SELF INTERSECTION NUMBERS OF MINIMAL, REGULAR MODELS OF MODULAR CURVES $X_0(p^2)$ OVER RATIONAL NUMBERS

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Abstract. For an odd prime $p > 7$, we compute the Arakelov self-intersection numbers of the relative dualizing sheaves for Edixhoven’s minimal, regular models for modular curves $X_0(p^2)$ over $\mathbb{Q}$. The computation of the self-intersection numbers should be useful to find an effective version of Bogolomov’s conjecture for the semi-stable models of modular curves $X_0(p^2)$ and obtain a bound on the stable Falting’s height for those curves [5].

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1. INTRODUCTION

For an integer $N > 1$ consider the following congruence subgroup:

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N} \right\}.$$

The congruence subgroup $\Gamma_0(N)$ acts on the upper half plane $\mathbb{H}$ and we denote by $Y_0(N) = \Gamma_0(N) \backslash \mathbb{H}$ the quotient space, this is a Riemann surface and let $X_0(N)$ be the compactification. In fact its equations can be written with $\mathbb{Q}$ coefficients and hence can be considered as a curve over $\mathbb{Q}$ [9]. The modular curve $X_0(N)$ is a compact Riemann surface and hence an algebraic curve over $\mathbb{C}$. Let $v_{\Gamma_0(N)}$ the volume of the modular curve $X_0(N)$ and $g_N$ be the genus of the Riemann surface $X_0(N)$.

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In this article, we wish to express the Arakelov self intersection number of the relative dualizing sheaf for the minimal regular model over $\mathbb{Z}$ of the modular curve $X_0(p^2)$ in terms of its genus $g_{p^2}$, for primes $p > 7$. For odd, square-free $N$, this quantity was computed for the congruence subgroups $\Gamma_0(N)$ [1] and $\Gamma_1(N)$ [23] and recently for the principal congruence subgroups $\Gamma(N)$ [13]. We impose the hypothesis on the prime $p$ to ensure that the genus of the modular curve is non-zero [cf. Prop. 3]. It is tempting to expect that similar results should be true for the modular curves $X_0(p^n)$ if $n > 2$ but the geometry of the modular curves become complicated in those cases. The generalization of our result should depend on careful analysis of the regular minimal model of $X_0(p^n)$ [6]. The possibility of extending the result to the modular curve $X_0(p^2)$ from the information about the modular curve $X(p)$ were evident in the work of Edixhoven and Bart de Smit [8]. We hope that our results will find applications in finding Fourier coefficients and residual Galois representations following the strategy outlined in [11].

The technical difficulty of this paper lies in the fact that for square free $N$ the special fiber of the modular curve is reduced and even semi-stable over $\mathbb{Q}$, while without this hypothesis the special fiber is non-reduced and not semi-stable. We manage to remove the square-free assumption in our paper because of careful analysis of the the regular but non-minimal models for the corresponding modular curves following Edixhoven [10].

The bound on the self-intersection number for the infinite place in terms of green functions has been achieved by the idea outlined by Zagier [28] using Selberg trace formula. Although, it must be well-known to the experts we included this for our particular modular curves for the sake of completeness. A better bound for these self-intersection numbers for square free level have been established by Ulmo-Michel [25] by using Rankin-Selberg methods.

Although the complete information about the the boundaries of the modular curves $X_0(p^2)$ [4] was necessary but the rationality of the special cusps 0 and $\infty$ plays a key role in our computations. The key ingredient in this computation is to relate this self-intersection numbers with the values of a Green function at cusps this again depends on the constant term of the non-holomorphic Eisenstein series.

The following Theorem is analogue of Theorem F of [1], Theorem 1 of [23].

**Theorem 1.** For $N = p^2$, the constant term $R^\Gamma_\infty(p^2)$ in the Laurent series expansion of the Rankin-Selberg transform for the Arakelov metric for the modular curve $X_0(p^2)$ is asymptotically given by:

$$R^\Gamma_\infty(p^2) = o\left(\frac{\log(p)}{g_{p^2}}\right).$$

Although, the above theorem is analogue of the Theorems in the loc. cit. but computation of hyperbolic [§ 5.1] and the parabolic [§ 5.3] contribution in the Selberg Trace formula is a bit different in the trace formula because of the condition we imposed on $N$. Our computation in the elliptic contribution [§ 5.2] is an adaptation in our setting of the strategy outlined by Zagier [29].

Being an algebraic curve over $\mathbb{Q}$, $X_0(p^2)$ has a minimal regular model over $\mathbb{Z}$ which we denote by $X_0(p^2)$. Let $\varpi_N$ be the relative dualizing sheaf of $X_0(N)$ equipped with the Arakelov metric and $\varpi_N^2 = \langle \varpi_N, \varpi_N \rangle$ be the Arakelov self-intersection number as defined in the Section 2. The following theorem is analogous to Proposition D of [1], Theorem 1 of [23] and Theorem 5. 2. 3 of [13] for the modular curve $X_0(p^2)$:
Theorem 2. For \( p > 7 \), we have the following asymptotic formula for the Arakelov self intersection numbers for the modular curve \( X_0(p^2) \):

\[
\omega_2^2 p^2 = 3g p^2 \log(p^2) + o(g p^2 \log(p)).
\]

We wish to generalize our work to semi-stable model of these modular curves in a future article [5]. Over \( \mathbb{Z} \) the regular minimal models are not semi-stable. For a semi-stable model we have to work over the ring of integers of a finite extension of \( \mathbb{Q} \). From the viewpoint of Arakelov theory, the main motivation for studying the self-intersection number is to give an effective Bogomolov’s conjecture for the particular modular curve \( X_0(p^2) \). Bogomolov’s conjecture has been proved by Ulmo [27] using Ergodic theory though the proof is not effective.

Unfortunately, we are not able to prove an effective version of Bogomolov’s conjecture in this article due to the lack of semi-stability which is required to calculate the admissible self-intersection in the sense of Zhang [30]. As far as infinite part of the Arakelov self-intersection is concerned, the computation carried out in this article should be used verbatim. However a careful study of the special fiber of the semi-stable model is required to carry out the computations for the finite part of the intersection. The semi-stable model is obtained by a base-change followed by normalisation and then resolving any remaining singularities. To keep the length of the current paper reasonable we choose to present those calculations along with the arithmetic applications in a separate article [5].

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2. Arakelov intersection pairing

Let \( K \) be a number field and and \( R \) be its ring of integers. Let \( \mathcal{X} \) be an arithmetic surface over \( R \) with the map \( f : \mathcal{X} \to \text{Spec} R \). Let \( X = \mathcal{X}_{(0)} \) be the generic fiber which is a smooth irreducible projective curve over \( K \). For each embedding \( \sigma : K \to \mathbb{C} \) we have a connected Riemann surface

\[
\mathcal{X}_\sigma = X \times_{\text{Spec} \mathbb{K}, \sigma} \text{Spec} \mathbb{C}.
\]

Collectively we denote

\[
\mathcal{X}_\infty = \bigsqcup_{\sigma : K \to \mathbb{C}} \mathcal{X}_\sigma.
\]

Any line bundle \( L \) on \( \mathcal{X} \) induces a line bundle on \( \mathcal{X}_\sigma \) which we denote by \( L_\sigma \). A metrized line bundle \( \bar{L} = (L, h) \) is a line bundle \( L \) on \( \mathcal{X} \) along with a hermitian metric \( h_\sigma \) on each \( L_\sigma \). Arakelov invented an intersection pairing for metrized line bundles which we describe now (this version is most probably due to
Let $\bar{L}$ and $\bar{M}$ be two metrized line bundles with non-trivial global sections $l$ and $m$ respectively such that the associated divisors do not have any common components, then

$$\langle \bar{L}, \bar{M} \rangle = \langle L, M \rangle_{\text{fin}} + \sum_{\sigma : K \to \mathbb{C}} \langle L_\sigma, M_\sigma \rangle.$$

The first summand is the algebraic part whereas the second summand is the analytic part of the intersection. For each closed point $x \in X$, $l_x$ and $m_x$ can be thought of as elements of $O_{X,x}$ via a suitable trivialization. If $X^{(2)}$ is the set of closed points of $X$, (the 2 here signifies the fact that a closed point is a 2 co-dimensional algebraic cycle on $X$), then

$$\langle L, M \rangle_{\text{fin}} = \sum_{x \in X^{(2)}} \log \#(O_{X,x}/(l_x, m_x)).$$

Now for the analytic part, we assume that the associated divisors of $l$ and $m$ which we denote by $\text{div}(l)_\sigma$ and $\text{div}(m)_\sigma$ on $X_\sigma$ do not have any common points, and that $\text{div}(l)_\sigma = \sum \alpha n_\alpha P_\alpha$ with $n_\alpha \in \mathbb{Z}$, then

$$\langle L_\sigma, M_\sigma \rangle = - \sum \alpha n_\alpha \log \|m(P_\alpha)\| - \int_{X_\sigma} \log \|l\| c_1(M_\sigma).$$

where $\| \cdot \|$ denotes the norm given by the hermitian metric on $L$ or $M$ respectively and is clear from the context. $c_1(M_\sigma)$ is the first Chern class of $M_\sigma$ and is a closed $(1,1)$ form on $X_\sigma$ (see for instance Griffith-Harris [14]).

This intersection product is symmetric in $\bar{L}$ and $\bar{M}$. Moreover if we consider the group of metrized line bundles up to isomorphisms, this is called the arithmetic Picard group denoted by $\hat{\text{Pic}}X$, then the arithmetic intersection product extends to a symmetric bilinear form on all of $\hat{\text{Pic}}X$. It can be extended by linearity to the rational arithmetic Picard group $\hat{\text{Pic}}_Q X = \hat{\text{Pic}}X \otimes \mathbb{Q}$.

For more details see Arakelov [3, 2] and Curilla [7].

Arakelov gave a unique way of attaching a hermitian metric to a line bundle on $X$, see for instance Faltings [12], section 3. We summarise the construction here. Note that the space $H^0(X_\sigma, \Omega^1)$ of holomorphic differentials on $X_\sigma$ has a natural inner product on it

$$\langle \phi, \psi \rangle = \frac{i}{2} \int_{X_\sigma} \phi \wedge \overline{\psi}.$$ 

We assume that the genus of $X$ is greater than 1, then choose an orthonormal basis of $H^0(X_\sigma, \Omega^1)$, $f_1^\sigma, \ldots, f_g^\sigma$. The canonical volume form on $X_\sigma$ is

$$\mu_{\text{can}}^\sigma = \frac{i}{2g} \sum_{j=1}^g f_j^\sigma \wedge \overline{f_j^\sigma}.$$ 

There is a hermitian metric on a line bundle $L$ on $X$ such that $c_1(L_\sigma) = \deg(L_\sigma) \mu_{\text{can}}^\sigma$ for each embedding $\sigma : K \to \mathbb{C}$. This metric is unique up to scalar multiplication. Such a metric is called admissible metric.
Let now \( X \) be a Riemann surface of genus greater than 1 and \( \mu_{\text{can}} \) the canonical volume form. The canonical Green’s function for \( X \) is the unique solution of the differential equation

\[
\partial_z \bar{\partial}_z g_{\text{can}}(z, w) = i\pi (\mu_{\text{can}}(z) - \delta_w(z))
\]

where \( \delta_w(z) \) is the Dirac delta distribution, with the normalization condition

\[
\int_X g_{\text{can}}(z, w) \mu_{\text{can}}(z) = 0.
\]

For \( Q \in X \) there is a unique admissible metric on \( L = \mathcal{O}_X(Q) \) such that the norm of the constant function 1, which is a section of \( \mathcal{O}_X(Q) \), at the point \( P \) is given by

\[
|1|(P) = \exp(g_{\text{can}}(p, q)).
\]

By tensoring we can get unique admissible metric on any line bundle on \( X \).

Let again \( X \) be an arithmetic surface over \( R \) as above. To any line bundle \( L \) on \( X \) we can associate in this way a unique hermitian metric on \( L \) for each \( \sigma \). This metric is called the Arakelov metric.

Let \( L \) and \( M \) be two line bundles on \( X \), we equip them with the Arakelov metrics to get metrized line bundles \( \bar{L} \) and \( \bar{M} \). The Arakelov intersection pairing of \( L \) and \( M \) is defined as arithmetic intersection pairing of \( \bar{L} \) and \( \bar{M} \)

\[
\langle L, M \rangle_{\text{Ar}} = \langle \bar{L}, \bar{M} \rangle.
\]

It relates to the canonical Green’s function as follows. Let \( l \) and \( m \) be global sections of \( L \) and \( M \) as above. Assume that the corresponding divisors don’t have any common components. Further more let

\[
\text{div}(l)_{\sigma} = \sum_{\alpha} n_{\alpha, \sigma} P_{\alpha, \sigma}, \quad \text{and} \quad \text{div}(m)_{\sigma} = \sum_{\beta} r_{\beta, \sigma} Q_{\beta, \sigma}
\]

then

\[
\langle L, M \rangle_{\text{Ar}} = \langle L, M \rangle_{\text{fin}} - \sum_{\sigma : K \to \mathbb{C}} \sum_{\alpha, \beta} n_{\alpha, \sigma} r_{\beta, \sigma} g_{\text{can}}(P_{\alpha, \sigma}, Q_{\beta, \sigma}).
\]

By \( \mathcal{O}_{X, \text{Ar}} \) we denote the relative dualizing sheaf on \( X \) (see Qing Liu [22], chapter 6, section 6.4.2) equipped with the Arakelov metric. We shall usually denote this simply by \( \mathcal{O} \) if the arithmetic surface \( X \) is clear from the context.

We are interested in a particular invariant of the modular curve \( X_0(p^2) \) which arises from Arakelov geometry and has applications in number theory. \( X_0(p^2) \) which is defined over \( \mathbb{Q} \) and has a minimal regular model \( X_0(p^2) \) over \( \mathbb{Z} \) for primes \( p > 5 \). In this paper we shall calculate the Arakelov self intersection \( \mathcal{O}^2 = \langle \mathcal{O}, \mathcal{O} \rangle \) of the relative dualizing sheaf on \( X_0(p^2) \).

We retain the notation \( K \) for a number field and \( R \) its ring of integers. If \( X \) is a smooth curve of over \( K \) then a regular model for \( X \) is an arithmetic surface \( p : X \to \text{Spec } R \) whose generic fiber \( X_{(0)} \) is isomorphic to \( X \). If genus of \( X \) is greater than 1 then there is a minimal regular model \( X_{\text{min}} \), which is unique. \( X_{\text{min}} \) is minimal among the regular models for \( X \) in the sense that any proper birational morphism to another regular model is an isomorphism. Another equivalent criterion for minimality is that \( X_{\text{min}} \) does not have any prime vertical divisor that can be blown down without introducing a singularity.
A regular model for $X_0(p^2)/\mathbb{Q}$ was constructed in Edixhoven [10]. We denote this model by $\overline{X}_0(p^2)/\mathbb{Z}$. This model is not minimal but a minimal model $X_0(p^2)$ is easily obtained by blowing down certain prime vertical divisors. We describe these constructions in Section 6.

We conclude this section with a formula for the genus of $X_0(p^2)$.

**Proposition 3.** The genus of $X_0(p^2)$ is given by

$$g_{p^2} - 1 = \frac{(p + 1)(p - 6) - 12c}{12}$$

where $c \in \{0, \frac{1}{2}, \frac{2}{3}, \frac{7}{6}\}$.

**Proof.** By [9, Theorem 3.1.1], the genus is given by

$$g_{p^2} = 1 + \frac{\mu}{12} - \frac{\nu_2}{4} - \frac{\nu_3}{3} - \frac{\nu_\infty}{2}$$

where $\mu = [\text{SL}_2(\mathbb{Z}) : \Gamma_0(p^2)]$, and $\nu_\infty = p + 1$ is the number of cusps of $X_0(p^2)$. The numbers $\nu_2$ and $\nu_3$ are the numbers of elliptic points of order 2 and 3 in $X_0(p^2)$ respectively and can be computed using the following formulae:

$$\nu_2 = 1 + \left(\frac{-1}{p}\right) = \begin{cases} 2 & \text{if } p \equiv 1 \pmod{4}, \\ 0 & \text{if } p \equiv 3 \pmod{4}, \end{cases}$$

$$\nu_3 = 1 + \left(\frac{-3}{p}\right) = \begin{cases} 2 & \text{if } p \equiv 1 \pmod{3}, \\ 0 & \text{if } p \equiv 2 \pmod{3}. \end{cases}$$

The proposition follows once we substitute $\nu_2$ and $\nu_3$ in (2.1).

### 3. Canonical Green’s functions at the rational cusps

In this section we compute the canonical Green’s function $g_{\text{can}}$ for $X_0(p^2)$ at the cusps in terms of the Eisenstein series. We begin by recalling the definition and some properties of this series. For a more elaborate discussion on Eisenstein series for general congruence subgroup, we refer to [13, p.10]. Let

$$\partial(X_0(p^2)) = \left\{ 0, \infty, \frac{1}{p}, \ldots, \frac{1}{kp} \right\}, \quad k = 1, \ldots, (p - 1)$$

be the complete set of coset representatives of the cusps of $X_0(p^2)$ (see [4]). For $P \in \partial(X_0(p^2))$, let $\Gamma_0(p^2)_P$ be the stabilizer of $P$ in $\Gamma_0(p^2)$. Also, denote by $\sigma_P$, any scaling matrix of the cusp $P$, i.e., $\sigma_P$ is an element of $\text{SL}_2(\mathbb{R})$ with the properties $\sigma_P(\infty) = P$ and

$$\sigma_P \Gamma_0(p^2)_P \sigma_P^{-1} = \Gamma_0(p^2)_\infty = \left\{ \pm \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \bigg| m \in \mathbb{Z} \right\},$$

and fix such a matrix. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R})$ and $k = 0, 2$, we consider the automorphic factor of weight $k$ to be

$$j_\gamma(z; k) = \frac{(cz + d)^k}{|cz + d|^k}.$$
We put:

\[ E_{p,k}(z, s) = \sum_{\gamma \in \Gamma_0(p^2) \setminus \Gamma_0(p^2)} (\text{Im}(\sigma_p^{-1} \gamma z))^s j_{p^{-1}}(\gamma ; k)^{-1}. \]  

The series \( E_{p,k}(z, s) \) is a holomorphic function of \( s \) in \( \text{Re} \ s > 1 \) and for each such \( s \), it is an automorphic function of \( z \) with respect to \( \Gamma_0(p^2) \). Moreover, it has a meromorphic continuation to the whole complex plane. Also, \( E_{p,k} \) is an eigenfunction of the hyperbolic Laplacian of weight \( k \) defined by

\[ \Delta_k = y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - i k(k - 1)y \frac{\partial}{\partial x}. \]

We note that \( E_{p,0} \) has a simple pole at \( s = 1 \) with residue \( 1/v_{\Gamma_0(p^2)} \) independent of the \( z \) variable. Being an automorphic function, \( E_{p,0}(z, s) \) has a Fourier series expansion at any cusp \( Q \), given by

\[ E_{p,0}(\sigma_Q(z), s) = \delta_{P,Q} y^s + \phi_{P,Q}^0(s) y^{1-s} + \sum_{n \neq 0} \phi_{P,Q}(n, s) W_s(\mu) \]

where

\[ \phi_{P,Q}^0(s) = \sqrt{\pi} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} \sum_c c^{-2s} S_{P,Q}(0, 0, c) \]

\[ \phi_{P,Q}(n, s) = \pi s \Gamma(s)^{-1} |n|^{s-1} \sum_c c^{-2s} S_{P,Q}(0, n, c), \]

\( S_{P,Q}(a, b, c) \) is the Kloosterman sum and \( W_s(\mu) \) is the Whittaker function (see [16]).

We put

\[ C_{P,Q}^{\Gamma_0(p^2)} := -2\pi \lim_{s \to 1} \left( \phi_{P,Q}^0(s) - \frac{1}{v_{\Gamma_0(p^2)} s - 1} \right) \]

as the constant term in the Laurent series expansion of \( \phi_{P,Q}^0(s) \).

The connection between the canonical Green’s function and the Eisenstein series is given by the following formula [1, Proposition E]: For \( \Gamma = \Gamma_0(p^2) \),

\[ g_{\text{can}}(\infty, 0) = -2\pi \lim_{s \to 1} \left( \frac{\phi_{\infty,0}^0(s)}{v_{\Gamma_0(p^2)} s - 1} \right) - \frac{2\pi}{v_{\Gamma_0(p^2)}} \]

\[ + 2\pi \lim_{s \to 1} \left( \frac{1}{v_{\Gamma_0(p^2)} s(s - 1)} + \int_{\Gamma_0(p^2) \times \Gamma_0(p^2)} G_s(z, w) \mu_{\text{can}}(z) \mu_{\text{can}}(w) \right) \]

\[ + 2\pi \lim_{s \to 1} \left( \int_{\Gamma_0(p^2)} E_{\infty,0}(z, s) \mu_{\text{can}}(z) + \int_{\Gamma_0(p^2)} E_{0,0}(z, s) \mu_{\text{can}}(z) - \frac{2}{v_{\Gamma_0(p^2)} s - 1} \right). \]

We put:

\[ R_{\infty}^{\Gamma_0(p^2)} = \frac{1}{2} \lim_{s \to 1} \left( \int_{\Gamma_0(p^2)} E_{\infty,0}(z, s) \mu_{\text{can}}(z) + \int_{\Gamma_0(p^2)} E_{0,0}(z, s) \mu_{\text{can}}(z) - \frac{2}{v_{\Gamma_0(p^2)} s - 1} \right). \]
Also from [18] and [19], we know that
\[
\lim_{s \to 1} \frac{1}{v_{\Gamma_0(p^2)}} s(s-1) + \int Y_{\Gamma_0(p^2)} G_s(z, w) \mu_{\text{can}}(z) \mu_{\text{can}}(w) = O \left( \frac{1}{g_{p^2}} \right).
\]
Thus we have
\[
(3.6) \quad g_{\text{can}}(\infty, 0) = C_{\Gamma_0(p^2)}^0 + 4\pi R_{\Gamma_0(p^2)} + O(p^{-2}).
\]

3.1. Computation of $C_{\infty, \infty}^{\Gamma_0(p^2)}$ and $C_{0, \infty}^{\Gamma_0(p^2)}$. We expand the terms $\phi_{\infty, \infty}^{\Gamma_0(p^2)}(s)$ and $\phi_{0, \infty}^{\Gamma_0(p^2)}(s)$ in the Laurent series expansion about $s = 1$. The below computations are inspired by [21].

Lemma 5. The Laurent series expansion of $\phi_{\infty, \infty}^{\Gamma_0(p^2)}(s)$ about $s = 1$ is given by
\[
\phi_{\infty, \infty}^{\Gamma_0(p^2)}(s) = \frac{1}{v_{\Gamma_0(p^2)}} s(s-1) + \frac{1}{v_{\Gamma_0(p^2)}} \left( 2\gamma_{EM} + \frac{a\pi}{6} - \frac{2(2p^2-1) \log p}{p^2 - 1} \right) + O(s-1),
\]
where $\gamma_{EM}$ is the Euler Gamma constant and $a$ is a constant.

Proof. We choose $\sigma_{\infty} = I$ as a scaling matrix of $\infty$. Then from [16, page 48],
\[
S_{\infty, \infty}(0, 0, c) = | \{ d \pmod{c} | (a \ b \ c \ d) \in \Gamma_0(p^2) \} | = \begin{cases} 1 & \text{if } p^2 \nmid c, \\ \phi(c) & \text{if } p^2 \mid c, \end{cases}
\]
and writing $c$ in the form $c = p^{k+2}n$ where $k \geq 0$ and $p \nmid n$,
\[
\phi(c) = \phi(p^{k+2}n) = \phi(p^{k+2})\phi(n) = (p-1)p^{k+1}\phi(n).
\]
Therefore,
\[
\sum c^{-2s} S_{\infty, \infty}(0, 0; c) = \sum_{n=1, p \nmid n}^{\infty} \sum_{k=0}^{\infty} (p^{k+2}n)^{-2s}(p-1)p^{k+1}\phi(n)
\]
\[
= p^{-4s+1}(p-1) \sum_{n=1, p \nmid n}^{\infty} n^{-2s} \phi(n) \sum_{k=0}^{\infty} p^{-2s+1)k}
\]
\[
= p^{-4s+1}(p-1) \left( \zeta(2s-1) p^{2s-p} - \frac{2}{\zeta(2s)} \right) \left( \frac{1}{1-p^{-2s+1}} \right)
\]
\[
= \frac{p(p-1)}{p^{2s}(p^{2s}-1)} \zeta(2s-1) \zeta(2s).
\]
Therefore,
\[
\phi_{\infty, \infty}^{\Gamma_0(p^2)}(s) = (p(p-1)) \left( \frac{1}{p^{2s}(p^{2s}-1)} \right) \left( \frac{1}{\pi} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)\zeta(2s)} \right) \zeta(2s - 1).
\]
The second term is holomorphic near $s = 1$ and has the Taylor series expansion
\[
(3.7) \quad \frac{1}{p^{2s}(p^{2s}-1)} = \frac{1}{p^2(p^2-1)} - \frac{2(2p^2-1) \log p}{p^2(p^2-1)^2} (s-1) + O((s-1)^2).
\]
The third term is holomorphic as well near $s = 1$ and has the Taylor series expansion
\[
(3.8) \quad \sqrt{\pi} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)\zeta(2s)} = \frac{6}{\pi} + a(s-1) + O((s-1)^2),
\]
where $a$ is the derivative at $s = 1$ of this function. Finally, $\zeta(2s - 1)$ is meromorphic near $s = 1$ with the Laurent series expansion

$$\zeta(2s - 1) = \frac{1}{2(s-1)} + \gamma_{EM} + O(s-1).$$

Multiplying these expansions, we see that

$$\phi_{\infty,0}(p^2)(s) = \frac{1}{\Gamma_0(p^2)} \left( \frac{2\gamma_{EM} + a\pi}{6} - \frac{2(2p^2-1)\log p}{p^2-1} \right) + O(s-1),$$

which gives the result observing that $v_\Gamma(p^2) = \frac{\pi}{2}p(p+1)$.

**Proposition 6.** The constant term in the Laurent series expansion of the Eisenstein series is given by

$$\phi_{\infty,0}(p^2) = \frac{1}{\Gamma_0(p^2)} \left( \frac{2\gamma_{EM} + a\pi}{6} - \frac{2(2p^2-1)\log p}{p^2-1} \right) + O(s-1).$$

**Lemma 7.** The Laurent series expansion of $\phi_{\infty,0}(p^2)$ about $s = 1$ is

$$\phi_{\infty,0}(p^2)(s) = \frac{1}{\Gamma_0(p^2)} \left( \frac{2\gamma_{EM} + a\pi}{6} - \frac{2(2p^2-1)\log p}{p^2-1} \right) + O(s-1).$$

**Proof.** Let $\sigma_0$ be the scaling matrix of the cusp 0 defined by

$$\sigma_0^{-1} = \frac{1}{\sqrt{p^2}} W_{p^2} \in SL_2(R),$$

where $W_{p^2} = \left( \begin{array}{cc} 1 & -1 \\
 p & 1 \end{array} \right)$ the Atkin-Lehner involution. For the cusp $\infty$, we take $\sigma_\infty = I$ as a scaling matrix. Then from [16, page 48], we have

$$S_{\infty,0}(0,0;c) = \left\{ \begin{array}{ll}
 d \pmod{c} | \left( \begin{array}{cc} a & b \\
 c & d \end{array} \right) = \left( \begin{array}{cc}
 p & -a' \\
 d & -p c' \end{array} \right), a', b', c', d' \in Z, |a'c' - b'p^2| = 1 \\
 \sigma(n) & \text{if } c = pn \text{ with } p \nmid n.
\end{array} \right.$$
Proposition 8. $C^\Gamma_0(p^2)$ is given by

$$C^\Gamma_0(p^2) = -\frac{2\pi}{v^\Gamma_0(p^2)} \left( 1 + 2\gamma_{EM} + \frac{a\pi}{6} - \frac{2(p^2 - p - 1)}{p^2 - 1} \log p \right).$$

3.2. Computation of $\mathcal{R}_\infty^\Gamma_0(p^2)$. We first show that $\mathcal{R}_\infty^\Gamma_0(p^2)$ is the constant term in the Laurent series expansion of the Rankin-Selberg transform of the canonical metric. We start by writing down the canonical volume form $\mu_{can}$ in coordinates. Let $S_2(\Gamma_0(p^2))$ be the space of holomorphic cusp forms of weight 2 with respect to $\Gamma_0(p^2)$. Then $S_2(\Gamma_0(p^2)) \cong H^0(X_0(p^2), \Omega^1)$ by $f(z) \mapsto f(z)dz$. The inner product on $S_2(\Gamma_0(p^2))$ induced by this isomorphism from $H^0(X_0(p^2), \Omega^1)$ is the Petersson inner product. Let $\{f_1, \ldots, f_{g_{p^2}}\}$ be an orthonormal basis of $S_2(\Gamma_0(p^2))$. Then it follows that

$$\mu_{can}(z) = \frac{i}{2g_{p^2}} \sum_{j=1}^{g_{p^2}} |f_j(z)|^2 dz \wedge d\bar{z} = F(z)\mu_{hyp},$$

where

$$F(z) = \frac{\text{Im}(z)^2}{g_{p^2}} \sum_{j=1}^{g_{p^2}} |f_j(z)|^2.$$

The following Lemma is similar to [23, Lemma 3.5].

Lemma 9. We have

$$\int_{Y_0(p^2)} E_{0,0}(z,s)\mu_{can}(z) = \int_{Y_0(p^2)} E_{\infty,0}(z,s)\mu_{can}(z).$$

Proof. Let $\sigma_0$ be the scaling matrix of the cusp 0 defined in (3.12). Then, since $\sigma_0^{-1}\Gamma_0\sigma_0 = \Gamma_\infty$,

$$E_{0,0}(z,s) = \sum_{\gamma \in \Gamma_0(p^2) \setminus \Gamma_0(\mathbb{R})} (\text{Im}(\sigma_0^{-1}\gamma z))^s = \sum_{\beta \in \Gamma_\infty \setminus \Gamma_0(\mathbb{R})} (\text{Im}(\beta \sigma_0^{-1} z))^s = \sum_{\beta \in \Gamma_\infty \setminus \Gamma_0(p^2)} (\text{Im}(\sigma_\infty^{-1} \beta (\sigma_0^{-1} z)))^s = E_{\infty,0}(\sigma_0^{-1} z, s),$$

by taking $\sigma_\infty = I$. Also, if $\{f_j\}_{j=1}^{g_{p^2}}$ is an orthonormal basis for $S_2(\Gamma_0(p^2))$, then

$$\mu_{can}(z) = F(z)\mu_{hyp}(z),$$

where $F(z)$ is given by (3.15). As $\mu_{hyp}$ is invariant under the automorphisms of $\mathbb{H}$, it suffices to prove that $F(z) = F(\sigma_0^{-1}(z))$. But

$$F(\sigma_0^{-1}(z)) = F(W_{p^2}z) = \frac{\text{Im}(W_{p^2}z)^2}{g_{p^2}} \sum_{j=1}^{g_{p^2}} |f_j(W_{p^2}z)|^2 = \frac{p^4}{g_{p^2}} \text{Im}(z)^2 \sum_{j=1}^{g_{p^2}} |f_j(W_{p^2}z)|^2.$$

A direct computation shows that

$$\left\{ \frac{p^2}{(p^2z)^2} f_j(W_{p^2}z) \right\}_{j=1}^{g_{p^2}}$$

is also an orthonormal basis for $S_2(\Gamma_0(p^2))$ and hence the right hand side of (3.16) must be equal to $F(z)$. This proves the lemma. $\square$
As a result we have the following simpler formula for $R_{\Gamma_0(p^2)}$ defined in 3.5:

$$R_{\Gamma_0(p^2)} = \lim_{s \to 1} \left( \int_{Y_0(p^2)} E_{\infty,0}(z,s) \mu_{\text{can}}(z) - \frac{1}{v_{\Gamma_0(p^2)}} \frac{1}{s-1} \right).$$

Writing $\mu_{\text{can}}(z) = F(z)\mu_{\text{hyp}}(z)$ in (3.17), where $F$ is the function defined in (3.15), we see that the integral is the Rankin-Selberg transform $R_F(s)$ of $F$ at the cusp $\infty$. To quickly recall the definition of Rankin-Selberg transform, let $f$ be a $\Gamma_0(p^2)$-invariant holomorphic function on $\mathbb{H}$ of rapid decay at the cusp $\infty$, i.e., the constant term of the Fourier series expansion of $f$ at the cusp $\infty$ given by

$$f(x + iy) = \sum_n a_n(y)e^{2\pi i n x},$$

satisfies $a_0(y) = O(y^{-M})$ for some $M > 0$ as $y \to \infty$. Then the Rankin-Selberg transform of $f$, denoted by $R_f(s)$, is defined by

$$R_f(s) = \int_{Y_0(p^2)} f(z)E_{\infty,0}(z,s)\mu_{\text{hyp}}(z) = \int_0^\infty a_0(y)y^{s-2} dy.$$

The function $R_f(s)$ is holomorphic for $\text{Re}(s) > 1$ and admits a meromorphic continuation to the whole complex plane and has a simple pole at $s = 1$ with residue

$$\frac{1}{v_{\Gamma_0(p^2)}} \int_{Y_0(p^2)} f(z)\mu_{\text{hyp}}(z).$$

Thus $R_F(s)$ has the following Laurent series expansion near $s = 1$

$$R_F(s) = \frac{1}{v_{\Gamma_0(p^2)}} \frac{1}{s-1} + R_{\Gamma_0(p^2)} + O(s-1).$$

The next two sections of the paper are devoted to finding an estimate of $R_{\Gamma_0(p^2)}$ using this formula.

4. Selberg Trace formula

To compute $R_F(s)$, we follow the strategy carried out in [1] and [23], namely consider on one hand the spectral expansions of certain automorphic kernels that consist of the term $F$ together with some well-behaved terms and on the other hand the contribution of various motions of $\Gamma_0(p^2)$ to these kernels. Computation of $R_F(s)$ then reduces to understanding the other terms in this identity which are easier to handle. To elaborate this process, let us recall the definitions of these kernels. For $t > 0$, let $h_t : \mathbb{R} \to \mathbb{R}$ be the test function

$$h_t(r) = e^{-t(\frac{1}{4} + r^2)}$$

which is a function of rapid decay.
4.1. Automorphic kernels of weight \( k = 0, 2 \). The automorphic kernels involves the inverse Selberg/Harishchandra transform \( \phi_k(t, \cdot) \) of \( h_t \) of weight \( k \) is given by

\[
g_t(v) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h_t(r)e^{-ivr} \, dr, \quad v \in \mathbb{R},
\]

\[
q_t(e^v + e^{-v} - 2) = g_t(v), \quad v \in \mathbb{R},
\]

\[
\phi_0(t, u) = -\frac{1}{\pi} \int_{-\infty}^{\infty} q_t'(u + v^2) \, dv, \quad u \geq 0,
\]

\[
\phi_2(t, u) = -\frac{1}{\pi} \int_{-\infty}^{\infty} q_t'(u + v^2) \frac{\sqrt{u + 4 + v^2} - v}{\sqrt{u + 4 + v^2} + v} \, dv, \quad u \geq 0.
\]

Now consider the following functions:

\[
u_k(t, \gamma; z) = j_\gamma(z, k) \pi_k(t, u(z, \gamma z))
\]

where \( j_\gamma \) is the automorphic factor defined in (3.1).

The automorphic kernel \( K_k(t, z) \) of weight \( k \) with respect to \( \Gamma_0(p^2) \) is defined as

\[
K_0(t, z) = \frac{1}{2} \sum_{\gamma \in \Gamma_0(p^2)} \nu_0(t, \gamma; z) = \frac{1}{2} \sum_{\gamma \in \Gamma_0(p^2)} \phi_0(t, u(z, \gamma z)),
\]

\[
K_2(t, z) = \frac{1}{2} \sum_{\gamma \in \Gamma_0(p^2)} \nu_2(t, \gamma; z) = \frac{1}{2} \sum_{\gamma \in \Gamma_0(p^2)} j_\gamma(z, 2) H(z, \gamma z) \phi_2(t, u(z, \gamma z)).
\]

Also denote the corresponding summations over the elliptic, hyperbolic and parabolic elements of \( \Gamma_0(p^2) \) by \( E_k \), \( H_k \) and \( P_k \) respectively, i.e.,

\[
E_k(t, z) = \frac{1}{2} \sum_{\substack{\gamma \in \Gamma_0(p^2) \ \mid \ \text{tr} \gamma < 2}} \nu_k(t, \gamma; z),
\]

\[
H_k(t, z) = \frac{1}{2} \sum_{\substack{\gamma \in \Gamma_0(p^2) \ \mid \ \text{tr} \gamma > 2}} \nu_k(t, \gamma; z),
\]

\[
P_k(t, z) = \frac{1}{2} \sum_{\substack{\gamma \in \Gamma_0(p^2) \ \mid \ \text{tr} \gamma = 2}} \nu_k(t, \gamma; z).
\]

4.2. Trace formula. Recall that the hyperbolic laplacian \( \Delta_k \) defined in (3.2) acts as a positive self-adjoint operator on \( L^2(\Gamma_0(p^2) \backslash \mathbb{H}, k) \)—the space of square integrable automorphic forms of weight \( k \). Both these operators have the same discrete spectrum [26] say

\[
0 = \lambda_0 < \lambda_1 \leq \lambda_2 \cdots
\]
We write $\lambda_j = 1/4 + r_j^2$ where $r_j$ is real or purely imaginary and let $\{u_j\}$ be an orthonormal basis of eigenspaces $L^2_{\lambda_j} (\Gamma_0(p^2) \backslash \mathbb{H}, k)$, $k = 0, 2$, corresponding to each eigenvalue $\lambda_j \neq 0$.

Via

$$E \left\{ \sum_{P \in \partial(X_0(p^2))} \int_{-\infty}^{\infty} h_t(r) \left| E_{P,0} \left( z, \frac{1}{2} + ir \right) \right|^2 \right| dr, $$

we obtain the identity

$$g_{p^2} F(z) + D(t, z) + C(t, z).$$

Then from Theorem 10 we have

$$K_2(t, z) - K_0(t, z) = g_{p^2} F(z) + (D_2(t, z) - D_0(t, z)) + (C_2(t, z) - C_0(t, z))$$

$$= g_{p^2} F(z) + D(t, z) + C(t, z).$$

On the other hand from (4.3)

$$K_2(t, z) - K_0(t, z) = (E_2(t, z) - E_0(t, z)) + (H_2(t, z) - H_0(t, z)) + (P_2(t, z) - P_0(t, z))$$

$$= E(t, z) + H(t, z) + P(t, z).$$

Combining these two we obtain the identity

$$g_{p^2} F(z) + D(t, z) + C(t, z) = H(t, z) + P(t, z) + E(t, z),$$

from which upon integration with respect to $E_\infty(z, s) \mu_{\text{hyp}}$ over $Y_0(p^2)$ we obtain

$$g_{p^2} R_F(s) = -R_D(t, s) + R_H(t, s) + R_E(t, s) + R_{P-C}(t, s),$$

as mentioned in the beginning of this section.

**Proposition 11.** We have

(i) $R_D(t, s)$ is holomorphic near $s = 1$ for all $t$ and

$$R^\text{dis}_0(t) = R_D(t, 1) \to 0$$

as $t \to \infty$. 

**Theorem 10.** [15, (6.11), p. 387] The spectral expansion of $K_k$ is given by

$$K_0(t, z) = \frac{1}{v_{\Gamma_0(p^2)}} + \sum_{j=1}^{\infty} h_t(r_j) |u_j(z)|^2 + \frac{1}{4\pi} \sum_{P \in \partial(X_0(p^2))} \int_{-\infty}^{\infty} h_t(r) \left| E_{P,0} \left( z, \frac{1}{2} + ir \right) \right|^2 dr,$$

$$K_2(t, z) = g_{p^2} F(z) + \sum_{j=1}^{\infty} \frac{h_t(r_j)}{\lambda_j} |\Lambda_0 u_j(z)|^2 + \frac{1}{4\pi} \sum_{P \in \partial(X_0(p^2))} \int_{-\infty}^{\infty} h_t(r) \left| E_{P,2} \left( z, \frac{1}{2} + ir \right) \right|^2 dr.$$
Then observe that
\[
R_{H}(t, s) \text{ is holomorphic near } s = 1 \text{ and }
\]
\[
R_{0,\text{hyp}}^{\text{hyp}}(t) = R_{H}(t, 1) = \mathcal{E}_{\text{hyp}}^{\text{hyp}}(t) - \frac{t - 1}{2\nu_{0}(p^{2})}
\]
where \( \lim_{t \to \infty} \mathcal{E}_{\text{hyp}}^{\text{hyp}}(t) = \frac{1}{2\nu_{0}(p^{2})}O_{c}(p^{2c}) \).

(iii) \( R_{E}(t, s) \) is of the form
\[
R_{E}(t, s) = \frac{R_{0,\text{ell}}}{s - 1} + R_{0,\text{ell}}^{\text{ell}}(t) + O(s - 1)
\]
near \( s = 1 \), where \( R_{0,\text{ell}}^{\text{ell}}(t) \) and \( R_{0,\text{ell}}^{\text{ell}}(t) \) have finite limits as \( t \to \infty \), and \( R_{0,\text{ell}}^{\text{ell}} = \lim_{t \to \infty} R_{0,\text{ell}}^{\text{ell}}(t) = o(\log(p)) \).

(iv) \( R_{P-S}(t, s) \) is given by
\[
R_{P-S}(t, s) = \frac{R_{0,\text{par}}^{\text{par}}(t)}{s - 1} + R_{0,\text{par}}^{\text{par}}(t) + O(s - 1).
\]
where \( R_{0,\text{par}}^{\text{par}}(t) = \frac{\ell + 1}{2\nu_{0}(p^{2})} + \mathcal{E}_{\text{par}}(t) \) with \( \lim_{t \to \infty} \mathcal{E}_{\text{par}}(t) = \frac{1}{4\pi} \log(p) + O\left(\log(p)\right) \).

The proof of this proposition will be given in the next section. In the rest of this section, we simplify the calculation of the Rankin-Selberg transforms of various terms above. To this end we introduce
\[
F_{k}^{l}(t, z) = \sum_{\gamma \in \Gamma_{0}(p^{2})} \nu_{k}(t, \gamma; z),
\]
\[
R_{k}^{l}(t, s) = \int_{\Gamma_{0}(p^{2})} E_{\infty}(z, s) F_{k}^{l}(t, z) d\mu_{\text{hyp}}(z).
\]
Then observe that \( R_{H}, R_{E} \) and \( R_{P} \) can be obtained by summing up \( R_{k}^{l} - R_{0}^{l} \) respectively over \( |l| > 2, < 2 \) and \( = 2 \). We will compute \( R_{k}^{l} \) by exploiting a connection between Epstein zeta function and Eisenstein series.

4.4. Epstein Zeta function and non-holomorphic Eisenstein series. We recall a connection between quadratic forms and elements of \( \Gamma_{0}(N) \). For \( a, b, c \in \mathbb{Z} \), let us denote by \( [a, b, c] \) the quadratic form
\[
\Phi(X, Y) = aX^{2} + bXY + cY^{2} = (X, Y) \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} (X, Y)^{t}.
\]
The discriminant of \( \Phi \) is by definition \( \text{dis} \Phi = b^{2} - 4ac \). For any integer \( l \) with \( |l| \neq 2 \), define
\[
Q_{l} = \{ \Phi : \text{dis} \Phi = l^{2} - 4 \},
\]
\[
Q_{l}(N) = \{ \Phi \in Q_{l} : \Phi = [aN, b, c] ; a, b, c \in \mathbb{Z} \}.
\]
The full modular group \( SL_{2}(\mathbb{Z}) \) acts on \( Q_{l} \) by
\[
SL_{2}(\mathbb{Z}) \times Q_{l} \to Q_{l}
\]
\[
(\delta, \Phi) \mapsto \Phi \circ \delta
\]
where \( \Phi \circ \delta(X, Y) = \Phi((X, Y) \delta)^{t} \). If \( \delta = \begin{pmatrix} x & y \\ z & t \end{pmatrix} \), then it can be shown that
\[
\Phi \circ \delta = [\Phi(x, z), b(xt + yz) + 2(axy + czt), \Phi(y, t)].
\]
Note that the above defines an action of $\Gamma_0(N)$ on $Q_l(N)$. It is worth mentioning that

**Proposition 12.** Let $\Phi \in Q_l$. If $|l| < 2$ then $\text{SL}_2(\mathbb{Z})_\Phi$ is finite and if $|l| > 2$ then

$$\text{SL}_2(\mathbb{Z})_\Phi = \{\pm M^n : n \in \mathbb{Z}\}$$

for a unique $M \in \text{SL}_2(\mathbb{Z})$ with positive trace. Moreover, if $\Phi \in Q_l(N)$, then

$$\Gamma_0(N)_\Phi = \text{SL}_2(\mathbb{Z})_\Phi.$$  

**Proof.** Let $\Phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\Delta = \text{dis} \Phi$. If $(x, y) \in \mathbb{Z}^2$ is a solution of the Pell’s equation $P_\Delta : x^2 - \Delta y^2 = 4$, then the matrix

$$U_\Phi(x, y) = \begin{pmatrix} xy - b & -cy \\ ay + bx & cy \end{pmatrix}$$

is an automorphism of $\Phi$ and all the automorphisms of $\Phi$ are of this form. Since $\Phi \in Q_l(N)$, we have $U_\Phi \in \Gamma_0(N)$ and hence $\text{SL}_2(\mathbb{Z})_\Phi = \Gamma_0(N)_\Phi$. \qed

For a quadratic form $\Phi \in Q_l$ with $|l| > 2$, we define its fundamental unit to be the largest eigenvalue of $M$ where $M$ is as in the proposition and denote it by $\epsilon_\Phi$.

On the other hand, let $\Gamma_0(N)_l = \{\gamma \in \Gamma_0(N) : \text{tr} \gamma = l\}$. Then $\text{SL}_2(\mathbb{Z})$ acts on this set by conjugation

$$\text{SL}_2(\mathbb{Z}) \times \Gamma_0(N)_l \to \Gamma_0(N)_l$$

$$(\delta, \gamma) \mapsto \delta^{-1} \gamma \delta.$$ 

Note that $\Gamma_0(N)_l$ also, as a subgroup of $\text{SL}_2(\mathbb{Z})$ acts on $\Gamma_0(N)_l$.

There is a $\Gamma_0(N)$ equivariant one-one correspondence between $\Gamma_0(N)_l$ and $Q_l(N)$ given by

$$\psi : \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \Phi_\gamma = [Nc, d - a, -b],$$

with inverse

$$\psi' : \Phi = [aN, b, c] \mapsto \gamma_\Phi = \begin{pmatrix} b & -c \\ aN & c \end{pmatrix}.$$ 

is the inverse of $\psi$. We observe that this correspondence induces a bijection

$$Q_l(N)/\Gamma_0(N) \cong \Gamma_0(N)_l/\Gamma_0(N).$$

Fix $\Phi \in Q_l$. Define

$$\Phi \cdot (m, n) = \Phi(n, -m), \quad (m, n) \in \mathbb{Z}^2,$$

$$M^\Phi = \{(m, n) \in \mathbb{Z}^2 : \Phi \cdot (m, n) > 0\}.$$ 

Observe that $\text{SL}_2(\mathbb{Z})_\Phi$ acts on $M^\Phi$ as

$$(\Phi \circ \delta) \cdot (m, n) \delta = \Phi \cdot (m, n)$$

**Definition 13.** The Epstein Zeta function associated to the quadratic form $\Phi$ is defined by

$$\zeta_\Phi(s) = \sum_{(m, n) \in M^\Phi/\text{SL}_2(\mathbb{Z})_\Phi} \frac{1}{\Phi \cdot (m, n)^s}, \quad \text{Re } s > 1.$$
The series converges absolutely for $\Re s > 1$ and defines a holomorphic function. It has a meromorphic extension to the entire complex plane and has a simple pole at 1 with residue

\begin{equation}
\text{Res}_{s=1} \zeta_{\Phi}(s) = \begin{cases} 
\frac{2\pi}{\sqrt{|\text{dis } \Phi|}} \frac{1}{|\text{SL}_2(\mathbb{Z})\Phi|} & \text{dis } \Phi < 0, \\
\frac{1}{\sqrt{|\text{dis } \Phi|}} \log(\epsilon_{\Phi}) & \text{dis } \Phi > 0,
\end{cases}
\end{equation}

where $\epsilon_{\Phi}$ is the fundamental unit of $\Phi$ (see [1, §3.2.2]) .

Now let $\Phi \in \mathbb{Q}_l(N)$ and $d \mid N$. Set

\[ M_d = \{(Nm, dn) \in \mathbb{Z}^2 \setminus \{0,0\}\}, \]
\[ M_{\Phi}^d = \{(Nm, dn) : \Phi \cdot (Nm, dn) > 0\}. \]

Note that $\text{SL}_2(\mathbb{Z})\Phi = \Gamma_0(N)\Phi$ acts on $M_{\Phi}^d$ by (4.11).

**Definition 14.** Let $\Phi \in \mathbb{Q}_l(N)$ and $d \mid N$. Define

\[ \zeta_{\Phi, d}(s) = \sum_{(m,n) \in M_{\Phi}^d / \text{SL}_2(\mathbb{Z})\Phi} \frac{1}{\Phi \cdot (m,n)^s}. \]

The residue of $\zeta_{\Phi, d}(s)$ can be calculated by expressing it in terms of a certain Epstein zeta function as in [1]. Indeed, consider the group homomorphism $*d : \Gamma_0(N) \to \text{SL}_2(\mathbb{Z})$ defined by

\[ \gamma = \begin{pmatrix} x & y \\
Nz & t \end{pmatrix} \rightarrow \gamma^{*d} = \begin{pmatrix} x & \frac{N}{d} y \\
z & t \end{pmatrix} \]

and note that this map induces the injection $*d : \mathbb{Q}_l(N) \to \mathbb{Q}_l$. Thus denoting the image under $*d$ by a superscript,

\[ (\Gamma_0(N)\Phi)^{*d} = \Gamma_0(d)_{\Phi^{*d}}. \]

Also $\Phi \cdot (Nm, dn) = (Nd)^{*d} \Phi^{*d} \cdot (m, n)$ and thus

\begin{equation}
\zeta_{\Phi, d}(s) = \sum_{(m,n) \in M_{\Phi}^d / \Gamma_0(N)\Phi} \frac{1}{\Phi \cdot (m,n)^s} = \sum_{(m,n) \in M_{\Phi^{*d}} / \Gamma_0(d)_{\Phi^{*d}}} \frac{1}{(Nd)^{*d} \Phi^{*d} \cdot (m,n)} = \frac{1}{(Nd)^{*d}} [\text{SL}_2(\mathbb{Z})_{\Phi^{*d}} : \Gamma_0(d)_{\Phi^{*d}}] \zeta_{\Phi^{*d}}(s).
\end{equation}

By Proposition 12, we have $[\text{SL}_2(\mathbb{Z})_{\Phi^{*d}} : \Gamma_0(d)_{\Phi^{*d}}] = 1$. It follows from (4.12) that for $\text{dis } \Phi > 0$ and hence,

\begin{equation}
\text{Res}_{s=1} \zeta_{\Phi, d}(s) = \frac{\log \epsilon_{\Phi^{*d}}}{Nd \sqrt{\text{dis } \Phi^{*d}}} = \frac{\log \epsilon_{\Phi}}{Nd \sqrt{\text{dis } \Phi}}.
\end{equation}
where the last equality is a consequence of the fact that \( *d \) preserves trace and of course determinant, and for \( \text{dis} \Phi < 0 \),

\begin{equation}
(4.15) \quad \text{Res}_{s=1} \zeta_{\Phi,d}(s) = \frac{2\pi}{Nd\sqrt{|\text{dis} \Phi|\text{det} \SL_2(\mathbb{Z})}} = \frac{2\pi}{Nd\sqrt{|\text{det} \Phi|\text{det} \SL_2(\mathbb{Z})}}.
\end{equation}

**Proposition 15.** We have

\begin{equation}
E_{\infty,0}(z, s) = \frac{1}{2} \frac{1}{\zeta(2s)} \left[ \sum_{(m,n)} \frac{y^s}{|p^2mz + n|^{2s}} - \sum_{(m,n)} \frac{y^s}{|p^2mz + pn|^{2s}} \right].
\end{equation}

**Proof.** Replacing \( (m, n) \) by \((dm, dn)\) with \( (m, n) = 1 \), the above sum is the same as

\begin{equation}
2(1 - p^{-2s}) E_{\infty,0}(z, s) = \sum_{(m,n)=1} \frac{y^s}{|p^2mz + n|^{2s}} - \sum_{(m,n)=1} \frac{y^s}{|p^2mz + pn|^{2s}}.
\end{equation}

The left hand side of (4.16) is equal to

\[
2(1 - p^{-2s}) \sum_{\gamma \in \Gamma_0(p^2) \setminus \Gamma_0(p^2)} \text{Im}(\gamma z)^s = 2(1 - p^{-2s}) \frac{1}{2} \sum_{(m,n)=1, \atop m \equiv 0(p^2)} \frac{y^s}{|mz + n|^{2s}}
\]

\[
= \sum_{(m,n)=1, \atop m \equiv 0(p^2)} \frac{y^s}{|mz + n|^{2s}} - \sum_{(m,n)=1, \atop m \equiv 0(p^2)} \frac{y^s}{|pmz + pn|^{2s}}
\]

\[
= \sum_{(p^2m,n)=1} \frac{y^s}{|p^2mz + n|^{2s}} - \sum_{(p^2m,n)=1} \frac{y^s}{|p^3mz + pn|^{2s}}
\]

\[
= A + B.
\]

The first term in the right hand side of (4.16) is equal to

\[
\sum_{(m,n)=1, \atop m \equiv 0(p^2)} \frac{y^s}{|p^2mz + n|^{2s}} + \sum_{(m,n)=1, \atop m \equiv 0(p^2)} \frac{y^s}{|p^2mz + pn|^{2s}}
\]

\[
= \sum_{(m,n)=1, \atop (m,n)=1} \frac{y^s}{|p^2mz + n|^{2s}} + \sum_{(m,n)=1} \frac{y^s}{|p^2mz + pn|^{2s}},
\]

and the second term in the right hand side of (4.16) is equal to

\[
\sum_{(m,n)=1, \atop (m,n)=1} \frac{y^s}{|p^2mz + pn|^{2s}} + \sum_{(m,n)=1, \atop (m,n)=1} \frac{y^s}{|p^3mz + pn|^{2s}}
\]

\[
= \sum_{(m,n)=1, \atop (m,n)=1} \frac{y^s}{|p^2mz + pn|^{2s}} + \sum_{(m,n)=1, \atop (m,n)=1} \frac{y^s}{|p^3mz + pn|^{2s}}.
\]
Therefore the right hand side of (4.16) is
\[
\sum_{(m,n)=1,p|m} \frac{y^s}{|p^2mz+n|^{2s}} - \sum_{(pm,n)=1,(m,n)=1} \frac{y^s}{|p^2mz + pn|^{2s}} = A' - B'.
\]
To complete the proof, we only need to observe that \((p^2m, n) = 1\) if and only if \((m, n) = 1\) and \(p \nmid n\), so that \(A = A'\) and similarly \((p^2m, n) = 1\) if and only if \((pm, n) = 1\) and \((m, n) = 1\), so that \(B = B'\).

Lemma 16. Let \(\gamma \in \Gamma_0(p^2)\) and suppose \(\Phi_\gamma \cdot (m, n) > 0\). Then
\[
\int \nu_k(t, \gamma, z) \frac{y^s}{|mz+n|^{2s}} \mu_{\text{hyp}}(z) = \frac{1}{\Phi_\gamma \cdot (m, n)^s} \int \nu_k(t, \gamma_{\pm l}, z) y^s \mu_{\text{hyp}}(z),
\]
where \(\gamma_{\pm l} = \left(\pm \frac{1}{2} \frac{z}{2^\pm 1} \right)\).

Proof. Let \(\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\). Then recall that \(\Phi_\gamma = [c, d - a, -b]\). Define
\[
T = \frac{1}{\Phi_\gamma \cdot (m, n)^{1/2}} \left( \begin{array}{cc} n & -(d-a)n^2 - bm \\ m & cn - (d-a)m^2 \end{array} \right).
\]
Then \(T \in SL_2(\mathbb{R})\), \(T^{-1} \gamma T = \gamma_l\) and
\[
\text{Im} Tw = \frac{\text{Im} w}{|mTw + n|^2} \frac{\text{Im} w}{\Phi_\gamma \cdot (m, n)^s}.
\]
Thus the equality of the integrals follows immediately once we substitute \(z = Tw\) and observe that \(\nu_k(t, \gamma, Tw) = \nu_k(t, T^{-1} \gamma T, w)\).

Definition 17. Let
\[
\zeta_{\Gamma_0(p^2)}(s, l) = \frac{1}{2\zeta(2s)(1-p^{-2s})} \sum_{d \in \{1, p\}} \mu(d) \sum_{\Phi \in \Gamma_0(p^2)/\Gamma_0(p^2)} \zeta_{\Phi, d}(s)
\]
where \(\mu\) is the Möbius function.

Proposition 18. We have
\[
R_k^l(t, s) = \zeta_{\Gamma_0(p^2)}(s, l) I_k(t, s, l),
\]
where \(I_k^\pm(t, s, l) = \int \nu_k(t, \gamma_{\pm l}, z) y^s \mu_{\text{hyp}}(z)\), and \(I_k(t, s, l) = I_k^+(t, s, l) - I_k^-(t, s, l)\).

Proof. From Proposition 15
\[
E_{\infty}(z, s) = \frac{1}{2\zeta(2s)(1-p^{-2s})} \sum_{d \in \{1, p\}} \mu(d) \sum_{(m,n) \in M_d} \frac{y^s}{|mz+n|^{2s}}.
\]
From (4.9) now
\[
R_k^l(s) = \frac{1}{2\zeta(2s)(1-p^{-2s})} \sum_{d \in \{1, p\}} \mu(d) \int_{\Gamma_0(p^2)} \sum_{(m,n) \in M_d} \nu_k(t, \gamma, z) \frac{y^s}{|mz+n|^{2s}} \mu_{\text{hyp}}(z).
\]
We now break the summations inside the integral into two parts as follows: Let
\[
S_d^\pm(l) = \left\{ (\Phi, (m, n)) : \Phi \in \Gamma_0(p^2), (m, n) \in M_d^{\pm \delta} \right\}.
\]
Then identifying $\gamma_\Phi$ with $\gamma$, and letting $\sigma_d(t, z) = \sum_{S^d} \nu_k(t, \gamma, z) \frac{y^s}{|mz + n||2s|}$, we have

$$\sum_{\text{tr} \gamma = l} \sum_{(m, n) \in M_d} \nu_k(t, \gamma, z) \frac{y^s}{|mz + n||2s|} = \sigma^+_d(t, z) + \sigma^-_d(t, z).$$

Now, $\Gamma_0(p^2)$ acts freely on $S^+_d$ component wise:

$$\alpha \cdot (\Phi, (m, n)) = (\Phi \circ \alpha, (m, n)\alpha),$$

and recalling that $\Phi \circ \alpha$ corresponds to $\alpha^{-1}\gamma\alpha$, and writing $(m_{\alpha}, n_{\alpha}) = (m, n)\alpha$ we have

$$\sigma^\pm_d(t, z) = \sum_{(\Phi, (m, n)) \in S^\pm_d/\Gamma_0(p^2)} \sum_{\alpha \in \Gamma_0(p^2)} \nu_k(t, \alpha^{-1}\gamma\alpha, z) \frac{y^s}{|m_{\alpha}z + n_{\alpha}||2s|}.$$

Therefore,

$$\int_{Y_0(p^2)} \sigma^+_d(t, z) \mu_{\text{hyp}}(z) = \sum_{(\Phi, (m, n)) \in S^+_d/\Gamma_0(p^2)} \int_{\mathbb{H}} \nu_k(t, \gamma, z) \frac{y^s}{|mz + n||2s|} \mu_{\text{hyp}}(z)$$

$$= \sum_{\Phi \in Q_l(p^2)/\Gamma_0(p^2)} \sum_{(m, n) \in M^d/\Gamma_0(p^2)} \frac{1}{\Phi\gamma \cdot (m, n)^s} \int_{\mathbb{H}} \nu_k(t, \gamma, z) y^s \mu_{\text{hyp}}(z)$$

$$= I^+_e(t, s, l) \zeta_{\Gamma_0(p^2), d}(s, l)$$

using

$$S^\pm_d/\Gamma_0(p^2) = \left\{ (\Phi, (m, n)) : \Phi \in Q_l(p^2)/\Gamma_0(p^2), (m, n) \in M^\pm_d/\Gamma_0(p^2) \right\}$$

and the previous lemma.

Finally, observing that $-\Phi_\gamma = \Phi_{-\gamma}$, and $S^-_d(l) = S^+_d(-l)$, we obtain

$$\int_{Y_0(p^2)} \sigma^-_d(t, z) \mu_{\text{hyp}}(z) = I^-_e(t, s, l) \zeta_{\Gamma_0(p^2), d}(s, l)$$

and the proposition now follows.

5. Constant term of Trace Formula

This section is devoted to proving Proposition 11 which then is used to get an expression for $R_{\infty}^{\Gamma_0(p^2)}$. This is then used to derive an asymptotic expression of $g_{\text{can}}(0, \infty)$ in terms of $p$. 

□
5.1. Hyperbolic Contribution.

**Proposition 19.** $R_H(t, s)$ is holomorphic near $s = 1$ and

$$R_H(t, 1) = -\frac{1}{2v\Gamma_0(p^2)} \int_0^t \Theta(\xi) \, d\xi$$

where $\Theta(\xi)$ is the theta function for $X_0(p^2)$ as defined in [24].

**Proof.** From Proposition 18,

$$R_H(t, s) = \sum_{|l| > 2} \zeta_{\Gamma_0(p^2)}(s, l) (I_2(t, s, l) - I_0(t, s, l)).$$

Observe that from (4.14)

$$\text{Res}_{s=1} \zeta_{\Gamma_0(p^2)}(s, l) = \frac{1}{\pi v\Gamma_0(p^2)} \sum_{\Phi \in Q_l(p^2) \setminus \Gamma_0(p^2)} \frac{\log \epsilon \Phi}{\sqrt{\epsilon^2 - 4}}.$$

Also, from [1, Propositions 3.3.2, 3.3.3], for $\text{Re } s < 1 + \delta$, where $\delta > 0$ is some constant,

$$I_2(t, s, l) - I_0(t, s, l) = \frac{C-1(t)}{(s-1)} + A_l(t) + O((s-1))$$

where

$$A_l(t) = -\frac{\pi}{2} \int_0^t \frac{1}{\sqrt{4\pi \xi}} e^{-\frac{\epsilon^2 + (\log n_l^2)^2}{4\xi}} \, d\xi$$

and $n_l = (l + \sqrt{l^2 - 4})/2$ is the larger eigenvalue of $\gamma_l$. It follows that $R_H(t, s)$ is holomorphic near $s = 1$ and

$$R_H(t, 1) = -\frac{1}{2v\Gamma_0(p^2)} \int_0^t \Theta_{\Gamma_0(p^2)}(\xi) \, d\xi$$

where

$$\Theta_{\Gamma_0(p^2)}(\xi) = \sum_{|l| > 2} \sum_{\Phi \in Q_l(p^2) \setminus \Gamma_0(p^2)} \frac{\log \epsilon \Phi}{\sqrt{\epsilon^2 - 4}} \frac{1}{\sqrt{4\pi \xi}} e^{-\frac{\epsilon^2 + (\log n_l^2)^2}{4\xi}}.$$

We note that this function is exactly equal to the one defined in [1, Page 54] as under the correspondence $\Phi \to \gamma_\Phi$, $N(\gamma_0) = \epsilon_0^2$ and $N(\gamma) = n_l^2$. □

**Proof of Proposition 11 (ii).** By the above proposition, we can write

$$R_{hyp}^0(t) = R_H(t, 1) = E_{hyp}(t) = \frac{t - 1}{2v\Gamma_0(p^2)}$$

where

$$E_{hyp}(t) = -\frac{1}{2v\Gamma_0(p^2)} \left( \int_0^t (\Theta(\xi) - 1) \, d\xi + 1 \right).$$

We note that from [1, Lemma 3.3.6]

$$\int_0^\infty (\Theta(\xi) - 1) \, d\xi = \lim_{s \to 1} \left( \frac{Z_{\Gamma_0(p^2)}(s)}{Z_{\Gamma_0(p^2)}(s)} - \frac{1}{s-1} \right) - 1$$

(5.2)
where $Z$ is the Selberg’s zeta function for $X_0(p^2)$ and from [17, p. 27]

\begin{equation}
\lim_{s \to 1} \left( \frac{Z'_{\Gamma_0(p^2)}}{Z_{\Gamma_0(p^2)}} - \frac{1}{s-1} \right) = O_e(p^{2r}).
\end{equation}

(5.3)

It follows that $\lim_{t \to \infty} E_{hyp}(t) = \frac{1}{2 \pi i \zeta_0(p^2)} O_e(p^{2r})$ as required. \hfill \square

5.2. Elliptic Contribution.

**Proof of Proposition 11 (iii).** The elliptic contribution is obtained by putting $l = 0, 1, -1$ in the Proposition 18. We have:

$$R_E(t, s) = \sum_{l = 0, 1, -1} (I_2(t, s, l) - I_0(t, s, l)) \zeta_{\Gamma_0(p^2)}(s, l).$$

Let

$$\zeta_{\Gamma_0(p^2)}(s, l) = \frac{a_{-1}(l)}{s-1} + a_0(l) + O(s-1),$$

$$I_2(t, s, l) - I_0(t, s, l) = b_0(t, l) + b_1(t, l)(s-1) + O((s-1)^2).$$

Then

$$R_{E, l}^{-1}(t) = \sum_{l = -1}^1 a_{-1}(l)b_0(t, l),$$

$$R_{E, 0}^{\text{ell}}(t) = \sum_{l = -1}^1 a_0(l)b_0(t, l) + a_{-1}(l)b_1(t, l).$$

Note that $b_i(t, l)$ differ from $C_{i,l}(t)$ of [1, §3.3.3, p 57] by a multiplicative function that does not depend on $t$ and therefore $\lim_{t \to \infty} b_i(t, l)$ exists, say $b_i(\infty, l)$. Also

(5.4) \quad |Q_i(p^2) \setminus \Gamma_0(p^2)| \leq |Q_i| |SL_2(\mathbb{Z})| |SL_2(\mathbb{Z}) : \Gamma_0(p^2)| \leq b_i(1) |SL_2(\mathbb{Z}) : \Gamma_0(p^2)|

and thus $a_{-1}(l) = o(1)$.

From Definition 17 and (4.13)

$$\zeta_{\Gamma_0(p^2)}(s, l) = \frac{1}{2\zeta(2s)(1 - p^{-2s})} \sum_{d \mid (1, p)} \mu(d) \frac{1}{(p^2d)^s} \sum_{\Phi \in \Gamma_0(p^2)/\Gamma_0(p^2)} \zeta_{\Phi^{*d}}(s)$$

$$= \frac{1}{2\zeta(2s)(1 - p^{-2s})} \frac{1}{p^{2s}} \sum_{\Phi \in \Gamma_0(p^2)/\Gamma_0(p^2)} \zeta_{\Phi^{*1}}(s) - \frac{1}{p^s} \sum_{\Phi \in \Gamma_0(p^2)/\Gamma_0(p^2)} \zeta_{\Phi^{*p}}(s)$$

$$= I_1(s) I_2(s)$$

with

$$I_1(s) = \frac{1}{2\zeta(2s)(1 - p^{-2s})} \frac{1}{p^{2s}}$$

and

$$I_2(s) = \sum_{\Phi \in \Gamma_0(p^2)/\Gamma_0(p^2)} \zeta_{\Phi^{*1}}(s) - \frac{1}{p^s} \sum_{\Phi \in \Gamma_0(p^2)/\Gamma_0(p^2)} \zeta_{\Phi^{*p}}(s).$$

By [13, p. 78], we note that $\zeta_{\Phi^{*p}}(s, l) = \zeta(s, A)$, the Zeta function associated to narrow ideal class $A$. That $\zeta_{\Phi^{*d}}(s)$ can be written as Linear combination of $L$ functions for imaginary quadratic fields $\mathbb{Q}[\sqrt{-d}]$.
with \(d \in \{1, 3\}\) follows from [29, p.108, (18)]. Since for the imaginary quadratic fields \(\mathbb{Q}[\sqrt{-d}]\) with \(d \in \{1, 3\}\), the class number is 1, \(\zeta(s, A) = \zeta_K(s)\), the Dedekind Zeta function associated to the suitable imaginary quadratic field \(K\). For \(d \in \{1p\}\), if

\[
J_d(s) = \sum_{\Phi \in Q_i(p^2)/\Gamma_0(p^2)} \zeta_{\Phi^*d}(s, l) = \frac{c_{-1,d}}{s-1} + c_{0,d} + O((s - 1))
\]

then \(|c_{i,d}| \leq \text{const. } h_l(1)|\text{SL}_2(\mathbb{Z}) : \Gamma_0(p^2)|\) where the constant is independent of \(p\). Now

\[
I_2(s) = J_1(s) - \frac{1}{p^s} J_p(s) = \frac{c_{-1,1}}{s-1} + c_{0,1} + O(s - 1) - \frac{1}{p^s} \left( \frac{c_{-1,1}}{s-1} + c_{0,1} + O(s - 1) \right)
\]

\[
= \frac{c_{-1,1}}{s-1} + c_{0,0} + O(s - 1) - \left( \frac{1}{p} - \log p \right) \left( \frac{c_{-1,1}}{s-1} + c_{0,1} + O(s - 1) \right)
\]

\[
= \frac{1}{s-1} \left( c_{-1,1} - \frac{c_{-1,1}}{p} \right) + \left( c_{0,1} - \frac{c_{0,1}}{p} + \log p \frac{c_{-1,1}}{p} + O(s - 1) \right)
\]

\[
= \frac{A_{-1}(p)}{s-1} + A_0(p) + O(s - 1)
\]

with \(A_{-1}(p) = c_{-1,1} - \frac{c_{-1,1}}{p}\) and \(A_0(p) = c_{0,1} - \frac{c_{0,1}}{p} + \log p \frac{c_{-1,1}}{p}\). Also, it is evident that

\[
\frac{1}{\zeta(2s)(1-p^{-2s})} \frac{1}{p^{2s}} = \frac{3}{\pi^2(p^2 - 1)} + \left( \frac{D_1 \log(p) + D_2}{p^2 - 1} - \frac{6 \log(p)}{\pi^2(p^2 - 1)} \right) (s - 1) + O(s - 1)^2
\]

where the constants \(D_1\) and \(D_2\) are independent of \(p\). We conclude that

\[
\zeta_{\Gamma_0(p^2)}(s, l) = \frac{1}{\zeta(2s)(1-p^{-2s})} \sum_{d \in \{1, p\}} \mu(d) \frac{1}{(p^2d)^s} \sum_{\Phi \in Q_i(p^2)/\Gamma_0(p^2)} \zeta_{\Phi^*d}(s) = I_1(s) I_2(s)
\]

\[
= \frac{A_{-1}(p)}{s-1} + A_0(p) + O(s - 1) \left( \frac{3}{\pi^2(p^2 - 1)} + \left( \frac{D_1 \log(p) + D_2}{p^2 - 1} - \frac{6 \log(p)}{\pi^2(p^2 - 1)} \right) (s - 1) + O(s - 1)^2 \right)
\]

\[
= \frac{3A_{-1}(p)}{\pi^2(p^2 - 1)} \frac{1}{p^{2s}} + \frac{3A_0(p)}{\pi^2(p^2 - 1)} \left( \frac{D_1 \log(p) + D_2}{p^2 - 1} - \frac{6 \log(p)}{\pi^2(p^2 - 1)} \right) A_{-1}(p) + O((s - 1)).
\]

Hence, \(a_{-1}(l) = \frac{3A_{-1}(p)}{\pi^2(p^2 - 1)}\) and \(a_0(l) = \frac{3A_0(p)}{\pi^2(p^2 - 1)} + \left( \frac{D_1 \log(p) + D_2}{p^2 - 1} - \frac{6 \log(p)}{\pi^2(p^2 - 1)} \right) A_{-1}(p)\). It is evident from the above expression that all the terms have \(p^2 - 1\) in the denominator and it is of same order in \(p\) as \(h_l(1)|\text{SL}_2(\mathbb{Z}) : \Gamma_0(p^2)|\). Since the terms \(b_i's\) are independent of \(p\), we get \(R_{01}^p = o(\log(p))\). \(\square\)
5.3. Parabolic and Spectral contributions. Recall that we are interested in the term $R_{p-C}$ of (4.8). For $z \in \mathbb{H}$ define the functions:

$$p_1(t, y, k) = \frac{1}{2} \int_{-1/2}^{1/2} \sum_{\gamma \in \Gamma_0(p^2)} \nu_k(t, \gamma; z) dx,$$

$$p_2(t, y, k) = \frac{1}{2} \int_{-1/2}^{1/2} \gamma \in \Gamma_0(p^2) \sum_{\gamma \in \Gamma_0(p^2)} \nu_k(t, \gamma; z) dx - \frac{y}{2\pi} \int_{-\infty}^{\infty} h(t, r) dr,$$

$$p_3(t, y, k) = -\frac{y}{2\pi} \int_{-\infty}^{\infty} h(t, r) \phi_{\infty, \infty} \left( \frac{1}{2} - ir \right) \left( \frac{1}{2} + ir \right)^{k/2} y^{2ir} dr - \frac{2 - k}{2} \frac{1}{v_p^2},$$

$$p_4(t, y, k) = -\frac{1}{4\pi} \sum_{q \in \partial \chi_0 / (p^2)} \int_{-1/2}^{1/2} \int_{-\infty}^{\infty} h(t, r) \left| \tilde{E}_{q,k} \left( x + iy, \frac{1}{2} + ir \right) \right|^2 dv dx,$$

where

$$\tilde{E}_{q,k}(z, s) = E_{q,k}(z, s) - a_0 \left( y, \frac{1}{2} + ir; q\infty, k \right)$$

$a_0(y, s; q\infty, k)$ being the zero-th Fourier coefficient of $E_{q,k}(z, s)$, and is given by

$$a_0(y, s; q\infty, 0) = \delta_{q,\infty} y^s + \phi_{q,\infty}(s) y^{1-s},$$

$$a_0(y, s; q\infty, 2) = \delta_{q,\infty} y^s + \phi_{q,\infty}(s) \frac{1-s}{s} y^{1-s}.$$ 

Also consider the Melin transform

$$\mathcal{M}_j(t, s) = \int_0^{\infty} \left( p_j(t, y, 2) - p_j(t, y, 0) \right) y^{s-2} dy.$$ 

Then by [13, Lemma 4.4.1] we have for $q_p^2 > 1$

$$R_{p-S}(t, s) = \mathcal{M}_1(t, s) + \mathcal{M}_2(t, s) + \mathcal{M}_3(t, s) + \mathcal{M}_4(t, s)$$

for Re $s > 1$.

Let

$$\mathcal{B} = \Gamma_0(p^2) \infty = \{ \pm \left( \begin{array}{c} 1 & m \\ 0 & 1 \end{array} \right) | m \in \mathbb{Z} \}$$

Note that $\mathcal{B} \subset \Gamma_0(N)$ for any $N$. For any $\sigma \in \mathcal{B} \setminus \Gamma_0(p^2)$ with $\sigma = \left( \begin{array}{c} a & b \\ c & d \end{array} \right)$, we denote $c(\sigma) = c$. This is independent of the choice of the right coset representative. From [1, p. 37], any matrix $\gamma \in \Gamma_0(p^2)$ with trace 2 is of the form $\sigma^{-1} \left( \begin{array}{c} 1 & m \\ 0 & 1 \end{array} \right) \sigma$ where $k \in \mathbb{Z}$ and $\sigma \in \mathcal{B} \setminus SL_2(\mathbb{Z})$ are unique. Considering the possibilities (i) $p \nmid m$, (ii) $p | m$ but $p^2 \nmid m$, and $p^2 | m$, we see that

$$p_1(t, y, k) = q_1(t, y, k) + q_p(t, y, k) + q_p^2(t, y, k)$$
where for \( d \in \{1, p\} \) and \( k = 0, 2 \), we have
\[
q_d(t, y, k) = \frac{1}{2} \int_{-1/2}^{1/2} \sum_{m \neq 0, (m, p) = 1} \sum_{\sigma \in \mathcal{B}(\Gamma_0(p)), \sigma \neq \mathcal{B}} \nu_k(t, \sigma^{-1} \begin{pmatrix} \pm 1 & m \cr 0 & \pm 1 \end{pmatrix} \sigma; z) \, dx
\]
\[
= \int_{-1/2}^{1/2} \sum_{m \neq 0, (m, p) = 1} \sum_{\sigma \in \mathcal{B}(\Gamma_0(p^2)/\mathcal{B}, \sigma \neq \mathcal{B}} \sum_{n = -\infty}^{\infty} \nu_k(t, \sigma^{-1} \begin{pmatrix} \pm 1 & m \cr 0 & \pm 1 \end{pmatrix} \sigma; z + n) \, dx
\]
and
\[
q_{p^2}(t, y, k) = \frac{1}{2} \int_{-1/2}^{1/2} \sum_{m \neq 0} \sum_{\sigma \in \mathcal{B}(\Gamma_0(p^2)/\mathcal{B}, \sigma \neq \mathcal{B}} \nu_k(t, \sigma^{-1} \begin{pmatrix} \pm 1 & mp \cr 0 & \pm 1 \end{pmatrix} \sigma; z) \, dx
\]
\[
= \int_{-1/2}^{1/2} \sum_{m \neq 0} \sum_{\sigma \in \mathcal{B}(\Gamma_0(p^2)/\mathcal{B}, \sigma \neq \mathcal{B}} \sum_{n = -\infty}^{\infty} \nu_k(t, \sigma^{-1} \begin{pmatrix} \pm 1 & mp \cr 0 & \pm 1 \end{pmatrix} \sigma; z + n) \, dx.
\]

**Lemma 20.** For any \( \tau = \begin{pmatrix} a & b \\
\pm 1 & d \end{pmatrix} \in \text{SL}(2, \mathbb{R}) - \mathcal{B} \) with \( \text{tr} \, \tau = \pm 2 \), we have
\[
\int_{\mathbb{H}} \nu_k(t, \tau; z) \text{Im}(z)^s \mu_{\text{hyp}}(z) = \frac{1}{|c|} \int_{\mathbb{H}} \nu_k(t, L; z) \text{Im}(z)^s \mu_{\text{hyp}}(z).
\]
where \( L \pm = \begin{pmatrix} \pm 1 & 0 \\
1 & \pm 1 \end{pmatrix} \).

**Proof.** Substitute \( z = Mw \) where \( M = \frac{1}{\sqrt{c}} \begin{pmatrix} 1 & \frac{a + d}{c} \\
0 & c \end{pmatrix} \). \qed

For any positive integer \( M \), define:
\[
\mathcal{L}_M(s) = \sum_{\sigma \in \mathcal{B}(\Gamma_0(M)/\mathcal{B}, \sigma = \begin{pmatrix} \ast & \ast \\
\ast & \ast \end{pmatrix}, \sigma \neq \mathcal{B}} \frac{1}{|c|^{2s}}
\]
and
\[
\zeta_M(s) = \sum_{k \geq 1, (k, M) = 1} \frac{1}{k^s}.
\]

**Proposition 21.** We have
\[
\mathcal{M}_1(t, s) = \frac{p}{p - 1} \left( 1 - \frac{1}{p^{2s}} \right) \zeta(s) \mathcal{L}_p(s) [I_2(t, s, 2) - I_0(t, s, 2)]
\]
where for \( k = 0, 2 \)
\[
I_k(t, s, 2) = \int_{\mathbb{H}} \nu_k(t, L; z) (\text{Im} \, z)^s \mu_{\text{hyp}}(z) + \int_{\mathbb{H}} \nu_k(t, L; z) (\text{Im} \, z)^s \mu_{\text{hyp}}(z).
\]
Proof. By Lemma 20
\[
\int_0^\infty q_1(t, y, k) y^{s-2} dy = \sum_{m \neq 0} \sum_{\sigma \in B \setminus \Gamma_0(p)/B, \sigma \neq B} \int_{\mathbb{H}} \nu_k(t, \sigma^{-1} \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \sigma, z) (\text{Im } z)^s \mu_{\text{hyp}}
\]
\[
= \zeta_p(s) I_k(t, s, 2) \mathcal{L}_p(s)
\]
\[
\int_0^\infty q_p(t, y, k) y^{s-2} dy = \sum_{m \neq 0} \sum_{\sigma \in B \setminus \Gamma_0(p)/B, \sigma \neq B} \int_{\mathbb{H}} \nu_k(t, \sigma^{-1} \begin{pmatrix} 1 & mp \\ 0 & 1 \end{pmatrix} \sigma, z) (\text{Im } z)^s \mu_{\text{hyp}}
\]
\[
= \frac{\zeta_p(s)}{p^{2s}} I_k(t, s, 2) \mathcal{L}_1(s)
\]
Summing up and recalling the identity \( \zeta_p(s) = \zeta(s)(1 - p^{-s}) \), we obtain
\[
\mathcal{M}_1(t, s) = \left[ \left( 1 - \frac{1}{p^{2s}} \right) \mathcal{L}_p(s) + \frac{1}{p^{2s}} \mathcal{L}_1(s) \right] \zeta(s) [I_2(t, s, 2) - I_0(t, s, 2)].
\]
Now by [16, p.49] we have
\[
\mathcal{L}_1(s) = \frac{\zeta(2s - 1)}{\zeta(2s)}.
\]
By the proof of Lemma 3.2.19 in [1] we have
\[
\phi_{\Gamma_0(p), \infty}(s) = \sqrt{\pi} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} \frac{p - 1}{p^{2s} - 1} \frac{\zeta(2s - 1)}{\zeta(2s)},
\]
and from [1, p.56]
\[
(5.6) \quad \phi_{\Gamma_0(p), \infty}(s) = \sqrt{\pi} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} \mathcal{L}_p(s).
\]
Comparing \( \mathcal{L}_1(s) = \frac{p^{2s} - 1}{p - 1} \mathcal{L}_p(s) \) and the result follows. \( \square \)

Proposition 22. The Laurent series expansion of \( \mathcal{M}_1(t, s) \) at \( s = 1 \) is given by
\[
\mathcal{M}_1(t, s) = \left( \frac{1}{\nu_{\Gamma_0(p)}} \frac{p + 1}{p} A_1(t) \right) \frac{1}{s - 1}
\]
\[
+ \frac{1}{\nu_{\Gamma_0(p)}} \frac{p + 1}{p} \left[ (3\gamma_{\text{EM}} + \frac{a\pi}{6} - 2\log p) A_1(t) + B_1(t) \right]
\]
\[
+ O(s - 1)
\]
where \( A_1 \) and \( B_1 \) are functions independent of \( p \), such that \( \lim_{t \to \infty} A_1(t) = 1/2 \) and \( B_1 \) has a finite limit as \( t \to \infty \), we call it \( B_1(\infty) \).
Proof. From Proposition 21 and (5.6)

\[ \mathcal{M}_1(t, s) = \frac{p}{p-1} \left( 1 - \frac{1}{p^2s} \right) \zeta(s) \phi_{\infty, \infty}^\Gamma(s) \left( \frac{\Gamma(s)}{\sqrt{\pi} \Gamma(s-\frac{1}{2})} [I_2(t, s, 2) - I_0(t, s, 2)] \right). \]

By [1, p.59] we have

\[ \phi_{\infty, \infty}^\Gamma(s) = \frac{1}{v_{\Gamma(p)}} \frac{1}{s-1} + \frac{1}{v_{\Gamma(p)}} \left[ 2\gamma_{\text{EM}} + a \frac{\pi}{6} - 2 \frac{p^2}{p^2-1} \log(p) \right] + O(s-1) \]

and

\[ \zeta(s) = \frac{1}{s-1} + \gamma_{\text{EM}} + O(s-1). \]

Also,

\[ \left( 1 - \frac{1}{p^2s} \right) = \left( 1 - \frac{1}{p^2} \right) + \frac{\log(p^2)}{p^2} (s-1) + O((s-1)^2). \]

By a verbatim generalization of [13, Lemma B.2.1] and [1, Proposition 3.3.4] we have:

\[ \frac{\Gamma(s)}{\sqrt{\pi} \Gamma(s-\frac{1}{2})} [I_2(t, s, 2) - I_0(t, s, 2)] = A_1(t)(s-1) + B_1(t)(s-1)^2 + O((s-1)^3). \]

where \( \lim_{t \to \infty} A_1(t) = 1/2. \) It also follows from [1, Lemma 3.3.10] that \( B_1 \) has a limit as \( t \to \infty. \) \qed

From [13, Lemma 4.4.7-9]

\begin{align*}
\mathcal{M}_2(t, s) &= \frac{1}{s-1} \left( A_2(t) + \frac{1}{4\pi} \right) + \left( 1 - \frac{\log(4\pi)}{4\pi} + \gamma_{\text{EM}} A_2(t) + B_2(t) \right) + O(s-1), \\
\mathcal{M}_3(t, s) &= \frac{1}{v_{\Gamma(p)^2}} \frac{1}{s-1} + \left( \frac{c_{\infty, \infty}^{\Gamma(p)}}{2} + \frac{t+1}{2v_{\Gamma(p)^2}} \right) + O(s-1) \tag{5.7} \\
\mathcal{M}_4(t, s) &= A_4(t) + O(s-1)
\end{align*}

where \( A_2, B_2 \) and \( A_4 \) depend only on \( t \) and tends to zero as \( t \to \infty. \) \( c_{\infty, \infty}^{\Gamma(p^2)} \) is the constant term of \( \phi_{\infty, \infty}^{\Gamma(p^2)} \) (see Lemma 5).

**Proof of Proposition 11 (iv).** Combining (5.5), Proposition 22 and (5.7) we get

\[ R_{p-S}(t, s) = \frac{R_{1-p}^{\text{par}}(t)}{s-1} + R_0^{\text{par}}(t) + O(s-1) \]

where

\[ R_{-1}^{\text{par}} = \frac{1}{4\pi} + \frac{1}{v_{\Gamma(p)^2}} \frac{p+1}{p} A_1(t) + A_2(t) \]

and

\[ R_0^{\text{par}} = \frac{1}{v_{\Gamma(p)}} \frac{p+1}{p} \left[ \left( 3\gamma_{\text{EM}} + \frac{a\pi}{6} - 2 \log(p) \right) A_1(t) + B_1(t) \right] \\
+ \frac{1 - \log(4\pi)}{4\pi} + \gamma_{\text{EM}} A_2(t) + B_2(t) + \frac{c_{\infty, \infty}^{\Gamma(p)}}{2} + \frac{t+1}{2v_{\Gamma(p)^2}} + A_4(t). \]
Moreover writing \( R_0^{\text{par}}(t) = \frac{t + 1}{2 \ell_{\Gamma_0(p^2)}} + \mathcal{E}^{\text{par}}(t) \) we have

\[
\lim_{t \to \infty} \mathcal{E}^{\text{par}}(t) = \frac{1}{2 \ell_{\Gamma_0(p^2)}} + \frac{p + 1}{p} \left( 3 \gamma_{\text{EM}} + \frac{a \pi}{6} - 2 \log p + 2 B_1(\infty) \right) + \frac{1 - \log(4 \pi)}{4 \pi} + \frac{C_{\ell_{\Gamma_0(p^2)}^{\infty}}}{2}.
\]

We only need to show that \( C_{\ell_{\Gamma_0(p^2)}^{\infty}} = O \left( \frac{\log(p)}{p^2} \right) \) which follows from our computation of \( \phi_{\ell_{\Gamma_0(p^2)}^{\infty}} \) in Lemma 5.

\( \square \)

This concludes the proof of Proposition 11.

5.4. Asymptotics of the canonical Green’s function. We now prove Theorem 1.

**Proof.** From (4.8) and Proposition 11 we have \( \mathcal{R}^{\Gamma_0(p^2)}_{\infty} = -R_0^{\text{dis}(t)} + R_0^{\text{hyp}(t)} + R_0^{\text{ell}(t)} + R_0^{\text{par}(t)} \), hence taking limit as \( t \to \infty \) and by the previous calculations we get

\[
g_{p^2} \mathcal{R}^{\Gamma_0(p^2)}_{\infty} = \frac{1}{\ell_{\Gamma_0(p^2)}} + \frac{1}{2 \ell_{\Gamma_0(p^2)}} \lim_{s \to 1} \left( \frac{Z^{\prime}_{\Gamma_0(p^2)}}{Z_{\Gamma_0(p^2)}^s} - \frac{1}{s - 1} \right) + R_0^{\text{ell}}
\]

\[
+ \frac{1}{2 \ell_{\Gamma_0(p^2)}} \frac{p + 1}{p} \left( 3 \gamma_{\text{EM}} + \frac{a \pi}{6} - 2 \log p + 2 B_1(\infty) \right) + \frac{1 - \log(4 \pi)}{4 \pi} + \frac{C_{\ell_{\Gamma_0(p^2)}^{\infty}}}{2}.
\]

Hence \( \mathcal{R}^{\Gamma_0(p^2)}_{\infty} = o \left( \frac{\log(p)}{g_{p^2}} \right) \), major contribution being from \( R_0^{\text{ell}} \). \( \square \)

**Proposition 23.** For \( p > 7 \)

\[
g_{\text{can}}(\infty, 0) = -\frac{\log(p)}{g_{p^2}} + o \left( \frac{\log(p)}{g_{p^2}} \right).
\]

**Proof.** By Equation 3.6, we have the canonical Green’s function at the cusp \( \infty \) and \( 0 \) is given by

\[
g_{\text{can}}(\infty, 0) = C^{\Gamma_0(p^2)}_{0, \infty} + 4 \pi \mathcal{R}^{\Gamma_0(p^2)}_{0, \infty} + O \left( \frac{1}{g_{p^2}} \right) = C^{\Gamma_0(p^2)}_{0, \infty} + o \left( \frac{\log(p)}{g_{p^2}} \right) = C^{\Gamma_0(p^2)}_{0, \infty} + o \left( \frac{\log(p)}{g_{p^2}} \right).
\]

Hence, asymptotically the main contribution for \( g_{\text{can}}(\infty, 0) \) comes from \( C^{\Gamma_0(p^2)}_{0, \infty} \). We use Prop 8 to obtain

\[
C^{\Gamma_0(p^2)}_{0, \infty} = \frac{2 \pi}{\ell_{g_{p^2}}} \left( 1 + 2 \gamma_{\text{EM}} + \frac{a \pi}{6} - \frac{2(p^2 - p - 1)}{p^2 - 1} \log p \right) = \frac{-12 \log p}{p(p + 1)} + o \left( \frac{\log(p)}{p^2} \right).
\]

noting that \( \ell_{g_{p^2}} = \frac{2}{\pi} p(p + 1) \). Finally from the expression for \( g_{p^2} \) in Proposition 3 we have the desired result. \( \square \)

**Remark 24.** In the parabolic part we needed \( g_{p^2} > 1 \), hence we need the assumption \( p > 7 \).
6. Minimal regular models of Edixhoven

For any primes $p \geq 7$, the modular curve $X_0(p^2)$ is an algebraic curve defined over $\mathbb{Q}$. For each prime $p$ Edixhoven constructed a regular model $\tilde{X}_0(p^2)$ [10]. $\tilde{X}_0(p^2)$ is an arithmetic surface over $\text{Spec } \mathbb{Z}$. These models are not minimal however. In this section we describe the regular models of Edixhoven and the minimal regular models obtained from them.

For any prime $q$ of $\mathbb{Z}$ such that $q \neq p$ the fiber $\tilde{X}_0(p^2)_{\mathbb{F}_q}$ is a smooth curve of genus $g_\Gamma$, the genus of $X_0(p^2)$. For the prime $p$ the fiber $\tilde{X}_0(p^2)_{\mathbb{F}_p}$ is reducible and non-reduced, of arithmetic genus $g_\Gamma$, and whose geometry depends on the class of $p$ in $\mathbb{Z}/12\mathbb{Z}$. To describe $\tilde{X}_0(p^2)$ it is thus enough to describe the special fiber.

The minimal regular model $X_0(p^2)$ is obtained from $\tilde{X}_0(p^2)$ by three successive blow downs of curves in the special fiber $\tilde{X}_0(p^2)_{\mathbb{F}_p}$ and we shall denote by $\pi : \tilde{X}_0(p^2) \to X_0(p^2)$ the morphism from Edixhoven’s model. The following theorem about the blow-down morphism $f : X \to Y$ for curves over $k$ is crucial for the computations in subsequent section. Let $\cdot$ be the usual intersection pairing as [22].

**Theorem 25** ([22], Theorem 2.12, p. 398). Let $C$ be a divisor on $X$ and $D$ be a divisor on $Y$. The following properties are true:

- For any divisor $E$ on $X$ such that $f(\text{supp}(E))$ is finite, we have $E \cdot f^*D = 0$.
- Suppose $C$ or $D$ is vertical, then
  \[ C \cdot f^*D = f_*C \cdot D \]
  where $f_*C$ is the Cartier divisor such that $[f_*C] = f_*[C]$.
- Let $C$ be a vertical divisor on $Y$ then $f^*C$ is a vertical divisor on $X$ and
  \[ f^*C \cdot f^*D = [k(X) : k(Y)][C \cdot D]. \]

In the following sections, we compute the vertical components of the blown down $X_0(p^2)$ and the corresponding intersection multiplicities. The Arakelov intersections in this case are obtained by simply multiplying the local intersection numbers by $\log(p)$.

The following Proposition is the key result of this section. Let $\pi : \tilde{X}_0(p^2) \to X_0(p^2)$ be the morphism obtained by a sequence of blow downs.

**Proposition 26.** The special fiber of $X_0(p^2)$ consists of two curves $C'_{2,0}$ and $C'_{0,2}$ and the intersection multiplicities are given by:

\[ C'_{2,0} \cdot C'_{0,2} = -(C'_{2,0})^2 = -(C'_{0,2})^2 = \frac{p^2 - 1}{24}. \]

We will prove this proposition in four different cases depending on $p$.

6.1. **Case $p \equiv 1 \pmod{12}$**. In this case the special fiber $V_p = \tilde{X}_0(p^2)_{\mathbb{F}_p}$ is described by Figure 1 as given in Edixhoven [10]. Each component is a $\mathbb{P}^1$ and the pair $(n,m)$ adjacent to each component denotes the multiplicity of the component $n$ and the local self-intersection number $m$. The genus is given by $g_\Gamma = 12k^2 - 3k - 1$ where $p = 12k + 1.$
Proposition 27. The local intersection numbers of the vertical components supported on the special fiber of $\tilde{X}_0(p^2)$ are given by:

$$
\begin{array}{c|ccccc}
 & C_{2,0} & C_{0,2} & C_{1,1} & E & F \\
\hline
C_{2,0} & -\frac{p(p-1)}{12} & \frac{p-1}{12} & \frac{p-1}{12} & 0 & 0 \\
C_{0,2} & \frac{p-1}{12} & -\frac{p(p-1)}{12} & \frac{p-1}{12} & 0 & 0 \\
C_{1,1} & \frac{p-1}{12} & \frac{p-1}{12} & -1 & 1 & 1 \\
E & 0 & 0 & 1 & -2 & 0 \\
F & 0 & 0 & 1 & 0 & -3 \\
\end{array}
$$

Proof. We have the following local intersection numbers (see Liu [22], Chapter 9) of the prime vertical divisors supported on the special fiber of $\tilde{X}_0(p^2)$. The only unknown quantities here are $C_{2,0}^2$ and $C_{0,2}^2$,
which are obtained from the fact that $V_p \cdot C_{2,0} = V_p \cdot C_{0,2} = 0$ where $V_p = C_{2,0} + C_{0,2} + (p-1)C_{1,1} + \frac{E-1}{2}E + \frac{p-1}{r}F$ is the linear combination of all the prime divisors of the special fiber counted with multiplicities. \(\square\)

We now prove Proposition 26 if $p \equiv 1 \pmod{12}$.

**Proof.** By Proposition 27, note that the component $C_{1,1}$ is rational and has self-intersection $-1$. By Castelnuovo’s criterion we can thus blow down $C_{1,1}$ without introducing a singularity, see [22], Chapter 9, Theorem 3.8, p. 416]. Let $X_0(p^2)'$ be the corresponding arithmetic surface and $\pi_1 : \tilde{X}_0(p^2) \to X_0(p^2)'$, be the blow down morphism.

For $E' = \pi_1(E)$, we see that $\pi_1^*E' = E + C_{1,1}$. In fact since $\pi_1^*E = E$ in this case, we must have $\pi_1^*E' = E + \mu C_{1,1}$ (see Liu [22], Theorem 2.18), then using Theorem 25, $0 = C_{1,1} \cdot \pi_1^*E = 1 - \mu$. Hence $(E')^2 = (\pi_1^*E')^2 = (E + C_{1,1}, E + C_{1,1}) = -1$. Thus $E'$ is a rational curve in the special fiber of $X_0(p^2)'$ with self intersection $-1$. It can thus be blown down again and the resulting scheme is again regular. Let $X_0(p^2)''$ be the blow down and $\pi_2 : \tilde{X}_0(p^2) \to X_0(p^2)''$ the morphism from $\tilde{X}_0(p^2)$.

Let $F' = \pi_2(F)$, and if $\pi_2^*F' = F + \mu C_{1,1} + \nu E$ for $\mu, \nu \in \mathbb{Q}$ then using the fact that $C_{1,1} \cdot \pi_2^*F' = E \cdot \pi_2^*F' = 0$ we find $\mu = 2$ and $\nu = 1$. This yields $\pi_2^*F' = F + 2C_{1,1} + E$ and hence $(F')^2 = -1$. We can thus blow down $F'$ further to arrive finally at an arithmetic surface $X_0(p^2)$. This is the minimal regular model of $X_0(p^2)$ since no further blow down is possible. Let $\pi$ : $\tilde{X}_0(p^2) \to X_0(p^2)$ be the morphism obtained by a sequence of blow downs.

The special fiber now consists of two curves $C_{2,0}'$ and $C_{0,2}'$ that are the images of $C_{2,0}$ and $C_{0,2}$ respectively, and they intersect with high multiplicity at a single point. To calculate the intersections we notice that

\[
\pi^*C_{2,0}' = C_{2,0} + \frac{p-1}{2}C_{1,1} + \frac{p-1}{4}E + \frac{p-1}{6}F,
\]

\[
\pi^*C_{0,2}' = C_{0,2} + \frac{p-1}{2}C_{1,1} + \frac{p-1}{4}E + \frac{p-1}{6}F,
\]

obtained as before from the fact that the intersections of $\pi^*C_{2,0}'$ and $\pi^*C_{0,2}'$ with $C_{1,1}$, $E$ and $F$ are all 0. This yields

\[
C_{2,0}' \cdot C_{0,2}' = -(C_{2,0}')^2 = -(C_{0,2}')^2 = \frac{p^2 - 1}{24}.
\]

\(\square\)

6.2. **Case** $p \equiv 5 \pmod{12}$. In this case the special fiber $V_p = \tilde{X}_0(p^2)e_p$ is described by Figure 2. Each component is a $\mathbb{P}^1$ and the genus is given by $g_F = 12k^2 + 5k$ where $p = 12k + 5$. 
Proposition 28. The local intersection numbers of the vertical components supported on the special fiber of $\tilde{X}_0(p^2)$ for $p \equiv 5 \pmod{12}$ are given as follows.

|    | $C_{2,0}$ | $C_{0,2}$ | $C_{1,1}$ | $E$ | $F$ |
|----|-----------|-----------|-----------|-----|-----|
| $C_{2,0}$ | $-\frac{p^2-p+4}{12}$ | $\frac{p-5}{12}$ | $\frac{p-5}{12}$ | 0 | 1 |
| $C_{0,2}$ | $\frac{p-5}{12}$ | $-\frac{p^2-p+4}{12}$ | $\frac{p-5}{12}$ | 0 | 1 |
| $C_{1,1}$ | $\frac{p-5}{12}$ | $\frac{p-5}{12}$ | $-1$ | 1 | 1 |
| $E$ | 0 | 0 | 1 | $-2$ | 0 |
| $F$ | 1 | 1 | 1 | 0 | $-3$ |

Proof. The quantities $C_{2,0}^2$ and $C_{0,2}^2$ are obtained by a calculation analogous to the previous case. □

We now prove Proposition 26 if $p \equiv 5 \pmod{12}$. 

Figure 2. The special fiber $\tilde{X}_0(p^2)_{F_p}$ when $p \equiv 5 \pmod{12}$. 

The special fiber $\tilde{X}_0(p^2)_{F_p}$ when $p \equiv 5 \pmod{12}$. 

The local intersection numbers of the vertical components supported on the special fiber of $\tilde{X}_0(p^2)$ for $p \equiv 5 \pmod{12}$ are given as follows.
Proof. The minimal regular model is obtained by blowing down $C_{1,1}$, then the image of $E$ and then the image of $F$ as in the previous section. We again denote the minimal regular model by $\tilde{X}_0(p^2)$, and by $\pi : \tilde{X}_0(p^2) \rightarrow X_0(p^2)$ the morphism obtained by the successive blow downs.

The special fiber consists of two curves $C_{2,0}'$ and $C_{0,2}'$ that are the images of $C_{2,0}$ and $C_{0,2}$ respectively under $\pi$ intersecting at a single point. Here we have

$$\pi^*C_{2,0}' = C_{2,0} + \frac{p-1}{2}C_{1,1} + \frac{p-1}{4}E + \frac{p+1}{6}F,$$

$$\pi^*C_{0,2}' = C_{0,2} + \frac{p-1}{2}C_{1,1} + \frac{p-1}{4}E + \frac{p+1}{6}F,$$

obtained as before from the fact that the intersections of $\pi^*C_{2,0}'$ and $\pi^*C_{0,2}'$ with $C_{1,1}$, $E$ and $F$ are all 0. This yields

$$C_{2,0}' \cdot C_{0,2}' = -(C_{2,0}')^2 = -(C_{0,2}')^2 = \frac{p^2-1}{24}.$$

□

6.3. Case $p \equiv 7 \pmod{12}$. In this case the special fiber $V_p = \tilde{X}_0(p^2)_{F_p}$ is described by Figure 3. Each component is a $\mathbb{P}^1$ occurring with the specified multiplicity. The genus is given by $g_T = 12k^2 + 9k + 1$ where $p = 12k + 7$. By a similar computation, we obtain:

![Figure 3. The special fiber $\tilde{X}_0(p^2)_{F_p}$ when $p \equiv 7 \pmod{12}$.](#)
Proposition 29. The local intersection numbers of the prime divisors supported on the special fiber of $\bar{X}_0(p^2)$ for $p \equiv 7 \pmod{12}$ are as follows:

|      | $C_{2,0}$ | $C_{0,2}$ | $C_{1,1}$ | $E$ | $F$ |
|------|-----------|-----------|-----------|-----|-----|
| $C_{2,0}$ | $-\frac{p^2 - p + 6}{12}$ | $\frac{p - 7}{12}$ | $\frac{p - 7}{12}$ | 1 | 0 |
| $C_{0,2}$ | $\frac{p - 7}{12}$ | $-\frac{p^2 - p + 6}{12}$ | $\frac{p - 7}{12}$ | 1 | 0 |
| $C_{1,1}$ | $\frac{p - 7}{12}$ | $\frac{p - 7}{12}$ | $-1$ | 1 | 1 |
| $E$ | 1 | 1 | 1 | $-2$ | 0 |
| $F$ | 0 | 0 | 1 | 0 | $-3$ |

We now prove Proposition 26 if $p \equiv 7 \pmod{12}$.

Proof. The minimal regular model is obtained by blowing down $C_{1,1}$, then the image of $E$ and then the image of $F$ as in the previous sub-section. Let $X_0(p^2)$ be the minimal regular model and $\pi : \bar{X}_0(p^2) \to X_0(p^2)$ the morphism obtained by the successive blow downs.

The special fiber consists of two curves $C'_{2,0}$ and $C'_{0,2}$ that are the images of $C_{2,0}$ and $C_{0,2}$ respectively under $\pi$ intersecting at a single point. Here

$$\pi^* C'_{2,0} = C_{2,0} + \frac{p - 1}{2} C_{1,1} + \frac{p + 1}{4} E + \frac{p - 1}{6} F,$$

$$\pi^* C'_{0,2} = C_{0,2} + \frac{p - 1}{2} C_{1,1} + \frac{p + 1}{4} E + \frac{p - 1}{6} F,$$

easily calculated using fact that the intersections of $\pi^* C'_{2,0}$ and $\pi^* C'_{0,2}$ with $C_{1,1}$, $E$ and $F$ are all 0. This yields

$$C'_{2,0} \cdot C'_{0,2} = -(C'_{2,0})^2 = -(C'_{0,2})^2 = \frac{p^2 - 1}{24}.$$

$\square$

6.4. Case $p \equiv 11 \pmod{12}$ In this case the special fiber $V_p = \bar{X}_0(p^2)_{F_p}$ is described by Figure 4. Each component is a $\mathbb{P}^1$. The genus is given by $g_r = 12k^2 + 17k + 6$ where $p = 12k + 11$. 
Figure 4. The special fiber $\tilde{X}_0(p^2)_{\mathbb{F}_p}$ when $p \equiv 11 \pmod{12}$.

**Proposition 30.** The local intersection numbers of the prime divisors supported on the special fiber of $\tilde{X}_0(p^2)$ for $p \equiv 11 \pmod{12}$ are given by:

|   | $C_{2,0}$ | $C_{0,2}$ | $C_{1,1}$ | $E$ | $F$ |
|---|---|---|---|---|---|
| $C_{2,0}$ | $-\frac{p^2-p+10}{12}$ | $\frac{p-11}{12}$ | $\frac{p-11}{12}$ | 1 | 1 |
| $C_{0,2}$ | $\frac{p-11}{12}$ | $-\frac{p^2-p+10}{12}$ | $\frac{p-11}{12}$ | 1 | 1 |
| $C_{1,1}$ | $\frac{p-11}{12}$ | $\frac{p-11}{12}$ | $-1$ | 1 | 1 |
| $E$ | 1 | 1 | 1 | $-2$ | 0 |
| $F$ | 1 | 1 | 1 | 0 | $-3$ |

We now prove Proposition 26 if $p \equiv 11 \pmod{12}$. 

Proposition 31. For $s_p = \frac{p^2 - 1}{24}$, we define $V_0 = -\frac{g_{p^2} - 1}{s_p} C'_0$ and $V_\infty = -\frac{g_{p^2} - 1}{s_p} C'_\infty$, then the divisors

$$D_m = K_{X_0(p^2)} - (2g_{p^2} - 2)H_m + V_m,$$

are orthogonal to all vertical divisors of $X_0(p^2)$ with respect to the Arakelov intersection pairing.

Proof. For any prime $q \neq p$ if $V$ is the corresponding fiber over $(q) \in \text{Spec} \mathbb{Z}$, then $\langle V_m, V \rangle = 0$ and $\langle K_{X_0(p^2)}, V \rangle = (2g_{p^2} - 2) \log(p)$. $H_0$ meets any fiber transversally at a smooth $\mathbb{F}_p$ rational point which gives $\langle D_m, V \rangle = 0$.

Note that

$$\langle K_{X_0(p^2)}, C'_0 + C'_\infty \rangle = (2g_{p^2} - 2) \log(p),$$

and on the other hand from the discussion of $X_0(p^2)$ it will be clear that $\langle K_{X_0(p^2)}, C'_0 \rangle = \langle K_{X_0(p^2)}, C'_\infty \rangle$, hence we have

$$\langle K_{X_0(p^2)}, C'_0 \rangle = \langle K_{X_0(p^2)}, C'_\infty \rangle = (g_{p^2} - 1) \log(p).$$
If \( m \in \{0, \infty\} \), then
\[
\langle D_m, C'_m \rangle = \left( g_{p^2} - 1 \right) \log(p) - (2g_{p^2} - 1) \log(p) + (g_{p^2} - 1) \log(p) = 0.
\]
Finally if \( n \in \{0, \infty\}, n \neq m \) then
\[
\langle D_m, C'_n \rangle = \left( g_{p^2} - 1 \right) \log(p) - (2g_{p^2} - 1) \log(p) = 0.
\]
This completes the proof. \( \square \)

**Proposition 32.** The line bundle \( \mathcal{O}(D_m) \) restricted to the generic fiber \( X_0(p^2)/\mathbb{Q} \) is torsion for \( m \in \{0, \infty\} \).

**Proof.** Consider the congruence subgroup
\[
\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, d \equiv 1 \pmod{N}, c \equiv 0 \pmod{N} \right\}
\]
and let \( X_1(N) \) be the corresponding compactified modular curve. There is a morphism
\[
f : X_1(p^2) \to X_0(p^2)
\]
due to the inclusion \( \Gamma_1(p^2) \subset \Gamma_0(p^2) \). We claim that \( f^* \mathcal{O}(D_m) \) is torsion, hence \( \mathcal{O}(D_m) \) is also torsion since \( f \) is a finite map.

Let \( \kappa_0, \kappa_{\infty} \in X_0(p^2)(\mathbb{Q}) \) be the points corresponding to the cusps 0 and \( \infty \). If \( K_{X_0(p^2)} \) is a canonical divisor of \( X_0(p^2) \), then on \( X_0(p^2) \)
\[
D_m \sim K_{X_0(p^2)} - (2g_{p^2} - 2)\kappa_m,
\]
here \( \sim \) denotes linear equivalence of divisors. If \( K_{X_1(p^2)} \) is a canonical divisor of \( X_1(p^2) \), then by Riemann-Hurwitz
\[
K_{X_1(p^2)} \sim f^* K_{X_0(p^2)} + R_f
\]
where \( R_f \) is the ramification divisor of \( f \) and is supported on cusps (see Chapter 3 of Diamond-Shurman [9]). Thus
\[
f^*D_m \sim K_{X_1(p^2)} - R_f - (2g_{p^2} - 2)f^*\kappa_m.
\]
It follows from Lemma 4.1.1 of Abbes-Ullmo [1], and Chapter 3 of Diamond-Shurman [9] that \( K_{X_1(p^2)} \) is supported on cusps. Hence \( f^*D_m \) is a cuspidal divisor of degree 0.

It then follows from a theorem of Manin and Drinfeld that the corresponding point in the Jacobian is torsion. \( \square \)

The following proposition is a consequence of Proposition 31. The proof is analogous to that of Proposition D of Abbes-Ullmo [1].

**Lemma 33.** With the notation as in Proposition 31 we have the following equality of intersection numbers
\[
(7.2) \quad (\omega_{X_0(p^2)})^2 = -4g_{p^2}(g_{p^2} - 1)(H_0, H_{\infty}) + \frac{1}{g_{p^2} - 1} \left[ g_{p^2}(V_0, V_{\infty}) - \frac{V_0^2 + V_{\infty}^2}{2} \right].
\]
Proof. It follows from Proposition 31 and Faltings [12], Theorem 4, that for \( m \in \{0, \infty\} \)

\[
\langle D_m, D_m \rangle = -2 \left( \text{Néron-Tate height of } \mathcal{O}(D_m) \right) = 0.
\]

The last equality is true since \( \mathcal{O}(D_m) \) is torsion. Since \( D_m \) is perpendicular to vertical divisors we obtain

\[
\langle D_m, K_{X_0(p^2)} - (2g_{p^2} - 2)H_m \rangle = 0,
\]

which expands to

\[
(\omega_{X_0(p^2)})^2 = -(2g_{p^2} - 2)^2 H_m^2 + 2(2g_{p^2} - 2) \langle K_{X_0(p^2)}, H_m \rangle - \langle K_{X_0(p^2)}, V_m \rangle + (2g_{p^2} - 2) \langle H_m, V_m \rangle.
\]

Now using the equality \( \langle D_m, V_m \rangle = 0 \) which yields \( \langle K_{X_0(p^2)}, V_m \rangle - (2g_{p^2} - 2) \langle H_m, V_m \rangle + V_m^2 = 0 \) and the adjunction formula \( \langle K_{X_0(p^2)}, H_m \rangle = -H_m^2 \), (see Corollary 5.6, Lang Chapter IV, Section 5), we obtain

\[
(\omega_{X_0(p^2)})^2 = -4g_{p^2}(g_{p^2} - 1)H_m^2 + V_m^2.
\]

This yields

\[
(\omega_{X_0(p^2)})^2 = -(2g_{p^2} - 2)^2 H_m^2 + 2(2g_{p^2} - 2) \langle K_{X_0(p^2)}, H_m \rangle - \langle K_{X_0(p^2)}, V_m \rangle + (2g_{p^2} - 2) \langle H_m, V_m \rangle.
\]

The divisor \( D_\infty - D_0 = \left[ H_0 - H_\infty + \frac{1}{2(2g_{p^2} - 2)}(V_\infty - V_0) \right] \) is perpendicular to all vertical divisors and is supported on cusps so by similar reasoning as above we have

\[
H_0^2 + H_\infty^2 = 2 \langle H_0, H_\infty \rangle + \frac{V_0^2 - 2\langle V_0, V_\infty \rangle + V_\infty^2}{(2g_{p^2} - 2)^2}.
\]

Hence substituting in (7.4) we obtain

\[
(\omega_{X_0(p^2)})^2 = -4g_{p^2}(g_{p^2} - 1)\langle H_0, H_\infty \rangle - \frac{1}{2(2g_{p^2} - 2)}(V_0^2 + V_\infty^2) + \frac{g_{p^2}}{g_{p^2} - 1} \langle V_0, V_\infty \rangle.
\]

\[\square\]

Finally we have the following theorem about the relative dualizing sheaf of the minimal regular model of \( X_0(p^2)/\mathbb{Q} \).

**Theorem 34.** The self-intersection of the relative dualizing sheaf on \( X_0(p^2) \) is given by

\[
(\omega_{X_0(p^2)})^2 = -4g_{p^2}(g_{p^2} - 1)\langle H_0, H_\infty \rangle - \frac{(g_{p^2}^2 - 1) \log(p)}{s_p}.
\]

The Theorem follows trivially from Lemma 33 and (7.1). Using the above results, we obtain the main theorem of the paper.
7.1. Proof of Theorem 2.

Proof. Using Theorem 34, we obtain:

\[
\left( \mathfrak{Z}_{X_0(p^2)} \right)^2 = -4g_p^2(g_p^2 - 1)\langle H_0, H_\infty \rangle + \frac{(g_p^2 - 1)\log(p)}{s_p}.
\]

By the explicit computation of genus [cf. Proposition 3], we have:

\[
g_p^2 - 1 = \frac{(p + 1)(p - 6) - 12c}{12}
\]

with \(c \in \{0, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{7}{6}\}\).

By Proposition 23 and from the above equation, we have

\[
(7.5) - 4g_p^2(g_p^2 - 1)g_{\text{can}}(\infty, 0) = 4g_p^2\log(p) + o(g_p^2\log(p)).
\]

From the same equation for \(g_p^2 - 1\), we also have:

\[
\frac{(g_p^2 - 1)\log(p)}{s_p} = \frac{(g_p^2 + 1)((g_p^2 - 1)\log(p))}{s_p}
\]

\[
= (g_p^2 + 1)\log(p)[2 + o(1)] = 2g_p^2\log(p) + o(g_p^2\log(p)).
\]

Hence, we obtain the Theorem. \(\square\)

References

[1] A. Abbes and E. Ullmo, Auto-intersection du dualisant relatif des courbes modulaires \(X_0(N)\), J. Reine Angew. Math. 484 (1997) 1–70.
[2] S. J. Arakelov, An intersection theory for divisors on an arithmetic surface, Izv. Akad. Nauk SSSR Ser. Mat. 38 (1974) 1179–1192.
[3] ———, Theory of intersections on the arithmetic surface, in Proceedings of the International Congress of Mathematicians (Vancouver, B.C., 1974), Vol. 1, 405–408, Canad. Math. Congress, Montreal, Que. (1975).
[4] D. Banerjee, A note on the Eisenstein elements of prime square level, Proc. Amer. Math. Soc. 142 (2014), no. 11, 3675–3686.
[5] D. Banerjee and C. Chaudhuri, Arithmetic applications of Self-intersection numbers of modular curves \(X_0(p^2)\), work in progress.
[6] R. F. Coleman, On the components of \(X_0(p^n)\), J. Number Theory 110 (2005), no. 1, 3–21.
[7] C. Curilla, Regular models of Fermat curves and applications to Arakelov theory, Ph.D. thesis, Universität Hamburg (2010).
[8] B. de Smit and B. Edixhoven, Sur un résultat d’Imin Chen, Math. Res. Lett. 7 (2000), no. 2-3, 147–153.
[9] F. Diamond and J. Shurman, A first course in modular forms, Vol. 228 of Graduate Texts in Mathematics, Springer-Verlag, New York (2005), ISBN 0-387-23229-X.
[10] B. Edixhoven, *Minimal resolution and stable reduction of $X_0(N)$*, Ann. Inst. Fourier (Grenoble) **40** (1990), no. 1, 31–67.

[11] ———, *Computing coefficients of modular forms*, in Computational aspects of modular forms and Galois representations, Vol. 176 of Ann. of Math. Stud., 383–398, Princeton Univ. Press, Princeton, NJ (2011).

[12] G. Faltings, *Calculus on arithmetic surfaces*, Ann. of Math. (2) **119** (1984), no. 2, 387–424.

[13] M. D. F. Grados, Arithmetic intersections on modular curves, Ph.D. thesis, Humboldt-Universität zu Berlin (2016).

[14] P. Griffiths and J. Harris, Principles of algebraic geometry, Wiley Classics Library, John Wiley & Sons, Inc., New York (1994), ISBN 0-471-05059-8. Reprint of the 1978 original.

[15] D. A. Hejhal, *The Selberg trace formula for $\text{PSL}(2, \mathbb{R})$*. Vol. 2, Vol. 1001 of Lecture Notes in Mathematics, Springer-Verlag, Berlin (1983), ISBN 3-540-12323-7.

[16] H. Iwaniec, *Spectral methods of automorphic forms*, Vol. 53 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI; Revista Matemática Iberoamericana, Madrid, second edition (2002), ISBN 0-8218-3160-7.

[17] J. Jorgenson and J. Kramer, *Bounds for special values of Selberg zeta functions of Riemann surfaces*, J. Reine Angew. Math. **541** (2001) 1–28.

[18] ———, *Bounds on canonical Green’s functions*, Compos. Math. **142** (2006), no. 3, 679–700.

[19] ———, *Bounds on Faltings’s delta function through covers*, Ann. of Math. (2) **170** (2009), no. 1, 1–43.

[20] N. M. Katz and B. Mazur, *Arithmetic moduli of elliptic curves*, Vol. 108 of Annals of Mathematics Studies, Princeton University Press, Princeton, NJ (1985), ISBN 0-691-08349-5; 0-691-08352-5.

[21] C. Keil, *Die Streumatrix fr Untergruppen der Modulgruppe*, Ph.D. thesis, Heinrich-Heine-Universität Düsseldorf, Düsseldorf.

[22] Q. Liu, *Algebraic geometry and arithmetic curves*, Vol. 6 of Oxford Graduate Texts in Mathematics, Oxford University Press, Oxford (2002), ISBN 0-19-850284-2. Translated from the French by Reinie Erné, Oxford Science Publications.

[23] H. Mayer, *Self-intersection of the relative dualizing sheaf on modular curves $X_1(N)$*, J. Théor. Nombres Bordeaux **26** (2014), no. 1, 111–161.

[24] H. P. McKean, *Selberg’s trace formula as applied to a compact Riemann surface*, Comm. Pure Appl. Math. **25** (1972) 225–246.

[25] P. Michel and E. Ullmo, *Points de petite hauteur sur les courbes modulaires $X_0(N)$*, Invent. Math. **131** (1998), no. 3, 645–674.

[26] W. Roelcke, *Das Eigenwertproblem der automorphen Formen in der hyperbolischen Ebene, I*, Math. Ann. **167** (1966), no. 4, 292–337.

[27] E. Ullmo, *Positivité et discrétion des points algébriques des courbes*, Ann. of Math. (2) **147** (1998), no. 1, 167–179.

[28] D. Zagier, *Eisenstein series and the Selberg trace formula. I*, in Automorphic forms, representation theory and arithmetic (Bombay, 1979), Vol. 10 of Tata Inst. Fund. Res. Studies in Math., 303–355,
Tata Inst. Fundamental Res., Bombay (1981).

[29] D. B. Zagier, *Zetafunktionen und quadratische Körper*, Springer-Verlag, Berlin-New York (1981), ISBN 3-540-10603-0. Eine Einführung in die höhere Zahlentheorie. [An introduction to higher number theory], Hochschultext. [University Text].

[30] S. Zhang, *Admissible pairing on a curve*, Invent. Math. **112** (1993), no. 1, 171–193.

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