Some Results for Beta Fréchet Distribution

WAGNER BARRETO-SOUZA¹, GAUSS M. CORDEIRO², AND ALEXANDRE B. SIMAS³

¹Departamento de Estatística, Universidade Federal de Pernambuco, Recife, Brazil
²Departamento de Estatística e Informática, Universidade Federal Rural de Pernambuco, Recife, Brazil
³Associação Instituto Nacional de Matemática Pura e Aplicada, Rio de Janeiro, Brazil

Nadarajah and Gupta (2004) introduced the beta Fréchet (BF) distribution, which is a generalization of the exponentiated Fréchet (EF) and Fréchet distributions, and obtained the probability density and cumulative distribution functions. However, they did not investigate the moments and the order statistics. In this article, the BF density function and the density function of the order statistics are expressed as linear combinations of Fréchet density functions. This is important to obtain some mathematical properties of the BF distribution in terms of the corresponding properties of the Fréchet distribution. We derive explicit expansions for the ordinary moments and L-moments and obtain the order statistics and their moments. We also discuss maximum likelihood estimation and calculate the information matrix which was not given in the literature. The information matrix is numerically determined. The usefulness of the BF distribution is illustrated through two applications to real data sets.

Keywords Beta distribution; Exponentiated Fréchet; Fréchet distribution; Information matrix; Maximum likelihood estimation.

Mathematics Subject Classification 60E05; 62E20.

1. Introduction

The Fréchet distribution has applications ranging from accelerated life testing through to earthquakes, floods, horse racing, rainfall, queues in supermarkets, sea currents, wind speeds, and track race records. Kotz and Nadarajah (2000) gave some applications in their book. In this article, we discuss the BF distribution which stems from the following idea. Eugene et al. (2002) defined the beta $G$ distribution from a
Beta Fréchet Distribution

quite arbitrary cumulative distribution function (cdf) $G(x)$ by

$$F(x) = \frac{1}{B(a, b)} \int_0^{G(x)} \omega^{a-1}(1 - \omega)^{b-1} d\omega,$$

(1)

where $a > 0$ and $b > 0$ are two additional parameters whose role is to introduce skewness and to vary tail weight and $B(a, b) = \int_0^1 \omega^{a-1}(1 - \omega)^{b-1} d\omega$ is the beta function. The class of distributions (1) has an increased attention after the works by Eugene et al. (2002) and Jones (2004). Application of $X = G^{-1}(V)$ to the random variable $V$ following a beta distribution with parameters $a$ and $b$, $V \sim \beta(a, b)$ say, yields $X$ with cdf (1).

Eugene et al. (2002) defined the beta normal (BN) distribution by taking $G(x)$ to be the cdf of the normal distribution and derived some of its first moments. General expressions for the moments of the BN distribution were derived (Gupta and Nadarajah, 2004). Nadarajah and Kotz (2004) also introduced the beta Gumbel (BG) distribution by taking $G(x)$ to be the cdf of the Gumbel distribution and provided closed-form expressions for the moments, the asymptotic distribution of the extreme order statistics and discussed the maximum likelihood estimation procedure. Nadarajah and Gupta (2004) introduced the BF distribution by taking $G(x)$ to be the Fréchet distribution, derived the analytical shapes of the probability density function (pdf) and the hazard rate function, and calculated the asymptotic distribution of the extreme order statistics. However, they do not investigate expressions for the moments and the information matrix which we do in this article. Also, Nadarajah and Kotz (2005) worked with the beta exponential (BE) distribution and obtained the moment generating function, the first four cumulants, the asymptotic distribution of the extreme order statistics and discussed the maximum likelihood estimation. We can write (1) as

$$F(x) = I_{G(x)}(a, b),$$

(2)

where $I_x(a, b) = B(a, b)^{-1} \int_0^x w^{a-1}(1 - w)^{b-1} dw$ denotes the incomplete beta function ratio, i.e., the cdf of the beta distribution with parameters $a$ and $b$. For general $a$ and $b$, we can express (2) in terms of the well-known hypergeometric function defined by

$$\text{$_2F_1$(}$x$, $\beta$, $\gamma$; $x$) = \sum_{i=0}^{\infty} \frac{(x)_i (\beta)_i}{(\gamma)_i i!} x^i,$$

where $(x)_i = x(x + 1) \cdots (x + i - 1)$ denotes the ascending factorial. We obtain

$$F(x) = \frac{G(x)^a}{aB(a, b)} \text{$_2F_1$(}$a, 1 - b, a + 1; G(x)$).$$

The properties of the cdf $F(x)$ for any beta $G$ distribution defined from a parent $G(x)$ in (1), could, in principle, follow from the properties of the hypergeometric function which are well established in the literature; see, for example, Sec. 9.1 of Gradshteyn and Ryzhik (2000).
The probability density function (pdf) corresponding to (1) can be written in the form
\[
f(x) = \frac{1}{B(a, b)} G(x)^{a-1} [1 - G(x)]^{b-1} g(x),
\]  
(3)

where \( g(x) = dG(x)/dx \) is the pdf of the parent distribution. The pdf \( f(x) \) will be most tractable when the functions \( G(x) \) and \( g(x) \) have simple analytic expressions as is the case of the Fréchet distribution. Except for some special choices for \( G(x) \) in (1), it would appear that the formula (3) will be difficult to deal with.

The cdf and pdf of the Fréchet distribution are, respectively,
\[
G_{a, \lambda}(x) = e^{-\left(\frac{x}{\lambda}\right)^{\frac{1}{a}}}, \quad x > 0,
\]  
(4)

and
\[
g_{a, \lambda}(x) = \lambda \sigma^x x^{-\left(\frac{x+1}{\lambda}\right)} e^{-\left(\frac{x}{\lambda}\right)^{\frac{1}{a}}}, \quad x > 0,
\]

where \( \sigma > 0 \) is the scale parameter and \( \lambda > 0 \) is the shape parameter. The \( r \)th moment of the Fréchet distribution for \( r < \lambda \) is \( \mu_r^{\prime} = \sigma^{r} \Gamma(1 - r/\lambda) \), and then the first four cumulants if \( \lambda > 4 \) are
\[
\kappa_1 = \sigma g_1, \quad \kappa_2 = \sigma^2 \left( g_2 - g_1^2 \right), \quad \kappa_3 = \frac{g_3 - 3g_1 g_2 + 2g_1^3}{\left( g_2 - g_1^2 \right)^{1/2}},
\]
\[
\kappa_4 = \frac{g_4 - 4g_1 g_3 + 6g_1^2 g_2 - 3g_1^4}{\left( g_2 - g_1^2 \right)^2},
\]

where \( g_k = \Gamma(1 - k/\lambda) \) for \( k = 1, \ldots, 4 \).

Nadarajah and Gupta (2004) gave the cdf of the BF distribution with parameters \( a > 0, b > 0, \sigma > 0, \) and \( \lambda > 0 \) (denoted by BF\((a, b, \sigma, \lambda)\)) in the same way from (1) by replacing the parent cdf \( G(x) \) by (4)
\[
F(x) = \frac{1}{B(a, b)} \int_0^{\exp\left(-\left(\frac{x}{\lambda}\right)^{\frac{1}{a}}\right)} \omega^{a-1} (1 - \omega)^{b-1} d\omega = I_{\exp\left(-\left(\frac{x}{\lambda}\right)^{\frac{1}{a}}\right)}(a, b), \quad x > 0.
\]  
(5)

They also give the corresponding pdf and hazard function, respectively, as
\[
f(x) = \frac{\lambda \sigma^x}{B(a, b)} x^{-\left(\frac{x+1}{\lambda}\right)} e^{-a \left(\frac{x}{\lambda}\right)^{\frac{1}{a}}} \left[1 - e^{-\left(\frac{x}{\lambda}\right)^{\frac{1}{a}}}\right]^{b-1}, \quad x > 0,
\]  
(6)

and
\[
h(x) = \frac{\lambda \sigma^x}{B(a, b)} x^{-\left(\frac{x+1}{\lambda}\right)} e^{-a \left(\frac{x}{\lambda}\right)^{\frac{1}{a}}} \left[1 - e^{-\left(\frac{x}{\lambda}\right)^{\frac{1}{a}}}\right]^{b-1}, \quad x > 0.
\]  
(7)

Figures 1 and 2 illustrate some of the possible shapes of the pdf (6) and hazard function (7), respectively, for selected parameter values, including the case of the Fréchet distribution. The BF distribution is easily simulated from (5) as follows: if \( V \sim \beta(a, b) \) then \( X = \sigma/(-\log V)^{1/2} \) has the BF\((a, b, \sigma, \lambda)\) distribution.
The BF distribution generalizes some known distributions. The exponentiated Fréchet (EF) distribution (Nadarajah and Kotz, 2003) is a special case when $a = 1$. The Fréchet distribution (with parameters $\sigma$ and $\lambda$) is also a special case of (6) when $a = b = 1$. Further, when $b = 1$ and $\lambda = 1$, (6) is an inverse gamma distribution with shape parameter 2 and scale parameter $a \sigma$. Since the BF distribution generalizes the Fréchet and EF distributions by adding two parameters and one parameter, respectively, it can be used by practitioners as an extra tool to analyze the data we would normally use with the last two distributions. The book of Kotz and Nadarajah (2000) demonstrates the applicability of the Fréchet distribution in several fields.

The rest of the article is organized as follows. Section 2 gives expansions for the pdf and cdf of the BF distribution and for the density of the order statistics depending on whether the parameters $b$ (or $a$) is real non integer and integer. We show that the density functions of the BF and the order statistics can be expressed as mixture of Fréchet density functions. The moments of this distribution and of the order statistics are not known and general expansions are derived in
Sec. 3 for the cases $b$ real non integer and integer. $L$-moments are expectations of certain linear combinations of order statistics and form the basis of a general theory which covers the summarization and description of theoretical probability distributions. In Sec. 4, we present expansions for the $L$-moments of the BF distribution. We discuss in Sec. 5 maximum likelihood estimation and calculate the elements of the information matrix. Section 6 provides two applications to real data sets. Section 7 ends with some conclusions.

2. Expansions for the Distribution and Density Functions

Here, we provide simple expansions for the cdf of the BF distribution depending on whether the parameter $b$ (or $a$) is real non integer or integer. We consider the series expansion

\[
(1 - z)^{b-1} = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(b)}{\Gamma(b-j) j!} z^j,
\]
valid for $|z| < 1$ and $b > 0$ real non integer. Application of (8) to (1) if $b$ is real non integer gives

$$F(x) = \frac{\Gamma(a + b)}{\Gamma(a)} \sum_{j=0}^{\infty} \frac{(-1)^j G_{\sigma,j}(x)^{a+j}}{\Gamma(b - j)! (a + j)}.$$

where $G_{\sigma,j}(x)$ comes from (4). Then, we have

$$F(x) = \frac{\Gamma(a + b)}{\Gamma(a)} \sum_{j=0}^{\infty} \frac{(-1)^j e^{-(a+j)(\xi)^j}}{\Gamma(b - j)! (a + j)}.$$  \hspace{1cm} (9)

For $b$ integer, the sum in (10) stops at $b - 1$. When $b = 1$, it follows $F(x) = e^{-a(\xi)^1}$.

It can be seen in the Wolfram Functions Site\(^1\) that for integer $b$

$$I_y(a, b) = \frac{y^a}{\Gamma(a)} \sum_{j=0}^{b-1} \frac{\Gamma(a + j)(1 - y)^j}{j!}$$

and for integer $a$

$$I_y(a, b) = 1 - \frac{(1 - y)^b}{\Gamma(b)} \sum_{j=0}^{a-1} \frac{\Gamma(b + j)}{j!} y^j.$$

Hence, if $b$ is integer, we obtain another equivalent form for (10)

$$F(x) = \frac{e^{-a(\xi)^1}}{\Gamma(a)} \sum_{j=0}^{b-1} \frac{\Gamma(a + j)(1 - e^{-(\xi)^j})^j}{j!}.$$

and, for integer values of $a$, we have

$$F(x) = 1 - \frac{(1 - e^{-(\xi)^1})^b}{\Gamma(b)} \sum_{j=0}^{a-1} \frac{\Gamma(b + j)}{j!} e^{-j(\xi)^1}.$$

If $a = 1$, the above expression reduces to

$$F(x) = 1 - (1 - e^{-(\xi)^1})^b,$$

which agrees with the cdf of the EF distribution.

The pdf in (6) is straightforward to compute using any statistical software. However, we show that the BF density can be expressed as an infinite (or finite) weighted linear combination of pdf’s of random variables having Fréchet distributions. This is important to provide some mathematical properties of the BF distribution directly from the corresponding properties of the Fréchet distribution. If $b$ is real non integer, and again using (8) we can rewrite (6) as

$$f(x) = \sum_{k=0}^{\infty} w_k S_{\sigma,j}(x), \hspace{1cm} \text{(11)}$$

\(^1\)http://functions.wolfram.com/
where

\[ w_k = \Gamma(a + b)(-1)^k / [\Gamma(a)\Gamma(b - k)k!(k + a)] \]

represent weighted constants such that \( \sum_{k=0}^{\infty} w_k = 1 \) and \( g_{a,b}(x) \) is a Fréchet density with scale parameter \( a_k = \sigma(k + a)^{1/k} \) and shape parameter \( \lambda \). In addition, if \( a = 1 \), (11) agrees with the corresponding result obtained by Nadarajah and Kotz (2003, Sec. 5). If \( b \) is integer, the sum in (11) is finite and stops at \( b - 1 \). Then, the ordinary, central, factorial moments, and the moment generating function of the BF distribution could in principle follow from the same weighted infinite (or finite if \( b \) is an integer) linear combination of the corresponding quantities for the Fréchet distribution.

We now give the density of the \( i \)th order statistic \( X_{i,n} \), \( f_{i,n}(x) \) say, in a random sample of size \( n \) from the BF distribution. It is well known that

\[ f_{i,n}(x) = \frac{1}{B(i, n - i + 1)} f(x)F^{i-1}(x)\{1 - F(x)\}^{n-i}, \]

for \( i = 1, \ldots, n \). Using (5) and (6), we can express \( f_{i,n}(x) \) in terms of the incomplete beta function ratio

\[ f_{i,n}(x) = \frac{n! g_{a,b}(x)}{(i-1)!(n-i)! B(a, b)} G_{a,b}(x)a^{i-1}\{1 - G_{a,b}(x)\}^{b-1} \]

\[ \times I_{G_{a,b}(x)}(a, b)^{i-1} I_{[1-G_{a,b}(x)]}(b, a)^{n-i}. \]

The cdf of the \( i \)th order statistic \( X_{i,n} \), \( F_{i,n}(x) \) say, is

\[ F_{i,n}(x) = \sum_{r=i}^{n} \binom{n}{r} I_{G_{a,b}(x)}(a, b)^{r} I_{[1-G_{a,b}(x)]}(b, a)^{n-r}. \]

Using the identity \( (\sum_{k=0}^{\infty} a_k x^k)^n = \sum_{k=0}^{\infty} c_{k,n} x^k \) (see Gradshteyn and Ryzhik, 2000), where \( c_{0,n} = a^n_0 \) and

\[ c_{k,n} = (ka_0)^{-1} \sum_{l=1}^{k} (nl - k + l) a_l c_{k-l,n} \]

for \( k = 1, 2, \ldots \) and (10), the pdf of the \( i \)th order statistic can be written for \( b \) real non integer

\[ f_{i,n}(x) = \sum_{i=0}^{\infty} \sum_{j=0}^{i+k-1} \frac{(-1)^j \binom{n-i}{k} \Gamma(b) B(a(i+k) + j, b)}{B(a, b)^{i+k} B(i, n - i + 1)} c_{i,j,k} f_{i,j,k}(x), \]  \hspace{1cm} (12)

where \( f_{i,j,k}(x) \) is the density of a BF(\( a(i+k) + j, b, \sigma, \lambda \)) distribution and the constants \( c_{i,j,k} \) are obtained recursively from

\[ c_{i,0,k} = \left\{ \frac{1}{a \Gamma(b)} \right\}^{i+k-1} \text{ and } c_{i,j,k} = a \Gamma(b) \sum_{l=1}^{j} \frac{(-1)^l \Gamma(l + k - j)}{\Gamma(b-l)! (a+l)} c_{i,j-l,k} \text{ for } j \geq 1. \]
For $b$ integer, expansion (12) is valid but the sum in $j$ stops at $(b - 1)(k + i - 1)$. Expansions (10)-(12) are the main results of this section.

An alternative expansion for the density of the order statistics follows from the identity $(\sum_{i=0}^{\infty} a_i)^k = \sum_{[m_1, \ldots, m_k]} a_{m_1} \cdots a_{m_k}$ for $k$ a positive integer. Using this identity and (10), we obtain for $b > 0$ real non integer

$$f_{\ell,n}(x) = \sum_{k=0}^{n-i} \sum_{m_1=0}^{\infty} \cdots \sum_{m_{i+k-1}=0}^{\infty} \delta_{i,k} f_{i,k}(x), \quad (13)$$

where $f_{i,k}(x)$ is the pdf of a $BF(a(i + k) + \sum_{j=1}^{i+k-1} m_j, b, \sigma, \lambda)$ distribution and

$$\delta_{i,k} = (-1)^{k+i+j} \sum_{m_j=0}^{\infty} B(a(i + k) + \sum_{j=1}^{i+k-1} m_j, b) \Gamma(b(i, n - i + 1) \prod_{j=1}^{i+k-1} \Gamma(b - m_j) m_j! (a + m_j).$$

The constants $\delta_{i,k}$ are easily obtained given $i, n, k$ and a sequence of indices $m_1, \ldots, m_{i+k-1}$. The sums in (13) extend over all $(i + k)$-tuples $(k, m_1, \ldots, m_{i+k-1})$ of non negative integers and is implementable on a computer. If $b \geq 0$ is an integer, Eq. (13) holds but the indices $m_1, \ldots, m_{i+k-1}$ vary from zero to $b - 1$. Expansion (12) is much simpler to be calculated and its CPU times are usually smaller than using (13).

3. Moments

As with any other distribution, many of the interesting characteristics and features of the BF distribution can be studied through the moments. We obtain immediately the $r^{th}$ moment $\mu_r'$ of the BF distribution from (11) if $r < \lambda$

$$\mu_r' = \frac{\sigma^r \Gamma(1 - r/\lambda) \Gamma(a + b)}{\Gamma(a)} \sum_{j=0}^{\infty} \frac{(-1)^j (a + j)^{r+j-1}}{\Gamma(b - j) j!}.$$

If $b > 0$ is integer and $r < \lambda$, the sum stops at $b - 1$. If $a = 1$ and $r < \lambda$, (14) gives the $r^{th}$ moment of the EF distribution with parameters $b, \sigma$, and $\lambda$ which is a new result for the EF distribution.

From (12), we obtain simple expansions for the moments of the order statistics. The $r^{th}$ moment of the $X_{\ell,n}$ for $b > 0$ real non integer is

$$E(X'_{\ell,n}) = \sum_{k=0}^{n-i} \sum_{j=0}^{\infty} \frac{(-1)^k (a + j)^{r+j-1}}{\Gamma(b - j) j!} B(a(i + k) + j, b) c_{i,j,k} E(X'_{i,j,k}), \quad (15)$$

where $X_{i,j,k} \sim BF(a(i + k) + j, b, \sigma, \lambda)$ and the constants $c_{i,j,k}$ were defined before. If $b$ is an integer, the sum in $j$ stops at $b - 1$.

From (13), we obtain an alternative expression for the moments of the order statistics valid for $b > 0$ real non integer

$$E(X'_{\ell,n}) = \sum_{k=0}^{n-i} \sum_{m_1=0}^{\infty} \cdots \sum_{m_{i+k-1}=0}^{\infty} \delta_{i,k} E(X'_{i,k}),$$
where $X_{i,k} \sim BF(a(i + k) + \sum_{j=1}^{i+k-1} m_j, b, \sigma, \lambda)$. For $b > 0$ integer, the indices $m_1, \ldots, m_{i+k-1}$ stop at $b-1$.

Graphical representation of skewness and kurtosis when $\lambda = 5$ and $\sigma = 1$, as a function of parameter $a$ for some choices of parameter $b$, and as a function of parameter $b$ for some choices of parameter $a$, are given in Figs. 3 and 4, respectively.

4. **L-Moments**

The $L$-moments are analogous to the ordinary moments but can be estimated by linear combinations of order statistics. They are linear functions of expected order statistics defined by (Hosking, 1990)

$$
\lambda_{r+1} = (r + 1)^{-1} \sum_{k=0}^{r} (-1)^k \binom{r}{k} E(X_{r+1-k|r+1}), \quad r = 0, 1, \ldots
$$

\[ \text{(16)} \]
The first four \(L\)-moments are: \(\lambda_1 = E(X_{11}),\ \lambda_2 = \frac{1}{2}E(X_{22} - X_{11}),\ \lambda_3 = \frac{1}{3}E(X_{33} - 2X_{23} + X_{13})\) and \(\lambda_4 = \frac{1}{4}E(X_{44} - 3X_{34} + 3X_{24} - X_{14})\). The \(L\)-moments have the advantage that they exist whenever the mean of the distribution exists, even though some higher moments may not exist, and are relatively robust to the effects of outliers.

From the expansion (15) for the moments of the order statistics we can obtain expansions for the \(L\)-moments of the BF distribution as weighted linear combinations of the means of suitable BF distributions.

5. Estimation and Information Matrix

We assume that \(Y\) follows the BF distribution and let \(\theta = (a, b, \sigma, \lambda)^T\) be the true parameter vector. The log-likelihood \(\ell = \ell(\theta)\) for a single observation \(y\) of \(Y\) is given by

\[
\ell = \log \lambda + \lambda \log(\sigma/y) - \log[B(a, b)] - a(\sigma/y)^{\lambda} + (b - 1) \log[1 - e^{-\lambda \sigma/y}].
\]

The components of the score vector \(U = U(\theta) = (\partial \ell/\partial a, \partial \ell/\partial b, \partial \ell/\partial \sigma, \partial \ell/\partial \lambda)^T\) for one observation are given by

\[
\frac{\partial \ell}{\partial a} = -\psi(a) + \psi(a + b) - (\sigma/y)^\lambda,
\]

\[
\frac{\partial \ell}{\partial b} = -\psi(b) + \psi(a + b) + \log[1 - e^{-\lambda \sigma/y}],
\]

\[
\frac{\partial \ell}{\partial \sigma} = \frac{\lambda}{\sigma} - \frac{\lambda \sigma^{\lambda-1}}{y^\lambda} \left\{ a - \frac{b - 1}{e^{\lambda \sigma/y} - 1} \right\},
\]

\[
\frac{\partial \ell}{\partial \lambda} = \frac{1}{\lambda} + \log \left( \frac{\sigma}{y} \right) \left[ 1 - \left( \frac{\sigma}{y} \right)^\lambda \right] \left\{ a - \frac{b - 1}{e^{\lambda \sigma/y} - 1} \right\}.
\]

From \(E(\partial \ell/\partial a) = 0\), we obtain

\[
E(X^{-\lambda}) = \frac{\psi(a + b) - \psi(a)}{\sigma^\lambda}.
\]

For interval estimation and hypothesis tests on the model parameters, we require the information matrix. The 4 \(\times\) 4 unit information matrix \(K = K(\theta)\) is

\[
K = \begin{pmatrix}
\kappa_{a,a} & \kappa_{a,b} & \kappa_{a,\sigma} & \kappa_{a,\lambda} \\
\kappa_{b,a} & \kappa_{b,b} & \kappa_{b,\sigma} & \kappa_{b,\lambda} \\
\kappa_{\sigma,a} & \kappa_{\sigma,b} & \kappa_{\sigma,\sigma} & \kappa_{\sigma,\lambda} \\
\kappa_{\lambda,a} & \kappa_{\lambda,b} & \kappa_{\lambda,\sigma} & \kappa_{\lambda,\lambda}
\end{pmatrix},
\]

whose elements are

\[
\kappa_{a,a} = \psi'(a) - \psi'(a + b), \quad \kappa_{b,b} = \psi'(b) - \psi'(a + b),
\]

\[
\kappa_{\sigma,a} = \frac{\lambda}{\sigma^2} [1 + a(\lambda - 1)](\psi(a + b) - \psi(a)) + (b - 1)(\lambda T_{1.1,2.0} - T_{1.1,1.0}),
\]

\[
\kappa_{\lambda,b} = \frac{1}{\lambda} [1 + a(\lambda - 1)](\psi(a + b) - \psi(a)) + (b - 1)(\lambda T_{1.1,2.0} - T_{1.1,1.0}).
\]
Here, we define a random variable $V$ following a Beta$(a, b)$ distribution and the expected value

$$T_{i,j,k,l} = E\left[V^i(1-V)^j(-\log V)^k\log(-\log V)^l\right],$$

where the integral obtained from the above definition is numerically determined using MAPLE and MATHEMATICA for any $a$ and $b$. For example, for $a = 1.5$ and $b = 2.5$ we easily calculated all $T$'s in the information matrix: $T_{1,1,2,0} = 0.51230070$, $T_{1,1,1,0} = 0.55296103$, $T_{0,0,1,2} = 0.62931802$, $T_{1,2,2,2} = 0.43145336$, $T_{1,1,1,2} = 0.32124774$, $T_{0,0,1,1} = 0.48641180$, $T_{1,1,1,1} = -0.16152763$ and $T_{1,2,2,0} = 0.86196008$.

For a random sample $y = (y_1, \ldots, y_n)^T$ of size $n$ from $Y$, the total log-likelihood is

$$\ell_n = \ell_n(\theta) = \sum_{i=1}^n \ell(i),$$

where $\ell(i)$ is the log-likelihood for the $i$th observation ($i = 1, \ldots, n$) as given before. The total score function is $U_n = U_n(\theta) = \sum_{i=1}^n \ell(i)$, where $U(i)$ for $i = 1, \ldots, n$ has the form given earlier and the total information matrix is $K_n(\theta) = nK(\theta)$.

The MLE $\hat{\theta}$ of $\theta$ is numerically determined from the solution of the nonlinear system of equations $U_n = 0$. Under conditions that are fulfilled for the parameter $\theta$ in the interior of the parameter space but not on the boundary, the asymptotic distribution of $\sqrt{n}(\hat{\theta} - \theta)$ is $N_4(0, K(\theta)^{-1})$. The asymptotic multivariate normal $N_4(0, K_\theta(\theta)^{-1})$ distribution of $\hat{\theta}$ can be used to construct approximate confidence regions for some parameters and for the hazard and survival functions. In fact, an $100(1-\gamma)$% asymptotic confidence interval for each parameter $\theta_i$ is given by

$$ACI_i = (\hat{\theta}_i - z_{\gamma/2}\sqrt{K^{\theta_i,\theta_i}}, \hat{\theta}_i + z_{\gamma/2}\sqrt{K^{\theta_i,\theta_i}}),$$

where $K^{\theta_i,\theta_i}$ denotes the $i$th diagonal element of $K_n(\hat{\theta})^{-1}$ for $i = 1, 2, 3, 4$ and $z_{\gamma/2}$ is the quantile $1-\gamma/2$ of the standard normal distribution. The asymptotic normality is also useful for testing goodness of fit of the four parameter BF distribution and for comparing this distribution with some of its special submodels using the likelihood ratio (LR) statistic.

We consider the partition $\theta = (\theta_1^T, \theta_2^T)^T$, where $\theta_1$ is a subset of the parameters of interest of the BF and $\theta_2$ is a subset of the remaining parameters. The LR statistic for testing the null hypothesis $H_0 : \theta_1 = \theta_1^{(0)}$ vs. the alternative hypothesis $H_1 : \theta_1 \neq \theta_1^{(0)}$ is given by $w = 2[\ell(\hat{\theta}) - \ell(\hat{\theta})]$, where $\hat{\theta}$ and $\hat{\theta}$ denote the MLEs under the null
and alternative hypotheses, respectively. The statistic \( w \) is asymptotically (as \( n \to \infty \)) distributed as \( \chi^2_k \), where \( k \) is the dimension of the subset \( \theta_i \) of interest. Then, we can compare for example a BF model against an EF model by testing \( H_0 : a = 1 \) vs. \( H_1 : a \neq 1 \). We can also compare a BF model against the Fréchet model by testing \( H_0 : a = b = 1 \) vs. \( H_1 : H_0 \) is false.

### 6. Applications

One major benefit of beta generalized class distributions is its ability of fitting skewed data that cannot be properly fitted by existing distributions. In this section, we fit the BF distribution to two examples of real data and test two types of hypotheses: \( H_0 : \text{Fréchet} \times H_1 : \text{BF} \) and \( H_0 : \text{EF} \times H_1 : \text{BF} \). The first example is an uncensored data set from Nichols and Padgett (2006) consisting of 100 observations on breaking stress of carbon fibers (in Gba):

\[
3.7, 2.74, 2.73, 2.5, 3.6, 3.11, 3.27, 2.87, 1.47, 3.11, 4.42, 2.41, 3.19, 3.22, 1.69, 3.28, \\
3.09, 1.87, 3.15, 4.9, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.53, 2.67, 2.93, 3.22, 3.39, \\
2.81, 4.2, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35, 2.55, 2.59, 2.38, 2.81, \\
2.77, 2.17, 2.83, 1.92, 1.41, 3.68, 2.97, 1.36, 0.98, 2.76, 4.91, 3.68, 1.84, 1.59, 3.19, \\
1.57, 0.81, 5.56, 1.73, 1.59, 2, 1.22, 1.12, 1.71, 2.17, 1.17, 5.08, 2.48, 1.18, 3.51, 2.17, \\
1.69, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.7, 2.03, 1.8, 1.57, 1.08, 2.03, \\
1.61, 2.12, 1.89, 2.88, 2.82, 2.05, 3.65.
\]

The MLEs and the maximized log-likelihood using the BF distribution are

\[
\hat{a} = 0.4108, \quad \hat{b} = 125.1891, \quad \hat{\lambda} = 0.7496, \quad \hat{\sigma} = 31.4556, \quad \hat{\ell}_{\text{BF}} = -142.9640,
\]

whereas for the EF and Fréchet distributions we obtain

\[
\hat{b} = 52.0491, \quad \hat{\lambda} = 0.6181, \quad \hat{\sigma} = 26.1730, \quad \hat{\ell}_{\text{EF}} = -145.0870,
\]

and

\[
\hat{\lambda} = 1.7690, \quad \hat{\sigma} = 1.8916, \quad \hat{\ell}_{\text{Fréchet}} = -173.1440,
\]

respectively.

The second data set is obtained from Smith and Naylor (1987). The data are the strengths of 1.5 cm glass fibres, measured at the National Physical Laboratory, England. Unfortunately, the measurement units are not given in the article. The data set consisting of 63 observations is:

\[
0.55, 0.93, 1.25, 1.36, 1.49, 1.52, 1.58, 1.61, 1.64, 1.68, 1.73, 1.81, 2.00, 0.74, 1.04, \\
1.27, 1.39, 1.49, 1.53, 1.59, 1.61, 1.66, 1.68, 1.76, 1.82, 2.01, 0.77, 1.11, 1.28, 1.42, \\
1.50, 1.54, 1.60, 1.62, 1.66, 1.69, 1.76, 1.84, 2.24, 0.81, 1.13, 1.29, 1.48, 1.50, 1.55, \\
1.61, 1.62, 1.66, 1.70, 1.77, 1.84, 0.84, 1.24, 1.30, 1.48, 1.51, 1.55, 1.61, 1.63, 1.67, \\
1.70, 1.78, 1.89.
\]
Fitting the BF, EF, and Fréchet distributions we obtain the MLEs and the maximized log-likelihood:

\[ \hat{a} = 0.3962, \quad \hat{b} = 225.7272, \quad \hat{\lambda} = 6.8631, \quad \hat{\alpha} = 1.3021, \quad \hat{\ell}_{\text{BF}} = -90.5180, \]
\[ \hat{b} = 112.5986, \quad \hat{\lambda} = 7.7859, \quad \hat{\alpha} = 0.9814, \quad \hat{\ell}_{\text{EF}} = -93.1962 \]

and

\[ \hat{\lambda} = 1.2643, \quad \hat{\alpha} = 2.8875, \quad \hat{\ell}_{\text{Fréchet}} = -117.7765, \]

respectively.

For the first data set, the values of the LR statistics for testing the hypotheses \( H_0 : \text{Fréchet} \times H_1 : \text{BF} \) and \( H_0 : \text{EF} \times H_1 : \text{BF} \) are: 60.36 (\( p\)-value = 7.81 × 10^{-14}) and 4.246 (\( p\)-value = 3.93 × 10^{-2}), respectively. For the second data set, we obtain the values of the LR statistics 54.5170 (\( p\)-value = 1.45 × 10^{-12}) and 5.3564 (\( p\)-value = 2.06 × 10^{-2}) for the hypotheses \( H_0 : \text{Fréchet} \times H_1 : \text{BF} \) and \( H_0 : \text{EF} \times H_1 : \text{BF} \), respectively. Hence, in both situations, we reject the null hypotheses in favor of the alternative hypothesis that the BF distribution is an adequate model at any usual significance level.

The plots of the estimated BF, EF, and Fréchet densities in Fig. 5 show that the BF distribution gives a better fit than the other two submodels for both data sets.

7. Conclusions

The BF distribution provides a rather general and flexible framework for statistical analysis of positive data. It unifies some previously proposed distributions, therefore yielding a general overview of these distributions for theoretical studies, and it also provides a rather flexible mechanism for fitting a wide spectrum of real world data sets. The BF distribution is motivated by the wide use of the Fréchet distribution in practice, and also for the fact that the generalization provides more flexibility...
to analyze skewed data. In fact, the BF distribution \((6)\) represents a generalization of some distributions previously considered in the literature such as the Fréchet and EF (Nadarajah and Kotz, 2003) distributions. This generalization provides a continuous crossover towards cases with different shapes for skewness and kurtosis.

The BF density can be expressed in the mixture form of Fréchet densities which allow to derive some expansions for the cdf and the ordinary and \(L\)-moments. We call attention for the fact that the moments of the EF are not known in the literature and they follow as a particular case of our results. The pdf of the BF order statistics can also be expressed in terms of a linear combination of Fréchet densities. We also derive the moments of the order statistics. We discuss the maximum likelihood estimation and obtain the information matrix, and consider the LR test which may be very useful in practice. We show that the formulae related with the BF are manageable, and with the use of modern computer resources with analytic and numerical capabilities, may turn into adequate tools comprising the arsenal of applied statisticians. Two numerical examples illustrate that the BF distribution provides better fits than the EF and Fréchet distributions. For data sets that are highly dispersed and/or highly skewed the benefit of the BF distribution over the EF and Fréchet distributions is more evident.

References

Eugene, N., Lee, C., Famoye, F. (2002). Beta-normal distribution and its applications. *Commun. Statist. Theor. Meth.* 31:497–512.

Gradshteyn, I. S., Ryzhik, I. M. (2000). *Table of Integrals, Series, and Products*. San Diego: Academic Press.

Gupta, A. K., Nadarajah, S. (2004). On the moments of the beta normal distribution. *Commun. Statist. Theor. Meth.* 33:1–13.

Hosking, J. R. M. (1990). \(L\)-moments: analysis and estimation of distributions using linear combinations of order statistics. *J. Roy. Statist. Soc. B* 52:105–124.

Jones, M. C. (2004). Families of distributions arising from distributions of order statistics. *Test* 13:1–43.

Kotz, S., Nadarajah, S. (2000). *Extreme Value Distributions: Theory and Applications*. London: Imperial College Press.

Nadarajah, S., Kotz, S. (2003). The exponentiated Fréchet distribution. InterStat. Available online at http://interstat.statjournals.net/YEAR/2003/abstracts/0312001.php

Nadarajah, S., Gupta, A. K. (2004). The beta Fréchet distribution. *Far East J. Theoret. Statist.* 14:15–24.

Nadarajah, S., Kotz, S. (2004). The beta Gumbel distribution. *Math. Probab. Eng.* 10:323–332.

Nadarajah, S., Kotz, S. (2005). The beta exponential distribution. *Reliab. Eng. Syst. Safety* 91:689–697.

Nichols, M. D., Padgett, W. J. (2006). A bootstrap control chart for Weibull percentiles. *Qual. Reliab. Eng. Int.* 22:141–151.

Smith, R. L., Naylor, J. C. (1987). A comparison of maximum likelihood and Bayesian estimators for the three-parameter Weibull distribution. *Appl. Statist.* 36:358–369.