Dynamic Modeling Of Spherical Variable-Shape Wave Energy Converters*

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ABSTRACT

In the recently introduced Variable-Shape heaving wave energy converters, the buoy changes its shape in response to changing incident waves actively. In this study, the dynamic model for a spherical Variable-Shape Wave Energy Converter is developed using the Lagrangian approach. The classical bending theory is used to write the stress-strain equations for the flexible body using Love’s first approximation. The elastic spherical shell is assumed to have an axisymmetric vibration behavior. The Rayleigh-Ritz discretization method is adopted to find approximate solution for the vibration model of the spherical shell. One-way Fluid-Structure Interaction simulations are performed using MATLAB to validate the developed dynamic model and to study the effect of using a flexible buoy in the wave energy converter on its trajectory and power production.

Nomenclature

\[ \begin{align*}
    c &= \text{Damping Coefficient (Ns/m)} \\
    E &= \text{Young Modulus of Elasticity (MPa)} \\
    h &= \text{Sphere thickness (m)} \\
    V &= \text{Volume (m}^3) \\
    t &= \text{Time (sec)} \\
    r &= \text{Sphere Radius (m)} \\
    \rho &= \text{Material Destiny (kg/m}^3) \\
    \nu &= \text{Poisson’s Ratio} \\
    \eta &= \text{Rayleigh-Ritz Coefficient} \\
    \omega &= \text{Frequency (rad/sec)} \\
    \mathbf{I} &= \text{Identity matrix} = [1, 1, 1] \\
    \text{hydro} &= \text{hydrodynamic} \\
    b &= \text{buoyancy} \\
    pto &= \text{Power Take-off Unit} \\
    w &= \text{Water} \\
    {}^\dagger &= \text{Pseudo Inverse}
\end{align*} \]

Subscripts and Superscripts

hydro = hydrodynamic
b = buoyancy
pto = Power Take-off Unit
w = Water

1. Introduction

Oceans are a colossal reservoir of energy of particularly high density [1]. The total theoretical ocean energy potential is estimated to be 29.5 PWh/yr [2], which is more than the US electric power needs in 2020. Despite its significant potential, ocean energy is still a very small portion of the overall renewable energy production [3].

One of the concepts that is widely used for harvesting wave power is the heaving Wave Energy Converter (WEC). This point absorber device, in its simplest form, may consist of a floating buoy connected to a vertical hydraulic cylinder
(spar) attached at the bottom to the seabed. As the buoy moves due to the wave and control forces, the hydraulic cylinders drive hydraulic motors, and the motors drive a generator [4]. The forces on the float are the excitation, radiation, and hydrostatic forces [5]. The excitation force is due to the wave field and the buoy’s geometry. The motion of the buoy itself creates waves which in turn create the radiation forces. The hydrostatic force accounts for the buoyancy force and weight of the buoy. In most of the current wave harvesting devices, the WEC has a Fixed-Shape Buoy (FSB). The equation of motion for a heave-only 1-DoF FSB WEC is [6]:

\[
m̈ \bar{z}(t) = \int_{-\infty}^{\infty} h_f(\tau) \eta(t - \tau, z) d\tau + f_s - \mu \ddot{z}(t) - \int_{-\infty}^{t} h_r(\tau) \ddot{z}(t - \tau) d\tau - u
\]

where \( m \) is the buoy mass, \( u \) is the control force, \( t \) is the time, \( z \) is the heave displacement of the buoy from the sea surface, and \( f_s \) is the hydrostatic force that reflects the spring-like effect of the fluid. The \( \eta \) is the wave surface elevation at buoy centroid, \( f_e \) is the excitation force, and \( h_f \) is the impulse response function defining the excitation force in heave. The radiation force is \( f_r \), where \( \mu \) is a frequency dependant added mass, and \( h_r \) is the impulse response function defining the radiation force in heave.

For a FSB, the radiation force \( f_r \) in Eq. (1) can be approximated using a state space model of \( N \) states, \( \bar{x}_r = [x_{r1}, \cdots, x_{rN}]^T \), which outputs the radiation force [7]:

\[
\dot{\bar{x}}_r = A_r \bar{x}_r + B_r \bar{z} \quad \text{and} \quad f_r = C_r \bar{x}_r,
\]

where the constant radiation matrices \( A_r \) and \( B_r \) are obtained by approximating the impulse response function in the Laplace domain, as detailed in several references such as [8].

Two important aspects impact the energy converted from oceans: the control force and the buoy’s shape. For a FSB WEC, linear dynamic models are widely used in control design e.g. [4, 9–14]. Usually, the control is designed so as to maximize the mechanical power of the WEC; and many reference adopt different approaches in optimal control to achieve that. Often, the resulting control forces have a spring-like component, in addition to the resisting force [15, 16]. Hence the Power Take-Off (PTO) unit needs to have a bidirectional power flow capability, which is typically complex and more expensive. In analyzing the shapes of FSB WECs, the use of any non-cylindrical shape requires the use of nonlinear hydro models [17]. This is the reason that most studies assume cylindrical shape of the FSB WEC.

From an economic prospective, the cost of having the complex bidirectional power flow PTO to maximize harvested energy is high. Moreover, the structure of a FSB WEC needs to be designed so as to stand very high loads, at peak times, despite operating at a much less load most of the time. This impacts the structural design and increases the cost. To mitigate this peak load, controlled-geometry OSWEC were recently proposed in references [18–21], where controllable surfaces, along with a wave-to-wave control, are used to maximize power capture, increase capacity factor, and reduce design loads. The latter controlled-geometry OSWEC changes shape only when the wave climate changes, and hence it can be considered similar to the case of a FSB WEC when it is not in the transition from one geometry to another. A geometry control of the overtopping WEC is proposed in reference [22]. The slope angle and crest free board of the device is made adaptive to the sea conditions by geometry control. Reference [23] proposed a variable flap angle pitching device. The resonance characteristics of the WEC can be altered by controlling the angle of the flap. Later, reference [24] proposed a floating airbag WEC which has a longer resonance period without implementing phase control.

The concept of variable-shape buoy WEC was recently introduced to reduce the complexity of the PTO. A VSB WEC changes its shape continuously. The wave/WEC interaction produced by the VSB WEC can be leveraged to produce more power without adding complexity to the PTO unit. Specifically, Zou et al. [25] proposed the Variable-Shape point absorber; their original design comprises a pressurized gas chamber attached to a set of multiple controllable moving panels. This VSB WEC is controlled by a simple linear damping PTO unit [25]. A low-fidelity dynamic model is derived in [25] to demonstrate the superiority of the VSB WEC compared to the FSB WEC. The average power harvested using the VSB WEC in [25] is about 18% more compared to the FSB WEC.

In another study, references [26, 27] present three-dimensional two-way Fluid-Structure Interaction (FSI) high fidelity simulations, using the ANSYS software package, to simulate a spherical VSB WEC. The WEC in [26, 28] has a hyper-elastic hollow shell of radius 2 m. The internal volume contains trapped gas that helps in creating a restoring moment. The device is simulated in a numerical wave tank (NWT) with dimensions \( 80 \times 60 \times 60 \) m\(^3\) and damping
regions at the sides and at outlet of the NWT. The free surface height was at 40 m, and a control valve is placed to control the pressure inside the solid shell. Their simulation captured the high non-linear behavior of the VSB WEC and showed an enhancement in the heave displacement and velocity for the VSB WEC compared to a similar-size FSB WEC. Reference [29] presents a study that uses a similar approach, but applies a passive control force. A concrete plate is attached to the middle section of the VSB WEC that divides the internal volume into two separate partitions. The PTO unit is assumed to provide only a damping control that is applied to the upper surface of the concrete plate. The PTO force is dependent on the WEC heave velocity ($F_{PTO} = -c \dot{x}$), where $c$ is a constant damping coefficient. The results showed an increase in the heave displacement and velocity for the VSB WEC over the similar-size FSB WEC. The results also show an increase in the harvested energy of about 8%.

As can be seen from the above discussion, studies on VSB WECs currently use either a high fidelity software tools or a rough approximate low fidelity tools, for numerical simulations. The high fidelity tools are numerically expensive; and the above low fidelity simulations cannot capture important features in this FSI phenomenon since rigid moving panels are used to mimic the flexible body dynamics. In this work, a more rigorous dynamic model, that accounts for the flexibility of the buoy, is developed using Lagrangian mechanics, and is programmed for low fidelity simulations of VSB WECs.

This paper is divided into 6 main sections. In section 2, the kinetic and potential energies of asymmetric free vibrating spherical shells is derived using a similar approach to the approach presented in [30, 31]. In section 3 the equations of motion for spherical shell buoys are derived using Lagrangian mechanics, for the free unconstrained case. The equation of motion for the forced case are derived in section 4. The numerical simulation results are discussed in section 5.

2. Kinetic and Potential Energies of Spherical Shell Buoys

The derivation of the equation of motion in the current study is carried out using Lagrangian mechanics; hence the calculation of the kinetic and potential energies are required. This section starts with the layout of the used reference frames, then a description for the domain discretization technique used in the current study. The kinematics of the system are derived in subsection 2.1; the result is then used in the calculation of kinetic and potential energies in subsections 2.2 and 2.3, respectively.

Consider a flexible buoy for which the non-deformed shape is spherical. As shown in Fig. (1), the inertial frame is centered at the infinitesimal mass on the surface such that $\hat{e}_3$ is normal to the surface. The angle $\psi$ is the angle between $\hat{e}_3$ and $\hat{e}_1$. In the analysis presented in this paper, it is assumed that the deformations are axi-symmetric about the $\hat{e}_3$ axis; hence the axis $\hat{e}_2$ is always perpendicular to the page. The axes $\hat{a}_2$, $\hat{a}_3$, and $\hat{e}_2$ are also perpendicular to the page. If the shape is not deformed from its original spherical shape, then the frames $\hat{e}$ and $\hat{v}$ coincide. The frames $\hat{e}$ and $\hat{v}$ become different, in general, when the shape is deformed. For the FSB WEC the reference frames $\hat{e}$ and $\hat{v}$ coincide; this applies for the VSB WEC at initial time before deformation.

The coordinate transformation matrix from the $\hat{a}$ frame to the $\hat{v}$ frame is computed in this paper using the the 3-2-1 Euler angle sequence as $C_{av}(\alpha, \beta, \gamma) = C_3(\alpha)C_2(\beta)C_3(\gamma)$. The cross product of two arbitrary vectors expressed in the same reference frame can be replaced with matrix multiplication: $\mathbf{u} \times \mathbf{v} = \mathbf{u}^\times \mathbf{v}$, where

$$
\mathbf{u}^\times = \begin{bmatrix} u_{a3} & u_{a2} & 0 \\ u_{a1} & 0 & -u_{a3} \\ -u_{a2} & u_{a1} & 0 \end{bmatrix}
$$
Since the changes in the buoy shape are assumed axisymmetric, we can express the deformation vector (displacement) as function of only the angle $\phi$ and the time $t$. This deformation vector can be expressed in the $\hat{e}$ frame as:

$$\vec{r}_{dmc}(\phi, t) = [u(\phi, t) \quad 0 \quad v(\phi, t)]^T$$

where the second component (normal to the page plane) is set to zero because of the axisymmetry of the deformation, $u(\phi, t)$ is the displacement component in the $\hat{e}_1$ direction, and $v(\phi, t)$ is the displacement component in the $\hat{e}_3$ direction. In this paper, each of these displacement components is assumed a series of separable functions; that is each term in their series can be expressed as a product of two functions, one of them depends only on $\phi$ and the other depends only on $t$. Moreover, the Rayleigh-Ritz approximation is used to obtain approximate solution for the displacement vector as discussed in the following section.

### 2.1. Kinematics of a Flexible Spherical Buoy - Free Vibration

This subsection is concerned with the calculation of the $\dot{\vec{r}}_{dma}$ vector as it is crucial for the calculation of the kinetic energy of the spherical shell due to the translation and rotational motions as well as the deformation of the sphere external shell. As shown in Fig. (1), the position vector of a point on the surface of the non-deformed sphere in the inertial frame "$\vec{a}$" is expressed as

$$\vec{r}_{dma} = \vec{r}_{sa} + \vec{r}_{cs} + \vec{r}_{dmc}$$

The velocity vector is expressed as

$$\dot{\vec{r}}_{dma} = \dot{\vec{r}}_{sa} + \dot{\vec{r}}_{cs} + \dot{\vec{r}}_{dmc}$$

Note that, the left superscript denotes the reference frame used to describe the vector. Applying the transport theorem knowing that $\dot{\vec{r}}_{cs} = 0$, we get:

$$\dot{\vec{r}}_{dma} = \dot{\vec{r}}_{sa} + (\ddot{\vec{r}}_{cs} + \vec{\omega}_{as} \times \dot{\vec{r}}_{cs}) + (\ddot{\vec{r}}_{dmc} + \vec{\omega}_{as} \times \dot{\vec{r}}_{dmc})$$
Finally, the kinetic energy of the system is expressed as:

\[ T = \frac{1}{2} \int_S \mathbf{a} \mathbf{\hat{r}}_{d_{ma}} \cdot \mathbf{\hat{r}}_{d_{ma}} d\mathbf{m} \]  

where \( S \) denotes the surface of the buoy. Substituting Eq. (11) in Eq. (12) to get,

\[ T = \frac{1}{2} \int_S \left[ C_{sa} - [C_{es}(\mathbf{e}_r \mathbf{r}_es + \mathbf{e}_r \mathbf{r}_d_{mc})]^\times C_{se} \right] \mathbf{\hat{x}} \]

\[ \cdot \left[ C_{sa} - [C_{es}(\mathbf{e}_r \mathbf{r}_es + \mathbf{e}_r \mathbf{r}_d_{mc})]^\times C_{se} \right] \mathbf{\hat{x}} \, d\mathbf{m} \]  

(13)

Let \( H = [C_{es}(\mathbf{e}_r \mathbf{r}_es + \mathbf{e}_r \mathbf{r}_d_{mc})]^\times \) such that Eq. (13) can be reduced to,

\[ T = \frac{1}{2} \int_S \mathbf{x}^T \begin{bmatrix} C_{sa} & -H & C_{se} \end{bmatrix} \mathbf{\hat{x}} \, d\mathbf{m} \]  

(14)

\[ = \frac{1}{2} \int_S \begin{bmatrix} a_{r_{sa}}^T \mathbf{\hat{r}}_{d_{mac}} & a_{r_{sa}}^T \mathbf{\hat{r}}_{d_{mc}} & a_{r_{es}}^T \mathbf{\hat{r}}_{d_{mc}} \end{bmatrix} \begin{bmatrix} C_{sa} & -C_{sa} \mathbf{H} C_{se} & C_{se} \end{bmatrix} \mathbf{\hat{x}} \, d\mathbf{m} \]  

(15)

\[ = \frac{1}{2} \int_S \begin{bmatrix} a_{r_{sa}}^T \mathbf{\hat{r}}_{d_{mac}} & a_{r_{sa}}^T \mathbf{\hat{r}}_{d_{mc}} & a_{r_{es}}^T \mathbf{\hat{r}}_{d_{mc}} \end{bmatrix} \begin{bmatrix} C_{sa} & -C_{sa} \mathbf{H} C_{se} & C_{se} \end{bmatrix} \mathbf{\hat{x}} \, d\mathbf{m} \]  

(16)

Finally, the kinetic energy of the system is expressed as:

\[ T = \frac{1}{2} \int_S \left\{ a_{r_{sa}}^T (\mathbf{a}_{r_{sa}} \mathbf{\hat{r}}_{d_{mac}} - C_{sa} \mathbf{H} \mathbf{a}_{r_{es}} + C_{sa} C_{se} \mathbf{e}_r \mathbf{r}_d_{mc}) ight. \\
+ \mathbf{a}_{r_{es}}^T (C_{es} C_{se} \mathbf{a}_{r_{es}} - C_{es} \mathbf{H}^T \mathbf{x}_{r_{es}} - H^T C_{se} \mathbf{e}_r \mathbf{r}_d_{mc}) \\
\left. + \mathbf{e}_r \mathbf{r}_{d_{mc}} (C_{sa} C_{se} \mathbf{a}_{r_{es}} - H^T C_{se} \mathbf{a}_{r_{es}} + \mathbf{e}_r \mathbf{r}_{d_{mc}}) \right\} \, d\mathbf{m} \]  

(18)
Assuming the kinetic energy components associated with the multiplication of any two of $\dot{\vec{r}}_{sa}$, $\vec{\omega}_{as}$ and $\dot{\vec{r}}_{dmc}$ is negligible, then Eq. (18) is reduced to:

$$
\mathcal{T} = \frac{1}{2} \int_S \left\{ a^T \dot{\vec{r}}_{sa} a + \omega_T^T \vec{H} \vec{H}^T \omega_{as} + \dot{\vec{r}}_{dmc} \dot{\vec{r}}_{dmc} dm \right\}
$$

$$
= \frac{1}{2} \frac{a^T m \dot{\vec{r}}_{sa} + \frac{1}{2} \omega_T^T J \omega_{as} + \frac{1}{2} \int_S \dot{\vec{r}}_{dmc} \dot{\vec{r}}_{dmc} dm}{T_s}
$$

$$
= \frac{1}{2} \frac{a^T m \dot{\vec{r}}_{sa} + \frac{1}{2} \omega_T^T J \omega_{as} + \frac{1}{2} \int_S \dot{\vec{r}}_{dmc} \dot{\vec{r}}_{dmc} dm}{T_s}
$$

(19)

where $m = 1m$ is the VSB mass matrix and $J_s = 1J_s$ is the second moment of inertia. Considering Eq. (19), the following is noted:

1. The first term yields the translational kinetic energy $T_s$ of the rigid buoy.
2. The second term describes the rotational kinetic energy $T_o$ of the rigid shell.
3. The third term accounts for the kinetic energy associated with the deformation of the shell $T_s$ and is simplified as follows:

$$
T_s = \frac{1}{2} \int_S \dot{\vec{r}}_{dmc} \dot{\vec{r}}_{dmc} dm = \frac{1}{2} \rho h \int_0^\pi \int_0^{2\pi} \dot{r}_{dmc} r^2 \sin \phi \, d\phi d\theta
$$

$$
= \frac{1}{2} \rho h \int_0^\pi \int_0^{2\pi} \left[ \dot{\vec{u}} \quad \dot{\vec{v}} \right] \left[ \begin{array}{c} \ddot{u} \\ 0 \\ \ddot{v} \end{array} \right] r^2 \sin \phi \, d\phi d\theta
$$

$$
= \frac{1}{2} \rho h \int_0^\pi \int_0^{2\pi} \left( \dot{\vec{u}}^2 + \dot{\vec{v}}^2 \right) r^2 \sin \phi \, d\phi d\theta
$$

(20)

(21)

(22)

where $h$ and $\rho$ are the thickness and the material density of the shell. Eqs. (22) describes the kinetic energy for spherical axisymmetric homogeneous thin shells.

### 2.3. Potential Energy for Flexible Spherical Buoyas

Considering the buoy as a spherical shell, the gravitational energy can be expressed as:

$$
\mathcal{G} = \int_S -\vec{g} \cdot \vec{r}_{sa} dm = mg\hat{a}_s T \vec{r}_{sa}
$$

(23)

where $\vec{g} = -g\hat{a}_s$. The strain energy-displacement expressions (membrane strains) for axisymmetric shells can be written as [32, 33]:

$$
\varepsilon_{\phi\phi} = \frac{1}{r} \left( \frac{\partial u}{\partial \phi} + v \right)
$$

(24)

$$
\varepsilon_{\theta\theta} = \frac{1}{r} (u \cot(\phi) + v)
$$

(25)

and the total strain energy can be expressed as [32]:

$$
\mathcal{U}_s = \frac{1}{2} \frac{Eh}{1 - v^2} \int_0^{2\pi} \int_0^\pi \left\{ \varepsilon_{\phi\phi}^2 + \varepsilon_{\theta\theta}^2 + 2v \varepsilon_{\phi\phi} \varepsilon_{\theta\theta} \right\} r^2 \sin(\phi) \, d\phi d\theta
$$

(26)

Combining the sphere elastic and gravitational potential energies in Eqs. (65) and (23) yields the total potential energy of the spherical shell buoy:

$$
\pi = U' + G = \frac{1}{2} \frac{Eh}{1 - v^2} \int_s \left\{ \varepsilon_{\phi\phi}^2 + \varepsilon_{\theta\theta}^2 + 2v \varepsilon_{\phi\phi} \varepsilon_{\theta\theta} \right\} r^2 \sin(\phi) \, dm + mg \frac{1}{3} T \vec{r}_{sa}
$$

(27)

$$
= \frac{1}{2} \frac{Eh}{1 - v^2} \int_0^{2\pi} \int_0^\pi \left\{ \varepsilon_{\phi\phi}^2 + \varepsilon_{\theta\theta}^2 + 2v \varepsilon_{\phi\phi} \varepsilon_{\theta\theta} \right\} r^2 \sin(\phi) \, d\phi d\theta + mg \frac{1}{3} T \vec{r}_{sa}
$$

(28)
3. Unconstrained Equations of Motion For Flexible Spherical Buoys

The unconstrained equations of motion are here derived as a first step towards writing the constrained equations of motion. The Lagrangian for this buoy system can be written as the summation of three quantities [34]:

\[\mathcal{L} = \mathcal{L}_D + \mathcal{L}_B + \int_s \dot{\mathcal{L}} d\phi\]  

(29)

where \(\mathcal{L}_D(t, \dot{x}, \ddot{x})\) is associated with the discrete coordinates, \(\dot{\mathcal{L}}\) is the Lagrangian density function and it is a function of the discrete and distributed parameter coordinates, \(\mathcal{L}_B\) is associated with the boundaries. In the this work there are no boundary terms in the Lagrangian equation, i.e., \(\mathcal{L}_B = 0\). First we will derive the equations of motion related to the discrete coordinates then the equations of motion related to the distributed parameter coordinates will be derived where Rayleigh-Ritz approximation will be summoned.

3.1. Equation of Motion Associated with Discrete Coordinates

The Lagrangian for the discrete coordinates is expressed as:

\[\mathcal{L}_D = \mathcal{T}_D - \pi_D = \frac{1}{2} \mathcal{Q} \tau^{Ta} m \omega \hat{\tau}_{sa} + \frac{1}{2} \omega^{Ta} \mathcal{J}_s \hat{\omega}_{as} - mg \mathcal{V} \hat{r}_{sa}\]  

(30)

The Lagrange Equation for the discrete coordinates is expressed as:

\[\frac{d}{dt} \left( \frac{\partial \mathcal{L}_D}{\partial \dot{\mathcal{X}}} \right) - \frac{\partial \mathcal{L}_D}{\partial \mathcal{X}} = 0\]  

(31)

To write the equations of motion of the discrete coordinates, we first write:

\[\frac{\partial \mathcal{L}_D}{\partial \hat{r}_{sa}} = -mg \mathcal{I}_3, \text{ and } \frac{\partial \mathcal{L}_D}{\partial \hat{\omega}_{sa}} = 0\]  

(32)

\[\frac{d}{dt} \left( \frac{\partial \mathcal{L}_D}{\partial \hat{\tau}_{sa}} \right) = m \omega \hat{\tau}_{sa}, \text{ and } \frac{d}{dt} \left( \frac{\partial \mathcal{L}_D}{\partial \hat{\omega}_{sa}} \right) = \mathcal{J}_s \hat{\omega}_{as} + \mathcal{J}_s \hat{\omega}_{as}\]  

(33)

The equations of motion for the translation and rotational motions become:

\[m \omega \hat{\tau}_{sa} + mg \mathcal{I}_3 = 0\]  

(34)

\[\mathcal{J}_s \hat{\omega}_{as} + \mathcal{J}_s \hat{\omega}_{as} = 0\]  

(35)

The equations of motion described by Eq. (34) and (35) can be extended to include damping coefficient matrices as follows:

\[m \omega \hat{\tau}_{sa} + D_x \omega \hat{\tau}_{sa} + mg \mathcal{I}_3 = 0\]  

(36)

\[\mathcal{J}_s \hat{\omega}_{as} + (\mathcal{J}_s + D_\omega) \hat{\omega}_{as} = 0\]  

(37)

where \(D_x\) and \(D_\omega\) are the damping matrices for the transnational and rotational motions, respectively.

3.2. Equations of Motion Associated with The Distributed Parameter Coordinates

The distributed parameters Lagrangian, \(\hat{\mathcal{L}}\), is expressed as:

\[\hat{\mathcal{L}} = \frac{1}{2} (2\pi \rho h) \left\{ u^2 + v^2 \right\} r^2 \sin \phi\]  

(38)

\[-\frac{1}{2} \frac{2\pi E h}{(1 - \nu^2)} \left\{ \left( \frac{\partial u}{\partial \phi} + v \right)^2 + (u \cot(\phi) + v)^2 + 2v \left( \frac{\partial u}{\partial \phi} + v \right) (u \cot(\phi) + v) \right\} r^2 \sin \phi,\]  

(39)
\[
\frac{d}{dt} \left( \frac{\partial \hat{L}}{\partial \dot{\bar{r}}_{dmc}} \right) - \frac{\partial L}{\partial \bar{r}_{dmc}} + \frac{\partial}{\partial \bar{x}} \left( \frac{\partial \hat{L}}{\partial \dot{\bar{x}}_{dmc}} \right) + \frac{\partial^2}{\partial \bar{x}^2} \left( \frac{\partial \hat{L}}{\partial \bar{x} \gamma_{dmc}} \right) = 0
\]  

(40)

where \( \bar{x} \) is the vector of spatial coordinates in the \( \bar{a} \) directions, such that the transformation from the cartesian coordinates to spherical coordinates is as follows:

\[
x_1 = r \sin \phi \Rightarrow \bar{x} = \frac{1}{r \cos \phi} \frac{\partial}{\partial x_1} = \frac{\partial}{\partial x_3} = \frac{-1}{r \sin \phi} \frac{\partial}{\partial x_3}
\]

(41, 42)

For \( \bar{r}_{dmc1} = u \):

\[
\frac{d}{dt} \left( \frac{\partial \hat{L}}{\partial \dot{u}} \right) = 2\pi \rho hr^2 \sin \phi \dot{u}
\]

(43)

\[
\frac{2\pi E h}{(1 - \nu^2)} \left\{ \left( \frac{\partial u}{\partial \phi} + v \right) + v \left( \frac{\partial \dot{u}}{\partial \phi} + v \right) \right\} r \tan \phi \]

(44)

\[
= -2\pi E h \left\{ \left( \frac{\partial u}{\partial \phi} + v \right) + v \left( \frac{\partial \dot{u}}{\partial \phi} + v \right) \right\} r \sec^2 \phi
\]

\[
-2\pi E h \left\{ \left( \frac{\partial^2 u}{\partial \phi^2} + \frac{\partial v}{\partial \phi} \right) + v \left( \frac{\partial u}{\partial \phi} \cot(\phi) - u \csc(2\phi) + \frac{\partial v}{\partial \phi} \right) \right\} r \tan \phi
\]

(45)

For \( \bar{r}_{dmc3} = v \):

\[
\frac{d}{dt} \left( \frac{\partial \hat{L}}{\partial \dot{v}} \right) = 2\pi \rho hr^2 \sin \phi \dot{v}
\]

(46)

\[
\frac{2\pi E h}{(1 - \nu^2)} \left\{ \left( \frac{\partial u}{\partial \phi} + v \right) + (u \cot(\phi) + v) + \left( \frac{\partial \dot{u}}{\partial \phi} + v \right) + (u \cot(\phi) + v) \right\} r \tan \phi
\]

(47)

\[
-\frac{1}{r \sin \phi} \frac{\partial}{\partial \phi} \left( \frac{\partial \hat{L}}{\partial \phi} \right)(0) = 0
\]

(48)

Accordingly, the equations of motion associated with the distributed parameter are expressed as:

\[
2\pi \rho hr^2 \sin \phi \ddot{u} - \frac{2\pi E h}{(1 - \nu^2)} \left\{ \left( \frac{\partial u}{\partial \phi} + v \right) + v \left( \frac{\partial \dot{u}}{\partial \phi} + v \right) \right\} r^2 \sin \phi
\]

\[
- \frac{2\pi E h}{(1 - \nu^2)} \left\{ \left( \frac{\partial u}{\partial \phi} + v \right) + v \left( \frac{\partial \dot{u}}{\partial \phi} + v \right) \right\} r \sec^2 \phi
\]

\[
- \frac{2\pi E h}{(1 - \nu^2)} \left\{ \left( \frac{\partial^2 u}{\partial \phi^2} + \frac{\partial v}{\partial \phi} \right) + v \left( \frac{\partial u}{\partial \phi} \cot(\phi) - u \csc(2\phi) + \frac{\partial v}{\partial \phi} \right) \right\} r \tan \phi = 0
\]

(49)

\[
2\pi \rho hr^2 \sin \phi \ddot{v} - \frac{2\pi E h}{(1 - \nu^2)} \left\{ \left( \frac{\partial u}{\partial \phi} + v \right) + (u \cot(\phi) + v) + \left( \frac{\partial \dot{u}}{\partial \phi} + v \right) + (u \cot(\phi) + v) \right\} r^2 \sin \phi = 0
\]

(50)

Equations (49) and (50) are the equations of motion of deformation of axisymmetric homogeneous spherical shell. These equations of motion can not be solved analytically, and an approximate method is used. Most of the equations of motions for continuous systems are usually difficult to obtain; this difficulty arises either from the difficulty in solving the governing equations or from imposing the boundary conditions. Here, an approximate method is implemented to convert the partial differential equations to ordinary differential equations.
Rayleigh-Ritz Approximation

The approximation method used in this work is the Rayleigh-Ritz method. Each component of the displacement vector $\mathbf{r}_{dmc}$ is assumed to have the following form:

$$
\begin{align*}
  u(\phi, t) &= \sum_{n=1}^{N} \Psi_{n}^{\phi}(\phi) \eta_{n}(t) = \left[ \begin{array}{c}
  \Psi_{1}^{\phi} \\
  \vdots \\
  \Psi_{N}^{\phi}
  \end{array} \right] \left[ \begin{array}{c}
  \eta_{1}(t) \\
  \vdots \\
  \eta_{N}(t)
  \end{array} \right] = \Psi_{e}^{\phi}(\phi) \eta(t) \\

  v(\phi, t) &= \sum_{n=1}^{N} \Psi_{n}^{r}(\phi) \eta_{n}(t) = \left[ \begin{array}{c}
  \Psi_{1}^{r} \\
  \vdots \\
  \Psi_{N}^{r}
  \end{array} \right] \left[ \begin{array}{c}
  \eta_{1}(t) \\
  \vdots \\
  \eta_{N}(t)
  \end{array} \right] = \Psi_{e}^{r}(\phi) \eta(t)
\end{align*}
$$

where the functions $\Psi_{n}^{\phi}$ and $\Psi_{n}^{r}$ are trial (admissible) functions of $\phi$ and the functions $\eta_{n}$ are functions of time $t$, $\forall n = 1, \cdots, N$. Therefore, the displacement vector can be expressed in the $\hat{e}$ frame as follows:

$$
\mathbf{r}_{dmc}(\phi, t) = \left[ \begin{array}{c}
  \Psi_{e}^{\phi}(\phi) \\
  0 \\
  \Psi_{e}^{r}(\phi)
  \end{array} \right] \eta(t) = \Phi_{e}(\phi) \eta(t)
$$

The Legendre functions of the first kind $P_{n}$ [35] can serve as shape functions for the Ritz-Rayleigh method to satisfy the essential geometrical (Dirichlet) boundary conditions [31, 33, 36–38], as follows:

$$
\Psi_{n}^{\phi}(\phi) = A \frac{d P_{n}(\cos(\phi))}{d\phi}, \text{ and } \Psi_{n}^{r}(\phi) = A \frac{1 + (1 + \nu) \Omega_{n}^{2}}{1 - \Omega_{n}^{2}} P_{n}(\cos(\phi))
$$

where the coefficients of the equations above form an eigenvector for the Legendre differential equation, i.e. the constant "A" can take any real value. $\Omega_{n}^{2}$ is a dimensionless frequency parameter expressed as [33]:

$$
\Omega_{n}^{2} = \frac{1}{2(1 - \nu^{2})} \left( A \pm \sqrt{A^{2} - 4mB} \right)
$$

where $\nu$ is the Poisson’s ratio, and

$$
\begin{align*}
  m &= n(n + 1) - 2, \quad n \in \mathbb{Z}^{+} \\
  B &= 1 + \nu^{2} + \frac{1}{12} [(m + 1)^{2} - \nu^{2}] \\
  A &= 3(1 + \nu) + m + \frac{1}{2} \left( \frac{h}{r} \right)^{2} (m + 3)(m + 1 + \nu)
\end{align*}
$$

From [33, 36] the natural frequencies in radians per second for spherical shells are calculated using Eq. (59)

$$
\omega_{n}^{2} = \frac{E}{r^{2} \rho \Omega_{n}^{2}}
$$

where $E$ is Young’s Modulus, $r$ is the non-deformed radius of the shell, and $\rho$ is the density of the shell material. When $n = 0$, the vibration mode corresponds to the breathing mode (volumetric or pulsating modes) which is a pure radial vibration mode [33, 38, 39]. For $n > 0$, the ± sign in Eq. (55) yields the modes corresponding to the membrane vibration modes and bending vibration modes. The bending vibration modes are obtained when using the negative sign;
these modes are sensitive to the $h/r$ ratio. On the other hand, the membrane modes are insensitive to the change in the $h/r$ ratio. Due to the extensional motion of the buoy, only the membrane vibration modes are used in the current work. To obtain the approximated equations of motion using Rayleigh-Ritz method, the approximated displacement vector needs to be substituted in the kinetic and strain energy equations as follows. The kinetic energy $T_s$ is approximated by substituting Eqs. (51) and (52) into Eq. (22); to get:

$$T_s = \frac{1}{2} \eta^T \left\{ 2\pi \rho h \int_0^\pi \left( \Psi_e^T \Psi_e + \Psi_e^T \Psi_e \right) r^2 \sin \phi \, \, d\phi \right\} \dot{\eta}$$

$$= \frac{1}{2} \eta^T M_{ee} \dot{\eta}$$

(60)

The strain energy is approximated by first substituting Eqs. (51) and (52) in Eqs. (24) and (25) to get:

$$\varepsilon_{\phi\phi} = \frac{1}{r} \left( \frac{\partial \Psi_e^T}{\partial \phi} + \Psi_e^T \right) \eta$$

$$= \frac{1}{r} \left( \frac{\partial \Psi_e^T}{\partial \phi} + \Psi_e^T \right) \eta$$

(62)

$$\varepsilon_{\theta\theta} = \frac{1}{r} \left( \frac{\partial \Psi_e^T}{\partial \phi} + \Psi_e^T \right) \eta$$

(63)

Then the strain energy equation in Eq. (26) becomes:

$$U_s = \frac{1}{2} \eta^T \left[ \frac{2\pi E h}{1 - \nu^2} \int_0^\pi \left\{ \left( \frac{\partial \Psi_e^T}{\partial \phi} + \Psi_e^T \right) \left( \frac{\partial \Psi_e^T}{\partial \phi} + \Psi_e^T \right) + \Psi_e^T \cot(\phi) + \Psi_e^T \right\} \sin(\phi) \, \, d\phi \right] \eta$$

$$= \frac{1}{2} \eta^T K_{ee} \eta$$

(64)

(65)

where

$$K_{ee} = \frac{2\pi E h}{1 - \nu^2} \int_0^\pi \left\{ \xi_{\phi\phi}^T \xi_{\phi\phi} + \xi_{\theta\theta}^T \xi_{\theta\theta} + \nu \left( \xi_{\phi\theta} \xi_{\phi\theta} + \xi_{\theta\phi} \xi_{\theta\phi} \right) \right\} \sin(\phi) \, \, d\phi$$

(66)

$$\xi_{\phi\phi} = \frac{\partial \Psi_e^T}{\partial \phi} + \Psi_e^T$$

(67)

$$\xi_{\theta\theta} = \frac{\partial \Psi_e^T}{\partial \phi} + \Psi_e^T$$

(68)

Using the above approximate expressions for the kinetic and strain energies, the Lagrangian $L_s$ is expressed as follows:

$$L_s = T_s - U_s = \frac{1}{2} \eta^T M_{ee} \eta - \frac{1}{2} \eta^T K_{ee} \eta$$

(69)

With this approximation, the Lagrangian can be used to write the equations of motion of the flexible buoy in the form:

$$\frac{d}{dt} \left( \frac{\partial L_s}{\partial \dot{\eta}} \right) - \frac{\partial L_s}{\partial \eta} = 0$$

(70)

From Eq. (69), one can write:

$$\frac{\partial L_s}{\partial \eta} = -K_{ee} \eta \quad \text{and} \quad \frac{\partial L_s}{\partial \dot{\eta}} = M_{ee} \dot{\eta}$$

$$= -K_{ee} \eta \quad \text{and} \quad M_{ee} \dot{\eta}$$
\[
\dot{d} \left( \frac{\partial L_s}{\partial \dot{\eta}} \right) = M_{ee} \ddot{\eta}
\]  
(71)

Hence, the equation of motion for the shell of a flexible buoy vibrating axisymmetrically is:

\[
M_{ee} \ddot{\eta} + K_{ee} \eta = 0
\]  
(72)

Equation (72) can be further extended to include Rayleigh damping as follows:

\[
M_{ee} \ddot{\eta} + D_{ee} \dot{\eta} + K_{ee} \eta = 0
\]  
(73)

where \(D_{ee}\) is a proportional damping matrix that is assumed to be a function of the mass and stiffness matrices as follows:

\[
D_{ee} = \alpha_d M_{ee} + \beta_d K_{ee}
\]

where the \(\alpha_d\) and \(\beta_d\) are real scalars called the mass and stiffness matrix multipliers with units 1/sec and sec, respectively [40–42]. Combining the equations of motion from Eqs. (34), (35) and (73) yields the equation of motion of the flexible buoy:

\[
M \dddot{\hat{x}} + (\dot{M} + D) \ddot{\hat{x}} + [mg1^T_3 \quad 0 \quad (K_{ee} \eta)^T]^T = 0
\]  
(74)

where

\[
\dot{\hat{x}} = \begin{bmatrix} \dot{\omega}_{sa}^T \\ \dot{\omega}_{sa}^T \end{bmatrix} \quad \eta^T_{(6+N) \times 1}
\]

\[
M = \text{diag}\{m \quad J_s \quad M_{ee}\}_{(6+N) \times (6+N)}
\]

\[
\dot{M} = \text{diag}\{0 \quad J_s \quad 0\}_{(6+N) \times (6+N)}
\]

\[
D = \text{diag}\{D_x \quad D_\omega \quad D_{ee}\}_{(6+N) \times (6+N)}
\]

It is noted here that the system mass matrix is a function of time because \(J_s\) is a function of time. The vector \(\dot{\omega}_{sa}\) describes the instantaneous body angular velocities in the body frame \(\dot{s}\) with respect to the inertial frame \(\dot{a}\). To avoid integrating the direction cosine matrices, the Euler angles are used for orientation (attitude) descriptions. Let \([B(\theta)]\) be the mapping matrix that converts the angular velocity \(\dot{\omega}_{sa}\) to Euler angle rates \(\dot{\theta}_{sa}\), hence we can write:

\[
\ddot{\omega}_{sa} = [B(\theta)]^{-1} \dot{\theta}_{sa}
\]  
(75)

\[
\dot{\omega}_{sa} = [B(\theta)]^{-1} \dot{\theta}_{sa} + [\dot{B}(\theta)]^{-1} \dot{\theta}_{sa}
\]  
(76)

\[
[B(\theta)]^{-1} = \begin{bmatrix} -\sin \theta_2 & 0 & 1 \\ \cos \theta_2 \sin \theta_3 & \cos \theta_3 & 0 \\ \cos \theta_2 \cos \theta_3 & -\sin \theta_3 & 0 \end{bmatrix}
\]  
(77)

\[
[\dot{B}(\theta)]^{-1} = \begin{bmatrix} \dot{\theta}_2 \cos \theta_2 & \dot{\theta}_2 \sin \theta_2 \sin \theta_3 - \dot{\theta}_3 \cos \theta_2 \cos \theta_3 & 0 & 0 \\ \dot{\theta}_2 \sin \theta_2 \sin \theta_3 + \dot{\theta}_3 \cos \theta_2 \cos \theta_3 & \dot{\theta}_3 \sin \theta_3 & 0 & 0 \\ -\dot{\theta}_2 \sin \theta_2 \cos \theta_3 + \dot{\theta}_3 \cos \theta_2 \sin \theta_3 & \dot{\theta}_3 \cos \theta_3 & 0 & 0 \end{bmatrix}
\]  
(78)

where Eqs. (77) and (78) are derived using a 3-2-1 Euler angles sequence. Substituting Eq. (76) into the inertia term in (74) to get:

\[
M \begin{bmatrix} \ddot{\omega}_{sa} \\ \dot{\omega}_{sa} \end{bmatrix} + (\dot{M} + D) \begin{bmatrix} \dot{\omega}_{sa} \\ \dot{\eta} \end{bmatrix} + a = M \begin{bmatrix} a_{\theta_{sa}} \\ a_{\omega_{sa}} \end{bmatrix} + M \begin{bmatrix} 0 \\ [B(\theta)]^{-1} \dot{\theta}_{sa} \end{bmatrix} + D \begin{bmatrix} \ddot{\omega}_{sa} \\ \dot{\omega}_{sa} \end{bmatrix} + D_{ee} [\omega_{sa} J_s] + K_{ee} \eta
\]  
(79)

\[
= M \dot{\bar{B}} \ddot{q} + M \dot{\bar{B}} \dot{q} + D \dot{q} + a
\]  
(80)
where \( q \) is the generalized coordinates vector and is defined by Eq. (81):

\[
q = [\mathbf{r}^T_{sa} \theta^T_{sa} \eta^T]_T
\]  

(81)

where

\[
\bar{B} = \text{diag}\{1 \quad [B(\theta)]^{-1} \quad 1\},
\]

(82)

\[
\bar{\dot{B}} = \text{diag}\{0 \quad [B(\theta)]^{-1} \quad 0\},
\]

(83)

\[
a = [mgI^T_s (\bar{\omega}_{sa} \bar{J}_s)^T (K_{ee}\eta)]_T
\]

(84)

Then the Equation of motion for free buoy is expressed as:

\[
\bar{B}^T M \bar{B}q + \bar{\dot{B}}^T (M \bar{B}q + D \bar{B}q + a) = 0
\]

(85)

The \( q \) is a \((6 + N)\) column vector of generalized coordinates, \( D \) is the system damping matrix, \( D_x \) and \( D_{\omega} \) are the translational and rotational damping coefficients matrices, respectively.

### 4. Forced Constrained Equations of Motion

The Lagrange equation for the forced motion for the discrete coordinates is expressed as:

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = Q
\]

(86)

where, \( Q \) is a column vector of generalized forces. The external generalized forces on a buoy are the generalized PTO force “\( Q_{pto} \)”, generalized buoyant force “\( Q_b \)”, generalized radiation forces \( Q_r \), and generalized hydrodynamic excitation forces “\( Q_e \)”.

The WECs can have an internal structure to install a PTO unit at an angle \( \phi_e \) on the shell. An example of this is the VSB WEC tested in [29], where a concert plate was attached at an angle \( \phi_e = 90^\circ \) to apply the PTO force. The elastic surface displacement \( \mathbf{r}_{dmc} \) at \( \phi_e = 90^\circ \) should equal to zero, resulting in the holonomic constraint in Eq. (87):

\[
\mathbf{r}_{dmc}(\phi_e) = \Phi_e \eta = 0
\]

(87)

where \( \Phi_e = \Phi_e(\phi_e) \).

Hence, the most general form for the Equation of motion for a Spherical VSB is shown in Eq. (88):

\[
\bar{B}^T M \bar{B}q + \bar{\dot{B}}^T (M \bar{B}q + D \bar{B}q + a) = Q_c + Q
\]

\[
= A^T \lambda + Q_{pto} + Q_b + Q_r + Q_e
\]

(88)

where \( Q_c = A^T \lambda \) is the generalized constraint vector, \( \lambda \) column matrix of Lagrange multipliers with dimensions \((3 \times 1)\), and \( A \) is the Jacobian constraint matrix.

To write the constraint in Eq. (87) in the generalized coordinates, we write the following transformation:

\[
0 = \Phi_e \eta \equiv \begin{bmatrix} 0 & 0 & \Phi_e \end{bmatrix}_A \bar{B}q = \Phi_e \bar{B}q
\]

(89)

Eq. (89) can be written as \( Aq = 0 \), where \( A = \Phi_e \bar{B} \). Reference [43] shows that the generalized constraint forces for ideal constraints can be expressed as:

\[
Q_c = M^{\frac{1}{2}} (AM^{-\frac{1}{2}})^{(b - AM^{-1}(Q - \bar{a}))}
\]

(90)

where \( b = -(\bar{A}q + 2\bar{A}q) \), and \( \bar{a} = \bar{B}^T (M \bar{B}q + D \bar{B}q + a) \).

It is common in WEC analysis to assume that the WEC is only heaving (moving only in the vertical direction,) and hence we can simplify the equations by assuming that the buoy is not rotating; that is \( \bar{B} = 0 \). In such case, the vector \( \bar{b} = 0 \). Next, the expression for each of the external forces is developed.
4.1. Power Take-off Unit Force

The PTO can be either active or passive [29], in this paper the PTO is assumed a passive damping. To apply the PTO effect for that case one can either include its effect in the $D$ matrix in Eq. (88), or as an external force in Eq. (91).

$$\vec{f}_{pto} = -c a \vec{r}_{sa}$$  \hspace{1cm} (91)

Where $\vec{f}_{pto}$ is the damping force and $c$ is the damping coefficient. Note that the equation above applies to heave only motion. To compute the generalized force, the general transformation takes the form [34]:

$$Q_j = \sum_{i=1}^{6+N} f_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}$$  \hspace{1cm} (92)

Hence, the generalized force corresponding to the PTO force takes the form:

$$Q_{pto}^j = \vec{f}_{pto} \cdot \frac{\partial \vec{r}_{sa}}{\partial q_j}, \ \forall j = 1, 2, \cdots 6 + N$$  \hspace{1cm} (93)

We can then write the PTO generalized forces using Eq. (81) and Eq. (91) as follows:

$$Q_{pto}^1 = \vec{f}_{pto} \cdot \frac{\partial \vec{r}_{sa}}{\partial \theta_1} = 0$$

$$Q_{pto}^2 = \vec{f}_{pto} \cdot \frac{\partial \vec{r}_{sa}}{\partial \theta_2} = 0$$

$$Q_{pto}^3 = \vec{f}_{pto} \cdot \frac{\partial \vec{r}_{sa}}{\partial \theta_3} = 0$$

$$Q_{pto}^4 = \vec{f}_{pto} \cdot \frac{\partial \vec{r}_{sa}}{\partial \eta_1} = 0$$

$$Q_{pto}^5 = \vec{f}_{pto} \cdot \frac{\partial \vec{r}_{sa}}{\partial \eta_2} = 0$$

$$Q_{pto}^6 = \vec{f}_{pto} \cdot \frac{\partial \vec{r}_{sa}}{\partial \eta_3} = 0$$

In a compact form,

$$Q_{pto} = \begin{bmatrix} c a \vec{r}_{sa,3}^T 1 \\ 0 \\ 0 \end{bmatrix}$$  \hspace{1cm} (94)

4.2. Hydro Forces

In this work, it is assumed that the excitation pressure $P_{ex}$ is uniformly distributed around the submerged buoy’s volume; such that the excitation force on a submerged partition is expressed as $f_{exi} = P_{ex} A_i$, where $A_i$ is the $i$th partition surface area, the calculation of the surface area is detailed in section 4.2.2.

The radiation forces are neglected in this study; radiation forces are usually in the form of damping and inertia forces. While radiation forces are important to model for accurate simulation of a given WEC, the focus of this study is the dynamic modeling of a VSB WEC, and the comparison between the VSB and FSB; for the purpose of this study the radiation forces can be neglected. In future work, the radiation forces will be included. Hence, here the generalized hydro force is the summation of the generalized excitation and the generalized hydrostatic force (or buoyant force). The buoyant force $\vec{f}_b$ is exerted on the buoy’s surface due to the displaced water by the submerged volume. For the case of a FSB, the buoyant force acts on the center of buoyancy along the $\hat{a}_3$ direction. On the other hand, for a VSB, the buoyant force is computed as integration of pressure over the surface. This buoyant pressure contributes to the shell’s deformation; hence Eq. (95) is used to compute the buoyant force at each node $i$:

$$\vec{f}_i^b = \frac{\rho_u V_{s,i} g}{\cos(\pi - \psi_i)} \hat{c}_3^T$$  \hspace{1cm} (95)
where $\rho_w$ is the density of the water, $V_{s,i}$ is the submerged volume corresponding to the node $i$, and the angle $\phi$ is defined as shown in Fig. 1. The calculation of the buoyant force for the VSB takes into account the continuous change in the buoy shape and the submerged volume. In the rest of the current subsection, the methodology followed to calculate the submerged volume is presented, and then the generalized hydrodynamic force is calculated.

4.2.1. Submerged Volume Calculation

The submerged volumes of the VSB and the FSB are calculated using the Riemann integrals approach [44] where the submerged volume is divided into a set of $n$ horizontal disks (partitions), and the total submerged volume of the buoy is the sum of volumes of disks below the water surface, as shown in Fig. 2. Let $\phi_1$ be the angle of the highest wet disk on the VSB surface as shown in Fig. 2-a; this can be written as:

$$\phi_1 = \phi(\max\{r_{\text{wet}}^{dms_{3}}\})$$  \hspace{1cm} (96)

Recall that $r_{dms_{3}}$ is the vertical component (in the $s_3$ direction) of the $\vec{r}_{dms}$ vector. Then it is clear that there exists a closed and bounded set that divides the circumference of the buoy into a set of partitions such that:

$$r_{dms_{3}}(\phi = \pi) \leq r_{dms_{3}}(\phi_{n-1}) \leq \cdots \leq r_{dms_{3}}(\phi_2) \leq r_{dms_{3}}(\phi_1)$$  \hspace{1cm} (97)

The height of the $i^{th}$ disk is calculated as $\Delta r_{dms_{3,i}} = r_{dms_{3,i}} - r_{dms_{3,i-1}}$. Let $\bar{R}_i$ be the supremum (sup) of $r_{dms_{3}}$ in
the $i^{th}$ disk, that is:

$$\bar{R}_i = \sup_{\phi \in [\phi_{i-1}, \phi_i]} \left( r_{dms}^i (\phi) \right).$$  \hfill (98)

Likewise, let $\bar{r}_i$ be the infimum (inf) of $r_{dms}$ in the $i^{th}$ disk, that is:

$$\bar{r}_i = \inf_{\phi \in [\phi_{i-1}, \phi_i]} \left( r_{dms}^i (\phi) \right).$$  \hfill (99)

Consider the volume of the submerged disks ($V_s$), there is a lower and upper limit for the volume as shown in Fig. 2-b. It is possible to compute the lower Riemann sum, $L(r_{dms}, V_s)$, and upper Riemann sum, $U(r_{dms}, V_s)$, for the submerged volume as follows:

$$U(r_{dms}, V_s) = \sum_{i=1}^{n} \pi \bar{R}_i^2 \Delta r_{dms}$$  \hfill (100)

$$L(r_{dms}, V_s) = \sum_{i=1}^{n} \pi \bar{r}_i^2 \Delta r_{dms}$$  \hfill (101)

The difference between $U(r_{dms}, V_s)$ and $L(r_{dms}, V_s)$ is bounded; that is:

$$U(r_{dms}, V_s) - L(r_{dms}, V_s) < \epsilon_{V_s}$$  \hfill (102)

where $\epsilon_{V_s} > 0$. An accurate calculation of the submerged volume would have a small $\epsilon_{V_s}$. Clearly, as $n \to \infty$, the $\epsilon_{V_s} \to 0$. However, as $n$ increases the computational cost increases significantly.

### 4.2.2. Submerged Area Calculation

The internal surface area of the buoy was calculated using Riemann sums as well. The internal area is divided into $n$ number of horizontal slices with infinitesimal heights. Fig. (3) shows a schematic for the $i^{th}$ area partition. The supremum $\bar{R}_i$, and infimum $\bar{r}_i$, of the radii of this $i^{th}$ infinitesimal partition are calculated using Eqs. (103) and (104)

$$\bar{R}_i = \sup_{\phi \in [\phi_{i-1}, \phi_i]} \| r_{dms} (\phi) \|$$  \hfill (103)

$$\bar{r}_i = \inf_{\phi \in [\phi_{i-1}, \phi_i]} \| r_{dms} (\phi) \|$$  \hfill (104)

The height of the $i^{th}$ partition ($\Delta r_{dms} = r_{dms}^i - r_{dms}^{i-1}$) and the lower and upper surface Riemann sums are calculated as follows:

$$U(f, A_i) = \sum_{i=1}^{n} 2\pi \bar{R}_i^2 \Delta r_{dms}$$  \hfill (105)

$$L(f, A_i) = \sum_{i=1}^{n} 2\pi \bar{r}_i^2 \Delta r_{dms}$$  \hfill (106)

$$\exists \epsilon_A > 0 \text{ such that}$$

$$U(f, A_i) - L(f, A_i) < \epsilon_A$$  \hfill (107)

To increase the accuracy of the discretized area calculation, the number of partitions $n$ has to be big enough such that $\epsilon_A \to 0$ as $n \to \infty$. The final surface area of any portion is calculated as:

$$A_i = \frac{U(r_{dms}, A_i) + L(r_{dms}, A_i)}{2}$$  \hfill (108)

Noting that, using either of the areas calculated in Eq. (105) and (106) produce a first order accurate area calculation, on the other hand, Eq. (108) produces a second order accurate area calculation.
4.2.3. Generalized Hydro Force

The hydrodynamic force at the $i^{th}$ disk on the buoy surface can be expressed as:

\[
\vec{f}^\text{hydro}_i = -F^\text{hydro}_i \hat{e}_{3i}
\]

\[
= - \left( f_{exi} \cos c + \frac{g}{\cos(\pi - \psi_i)} \right) \hat{e}_{3i}
\]

where $f_{exi}$ is the buoyant force on the $i^{th}$ disk of the submerged volume. The generalized hydro force can be written in the following form:

\[
Q^\text{hydro}_j = \vec{f}^\text{hydro} \cdot \frac{\partial \vec{r}_{dma}}{\partial q_j}, \quad j = 1, 2, ..., 6 + N
\]

We can then write the hydro generalized forces using Eq. (109) and Eq. (111) as follows:

\[
Q_{1,i}^\text{hydro} = \vec{f}^\text{hydro} \cdot \frac{\partial \vec{r}_{dma}}{\partial \theta_1} = -F^\text{hydro}_i \hat{e}_{3i} \cdot 1 = -F^\text{hydro}_i 1^T \hat{e}_{3i} = -F^\text{hydro}_i \sin \psi_i
\]

\[
Q_{2,i}^\text{hydro} = \vec{f}^\text{hydro} \cdot \frac{\partial \vec{r}_{dash1}}{\partial \theta_1} = -F^\text{hydro}_i \hat{e}_{3i} \cdot 2 = 0
\]

\[
Q_{3,i}^\text{hydro} = \vec{f}^\text{hydro} \cdot \frac{\partial \vec{r}_{dash3}}{\partial \theta_3} = -F^\text{hydro}_i \hat{e}_{3i} \cdot 3 = -F^\text{hydro}_i 1^T \hat{e}_{3i} = -F^\text{hydro}_i \cos \psi_i
\]

\[
Q_{4,i}^\text{hydro} = \vec{f}^\text{hydro} \cdot \frac{\partial \vec{r}_{dash4}}{\partial \theta_4} = 0, \quad Q_{5,i}^\text{hydro} = \vec{f}^\text{hydro} \cdot \frac{\partial \vec{r}_{dash5}}{\partial \theta_5} = 0, \quad Q_{6,i}^\text{hydro} = \vec{f}^\text{hydro} \cdot \frac{\partial \vec{r}_{dash3}}{\partial \theta_3} = 0
\]

\[
Q_{7,i}^\text{hydro} = \vec{f}^\text{hydro} \cdot \frac{\partial \vec{r}_{dash7}}{\partial \eta_1} = -F^\text{hydro}_i \hat{e}_{3i} \cdot \Phi_j(:, 1), \quad ..., \quad Q_{6+N,i}^\text{hydro} = \vec{f}^\text{hydro} \cdot \frac{\partial \vec{r}_{dash3}}{\partial \eta_N} = -F^\text{hydro}_i \hat{e}_{3i} \cdot \Phi_j(:, N)
\]

The generalized hydro force on the buoy’s shell is then expressed as:

\[
Q^\text{hydro} = - \sum_{i=1}^{m} \begin{bmatrix} 0 & 0 & -F^\text{hydro}_i \cos \psi_i \end{bmatrix}^T \begin{bmatrix} [F^\text{hydro}_i \hat{e}_{3i} \cdot \Phi_j(:, 1) & ... & F^\text{hydro}_i \hat{e}_{3i} \cdot \Phi_j(:, N)]^T \end{bmatrix}
\]

where $m$ is the number of partitions on the buoy’s shell.

5. Results and Discussion

Simulation results for the dynamic model of a spherical axisymmetric VSB WEC are presented in this section. The VSB is assumed to be made of a flexible hollow shell vented to the atmosphere. The simulations were carried for both the FSB WEC and VSB WEC; a free vibrating shell was tested as well as shells constrained from deforming at the top and middle horizontal section. The simulations show also comparison between the energy converted by the VSB WEC and the FSB WEC. The equations of motion are solved using the MATLAB function ode45 which uses a six-step, fifth-order, Runge-Kutta method with variable time step.

The radius of the buoy is 2 m and the shell thickness is 0.01 m. The modulus of elasticity and the poissons ratio are $10^7$ MPa and 0.3, respectively. The total mass of the buoy is 17170 kg, and the wave excitation pressure is 1800 Pa with a period of 2.5 sec. The damping coefficient of the PTO unit is set to $c = 6000$ Ns/m. All the initial conditions for the simulation were set to zeros except for $\dot{r}_{sa} = [0 \ 0 \ -0.8]^T$ and the Legendre polynomial is truncated in 7 terms, i.e. $N = 7$.

The natural frequencies for the VSB WEC surface obtained numerically via the Rayleigh–Ritz method are compared to the theoretical values, obtained by solving Eqs. (55) and (59), in Table 1. It is observed that the numbers are identical for the breathing mode ($n = 0$), while for the other modes, there are negligible discrepancies.
Table 1
Natural Frequencies Resulted From Rayleigh-Ritz Method and The Analytical Full Formulation

| n | Rayleigh-Ritz (rad/sec) | Full Formulation (rad/sec) |
|---|------------------------|---------------------------|
| 0 | 51.4344987369114539941074327 | 51.43449873696200911258347325 |
| 1 | 62.1537718935904973224934563 | 62.99414453669143501883809221908 |
| 2 | 86.525268328571727015146252364 | 86.8239497354184130361012648791 |
| 3 | 115.855327987859833194701001 | 115.9418953668235019449639369397 |
| 4 | 146.5801646297458285379821900278 | 146.61008798738586733634292613715 |
| 5 | 177.8200397827077381407434864356 | 177.83264073210156652748992200941 |
| 6 | 209.2984266626109753725375048816 | 209.3048568518272746041475329548 |

Figure 4: Heave Displacement and The vertical Component of the Excitation Force

Three different VSB WEC designs are simulated in this article; the first design has an unconstrained buoy and is denoted in this section as VSWEC, the second design has buoy with constrained shells an angle $\phi_c = 0^\circ$, and is denoted by VSWEC$^0$, while the third design is constrained at angle $\phi_c = 90^\circ$ and is denoted as VSWEC$^9/2$.

Figure 4 shows the heave displacement and the vertical component of the excitation forces for all the four cases. The transient effects dies out after almost 25 seconds in the simulation, and the plots show the interval from 40 to 60 seconds. It is observed that the three VSB WECs designs have higher displacements compared to the FSB WEC. Larger motions are usually associated with higher energy conversion, and hence higher displacements are usually desirable. These results support the hypothesis of this research which is that flexible buoys would leverage the waves and behave like a rigid buoy that has reactive power; indeed the reactive power in this case is obtained from the waves themselves.
Table 2  
Comparison Between the Overall Performance of the Different Designs for WECs at Steady State

| Design      | Displacement (m pk-pk) | Velocity (m/sec pk-pk) | Control Force (N pk) | Power (Watt) | Generated Energy (KJ) |
|-------------|------------------------|------------------------|----------------------|--------------|-----------------------|
| FSWEC       | 1.1632                 | 0.9732                 | 109480               | 1421         | 43.42                 |
| VSWEC$^{\pi/2}$ | 1.1122               | 1.081                  | 104360               | 1757         | 53.31                 |
| VSWEC       | 1.019                  | 1.2266                 | 113160               | 2253         | 68.27                 |
| VSWEC$^0$   | 0.9844                 | 1.2338                 | 111220               | 2283         | 69.05                 |

It is also noticed that the peak-to-peak (pk-pk) displacement of the VSWEC$^0$ is slightly higher than that of the VSWEC; this is due to the bigger deformations of the bottom half of the VSWEC$^0$ buoy. On the other hand, the peak-to-peak displacement of the VSWEC$^{\pi/2}$ is less than the other VSB WEC designs and higher that the FSB WEC because the former allows for a pivoting point at angle $\phi_c = 90^\circ$. Also a phase shift of $\pi/4$ is noticed between the vertical component of the excitation force and the heave displacement occurred due to the high non-linearity of the VSB WEC.

Figure 6 shows the shapes of the FSB and the VSBs at their state of minimum vertical deformation (for highlighting the difference in deformations, the deformation vector is multiplied by a factor of 10), in the $-\hat{c}_3$ direction. Fig 6 demonstrates the imposed constraint on the VSWEC$^0$ since it coincides with the FSB (i.e. no deformation) at the top point at $\phi_c = 0^\circ$, which is the constraint point. Likewise, Fig. 6 shows that the VSWEC$^{\pi/2}$ coincides with the FSB WEC (i.e. no deformation) at $\phi_c = 90^\circ$ which is the location of the imposed no-deformation constraint.

The heave velocities of the VSW WECs are higher than the FSB WEC as shown in Fig. 5-a. At steady state response, the waveform for the heave velocity for the VSWEC and the VSWEC$^0$ coincide over each other The PTO force shown in Fig. 5-b is calculated by multiplying $-c$ by the heave velocity as expressed in Eq.(91).

The total volume and total surface area change over time are shown in Figures 7-a and 7-b. The change in the volume and surface areas corresponds to the change in the buoyant force and excitation forces as discussed in subsection 4.2. It is noticed that the VSWEC$^0$ has the highest pk-pk change for the volume and areas change, and the VBWEC$^\pi$ has the least change in volume and surface area.

The generated power peaks for all the VSB WEC designs are higher than the FSB WEC as shown in Fig. 8-a; the lowest power peaks are generated by the FSB WEC. Moreover, the peaks of the VSWEC$^0$ and VSWEC are almost overlapping. Fig. 8-b shows the total harvested energy over a period of 60 seconds for all the four cases. Clearly there is multiple-fold increase in the harvested energy of a VSB WEC compared to a FSB WEC. The VSWEC$^0$ harvests 59% more energy with reference to the FSB WEC; the VSWEC and the VBWEC$^{\pi/2}$ harvested 57% and 22.8% more energy, respectively. The discussion of the effect of material properties on the VSB WEC is beyond the scope of the current article; however, its worth noting that several values of the modulus of elasticity were tested, and it is observed that the softer the buoy material, the more energy is harvested from the waves. Finally, Table 2 summarizes all the performance measures for the four test cases, where the generated energy is the energy harvested over a 60 seconds of simulation period.
Figure 5: Heaving Velocity and PTO Force
Figure 6: shapes of the FSB and the VSBs at their state of minimum vertical deformation
Figure 7: Volume and surface area cover time
Figure 8: Generated Power and Harvested Energy
6. Conclusion

The equations of motion of spherical Variable-Shape Buoy Wave Energy Converters, with axisymmetric deformations, were derived using a Lagrangian formulation in this paper. A Rayleigh-Ritz method along with the classical bending theory for stress-strain relations were used to approximate the equations of motion to finite-dimension equations of motion, in the six degrees of freedom. Holonomic constraints were imposed to limit the buoy to only-heave motion, and to enforce no-deformation at specific locations, to account for the power take-off unit installation flanges. The inner volume of the VSB WEC is assumed to be vented to the atmosphere to exclude any internal pressure variation effect on the buoys’ shell. The numerical results support the hypothesis of this work which is that a VSB WEC would harvest energy at a significantly higher rate compared to that of a FSB WEC, when both WECs use no reactive power. The VSB WEC with zero deformation at its highest vertical location harvested more energy than the VSB WEC with unconstrained shell deformations and the VSB WEC with shell constraint at the horizontal midsection.

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