Experimental Demonstration of a Reconfigurable Coupled Oscillator Platform to Solve the Max-Cut Problem

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ABSTRACT In this work, we experimentally demonstrate an integrated circuit (IC) of 30 relaxation oscillators with reconfigurable capacitive coupling to solve the NP-Hard maximum cut (Max-Cut) problem. We show that under the influence of an external second-harmonic injection signal, the oscillator phases exhibit a bipartition that can be used to calculate a high-quality approximate Max-Cut solution. Leveraging the all-to-all reconfigurable coupling architecture, we experimentally evaluate the computational properties of the oscillators using randomly generated graph instances of varying size and edge density (\(\eta\)). Furthermore, comparing the Max-Cut solutions with the optimal values, we show that the oscillators (after simple postprocessing) produce a Max-Cut that is within 99% of the optimal value in 28 of the 36 measured graphs; importantly, the oscillators are particularly effective in dense graphs with the Max-Cut being optimal in seven out of nine measured graphs with \(\eta = 0.8\). Our work marks a step toward creating an efficient, room-temperature-compatible non-Boolean hardware-based solver for hard combinatorial optimization problems.

INDEX TERMS Analog, coupled oscillators, integrated circuit (IC), Ising machines, maximum cut (Max-Cut).

I. INTRODUCTION

DIGITAL computing has been the backbone of modern information processing technology. Despite its tremendous strides, there is a class of computational problems, commonly referred to as NP-Hard problems, which are still considered fundamentally intractable to compute using digital computers. A case in point, and the focus of this work, is computing the maximum cut (Max-Cut) of a graph \(G(V,E)\) (\(V\): vertices and \(E\): edges), which is a cut that divides \(G\) into two sets such that the number of common edges between them is as large as possible; the number of common edges is the value of the Max-Cut. The Max-Cut problem is an archetypal NP-Hard problem [1] that finds extensive use in areas ranging from statistical physics [2]–[4] and medicine discovery to VLSI design [5]. However, solving the problem using conventional digital computers entails an exponential increase in computational resources as the size of the problems increase. Subsequently, this has motivated the search for alternate computing platforms [6]–[11], such as Ising machines (based on the Ising model) evaluated here, which can potentially provide a more efficient pathway to solving such problems. It is worth emphasizing that the successful realization of such a non-Boolean platform (e.g., the coupled oscillators explored here) is likely to benefit the broader class of such problems since many such problems can be formulated in terms of the underlying Ising model [9] through polynomial-time transformations.

In this work, we experimentally develop an integrated circuit (IC) of CMOS-compatible coupled relaxation oscillators and demonstrate its functionality as an Ising machine to compute the Max-Cut of a graph. The Max-Cut problem can be mapped directly to an Ising Hamiltonian: \(H = -\sum_{ij} J_{ij} \sigma_i \sigma_j\), where each spin \(\sigma\) corresponds to a node of the graph and can take binary values \(\sigma \in \{\pm 1\}\), \(N\) is
the total number of nodes in the problem, and $J_{ij}$ is the interaction coefficient between nodes $i$ and $j$. Computing the Max-Cut solution then corresponds to minimizing $H$ [12]. Consequently, there has been an active research effort to realize a physical “Ising machine” that inherently evolves to minimize its energy and, thus, naturally computes the Max-Cut solution. Examples of such demonstrations include the D-Wave quantum annealer [13]–[15], optical parametric oscillator-based coherent Ising machines (CIMs) [16]–[18], and SRAM-based Ising machines that use CMOS annealing [19], as well as the new CMOS annealing processors that use processing—in-/near—memory [20]–[22]. Coupled oscillators have also been explored as an alternate non-Boolean approach to solving computationally hard problems [23]–[35] and, more importantly, have recently been shown to behave as Ising machines [36]–[39] relevant to solving the Max-Cut problem. Wang and Roychowdhury [36], Wang et al. [37], and Chou et al. [38] recently demonstrated Ising machines using resistively coupled sinusoidal oscillators operating under the influence of a second-harmonic injection signal, and Dutta et al. [39] showed a similar functionality in four capacitively coupled injection-locked VO$_2$ oscillators. Furthermore, Ahmed et al. [40] recently demonstrated a scaled IC of 560 hexagonally connected CMOS-based ring oscillators to solve the Max-Cut in large planar graphs. These works attest to the increasing interest in exploring coupled oscillators to solve computationally challenging problems.

Here, we demonstrate a coupled relaxation oscillator IC to solve the Max-Cut problem (see Fig. 1) in nonplanar graphs. Our platform incorporates: 1) 30 programmable CMOS Schmitt-trigger-based relaxation oscillators that operate under the influence of a second-harmonic injection signal ($f_{inj} \cong 2f_R$, where $f_R$ is the resonant frequency) and 2) reconfigurable and symmetric capacitive coupling among the oscillators, i.e., any oscillator can be coupled to any and all other oscillators that allow us to process a graph (up to 30 nodes) with arbitrary connectivity.

Subsequently, we show that by creating a network that is topologically equivalent to the input graph, i.e., each oscillator is mapped to a node of the graph and every coupling capacitor corresponds to an edge, the resulting phase dynamics of the oscillators can be empirically used to compute an approximate Max-Cut solution. The oscillator phases exhibit a bipartition, i.e., $0^\circ$ or $180^\circ$, which corresponds to the two subsets created by the Max-Cut. We note that the external subharmonic signal helps induce the bipartition relevant to the Max-Cut; without this signal, the oscillators exhibit a continuous phase ordering (as shown in our prior work [26] and also observed in this IC but not shown here). The advantage of the developed hardware is that besides being compatible with state-of-the-art CMOS foundry processes, it is compact (unlike other Ising implementations, such as CIM) [41] and suited for room temperature operation (unlike quantum annealing) [42]. Furthermore, the reconfigurability incorporated in the design gives us a unique opportunity to characterize and evaluate the dynamics and the computational properties of the system over a range of graph sizes and connectivity.

II. COUPLED OSCILLATOR IC

The coupled oscillator IC is fabricated using bulk CMOS 65-nm node technology [see Fig. 1(a)]. Each oscillator is implemented using a Schmitt trigger inverter with a negative $RC$ feedback [see Fig. 1(b)]; the feedback resistor is implemented using a switched capacitor. Furthermore, current mirrors are implemented at the header and the footer of each oscillator to control the oscillation frequency and, importantly, also enable injection of the second-harmonic signal. The external injection signal is a sinusoidal signal with a peak-to-peak amplitude of 150 mV, dc offset (footer: 0.5 V and header: 0.3 V), and a frequency ($f_{inj}$) approximately twice the resonant frequency of the coupled circuit, i.e., $f_{inj} \cong 2f_R$. This signal is generated using an external function generator (two separate channels were used to achieve the different dc offsets required for the header and footer) and injected to the header and the footer circuit. To sense the resonant frequency, the coupled oscillator circuit (corresponding to the graph) is first measured without the injection signal. Each oscillator output is buffered and digitized using a hysteretic Schmitt-trigger buffer to facilitate read-out while preserving the phase information. Fig. 1(c) shows the output from a single oscillator. The coupling architecture shown in Fig. 1(b) is implemented as a 30-line bus wherein an oscillator can be coupled to any and all other oscillators through the bus using a capacitor in series with a T-gate, which is used to program (on/off) the coupling between any two oscillators; the coupling capacitor is implemented as a metal–insulator–metal capacitor with an area of 14.78 µm$^2$ and having a value of 32.5 fF. This value of coupling capacitance was chosen since it was in the range of values wherein the system demonstrated the desired phase dynamics. We observed using simulations that, for very small values of coupling capacitance (<5 fF), the oscillators may fail to lock for certain coupling configurations. In contrast, for large capacitances (>120 fF), the system exhibited in-phase locking for certain graphs. A total of 870 coupling elements enable programmable and symmetric coupling between any and all other oscillators in the network. Serial-in–parallel-out registers are used to program the oscillators and the coupling elements; a 32:1 MUX is used for serial read-out. The output of one of the oscillators is also tapped directly (besides passing through the MUX) and serves as the reference to which the phases of the other oscillators are compared. The power dissipated in the chip is measured to be 1.76 mW.

III. RESULTS

A. COMPUTING MAX-CUT USING COUPLED OSCILLATORS

To compute the Max-Cut of a graph using the coupled oscillators, we start with the adjacency matrix $A$ of the graph, where $A_{ij}$ indicates the presence ($A_{ij} = 1$) or absence ($A_{ij} = 0$) of
The oscillator is implemented using an inverting Schmitt-trigger design with feedback. The programmable coupling element consists of a capacitor in series with a T-gate. The coupling architecture enables each oscillator to be coupled to any and all other oscillators in the circuit. (c) Output of the free-running relaxation oscillator. (d) Experimentally measured dynamics of the coupled oscillators (operating under the influence of a second-harmonic injection locking signal) for a representative six-node graph. It can be observed from the polar plot that the oscillator phases exhibit a bipartition, which can be used to compute a high-quality Max-Cut of the graph.

We test our hardware on randomly generated graph instances with $V = 10, 20$, and $30$ nodes, and having edge density, $\eta = 0.2, 0.4, 0.6, 0.8$ ($\eta$ is the ratio of the number of edges in the graph to the number of edges in an all-to-all connected graph of the same size); three graphs are tested for

FIGURE 1. (a) Die photograph and the operating parameters of the fabricated IC. (b) Circuit schematic of the coupled oscillator circuit. The oscillator is implemented using an inverting Schmitt-trigger design with feedback. The programmable coupling element consists of a capacitor in series with a T-gate. The coupling architecture enables each oscillator to be coupled to any and all other oscillators in the circuit. (c) Output of the free-running relaxation oscillator. (d) Experimentally measured dynamics of the coupled oscillators (operating under the influence of a second-harmonic injection locking signal) for a representative six-node graph. It can be observed from the polar plot that the oscillator phases exhibit a bipartition, which can be used to compute a high-quality Max-Cut of the graph.

an edge between node $i$ and $j$ of the graph. Since we consider undirected graphs here, $A_{ij} = A_{ji}$. Each node of the graph is mapped to an oscillator and every edge (represented by $A_{ij} = A_{ji}$) to a coupling capacitor; node $\equiv$ oscillator; edge $\equiv$ coupling capacitor; and oscillator phase $\equiv$ set (created by the cut) to which the node belongs. In the context of the proposed hardware, the number of rows (or columns) in $A$ represents the number of oscillators required to process the graph, and $A_{ij}$ is used to configure the corresponding coupling among the oscillators. The capacitors couple the oscillators negatively [39], i.e., oscillators exhibit phase repulsion when capacitively coupled, and have a negative relationship to the edge weight. The matrix $A$ is passed on to the SIPO registers to initialize a topologically similar oscillator network. Fig. 1(d) shows the experimentally measured oscillator outputs along with the corresponding phase plot for a representative six-node graph. A bipartition in the oscillator phases observed in the polar plot corresponds to the two subsets (Sets I and II) created by the (Max-)Cut; the Max-Cut value can subsequently be computed by counting the number of common edges (8 in this example) between the sets.

We test our hardware on randomly generated graph instances with $V = 10, 20$, and $30$ nodes, and having edge density, $\eta = 0.2, 0.4, 0.6, 0.8$ ($\eta$ is the ratio of the number of edges in the graph to the number of edges in an all-to-all connected graph of the same size); three graphs are tested for
FIGURE 2. (a) Bar plot showing the measured Max-Cut solutions for 36 randomly generated graph instances as a function of their size and edge density. (b) Bubble plot comparing (best case) Max-Cut solution obtained from the oscillators with the optimal Max-Cut of the graph. (c) Variation of cluster separation (i.e., the angular separation between the two oscillator phases) with graph size and edge density.

each combination of $V$ and $\eta$ [see Fig. 2(a)] with each graph being measured ten separate times. While the best solution has been considered in Fig. 2, the distribution of solutions over the ten runs is shown in Fig. 1 in the Supplemental Material. It is evident that larger and denser graphs have larger Max-Cuts and, consequently, are more challenging to solve [1]. Fig. 2(b) shows a bubble plot comparing the value of the measured Max-Cut (best case) using the oscillators with the optimal solution obtained using the BiqMac solver developed by Rendl et al. [1], [43]; comparison of the mean value of the Max-Cut solution computed by the oscillators is shown in Fig. 2 in the Supplemental Material. It can be observed that the solution to most of the analyzed graphs lies near or on the identity line ($y = x$) although larger graphs tend to show higher deviations from the optimal solution. As measured, the oscillator solution is within 99% of the optimal solution in 12 of the 36 graphs. We note that the abovementioned experiments were performed without using a specialized annealing schedule, and the accuracy can be improved in the future by incorporating an annealing scheme [38]. Furthermore, as described in Section III-B, the solution can be dramatically improved by using a simple polynomial-time local search scheme such that the solution is within the 99% of the optimal Max-Cut in 28 graphs and equals the optimal Max-Cut in 26 of the 36 measured graphs.

B. IMPROVING MEASURED MAX-CUT SOLUTION

To improve the Max-Cut solution obtained from the oscillators, we explore a simple polynomial-time ($O(n^2)$) scheme based on local search as shown in the flowchart in Fig. 3(a). Using the Max-Cut solution computed from the oscillators, the scheme proceeds by moving nodes between the sets if and only if the move increases the value of the cut. This process is repeated until no more nodes can be found that can increase the value of the cut. The cumulative graph count distribution as a function of the distance from optimal solution (i.e., the difference between the Max-Cut solution obtained using the coupled oscillators and optimal Max-Cut) before and after postprocessing shows the corresponding improvement in the solution for the experimentally measured graphs. The hardware–software approach produces the optimal Max-Cut in 26 (∼72%) of the 36 graphs. Moreover, the oscillators are also effective in solving challenging dense graphs where they produce optimal solutions in seven out of the nine measured graphs with edge density $\eta = 0.8$.

C. COMPARISON WITH OTHER APPROACHES

Fig. 4 shows a table comparing this work with other alternate approaches being explored to solve such computationally hard problems. While CMOS-based classical implementations do not reduce the fundamental complexity of the problem (as expected with quantum mechanical systems [41]), they offer a room temperature solution that can still provide a significant speed in comparison to digital computers owing to the inherent parallelism of the approach. Of these, coupled electronic oscillators provide a potentially promising ideal 180° phase difference) in comparison to smaller and sparser graphs implying that the system finds it increasingly challenging to attain the global energy minima corresponding to the optimal Max-Cut solution. In addition, the effect of $V_{DD}$ variation and temperature is shown in Fig. 3 in the Supplemental Material.
low-power, integrated, and compact solution. While larger coupled oscillator systems have been recently demonstrated [40], this work differentiates itself by exploring the computational dynamics of the oscillators in nonplanar graphs with a wide range of edge densities, enabled through the implementation of an all-to-all reconfigurable capacitive coupling scheme.

Finally, we would also like to point out that while the coupled oscillator-based approach is promising, system scalability will be a critical factor in deciding its eventual success. Scaled systems will require the design and implementation of specialized annealing schemes/schedules, which can prevent the system from getting trapped in local minima and producing suboptimal solutions; the role of annealing will be particularly critical in larger systems that have an increasingly complex solution space and will be investigated in the future. Furthermore, scaled systems are also likely to require optimization of the coupling architecture including additional design considerations, such as managing the delay between the coupling elements. For instance, since practical graphs are unlikely to be very dense [45], the coupling architecture in larger systems could be optimized to have densely connected oscillator clusters with relatively sparse connectivity amongst them to achieve an optimal tradeoff between functionality and reconfigurability.

IV. CONCLUSION

In summary, we have experimentally investigated the computational properties of coupled relaxation oscillators to solve the NP-Hard Max-Cut problem by developing a prototype integrated IC of 30 relaxation oscillators with reconfigurable all-to-all coupling. Using minimal postprocessing, we show that the oscillator-based approach computes high-quality approximate Max-Cut solutions even in nonplanar graphs.

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