Probabilistic linear widths of Sobolev space with Jacobi weights on $[-1, 1]$

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Abstract

Optimal asymptotic orders of the probabilistic linear $(n, \delta)$-widths of $\lambda_{n, \delta}(W^{r}_{\alpha, \beta}, \nu, L^{q}_{\alpha, \beta})$ of the weighted Sobolev space $W^{r}_{\alpha, \beta}$ equipped with a Gaussian measure $\nu$ are established, where $L^{q}_{\alpha, \beta}$, $1 \leq q \leq \infty$, denotes the $L^{q}$ space on $[-1, 1]$ with respect to the measure $(1 - x)^{\alpha}(1 + x)^{\beta}$, $\alpha, \beta > -1/2$.

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1 Introduction

This paper mainly focuses on the study of probabilistic linear $(n, \delta)$-widths of a Sobolev space with Jacobi weights on the interval $[-1, 1]$. This problem has been investigated only recently. For calculation of probabilistic linear $(n, \delta)$-widths of the Sobolev spaces equipped with Gaussian measure, we refer to [1–5]. Let us recall some definitions.

Let $K$ be a bounded subset of a normed linear space $X$ with the norm $\|\cdot\|_X$. The linear $n$-width of the set $K$ in $X$ is defined by

$$\lambda_n(K, X) = \inf_{L_n} \sup_{x \in K} \|x - L_n x\|_X,$$

where $L_n$ runs over all linear operators from $X$ to $X$ with rank at most $n$.

Let $W$ be equipped with a Borel field $B$ which is the smallest $\sigma$-algebra containing all open subsets. Assume that $\nu$ is a probability measure defined on $B$. Let $\delta \in [0, 1)$. The probabilistic linear $(n, \delta)$-width is defined by

$$\lambda_{n, \delta}(W, \nu, X) = \inf_{G_\delta} \lambda_n(W \setminus G_\delta, X),$$

where $G_\delta$ runs through all possible $\nu$-measurable subsets of $W$ with measure $\nu(G_\delta) \leq \delta$.

Compared with the classical case analysis (see [2] or [6]), the probabilistic case analysis, which reflects the intrinsic structure of the class, can be understood as the $\nu$-distribution of the approximation on all subsets of $W$ by $n$-dimensional subspaces and linear operators with rank $n$.

In his recent paper [7], Wang has obtained the asymptotic orders of probabilistic linear $(n, \delta)$-widths of the weighted Sobolev space on the ball with a Gaussian measure in a...
weighted $L_q$ space. Motivated by Wang’s work, this paper considers the probabilistic linear $(n, \delta)$-widths on the interval $[-1, 1]$ with Jacobi weights and determines the asymptotic orders of the probabilistic linear $(n, \delta)$-widths. The difference between the work of Wang and ours lies in the different choices of the weighted points for the proofs of discretization theorems.

2 Main results

Consider the Jacobi weights

$$w_{\alpha, \beta}(x) := (1 - x)^{\alpha}(1 + x)^{\beta}, \quad \alpha, \beta > -1/2.$$ 

Denote by $L_{p;\alpha, \beta} \equiv L_{p}(w_{\alpha, \beta})$, $1 \leq p < \infty$, the space of measurable functions defined on $[-1, 1]$ with the finite norm

$$\|f\|_{p;\alpha, \beta} := \left( \int_{-1}^{1} |f(x)|^p w_{\alpha, \beta}(x) \, dx \right)^{1/p}, \quad 1 \leq p < \infty,$$

and for $p = \infty$ we assume that $L_{\infty;\alpha, \beta}$ is replaced by the space $C[-1, 1]$ of continuous functions on $[-1, 1]$ with the uniform norm. Let $\Pi_n$ be the space of all polynomials of degree at most $n$. Denote by $\mathbb{P}_n$ the space of all polynomials of degree $n$ which are orthogonal to polynomials of low degree in $L_2(w_{\alpha, \beta})$. It is well known that the classical Jacobi polynomials $\{P_n^{(\alpha, \beta)}\}_{n=0}^\infty$ form an orthogonal basis for $L_2([-1, 1], w_{\alpha, \beta})$ and are normalized by $P_n^{(\alpha, \beta)}(1) = \binom{n+\alpha}{n}$ (see [8]). In particular,

$$\int_{-1}^{1} P_n^{(\alpha, \beta)}(x)P_n^{(\alpha, \beta)}(y)w_{\alpha, \beta}(x) \, dx = \delta_{n,m}h_n(\alpha, \beta),$$

where

$$h_n(\alpha, \beta) = \frac{\Gamma(\alpha + \beta + 2)}{\Gamma(\alpha + 1)\Gamma(\beta + 1)} \frac{\Gamma(n + \alpha + 1)\Gamma(n + \beta + 1)}{\Gamma(n + 1)\Gamma(n + \alpha + \beta + 1)} \sim n^{-1}$$

with constants of equivalence depending only on $\alpha$ and $\beta$. Then the normalized Jacobi polynomials $P_n(x)$, defined by

$$P_n(x) = \left(h_n^{(\alpha, \beta)}\right)^{-1/2} P_n^{(\alpha, \beta)}(x), \quad n = 0, 1, \ldots,$$

form an orthonormal basis for $L_2(\alpha, \beta)$, where the inner product is defined by

$$\langle f, g \rangle := \int_{-1}^{1} f(x)\overline{g(x)}w_{\alpha, \beta}(x) \, dx.$$ 

Denote by $S_n$ the orthogonal projector of $L_2(\alpha, \beta)$ onto $\Pi_n$ in $L_2(\alpha, \beta)$, which is called the Fourier partial summation operator. Consequently, for any $f \in L_2(\alpha, \beta)$,

$$f = \sum_{l=0}^{\infty} \langle f, P_l \rangle P_l, \quad S_n f := \sum_{l=0}^{n} \langle f, P_l \rangle P_l.$$  (2.1)
It is well known that (see Proposition 1.4.15 in [9]) \( p_n^{(\alpha, \beta)} \) is just the eigenfunction corresponding to the eigenvalues \(-n(n + \alpha + \beta + 1)\) of the second-order differential operator

\[
D_{\alpha, \beta} := (1 - x^2)D^2 - (\alpha - \beta + (\alpha + \beta + 2)x)D,
\]

which means that

\[
D_{\alpha, \beta} p_n^{(\alpha, \beta)}(x) = -n(n + \alpha + \beta + 1) p_n^{(\alpha, \beta)}(x).
\]

Given \( r > 0 \), we define the fractional power \((-D_{\alpha, \beta})^{r/2}\) of the operator \(-D_{\alpha, \beta}\) on \( f \) by

\[
(-D_{\alpha, \beta})^{r/2}(f) = \sum_{k=0}^{\infty} (k(k + \alpha + \beta + 1))^{r/2} \langle f, P_k \rangle P_k,
\]

in the sense of distribution. We call \( f^{(r)} := (-D_{\alpha, \beta})^{r/2} \) the \( r \)th order derivative of the distribution \( f \). It then follows that for \( f \in L_{2,\alpha,\beta}, r \in \mathbb{R} \), the Fourier series of the distribution \( f^{(r)} \) is

\[
f^{(r)} = \sum_{k=1}^{\infty} (k(k + \alpha + \beta + 1))^{r/2} \langle f, P_k \rangle P_k.
\]

Using this operator, we define the weighted Sobolev class as follows: For \( r > 0 \) and \( 1 \leq p \leq \infty \),

\[
W_{p,\alpha,\beta}^{r}([-1,1]) \equiv W_{p,\alpha,\beta}^{r} := \{ f \in L_{p,\alpha,\beta} : \| f \|_{W_{p,\alpha,\beta}^{r}} := \| f \|_{p,\alpha,\beta} + \| (-D_{\alpha, \beta})^{r/2}(f) \|_{p,\alpha,\beta} < \infty \},
\]

while the weighted Sobolev class \( BW_{p,\alpha,\beta}^{r} \) is defined to be the unit ball of \( W_{p,\alpha,\beta}^{r} \). When \( p = 2 \), the norm \( \| \cdot \|_{W_{2,\alpha,\beta}^{r}} \) is equivalent to the norm \( \| \cdot \|_{L_{2,\alpha,\beta}^{r}} \), and we can rewrite \( W_{2,\alpha,\beta}^{r} \) as

\[
W_{2,\alpha,\beta}^{r} = \overline{W}_{2,\alpha,\beta}^{r} := \left\{ f(x) = \sum_{k=0}^{\infty} \langle f, P_k \rangle P_k(x) : ||f||_{W_{2,\alpha,\beta}^{r}} := ||f, P_0||^2 + \langle f^{(r)}, f^{(r)} \rangle \right\}
\]

\[
= \langle f, P_0 \rangle^2 + \sum_{k=1}^{\infty} (k(k + \alpha + \beta + 1))^{r/2} \langle f, P_k \rangle^2 < \infty
\]

with the inner product

\[
\langle f, g \rangle_r := \langle f, P_0 \rangle \langle g, P_0 \rangle + \langle f^{(r)}, g^{(r)} \rangle.
\]

Obviously, \( \overline{W}_{2,\alpha,\beta}^{r} \) is a Hilbert space. We equip \( \overline{W}_{2,\alpha,\beta}^{r} = \overline{W}_{2,\alpha,\beta}^{s} \) with a Gaussian measure \( \nu \) whose mean is zero and whose correlation operator \( C_{\nu} \) has eigenfunctions \( P_l(x), l = 0,1,2,\ldots \), and eigenvalues

\[
\lambda_0 = 1, \quad \lambda_l = (l(l + \alpha + \beta + 1))^{-s/2}, \quad l = 1,2,\ldots,s > 1,
\]
that is,
\[ C_ν P_0 = P_0, \quad C_ν P_l = λ_l P_l, \quad l = 1, 2, \ldots. \]

Then (see [10], pp. 48-49),
\[ \langle C_ν f, g \rangle_r = \int_{W_{2,α,β}^r} \langle f, h \rangle_r \langle g, h \rangle_r ν(dh). \]

By Theorem 2.3.1 of [10] the Cameron-Martin space \( H(ν) \) of the Gaussian measure \( ν \) is \( W_{2,α,β}^{r+1/2} \), i.e.,
\[ H(ν) = W_{2,α,β}^{r+1/2}. \]

See [10] and [11] for more information about the Gaussian measure on Banach spaces. Throughout the paper, \( A(n, δ) ≍ B(n, δ) \) means \( A(n, δ) ≪ B(n, δ) \) and \( A(n, δ) \gg B(n, δ) \), \( A(n, δ) ≪ B(n, δ) \) means that there exists a positive constant \( c \) independent of \( n \) and \( δ \) such that \( A(n, δ) \leq cB(n, δ) \). If \( 1 \leq q \leq ∞ \), \( r > (2 + 2 \min[0, \max(α, β)])/(1/p − 1/q) \), the space \( W_{p,α,β}^r \) can be continuously embedded into the space \( L_{q,α,β} \) (see Lemma 2.3 in [12]).

Set \( ρ = r + \frac{1}{2} \). The main result of this paper can be formulated as follows.

**Theorem 2.1** Let \( 1 \leq q \leq ∞ \), \( δ \in (0, 1/2] \), and let \( ρ > 1/2 + (2 \max(α, β) + 1)(1/2 + 1/q) \). Then
\[ λ_{n, δ}(W_{2,α,β}^r, v, L_{q,α,β}) \approx \begin{cases} n^{1/2−p}(1 + n^{−\min[1/2,1/q]})(\ln(\frac{1}{δ}))^{1/2}, & 1 \leq q < ∞, \\ n^{1/2−p}(\ln(\frac{1}{δ}))^{1/2}, & q = ∞. \end{cases} \] (2.2)

For the proof of Theorem 2.1, the discretization technique is used (see [1, 4, 13, 14]). Since the known results of the probabilistic linear widths of the identity matrix on \( ℓ_m^\infty \) are inappropriate here, the probabilistic linear widths of diagonal matrices on \( ℓ_m^\infty \) are adopted for the proof of the upper estimates.

### 3 Main lemmas

Let \( ℓ_q^m \) \( (1 \leq q \leq ∞) \) denote the space \( ℓ_q^m \) equipped with the \( ℓ_q^m \)-norm defined by
\[ \|x\|_{ℓ_q^m} := \begin{cases} \left(\sum_{i=1}^m |x_i|^q\right)^{1/q}, & 1 \leq q < ∞, \\ \max_{1 \leq i \leq m} |x_i|, & q = ∞. \end{cases} \]

We identify \( ℓ_q^m \) with the space \( ℓ_2^m \), denote by \( ⟨x, y⟩ \) the Euclidean inner product of \( x, y \in ℓ_2^m \), and write \( \|·\|_2 \) instead of \( \|·\|_{ℓ_2^m} \).

Consider in \( ℓ_2^m \) the standard Gaussian measure \( γ_m \), which is given by
\[ γ_m(G) = (2π)^{−m/2} \int_G \exp\left(-\frac{|x|^2}{2}\right) dx, \]
where $G$ is any Borel subset in $\mathbb{R}^m$. Let $1 \leq q \leq \infty$, $1 \leq n < m$, and $\delta \in [0,1)$. The probabilistic linear $(n, \delta)$-width of a linear mapping $T : \mathbb{R}^m \to \ell_q^n$ is defined by

$$\lambda_{n,\delta}(T : \mathbb{R}^m \to \ell_q^n, \gamma_m) = \inf_{G_\delta} \inf_{T_n : \mathbb{R}^m \to \mathbb{R}^n} \sup_{x \in \mathbb{R}^m \setminus G_\delta} \|Tx - T_n x\|_{\ell_q^n},$$

where $G_\delta$ runs over all possible Borel subsets of $\mathbb{R}^m$ with measure $\gamma_m(G_\delta) \leq \delta$, and $T_n$ runs over all linear operators from $\mathbb{R}^m$ to $\ell_q^n$ with rank at most $n$.

Throughout the paper, $D$ denotes the $m \times m$ real diagonal matrix $\text{diag}(d_1, \ldots, d_m)$ with $d_1 \geq d_2 \geq \cdots \geq d_m > 0$, $D_n$ denotes the $m \times m$ real diagonal matrix $\text{diag}(d_1, \ldots, d_n, 0, \ldots, 0)$ with $1 \leq n \leq m$, and $I_m$ denotes the $m \times m$ identity matrix. Moreover, $(e_1, \ldots, e_m)$ denotes the standard orthonormal basis in $\mathbb{R}^m$:

$$e_1 = (1, 0, \ldots, 0), \quad \ldots, \quad e_m = (0, \ldots, 0, 1).$$

Now, we introduce several lemmas which will be used in the proof of Theorem 2.1.

**Lemma 3.1** (See [1]) If $1 \leq q \leq 2$, $m \geq 2n$, $\delta \in (0,1/2]$, then

$$\lambda_{n,\delta}(I_m : \mathbb{R}^m \to \ell_q^n, \gamma_m) \approx m^{1/q} + m^{1/q - 1/2} \sqrt{\ln(1/\delta)}. \quad (3.1)$$

(2) (See [4]) If $2 \leq q < \infty$, $m \geq 2n$, $\delta \in (0,1/2]$, then

$$\lambda_{n,\delta}(I_m : \mathbb{R}^m \to \ell_q^n, \gamma_m) \approx m^{1/q} + \sqrt{\ln(1/\delta)}. \quad (3.2)$$

(3) (See [5]) If $q = \infty$, $m \geq 2n$, $\delta \in (0,1/2]$, then

$$\lambda_{n,\delta}(I_m : \mathbb{R}^m \to \ell_q^n, \gamma_m) \approx \sqrt{\ln((m-n)/\delta)} \approx \sqrt{\ln m + \ln(1/\delta)}. \quad (3.3)$$

**Lemma 3.2** (See [7]) Assume that

$$\sum_{i=1}^m d_i^\beta \leq C(m, \beta) \quad \text{for some } \beta > 0.$$

Then, for $2 \leq q \leq \infty$, $m \geq 2n$, $\delta \in (0,1/2]$, we have

$$\lambda_{n,\delta}(D : \mathbb{R}^m \to \ell_q^n, \gamma_m) \ll \left( \frac{C(m, \beta)}{n + 1} \right)^{1/2} \begin{cases} (m^{1/q} + \sqrt{\ln(1/\delta)}), & 2 \leq q < \infty, \\ \sqrt{\ln m + \ln(1/\delta)}, & q = \infty. \end{cases} \quad (3.4)$$

Let $\xi_j = \cos \theta_j$, $1 \leq j \leq 2n$, denote the zeros of the Jacobi polynomial $p_{2n}^{(\alpha, \beta)}(t)$, ordered so that

$$0 =: \theta_0 < \theta_1 < \cdots < \theta_{2n} < \theta_{2n+1} := \pi.$$

Let $\lambda_{2n}(t)$ be the Christoffel function and $b_j = \lambda_{2n}(\xi_j)$. Denote

$$W(n; \xi_j) = (1 - x + n^{-2})^\alpha \frac{1}{2} (1 - x + n^{-2})^\beta \frac{1}{2}.$$
It is well known uniformly (see [15])
\[ \theta_{j+1} - \theta_j \asymp n^{-1}, \quad \theta_j \asymp j n^{-1} \quad (1 \leq j \leq 2n), \]
and also
\[ b_j \asymp n^{-1} w_{\alpha, \beta}(\xi_j) (1 - \xi_j^2)^{1/2} \asymp n^{-1} W(n; \xi_j), \]
where the constants of equivalence depend only on \( \alpha, \beta \) (see [16] or [17]).

The following lemma is well known as Gaussian quadrature formulae.

**Lemma 3.3** (See [8]) For each \( n \geq 1 \), the quadrature
\[
\int_{-1}^{1} f(x) w_{\alpha, \beta}(x) \, dx \asymp \sum_{j=1}^{2n} b_j f(\xi_j)
\]
(3.5)
is exact for all polynomials of degree \( 4n - 1 \). Moreover, for any \( 1 \leq p \leq \infty, f \in \Pi_n \), we have
\[
\|f\|_{P, \alpha, \beta} \asymp \left( \sum_{j=1}^{2n} b_j |f(\xi_j)|^p \right)^{1/p} .
\]
(3.6)
An equivalence like (3.6) is generally called a Marcinkiewicz-Zygmund type inequality.

**Lemma 3.4** (See [12], Lemma 2.7) Let \( \alpha, \beta > -1/2, \sigma \in (0, \frac{1}{\max(\alpha, \beta) + 1}) \) and let \( b_j, 1 \leq j \leq n \), be defined as in Lemma 3.3. Then
\[
\sum_{j=1}^{n} b_j^{-\sigma} \ll n^{1+\sigma} .
\]
(3.7)

Let
\[
L_n(x, y) := \sum_{j=0}^{\infty} \eta \left( \frac{j}{n} \right) P_j(x) P_j(y), \quad x, y \in [-1, 1],
\]
(3.8)
where \( \eta \in C^\infty(\mathbb{R}) \) is a nonnegative \( C^\infty \)-function on \([0, \infty)\) supported in \([0, 2]\) with the properties that \( \eta(t) = 1 \) for \( 0 \leq t \leq 1 \) and \( \eta(t) > 0 \) for \( t \in [0, 2) \). For any \( f \in L_{2, \alpha, \beta} \), we define
\[
\delta_1(f) = S_2(f), \quad \delta_k(f) = S_{2^k}(f) - S_{2^{k-1}}(f) \quad \text{for } k = 2, 3, \ldots ,
\]
(3.9)
where \( S_n \) is given in (2.1). Denote by
\[
M_k(x, y) = \sum_{l=2^{k-1}+1}^{2^k} P_l(x) P_l(y)
\]
(3.10)
the reproducing kernel of the Hilbert space \( L_{2, \alpha, \beta} \cap \bigcap_{n=2^{k-1}+1}^{2^k} \Pi_n \). Then, for \( x \in [0, 1] \),
\[
\delta_k(f)(x) = \sum_{l=2^{k-1}+1}^{2^k} \int_{-1}^{1} f(x) P_l(x) P_l(y) w_{\alpha, \beta}(y) \, dy = \langle f, M_k(\cdot, x) \rangle .
\]
For $f \in \bigoplus_{n=2^{k-1}+1}^{2^k} \mathbb{P}_n$, 
\[ f(x) = \delta_k(f)(x) = [f, M_k(., x)]. \]

By Lemma 3.3, there exists a sequence of positive numbers $w_i = b_i \sim n^{-1} W_{\alpha, \beta}(n; \xi_i), 1 \leq i \leq 2^{k+1}$, for which the following quadrature formula holds for all $f \in \Pi_{2^{k+1}}$:
\[ \int_{-1}^{1} f(t) W_{\alpha, \beta}(t) dt = \sum_{i=1}^{2^{k+1}} w_i f(\xi_i). \tag{3.11} \]

Moreover, for any $1 \leq p \leq \infty, f \in \Pi_2$, we have
\[ \|f\|_{p, \alpha, \beta} \asymp \left( \sum_{i=1}^{2^{k+1}} w_i |f(\xi_i)|^p \right)^{1/p} = \|U_k(f)\|_{p, \alpha, \beta}, \]
where $w = (w_1, \ldots, w_{2^{k+1}}), U_k : \Pi_2 \mapsto \mathbb{R}^{2^{k+1}}$ is defined by
\[ U_k(f) = (f(\xi_1), \ldots, f(\xi_{2^{k+1}})), \tag{3.12} \]
and for $x \in \mathbb{R}^{2^{k+1}}$,
\[ \|x\|_{p, w}^{2^{k+1}} := \begin{cases} \left( \sum_{i=1}^{2^{k+1}} |x_i|^p w_i \right)^{1/p}, & 1 \leq p < \infty, \\ \max_{1 \leq i \leq 2^{k+1}} |x_i|, & p = \infty. \end{cases} \]

Let the operator $T_k : \mathbb{R}^{2^{k+1}} \mapsto \Pi_{2^{k+1}}$ be defined by
\[ T_k a(x) := \sum_{i=1}^{2^{k+1}} a_i w_i L_{2^{k+1}}(x, \xi_i), \tag{3.13} \]
where $a := (a_1, \ldots, a_{2^{k+1}}) \in \mathbb{R}^{2^{k+1}}$. It is shown in [12] that for $1 \leq q \leq \infty$,
\[ \|T_k a\|_{q, \alpha, \beta} \ll \|v\|_{q, w}^{2^{k+1}}. \tag{3.14} \]

For $f \in \Pi_{2^{k+1}}$, we have
\[ f(x) = \int_{-1}^{1} f(y) L_{2^{k+1}}(x, y) W_{\alpha, \beta}(x, y) dy = \sum_{i=1}^{2^{k+1}} w_i f(\xi_i) L_{2^{k+1}}(x, \xi_i) = T_k U_k(f)(x). \]

In what follows, we use the letters $S_k, R_k, V_k$ to denote $u_k \times u_k$ real diagonal matrixes as follows:
\[ S_k = \text{diag}(w_1^{1 \over 2}, \ldots, w_{2^{k+1}}^{1 \over 2}), \]
\[ R_k = \text{diag}(w_1^{1 \over q}, \ldots, w_{2^{k+1}}^{1 \over q}), \tag{3.15} \]
\[ V_k = \text{diag}(w_1^{1 \over 2+1 \over q}, \ldots, w_{2^{k+1}}^{1 \over 2+1 \over q}), \]
and use the letter $R_k^{-1}$ to represent the inverse matrix of $R_k$. 

Lemma 3.5 For any $z = (z_1, \ldots, z_{2^k+1}) \in \mathbb{R}^{2^k+1}$, we have

$$\left\| \sum_{j=1}^{2^k+1} w_j^{1/2} z_j M_k(\cdot, \xi_j) \right\|_{2, \alpha, \beta} \ll \|z\|_{2^{k+1}}.$$

(3.16)

where $M_k(x, y)$ is given in (3.10), and $(\xi_1, \ldots, \xi_{2^k+1})$ is defined as above.

Proof Denote by $K$ the set

$$\left\{ g \in \bigoplus_{j=2^{k-1}}^{2^k} P_j : \|g\|_{2, \alpha, \beta} \leq 1 \right\}.$$

Since

$$\sum_{j=1}^{2^k+1} w_j^{1/2} z_j M_k(\cdot, \xi_j) \in L_{2, \alpha, \beta} \cap \left( \bigoplus_{j=2^{k-1}}^{2^k} P_j \right).$$

By the Riesz representation theorem and the Cauchy-Schwarz inequality, we have

$$\left\| \sum_{j=1}^{2^k+1} w_j^{1/2} z_j M_k(\cdot, \xi_j) \right\|_{2, \alpha, \beta} = \sup_{g \in K} \left| \sum_{j=1}^{2^k+1} w_j^{1/2} z_j M_k(\cdot, \xi_j), g \right|$$

$$= \sup_{g \in K} \left| \sum_{j=1}^{2^k+1} w_j^{1/2} z_j g(\xi_j) \right|$$

$$\leq \sup_{g \in K} \left( \sum_{j=1}^{2^k+1} |\xi_j|^2 \right)^{1/2} \left( \sum_{j=1}^{2^k+1} w_j |g(\xi_j)|^2 \right)^{1/2}$$

$$\ll \sup_{g \in K} \left( \sum_{j=1}^{2^k+1} |\xi_j|^2 \right)^{1/2} \|g\|_{2, \alpha, \beta} \leq \|z\|_{2^{k+1}}.$$

\[ \square \]

4 Proofs of main results

Before Theorem 2.1 is proved, we establish the discretization theorems which give the reduction of the calculation of the probabilistic widths.

Theorem 4.1 Let $1 \leq q < \infty$, $\alpha \in (0, 1)$, and let the sequences of numbers $\{n_k\}$ and $\{\sigma_k\}$ be such that $0 \leq n_k \leq 2^{k+1} =: m_k$, $\sum_{k=1}^{\infty} n_k \leq n$, $\sigma_k \in (0, 1)$, $\sum_{k=1}^{\infty} \sigma_k \leq \sigma$. Then

$$\lambda_{n, \sigma} \left( W_{2, \alpha, \beta}^{q}, L_q, \alpha, \beta \right) \leq \sum_{k=1}^{\infty} 2^{-k \sigma} \lambda_{n_k, \alpha_k} \left( V_k : \mathbb{R}^{m_k} \to l_q^{m_k}, \gamma_{m_k} \right).$$

(4.1)

Proof For convenience, we write

$$\lambda_{n_k, \alpha_k} := \lambda_{n_k, \alpha_k} \left( V_k : \mathbb{R}^{m_k} \to l_q^{m_k}, \gamma_{m_k} \right),$$
where $\gamma_{\nu k}$ is the standard Gaussian measure in $\mathbb{R}^{\nu k}$. Denote by $L_k$ a linear operator from $\mathbb{R}^{\nu k}$ to $\mathbb{R}^{\nu k}$ such that the rank of $L_k$ is at most $\nu_k$ and

$$
\gamma_{\nu k} \{ \{ y \in \mathbb{R}^{\nu k} | \| V_k y - L_k y \| > 2 \lambda_{\nu_k, \sigma_k} \} \} \leq \sigma_k.
$$

Then, for any $f \in W^r_{2,\alpha,\beta}$, by (3.8)-(3.10), (3.14) and (3.15) we have

$$
\| \delta_k(f) - T_k R_k^{-1} L_k S_k U_k \delta_k(f) \|_{q, \alpha, \beta} = \| T_k U_k \delta_k(f) - T_k R_k^{-1} L_k S_k U_k \delta_k(f) \|_{q, \alpha, \beta}
$$

$$
\leq \| U_k \delta_k(f) - R_k^{-1} L_k S_k U_k \delta_k(f) \|_{q, \alpha, \beta}
$$

$$
= \| V_k S_k U_k \delta_k(f) - L_k S_k U_k \delta_k(f) \|_{q, \alpha, \beta}.
$$

(4.2)

Let $y = S_k U_k \delta_k(f) = (w_1^k \delta_k(f)(\xi_1), \ldots, w_{\nu k}^k \delta_k(f)(\xi_{\nu k})) \in \mathbb{R}^{\nu k}$, for $x \in [-1, -1]$, $\delta_k(f)(x) = [f, M_k(x, \cdot)]_r = [f, M_k^{(-r, 0)}(\cdot, x)]_r$, where $M_k^{(r, 0)}(x, y)$ is the $r_1$-order partial derivative of $M_k(x, y)$ with respect to the variable $x$, $r_1 \in \mathbb{R}$. Since the random vector $f$ in $W^r_{2,\alpha,\beta}$ is a centered Gaussian random vector with a covariance operator $C_v$, the vector

$$
y = S_k U_k \delta_k(f) = (f, w_1^k M_k^{(-2r, 0)}(\cdot, \xi_1), \ldots, w_{\nu k}^k M_k^{(-2r, 0)}(\cdot, \xi_{\nu k}))_r
$$

in $\mathbb{R}^{\nu k}$ is a random vector with a centered Gaussian distribution $\gamma$ in $\mathbb{R}^{\nu k}$, and its covariance matrix $C_v$ is given by

$$
C_v = ([C_v(w_j^k M_k^{(-2r, 0)}(\cdot, \xi_i)), w_j^k M_k^{(-2r, 0)}(\cdot, \xi_i)]_r)_{i,j=1}^{\nu k}.
$$

Since for any $z = (z_1, \ldots, z_{\nu k}) \in \mathbb{R}^{\nu k}$,

$$
\sum_{j=1}^{\nu k} w_j^k \xi_j M_k^{(-2r, 0)}(\cdot, \xi_i) \in \bigoplus_{j=2^{k+1}+1}^{2^k} \mathbb{R}_j
$$

and

$$
[C_v(w_j^k M_k^{(-2r, 0)}(\cdot, \xi_i)), w_j^k M_k^{(-2r, 0)}(\cdot, \xi_i)]_r = [w_j^k M_k^{(-2r, 0)}(\cdot, \xi_i), w_j^k M_k^{(-2r, 0)}(\cdot, \xi_i)]_r
$$

$$
= [w_j^k M_k^{(-r, 0)}(\cdot, \xi_i), w_j^k M_k^{(-r, 0)}(\cdot, \xi_i)],
$$

by Lemma 3.5 we get

$$
\int_{\mathbb{R}^{\nu k}} (y, z)^2 \gamma(dy) = z C_v z^T = \sum_{i,j=1}^{\nu k} z_i z_j [w_j^k M_k^{(-r, 0)}(\cdot, \xi_i), w_j^k M_k^{(-r, 0)}(\cdot, \xi_i)]_r
$$

$$
= \left( \sum_{j=1}^{\nu k} w_j^k \xi_j M_k^{(-r, 0)}(\cdot, \xi_i), \sum_{j=1}^{\nu k} w_j^k \xi_j M_k^{(-r, 0)}(\cdot, \xi_i) \right)
$$
\[
= \left\| \sum_{j=1}^{m_k} w_j^\frac{1}{2} M_k(\cdot, \xi_j) \right\|_2^2 \approx 2^{-2k\rho} \left\| \sum_{j=1}^{m_k} w_j^\frac{1}{2} z_j M_k(\cdot, \xi_j) \right\|_2^2 \\
\ll 2^{-2k\rho} \|z\|_{L^q}^{m_k} = 2^{-2k\rho} \int_{\mathbb{R}^{m_k}} \|y\|^2 \gamma_{m_k}(dy).
\]

(4.3)

Now we consider the subset of \(W^r_{2,a,\beta}\)

\[G_k := \{ f \in W^r_{2,a,\beta} | \| \delta_k(f) - T_k R_k^{-1} L_k S_k U_k \delta_k(f) \|_{L^q} > 2c_1 c_2 2^{-k\rho} \lambda_{m_k, \sigma_k} \},\]

where \(c_1, c_2\) are the positive constants given in (4.2), (4.3). Then by (4.2) we get

\[v(G_k) \leq v(\{ f \in W^r_{2,a,\beta} | \| V_k S_k U_k \delta_k(f) - L_k S_k U_k \delta_k(f) \|_{L^q} > 2c_2 2^{-k\rho} \lambda_{m_k, \sigma_k} \})
\]

\[= \gamma(\{ y \in \mathbb{R}^{m_k} | \| V_k y - L_k y \|_{L^q} > 2c_2 2^{-k\rho} \lambda_{m_k, \sigma_k} \}).\]

Note that for any \(t > 0\), the set \(\{ y \in \mathbb{R}^{m_k} : \| V_k y - L_k y \|_{L^q} \leq t \}\) is convex symmetric. It then follows by Theorem 1.8.9 in [10] and (4.3), we have

\[v(G_k) \leq \gamma(\{ y \in \mathbb{R}^{m_k} : \| V_k y - L_k y \|_{L^q} > 2c_2 2^{-k\rho} \lambda_{m_k, \sigma_k} \})
\]

\[\leq \lambda(\{ y \in \mathbb{R}^{m_k} : \| V_k y - L_k y \|_{L^q} > 2c_2 2^{-k\rho} \lambda_{m_k, \sigma_k} \})
\]

\[\leq \gamma_{m_k}(\{ y \in \mathbb{R}^{m_k} : \| V_k y - L_k y \|_{L^q} > 2c_2 2^{-k\rho} \lambda_{m_k, \sigma_k} \}) \leq \sigma_k,\]

where \(\lambda\) is a centered Gaussian measure in \(\mathbb{R}^{m_k}\) with covariance matrix \(c_2^2 2^{-2k\rho} I_{m_k}\). Consider \(G = \bigcup_{k=1}^{\infty} G_k\) and the linear operator \(\tilde{T}_n\) on \(W^r_{2,a,\beta}\) which is given by

\[\tilde{T}_n f = \sum_{k=1}^{\infty} T_k R_k^{-1} L_k S_k U_k \delta_k(f).\]

Then

\[v(G) = v\left(\bigcup_{k=1}^{\infty} G_k\right) \leq \sum_{k=1}^{\infty} v(G_k) \leq \sum_{k=1}^{\infty} v(\sigma_k) \leq \sigma,\]

and

\[\text{rank } \tilde{T}_n \leq \sum_{k=1}^{\infty} \text{rank}(T_k R_k^{-1} L_k S_k U_k \delta_k) \leq \sum_{k=1}^{\infty} n_k \leq n.\]

Thus, according to the definitions of \(G, \tilde{T}_n,\) and \(L_k\), we obtain

\[\lambda_{n,\beta}(W^r_{2,a,\beta}, v, \| \cdot \|_{L^q}) = \sup_{f \in W^r_{2,a,\beta}, G} \| f - \tilde{T}_n f \|_{L^q} \]

\[\leq \sup_{f \in W^r_{2,a,\beta}, G} \sum_{k=1}^{\infty} \| \delta_k(f) - T_k R_k^{-1} L_k S_k U_k \delta_k(f) \|_{L^q} \]

\[\leq \sup_{f \in W^r_{2,a,\beta}, G} \sum_{k=1}^{\infty} \| \delta_k(f) - T_k R_k^{-1} L_k S_k U_k \delta_k(f) \|_{L^q} \]
\[
\leq \sum_{k=1}^{\infty} \sup_{f \in W_{2;\alpha}\beta} \|\delta_k(f) - T_k R_k L_k S_k U_k \delta_k(f)\|_{q;\alpha,\beta}
\ll \sum_{k=1}^{\infty} 2^{-kp} \chi_{m,\alpha_k},
\]
which completes the proof of Theorem 4.1.

Now we turn to the lower estimates. Assume that \(m \geq 6\) and \(b_1 m \leq n \leq 2b_1 m\) with \(b_1 > 0\) being independent of \(n\) and \(m\). Set \(\{x_j\}_{j=1}^{N} \subset \{x \in [-1,1] : |x| \leq 2/3\}\) and \(x_{j_1} - x_j = 3/m, j = 1, \ldots, N - 1\). Then \(M = N\) and
\[
\{x \in [-1,1] : |x - x_j| \leq 1/m\} \cap \{x \in [-1,1] : |x - x_i| \leq 1/m\} = \emptyset, \quad \text{if } i \neq j.
\]
We may take \(b_1 > 0\) sufficiently large so that \(N \geq 2n\). Let \(\psi^1\) be a \(C^\infty\)-function on \(\mathbb{R}\) supported in \([-1,1]\), and be equal to 1 on \([-2/3,2/3]\). Let \(\psi^2\) be a nonnegative \(C^\infty\)-function on \(\mathbb{R}\) supported in \([-1/2,1/2]\), and be equal to 1 on \([-1/4,1/4]\). Define
\[
\psi_j(x) = \psi^1(m(x - x_j)) - c_j \psi^2(m(x - x_j)),
\]
for some \(c_j\) such that \(\int_{-1}^1 \psi_i(x) W_{\alpha,\beta}(x) \, dx = 0, i = 1, \ldots, N\). Set
\[
A_N := \text{span}\{\psi_1, \ldots, \psi_N\} = \{F_a(x) = \sum_{j=1}^{N} a_j \psi_j(x) : a = (a_1, \ldots, a_N) \in \mathbb{R}^N\}.
\]
Clearly,
\[
\psi_j \in W_{2;\alpha}\beta, \quad \text{supp } \psi_j \subset \{x \in [-1,1] : |x - x_j| \leq 1/m\} \subset \{x \in [-1,1] : |x| \leq 5/6\},
\]
\[
\|\psi_j\|_{q;\alpha,\beta} \asymp \left(\int_{-2/3}^{2/3} |\psi_j(x)|^q \, dx\right)^{1/q} = \left(\int_{-2/3}^{2/3} |\psi^1(m(x - x_j)) - c_j \psi^2(m(x - x_j))|^q \, dx\right)^{1/q}
\]
\[
\asymp m^{-1/q}, \quad 1 \leq q \leq \infty, j = 1, \ldots, N,
\]
and
\[
\text{supp } \psi_j \cap \text{supp } \psi_i = \emptyset, \quad (i \neq j).
\]
It follows that for \(F_a \in A_n, a = (a_1, \ldots, a_N) \in \mathbb{R}^N\), we have
\[
\|F_a\|_{q;\alpha,\beta} \asymp \left(m^{-1} \sum_{j=1}^{N} |a_j|^q\right)^{1/q} = m^{-1/q} \|a\|_{2;\alpha,\beta}^q.
\]
(4.4)

For a nonnegative integer \(\nu = 0, 1, \ldots\), and \(F_a \in A_N, a = (a_1, \ldots, a_N) \in \mathbb{R}^N\), it follows from the definition of \(-D_{\alpha,\beta}\) that
\[
\text{supp}(-D_{\alpha,\beta})^{\nu} (\psi_j) \subset \{x \in [-1,1] : |x - x_j| \leq 1/m\}
\]
and

\[ \| (D_{a,\beta})^r (\varphi_i) \|_{q,a,\beta} \leq N^{2r-1/q}. \]

Hence, for \( 1 \leq q \leq \infty \) and \( F_a = \sum_{i=1}^{N} a_i \varphi_i \in A_N, \)

\[ \| (D_{a,\beta})^r (F_a) \|_{q,a,\beta} \leq m^{2r-1/q} \| a \|_{q,a,\beta}^r. \]

It then follows by the Kolmogorov type inequality (see Theorem 8.1 in [18]) that

\[ \left\| \mathcal{F}_a^{(\alpha)} \right\|_{q,a,\beta} = \left\| (D_{a,\beta})^{\alpha/2} (F_a) \right\|_{q,a,\beta} \]

\[ \leq \left\| (D_{a,\beta})^{1+|\alpha|} (F_a) \right\|_{q,a,\beta}^{1-\frac{\alpha}{q},m} \| F_a \|_{q,a,\beta}^{1+\frac{\alpha}{q},m} \]

\[ \leq m^{|\alpha|/q} \| a \|_{q,a,\beta}^{\alpha}. \]

(4.5)

For \( f \in L_{1,a,\beta} \) and \( x \in [-1,1], \) we define

\[ P_N(f)(x) = \sum_{j=1}^{N} \frac{\varphi_j(x)}{\| \varphi_j \|_{2,a,\beta}^2} \int_{-1}^{1} f(y) \varphi_j(y) W_{a,\beta}(y) \, dy \]

and

\[ Q_N(f)(x) = \sum_{j=1}^{N} \frac{\varphi_j(x)}{\| \varphi_j \|_{2,a,\beta}^2} \int_{-1}^{1} f(y) \varphi_j^{(\alpha)}(y) W_{a,\beta}(y) \, dy. \]

Clearly, the operator \( P_N \) is the orthogonal projector from \( L_{2,a,\beta} \) to \( A_N, \) and if \( f \in W^p_{2,a,\beta}, \)

then \( Q_N(f)(x) = P_N(f^p)(x). \) Also, using the method in [19], we can prove that \( P_N \) is the bounded operator from \( L_{q,a,\beta} \) to \( A_N \cap L_{q,a,\beta} \) for \( 1 \leq q \leq \infty, \)

\[ \| P_N(f) \|_{q,a,\beta} \ll \| f \|_{q,a,\beta}. \]

(4.6)

Since \( Q_N(f) \in A_N \) for \( f \in W^p_{2,a,\beta}, \) we have

\[ \| Q_N(f)^{(\alpha)} \|_{2,a,\beta} \ll m^{\alpha} \| Q_N(f) \|_{2,a,\beta} = m^{\alpha} \| P_N(f)^{(\alpha)} \|_{2,a,\beta} \ll m^{\alpha} \| f^{(\alpha)} \|_{2,a,\beta}. \]

(4.7)

**Theorem 4.2** Let \( 1 \leq q \leq \infty, \) \( \delta \in (0,1), \) and let \( N \) be given above. Then

\[ \lambda_{n,\delta}(W^r_{2,a,\beta}, L_{q,a,\beta}) \geq n^{1/2-p-1/q} \lambda_{n,\delta}(I_N : \mathbb{R}^N \to L^N_q, \gamma_N), \]

where \( N \approx n, N \geq 2n \) and \( \gamma_N \) is the standard Gaussian measure in \( \mathbb{R}^N. \)

**Proof** Let \( T_a \) be a bounded linear operator on \( W^r_{2,a,\beta} \) with rank \( T_a \leq n \) such that

\[ \nu(\{ f \in W^r_{2,a,\beta} : \| f - T_a f \|_{q,a,\beta} > 2\lambda_{n,\delta} \}) \leq \delta, \]
Applying (4.9) we assert that

\[ R_\nu(f)(f) = \langle A^* C_f, A^* C_f \rangle_{H(v)}, \quad f \in W_{2,\alpha,\beta}^\nu, \]

where \( C_f \) is the covariance of the measure \( \nu \), \( H(v) = W_{2,\alpha,\beta}^\rho \) is the Camera-Martin space of \( \nu \), and \( A^* \) is the adjoint of \( A \) in \( H(v) \) (see Theorem 3.5.1 of [10]). Furthermore, if the operator \( A \) also satisfies

\[ \| Af \|_{H(v)} \leq \| f \|_{H(v)}, \]

then

\[ R_\nu(f)(f) = \| A^* C_f \|_{H(v)}^2 \leq \| A^* \|^2 \| C_f \| \leq \langle C_f, C_f \rangle_{H(v)} = R_\nu(f)(f). \]

By Theorem 3.3.6 in [10], we get that for any absolutely convex Borel set \( E \) of \( W_{2,\alpha,\beta}^\nu \) there holds the inequality

\[ \nu(E) \leq \lambda(E). \]

Applying (4.7) we assert that

\[ \| Q_N(f) \|_{H(v)} = \| (Q_N(f))^{(\rho)} \|_{2,\alpha,\beta} \ll m^\rho \| f^{(\rho)} \|_{2,\alpha,\beta} = m^\rho \| f \|_{H(v)}. \]

Then there exists a positive constant \( c_3 \) such that

\[ \frac{1}{c_3 m^\rho} Q_N(f) \|_{H(v)} \leq \| f \|_{H(v)}. \]

Note that, for any \( t > 0 \), the set \( \{ f \in W_{2,\alpha,\beta}^\nu : \| f - T_n f \|_{q,\alpha,\beta} \leq t \} \) is absolutely convex. It then follows that

\[ \nu(\{ f \in W_{2,\alpha,\beta}^\nu : \| f - T_n f \|_{q,\alpha,\beta} < 2\lambda_n, \beta \}) \leq \lambda(\{ f \in W_{2,\alpha,\beta}^\nu : \| f - T_n f \|_{q,\alpha,\beta} < 2\lambda_n, \beta \}), \]

which leads to

\[ \nu(\{ f \in W_{2,\alpha,\beta}^\nu : \| f - T_n f \|_{q,\alpha,\beta} > 2\lambda_n, \beta \}) \geq \nu(\{ f \in W_{2,\alpha,\beta}^\nu : \| Q_N f - T_n Q_N f \|_{q,\alpha,\beta} > 2c_3 m^\rho \lambda_n, \beta \}). \]

Let \( L_N : \mathbb{R}^N \rightarrow A_N \) and \( J_N : A_N \rightarrow \mathbb{R}^N \) be defined by

\[ L_N(a)(x) = \sum_{i=1}^N a_i \psi_i(x), \quad a = (a_1, \ldots, a_N) \in \mathbb{R}^N \]

and

\[ J_N(F_a) = (a_1 \| \psi_1 \|_{2,\alpha,\beta}, \ldots, a_N \| \psi_N \|_{2,\alpha,\beta}), \quad F_a \in A_N. \]
We see at once that $L_N f_N(F_\alpha) = F_\alpha$ for any $F_\alpha \in A_N$. Set $y = (y_1, \ldots, y_N) \in \mathbb{R}^N$, where $y_j = \frac{1}{\|y_j\|^{2\alpha}} f_j(y_j^{(0)}).$ Then $y = f_N Q_N(f)$. Thus by (4.4) and $\|\psi_0\|_{2\alpha, \beta} \asymp m^{-\frac{1}{2}},$ we obtain

$$\|L_N(a)\|_{2\alpha, \beta} \asymp m^{-\frac{1}{4} + \frac{1}{2}} \|a\|_q^N. \tag{4.8}$$

Combining (4.6) with (4.8), we conclude that for any $f \in W^r_{2\alpha, \beta}$,

$$\|Q_N(f) - T_N Q_N(f)\|_{q, \alpha, \beta} \gg \|P_N(Q_N(f)) - P_N T_N Q_N(f)\|_{q, \alpha, \beta} = \|L_N J_N Q_N(f) - L_N J_N P_N T_N L_N J_N Q_N(f)\|_{q, \alpha, \beta},$$

$$\gg m^{-\frac{1}{4} + \frac{1}{2}} \|J_N Q_N(f) - J_N P_N T_N L_N J_N Q_N(f)\|_q^N,$$

$$\gg m^{-\frac{1}{4} + \frac{1}{2}} \|y - J_N P_N T_N L_N y\|_q^N.$$

Remark that $g_k = \frac{\|k\|_{2\alpha, \beta}}{\|\phi\|_{2\alpha, \beta}}$, $k = 1, 2, \ldots, N$, is an orthonormal system in $L_{2\alpha, \beta}$ and $g_k \in H(v) = W^r_{2\alpha, \beta}$. Then the random vector $(f, g_1^{(0)}, \ldots, f, g_N^{(0)}) = y$ in $\mathbb{R}^N$ on the measurable space $(W^r_{2\alpha, \beta}, \nu)$ has the standard Gaussian distribution $r_N$ in $\mathbb{R}^N$. It then follows that

$$v\{f \in W^r_{2\alpha, \beta} : \|Q_N(f) - T_N Q_N(f)\|_{q, \alpha, \beta} > 2c_3 m^\rho \lambda_{n, \delta}\}$$

$$\geq v\{f \in W^r_{2\alpha, \beta} : \|y - T_N P_N T_N L_N y\|_q > c_4 m^\rho \lambda_{n, \delta}\}$$

$$= r_N\{y \in \mathbb{R}^N : \|y - T_N P_N T_N L_N y\|_q > c_4 m^\rho \lambda_{n, \delta}\}$$

$$= r_N(G),$$

where $c_4$ is a positive constant. Clearly, $\text{rank}(J_N P_N T_N L_N) \leq n$ and

$$r_N(G) \leq v\{f \in W^r_{2\alpha, \beta} : \|f - T_N f\|_{q, \alpha, \beta} > 2\lambda_{n, \delta}\} \leq \delta.$$

Consequently,

$$\lambda_{n, \delta}(I_N : \mathbb{R}^N \to \mathbb{R}^N_q, r_N) = \inf_G \sup_{x \in \mathbb{R}^N_G} \|I_N x - T_N x\|_q^N \leq \sup_{y \in \mathbb{R}^N_G} \|I_N y - J_N P_N T_N L_N y\|_q^N \ll m^{\rho - \frac{1}{4} + \frac{1}{2} \lambda_{n, \delta}},$$

which implies

$$\lambda_{n, \delta}(W^r_{2\alpha, \beta}, L_{q, \alpha, \beta}) \ll m^{\rho - \frac{1}{4} + \frac{1}{2} \lambda_{n, \delta}}(I_N : \mathbb{R}^N \to \mathbb{R}^N_q, r_N),$$

$$\asymp N^{\rho - \frac{1}{4} + \frac{1}{2} \lambda_{n, \delta}}(I_N : \mathbb{R}^N \to \mathbb{R}^N_q, r_N).$$

This completes the proof of Theorem 4.2. \hfill \Box

Now, we are in a position to prove Theorem 2.1.
Proof. For the lower estimates, using Theorem 4.2 and Lemma 3.1, we have for $1 \leq q \leq 2$

$$
\lambda_{n, \delta}(W^r_{2a, b, 1}, L_{q,a,b}) \gg n^{-e+1/2-1/q}\lambda_{n, \delta}(I_N : \mathbb{R}^N \rightarrow l^N_q , \gamma_N)
\asymp n^{-e+1/2-1/q}\left(N^{1/q} + N^{1/q-1/2} \left(\ln \left( \frac{1}{\delta} \right) \right)^{1/2}\right)
\asymp n^{1/2-\rho}\left(1 + n^{-1/2} \left(\ln \left( \frac{1}{\delta} \right) \right)^{1/2}\right).
$$

For $2 \leq q < \infty$, we have

$$
\lambda_{n, \delta}(W^r_{2a, b, 1}, L_{q,a,b}) \gg n^{-e+1/2-1/q}\left(n^{1/q} + \left(\ln \left( \frac{1}{\delta} \right) \right)^{1/2}\right)
\asymp n^{1/2-\rho}\left(1 + n^{-1/2} \left(\ln \left( \frac{1}{\delta} \right) \right)^{1/2}\right).
$$

And for $q = \infty$,

$$
\lambda_{n, \delta}(W^r_{2a, b, 1}, L_{q,a,b}) \gg n^{-e+1/2-1/q}\left(\ln m + \ln \left( \frac{1}{\delta} \right) \right)^{1/2}
= n^{1/2-\rho}\left(\ln \left( \frac{m}{\delta} \right) \right)^{1/2}.
$$

It remains to prove the upper estimates. For $2 \leq q \leq \infty$ and any fixed natural number $n$, assume $C_12^m \leq n \leq C_12^m$ with $C_1 > 0$ to be specified later. We may take sufficiently small positive numbers $\varepsilon > 0$ such that $\rho > \frac{1}{2} + (1 + \varepsilon)(2\max\{\alpha, \beta\} + 1 + \varepsilon)(\frac{1}{2} - \frac{1}{4})$. Set

$$
n_j = \begin{cases} 
2^{j+1}, & \text{if } j \leq m, \\
2^{j+1}2^{(1+\varepsilon)(m-j)-1}, & \text{if } j > m,
\end{cases}
$$

and

$$
\delta_j = \begin{cases} 
0, & \text{if } j \leq m, \\
\delta 2^{m-j}, & \text{if } j > m.
\end{cases}
$$

Then

$$
\sum_{j \geq 0} n_j \ll \sum_{j \leq m} 2^j + \sum_{j > m} 2^{m(1+\varepsilon)-j} \ll 2^m
$$

and

$$
\sum_{j \geq 0} \delta_j = \delta \sum_{j \leq m} 2^{m-j} \leq \delta.
$$

Thus, we can take $C_1$ sufficiently large so that

$$
\sum_{j=0}^\infty n_j \leq C_1 2^m \leq n.
$$
It follows from Lemma 3.4 for \( \tau \in (0, \frac{1}{(2 \max(\alpha, \beta) + 1)/2 - 1/2 - 1/q}) \), \( 2 \leq q \leq \infty \),

\[
\sum_{j=1}^{n} b_j^{-\tau(1/2 - 1/q)} \ll 2^{k(1 + \tau(1/2 - 1/q))} = 2^{-k + \tau(1/2 - 1/q)}.
\]

If \( j \leq m \), then \( n_j = 2^{j+1} \), and hence \( \lambda_{n_j}(V_j : \mathbb{R}^{2^{j+1}} \to \ell^q_{2^{j+1}}, \gamma_{2^{j+1}}) = 0 \). If \( j > m \), then taking \( \frac{1}{\tau} = (2 \max(\alpha, \beta) + 1 + \varepsilon)(1/2 - 1/q) \) and applying Lemma 3.2, Theorem 4.1, we obtain for \( 2 \leq q < \infty \),

\[
\lambda_{n_j}(V_j : \mathbb{R}^{2^{j+1}} \to \ell^q_{2^{j+1}}, \gamma_{2^{j+1}})
\ll \left( \frac{C(m, \tau)}{n_j + 1} \right)^{1/\tau} \left( 2^{(j+1)/q} + \sqrt{\frac{1}{\delta}} \right)
\ll 2^{(1/2 - 1/q)(2 - (1 + \varepsilon)(m - j)/(2 \max(\alpha, \beta) + 1 + \varepsilon))(1/2 - 1/q)} \left( 2^{\frac{j}{q}} + \sqrt{\frac{1}{\delta}} \right),
\]

which yields

\[
\lambda_{n, \delta}(W_{2, \alpha, \beta}^r, \nu, L_{\alpha, \beta})
\ll \sum_{j=m+1}^{\infty} 2^{-j/p} 2^{(1/2 - 1/q)(2 - (1 + \varepsilon)(m - j)/(2 \max(\alpha, \beta) + 1 + \varepsilon))(1/2 - 1/q)} \left( 2^{\frac{j}{q}} + \sqrt{\frac{1}{\delta}} \right)
\ll 2^{-m(p - 1/2)} \left( 2^m + \sqrt{\frac{1}{\delta}} \right) \approx n^{1/2 - p} \left( 1 + n^{-1/q} \sqrt{\frac{1}{\delta}} \right). \tag{4.9}
\]

For \( q = \infty \), by Lemma 3.2 we get

\[
\lambda_{n, \delta}(V_j : \mathbb{R}^{2^{j+1}} \to \ell^\infty_{2^{j+1}}, \gamma_{2^{j+1}}) \ll \left( \frac{C(2^{j+1}, \tau)}{n_j + 1} \right)^{1/\tau} \sqrt{\ln 2^{j+1} + \frac{1}{\delta}}
= 2^{j/2 - (1 + \varepsilon)(m - j)/(2 \max(\alpha, \beta) + 1 + \varepsilon)/2} \sqrt{j + \ln \frac{1}{\delta}},
\]

then applying Theorem 4.1, we obtain

\[
\lambda_{n, \delta}(W_{2, \alpha, \beta}^r, \nu, L_{\infty, \alpha, \beta}) \ll \sum_{j=m+1}^{\infty} 2^{-j/p} 2^{j/2 - (1 + \varepsilon)(m - j)/(2 \max(\alpha, \beta) + 1 + \varepsilon)/2} \sqrt{j + \ln \frac{1}{\delta}}
\ll 2^{-m(p - 1/2)} \sqrt{m + \ln \frac{1}{\delta}} \approx n^{1/2 - p} \sqrt{\frac{n}{\delta}}. \tag{4.10}
\]

To finish the proof of the upper estimates, we only need to show that, for \( 1 \leq q < 2 \),

\[
\lambda_{n, \delta}(W_{2, \alpha, \beta}^r, \nu, L_{q, \alpha, \beta}) \ll \lambda_{n, \delta}(W_{2, \alpha, \beta}^r, \nu, L_{2, \alpha, \beta}) \ll n^{1/2 - p} \left( 1 + n^{-1} \sqrt{\frac{1}{\delta}} \right)^{1/2}.
\]

Theorem 2.1 is proved. \( \Box \)
5 Conclusions

In this paper, optimal estimates of the probabilistic linear \((n, \delta)\)-widths of the weighted Sobolev space \(W^{r}_{\alpha, \beta}[\mathbb{R}^d, \mu] \) on \([-1, 1]\) are established. This kind of estimates play an important role in the widths theory and have a wide range of applications in the approximation theory of functions, numerical solutions of differential and integral equations, and statistical estimates.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this article. All the authors read and approved the final manuscript.

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