Confining Properties of Abelian(-Projected) Theories

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Abstract

Representations of the Abelian-projected $SU(2)$- and $SU(3)$-gluodynamics in terms of the magnetic monopole currents are derived. Besides the quadratic part, the obtained effective actions contain interactions of these currents with the world-sheets of electric strings in 4D or electric vortex lines in 3D. Next, we illustrate that 3D compact QED is a small gauge boson mass limit of 3D Abelian Higgs model with external monopoles and give a physical interpretation to the confining string theory as the integral over the monopole densities. Finally, we derive the bilocal field strength correlator in the weak-field limit of 3D compact QED, which turns out to be in line with the one predicted by the Stochastic Vacuum Model.

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1 Introduction

During several last years, there has appeared a vast amount of papers devoted to the description of confinement in Abelian-projected theories [1] (see e.g. [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13] and Refs. therein). The main goal of most of these papers is a derivation of the so-called string representation of such theories, i.e. a reformulation of their partition functions in terms of the integral over the world-sheets of the Abrikosov-Nielsen-Olesen (ANO) strings [14] with a certain nonlocal (i.e. depending on a relative distance between two points in the target space) action. Such a representation then enables one to get the coupling constants of the corresponding string theory including higher order derivative terms and to evaluate correlators of the dual field strength tensors [11, 12], which play a major role in the so-called Stochastic Vacuum Model (SVM) [15, 16]. In the case when there are no external quarks in the underlying non-Abelian theory, the corresponding Abelian-projected theory is some kind of a dual Abelian Higgs model with magnetic Higgs fields, which describe the condensates of monopole Cooper pairs. This model possesses classical solutions, which in 4D are just the electric ANO strings, whereas in 3D they are simply closed electric vortex lines [7]. It is therefore intuitively clear that there should exist some interaction between magnetic monopoles and ANO strings (vortex lines), and we shall demonstrate below that such an interaction really exists. Besides the interaction of the monopole Cooper pairs with strings or vortex lines, we shall obtain the full effective action of the monopole currents for the $SU(2)$- and $SU(3)$ Abelian-projected theories. Notice, that an action of this kind for the case of the usual AHM on a lattice has been presented in Ref. [9] in a form where the above mentioned interaction of the monopole currents with the string world-sheets has not, however, been manifestly shown.

Another, known for even a longer time theory, which allows for an analytic description of confinement, is 3D compact QED [1, 17]. There, similarly to the Abelian-projected theories, confinement also occurs due to the monopole condensation. The problem of string representation of this model has been addressed in Ref. [4], where it has been argued that the desired string theory is formulated in terms of a massive antisymmetric tensor field (usually called Kalb-Ramond field [18]) interacting with the string world-sheet. The complete form of the action of this field turned out to be quite a nonlinear one and was found recently in Ref. [19]. After that, the resulting theory, which is usually referred to as the confining string theory, has undergone intensive developments [20]. Notice, that the weak-field effective action of this theory occurs to have the linear form of the massive Kalb-Ramond field action, coinciding with the one of the dual version of AHM in the London limit. This reflects the fact that there should exist a correspondence between the direct formulations of compact QED and AHM as well. Below, we shall illustrate this to be really the case, namely that 3D compact QED is nothing else, but the small gauge boson mass limit of 3D AHM with external monopoles. Besides that, we show that the confining string theory is actually simply the integral over the monopole densities, which gives physical interpretation to the Kalb-Ramond field as a sum of the monopole and photon field strength tensors. Finally, we evaluate field correlators in the weak-field limit of 3D compact QED, and argue that their large-distance asymptotic behaviours are in line with the ones observed in the lattice experiments [21] within SVM.

The organization of the paper is as follows. In the next Section, we study the topic of representation of Abelian-projected theories in terms of the magnetic monopole currents. In Section 3, we revisit 3D compact QED and its string representation, after which it is demonstrated how this theory can be obtained by a limiting procedure from 3D AHM with monopoles. In the Appendix,
we outline some details of the path-integral duality transformation.

2 Representation of the Abelian-Projected Theories in Terms of Monopole Currents

Let us start with the 4D \( SU(2) \) Abelian-projected gluodynamics, which is argued to be just DAHM, whose partition function in the London limit has the form (see e.g. [11])

\[
Z_{4D \text{DAHM}} = \int DB_\mu D\theta^{\text{sing.}} D\theta^{\text{reg.}} \exp \left\{ - \int d^4x \left[ \frac{1}{4} F_{\mu\nu}^2 + \frac{\eta^2}{2} (\partial_\mu \theta - 2gB_\mu)^2 \right] \right\}.
\]

Here, \( F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \) is a field strength tensor of the dual vector potential \( B_\mu \), and \( g \) is a magnetic coupling constant (\( 2g \) is the magnetic charge of the monopole Cooper pair). Next, \( \eta \) denotes the v.e.v. of the magnetic Higgs field, whose phase has the form \( \theta = \theta^{\text{sing.}} + \theta^{\text{reg.}} \), where \( \theta^{\text{sing.}} \) describes a given electric string configuration, whereas \( \theta^{\text{reg.}} \) stands for a single-valued fluctuation around this configuration. The singular part of the phase of the magnetic Higgs field is related to the (closed) world-sheet \( \Sigma \) of the electric ANO string according to the equation

\[
\varepsilon_{\mu\nu\lambda\rho} \partial_\lambda \partial_\rho \theta^{\text{sing.}}(x) = 2\pi \Sigma_{\mu\nu}(x) \equiv 2\pi \int_\Sigma d\sigma_{\mu\nu}(x(\xi)) \delta(x - x(\xi))
\]

where \( \Sigma_{\mu\nu} \) is usually referred to as a vorticity tensor current [1], and \( \xi = (\xi^1, \xi^2) \) stands for the two-dimensional coordinate. Performing the path-integral duality transformation [3, 4, 11], we arrive at the following representation for the partition function (1) (see Appendix A)

\[
Z_{4D \text{DAHM}} = \int DA_\mu Dx_\mu(\xi) Dh_{\mu\nu} \exp \left\{ - \int d^4x \left[ \frac{1}{12\eta^2} H_{\mu\nu\lambda}^2 - i\pi h_{\mu\nu} \Sigma_{\mu\nu} + (gh_{\mu\nu} + \partial_\mu A_\nu - \partial_\nu A_\mu)^2 \right] \right\}.
\]

Here, \( A_\mu \) is the usual gauge field dual to the vector potential \( B_\mu \), and \( H_{\mu\nu\lambda} \equiv \partial_\mu h_{\nu\lambda} + \partial_\lambda h_{\mu\nu} + \partial_\nu h_{\lambda\mu} \) is the field strength tensor of a massive antisymmetric tensor field \( h_{\mu\nu} \) (the so-called Kalb-Ramond field [18]). This antisymmetric spin-1 tensor field describes a massive dual vector boson. Thus, the path-integral duality transformation is just a way of getting a coupling of this boson to a string world-sheet, rather than to a world-line (as it takes place in the usual case of the Wilson loop). In particular, carrying out in Eq. (3) the Gaussian integration over the Kalb-Ramond field, one gets a representation of the partition function in terms of an interaction of the elements of the world-sheet \( \Sigma \), mediated by the propagator of this field [11]. In what follows, we shall derive another useful representation for the partition function, where it will be expressed directly in terms of magnetic monopole currents.

Notice that according to the equation of motion for the field \( A_\mu \), the absence of external electric currents is expressed by the equation \( \partial_\mu F_{\mu\nu} = 0 \), where \( F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + gh_{\mu\nu} \). Regarding \( F_{\mu\nu} \) as a full electromagnetic field strength tensor, one can write for it the corresponding Bianchi identity modified by the monopoles, \( \partial_\mu \tilde{F}_{\mu\nu} = g\partial_\rho \tilde{h}_{\rho\nu} \). This identity means that the monopole current can be written in terms of the Kalb-Ramond field \( h_{\mu\nu} \) as

\[
j_\mu = g\partial_\nu \tilde{h}_{\nu\mu},
\]
which manifests its conservation.

It is also instructive to write down the equation of motion for the Kalb-Ramond field in terms of the introduced full electromagnetic field strength tensor. This equation has the form

$$\mathcal{F}_{\nu\lambda} = \frac{g}{m^2} \partial_\mu H_{\mu\nu\lambda} + \frac{i\pi}{2g} \Sigma_{\nu\lambda},$$

where \( m = 2g\eta \) stands for the mass of the dual gauge boson (equal to the mass of the Kalb-Ramond field). By virtue of conservation of the vorticity tensor current for the closed string world-sheets, \( \partial_\mu \Sigma_{\mu\nu} = 0 \), this equation again yields the condition of absence of external electric currents, \( \partial_\mu \mathcal{F}_{\mu\nu} = 0 \).

Let us now turn ourselves to a derivation of the monopole current representation for the partition function of DAHM. To this end, we shall first resolve the equation \( g^2 \varepsilon_{\mu\nu\lambda\rho} \partial_\nu h_{\lambda\rho} = -j_\mu \) w.r.t. \( h_{\mu\nu} \), which yields

$$h_{\mu\nu}(x) = -\frac{1}{2\pi^2 g^2} \varepsilon_{\mu\nu\lambda\rho} \int d^4y \frac{(x - y)_\lambda}{|x - y|^4} j_\rho(y).$$

Next, we get the following expressions for various terms on the R.H.S. of Eq. (3)

$$H^2_{\mu\nu\lambda} = \frac{6}{g^2} j^2_\mu, \quad \int d^4x h^2_{\mu\nu} = \frac{1}{2\pi^2 g^2} \int d^4x d^4y j_\mu(x) \frac{1}{(x - y)^2} j_\mu(y).$$

Bringing all this together and performing in Eq. (3) the hypergauge transformation \( h_{\mu\nu} \rightarrow h_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu \) with the gauge function \( \Lambda_\mu = -\frac{1}{g} A_\mu \), which eliminates the field \( A_\mu \), we finally arrive at the desired monopole current representation, which has the form

$$\mathcal{Z}_{4D \text{DAHM}} = \int Dx_\mu(\xi) Dh_{\mu\nu} \exp \left\{ -\frac{1}{2\pi^2} \int d^4x d^4y j_\mu(x) \frac{1}{(x - y)^2} j_\mu(y) - \frac{2}{m^2} \int d^4x j^2_\mu + \frac{2\pi i}{g} S_{\text{int.}}(\Sigma, j_\mu) \right\}. \quad (5)$$

The first term in the exponent on the R.H.S. of Eq. (5) has the form of the Biot-Savart energy of the electric field generated by monopole currents \[7\], the second term corresponds to the (gauged) kinetic energy of Cooper pairs, and the term

$$S_{\text{int.}}(\Sigma, j_\mu) = \frac{1}{4\pi^2} \varepsilon_{\mu\nu\lambda\rho} \int d^4x d^4y j_\mu(x) \frac{(y - x)_\nu}{|y - x|^4} \Sigma_{\lambda\rho}(y) \quad (6)$$

describes the interaction of the string world-sheet with the monopole current \( j_\mu \). This interaction can obviously be rewritten in the form \( S_{\text{int.}} = \int d^4x j_\mu H^\text{str.}_\mu \), where \( H^\text{str.}_\mu \) is the four-dimensional analogue of the magnetic induction, produced by the electric string according to the equation

$$\varepsilon_{\mu\nu\lambda\rho} \partial_\lambda H^\text{str.}_\rho = \Sigma_{\mu\nu}. \quad (7)$$

Notice, that if one includes an additional current describing an external monopole,

$$j^{\text{ext.}}_\mu(x) = g \oint \frac{dx_\mu(\tau) \delta(x - x(\tau))}{r}, \quad (8)$$
there arises among others an interaction term \(\hat{\mathcal{G}}\), which in this case takes the form \(S_{\text{int.}} = g L(\Sigma, \Gamma)\), where \(L(\Sigma, \Gamma)\) is simply the Gauss linking number of the world-sheet \(\Sigma\) with the contour \(\Gamma\) \(^1\).

Clearly, the functional integral over the Kalb-Ramond field in Eq. (3) has to be evaluated at the saddle-point

\[
h_{\mu\nu}^{\text{s.p.}}(x) = \frac{i g \eta^3}{\pi} \int d\sigma_{\mu\nu}(x(\xi)) \frac{K_1(m|x - x(\xi)|)}{|x - x(\xi)|},
\]

where from now on \(K_n\)'s, \(n = 0, 1, 2\), stand for the modified Bessel functions. By virtue of Eq. (4), the monopole current can then be expressed via the string world-sheet \(\Sigma\) as follows

\[
j_{\mu}(x) = \frac{i m^2 \eta}{8\pi} \varepsilon_{\mu\nu\rho} \int d\sigma_{\mu\nu}(x(\xi))(x - x(\xi))^{\nu} \frac{(x - x(\xi))^{\nu}}{(x - x(\xi))^2} \}
\]

It is straightforward to extend the above analysis to the case of the Abelian-projected \(SU(3)\)-gluodynamics \(\hat{\mathcal{G}}\). There, the partition function (3) is replaced by

\[
\mathcal{Z}_{SU(3)} = \int D\vec{B}_{\mu} D\theta^a_D \exp \left\{ - \int d^4x \left[ \frac{1}{4} \vec{F}^2_{\mu\nu} + \frac{\eta^2}{2} \sum_{a=1}^3 (\partial_{\mu} \theta_a - g \varepsilon_a \vec{B}_{\mu})^2 \right] \right\}
\]

with \(\vec{F}_{\mu\nu} = \partial_{\mu} \vec{B}_{\nu} - \partial_{\nu} \vec{B}_{\mu}\) standing for the field strength tensor of the Abelian vector potential \(\vec{B}_{\mu} \equiv (B_\mu^3, B_\mu^8)\) dual to the usual vector potential \(\vec{A} \equiv (A^3_\mu, A^8_\mu)\). Next, on the R.H.S. of Eq. (5),

\[
\varepsilon_1 = (1, 0), \quad \varepsilon_2 = \left( -\frac{1}{2}, -\frac{\sqrt{3}}{2} \right), \quad \varepsilon_3 = \left( -\frac{1}{2}, \frac{\sqrt{3}}{2} \right)
\]

denote the so-called root vectors, and the phases of three magnetic Higgs fields, \(\theta_a\)'s, obey the constraint \(\sum_{a=1}^3 \theta_a = 0\), which is due to the fact that the unitary group under study is special. The singular parts of these fields are related to the world-sheets of the strings of three types as

\[
\varepsilon_{\mu\nu\rho} \partial_{\lambda} \partial_{\mu} \theta^a_D(\sigma) = 2\pi \sum_{a=1}^3 \delta(x - x(\xi)) \}
\]

where \(x_a \equiv x_a(\xi)\) is a four-vector parametrizing the world-sheet \(\Sigma_a\).

Performing the path-integral duality transformation of Eq. (3) by making use of Eq. (10), we obtain (see Ref. [12] for the details)

\[
\mathcal{Z}_{SU(3)} = \int D\sigma_{\mu\nu}(x) \frac{1}{12\eta^2} \left( H_{\mu\nu\lambda}^a \right)^2 -
\]

\(^1\)Topological interactions of this kind are sometimes interpreted as a 4D analogue of the Aharonov-Bohm effect. In particular, this interaction, albeit for the current of an external electrically charged particle with the string world-sheet, emerges in the string representation for the Wilson loop of this particle in AHM.\(^3\).
\[-i\pi h^a_\mu \Sigma^a_{\mu\nu} + \left( g \frac{\sqrt{3}}{2\sqrt{2}} h^a_\mu + \partial_\mu A^a_\nu - \partial_\nu A^a_\mu \right)^2 \}\right\},

(11)

where \(A^a_\mu \equiv \varepsilon_a \tilde{A}_\mu\). Eq. (11) means that the three monopole currents can be expressed in terms of three Kalb-Ramond fields as \(j^a_\mu = g \frac{\sqrt{3}}{2\sqrt{2}} \partial_\mu \tilde{h}^a_\mu\) (cf. Eq. (1)). Finally, rewriting Eq. (11) via these currents and resolving the constraint \(\sum_3 a=1 \Sigma^a_{\mu\nu} = 0\) by integrating over one of the world-sheets (for concreteness, \(x^3_\mu(\xi)\)), we obtain

\[ Z_{SU(3)} = \int Dx_\mu^1(\xi) Dx_\mu^2(\xi) Dh^a_\mu \exp \left\{ -\frac{1}{2\pi^2} \int d^4x d^4y j^a_\mu(x) \frac{1}{(x-y)^2} j^a_\mu(y) - \frac{2}{m_B^2} \int d^4x (j^a_\mu)^2 + +4\pi i \sqrt{\frac{2}{3}} g \left[ S_{\text{int.}} \left( \Sigma^1, j^1_\mu \right) + S_{\text{int.}} \left( \Sigma^2, j^2_\mu \right) - S_{\text{int.}} \left( \Sigma^1, j^3_\mu \right) - S_{\text{int.}} \left( \Sigma^2, j^3_\mu \right) \right] \right\}, \]

(12)

where \(m_B = \sqrt{\frac{3}{2} g \eta}\) stands for the mass of the fields \(B^3_\mu\) and \(B^8_\mu\), which they acquire due to the Higgs mechanism. Eq. (12) is the desired representation for the partition function of the Abelian-projected \(SU(3)\)-gluodynamics in terms of three monopole currents, which should be evaluated at the saddle-point. The terms in square brackets on its R.H.S. yield an interference between various possibilities of the interaction between the string world-sheets and monopole currents in this model to occur.

For illustrations, let us establish a correspondence of the above results to the 3D ones. Namely, let us derive a 3D analogue of Eq. (3), i.e. find a representation in terms of the monopole currents of the dual Ginzburg-Landau model. There, Eq. (2) is replaced by

\[ \varepsilon_{\mu\nu\lambda} \partial_\nu \partial_\lambda \theta^{\text{sing.}}(\bar{x}) = 2\pi \delta_\mu(\bar{x}). \]

(13)

Here, on the R.H.S. stands the so-called vortex density with \(\delta_\mu(\bar{x}) \equiv \int L \delta(x - \bar{y}(\tau))\) being the transverse \(\delta\)-function defined w.r.t. the electric vortex line \(L\), parametrized by the vector \(\bar{y}(\tau)\). This line is closed in the case under study, i.e. in the absence of external quarks, which means that \(\partial_\mu \delta_\mu = 0\). Performing again by virtue of Eq. (13) the path-integral duality transformation of the partition function (1) with the 3D action, we get for it the following representation

\[ Z_{3D\text{DAHM}} = \int D\varphi Dy_\mu(\tau) Dh_\mu \exp \left\{ -\int d^3x \left[ \frac{1}{4\eta^2} (\partial_\nu h_\mu - \partial_\mu h_\nu)^2 - -2\pi i h_\mu \delta_\mu + \left( g \sqrt{2} h_\mu + \partial_\mu \varphi \right)^2 \right] \right\}. \]

(14)

Notice, that the Kalb-Ramond field has now reduced to a massive one-form field \(h_\mu\) with the mass \(m = 2g\eta\), as well as the \(A_\mu\)-field reduced to a scalar \(\varphi\). Analogously to the 4D case, the field \(\mathcal{E}_\mu \equiv g \sqrt{2} h_\mu + \partial_\mu \varphi\) can be regarded as a full electric field, defined via the full dual electromagnetic field strength tensor as \(\mathcal{E}_\mu = \frac{1}{2} \varepsilon_{\mu\nu\lambda} \mathcal{F}_{\nu\lambda}\). The absence of external quarks is now expressed by the equation \(\partial_\mu \mathcal{E}_\mu = 0\), following from the equation of motion for the field \(\varphi\). Correspondingly, the monopole currents are defined as \(j_\mu = \partial_\nu \mathcal{F}_{\nu\mu} = g \sqrt{2} \varepsilon_{\mu\nu\lambda} \partial_\nu h_\lambda\) and are manifestly conserved. Notice
also, that the condition of closeness of the vortex lines, \( \partial_\mu \delta_\mu = 0 \), unambiguously exhibits itself as a condition of absence of external quarks, \( \partial_\mu \mathcal{E}_\mu = 0 \), by virtue of equation of motion for the field \( h_\mu \), which can be written in the form

\[
\mathcal{E}_\mu = \frac{1}{g \sqrt{2}} \left[ \frac{i}{2\pi} \partial_\nu (\partial_\nu h_\mu - \partial_\mu h_\nu) + i\pi \delta_\mu \right].
\]

Next, after performing the gauge transformation \( h_\mu \to h_\mu + \partial_\mu \gamma \) with the gauge function \( \gamma = -\frac{1}{g \sqrt{2}} \varphi \), the field \( \varphi \) drops out. Expressing \( h_\mu \) via \( j_\mu \) as follows

\[
h_\mu(\vec{x}) = -\frac{1}{4 \sqrt{2} \pi g} \varepsilon_{\mu \nu \lambda} \frac{\partial}{\partial x_\nu} \int d^3 y \frac{j_\lambda(y)}{|x - y|},
\]

and substituting this expression into the R.H.S. of Eq. (14), we finally arrive at the desired representation for the partition function of 3D DAHM in terms of the monopole currents

\[
Z_{3D \text{DAHM}} = \int Dy_\mu(\tau) Dh_\mu \exp \left\{ -\frac{1}{4\pi} \int d^3 x d^3 y j_\mu(\vec{x}) \frac{1}{|x - y|} j_\mu(\vec{y}) + \frac{1}{m^2} \int d^3 x j_\mu^2 + \frac{i\sqrt{2} \pi}{g} S_{\text{int.}}(L, j_\mu) \right\}.
\]

The interaction term of the electric vortex line with the monopole current now takes the form

\[
S_{\text{int.}}(L, j_\mu) = \frac{1}{4\pi} \varepsilon_{\mu \nu \lambda} \int d^3 x d^3 y j_\mu(\vec{x}) \frac{(\vec{y} - \vec{x}) \cdot \varepsilon_\lambda(y)}{|\vec{y} - \vec{x}|} S_{\text{int.}}(L, j_\mu).
\]

This interaction term can be again rewritten as

\[
S_{\text{int.}} = \int d^3 x j_\mu H_\mu^\text{nor},
\]

where the magnetic induction, generated by the electric vortex line, obeys the equation \( \varepsilon_{\mu \nu \lambda} \partial_\nu H_\lambda^\text{nor} = \delta_\mu \). In the particular case, when one introduces an external current of the form (8), there emerges a term \( S_{\text{int.}} = g\tilde{L}(L, \Gamma) \) with \( \tilde{L}(L, \Gamma) \) standing for the Gauss linking number of the contours \( L \) and \( \Gamma \). The functional integral over the field \( h_\mu \) in Eq. (15) should again be evaluated at the saddle-point \( h_\mu^{\text{s.p.}} \), which is determined by the classical equation of motion, following from Eq. (14) after gauging away the field \( \varphi \). This saddle-point has the form

\[
h_\mu^{\text{s.p.}}(\vec{x}) = \frac{i \eta}{2} \oint_L d\mu(\tau) e^{-m|\vec{x} - \vec{y}(\tau)|}.
\]

which yields the following expression for the monopole current

\[
j_\mu(\vec{x}) = \frac{i \eta}{\sqrt{2}} \varepsilon_{\mu \nu \lambda} \oint_L d\mu(\tau) \frac{(\vec{x} - \vec{y}(\tau))_\mu}{|\vec{x} - \vec{y}(\tau)|} \left( m + \frac{1}{|\vec{x} - \vec{y}(\tau)|} \right) e^{-m|\vec{x} - \vec{y}(\tau)|}.
\]

In the next Section, we shall investigate the relation between 3D AHM with external monopoles and 3D compact QED.

### 3 Vacuum Correlators and String Representation of 3D Compact QED

In this Section, we shall revisit 3D compact QED and find its string representation in the form of an integral over the monopole densities. Besides that, we shall investigate vacuum correlators in the weak-field limit, and demonstrate the relation of this theory to 3D AHM with monopoles.
The most important feature of 3D compact QED, which distinguishes it from the noncompact case, is the existence of magnetic monopoles. Their general configuration is the Coulomb gas with the action \[ S_{\text{mon.}} = g^2 \sum_{a < b} q_a q_b \left( \Delta^{-1} \right) (\vec{z}_a, \vec{z}_b) + S_0 \sum_a q_a^2, \] (16)

where \( \Delta \) is the 3D Laplace operator, and \( S_0 = \text{const} \cdot e^2 \). Here, similarly to Ref. \[17\], we have adopted standard Dirac notations, where \( eg = 2\pi n \), restricting ourselves to the monopoles of the minimal charge, i.e. setting \( n = 1 \). Then, the partition function of the grand canonical ensemble of monopoles associated with the action (16) reads

\[ Z_{\text{mon.}} = \sum_{N=1}^{+\infty} \sum_{q_a = \pm 1} \frac{\zeta^N}{N!} \prod_{i=1}^N \int d^3 z_i \exp \left[ -\frac{\pi}{2e^2} \int d^3 x d^3 y \rho_{\text{gas}} (\vec{x}) \frac{1}{|\vec{x} - \vec{y}|} \rho_{\text{gas}} (\vec{y}) \right], \] (17)

where \( \rho_{\text{gas}} (\vec{x}) = \sum_a q_a \delta (\vec{x} - \vec{z}_a) \) is the monopole density, corresponding to the gas configuration. Here, a single monopole weight \( \zeta \propto \exp (-S_0 \sum_a q_a^2) \) has the dimension of mass \( 3 \) and is usually referred to as fugacity. Notice also that, as usual, we have restricted ourselves to the values \( q_a = \pm 1 \), since at large values of the magnetic coupling constant \( g \), monopoles with \( |q| > 1 \) turn out to be unstable and tend to dissociate into the monopoles with \( |q| = 1 \). Below in this Section, it will be demonstrated that the limit of a small gauge boson mass (which takes place e.g. at large \( g \)) is just the case, when 3D compact QED follows from 3D AHM with external monopoles.

Next, Coulomb interaction can be made local, albeit nonlinear one, by introduction of an auxiliary scalar field \[17\]

\[ Z_{\text{mon.}} = \int D\chi \exp \left\{ - \int d^3 x \left[ \frac{1}{2} \left( \partial_\mu \chi \right)^2 - 2\zeta \cos (g\chi) \right] \right\}. \] (18)

The magnetic mass \( m = g \sqrt{2\zeta} \) of the field \( \chi \), following from the quadratic term in the expansion of the cosine on the R.H.S. of Eq. (18), is due to the Debye screening in the monopole plasma. The next, quartic, term of the expansion determines the coupling constant of the diagrammatic expansion for the monopole gas, which is therefore exponentially small and proportional to \( g^4 \exp (-\text{const} \cdot g^2) \).

Let us now cast the partition function (18) into the form of an integral over the monopole densities. This can be done by introducing into Eq. (17) a unity of the form

\[ \int D\rho \delta (\rho (\vec{x}) - \rho_{\text{gas}} (\vec{x})) = \int D\rho D\mu \exp \left\{ i \int d^3 x \mu \rho - \sum_a q_a \mu (\vec{z}_a) \right\}. \]

Then, passing to the representation of the partition function in terms of the field \( \chi \), changing the integration variable, \( \mu \to \phi \equiv g\chi - \mu \), and integrating the field \( \chi \) out, we get

\[ Z_{\text{mon.}} = \int D\rho D\phi \exp \left\{ -\frac{\pi}{2e^2} \int d^3 x d^3 y \rho (\vec{x}) \frac{1}{|\vec{x} - \vec{y}|} \rho (\vec{y}) + \int d^3 x \left( 2\zeta \cos \phi - i\phi \rho \right) \right\}. \] (19)

Finally, integrating over the field \( \phi \) by resolving the corresponding saddle-point equation,

\[ \sin \phi = -\frac{i\rho}{2\zeta}, \] (20)
we arrive at the desired representation for the partition function

\[ Z_{\text{mon.}} = \int D\rho \exp \left\{ - \frac{\pi}{2 e^2} \int d^3 x d^3 y \rho(\vec{x}) \frac{1}{|\vec{x} - \vec{y}|} \rho(\vec{y}) + V[\rho] \right\}, \quad (21) \]

where \[ V[\rho] = \int d^3 x \left\{ \rho \ln \left[ \frac{\rho}{2\zeta} + \sqrt{1 + \left( \frac{\rho}{2\zeta} \right)^2} \right] - 2\zeta \sqrt{1 + \left( \frac{\rho}{2\zeta} \right)^2} \right\} \quad (22) \]
is the parabolic-type effective monopole potential, whose asymptotic behaviours at \( \rho \ll \zeta \) and \( \rho \gg \zeta \) read

\[ V[\rho] \to \int d^3 x \left( -2\zeta + \frac{\rho^2}{4\zeta} \right) \quad (23) \]

and

\[ V[\rho] \to \int d^3 x \left[ \rho \left( \ln \frac{\rho}{\zeta} - 1 \right) \right], \]
respectively. Notice, that during the integration over the field \( \phi \) in Eq. (19), we have chosen only the real branch of the solution to the saddle-point equation (20) and disregarded the complex ones.

The obtained representation for the partition function in terms of the monopole densities can be immediately applied to the calculation of the coefficient function \( D_{\text{mon.}}(x^2) \), related to the bilocal correlator of the field strength tensors as follows \[ 15, 16 \]

\[ \langle F_{\lambda \nu}(\vec{x}) F_{\mu \rho}(0) \rangle_{A_{\mu, \rho}} = \left( \delta_{\lambda \mu} \delta_{\nu \rho} - \delta_{\lambda \rho} \delta_{\nu \mu} \right) D_{\text{mon.}}(x^2) + \]

\[ + \frac{1}{2} \left[ \partial_\lambda \left( x_\mu \delta_{\nu \rho} - x_\rho \delta_{\nu \mu} \right) + \partial_\nu \left( x_\rho \delta_{\lambda \mu} - x_\mu \delta_{\lambda \rho} \right) \right] D_{\text{full}}^1(x^2), \quad (24) \]

where the average over the monopole densities is defined by the partition function (21), whereas the \( A_{\mu} \)-average is defined as

\[ \langle \ldots \rangle_{A_{\mu}} \equiv \int DA_{\mu} (\ldots) \exp \left( -\frac{1}{4e^2} \int d^3 x F_{\mu \nu}^2 \right). \]

In Eq. (24), \( F_{\mu \nu} = F_{\mu \nu} + F_{\mu \nu}^M \) stands for the full electromagnetic field strength tensor, which includes also the monopole part

\[ F_{\mu \nu}^M(\vec{x}) = -\frac{1}{2} \varepsilon_{\mu \nu \lambda} \frac{\partial}{\partial x_\lambda} \int d^3 y \frac{\rho(\vec{y})}{|\vec{x} - \vec{y}|}, \]

This monopole part yields the R.H.S. of the Bianchi identities modified by the monopoles,

\[ \partial_\mu \mathcal{H}_\mu = 2\pi \rho, \quad (25) \]

where \( \mathcal{H}_\mu = \frac{1}{2} \varepsilon_{\mu \nu \lambda} F_{\nu \lambda} \) stands for the full magnetic induction. Eqs. (24) and (25) then lead to the following equation for the function \( D_{\text{mon.}} \).
\[ \Delta \mathcal{D}^{\text{mon.}}(x^2) = -4\pi^2 \langle \rho(x) \rho(0) \rangle_\rho, \]  

which in fact is a 3D analogue of the 4D equation [14]

\[ \left( \partial_\mu \partial_\nu - \partial^2 \delta_{\mu\nu} \right) \mathcal{D}^{\text{mon.}}(x^2) = \langle j_\mu(x) j_\nu(0) \rangle. \]

The correlator standing on the R.H.S. of Eq. (26) can be found in the limit of small monopole densities, \( \rho \ll \zeta \). By making use of Eqs. (21) and (23), we obtain

\[ \langle \rho(x) \rho(0) \rangle = -\frac{\zeta}{2\pi} \Delta e^{-m|x|}. \]

Then, demanding that \( \mathcal{D}^{\text{mon.}}(x^2 \to \infty) \to 0 \), we get by the maximum principle for the harmonic functions the desired expression for the function \( \mathcal{D}^{\text{mon.}} \) in the low-density limit

\[ \mathcal{D}^{\text{mon.}}(x^2) = 2\pi \zeta e^{-m|x|}. \]  

We see that in the model under study, the correlation length of the vacuum [13, 16] \( T_g \), i.e. the distance at which the function \( \mathcal{D}^{\text{mon.}} \) decreases, corresponds to the inverse mass of the field \( \chi \), \( m^{-1} \) (cf. the case of Abelian-projected theories, studied in Refs. [11, 12]). The coefficient function \( \mathcal{D}^{\text{full}}_1(x^2) \) will be derived later on.

Let us now proceed to the problem of string representation of 3D compact QED. To this end, let us consider an expression for the Wilson loop and try to represent it as an integral over the world-sheets \( \Sigma \)'s, bounded by the contour \( C \). By virtue of the Stokes theorem, the Wilson loop can be rewritten in the form

\[ \langle W(C) \rangle = \left\langle \exp \left( \frac{i}{2} \int_{\Sigma} d\sigma_{\mu\nu} F_{\mu\nu} \right) \right\rangle_{A_{\mu,\rho}} = \left\langle \exp \left( i \int_{\Sigma} d\sigma_{\mu} H_{\mu} \right) \right\rangle_{A_{\mu,\rho}} = \langle W(C) \rangle_{A_{\mu}} \left\langle \exp \left( \frac{i}{2} \int d^3x \rho(x) \eta(x) \right) \right\rangle_\rho, \]  

where the free photon contribution reads

\[ \langle W(C) \rangle_{A_{\mu}} = \left\langle \exp \left( i \oint_{C} A_{\mu} dx_{\mu} \right) \right\rangle_{A_{\mu}} = \exp \left( -\frac{e^2}{8\pi} \oint_{C} dx_{\mu} \oint_{C} dy_{\mu} \frac{1}{|x-y|} \right). \]  

In Eq. (28), \( d\sigma_{\mu} \equiv \frac{1}{2} \varepsilon_{\mu\nu\lambda} d\sigma_{\nu\lambda} \), and \( \eta(x) = \oint_{\Sigma} d\sigma_{\mu} \left( \frac{\delta}{\delta A_{\mu}^{\nu}} \eta(\vec{y}) \right) \frac{1}{|x-\vec{y}|} \) stands for the solid angle under which the surface \( \Sigma \) is seen by an observer at the point \( \vec{x} \). Notice that due to the Gauss law, in the case when \( \Sigma \) is a closed surface surrounding the point \( \vec{x} \), \( \eta(\vec{x}) = 4\pi \), which is the standard result for the total solid angle in 3D.

Eq. (28) seems to contain some discrepancy, since its L.H.S. depends only on the contour \( C \), whereas the R.H.S. depends on an arbitrary surface \( \Sigma \), bounded by \( C \). However, this actually occurs to be not a discrepancy, but a key point in the construction of the desired string representation. The resolution of the apparent paradox lies in the observation that during the derivation of the effective monopole potential (22), we have accounted only for the one, namely real, branch of the solution to the saddle-point equation (20). Actually, however, one should sum up over all
the (complex-valued) branches of the integrand of the effective potential \( \mathcal{V} \) at every space point \( \vec{x} \). This requires to replace \( V[\rho] \) by

\[
V_{\text{total}}[\rho] = \sum_{\text{branches}} \int d^3x \left\{ \pm \rho \ln \left[ \frac{\rho}{2\zeta} + \sqrt{1 + \left( \frac{\rho}{2\zeta} \right)^2} \right] \mp 2\zeta \sqrt{1 + \left( \frac{\rho}{2\zeta} \right)^2} + \left( 0 + i\pi \right) + 2\pi in \right\},
\]

\( n = 0, \pm 1, \pm 2, \ldots \), where adding of 0 or \( i\pi \) corresponds to choosing of upper or lower sign, respectively. Such a summation over the branches of the multivalued potential in the expression for the Wilson loop

\[
\langle W(C) \rangle = \langle W(C) \rangle_{\text{A}} \int D\rho \exp \left\{ - \left[ \frac{\pi}{2e^2} \int d^3xd^3y \rho (\vec{x}) \frac{1}{|\vec{x} - \vec{y}|} \rho (\vec{y}) + V_{\text{total}}[\rho] - \frac{i}{2} \int d^3x \rho (\vec{x}) \eta (\vec{x}) \right] \right\}
\]

thus restores the independence of the choice of the world-sheet. Notice, that from now on we omit an inessential normalization factor, implying everywhere the normalization \( \langle W(0) \rangle = 1 \).

It is worth noting that the obtained string representation (30) has been for the first time derived in another, more indirect, way in Ref. [19]. It is therefore instructive to establish a correspondence between our interpretation and the one of that paper.

The main idea of Ref. [19] was to calculate the Wilson loop starting with the direct definition of this average in a sense of the partition function (17) of the monopole gas. The corresponding expression has the form

\[
\langle W(C) \rangle_{\text{mon.}} = \sum_{N=1}^{\infty} \sum_{q_{a}=\pm 1} \zeta^N \prod_{i=1}^{N} \int d^3z_i \exp \left[ - \frac{\pi}{2e^2} \int d^3xd^3y \rho_{\text{gas}} (\vec{x}) \frac{1}{|\vec{x} - \vec{y}|} \rho_{\text{gas}} (\vec{y}) + i \int d^3x \rho_{\text{gas}} (\vec{x}) \eta (\vec{x}) \right] = \int D\chi \exp \left\{ - \int d^3x \left[ \frac{1}{2} (\partial_\mu \chi)^2 - 2\zeta \cos \left( g\chi + \eta \right) \right] \right\} = \int D\varphi \exp \left\{ - \int d^3x \left[ \frac{e^2}{8\pi^2} \left( \partial_\mu \varphi - \frac{1}{2} \partial_\mu \eta \right)^2 - 2\zeta \cos \varphi \right] \right\},
\]

where \( \varphi \equiv g\chi + \eta \).

Next, one can prove the following equality

\[
\exp \left[ - \frac{e^2}{8\pi^2} \oint dx_\mu \oint dy_\mu \frac{1}{|\vec{x} - \vec{y}|} - \frac{e^2}{8\pi^2} \int d^3x \left( \partial_\mu \varphi - \frac{1}{2} \partial_\mu \eta \right)^2 \right] = \int Dh_{\mu\nu} \exp \left[ - \int d^3x \left( i\varphi \varepsilon_{\mu\nu\lambda} \partial_\mu h_{\nu\lambda} + g^2 h_{\mu\nu}^2 - 2\pi ih_{\mu\nu} \Sigma_{\mu\nu} \right) \right],
\]

which makes it possible to represent the contribution of the kinetic term on the R.H.S. of Eq. (31) and the free photon contribution (29) to the Wilson loop as an integral over the Kalb-Ramond
The only nontrivial point necessary to prove this equality is an expression for the derivative of the solid angle. One has

$$\partial_\lambda \eta (\vec{x}) = \int_\Sigma \left( d\sigma_\mu (\vec{y}) \frac{\partial}{\partial y_\lambda} - d\sigma_\lambda (\vec{y}) \frac{\partial}{\partial y_\mu} \right) \frac{1}{|\vec{x} - \vec{y}|} + \int_\Sigma d\sigma_\lambda (\vec{y}) \Delta \frac{1}{|\vec{x} - \vec{y}|}. \quad (33)$$

Applying to the first integral on the R.H.S. of Eq. (33) the Stokes theorem in the operator form,

$$d\sigma_\mu \frac{\partial}{\partial y_\lambda} - d\sigma_\lambda \frac{\partial}{\partial y_\mu} \rightarrow \epsilon_{\mu\nu\lambda} dy_\nu,$$

one finally obtains

$$\partial_\lambda \eta (\vec{x}) = \epsilon_{\lambda\mu\nu} \frac{\partial}{\partial x_\mu} \oint_C dy_\nu \frac{1}{|\vec{x} - \vec{y}|} - 4\pi \int_\Sigma d\sigma_\lambda (\vec{y}) \delta (\vec{x} - \vec{y}).$$

Making use of this result and carrying out the Gaussian integral over the field $h_{\mu\nu}$, one can demonstrate that both sides of Eq. (32) are equal to

$$\langle W (C) \rangle_{\text{weak-field}} = \int D h_{\mu\nu} \exp \left\{- \frac{e^2}{2} \left[ \frac{1}{4\pi^2} \int d^3 x (\partial_\mu \varphi)^2 + \frac{1}{\pi} \int d\sigma_\mu \partial_\mu \varphi + \int \Sigma \left( d\sigma_\mu (\vec{x}) \right) \int \Sigma d\sigma_\lambda (\vec{y}) \delta (\vec{x} - \vec{y}) \right] \right\}.$$ 

(34)

making use of this result and carrying out the Gaussian integral over the field $h_{\mu\nu}$, one can demonstrate that both sides of Eq. (32) are equal to

$$\langle W (C) \rangle_{\text{weak-field}} = \int D h_{\mu\nu} \exp \left\{- \frac{e^2}{2} \left[ \frac{1}{4\pi^2} \int d^3 x (\partial_\mu \varphi)^2 + \frac{1}{\pi} \int d\sigma_\mu \partial_\mu \varphi + \int \Sigma \left( d\sigma_\mu (\vec{x}) \right) \int \Sigma d\sigma_\lambda (\vec{y}) \delta (\vec{x} - \vec{y}) \right] \right\}.$$ 

(34)

where the world-sheet independence of the R.H.S. is again provided by the summation over the branches of the multivalued action, which is now the action of the Kalb-Ramond field.

Comparing now Eqs. (34) and (34), we see that the Kalb-Ramond field is indeed related to the monopole density via the equation $\epsilon_{\mu\nu\lambda} \partial_\mu h_{\nu\lambda} = \rho$. Thus, a conclusion following from the representation of the full Wilson loop in terms of the integral over the Kalb-Ramond field is that this field is simply related to the sum of the photon and monopole field strength tensors as $h_{\mu\nu} = \frac{1}{4\pi} F_{\mu\nu}$. In the formal language, such a decomposition of the Kalb-Ramond field is just the essence of the Hodge decomposition theorem.

Let us now consider the weak-field limit of Eq. (34) and again restrict ourselves to the real branch of the effective potential, i.e. replace $V_{\text{total}}[\epsilon_{\mu\nu\lambda} \partial_\mu h_{\nu\lambda}]$ by $V[\epsilon_{\mu\nu\lambda} \partial_\mu h_{\nu\lambda}]$. This yields the following expression for the Wilson loop

$$\langle W (C) \rangle_{\text{weak-field}} = \int D h_{\mu\nu} \exp \left\{- \frac{1}{6\zeta} H_{\mu\nu}^2 + \frac{1}{2\pi i} h_{\mu\nu} \Sigma_{\mu\nu} \right\}.$$ 

(35)
Notice, that the mass of the Kalb-Ramond field resulting from this equation is equal to the mass \( m \) of the field \( \chi \) from Eq. (18).

One can now see that Eq. (35) is quite similar to the 3D version of Eq. (3) (with the \( A_\mu \)-field gauged away) we had in the DAHM case. However, the important difference from DAHM is that restricting ourselves to the real branch of the potential, we have violated the surface independence of the R.H.S. of Eq. (35). This problem is similar to the one which appears in SVM \([15, 16]\), where in the expression for the Wilson loop, written via the non-Abelian Stokes theorem and cumulant expansion, one disregards all the cumulants higher than the bilocal one (the so-called bilocal approximation). There, the surface independence is restored by replacing \( \Sigma \) by the surface of the minimal area, \( \Sigma_{\text{min}} = \Sigma_{\text{min}}[C] \), bounded by the contour \( C \). Let us follow this recipe, after which the quantity

\[
S_{\text{str}} = - \ln \langle W(C) \rangle_{\text{weak-field}} \bigg|_{\Sigma \to \Sigma_{\text{min}}},
\]

(36)
can be considered as a weak-field string effective action of 3D compact QED.

The integration over the Kalb-Ramond field in Eq. (35) is now almost the same as the one of Ref. \([11]\) and yields

\[
\langle W(C) \rangle_{\text{weak-field}} \bigg|_{\Sigma \to \Sigma_{\text{min}}} = \exp \left\{ - \frac{1}{8} \int_{\Sigma_{\text{min}}} d\sigma_{\lambda\nu} (\vec{x}) \int_{\Sigma_{\text{min}}} d\sigma_{\mu\rho} (\vec{y}) (F_{\lambda\nu}(\vec{x}) F_{\mu\rho}(\vec{y}))_{A_\mu,\rho} \right\},
\]

which is consistent with the result following directly from the cumulant expansion of Eq. (28). Here, the bilocal correlator is defined by Eq. (24) with the function \( D_{\text{mon}} \) given by Eq. (27) and \( \mathcal{D}_{1}^{\text{full}} = \mathcal{D}_{1}^{\text{phot}} + \mathcal{D}_{1}^{\text{mon}}, \) where the photon and monopole contributions read

\[
\mathcal{D}_{1}^{\text{phot}}(x^2) = \frac{e^2}{2\pi |\vec{x}|^3}
\]

and

\[
\mathcal{D}_{1}^{\text{mon}}(x^2) = \frac{e^2}{4\pi x^2} \left(m + \frac{1}{|\vec{x}|} \right) e^{-m|\vec{x}|},
\]

(37)
respectively. Since the approximation \( \rho \ll \zeta \), in which Eq. (27) has been derived, is just the weak-field limit, in which Eq. (35) follows from Eq. (14), coincidence of the function \( \mathcal{D}^{\text{mon}} \), following from the propagator of the Kalb-Ramond field, with the one of Eq. (27) confirms the consistency of our calculations.

Notice that by performing an expansion of the nonlocal string effective action (36) in powers of the derivatives w.r.t. the world-sheet coordinates \( \xi \), one gets the string tension of the Nambu-Goto term and the inverse bare coupling constant of the rigidity term, which read

\[
\sigma = \pi^2 \sqrt{\frac{2\zeta}{g}} \quad \text{and} \quad \frac{1}{\alpha_0} = - \frac{\pi^2}{8\sqrt{2\zeta}g^3},
\]

(38)
respectively. Similarly to the corresponding quantities in the Abelian-projected \( SU(2) \)- and \( SU(3) \)-gluodynamics, found in Refs. \([11]\) and \([12]\), both of them are nonanalytic in \( g \), which manifests the nonperturbative nature of string representation of all the three theories. Notice also, that the negative sign of \( \alpha_0 \) is important for the stability of the string world-sheets \([20]\).
We see that the long- and short distance asymptotic behaviours of the functions \( \mathcal{D} \) and \( \mathcal{D}_i \) have the same properties as the ones of the corresponding functions in QCD within SVM [21]. Namely, at large distances both of the functions \( \mathcal{D} \) and \( \mathcal{D}_i \) decrease exponentially with the correlation length \( m^{-1} \), and at such distances \( \mathcal{D}^{\text{mon}} \ll \mathcal{D}_i^{\text{mon}} \) due to the preexponential factor. In the same time, in the opposite case \( |\vec{x}| \ll m^{-1} \), the function \( \mathcal{D}^{\text{mon}}_i \) is much larger than the function \( \mathcal{D}^{\text{mon}} \), which also parallels the SVM results. Notice, however, that the short-distance similarity takes place only to the lowest order of perturbation theory in QCD, where its specific non-Abelian properties are not important.

It is also worth noting, that the above described asymptotic behaviours of the functions \( \mathcal{D}^{\text{mon}} \) and \( \mathcal{D}_i^{\text{mon}} \) match those of the corresponding functions in QCD within SVM [21]. This similarity, as well as the similarity of Eqs. (8) and (35), tells us that there should exist some relation between 3D compact QED and 3D AHM. In what follows, we shall demonstrate that such a relation really exists, namely 3D compact QED corresponds to the case of small gauge boson mass in the London limit of 3D AHM. Let us stress that in 3D, monopoles are considered as particles at rest, contrary to the 4D case, where they are generally treated as world-lines of moving particles. That is why, in order to end up with 3D compact QED (i.e. the partition function (17) of the monopole gas), one should start with 3D AHM with the scalar density \( \rho_{\text{gas}} \) of external monopoles at rest, rather than with DAHM. The corresponding partition function has the form

\[
\mathcal{Z}_{3\text{D AHM}} = \int DA_\mu D\theta^{\text{sing.}} D\theta^{\text{reg.}} \exp \left\{ - \int d^3x \left[ \frac{1}{4e^2} F_{\mu\nu}^2 + \frac{\eta^2}{2} (\partial_\mu A_\nu - A_\mu A_\nu)^2 \right] \right\}, \tag{39}
\]

Here, the full field strength tensor again reads \( F_{\mu\nu} = F_{\mu\nu}^M + F_{\mu\nu}^\ast \), where the monopole part obeys the equation (25) with the replacement \( \rho \rightarrow \rho_{\text{gas}} \). Making use of the relation (13) (where \( L \)'s are now open lines of magnetic vortices, ending at monopoles and antimonopoles), one can perform the path-integral duality transformation of the partition function (39), which yields

\[
\mathcal{Z}_{3\text{D AHM}} = \int D\varphi D\mu(\tau) Dh_{\mu} \exp \left\{ - \int d^3x \left[ \frac{1}{4\eta^2} (\partial_\mu h_{\mu} - \partial_\nu h_{\mu})^2 - 2\pi i h_{\mu} \delta_{\mu} + \left( \frac{e}{\sqrt{2}} h_{\mu} + \partial_\mu \varphi \right)^2 + \frac{i}{\sqrt{2}} \left( \frac{e}{\sqrt{2}} h_{\mu} + \partial_\mu \varphi \right) \frac{\partial}{\partial x_\mu} \int d^3y \frac{\rho_{\text{gas}} (\vec{y})}{|\vec{x} - \vec{y}|} \right] \right\}
\]

(cf. Eq. (41)). Performing again the gauge transformation, which eliminates the field \( \varphi \), and integrating over the field \( h_{\mu} \), we obtain

\[
\mathcal{Z}_{3\text{D AHM}} = \int Dy_{\mu}(\tau) \exp \left\{ - \frac{\pi \eta^2}{8} \int d^3xd^3y \frac{e^{-m_{A} |\vec{x} - \vec{y}|}}{|\vec{x} - \vec{y}|} \left[ 4\pi \left( \frac{1}{m_{A}^2} \rho_{\text{gas}} (\vec{x}) \rho_{\text{gas}} (\vec{y}) + \delta_{\mu} (\vec{x}) \delta_{\mu} (\vec{y}) \right) + \int d^3z \frac{\rho_{\text{gas}} (\vec{x}) \rho_{\text{gas}} (\vec{z})}{|\vec{y} - \vec{z}|} \right] \right\},
\]

where \( m_{A} = e\eta \) stands for the gauge boson mass. The integral

\[
\int d^3y \frac{e^{-m_{A} |\vec{x} - \vec{y}|}}{|\vec{x} - \vec{y}| |\vec{y} - \vec{z}|} = \int d^3u \frac{e^{-m_{A} |\vec{u}|}}{|\vec{u}| |\vec{x} - \vec{z} - \vec{u}|}
\]
can easily be calculated by expanding \( \frac{1}{|\vec{x} - \vec{y}|} \) in Legendre polynomials, and the result reads

\[
\frac{4\pi}{m_A^2 |\vec{x} - \vec{z}|} \left( 1 - e^{-m_A|\vec{x} - \vec{z}|} \right).
\]

Taking this into account, we can write down the final expression for the partition function of 3D AHM with external monopoles, which has the following simple form

\[
Z_{3D\text{ AHM}} = \int D\mu(\tau) \exp \left\{ -\frac{\pi \eta^2}{2} \int d^3x d^3y \left[ \frac{e^{-m_A|\vec{x} - \vec{y}|}}{|\vec{x} - \vec{y}|} \delta_\mu(\vec{x}) \delta_\mu(\vec{y}) + \frac{1}{m_A^2} \rho_{\text{gas}}(\vec{x}) \rho_{\text{gas}}(\vec{y}) \right]\right\}. \tag{40}
\]

The first term in square brackets on the R.H.S. of Eq. (40) represents again the Biot-Savart interaction between the points of the magnetic vortex (and also interaction between vortices, if we included several ones), which is Yukawa-type, i.e. their Coulomb interaction is screened by the condensate of electric Cooper pairs. Contrary, the interaction between \textit{external} monopoles, represented by the last term on the R.H.S. of Eq. (40) remains to be unscreened.

We now see, that when the gauge boson mass becomes small (i.e., for example, the magnetic coupling constant \( g = \frac{2\pi}{e} \) becomes large), the Biot-Savart term can be disregarded w.r.t. the interaction of external monopoles. In this limit,

\[
Z_{3D\text{ AHM}} \rightarrow \exp \left[ -\frac{\pi}{2e^2} \int d^3x d^3y \frac{\rho_{\text{gas}}(\vec{x}) \rho_{\text{gas}}(\vec{y})}{|\vec{x} - \vec{y}|} \right],
\]

which is just the statistical weight of the partition function of the monopole gas (17). Clearly, this result is in agreement with that of the corresponding limiting procedure applied directly to Eq. (39).

4 Summary

In the present paper, we have addressed two problems. The first of them was the investigation of the relation between confinement in the Abelian-projected \( SU(2) \)- and \( SU(3) \)-gluodynamics and the interactions between magnetic monopole currents and electric strings. To study this problem, we have casted the partition function of the 4D Abelian-projected \( SU(2) \)-gluodynamics, which is argued to be just the dual Abelian Higgs model, into the form of the integral over the monopole currents. Besides the part, quadratic in these currents, the resulting monopole effective action turned out to contain also a term, which described the interaction of a monopole current with the electric ANO string. Then, we have extended our analysis to the case of the Abelian-projected 4D \( SU(3) \)-gluodynamics, where the resulting representation turned out to contain three monopole currents linked to two independent string world-sheets in a certain way. Finally, for illustrations, we have also performed the corresponding calculation in 3D, where the role of the moving string world-sheets is played by the static electric vortex lines, and the found expressions are more transparent.

The second topic, studied in this paper, was the investigation of 3D compact QED and its relation to SVM and 3D Abelian Higgs model with external monopoles. Firstly, we have demonstrated that the string representation of 3D compact QED (the so-called confining string theory) is nothing else, but the integral over the monopole densities. Secondly, in the weak-field limit of 3D compact QED, we have calculated two coefficient functions, which parametrize the bilocal
correlator of the field strength tensors in the analogous case of SVM. One of them has been found by two methods: from the correlator of the monopole densities and by virtue of the weak-field limit of the confining string theory. Coincidence of both results thus confirms the consistency of our calculations. By making use of this function, we have then obtained the string tension of the Nambu-Goto term and the inverse bare coupling constant of the rigidity term, corresponding to the weak-field effective action of the confining string theory. The large-distance asymptotic behaviours of both coefficient functions correspond to the ones parametrizing the bilocal correlator of the field strength tensors in QCD within SVM. The obtained asymptotic behaviours are also similar to those of the corresponding functions in DAHM. Finally, we have proved that the latter similarity is not accidental, namely 3D compact QED is related to the small gauge boson mass limit of 3D AHM with external monopoles.

In conclusion, the obtained results shed some light on the mechanisms of confinement in various Abelian-projected theories. Furthermore, they prove the relevance of concepts of SVM to the description of confinement in 3D compact QED.

5 Acknowledgments

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Appendix A. Path-Integral Duality Transformation

In this Appendix, we shall outline some details of a derivation of Eq. (3). Firstly, let us linearize the term \( \frac{\eta^2}{2} (\partial_\mu \theta - 2gB_\mu)^2 \) in the exponent on the R.H.S. of Eq. (1) and carry out the integral over \( \theta^{\text{reg}} \) as follows

\[
\int D\theta^{\text{reg}} \exp \left\{ -\frac{\eta^2}{2} \int d^4x (\partial_\mu \theta - 2gB_\mu)^2 \right\} =
\]

\[
= \int DC_\mu D\theta^{\text{reg}} \exp \left\{ \int d^4x \left[ -\frac{1}{2\eta^2} C_\mu^2 + iC_\mu (\partial_\mu \theta - 2gB_\mu) \right] \right\} =
\]

\[
= \int DC_\mu \delta (\partial_\mu C_\mu) \exp \left\{ \int d^4x \left[ -\frac{1}{2\eta^2} C_\mu^2 + iC_\mu (\partial_\mu \theta^{\text{sing}} - 2gB_\mu) \right] \right\}. \tag{A.1}
\]

Next, one can resolve the constraint \( \partial_\mu C_\mu = 0 \) by representing \( C_\mu \) in the form \( C_\mu = \partial_\nu \tilde{h}_{\nu\mu} \equiv \frac{1}{2} \varepsilon_{\mu\nu\lambda\rho} \partial_\nu h_{\lambda\rho} \), where \( h_{\lambda\rho} \) stands for an antisymmetric tensor field.

Notice, that the field \( C_\mu \) is related to the monopole current (4) as \( C_\mu = -\frac{1}{g} j_\mu \). This means that the \( \delta \)-function in the last equality on the R.H.S. of Eq. (A.1) just imposes the conservation of this current.

Next, taking into account the relation (4) between \( \theta^{\text{sing}} \) and \( \Sigma_{\mu\nu} \), we get
\[
\int D\theta^{\text{sing}} D\theta^{\text{reg}} \exp \left\{ -\frac{\eta^2}{2} \int d^4x \left( \partial_\mu \theta - 2g B_\mu \right)^2 \right\} = \\
= \int Dx_\mu(\xi) Dh_{\mu\nu} \exp \left\{ \int d^4x \left[ -\frac{1}{12\eta^2} H_{\mu\nu\lambda}^2 + i\pi h_{\mu\nu} \Sigma_{\mu\nu} - ig \varepsilon_{\mu\nu\lambda\rho} B_\mu \partial_\nu h_{\lambda\rho} \right] \right\}.
\] (A.2)

In the derivation of Eq. (A.2), we have replaced \( D\theta^{\text{sing}} \) by \( Dx_\mu(\xi) \) (since the surface \( \Sigma \), parametrized by \( x_\mu(\xi) \), is just the surface, at which the field \( \theta \) is singular), discarding for simplicity the Jacobian \[\text{[8]}\] arising during such a change of the integration variable.

Bringing together Eqs. (1) and (A.2), we obtain

\[
Z = \int DB_\mu Dx_\mu(\xi) Dh_{\mu\nu} \exp \left\{ -\int d^4x \left[ -\frac{1}{12\eta^2} H_{\mu\nu\lambda}^2 + \frac{1}{4} F_{\mu\nu}^2 - i\pi h_{\mu\nu} \Sigma_{\mu\nu} + ig \tilde{F}_{\mu\nu} h_{\mu\nu} \right] \right\}. \tag{A.3}
\]

Let us now integrate over the field \( B_\mu \). To this end, we find it convenient to rewrite

\[
\exp \left( -\frac{1}{4} \int d^4x F_{\mu\nu}^2 \right) = \int DG_{\mu\nu} \exp \left\{ \int d^4x \left[ -G_{\mu\nu}^2 + i\tilde{F}_{\mu\nu} G_{\mu\nu} \right] \right\},
\]

after which the \( B_\mu \)-integration yields

\[
\int DB_\mu \exp \left\{ -\int d^4x \left[ \frac{1}{4} F_{\mu\nu}^2 + ig \tilde{F}_{\mu\nu} h_{\mu\nu} \right] \right\} = \\
= \int DG_{\mu\nu} \exp \left( -\int d^4x G_{\mu\nu}^2 \right) \delta \left( \varepsilon_{\mu\nu\lambda\rho} \partial_\lambda \left( G_{\mu\nu} - g h_{\mu\nu} \right) \right) = \\
= \int DA_\mu \exp \left\{ -\int d^4x \left( g h_{\mu\nu} + \partial_\mu A_\nu - \partial_\nu A_\mu \right)^2 \right\}. \tag{A.4}
\]

Here, \( A_\mu \) is just the usual gauge field, dual to the dual vector potential \( B_\mu \). Finally, by substituting Eq. (A.4) into Eq. (A.3), we arrive at Eq. (3) of the main text.

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