BOUND ON THE DIMENSION OF LINEAR SERIES ON STABLE CURVES

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ABSTRACT. We study the dimension of linear series on stable curves. In the first part, we show that a general linear series with semistable multidegree is not special, and obtain results on the dimension of the special loci in the Picard scheme. In the process, we give a new characterization of semistability when the total degree equals the genus of the curve. In the second part, we give a generalization of Clifford’s inequality to linear series of uniform multidegree and show that the new bound is achieved on every stable curve.

1. INTRODUCTION

The study of linear series on smooth curves has long used degeneration to singular curves as an important tool. Of particular interest are degenerations to stable curves, since they form the boundary of the Deligne-Mumford compactification of the moduli space of smooth curves. Rather than starting from smooth curves, another approach is to study linear series on the stable curves themselves. It is necessary, whenever one wants to extend results about linear series. Our knowledge here is limited to special cases and many basic questions remain open. The difficulties mainly arise, since the degree $d$ of a line bundle splits into a tuple of numbers, its multidegree $d$, consisting of the degree on each irreducible component. This introduces many pathologies, and to say anything meaningful one needs to fix a class of such multidegrees. Often, this choice depends on the properties one would like to obtain, and there seems to be no overall well-behaved choice. In this paper, we are interested in semistable and uniform multidegrees.

One special case that has received considerable attention is the following: degree $g-1$ line bundles with semistable multidegree on a stable curve of genus $g$. Beauville [Bea77] showed that in this case the effective locus defines a divisor in the corresponding component of the Picard scheme (note however the small caveat discussed below). This is in analogy to the Theta divisor on smooth curves. He then used this divisor to define a polarization on generalized Prym varieties. In a similar direction, Alexeev [Ale04] used the description to define a Theta divisor on the compactified Jacobian of a stable curve. Using this divisor, he described a new extension of the classical Torelli map. Caporaso and Viviani [CV11] gave a generalization also of the Torelli theorem in this setup. Caporaso [Cap09] showed that if the multidegree $d$ is stable, then the divisor is irreducible. Coelho-Esteves [CE15] gave an explicit description of the irreducible components if $d$ is semistable but not stable.

The effective loci discussed above, or rather their complements, give a lower bound on the dimension of a general linear series. The basic upper bound for any linear series on smooth curves is given by the Clifford inequality. The behaviour of this inequality on singular curves has been studied as well, and some special cases are known: Eisenbud-Koh-Stillman [EKS88] showed in an appendix with Harris, that the inequality still holds for irreducible curves. Caporaso [Cap11] showed that it holds for stable curves $X$ and semistable multidegrees $d$ in the following cases: if

\[ \text{The author is supported by the Israel Science Foundation (grant No. 821/16) and by the Center for Advanced Studies at BGU.} \]
X has 2 irreducible components; if the degree is 0 or 2g − 2; and if X has no separating nodes and the degree is at most 4. Franciosi-Tenni [FT14] (see also Franciosi [Fra19]) showed, that the inequality still holds if X has no separating nodes, the multidegree is uniform and L is in the image of the rational Abel map. A result of a somewhat different flavour was established by Caporaso-Len-Melo [CLM15] using the Baker-Norine rank [BN07] of the multidegrees. They showed that every multidegree is equivalent via chip-firing to a multidegree for which every line bundle satisfies the Clifford inequality.

**Results.** Recall that a line bundle L on a curve X is called special if \( h^0(X, L) > \max\{0, g - d + 1\} \). Motivated by the results mentioned above, we consider special loci for semistable multidegrees (see Definition 3.2). They play a central role in the construction of (universal) compactified Jacobians, which extend Jacobians of smooth curves to stable curves. For general degree \( d \), the special loci need not have expected dimension, even if \( \mathbf{d} \) is semistable. We show however that the generic behaviour of line bundles with semistable multidegree is as for smooth curves:

**Theorem A.** Let X be a stable curve and \( \mathbf{d} \) a semistable multidegree. Then a general line bundle of multidegree \( \mathbf{d} \) is not special. If \( \mathbf{d} \) is in addition uniform, the special locus contains an irreducible component of expected dimension.

See Theorem 3.21 for a slightly stronger statement (the case \( d \geq g - 1 \) follows by taking residuals). This complements Caporaso’s result [Cap11, Theorem 2.3], where it is shown that outside the range \( 0 \leq d \leq 2g - 2 \) every line bundle with semistable multidegree is non-special. We obtain stronger results in three special cases, that are of independent interest:

- The case \( d = g - 1 \). This is the case already mentioned above, and the main part here is due to Beauville [Bea77]. Our contribution is to fix a gap in the corresponding statement [Bea77, Proposition 2.2]: the effective locus can be empty (see Example 3.6) and we characterize in Proposition 3.8 semistable multidegrees \( \mathbf{d} \) for which it is. Namely, this happens when X has only rational components and \( \mathbf{d} \) is given by an acyclic orientation. Fortunately, the effective locus is not empty for multidegrees of Theta characteristics and stable multidegrees, and thus the results about generalized Prym varieties and Theta divisors of compactified Jacobians seem unaffected.

- The cases \( d = g - 2 \) and \( d = g \). In this case, we show in Lemma 3.14 that if \( \mathbf{d} \) is semistable, then the special locus is either empty or of pure dimension \( g - 2 \), as expected. For \( d = g \) we show in Lemma 3.16 that \( \mathbf{d} \) is semistable if and only if there is a line bundle \( L \) of multidegree \( \mathbf{d} \) such that \( h^0(X, L) = 1 \) and the base locus of \( L \) is zero-dimensional. This, together with the characterization given by Caporaso and the author in [CC19], generalizes both characterizations given in [Bea77, Lemma 2.1] (cited below as Proposition 3.7) to the case \( d = g \).

In the second part of the paper, we study the Clifford inequality which for smooth curves is \( h^0(X, L) \leq \frac{d}{2} + 1 \). The multidegrees \( \mathbf{d} \) we consider are uniform ones, generalizing the numerical condition \( 0 \leq d \leq 2g - 2 \) on smooth curves (see Definition 4.1). A central role is played by the graph of 2-edge-connected components \( G^2_X \); it is the graph that has a vertex for every maximal subcurve of X containing no separating nodes and an edge for each separating node (see Definition 4.5). The main result is the following (see Proposition 4.6 and Theorem 4.13):

**Theorem B.** Let X be a stable curve. All line bundles L on X with uniform multidegree satisfy

\[
h^0(X, L) \leq \frac{d}{2} + 1 + \sum_{v \in V(G^2_X)} \max\left\{0, \frac{\text{val}(v) - 2}{2}\right\}.
\]

Conversely, for every stable curve X, there is a line bundle L with uniform multidegree that achieves the above bound.
In particular, if \( X \) has no separating nodes, then every \( L \) with uniform multidegree satisfies the classic Clifford inequality \( h^0(X, L) \leq \frac{d^2}{2} + 1 \). In addition, we also show in Theorem 4.13 also that a general \( L \) with uniform multidegree satisfies the classic Clifford inequality.

We conclude with some remarks about the relation of Theorem B to results in the literature: The theorem is a stronger version for stable curves of the inequality given in [FT14, Theorem A and Theorem 3.8] (though we do not attempt to classify cases in which equality is achieved). The two statements [Cap11, Proposition 3.1] and [Fra19, Theorem 3.14] are similar to Theorem B, in that they also concern uniform multidegrees on nodal curves. The respective claims are however somewhat too optimistic (see Example 4.2 and Remark 4.3). As mentioned above, there is a class of multidegrees such that every line bundle satisfies the classic Clifford inequality by [CLM15], and we hope that our results can help to eventually shed some light on it.

**Structure of the paper.** In Section 2 we fix conventions and recall some background. In Section 3 we study special loci for semistable multidegrees. Most results are obtained by reducing the claims to the case \( d = g - 1 \). The main statement is Theorem 3.21. In Section 3.5 we give counterexamples to some possible strengthenings of the results. In Section 4, we study the Clifford inequality for uniform multidegrees. Our approach here is close in spirit to [Cap11] – a mixture of graph theory and calculations in sheaf cohomology. An important step is Theorem 4.10, in which we show by induction that on a stable curve without separating nodes the classic Clifford inequality is satisfied. The main statement, Theorem 4.13, then follows by reducing to this case. We discuss the relationship of uniform and semistable multidegrees in Section 4.5.

**Acknowledgements.** Many discussions with Lucia Caporaso and Ilya Tyomkin helped shape this paper and I am grateful for the insights and suggestions they provided.

2. **Preliminaries**

In this section, we fix some conventions and recall the basic objects considered in the paper. Throughout the paper, we work over an algebraically closed field \( k \) of characteristic 0.

2.1. **Nodal curves and dual graphs.** We consider fixed curves \( X \), which we will always assume to be reduced and either smooth or nodal. If not stated otherwise, we assume \( X \) to be in addition connected. We denote the weighted dual graph of \( X \) by \( G_X \). That is, \( G_X \) contains a vertex \( v \) for every irreducible component \( X_v \) of \( X \); edges of \( G_X \) correspond to nodes of \( X \); and each vertex \( v \) is assigned the weight \( g_v \) given by the geometric genus of \( X_v \). We denote by \( V(G_X) \) and \( E(G_X) \) the sets of vertices and edges of \( G_X \), respectively. The genus of \( G_X \) is defined as

\[
g(G_X) = b_1(G_X) + \sum_{v \in V(G_X)} g_v,
\]

where \( b_1(G_X) = |E(G_X)| - V(G_X) + 1 \) denotes the first Betti number of \( G_X \). The genus of \( G_X \) equals the arithmetic genus \( g(X) \) of \( X \). We write \( g := g(X) \) if \( X \) is clear from the context.

We denote by \( \text{val}(v) \) the valence of \( v \in G_X \); that is, the number of edges adjacent to \( v \), with loops counted twice. For a subset of vertices \( Z \subset V(G_X) \) we define the induced subgraph \( [Z] \) to be the graph with vertices \( Z \subset V(G_X) \) and edges the ones of \( G_X \) whose adjacent vertices are contained in \( Z \). We write \( g(Z) \) for the genus of \( [Z] \). Notice that \( [Z] = G_Y \) where \( Y \) is the subcurve of \( X \) whose irreducible components correspond to vertices in \( Z \). We set \( Z^c = V(G_X) \setminus Z \) and, with a slight abuse of notation,

\[
Y^c = \overline{X \setminus Y}.
\]
We write \((Z, Z^c) ∈ E(G_X)\) for the cut defined by \(|Z|\); that is, \((Z, Z^c)\) are edges of \(G_X\) with one adjacent vertex in \(Z\) and one in \(Z^c\). Clearly, \(|(Z, Z^c)| = |Y ∩ Y^c|\).

A curve \(X\) is called semistable, if \(\text{val}(v) + g_v ≥ 2\) for all \(v ∈ V(G_X)\). It is called stable if it is semistable and whenever \(\text{val}(v) = 2\), we have \(g_v ≠ 0\). A node \(z\) of \(X\) will be called separating, if the partial normalization of \(X\) at \(z\) has more connected components than \(X\). A separating node corresponds to a bridge of \(G_X\). If \(G_X\) is connected and has no bridges (and hence \(X\) no separating nodes), it is called 2-edge-connected. The graph \(G_X\) is called a tree if it is connected and every edge is a bridge. In this case, we call the vertices of valence \(1\) in \(G_X\) its leaves. The graph \(G_X\) is called a chain if it is a tree with at most two leaves. A cycle of \(G_X\) is a connected subgraph in which every vertex has valence \(2\).

### 2.2. Line bundles and multidegrees.

Our main object of study are line bundles \(L\) (or equivalently, invertible sheaves) on a nodal curve \(X\). We denote by \(\deg(L)\) the multidegree of \(L\); it is an element of the free \(Z\)-module on \(V(G_X)\) with coefficient at \(v\) given by \(\deg(L)|_X\). For a multidegree \(d\) we denote by \(d_v\) its coefficient at \(v\). We set \(|d| = ∑_{v∈V(G_X)} d_v\); if \(d = \deg(L)\), \(|d|\) equals the total degree \(\deg(L)\) of \(L\) and accordingly we will call \(|d|\) the total degree of \(d\). For \(Z ⊂ V(G_X)\) we denote by \(d_Z\) the restriction of \(d\) to \(|Z|\); that is, the multidegree of \(L|_Z\) where \(Y\) is the subcurve of \(X\) corresponding to \(|Z|\). A multidegree is called effective, if \(d_v ≥ 0\) for all vertices \(v\). For two multidegrees \(d\) and \(d'\) we will write \(d ≤ d'\) if \(d_v ≤ d'_v\) for all vertices \(v ∈ V(G_X)\). We denote the dualizing sheaf of \(X\) by \(ω\) and its multidegree by \(ω\); that is, \(ω_v = 2g_v - 2 + \text{val}(v)\) and \(|ω| = 2g(Z) - 2 + |(Z, Z^c)|\).

We denote by \(\text{Pic}^d(X)\) the connected component of the Picard scheme of \(X\) that parametrizes line bundles of multidegree \(d\). We write \([L]\) for the point in \(\text{Pic}^d(X)\) parametrizing the line bundle \(L\). Then \(\text{Pic}^d(X)\) is a \(g\)-dimensional semi-abelian variety via the presentation

\[
0 → (k^*)^h(G_X) → \text{Pic}^d(X) → \text{Pic}^d(X^c) → 0,
\]

where \(X^c\) is the normalization of \(X\). To specify \(L\) on \(X\), one thus needs to specify a line bundle of degree \(d_v\) on the normalization of each irreducible component \(X_v\) of \(X\) plus an isomorphism \(Ω_{p_1} → Ω_{p_2}\), where \(p_1, p_2\) are the two preimages of a node in the normalization of \(X\); different choices of this gluing data over separating nodes gives isomorphic line bundles.

As usual, we write \(h^0(X, L)\) for the dimension of the space of global sections \(H^0(X, L)\). We will often consider the contributions to \(h^0(X, L)\) on closed subsets of \(X\) and to unify notation will use the following convention.

**Convention 2.1.** For a closed subset \(W\) of \(X\) we will write \(h^0(X, L)|_W\) and \(h^0(X, L)|_W\) for the image of the restriction map \(H^0(X, L) → H^0(W, L)|_W\), respectively its dimension.

A base point \(p\) of \(L\) is a point at which every global section of \(L\) vanishes; thus if \(p\) is a smooth base point, \(h^0(X, L) = h^0(X, L(−p))\). A neutral pair of \(L\) is a pair of smooth points \(p_1, p_2\) such that none of them is a base point of \(L\) and \(h^0(X, L(−p_1)) = h^0(X, L(−p_2)) = h^0(X, L(−p_1 − p_2))\) (following [Cap11, Section 1.2]; though we do not call \(p_1, p_2\) a neutral pair if both are base points). Notice that when \(p_1\) and \(p_2\) are contained in different connected components of \(X\), they are never a neutral pair. The following lemma gives a very useful way to inductively calculate \(h^0(X, L)\); we will use it explicitly only in Section 4, but mention it already since it allows to easily calculate \(h^0(X, L)\) in the examples throughout the paper.

**Lemma 2.2.** [Cap11, Lemma 1.4] Let \(z\) be a node of a nodal curve \(X\). Let \(v : X^c_z → X\) be the partial normalization of \(X\) at \(z\) and \(z_1, z_2\) the two preimages of \(z\). Let \(L\) be a line bundle on \(X^c_z\).
(1) If \( z_1 \) and \( z_2 \) are base points of \( L \), every line bundle \( L' \) on \( X \) with \( \nu^* L' = L \) satisfies \( h^0(X^1, L) = h^0(X, L) \);
(2) if \( z_1 \) and \( z_2 \) are a neutral pair of \( L \), there is a unique \( L' \) on \( X \) with \( \nu^* L' = L \) and \( h^0(X^1, L) = h^0(X, L) \); all other \( L' \) with \( \nu^* L' = L \) satisfy \( h^0(X^1, L) = h^0(X, L') + 1 \);
(3) in all other cases, every \( L' \) with \( \nu^* L' = L \) satisfies \( h^0(X^1, L) = h^0(X, L') + 1 \).

3. EFFECTIVE LOCI FOR SEMISTABLE MULTIDEGREES

In this section, we study the dimension of effective loci on stable curves. For a smooth curve \( X \), the locus of effective line bundles is the image of the Abel map
\[
\alpha_d : X^d \rightarrow \text{Pic}^d(X),
\]
which has dimension \( \min\{d, g\} \). For reducible curves, the picture is more complicated. First of all, the Abel map generalizes only if the multidegree is effective. As we will see, if the multidegree \( d \) is not effective, the effective locus in \( \text{Pic}^d(X) \) can be empty even if the total degree \( d \) is nonnegative. If the multidegree \( d \) is the other hand is effective, one can define a rational Abel map. But not every line bundle \( L \) with \( h^0(X, L) \geq 1 \) lies in the closure of its image. As a consequence, the effective locus in \( \text{Pic}^d(X) \) can have dimension larger than \( \min\{d, g\} \) and need not be irreducible. By [CE15, Proposition 2.1] this phenomenon comes from Abel maps on subcurves of \( X \), and we will describe the relevant construction below.

The main result is Theorem 3.21. We show there that a general line bundle of semistable multidegree \( d \) and total degree \( d \leq g - 2 \) is not effective, and if \( d \) is semistable and effective, then there is an irreducible component of the effective locus that has expected dimension. After some preliminaries in Section 3.1, we will successively study the cases \( d = g - 1 \), \( d = g - 2 \) or \( d = g \), and finally \( d \leq g - 2 \). In the last part, Section 3.5, we collect some counterexamples to possible strengthenings of the claims in this section.

3.1. Semistable multidegrees and Brill-Noether loci. We begin with an example to illustrate that if \( X \) is reducible, then one needs to restrict the class of multidegrees \( d \) considered to say anything meaningful about \( h^0(X, L) \) in terms of the total degree \( d \).

Example 3.1. Let \( X \) be a reducible nodal curve and \( d \in \mathbb{N} \). Then for any \( r \in \mathbb{N} \), there are (infinitely many) multidegrees \( d \) of total degree \( d \) such that for every line bundle \( [L] \in \text{Pic}^d(X) \) we have \( h^0(X, L) > r \). For example, if \( v, \omega \in V(G_X) \) set \( d_\omega \geq g_v + \text{val}(\nu) + r, \ d_\omega = d - d_\nu \), and \( d \) zero on all other vertices. Then \( h^0(X, L) > r \) for all \( [L] \in \text{Pic}^d(X) \).

The multidegrees \( d \) we will consider in this section are the semistable ones, in the sense of [Cap94]. Recall that for a subset of vertices \( Z \subset V(G_X) \), we denote by \( (Z, Z^c) \) the associated cut; that is, the set of edges that are adjacent to one vertex in \( Z \) and one in \( Z^c \).

Definition 3.2. Let \( X \) be a stable curve and \( \omega \) the multidegree of its dualizing sheaf. A multidegree \( d \) of total degree \( d \) is called semistable if for any \( Z \subset V(G_X) \) we have
\[
g(Z) - 1 + (d - g + 1) \frac{|\omega|}{2g - 2} \leq |d_\omega| \leq g(Z) - 1 + (d - g + 1) \frac{|\omega|}{2g - 2} + |(Z, Z^c)|.
\]
The multidegree \( d \) is called stable if the inequalities above are strict for every \( Z \subset V(G_X) \).

Remark 3.3. It is well known that the condition in Definition 3.2 is equivalent to imposing only either the upper or lower bound on \( |d_\omega| \) for all \( Z \). Indeed, one readily checks that \( |d_\omega| \) satisfies the upper bound if and only if \( |d_\nu| \) satisfies the lower bound. In addition, it is sufficient to check
the condition for subsets of vertices $Z$ such that the induced subgraph $[Z]$ is connected, since both bounds are additive on connected components of induced subgraphs. Finally, another easy calculation shows that $\omega$ is semistable if and only if the residual multidegree $\omega - \underline{d}$ is semistable.

Next, we introduce the main object of interest for this section, namely the effective loci $W(d)(X)$ in $\text{Pic}^{-d}(X) = \{ [L] \in \text{Pic}(X) | \text{deg}(L) = d \}$. More generally, we set:

**Definition 3.4.** The Brill-Noether loci $W^d_r(X)$ are the subsets of $\text{Pic}^{-d}(X)$ given by

$$W^d_r(X) := \{ [L] \in \text{Pic}^{-d}(X) | h^0(X, L) - 1 \geq r \}.$$  

We will write $W_d(X) := W^0_d(X)$ for the effective locus.

The next lemma is the standard observation that the $W^0_d(X)$ can be realized as degeneracy loci of a map between vector bundles. We include the argument for the convenience of the reader and follow the presentation in the proof of [Bea77, Proposition 2.2].

**Lemma 3.5.** Let $X$ be a semistable curve and $d$ a multidegree of total degree $d$. Then $W^0_d(X)$ is a closed subset of $\text{Pic}^{-d}(X)$ with either $W^0_d(X) = \emptyset$, or each irreducible component of $W^0_d(X)$ has dimension at least $\min \{ g, g - (r + 1)(g - d + r) \}$.

**Proof.** Let $\mathcal{P} \rightarrow \text{Pic}^{-d}(X) \times X$ be a Poincaré bundle, that is, a line bundle that restricts to $[L] \times X$. Denote by $\text{pr}_1$ and $\text{pr}_2$ the projections from $\text{Pic}^{-d}(X) \times X$ to $\text{Pic}^{-d}(X)$ and $X$, respectively.

We choose $D = \sum_{i=1}^k p_i$ where the $p_i \in X$ are $k$ smooth points, such that $h^0(X, \omega \otimes L^{-1} \otimes \mathcal{O}_X(D)) = g - 1 - d + k$ for all $[L] \in \text{Pic}^{-d}(X)$ (which is always possible, see [CF96, Lemma 2.1] or [Cap11, Lemma 2.5]). We get a short exact sequence

$$0 \rightarrow \mathcal{P} \otimes \text{pr}_2^* \mathcal{O}_X(-D) \rightarrow \mathcal{P} \rightarrow \mathcal{P} \otimes \text{pr}_2^* \mathcal{O}_D \rightarrow 0$$

We apply $(\text{pr}_1)_*$ to this sequence and set $E_1 = (\text{pr}_1)_* (\mathcal{P} \otimes \text{pr}_2^* \mathcal{O}_D)$, which is locally free of rank $k$, and $E_2 = R^1((\text{pr}_1)_* (\mathcal{P} \otimes \text{pr}_2^* \mathcal{O}_X(-D)))$, which is locally free of rank $g - 1 - d + k$. Since $R^1((\text{pr}_1)_* (\mathcal{P} \otimes \text{pr}_2^* \mathcal{O}_D)) = 0$ the higher direct image sequence induces an exact sequence

$$E_1 \xrightarrow{\mu} E_2 \rightarrow R^1((\text{pr}_1)_* \mathcal{P}) \rightarrow 0.$$  

Over $[L] \in \text{Pic}^{-d}(X)$ it restricts to

$$(E_1)_{[L]} \xrightarrow{\mu_{[L]}} (E_2)_{[L]} \rightarrow H^1(X, L) \rightarrow 0.$$  

By Riemann-Roch, $h^0(X, L) \geq r + 1$ if and only if $h^1(X, L) \geq g - d + r$. On the other hand, the exact sequence gives $h^1(X, L) = g - 1 - d + k - \dim(\text{Im}(\mu_{[L]}))$. Thus $W^0_d(X)$ is the locus where $u$ has rank at most $k - 1 - r$. The expected dimension of this locus is the Brill-Noether number

$$\rho = g - (k - k + 1 + r)(g - 1 - d + k - k + 1 + r) = g - (r + 1)(g - d + r).$$

The claim now follows by the observation that a degeneracy locus is either empty, the whole space, or each of its irreducible components has dimension at least the expected dimension (see, e.g., [ACGH85, Section 2.4]).

In particular, if $d \leq g - 1$, then the effective locus $W_d(X)$ is either empty or each irreducible component has dimension at least $d$. The effective locus can be empty, even if $d \geq 0$:
Example 3.6. Consider a curve $X$ with dual graph $\mathcal{G}_X$ that has two vertices $v, w$ of weight 0 that are joined by $k$ edges. Let the multidegree $\underline{d}$ be given as $(-1, k - 1)$. That is, we have for the total degree $|\underline{d}| = g - 1 = k - 2$ and one checks that $\underline{d}$ is semistable. Let $[L] \in \text{Pic}\underline{d}(X)$. Then any global section of $L$ vanishes on $X_v$ and thus $H^0(X, L)|_{X_w}$ consists of sections in $H^0(X_w, L|_{X_w})$ vanishing at $k$ distinct points of $X_w$. Since $L|_{X_w}$ has degree $k - 1$, this is only possible for the zero section. Thus $h^0(X, L) = 0$ for all line bundles $[L] \in \text{Pic}\underline{d}(X)$ and $W_\underline{d}(X) = \emptyset$, whereas the expected dimension of the effective locus $W_\underline{d}(X)$ is $g - 1 = k - 2$.

3.2. **Theta divisors.** In this section, we consider the case when the total degree is $d = g - 1$. This case is particularly well-behaved, as we will explain below.

3.2.1. **Dimension of effective loci.** Recall that a multidegree $\underline{d}$ is **orientable**, if there is an orientation $O$ of the edges of $\mathcal{G}_X$ such that $d_v = g_v - 1 + k$ where $k$ is the number of edges adjacent to $v$ that are oriented towards $v$. We will write $\underline{d} = \underline{d}_O$ in this case (note however that $O$ is in general not unique). It is not difficult to see that every orientable multidegree has total degree $g - 1$. The key observation in case of $d = g - 1$ is the following:

**Proposition 3.7.** [Bea77, Lemma 2.1] Let $X$ be a stable curve and $\underline{d}$ a multidegree of total degree $g - 1$. Then the following are equivalent:

1. $\underline{d}$ is semistable,
2. there is a line bundle $L$ of multidegree $\underline{d}$ with $h^0(X, L) = 0$ and,
3. $\underline{d}$ is orientable.

We will need some additional properties of orientations $O$ on $\mathcal{G}_X$: A directed cut in $O$ is a cut $(Z, Z^c)$ such that every edge in the cut is directed towards $[Z]$. A directed cycle in $O$ is a cycle in $\mathcal{G}_X$, such that every vertex has exactly one edge directed towards it in the restriction of $O$ to the cycle; in particular, a loop edge is a directed cycle in every orientation. It is a standard fact in graph theory that for every orientation $O$, an edge of $\mathcal{G}_X$ is contained in a directed cut or a directed cycle, but not both. We say $O$ is **acyclic** if it has no directed cycles and hence every edge is contained in a directed cut.

The next proposition is a small correction of [Bea77, Proposition 2.2], taking into account that the degeneracy locus in the proof of Lemma 3.5 can be empty.

**Proposition 3.8.** Let $X$ be a stable curve and $\underline{d}$ a multidegree of total degree $g - 1$. Then we have for the effective locus $W_\underline{d}(X)$:

1. $W_\underline{d}(X)$ is empty or has pure dimension $g - 1$ if and only if $\underline{d}$ is semistable.
2. $W_\underline{d}(X)$ is empty if and only if all irreducible components of $X$ are rational and $\underline{d} = \underline{d}_O$ with $O$ acyclic.

**Proof.** The first claim follows by Lemma 3.5 and Proposition 3.7. For the second claim, we show more generally that for an orientable multidegree $\underline{d} = \underline{d}_O$ on a nodal curve $X$, not necessarily stable or connected, $W_\underline{d}(X) = \emptyset$ if and only if $g_v = 0$ for all $v \in V(\mathcal{G}_X)$ and $O$ acyclic. The second claim then follows from Proposition 3.7. Notice that for any effective multidegree $\underline{d}$ we have $W_\underline{d}(X) \neq \emptyset$, since $[\mathcal{O}_X(D)] \in W_\underline{d}(X)$ where $D$ is a divisor consisting of a sum of smooth points of $X$, with $d_v$ of them contained in $X_v$ for each vertex $v \in V(\mathcal{G}_X)$. Similarly, if $\underline{d}$ is effective on a connected component of $\mathcal{G}_X$, we have $W_\underline{d}(X) \neq \emptyset$.

So suppose there is no connected component of $\mathcal{G}_X$ on which $\underline{d}$ is effective. Observe that $d_v < 0$ if and only if $g_v = 0$ and either there are no edges adjacent to $v$ or all edges adjacent to $v$ in $\mathcal{G}_X$ are oriented away from $v$ in $O$ (in which case they in particular form a directed cut).
In the latter case, set $X_1 = X^0_\nu$, $G_1 = [V(\mathcal{G}_X) \setminus \{\nu\}]$, the dual graph of $X_1$, and for a line bundle $[L] \in \text{Pic}^d(X)$ let $L_1 = L|_{X_1}(-X_\nu \cap X_1)$. Since any global section of $L$ vanishes on $X_\nu$, we have $h^0(X, L) = h^0(X_1, L_1)$. For $d_1 = \deg(L_1)$ we have $d_1 = d_{\nu \mid G_1}$ and thus $d_1$ is orientable. Since any oriented cycle of the restriction $O_{G_1}$ to $G_1$ corresponds to an oriented cycle of $O$ on $G_X$, and we only remove edges contained in a directed cut, $O_{G_1}$ is acyclic if and only if $O$ is. If there is no connected component of $G_1$ on which $d_1$ is effective and $E(G_1) \neq \emptyset$, we apply the construction again to obtain $d_2$ from $d_1$. Eventually, we obtain $d_k$ with either $(\ast)$ $E(G_k) \neq \emptyset$ and $d_k$ effective on a connected component of $G_k$ or $(\dagger)$ $E(G_k) = \emptyset$ and $(d_k)_\nu = g_\nu - 1$ for all vertices $\nu \in V(G_k)$. Furthermore, we still have $h^0(X, L) = h^0(X_k, L_k)$ and thus obtain a surjective map

$$\text{Pic}^d(X) \to \text{Pic}^d_k(X_k),$$

(1)

that preserves the dimension of $H^0$.

If we are in case $(\ast)$, by the observation in the beginning of the proof, $W_d^d(X_\nu) \neq \emptyset$ and hence $W_d^d(X) \neq \emptyset$ by (1). If we are in case $(\dagger)$, $W_d^d(X_\nu) = \emptyset$ (and hence $W_d^d(X) = \emptyset$) if and only if $g_\nu = 0$ for all $\nu \in V(G_k)$. The claim now follows by observing the following: If $O$ is acyclic, there is always at least one vertex such that all edges are oriented away from the vertex. If furthermore $g_\nu = 0$ for all $\nu \in V(G_X)$, we thus need to end up in case $(\dagger)$, since at each step the obtained orientation remains acyclic. Hence $W_d^d(X) = \emptyset$ in this case. Conversely, if $O$ is not acyclic, it remains non acyclic at each step and thus we end up in case $(\ast)$ and $W_d^d(X) \neq \emptyset$. If $g_\nu \neq 0$ for some $\nu \in V(G_X)$, both cases are possible but in each case $W_d^d(X) \neq \emptyset$, as claimed. \hfill $\square$

3.2.2. Rational Abel maps and components of Theta divisors. We recall some more results for the case $d = g - 1$, which will be needed in the next section. For an irreducible component $X_\nu$ of $X$, denote by $X^\text{sm}_\nu$ the locus of points in $X_\nu$ that are smooth points of $X$. Following [Cap09, Section 1.2.7], we define the rational Abel map associated to an effective multidegree $d$ of total degree $d$ as the map

$$\alpha^d : \prod_{\nu \in V(G_X)} (X^\text{sm}_\nu)^{d_\nu} \to \text{Pic}^d(X),$$

given by sending a $d$-tuple of points $(p_1, \ldots, p_d)$ to $[\mathcal{O}_X(\sum_{i=1}^d p_i)]$. We denote the image of $\alpha^d$ by $A^d_d(X)$; clearly, it is irreducible of dimension at most $\min\{d, g\}$.

Lemma 3.9. Let $d$ be an effective multidegree of total degree $d$ and suppose a general $[L] \in A^d_d(X)$ satisfies $h^0(X, L) = 1$. Then the dimension of $A^d_d(X)$ is $d$.

Proof. Note that by Riemann-Roch the assumption $h^0(X, L) = 1$ can only be satisfied for $d \leq g$. The dimension of $\prod_{\nu \in V(G_X)} (X^\text{sm}_\nu)^{d_\nu}$ is $d$ and the fiber of $\alpha^d$ over a point $[L] = [\mathcal{O}_X(\sum_{i=1}^d p_i)]$ consists of tuples $(p_1', \ldots, p_d')$ such that $\mathcal{O}_X(\sum_{i=1}^d p_i) = \mathcal{O}_X(\sum_{i=1}^d p_i')$. In particular, if $p_1 \neq p_1'$, then $L$ has two linearly independent global sections. Thus $\alpha^d_d$ is generically injective by the assumption that $h^0(X, L) = 1$ for a general $L$, and the claim follows since $A^d_d(X)$ is irreducible. \hfill $\square$

We will need to refine the construction of $A^d_d(X)$, following [CE15, Section 2.2]. Roughly speaking, the locus $W_d^d(Z)(X)$ they construct consists of line bundles that are in the image of a rational Abel map on a subcurve $Y$ of $X$ and arbitrary outside of $Y$; here $Y$ corresponds to the subset of vertices $Z \subset V(G_X)$.

More precisely, let $d$ be a multidegree and $Z \subset V(G_X)$ a subset of vertices, such that $d - z$ is an effective multidegree on the induced subgraph $[Z]$; here $z$ is the multidegree on $[Z]$ whose value at $\nu \in Z$ is the number of edges in $(Z, Z' \setminus \nu)$ adjacent to $\nu$. Let $Y$ be the subcurve of $X$
corresponding to $Z$ and $v_{Y \cap Y^c} : X^v_{Y \cap Y^c} \to X$ the partial normalization of $X$ at nodes in $Y \cap Y^c$. We define $V \subset \text{Pic}^d_{\mathcal{Z}}(Y)$ as the locus

$$\left\{ [L \otimes \mathcal{O}_Y(Y \cap Y^c)] \mid [L] \in A_{d,e}(Y) \right\}$$

and set

$$W_d^{\mathcal{Z}}(X) := (V_{Y \cap Y^c})^{-1} \left( \text{Pic}^d_{\mathcal{Z}}(Y^e) \times V \right) \subset \text{Pic}^d_{\mathcal{Z}}(X).$$

The motivation for this construction is the following proposition, which we rephrase for our setup.

**Proposition 3.10.** [CE15, Theorem 3.6] Let $X$ be a stable curve and $d = d_O$ a semistable multidegree of total degree $g - 1$. Then

$$W_d(X) = \bigcup_Z W_d^{\mathcal{Z}}(X),$$

and the $W_d^{\mathcal{Z}}(X)$ are irreducible; the union is over all subsets of vertices $Z \subset V(\mathbb{G}_X)$ such that the induced subgraph $[Z]$ is connected, all edges in $(Z, Z^c)$ are oriented towards $[Z]$ in $O$ and $d_{OZ}$ is an effective multidegree on $[Z]$.

**Remark 3.11.** Proposition 3.8 is consistent with Proposition 3.10; namely, if $X$ has only rational components and $O$ is acyclic, then there exists no subset of vertices $Z \subset V(\mathbb{G}_X)$ as required in Proposition 3.10.

**Lemma 3.12.** Let $W_d^{\mathcal{Z}}(X)$ be as in Proposition 3.10 and $p \in X_v$ a smooth point with $v \in Z$. Then a general line bundle $[L] \in W_d^{\mathcal{Z}}(X)$ satisfies $h^0(X, L) = 1$ and $p$ is not a base point of $L$.

**Proof.** The first claim, that $h^0(X, L) = 1$ for general $L$, is part of [CE15, Proposition 3.5]. Let $Y$ be as above the subcurve of $X$ corresponding to $Z$. The second claim follows, since the subsheaf of sections of $L$ vanishing on $Y^c$ is by construction of the form $\mathcal{O}_Y(\sum_{i=1}^k p_i)$ for varying smooth points $p_i$, where $k = |d_{\mathcal{Z}}| - |(Z, Z^c)|$. \hfill \qed

### 3.3. The cases $d = g - 2$ and $d = g$.

In this section, we consider multidegrees of total degrees $d = g - 2$ and $d = g$. The following partial analogue of Proposition 3.7 is the residual statement of [CC19, Lemma 3.3.2].

**Lemma 3.13.** Let $X$ be a stable curve and $d$ a multidegree of total degree $g - 2$. Then $d$ is semistable if and only if for every vertex $v \in V(\mathbb{G}_X)$ we have $d = d_O - v$ where $O$ is an orientation on $\mathbb{G}_X$ such that every directed cut in $O$ is oriented towards the subgraph containing $v$.

**Proof.** This follows from [CC19, Lemma 3.3.2], Remark 3.3 and the observation that the residual multidegree $\omega - d_O$ is $d_O'$, where edges are oriented in $O'$ opposite to their orientation in $O$. \hfill \qed

**Lemma 3.14.** Let $X$ be a stable curve and $d$ a semistable multidegree of total degree $g - 2$. Then the effective locus $W_d(X)$ is empty, or of pure dimension $g - 2$.

**Proof.** If $W_d(X) = \varnothing$, there is nothing to show. Otherwise, let $W \subset W_d(X)$ be an irreducible component. Let $d_O$ and $v \in V(\mathbb{G}_X)$ be as in Lemma 3.13; that is, $d = d_O - v$ and every directed cut in $O$ is oriented towards the subgraph containing $v$. Choose a smooth point $p \in X_v$ and consider the isomorphism

$$\varphi_p : \text{Pic}^d(X) \to \text{Pic}^{d_O}(X),$$

given by sending $[L]$ to $[L(p)]$. Since $d_O$ is semistable, the image $\varphi_p(W)$ is contained in an irreducible component $W_{d_O}^{\mathcal{Z}}(X)$ for some subset of vertices $Z \subset V(\mathbb{G}_X)$ by Proposition 3.10. Since every directed cut of $O$ is oriented towards the subgraph containing $v$, Proposition 3.10
Furthermore implies \( v \in Z \). Thus we can apply Lemma 3.12 and for a general line bundle \([L'] \subset W_{d}^{Z}(X)\), we have \( h^{0}(X,L') = 1 \) and \( p \) is not a base point of \( L' \); hence \( \varphi_{p}(W) \neq W_{d}^{Z}(X) \). By Proposition 3.8, we have \( \dim(W_{d}^{Z}(X)) = g - 1 \). Since \( W_{d}^{Z}(X) \) is irreducible and \( \varphi_{p}(W) \) is a closed subset of \( W_{d}^{Z}(X) \), this implies \( \dim(W) \leq g - 2 \). The claim now follows from Lemma 3.5, since the expected dimension of \( W \) is \( d = g - 2 \). 

We next give a characterization of semistability if \( d = g \), analogous to Proposition 3.7 (2). We will need the following observation; it is a variation of Lemma 3.13 and appears in different form already in [CPS19, Lemma 5.13]:

**Lemma 3.15.** Let \( X \) be a stable curve and \( \underline{d} \) a multidegree of total degree \( g \). Then \( \underline{d} \) is semistable, if and only if \( \underline{d} - \nu \) is semistable for all vertices \( \nu \in V(G_{X}) \).

**Proof.** Observe first that for every subset of vertices \( Z \subset V(G_{X}) \) we have \( 0 < \frac{|\omega_{X}|}{2g - 2} < 1 \) since \( X \) is stable. Thus by Definition 3.2 and Remark 3.3, \( \underline{d} \) is semistable if and only if \( |d_{X}| \geq g(Z) \) for all subsets of vertices \( Z \subset V(G_{X}) \). On the other hand, \( \underline{d} - \nu \) is semistable if and only if \( |(\underline{d} - \nu)_{Z}| \geq g(Z) - 1 \) for all subsets of vertices \( Z \subset V(G_{X}) \), again by Definition 3.2 and Remark 3.3. Thus the claim follows.

Recall that we write \( h^{0}(X,L)_{X_{\nu}} \) for the dimension of the image of the restriction map on global sections in \( H^{0}(X_{\nu},L|_{X_{\nu}}) \) (see Convention 2.1).

**Lemma 3.16.** Let \( X \) be a stable curve and \( \underline{d} \) a multidegree of total degree \( g \). Then \( \underline{d} \) is semistable if and only if there is a line bundle \([L] \in \text{Pic}\underline{d}(X)\) such that \( h^{0}(X,L) = 1 \) and \( h^{0}(X,L)_{X_{\nu}} = 1 \) for every irreducible component \( X_{\nu} \) of \( X \).

**Proof.** If there is a line bundle \([L] \in \text{Pic}\underline{d}(X)\) such that \( h^{0}(X,L) = 1 \) and \( h^{0}(X,L)_{X_{\nu}} = 1 \) for all vertices \( \nu \in V(G_{X}) \), there is a smooth point \( p_{\nu} \in X_{\nu} \) for every irreducible component \( X_{\nu} \) such that \( h^{0}(X,L(-p_{\nu})) = 0 \). Thus \( \underline{d} - \nu \) is semistable for every \( \nu \in V(G_{X}) \) by Proposition 3.7 and \( \underline{d} \) is semistable by Lemma 3.15.

Conversely, suppose \( \underline{d} \) is semistable. Then \( \underline{d} - \nu \) is semistable for all vertices \( \nu \in V(G_{X}) \) by Lemma 3.15. Choose a smooth point \( p_{\nu} \in X_{\nu} \) for every \( \nu \in V(G_{X}) \). By Proposition 3.8, there is a dense open set \( U_{\nu} \subset \text{Pic}\underline{d}(X) \) such that for all line bundles \([L] \in U_{\nu} \) we have \( h^{0}(X,L(-p_{\nu})) = 0 \). Choose \([L] \subset \text{Pic}\underline{d}(X) \) in the intersection of the finitely many \( U_{\nu} \). Then by construction \( h^{0}(X,L) \leq 1 \) and hence \( h^{0}(X,L) = 1 \) by Riemann-Roch. Furthermore, none of the \( p_{\nu} \) are base points of \( L \) and hence \( h^{0}(X,L)_{X_{\nu}} = 1 \) for all \( \nu \in V(G_{X}) \), as claimed.

### 3.4. Effective loci for general semistable multidegrees

In this section, we prove the main result, Theorem 3.21, in which we consider effective loci for semistable multidegrees of total degree \( d \leq g - 2 \).

#### 3.4.1. Combinatorial considerations

We will need a combinatorial observation, Lemma 3.18, which in turn requires the following:

**Lemma 3.17.** Let \( X \) be a stable curve and \( \underline{d} \) a multidegree of total degree \( d < g - 1 \). Suppose \( |d_{Z}| \leq g(Z) - 1 + |(Z,Z')| \) for all \( Z \subset V(G_{X}) \). Then there is a vertex \( \nu \in V(G_{X}) \) such that for all \( Z \) with \( \nu \in Z \) the strict inequality \( |d_{Z}| < g(Z) - 1 + |(Z,Z')| \) holds.

**Proof.** To ease notation, we show the residual claim, which is easily seen to be equivalent: suppose \( d > g - 1 \) and \( |d_{Z}| \geq g(Z) - 1 \) for all subsets of vertices \( Z \subset V(G_{X}) \); then we claim that there is a vertex \( \nu \in V(G_{X}) \) such that \( |d_{Z}| > g(Z) - 1 \) for all subsets of vertices \( Z \subset V(G_{X}) \) with \( \nu \in Z \).
Let \( Z_1, Z_2 \subset V(G_X) \) and assume \(|d_{Z_1}| = g(Z_1) - 1 \) and \(|d_{Z_2}| = g(Z_2) - 1 \). Since \( d > g - 1 \), both \( Z_i \) are proper subsets of \( V(G_X) \). We first show that we then need to also have \(|d_{Z_1 \cup Z_2}| = g(Z_1 \cup Z_2) - 1 \).

Indeed, on the one hand we have
\[
|d_{Z_1 \cup Z_2}| = g(Z_1) - 1 + g(Z_2) - 1 - |d_{Z_1 \cap Z_2}|.
\]
Write \((Z_1, Z_2) \subset E(G_X)\) for the set of edges with one adjacent vertex in \( Z_1 \) and the other in \( Z_2 \). Then we have on the other hand
\[
g(Z_1) - 1 + g(Z_2) - 1 = g(Z_1 \cup Z_2) - 1 - |(Z_1, Z_2)| + g(Z_1 \cap Z_2) - l,
\]
where \( l = 0 \) if \( Z_1 \cap Z_2 = \emptyset \) and \( l = 1 \) otherwise. Substituting (3) in (2) and using the assumption on \( d \) applied to \(|d_{Z_1 \cup Z_2}| \) gives
\[
|d_{Z_1 \cup Z_2}| = g(Z_1 \cup Z_2) - 1 - |(Z_1, Z_2)| + g(Z_1 \cap Z_2) - l - |d_{Z_1 \cap Z_2}|
\]
\[
\leq g(Z_1 \cup Z_2) - 1 - |(Z_1, Z_2)|.
\]
By the assumption on \( d \) applied to \(|d_{Z_1 \cup Z_2}| \), this gives \(|d_{Z_1 \cup Z_2}| = g(Z_1 \cup Z_2) - 1 \), as claimed.

Now if we had for every vertex \( v_i \in V(G_X) \) a subset of vertices \( Z_i \subset V(G_X) \) containing \( v_i \) and such that \(|d_{Z_i}| = g(Z_i) - 1 \), the union of the \( Z_i \) would cover \( V(G_X) \). Applying what we showed above, this would imply \( d = g - 1 \), contradicting the assumption. \( \square \)

The next lemma is a partial generalization of Lemma 3.13 to arbitrary degrees; it is however far from giving a characterization of semistability.

**Lemma 3.18.** Let \( X \) be a stable curve and \( d \) a semistable multidegree of total degree \( d \leq g - 1 \). Then there is an effective multidegree \( \omega \) such that \( d + \omega \) is semistable of total degree \( g - 1 \).

**Proof.** If \( d = g - 1 \), there is nothing to show. So assume \( d < g - 1 \). Since \( d \) is semistable, we have for all subsets of vertices \( Z \subset V(G_X) \):
\[
|d_Z| \leq g(Z) - 1 + (d - g + 1) \frac{|\omega_Z|}{2g - 2} + |(Z, Z^c)|.
\]

By assumption, \( d - g + 1 \leq -1 \) and since \( X \) is stable we have \( 0 < \frac{|\omega_Z|}{2g - 2} \). So we have for all subsets of vertices \( Z \subset V(G_X) \)
\[
|d_Z| \leq g(Z) - 1 + |(Z, Z^c)|.
\]

Thus we can apply Lemma 3.17 and there is a vertex \( v_1 \in V(G_X) \) such that \(|d_{v_1}| < g(Z) - 1 + |(Z, Z^c)| \) whenever \( v_1 \in Z \). We then have for every subset of vertices \( Z \subset V(G_X) \):
\[
|(d + v_1)_Z| \leq g(Z) - 1 + |(Z, Z^c)|.
\]

If \( d = g - 2 \), then \( d + v_1 \) is semistable of degree \( g - 1 \) (cf. Remark 3.3). Otherwise, \( d + v_1 \) again satisfies the assumption of Lemma 3.17 by (4). We obtain \( v_2 \) such that \( d + v_1 + v_2 \) satisfies an inequality as in (4). Repeating this procedure \( g - 1 - d \) times gives \( \omega = \sum_{i=1}^{g-1-d} v_i \) with \( d + \omega \) semistable as claimed. \( \square \)

**Remark 3.19.** It follows from Lemma 3.13 that the claim of Lemma 3.18 is true also for \(|d + \omega| = g - 2 \); that is, for \( d \) semistable of total degree \( d \leq g - 2 \) there is an effective multidegree \( \omega \) of degree \( g - 2 - d \) such that \( d + \omega \) is semistable.

For general total degrees of \( \omega \), this is not the case, as the next example shows.
Example 3.20. Let $G_X$ be the graph with three vertices joined by two pairs of double edges. Let the weight of the vertices be $(2,1,2)$ and consider the multidegree $d = (0,3,0)$, where the middle entry in both cases corresponds to the vertex in $G_X$ of valence 4. Then the total degree is $|d| = 3 = g - 4$ and one checks that $d$ is semistable. But $d + v$ is not semistable for any vertex $v$ of $G_X$ (however, adding $e = (1,0,1)$ gives a semistable multidegree of total degree $g - 2 = 5$).

3.4.2. Effective loci for semistable multidegrees. We are ready to prove the main statement of this section. While we give examples in the next section that show that $W_d(X)$ can have dimension larger than the expected dimension even if $d$ is semistable, the upper bound $g - 2$ given in the next theorem is in general not strict.

Theorem 3.21. Let $X$ be a stable curve and $d$ a semistable multidegree of total degree $d \leq g - 2$. Then each irreducible component of $W_d(X)$ has dimension at most $g - 2$. If $d$ is in addition effective, then the effective locus $W_d^{\text{eff}}(X)$ contains an irreducible component of dimension $d$.

Proof. By Lemma 3.18 and Remark 3.19, there is an effective multidegree $e$ and a semistable multidegree $d'$ of total degree $g - 2$ such that $d + e = d'$. Fix a line bundle $[L_e] \in A_e(X)$, where $A_e(X)$, as before, denotes the image of the rational Abel map. Consider the isomorphism

$$\phi: \text{Pic}^d(X) \to \text{Pic}^{d'}(X),$$

given by sending $[L]$ to $[L \otimes L_e]$. Since $[L_e] \in A_e(X)$, we have $\phi([L]) = [L(p_1 + \cdots + p_{g - 2} - d)]$ for the collection of smooth points $p_i$ giving $L_e$. In particular, $h^0(X, L) \leq h^0(X, \phi(L))$ and $\phi(W_d(X)) \subset W_{d'}(X)$. Thus the first claim follows by Lemma 3.14.

For the second claim, since $d$ is assumed effective, we can consider the rational Abel map for $d$. Its image $A_d(X) \subset \text{Pic}^d(X)$ is irreducible and contained in an irreducible component $W$ of $W_d(X)$. Let $d + e = d'_0$ as in Lemma 3.18, that is, $e$ is an effective multidegree and $d'_0$ is a semistable multidegree of total degree $g - 1$. Consider the map

$$\phi: W \times A_e(X) \to V,$$

given by tensor product, that is, $\phi$ sends $[L] \in W$ and $[L_e] \in A_e(X)$ to $[L \otimes L_e]$; here $V$ denotes an irreducible component of the effective locus $W_d^{\text{eff}}(X)$ containing the image of $\phi$. Since $A_d(X) \subset W$, we have $A_{d'_0}(X) \subset V$ and $A_e(X)$ has dimension $g - 1$ by [Cap09, Proposition 3.2.1] (clearly $d'_0$ is effective if $d$ is).

We claim that $\dim(A_e(X)) = g - 1 - d$. Indeed, the restriction of $\phi$ to $A_d(X) \times A_e(X) \to A_{d'_0}(X)$ is surjective. On the other hand, a general line bundle $[L'] \in A_{d'_0}(X)$ satisfies $h^0(X, L') = 1$ by Lemma 3.12. Since $h^0(X, L_e) \leq h^0(X, L_e \otimes L)$ for line bundles $[L_e] \in A_e(X)$ and $[L] \in A_d(X)$, it follows that $h^0(X, L_e) = 1$ for a general line bundle $[L_e] \in A_e(X)$. Thus $\dim(A_e(X)) = |e| = g - 1 - d$ by Lemma 3.9.

Choose $[L] \in W$ and $[L_e] \in A_e(X)$. The restrictions of $\phi$ to $[L] \times A_e(X)$ and $W \times [L_e]$ are injective. Denote by $W_L$ and $W_{L_e}$ the respective images. By injectivity, $W_L \cap W_{L_e} = \{[L \otimes L_e]\}$. Furthermore, since the intersection $W_L \cap W_{L_e}$ is not empty, it has dimension at least $\dim(W) + \dim(A_e) - \dim(A_{d'_0}) = \dim(W) - d$. Thus $\dim(W) \leq d$, and the claim follows from Lemma 3.5.

3.5. Further counterexamples. We conclude our discussion of effective loci with a handful more examples, giving counterexamples to possible strengthenings of the results in this section. In the examples we will mention stable multidegrees, a notion otherwise not used in this section (see
Definition 3.2). We do so, since one might hope\(^1\) that restricting to stable multidegrees remedies the illustrated issues – which it does not.

The first two examples we consider show that the converse of Lemma 3.14 is not true. That is, if \(d = g − 2\) and the effective locus \(W_d(X)\) is either empty or equidimensional of dimension \(g − 2\) does not imply that the multidegree \(d\) is semistable.

**Example 3.22.** Let \(G_X\) be the graph with three vertices \(v_1, v_2, v_3\) of weight \((0, 0, 1)\), a triple edge between \(v_1\) and \(v_2\), and a single edge between \(v_2\) and \(v_3\). Consider the multidegree \(d = (2, −1, 0)\). Its total degree is \(|d| = 1 = g − 2\) and one checks that \(d\) is not semistable. However, the effective locus \(W_d(X)\) is empty in this case. Let us also mention in passing that the multidegree \((-1, 2, 0)\) on the same graph is stable with empty effective locus.

**Example 3.23.** Let \(G_X\) be the graph with vertices \(v_1, v_2, v_3\), all of weight 0, and three edges between each of the pairs \(v_1, v_2\) and \(v_2, v_3\). Denote by \(X_i\) the irreducible component of \(X\) corresponding to \(v_i\). Consider the multidegree \(d = (2, 0, 0)\). Its total degree is \(|d| = 2 = g − 2\) and one checks that \(d\) is not semistable. Effective line bundles \([L] ∈ \text{Pic}^d(X)\) are exactly those with \(L|_{X_1∪X_2} = O_{X_1∪X_2}\). Thus \(W_d(X)\) is irreducible and \(2\)-dimensional, the two parameters being given by the gluing data along the three nodes in \(X_1 ∩ X_2\).

We next turn to Theorem 3.21. Since we used in the proof only that \(d = d′ − e\) with \(d′\) semistable and \(e\) effective, it is easy to see that the claim in Theorem 3.21 does not characterize semistability. Instead, we give an example, where \(d\) is semistable, but the effective locus \(W_d(X)\) is irreducible of dimension greater than the expected dimension. Thus requiring \(d\) effective for the second claim in Theorem 3.21 is necessary.

**Example 3.24.** Let \(G_X\) be the graph with two vertices \(v_1\) and \(v_2\), of respective weights 1 and 5, and three edges between them. Consider the multidegree \(d = (−1, 3)\), which one checks to be semistable (in fact, stable). Denote by \(X_1\) and \(X_2\) the irreducible components of \(X\). The effective locus \(W_d(X)\) is then given as follows: any choice for \(L|_{X_1}\), any choice of gluing data, and \(L|_{X_2} = O_{X_2}(X_1 ∩ X_2)\). Thus \(W_d(X)\) is irreducible of dimension \(3 > 2 = |d|\).

Finally, we give an example where the multidegree \(d\) is semistable and effective, but \(W_d(X)\) has a component of dimension greater than the expected dimension. Thus in the second claim of Theorem 3.21, not all components need to be of expected dimension.

**Example 3.25.** Let \(G_X\) be the graph with two vertices \(v_1\) and \(v_2\), of respective weight 3 and 4, and a single edge between them. Consider the multidegree \(d = (1, 2)\), which is effective and semistable (in fact, stable). Denote by \(X_1\) and \(X_2\) the irreducible components of \(X\) and set \(p = X_1 ∩ X_2\). The effective locus \(W_d(X)\) has two irreducible components in this case: the first is the closure of \(A_d(X)\), which has dimension 3, equal to the expected dimension, and the second is \(\text{Pic}^1(X_1) × \{O_{X_2}(p + q)|q ∈ X_2\}\), which has dimension 4.

4. **Clifford inequality for uniform multidegrees**

In this section, we study the behaviour of the Clifford inequality. Recall that on a smooth curve \(X\) every line bundle \(L\) with degree \(0 ≤ d ≤ 2g − 2\) satisfies \(h^0(X, L) ≤ \frac{d}{2} + 1\) (see, e.g., [ACGH85, Clifford’s Theorem, p.107]). This is a consequence of the Riemann-Roch theorem and the fact that the natural map

\[
\mu_0 : H^0(X, L) ⊗ H^0(X, K ⊠ L^{-1}) → H^0(X, K)
\]

\(^1\)In [Cap09] stability as opposed to semistability is crucial in showing irreducibility of the Theta divisor.
satisfies \( \mu_0(s_1 \otimes s_2) \neq 0 \) for \( 0 \neq s_1 \in H^0(X, L) \) and \( 0 \neq s_2 \in H^0(X, K \otimes L^{-1}) \); here \( K \) denotes the canonical sheaf on \( X \). If \( X \) is reducible, the assertion about \( \mu_0 \) often fails, since the sections can be non-zero but still vanish along some irreducible components of \( X \) (see also Example 3.1).

As in Section 3, we address this by restricting the multidegrees of given total degree \( d \). Other than there, we consider in this section uniform multidegrees, since semistable multidegrees need not behave well with respect to the Clifford inequality (see Section 4.5).

**Definition 4.1.** Let \( X \) be a semistable curve. A multidegree \( \underline{d} \) will be called \emph{uniform} if \( 0 \leq d_v \leq 2g(v) - 2 + \text{val}(v) \) for all vertices \( v \in V(G_X) \).

We construct line bundles that do not satisfy the classic Clifford inequality in Section 4.1, showing in Proposition 4.6 that the new upper bound of Theorem B is always achieved. Then we describe the behaviour under additional restrictions on \( X \), first having a chain as dual graph and then having no separating nodes, leading to Theorem 4.10. In Section 4.4 we will prove the main result in Theorem 4.13.

4.1. **Counterexamples.** In this section, we construct examples of uniform multidegrees that do not satisfy the classic Clifford inequality.

**Example 4.2.** Let \( X \) be the stable curve consisting of three irreducible genus 1 components \( X_i \) each attached along a single node \( p_i \) to an irreducible rational component \( X_\nu \). Consider the multidegree \( \underline{d} \) with \( d_\nu = 0 \) and \( d_{v_i} = 1 \) where \( v_i \) is the vertex corresponding to \( X_i \). Let \( L \in \text{Pic}(X) \) be the line bundle whose restriction to \( X_i \) is \( \mathcal{O}_{X_i}(p_i) \). Then \( \underline{d} \) is uniform, but \( h^0(X, L) = 3 > \frac{g}{2} + 1 \).

**Remark 4.3.** Example 4.2 contradicts the claims in [Cap11, Proposition 3.1] \(^2\) and [Fra19, Theorem 3.14]. We will give a corrected statement in Theorem 4.13 below.

Example 4.2 works without changes for any choice of genera of the irreducible components, as long as \( g_{v_i} \geq 1 \). The next example and the construction in the proof of Proposition 4.6 give two other classes of examples that generalize Example 4.2.

**Example 4.4.** Let \( X \) be a semistable curve and \( \omega \) its dualizing sheaf. Suppose \( \omega \) has a smooth base point \( p \in X \) (in Example 4.2, all global sections of \( \omega \) vanish along \( X_p \)). We then have

\[
h^0(X, \omega(-p)) = h^0(X, \omega) = g > g - \frac{1}{2} = \frac{2g - 3}{2} + 1.
\]

Thus \( \omega(-p) \) does not satisfy the Clifford inequality. On the other hand, the multidegree \( \underline{d} \) of \( \omega \) is uniform since \( X \) is stable. By [Cat82, Theorem D, p. 75], the dualizing sheaf has a smooth base point if and only if \( X \) contains a smooth rational component \( X_\nu \) such that all points in \( X_\nu^c \cap X_\nu \) are separating nodes. In this case, all global sections of \( \omega \) vanish along \( X_\nu \). Thus we can subtract up to \( \text{val}(\nu) - 2 \) arbitrary smooth points of \( X_\nu \) from \( \omega \), obtaining in each case a line bundle with uniform multidegree not satisfying the classic Clifford inequality.

Before we can state Proposition 4.6, we will need one more combinatorial tool used throughout the section, the graph \( G^2_X \) associated to a curve \( X \). Its edges correspond to separating nodes and its vertices to connected components of the partial normalization of \( X \) at separating nodes.

**Definition 4.5.** Let \( G_X \) be a graph. The associated \emph{graph of 2-edge-connected components} \( G^2_X \) is the graph obtained from \( G_X \) by contracting all edges of \( G_X \) that are not bridges.

Note that each edge in \( G^2_X \) is by construction a bridge and thus \( G^2_X \) is a tree.

\(^2\)Note however that [Cap11, Proposition 3.1] uses a slightly different notion of uniform multidegrees.
Proposition 4.6. Let \( X \) be a semistable curve. Then there is a uniform multidegree \( d \) of total degree \( d \) and a line bundle \( [L] \in \text{Pic}^d(X) \) such that

\[
h^0(X, L) = \frac{d}{2} + 1 + \sum_{v \in V(G_X)} \max \left\{ 0, \frac{\text{val}(v) - 2}{2} \right\}.
\]

Proof. If \( \text{val}(v) \leq 2 \) for all vertices \( v \in \nu(G_X^2) \), we may choose \( L = \omega \), the dualizing sheaf of \( X \), for which the claim follows by Riemann-Roch. Otherwise, let \( v_i \in V(G_X^2) \) with \( 1 \leq i \leq l \) denote the \( l \geq 3 \) leaves of \( G_X^2 \) and \( Y_i \subset X \) the subcurves corresponding to these leaves. Let \( Y = (\bigcup_i Y_i)^c \) be the complementary subcurve, which is not empty since \( G_X^2 \) contains a 3-valent vertex. Set \( p_i = Y_i \cap Y^c \), a collection of \( l \) separating nodes of \( X \).

We define \( L \) by setting \( L|_Y = \mathcal{O}_Y \) and \( L|_{Y_i} = \omega_{Y_i}(p_i) = \omega|_{Y_i} \) where \( \omega \) and \( \omega_{Y_i} \) denote the dualizing sheaf of \( X \) and \( Y_i \), respectively. By construction, \( L \) has uniform multidegree. By Riemann-Roch, \( p_i \) is a base point of \( L|_{Y_i} \) and thus global sections of \( L \) vanish along \( Y \) and \( h^0(X, L)|_{Y_i} = h^0(Y_i, \omega_{Y_i}) \). Thus

\[
h^0(X, L) = \sum_{i=1}^l g(Y_i).
\]

On the other hand,

\[
\deg(L) = \sum_{i=1}^l (2g(Y_i) - 1) = 2 \sum_{i=1}^l g(Y_i) - l.
\]

Since \( G_X^2 \) is a tree, we have for the number of leaves \( l \):

\[
l = 2 + \sum_{v \in V(G_X^2)} \max \{0, \text{val}(v) - 2\}.
\]

Combining the three equations gives the claim. \( \square \)

The construction in the proof of Proposition 4.6 is in general not unique. For example, replacing \( \omega_{Y_i}(p_i) \) with \( \mathcal{O}_{Y_i}(p_i) \) gives other ways to construct \( d \) and \( L \) as claimed. In addition, the choice of \( Y \) allows for variations.

4.2. Curves of compact type. Proving the main theorem of this section, Theorem 4.13, requires several steps. We begin by considering the special case where \( X \) is of compact type, that is, \( G_X \) is a tree. Recall that \( G_X \) is called a chain if it is a tree with at most two leaves. Recall furthermore that we denote by \( \omega \) the multidegree of the dualizing sheaf on \( X \) and set \( 0 \) for the multidegree with value 0 on each vertex.

Lemma 4.7. Let \( X \) be a nodal curve with dual graph \( G_X \) a chain. Denote by \( v_1 \) and \( v_n \) the two leaves of \( G_X \) if \( G_X \) has more than one vertex, and \( v_1 = v_n \) if it has only one vertex. Let \( d \) be a multidegree of total degree \( d \) and such that \( 0 - v_1 - v_n \leq d \leq \omega + v_1 + v_n \). Then \( h^0(X, L) \leq \frac{d}{2} + 1 \) for every line bundle \( [L] \in \text{Pic}^d(X) \).

Proof. We will prove the claim by induction on the number of vertices of \( G_X \). If \( G_X \) has only one vertex, \( X \) is a smooth curve. In this case the claim follows from the Clifford Theorem for smooth curves (usually this theorem is stated for \( 0 \leq d \leq 2g - 2 \) but Riemann-Roch gives the range \( -2 \leq d \leq 2g \) required for our claim).

For the inductive step, suppose \( G_X \) has \( n \) vertices. Denote by \( X_n \) the component of \( X \) corresponding to the leaf \( v_n \) and let \( [L] \in \text{Pic}^d(X) \) with \( d \) as in the assumptions. Let \( p = X_n \cap X_n^c \) and set \( X' = X_n^c \) and \( L' = L|_{X_n^c} \). Since \( v_n \) is a leaf of \( G_X, G_{X'} \) is a chain.
If \( p \) is a base point of \( L' \) and \( L|_{X_n} \), we get \( h^0(X, L) = h^0(X', L') + h^0(X_n, L|_{X_n}) \) by Lemma 2.2. Since \( L'(-p) \) and \( L|_{X_n}(-p) \) are in range of the assumptions on \( X' \) and \( X_n \), respectively, we have
\[
h^0(X', L') = h^0(X', L'(-p)) \leq \frac{d'}{2} + \frac{1}{2}
\]
and
\[
h^0(X_n, L|_{X_n}) = h^0(X_n, L|_{X_n}(-p)) \leq \frac{d_{v_n}}{2} + \frac{1}{2},
\]
where \( d' \) is the total degree of \( L' \). Thus the claim follows since \( d = d' + d_{v_n} \).

If \( p \) is not a base point of \( L' \), we have \( h^0(X, L) = h^0(X', L') + h^0(X_n, L|_{X_n}) - 1 \) by Lemma 2.2. Since \( L' \) and \( L|_{X_n} \) are in range of the assumptions on \( X' \) and \( X_n \), respectively, we get \( h^0(X', L') \leq \frac{d'}{2} + 1 \) and \( h^0(X_n, L|_{X_n}) \leq \frac{d_{v_n}}{2} + 1 \). Thus the claim follows. \( \square \)

**Lemma 4.8.** Let \( X \) be a semistable curve of compact type and \( d \) a uniform multidegree of total degree \( d \). Then the general \([L] \in \text{Pic}^d(X)\) satisfies \( h^0(X, L) \leq \frac{d}{2} + 1 \).

**Proof.** We show the claim by induction on \(|V(\mathbb{G}_X)|\). If \(|V(\mathbb{G}_X)| = 1\), \( X \) is smooth and the claim follows from Clifford’s theorem. For the induction step, let \( v \in V(\mathbb{G}_X) \) be a leaf of \( \mathbb{G}_X \). Let \( X_1 \) be the connected subcurve of \( X \) containing \( X_v \) and the (possibly empty) chain of rational components \( X_w \) with \( \text{val}(w) = 2 \) in \( \mathbb{G}_X \). Let \( X_2 = X_1^c \) and \( p = X_1 \cap X_2 \) (since \( X \) is semistable, \( X_1 \neq X \)). Then \( X_2 \) is semistable and \( \mathbb{G}_{X_2} \) a tree. Denote by \( L_1, L_2 \) and \( L_v \) the restriction of \( L \) to \( X_1, X_2 \) and \( X_v \), respectively. Since \( d \) is uniform, it is zero on any component of \( X_1 \) other than \( X_v \).

Thus \( h^0(X_1, L_1) = h^0(X_v, L_v) \) and \( p \) is a base point of \( L_1 \) if and only if \( X_v \cap X_2^c \) is a base point of \( L_v \). Hence, for a general \( L_v \in \text{Pic}^d(X_v) \), either \( h^0(X_v, L_v) = 0 \) or \( p \) is not a base point of \( L_1 \). Note that \( L_2 \) might not have uniform multidegree but if it does not, then \( L_2(-p) \) does; thus in any case we have by induction \( h^0(X_2, L_2) \leq \frac{d}{2} + \frac{3}{2} \).

Case 1: \( h^0(X_v, L_v) = h^0(X_1, L_1) = 0 \). In this case, \( h^0(X, L) \leq h^0(X_2, L_2) \). If \( p \) is a base point of \( L_2 \), then by induction in fact \( h^0(X_2, L_2) \leq \frac{d}{2} + 1 \) and the claim follows. If \( p \) is not a base point of \( L_2 \), then \( h^0(X, L) = h^0(X_2, L_2) - 1 \) by Lemma 2.2 and the claim again follows.

Case 2: \( p \) is not a base point of \( L_1 \). Then by assumption and Lemma 2.2, \( h^0(X, L) = h^0(X_1, L_1) + h^0(X_2, L_2) - 1 \). Since \( d \) is uniform, we have \( 0 \leq d_{v} \leq g_{v} - 1 \). Thus we have \( h^0(X_v, L_v) \leq \frac{d}{2} + \frac{1}{2} \) for a general \([L_v] \in \text{Pic}^d(X_v)\) by Riemann-Roch and the second part of Clifford’s theorem (cf., e.g., [ACGH85, Clifford’s Theorem, p. 107]). This proves the claim. \( \square \)

### 4.3. Semistable curves without separating nodes

We continue with the next special case, namely if \( X \) has no separating nodes. This case is central for our discussion of the general case in the next section. To do so, we will need a variation of a graph-theoretic result by Fleischner [Fle76, Satz 1], known as the splitting lemma for 2-edge-connected graphs:

**Lemma 4.9.** Let \( \mathbb{G}_X \) be a 2-edge-connected graph and \( v \) and \( w \) two vertices. Suppose that there are no loops based at \( v \), \( \text{val}(w) \geq 3 \), and not all edges adjacent to \( w \) are also adjacent to \( v \). Then there is an edge \( e \) adjacent to vertices \( w \) and \( w' \neq v \) such that the graph obtained by deleting \( e \) and adding an edge between \( w' \) and \( v \) is still 2-edge-connected.

**Proof.** Suppose first there is an edge \( e_{vw} \) in \( \mathbb{G}_X \) adjacent to \( v \) and \( w \). By assumption, there is a vertex \( w' \neq v \) adjacent to \( w \). In this case, the graph \( G' \) obtained by deleting an edge \( e_{wv} \) between \( w' \) and \( w \) and adding an edge \( e_{vw} \) between \( w' \) and \( v \) is 2-edge connected. Indeed, a graph is 2-edge-connected if and only if every edge is contained in a cycle. Let \( C \) be a cycle of \( \mathbb{G}_X \) containing \( e_{wv} \). Let \( C' \) be the cycle of \( G' \) obtained from \( C - e_{wv} \) as follows: if \( e_{vw} \in C \), subtract \( e_{vw} \) and...
add $e_{vw}$; if $e_{vw} \notin C$, add $e_{vw}$ and $e_{vw}'$. One checks that $C'$ indeed is a cycle and since it contains $e_{vw}'$, we get that $e_{vw}'$ is not a bridge of $G'$. Thus we may choose a spanning tree $T$ of $G'$ that has $w$ as a leaf with adjacent edge $e_{vw}$, and does not contain $e_{vw}'$. The cycles of $G'$ correspond to edges of $G$ not contained in $T$. Since $G_X$ is 2-edge-connected and $T$ is also a spanning tree of $G_X$, the only possible bridge of $G'$ is $e_{vw}$. This is not the case, since by assumption $\text{val}(w) \geq 3$ and thus there is an edge adjacent to $w$ in $G'$ in addition to $e_{vw}$.

Now assume that there is no edge between $v$ and $w$. We construct an auxiliary graph $G_1$ as follows: we add an edge $e_{vw}$ adjacent to $v$ and $w$; then we delete 2 of the edges $e_{vw_1}$ and $e_{vw_2}$ adjacent to $w$ and replace each by an edge $e_{vw_1}$ adjacent to $v$ and the vertex $w_i$ the deleted edge was adjacent to. Arguing as above, the assumptions ensure that $G_1$ is still 2-edge-connected. Then the valence of $v$ in $G_1$ is at least 4 and thus by the splitting lemma [Fle76, Satz 1] there is a 2-edge-connected graph $G_2$ obtained by adding a new vertex $w''$, deleting $e_{vw}$ and one of the edges $e_{vw_1}$, and adding two edges $e_{wv''}$ and $e_{w''v_i}$. Contracting $e_{wv''}$ gives a graph as in the claim. □

**Theorem 4.10.** Let $X$ be a semistable curve without separating nodes and a uniform multidegree of total degree $d$. Then all line bundles $[L] \in \text{Pic}^d(X)$ satisfy $h^0(X, L) \leq \frac{d^2}{2} + 1$.

**Proof.** If the curve $X$ has no separating nodes, its dual graph $G_X$ is 2-edge-connected. It is a standard result in graph theory that every 2-edge-connected graph admits an ear decomposition. That is, there is a sequence of graphs $G_0, \ldots, G_n$ such that $G_0$ has a single vertex and no edges, $G_n = G_X$ and $G_i$ is obtained from $G_{i-1}$ by either adding an edge or attaching a chain along two edges, one at each leaf of the chain (or both at the same vertex if the chain is a single vertex; see Figure 1). In particular, each $G_i$ is 2-edge-connected. For the base case $G_0$, the corresponding curves are smooth and the claim follows from Clifford’s theorem. We need to show that attaching edges and chains as described above preserves the claim. Note that gluing two points of the curve (that is, adding an edge to the dual graph) has the same effect as attaching a component $X_v$ at the two points that get glued and extending the line bundles by $\mathcal{O}_{X_v}$ on $X_v$. Thus it suffices to check that the claim is preserved if we attach chains. In Case 3 below we will need to modify the curve the chain is attached to, thus we perform an induction on the number of edges $|E(G_X)|$.

![Four examples of attaching chains and edges to the bold triangle.](image)

**Adding chains.** Let $X' = X \cup C$ where $X'$ has no separating nodes and $G_C$ is a chain. Assume that every line bundle on $X$ with uniform multidegree satisfies the claim. We need to show that the same holds for $X'$. Let $L'$ be a line bundle on $X'$ with uniform multidegree $d'$ and total degree $d'$. Set $L = L'|_X$ and $L_C = L'|_C$ and denote by $d$ and $d_C$ their respective total degrees. Let $(p_1, p_n) = X \cap C$ be the two attaching points, contained in irreducible components of $C$ corresponding to the two leaves of $G_C$ (except if $C$ is irreducible).

Recall that we write $H^0(X', L')|_{(p_1, p_n)}$ and $h^0(X', L')|_{(p_1, p_n)}$ for the image of the restriction map on global sections in $k^2$, respectively its dimension (Convention 2.1). We have an exact sequence

$$0 \to H^0(X, L(-p_1 - p_n)) \times H^0(C, L_C(-p_1 - p_n)) \to H^0(X', L') \to H^0(X', L')|_{(p_1, p_n)} \to 0.$$  \hspace{1em} (5)
and thus
\[ h^0(X', L') = h^0(X, L(-p_1 - p_n)) + h^0(C, L_C(-p_1 - p_n)) + h^0(X', L')|_{(p_1, p_n)}. \]  
(6)

We need to show \( h^0(X', L') \leq \frac{d'}{2} + 1 \) where \( d' = d + d_C \). Notice that since \( p_1 \) and \( p_2 \) are contained in irreducible components of \( C \) corresponding to leaves of \( G_C \) and \( L' \) is uniform, the assumptions of Lemma 4.7 are satisfied for \( L_C, L_C(-p_1) \) and \( L(-p_1 - p_n) \). The main difficulty in the proof comes from the fact that \( L \) need not have uniform multidegree on \( X \); the following three cases cover all possibilities, up to renaming \( p_1 \) and \( p_n \) in Case 3:

Case 1: \( L \) has uniform multidegree on \( X \). By assumption, we have \( h^0(X, L) \leq \frac{d}{2} + 1 \). On the other hand, by Lemma 4.7 we have \( h^0(C, L_C(-p_1 - p_n)) \leq \frac{d}{2} + 1 \). Since \( h^0(X, L) = h^0(X, L(-p_1 - p_n)) + h^0(X, L)|_{p_1, p_n} \) and \( h^0(X', L')|_{p_1, p_n} \leq h^0(X, L)|_{p_1, p_n} \), this implies the claim by (6).

Case 2: \( L(-p_1 - p_n) \) has uniform multidegree on \( X \). Thus \( h^0(X, L(-p_1 - p_n)) \leq \frac{d}{2} \) by assumption and \( h^0(C, L_C) \leq \frac{d}{2} + 1 \) by Lemma 4.7. Since \( h^0(C, L_C) = h^0(C, L_C(-p_1 - p_n)) + h^0(C, L_C)|_{p_1, p_n} \) and \( h^0(X', L')|_{p_1, p_n} \leq h^0(C, L_C)|_{p_1, p_n} \), this implies the claim.

Case 3: \( L(-p_1) \) has uniform multidegree on \( X \). By assumption we have
\[ h^0(X, L) = h^0(X, L(-p_1)) + h^0(X, L)|_{p_1} \leq \frac{d}{2} + \frac{1}{2} + h^0(X, L)|_{p_1}. \]

Thus if \( h^0(X, L)|_{p_1} = 0 \), we can argue as in Case 1. If \( h^0(X, L)|_{p_1} = 1 \) and \( h^0(X, L)|_{p_2} = 0 \), we get
\[ h^0(C, L_C(-p_1 - p_n)) + h^0(X', L')|_{p_1, p_n} = h^0(C, L_C(-p_n)). \]

By assumption and Lemma 4.7, \( h^0(X, L(-p_1 - p_n)) \leq \frac{d}{2} + \frac{1}{2} \) and \( h^0(C, L_C(-p_n)) \leq \frac{d}{2} + \frac{1}{2} \); thus the claim follows. If \( h^0(X, L)|_{p_1, p_n} = 2 \), then \( L(-p_1 - p_n) \) is uniform and we proceed in Case 2.

Finally, suppose \( h^0(X, L)|_{p_1, p_n} = 1 \) but both \( p_1 \) and \( p_n \) are not a base point of \( L \); that is, they are a neutral pair for \( L \), the most delicate part of the calculation. Arguing as in Case 1, it suffices to show \( h^0(X, L) = h^0(X, L(-p_1)) + 1 \leq \frac{d}{2} + 1 \). Note that by assumption we have in any case \( h^0(X, L) \leq \frac{d}{2} + \frac{d}{2} \). Denote by \( p_1 \in X_v \) and \( p_n \in X_v \) the irreducible components of \( X \) containing \( p_1 \) and \( p_n \). We may assume that \( L \) and \( L(-p_1 - p_n) \) do not have uniform multidegree, otherwise proceeding with Case 1 or 2. Since \( L(-p_1) \) has uniform multidegree, this implies
\[ (d - v)_w = 0 \text{ and } d_v = 2g_v - 1 + \text{val}(v). \]

Suppose first \( X_v = X_w \). In this case, we need to have \( d_v = 1 \), \( g_v = 0 \) and \( \text{val}(v) = 2 \) by the above assumptions and since \( G_X \) is 2-edge-connected. If \( h^0(X_v, L|_{X_v} = 2 \) we also have \( h^0(X, L)|_{X_v} = 2 \) since \( g_v = 0 \) and \( d_v = 1 \). Since \( L(-p_1) \) has degree zero on \( X_v \), in this case \( p_n \) cannot be a base point of \( L(-p_1) \), contrary to our assumptions. Similarly, if \( h^0(X_v, L|_{X_v} = 0 \) and \( p_n \) are base points of \( L \) and hence not a neutral pair. Thus \( h^0(X_v, L|_{X_v} = 1 \). If \( X_v \) are a neutral pair of \( L|_{X_v} \), we can identify the two points \( X_v \cap X_v \) to obtain \( X \). By Lemma 2.2, there is a line bundle \( L \) on \( X \) with \( h^0(X, L) = h^0(X, L) \). Since \( L \) has uniform multidegree on \( X \), the claim follows by induction. It remains the case in which \( p_1 \) is a base point of \( L|_{X_v} \) but \( p_n \) is not. In this case, we replace \( X \) \( X \) obtained by attaching a curve \( X_v \) of genus 1 along \( X_v \cap X_v \) to \( X_v \). We replace \( L \) with a line bundle \( L \) on \( X \), whose restriction to \( X_v \) is \( L|_{X_v} \) and whose restriction to \( X_v \) is \( X_v \). Using Lemma 2.2, we get \( h^0(X, L) = h^0(X, L) \). But \( L \) is uniform on \( X \), and the claim follows by induction.

Suppose from now on \( X_v \neq X_w \) and recall that we have \( d_w = 0 \) in this case. If \( \text{val}(v) = 2 \), we replace \( X_{w v} \) by a rational component \( X_{w v} \). For an appropriate modification of \( L \) as above, this can only increase the dimension of \( h^0(X, L) \) and \( L \) is still uniform. Thus we may assume \( g_w = 0 \).
Consider the residual $\omega \otimes L^{-1}$ of $L$. It is immediate that $L$ is uniform if and only if $\omega \otimes L^{-1}$ is and a straightforward calculation using Riemann-Roch shows that $L$ satisfies the claim if and only if $\omega \otimes L^{-1}$ does. Since $p_1$ is not a base point of $L$ we have by Riemann-Roch that it is a base point of $(\omega \otimes L^{-1})(p_1)$. Similarly, since $p_n$ is a base point of $L(-p_1)$, it is not a base point of $(\omega \otimes L^{-1})(p_1 + p_n)$. Thus $h^0(X, \omega \otimes L^{-1}) = h^0(X, (\omega \otimes L^{-1})(p_1 + p_n)) - 1$. Since $(\omega \otimes L^{-1})(p_1 + p_n)$ has degree 1 on $X_w$ and all global sections vanish along $X_w$, we are in the same situation as in case $X_w = X_v$. Arguing as there, we get $h^0(X, (\omega \otimes L^{-1})(p_1 + p_n)) \leq \frac{2g - d}{2} + 1$ and thus $h^0(X, \omega \otimes L^{-1}) \leq \frac{2g - d}{2} + 1$, which implies the claim.

It remains the case $\text{val}(w) \geq 3$. If $G_X$ contains loops at $w$, we pass to the partial normalization at the corresponding nodes, which can only increase the number of sections and the pullback of $L$ still is uniform since $d_w = 0$. In addition, the partial normalization still has no separating nodes. If $\text{val}(w) = 2$ in the partial normalization, we proceed as above. Thus assume $X_w$ is smooth and $\text{val}(w) \geq 3$. Suppose first that all edges of $w$ are adjacent to $v$. In this case $X_w$ has still no separating nodes and $h^0(X, L(-p_n)) = h^0(X_w, L|_{X_w}(-X_w \cap X_w'))$. If $L|_{X_w}(-X_w \cap X_w')$ is not uniform, we need to have $X = X_w \cup X_v$ since $d_w = 2g - 1 + \text{val}(v)$ and this case is readily checked. Otherwise, we have by induction

\[ h^0(X, L) = h^0(X, L|_{X_w}(-X_w \cap X_w')) \leq \frac{d - \text{val}(w)}{2} + 1. \]

Since $\text{val}(w) \geq 3$, we get $h^0(X, L(-p_n)) \leq \frac{d}{2}$ and the claim follows.

In the remaining case, the assumptions of Lemma 4.9 are satisfied. Thus there is an edge $e$ adjacent to vertices $w$ and $w'$, such that removing $e$ and instead adding an edge between $w'$ and $v$ gives a graph that is still 2-edge-connected. Let $z$ be the node corresponding to $e$ and $p \in X_w$ and $p' \in X_w$ be the two preimages of $z$ in the partial normalization $X_w^c$ of $X$ at $z$. Let $L^c_{X_w}$ be the pullback of $L$ to $X_w^c$. By assumption, $p_1$ and $p'$ are not a base point of $L^c_{X_w}$. We first claim that if $p'$ and $p_1$ are not a neutral pair of $L^c_{X_w}$, then $h^0(X, L) = h^0(X^c_{X_w}, L^c_{X_w}) - 1$. Indeed, if $p'$ and $p_1$ are not a neutral pair, then by definition $p'$ is not a base point of $L^c_{X_w}(-p_1)$. By assumption, $z$ on the other hand is a base point of $L(-p_1)$. Since $p_1$ is not a base point of either $L$ or $L^c_{X_w}$, this implies $h^0(X, L) < h^0(X^c_{X_w}, L^c_{X_w})$ and thus $h^0(X, L) = h^0(X^c_{X_w}, L^c_{X_w}) - 1$. Let $\tilde{X}$ be the curve obtained from $X^c_{X_w}$ by gluing $p'$ and $p_1$. By construction, its dual graph is the one obtained from Lemma 4.9 and thus $\tilde{X}$ has no separating nodes. We choose a line bundle $\tilde{L}$ on $\tilde{X}$ that pulls back to $L^c_{X_w}$; the gluing is arbitrary if $p'$ and $p_1$ are not a neutral pair of $L^c_{X_w}$ and otherwise the unique gluing which preserves the dimension of the space of global sections as in Lemma 2.2. Thus in any case $h^0(X, L) = h^0(\tilde{X}, \tilde{L})$. By construction, $\tilde{L}$ is uniform on $X \cup E(G_X) = \text{val}(G_X)$. By induction we thus have $h^0(X, L) = h^0(\tilde{X}, \tilde{L}) \leq \frac{d}{2} + 1$, as claimed.

**Lemma 4.11.** Let $X$ be a nodal curve, $z \in X$ a non–separating node and $X^c_z$ the partial normalization of $X$ at $z$. Suppose for any uniform multidegree $d$, a general $\{L^c_z\} \in \text{Pic}(X^c_z)$ satisfies $h^0(X^c_z, L^c_z) \leq \frac{d}{2} + 1$. Then the same holds for $X$.

**Proof.** Let $L^c_z$ be a general line bundle on $X^c_z$ such that line bundles $L$ on $X$ that pull back to $L^c_z$ have uniform multidegree. Let $z_v \in X_v$ and $z_w \in X_w$ be the preimages of the node $z$ in $X^c_z$, contained in (possibly identical) irreducible components $X_v$ and $X_w$ of $X^c_z$.

Case 1: $L^c_z$ has uniform multidegree on $X^c_z$. In this case, the claim follows by assumption since $h^0(X, L) \leq h^0(X^c_z, L^c_z)$.

Case 2: $L^c_z(-z_v - z_w)$ has uniform multidegree on $X^c_z$. By assumption, we have

\[ h^0(X^c_z, L^c_z) \leq \frac{d}{2} + h^0(X^c_z, L^c_z)(z_v, z_w). \]
If $h^0(X^v, L^v)|_{z_v} \leq 1$, the claim follows. If $h^0(X^v, L^v)|_{z_v} = 2$, we get $h^0(X^v, L^v) = h^0(X, L) + 1$ by Lemma 2.2 and the claim also follows in this case.

Case 3: $L^v(-z_v)$ has uniform multidegree on $X^v$. By assumption, we have

$$h^0(X^v, L^v) \leq \frac{d}{2} + \frac{1}{2} + h^0(X^v, L^v)|_{z_v}.$$

If $h^0(X^v, L^v)|_{z_v} = 0$, the claim is immediate. If $h^0(X^v, L^v)|_{z_v} = 1$, we have $h^0(X, L) = h^0(X^v, L^v) - 1$ for a general $L$ that pulls back to $L$ by Lemma 2.2 and the claim follows.

### 4.4. The general case.

We are now ready to prove Theorem B. Recall that we denote by $G^2_X$ the tree obtained from the dual graph $G_X$ by contracting all edges that are not bridges.

**Lemma 4.12.** Let $X$ be a semistable curve and assume $G^2_X$ is a chain. Then $h^0(X, L) \leq \frac{d}{2} + 1$ for any line bundle $L$ with uniform multidegree and total degree $d$.

**Proof.** If $G^2_X$ consists of a single vertex, the claim follows by Theorem 4.10. Let $X_1$ and $X_2$ be two irreducible components of $X$ whose corresponding vertices in $G_X$ get mapped to the two leaves of $G^2_X$ under the contraction map $G_X \to G^2_X$. Let $p_i \in X_i$ be two smooth points of $X$ and denote by $\bar{X}$ the curve obtained from $X$ by gluing $p_1$ and $p_2$. Then $\bar{X}$ has no separating nodes by construction. Let $L$ be a line bundle on $X$ with uniform multidegree and $\bar{L}$ a line bundle on $\bar{X}$ whose pullback to $X$ is $L$. Then $\bar{L}$ has uniform multidegree on $\bar{X}$, as well, and $h^0(X, L) \leq h^0(\bar{X}, \bar{L}) + 1$ by Lemma 2.2. Thus by Theorem 4.10,

$$h^0(X, L) \leq \frac{d}{2} + 2$$

(7)

for any $X$ with $G^2_X$ a chain and $L$ with uniform multidegree of total degree $d$.

To decrease the bound of (7) by 1, we proceed as follows: Let $X_i$ and $p_i$ be as above and denote by $\tilde{X}$ the curve obtained by gluing two copies of $X$ along $p_1$ on one copy and $p_2$ on the other. Denote by $\tilde{L}$ the line bundle on $\tilde{X}$ that restricts to $L$ on both copies of $X$. More generally, repeatedly applying this construction, denote by $kX$ the curve consisting of $k$ copies of $X$ (see Figure 2). Let $kL$ be the corresponding line bundle on $kX$; that is, $G^2_{kX}$ is a chain with $k$ times the number of vertices of $G^2_X$ and $\deg(kL) = k \deg(L) = kd$.

![Figure 2](image)

*Figure 2.* In black, the dual graph of a curve $3X$ as in the proof of Lemma 4.12. The dashed black edges are those added to the three copies of $G_X$. The red edge is the one added to obtain the dual graph of $\bar{X}$.

By Lemma 2.2, we have $kh^0(X, L) - (k - 1) \leq h^0(kX, kL)$. Since $G^2_{kX}$ is a chain and $kL$ has uniform multidegree, we have by (7): $h^0(kX, kL) \leq \frac{kd}{2} + 2$. Combining the two estimates we get

$$h^0(X, L) \leq \frac{d}{2} + 1 + \frac{1}{k},$$

which implies the claim by choosing $k \geq 3$. 

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Theorem 4.13. Let $X$ be a semistable curve and $d$ a uniform multidegree of total degree $d$. Then all $[L] \in \text{Pic}^d(X)$ satisfy

$$h^0(X, L) \leq \frac{d}{2} + 1 + \sum_{v \in V(G^2_X)} \max \left\{ 0, \frac{\text{val}(v) - 2}{2} \right\}.$$ 

Furthermore, a general $[L] \in \text{Pic}^d(X)$ satisfies $h^0(X, L) \leq \frac{d}{2} + 1$.

Proof. For the first claim, denote by $l$ the number of leaves of $G^2_X$. Choose $l$ smooth points $p_i \in X_i$ with $X_i$ an irreducible component of $X$ such that the vertex in $G_X$ corresponding to $X_i$ gets mapped to the $i$-th leaf of $G^2_X$ under the contraction map $G_X \rightarrow G^2_X$. Denote by $\bar{X}$ the curve obtained by pairwise identifying the first $l - 1$ of the $p_i$ in two copies of $X$ (see Figure 3). Then $G^2_{\bar{X}}$ is a chain.

Let $\bar{L}$ be an invertible sheaf on $\bar{X}$ that restricts to $L$ on each copy of $X$. By Lemma 2.2 we have

$$2h^0(X, L) - (l - 1) \leq h^0(\bar{X}, \bar{L}).$$

Furthermore, since $L$ has uniform multidegree, so does $\bar{L}$, and, by construction, $\deg(\bar{L}) = 2 \deg(L) = 2d$. Since $G^2_{\bar{X}}$ is a chain, we get by Lemma 4.12

$$h^0(\bar{X}, \bar{L}) \leq \frac{2d}{2} + 1.$$ 

Combining the two estimates we obtain

$$h^0(X, L) \leq \frac{d}{2} + \frac{l}{2}.$$ 

The first claim then follows since the number of leaves of the tree $G^2_{\bar{X}}$ is given by

$$l = 2 + \sum_{v \in V(G^2_{\bar{X}})} \max \{0, \text{val}(v) - 2\}.$$ 

We show the second claim by induction on $|E(G_X)|$. As an extended base we use Lemma 4.8, that is, $X$ of compact type. For the induction step, we choose a non-separating node $z$ and consider $X'_z$, the partial normalization of $X$ at $z$. If $X'_z$ is semistable, a general line bundle $L'_z$ on $X'_z$ with uniform multidegree of total degree $d$ satisfies $h^0(X, L) = \frac{d}{2} + 1$ by induction. Hence the claim for $X$ follows by Lemma 4.11.
If $X'_\nu$ is not semistable, $z$ is contained in a connected chain of rational components $X_1 \subset X$. Let $L$ be a line bundle on $X$ with uniform multidegree. In particular, $L$ has degree 0 on each component of $X_1$. Thus if $X_1 = X$, $h^0(X, L) \leq 1$ and the claim is immediate. If $X_1 \neq X$, let $\{p_1, p_2\} = X_1 \cap X'_1$ and let $X'$ be obtained from $X'_1$ by gluing $p_1$ and $p_2$. Let $L'$ be such that its pullback to $X'_1$ is the restriction of $L$ to $X'_1$ and with gluing datum over $p_1$ and $p_2$ the one of $L$. Since $L$ has degree 0 on each irreducible component of $X_1$, we get $h^0(X, L) = h^0(X', L')$. Furthermore, $X'$ is semistable and $L'$ has uniform multidegree. Since $|E(G_{X'})| < |E(G_X)|$, the claim follows by induction. □

4.5. Clifford inequality and semistable multidegrees. We conclude the section by relating our discussion of semistable multidegrees in Section 3 to the results on the Clifford inequality in this section. Of course we immediately get from Theorem 3.21 and its residual statement:

**Corollary 4.14.** Let $X$ be a stable curve and $d$ a semistable multidegree whose total degree satisfies $0 \leq d \leq 2g - 2$. Then a general $[L] \in \text{Pic}^d(X)$ satisfies the $h^0(X, L) \leq \frac{d^2}{2} + 1$.

In general, semistable multidegrees are not uniform and vice versa. By Proposition 3.7, uniform multidegrees need not behave well with respect to the dimension of special loci (it is not difficult to construct examples for total degrees different than $g - 1$). On the other hand, the following variation of Example 4.2 shows that semistable multidegrees in turn need not satisfy the claims of this section with regard to the Clifford inequality (see also [Cap11, Example 4.15 and 4.17]).

**Example 4.15.** Let $G_X$ and $d$ be as in Figure 4. Let $v_1$ be the 9-valent vertex in the middle and $v_i$, $2 \leq i \leq 10$, the 2-valent vertices of weight 1. Denote by $X_i$ the irreducible components of $X$ corresponding to the vertices $v_i$ and let $p_i = X_1 \cap X_i$. Then the line bundle $L$ given by $L|_{X_i} = O_{X_i}(p_i)$ for $i \geq 2$ and an arbitrary choice of gluing datum over $p_1$ and $p_2$ the one of $L$. Since $L$ has degree 0 on each irreducible component of $X_1$, we get $h^0(X, L) = h^0(X', L')$. Furthermore, $X'$ is semistable and $L'$ has uniform multidegree. Since $|E(G_{X'})| < |E(G_X)|$, the claim follows by induction. □

**Figure 4.** The graph $G_X$ and the multidegree $d$ of Example 4.15. Bold vertices have weight 1 and vertices drawn as circles weight 0. We included the generalized orientation giving $d$, and stability of $\underline{d}$ follows from [CC19, Lemma 3.3.2].

There are two special cases, for which the results of this section apply:

**Lemma 4.16.** Let $X$ be a stable curve and $d$ a multidegree of total degree $d$.

1. If $d = g - 1$ and $\underline{d}$ is stable, then $\underline{d}$ is uniform.
2. If $d \in \{g - 2, g - 1, g\}$, $\underline{d}$ is semistable and $X$ has no irreducible components of genus 0, then $\underline{d}$ is uniform.
Proof. In the first case, \( g_v - 1 < d_v < g_v - 1 + \text{val}(v) \) for all \( v \in V(G_X) \) (cf. Definition 3.2). Thus \( g_v \leq d_v \leq g_v - 2 + \text{val}(v) \) for all \( v \in V(G_X) \), which clearly implies \( d \) uniform.

If \( X \) has no components of genus 0, we have \( g_v - 1 \geq 0 \) and \( g_v - 1 \leq 2g_v - 2 \). For \( d = g - 1 \) or \( d = g \) and \( d \) semistable, we have \( g_v - 1 \leq d_v \leq g_v - 1 + \text{val}(v) \) by Definition 3.2, respectively Lemma 3.15. Thus the claim follows. The case \( d = g - 2 \) is residual to \( d = g \) and \( d \) is semistable (respectively, uniform) if and only if \( \omega - d \) is semistable (respectively, uniform).

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\square
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