Subsystems and time in quantum mechanics

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This paper investigates the relationship between subsystems and time in a closed nonrelativistic system of interacting bosons and fermions. It is possible to write any state vector in such a system as an unentangled tensor product of subsystem vectors, and to do so in infinitely many ways. This requires the superposition of different numbers of particles, but the theory can describe in full the equivalence relation that leads to a particle-number superselection rule in conventionally defined subsystems. Time is defined as a functional of subsystem changes, thus eliminating the need for any reference to an external time variable. The dynamics of the unentangled subsystem decomposition is derived from a variational principle of dynamical stability, which requires the decomposition to change as little as possible in any given infinitesimal time interval, subject to the constraint that the state of the total system satisfy the Schrödinger equation. The resulting subsystem dynamics is deterministic. This determinism is regarded as a conceptual tool that observers can use to make inferences about the outside world, not as a law of nature. The experiences of each observer define some properties of that observer’s subsystem during an infinitesimal interval of time (i.e., the present moment); everything else must be inferred from this information. The overall structure of the theory has some features in common with quantum Bayesianism, the Everett interpretation, and dynamical reduction models, but it differs significantly from all of these. The theory of information described here is largely qualitative, as the most important equations have not yet been solved. The quantitative level of agreement between theory and experiment thus remains an open question.

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I. INTRODUCTION

*It seems to me that for a systematic foundation of quantum mechanics one needs to begin with the composition and decomposition of quantum systems.*

— W. Pauli [1–3]

In physics, as in everyday life, we must divide the world conceptually into subsystems before we can say anything about it. We cannot comprehend the undivided world; we can only talk about how its parts differ from or change relative to each other. In quantum mechanics the role of subsystem decompositions is, if anything, even more important. As Wheeler has said, “we are first able to play the game when with chalk we have drawn a line across the empty courtyard” [4]. If we decline to draw such a line, “then physics has vanished, and only a mathematical scheme remains” [5].

The “mathematical scheme” that Heisenberg refers to is the Schrödinger equation for the time evolution of a closed system. If the world is not divided into subsystems, time may pass on a sheet of paper, but nothing can be said to happen, as there is no point of contact between theory and experiment. Indeed it is not even clear what the time variable in the Schrödinger equation means if one cannot talk about relative changes between different parts of the system.

This paper explores the relationship between subsystems and time in nonrelativistic quantum mechanics. It springs from an examination of four interrelated questions: (1) how to construct subsystem decompositions without entanglement; (2) how to define time in a closed system; (3) how to define a dynamics of interacting subsystems without entanglement; and (4) how to extract information from these subsystems.

Cursory answers to these four questions are as follows: (1) Unentangled subsystem decompositions can be constructed in Fock space, using superpositions of different numbers of particles. (2) Time can be defined as a functional of subsystem changes. (3) An entanglement-free dynamics can be derived by maximizing the stability of the subsystem decomposition. (4) An observer obtains information from only one subsystem, and only during the present moment of time. All else must be inferred.

A brief introduction to each of these four subject areas is given below. It must be emphasized at the outset that this paper is exploratory in nature; the theory of information presented here has not yet reached a stage of development that would permit a direct comparison with experiment.

### A. Definition of subsystems

In quantum mechanics, subsystems are traditionally defined using a tensor product of vector spaces [6]. Subsystems defined in this way inevitably become entangled by interactions [7]. Although entanglement is now commonly regarded as a resource [8], its consequences still leave many people uneasy with the foundations of quantum mechanics [9–13].

But is this definition necessary? Another approach has been advocated by Primas and Amann [14–28]. It is based on the concept of a quantum object, which Primas defines as “an open quantum system which in spite of its interaction with the environment is not [entangled] with the environment” [15]. Objects may change their properties but they are required to “keep their identity in the course of time” [15]. The concept of an object is similar to that of a quasiparticle [29–31]. However, quasiparticles are designed to minimize interactions, whereas objects are designed to minimize entanglement.

Algebraic “dressing” techniques have been developed for the construction of quantum objects in some simple models, but only in the context of certain approximations and asymptotic limits [14–28]. One approximation considered essential by Primas is neglecting the Pauli exclusion principle at the level of the interactions between subsystems. Thus, electrons in different subsystems are treated as distinguishable. However, this contradicts a basic principle of elementary-particle physics.

Is it possible to construct quantum objects without neglecting the Pauli exclusion principle? The answer is yes, if we define a subsystem decomposition as a tensor product of vectors rather than a tensor product of vector spaces. Of course, the answer also depends on how we define a system.

The system studied in this paper is a closed nonrelativistic system of interacting fermions and bosons. The dimensions of the underlying single-particle Hilbert
spaces are assumed to be finite; the dimension of the resulting fermion Fock space is then finite too, but that of the boson Fock space is infinite. For example, one might consider a finite volume in coordinate space and cut off wavelengths shorter than some given value. The use of such a cutoff is congruent with the current understanding of the standard model of elementary-particle physics as an effective field theory [32]. Bell has called this type of system a “serious part of quantum mechanics” that covers a “substantial fragment of physics” [33], thus making it (in his opinion) a worthy object of study in the field of quantum foundations [33, 34].

It is shown in Secs. II and III that any state vector $|\psi\rangle$ in such a system can be written exactly as an unentangled tensor product of an arbitrary number of subsystem vectors $|u_k\rangle$. Furthermore, for any given number of subsystems, this can be done in infinitely many ways. Despite the lack of entanglement, knowledge of the whole $(|\psi\rangle)$ does not imply knowledge of the parts $(|u_k\rangle)$, because the subsystem vectors are not confined to subspaces. On the contrary, $|u_k\rangle$ occupies the same Fock space as $|\psi\rangle$.

B. Definition of time

Such a decomposition is only meaningful if it persists over time. A crucial question that must be addressed is how to define time in a closed system without making reference to an external time variable [35–51]. The given subsystem decomposition is very useful in this regard. It allows time to be defined as a functional that organizes information about subsystem changes.

Section IV begins the preparatory work for this definition by developing ways to quantify differences between subsystem decompositions. The main focus is on geometric concepts related to the Fubini–Study metric in Hilbert space. Section V then defines time as a functional of two infinitesimally different (but otherwise arbitrary) subsystem decompositions. This functional is defined so as to maximize the amount of change that can be expressed in the form of Schrödinger dynamics. It is a functional of the entire subsystem decomposition; this eliminates the need to single out any particular subsystem as a clock.

C. Subsystem dynamics

The time functional can be used to formulate a dynamics of subsystems by means of a variational principle of dynamical stability. This principle requires the subsystem decomposition to change as little as possible in any given infinitesimal time interval, subject to the constraint that the total state vector $|\psi\rangle$ of the closed system satisfy the Schrödinger equation. However, the interacting subsystems derived from this principle do not satisfy the Schrödinger equation.

The concept of subsystem dynamics does not even exist in the usual tensor-product-of-vector-spaces formulation of subsystems. The tensor-product decomposition is just given arbitrarily, with no connection (in principle) between different times (although it is often taken to be time independent). The subsystem dynamics developed here (in Sec. VI) therefore has no parallel in orthodox quantum mechanics.

Dynamically stable subsystems are quantum objects in the sense defined above. Does this mean that their observable properties can be regarded as elements of a “free-standing reality” [52]? Deriving such a description was in fact a large part of the original motivation for this study. However, the answer turns out to be an emphatic no. Dynamically stable subsystems still have an unavoidable element of subjectivity.

There are two fundamental reasons for this. One is that the subjective choice of a number of subsystems is essential to the dynamics. The results depend explicitly on this number, and it has to be put in by hand. Its value cannot be derived from the principle of dynamical stability itself.

The second reason is that the resulting dynamics is deterministic. This investigation began with a vague expectation that the dynamical stability problem might not have unique solutions, thereby necessitating the introduction of “objective” probabilities at a fundamental level. This could be regarded as a new type of decoherence mechanism that is inherent in the dynamics of quantum objects.

However, this expectation proved to be false. The variation problem has a unique solution, so the resulting subsystem dynamics is deterministic. This means that the principle of dynamical stability cannot explain the lack of determinism exemplified by the “quantum jumps” of orthodox quantum mechanics [55, 56].

If the principle of dynamical stability cannot be regarded as the foundation for a law of nature, what is it good for? It is essentially just a tool for observers to use. They can use it to infer something about the properties of subsystems in the past or the future from whatever information they have about those subsystems now. There is no guarantee that these inferences will agree with their experiences, because the subsystem dynamics is not viewed as a law governing the behavior of anything “real.” It is instead viewed as an instantiation of Wheeler’s aphorism that “the only law is the law that there is no law” [56]. That is, nature is not governed by laws; laws are just useful conceptual tools.

How are these results affected by the inclusion of superselection rules [57, 60]? One must be careful in answering this question, because the standard rules are derived from the tensor-product-of-subspaces definition of subsystems, which is not relevant in the present context. (The standard rules are also highly controversial even in the proper context [52, 61, 74].) One must therefore go back to the underlying cause of a superselection rule (i.e., the lack of an external reference frame [74]) and reexamine its consequences for the definition of subsystems used here.

In the absence of an external reference frame, cer-
tain subsystem decompositions become observationally indistinguishable from one another. One can account for this indistinguishability by introducing equivalence classes \([70]\) of subsystem decompositions. For the reference frame that gives rise to a particle-number superselection rule, the subsystem dynamics with or without such equivalence classes is qualitatively the same, as shown in Sec. [VII]. That is, the dynamics remains fully deterministic. However, the quantitative dynamics can be quite different in these two cases.

An intriguing consequence of the Pauli exclusion principle is that the subsystem dynamics depends on the order in which the subsystem states \([u_k]\) are multiplied. Ignoring this ordering would give rise to an apparent decoherence effect. However, when the ordering of subsystems is accounted for, the subsystem dynamics remains deterministic, as shown in Sec. [VIII].

D. Information about subsystems

Section [IX] addresses the question of precisely how inferences are to be drawn from information about the properties of subsystems. Because the subsystems are not entangled, we can use the methods of Bayesian inference familiar from classical probability theory \([77, 82]\). This automatically ensures compliance with Bohr’s injunction that “however far the phenomena transcend the scope of classical physical explanation, the account of all evidence must be expressed in classical terms” \([83, 84]\).

These subsystems are nonclassical, but due to their lack of entanglement, their properties can be regarded as “beables” rather than “observables” (in the sense of Bell \([13]\)). That is, the subsystems can be considered to have certain properties, independently of whether they are “measured.” However, it should be emphasized that these properties are in general only inferred to exist, and that they are contextual in the sense that they depend on the subjective choice of a number of subsystems. As noted above, they cannot be regarded as elements of a free-standing reality.

How are we to reconcile the determinism of the subsystem dynamics with the “quantum jumps” of textbook quantum mechanics? This can be done by imposing limitations on the information an observer has access to. An observer is assumed to experience directly only those beables associated with one subsystem, and only for an infinitesimal interval of time. This interval is called the present moment of time. The properties of all other subsystems, as well as those of the observer’s own subsystem in the past and future of the present moment, must be inferred from this information.

These limitations are not due to an observer’s ignorance, in the literal sense of ignoring possibly accessible information. The observer simply has no access to anything other than her own experiences. These experiences are not predicted by anything in the theory. The degree to which an observer’s inferences of an outside world match her experiences in successive present moments determines how useful her image of this outside world is. The degree of mismatch can be regarded as an effective “quantum jump.”

The conceptual structure of this theory of information draws heavily upon the ideas of quantum Bayesianism (or QBism) developed by Caves, Fuchs, Schack, and Mermin \([35, 33, 34]\). The main difference is that an observer is treated not as a black box but as a subsystem like any other. This allows multiple observers to be treated as equals, thus providing a framework for meaningful descriptions of intersubjective agreement. “Objective” properties can then be defined as those features of the common worldview that survive when more observers are added to the framework.

The theory of information is developed only to the level of a bare skeleton, however, because I have not solved the resulting system of equations. The discussion of this part of the theory is therefore mostly qualitative, and no proof can be offered that it leads to predictions in agreement with experiment. The challenge of adding flesh to this skeleton nevertheless opens up many promising avenues for future research.

E. A guide for the reader

As an aid to navigation, this subsection indicates which parts of the paper are most essential on a first reading. Overall, the most difficult parts are those dealing with technical details of boson vector spaces; these are largely relegated to a series of appendices. On a first reading, one should focus attention primarily on the properties of systems containing only fermions.

The beginning of Sec. [III] shows that, contrary to statements in the literature, systems of indistinguishable particles do have a tensor product structure. Readers who are willing to take this for granted can go straight to Eqs. (2.22)–(2.25), referring back to earlier material as needed for definitions of notation. The paragraph below Eq. (2.29) establishes when a fermion creator is invertible. Equation (2.34) shows that for systems containing bosons, the complex numbers in the fermion matrix (2.29) are replaced by commuting operators.

Section [III] explains why invertibility of subsystems is important and develops some mathematical tools for inversion. Readers who are willing to take invertibility for granted can go straight to the exponential representation of Eqs. (3.16) and (3.19). The definition of the word “quasiclassical” and its relation to classical probability theory in Sec. [III] are also worth noting.

Section [IV] deals with various kinematical aspects of the definition of subsystems. The definition of observable quantities in Secs. [IVA] and [IVB] should be included in a first reading, but Secs. [IVC] and [IVD] can be glossed over and referred back to when needed. The geometric properties defined in Sec. [IVE] are essential as they are used throughout the rest of the paper.
Section V develops various connections between subsystems and time. Those willing to accept that interacting subsystems never satisfy the Schrödinger equation can skip Sec. V.A. The remaining subsections all contain essential material. The key ideas of Sec. V.C are the conceptual foundation for the time functional in Eqs. (5.9) and (5.11) and the resulting functional in Eq. (5.16). The central result of Sec. V.D is the inequality (5.29).

The topic of Sec. VI is subsystem dynamics. The concept of dynamical stability is quantified in Eq. (6.2) of Sec. VI.A. The simple case of a time-independent total system state $|\psi\rangle$ is considered first in Sec. VI.B. The solution for the unique maximum of the dynamical stability functional is given in Eq. (6.15). The extension to time-dependent $|\psi\rangle$ in Secs. VI.C, VI.D, and VI.E can be glossed over on a first reading. The main result, shown in Eq. (6.46), is qualitatively the same as the previous solution (6.15), apart from a necessary change of notation. Section VI.F can be skipped on a first reading, but the results of Sec. VI.G are essential for understanding the results of Sec. V.D. The simple case of a time-independent total system state $|\psi\rangle$ is considered first in Sec. VI.B. The solution (6.15), apart from a necessary change of notation, is qualitatively the same as the previous solution (6.15). This is an important result, but the details of the derivation are not needed in any later sections of the paper. The same is true for Sec. VII which deals with the effect of subsystem permutations on subsystem dynamics.

Section XI discusses a system lacking a phase reference, which would lead to a particle-number superselection rule in textbook quantum mechanics. In such a system, the time functional (5.10) can be replaced with the renormalized functional (7.27). After a similar renormalization of other variables, the overall solution for the dynamically stable subsystem change has the same form as before [i.e., Eq. (6.46)]. This is an important result, but the details of the derivation are not needed in any later sections of the paper. The same is true for the conclusions in Sec. XI.

II. TENSOR PRODUCTS AND INDISTINGUISHABLE PARTICLES

The purpose of this section is to define clearly what is meant by the tensor product of many-particle quantum states. Only a small part of the material presented here is entirely new, but establishing a clear notation at the outset helps to simplify calculations in subsequent sections of the paper.

The most convenient tensor product for systems of many indistinguishable particles has an additional algebraic structure that accounts for symmetry or antisymmetry with respect to particle permutations. The precise form of this algebra is defined uniquely by the geometry of Hilbert space, in the form of a cluster decomposition property for the inner product in Fock space. The resulting tensor algebra is the same as that of particle creation operators in Fock space. Special care is needed to ensure closure of the algebra for systems containing bosons.

A. Definition of a tensor product

Given two vector spaces $V_1$ and $V_2$, the tensor product of vectors $|u\rangle \in V_1$ and $|v\rangle \in V_2$ is written as $|u\rangle \otimes |v\rangle$. The tensor product is bilinear but not commutative. That is, it must be distributive over vector addition:

$$|u\rangle \otimes (|v\rangle + |w\rangle) = |u\rangle \otimes |v\rangle + |u\rangle \otimes |w\rangle,$$

(2.1a)

$$(|u\rangle + |z\rangle) \otimes |v\rangle = |u\rangle \otimes |v\rangle + |z\rangle \otimes |v\rangle,$$

(2.1b)

as well as bilinear with respect to scalar multiplication:

$$\alpha(|u\rangle \otimes |v\rangle) = (\alpha|u\rangle) \otimes |v\rangle = |u\rangle \otimes (\alpha|v\rangle).$$

(2.2)

The set of linear combinations of such tensor products defines the vector space $V_1 \otimes V_2$. The tensor product of three or more vectors is associative:

$$((|u\rangle \otimes |v\rangle) \otimes |w\rangle) = |u\rangle \otimes (|v\rangle \otimes |w\rangle),$$

(2.3)

thus defining the vector space $V_1 \otimes V_2 \otimes V_3$ uniquely.

B. Indistinguishable particles

The vector space for a system of $n$ identical particles is

$$H^n = \mathcal{H} \otimes \mathcal{H} \otimes \cdots \otimes \mathcal{H},$$

(2.4)

in which $\mathcal{H}$ is the Hilbert space of a single particle. The Fock space $\mathcal{F}(\mathcal{H})$ for a system with an indefinite number of identical particles is defined as the direct sum

$$\mathcal{F}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} H^n = H^0 \oplus H \oplus H^2 \oplus \cdots,$$

(2.5)

where the zero-particle space $H^0$ consists of all scalar multiples of the vacuum state $|0\rangle$ (the null vector is written as 0). In the tensor algebra of Fock space, $|0\rangle$ is the multiplicative identity:

$$|0\rangle \otimes |\psi\rangle = |\psi\rangle \otimes |0\rangle = |\psi\rangle \quad \forall |\psi\rangle \in \mathcal{F}(\mathcal{H}).$$

(2.6)

By construction, $\mathcal{F}(\mathcal{H})$ is therefore closed under tensor multiplication:

$$\mathcal{F}(\mathcal{H}) \otimes \mathcal{F}(\mathcal{H}) = \mathcal{F}(\mathcal{H}).$$

(2.7)

According to the symmetrization postulate, the only physically meaningful states in $\mathcal{F}(\mathcal{H})$ are those satisfying the symmetry condition

$$S|\psi\rangle = |\psi\rangle,$$

(2.8)
where $S$ is a projector for states of appropriate symmetry (i.e., totally symmetric for bosons and totally antisymmetric for fermions). It is defined by

$$S = \sum_{n=0}^{\infty} \Pi_n S_n \Pi_n, \quad S_n = \frac{1}{n!} \sum_{\sigma} \varepsilon(\sigma) \sigma,$$

(2.9)
in which $\Pi_n$ is the projector for the $n$-particle subspace $H^n$ and $S_n$ is the projector for the symmetric or antisymmetric states of $H^n$ (with $S_0 = S_1 = 1$). The sum over $\sigma$ covers the $n!$ permutation operators in $H^n$. For fermions, $\varepsilon(\sigma)$ is the sign of the permutation $\sigma$:

$$\varepsilon(\sigma) = \begin{cases} +1 & \text{if } \sigma \text{ is even}, \\ -1 & \text{if } \sigma \text{ is odd}. \end{cases}$$

(2.10)

For bosons, $\varepsilon(\sigma) = 1$. The subspace of vectors in $F(H)$ that satisfy the symmetry constraint (2.8) is denoted $F_s(H)$, while the corresponding subspace of $H^n$ is denoted $S(H^n)$.

C. The $\psi$ product

It is convenient at this point to introduce another tensor product that automatically accounts for all symmetry requirements. This binary operator is defined by

$$|u^{(p)}\rangle \otimes |v^{(q)}\rangle = c(p, q) S(|u^{(p)}\rangle \otimes |v^{(q)}\rangle),$$

(2.11)
in which $|u^{(p)}\rangle \in S(H^p)$, $|v^{(q)}\rangle \in S(H^q)$, and $c(p, q)$ is a numerical coefficient to be defined below. The linearity of $S$ and bilinearity of $|u\rangle \otimes |v\rangle$ then give a unique extension of $|u\rangle \otimes |v\rangle$ to the case of general $|u\rangle, |v\rangle \in F_s(H)$.

For fermions, $|u\rangle \otimes |v\rangle$ is the same as the exterior or wedge product $|u\rangle \wedge |v\rangle$ [95, 102, 106], while for bosons, $|u\rangle \otimes |v\rangle$ is known as the symmetric product [95, 104–106]. For brevity, the name “$\psi$ product” is used here as an umbrella term covering both cases in the context of many-particle quantum states of generic symmetry. Note that the $\psi$ product is commutative only for bosons, since [95, 102, 106]

$$|u^{(p)}\rangle \otimes |v^{(q)}\rangle = \zeta^{pq} |v^{(q)}\rangle \otimes |u^{(p)}\rangle,$$

(2.12)

where $\zeta = +1$ for bosons and $\zeta = -1$ for fermions.

The coefficient $c(p, q)$ is partially defined by requiring the $\psi$ product to be associative. As shown in Appendix A, this requires $c(p, q)$ to have the form [104]

$$c(p, q) = \frac{f(p+q)}{f(p)f(q)},$$

(2.13)
in which $f(1) = 1$ but $f(n)$ is otherwise arbitrary. Most authors choose either $f(n) = 1$ [95, 105] or $f(n) = n!$ [102, 104, 106].

However, for applications in quantum mechanics, it is more convenient to choose $f(n) = \sqrt{n!}$ [107], due to the following cluster decomposition theorem. The theorem refers to a case in which $H = H_1 \oplus H_2$, where the subspaces $H_1$ and $H_2$ are orthogonal. The Fock space therefore factors as $F_s(H) = F_s(H_1) \otimes F_s(H_2)$.

Theorem 1 (Cluster decomposition) Let $|st\rangle = |s\rangle \otimes |t\rangle$ and $|uv\rangle = |u\rangle \otimes |v\rangle$ be vectors in $F_s(H)$, where $|s\rangle, |u\rangle \in F_s(H_1)$ and $|t\rangle, |v\rangle \in F_s(H_2)$. Then the inner product $\langle st|uv\rangle$ factors as

$$\langle st|uv\rangle = \langle s|u\rangle \langle t|v\rangle$$

(2.14)

for all such vectors if and only if $|f(n)| = \sqrt{n!}$.

The factorization (2.14) is called a cluster decomposition [108] because the subspaces $H_1$ and $H_2$ typically refer to different regions in coordinate space. Theorem 1 is proved in Appendix B.

The phase of $f(n)$ is not determined by this theorem. However, choosing $f(n)$ to be real and positive means that the coefficient of $|u^{(p)}\rangle \otimes |v^{(q)}\rangle$ in $|u^{(p)}\rangle \otimes |v^{(q)}\rangle$ is also real and positive.

The $\psi$ product is required here to have the cluster decomposition property and to satisfy this phase convention. The coefficient in equation (2.11) is thus given uniquely by

$$c(p, q) = \sqrt{\frac{(p+q)!}{p!q!}}.$$
Consider now the case in which the set \( \{ |\alpha_k\} \) is orthonormal, with repetition of the same state permitted. Let \( n_\alpha \) be the number of times a particular state \( |\alpha\rangle \in \mathcal{H} \) occurs in equation (2.16), with \( \sum_\alpha n_\alpha = n \). For fermions, \( |\alpha\rangle \otimes |\alpha\rangle = 0 \), so values of \( n_\alpha > 1 \) merely give rise to the null vector. Excluding such cases, the normalization of the vector (2.16) is given by \[ (2.19) \]

\[
\langle \alpha_1 \cdots \alpha_n | \alpha_1 \cdots \alpha_n \rangle = \prod_\alpha n_\alpha^{1/2}.
\]

That is, nonzero fermion states are normalized to unity, but this is true for bosons only if no single-particle state is repeated. Unit vectors are useful in some contexts (e.g., Appendix C), but here the normalization (2.19) is more convenient.

The particle creation operator \( a_\lambda^\dagger \) can now be defined as \[ \text{(2.20)} \]

\[
a_\lambda^\dagger |\lambda_1 \cdots \lambda_n\rangle = |\lambda\rangle \otimes |\lambda_1 \cdots \lambda_n\rangle = |\lambda \lambda_1 \cdots \lambda_n\rangle,
\]

which maps \( \mathcal{H}^n \) into \( \mathcal{H}^{n+1} \). The vectors \( \{ |\lambda_\nu\rangle \} \) need not be orthonormal, although the operator commutation relations are simpler if they are. A product of creation operators can thus be used to generate any simple product state from the vacuum:

\[
|\lambda_1 \cdots \lambda_n\rangle = a_\lambda^\dagger \cdots a_{\lambda_n^\dagger} |0\rangle.
\]

(2.21)

The absence of numerical factors in this equation is due to the fact that no normalization convention is imposed on \( |\lambda_1 \cdots \lambda_n\rangle \).

Any vector \( |u\rangle \in \mathcal{F}_s(\mathcal{H}) \) can therefore be generated from the vacuum by

\[
|u\rangle = U |0\rangle,
\]

(2.22)

in which \( U \) is a linear combination of products of creation operators, including (in general) a scalar term for the creation of no particles. For convenience, the operator \( U \) is called the creator of the state \( |u\rangle \). The lengthier phrase “creation operator” is reserved for the creator \( a_\lambda^\dagger \) of a single-particle state \( |\lambda\rangle \); thus, the set of creation operators is a proper subset of the creators.

Given another such state \( |v\rangle = V |0\rangle \), the \( \psi \) product of \( |u\rangle \) and \( |v\rangle \) is

\[
|u\rangle \otimes |v\rangle = (U |0\rangle) \otimes V |0\rangle
= U (|0\rangle \otimes |v\rangle)
= U |v\rangle
= UV |0\rangle.
\]

(2.23a-d)

The algebra of the vectors \( |u\rangle \) and \( |v\rangle \) is therefore the same as the algebra of the creators \( U \) and \( V \). This can be seen even more clearly when a redundant vacuum state is appended to \( |v\rangle \):

\[
|u\rangle \otimes |v\rangle \otimes |0\rangle = UV |0\rangle.
\]

(2.24)

This result can be extended to any number of \( \psi \) products; for example, if \( |w\rangle = W |0\rangle \), then

\[
|u\rangle \otimes |v\rangle \otimes |w\rangle \otimes |0\rangle = UVW |0\rangle.
\]

(2.25)

Of course, it is only meaningful to write such equations if the vector defined by this product is normalizable. No difficulty arises for fermions, because the dimension of the fermion Fock space is finite; fermion creation operators are therefore bounded. However, boson creators are unbounded (with respect to the topology defined by the usual Hilbert space norm); the \( \psi \) product of two normalized boson vectors could therefore be unnormalizable. One must take care to choose a boson vector space that is closed under the \( \psi \) product.

The construction of such a vector space is described in Appendix C. The result, denoted \( \mathcal{F}_s(\mathcal{H}_\ell) \), is a dense subspace of \( \mathcal{F}_s(\mathcal{H}_b) \), where \( \mathcal{H}_b \) is the Hilbert space of a single boson. It should be noted that \( \mathcal{F}_s(\mathcal{H}_b) \) is a Fréchet space \[ \text{(2.26)} \]

\[
\mathcal{F}_s(\mathcal{H}_b) \]

rather than a Hilbert space. This distinction can be ignored for many purposes, because the inner product from \( \mathcal{F}_s(\mathcal{H}_b) \) remains well defined in the subspace \( \mathcal{F}_s(\mathcal{H}_b) \); the only change is that the Fréchet-space topology is not defined by this inner product.

### E. Matrix notation for fermions

In a fermion system, the Fock space has a finite dimension \( 2^d \), in which \( d \) is the dimension of the single-fermion Hilbert space \( \mathcal{H}_t \). It is therefore convenient to introduce matrix representations for the (always bounded) fermion creators \( U \) and \( V \).

Let \( \mathcal{H}_t \) be spanned by an orthonormal basis \( \{ |e_k\rangle \} \), in which \( k \in \{ 0, 1, 2, \ldots, d - 1 \} \). The Fock space \( \mathcal{F}_s(\mathcal{H}_t) \) is then spanned by the basis \( \{ |f_i\rangle \} \), where the integer \( i \in \{ 0, 1, 2, \ldots, 2^d - 1 \} \) has the binary representation

\[
|i\rangle = \sum_{k=0}^{d-1} i_k 2^k \quad (i_k \in \{ 0, 1 \}),
\]

(2.26)

in which \( i_k \) is the \( k \)th binary digit of \( i \). If \( i_k = 1 \), the state \( |e_k\rangle \) is occupied in \( |f_i\rangle \); otherwise, it is unoccupied. Thus, for example, when \( d = 4 \), the basis vector \( |f_5\rangle \) can be written in various ways as

\[
|f_5\rangle = |0101\rangle = |e_2\rangle \otimes |e_0\rangle.
\]

(2.27)

In this notation, \( |f_0\rangle \) is just the vacuum state \( |0\rangle \).

As a simple example, consider the case \( d = 2 \), for which a general vector \( |u\rangle \in \mathcal{F}_s(\mathcal{H}_t) \) can be written as

\[
|u\rangle = \sum_{i=0}^{3} c_i |f_i\rangle, \quad c_i = \langle f_i | u \rangle.
\]

(2.28)

The \( \psi \) products of \( |u\rangle \) with the basis vectors of \( \mathcal{F}_s(\mathcal{H}_t) \) are then given by

\[
|u\rangle \otimes |f_0\rangle = |u\rangle, \quad |u\rangle \otimes |f_1\rangle = c_0 |f_1\rangle + c_2 |f_3\rangle,
\]

(2.29)
|u⟩ ⊙ |f_2⟩ = c_0 |f_2⟩ - c_1 |f_3⟩, and |u⟩ ⊙ |f_3⟩ = c_0 |f_3⟩. The matrix representing U in this basis is therefore

\[
U = \begin{pmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 \\
3 & 2 & -1 & 0
\end{pmatrix}.
\] (2.29)

In general, U has the form of a lower triangular matrix whenever the basis \{⟨f_i⟩\} is arranged in order of nondecreasing particle number |i| ≡ ∑_{k=0}^{d-1} i_k. (Such an arrangement is called graded lexicographic ordering [111].) Hence, U is invertible if and only if c_0 ≠ 0, since the determinant of a triangular matrix is just the product of its diagonal elements.

### F. Different types of particles

Up to this point it has tacitly been assumed that the system under consideration contains only one type of particle. To combine different types of either bosons or fermions, one can simply take a direct sum of the single-particle Hilbert spaces before constructing the Fock space [112]. All boson (fermion) creation operators then commute (anticommute) with each other. Such a description is possible because the operators for different fermions can be chosen arbitrarily to either commute or anticommutate [113,114]. (These two choices are related by a Jordan–Wigner transformation [115–119].)

In this mode of description, which is commonly used in the treatment of isospin [100,112,113], there are only two types of particles: bosons and fermions. Different types of bosons or different types of fermions are treated as formally indistinguishable; the distinction is maintained only at the level of quantum numbers within the single-particle Hilbert spaces \(H_b\) and \(H_f\) [120].

Bosons and fermions are combined together into one system by means of an ordinary (unsymmetrized) tensor product. A general vector in the tensor-product space

\[\mathcal{E} = \mathcal{F}_\psi(H_b) \otimes \mathcal{F}_s(H_f)\]

is thus of the form

\[|u⟩ = \sum_{j=0}^{\infty} \sum_{i=0}^{2^d-1} c_{ji} |b_j⟩ \otimes |f_i⟩,\] (2.31)

in which \{⟨b_j⟩\} and \{⟨f_i⟩\} are fixed orthonormal bases for the boson and fermion Fock spaces (see Sec. II E for the definition of \{⟨f_i⟩\}). This can also be written as

\[|u⟩ = \sum_{i=0}^{2^d-1} |u_i⟩ \otimes |f_i⟩,\] (2.32)

in which \(|u_i⟩ = \sum_j c_{ji} |b_j⟩\). This is not a Schmidt decomposition, since the basis \{⟨f_i⟩\} is independent of \(|u⟩\); the set \{⟨u_i⟩\} is therefore generally not orthonormal.

In the matrix notation of Sec. II E, \(|u⟩\) can be written as a 2^d-component “spinor,” given here explicitly (in decimal and binary) for the case \(d = 2\):

\[|u⟩ = \begin{pmatrix}
|u_0⟩ \\
|u_1⟩ \\
|u_2⟩ \\
|u_3⟩
\end{pmatrix} = \begin{pmatrix}
|u_{00}⟩ \\
|u_{01}⟩ \\
|u_{10}⟩ \\
|u_{11}⟩
\end{pmatrix}.\] (2.33)

The boson-fermion creator U then has the form

\[U = \begin{pmatrix}
U_0 & 0 & 0 & 0 \\
U_1 & U_0 & 0 & 0 \\
U_2 & 0 & U_0 & 0 \\
U_3 & U_2 & -U_1 & U_0
\end{pmatrix},\] (2.34)

in which the constants c_0 of Eq. (2.29) are replaced by boson creators \(U_i\) (defined by \(|u_i⟩ = U_i|0⟩_b\)).

As shown in Appendix E, the definition of the \(\psi\) product \(|u⟩ \circ |v⟩\) can easily be extended to the vector space \(\mathcal{E}\). All of the main conclusions of Secs. II C and II D, including the cluster decomposition property and the algebraic equivalence shown in Eq. (2.25), remain valid in \(\mathcal{E}\).

### III. BASIC TOOLS FOR SUBSYSTEM ANALYSIS

Let us turn now to an investigation of the conditions under which a general ket \(|ψ⟩ \in \mathcal{E}\) can be written as a \(\psi\) product of the form \(|ψ⟩ = |u⟩ \circ |v⟩\). This equation is equivalent to the operator product

\[Ψ = UV,\] (3.1)

in which \(Ψ, U,\) and \(V\) are the creators of the states \(|ψ⟩, |u⟩,\) and \(|v⟩\), respectively.

#### A. Invertible subsystem creators

Such an equation can clearly be written for any invertible subsystem creator \(U\), since one can then take

\[V = U^{-1}Ψ\] (3.2)

for arbitrary \(Ψ\). Likewise, if \(V\) is invertible, one can always let \(U = ΨV^{-1}\). This approach can easily be extended to more general products such as \(Ψ = UVW\), since (for example) if \(U\) and \(W\) are invertible one can always let \(V = U^{-1}ΨW^{-1}\).

Under what conditions is a creator invertible? This question has already been answered for fermion systems; in Sec. II E it was noted that a fermion creator \(U\) is invertible if and only if \(c_0 ≠ 0\), where \(c_0 = ⟨f_0|u⟩ = ⟨0|u⟩\) is the vacuum component of \(|u⟩\). Invertibility is thus a weak constraint: the set of invertible fermion creators is uncountable and forms a smooth manifold.

A similar condition can be derived for systems containing both bosons and fermions. The main result is expressed here in the form of a theorem (proved in Appendix E, using the notation of Sec. II E).
Theorem 2 (Invertibility) For a boson-fermion creator $U$, the following statements are equivalent: (a) The linear map $U : \mathcal{E} \to \mathcal{E}$ is invertible. (b) The associated boson creator $U_0 : \mathcal{F}_\psi \to \mathcal{F}_\psi$ is invertible. (c) The corresponding boson state $|u_0\rangle = U_0|0\rangle_b$ is a coherent state.

The coherent states mentioned in part (c) of the theorem are just the familiar Glauber states [96, 121–126], as defined in Eq. (C2) of Appendix C.

The constraint imposed by Theorem 2 remains very weak. For $U$ to be invertible, the boson state $|u_0\rangle$ associated with the fermion vacuum $|f_0\rangle = |0\rangle_f$ must be a coherent state, but all of the other kets $|u_i\rangle$ in Eq. (2.32) (i.e., those with $i \neq 0$) are completely arbitrary.

B. Why invertibility is important

Equation (3.2) is not the most general solution of Eq. (3.1). The most general solution has the form $[127, 130]

V = U^{-1} \Psi + (1 - U^{-1}) Y, \quad (3.3)

in which $U^{-1}$ is a generalized inverse of $U [127, 130]$ and the creator $Y$ is arbitrary. For finite-dimensional matrices, such a solution exists if and only if $\Psi \in \text{im} U$, where $\text{im} U$ denotes the image (or range) of $U$. When $U$ is invertible, Eq. (3.3) reduces to Eq. (3.2), but invertibility of $U$ is not necessary for the existence of the solution (3.3).

Nevertheless, all subsequent analysis in this paper is based on the study of invertible subsystem creators. This is done for two reasons. The first is simply the pragmatic reason that the algebra of the subsystem creators is much simpler when invertibility is assumed at the outset.

The second reason is more technical but also more compelling. For simplicity, consider the Moore–Penrose generalized inverse $A^-$ of a finite-dimensional square matrix $A [127, 129]$. In this case, $(A + \delta A)^-$ is well known to be continuous at $\delta A = 0$ if and only if the rank of $(A + \delta A)$ is the same as the rank of $A$ for all perturbations $\delta A$ in some finite neighborhood of $\delta A = 0 [127, 129]$. But if $A$ does not have full rank, there exist matrices $A + \delta A$ with rank$(A + \delta A) > \text{rank} A$ for arbitrarily small $||\delta A||$ > 0. Hence, $(A + \delta A)^-$ is continuous at $\delta A = 0$ if and only if $A$ has full rank—i.e., if and only if $A$ is invertible.

Continuity of the subsystem (3.3) is essential because the subsystem dynamics in Sec. VII is derived from a variational principle. In order for subsystems to be stationary states of the dynamical stability functional, they must first be continuous with respect to small variations. Subsystems derived from generalized inverses are consequently not used in this paper.

A simple corollary of Theorem 2 is that all invertible subsystems must have a vacuum component. This implies that all nontrivial (i.e., not purely vacuum) invertible subsystems must have an indefinite number of particles. A pragmatic reason for allowing this type of subsystem was pointed out by Bell [131].

The real world is made of electrons and protons and so on, and as a result the boundaries of natural objects are fuzzy, and some particles in the boundary can only doubtfully be assigned to either object or environment. I think that fundamental physical theory should be so formulated that such artificial divisions are manifestly inessential.

The superposition of different numbers of particles is therefore essential to the definition of the subsystems considered here. This does not imply that the physical limitations leading to particle-number superselection rules must be ignored in this theory. A detailed discussion of this topic is, however, postponed until Sec. VII.

C. Functions of subsystem creators

Before writing down any formulas for the inverse of a creator, it is helpful to start by establishing some general properties of creators and functions of creators. Creators are actually easier to work with than generic operators. The reason for this is that any creator $A$ can be decomposed into even and odd parts:

\[ A = A_+ + A_, \quad (3.4) \]

in which $A_+ (A_-)$ comprises all terms with an even (odd) number of fermion creation operators. From Eq. (2.12) we see that an even creator commutes with any creator, whereas two odd creators anticommute:

\[ A_+ B_\pm = B_\pm A_+, \quad A_- B_- = -B_- A_. \quad (3.5) \]

The commutator $[A, B] = AB - BA$ of two creators is therefore given by

\[ [A, B] = 2A_- B_-, \quad (3.6) \]

which commutes with any other creator $C$, since $A_- B_-$ is even. Equation (3.5) also implies that all odd creators are nilpotent:

\[ A_-^2 = 0. \quad (3.7) \]

The binomial expansion of any integer power of a creator thus contains only two terms:

\[ A^n = (A_+ + A_-)^n = A_+^n + nA_+^{n-1}A_. \quad (3.8) \]

All functions $f(A)$ defined by a power series can be expanded likewise as

\[ f(A) = f(A_+) + A_- f'(A_+), \quad (3.9) \]

in which $f'(x) = df/dx$. 
D. Inversion formula

Let us now return to the case of an invertible creator \( U \). Given that \( U_0 \) is invertible, we can write \( U \) as

\[
U = U_0(1 + Z) = (1 + Z)U_0, \tag{3.10}
\]

in which \( Z = U_0^{-1} - 1 = UU_0^{-1} - 1 \). Calculating \( U_0^{-1} \) is trivial, because the operator \( U_0 \) for a coherent state is an exponential function [see Eq. (3.10)]. In the matrix notation of Eq. (2.34), \( Z \) has the same lower-triangular form as \( U \):

\[
Z = \begin{pmatrix}
Z_0 & 0 & 0 & 0 \\
Z_1 & Z_0 & 0 & 0 \\
Z_2 & Z_0 & Z_0 & 0 \\
Z_3 & Z_2 & -Z_1 & Z_0
\end{pmatrix} \quad \text{(for } d = 2). \tag{3.11}
\]

The key difference is that \( Z_0 = 0 \), by definition of \( Z \). The operator \( Z \) is therefore nilpotent—i.e., \( Z^k = 0 \) for some finite integer \( k \). Since the matrix size is \( 2^d \times 2^d \), we see immediately that \( k \leq 2d \).

However, we can obtain a stronger bound on \( k \) by noting from Eq. (3.7) that the square of any monomial function of the fermion creation operators is also zero. Since the maximum degree of any such monomial is \( d \), inspection of Eq. (3.8) shows that

\[
Z^k = 0 \quad \forall k \geq \lceil (d/2) \rceil, \tag{3.12}
\]

in which \( \lceil x \rceil \) is the ceiling function. One can then use the geometric series to obtain the inversion formula

\[
U^{-1} = U_0^{-1} \sum_{n=0}^{\lceil (d/2) \rceil} (-1)^n Z^n. \tag{3.13}
\]

E. The exponential representation

In a similar fashion, we can use Eq. (3.10) to calculate the logarithm of \( U \):

\[
\ln U = \ln U_0 + \ln(1 + Z), \tag{3.14}
\]

in which \( \ln(1 + Z) \) is given by the power series

\[
\ln(1 + Z) = \sum_{n=1}^{\lceil (d/2) \rceil} \frac{(-1)^{n+1}}{n} Z^n. \tag{3.15}
\]

Every invertible creator can therefore be represented as an exponential function:

\[
U = \exp X = e^{X_0 e^{(X - X_0)}} = e^{X_0 \sum_{n=0}^{\lceil (d/2) \rceil} \frac{(X - X_0)^n}{n!}}, \tag{3.16}
\]

in which \( X = \ln U \). The intermediate steps in Eq. (3.16) used the facts that \( X_0 = \ln U_0 \) is even and that \( X - X_0 \) is nilpotent.

The exponential representation (3.16) plays a crucial role in the remainder of this paper. For both computation and analysis, it is preferable to take the creator \( X \) as fundamental and define the subsystem \( U \) as \( U = \exp X \). This guarantees that regardless of any changes in \( X \), \( U \) always remains invertible, its inverse being given simply by \( U^{-1} = \exp(-X) \). In this approach, the inversion formula (3.15) becomes redundant.

The Campbell–Baker–Hausdorff formula for two creators \( A \) and \( B \) can now be used to show that

\[
e^A e^B = e^{A+B+\frac{[A,B]}{2}} = e^{A+B+2/(1+\frac{A-B}2)}, \tag{3.17}
\]

in which Eq. (3.6) and \( (A-B)^2 = 0 \) were used. Adding \( e^A e^B \) and \( e^B e^A \) then yields the very useful formula

\[
e^{A+B} = \{e^A, e^B\}, \tag{3.18}
\]

in which \( \{A, B\} = \frac{1}{2}(AB + BA) \) denotes the symmetrized product. As a special case of Eq. (3.18), note that if \( A \) is varied by \( \delta A \), the corresponding first-order variation in \( e^A \) is given by

\[
\delta(e^A) \equiv e^{A+\delta A} - e^A = \{\delta A, e^A\} = \{\delta A, e^A\}, \tag{3.19}
\]

in which terms of second and higher order in \( \delta A \) were discarded in the final step. This yields a simple expression for the derivative of \( e^A \) with respect to a parameter \( s \):

\[
\frac{\partial}{\partial s} (e^A) = \left\{ \frac{\partial A}{\partial s}, e^A \right\}. \tag{3.20}
\]

This result is much simpler than the corresponding formulas for a general operator \( A \) [122]. Several other identities for the symmetrized product of creators are collected together in Appendix F.

F. Quasiclassical subsystems

The representation of quantum states by exponential functions has a long history, dating back at least to the WKB approximation of 1926. The modern theory of generalized coherent states [122, 124–126] also relies heavily on exponential representations. In the latter approach, the coherent states are tied to a particular Lie group (namely, the dynamical group associated with the Hamiltonian of the total system), and the exponential functions used to construct the coherent states are unitary operators involving both creation and annihilation operators. Such unitary operators have the advantage of convenient normalization properties, but they are not useful in the present context because their algebra is not isomorphic to the algebra of the \( \psi \) product.

The exponential representation (3.16) can be viewed as a further generalization of the coherent-state concept, in
that the creator $X$ is not tied to any Lie group. Indeed, $X$ is almost entirely arbitrary, the only restrictions being the algebraic closure constraint of Appendix C (which requires all $\psi$ products of subsystems to be normalizable) and the invertibility constraint requiring $U_0 = \exp X_0$ to be a Glauber coherent state $C^{10}$ (which implies that $X_0$ is a linear function of the boson creation operators). But the latter constraint is not independent of the former, because the algebraic closure condition of Appendix C was used in Appendix D to derive the linearity of $X_0$.

Subsystems permitting an exponential representation $U = \exp X$ will be referred to as quasiclassical subsystems in this paper. However, since the word “quasiclassical” has various other connotations (including the WKB approximation and the Glauber coherent states $C^6$), it is important to be clear about the sense in which this label is being used here. In this paper, “quasiclassical” is just a synonym for “invertible.”

This sense of the word quasiclassical does not imply the use of any approximation. As emphasized in Sec. IIIA, the product $\Psi = UV$ provides an exact representation for arbitrary states $\Psi$. As discussed in Sec. IIIIB, invertibility of $U$ is the minimal restriction needed to ensure continuity of $V$ when $U$ is varied and $\Psi$ is held constant. Note that when $U$ is taken to be quasiclassical, $V$ is quasiclassical if and only if $\Psi$ is quasiclassical—but $\Psi$ need not be quasiclassical.

The fact that all coherent states are invertible is one possible reason for designating the latter as quasiclassical. A more significant reason is that the existence of an unentangled product $|\psi\rangle = |u\rangle \circ |v\rangle$ permits the use of classical probability theory. Since different subsystem decompositions $|u\rangle \circ |v\rangle$ are mutually exclusive alternatives for the representation of $|\psi\rangle$, one’s state of knowledge or belief about the suitability of various decompositions can be described using ordinary Bayesian probability theory. The implications of this idea are developed further in the next section.

**IV. KINEMATICS OF SUBSYSTEMS WITHOUT SUBSPACES**

This section explores several topics that are independent of and logically prior to any concept of subsystem dynamics, thus falling into the category of kinematics. These include the definition of observable quantities, the description of differences between two subsystem decompositions, and subsystem geometry.

For generality, let $|\psi\rangle$ be decomposed into a product of $m$ subsystems, where $m \geq 2$:

$$|\psi\rangle = |u_1\rangle \circ |u_2\rangle \circ \cdots \circ |u_m\rangle. \quad (4.1)$$

Here and below, the subscript $k$ in $|u_k\rangle$ is just a label for the $m$ different subsystems ($k \in \{1, 2, \ldots, m\}$). The subscripts introduced previously in Eq. (2.32) are henceforth retired as they have no further use.

**A. Observables and beables**

Given such a subsystem decomposition, how are we to define the observable quantities of the theory?

A fundamental hypothesis of the present paper is that all observables are calculated from the subsystem vectors $|u_k\rangle$. In other words, no observable quantity is calculated directly from the total system vector $|\psi\rangle$. This is not to say that $|\psi\rangle$ is meaningless; the subsystem dynamics depends on $|\psi\rangle$. However, $|\psi\rangle$ does not appear anywhere in the definition of observables.

Apart from this change, the mathematical apparatus used to define observables is nearly the same as that in ordinary many-particle quantum mechanics $[96, 123, 133]$. That is, observables are represented by hermitian operators $A$ that are totally symmetric with respect to permutations of identical particles. For each such operator, one can calculate a number

$$\langle A \rangle_k = \frac{\langle u_k | A | u_k \rangle}{\langle u_k | u_k \rangle} \quad (4.2)$$

for each subsystem $|u_k\rangle$. In ordinary quantum mechanics, only the eigenvalues of $A$ are observable; the numbers $\langle A \rangle_k$ are therefore interpreted as mean values.

Here, however, the numbers $\langle A \rangle_k$ are taken to be directly observable. This means that observables are defined using what Bell has called the “density of stuff” interpretation $[39]$, first introduced by Schrödinger $[134, 135]$. A similar mass-density interpretation has been used in the dynamical reduction theory of Ghirardi et al. $[136, 138]$, but here $\langle A \rangle_k$ can describe properties other than mass density.

According to Bell $[13]$, the numbers $\langle A \rangle_k$ can then be classified as “beables” rather than observables. That is, subsystem $|u_k\rangle$ is taken to possess the property $\langle A \rangle_k$ independently of any measurement (the concept of “measurement” having no place in the fundamental postulates of the theory).

Such an interpretation of $\langle A \rangle_k$ is possible for the reason already explained in Sec. IIIIB—namely, that different subsystem decompositions can be described using classical probability theory. Without entanglement, there is no need for the properties of the subsystems to be described as potential rather than actual.

Of course, it remains to be seen whether this interpretation of $\langle A \rangle_k$ can give rise to experimental predictions similar to those obtained from ordinary quantum mechanics. This is a difficult problem for which only qualitative results are obtained in this paper. Further discussion of this topic is presented in Sec. IX.

**B. Relational properties**

One aspect of observables that deserves special attention is that not all quantities $\langle A \rangle_k$ are necessarily observable. For example, since the total system is taken to be
closed (i.e., not interacting with anything else), quantities such as absolute position or orientation in space have no meaning within the theory. One can of course calculate numbers for these quantities within a given model, but such numbers are meaningless due to the lack of any external reference frame.

The only meaningful properties are therefore relational properties. For example, although the absolute position of the center of mass of a given subsystem is meaningless, it is meaningful to talk about the relative distance between the centers of mass of two subsystems. Another meaningful quantity would be the mass density of a subsystem relative to the position of its own center of mass.

This type of restriction has received much attention lately in regard to the connection between reference frames (or their lack) and superselection rules [74]. The consequences of such restrictions within the present theory will be investigated in detail (for a specific example) in Sec. VII. Until then, however, it will be assumed that there are no restrictions (in principle) on the observable quantities \( (A)_k \).

### C.Subsystem permutations

Let us now return to the subsystem decomposition \([4.1]\), expressed in terms of creators:

\[
\Psi = U_1 U_2 \cdots U_m. \tag{4.3}
\]

If the only observables are the numbers \([4.2]\), no observable quantity depends on the order in which the subsystems are multiplied. Any permutation

\[
\Psi_\pi = U_{\pi(1)} U_{\pi(2)} \cdots U_{\pi(m)} \tag{4.4}
\]

yields the same observables, where \( \pi \) denotes one of the \( m! \) permutations of the integers \( (1, 2, \ldots, m) \). The value of the product does, however, depend on this order, as indicated by the subscript on \( \Psi_\pi \).

According to the results of Sec. [III F] all but one of the subsystems \( U_k \) are taken to be quasiclassical. For definiteness, let \( U_1 \) be the one subsystem that need not be quasiclassical. In the permutation \([4.4]\), \( U_1 \) is at position \( k = k_v \), where

\[
\pi(k_v) = 1, \quad k_v = \pi^{-1}(1). \tag{4.5}
\]

We can then solve Eq. \([4.4]\) for \( U_1 \), obtaining

\[
V \equiv U_1 = U_{\pi(k_v-1)}^{-1} \cdots U_{\pi(1)}^{-1} \Psi_\pi U_{\pi(m)}^{-1} \cdots U_{\pi(k_v+1)}^{-1}. \tag{4.6}
\]

As noted here, the symbol \( V \) or \( |v\rangle = V|0\rangle \) will often be used to refer to this non-quasiclassical subsystem.

### D.Subsystem differences

The next topic is how to describe small differences between two subsystem decompositions. Consider two sets of subsystems, \( \{U_k\} \) and \( \{U'_k\} \), whose products are \( \Psi \) and \( \Psi' \), respectively. The quasiclassical subsystems can be given an exponential representation \([3.16]\), for which the difference \( \Delta X_k \equiv X'_k - X_k \) is assumed to be small. The corresponding difference \( \Delta U_k \equiv U'_k - U_k \) is then given (for \( k \neq 1 \)) by Eq. \([3.19]\):

\[
\Delta U_k = \exp(X_k + \Delta X_k) - \exp X_k = \{\Delta X_k, U_k\}, \tag{4.7}
\]

in which terms beyond the first order in \( \Delta X_k \) have been neglected. Likewise, the first-order difference between \((U_k^{-1})^{-1} \) and \( U_k^{-1} \) is (for \( k \neq 1 \))

\[
\Delta U_k^{-1} = \exp(-X_k - \Delta X_k) - \exp(-X_k) = -\{\Delta X_k, U_k^{-1}\}. \tag{4.8}
\]

The difference \( \Delta V = V' - V = U'_1 - U_1 \) is taken to be determined by the values of \( \{\Delta U_k^{-1}\} \) and \( \Delta \Psi = \Psi' - \Psi \). To define \( \Delta V \), it is helpful to introduce linear functionals \( V_\Psi \) and \( \tilde{V}_k[Y] \) such that when \( X = \Psi \) and \( Y = U_k^{-1} \) we have

\[
V_\Psi = V = \tilde{V}_k[U_k^{-1}] \quad (k \neq 1). \tag{4.9}
\]

To first order in small quantities, \( \Delta V \) is then given by

\[
\Delta V = V \Delta \Psi + \sum_{k=2}^{m} \tilde{V}_k[\Delta U_k^{-1}]. \tag{4.10}
\]

Note that \( \Delta V \) depends (implicitly) on the choice of permutation \( \pi \) in Eq. \([4.4]\).

For practical calculations, the exponents \( |x_k⟩ = X_k|0⟩ \) are usually expanded in some orthonormal basis \( \{|e_{ki}\}\}:

\[
|x_k⟩ = \sum_i c_{ki} |e_{ki}\rangle \quad (k \neq 1), \tag{4.11}
\]

in which \( c_{ki} = \langle e_{ki} | x_k \rangle \). The corresponding change \( \Delta x_k \) is thus

\[
|\Delta x_k⟩ = \sum_i \Delta c_{ki} |e_{ki}\rangle, \quad \Delta c_{ki} = \langle e_{ki} | \Delta x_k \rangle. \tag{4.12}
\]

Combining this expression with Eq. \([4.7]\) then gives

\[
|\Delta u_k⟩ = \sum_i \Delta c_{ki} |f_{ki}\rangle, \tag{4.13}
\]

in which the creator of \( |f_{ki}\rangle \) is defined to be

\[
f_{ki} = \{e_{ki}, U_k\}. \tag{4.14}
\]

For a given value of \( k \), the set \( \{|f_{ki}\}\} \) is generally not orthonormal, but it is linearly independent if and only if the set \( \{|e_{ki}\}\} \) is. [This can be shown easily using Eq. \([F3]\)] From Eq. \([4.13]\), we now see that \( |\Delta u_k⟩ \) and \( |\Delta x_k⟩ \) are related by

\[
|\Delta u_k⟩ = \left( \sum_i |f_{ki}\rangle \langle e_{ki}| \right) |\Delta x_k⟩. \tag{4.15}
\]
A similar expression for $|\Delta v| = \Delta V|0\rangle$ can be derived from Eq. (4.10):

$$|\Delta v| = |v_{\Delta \psi}| + \sum_{k=2}^{m} \sum_{i} \Delta c_{ki}|g_{ki}|,$$  

(4.16a)

$$= |v_{\Delta \psi}| + \sum_{k=2}^{m} \left( \sum_{i} |g_{ki}\rangle \langle e_{ki}| \right) |\Delta x_{k}|,$$  

(4.16b)

in which the creators of $|g_{ki}\rangle$ are defined as

$$g_{ki} = -\hat{V}_{k}[\{e_{ki}, U_{k}^{-1}\}].$$  

(4.17)

E. Subsystem geometry

The preceding expressions for subsystem differences are useful primarily in the context of geometric structures that allow us to measure the distance between neighboring subsystem decompositions. Such a metric is essential for both the definition of the time functional in Sec. [VI] and the variational formulation of dynamical stability in Sec. [VI].

There are many ways to define such a distance, but the most suitable measure for the present purposes is the Hilbert–Schmidt distance [139, 140]. This distance is based on the Hilbert–Schmidt inner product and norm

$$(A, B) = \frac{1}{2} \text{tr}(A^\dagger B), \quad \|A\| = \sqrt{(A, A)},$$  

(4.18)

in which $A$ and $B$ are operators. The Hilbert–Schmidt distance $D$ is then defined as

$$D(A, B) = \|A - B\|.$$  

(4.19)

The operators of interest are the subsystem projectors

$$\rho_{k} = |u_{k}\rangle\langle u_{k}|, \quad \rho'_{k} = |u'_{k}\rangle\langle u'_{k}|.$$  

(4.20)

The square of the Hilbert–Schmidt distance between the subsystem states $|u_{k}\rangle$ and $|u'_{k}\rangle$ is thus given by

$$D^2(\rho_{k}, \rho'_{k}) = 1 - \text{tr}(\rho_{k}\rho'_{k}) = \frac{\langle u'_{k}|(1 - \rho_{k})|u'_{k}\rangle}{\langle u_{k}|u_{k}\rangle}.$$  

(4.21b)

This satisfies $0 \leq D^2(\rho_{k}, \rho'_{k}) \leq 1$, which is the reason for introducing the factor of $1/2$ in Eq. (4.18).

Our primary interest is in the value of $D^2(\rho_{k}, \rho'_{k})$ for small subsystem differences $|\Delta u_{k}| = |u'_{k}\rangle - |u_{k}\rangle$. Noting that $(1 - \rho_{k})|u_{k}\rangle = 0$, we can rewrite Eq. (4.21) as

$$D^2(\rho_{k}, \rho'_{k}) = \frac{\langle \Delta u_{k}|(1 - \rho_{k})|\Delta u_{k}\rangle}{\langle u_{k}|u_{k}\rangle},$$  

(4.22)

which shows that $D^2(\rho_{k}, \rho'_{k})$ is of order $|\Delta u_{k}|^2 \equiv \langle \Delta u_{k}|\Delta u_{k}\rangle$. Indeed, since $\langle u'_{k}|u'_{k}\rangle = \langle u_{k}|u_{k}\rangle + O(|\Delta u_{k}|)$, we have

$$D^2(\rho_{k}, \rho'_{k}) = \frac{\langle \Delta u_{k}|(1 - \rho_{k})|\Delta u_{k}\rangle}{\langle u_{k}|u_{k}\rangle} + O(|\Delta u_{k}|^3).$$  

(4.23)

The leading term in this expression is the familiar Fubini–Study metric [139, 140]. The Hilbert–Schmidt distance is only one of several large-scale measures of distance that lead to the Fubini–Study metric in the limit of infinitesimal $|\Delta u_{k}|$ [139, 140], but it is generally the easiest of these to work with.

The next step is to extend this measure of distance to the subsystem decomposition [1, 4] as a whole. The simplest way to do this is to construct a direct sum of the projectors $\rho_{k}$:

$$\rho \equiv \bigoplus_{k=1}^{m} \rho_{k} = \rho_{1} \oplus \rho_{2} \oplus \cdots \oplus \rho_{m}.$$  

(4.24)

The resulting operator $\rho$ is also a projector, since $\rho^2 = \rho \rho'$ implies $\rho^2 = \rho$. The Hilbert–Schmidt distance (4.19) between $\rho$ and $\rho'$ is then

$$D^2(\rho, \rho') = m - \text{tr}(\rho \rho') = \sum_{k=1}^{m} D^2(\rho_{k}, \rho'_{k}),$$  

(4.25b)

which is clearly independent of the choice of permutation $\pi$ in Eq. (4.4). In the limit of infinitesimal $|\Delta u_{k}|$, this reduces to

$$D^2(\rho, \rho') = \sum_{k=1}^{m} \frac{\langle \Delta u_{k}|(1 - \rho_{k})|\Delta u_{k}\rangle}{\langle u_{k}|u_{k}\rangle},$$  

(4.26)

which is the Fubini–Study metric for the entire subsystem decomposition.

This result can be expressed more concisely by using a direct-sum representation of vectors. For arbitrary subsystem kets $|\varphi_{k}\rangle$ and $|\chi_{k}\rangle$, let their direct sum be denoted by the same symbol without the subscript $k$:

$$|\varphi\rangle \equiv \bigoplus_{k=1}^{m} |\varphi_{k}\rangle, \quad |\chi\rangle \equiv \bigoplus_{k=1}^{m} |\chi_{k}\rangle.$$  

(4.27)

It is convenient also to bury the normalization factor $\langle u_{k}|u_{k}\rangle$ inside the definition of the inner product:

$$\langle \varphi|\chi\rangle \equiv \sum_{k=1}^{m} \langle \varphi_{k}|\chi_{k}\rangle.$$  

(4.28)

This allows Eq. (4.26) to be written simply as

$$\eta \equiv D^2(\rho, \rho') = \langle \Delta u|(1 - \rho)|\Delta u\rangle,$$  

(4.29)

in which the letter $\eta$ is introduced as a concise symbol for this functional of $\rho$ and $|\Delta u\rangle$.

V. TIME AS A FUNCTIONAL

With this measure of distance in hand, we can now turn to the topic of subsystem dynamics, beginning with
the concept of time. This section starts by explaining why, in a system of interacting particles, it is impossible for both $|\psi\rangle$ and the subsystems $|u_k\rangle$ to satisfy the Schrödinger equation. Next, it is argued that in a closed system, the external time parameter $t$ of conventional quantum mechanics is meaningless. Instead, time should be defined internally via the relations between changes in subsystems. This is then used to construct a time functional, which will be applied to calculations of subsystem dynamics in Sec. VI.

### A. Why interacting subsystems cannot satisfy the Schrödinger equation

Let us start by examining whether it is possible for a closed system and its subsystems to satisfy the Schrödinger equation. It is sufficient for this purpose to study a two-subsystem decomposition $|\psi\rangle = |u\rangle \otimes |v\rangle$. The Schrödinger equation for the total system is

$$i\partial_t|\psi\rangle = H|\psi\rangle = HU|\psi\rangle,$$

(5.1)

in which $H$ is the Hamiltonian. But this is the same as

$$i\partial_t|\psi\rangle = [H,U]|\psi\rangle + UH|\psi\rangle$$

(5.2a)

and

$$= [H,U]|\psi\rangle + |u\rangle \otimes (H|v\rangle).$$

(5.2b)

If the operator $[H,U]$ is a creator, then $[H,U]|\psi\rangle = ([H,U]|0\rangle) \otimes |v\rangle$, in which

$$[H,U]|0\rangle = HU|0\rangle - UH|0\rangle = (H - E_0)|u\rangle,$$

(5.3)

where $E_0$ is the energy of the vacuum. Assuming that $E_0 = 0$ (which is necessary if $|0\rangle$ is to act as a time-independent multiplicative identity), Eq. (5.2b) can thus be written as

$$i\partial_t|\psi\rangle = (H|u\rangle) \otimes |v\rangle + |u\rangle \otimes (H|v\rangle).$$

(5.4)

A comparison with the differential identity

$$\partial_t|\psi\rangle = (\partial_t|u\rangle) \otimes |v\rangle + |u\rangle \otimes (\partial_t|v\rangle)$$

(5.5)

then shows that both $|u\rangle$ and $|v\rangle$ can satisfy the Schrödinger equation.

But when is it true that $[H,U]$ is a creator? If $H$ conserves particle number, it can be written as a polynomial (usually quadratic) function of the hopping operators $a_i^\dagger a_j$ (which reduce to number operators $N_i = a_i^\dagger a_i$ when $i = j$). Now the commutator of a hopping operator and a creation operator is just another creation operator:

$$[a_i^\dagger a_j, a_k^\dagger] = \delta_{jk}a_i^\dagger.$$  

(5.6)

Assuming that $H$ is linear in the hopping operators—as would be the case for a system of noninteracting particles—this shows that the commutator of $H$ with a creator always generates another creator. The subsystems $|u\rangle$ and $|v\rangle$ can therefore satisfy the Schrödinger equation in this case.

However, if pairs of particles interact, then $H$ also includes the pair distribution operator

$$P_{ij} = N_i N_j - \delta_{ij} N_i,$$

(5.7)

for which

$$[P_{ij}, a_k^\dagger] = \delta_{ik} a_k^\dagger N_j + \delta_{jk} a_k^\dagger N_i.$$

(5.8)

This contains both creation and annihilation operators. Hence, in a system of interacting particles, $[H,U]$ is not a creator, and it is impossible in general for all of $|\psi\rangle$, $|u\rangle$, and $|v\rangle$ to satisfy the Schrödinger equation.

### B. Relational time in quantum mechanics

An extensive literature on the topic of relational time in quantum mechanics also casts serious doubt on whether the time parameter $t$ in Eq. (5.1) can have any meaning in a closed system (see, e.g., Refs. 35–51). The argument is very similar to that already given in Sec. IV B. Namely, within a closed system, one can observe only changes in the relations between various subsystems; one does not have access to any hypothetical absolute external time variable.

Page and Wooters 30 37 31 have argued that this gives rise to an effective energy superselection rule in which a coherent superposition of different energy eigenstates is experimentally indistinguishable from a statistical mixture. Page 11, Poulin 17, and Milburn and Poulin 48 have extended this approach by using group averaging of density operators to eliminate the external time parameter $t$, thereby reducing a general unmixed state to a statistical mixture of energy eigenstates.

A common strategy in this type of approach is to identify one subsystem as a clock and measure time via correlations between the clock subsystem and other subsystems. This gives rise to an effective decoherence mechanism if the clock is of finite size 44 45 47 48 44.

Barbour 42 has argued that such approaches do not agree with how time is defined operationally. In practice, we define time not by looking at a single clock, but by using the time parameter $t$ to achieve the best fit to all of the experimental information at our disposal. This ephemeris time concept was developed long ago by astronomers, but even today it is how time is defined from a network of atomic clocks, all of which operate in different environments and run at slightly different rates. From this perspective, “ultimately the universe is the only clock” 42.

Time is defined in the present paper by using the concept of ephemeris time in the context of the geometric approach to quantum mechanics developed by Anandan and Aharonov 144 145. These authors have alluded to this concept themselves, even going so far as to say that “The parameter $t$ represents correlation between the Fubini–Study distances determined by different clocks” 144.
However, the correlation between different clocks is found only in these words, not in any of their equations. Their equations establish only a relationship between external time and the Fubini–Study distance traveled by a system evolving according to Schrödinger’s equation [134] [135]. During a “measurement,” however, the system can move a finite distance in the projective Hilbert space during a time interval of zero [136]. It is not clear how these disparate Hilbert-space transport mechanisms are to be reconciled. But this is of course just the old conundrum posed by von Neumann’s axioms of time evolution [53].

This paper implements Anandan and Aharonov’s idea mathematically by introducing a time functional that is optimized to achieve the best fit between Schrödinger dynamics and the changes that occur in all subsystems. The actual value of these changes is not determined by this functional; that task is left to the principle of dynamical stability, to be discussed below in Sec. VI.

C. Definition of the time functional

The time functional can be defined using a simple extension of the geometric concepts introduced previously in Sec. [135]. Consider two slightly different subsystem decompositions, \( \rho \) and \( \rho' \). If \( \rho' \) differs from \( \rho \) only by a Schrödinger time evolution, the two decompositions must be related by

\[
\rho' \triangleq e^{-i\hat{H}\Delta t} \rho e^{i\hat{H}\Delta t} \tag{5.9}
\]

for some time interval \( \Delta t \), in which

\[
\hat{H} \equiv \bigoplus_{k=1}^m H \tag{5.10}
\]

is the Hamiltonian in the direct-sum formalism. Of course, for arbitrary \( \rho \) and \( \rho' \), Eq. (5.9) will not be true, but we can try to get as close as possible to such a description by minimizing the Hilbert–Schmidt distance between the two sides of the equation. In other words, we can define a function

\[
\lambda(\Delta t) \equiv D^2(e^{-i\hat{H}\Delta t} \rho e^{i\hat{H}\Delta t}, \rho') \tag{5.11a}
\]

\[
= D^2(\rho, e^{i\hat{H}\Delta t} \rho' e^{-i\hat{H}\Delta t}) \tag{5.11b}
\]

and seek the value of \( \Delta t \) that minimizes this function. This special value, denoted \( \Delta t = \Delta t \), provides the best fit between \( \rho \) and \( \rho' \) that can be expressed in the language of Schrödinger dynamics.

Our only concern is the case of infinitesimal differences \( ||\Delta u_k|| \), for which \( \Delta t \) is also infinitesimal. We can therefore use the Fubini–Study metric of Eqs. (4.26) and (4.29) to write

\[
\lambda(\Delta t) = \langle u' | e^{-i\hat{H}\Delta t} (1 - \rho) e^{i\hat{H}\Delta t} | u' \rangle \tag{5.12a}
\]

\[
= \sum_{k=1}^m \frac{\langle u'_k | e^{-i\hat{H}\Delta t} (1 - \rho_k) e^{i\hat{H}\Delta t} | u'_k \rangle}{\langle u_k | u_k \rangle}. \tag{5.12b}
\]

Here the exponentials can be expanded in the usual way:

\[
e^{-i\hat{H}\Delta t} (1 - \rho) e^{i\hat{H}\Delta t} = (1 - \rho) - i\Delta t [\hat{H}, 1 - \rho] \tag{5.13}
\]

\[
- \frac{1}{2} \Delta t^2 [\hat{H}, [\hat{H}, 1 - \rho]] + \cdots.
\]

If we treat \( \Delta t \) and \( ||\Delta u|| \) as of the same order and work to second order overall, the final result can be written as

\[
\lambda(\Delta t) = \eta - 2\Delta t \text{Im}(\Delta u|H) + \Delta t^2 (H|H), \tag{5.14}
\]

in which \( \lambda(0) = \eta \) was already defined in Eq. (4.29). The vector \( |H\rangle \) in this expression is defined as

\[
|H\rangle \equiv (1 - \rho)\hat{H}|u\rangle \tag{5.15a}
\]

\[
= \bigoplus_{k=1}^m (1 - \rho_k)H|u_k\rangle. \tag{5.15b}
\]

The minimum of the quadratic function (5.14) occurs at \( \Delta t = \Delta t \), in which

\[
\Delta t = \frac{\text{Im}(\Delta u|H)}{<H|H>}. \tag{5.16}
\]

This is the desired expression giving the optimal time interval \( \Delta t \) as a functional of the subsystem change \( |\Delta u|\).

D. Properties of the time functional

Note that the solution (5.16) can be used to rewrite Eq. (5.14) as

\[
\lambda(\Delta t) = \eta + \Delta t(\Delta t - 2\Delta t)(H|H). \tag{5.17}
\]

When \( \Delta t = \Delta t \), this function attains its minimum value

\[
\lambda(\Delta t) = \eta - \Delta t^2 (H|H) \tag{5.18a}
\]

\[
= \eta - \frac{(\text{Im}(\Delta u|H))^2}{<H|H>}. \tag{5.18b}
\]

Given the definition (5.11) of \( \lambda(\Delta t) \) as the square of a distance, it seems obvious that this minimum value must satisfy \( \lambda(\Delta t) \geq 0 \). However, it is not immediately clear from Eq. (5.18) that this is in fact the case.

To see explicitly that \( \lambda(\Delta t) \) is indeed nonnegative, note that

\[
(\text{Im}(\Delta u|H))^2 \leq ||\Delta u|H||^2. \tag{5.19}
\]

This inequality in conjunction with Eq. (5.18b) implies that

\[
\lambda(\Delta t) \geq \eta - \frac{||\Delta u|H||^2}{<H|H>}. \tag{5.20a}
\]

\[
= <\Delta u|(1 - \rho - \Pi_H)|\Delta u>, \tag{5.20b}
\]

in which \( \Pi_H \) is the projector

\[
\Pi_H \equiv |H><H|<H|H>. \tag{5.21}
\]
Because $\rho$ and $\Pi_H$ are orthogonal, the operator $(1 - \rho - \Pi_H)$ is also a projector. This can be used to write
\[ \langle \Delta u | (1 - \rho - \Pi_H) | \Delta u \rangle = \langle w | w \rangle \geq 0, \quad (5.22) \]
in which
\[ |w\rangle = (1 - \rho - \Pi_H) | \Delta u \rangle. \quad (5.23) \]
This proves that $\lambda(\Delta t) \geq 0$, and furthermore that a necessary condition for $\lambda(\Delta t) = 0$ is $|w\rangle = 0$ or
\[ (1 - \rho) | \Delta u \rangle = \Pi_H | \Delta u \rangle. \quad (5.24) \]
However, this condition is not sufficient. Tracing back to the previous inequality $\langle 5.19 \rangle$, we see that $\text{Re} \langle \Delta u | H \rangle = 0$ is also required. Hence, in order to achieve $\lambda(\Delta t) = 0$, it is necessary and sufficient that
\[ (1 - \rho) | \Delta u \rangle = iC|H\rangle, \quad (5.25) \]
in which $C$ is a real constant. In other words, the component of $|\Delta u\rangle$ that is orthogonal to $|u\rangle$ must be proportional to $|H\rangle$, with an imaginary coefficient.

What is the significance of this? According to the definition $\langle 5.15 \rangle$, the vector $|H\rangle$ is just the component of $\hat{H}|u\rangle$ that is orthogonal to $|u\rangle$. Hence, the condition $\langle 5.25 \rangle$ says that in order to achieve $\lambda(\Delta t) = 0$, all subsystems must satisfy the Schrödinger equation, but only insofar as the component of $|\Delta u\rangle$ orthogonal to $|u\rangle$ is concerned. [The component of $|\Delta u\rangle$ that is parallel to $|u\rangle$ does not contribute to the distance $\langle 4.29 \rangle$.]

Another way of expressing $|H\rangle$ is
\[ |H\rangle = \bigoplus_{k=1}^{\infty} (H - \langle H \rangle_k)|u_k\rangle, \quad (5.26) \]
in which $\langle H \rangle_k$ is the mean value [cf. Eq. $\langle 4.12 \rangle$]
\[ \langle H \rangle_k \equiv \frac{\langle u_k | H | u_k \rangle}{\langle u_k | u_k \rangle}. \quad (5.27) \]
Hence, the inner product $\langle H | H \rangle$ can be written as
\[ \langle H | H \rangle = \sum_{k=1}^{m} \frac{\langle u_k | (H - \langle H \rangle_k)^2 | u_k \rangle}{\langle u_k | u_k \rangle}. \quad (5.28) \]
This provides a simple physical interpretation of $\langle H | H \rangle$: it is the combined energy variance of all subsystems. The corresponding standard deviation is denoted $\Delta E \equiv \sqrt{\langle H | H \rangle}$.

The inequality $\lambda(\Delta t) \geq 0$ can thus be written in the alternative form [cf. Eq. $\langle 5.18a \rangle$]
\[ \Delta E^2 \Delta t^2 \leq \eta. \quad (5.29) \]
This looks vaguely like a time–energy uncertainty relation, except that the inequality is pointing in the wrong direction—so actually it is nothing of the kind. It simply says that, for a given squared distance $\eta = D^2(\rho, \rho')$, there is an upper bound on the optimal value of $\Delta t$ that can be fitted to $\rho$ and $\rho'$ using Eq. $\langle 5.9 \rangle$. Furthermore, because the optimal value $\Delta t = \Delta t_{\text{opt}}$ is intimately related to Schrödinger dynamics, the numerical value of $\Delta t$ depends on the energy scale $\Delta E$ determined by the subsystem decomposition $\rho$.

To conclude this section, we may note that the time functional $\langle 5.16 \rangle$ offers a very simple way of implementing the idea that “ultimately the universe is the only clock.” But of course, as mentioned previously, the definition of such a clock tells us nothing about how the subsystems evolve in time. Finding a way to define this time evolution is the subject of the next section.

VI. DYNAMICAL STABILITY OF SUBSYSTEMS

The most obvious criterion for defining subsystem dynamics is to maximize the stability of the subsystem decomposition. In other words, we should choose the dynamics such that the decomposition “hops about the least” $\langle 147 \rangle$. This concept of dynamical stability has a long history. In the early days of quantum mechanics, Schrödinger used it in an attempt to interpret particles as stable wave packets $\langle 144 \rangle \langle 135 \rangle$. More recently, its application for decoherence theory has been repeatedly emphasized by Zeh $\langle 108 \rangle \langle 148 \rangle \langle 155 \rangle$. The basic idea has been developed extensively by Zurek under such names as the predictability sieve $\langle 156 \rangle \langle 157 \rangle$, one-section $\langle 69 \rangle \langle 158 \rangle \langle 159 \rangle$, the existential interpretation $\langle 156 \rangle \langle 158 \rangle \langle 159 \rangle$, and quantum Darwinism $\langle 159 \rangle \langle 160 \rangle$.

In this section, the concept of dynamical stability is defined for the subsystem decomposition $\langle 4.4 \rangle$ in terms of a dynamical stability functional $\chi$. The time evolution of the subsystems is then determined by maximizing $\chi$. For simplicity, the total system state $|\psi\rangle$ is initially assumed to be independent of time. This analysis is then extended to the case of time-dependent $|\psi\rangle$.

A. Dynamical stability functional

To simplify the description of dynamical stability, it is convenient to introduce the dimensionless variable
\[ \sigma \equiv \Delta E \Delta t = \frac{\text{Im}(\Delta u | H \rangle)}{\Delta E}. \quad (6.1) \]
The dynamical stability functional $\chi$ is then defined as
\[ \chi \equiv \frac{\sigma^2}{\eta} = \frac{\Delta E^2 \Delta t^2}{D^2(\rho, \rho')} \quad (\eta \neq 0). \quad (6.2) \]
The foundation for this definition is the inequality $\langle 5.29 \rangle$, which says that $0 \leq \chi \leq 1$. The principle of dynamical stability is implemented by holding $\rho$ fixed and varying $\rho'$ so as to maximize the value of $\chi$. The decompositions $\rho$ and $\rho'$ could thus be regarded as “initial” and “final,” respectively.
although this has the potential to be misleading because it has nothing to do with the sign of $\Delta t$.

Maximizing $\chi$ with respect to variations in $\rho'$ simply requires that the subsystems change as little as possible (as measured by the Fubini–Study metric) in a given infinitesimal time interval $\Delta t$. According to the results of Sec. IV D if $|\psi\rangle$ is assumed to satisfy the Schrödinger equation, the upper limit $\chi = 1$ is generally unattainable in a system of interacting particles. Suppose now that $\rho'$ is varied by a small amount $\delta \rho$. (This should perhaps be written as $\delta \rho'$, but the prime symbol can be omitted because $\rho$ itself is not varied.) This will give rise to corresponding variations $\delta \sigma$, $\delta \eta$, and $\delta \chi$, which are related by

$$
\delta \chi = \frac{(\sigma + \delta \sigma)^2}{\eta + \delta \eta} - \frac{\sigma^2}{\eta} = 2\sigma \delta \sigma - \chi \delta \eta + \delta \sigma^2.
$$

(6.3)

To first order in small quantities, this reduces to

$$
\delta \chi = \eta^{-1}(2\sigma \delta \sigma - \chi \delta \eta),
$$

(6.4)

in which all variations are evaluated to first order in $\delta \rho$. The stationary states of the dynamical stability functional are then given by $\delta \chi = 0$ or

$$
\delta \eta = \frac{\delta \sigma}{\sigma} (\sigma \neq 0),
$$

(6.5)

in which $\sigma \neq 0$ can always be assumed because we have no interest in the minima of $\chi$.

### B. Time-independent $\Psi$

Let us now apply the principle of dynamical stability to the special case in which the total system state $\Psi$ is assumed to be independent of time, so that $V \Psi = \Psi' - \Psi = 0$. This implies that $V_{\Delta \Psi} = 0$ in the general expression (4.10) for $\Delta V \equiv \Delta U_1$, which simplifies the analysis considerably.

The quantities to be varied are the “final” subsystem exponents $|x'_k\rangle = X'_k(0)$ for the quasiclassical subsystems ($k \neq 1$). As before, it is convenient to use a direct-sum representation for the differences $|\Delta x_k\rangle = |x'_k\rangle - |x_k\rangle$:

$$
|\Delta x\rangle = \bigoplus_{k=2}^{m} |\Delta x_k\rangle.
$$

(6.6)

In contrast to the definition of $|\Delta u\rangle$ [see Eq. (4.27)], the value $k = 1$ is not included in the definition of $|\Delta x\rangle$. There are two reasons for this. First, as noted in Sec. IV D only the subsystems with $k \neq 1$ are treated as independently variable; the subsystem $k = 1$ is entirely determined by $\Psi$ and the other subsystems. Second, it is not generally even possible to define $X_1 = \ln U_1$, since $\Psi$ need not be quasiclassical.

The dimensionless time interval $\delta \rho$ can now be expressed in terms of the independent variables $|\Delta x\rangle$ as

$$
\delta \rho = \frac{\text{Im}\langle \Delta u | H \rangle}{\Delta E} \equiv \text{Im}\langle \Delta x | \sigma \rangle,
$$

(6.7)

in which the components of the vector $|\sigma\rangle$ are given by [cf. Eqs. (4.15), (4.16), (5.15)]

$$
\langle e_{k1} | \sigma \rangle = \frac{1}{\Delta E} \left[ \langle f_{k1} | (1 - \rho_k) H | u_k \rangle \langle u_k | u_k \rangle + \langle g_{k1} | (1 - \rho_v) H | v \rangle \langle v | v \rangle \right].
$$

(6.8)

In this expression, $\rho_v \equiv \rho_1$ and $|v\rangle \equiv |u_1\rangle$. The squared distance $\eta$ can be written likewise as

$$
\eta = \langle \Delta x | \eta | \Delta x \rangle,
$$

(6.9)

in which the matrix elements of the operator $\hat{\eta}$ are given by [cf. Eq. (4.29)]

$$
\langle e_{k1} | \hat{\eta} | e_{k'1} \rangle = \delta_{kk'} \frac{\langle f_{k1} | (1 - \rho_k) f_{k'1} \rangle}{\langle u_k | u_k \rangle} + \frac{\langle g_{k1} | (1 - \rho_v) g_{k'1} \rangle}{\langle v | v \rangle}.
$$

(6.10)

The operator $\hat{\eta}$ is clearly positive ($\hat{\eta} \geq 0$), and it can be made positive definite ($\hat{\eta} > 0$) if we agree to exclude the vacuum state $\{1\}$ from the orthonormal basis $\{|e_{k1}\rangle\}$ used to define the subsystem exponents in Eq. (4.11).

If we now vary $|x'_k\rangle$ by $|\delta x\rangle$ (holding $|x\rangle$ fixed), $|\Delta x\rangle$ also varies by $|\delta x\rangle$. The resulting first-order variations in $\sigma$ and $\eta$ are

$$
\delta \sigma = \text{Im}\langle \delta x | \sigma \rangle = \frac{\langle \delta x | \sigma \rangle - \langle \sigma | \delta x \rangle}{2i}, \quad (6.11a)
$$

$$
\delta \eta = \langle \delta x | \eta | \Delta x \rangle + \langle \Delta x | \eta | \delta x \rangle. \quad (6.11b)
$$

Substituting these expressions into the stationary-state condition (6.5) gives

$$
\langle \delta x | \eta | \Delta x \rangle + \langle \Delta x | \eta | \delta x \rangle = -iC(\langle \delta x | \sigma \rangle - \langle \sigma | \delta x \rangle), \quad (6.12)
$$

in which $C \equiv \eta/\sigma$ is real. Because the variation $|\delta x\rangle$ is arbitrary, we can partition this equation in the usual way [161] pp. 764–765] to obtain

$$
\langle \delta x | \eta | \Delta x \rangle = -iC \langle \delta x | \sigma \rangle, \quad (6.13)
$$

together with its complex conjugate. Removing the arbitrary vector $\langle \delta x |$ gives the linear algebraic equation

$$
\hat{\eta} \langle \Delta x | = -iC \langle \sigma |, \quad (6.14)
$$

in which $\hat{\eta}$ is positive definite and therefore invertible. All stationary states of the dynamical stability functional $\chi$ with $\chi > 0$ are thus given explicitly by

$$
|\Delta x\rangle = -iC\hat{\eta}^{-1} |\sigma\rangle, \quad (6.15)
$$

Upon substituting this result back into the definitions of $\sigma$, $\eta$, and $\chi$, we find

$$
\sigma = C|\hat{\eta}^{-1}| \sigma \rangle, \quad (6.16a)
$$

$$
\eta = C^2|\hat{\eta}^{-1}| \sigma \rangle, \quad (6.16b)
$$

$$
\chi = |\sigma| \hat{\eta}^{-1} |\sigma\rangle. \quad (6.16c)
$$
The only degree of freedom in the solution \((6.15)\) is the value of the real constant \(C = \eta/\sigma = \sigma/\chi\). Since \(\chi\) is independent of \(C\), one can find the value of \(C\) from a given time interval \(\Delta t\) simply by calculating \(C = \Delta E\Delta t/\chi\). The sign of \(\Delta t\) can be positive or negative, but its magnitude should always be chosen small enough that \(\eta \ll 1\) (or else the approximations used in deriving the basic equations are no longer valid).

Thus, for a given sign and magnitude of \(\Delta t\), there is only one stationary state of the dynamical stability functional with \(\chi > 0\). This strongly suggests that this stationary state is the unique global maximum of \(\chi\). A proof of this conjecture is given in Appendix G.

The solution \((6.15)\) can be viewed as a differential equation for \(|x\rangle\), since
\[
\frac{\partial |x\rangle}{\partial t} = \lim_{\Delta t \to 0} \frac{|\Delta x\rangle}{\Delta t} = -i\Delta E \frac{\hat{\eta}^{-1}|\sigma\rangle}{\langle \sigma|\hat{\eta}^{-1}|\sigma\rangle},
\]
in which the limit \(\Delta t \to 0\) is somewhat redundant because it has been assumed throughout the derivation. This equation can be integrated to obtain \(|x\rangle\) as a function of \(t\); in practice, this is done by using Eq. \((6.15)\) repeatedly for small but finite intervals \(\Delta t\). Numerical calculations on simple models (see Sec. [VIE]) show that the change in \(|x\rangle\) over a fixed time interval \(T\) does indeed converge (quadratically) to a definite value in the limit \(\Delta t \to 0\).

The time evolution generated by Eq. \((6.17)\) is deterministic. That is, the final subsystem decomposition is uniquely determined by the initial one, and if the subsystems are propagated forward and backward over a finite interval (by changing the sign of \(\Delta t\) at the far end), the returning solution always converges to its initial value. Hence, even though the differential equation \((6.17)\) is nonlinear, it does not exhibit any of the lack of determinism so characteristic of standard quantum mechanics.

### C. Time-dependent \(\Psi\)

The next step is to lift the restriction \(\Delta \Psi = 0\) that was imposed in Sec. [VII]. Because the total system is closed, \(|\Delta \psi\rangle\) is assumed to follow the Schrödinger equation (to first order in the time functional \(\Delta t\):
\[
|\Delta \psi\rangle = -i\Delta t H|\psi\rangle = -i\Delta t H \Psi |0\rangle.
\]
This does not imply that \(\Delta \Psi = -i\Delta t H \Psi\), because \(H \Psi\) is not a creator. Rather, we have
\[
\Delta \Psi = -i\Delta t (H \cdot \Psi),
\]
in which \(H \cdot \Psi\) denotes the creator defined by \((H \cdot \Psi)|0\rangle \equiv H|\psi\rangle\). Upon substituting this result into Eq. \((4.16)\), we obtain
\[
|\Delta \psi\rangle = -i\Delta t |v_{H \cdot \Psi}\rangle + \sum_{k=2} m \left( \sum_i (g_{ki}) \langle e_{ki} | \right) |\Delta x_k\rangle.\]
The dimensionless interval \(\sigma\) then takes the form
\[
\sigma = \Delta E \Delta t = \frac{\text{Im}(\Delta u|H)}{\Delta E} = \frac{\text{Im}(\Delta \tau|\sigma_0)}{\omega} \text{Re} \left( \frac{\langle v_{H \cdot \Psi}|(1 - \rho_v)H|v_{H \cdot \Psi}\rangle}{\langle v|v\rangle} \right), (6.21)
\]
in which \(|\sigma_0\rangle\) relabels the vector introduced previously in Eq. \((6.8)\). Noting that \(\Delta t\) now appears on both sides of the equation, we can combine these terms to obtain
\[
\omega \Delta E \Delta t = \text{Im}(\Delta \tau|\sigma_0), \quad (6.22)
\]
in which \(\omega\) is the real constant
\[
\omega \equiv 1 - \frac{1}{\Delta E^2} \text{Re} \left( \frac{\langle v_{H \cdot \Psi}|(1 - \rho_v)H|v_{H \cdot \Psi}\rangle}{\langle v|v\rangle} \right), (6.23)
\]
At this point it is convenient to redefine the vector \(|\sigma\rangle\) so as to obtain the same outward appearance as Eq. \((6.7)\):
\[
|\sigma\rangle = \frac{\text{Im}(\Delta \tau|\sigma_0)}{\omega} \equiv \text{Im}(\Delta \tau|\sigma), \quad (6.24)
\]
in which \(|\sigma\rangle\) is just a renormalized version of Eq. \((6.8)\):
\[
\langle e_{ki}|\sigma\rangle = \frac{1}{\omega \Delta E} \left[ \frac{\langle f_{ki}|(1 - \rho_v)H|u_k\rangle}{\langle u_k|u_k\rangle} + \frac{\langle g_{ki}|(1 - \rho_v)H|v\rangle}{\langle v|v\rangle} \right]. (6.25)
\]
Hence, the time evolution of \(\Psi\) affects the variable \(\sigma\) only through the renormalization factor \(\omega\).

However, its effect on \(\eta\) is more profound. Because \(|\Delta \psi\rangle\) is now linear in \(\Delta t\), \(\eta\) in Eq. \((4.29)\) becomes a quadratic function of \(\Delta t\). When expressed in terms of \(\sigma\), this quadratic dependence takes the form
\[
\eta(\Delta t) = \eta_0 + 2\beta \sigma + \kappa \sigma^2, \quad (6.26)
\]
in which \(\eta(\Delta t) = D^2(\rho_v, \rho_v) = \langle \Delta \psi|\rho|\Delta \psi\rangle\) is the function defined in Eq. \((4.29)\) and \(\eta_0 \equiv \eta(0) = \langle \Delta x|\hat{\eta}|\Delta x\rangle\) relabels the quantity defined earlier in Eq. \((6.9)\). Although \(\eta\) now depends on \(\Delta t\), one should note carefully that \(\eta(\Delta t) \neq \lambda(\Delta t)\) [see Eq. \((5.18)\)].

The new functional \(\beta\) in Eq. \((6.26)\) is defined by
\[
\beta = \text{Im}(\Delta \tau|\beta), \quad (6.27)
\]
in which \(|\beta\rangle\) is the vector
\[
\langle e_{ki}|\beta\rangle = \frac{1}{\Delta E} \frac{\langle g_{ki}|(1 - \rho_v)|v_{H \cdot \Psi}\rangle}{\langle v|v\rangle}. (6.28)
\]
The dimensionless constant \(\kappa\) in Eq. \((6.26)\) is defined by
\[
\kappa = \frac{1}{\Delta E^2} \frac{\langle v_{H \cdot \Psi}|(1 - \rho_v)|v_{H \cdot \Psi}\rangle}{\langle v|v\rangle}. (6.29)
\]
With these results, we can now construct the dynamical stability functional \(\chi = \sigma^2/\eta\) just as before, in which \(\eta \equiv \eta(\Delta t)\).
D. Real matrix representation

There are several ways to solve the variation problem for \( \chi \). The method used here has the advantage of requiring very little modification when the definition of distance is changed below in Sec. VII.

When \( \eta \) in Eq. (6.26) is expressed as a function of the expansion coefficients \( \Delta c_{ki} = \langle \epsilon_{ki} | \Delta x \rangle \) introduced in Eq. (4.12), the result can be written as

\[
\eta = \mu + \nu, \tag{6.30}
\]

in which \( \mu \) has the same form as \( \eta_0 = \langle \Delta x | \hat{\eta} | \Delta x \rangle \):

\[
\mu = \sum_{ki} \sum_{k'i'} \Delta c^e_{ki} \Delta c^\prime_{k'i'} \mu_{ki,k'i'}. \tag{6.31}
\]

However, \( \nu \) is qualitatively different:

\[
\nu = \text{Re} \left( \sum_{ki} \sum_{k'i'} \Delta c^e_{ki} \Delta c^\prime_{k'i'} \nu_{ki,k'i'} \right). \tag{6.32}
\]

Here the matrix elements \( \mu_{ki,k'i'} \) and \( \nu_{ki,k'i'} \) are given by

\[
\mu_{ki,k'i'} = \delta_{ki} \frac{\langle f_k | (1 - \rho_k) f_{k'i'} \rangle}{\langle u_k | u_k \rangle} + \frac{\langle g_k | (1 - \rho_k) g_{k'i'} \rangle}{\langle v | v \rangle} + \frac{1}{2} (\beta_{ki} \sigma^e_{k'i'} + \sigma_{ki} \beta^e_{k'i'} + \kappa \sigma_{ki} \sigma^e_{k'i'}) \tag{6.33}
\]

and

\[
\nu_{ki,k'i'} = -\frac{1}{2} (\beta_{ki} \sigma^e_{k'i'} + \sigma_{ki} \beta^e_{k'i'} + \kappa \sigma_{ki} \sigma^e_{k'i'}), \tag{6.34}
\]

in which \( \beta_{ki} = \langle \epsilon_{ki} | \beta \rangle \) and \( \sigma_{ki} = \langle \epsilon_{ki} | \sigma \rangle \). Because \( \nu \), unlike \( \mu \) and \( \eta_0 \), is not a sesquilinear form, the stationary-state equation \( (6.5) \) no longer reduces to a linear algebraic equation for the complex coefficients \( \Delta c_{ki} \).

However, one can put Eq. (6.5) into the form of a linear algebraic equation simply by separating

\[
\Delta c_{ki} = \Delta c^e_{ki} + i \Delta c^\prime_{ki}, \tag{6.35}
\]

and working with the real variables \( \Delta c^e_{ki} \) and \( \Delta c^\prime_{ki} \). If the real and imaginary parts of \( \mu_{ki,k'i'} \) and \( \nu_{ki,k'i'} \) are likewise separated, \( \eta \) can be expressed in block matrix notation as

\[
\eta = \begin{pmatrix} \Delta c^e & \Delta c^\prime \\ \Delta c^\prime & \Delta c^e \end{pmatrix} \begin{pmatrix} \mu' + \nu' & -\mu'' + \nu'' \\ -\mu'' + \nu'' & \mu' - \nu' \end{pmatrix} \begin{pmatrix} \Delta c^e \\ \Delta c^\prime \end{pmatrix}, \tag{6.36}
\]

in which all matrix elements are real.

It is convenient to write this equation in a quasi-Dirac notation:

\[
\eta = \langle \Delta x | \hat{\eta} | \Delta x \rangle, \tag{6.37}
\]

in which the rounded ket vector is represented by the real column matrix

\[
| \Delta x \rangle = \begin{pmatrix} \Delta c^e \\ \Delta c^\prime \end{pmatrix}, \tag{6.38}
\]

and the operator \( \hat{\eta} \) is represented by the real symmetric matrix

\[
\hat{\eta} = \begin{pmatrix} \mu' + \nu' & -\mu'' + \nu'' \\ -\mu'' + \nu'' & \mu' - \nu' \end{pmatrix}. \tag{6.39}
\]

This matrix is symmetric because \( \mu' \), \( \nu' \), and \( \nu'' \) are symmetric, whereas \( \mu'' \) is antisymmetric.

A similar representation can be introduced for the dimensionless time interval

\[
\sigma = \text{Im} \langle \Delta x | \sigma \rangle = \text{Im} \left( \sum_k \Delta c^e_{ki} \sigma_{ki} \right), \tag{6.40}
\]

if we separate \( \sigma_{ki} = \sigma'_{ki} + i \sigma''_{ki} \) just as for \( \Delta c_{ki} \). This is written in quasi-Dirac notation as

\[
\sigma = (\Delta x | \sigma \rangle = (\sigma | \Delta x \rangle, \tag{6.41}
\]

in which the matrix representation for \( \sigma \) is

\[
| \sigma \rangle = \begin{pmatrix} \sigma'' \\ -\sigma' \end{pmatrix}. \tag{6.42}
\]

E. Dynamically stable subsystem changes

The dynamical stability functional \( \chi \) now has a form very similar to that found in Sec. VII:

\[
\chi = \frac{\sigma^2}{\eta} = \frac{\langle \Delta x | \sigma \rangle^2}{\langle \Delta x | \hat{\eta} | \Delta x \rangle}. \tag{6.43}
\]

When \( \langle \Delta x \rangle \) is varied by \( \delta \langle x \rangle \), the stationary states are determined by \( \delta \chi = 0 \) or [cf. Eq. (6.13)]

\[
\langle \delta \langle x \rangle | \Delta x \rangle = C \langle \delta \langle x \rangle | \sigma \rangle, \tag{6.44}
\]

in which \( C \equiv \sigma / \chi = \eta / \sigma \). Because \( \langle \delta \langle x \rangle \rangle \) can range over the whole vector space, this is equivalent to

\[
\langle \delta \langle x \rangle \rangle = C \langle \delta \langle x \rangle | \sigma \rangle, \tag{6.45}
\]

which can be solved as before to obtain the dynamically stable subsystem change

\[
\langle \Delta x \rangle = C \langle \delta \langle x \rangle | \sigma \rangle. \tag{6.46}
\]

Note that the factor of \(-i\) in Eq. (6.15) is absent here because it is embedded into the definition (6.42) of \( | \sigma \rangle \).

The qualitative properties of this solution are again very similar to the solution (6.15) found in Sec. VII:

In particular, the time evolution generated by Eq. (6.46) remains deterministic.

If \( \Psi \) happens to be an energy eigenstate with energy \( E \), we have \( H \cdot \Psi = E \Psi \) and thus

\[
| v_{H \cdot \Psi} \rangle = E | v_{\Psi} \rangle = E | v \rangle. \tag{6.47}
\]

In this case

\[
(1 - \rho_v) | v_{H \cdot \Psi} \rangle = E (1 - \rho_v) | v \rangle = 0, \tag{6.48}
\]
which implies that $\omega = 1$, $|\beta\rangle = 0$, and $\kappa = 0$. Consequently $\sigma$ and $\eta$ are exactly the same as when $\Delta \Psi = 0$, and there is no difference between the present results and those of Sec. VI B. This is reassuring because it is precisely what we would expect when time evolution does not change the ray that $|\psi\rangle$ belongs to.

F. Model calculations and special cases

As a tool for developing insight, it is helpful to run some numerical calculations on simple models and see how well the general principles of the theory hold up in practice. The model used here was the extended Hubbard model [162] for small one-dimensional lattices of interacting fermions. Tests were run on both spinless fermions (with nearest-neighbor interactions) and spin 1/2 fermions (with on-site and nearest-neighbor interactions). For fermions, the algebra of the $\psi$ product can easily be implemented using bitwise operations in the binary representation of Eq. (2.26).

As noted already at the end of Sec. VI B convergence tests of evolution over finite time intervals show that the subsystem dynamics is indeed deterministic, with the solutions converging quadratically in $\Delta t$. This remains true for the case of time-dependent $\Psi$.

An interesting test case is obtained by setting all terms derived from $|\psi\rangle$ equal to zero in Eqs. (6.10), (6.23), (6.25), (6.28), (6.29), (6.32), and (6.34). This eliminates all constraints on $\Delta \Psi$, thereby converting the constrained variation problem to an unconstrained variation. With no constraints on the quasiclassical subsystems, one would expect the solutions of Eq. (6.46) to have the absolute maximum value of $\chi = 1$, corresponding to the limiting case of Schrödinger dynamics for all subsystems [cf. Eq. (5.25)]. This is precisely what happens.

A similar result is obtained if one keeps all terms derived from $|\psi\rangle$ but sets the particle interaction potential to zero. This again yields $\chi = 1$ and Schrödinger dynamics for all subsystems. As noted by Wiseman [163], such a limit is physically uninteresting because it turns each particle into an isolated universe having no connection with anything else. However, it is a crucial test for the logical coherence of the theory, in that it establishes the consistency of assuming Schrödinger dynamics for the state $|\psi\rangle$ of a closed system.

G. The number of subsystems is dynamically essential

The number of subsystems $m$ has so far been treated as an arbitrary parameter. But what is the significance of this number? Does it play an active part in determining the subsystem dynamics, or is its role more passive?

This question addresses the distinction between trivial and nontrivial subsystems. A trivial subsystem is one that remains in a pure vacuum state ($|u_k\rangle = |0\rangle$) as time evolves. A trivial subsystem has no observable properties (see Sec. IV A), so it makes no difference whether it is included in the subsystem decomposition. The value of $\chi$ is also unchanged by the addition of trivial subsystems. If trivial subsystems are allowed by the principle of dynamical stability, then the number $m$ plays no essential role in the dynamics, because one can add trivial subsystems to any given subsystem decomposition without changing any observable property.

However, vacuum subsystems are not dynamically stable in systems of interacting particles. All quasiclassical subsystems, including vacuum subsystems, are coupled to each other by the terms derived from $|v\rangle$ in the matrix (6.10). A vacuum subsystem satisfies the time-dependent Schrödinger equation, and we know already from Secs. VA and VIA that such a time dependence is not dynamically stable in a system of interacting particles. Because $\chi < 1$ (see Sec. VIA), there is always room to increase $\chi$ by allowing an initial vacuum subsystem to evolve into a nonvacuum final subsystem. Hence, in a system of interacting particles, the number $m$ plays an essential role in the subsystem dynamics, because there are no trivial subsystems.

On the other hand, for noninteracting particles, dynamically stable subsystem decompositions always have $\chi = 1$. The most general such decomposition can be obtained by choosing an independent solution of the Schrödinger equation for each subsystem. Because the vacuum state satisfies the Schrödinger equation, trivial subsystems are dynamically stable. Hence, for noninteracting particles, the number $m$ need not be the same as the number of nontrivial subsystems. (However, this case is physically uninteresting, as noted in Sec. VI F.)

In conclusion, for the physically interesting case of interacting particles, the number of subsystems $m$ is an essential determining factor for the subsystem dynamics. The value of $m$ is arbitrary, but some number must be chosen in order to apply the principle of dynamical stability.

VII. Reference frames and superselection rules

Thus far, we have seen no sign of any deviation from strict determinism in the subsystem dynamics. This result seems to be in tension with the lack of determinism exhibited by ordinary quantum mechanics. However, up to this point it has also been assumed that there are in principle no restrictions on observable quantities. It is therefore of interest to consider the effect of restrictions arising from the lack of an external reference frame, which were discussed briefly in Sec. VI B. In standard quantum mechanics, it is well known that the lack of a reference frame generally gives rise to a superselection rule [74] together with associated classical variables.
A. Lack of phase reference

Rather than discussing reference frames in general, this paper focuses on a particular example relevant to nonrelativistic fermions, namely the number superselection rule arising from the lack of a phase reference \[14\]. Lack of a phase reference simply means that the phase transformation

$$\langle \psi \rangle \rightarrow e^{iN\phi} |\psi\rangle$$

has no observable consequences, where \(N\) is the operator for the total number of particles and \(\phi\) is any real number. If particle number is conserved \([H, N] = 0\), then this symmetry is maintained over time, since

$$e^{-iHt}|\psi\rangle = e^{-iN\phi} e^{-iHt} e^{iN\phi} |\psi\rangle.$$  

(7.2)

Of course, in the present theory, observables are associated with the subsystems rather than \(|\psi\rangle\) (Sec. \[VA\]). To see the effect on the subsystems, note that the transformation (7.1) is equivalent to

$$\Psi \rightarrow e^{iN\phi} \Psi e^{-iN\phi},$$

(7.3)

because \(N|0\rangle = 0\). But this is equivalent to applying the same phase transformation to every subsystem:

$$U_k \rightarrow e^{iN\phi} U_k e^{-iN\phi} \quad (k = 1, 2, \ldots, m).$$

(7.4)

A crucial difference between this phase shift and the Schrödinger dynamics problem studied in Sec. \[VA\] is that both sides of the mapping (7.4) are creators [see Eq. (5.6)]. Hence, a phase shift applied to the total state \(\Psi\) propagates directly to all of the subsystems. This is analogous to Lubkin’s description of superselection rules in standard quantum mechanics \[67\].

B. Equivalence classes of subsystem decompositions

In the direct-sum formalism, applying the phase shift \(|u_k\rangle \rightarrow e^{iN\phi}|u_k\rangle\) to all subsystems \(k\) is the same as applying the phase shift \(|u\rangle \rightarrow e^{iN\phi}|u\rangle\) to the direct sum of subsystems, in which [cf. Eq. (5.10)]

$$\hat{N} \equiv \bigoplus_{k=1}^{m} N.$$  

(7.5)

If this phase shift has no observable consequences, the relevant mathematical object is not the individual subsystem decomposition \(|u\rangle\) but rather the equivalence class

$$|u\rangle \equiv \{\exp(i\hat{N}\phi)|u\rangle : 0 \leq \phi < 2\pi\}.$$  

(7.6)

The corresponding equivalence class for a subsystem projector \(\rho\) is

$$|\rho\rangle \equiv \{\exp(i\hat{N}\phi)\rho \exp(-i\hat{N}\phi) : 0 \leq \phi < 2\pi\}.$$  

(7.7)

Such equivalence classes are also referred to as phase orbits \[164\]. In a system without a phase reference, all of the preceding theory must be reformulated in terms of orbits rather than individual subsystem decompositions.

C. Distance between phase orbits

The first step is to define a suitable measure of distance between phase orbits. The distance between \(|\rho\rangle\) and \(|\rho'\rangle\) can be defined simply as the minimum distance \[165\] between any two elements of these orbits:

$$D^2(|\rho\rangle, |\rho'\rangle) \equiv \min_{\theta, \phi} D^2(e^{i\hat{N}\theta} \rho e^{-i\hat{N}\theta}, e^{i\hat{N}\phi} \rho' e^{-i\hat{N}\phi}).$$  

(7.8)

One of these phase shifts is redundant, so we can write this definition more simply as

$$D^2(|\rho\rangle, |\rho'\rangle) = \min_{\phi} \lambda(\phi),$$  

(7.9)

in which

$$\lambda(\phi) = D^2(\rho, e^{i\hat{N}\phi} \rho' e^{-i\hat{N}\phi}).$$

(7.10)

Here it is worthwhile to pause and note the similarity between \(\lambda(\phi)\) and the function \(\lambda(\Delta \tau)\) introduced previously in Eq. (5.11). This similarity means that much of the following derivation will be almost identical to that given in Sec. \[VC\]. Consequently, only a brief outline of the results is presented.

For small changes \(||\Delta u||\), Eq. (7.10) reduces to

$$\lambda(\phi) = \langle u_\prime| e^{-i\hat{N}\phi}(1 - \rho) e^{i\hat{N}\phi}|u_\prime\rangle.$$  

(7.11)

After expanding the right-hand side to second order in small quantities, we obtain the quadratic function

$$\lambda(\phi) = \eta_0 - 2\phi \text{Im} \langle \Delta u | N \rangle + \phi^2 \langle N | N \rangle,$$  

(7.12)

in which \(\eta_0 = \lambda(0)\) and \(\langle N \rangle \equiv (1 - \rho)\hat{N}|u\rangle\). The function (7.12) has a minimum at \(\phi = \varphi\), in which

$$\varphi = \frac{\langle \Delta u | N \rangle}{\langle N | N \rangle}.$$  

(7.13)

The value of \(\lambda(\phi)\) at the minimum is

$$\lambda(\varphi) = \eta_0 - \frac{(\text{Im} \langle \Delta u | N \rangle)^2}{\langle N | N \rangle}.$$  

(7.14)

But this minimum value is just the desired distance \(7.9\) between the two orbits:

$$D^2(|\rho\rangle, |\rho'\rangle) = \langle \Delta u | (1 - \rho) | \Delta u \rangle - \frac{(\text{Im} \langle \Delta u | N \rangle)^2}{\langle N | N \rangle}.$$  

(7.15)

As before, the symbol \(\eta\) is used to refer to the square of the basic measure of distance:

$$\eta \equiv D^2(|\rho\rangle, |\rho'\rangle) = \lambda(\varphi).$$  

(7.16)

It is convenient to write this more concisely as

$$\eta = \eta_0 - \xi^2,$$  

(7.17)
in which $\xi$ is the functional

$$\xi \equiv \frac{\text{Im}(\Delta u|N)}{\Delta N} = \text{Im} \langle \Delta x|\xi \rangle, \quad \Delta N \equiv \sqrt{\langle N|N \rangle}, \quad (7.18)$$

and $|\xi\rangle$ is the vector

$$\langle e_{ki}|\xi \rangle = \frac{1}{\Delta N} \left[ \langle j_{ki}|(1 - \rho_k)N|u_k \rangle \langle u_k|u_k \rangle + \langle g_{ki}|(1 - \rho_v)N|v \rangle \langle v|v \rangle \right]. \quad (7.19)$$

Again, it is worth noting the close similarity between these quantities and those defined in Secs. V and VI.

Sometimes it is necessary to calculate $D^2(|\rho|, |\rho'|)$ in situations where $\|\Delta u\|$ is not small. (See, for example, the last paragraph in Sec. VII E.) This case is considered in Appendix H.

### D. Time functional for phase orbits

A time functional suitable for phase orbits can now be derived by minimizing the function [cf. Eqs. (5.11), (7.10)]

$$\lambda(\phi, \Delta \tau) \equiv D^2(\rho, e^{i\hat N\phi}e^{i\hat H\Delta \tau}\rho^*e^{-i\hat H\Delta \tau}e^{-i\hat N\phi}) \quad (7.20)$$

with respect to both $\phi$ and $\Delta \tau$. Given that $|H, N\rangle = 0$, this reduces in the case of small $\|\Delta u\|$ to

$$\lambda(\phi, \Delta \tau) = \langle u'|e^{-i\hat A}(1 - \rho)e^{i\hat A}|u'\rangle, \quad (7.21)$$

in which the operator $\hat A$ is defined by

$$\hat A(\phi, \Delta \tau) \equiv \hat H\Delta \tau + \hat N\phi. \quad (7.22)$$

When $\|\Delta u\|$ is small, $\phi$ and $\Delta \tau$ can also be treated as small quantities of the same order, and Eq. (7.21) can be expanded as usual to obtain the quadratic approximation

$$\lambda(\phi, \Delta \tau) = \eta_0 - 2 \text{Im}(\Delta u|A\rangle + \langle A|\Delta u\rangle, \quad (7.23)$$

in which $\eta_0 = \lambda(0, 0)$ and $|A\rangle = (1 - \rho)|\Delta u\rangle$.

The minimum of this function occurs at $(\phi, \Delta \tau) = (\varphi, \Delta t)$, in which $\varphi$ and $\Delta t$ satisfy the system of equations

$$\begin{pmatrix} \langle N|N \rangle & \langle N|H \rangle \\ \langle H|N \rangle & \langle H|H \rangle \end{pmatrix} \begin{pmatrix} \varphi \\ \Delta t \end{pmatrix} = \begin{pmatrix} \text{Im}(\Delta u|N\rangle) \\ \text{Im}(\Delta u|H\rangle) \end{pmatrix}. \quad (7.24)$$

The matrix on the left is real and symmetric, because $[H, N] = 0$ implies $\langle N|H \rangle = \langle H|N \rangle$. Upon inverting this matrix, we find the desired time functional

$$\Delta t = \frac{\langle N|N \rangle \text{Im}(\Delta u|H\rangle) - \langle H|N \rangle \text{Im}(\Delta u|N\rangle)}{\langle N|N \rangle \langle H|H \rangle - \langle N|H \rangle \langle H|N \rangle}. \quad (7.25)$$

This solution is well defined as long as $\langle N|N \rangle > 0$, $\langle H|H \rangle > 0$, and (by the Schwarz inequality) $|H\rangle \neq c|N\rangle$,

where $c$ is any constant. If $|H\rangle = c|N\rangle$, this simply means that Schrödinger dynamics cannot move the subsystems out of the initial phase orbit, so $\Delta t$ is ill defined (at least to first order in small quantities).

Equation (7.25) looks considerably more complicated than the previous functional (5.16), but it can be simplified by introducing the operator

$$K \equiv H - \frac{\langle H|N \rangle}{\langle N|N \rangle} N. \quad (7.26)$$

Here $K$ is just the component of $H$ that is orthogonal to $N$, in the sense that $\langle K|N \rangle = 0$. In terms of $K$, the time functional (7.25) is simply

$$\Delta t = \frac{\text{Im}(\Delta u|K\rangle)}{\langle K|K \rangle}. \quad (7.27)$$

The denominator satisfies $\langle K|K \rangle \leq \langle H|H \rangle$, equality occurring if and only if $\langle H|N \rangle = 0$. In geometric terms, $\Delta K \equiv \sqrt{\langle K|K \rangle}$ is the length of $|K\rangle$, which is the component of $|H\rangle$ orthogonal to $|N\rangle$. Physically, $\Delta K$ is a renormalized energy uncertainty, just what remains of $\Delta E$ after the unphysical part of $H$ is removed (“unphysical” because it has no observable consequences for this particular $|u\rangle$).

The phase angle $\varphi$ at the minimum of $\lambda(\phi, \Delta \tau)$ can be written likewise as

$$\varphi = \frac{\text{Im}(\Delta u|L\rangle)}{\langle L|L \rangle}, \quad L \equiv N - \frac{\langle N|H \rangle}{\langle H|H \rangle} H. \quad (7.28)$$

The value of $\lambda(\phi, \Delta \tau)$ at the minimum is

$$\lambda(\varphi, \Delta t) = \langle \Delta u|(1 - \rho)|\Delta u\rangle - \frac{\text{Im}(\Delta u|N\rangle)^2}{\langle N|N \rangle} - \frac{\text{Im}(\Delta u|K\rangle)^2}{\langle K|K \rangle}, \quad (7.29)$$

which is similar to Eqs. (5.18) and (7.14). This can also be written as

$$\lambda(\varphi, \Delta t) = \lambda(\varphi, 0) - \Delta K^2 \Delta t^2, \quad (7.30)$$

in which $\lambda(\varphi, 0)$ is the same as Eq. (7.14):

$$\lambda(\varphi, 0) = \langle \Delta u|(1 - \rho)|\Delta u\rangle - \frac{\text{Im}(\Delta u|N\rangle)^2}{\langle N|N \rangle}. \quad (7.31)$$

If we follow the argument used in Sec. V D it is easy to prove that $\lambda(\varphi, \Delta t) \geq 0$, in which a necessary condition for equality is

$$(1 - \rho)|\Delta u\rangle = (\Pi_K + \Pi_N)|\Delta u\rangle, \quad (7.32)$$

where $\Pi_K$ and $\Pi_N$ are the projectors for $|K\rangle$ and $|N\rangle$. A necessary and sufficient condition for $\lambda(\varphi, \Delta t) = 0$ is

$$(1 - \rho)|\Delta u\rangle = iC_K|K\rangle + iC_N|N\rangle, \quad (7.33)$$

in which the numbers $C_K$ and $C_N$ are real.
E. Dynamical stability of phase orbits

With the modified time functional (7.27) in hand, we can now apply the principle of dynamical stability in much the same way as before (see Sec. VI). Many parts of the previous analysis can be carried over to the case of phase orbits simply by replacing $H \rightarrow K$ and $\Delta E \rightarrow \Delta K$. Thus, for example, the dimensionless time interval $\sigma$ is redefined as [cf. Eqs. (6.7), (6.24)]

$$\sigma \equiv \Delta K \Delta t = \frac{\text{Im}\langle \Delta u|K \rangle}{\Delta K} = \text{Im}\langle \Delta x|\sigma \rangle, \quad (7.34)$$

in which the components of $|\sigma \rangle$ are [cf. Eq. (6.25)]

$$\langle k_i|\sigma \rangle = \frac{1}{\omega \Delta K} \frac{\langle f_{ki}[(1 - \rho_k)K]u_k \rangle}{\langle u_k|u_k \rangle} + \frac{\langle g_{ki}[(1 - \rho_v)K]v \rangle}{\langle v|v \rangle}, \quad (7.35)$$

and the renormalization factor $\omega$ is [cf. Eq. (6.23)]

$$\omega = 1 - \frac{1}{\Delta K^2} \text{Re} \left[ \langle v H \Psi(1 - \rho_v)K|v \rangle \right]. \quad (7.36)$$

Likewise, $|\beta \rangle$ and $\kappa$ in Eqs. (6.28) and (6.29) are redefined as

$$\langle k_i|\beta \rangle = \frac{1}{\Delta K} \frac{\langle g_{ki}[(1 - \rho_v)\beta H \Psi]v \rangle}{\langle v|v \rangle}, \quad (7.37)$$

$$\kappa = \frac{1}{\Delta K^2} \frac{\langle v H \Psi(1 - \rho_v)\beta H \Psi|v \rangle}{\langle v|v \rangle}. \quad (7.38)$$

The only truly new parameter to arise is

$$\theta \equiv \frac{1}{\Delta N \Delta K} \text{Re} \left[ \frac{\langle v H \Psi(1 - \rho_v)N|v \rangle}{\langle v|v \rangle} \right]. \quad (7.39)$$

Given these definitions, the squared distance between neighboring phase orbits is [cf. Eq. (6.26)]

$$\eta = \eta_0 - \xi^2 + \sigma^2(\beta - \theta \kappa) + \sigma^2(\kappa - \theta^2), \quad (7.40)$$

in which $\eta_0 = \langle \Delta x|\eta\Delta x \rangle$ and $\xi$ is defined in Eq. (7.18). Here it should be noted that $\Psi$ is treated as time dependent (see Sec. VI C) and $\eta$ refers to the quantity

$$\eta = \eta(\Delta t) \equiv \lambda(\varphi,0) \neq \lambda(\varphi, \Delta t), \quad (7.41)$$

in which $\lambda(\varphi,0)$ is given in Eq. (7.31).

It is convenient now to follow the approach of Sec. VIDA and write $\eta = \mu + \nu$, in which $\mu$ and $\nu$ are defined in Eqs. (6.31) and (6.32). The only difference is that the matrix elements $\mu_{ki,k'i'}$ and $\nu_{ki,k'i'}$ in Eqs. (6.33) and (6.34) are modified to become

$$\nu_{ki,k'i'} = -\frac{1}{2} \left[ (\beta_{ki}\sigma_{k'i'} + \sigma_{ki}\beta_{k'i'}) + (\kappa - \theta^2)\sigma_{ki}\sigma_{k'i'} - \xi_{ki}\xi_{k'i'} - \theta(\xi_{ki}\sigma_{k'i'} + \sigma_{ki}\xi_{k'i'}) \right]. \quad (7.43)$$

Aside from this change of definition, all of the subsequent analysis in Secs. VIDA and VIE follows through in the same way as before. Because the solution (6.46) has the same mathematical structure as before, it gives rise to the same qualitative behavior too. That is, the dynamics of phase orbits is also deterministic.

Of course, this does not mean that the subsystem dynamics is totally unchanged. The main difference arises because $\Delta K$ is generally smaller than $\Delta E$. The subsystems therefore tend to evolve in time more slowly, because all physically unobservable changes (lying entirely within a given orbit) have been filtered out.

Test calculations show that the phase-orbit dynamics obtained by integrating Eq. (6.46) over a finite time interval has the correct limiting behavior (i.e., Schrödinger subsystem dynamics) for the special cases discussed in Sec. VI F. To demonstrate this, one must use the phase-orbit distance formulas derived in Appendix I for the case of large $\|\Delta u\|$.

VIII. SUBSYSTEM PERMUTATIONS

Thus far we have considered only one of the $m!$ possible permutations of the subsystems in Eq. (4.4). The next step is to consider the set of all permutations.

A. Influence of permutations on dynamics

According to the definition of an observable in Sec. IV A, a permutation of the subsystems merely rearranges the labels $k$ on the numbers $\langle A \rangle_k$ in Eq. (4.2). But the subsystem labels $k$ are not themselves observable, so a permutation cannot affect the value of any observable quantity at any given time.

On the other hand, permutation of the subsystems does affect the value of the product $\Psi_\pi$ in Eq. (4.4). This has no direct effect on any observable quantity, but $\Delta \Psi_\pi = \Psi_\pi - \Psi_\pi$ determines the value of the subsystem change $\Delta V$ [see Eq. (4.10)] generated by given changes $\Delta U_k$ in the quasiclassical subsystems ($k \neq 1$).

This means that subsystem permutations do alter the subsystem dynamics. Starting from a given initial subsystem decomposition $\rho$ or phase orbit $|\rho \rangle$, a single time step (6.46) will in general carry each permutation into a different final state. In other words, the permutation symmetry of observables is broken by the dynamics. Subsystem permutations are thus qualitatively different from the phase transformations considered in Sec. VII A.

But this suggests that it might be possible, at least in principle, to use the effect of permutations on dynamics to obtain information about subsystem permutations from the time evolution of observables. Given unlimited information about the observables of all subsystems
(which is an unrealistic assumption, as will be discussed in Sec. IX), one might even be able to deduce which individual permutation was consistent with a given set of experimental data.

B. Subsystem ordering in orthodox quantum mechanics

It is important to pause here and note that this effect of subsystem permutations on dynamics is not limited to the context of the present dynamical stability formalism. It is a general consequence of the noncommutative algebra of fermion creation operators that holds even in orthodox quantum mechanics, although this aspect of the theory has received little prior attention.

Consider, for example, the products \( \Psi_a = U_1 U_2 \) and \( \Psi_b = U_2 U_1 \) of two creators \( U_1 \) and \( U_2 \). Since \( \Psi_a \) and \( \Psi_b \) generally belong to different rays in projective Hilbert space, they will of course evolve in different ways under the Schrödinger equation. That is really all there is to it.

An obvious objection to this conclusion is that \( \Psi_a \) and \( \Psi_b \) do not belong to different rays if \( U_1 \) and \( U_2 \) have a definite number of fermions. In that case, \( \Psi_a \) differs from \( \Psi_b \) by at most a physically meaningless overall sign [see Eq. (2.12)]. More generally, this is also true if every subsystem \( U_k \) is even or odd [see Eq. (3.5)] — or, in other words, if each \( U_k \) has a definite univalence. Where the univalence \( W \) is the number of fermions modulo 2:

\[
W \equiv N_f \pmod{2}. \tag{8.1}
\]

Therefore, if the subsystems in orthodox quantum mechanics are required to satisfy a fermion-number or univalence superselection rule [57–60], the Schrödinger dynamics of \( \Psi \) does not depend on the order in which the subsystems are multiplied.

An answer to this objection can be found in the growing consensus [82] that most (if not all) superselection rules are pragmatic expressions of practical limitations on experimental capabilities rather than fundamental laws of nature. This suggests that, at the level of fundamental theory, the superposition principle should be taken seriously, not lightly brushed aside [153].

The practical limitation that gives rise to a particle-number superselection rule in orthodox quantum mechanics is just the lack of a phase reference discussed in Sec. VIIIA [74]. Since this limitation was already taken into account in the analysis of Sec. VII and the discussion of Sec. VIIIA, it does not alter the conclusion that subsystem permutations can have an observable effect on subsystem dynamics.

C. Significance of a univalence superselection rule

Thus, if one wishes to eliminate the dependence of dynamics on subsystem permutations, one must introduce the univalence superselection rule as an independent axiom; it cannot be derived from the phase symmetry (7.4). This rule would require all quasiclassical subsystems to be even, but subsystem \( U_1 \) could be even or odd. One can easily verify that the univalence of all subsystems is conserved by the time evolution (6.46) if \( [H, W] = 0 \).

Introducing such a rule makes it easier to combine two subsystems into one. Suppose, for example, we have two subsystems localized in adjacent regions of coordinate space. In such a case it seems natural to talk about using the \( \psi \) product to merge these subsystems. However, this cannot be done if they are separated in the product (4.4) by a subsystem that commutes with neither of them, even if that subsystem is localized far away in coordinate space. The univalence superselection rule therefore provides a natural framework for discussing composition and decomposition of subsystems.

However, at present the question of whether to introduce such an axiom is simply left open. The basic structure of the theory that follows does not depend on this choice.

IX. A BARE-BONES THEORY OF INFORMATION

At this stage the basic theory of subsystem dynamics is more or less complete. The next step is to construct a theory of information (or, in other words, a theory of measurement) that connects the subsystem dynamics to the experiences of observers. This paper develops such a theory only at a very rudimentary level, focusing mostly on qualitative questions such as “Whose information?” and “Information about what?” [33]. Detailed investigations of this theory of information are left for future work.

A. Bayesian inference in the present moment

The present theory of information is essentially just a theory of Bayesian inference for individual observers treated as subsystems of a closed system. The use of Bayesian inference means that this theory has much in common with QBism [52, 85, 94]. However, it differs from QBism in its assignment of subsystem vectors \( |u_k\rangle \) to all observers whose experiences are being described.

Here the implementation of Bayesian inference is controlled by two fundamental principles: (1) Each observer experiences directly only the beables associated with one subsystem \( |u_k\rangle \). All else must be inferred. In particular, the existence of other subsystems is inferred from the influence of those subsystems on the dynamics of \( |u_k\rangle \). (2) Each observer experiences directly only the beables associated with a single moment of time (a moment being defined as an infinitesimal interval of time). All else must be inferred. In particular, the dynamics of all subsystems in the past and future of the present moment is purely an inference.
The meaning, significance, and historical context of these statements are elaborated in this subsection. Their mathematical implications are discussed in Sec. IX B.

Let us start with the case in which inferences are drawn from the experiences of only one observer. In this case, principle (1) says that the existence of other subsystems is inferred from their influence on the dynamics of the observer’s subsystem \( |u_k \rangle \). Interactions between particles are crucial in this regard. If the particles do not interact, the time evolution of \( |u_k \rangle \) is independent of the other subsystems, so nothing at all can be inferred about the properties of other subsystems.

The idea that an observer can only “measure” the state of his own subsystem was proposed long ago by London and Bauer [166, 167]. They described this capacity of an observer as a “faculty of introspection.” London and Bauer’s concept of measurement is therefore very different from that of von Neumann [55], in which the consciousness of the observer somehow directly perceives the state of the outside world, even though the observer is expressly excluded from this state.

In the present theory, the “faculty of introspection” does not lead to any collapse of the total state vector \( |\psi\rangle \). Instead, the observer merely takes note of the beables for subsystem \( |u_k\rangle \) (or some subset thereof) during the present moment of time. This information is then used by the observer to perform a Bayesian updating of the probabilities that he ascribes to the various possible subsystem decompositions of \( |\psi\rangle \). Here Bayesian updating is just the usual process of replacing prior probabilities with posterior probabilities, in which the words “prior,” “posterior,” and “updating” refer to a direction of logical inference, not to a direction in time. The total state \( |\psi\rangle \) is not assumed to be known, since all subsystems are treated as initially unknown.

The idea that all observations are fundamentally self-observations may seem strange from the perspective of orthodox measurement theory [55], which requires an observer to always be outside the observed system [168]. Indeed the orthodox description is ostensibly the most natural one, since we intuitively regard our sense of vision as a direct perception of the world around us. However, our susceptibility to optical illusions shows clearly that the three-dimensional world we see is actually an inference based on very incomplete two-dimensional information provided by the retinas [169].

The fact that observers are treated as subsystems does not mean that the problem of consciousness must be solved before this theory can be used. The theory merely limits what can be known by any observer to the properties of a single subsystem. However, the consequences of imposing such a limit can be evaluated by studying simple non-biological subsystems. According to Wheeler [4], it is “not consciousness but the distinction between the probe and the probed [that is] central to the elementary quantum act of observation.” A similar remark was made by Heisenberg [5]: “The observing system need not always be a human being; it may also be an inanimate apparatus, such as a photographic plate.”

Principle (2) bridges the gap between the “block universe” concept of time [170, 172] used in most physical theories and the subjective flow of time that each of us experiences. According to Carnap [173], Bergson’s criticism of the block universe picture [174, 176] was deeply troubling to Einstein:

> Once Einstein said that the problem of the Now worried him seriously. He explained that the experience of the Now means something special for man, something essentially different from the past and the future, but that this important difference does not and cannot occur within physics. That this experience cannot be grasped by science seemed to him a matter of painful but inevitable resignation. . . . We both agreed that this was not a question of a defect for which science could be blamed, as Bergson thought.

The idea that the present moment does play a crucial role in physics was emphasized by Wheeler in connection with his “delayed-choice” experiments [177, 179]:

> The “past” is theory. The past has no existence except as it is recorded in the present.

Wheeler’s central message is that everything we say about the past is necessarily an inference based on records in the present. This is of course a platitude for historians and paleontologists, but it carries important lessons for experimental physicists as well [180].

Everett [181, 183], Bell [131, 184, 185], Barbour [43, 186, 187], and Page [41] have suggested ways of taking this into account by restructuring quantum theory around the records contained in the present value of \( |\psi\rangle \). Mermin [91, 92, 188], Hartle [189], and Smolin [190] have also called attention to the physical significance of the present. But Zeh has remarked that “physics does not even offer any conceptual means for deriving the concept of a present that would objectively separate the past from the future” [172].

Here this concept is not derived; rather, it is built into the theory of information as a fundamental axiom about the subjective experiences of observers. The subjective nature of the Now was stressed by Zeh [172], and its importance for the foundations of both classical and quantum physics has been emphasized especially by Mermin [91, 92].

It should be noted that the infinitesimal time interval referred to in principle (2) is different from the finite time interval (of “perhaps a few tenths of a second” [158]) that is associated with our intuitive perception of the present. As discussed by Barbour [43, 186] and Hartle [189], the perceived duration of the present moment is probably related to the way in which information about the immediate past is stored and processed in the brain.
B. Mathematical backbone for inferences

Let us now examine some mathematical implications of the fundamental principles discussed in Sec. [VIA]. The first step is to consider ideal observers, each of whom is fully aware of every detail of the state of her subsystem. This is of course unrealistic, as an observer’s experience will in general tell her only the values of some subset of the beables associated with her subsystem. The consequences of this further restriction are discussed below in Sec. [VIEL]

To begin, let us review the way in which the principle of dynamical stability was used in Sec. [VI]. The problem solved there was a time evolution problem. The initial subsystems $|u_k\rangle$ were all assumed to be known; the unknown quantities were the final subsystems $|u'_k\rangle$. The initial and final subsystems were assumed to differ only infinitesimally. The known subsystems were held fixed and the unknown subsystems were varied so as to maximize the dynamical stability functional $\chi$, subject to the constraint that the total states $|\psi\rangle$ and $|\psi'\rangle$ satisfy the Schrödinger equation (6.18).

The problem of concern now is an environmental stabilization problem, in which environmental subsystems are used to stabilize the overall subsystem decomposition. The difference lies in the classification of known and unknown subsystem states. Let us write the total number of subsystems as

$$m = s + r,$$

in which $s$ is the number of subsystems that are classified as observers; the remaining $r$ subsystems are regarded as parts of the environment. For the environmental stabilization problem, all observer states $|u_k\rangle$ and $|u'_k\rangle$ are treated as known quantities (defined by the experiences of the observers), whereas the environmental states $|u_k\rangle$ and $|u'_k\rangle$ are treated as unknown. As before, the initial and final subsystems are assumed to differ only infinitesimally. As before, the known subsystems are held fixed and the unknown subsystems are varied so as to maximize $\chi$, subject to the constraint that $|\psi\rangle$ and $|\psi'\rangle$ satisfy the Schrödinger equation.

The solutions of the environmental stabilization problem are pairs $(|u\rangle, |u'\rangle)$ of subsystem decompositions. Each such pair has its own (infinitesimal) time interval $\Delta t$, which is the duration of the present moment associated with that pair. This is always an inferred quantity because $\Delta t$ is defined only for the subsystem decomposition as a whole, not for individual subsystems. Once the environmental stabilization problem has been solved, we can use time evolution to extrapolate any given pair $(|u\rangle, |u'\rangle)$ into the past and future of $\Delta t$.

From their definition as variation problems, the time evolution and environmental stabilization problems each lead to a set of equations in which the number of equations is equal to the number of unknown variables. However, there is a big difference in the difficulty of these sets of equations. The time evolution problem leads to a set of linear equations (see Sec. [VI][F]), whereas the environmental stabilization problem leads to a set of nonlinear equations.

The latter set of equations is not written out explicitly here, but its qualitative properties can easily be seen by returning to the model system introduced in Sec. [VI][F] (i.e., a system of interacting fermions). In this model, each state $|u_k\rangle$ or $|u'_k\rangle$ can be regarded as a function of $(2^d - 1)$ independent complex variables or $2(2^d - 1)$ independent real variables, where $d = \dim H_t$. The dynamical stability functional $\chi$ is a rational function of these variables. The environmental stabilization problem therefore requires one to find the common zeros of a set of rational functions. This is a difficult but well defined problem in algebraic geometry [III].

However, since the author has no training in this field, the surest route to progress is to publicize the problem and invite experts in algebraic geometry to work on it. For this reason, the quest for explicit solutions shall not be pursued any further here. The existence of solutions is simply taken for granted. Indeed, since the environmental stabilization problem is nonlinear, it may have many solutions (in contrast to the time evolution problem, for which the set of linear equations (6.45) has a unique solution (6.46)). If no univalence superselection rule is imposed (see Sec. [VII][C]), there will in general be a different set of solutions for each possible permutation of the subsystems, thus further increasing the total number of solutions.

Assuming that the environmental stabilization problem can be solved, how do we use its solutions to perform Bayesian inference? Consider the set of pairs $(|u\rangle, |u'\rangle)$ of subsystem decompositions that satisfy the environmental stabilization problem for all possible choices of observer states. In this set, the observer states are not required to agree with the experiences of the observers. However, to fix the scale of the subsystem changes, the Fubini–Study distance between the sets of observer states in $|u\rangle$ and $|u'\rangle$ is required to be the same as that given by the experiences of the observers.

The first step in the Bayesian inference problem is to assign a prior probability to each pair $(|u\rangle, |u'\rangle)$ in this set. The choice of prior probability is subjective [80][82], although with some work it may be possible to reduce the degree of subjectivity to a choice of symmetry principle [77][78]. Posterior probabilities are then obtained simply by setting to zero the probabilities of all pairs $(|u\rangle, |u'\rangle)$ that have the wrong observer states (i.e., states inconsistent with the experiences of the observers) and renormalizing.

The outcome of this inference problem is reminiscent of Wheeler’s definition of reality [4]:

The vision of what we call “reality” is in large measure of a pale and theoretic hue. It is framed by a few iron posts of true observation—themselves also resting on theory for their meaning—but most of the walls and towers in the vision are of papier-mâché,
plastered in between those posts by an immense labor of imagination and theory.

Here the iron posts are the observer subsystems in the present moment; everything else is inferred. In general, an observer’s experiences in the present moment have a definiteness that is lacking in her memory of her inferences about that moment [191]. This plays a role similar to the “reduction of the wave packet” in orthodox quantum mechanics [6].

It must be emphasized that this is only an analogy. No actual “jump” is supposed to take place; there is only a contrast between expectations and experiences. The analogy with the orthodox description of wave-packet reduction [6] is actually rather distant. A closer analogy can be found in the dynamical reduction models of Ghirardi and others [138]. In these models the reduction process is continuous, and the beables are defined in terms of expectation values (rather than eigenvalues) of operators, just as in Sec. [IV.A] In the present paper, however, the beables are not limited to mass density in coordinate space, and the effective “reduction” takes place only at the subsystem level, leaving the Schrödinger dynamics of the total system untouched.

C. How many subsystems?

Are there any general criteria for choosing the subsystem numbers \( s \) and \( r \)? This question is easiest to answer for the number of environmental subsystems \( r \). As argued below, the sharpness of Bayesian inferences can be maximized by choosing the smallest possible value: namely, \( r = 1 \) [192]. The choice of the number of observers \( s \) requires a longer answer, as it occupies the borderland between subjectivity and objectivity.

The value \( r = 1 \) is favored by Occam’s razor, which can be regarded as a corollary of Bayesian probability theory [193]. The basic argument is that, for any given value of \( s \), the number of solutions to the environmental stabilization problem can be expected to increase rapidly as a function of \( r \). This statement is plausible because both the number of equations and the degree of the polynomials involved are increasing functions of \( r \). It is, however, not possible to prove this claim without actually solving the environmental stabilization problem. The sharpest Bayesian inferences are thus expected to be given by \( r = 1 \).

The most obvious value to choose for the number of observers is also \( s = 1 \). This choice seems to be consistent with each person’s intuitive view of the world. If we assume that Darwinian natural selection has instilled in us a rough facsimile of the above Bayesian inference process as the basis for our perception of the world around us (which is, admittedly, a big assumption), then it would be difficult to explain how natural selection could settle on any value other than \( s = 1 \).

This instinctive value—the one used for the automatic subconscious inferences that underlie our perception of the world—cannot be changed. We cannot separate these subconscious inferences from our raw sense impressions any more than we could measure the bare (unrenormalized) charge of an electron in quantum electrodynamics. However, the value of primary concern here is not this instinctive value but the value of \( s \) used for conscious mathematical calculations in quantum mechanics.

The value most commonly chosen in this context is also \( s = 1 \). A division of the world into observer and observed is the foundation for the quantum theory of measurement used by Heisenberg [5], von Neumann [55], and many others. As Zeh has noted [154], it is never strictly necessary to introduce any other subsystems.

However, this choice leaves one open to a charge of solipsism, because each of us has experiences that include encounters with other human beings. Choosing \( s = 1 \) confines each observer to a hermitage, within which the experiences of others are treated only as inferences, never as primary data. Different observers will therefore always base their worldviews on mutually exclusive subsystem decompositions. This leads inevitably to Wheeler’s question [179, 195]:

What keeps these images of something “out there” from degenerating into separate and private universes: one observer, one universe; another observer, another universe?

Wheeler’s answers are cryptic but instructive:

That is prevented by the very solidity of those iron posts, the elementary acts of observership-participancy.

That is the importance of Bohr’s point that no observation is an observation unless we can communicate the results of that observation to others in plain language. [196]

Translated from Wheelerian poetry into the language of the present theory, the first answer says that although different observers always have different experiences, these experiences—the iron posts—are not affected by the choice of \( s \). A subsystem decomposition is just a tool observers use to draw inferences from their experiences. The only thing affected by the choice of \( s \) is the set of inferences. Choosing \( s > 1 \) facilitates intersubjective agreement by allowing multiple observers to have equal status within the theory.

The second answer says that if we are to believe that the image of an outside world depicted by these inferences is anything more than a mirage or a hallucination, the essential features of this image must be independent of the value of \( s \) used to perform the inferences. Indeed, we can take the set of those features that are robust under increases of \( s \) as the definition of what is objective. Here the word “objective” is used in roughly the sense defined by Primas [13, p. 352]:

That is, objectivity can never mean anything else but conditional intersubjective agree-
The idea that “objective” properties are necessarily contextually dependent dates back at least to Bohr’s thoughts on complementarity \[196,197\]. Here the “jointly accepted context” is the entire structure of the present theory of information. The “perceptions” that can be shared are the common features of the worldview that emerges in the limit of many observers. Communication between observers therefore plays a central role in establishing which elements of the theory are meaningful \[179,195,198,199\].

To flesh out these answers, it is necessary to explain how the theory works when \(s > 1\). The most egalitarian approach would be to assign all probabilities as a team. A minimum requirement for forming such a team is to get all teammates to agree to assign unit probability to the hypothesis that “you are a being very much like myself, with your own private experience” \[93\]. Prior probabilities are assigned by team consensus and can be regarded as an expression of gambling commitments by the team as a whole. Inferences drawn from such calculations can then be used as the basis for group decisions.

Of course, this egalitarian approach is somewhat in tension with the subconscious inferences that define each observer’s intuitive worldview. Members of such a team might still be wise to make personal decisions by assigning less than unit probability to the hypothesis that the other teammates always tell the truth and never make mistakes. If the reported experiences of these teammates are weighted accordingly, the resulting monarchical structure is no longer fully egalitarian. However, it is not solipsist either, as long as these weights are nonzero.

This type of monarchy is the closest point of approach between QBism and the present theory. QBism requires quantum mechanics to be a “single-user theory” \[87,89,90,93\] but also emphasizes that communications with others can (and should) form part of the basis for single-user quantum state assignments. The two theories are not directly comparable because the words “user” (in QBism) and “observer” (in this paper) have different mathematical implications. In particular, the single-user theory of the QBist is not equivalent to choosing \(s = 1\) in the present theory, because according to the principles of Sec. \[IX.A\] (which are not part of QBism), the choice \(s = 1\) does not allow the experiences of more than one observer to be included in any way.

It should also be noted that the QBist arguments for requiring quantum mechanics to be a single-user theory do not rule out the possibility of choosing \(s > 1\) in the present theory. The basic argument is that “my internal personal experience is inaccessible to you except insofar as I attempt to represent it to you verbally, and vice-versa.” \[93\]. But this is irrelevant here, because neither classical nor quantum Bayesian theory has any direct contact with internal personal experience; it can only deal with what can be distilled out of that experience and represented mathematically \[200\]. But what can be described mathematically can also be communicated to another person “in plain language” \[196\]. So there is no reason why these mathematical representations of personal experiences cannot be pooled and used as the basis for group inferences, in the manner described above.

### D. Strong dynamical stability

As shown in Sec. \[IX.C\], the sharpness of Bayesian inferences can be optimized with respect to the number of environmental subsystems by choosing \(r = 1\). However, given that the environmental stabilization problem of Sec. \[IX.B\] has not yet been solved, it is not possible at this stage to say whether the resulting inferences would be sharp enough to be comparable to the predictions of orthodox quantum mechanics. The situation could be similar to that pointed out by Zurek \[156\] in connection with the consistent histories formulation of quantum mechanics \[201-204\], where the consistency conditions alone are not sufficiently selective to single out emergent classical behavior.

Given this possibility, it is of interest to consider whether there are any means available for further sharpening of inferences. The method discussed here is based on a strengthening of the principle of dynamical stability. The basic idea is similar to the concept of the “predictability sieve” introduced by Zurek \[156,157\].

The environmental stabilization problem of Sec. \[IX.B\] involves holding the observer states fixed and maximizing \(\chi\) with respect to variations in the environmental states. The observer states can be chosen arbitrarily, subject only to the (implicit) requirement of leading to a mathematically well defined variation problem.

But is this complete freedom of choice necessary? Might it be possible to narrow down the choices by maximizing \(\chi\) with respect to all subsystem states, including those of the observers? The choice of observer states would then no longer be completely arbitrary, but if this variation problem has a sufficiently large number of solutions, it may still be possible to account for the actual experiences of observers.

For obvious reasons, the variational principle thus defined is called the strong principle of dynamical stability. Subsystem decompositions derived from this principle are maximally stable, in the sense that they are constrained solely by the requirement that \(|\psi\rangle\) and \(|\psi'\rangle\) satisfy the Schrödinger equation. Given that our experience of the world does have a certain stability, it is at least plausible that this experience is congruent with strong dynamical stability.

The selective power of strong dynamical stability is most noticeable in a context where the observer’s perception of her state is assumed not to be infinitely precise. Given that an observer’s experience is only capable of selecting a certain continuous range of states, strong dynamical stability may be able to narrow the possibilities down to a much smaller (perhaps finite) subset. This
would lead to sharper Bayesian inferences.

E. The rest of the skeleton: Complementarity and “phenomenon”

Let us now consider the effect of removing the restriction to ideal observers imposed in Sec. [X B]. What an observer actually perceives is not the state of her subsystem but some subset of the beables for that subsystem. The perceived values of these beables may be consistent with more than one state $|u_k\rangle$, so the state must be inferred from the beables. This imposes another layer of inference, thus making the inferences drawn by a real observer less sharp than those drawn by an ideal observer.

The reason why all beables are not perceived is probably Darwinian. Survival requires an efficient mechanism for drawing inferences about the outside world, and it is most efficient to focus attention on those beables with the greatest signal-to-noise ratio and ignore the rest. This signal-to-noise ratio can be greatly enhanced by sense organs, so the set of perceived beables may vary between organisms with a different evolutionary history. However, the degree of variation is also constrained by efficiency considerations, since slowly changing beables are easier to keep track of than rapidly changing ones. All beables change more slowly under the strong principle of dynamical stability (Sec. [IX D]), but some still change more slowly than others. This limits the possible sets of beables that it is worth developing sense organs for.

The question of which beables are most stable—thus readily seized upon by Darwinian natural selection—is essentially just the “preferred basis problem” of decoherence theory. Given that particle interactions are typically functions of number operators in coordinate space, the expectation values of such operators (see Sec. [IV A]) will play the most prominent role in a strongly interacting system. This yields a description similar to the mass-density beables of dynamical reduction theories [136–138]. However, in general one must consider the dynamics generated by the total Hamiltonian, not just the interaction Hamiltonian. This typically shifts the arena from coordinate space to phase space [205].

Bohr’s principle of complementarity [196, 197] is an immediate corollary of these Darwinian restrictions on the subset of perceived beables. Given that all inferences are performed in the present moment, it is simply impossible for an observer to infer definite values for all beables of her environment from a limited set of her own beables. But the set of sharply inferred environmental beables may change with her experiences in the present moment—depending, for example, on which piece of measurement apparatus she is looking at.

The fact that all subsystem beables are defined in the present theory, but not all of them play an active role in the worldview of observers, is similar to ideas used previously in modal interpretations of quantum mechanics. As noted by Vink [206] and Bub [207, 208], in modal interpretations it is possible to assign definite values to all possible observables without violating the Bell–Kochen–Specker theorem [209–211] for the measured values of observables. The axiomatic foundations of the theory can thus be simplified by allowing the process of measurement (or, in the present case, Bayesian inference from a Darwinian subset) to take up some of the burden that would otherwise be shouldered by a restriction of the allowed beables.

As discussed in Sec. [IX C], our conscious experiences are largely based on subconscious inferences about the structure of our environment. We seem to perceive a “real” three-dimensional butterfly rather than a pair of two-dimensional butterfly-shaped patterns on our retinas. In order to match our conscious experiences to the mathematical structure of the present theory of information, we have to infer the latter description from the former. This additional layer of (inverse) inference makes the theory even more complicated. However, such a description is arguably more reasonable than assuming that we directly perceive the three-dimensional butterfly, which is the approach used in orthodox quantum mechanics.

This type of automatic subconscious inference (i.e., an inference about the environment based on a Darwinian-restricted set of observer beables) is an example of what Bohr and Wheeler call a “phenomenon” [89, 177, 178, 195]. A phenomenon is subjective in the sense that it depends on the sensory apparatus of the observer. For example, although the inferred “blackening of a grain of silver bromide” constitutes a phenomenon for us [178], it presumably would not be regarded as such by a Kentucky cave shrimp, which lives its entire life underground and has no eyes.

F. Comparison with other quantum theories

The essence of the theory of information outlined here is its reliance upon inferences from the properties of a limited number of subsystems in the present moment. In some ways this is similar to the Everett interpretation [181, 182], which also uses the properties of one subsystem to make inferences about others and also relies on memories and records in the present to make inferences about the past. However, Everett defines subsystems using the traditional tensor product of vector spaces, and he defines the present to be an instant rather than a moment (i.e., a point on the real line rather than an interval).

Because Everett’s subsystems are entangled, he introduces the relative-state concept as a new axiom [58] that allows him to infer the state of the outside world from the instantaneous state of the observer. However, this inference assumes knowledge of the total state $|\psi\rangle$. It is not clear how such an instantaneous inference scheme can get off the ground if the observer starts in a state of ignorance of $|\psi\rangle$. That is, it is not clear why an observer using such a scheme would have any justification for believing that knowledge of his own present state says
anything about the state of his environment.

By contrast, the subsystems in the present theory are defined to be unentangled quantum objects, and inferences are based on an infinitesimal time interval rather than a single point in time. The information about dynamics contained in this interval then allows an observer to infer something about the state of the environment even when the observer has no prior information about the environment. Of course, it is not clear whether this scheme can get off the ground either; at this stage, it is only a conjecture that the solution of the environmental stabilization problem (in either the original form of Sec. [X.B] or the strong form of Sec. [X.D]) will provide inferences sharp enough to say anything significant about the environment. However, it is reasonable to assume that inferences from a moment in time would be considerably sharper than instantaneous inferences.

Bell has said that the really novel element in Everett’s theory is “a repudiation of the concept of the ‘past’,” although he acknowledges that this interpretation might not be accepted by Everett [153]. Bell did not endorse this interpretation, which Kuchař has described as “solipsism of an instant” [212]. For the reasons described above, this label might indeed be warranted for a truly instantaneous inference scheme.

However, the momentary inference scheme adopted here does not require a wholesale repudiation of the concept of the past. Rather, it places limits on what can be said about the past. It remains to be seen whether these limits are in quantitative agreement with the experiences of observers.

This approach also places limits on what can be said about the future. As noted near the end of Sec. [X.B] the contrast between an observer’s inferences about the future and her experiences in successive present moments has an effect roughly comparable to the quantum jumps of orthodox quantum theory.

The definition of subsystems used by Everett is the same as that used in nearly all other formulations of quantum mechanics. The vector spaces entering into the tensor product have no dynamics and can be chosen arbitrarily at different times. Because the resulting subsystem states are not objects (see Sec. [IA]), it is meaningless to compare their observable properties directly with those of the present theory. The subsystems in the two theories simply refer to different things. Given this qualitative difference, perhaps the only meaningful test of the present theory would be a direct comparison between theory and experiment.

G. Is anything missing?

When one is contemplating the possible outcomes of such a comparison, two other qualitative differences between this theory and standard quantum mechanics stand out. One is that the quasiclassical subsystems used here have indefinite particle numbers (see Sec. [II.B]. Therefore, they cannot directly replicate standard textbook problems for distinguishable subsystems with definite particle numbers. Instead, these subsystems seem to enforce a compliance with Bohr’s emphasis on the “whole experimental arrangement” [33]. That is, they would only be able to describe a hydrogen atom as a part of a larger subsystem, not as a subsystem by itself.

Another difference is the way in which observable (or beable) quantities were defined in Sec. [IV.A] This definition is based on what would conventionally be described as expectation values rather than eigenvalues. The physical picture is closest to the “density of stuff” interpretation [33] used in the dynamical reduction models of Ghirardi and others [136, 138], but with the continuous “reduction” occurring in the subsystem states |ψu⟩ rather than the total system state |ψ⟩ (see Sec. [X.B]).

Taken together, these differences raise doubts as to whether the present theory would be able to replicate the innumerable successful predictions of orthodox quantum mechanics. A particular concern is whether there is any element of this theory comparable to the discrete eigenvalue spectra predicted for the results of ideal measurements in orthodox quantum mechanics.

However, the predictions of dynamical reduction models are well known to be experimentally indistinguishable (given the current state of experimental capabilities) from those of orthodox quantum mechanics [138]. This shows it is unnecessary to define beables as eigenvalues. Since the definition of beables used in Sec. [IV.A] is just a generalization of the one used in dynamical reduction models, it is not unreasonable to expect it to yield comparable results. Testing this conjecture is a key challenge for future research.

X. CONCLUSIONS

This paper arose from the observation that ordinary many-particle quantum mechanics has a hitherto unnoticed mathematical structure that can be interpreted as an unentangled subsystem decomposition. This structure relies on the superposition of different numbers of particles, but it also permits a full description of the equivalence relation that leads to a particle-number superselection rule in orthodox quantum mechanics. The goal of the paper was to take this structure seriously and see what it leads to. Can the elimination of entanglement between subsystems help to resolve some of the conceptual difficulties at the heart of quantum mechanics?

To build on this foundation, one must link the subsystem states to the experiences of observers. The first step is the definition of time as a functional of subsystem changes. This functional can then be embedded into a dynamical stability functional that describes the subsystem dynamics. The resulting subsystem decompositions change by the smallest amount consistent with Schrödinger dynamics for the total system. This change is deterministic.
The observable or “beable” properties of subsystems are defined as expectation values of the conventional operators in many-particle quantum mechanics. These beables include, but are not limited to, the mass density functions used in dynamical reduction theories. An observer could in principle experience any beable associated with her subsystem. However, for Darwinian reasons it is assumed that only those beables with the greatest signal-to-noise ratio are experienced.

An observer’s experiences are also limited to the present moment of time. These experiences are the “iron posts” upon which our concept of reality is based. From them, an observer can infer the existence of an outside world together with a past and future of the present moment. These inferences are extremely useful tools that are indispensable for us to “orient ourselves in the labyrinth of sense impressions,” but they always remain an “arbitrary creation of the human (or animal) mind” \[ \text{[213–215].} \] The resulting image of the world is thus unavoidably subjective. Objectivity emerges only in the limit of many observers.

Much work remains to be done in order to fill in the details of the theory of information outlined here. The most important task is to solve the variation problem in which an environmental subsystem is used to stabilize the dynamics of the observer subsystems in the present moment. Only then will it be possible to see whether the inferences derived in this way are in sufficiently close agreement with experience to be useful.

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**Appendix A: Associativity of the \( \psi \) product**

The \( \psi \) product (2.11) is associative if

\[
(|u^{(k)}\rangle \otimes |v^{(l)}\rangle) \otimes |w^{(m)}\rangle = |u^{(k)}\rangle \otimes (|v^{(l)}\rangle \otimes |w^{(m)}\rangle) \tag{A1}
\]

for all \( |u^{(k)}\rangle \in \mathcal{H}^{k} \), \( |v^{(l)}\rangle \in \mathcal{H}^{l} \), and \( |w^{(m)}\rangle \in \mathcal{H}^{m} \). The explicit form of the associativity condition given by the definition (2.11) is

\[
c(k,l)c(k+l,m)S(|u^{(k)}\rangle \otimes |v^{(l)}\rangle) \otimes |w^{(m)}\rangle) = c(k,l+m)c(l,m)S(|u^{(k)}\rangle \otimes |v^{(l)}\rangle) \otimes |w^{(m)}\rangle). \tag{A2}
\]

This appendix examines the implications of this constraint for the function \( c(k,l) \).

Now it is well known that \[ |u\rangle \otimes |v\rangle \rangle = S(|u\rangle \otimes |v\rangle \rangle) = S(|u\rangle \otimes |v\rangle \rangle) \] for all \( |u\rangle \in \mathcal{H}^{u} \) and \( |v\rangle \in \mathcal{H}^{v} \). Equation (A2) consequently reduce to the condition

\[
c(k,l)c(k+l,m) = c(k,l+m)c(l,m), \tag{A4}
\]

which must be satisfied in order for the \( \psi \) product to be associative.

To solve this equation, it is convenient to follow the procedure suggested on p. 401 of Ref. [104]. Define a function \( f(l) \) recursively by the relation

\[
f(l+1) = c(1,l)f(l). \tag{A5}
\]

In the special case \( k = 1 \), Eq. (A4) gives

\[
\frac{c(l+1,m)}{c(l,m)} = \frac{c(l+1,m)}{c(l,m)} = \frac{f(l+m+1)}{f(l+m)}f(l+1), \tag{A6}
\]

in which \( c(1,l) = f(l+1)/f(l) \) was used in the last step. But this is just a special case of a more general relation

\[
\frac{c(l+k,m)}{c(l,m)} = \frac{f(l+m+k)}{f(l+m)}f(l+k), \tag{A7}
\]

which can be proved by mathematical induction. In the identity

\[
\frac{c(l+k+1,m)}{c(l,m)} = \frac{c(l+k+1,m)}{c(l+k,m)} \frac{c(l+k,m)}{c(l,m)}, \tag{A8}
\]

one can replace the first term using Eq. (A6) (with \( l \to l+k \)) and the second term using Eq. (A7). This gives

\[
\frac{c(l+k+1,m)}{c(l,m)} = \frac{f(l+k+m+1)}{f(l+k+m+1)}f(l+k) \times \frac{f(l+k+m+1)}{f(l+k+m+1)}f(l+1) \times \frac{f(l+k+m+1)}{f(l+k+m+1)}f(l+k) \times \frac{f(l+k+m+1)}{f(l+k+m+1)}f(l+k), \tag{A9}
\]

which shows that Eq. (A7) holds for \( k+1 \) whenever it holds for \( k \). The initial condition (A6) then establishes that (A7) holds for all \( k \geq 1 \). (Of course, it is trivially valid when \( k = 0 \) too.)

Substituting \( k = q - l \) in Eq. (A7) then gives

\[
\frac{c(q,m)}{c(l,m)} = \frac{f(q+m)f(l)}{f(l+m)f(q)}, \tag{A10}
\]

which becomes

\[
\frac{c(q,m)}{c(l,m)} = \frac{f(q+m)f(l)}{f(q+m)f(q)} \tag{A11}
\]

when \( l = 1 \). Replacing \( c(1,m) = f(m+1)/f(m) \) then yields the desired solution

\[
\frac{c(q,m)}{c(q,m)} = \frac{f(q+m)f(1)}{f(1)f(m)}, \tag{A12}
\]

for all \( q, m \) and \( l \).
showing that \( c(q, m) = c(m, q) \).

When \( m = 0 \), this reduces to

\[
e(q, 0) = c(0, q) = \frac{f(1)}{f(0)}, \tag{A13}
\]

which is independent of \( q \). The value of \( c(q, 0) \) can be fixed by requiring the vacuum state \( |0\rangle \) to serve as a multiplicative identity for the \( \psi \) product [cf. Eq. (2.6)]:

\[
|0\rangle \otimes |\Psi\rangle = |\Psi\rangle \otimes |0\rangle = |\Psi\rangle \quad \forall |\Psi\rangle \in F_c(H). \tag{A14}
\]

Imposing this condition when \( |\Psi\rangle = |u(q)\rangle \) gives \( c(q, 0) = 1 \) or \( f(0) = f(1) \). The result \( |f(0)| = |f(1)| \) can also be derived from cluster decomposition, as shown in Appendix B.

The recursive definition \((A5)\) of \( f(l) \) leaves one value of \( f(l) \) that can be chosen arbitrarily. It is convenient to choose \( f(1) = 1 \), thus reducing Eq. \((A12)\) to Eq. \((2.13)\), which was to be proved.

### Appendix B: Cluster decomposition theorem

This appendix contains a proof of Theorem 1, which is about the cluster decomposition property \( \langle st|uv \rangle = \langle s|u\rangle \langle t|v \rangle \) of the inner product in \( F_c(H) \). This inner product is derived from the inner product in \( F(H) \) and the definition of the \( \psi \) product in Eqs. (2.11) and (2.13).

To define the inner product in \( F(H) \), let

\[
|\alpha_1 \alpha_2 \cdots \alpha_n \rangle = |\alpha_1\rangle \otimes |\alpha_2\rangle \otimes \cdots \otimes |\alpha_n\rangle \tag{B1}
\]

denote a general tensor-product state in \( H^n \), where \( |\alpha_k\rangle \in H \). The set \( \{ |\alpha_k\rangle \} \) is not assumed to be linearly independent or normalized. The rounded bracket on the ket \( |\alpha_1 \alpha_2 \cdots \alpha_n\rangle \) distinguishes this unsymmetrized tensor product from the symmetrized product \( |\alpha_1 \alpha_2 \cdots \alpha_n\rangle \) defined in Eqs. \((2.16)\) and \((2.17)\).

The inner product of two such tensor products is defined in the usual way as \( [6] \)

\[
(\alpha_1 \cdots \alpha_n |\beta_1 \cdots \beta_n \rangle = \langle \alpha_1 |\beta_1 \rangle \cdots \langle \alpha_n |\beta_n \rangle, \tag{B2}
\]

where \( \langle \alpha_k |\beta_k \rangle \) is the inner product in \( H \). Now let \( \{ |e_i\rangle \} \) be an orthonormal basis in \( H \). The corresponding tensor products

\[
|e_{i_1} \cdots e_{i_n}\rangle = |e_{i_1}\rangle \otimes \cdots \otimes |e_{i_n}\rangle \tag{B3}
\]

therefore form an orthonormal basis in \( H^n \), since

\[
|e_{i_1} \cdots e_{i_n}|e_{j_1} \cdots e_{j_n}\rangle = \delta_{i_1 j_1} \cdots \delta_{i_n j_n}. \tag{B4}
\]

These elementary results can now be used to evaluate the inner product \( \langle st|uv \rangle \) of the vectors \( |st\rangle = |s\rangle \otimes |t\rangle \) and \( |uv\rangle = |u\rangle \otimes |v\rangle \) in Theorem 1. The kets \( |s\rangle \in F_c(H_1) \) and \( |t\rangle \in F_c(H_2) \) are expanded as

\[
|s\rangle = \sum_k |s^{(k)}\rangle, \quad |t\rangle = \sum_l |t^{(l)}\rangle, \tag{B5}
\]

in which \( |s^{(k)}\rangle \in S(H^k) \) and \( |t^{(l)}\rangle \in S(H^l) \). Hence

\[
|st\rangle = \sum_k \sum_l |s^{(k)}\rangle \otimes |t^{(l)}\rangle = \sum_k \sum_l |s^{(k)} t^{(l)}\rangle, \tag{B6}
\]

in which \( |s^{(k)} t^{(l)}\rangle \in S(H^{k+l}) \). With a similar expansion for \( |uv\rangle \), we can write

\[
\langle st|uv \rangle = \sum_{kl} \langle s^{(k)} t^{(l)}|u^{(m)} v^{(n)}\rangle. \tag{B7}
\]

Here \( \langle s^{(k)} t^{(l)}|u^{(m)} v^{(n)}\rangle \) is zero unless \( k + l = m + n \), since states with different numbers of particles are orthogonal.

But due to the orthogonality of the subspaces \( H_1 \) and \( H_2 \), a stronger restriction is possible: \( \langle s^{(k)} t^{(l)}|u^{(m)} v^{(n)}\rangle \) is zero unless \( k = m \) and \( l = n \). Thus

\[
\langle st|uv \rangle = \sum_{kl} \langle s^{(k)} t^{(l)}|u^{(k)} v^{(l)}\rangle. \tag{B8}
\]

The definition \((2.11)\) of the \( \psi \) product can now be used to write

\[
|u^{(k)} v^{(l)}\rangle = c(k, l) S(|u^{(k)}\rangle \otimes |v^{(l)}\rangle). \tag{B9}
\]

Here associativity requires \( c(k, l) \) to have the form derived in Appendix A

\[
c(k, l) = \frac{f(k + l)}{f(k) f(l)}, \quad f(1) = 1, \tag{B10}
\]

but the function \( f(k) \) has not yet been determined.

The inner product \((B8)\) is therefore

\[
\langle st|uv \rangle = \sum_{kl} |c(k, l)|^2 \langle s^{(k)} \otimes t^{(l)}|S^* S(|u^{(k)}\rangle \otimes |v^{(l)}\rangle). \tag{B11}
\]

Since \( S \) is an orthogonal projector, \( S^* S = S^2 = S \), thus

\[
\langle st|uv \rangle = \sum_{kl} |c(k, l)|^2 \langle s^{(k)} \otimes t^{(l)}|S(|u^{(k)}\rangle \otimes |v^{(l)}\rangle). \tag{B12}
\]

Here \( \langle u^{(k)} \otimes |v^{(l)}\rangle \in H^{k+l} \), so the definition of \( S \) gives

\[
S(|u^{(k)}\rangle \otimes |v^{(l)}\rangle) = \frac{1}{(k + l)!} \sum_\sigma \varepsilon(\sigma) \sigma(|u^{(k)}\rangle \otimes |v^{(l)}\rangle). \tag{B13}
\]

Now if the product \( \langle u^{(k)} \otimes |v^{(l)}\rangle \) is expanded in the unsymmetrized basis of Eq. \((B3)\) with \( n = k + l \), the first \( k \) vectors \( |e_i\rangle \) will be from \( H_1 \) and the last \( l \) vectors will be from \( H_2 \). The same is true for the product \( \langle s^{(k)} \otimes |t^{(l)}\rangle \). The only permutations \( \sigma \) in equation \((B13)\) that contribute nonvanishing terms to equation \((B12)\) are therefore those that do not interchange any of the first \( k \) vectors with the last \( l \) vectors. These permutations are of the form \( \sigma = \sigma_1 \sigma_2 \), where \( \sigma_1 \) is any permutation of the first \( k \) vectors and \( \sigma_2 \) is any permutation of the last \( l \) vectors.
The orthogonality of the basis vectors \((\mathcal{B}_3)\) therefore reduces Eq. \((\mathcal{B}_12)\) to

\[
\langle st|uv \rangle = \sum_{kl} \frac{|c(k, l)|^2}{(k + l)!} \sum_{\sigma_1} \varepsilon(\sigma_1) \langle s(k)|\sigma_1|u(k)\rangle \\
\times \sum_{\sigma_2} \varepsilon(\sigma_2) \langle t(l)|\sigma_2|v(l)\rangle,
\]

in which \(\varepsilon(\sigma_1) \varepsilon(\sigma_2) = \varepsilon(\sigma_1) \varepsilon(\sigma_2)\) was used. Now \(|u(k)\rangle \in S(H^k)\), so \(\sigma_1|u(k)\rangle = \varepsilon(\sigma_1)|u(k)\rangle\), and likewise \(\sigma_2|v(l)\rangle = \varepsilon(\sigma_2)|v(l)\rangle\). But \(\varepsilon(\sigma)|^2 = 1\) for any \(\sigma\), hence

\[
\langle st|uv \rangle = \sum_{kl} \frac{|c(k, l)|^2}{(k + l)!} \sum_{\sigma_1} \langle s(k)|u(k)\rangle \sum_{\sigma_2} \langle t(l)|v(l)\rangle (B15)
\]

\[
= \sum_{kl} \left[ \frac{f(k + l)}{f(k)f(l)} \right] \frac{k!!l!!}{(k + l)!} \langle s(k)|u(k)\rangle \langle t(l)|v(l)\rangle.
\]

At this point, choosing \(|f(k)| = \sqrt{k!}\) eliminates all of the numerical factors, yielding

\[
\langle st|uv \rangle = \sum_k \langle s(k)|u(k)\rangle \sum_l \langle t(l)|v(l)\rangle.
\]

Each factor can be rewritten as

\[
\sum_k \langle s(k)|u(k)\rangle = \sum_k \sum_m \langle s(k)|u(m)\rangle = \langle s|u\rangle,
\]

since \(\langle s(m)|u(m)\rangle = 0\) when \(k \neq m\). Therefore \(\langle st|uv \rangle = \langle s|u\rangle \langle t|v\rangle\), thus proving the “iii” part of the cluster decomposition theorem 1

Conversely, suppose it is given that \(\langle st|uv \rangle = \langle s|u\rangle \langle t|v\rangle\) for all \(|s|, |u\rangle \in \mathcal{F}_s(H_1)\) and all \(|t|, |v\rangle \in \mathcal{F}_s(H_2)\), where \(H_1\) and \(H_2\) are orthogonal subspaces of \(H\). What does this tell us about \(f(k)\)?

Under the given conditions, we are free to choose \(|u\rangle = |u(k)\rangle\) and \(|v\rangle = |v(l)\rangle\) for any values of \(k\) and \(l\). Equation (B16) then reduces to

\[
\langle st|uv \rangle = \frac{f(k + l)}{f(k)f(l)} \left[ \frac{k!!l!!}{(k + l)!} \langle s(k)|u(k)\rangle \langle t(l)|v(l)\rangle \right],
\]

with no summation on \(k\) and \(l\). Likewise

\[
\langle st|uv \rangle = \langle s|u\rangle \langle t|v\rangle\in all such cases only if
\]

\[
\left[ \frac{f(k + l)}{f(k)f(l)} \right] = \frac{(k + l)!}{k!!l!!} \forall k, l \geq 0.
\]

Setting \(l = 1\) and using \(f(1) = 1\) then gives

\[
|f(k + 1)|^2 = (k + 1)!|f(k)|^2.
\]

This defines \(|f(k)|\) recursively as

\[
|f(k)|^2 = k!|f(1)|^2 = k! \quad (k \geq 1),
\]

while \(k = 0\) gives \(|f(1)|^2 = |f(0)|^2\). Hence, \(|f(k)| = \sqrt{k!}\) for all \(k\), thus completing the proof of Theorem 1.

**Appendix C: Algebraic closure conditions for bosons**

This appendix describes the construction of the boson vector space \(\mathcal{F}_\psi(H_b)\) mentioned at the end of Sec. [11D]. The objective is to identify a subspace of \(\mathcal{F}_\psi(H_b)\) that is a good match for the algebra of the boson \(\psi\) product. This algebra is easiest to describe using the Segal–Bargmann representation of Fock space \([216, 219]\), which is often used in the definition of coherent boson states \([96, 121, 122]\).

Let us start by establishing a concise notation. Standard basis kets in Fock space are written as \(|n\rangle\), where \(n = (n_1, n_2, \ldots, n_b) \in \mathbb{N}^b\) is a vector of nonnegative integers, \(b\) is the dimension of the single-boson Hilbert space \(H_b\), and \(n_i\) is the number of bosons in the single-particle state \(i\). In multi-index notation \([97, 216]\), powers of a complex vector \(z = (z_1, \ldots, z_b) \in \mathbb{C}^b\) are written as \(z^n = z_1^{n_1} \cdots z_b^{n_b}\), and likewise for powers of the vector \(a^\dagger = (a_1^\dagger, \ldots, a_b^\dagger)\) of boson creation operators. This allows the normalized basis ket \(|n\rangle\) to be written simply as \(|n\rangle = (n!)^{-1/2}\langle a^\dagger \rangle^n|0\rangle\) [see Eqs. (2.19) and (2.21)]. For reasons to be explained below, the Hilbert–Fock space defined in this way is also written as \(\mathcal{H}_1\).

A general vector in \(\mathcal{F}_\psi(H_b)\) has the form \(|f\rangle = \sum_n c_n|n\rangle\), where \(c_n = \langle n|f\rangle\). The Fock space \(\mathcal{F}_\psi(H_b)\) is also required to be a Hilbert space, the members of which must satisfy \(||f|| < \infty\), in which \(||f||^2 = \langle f|f\rangle = \sum_n |c_n|^2\). For reasons to be explained below, the Hilbert–Fock space defined in this way is also written as \(\mathcal{H}_1\).

The Segal–Bargmann representation of \(|f\rangle\) is defined by the expression [cf. Eq. (2.22)]

\[
|f\rangle = F(a^\dagger)|0\rangle,
\]

in which the function \(F(z)\) is defined by the power series

\[
F(z) = \sum_n \frac{c_n}{\sqrt{n!}} z^n.
\]

If \(||f|| < \infty\), \(F(z) \in \mathbb{C}\) is an entire holomorphic function of \(z = x + iy\), where \(x, y \in \mathbb{R}\). This representation of ket vectors by entire functions is a powerful advantage of the Segal–Bargmann theory.

In the Segal–Bargmann representation, the inner product of two vectors \(|f\rangle\) and \(|g\rangle\) is given by the integral

\[
\langle f|g \rangle = \int F^*_\gamma(z)G^*(z)\rho(z) d^{2b}z,
\]

in which \(\rho(z) = \pi^{-b} \exp(-|z|^2)\), \(|z|^2 = |z_1|^2 + \cdots + |z_b|^2\), and \(d^{2b}z = dx_1 dy_1 \cdots dx_b dy_b\). Functions of the form \(F(z) = \exp\left(\frac{1}{2}\gamma z^2 + \alpha \cdot z\right)\) are of special interest, where \(\gamma \in \mathbb{C}\), \(\alpha \in \mathbb{C}^b\), \(\alpha \cdot z = \alpha_1 z_1 + \cdots + \alpha_b z_b\), and \(z^2 = z \cdot z\). This function is normalizable (i.e., \(||f|| < \infty\)) if and only if \(||\gamma|| < 1\) \([216]\). A general bound on all normalizable states is given by the Schwarz inequality \([216]\):

\[
||F(z)|| \leq \exp\left(\frac{1}{2}|\gamma|^2\right)||f|| \forall z \in \mathbb{C}^b.
\]

This implies that the Hilbert–Fock space \(\mathcal{H}_1 = \mathcal{F}_\psi(H_b)\) is a poor match for the algebra of the \(\psi\) product, since
implies that $F$ can be used to provide a rigorous justification for the inner product $\langle H \rangle$. The set of vectors with $\parallel k \parallel \in H$ is represented by the product $(z \exp(z)) \parallel u \rangle \parallel v \rangle$ as the product of two simple ten-

Let us now define a vector space $F_\psi = F_\psi(\mathbb{H}_k)$ as the intersection of the Hilbert spaces $\mathbb{H}_k$ for all $k \in \mathbb{N}_+$. $F_\psi$ is defined by the countable family of norms $\parallel f \parallel_k$ or $\parallel f \parallel$, which cannot be defined by any single norm. This space is therefore a Fréchet space [27, 110], not a Hilbert space. Such vector spaces are familiar from the rigged Hilbert space formalism of quantum mechanics [220–223], which can be used to provide a rigorous justification for the Dirac bra-ket formalism.

The subscript on $F_\psi$ is intended to suggest that this space is a suitable arena for the algebra of the $\psi$ product. To show this, we need to prove that $|h \rangle = |f \rangle \otimes |g \rangle$ belongs to $F_\psi$ whenever $|f \rangle$ and $|g \rangle$ do. In other words, we must show that $|h \rangle \in \mathbb{H}_k$ for all $k \in \mathbb{N}_+$ whenever $|f \rangle \in \mathbb{H}_n$ and $|g \rangle \in \mathbb{H}_n$ for all $q \in \mathbb{N}_+$. But this is easily done, since $\parallel f \parallel$ and $|g \rangle$ give inequalities $\parallel f \parallel \leq \exp(\parallel z \parallel^2/2k^2) \parallel f \parallel$ and $\parallel g \parallel \leq \exp(\parallel z \parallel^2/2k^2) \parallel g \parallel$; the product $H(z) = F(z) \parallel G(z) \parallel$ thus satisfies $\parallel H(z) \parallel \leq A \exp(\parallel z \parallel^2/2k^2)$, where $A = \parallel f \parallel \parallel g \parallel$ and $\lambda = 2k^2/2q^2$. According to $\parallel C_8 \parallel$, this implies that $|h \rangle \in \mathbb{H}_k$ as long as we are free to choose $\lambda < 1$, i.e., $\kappa > \sqrt{2k}$. But this can be done for any $k \in \mathbb{N}_+$, by the definition of $F_\psi$.

It should be clear from the above derivation that the algebra of the $\psi$ product cannot be accommodated within any vector space defined by a finite number of norms. Hence, the move from Hilbert space to Fréchet space is necessary for boson systems.

What type of vectors belong to $F_\psi$? It was noted above that $F(z) = \exp(\parallel z \parallel^2/2 + \alpha \cdot z)$ is in $\mathbb{H}_1$ if and only if $\parallel z \parallel^2 < 1$. However, Eqs. (60) and (61) show that it belongs to $F_\psi$ if and only if $\gamma = 0$. The only exponential functions in $F_\psi$ are therefore those of the form $F(z) = \exp(\alpha \cdot z)$, for arbitrary $\alpha \in \mathbb{C}^b$. But these are just the coherent states $|\alpha \rangle = \exp(\alpha \cdot a^\dagger)|0\rangle$, which can be defined as eigenvectors of the boson annihilation operators $a_i$ (i.e., $a_i|\alpha \rangle = \alpha_i|\alpha \rangle$) [221]. Note that the operator $\exp(\alpha \cdot a^\dagger)$ in Eq. (60) is easy to invert; its inverse is $\exp(\kappa \cdot a^\dagger)$.

Bargmann called the functions $F(z) = \exp(\alpha \cdot z)$ “principal vectors” and showed that they are complete (although not orthogonal), in the sense that linear combinations of them are dense in $\mathbb{H}_1$ [216]. This completeness is usually expressed as the integral [217]

$$\frac{1}{\pi} \int |\alpha \rangle \langle \alpha | \exp(-\|z\|^2) \, d^b z = 1.$$  

The monomials $F(z) = (n!)^{-1/2} z^n$ also form a complete orthonormal basis corresponding to the original basis $|n \rangle = (n!)^{-1/2}(a^\dagger)^n|0\rangle$ in Fock space.

Finally, note from Eq. (61) that if $|f \rangle \in F_\psi$, then $|n \rangle \rightarrow \infty$, $\langle n | f \rangle$ must decrease faster than $\exp(-\kappa |n|)$ for any positive value of $\kappa$. This rate of decrease is even faster than that of the sequences of rapid descent encountered in connection with Schwartz spaces $S$ by $S$.

Appendix D: Different types of particles

Consider a system containing two types of particles, labeled $A$ and $B$. If the corresponding single-particle Hilbert spaces are $\mathbb{H}_A$ and $\mathbb{H}_B$, the vector space of the whole system can be defined as the tensor-product space $\mathcal{G} = F_\psi(\mathbb{H}_A) \otimes F_\psi(\mathbb{H}_B)$. That is, a general vector $|u \rangle \in \mathcal{G}$ is a linear combination of tensor and product states $|u_A \rangle \otimes |u_B \rangle$, where $|u_A \rangle \in F_\psi(\mathbb{H}_A)$ and $|u_B \rangle \in F_\psi(\mathbb{H}_B)$.

One can define a $\psi$ product in $\mathcal{G}$ by letting the $\psi$ product of Sec. 12 act in parallel on the subspaces $F_\psi(\mathbb{H}_A)$ and $F_\psi(\mathbb{H}_B)$. That is, the $\psi$ product of two simple tensor states $|u \rangle = |u_A \rangle \otimes |u_B \rangle$ and $|v \rangle = |v_A \rangle \otimes |v_B \rangle$ is defined to be

$$|u \rangle \otimes |v \rangle \rangle = (|u_A \rangle \otimes |v_A \rangle) \otimes (|u_B \rangle \otimes |v_B \rangle).$$

Therefore, the product of two simple tensor states $|u \rangle = |u_A \rangle \otimes |u_B \rangle$ and $|v \rangle = |v_A \rangle \otimes |v_B \rangle$ is defined to be

$$|u \rangle \otimes |v \rangle \rangle = (|u_A \rangle \otimes |v_A \rangle) \otimes (|u_B \rangle \otimes |v_B \rangle).$$  

(D2)
This is then extended to arbitrary vectors $|u⟩, |v⟩ ∈ G$ by multilinearity. The algebra thus defined is associative, which follows directly from the associativity of the $ψ$ product in $F_s(ℋ_A)$ and $F_s(ℋ_B)$. From this definition, it is a simple exercise to show that the cluster decomposition property of Theorem 1 is valid in $G$ if it holds in both $F_s(ℋ_A)$ and $F_s(ℋ_B)$. The equivalence between the algebra of the $ψ$ product and the algebra of creation operators discussed in Sec. IID likewise remains valid in systems with more than one type of particle.

**Appendix E: Invertibility theorem**

The first step in the proof of Theorem 2 is to show that a boson-fermion creator $U : E → E$ is invertible if and only if its associated boson creator $U_0 : F_ψ(ℋ_B) → F_ψ(ℋ_B)$ is invertible. The basic reason for this can be seen intuitively from the matrix of creators (2.34). The determinant of this triangular matrix is $det U = (U_0)^{2d}$, in which $d = dim ℋ_t$. If we assume that the standard theorems of matrix algebra can be extended to this matrix of commuting operators, then $U^{-1}$ exists if and only if $U_0^{-1}$ exists.

A more explicit argument is as follows. Because $U_0$ is a boson creator, we can use the multi-index notation of Appendix C to write $U_0 = F(a!1)$, in which $F(z)$ is an entire function of $z ∈ ℂ^b$ and $b = dim ℋ_t$. Invertibility of $U_0$ thus requires that $1/F(z)$ is also an entire function, or that $F(z) ≠ 0$ for finite $z$. But if $F(z) = 0$ at $z = α^*$, the coherent state $|α⟩$ in Eq. (C10) is orthogonal to every vector in the image of $U_0$, as shown below. Hence, $U_0$ cannot be surjective (or onto) in this case. This also implies that $U$ is not surjective, because $|α⟩ ⊗ |0⟩_t$ is orthogonal to the image of $U$. Invertibility of $U_0$ is therefore necessary for invertibility of $U$. Its sufficiency follows immediately from Eqs. (3.10) and (3.13).

To clarify the orthogonality relation mentioned above, let us start by writing $U_0^1 = F(α)$, in which $a = (a_1, ..., a_b)$ is a vector of boson annihilation operators and $F(α^*) ≡ [F(α)]^*$ is an entire function of $α^*$. Given that $F(0) = 0$ at $z = α^*$, we have $F(α) = 0$ and thus $U_0^1|α⟩ ≡ U_0^1|0⟩ = F(α)|0⟩ = F(α)|0⟩ = 0$, because $|0⟩$ is an eigenket of $a$. For any $|z⟩ ∈ F_ψ(ℋ_B)$ we then have $(U_0^1|α⟩|z⟩ = ⟨α|U_0|z⟩ = 0$. But $(⟨α|U_0|z⟩ = 0$ for all $|z⟩ ∈ F_ψ(ℋ_B)$ is precisely the statement that $|α⟩$ is orthogonal to every vector in the image of $U_0$.

The second step in the proof of Theorem 2 is to show that $U_0$ is invertible if and only if $|u⟩_0$ is a coherent state. The starting point is the condition $F(z) ≠ 0$ established above. Now it is well known in the theory of a single complex variable $z ∈ ℂ$ that every entire function $F(z)$ with no zeros can be written as $F(z) = exp(G(z))$, where $G(z)$ is another entire function [229][231]. This is equivalent to the existence of a global logarithm of such a function $F(z)$, which depends essentially on whether the domain of $F$ is simply connected. The single-variable proof given in Ref. 231 can also be extended to the case of entire functions of several complex variables $z ∈ ℂ^b$. In order for $U_0$ to be invertible, it is therefore necessary that $F(z) = exp(G(z))$ for some entire function $G(z)$.

However, according to the results of Appendix C if $|u⟩_0 ∈ F_ψ(ℋ_b)$, then $G(z)$ can only be a linear function of $z$. That is, $F(z)$ must be proportional to $exp(α · z)$ for some $α ∈ ℂ^b$, and $|u⟩_0$ must be proportional to one of the coherent states $|α⟩$ defined in Eq. (C10). The necessity of $|u⟩_0$ being a coherent state is therefore established.

To demonstrate its sufficiency, we only need to note that the operator $exp(α · a!1)$ appearing in Eq. (C10) is invertible, its inverse being given by $exp(−α · a!1)$. Thus, $U_0$ is invertible whenever $|u⟩_0$ is a coherent state. This concludes the proof of Theorem 2.

**Appendix F: Creator identities**

A useful identity for the symmetrized product of three creators $A, B$, and $C$ is

$$\{A, \{B, C\}\} = \{\{A, B\}, C\}. \tag{F1}$$

This can be derived simply by writing out the definition of the symmetrized products, which leads to the general operator identity

$$\{A, \{B, C\}\} − \{\{A, B\}, C\} = \frac{1}{4}[[A, C], B]. \tag{F2}$$

Given that $A, B$, and $C$ are creators, the right-hand side vanishes due to Eq. (3.6), yielding the identity in Eq. (F1).

A useful corollary of this identity is the equivalence

$$B = \{U, A\} \iff A = \{U^{-1}, B\}, \tag{F3}$$

in which $A$ and $B$ are creators and $U$ is an invertible creator. For example, the leftward implication can be derived from

$$\{U, A\} = \{U, \{U^{-1}, B\}\} = \{\{U, U^{-1}\}, B\} = B, \tag{F4}$$

since $\{U, U^{-1}\} = 1$.

**Appendix G: Proof that $χ$ has a global maximum**

In Sec. IV B it was shown that, for a given value of $∆t$, the dynamical stability functional has only one stationary state with $χ > 0$. To prove that this is indeed the global maximum of $χ$, we can follow the approach used in Eq. (4.19) to obtain the inequality

$$χ = \frac{(Im(∆x|σ)|^2}{(Δx|^2)} ≤ \frac{⟨σ⟩ |Δx|^2}{(Δx|^2)} = γ. \tag{G1}$$
in which $\Sigma \equiv |\sigma\rangle\langle\sigma|$. Varying the functional $\gamma$ leads to
the generalized eigenvalue equation
\[ \Sigma|\Delta x\rangle = \gamma \hat{\eta}|\Delta x\rangle, \] (G2)
which is well defined because $\hat{\eta} > 0$. However, because $\Sigma$ is a projector of rank one, it has only one eigenvector with eigenvalue $\gamma > 0$. This is just
\[ |\Delta x\rangle = -iC\hat{\eta}^{-1}|\sigma\rangle, \] (G3)
where $C$ is an arbitrary complex number. Since the functional $\gamma$ is bounded from above by its maximum eigenvalue, this eigenvalue is the global maximum of $\gamma$.

Looking back now at Eq. (G1), we see that choosing $C$ to be real turns the inequality $\chi \leq \gamma$ into an equality, and also makes the eigenvector (G3) identical to the stationary state (6.15) of $\chi$. Hence, the global maximum of $\gamma$ is also the global maximum of $\chi$, and the conjecture is proved.

Appendix H: Distance between phase orbits

The calculation of $D^2(\{\rho_0,\rho_1\})$ in Sec. VII C was based on the assumption that $\|\Delta u\|$ is small. If this is not true, we must return to Eqs. (7.9) and (7.10) and calculate the function
\[ \lambda(\phi) = m - \sum_{n,n'} M_{nn'} e^{i(n-n')\phi}, \] (H3)
in which
\[ M_{nn'} \equiv \sum_{k=1}^{m} \text{tr}(\rho_k \Pi_n \rho_k^* \Pi_{n'}) \] (H4)
\[ = \sum_{k=1}^{m} \frac{\langle u_k | \Pi_n | u_k' \rangle \langle u_k' | \Pi_{n'} | u_k \rangle}{\langle u_k | u_k \rangle \langle u_k' | u_k' \rangle}. \] (H5)

This matrix is hermitian, as can be seen from Eq. (H4). The function (H3) can therefore be written as $\lambda(\phi) = m - G_0 + 2g(\phi)$, in which $G_l \equiv \sum_n M_{n+l,n}$ and
\[ g(\phi) = -\sum_{l>0} \text{Re}(G_l) \cos(l\phi) + \sum_{l>0} \text{Im}(G_l) \sin(l\phi). \] (H6)

Hence, in Eq. (7.9), minimizing $\lambda(\phi)$ is the same as minimizing $g(\phi)$. This is easy to do in fermion systems with small $d = \dim H_1$, because $l \leq d$. The minimum of $g(\phi)$ can then be found quickly using a simple grid search and Newton’s method.

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