Equivariant localization, parity sheaves, and cyclic base change

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Global Langlands correspondence

Notation:

- $F = \text{global function field, e.g. } \mathbb{F}_\ell(t)$
- $G = \text{reductive group over } F, \text{ e.g. } \text{SL}_n$
- $k = \overline{\mathbb{F}}_p \text{ (coefficients), } p \neq \text{char}(F)$

Vincent Lafforgue constructed

\[
\left\{ \begin{array}{c}
\text{irreducible cuspidal} \\
\text{automorphic representations} \\
\text{of } G \text{ over } k
\end{array} \right\} \rightarrow \left\{ \begin{array}{c}
\text{Langlands parameters} \\
\text{Gal}(F^s/F) \rightarrow \mathcal{L}^* G(k)/\sim
\end{array} \right\}.
\]

Does it have expected properties?
Langlands functoriality:

\[ f_H \xrightarrow{\sim} e_f \]

\[ f_c \xrightarrow{\sim} \phi \circ pf \]

\[ L \Gamma \]
Global base change functoriality: Suppose

- $H$ reductive over $F$,
- $E/F$ field extension, $G := \text{Res}_{E/F}(H_E)$.

Let $\phi : L^1 H \to L^1 G$

$\hat{H} \xrightarrow{\Delta} \hat{E} = \hat{F}^{E:F}$

+ conditional on $\chi_{\nu E}$



Thm $[E:F] = p$, cyclic. $p$ odd, good for $G$

Then $\rho_H$ automorphic $\Rightarrow \phi \circ \rho_H$ autom.
Previous proofs of base change (for GL\(_n\)) are based on the trace formula.

\((\text{e.g. } \text{SL}_n \ \forall n \geq 2)\)

**Novelty for general \(G\):** can have

\(f, f'\) generating *isomorphic* automorphic representations \(\pi_{\xi}, \pi_{\xi'}\)

→ different \(L\)-parameters.

Indistinguishable by the trace formula!
Local Langlands correspondence

Notation:

- $F_v = \text{local function field of } \text{char } \neq p$, e.g. $\mathbb{F}_\ell((t))$.
- $H = \text{reductive group over } F_v$.

Genestier-Lafforgue constructed:

$$\left\{ \begin{array}{c} \text{irreducible representations} \\ \text{of } H(F_v) \text{ over } k \end{array} \right\} \sim \rightarrow \left\{ \begin{array}{c} \text{semi-simple} \\ \text{Langlands parameters} \\ \text{Weil}(F_v) \rightarrow L^1 H(k)/\sim \end{array} \right\}.$$

Does it have expected properties?
We will investigate local base change:

- $E_v/F_v$ extension, $\text{Gal}(E_v/F_v) \approx \mathbb{Z}/p = \langle \sigma \rangle$.
- $G = \text{Res}_{E_v/F_v}(H_{E_v})$

$$\sigma \begin{cases} \text{irreducible representations} \quad \overset{\text{BC}}{\longrightarrow} \quad \text{irreducible representations} \\ \text{of } H(F_v) \text{ over } k \quad \text{of } G(F_v) \text{ over } k \end{cases}$$

$$\sigma - \text{fixed if } \prod \cong \prod^\sigma$$

$$\mathcal{B} C^{-1}(\prod) \asymp ?$$
Tate cohomology

\[ G \xrightarrow{G} \prod \mathcal{O} (\mathfrak{m}/p \mathfrak{m}) = \langle \sigma \rangle \]

\[ 0 = \sigma^{p-1} = (\sigma-1) \left( 1 + \sigma + \ldots + \sigma^{p-1} \right) \in \mathbb{Z}[\sigma] \]

\[ N. \]

\[ \pi \xrightarrow{N} \pi \xrightarrow{1-\sigma} \pi \xrightarrow{N} \pi \xrightarrow{1-\sigma} \ldots \]

\[ \prod \]

\[ \text{ker}(N) \]

\[ \text{Im}(\sigma-1) \]
Conjecture (Treumann-Venkatesh)

Let $\Pi$ be an irreducible $\sigma$-fixed representation of $G(F_v)$. Then any irreducible subquotient of $T^i(\Pi)$ transfers under LLC to

$$\Pi(p) := \Pi \otimes_{k, \text{Frob}_p} k.$$
\[ X = \frac{\prod \left(1 + \sigma_{r-1}\right)}{\prod} \]

\[ G^{-1/\rho}H = G^{2/\rho} \]

\[ \rightarrow \text{analogous results} \]

\[ B_{m_{\alpha}} = B_{m_{H}} \]

\[ \text{act and } \alpha \text{ in } \text{Act}_{up} \]
Conjecture (Treumann-Venkatesh)

Let $\Pi$ be an irreducible $\sigma$-fixed representation of $G(F_v)$. Then any irreducible subquotient of $T^i(\Pi)$ transfers under LLC to

$$\Pi(p) := \Pi \otimes_{k, \text{Frob}_p} k.$$ 

Theorem (F.)

Assume $p$ is odd and good for $\hat{G}$. Then any irreducible subquotient of $T^i(\Pi)$ transfers under the Genestier-Lafforgue correspondence to $\Pi(p)$.

$$T^i(\Pi) \subset \text{BC}^{-}\ell(\Pi(p))$$

- Previously proved by Ronchetti for depth zero supercuspidals of $GL_n$ induced from cuspidal Deligne-Lusztig representations.
Plan

1. Statement of the results. ✓
2. Summary of Lafforgue’s idea.
3. Equivariant localization.
4. Modular representation theory.
Let $\Gamma$ be a group, $\hat{G}$ a reductive group over $k$.

\[
\text{comm } \text{Exc}(\Gamma, \hat{G}) \sim \text{ "functions on " } \begin{cases} \Gamma \rightarrow \hat{G} / \sim \end{cases}
\]

\[
\text{key property } \chi_f \leftrightarrow \text{ Pf } \begin{cases} \text{ chars Exc}(\Gamma, \hat{G}) \rightarrow k \\
\text{ semisimple } \begin{cases} \Gamma \rightarrow \hat{G} / \sim \end{cases} \end{cases}
\]

Want construct $\text{Exc}(\Gamma, \hat{G}) \cap \{ \text{ aut } \text{ functions } \}$

\[
\chi_f \rightarrow k
\]
Can present $\text{Exc}(\Gamma, \hat{G})$ explicitly by generators and relations.

Generators: $S_n, f, (\gamma_i)_{i=1, \ldots, n}$

- $n \geq 0$
- $f \in \mathcal{A} \left( \frac{\mathbb{Z}}{\mathbb{Z}/\mathcal{A}_n} \right)$
- $\gamma_i \in \hat{\Gamma}$

"Value" at $\rho : \Sigma \to \hat{\mathcal{A}}_n$ is

$$f \left( \rho \left( \gamma_i, \gamma_{i+1}, \ldots, \gamma_{i+p}, \gamma_{i+p+1} \right) \right)$$
4.2.2. Relations. Next we describe the relations. (Compare [Laf18a] §10.)

(i) \( S_{\emptyset,f,*} = f(1_G) \).

(ii) The map \( f \mapsto S_{I,f,(\gamma_i)_{i \in I}} \) is a \( k \)-algebra homomorphism in \( f \), i.e.

\[
S_{I,f+f',(\gamma_i)_{i \in I}} = S_{I,f,(\gamma_i)_{i \in I}} + S_{I,f',(\gamma_i)_{i \in I}},
\]

\[
S_{I,ff',(\gamma_i)_{i \in I}} = S_{I,f,(\gamma_i)_{i \in I}} \cdot S_{I,f',(\gamma_i)_{i \in I}},
\]

and

\[
S_{I,\lambda f,(\gamma_i)_{i \in I}} = \lambda S_{I,f,(\gamma_i)_{i \in I}} \quad \text{for all } \lambda \in k.
\]

(iii) For all maps of finite sets \( \zeta : I \to J \), all \( f \in \mathcal{O}(\hat{G}_k \backslash (L G^\text{alg}_k)^I / \hat{G}_k) \), and all \( (\gamma_j)_{j \in J} \in \Gamma^J \), we have

\[
S_{J,f\zeta,(\gamma_j)_{j \in J}} = S_{I,f,(\gamma_{\zeta(i)})_{i \in I}}
\]

where \( f\zeta \in \mathcal{O}(\hat{G}_k \backslash (L G^\text{alg}_k)^J / \hat{G}_k) \) is defined by \( f\zeta((g_j)_{j \in J}) := f((g_{\zeta(i)})_{i \in I}) \).

(iv) For all \( f \in \mathcal{O}(\hat{G}_k \backslash (L G^\text{alg}_k)^I / \hat{G}_k) \) and \( (\gamma_i)_{i \in I}, (\gamma'_i)_{i \in I}, (\gamma''_i)_{i \in I} \in \Gamma^I \), we have

\[
S_{I \sqcup I \sqcup I, \tilde{f}((\gamma_i)_{i \in I} \times (\gamma'_i)_{i \in I} \times (\gamma''_i)_{i \in I})} = S_{I,f,(\gamma_i (\gamma'_i)^{-1} \gamma''_i)_{i \in I}},
\]

where \( \tilde{f} \in \mathcal{O}(\hat{G}_k \backslash (L G^\text{alg}_k)^{I \sqcup I \sqcup I} / \hat{G}_k) \) is defined by

\[
\tilde{f}((g_i)_{i \in I} \times (g'_i)_{i \in I} \times (g''_i)_{i \in I}) = f((g_i(g'_i)^{-1}g''_i)_{i \in I}).
\]
Actions of the excursion algebra

How to construct $\text{Exc}(\Gamma, \hat{G}) \sim V$

Tannakian construction: Given

Family of functors

Output action of $\text{Exc}(\Gamma, \hat{G})$
on $V = H^{\text{El}}(\mathbb{1})$
Summary of Lafforgue’s correspondence

Where does this structure come from?

\[ F \leftrightarrow X \text{ smooth projective curve.} \]

\[ \Psi : \text{Rep}(\hat{\mathfrak{m}}^I) \to \text{Set} \]

\[ \pi_I^* : \text{Rep}(\hat{\mathfrak{m}}^I) \to \pi_c(X)^I \]

\[ \pi_I^*(\text{Sh}^I \text{ on } \mathfrak{m}^I) \]

\[ \pi_c(X)^I \]

\[ \mathfrak{m}^I = \pi_c(X) \]

\[ \pi_I \]

\[ \pi_c(X)^I \]

\[ X^I \]
Summary of Lafforgue’s correspondence

Where does this structure come from?

Geometric Satake equivalence:

\[ \text{Rep}_k(\hat{G}) \cong P_G(O_v) \left( \frac{G(F_v)/G(O_v)}{\text{affine Grassmannian } \text{Gr}_G} \right). \]

Viewed as sheaves on moduli spaces of shtukas:
\[ E_{\infty}(\pi, \omega) \subset H^c_0(\text{Surf}_\phi, \kappa) = \text{Fun}^c(\text{BM}_\alpha(F^e)) \]

\[ \text{BM}_\alpha(F^e) \sim C(CP) \backslash C(AP) / \mathbb{Q} \]
Summary

- **Source of Galois representations**: cohomology of moduli spaces of shtukas.

- **Excursion operators**: endomorphisms of automorphic forms coming from “combinatorial” pattern of maps between cohomology groups.

- **Langlands parametrization** comes from having sheaves indexed by $\text{Rep}(\hat{G})$. 

(Rep. Theory)
Suppose we want Langlands functoriality between $H$ and $G$:

- (Topology) Need mechanism to relate cohomology of shtukas for $G$ and for $H \leftarrow$ **equivariant localization**.

- (Representation Theory) Need mechanism to relate sheaves indexed by $\text{Rep}(\hat{G})$ and by $\text{Rep}(\hat{H}) \leftarrow$ **sheaf-theoretic Smith theory**.
Equivariant localization \implies \text{relationship between the cohomology of a space and its fixed point subspace under a group action.}

\[
\mathcal{E}_X(S) \Rightarrow 2 \{ \sigma \} \\
T^\sigma \left( \text{Fun}^c(S, k) \right) = \frac{\text{Fun}^c(S, k)^\sigma}{N \cdot \text{Fun}^c(S, k)} \\
\text{Fun}^c(S^T, k)^N
\]
Base change situation: $G = \text{Res}_{E/F}(H_E)$.

$$s = \text{BM}_a(F_e), \quad s^0 = \text{BM}_{H}(F_e)$$

$$H^k(C_a) \rightarrow H^k(H)$$

$$\cong$$

$$\cong$$

$$\cong$$

$$\cong$$

Prove: $\text{Exc}(\Gamma, \mathfrak{w})'$ big enough.
For **global base change**, we need to transfer eigensystems for the excursion operators.

- Need to study possible extensions of a character of $\text{Exc}(\Gamma, \hat{G})'$ to all of $\text{Exc}(\Gamma, \hat{G})$.

*Lemma*: $\exists!$ extension.
For local base change, we need to examine the construction of the Genestier-Lafforgue correspondence.

- Study $S_{n,f,\gamma_i}$ for $\{\gamma_i\} \subset \text{Weil}(\overline{F}_v/F_v) \subset \text{Gal}(\overline{F}/F)$.

Details omitted here.
Equivariant localization and excursion operators

Excursion operators:

\[
\begin{align*}
G & \quad T^0 H^0_c(\text{Sht}_G; \text{Sat}(\mathbb{1})) \quad \longrightarrow \quad T^0(\text{Sht}_G; \text{Sat}(W)) \quad \longrightarrow \quad \ldots \\
H & \quad T^0 H^0_c(\text{Sht}_H; \text{Sat}(\mathbb{1})) \quad \longrightarrow \quad T^0(\text{Sht}_H; \text{Sat}(\text{Res}(W))) \quad \longrightarrow \quad \ldots \\
\end{align*}
\]

Want to identify all steps of the excursion. \(\text{Lemma: } \text{Sht}_G \rightarrow \text{Sht}_H\)

- **Topological** aspect: relate (Tate) cohomology of a space with (Tate) cohomology of its fixed points.

- **Representation-theoretic** aspect: geometric interpretation of restriction functor \(\text{Rep}_k(\mathcal{L} G) \xrightarrow{\text{Res}} \text{Rep}_k(\mathcal{L} H)\).
Tate cohomology

Suppose \( \langle \sigma \rangle \cong \mathbb{Z}/p \cong X, \mathcal{F} \in D^b_\sigma(X; k) \).

\[
\sigma \in C^\cdot(X; \mathcal{F})
\]

\[
T^i(X; \mathcal{F}) := H^i(\text{Tot}(... \xrightarrow{\mathcal{N}} C^\cdot(X; \mathcal{F}) \xrightarrow{1-\sigma} C^\cdot(X; \mathcal{F}) \xrightarrow{\mathcal{N}} C^\cdot(X; \mathcal{F}) \xrightarrow{1-\sigma} ...)))
\]
Equivariant localization (Smith, Quillen, Treumann)

\[ T^i(X; F) \cong T^i(X^\sigma; F|_{X^\sigma}). \]

Can apply this to shtukas because (Lemma): \( Sht^G = Sht_H. \)

\[ X = X^\sigma \cup (X - X^\sigma) \]

\[ \sim \quad \sim \]

\[ C^*(x - x^\sigma) \]

\[ \sim \quad \sim \]

\[ \{ T^i(\cdot) \}

\[ 0 \]

Tate cohomology kills perfect complexes.
We have explained the topological input into functoriality.

Now zoom in on the representation-theoretic input:

\[
\begin{array}{ccc}
\text{Hecke category for } G & \longrightarrow & \text{Hecke category for } H \\
\| & & \| \\
\text{Rep}_k(L G) & \longrightarrow & \text{Rep}_k(L H)
\end{array}
\]
Smith theory

Notation:

- $H = \text{reductive group over } F_v$.
- $E_v/F_v$ extension, $\text{Gal}(E_v/F_v) \cong \mathbb{Z}/p = \langle \sigma \rangle$.
- $G = \text{Res}_{E_v/F_v}(H_{E_v})$.
- (Coefficients) $k = \overline{F}_p$.

Treumann-Venkatesh construct **Brauer homomorphism**

$k$-Hecke algebra for $G \overset{\text{br}}{\longrightarrow} k$-Hecke algebra for $H$

$\text{Sat iso} \overset{\text{Res}}{\longrightarrow} \text{K}_0 \text{Rep}_k(LG) \overset{\text{Res}}{\longrightarrow} \text{K}_0 \text{Rep}_k(LH)$
With Gus Lonergan we construct a categorification $k$-Hecke category for $G$ $\xrightarrow{\text{BC}}$ $k$-Hecke category for $H$

\[
\begin{array}{c}
\text{Geom. Sat} \\
\text{Rep}_k(^L G) \\
\end{array}
\xrightarrow{\text{Res}}
\begin{array}{c}
\text{Res} \\
\text{Rep}_k(^L H) \\
\end{array}
\]

using recent tools in geometric representation theory:

- parity sheaves (Juteau-Mautner-Williamson),
- Smith-Treumann theory (Treumann, Leslie-Lonergan, Riche-Williamson).
Brauer homomorphism

Assume: $\left[ \frac{G(F_v)}{G(O_v)} \right]^\sigma = \left[ \frac{H(F_v)}{H(O_v)} \right] = \left[ H \right]$

$\Lambda$ $\sigma$ $k$-Hecke algebra for $G$ $\xrightarrow{br}$ $k$-Hecke algebra for $H$

$\text{Fun}^c_{G(F_v)} \left( \frac{G(F_v)}{G(O_v)} \times \frac{G(F_v)}{G(O_v)}, k \right)$ $\xrightarrow{\phi}$ $\text{Fun}^c_{H(F_v)} \left( \frac{H(F_v)}{H(O_v)} \times \frac{H(F_v)}{H(O_v)}, k \right)$

$\sum_{y_0 \in \mathcal{A}_\mathcal{A}} \phi \left( x, y \right)$ $\phi^{-1} \left( y_1, 2 \right)$
\[ \sum_{y \in \text{case object}} \phi(x, y) \quad \phi^*(y, z) = 0 \]

\( \Rightarrow \) acts freely

\[ \phi(x, \sigma y) \quad \phi^*(\sigma y, z) \]

\[ [\ast \quad \text{Fr}^{-1} \quad \text{linearize}] \]
Base change functor

\[ k\text{-Hecke category for } G \xrightarrow{BC} k\text{-Hecke category for } H \]

\[ P_G(O_v)(Gr_G) \longrightarrow P_H(O_v)(Gr_H) \]

\[ D_{\sigma}(X_{\sigma,k}) \cong \mathbb{D}(X^\sigma, k[\sigma]) \subset \text{Perf}(k[\sigma]) \]
Define \( \{ x_0 \} \) as the category of \( x_0 = \frac{D_\sigma(x_0, x)}{\text{Perf}(k \Sigma_{\sigma})} \).

\[
\begin{aligned}
D(x) \xrightarrow{\sigma} D_\sigma(x_0) \\
\text{res} \quad \Downarrow \quad \text{res}
\end{aligned}
\]

\[
\Rightarrow D_\sigma(x_0) \quad \Downarrow \quad D_\sigma(x_0) / \text{perf}
\]

Example: \( x_0 = \text{perf} \).

\[
k \to k \{ x_0 \} \to k \{ \Sigma_{\sigma} \} \to k
\]

\[
\Rightarrow k = k \{ \Sigma_{\sigma} \}, \quad \text{perf} = \text{perf} \{ \Sigma_{\sigma} \}
\]
Perverse sheaves

- conditions on $*$ and $!$ stalks.

- cut out by $\leq, \geq$ on coh. degrees

Parity sheaves

- congruences mod 2
Parity sheaves

Given a suitable stratification:

- (Juteau-Mautner-Williamson) **Parity sheaves** are $\mathcal{K} \in D^b$ whose $\ast$-stalks and $!$-stalks have cohomology concentrated either in even degrees or odd degrees.

- (Leslie-Lonergan) **Tate-Parity sheaves** are $\mathcal{K} \in D^b/\text{Perf}$ whose $\ast$-stalks and $!$-stalks have Tate cohomology concentrated in either even degrees or odd degrees.

**Feature:** (Tate-)Parity sheaves enjoy strong rigidity properties.
Parity^0_{G(O)}(Gr_G) \xrightarrow{\text{Res}} \text{Parity}^0_{G(O) \times \sigma}(Gr_G) \xrightarrow{\sigma} D_{H(O) \times \sigma}(Gr_H)

\mathcal{F} \xrightarrow{\sigma} \mathcal{F} \ast \sigma \mathcal{F} \ast \ldots \ast \sigma^{p-1} \mathcal{F}

\text{Thm} \quad \text{restrict}

\text{Late-Parity}^0 \quad \subset \quad D_{H(O) \times \sigma}(Gr_H)/\text{Perf}

\text{Thm} \quad \text{left}

\text{Parity}^0(Gr_H) \quad \subset \quad P_{H(O)}(Gr_H)
This is interesting even for $H = \text{GL}(1)$, for which $\text{Gr}_H$ is (étale homotopic to) a discrete union of points.
P \text{ good}

\text{MacHaven Brief}

\text{Positively } (a_{\xi}) < P_{\alpha(\omega)} (a_{\nu})

\text{Tilt} (a^{\sim}) \subset \text{Rep} (\hat{a})

\text{Fact: } \exists \text{ enough Tilting modules}
Pims k che 0 Party $\Rightarrow$ Perverse
