AN EQUIVALENCE THEOREM FOR A CLASS OF MINKOWSKI SPACES

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ABSTRACT. In this paper, as a generalization of $(\alpha, \beta)$-norms we define semi-C-reducible Minkowski norms which are different from literatures. Then we prove an equivalence theorem for a class of semi-C-reducible Minkowski spaces. As a consequence, a class of semi-C-reducible Finsler metrics with vanishing Landsberg curvatures are Berwaldian.

INTRODUCTION

In this paper, a Minkowski space is an $n$ dimensional real vector space $V$ equipped with a Minkowski norm $F$ which is smooth and strongly convex. Hence the fundamental form $\hat{g}$ is positive definite and $(V_0, \hat{g})$ is a Riemannian manifold where $V_0 := V \setminus \{0\}$. $\hat{g}$ and the Cartan tensor $\hat{A}$ are basic geometric invariants on $V_0$ for a given Minkowski space. The trace of $\hat{A}$ with respect to $\hat{g}$ is the Cartan form $\eta$. For the study of the geometry of $(V, F)$ via these tensors one refers to \cite{BaoCS}.

Let $(M, F)$ be a smooth Finsler manifold. The restriction of $F$ on the tangent space at each point of $M$ is a Minkowski norm. A long-standing problem in Finsler geometry is the so called “unicorn problem”: Is there any Landsberg metric which is not Berwaldian?.

For general comments on this problem, one refers to \cite{Aik, Bao, Matv, Sza1, Sza2, Tor}, etc. It is proved in \cite{She} that Landsberg $(\alpha, \beta)$-metrics are Berwaldian when dimension $n \geq 3$. In \cite{Asa1, Asa2}, some non-regular $(\alpha, \beta)$-metrics are constructed, which are Landsberg but not Berwaldian. Recently, it is proved in \cite{ZouC} that weakly Landsberg $(\alpha, \beta)$-manifolds are Berwald manifolds when dimension $n \geq 3$. There are some results in \cite{MatC} about this problem for semi-C-reducible Finsler spaces, which may not be regular metrics.

Parallel transports of a Finsler manifold are norm preserving maps which are linear along rays. If the Finsler metric is Landsberg, then the parallel transports are Riemannian isometries between the punched tangent spaces. Similarly, a Finsler manifold is Berwaldian if and only if the parallel transports are linear isomorphisms. These properties are our motivation to investigate the equivalence problem for Minkowski spaces. One refers to \cite{Bao, CheS, L} for more details.

Combining Proposition 4.2 in \cite{BaoCS} and Theorem 1 in \cite{L}, we have the following theorem.

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Theorem 1. Let $(V_1, F_1)$ and $(V_2, F_2)$ be two Minkowski spaces of dimension $n \geq 2$, respectively. Let $f : V_1 \to V_2$ be a norm preserving map which is a diffeomorphism between $(V_1)_0$ and $(V_2)_0$, and satisfies $f(tv) = tf(v)$, $\forall v \in V_1$, $\forall t > 0$.

Then $g_1 = f^*g_2$ and $\eta_1 = f^*\eta_2$ if and only if there exists a nondegenerate linear homomorphism $L \in \text{Hom}(V_1, V_2)$, such that $f = L$ and $F_1 = F_2 \circ L$.

Some results related to the unicorn problem based on Theorem 1 can be found in [L]. A nature question is that can one prove Theorem 1 under weaker conditions. For examples, whether or not only one of the conditions $g_1 = f^*g_2$ and $\eta_1 = f^*\eta_2$ is sufficient to guarantee that the conclusion of Theorem 1 is still valid. In this paper, we will prove the following theorem.

Theorem 2. Let $(V, F)$ and $(\tilde{V}, \tilde{F})$ be $n \geq 4$ dimensional semi-C-reducible Minkowski spaces with characteristic functions $q$ and $\tilde{q}$, respectively. Assume that $q$ and $\tilde{q}$ both take value 1 on subsets with empty interior. Let $f : V \to \tilde{V}$ be a norm preserving map which is diffeomorphism between $V_0$ and $\tilde{V}_0$, and satisfies $f(tv) = tf(v)$, $\forall v \in V_0$, $t > 0$.

If $\tilde{g} = f^*\tilde{g}$, then $\tilde{A} = \pm f^*\tilde{A}$.

Our definition of semi-C-reducible Minkowski spaces is a generalization of the one in [Mat2; MatC]. In fact we will show that all of $(\alpha, \beta)$-norms are semi-C-reducible in our sense.

A Finsler metric $F$ is called semi-C-reducible if the restriction of $F$ on each tangent space is semi-C-reducible. A direct consequence of Theorem 2 in Finsler geometry is the following result related to the unicorn problem.

Theorem 3. Let $(M, F)$ be a semi-C-reducible Finsler manifold of dimensional $n \geq 4$. Assume that the characteristic function $q$ takes value 1 on a subset with empty interior along each fiber. If the Landsberg curvature $L = 0$, then $(M, F)$ is a Berwald space.

In [She] Shen proves that Landsberg $(\alpha, \beta)$-metrics $F = \alpha\phi(\beta/\alpha)$ are Berwaldian, in which $\phi$ is independent to the points of the ground manifold. However, Theorem 3 involves similar metrics of type $F = \alpha\phi(x, \beta/\alpha)$, which are called general $(\alpha, \beta)$-metrics by Yu and Zhu in [YuZ]. So a consequence is that general $(\alpha, \beta)$-metrics of $L = 0$ are Berwaldian under the restriction on $q$ and $n$ as in Theorem 3.

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1. $(\alpha, \beta)$-norms and semi-C-reducible norms

1.1. The Cartan tensors of $(\alpha, \beta)$-norms. Let $V$ be an $n$ dimensional real vector space. Let $F$ be a Minkowski norm on $V$. Let us recall the following geometric
invariants induced by $F$ on $V_0$:
\[
\bar{g} = F^2 \tilde{g} = (F_{y^i} F_{y^j} + FF_{y^i y^j}) dy^i \otimes dy^j =: g_{ij} dy^i \otimes dy^j,
\]
\[
h = h_{ij} dy^i \otimes dy^j = \bar{g} - dF \otimes dF = FF_{y^i y^j} dy^i \otimes dy^j,
\]
\[
\hat{A} := A_{ijk} dy^i \otimes dy^j \otimes dy^k = F^2 \frac{\partial g_{ij}}{\partial y^k} dy^i \otimes dy^j \otimes dy^k
\]
\[
= \frac{F}{2} \left( F_{y^i} F_{y^j y^k} + F_{y^j} F_{y^i y^k} + F_{y^k} F_{y^i y^j} + FF_{y^i y^j y^k} \right) dy^i \otimes dy^j \otimes dy^k.
\]
\[(1.1)\]
\[
\eta := \frac{F}{2} d\log \det(\bar{g}) = g^{jk} A_{ijk} dy^i =: A_i dy^i,
\]
which are the fundamental tensor, the angular metric, the Cartan tensor and the Cartan form, respectively.

An $(\alpha, \beta)$-functional on a vector space $V$ is defined by using a Euclidean norm $\alpha$, a linear functional $\beta$ on $V$ and a $C^\infty$ positive function $\phi = \phi(s)$ on an open interval $(-\epsilon, \epsilon)$ in the following form
\[(1.2)\]
\[
F = \alpha \left[ \phi \circ \frac{\beta}{\alpha} \right],
\]
provided $b := \|\beta\|_\alpha < \epsilon$.

**Lemma 1** ([Sh], [CheS]). The functional (1.2) is a Minkowski norm if and only if the following inequalities
\[(1.3)\]
\[
\phi(s) - s\phi'(s) > 0, \quad \left[ \phi(s) - s\phi'(s) \right] + (b^2 - s^2)\phi''(s) > 0,
\]
hold for any $s$ in $[-b, b]$. The determinate of $\bar{g}$ is given by
\[(1.4)\]
\[
\det \bar{g} = \phi^{n+1}(\phi - s\phi')\phi^n - \left[ (\phi - s\phi') + (b^2 - s^2)\phi'' \right] \det \alpha^2.
\]

If (1.2) is a Minkowski norm, it is clear that $F = \alpha \left[ \phi \circ \frac{\lambda \beta}{\alpha} \right]$ is also a Minkowski norm for any $\lambda \in [-1, 1]$.

Fixing an orthonormal basis $\{e_1, \ldots, e_n\}$ of $\alpha$, for any $y = y^i e_i$ we have
\[
\alpha(y) = \alpha(y^1, \ldots, y^n) = \sqrt{\delta_{ij} y^i y^j} = |y|, \quad \text{and} \quad \beta(y) = b_i y^i.
\]

We would like to denote that
\[
s(y) = \frac{\beta(y)}{\alpha(y)} = \frac{b_i y^i}{|y|}.
\]
Under this coordinate system, the norm (1.2) is expressed as
\[(1.5)\]
\[
F(y) = \alpha \phi(s) = |y| \phi \left( \frac{b_i y^i}{|y|} \right).
\]
For our purpose we will calculate the Cartan tensors of the $(\alpha, \beta)$ norms.
Lemma 2. The Cartan tensor of the \((\alpha, \beta)\)-norm \((1.2)\) can be formulated as following

\[
A_{ijk} = v(h_{ij}Y_k + h_{jk}Y_i + h_{ki}Y_j) + \frac{u - (n + 1)v}{\|Y^z\|^2}Y_iY_jY_k.
\]

The terms \(u, v\) and \(Y^z\) are defined as follows:

\[
Y^z := Y_idy^i := (b_i - sa_y^i)dy^i, \quad v := \frac{\phi}{2} \{ \log [\phi(\phi - s\phi')] \}^0, \quad u := \frac{\phi}{2} \{ \log \left[ \phi^{n+1}(\phi - s\phi') - (b^2 - s^2)\phi'' \right] \}^0.
\]

Furthermore, the set \(\{ s | v(s) = 0, u(s) \neq 0 \}\) has empty interior.

Proof. Routine calculation shows

\[
F_{y'} = \phi\alpha_{y'} + \phi'(b_i - s\alpha_{y'})^i, \quad F_{y'y'} = \frac{1}{\alpha} \left[ (\phi - s\phi') (\delta_{ij} - \alpha_{y'} \alpha_{y'}^i) + \phi'' (b_i - s\alpha_{y'})(b_j - s\alpha_{y'}) \right]
\]

\[
F_{y'y'y'} = -\frac{1}{\alpha^2} (\phi - s\phi') \left[ \alpha_{y'} (\delta_{jk} - \alpha_{y'} \alpha_{y'}) + \alpha_{y'} (\delta_{ki} - \alpha_{y'} \alpha_{y'}) + \alpha_{y'} (\delta_{ij} - \alpha_{y'} \alpha_{y'}) \right]
\]

\[
- \frac{1}{\alpha^2} \phi'' \left[ (b_i - s\alpha_{y'})(\delta_{jk} - \alpha_{y'} \alpha_{y'}) + (b_j - s\alpha_{y'})(\delta_{ki} - \alpha_{y'} \alpha_{y'}) + (b_k - s\alpha_{y'})(\delta_{ij} - \alpha_{y'} \alpha_{y'}) \right]
\]

\[
+ \frac{1}{\alpha^2} \phi''' (b_i - s\alpha_{y'})(b_j - s\alpha_{y'})(b_k - s\alpha_{y'})
\]

where \(\alpha_{y'} = |y|^{-1} \delta_{ij} y^i\). By definition, we have

\[
h_{ij} = F_{y'y'} = \phi(\phi - s\phi')(\delta_{ij} - \alpha_{y'} \alpha_{y'}) + \phi'' (b_i - s\alpha_{y'})(b_j - s\alpha_{y'})
\]

and

\[
A_{ijk} = \frac{1}{2} \left( F_{y'} F_{y'y'} + F_{y'} F_{y'y'} + F_{y} F_{y'y'} + FF_{y'y'y'} \right)
\]

\[
= \phi \left[ (\phi - s\phi') - s\phi'' \right] \left[ (b_i - s\alpha_{y'})(\delta_{jk} - \alpha_{y'} \alpha_{y'}) + (b_j - s\alpha_{y'})(\delta_{ki} - \alpha_{y'} \alpha_{y'}) \right]
\]

\[
+ (b_k - s\alpha_{y'})(\delta_{ij} - \alpha_{y'} \alpha_{y'}) \right] + \phi \left( 3\phi'' + \phi''' \right) (b_i - s\alpha_{y'})(b_j - s\alpha_{y'})(b_k - s\alpha_{y'}).
\]

The one form \(Y^z := (b_i - s\alpha_{y'})dy^i\) defined on \(V_0\) is essential. With respect to \(\alpha\), its dual vector field \(Y\) at \(y \in V \setminus \{0\}\) is just the projection of the dual of \(\beta\) on the \(y^\perp\). By Lemma \(\Box\) we know that \(\phi(\phi - s\phi') > 0\). Then we have

\[
\delta_{ij} - \alpha_{y'} \alpha_{y'} = \frac{1}{\phi(\phi - s\phi')} [h_{ij} - \phi'' Y_iY_j].
\]
Using (1.7), the Cartan tensor reduces to
\[
A_{ijk} = \frac{\phi}{2} \{ \log [\phi(\phi - s\phi')] \}' (h_{ij}Y_k + h_{jk}Y_i + h_{ki}Y_j) \\
+ \frac{\phi}{2} \{ 3\phi'\phi'' + \phi\phi''' - 3\phi\phi'' \{ \log [\phi(\phi - s\phi')] \}' \} Y_iY_jY_k
\]
(1.8)
\[
= v(h_{ij}Y_k + h_{jk}Y_i + h_{ki}Y_j) + w(Y_iY_jY_k),
\]
where we have denoted
\[
v := \frac{\phi}{2} \{ \log [\phi(\phi - s\phi')] \}' , \quad w := \frac{\phi}{2} \{ 3\phi'\phi'' + \phi\phi''' - 3\phi\phi'' \{ \log [\phi(\phi - s\phi')] \}' \} .
\]

Then the Cartan form is given by
\[
A_i = g_{jk}A_{ijk} = ((n + 1)v + w\|Y^2\|) Y_i. \tag{1.9}
\]
On the other hand, from (1.1) and (1.4) the Cartan form has another expression
\[
A_i = \frac{\phi}{2} \{ \log [\phi^{n+1}(\phi - s\phi')^{n-2} [(\phi - s\phi') + (b^2 - s^2)\phi'']] \}' Y_i =: uY_i, \tag{1.10}
\]
where we have denoted
\[
u := \frac{\phi}{2} \{ \log [\phi^{n+1}(\phi - s\phi')^{n-2} [(\phi - s\phi') + (b^2 - s^2)\phi'']] \}'.
\]

By (1.9) and (1.10), on the points where \(Y^2 \neq 0\) we have
\[
w = [u - (n + 1)v\|Y^2\|^{-2} . \tag{1.11}
\]
Plugging (1.11) into (1.8), one proved (1.6).

Suppose that \(v \equiv 0\) in a neighborhood of a point \(s_0 \in (-\epsilon, \epsilon)\). we will prove that \(\phi^2\) is a quadratic function of \(s\) on this neighborhood. In fact, \(v = 0\) and \(\phi > 0\) implies that
\[
\phi(\phi - s\phi') = C_0 > 0
\]
of which the general solutions are
\[
\phi = \sqrt{C_0 + C_1 s^2}, \tag{1.12}
\]
where \(C_0 > 0\) and \(C_1\) are constants. On the corresponding area the norm (1.2) is quadratic, hence \(\hat{A} = 0\) and \(u \equiv 0\). So the area on which
\[
A_{ijk} = \frac{u}{\|Y^2\|^2} Y_iY_jY_k \neq 0
\]
has no interior points. \(\square\)

Setting \(n + q - 1 := uv^{-1}\) on the set of points where \(uv \neq 0\), the Cartan tensor is expressed as
\[
A_{ijk} = \frac{1}{n + q - 1} \left( h_{ij}A_k + h_{jk}A_i + h_{ki}A_j + \frac{q - 2}{\|\eta\|^2} A_iA_jA_k \right) . \tag{1.13}
\]
If we regard (1.6) as the limit of (1.13) when \(q\) takes value \(1 - n\) or \(\infty\), then (1.13) holds for general \((\alpha, \beta)\)-norms. We call \(q = 1 - n + uv^{-1}\) the characteristic function of the \((\alpha, \beta)\)-norm (1.2).
The equation \( n + q - 1 = uv^{-1} \) is in fact an third order non-linear equation of \( \phi \).

The solution exists locally and satisfies (1.3) provided that \( b \) is sufficiently small and \( q \) is a smooth function near zero.

**Lemma 3.** There exist \((\alpha, \beta)\)-norms with \( q \equiv 1 \).

**Proof.** We are going to prove that there exist positive functions \( \phi \) which satisfies (1.3) such that \( q \equiv 1 \). We have to solve the following equation for a constant \( b > 0 \)

\[
 n \log \phi(\phi - s\phi') = \log \left( \phi^{n+1}(\phi - s\phi')^{n-2} \left[ (\phi - s\phi') + (b^2 - s^2)\phi'' \right] \right) + c_0,
\]

or equivalently

\[
(\phi - s\phi')^2 = c\phi \left[ (\phi - s\phi') + (b^2 - s^2)\phi'' \right],
\]

where \( c_0 \) is a constant and \( c = e^{c_0} > 0 \). We note that a positive solution of (1.15) satisfies the second inequality (1.3) automatically. So one can use \( \phi \) to construct \((\alpha, \beta)\) norms provided \( \|\beta\|_\alpha \) sufficiently small.

Setting \( \psi = (\log \phi)' \), the equation (1.15) changes to

\[
(1 - s\psi)^2 = c \left[ 1 - s\psi + (b^2 - s^2)(\psi' + \psi^2) \right].
\]

On the interval \( s \neq \pm b \), (1.16) is a Riccati equation

\[
\psi' = \frac{(1 + c^{-1})s^2 - b^2}{b^2 - s^2} \psi^2 + \frac{(1 - 2c^{-1})s}{b^2 - s^2} \psi + \frac{c^{-1} - 1}{b^2 - s^2},
\]

and is solvable on \((-b, b)\). Hence we have proved that there exists \((\alpha, \beta)\)-norms with characteristic function \( q \equiv 1 \).

Form the above discussion, one can imagine that the complexity of \((\alpha, \beta)\)-norms with \( q \equiv 1 \). However in the next section we will show that they are all Euclidean norms provided the dimension \( n \geq 4 \).

1.2. **Semi-C-reducible norms.** In literatures [Mat1, Mat2], a norm is semi-C-reducible if the Cartan tensor and Cartan form satisfy (1.13). when \( q = 2 \) in (1.13), the norm is called C-reducible also by Matsumoto [Mat1].

According to the discussion on the \((\alpha, \beta)\)-norms, we slightly generalize Matsumoto’s semi-C-reducibility by the following definition.

**Definition 1.** Let \((V, F)\) be a Minkowski norm. Suppose \( F \) satisfies the following conditions:

(a) The Cartan form \( \eta = uY^2 \);

(b) the Cartan tensor is given by

\[
A_{ijk} = v \left( h_{ij}Y_k + h_{jk}Y_i + h_{ki}Y_j \right) + \frac{u - (n + 1)v}{\|Y^2\|^2} Y_iY_jY_k;
\]
(c) \(u\) and \(v\) are functions of positively homogeneous of degree 0.Restricting on \(I_F\), the interior of the zero set of \(v\) is contained in the zero set of \(u\);
(d) The zero set of \(Y^\sharp := i_F^* Y^\sharp\) has no interior points.

Set
\[
\frac{1}{n + q - 1} := \frac{v}{u}.
\]
On the zero set of \(v\), we agree that \(q = \infty\). Then the Cartan tensor is expressed formally as
\[
A_{ijk} = \frac{1}{n + q - 1} \left( h_{ij} A_k + h_{jk} A_i + h_{ki} A_j + q - 2 \frac{\|\eta\|^2}{\|T^\sharp\|^2} A_i A_j A_k \right).
\]
\((V, F)\) is called a semi-C-reducible space with characteristic function \(q\).

So \((\alpha, \beta)\)-norms are semi-C-reducible in the sense of Definition 1. Of course one can solve the equation \(q \equiv 2\) to get all of the C-reducible \((\alpha, \beta)\)-norms. But it is better to recall that Matsumoto proved that C-reducible norms are all Randers norms [Mat1, SheS]. An alternate proof can also be found in [SimSV]. Comparing to the case \(q \equiv 2\), we have the following theorem.

**Theorem 4.** Let \(F\) be a Minkowski norm on a vector space of dimension \(n \geq 4\). If \(F\) is semi-C-reducible with characteristic function \(q \equiv 1\), then \(F\) is a Euclidean norm.

**Proof.** By (1.19) the cubic form \(C\) and the Tchebychev form \(T^\sharp\) of the indicatrix \(I_F\) satisfy
\[
C_{\alpha\beta\gamma} = \frac{n - 1}{n} \left( \delta_{\alpha\beta} T_\gamma + \delta_{\beta\gamma} T_\alpha + \delta_{\gamma\alpha} T_\beta \right) - \frac{1}{\|T^\sharp\|^2} T_\alpha T_\beta T_\gamma
\]
with respect to a local orthonormal frame of the affine metric \(h\). Recall that the curvature tensor of \(h\) is given by
\[
R_{\beta\gamma\mu\nu} = -(\delta_{\beta\mu} \delta_{\gamma\nu} - \delta_{\beta\nu} \delta_{\gamma\mu}) + \sum_\alpha \left( C_{\alpha\beta\mu} C_{\alpha\gamma\nu} - C_{\alpha\beta\nu} C_{\alpha\gamma\mu} \right).
\]
Plugging (1.20) into (1.21) we get
\[
R_{\beta\gamma\mu\nu} = -\left( 1 - \frac{(n - 1)^2}{n^2} \|T^\sharp\|^2 \right) \left( \delta_{\beta\mu} \delta_{\gamma\nu} - \delta_{\beta\nu} \delta_{\gamma\mu} \right).
\]
So the metric \(h\) has isotropic sectional curvature. According to the assumption on the dimension, Schur’s lemma implies that \(\|T^\sharp\|^2\) is constant on \(I_F\). \(T^\sharp\) is an exact one form and always has zeros. Hence \(T^\sharp \equiv 0\). The Blaschke-Deicke theorem implies that \(F\) is a Euclidean norm. \(\square\)

2. **Proof of Theorem 2 and Theorem 3**

**Proof of Theorem 2** On the hypersurfaces \(I_F\) and \(\tilde{I}_F\), we have the cetroaffine geometric structures induced by the identity maps. Since that \(f : I_F \to \tilde{I}_F\) is a diffeomorphism, the geometric invariants on \(\tilde{I}_F\) have been pulled back to \(I_F\) via \(f\).
Denote the union of the following sets
\{q = 1\}, \{\bar{q} = 1\}, \{Y^2 = 0\}, \{\bar{Y}^2 = 0\}, \{v = 0 \text{ and } u \neq 0\}, \{\bar{v} = 0 \text{ and } \bar{u} \neq 0\}
by \(U\). By the assumption, the complement \(U^c\) in \(I_F\) is dense. Hence we only need to prove the theorem on the set \(U^c\).

Because that \(\hat{g} = f^* \tilde{h}\), the induced affine metrics satisfy \(h = f^* \hat{h}\), So the Riemannian curvature tensor of \(h\) and \(f^* \hat{h}\) coincide. Let \(e_1, \ldots, e_{n-1}\) be an arbitrary local orthonormal frame field of the metric \(h\) around a point \(x \in U^c\). From (1.21), we have

\[
(2.1) \quad \sum_{\alpha}(C_{\alpha \beta \gamma}C_{\alpha \gamma \mu} - C_{\alpha \beta \gamma \mu}) = \sum_{\alpha}(\tilde{C}_{\alpha \beta \gamma}C_{\alpha \gamma \mu} - \tilde{C}_{\alpha \beta \gamma \mu}),
\]

where \(C\) and \(\tilde{C}\) are the cubic forms of \(I_F\) and \(I_{\tilde{F}}\), respectively. Consequently, from (2.1) we have

\[
(2.2) \quad (n - 1) \sum_{\alpha}T_{\alpha}C_{\alpha \gamma \nu} - \sum_{\alpha, \beta}C_{\alpha \beta \gamma}C_{\alpha \gamma \beta} = (n - 1) \sum_{\alpha}\tilde{T}_{\alpha}\tilde{C}_{\alpha \gamma \nu} - \sum_{\alpha, \beta}\tilde{C}_{\alpha \beta \gamma}C_{\alpha \gamma \beta},
\]

and

\[
(2.3) \quad (n - 1)^2\|T^\sharp\|^2 - \|C\|^2 = (n - 1)^2\|\tilde{T}^\sharp\|^2 - \|\tilde{C}\|^2,
\]

where \(T^\sharp\) and \(\tilde{T}^\sharp\) are the Tchebychev forms of \(I_F\) and \(I_{\tilde{F}}\), respectively.

According to the assumption of semi-C-reducibility, (1.17) implies that

\[
C_{\alpha \beta \gamma} = v(\delta_{\alpha \beta}Y_\gamma + \delta_{\beta \gamma}Y_\alpha + \delta_{\gamma \alpha}Y_\beta) + \frac{u - (n + 1)vY_\alpha Y_\beta Y_\gamma}{\|Y^2\|^2},
\]

\[
\tilde{C}_{\alpha \beta \gamma} = \bar{v}(\delta_{\alpha \beta}\bar{Y}_\gamma + \delta_{\beta \gamma}\bar{Y}_\alpha + \delta_{\gamma \alpha}\bar{Y}_\beta) + \frac{\bar{u} - (n + 1)\bar{v}\bar{Y}_\alpha \bar{Y}_\beta \bar{Y}_\gamma}{\|\bar{Y}^2\|^2},
\]

respectively. From (2.4), a direct calculation gives

\[
(2.5) \quad \|C\|^2 = (n - 1)^2[3(n - 2)u^2 + (u + (2 - n)v)^2]\|Y^2\|^2,
\]

\[
(2.6) \quad \|\tilde{C}\|^2 = (n - 1)^2[3(n - 2)\bar{u}^2 + (\bar{u} + (2 - n)\bar{v})^2]\|\bar{Y}^2\|^2.
\]

Plugging (2.5) into (2.3) and using the assumption of the semi-C-reducibility, we find

\[
(2u - (n + 1)v)\|Y^2\|^2 = (2\bar{u} - (n + 1)\bar{v})\|\bar{Y}^2\|^2.
\]

Similarly plugging (2.4) in (2.2), one has

\[
(2.7) \quad (n - 3)(u - nv)Y_\gamma Y_\nu + (u - 2v)\delta_{\gamma \nu}\|Y^2\|^2 = (n - 3)(\bar{u} - n\bar{v})\bar{Y}_\gamma \bar{Y}_\nu + (\bar{u} - 2\bar{v})\bar{v}\delta_{\gamma \nu}\|\bar{Y}^2\|^2.
\]

As \(n > 3\), choosing indices \(\gamma \neq \nu\) in (2.7) gives

\[
(2.8) \quad (u - nv)Y_\gamma Y_\nu = (\bar{u} - n\bar{v})\bar{v}Y_\gamma \bar{Y}_\nu.
\]

**Case 1:** Assume that \(\bar{v}(x) = 0\) and \(\bar{u}(x) = 0\).

By (2.8), \(Y^2 \neq 0\) and \(q \neq 1\), we get \(v(x) = 0\), then \(u(x) = 0\). In this case, \(C(x) = \tilde{C}(x) = 0\).

**Case 2:** Assume that \(\bar{v}(x) \neq 0\) and \(\bar{u}(x) = 0\).
According to Case 1, \( v(x) \neq 0 \). Now we are going to prove \( u(x) = 0 \). We rewrite (2.8) as following

\[
(2.9) \quad (u - n v) v \gamma \gamma = -n \tilde{v}^{2} \tilde{Y}_{\gamma} \tilde{Y}_{\gamma}.
\]

As \( \tilde{Y}^{2}(x) \neq 0 \), we can choose the frame field \( e_{1}, \ldots, e_{n-1} \) such that \( \tilde{Y}_{\gamma} \neq 0 \) for \( \gamma = 1, \ldots, n-1 \). From (2.9), \( q \neq 1 \) and \( \tilde{v}(x) \neq 0 \), we know that \( Y_{\gamma} \neq 0 \) for \( \gamma = 1, \ldots, n-1 \). Since \( n > 3 \), for each index \( \gamma \) we can choose indices \( \mu \) and \( \nu \) such that they are not equal to each other. Then one has

\[
(2.10) \quad (u - n v) v \gamma \mu = -n \tilde{v}^{2} \tilde{Y}_{\gamma} \tilde{Y}_{\mu},
\]

and

\[
(2.11) \quad (u - n v) v \mu \nu = -n \tilde{v}^{2} \tilde{Y}_{\mu} \tilde{Y}_{\nu}.
\]

From (2.9)-(2.11), one gets

\[
(2.12) \quad (u - n v) v \gamma^{2} = -n \tilde{v}^{2} \tilde{Y}_{\gamma}^{2}, \quad \gamma = 1, \ldots, n-1.
\]

Hence

\[
(2.13) \quad (u - n v) v \|Y^{2}\|^{2} = -n \tilde{v}^{2} \|\tilde{Y}^{2}\|^{2}.
\]

Comparing (2.6) and (2.12), we get

\[
(2.14) \quad v(x) Y^{2}(x) = \pm \tilde{v}(x) \tilde{Y}^{2}(x).
\]

So (2.4) and (2.14) imply \( C(x) = \pm \tilde{C}(x) \).

**Case 3:** Assume that \( \tilde{v}(x) \neq 0 \) and \( \tilde{u}(x) \neq 0 \).

According to Case 1 and 2, \( v(x) \) and \( u(x) \) are both not zero. Substituting (1.18) into (2.6) and (2.8), one gets

\[
(2.15) \quad \frac{n - 3 + 2q}{(n + q - 1)^{2}} \|T\|^{2} = \frac{n - 3 + 2\tilde{q}}{(n + \tilde{q} - 1)^{2}} \|\tilde{T}\|^{2},
\]

and

\[
(2.16) \quad \frac{q - 1}{(n + q - 1)^{2}} T^{}_{\gamma} T^{}_{\nu} = \frac{\tilde{q} - 1}{(n + \tilde{q} - 1)^{2}} \tilde{T}_{\gamma} \tilde{T}_{\nu}, \quad \gamma \neq \nu.
\]

By a similar argument as Case 2, one concludes \( T^{2}(x) = \pm \tilde{T}^{2}(x) \) and \( C(x) = \pm \tilde{C}(x) \).

As the tensor \( \hat{A} \) is zero on directions along the rays emitted from zero, we conclude that \( \hat{A} = \pm \tilde{A} \).

\[\square\]

**Remark 1.** Any polar pair of hypersurfaces in a Euclidean space has coincide induced cetroaffine metrics, but their cubic forms differ by a sign (Sim). So there exist Minkowski spaces with the same metric \( \hat{g} \), but different Cartan tensors.
Proof of Theorem 3. We would like to follow \cite{Li} terminologically. Without loss of
generality, we can assume that $M$ is connected. The assumption $L = 0$ means that
\( \hat{g} \) is preserved along any parallel transport. By Theorem 2 and the connectedness, \( \hat{A} \)
is also preserved along any parallel transport. So $M$ is a Berwald space followed by
Theorem \cite{Li} and Proposition 6 in \cite{Li}.

\[ \square \]

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