INVERSE OBSTACLE SCATTERING FOR MAXWELL’S EQUATIONS IN AN UNBOUNDED STRUCTURE

PEIJUN LI∗, JUE WANG†, AND LEI ZHANG‡

Abstract. This paper is concerned with analysis of electromagnetic wave scattering by an obstacle which is embedded in a two-layered lossy medium separated by an unbounded rough surface. Given a dipole point source, the direct problem is to determine the electromagnetic wave field for the given obstacle and unbounded rough surface; the inverse problem is to reconstruct simultaneously the obstacle and unbounded rough surface from the electromagnetic field measured on a plane surface above the obstacle. For the direct problem, a new boundary integral equation is proposed and its well-posedness is established. The analysis is based on the exponential decay of the dyadic Green function for Maxwell’s equations in a lossy medium. For the inverse problem, the global uniqueness is proved and a local stability is discussed. A crucial step in the proof of the stability is to obtain the existence and characterization of the domain derivative of the electric field with respect to the shape of the obstacle and unbounded rough surface.

Key words. Maxwell’s equations, inverse scattering problem, unbounded rough surface, domain derivative, uniqueness, local stability

AMS subject classifications. 78A46, 78M30

1. Introduction. Consider the electromagnetic scattering of a dipole point source illumination by an obstacle which is embedded in a two-layered medium separated by an unbounded rough surface in three dimensions. An obstacle is referred to as an impenetrable medium which has a bounded closed surface; an unbounded rough surface stands for a nonlocal perturbation of an infinite plane surface such that the perturbed surface lies within a finite distance of the original plane. Given the dipole point source, the direct problem is to determine the electromagnetic wave field for the known obstacle and unbounded rough surface; the inverse problem is to reconstruct both of the obstacle and the unbounded rough surface, from the measured wave field. The scattering problems arise from diverse scientific areas such as radar and sonar, geophysical exploration, nondestructive testing, and medical imaging. In particular, the obstacle scattering in unbounded structures has significant applications in radar based object recognition above the sea surface and detection of underwater or underground mines.

As a fundamental problem in scattering theory, the obstacle scattering problem, where the obstacle is embedded in a homogeneous medium, has been examined extensively by numerous researchers. The details can be found in the monographs [6, 27] and [5, 7, 16] on the mathematical and numerical studies of the direct and inverse problems, respectively. The unbounded rough surface scattering problems have also been widely examined in both of the mathematical and engineering communities. We refer

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for various solution methods including mathematical, computational, approximate, asymptotic, and statistical methods. The scattering problems in unbounded structures are quite challenging due to two major issues: the usual Silver–Müller radiation condition is no longer valid; the Fredholm alternative argument does not apply due to the lack of compactness result. The mathematical analysis can be found in [10, 11, 18, 22, 32] and [13, 20, 23] on the well-posedness of the two-dimensional Helmholtz equation and the three-dimensional Maxwell equations, respectively. The inverse problems have also been considered mathematically and computationally for unbounded rough surfaces in [1, 2, 3, 24].

In this paper, we study the electromagnetic obstacle scattering for the three-dimensional Maxwell equations in an unbounded structure. Specifically, we consider the illumination of a time-harmonic electromagnetic wave, generated from a dipole point source, onto a perfectly electrically conducting obstacle which is embedded in a two-layered medium separated by an unbounded rough surface. The obstacle is located either above or below the surface and may have multiple disjoint components. For simplicity of presentation, we assume that the obstacle has only one component and is located above the surface. The free spaces are assumed to be filled with some homogeneous and lossy materials accounting for the energy absorption. The problem has received much attention and many computational work have been done in the engineering community [14, 17, 19]. However, the rigorous analysis is very rare, especially for the three-dimensional Maxwell equations.

In this work, we introduce an energy decaying condition to replace the Silver–Müller radiation condition in order to ensure the uniqueness of the solution. The asymptotic behaviour of dyadic Green’s function is analyzed and plays an important role in the analysis for the well-posedness of the direct problem. A new boundary integral equation is proposed for the associated boundary value problem. Based on some energy estimates, the uniqueness of the solution for the scattering problem is established. For the inverse problem, we intend to answer the following question: what information can we extract about the obstacle and the unbounded rough surface from the tangential trace of the electric field measured on the plane surface above the obstacle? The first result is a global uniqueness theorem. We show that any two obstacles and unbounded rough surfaces are identical if they generate the same data. The proof is based on a combination of the Holmgren uniqueness, unique continuation, and a construction of singular perturbation. The second result is concerned with a local stability: if two obstacles are “close” and two unbounded rough surfaces are also “close”, then for any $\delta > 0$, the measurements of the two tangential trace of the electric fields being $\delta$-close implies that both of the two obstacles and the two unbounded rough surfaces are $O(\delta)$-close. A crucial step in the stability proof is to obtain the existence and characterization of the domain derivative of the electric field with respect to the shape of the obstacle and unbounded rough surface.

The paper is organized as follows. In Section 2, we introduce the model problem and present some asymptotic analysis for dyadic Green’s function of the Maxwell equations. Section 3 is devoted to the well-posedness of the direct scattering problem. An equivalent integral representation is proposed for the boundary value problem. A new boundary integral equation is developed and its well-posedness is established. In Sections 4 and 5, we discuss the global uniqueness and local stability of the inverse problem, respectively. The domain derivative is studied. The paper is concluded with some general remarks in Section 6.
2. Problem formulation. Let us first specify the problem geometry which is shown in Figure 2.1. Let $S$ be an unbounded rough surface given by

$$S = \{ \mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = f(x_1, x_2) \},$$

where $f \in C^2(\mathbb{R}^2)$. The surface $S$ divides $\mathbb{R}^3$ into $\Omega^+_1$ and $\Omega_2$, where

$$\Omega^+_1 = \{ \mathbf{x} \in \mathbb{R}^3 : x_3 > f(x_1, x_2) \}, \quad \Omega_2 = \{ \mathbf{x} \in \mathbb{R}^3 : x_3 < f(x_1, x_2) \}.$$

Let $D$ be a bounded obstacle with $C^2$ boundary $\Gamma$. The obstacle is assumed to be a perfect electrical conductor which is located either in $\Omega^+_1$ or in $\Omega_2$. For instance, we may assume that $D \subset \subset \Omega^+_1$. Define $\Omega_j = \Omega^+_1 \setminus D$. The domain $\Omega_j$ is assumed to be filled with some homogeneous, isotropic, and absorbing medium which may be characterized by the dielectric permittivity $\varepsilon_j > 0$, the magnetic permeability $\mu_j > 0$, and the electric conductivity $\sigma_j > 0$, $j = 1, 2$.

In $\Omega_j$, the electromagnetic waves satisfy the time-harmonic Maxwell equations (time dependence $e^{-i\omega t}$):

$$\begin{align*}
\nabla \times \mathbf{E}_j &= i\omega\mu_j \mathbf{H}_j, \\
\nabla \times \mathbf{H}_j &= -i\omega\varepsilon_j \mathbf{E}_j + \mathbf{J}_j, \\
\nabla \cdot \varepsilon_j \mathbf{E}_j &= \rho_j, \\
\nabla \cdot (\mu_j \mathbf{H}_j) &= 0,
\end{align*}$$

where $\omega > 0$ is the angular frequency, $\mathbf{E}_j$, $\mathbf{H}_j$, $\mathbf{J}_j$ denote the electric field, the magnetic field, the electric current density, respectively, and $\rho_j = (i\omega)^{-1} \nabla \cdot \mathbf{J}_j$ is the electric charge density. The external current source is assumed to be located in $\Omega_1$.

The relation between the electric current density and the electric field is given by

$$\begin{align*}
\mathbf{J}_1 &= \sigma_1 \mathbf{E}_1 + \mathbf{J}_{cs} \quad \text{in } \Omega_1, \\
\mathbf{J}_2 &= \sigma_2 \mathbf{E}_2 \quad \text{in } \Omega_2,
\end{align*}$$

where $\mathbf{J}_{cs}$ stands for the current source.
Using the above constitutive relation, we obtain coupled systems

\[
\begin{align*}
\nabla \times E_1 &= i\omega \mu_1 H_1, \\
\nabla \times H_1 &= -i\omega \left( \varepsilon_1 + i\frac{\sigma_1}{\omega} \right) E_1 + J_{cs}, \\
\left( \varepsilon_1 + i\frac{\sigma_1}{\omega} \right) \nabla \cdot E_1 &= \frac{1}{i\omega} \nabla \cdot J_{cs}, \\
\nabla \cdot (\mu_1 H_1) &= 0,
\end{align*}
\]

in $\Omega_1$, and

\[
\begin{align*}
\nabla \times E_2 &= i\omega \mu_2 H_2, \\
\nabla \times H_2 &= -i\omega \left( \varepsilon_2 + i\frac{\sigma_2}{\omega} \right) E_2, \\
\left( \varepsilon_2 + i\frac{\sigma_2}{\omega} \right) \nabla \cdot E_2 &= 0, \\
\nabla \cdot (\mu_2 H_2) &= 0,
\end{align*}
\]

in $\Omega_2$.

Eliminating the magnetic field $H_1$ in (2.1), we obtain a decoupled equation for the electric field $E_1$:

\[
\nabla \times (\nabla \times E_1(x)) - \kappa_1^2 E_1(x) = i\omega \mu_1 J_{cs}(x), \quad x \in \Omega_1.
\]

Similarly, it follows from (2.2) that we may deduce a decoupled Maxwell system for the electric field $E_2$:

\[
\nabla \times (\nabla \times E_2(x)) - \kappa_2^2 E_2(x) = 0, \quad x \in \Omega_2.
\]

Here $\kappa_j = \omega \sqrt{\left( \varepsilon_j + i\frac{\sigma_j}{\omega} \right) \mu_j}$ is the wave number in $\Omega_j$, $j = 1, 2$. Since $\varepsilon_j, \mu_j, \sigma_j$ are positive constants, $\kappa_j$ is a complex constant with $\Re \kappa_j > 0, \Im \kappa_j > 0$, which accounts for the energy absorption.

By the perfect conductor assumption for the obstacle, it holds that

\[
\nu_T \times E_1 = 0 \quad \text{on } \Gamma,
\]

where $\nu_T$ denotes the unit normal vector on the boundary $\Gamma$ directed into the exterior of $D$. The usual continuity conditions need to be imposed, i.e., the tangential traces of the electric and magnetic fields are continuous across $S$:

\[
\nu_S \times E_1 = \nu_S \times E_2, \quad \nu_S \times H_1 = \nu_S \times H_2 \quad \text{on } S,
\]

where $\nu_S$ denotes the unit normal vector on $S$ pointing from $\Omega_2$ to $\Omega_1$.

The incident electromagnetic fields $(E^i, H^i)$ satisfy Maxwell’s equations

\[
\begin{align*}
\nabla \times (\nabla \times E^i(x)) - \kappa_i^2 E^i(x) &= i\omega \mu_1 J_{cs}(x), \\
\nabla \times (\nabla \times H^i(x)) - \kappa_i^2 H^i(x) &= \nabla \times J_{cs}(x),
\end{align*}
\]

in $\Omega_1$. In $\Omega_1$, the total electromagnetic fields $(E_1, H_1)$ consist of the incident fields $(E^i, H^i)$ and the scattered fields $(E^s, H^s)$. In $\Omega_2$, the electromagnetic fields $(E_2(x), H_2(x))$ are called the transmitted fields.

In addition, we propose an energy decaying condition

\[
\lim_{r \to +\infty} \int_{\partial B_r^+} |E^s|^2 ds = 0, \quad \lim_{r \to +\infty} \int_{\partial B_r^+} |H^s|^2 ds = 0
\]
and

\begin{equation}
\lim_{r \to +\infty} \int_{\partial B_r} |E_2|^2 \text{d}s = 0, \quad \lim_{r \to +\infty} \int_{\partial B_r} |H_2|^2 \text{d}s = 0,
\end{equation}

where \( \partial B_r^\pm \) denotes the hemisphere of radius \( r \) above or below \( S \).

The dyadic Green function is defined by the solution of the following equation

\begin{equation}
\nabla_x \times (\nabla_x \times G_j(x - y)) - \kappa_j^2 G_j(x - y) = \delta(x - y) I \quad \text{in } \Omega_j,
\end{equation}

where \( I \) is the unitary dyadic and \( \delta \) is the Dirac delta function. It is known that the dyadic Green function is given by

\begin{equation}
G_j(x - y) = \left[ I + \frac{\nabla_y \nabla_y}{\kappa_j^2} \right] \frac{\exp(i\kappa_j|x - y|)}{4\pi|x - y|}.
\end{equation}

We assume that the dipole point source is located at \( x_s \in \Omega_1 \) and has a polarization \( q \in \mathbb{R}^3, \|q\| = 1 \). Induced by this dipole point source, the incident electromagnetic fields are

\begin{equation}
E'(x) = G_1(x - x_s)q, \quad H'(x) = \frac{1}{\omega \mu_1} (\nabla \times E'(x)), \quad x \in \Omega_1.
\end{equation}

Hence the current source \( J_{cs} \) satisfies

\[ \omega \mu_1 J_{cs}(x) = q \delta(x - x_s), \quad x \in \Omega_1. \]

Denote by \( \mathcal{T}_j \) the set of functions \( \psi \in C^2(\Omega_j) \cap C^{0,\alpha}(\overline{\Omega_j}), j = 1, 2 \). The direct scattering problem can be stated as follows.

**Problem 2.1.** Given the incident field \( E' \) in (2.12), the direct problem is to determine \( E^s \in \mathcal{T}_1 \) and \( E_2 \in \mathcal{T}_2 \) such that

(i) The electric fields \( E_1 = E^s + E' \) and \( E_2 \) satisfy (2.8) and (2.4), respectively;

(ii) The electric field \( E_1 \) satisfies the boundary condition (2.3);

(iii) The electromagnetic fields \( (E_1, H_1), j = 1, 2 \) satisfy (2.4);

(iv) The scattered fields \( (E^s, H^s) \) and the transmitted fields \( (E_2, H_2) \) satisfy the radiation conditions (2.8) and (2.4), respectively.

It requires to study the dyadic Green function in order to find the integral representation of the solution for the scattering problem. The details may be found in [4] on the general properties of the dyadic Green function.

**Lemma 2.2.** For each fixed \( y \in \Omega_j \), the dyadic Green function \( G_j \) given in (2.11) admits the asymptotic behaviour

\[ G_j(x - y) = \mathcal{O} \left( \frac{\exp(-\Im(\kappa_j)|x|)}{|x|} \right) \hat{I} \quad \text{as } |x - y| \to \infty, \]

\[ \nabla_x \times G_j(x - y) = \mathcal{O} \left( \frac{\exp(-\Im(\kappa_j)|x|)}{|x|} \right) \hat{I} \quad \text{as } |x - y| \to \infty, \]

where \( \hat{I} := \hat{e} \hat{e} \) and \( \hat{e} = (1, 1, 1)^\top \).

**Proof.** Following

\[ |x - y| = \sqrt{|x|^2 - 2x \cdot y + |y|^2} = |x| - \hat{x} \cdot \hat{y} + \mathcal{O} \left( \frac{1}{|x|} \right) \quad \text{as } |x| \to \infty, \]
where \( \hat{x} = x / |x| \), we have

\[
\frac{\exp(i\kappa_j|\hat{x}-y|)}{|x-y|} = \frac{\exp(i\kappa_j|x|)}{|x|} \times \left\{ \exp(-i\kappa_j \hat{x} \cdot y) + O\left(\frac{1}{|x|}\right) \right\} \quad \text{as } |x| \to \infty
\]

(2.13)

uniformly for all \( y \) satisfying \( |x - y| \to \infty \). By (2.13), for \( \exists \kappa_j > 0 \), we obtain for \( |x| \to \infty \) that

\[
G_j(x - y) = \left[ I + \frac{\nabla_y \nabla_y}{\kappa_j^2} \right] \frac{\exp(i\kappa_j|x|)}{4\pi|x|} \left\{ \exp(-i\kappa_j \hat{x} \cdot y) + O\left(\frac{1}{|x|}\right) \right\} 
\]

\[
= \frac{\exp(i\kappa_j|x|)}{4\pi|x|} \left\{ [I - \hat{x} \hat{x}] \exp(-i\kappa_j \hat{x} \cdot y) + O\left(\frac{1}{|x|}\right) \right\} 
\]

\[
= O\left(\frac{|\exp(-\Im(\kappa_j)|x|)|}{|x|}\right) \hat{i}
\]

and

\[
\nabla_x \times G_j(x - y) = -\nabla_y \times G_j(x - y)
\]

\[
= \frac{\exp(i\kappa_j|x|)}{4\pi|x|} \left\{ -\nabla_y \times [I - \hat{x} \hat{x}] \exp(-i\kappa_j \hat{x} \cdot y) + O\left(\frac{1}{|x|}\right) \right\} 
\]

\[
= i\kappa_j \frac{\exp(i\kappa_j|x|)}{4\pi|x|} \left\{ \hat{x} \times [I - \hat{x} \hat{x}] \exp(-i\kappa_j \hat{x} \cdot y) + O\left(\frac{1}{|x|}\right) \right\} 
\]

\[
= O\left(\frac{|\exp(-\Im(\kappa_j)|x|)|}{|x|}\right) \hat{i},
\]

which completes the proof. \( \square \)

We introduce some Banach spaces. For \( V \subset \mathbb{R}^3 \), denote by \( BC(V) \) the set of bounded and continuous functions on \( V \), which is a Banach space under the norm

\[
\| \phi \|_\infty = \sup_{x \in V} |\phi(x)|.
\]

For \( 0 < \alpha \leq 1 \), denote by \( C^{0,\alpha}(V) \) the Banach space of functions \( \phi \in BC(V) \) which are uniformly Hölder continuous with exponent \( \alpha \). The norm \( \| \cdot \|_{C^{0,\alpha}(V)} \) is defined by

\[
\| \phi \|_{C^{0,\alpha}(V)} = \| \phi \|_\infty + \sup_{x,y \in V} \max_{x \neq y} \frac{|\phi(x) - \phi(y)|}{|x - y|^\alpha}.
\]

Let \( C^{1,\alpha}(V) = \{ \phi \in BC(V) \cap C^1(V) : \nabla \phi \in C^{0,\alpha}(V) \} \), which is a Banach space under the norm

\[
\| \phi \|_{C^{1,\alpha}(V)} = \| \phi \|_\infty + \| \nabla \phi \|_{C^{0,\alpha}(V)}.
\]

3. Well-posedness of the direct problem. In this section, we show the existence and uniqueness of the solution to Problem 2.1 by using the boundary integral equation method. First we derive an integral representation for the solution of Prob-
Applying the vector dyadic Green's second theorem to 

where 

\[ E \left( \omega \mu G_1(x - y) \right) \cdot \left[ \nu_S(y) \times H_1(y) \right] + \] 

\[ \left[ \nabla_x \times G_1(x - y) \right] \cdot \left[ \nu_S(y) \times E_1(y) \right] \] 

dy 

(3.1) 

+ \int_\Gamma \left[ \left[ \omega \mu_1 G_1(x - y) \right] \cdot \left[ \nu_T(y) \times H_1(y) \right] \right] ds_y, \quad x \in \Omega_1, 

and 

\[ E_2(x) = -\int_S \left[ \left[ \omega \mu_1 G_2(x - y) \right] \cdot \left[ \nu_S(y) \times H_2(y) \right] \right] ds_y, \quad x \in \Omega_2. \] 

(3.2) 

**Proof.** Let \( B_r = \{ x \in \mathbb{R}^3 : |x| < r \} \). Denote \( \Omega_r = B_r \cap \Omega_1 \) with the boundary \( \partial \Omega_r = \partial B_r \cap \Omega_1 + \Gamma \cup S_r \), where \( \partial B_r^+ = \partial B_r \cap \Omega_1 \) and \( S_r = S \cap B_r^+ \). For each fixed \( x \in \Omega_r \), applying the vector dyadic Green's second theorem to \( E_1 \) and \( G_1 \) in the region \( \Omega_r \), we obtain 

\[ \int_{\Omega_r} \left[ E_1(y) \cdot \nabla_y \times G_1(y - x) \right] - \left[ \nabla_y \times E_1(y) \right] \cdot G_1(y - x) \] 

\[ = -\int_{\partial \Omega_r} \left[ \nu(y) \times (\nabla_y \times E_1(y)) \right] \cdot G_1(y - x) 

+ \left[ \nu(y) \times E_1(y) \right] \cdot \left[ \nabla_y \times G_1(y - x) \right] ds_y, \] 

(3.3) 

where \( \nu = \nu(y) \) stands for the unit normal vector at \( y \in \partial \Omega_r \) pointing out of \( \Omega_r \). 

It follows from (2.8) and (2.10) that 

\[ \int_{\Omega_r} \left[ E_1(y) \cdot \nabla_y \times G_1(y - x) \right] - \left[ \nabla_y \times E_1(y) \right] \cdot G_1(y - x) \] 

\[ = \int_{\Omega_r} \left[ E_1(y) \cdot \nabla_y \times G_1(y - x) - \kappa^2 G_1(y - x) \right] ds_y \] 

\[ - \int_{\Omega_r} \left[ \nabla_y \times E_1(y) - \kappa^2 E_1(y) \right] \cdot G_1(y - x) ds_y \] 

\[ = \int_{\Omega_r} \left[ E_1(y) \cdot (\delta(y - x)) \right] ds_y - \int_{\Omega_r} \left[ \omega \mu_1 J_{cs}(y) \cdot \nabla \times G_1(y - x) \right] ds_y \] 

\[ = E_1(x) - \int_{\Omega_r} \left[ \omega \mu_1 J_{cs}(y) \cdot G_1(y - x) \right] ds_y, \] 

where 

\[ \lim_{r \to +\infty} \int_{\Omega_r} \left[ \omega \mu_1 J_{cs}(y) \cdot G_1(y - x) \right] ds_y = \int_{\Omega_1} \left[ q \delta(y - x) \cdot G_1(y - x) \right] ds_y \] 

\[ = G_1(x - x_s)q = E'(x). \] 

(3.4)
Hence, letting \( r \to +\infty \), with the aid of (3.3)–(3.4), we have

\[
E_1(x) - E'(x) = - \int_{\partial \Omega_1} \left\{ [\nu(y) \times (\nabla_y \times E_1(y))] \cdot G_1(y - x) + [\nu(y) \times E_1(y)] \cdot [\nabla_y \times G_1(y - x)] \right\} ds_y
\]

\[
= - \left( \int_S + \int_\Gamma + \lim_{r \to +\infty} \int_{\partial B^r_+} \right) \left\{ [\nu(y) \times (\nabla_y \times E_1(y))] \cdot G_1(y - x) + [\nu(y) \times E_1(y)] \cdot [\nabla_y \times G_1(y - x)] \right\} ds_y.
\]

(3.5)

Following Lemma 2.2 and (2.8)–(2.9), we obtain for \( r \to +\infty \) that

\[
\left| \int_{\partial B^r_+} \left\{ [\nu(y) \times (\nabla_y \times E'(y))] \cdot G_1(y - x) + [\nu(y) \times E'(y)] \cdot [\nabla_y \times G_1(y - x)] \right\} ds_y \right|
\]

\[
\leq \left[ \omega^2 \mu^2 \int_{\partial B^r_+} |H'(y)|^2 ds_y \right]^{\frac{1}{2}} \cdot \left[ \int_{\partial B^r_+} |G_1(y - x)|^2 ds_y \right]^{\frac{1}{2}}
\]

\[
+ \left[ \int_{\partial B^r_+} |E'(y)|^2 ds_y \right]^{\frac{1}{2}} \cdot \left[ \int_{\partial B^r_+} |\nabla_y \times G_1(y - x)|^2 ds_y \right]^{\frac{1}{2}} \to 0.
\]

(3.6)

By Lemma 2.2 and the definition of incident field \( E^i \), we have for \( r \to +\infty \) that

\[
\left| \int_{\partial B^r_+} \left\{ [\nu(y) \times (\nabla_y \times E^i(y))] \cdot G_1(y - x) + [\nu(y) \times E^i(y)] \cdot [\nabla_y \times G_1(y - x)] \right\} ds_y \right|
\]

\[
\leq \left[ \omega^2 \mu^2 \int_{\partial B^r_+} |H^i(y)|^2 ds_y \right]^{\frac{1}{2}} \cdot \left[ \int_{\partial B^r_+} |G_1(y - x)|^2 ds_y \right]^{\frac{1}{2}}
\]

\[
+ \left[ \int_{\partial B^r_+} |E^i(y)|^2 ds_y \right]^{\frac{1}{2}} \cdot \left[ \int_{\partial B^r_+} |\nabla_y \times G_1(y - x)|^2 ds_y \right]^{\frac{1}{2}} \to 0.
\]

(3.7)

Using (3.5)–(3.7) and conditions (ii), (iv) in Problem 2.1 and letting \( r \to +\infty \), we have for each fixed \( x \in \Omega_1 \) that

\[
E_1(x) - E'(x) = - \int_S \left\{ [\nu(y) \times (\nabla_y \times E_1(y))] \cdot G_1(y - x) + [\nu(y) \times E_1(y)] \cdot [\nabla_y \times G_1(y - x)] \right\} ds_y
\]

\[
= \int_S \left\{ [\nu(y) \times (\nabla_y \times E_1(y))] \cdot G_1(y - x) + [\nu(y) \times E_1(y)] \cdot [\nabla_y \times G_1(y - x)] \right\} ds_y
\]

\[
+ \int_\Gamma \left\{ [\nu(y) \times (\nabla_y \times E_1(y))] \cdot G_1(y - x) + [\nu(y) \times E_1(y)] \cdot [\nabla_y \times G_1(y - x)] \right\} ds_y.
\]
Similarly, for each fixed $x \in \Omega_2$, we have
\[ E_2(x) = -\int_{\Gamma} \left\{ [G_2(x - y)] \cdot [\nu_S(y) \times (\nabla_y \times E_2(y))] + [\nabla_x \times G_2(x - y)] \cdot [\nu_S(y) \times E_2(y)] \right\} ds_y \]
\[ = -\int_{\Gamma} \left\{ [i\omega \mu_2 G_2(x - y)] \cdot [\nu_S(y) \times H_2(y)] + [\nabla_x \times G_2(x - y)] \cdot [\nu_S(y) \times E_2(y)] \right\} ds_y, \]
where
\[ \nu_S(y) \times E_j(y) = \lim_{h \to +0} \nu_S(y) \times E_j(y) + (-1)^j h \nu_S(y), \]
\[ \nu_S(y) \times [\nabla_y \times E_j(y)] = \lim_{h \to +0} \nu_S(y) \times [\nabla_y \times E_j(y) + (-1)^j h \nu_S(y)] \]
are to be understood in the sense of uniform convergence on $S$, and $j = 1, 2$.

Finally, from the jump relations and (2.5), we note that the integral representations (3.11) lead to the boundary integral equations:
\[ \frac{1}{2} \nu_S(x) \times E_1(x) = \nu_S(x) \times E'(x) + \int_{\Gamma} \left\{ [i\omega \mu_1 \nu_S(x) \times G_1(x - y)] \cdot [\nu_S(y) \times H_1(y)] + [\nu_S(x) \times (\nabla_x \times G_1(x - y))] \cdot [\nu_S(y) \times E_1(y)] \right\} ds_y \]
\[ + \int_{\Gamma} \left\{ [i\omega \mu_1 \nu_S(x) \times G_1(x - y)] \cdot [\nu_S(y) \times H_1(y)] \right\} ds_y, \quad x \in S, \tag{3.8} \]
\[ 0 = \nu_S(x) \times E'(x) + \int_{\Gamma} \left\{ [i\omega \mu_1 \nu_S(x) \times G_1(x - y)] \cdot [\nu_S(y) \times H_1(y)] + [\nu_S(x) \times (\nabla_x \times G_1(x - y))] \cdot [\nu_S(y) \times E_1(y)] \right\} ds_y \]
\[ + \int_{\Gamma} \left\{ [i\omega \mu_1 \nu_S(x) \times G_1(x - y)] \cdot [\nu_S(y) \times H_1(y)] \right\} ds_y, \quad x \in \Gamma, \tag{3.9} \]
and
\[ \frac{1}{2} \nu_S(x) \times E_2(x) = -\int_{\Gamma} \left\{ [i\omega \mu_2 \nu_S(x) \times G_2(x - y)] \cdot [\nu_S(y) \times H_2(y)] + [\nu_S(x) \times (\nabla_x \times G_2(x - y))] \cdot [\nu_S(y) \times E_2(y)] \right\} ds_y, \quad x \in S. \tag{3.10} \]

Hence, the electric fields $\{E_1, E_2\}$ satisfy the boundary integral equations (3.8–3.10) and the continuity conditions
\[ \nu_S \times E_1 = \nu_S \times E_2, \quad \nu_S \times H_1 = \nu_S \times H_2 \quad \text{on } S, \tag{3.11} \]
which completes the proof. \[ \]

To show the well-posedness of the boundary integral equations (3.8–3.10), we introduce the normed subspace of continuous tangential fields
\[ \mathcal{H}(S) := \{ \psi \in C(S) : \nu_S \cdot \psi = 0 \}, \]
and the normed space of uniformly Hölder continuous tangential fields
\[ \mathcal{H}^{0,\alpha}(S) := \{ \psi \in \mathcal{H}(S) : \psi \in C^{0,\alpha}(S) \}. \]
We consider the integral operator $T : \mathcal{T}^{0,\alpha}(S) \to \mathcal{T}^{0,\alpha}(S)$ defined by

$$(T \Psi)(x) = \int_S [i \omega_1 \nu_S(x) \times G_1(x - y)] \cdot [\Psi(y)] ds_y$$

$$= \int_{\mathbb{R}^2} [i \omega_1 \nu_S(x) \times G_1(x - y)] \cdot [\Psi(y)] |_{y_3 = f(y_1, y_2)} (1 + f_{y_1}^2 + f_{y_2}^2)^{1/2} dy_1 dy_2,$$

and the integral operator $K : \mathcal{T}^{0,\alpha}(S) \to \mathcal{T}^{0,\alpha}(S)$ defined by

$$(K \Phi)(x) = \int_S [\nu_S(x) \times (\nabla \times G_1(x - y))] \cdot [\Phi(y)] ds_y$$

$$= \int_{\mathbb{R}^2} [i \omega_1 \nu_S(x) \times G_1(x - y)] \cdot [\Psi(y)] |_{y_3 = f(y_1, y_2)} (1 + f_{y_1}^2 + f_{y_2}^2)^{1/2} dy_1 dy_2.$$

For each $n \in \mathbb{Z}^+$, define the truncated operator $T_n : \mathcal{T}^{0,\alpha}(S_n) \to \mathcal{T}^{0,\alpha}(S_n)$ by

$$(T_n \Psi)(x) = \int_{-n}^{n} \int_{-n}^{n} [i \omega_1 \nu_S(x) \times G_1(x - y)] \cdot [\Psi(y)] |_{y_3 = f(y_1, y_2)}$$

$$\times (1 + f_{y_1}^2 + f_{y_2}^2)^{1/2} dy_1 dy_2,$$

and the operator $K_n : \mathcal{T}^{0,\alpha}(S_n) \to \mathcal{T}^{0,\alpha}(S_n)$ by

$$(K_n \Phi)(x) = \int_{-n}^{n} \int_{-n}^{n} [i \omega_1 \nu_S(x) \times G_1(x - y)] \cdot [\Psi(y)] |_{y_3 = f(y_1, y_2)}$$

$$\times (1 + f_{y_1}^2 + f_{y_2}^2)^{1/2} dy_1 dy_2,$$

where $S_n = \{ x \in S : |x_j| \leq n, j = 1, 2 \}$.

It follows from [6] Theorems 2.32 and 2.33 that the integral operators $T_n$ and $K_n$ are compact. We show that the integral operators $T$ and $K$ are also compact. Hence the boundary integral equations \((3.8) - (3.10)\) are of the Fredholm type, i.e., the existence of the solution follows immediately from the uniqueness of the solution.

**Lemma 3.2.** The integral operators $T$ and $K$ are compact.

**Proof.** For each fixed $x \in S_n$, it follows from \((3.12)\) and \((3.13)\) that

$$(T \Psi)(x) - (T_n \Psi)(x) = \left( \int_{-n}^{+n} \int_{-n}^{+n} \int_{-n}^{+n} \int_{-n}^{+n} \right) \varphi(x, y_1, y_2) dy_1 dy_2$$

$$= I_1 + I_2 + I_3 + I_4,$$

where

$$\varphi(x, y_1, y_2) = [i \omega_1 \nu_S(x) \times G_1(x - y)] \cdot [\Psi(y)] |_{y_3 = f(y_1, y_2)} (1 + f_{y_1}^2 + f_{y_2}^2)^{1/2}.$$
By Lemma 2.2 we have for $n \to +\infty$ that

$$|I_1| \leq \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |\varphi(x, y_1, y_2)| dy_1 dy_2 \leq C \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left|G_1(x-y) \cdot |\Psi(y)|_{y_3=f(y_1,y_2)}\right| dy_1 dy_2$$

$$\leq C \left\|\Psi\right\|_{C^{0,\alpha}(S)} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left(\frac{\exp(-3(\kappa_1)|y|)}{|y|}ight)_{y_3=f(y_1,y_2)} dy_1 dy_2 \leq C \left\|\Psi\right\|_{C^{0,\alpha}(S)} \int_{0}^{+\infty} \exp \left(-\frac{1}{2}3(\kappa_1)\right)_{y} dy_1 \int_{n}^{+\infty} \exp \left(-\frac{n}{2}3(\kappa_1)\right)_{y} dy_2$$

$$= C \left\|\Psi\right\|_{C^{0,\alpha}(S)} \left(\frac{2}{3(\kappa_1)}\right)^2 \left(\frac{1}{n} \exp \left(-\frac{n}{2}3(\kappa_1)\right)\right) \to 0,$$

where $C$ is a positive constant and may change from step to step. Similarly, we may show for $j = 2, 3, 4$ that

$$|I_j| \leq C \left\|\Psi\right\|_{C^{0,\alpha}(S)} \left(\frac{1}{n} \exp \left(-\frac{n}{2}3(\kappa_1)\right)\right) \to 0 \text{ as } n \to +\infty.$$  

Combining (3.16), (3.18) leads to

$$|(T\Psi)(x) - (T_n\Psi)(x)| \leq \sum_{j=1}^{4} |I_j| \leq C \left\|\Psi\right\|_{C^{0,\alpha}(S)} \left(\frac{1}{n} \exp \left(-\frac{n}{2}3(\kappa_1)\right)\right) \to 0 \text{ as } n \to +\infty.$$  

Hence we have

$$\|T - T_n\Psi\|_{\infty} \leq C \left\|\Psi\right\|_{C^{0,\alpha}(S)} \left(\frac{1}{n} \exp \left(-\frac{n}{2}3(\kappa_1)\right)\right) \to 0 \text{ as } n \to +\infty.$$  

For each fixed $x, \bar{x} \in S_n$ and $x \neq \bar{x}$, it follows from (3.15) and (3.14) that

$$((T - T_n)\Psi)(x) - ((T - T_n)\Psi)(\bar{x}) = \left(\int_{n}^{+\infty} \int_{n}^{+\infty} \int_{n}^{+\infty} \int_{n}^{+\infty} \int_{n}^{+\infty} \int_{n}^{+\infty} \int_{n}^{+\infty} \right) \left[\varphi(x, y_1, y_2) - \varphi(\bar{x}, y_1, y_2)\right] dy_1 dy_2$$

$$= I_5 + I_6 + I_7 + I_8.$$  

From Lemma 2.2 and the mean value theorem, we get

$$|G_1(x-y) - G_1(\bar{x} - y)| \leq C \frac{\exp(-3(\kappa_1)|y|)}{|y|} |x - \bar{x}|.$$
Therefore

\[ |I_5| \leq \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |\varphi(x, y_1, y_2) - \varphi(\bar{x}, y_1, y_2)| dy_1 dy_2 \]

\[ \leq C \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left\| G_1(x - y) - G_1(\bar{x} - y) \right\| \cdot |\Psi(y)|_{y_1 = f(y_1, y_2)} dy_1 dy_2 \]

\[ \leq C(|x - \bar{x}|) \sup_{y \in S} |\Psi(y)| \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left( \frac{\exp(-3(\kappa_1)|y|)}{|y|} \right)_{y_1 = f(y_1, y_2)} dy_1 dy_2 \]

(3.21) \leq C(|x - \bar{x}|) \|\Psi\|_{C^{0, \alpha}(S)} \left( \frac{2}{3(\kappa_1)} \right)^2 \left( \frac{1}{n} \exp \left( -n \frac{\alpha}{2} 3(\kappa_1) \right) \right).

Similarly, for \( j = 6, 7, 8 \), we also have

(3.22) \quad |I_j| \leq C(|x - \bar{x}|) \|\Psi\|_{C^{0, \alpha}(S)} \left( \frac{1}{n} \exp \left( -\frac{n}{2} 3(\kappa_1) \right) \right).

Combining (3.20) and (3.22) and noting \( 0 < \alpha \leq 1 \), we obtain

\[ \left\| ((T - T_n)\Psi)(x) - ((T - T_n)\Psi)(\bar{x}) \right\|_{x - \bar{x}|^n} \leq \sum_{j=5}^{8} |I_j| \leq C(|x - \bar{x}|^{1-\alpha}) \|\Psi\|_{C^{0, \alpha}(S)} \left( \frac{1}{n} \exp \left( -\frac{n}{2} 3(\kappa_1) \right) \right) \]

\[ \leq C(n^{1-\alpha}) \|\Psi\|_{C^{0, \alpha}(S)} \left( \frac{1}{n} \exp \left( -\frac{n}{2} 3(\kappa_1) \right) \right) \]

(3.23) = C\|\Psi\|_{C^{0, \alpha}(S)} \left( n^{-\alpha} \exp \left( -\frac{n}{2} 3(\kappa_1) \right) \right) \rightarrow 0 \quad \text{as} \quad n \rightarrow +\infty.

For \( 0 < \alpha \leq 1 \), it can be deduced from (3.19) and (3.23) that

\[ \|T - T_n\|_{C^{0, \alpha}(S)} = \sup_{\|\Psi\|_{C^{0, \alpha}(S)} \neq 0} \frac{\|(T - T_n)\Psi\|_{C^{0, \alpha}(S)}}{\|\Psi\|_{C^{0, \alpha}(S)}} \]

\[ = \sup_{\|\Psi\|_{C^{0, \alpha}(S)} \neq 0} \frac{1}{\|\Psi\|_{C^{0, \alpha}(S)}} \left[ \|(T - T_n)\Psi\|_{\infty} \right. \]

\[ + \sup_{x, \bar{x} \in S} \left. \frac{|((T - T_n)\Psi)(x) - ((T - T_n)\Psi)(\bar{x})|}{|x - \bar{x}|^n} \right] \]

\[ \leq C \left( n^{-\alpha} \exp \left( -\frac{n}{2} 3(\kappa_1) \right) \right) \rightarrow 0 \quad \text{as} \quad n \rightarrow +\infty, \]

which shows that the operator \( T \) is compact on \( \mathbb{T}^{0, \alpha}(S) \). Similarly, it can be shown from (3.13) and (3.15) that operator \( K \) is also compact on \( \mathbb{T}^{0, \alpha}(S) \). \( \square \)

**Theorem 3.3.** Let \( E^* \in T_1, E_2 \in T_2 \) have the integral representations (3.1)–(3.2) and satisfy the boundary integral equations (3.8)–(3.10) with the continuity conditions (3.11). Then \( (E_1, E_2) \) are the solutions of Problem 2.7.

**Proof.** We only show the proof for the field \( E_1 \). If the field \( E^* \in T_1 \) has the
It follows from (2.7) and (3.25) that
equations with the aid of (2.10), we obtain
\[ E^s(x) = \int_S \left\{ [i\omega \mu_1 G_1(x - y)] \cdot [\nu_S(y) \times H_1(y)] \right. \\
+ [\nabla_y \times G_1(x - y)] \cdot [\nu_S(y) \times E_1(y)] \right\} ds_y \\
\] (3.24) \\
\[ + \int_{\Gamma} \left\{ [i\omega \mu_1 G_1(x - y)] \cdot [\nu_T(y) \times H_1(y)] \right\} ds_y, \quad x \in \Omega_1. \]

It is easy to verify that \( \nu_T(y) \times E_1(y) \rvert_{L} = 0 \), i.e., \( E_1 = E^s + E^i \) satisfies the boundary condition (ii) of Problem 2.1.

Noting that for any \( x \in \Omega_1 \) and \( y \in S \cup \Gamma \), we have \( x \neq y \). Taking double curl of (3.24), multiplying (3.24) by \(-\kappa_i^2 = -\omega^2 \mu_1 (\varepsilon_1 + i\omega \sigma)\), and adding the resulting two equations with the aid of (2.10), we obtain
\[ \nabla \times (\nabla \times E^s(x)) - \kappa_i^2 E^s(x) \]
\[ = \int_S \left\{ i\omega \mu_1 [\nabla_x \times \nabla_x \times G_1(x - y) - \kappa_i^2 G_1(x - y)] \cdot [\nu_S(y) \times H_1(y)] \right. \\
+ [\nabla_x \times (\nabla_x \times \nabla_x G_1(x - y) - \kappa_i^2 G_1(x - y))] \cdot [\nu_S(y) \times E_1(y)] \right\} ds_y \\
\[ + \int_{\Gamma} \left\{ i\omega \mu_1 [\nabla_x \times \nabla_x \times G_1(x - y) - \kappa_i^2 G_1(x - y)] \cdot [\nu_T(y) \times H_1(y)] \right\} ds_y \]
(3.25) = 0, \quad x \in \Omega_1.

It follows from (2.7) and (3.25) that
\[ \nabla \times (\nabla \times E_1(x)) - \kappa_i^2 E_1(x) \]
\[ = [\nabla \times (\nabla \times E^s(x)) - \kappa_i^2 E^s(x)] + [\nabla \times \nabla \times E^i(x) - \kappa_i^2 E^i(x)] \]
\[ = i\omega \mu_1 J_{\varepsilon_1}(x), \quad x \in \Omega_1. \]

Furthermore, with the help of Lemma 3.2 and (3.24), we deduce that
\[ |E^s(x)| \leq C \left[ \int_S |G_1(x - y)| \cdot |\nu_S(y) \times H_1(y)| ds_y \right. \\
\[ + \int_S |\nabla_x \times G_1(x - y)| \cdot |\nu_S(y) \times E_1(y)| ds_y \right. \\
\[ + \int_{\Gamma} |G_1(x - y)| \cdot |\nu_T(y) \times H_1(y)| ds_y \right] \]
\[ \leq C \left[ \|\nu_S \times H_1\|_{C^{0,\alpha}(S)} \int_S |G_1(x - y)| ds_y \right. \\
\[ + \|\nu_S \times E_1\|_{C^{0,\alpha}(S)} \int_S |\nabla_x \times G_1(x - y)| ds_y \right. \\
\[ + \|\nu_T \times H_1\|_{C^{0,\alpha}(\Gamma)} \int_{\Gamma} |G_1(x - y)| ds_y \right] \]
\[ \leq C \left[ \lim_{n \to +\infty} \int_{S_n} |G_1(x - y)| ds_y + \lim_{n \to +\infty} \int_{S_n} |\nabla_x \times G_1(x - y)| ds_y \right. \\
\[ + \int_{\Gamma} |G_1(x - y)| ds_y \]. \] (3.26)
For each fixed $n \geq 1$, as $|x| \to +\infty$, by Lemma 2.2 we have

$$
\int_{S_n} |G_1(x - y)| ds_y \leq C \int_{S_n} \frac{\exp \left( \frac{1}{2} i \kappa_1 |x - y| \right)}{|x|} \exp \left( -\frac{1}{2} i \kappa_1 |x - y| \right) ds_y
$$

$$
\leq C \frac{\exp \left( -\frac{1}{2} \Im(\kappa_1) |x| \right)}{|x|} \int_{S_n} \exp \left( -\frac{1}{2} \Im(\kappa_1) |y| \right) ds_y
$$

$$
\leq C \frac{\exp \left( -\frac{1}{2} \Im(\kappa_1) |x| \right)}{|x|} \left( \int_0^n \exp \left( -\frac{1}{4} \Im(\kappa_1) y_1 \right) dy_1 \right)^2
$$

$$
\leq C \frac{\exp \left( -\frac{1}{2} \Im(\kappa_1) |x| \right)}{|x|} \left( 1 - \exp \left( -\frac{n}{4} \Im(\kappa_1) \right) \right)^2
$$

(3.27)

and

$$
\int_{S_n} |\nabla_x \times G_1(x - y)| ds_y \leq C \frac{\exp \left( -\frac{1}{2} \Im(\kappa_1) |x| \right)}{|x|} \left( 1 - \exp \left( -\frac{n}{4} \Im(\kappa_1) \right) \right)^2.
$$

(3.28)

Similarly, we can obtain

$$
\int_{\Gamma} |G_1(x - y)| ds_y \leq C \frac{\exp \left( -\Im(\kappa_1) |x| \right)}{|x|} \int_{\Gamma} \left| \exp \left( -i \kappa_1 \hat{x} \cdot y \right) |I - \hat{x} \hat{y}| + O \left( \frac{1}{|x|} \right) \right| ds_y
$$

$$
\leq C \frac{\exp \left( -\Im(\kappa_1) |x| \right)}{|x|}.
$$

(3.29)

Combining (3.26) – (3.29), we have for $\Im(\kappa_1) > 0$ that

$$
|E^s(x)| = O \left( \frac{\exp \left( -\frac{1}{2} \Im(\kappa_1) |x| \right)}{|x|} \right) \text{ as } |x| \to +\infty
$$

and

$$
\int_{\partial B_r^+} |E^s|^2 ds_x \leq C \int_{\partial B_r^+} \frac{\exp \left( -\Im(\kappa_1) r \right)}{r^2} ds_x
$$

$$
\leq C \left( \frac{\exp \left( -\Im(\kappa_1) r \right)}{r^2} \right) \frac{4 \pi r^2} = C \exp \left( -\Im(\kappa_1) r \right) \to 0 \text{ as } r \to +\infty,
$$

where $C$ is a positive constant independent of $r$.

Similarly, we can also show that

$$
\lim_{r \to +\infty} \int_{\partial B_r^+} |H^s|^2 ds_x = \lim_{r \to +\infty} \int_{\partial B_r^-} |E_2|^2 ds_x = \lim_{r \to +\infty} \int_{\partial B_r^-} |H_2|^2 ds_x = 0,
$$

which complete the proof. \qed

It can be seen from Theorems 3.1 and 3.3 that there exits a solution of Problem 2.1 by using the boundary integral equation method. To prove the uniqueness, it suffices to show that $E_1 = E^s$ and $E_2$ vanish identically in $\Omega_1$ and $\Omega_2$ if $E^s = 0$. For the sake of brevity for the proof, we consider the homogeneous Maxwell’s equations

$$
\nabla \times (\nabla \times E_j) - \kappa_j^2 E_j = 0 \text{ in } \Omega_j,
$$

(3.30)

along with the boundary condition

$$
\nu_G \times E_1 = 0 \text{ on } \Gamma.
$$

(3.31)
and the continuity conditions
\begin{equation}
\nu_S \times E_1 = \nu_S \times E_2, \quad \nu_S \times H_1 = \nu_S \times H_2 \quad \text{on } S,
\end{equation}
and the radiation conditions
\begin{equation}
\lim_{r \to +\infty} \int_{\partial B^+} |E_1|^2 \, ds = \lim_{r \to +\infty} \int_{\partial B^+} |H_1|^2 \, ds = 0.
\end{equation}

**Theorem 3.4.** Let \((E_1, E_2)\) be the solutions of the problem (3.30)–(3.33). Then \((E_1, E_2)\) vanish identically.

**Proof.** Denote \(\Omega_r = (B_r \cap \Omega_1)\) with boundary \(\partial \Omega_r = \partial B^+_r \cup \Gamma \cup S_r\), where \(\partial B^+_r = \partial B_r \cap \Omega_1\) and \(S_r = S \cap B_r\). For each fixed \(x \in \Omega_r\), applying the vector Green first theorem to \(E_1\) in \(\Omega_r\), we have
\begin{equation}
\int_{\Omega_r} \left[ (\nabla \times E_1)^2 - E_1 \cdot (\nabla \times \nabla \times E_1) \right] \, dx
= \int_{\partial \Omega_r} \nu \cdot [E_1 \times (\nabla \times E_1)] \, ds_x
= \int_{\partial \Omega_r} (\nabla \times E_1) \cdot [\nu \times E_1] \, ds_x
\end{equation}
where \(\nu = \nu(x)\) stands for the unit normal vector at \(x \in \partial \Omega_r\) pointing out of \(\Omega_r\). Letting \(r \to +\infty\), we have from (3.31), (3.33), and (3.34) that
\begin{equation}
\int_{\Omega_1} \left[ (\nabla \times E_1)^2 - E_1 \cdot (\nabla \times \nabla \times E_1) \right] \, dx
= -i \omega \mu_1 \int_S \bar{H}_1 \cdot (\nu_S \times E_1) \, ds_x,
\end{equation}
where \(\nu_S = \nu_S(x)\) denotes the unit normal vector at \(x \in S\) pointing from region \(\Omega_2\) to region \(\Omega_1\).

Using (3.35) and (3.30) yields
\begin{equation}
\int_{\Omega_1} \left[ (\nabla \times E_1)^2 - E_1 \cdot (\nabla \times \nabla \times E_1) \right] \, dx
= -i \omega \mu_1 \int_S \bar{H}_1 \cdot (\nu_S \times E_1) \, ds_x,
\end{equation}
which gives by taking the imaginary part of (3.30) that
\begin{equation}
-\Re \left[ \int_S \bar{H}_1 \cdot (\nu_S \times E_1) \, ds_x \right] = \sigma_1 \int_{\Omega_1} |E_1|^2 \, dx \geq 0.
\end{equation}
Similarly, we may show that
\begin{equation}
\Re \left[ \int_S \bar{H}_2 \cdot (\nu_S \times E_2) \, ds_x \right] = \sigma_2 \int_{\Omega_2} |E_2|^2 \, dx \geq 0.
\end{equation}
Noting the continuity conditions (3.37) and \(H_j = \nu_S \times (H_j \times \nu_S) + (\nu_S \cdot H_j) \nu_S, j = 1, 2\) on \(S\), we have \(\overline{\Pi}_1 \cdot (\nu_S \times E_1) = \overline{\Pi}_2 \cdot (\nu_S \times E_2)\) on \(S\), and

\[
(3.39) \quad \int_S \overline{\Pi}_1 \cdot (\nu_S \times E_1) \, dx = \int_S \overline{\Pi}_2 \cdot (\nu_S \times E_2) \, dx.
\]

It follows immediately from combining (3.37) and (3.39) and \(\sigma_j > 0\) that

\[
\int_{\Omega_1} |E_1|^2 \, dx = \int_{\Omega_2} |E_2|^2 \, dx = 0,
\]

which implies that \(E_1 = 0\) in \(\Omega_1\) and \(E_2 = 0\) in \(\Omega_2\). \(\Box\)

### 4. Uniqueness of the inverse problem.

This section addresses the uniqueness of the inverse hybrid surface scattering problem. For the given incident field, we show that the obstacle and the unbounded rough surface can be uniquely determined by the tangential trace of the electric field \(E_{\Gamma_H} \times E_1|_{\Gamma_H}\), where \(\Gamma_H = \{x \in \mathbb{R}^3 \mid x_3 = H\}\) is a plane surface above the obstacle and unbounded rough surface and \(E_{\Gamma_H} = (0, 0, 1)^T\).

Let \(\tilde{S} \subset C^2\) be an unbounded rough surface which divides \(\mathbb{R}^3\) into the upper half space \(\tilde{\Omega}_1^+\) and the lower half space \(\tilde{\Omega}_2\). Let \(\tilde{D} \subset C^2 \tilde{\Omega}_1^+\) be a bounded domain with the boundary \(\tilde{\Gamma} \subset C^2\). Define \(\Omega_1^+ = \tilde{\Omega}_1^+ \setminus \tilde{D}\). Let \((\tilde{E}_1, \tilde{E}_2)\) be the unique solutions of Problem 2.1 with the hybrid surface \((D, S)\) replaced by \((\tilde{D}, \tilde{S})\) but for the same incident field \(E'\) satisfying (2.12). The point dipole source is assumed to be located at \(x_s \in \Omega_1 \cap \tilde{\Omega}_1\).

**Theorem 4.1.** Assume that \(\nu_{\Gamma_H} \times E_1|_{\Gamma_H} = \nu_{\Gamma_H} \times \tilde{E}_1|_{\Gamma_H}\), then \(D = \tilde{D}, S = \tilde{S}\).

**Proof.** We prove it by contradiction and assume that \(D \neq \tilde{D}, S \neq \tilde{S}\). The problem geometry is shown in Figure 4.1. Let \(\tilde{E} = E_1 - \tilde{E}_1\), then \(\tilde{E}\) satisfies Maxwell’s equation

\[
\nabla \times \nabla \times \tilde{E} - \kappa_1^2 \tilde{E} = 0 \quad \text{in} \quad \Omega_1 \cap \tilde{\Omega}_1.
\]

By the assumption \(\nu_{\Gamma_H} \times E_1|_{\Gamma_H} = \nu_{\Gamma_H} \times \tilde{E}_1|_{\Gamma_H}\) and the uniqueness result for the direct scattering problem, it follows that \(E_1(x) = \tilde{E}_1(x)\) for all \(x \in \Omega_H = \{x \in \mathbb{R}^3 \mid x_3 \geq H\}\). By the analytic continuation, we get that \(E_1(x) = \tilde{E}_1(x)\) for all \(x \in \Omega_1 \cap \tilde{\Omega}_1\). Since \(\tilde{E} \in C^2(\Omega_1 \cap \tilde{\Omega}_1) \cap C^{0, a}(\Omega_1 \cap \tilde{\Omega}_1)\), we have

\[
\tilde{E}(x) = E_1(x) - \tilde{E}_1(x) = 0, \quad x \in \Omega_1 \cap \tilde{\Omega}_1.
\]

In particular, we have

\[
(4.1) \quad \tilde{E}|_{\partial(\Omega_1 \cap \tilde{\Omega}_1)} = E_1|_{\partial(\Omega_1 \cap \tilde{\Omega}_1)} - \tilde{E}_1|_{\partial(\Omega_1 \cap \tilde{\Omega}_1)} = 0.
\]

First, we prove that the obstacle can be uniquely determined. In the case when \(D \neq \tilde{D}\) which include \(D \cap \tilde{D} \neq \emptyset\) and \(D \cap \tilde{D} = \emptyset\), without loss of generality, let us denote the region between \(D\) and \(D \cap \tilde{D}\) by \(\tilde{Q} = D \setminus (D \cap \tilde{D})\), then we have \(\tilde{Q} \subset D\) and \(\tilde{Q} \notin \tilde{D}\) with the boundary \(\partial \tilde{Q} = \Gamma_p \cup \tilde{\Gamma}_p\), where \(\Gamma_p\) and \(\tilde{\Gamma}_p\) denote the part of the boundary \(\Gamma\) and \(\tilde{\Gamma}\), respectively. Thus, from (4.1) and (2.5), we obtain

\[
(4.2) \quad \nu_{\Gamma_p} \times \tilde{E}_1|_{\Gamma_p} = \nu_{\Gamma_p} \times \tilde{E}_1|_{\tilde{\Gamma}_p} = 0.
\]
Applying vector Green’s first theorem to $\mathbf{E}_1$ in $\bar{Q}$, we have from (1.2) that
\[
\int_{\bar{Q}} \left( |\nabla \times \mathbf{E}_1|^2 - \mathbf{E}_1 \cdot (\nabla \times \nabla \times \mathbf{E}_1) \right) \, dx = \int_{\partial \bar{Q}} \mathbf{\nu} \cdot [\mathbf{E}_1 \times (\nabla \times \mathbf{E}_1)] \, ds_x
\]
\[
= \int_{\partial \bar{Q}} (\nabla \times \mathbf{E}_1) \cdot [\mathbf{\nu} \times \mathbf{E}_1] \, ds_x = 0.
\]
On the other hand, note that the incident field is a point dipole source located at $x_s \in \Omega_1 \cap \Omega_1$, then we have $i\omega \mu_1 \mathbf{J}_{cs}(x) = \mathbf{I} \delta(x - x_s) = 0$ in $\bar{Q}$. By (2.3), we have
\[
\int_{\bar{Q}} \left( |\nabla \times \mathbf{E}_1|^2 - \mathbf{E}_1 \cdot (\nabla \times \nabla \times \mathbf{E}_1) \right) \, dx
\]
\[
= \int_{\bar{Q}} \left( |\nabla \times \mathbf{E}_1|^2 - \mathbf{E}_1^2 + i\omega \mu_1 \mathbf{E}_1 \cdot \mathbf{J}_{cs} \right) \, dx
\]
\[
= \int_{\bar{Q}} \left( |\nabla \times \mathbf{E}_1|^2 - \omega^2 \mu_1 \varepsilon_1 |\mathbf{E}_1|^2 + i\omega \mu_1 \sigma_1 |\mathbf{E}_1|^2 \right) \, dx.
\]
(4.3)

For $\omega \mu_1 \sigma_1 > 0$, taking the imaginary part of (4.3), we obtain $\int_{\bar{Q}} |\mathbf{E}_1|^2 \, dx = 0$, which implies that $\mathbf{E}_1 = 0$ in $\bar{Q}$. It follows from Theorem 3.4 and $\mathbf{E}_1 \in \mathcal{T}_1(\bar{\Omega}_1)$ that we have $\mathbf{E}_1 = 0$ in $\bar{\Omega}_1$. This is a contradiction because the total field $\mathbf{E}_1$ is a nontrivial solution of the inhomogeneous equation (2.3) in $\bar{\Omega}_1$. Hence, $D = \bar{D}$.

Next we show that the unbounded rough surface can also be uniquely determined. In the case when $S \neq S$ which includes $S \cap S \neq \emptyset$ and $S \cap S = \emptyset$. If $S_1$ is a segment of $S$, we may assume without loss of generality that $S_1$ is located above $\bar{S}$. Let $x^* \in S_1$, choose $\varepsilon > 0$ such that $x := x^* + \varepsilon e_3 \in \Omega_1 \cap \Omega_1$, where $e_3 = (0, 0, 1)^T$. Assuming that the incident field is given by a point dipole source located at $x_s$ with the unit polarization vector $q_e$, we take
\[
\mathbf{E}'(x) = \mathbf{G}_1(x - x_s)q_e \quad \text{in } \Omega^+_1 \cap \Omega^+_1.
\]
(4.4)
Let $S_e = S_1 \cap B_e(x^*)$ be $S$, where $B_e(x^*)$ denotes the sphere centered at the origin $x^*$ with radius $\varepsilon$ and $\bar{S} \cap B_e(x^*) = \emptyset$. Then, from $\mathbf{E}' + \mathbf{E}^s = \mathbf{E}_1$ and (4.4), we have
\[
\| \mathbf{G}_1 q_e \|_{L^\infty(S_e)} = \| \mathbf{E}_1 - \mathbf{E}^s \|_{L^\infty(S_e)} \leq \| \mathbf{E}_1 \|_{L^\infty(S_e)} + \| \mathbf{E}^s \|_{L^\infty(S_e)}.
\]
(4.5)
Because $x^*$ has a positive distance from $\bar{S}$, the well-posedness of the direct problem implies that there exists $C_1 > 0$ (independent of $\varepsilon$) such that $\mathbf{E}_1|_{S_e} = \mathbf{E}_1|_{S_e}$ and $\mathbf{E}^s|_{S_e} = \mathbf{E}^s|_{S_e}$ satisfy the estimate
\[
\| \mathbf{E}_1 \|_{L^\infty(S_e)} + \| \mathbf{E}^s \|_{L^\infty(S_e)} \leq C_1 < +\infty.
\]
(4.6)
It follows from (4.3) and (4.6) that

\[ \|G_1q_\varepsilon\|_{L^\infty(S_\varepsilon)} \leq C_1. \]

This is a contradiction because the left-hand side of the above inequality (4.7) goes to infinity as \( \varepsilon \to 0 \). Hence, \( S = \bar{S} \).

5. Local stability. In this section, we present a local stability result. Let us begin with the calculation of domain derivative which plays an important role in the stability analysis.

Let \( I : \mathbb{R}^3 \to \mathbb{R}^3 \) be the identity mapping and let \( \theta : \Gamma \cup S \to \mathbb{R}^3 \) be an admissible perturbation, where \( \theta \) is assumed to be an admissible perturbation in \( C^2(\Gamma \cup S, \mathbb{R}^3) \) and has a compact support. For \( \theta \in C^2(\Gamma \cup S, \mathbb{R}^3) \), we can extend the definition of function \( \theta(x) \) to \( \bar{\Omega}_j \) by satisfying: \( \theta(x) \in C^2(\Omega_j, \mathbb{R}^3) \cap C(\bar{\Omega}_j) \); \( I + \theta : \Omega_j \to \Omega_{j, \theta}, j = 1, 2 \). Here the region \( \Omega_{j, \theta} \) bounded by \( \Gamma_\theta \) and \( S_\theta \), where

\[ \Gamma_\theta = \{ x + \theta(x) : x \in \Gamma \}, \quad S_\theta = \{ x + \theta(x) : x \in S \}. \]

Let \( \theta(x) = (\theta_1(x), \theta_2(x), \theta_3(x))^T \). Clearly, \( \Omega_{j, \theta} \) is an admissible perturbed configuration of the reference region \( \Omega_j \). Note that \( \Omega_{j, 0} = \Omega_j \), \( \Gamma_0 = \Gamma \), and \( S_0 = S \). According to Theorem 3.4, there exist the unique solutions \((E_{1, \theta}, E_{2, \theta})\) to Problem 2.1 corresponding to the region \( \Omega_{j, \theta} \) for any small enough \( \theta \). Note that this function \( E_{j, \theta} = E_j(\theta, x) \) cannot be differentiated with respect to \( \theta \) in the classical sense. For this reason, we adopt the following concept of a domain derivative.

Denote by

\[ E'_j = \frac{\partial E_{j, \theta}}{\partial \theta}(0)p \]

the domain derivative of \( E_{j, \theta} \) at \( \theta = 0 \) in the direction \( p(x) = (p_1(x), p_2(x), p_3(x))^T \in C^2(\Gamma \cup S, \mathbb{R}^3) \). Define a nonlinear map

\[ Y : \Gamma_\theta \cup S_\theta \to \nu_{\Gamma_H} \times E_{1, \theta}|_{\Gamma_H}. \]

The domain derivative of the operator \( Y \) on the boundary \( \Gamma \cup S \) along the direction \( p \) is defined by

\[ Y'(\Gamma \cup S, p) := \nu_{\Gamma_H} \times E'_1|_{\Gamma_H}. \]

We introduce the notations

\[ V_{\Gamma_r} = \nu_{\Gamma} \times (V \times \nu_{\Gamma}), \quad V_{\Gamma_r} = \nu_{\Gamma} \cdot V, \quad V_{S_r} = \nu_S \times (V \times \nu_S), \quad V_{S_r} = \nu_S \cdot V, \]

which are the tangential and the normal components of a vector \( V \) on the boundary \( \Gamma \) and \( S \), respectively. It is clear to note that \( V = V_{\Gamma_r} + V_{\Gamma_\nu} \nu_{\Gamma} \) on \( \Gamma \) and \( V = V_{S_r} + V_{S_\nu} \nu_S \) on \( S \). Denote by \( \nabla_{\Gamma_r} \) and \( \nabla_{S_r} \) the surface gradient on \( \Gamma \) and \( S \), and denote by \( \partial_{\nu_\Gamma} \) and \( \partial_{\nu_S} \) the normal derivative on \( \Gamma \) and \( S \), respectively.

Define the jump

\[ [E] = \lim_{\Omega_j \to 0} E_1(x + a_1) - \lim_{\Omega_j \to 0} E_2(x + a_2), \quad x \in S, \]

\[ |E| = \lim_{\Omega_j \to 0} |E_1(x + a_1)| - \lim_{\Omega_j \to 0} |E_2(x + a_2)|, \quad x \in S. \]
of the continuous extension of a function $E$ to the boundary from $\Omega_1$ and $\Omega_2$, respectively.

**Theorem 5.1.** Let $(E_1, E_2)$ be the solutions of Problem 2.1. Given $p \in C^2(\Gamma \cup S, \mathbb{R}^3)$, the domain derivatives $(E_1', E_2')$ of $(E_1, E_2)$ are the radiation solutions of the following problem:

\[
\begin{aligned}
\nabla \times \nabla \times E_1' - \kappa_1^2 E_1' &= 0 \quad &\text{in } \Omega_1, \\
\nabla \times \nabla \times E_2' - \kappa_2^2 E_2' &= 0 \quad &\text{in } \Omega_2, \\
\nu \times E_1' &= \left[p_{\Gamma} \left(\partial_{\nu_1} E_{1,\Gamma} \right) + E_{1,\Gamma} \left(\nabla \nu \times p_{\Gamma} \right)\right] \times \nu \quad &\text{on } \Gamma, \\
\nu \times E_2' &= -i\omega \left[\mu H_{S_1} \right] p_{S_1} - \left[\nu \times \left(\nabla S_1 \left(p_{S_1} E_{S_1} \right)\right)\right] \quad &\text{on } S, \\
\nu \times H' &= i\omega \left[\epsilon + i\sigma \right] E_{S_1} \times p_{S_1} - \left[\nu \times \left(\nabla S_1 \left(p_{S_1} H_{S_1} \right)\right)\right] \quad &\text{on } S.
\end{aligned}
\]

**Proof.** Define the operator $A = \nabla \times (\nabla \times - \kappa_1^2 I)$ and let

\[
\omega_\theta = A E_{1,\theta},
\]

where $E_{j,\theta}$ is a solution of Problem 2.1 corresponding to the region $\Omega_{j,\theta}$, $j = 1, 2$ for sufficiently small $\theta$. Then, we have

\[
\omega_\theta = q\delta \quad \text{in } \Omega_{1,\theta}
\]

and

\[
\omega_\theta (I + \theta) = q\delta \quad \text{in } \Omega_1.
\]

Since $A$ is a linear and continuous operator from $H(\text{curl}, \Omega_1) = \{u \in L^2(\Omega_1)^3 : \nabla \times u \in L^2(\Omega_1)^3\}$ into $D'(\Omega_1)$, $A$ is differentiable in the distribution sense, i.e., $v \mapsto \langle Av, \psi \rangle$ is differentiable for each $\psi \in D(\Omega_1)$ and

\[
\frac{\partial A}{\partial v} = A.
\]

Here $D(\Omega_1)$ is the standard space of infinitely differentiable functions with compact support in $\Omega_1$ and $D'(\Omega_1)$ is the standard space of distributions. Therefore, it follows from the differentiability of $\theta \mapsto E_{1,\theta}(I + \theta)$ and $\theta \mapsto E_{1,\theta}$ that $\theta \mapsto \omega_\theta (I + \theta)$ is continuously Fréchet differentiable at $\theta = 0$ in the direction $p \in C^2(\Gamma \cup S, \mathbb{R}^3)$. Moreover, for an admissible perturbation $\theta$, their derivatives satisfy

\[
\frac{\partial}{\partial \theta} (\omega_\theta (I + \theta))(0)p = \frac{\partial \omega_\theta}{\partial \theta}(0)p + (p \cdot \nabla)\omega \quad \text{in } \Omega_1.
\]

We deduce from (5.3) and (5.7) that

\[
\frac{\partial \omega_\theta}{\partial \theta}(0)p = \frac{\partial A}{\partial E_{1,\theta}} \frac{\partial E_{1,\theta}}{\partial \theta}(0)p = \frac{\partial A}{\partial E_1} E_1'
\]

\[
= \frac{\partial}{\partial \theta} (\omega_\theta (I + \theta))(0)p - (p \cdot \nabla)\omega
\]

\[
= (p \cdot \nabla)q\delta - (p \cdot \nabla)q\delta = 0 \quad \text{in } \Omega_1.
\]

It follows from (5.6) and (5.8) that

\[
AE_1' = \nabla \times (\nabla \times E_1') - \kappa_1^2 E_1' = 0 \quad \text{in } \Omega_1.
\]
For the boundary condition, we may follow the same steps as those in [21] and obtain
\[
\mathbf{v}_\Gamma \times E'_1 = [p_{\Gamma_\nu}(\partial_{\Gamma_\nu}E_{1,\Gamma_\nu}) + E_{1,\Gamma_\nu}(\nabla_{\Gamma_\nu}p_{\Gamma_\nu})] \times \mathbf{v}_\Gamma \quad \text{on } \Gamma.
\]

Furthermore, for every perturbation \( \theta \in C^2(\Gamma \cup S, \mathbb{R}^3) \), the tangential traces of the electric fields are assumed to be continuous across \( S \), i.e.,
\[
(5.9) \quad \mathbf{v}_\theta \times E_{1,\theta} = \mathbf{v}_\theta \times E_{2,\theta} \quad \text{on } S_\theta.
\]
Hence, we have
\[
(5.10) \quad [\mathbf{v}_\theta(\mathbb{I} + \theta)] \times [E_{1,\theta}(\mathbb{I} + \theta)] = [\mathbf{v}_\theta(\mathbb{I} + \theta)] \times [E_{2,\theta}(\mathbb{I} + \theta)] \quad \text{on } S.
\]
Moreover, it follows from [9, Lemma 3] and [26, Lemma 4.8] that
\[
(5.11) \quad \mathbf{v}_\theta(\mathbb{I} + \theta) = \frac{1}{\|g(\theta)\mathbf{v}_S\|_{L^2(S)}} g(\theta) \mathbf{v}_S \quad \text{on } S,
\]
where the matrix \( g(\theta) = (\mathbb{I} + \frac{\partial \theta}{\partial x})^{-\top} \) satisfies
\[
g(0) = \mathbb{I}, \quad \frac{\partial g(\theta)}{\partial \theta}(0)\mathbf{p} = - (\nabla \mathbf{p})^\top.
\]
By (5.10) and (5.11), we have
\[
(5.12) \quad [g(\theta)\mathbf{v}_S] \times [E_{1,\theta}(\mathbb{I} + \theta)] = [g(\theta)\mathbf{v}_S] \times [E_{2,\theta}(\mathbb{I} + \theta)] \quad \text{on } S
\]
and
\[
(5.13) \quad \frac{\partial}{\partial \theta} \{[g(\theta)\mathbf{v}_S] \times [E_{1,\theta}(\mathbb{I} + \theta)]\}(0)\mathbf{p} = \frac{\partial}{\partial \theta} \{[g(\theta)\mathbf{v}_S] \times [E_{2,\theta}(\mathbb{I} + \theta)]\}(0)\mathbf{p} \quad \text{on } S.
\]
Using the chain rule, we deduce from (5.13) that
\[
(5.14) \quad \frac{\partial}{\partial \theta} \{[g(\theta)\mathbf{v}_S] \times [E_{1,\theta}(\mathbb{I} + \theta)]\}(0)\mathbf{p} = -(\nabla \mathbf{p})^\top \mathbf{v}_S \times E_j + \mathbf{v}_S \times [E_j' + (\mathbf{p} \cdot \nabla)E_j] \quad \text{on } S, \quad j = 1, 2.
\]
Since on \( S \) we have
\[
(5.15) \quad ((\nabla \mathbf{p})^\top \mathbf{v}_S) \times E_j = [\mathbf{v}_S \times (\nabla \times \mathbf{p}) + (\mathbf{v}_S \cdot \nabla)\mathbf{p}] \times E_j = [\mathbf{v}_S \times (\nabla \times \mathbf{p})] \times E_j + (\mathbf{v}_S \cdot \nabla)\mathbf{p} \times E_j = - \mathbf{v}_S \times [E_j \times (\nabla \times \mathbf{p})] - (\nabla \times \mathbf{p}) \times (\mathbf{v}_S \times E_j) + (\mathbf{v}_S \cdot \nabla)\mathbf{p} \times E_j = - \mathbf{v}_S \times [E_j \times (\nabla \times \mathbf{p})] - \mathbf{v}_S \times [(E_j \cdot \nabla)\mathbf{p}] - (\nabla \mathbf{p}) (\mathbf{v}_S \times E_j) + (\nabla \cdot \mathbf{p}) (\mathbf{v}_S \times E_j), \quad j = 1, 2.
\]
With the aid of (5.14) and (5.15), we obtain
\[
\frac{\partial}{\partial \theta} \{ [g(\theta)\nu_S] \times [E_{\lambda,\theta}(I + \theta)] \} (0) p
\]
\[
= -\{ -\nu_S \times [E_j \times (\nabla \times p)] - \nu_S \times [(E_j \cdot \nabla)p] - (\nabla p)(\nu_S \times E_j)
+ (\nabla \cdot p)(\nu_S \times E_j) \} + \nu_S \times E'_j + \nu_S \times [(p \cdot \nabla)E_j]
\]
\[
= \{ \nu_S \times [E_j \times (\nabla \times p)] + \nu_S \times [(E_j \cdot \nabla)p] + \nu_S \times [(p \cdot \nabla)E_j] \}
+ \nu_S \times E'_j + (\nabla p)(\nu_S \times E_j) - (\nabla \cdot p)(\nu_S \times E_j)
\]
\[
= \nu_S \times [E_j \times (\nabla \times p) + (E_j \cdot \nabla)p + (p \cdot \nabla)E_j]
+ \nu_S \times E'_j + (\nabla p)(\nu_S \times E_j) - (\nabla \cdot p)(\nu_S \times E_j)
\]
\[
= \nu_S \times [(\nabla \times E_j) \times p] + \nu_S \times [p \times (\nabla \times E_j) + E_j \times (\nabla \times p) + (E_j \cdot \nabla) p
+ (p \cdot \nabla)E_j] + \nu_S \times E'_j + (\nabla p)(\nu_S \times E_j) - (\nabla \cdot p)(\nu_S \times E_j)
\]
\[
= \nu_S \times [(\nabla \times E_j) \times p] + \nu_S \times [(\nabla(p \cdot E_j)] + \nu_S \times E'_j
+ (\nabla p)(\nu_S \times E_j) - (\nabla \cdot p)(\nu_S \times E_j)
\]
\[
= i\omega [\nu_S \times ((\mu H_j) \times p)] + [\nu_S \times (\nabla(p \cdot E_j)]
\]
\[
(5.16)
\]
\[
+ \nu_S \times E'_j + (\nabla p)(\nu_S \times E_j) - (\nabla \cdot p)(\nu_S \times E_j) \quad \text{on } S, \quad j = 1, 2.
\]

By taking into account of the continuous conditions (2.6) and \( p \in C^2(\Gamma \cup S, \mathbb{R}^3) \), from (5.14) and (5.15), the jump relations read
\[
(5.17)
[\nu_S \times E'] = -i\omega [\nu_S \times ((\mu H) \times p)] - [\nu_S \times (\nabla(p \cdot E))].
\]

For the first term of in the right hand side of (5.17), we conclude from the jump condition \([\mu H_{S_r}] = 0\) that
\[
i\omega [\nu_S \times ((\mu H) \times p)] = i\omega [(\mu H)(\nu_S \cdot p) - p(\nu_S \cdot (\mu H))]
\]
\[
= i\omega [\mu (H_{S_r} + H_{S_\nu} \nu_S)(p_{S_r} - (p_{S_r} + p_{S_\nu} \nu_S)(\mu H_{S_r})]
\]
\[
= i\omega [\mu H_{S_r} p_{S_r} - \mu H_{S_\nu} p_{S_\nu}]
\]
\[
= i\omega [\mu H_{S_r}] p_{S_r} - i\omega [\mu H_{S_\nu}] p_{S_\nu}
\]
\[
\text{on } S.
\]
\[
(5.18)
\]

It follows from \([\nu_S \times E] = [\nu_S \times E_{S_r}] = 0\) and the definition of the surface gradient \(\nabla_{S_r}\), that we obtain \([\nu_S \times (\nabla_{S_r} (p_{S_r} \cdot E_{S_r}))] = 0\). Thus, the second term in the right hand side of (5.17) reduces to
\[
[\nu_S \times (\nabla(p \cdot E))] = [\nu_S \times (\nabla_{S_r} (p \cdot E))]
\]
\[
= [\nu_S \times (\nabla_{S_r} ((p_{S_r} + p_{S_\nu} \nu_S) \cdot (E_{S_r} + E_{S_\nu} \nu_S)))]
\]
\[
= [\nu_S \times (\nabla_{S_r} (p_{S_r} \cdot E_{S_r} + p_{S_\nu} E_{S_\nu}))]
\]
\[
= [\nu_S \times (\nabla_{S_r} (p_{S_r} E_{S_r}))] \quad \text{on } S.
\]
\[
(5.19)
\]
Finally, by (5.17) - (5.19), we have the boundary condition
\[
[\nu_S \times E'] = -i\omega [\mu H_{S_r}] p_{S_r} - [\nu_S \times (\nabla_{S_r} (p_{S_r} E_{S_r}))] \quad \text{on } S.
\]

Similarly, we can obtain
\[
[\nu_S \times H'] = i\omega [(\varepsilon + i\sigma) E_{S_r}] p_{S_r} - [\nu_S \times (\nabla_{S_r} (p_{S_r} H_{S_r}))] \quad \text{on } S.
\]
Based on the existence of the domain derivatives $E_j'$, the proof of the the integral representations for $E_j'$ follow in the same manner as for the the integral representation of $E_j$. Therefore, the asymptotic behavior to the domain derivative $E_j'$ has the same form as $E_j$. This means that the domain derivatives $(E_1', E_2')$ are the radiation solutions of the problem (5.2). \[\square\]

Introduce the domain $\Omega_{1,h}$ bounded by $\Gamma_h$ and $S_h$, where

$$\Gamma_h = \{ x + h p(x) \nu : x \in \Gamma \}, \quad S_h = \{ x + h p(x) \nu_S : x \in S \},$$

where $p \in C^2(\mathbb{R}^3, \mathbb{R})$ and $h > 0$. For any two domains $\Omega_1$ and $\Omega_{1,h}$ in $\mathbb{R}^3$, define the Hausdorff distance

$$\text{dist}(\Omega_1, \Omega_{1,h}) = \max\{ \rho(\Omega_{1,h}, \Omega_1), \rho(\Omega_1, \Omega_{1,h}) \},$$

where

$$\rho(\Omega_1, \Omega_{1,h}) = \sup_{x \in \Omega_1} \inf_{y \in \Omega_{1,h}} |x - y|.$$

It can be easily seen that the Hausdorff distance between $\Omega_{1,h}$ and $\Omega_1$ is of the order $h$, i.e., $\text{dist}(\Omega_1, \Omega_{1,h}) = O(h)$. We have the following local stability result.

**Theorem 5.2.** If $p \in C^2(\Gamma \cup S, \mathbb{R})$ and $h > 0$ is sufficiently small, then

$$\text{dist}(\Omega_1, \Omega_{1,h}) \leq C\| \nu_{\Gamma_h} \times E_{1,h} - \nu_{\Gamma_h} \times E_1 \|_{C^{0,\alpha}(\Gamma_h)},$$

where $E_{1,h}$ and $E_1$ is the solution of Problem 2.1 corresponding to the domain $\Omega_{1,h}$ and $\Omega_1$, respectively, and $C$ is a positive constant independent of $h$.

**Proof.** Assume by contradiction that there exists a subsequence from $\{E_{1,h}\}$, which is still denoted as $\{E_{1,h}\}$ for simplicity, such that

$$\lim_{h \to 0} \frac{\| \nu_{\Gamma_h} \times E_{1,h} - \nu_{\Gamma_h} \times E_1 \|_{C^{0,\alpha}(\Gamma_h)}}{h} = \| \nu_{\Gamma_h} \times E_1' \|_{C^{0,\alpha}(\Gamma_h)} = 0 \quad \text{as} \quad h \to 0,$$

which yields $\nu_{\Gamma_h} \times E_1' = 0$ on $\Gamma_h$. Following a similar proof of Theorem 3.4, we can show the uniqueness of the solution for problem (5.2). An application of the uniqueness for problem (5.2) yields that $E_j' = 0$ in $\Omega_j, j = 1, 2$. Noting the boundary condition of $E_1'$ in problem (5.2) gives

$$\nu_{\Gamma} \times E_1' = [p(x) \nu_{\Gamma} p(x) (\partial_{\nu_{\Gamma}} E_{1,\Gamma}) + E_{1,\Gamma} \nu_{\Gamma} (p(x) \nu_{\Gamma}) \nu_{\Gamma}] \times \nu_{\Gamma}$$

$$= [p(\partial_{\nu_{\Gamma}} E_{1,\Gamma}) + E_{1,\Gamma} \nu_{\Gamma} (p(x) \nu_{\Gamma}) \nu_{\Gamma}] \times \nu_{\Gamma} = 0 \quad \text{on} \quad \Gamma.$$  

(5.20)

Since $p$ is arbitrary in (5.20), we have

$$\partial_{\nu_{\Gamma}} E_{1,\Gamma} = \partial_{\nu_{\Gamma}} [\nu_{\Gamma} \times (E_1 \times \nu_{\Gamma})]$$

$$= \partial_{\nu_{\Gamma}} E_1 - \partial_{\nu_{\Gamma}} [ (\nu_{\Gamma} \times E_1) \nu_{\Gamma}] = 0 \quad \text{on} \quad \Gamma$$

and

$$E_{1,\Gamma} = \nu_{\Gamma} \cdot E_1 = 0 \quad \text{on} \quad \Gamma.$$  

(5.21)

(5.22)

It follows from (5.21) and (5.22) that

$$\partial_{\nu_{\Gamma}} E_1 = 0 \quad \text{on} \quad \Gamma.$$  

(5.23)
With the aid of \( \nu_\Gamma \times E_1|_\Gamma = 0 \) and \( \nu_\Gamma \cdot E_1|_\Gamma = 0 \), we have
\[
E_1 = 0 \quad \text{on } \Gamma.
\]
Therefore, combining (5.23) and (5.24), we infer by unique continuation that
\[
E_1 = 0 \quad \text{in } \Omega_1,
\]
which is a contradiction to the
\[
\nabla \times (\nabla \times E_1) - \kappa_1^2 E_1 = i\omega \mu_1 J_{cs} \neq 0 \quad \text{in } \Omega_1.
\]
The proof is completed. \( \square \)

6. Conclusion. In this paper, we have studied the direct and inverse electromagnetic obstacle scattering problems for the three-dimensional Maxwell equations in an unbounded structure. We present an equivalent integral equation to the boundary value problem and show that it has a unique solution. For the inverse problem, we prove that the obstacle and unbounded rough surface can be uniquely determined by the tangential component of the electric field measured on the plane surface above the obstacle. The local stability shows that the Hausdorff distance of the two regions, corresponding to small perturbations of the obstacle and the unbounded rough surface, is bounded by the distance of corresponding tangential trace of the electric fields if they are close enough. To prove the stability, the domain derivative of the electric field with respect to the change of the shape of the obstacle and unbounded rough surface is examined. In particular, we deduce that the domain derivative satisfies a boundary value problem of the Maxwell equations, which is similar to the model equation of the direct problem.

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