HYPERSPACES OF DIMENSION 1

ALFREDO ZARAGOZA

Abstract. In a previous paper the author asked if there exists a one-dimensional space $X$ that is not almost zero-dimensional, such that the dimension of the hyperspace of compact subsets of $X$ is one-dimensional. In this short note we give examples of spaces $X$ that are not almost zero-dimensional such that $X$ is one-dimensional and their hyperspace of compacta of $X$ also is one-dimensional.

1. Introduction

All spaces will be assumed to be separable and metrizable. A space $X$ is zero-dimensional if it has a base of clopen sets. A space $X$ is one-dimensional if and only if it has a base $\beta$ of neighborhoods such that $bd_X(U)$ and is zero-dimensional and nonempty for any $U \in \beta$. If $X$ has dimension one we write $dim(X) = 1$. In general we can define the dimension of a space $X$ for any $n \in \mathbb{N}$ (see [7]) but in this work we will only use the definition of dimension 0 and 1. For a space $X$, $K(X)$ denotes the hyperspace of non-empty compact subsets of $X$ with the Vietoris topology; for any $n \in \mathbb{N}$, $\mathcal{F}_n(X)$ is the subspace of $K(X)$ consisting of all the non-empty subsets that have cardinality less or equal to $n$; and $\mathcal{F}(X)$ is the subspace of $K(X)$ of finite subsets of $X$. For $n \in \mathbb{N}$ and subsets $U_1, \ldots, U_n$ of a topological space $X$, we denote by $\langle U_1, \ldots, U_n \rangle$ the collection $\{F \in K(X) : F \subset \bigcup_{k=1}^{n} U_k, F \cap U_k \neq \emptyset \text{ for } k \leq n \}$. Recall that the Vietoris topology on $K(X)$ has as its canonical base all the sets of the form $\langle U_1, \ldots, U_n \rangle$ where $U_k$ is a non-empty open subset of $X$ for each $k \leq n$. Note that if $X$ is a separable metrizable space, then they every subspace of $K(X)$ is also a separable metrizable space (see [5] Theorem 3.3 and Propositions 4.4 and 4.5.2). In [9] it was shown that if $X$ is a almost-zero dimensional space, then $\dim(X) = \dim(K(X))$. We are going to show spaces $X$ of one dimension that are not almost zero dimensional such that $\dim(K(X)) = 1$. The main results of this work are:

1.1. Theorem There exists a connected space $X$ such that $\dim(X) = \dim(K(X)) = 1$.

1.2. Theorem There exists a totally disconnected space $X$ which is not AZD such that $\dim(X) = \dim(K(X)) = 1$.

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2. Preliminaries

A space \((X, T)\) is almost zero-dimensional (AZD) if there is a zero-dimensional topology \(W\) in \(X\) such that \(W\) is coarser than \(T\) and has the property that every point in \(X\) has a local neighborhood base consisting of sets that are closed with respect to \(W\). This concept was introduced by Oversteegen and Tymchatyn in [3]. They proved that almost zero-dimensional spaces are at most 1-dimensional. Recall that Erdős space is defined as:

\[
E = \{(x_n)_{n \in \omega} \in \ell^2 : x_i \in \mathbb{Q}, \text{for all } i \in \omega\};
\]

and complete Erdős space as

\[
E_c = \{(x_n)_{n \in \omega} \in \ell^2 : x_i \in \{0\} \cup \{1/n : n \in \mathbb{N}\}, \text{for all } i \in \omega\}
\]

when \(\ell^2\) is the Hilbert space of all square summable real sequences. It’s known that Erdős space, and complete Erdős space are almost zero-dimensional spaces which are not zero-dimensional spaces (see [1]). A space \(X\) is called cohesive if every point of the space has a neighborhood that does not contain nonempty proper clopen subsets of \(X\).

2.1. Lemma[2] \(E\) and \(E_c\) are cohesive spaces.

A one-point connectification of a space \(X\) is a connected extension \(Y\) of the space such that the remainder \(Y \setminus X\) is a singleton.

2.2. Example Let \(p\) be a point outside \(E_c\), consider \(E_c^+ = E_c \cup \{p\}\) whose neighbourhoods of \(\{p\}\) are the complements of closed bounded sets of \(E_c\). Then \(E_c^+\) is metric separable connected space.

It is known that if a space admits a one-point connectification, then it is cohesive. Moreover if an almost zero-dimensional space is cohesive, then it admits a one point connectification (see [1] Proposition 5.4, p. 22)

Let \(X\) be an AZD and cohesive space (for example \(E, E_c\)), then \(X\) has a one-point connectification. Suppose that \(Y = \{p\} \cup X\) where \(p \notin X\). Since \(Y\) is connected then \(Y\) is not an AZD space.

Now let us consider \(E_c^+\) of example 2.2 and \(N = \{0\} \cup \{1/n : n \in \mathbb{N}\}\). Let

\[
P = [E_c^+ \times \{1/n : n \in \mathbb{N}\}] \cup (p, 0)
\]

with the topology inherited from \(E_c^+ \times N\), then is a totally disconnected and is not an AZD space (see [1] Example 3.6). The spaces \(Y\) and \(P\) are the spaces we will use to prove the main result.

Another important result for proving the main Theorems is the following:

2.3. Proposition [3] Proposition 2.2] \(X\) is an AZD space if only if \(K(X)\) is an AZD space.

3. Proof of main Theorems

Let \(Z \in \{P, Y\}\) and \(d\) a metric for \(Z\). For each \(n \in \mathbb{N}\), let \(B_n = \{z \in Z : d(z, q) < 1/n\}\) and let \(E_n = Z \setminus B_n\), where \(q = p\) if \(Z = Y\) or \(q = (p, 0)\) if \(Z = P\).
Note that for any \( n \in \mathbb{N} \), \( E_n \) is an AZD space and, by Proposition 2.3, \( K(E_n) \) is an AZD space. Let \( \mathcal{N} = \prod_{n} \{ K(E_n) \cup \{ \emptyset \} \} \), then \( \mathcal{N} \) is an AZD space (let's consider the set \( \{ \emptyset \} \) as an isolated point of \( K(E_n) \cup \{ \emptyset \} \)).

Let 
\[
\mathcal{L} = \{ (K_1, K_2, \ldots) \in \mathcal{N} : \text{for } m \geq n, K_m \cap E_n = K_n \}, \text{ and } \\
\mathcal{S} = \{ H \in K(Z) : q \in H \}
\]

Let's consider the following functions \( \mathcal{G} : \mathcal{L} \to \mathcal{S} \) and \( \mathcal{G}_n : \mathcal{S} \to \pi_n[\mathcal{L}] \) (where \( \pi_n \) is the projection to the \( n \)-th coordinate) such that
\[
\mathcal{G}(K_1, K_2, \ldots) = \{ q \} \cup \bigcup_{n} K_n, \text{ and } \\
\mathcal{G}_n(K) = K \cap E_n
\]

### 3.1. Lemma

\( \mathcal{G} \) is well defined and is a homeomorphism.

**Proof.** To prove that \( \mathcal{G} \) is well defined, let \( \mathcal{U} \) be an open cover of \( \{ q \} \cup \bigcup_{n} K_n \) in \( Z \). Since \( q \in \{ q \} \cup \bigcup_{n} K_n \), we can suppose that \( B_m \in \mathcal{U} \) for some \( m \). Note that \((\{ q \} \cup \bigcup_{n} K_n) \setminus K_m \subset B_m \), since \( Z \setminus K_m \subset B_m \). As \( \{ U \cap E_m : U \in \mathcal{U} \} \) is an open cover of \( K_m \) and \( K_n \) is a compact subset of \( E_m \), then there exists \( U_1, \ldots, U_k \), such that \( K_m \subset \bigcup_{i \leq k} (U_i \cap E_m) \). Therefore \( \{ B_m, U_1, \ldots, U_k \} \) is a finite subcover of \( \mathcal{U} \), that is \( \{ q \} \cup \bigcup_{n} K_n \) is a compact subset of \( Z \). Therefore \( \mathcal{G} \) is well defined.

Let's prove that \( \mathcal{G} \) is injective, let \( \hat{K} = (K_1, K_2, \ldots), \hat{H} = (H_1, H_2, \ldots) \in \mathcal{L} \) such that \( \hat{H} \neq \hat{K} \), then there exists \( k \in \mathbb{N} \) so that \( H_k \neq K_k \). Therefore there exists \( x \in H_k \setminus K_k \), then \( x \in \mathcal{G}(\hat{H}) \) and \( x \notin \mathcal{G}(\hat{K}) \). That is, \( \mathcal{G} \) is injective. Now let's see that \( \mathcal{G} \) is surjective, let \( \mathcal{K} \in \mathcal{S} \). We define \( K_n = \mathcal{K} \cap E_n \), since \( E_n \) is a closed subset in \( Z \), then \( K_n \) is empty or is a compact subset of \( E_n \). Then \( K_n \in K(E_n) \cup \{ \emptyset \} \) for each \( n \in \mathbb{N} \), therefore \( (K \cap E_1, \ldots) \in \mathcal{L} \) and \( \mathcal{G}((K \cap E_1, \ldots)) = \mathcal{K} \). That is, \( \mathcal{G} \) is surjective. Before proving that \( \mathcal{G} \) is a homeomorphism, let's show that
\[
\mathcal{B} = \{ \langle U_1, \ldots, U_n, B_k \rangle \cap \mathcal{S} : n, k \in \mathbb{N} \text{ and } U_1, \ldots, U_n \text{ are open subsets of } Z \setminus \{ p \} \} \cup \{ \langle B_k \rangle : k \in \mathbb{N} \}
\]
is a basis for \( \mathcal{S} \).

Let \( \mathcal{K} \in \mathcal{S} \) and \( \mathcal{W} = \{ W_1, \ldots, W_n \} \) an open subset of \( K(Z) \) such that \( \mathcal{K} \in \mathcal{W} \). If \( K_n \neq \emptyset \) for some \( n \in N \), then \( H_r \neq \emptyset \) for \( r \geq n \), without loss of generality we can assume that \( n = 1 \). As \( p \in K \in \mathcal{W} \), then there exist \( j \leq n \) such that \( p \in \bigcap \{ W_j : j \leq n, p \in W_j \} \), and \( k \in \mathbb{N} \) such that \( p \in B_k \subset \bigcap \{ W_j : j \leq n, p \in W_j \} \). To find an element \( \mathcal{V} \) of the base \( \mathcal{B} \) such that \( \mathcal{K} \in \mathcal{V} \subset \mathcal{W} \), we consider two cases. If \( K \setminus B_k = \emptyset \) or if \( K \setminus B_k \neq \emptyset \). If \( K \setminus B_k = \emptyset \) then \( K \subset B_k \). Therefore \( K \in \langle B_k \rangle \subset \mathcal{W} \). If \( K \setminus B_k \neq \emptyset \) then for each \( x \in K \setminus B_k \) there exist \( U_x \) such that \( x \in U_x \subset \bigcap \{ W_j : j \leq n, x \in W_j \} \), as \( K \setminus B_k \) is compact and \( \{ U_x : x \in K \setminus B_k \} \) is an open cover of \( K \setminus B_k \), there exist \( x_1, \ldots, x_i \) such that \( K \setminus B_k \subset \{ U_{x_1}, \ldots, U_{x_i} \} \). Let \( \mathcal{V} = \{ U_{x_1}, \ldots, U_{x_i}, B_k \} \cap \mathcal{S} \), note that \( \mathcal{K} \in \mathcal{V} \), and \( \mathcal{V} \subset \mathcal{W} \cap \mathcal{S} \). On the other hand if \( K = \{ p \} \), there exist \( k \in \mathbb{N} \) such that \( p \in B_k \) and \( p \in B_k \subset \bigcap \{ W_j : j \leq n, p \in W_j \} \) this implies that \( K \in \langle B_k \rangle \subset \mathcal{W} \). Therefore \( \mathcal{B} \) is a basis for \( \mathcal{S} \).

Let \( \mathcal{K} = (H_1, \ldots, H_n, \ldots) \in \mathcal{L} \), and \( \mathcal{U} \in \mathcal{B} \) such that \( \mathcal{K} = \mathcal{G}(K) \in \mathcal{U} \). If \( H = \{ p \} \), then \( \mathcal{U} = \langle B_k \rangle \) for some \( k \in \mathbb{N} \) and \( H_n = \emptyset \) for each \( n \in \mathbb{N} \). Let \( \mathcal{W} = \{ \emptyset \} \times \prod_{k} [\langle K(E_{k+1}) \cup \{ \emptyset \} \rangle] \times \prod_{m, k+1} [\langle K(E_m) \cup \{ \emptyset \} \rangle] \), note that \( \mathcal{W} \in \mathcal{W} \), and \( \mathcal{G}(\mathcal{W}) \subset \mathcal{U} \). If \( H_i \neq \emptyset \) for some \( i \in \mathbb{N} \), then \( H_r \neq \emptyset \) for \( r \geq n \), without loss
of generality we can assume that $i = 1$, and that $U = \langle U_1, \ldots, U_n, B_k \rangle$ for some $n, k \in \mathbb{N}$. Let $A = \{ j \in \mathbb{N} : U_l \cap H_j \neq \emptyset \text{ for all } l \leq n \}$ and as \{ $H_k : k \in \mathbb{N} \}, \text{ is not finite, then } A \neq \emptyset$. Let $r = \min A$, if $r < k$, then $F_k \cap U_j = \emptyset$ for some $j \leq n$, so $F_r \cap U_j \neq \emptyset$ and $F_r \cap U_j \subset B_k$. Let

$$N = \{(F_1, F_2, \ldots) \in \mathcal{L} : F_r \in \langle U_1, \ldots, U_n, B_k \rangle \}.$$ 

Note that $K \in N$, if $F = (F_1, F_2, \ldots) \in N$ and $G(F_1, F_2, \ldots) = F$, then $p \in F \setminus F_k \subset B_r \subset B_k$, so $F \in \mathcal{U}$. If $r \leq k$, then $H_k \setminus H_r \subset \bigcup_{j \leq n} U_j$ and $H \setminus H_k \subset B_k$. Let

$$N = \{(F_1, F_2, \ldots) \in \mathcal{L} : F_k \in \langle U_1, \ldots, U_n \rangle \}.$$ 

Note that $K \in N$. If $(F_1, \ldots, F_k, \ldots) \in N$, and $G(F_1, F_2, \ldots) = F$, then $p \in F \setminus F_k \subset B_k$, so $F \in \mathcal{U}$. This implies that $G$ is a continuous function.

Finally we will show that $G^{-1}$ is a continuous function, if $U$ is a basic open subset of $\mathcal{L}$, then $U = (\bigcap_{j \in F} \pi_j^n[W_j]) \cap \mathcal{L}$, where $W_j$ is an open subset of $K(E_j) \cup \{\emptyset\}$ and $F$ is a finite subset of $N$. Hence

$$(G^{-1})^{-1}[U] = \bigcap_{j \in F} (G^{-1})^{-1}[\pi_j^n[W_j]] \cap \mathcal{S} = \bigcap_{j \in F} G^{-1}_n[W_j].$$

So that is enough to show the continuity of $G_n$ for any $n$. To prove that $G_n$ is continuous, it is sufficient to show that $G_n^{-1}[(U_1, \ldots, U_k) \cap K(E_n)]$ and $G_n^{-1}(\{\emptyset\})$ are open subsets of $\mathcal{S}$, where $U_1, \ldots, U_k$ are open subsets of $Z \setminus \{p\}$ such that $E_n \cap U_j \neq \emptyset$ for each $j \leq k$. We will show that

$$G_n^{-1}[(U_1, \ldots, U_k) \cap K(E_n)] = \mathcal{S} \cap \langle U_1, \ldots, U_k, B_n \rangle$$

and that

$$G_n^{-1}(\{\emptyset\}) = \mathcal{S} \cap (B_n).$$

Let $H \in \mathcal{S} \cap \langle U_1, \ldots, U_k, B_n \rangle$, then $H_n \neq \emptyset$, $H \setminus H_n \subset B_n$, and $H_n \in [(U_1, \ldots, U_k) \cup \{\emptyset\}] \cap K(E_n)$, thus $H \in G_n^{-1}[(U_1, \ldots, U_k) \cap K(E_n)]$. Let $F \in G_n^{-1}[(U_1, \ldots, U_k) \cap K(E_n)]$, then $F_n = F \cap E_n \in (U_1, \ldots, U_k) \cap K(E_n)$ and $F \setminus F_n \subset B_n$, thus $F \in \langle U_1, \ldots, U_k, B_n \rangle$. Let $H \in \mathcal{S} \cap (B_n)$, then $H_n = \emptyset$, therefore, $H_n \in \{\emptyset\}$, thus $H \in G_n^{-1}(\{\emptyset\})$. Let $F \in G_n^{-1}(\{\emptyset\})$, then $F_n = F \cap E_n = \emptyset$, thus $F \subset B_n$ then $F \in (B_n)$. This implies that $G_n$ is a continuous function. Therefore $G$ is a homeomorphism.

\[ \square \]

3.2. Theorem $\dim(Z) = \dim(K(Z)) = 1$

\textbf{Proof.} Note that $K(Z) = K(Z \setminus \{q\}) \cup \mathcal{S}$. As $K(Z \setminus \{q\})$ is an AZD cohesive space, then $\dim(K(Z \setminus \{q\})) = 1$. By Theorem 3.1 $S$ is homeomorphic to $\mathcal{L}$ and $\dim(\mathcal{L}) = 1$ because $\mathcal{L}$ is an AZD, but $\mathcal{L}$ is not zero dimensional space. This implies that $\dim(\mathcal{S}) = 1$. Thus $\dim(K(Z)) = 1$. \[ \square \]

Proof of Theorem 1.1

\textbf{Proof.} Let $Y$ be a one-point connectification of $\mathcal{E}$. By Theorem 3.2 we have the result. \[ \square \]

3.3. Corollary Let $X$ be an cohesive and AZD space. If $Y$ is a one-point connectification of $X$, then $\dim(Y) = \dim(K(Y)) = 1$.

\textbf{Proof.} By Theorem 3.2 we have the result. \[ \square \]
Proof of Theorem 1.2

Proof. Consider the space $P$. By Theorem 3.2 we have the result. \hfill \square

Note that the spaces given in the Corollaries 1.1 and 1.2 are unions of AZD spaces. A natural question is:

Does every space $Z$ of dimension 1 that is not AZD and is a finite union of subspaces AZD satisfy that $\dim(K(Z)) = 1$? The answer to this question is negative because $[0, 1]$ is not an AZD space, but is a union of $\mathbb{Q} \cap [0, 1]$ and $\mathbb{P} \cap [0, 1]$ which are AZD spaces, and $\dim(K([0, 1]))$ is not 1.

On the other hand it is known that if $X$ is a compact space of dimension 1, then the dimension of $K(X)$ is not finite (see [8, pag 123]). This implies that if a space $X$ has a compact subset of dimension 1 then the dimension of $K(X)$ is not finite. Then for $K(X)$ to have dimension 1 each $A \in K(X)$ must have dimension zero. With the following Theorem, we will show that it is not enough that the compact subsets of a space $X$ of dimension 1 have dimension 0 for that hyperspace of compact subsets of $X$ to have dimension 1.

3.4. Theorem [6, Theorem 4.1] There exists a space $X$ of dimension 1 such that all its compacta have dimension 0 and $\dim(X^2) = 2$.

3.5. Example Let $Y = X \times \{0, 1\}$ where $X$ is as in Theorem 3.3 then $\dim(Y) = 1$ and for each compact subset $F$ of $X$ we have that $\dim(F) = 0$. Let $f : X^2 \to K(X)$ given by $f(x, y) = \{(x, 0), (y, 1)\}$. Note that $f$ is an embedding. This implies that $\dim(K(Y)) \geq 2$.

3.6. Question Let $X$ be a space of dimension 1, such that $\dim(X^\omega) = 1$ and for each $A \in K(X)$, $\dim(A) = 0$. Does $K(X)$ have dimension 1?

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(A. Zaragoza) Departamento de Matemáticas, Facultad de Ciencias, Universidad Nacional Autónoma de México, Circuito Exterior s/n, Ciudad Universitaria, Coyoacán, 04510, Mexico City, Mexico

Email address, A. Zaragoza: soad151192@icloud.com