RESEARCH ARTICLE

Overnight Index Rate: Model, calibration and simulation

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Overnight Index Rate: Model, calibration and simulation

Olga Yashkir¹ and Yuri Yashkir¹*

Abstract: In this study, the extended Overnight Index Rate (OIR) model is presented. The fitting function for the probability distribution of the OIR daily returns is based on three different Gaussian distributions which provide modelling of the narrow central peak and the wide fat-tailed component. The calibration algorithm for the model is developed and investigated using the historical OIR data.

Keywords: Overnight Index Rate, fat-tailed distribution, autocorrelation, calibration, interest rate simulation, stress testing

1. Introduction
The development of OIR models is very important. There are several publications on this topic, such as Poisson–Gaussian models (Das, 2002) for the fed funds rates, (Benito, León, & Nave, 2006) for Eonia, a OIR model based on jump-diffusion process (Raudaschl, 2012) and the OIR model based on short-term “memory” (auto-correlation) and its highly leptokurtic nature in (Yashkir & Yashkir, 2003). The OIR is used in overnight indexed swaps valuations, and is considered as the risk-free rate for valuation of collateralized portfolios (Hull & White, 2013). In the present study, we introduce the extended OIR model that was developed and validated. The model is based on auto-correlated daily log returns with the special stochastic driver represented by the weighted mix of three different Gaussian processes. The density distribution of this stochastic driver provides flexible modelling of the narrow central peak, the medium width component and the wide fat-tailed band. The calibration algorithm is developed, tested and validated using both “in-sample” and “out-of-sample” OIR simulations.
2. OIR model
The OIR $r$ for a Monte Carlo scenario $s$ is modelled as follows:

\[
\begin{align*}
& r_0^{(s)} = r_0 \\
& r_{i+1}^{(s)} = r_i^{(s)} \left( 1 + x_{i+1}^{(s)} \right) \quad (i = 0, \ldots, n) \\
& x_{i+1}^{(s)} = \sum_{k=1}^{\min(n/m)} \beta_k \epsilon_{i-k+1}^{(s)} \mathbf{q} \quad (i = 0, \ldots, n)
\end{align*}
\]  

The OIR daily return $x_{i+1}^{(s)}$ at a time point $t_{i+1}$ is correlated to $m$ previous daily returns. It is accounted for by the weighted sum of corresponding random drivers $\epsilon^i(\mathbf{q})$. The probability distribution function $g(x, \mathbf{q})$ of the random drivers $\epsilon^i(\mathbf{q})$ is introduced as the linear combination of three normal distributions:

\[
g(x, \mathbf{q}) = w_1 G(x, \mu_1, \sigma_1) + w_2 G(x, \mu_2, \sigma_2) + w_3 G(x, \mu_3, \sigma_3)
\]  

\[
G(x, \mu, \sigma) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}
\]  

where

\[
\begin{align*}
& t_i = i \Delta t \quad \text{time points } (i = 0, \ldots, n) \\
& \mathbf{q} = (\sigma_1, \sigma_2, \sigma_3, w_1, w_2, \mu_1, \mu_2, \mu_3) \quad \text{parameters to be calibrated} \\
& \sigma_{1,2,3} \quad \text{standard deviations of Gaussian functions } G(x, \mu, \sigma) \\
& w_{1,2} \quad \text{weight coefficients} \\
& w_3 = 1 - w_1 - w_2 \\
& \mu_{1,2,3} \quad \text{centering parameters} \quad (3)
\end{align*}
\]

The proposed distribution function (2) has enough flexibility to fit a typical historical distribution with a narrow central peak, and fat tails. The possible upward/downward rate drifts are reflected in non-zero values of $\mathbf{q}$. The auto-correlations of the daily returns of the OIR model (1) should satisfy the historical auto-correlations $\bar{\rho}$. Therefore, the auto-correlation factors $\bar{\rho}$ must satisfy the following equation:

\[
\sum_{k=1}^{m-\rho+1} \beta_k \rho_{k+p-1} = \rho_p \quad (p = 1, \ldots, m) \quad (4)
\]

where $\bar{\rho}$ is the historical auto-correlation vector (the overline indicates averaging by $\bar{i}$):

\[
\rho_p = \frac{(x_i - \bar{x})(x_{i+p-1} - \bar{x})}{(x_i - \bar{x})^2} \quad (5)
\]

The OIR model calibration is based on fitting of the model distribution (2) and of the model autocorrelation factors $\bar{\rho}$, to the historical data.

Given the set of historical overnight rates $\tilde{y}^{(h)}$ for a chosen time period, we calculate the historical density distribution $y^{(h)}(x)$ of overnight returns $\tilde{x}^{(h)}$. The calibration of the distribution $g(x, \mathbf{q})$ (2) is obtained by minimizing the objective function $H(\mathbf{q})$:

\[
\begin{align*}
& \mathbf{q}^{(\star)} = \arg \min_{\mathbf{q}} H(\mathbf{q}) \\
& H(\mathbf{q}) = \sum_{i} \left( \tilde{y}^{(h)}(x_i) - g(x_i, \mathbf{q}) \right)^2
\end{align*}
\]  

\[\]
where \( Q \) is a user-defined argument hyper-box:

\[
Q = \begin{cases} 
\sigma_k^{(\text{min})} < \sigma_k < \sigma_k^{(\text{max})} & k = 1, 2, 3 \\
 w_k^{(\text{min})} < w_k < w_k^{(\text{max})} & k = 1, 2 \\
 \mu_k^{(\text{min})} < \mu_k < \mu_k^{(\text{max})} & k = 1, 2, 3 
\end{cases} 
\] (7)

The usage of constraints defined by Equation 7 is in fact a method of regularization of the optimization procedure. A proper choice of the argument hyper-box based on user's experience (and intuition) makes the optimization algorithm convergence more reliable. In some cases, the hyper-box limits must be widened to ensure that optimal values of the model parameters are within limits of the hyper-box. The hyper-box limits do not affect the calibration parameter values as long as these values remain within the hyper-box.

The auto-correlation coefficients \( \vec{\rho}^{(h)} \) are calculated using \( \vec{x}^{(h)} \) in Equation 5. Taking into account (4) the factors \( \vec{\beta} \) are obtained by minimizing the objective function \( V(\vec{\beta}) \):

\[
\vec{\beta}^{(*)} = \arg \min_{\vec{\beta}} V(\vec{\beta}) \\
V(\vec{\beta}) = \sum_{p=1}^{m} \left( \sum_{k=1}^{m-p+1} \beta_k \beta_{k+p-1} - \rho_p^{(h)} \right)^2 
\] (8)

The optimization procedures (6) and (8) can be performed using the method “L-BFGS-B” that incorporates the box constraints.\(^1\)

The simulation of the OIR requires a special random driver function which generates random sequences distributed according to the function (2). The following random number generator was used:

\[
\epsilon(\vec{q}) = \eta_1 \cdot \gamma_1 + (1 - \eta_1) \cdot \eta_2 \cdot \gamma_2 + (1 - \eta_1) \cdot (1 - \eta_2) \cdot \gamma_3 
\] (9)

where the function \( \eta_i \) returns 1 with probability \( w_i \) or 0 with probability \( 1 - w_i \). The function \( \gamma_k \) generates normally distributed random numbers centred at \( \mu_k \) with standard deviation of \( \sigma_k \).

3. Historical data

The historical overnight rate data\(^2\) (4 January 1999 to 5 June 2013) were used as follows.

The long-time period data-set (4 January 1999 to 11 July 2012; Long Period A) covering 3464 time points, the short-time period data-set (11 July 2011 to 11 July 2012; Short Period B) corresponding to 259 time points, and the medium-time period data-set (4 January 1999 to 31 December 2004; Medium Period C; 1534 time points) were chosen for the calibration of the model.

The out-of-sample simulations of overnight rates were tested for two different time periods: from 1 January 2005 to 30 December 2011 (using calibration from the period C) and from 12 July 2012 to 5 June 2013 (using for comparison the two cases of calibration—the period A and the period B).

The time dependence of Eonia rates and daily returns \( \vec{x}^{(h)} \) is presented in Figures 1 and 2.

The autocorrelation function (ACF) analysis of the Eonia daily rate returns is presented in Figure 3. The process is clearly stationary because autocorrelation coefficients decline rapidly as the time lag increases. At the same time, the OIR time series is a non-stationary process: the ACF (Figure 4) is decreasing very slowly.\(^1\)
4. The OIR model calibration

The OIR model calibration based on the Long Period A data (4 January 1999 to 11 July 2012) begins from the choice of the hyper-box for finding random driver parameters $\vec{q}$:

$$Q = \begin{cases} 
0.0001 < \sigma_1 < 0.01 \\
0.0001 < \sigma_2 < 0.02 \\
0.0001 < \sigma_3 < 0.95 \\
0 < w_k < 0.5 & k = 1, 2 \\
w_3 = 1 - w_1 - w_2 \\
0 < \mu_k < 0.003 & k = 1, 2, 3
\end{cases}$$
The result of the optimization procedure (6) was reached after eight iterations (from $H = 535.2$ to $\min H = 53.8$), and the resulting vector $\vec{q}$ is presented in Table 1 (Long Period A). The process of the convergence is shown in Figure 5.

The Long Period A calibration results demonstrate that the best-fit distribution has a narrow $(\sigma_1 = .38\%$) peak ($w_1 = 45\%$ weight), a wide band ($\sigma_2 = 2.0\%$ with the $w_2 = 45\%$ weight) and a fat-tail band ($\sigma_3 = 9.25\%$ with the $w_3 = 9.7\%$ weight). The optimal fit of the calibrated probability distribution function (2) to the historical density distribution $y^{(h)}$ is shown in Figure 6.
Using the algorithm (8), we obtained the $\vec{\beta}$ values (Table 1, Long Period A). Note that the one-day log auto-correlation coefficient is negative which reflects the auto-compensation feature of the OIR time dynamics.

The efficiency of the calibration, and of the model itself, can be verified by the “in-sample” backtesting procedure. This backtesting procedure consists in the simulation of the OIR using the calibrated model and in comparing simulation results with historical OIR series. We assume that the model performs well if the historical OIR time series lies between low- and high-confidence levels of simulated rates. The backtesting was done using the OIR model (1) with calibration parameters presented in Table 1 (Long Period A). The number of Monte Carlo scenarios was $N=5,000$. Results of the simulation are presented in Figure 7.
The historical OIR time series is mostly covered by low/high quantiles of simulated rates in spite of a very wide range of rate changes (the historical ratio of the highest rate to the lowest rate is equal to 5.75%/.131% > 40!).

The similar calibration of the OIR model for the Short Period B and for the Medium Period C was performed with results presented in Table 1. The “in-sample” backtesting results for these cases are illustrated in Figures 8 and 9.

The historical OIR time series is mostly covered by low/high quantiles of simulated rates in spite of very strong upward/downward rate drift periods and long periods with relatively stable rates.

The results of the OIR calibration based on different data-sets are summarized in Table 1.

5. OIR simulation using the “out-of-sample” calibration

5.1. The short-term OIR simulation
The “out-of-sample” OIR simulation for a short term (11 July 2012 to 5 June 2013; 230 days) was done:

| Case | Time period | From | To | $\sigma_1$ | $\sigma_2$ | $\sigma_3$ | $\beta_1$ | $\beta_2$ | $\beta_3$ | $\beta_4$ |
|------|-------------|------|----|------------|------------|------------|----------|----------|----------|----------|
| A    | 4 January 1999 | 11 July 2012 | .0038 | .0200 | .0925 | .4516 | .4515 | .0969 | .9656 | -.2333 | -.0760 | -.0594 |
| B    | 11 July 2011 | 11 July 2012 | .0230 | .0142 | .1585 | .4000 | .3968 | .2032 | .9750 | -.2050 | -.0212 | +.0142 |
| C    | 4 January 1999 | 31 December 2004 | .0092 | .0019 | .0762 | .3680 | .3680 | .2640 | .9445 | -.2520 | -.1925 | -.0697 |

Table 1. OIR Model Calibration Results

Figure 7. Backtesting (Long Period A): historical OIR (red), the upper/lower percentiles (99 and 1%) of simulated rates and the OIR simulated average (dashed curve).
• using the Long Period A calibration (Table 1, case A). Results of the simulation are presented in Figure 10; and

• using the Short Period B calibration (Table 1, case B). Results of the simulation are presented in Figure 11.

Simulated rates are presented in Figures 10 and 11 by upper/lower percentiles (99%/1%) and by the average of simulated rates. Historical rates (not used for calibration) are plotted as dots. In both cases, historical rates do not deviated far from the simulated averages. In both cases (Figures 10 and 11) the historical “out-of-sample” rates lie within the quantile envelope (99–1%).
5.2. The long-term OIR simulation

The “out-of-sample” OIR simulation for a long-term (31 December 2004 to 30 December 2011; 1,796 days) was done using Long Period C calibration (Table 1). Results of the simulation are presented in Figure 12.

In spite of the strong upward/downward drifts of the rate during certain periods of time, the envelope of upper/lower quantiles covers most of historical rate changes. The simulated OIR average and the historical rates have similar time dependence tendencies.

6. Summary

The extended OIR model was developed and validated. The model is based on auto-correlated daily log returns with the special stochastic driver (represented by the mix of three different Gaussian processes). The density distribution of this stochastic driver provides flexible modelling of the narrow central peak, the medium width component and the wide fat-tailed band. The calibration algorithm...
was developed, tested and validated using both “in-sample” and “out-of-sample” OIR simulations. The model is well suited for the OIR simulation in both quiet and stressed market conditions. The model can be used for OIR forward estimation, pricing of OIR-based derivatives (such as overnight interest rate swaps) and for stress testing after being calibrated on stressed market conditions.

Figure 12. The long-term OIR simulation using the Long Period C calibration.

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