Effect of magnetohydrodynamic Casson fluid flow and heat transfer past a stretching surface in porous medium with slip condition

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Abstract. In this paper, we investigate the flow of magnetohydrodynamic Casson fluid over a linear stretching surface in porous medium with slip condition. The governing partial differential equations are reduced to non-linear ordinary differential equations with the aid of similarity transformation. The transformed equations and boundary conditions are then solved by using exact analytical method. The flow field is affected by the presence of physical parameters, such as Casson fluid parameter, magnetohydrodynamic parameter, velocity slip parameter, porosity parameter, and suction/injection, whereas the temperature field is additionally affected by magnetohydrodynamic, thermal radiation, Prandtl and Eckert numbers. The effects of the pertinent physical parameters on the velocity and temperature fields are presented through graphs and discussed. Skin friction and heat transfer coefficients are tabulated and analyzed.

1. Introduction
Since few decades ago, the study about the boundary layer and heat transfer of non-Newtonian fluids has been carried out by many researchers. Many discoveries have been made include the invention of new models and the improvement of the method to solve the problem either in numerical or analytical way. Casson fluid can be categorized as a non-Newtonian fluid model that becomes the major subject of interest among the scientists in conducting the research as it is very significant in real life application, especially in the area of engineering. It was first discovered and invented in 1959 by Casson for the prediction on the flow behavior of pigment oil suspensions [1] and until today many investigations regarding Casson fluid have been conducted.

Mukhopadhayay et al. [2] have numerically investigated about the flow and heat transfer of Casson fluid at an exponentially stretching permeable surface. They have found out that the high value of Casson parameter slower the velocity field but enhances the temperature field. Later, Bhattacharyya et al. [3] have considered both permeable stretching and shrinking sheet for boundary layer flow of Casson fluid. Through the exact solutions obtained, they revealed that...
stronger mass suction is needed for the steady flow of Casson fluid. Then, Raju et al. [4] have considered magnetohydrodynamic (MHD) and several other effects in their study, and focused on the heat and mass transfer analysis for the Casson fluid bounded by an exponentially permeable stretching surface and they have solved the problem numerically. The natural convection flow of Casson nanofluid is investigated by Ullah et al. [5] in the presence of thermal radiation and chemical reaction through the porous medium. By considering a stretching cylinder, Tamoor et al. [6] have discovered that the curvature parameter gives the same impacts on the physical profiles, and also the skin friction and heat transfer coefficients in the study of MHD flow of Casson fluid. Moreover, the study on the viscoplastic Casson fluid flow bounded by a stretching surface has been carried out by Hussanan et al. [7] by considering the suction/injection and also the viscous dissipation effects. Nawaz et al. [8] then studied on the axisymmetric Casson fluid flow in the presence of hydromagnetic to figure out the impacts of the conductivity of variable thermal. The effects of Newtonian heating has been considered by Ahmad et al. [9] in their analysis towards the Casson fluid flow and heat transfer on the stretchable sheet. Recently, the forced convection on Casson fluid has been studied by Qawasmeh et al. [10] by using Darcy Forchheimer Brinkman model. Also, the heat generation/absorption and chemical reaction impacts towards MHD Casson fluid has been studied by Kataria and Patel [11] by using Laplace transform technique.

Thus, in this study we want to construct a mathematical model for the boundary layer flow and heat transfer of Casson fluid that is bounded by a stretching surface with the presence of viscous dissipation, where we have extended the previous study by Hussanan et al. [7]. The main highlight of this study is that, we want to discover the impacts of additional parameters which are the magnetohydrodynamic, porosity and velocity slip parameters towards the velocity and temperature profiles, as well as the skin friction and heat transfer coefficients of this Casson fluid flow model, by implementing exact analytical method. The results are presented in graphical and tabular form.

2. Modeling

We consider two-dimensional (2D) flow of Casson fluid bounded by a linear stretching surface with the presence of viscous dissipation. The x-axis is taken along the continuous stretching sheet, where we let \( u_w(x) = cx \) illustrates the stretching velocity along the \( x \)-direction. The \( y \)-axis is perpendicular to the surface. Magnetohydrodynamic, porosity and velocity slip parameters are being considered to extend the previous work by Hussanan et al. [7]. Under these assumptions, the boundary layer equations governing the flow and heat transfer of Casson fluid can be expressed as follows:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}
\]

\[
u \frac{u}{\partial x} + v \frac{u}{\partial y} = \nu \left(1 + \frac{1}{\beta}\right) \frac{\partial^2 u}{\partial y^2} - \frac{\sigma B_0^2 u}{\rho} - \frac{\nu}{K^*} u, \tag{2}\]

\[
u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{\mu}{\rho C_p} \left(1 + \frac{1}{\beta}\right) \left(\frac{\partial u}{\partial y}\right)^2 + \frac{k}{\rho C_p} \left(\frac{\partial^2 T}{\partial y^2}\right) + \frac{\sigma B_0^2 u^2}{\rho C_p} - \frac{1}{\rho C_p} \frac{\partial q_r}{\partial y}, \tag{3}\]

where \( u \) and \( v \) are the velocity components in \( x \)-and \( y \)-directions accordingly, while \( \nu = \frac{\mu}{\rho} \) is the kinematic viscosity, where \( \mu \) is the coefficient of fluid viscosity and \( \rho \) is the fluid density, \( \beta \) is the Casson parameter, \( \sigma \) is for fluid electrical conductivity, \( C_p \) stands for heat capacity at constant pressure, \( B_0 \) is the uniform magnetic field, \( K^* \) for permeability of porous media, \( T \) for temperature and \( k \) stands for thermal conductivity fluid. Also, by applying Rosseland
approximation [12] for radiation we may write the radiative heat flux \( q_r \) as follows

\[
q_r = -\frac{4\sigma^* T^4}{3k^*} \frac{\partial T}{\partial y},
\]

(4)

where \( \sigma^* \) and \( k^* \) are the Stefan-Boltzmann constant and the absorption coefficient accordingly. We assume that the difference in temperature within the flow is such that the term \( T^4 \), which can be expanded in a Taylor series about \( T_\infty \) and also can be expressed as a linear function of temperature as shown below

\[
T^4 \approx 4T_\infty^3 T - 3T_\infty^4.
\]

(5)

Applying the above approximation to (4), we have

\[
q_r = -\frac{16\sigma^* T_\infty^3}{3k^*} \frac{\partial T}{\partial y}.
\]

(6)

The associated boundary conditions are

\[
u = \nu_w(x), \quad v = v_w(x), \quad T = T_w = T_\infty + Ax^2 \quad \text{at} \quad y = 0,
\]

\[
u \to 0, \quad T \to T_\infty, \quad \text{as} \quad y \to \infty,
\]

(7)

in which \( c \) stands for positive stretching rate constant, \( l \) is the slip parameter, \( T_w \) stands for the temperature of the stretching sheet, \( T_\infty \) is for the temperature far away from the stretching sheet and \( A \) is a constant. Setting

\[
u = cx f'(\eta), \quad v = -(cw)^{\frac{1}{2}} f(\eta), \quad \eta = \left(\frac{c}{\nu}\right)^{\frac{1}{2}} y, \quad \theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty},
\]

(8)

Equation (1) is satisfied and Equations (2), (3) become

\[
\left(1 + \frac{1}{\beta}\right) f''' + ff'' - (M + K)f' - f'^2 = 0,
\]

(9)

\[
(1 + R_d) \theta'' + Pr f' \theta' - 2Pr f' \theta + Pr Ec \left(M f'^2 + \left(1 + \frac{1}{\beta}\right) f'^2\right) = 0.
\]

(10)

The new transformed boundary conditions are

\[
f(\eta) = s, \quad f'(\eta) = 1 + LF''(\eta), \quad \theta(\eta) = 1, \quad \text{at} \quad \eta = 0,
\]

\[
f'(\eta) \to 0, \quad \theta(\eta) \to 0, \quad \text{as} \quad \eta \to \infty,
\]

(11)

where \( f \) and \( \theta \) are the dimensionless stream function and temperature function respectively, and the prime denotes differentiation with respect to \( \eta \). Here, \( M, K, R_d, s, \) and \( L \) stands for magnetohydrodynamic, porosity, thermal radiation, suction or injection, and velocity slip parameters, accordingly. \( Pr \) and \( Ec \) are the Prandtl and Eckert numbers. Refer below for the precise definition:

\[
Pr = \frac{\nu}{\alpha}, \quad M = \frac{\sigma B_0^2}{\rho c}, \quad R_d = \frac{16\sigma^* T_\infty^3}{3k^*k}, \quad Ec = \frac{c^2}{AC_p}, \quad L = l \left(\frac{c}{\nu}\right)^{\frac{1}{2}}
\]

(12)

Following the work of Hussanan et al. [7], the dimensionless expression of skin friction coefficient and local Nusselt number can be written as below, respectively.

\[
Re_{xx}^{\frac{1}{2}} C_f = \left(1 + \frac{1}{\beta}\right) f''(0),
\]

(13)

\[
Re_{xx}^{\frac{1}{2}} Nu_x = -\theta'(0),
\]

(14)

where \( Re_x \) is the local Reynolds number.
3. Exact analytical solutions

3.1. Flow field

As according to Crane [13], the exact formula representing the boundary layer flow can be found, where we can assume that equation (9) possesses solution of exponential type

\[ f(\eta) = s + \frac{1}{c} (1 - e^{-zn}), \]

where \( c \) is the slip condition produces the relation

\[ c = z(1 + Lz). \]

By using the boundary conditions and some steps of substitution, we then obtain another relation in the form of cubic algebraic equation

\[ z^3 L \left(1 + \frac{1}{\beta}\right) + z^2 \left(\left(1 + \frac{1}{\beta}\right) - Ls\right) - z \left(s + L(K + M)\right) - K - M - 1 = 0. \]

Solving the above cubic algebraic equation, we obtain the positive value of \( z \) and the velocity field is determined to be

\[ f'(\eta) = \frac{1}{1 + Lz} e^{-zn}. \]

3.2. Temperature field

We suppose transformation, where \( t = e^{-zn} \) and the relations between the derivatives with respect to \( \eta \) and the derivatives with respect to \( t \) [14]:

\[ \frac{\partial \theta}{\partial \eta} = -zt \frac{\partial \theta}{\partial t}, \quad \frac{\partial^2 \theta}{\partial \eta^2} = - \frac{1}{t} \left( t^2 \frac{\partial^2 \theta}{\partial t^2} + \frac{\partial \theta}{\partial t}\right). \]

By applying the transformation to Equation (10), we obtain

\[ t\theta''(t) + (n - mt)\theta'(t) + 2m\theta(t) + \frac{P Ec}{c^2} \left(\left(1 + \frac{1}{\beta}\right) tz^2 + Mt\right) = 0, \]

with the appropriate boundary conditions

\[ \theta(0) = 0, \quad \theta(1) = 1. \]

Therefore, solving the ordinary differential equation (20), we obtain the exact solution for temperature field

\[ \theta(\eta) = -\frac{1}{2} \frac{m^2 (e^{-zn})^2 - 2me^{-zn}(n + 1) + n^2 + n) Ec P(\beta z^2 + M \beta + z^2)}{\beta c^2 m^2 (n + 1)} + \frac{1}{2} \frac{H(e^{-zn})^{-n+1}}{\beta c^2 (n + 1)} \left( Ec P\beta z^2 + Ec PM \beta + Ec Pz^2 + 2\beta c^2 n + 2\beta c^2\right) - \frac{(z^2 + M) \beta + z^2}{\beta c^2 m^2} \text{hypergeometric}([-1-n,[-n+2],me^{-zn})} \]

where \( H = \frac{\text{hypergeometric}([-1-n],[-n+2],me^{-zn})}{\text{hypergeometric}([-1-n],[-n+2],m)}, \quad n = 1 - \frac{P}{ze} - \frac{P_s}{z}, \quad m = -\frac{P}{ze}, \quad \text{and} \quad P = \frac{Pr}{(1+Kd)}.\)
4. Results and Discussions

The process of calculating and solving the problem as well as plotting the graphs are aided by Maple software. So, in this section we will explore the impacts of various relevant parameters on the velocity \(f'(\eta)\) and temperature \(\theta(\eta)\) profiles as well as the skin friction \(-f''(0)\) and heat transfer \(-\theta'(0)\) coefficients.

It can be seen that from Figure 1, the rise in Casson parameter \(\beta\) causes depletion in the velocity profile \(f'(\eta)\). The same trend also goes to the velocity slip parameter \(L\) in figure 2, as the slip resists the fluid flow, causes the velocity to reduce. In figures 3 and 4, it can be seen that the increment of magnetic \(M\) and porosity \(K\) parameters also causes decrement in the velocity profile \(f'(\eta)\). Moreover, as illustrated in figure 5, the increment in suction parameter \(s\) causes the velocity profile \(f'(\eta)\) to reduce, as suction is also an agent that provides resistance to the flow of the fluid.

Meanwhile, for the temperature profile \(\theta(\eta)\), the high value of Casson parameter \(\beta\) and velocity slip parameter \(L\) causes the increment in the temperature profile \(\theta(\eta)\) as shown in figures 6 and 7 respectively. Then, figure 8 illustrates that the increment in Prandtl number \(\text{Pr}\) causes the temperature profile \(\theta(\eta)\) to decrease which means that the Prandtl number \(\text{Pr}\) can be used as a cooling agent in conducting flows.

For figure 9 until figure 12, we can see the impacts of thermal radiation \(R_d\), Eckert number \(\text{Ec}\) which is the viscous dissipation parameter, magnetic \(M\) and porosity \(K\) parameters towards the temperature profile \(\theta(\eta)\) respectively. It reveals that the increment in all of these parameters increase the temperature profile \(\theta(\eta)\). The Lorentz force that is developed due to the increment in magnetic field \(M\) causes the velocity to reduce, as this force is in the opposite direction to the flow which then enhances the temperature. Then, figure 13 displays that the presence of suction parameter \(s\) causes the temperature profile \(\theta(\eta)\) to decrease.

For the numerical data, tables 1-3 show the numerical values for the skin friction \(-f''(0)\) and heat transfer \(-\theta'(0)\) coefficients. In table 1, it is noticed that the increment of velocity slip parameter \(L\) causes the skin friction coefficient \(-f''(0)\) and heat transfer coefficient \(-\theta'(0)\) to decrease. Besides, the skin friction coefficient \(-f''(0)\) increases as the value of magnetic \(M\) and porosity \(K\) parameters increase. Meanwhile, the opposite trend occurs for magnetic \(M\) and porosity \(K\) parameters towards the heat transfer coefficient \(-\theta'(0)\).

In table 2, it can be analyzed that the higher the Casson parameter \(\beta\), the higher the skin friction coefficient \(-f''(0)\). Moreover, the presence of suction \((s > 0)\) causes the skin friction \(-f''(0)\) to increase, and the opposite behaviour for injection \((s < 0)\). As to be compared with the previous analytical results by Hussanan et al. [7], it can be seen that the values of the present skin friction coefficient \(-f''(0)\) are well coincided. It should be noted that Hussanan et al. [7] have used both numerical and analytical method for comparison purpose, but they only presented the analytical results.

Furthermore, from table 3, it is noticed that the increment of Casson parameter \(\beta\), thermal radiation parameter \(R_d\) and Eckert number \(\text{Ec}\) have reduced the local Nusselt number which is the heat transfer coefficient \(-\theta'(0)\). Meanwhile, different from Prandtl number \(\text{Pr}\), its increment enhances the heat transfer coefficient \(-\theta'(0)\). Last but not least, the presence of suction \((s > 0)\) leads to increment in the heat transfer coefficient \(-\theta'(0)\), and the opposite trend for injection \((s < 0)\) case.

5. Conclusion

The Casson fluid flow and heat transfer bounded by a linear stretching surface in the presence of viscous dissipation has been investigated. The additional parameters which are the velocity slip, magnetohydrodynamic and porosity parameters have been considered. It is noticed that the present results coincide well with the previous research work and are in a good agreement, which thus validate the calculation and the technique we used. Based on the results, it reveals
that the velocity profile $f'(\eta)$ decreased in the presence of velocity slip, magnetohydrodynamic and porosity parameters. Also, the temperature profile $\theta(\eta)$ is enhanced in the presence of velocity slip, magnetic and porosity parameters. Meanwhile, the skin friction coefficient $-f''(0)$ is increasing as the value of magnetohydrodynamic and porosity parameters increasing, and decreasing as the value of velocity slip parameter increasing. The heat transfer coefficient $-\theta'(0)$ is decreasing as the value of magnetohydrodynamic, porosity and velocity slip parameters increasing.

Figure 1. Velocity profile for Casson parameter $\beta$, when $s=0$.

Figure 2. Velocity profile for velocity slip parameter $L$, when $s=0$.

Figure 3. Velocity profile for magnetic parameter $M$, when $s=0$.

Figure 4. Velocity profile for porosity parameter $K$, when $s=0$. 
Figure 5. Temperature profile for suction parameter $s$.

Figure 6. Temperature profile for Casson parameter $\beta$, when $s=0$.

Figure 7. Temperature profile for velocity slip parameter $L$, when $s=0$.

Figure 8. Temperature profile for Prandtl number $Pr$, when $s=0$. 
Figure 9. Temperature profile for thermal radiation parameter $R_d$, when $s=0$.

Figure 10. Temperature profile for Eckert number $Ec$, when $s=0$.

Figure 11. Temperature profile for magnetic parameter $M$, when $s=0$.

Figure 12. Temperature profile for porosity parameter $K$, when $s=0$. 
Figure 13. Temperature profile for suction parameter $s$.

Table 1. The values of skin friction coefficient $-f''(0)$ and local Nusselt number $-\theta'(0)$ for distinct values of $L$, $K$ and $M$ when $\beta = 0.5$, $Ec = Pr = Re = 1$, $s = 0$.

| $L$ | $K$ | $M$ | $-f''(0)$ | $-\theta'(0)$ |
|-----|-----|-----|-----------|--------------|
| 1.0 | 0.1 | 1.0 | 0.427480  | 0.431339     |
| 2.0 |     |     | 0.293423  | 0.414981     |
| 3.0 |     |     | 0.224695  | 0.382240     |
| 1.0 | 0.5 | 1.0 | 0.452401  | 0.394038     |
|     |     | 1.0 | 0.478304  | 0.353030     |
|     | 1.3 |     | 0.491745  | 0.314367     |
|     | 1.5 |     | 0.500000  | 0.291058     |

Table 2. Comparison of skin friction coefficient $-f''(0)$ for distinct values of $\beta$ and $s$ with previously published results.

| $\beta$ | $s$   | $-f''(0)$  |
|---------|-------|------------|
|         | Hussanan et al. [7] | Present   |
| 0.5     | 1.0   | 0.767722   | 0.767592   |
| 1.0     | 1.000018 | 1.000000  |
| 3.0     | 1.318730 | 1.318729  |
| 0.5     | 1.073597 | 1.073590  |
| −0.5    | 0.698647 | 0.698590   |
| 0.0     | 0.866053 | 0.866025   |
Table 3. Comparison of local Nusselt number $-\theta'(0)$ for distinct values of $\beta$, Pr, Ec, $R_d$ and $s$ with previously published results.

| $\beta$ | Pr | Ec | s | $R_d = 0$ | $R_d = 2$ |
|---|---|---|---|---|---|
|    | Hussanan et al. [7] Present | Hussanan et al. [7] Present |
| 0.5 | 3.0 | 0.3 | 1.0 | 4.09342 | 4.014527 | 1.971559 | 1.817597 |
| 1.0 | 4.277934 | 3.989199 |
| 3.0 | 4.090609 | 3.949983 | 1.777242 | 1.724577 |
| 1.0 | 1.777242 | 1.724577 | 0.763155 | 0.742716 |
| 3.0 | 4.090609 | 3.949983 | 1.777242 | 1.724577 |
| 10.0 | 10.899393 | 10.479298 | 4.440807 | 4.286236 |
| 3.0 | 3.106213 | 2.637459 | 1.408578 | 1.233026 |
| 2.0 | 1.699934 | 0.762424 | 0.881914 | 0.530809 |
| 0.3 | 1.761117 | 1.725946 | 1.102123 | 1.084125 |
| 0.0 | 2.362104 | 2.301159 | 1.289373 | 1.262301 |
| 0.5 | 3.148674 | 3.051763 | 1.513598 | 1.474815 |

Acknowledgments

The financial support received from Universiti Putra Malaysia in the form of Putra Grant [9619000] are gratefully acknowledged.

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