MOLECULAR DECOMPOSITION AND A CLASS OF FOURIER MULTIPLIERS FOR BI-PARAMETER MODULATION SPACES

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Abstract. In this paper, we investigate bi-parameter modulation spaces on the product of two Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ via uniform decompositions of each factor. A molecular decomposition of these bi-parameter spaces are given, which generalizes the related single-parameter result of Kobayashi and Sawano [33]. Furthermore, we prove the boundedness of a class of Fourier multipliers on bi-parameter modulation spaces, generalizing the results of Bényi et al. [2] and Feichtinger and Narimani [17].

1. Introduction and statement of main results. Modulation spaces, introduced by Feichtinger [15] in the early 1980s, were originally used to measure smoothness of functions and to analyze local properties of frequency space. During the last decades, modulation spaces have been studied systematically; see, for instance, [2,16,17,19,20,33,35,49] and the references therein. These spaces have turned out to be useful in the study of pseudo-differential operators, partial differential equations, signal analysis and quantum mechanics. For instance, by using modulation spaces, Sjöstrand [47] and Tachizawa [50] generalized the Calderón-Vaillancourt theorem; see also the work of Gröchenig and Heil [20]. Moreover, Tomita [51] generalized the Hörmander multiplier theorem by using modulation spaces. In recent years, modulation spaces were also applied to study the global well-posedness of solutions to partial differential equations [3,53].

On the other hand, harmonic analysis is closely related to the dilation structures on the settings. For example, the usual single-parameter dilations on $\mathbb{R}^n$ are given by

$$\delta x = \delta(x_1, \ldots, x_n) = (\delta x_1, \ldots, \delta x_n)$$

for $\delta \in \mathbb{R}_+$ and $x \in \mathbb{R}^n$, and associated to these single-parameter dilations is the Hardy-Littlewood maximal function

$$Mf(x) := \sup_{Q \ni x, Q \text{ cube}} \frac{1}{|Q|} \int_Q |f(y)|dy, \quad x \in \mathbb{R}^n.$$
and the classical Calderón-Zygmund theory. However, if one considers bi-parameter
dilations on \( \mathbb{R}^n \times \mathbb{R}^m \), given by
\[
\delta(x, x') = (\delta_1 x, \ldots, \delta_1 x_n, \delta_2 x_1', \ldots, \delta_2 x_m'),
\]
for \( \delta = (\delta_1, \delta_2) \in \mathbb{R}^+ \times \mathbb{R}^+ \) and \((x, x') \in \mathbb{R}^n \times \mathbb{R}^m\), then associated to these bi-parameter
dilations are the strong maximal function [30], given by
\[
\mathcal{M}_s(f)(x, x') = \sup_{Q \times Q' \ni (x, x')} \frac{1}{|Q|} \int_{Q \times Q'} |f(y, y')| dy dy', \quad (x, x') \in \mathbb{R}^n \times \mathbb{R}^m,
\]
and the bi-parameter/product singular integrals and Fourier/spectral multipliers.
During the past forty years, bi-parameter (and more generally, multi-parameter)
theory of harmonic analysis was systematically developed; see the works of Gundy
and Stein [22], R. Fefferman and Stein [14], Chang and R. Fefferman [4–6], R.
Fefferman [12], Journe [31, 32], Pipher [45], R. Fefferman and Pipher [13], M"{u}ller,
Ricci and Stein [40,41], Nagel, Ricci and Stein [42], Ferguson and Lacey [18], Muscalu
et al. [38, 39], Street [48], Nagel et al. [43, 44], Han and Lu [24], Han, Lu and
Sawyer [26], Han, Lu and Ruan [25], Lu et al. [36, 37]. For more recent develop-
ments on multi-parameter harmonic analysis, we refer to [7–11, 23, 27–29, 46] and
the references therein.

The main purpose of this paper is to investigate bi-parameter modulation spaces,
in particular, to derive a molecular decomposition for these spaces and to prove the
boundedness of a class of Fourier multipliers on these spaces. Recall that there are
two equivalent definitions for the classical (single-parameter) modulation spaces.
One definition makes use of the short-time Fourier transform and window functions
(see, e.g., [15, 19]), while the other one is based on a uniform decomposition of the
Euclidean spaces. To recall the latter definition, we introduce some notation. For
any \( \xi \in \mathbb{R}^n \), we denote \( (\xi) := (1 + |\xi|^2)^{1/2} = (1 + |\xi_1|^2 + \cdots + |\xi_n|^2)^{1/2} \). Let
\[
Q_0 = \{ \xi \in \mathbb{R}^n : -1/2 \leq \xi_i < 1/2 \text{ for } i = 1, \ldots, n \}
\]
be the unit cube in \( \mathbb{R}^n \) and let \( Q_k := k + Q_0 \) for \( k \in \mathbb{Z}^n \). Then the \( Q_k \)'s are pairwise
disjoint and we have \( \mathbb{R}^n = \cup_{k \in \mathbb{Z}^n} Q_k \), which is called the uniform decomposition of \( \mathbb{R}^n \).
Let \( \{ \varphi_k \}_{k \in \mathbb{Z}^n} \) be a sequence of Schwartz functions and assume that it is
exclusive (see Section 2 for definition). Define the operators
\[
\Box_k f := F^{-1}(\varphi_k F f), \quad k \in \mathbb{Z}^n.
\]
Then, for \( s \in \mathbb{R} \) and \( 0 < p, q \leq \infty \), the (single-parameter) modulation
space \( M^s_{p,q}(\mathbb{R}^n) \) is defined as the collection of all distributions \( f \in \mathcal{S}'(\mathbb{R}^n) \) for which
\[
\|f\|_{M^s_{p,q}(\mathbb{R}^n)} := \left( \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{sq} \| \Box_k f \|_{L^p(\mathbb{R}^n)}^q \right)^{1/q} < \infty.
\]
It is well known that the scale of \( M^s_{p,q}(\mathbb{R}^n) \) is independent of the choice of the exclusive sequence \( \{ \varphi_k \}_{k \in \mathbb{Z}^n} \).

Bi-parameter modulation spaces were first introduced by Kobayashi, Sugimoto
and Tomita [34]. Indeed, they introduced a more general notion, which is called bi-
parameter \( \alpha \)-modulation spaces and denoted by \( \mathcal{BM}_p^s(\mathbb{R}^n \times \mathbb{R}^m) \), for \( 0 \leq \alpha \leq 1 \),
\( s = (s_1, s_2) \in \mathbb{R} \times \mathbb{R} \) and \( 1 \leq p, q \leq \infty \). When \( \alpha = 0 \), these spaces reduce to the
bi-parameter modulation spaces \( \mathcal{BM}_p^s(\mathbb{R}^n \times \mathbb{R}^m) \). Recently, Xu and Huang [54]
extend the definition of bi-parameter \( \alpha \)-modulation spaces \( \mathcal{BM}_p^s(\mathbb{R}^n \times \mathbb{R}^m) \) to
the case \( 0 < p, q \leq \infty \), and proved the boundedness of bi-parameter pseudo-differential
operators on \( \mathcal{BM}_p^s(\mathbb{R}^n \times \mathbb{R}^m) \). See also [55] for the study of boundedness of
bi-parameter fractional integrals on bi-parameter modulation spaces.
Let us recall the definition of bi-parameter modulation spaces. More details will be given in Section 2. We fix two exclusive sequences \( \{ \varphi_k \}_{k \in \mathbb{Z}^n} \) and \( \{ \psi_{k'} \}_{k' \in \mathbb{Z}^m} \) with respect to the uniform decompositions of \( \mathbb{R}^n \) and \( \mathbb{R}^m \), respectively. Define the operators
\[
\square_{k,k'} f := F^{-1} \left( \langle \varphi_k \otimes \psi_{k'} \rangle \hat{f} \right)
\]
for \( k \in \mathbb{Z}^n \) and \( k' \in \mathbb{Z}^m \), where \( \varphi_k \otimes \psi_{k'} \) is a function on \( \mathbb{R}^n \times \mathbb{R}^m \) defined by
\[
(\varphi_k \otimes \psi_{k'})(\xi,\zeta') := \varphi_k(\xi)\psi_{k'}(\zeta').
\]
For \( s = (s_1, s_2) \in \mathbb{R} \times \mathbb{R} \) and \( 0 < p, q \leq \infty \), we define the bi-parameter modulation space \( \mathcal{BM}^{s}_{p,q}(\mathbb{R}^n \times \mathbb{R}^m) \) as the collection of all \( f \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^m) \) for which
\[
\|f\|_{\mathcal{BM}^{s}_{p,q}(\mathbb{R}^n \times \mathbb{R}^m)} := \left( \sum_{k \in \mathbb{Z}^n} \sum_{k' \in \mathbb{Z}^m} \langle k \rangle^{s_1q} \langle k' \rangle^{s_2q} \| \square_{k,k'} f \|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)}^q \right)^{1/q} < \infty,
\]
where \( \langle k \rangle := (1 + |k_1|^2 + \ldots + |k_n|^2)^{1/2} \). In Section 2 we shall show that the scale of \( \mathcal{BM}^{s}_{p,q}(\mathbb{R}^n \times \mathbb{R}^m) \) is independent of the choice of \( \{ \varphi_k \}_{k \in \mathbb{Z}^n} \) and \( \{ \psi_{k'} \}_{k' \in \mathbb{Z}^m} \), as long as they are exclusive.

One main objective of the present paper is to derive a molecular decomposition for the bi-parameter modulation spaces \( \mathcal{BM}^{s}_{p,q}(\mathbb{R}^n \times \mathbb{R}^m) \). We point out that various definitions of molecules of (single-parameter) modulation spaces has been introduced in \([1,21,33]\). Our definition of molecules for bi-parameter modulation spaces is inspired by \([33]\).

**Definition 1.1** (Molecule). Let \( s = (s_1, s_2) \in \mathbb{R} \times \mathbb{R}, k = (k, k') \in \mathbb{Z}^n \times \mathbb{Z}^m \) and \( l = (l, l') \in \mathbb{Z}^n \times \mathbb{Z}^m \). Suppose that \( K \in \mathbb{N} \) and \( N, N' \in \mathbb{R}_+ \) are large enough and fixed. A \( C^K \) function \( \Psi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{C} \) is said to be an \((s; k, l)\)-molecule, if it satisfies
\[
|\partial_x^\alpha \partial_{x'}^\beta \left[ e^{-ik \cdot x} e^{-ik' \cdot x'} \Psi(x, x') \right]| \leq \langle k \rangle^{-s_1} \langle k' \rangle^{-s_2} \langle x - l \rangle^{-N} \langle x' - l' \rangle^{-N'}
\]
for all \( \alpha \in \mathbb{Z}_n^+ \) and \( \beta \in \mathbb{Z}_m^+ \) such that \(|\alpha| + |\beta| \leq K\).

For \( 0 < p, q \leq \infty \) and a sequence \( \{ f_j \} \) of functions on \( \mathbb{R}^n \times \mathbb{R}^m \), we define
\[
\| \{ f_j \} \|_{\ell^q_{L^p}(\mathcal{BM}^{s}_{p,q}(\mathbb{R}^n \times \mathbb{R}^m))} := \left( \sum_j \| f_j \|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)}^q \right)^{1/q}.
\]

**Definition 1.2** (Sequence space \( m_{p,q} \)). Let \( 0 < p, q \leq \infty \). Given a sequence \( \lambda = \{ \lambda_{k,l} : k = (k, k') \in \mathbb{Z}^n \times \mathbb{Z}^m, l = (l, l') \in \mathbb{Z}^n \times \mathbb{Z}^m \} \) of complex numbers, we define
\[
\| \lambda \|_{m_{p,q}} := \left\| \left\{ \sum_{k \in \mathbb{Z}^n \times \mathbb{Z}^m} \lambda_{k,l} (\chi_{Q_l} \otimes \chi_{Q'_{l'}}) \right\} \right\|_{\ell^q_{L^p}(\mathbb{R}^n \times \mathbb{R}^m)}
\]
where \( \mathbb{R}^n = \bigcup_{l \in \mathbb{Z}^n} Q_l \) and \( \mathbb{R}^m = \bigcup_{l' \in \mathbb{Z}^m} Q'_{l'} \) are uniform decompositions of \( \mathbb{R}^n \) and \( \mathbb{R}^m \), respectively, and \( (\chi_{Q_l} \otimes \chi_{Q'_{l'}})(\xi, \zeta') := \chi_{Q_l}(\xi)\chi_{Q'_{l'}}(\zeta') \).

Our result concerning molecular decomposition for bi-parameter modulation spaces can be stated as follows.

**Theorem 1.3**. Let \( s = (s_1, s_2) \in \mathbb{R} \times \mathbb{R} \) and \( 0 < p, q \leq \infty \).

(i) For every \( f \in \mathcal{BM}^{s}_{p,q}(\mathbb{R}^n \times \mathbb{R}^m) \), there exist a sequence \( \lambda = \{ \lambda_{k,l} \}_{k,l \in \mathbb{Z}^n \times \mathbb{Z}^m} \in m_{p,q} \) and a sequence \( \{ \Psi_{k,l} \}_{k,l \in \mathbb{Z}^n \times \mathbb{Z}^m} \), where each \( \Psi_{k,l} \) is an \((s; k, l)\)-molecule, such that
\[
f = \sum_{k,l \in \mathbb{Z}^n \times \mathbb{Z}^m} \lambda_{k,l} \Psi_{k,l} \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^m)
\]
and
\[ \| \lambda \|_{m,p,q} \leq C\| f \|_{B_{m,p,q}(\mathbb{R}^n \times \mathbb{R}^m)}, \]

where \( C \) is a constant independent of \( f \).

(ii) Conversely, if \( \lambda = \{ \lambda_{k,1} \}_{k,1 \in \mathbb{Z}^n \times \mathbb{Z}^m} \in m_{p,q} \) and \( \{ \Psi_{k,1} \}_{k,1 \in \mathbb{Z}^n \times \mathbb{Z}^m} \) is a sequence such that each \( \Psi_{k,1} \) is an \((s,k,1)\)-molecule with \( K, N, N' \) in Definition 1.1 satisfying \( K > 4 \max(n,m) \max(1,1/p) + 2 \max(|s_1|,|s_2|) + 2, N > n/\min\{p,1\} \) and \( N' > m/\min\{p,1\} \), then the sum \( \sum_{k,1 \in \mathbb{Z}^n \times \mathbb{Z}^m} \lambda_{k,1} \Psi_{k,1} \) converges in \( S'(\mathbb{R}^n \times \mathbb{R}^m) \), and
\[ \left\| \sum_{k,1 \in \mathbb{Z}^n \times \mathbb{Z}^m} \lambda_{k,1} \Psi_{k,1} \right\|_{B_{m,p,q}(\mathbb{R}^n \times \mathbb{R}^m)} \leq C\| \lambda \|_{m,p,q}. \]

We also consider Fourier multipliers for bi-parameter modulation spaces. Bényi, Grafakos, Gröchenig and Okoudjou [2] studied a class of Fourier multipliers of the form
\[ m(\xi) = -2\imath \sum_{j \in \mathbb{Z}} c_j \chi_{(b_j,b_j+1)}(\xi), \]
where \( b > 0 \) and \( \{ c_j \}_{j \in \mathbb{Z}} \) is a sequence of bounded complex numbers. They proved that if \( m(\xi) \) is as above then for \( 1 < p < \infty \) and \( 1 \leq q \leq \infty \), one has
\[ \| \mathcal{F}^{-1}(m \cdot \mathcal{F} f) \|_{M_{p,q}(\mathbb{R})} \leq C\| f \|_{M_{p,q}(\mathbb{R})}. \]

Feichtinger and Narimani [17] reproved this result using a different approach. In the present paper, we generalize this result to the bi-parameter setting. For the sake of simplicity we only formulate our result in the case \( n = m = 1 \), i.e., for bi-parameter modulation spaces on \( \mathbb{R} \times \mathbb{R} \).

**Theorem 1.4.** Assume that \( \{ R_j \}_j \) is a sequence of rectangles such that
\[ \bigcup_j R_j = \mathbb{R} \times \mathbb{R}, \quad R_j \cap R_l = \emptyset \quad (j \neq l) \quad \text{and} \quad \inf \delta(R_j) > 0, \]
where \( \delta(R) := \min\{b-a,e-d\} \) if \( R = [a,b] \times [d,e] \). Let \( s = (s_1,s_2) \in \mathbb{R} \times \mathbb{R} \), \( 1 < p < \infty \) and \( 0 < q \leq \infty \). Then, for any sequence \( \{ c_j \} \) of bounded complex numbers, the function \( m : \mathbb{R} \times \mathbb{R} \to \mathbb{C} \) given by
\[ m(\xi_1,\xi_2) = \sum_j c_j \chi_{R_j}(\xi_1,\xi_2) \]
is a Fourier multiplier on \( B_{m,p,q}(\mathbb{R} \times \mathbb{R}) \). Moreover,
\[ \| \mathcal{F}^{-1}(m \cdot \mathcal{F} f) \|_{B_{m,p,q}(\mathbb{R} \times \mathbb{R})} \leq C\| \{ c_j \}_j \|_{\ell^\infty} \| f \|_{B_{m,p,q}(\mathbb{R} \times \mathbb{R})}, \]
where \( C \) is a constant independent of \( f \).

The rest of this paper is organized as follows. In Section 2 we give the definition of bi-parameter modulation spaces and show that they are well-defined. In Section 3 we present some fundamental properties of bi-parameter modulation spaces. Sections 4 and 5 are devoted to the proofs of Theorems 1.1 and 1.2, respectively.

**Notation.** We use \( \mathcal{F} f \) (or \( \hat{f} \)) and \( \mathcal{F}^{-1} f \) (or \( f' \)) to denote the Fourier transform and the inverse Fourier transform of a function or distribution \( f \), respectively. The letter \( C \) will denote positive constants, which may vary at every occurrence. If \( a \) and \( b \) are two quantities (typically nonnegative), we use \( a \lesssim b \) or \( b \gtrsim a \) to denote that there exists a positive constant \( C \) such that \( a \leq Cb \). By writing \( a \sim b \) we mean that \( a \lesssim b \lesssim a \).
2. Definition of bi-parameter modulation spaces. In this section, we introduce bi-parameter modulation spaces on \( \mathbb{R}^n \times \mathbb{R}^m \) for full range of indices, and show that these spaces are well-defined. It should be pointed out that Xu and Huang introduced the notion of bi-parameter of \( \alpha \)-modulation spaces for full range of indices in their recent work [54].

First we make some conventions. For \( r > 0, k \in \mathbb{Z}^n \) and \( k' \in \mathbb{Z}^m \), set
\[
Q(r) := \{ \xi \in \mathbb{R}^n : -r \leq \xi_i < r \text{ for } i = 1, \ldots, n \},
\]
\[
Q'(r) := \{ \xi' \in \mathbb{R}^m : -r \leq \xi'_i < r \text{ for } i = 1, \ldots, m \}.
\]

Let \( Q_0 := Q(1/2) \) and \( Q'_0 := Q'(1/2) \) be unit cubes centered at the origins of \( \mathbb{R}^n \) and \( \mathbb{R}^m \), respectively. Let \( Q_k := k + Q_0 \) for \( k \in \mathbb{Z}^n \) and \( Q'_{k'} := k' + Q'_0 \) for \( k' \in \mathbb{Z}^m \). Then we have \( \mathbb{R}^n = \bigcup_{k \in \mathbb{Z}^n} Q_k \) and \( \mathbb{R}^m = \bigcup_{k' \in \mathbb{Z}^m} Q'_{k'} \), which are called the uniform decompositions of \( \mathbb{R}^n \) and \( \mathbb{R}^m \), respectively.

The Fourier and inverse Fourier transform on \( \mathbb{R}^n \) are defined respectively by
\[
\mathcal{F}f(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi} dx \quad \text{and} \quad \mathcal{F}^{-1}f(x) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(\xi)e^{ix\cdot\xi} d\xi.
\]

In order to introduce bi-parameter modulation spaces, we need the notion of exclusive sequence of functions associated to the uniform decomposition.

**Definition 2.1.** A sequence \( \{ \varphi_k \}_{k \in \mathbb{Z}^n} \) of Schwartz functions on \( \mathbb{R}^n \) is said to be exclusive if it satisfies the following properties:

(i) There exists \( c > 0 \) such that \( |\varphi_k(\xi)| \geq c \) for all \( \xi \in Q_k \) and \( k \in \mathbb{Z}^n \);

(ii) \( \text{supp } \varphi_k \subseteq k + Q(1) \);

(iii) \( \sum_{k \in \mathbb{Z}^n} \varphi_k(\xi) = 1 \) for all \( \xi \in \mathbb{R}^n \);

(iv) For any \( \alpha \in \mathbb{Z}^+_n \), there exists \( C_\alpha > 0 \) such that \( \|\partial^\alpha \varphi_k\|_{L^\infty} \leq C_\alpha \) for all \( k \in \mathbb{Z}^n \).

We denote
\[
\Upsilon_n := \{ \{ \varphi_k \}_{k \in \mathbb{Z}^n} : \{ \varphi_k \}_{k \in \mathbb{Z}^n} \text{ is exclusive on } \mathbb{R}^n \}.
\]

Note that the set \( \Upsilon_n \) is not empty. Indeed, take a function \( \rho \in \mathcal{S}(\mathbb{R}^n) \) such that \( 0 \leq \rho(\xi) \leq 1 \) for all \( \xi \in \mathbb{R}^n \), \( \text{supp } \rho \subseteq Q(1) \), and \( \rho(\xi) = 1 \) if \( \xi \in Q(1/2) \). For any \( k \in \mathbb{Z}^n \), let \( \rho_k(\xi) := \rho(\xi - k) \) and \( \varphi_k(\xi) := \rho_k(\xi)\left( \sum_{j \in \mathbb{Z}^n} \rho_j(\xi) \right)^{-1} \). Then it is straightforward to verify that the sequence \( \{ \varphi_k \}_{k \in \mathbb{Z}^n} \) satisfies properties (i)–(iv) in Definition 2.1.

Given two sequences \( \{ \varphi_k \}_{k \in \mathbb{Z}^n} \in \Upsilon_n \) and \( \{ \psi_{k'} \}_{k' \in \mathbb{Z}^m} \in \Upsilon_m \), we define the operators
\[
\Box_{k,k'} := \mathcal{F}^{-1}\left[ (\varphi_k \otimes \psi_{k'})\mathcal{F}f \right], \quad k \in \mathbb{Z}^n, \quad k' \in \mathbb{Z}^m,
\]
where \( (\varphi_k \otimes \psi_{k'})(\xi,\xi') := \varphi_k(\xi)\psi_{k'}(\xi') \), for \( (\xi,\xi') \in \mathbb{R}^n \times \mathbb{R}^m \).

Now we are ready to introduce bi-parameter modulation spaces.

**Definition 2.2.** Let \( s = (s_1, s_2) \in \mathbb{R} \times \mathbb{R} \) and \( 0 < p, q \leq \infty \). The bi-parameter modulation space \( \mathcal{B}M_{p,q}^s(\mathbb{R}^n \times \mathbb{R}^m) \) is defined as the collection of all \( f \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^m) \) for which
\[
\|f\|_{\mathcal{B}M_{p,q}^s(\mathbb{R}^n \times \mathbb{R}^m)} := \left( \sum_{k \in \mathbb{Z}^n} \sum_{k' \in \mathbb{Z}^m} \langle k \rangle^{sa} \langle k' \rangle^{s_2a} \|\Box_{k,k'} f\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)}^q \right)^{1/q} < \infty,
\]
with the obvious modification when \( q = \infty \). Recall that \( \langle k \rangle := (1 + |k_1|^2 + \cdots + |k_n|^2)^{1/2} \).
Next we show that the bi-parameter modulation spaces introduced above are independent of the choice of the sequences \( \{ \varphi_k \}_{k \in \mathbb{Z}^n} \in \mathcal{Y}_n \) and \( \{ \psi_{k'} \}_{k' \in \mathbb{Z}^m} \in \mathcal{Y}_m \). For our purpose we need some preparations.

If \( 0 < p < \infty \) and \( \Omega \) is a compact subset of \( \mathbb{R}^n \times \mathbb{R}^m \), we denote
\[ L^p_\Omega(\mathbb{R}^n \times \mathbb{R}^m) := \{ f \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^m) : \text{supp} \mathcal{F} f \subset \Omega, \| f \|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} < \infty \} \]

The following result can be found in [52, §1.5.1].

**Lemma 2.3.** Let \( \Omega \) and \( \Gamma \) be compact subsets of \( \mathbb{R}^n \times \mathbb{R}^m \), \( 0 < p \leq \infty \) and \( p_0 = \min\{1, p\} \). Then there exists a constant \( C > 0 \) such that
\[ \| \mathcal{F}^{-1}(\varphi \mathcal{F} f) \|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \leq C \| \mathcal{F}^{-1} \varphi \|_{L^{p_0}(\mathbb{R}^n \times \mathbb{R}^m)} \| f \|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \]
for all \( f \in L^p_\Omega(\mathbb{R}^n \times \mathbb{R}^m) \) and all \( \mathcal{F}^{-1} \varphi \in L^{p_0}(\mathbb{R}^n \times \mathbb{R}^m) \).

For \( \alpha = (\alpha_1, \alpha_2) \in \mathbb{R} \times \mathbb{R} \), the product sobolev space \( H^2(\mathbb{R}^n \times \mathbb{R}^m) \) is defined as the collection of all tempered distributions \( f \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^m) \) for which
\[ \| f \|_{H^2(\mathbb{R}^n \times \mathbb{R}^m)} := \| (1 + |\xi|^2)^{\alpha_1/2} (1 + |\xi'|^2)^{\alpha_2/2} \mathcal{F} f(\xi, \xi') \|_{L^2(\mathbb{R}^n \times \mathbb{R}^m)} < \infty. \]

**Lemma 2.4.** Let \( \alpha = (\alpha_1, \alpha_2) \in \mathbb{R} \times \mathbb{R} \), \( 0 < p \leq \infty \), \( p_0 = \min\{1, p\} \), \( \sigma_p = n\left(1 - \frac{1}{p_0} - \frac{1}{2}\right) \) and \( \sigma'_p = m\left(1 - \frac{1}{p_0} - \frac{1}{2}\right) \). Let \( \Omega \) be a compact subset of \( \mathbb{R}^n \times \mathbb{R}^m \). If \( \alpha_1 > \sigma_p \) and \( \alpha_2 > \sigma'_p \), then there exists a constant \( C > 0 \) such that
\[ \| \mathcal{F}^{-1}(\omega \mathcal{F} f) \|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \leq C \| \omega \|_{H^2(\mathbb{R}^n \times \mathbb{R}^m)} \| f \|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)}, \]
for all \( f \in L^p_\Omega(\mathbb{R}^n \times \mathbb{R}^m) \) and all \( \omega \in H^2(\mathbb{R}^n \times \mathbb{R}^m) \).

**Proof.** Let \( \Gamma = \{ (y, y') \in \mathbb{R} \times \mathbb{R} : |x - y| \leq 1, |x' - y'| \leq 1 \} \) for some \( (x, x') \in \Omega \). Choose a function \( \theta \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m) \) such that \( \text{supp} \theta \subset \Gamma \) and \( \theta \equiv 1 \) on \( \Omega \). For any \( \omega \in H^2(\mathbb{R}^n \times \mathbb{R}^m) \) and \( f \in L^p_\Omega(\mathbb{R}^n \times \mathbb{R}^m) \), from the definition of \( L^p_\Omega(\mathbb{R}^n \times \mathbb{R}^m) \) we see that
\[ \mathcal{F}^{-1}(\omega \mathcal{F} f) = \mathcal{F}^{-1}(\theta \mathcal{F} f). \]

Hence it follows from Lemma 2.3 that
\[ \| \mathcal{F}^{-1}(\omega \mathcal{F} f) \|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} = \| \mathcal{F}^{-1}(\theta \mathcal{F} f) \|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \leq \| \mathcal{F}^{-1}(\theta \omega) \|_{L^{p_0}(\mathbb{R}^n \times \mathbb{R}^m)} \| f \|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)}. \]

Therefore, it suffices to show \( \| \mathcal{F}^{-1}(\theta \omega) \|_{L^{p_0}(\mathbb{R}^n \times \mathbb{R}^m)} \lesssim \| \omega \|_{H^2(\mathbb{R}^n \times \mathbb{R}^m)}. \)

To see the latter inequality, we consider the following two cases.

**Case 1.** \( 0 < p \leq 1 \). In this case, \( p_0 = \min\{p, 1\} = p \) thus we need to show
\[ \| \mathcal{F}^{-1}(\theta \omega) \|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \lesssim \| \omega \|_{H^2(\mathbb{R}^n \times \mathbb{R}^m)}. \]

To this end, set
\[ E_0 = \{ x \in \mathbb{R}^n : |x| \leq 1 \}, \]
\[ E_\mu = \{ x \in \mathbb{R}^n : 2^{\mu - 1} < |x| \leq 2^\mu \}, \quad \mu = 1, 2, \cdots, \]
\[ F_0 = \{ x' \in \mathbb{R}^m : |x'| \leq 1 \}, \]
\[ F_\nu = \{ x' \in \mathbb{R}^m : 2^{\nu - 1} < |x'| \leq 2^\nu \}, \quad \nu = 1, 2, \cdots. \]

By Hölder’s inequality, we have
\[ \| \mathcal{F}^{-1}(\theta \omega) \|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \leq \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\infty} 2^{\mu n(2-p)/2} 2^{\nu m(2-p)/2} \left( \int_{E_\mu \times F_\nu} |\mathcal{F}^{-1}(\theta \omega)|^2 dx dx' \right)^{p/2}. \]
Proof. Suppose Proposition 2.1. From this and the fact that \( \alpha_1 > \sigma_p \) and \( \alpha_2 > \sigma_p' \), it follows that

\[
\|\mathcal{F}^{-1}(\theta\omega)\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \leq \left( \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\infty} 2^{\mu\rho p} 2^{\nu\rho p'} \left( \int_{E_\mu \times F_\nu} |\mathcal{F}(\theta\omega)|^2 \, dx \, dx' \right)^{\frac{p}{2}} \right)^\frac{1}{2} 
\times \left( \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\infty} 2^{\mu\rho p} 2^{\nu\rho p'} \int_{E_\mu \times F_\nu} |\mathcal{F}(\theta\omega)|^2 \, dx \, dx' \right) \leq \left( \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\infty} 2^{\mu\rho p} 2^{\nu\rho p'} \int_{E_\mu \times F_\nu} |\mathcal{F}(\theta\omega)|^2 \, dx \, dx' \right) \leq \left( \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\infty} 2^{\mu\rho p} 2^{\nu\rho p'} \int_{E_\mu \times F_\nu} |\mathcal{F}(\theta\omega)|^2 \, dx \, dx' \right).
\]

Observe that \((1 + |x|^2)^{\alpha_1/2} \sim 1\) if \(x \in E_\mu\), and \((1 + |x|^2)^{\alpha_1} \sim 2^{\mu \alpha_1} 1\) if \(x \in E_\mu\), \(\mu = 1, 2, \ldots\). Similar properties are also satisfied by the \(x' \in F_\nu\). Therefore,

\[
\|\mathcal{F}^{-1}(\theta\omega)\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \leq \left( \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\infty} \int_{E_\mu \times F_\nu} (1 + |x|^2)^{\alpha_1} (1 + |x'|)^{\alpha_2} |\mathcal{F}(\theta\omega)(x, x')|^2 \, dx \, dx' \right)^{\frac{p}{2}} \leq \left( \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\infty} \int_{E_\mu \times F_\nu} |\mathcal{F}(\theta\omega)(x, x')|^2 \, dx \, dx' \right)^{\frac{p}{2}} \leq \left( \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\infty} \int_{E_\mu \times F_\nu} |\mathcal{F}(\theta\omega)(x, x')|^2 \, dx \, dx' \right)^{\frac{p}{2}}.
\]

as desired.

Case 2. \(p > 1\). In this case \(p_0 = \min(p, 1) = 1\). The same argument as in Case 1 yields

\[
\|\mathcal{F}^{-1}(\theta\omega)\|_{L^1(\mathbb{R}^n \times \mathbb{R}^m)} \lesssim \|\omega\|_{H^p_2(\mathbb{R}^n \times \mathbb{R}^m)}.
\]

Thus the proof of the lemma is complete.

With the above preparation, we are ready to show the well-definedness of our bi-parameter modulation spaces, which is stated in the following proposition.

Proposition 2.1. Suppose \(\{\varphi_k\}_{k \in \mathbb{Z}^n}, \{\tilde{\varphi}_j\}_{j \in \mathbb{Z}^m} \in \mathcal{Y}_n\) and \(\{\psi_{k'}\}_{k' \in \mathbb{Z}^m}, \{\tilde{\psi}_{j'}\}_{j' \in \mathbb{Z}^m} \in \mathcal{Y}_m\). Then for \(f \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^m)\), we have

\[
\left( \sum_{k \in \mathbb{Z}^n} \sum_{k' \in \mathbb{Z}^m} \langle k \rangle^{s_1} \langle k' \rangle^{s_2} \|\Box_{k, k'} f\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \right)^{\frac{1}{2}} \lesssim \left( \sum_{j \in \mathbb{Z}^n} \sum_{j' \in \mathbb{Z}^m} \langle j \rangle^{s_1} \langle j' \rangle^{s_2} \|\Box_{j, j'} f\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \right)^{\frac{1}{2}},
\]

where \(\Box_{k, k'} f = \mathcal{F}^{-1}[(\varphi_k \otimes \psi_{k'}) f]\) and \(\Box_{j, j'} f = \mathcal{F}^{-1}[(\tilde{\varphi}_j \otimes \tilde{\psi}_{j'}) f]\).

Proof. For convenience, we denote

\[
\Lambda_{k, k'} := \{(j, j') \in \mathbb{Z}^n \times \mathbb{Z}^m : \Box_{k, k'} \mathcal{F} f \neq 0\}.
\]

Then for any \((j, j') \in \Lambda_{k, k'}\), we have \(\varphi_k(\xi) \tilde{\varphi}_j(\xi') \not= 0\) and \(\psi_{k'}(\xi') \tilde{\psi}_{j'}(\xi') \not= 0\), and hence

\[
\langle k \rangle \sim \langle j \rangle, \quad \langle k' \rangle \sim \langle j' \rangle.
\]

Moreover, the number of the elements in \(\Lambda_{k, k'}\) is uniformly bounded.
By the properties of \( \{\varphi_k\} \), \( \{\tilde{\varphi}_k\} \), \( \{\psi_k\} \) and \( \{\tilde{\psi}_k\} \) and Lemma 2.4, we have
\[
\|\square_{k,k'} f\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \leq \sum_{(j,j') \in \Lambda_{k,k'}} \|\square_{j,j'} f\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \leq \|\mathcal{F}^{-1} \left[ (\varphi_k \otimes \psi_k') \mathcal{F}(\varphi_k \otimes \psi_k') \right]\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \approx \sum_{(j,j') \in \Lambda_{k,k'}} \|\varphi_k \otimes \psi_k'\|_{H^s(\mathbb{R}^n \times \mathbb{R}^m)} \|\square_{j,j'} f\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} .
\]

Since \( \mathcal{F}(\varphi_k \otimes \psi_k')(\xi, \xi') = \mathcal{F}(\varphi_k)(\xi) \mathcal{F}(\psi_k')(\xi') \), we have
\[
\|\varphi_k \otimes \psi_k'\|_{H^s(\mathbb{R}^n \times \mathbb{R}^m)} = \|\varphi_k H_{s,1}^s(\mathbb{R}^n)\|_{\mathcal{F}(\psi_k) H_{s,2}^s(\mathbb{R}^m)} \lesssim 1.
\]

Combining (7) and (8), we get
\[
\|\square_{k,k'} f\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \lesssim \sum_{(j,j') \in \Lambda_{k,k'}} \|\square_{j,j'} f\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} .
\]

Hence, using (6) and the fact that the number of the elements in \( \Lambda_{k,k'} \) is uniformly bounded, we further deduce that
\[
\sum_{k \in \mathbb{Z}^n} \sum_{k' \in \mathbb{Z}^m} \langle k \rangle^{s_1 q} \langle k' \rangle^{s_2 q} \|\square_{k,k'} f\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \lesssim \sum_{k \in \mathbb{Z}^n} \sum_{k' \in \mathbb{Z}^m} \langle k \rangle^{s_1 q} \langle k' \rangle^{s_2 q} \left( \sum_{(j,j') \in \Lambda_{k,k'}} \|\square_{j,j'} f\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \right)^q \lesssim \sum_{j \in \mathbb{Z}^n} \sum_{j' \in \mathbb{Z}^m} \langle j \rangle^{s_1 q} \langle j' \rangle^{s_2 q} \|\square_{j,j'} f\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} .
\]

This verifies the direction \( \lesssim \) in (5). By symmetry, the converse direction \( \gtrsim \) also holds. Thus the proof of Proposition 2.1 is complete. \( \square \)

At the end of this section we list some fundamental properties of bi-parameter modulation spaces. Since the proofs of these properties are analogous to those in the single-parameter setting, we shall skip the details.

Proposition 2.2. Let \( s = (s_1, s_2) \in \mathbb{R} \times \mathbb{R} \) and \( 0 < p, q \leq \infty \).

(i) \( BM^s_{p,q}(\mathbb{R}^n \times \mathbb{R}^m) \) is a quasi-Banach space, and a Banach space if \( 1 \leq p, q \leq \infty \).
(ii) \( S(\mathbb{R}^n \times \mathbb{R}^m) \subseteq BM^s_{p,q}(\mathbb{R}^n \times \mathbb{R}^m) \subseteq S'(\mathbb{R}^n \times \mathbb{R}^m) \), and the inclusion maps are continuous.
(iii) If \( 0 < p, q < \infty \), \( S(\mathbb{R}^n \times \mathbb{R}^m) \) is dense in \( BM^s_{p,q}(\mathbb{R}^n \times \mathbb{R}^m) \).

Proposition 2.3. Let \( s = (s_1, s_2) \in \mathbb{R} \times \mathbb{R} \) and \( 0 < p, q \leq \infty \). Then
\[
(BM^s_{p,q}(\mathbb{R}^n \times \mathbb{R}^m))^* = BM^{-s}_{p,q}(\mathbb{R}^n \times \mathbb{R}^m) .
\]
Proposition 2.4. Let \( s = (s_1, s_2), r = (r_1, r_2) \in \mathbb{R} \times \mathbb{R} \) and \( p, q, \tilde{p}, \tilde{q} \in (0, \infty) \).

(i) If \( s_1 \geq r_1, s_2 \geq r_2, p \leq \tilde{p} \) and \( q \leq \tilde{q} \), then \( BM_{p,q}^s(\mathbb{R}^n \times \mathbb{R}^m) \subseteq BM_{\tilde{p},\tilde{q}}^r(\mathbb{R}^n \times \mathbb{R}^m) \).

(ii) If \( q > \tilde{q}, s_1 - r_1 > \frac{n}{1/\tilde{q} - 1/q}, s_2 - r_2 > \frac{m}{1/\tilde{q} - 1/q} \), then

\[
BM_{p,q}^s(\mathbb{R}^n \times \mathbb{R}^m) \subseteq BM_{\tilde{p},\tilde{q}}^r(\mathbb{R}^n \times \mathbb{R}^m).
\]

3. Molecular decomposition of bi-parameter modulation spaces. The aim of this section is to prove Theorem 1.3. We follow the idea of Kobayashi and Sawano [33]. For our purpose, we will need the following lemma, which is a bi-parameter analogue of the Theorem in [52, §1.3.1]. Since its proof is parallel to that of Theorem in [52, §1.3.1], we omit the details here.

Lemma 3.1. Let \( \Omega \) be a compact subset in \( \mathbb{R}^n \times \mathbb{R}^m \) and \( 0 < r < \infty \). Then for all \( f \in S'(\mathbb{R}^n \times \mathbb{R}^m) \) with \( \text{supp} \mathcal{F}f \subset \Omega \), we have

\[
\sup_{(y,y')\in \mathbb{R}^n \times \mathbb{R}^m} \langle y \rangle^{-n/r} \langle y' \rangle^{-m/r} |f(x-y, x'-y')| \leq C [\mathcal{M}_s(|f|')(x,x')]^{\frac{1}{r}},
\]

where \( C \) is a positive constant depending only on \( n, m \) and \( \text{diam}(\Omega) \), and \( \mathcal{M}_s \) is the strong maximal operator defined by (1).

Before going to the next lemma, we need to introduce some notation. For \( f \in S'(\mathbb{R}^n) \), \( a, b \in \mathbb{R}^n \), we denote

\[
T_a f(x) := f(x-a) \quad \text{and} \quad M_b f(x) := e^{ib \cdot x} f(x).
\]

Also, for function \( \phi_1 \in S(\mathbb{R}^n) \), \( \phi_2 \in S(\mathbb{R}^m) \) and \( \theta \in S(\mathbb{R}^n \times \mathbb{R}^m) \), we define the operators

\[
\phi_1(D_1) g := \mathcal{F}^{-1}(\phi_1 \mathcal{F} g), \quad g \in S'(\mathbb{R}^n),
\]

\[
\phi_2(D_2) h := \mathcal{F}^{-1}(\phi_2 \mathcal{F} h), \quad h \in S'(\mathbb{R}^m)
\]

and

\[
\theta (D_1, D_2) f := \mathcal{F}^{-1}(\theta \mathcal{F} f), \quad f \in S'(\mathbb{R}^n \times \mathbb{R}^m).
\]

Lemma 3.2. Suppose \( f \in S'(\mathbb{R}^n \times \mathbb{R}^m) \) such that \( \text{supp} \mathcal{F}f \subset Q(1) \times Q'(1) \). Suppose further that \( \omega \in S(\mathbb{R}^n) \), \( \tilde{\omega} \in S(\mathbb{R}^m) \), \( \text{supp}(\omega \otimes \tilde{\omega}) \subset Q(2) \times Q'(2) \) and \( (\omega \otimes \tilde{\omega})(x, x') = 1 \) for \( (x, x') \in Q(1) \times Q'(1) \). Then we have

\[
f = \frac{1}{(2\pi)^{(n+m)/2}} \sum_{l \in \mathbb{Z}^n} \sum_{l' \in \mathbb{Z}^m} f(l, l')(T_l \mathcal{F}^{-1} \omega) \otimes (T_{l'} \mathcal{F}^{-1} \tilde{\omega}).
\]

Proof. Take arbitrary \( \theta \in S(\mathbb{R}^n \times \mathbb{R}^m) \). Then by the support conditions we have

\[
\langle \mathcal{F} f, \theta \rangle = \langle \mathcal{F} f, (\omega \otimes \tilde{\omega}) \cdot (\omega \otimes \tilde{\omega}) \cdot \theta \rangle.
\]

Consider the function

\[
\sigma(x, x') := \sum_{k \in \mathbb{Z}^n} \sum_{k' \in \mathbb{Z}^m} (\omega \otimes \tilde{\omega})(x - 2\pi k, x' - 2\pi k') \theta(x - 2\pi k, x' - 2\pi k').
\]

which is \( 2\pi (\mathbb{Z}^n \times \mathbb{Z}^m) \)-periodic on \( \mathbb{R}^n \times \mathbb{R}^m \). Expanding \( \sigma \) to the Fourier series, we obtain

\[
\sigma(x, x') = \sum_{l \in \mathbb{Z}^n} \sum_{l' \in \mathbb{Z}^m} a_{l, l'} \exp(i x \cdot l + i x' \cdot l'),
\]

where the coefficients is given by

\[
a_{l, l'} = \frac{1}{(2\pi)^{n+m}} \int_{Q(1) \times Q'(1)} \sigma(x, x') \exp(-i x \cdot l - i x' \cdot l') dx dx'.
\]
To see this, one only needs to replace \( f \) by \( \omega \). Taking the inverse Fourier transform on both sides, and using (10), we get

\[
\sum_{k} \sum_{k'} \sum_{l} \sum_{l'} (\omega \otimes \bar{\omega})(x - 2\pi k, x' - 2\pi k') \theta(x - 2\pi k, x' - 2\pi k') \cdot \exp(-ix \cdot l - ix' \cdot l') dx dx'.
\]

Substituting (12) and (11) into (10) gives

\[
\langle F f, \theta \rangle = \langle F f, (\omega \otimes \bar{\omega}) \cdot \theta \rangle.
\]

Since \( \text{supp}(\omega \otimes \bar{\omega}) \subset Q(2) \times Q'(2) \), we see that

\[
(\omega \otimes \bar{\omega})(x, x') \cdot (\omega \otimes \bar{\omega})(x - 2\pi k, x' - 2\pi k') = 0
\]

whenever \((k, k') \neq (0, \ldots, 0), (0, \ldots, 0)\). It follows that

\[
\begin{align*}
(\omega \otimes \bar{\omega})(x, x') \cdot (\omega \otimes \bar{\omega})(x, x') \cdot \theta(x, x') \\
= (\omega \otimes \bar{\omega})(x, x') \cdot \sigma(x, x') \\
= \sum_{l} \sum_{l'} a_{l, l'} \cdot (\omega \otimes \bar{\omega})(x, x') \cdot \exp(ix \cdot l + ix' \cdot l').
\end{align*}
\]

Substituting (12) and (11) into (10) gives

\[
\langle F f, \theta \rangle = \langle F f, (\omega \otimes \bar{\omega}) \cdot (\omega \otimes \bar{\omega}) \cdot \theta \rangle
\]

\[
= \left\langle \sum_{l} \sum_{l'} a_{l, l'} \cdot (\omega \otimes \bar{\omega}) \cdot \exp(\ast(l, l')i) \right\rangle
\]

\[
= \sum_{l} \sum_{l'} a_{l, l'} \left\langle F f, (\omega \otimes \bar{\omega}) \cdot \exp(\ast(l, l')i) \right\rangle
\]

\[
= \sum_{l} \sum_{l'} \frac{1}{(2\pi)^{m+n}} \text{supp}(\omega \otimes \bar{\omega}) \cdot \exp(-\ast(l, l')i) \cdot \langle F f, (\omega \otimes \bar{\omega}) \cdot \exp(\ast(l, l')) \rangle
\]

\[
= \left\langle \sum_{l} \sum_{l'} \frac{1}{(2\pi)^{(n+m)/2}} f(l, l') \cdot (\omega \otimes \bar{\omega}) \cdot \exp(-\ast(l, l')i) \right\rangle \cdot \theta,
\]

where we used that \( \langle F f, (\omega \otimes \bar{\omega}) \exp(\ast(l, l')i) \rangle = (2\pi)^{(n+m)/2} f(l, l') \). Since \( \theta \) is arbitrary, we obtain

\[
F f(x, x') = \frac{1}{(2\pi)^{(n+m)/2}} \sum_{l} \sum_{l'} f(l, l') \omega(x) \bar{\omega}(x') \exp(-ix \cdot l - ix' \cdot l').
\]

Taking the inverse Fourier transform on both sides, and using \( F^{-1}[\omega(x)e^{-ix \cdot l}] = T_l F^{-1} \omega \), we arrive at (9). This completes the proof. \qed

**Remark 1.** If \( f \in S'(\mathbb{R}^n \times \mathbb{R}^m) \) such that \( \text{supp} f \subset (k + Q(1)) \times (k' + Q'(1)) \), and \( \omega, \bar{\omega} \) are functions exactly the same as in Lemma 3.2, then \( f \) has the representation

\[
f = \frac{1}{(2\pi)^{(n+m)/2}} \sum_{l} \sum_{l'} f(l, l')(T_l M_k \omega) \otimes (T_{l'} M_{k'} \bar{\omega}).
\]

To see this, one only needs to replace \( \omega \) and \( \bar{\omega} \) in the proof of Lemma 3.2 with \( \omega(\cdot - k) \) and \( \omega(\cdot - k') \) respectively, and use the fact that \( F^{-1}[\omega(\cdot - k)] = M_k F^{-1} \omega \).

We will also need the following convolution type lemma, whose proof is standard.
Lemma 3.3. Let $0 < p, q \leq \infty$. Let $\{F_{k,k'}\}_{(k,k') \in \mathbb{Z}^n \times \mathbb{Z}^m}$ be a sequence of positive measurable functions. Set
\[
G_{j,j'} = \sum_{k \in \mathbb{Z}^n} \sum_{k' \in \mathbb{Z}^m} (j-k)^{-\delta} (j'-k')^{-\delta'} F_{k,k'}, \quad (j,j') \in \mathbb{Z}^n \times \mathbb{Z}^m,
\]
where $\delta > 2n \max\{1,1/p\}$ and $\delta' > 2m \max\{1,1/p\}$. Then we have
\[
\|\{G_{j,j'}\}_{(j,j') \in \mathbb{Z}^n \times \mathbb{Z}^m}\|_{\ell^p(L^q)} \leq C \|\{F_{k,k'}\}_{(k,k') \in \mathbb{Z}^n \times \mathbb{Z}^m}\|_{\ell^p(L^q)}.
\]

With these preparations, we are now in a position to prove Theorem 1.3.

Proof of Theorem 1.3. First we prove (i). Let $f \in \mathcal{BM}^s_{p,q}(\mathbb{R}^n \times \mathbb{R}^m)$, $\{\varphi_{j}\} \in \mathcal{T}(n)$ and $\{\psi_{j'}\} \in \mathcal{T}(m)$. Fix two functions $\omega, \tilde{\omega}$ which satisfy the hypothesis of Lemma 3.2. Then using that $\sum_{k \in \mathbb{Z}^n \times \mathbb{Z}^m} \varphi_{k}(\xi) \psi_{k'}(\xi') \equiv 1$ and Remark 3.1, we have
\[
f = \sum_{k \in \mathbb{Z}^n \times \mathbb{Z}^m} \varphi_{k}(D_1) \otimes \psi_{k'}(D_2) \int f(l,l') [T_l M_k \mathcal{F}^{-1} \omega] \otimes [T_{l'} M_{k'} \mathcal{F}^{-1} \tilde{\omega}].
\]
Thus, if we set
\[
\Psi^s_{k,1} := C^{-1} (k)^{-s_1} (k')^{-s_2} [T_l M_k \mathcal{F}^{-1} \omega] \otimes [T_{l'} M_{k'} \mathcal{F}^{-1} \tilde{\omega}]
\]
and
\[
\lambda_{k,1} := C \cdot (2\pi)^{-\frac{(m+n)}{2}} (k)^{s_1} (k')^{s_2} \varphi_{k}(D_1) \otimes \psi_{k'}(D_2) \int f(l,l'),
\]
we have $f = \sum_{k \in \mathbb{Z}^n \times \mathbb{Z}^m} \lambda_{k,1} \Psi^s_{k,1}$. Obviously, if $C$ is sufficiently large, then each $\Psi^s_{k,1}$ is an $(s; k, 1)$-molecule. Let us show that the coefficients $\{\lambda_{k,1}\}$ satisfies (3). Indeed, for fixed $k$ and $(x, x')$, there is a unique $l_0 = (l_0, l_{0}') \in \mathbb{Z}^n \times \mathbb{Z}^m$ such that $(x, x') \in Q_{l_0} \times Q_{l_{0}'}$. This is because $\{Q_l\}_{l \in \mathbb{Z}^n}$ and $\{Q_{l'}\}_{l' \in \mathbb{Z}^m}$ are pairwise disjoint sequences of sets. Thus, fixing $0 < r < \min(1,p)$, we have
\[
\left| \sum_{l \in \mathbb{Z}^n \times \mathbb{Z}^m} \lambda_{k,1}(\chi_{Q_l} \otimes \chi_{Q_{l}'}) (x, x') \right| = \left| \lambda_{k,1}(\chi_{Q_{l_0}} \otimes \chi_{Q_{l_0}'}) (x, x') \right|
\]
\[
= |\lambda_{k,1}| \lesssim \langle k \rangle^{s_1} \langle k' \rangle^{s_2} \sup_{(y,y') \in Q_{l_0} \times Q_{l_{0}'}} \left| \varphi_{k}(D_1) \otimes \psi_{k'}(D_2) \int f(l, l') \right| \lesssim \langle k \rangle^{s_1} \langle k' \rangle^{s_2} \lesssim \langle k \rangle^{s_1} \langle k' \rangle^{s_2} \lesssim 1,
\]
where for the third line we used the fact that
\[
(x, x'), (y, y') \in Q_{l_0} \times Q_{l_{0}'} \implies (1 + |x-y|)^{n/r}(1 + |x'-y'|)^{m/r} \lesssim 1
\]
and for the last line we applied Lemma 3.1. Finally, by the vector-valued inequality for strong maximal function we obtain
\[
\|\lambda\|_{\mathcal{BM}^s_{p,q}(\mathbb{R}^n \times \mathbb{R}^m)} \leq \|f\|_{\mathcal{BM}^s_{p,q}(\mathbb{R}^n \times \mathbb{R}^m)}.
\]
Next we prove (ii). Let \( \lambda = \{ \lambda_k, \ell \}_{k, \ell} \in \mathbb{Z}^n \times \mathbb{Z}^m \in \mathfrak{m}_{p,q} \) and let \( \{ \Psi_{k,\ell} \}_{k,\ell} \in \mathbb{Z}^n \times \mathbb{Z}^m \) be a sequence such that each \( \Psi_{k,\ell} \) is a \((\ell, k, 1)\)-molecule with \( K, N', N'' \) in Definition 1.1 satisfying

\[
K > 4n \max(1, 1/p) + 2|s_1| + 2, \quad K > 4m \max(1, 1/p) + 2|s_2| + 2, \quad N > n \min\{p, 1\} \text{ and } N'' > m \min\{p, 1\}. \]

Define \( f = \sum_{k,\ell} \lambda_k \Psi_{k,\ell} \). Then we must show that

\[
\| \{ (j)^s \langle j' \rangle \times \square_{j,j'} f \} \}_{(j, j') \in \mathbb{Z}^n \times \mathbb{Z}^m} \|_{\ell^1(L^p)} \lesssim \| \lambda \|_{\mathfrak{m}_{p,q}}.
\]

To this end, let \( k = (k, k'), \ell = (\ell, \ell'), j = (j, j') \in \mathbb{Z}^n \times \mathbb{Z}^m \), and \( K_0 := \lceil K/4 \rceil \). Then by elementary Fourier analysis, we have

\[
\square_{j,j'}(\Psi_{k,\ell})(x, x') = [\mathcal{F}^{-1}(\varphi_j \otimes \psi_{j'})] \ast \Psi_{k,\ell}(x, x')
\]

\[
= \iint_{\mathbb{R}^n \times \mathbb{R}^m} \mathcal{F}^{-1}((\varphi_j \otimes \psi_{j'})(y, y')) \Psi_{k,\ell}(x - y, x' - y') dy dy'
\]

\[
= \iint_{\mathbb{R}^n \times \mathbb{R}^m} e^{i(y \cdot y')} e^{i(j') \cdot y'} \mathcal{F}^{-1} \left( (T_{-j} \varphi_j) \otimes (T_{-j'} \psi_{j'}) \right)(y, y') \Psi_{k,\ell}(x - y, x' - y') dy dy'
\]

\[
= \iint_{\mathbb{R}^n \times \mathbb{R}^m} e^{i(j \cdot y)} e^{i(j' \cdot y')} \mathcal{F}^{-1} \left( (T_{-j} \varphi_j) \otimes (T_{-j'} \psi_{j'}) \right)(x - y, x' - y') \Psi_{k,\ell}(y, y') dy dy'
\]

\[
e^{i(j \cdot x)} e^{i(j' \cdot x')} \iint_{\mathbb{R}^n \times \mathbb{R}^m} e^{-i(j \cdot y)} e^{-i(j' \cdot y')} \mathcal{F}^{-1} \left( (T_{-j} \varphi_j) \otimes (T_{-j'} \psi_{j'}) \right)(x - y, x' - y')
\]

\[
\quad \times e^{-i(k \cdot y)} e^{-i(k' \cdot y')} \Psi_{k,\ell}(y, y') dy dy',
\]

where we used integration by parts and the fact that

\[
(I - \Delta_y)^{K_0} (I - \Delta_{y'})^{K_0} \left[ e^{-i(j \cdot y)} e^{-i(j' \cdot y')} \right]
\]

\[
= (j \cdot k)^{2K_0} (j' \cdot k)^{2K_0} \left[ e^{-i(j \cdot y)} e^{-i(j' \cdot y')} \right].
\]

Note that for any \( \alpha, \beta \in \mathbb{Z}^n_+ \) and \( \beta \in \mathbb{Z}^m_+ \),

\[
|\partial_{\xi}^\alpha \partial_{\xi'}^\beta \left( \mathcal{F}^{-1} \left( (T_{-j} \varphi_j) \otimes (T_{-j'} \psi_{j'}) \right)(x - y, x' - y') \right)|
\]

\[
= \frac{1}{(2\pi)^{(n+m)/2}} \left| \partial_{\xi}^\alpha \partial_{\xi'}^\beta \left( \iint_{Q(1)} \varphi_j(\xi + j) \psi_{j'}(\xi' + j') e^{i(\xi \cdot x - y)} e^{i(\xi' \cdot x' - y')} \xi d\xi d\xi' \right) \right|
\]

\[
= \left| \iint_{Q(1) \times Q'(1)} \varphi_j(\xi + j) \psi_{j'}(\xi' + j') e^{i(\xi \cdot x - y)} \xi e^{i(\xi' \cdot x' - y')} \xi' \xi d\xi d\xi' \right|
\]

\[
= \left| (I - \Delta_\xi)^{L}(I - \Delta_{\xi'})^{L'} \left[ \xi^\alpha \xi'^\beta \varphi_j(\xi + j) \psi_{j'}(\xi' + j') e^{i(\xi \cdot x - y)} \xi e^{i(\xi' \cdot x' - y')} \xi' \xi d\xi d\xi' \right] \right|
\]

\[
= \left| (I - \Delta_\xi)^{L}(I - \Delta_{\xi'})^{L'} \left[ \xi^\alpha \xi'^\beta \varphi_j(\xi + j) \psi_{j'}(\xi' + j') e^{i(\xi \cdot x - y)} \xi e^{i(\xi' \cdot x' - y')} \xi' \xi d\xi d\xi' \right] \right|
\]

\[
= \left| (I - \Delta_\xi)^{L}(I - \Delta_{\xi'})^{L'} \left[ \xi^\alpha \xi'^\beta \varphi_j(\xi + j) \psi_{j'}(\xi' + j') e^{i(\xi \cdot x - y)} \xi e^{i(\xi' \cdot x' - y')} \xi' \xi d\xi d\xi' \right] \right|
\]

\[
\quad \times \left| (I - \Delta_\xi)^{L}(I - \Delta_{\xi'})^{L'} \left[ \xi^\alpha \xi'^\beta \varphi_j(\xi + j) \psi_{j'}(\xi' + j') e^{i(\xi \cdot x - y)} \xi e^{i(\xi' \cdot x' - y')} \xi' \xi d\xi d\xi' \right] \right|
\]

\[
= \left| (I - \Delta_\xi)^{L}(I - \Delta_{\xi'})^{L'} \left[ \xi^\alpha \xi'^\beta \varphi_j(\xi + j) \psi_{j'}(\xi' + j') e^{i(\xi \cdot x - y)} \xi e^{i(\xi' \cdot x' - y')} \xi' \xi d\xi d\xi' \right] \right|
\]
\[ \leq C_{\alpha,\beta,L,L'} \langle x-y \rangle^{-2L} \langle x'-y' \rangle^{-2L'}, \]

where for the last inequality we used the property (iv) in Definition 2.1. Here, \( L \) and \( L' \) are positive integers, which can be chosen arbitrarily large.

We now choose \( L, L' \) such that \( 2L - N > n \) and \( 2L' - N' > m \). Then from the estimate (14) and the definition of molecules (recall that \( K_0 = [K/4] \)), it follows that

\[
\left| (I - \Delta_y)^{K_0} (I - \Delta_{y'})^{K_0} \left\{ \mathcal{F}^{-1} \left[ (T_{j,j'} \phi_j) \otimes (T_{j,j'} \psi_{j'}) \right] (x-y, x'-y') e^{-ik \cdot y} e^{-ik' \cdot y'} \Psi_{k,1}(y, y') \right\} \right|
\leq (x-y)^{-2L} \langle x'-y' \rangle^{-2L'} \langle k \rangle^{-s_1} \langle k' \rangle^{-s_2} \langle y-l \rangle^{-N} \langle y'-l' \rangle^{-N} (15)
\leq \langle k \rangle^{-s_1} \langle k' \rangle^{-s_2} \langle x-y \rangle^{-(2L'-N)} \langle x'-y' \rangle^{-(2L'-N')} \langle x-l \rangle^{-N} \langle x'-l' \rangle^{-N'},
\]

where for the last estimate we used the elementary inequalities

\[ \langle x-y \rangle^{-1} \langle y-l \rangle^{-1} \leq 2 \langle x-l \rangle^{-1} \quad \text{and} \quad \langle x'-y' \rangle^{-1} \langle y'-l' \rangle^{-1} \leq \sqrt{2} \langle x'-l' \rangle^{-1} \]

Inserting (15) into (13), and using that

\[ \iint_{\mathbb{R}^n \times \mathbb{R}^m} (x-y)^{-(2L'-N)} (x'-y')^{-(2L'-N')} dydy' \leq 1, \]

we obtain

\[ \left| \Box_{j,j'}(\Psi_{k,1}^u)(x, x') \right| \leq \langle k \rangle^{-s_1} \langle k' \rangle^{-s_2} \langle j-k \rangle^{-2K_0} \langle j'-k' \rangle^{-2K_0} \langle x-l \rangle^{-N} \langle x'-l' \rangle^{-N'}. \]

It then follows that

\[
\rho \left( \sum_{k, l \in \mathbb{Z}^n \times \mathbb{Z}^m} \lambda_{k,l} \Psi_{k,l} \right)_{BM_{L,\rho}(\mathbb{R}^n \times \mathbb{R}^m)}^q \leq \sum_{j \in \mathbb{Z}^n \times \mathbb{Z}^m} \left\{ \int_{\mathbb{R}^n \times \mathbb{R}^m} \left( \langle j \rangle^{-s_1} \langle j' \rangle^{-s_2} \sum_{k, l \in \mathbb{Z}^n \times \mathbb{Z}^m} |\lambda_{k,l}| \left| \Box_{j,j'}(\Psi_{k,1}^u)(x, x') \right| \right)^p dx dx' \right\}^{q/p}
\leq \sum_{j \in \mathbb{Z}^n \times \mathbb{Z}^m} \left\{ \int_{\mathbb{R}^n \times \mathbb{R}^m} \left( \langle j \rangle^{-s_1} \langle j' \rangle^{-s_2} \sum_{k, l \in \mathbb{Z}^n \times \mathbb{Z}^m} |\lambda_{k,l}| \langle k \rangle^{-s_1} \langle k' \rangle^{-s_2} \right. \right.
\left. \times \langle j-k \rangle^{-2K_0} \langle j'-k' \rangle^{-2K_0} \langle x-l \rangle^{-N} \langle x'-l' \rangle^{-N'} \right)^p dx dx' \right\}^{q/p}. \quad (16)
\]

To estimate the last sum in (16), we fix a number \( r \) such that

\[ \max\{n/N, m/N'\} < r < \min\{p, 1\}. \]
This is possible since $N > n/\min\{p, 1\}$ and $N' > m/\min\{p, 1\}$. Then we note that
\[
\sum_{l \in \mathbb{Z}^n} \sum_{l' \in \mathbb{Z}^m} \frac{|\lambda_{k,l}|}{(x-l)^N(x'-l')^{N'}} \\
= \sum_{\mu, \nu \in \mathbb{N}} \sum_{l \in \mathbb{Z}^n : 2^{\mu-1} \leq (x-l) \leq 2^\mu} \sum_{l' \in \mathbb{Z}^m : 2^{\nu-1} \leq (x'-l') \leq 2^\nu} \frac{|\lambda_{k,l}|}{(x-l)^{N+\nu}(x'-l')^{N'}} \\
\leq \sum_{\mu, \nu \in \mathbb{N}} \left( \sum_{l \in \mathbb{Z}^n : 2^{\mu-1} \leq (x-l) \leq 2^\mu} \sum_{l' \in \mathbb{Z}^m : 2^{\nu-1} \leq (x'-l') \leq 2^\nu} \frac{|\lambda_{k,l}|^r}{(x-l)^{N+\nu}(x'-l')^{N'}} \right)^{1/r} \\
\leq \sum_{\mu, \nu \in \mathbb{N}} \frac{1}{2^\mu 2^\nu (N-n/r) 2^\nu (N'-m/r)} \left( \sum_{l \in \mathbb{Z}^n : 2^{\mu-1} \leq (x-l) \leq 2^\mu} \sum_{l' \in \mathbb{Z}^m : 2^{\nu-1} \leq (x'-l') \leq 2^\nu} |\lambda_{k,l}|^r \right)^{1/r}.
\]

Using the facts that
\[
\langle x-l \rangle \leq 2^\mu, \langle x'-l' \rangle \leq 2^\nu \implies Q_i \times Q_i' \subset B(x, 3 \cdot 2^\mu) \times B(x', 3 \cdot 2^\nu),
\]
and $|Q_i \times Q_i'| = 1$, we infer
\[
\frac{1}{2^\mu 2^\nu m} \sum_{l \in \mathbb{Z}^n : 2^{\mu-1} \leq (x-l) \leq 2^\mu} \sum_{l' \in \mathbb{Z}^m : 2^{\nu-1} \leq (x'-l') \leq 2^\nu} \left| \chi_{Q_i \times Q_i'}(y, y') \right| dydy' \\
= \frac{1}{2^\mu 2^\nu m} \iint_{B(x, 3 \cdot 2^\mu) \times B(x', 3 \cdot 2^\nu)} \left| \chi_{Q_i \times Q_i'}(y, y') \right| dydy' \\
\leq \frac{1}{2^\mu 2^\nu m} \iint_{B(x, 3 \cdot 2^\mu) \times B(x', 3 \cdot 2^\nu)} \left( \sum_{(l,l') \in \mathbb{Z}^n \times \mathbb{Z}^m} |\lambda_{k,l}|^r \chi_{Q_i \times Q_i'}(y, y') \right) dydy' \\
\lesssim M_s \left( \sum_{(l,l') \in \mathbb{Z}^n \times \mathbb{Z}^m} |\lambda_{k,l}|^r \chi_{Q_i \times Q_i'}(x, x') \right).
\]

Inserting this into (17) and using that $Nr - n > 0$ and $N' - m > 0$ we obtain
\[
\sum_{l \in \mathbb{Z}^n \times \mathbb{Z}^m} \frac{|\lambda_{k,l}|}{(x-l)^N(x'-l')^{N'}} \lesssim \left[ M_s \left( \sum_{l \in \mathbb{Z}^n \times \mathbb{Z}^m} |\lambda_{k,l}|^r \chi_{Q_i \times Q_i'} \right)(x, x') \right]^{1/r}.
\]

Meanwhile, by the elementary inequality $\langle j \rangle^s \langle k \rangle^{-s} \leq (\sqrt{2})^{|s|} |\langle j-k \rangle|^{s_1}$, we deduce that
\[
\langle j \rangle^{s_1} \langle j' \rangle^{s_2} (k)^{-s_1} \langle k' \rangle^{-s_2} (j-k)^{-2K_0} (j'-k')^{-2K_0} \\
\lesssim (j-k)^{-2K_0+|s_1|} (j'-k')^{-2K_0+|s_2|}.
\]

Inserting (19) and (18) into (16), and using Lemma 3.3 (taking into account that $2K_0 - |s_1| > 2n \max(1, 1/p)$ and $2K_0 - |s_2| > 2m \max(1, 1/p)$) and the vector-valued inequality for strong maximal function, we obtain
\[
\left\| \sum_{k,l \in \mathbb{R}^n \times \mathbb{R}^m} \lambda_{k,l} \Psi_{k,l}^q \right\|_{BM_{p,q}(\mathbb{R}^n \times \mathbb{R}^m)}^q.
\]
as we see that there exist finitely many integers $C$ such that for any $\xi, \xi' \neq 0$, $\forall \mu, \nu \in \{0, 1\}$. Then there is a constant $C$ such that for all $f \in L^p(\mathbb{R} \times \mathbb{R})$, 

$$
\|F^{-1}(\gamma f)\|_{L^p(\mathbb{R} \times \mathbb{R})} \leq C\|f\|_{L^p(\mathbb{R} \times \mathbb{R})}.
$$

Proof of Theorem 1.4. Let $\{\varphi_k\}_{k \in \mathbb{Z}}, \{\psi_{k'}\}_{k' \in \mathbb{Z}} \in \mathcal{F}(1)$. Then 

$$
\|F^{-1}(m \cdot f)\|_{BM_{p,q}^r(\mathbb{R} \times \mathbb{R})} = \left(\sum_{k \in \mathbb{Z}} \sum_{k' \in \mathbb{Z}} (k)^{-s_1} (k')^{-s_2} \|F^{-1}[m \cdot (\varphi_k \otimes \psi_{k'}) f]\|_{L^p(\mathbb{R} \times \mathbb{R})}\right)^{1/q}
$$

Hence, it suffices to show that there is a constant $C$ such that for all $k, k' \in \mathbb{Z}$, 

$$
\|F^{-1}[m \cdot (\varphi_k \otimes \psi_{k'}) f]\|_{L^p(\mathbb{R} \times \mathbb{R})} \leq C \|F^{-1}[(\varphi_k \otimes \psi_{k'}) f]\|_{L^p(\mathbb{R} \times \mathbb{R})}.
$$

To this end, we fix arbitrary $k, k' \in \mathbb{Z}$. By the support conditions of $\varphi_k$ and $\psi_{k'}$, we see that there exist finitely many integers $j_{t_1}, j_{t_2}, \ldots, j_{t_N}$ such that 

$$
m \cdot (\varphi_k \otimes \psi_{k'}) = c_{j_{t_1}} \chi_{R_{j_{t_1}}} \cdot (\varphi_k \otimes \psi_{k'}) + c_{j_{t_2}} \chi_{R_{j_{t_2}}} \cdot (\varphi_k \otimes \psi_{k'}) + \ldots + c_{j_{t_N}} \chi_{R_{j_{t_N}}} \cdot (\varphi_k \otimes \psi_{k'})
$$

Moreover, the integer $N$ is independent of $k$ and $k'$. Thus, it suffices to show that there is a constant $C$ such that for any $j \in \mathbb{Z}$, 

$$
\|F^{-1}[\chi_{R_j} \cdot (\varphi_k \otimes \psi_{k'}) f]\|_{L^p(\mathbb{R} \times \mathbb{R})} \leq C \|F^{-1}[(\varphi_k \otimes \psi_{k'}) f]\|_{L^p(\mathbb{R} \times \mathbb{R})}.
$$

(21) 

Fix $j \in \mathbb{Z}$ arbitrarily, and write $R_j = [a_j, b_j] \times [d_j, e_j]$. Then we can write $\chi_{R_j}$ as 

$$
\chi_{R_j}(\xi, \xi') = \chi(a_j, b_j)(\xi) \chi(d_j, e_j)(\xi') = \left[\frac{1}{2} (\chi(a_j, +\infty)(\xi) - \chi(-\infty, a_j)(\xi)) - \frac{1}{2} (\chi(b_j, +\infty)(\xi) - \chi(-\infty, b_j)(\xi))\right]
$$
Obviously, \( g_1(\xi, \xi') - g_2(\xi, \xi') - g_3(\xi, \xi') + g_4(\xi, \xi') \) satisfy (20). Hence, applying Lemma 4.1 yields (21). This completes the proof of Theorem 1.4.

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REFERENCES

[1] R. Balan, P. G. Casazza, C. Heil and Z. Landau, Density, overcompleteness, and localization of frames, II, Gabor systems, *J. Fourier Anal. Appl.*, 12 (2006), 309–344.
[2] Á. Bényi, L. Grafakos, K. Gröchenig and K. Okoudjou, A class of Fourier multipliers for modulation spaces, *Appl. Comput. Harmon. Anal.*, 19 (2005), 131–139.
[3] Á. Bényi and K. Okoudjou, Local well-posedness of nonlinear dispersive equations on modulation spaces, *Bull. Lond. Math. Soc.*, 41 (2009), 549–558.
[4] S-Y. A. Chang and R. Fefferman, Some recent developments in Fourier analysis and \( H^p \) theory on product domains, *Bull. Amer. Math. Soc.*, 12 (1985), 1–43.
[5] S-Y. A. Chang and R. Fefferman, The Calderón-Zygmund decomposition on product domains, *Amer. J. Math.*, 104 (1982), 455–468.
[6] S-Y. A. Chang and R. Fefferman, A continuous version of duality of \( H^1 \) with \( BMO \) on the bidisk, *Ann. of Math.*, 112 (1980), 179–201.
[7] J. Chen, Hörmander type theorem for Fourier multipliers with optimal smoothness on Hardy spaces of arbitrary parameters, *Acta Math. Sin. (Engl. Ser.),* 33 (2017), 1083–1106.
[8] J. Chen and G. Lu, Hörmander type theorems for multi-linear and multi-parameter Fourier multiplier operators with limited smoothness, *Nonlinear Anal.*, 101 (2014), 98–112.
[9] J. Chen and G. Lu, Hörmander type theorem on bi-parameter Hardy spaces for bi-parameter Fourier multipliers with optimal smoothness, *Rev. Mat. Iberoam.*, 34 (2018), 1541–1561.
[10] W. Ding and G. Lu, Duality of multi-parameter Triebel-Lizorkin spaces associated with the composition of two singular integral operators, *Trans. Amer. Math. Soc.*, 368 (2016), 7119–7152.
[11] Y. Ding, G. Lu and B. Ma, Multi-parameter Triebel-Lizorkin and Besov spaces associated with flag singular integrals, *Acta Math. Sin. (Engl. Ser.),* 26 (2010), 603–620.
[12] R. Fefferman, Harmonic Analysis on product spaces, *Ann. of Math.,* 126 (1987), 109–130.
[13] R. Fefferman and J. Pipher, Multiparameter operators and sharp weighted inequalities, *Amer. J. Math.*, 119 (1997), 337–369.
[14] R. Fefferman and E. M. Stein, Singular integrals on product spaces, *Adv. Math.*, 45 (1982), 117–143.
[15] H. G. Feichtinger, Modulation spaces on locally compact abelian groups, Technical report, University Vienna, January 1983.
[16] G. Feichtinger and K. Gröchenig, Gabor frames and time-frequency analysis of distributions, *J. Funct. Anal.*, 146 (1997), 464–495.
[17] H. G. Feichtinger and G. Narimani, Fourier multipliers of classical modulation spaces, *Appl. Comput. Harmon. Anal.*, 21 (2006) 349–359.
[18] S. H. Ferguson and M. T. Lacey, A characterization of product BMO by commutators, *Acta Math.*, 189 (2002), 143–160.
[19] K. Gröchenig, *Foundations of Time-frequency Analysis*, Applied and Numerical Harmonic Analysis, Birkhäuser Boston, 2001.
[20] K. Gröchenig and C. Heil, Modulation spaces and pseudodifferential operators, *Integral Equations Operator Theory*, 34 (1999), 439–457.
[21] K. Gröchenig and Z. Rzeszotnik, Banach algebras of pseudodifferential operators and their almost diagonalization, *Ann. Inst. Fourier (Grenoble)*, 58 (2008), 2279–2314.
[22] R. Gundy and E. M. Stein, $H^p$ theory for the polydisk, *Proc. Nat. Acad. Sci.*, 76 (1979), 1026–1029.
[23] Y. Han, J. Li and G. Lu, Multiparameter Hardy space theory on Carnot-Carathéodory spaces and product spaces of homogeneous type, *Trans. Amer. Math. Soc.*, 365 (2013), 319–360.
[24] Y. Han and G. Lu, Discrete Littlewood-Paley-Stein theory and multi-parameter Hardy spaces associated with the flag singular integrals, preprint, arXiv:0801.1701.
[25] Y. Han, G. Lu and Z. Ruan, Boundedness criterion of Journé’s class of singular integrals on multiparameter Hardy spaces, *J. Funct. Anal.*, 264 (2013), 1238–1268.
[26] Y. Han, G. Lu and E. Sawyer, Flag Hardy spaces and Marcinkiewicz multipliers on the Heisenberg group, *Anal. PDE*, 7 (2014), 1465–1534.
[27] Q. Hong and G. Lu, Weighted $L^p$ estimates for rough bi-parameter Fourier integral operators, *J. Differential Equations*, 265 (2018), 1097–1127.
[28] Q. Hong, G. Lu and L. Zhang, $L^p$ boundedness of rough bi-parameter Fourier integral operators, *Forum Math.*, 30 (2018), 87–107.
[29] Q. Hong and L. Zhang, $L^p$ estimates for bi-parameter and bilinear Fourier integral operators, *Acta Math. Sin. (Engl. Ser.)*, 33 (2017), 165–186.
[30] B. Jessen, J. Marcinkiewicz and A. Zygmund, Note on the differentiability of multiple integrals, *Fundamenta Mathematicae*, 25 (1935), 217–234.
[31] J. L. Journé, Calderón-Zygmund operators on product spaces, *Rev. Mat. Iberoamericana*, 1 (1985), 55–91.
[32] J. L. Journé, Two problems of Calderón-Zygmund theory on product spaces, *Ann. Inst. Fourier (Grenoble)*, 38 (1988), 111–132.
[33] M. Kobayashi and Y. Sawano, Molecular decomposition of the modulation spaces, *Osaka J. Math.*, 47 (2010), 1029–1053.
[34] M. Kobayashi, M. Sugimoto and N. Tomita, Trace ideals for pseudo-differential operators and their commutators with symbols in $\alpha$-modulation spaces, *J. Anal. Math.*, 107 (2009), 141–160.
[35] M. Kobayashi and M. Sugimoto, The inclusion relation between Sobolev and modulation spaces, *J. Funct. Anal.*, 260 (2011), 3189–3208.
[36] G. Lu and Z. Ruan, Duality theory of weighted Hardy spaces with arbitrary number of parameters, *Forum Math.*, 26 (2014), 1429–1457.
[37] G. Lu and Y. Zhu, Singular integrals and weighted Triebel-Lizorkin and Besov spaces of arbitrary number of parameters, *Acta Math. Sin. (Engl. Ser.)*, 29 (2013), 39–52.
[38] C. Muscalu, J. Pipher, T. Tao and C. Thiele, Bi-parameter paraproducts, *Rev. Mat. Iberoam.*, 22 (2006), 963–976.
[39] D. Müller, F. Ricci and E. M. Stein, Marcinkiewicz multipliers and multi-parameter structure on Heisenberg (-type) groups, *Invent. Math.*, 119 (1995), 119–233.
[40] D. Müller, F. Ricci, and E. M. Stein, Marcinkiewicz multipliers and multi-parameter structure on Heisenberg(-type) groups, *Ill. Math.*, 221 (1996), 267–291.
[41] A. Nagel, F. Ricci and E. M. Stein, Singular integrals with flag kernels and analysis on quadratic CR manifolds, *J. Funct. Anal.*, 181 (2001), 29–118.
[42] A. Nagel, F. Ricci and E. M. Stein, Singular integrals with flag kernels and analysis on homogeneous groups, *Rev. Mat. Iberoam.*, 28 (2012), 631C-722.
[43] A. Nagel, F. Ricci, E. M. Stein and S. Wainger, Singular integrals with flag kernels on homogeneous groups, *Mem. Amer. Math. Soc.*, 256 (2018), no. 1230, vii+141 pp.
[44] J. Pipher, Journé’s covering lemma and its extension to higher dimensions, *Duke Math. J.*, 53 (1986), 683–690.
[45] Z. Ruan, The Calderón-Zygmund decomposition and interpolation on weighted Hardy spaces, *Acta Math. Sin. (Engl. Ser.)*, 27 (2011), 1967–1978.
[46] J. Sjöstrand, An algebra of pseudodifferential operators, *Math. Res. Lett.*, 1 (1994), 185–192.
[48] B. Street, *Multi-parameter Singular Integrals*, Annals of Mathematics Studies, 189, Princeton University Press, Princeton, NJ, 2014.

[49] M. Sugimoto and N. Tomita, The dilation property of modulation spaces and their inclusion relation with Besov spaces, *J. Funct. Anal.*, **248** (2007), 79–106.

[50] K. Tachizawa, The boundedness of pseudodifferential operators on modulation spaces, *Math. Nachr.*, **168** (1994), 263–277.

[51] N. Tomita, On the Hörmander multiplier theorem and modulation spaces, *Appl. Comput. Harmon. Anal.*, **26** (2009), 408–415.

[52] H. Triebel, *Theory of Function Spaces*, Monographs in Mathematics, 78, Birkhäuser Verlag, Basel, 1983.

[53] B. Wang and H. Hudzik, The global Cauchy problem for the NLS and NLKG with small rough data, *J. Differential Equations*, **231** (2007), 36–73.

[54] C. Xu and L. Huang, Boundedness of bi-parameter pseudo-differential operators on bi-parameter α-modulation spaces, *Nonlinear Anal.*, **180** (2019), 20–40.

[55] C. Xu, Boundedness of bi-parameter fractional integrals on bi-parameter modulation spaces, preprint.

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