Linear MIMO Precoding in Jointly-Correlated Fading Multiple Access Channels with Finite Alphabet Signaling

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Abstract—In this paper, we investigate the design of linear precoders for multiple-input multiple-output (MIMO) multiple access channels (MAC). We assume that statistical channel state information (CSI) is available at the transmitters and consider the problem under the practical finite alphabet input assumption. First, we derive an asymptotic (in the large-system limit) weighted sum rate (WSR) expression for the MIMO MAC with finite alphabet inputs and general jointly-correlated fading. Subsequently, we obtain necessary conditions for linear precoders maximizing the asymptotic WSR, and propose an iterative algorithm for determining the precoders of all users. In the proposed algorithm, the search space of each user for designing the precoding matrices is its own modulation set. This significantly reduces the dimension of the search space for finding the precoding matrices of all users as compared to the conventional precoding design for the MIMO MAC with finite alphabet inputs, where the search space is the combination of the modulation sets of all users. As a result, the proposed algorithm decreases the computational complexity of the search space for finding the precoding matrices of all users in terms of both the sum rate and the coded bit error rate.

I. INTRODUCTION

In recent years, the channel capacity and the design of optimum transmission strategies for multiple-input multiple-output (MIMO) multiple access channels (MAC) have been widely studied [1, 2]. However, most works on MIMO MAC rely on the critical assumption of Gaussian input signals. Although Gaussian inputs are optimal in theory, they are rarely used in practice. Rather, it is well-known that practical communication signals usually are drawn from finite constellations, such as pulse amplitude modulation (PAM), phase shift keying (PSK) modulation, and quadrature amplitude modulation (QAM). These finite constellation sets differ significantly from the Gaussian idealization [3]. Accordingly, transmission schemes designed based on the Gaussian input assumption may result in substantial performance losses when finite alphabet inputs are used for transmission [4–8]. For the case of the two-user single-input single-output MAC with finite alphabet inputs, the optimal angle of rotation and the optimal power division between the transmit signals were found in [9] and [10], respectively. For the MIMO MAC with an arbitrary number of users and generic antenna configurations, an iterative algorithm for searching for the optimal precoding matrices of all users was proposed in [5].

The transmission schemes in [4–10] require accurate instantaneous channel state information (CSI) available at the transmitters for precoder design. However, in some applications, the obtained instantaneous CSI at the transmitters might be outdated. Therefore, for these scenarios, it is more reasonable to exploit the channel statistics at the transmitter for precoder design, as they change much slower than the instantaneous channel parameters [11]. For finite alphabet inputs, for point-to-point systems, an efficient precoding algorithm for maximization of the ergodic capacity over Kronecker fading channels was developed in [12]. Also, in [13], asymptotic (in the large-system limit) expressions for the mutual information of the MIMO MAC with Kronecker fading were derived. Despite these previous works, the study of the MIMO MAC with statistical CSI at the transmitter and finite alphabet inputs remains incomplete, for two reasons: First, the Kronecker fading model characterizes the correlations of the transmit and the receive antennas separately, which is often not in agreement with measurements [14]. In contrast, jointly-correlated fading models, such as Weichselberger’s model [14], do not only account for the correlations at both ends of the link, but also characterize their mutual dependence. As a consequence, Weichselberger’s model provides a more general representation of MIMO channels. Second, systematic precoder designs for statistical CSI at the transmitter for the MIMO MAC with finite alphabet inputs have not been reported yet, even for the Kronecker fading model.

In this paper, we investigate the linear precoder design for the K-user MIMO MAC assuming Weichselberger’s fading model, finite alphabet inputs, and availability of statistical CSI at the transmitter. By exploiting a random matrix theory tool from statistical physics, called the replica method1, we first derive an asymptotic expression for the weighted sum rate (WSR) of the MIMO MAC for Weichselberger’s fading model in the large-system regime where the numbers of transmit and receive antenna both approach infinity. The derived expression indicates that the WSR can be obtained asymptotically by calculating the mutual information of each user separately over equivalent deterministic channels. This property significantly reduces the computational effort for calculation of the WSR. Furthermore, exploiting Karush-Kuhn-Tucker (KKT) analysis, we establish necessary conditions for the optimal precoding matrices for asymptotic WSR maximization. This analysis facilitates the derivation of an efficient iterative gradient descent

1We note that the replica method has been applied to communications problems before [13, 15, 16].
algorithm\(^2\) for finding the optimal precoders of all users. In the proposed algorithm, the search space for the design of the precoding matrix of each user is only the user’s own modulation set. Accordingly, denoting the number of transmit antennas and the size of the modulation set of user \(k\) by \(N_t\) and \(Q_k\), respectively, the dimensionality of the search space for finding the precoding matrices of all users with the proposed algorithm is \(\sum_{k=1}^{K} Q_k^{N_t}\), whereas the dimensionality of the search space of the algorithm employing instantaneous CSI at the transmitter in [5] is \(\prod_{k=1}^{K} Q_k^{N_t}\). This indicates that the proposed algorithm does not only provide a systematic precoder design method for the MIMO MAC with statistical CSI at the transmitter, but also reduces the implementation complexity by several orders of magnitude compared to the precoder design for instantaneous CSI. Moreover, the precoder designed for statistical CSI has to be updated much less frequently than the precoder designed for instantaneous CSI as the channel statistics change very slowly compared to the instantaneous CSI. In addition, unlike the algorithm in [5], the proposed algorithm does not require the computationally expensive simulation over each channel realization. Numerical results demonstrate that the proposed design provides substantial performance gains over systems without precoding and systems employing precoders designed under the Gaussian input assumption.

The following notations are adopted throughout the paper: Column vectors are represented by lower-case boldface letters, and matrices are represented by upper-case boldface letters. Superscripts \((\cdot)^T\), \((\cdot)^H\), and \((\cdot)^H\) stand for the matrix/vector transpose, conjugate, and conjugate-transpose operations, respectively. \(\det(\cdot)\) and \(\text{tr}(\cdot)\) denote the matrix determinant and trace operations, respectively. \(\text{diag}(\cdot)\) and \(\text{blockdiag}(\cdot)\) denote diagonal matrix and block-diagonal matrix containing in the main diagonal (or block diagonal) the elements of vector \(\cdot\) and matrices \(\cdot_{k}\). \(\mathbf{1} \times N\) denote the matrix with entries one, and \(\mathbf{0} \times N\) the zero vector or matrix, respectively. \(\otimes\) and \(\otimes\) denote the element-wise product and the Kronecker product of two matrices. \(\text{vec}(\mathbf{A})\) returns a column vector whose entries are the ordered stack of columns of \(\mathbf{A}\). \(\|\mathbf{x}\|_p\) denotes the Frobenius norm of matrix \(\mathbf{X}\). \(\mathbf{I}_M\) denotes an \(M \times M\) identity matrix, and \(E_V[\cdot]\) represents the expectation with respect to random variable \(V\), which can be a scalar, a vector, or a matrix.

### II. SYSTEM MODEL

Consider a MIMO MAC system with \(K\) independent users. We suppose each of the \(K\) users has \(N_t\) transmit antennas and the receiver has \(N_r\) antennas. Then, the received signal \(\mathbf{y} \in \mathbb{C}^{N_r \times 1}\) is given by

\[
\mathbf{y} = \sum_{k=1}^{K} \mathbf{H}_k \mathbf{x}_k + \mathbf{v}
\]

(1)

where \(\mathbf{x}_k \in \mathbb{C}^{N_t \times 1}\) and \(\mathbf{H}_k \in \mathbb{C}^{N_r \times N_t}\) denote the transmitted signal and the channel matrix of user \(k\), respectively. \(\mathbf{v} \in \mathbb{C}^{N_r \times 1}\) is a zero-mean complex Gaussian noise vector with covariance matrix \(\mathbf{I}_N\). Furthermore, we make the common assumption (as e.g. [11,17]) that the receiver has the instantaneous CSI of all users, and each transmitter has the statistical CSI of all users.

The transmitted signal vector \(\mathbf{x}_k\) can be expressed as

\[
\mathbf{x}_k = \mathbf{B}_k \mathbf{d}_k
\]

(2)

\(^1\)It is noted that although we derive the asymptotic WSR in the large-system regime, the proposed algorithm can also be applied for systems with a finite number of antennas.

\(^2\)To simplify our notation, in this paper, without loss of generality, we normalize the power of the noise to unity.
A. Some Useful Definitions

We first provide some useful definitions. Consider a virtual MIMO channel defined by

$$ \textbf{T}_A \in \mathbb{C}^{K_t N_r \times K_r N_t} \text{ is given by } \textbf{T}_A = \text{blockdiag} \left( \textbf{T}_{i_1}, \textbf{T}_{i_2}, \ldots, \textbf{T}_{i_K} \right) \in \mathbb{C}^{K_t N_r \times K_r N_t}, \text{ where } \textbf{T}_{i_k} \in \mathbb{C}^{N_r \times N_t}, $$

denotes a deterministic matrix, $k = 1, 2, \ldots, K_1$. $\psi_A \in \mathbb{C}^{K_r N_t \times 1}$ is a standard complex Gaussian random vector with i.i.d. elements. The minimum mean square error (MMSE) estimate of signal vector $\textbf{d}_A$ given (9) can be expressed as

$$ \textbf{d}_A = E_{\textbf{d}_A} \left( \textbf{E}_{\textbf{d}_A} \left[ \sqrt{T} \right] \textbf{B}_A \right). $$

(10)

Define the following mean square error (MSE) matrix

$$ \Omega_{i_k} = \textbf{B}_A E_{\textbf{d}_A} \left[ \textbf{d}_A \left( \textbf{d}_A - \textbf{d}_A \right)^H \right] \textbf{B}_A \text{.} $$

(11)

Also, define the MSE matrix of the $i_k$th ($i_1 \leq i_k \leq i_K$) user as

$$ \Omega_{i_k} = \langle \Omega_{A} \rangle_k \text{.} $$

where $\langle X \rangle \in \mathbb{C}^{N_r \times N_t}$ denotes a submatrix obtained by extracting the $((k-1)N_t + 1)$th to the $(kN_t)$th row and column elements of matrix $X$.

**Definition 1:** Define vectors

$$ \gamma_{i_k} = \left[ \gamma_{i_k,1}, \gamma_{i_k,2}, \ldots, \gamma_{i_k,N_t} \right]^T \text{ and } \psi_{i_k} = \left[ \psi_{i_k,1}, \psi_{i_k,2}, \ldots, \psi_{i_k,N_t} \right]^T \text{.} $$

Define the following matrices

$$ \textbf{R}_{i_k} = \Upsilon_{i_k} \text{diag} \left( \Gamma_{i_k}^T \right) \Gamma_{i_k}^H \in \mathbb{C}^{N_r \times N_r}, \text{ and } \textbf{r}_{i_k} = \Upsilon_{i_k} \text{diag} \left( \psi_{i_k} \right) \psi_{i_k} \in \mathbb{C}^{N_r \times N_r} \text{.} $$

(12)

where $\gamma_{i_k,n}$ and $\psi_{i_k,m}$ satisfy the following equations

$$ \begin{align*}
\gamma_{i_k,n} &= \mu_{i_k,n}^H (I_{N_r} + \textbf{R}_A)^{-1} \Upsilon_{i_k,n}, \text{ for } n = 1, 2, \ldots, N_r, \\
\psi_{i_k,m} &= \mu_{i_k,m}^H (I_{N_t} + \textbf{B}_A)^{-1} \Upsilon_{i_k,m}, \text{ for } m = 1, 2, \ldots, N_t.
\end{align*} $$

(13)

B. Asymptotic Mutual Information

Now, we are ready to provide a simplified asymptotic expression of (8).

**Proposition 1:** For the MIMO MAC model (1), when $N_r$ and $N_t$ both approach infinity but the ratio $\beta = N_t/N_r$ is fixed, the mutual information in (8) can be asymptotically approximated by

$$ I \left( \textbf{d}_A; \textbf{y} \left| \textbf{d}_A \right. \right) \simeq I \left( \textbf{d}_A; \textbf{z}_A \left| \square \textbf{T}_A \textbf{B}_A \right. \right) - \log_2 \det \left( I_{N_r} + \textbf{R}_A \right) - \log_2 \sum_{i_k=1}^{K_1} \gamma_{i_k}^T \Gamma_{i_k} \psi_{i_k} \text{.} $$

(14)

where $I \left( \textbf{d}_A; \textbf{z}_A \left| \sqcup \textbf{T}_A \textbf{B}_A \right. \right)$ represents the mutual information between $\textbf{d}_A$ and $\textbf{z}_A$ of channel model (9).

**Proof:** Please refer to Appendix A.

Suppose the transmit signal $\textbf{d}_A$ is taken from a discrete constellation with cardinality $Q_k$. Define $M_k = Q_k N_t$. $S_k$ denotes the constellation set of user $k$, $a_{k,j}$ denotes the $j$th element in the constellation set $S_k$, $k = 1, 2, \ldots, K_1$, $j = 1, 2, \ldots, M_k$. Then, based on the definition of $\textbf{T}_A$, $\textbf{B}_A$, and model (9), (14) can be further simplified as

$$ I \left( \textbf{d}_A; \textbf{y} \left| \textbf{d}_A \right. \right) \simeq \sum_{i_k=1}^{K_1} \sum_{i_k=1}^{M_k} \log_2 \sum_{a_{k,j} \in S_k} \left( \sqrt{T_{i_k}} \textbf{B}_i \right) \textbf{d}_{i_k} \text{.} $$

(15)

Now, we can express $I \left( \textbf{d}_A; \textbf{y} \left| \textbf{d}_A \right. \right)$ as

$$ \frac{1}{\lambda} \sum_{i_k=1}^{K_1} \sum_{i_k=1}^{M_k} \text{E} \left( \log_2 \sum_{a_{k,j} \in S_k} \left( \sqrt{T_{i_k}} \textbf{B}_i \right) \textbf{d}_{i_k} \right) $$

(16)

Eq. (15) implies that, in the large-system regime, the mutual information $I \left( \textbf{d}_A; \textbf{y} \left| \textbf{d}_A \right. \right)$ can be evaluated by calculating the sum of the mutual informations of all individual users.

over the equivalent channel $\textbf{T}_A$, $k = 1, 2, \ldots, K_1$. Compared to the conventional method of calculating mutual information which requires a search over all possible combinations of all users’ signal sets [5], the asymptotic expression in Proposition 1 has a significantly lower implementation complexity. Moreover, given statistical channel knowledge (i.e., $\left\{ U_{T_{t_k}} \mid k \right\}, \left\{ R_{U_{B_k}} \mid k \right\}, \left\{ G_{k} \right\}$), the asymptotic mutual information can be obtained from Proposition 1, without knowing the actual channel realization. Thus, the derived asymptotic expression can be used to design transceivers which only require knowledge of the channel statistics, see Section IV.

C. WSR Problem

It is well known that the capacity region of the MIMO MAC $(R_1, R_2, \ldots, R_K)$ can be achieved by solving the WSR optimization problem [1]. Without loss of generality, assume weights $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_K \geq \mu_{K+1} = 0$, i.e., users are decoded in the order $K, K-1, \ldots, 1$ [5]. Then, the WSR problem can be expressed as

$$ R_{\text{sum}}^w (B_1, B_2, \ldots, B_K) = \max_{B_1, B_2, \ldots, B_K} \sum_{k=1}^{K} \Delta_k f (B_1, B_2, \ldots, B_K) \text{.} $$

(17)

$$ \text{tr} \left( \textbf{B}_k^H \textbf{B}_k \right) \leq P_k, \quad k = 1, 2, \ldots, K \text{.} $$

(18)

where $\Delta_k = \mu_k - \mu_{k+1}$, $k = 1, 2, \ldots, K$.

$f (B_1, B_2, \ldots, B_K) = \text{tr} (\textbf{d}_1 \left( \textbf{d}_1^H \right) \ldots \text{tr} (\textbf{d}_K \left( \textbf{d}_K^H \right) \right)$ can be evaluated based on Proposition 1. When $\mu_1 = \mu_2 = \ldots = \mu_K = 1$, (17) reduces to the sum-rate maximization.

IV. LINEAR PRECODING DESIGN FOR MIMO MAC

A. Necessary Conditions for Asymptotically Optimal Precoders

**Proposition 2:** The asymptotically optimal precoders for maximization of the WSR in (17) satisfy the following conditions

$$ \kappa_l B_l = \log_2 e \sum_{k=1}^{K} \Delta_k \left( \sum_{l=1}^{K} \left( \Theta_{1,k,l,m,n} - \Theta_{2,k,l,m,n} + \Theta_{3,k,l,m,n} \right) \right), $$

for $l = 1, 2, \ldots, K \text{.}$

$$ \kappa_l \left( \text{tr} \left( \textbf{B}_k^H \textbf{B}_k - \textbf{P}_l \right) \right) = 0, \quad l = 1, 2, \ldots, K \text{.} $$

$$ \kappa_l \left( \text{tr} \left( \textbf{B}_k^H \textbf{B}_k \right) - \text{tr} \left( \textbf{P}_l \right) \right) \leq 0, \quad l = 1, 2, \ldots, K \text{.} $$

where $\Theta_{1,k,l,m,n} \in \mathbb{C}^{N_r \times N_r}$ are matrices with elements

$$ \Theta_{1,k,l,m,n} = \left( \Theta_{2,k,l,m,n} \right)^{\dagger} $$

(19)

(20)

(21)

(22)

(23)

(24)

(25)

(26)

(27)

(28)

(29)

It notes that the asymptotic expression obtained based on the replica method is also useful for systems with a finite number of antennas [13].

It is noted that the asymptotic expression obtained based on the replica method is also useful for systems with a finite number of antennas [13].
Here, \( e_m \) is a unit-vector with the \( m \)th element being one and all other elements zeros, and \( \sigma[x] \) denotes the Kronecker delta function where \( \sigma[x] = 1 \) if \( x = 0 \), and \( \sigma[x] = 0 \), otherwise. Also, \( T^{(k)}_l \), \( R^{(k)}_l \), \( \gamma^{(k)}_l \) and \( \Omega^{(k)}_l \) are obtained based on Definition 1 and (11) by setting \( A = \{ 1, 2, \cdots, k \} \), \( t = 1, 2, \cdots, k \). Furthermore, matrix \( R_{A_k} \in \mathbb{C}^{N_t \times N_t} \) in (27) is given by \( R_{A_k} = \sum_{t=1}^{k} R^{(k)}_t \). Moreover, \( \Delta_{k,m,n} \in \mathbb{C}^{N_t \times N_t} \) in (30) is a matrix where element \( \Delta_{k,m,n} \) is taken from row \( p = i + (j-1)N_t \) and column \( q = m + (n-1)N_t \) of matrix \( \Xi_{k,t} \in \mathbb{C}^{N_t^2 \times N_t^2} \), \( 1 \leq i \leq N_t \), \( 1 \leq j \leq N_t \), defined as

\[
\Xi_{k,t} = -E_{d_t} \left[ \Phi_{k,d_t,d_t^H} \otimes \Phi_{k,d_t,d_t^H}^T B_t \sqrt{\left( T^{(k)}_t \right)^T \left( T^{(k)}_t \right)} \right] - E_{d_t} \left[ \Psi_{k,d_t,d_t^T} \otimes \Psi_{k,d_t,d_t^T}^T B_t \sqrt{\left( T^{(k)}_t \right)^T \left( T^{(k)}_t \right)} \right].
\]

Here, \( \Xi_{k,t} \in \mathbb{C}^{N_t^2 \times N_t^2} \) denotes a communication matrix [27], and

\[
\Phi_{k,d_t,d_t^H} = \left( d_t - \hat{d}_t^{(k)} \right) \left( d_t - \hat{d}_t^{(k)} \right)^H \quad (32)
\]

\[
\Phi_{k,d_t,d_t^T} = \left( d_t - \hat{d}_t^{(k)} \right) \left( d_t - \hat{d}_t^{(k)} \right)^T \quad (33)
\]

\[
E_{d_t} = E_{d_t} \left[ \left( d_t - \hat{d}_t^{(k)} \right) \left( d_t - \hat{d}_t^{(k)} \right)^H \right]. \quad (34)
\]

Vectors \( z_l^{(k)} \) and \( \hat{d}_l^{(k)} \) of the \( l \)th user are obtained based on (9), (10), and (T^{(k)})_l, \( l = 1, 2, \cdots, k \).

**Proof:** In order to solve the WSR optimization problem in (17), we can establish a Lagrangian cost function for the precoding matrices. Then, based on the KKT conditions and the matrix derivation technique [28], we can obtain Proposition 2. Due to the space limitation, details of the proof are omitted here, and will be given in an extended journal version of this paper.

### B. Iterative Algorithm for Weighted Sum Rate Maximization

The necessary condition in (19) indicates that the precoding matrices of different users depend on each other. Thus, the optimal precoding matrices \( B_l \), \( l = 1, 2, \cdots, K \) have to be found numerically. Problem (17) is a multi-variable optimization problem. Therefore, we employ the alternating optimization method which iteratively updates one precoder at a time with the other precoders being fixed. This is a commonly used approach in handling multi-variables optimization problems [5]. In each iteration step, we optimize the precoders along the gradient descent direction which corresponds to the partial derivative of the WSR (17) with respect to \( B_l \), \( l = 1, 2, \cdots, K \). The partial derivative is given by the right

\[
\text{Algorithm 1: Gradient descent algorithm for WSR maximization with respect to } \{ B_1, B_2, \cdots, B_K \}
\]

1. Initialize \( B^{(1)}_l \), \( l = 1, 2, \cdots, K \), with

\[
\text{tr} \left( B^{(1)}_l \right)^H B^{(1)}_l = P_l, \quad l = 1, 2, \cdots, K.
\]

2. Using Definition 1, compute \( T^{(k)}_l \), \( \hat{d}_l^{(k)} \), \( \gamma_l^{(k)} \), \( \sigma_l^{(k)} \), \( \psi_l^{(k)} \), \( \gamma_l^{(k)}(n-1) \), \( \psi_l^{(k)}(n-1) \), \( k = 1, 2, \cdots, K \).

3. Using (17), compute the asymptotic value

\[
\frac{R_{w,n}(B_1, B_2, \cdots, B_K)}{F(B_1, B_2, \cdots, B_K)}
\]

4. Using (19), compute the asymptotic gradient

\[
\frac{\partial F(B_1, B_2, \cdots, B_K)}{\partial B_l} = \frac{\partial R_{w,n}(B_1, B_2, \cdots, B_K)}{\partial B_l}, \quad l = 1, 2, \cdots, K.
\]

5. Set \( l := 1 \).

6. Set the step size \( u := 1 \).

7. Evaluate \( c = \alpha u \left\| \nabla F(B_l) \right\|_F \). If \( c \) is smaller than a threshold, then go to step 13.

8. Compute \( B_l^{(n+1)} = B_l^{(n)} + u \nabla F(B_l) \).

9. If \( \text{tr} \left( B_l^{(n+1)} \right)^H B_l^{(n)} > P_l \), update \( B_l^{(n+1)} = B_l^{(n)} \) otherwise, \( B_l^{(n+1)} = B_l^{(n+1)} \).

10. Using Definition 1, compute \( T_l^{(k)} \), \( \hat{d}_l^{(k)} \), \( \gamma_l^{(k)}(n) \), and \( \psi_l^{(k)}(n) \), \( k = 1, 2, \cdots, K \).

11. Set \( u := \beta u \). If \( R_{w,n} \leq R_{w,n} + c \), go to step 7.

12. If \( l < K \), \( l := l + 1 \), go to step 6.

13. If \( R_{w,n} < R_{w,n} \), go to step 2, otherwise, stop the algorithm.

The backtracking line search method is incorporated to determine the step size for each gradient update [21]. In addition, if the updated precoder exceeds the power constraint \( \text{tr} \left( B_l \right)^H B_l > P_l \), we project \( B_l \) onto the feasible set through a normalization step: \( B_l := \sqrt{\text{tr} B_l^H} \left\{ B_l \right\} \) [22]. The resulting algorithm is given in Algorithm 1.

The computational complexity of linear precoder design algorithms for the MIMO MAC with finite alphabet inputs is determined by the required number of summations in calculating the mutual information and the MSE matrix (e.g., (16), (31) or [5, Eq. (5)]). The conventional precoder design for instantaneous CSI in [5] requires summations over all possible transmit vectors of all users. For this reason, the computational complexity of the conventional precoding design scales linearly with \( \left( N_t^K \right)^{2N_t} \). However, (15) and (19) imply that Algorithm 1 only requires summations over each user’s own possible transmit vectors to design the precoders. Accordingly, the computational complexity of the proposed Algorithm 1 for statistical CSI grows linearly with \( \sum_{k=1}^{K} Q_{k}^{2N_t} \). As a result, the computational complexity of Algorithm 1 is several orders of magnitude lower than that of the conventional design. To exemplify this more clearly, we give an example. We consider a practical massive MIMO MAC system where the base station is equipped with a large number of antennas and serves multiple users having much smaller numbers of antennas. In particular, we assume \( N_t = 64 \), \( N_l = 4 \), \( K = 4 \), \( \mu_1 = \mu_2 = \mu_3 = \mu_4 \), and all users employ the same modulation constellation. The numbers of summations required for calculating the mutual information and the MSE
We observe from Table I that Algorithm 1 significantly reduces number of summations required for MIMO MAC precoder design for finite alphabet inputs. Moreover, since Algorithm 1 is based on the channel statistics \( \{ U_{T_k} \}_{k=1}^{K}, \{ U_{R_k} \}_{k=1}^{K}, \{ G_k \}_{k=1}^{K} \), it avoids the time-consuming averaging process over each channel realization for the mutual information in (7). In addition, Algorithm 1 is executed only once since the precoders are constant as long as the channel statistics do not change, whereas the algorithm in [5] has to be executed for each channel realization. Due to the non-convexity of the objective function \( R_{\text{sum}}(B_1, B_2, \cdots, B_K) \) in general, Algorithm 1 will find a local maximizer of the WSR as the final design solution [5, 6].

V. NUMERICAL RESULTS

In this section, we provide examples to illustrate the performance of the proposed iterative optimization algorithm. We consider a two-user MIMO MAC system with two transmit antennas and two receive antennas for each user. We assume equal user powers \( P_1 = P_2 = P, \mu_1 = \mu_2 = 1, \) and the same modulation format for both users. The average signal-to-noise ratio (SNR) for the MIMO MAC with statistical CSI at the transmitter is given by SNR = \( \frac{E[|I_n|^2]}{NH_1} \).

For illustrative purpose, we consider an example of jointly correlated fading channel matrices for two users. The channel statistics in (4) are given by

\[
U_{T_1} = \begin{bmatrix}
-0.7830 & 0.6196 + 0.0547j \\
-0.6196 + 0.0547j & -0.7830 
\end{bmatrix},
\]

\[
U_{R_1} = \begin{bmatrix}
0.9513 & 0 - 0.0364j \\
0 - 0.0364j & 0.9513 
\end{bmatrix},
\]

\[
G_1 = \begin{bmatrix}
1.8366 & 0.3979 \\
0.6122 & 0.3061 
\end{bmatrix},
\]

and

\[
U_{T_2} = \begin{bmatrix}
-0.9628 & 0.2683 - 0.0313j \\
-0.2683 + 0.0313j & 0.9628 
\end{bmatrix},
\]

\[
U_{R_2} = \begin{bmatrix}
0.7757 & 0 - 0.4079j \\
0 - 0.4079j & 0.7757 
\end{bmatrix},
\]

\[
G_2 = \begin{bmatrix}
1.1242 & 1.5151 \\
0.862 & 1.5151 
\end{bmatrix}.
\]

Figure 1 plots the sum-rate curves for different transmission schemes and QPSK inputs. We employ the Gauss-Seidel algorithm together with stochastic programming to obtain the optimal covariance matrices of both users under the Gaussian input assumption [11]. Then, we decompose the obtained optimal covariance matrices \( \{ Q_1, Q_2, \cdots, Q_K \} \) as \( Q_k = U_{K,k} A_k U_{K,k}^H \) and set \( B_k = U_{K,k} A_k \), \( k = 1, 2, \cdots, K \). Finally, we calculate the average sum-rate of this precoding design under finite alphabet constraints. We denote the corresponding sum-rate as “GP with QPSK inputs”. For the case without precoding, we set \( B_1 = B_2 = \sqrt{\frac{N_t}{N_r}} I_{N_t} \). We denote the corresponding sum-rate as “NP with QPSK inputs”. Also, the sum-rates achieved with the Gauss-Seidel algorithm and without precoding for Gaussian inputs are also plotted in Figure 1, and denoted as “GP with Gaussian input” and “NP with Gaussian input”, respectively. For comparison purpose, we plot the average sum rate achieved by Algorithm 1 in [5] with instantaneous CSI, which is denoted as “AL in [12] with QPSK inputs”. We denote the proposed design in Algorithm 1 as “FAP with QPSK inputs”. We observe from Figure 1 that for QPSK inputs, the proposed algorithm achieves a better sum-rate performance than the other precoding schemes with statistical CSI. In particular, at a sum-rate of 4 b/s/Hz, the SNR gains of the proposed algorithm over the “NP with Gaussian input” design and the “GP with Gaussian input” design are 2.5 dB and 11 dB, respectively. The sum rate achieved by the proposed algorithm with statistical CSI is close to the sum rate achieved by Algorithm 1 in [5] with instantaneous CSI. At a target sum rate of 4 b/s/Hz, the SNR gap between the proposed algorithm and Algorithm 1 in [5] is less than 1 dB. The sum-rates achieved by the “GP with Gaussian input” design almost remain unchanged for SNRs between 10 dB and 20 dB. Similar to the point-to-point MIMO case [12], this is because the Gauss-Seidel algorithm design implements a “water filling” power allocation policy within this SNR region. As a result, when the SNR is smaller than a threshold (e.g., 20 dB in this case), the precoders allocate most energy to the strongest subchannels and allocates little to the weaker subchannels. Therefore, one eigenvalue of \( Q_k \) may approach zero. For finite alphabet inputs, this power allocation policy may result in allocating most of the power to subchannels that are close to saturation. It will lead to a waste of transmission power and impede the further improvement of the sum-rate performance.

Next, we verify the performance of the proposed precoding design in a practical communication system. To this end, we employ the low density parity check encoder and decoder simulation packages from [24], with code rate 1/2 and code length \( L = 9000 \). We employ the same transceiver structure as in [5]. Figure 2 depicts the average coded BER performance of different precoding designs for QPSK inputs. We observe that for a target BER of \( 10^{-4} \), the proposed “FAP” design achieves 2.5 dB SNR gain over the “NP” design. It is noted that code rate 1/2 corresponds to a targeted sum-rate of 4 b/s/Hz for a two-user MIMO MAC system with two transmit antennas and QPSK inputs. Therefore, the SNR gain for the coded BER matches the SNR gain for the sum-rate. Also, for

\[\text{Fig. 1: Average sum-rate of two-user MIMO MAC with QPSK modulation.}\]
the coded BER, the “FAP” design yields a 28 dB SNR gain over the “GP” design, which is larger than that for the sum-rate in Figure 1. This is because for SNRs between 10 dB and 20 dB, the “GP” design results in a beamforming structure which allocates most power to the stronger subchannel. Thus, the overall coded BER is high.

VI. CONCLUSION

In this paper, we have studied the linear precoder design for the $K$-user MIMO MAC with statistical CSI at the transmitter. We formulated the problem from the standpoint of finite alphabet inputs based on a very general jointly-correlated fading model. We first obtained the WSR expression for a MIMO MAC system assuming a jointly-correlated fading model for the asymptotic large-system regime under finite alphabet input constraints. Then, we established a set of necessary conditions for the precoding matrices which maximize the asymptotic WSR. Subsequently, we proposed an iterative algorithm to find the precoding matrices of all users with statistical CSI at the transmitter. In the proposed algorithm, the search space for each user is its own modulation set, which significantly reduces the dimension of the search space compared to a previously proposed precoding design method for MIMO MAC with finite alphabet inputs and instantaneous CSI at the transmitter. Numerical results showed that, for finite alphabet inputs, precoders designed with the proposed iterative algorithm achieve substantial performance gains over the precoders designed based on the Gaussian input assumption and transmissions without precoding.

APPENDIX A

PROOF OF PROPOSITION 1

Due to space limitations, we only outline the main steps leading to Proposition 1. Details of the proof will be given in an extended journal version of this paper. First, we consider the case $K_1 = K$. Define $H = [H_1 \ H_2 \ \ldots \ H_K]$, $B = \text{blockdiag} \{B_1, B_2, \ldots, B_K\}$, $x = [x_1^T \ x_2^T \ \ldots \ x_K^T]^T$, and $d = [d_1^T \ d_2^T \ \ldots \ d_K^T]^T$. From (8), the mutual information of the MIMO MAC can be expressed as $I(d; y) = F = N_k \log_2 e$, where $F = -E_y[H \log_2 Z(y, H)]$ and $Z(y, H) = E_x[e^{-\|y - Hx\|^2}]$. The expectations over $y$ and $H$ are difficult to perform because the logarithm appears inside the average. The replica method, nevertheless, circumvents the difficulties by rewriting $F$ as

$$F = -\log_2 e \lim_{r \to 0} \frac{\partial}{\partial r} \ln E_{y,H}[Z(y, H)^r]$$

(35)

The reformulation is very useful because it allows us to first evaluate $E_{y,H}[Z(y, H)^r]$ for an integer-valued $r$, before considering $r$ in the vicinity of 0. This technique is called the replica method [26], and has been widely adopted in the field of statistical physics [29].

Basically, to compute the expectation over $Z(y, H)$, it is useful to introduce $r$ replicated signal vectors $x_k^{(\alpha)}$, for $\alpha = 0, 1, \ldots, r$, yielding

$$E_{y,H}[Z(y, H)^r] = E_{y,x} \left[ \int dy \ e^{-\|y - \sum_{k=1}^K H_k x_k^{(\alpha)}\|^2} \right]$$

(36)

where $X = \{X_1^T \ X_2^T \ \ldots \ X_K^T\}$, $x_k = \{x_k^{(0)} \ x_k^{(1)} \ \ldots \ x_k^{(r)}\}$, and $\{x_k^{(\alpha)}\}$ are i.i.d. with distribution $p(x_k)$. Now, the expectation over $y$ can be performed because it is reduced to the Gaussian integral. However, the expectation over $H$ involves interactions among the replicated signal vectors. Define a set of random vectors: $V = \{V_1 \ V_2 \ \ldots \ V_K\}$,

$$V_k = \begin{bmatrix} v_{k,1}^T \ v_{k,2}^T \ \ldots \ v_{k,N_1}^T \end{bmatrix}^T$$

where $V_k = \begin{bmatrix} v_{k,1} \ v_{k,2} \ \ldots \ v_{k,N_1} \end{bmatrix}$ and $v_{k,n} = \sum_m v_{k,n,m}$. Thus,

$$v_{k,n,m} = \sum_{\alpha} \frac{1}{2} v_{k,n,m}^{(\alpha)}$$

(37)

and $v_{k,n,m}^{(\alpha)}$ are i.i.d. with distribution $p(v_{k,n,m})$. For ease of notation, we further define $T_{k,m} = u_{T,k,m}^H u_{T,k,m}^H$ and $R_{k,n} = u_{R,k,n}^H u_{R,k,n}^H$. Therefore, we have $Q_{k,n,m}^{(\alpha)} = g_{k,n,m} \left[ v_{k,n,m}^{(\alpha)} H \right] T_{k,m} x_k^{(\beta)}$. Let $Q = \{Q_{k,n,m}\} \forall k,n,m$, where $\forall k,n,m \in \mathbb{C}^{(r+1) \times (r+1)}$. It is useful to separate the expectation over $X$ in (36) into the expectation over $Q$, and then all possible $x_k^{(\alpha)}$ configurations for a given $Q$ by introducing a $\delta$-function,

$$E_{y,H}[Z(y, H, \sigma)^r] = E_{Q}(0) \mu(Q)$$

(38)

Using the inverse Laplace transform of the $\delta$-function, we can show that if $N_{r}$ is large, then $\mu(Q)$ is dominated by the exponent term as

$$J(Q) = \max_{Q} \left\{ \sum_{k,n,m} \ln \left[ \sum_{\alpha=1}^{(r+1)} (\sum_{\beta=1}^{r} v_{k,n,m}^{(\alpha)} H T_{k,m} x_k^{(\beta)}) \right] \right\}$$

(39)

We define the set $\bar{Q} = \{\bar{Q}_{k,n,m}\} \forall k,n,m$ and $\bar{Q}_{k,n,m} \in \mathbb{C}^{(r+1) \times (r+1)}$ is a Hermitean matrix. As a result, by applying
The extremum over $\tilde{Q}$ and $Q$ in (40) and (41) can be obtained via the saddle point method, yielding a set of self-consistent equations. To avoid searching for the saddle-points over all possible $\tilde{Q}$ and $Q$, we make the following replica symmetry (RS) assumption for the saddle point:

$$Q_{k,n,m} = q_{k,n,m} 11^H + (c_{k,n,m} - q_{k,n,m}) I_{11}$$

$$Q_{k,n,m} = q_{k,n,m} 11^H + (c_{k,n,m} - q_{k,n,m}) I_{11}$$

where $1 = (1,1)\times1$ is a vector with all elements equaling to one. This RS assumption has been widely accepted in physics [29], and was also used in communications [11, 13, 15, 16].

After some tedious algebraic manipulations, we obtain the RS solution of $F$ as

$$F = \lim_{r \to 0} E_n \mathbb{E} \left[ \left( Z(y,H) \right)^T \right] \approx -\max_{Q} \left\{ S(Q) - J(Q) \right\}$$

$$F = \lim_{r \to 0} \frac{\partial}{\partial r} \mathbb{E} \left[ \left( Z(y,H) \right)^T \right] \approx -\max_{Q} \left\{ S(Q) - J(Q) \right\}$$

where $0 < r < 1$. The case with arbitrary value $F$ can be proved following a similar approach as above.

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