Topological Connectedness and Behavioral Assumptions on Preferences: A Two-Way Relationship

M. Ali Khan† and Metin Uyanık‡

October 25, 2018

Abstract: This paper offers a comprehensive treatment of the question as to whether a binary relation can be consistent (transitive) without being decisive (complete), or decisive without being consistent, or simultaneously inconsistent or indecisive, in the presence of a continuity hypothesis that is, in principle, non-testable. It identifies topological connectedness of the (choice) set over which the continuous binary relation is defined as being crucial to this question. Referring to the two-way relationship as the Eilenberg-Sonnenschein (ES) research program, it presents four synthetic, and complete, characterizations of connectedness, and its natural extensions; and two consequences that only stem from it. The six theorems are novel to both the economic and the mathematical literature: they generalize pioneering results of Eilenberg (1941), Sonnenschein (1965), Schmeidler (1971) and Sen (1969), and are relevant to several applied contexts, as well as to ongoing theoretical work. (140 words)

Journal of Economic Literature Classification Numbers: C00, D00, D01

2010 Mathematics Subject Classification Numbers: 91B55, 37E05.

Key Words: Connected, 2-connected, k-connected, k-non-triviality, continuous, complete, transitive, semi-transitive, pseudo-transitive, fragile, flimsy

Running Title: Topological Connectedness and Preferences

*A previous version of this paper was completed while Uyanık was a post-doctoral fellow at the Allen Wallis Institute of Political Economy at the University of Rochester; earlier versions were presented at Rutgers University (April 19, 2017), the University of Rochester (May 4, 2017), the University of Queensland (August 8, 2018), the 17th SAET Conference in Faro (June 30, 2017), and the 36th Australasian Economic Theory Workshop in Canberra (February 8, 2018). We are grateful to Rich McLean for the SAET invitation, and thank him for detailed and indispensable advice on the presentation of our results. We also thank Paul Anand, Han Bleichrodt, Ying Chen, John Duggan, Hülya Eraslan, Filippo Massari, Andy McLennan, Onur Ozgur, John Quah, Eddie Schlee and Rajiv Vohra for encouragement and stimulating conversation.

†Department of Economics, Johns Hopkins University, Baltimore, MD 21218.
‡School of Economics, University of Queensland, Brisbane, QLD 4072.
## Contents

1 Introduction 2

2 Notational and Conceptual Preliminaries 5

3 The Theory 8
   3.1 A General Result on $k$-Connectedness 8
   3.2 Specializations: Connectedness and 2-Connectedness 9
   3.3 A Return to Connectedness 14
   3.4 Sen’s Deconstruction of the Transitivity Postulate 16

4 Applications of the Theory: A Brief Excursus 22
   4.1 Real-valued Representation of Preferences 24
   4.2 Shafer’s Non-Transitive Consumer 25
   4.3 Walrasian Economies and Normal-Form Games 26
   4.4 Other Potential Applications 28

5 Concluding Remarks 30

6 Proofs of the Results 31
Just as it is possible to speak prose without noticing that fact ..., it is possible also to be talking topology without sensing any topology around us.

The fear of being ‘unmeasurable’ can be a rather raw worry (more polemical than illuminating) and such a diagnosis can serve as a reactionary diversion from reasoning (reasoning that we can sensibly use). ... [it] explains why set theory (and, based on that, topological ideas) can be very useful in practical economic and social evaluation.  

1 

Introduction

The postulates of transitivity, completeness and continuity of a binary relation defined over a (choice) set are basic to modern microeconomic theory, and this paper addresses itself to the logical implications of these properties in an exclusively topological register. In particular, it asks whether an agent can be decisive, in the formal sense of having complete preferences, without being consistent, again in the formal sense of having transitive preferences? Or to turn the matter on its head, whether she can be consistent without being decisive? Or even, in the case of an anti-symmetric binary relation, inconsistent or indecisive or both? It shows, to put it in a nutshell, that all these possibilities are foreclosed under the standard technical assumption of continuity (typically taken to be an innocuous technical property that cannot be falsified by any finite number of observations) in any topology in which the choice set is connected. Indeed, the paper goes beyond these three basic questions to consider additional complementary queries, and in their turn, settles them by appealing to the interplay of the topological properties of continuity and connectedness. As such, the results raise a subtle but striking challenge to the modeling of agency in the theory of binary choice.

To be sure, the questions we ask have a venerable history in economic theory and in mathematics, though they have not been posed and investigated in the way that we do here. The treatment in the antecedent literature has been piecemeal. Thus, in the context of a connected choice set, and a correspondingly continuous binary relation,

1. Eilenberg (1941) showed that completeness and the impossibility of indifference between any two distinct items (a sort of extreme decisiveness), imply transitivity,

2. Sonnenschein (1965) showed that completeness and semi-transitivity of the binary relation imply its transitivity, 

3. The first sentence is taken from p. 298, and the second from p. 367 in Sen (2017). Sen fills the ellipsis in the first by referring to Molière, and in the second, by the Aristotelian counsel that one aspire to as much precision as the subject deserves. This is also the place to acknowledge our substantial indebtedness to Sonnenschein (1965) and Sen (2017) for rekindling our interest in the subject on which we report in this paper: in fact, the conclusion of the former could be read as spelling out the basic motivation for this paper as well.

2 See, for example, Barten and Böhm (1982), a survey that still remains fresh and worth reading. The authors cite Eilenberg, Sonnenschein and Schmeidler, reproduce the proof of Schmeidler’s result, differentiate points of departure based on whether strict or weak preferences as used as primitive, but do not make the connections that we do here. The properties of the binary relation referred to below constitute standard textbook material, as in Barten and Böhm (1982) or Sen (2017) for example, but precise definitions are spelt out for the reader’s convenience in the section to follow.

3 We may point out here that Sonnenschein (1965) is an elaboration of his Ph.D. dissertation that dispensed with Eilenberg’s requirement that the binary relation be anti-symmetric, that preferences not be constrained by the requirement of singleton indifference sets. We return to this point in the sequel. In fact, a careful reader
3. Schmeidler (1971) showed that transitivity and non-triviality of the binary relation imply its completeness. Eilenberg (1941) also furnished necessary and sufficient conditions for a connected topological space to be ordered, which is to say, conditions under which it admits an anti-symmetric, transitive, complete and continuous relation on it.

Connected topologies are of course ubiquitous in economic theory and these pioneering contributions testify to the substantive restrictions they engender when the continuity property of the ambient binary relation is indexed by them. The sufficiency theorems they offer move forward from connectedness, and they all question the opinion, still pervasive in some circles, that topological assumptions on the choice set, and on the objectives that are defined over that set, have no substantive and behavioral content. They are simply seen as innocuous technical requirements made to guarantee the existence of an optimal choice. Thus Wakker (1988a) writes:

[...] by themselves these topological assumptions are merely “technical.” They have no empirical implications and cannot be verified or falsified by observations. Technical assumptions, while not very satisfactory, are not very bothersome either. They merely serve to make mathematical machinery work smoothly. They are void of empirical meaning, and so do not entail obscurity.

It was perhaps Schmeidler who offered, though more understatedly than warranted, the sharpest counter to this prevailing opinion. He wrote:

Order properties of preference relations have intuitive meaning in the context of the behavioral sciences. This is not the case with topological conditions. They are only assumed in order to utilize the mathematical tools applied in the analysis of some problems in the behavioral sciences. They may imply, however, as in the theorem, a very restrictive condition of plausible nature.

In an article scarcely over a page long, and without any references to its precursors in the work not only of Eilenberg and Sonnenschein, but also Rader (1963) and Sen (1969), Schmeidler presented a theorem that confined itself solely to the topological register. With all linear structures, in particular, dispensed with, he articulated the claim that any topology that ensured connectedness of the choice set, and the continuity of a non-trivial and transitive relation over that set, necessarily guarantees the completeness of the relation – a sharp and influential example of a technical requirement leading to a substantive conclusion of a behavioral nature.

However, one can ask whether Schmeidler’s results, as well as those of the others, can be framed under a broader rubric that enables one to see how one result can be reinforced by another, a mutual strengthening that offers a synthetic and more comprehensive overview. Put differently, we would have already noted Sonnenschein making this point in the very first paragraph of his paper.

4 We return to this theorem below. It is perhaps worth reminding the reader that Nachbin’s pioneering monograph is titled “Topology and Order;” see Ward (1954) and Bridges and Mehta (1995) for details.

5 We need hardly remind the reader that finite-dimensional Euclidean spaces, the simplex of (objective) lotteries over a finite set, and infinite-dimensional topological vector spaces are some of the most obvious examples.

6 Some would say that this is almost professional folk-wisdom still: see for example page 80 (last paragraph) in Gilboa (2009).

7 By truncating Wakker’s words as we do, we take them out of context; we return to them, and to Narens (1985), in Section 4 below.
can one ask for necessity results that move \textit{backward} to connectedness? Indeed, in spite of his focus on the question of a numerical representation of a given preference relation over the choice space, Eilenberg had already furnished necessary and sufficient conditions for the existence of topology that provided a simultaneous guarantee both of the connectedness of the choice set and the continuity of the relation defined over the set. And Sonnenschein’s work can be seen as a further elaboration of this line of inquiry to a setting in which binary relations are unconstrained by Eilenberg’s singleton indifference sets, a context of primary interest for microeconomic theory. And surely, the next step in the line of development is Sen’s. Even though his primary emphasis is on choice functions and problems of social choice, his forensic examination of the transitivity postulate, one that abstains from topological considerations, draws on Eilenberg, Sonnenschein and Lorimer as a subtext, and thereby leaves open the question as to the consequences of putting continuity considerations back in. In any case, a principal motivation behind this paper is to focus on this simultaneity: an inquiry that moves forward and backward in its investigation of not only what topological connectedness implies for a binary relation continuous in that topology, but also what is implied in its turn by the substantive properties of that class of relations. It is to focus on both directions of a two-way relationship between topological and order structures. As such, we refer to this investigation as the Eilenberg-Sonnenschein (ES) research program, recognizing that even though it is the forward (sufficiency) direction that is of primary interest to economists, and the backward (necessary) direction of primary interest to topologists, the four equivalence theorems that we present below are perhaps two sides of the same coin – a productive two-way relationship in which assumptions on the choice set, and the objectives that are defined on that set, have strong and obvious implications for each other. Eilenberg saw this more than seventy-five years ago.

In summary, without being overly pompous and pedantic, and motivated primarily to make an exploratory essay in pure theory, reader-friendly, we list what we see to be the six contributions of this paper, and leave it to the reader to separate the primary from the secondary.

1. Gathering and connecting under one rubric the three foundational results enumerated above on how continuity and connectedness in a given topology logically link transitivity and completeness.

2. Remaining within the parametric ambit of these foundational results, a generalization of Eilenberg’s theorem that non-triviality and no-indifference between distinct items imply both transitivity and completeness.

3. Turning the foundational results around by asking for the type of topologies under which...
the logical implications on the binary relations hold, and providing complete answers.

4. Generalizing the collectivity of these results by deconstructing notions of transitivity, completeness and connectedness, and establishing the tightness of these generalizations.

5. Investigating ancillary notions such as *fragility, flimsiness* and *separability* that fall within the ES program, even though not specifically related to transitivity and completeness.

6. Drawing the implications of the results for a variety of applied contexts under the headings of *redundancy* and *hiddenness*.

Once the basic motivation of the paper is understood, and the results outlined, the presentation of the rest of the paper follows rather naturally. Section 2 develops the basic notation and vocabulary of the subject and Section 3 presents the results in four subsections: the four equivalence theorems in the first three, and a sufficiency theorem that casts the results of Sen and Sonnenschein in the framework of incomplete preferences. The latter, and its affiliated proposition, while of interest for its own sake, also undergirds the proofs of the equivalence theorems that precede it. Section 4 is devoted to a somewhat hurried discussion and application of the results to a variety of applied registers. Section 5 recollects the strands already laid out in this introduction, and concludes with two observations for further work. Section 6 is devoted to the proofs.

2 Notational and Conceptual Preliminaries

Let $X$ be a set and a *binary relation* $R$ on it as a subset $R \subset X \times X$. We define

$$
R(x) = \{ y \mid (x, y) \in R \},
$$

$$
R^{-1}(x) = \{ y \mid (y, x) \in R \},
$$

$$
R^{-1} = \{(x, y) \mid (y, x) \in R \},
$$

where $R(x)$ denotes the *upper section* of $R$ at $x$, $R^{-1}(x)$ the *lower section* of $R$ at $x$ and $R^{-1}$ the *transpose* of $R$. Let $\Delta = \{(x, x) \mid x \in X \}$ and $R^c$ the complement of $R$. We say that $R$ has *open* (closed) *sections* if its upper and lower sections are open (closed) in the topology that $X$ is endowed with. We call $R$ *continuous* if its sections are closed and the sections of its asymmetric part $P = R \setminus R^{-1}$ are open. We shall also denote $R$ by $\preceq$, as is standard especially in the economics literature.

The descriptive adjectives pertaining to a relation are presented in a tabular form for the convenience of the reader in Table 1 below.

As is conventional, we denote the *symmetric part* of the binary relation $R$ by $I = R \cap R^{-1}$ and its *asymmetric part* by $P = R \setminus R^{-1}$. In terms of Sen’s (1969) felicitous notation, let $PI$ be the containment $P^{-1}(x) \times I(x) \subset P$ for all $x \in X$, $IP$ the containment $I^{-1}(x) \times P(x) \subset P$ for all $x \in X$. Note, for example, that $PI$ amounts to the requirement that $y \prec x \sim z$ implies $y \prec z$ and $IP$ that $y \sim x \prec z$ implies $y \prec z$, where $\sim$ denotes the symmetric part of $\preceq$ and $\prec$ its asymmetric part. We shall also use the abbreviations $T$, $PP$, $II$ and $NP$, where the first three refer to the transitivity of $R$, $P$ and $I$, respectively, and $NP$ the negative transitivity of $P$.

**Definition 1.** A binary relation is said to be semi-transitive if $PI$ and $IP$ hold.
beware, however, that shows that spaces with finitely many components are well-known and studied extensively. The reader should see if the results in this paper generalize to quasi-components.

Definition 2. A topological space is said to be connected if it is not the union of two non-empty, disjoint open sets. Equivalently, X is connected if the only subsets of X which are both open and closed are \( \emptyset \) and \( X \). The space X is disconnected if it is not connected. A subset of X is connected if it is connected as a subspace.

All this is routine so far. We now break new ground by considering the concepts of k-connectedness of a set and k-nontriviality of a binary relation on that set.\(^{13}\)

Definition 3. A component of a topological space is a maximal connected set in the space; that is, a connected subset which is not properly contained in any connected subset.\(^{14}\) For any natural number k, a topological space is k-connected if it has at most k components.

The concept of k-connectedness provides a quantitative measure of the degree of disconnectedness of a topological space. Note also that it is an adjective for space, and not for the topology it is endowed with. It is also clear that 1-connectedness is equivalent to connectedness and that any k-connected space is l-connected for all \( l \geq k \).

Next, we present a generalization of non-triviality.

Definition 4. A binary relation \( R \) on a topological space X is componentwise non-trivial if

(i) for any component \( C \) of X, there exists \( x, y \in C \) such that \( (x, y) \in P \),

(ii) for any distinct components \( C, C' \) of X, there exist \( x \in C, y \in C' \) such that \( (x, y) \in R \cup R^{-1} \).

It is easy to see that the concept requires strict comparability within the components and weak comparability across the components. We now give it a quantitative cast by presenting a formal definition of k-nontriviality.

Table 1: Properties of Binary Relations

| Property          | Set-theoretic notation | Relational notation |
|-------------------|------------------------|--------------------|
| reflexive         | \( \Delta \subset R \)  | \( x \leq x \forall x \in X \) |
| complete          | \( X \times X = R \cup R^{-1} \) | \( x \leq y \text{ or } y \leq x \forall x, y \in X \) |
| symmetric         | \( R = R^{-1} \) | \( x \leq y \text{ implies } y \leq x \forall x, y \in X \) |
| asymmetric        | \( R \cap R^{-1} = \emptyset \) | \( x \leq y \text{ implies } y \not\leq x \forall x, y \in X \) |
| anti-symmetric    | \( R \cap R^{-1} \subset \Delta \) | \( x \leq y \text{ implies } x = y \forall x, y \in X \) |
| non-trivial       | \( R \cup R^{-1} \neq \emptyset \) | \( \exists x, y \in X \text{ such that } x \leq y \text{ and } y \not\leq x \forall x, y \in X \) |
| transitive        | \( R^{-1}(y) \times R(y) \subset R \forall y \in X \) | \( x \leq y \leq z \text{ implies } x \leq z \forall x, y, z \in X \) |
| negatively transitive | \( R^{-1} \) is transitive | \( x \not\leq y \not\leq z \text{ implies } x \not\leq z \forall x, y, z \in X \) |

\(^{13}\)We do not mean to overemphasize the novelty: a cursory reference to Wilder (1949), or to Dugundji (1966), shows that spaces with finitely many components are well-known and studied extensively. The reader should beware, however, that k-connectedness is used in algebraic topology with a totally different meaning.

\(^{14}\)There is the weaker concept of a quasi-component of a topological space. In a compact Hausdorff space, the two are identical; see Wilder (1949). Even though many economic settings assume the Hausdorff separation axiom, compactness may fail, like the consumption set of a consumer in a Walrasian economy. Hence, it may be of interest to see if the results in this paper generalize to quasi-components.
(a) there exists \((x, y) \in C_{m_i} \times C_{n_j}\) such that \((x, y) \in P \cup P^{-1},\)
(b) there exists \((x, y) \in (C_{m_i} \times C_{n_j}) \cup (C_{m_j} \times C_{n_i})\) such that \((x, y) \in R \cup R^{-1}.)

First, the elementary observation that in any space, non-triviality and 1-non-triviality of a binary relation are equivalent. Furthermore, for \(k\)-connected spaces, for all \(i \leq k\), \(k\)-non-triviality of a binary relation can be qualitatively conceived of as being componentwise non-trivial since \(m_i = n_i = i\). Nevertheless, the concept is not straightforward, and it is worthwhile to discuss it in the context of simple examples: we illustrate 1-, 2- and 3-non-triviality in the context of a space with three components.

Example 1. In Figure 1, \(X = C_1 \cup C_2 \cup C_3\) is the union of three non-degenerate open intervals in the real line which is endowed with its standard topology. It is clear that \(C_1, C_2, C_3\) are the components of \(X\). Panels (a), (b) and (c) of Figure illustrate three distinct binary relations \(R_a, R_b, R_c\) defined on \(X\), with points labeled with a filled circle illustrating condition (a) of Definition 5, and the ones with an empty circle illustrating condition (b). Hence, \(R_a = P_a, R_b = P_b\) and \(R_c = P_c\). We now turn the detailed explanation of each relation illustrated in the panels.

We first show that the relation \(R_a\) is 1-non-trivial. In Definition 5 set \(k = 1, m_1 = 2, n_1 = 3\). As illustrated, \((x_1, x_2) \in C_{m_1} \times C_{n_1}\). Since \((x_1, x_2) \in R_a\) and \((x_2, x_1) \notin R_a\), therefore \((x_1, x_2) \in P_a\). Therefore, condition (a) of the definition is satisfied. Condition (b) does not have any bite since there are no distinct \(i, j \leq k = 1\).

Second, we show that the relation \(R_b\) is 2-non-trivial. In this example, it is clear that \(R_b = P_b\). Set \(k = 2, m_1 = 2, m_2 = 3\) and \(n_1 = 1, n_2 = 3\). Since each of \(C_{m_1} \times C_{n_1}\) and \(C_{m_2} \times C_{n_2}\) contains a pair of strictly comparable alternatives, \((x_1, x_2)\) and \((y_1, y_2)\) respectively, \(R_b\) satisfies condition (a) of the Definition 5. The only \(i, j\) that satisfy \(i < j \leq 2\) are \(i = 1, j = 2\). It follows from \((z_1, z_2) \in P_b \cap C_{m_2} \times C_{n_1}\) that condition (b) also holds. Hence, \(R_b\) is 2-non-trivial.

Finally, we show that the relation \(R_c\) is 3-non-trivial. In this example, it is clear that \(R_c = P_c\). Set \(k = 3\). In this case, there exists only one ordering that satisfies \(1 \leq l_1 < l_2 < l_3 \leq 3\) which is \(1 < 2 < 3\). Hence \(m_i = n_i = i\) for \(i = 1, 2, 3\) in the definition above. Since for each \(i\), each \(C_i\) contains a pair of strictly comparable points \((one of (x_1, x_2), (y_1, y_2) and (z_1, z_2))\), condition (a) of Definition 5 holds. For \(i = 1 < 2 = j\), \((t_1, t_2) \in C_{m_j} \cap C_{n_i}\), for \(i = 1 < 3 = j\), \((w_1, w_2) \in C_{m_i} \cap C_{n_j}\), and for \(i = 2 < 3 = j\), \((v_1, v_2) \in C_{m_1} \cap C_{n_j}\). Therefore, condition (b) also holds. Example 1 is complete.

Figure 1: \(k\)-nontriviality
3 The Theory

In this section, we present the six basic results of this paper along with two supplementary propositions. The first three subsections present equivalence results characterizing connectedness and its natural generalizations; while the fourth is devoted to two sufficiency theorems. These two theorems set Sen’s topologically-free results in a topological register, and elaborate Sonnenschein’s invocation of the Phragmen-Brouwer (topological) property, all in the setting of incomplete preferences. These six theorems all have constitutive implications for what we are referring to as the ES research program.

3.1 A General Result on $k$-Connectedness

We now present our first equivalence theorem, deferring to the next subsection its relationship to the antecedent literature.

**Theorem 1.** For any natural number $k$, the following statements are equivalent for a $k$-non-trivial and continuous binary relation defined on any topological space with at least $k$-components.

(a) The space is $k$-connected.
(b) Any transitive relation is complete.
(c) Any anti-symmetric relation is complete.
(d) Any relation whose symmetric part is transitive with connected sections, is complete.
(e) Any semi-transitive relation with transitive symmetric part is complete.

Even though Theorem 1 achieves a symbol-free and clear expression, a further discussion of it in terms of the vernacular used in the introduction may be worthwhile. But before that, note that the theorem works with a class of topologies, and that statement (a), together with the specification that the space has at least $k$-components, implies that the space has exactly $k$ components. Moving on to the other assertions, under the topological specifications of connectedness and continuity, condition (b) asserts the impossibility of a consistent agent being indecisive, and condition (c) insists that she, when never indifferent between distinct items, must be decisive irrespective of being consistent or not. The third assertion drops consistency and adds extreme decisiveness regarding comparable items. Indeed, one can push the latter claim a little further. Even when she can “choose and not only pick” which is to say, when she has a fine enough taste to discriminate between distinct items and to be possibly indifferent among them (has possibly non-singleton transitive indifference classes), she must be consistent if these classes are either connected (condition (d)), or can be compared in the sense of being semi-transitive (condition (e)). As Figure 2 makes clear, there is a relationship of inclusion in terms of conditions (b) to (e) of Theorem 1, and so, at least as far as the forward direction from condition (a) is concerned, the statement (a) $\Rightarrow$ (e) is all that needs to be shown. However, the very generality of condition (e) goes towards moving away from it when we consider the backward direction.

---

15In terms of a reader’s guide, and especially for a first reading, one may read these theorems for the case $k = 1$, and instead of the full equivalence, even limit oneself to the forward direction as in the foundational papers.
16We are indebted to Ullman-Margalit and Morgenbesser (1977) for this terminological distinction; also taken up in Sen (1993), and under the heading of “the idea of internal consistency of choice,” in Sen (2017, pp. 309-312).
17We shall return to this below in Proposition 2 and in Theorems 5 and 6.
The sets (b)–(e) correspond to the sets of binary relations which satisfy the assumptions of assertions (b)–(e) of Theorem 1 respectively. In particular, the set (e) denotes the set of all $k$-non-trivial, semi-transitive and continuous binary relations $R$ whose symmetric part $I$ is transitive, (d) is the subset of (e) such that the sections of $I$ are connected, (c) is the subset of (e) such that $R$ is anti-symmetric and (b) is the subset of (e) such that $R$ is transitive.

Figure 2: The inclusion relationship between assertions (b)–(e) of Theorem 1

3.2 Specializations: Connectedness and 2-Connectedness

For the special cases of connected and 2-connected spaces, we have more interesting equivalence results. In terms of the forward direction, we not only obtain completeness but also transitivity for free!

**Theorem 2.** For any natural number $k \leq 2$, the following statements are equivalent for a $k$-non-trivial and continuous binary relation defined on any topological space with at least $k$-components.

(a) The space is $k$-connected.

(b) Any transitive relation is complete.

(c) Any anti-symmetric relation is complete and transitive.

(d) Any relation whose symmetric part is transitive with connected sections, is complete and transitive.

(e) Any semi-transitive relation with a transitive symmetric part is complete and transitive.

Thus, once we specialize to connectedness or to 2-connectedness, we can strengthen Theorem 1 to obtain consistency and decisiveness instead of only decisiveness. As already emphasized in the introduction, a simple gathering of the foundational results has led to an equivalence theorem, and thereby also to a characterization of connectedness and 2-connectedness of the choice set. In terms of the relationship of Theorem 2 to the antecedent literature, it takes a piecemeal treatment into a mutually reinforcing one, and thereby in indissolubly connecting the behavioral and mathematical registers, testifies to the analytical depth of the ES research program.

Next, moving to a blow-by-blow account of a comparison with the theorems of Eilenberg, Sonnenschein and Schmeidler, it is perhaps easiest to begin with the concept of semi-transitivity,

---

18We may point out here in anticipation that our joint treatment of these foundational results has also allowed us to provide alternative proofs of the results of Eilenberg and Sonnenschein. Our method of proof is inspired by Schmeidler, and is totally different compared to theirs; see this observation formally made in the paragraph preceding the proof of this theorem, and in the remark following it.
pioneered, though not named as such,\textsuperscript{19} by Rader (1963, p. 232), and formally presented as Definition 1 above. He used it to present what he saw to be a “remarkable result, due, in essence to Eilenberg,” and referred to it as a condition under which the choice set

may be decomposed into indifference classes and that these classes may be compared as more or less preferable. No strict transitivity assumption is made, although two indifference classes are never allowed to be indifferent to each other.

In the specialization of Theorem 2 to $k = 1$, note that

(i) $[a] \Rightarrow [e]$ drops the completeness and anti-symmetry assumptions of Eilenberg (1941, 2.1) and the completeness assumption of Sonnenschein (1965, Theorem 3), weakens the transitivity assumption of Schmeidler (1971, Theorem);

(ii) $[a] \Rightarrow [d]$ drops the completeness and anti-symmetry assumptions of Eilenberg (1941, 2.1) and the completeness assumption of Sonnenschein (1965, Theorem 4);

(iii) $[a] \Rightarrow [c]$ drops the completeness assumption of Eilenberg (1941, 2.1);

(iv) $[a] \Rightarrow [b]$ is due to Schmeidler (1971, Theorem).

All this raises a larger point that some of the assumptions that we make salient are hidden in the statements of the theorems of Eilenberg, Sonnenschein and Schmeidler. For example, Schmeidler’s transitivity assumption already implies that the relation is semi-transitive and its symmetric part is transitive. As another example, Sonnenschein does not assume that the symmetric part of a relation is transitive, but under the completeness assumption, this already follows from the relation itself being semi-transitive.\textsuperscript{20} The reader should also note that the theorem of Eilenberg, as well as that of Sonnenschein, does not assume non-triviality; but cases where non-triviality does not hold, yield trivialities in themselves.\textsuperscript{21} If the preference relation is trivial in Eilenberg’s result, then the choice space can consist of at most one element. And in Sonnenschein’s result, it requires all elements to be indifferent to each other. In both cases, therefore, transitivity of the preference relation is equivalent to the transitivity of the indifference relation which is already assumed. This issue of hidden assumptions goes beyond these foundational papers\textsuperscript{22} and in our consideration of other work, we shall return to this theme below in Section 4.

Two final remarks concerning Theorem 2. First, assertions $[c][e]$ in Theorem 2 reproduce assertions $[c][e]$ in Theorem 1 but with the strengthened form that substitutes completeness and transitivity for completeness. The reader should note that this strengthening does not necessarily hold for $k > 2$. The following Example settles this issue.

\textbf{Example 2.} Let $X = (0,1) \cup (1,2) \cup (2,3)$ and the topology is the Euclidean metric. Then, $k = 3$. Now let $R$ be an anti-symmetric binary relation defined as follows: $(x, y) \in R$ if $x, y \in C_k, x \leq y$, if $x \in (0,1)$ and $y \in (1,2)$, if $x \in (1,2)$ and $y \in (2,3)$, and if $x \in (2,3)$ and $y \in (0,1)$. It is clear that $R$ is complete and has closed sections. However, it is non-transitive.

\textsuperscript{19}Note that the usage here differs from that of Houthisaker’s concept; see Sen (2017, pp. 94, 295-298, 299).

\textsuperscript{20}On this, see Figure 2 and Proposition 2. These relationships are partially available in Sonnenschein (1965, Theorem 3), Lorimer (1967, Theorem 2) and Sen (1969, Theorem I).

\textsuperscript{21}We return to this in Section 4.1.

\textsuperscript{22}To be sure, the discussion in this paragraph assumes $k = 1$; the foundational literature is silent on higher values of the natural number $k$. 

10
Our second remark concerns assertion \((b)\). Even though it already assumes transitivity and is included in Theorem \([1]\), it does not mean it is redundant because of the basic issue of equivalence.

In Theorems 1 and 2, the implication \((a) \Rightarrow (b)\) is a literal rendering of Schmeidler’s claim of completeness, in the case \(k = 1\). Moreover, the Eilenberg-Sonnenschein claim of transitivity is ignored in this first result and bundled with completeness in the second. The question then arises as to whether there is an equivalence claim that is focused only on transitivity. This is to ask whether in assertions (c)–(e) in Theorem 2, completeness can be shifted from the conclusion to the hypothesis. The answer is negative. In order to see this, let \(X = \{0, 1\}\) be endowed with a discrete topology. Then it is clear that \(X\) is disconnected and every binary relation on \(X\) has both closed and open sections, and is transitive. Therefore, even though connectedness of the space is a sufficient condition for the results of Eilenberg-Sonnenschein, it is not a necessary condition. However, in our simple example, the space is 2-connected, and this gives a hint that a version of Theorem \([2]\) may be true for 2-connected spaces when the completeness of the relation is correspondingly shifted. It is indeed so, but involves a subtlety to which we turn next.

As already emphasized in the introduction, Eilenberg’s paper pioneered the question of the representation of a preferences relation on a set in terms of a real-valued function on the same set, but it also investigated the question of conditions under which a binary relation with attractive natural properties exists on the set. \([23]\) Introducing a notion of an ordered space as one that admits an anti-symmetric, complete, transitive and continuous relation, he presented the following result.

\[
A \text{ connected topological space } X \text{ containing at least two elements can be ordered if and only if } P(X) \text{ is disconnected, where } P(X) \text{ consists of } (x, y) \in X \times X \text{ such that } x \neq y.
\]

In the move to our next result, we introduce the following definition which drops the transitivity requirement in Eilenberg’s (1941) characterization of an ordered space \([24]\).

**Definition 6.** A topological space is quasi-ordered if there exists an anti-symmetric, complete and continuous binary relation on it.

We are now ready to state the third equivalence result of the paper.

**Theorem 3.** The following statements are equivalent for a complete and continuous binary relation on any quasi-ordered topological space.

(a) The space is 2-connected.

(b) Any anti-symmetric relation is transitive.

(c) Any relation whose symmetric part is transitive with connected sections, is transitive.

(d) Any semi-transitive relation is transitive.

\([23]\) We have already emphasized the first question in Footnote \([4]\) and in Footnote \([8]\), but what is being emphasized here is the existence of the relation rather than a function, a question that is taken up in the social choice literature; see Khan and Uyanık (2018) for an exploration of this question and an elaboration of this connection. At any rate, as we shall see, it is essential to the result we present below.

\([24]\) The reader is again warned about the lack of a uniform terminology in the literature; see Sen (2017, p. 54).
The subtlety referred to above involves the restriction of the result to a quasi-ordered topological space. To begin with, note at the outset that the restriction is required only for the backward direction, and that to ensure that the assertions guaranteeing 2-connectedness are not vacuous, that there indeed exist binary relations satisfying the properties required of them. Eilenberg’s theorem guarantees their existence in an ordered space, and a fortiori, in a quasi-ordered space. Note this is not an issue in Theorems 1 and 2 since we can construct a $k$-non-trivial continuous binary relation without any assumption on the topological space.

Proposition 3 in Section 4 implies that the topology on a finite topological space with an anti-symmetric, complete and continuous binary relation (quasi-order) has to be discrete. This shows that there does not exist a quasi-order on any finite topological space which is not discrete. We next illustrate an example of such a topological space which has three components. On this space there does not exist a quasi-order, hence it is vacuously true that any quasi-order is transitive. But the space is not 2-connected.

**Example 3.** Let $X = \{x, y, z, w\}$ and the collection $\{\emptyset, \{x\}, \{y\}, \{z, w\}\}$ is a basis for the topology defined on $X$. For any complete and anti-symmetric binary relation $R$ on $X$, either $(z, w) \in P$, or $(w, z) \in P$, where $P$ is the asymmetric part of $R$. Then, either $w \in P(z)$, or $w \in P^{-1}(z)$. Note that $z \notin P(z) \cup P^{-1}(z)$ by definition. Then, either $P(z)$, or $P^{-1}(z)$ contains $w$ but excludes $z$. Since continuity of $R$ requires $P$ has open sections, any complete and anti-symmetric relation on $X$ cannot be continuous.

For an example of a general topological space, we know by Proposition 3 that there does not exist a quasi-order on any non-Hausdorff topological space. Such examples are analogous to the one above.

And while we are on the asymmetry between Theorem 3 and the preceding Theorems 1 and 2, let us also note that for a symmetrical treatment, we can simply include assertion (b) in the theorem by specifying attention to 2-non-trivial binary relations, and taking the requirement of completeness down to the other assertions. In Theorem 3, the statements (a) $\Rightarrow$ (b), (c), (d) are generalizations of Schmeidler (1971, Theorem), Eilenberg (1941, 2.1) and Sonnenschein (1965, Theorems 4 and 3) to 2-connected spaces, respectively. For the reader’s convenience, Table 2 summarizes the statements of Theorems 1–2 as well as those of Eilenberg, Sonnenschein and Schmeidler which are listed in Theorem 3.

Lorimer (1967) claims that Sonnenschein’s theorems are not really theorems of topology, but theorems of set theory that can be proved by set-theoretic considerations alone. In his reply to Lorimer, Sonnenschein (1967) provides a version of assertion (a) $\Leftrightarrow$ (d) of Theorem 3 for subsets of the real line in order to highlight the necessity of connectedness. Sonnenschein also adds that Lorimer’s conditions “would have been very unnatural and generally of little

---

25 This construction is illustrated in the backward direction of the proofs of Theorems 1 and 2; see the penultimate paragraph of the proof of Theorem 1 presented below. As regards Theorem 2, the cases $k = 1$ and $k = 2$ are considered separately; see the last paragraph of the proof in each case.

26 We do not do this because Theorem 3 is primarily motivated by the transitivity claim of Eilenberg-Sonnenschein.

27 We present here a slightly stronger version of Sonnenschein’s theorems in order to underscore the comparison. He assumes the connectedness of $X \setminus I$ instead of connectedness of $X$. It is easy to prove a version of our results under this assumption. A result similar to Sonnenschein’s theorem is provided by Rader (1963, Lemma) under a stronger connectedness assumption which requires the upper sections of the weak preference relation to be connected.
In this table, if one property directly follows from the other assumed properties, then we mark it as ✓. For example, Eilenberg’s theorem assume the preference relation is complete and anti-symmetric, which directly imply semi-transitivity of the preference relation along with the transitivity of its symmetric part. Hence we mark that these two additional properties are also assumed.

* Even though theorems of Eilenberg and Sonnenschein do not assume (2)-non-triviality, without this assumption their results become triviality, see Section 3.2 for details.

Table 2: Comparison of the Results

|                  | Eil | Son | Sch | T1(b) | T1(e) | T1(d) | T1(e) | T1(e) | T2(c) | T2(d) | T2(e) |
|------------------|-----|-----|-----|-------|-------|-------|-------|-------|-------|-------|-------|
| X : (2-)connected| ✓   | ✓   | ✓   | ×     | ✓     | ×     | ×     | ✓     | ✓     | ✓     | ✓     |
| k-connected      | ✓   | ✓   | ✓   | ✓     | ✓     | ✓     | ✓     | ✓     | ✓     | ✓     | ✓     |
| R : complete     | ✓   | ✓   | ×   | ×     | ×     | ×     | ×     | ×     | ×     | ×     | ×     |
| transitive       | ×   | ×   | ✓   | ✓     | ✓     | ✓     | ✓     | ✓     | ✓     | ✓     | ✓     |
| semi-transitive  | ✓   | ✓   | ✓   | ✓     | ✓     | ✓     | ✓     | ✓     | ✓     | ✓     | ✓     |
| anti-symmetric   | ✓   | ×   | ×   | ×     | ✓     | ✓     | ✓     | ✓     | ✓     | ✓     | ✓     |
| continuous       | ✓   | ✓   | ✓   | ✓     | ✓     | ✓     | ✓     | ✓     | ✓     | ✓     | ✓     |
| (2-)non-trivial* | ✓   | ✓   | ✓   | ✓     | ✓     | ✓     | ✓     | ✓     | ✓     | ✓     | ✓     |
| k-non-trivial    | ×   | ×   | ×   | ✓     | ✓     | ✓     | ✓     | ✓     | ×     | ×     | ×     |
| I : transitive   | ✓   | ✓   | ✓   | ✓     | ✓     | ✓     | ✓     | ✓     | ✓     | ✓     | ✓     |
| connected sections | ✓ | × | × | × | ✓ | ✓ | ✓ | ✓ | × | × | × |

This proposition shows that if there are comparable alternatives within and across components, then all alternatives are comparable. The proof of the necessity of completeness follows from the proof of Theorem 1 since its proof requires only componentwise non-triviality and does

28It should be noted, however, that Lorimer also shows that these conditions are not only sufficient for transitivity of a complete relation, but also necessary, and hence there is something extra and of consequent use in his paper.
not hinge on the finiteness of the number of components. The necessity of transitivity follows from Theorem 2. Moreover, Theorem 3(d) implies that semi-transitivity can be substituted with further topological assumptions on the binary relation; see Section 3.4 below for details, in particular Theorems 5 and 6.

### 3.3 A Return to Connectedness

A natural question to ask at this stage is how Theorem 2 can be further sharpened if we limit ourselves to the setting original to the foundational papers. This is to say, to 1-connectedness of the topological space. This, by necessity, brings us to current work, and allows a re-framing and a generalization of it.

Towards this end, we begin with a stronger non-triviality concept.

**Definition 7.** A binary relation \( R \) on a set \( X \) is called strongly non-trivial if there exists \( x \in X \) such that \( P(x) \neq \emptyset \), and \( R(x') \cap R(y') \neq \emptyset \) for all \( x', y' \in P(x) \).

In terms of relational notation, strong non-triviality amounts to the requirement that (i) there exist \( x, y \in X \) with \( x \prec y \), and that (ii) for all \( x', y' \in X \) with \( x \prec x' \) and \( x \prec y' \), there exists \( z \in X \) such that \( x' \prec z \) and \( y' \prec z \).

It is not surprising that weakening the continuity assumptions in the hypothesis of Theorems 1 to 3 falsifies their conclusions. In order to elaborate on this, we consider the concept of **fragility** due to Gerasimou (2013), and supplement it by a parallel concept of **flimsiness**: both are useful for the analysis of the structure of the preferences under weaker continuity assumptions. The first assumes that every neighborhood of some strictly comparable alternatives of a non-trivial binary relation on the choice space contains non-comparable alternatives, Gerasimou (2013, p. 161) motivates his concept as follows:

From a normative point of view, fragility of a preference relation is an undesirable property, both when this preference relation is that of an individual and also when it represents the preferences over social alternatives. Indeed, one would expect that when decision makers express strict preference for one alternative over another, marginal changes in these two alternatives should not result in them becoming incomparable. If they do, then doubt should perhaps be cast on the validity of the strict-preference comparison between the original alternatives. Finally, introspection and casual empiricism do not seem favorable for the property’s descriptive accuracy either.

**Definition 8.** We call a binary relation \( R \) on a topological space \( X \) fragile if there exist \( x, y \in X \) such that (i) \( (x, y) \in P \), and that (ii) any open neighborhood of \( (x, y) \) contains \( (x', y') \) such that \( (x', y') \notin R \cup R^{-1} \).

In terms of relational notation, fragility amounts to the requirement that (i) there exists \( x, y \in X \) with \( x \prec y \), and that (ii) any open neighborhood of \( (x, y) \) contains \( (x', y') \) such that \( x' \nleq y' \) and \( y \nleq x' \).

Next, we provide a definition which complements fragility.  

---

\(^{29}\)We are primarily motivated by forging connections to current work; there is little doubt that results of several of the papers that we connect to can be generalized to the setting of \( k \)-connected sets for any natural number \( k \), and thereby to reformulate the theorem presented below along the lines of Theorem 1.
Definition 9. A binary relation $R$ on a topological space $X$ is flimsy if there exist $x, y \in X$ with $(x, y) \notin R \cup R^{-1}$ such that every open neighborhood of $(x, y)$ contains $(x', y') \in R \cup R^{-1}$.

In terms of relational notation, flimsiness amounts to the requirement that there exists $x, y \in X$ with $x \not\preceq y$ and $y \not\preceq x$ such that every open neighborhood of $(x, y)$ contains $(x', y')$ with $x' \preceq y'$ or $y' \preceq x'$.

Gerasimou (2013) Corollary 3) showed that every incomplete, non-trivial and transitive binary relation with closed sections defined on a connected topological space, is fragile. Therefore, dropping one of the continuity assumption of Schmeidler’s theorem yields an undesirable case of incompleteness from the normative point of view. We show that his result is equivalent to topological connectedness of the space. And we supplement his result by other affiliated concepts; but before turning to them, we recall for the reader three ways of taking an asymmetric relation and associating its reflexive hull with it.

Definition 10. For any asymmetric binary relation $P$ on a topological space $X$, define its reflexive hull as $R = (P^c)^{-1} = \{(x, y) \mid (y, x) \notin P\}$. Moreover, define the lower covering relation $R_\ell$ of $R$ and its upper covering relation $R_u$ as follows:

$$R_\ell = \{(x, y) \mid R^{-1}(x) \subset R^{-1}(y)\} \quad \text{and} \quad R_u = \{(x, y) \mid R(y) \subset R(x)\},$$

We call the pair $(R_\ell, R_u)$ the covering relations of $R$. In terms of relational notation, the reflexive hull of an asymmetric relation $\prec$ is defined as follows: $x \preceq y$ if and only if $x \not\prec y$. The lower covering relation $\preceq_\ell$ of $\preceq$ and its upper covering relation $\preceq_u$ are defined as follows: $x \preceq_\ell y$ if and only if $z \preceq x$ implies $z \preceq y$, and $x \preceq_u y$ if and only if $y \preceq z$ implies $x \preceq z$.

Finally, we present two additional concepts due to Chateauneuf (1987), the second being his strengthening of the standard notion of a separable relation.

Definition 11. Let $P$ be an asymmetric binary relation $P$ on a topological space $X$ and $R = (P^c)^{-1}$ denote its reflexive hull. Then, (also with relational notation in braces),

(i) $R$ is called pseudo-transitive if $(x, x') \in P$, $(x', y') \in R$ and $(y', y) \in P$ imply $(x, y) \in P$ for all $x, y, x', y' \in X$, $(x \prec x' \preceq y' \prec y$ implies $x \prec y$).

(ii) $P$ is called separable if there exists a countable subset $A$ of $X$ such that $(x, y) \in P$ implies there exists $x' \in A$ such that $(x, x') \in P$ and $(x', y) \in P$, $(x \prec y$ implies $\exists x' \in A$ such that $x \prec x' \prec y$).

(iii) $P$ is called strongly separable if there exists a countable subset $A$ of $X$ such that $(x, y) \in P$ implies there exist $x', y' \in A$ such that $(x, x') \in P$, $(x', y') \in R$ and $(y', y) \in P$, $(x \prec y$ implies $\exists x', y' \in A$ such that $x \prec x' \preceq y' \prec y$).

(iv) $P$ has a continuous dual-representation if there exist two continuous real valued functions $u$ and $v$ on $X$ such that $(x, y) \in P$ if and only if $u(x) < v(y)$ for all $x, y \in X$, $(x \prec y$ if and only if $u(x) < v(y)$.)

---

These “covering” type of derived relations have important applications and implications in decision theory as well as in social choice theory. In particular, they have been used for the numerical representation of incomplete preferences: see for example Chateauneuf (1987) for representation of interval orders, and Peleg (1970) and Galaabaatar and Karni (2013) for expected utility representation of incomplete preferences. In social choice theory, the “covering relation” has been used since mid-twentieth century; see Duggan (2013) for a comprehensive survey on covering relation.
Theorem 4. The following statements are equivalent for a binary relation defined on any quasi-ordered topological space which contains more than two elements.

(a) The space is connected.

(b) Any strongly non-trivial and transitive relation with closed upper sections, and whose asymmetric part is negatively transitive with open upper sections, is complete and continuous.

(c) Any two anti-symmetric, non-trivial and continuous relations on $X$ are either identical or inverse to each other.

(d) Any incomplete, non-trivial and transitive relation with closed sections, is fragile.

(e) Any incomplete, non-trivial and transitive relation whose asymmetric part has open sections, is flimsy.

(f) Any asymmetric relation with a continuous dual-representation is strongly separable.

(g) Any asymmetric relation has a continuous dual-representation if and only if it is strongly separable, its dual is pseudo-transitive and its covering relations have closed sections.

Note that unlike Theorems 1 to 3, continuity of the given relation is not a standing hypothesis for Theorem 4, and thus the theorem can be read as an attempt to deconstruct the continuity postulate. When we substitute the strong form of transitivity for continuity, we almost necessitate continuity. Again, the quasi-order assumption is only used in the proof of the implication (c) $\Rightarrow$ (a).

The statement (a) $\Rightarrow$ (c) in the above theorem generalizes Eilenberg (1941, Theorem II) by dropping completeness and transitivity; and the statements (a), (d), (f) and (g) are due to Gerasimou (2013, Corollary 3) and Chateauneuf (1987, Fundamental Lemma and Theorem), respectively. The statement (a) $\Rightarrow$ (b) shows that the continuity assumption in Theorem 2 can be weakened by strengthening the non-triviality assumption. Note that even though we did not explicitly assume negative transitivity in Theorem 2, it follows from Theorem 4 that the assumptions of the hypothesis of the theorem imply that $P$ is negatively transitive.

We conclude this section with the elementary observation that Theorems 2 and 4 can be collapsed into a single portmanteau result characterizing topological connectedness. Such a portmanteau theorem, on its own, or in conjunction with Theorems 1 and 3, can perhaps be read as an up-to-date survey of the two-way relationship brought to light by topological assumptions on the binary relation and the set over which it is defined.

3.4 Sen’s Deconstruction of the Transitivity Postulate

The results presented in this section can be introduced in two alternative ways: (i) as an answer to questions that are naturally raised by the four equivalence theorems presented above, or (ii) as a generalization of Sen’s (1969) non-topological rendering of the results of Sonnenschein to the topological register. In terms of the first, let us consider what Theorems 1 to 4 bring to the table in terms of giving an underlying basis for the transitivity postulate. As already noted, Theorem

---

31 We return to this issue in Section 5 below; see Footnote 56 and the text it footnotes.
1 is primarily concerned with completeness, and Theorem 4 with separability, fragility, and flimsiness. Theorem 2 does offer transitivity as a conclusion but bundles it with completeness, while Theorem 3 uses completeness of the relation as a hypothesis. Natural questions then arise as to whether the dual conclusion can be unbundled, whether the completeness hypothesis can be dispensed with, and whether the role of connectedness pinned down. In short, this is to ask for some sort of minimal setting under which transitivity obtains, and consistency is minimally unencumbered by decisiveness? Theorems 5 and 6 below respond to these questions.32

But an alternative presentation of our results can be furnished in terms of what we are referring to as Sen’s deconstruction of the transitivity postulate. Towards this end, we refer the reader back to Sen’s notation for the six different transitivity concepts $T, NP, II, PP, IP, PI$, already presented in Section 2. Referring to two transitivity conditions as independent if there exists a relation which satisfies one and not the other, and interdependent otherwise, Sen (1969) has offered a non-topological examination of the transitivity postulate. In his Theorem I, he provides a synthetic treatment of the interdependence between the various transitivity conditions when the underlying primitive binary relation is complete; see his Figure 1, also reproduced in Sen (2017, p. 66), and as Figure 3(b) below. Under completeness of $R$, which Sen assumed, $PP$ and any one of $PI, IP$ and $II$ imply $T$, as the reader can see from Theorem 3b.34 The point of departure for our results is how Figure 3(b) unravels without the completeness postulate on $R$ (see Figure 3(a)), and how most of these interdependence relations can be recovered, and new relationships emerge, if the underlying space and the preference relation is assumed to satisfy some suitable topological properties. As such, this inquiry falls very much within the ES program, and in this subsection we provide its elaboration in this direction.35

We begin with the following.

**Proposition 2.** For any binary relation $R$ on a set containing at least four elements, and with $I$ and $P$ denoting its symmetric and asymmetric parts, the following statements are valid.

(a) $T \iff PP \land PI \land IP \land II$.

(b) $NP \Rightarrow PP \land PI \land IP$.

(c) $T$ is independent of $NP$.

---

32Since we exclusively work with a topological structure, we do not investigate the implications of the added assumptions related to linear structure on the ES program. Uzawa (1960) pioneered this line of research by showing that convexity of a complete preference relation defined on a convex subset of a topological vector space with closed upper sections and a transitive asymmetric part implies that the relation itself is transitive. This result is reproduced and generalized by Sonnenschein (1965, Theorems 5 and 6) and picked up by Galaabaatar, Khan, and Uyanik (2018).

33Note that to say that the symmetric part of a relation is complete renders the relation trivial, whereas to say that the asymmetric part is complete furnishes the contradiction that an element of the space is preferred to itself. In Sen (1969), $PP$ is the crucial transitivity condition and he found it convenient to give it a more usable name: “quasi-transitivity”. In this connection, also see the subject index in Sen (2017) for additional discussion of his named concept.

34For example, to see $PP$ and $IP$ imply $T$ from Figure 3(b), simply note that $PP$ and $II$ imply $PI$, which combined with $PP$ then implies $T$. And so on for the other implications.

35There is a rich philosophical literature on the discussion of non-transitivity, already referred to in Footnote 35. One may add that Tullock (1964) defends transitivity by using simple logical arguments and criticizes the experiments which argue evidence against transitivity. Anand (1993) criticizes Tullock for not considering ternary relation; also see also Luce (1950), Fishburn (1970) and Eliaz and Ok (2006).
any subcollection of $PP, PI, IP, II$ is independent of the remaining collection, severally and collectively.

Proposition 2 implies that, in the absence of completeness, four of the transitivity conditions, $PP, IP, PI$ and $II$, are independent of each other, severally and collectively; and $T$ is independent of $NP$. Therefore, the relationship between different transitivity concepts that Sen illustrates falls apart without the completeness assumption.

We now turn to showing that under suitable topological conditions, $PI$ together with $IP$ play an essential role for the completeness and the transitivity of $R$, and are thereby led to the notion of “semi-transitivity”[36]. But before the formalities, we present a picture of the interdependence between the transitivity conditions without referring to the completeness assumption. Panel (a) of Figure 3 illustrates the interdependence between different transitivity conditions for an arbitrary preference relation; panel (b) (as already mentioned above) illustrates the relationship under the completeness assumption; panel (c) remains with the incompleteness assumption but introduces continuity and connectedness assumptions; panel (d) includes completeness as well as continuity and connectedness assumptions. Label CC denotes that the underlying space is connected and that the binary relation defined on it is continuous. Label CIC keeps the continuity assumption of CC and replaces the connectedness of the space with the connectedness of the sections of the symmetric part of the binary relation. To be sure, transitivity and completeness are totally different conditions and completeness has implications on transitivity, but what our result brings out is that transitivity has implication for completeness. As an example, negative transitivity of $\prec$ is equivalent to the following: if $x \prec y$, then $z \prec y$ or $x \prec z$ for all $z$ in the space. This conditions puts a limit on the level of incompleteness of the relation.[37]

Theorems 5 and 6 presented below are an attempt to identify the most parsimonious setting to obtain transitivity, one that has no reference to completeness at all. As alluded to in the beginning of this subsection, they are the next step from the point where Theorems 1 to 4 have brought us. They extract what all can be said about transitivity without any other considerations: suitable topological assumptions on the choice set and on the preferences defined on it, allow us to both reconstruct some of these relationships and also bring new ones to light. But in their single-minded concern with the postulate of transitivity, as in Eilenberg (1941), Sonnenschein (1965, 1967) and Sen (1969), one of our findings, through decisive counterexamples, is that equivalence has to be necessarily jettisoned. As Proposition 1 above, they limit themselves only to the forward direction.

**Theorem 5.** For any continuous binary relation $R$ on a topological space, with $I$ and $P$ denoting its symmetric and asymmetric parts, the following statements are valid.

(a) If the space is connected, then

(i) semi-transitivity of $R$ is equivalent to negative transitivity of $P$ ($PI \land IP \Leftrightarrow NP$),
(ii) semi-transitivity of $R$ implies transitivity of $P$ ($PI \land IP \Rightarrow PP$),
(iii) transitivity of $R$ implies negative transitivity of $P$ ($T \Rightarrow NP$),
(iv) transitivity of $R$ is equivalent to its semi-transitivity and transitivity of $I$ ($T \Leftrightarrow PI \land IP \land II$).

[36] We warn the reader of the difference in terminology; see Footnote 19 above.
[37] As we shall have occasion to see below, all this supports the quotation from Wakker in the next section about the importance of judging the conditions in combination.
Figure 3: The Interdependence of Different Transitivity Conditions

(b) If the sections of $I$ are connected, then

(i) transitivity of $I$ implies semi-transitivity of $R$ ($II \Rightarrow PI \land IP$),

(ii) transitivity of $R$ is equivalent to transitivity of $P$ and of $I$ ($T \Leftrightarrow PP \land II$).

Theorem 5(a) illustrates the essentially of semi-transitivity by showing that a continuous and semi-transitive relation defined on a connected space implies $PP$ and $NP$. Whereas Theorem 5(b) illustrates the essentially of the transitivity of $I$ by showing that for any continuous relation whose $I$ has connected sections, $II$ implies semi-transitivity. Sonnenschein (1965, Theorem 4) uses connectedness of the sections of the indifference relation in order to obtain semi-transitivity of a complete and continuous relation whose symmetric part is transitive. All in all, Theorem 5 provides a complete picture of the relationship between different transitivity conditions without using the completeness of the relation. This is all to say that under suitable topological assumptions, the transitivity of $I$ implies the remaining transitivity conditions.

A natural question arises regarding the backward direction, and hence the possibility of equivalence. Such an equivalence can be written in two parts, one for the setting where $X$ is connected, and the other when for the sections of $I$ are connected. The following simple example suggests that both are false.

Example 4. Assume $X = \{a, b\}$ is endowed with the discrete topology. Then any relation is continuous and satisfy all of the six transitivity properties we use except $T$ and $II$. Hence conditions (i) to (iii) in part (a) hold. It follows from $PP$, $IP$ and $PI$ hold for any relation on $X$ that $T$ is equivalent to $II$, hence condition (iv) holds. However, it is clear that $X$ is disconnected.
The statement of the converse related to the sections of $I$ needs some elaboration. We can state the equivalence theorem for this part as follows: “For any continuous binary relation $R$ on a topological space, the sections of its $I$ are connected if and only if $R$ satisfies (i) and (ii).” The relation $I = X^2$ satisfies both conditions, however its sections are disconnected.

We now move on to the formalities required for the formulation of our interdependence result, and follow Sonnenschein (1965) in invoking two additional concepts: path-connectedness and the so-called Phragmen-Brouwer property, henceforth PBP. To be sure, Theorem 5 above also connects to both Theorems 3 and 4 (without the completeness assumption of course) of Sonnenschein. In his Theorem 7, Sonnenschein (1965) uses path-connectedness of the upper sections of the relation in order to obtain semi-transitivity of a complete and continuous relation with a transitive symmetric part defined on a topological space with the PBP. Theorem 6, like Theorem 5, provides a complete picture of the relationship between different transitivity conditions without using the completeness of the relation. Note that unlike Sonnenschein, we assume both upper and lower sections of the relation are path-connected. When a relation is complete, $IP$ and $PI$ are equivalent. It is easy to see that path-connectedness of the upper sections implies the latter property, but we need it for the lower sections to establish the former. Noting that an “example of a class of sets for which the PBP holds is the collection of all convex sets in Euclidean $n$-space,” Sonnenschein nevertheless proves a special case of his result that “does not rely on the rather deep PBP.”

A comprehensive discussion of the PBP, a property nothing if not elusive, is furnished in Wilder (1949, Chapter II.4), and we refer the reader to it.

For the purposes of this work we shall make do with the following definitions.

**Definition 12.** We call two non-empty subsets $A, B$ of a topological space $X$ separated if $\bar{A} \cap B = \emptyset = A \cap \bar{B}$. We say that a set $A \subset X$ separates $x$ from $y$ if $x$ and $y$ lie in different components of $X \setminus A$. We say that $X$ has Phragmen-Brouwer property if for all separated open sets $A, B \subset X$ and all $x \in A, y \in B$, there exists a connected subset of $X \setminus (A \cup B)$ which separates $x$ and $y$. A path in $A \subset X$ is a continuous function $s : [0, 1] \to A$, where $[0, 1]$ is endowed with the usual topology. A set $B \subset X$ is path-connected if for all $x, y \in B$ there exists a path $s : [0, 1] \to B$ such that $s(0) = x, s(1) = y$.

The following example illustrates a subset of Euclidean space which does not have the PBP. It is well-known that in addition to a convex subset, the Euclidean $n$-sphere, $n > 1$ satisfies this property, and so by necessity, our example is based on $n = 1$.

**Example 5.** Let $X = \{x \in \mathbb{R}^2 | x_1^2 + x_2^2 = 1\}$ be endowed with the usual Euclidean topology. Let $x, y$ be two distinct points in $X$. Let $A$ and $B$ be the two components of $X \setminus \{x, y\}$. Then $A$ and $B$ are separated open sets. However, any connected subset of the subspace $\{x, y\}$ does not separate any $a \in A$ and $b \in B$. 

---

38 See Sonnenschein (1965) Footnote 4 and Theorem 7A) where the alternative proof of the special case is justified on the grounds of furnishing “an interesting technique.”

39 See, in particular, (Wilder, 1949, Theorems 4:12 and 9.3); also Dickman Jr. (1984), Brown (2006), Brown and Camarena (2015) and their references for modern work on the property. Note that the PBP is not a strengthening of the connectedness assumption since any space with the discrete topology satisfies it, but rather a different separation property.
With all the preliminaries behind us, we can finally present the result showing that the connectedness of the sections of \( I \) in Theorem 5(b) can be replaced by the assumptions that \( R \) has path connected sections and that the space has PB property.

**Theorem 6.** Let \( X \) be a topological space with the Phragmen-Brouwer property. Then for any continuous binary relation \( R \) on it, with \( I \) and \( P \) denoting its symmetric and asymmetric parts, the following statements are valid:

(i) If \( R \) has path-connected upper sections, then transitivity of \( I \) implies \( PI \) (\( II \Rightarrow PI \)).

(ii) If \( R \) has path-connected lower sections, then transitivity of \( I \) implies \( IP \) (\( II \Rightarrow IP \)).

(iii) If \( R \) has path-connected sections, then transitivity of \( R \) is equivalent to transitivity of \( P \) and of \( I \) (\( T \Leftrightarrow PP \land II \)).

Again, a natural question arises regarding the backward direction, and hence the possibility of equivalence. We can formulate an equivalence conjecture as follows: "A topological space has the PBP if and only if any continuous binary relation \( R \) on it satisfies (i) to (iii)." Unlike connectedness, working with a topology with a lot of open sets will not yield a counterexample to PBP; in fact we have to work with a topology with few open sets. But we cannot also work with a very poor topology in order to obtain a counterexample to the property since there may not be enough separated sets. In any case, the following example shows that the converse of Theorem 6 is false.

**Example 6.** Let \( X = \{a, b, c, d\} \) be the choice set which is endowed with the topology \( \tau = \{\emptyset, \{a\}, \{c\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}, X\} \). Note that \( \{a\} \) and \( \{c\} \) are separated open sets. Moreover, \( \{b, d\} \) is disconnected and \( \{a, b, c\}, \{a, c, d\}, X \) are connected. Then any connected subset of \( \{b, d\} \) does not separate \( a \) and \( b \). Therefore, \( X \) does not have the Phragmen-Brouwer property.

Next, we show that any continuous binary relation \( R \) on \( X \) satisfies conditions (i) to (iii) of Theorem 6. Note that if we prove conditions (i) and (ii), then Proposition 2 implies condition (iii). To this end, assume that the symmetric part of the relation is transitive. We first show that path-connectedness of the upper sections of \( R \) implies that \( PI \) holds. Note that the collection of closed sets in \( X \) is \( \{\emptyset, \{b\}, \{d\}, \{b, d\}, \{b, c, d\}, \{a, b, d\}, X\} \). It is easy to show that among the closed sets, only \( \{b, d\} \) is not path-connected. Therefore, the upper sections of \( R \) must belong to the collection \( \{\emptyset, \{b\}, \{d\}, \{b, c, d\}, \{a, b, d\}, X\} \). Since \( I(x) = R(x) \cap R^{-1}(x) \) for all \( x \in X \) and \( R \) has closed sections, therefore \( I(x) \) belongs to the collection \( \{\emptyset, \{b\}, \{d\}, \{b, d\}, \{b, c, d\}, \{a, b, d\}, X\} \) for all \( x \in X \). Note that all members of this collection is connected except \( \{b, d\} \). Then for any relation \( R \) on \( X \) with \( I(x) \neq \{b, d\} \) for all \( x \in X \), it follows from Theorem 5(b) that \( R \) is semitransitive, hence condition (i) holds.

We now show that if a continuous binary relation \( R \) on \( X \) has path-connected upper sections and a transitive symmetric part, then \( I(x) \neq \{b, d\} \) for all \( x \in X \). Assume that for some \( R \) there exists \( x \in X \) such that \( I(x) = \{b, d\} \). Then the transitivity of \( I \) implies that \( x = b \) or \( x = d \). Let \( x = b \). Since \( R(b) \) is closed, path-connected and \( \{b, d\} \subset R(b) \), therefore \( R(b) \)

---

40In a connected and locally connected space, path-connectedness of the sections of a complete relation \( R \) implies that \( I \) has connected sections. Moreover, connectedness of the sections of \( I \) in a connected space implies that \( R \) has connected sections, which is weaker than path-connectedness. See [Wilder, 1949] Theorem 4.5 on p. 49 and Theorem 9.9 on p. 20 for the proofs.
is equal to one of \{b, c, d\}, \{a, b, d\} and \(X\). Let \(R(b) = \{b, c, d\}\). Then it follows from \(R\) has closed sections and \(I(b) = \{b, d\}\) that \(R^{-1}(b)\) is equal to either \{b, d\} or \{a, b, d\}. In both cases, \(P(b) = \{c\}\). Since \(b \in P^{-1}(c)\) and \(P\) has open sections, therefore \(P^{-1}(c)\) must contain \(c\), which furnishes us a contradiction. Now let \(R(b) = \{a, b, d\}\). Then \(R^{-1}(b)\) is equal to either \{b, d\} or \{b, c, d\}. In both cases, \(P(b) = \{a\}\). Since \(b \in P^{-1}(a)\) and \(P\) has open sections, therefore \(P^{-1}(a)\) must contain \(a\), which furnishes us a contradiction. Finally, let \(R(b) = X\). Then \(R^{-1}(b) = \{b, d\}\). Hence, \(P(b) = \{a, c\}\). Since \(b \in P^{-1}(a)\) and \(P\) has open sections, therefore \(P^{-1}(a)\) must contain \(a\), which furnishes us a contradiction. Therefore, \(x \neq b\). Analogously, the case \(x = d\) yields contradiction.

An analogous argument shows that path-connectedness of the lower sections of \(R\) implies that \(IP\) holds. Therefore, the example is complete.

Theorems 5 and 6 are in-line with the four theorems we present in the previous section. In the forward direction of those theorems we show that under suitable topological assumptions on the choice space and the preferences defined on it, non-triviality and weak forms of transitivity imply full transitivity as well as the completeness of the preferences. The proposition above and these two theorems present the relationship among different transitivity conditions without referring to the completeness assumption. In particular, assertions (a) and (b) in Theorem 5 are analogous to assertions (c) and (d) of Theorems 1 and 2 above.

We conclude this section by observing that Theorems 5 and 6 have implications for the topological structure of the graph of a binary relation. A binary relation \(P\) on \(X\) is said to have open graph if \(P\) is open in the product topology on \(X \times X\). Bergstrom, Parks, and Rader (1976, Theorem 2 and Corollary) proved that an asymmetric and negatively transitive binary relation on a topological space has open sections if and only if it has open graph. The two theorems above show that we can weaken negative transitivity by strengthening the topological assumptions on preferences and on the space.

4 Applications of the Theory: A Brief Excursus

Results in pure theory are to be judged not only for their application to real-world problems, but also by new light they cast on the earlier theoretical results themselves: they allow, indeed enable and empower, one to really see what was already seen before. In this section, we do this through the help of two lenses: redundancy and hiddenness of hypotheses in a rather extensive antecedent literature. The postulates of completeness and transitivity serve as hypotheses to conclusions concerning results (i) on the representation of a preference relation by a real-valued (utility) function, (ii) in neoclassical consumer theory, (iii) in Walrasian and Cournot-Nash equilibrium theory, and (iv) in social choice theory; and it stands to reason that our six results and the examples would have some impact on these results. Redundancy, the removal of hypotheses unnecessary for a conclusion, is simply bringing Occam’s razor into play, a procedure with a long and rich lineage in mathematical investigation. It goes into what one means by generalizing a theorem. Hiddenness is somewhat more subtle, and more far-reaching. Rather then saying that a hypothesis can be eliminated from an assertion, it says that it is already incorporated in other

\[\text{Footnote 32}\] Gerasimou (2015) shows that reflexive, transitive and additive binary relation with closed upper sections has a closed graph, and hence, convexity of a relation has implications on its continuity; see Footnote 32 for further references and discussion on convexity.

22
hypotheses already assumed. We say that a particular assumption is hidden in a theorem when it is already implied by the others in the battery of assumptions constituting the hypothesis of the theorem. As such, showing that it can be removed does not lead to a generalization of the result, but merely to obtaining its clearer formulation and a more parsimonious expression. But it also includes making an implicit assumption or drawing out its fuller meaning of an assumption in the sense of uncovering the implications that are hidden in it. In this section, we work this distinction through the antecedent literature listed above.

With this background we return to Wakker’s (1988a) assertion that technical assumptions by themselves “do not entail obscurity and are not very bothersome.” His text continues beyond our truncated citation.

However, when one requires a list of conditions, then one should not judge each condition separately, but one should judge the conditions in combination. We give an example where, paradoxically, each individual condition involved is not falsified by the observations, but the combination of the conditions is falsified. As it turns out in this example, continuity, in the presence of other conditions, may have empirical meaning. It is very bothersome that usually the exact empirical meaning of simplifying non-necessary conditions such as continuity is unclear.

However, rather than Wakker’s example, focus simply on a decision-maker with a choice set of $\mathbb{R}_+$, and who, when faced with the options $\{0, 1\}$, $\{1, 2\}$, and $\{2, 0\}$, chooses 1, 2 and 0 respectively. Since his choices do not falsify anti-symmetry and non-triviality, and, irrespective of the number of additional but finite observations, cannot falsify continuity and exhibit non-transitivity, an appeal to the implication $\alpha \Rightarrow (c)$ in Theorem 2 and a presumption of anti-symmetry would falsify continuity. And this is all that is being asserted about continuity, in the presence of other conditions, having both behavioral and empirical implications.

Finally, before taking our theorems to the antecedent literature, we take notice of three rather current references that engage with Schmeidler’s theorem in directions somewhat oblique to ours. Cerreia-Vioglio and Ok (2018) work on the structure of an incomplete and non-transitive preference relation by identifying and examining the properties of largest transitive sub-relation of it. They assume partial continuity of the relation, and are well aware that full continuity would otherwise imply completeness. However, they make no connection to the work of Eilenberg, Sonnenschein and Sen. Nishimura and Ok (2018) work with a preference structure consisting of two relations, one transitive but incomplete and the other complete but non-transitive. In this interesting paper, they redo notions of maximality, social choice and decision theory. It would be interesting to see whether the results reported above have any relevance to these lines of work.

A celebrated example of the former is Malinvaud (1952); also see Footnote 51 below. An example of the latter is a reconsideration of Shafer’s non-transitive consumer that we present in Section 4.2, and of Schmeidler’s existence theorem in Section 4.3; also see Footnote 23 below. What we do not mean by the word hiddenness is the sense that is given to it by Tourky and Yannelis (2000). Their usage is orthogonal to ours, and perhaps also to redundancy: they refer to an assumption being “hidden” as one that is explicit but essential in the sense that eliminating it would require additional hypotheses to obtain the same consequences.

The italics are the author’s, and the example is Wakker (1988a, Example 7.3). For further discussion on the empirical implication of technical assumptions, see Pfanzagl (1971, pp. 107-108) and Narens (1985, pp. 83-84).
We now turn more sure-footedly to papers that are directly impacted by our theorems. However, it is important to keep in mind that it is not our intention to give a comprehensive catalogue of each and every result to which our six theorems can be fruitfully applied. We content ourselves with laying out excursionary directions, and leaving it to the reader to pursue in more detail the direction that interests him or her. Furthermore, we emphasize the “forward” direction in our applications.

4.1 Real-valued Representation of Preferences
In Barten and Böhm (1982, Section 5), the authors present a representation theorem that they ascribe to Debreu, Eilenberg and Rader. We begin with the simple version due to Eilenberg (1941, 6.1)” in his seminal paper.44

Every non-trivial, anti-symmetric, complete, transitive and continuous preference relation on a connected and separable topological space has a continuous utility representation.

It follows from implication \( (a) \Rightarrow (c) \) in Theorem 2 that both completeness and transitivity follow from the remaining assumptions of his theorem, hence they are hidden assumptions of his statement. Therefore, we can equivalently state his theorem by dropping both completeness and transitivity assumptions. Moving on to the variant of Eilenberg’s result in Debreu (1954, Theorem I), one that dispenses with his anti-symmetry assumption, it is already well-understood that as the consequence of the theorems of of Sonnenschein and Schmeidler, either completeness or full transitivity is hidden in Debreu’s assertion. The point that we wish to emphasize here is that as a consequence of \( (a) \Rightarrow (e) \) in Theorem 2 not one but that both completeness and full transitivity are hidden in Debreu’s theorem. Moreover, Wakker (1989, p. 42), as well as Fishburn (1972, pp. 65-66), observe the hiddenness of either one of completeness and transitivity for the existence of a utility representation, and our point is again that both of these assumptions are hidden in this line of the literature.46

This being said, it is important to emphasize that applications that work with strict preference relations as their primitive, such as Peleg (1970) and Majumdar and Sen (1976), or those that simply drop both completeness and transitivity such as Nishimura and Ok (2016), also do not fall under the ambit of this paper. The same is true for applications relying on linear structures: they do not fall under the exclusively topological rubric of this paper, and require different mathematical techniques and tools, and we investigate such structures elsewhere; see Galaabaatar, Khan, and Uyanık (2018). But again, just because the setting is one of uncertainty, it does not mean that the results are necessarily out of the bounds of our treatment.

44 Also see Wakker (1988b). For comprehensive treatments, see Bridges and Mehta (1995), Mehta (1998), Hervés-Beloso and del Valle-Inclán Cruces (2018) and their references.

45 Note that Eilenberg (1941, 2.1) observes the hiddenness of transitivity. Moreover, our insertion of non-triviality in the statement of Eilenberg’s theorem do not do violence to his original statement because of the consideration emphasized in Footnote 21 and in the text it footnotes.

46 For additional references, the interested reader may see, for example, Debreu (1960), Gorman (1968), Bridges and Mehta (1995) and Vind (2003).

47 While Peleg’s representation provides only forward direction representation result for incomplete preferences, Majumdar-Sen’s approach provides both directions.

48 This literature stems from the classic work of Herstein and Milnor (1953), where the topological structure is restricted to that on the unit interval, and the overall thrust is algebraic.
elaborated in this paper. In those that rely on the continuity assumptions which we use in this paper, as, for example [Karni 2014, Theorem 1.3], the implication $(a) \Rightarrow (e)$ in Theorem 2 yet again implies that both completeness and full transitivity are hidden assumptions.

There has been a surge of recent work on decision theory without the completeness postulate in a setting where the choice set is convex, and therefore connected. Our results, specifically Theorem 2, then suggest that any result for a setting with incomplete preferences, must of necessity weaken the continuity assumption. In this line of literature, most papers drop the assumption of open sections; see for example [Ghirardato, Maccheroni, Marinacci, and Siniscalchi 2003, Dubra, Maccheroni, and Ok 2004] and [Evren 2014]. The implication $(a) \Rightarrow (d)$ in Theorem 4 then yields the consequence that the preference relation must be fragile! To be specific, [Dubra, Maccheroni, and Ok 2004, Expected Multi-Utility Theorem] satisfies all of the assumptions of the implication $(a) \Rightarrow (d)$ in Theorem 4, and hence the preference relation the authors work with is fragile, i.e., there exist $(x, y) \in P$ such that every open neighborhood of $(x, y)$ contains non-comparable elements. We can, moreover, use the other assumptions of their theorem in order to obtain more information about the structure of their incomplete preferences as follows. Their theorem implies that there exists a closed and convex set $U$ of continuous utility functions that represents $R$. Fragility implies that for all $u \in U$, $u(x) < u(y)$ and any open neighborhood of $(x, y)$ contains $(x', y') \in V$ such that $u'(x') > u'(y')$ (note that weak inequality contradicts fragility) for some $u' \in U$. Since the space of utility functions is endowed with the sup-norm topology, it is easy to show that there exists an open neighborhood $U$ of $u'$ such that $v'(x') > v'(y')$ for all $v' \in U$. Hence, there is an open set of utility functions that rank $x'$ and $y'$ opposite of the ranking of $x$ and $y$.

4.2 Shafer’s Non-Transitive Consumer

In a direction initiated by [Sonnenschein 1971], Shafer [1974] re-works the neoclassical theory of demand for a consumer with incomplete and possibly non-transitive preferences. Our results impact his work not by bringing out any redundancies, but bringing in what non-transitivity assumption fully entails in the light of his other hypotheses. We show that if the preferences of Shafer’s non-transitive consumer satisfies a little bit of consistency, then it is fully destructive of all non-transitivity. This is to say that, if any of the four dis-aggregations $II$, $IP$, $PI$ and $PP$ of $R$ holds, then the remaining three also hold under his assumptions. And so Shafer’s non-transitive consumer has to be, by necessity, a fundamentally non-transitive agent. We elaborate this claim in the following paragraph; it seems to have been missed in the literature.

Shafer assumes that the consumer has a complete, continuous and strictly convex preference relation $R$ on $\mathbb{R}_+^n$. Strict convexity implies that $R$ is non-trivial and has path-connected upper sections, and convexity of $\mathbb{R}_+^n$ implies that the choice set is connected. We can now develop the argument for our assertion under three cases. First, if $R$ satisfies $II$, then by Theorem 6(i), $II$ implies that $PI$ holds. Then it follows from completeness of $R$ and [Sen 1969, Theorem I], which we illustrate in Figure 3 that $IP$ also holds. Hence $R$ is semi-transitive and its symmetric part is transitive. Then the implication $(a) \Rightarrow (e)$ in Theorem 2 implies that $R$ is transitive. Second, assume $R$ satisfies $PI$. Then Sen’s theorem implies that $IP$ and $II$ hold. As above, the transitivity of $R$ follows from Theorem 2. The proof is analogous if $R$ satisfies $IP$. Finally, if $R$ satisfies $PP$, the transitivity of $R$ follows from [Sonnenschein 1965, Theorem 5].

[Gerasimou 2010] re-works Shafer’s theory of the non-transitive consumer by also elimi-
nating the completeness postulate and weakening the continuity assumption. Under this sparser structure, a consumer can satisfy some form of transitivity without destroying all forms of transitivity. A consumer can admit consistency in some registers without consistency in all. However, what emerges is the essentiality of the transitivity of $II$: even if we drop completeness in Shafer’s model, under full continuity assumption (the sections of $R$ are closed and of $P$ are open) and the convexity assumption of Shafer, our Theorems 2 and 6 that if $II$ holds then the relation $R$ has to be transitive. Therefore, Shafer’s non-transitive consumer necessarily violates the transitivity of $I$ even if she has incomplete preferences.

We conclude this section with the observation that the points that we make in the first two paragraphs above could already have been made in 1965 drawing only on the results of Sonnenschein (1965); as such, they do not require the full power of our results. In any case, they shed new light on what hiddenness may entail.

4.3 Walrasian Economies and Normal-Form Games

In this section, we first illustrate the hiddenness of completeness and full transitivity assumptions in the results on the existence of an equilibrium in Walrasian economies and in normal form games. Then we show how these observations carry over to the economies with indivisibilities.

The classical equilibrium existence results in Walrasian economies assume that the choice set of each consumer is a convex subset of the Euclidean space and that each consumer has a complete, transitive and continuous preference relation on the choice set; see for example Arrow and Debreu (1954, Theorem 1) and Debreu (1982, Theorems 5 and 8). Moreover, one of the following two properties is assumed: monotonicity or non-satiation. Since the former implies more is better for each consumer and the latter that no consumer has a best element in her consumption set, the preference relation of each consumer is non-trivial. Since convexity of the choice set implies its connectedness, all of the assumptions of the implication (a) ⇒ (e) in Theorem 2 are satisfied. Hence both of the completeness and full transitivity postulates are hidden for the existence of a Walrasian equilibrium. However hiddenness is also present in many other results in this literature. We invite the reader to check out theorems on the existence of a Walrasian equilibrium with continuum of agents, or externalities, or public goods, or infinite dimensional commodity spaces; see for example Mas-Colell and Zame (1991), Khan and Sun (2002), and McKenzie (2005). However, one can go beyond hiddenness to make points akin to that made above regarding Shafer’s non-transitive consumer. If, for example, the continuity and transitivity assumptions made in Schmeidler (1971) are also made in Schmeidler (1969), the existence of competitive equilibria in markets with a continuum of traders and incomplete preferences follows as a straightforward consequence of Aumann’s existence theorem, and hardly requires an additional independent proof.

In finite games, Nash (1950b) assumes each player has a finite number of pure strategies

---

49This connects us to Luce’s (1956) semi-order, which is further elaborated in Fishburn (1970), a connection that we hope to explore in future work.

50We also single out in this connection, Moldau (1996, Proposition 1). This claims that transitivity on every closed interval of a preference ration defined on a linear space follows from a weak convexity assumption, and as such, hidden by the convexity postulate. Since we are limiting ourselves to the topological register, we hope to engage this claim elsewhere; also see footnote 32 for further references and discussion regarding the convexity assumption.

26
and that the preferences of each player defined on the set of all probability distributions on the set of pure strategies satisfy the axioms of the classic expected utility representation theorem.\textsuperscript{51} Even though he assumes a weaker continuity assumption than we use, in the presence of other assumptions it implies our stronger continuity assumption; see Dubra (2011). Therefore, both completeness and full transitivity assumptions are hidden in Nash’s results. For games with continuum of actions, Debreu (1952, Theorem) follows a different path (which eliminates randomization) and proposes a generalizes Nash’s theorem by assuming the choice space is a convex and compact subset of a Euclidean space and the preferences defined on it are complete, transitive, continuous and convex.\textsuperscript{52} Then, as illustrated above, the implication $(a) \Rightarrow (e)$ in Theorem 2 suggests the hiddenness of both completeness and full transitivity assumptions.\textsuperscript{53} Therefore, we can, equivalently re-state Debreu’s theorem by dropping the completeness and weakening the transitivity assumption.

Although connectedness, or more precisely convexity, of the choice space is a common assumption in economics, an important class of models which study markets with indivisibilities naturally assume disconnected choice spaces; see for example Dierker (1971), Broome (1972), Mas-Colell (1975, 1977), Khan and Yamazaki (1981) and Thomson (2011). In these models, there are $\ell$ indivisible goods and 1 divisible good. For illustration, assume $\ell = 1$. In particular, assume the consumption set of a consumer $X = \mathbb{R}_+ \times \mathbb{Z}_+$ consists of money, which is perfectly divisible, and an indivisible good. Let $\succeq$ denote the preferences of the consumer on $X$. Let $\sim$ and $\succ$ denote the symmetric and asymmetric parts of $\succeq$, respectively. The following assumptions are standard in these models.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{economies.png}
\caption{Economies with Indivisible Commodities}
\end{figure}

\textsuperscript{51}As shown by Malinvaud (1952), von Neumann and Morgenstern (1947) do not state the independence axiom in their axiomatization. Nash (1950a) and Marschak (1950) independently provide complete axiomatization of expected utility; see Bleichrodt, Li, Moscati, and Wakker (2016), and also Footnote 42 above.

\textsuperscript{52}We summarize the assumptions of the version of Debreu’s theorem re-stated in Arrow and Debreu (1954, Lemma).

\textsuperscript{53}Note that without non-triviality, the existence of an equilibrium is triviality, hence the non-triviality assumption is non-restrictive.
(A1) For all \( a \in \mathbb{Z}_+ \), \((m, a) \succ (m', a)\) whenever \( m > m' \).

(A2) For all \( a, a' \in \mathbb{Z}_+ \), there exist \( m, m' \in \mathbb{R}_+ \) such that \((m, a) \sim (m', a')\).

(A2') For all \((m, a), (m', a') \in \mathbb{R}_+ \times \mathbb{Z}_+ \), there exists \( \lambda \in \mathbb{R}_+ \) such that \((m + \lambda, a) \succeq (m', a')\).

Assumption (A1) is called strict monotonicity in the divisible commodity, (A2) possibility of compensation and (A2') overriding desirability of the divisible commodity. The latter two assumptions are substitutes and used interchangeably in the literature. Figure 4 illustrates an economy with indivisible goods which satisfy these standard assumptions.

**Broome (1972) Theorem 4.11** shows that if a complete, transitive and continuous preference relation on \( X \) satisfies Assumptions (A1) and (A2) along with the assumptions that the consumption set and the preferences satisfy some additional topological and linear structure properties, then there exists an approximate equilibrium. We now illustrate that completeness and full transitivity are hidden in the hypothesis of Broome’s theorem. Consider a semi-transitive and continuous preference relation \( \succeq \) on \( X \) whose symmetric part is transitive. It is clear that \( X \) is disconnected – for each \( a \in \mathbb{Z}_+, C_a = \mathbb{R}_+ \times \{a\} \) is a component of \( X \). It follows from (A1) that there are strictly comparable elements within each component and from any of (A2) or (A2') that there are weakly comparable elements across different components. Therefore, (A1) along with any of (A2) or (A2') imply \( \succeq \) is componentwise non-trivial. Therefore, Proposition 1 implies that both completeness and full transitivity are hidden assumptions in Broome’s theorem.

In this line of research, [Mas-Colell (1974), Shafer and Sonnenschein (1975)] and their followers work with incomplete and non-transitive preferences, and assume only a strict preference relation, one with open graph or open sections. Hence, our results do not have implications for their work, as indeed, they also do not impact the literature on games with discontinuous preferences pioneered by [Dasgupta and Maskin (1986) and Reny (1999)], or the reformulation of Cournot-Nash equilibria presented in [Khan and Sun (1990)].

### 4.4 Other Potential Applications

Our final tripartite subsection is as much an invitation to the reader to apply the results reported above, as it is a laying-out of directions for further work. We begin by asking whether the added specification of finite choice sets allows, if not a sharpening of the results, an opening into other productive directions. We then move on to other potential applications to classical social choice theory and to graphs and networks.

**Finite Choice Sets**

Eilenberg’s remarkable paper notwithstanding, modern decision and social-choice theory has focused on compactness rather than on connectedness, and has limited itself to a finite setting for an exploration of ideas. It is thus natural to ask whether the results reported in Section 3 of the paper can be extended to provide sharper results simply by seeing a finite set of \( k \) alternatives as a \( k \)-connected set. One rather obvious impediment to this is the fact that the hypotheses, say of Theorem 1, require \( P \) to have a strict relationship within every component, a property precluded by singletons in a finite setting. The question then reduces to whether indifferent alternatives can simply be “factored out”?

---

\[54\]See Assumptions 2.1 (excluding (e) and (f)), 2.6, 2.7 and 2.8 in [Broome (1972)].
We present a result that gives the negative answer to this question: we show that one cannot define a non-trivial, complete, semi-transitive and continuous binary relation on a finite and connected topological space.

**Proposition 3.** If \( R \) is a complete, semi-transitive and continuous binary relation on a topological space \( X \), then the quotient topology on \( X|I \) is Hausdorff.

For a finite choice set, this proposition shows that the quotient topology with respect to the symmetric part of a continuous binary relation is discrete, hence for any connected set in the original space, all of its the elements must be indifferent to each other. As such this direction is a dead end and substantial rethinking is needed, perhaps along the lines of the literature stemming from the application of convex geometry recently developed in the imaginative contribution of Richter and Rubinstein (2015); see Edelman and Jamison (1985) for an early survey.

**Collective Choice and Social Welfare**

Moving on to the substance of the theory itself, the last two decades have seen a substantial maturing of the theory of social choice and welfare; see Arrow, Sen, and Suzumura (1995, 1996), Arrow, Sen, and Suzumura (2002, 2011), Fleurbaey, Salles, and Weymark (2006), Fleurbaey and Blanchet (2013), Suzumura (2016), Sen (2017) and their references.

Leaving aside the rich philosophical and technical subtlety of this literature, the point is that it involves establishing the consistency, aggregation if one prefers to be more specific, of two types of binary relations: preferences of individuals and those of the group that those individuals constitute. The question reductively reduces to asking whether group and social preferences are “nice” when the individual preferences are “nice”? And to be sure, the adjectives formalizing these valorizations necessarily reduce to postulates such as completeness and transitivity, and thereby bring into play the six theorems that we present above. To get down to specifics, one can delineate how the the hiddenness and redundancy criteria impact the theorems of Harsanyi (1955) and Samuelson (1981), or factor into the recent exercise that Magyarkuti (2010) has carried out also in a purely topological register.

Some what more obliquely, but perhaps even more promisingly, Bernheim and Rangel (2009), propose “replacing the standard revealed preference relation with an unambiguous choice relation: roughly, \( x \) is (strictly) unambiguously chosen over \( y \) (written \( xP^*y \)) iff \( y \) is never chosen when \( x \) is available.” They write

If one thinks of \( P^* \) as a preference relation, then our notion of a weak generalized Pareto optimum coincides with existing notions of social efficiency when consumers have incomplete and/or intransitive preferences ... [t]hough \( P^* \) need not be transitive, it is always acyclic, and therefore suitable for rigorous welfare analysis.\(^5\)

As such, the authors do not take a stand on a particular story about why choices violate, for example, WARP, the weak axiom of revealed preferences; but by subsuming behavioral economics as theories that violate WARP, present explorations that surely fall within the rubric of the ES program.

\(^5\)See Bernheim and Rangel (2009, pp. 75-76); also their Theorem 7 which they see as following directly from standard results of Fon, Mandler, Otani, Rigotti and Shannon.
Reverting to finite choice sets and to convex geometry and its implications for partially ordered sets and in graphs, we close this subsection on potential applications by drawing the reader’s attention to the analysis of networks, be they social, economic, political or anthropological. This is a very active field of microeconomic theory; see, for example, Jackson (2008) and his references.

A simple application of our result on $k$-connectedness to information transmission networks, as culled from Jackson (2008, Chapter 7 and Section 13.2) and Newman (2006), draws on the observation that transmission of information in small communities, villages and such, can be seen as a network whose node can be seen as a component in the technical sense that we have given to the word in the work reported in this paper. Completeness and transitivity of communication relationships, in the sense of a node $x$ directly or indirectly communicating with the node $y$, seems formalizable and directly relevant. The continuity of the communication relationship in terms of the distance between villages, the topology being defined by this distance. In this simple formulation, let $X \subset \mathbb{R}^2$ be the set of individuals which is endowed with the Euclidean topology. The distance between different points represent the distance of the individuals. Let each node (village) denoting a component of $X$. Define a binary relation $\succeq$ on $X$ as follows: $x \succeq y$ if and only if $x$ (directly or indirectly) transmits information to $y$. Under the usual assumptions, if there are pairs of individuals within and across villages who communicate (receive or transmit information to the others), then everybody in the society communicates and the information transmission relation is transitive. On the other hand, if we assume transitivity, then we can weaken the assumption of the existence of a communication link between any pair of villages with that of a chain of communication links among all the villages. To be sure, these are fragmented observations that cry out for a systematic treatment.

5 Concluding Remarks

Looked at from far enough, this paper can be easily summarized as revolving around two-and-a-half theorems: the first, second and fourth can be combined and collectively presented for the case $k = 1$, as one big portmanteau equivalence theorem offering a characterization of topological connectedness; the third result, combined with the preceding two for the case $k = 2$, as a characterization of 2-connectedness. The fifth and sixth results, in giving sufficient conditions for the refinements of transitivity, can be seen as half a theorem of an equivalence that does holds only in one direction.

But one or many, the six theorems, and the propositions that supplement them, furnish an overview of a diverse literature in microeconomic theory that is bracketed by a rich mathematical and philosophical literature. Our synthetic treatment of the three remarkable contributions that we sight as foundational, facilitates the reading of past, somewhat neglected, work as well as allows a deeper appreciation of how current contributions fit into lines of inquiry with a long-established lineage. In showing the sufficiency and necessity of topological connectedness for both completeness and/or transitivity under one rubric, we generalize and unify these three

---

56 In some sense, we have kept doing versions that approach such a portmanteau theorem.
57 The reader is referred to the counterexample a little above the statement of Theorem 3.
58 The reader is referred to Temkin (2015), Sen (2017) and their references; also to Anand (1987) and Anand (1993).
foundational theorems. To be specific, and perhaps to overly belabour the point, we are not aware of any paper in the literature following Schmeidler that asks for conditions on the topology that follow from his, and Eilenberg’s and Sonnenschein’s, behavioral consequences. As such, our results in providing a characterization of topological connectedness for behavior, and therefore for its indispensability, are of interest in and of themselves. We are also not aware of topologizing, and thereby bringing into a productive relationship, the influential non-topological results of Sen.

Modern decision theory, as charted out by von Neumann, Savage and Anscombe-Aumann, is proving fundamental for both behavioural economics and more generally in issues of empirical inference but to the extent that it can be projected to the question of a numerical representation of a preference relation over a set of objects, the ancillary structures both on the set of objects and on the preferences under investigation assume a paramount role. The objects could be probability measures, as in von Neumann-Morgenstern; or functions from a state space to a space of consequences, as in Savage; or functions from a state space to a space of probability measures on a space of consequences, as in Aumann-Anscombe; or induced relations on the subsets of a space of consequences, as in the menu choices of Kreps; or n-tuples of preferences over n-products of probability measures, as in the temporal lotteries of Kreps-Porteus; but whatever the objects, successful analysis has, of necessity, to involve a play on the assumptions made on the preferences and the objects on which these preferences may be defined. We have limited ourselves solely to the topological register, but already in 1967, Sonnenschein was bringing to bear linear considerations to his deductions concerning transitivity of preferences. To be sure, applications come with a veritable variety of structures, including the linear-algebraic one, with or without a finiteness specification, and how these all interact with each other is a fascinating question that will surely build on the topological treatment explored herein. What also merits emphasis is that this literature naturally dovetails into work on empirical microeconomic theory, experimental psychology, and philosophical investigation into the very meaning of transitivity, completeness and non-satiation, and thereby into specific formalizations of “rationality,” one of the more vexatious words of our times. We hope to take our results to this theoretical and applied subject matter next.

6 Proofs of the Results

The presentation of our results in Section 3 proceeded from the general to the particular: from full equivalence of the k-connected case to the 2-connected and connected cases (Theorems 1 to 59). As already pointed out in Footnote 3 in the context of the mathematical literature, Ward (1954) and Bridges and Mehta (1995) are important references subsequent to Nachbin’s pioneering monograph on “Topology and Order;” and McGehee (1992) for application to dynamical systems. The philosophical literature is immense but Anand (1987, 1993) furnish an admirable entry into issues concerning the rationality or the irrationality of transitivity and completeness; in addition to the references in Sen (2017) and Temkin, see Tullock (1964). To be sure, any sharp lines to differentiate between the substantive and the technical eventually leads to sterility.

For the first see Bernheim and Rangel (2009) and their references; and for the second, see Pfanzagl (1971) and Narens (1985).

All this is now so much part of the folklore that detailed references are hardly necessary. But the reader can do no worse than begin with Fishburn (1972) and Gilboa (2009) on the one hand, and Mehta (1998) and Bridges and Mehta (1995) on the other. For current activity in the field, in addition to the three references with which we began Section 4, we refer the reader to Gerasimou (2017) and Strzalecki (2013) and their references.
and then finally to the forward (sufficiency) case establishing transitivity in Theorem 5. The motivation for this is simply that it simply gives the readership the option to read Theorems 1 and 2 for for the case $k = 1$, and indeed limit themselves only to the forward case. This enables it to see how more restricted settings lead to sharper conclusions. However this is not the best strategy for doing, and presenting, the proofs. It is the more concrete cases that are generalized. As such, we begin with the ancillary Proposition 2 to follow, and use it to prove Theorems 5 and 6. And then Theorem 5 is used in the proof of Theorem 1. These two theorems are used as an input in the proof of Theorem 2. The latter is then used to prove Theorems 3 and 4.

Proof of Proposition 2. We take each claim in turn.

(a) The sufficiency of $T$ is due to Sen (1969, Theorem I, assertion I.1). In order prove the necessity of $T$, assume $y \in R(x)$ and $z \in R(y)$. If $y \in R^{-1}(x)$ and $z \in R^{-1}(y)$, then $II$ implies $z \in I(x)$, hence $z \in R(x)$. If $y \notin R^{-1}(x)$, or $z \notin R^{-1}(y)$, or both, then it follows from $PP, PI, IP$ that $z \in P(x)$, hence $z \in R(x)$.

(b) Assume $y \in P(x)$ and $z \in P(y)$. It follows from $y \in P(x)$ and $NP$ that $z \in P(x) \cup P^{-1}(y)$. Since $z \in P(y)$ and $P$ is asymmetric, therefore $z \in P(x)$, hence $PP$ holds. Now, assume $y \in P(x), z \in I(y)$ and $z \notin P(x)$. It follows from $z \in I(y)$ that $y \notin P(z)$. Then $NP$ implies $y \notin P(x)$. This furnishes us a contradiction. Hence, $PI$ holds. An analogous argument implies $IP$.

(c) Let $X$ be a set with at least three elements and $R = \{(x, y)\}$ where $x, y \in X$ and $x \neq y$. It is clear that $R$ is transitive and $P = R$. It follows from $(x, z) \notin P, (z, y) \notin P$ and $(x, y) \in P$ for $z \neq x, y$ that $NP$ is not satisfied. Now define $R' = \{(x, y), (y, x)\}$. Then, $P' = \emptyset$, hence $NP$ holds. Since $(x, x) \notin R'$, therefore $T$ is not satisfied.

(d) We provide a proof by considering examples. Let $x, y, z, w$ be distinct elements of $X$. We first show $II$ is independent of $PP, PI, IP$. Define $R = \{(x, y), (y, x), (x, x), (y, y), (y, z), (z, x)\}$. It is clear that this violates $PP, PI, IP$, but not $II$. Next consider $R = \{(x, y), (y, x)\}$. This violates $II$, but not $PP, PI, IP$. Second, we show that $PI$ is independent of $PP, IP, II$. Define $R = \{(x, y), (y, z), (x, w), (w, x)\}$. This violates $PP, IP, II$, but not $PI$. Next consider $R = \{(x, y), (y, z), (z, y), (y, y), (z, z)\}$. This violates $PI$, but not $PP, IP, II$. The independence of $IP$ is analogously proved and the independence of $PP$ is illustrated in Sen (1969, Theorem I, assertion I.2). Third, we show that $PP, II$ are independent of $IP, PI$. Define $R = \{(x, y), (y, z), (z, y), (y, y), (z, z), (z, w)\}$. This violates $IP, PI$, but not $PP, II$. Next consider $R = \{(x, y), (y, x), (y, z), (x, z), (z, w)\}$. This violates $PP, II$, but not $IP, PI$. Fourth, we show that $II, PI$ are independent of $PP, IP$. Define $R = \{(x, w), (w, x), (x, x), (w, w), (x, y), (y, z)\}$. This violates $PP, IP$, but not $II, PI$. Next consider $R = \{(x, y), (y, z), (z, y)\}$. This violates $II, PI$, but not $PP, IP$. The independence of $II, IP$ and $PP, PI$ can be shown analogously.

The proof of Proposition 2 is complete.

Next we turn to the proof of Theorem 5. Before that we need the following definition and a lemma.

Definition 13. A partition of a set $X$ is a collection of non-empty and pairwise disjoint sets $\{A_{\lambda} \subset X \mid \lambda \in \Lambda\}$ such that $\bigcup_{\lambda \in \Lambda} A_{\lambda} = X$. A partition is open if all of its members are open, and a partition is closed if all of its members are closed.
Lemma 1. For all semi-transitive and continuous binary relation \( R \) on a topological space \( X \), all components \( C, C' \) of \( X \) and all \( x \in C, y \in C' \), if \( (x, y) \in P \), then \( C \cup C' \subset P(x) \cup P^{-1}(y) \).

**Proof of Lemma.** Pick, possibly identical, two components \( C, C' \) of \( X \) and \( x \in C, y \in C' \) such that \((x, y) \in P \). Then, \( x \in P^{-1}(y) \) and \( y \in P(x) \). Hence \( P(x) \cup P^{-1}(y) \) has non-empty intersections with both \( C \) and \( C' \). It follows from \( P \) has open sections that \( P(x) \cup P^{-1}(y) \) is open. Since \( R \) has closed sections, therefore \( R(x) \cup R^{-1}(y) \) is closed. If \( P(x) \cup P^{-1}(y) = R(x) \cup R^{-1}(y) \), then we have a subset of \( X \) which is both open and closed, and has non-empty intersection with both \( C \) and \( C' \). Since \( C \) and \( C' \) are components of \( X \), therefore \( C \cup C' \subset P(x) \cup P^{-1}(y) \). It remains to prove \( P(x) \cup P^{-1}(y) = R(x) \cup R^{-1}(y) \).

It is clear that \( P(x) \cup P^{-1}(y) \subset R(x) \cup R^{-1}(y) \). In order to show the reverse inclusion assume there exists \( z \in R(x) \cup R^{-1}(y) \) such that \( z \notin P(x) \) and \( z \notin P^{-1}(y) \). If \( z \in R(x) \), then it follows from \( z \notin P(x) \) that \( x \in R(z) \). Hence \((z, x) \in I \). It follows from \( IP \) and \( (x, y) \in P \) that \( z \in P^{-1}(y) \). This furnishes us a contradiction. If \( z \in R^{-1}(y) \), then it follows from \( z \notin P^{-1}(y) \) that \( z \in R(y) \). Hence \((y, z) \in I \). It follows from \( PI \) and \((x, y) \in P \) that \( z \in P(x) \). This furnishes us a contradiction. Therefore \( R(x) \cup R^{-1}(y) \subset P(x) \cup P^{-1}(y) \).

**Proof of Theorem.** We assume that \( R \) is a continuous binary relation on a topological space \( X \) and begin the proof of each claim in (a) under the assumption that the topology on \( X \) is connected.

\[ \text{(i) Note that } P \text{ is negatively transitive if and only if for all } x, y, z \in X, (x, y) \in P \text{ implies either } (x, z) \in P \text{ or } (z, y) \in P. \text{ Pick } x, y \in X \text{ such that } (x, y) \in P. \text{ Since } X \text{ is connected, it follows from Lemma 1 that } X \subset P(x) \cup P^{-1}(y). \text{ Hence, } P \text{ is negatively transitive. The backward direction follows from the assertion (b) in Proposition 2.} \]

\[ \text{(ii)(iii)(iv) The proofs follow from the assertions (i) above, and (a), (b) in Proposition 2.} \]

Next, we turn to the proof of each claim in (b) under the assumption that the sections of \( I \) are connected.

\[ \text{(i) Pick } x, y, z \in X \text{ such that } y \in P(x) \text{ and } z \in I(y). \text{ Assume } z \notin P(x). \text{ Then, it follows from II that } I(x) \cap I(z) = \emptyset. \text{ Then } X = I(x) \cup P(x) \cup P^{-1}(x) \cup (R(x) \cup R^{-1}(x))^c \text{ implies that} \]

\[ I(z) = [P(x) \cap I(z)] \cup [P^{-1}(x) \cap I(z)] \cup [(R(x))^c \cap (R^{-1}(x))^c \cap I(z)]. \]

It is clear that the three sets in square brackets are pairwise disjoint. Since \( P \) has open sections and \( R \) has closed sections, the three sets in square brackets are open in \( I(z) \). Since \((z, y) \in I \) and \( I \) is symmetric, therefore \((y, z) \in I \). Then II implies \((z, z) \in I \). Therefore, \( y, z \in I(z) \). It is clear that \( y \in P(x) \cap I(z) \). Since we assume above that \( z \notin P(x) \), therefore \( z \) is either in \( P^{-1}(x) \) or in \((R(x))^c \cap (R^{-1}(x))^c \), but not in both since these two sets are disjoint. If \( z \in P^{-1}(x) \), then \( P(x) \cap I(z) \) and the union of the remaining two sets in square brackets above form an open partition of \( I(z) \) which contradicts the connectedness of \( I(z) \). Analogously, \( z \in (R(x))^c \cap (R^{-1}(x))^c \) furnishes us a contradiction to the connectedness of \( I(z) \). Therefore, \( z \in P(x) \), and hence \( PI \) holds. An analogous argument implies \( IP \) holds.

\[ \text{(ii) The proof follows from assertions (i) above and (a) in Proposition 2.} \]

The proof of Theorem is complete.

We now turn to the proof of Theorem 6.
Proof of Theorem 6. Let $X$ be a topological space with the Phragmen-Brouwer property and $R$ a continuous relation on it with path-connected sections. We take each claim in turn.

[1] Assume the section of the symmetric part $I$ of $R$ is transitive. Now consider the following claim.

Claim 1. For all $z \in X$, the sets $P(z)$ and $P^{-1}(z) \cup (R(z) \cup R^{-1}(z))^c$ are separated.

pick $x, y, z \in X$ such that $x \in I(y)$ and $y \in P(z)$. Assume $x \not\in P(z)$. If $x \in I(z)$, then it follows from the transitivity of $I$ that $y \in I(z)$, which contradicts with $y \in P(z)$. Therefore, either $x \in P^{-1}(z)$ or $x \in (R(z) \cup R^{-1}(z))^c$. Recall that $x \in P^{-1}(z) \cup (R(z) \cup R^{-1}(z))^c, y \in P(z)$ and

$$X \setminus I(z) = P(z) \cup P^{-1}(z) \cup (R(z) \cup R^{-1}(z))^c.$$  

Claim 1 implies that the sets $P(z)$ and $P^{-1}(z) \cup (R(z) \cup R^{-1}(z))^c$ are separated. Then it follows from $X$ has the Phragmen-Brouwer property that there exists a connected subset $I_z$ of $I(z)$ which separates $x$ and $y$. Note that $I$ and $(x, y) \in I$ imply that $x, y \in R(y)$. Since $R(y)$ is path-connected, therefore there exists a continuous function $s : [0, 1] \to R(y)$ such that $s(0) = x$ and $s(1) = y$. Note that $s([0, 1])$ is connected and $x, y$ are contained in different components of $X \setminus I_z$, therefore there exists $z \in [0, 1]$ such that $s(z) = z' \in I_z \cap R(y)$. Since $I_z \subset I(z)$, it follows from $I$ and $y \in P(z)$ that $I_z \cap I(y) = \emptyset$. Hence $z' \in P(y)$ and

$$I_z = [P(y) \cap I_z] \cup [P^{-1}(y) \cap I_z] \cup [(R(y) \cup R^{-1}(y))^c \cap I_z].$$

Since $I_z$ is connected and $z' \in P(y)$, therefore $I_z \subset P(y)$.

Claim 1 implies that $P(y)$ and $P^{-1}(y) \cup (R(y) \cup R^{-1}(y))^c$ are separated subsets of $X$. Then, it follows from $z' \in P(y)$ and $z \in P^{-1}(y)$ that there exists a connected subset $I_y$ of $I(y)$ which separates $z'$ and $z$. Note that $z' \in I_z \subset I(z)$ and $I$ imply that $(z, z) \in I$. Then $z, z' \in R(z')$ and $R(z')$ is path-connected imply that there exists a continuous function $s' : [0, 1] \to R(z')$ such that $s'(0) = z$ and $s'(1) = z'$. Note that $s'([0, 1])$ is connected and $z, z'$ are contained in different components of $X \setminus I_y$, therefore there exists $\lambda \in [0, 1]$ such that $s'(\lambda) = y' \in I_y$. It follows from the transitivity of $I$ that $I_y \cap I(z') = \emptyset$, therefore

$$I_y = [P(z') \cap I_y] \cup [P^{-1}(z') \cap I_y] \cup [(R(z') \cup R^{-1}(z'))^c \cap I_y].$$

Since $I_y$ is connected and $y' \in P(z')$, therefore $I_y \subset P(z')$.

It follows from $z' \in P^{-1}(y')$ and $I_z$ is connected that $I_z \subset P^{-1}(y')$. Therefore, $R(y') \subset X \setminus I_z$. Then $x, y \in R(y')$ and $x, y$ are contained in distinct components of $X \setminus I_z$, $R(z')$ is connected with $I_z \subset P(z')$. Therefore, $x \in P(z)$, hence $PF$ holds.

It remains to prove Claim 1 in order to finish the proof of (i).
Proof of Claim 1. Pick \( z \in X \) and define \( A = P^{-1}(z) \cup (R(z) \cup R^{-1}(z))^c \). Since \( R(z) \) is closed, therefore \( \overline{P(z)} \subset R(z) \). It is clear that \( R(z) \cap A = \emptyset \), hence \( \overline{P(z)} \cap A = \emptyset \). It remains to show that \( \overline{A} \cap P(z) = \emptyset \). It follows from \( P(z) \) is open, \( A \cup I(z) \cup P(z) = X \) and \((A \cup I(z)) \cap P(z) = \emptyset\) that \( A \cup I(z) = (P(z))^c \) and \( A \cup I(z) \) is closed. Therefore \( \overline{A} \subset A \cup I(z) \), and hence \( \overline{A} \cap P(z) = \emptyset \). Therefore, \( P(z) \) and \( P^{-1}(z) \cup (R(z) \cup R^{-1}(z))^c \) are separated sets in \( X \). \( \blacksquare \)

\( \text{(ii)} \) Replacing \( R \) and \( P \) with \( R^{-1} \) and \( P^{-1} \) in the proof of claim (i) above completes the proof.

\( \text{(iii)} \) The proof follows from claims (i) and (ii) above and Proposition 2(a).

The proof of Theorem 3 is complete.

We now turn to the proof of Theorem 1. Before that we need the following two lemmata.

**Lemma 2.** Any non-empty, closed and open subset of a topological space is a union of the components of the space.

**Proof of Lemma 2.** Pick a non-empty, closed and open subset \( V \) of a topological space \( X \). It follows from [Dugundji 1966, Theorem 3.2] that the components of \( X \) form a closed partition of \( X \). Let \( C_x \) denote the component of \( X \) containing \( x \in X \). We next show that \( V = \bigcup_{x \in V} C_x \). It is clear that \( V \subseteq \bigcup_{x \in V} C_x \). In order to show the reverse inclusion, pick \( x \in V \), define \( A_x = C_x \cap V \) and \( B_x = C_x \cap V^c \). Since \( C_x, V, V^c \) are closed, therefore \( A_x, B_x \) are closed. Moreover, \( A_x \cup B_x = C_x \) and \( A_x \cap B_x = \emptyset \). Then \( A_x = B_x^c \cap C_x \) and \( B_x = A_x^c \cap C_x \). Hence, \( A_x \) and \( B_x \) are both open and closed in the subspace \( C_x \). Since \( C_x \) is connected, therefore either \( A_x = \emptyset \) or \( B_x = \emptyset \). It follows from \( x \in A \) that \( B_x = \emptyset \). Then \( A_x = C_x \). Since \( A_x = C_x \cap V \), therefore \( C_x \subseteq V \). \( \blacksquare \)

**Lemma 3.** Any topological space with at least \( k \) components has a partition consisting of \( k \) sets which are both open and closed.

**Proof of Lemma 3.** Pick a natural number \( k \) and assume \( X \) is a topological space with at least \( k \) components. If \( k = 1 \), then setting \( \{X\} \) as the partition of \( X \) completes the proof. Otherwise, \( X \) is disconnected, hence there exist \( A_1, A_1^c \) which form a partition of \( X \) which is both open and closed. If \( k = 2 \), then the proof is complete. Otherwise, Lemma 2 implies that \( A_1 \) and \( A_1^c \) can be written as a union of the components of \( X \). Since the components of \( X \) are disjoint, therefore \( A_1 \) and \( A_1^c \) are the unions of distinct components. Let \( A_i^c \) be the union of at least two components. Then, \( A_i^c \) is disconnected, hence there exist non-empty subsets \( A_2, A_2^c \) of \( A_i^c \) which are both open and closed in the subspace \( A_i^c \). Since \( A_i^c \) is open and closed in \( X \), therefore \( A_2, A_2^c \) are also open and closed in \( X \). Then, \( A_1, A_2, A_2^c \) form a partition of \( X \) which is both open and closed. If \( k = 3 \), then the proof is complete. Otherwise, repeating this procedure \((k - 1)\)-many times yields an open partition of \( X \) consisting of \( k \) sets. \( \blacksquare \)

**Proof of Theorem 1.** We begin the proof with (a) \( \Rightarrow \) (e). The proof rests on three claims which we state and use, and prove only after the proof of the implication (a) \( \Rightarrow \) (e) is complete. Assume \( X \) is \( k \)-connected and \( \{C_1, \ldots, C_k\} \) denote the set of components of \( X \). Define \( K = \{1, \ldots, k\} \). Let \( R \) be a \( k \)-non-trivial, semi-transitive and continuous binary relation on \( X \) such that its symmetric part is transitive. The following claim shows that every pair of components contains strictly comparable elements.

**Claim 2.** For all \( i, j \in K \), there exists \( x_i \in C_i, x_j \in C_j \) such that \( (x_i, x_j) \in P \cup P^{-1} \).
Assume there exist \( x, y \in X \) such that \((x, y) \notin R \cup R^{-1}\). Then \( x \in C_i \) and \( y \in C_j \) for some \( i, j \in K \). Claim 2 implies that there exist \( x_i \in C_i, x_j \in C_j \) such that \((x_i, x_j) \in P \cup P^{-1}\). Without loss of generality, assume \((x_i, x_j) \in P\). Then, it follows from Lemma 1 that \( x, y \in P(x_i) \cup P^{-1}(x_j)\). The following claim shows that both \( x \) and \( y \) are contained in at least one of these two sets.

**Claim 3.** \( \{x, y\} \subset P(x_i) \) or \( \{x, y\} \subset P^{-1}(x_j) \).

It follows from Claim 3 that \( x_i \in P^{-1}(x) \cap P^{-1}(y) \) or \( x_j \in P(x) \cap P(y) \). Therefore, \([P^{-1}(x) \cap P^{-1}(y)] \cap C_i \neq \emptyset \) or \([P(x) \cap P(y)] \cap C_j \neq \emptyset \). It follows from \( x \in C_i, y \in C_j, x, y \notin P^{-1}(x) \cap P^{-1}(y) \) and \( x, y \notin P(x) \cap P(y) \) that \( C_i, C_j \notin P^{-1}(x) \cap P^{-1}(y) \) and \( C_i, C_j \notin P(x) \cap P(y) \).

**Claim 4.** \( P^{-1}(x) \cap P^{-1}(y) = R^{-1}(x) \cap R^{-1}(y) \) and \( P(x) \cap P(y) = R(x) \cap R(y) \).

It follows from continuity of \( R \) and Claim 4 that \( P(x) \cap P(y) \) and \( P^{-1}(x) \cap P^{-1}(y) \) are both open and closed. Then \( \{P^{-1}(x) \cap P^{-1}(y) \cap C_i, [P^{-1}(x) \cap P^{-1}(y)]^c \cap C_i\} \) is an open partition of \( C_i \) or \( \{P(x) \cap P(y) \cap C_j, [P(x) \cap P(y)]^c \cap C_j\} \) is an open partition of \( C_j \). This furnishes us a contradiction to \( C_i \) and \( C_j \) being components of \( X \). Therefore, \( R \) is complete and hence the proof of assertion \( (a) \Rightarrow (c) \) is complete.

It now remains to prove Claims 2, 3, 4.

**Proof of Claim 2.** First, pick \( i, j \in K \) such that \( i = j \). It follows from \( k \)-non-triviality that there exist \( \bar{x}, \bar{y} \in C_j \) such that \((\bar{x}, \bar{y}) \in P\). Then Lemma 1 implies that \( C_j \subset P(\bar{x}) \cup P^{-1}(\bar{y}) \). Therefore, \( \bar{x}, \bar{y} \in P \) if \( \text{Claim 2} \) implies that \( \{\bar{x}, \bar{y}\} \subset P(\bar{x}) \cup P^{-1}(\bar{y}) \). Hence, \( \bar{x} \in P(\bar{x}) \) and \( \bar{y} \in P^{-1}(\bar{y}) \). Since \( \text{Claim 2} \) implies that \( \{\bar{x}, \bar{y}\} \subset P(\bar{x}) \cup P^{-1}(\bar{y}) \), \( \bar{x} \) and \( \bar{y} \) are contained in at least one of these two sets.

**Proof of Claim 3.** If \( x \in P^{-1}(x_j) \), then Lemma 1 implies that \( y \in P(x) \cup P^{-1}(x_j) \). Since \( y \notin P(x) \), \( y \in P^{-1}(x_j) \). Therefore, \( y \in P(x) \cup P^{-1}(x_j) \). If \( x \in P(x_i) \), then it follows from \( x_i \in C_i \) and Lemma 1 that \( P(x_i) \cup P^{-1}(x) \) is both open and closed and contains \( C_i \). Since \( x_j \in P(x_i) \cap C_j \), therefore \( P(x_i) \cup P^{-1}(x) \) has a non-empty intersection with \( C_j \). Then, it follows from \( P(x_i) \cup P^{-1}(x) \) is both open and closed, and \( C_j \) is a component of \( X \) that \( C_i \cup C_j \subset P(x_i) \cup P^{-1}(x) \). Hence, \( x \notin P(x_i) \cup P^{-1}(x) \). Therefore, \( y \notin P(x_i) \cup P^{-1}(x) \).

**Proof of Claim 4.** It follows from the reverse inclusion assumption that \( z \in R(x) \cap R(y) \) such that \( z \notin P(x) \) or \( z \notin P(y) \). If \( z \notin P(x) \), then it follows from \( z \in R(x) \) that \( (z, x) \in I \). Then, \( z \in R(y) \), \( II \) and PI imply either \( (y, x) \in I \) or \( (y, x) \in P \). This furnishes us a contradiction to \( (x, y) \notin R \cup R^{-1} \). If \( z \notin P(y) \), then it follows from \( z \in R(y) \) that \( (z, y) \in I \). Then, \( z \in R(x) \), \( II \) and PI imply either \( (x, y) \in I \) or \( (x, y) \in P \). This furnishes us a contradiction to \( (x, y) \notin R \cup R^{-1} \). Therefore, \( R(x) \cap R(y) \subset P(x) \cap P(y) \).

Replacing \( R \) with \( R^{-1} \) and \( P \) with \( P^{-1} \) in the argument above implies that \( P^{-1}(x) \cap P^{-1}(y) = R^{-1}(x) \cap R^{-1}(y) \).

Next we turn to the other implications in Theorem 1.

\( (a) \Rightarrow (d) \) Theorem 3(b) implies \( R \) is semi-transitive. Then \( (a) \Rightarrow (e) \) above completes the proof.

\( (a) \Rightarrow (c) \) The proof follows from \( (a) \Rightarrow (e) \) above and the observation that the anti-symmetry of \( R \) implies that \( R \) is semi-transitive and \( I \) is transitive. In order to see this note that it follows from the anti-symmetry of \( R \) that \( I(x) \subset \{x\} \). Then, if \( y \in P(x) \) and \( z \in I(y) \), then \( z = y \).
hence $z \in P(x)$. If $y \in I(x)$ and $z \in P(y)$, then $x = y$, hence $z \in P(x)$. Similarly, if $y \in I(x)$ and $z \in I(y)$, then $z = y = x$, hence $z \in I(x)$.

(a) $\Rightarrow$ (b) The proof follows from (a) $\Rightarrow$ (e) since Proposition 3(a) implies that $R$ is semi-transitive and its symmetric part is transitive.

(e) (d) (c) (b) $\Rightarrow$ (a) Assume $X$ has at least $k + 1$ components. Then Lemma 3 implies that there exists a partition $\{Y_1, Y_2, \ldots, Y_{k+1}\}$ of $X$ which is both open and closed. Define a binary relation $R$ on $X$ as

$$R = \bigcup_{i=1}^{k} \left( \bigcup_{j=i+1}^{k+1} Y_i \times Y_j \right).$$

Then the symmetric part of $R$ is $I = \emptyset$ and its asymmetric part is $P = R$. By construction, the sections of $R$ are closed and the sections of $P$ are open. Moreover, $R$ is transitive, semi-transitive and anti-symmetric, and $I$ is transitive. Defining $m_i = i$ and $n_i = i + 1$ for all $i \leq k + 1$ imply that $R$ is $k$-non-trivial. Finally, it is clear that $R$ is incomplete.

The proof of Theorem 1 is complete.

Before turning to the proof of Theorem 2, we comment on our proof-technique. Eilenberg uses completeness of an anti-symmetric and continuous relation on a connected space in order to obtain transitivity. Sonnenschein exploits his standard quotient-space construction to drop the anti-symmetry assumption in Eilenberg’s theorem. He bring into prominence the assumption of semi-transitivity of the relation which is satisfied by any anti-symmetric relation, and then provides a series of sufficient conditions for semi-transitivity by imposing further topological assumptions on preferences. On the other hand Schmeidler uses transitivity of a non-trivial and continuous relation on a connected space in order to obtain completeness. Although both use the connectedness of the space and the continuity of the relation, the proof techniques of Schmeidler and of Eilenberg-Sonnenschein are quite different. In the proofs of Theorems 1 and 5 above, the latter has transitivity and the former, completeness as its necessary condition. Our proof-technique is inspired by that of Schmeidler, and we use it to obtain independently each of the completeness and the transitivity properties in the forward-direction of Theorem 2. This alternative proof of the results of Eilenberg and Sonnenschein may have some independent interest.

We are now ready to prove Theorem 2.

**Proof of Theorem 2.** We first prove the forward direction and then turn to the proof of the backward direction. In the forward direction, for each assertion, completeness follows from its counterpart in Theorem 1. Hence, the proof of the implication (a) $\Rightarrow$ (b) is complete. We next prove that transitivity holds in the remaining implications without using the completeness property. If $X$ has only one component, then transitivity follows from Proposition 2 and Theorem 5. Hence, assume $X$ has two components $C_1, C_2$ which form a partition of $X$ that is both open and closed.

(a) $\Rightarrow$ (e) Assume $R$ is a 2-non-trivial, semi-transitive and continuous binary relation on $X$ with a transitive symmetric part. Pick $x, y, z \in X$ such that $y \in R(x)$ and $z \in R(y)$. If $x \in R(y)$ or $y \in R(z)$ holds, then the semi-transitivity of $R$ and the transitivity of $I$ imply $z \in R(x)$. Hence, assume $y \in P(x)$ and $z \in P(y)$. Since $C_1$ and $C_2$ form a partition of $X$, each of $x, y, z$ is
contained in one and only one of these two components. Then the following four cases cover all possibilities: (i) \( x, y, z \in C_i \), (ii) \( x, y \in C_i, z \in C_j \), (iii) \( x, z \in C_i, y \in C_j \) and (iv) \( x \in C_i, y, z \in C_j \) where \( i = 1, 2, i \neq j \).

Before elaborating the cases note that it follows from Lemma 1 that for all \( i, j = 1, 2 \), if there exist \( x_i \in C_i \) and \( y_j \in C_j \) such that \((x_i, y_j) \in P\), then \( C_i \cup C_j \subseteq P(x_i) \cup P^{-1}(x_j) \). Then in cases (i), (ii) and (iii), Lemma 1 implies that \( x \in P(y) \cup P^{-1}(z) \). Then it follows from \( y \in P(x) \) that \( z \in P(x) \). Similarly, in case (iv), Lemma 1 implies that \( z \in P(x) \cup P^{-1}(y) \). Then it follows from \( z \in P(y) \) that \( z \in P(x) \). Therefore, \( R \) is transitive.

\[ (a) \Rightarrow (d) \] The proof follows from \( (a) \Rightarrow (e) \) above and Theorem 3(b).

\[ (a) \Rightarrow (e) \] The proof follows from \( (a) \Rightarrow (e) \) above and the observation that any anti-symmetric relation is semi-transitive and its symmetric part is transitive.

The proof of the forward direction is complete. We provide the proof of the backward direction by considering cases \( k = 1 \) and \( k = 2 \) separately.

We begin with the case \( k = 1 \).

\( (e) \rightarrow (d) \rightarrow (c) \rightarrow (b) \) Assume \( X \) is disconnected. Then there exists an open partition of \( X \) consisting of a set \( Y \) and its complement \( Y^c \). Define \( R = Y \times Y^c \). Then it is asymmetric, hence \( I = \emptyset \) and \( P = R \). Since \( Y \) and \( Y^c \) are both open and closed, therefore \( R(x) = Y^c = P(x) \) and \( R^{-1}(x) = Y = P^{-1}(x) \) are both open and closed for all \( x \in X \). Since \( P \neq \emptyset \), \( R \) is non-trivial. It follows from \( P = R \) that \( R \) is anti-symmetric. It is clear that \( R \) is transitive, hence semi-transitive. Moreover, since \( I = \emptyset \), therefore it is transitive and its sections are connected. Since \( Y \) and \( Y^c \) are non-empty, therefore \( R \cup R^{-1} \neq X \times X \), i.e. \( R \) is not complete. This furnishes us a contradiction.

Next we turn to the case \( k = 2 \).

\( (e) \rightarrow (d) \rightarrow (c) \rightarrow (b) \rightarrow (a) \) Assume \( X \) has at least three components. Then Lemma 3 implies that there exists a partition \( \{Y_1, Y_2, Y_3\} \) of \( X \) which is both open and closed. Define a binary relation on \( X \) as \( R = (Y_1 \times Y_2) \cup (Y_1 \times Y_3) \cup (Y_2 \times Y_3) \). Then it is asymmetric, hence \( I = \emptyset \) and \( P = R \). By construction, the sections of \( R \) is closed, of \( P \) are open and of \( I \) are connected. Moreover, \( R \) is transitive, semi-transitive and anti-symmetric, and \( I \) is transitive. Defining \( C^1 = \{Y_1, Y_2\} \) and \( C^2 = \{Y_2, Y_3\} \) implies \( R \) is 2-non-trivial. Finally, it is clear that \( R \) is incomplete.

The proof of Theorem 2 is complete.

Remark: We could have also relied on the proof-technique of Eilenberg-Sonnenschein in order to prove Theorem 2. This would require reflexivity of the preference relation. Since the assumptions of the theorem imply the completeness of the relation, and hence its reflexivity, this is not a restrictive assumption. Moreover, this method requires the following intermediate result on the relation between the number of the components of a space and of its quotient space, and whose proof is an easy consequence of Lemma 3.

Let \( X \) be a topological space, \( I \) an equivalence relation on it and \( X/I \) the quotient space of it with respect to \( I \). If \( X \) has \( k \) components, then \( X/I \) has at most \( k \) components. Moreover, if \( X/I \) has \( k \) components, then \( X \) has at least \( k \) components.

If the section of \( I \) are connected, then this result suggests that the connectedness of a space \( X \) is equivalent to the connectedness of its quotient \( X/I \). Therefore, the weaker connectedness
assumption of Sonnenschein (1965). Theorem 4) \((X \mid I)\) is connected) is equivalent to the stronger connectedness assumption \((X \text{ is connected})\).

We now turn to the proof of Theorem 3.

**Proof of Theorem 3.** In the forward direction, each assertion is a special case of its counterpart in Theorem 2, hence the proof of this direction is complete. Note that we have not used the assumption that the space is quasi-ordered yet.

\(\square\) Assume \(X\) is a quasi-ordered space and has at least three components. Then Lemma 3 implies that there exists a partition \(\{Y_1, Y_2, Y_3\}\) of \(X\) which is both open and closed. Let \(Q\) be a complete, anti-symmetric and continuous binary relation (since the space is quasi-ordered, such relation exists). Define a binary relation \(R\) on \(X\) as follows.

\[
\forall x, y \in Y_i, \quad x \in R(y) \iff x \in Q(y), \quad i = 1, 2, 3,
\]

\[
\forall x \in Y_1, \forall y \in Y_2, \quad y \in R(x),
\]

\[
\forall y \in Y_2, \forall z \in Y_3, \quad z \in R(y),
\]

\[
\forall x \in Y_1, \forall z \in Y_3, \quad x \in R(z).
\]

We next show that \(R\) is complete and anti-symmetric with closed sections, but non-transitive. It follows from \(Q\) is complete and anti-symmetric that \(R\) is complete and anti-symmetric. Pick \(x \in Y_1, y \in Y_2, z \in Y_3\). Then

\[
R(x) = (Q(x) \cap Y_1) \cup Y_2 \quad \text{and} \quad R^{-1}(x) = (Q^{-1}(x) \cap Y_1) \cup Y_3,
\]

\[
R(y) = (Q(y) \cap Y_2) \cup Y_3 \quad \text{and} \quad R^{-1}(y) = (Q^{-1}(y) \cap Y_2) \cup Y_1,
\]

\[
R(z) = (Q(z) \cap Y_3) \cup Y_1 \quad \text{and} \quad R^{-1}(z) = (Q^{-1}(z) \cap Y_3) \cup Y_2.
\]

It follows from \(Q\) has closed sections and \(Y_1, Y_2, Y_3\) are closed that \(R\) has closed sections. Since \(x \in Y_1, y \in Y_2, z \in Y_3\) implies \(y \in R(x), z \in R(y)\) and \(x \in R(z)\), the relation \(R\) is non-transitive. This furnishes us a contradiction. Hence \(X\) is 2-connected.

The proof of Theorem 3 is complete.

\(\blacksquare\) We now turn to the proof of Theorem 4.

**Proof of Theorem 4.** Assume \(R\) is incomplete. Let \(P\) denote its asymmetric part. Note that it follows from the negative transitivity of \(P\) that

\[
\text{if } (x, y) \in P, \text{ then } P(x) \cup P^{-1}(y) = X. \tag{1}
\]

Since \(R\) is incomplete, there exists \(x, y \in X\) such that \((x, y) \notin R \cup R^{-1}\). Then \(P^{-1}(x) \cap P^{-1}(y)\) and \(P(x) \cap P(y)\) are proper subsets of \(X\). Since \(R\) is strongly non-trivial, there exist \((\bar{x}, \bar{y}) \in P\) such that \(R(x') \cap R(y') \neq \emptyset\) for all \(x', y' \in P(\bar{x})\). It follows from \(1\) above that \(x \in P(\bar{x}) \cup P^{-1}(\bar{y})\).

\(\square\) If \(x \in P(\bar{x})\), then \(1\) implies that \(y \in P(\bar{x}) \cup P^{-1}(x)\). Since \(y \notin P^{-1}(x)\), therefore \(y \in P(\bar{x})\).

Since \(x, y \in P(\bar{x})\), there exists \(z \in X\) such that \(z \in R(x) \cap R(y)\). If \(x \in P^{-1}(\bar{y})\), then \(1\) implies that \(y \in P(x) \cup P^{-1}(\bar{y})\). Since \(y \notin P(x)\), \(y \in P^{-1}(\bar{y})\). Hence, \(\bar{y} \in P(x) \cap P(y)\). Therefore, we established that \(R(x) \cap R(y) \neq \emptyset\).

Since \(R\) is semi-transitive and \(I\) is transitive, therefore Claim 1 in the proof of Theorem 1 implies that \(R(x) \cap R(y) = P(x) \cap P(y)\). Then there exists a non-empty and proper subset of \(X\) which is both open and closed. This furnishes us a contradiction to the connectedness of \(X\). Therefore, \(R\) is complete. Then, the completeness of \(R\) and the negative transitivity of \(P\) implies the transitivity of \(R\). Since \(R^{-1}(z) = (P(z))^c\) for all \(z \in X\) when \(R\) is complete, therefore \(R\) has closed sections and \(P\) has open sections, hence \(R\) is continuous.
Theorem 2 implies that the relation is complete and transitive. Then the proof follows from Eilenberg (1941, Theorem II).

Let $R$ be a binary relation $R$ on topological space $X$ satisfying the assumptions of the hypothesis of the implication and assume it is not sensitive. Then, for all $(x, y) \notin R \cup R^{-1}$ there exists an open neighborhood $V$ of $(x, y)$ such that $V \cap (R \cup R^{-1}) = \emptyset$. Then $(R \cup R^{-1})^c$ is open, hence it has open sections. Note that $X = P(z) \cup R^{-1}(z) \cup (R \cup R^{-1})^c(z) = P^{-1}(z) \cup R(z) \cup (R \cup R^{-1})^c(z)$ for any $z \in X$. Then it follows from $P$ has open sections that $R$ has closed sections. Then Theorem 2 implies that $R$ is complete. This furnishes us a contradiction.

$\text{(a)} \implies \text{(d)}$ is due to Gerasimou (2013, Corollary 3) and $\text{(a)} \implies \text{(f)}$ is due to Chateauneuf (1987, Fundamental Lemma) and $\text{(a)} \implies \text{(g)}$ is due to Chateauneuf (1987, Theorem).

Assume $X$ is disconnected. Then there exists an open partition of $X$ consisting of a set $Y$ and its complement $Y^c$. Define $R = X \times Y^c$. Then, $P = Y \times Y^c$ is its asymmetric part and $I = Y^c \times Y^c$ its symmetric part. For all $x \in X$, $R(x) = Y^c$ and $P(x) = Y^c$ or $P(x) = \emptyset$. Since $Y^c$ and $\emptyset$ are open, therefore $R$ has closed upper sections and $P$ has open upper sections. The strong non-triviality of $R$ follows from $P(x) = Y^c$ for all $x \in Y$ and $R(x') = Y^c$ for all $x' \in Y^c$. In order to see that $R$ is transitive, pick $(x, y), (y, z) \in R$. Then $y, z \in Y^c$. Hence $(x, z) \in R$. It follows from $P(x) \cup P^{-1}(y) = Y^c \cup Y$ for all $(x, y) \in P$ that $P$ is negatively transitive. Finally, it is clear that $R$ is incomplete.

Assume $X$ is disconnected. Then there exists an open partition of $X$ consisting of a set $Y$ and its complement $Y^c$. Let $Q$ be an anti-symmetric, complete and continuous binary relation on $X$ (since the space is quasi-ordered, such relation exists). Then for all $x \in Y$ and $y \in Y^c$, either $(x, y) \in Q$ or $(y, x) \in Q$. Assume without loss of generality that there exist $x \in Y$ and $y \in Y^c$ such that $(y, x) \in Q$. Define another binary relation $R$ on $X$ as follows.

\[
\forall x, y \in Y, \ x \in R(y) \iff x \in Q(y), \\
\forall x, y \in Y^c, \ x \in R(y) \iff x \in Q(y), \\
\forall x \in Y, \forall y \in Y^c, \ y \in R(x).
\]

It is clear that $R$ is anti-symmetric and complete. For all $x \in Y$, $R(x) = Q(x) \cup Y^c$ and $R^{-1}(x) = Q^{-1}(x) \cap Y$. For all $y \in Y^c$. Then $R(y) = Q(y) \cap Y^c$ and $R^{-1}(y) = Q^{-1}(y) \cup Y$. Since $Q$ has closed sections and $Y, Y^c$ are closed, therefore $R$ has closed sections. Then, completeness implies $R$ is continuous.

Note that $Q$ and $R$ has identical ordering both on $Y$ and on $Y^c$. Moreover, it follows from $(y', x') \in Q$ for some $(x', y') \in Y \times Y^c$ and $(x, y) \in R$ for all $(x, y) \in Y \times Y^c$ that $Q$ and $R$ have different ordering among the elements of $Y$ and $Y^c$. Since $X$ has more than two elements, therefore $Q$ and $R$ are neither identical, nor inverse to each other.

Assume $X$ is disconnected. Then there exists an open partition of $X$ consisting of a set $Y$ and its complement $Y^c$. Define $R = Y \times Y^c$ as in the proof of the backward direction implication for $k = 1$ in Theorem 2 Then $R$ is incomplete, non-trivial, transitive, has closed sections and its asymmetric part $P$ has open sections. Now pick an arbitrary pair $(x, y) \in P = Y \times Y^c$. Since $Y$ and $Y^c$ are open, $Y \times Y^c$ is open. Hence $Y \times Y^c$ is an open neighborhood.

\footnote{Gerasimou (2013, Corollary 3) showed that this statement is true provided that the space is connected and $R$ is reflexive. However, the statement is true without the reflexivity assumption and the construction in his proof directly follows. Hence, we drop reflexivity.}
of \((x, y)\). By construction \(P = R = Y \times Y^c\). Hence \((x', y') \in R\) for all \((x', y') \in Y \times Y^c\). This furnishes us a contradiction to the fragility of \(R\). This completes the proof of (d) \(\Rightarrow\) (a). Now, pick an arbitrary pair in \((x, y) \in P^c = R^c = (Y \times Y) \cup (Y^c \times Y^c) \cup (Y \times X^c)\). Since \(P^c\) is open, this furnishes us a contradiction to sensitivity of \(R\). This completes the proof of (e) \(\Rightarrow\) (a).

\[(f) = (a)\] Assume \(X\) is disconnected. Then there exist \(Y, Y^c\) non-empty and open subsets of \(X\). Define \(P = Y \times Y^c\). It is clear that \(P\) is asymmetric. Define a function \(u : X \to \mathbb{R}\) as \(u(x) = 0\) if \(x \in Y\) and \(u(x) = 1\) if \(x \in Y^c\). Since \(Y\) and \(Y^c\) are open, \(u\) is continuous. Moreover \(u(x) < u(y)\) if and only if \((x, y) \in Y \times Y^c = P\), hence \(P\) has a continuous dual-representation. We will now show that \(P\) is not strongly separable. The relation \(R = \{(x, y) \mid (y, x) \notin P\}\) is then defined as \(R = (Y \times Y) \cup (Y^c \times Y^c) \cup (Y \times X^c)\). Pick \((x, y) \in P\). Then \(x \in Y\) and \(y \in Y^c\). By construction of \(R\), for all \(x' \in X\), if \(x' \in P(x)\), then \(x' \in Y^c\). Similarly, for all \(y' \in X\), if \(y' \in P^{-1}(y)\), then \(y' \in Y\). Hence \((x', y') \in Y^c \times Y\) for all \(x' \in P(x)\) and \(y' \in P^{-1}(y)\). It follows from \(Y^c \times Y = R^c\) that \((x', y') \notin R\) for all \(x' \in P(x)\) and \(y' \in P^{-1}(y)\). Therefore \(P\) is not strongly separable.

\[(g) = (a)\] The proof follows from \((f) = (a)\) above.

The proof of Theorem 4 is complete.

We finally turn to the proof of Proposition 3. Before that we need the following notation and a lemma. A binary relation is said to be an equivalence relation if it is reflexive, symmetric and transitive. Let \(I\) be an equivalence relation on a set \(X\), \(\{x' \in X \mid \{x, x'\} \in I\}\) denote the equivalence class of \(x\) and the space \(X/I\) the quotient space of \(X\) with respect to the relation \(I\). Let \(\pi : X \to X/I\) denote the corresponding quotient map defined as \(\pi(x) = [x]\). Let \(\tau\) be a topology on \(X/I\). Then the quotient topology is defined as \(\tau = \{U \subset X/I \mid \pi^{-1}(U) \in \tau\}\). Hence, a set \(A\) in \(X/I\) is open if and only if \(\pi^{-1}(A)\) is open (in \(X\)). Equivalently, a set \(A\) in \(X/I\) is closed if and only if \(\pi^{-1}(A)\) is closed. For any binary relation \(R\) on a set \(X\) whose symmetric part \(I\) is an equivalence relation, define an induced relation \(\hat{R}\) on \(X/I\) as \([x, y] \in \hat{R}\) if \((x', y') \in R\) for all \(x' \in [x]\) and \(y' \in [y]\). Define \(\hat{I}\) as the symmetric part of \(\hat{R}\) and \(\hat{P}\) as its asymmetric part.

**Lemma 4.** Let \(R\) be a semi-transitive and continuous binary relation on a topological space \(X\) such that its symmetric part \(I\) is an equivalence relation. Then the induced relation \(\hat{R}\) on \(X/I\) is anti-symmetric and continuous.

**Proof of Lemma 4.** The semi-transitivity of \(R\) implies that \([x, y] \in \hat{R}\) if and only if \((x, y) \in R\). One of the directions is implied by the definition of \(\hat{R}\). In order to prove the other direction, assume \((x, y) \in R\). Then either \((y, x) \in R\) or \((y, x) \notin R\). If \((y, x) \in R\), then \((x, y) \in I\). Since \(I\) is transitive, \((x', y') \in I\) for all \(x' \in [x]\) and \(y' \in [y]\). If \((y, x) \notin R\), then \((x, y) \in P\). Since \(\hat{R}\) is semi-transitive, therefore \((x', y') \in P\) for all \(x' \in [x]\) and \(y' \in [y]\). Hence, the definitions of \(I\) and \(P\) imply that \((x', y') \in R\) for all \(x' \in [x]\) and \(y' \in [y]\). Therefore, \([x, y] \in \hat{P}\) if and only if \((x, y) \in P\). The anti-symmetry of \(\hat{R}\) directly follows from \([x, y] \in \hat{R}\) if and only if \((x, y) \in R\). We recall that the quotient map \(\pi\) is defined as \(\pi(x) = [x]\). Then \(\pi^{-1}(\hat{R}([x])) = \{y : ([x], [y]) \in \hat{R}\} = \{y : (x, y) \in R\} = R(x)\).

Analogously, \(\pi^{-1}(\hat{R}^{-1}([x])) = R^{-1}(x), \pi^{-1}(\hat{P}([x])) = P(x)\) and \(\pi^{-1}(\hat{P}^{-1}([x])) = P^{-1}(x)\) for all \(x \in X\). Therefore, \(\hat{R}\) has closed sections and \(\hat{P}\) has open sections.

**Proof of Proposition 3.** Assume \(X\) is a topological space and \(R\) is a complete, semi-transitive and continuous binary relation on it. It follows from [Sen 1969, Theorem I] that its symmetric
part $I$ is transitive. Define a binary relation $\hat{R}$ on the quotient space $X/I$ as $([x], [y]) \in \hat{R}$ if and only if $(x', y') \in R$ for all $x' \in [x]$ and all $y' \in [y]$. The definition of the induced relation $\hat{R}$ on the quotient space implies its completeness. It follows from Lemma 4 that $\hat{R}$ is anti-symmetric and continuous. Pick $[x], [y] \in X/I$ such that $[x] \neq [y]$. Assume without loss of generality that $[x] \in \hat{P}([y])$. If there exists $z \in X$ such that $[z] \in \hat{P}([y]) \cap \hat{P}^{-1}([x])$, then $[x] \in \hat{P}([z])$ and $y \in P^{-1}(z)$. Hence, $\hat{P}([z])$ and $\hat{P}^{-1}([z])$ are disjoint and open neighborhoods of $[x]$ and $[y]$, respectively. If $\hat{P}([y]) \cap \hat{P}^{-1}([x]) = \emptyset$, then $[x] \in \hat{P}([y])$ and $[y] \in \hat{P}^{-1}([x])$ imply $\hat{P}([y])$ and $\hat{P}^{-1}([x])$ are disjoint and open neighborhoods of $[x]$ and $[y]$, respectively.\[64\]

References

Anand, P. (1987): “Are the preference axioms really rational?,” *Theory and Decision*, 23(2), 189–214.

——— (1993): “The philosophy of intransitive preference,” *The Economic Journal*, 103(417), 337–346.

Arrow, K. J., and G. Debreu (1954): “Existence of an equilibrium for a competitive economy,” *Econometrica*, pp. 265–290.

Arrow, K. J., A. K. Sen, and K. Suzumura (1995, 1996): *Social Choice Re-examined*, vol. 1 and 2. New York: St. Martin’s Press.

——— (2002, 2011): *Handbook of Social Choice and Welfare*, vol. 1 and 2. North Holland: Elsevier.

Barten, A. P., and V. Böhm (1982): “Consumer theory,” *Handbook of Mathematical Economics*, 2, 381–429.

Bergstrom, T. C., R. P. Parks, and T. Rader (1976): “Preferences which have open graphs,” *Journal of Mathematical Economics*, 3(3), 265–268.

Bernheim, B. D., and A. Rangel (2009): “Beyond revealed preference: choice-theoretic foundations for behavioral welfare economics,” *Quarterly Journal of Economics*, 124(1), 51–104.

Bleichrodt, H., C. Li, I. Moscati, and P. P. Wakker (2016): “Nash was a first to axiomatize expected utility,” *Theory and Decision*, 81(3), 309–312.

Bridges, D. S., and G. B. Mehta (1995): *Representations of Preference Orderings*. Berlin: Springer-Verlag.

Broome, J. (1972): “Approximate equilibrium in economies with indivisible commodities,” *Journal of Economic Theory*, 5(2), 224–249.

Brown, R. (2006): “Groupoids, the Phragmen-Brouwer property, and the Jordan curve theorem,” *Journal of Homotopy and Related Structures*, 1(1).

\[64\] A similar result for anti-symmetric binary relations is provided by Eilenberg (1941, 1.4).
Brown, R., and O. A. Camarena (2015): “Erratum to: Groupoids, the Phragmen–Brouwer property, and the Jordan curve theorem,” *Journal of Homotopy and Related Structures*, 10(3), 669–672.

Cerreia-Vioglio, S., and E. A. Ok (2018): “The Rational Core of Preference Relations,” *Working Paper*.

Chateauneuf, A. (1987): “Continuous representation of a preference relation on a connected topological space,” *Journal of Mathematical Economics*, 16(2), 139–146.

Dasgupta, P., and E. Maskin (1986): “The existence of equilibrium in discontinuous economic games, I: Theory, II: Applications,” *The Review of Economic Studies*, 53(1), 1–41.

Debreu, G. (1952): “A social equilibrium existence theorem,” *Proceedings of the National Academy of Sciences*, 38(10), 886–893.

——— (1954): “Representation of a preference ordering by a numerical function,” in *Decision Processes*, ed. by M. Thrall, R. Davis, and C. Coombs, pp. Chapter 11, 159–165. New York: Wiley.

——— (1960): “Topological methods in cardinal utility theory,” in *Mathematical Methods in the Social Sciences*, ed. by K. Arrow, S. Karlin, and P. Suppes, pp. 16–26. California: Stanford University Press.

——— (1982): “Existence of competitive equilibrium,” *Handbook of Mathematical Economics*, 2, 697–743.

Dickman Jr., R. F. (1984): “A strong form of the Phragmen-Brouwer theorem,” *Proceedings of the American Mathematical Society*, 90(2), 333–337.

Dierker, E. (1971): “Equilibrium analysis of exchange economies with indivisible commodities,” *Econometrica*, pp. 997–1008.

Dubra, J. (2011): “Continuity and completeness under risk,” *Mathematical Social Sciences*, 61(1), 80–81.

Dubra, J., F. Maccheroni, and E. A. Ok (2004): “Expected utility theory without the completeness axiom,” *Journal of Economic Theory*, 115(1), 118–133.

Duggan, J. (2013): “Uncovered sets,” *Social Choice and Welfare*, 41(3), 489–535.

Dugundji, J. (1966): *Topology*. Boston: Allyn and Bacon.

Edelman, P. H., and R. E. Jamison (1985): “The theory of convex geometries,” *Geometriae Dedicata*, 19(3), 247–270.

Eilenberg, S. (1941): “Ordered topological spaces,” *American Journal of Mathematics*, 63(1), 39–45.

Eliaz, K., and E. A. Ok (2006): “Indifference or indecisiveness? Choice-theoretic foundations of incomplete preferences,” *Games and Economic Behavior*, 56(1), 61–86.

Evren, Ö. (2014): “Scalarization methods and expected multi-utility representations,” *Journal of Economic Theory*, 151, 30–63.

Fishburn, P. C. (1970): “Intransitive indifference in preference theory: a survey,” *Operations
Research, 18(2), 207–228.

—– (1972): Mathematics of Decision Theory. The Hague: Mouton.

Fleurbaey, M., and D. Blanchet (2013): Beyond GDP: Measuring Welfare and Assessing Sustainability. Oxford: Oxford University Press.

Fleurbaey, M., M. Salles, and J. Weymark (2006): Justice, Political Liberalism, and Utilitarianism. Oxford: Oxford University Press.

Galaabaatar, T., and E. Karni (2013): “Subjective expected utility with incomplete preferences,” Econometrica, 81(1), 255–284.

Galaabaatar, T., M. A. Khan, and M. Uyanık (2018): “Completeness and transitivity of preferences on mixture sets,” working paper.

Gerasimou, G. (2010): “Consumer theory with bounded rational preferences,” Journal of Mathematical Economics, 46(5), 708–714.

—– (2013): “On continuity of incomplete preferences,” Social Choice and Welfare, 41(1), 157–167.

—– (2015): “(Hemi) continuity of additive preference preorders,” Journal of Mathematical Economics, 58, 79–81.

—– (2017): “Indecisiveness, undesirability and overload revealed through rational choice deferral,” The Economic Journal, Online.

Ghirardato, P., F. Maccheroni, M. Marinacci, and M. Siniscalchi (2003): “A subjective spin on roulette wheels,” Econometrica, 71(6), 1897–1908.

Gilboa, I. (2009): Theory of Decision Under Uncertainty. Cambridge: Cambridge University Press.

Gorman, W. M. (1968): “The structure of utility functions,” The Review of Economic Studies, 35(4), 367–390.

Gorno, L. (2018): “The structure of incomplete preferences,” Economic Theory, 66(1), 159–185.

Harsanyi, J. C. (1955): “Cardinal welfare, individualistic ethics, and interpersonal comparisons of utility,” Journal of Political Economy, 63(4), 309–321.

Herstein, I. N., and J. Milnor (1953): “An axiomatic approach to measurable utility,” Econometrica, 21(2), 291–297.

Hervés-Beloso, C., and H. del Valle-Inclán Cruces (2018): “Continuous preferfence orderings representable by utility functions,” Journal of Economic Surveys.

Jackson, M. O. (2008): Social and Economic Networks. Princeton: Princeton University Press.

Karni, E. (2014): “Axiomatic foundations of expected utility and subjective probability,” in Handbook of the Economics of Risk and Uncertainty, vol. 1, pp. 1–39. Elsevier.

Khan, M. A., and Y. Sun (1990): “On a reformulation of Cournot-Nash equilibria,” Journal of Mathematical Analysis and Applications, 146(2), 442–460.

—– (2002): “Non-cooperative games with many players,” Handbook of Game Theory with
Khan, M. A., and M. Uyanık (2018): “On the existence of continuous binary relations on a topological space,” University of Queensland, mimeo.

Khan, M. A., and A. Yamazaki (1981): “On the cores of economies with indivisible commodities and a continuum of traders,” Journal of Economic Theory, 24(2), 218–225.

Lorimer, P. (1967): “A note on orderings,” Econometrica, pp. 537–539.

Luce, R. D. (1956): “Semiorders and a theory of utility discrimination,” Econometrica, pp. 178–191.

Magyarkuti, G. (2010): “Revealed preferences: A topological approach,” Journal of Mathematical Economics, 46(3), 320–325.

Majumdar, M., and A. Sen (1976): “A note on representing partial orderings,” The Review of Economic Studies, 43(3), 543–545.

Malinvaud, E. (1952): “Note on von Neumann-Morgenstern’s strong independence axiom,” Econometrica, p. 679.

Marschak, J. (1950): “Rational behavior, uncertain prospects, and measurable utility,” Econometrica, pp. 111–141.

Mas-Colell, A. (1974): “An equilibrium existence theorem without complete or transitive preferences,” Journal of Mathematical Economics, 1(3), 237–246.

Mas-Colell, A. (1975): “A model of equilibrium with differentiated commodities,” Journal of Mathematical Economics, 2(2), 263–295.

——— (1977): “Indivisible commodities and general equilibrium theory,” Journal of Economic Theory, 16(2), 443–456.

Mas-Colell, A., and W. R. Zame (1991): “Equilibrium theory in infinite dimensional spaces,” Handbook of Mathematical Economics, 4, 1835–1898.

McGehee, R. (1992): “Attractors for closed relations on compact Hausdorff spaces,” Indiana University Mathematics Journal, 41(4), 1165–1209.

McKenzie, L. W. (2005): Classical General Equilibrium Theory, vol. 1. Cambridge: The MIT Press.

Mehta, G. B. (1998): “Preference and utility,” Handbook of Utility Theory: Principles, 1, 1–47.

Moldau, J. H. (1996): “A simple existence proof of demand functions without standard transitivity,” Journal of Mathematical Economics, 25(3), 325–333.

Narens, L. (1985): Abstract Measurement Theory. Cambridge: MIT Press.

Nash, J. F. (1950a): “The bargaining problem,” Econometrica, pp. 155–162.

——— (1950b): “Equilibrium points in n-person games,” Proceedings of the National Academy of Sciences, 36(1), 48–49.

Newman, M. E. J. (2006): “Modularity and community structure in networks,” Proceedings of the National Academy Sciences, USA, 103(23).
Nishimura, H., and E. A. Ok (2016): “Utility representation of an incomplete and nontransitive preference relation,” Journal of Economic Theory, 166, 8577–8582.
——— (2018): “Preference structures,” Discussion paper, Mimeo.

Peleg, B. (1970): “Utility functions for partially ordered topological spaces,” Econometrica, 38(1), 93–96.

Pfanzagl, J. (1971): Theory of Measurement. Berlin: Springer-Verlag.

Rader, T. (1963): “The existence of a utility function to represent preferences,” The Review of Economic Studies, 30(3), 229–232.

Reny, P. J. (1999): “On the existence of pure and mixed strategy Nash equilibria in discontinuous games,” Econometrica, 67(5), 1029–1056.

Richter, M., and A. Rubinstein (2015): “Back to fundamentals: Equilibrium in abstract economies,” American Economic Review, 105(8), 2570–94.

Samuelson, P. A. (1981): “Bergsonian welfare economics,” in Economic welfare and the economics of Soviet socialism: essays in honor of Abram Bergson, ed. by S. Rosefields, pp. 223–266. New York: Cambridge University Press.

Schmeidler, D. (1969): “Competitive equilibria in markets with a continuum of traders and incomplete preferences,” Econometrica, pp. 578–585.
——— (1971): “A condition for the completeness of partial preference relations,” Econometrica, 39(2), 403–404.

Sen, A. (1969): “Quasi-transitivity, rational choice and collective decisions,” The Review of Economic Studies, 36(3), 381–393.
——— (1993): “Internal consistency of choice,” Econometrica, 61(3), 495–521.
——— (2017): Collective Choice and Social Welfare: An Expanded Edition. Massachusetts: Harvard University Press.

Shafer, W. (1974): “Equilibrium in abstract economies without ordered preferences,” Econometrica, 42(5), 913–919.

Shafer, W., and H. Sonnenschein (1975): “Equilibrium in abstract economies without ordered preferences,” Journal of Mathematical Economics, 2(3), 345–348.

Sonnenschein, H. (1965): “The relationship between transitive preference and the structure of the choice space,” Econometrica, 33(3), 624–634.
——— (1967): “Reply to “A Note on Orderings”,” Econometrica, 35(3, 4), 540.
——— (1971): “Demand theory without transitive indifference with applications to the theory of competitive equilibrium,” in Preferences, Utility and Demand: A Minnesota Symposium, ed. by J. Chipman, L. Hurwicz, M. Richter, and H. Sonenschein, pp. 215–234. New York: Harcourt Brace Jovanovich.

Strzalecki, T. (2013): “Temporal resolution of uncertainty and recursive models of ambiguity aversion,” Econometrica, 81(3), 1039–1074.

Suzumura, K. (2016): Choice, Preferences and Procedures. Cambridge: Harvard University
TEMKIN, L. S. (2015): *Rethinking the Good: Moral Ideals and the Nature of Practical Reasoning*. Oxford: Oxford University Press.

THOMSON, W. (2011): “Fair allocation rules,” in *Handbook of Social Choice and Welfare*, vol. 2, pp. 393–506. Elsevier.

TOURKY, R., AND N. C. YANNELIS (2000): “Markets with many more agents than commodities: Aumann’s “hidden” assumption,” *Journal of Economic Theory*, 101(1), 189–221.

TULLOCK, G. (1964): “The irrationality of intransitivity,” *Oxford Economic Papers*, 16(3), 401–406.

ULLMAN-MARGALIT, E., AND S. MORGENBESSER (1977): “Picking and choosing,” *Social Research*, 44(4), 757–785.

UZAWA, H. (1960): *Preference and Rational Choice in the Theory of Consumption*. Stanford University Press.

VIND, K. (2003): *Independence, Additivity, Uncertainty*. With contributions by B. Grodal. Berlin: Springer.

VON NEUMANN, J., AND O. MORGENSTERN (1947): *Theory of Games and Economic Behavior*. New Jersey: Princeton University Press, 2nd edn.

WAKKER, P. (1988a): “The algebraic versus the topological approach to additive representations,” *Journal of Mathematical Psychology*, 32(4), 421–435.

——— (1988b): “Continuity of preference relations for separable topologies,” *International Economic Review*, 29(1), 105–110.

WAKKER, P. P. (1989): *Additive Representations of Preferences: A New Foundation of Decision Analysis*. Boston: Kluwer Academic Publishers.

WARD, L. (1954): “Partially ordered topological spaces,” *Proceedings of the American Mathematical Society*, 5(1), 144–161.

WILDER, R. L. (1949): *Topology of Manifolds*. Berlin: American Mathematical Society Colloquium Publications XXXII.