Topological-Berry-phase-induced spin torque current in a two-dimensional system with generic $k$-linear spin-orbit interaction

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The Berry phase on the Fermi surface and its influence on the conserved spin current in a two-dimensional system with generic $k$-linear spin-orbit interaction are investigated. We calculate the response of the effective conserved spin current to the applied electric field, which is composed of conventional and spin torque currents, by using the Kubo formula. We find that the conventional spin current is not determined by the Berry phase effect. Remarkably, the spin torque Hall current is found to be proportional to the Berry phase, and the longitudinal spin torque current vanishes because of the Berry phase effect. When the $k$-linear spin-orbit interaction dominates the system, the Berry phase on the Fermi surface maintains two invariant properties. One is that the magnitude of the spin torque current protected by the Berry phase is unchanged by a small fluctuation of energy dispersion. The other one is that the change in the direction of the applied electric field does not change the magnitude of the spin torque current even if the energy dispersion is not spherically symmetric; i.e., the Berry phase effect has no dependence on the two-dimensional material orientation. The spin torque current is a universal value for all $k$-linear systems, such as Rashba, Dresselhaus, and Rashba-Dresselhaus systems. The topological number attributed to the Berry phase on the Fermi surface represents the phase of the orbital chirality of spin in the $k$-linear system. The change in the topological number results in a phase transition in which the orbital chirality of spin $s_z$ and $-s_z$ is exchanged. We found that the spin torque current can be experimentally measured.

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I. INTRODUCTION

The spin Hall effect, a phenomenon in which an applied electric field generates a lateral spin current, has become an important field in spintronics [1, 2]. In the extrinsic spin Hall effect, the transverse spin current occurs via spin-dependent scattering of carriers with localized impurities via the spin-orbit interaction [3]. An intrinsic spin Hall effect has also been proposed: Here, the applied longitudinal electric field in an intrinsic spin-orbit-coupled system such as a p-type or an n-type semiconductor generates the transverse spin current [4, 5]. Both intrinsic and extrinsic spin Hall effects have been experimentally detected by various optical measurements of spin accumulation [6] and electrical measurements through the inverse spin Hall effect [7]. The comparison of experimental results and theoretical calculations has also been thoroughly investigated by many authors [8].

One of the potential applications of the spin Hall effect is to make the spin current, if caused by the Berry curvature [9], dissipationless or independent of the impurity scattering. The Berry phase stems from the accumulation by adiabatic motion of electrons or holes on the Fermi surface, which is the integration of the Berry curvature over the Brillouin zone. If the Fermi level lies on the conduction band or valence bands, the Berry curvature could provide a transverse Lorentz force in momentum space such that the transverse spin current does not cause Ohmic heat. When the bulk is gapped by the spin-orbit interaction, the edge states caused by the Berry curvature can carry spin current, which leads to the topological quantization of the spin Hall effect [10].

In the Rashba and Rashba-Dresselhaus systems [11, 12], it has been shown that the conventional definition of the spin current, which is the symmetrical product of the velocity and spin, is closely related to the Berry phase [13, 14]. Nevertheless, as has been shown, a small Dresselhaus spin-orbit interaction added to the Rashba system leads to a nonvanishing longitudinal part of the conventional spin conductivity [15]. The influence of the Berry phase on the bulk spin current depends not only on the definition of the spin current but also on the position of the Fermi level. Since the total spin is not conserved in a spin-orbit-coupled system, the definition of a spin current is not unique [16]. Moreover, it remains unclear whether the topological-Berry-phase-protected bulk spin current should be unaltered by a small fluctuation of energy dispersion when the Fermi level lies on the conduction or valence bands.

Recently, it has been shown that a nontrivial $\pi$ Berry phase in the bulk Rashba semiconductor BiTeI can be detected by an analysis of the Shubnikov–de Haas effect [17]. It is important to investigate whether the distortion of energy dispersion on the Fermi surface would break the Berry phase.

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The topological number attributed to the Berry phase plays an important role in the phase transition of the spin Hall system. It has been shown that, similar to the quantum Hall effect [18], several plateau states exhibited by the spin current in the kagome lattice can be ascribed to the Berry phase effect on the Fermi surface [19]. However, the mechanism of the phase transition exhibited by the change in the topological number is still unclear.

Motivated by these issues, in this paper, we focus on the effective conserved bulk spin current proposed in Ref. [20], which is composed of conventional spin current \( J_a^z = \frac{1}{2} \{ s_a \cdot v_a \} \) and spin torque current \( J_a^x = \frac{1}{2} \{ d_a^x / d t, x_a \} \). The total spin current \( J_a^z = J_a^x + J_a^y \) effectively satisfies the continuity equation

\[
\frac{d S_a}{dt} + \nabla_a J_a^z = 0, \tag{1}
\]

where \( S_a = \Psi^\dagger s_a \Psi \) is the spin density and \( J_a^z = \text{Re} (\Psi^\dagger J_a^z \Psi) \) is the spin current density. The wave function \( \Psi \) satisfies the Schrödinger equation \( H \Psi = i \hbar \partial \Psi / \partial t \), where \( H = H_0 + e E \cdot x \), with \( E \) being the applied electric field. We also focus on the two-dimensional spin-orbit-coupled system with a Fermi level lying on the conduction or valence band. The Berry phase is obtained on the Fermi surface. We study the generic \( k \)-linear spin-orbit-coupled system in which energy dispersion could be non-spherical and the fluctuation of energy dispersion can be studied by changing the spin-orbit strength. We find that the Berry-phase-protected response is invariant under a fluctuation of energy dispersion. A small fluctuation of the energy dispersion changes the magnitude of the conventional spin Hall response and breaks the antisymmetric properties of the conventional spin conductivity. The conventional spin conductivity is closely related to the Berry phase but not protected by the Berry phase effect. Remarkably, we find that the spin torque current is protected by the Berry phase effect. Moreover, we find that the topological number induced by the Berry phase on the Fermi surface manifests the phase transition of the orbital chirality of spin.

Our present paper is organized as follows. In Sec. II to simplify the calculation, we rewrite the Kubo formulae of conventional and spin torque conductivity tensors in terms of the spin-orbit interaction of the two-dimensional system. In Sec. III, we study the generic \( k \)-linear system and calculate the Berry phase on the Fermi surface. We show that the Berry phase on the Fermi surface manifests two invariant properties. We also show that the topological number induced by the Berry phase governs the orbital chirality phase transition of the spin current. In Sec. IV, we calculate the conventional and spin torque conductivities. We find that the conventional spin current is not protected by the Berry phase. The spin torque conductivity is shown to be proportional to the Berry phase and the longitudinal spin torque conductivities vanish. Furthermore, the spin torque Hall conductivity satisfies antisymmetric properties. The effect of disorder is also addressed. Our conclusions are presented in Sec. V. Some calculations are provided in the Appendices.

## II. CONSERVED SPIN CONDUCTIVITY TENSOR

In this section, we simply review the Kubo formulae of the conventional spin current and spin torque current in the two-dimensional spin-orbit-coupled system. The system Hamiltonian in the presence of an applied electric field is given by

\[
H = H_0 + e E \cdot x, \tag{2}
\]

where the external perturbation is the in-plane electric field \( e E \cdot x \), where \( e > 0 \) and \( -e \) is the electric charge of an electron. The unperturbed Hamiltonian \( H_0 \) is the two-dimensional spin-orbit-coupled system,

\[
H_0 = \varepsilon_k + \sigma_x d_x + \sigma_y d_y, \tag{3}
\]

where \( \varepsilon_k = \hbar^2 k^2 / 2m \), \( \sigma_i \) \((i = x, y, z)\) are Pauli matrices, and \( d_i \) \((i = x, y)\) are functions of \( k \) and the spin-orbit interaction. The eigenvalue equation of Eq. (3), \( H_0 | n k \rangle = \varepsilon_{n k} | n k \rangle \), can be solved exactly. The eigenvalue is given by

\[
\varepsilon_{n k} = \varepsilon_k - n d \tag{4}
\]

with \( d = \sqrt{d_x^2 + d_y^2} \) and \( n = \pm 1 \). The eigenstate of the Hamiltonian (3) can be chosen as

\[
| n k \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i \theta} & i \end{pmatrix} \text{Im} \langle n k | \langle \xi | n k \rangle, \tag{5}
\]

where \( \theta = \text{tan}^{-1}(-d_y / d_x) \) is also a function of \( k \). It can be shown that \( \langle n k | \sigma_{z} | n k \rangle = 0 \) and \( \langle n k | \sigma_{y} | n k \rangle = -n d / d \). In the absence of an electric field, the direction of spin on the Fermi surface is along the vector \( d = d_x \hat{e}_x + d_y \hat{e}_y \).

The total spin current linear response to the electric field is given by

\[
\begin{pmatrix} J_x^z \\ J_y^z \end{pmatrix} = \begin{pmatrix} \sigma_{xx}^z & \sigma_{xy}^z \\ \sigma_{yx}^z & \sigma_{yy}^z \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix}, \tag{6}
\]

where the total spin conductivity tensor \( \sigma_{ab}^z \) is composed of the conventional spin conductivity tensor \( \sigma_{ab}^z \) and the spin torque conductivity tensor \( \sigma_{ab}^t \):

\[
\sigma_{ab}^z = \sigma_{ab}^t + \sigma_{ab}^\tau. \tag{7}
\]

The Kubo formula for the conventional spin conductivity tensor is given by

\[
\sigma_{ab}^z = \frac{e \hbar}{V} \sum_{n \neq n'} \sum_{k} \frac{f_{nk} - f_{n'k}}{(|E_{nk} - E_{n'k}|)^2} \times \text{Im} \langle n k | J_a^z | n' k \rangle \langle n' k | e_b | n k \rangle. \tag{8}
\]
The Kubo formula for the spin torque conductivity tensor is given by [21]

$$
\sigma_{ab}^s = \lim_{q_0 \to 0} \frac{1}{\hbar V} \sum_{n \neq n'} \sum_{\mathbf{k}} \frac{f_{\mathbf{n} \mathbf{k}} - f_{\mathbf{n}' \mathbf{k} + \mathbf{q}}}{(E_{\mathbf{n} \mathbf{k}} - E_{\mathbf{n}' \mathbf{k} + \mathbf{q}})^2} \times \text{Re}(\langle \mathbf{n} | \tau_z(\mathbf{k}, \mathbf{q}) | \mathbf{n}' \mathbf{k} + \mathbf{q} \rangle \langle \mathbf{n}' \mathbf{k} + \mathbf{q} | \mathbf{v}_b(\mathbf{k}, \mathbf{q}) | \mathbf{n} \mathbf{k} \rangle),
$$

(9)

where \(\tau_z(\mathbf{k}, \mathbf{q}) = [\tau_z(\mathbf{k}) + \tau_z(\mathbf{k} + \mathbf{q})]/2\) and \(\mathbf{v}_b(\mathbf{k}, \mathbf{q}) = [\mathbf{v}_b(\mathbf{k}) + \mathbf{v}_b(\mathbf{k} + \mathbf{q})]/2\) with torque \(\tau_z(\mathbf{k}) = (1/\hbar)|s_z, H_0|\) and velocity \(\mathbf{v}_b(\mathbf{k}) = \partial H_0/\partial \mathbf{k}_b\).

It can be shown that the conventional spin conductivity [Eq. 8] can be written in terms of \(d^2\) as

$$
\sigma_{ab}^c = \frac{e}{V} \sum_{\mathbf{k}} (f_{\mathbf{k}+} - f_{\mathbf{k}-}) \left( \frac{\partial f_{\mathbf{n} \mathbf{k}}}{\partial \mathbf{d}_{\mathbf{k}a}} \right) \frac{1}{4d^2} \left( d_x \frac{\partial d_y}{\partial \mathbf{k}_b} - d_y \frac{\partial d_x}{\partial \mathbf{k}_b} \right),
$$

(10)

It has been shown that the polarized spin response,

$$
\left. \mathcal{Q}_a^* \right| = \frac{e \hbar}{4d^2} \left( d_x \frac{\partial d_y}{\partial \mathbf{k}_a} - d_y \frac{\partial d_x}{\partial \mathbf{k}_a} \right),
$$

(11)

is equivalent to the magnitude of the spin projecting on the out-of-plane magnetic field, \(\mathcal{Q}_a^* \equiv \langle \hbar/2 \rangle B_z/|\mathbf{B}|\), with \(|\mathbf{B}| = d + B_z = e|\mathbf{d} \times \partial d/\partial \mathbf{k}_a|/2d^2\), which can reproduce the result of Refs. 3, 14 in the Rashba system.

The spin torque conductivity tensor [Eq. 9] can also be written in terms of \(d^2\) and simplified to the following form (see Appendix A):

$$
\sigma_{ab}^s = -2\sigma_{ab}^c + \sigma_{ba}^s + \Sigma_{ab}^s,
$$

(12)

where the pure spin torque conductivity \(\Sigma_{ab}^s\) is given by

$$
\Sigma_{ab}^s = \frac{e}{2V} \sum_{\mathbf{n} \mathbf{k}} \frac{\partial f_{\mathbf{n} \mathbf{k}}}{\partial \mathbf{k}_a} \frac{1}{2d^2} \left( d_x \frac{\partial d_y}{\partial \mathbf{k}_b} - d_y \frac{\partial d_x}{\partial \mathbf{k}_b} \right).
$$

(13)

Equation 12 has also been found in Ref. 19 in the two-band model with Rashba spin-orbit interaction. In fact, from the Kubo formula 9, it can be shown that the response \(\Sigma_{ab}^s\) is obtained from the following Kubo formula near the Fermi surface:

$$
\Sigma_{ab}^s = -\frac{e \hbar}{V} \sum_{n \neq n'} \sum_{\mathbf{k}} \frac{\partial f_{\mathbf{n} \mathbf{k}}}{\partial \mathbf{k}_a} \text{Re}(\langle \mathbf{n} \mathbf{k} | \tau_z(\mathbf{k}, \mathbf{q}) | \mathbf{n}' \mathbf{k}+\mathbf{q} \rangle (\mathbf{E}_{\mathbf{n} \mathbf{k}} - \mathbf{E}_{\mathbf{n}' \mathbf{k}+\mathbf{q}})^2),
$$

(14)

which is the electric-field-induced pure spin torque \(\tau_z\) on the Fermi surface. We note that the spin torque conductivity tensor is in general not twice as large as the conventional spin conductivity tensor and opposite in sign 23. In the following sections, we first consider a generic \(k\)-linear spin-orbit-coupled system, in which the energy dispersion is nonspherical. Furthermore, we investigate the relationship between the Berry phase and the spin current by using Eqs. 10 and 12.

III. BERRY PHASE IN A \(k\)-LINEAR SPIN-ORBIT-COUPLED SYSTEM

A. Generic \(k\)-linear system

Consider a generic \(k\)-linear spin-orbit-coupled system, where \(d_x = \beta_{xx} x + \beta_{xy} y\), \(d_y = \beta_{yx} x + \beta_{yy} y\), and \(d = k \Gamma (\phi)\) with

$$
\Gamma(\phi)^2 = (\beta_{xx}^2 + \beta_{xy}^2) \cos^2 \phi + (\beta_{yx}^2 + \beta_{yy}^2) \sin^2 \phi + (\beta_{xx} \beta_{yy} - \beta_{yx} \beta_{xy}) \sin(2\phi).
$$

(15)

The spinorial interaction term \(\sigma_{ax} d_x + \sigma_{ay} d_y\) can be written as \(\sum_{ij} \sigma_i \beta_{ij} k_j\), where the spinor-orbit matrix \(\beta_{ij}\) represents the spin-orbit interactions in the system. Some mechanisms could result in \(k\)-linear spin-orbit interaction. Structure inversion asymmetry (SIA) results in a pure Rashba spin-orbit interaction 11, and the spin-orbit matrix elements are \(\beta_{xx} = \beta_{yy} = 0\) and \(\beta_{xy} = -\beta_{yx} = \alpha\). Bulk inversion asymmetry (BIA) results in pure Dresselhaus spin-orbit interactions 12, and the spin-orbit matrix elements are \(\beta_{xx} = -\beta_{yy} = \gamma\) and \(\beta_{xy} = \beta_{yx} = 0\). The strain effect in SIA and BIA systems can also induce \(k\)-linear spin splitting that is linear in momentum 24.

The terms \(d_x \partial d_y / \partial k_b - d_y \partial d_x / \partial k_b\) appearing in the conventional and spin torque conductivities in the \(x\) and \(y\) components are

$$
d_x \frac{\partial d_y}{\partial k_x} - d_y \frac{\partial d_x}{\partial k_x} = -\text{det}(\tilde{\beta}) k_y, \quad d_x \frac{\partial d_y}{\partial k_y} - d_y \frac{\partial d_x}{\partial k_y} = \text{det}(\tilde{\beta}) k_x,
$$

(16)

where

$$
\text{det}(\tilde{\beta}) = \text{det} \left( \begin{array}{cc} \beta_{xx} & \beta_{xy} \\ \beta_{yx} & \beta_{yy} \end{array} \right) = \beta_{xx} \beta_{yy} - \beta_{yx} \beta_{xy}.
$$

(17)

Both the conventional spin current and the spin torque current contain Eq. 13 in the integral. When \(\text{det}(\tilde{\beta}) = 0\), both conventional spin and spin torque currents should
vanish. This can be seen as follows. When degeneracy occurs by tuning the spin-orbit strength such that 
\[ \Gamma(\phi_0) = 0 \] at some angle \( \phi_0 \), as shown in Fig. 1 [23], we find that the angle \( \phi_0 \) is determined by 
\[ \tan(\phi_0) = -[(\beta_{xx}\beta_{yy} + \beta_{yx}\beta_{yy}) + \sqrt{|\det(\beta)|^2/|\beta_{xx}^2 + \beta_{yy}^2|}}. \] Therefore, the occurrence of degeneracy that leads to the result \( \det(\beta) = 0 \) implies that the system should have an additional conserved quantity. We find that the conserved quantity (a unitary matrix), up to a global phase, is given by 
\[ U = \begin{pmatrix} 1 & A \\ -A & -1 \end{pmatrix} \sigma_x + \frac{B}{|B|} \sigma_y, \] (18)
where \( A = (\beta_{xy}\beta_{xx} - \beta_{yx}\beta_{yy})/(\beta_{xy}\beta_{xx} + \beta_{yx}\beta_{yy}) \) and 
\( B = (\beta_{xx}\beta_{yy} + \beta_{yx}\beta_{yy})/(\beta_{xy}\beta_{xx} + \beta_{yx}\beta_{yy}) \). It can be shown that 
\[ |B| = \sqrt{1 - A^2}. \] For example, in the Rashba-Dresselhaus system, the spin-orbit matrix elements are \( \beta_{xx} = \beta, \beta_{xy} = \alpha, \beta_{yx} = -\alpha, \beta_{yy} = -\beta \), and we have \( A = 0 \) and \( B = -\beta/\alpha \). For \( \alpha = \beta \), we have \( U = (\sigma_x - i\sigma_y)/\sqrt{2} \), and, for \( \alpha = -\beta \), we have \( U = (\sigma_x + i\sigma_y)/\sqrt{2} \). The system Hamiltonian with \( \det(\beta) = 0 \) is invariant under the unitary transformation, i.e., \( UH_0U^\dagger = H_0 \). We find that the in-plane spin \( (\sigma_x, \sigma_y) \) under the unitary transformation [Eq. (18)] can be given by 
\[ \begin{align*} 
U\sigma_xU^\dagger &= A\sigma_x + B\sigma_y, \\
U\sigma_yU^\dagger &= B\sigma_x - A\sigma_y. 
\end{align*} \] (19)
For the \( z \) component, interestingly, we find that under the unitary transformation \( \sigma_z \) is simply replaced by \( -\sigma_z \), i.e.,
\[ U\sigma_zU^\dagger = -\sigma_z. \] (20)
This implies that the spin current \( J^z \) (transverse and longitudinal) in the original basis has the same magnitude as that in the transformed basis but they are opposite in sign. Nevertheless, the system with \( \det(\beta) = 0 \) has the same Hamiltonian in both basis, and, thus, this leads to the result that the spin current \( J^z \) must vanish when \( \det(\beta) = 0 \).

The vanishing spin current can also be seen as follows. It has been shown that, in the generic \( k \)-linear system, the effective coupling between spin \( z \) component \( s_z \) and orbital angular momentum \( L_z \) is given by 
\[ (-2m/h^2)\text{det}(\beta)s_zL_z \] (21, 27). When \( \text{det}(\beta) = 0 \), the effective coupling between spin and orbital angular momentums in the \( z \) component vanishes; thus, both conventional and spin torque currents must be zero. The effective coupling \( (-2m/h^2)\text{det}(\beta) \) will play an important role in the interpretation of phase transition caused by the Berry phase, as shown in the following section.

\[ A_a = \langle nk|\frac{\partial}{\partial k_a}|nk\rangle \]
\[ = \frac{1}{2} \frac{\partial}{\partial k_a} \left( d_x \frac{\partial d_y}{\partial k_a} - d_y \frac{\partial d_x}{\partial k_a} \right). \] (21)

Equation (21) also implies that the Berry vector potential [Eq. (10)] is given by
\[ A = \frac{1}{2} \frac{\partial}{\partial k} \frac{\det(\beta)(-k_y \hat{e}_x + k_x \hat{e}_y)}{2k^2 \Gamma(\phi)^2}. \] (22)

However, in spherical coordinates, the Berry vector potential can be written as 
\[ A = A_\phi = (k \times A)_z/k = \text{det}(\beta)/2k\Gamma(\phi). \]

Therefore, in a generic Dirac Hamiltonian the Berry vector potential has only an \( A_\phi \) component. By using Eq. (22) and the line element \( d\ell = dk\hat{e}_\rho + kd\phi \hat{e}_\phi \), the Berry phase \( \Phi \) is given by
\[ \Phi = \int A \cdot d\ell = \frac{1}{2} \text{det}(\beta) \int_0^{2\pi} d\phi \Gamma(\phi)^2 = \frac{\pi}{|\det(\beta)|} \text{det}(\beta). \] (23)

We note that the integral is performed on the Fermi surface. In obtaining Eq. (23), we have used the result 
\[ \int_{0}^{2\pi} d\phi (1/\Gamma(\phi)^2) = 2\pi/|\text{det}(\beta)|, \]
which can be obtained by using the residue method to the two poles \( \lambda_1 = (\beta_{xx} + \beta_{yy}) + i(\beta_{yx} - \beta_{xy}) \) and \( \lambda_2 = (\beta_{xx} - \beta_{yy}) + i(\beta_{yx} + \beta_{xy}) \). The Berry curvature \( F^z_{ab} \) is defined as
\[ F^z_{ab} = \frac{\partial A_b}{\partial k_a} - \frac{\partial A_a}{\partial k_b}. \] (24)

For \( k \neq 0 \), we have \( \nabla_k \times A = 0 \), and the Berry curvature vanishes everywhere except at \( k = 0 \). Using the divergence theorem in two dimensions and taking the small closed curve around the point \( k = 0 \), we obtain
\[ F^z_{xy} = -F^z_{yx} = \Phi \delta(k). \] (25)

Since the Berry curvature in the generic \( k \)-linear system is a delta function peaked at \( k = 0 \), when \( k \neq 0 \), the Berry curvature vanishes. As a result, the transverse

\[ 1 \]

If we use a different basis, the Berry vector potential could depend on the band index. However, it does not change the physics conclusion in this paper since the integrant in Eq. (23) is invariably unchanged.
current of spin cannot be caused by the Lorentz force in momentum space. However, the spin torque current by definition does not require the charge current in which $k = 0$ states would have a large contribution to the spin torque current, as will be shown in the following section.

The Berry phase $\Phi$ is an invariant quantity in the sense that $\text{det}(\tilde{\beta})$ is invariant under rotation. That is, if the response is caused by the Berry phase, the response remains a universal constant regardless of the change in the direction of applied electric field. Furthermore, the Berry phase $\Phi$ is a constant $\pm \pi$, which is independent of spin-orbit strength. This means that a small change in the spin-orbit strength $\beta_{ij}$ does not change the value of $\Phi$ and so the Berry-phase-induced response does not change its magnitude under a fluctuation of energy dispersion.

In the quantum Hall effect, the topological number exhibited by the Berry phase divided by $\pi$ is an integer. Similar to the quantum Hall effect, the Berry phase divided by $\pi$ is also an integer: $\text{det}(\tilde{\beta})/|\text{det}(\tilde{\beta})| = \pm 1$. The physical meaning of the integer in the quantum Hall effect is the number of edge states, in which the change in the topological number results in a phase transition of a quantum Hall system from $n$ to $n \pm 1$ edge states. The physical meaning of the topological number $\text{det}(\tilde{\beta})/|\text{det}(\tilde{\beta})| = \pm 1$ is as follows. It has been shown that $\text{det}(\tilde{\beta})$ governs the effective coupling of $s_z$ and $L_z$ via $\text{det}(\tilde{\beta})s_zL_z$ and, thus, represents the orbital chirality of the spin $s_z$. When $\text{det}(\tilde{\beta})/|\text{det}(\tilde{\beta})|$ changes from $+1$ to $-1$, the current carrying $s_z$ changes orbital chirality from $L_z$ to $-L_z$, and $-s_z$ changes orbital chirality from $-L_z$ to $+L_z$. The phase transition is summarized in Fig. 2. When $\text{det}(\tilde{\beta}) = 0$, the spin Hall effect vanishes, as mentioned above. Therefore, the change in the topological number of $\text{det}(\tilde{\beta})/|\text{det}(\tilde{\beta})|$ results in the phase transition of exchanging the orbital chirality of spin $s_z$ and $-s_z$ in the spin Hall system.

In the following section, we will show that the conventional spin conductivity is not protected by the Berry phase and that the spin torque conductivity is truly due to the Berry phase effect.

**IV. BERRY PHASE AND SPIN CURRENT**

By using Eqs. (10) and (16), we obtain the four components of the conventional spin conductivity tensor:

$$
\sigma_{xy}^s = \frac{e}{8\pi^2} \text{det}(\tilde{\beta}) \int_0^{2\pi} \frac{\cos^2 \phi}{\Gamma(\phi)^2} d\phi,
$$

$$
\sigma_{yy}^s = -\frac{e}{8\pi^2} \text{det}(\tilde{\beta}) \int_0^{2\pi} \frac{\sin^2 \phi}{\Gamma(\phi)^2} d\phi,
$$

$$
\sigma_{xx}^s = -\sigma_{yy}^s = -\frac{e}{8\pi^2} \text{det}(\tilde{\beta}) \int_0^{2\pi} \frac{\sin \phi \cos \phi}{\Gamma(\phi)^2} d\phi.
$$

Comparing Eq. (24) with Eq. (28), we find that the conventional spin conductivity is not determined by the Berry phase. However, it is interesting to evaluate each integral in Eq. (26). The integral can be performed by using the residue method to the two poles $\lambda_1 = (\beta_{xx} + \beta_{yy}) + i(\beta_{xy} - \beta_{yx})$ and $\lambda_2 = (\beta_{xx} - \beta_{yy}) + i(\beta_{xy} + \beta_{yx})$, and the result is given by

$$
\left(\begin{array}{ll}
\sigma_{xx}^s & \sigma_{xy}^s \\
\sigma_{yx}^s & \sigma_{yy}^s
\end{array}\right) = \frac{|\text{det}(\tilde{\beta})|}{8\pi} \frac{e}{\text{det}(\tilde{\beta})} \left(\begin{array}{cc}
\text{Im}\left(\frac{\lambda_1}{\lambda_2}\right) & 1 - \text{Re}\left(\frac{\lambda_1}{\lambda_2}\right) \\
-1 + \text{Re}\left(\frac{\lambda_1}{\lambda_2}\right) & -\text{Im}\left(\frac{\lambda_1}{\lambda_2}\right)
\end{array}\right),
$$

where $\lambda_1$ and $\lambda_2$ are taken from the relative maximum (minimum) value of $(|\lambda_1|, |\lambda_2|)$. That is, if $|\lambda_1| > |\lambda_2|$ then $\lambda_1 = \lambda_1$ and $\lambda_2 = \lambda_2$, and vice versa. The result shows that the conventional spin conductivity seems to be affected by the Berry phase. Because $\text{Re}(\lambda_1/\lambda_2)$ and $\text{Im}(\lambda_1/\lambda_2)$ in Eq. (27) depend on the spin-orbit strength $\beta_{ij}$, we find that $\sigma_{\alpha\beta}^s$ is not purely caused by the Berry phase. The Berry-phase-induced spin current should have vanishing longitudinal conductivity.

In the pure Rashba system, we have $\text{Im}(\lambda_1/\lambda_2) = 0$ and $\text{Re}(\lambda_1/\lambda_2) = 0$. It seems that the conventional spin conductivity is protected by the Berry phase. Nevertheless, a small Dresselhaus spin-orbit interaction breaks the spherically symmetric energy dispersion and leads to $\text{Im}(\lambda_1/\lambda_2) \neq 0$, which depends on the ratio $\beta/\alpha$ or $\alpha/\beta$; i.e., the longitudinal term is not zero.

In the Rashba-Dresselhaus system $H_0 = \varepsilon_k + \alpha(\sigma_x k_y - \sigma_y k_x) + \gamma(\sigma_x k_z - \sigma_y k_y)$, the determinant of the spin orbit matrix is $\alpha^2 - \gamma^2$, and $\lambda_1 = -2i\alpha$ and $\lambda_2 = 2\gamma$ (if the electric field is applied in the direction [010]). We have $\text{Re}(\lambda_1/\lambda_2) = 0$; however, $\text{Im}(\lambda_1/\lambda_2) \neq 0$. Although the conventional spin Hall conductivity in the Rashba-Dresselhaus system is numerically equal to the Berry conductivity, the Berry-phase-induced spin current should have vanishing longitudinal conductivity.
phase (i.e., the system has nonzero longitudinal terms), a small change in the direction of the applied electric field actually results in $\text{Re}(\lambda_\perp/\lambda_\parallel) \neq 0$. For example, when the electric field is applied in [110], we change the coordinate $(k_x, k_y)$ to $(k'_x, k'_y)$ such that $k'_x$ and $k'_y$ are parallel to [110] and [110], respectively. The resulting spin-orbit matrix elements are $\beta_{x'x'} = (\alpha - \gamma)/\sqrt{2}$, $\beta_{x'y'} = (\alpha + \gamma)/\sqrt{2}$, and $\beta_{y'y'} = -(\alpha - \gamma)/\sqrt{2}$, and $\beta_{y'y'} = (\alpha + \gamma)/\sqrt{2}$. It can be shown that the determinant of the new spin-orbit matrix is still $\alpha^2 - \gamma^2$. We have $\lambda_1 = \sqrt{2}\alpha(1 + i)$ and $\lambda_2 = -\sqrt{2}\alpha(1 - i)$, and $\text{Re}(\lambda_\perp/\lambda_\parallel) \neq 0$. Therefore, the conventional spin current in the Rashba-Dresselhaus system does not behave as an isotropic system.

If the conventional spin current was protected by the Berry phase, then the resulting conventional spin current should have maintained a universal value under a small fluctuation of energy dispersion (without breaking time-reversal symmetry). However, as we have shown above, the conventional spin current depends on the shape of the energy dispersion. The conventional spin current is not purely caused by the Berry phase.

We now consider the spin torque conductivity tensor. We first calculate $\Sigma_{xy}^{\tau}$ by using Eq. (12). It can be shown that (see Appendix B8)

$$
\Sigma_{xx}^{\tau} = -\Sigma_{yy}^{\tau} = -\frac{e^2}{8\pi^2} \text{det}(\bar{\beta}) \int_0^{2\pi} \sin \phi \cos \phi \frac{1}{\Gamma(\phi)^2} d\phi,
$$

$$
\Sigma_{xy}^{\tau} = +\frac{e^2}{8\pi^2} \text{det}(\bar{\beta}) \int_0^{2\pi} \cos^2 \phi \frac{1}{\Gamma(\phi)^2} d\phi,
$$

$$
\Sigma_{yx}^{\tau} = -\frac{e^2}{8\pi^2} \text{det}(\bar{\beta}) \int_0^{2\pi} \sin^2 \phi \frac{1}{\Gamma(\phi)^2} d\phi.
$$

Comparing Eq. (28) with Eq. (26), we find that $\sigma_{xy}^{\tau} = \Sigma_{xy}^{\tau}$, $\sigma_{yx}^{\tau} = \Sigma_{yx}^{\tau}$, and $\sigma_{xx}^{\tau} = \sigma_{yy}^{\tau} = 0$. Substituting Eqs. (26) and (28) into (12) and performing some straightforward calculations, we have

$$
\sigma_{xy}^{\tau} = -\sigma_{yx}^{\tau} = -\frac{e}{8\pi^2} \text{det}(\bar{\beta}) \int_0^{2\pi} \frac{1}{\Gamma(\phi)^2} d\phi,
$$

$$
\sigma_{xx}^{\tau} = \sigma_{yy}^{\tau} = 0.
$$

Comparing Eq. (29) with the Berry phase given by Eq. (23), we see that the spin torque Hall conductivity is closely related to the Berry phase effect. By using Eq. (23) we obtain

$$
\begin{pmatrix}
\sigma_{xx}^{\tau} & \sigma_{xy}^{\tau} \\
\sigma_{yx}^{\tau} & \sigma_{yy}^{\tau}
\end{pmatrix} = \frac{e}{4\pi^2} \frac{\Gamma(\phi)}{\Lambda(\phi)} \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}.
$$

The magnitude of the spin torque Hall conductivity is independent of the direction of the applied electric field. The spin torque response of the system behaves like an isotropic system. That is, a small change in the direction of the applied electric field does not change the magnitude of the spin torque current. Furthermore, the longitudinal conductivities are zero. By using the Berry curvature [Eq. (25)] and $\sigma_{ab}^{\tau} = -i\omega_0 \sum_{\mathbf{k},\mathbf{k}'} f_{\mathbf{k},\mathbf{k}'} F_{ab}^z$ with $\mu > 0$, we can also reproduce the result of Eq. (30). This also explains why the longitudinal spin torque conductivities vanish and the transverse part satisfies the antisymmetric property.

However, because the Fermi level does not lie in the true gap, the conserved spin current may be affected by impurity scattering. The conventional spin conductivity in the Rashba system is shown to be significantly influenced by impurity scattering [25]. Impurity scattering in the Rashba system with the spin torque current taken into account has been studied by Sugimoto et al. [26]. By using the Keldysh formalism, the conserved spin Hall current is shown to be zero in the Rashba system with a delta impurity potential and remains a finite value with a finite range potential.

In short, the spin torque current protected by the Berry phase is unaltered by a fluctuation of energy dispersion and a change in the direction of the applied electric field. The Berry-phase-induced spin torque current does not contribute to dissipation. Unlike the quantum Hall effect, the topological number may not be invariant under the influence of impurity scattering.

We propose an experiment for detecting our prediction of rotationally invariant spin torque current. By using Eqs. (27) and (30), the total spin-Hall conductivity $\sigma_{xy}^{\tau} = \sigma_{xy}^{\tau} + \sigma_{yx}^{\tau}$ satisfies the following equation

$$
\sigma_{xy}^{\tau} + \sigma_{yx}^{\tau} = \lambda,
$$

where $\lambda = \text{Re}(\lambda_\perp/\lambda_\parallel)$. The experimental value of the spin current is assumed to be the sum of conventional and spin torque currents, and we could obtain the corresponding experimental value $\lambda_{\text{exp}}$. The theoretical value of the conventional spin-Hall conductivity should be $\sigma_{xy,\text{th}}^{\tau} = \pm(e/8\pi)(1 - \lambda_{\text{exp}})$. We can calculate the theoretical value of the spin torque-Hall conductivity

$$
\sigma_{xy,\text{th}}^{\tau} = \sigma_{xy,\text{exp}}^{\tau} - \left(\frac{e}{8\pi}\right)(1 - \lambda_{\text{exp}}).
$$

If we rotate the direction of an applied electric field, we should obtain different values of $\lambda_{\text{exp}}$ and $\sigma_{xy,\text{exp}}^{\tau}$ in a linear momentum dominate regime with nonspherical symmetric energy dispersion, such as the Rashba system with a small correction of Dresselhaus spin-orbit strength in II-VI semiconductors [2, 11]. However, the resulting $\sigma_{xy,\text{th}}^{\tau}$ should be rotationally invariant. We predict that $\sigma_{xy,\text{th}}^{\tau}$ maintains a universal constant and is experimentally measurable.

So far we have not addressed the contribution of higher momentum to the Berry phase and spin torque current. The Berry curvature in a two-dimensional system is always a Dirac delta function at $k = 0$ since the Berry vector potential is a gradient of the scalar function $\theta(\mathbf{k})$ [see Eq. (21)]. For the k-cubic Rashba system [30], we have $d_x = \alpha_b k^3 \sin(3\phi)$ and $d_y = -\alpha_b k^3 \cos(3\phi)$, and the Berry phase is found to be $3\pi$. However, only at low density approximation, the spin torque-Hall conductivity may be numerically determined by the Berry
phase [21]. For the $k$-cubic Rashba-Dresselhaus system [31], we have $d_x = [\alpha_h \sin(3\phi) + \beta_h \cos \phi][k^3]$ and $d_y = [-\alpha_h \sin(3\phi) + \beta_h \sin \phi][k^3]$. The corresponding Berry phase is $3\pi$ for $\alpha_h^2 > \beta_h^2$ and $\pi$ for $\alpha_h^2 < \beta_h^2$. It has been found that the spin torque-Hall conductivity at low density approximation is $-9e/4\pi$ for $\alpha_h^2 > \beta_h^2$ and $-3e/4\pi$ for $\alpha_h^2 < \beta_h^2$ [32], which can be numerically determined by the Berry phase. That is, the relationship between the spin torque current and the Berry phase would depend on the position of the Fermi level when the higher momentum is included in the system. Interestingly, it has recently been shown that the Fermi surface contribution of the pure spin torque response Eq. (13) in two-dimensional systems exhibiting higher momentum always exhibits quantized conductivity [33].

On the other hand, it seems that a non-vanishing Berry phase on the Fermi surface implies a non-vanishing charge Hall conductance in a time-reversal symmetric system. We can choose a new basis vector such that the Berry phase vanishes [34], however, the physics conclusion Eq. (30) [see also Eq. (29)] is still unchanged. The only change is that the Berry phase in the new basis would depend on the band index, which could be experimentally determined [17].

V. CONCLUSION

For a generic $k$-linear spin-orbit-coupled system, we calculated the Berry phase on the Fermi surface, and the conventional and spin torque conductivities by using the Kubo formula. The conventional spin Hall current in general depends on the spin-orbit strength and is not proportional to the Berry phase. The longitudinal term of the conventional spin conductivity also does not vanish.

We found that the spin torque Hall current is proportional to the Berry phase and that the longitudinal spin torque current vanishes. Since the Berry phase effect prohibits the longitudinal response and results in antisymmetric properties in transverse conductivities, in this sense, we found that the spin torque conductivity is truly caused by the Berry phase effect on the Fermi surface. We showed that the Berry phase on the Fermi surface manifests two invariant quantities. The Berry phase is a sign function of the spin-orbit matrix, which is invariant under rotation. This means that the magnitude of the spin torque current response is independent of the direction of the applied electric field. The Berry phase also implies that a small fluctuation of spin-orbit strength (if it does not cause the phase transition) does not change the magnitude of the spin torque current. The Berry phase divided by $\pi$ is an integer, $+1$ or $-1$, which is a topological number. The topological number $\pm 1$ due to the Berry phase not only represents the different phase of the spin Hall system but also reflects the rotationally invariant property. The change in the topological number results in the phase transition of exchanging the orbital chirality of spin $s_z$ and $-s_z$ in the spin Hall system. The phase transition phenomenon occurs in conventional spin and spin torque currents.

Because the Fermi level does not lie in the true gap, the topological number would be destroyed by the influence of impurity scattering, which depends on the scattering mechanism.

Hopefully, our interesting predictions of the invariant properties of the spin torque current will stimulate measurements in two-dimensional semiconductor systems in the near future.

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Appendix A: Torque-Spin conductivity

In this appendix, we derive Eq. (12). Since in obtaining $\sigma'_{ab}$, we have to take the limit $q_a \to 0$, we expand each term to first order of $q_a$. By calculating the matrix element $\langle n|\tau_z(k, q)|-n-k+q\rangle$, where we have replaced $n'$ by $-n$,

$$\langle n|\tau_z(k, q)|-n-k+q\rangle = -i n \left( d + \frac{q_a}{2} \frac{\partial d}{\partial k_a} \right) + q_a \frac{\partial d}{\partial k_a} + o(q_a^2). \tag{A1}$$

For the matrix element $\langle -n-k+q|\tau_0(k, q)|n+k-q\rangle$, we have

$$\langle -n-k+q|\tau_0(k, q)|n+k-q\rangle = \frac{\text{ind} d \frac{\partial \theta}{\partial k_b}}{\hbar} - \frac{i}{2} q_a \left( \frac{d_y}{\hbar} \frac{\partial^2 d_x}{\partial k_a \partial k_b} + \frac{d_x}{\hbar} \frac{\partial^2 d_y}{\partial k_a \partial k_b} \right) + q_a \frac{\partial \theta}{\partial k_a} \left( \frac{i \partial \epsilon}{2 \hbar} - \frac{n \partial d_x}{\hbar} e^{i\theta} + \frac{\partial d_y}{\hbar} e^{i\theta} + o(q_a^2) \right). \tag{A2}$$

On the other hand, we have

$$\frac{f_{nk} - f_{-nk+q}}{(\epsilon_{nk} - \epsilon_{-nk+q})^2} = \frac{f_{nk} - f_{-nk}}{(\epsilon_{nk} - \epsilon_{-nk})^2} + \frac{\partial \epsilon}{\partial k_a} \left[ 2 \frac{f_{nk} - f_{-nk}}{(\epsilon_{nk} - \epsilon_{-nk})^3} \frac{\partial f_{-nk}}{\partial \epsilon_{-nk}} + o(q_a^2) \right]. \tag{A3}$$
Inserting Eqs. (A1), (A2) and (A3) into Eq. (9) and after straightforward calculations, we have

$$\sigma_{ab}^{s} = -\frac{e}{V} \sum_{k} \frac{1}{4} \frac{\partial f_{k+}}{\partial \varepsilon_{k}^{+}} \sum_{n} \left( \frac{\partial f_{k+}}{\partial \varepsilon_{k}^{+}} \frac{\partial f_{k-}}{\partial \varepsilon_{k}^{-}} + \frac{\partial f_{k-}}{\partial \varepsilon_{k}^{-}} \frac{\partial f_{k-}}{\partial \varepsilon_{k}^{+}} \right)$$

$$+ \frac{e}{V} \sum_{k} \frac{f_{k+} - f_{k-}}{2d} \frac{\partial \varepsilon_{k}^{+}}{\partial k_{a}} \frac{\partial \varepsilon_{k}^{-}}{\partial k_{b}},$$

\[\text{(A4)}\]

The first term of Eq. (A4) is the pure spin torque conductivity \(\Sigma_{ab}^{z}\) [see Eq. (13)]. The second and third terms of Eq. (A4) equals to \(\sigma_{ba}\), and \(-2\sigma_{ab}\), respectively.

**Appendix B: Calculation of \(\Sigma_{ab}^{z}\)**

In this appendix, we want to calculate the response \(\Sigma_{ab}^{z}\) [Eq. (13)] in the system with generic \(k\)-linear Hamiltonian. At zero temperature, the Fermi level \(\mu\) determines the Fermi momentum \(k_{F}^{+}\) for the two bands by

$$\mu = \frac{\hbar^{2}(k_{F}^{+})^{2}}{2m} - nk_{F}^{+}\Gamma.$$  

\[\text{(B1)}\]

The term \(\partial f_{n} / \partial \varepsilon_{nk}\) is given by the delta function \(-\delta(\mu - \varepsilon_{nk})\). The term \(\mu - \varepsilon_{nk}\) can be written as

\(-\frac{k_{n}^{2}}{2m}(k^{+} - k_{F}^{+})(k + k_{F}^{+} - 2m\Gamma / \hbar^{2})\). Consider \(n = 1\), we have \((k_{F}^{+} - 2m\Gamma / \hbar^{2}) = (m\sqrt{\Gamma^{2} + 2\mu^{2} / m^{2}} - m\Gamma / \hbar^{2}) = k_{F}^{+}\). On the other hand, when \(n = -1\), we have \((k_{F}^{+} - 2m\Gamma / \hbar^{2}) = k_{F}^{-}\). Using \(\delta(x-a)(x-b) = \delta(x-a) - \delta(x-b)\) and Eq. (B1), we have

$$\frac{\partial f_{n}}{\partial \varepsilon_{nk}} = -\delta(\mu - \varepsilon_{nk})$$

$$= -\frac{1}{\sqrt{\Gamma^{2} + 2\mu^{2} / m^{2}}} \left[ \delta(k - k_{F}^{+}) - \delta(k - k_{F}^{-}) \right].$$  

\[\text{(B2)}\]

Since \(k > 0\), the second term of Eq. (B2) vanishes, and thus, we have

$$\frac{\partial f_{n}}{\partial \varepsilon_{nk}} = -\frac{\delta(k - k_{F}^{+})}{\sqrt{\Gamma^{2} + 2\mu^{2} / m^{2}}} = \frac{2m}{\hbar^{2}} \frac{\delta(k - k_{F}^{+})}{k_{F}^{+} + k_{F}^{-}}.$$  

\[\text{(B3)}\]

Using Eqs. (B3), (13) and (22), the term \(\Sigma_{ab}^{z}\) is given by

$$\Sigma_{ab}^{z} = -\frac{e}{2V} \sum_{nk} \frac{\partial f_{nk}}{\partial \varepsilon_{nk}} A_{b}$$

$$= -\frac{e}{2V} \sum_{k} \left( \frac{\partial f_{k+}}{\partial \varepsilon_{k+}} \frac{\partial f_{k-}}{\partial \varepsilon_{k-}} + \frac{\partial f_{k-}}{\partial \varepsilon_{k-}} \frac{\partial f_{k-}}{\partial \varepsilon_{k+}} \right) A_{b}$$

$$= \frac{e}{8\pi^{2}} \int_{0}^{2\pi} d\phi \frac{2m}{k_{F}^{+} + k_{F}^{-}} \left\{ f_{1}(\phi) - f_{2}(\phi) \right\},$$  

\[\text{(B4)}\]

where

$$f_{1}(\phi) = \int kdk \frac{\partial \varepsilon_{k}^{+}}{\partial k_{a}} A_{b} \left[ \delta(k - k_{F}^{+}) + \delta(k - k_{F}^{-}) \right],$$

$$f_{2}(\phi) = \int kdk \frac{\partial \varepsilon_{k}^{-}}{\partial k_{a}} A_{b} \left[ \delta(k - k_{F}^{+}) - \delta(k - k_{F}^{-}) \right].$$  

\[\text{(B5)}\]

It can be straightforwardly shown that \(f_{2}(\phi)\) always vanishes for \(a = x, y\) and \(b = x, y\). This can also be seen as follows. Because we use the \(k\)-linear system, we have \(d \sim k\) and \(A_{b} \sim 1/k\), and thus, the term \(k(\partial \varepsilon / \partial k_{a}) \phi \) is a function of \(\phi\) only. Therefore, for \(\Sigma_{ab}^{z}\), we have

$$\Sigma_{ab}^{z} = \frac{e}{8\pi^{2}} \int_{0}^{2\pi} d\phi \frac{2}{k_{F}^{+} + k_{F}^{-}} \int kdkk_{a} A_{b}$$

$$\times \left[ \delta(k - k_{F}^{+}) - \delta(k - k_{F}^{-}) \right].$$  

\[\text{(B6)}\]

Using the Berry vector potential for \(k\)-linear system [see Eq. (22)], the matrix formed by \(k_{a} A_{b}\) is given by

$$\begin{pmatrix} k_{x} A_{x} & k_{x} A_{y} \\ k_{y} A_{x} & k_{y} A_{y} \end{pmatrix} = \frac{\det(\vec{\beta})}{21}\begin{pmatrix} -\sin\phi \cos\phi & \cos^{2}\phi \\ -\sin^{2}\phi & \sin\phi \cos\phi \end{pmatrix}.$$  

\[\text{(B7)}\]

Inserting Eq. (B7) into Eq. (B6), we have

$$\begin{pmatrix} \Sigma_{xx}^{z} & \Sigma_{x}^{z} \\ \Sigma_{yy}^{z} & \Sigma_{y}^{z} \end{pmatrix} = \frac{e}{8\pi^{2}} \det(\vec{\beta}) \int_{0}^{2\pi} d\phi \frac{1}{\Gamma^{2}} \begin{pmatrix} -\sin\phi \cos\phi & \cos^{2}\phi \\ -\sin^{2}\phi & \sin\phi \cos\phi \end{pmatrix}.$$  

\[\text{(B8)}\]

Compare Eq. (B8) with Eq. (26), we find that \(\Sigma_{ab}^{z}\) is equal to \(\sigma_{ab}^{s}\) for \(a = x, y\) and \(b = x, y\).
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