Non-isothermal flow of power-law fluid in a pipe with sudden expansion

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Abstract. In this paper, numerical simulation of steady power-law fluid flow through axisymmetric sudden expansion under non-isothermal conditions is implemented. Mathematical model of the flow accounts for viscous dissipation effects and temperature-dependent consistency coefficient in the Ostwald de Waele power law. The problem formulated is solved using the finite-difference method. Two-dimensional flow zones are found in the vicinity of expansion plane including recirculating region in the corner. Variation in the length of these zones is analysed in respect to power-law index and dimensionless criteria of the problem. The effect of viscous dissipation on the flow structure formation and viscosity and temperature distribution is estimated in a wide range of the main parameters.

1. Introduction
Since the middle of last century, a number of studies have been done on the fluid flow through sudden expansion, which are described in reviews [1,2]. Such a profound interest is justified by prevalence of sudden expanding elements in industrial equipment applied for liquid medium transportation and processing of liquid materials such as polymeric melts, pastes, slurries, petroleum products, etc. The most simplified rheological model for these liquids is Newtonian model. Distinctive aspects of Newtonian fluid flow in the channel with sudden expansion are considered in [3-6]. However, accounting for real complex rheological properties of the mentioned materials requires applying the non-Newtonian models. One among many is the Ostwald de Waele power law which allows obtaining viscosity of shear-thinning, shear-thickening, and Newtonian fluids setting the power-law index to the values of less than unity, more than unity, and unity, respectively. Accounting for non-Newtonian fluid behavior requires significant efforts for effective realization of computational technologies. On the one hand, appearance of additional nonlinear terms in the constitutive equations and boundary conditions and the variation of effective viscosity value in the solution domain both significantly affect the conditions and rate of convergence of computational algorithms. On the other hand, in the problems of non-Newtonian fluid flow, it is often necessary to cope with singularities of the rheological model, for example, during implementation of the shock-capturing calculation of the viscoplastic fluid flow without distinguishing the unyielded regions. In the case of the Ostwald de Waele power-law model, for pseudoplastic medium, the value of apparent viscosity tends to infinity in the flow regions where the values of intensity of strain-rate tensor are little. A detailed discussion on the flow structure, flow kinematics, and pressure losses depending on the Reynolds number and expansion ratio for power-law fluid flow through expansions is presented in [7,8]. Most of these works are devoted to isothermal cases, while, in fact, in industrial processes, the flow of non-Newtonian
fluids is found to be non-isothermal. Accounting for non-isothermality leads to a more complex mathematical model that takes into account mechanical energy dissipation and dependence of the rheological parameters on the temperature. Moreover, a possibility of hydrodynamic thermal explosion should be excluded to provide flow stability [9,10].

The aim of present work is to solve numerically the problem of laminar steady non-isothermal power-law fluid flow in a pipe with sudden expansion in order to obtain distribution of the flow kinematic and dynamic characteristics and evaluate the influence of viscous dissipation and variation in the basic parameters. The results of the research are supposed to be used for analysis of the flow of high-energy polymer compounds in the elements of engineering facilities which are characterized by low Reynolds numbers.

2. Formulation of the Problem

The laminar steady flow of power-law fluid in the pipe with sudden expansion under non-isothermal conditions is considered. Assuming the flow symmetry, a schematic solution domain is shown in figure 1.

![Figure 1. Solution domain.](image)

The mathematical statement of the problem is presented in terms of dimensionless stream function ($\psi$), vorticity ($\omega$), and temperature ($\theta$) [11]:

$$\frac{\partial (\nu \omega)}{\partial r} + \frac{\partial (u \omega)}{\partial z} = \frac{2^n \cdot B}{Re} \left( \frac{\Delta \omega - \omega}{r^2} \right) + \frac{2^n \cdot S}{Re},$$

$$\Delta \psi - \frac{2}{r} \frac{\partial \psi}{\partial r} = -r \omega,$$

$$\frac{\partial (v \theta)}{\partial r} + \frac{\partial (u \theta)}{\partial z} = \frac{2}{Pe} \left( \Delta \theta + 2^{n-1} A^2 B \cdot Br \right) \frac{v \theta}{r}.$$

In these equations, $v, u$ are the radial and axial velocity components, respectively, $\theta = \beta (T - T_i)$ is the dimensionless temperature, $\beta$ is the temperature dependency coefficient, $T$ and $T_i$ are the dimensional temperatures of the fluid in the flow and on the solid wall, respectively, $B$ is the apparent viscosity, and $n$ is the power-law index. The source term ($S$) and intensity of strain rate tensor ($A$) are specified as

$$S = 2 \frac{\partial^2 B}{\partial r \partial z} \left( \frac{\partial v}{\partial r} - \frac{\partial u}{\partial z} \right) + \left( \frac{\partial^2 B}{\partial r^2} - \frac{\partial^2 B}{\partial z^2} \right) \left( \frac{\partial u}{\partial r} + \frac{\partial v}{\partial z} \right) + 2 \frac{\partial B}{\partial r} \cdot \frac{\partial \omega}{\partial r} + 2 \frac{\partial B}{\partial r} \cdot \frac{\partial \omega}{\partial r} + \frac{\partial B}{\partial r} \cdot \omega,$$

$$A = \left[ 2 \left( \frac{\partial u}{\partial z} \right)^2 + 2 \left( \frac{\partial v}{\partial r} \right)^2 + 2 \left( \frac{v}{r} \right)^2 + \left( \frac{\partial u}{\partial r} + \frac{\partial v}{\partial z} \right)^2 \right]^{3/2}.$$
Governing equations include such dimensionless criteria as Reynolds, Peclet, and Brinkman numbers, which are determined as

\[
Re = \frac{\rho U^2 \alpha D^{\alpha}}{k_1}, \quad Pe = \frac{c \rho U D}{\lambda}, \quad Br = \frac{k_1 D^2 \beta (U / D)^{n+1}}{\lambda}.
\]

Here, \(k_1 = k_0 \exp[-\beta(T - T_0)]\) is the consistency coefficient at \(T_1\), \(k_0\) is the consistency coefficient at \(T_0\), \(D = 2R_i\) is the upstream pipe diameter, \(\rho\) is the fluid density, \(c\) is the heat capacity, and \(\lambda\) is the thermal conductivity. The properties of the medium (\(\rho, c, \lambda\)) are considered to be constant.

According to the definition, the stream function and vorticity are as follows:

\[
v = -\frac{1}{r} \frac{\partial \psi}{\partial z}, \quad u = \frac{1}{r} \frac{\partial \psi}{\partial r}, \quad \omega = \frac{\partial v}{\partial z} - \frac{\partial u}{\partial r}.
\]

The rheological behavior of the fluid is defined by the Ostwald de Waele power law with temperature-dependent apparent viscosity given by \([11]\)

\[
B = \exp[-\theta] A^{n+1}.
\]

At the inlet section (\(\Gamma_1\)), the velocity and temperature profiles are calculated corresponding to a fully developed one-dimensional non-isothermal fluid flow with specified constant flow rate in the infinite pipe \([12]\). Based on the obtained velocity field, the stream function and vorticity are calculated by equations \((4,5)\) and assigned as inlet boundary conditions. On the rigid walls (\(\Gamma_2\)), the no-slip boundary conditions are applied, and the dimensionless temperature of the fluid is equal to zero. At the output section (\(\Gamma_3\)), the derivatives of stream function, vorticity, and temperature with respect to \(z\) are set to zero. The inlet and outlet sections are supposed to be far away from expansion to exclude the effect of the latter on the flow behavior in the vicinity of pipe inlet and outlet. Along the symmetry axis (\(\Gamma_4\)), the symmetry condition is used.

Thus, the boundary conditions are written as follows:

\[
\Gamma_1: u = f_i(r), \quad v = 0, \quad \psi = \int_0^r u dr, \quad \omega = -\frac{\partial u}{\partial r}, \quad \theta = f_2(r), \quad 0 \leq r \leq 1, \quad z = 0;
\]

\[
\Gamma_2: \psi = \text{const}, \quad \omega = -\frac{\partial^2 \psi}{\partial r^2}, \quad \theta = 0, \quad r = 1, \quad 0 \leq z \leq \frac{L_1}{R_i},
\]

\[
\psi = \text{const}, \quad \omega = \frac{1}{R_i^2} - \frac{\partial^2 \psi}{\partial r^2}, \quad \theta = 0, \quad 1 \leq r \leq \frac{R_2}{R_i}, \quad z = \frac{L_1}{R_i};
\]

\[
\psi = \text{const}, \quad \omega = \frac{1}{R_i^2} - \frac{\partial^2 \psi}{\partial r^2}, \quad \theta = 0, \quad r = \frac{R_2}{R_i}, \quad \frac{L_1}{R_i} \leq z \leq \frac{L_1}{R_i} + \frac{L_2}{R_i};
\]

\[
\Gamma_3: \frac{\partial \psi}{\partial z} = 0, \quad \frac{\partial \omega}{\partial z} = 0, \quad \frac{\partial \theta}{\partial z} = 0, \quad 0 \leq r \leq \frac{R_2}{R_i}, \quad z = \frac{L_1}{R_i};
\]

\[
\Gamma_4: \psi = 0, \quad \omega = 0, \quad \frac{\partial \theta}{\partial r} = 0, \quad r = 0, \quad 0 \leq z \leq \frac{L_1}{R_i} + \frac{L_2}{R_i}.
\]
3. Method of Solution

A numerical simulation of the laminar steady power-law fluid flow through axisymmetric sudden expansion was carried out. The asymptotic time solution to unsteady flow equations was obtained in order to yield a steady-state solution to the initial problem [13]. The finite-difference method based on the alternative directions scheme was used to discretize the governing equations [14]. A resulting system of nonlinear equations was solved using the sweep method [13].

For verification of numerical algorithm, the approximating convergence is tested on the set of square grids. The maximum axial velocity and maximum temperature obtained both in the expansion plane and at the outlet section at various grid steps \( h \) are shown in table 1. According to this data, the approximating convergence of the algorithm is observed. All the following calculations are performed using difference grid with \( h=0.025 \).

| \( h \) | \( u_{\text{max}}(\text{expansion}) \) | \( \theta_{\text{max}}(\text{expansion}) \) | \( u_{\text{max}}(\text{outlet}) \) | \( \theta_{\text{max}}(\text{outlet}) \) |
|------|-------------------------------|-------------------|-----------------|-------------------|
| 0.1  | 1.87749                       | 0.54034           | 0.47770         | 0.06046           |
| 0.05 | 1.90238                       | 0.56226           | 0.47786         | 0.06090           |
| 0.025| 1.91194                       | 0.56945           | 0.47784         | 0.06100           |
| 0.0125| 1.91620                      | 0.57208           | 0.47783         | 0.06102           |

4. Results and Discussion

In the stated problem of non-isothermal power-law fluid flow in a circular pipe with sudden expansion, the expansion ratio is uniform \( (R_2/R_1=2) \). The lengths of upstream and downstream pipes are taken as \( L_1/R_1=8 \) and \( L_2/R_1=40 \), respectively. The values of Reynolds and Brinkman numbers are fixed at 1.

The structure of the flow through sudden pipe expansion is found to consist of one-dimensional and two-dimensional flow regions under both isothermal and non-isothermal conditions. One-dimensional flow zones are characterized by fully developed flow observed in the upstream and downstream pipes except for the region in the vicinity of expansion plane. According to the flow pattern depicted in figure 2, in these zones the streamlines are parallel to the pipe walls. At the distance of \( l_1 \) upstream of expansion plane, the radial velocity component appears, and two-dimensional flow occurs. Right after the sudden expansion, a recirculating flow region with the length of \( L \) is observed in the corner. Two-dimensional flow zone extends downstream of expansion plane over a distance of \( l_2 \).

![Figure 2. Distribution of the streamlines along the pipe with sudden expansion (Pe=100, n=0.8).](image-url)

The parametric study results presented in table 2 expose the influence of power-law index, thermal conditions, and Peclet number on the lengths of distinctive regions in the vicinity of expansion plane.
Table 2. The lengths of distinctive regions at various power-law indexes for isothermal and non-isothermal cases.

| n    | 0.8  | 1.0  | 1.2  | 0.8  | 1.0  | 1.2  | 0.8  | 1.0  | 1.2  |
|------|------|------|------|------|------|------|------|------|------|
| \(l_1\) | 1.3000 | 1.0000 | 0.8500 | 1.4750 | 1.2000 | 0.9250 | 1.4250 | 1.1000 | 0.8750 |
| \(l_2\) | 4.0250 | 3.4000 | 2.9000 | 6.8000 | 7.1500 | 7.5500 | 6.7500 | 8.9250 | 11.275 |
| \(L\)  | 0.4604 | 0.5663 | 0.6306 | 0.3953 | 0.4395 | 0.4942 | 0.3828 | 0.4260 | 0.4834 |

Focusing on the similarities and differences between the values calculated for isothermal and non-isothermal flows, it is revealed that, in both cases, the length of upstream two-dimensional flow zone \((l_1)\) decreases and the length of recirculation region \((L)\) increases with an increase in the power-law index. However, variation in the length of downstream two-dimensional flow region \((l_2)\) with the power-law index for isothermal flow is different from those for non-isothermal flow. In the first case, the length \(l_2\) becomes less as the \(n\) is increased, while, in the second case, it increases with \(n\), and this growth is more significant at high \(Pe\).

Figure 3. Distribution of the apparent viscosity \((a,b)\) and temperature \((c,d)\) in the vicinity of expansion plane at \(n=0.8\): \((a,c)\) \(Pe=10\) and \((b,d)\) \(Pe=100\).
Analyzing the effect of Peclet number and power-law index on the apparent viscosity and temperature distribution along the pipe, the main attention is concentrated on the expansion plane and its vicinity due to the fact that the flow undergoes considerable changes in this region. In figures 3, 4, the contour plots for shear-thinning and shear-thickening fluids are demonstrated at $Pe=10$ and $Pe=100$. It is notable that increased heat transfer due to convection (as compared to conduction), i.e. increased Peclet number, leads to a significant change in the temperature field, which is observed as displacement of the heated region towards outlet section. Therefore, the higher $Pe$, the longer the distance from expansion plane to the pipe section, where the fully developed flow state is reached.

5. Conclusion
The non-isothermal steady power-law fluid flow through a circular pipe expansion was numerically simulated using the finite-difference method. The flow structure including two-dimensional flow zones in the vicinity of expansion plane was analyzed in respect to varying power-law index and Peclet number. It was found that increased power-law index leads to a decrease in the upstream two-dimensional flow zone and an increase in the downstream two-dimensional flow zone and recirculation region in the corner. The effect of accounted viscous dissipation on the flow kinematics was estimated comparing non-isothermal power-law fluid flow with isothermal one. In these two cases, the behavior of downstream two-dimensional flow zones in respect to power-law index was revealed to be directly opposite. Based on the calculated data, the influence of the flow thermal conditions and fluid rheology on the apparent viscosity and temperature distribution in the vicinity of expansion plane was demonstrated. It was concluded that an increase in the Peclet number provided displacement of the heated region towards the pipe outlet and, thus, the longer downstream pipe was needed in order to reach a fully developed flow.

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