Perturbations of Simultaneously Growing Multiple Schramm-Loewner Evolutions

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Abstract

In this article we study multiple $SLE_\kappa$, for $\kappa \in (0, 4]$, driven by Dyson Brownian motion. This model was introduced in the unit disk by Cardy in connection with the Calogero-Sutherland model. We prove the Carathéodory convergence of perturbed Loewner chains under different initial conditions and under different diffusivity $\kappa \in (0, 4]$ for the case of $N = 2$ driving forces. Our proofs use the analysis of Bessel processes and estimates on Loewner differential equation with multiple driving forces. In the last section, we estimate the Hausdorff distance of the hulls under perturbations of the driving forces, with assumptions on the modulus of the derivative of the multiple Loewner maps.

1 Introduction

The forward multiple Loewner chain encodes the dynamics of a family of conformal maps $g_t(z)$ defined on simply connected domains $\mathbb{H}\setminus K_t$ of the upper-half plane $\mathbb{H}$, where $K_t$ are growing hulls ([5] Sec. 4.1.2) in the sense that $K_s \subset K_t$ for all $0 \leq s \leq t$. In this work we study a Loewner chain generated by $N \in \mathbb{N}$ continuous driving forces $\{\lambda_1(t), \lambda_2(t), \ldots, \lambda_N(t)\}$ from $\mathbb{R}$ to $\mathbb{R}$. We denote these driving functions by $\lambda_j : [0, T] \rightarrow \mathbb{R}$, $j = 1, \ldots, N$. We have

$$\partial_t g_t(z) = \frac{1}{N} \sum_{j=1}^{N} \frac{2}{g_t(z) - \lambda_j(t)},$$

(1.1)
with \( g_0(z) = z \). This work is motivated by [13], where the author establishes a connection between the Calogero-Sutherland model and the multiple SLE driven by Dyson Brownian motion. This model was studied in more details in [12]. In this paper, we treat further the multiple SLE with Dyson Brownian motion as driving forces. In order to define this object, we consider the Weyl chamber ([2] Sec. 4) given by

\[ M_N := \{ x \in \mathbb{R}^N; x_1 < x_2 < \cdots < x_N \}. \]  

(1.2)

Throughout the paper we work with \((\Omega, \mathcal{F}, \mathbb{P})\), the standard probability space. Let \( B_j(t), j = 1, \ldots, N \) be one-dimensional standard independent Brownian motions defined on this space. The Dyson Brownian motion with diffusivity parameter \( \kappa \in (0, 4] \) is defined by a system of differential equations in the following

\[ d\lambda_j(t) = \frac{1}{\sqrt{2}} dB_j(t) + \frac{2}{\kappa} \sum_{1 \leq k \leq N, k \neq j} dt \frac{\lambda_j(t) - \lambda_k(t)}{\lambda_j(t) - \lambda_k(t)}, \]  

(1.3)

with \((\lambda_1(0), \ldots, \lambda_N(0)) \in M_N\), for all \( t \in \mathbb{R}_+ \) and \( j = 1, \ldots, N \). We refer the reader to [50] for more details. In the literature on Dyson Brownian motion, the real parameter of the system is usually denoted by \( \beta > 0 \). The relation between this \( \beta > 0 \) and the diffusivity parameter \( \kappa \in \mathbb{R}_+ \) in SLE is \( \beta = \frac{8}{\kappa} \) ([31]). The Dyson Brownian motion has a unique strong solution ([32] Thm. 12.2) that we use as the simultaneous driving force for our multiple SLE. An intuitive picture is that \( \{\lambda_1(t), \lambda_2(t), \ldots, \lambda_N(t)\} \) describes an ensemble of diffusing particles ([3] Rmk. 2.4) in which these particles repel each other via a Coulomb force. It is known that when \( \kappa \in (0, 4] \), no two Dyson Brownian particles will collide (i.e. touch \( \partial M_N \)) almost surely. To be precise, denote by

\[ \tau_N := \inf\{0 \leq t \leq T; \exists i, j \text{ s.t. } |\lambda_i(t) - \lambda_j(t)| = 0\}. \]  

(1.4)

Then \( \tau_N = \infty \) almost surely as in ([16] Prop. 3.1). This result also justifies our choice of an arbitrary time interval \([0, T]\). Also, it is known ([1] Thm. 1.3) that when \( \kappa \in (0, 4] \), the transformations \( g_t(z) \) map a simply connected subset \( \mathbb{H} \setminus K_t \) conformally onto the upper-half plane \( \mathbb{H} \), where \( K_t \) consists of the image of \( N \) non-intersecting simple curves, that is \( g_t : \mathbb{H} \setminus \bigcup_{j=1}^N \gamma_t^j \to \mathbb{H} \). Each curve corresponds to a driving force \( \lambda_j(t), j \in N \). We focus on this case and throughout this article we assume \( \kappa \in (0, 4] \).
In the last years, there have been many results on the multiple SLE model (see [19], [20], [21], [23], [24], [25], [26], [27], [28], [30], [31], [29], [58], [59], [60] for a non-exhaustive list of papers where the model is studied in the upper half-plane, unit disk, either in the simultaneous growth case, or in the non-simultaneous growth case). There is also literature on the connection between multiple SLE and Conformal Field Theory (CFT). We refer to [20], [22].

In this paper, we mainly focus on how the forward Loewner chain \( g_t(z) \) behaves when the system is under different perturbations. In the following sections, we propose an estimate of such perturbations in the sense of Carathéodory convergence. Then we study, via Carathéodory convergence, what happens when either the initial value of the driving forces \( \lambda_j(t) \)'s or their diffusivity parameter \( \kappa \) are perturbed in the \( N = 2 \) driving forces case. In [60], the authors study the deterministic case with non-constant weights, under assumptions on them and on the driving forces. In our work, we study the random driving forces case with constant weights. We use probabilistic properties of the driving forces (similarly, in the one curve case where the study of the Brownian motion brings new probabilistic techniques compared with the deterministic forces). Also, our deterministic intermediate results are performed for general \( N \) case and constant weights using an in integrating factor, compared with [60] where the analysis is detailed in the case \( N = 2 \) and non-constant weights (under assumptions) using Grönwall’s inequality.

The analysis in this paper can be thought as a first-step towards the general \( N \) case, and asymptotic \( N \to \infty \), where there are techniques that involve the study of local statistical properties such as the study of the gaps between particles, and other local statistics developed in random matrix theory (see [36], [35], [34], [33], [32], for a non-exhaustive list). We plan to investigate this in the future. For more works in random matrix theory on other models, we refer the reader to the non-exhaustive list [57], [52], [53], [54], [55], [56], [51].

The paper is divided in several sections. In Section 2, we introduce the Carathéodory convergence of multiple Loewner chains and obtain some preliminary estimates. In Section 3, we analyze two types of perturbations of the multiple SLE. The first type of perturbation is on the initial value of driving forces; the second type of perturbation is on the diffusivity parameter \( \kappa \in (0.4] \). For the first situation, it is a natural question to consider the stability of Statistical Physics models under boundary perturbations. For example, for the case of \( \kappa = 3 \), this leads to the question of stability under boundary perturbations of
multiple Ising model interfaces. The first type of perturbation can be thought as an initial step in the study of the large $N$ driving forces with techniques that were developed for the proof of the universality of certain random matrices ensembles in random matrix theory. In that context, Dyson Brownian motion appears as a tool in the proof of the universality using the 'three steps strategy', see [32] for a detailed exposition of these methods. For a different approach we refer the reader to [49]. The second type of perturbation, that is the perturbation in the parameter $\kappa$ is studied in the sup-norm in a sequence of papers in the one curve case (see [37], [38], [39], [40], and [41] for the continuity in the parameter $\kappa \leq 4$ of the welding homeomorphism). In particular, the a.s. continuity in $\kappa$ of the $SLE$ curves was shown for $\kappa \in (0, 8(2 - \sqrt{3})) \cup (8(2 + \sqrt{3}), \infty)$ in [37] and for $\kappa \in (0, 8/3)$ in [38], [39]. Motivated by these works, in this paper we study the perturbation in the parameter $\kappa \in (0, 4]$ in the Carathéodory sense, for $N = 2$ driving forces. We plan to study the problem considering stronger topology in the future. In the last section, we perform a variant of this analysis in which under a natural assumption on the derivative of the maps, we estimate the Hausdorff distance between the perturbation of the hulls under the multiple backward Loewner differential equation.

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2 Carathéodory convergence of Loewner chains

Throughout this paper, we use $||\cdot||_{[0,T]}$ for the uniform norm on the interval $[0,T]$, and denote by $||\cdot||_{[0,T] \times G}$ the uniform norm on the product space $[0,T] \times G$, where $G \in \mathbb{H}$ is compact. Also, throughout the paper we consider the coupling of the Loewner chains in which both chains are driven by Dyson Brownian motion with the same Brownian motions.

In this section, we propose an estimate to the perturbation of forward Loewner chain $g_t(z)$ in the sense of Carathéodory convergence. The central idea is convergence on compact sets. This type of convergence is useful. For example in complex analysis, we know ([4] Thm. 10.28) that when a sequence of
holomorphic functions Carathéodory converges to a limit function, then taking the limit preserves the holomorphicity, hence the limit function is holomorphic. This article follows the convention in ([5] Sec. 6.1.1).

One can study the Carathéodory convergence in the setting when $G$ is a subset of $\mathbb{H}$. Indeed, the result in ([7] Lem. 3.1) tells us that height $K_t \leq 2\sqrt{t}$ for all $t \in [0, T]$. Hence we could simply subtract the box $\mathbb{R} \times [0, iT]$, which we denote by $K_0$, from $\mathbb{H}$. We know that $K_t \subset K_0$ for all $t \in [0, T]$ almost surely. And then one can restrict $g_t(z)$ to the simply connected domain $\mathbb{H} \setminus K_0$. This methodology gives a uniform bound to perturbation of forward Loewner chains. But if we want a Carathéodory estimate where compact sets $G$ could run freely over a larger domains (depending on $\omega \in \Omega$), then the results must be discussed pathwisely.

**Definition 2.1.** Denote by $D \subset \mathbb{H}$ a simply connected domain. Let $f_n(t, z) : [0, T] \times D \rightarrow \mathbb{H}$ be a sequence of conformal maps, and let $f(t, z) : [0, T] \times D \rightarrow \mathbb{H}$ be a conformal map. We say $f_n$ converges in the Carathéodory sense to $f$, or $f_n \xrightarrow{\text{Car}} f$, if for each compact $G \subset D$, the sequence $(f_n)_{n \in \mathbb{N}}$ converges to $f$ uniformly on $[0, T] \times G$.

The estimate on the $g_t(z) : \mathbb{H} \setminus K_t \rightarrow \mathbb{H}$ corresponding to different inputs is based on Definition 3.1. We will give a proposition regarding estimating the difference between two forward Loewner chains. For notational convenience, we will write $g_t(z)$ as $g(t, z)$ from now on. Consider two forward Loewner chains $g_1(t, z)$ and $g_2(t, z)$ defined on $[0, T] \times \mathbb{H} \setminus K_t$, where $K_t$ is the union of the hulls of $g_1(t, z)$ and $g_2(t, z)$, that is $K_t = \bigcup_{i=1}^{n_i} K_i$. Suppose they are generated respectively by $N$ continuous driving forces $\{V_{k,1}(t), \ldots, V_{k,N}(t)\}$ with $k = 1, 2$, and restrict $g_k(t, z)$ to the domain $[0, T] \times K_T$. Then we have the following estimate.

**Proposition 2.2.** For an arbitrary compact $G \subset \mathbb{H} \setminus K_T$, for all $z \in G$, there exists a constant $C(T, G) > 0$ such that we have

$$\|g_1(t, z) - g_2(t, z)\|_{[0,T] \times G} \leq C(T, G) \sum_{j=1}^{N} \|V_{1,j}(t) - V_{2,j}(t)\|_{[0,T]}.$$  \hfill (2.1)

**Proof.** It is by Eqn. (1.1) that we have the constraint

$$\partial_t g_k(t, z) = \frac{1}{N} \sum_{j=1}^{N} \frac{2}{g_k(t, z) - V_{k,j}(t)}.$$  \hfill (2.2)
with \( k = 1, 2 \). Choose arbitrarily \( z_1, z_2 \in G \). Let \( \psi(t) := g_1(t, z_1) - g_2(t, z_2) \).

And we have

\[
\frac{d}{dt} \psi(t) = \partial_t g_1(t, z_1) - \partial_t g_2(t, z_2)
\]

\[
= \frac{1}{N} \sum_{j=1}^{N} \left( \frac{2}{g_1(t, z_1) - V_{1,j}(t)} - \frac{2}{g_2(t, z_2) - V_{2,j}(t)} \right)
\]

\[
= \frac{1}{N} \sum_{j=1}^{N} \xi_j(t) \left( g_1(t, z_1) - V_{1,j}(t) - g_2(t, z_2) + V_{2,j}(t) \right),
\]

where we define

\[
\xi_j(t) := \frac{-2}{(g_1(t, z_1) - V_{1,j}(t)) \cdot (g_2(t, z_2) - V_{2,j}(t))},
\]

for \( j = 1, \ldots, N \). Additionally we define \( D_j(t) := V_{1,j}(t) - V_{2,j}(t) \) for each \( j \).

Combined with Eqn. (2.3), then we have

\[
\frac{d}{dt} \psi(t) = \frac{1}{N} \sum_{j=1}^{N} \xi_j(t) (\psi(t) - D_j(t)).
\]

At this moment, we observe that

\[
\frac{d}{dt} \left( e^{\frac{1}{N} \sum_{j=1}^{N} \int_0^t \xi_j(s) \, ds} \cdot \psi(t) \right) = \frac{1}{N} \sum_{j=1}^{N} \xi_j(t) D_j(t) \cdot e^{\frac{1}{N} \sum_{j=1}^{N} \int_0^t \xi_j(s) \, ds},
\]

and consequently

\[
\psi(t) = e^{\frac{1}{N} \sum_{j=1}^{N} \int_0^t \xi_j(s) \, ds} \cdot \psi(0) - \frac{1}{N} \sum_{j=1}^{N} \int_0^t du \cdot \xi_j(u) D_j(u) \cdot e^{\frac{1}{N} \sum_{j=1}^{N} \int_0^t \xi_j(s) \, ds}.
\]

On the other hand, we have the following inequality

\[
\left| e^{\frac{1}{N} \sum_{j=1}^{N} \int_0^t \xi_j(s) \, ds} \right| \leq e^{\frac{1}{N} \sum_{j=1}^{N} \int_0^t |\xi_j(s)| \, ds}.
\]
Then, we know that
\[
\left| \frac{1}{N} \sum_{j=1}^{N} \int_0^t du \cdot \xi_j(u) D_j(u) \cdot e^{\frac{\sum_{j=1}^{N} f_j(s)}{N}} \right| \\
\leq \frac{1}{N} \sum_{j=1}^{N} \|D_j(t)\|_{[0,T]} \cdot \int_0^t du \cdot |\xi_j(u)| e^{\frac{\sum_{j=1}^{N} f_j(s)}{N}} ds \\
\leq \left( \sum_{j=1}^{N} \|D_j(t)\|_{[0,T]} \right) \cdot \int_0^t du \cdot \frac{1}{N} \sum_{j=1}^{N} |\xi_j(u)| \cdot e^{\frac{\sum_{j=1}^{N} f_j(s)}{N}} ds \\
= \left( \sum_{j=1}^{N} \|D_j(t)\|_{[0,T]} \right) \cdot \left( e^{-\frac{\sum_{j=1}^{N} f_j(s)}{N}} - 1 \right). 
\]

Moreover, by the Cauchy-Schwartz inequality, we have
\[
\frac{1}{N} \sum_{j=1}^{N} \int_0^t |\xi_j(s)| ds \leq \frac{1}{N} \sum_{j=1}^{N} \sqrt{I_{1,j} \cdot I_{2,j}}, 
\]
where we define \( I_{k,j} \) for \( k = 1, 2 \) and \( j = 1, \ldots, N \) in the following
\[
I_{k,j} := \int_0^t \frac{2}{|g_k(s, z_k) - V_{k,j}(s)|^2} ds \leq N \cdot \log \frac{\text{Im} z_k}{\text{Im} g_k(t, z_k)}, 
\]
because \( \partial_t \text{Im} g_k(t, z_k) = \frac{1}{N} \sum_{j=1}^{N} \frac{-\text{Im} g_k(t, z_j)}{|g_k(t, z_j) - V_{k,j}(t)|} \) similar to the one-curve case in ([5] Sec. 6.2.2). In fact, with the compact \( G \subset \mathbb{H} \setminus K_T \), there exists \( \delta_1(G) > 0 \) such that \( \text{Im} g_k(T, z) \geq \delta_1(G) \) for all \( z \in G, k = 1, 2 \). Hence, we have
\[
I_{k,j} \leq N \cdot \log \frac{\text{Im} z_k}{\max \left\{ \delta_1(G), \sqrt{((\text{Im} z_k)^2 - 4t)^+} \right\}}, 
\]
where \( x^+ = \max\{x, 0\} \). Since \( t \in [0, T] \) and \( z_1, z_2 \in G \) where \( G \) is compact in \( \mathbb{H} \setminus K_T \), we could choose \( \delta_2(G) := \text{dist}(G, \mathbb{R}) > 0 \) and define a constant by
\[
C(T, G) := \frac{\delta_2(G)}{\max \left\{ \delta_1(G), \sqrt{((\delta_2(G)^2 - 4t)^+} \right\}}. 
\]
Here we have \( I_{k,j} \leq \log C(T, G) \) for all \( k \) and \( j \). Hence, we know that
\[
|\psi(t)| \leq e^{\frac{\sum_{j=1}^{N} \log C(T, G)}{N}} \cdot |\psi(0)| + \left( \sum_{j=1}^{N} \|D_j(t)\|_{[0,T]} \right) \cdot (C(T, G) - 1). 
\]
Therefore, we conclude
\[ |g_1(t, z_1) - g_2(t, z_2)| \leq C(T, G) \cdot \left( \sum_{j=1}^{N} \|V_{1,j}(t) - V_{2,j}(t)\|_{[0,T]} + |z_1 - z_2| \right), \] (2.15)
for all \( t \in [0, T] \) and \( z_1, z_2 \in G \). Now choose \( z_1 = z_2 = z \) and take supremum over the left side, we arrive at our final result
\[ \|g_1(t, z) - g_2(t, z)\|_{[0,T] \times G} \leq C(T, G) \cdot \sum_{j=1}^{N} \|V_{1,j}(t) - V_{2,j}(t)\|_{[0,T]}. \] (2.16)

Remark 2.3. With slight changes, the above argument can be adapted to the multiple backward Loewner maps, and a similar Carathéodory estimate ([5] Lem. 6.1) can be obtained.

3 Perturbations

We are particularly interested in the forward Loewner map driven by Dyson Brownian motion. In this section, we restrict our attention to the \( N = 2 \) case. We plan to study the general \( N \)-curve case in future works. When \( N = 2 \), we have two driving forces \( \{\lambda_1(t), \lambda_2(t)\} \) that are interacting diffusions modelled by Dyson Brownian motion. Their evolution is described in the following equation
\[
\begin{align*}
d\lambda_1(t) &= \frac{2}{\kappa} \frac{dt}{\lambda_1(t) - \lambda_2(t)} + \frac{1}{\sqrt{2}} dB_1(t), \\
\frac{dt}{\lambda_2(t)} &= \frac{2}{\kappa} \frac{dt}{\lambda_2(t) - \lambda_1(t)} + \frac{1}{\sqrt{2}} dB_2(t),
\end{align*}
\] (3.1)
with \( \lambda_1(0) = a_1, \lambda_2(0) = a_2, \) \( a_1 > a_2, \) where \( B_1(t) \) and \( B_2(t) \) are independent one-dimensional Brownian motions (see [31] where the corresponding Dyson dynamics is multiplied with \( \sqrt{2} \), as well as Proposition 5.2 in [42] for a radial version of these). Here we consider only \( \kappa \in (0, 4] \). In this case, these two particles \( \lambda_1(t) \) and \( \lambda_2(t) \) never collide on \( \mathbb{R} \). In other words, the stopping time defined in Eqn. (1.4) satisfies \( \tau_2 = \infty \) almost surely as in ([6] Prop. 1.) because we consider Bessel processes of dimension \( d = 1 + \frac{4}{\kappa} \geq 3, \) for \( \kappa \leq 4 \).

Let \( X_t := \lambda_1(t) - \lambda_2(t) \). Based on the above observations, we know \( X_t > 0 \)
for all $t \in [0, T]$ almost surely. We further observe that

$$dX_t = \frac{4}{\kappa} \cdot \frac{dt}{X_t} + dW_t,$$

(3.2)

with $X_0 = a_1 - a_2$ and $W_t := \frac{1}{\sqrt{2}}(B_1(t) - B_2(t))$ is a Wiener process. Choose $d = 1 + \frac{8}{\kappa}$, then $X_t$ admits the canonical form of $d$-dimensional Bessel process with

$$dX_t = \frac{d - 1}{2} \cdot \frac{dt}{X_t} + dW_t.$$

(3.3)

In this section we discuss two types of perturbations. The first type of perturbation is varying the initial value of driving forces. The second type is varying the diffusivity parameter $\kappa \in (0, 4]$. The study in both cases involves the analysis of transient Bessel processes with dimension $d \geq 3$.

### 3.1 Perturbation of the initial value

The first type of perturbation is to slightly change the initial value of $\lambda_k(0)$ for $k = 1, 2$. With the initial value under perturbation, we get a different set of Dyson Brownian motion. Our goal is to estimate the difference of the forward Loewner chains driven by these varying forces. Dyson Brownian motions with different initial conditions appear naturally in the context of the study of the universality of certain random matrices ensembles, as mentioned in the Introduction. The analysis presented in this section can be thought as a first step in this direction, as the analysis in the case of general $N$ Dyson particles, and random initial conditions is much more involved. We plan to do this in the future.

To be precise, choose $0 < \epsilon < \frac{1}{4}(a_1 - a_2)$ and select $b_k$ in the $\epsilon$-ball of $a_k$ for $k = 1, 2$ to be the perturbed initial value of the Dyson Brownian motion. Then $b_1 > b_2$ and we arrive at another set of perturbed Dyson Brownian motion $\{\eta_1(t), \eta_2(t)\}$ with

$$d\eta_1(t) = \frac{2}{\kappa} \cdot \frac{dt}{\eta_1(t) - \eta_2(t)} + \frac{1}{\sqrt{2}}dB_1(t),$$

$$d\eta_2(t) = \frac{2}{\kappa} \cdot \frac{dt}{\eta_2(t) - \eta_1(t)} + \frac{1}{\sqrt{2}}dB_2(t),$$

(3.4)

with $\eta_1(0) = b_1$ and $\eta_2(0) = b_2$. Notice that the process $\eta_k(t)$ is still driven by the same Brownian motion $B_k(t)$, because we consider perturbation only on the initial value.
In this two-force case, we denote by \( g_\lambda(t, z) \) the original forward Loewner chain generated by forces \( \{\lambda_1(t), \lambda_2(t)\} \) and by \( g_\eta(t, z) \) the perturbed forward Loewner chain generated by forces \( \{\eta_1(t), \eta_2(t)\} \). Hence, we have

\[
\begin{align*}
\partial_t g_\lambda(t, z) &= \frac{1}{g_\lambda(t, z) - \lambda_1(t)} + \frac{1}{g_\lambda(t, z) - \lambda_2(t)}, \\
\partial_t g_\eta(t, z) &= \frac{1}{g_\eta(t, z) - \eta_1(t)} + \frac{1}{g_\eta(t, z) - \eta_2(t)},
\end{align*}
\]

with \( g_\lambda/\eta(0, z) = z \), for all \( z \in \mathbb{H} \). We continue using \( X_t = \lambda_1(t) - \lambda_2(t) \) to denote the gap between two interacting Brownian forces \( \lambda_k(t), \ k = 1, 2 \). As shown in Eqn. (3.2), \( X_t \) is a Bessel process with dimension \( 1 + \frac{8}{k} \) and initial value \( X_0 = a_1 - a_2 \). Denote by \( Y_t := \eta_1(t) - \eta_2(t) \) the gap between \( \eta_k(t), \ k = 1, 2 \). Then \( Y_t \) is a Bessel process with the same dimension \( d = 1 + \frac{8}{k} \) and satisfies

\[
dY_t = \frac{4}{\kappa} \cdot \frac{dt}{Y_t^2} + dW_t,
\]

with \( Y_0 = b_1 - b_2 \). Observe the Bessel processes \( X_t \) and \( Y_t \) are driven by the same Wiener process \( W_t \). Hence their difference \( X_t - Y_t \) satisfies

\[
d(X_t - Y_t) = -\frac{4}{\kappa} \cdot \frac{X_t - Y_t}{X_t Y_t} dt.
\]

Denote by \( a := a_1 - a_2 \) and \( b := b_1 - b_2 \). Integrate both sides on Eqn. (3.7) and we see that

\[
X_t - Y_t = (a - b) \cdot e^{-\frac{4}{\kappa} \int_0^t \frac{1}{X_s Y_s} ds}.
\]

Notice that we cannot ascertain \( X_t - Y_t \) to be whether deterministic at this moment. In fact, the term \( \frac{1}{X_t Y_t} \) might evolve stochastically. Still, for \( k = 1, 2 \), we could observe that

\[
d\lambda_k(t) - d\eta_k(t) = \frac{2}{\kappa} \left( \frac{1}{\lambda_k(t) - \lambda_{3-k}(t)} - \frac{1}{\eta_k(t) - \eta_{3-k}(t)} \right) dt \\
= (-1)^k \frac{2}{\kappa} \cdot \frac{X_t - Y_t}{(\lambda_k(t) - \lambda_{3-k}(t)) \cdot (\eta_k(t) - \eta_{3-k}(t))} dt.
\]

Hence, we have

\[
d\lambda_k(t) - d\eta_k(t) = (-1)^k (a - b) \frac{2}{\kappa} \cdot e^{-\frac{4}{\kappa} \int_0^t \frac{1}{X_s Y_s} ds} \cdot \frac{1}{X_t Y_t} dt,
\]
for \( k = 1, 2 \). The above equation admits an integral form

\[
\lambda_k(t) - \eta_k(t) = a_k - b_k + (-1)^k(a - b) \frac{2}{\kappa} \int_0^t e^{-\frac{2}{\kappa} \int_s^t \frac{1}{X_u Y_u} ds} \cdot \frac{1}{X_s Y_s} ds
\]

\[
= a_k - b_k + \frac{1}{2} (-1)^{3-k} (a - b) \left( e^{-\frac{2}{\kappa} \int_0^t \frac{1}{X_s Y_s} ds} - 1 \right). \tag{3.11}
\]

At this point, we have an explicit form to \( \lambda_k(t) - \eta_k(t) \). Looking back to Proposition 2.2, we naturally want to have an estimate to \( g_\lambda(t, z) - g_\eta(t, z) \) in the Carathéodory sense.

**Proposition 3.1.** For all \( 0 < \epsilon < \frac{a_3}{3} \), let \( H_T = \mathbb{H}\setminus K_T(\omega) \), where \( K_T(\omega) = \bigcup_{j=1}^{n_T} K_j^T(\omega) \). Choose \( b_k \in \mathbb{R} \) with \( |a_k - b_k| < \epsilon \) for \( k = 1, 2 \). Let \( g_\lambda(z) \) and \( g_\eta(z) \) be two multiple Loewner chains induced by Dyson Brownian motion \( \{\lambda_1(t), \lambda_2(t)\} \) and \( \{\eta_1(t), \eta_2(t)\} \), respectively. Suppose \( \lambda_k(0) = a_k \) and \( \eta_k(0) = b_k \) for \( k = 1, 2 \).

Then almost surely we have

\[
\|g_\lambda(t, z) - g_\eta(t, z)\|_{[0,T] \times G} < 4C(T, G) \cdot \epsilon, \quad \forall \ G \subseteq H_T. \tag{3.12}
\]

**Proof.** At this moment, we already know \( X_t, Y_t > 0 \) for all \( t \in [0, T] \) almost surely using properties of Bessel processes of dimension \( d \geq 3 \). Inspect Eqn. (3.11), we know for \( k = 1, 2 \) that

\[
|\lambda_k(t) - \eta_k(t)| \leq |a_k - b_k| + \frac{1}{2} |a - b| \cdot \left( 1 - e^{-\frac{2}{\kappa} \int_0^t \frac{1}{X_s Y_s} ds} \right) < 2\epsilon. \tag{3.13}
\]

By Proposition 2.2, we know that

\[
\|g_\lambda(t, z) - g_\eta(t, z)\|_{[0,T] \times G} \leq C(T, G) \cdot \sum_{k=1}^{2} \|\lambda_k(t) - \eta_k(t)\|_{[0,T]}
\]

\[
< 4C(T, G) \cdot \epsilon. \tag{3.14}
\]

And the proposition is verified.

So far we have estimated \( g_\lambda(t, z) - g_\eta(t, z) \) in the Carathéodory sense under a perturbation of initial value of driving forces. In practice, when we compute a multiple forward Loewner chain driven by Dyson Brownian motion, we could approximate its initial value and it turns out the approximated Loewner chains converge in the Carathéodory sense. Indeed, we have the following result.

**Corollary 3.2.** Suppose \( g_\lambda(z) : \mathbb{H}\setminus K_T(\omega) \rightarrow \mathbb{H} \), where \( K_T(\omega) = \bigcup_{j=1}^{n_T} K_j^T(\omega) \) is
a forward Loewner chain induced by Dyson Brownian motion \( \{ \lambda_1(t), \lambda_2(t) \} \) with initial value \( \lambda_1(0) > \lambda_2(0) \). Additionally, suppose there is a sequence of forward Loewner chains \( g^n_t(z) : H \setminus K_T(\omega) \to \mathbb{H} \), induced by Dyson Brownian motion \( \{ \lambda^n_1(t), \lambda^n_2(t) \} \) with \( \lambda^n_1(0) > \lambda^n_2(0) \) and approaching initial value \( \lambda^n_k(0) \sim \lambda_k(0) \). Then we have

\[
P(g^n_T(z) \xrightarrow{\text{Cara}} g_T(z)) = 1. \tag{3.15}
\]

**Remark 3.3.** The next step of this analysis is to consider random initial conditions motivated by the study of Statistical Physics models with random boundary conditions.

### 3.2 Perturbation of the diffusivity parameter \( \kappa \in (0, 4] \)

The second type of perturbation is with respect to the diffusivity parameter \( \kappa \in (0, 4] \). This type of perturbation is a natural problem and was considered extensively in the one-curve case as mentioned in the introduction. We recall that in the current work we have always chosen \( \kappa \in (0, 4] \) so that there is no phase transition ([8] Sec. 3.) corresponding to the \( (1 + \frac{8}{\kappa}) \)-dimensional Bessel process. When there is perturbation, \( \kappa \) is varied and we have a new diffusivity parameter \( \kappa^* \in (0, 4] \) such that \( \kappa^* \neq \kappa \). The difference in parameter results in different Dyson Brownian motions, and therefore different forward Loewner chains.

To simplify the model, we assume \( \kappa^* > \kappa \) without loss of generality. Denote by \( \{ \lambda_1(t), \lambda_2(t) \} \) the original Dyson Brownian motion. Their dynamics is described in Eqn. (3.1) with initial value \( \lambda_1(0) = a_1, \lambda_2(0) = a_2, a_1 > a_2 \). Denote by \( \{ \lambda^*_1(t), \lambda^*_2(t) \} \) the perturbed Dyson Brownian motion. They respect the following equations

\[
\begin{align*}
d\lambda^*_1(t) &= \frac{2}{\kappa^*} \frac{dt}{\lambda^*_1(t) - \lambda^*_2(t)} + \frac{1}{\sqrt{2}} dB_1(t), \\
d\lambda^*_2(t) &= \frac{2}{\kappa} \frac{dt}{\lambda^*_2(t) - \lambda^*_1(t)} + \frac{1}{\sqrt{2}} dB_2(t),
\end{align*} \tag{3.16}
\]

with initial value \( \lambda^*_k(0) = \lambda_k(0) = a_k, k = 1, 2 \). Let \( K_T(\omega) = \bigcup_{j=1}^2 K_T^j(\omega) \), with \( j = 1 \) corresponding to the parameter \( \kappa \in (0, 4] \) and \( j = 2 \) corresponding to the parameter \( \kappa^* \in (0, 4] \). We have \( g(t,z) : [0,T] \times \mathbb{H} \setminus K_T(\omega) \to \mathbb{H} \) the original Loewner chain generated by forces \( \{ \lambda_1(t), \lambda_2(t) \} \). And we denote by \( g^*(t,z) : [0,T] \times \mathbb{H} \setminus K_T(\omega) \to \mathbb{H} \) the perturbed Loewner chain generated by
\{\lambda_1^*(t), \lambda_2^*(t)\}.

The evolution respects

$$\partial_t g^*(t, z) = \frac{1}{g^*(t, z) - \lambda_1^*(t)} + \frac{1}{g^*(t, z) - \lambda_2^*(t)}, \quad (3.17)$$

with \(g^*(0, z) = z\) for all \(z \in \mathbb{H} \setminus K_T(\omega)\). Denote by \(X_t\) the gap between \(\lambda_1(t)\) and \(\lambda_2(t)\). Then \(X_t\) is a \((1 + \frac{8}{\kappa})\)-dimensional Bessel process with initial value \(X_0 = a\). Its evolution is described in Eqn. (3.2). At the same time, let \(X_t^* := \lambda_1^*(t) - \lambda_2^*(t)\) the gap of the two perturbed driving forces. The gap respects the following equations

$$dX_t^* = \frac{4}{\kappa^*} \cdot \frac{dt}{X_t^*} + dW_t, \quad (3.18)$$

with \(X_0^* = X_0 = a\) and where \(W_t\) is the Wiener process defined in Eqn. (3.2). Notice here \(X_t^*\) is a \((1 + \frac{8}{\kappa})\)-dimensional Bessel process.

Our main goal is to give an probabilistic estimate of \(g(t, z) - g^*(t, z)\) in the Carathéodory sense. Following Proposition 2.2, we need first estimate the sup-norm of \(\lambda_k(t) - \lambda_k^*(t)\) for \(k = 1, 2\). Indeed, define the indices of the Bessel processes by \(\nu := \frac{4}{\kappa} - \frac{1}{2}, \nu^* := \frac{4}{\kappa^*} - \frac{1}{2}\). Before proving our result, we have the following lemma. Elements of this lemma were kindly provided by H. Elad Altman in a private communication.

**Lemma 3.4.** Given a \((1 + \frac{8}{\kappa})\)-dimensional Bessel process \(X_t^*\) and a \((1 + \frac{8}{\kappa^*})\)-dimensional Bessel process \(X_t\) with \(4 \geq \kappa^* > \kappa > 0\) and the same initial value \(X_0^* = X_0 = a > 0\), we have almost surely that

$$\sup_{0 \leq s \leq t} (X_s^* - X_s)^2 \leq \frac{4t}{\kappa^*} (\kappa^* - \kappa). \quad (3.19)$$

**Proof.** Observe Eqn. (3.2) and Eqn. (3.17), we see

$$X_t^* - X_t = \frac{4}{\kappa^*} \int_0^t ds \frac{1}{X_s^*} - \frac{4}{\kappa} \int_0^t ds \frac{1}{X_s}. \quad (3.20)$$

Using Itô’s lemma, we have

$$d(X_t^* - X_t)^2 = 2(X_t^* - X_t) \left( \frac{4}{\kappa^* X_t^*} - \frac{4}{\kappa X_t} \right) dt$$

$$= \frac{4}{\kappa^* \kappa} (\kappa - \kappa^*) \cdot \frac{X_t^* - X_t}{X_t^*} dt + \frac{8}{\kappa} (X_t^* - X_t) \cdot \frac{1}{X_t^*} dt. \quad (3.21)$$

At the same time, we have that \((X_t^* - X_t) \cdot (\frac{1}{X_t^*} - \frac{1}{X_t}) \leq 0\) for all \(t \in [0, T]\)
almost surely. Integrating both sides and we obtain
\[
(X_\ast^* - X_t)^2 \leq (\kappa - \kappa^*) \frac{4}{\kappa^* \kappa} \int_0^t \frac{(X_\ast^* - X_s)_+ ds}{X_\ast^t}. \tag{3.22}
\]
On the other hand, we have that \((X_\ast^* - X_t)_+ \leq X_t^\ast\). By considering \(\kappa^* \leq \kappa\), we have the conclusion
\[
\sup_{0 \leq t \leq T} (X_\ast^* - X_t)^2 \leq \frac{4T}{\kappa^2}(\kappa - \kappa^*). \tag{3.23}
\]

We denote by \(S_t = \sup_{0 \leq s \leq t} W_t\) the supremum Brownian motion. We are ready to state the main result.

**Theorem 3.5.** Let \(g(t, z)\) and \(g^*(t, z)\) be two multiple Loewner chains for the parameters \(\kappa, \kappa^* \in (0, 4]\), resp. Choose arbitrary compacts \(G \subset H_T = \mathbb{H}\setminus K_T(\omega)\), where \(K_T(\omega) = \bigcup_{j=1}^\infty K_T^j(\omega)\), with \(K_T^1(\omega)\) corresponding to the parameter \(\kappa \in (0, 4]\) and \(K_T^2(\omega)\) to the parameter \(\kappa^* \in (0, 4]\). There exist \(\alpha_1, \alpha_2, \alpha_3 > 0\) depending on \((T, G, a, \kappa)\) such that if we further define
\[
\varphi(x) := \alpha_1 x^{1/8} + \alpha_2 x^{1/4} + \alpha_3 x^{7/8},
\]
\[
\zeta(x) := 2 x^{\nu/8} + 2 x^{3/4} e^{-1/2} T x^{3/2}, \tag{3.24}
\]
for all \(x \in \mathbb{R}_+\), then \(\lim_{x \to 0^+} \varphi(x) = 0, \lim_{x \to 0^+} \zeta(x) = 0\) almost surely and we have
\[
P\left(\|g(t, z) - g^*(t, z)\|_{[0, T] \times G} > \varphi(\kappa^* - \kappa), \forall G \subset H_T \text{ compact} \right) < \zeta(\kappa^* - \kappa). \tag{3.25}
\]

**Proof.** From Eqn. (3.1) and Eqn. (3.16), we see for \(k = 1, 2\) that
\[
d\lambda_k(t) - d\lambda_k^\ast(t) = \frac{2}{\kappa} \frac{dt}{\lambda_k(t) - \lambda_{3-k}(t)} - \frac{2}{\kappa^*} \frac{dt}{\lambda_k^\ast(t) - \lambda_{3-k}^\ast(t)}
\]
\[
= (-1)^{3-k} \frac{2}{\kappa^* \kappa} \frac{\kappa^* X_t^\ast - \kappa X_t}{X_t^\ast X_t} dt. \tag{3.26}
\]
To obtain an expression of \(\lambda_k(t) - \lambda_k^\ast(t)\), we need to express the process \(\kappa^* X_t^\ast - \kappa X_t\). Indeed, we have
\[
\kappa^* dX^\ast_t - \kappa dX_t = 4 \left( \frac{1}{X_t^\ast} - \frac{1}{X_t} \right) dt + (\kappa^* - \kappa) dW_t. \tag{3.27}
\]
Integrate both sides, we write

\[ \kappa^* X_t^* - \kappa X_t = (\kappa^* - \kappa) \cdot a + 4 \int_0^t \frac{X_s - X_t^*}{X_t^* X_s} ds + (\kappa^* - \kappa) W_t. \] (3.28)

On the other hand, inspecting the above equation, we see another term \( X_t^* - X_t \) appears in the integrand. Based on Lemma 3.4, we have

\[ \sup_{0 \leq s \leq t} |X_t^* - X_s| \leq \frac{2\sqrt{t}}{\kappa} (\kappa^* - \kappa)^{1/2}. \] (3.29)

At this moment, we have obtained an explicit form of \( \kappa^* X_t^* - \kappa X_t \), which is contained in the expression of \( d\lambda_k(t) - d\lambda_k^*(t) \). Define \( M_t := \inf_{0 \leq s \leq t} X_s \) as the running infimum of the Bessel process \( X_t \). The running infimum \( M_t^* \) of \( X_t^* \) is similarly defined. We further denote by \( M_\infty = \lim_{t \to \infty} M_t \) the infimum of \( X_t \). And similarly, we denote by \( M_\infty^* = \lim_{t \to \infty} M_t^* \) the infimum of \( X_t^* \). Indeed, from (9) Eqn. 2.1) and the Brownian scaling property that \( X_t \) being a Bessel process starting from \( a \) implies \( a^{-1} X_{a^2 t} \) being a Bessel process starting from 1, we know that

\[ \mathbb{P}(M_\infty < y) = \frac{y^{2\nu}}{a^{2\nu}} \cdot 1_{y \in [0,a]}, \]
\[ \mathbb{P}(M_\infty^* < y) = \frac{y^{2\nu^*}}{a^{2\nu^*}} \cdot 1_{y \in [0,a]}. \] (3.30)

Combining Eqn. (3.20), Eqn. (3.26) and Eqn. (3.28), we have that

\[ |\lambda_k(t) - \lambda_k^*(t)| \leq (\kappa^* - \kappa) \frac{2a}{\kappa^* \kappa} \cdot \frac{t}{M_t^* M_t} + (\kappa^* - \kappa)^{1/2} \frac{16}{\kappa^* \kappa^2} \cdot \frac{t^{3/2}}{(M_t^* M_t)^2} \]
\[ + (\kappa^* - \kappa) \frac{2}{\kappa^* \kappa} \cdot \frac{t}{M_t^* M_t} \sup_{0 \leq s \leq T} |W_s|. \] (3.31)

Considering \( \kappa^* > \kappa \), we have that

\[ \sup_{t \in [0,T]} |\lambda_k(t) - \lambda_k^*(t)| \leq (\kappa^* - \kappa) \frac{2a}{\kappa^* \kappa} \cdot \frac{T}{M_\infty^* M_\infty} + (\kappa^* - \kappa)^{1/2} \frac{16}{\kappa^* \kappa^2} \cdot \frac{T^{3/2}}{(M_\infty^* M_\infty)^2} \]
\[ + (\kappa^* - \kappa) \frac{2}{\kappa^2} \cdot \frac{T}{M_\infty^* M_\infty} \sup_{0 \leq s \leq T} |W_s|. \] (3.32)
Based on Eqn. (3.30), define the following events

\[ E_1 := \{ M_\infty \geq (\kappa^* - \kappa)^{\frac{1}{16}} \}, \]
\[ E_2 := \{ M_\infty^* \geq (\kappa^* - \kappa)^{\frac{1}{16}} \}, \]
\[ E_3 := \left\{ \sup_{0 \leq s \leq T} |W_s| \leq \frac{1}{(\kappa^* - \kappa)^{\frac{3}{4}}} \right\}. \] (3.33)

Since \( \kappa^* > \kappa \) and by Eqn. (3.30), we know that

\[ P(E_1) = 1 - \frac{(\kappa^* - \kappa)^{\frac{1}{8}}}{a^{\nu}}, \]
\[ P(E_2) = 1 - \frac{(\kappa^* - \kappa)^{\frac{3}{8}}}{a^{\nu^*}}. \] (3.34)

From ([10], Cor. 2.2), we know the supremum Brownian motion \( S_t \) admits the distribution

\[ P(S_t \leq x) = 2\Phi\left(\frac{x}{\sqrt{t}}\right) - 1, \] (3.35)
for all \( x \geq 0 \) and where \( \frac{d}{dx}\Phi(x) := e^{-x^2/2}/\sqrt{2\pi} \) is the density of standard normal variable. It follows from the reflection principle that

\[ 1 - P(E_3) = 2P(S_T \geq (\kappa^* - \kappa)^{-\frac{3}{4}}) \leq 2\sqrt{\frac{2}{\pi}} (\kappa^* - \kappa)^{\frac{1}{8}} \cdot e^{-\frac{1}{2}T(\kappa^* - \kappa)^{3/2}}. \] (3.36)

Choose \( \alpha_1 := C(T,G)^{\frac{1}{16}}, \alpha_2 := C(T,G)^{\frac{32T^{\frac{1}{2}}}{\kappa^*}}, \alpha_3 := C(T,G)^{\frac{4T^{\frac{1}{2}}}{\kappa^*}} \). It follows from Proposition 2.2 and Eqn. (3.32) that on the event \( E_1 \cap E_2 \cap E_3 \subset \Omega \), we have the estimate

\[ \|g(t,z) - g^*(t,z)\|_{[0,T] \times G} \leq C(T,G) \sum_{k=1}^2 \|\lambda_k(t) - \lambda_k^*(t)\|_{[0,T]} \leq \alpha_1(\kappa^* - \kappa)^{1/8} + \alpha_2(\kappa^* - \kappa)^{1/4} + \alpha_3(\kappa^* - \kappa)^{7/8}. \] (3.37)

On the other hand, from Eqn. (3.33) and Eqn. (3.35) we have

\[ P(E_1 \cap E_2 \cap E_3) \geq 1 - 2\frac{(\kappa^* - \kappa)^{\frac{3}{4}}}{a^{\nu}} - 2(\kappa^* - \kappa)^{\frac{3}{4}} \cdot e^{-\frac{1}{2}T(\kappa^* - \kappa)^{3/2}}, \] (3.38)

where \( a = a_1 - a_2 > 0 \). Hence, the result is verified.

**Corollary 3.6.** Suppose there is a sequence of forward Loewner chains \( g^n(t,z) : \mathbb{H} \setminus K_T(\omega) \to \mathbb{H} \) generated by Dyson Brownian motion \( \{\lambda^n_1(t), \lambda^n_2(t)\} \) with parame-
\( e \) such that \( \lim_{n \to \infty} \kappa_n = \kappa \in (0, 4] \), and let \( K_T(\omega) = \bigcup_{j=1}^n K^n_T(\omega) \). Suppose \( g_t(z) : \mathbb{H} \setminus K_T(\omega) \to \mathbb{H} \) is a forward Loewner chain generated by Dyson Brownian motion \( \{\lambda_1(t), \lambda_2(t)\} \) with diffusivity parameter \( \kappa \in (0, 4] \).

Then, for all \( \epsilon > 0 \), there exists \( N \in \mathbb{N} \) such that with \( n > N \), we have
\[
\mathbb{P}(g^n_t(z) \xrightarrow{\text{Car}} g_t(z)) \geq 1 - \epsilon. \quad (3.39)
\]

4 Variant estimate on the Hausdorff distance

In this section, we analyze the Hausdorff convergence under assumptions on the behaviour of the derivative of the map. The method follows the one curve strategy from \([14]\). First, we define the notion of Hausdorff distance. For any two compacts \( A, B \subset \mathbb{C} \), define the Hausdorff metric ([12] Sec. 6.1) by
\[
d_H(A, B) := \inf \left\{ \epsilon > 0; A \subset \bigcup_{z \in B} B(z, \epsilon), B \subset \bigcup_{z \in A} B(z, \epsilon) \right\},
\]
where \( B(z, \epsilon) \) is the \( \epsilon \)-ball centered at \( z \in \mathbb{C} \). In this section, we prove a variant pathwise perturbation estimate in Hausdorff distance of the hulls \( K_t \) generated by the forward Loewner flow. Following \([1]\), we have that for \( \kappa \in (0, 4] \), the multiple \( \text{SLE} \) curves are a.s. simple and non-intersecting. This serves as a motivation to understand the Hausdorff distance convergence following the analysis of the Caratheodory type convergence. In general, Hausdorff distance convergence is stronger than the Caratheodory convergence, however we are a.s. in the case of simple non-intersecting curves.

Going back to the \( N \)-curve case. First consider a forward Loewner chain \( g_t(z) : \mathbb{H} \setminus K_t \to \mathbb{H} \) driven by forces \( t \mapsto (\lambda_1(t), \ldots, \lambda_N(t)) \). Denote the inverse map corresponding to \( g_t(z) \) by \( f_t(z) : \mathbb{H} \to \mathbb{H} \setminus K_t \), with \( g_t(f_t(z)) = z \), for all \( z \in \mathbb{H} \). On the other hand, consider the time-reversed forward Loewner chain generated by the time-reversed forces \( t \mapsto (\lambda_1(T - t), \ldots, \lambda_N(T - t)) \). Denote this forward Loewner chain by \( h_t(z) \) for \( z \in \mathbb{H} \). Then it satisfies
\[
\partial_t h_t(z) = \frac{1}{N} \sum_{j=1}^N \frac{-2}{h_t(z) - \lambda_j(T - t)}, \quad (4.2)
\]
with \( h_0(z) = z \) for all \( z \in \mathbb{H} \). Similar to the \( N = 1 \) case as in ([13] Sec. 2.), we could verify that \( f_T(z) = h_T(z) \) for all \( z \in \mathbb{H} \). When the system is under
perturbation, we need to compare a Loewner chain with its perturbed counterpart. Indeed, denote by $f_k(t, z)$ and $g_k(t, z)$ the Loewner chains driven by \{${V_{k,1}(t), \ldots, V_{k,N}(t)}$\} for $k = 1, 2$. Denote by $h_k(t, z)$ the backward Loewner chains driven by \{{$V_{k,1}(T-t), \ldots, V_{k,N}(T-t)$}\} for $k = 1, 2$. The following lemma estimates pathwisely the backward Loewner chain.

**Lemma 4.1.** For all $\delta > 0$, there exists a constant $C(\delta, T) = \sqrt{1 + 4T/\delta^2}$ such that whenever $\text{Im} \ z \geq \delta$, we have

$$|h_1(T, z) - h_2(T, z)| \leq C(\delta, T) \sum_{j=1}^{N} |V_{1,j}(T-t) - V_{2,j}(T-t)||_{[0,T]}.$$  

(4.3)

**Proof.** The proof is similar to Proposition 2.2. Take $I_{k,j} \leq \log \text{Im} \ h_k(t,z)$. We also need the following Koebe distortion theorem, see ([14] Lem. 2.1)

**Lemma 4.2.** Let $D$ be a simply connected domain and assume $f : D \to \mathbb{C}$ is conformal map. Let $d = \text{dist}(z, \partial D)$ for $z \in D$. If $|z-w| \leq rd$ for some $0 < r < 1$, then

$$\frac{|f'(z)|}{(1+r)^2}|z-w| \leq |f(z) - f(w)| \leq \frac{|f'(z)|}{(1-r)^2}|z-w|.$$  

(4.4)

**Proposition 4.3.** Let $g_k(t, k) : [0, T] \times \mathbb{H} \setminus K_{k,t}$ be two forward Loewner chains driven by forces $t \mapsto (V_{k,1}(t), \ldots, V_{k,N}(t))$ with hulls $K_{k,t}$, for $k = 1, 2$. Let $f_k(t, z)$ be their inverse so that $g_k(t, f_k(t, z)) = z$. Write $f_k(z) := f_k(T, z)$, for $k = 1, 2$. Suppose that

$$\sum_{j=1}^{N} \sup_{0 \leq t \leq T} |V_{1,j}(t) - V_{2,j}(t)| < \epsilon,$$  

(4.5)

where $\epsilon > 0$ is taken sufficiently small. Suppose further there exists $\theta \in (0, 1)$ such that for all $\zeta \in \mathbb{R}$, we have

$$|f'_{1}(\zeta + i\delta)| \leq \delta^{-\theta},$$  

(4.6)

for all $\delta \leq 4\sqrt{T}\epsilon$. Then, we have the Hausdorff metric estimate

$$d_H(K_{1,T} \cup \mathbb{R}, K_{2,T} \cup \mathbb{R}) \leq 8(T\epsilon)^{1-\theta} + 3\sqrt{\epsilon(1+\epsilon)}.$$  

(4.7)

**Proof.** Denote by $h_k(t, z)$ the time-reversed Loewner chains driven by $\{V_{k,1}(T-
\(t, \ldots, V_{k,N}(T-t)\) for \(k = 1, 2\). Based on Lemma 4.1 and the observation that 
\(f_k(z) = h_k(T, z)\), we know

\[
|f_1(z) - f_2(z)| \leq \epsilon \cdot \sqrt{1 + 4T/\delta^2},
\]
whenever \(\text{Im} \; z \geq \delta\). Take \(\delta_0 = 4\sqrt{T \epsilon}\), we have

\[
\sup_{\text{Im} \; z \geq \frac{\delta_0}{2}} |f_1(z) - f_2(z)| \leq \sqrt{\epsilon(1 + \epsilon)}.
\]

Hence, Cauchy’s integral formula implies

\[
\sup_{\text{Im} \; z \geq \delta_0} |f_1'(z) - f_2'(z)| \leq \frac{1}{2\pi} \int_{\partial B(z, \delta/2)} \frac{d\zeta}{|z - \zeta|^2} \leq \frac{1 + \epsilon}{4T}.
\]

For notational convenience, we write \(\hat{K}_k := K_{k,T} \cup \mathbb{R}\), for \(k = 1, 2\). Fix \(\zeta \in \mathbb{R}\), by Lemma 4.2, we have

\[
|f_1(\zeta + i0^+) - f_1(\zeta + i\delta)| \leq \delta \cdot |f_1'(\zeta + i\delta)| \leq \delta^{1-\theta} \leq (16T \epsilon)^{\frac{1-\theta}{2}}.
\]

Hence, we have

\[
f_1(\{\text{Im} \; z \leq \delta_0\}) \subset \bigcup_{z \in \hat{K}_1} B(z, (16T \epsilon)^{\frac{1-\theta}{2}}).
\]

We have that

\[
\hat{K}_2 \subset f_2(\{\text{Im} \; z \leq \delta_0\}).
\]

For the above fixed \(\zeta \in \mathbb{R}\), write \(w := f_1(\zeta + i0^+) \in \hat{K}_1\). Choose \(\hat{w} \in \hat{K}_2\) be the point in \(\hat{K}_2\) nearest to \(f_2(\zeta + i\delta_0)\).

By Lemma 4.2 again

\[
|\hat{w} - f_2(\zeta + i\delta_0)| \leq |f_2(\zeta + i0^+) - f_2(\zeta + i\delta_0)| \leq \delta_0 \cdot |f_2'(\zeta + i\delta_0)|
\]

\[
\leq \delta_0 \cdot \left(|f_1'(\zeta + i\delta_0)| + |f_1'(\zeta + i\delta_0) - f_2'(\zeta + i\delta_0)|\right) \leq (16T \epsilon)^{\frac{1-\theta}{2}} + \sqrt{4\epsilon(1 + \epsilon)}.
\]
Hence, we see that
\[
|w - \hat{w}| \leq |w - f_1(\zeta + i\delta_0)| + |f_1(\zeta + i\delta_0) - f_2(\zeta + i\delta_0)| + |\hat{w} - f_2(\zeta + i\delta_0)| \\
\leq (16T\epsilon)^\frac{b}{2} + \sqrt{\epsilon(1 + \epsilon)} + (16T\epsilon)^\frac{b}{2} + \sqrt{4\epsilon(1 + \epsilon)} \\
\leq 8(T\epsilon)^\frac{b}{2} + 3\sqrt{\epsilon(1 + \epsilon)}.
\]
(4.15)
Hence the result is verified.

**Remark 4.4.** When the forward Loewner chain \(g_t(z)\), with inverse \(f_t(z)\), is driven by a single Brownian motion, from ([15], Cor. 3.5) we have the following derivative estimate.

Let \(b \in [0, 1 + 4\kappa^{-1}]\), and let \(\lambda\) and \(a\) be explicitly dependent on \((b,\kappa)\). There is a constant \(C(\kappa, b)\), depending only on \(\kappa\) and \(b\), such that the following estimate holds for all \(t \in [0,1], y, \delta \in (0,1]\) and \(x \in \mathbb{R}\)

\[
\mathbb{P}\left(|f'_t(x + \xi(t) + iy)| \geq \delta y^{-1}\right) \leq C(\kappa, b) \left(1 + x^2/y^2\right)^b (y/\delta)^\lambda \vartheta(\delta, a - \lambda)
\]
(4.16)
where \(f_t(z)\) is the \(z\)-inverse of the one-curve forward Loewner map with parameter \(\kappa \neq 8\), driven by the single continuous force \(\xi(t)\).

This type of derivative estimate gives Ineq. (4.6). We expect that a similar inequality holds for the multiple SLE\(_\kappa\) case in the following form

\[
\mathbb{P}(|f'_t(\zeta + i\delta)| \geq \delta^{-\theta}) \leq C_{\kappa, \theta} \cdot u(\delta),
\]
(4.18)
where \(C_{\kappa, \theta} > 0\) depends only on \(\kappa, \theta\), and \(u(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+\) is a proper function.
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