Giant number fluctuations in dry active polar fluids: A shocking analogy with lightning rods

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Abstract

The hydrodynamic equations of dry active polar fluids (i.e., moving flocks without momentum conservation) are shown to imply giant number fluctuations. Specifically, the rms fluctuations $\sqrt{\langle (\delta N)^2 \rangle}$ of the number $N$ of active particles in a region containing a mean number of active particles $\langle N \rangle$ scales according to the law $\sqrt{\langle (\delta N)^2 \rangle} = K' \sqrt{\langle N \rangle}$ with $\phi(d) = \frac{7}{10} + \frac{1}{5d}$ in $d \leq 4$ spatial dimensions. This is much larger than the “law of large numbers” scaling $\sqrt{\langle (\delta N)^2 \rangle} = K \sqrt{\langle N \rangle}$ found in most equilibrium and non-equilibrium systems. In further contrast to most other systems, the coefficient $K'$ also depends singularly on the shape of the box in which one counts the particles, vanishing in the limit of very thin boxes. These fluctuations arise not from large density fluctuations - indeed, the density fluctuations in polar ordered dry active fluids are not in general particularly large - but from long ranged spatial correlations between those fluctuations. These are shown to be closely related in two spatial dimensions to the electrostatic potential near a sharp upward pointing conducting wedge of opening angle $\frac{3\pi}{8} = 67.5^\circ$, and in three dimensions to the electrostatic potential near a sharp upward pointing charged cone of opening angle $37.16^\circ$. This very precise prediction can be stringently tested by alternative box counting experiments that directly measure this density-density correlation function.

I. INTRODUCTION

Non-equilibrium systems in general, and active matter in particular, can exhibit many novel behaviors impossible in equilibrium systems. One of the most striking examples is the existence of long-ranged order associated with a broken continuous symmetry in two
dimensions (2D) – a phenomenon forbidden in equilibrium systems by the Mermin-Wagner theorem [1]. Collective motion, or “flocking”, can therefore exist in active matter, even in two dimensions [2–6].

Another striking phenomenon that can occur in active matter is Giant number fluctuations (GNF). These were first predicted to occur for dry (i.e., non-momentum conserving) apolar active fluids (also known as “active nematics”) [7, 8]. It was later noted that these should also occur in dry polar active fluids (also known as “ferromagnetic flocks”) [9]).

The phenomenon of Giant number fluctuations can be detected simply by counting, as follows:

Within a large polar ordered dry active fluid, identify some smaller sub-volume, which I’ll hereafter call the “counting box”, that is still large enough to contain an enormous number $N$ of particles. Count the number of particles in it. Repeat this count in the same volume many times, as the system evolves. Once enough statistics have been collected, determine the mean number $<N>$ of particles, and its rms fluctuation $\sqrt{< (\delta N)^2 >}$, where $\delta N \equiv N - <N>$. Now repeat this process with a sequence of progressively larger boxes. As one does so, both the mean number of particles $<N>$, and its variance $\sqrt{< (\delta N)^2 >}$, will increase.

In virtually all equilibrium systems [10], and most non-equilibrium systems, the result of such an analysis will be the so-called “law of large numbers”: $\sqrt{< (\delta N)^2 >} \propto \sqrt{N}$. But in polar ordered dry active fluid, I find that these fluctuations are far larger.

Specifically, I derive the existence of these Giant number fluctuations directly from the hydrodynamic equations [2–6] of polar ordered dry active fluids. Making a plausible conjecture about the scaling laws implied by those hydrodynamic equations, I find that the rms number fluctuations $\sqrt{< (\delta N)^2 >}$ of the number $N$ of active particles in a region containing a mean number of active particles $<N>$ scales according to the law

$$\sqrt{< (\delta N)^2 >} = K' <N>^{\phi(d)}$$  \hspace{1cm} (1.1)

with

$$\phi(d) = \frac{7}{10} + \frac{1}{5d}$$  \hspace{1cm} (1.2)

in $d \leq 4$ spatial dimensions. Since $\phi(d) > 1/2$ in all $d < 4$ [11], this is much larger the “law of large numbers” scaling $\sqrt{< (\delta N)^2 >} \propto \sqrt{N}$ found in most equilibrium and non-equilibrium systems.
Even stranger is the fact that, in further contrast to most other systems, the coefficient $K'$ also depends singularly on the *shape* of the box in which one counts the particles, vanishing in the limit of very thin boxes.

Considering, for example, a three dimensional system in which I count particles in a box that I call a “needle shaped” counting volume. I define this as a long thin cylinder with its axis along the direction of the mean average velocity $<\mathbf{v}>$ of the polar ordered dry active fluid, with aspect ratio $\beta \equiv \frac{L_\parallel}{L_\bot} \gg 1$, where $L_\parallel$ and $L_\bot$ are respectively the length of the cylinder axis, and its radius respectively. For such a shape, I find

$$K' \propto \beta^{-23/30}.$$  

(1.3)

In contrast, for a “pancake shaped” box, which I define as a squat cylinder again with its axis along the direction of the mean average velocity $<\mathbf{v}>$ of the polar ordered dry active fluid, a direction I will hereafter refer to as $\hat{x}_\parallel$, but now with aspect ratio $\beta \equiv \frac{L_\parallel}{L_\bot} \ll 1$, where $L_\parallel$ and $L_\bot$ are respectively the length of the cylinder axis, and its radius respectively. For this shape, I find

$$K' \propto \beta^{8/15}.$$  

(1.4)

Similar results hold in $d = 2$. Here, I consider a rectangular counting box aligned with two of its edges parallel to $\hat{x}_\parallel$. In the “needle” limit, this will be the long axis, while for the “pancake” limit (which in $d = 2$ is just the needle rotated by 90 degrees), it will be the short axis.

Continuing to define $\beta \equiv \frac{L_\parallel}{L_\bot}$ in all cases, for the needle case $\beta \gg 1$, I find

$$K' \propto \beta^{-1/5},$$  

(1.5)

while for the pancake case $\beta \ll 1$, I find

$$K' \propto \beta^{1/5}.$$  

(1.6)

This has the appealingly symmetric feature that the giant number fluctuations are comparable if obtained from a long cylindrical counting box aligned with its long axis along $\hat{x}_\parallel$ as in the same box rotated by 90° to align with its long axis perpendicular to $\hat{x}_\parallel$. 
These fluctuations arise not from large density fluctuations - indeed, the density fluctuations in polar ordered dry active fluids are not in general particularly large - but from long ranged spatial correlations between those fluctuations. I find that in two spatial dimensions, these are related, by a simple anisotropic spatial rescaling, in the upper half plane, from the electrostatic potential near a sharp upward pointing conducting wedge of opening angle \( \frac{3\pi}{8} = 67.5^\circ \). Likewise, in three dimensions, an identical rescaling connects density correlations in the upper half space to the electrostatic potential near a sharp upward pointing charged cone of opening angle \( 37.16^\circ \).

Here, by upper, I mean in the direction of the mean velocity \(<v> \equiv v_0 \hat{x}_\parallel\) of the polar ordered dry active fluid, which is by definition non-zero in the ordered state, to which all these results are limited.

Specifically, I find

\[
C_\rho(r) = r^{-\alpha(d)}G_d(\theta_r) \tag{1.7}
\]

where \( r \) is the magnitude of \( r \) \((r = |r|)\), \( \theta_r \) is the angle between \( r \) and the direction of mean flock motion \( \hat{x}_\parallel \) (defined in \( d = 2 \) to run from \(-\pi\) to \( \pi \), and in \( d = 3 \) to run between \( 0 \) and \( \pi \)), and

\[
\alpha(d) = \frac{3d - 2}{5}. \tag{1.8}
\]

The function \( G_d(\theta_r) \) is given by

\[
G_d(\theta_r) \equiv \frac{\Upsilon_d(\theta_R)}{\left[ \frac{c^2}{(\gamma - v_2)^2} \cos^2 \theta + \sin^2 \theta \right]^{\alpha(d)/2}}, \tag{1.9}
\]

where

\[
\theta_R = \tan^{-1} \left( \frac{|\gamma - v_2|}{c} \tan(\theta_r) \right), \tag{1.10}
\]

with \( c, \gamma, \) and \( v_2 \) system dependent parameters defined in section II, and the function \( \Upsilon_d \) given by

\[
\Upsilon_2(\theta_R) = \begin{cases} 
B_2 \cos(4\theta_R/5) & , \ |\theta_R| < \frac{\pi}{2}, \\
\Upsilon_2(\pi - \theta_R) & , \ |\theta_R| > \frac{\pi}{2},
\end{cases} \tag{1.11}
\]
in two dimensions, and by

$$\Upsilon_3(\theta_R) = \begin{cases} 
B_3 P_{\frac{3}{2}}(\cos \theta_R), & \theta_R < \frac{\pi}{2}, \\
Y_3(\pi - \theta_R), & \theta_R > \frac{\pi}{2}.
\end{cases} \quad (1.12)$$

in three dimensions, where $P_\nu$ is the generalized Legendre function of non-integer index. Here $B_{2,3}$ are non-universal (i.e., system-dependent) constants.

This correlation function can be measured directly in a box counting experiment by correlating the number fluctuations in one box with those in a different box separated from the first by a displacement $\delta r$ whose magnitude $|\delta r|$ is much greater than the largest linear extent of either box.

In the next section, I’ll derive these results.

II. DERIVATION

My starting point is the continuum theory for a collection of self-propelled active particles moving without momentum conservation (i.e., a polar ordered dry active fluid) introduced in refs. [2–6]. This theory takes the form of the following equations of motion for the velocity field $v$ and number density $\rho$ of the active particles:

$$\partial_t v + \lambda_1 (v \cdot \nabla) v + \lambda_2 (\nabla \cdot v) v + \lambda_3 \nabla (|v|^2) = U(|v|, \rho) v - \nabla P + D_B \nabla (\nabla \cdot v) + D_T \nabla^2 v + D_2 (v \cdot \nabla)^2 v + f \quad (2.1)$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (v \rho) = 0 \quad (2.2)$$

where all of the parameters $\lambda_i (i = 1 \to 4)$, $U(|v|)$, $D_{B,T,2}(\rho, |v|)$ and the “isotropic Pressure” $P(\rho, |v|)$ and the “anisotropic Pressure” $P_2(\rho, |v|)$ are, in general, functions of the density $\rho$ and the magnitude $|v|$ of the local velocity. Since I am interested in an ordered, moving state with a non-zero average velocity, I assume the $U$ term makes the local $v$ have a nonzero magnitude $v_0$ in the steady state, by the simple expedient of having $U > 0$ for $|v| \equiv v < v_0$, $U = 0$ for $v = v_0$, and $U < 0$ for $v > v_0$.

The diffusion constants (or viscosities) $D_{B,T,2}$ reflect the tendency of a localized fluctuation in the velocities to spread out because of the coupling between neighboring “birds”.

The $f$ term is a random driving force representing the noise. I assume it is Gaussian with white noise correlations:

$$< f_i(r, t) f_j(r', t') > = \Delta \delta_{ij} \delta^d(r - r') \delta(t - t') \quad (2.3)$$

where $\Delta$ is a constant, and $i, j$ denote Cartesian components. The pressure $P$ tends, as in an equilibrium fluid, to maintain the local number density $\rho(r)$ at its mean value $\rho_0$, and $\delta \rho = \rho - \rho_0$. The “anisotropic pressure” $P_2(\rho, |v|)$ in (2.1) is only allowed due to the non-equilibrium nature of the flock; in an equilibrium fluid such a term is forbidden, since Pascal’s Law ensures that pressure is isotropic. In the non-equilibrium steady state of a flock, no such constraint applies.

The final equation (2.2) is just conservation of bird number: we don’t allow our birds to reproduce or die on the wing. The interesting and novel results that arise when this constraint is relaxed by allowing birth and death while the flock is moving has been discussed elsewhere [12].

Since I am interested in an ordered, moving state with a non-zero average velocity, I assume the $U$ term makes the local $v$ have a nonzero magnitude $v_0$ in the steady state, by the simple expedient of having $U > 0$ for $v < v_0$, $U = 0$ for $v = v_0$, and $U < 0$ for $v > v_0$.

The hydrodynamic model embodied in equations (2.1), (2.2), and (2.3) is equally valid in both the “disordered” (i.e., non-moving) and “ferromagnetically ordered” (i.e., moving) state. Here I am interested in the “ferromagnetically ordered”, broken-symmetry phase which occurs when $U > 0$ for $v < v_0$, $U = 0$ for $v = v_0$, and $U < 0$ for $v > v_0$, as discussed earlier. In this state, I can expand the equations of motion (2.1) and (2.2) for small fluctuations $\delta v$ and $\delta \rho$ of the velocity and density about their mean values. That is, I write the velocity and density fields as:

$$v = (v_0 + \delta v_\parallel) \hat{x}_\parallel + v_\perp, \quad (2.4)$$

and

$$\rho = \rho_0 + \delta \rho, \quad (2.5)$$

where $v_0 \hat{x}_\parallel = < v >$ is the spontaneous average value of $v$ in the ordered phase, and the fluctuations $\delta v_\parallel$ and $v_\perp$ of $v$ about this mean velocity along and perpendicular to the direction of the mean velocity are assumed to be small, as are the fluctuations $\delta \rho$ of the density.
Expanding the equation of motion (2.1) in these small quantities $\delta v_\parallel, v_\bot$ and $\delta \rho$, and then eliminating the “fast” variable $\delta v_\parallel$ proves to be quite subtle; see [6] for details. The result is a pair of coupled equations of motion for the fluctuation $v_\bot(r, t)$ of the local velocity of the flock perpendicular to the direction of mean flock motion (which mean direction will hereafter denoted as ”$\parallel$”), and the departure $\delta \rho(r, t)$ of the density from its mean value $\rho_0$

$$\partial_t v_\bot + \gamma \partial_\parallel v_\bot + \lambda (v_\bot \cdot \nabla_\bot) v_\bot = -g_1 \delta \rho \partial_\parallel v_\bot - g_2 v_\bot \partial_\parallel \delta \rho - g_3 v_\bot \partial_t \delta \rho - \frac{c_0^2}{\rho_0} \nabla_\bot \delta \rho - g_4 \nabla_\bot (\delta \rho^2) + D_B \nabla_\bot (\nabla_\bot \cdot v_\bot) + D_T \nabla_\bot^2 v_\bot + D_\parallel \partial_\parallel^2 v_\bot + \nu_t \partial_\parallel \nabla_\bot \delta \rho + \nu_\parallel \partial_\parallel \nabla_\bot \delta \rho + f_\bot, \quad (2.6)$$

$$\partial_t \delta \rho + \rho_0 \nabla_\bot \cdot v_\bot + \lambda_\rho \nabla_\bot \cdot (v_\bot \delta \rho) + v_2 \partial_\parallel \delta \rho = D_\parallel \partial_\parallel^2 \delta \rho + D_{\rho_\parallel} \partial_\parallel (\nabla_\bot \cdot v_\bot) + \phi \partial_\parallel \partial_\parallel \delta \rho + w_1 \partial_\parallel (\delta \rho^2) + w_2 \partial_\parallel (|v_\bot|^2), \quad (2.7)$$

where $\gamma, \lambda, \lambda_\rho, c_0^2, g_{1,2,3,4}, w_{1,2}, D_{B,T,\parallel,\rho,\rho_\parallel}, \nu_t, v_2, \phi$, and $\rho_0$ are all phenomenological constants, which can be expressed in terms of the expansion coefficients of the various functions of $|v|$ and $\rho$ in (2.1). The interested reader is referred to [6] for those expressions.

The important fact about these equations is that they have many non-linearities that are relevant in the renormalization group (RG) sense. What “relevant in the RG sense” means in plain English is that these specific non-linear terms lead to different scaling behavior at long distances and times than predicted by the linearized version of those equations, which of course drop those terms. This modified scaling can be encapsulated by three anomalous scaling exponents: a “dynamical” exponent $z$, an “anisotropy” exponent $\zeta$, and a “roughness” exponent $\chi$ which characterizing the scalings of time $t$, $r_\parallel$ (distance along the direction of flock motion), and velocity $v_\bot$ with distance $r_\bot$ perpendicular to the direction of flock motion.

It proves prohibitively difficult to calculate these universal exponents $z, \zeta, \chi$ in $d < 4$, where the non-linear effects described above become important. However, if one is willing to conjecture that the dominant non-linearity in Eqn. (2.6) is $\lambda_0^1$, and that all of the other non-linearities are irrelevant, in the RG sense, below $d = 4$ (a possibility which is not ruled out by any calculation that has been done), then one can show [6] that the exponents $z, \zeta, \chi$ are given exactly, in $d = 2$, by

$$z = \frac{6}{5} \quad (2.8)$$
\[ \zeta = \frac{3}{5} \]  

(2.9)

\[ \chi = -\frac{1}{5} \]  

(2.10)

Furthermore, it can also be shown that for both “Malthusian” flocks\[12\] (that is, polar ordered dry active fluids without number conservation due to “birth and death” of the active particles) and incompressible polar ordered dry active fluids\[13\], for all dimensions \(2 \leq d \leq 4\), these exponents are given by

\[ z = 2\left(\frac{d + 1}{5}\right) \]  

(2.11)

\[ \zeta = \frac{d + 1}{5} \]  

(2.12)

\[ \chi = \frac{3 - 2d}{5} \]  

(2.13)

Note that these reduce to the values obtained by the aforementioned conjecture in \(d = 2\) for number conserving polar ordered dry active fluids.

It is therefore tempting to conjecture that these “canonical” exponents \(2.11\), \(2.12\), and \(2.13\) apply for compressible, number conserving flocks as well. For the remainder of this paper, I will do so, and use their values to obtain the scaling of real space and time density and number fluctuations for \(2 \leq d \leq 4\).

In general, the equal-time, spatially Fourier-transformed density-density correlation function predicted by these equations is

\[ C_\rho(q) \equiv \langle |\rho(q, t)|^2 \rangle \]  

(2.14)

is given by:

\[ C_\rho(q) = \frac{q^{2(1-\zeta)}}{(\gamma - v_2)^2 q_\parallel + c^2 q_\perp^2} f \left( \frac{q_\parallel}{\Lambda^{1-\zeta} q_\perp^\zeta} \right), \]  

(2.15)

where the scaling function \(f(x)\) has the limits

\[ f(x) \to \begin{cases} 
\text{constant} & \text{, } x \ll 1 \\
\text{a different constant } \times x^2 & \text{, } x \gg 1 
\end{cases} \]  

(2.16)
where $\Lambda$ is an ultraviolet cutoff wavevector[14]. This scaling function is proportional to the noise strength $\Delta$; see [6] for more details.

Multiplying both sides of (2.15) by $(\gamma - v_2)^2 q^2 + c^2 q^2_\perp$ and Fourier transforming back to real space shows that $C_\rho(r)$ obeys an anisotropic Poisson equation

\[
\left[(\gamma - v_2)^2 \partial^2_{||} + c^2 \nabla^2_\perp\right] C_\rho(r) = \nabla^2_\perp G(r),
\]

where the source term $G(r)$ is the Fourier transform of $q^2_\perp f\left(\frac{q_{\perp}}{\Lambda^1 - c q_{\perp}^2}\right)$. That is,

\[
G(r) = \int \frac{d^{d-1}q_{\perp} dq_{||}}{(2\pi)^d} q^{-2\zeta(d)} f\left(\frac{q_{||}}{\Lambda^{1 - c q_{\perp}^2}}\right) e^{i[q_{\perp} \cdot r_{\perp} + q_{||} r_{||}]}.
\]

It is straightforward to show from this expression that $G(r) = G(r_{\perp}, r_{||})$ itself has a simple scaling form. To see this, make the linear change of variables of integration in (2.18) from $q_{\perp}$ and $q_{||}$ to $Q_{\perp}$ and $Q_{||}$ defined via $q_{||} \equiv \frac{Q_{||}}{|r_{||}|}$ and $q_{\perp} \equiv \frac{Q_{\perp}}{(\Lambda |r_{||}|) r_{||}}$. This gives

\[
G(r) = |r_{||}|^{(1-d)/\zeta + 1} h \left(\frac{\Lambda r_{\perp}}{(\Lambda |r_{||}|)^{1/\zeta}}\right) = |r_{||}|^{2\chi/\zeta} h \left(\frac{\Lambda r_{\perp}}{(\Lambda |r_{||}|)^{1/\zeta}}\right),
\]

where the scaling function

\[
h(x) \equiv \int \frac{d^{d-1}Q_{\perp} dQ_{||}}{(2\pi)^d} Q^{-2\zeta} f\left(\frac{Q_{||}}{Q_{\perp}^{\chi}}\right) e^{i[Q_{\perp} \cdot \hat{r}_{\perp} x + Q_{||}] \Lambda^{-\omega}},
\]

with the utterly unimportant exponent $\omega = (1 - 1/\zeta)(d - 1 - 2\zeta)$. In deriving the second equality in (2.19), I’ve used the values of the canonical exponents to obtain $(1 - d)/\zeta + 1 = 2\chi/\zeta$, as the algebraically inclined reader can verify for herself using the expressions (2.11), (2.12), and (2.13) for the canonical values of the exponents.

Note that I expect $G(r)$ to depend only on $r_{||}$ when $\Lambda |r_{||}| \gg (\Lambda r_{\perp})^\zeta$, as it is at large $r$ for almost all directions of $r$. Hence, for most directions of $r$, $\nabla^2_\perp G(r)$ vanishes. The only exception to this is the thin sliver $|r_{||}| \lesssim (\Lambda r_{\perp})^\zeta \Lambda^{-1}$, which gets very thin compared to $r_{\perp}$ for $r_{\perp} \gg \Lambda^{-1}$, since $\zeta < 1$ for all $d < 4$.

This means that our anisotropic Poisson equation (2.17) has as a source on the right hand side a thin layer of charge lying very near the plane (or the line, in $d = 2$) perpendicular to the mean velocity. I will therefore model it as an \textit{infinitesimally} thin layer of charge, with charge density given by

\[
\sigma(r_{\perp}) = \int_{-\infty}^{\infty} \nabla^2_\perp G(r_{\perp}, r_{||}) dr_{||}.
\]
Using the scaling expression (2.19) for \( G(\mathbf{r}) \), I find

\[
\nabla^2 \perp G(r_\perp, r_\parallel) = |r_\parallel|^{2\chi/\zeta} \nabla^2 \perp h \left( \frac{\Lambda r_\perp}{(\Lambda |r_\parallel|)^{1/\zeta}} \right) = |r_\parallel|^{2\chi/\zeta} \frac{\Lambda^2}{(\Lambda |r_\parallel|)^{2/\zeta}} \left[ \frac{(d-2)}{u} h'(u) + h''(u) \right]
\]

\[
\equiv |r_\parallel|^{-4} Y(u),
\]

(2.22)

where I’ve defined the scaling variable

\[
u \equiv \frac{\Lambda r_\perp}{(\Lambda |r_\parallel|)^{1/\zeta}}
\]

(2.23)

and the scaling function

\[
Y(u) \equiv \Lambda^{2(1-1/\zeta)} \left[ \frac{(d-2)}{u} h'(u) + h''(u) \right].
\]

(2.24)

I’ve also used the fact that \( \frac{2(\chi-1)}{\zeta} = -4 \), as the skeptical reader can verify for himself by once again using the expressions (2.11), (2.12), and (2.13) for the canonical values of the exponents.

Using the last equality in (2.22) in my expression (2.21) for \( \sigma(r_\perp) \)) gives

\[
\sigma(r_\perp) = 2 \int_0^\infty Y(u) r_\parallel^{-4} dr_\parallel,
\]

(2.25)

where the scaling variable \( u \) continues to be given by (2.23), and I’ve used the fact that the scaling function \( Y(u) \) is an even function of \( r_\parallel \) to replace the integral over \( r_\parallel \) over the range \([-\infty, \infty]\) with twice the integral over the range \([0, \infty]\).

Solving (2.23) for \( r_\parallel \) in terms of \( u \) gives (for positive \( r_\parallel \), which is all I need)

\[
r_\parallel = \Lambda^{-1} \left( \frac{\Lambda r_\perp}{u} \right)^\zeta.
\]

(2.26)

Now changing variables of integration in (2.25) from \( r_\parallel \) to \( u \) (keeping in mind that the integral over \( r_\parallel \) is at constant \( r_\perp \)) gives

\[
\sigma(r_\perp) = A r_\perp^{-3\zeta},
\]

(2.27)

where I’ve defined the constant

\[
A = 2 \int_0^\infty Y(u) u^{3\zeta-1} du.
\]

(2.28)

Note that this constant is non-universal - that is, it depends on the hydrodynamic parameters of the particular flock we’re studying (through both the noise strength \( \Delta \) and the
ultraviolet cutoff \( \Lambda \). But for a given type of flocker, it is independent of position \( \mathbf{r} \), time \( t \), and the size of the flock.

Thus, our anisotropic Poisson equation (2.17) can be rewritten as

\[
\left[ (\gamma - v_2)^2 \partial^2 \parallel + c^2 \nabla^2 \perp \right] C_\rho(\mathbf{r}) = A r_\perp^{-3\zeta} \delta(r_\parallel). \tag{2.29}
\]

I recognize that at this point the more skeptical reader may be doubting this “infinitely thin sheet” approximation. In particular, she may be wondering whether the value of \( C_\rho(\mathbf{r}) \) found this way might be invalid within the region

\[
|r_\parallel| \lesssim (\Lambda r_\perp)^4 \Lambda^{-1}, \tag{2.30}
\]

within which the “charge” distribution does not look like a thin sheet. However, I will show later that this is no more a problem here than it is for real electrostatic problems involving charge layers, for which it is not necessary to consider the finite thickness of a real charge layer, since the “potential” cannot change appreciably over that thickness. I will likewise show a posteriori here that the correlation function \( C_\rho(\mathbf{r}) \) does not change appreciably (for large \( \mathbf{r} \)) over the thin sheet (2.30); hence, I can use the result of this thin sheet calculation for all \( \mathbf{r} \), even those within the sheet.

A consequence of this, as we’ll see, is that even though \( C_\rho(\mathbf{q}) \) is, as equation (2.15) shows, strongly anisotropic—indeed, it exhibits anisotropic scaling—\( C_\rho(\mathbf{r}) \) is nearly isotropic, and in particular is completely isotropic in its scaling.

Before proceeding, in the interests of making the analogy with electrostatics more perfect, I will anisotropically rescale lengths to make (2.29) an isotropic Poisson equation. Specifically, I’ll define a new vector \( \mathbf{R} \) via

\[
\mathbf{R}_\perp = \mathbf{r}_\perp, \quad R_\parallel = \frac{c R_\parallel}{|\gamma - v_2|}, \tag{2.31}
\]

In this new variable \( \mathbf{R} \), equation (2.29) becomes a completely isotropic Poisson equation:

\[
\nabla^2 \mathbf{R} C_\rho(\mathbf{R}) = A' R_\perp^{-3\zeta} \delta(R_\parallel), \tag{2.32}
\]

where I’ve defined \( A' = \frac{A}{|\gamma - v_2|^4} \).

By inversion symmetry, \( C_\rho(\mathbf{R}) \) must remain unchanged when \( R_\parallel \to -R_\parallel \). This will lead to a gradient discontinuity in \( C_\rho(\mathbf{R}) \) at the equatorial plane \( \theta = \pi/2 \). By the usual
“Gaussian pillbox” argument of electrostatics, the presence of a charge sheet is equivalent to a boundary condition at the equatorial plane:

$$\left( \nabla_R \right)_N C_\rho(R) = -\frac{1}{R} \left( \frac{\partial C_\rho}{\partial \theta_R} \right)^- (R, \theta_R = \pi/2) = \frac{A'}{2} R^{-3\zeta}.$$  \hspace{1cm} (2.33)

Here the superscript “−” on $$\left( \frac{\partial C_\rho}{\partial \theta_R} \right)^- (R, \theta_R = \pi/2)$$ denotes a derivative evaluated as $$\theta_R \to \pi/2$$ from below. The derivative as $$\theta_R \to \pi/2$$ from above has the opposite sign, due to the inversion symmetry of $$C_\rho$$.

So now I must satisfy the Poisson equation (2.32) subject to the boundary condition (2.33). I will seek a separable solution of the form

$$C_\rho(R) = R^{-\alpha} \Upsilon_d(\theta_R).$$ \hspace{1cm} (2.34)

By the inversion symmetry of $$C_\rho(R)$$, I know that

$$\Upsilon_d(\pi - \theta_R) = \Upsilon_d(\theta_R).$$ \hspace{1cm} (2.35)

This will lead to a slope discontinuity in $$\Upsilon_d(\theta_R)$$ at $$\theta = \pi/2$$, which is, of course, precisely what is generated by the thin charge layer.

Inserting the ansatz (2.34) into the boundary condition (2.33) gives

$$R^{-\alpha - 1} \Upsilon_d'(\theta_R = \pi/2) = \frac{A'}{2} R^{-3\zeta},$$ \hspace{1cm} (2.36)

which implies

$$\alpha + 1 = 3\zeta.$$ \hspace{1cm} (2.37)

This is obviously trivially solved to give:

$$\alpha(d) = 3\zeta(d) - 1 = \frac{3d - 2}{5},$$ \hspace{1cm} (2.38)

where in the last equality I have used the canonical value (2.12) for $$\zeta(d)$$.

So far, I have worked in completely general spatial dimension $$d$$. To proceed, I’ll now deal specifically with the two physical cases $$d = 2$$ and $$d = 3$$.

In $$d = 2$$, (2.38) gives $$\alpha = \frac{4}{5}$$. Requiring that the ansatz (2.34) obeys Laplace’s equation (2.32) away from the plane $$R_\parallel = 0$$ determines $$\Upsilon_2$$:

$$\Upsilon_2(\theta_R) = B_2 \cos(\alpha \theta_R) = B_2 \cos(4\theta_R/5),$$ \hspace{1cm} (2.39)
which is the result (1.11) quoted in the introduction. Fixing $B_2$ using the boundary condition (2.33) gives

$$B_2 = \frac{A'}{\sin\left(\frac{2\pi}{5}\right)} \approx 1.05A'. \quad \text{(2.40)}$$

Note that $C_\rho(R)$ is identical in the upper half plane (i.e., $-\frac{\pi}{2} < \theta_R < \frac{\pi}{2}$) to the solution for the electrostatic potential near a sharp upward pointing conducting wedge\[15\] of opening angle $\frac{2\pi}{5} = 67.5^\circ$. Of course, in the lower half plane $C_\rho(R)$ is just the mirror image of the upper half plane solution, since $C_\rho(R)$ is symmetric about the axis $R_\parallel = 0$.

In $d = 3$, I have $\alpha = \frac{7}{5}$, and requiring that the ansatz (2.34) obeys Laplace’s equation away from the plane $R_\parallel = 0$ determines $\Upsilon_3$:

$$\Upsilon_3(\theta_R) = B_3P_{\alpha-1}(\cos\theta_R) = B_3P_{\frac{2}{5}}(\cos\theta_R), \quad \text{(2.41)}$$

where $P_\nu$ is the generalized Legendre function of non-integer index. This is, of course, just the result (1.12) quoted in the introduction.

Fixing $B_3$ using the boundary condition (2.33) gives

$$B_3 = \left. -\frac{A'}{dP_{\frac{2}{5}}(\theta)} \right|_{\theta = \frac{\pi}{2}} = -\frac{5\Gamma\left(\frac{17}{10}\right)\Gamma\left(-\frac{1}{5}\right)}{7\sqrt{\pi}}A' \approx 2.13156A'. \quad \text{(2.42)}$$

Note that $C_\rho(R)$ is now identical in the upper half space (i.e., $0 < \theta_R < \frac{\pi}{2}$) to the electrostatic potential near a sharp upward pointing charged cone\[15\] of opening angle $37.16^\circ$. Of course, in the lower half space $C_\rho(R)$ is just the mirror image of the upper half space solution, since $C_\rho(R)$ is symmetric about the plane $R_\parallel = 0$.

Using the coordinate transformation (2.31) to rewrite the above results in terms of the real coordinates $r$, I have

$$C_\rho(r) = r^{-\alpha(d)}G_d(\theta_r) \quad \text{(2.43)}$$

where $r$ is the magnitude of $r$ ($r = |r|$), $\theta_r$ is the angle between $r$ and the direction of mean flock motion $\hat{x}_\parallel$,

$$G_d(\theta_r) \equiv \frac{\Upsilon_d(\theta_R)}{\left[\frac{c^2}{(\gamma-v_2)c^2} \cos^2 \theta + \sin^2 \theta\right]^{\alpha(d)/2}}, \quad \text{(2.44)}$$

which is just equation (1.9) of the introduction, with

$$\theta_R = \tan^{-1}\left(\frac{R_\perp}{R_\parallel}\right) = \tan^{-1}\left(\frac{\gamma - v_2}{c}\frac{r_\perp}{r_\parallel}\right) = \tan^{-1}\left(\frac{|\gamma - v_2|}{c}\tan(\theta_r)\right). \quad \text{(2.45)}$$
which is just (1.10) of the introduction, with the function $\Upsilon_d$ given by equations (1.11) and (1.12) quoted in the introduction for $d = 2$ and $d = 3$ respectively, and $\alpha(d)$ given by (2.38).

This result summarized by (2.43), (2.38), (1.11), (1.12), and (1.9) for the real space, equal-time density-density correlation function are the basis of derivation of the giant number fluctuations I am about to perform. To complete that derivation, I must first complete the a posteriori argument made earlier that the departures of $C_\rho$ from the “infinitely thin sheet” approximation are negligible.

This is quite straightforward to do using the electrostatic analogy. Within the sheet, whose thickness $|r_\parallel|$, I remind the reader, is given by (2.30), which says $|r_\parallel| \lesssim (\Lambda r_\perp)^\zeta \Lambda^{-1}$, the “electric field” $(\nabla_R)_N C_\rho(R)$ will always be less, by the “Gaussian pillbox” argument, than that just above the thin sheet, since a Gaussian pillbox that starts at the equatorial plane and ends within the thin sheet will always contain less charge than on that spans the entire thickness of the sheet. Therefore, the “potential” - which is actually the correlation function $C_\rho$ - can change within the sheet by no more than $(\nabla_R)_N C_\rho(R)$ evaluated just outside the sheet, times the thickness of the sheet. Since $(\nabla_R)_N C_\rho(R) \sim C_\rho/r_\perp$, and the thickness of the sheet is $|r_\parallel| \lesssim (\Lambda r_\perp)^\zeta \Lambda^{-1} \ll r_\perp$, the last inequality holding for all $r_\perp \gg \Lambda^{-1}$ since $\zeta = \frac{d+1}{2} < 1$ for all $d < 4$, it follows that the change $\delta C_\rho$ in the “potential”, - that is, in $C_\rho$ across the thickness of the thin sheet obeys $\delta C_\rho < ((\nabla_R)_N C_\rho(R)|r_\parallel| \lesssim C_\rho/r_\perp)(\Lambda r_\perp)^\zeta \Lambda^{-1} \propto C_\rho r_\perp^{\zeta-1}$. Since $\zeta < 1$, this is much less than $C_\rho$ itself, so the change in $C_\rho$ across the thickness of the thin sheet is indeed negligible, as I assumed.

Since $\alpha(d = 2) = \frac{4}{5} = .8$, and $\alpha(d = 3) = \frac{7}{5} = 1.4$, are quite different, it would appear to be quite straightforward to see the difference between the scaling behavior of density fluctuations in two and three dimensions in simulations or experiments.

Unfortunately, things are not quite so simple. The most natural quantity to look at when studying density fluctuations is the fluctuations of the number of particles in an imaginary “counting box” (which need not be a rectangular, or even polyhedral, but could, for example, be a (hyper)sphere or an ellipsoid, etc.) of some volume $V_{\text{box}}$ inside a flock of volume $V_{\text{flock}} \gg V_{\text{box}}$. The mean squared number fluctuations $\langle (\delta N)^2 \rangle \equiv \langle N^2 \rangle - \langle N \rangle^2$ can readily be related to the real space correlations $C_\rho(r)$:

$$\langle (\delta N)^2 \rangle = \int_V d^d r d^d r' \langle \delta \rho(r) \delta \rho(r') \rangle = \int_V d^d r d^d r' C_\rho(r - r')$$
where the subscript $V$ denotes that the integrals are over $\mathbf{r}$ and $\mathbf{r}'$'s contained within our experimental “counting box”. Using our expression \((2.43)\) for $C_{\rho}(\mathbf{r} - \mathbf{r}')$ gives

$$\langle (\delta N)^2 \rangle = \int_V d^d r d^d r' |\mathbf{r} - \mathbf{r}'|^{-\alpha(d)} G_d (\theta_{\mathbf{r}-\mathbf{r}'}))$$

(2.47)

Now let’s take our “box” to be an arbitrary shape with total volume $V = L^d$. Making the changes of variables $r \equiv xL, r' \equiv x'L$, I obtain

$$\langle (\delta N)^2 \rangle = L^{2d-\alpha(d)} \int_{V_1} d^d x d^d x' |\mathbf{x} - \mathbf{x}'|^{-\alpha(d)} G_d (\theta_{\mathbf{x}-\mathbf{x}'})$$

(2.48)

where $V_1$ denotes that the integrals are over $\mathbf{x}$ and $\mathbf{x}'$ contained in a unit volume of the same shape as our original counting box. Clearly, this integral has no dependence on $L$. Therefore \((2.48)\) implies

$$\langle (\delta N)^2 \rangle = L^{2d-\alpha(d)} \times K \left( \frac{c}{|\gamma - v_2|}, \text{shape} \right)$$

(2.49)

where the constant

$$K \left( \frac{c}{|\gamma - v_2|}, \text{shape} \right) \equiv \int_{V_1} d^d x d^d x' |\mathbf{x} - \mathbf{x}'|^{-\alpha(d)} G_d (\theta_{\mathbf{x}-\mathbf{x}'})$$

(2.50)

depends on the shape of the box (as well as the ratio $\frac{c}{|\gamma - v_2|}$, which enters both explicitly in equation \((1.9)\) for $G_d$ and implicitly through the relation \((1.10)\) between $\theta_{\mathbf{r}}$ and $\theta_{\mathbf{R}}$), but is independent of its size $L$. This can be rewritten in terms of the mean number $\langle N \rangle$ of critters in the counting box, using the fact that the average density $\rho_0$ is well-defined. Hence, $\langle N \rangle = \rho_0 L^d$, or $L = \left( \frac{\langle N \rangle}{\rho_0} \right)^{\frac{1}{d}}$. Using this in \((2.49)\) and taking the square root of both sides gives:

$$\sqrt{\langle (\delta N)^2 \rangle} = K' < N >^{\phi(d)}$$

(2.51)

with

$$\phi(d) = \frac{2d - \alpha(d)}{2d} = \frac{7}{10} + \frac{1}{5d}.$$ 

(2.52)

The coefficient

$$K' \equiv \rho_0^{-\phi(d)} \sqrt{K \left( \frac{c}{|\gamma - v_2|}, \text{shape} \right)}$$

(2.53)

also depends on the shape of the box and the ratio $\frac{c}{|\gamma - v_2|}$. Equation \((2.51)\) is just \((1.1)\) of the introduction.
Eqn. (2.52) gives

\[ \phi(d = 2) = 0.8 \] \hspace{1cm} (2.54)

and

\[ \phi(d = 3) = \frac{23}{30} = 0.7666666... \] \hspace{1cm} (2.55)

Note that in all dimensions \( d \), even \( d > 4 \), where there is no “anomalous hydrodynamics”, the scaling of number fluctuations with mean number violates the “law of large numbers”: the general rule that rms number fluctuations scale like the square root of mean number. The fluctuations eqn. (1.1) are infinitely larger than this prediction in the limit of mean number \( \langle N \rangle \rightarrow \infty \) for all spatial dimensions \( d \); hence, they are much larger than those found in most equilibrium \([10]\) and most non-equilibrium systems, since most of those obey the law of large numbers. Giant number fluctuations like those found here, but even larger, are predicted theoretically \([7]\) and observed experimentally \([16]\) in “nematic” flocks, in which active creature align their long axes, but are equally likely to be moving in either direction along that axis, so that the net velocity is zero.

In addition to obeying a different scaling law, number fluctuations in polar ordered dry active fluids exhibit another phenomenon not present in most other systems: the number fluctuations depend not only on the mean number \( \langle N \rangle \) of particles in the box, but also on its shape, as embodied in the coefficient \( K' = \sqrt{K \left( \frac{c}{\gamma - v^2}, \text{shape} \right)} \) in (1.1).

This dependence is singular in the limit of a “needle shaped” counting box; that is, one that is much longer along the direction \( \hat{x}_\parallel \) of flock motion than perpendicular to it. I mean singular in the sense that the coefficient \( K' = K \left( \frac{c}{\gamma - v^2}, \text{shape} \right) \) actually vanishes in the limit that the aspect ratio \( \beta \equiv \frac{L_\parallel}{L_\perp} \) of the box goes to infinity (\( \beta \rightarrow \infty \)), where \( L_\parallel \) and \( L_\perp \) are respectively the linear extents of the counting box along and perpendicular to the direction of flock motion. I will illustrate this first in \( d = 3 \), with the example of a counting box that is a cylinder with its axis along the \( \hat{x}_\parallel \) direction, with height \( L_\parallel \) and radius \( L_\perp \). The volume of this cylinder is clearly \( \pi L_\parallel^2 L_\perp \), and, hence, the mean number of particles in it is

\[ \langle N \rangle_{\text{cylinder}} = \rho_0 \pi L_\parallel^2 L_\perp = \rho_0 \pi L_\perp^3 \beta. \] \hspace{1cm} (2.56)

Our general expression expression (2.47) for \( \langle (\delta N)^2 \rangle \) reads for this case

\[ \langle (\delta N)^2 \rangle = \int_{r_\perp < L_\perp} d^2 r_\perp \int_{r_\perp' < L_\perp} d^2 r_\perp' \int_0^{L_\parallel} dr_\parallel \int_0^{L_\parallel} dr_\parallel' |r - r'|^{-7/5} G_3 (\theta_{r-r'}). \] \hspace{1cm} (2.57)
I note that the integrals over \( r_\parallel \) and \( r'_\parallel \) both converge in the limit \( L_\parallel \to \infty \). This follows from the fact that the integrand \( |r - r'|^{-7/5} G_3(\theta_{r-r'}) \propto r_\parallel^{-7/5} \) as \( r_\parallel \to \infty \), and likewise for \( r'_\parallel \). This result also uses the fact that \( G_3(\theta) \) is finite and non-zero for all \( \theta \), and in particular for \( \theta \to 0 \).

Since this falloff with \( r_\parallel \) and \( r'_\parallel \) is faster than \( 1/r_\parallel \), the integrals over \( r_\parallel \) and \( r'_\parallel \) both converge in the limit \( L_\parallel \to \infty \). Note that this will not be true in \( d = 2 \), where \( \alpha(2) = 4/5 < 1 \), so the analogous integral will not converge.

This means that if the aspect ratio \( \beta \) is \( \gg 1 \) - that is, \( L_\parallel \gg L_\perp \) - I can accurately approximate the value of the integral in (2.57) by taking \( L_\parallel \to \infty \). Thus I get

\[
\left\langle (\delta N)^2 \right\rangle = K_{cyl} L_\perp^{23/5} \tag{2.59}
\]

where I’ve defined

\[
K_{cyl} \equiv \int_{|x_\perp|<1} d^2 x_\perp \int_{|x'_\perp|<1} d^2 x'_\perp \int_0^\infty dx_\parallel \int_0^\infty dx'_\parallel |x - x'|^{-7/5} G_3(\theta_{x-x'}) , \tag{2.60}
\]

which I remind the reader is a perfectly finite function of the ratio \( \frac{c}{|x_\parallel - y_\parallel|} \) (which is buried in \( G_3 \)). It is also independent of the aspect ratio \( \beta \).

Solving my expression (2.56) for \( L_\perp (\langle N \rangle, \beta) \) gives

\[
L_\perp = \left( \frac{\langle N \rangle}{\pi \rho_0 \beta} \right)^{1/3} . \tag{2.61}
\]

Using this, I can rewrite (2.59) (or, more precisely, its square root) in terms of \( \langle N \rangle \) and the aspect ratio \( \beta \):

\[
\sqrt{\langle (\delta N)^2 \rangle} = K' < N >^{23/30} \tag{2.62}
\]

with

\[
K' = \sqrt{K_{cyl}/(\pi \rho_0 \beta)^{23/30}} \propto \beta^{-23/30} , \tag{2.63}
\]

which, as claimed, vanishes as the cylinder gets very long (i.e., as \( \beta \to \infty \)). Note also that I’ve recovered the general \( \frac{23}{30} \) scaling law for \( \sqrt{\langle (\delta N)^2 \rangle} \) with \( \langle N \rangle \) in \( d = 3 \).
It is straightforward to see that for most three dimensional "needle" shapes (e.g., an ellipsoid of revolution about the \( \hat{x}_\parallel \) direction, with its long axis in that direction), the same "\( 23/30 \)" scaling law \((2.63)\) for the coefficient \( K' \) in \((2.62)\) with aspect ratio \( \beta \) (which in the ellipsoid case will be the ratio of semi-major to semi-minor axis) will apply.

I can also obtain a simple expression for the ratio the value of \( K' \) for a "pancake" shaped counting volume, by which I mean a volume much shorter along the direction \( \hat{x}_\parallel \) of flock motion than perpendicular to it.

Consider in particular a very squat cylinder with its axis along \( \hat{x}_\parallel \). For such a shape, I can now approximate \( r \) with \( r_\perp \), \( r' \) with \( r'_\perp \), and \( \theta_r = \theta_{r'} \approx \frac{\pi}{2} \) for the range of \( r \) and \( r' \) that dominate the integral. This gives

\[
\langle (\delta N)^2 \rangle = \frac{\pi}{2} \int_0^{L_\parallel} dr_\parallel \int_0^{L_\parallel} dr'_\parallel \int_{r_\perp < L_\perp} d^2 r_\perp \int_{r'_\perp < L_\perp} d^2 r'_\perp \ |r_\perp - r'_\perp|^{-7/5}. \tag{2.64}
\]

Using the change of variables \( r \equiv x L_\perp \), \( r' \equiv x' L_\perp \) for the \( r_\perp \) and \( r'_\perp \) integrals, and doing the trivial integrals over \( r_\parallel \) and \( r'_\parallel \), I get

\[
\langle (\delta N)^2 \rangle = K_{\text{pan}} L_{\perp}^{13/5} L_\parallel^2 \tag{2.65}
\]

where I’ve defined

\[
K_{\text{pan}} \equiv G_3 \left( \frac{\pi}{2} \right) \int_{|x_\perp| < 1} d^2 x_\perp \int_{|x_\perp| < 1} d^2 x'_\perp \ |x_\perp - x'_\perp|^{-7/5} \approx 38.651 A', \tag{2.66}
\]

where I’ve used equation \((1.9)\) for \( G_3 \), which implies

\[
G_3 \left( \frac{\pi}{2} \right) = \Upsilon_3 \left( \frac{\pi}{2} \right) = B_3 P_\frac{3}{2} \left( 0 \right) = \frac{25 \Gamma \left( \frac{17}{10} \right) \Gamma \left( \frac{4}{5} \right)}{7 \Gamma \left( \frac{6}{5} \right) \Gamma \left( \frac{3}{10} \right)} A' \approx 1.3755 A', \tag{2.67}
\]

the first two equalities following from equations \((1.9)\) and \((1.12)\), respectively. The penultimate equality follows from known properties of the generalized Legendre functions \([17]\). I’ve also numerically evaluated the four dimensional integral displayed explicitly in \((2.66)\) (it’s equal to 28.1).

Since I’m still dealing with a cylinder here, the expression \((2.61)\) for \( L_\perp (N, \beta) \) continues to hold. Using this and \( L_\parallel = \beta L_\perp \) in \((2.65)\) and taking the usual square root gives for the rms number fluctuations

\[
\langle (\delta N)^2 \rangle = \frac{\sqrt{K_{\text{pan}}}}{(\pi \rho_0)^{23/30}} < N >^{23/30} \beta^{8/15}, \tag{2.68}
\]
which vanishes as $\beta \to 0$. The scaling of this result with the aspect ratio $\beta$ is the result (1.3) quoted in the introduction.

To summarize what I’ve shown, in three dimensions the coefficient $K'$ of $<N>^{23/30}$ in the scaling law (1.1) for the rms fluctuations $<(\delta N)^2>$ vanishes as the aspect ratio $\beta \to 0$ like $\beta^{8/15}$, and as $\beta^{-23/30}$ as $\beta \to \infty$. Thus, there must be an optimal aspect ratio $\beta \sim 1$ where this coefficient is maximized. Thus, somewhat surprisingly given the anisotropy of the Fourier transformed correlation function, the optimal box for observing the largest possible giant number fluctuations proves to be roughly isotropic (e.g., a cube or a sphere). The precise value of the optimal ratio will depend on the ratio of hydrodynamic parameters $c/|\gamma-v_2|$.

The same qualitative behavior with aspect ratio proves to hold in two dimensions as well. I’ll show this by considering a rectangular counting box aligned with two of its edges parallel to $\hat{x}_\parallel$. In the “needle” limit, this will be the long axis, while for the “pancake” limit (which in $d=2$ is just the needle rotated by 90 degrees), it will be the short axis.

Continuing to define $\beta \equiv L_\parallel/L_\perp$ in all cases, I’ll now focus first on the needle case $\beta \gg 1$.

Because $\alpha = 4/5 < 1$ in $d = 2$, the double integral in (2.47) does not converge at large $r_\parallel$; therefore, it that integral is dominated by widely separated values of $r_\parallel$. This implies that those integrals will be dominated, for the needle geometry, by values of $r$ and $r'$ such that $\theta_{r-r'} \ll 1$. Furthermore, for these values of $r$ and $r'$, $|r-r'| \approx |r_\parallel - r'_\parallel|$. With these approximations, which become exact in the limit of the aspect ratio $\beta \to \infty$, I can therefore write

$$\langle (\delta N)^2 \rangle = B_2 \left( \frac{|\gamma-v_2|}{c} \right)^{4/5} \int_0^{L_\parallel} dr_\parallel \int_0^{L_\parallel} dr'_\parallel \int_0^{L_\perp} dr_\perp \int_0^{L_\perp} dr'_\perp |r_\parallel - r'_\parallel|^{-4/5}. \tag{2.69}$$

All of the integrals in this expression are elementary; doing them gives

$$\langle (\delta N)^2 \rangle = J L_\perp^2 L_\parallel^{6/5}, \tag{2.70}$$

where I’ve defined

$$J \equiv \frac{25}{3} B_2 \left( \frac{|\gamma-v_2|}{c} \right)^{4/5}. \tag{2.71}$$

To re-express the number fluctuations (2.70) in terms of the mean number of particles $<N>$, I can use the fact that

$$<N>_{\text{rectangle}} = \rho_0 L_\perp L_\parallel = \rho_0 L_\parallel^2 \beta, \tag{2.72}$$
which, when combined with (2.70) and \(L_\parallel = \beta L_\perp\) gives
\[
\sqrt{\langle (\delta N)^2 \rangle} = \sqrt{J_\rho_0^{-8/5}} < N >^{8/5} \beta^{-1/5}.
\] (2.73)

For the pancake, I have
\[
\langle (\delta N)^2 \rangle = G_2 \left(\frac{\pi}{2}\right) \int_0^{L_\parallel} dr_\parallel \int_0^{L_\perp} dr_\perp \int_0^{L_\parallel} dr_\parallel' \int_0^{L_\perp} dr_\perp' |r_\perp - r_\perp'|^{-4/5} = J_\rho L^2_\parallel L^{6/5}_\perp,
\] (2.74)
where I’ve defined
\[
J_\rho \equiv \frac{25}{3} G_2 \left(\frac{\pi}{2}\right) = \frac{25}{6\tau} B_2,
\] (2.75)
where \(\tau = \frac{\sqrt{5}+1}{2} = 1.6180...\) is the Golden mean.

Using (2.72) and \(L_\parallel = \beta L_\perp\) again gives
\[
\sqrt{\langle (\delta N)^2 \rangle} = \sqrt{J_\rho_0^{-8/5}} < N >^{8/5} \beta^{-1/5}.
\] (2.76)

For such a needle shaped box, almost all angles \(\theta_{r-r'}\) between two randomly chosen points \(r\) and \(r'\) in the integrals in (2.47) obey \(\theta_{x-x'} \ll 1\).

Note that these giant number fluctuations are not due to giant density fluctuations. In fact, the fluctuations in the density at any single point are perfectly finite, and not necessarily bigger than those in some equilibrium systems. It is the long-ranged correlations of those density fluctuations that give rise to giant number fluctuations.

Dramatic as the large fluctuations predicted by eqn. (1.1) are, the exponents \(\phi(2) = .8\) in \(d = 2\) and \(\phi(3) = 23/30 = .766666...\) in \(d = 3\) are numerically too close to each other for the difference between the behavior in the two different dimensions to be easily detectable experimentally. A more direct measure of \(\alpha(d)\), which differs considerably between two \((\alpha(2) = .8)\) and three \((\alpha(3) = 1.4)\) dimensions, would clearly be more useful.

One way to do so is to correlate the number fluctuations in one box with those in a different box separated from the first by a displacement \(\delta r\) whose magnitude \(|\delta r|\) is much greater than the largest linear extent of either box. In this case, the correlations are given by
\[
\langle \delta N_1 \delta N_2 \rangle = \int_{V_1} d^d r \int_{V_2} d^d r' C_\rho(r - r')
\] (2.77)
where the subscripts \(V_1\) and \(V_2\) denote that \(r\) and \(r'\) run over boxes 1 and 2 respectively. Since these two boxes are separated by a \(\delta r\) that is much greater than the linear extent of
either box, \( r - r' \) is, to a good approximation, equal to \(|\delta r|\) over the entire range of both integrals in (2.77). I can therefore replace \( r - r' \) with \( \delta r \) in (2.77), pull \( C_\rho(\delta r) \) (which is now independent of \( r \) and \( r' \)) out of the integrals, and perform the integrals over \( r \) and \( r' \). Doing so gives

\[
\langle \delta N_1 \delta N_2 \rangle = C_\rho(\delta r) \int_{V_1} d^d r \int_{V_2} d^d r' = C_\rho(\delta r)V_1V_2
\]

(2.78)

where \( V_{1,2} \) denote the volumes of boxes 1 and 2. Rewriting this in terms of the mean numbers \( \langle N_1 \rangle \) and \( \langle N_2 \rangle \) of particles in each of the two boxes, and then using our earlier expression (2.43) for \( C_\rho(r) \), gives:

\[
< \delta N_1 \delta N_2 > = \langle N_1 \rangle \langle N_2 \rangle |\delta r|^{-\alpha(d)}G_d(\theta_\delta r)/\rho_0^2.
\]

(2.79)

Here \( N_{1,2} \) are the particle numbers in box number 1 and 2, respectively, and \( \delta N_{1,2} \) are their fluctuations about their mean.

Thus, this two box measurement provides a direct measure of \( \alpha(d) \), which, as noted earlier, changes appreciably between two and three dimensions. It also provides the opportunity to directly test my extremely detailed predictions (1.9), (1.11), and (1.12) for the functional form of \( G_d(\theta_r) \).

III. SUMMARY

I have used hydrodynamic equations of dry active polar fluids[2–6] to show that these systems exhibit giant number fluctuations, much larger the “law of large numbers” scaling \( \sqrt{\langle N \rangle} \) scaling of number fluctuations in virtually all other equilibrium and non-equilibrium systems studied to date. Furthermore, I’ve shown that, again unlike most other systems, the number fluctuations also depend singularly on the shape of the box in which one counts the particles, vanishing in the limit of very thin boxes.

These fluctuations arise not from large density fluctuations (which are not, in fact, expected in general in polar ordered dry active fluids), but from long ranged spatial correlations between those fluctuations. These can be determined by a surprising electrostatic analogy. Specifically, in two spatial dimensions the correlations can be obtained by a simple rescaling of lengths from the electrostatic potential near a sharp upward pointing conducting wedge of opening angle \( \frac{3\pi}{8} = 67.5^\circ \), while in three dimensions they can likewise be obtained
in the same manner from the electrostatic potential near a sharp upward pointing charged cone of opening angle $37.16^\circ$. This very precise prediction can be stringently tested by alternative correlating number counts in two widely separated boxes.

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