Geometrical measurements in three-dimensional quantum gravity

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Abstract

A set of observables is described for the topological quantum field theory which describes quantum gravity in three space-time dimensions with positive signature and positive cosmological constant. The simplest examples measure the distances between points, giving spectra and probabilities which have a geometrical interpretation. The observables are related to the evaluation of relativistic spin networks by a Fourier transform.

1 Distances

In general relativity we can measure the distance $R$ between a pair of points by considering the length of a geodesic between them (Figure 1).

In quantum gravity the metric fluctuates, so we expect only to be able to say what the possible values for $R$ are, and their probabilities. In general we

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might expect these to depend on the topology of the manifold $M$ in which the points $p$ and $q$ lie.

The Turaev–Viro state sum model [11] gives a theory of quantum gravity in 3 dimensions where the metric has positive signature $[1]$; it gives a concrete method for calculating the functional integral for three-dimensional gravity [14]. You can think of this as related to a 4-dimensional theory with $-+++ \text{ signature metrics}$ where the time dimension ($-$) has been dropped. Although this is not entirely realistic, it does give us a model in which the relationship between classical geometry and quantum gravity can be explored.

The model is specified by an integer $r \geq 3$. Given points $p, q \in M$ connected by a curve then an observable can be defined which takes values in the set of spins

$$j \in \left\{0, \frac{1}{2}, 1, \ldots, \frac{r-2}{2} \right\}.$$ 

The probability that the spin takes value $j$ can be calculated to be

$$P_j = \frac{(\dim_q j)^2}{N},$$

with $\dim_q j$ the quantum dimension of the spin $j$ representation of $U_q sl(2)$ for $q = e^{i\pi/r}$, and $N$ a normalisation constant. The formula for the quantum dimension is

$$\dim_q j = (-1)^{2j} \left( \frac{\sin \frac{\pi}{r}(2j+1)}{\sin \frac{\pi}{r}} \right)$$

and $N = \sum_j (\dim_q j)^2$ is the constant which ensures that $\sum_j P_j = 1$. Actually in this model the distance measurements only depend on the topology of $p$, $q$ and $M$ to the extent that the points $p$ and $q$ are required to be in the same connected component of $M$. The topology of $M$ comes into generalizations of the formula considered further below. First I will describe how the probability formula is calculated, and then its physical interpretation.
2 Calculation

The Turaev-Viro state sum for a closed compact manifold $M$ is a formula for an invariant $Z(M) \in \mathbb{R}$. This is defined with the aid of a triangulation of the manifold; however the value of $Z(M)$ is independent of the triangulation chosen and depends only on the topology of $M$.

![Figure 2: State for a triangle](image)

A state for this state sum model is the assignment of a spin $i,j,k,\ldots$ to each edge of the triangulation (Figure 2) such that the following ‘admissibility’ conditions for each triangle are satisfied.

\begin{align*}
i &\leq j + k \quad (1) \\
j &\leq k + i \quad (2) \\
k &\leq i + j \quad (3) \\
i + j + k &\leq r - 2 \quad (4) \\
i + j + k &\equiv 0 \mod 1 \quad (5)
\end{align*}

Given a state, each simplex is assigned a weight, a real number. This number depends on the spin labels for the edges in that simplex. The weights are calculated using the spin network evaluation based on the Kauffman bracket $\langle \rangle$ with $A = e^{i\pi r/2}$ as follows [6]:

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For the tetrahedral simplex, the spin network in the right-hand column is the graph which is dual to the edges of the tetrahedron.

The state sum formula is

\[ Z(M) = \sum_{\text{states}} \prod_{\text{simplexes}} \text{weights}. \]

The probability formula is calculated by finding a triangulation of \( M \) such that \( p \) and \( q \) are the two vertices of a single edge in the triangulation. The probability \( P_j \) is just the probability that the distinguished edge has its spin equal to \( j \), i.e.

\[ P_j = \frac{Z(M, j)}{Z(M)}, \]

where \( Z(M, j) \) is the sum over the subset of states that have spin \( j \) on the distinguished edge. Clearly \( Z(M) = \sum_j Z(M, j) \), so the \( P_j \) sum to 1.

The proof that the formula for \( P_j \) is correct is to calculate it explicitly for a particular triangulation, and then use the triangulation invariance of the state sum formula to show that the formula holds for all triangulations which have an edge that runs from \( p \) to \( q \). The calculation can be done easily for \( M = S^3 \) using the singular triangulation of \( S^3 \) with two tetrahedra. It also follows from the Fourier transform result proved below. The proof that
the result is the same for any triangulation will appear elsewhere. The fact that the answer is the same for any manifold follows from the connected sum formula for the Turaev-Viro invariant and the fact that the edge is contained in a ball in $M$.

The positivity of the probabilities is not immediately obvious from the definition of the state sum since the total weight for a state of $M$ can have either sign. However it does follow from the fact that the Turaev-Viro model has state spaces on surfaces which are Hilbert spaces, and the state sum formula for $P_j$ is the expectation value of a positive operator (a projector) on the Hilbert space of $S^2$.

3 Geometrical Models

A physical interpretation is most apparent in the limiting case $r \to \infty$ (the Ponzano–Regge model [8]). Then,

$$P_j = \frac{(2j + 1)^2}{N}.$$  

There is however no value for $N$ which normalises $\sum P_j$ to 1 so this limit is somewhat degenerate. Nevertheless, Ponzano and Regge discovered that the asymptotic formulae for the state sum in the limit $j \to \infty$ have a geometric interpretation if one takes $j + \frac{1}{2}$ to be the length of the edge in 3-dimensional Euclidean space $\mathbb{R}^3$. Also, they suggested that the semi-classical configurations of the state sum model are given by mapping the simplicial complex to $\mathbb{R}^3$ with an approximate uniform measure for the position of the vertices in $\mathbb{R}^3$. This is also consistent with the gauge theory interpretation of the model in which the gauge group is the semi-direct product of $SU(2)$ and $\mathbb{R}^3$ [15].

These considerations suggest that for spin $j$, the distance between $p$ and $q$ is $j + \frac{1}{2}$ and the probability $P_j$ is proportional to the area of the 2-sphere of radius $j + \frac{1}{2}$. In other words, a geometric model for the probabilities $P_j$ is to consider displacement vectors in $\mathbb{R}^3$ which have length $R = j + \frac{1}{2}$ but undetermined direction. The measure $P_j$ is a discrete version of the uniform measure $4\pi R^2 dR$ in three-dimensional Euclidean space. This gives a model for the $P_j$ in terms of probability measures for points moving in the classical geometry.

Now to return to the Turaev–Viro model. The Lagrangian quantum field theory view is that this model is a version of quantum gravity with a positive cosmological constant $\Lambda$, whereas the Ponzano–Regge model has $\Lambda = 0$. The classical solutions are locally a 3-sphere, with radius $\sqrt{1/\Lambda}$. Obviously as $\Lambda \to \infty$ this degenerates to the Euclidean space $\mathbb{R}^3$ of the Ponzano-Regge
model. This suggests that the physical interpretation of the probabilities $P_j$ should be based on configurations in $S^3$. Indeed the area of a 2-sphere of radius $j + \frac{1}{2}$ in $S^3$ is

$$\text{area} = 4\pi \sin^2 \frac{\pi}{r} (2j + 1),$$

(6)
i.e., proportional to $P_j$, if the 3-sphere has radius $r/2\pi$. If the point $p$ is fixed at the ‘north pole’ then the possible positions for $q$ lie on the 2-spheres indicated on Figure 3 with probability proportional to the area. In the figure,

Figure 3: Possible orbits for $q$

the 3-sphere is projected to a disk on the plane and the 2-sphere of constant height is shown in its projection as a horizontal line.

In this way the range of values for the spin also has a natural explanation in terms of the 3-sphere: the lengths take all possible half-integer values for distances on the 3-sphere of radius $r/2\pi$. This only works because of the $' + \frac{1}{2}'$ in the relation between spin and distance. The minimum distance is then $1/2$ and the maximum $(r - 1)/2$. There are two other possible half-integral values for the distance between a pair of points, namely 0 and the half-circumference $r/2$. However the corresponding probabilities in this picture are zero, and so these possibilities don’t occur.
4 Generalizations

In a similar way we can calculate the probability in the state sum model for three points \( p, q, s \) which are the vertices of an embedded triangle in \( M \) to be separated by distances \( i + \frac{1}{2}, j + \frac{1}{2}, k + \frac{1}{2} \) (Figure 4). The result is

\[
P(i, j, k) = \frac{Z(M; i, j, k)}{Z(M)} = \begin{cases} 
N^{-2} \dim_q i \dim_q j \dim_q k & \text{if } (i, j, k) \text{ admissible}, \\
0 & \text{else}
\end{cases}
\]

In carrying out this calculation, the topological configuration is important. One has to specify curves which connect each pair of points. What is important is that the loop of the three edges is unknotted and is a contractible loop in \( M \), in other words that the three edges do indeed bound a triangle in \( M \). The non-zero part of this formula is

\[
\sin \frac{\pi}{r} (2i + 1) \sin \frac{\pi}{r} (2j + 1) \sin \frac{\pi}{r} (2k + 1)
\]

which is positive, the signs cancelling due to the admissibility condition (5). The other four admissibility conditions (1)–(4) have a geometrical interpretation when they are rewritten in terms of the lengths:

\[
i + \frac{1}{2} < \left( j + \frac{1}{2} \right) + \left( k + \frac{1}{2} \right) \\
j + \frac{1}{2} < \left( k + \frac{1}{2} \right) + \left( i + \frac{1}{2} \right) \\
k + \frac{1}{2} < \left( i + \frac{1}{2} \right) + \left( j + \frac{1}{2} \right) \\
\left( i + \frac{1}{2} \right) + \left( j + \frac{1}{2} \right) + \left( k + \frac{1}{2} \right) < r
\]
The first three are interpreted as the conditions for the edge-lengths of a non-degenerate triangle in a metric space geometry. A triangle is degenerate if there is a vertex whose location is uniquely determined by the location of the other two vertices. However the fourth condition is again specific to a sphere: a geodesic triangle with sides $R_1, R_2, R_3$ on a sphere (in any dimension) of radius $r/2\pi$ satisfies the inequality $R_1 + R_2 + R_3 \leq r$. The proof of this is very simple. The triangle inequalities for $tqs$ (Figure 5) give

$$R_1 \leq (r/2 - R_2) + (r/2 - R_3) \quad \text{or} \quad R_1 + R_2 + R_3 \leq r.$$ 

![Figure 5: Geodesic triangle on a sphere](image)

However in the case $R_1 + R_2 + R_3 = r$ the three points lie on a diameter and one of the points is determined uniquely by the location of other two. Such a triangle is therefore degenerate. The overall result is that the conditions (1-4) are the conditions for a non-degenerate triangle on $S^3$.

The geometrical model (6) for the single edge can be extended to this case. Consider three points $p$, $q$ and $s$ on $S^3$ with a uniform probability distribution. The probability that the distances between them are $R_1$, $R_2$ and $R_3$, as in Figure 5, is proportional to

$$\sin \frac{2\pi R_1}{r} \sin \frac{2\pi R_2}{r} \sin \frac{2\pi R_3}{r} \ dR_1 \ dR_2 \ dR_3$$
as long as the inequalities for a triangle are satisfied (Appendix 1). This formula is the continuum analogue of (7), and in fact (7) is obtained by substituting $R_1 = i + 1/2$, $R_2 = j + 1/2$, $R_3 = k + 1/2$ in this probability density. This means that the geometrical model reproduces the measure $P(i, j, k)$ under the additional assumption that all edge length are required to be a half-integer.

In a similar way one can analyse an embedded polygon in $M$, obtaining probabilities which can be considered as a measure of the volume of configurations of an unknotted circular loop of rods of fixed length in $S^3$. It is an interesting problem to relate this to other measures of the volume of these configurations, such as the symplectic volume measure provided in the flat $(r \to \infty)$ case by the Riemann–Roch theorem [3 4].

These simple examples may give the misleading impression that the classical geometry is always the standard metric 3-sphere. However this is not the case, as the observable is sensitive to knotting and linking. The general situation is studied in the next section.

5 Fourier transform

In general one can consider the set of edges on which the spins are fixed to form an embedded graph $\Gamma$ in $M$. Then the state sum invariant with these spins fixed gives an invariant of the embedded graph under motions of the graph in the manifold (ambient isotopies).\footnote{A different set of observables to the ones investigated here were defined in [12 5].}

In the case of $M = S^3$ there is another invariant of embedded graphs with edges labelled by spins, the relativistic spin network invariant defined by Yetter [16 2]. In this section it is shown that the two invariants are related by a Fourier transform of the spin labels. This substantially generalises the $\mathbb{Z}_2$ Fourier transform of [10 13].

The definition of the relativistic spin network invariant is as follows. Let $\Gamma(i_1, i_2, \ldots, i_n)$ be a graph embedded in $S^3$, and its edges labelled with spins $i_1, i_2, \ldots, i_n$ (in a fixed order). First, the invariant is defined in the case of trivalent graphs, then this will be generalised to arbitrary vertices. For each vertex of a trivalent graph there are three spin labels $(i, j, k)$ on the three edges meeting the vertex. The invariant is defined to be zero unless each triple satisfies the admissibility conditions (1)–(5). Suppose that these
conditions are satisfied for each vertex. Put

$$\Theta = \left\langle \begin{array}{c} i \\ j \\ k \end{array} \right\rangle$$

Then the relativistic invariant $\langle \Gamma(i_1, i_2, \ldots, i_n) \rangle_R$ is defined in terms of the Kauffman bracket invariant of the diagram given by projecting the graph in $S^3$ to $S^2$ by

$$\langle \Gamma \rangle_R = \frac{|\langle \Gamma \rangle|^2}{\prod_{\text{vertices}} \Theta}.$$ 

This definition is extended to arbitrary graphs by the relations

$$\left\langle \begin{array}{c} j \\ k \end{array} \right\rangle_R = \sum_j \left\langle \begin{array}{c} j \\ \cdot \end{array} \right\rangle_R \dim_q j$$

which defines an $n$-valent vertex recursively, for $n > 3$,

$$\left\langle \begin{array}{c} j \\ \cdot \end{array} \right\rangle_R = \delta_{j0} \left\langle \begin{array}{c} \cdot \\ \cdot \end{array} \right\rangle_R$$

for 1-valent vertices, and

$$\left\langle \begin{array}{c} j \\ k \end{array} \right\rangle_R = \frac{1}{\dim_q j} \delta_{jk} \left\langle \begin{array}{c} j \\ \cdot \end{array} \right\rangle_R$$

for 2-valent vertices.

The relation between the state sum invariant of a graph $Z(S^3, \Gamma)$ and the relativistic invariant $\langle \Gamma \rangle_R$ is given by a Fourier transform in the spin labels, using the kernel

$$K_b(a) = (-1)^{2b} \frac{\sin \frac{\pi}{r}(2a + 1)(2b + 1)}{\sin \frac{\pi}{r}(2a + 1)}.$$

The result is

**Theorem.**

$$\sum_{j_1, j_2, \ldots, j_n} \frac{Z(S^3, \Gamma(j_1, j_2, \ldots, j_n))}{Z(S^3)} K_{i_1}(j_1) K_{i_2}(j_2) \cdots K_{i_n}(j_n)$$

$$= \langle \Gamma(i_1, i_2, \ldots, i_n) \rangle_R. \quad (12)$$
A general proof of this result will appear elsewhere. However I will prove a particular special case which is interesting, as the result implies some new identities among quantum $6j$-symbols (Appendix 2). This example is also sufficient to provide a proof of the results for the edge and the triangle given earlier.

The example is the tetrahedral graph embedded in $S^3$. The definition of the state sum invariant is

$$Z \left( S^3, j_1 \ldots j_6 \right) = \dim q_{j_1} \ldots \dim q_{j_6} \frac{\Theta(j_1, j_2, j_3) \Theta(j_1, j_5, j_6) \Theta(j_3, j_4, j_5) \Theta(j_2, j_4, j_6)}{N^4} \left\langle j_{123456} \right\rangle^2,$$

(13)

since $S^3$ can be ‘triangulated’ with two tetrahedra. The following calculations prove the theorem for this example.

Using Roberts’ chain mail [9], the square of the spin network evaluation on the right-hand side can be expressed as a link diagram in which some components are labelled with the formal linear combination

$$\Omega = \sum_j (\dim q j) j$$

of spins.
\[ \frac{1}{\Theta(j_1, j_2, j_3) \Theta(j_1, j_5, j_6) \Theta(j_3, j_4, j_5) \Theta(j_2, j_4, j_6)} \left\langle \begin{array}{c} j_s \\
 & j_i \\
 & j_i \\
 & j_i \\
 & j_i \end{array} \right\rangle^2 \]

using the handleslide identity for \( \Omega \).

The Fourier transform kernel is related to the Hopf link

\[ K_i(j) = \left\langle i \begin{array}{c} j_s \\
 & j_i \\
 & j_i \\
 & j_i \\
 & j_i \end{array} j \right\rangle \frac{1}{\text{dim}_q j}, \]

and the action of the Fourier transform on an edge of a spin network is given by the replacement

\[ \sum_j K_i(j) \text{dim}_q j \begin{array}{c} j \\
 & \end{array} \Omega \]

(14)
Applying the Fourier transform to (14) gives

\[
\sum_{j_1 j_2 \ldots j_6} Z \left( S^3, j_1, j_2, j_3, j_4, j_5, j_6 \right) K_{i_1}(j_1) K_{i_2}(j_2) \ldots K_{i_6}(j_6)
\]

\[
= \frac{1}{N^7} \left\langle \begin{array}{ccccc}
\Omega & & \Omega & & \Omega \\
i_1 & \omega & i_2 & \omega & i_3 \\
i_2 & \omega & \omega & \omega & \omega \\
i_6 & \omega & i_4 & \omega & \omega \\
i_5 & \omega & \omega & \omega & \omega \\
i_4 & \omega & \omega & \omega & \omega
\end{array} \right\rangle = \frac{1}{N^4} \left\langle \begin{array}{ccc}
\Omega & & \\
i_1 & \omega & i_2 \\
i_2 & \omega & \omega \\
i_6 & \omega & i_4 \\
i_5 & \omega & \omega \\
i_3 & \omega & \omega
\end{array} \right\rangle
\]

\[
= \frac{1}{N \Theta(i_1, i_2, i_6) \Theta(i_2, i_3, i_4) \Theta(i_1, i_3, i_5) \Theta(i_4, i_5, i_6)} \left\langle \begin{array}{ccc}
i_2 & i_1 & i_6 \\
i_3 & i_2 & i_4 \\
i_4 & i_3 & i_5 \\
i_5 & i_4 & i_6 \\
i_6 & i_5 & i_1
\end{array} \right\rangle^2
\]

\[
= Z(S^3) \left\langle \begin{array}{ccc}
i_1 & i_2 & i_3 \\
i_2 & i_3 & i_4 \\
i_3 & i_4 & i_5 \\
i_4 & i_5 & i_6 \\
i_5 & i_6 & i_1 \\
i_6 & i_1 & i_2
\end{array} \right\rangle^2. \quad (15)
\]

The graph in the final relativistic spin network is the same as the graph of edges in the original partition function \( Z \). But now the admissibility conditions apply to triples of spins meeting at a vertex of the graph, whereas they applied to triples around a triangular circuit of the original graph in \( Z \).

From this example it is possible to prove the theorem very easily also for sub-graphs of the tetrahedron. Setting, for example, \( i_1 = 0 \) in (15) gives, on the left-hand side, a summation over \( j_1 \) weighted with \( K_0(j_1) = 1 \), which gives the correct state sum formula for the graph with this edge.
removed, whilst on the right-hand side this gives the relativistic invariant for the graph also with this edge removed. The results at the beginning of the paper can be checked very easily. For example, the relativistic spin network evaluation for $\bullet \quad i \quad \bullet$ is $\delta_{i0}$ and inverting the transform gives $P_j = \sum_i \dim_q^2 j K_i(j) \delta_{i0} = \dim_q^2 j$.

There is a curious analogy between the Fourier transform and the duality between position and momentum variables of a particle in quantum theory. In fact the kernel $K_j(a)$ of the Fourier transform is a discrete version of the ‘zonal spherical function’ on $S^3$. The Laplace operator on $S^3$ (with radius $r/2\pi$) has eigenvalues

$$\nabla^2 \phi = -\frac{16\pi^2}{r^2} j(j + 1)\phi$$

for non-negative half-integer $j$; the eigenfunction that is spherically symmetric about $p \in S^3$ (the zonal spherical function) is

$$G_j(R) = (-1)^{2j} \frac{\sin \frac{2\pi}{r}(2j + 1)R}{\sin \frac{2\pi}{r} R},$$

where $R$ is the distance from $p$.

Putting $R = a + \frac{1}{2}$ shows that at half-integer values, $G$ coincides with the Fourier transform kernel

$$K_j(a) = G_j(a + \frac{1}{2})$$

so that the Fourier transform can be interpreted as a transition to a sort of ‘momentum’ or ‘mass’ representation for the quantum probabilities.

**Appendix 1. 3 points on $S^3$**

If three points are distributed on $S^3$ with uniform probability, then this determines a probability distribution on the space of distances between these three points.

The 3-sphere has standard spherical coordinates $(\chi, \theta, \phi)$ which determine points in $S^3 \subset \mathbb{R}^4$ by

$$\frac{r}{2\pi} (\cos \chi, \sin \chi \cos \theta, \sin \chi \sin \theta \cos \phi, \sin \chi \sin \theta \sin \phi).$$

Using the rotational symmetry, three points on $S^3$ can be assumed to be at
(χ, θ, φ) coordinates

\[ p = (0, 0, 0) \]
\[ q = (\chi_2, 0, 0) \]
\[ s = (\chi_3, \theta_3, 0) \]

The probability is thus

\[ dP = \frac{2}{\pi} \sin^2 \chi_2 \, d\chi_2 \frac{1}{\pi} \sin^2 \chi_3 \, \sin \theta_3 \, d\chi_3 \, d\theta_3. \]

For three points on a 3-sphere of radius \( r/2\pi \), the distances between them (Figure 5) are given by

\[ R_2 = \frac{r}{2\pi} \chi_2 \]
\[ R_3 = \frac{r}{2\pi} \chi_3 \]

\[ \cos \left( \frac{2\pi}{r} R_1 \right) = \cos \chi_2 \cos \chi_3 + \sin \chi_2 \sin \chi_3 \cos \theta_3, \]

the last equation being the cosine law for the spherical triangle \( pqs \) with \( \theta_3 \) the angle at \( p \).

Differentiating these relations gives

\[ dP = \frac{16\pi}{r^3} \sin \frac{2\pi R_1}{r} \sin \frac{2\pi R_2}{r} \sin \frac{2\pi R_3}{r} \, dR_1 \, dR_2 \, dR_3 \]

when the inequalities for a spherical triangle are satisfied, and zero otherwise.

**Appendix 2. Identity for 6j-symbols**

The 6\(j\)-symbols are defined to be normalised versions of the tetrahedral spin network evaluation [7]:

\[ \{ j_1 \, j_2 \, j_3 \} \{ j_4 \, j_5 \, j_6 \}_q = \frac{1}{\sqrt{\Theta(j_1, j_2, j_3)\Theta(j_1, j_5, j_6)\Theta(j_3, j_4, j_5)\Theta(j_2, j_4, j_6)}} \]

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Using this definition, the identity proved after the statement of the theorem is

$$\frac{1}{N^3} \sum_{j_1,...,j_6} \left\{ \begin{array}{ccc} j_1 & j_5 & j_6 \\ j_1 & j_2 & j_3 \end{array} \right\}^2 H(j_1, i_1) \ldots H(j_6, i_6) = \left\{ \begin{array}{ccc} i_1 & i_2 & i_3 \\ i_4 & i_5 & i_6 \end{array} \right\}^2_q \,.$$ 

where

$$H(j, i) = K_i(j) \, \dim_q j = \frac{\sin \frac{\pi}{r}(2i + 1)(2j + 1)}{\sin \frac{\pi}{r}(-1)^{2i+2j}}.$$ 

The identity does not appear to have a classical ($q = 1$) analogue.

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