution is obtained by averaging equation (39) over an ensemble of realizations of the single-particle trajectory to obtain the Mittag-Leffler function:

\[
\langle \phi(t) \rangle = \phi(0) E_{\mu-1} \left( - [\lambda t]^{\mu-1} \right)
\]

From the series form of the Mittag-Leffler function, it is evident that for \( \mu = 2 \), the average probability difference would be an exponential. Consequently, the influence of the network on the behavior of the individual in this case would be essentially that of uncorrelated random noise, and therefore would not
In summary, the last few years have seen substantial attention focused on the role of criticality [19] in explaining the function of complex networks, from flocks of birds [6] to neural networks [11] to the brain [7]. At criticality,
the short-range links of Ising-like cooperative models are converted into long-range interactions, turning a set of $N$ distinct elements into an organized network that behaves as a single individual with extended cognition [3, 30]. A complex network at criticality generates 1/f noise [17], which is thought to be relevant to cognition [31], with the interesting property of maximizing information transport [1, 33]. Moreover, the network dynamics has a subtle but profound influence on the behavior of each individual within the network.

The numerical solution of the DMM on a $100 \times 100$ lattice with elements at each of the nodes and with nearest-neighbor interactions gives rise to an inverse power-law survival probability for the global variable introduced in section 2. Using the theory of subordination, that being the time experienced by an individual with the influence of the network entering into the individual’s dynamics through the distribution of critical events, the dynamics of an individual is determined by a fractional Langevin equation.

The explicit form of the fractional Langevin equation depends on whether the network dynamics is in the subcritical, critical, or supercritical domain. In all three domains, the Mittag-Leffler function solution to the fractional Langevin equation is sufficient to describe the dynamic response of an individual to the other 9,999 dynamic elements of the network. In the subcritical and critical domains, the solutions could be modified to include truncation effects evident in the numerical data.

The lesson to be learned from the combination of computation and analysis presented herein is that complex networks of finite size whose dynamics are members of the Ising universality class, such as those described by the DMM, have both an analytic and a numerical description. Instead of confining the dynamic description to that of the macroscopic variable, that being the global or average state of the network, one can also investigate how individual members of the network respond to the influence of the network as a whole. If we consider the fluctuations in the global dynamics to be microscopic and the potential of the global variable to be macroscopic, then the real-time dynamic description of the individuals is mesoscopic. In general, the mesoscopic dynamics can be described by a fractional stochastic differential equation.

Coupling two or more such fractional stochastic equations to model the across-scale coupling within the brain suggests itself. This is presently an active area of investigation.

Acknowledgments

The authors would like to thank the US Army Research Office for supporting this research. P.G. warmly thanks the ARO and the Welch Foundation for their support through Grants No. W911NF-11-1-0478 and No.B-1577, respectively. MT thanks the ARO for their support through Grant No. W911NF-04-D-0001.

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