The Central Decomposition of $FD_{01}(n)$

Peter Köhler

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Abstract
The paper presents a method of composing finite distributive lattices from smaller pieces and applies this to construct the finitely generated free distributive lattices from appropriate Boolean parts.

Keywords Free distributive lattices · Triple sum

1 Introduction

The free distributive lattice $FD_{01}(3)$ on three generators as drawn in Fig. 1 can be viewed as a sort of composition of four Boolean lattices layered on top of each other, with the three generators $a$, $b$, $c$ serving as additional merging points. Another way of seeing this is via the 'central' elements 0, $p$, $q$, $r$, 1, where the intervals $[0, p]$, $[p, q]$, $[q, r]$, $[r, 1]$ form the respective Boolean lattices.

In this paper we will show, that this behaviour can be found in all the finitely generated free distributive lattices. Moreover we will give a nonrecursive construction of these lattices from their Boolean lattice building blocks.

The original hope that this might provide a better way to compute their cardinalities did not materialize. As in the known recursive attempts (see e.g. [2, 11]) also this approach requires an addition of interval sizes which for larger values of $n$ goes beyond the capacities of current computers.

2 Lattices

In the following all lattices $L$ are finite distributive lattices with a 0-element $0_L$ and a 1-element $1_L$. By $B_n$ we denote the Boolean lattice with elements 0,..,$2^n - 1$ and binary join and meet. In particular $0_{B_n} = 0$ and $1_{B_n} = 2^n - 1$, and the atoms of $B_n$ are 1, 2, ..., $2^n - 1$.

For elements $a$ and $b$ of a lattice $L$ we denote by $(a)$ the principal ideal (or 'downset') \{x | x \in L, x \leq a\}, by $[a)$ the principal filter (or 'upset') \{x | x \in L, x \geq a\} and by $[a, b]$ the interval \{x | x \in L, a \leq x \leq b\}.

Peter Köhler
Peter.Koehler@math.uni-giessen.de; peter@koehler-weilburg.de

1 Mathematisches Institut, Justus-Liebig-Universität Giessen, Giessen, Germany
We start with an easy observation belonging to the folklore of distributive lattices (see e.g. [1, 6]):

**Theorem 1** Let $L$ be a finite distributive lattice and $a \in L$. Then

(i) $[a]$ is a dual ideal of $L$.

(ii) The relation $\Theta_a$ defined by

$$x \Theta_a y \overset{\text{def}}{=} x \land a = y \land a$$

is a congruence of $L$.

(iii) $L/\Theta_a = [a]$.

What is less known is the following reverse construction, which has its origin in the general theory of ’triple sums’ originally developed in [4, 5] and later extended in [7–10].

**Theorem 2** Let $L, M, N$ be finite distributive lattices and $\phi : L \rightarrow M$ be a 1-meet-preserving mapping. Let $L \otimes^\phi M$ be the set of all pairs

$$L \otimes^\phi M = \{(l, m) | l \in L, m \in M, m \leq \phi(l)\}.$$

Then

(i) The composition $L \otimes^\phi M$ is a distributive lattice (as a sublattice of $L \times M$).
(ii) There is an element \( a \in L \otimes \phi M \) such that \( (a) \cong L \) and \( [a] \cong M \).
(iii) For each \( a \in N \) there exists an \( 1 \)-meet-preserving mapping \( \phi : (a) \to [a] \) such that \( N \cong (a) \otimes [\phi(a)] \).

**Proof** (i) is obvious. For (ii) let \( a = (1_L, 0_M) \). Then \( a \in L \otimes \phi M \) and the mappings \( x \mapsto (x, 0_M) \) and \( y \mapsto (1_L, y) \) are clearly isomorphisms from \( L \) to \( [a] \) and \( M \) to \([a]\).

For (iii) define \( \phi \) by \( \phi(x) = a * x \) for all \( x \leq a \), where \(* \) denotes the relative pseudocomplement, i.e. \( a * x = \sqrt{\{z \mid z \in L, z \wedge a \leq x\}} \). Then it is well known from the theory of pseudocomplemented lattices that \( \phi \) has the required properties.

Let us note that the theorem above could have also formulated using the notion of split exact sequences (see [9, 10]).

There are three well known special cases:

(i) If \( \phi \) is the ‘1-mapping’, i.e. \( \phi(x) = 1_M \) for all \( x \in L \), then \( L \otimes \phi M \) is the direct product \( L \times M \).

(ii) If \( \phi \) is the ‘0-mapping’, i.e. \( \phi(x) = 0_M \) for all \( x \in L \setminus \{1_L\} \) and \( \phi(1_L) = 1_M \), then \( L \otimes \phi M \) is the ordinal sum \( L \oplus M \) with \( 1_L \) and \( 0_M \) amalgamated.

(iii) And if \( L = M \) and \( \phi \) is the identity mapping, then \( L \otimes \phi L \) is the ‘skew square’ \( L \triangle L \) of \( L \), which is used in the recursive construction of \( FD_{01}(n) \) via \( FD_{01}(n) \cong FD_{01}(n - 1) \triangle FD_{01}(n - 1) \). (see e.g. [2])

Another use of the skew square can be seen in the following easy observation:

**Theorem 3** Let \( B_n \) be the Boolean lattice of order \( 2^n \), and let \( C_3 \) be the three element chain. Then \( B_n \triangle B_n \cong C_3^n \).

**Proof** This is obvious for \( n = 1 \), the rest follows by an easy induction argument, enumerating pairs of pairs in two different ways.

Another interesting observation concerning the ‘skew square’ of a composition is:

**Lemma 1** Let \( L, M \) be distributive lattices and let \( \phi : L \to M \) be a 1-meet-preserving map. Then

\[
(L \otimes \phi M) \triangle (L \otimes \phi M) \cong L \otimes \psi (M \times L) \otimes \chi M,
\]

where the mappings \( \psi : L \to M \times L \) and \( \chi : M \times L \to M \) are defined by:

\[
\psi(x) = (\phi(x), x) \text{ for all } x \in L,
\]

\[
\chi((y, x)) = \phi(x) \wedge y \text{ for all } x \in L \text{ and } y \in M.
\]

**Proof** Obviously \( \psi \) and \( \chi \) are 1-meet-preserving. Now by definition \( (L \otimes \phi M) \triangle (L \otimes \phi M) = \{((x_1, y_1), (x_2, y_2)) \mid y_1 \leq \phi(x_1), y_2 \leq \phi(x_2), x_2 \leq x_1, y_2 \leq y_1\} \) whereas \( L \otimes \psi (M \times L) \otimes \chi M = \{(x_1, (y_1, x_2), y_2) \mid (y_1, x_2) \leq \psi(x_1), y_2 \leq \chi((y_1, x_2))\} \) and by the definition of \( \psi \) and \( \chi \) these conditions coincide.

So far we have only considered the composition of pairs of distributive lattices. Now if \( L \otimes \phi M \) and \( M \otimes \psi N \) are two such compositions, then these give rise to two combinations, namely \( (L \otimes \phi M) \otimes \psi^* N \) and \( L \otimes \psi^* (M \otimes \psi N) \) where \( \psi^* \) and \( \psi^* \) are the natural extensions of \( \phi \) and \( \psi \) defined by

\[
\phi^*(l) = (\phi(l), \psi(\phi(l))) \text{ for } l \in L,
\]

\[
\psi^*(m, n) = \psi(m) \text{ for } (m, n) \in M \otimes \psi N.
\]
Obviously both compositions amount to the same set, namely \( \{(l, m, n) | l \in L, m \in M, n \in N, m \leq \phi(l), n \leq \psi(m) \} \). Therefore it makes sense to introduce the notion of a triple composition \( L \otimes^\phi M \otimes^\psi N \), and more generally that of an \( n \)-fold composition

\[
L_0 \otimes^\phi_0 L_1 \otimes^\phi_1 \ldots \otimes^\phi_{n-2} L_{n-1}.
\]

And as such an \((n+1)\)-fold composition we will construct \( FD_{01}(n) \).

However, before turning to the general case we describe the construction of \( FD_{01}(3) \) as a quadruple

\[
B_1 \otimes^{\phi_0} B_3 \otimes^{\phi_1} B_3 \otimes^{\phi_2} B_1,
\]

where \( \phi_0 \) and \( \phi_2 \) are the 0-mappings and \( \phi_1 : B_3 \to B_3 \) is defined by \( \phi_1(7) = 7 \), \( \phi_1(6) = 4 \), \( \phi_1(5) = 2 \), \( \phi_1(3) = 1 \) and \( \phi_1(x) = 0 \) for all other \( x \in \{0, 1, \ldots, 7\} \).

That this really gives \( FD_{01}(3) \), can be seen from its diagram in the canonical numbering as in Fig. 2 and the expression of the element numbers as 4-tuples as in Table 1, where the correspondence is given by \( (c_0, c_1, c_2, c_3) \mapsto c_0 \ast 2^0 + c_1 \ast 2^1 + c_2 \ast 2^4 + c_3 \ast 2^7 \):

For the general case of \( n \in \mathbb{N} \) this suggests to use the Boolean lattices \( B_{(\binom{n}{k})} \) corresponding to the binomial coefficients \( \binom{n}{k} \) for \( k = 0, \ldots, n \) as building blocks.

**Theorem 4** Let \( n \in \mathbb{N} \). For \( k = 0, \ldots, n \) let \( L_k \) be the Boolean lattice \( L_k = B_{(\binom{n}{k})} \). Then there exist 1-meet-preserving mappings \( \phi_k : L_k \to L_{k+1} \) for \( k = 0, \ldots, n-1 \) such that

\[
FD_{01}(n) \cong L_0 \otimes^{\phi_0} L_1 \otimes^{\phi_1} \ldots \otimes^{\phi_{n-1}} L_n.
\]
Table 1 Numbers as 4-tuples

|       | 0       | 1       | 3       | 5       | 7       |
|-------|---------|---------|---------|---------|---------|
| (0,0,0,0) | (1,0,0,0) | (1,1,0,0) | (1,2,0,0) | (1,3,0,0) |
| 9     | (1,4,0,0) | (1,5,0,0) | (1,6,0,0) | (1,7,0,0) | (1,3,1,0) |
| 43    | (1,5,2,0) | (1,6,4,0) | (1,7,1,0) | (1,7,2,0) | (1,7,3,0) |
| 79    | (1,7,4,0) | (1,7,5,0) | (1,7,6,0) | (1,7,7,0) | (1,7,7,1) |

The key to the proof is the following generalization of Lemma 1:

Lemma 2 Let \( n \in \mathbb{N} \), let \( L_0, L_1, ..., L_n \) be distributive lattices and for \( 0 \leq i < n \) let \( \phi_i : L_i \rightarrow L_{i+1} \) be 1-meet-preserving maps. Then

\[
(L_0 \otimes \phi_0 \otimes \phi_1 ... \otimes \phi_{n-1} L_n) \triangleq (L_0 \otimes \phi_0 \otimes \phi_1 ... \otimes \phi_{n-1} L_n) \cong
L_0 \otimes \psi_0 (L_1 \otimes \phi_0 \otimes \phi_1 ... \otimes \phi_{n-1} L_n) \cong

\]

where the mappings \( \psi_1 : L_0 \rightarrow L_1 \otimes L_0, \psi_i : (L_i \otimes L_{i-1}) \rightarrow (L_{i+1} \otimes L_i) \) for \( i = 1, ..., n-1 \) and \( \psi_n : (L_n \times L_{n-1}) \rightarrow L_n \) are defined by:

\[
\psi_0(x_0) = (\phi_0(x_0), x_0) \text{ for all } x_0 \in L_0
\]

\[
\psi_i((y_{i-1}, x_i)) = (\phi_i(x_i), x_i \land \phi_{i-1}(y_{i-1})) \text{ for all } x_i \in L_i, y_{i-1} \in L_{i-1}, 0 < i < n
\]

\[
\psi_n((y_{n-1}, x_n)) = x_n \land \phi_{n-1}(y_{n-1}) \text{ for all } x_n \in L_n, y_{n-1} \in L_{n-1}.
\]

To prove this, obviously a similar argument as in Lemma 1 shows that the conditions for the elements \( ((x_0, ..., x_n), (y_0, ..., y_n)) \) on the left hand side and \( ((x_0, (y_0, x_1), ..., (y_{n-1}, x_n), y_n)) \) on the right hand side coincide:

Proof of Theorem 4 The result is immediate for \( n = 1 \) with \( FD_{0,1}(1) \cong \mathbb{C}_3 \cong B_1 \otimes \phi_0 B_1 \), where \( \phi_0 \) is the 0-map. Now assume that the result holds for \( n \geq 1 \). As in Example (iii) on page 3 we have that \( FD_{01}(n+1) \cong FD_{01}(n) \triangleq FD_{01}(n) \). By the induction hypothesis and Lemma 2 we get \( FD_{01}(n+1) \cong B_{(n)} \otimes \psi_0 (B_{(n)} \times B_{(n)} \otimes \psi_1 (B_{(n)} \times B_{(n)} \otimes \psi_2 ... \otimes \psi_{n-1} (B_{(n)} \times B_{(n-1)} \otimes \psi_n B_{(n)} \right) \triangleq \psi_0 (B_{(n)} \times B_{(n)} \otimes \psi_1 (B_{(n)} \times B_{(n)} \otimes \psi_2 ... \otimes \psi_{n-1} (B_{(n)} \times B_{(n-1)} \otimes \psi_n B_{(n)} \right) \right) \triangleq \psi_0 (B_{(n)} \times B_{(n)} \otimes \psi_1 (B_{(n)} \times B_{(n)} \otimes \psi_2 ... \otimes \psi_{n-1} (B_{(n)} \times B_{(n-1)} \otimes \psi_n B_{(n)} \right) \right)

\]

Now the fact that \( B_i \times B_j \cong B_{i+j} \) for all \( i, j \in \mathbb{N} \) and the addition rules for the binomial coefficients show that the statement of the theorem holds also for \( n + 1 \).

In this proof the crucial mappings \( \psi_0, ..., \psi_n \) are defined recursively. It is, however, possible to give a direct definition. We defer this to the next section.

3 Posets

An element \( x \) of a lattice \( L \) is called meet irreducible, if it cannot be expressed as a meet of greater elements, i.e. \( x = y \land z \) implies \( x = z \) or \( x = y \). In particular, \( 1_L \) is not meet irreducible. The poset of meet irreducible elements of \( L \) is denoted by \( \mathcal{M}(L) \).
A subset $I$ of a poset $P$ is called an ideal, if it it “downward closed”, i.e. $p \in I$ and $q \leq p$ implies $q \in I$. In particular, $\emptyset$ and $P$ are ideals of $P$. By $\mathcal{I}(P)$ we denote the set (lattice) of ideals of $P$.

We start this section with the poset counterpart of the triple construction for lattices:

**Theorem 5** Let $P$, $Q$ be finite posets and $\alpha : Q \to \mathcal{I}(P)$ be an order preserving mapping. Then the set

$$P \oplus^\alpha Q = P \cupdot Q$$

equipped with the relation $\leq$ defined by

$$x \leq y =_{\text{def}} \begin{cases} x \leq y & \text{if } x, y \in P \\ x \leq y & \text{if } x, y \in Q \\ x \in \alpha(y) & \text{if } x \in P \text{ and } y \in Q \end{cases}$$

is a poset.

**Proof** Clearly $\leq$ is reflexive and antisymmetric. To show that it is transitive too, it suffices to consider three elements $x, y, z$ with $x \leq y$ and $y \leq z$ and the two nontrivial cases (i) $x \in P, y \in Q, z \in Q$ and (ii) $x \in P, y \in P, z \in Q$. Now for (i) transitivity comes from the fact that $\alpha$ is order preserving, and for (ii) from the fact that $\alpha(z)$ is an ideal. □

To illustrate this consider two 3-element antichains $P = \{a, b, c\}, Q = \{d, e, f\}$ and define the mapping $\alpha : Q \to \mathcal{I}(P)$ by $\alpha(d) = \{a, b\}, \alpha(e) = \{a, c\}, \alpha(e) = \{b, c\}$. Then the poset $P \oplus^\alpha Q$ has the diagram of Figure 3:

![Diagram of Figure 3](image)

**Lemma 3** Let $L$, $M$ be finite distributive lattices and $\phi : L \to M$ be $1$-meet-preserving. Then $(x, y) \in L \otimes^\phi M$ is meet irreducible if and only if either $x$ is meet irreducible in $L$ and $y = \phi(x)$ or $x = 1$ and $y$ is meet irreducible in $M$.

**Theorem 6** Let $L$, $M$ be finite distributive lattices and $\phi : L \to M$ be $1$-meet-preserving and let $P = \mathcal{M}(L)$ and $Q = \mathcal{M}(M)$ be their posets of meet irreducible elements. Then the mapping $\alpha : Q \to \mathcal{I}(P)$ defined by

$$\alpha(y) = \{x \mid x \in P, \phi(x) \leq y\} \; y \in Q$$

is order preserving and

$$\mathcal{M}(L \otimes^\phi M) \cong P \oplus^\alpha Q.$$
so it remains to show that \((x, \phi(x)) \leq (1, y)\) if and only if \(\phi(x) \in \alpha(y)\), but this is just the definition of \(\alpha\).

**Theorem 7** Let \(P, Q\) be finite posets and \(\alpha : Q \to \mathcal{I}(P)\) be an order preserving map. Then the mapping \(\phi : \mathcal{I}(P) \to \mathcal{I}(Q)\) defined by
\[
\phi(X) = \{q \in Q, \alpha(q) \subseteq X\}
\]
is 1-meet-preserving and
\[
\mathcal{I}(P \oplus^\alpha Q) \cong \mathcal{I}(P) \otimes^\phi \mathcal{I}(Q).
\]

**Proof** Clearly \(\phi(X)\) is an ideal of \(Q\) for every \(X \in \mathcal{I}(P)\), so \(\phi\) is a mapping. It is 1-meet-preserving as well. We now observe that for any \((X, Y) \in \mathcal{I}(P) \otimes^\phi \mathcal{I}(Q)\) the set \(X \cup Y\) is an ideal of \(P \oplus^\alpha Q\). In fact let \(y \in X \cup Y\) and \(x \leq y\). In order to show that \(x \in X \cup Y\) too, it suffices to consider the case \(x \in P\) and \(y \in Q\). But then we have \(x \in \alpha(y)\) and hence \(x \in X\).

This implies we can define a mapping \(\chi : \mathcal{I}(P) \otimes^\phi \mathcal{I}(Q) \to \mathcal{I}(P \oplus^\alpha Q)\) by
\[
\chi(X, Y) = X \cup Y.
\]
Its inverse is given by \(Z \mapsto (Z \cap P) \cup (Z \cap Q)\) and since both are order preserving they are lattice isomorphisms too.

As already indicated, we will apply this result to obtain a nonrecursive definition of the composition mappings \(\phi_k\) of Theorem 2. In order to facilitate this we introduce some notation:

For \(n \in \mathbb{N}\) let
\[
P_n = \mathcal{P}([0, 1, ..., n-1])
\]
be the (Boolean) poset of all subsets of \([0, 1, ..., n-1]\) with set inclusion as ordering. More generally, for any set \(X\) let
\[
\mathcal{P}_n(X) = \{Y | Y \subseteq X, |Y| = n\}
\]
be the set of all \(n\)-element subsets of \(X\).

For \(n \in \mathbb{N}\) and \(k = 0, ..., n\) let
\[
S_{n,k} = \mathcal{P}_k([0, 1, ..., n-1])
\]
be the set of all \(k\)-element subsets of \(P_n\). Then \(P_n\) can be decomposed into antichain layers as
\[
P_n = S_{n,0} \cup S_{n,1} \cup ... \cup S_{n,n}.
\]
With the mappings \(\alpha_k : S_{n,k+1} \to \mathcal{I}(S_{n,k})\) defined by
\[
\alpha_k(X) = \{Y | Y \in S_{n,k}, Y \subseteq X\}
\]
we can even generalize the composition to
\[
P_n = S_{n,0} \oplus^{\alpha_0} S_{n,1} \oplus^{\alpha_1} ... \oplus^{\alpha_{n-1}} S_{n,n},
\]
where we tacitly extend the poset triple sum to an \(n\)-fold sum.

Repeatedly applying Theorem 4 we arrive at:

**Theorem 8** For \(n \in \mathbb{N}\)
\[
FD_01(n) \cong \mathcal{P}(S_{n,0}) \otimes^{\phi_{n,0}} \mathcal{P}(S_{n,1}) \otimes^{\phi_{n,1}} ... \otimes^{\phi_{n,n-1}} \mathcal{P}(S_{n,n}),
\]
where for \(k = 0, ..., n-1\) the 1-meet-preserving mappings \(\phi_{n,k} : \mathcal{P}(S_{n,k}) \to \mathcal{P}(S_{n,k+1})\) are defined by
\[
\phi_{n,k}(X) = \{Y | Y \in S_{n,k+1}, \mathcal{P}_k(Y) \subseteq X\}.
\]
Proof It is well known that $FD_{01}(n) \cong \mathcal{I}(P_n)$ (see e.g. [3]). Moreover, as $S_{n,k}$ is an antichain, it is clear that $\mathcal{I}(S_{n,k}) = \mathcal{P}(S_{n,k})$. So the only thing that remains to be shown, is that the formula given for $\phi_{n,k}$ is equivalent to the one obtained from Theorem 7 - but that is obvious too.

To illustrate the definition of $\phi_{n,k}$ we list some values for $n = 4$ in Table 2, where we restrict ourselves to list the mapping values for the topmost elements, i.e. the 1-element and the dual atoms:

### 4 Computations

Even though Theorem 8 gives a direct, nonrecursive construction, its application to determine the cardinalities for larger values of $n$ fails with respect to the slowness of computing the ‘downsets’ of the partial compositions.

To see this in some more detail let us recall that the definition of the composition $L \otimes^\phi M$ implies that for any $(x, y) \in L \otimes^\phi M$ we have

$$|((x, \phi(x))] = \sum_{a \in L, a \leq x} |((\phi(a))]$$

and in particular

$$|L \otimes^\phi M| = |((1_L, \phi(1_L))] = \sum_{x \in L} |((\phi(x)].$$

Applying this repeatedly to the formula of Thereom 2 we end up with

$$|FD_{01}(n)| = \sum_{i_0 \in L_0} \sum_{i_1 \in L_1} \ldots \sum_{i_n \in L_n} 2^{i_n}$$

| Table 2 | Mappings for $n = 4$ |
|---|---|
| $S_{4,0}$ | $\{\emptyset\}$ |
| $S_{4,1}$ | $\{\{0\}, \{1\}, \{2\}, \{3\}\}$ |
| $S_{4,2}$ | $\{\{0\}, \{0, 2\}, \{1, 2\}, \{0, 3\}, \{1, 3\}, \{2, 3\}\}$ |
| $S_{4,3}$ | $\{\{0, 1, 2\}, \{0, 1, 3\}, \{0, 2, 3\}, \{1, 2, 3\}\}$ |
| $S_{4,4}$ | $\{\{0, 1, 2, 3\}\}$ |

| $\phi_0$ | $S_{4,0}$ |
| $\phi_1$ | $S_{4,1}$ |
| $\phi_2$ | $S_{4,2}$ |
| $\phi_3$ | $S_{4,3}$ |

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| $S_{4,3}$ | $\{\{0, 1, 2\}, \{0, 1, 3\}, \{0, 2, 3\}, \{1, 2, 3\}\}$ |
| $S_{4,4}$ | $\{\{0, 1, 2, 3\}\}$ |

| $\phi_0$ | $S_{4,0}$ |
| $\phi_1$ | $S_{4,1}$ |
| $\phi_2$ | $S_{4,2}$ |
| $\phi_3$ | $S_{4,3}$ |
Table 3  \( \phi_{k,k}, \ k = 0, \ldots, 3 \)

| \( \phi_{4,0} \) | \( \phi_{4,1} \) | \( \phi_{4,2} \) | \( \phi_{4,3} \) |
|---|---|---|---|
| 1 | 15 | 15 | 63 | 15 | 46 | 4 | 15 | 1 |
| 14 | 52 | 62 | 12 | 43 | 4 |
| 13 | 42 | 61 | 10 | 42 | 4 |
| 12 | 32 | 60 | 8 | 39 | 1 |
| 11 | 25 | 59 | 6 | 31 | 3 |
| 10 | 16 | 58 | 4 | 29 | 2 |
| 9 | 8 | 57 | 2 | 27 | 2 |
| 7 | 7 | 55 | 9 | 25 | 2 |
| 6 | 4 | 54 | 8 | 23 | 1 |
| 5 | 2 | 53 | 8 | 15 | 1 |
| 3 | 1 | 52 | 8 | 7 | 1 |
| 47 | 5 |

(since \( L_n = B_1 \) and therefore \( i_n \) has only the two choices \( i_n = 0 \) and \( i_n = 1 \)).

Taking into account the number of necessary summations, which alone for the largest component is \( 2^{\binom{n}{2}} \), it is clear that this computation can be carried out only up to \( n = 6 \) using currently available computers.

But to be more precise:

We define for each \( n \in \mathbb{N} \) a sequence of functions \( c_0 : L_0 \to \mathbb{N}, \ldots, c_n : L_n \to \mathbb{N} \) recursively by:

\[
c_n(0) = 1, \quad c_n(1) = 2
\]

\[
c_{k-1}(x) = \sum_{y \in L_{k-1}, y \leq x} c_k(\phi_{k-1}(y)) \text{ for } k = n, \ldots, 1
\]

and the the formulas above finally yield

\[
c_0(1) = |FD_{01}(n)|.
\]

Table 4  c-values for \( n = 4 \)

| \( B_1 \) | \( B_4 \) | \( B_6 \) | \( B_4 \) | \( B_4 \) | \( B_1 \) |
|---|---|---|---|---|---|
| 1 | 168 | 1 | 15 | 167 | 1 | 63 | 114 | 1 | 15 | 17 | 1 | 12 | 1 |
| 0 | 1 | 1 | 7 | 19 | 4 | 31 | 41 | 6 | 7 | 8 | 4 | 0 | 1 | 1 |
| 3 | 5 | 6 | 15 | 18 | 12 | 3 | 4 | 6 |
| 1 | 2 | 4 | 30 | 16 | 3 | 1 | 2 | 4 |
| 0 | 1 | 1 | 7 | 9 | 4 | 0 | 1 | 1 |
| 11 | 8 | 16 |
| 3 | 4 | 15 |
| 1 | 2 | 6 |
| 0 | 1 | 1 |
| 2 | 16 | 64 | 16 | 2 |
We have carried out a computer calculation of these sequences up to \( n = 6 \). Tables 3 and 4 list the values of the mappings \( \phi_{n,k} \) and the respective c-values for \( n = 4 \). Note that in Table 3 the columns contain the nonzero function values and in Table 4 the three columns for each of the Boolean lattices contain representative elements, their c-value and the number of elements with the same value.

**Concluding remarks** It might be worthwhile to try to use some insight into the known structure of the Boolean lattices \( L_0, \ldots, L_n \) to speed up the computation.

An easy result in that direction is that

\[
c_{n-1}(2^{n-1}) = 2^n + 1
\]

\[
c_{n-1}(x) = 2^{\text{bitsize}(x)} \text{ for } x = 0, \ldots, 2^{n-1} - 1,
\]

which is simply due to the fact that \( \phi_{n-1} \) is the 0-mapping.

Another speedup approach would be the use of the induced action of the symmetric group \( S_n \) on the lattices \( L_1, \ldots, L_{n-1} \), as this was successfully done in [11].

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**Declarations**

**Conflict of Interests** The author declares that he has no conflict of interest.

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