ENTANGLEMENT - A QUEER IN THE WORLD OF QUANTUM: SOME PERSPECTIVES

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Chapter 1

Preface

1.1 General Introduction

Quantum Information and Computation is a recent and exciting field of interdisciplinary research largely connected with quantum physics, advances in mathematics and computer science. This is a rapidly growing field and theoretical development of the subject becomes very important. This subject is originated and related with other fields of research like Quantum Foundations [13, 30, 116], Quantum Optics [133], and Quantum Dots. Surprisingly, a special kind of quantum non-locality termed as Entanglement, plays a key role with its vast applicability on all those areas [3, 14, 26, 28]. After Bell’s work [11, 12] that addresses EPR paradox [58], the power of quantum entanglement reflected through many experiments. Bell’s work encourages a lot of people to thought about other proofs of non-locality to prove or disprove Contextual Hidden Variable theory. It took really a long time to draw the attention of the physical community about the features of Entangled systems. In quantum information and computation, general interests for using the structure of entangled systems grew after the very famous work of Bennett group that proposes a secure cryptographic protocol [15]. Some of the major areas of research about the theory of entanglement are related with the quantification [119, 120, 130, 131], characterization [28, 117, 134] and also the uses of entanglement for performing different computational tasks [16, 17, 20]. Specifically, the second and partly the third area are suitable to experience various counter intuitive phenomena of quantum mechanics. Often the behavior of entangled system surprises the scientific society in many senses. Surprise is not much surprising for results of quantum experiments. Entanglement is gradually becoming a queer in the physical world.

In this thesis our intention is to investigate two peculiar features of entanglement. First is the notion of Incomparability [105] and second one is the Activable Bound Entanglement [9, 127]. Existence of incomparable pair of pure bipartite states is a peculiarity of pure state entanglement manipulation by Local Operations and Classical Communications (in short, LOCC), starting from $3 \times 3$ systems. Incomparability shows that in such a simple structure of the state space,
there lies some unexplained phenomena regarding local evolution of non-local systems. In this simplest possible structure the incomparability is so strong that all the earlier methods for resolving incomparability with certainty, became unsuccessful. We have investigated about such unresolved classes. Incorporating free entanglement, it became possible to transform deterministically one pure bipartite entangled state to another, which are incomparable in nature, by collective LOCC. Obviously not only the presence but the non-recoverable use of entanglement enhances the process.

Next we use the phenomena of existence of incomparable pair of states, to show some impossibilities regarding the allowable local operations defined on quantum systems. Firstly, we consider the exact spin-flipping operation defined on three arbitrary input qubits. Then we consider two large classes of operations, one is general anti-unitary operations and the other one is general angle-preserving operations and both are detected to be impossible by the existence of incomparable pair of pure bipartite states. Thus we propose incomparability as a new detector of impossible operations in quantum mechanics.

Proceeding further to mixed entanglement states we found a general class of bound entangled states. Bound entanglement is an interesting observation in some mixed quantum systems. Though it is unusable in the sense of distilling out the hidden entanglement by LOCC, but still capable in performing some quantum computational tasks. Surprisingly it is seen to be more powerful than free entanglement in some cases. We found there exists exactly four orthogonal activable bound entangled states in any even number of qubit systems [9]. They have some peculiar characteristics including local indistinguishability. As an application, we have tried to use the general class of activable bound entangled states to a data hiding scheme to hide two classical bits of information [38]. However, the scheme has some limitations regarding security against possible classical or quantum attack. An important feature of our protocol is, the number of parties concerned can be increased in pairs as large as possible. A nice Bell correlation is responsible for that feature. This work is another example of using multipartite bound entanglement in information processing tasks.

In short our work represents some interesting phenomena in quantum information and computation theory through the notion of entangled systems. The work is directed to find new characteristics of entanglement and we use it to perform some tasks of information processing. In many perspectives of information theory, local manipulations of a non-local system is not always very predictable. The work is largely related with the investigation on evolution of entangled systems under LOCC. In conclusion, our work highlights on some special features of entangled systems in both pure and mixed level together with some new implementations.
1.2 Outline of the Thesis

In the second chapter, we describe the notations and basic ideas of quantum information and computation theory. Those prior knowledge will provided here for independent study of this work for an uninitiated reader. In the third chapter, we introduce the concept of entanglement, with elementary characteristics and applicability as an information theoretic resource of the system. We also describe the constraints on possible local evolution of entangled system. Examples are shown to emphasize on the importance of those constraints to detect impossible local operations defined on single system, via the evolutions of joint system under those operations. In the fourth chapter, we describe majorization process and its applicability on quantum mechanics. Then we describe Nielsen’s criteria for inter-conversion of a pair of pure bipartite states by LOCC and existence of incomparable pair of states. We provide here two methods for using free entanglement as a resource, to transform pairs of incomparable states [34]. Next, we investigate the root of existence of such incomparable pairs and their power to sense various local operations physical or non-physical. In the fifth chapter, we consider flipping operations defined on a single system. Here, we show the idea how incomparability of two pure bipartite states may used to detect various classes of impossible operations on a single system. We consider the exact flipping operation defined on a minimum number of three arbitrary qubit states and show that the operation is a valid quantum operation only when the three input states lies on a great circle of a qubit [36]. To investigate whether the anti-unitary nature of flipping operation is responsible for this feature, we further consider the general class of anti-unitary operators defined on only three arbitrary qubits, which again shows the same result in the sixth chapter [37]. We also consider here a general class of angle-preserving operations, that also give rise to a similar result, i.e., the operation is physical only on a great circle of the Bloch sphere. Next, we describe the notion of local indistinguishability of a set of orthogonal states of composite system in the seventh chapter. The states are Activable Bound Entangled [9] in nature. We emphasis on the difference between discrimination of a whole set of states with discrimination of some subsets of the whole set. Local indistinguishability of quantum systems exhibits various surprising notions. It reveals other non-local characters of quantum systems beside of entanglement. Our multi-partite activable bound entangled states also enriched of the local indistinguishable kind of non-locality [38]. Thus, we search with this nature of our system and attempts to apply this feature to build some secure data hiding protocols. Lastly, we discuss the aspects of data hiding and limitations regarding our protocol in the eighth chapter.
2.1 Mathematical Preliminaries

Before describing the main work, we first briefly mention some of the basic notions of mathematical terms widely used in quantum information theory [63, 106].

**Linear vector space:** A linear vector space is a set of elements, called vectors, which is closed under addition and multiplication by scalars. Thus, if we denote, \( |\psi\rangle, |\phi\rangle \) as vectors belonging to a certain vector space, then their superposition \( a|\psi\rangle + b|\phi\rangle \), is also a vector, where \( a, b \) are any scalars. We shall consider scalers in general complex numbers.

**Linearly independent vectors:** A set of non-zero vectors \( \{ |\phi_i\rangle; i = 1, 2, \cdots, n \} \) are linearly independent iff there does not exists a set of scalars \( \{ a_i; i = 1, 2, \cdots, n \} \) not all zero such that \( \sum_{i=1}^{n} a_i|\phi_i\rangle = 0 \)(zero vector).

The maximum number of linearly independent vectors of a linear vector space is called the dimension of the space and any maximal set of linearly independent vectors is called a basis of that space.

**Norm:** Norm is a function \( \| \cdot \| \) that associates to each vector of a linear vector space \( V \) a non-negative value which satisfies the following conditions:

(i) For any vector \( |v\rangle \in V \), \( \| |v\rangle \| \geq 0 \), where equality holds if and only if the vector is the zero(null) vector.

(ii) For any vector \( |v\rangle \in V \) and scaler \( a \), \( \| a|v\rangle \| = |a|\| |v\rangle \| \).

(iii)(Triangle inequality:) For any two vectors \( |v\rangle, |w\rangle \in V \), \( \| |v\rangle + |w\rangle \| \leq \| |v\rangle \| + \| |w\rangle \| \).

A linear vector space endowed with a norm is known as a normed linear space.

**Linear operator:** A linear operator from a vector space \( V \) to another vector space \( W \) is defined by a map \( A : V \to W \) which is linear in its inputs, i.e., for any set of vectors \( \{ |v_i\rangle; i = \)}
1, 2, · · · , n} and scalars \( \{a_i; i = 1, 2, \cdots, n\} \) we have,

\[
A(\sum_{i=1}^{n} a_i|v_i\rangle) = \sum_{i=1}^{n} a_i A(|v_i\rangle)
\] (2.1)

All matrices are linear operators acting on some suitable vector spaces.

**Inner product:** For any linear vector space \( V \) over the field of scalars \( F \) (real or complex) inner product is a mapping from \( V \times V \rightarrow F \) which associates a scalar denoted by \( \langle \psi, \phi \rangle \) (or, \( \langle \psi|\phi \rangle \)), with every ordered pair of vectors \( (|\psi\rangle, |\phi\rangle) \) of this space. It must also satisfy the following properties stated below.

(I) Positivity: \( \langle \psi, \psi \rangle \geq 0 \), for any \( |\psi\rangle \in V \) where equality holds if and only if \( |\psi\rangle \) is zero vector.

(II) Skew-symmetry: \( \langle \psi, \phi \rangle = \langle \phi, \psi \rangle^* \), where \( ^* \) denotes complex conjugation.

(III) Linearity: \( \langle \psi, c_1\phi_1 + c_2\phi_2 \rangle = c_1\langle \psi, \phi_1 \rangle + c_2\langle \psi, \phi_2 \rangle \) for every \( c_1, c_2 \in F \).

Clearly, inner product as defined above is anti-linear in it’s first argument. A linear vector space endowed with an inner product is generally known as inner product space. Every inner product space is naturally a normed linear space. Two vectors \( |\psi\rangle, |\phi\rangle \), are said to be orthogonal if and only if \( \langle \psi, \phi \rangle = 0 \). A basis of an inner product space is said to be an orthonormal basis if all vectors in the basis are mutually orthogonal and with unit norm. We now describe some of the linear operators that are required in our work.

**Adjoint of an operator:** Corresponding to any linear operator \( A \) acting on an inner product space \( H \), there exists a unique linear operator \( A^\dagger \) acting on \( H \), known as adjoint operator, so that

\[
\langle u|A|v\rangle = \langle (A^\dagger u)|v\rangle, \quad \forall \; |u\rangle, |v\rangle \in H.
\] (2.2)

**Normal operator:** An operator \( A \) is said to be Normal if \( AA^\dagger = A^\dagger A \).

**Hermitian operator:** An operator \( A \) is said to be hermitian or self-adjoint if \( A^\dagger = A \).

A normal operator is hermitian if and only if all eigenvalues of this operator are real. All eigenvalues of a hermitian operator are real. All hermitian operators are by definition normal.

**Unitary operator:** A normal operator \( U \) is said to be unitary if \( U^\dagger U = I \)

If \( U \) is an unitary operator, then it is possible to express it as \( U = e^{iA} \) where \( A \) is a hermitian operator. Unitary operators have eigenvalues of the form \( \exp(i\alpha) \) for some real \( \alpha \).

**Positive operator:** An operator \( A \) acting on an inner product space \( H \) is said to be a positive operator if \( \langle v|A|v\rangle \) is a real non-negative number for any vector \( |v\rangle \in H \).

A positive operator is necessarily hermitian and eigenvalues are nonnegative.

**Commutator:** Commutator of two operators \( A \) and \( B \) acting on the same vector space is defined as

\[
[A, B] = AB - BA
\] (2.3)

Two operators are said to commute each other, if \( [A, B] = 0 \), i.e., if \( AB = BA \).
2.1. Mathematical Preliminaries

**Anti-commutator:** Anti-Commutator of two operators $A$ and $B$, defined on the same vector space is

$$\{A, B\} = AB + BA$$  \hspace{1cm} (2.4)

Thus two operators are said to anti-commute with one another, if $\{A, B\} = 0$.

**Pauli matrices:** Pauli matrices are three $2 \times 2$ matrices defined as

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$  \hspace{1cm} (2.5)

**Properties of Pauli matrices:**

(i) $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = I$

(ii) $\sigma_x \sigma_y = i \sigma_z, \sigma_x \sigma_z = i \sigma_y, \sigma_y \sigma_z = i \sigma_x$

(iii) $\{ \sigma_i, \sigma_j \} = \delta_{ij} 2I \forall i, j \in \{x, y, z\}$, i.e., $\{ \sigma_i, \sigma_j \} = 0$ if $i \neq j$.

We now state some important theorems which are required for our discussions and have many applications in various disciplines including quantum information theory \[121\].

**Spectral decomposition theorem:** Any normal operator $N$ acting on an inner product space $V$ is diagonal with respect to some orthonormal basis of $V$. Also, every diagonalizable operator is normal.

**Simultaneous diagonalization theorem:** Any two hermitian operator $A$ and $B$ acting on an inner product space $V$ will commute with each other (i.e., $[A, B] = 0$), if and only if they are simultaneously diagonalizable (i.e., there exists an orthonormal basis of $V$ so that both of $A$ and $B$ are diagonal with respect to this basis).

**Polar and Singular value decompositions:** These are two prescription for decomposing a linear operator into simpler parts \[106\]. The structure of a general linear operator is quite complicated to study, by reducing it in terms of positive operators and unitary operators their action can be realized in a much more physical way.

**Polar decomposition theorem:** For every linear operator $A$ there exists an unitary operator, say $U$, and correspondingly two positive operators $P$ and $Q$, such that $A$ can be uniquely represented by,

$$A = UP = QU; \quad P \equiv \sqrt{A^\dagger A}, \quad Q \equiv \sqrt{AA^\dagger}$$  \hspace{1cm} (2.6)

**Singular value decomposition theorem:** For every square matrix $A$ there exists two unitary matrices $U$, $V$ and a non-negative diagonal matrix $D$ such that $A$ can be decomposed as, $A = UDV$.

The diagonal elements of $D$ are known as singular values.

**Linear functional:** Corresponding to a normed linear space $V$ there exists a *dual space of linear functionals* defined on $V$. A linear functional $F$ assigns a scalar value (here, real or complex) denoted by $F(\phi)$ to every vector $|\phi\rangle \in V$, such that

$$F(a \psi + b \phi) = a F(\psi) + b F(\phi)$$  \hspace{1cm} (2.7)
for every pair of vectors $|\psi\rangle, |\phi\rangle \in V$ and any pair of scalars $a, b \in F$. The set of all bounded (equivalently, continuous) linear functionals acting on $V$, forms a linear space, say, $V'$ which is also a normed linear space (actually, a Banach Space) where the sum of two functionals is defined as $(F_1 + F_2)(\phi) = F_1(\phi) + F_2(\phi)$.

**Riesz theorem:** There is an one to one correspondence between the linear functionals $F \in V'$ and the vectors $|\psi\rangle \in V$ of an inner product space $V$, such that

$$F(\phi) = \langle \psi, \phi \rangle, \quad \forall |\phi\rangle \in V$$

(2.8)

**Use of Dirac’s Bra and Ket notation:** Dirac introduced the bra and ket notation (by splitting the word ‘bracket’) in quantum mechanics. The vectors in the linear space are called ket vectors, denoted by $|\psi\rangle$ (which we have considered here from beginning) and the linear functionals in the dual space are called bra vectors and are denoted by $\langle F |$ whose numerical value is determined as $F(\psi) = \langle F | \psi \rangle$.

Now according to Riesz theorem there is an one-to-one correspondence between bras and kets. Therefore it is traditional to use same algebraic characters for the functional and the particular vector to which it corresponds differing only in bra and ket sign which indicates from which space it belongs to. The linear vector spaces where the rule of Riesz theorem always valid, are Hilbert spaces. They are the building blocks of quantum systems. We now formally state the notion of a Hilbert space.

**Hilbert space:** A Hilbert space is a linear vector space $H$ over the complex field $C$, such that an inner product $\langle \cdot | \cdot \rangle$ is defined on the linear space and the space is complete in its norm defined in the manner

$$\|\psi\| = \sqrt{\langle \psi | \psi \rangle}, \quad \forall |\psi\rangle \in H$$

(2.9)

### 2.2 Logical formalism of quantum mechanics

Every physical theory involves some basic physical concepts, a mathematical formalism, and set of correspondence rules which maps the physical concepts onto the mathematical objects that will represent them. Those correspondence rules express a physical problem in mathematical terms, so that the problem may be solved by purely mathematical techniques that need not have any physical interpretation. In this context we first recall some basic concepts [106 121] regarding mathematical description of any quantum mechanical system.

**Basic postulates:**

**Postulate-1:** Every physical system is associated with a separable complex Hilbert space $H$.

**Postulate-2:** Every state of a physical system corresponds to an unique state operator, known as density operator, which is hermitian, non-negative and of unit trace.
**Postulate-3:** To each dynamical variable (which is a physical quantity) there corresponds a linear operator (which is a mathematical object), and the possible values of the dynamical variable are the eigenvalues of the operator.

Usually, in logical formalism all physical quantities are known as observables. Every observable $A$ of the physical system is associated with a self-adjoint (i.e., hermitian) operator $\Gamma : H \to H$. Outcomes of any measurements of the observable $A$ is one of the eigenvalues of the corresponding operator $\Gamma$.

A hermitian or self-adjoint operator in a Hilbert space has a spectral representation (by spectral decomposition theorem) as $\Gamma = \sum_n a_n P_n$. Each $a_n$ is an eigenvalue of $\Gamma$ and $P_n \equiv |n\rangle \langle n|$ is the corresponding orthogonal projection onto the space of eigenvectors with eigenvalue $a_n$. The normalized eigenvectors of a hermitian operator forms a complete orthonormal basis in $H$.

**Postulate-4:** If, the outcome of the measurement of an observable $A$ on a physical system be $a_n$, then just after the measurement, the system jumps into one of the normalized states belonging to the support of the corresponding projector $P_n$. This rule is often designated as the **Collapse Postulate**.

**Postulate-5:** If the initial state of a quantum mechanical system is $\rho$ and a measurement of the observable $A$ is performed on the system, then the outcome $a_n$ is obtained with the probability

$$\text{Prob}(a_n) = \text{tr}(P_n \rho)$$

If outcome of the measurement is $a_n$, then the final quantum state of the system will be $P_n \rho P_n^\dagger / \text{tr}(P_n \rho)$. This is known as the **Born’s Rule**.

**Postulate-6:** The time evolution of any state of a quantum system is governed by a unitary operator, defined by the Hamiltonian $H$ acting on the system. The dynamics of the system, i.e., the evolution of the system described by the state vector $|\psi(t)\rangle$ with time, is governed by the **Schrödinger equation**,

$$\frac{d}{dt} |\psi(t)\rangle = \frac{-i}{\hbar} H |\psi(t)\rangle$$

### 2.2.1 Density operator

Density operator $\rho$ is a linear operator acting on the Hilbert space that corresponds to the state of the physical system. It obeys the following rules:

1. $\rho$ is non-negative,
2. $\rho$ is self-adjoint,
3. $\text{Tr}(\rho) = 1$. 

The set of all density operators acting on a Hilbert space forms a convex set. Furthermore, they satisfy the relation, $\rho^2 \leq \rho$. According to this relation we have the following two classifications of states in a physical system.

1. **Pure state**: If the density matrix $\rho$ corresponding to the state satisfies the relation $\rho^2 = \rho$, then the states are known as pure states. In this case, there is a nice correspondence between each density operator $\rho$ represents a pure state and the vectors of the Hilbert space $H$, viz., $\rho^2 = \rho$ if and only if $\rho = |\psi\rangle \langle \psi |$ where $|\psi\rangle \in H$. Thus pure states are described by rays of the Hilbert space, whether normalized or not. Pure states can not be written as a convex combination of the other states. They are the extreme points of the convex set of all density operators.

   **Phase invariance**: Two pure states $|\varphi\rangle$ and $e^{i\alpha} |\varphi\rangle$, differing only in some relative phase $e^{i\alpha}$, where $\alpha$ is a real number, are said to be physically equivalent up to a global phase factor. Interesting to note that statistical predictions of any allowable quantum measurement for these two states are all same. As the observed properties are equal so states are same from physical point of view. In this sense global phases are not physically important.

2. **Mixed state**: All other states which are not pure, called mixed states. In other words, for every mixed state $\rho$, we have $\rho^2 < \rho$. Any mixed state can be represented as a convex combination of some pure states as, $\rho = \sum_i |\psi_i\rangle \langle \psi_i |$, where each $|\psi_i\rangle \in H$.

   **A criterion to detect mixedness of a density operator**: If $\rho$ be any density operator then $\text{Tr}(\rho^2) \leq 1$. This density operator will represent a pure state if and only if $\text{Tr}(\rho^2) = 1$. Now we consider the simplest quantum system described by qubits.

### 2.2.2 Bloch sphere representation of Qubits

**Qubits**: In accordance with cbits as unit of classical information, the unit of quantum information is denoted by quantum bits, in short ‘qubit’. This is the simplest possible state of a quantum system corresponding to the smallest possible non-trivial Hilbert space of two dimension. An orthonormal basis of this space can be denoted by $\{|0\rangle, |1\rangle\}$. Since, superposition of pure states are also states of the system. Therefore, any pure state of a qubit system can be represented as

$$|\varphi\rangle = \alpha |0\rangle + \beta |1\rangle ; \quad |\alpha|^2 + |\beta|^2 = 1 \quad (2.10)$$

**Bloch sphere representation**: Bloch sphere [121] is a geometrical way of representing the state of a single qubit system. This is a unit ball with the principle axes along the direction of spin polarization of the three Pauli operators. The four orthogonal operator $I$, $\sigma_x$, $\sigma_y$, $\sigma_z$ forms an alternative basis of the density operators acting on the two-dimensional Hilbert space for qubits. Any density matrix of the qubit state can be expressed as

$$\rho = \frac{I + \vec{n} \cdot \vec{\sigma}}{2} ; \quad |\vec{n}| \leq 1 \quad (2.11)$$

where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ and $\vec{n}$ denotes the vector of the direction of the spin polarization of the state $\rho$, and known as **Bloch vector** for $\rho$. The case, $|\vec{n}| = 1$ represents pure qubits. If we
2.3. Composite system

A composite system consists of more than one distinct physical subsystems. The state space of a composite system is the tensor product of Hilbert spaces corresponding to its subsystems. Consider a composite system consists of two subsystems, i.e., a bipartite system. Suppose \( H_A \) be the Hilbert of the subsystem of the first party that has an orthonormal basis \( \{|\mu_1\rangle, |\mu_2\rangle, \cdots, |\mu_n\rangle\} \) and the Hilbert space corresponding to the second party is \( H_B \) which has another orthonormal basis \( \{|\nu_1\rangle, |\nu_2\rangle, \cdots, |\nu_m\rangle\} \). Then the combined system of these two subsystems is \( H_A \otimes H_B \) with dimension \( n \times m \) and it has a basis given by \( \{|\mu_1\rangle \otimes |\nu_1\rangle, |\mu_2\rangle \otimes |\nu_1\rangle, \cdots, |\mu_n\rangle \otimes |\nu_1\rangle, |\mu_1\rangle \otimes |\nu_2\rangle, |\mu_2\rangle \otimes |\nu_2\rangle, \cdots, |\mu_n\rangle \otimes |\nu_2\rangle, \cdots, |\mu_1\rangle \otimes |\nu_m\rangle, |\mu_2\rangle \otimes |\nu_m\rangle, \cdots, |\mu_n\rangle \otimes |\nu_m\rangle\} \).

In general if the composite systems consist of \( k (\geq 2) \) number of subsystems, we call them usually multipartite systems. Further if the states of the \( i^{th} \) local subsystem will be \( |\psi_i\rangle \) for all \( i = 1, 2, \cdots, k \), then the joint pure product state of the composite system will be of the form \( |\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_n\rangle \). Here after, for simplicity, in most of the cases, we shall drop the tensor product notations in writing composite states.

2.3.1 Schmidt decomposition of pure bipartite states

Pure bipartite states has a standard form, known as Schmidt decomposition. This form is used largely in all kind of information theoretic tasks performed on pure states including various entanglement manipulation procedure.

Let \( |\psi\rangle_{AB} \) be any pure state shared between two separated parties, Alice and Bob. If the Hilbert spaces corresponding to the local system of Alice and Bob be \( H_A \) and \( H_B \), then there exists orthonormal bases \( \{|i_A\rangle\} \) and \( \{|i_B\rangle\} \) of \( H_A \) and \( H_B \) respectively, such that \( |\psi\rangle_{AB} \) can be expressed as

\[
|\psi\rangle_{AB} = \sum_{i=1}^{k} a_i |i_A\rangle \otimes |i_B\rangle
\] (2.13)
Where the number of non-zero coefficients \( a_i \) in Schmidt decomposition is a unique number, satisfying \( k \leq \min\{ \dim H_A, \dim H_B \} \). It is called Schmidt number of the state and is constant for every pure state. The Schmidt coefficients can be taken non-negative real numbers satisfying \( 0 \leq a_i \leq 1 \) with \( \sum_{i=1}^{k} a_i = 1 \). A classification of pure bipartite states immediately follows from the number of Schmidt coefficients. If the number of Schmidt terms in a decomposition is one then it is a pure product state and if it is greater than one, the state is called an entangled one. It is well-known fact that all fundamental characteristics of any bipartite pure state is completely determined by its Schmidt coefficients. Study of all possible kind of local evolutions of the states are possible by using only the Schmidt vector, a vector formed by Schmidt coefficients taken in decreasing order. Thus it is traditional to specify a pure bipartite state by its Schmidt vector (sometimes it is even unnecessary to specify the Schmidt bases \( \{| i_A \rangle \}, \{| i'_B \rangle \} \) of the local subsystems) as,

\[
| \psi \rangle_{AB} \equiv (a_1, a_2, \ldots, a_k),
\]

where Schmidt coefficients are taken in decreasing order.

Now a more complicated case arises when the state of all the local subsystems are not pure. Rather we may think that it may be a matter of search, whether the state of the subsystems are pure or mixed. Given the most general form of a composite system in terms of density matrix representation, the corresponding state of any local subsystem is obtained by computing the reduced density matrices. To describe the process of finding the reduced density operator, we first describe the calculation of partial traces.

### 2.3.2 Partial trace and Reduced density matrices

Consider a bipartite system in the space \( H_A \otimes H_B \). Suppose the density operator representing the state of the joint system is \( \rho_{AB} \) and \( \{| \psi_i \rangle, i = 1, \cdots, m \}, \{| \phi_j \rangle, j = 1, \cdots, n \} \), be any orthonormal bases of the subsystems \( A \) and \( B \) respectively. Then the partial trace on the joint system is nothing but tracing over any of the subsystem with respect to an orthonormal basis and after tracing over, the remaining subsystem is the reduced density matrices of that subsystem. Usually they are denoted by,

\[
\rho_A = \text{Tr}_B(\rho_{AB}) = \sum_j \langle \phi_j | \rho_{AB} | \phi_j \rangle,
\]

\[
\rho_B = \text{Tr}_A(\rho_{AB}) = \sum_i \langle \psi_i | \rho_{AB} | \psi_i \rangle.
\]

In particular, if we consider the state of the joint system is a pure bipartite state \( | \Psi \rangle_{AB} \). Then there exists orthonormal bases \( \{| i_A \rangle \} \) and \( \{| i'_B \rangle \} \) of \( H_A \) and \( H_B \) respectively, such that \( | \Psi \rangle_{AB} \) can be expressed in its Schmidt form as,

\[
| \Psi \rangle_{AB} = \kappa \sum_{i=1}^{\kappa} a_i | i_A \rangle | i'_B \rangle ; \quad \kappa \leq \min\{ \dim(H_A), \dim(H_B) \}
\]
The reduced density matrices of the subsystems $A$ and $B$ respectively are,

$$
\rho_A = \text{Tr}_B(\ket{\Psi}_{AB}\bra{\Psi}) = \sum_{i=1}^{\infty} |a_i|^2 \ket{i_A}\bra{i_A},
$$
$$
\rho_B = \text{Tr}_A(\ket{\Psi}_{AB}\bra{\Psi}) = \sum_{i'=1}^{\infty} |a_i'|^2 \ket{i'_B}\bra{i'_B},
$$

(2.17)

where partial traces are taken over Schmidt bases. Clearly, the reduced density matrices are in general mixed for a pure bipartite composite states, except when the joint state is a product one.

Naturally, there arises the cases when the state of all the local subsystems are not pure. Rather it may be a matter of search to determine whether the state of the subsystem is pure or mixed. Given the most general form of a composite system (i.e., a multipartite system) in terms of density matrix representation, the corresponding state of any local subsystem is obtained by computing the reduced density matrices. This is done by tracing out the parts corresponding to all other subsystems from the density matrix of the joint composite system. It is interesting to note that the partial trace operation is the unique operation that provides correct description of the observable quantities for subsystems of a composite system. However, the reduced density matrices reveals only a very limited characteristics of the joint system, as the correlation between the disjoint subsystems are totally lost in computing the reduced density matrices.

### 2.4 Quantum operations

#### 2.4.1 POVM measurements:

Let us consider a physical observable described by the complete set of general measurements $\{M_i\}$. This observable acts on the system which is originally in the state $\ket{\psi}$. The probability of the outcome $i$ is, from Born rule, $p_i = \bra{\psi} M_i^\dagger M_i \ket{\psi}$. Thus the set of operators $E_i = M_i^\dagger M_i$ is a complete set of positive operators, sufficient to determine the probability of different measurement outcomes.

This may also be characterized as a partition of the unity operator into some non-negative operators. The positivity is required to ensure the existence of the probability representation given by the Born rule. To elaborate the concept of performing a generalized measurement in a joint space $H$, we express it as an extension of the space $H_A$, on which its action is observed. Now orthogonal measurements on the larger space $H = H_A \otimes H_B$, a direct product space, is not orthogonal on the system of $A$.

Thus the measurement prepare the system in one of the non-orthogonal states. We may express the orthogonal measurements on the joint space $H$ as a set of one-dimensional projectors $E_{\mu} = \ket{\mu}\bra{\mu}$, where $\ket{\mu}$ be a normalized state vector in $H$. We denote the orthogonal projection operator $M$, that takes the states from the space $H$ to its subspace $H_A$. Then we construct the POVM as $F_{\mu} = M E_{\mu} M$, each of the $F_{\mu}$ is a hermitian, non-negative operator [116, 121]. Next we consider symmetry of the transformations and the corresponding rules for certain class of operations.
2.4.2 Transformation symmetry:

The laws of nature are believed to be invariant under certain space-time symmetry operations including displacements, rotations and transformations between frame of references in uniform relative motion. Then some specific properties of nature expressed as relations corresponding the observable and states of the system, must be preserved under these transformations. For example, the eigenvalues of an observable must be same with that of the equivalent observable in the new reference frame. Similarly, the inner product of two state vectors must be always remain invariant after such a symmetry transformation, which is actually reflected by Wigner’s theorem.

Wigner Theorem:

Any mapping on the inner product space \( V \) onto itself that preserves the absolute value of the inner product \( \langle \phi | \psi \rangle \) of the two vectors \( | \psi \rangle, | \phi \rangle \in V \), may be implemented by an operator \( U \) as,

\[
\begin{align*}
| \psi \rangle & \quad \rightarrow \quad | \psi' \rangle = U | \psi \rangle \\
| \phi \rangle & \quad \rightarrow \quad | \phi' \rangle = U | \phi \rangle
\end{align*}
\]  

(2.18)

Then, the operator \( U \) is either unitary or anti-unitary.

Case-1 : When \( U \) is an unitary operator, then by definition, \( UU^\dagger = U^\dagger U = I \), the identity operator. Thus inner product between transformed and original vectors remain same. i.e.,

\[ \langle \phi' | \psi' \rangle = (\langle \phi | U^\dagger \rangle \langle U | \psi \rangle) = \langle \phi | (U^\dagger U) | \psi \rangle = \langle \phi | \psi \rangle. \]

Thus an unitary transformation preserves the complex value of an inner product, not merely its absolute value.

Case-2 : If \( U \) is anti-unitary then \( U (c | \psi \rangle) = c^* U | \psi \rangle \), for any complex number \( c \). Thus in this case the change in inner product of two vectors will be:

\[ \langle \phi' | \psi' \rangle = (\langle \phi | U^\dagger \rangle (U | \psi \rangle) = (\langle \phi | \psi \rangle)^* = \langle \psi | \phi \rangle, \]

that preserves the absolute value of inner product.

Transformation of operators:

Now, any transformation of state vectors of the form \( 2.18 \) will be accompanied by a transformation \( A \rightarrow A' \) of the operator corresponding to some physical observable. The eigenvalues of the operator must remain invariant. Thus if originally \( A | \phi_n \rangle = a_n | \phi_n \rangle \), then finally the eigenvalue equation changes to, \( A' | \phi_n' \rangle = a_n | \phi_n' \rangle \). Assuming the evolution of the state vector to be unitary, i.e., \( | \phi_n' \rangle = U | \phi_n \rangle \), the corresponding change in the observable will be \( A \rightarrow A' = UAU^{-1} \).

2.4.3 Physical operation

Any physical operation will correspond to a Superoperator \( \mathcal{S} \) acting on the state \( \rho \) of the physical system associated with the Hilbert space \( H \), and can be realized by an unitary evolution on a larger space.
Superoperator:

A superoperator can be regarded as a linear map that takes density operators to density operators. To fulfill the requirement of preserving density operator, such mappings must have the operator sum representation (or Kraus representation), given by

$$\rho \rightarrow \mathcal{S}(\rho) \equiv \rho' = \sum_{\mu} M_{\mu} \rho M_{\mu}^{\dagger}$$

(2.19)

where $\sum_{\mu} M_{\mu}^{\dagger} M_{\mu} = 1$.

The properties of this linear map are as follows:

1. $\rho'$ is hermitian, if $\rho$ is so,

$$\rho'^{\dagger} = \sum_{\mu} M_{\mu} \rho^{\dagger} M_{\mu}^{\dagger} = \rho'$$

(2.20)

2. $\rho'$ has unit trace, as

$$\text{Tr}\rho' = \sum_{\mu} \text{Tr}(\rho M_{\mu}^{\dagger} M_{\mu}) = \text{Tr}\rho = 1$$

(2.21)

3. $\rho'$ is a positive operator, as for any state $|\psi\rangle$,

$$\langle \psi | \rho' | \psi \rangle = \sum_{\mu} \text{Tr}(\langle \psi | M_{\mu}^{\dagger} \rho M_{\mu} | \psi \rangle) \geq 0$$

(2.22)

It is to be noted that any linear mapping $\mathcal{S} : \rho \rightarrow \rho'$, satisfying the above three conditions (1), (2), (3) does not have a operator sum representation. For obtaining that representation we need an additional requirement that $\mathcal{S}$ is completely positive.

**Kraus Representation Theorem:** Any linear mapping $\mathcal{S} : \rho \rightarrow \rho'$ on the states space, that takes a density $\rho$ matrix to another density matrix $\rho'$ has a operator-sum representation in the following form,

$$\mathcal{S}(\rho) = \sum_{\mu} M_{\mu} \rho M_{\mu}^{\dagger} ; \sum_{\mu} M_{\mu} M_{\mu}^{\dagger} = 1$$

(2.23)

for any density matrix $\rho$.

In general a quantum operation $\mathcal{S}$ need not to be trace preserving. It may be successful with some probability that does not increase the trace. For that we require only projective measurements possibly with postselection. Thus, in an alternate way, a quantum operation $\mathcal{S}$, in general, is a physical operation that transforms a state $\rho$ on $H$ into another state $\rho'$ on another Hilbert space $K$, consists of four elementary operations [121] on quantum states: (i) unitary transformations, (ii) adding an uncorrelated ancilla, (iii) tracing out part of the system and (iv) projective measurements possibly with postselection. It is clear that the operations (i)-(iii) can be represented by trace preserving completely positive maps. Now in the next section, we describe some fundamental concepts of quantum information theory alongwith some no-go theorems.
2.5 Distinguishability and related issues

In quantum information theory every information about the physical system, will be encoded in some quantum states. Suppose, there are \(n\) number of results of a random experiment, expressed as \(1, 2, 3, \ldots, n\). The \(i^{th}\) result is encoded in the state |\(\phi_i\rangle\). If it is then required to reveal the result, having in hand its encoded version, one has to construct a process of distinguishing the states or simply, to know which state is given. Thus discrimination of a set of states is directly related with revealing and processing of the information \([96]\) about the associated physical system.

2.5.1 Existence of non-orthogonal states

If the dimension of the Hilbert space associated with the system be \(d\), then the system has exactly \(d\) number of linearly independent states, orthogonal to each other. Superposition principle states that any superposition of these states will also represent a physical state of the system. In any physical system \((d \geq 2)\) there exists at least two linearly independent non-orthogonal states. This is a very special and entirely quantum mechanical phenomenon not seen in classical physics. In classical physics any two different states of the system are orthogonal which is equivalent to the statement that any two classical objects are always distinguishable. In quantum physics the peculiarity of non-orthogonal states arise and in principle there is no allowable quantum operation which can perfectly (with probability one) distinguish any two non-orthogonal states \([39]\). The immediate consequences of the existence of non-orthogonal states are as follows:

The three statements that two states are non-orthogonal or, two states can not be distinguished with probability one or, two states can not be perfectly cloned by any quantum machine, are equivalent.

2.5.2 Distinguishability of quantum states

Suppose a state is chosen at random from a known set of states \(|\psi_i\rangle; \ i = 1, 2, \ldots, n\). Then the task of determining the state given, is equivalent with the idea of discriminating the whole set of \(n\) states \([114]\). It is obvious that the task is realizable with certainty, only when the set of \(n\) states are orthonormal. If the states are orthonormal, then we can extend this set to a basis of the associated Hilbert space \(H\), by adjoining \(m - n\) orthonormal states \(|\psi_i\rangle; \ i = n + 1, 2, \ldots, m\) where \(m(\geq n)\) is the dimension of \(H\). So, we can construct a projective measurement

\[
\mathcal{F}_n = \sum_{i=1}^{m} \lambda_i |\psi_i\rangle\langle\psi_i|
\]

(2.24)

And after operating this measurement on the given unknown state, if we find the outcome of the measurement be \(k\), then we can determine with certainty that the given state is \(|\psi_k\rangle\). Since the aspect of distinguishing a set of states is closely related with ability of copying quantum
information encoded in quantum states, therefore, in the next section we review in brief the aspect of quantum cloning.

2.5.3 Cloning of quantum states

The quantum cloning operation is performed for copying the information of a given quantum state on some suitably chosen blank state \([27, 150]\). Operating the cloning machine on the system of input state and the blank state, it will produce two copies of the input state. Operationally, we may describe the situation as,

\[
\Gamma(|i\rangle|b\rangle|M\rangle) = |i\rangle|i\rangle|M_i\rangle
\]  

(2.25)

where \(\Gamma\) is the cloning operation, \(\{|i\rangle; \forall i\}\) be the set of all input states, \(|b\rangle\) is a suitably chosen blank state, \(|M\rangle\) is the initial machine state and \(|M_i\rangle\) is the final states of the machine after the cloning operation, corresponding to the different input states \(|i\rangle\).

As every state represents some information encoded in itself, and larger number of copies of same state will provide the ability of recovering more information from the state, therefore cloning operation is directly related with the aspect of state estimation \([5]\). Having one copy of the input state and sufficient number of copies of the blank state, if one repeatedly apply a properly designed cloning machine, then it produces a large number of copies of the input state. Thus, it is practically equivalent with estimating the input state \([124]\). Now naturally one may ask that whether there exists any quantum cloning machine which can copy exactly arbitrary quantum information or not? The answer is no for deterministic \([147]\) as well as probabilistic \([110]\) exact cloning. However, universal inexact cloning machine exists \([27]\). Here we now describe only the famous no-cloning theorem alongwith another famous no-go theorem, the no-deletion theorem which may be viewed as somewhat reverse kind of operation than that of cloning. In quantum deletion, the task is whether it is possible to delete arbitrary quantum information or not \([109]\)?

**No-Cloning and No-Deleting Theorems**

**No-Cloning theorem:** Universal Exact Cloning is not possible \([110, 150]\).

**No-Deletion theorem:** Exact Deletion of arbitrary quantum state is not possible \([109, 113]\).

Both the no-cloning and no-deleting theorems are closely related with the aspect of indistinguishability of states. Though it is found that being equivalent with the indistinguishability criteria, we should note that both the exact cloning and exact deletion of a set of orthogonal states are quantum mechanically allowable operations and are always achieved by some unitary machines. Further more stronger versions of these two no-go theorems are proposed as,

**No-Cloning theorem:** Any two non-orthogonal states can not be cloned exactly by any quantum operation.

**No-Deletion theorem:** Any two non-orthogonal states can not be deleted exactly by any quantum operation.
The word 'exactly’ is very much crucial here. Those restrictions are only imposed for deterministic performance of the cloning operation. Thus the condition for calling a machine to be faithful is that, in anyway, it is possible to distinguish each of the output states by any statistical tests, for any preparation of the input states. As a probabilistically exact machine, the set of input states, on which a single quantum exact cloning machine can be defined, will be enlarged from set of orthogonal states to the set of linearly independent states.

The above two theorems have some interesting connections with different directions of the mathematical construction of quantum mechanics. It is shown that linear dynamics provides these restrictions over the allowable operations performed on the system. Unitary evolutions of the systems also restrict such impossible operations. More fundamental is that if someone assumes the possibility of such impossible operations, then it is also possible to send signals faster than the speed of light [69, 113].

2.6 Some ideas from Classical Information Theory

Subject of Information theory evolves with the very practical requirement of protecting and conveying information [47]. In a more lucid way one may define study of sending, exchanging or expressing some data or messages as information theory. Classical information is thus the information that may be represented by some classical system. For example, result of any random classical experiment (such as, tossing a coin, or results of a football match) are always represented by some classical system and they are essentially known as classical information.

2.6.1 Shannon Entropy

Claude Shannon establishes the Shannon entropy as a quantification rule for data compression, known as ‘Noiseless coding theorem’. This expresses, up to what extent a given message can be compressed. The entropy function is described as,

\[ H(p) = - \sum_{i=1}^{k} (p_i \log p_i) \leq \log k, \]  

(2.26)

with respect to a probability distribution \( p \equiv \{p_i ; \ i = 1,2,\cdots,k\} \), where logarithms are taken with base 2.

For \( k = 2 \), the entropy of the probability distribution \( p_x \equiv \{x, 1-x\} \), is called binary entropy and is given by,

\[ h(x) = -x \log x - (1-x) \log (1-x) \]

(2.27)
2.6.2 Mutual information:

It quantifies the amount of common information between two messages. From Bayes probability rule this is given by,

\[ I(X;Y) \equiv H(X) - H(X \mid Y) \]
\[ = H(X) + H(Y) - H(X,Y) \]
\[ = H(Y) - H(Y \mid X), \]

where \( H(X,Y) \), is the joint entropy function of a pair of random variables \( X \) and \( Y \). Mutual information is symmetric under interchanges of \( X \) and \( Y \). This expresses a classical correlation between classical messages. When \( I(X;Y) = 0 \), then the there is no classical correlation between \( X \) and \( Y \). Thus in this situation learning about one can never provide some extra knowledge about the other.

2.6.3 Relative Entropy of Information

It expresses the closeness between two probability distributions \( p \equiv \{p_i\}_{i=1}^{k} \) and \( q \equiv \{q_i\}_{i=1}^{k} \) in the form,

\[ D(p \parallel q) = -\sum_{i=1}^{k} \{ p_i \log(q_i) \} \geq 0 \]

where the equality \( D(p \parallel q) = 0 \) holds only when \( p = q \). This is a measure of the classical information, done by comparing the difference between two probability distributions. The relative entropy \( D(p \parallel q) \) can be viewed as a measure of inefficiency of assuming that the distribution is \( q \), when the actual distribution is \( p \).

**Conclusion:** The research of quantum information and computation has the background of quantum foundations, information theory and quantum logic. After the ideas of Bennett *et al.* [15, 16, 17], the use of quantum states in computational schemes are enhanced. Thus the research in the field of quantum information theory started with the goal to exploit various quantum states in different computational tasks. As expected some states having purely quantum nature, found to give better result in developing schemes that are powerful than classical computational schemes. In composite systems, the invention of entangled states further enhanced the rapid progress in the field of quantum information and computation. In recent days some people also designate quantum information theory as the theory of entanglement. In the next chapter we will discuss about entanglement in detail.
3.1 EPR paradox and Bell’s resolution

Nature often shows contradiction with our common intuitions. Quantum mechanics evolved through the study of such apparently peculiar features of nature explored beyond the predictability of classical world. It happens to be a stochastic theory rather than a deterministic one. Afterwards it is found to be something more unpredictable than just a probabilistic theory. This unpredictability evolved through some special kind of nonlocal-correlation, that is absent in any classical system. More specifically, there are quantum correlations between subsystems that can not be described or interpreted by the laws of classical mechanics. The correlation, known as entanglement, causes many nonlocal phenomena. It usually described as non-locality of quantum systems, but as we will discuss later, there are other resources of non-locality in a quantum system.

In 1935 Einstein, Podolsky and Rosen proposed the famous EPR paradox [58] which raises a question on the completeness of quantum mechanics as a physical theory. EPR argument is very much concerned about the term defined as elements of reality. In that paper elements of reality is described in the following manner:

“If, without in any way disturbing a system we can predict with certainty . . . the values of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity”.

They considered a pure state describing two particles, say $S_1$ and $S_2$ which are well-separated by distance but correlated in some manner. Years after this correlation was termed as Entanglement by Schrödinger. The state has the property that, if one measures the position of the particle
S_1, one could determine with arbitrary accuracy, the position of the other particle S_2 and also for the case of momenta. Thus a distant observer can find out the position of the particle S_2 by a measurement on S_1. Locality argument says that this measurement will not disturb in any way the subsystem of S_1. Thus there exists an element of physical reality corresponding to the position of S_2. Similarly one could conclude that there exists also an element of physical reality corresponding to the momentum of S_2. Thus from EPR state one could ascribe both the position and momentum with arbitrarily large probability. This contradicts Copenhagen’s interpretation of quantum mechanics that position and momenta of a particle can not be determined simultaneously. Thus EPR paradox interprets that quantum mechanics (rather Copenhagen’s quantum mechanics) is an incomplete theory.

This counter-intuitive ideas and many attempts to pose some answers to it, from the viewpoint of Hidden variable models that runs for years. Then in 1964 comes the Bell’s theorem. It successfully shows the non-locality of all realistic quantum theory. Bell proved that even considering statistical quantum mechanics to be incomplete, it will violate local realism. Thus any realistic hidden variable theory must be non-local in nature.

In short, the results of Bell experiments concludes with only two possibilities that either, (1) the world is nonlocal:- which implies that events happen in nature, violating the principle of reality or, (2) objective reality does not exist:- there is no matter of fact about distant events.

However, in essence, with the EPR paradox modern quantum mechanics observes the birth of a new concept for composite system, known as entanglement which is responsible for all queers and counter intuitive ideas in quantum foundations, quantum information and computation theory.

### 3.2 Separability and concept of entanglement

Consider a multipartite system shared between distinct parties A, B, C, ··· etc. Each party is situated at a distant place, and suppose, H_A denote the Hilbert space of the local system of party A, and similarly, H_B, H_C, etc. The state of the joint system corresponds with the joint Hilbert space H_A \otimes H_B \otimes H_C \otimes ···. The first basic classification of quantum states is proposed as,

1. **Separable state**: Suppose the general form of a multipartite state is represented by convex combination of product states as follows:

   \[ \rho = \sum_i \omega_i \rho_A^i \otimes \rho_B^i \otimes \rho_C^i \otimes ··· ; 0 \leq \omega_i \leq 1, \sum_i \omega_i = 1, \]

then, we call it a multipartite separable state.

Pure separable states have the simplest form of product states, i.e., tensor product of local pure states associated with the different parties, as stated below,

\[ | \psi \rangle = | \chi \rangle_A \otimes | \phi \rangle_B \otimes | \varphi \rangle_C \otimes ··· \]
The physical interpretation of composite systems of two parties, in product state $|\psi\rangle = |\chi\rangle_A \otimes |\phi\rangle_B$ is that when the first system is in the state $|\chi\rangle_A$, the state of the second system is certainly determined by $|\phi\rangle_B$. So the systems in some separable state always specifies deterministically the state of one subsystem corresponding to the state of another subsystem. While for an entangled composite system, as we will see below, it is really complicated to assign the properties of individual subsystems.

2. Entangled state: A state of a multipartite system which cannot be expressed in anyway (i.e., considering any basis of the concerned local Hilbert spaces) in the general form of a separable state given in Eqs. (3.1), is defined to be an entangled state. There is always a special type of correlation between the different subsystems in an entangled state that cannot be interpreted classically.

One of the basic reasons for the above classification scheme is that the separable states shared between distant parties can be prepared locally. In other words, it is possible for some distant persons, holding separate local systems in their own places, to prepare any separable state of a composite system, by a sequence of local operations done on their local subsystems and communications between them through some classical channels shared among the concerned parties (i.e., by LOCC). This can never be possible for sharing any entangled state if initially the parties are separated [23]. This phenomena may be described as impossibility of creating non-local feature in a global system by some local manipulation. In the next section we will show some power of this non-local feature.

### 3.3 Entanglement as a resource of the system

Recent development of quantum information theory inspired us to manipulate entangled systems for implementing various computational and communication tasks [14]. Emphasis is given on the use of entanglement for performing such tasks with better precision comparing to any kind of classical schemes proposed for the same purposes. Entanglement is now-a-days considered as a valuable resource of the quantum systems. Quantum teleportation, dense coding and quantum key distributions are some successful applications of this powerful correlation.

#### 3.3.1 Quantum Teleportation:

The basic task in quantum teleportation is to send an unknown information encoded in a quantum state, say qubit, from one place to another through some quantum channel, without actually sending the qubit itself, using only local operations along with classical communications on the subsystems concerned. The protocol is given by Bennett group [17], [18] for the qubit system as well as for the qudit systems. The success relies on the quantum channel considered, taken usually the maximally entangled states. The interesting fact about quantum teleportation is that
Here both the sender as well as the receiver do not know the state transferred between them. Now, an unknown qubit information is equivalent with an infinite amount of classical bits. However, the process of teleporting a qubit information requires only two cbits of classical information together with the use of one e-bit of entanglement (this notion of e-bit will be described later). The situation is different when the sender knows the information encoded in the qubit and require to prepare a qubit in receiver’s side that contains the same information. This task is known as remote state preparation [23]. Such kind of tasks are impossible if the joint state (i.e., the quantum channel) is taken to be separable. The non-local character of entangled states made the tasks possible.

3.3.2 Dense Coding:

Dense coding is one of the most simple and elegant example to show the power of entangled state to transmit classical information encoded in quantum states. It is shown that, by sharing one Bell state, two distant parties can communicate with certainty, two classical bits of information from one place to another, by sending a single quantum bit and of course by destroying the entanglement shared between them [16]. From these two protocols of Teleportation and Dense Coding in contrast of one another, an equivalence in quantum information theory is stated as,

\[
1 \text{ Qubit} \leftrightarrow 1 \text{ e-bit} \quad 2 \text{ cbit}
\]

3.3.3 Quantum Key Distribution:

To prepare and share a secret communication channel between two distant labs so that any unwanted third party can not eavesdrop the secret, entanglement is used very efficiently. In BB84 protocol, proposed by Bennett and Brassard in 1984 [15], two distant parties, say Alice and Bob generate a secret key by sharing one Classical public channel (Noiseless and Unjammable) together with Quantum communication channel, which is assumed to be insecure in terms of an eavesdropper, who can manipulate the quantum signals in some senses. The protocol is proposed in such a way that the transmission of secret information is completely protected against Eave’s success of knowing the secret data without being detected. Alice and Bob, by their recurring random tests can always detect the eavesdropping and therefore just by destroying the corrupted channel and starting refreshed with a new set of data, they would able to communicate secret information. The protocol, given by Bennett and Brassard, is based on another invention of modern quantum mechanics, i.e., the existence of non-orthogonal states. After this, Ekert gave a protocol [60] based on the use of entangled states (i.e., Bell states). All the above schemes show how the shared entanglement may efficiently perform various computational tasks.
3.4 LOCC and Entanglement

Entanglement of quantum system is a peculiar kind of quantum mechanical correlation, that can be sensed through some operational criteria. It is generally seen in composite quantum systems shared between more than one party who are at distant positions. For this reason, it is sometimes termed as quantum non-locality. By sharing entanglement between several parties we usually mean some quantum communications, as there is an equivalence between perfect entanglement distribution and perfect quantum communication. For example, if we can transport any qubit without any decoherence (i.e., disturbance or error formation), then any entanglement shared by that qubit as a part of a joint system, will also be distributed perfectly. Thus distribution of entanglement is closely related with the achievement of a teleportation protocol.

Now, in implementing any kind of computational tasks or information processing tasks using entangled systems, Local Operations are often very important to perform. By allowable local operations on a joint system we define ensemble of operations that are expressed as tensor product of quantum mechanically allowable operations, i.e., POVM done on local subsystems of single parties. Local operations are largely used, and encouraged in performing any tasks as they can be performed in well-controlled environments without the decoherence induced by communication over long distances. Though almost all kind of tasks performed on joint system, requires interdependence of operations performed in separate places. Thus the results of operations performed on local laboratories are communicated through some classical channels (by any standard telecom technology which can communicate a classical data perfectly). Studying the status and ability of this general kind of Local Operations and Classical Communications, in short LOCC, is thus an important motivation of quantum information theory. The relation between LOCC and entanglement can be expressed as laws of quantum information processing tasks.

Fundamental laws of quantum information processing

Two parties, say, Alice and Bob can not create any amount of entanglement between them by some local operations on their own subsystems and communicating through some classical channel, if initially they are disentangled. By local operations and classical communications, two distant parties, say, Alice and Bob, can not increase the total amount of entanglement shared between them.

In quantum information theory, a question arises due to some closely related operations with the LOCC. Those operations are known as separable superoperators and the question is whether all separable superoperators can be represented by some LOCC or not. All LOCC is a kind of separable superoperator but the reverse is not yet clear. In Kraus form, a physical operation (may or may not be trace preserving) takes a density matrix shared between different parties A, B, C, D, ... to another density matrix is known as a separable superoperator, if it has the operator-sum representation.
in the following form,
\[ \rho'_{ABCD...} = \Im(\rho_{ABCD...}) = \sum_{\mu} M_{\mu} \rho_{ABCD...} M^\dagger_{\mu}; \quad (3.4) \]

where \( \sum_{\mu} M_{\mu} M^\dagger_{\mu} \leq 1 \) and each \( M_{\mu} \) is of the form \( M_{\mu} = A_{\mu} \otimes B_{\mu} \otimes C_{\mu} \otimes D_{\mu} \otimes \ldots \) with \( A_{\mu}, B_{\mu}, C_{\mu}, D_{\mu}, \ldots \) are positive linear operators. Now, before going to analyze further the connections between LOCC and entanglement, we first discuss some detection procedures and quantification methods of entanglement.

### 3.5 Detection of entanglement

Detection of a state to be entangled or not is not only a classification task but it is extremely useful for many practical purposes.

Here we will mention only some of the good detectors of entanglement used largely in information theory, like Bell violation, Partial transposition, Reduction criteria, Maximal entangled fraction, etc.

**Bell violation:** If the state \( \rho \) of a bipartite quantum system will violate Bell-CHSH inequality,
\[ \text{Tr}(\rho B) \leq 2 \quad (3.5) \]

where the Bell-CHSH observable \( B \) is defined as
\[ B = \tilde{a}\tilde{\sigma} \otimes (\tilde{b} - \tilde{b}')\tilde{\sigma} + \tilde{a}'\tilde{\sigma} \otimes (\tilde{b} + \tilde{b}')\tilde{\sigma} \quad (3.6) \]

then it is necessarily entangled [3]. \( \tilde{a}, \tilde{a}', \tilde{b}, \tilde{b}' \) are arbitrary unit vectors of \( \mathbb{R}^3 \) and \( \tilde{\sigma} = \{ \sigma_x, \sigma_y, \sigma_z \} \).

For \( 2 \times 2 \) states \( \rho \), the above mentioned Bell inequality (3.5) takes the simplified form [78],
\[ M(\rho) \leq 1 \quad (3.7) \]

where \( M(\rho) \) is the sum of the two larger eigenvalues of \( \Gamma^\dagger \Gamma \) and the \( 3 \times 3 \) real matrix \( \Gamma \) is constructed for the state \( \rho \) by the following prescription,
\[ \Gamma = \{ \Gamma_{ij} = \text{Tr}(\rho \sigma_i \otimes \sigma_j) \}_{3\times3} \quad (3.8) \]

Here we use the notations, \( \sigma_1 = \sigma_x, \sigma_2 = \sigma_y, \sigma_3 = \sigma_z \), for the Pauli operators.

The Bell-inequalities are not sufficient to detect all entangled states. All pure entangled states violate some Bell-inequalities. However, there are mixed entangled states that satisfy all the standard Bell inequalities. In the simplest dimension, i.e., in \( 2 \times 2 \) the so-called Werner states \( (U \otimes U \text{ invariant states) provide us an example that satisfies Bell-CHSH inequalities but} \]
entangled for a large region of a parameter \cite{146}. The entanglement is found initially by the flip-operator. The Werner states in $2 \times 2$, are given by,

$$\rho_W = p|\psi^\rangle\langle\Psi^-| + \frac{1-p}{4}I \tag{3.9}$$

where $\frac{1}{4} \leq p \leq 1$ and $|\Psi^-\rangle$. $I$ are respectively the singlet state and the identity operator in $2 \times 2$. It is easy to check that, for $p > \frac{1}{\sqrt{2}}$, $\rho_W$ violets Bell-inequalities. However, for $\frac{1}{3} < p \leq \frac{1}{\sqrt{2}}$, they remain entangled. We now present the most useful detector of entanglement via partial transposition.

**Partial transposition criteria:** This separability criteria connected with the partial transposition operation on composite systems. Let us consider the general form of a bipartite state as

$$\rho_{AB} = \sum_{i,j=1}^{m} \sum_{k,l=1}^{n} \alpha_{ijkl} |i\rangle_A \langle j| \otimes |k\rangle_B \langle l| \tag{3.10}$$

where \{|i\rangle_A, i = 1, 2, \ldots, m\} and \{|k\rangle_B, k = 1, 2, \ldots, n\} are orthonormal bases corresponding to the subsystems A and B respectively. Then the partial transposition of this density matrix with respect to the subsystem B, denoted by $\rho_{AB}^{T_B}$, is defined as

$$\rho_{AB}^{T_B} = \sum_{i,j=1}^{m} \sum_{k,l=1}^{n} \alpha_{ijkl} |i\rangle_A \langle j| \otimes |l\rangle_B \langle k| \tag{3.11}$$

In linear operator theory, the operation transposition is a positive operator but not a completely positive one. The consequence of not a completely positive map \cite{3}, partial transposition is unable to preserve positivity of density matrices/operators, but preserves hermiticity. So, after partial transposition a state cannot be remain in general a state, it may have some negative eigenvalues. We then classify the system of bipartite states into two distinct classes:

(i) **states with positive partial transposition** or, PPT states, i.e., after partial transposition they remain positive operator; and

(ii) **states with negative partial transposition** or, NPT states, i.e., after partial transposition they have at least one negative eigenvalues.

**Separability criteria:** Now partial transposition of a density matrix is proved to be a good detector of entanglement \cite{79, 80, 115}. In lower dimensional bipartite states (in $2 \times 2$ and $2 \times 3$), it is found to be necessary as well as sufficient condition for separability \cite{79, 80, 115}. Peres \cite{115} shows that for every separable state $\rho_{AB}$, both of its partial transpose are positive (i.e., $\rho_{AB}^{T_A} > 0$, $\rho_{AB}^{T_B} > 0$). Later, for $2 \times 2$ and $2 \times 3$ system of states, Horodecki group \cite{79} found that it is also sufficient one. The result thus known in literature as Peres-Horodecki theorem. Though for higher dimensional systems it is not sufficient in general \cite{3}. The following are precisely the results concerning separability and partial transposition.

**Theorem-1:** Let $\rho_{AB}$ be a state of the composite system described by the Hilbert space $H_A \otimes H_B$. Then $\rho_{AB}$ is separable if and only if for any positive map $\Lambda : H_B \rightarrow H_B$ the operator $(I \otimes \Lambda)\rho_{AB}$ is positive.
Theorem-2: A state $\rho_{AB}$ of a $2 \times 2$ or $2 \times 3$ system is separable if and only if its partial transposition is a positive operator [79, 80, 115].

Immediate consequences of the above results are as follows:
(a) all NPT states are entangled;
(b) all separable states are PPT states.

However, there are entangled states which are PPT states. They have another interesting property of bound entanglement, for which we require some notions of quantification procedures for entanglement. One can now check that the Werner states in $2 \times 2$ are entangled if $p > \frac{1}{3}$ and they are separable for $p \leq \frac{1}{3}$. Thus partial transposition criteria is found to be a good detector of entanglement in some cases.

Here we should note that the finding of good entanglement witness is closely related with the problem of characterizing positive operators acting on Hilbert spaces. Next, we just mention other two good detectors of entanglement for a large class of bipartite states.

**Reduction criteria:** Separable states in bipartite systems shared between two parties A and B, must satisfy the following two inequalities [82]

$$I \otimes \rho_B - \rho_{AB} \geq 0, \quad \rho_A \otimes I - \rho_{AB} \geq 0$$

(3.12)

If any one of the above two conditions is violated then the state $\rho_{AB}$ will be entangled. This criteria is very much useful to detect entanglement. However, there are mixed entangled states (Werner class of states in $d \times d$) that also satisfy reduction criteria [146]. So, the inequality is not tight enough.

**Maximal entangled fraction:** The maximally entangled fraction of a state $\rho_{AB}$ of $d \times d$ system is given by

$$F_{max} = \max_{\Psi} \langle \Psi | \rho_{AB} | \Psi \rangle$$

(3.13)

where $|\Psi\rangle$ is any maximally entangled state of the $d \times d$ system shared between A and B. Now, it is found that if $F_{max} > \frac{1}{d}$, then $\rho_{AB}$ is certainly entangled [3].

Both the above detection criteria are not only detects entanglement, but they are useful in characterizing another aspect of entanglement related with quantification, i.e., whether distillable or not. In the next section, we shall describe in brief some of the quantification procedures of entanglement.

### 3.6 Measure of entanglement

**Entanglement is a quantifiable property of the system**

Entanglement is a property of the quantum system, that can be used to perform various task which are otherwise impossible, or can enhance the performance of some tasks in quantum information theory. In practical cases, to apply this resource perfectly the quantification is very
much necessary. The subjects like, amplification, purification [131], concentration [102], distillation and manipulation are associated with the use of entanglement. Before going to describe some measures of entanglement, we first describe the properties of being a good measure of entanglement [85, 119].

**Basic properties of being a measure of entanglement:**

1. A bipartite entanglement measure denoted by $E(\rho_{AB})$, is a mapping from density matrices into non-negative real numbers.

2. For any separable state $\sigma_{AB}$, the measure should give the value zero, i.e.,

$$E(\sigma_{AB}) = 0$$ (3.14)

3. For any bipartite state $\sigma_{AB}$, its entanglement should remain unchanged by the action of any local unitary transformation of the form $U_A \otimes U_B$, i.e.,

$$E(\sigma_{AB}) = E(U_A \otimes U_B \sigma_{AB} U_A^\dagger \otimes U_B^\dagger)$$ (3.15)

4. Local operations, classical communications and sub-selections cannot increase the expected value of the entanglement, i.e., if we start with an ensemble in a state $\sigma_{AB}$ and end up with probability $p_i$ in sub-ensembles in states $\sigma_i$ then we have,

$$E(\sigma_{AB}) \geq \sum_i p_i E(\sigma_i)$$ (3.16)

Instead of those above basic requirements, there are some additional properties which are, though not necessary criteria but sometimes imposed on entanglement measures for better physical implications. Such as,

**Additivity.** Given two pairs of entangled particles in the total state $\sigma = \sigma_1 \otimes \sigma_2$, where $\sigma_1, \sigma_2$ are bipartite states, by additivity, we mean $E(\sigma) = E(\sigma_1) + E(\sigma_2)$. Sometimes instead of additivity we find an entangled measure $E$ satisfies partial additivity, i.e., for any bipartite entangled state $\rho$, $E(\rho^\otimes n) = nE(\rho)$. Also, in asymptotic region we want to observe whether the following limit exists or not;

$$E^\infty(\rho) = \lim_{n \to \infty} \frac{E(\rho^\otimes n)}{n}$$ (3.17)

for any bipartite entangled state $\rho$.

**Convexity.** For a set of bipartite density matrices $\{\rho_i\}$ the entanglement of any convex combination of the states will satisfy,

$$E(\sum_i p_i \rho_i) \leq \sum_i p_i E(\rho_i)$$ (3.18)
Continuity. Suppose $H_n = H_A^n \otimes H_B^n$ be a sequence of bipartite Hilbert spaces and $\rho_n$ and $\sigma_n$ be sequences of states from $H_n$ corresponding to each positive integer $n$. Then we have,

$$\|\rho_n - \sigma_n\| \to 0 : \lim_{n \to \infty} \frac{E(\rho_n) - E(\sigma_n)}{1 + \log_2 \text{dim}(H_n)} \to 0$$

(3.19)

Sometimes instead of arbitrary Hilbert spaces $H_n = H_A^n \otimes H_B^n$, we consider the continuity behavior of the measure $E$ only on $H_n = (H_A)^{\otimes n} \otimes (H_B)^{\otimes n}$.

Now we will discuss about some of the important measures of entanglement [117].

### 3.6.1 Von-Neumann Entropy and Entropy of Entanglement

The Von-Neumann entropy for any state whose density matrix is $\rho$, is given by

$$S(\rho) = -tr(\rho \log_2 \rho)$$

(3.20)

Thus if $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the $n$ eigenvalues of the state $\rho$ (including multiplicity), then we can express the Von-Neumann entropy as, $S(\rho) = -\sum \lambda_i \log_2 \lambda_i$.

Concerning physical importance of this measure, some properties of the Von-Neumann entropy are discussed below:

1. **Purity and Upper Bound:** Von-Neumann entropy is bounded by the relation $0 \leq S(\rho) \leq \log_2 d$, where $d$ is the dimension of the Hilbert space $H$. The lowest bound is achievable (i.e., $S(\rho) = 0$) if and only if $\rho$ is a pure state so that we may express the state as, $\rho = |\psi\rangle\langle\psi|$. Also, the upper bound $S(\rho) = \log_2 d$ is achievable if and only if $\rho = \frac{1}{d}I$, i.e., all the non-zero eigenvalues of the state are equal. Thus, the entropy of a state is maximum when the state chosen is completely random.

2. **Basis Invariance:** Entropy remains unchanged by any change in the basis of the system, attained by some unitary transformation over the system; i.e.,

$$S(U \rho U^\dagger) = S(\rho)$$

(3.21)

This is because entropy depends only on the eigenvalues of the density matrix.

3. **Concavity:** For any real numbers $\alpha_1, \alpha_2, \cdots, \alpha_n \geq 0$ such that $\sum_{i=1}^{n} \alpha_i = 1$ (i.e., $0 \leq \alpha_i \leq 1$, $\forall i$), we have

$$S\left(\sum_{i=1}^{n} \alpha_i \rho_i\right) = -\sum_{i=1}^{n} \alpha_i \log_2 \alpha_i + \sum_{i=1}^{n} \alpha_i S(\rho_i) \geq \sum_{i=1}^{n} \alpha_i S(\rho_i)$$

(3.22)

This result can be described as a relation between entropy of a system and its preparation. Here we observe that the Von-Neumann entropy increases when the information about the state preparation decreases.
(4) **Entropy of measurement:** If we measure the observable \( A = \sum_{i=1}^{n} a_i |\psi_i \rangle \langle \psi_i| \) in the input state \( \rho \), then the outcome \( a_i \) occurs with probability

\[
p_i = \langle \psi_i | \rho | \psi_i \rangle
\]  

(3.23)

Now the Shannon entropy defined by, \( H(Y) = -\sum_{i=1}^{n} p_i \log_2 p_i \), corresponding to the ensemble of measurement outcomes \( Y = \{p_i; i = 1, 2, \ldots, n\} \), will satisfy \( H(Y) \geq S(\rho) \). The equality holds only when, the operator \( A \) and \( \rho \) will commute. This can be interpreted physically as the randomness of the measurement outcomes will be minimized if we choose to measure an observable that commutes with the density matrix of the state.

(5) **Subadditivity:** For any bipartite system in the state \( \rho_{AB} \),

\[
S(\rho_{AB}) \leq S(\rho_A) + S(\rho_B)
\]  

(3.24)

(6) **Strong subadditivity:** For any state \( \rho_{ABC} \) of a tripartite system,

\[
S(\rho_{ABC}) + S(\rho_B) \leq S(\rho_{AB}) + S(\rho_{BC})
\]  

(3.25)

(7) **Triangle inequality:** For a bipartite system in a state \( \rho_{AB} \),

\[
S(\rho_{AB}) \geq |S(\rho_A) - S(\rho_B)|
\]  

(3.26)

Now, for a pure bipartite state \( |\Psi\rangle_{AB} \), the **entanglement or entropy of entanglement** \( E(|\Psi\rangle_{AB}) \) of the state is defined by the Von-Neumann entropy of any of its reduced density matrices. i.e.,

\[
E(|\Psi\rangle_{AB}) = S(\rho_A) = S(\rho_B)
\]  

(3.27)

where \( \rho_i = \text{Tr}_i(|\Psi\rangle_{AB} \langle \Psi|) \), \( i = A, B \).

If \( |\Psi\rangle_{AB} \) has the Schmidt decomposition, \( |\Psi\rangle_{AB} = \sum_{i=1}^{k} \sqrt{\lambda_i} |i_A\rangle |i_B\rangle \) where \( k \leq \min\{\dim H_A, \dim H_B\} \), \( 0 \leq \lambda_i \leq 1 \), \( \sum_{i=1}^{k} \lambda_i = 1 \), then we can express the entanglement of \( |\Psi\rangle_{AB} \) as,

\[
E(|\Psi\rangle_{AB}) = S(\rho_A) = S(\rho_B) = -\sum_{i=1}^{k} \lambda_i \log_2 \lambda_i
\]  

(3.28)

For a pure product state \( |\psi\rangle_{AB} = |\phi\rangle_A |\chi\rangle_B \), it is easy to calculate that \( E(|\psi\rangle_{AB}) = 0 \). Also, for a state of the form, \( |\Phi\rangle_{AB} = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i_A\rangle |i_B\rangle \) where \( \{ |i_A\rangle, |i_B\rangle, i = 0, 1, 2, \ldots, d - 1 \} \) are two orthonormal bases of subsystems A and B respectively, \( E(|\Phi\rangle_{AB}) = \log_2 d \), \( d = \dim H_A = \dim H_B \). This is the maximal possible value of entanglement of an entangled state in a two qudit system. For that reason the states of the kind \( |\Phi\rangle_{AB} \) are called maximally entangled states. All the states that are locally unitarily connected with \( |\Phi\rangle_{AB} \) have also the same entanglement. Clearly,
for two qubit system maximum possible value of the entanglement is 1. All the Bell states in two qubit system,  
\[ |\Phi^{\pm}\rangle = \frac{|00\rangle \pm |11\rangle}{\sqrt{2}}, \quad |\Psi^{\pm}\rangle = \frac{|01\rangle \pm |10\rangle}{\sqrt{2}} \]  
(3.29)

where first qubit is for system A and second for B, are pure maximally entangled states with entanglement 1. Thus, this value of the entanglement is known as the unit of entanglement and we usually denote it by 1-ebit. Interestingly, entropy of entanglement is the unique measure of pure state entanglement [120].

Quantification of mixed entangled states is a very hard task in quantum information theory. There is no single good measure of entanglement for mixed states. Next, we will proceed to some measures of entanglement that are proposed from some physically relevant processes. First, we describe the measure connected with the preparation process and then turn up to measures related with extraction or purification processes.

3.6.2 Entanglement of formation

In historical sense, this is the first measure proposed for quantifying entanglement of a system, by the observation that in asymptotic limit, the entanglement cost of any pure state in terms of Bell states is the entropy of entanglement [148, 149]. It gives an upper bound of efficiency of purification process.

**Entanglement of Formation** \( E_F \) of a bipartite mixed state \( \rho_{AB} \) is defined to be the minimum value of convex sum of pure state entanglements over all possible ensembles of pure states which realizes the mixed state \( \rho_{AB} \). If \( \rho_{AB} \) can be prepared from a mixture of the ensemble \( \xi \equiv \{ p_i, |\Psi_i\rangle \} \), as \( \rho_{AB} = \sum_i p_i |\Psi_i\rangle \langle \Psi_i | \), then \( E_F(\rho_{AB}) = \min _{\xi} \{ E(\xi) \} \); where \( E(\xi) = \sum_i p_i E(|\Psi_i\rangle) \).

As this minimization is taken over all possible decompositions of the density matrix \( \rho_{AB} \) into pure states, thus it is extremely difficult to compute numerically.

**Entanglement Cost:** The entanglement cost of a mixed state \( \rho_{AB} \) is described as [141],

\[
E_C(\rho_{AB}) = \lim_{n \to \infty} \frac{E_F(\rho_{AB} \otimes n)}{n}
\]  
(3.30)

For pure states it is equal to entropy of entanglement.

3.6.3 Distillability

The problem of distillation is associated with the aspect of extracting entanglement from the system. Thus distillability is tested only for entangled systems [2, 18, 19, 20, 33, 54, 125]. The subject is based on the idea of using entanglement as a resource of the system. Idea for using
entanglement as a resource of the system is built out of the quantum computational protocols, proposed on the maximally entangled states like, teleportation, dense coding, etc. In particular, for a number of spatially separated parties, entanglement is a physical resource with which we can overcome the practical restrictions of allowing only local operations and classical communications (i.e., LOCC).

Consider the situation where a source produces some quantum system in a particular state $\rho_{AB}$. Then the state $\rho_{AB}$ is distillable if using a large number of copies of (say $N$ copies of the state, it is possible to prepare a smaller number (say, $M$) of copies of a maximally entangled state by performing LOCC on the system. Formally, the distillable entanglement is defined as:

**Distillable entanglement:** The distillable entanglement $E_D(\rho_{AB})$ of a bipartite state $\rho_{AB}$ is given by,

$$E_D(\rho_{AB}) = \sup_P \zeta_P \text{ with } \zeta_P \equiv \lim_{n \to \infty} \frac{m}{n}$$

where $m$ copies of Bell state $|\Phi^+\rangle \equiv \frac{|00\rangle + |11\rangle}{\sqrt{2}}$ can be extracted from $n$ copies of $\rho_{AB}$ and $P$ be any LOCC protocol.

Now this definition is also in asymptotic sense and it is really hard to calculate. For product or separable states, distillable entanglement is zero. But, there are also mixed entangled states [81] with zero distillable entanglement. This result of distillability give birth of a fundamental classification scheme for entangled system. (i) States that have non-zero distillable entanglement (i.e., having free entanglement) and (ii), the states with zero distillable entanglement (known as bound entangled states). This classification is not complete yet. There are examples of PPT bound entangled states, but it is not known whether there exists NPT bound entangled states or not [49, 82]. However, to tackle this problem we need to define distillability of any bipartite mixed entangled state in an alternate way. A bipartite mixed entangled state $\rho_{AB}$ is distillable iff there exists a positive integer $k$ and a Schmidt rank 2 pure entangled state $|\psi\rangle$, such that

$$\langle \psi | (\rho^T_B)^{\otimes k} |\psi\rangle < 0 \quad (3.32)$$

Thus to check whether a bipartite mixed state is distillable, one has to start from $k = 1$ and search for the existence any rank 2 pure state $|\psi\rangle$, so that the above condition is satisfied. If one fails for $k = 1$, then it requires to check for $k = 2, 3, \cdots$. This definition is given on the basis that if a quantum state is distillable then projecting it onto a $2 \times 2$ subspace, we may able to check whether it is entangled or not. By Peres-Horodecki criteria, the $2 \times 2$ states are entangled if it has negative partial transpose.

Regarding the relation between distillable entanglement $E_D$, entanglement cost $E_C$ and any other measure of entanglement $E$ we have [120],

$$E_D \leq E \leq E_C$$

It is found that there are zero distillable entanglement states with non-zero entanglement cost [141], an irreversibility in quantum information processing.
The definition of distillable entanglement proposed only for bipartite states and but there is no immediate extension for multipartite entangled states. There are some other well known measures like, Relative entropy of Entanglement [130], Logarithmic Negativity [118], Squashed Entanglement [44], Concurrence [149], etc. For some specific tasks they are very much usable.

**Concurrence:** The entanglement of formation for two-qubit system is exactly calculable by a special method. Wootters provide [149] a new measure of entanglement known as concurrence, defined by,

\[
C_\rho = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}
\]

(3.34)

where \(\{\lambda_i; \ i = 1,2,3,4\}\) are the eigenvalues of the matrix \(R \equiv \sqrt{\rho \tilde{\rho}} \sqrt{\rho}\) arranged in decreasing order (i.e., \(\lambda_1\) is the maximum eigenvalue). The matrix \(\tilde{\rho}\) is formed as,

\[
\tilde{\rho} = (\sigma_y \otimes \sigma_y) \rho^* (\sigma_y \otimes \sigma_y)
\]

(3.35)

where \(\rho^*\) is formed by taking complex conjugation of each elements of the density matrix \(\rho\).

Then the entanglement of formation \(E_F(\rho)\) for the \(2 \times 2\) state \(\rho\) is given by,

\[
E_F(\rho) = h(\Theta); \quad \Theta = \frac{1 + \sqrt{1 - C_\rho^2}}{2}
\]

(3.36)

where the binary entropy function \(h(\cdot)\) is defined by

\[
h(x) = -x \log_2 x - (1 - x) \log_2 (1 - x)
\]

(3.37)

Thus, for two-qubit system \(E_F(\rho)\) is a monotonically increasing function of \(C_\rho\). The function \(C_\rho\) is an entanglement monotone, ranges from 0 to 1. Also, for pure two-qubit states \(E_F(\rho)\) is exactly equal to the entropy of reduced density matrices. The general formula of concurrence for any pure bipartite state \(|\Psi\rangle_{AB} \in H_A \otimes H_B\) is given by [43, 99, 100, 149],

\[
C(|\Psi\rangle_{AB}) = \sqrt{2(1 - \text{Tr}(\rho_S^2))}
\]

(3.38)

where \(\rho_S = \text{Tr}_S|\Psi\rangle_{AB}\langle\Psi|; \ S = A \text{ or } B\). For mixed bipartite states \(\rho_{AB}\), concurrence is given by the convex roof,

\[
C(\rho_{AB}) = \inf \sum p_i C(|\Psi_i\rangle_{AB})
\]

(3.39)

for all possible decomposition of \(\rho_{AB} = \sum_i p_i |\Psi_i\rangle_{AB}\langle\Psi_i|\) where \(0 \leq p_i \leq 1, \sum_i p_i = 1\). For separable states it gives exactly the value zero. However, for any general mixed states the calculation requires some optimization procedures.
3.6. Measure of entanglement

**Relative entropy of entanglement:** This is a generalization of the concept of classical relative entropy. The relative entropy between two states $\sigma$ and $\rho$ is given by,

$$D(\sigma \| \rho) = \text{Tr} \left[ \sigma (\log_2 \sigma - \log_2 \rho) \right], \text{ if } \text{Supp } \rho \subset \text{Supp } \sigma$$

$$= \infty, \text{ otherwise}$$

(3.40)

The relative entropy $D(\sigma \| \rho) \geq 0 \forall \rho, \sigma$ where equality holds if and only if $\rho = \sigma$. The relative entropy of entanglement of a bipartite state $\rho_{AB}$ is then defined by,

$$E(\rho_{AB}) = \inf_{\sigma_{AB} \in S} D(\rho_{AB} \| \sigma_{AB})$$

(3.41)

where $S$ is the set of all separable states [130]. The motivation of this measure of entanglement is to find the distance between the entangled state $\rho_{AB}$ with the set of all separable states. In other words, the amount of entanglement is quantified by the distinguishability between $\rho_{AB}$ and the closest possible separable state. It is an upper bound of distillable entanglement.

**Logarithmic negativity:** The quantity known as Negativity is the first one that attempts to find a computable measure of entanglement, and is defined by,

$$N(\rho_{AB}) = \frac{\|\rho_{AB}^T\| - 1}{2}$$

(3.42)

where $\|X\| = \text{tr} \sqrt{X^\dagger X}$ is the trace norm.

This definition suffers from the deficiency that it is not an additive measure. Thus an improvised form of this measure is proposed as Logarithmic Negativity [118, 142] and defined by,

$$E_N(\rho_{AB}) = \log_2 \|\rho_{AB}^T\|$$

(3.43)

It is an additive measure. Interestingly, logarithmic negativity is a full entanglement monotone that is not convex.

We have discussed before some of the important characteristics of a good measure of entanglement. Certainly, additivity is one of them. In almost all aspects of using entanglement as a resource, the additivity is a natural requirement. However, there are only very few measures of entanglement with additivity property. Logarithmic negativity is one and the next measure of entanglement also satisfies the additivity requirement.

**Squashed Entanglement:** Squashed entanglement [44] for a bipartite state $\rho_{AB}$ is defined by,

$$E_S(\rho_{AB}) = \inf \left[ \frac{1}{2} I(\rho_{ABE}) : \text{Tr}_E \{\rho_{ABE}\} = \rho_{AB} \right]$$

(3.44)

where the $\rho_{ABE}$ is any possible extension of $\rho_{AB}$ with the third subsystem $E$ (may be considered as the environment) and $I(\rho_{ABE})$ is the quantum conditional mutual information of $\rho_{ABE}$, defined as

$$I(\rho_{ABE}) = S(\rho_{AE}) + S(\rho_{BE}) - S(\rho_{ABE}) - S(\rho_E)$$

(3.45)
Squashed entanglement has nice properties like, it is zero for all separable states and coincides with the entropy of entanglement for the pure states. It is an entanglement monotone with superadditivity in general and additive on tensor products. Also, it is convex and continuous almost everywhere and bounded by $E_D$ and $E_F$.

### 3.7 Thermodynamical aspect of entanglement

Physically it is much more interesting to deal with open systems, i.e., quantum systems in contact with environment. In that case, we are largely concerned with the evolution of the open systems without monitoring the environment [86]. Now, like energy, entanglement is considered as a resource of the system. Thus for proper utilization of it, the quantification of this resource is required for information processing. Related with the quantification problem, there is a fundamental restriction on the evolution of a physical system consists of different non-interacting distant subsystems. The laws of nature itself evoke an irreversibility on the joint system [138]. It is found that in any process performed on any entangled system the amount of entanglement invested for the process is always greater than or equal to the amount of entanglement recovered from the process. For example, the amount of entanglement required to prepare a state (by any LOCC protocol) is always greater than or equal to the amount of entanglement that can be extracted from it [120]. Simply, we may describe it as, ”resource cannot be created out of nothing”. The purification process of entangled systems also exhibit such irreversibility [139]. Again, as observed, entanglement of a joint system has a relation with the concept of entropy [86]. Even in quantification process, we find, entanglement is directly connected with the idea of classical Shannon’s entropy. And the concept of entropy had evolved through the knowledge of thermodynamics [120]. Interestingly, there is always an underlying irreversibility that occurs in quantum information processing tasks.

### 3.7.1 Characteristics of Entanglement

Thus we have discussed in this chapter some of the fundamental properties of entangled system evolved directly through the definition of entanglement [28, 136, 138]. In short, we enlist them as follows:

- Separable states contain no entanglement.

- All non-separable states allow some tasks to achieve better results by some global operations rather than by LOCC alone, hence all non-separable states are defined to be entangled.

- The entanglement of the states does not increase under any sort of LOCC.

- In particular, entanglement of a system does not change under Local Unitary Operations.
• There are maximally entangled states in every dimension. In a $d \times d$ pure bipartite system, any state, locally unitarily connected with a state of the form

$$|\Phi^+_d\rangle_{AB} = \frac{1}{\sqrt{d}} \sum_i |i_A \rangle \otimes |i_B \rangle$$ (3.46)

is maximally entangled.

### 3.8 Discussions

Entangled states are used for performing various information processing tasks. Thus it is very important to investigate the properties of those states together with the process of preparing and sharing them among some distant parties. Different measures of entanglement are proposed in view of either how much amount of some known entangled state (like, Bell states having one e-bit of entanglement) is sufficient to obtain the required state or how many copies of a known entangled state are recoverable from the state, or sometimes how good the state is, for performing some information processing tasks relative to other known states. This is a very recent and growing field. A vast area of research is wide open. One of the interesting topic is manipulating entanglement by local means. In the next chapter, some ideas of manipulating bipartite pure entangled states by local operations on the subsystems with classical communications between them, will be discussed.
Chapter 4

Majorization and Incomparability

4.1 Concept of majorization

Given two sets of real numbers, i.e., two vectors of equal length as \( x \equiv (x_1, x_2, \cdots, x_d) \) and \( y \equiv (y_1, y_2, \cdots, y_d) \), to measure the disorder between these vectors we need a specific mathematical modeling. Majorization is a mathematical process of comparing the vectors in the most general and elegant way \([25, 77, 95, 137]\). It is widely applicable in various fields like, computer science, information theory, etc.

If the lengths of both the vectors are not same, we equate them by assuming the other elements of the short length vector to be all zero. To describe the process, we formulate two new vectors \( x^\downarrow \equiv (x_1^\downarrow, x_2^\downarrow, \cdots, x_d^\downarrow) \) and \( y^\downarrow \equiv (y_1^\downarrow, y_2^\downarrow, \cdots, y_d^\downarrow) \) by re-arranging the components of the \( d \)-dimensional vectors \( x \) and \( y \) in a decreasing way such as,

\[
x_1^\downarrow \geq x_2^\downarrow \geq \cdots \geq x_d^\downarrow, \quad y_1^\downarrow \geq y_2^\downarrow \geq \cdots \geq y_d^\downarrow
\] (4.1)

Then the vector \( x \) is said to be majorized by the vector \( y \) (or, alternately, \( y \) majorizes \( x \)), denoted by \( x \prec y \), if and only if

\[
\sum_{i=1}^{k} x_i^\downarrow \leq \sum_{i=1}^{k} y_i^\downarrow, \quad \forall \ k = 1, 2, \cdots, d - 1
\] (4.2)

and

\[
\sum_{i=1}^{d} x_i^\downarrow = \sum_{i=1}^{d} y_i^\downarrow
\] (4.3)

\(^1\)Some portions of this chapter is published in Quantum Information and Computation, 5(3), 247-257 (2005).
Example: If \( x_i \geq 0 \) with \( \sum_{i=1}^{d} x_i = 1 \), then
\[
\left( \frac{1}{d}, \frac{1}{d}, \cdots, \frac{1}{d} \right) \prec (x_1^1, x_2^1, \cdots, x_d^1) \prec (1, 0, \cdots, 0)
\] (4.4)

An alternative definition: If \( x^\uparrow \) be the vector obtained from \( x \) by rearranging the elements in an increasing order then, \( x^\uparrow_j = x_{d-j+1} \), \( \forall j \) such that \( 1 \leq j \leq d \). Thus we can write,
\[
\sum_{i=1}^{k} x^\uparrow_i = \sum_{i=1}^{d} x_i - \sum_{i=1}^{d-k} x_i
\]

So the majorization relation in equation (4.2) can also be expressed as, \( x \prec y \) if and only if
\[
\sum_{i=1}^{k} x^\uparrow_i \geq \sum_{i=1}^{k} y_i, \quad \forall k = 1, 2, \cdots, d-1
\] (4.5)

and
\[
\sum_{i=1}^{d} x^\uparrow_i = \sum_{i=1}^{d} y_i
\] (4.6)

Equations (4.3) and (4.6) are both equivalent with the condition \( \sum_{i=1}^{d} x_i = \sum_{i=1}^{d} y_i \).

Trace condition of majorization: Let us consider the vector \( |e\rangle = (1, 1, \cdots, 1) \) of dimension \( n \). Then given any vector \( |x\rangle \in \mathbb{R}^n \), we can define
\[
tr(x) = \sum_{j=1}^{n} x_j = \langle x, e \rangle
\] (4.7)

where \( \langle \cdot, \cdot \rangle \) denotes the inner product of two vectors in \( \mathbb{R}^n \).

Now, for any subset \( I \) of \( \{1, 2, \cdots, n\} \), the vector \( |e^I\rangle \in \mathbb{R}^n \) is defined as, \( |e^I\rangle = (e_1, e_2, \cdots, e_n) \), where \( e_j = 1 \) for all \( j \in I \) and \( e_j = 0 \), otherwise. Then we have a relation which provide us another definition of majorization between two vectors:

For any vectors \( x, y \in \mathbb{R}^n \), \( x \prec y \) if and only if for each subset \( I \) of \( \{1, 2, \cdots, n\} \) there exists another subset \( J \) of \( \{1, 2, \cdots, n\} \), with \( |I| = |J| \) such that
\[
\langle x, e^I \rangle \leq \langle y, e^J \rangle
\] (4.8)

and
\[
tr(x) = tr(y).
\] (4.9)

### 4.2 Doubly stochasticity and majorization

Linear mapping on vector spaces: Let \( \mathcal{L}(V,W) \) be the space of all linear mappings from a vector space \( V \) to another vector space \( W \). Then corresponding to every basis of \( V, W \), each linear mapping belonging to \( \mathcal{L}(V,W) \) has a unique matrix representation.

If \( H, K \) are Hilbert spaces with dimensions \( n, m \) respectively and the bases of the Hilbert spaces \( \{|e_1\rangle, |e_2\rangle, \cdots, |e_n\rangle\} \) and \( \{|f_1\rangle, |f_2\rangle, \cdots, |f_m\rangle\} \) are chosen to be orthonormal, then the
matrix representation of each operator $A \in \mathcal{S}(H,K)$ is obtained by, $A = (a_{ij})_{m \times n}$ where $a_{ij} = \langle f_i | A e_j \rangle$.

Now we define some characteristics of linear maps $A$ defined on $C^n$.

A linear maps $A$ on $C^n$ is called **positivity-preserving** if it transforms any vector with non-negative coordinates to some vector with non-negative coordinates.

A linear maps $A$ on $C^n$ is called **trace-preserving** if $\text{tr}(Ax) = \text{tr}(x)$ for all $x \in C^n$.

A linear maps $A$ on $C^n$ is called **unital** if $Ae = e$ where $e$ is a unit vector.

The majorization relation between two vectors is associated with a very special kind of matrix defined below.

**Doubly stochastic matrix related with majorization:** An $n \times n$ matrix $A = (a_{ij})_{n \times n}$ is said to be doubly stochastic, iff

$$
\begin{align*}
    a_{ij} \geq 0, & \quad \forall \ i, j, \\
    \sum_{i=1}^{n} a_{ij} = 1, & \quad \forall \ j, \\
    \sum_{j=1}^{n} a_{ij} = 1, & \quad \forall \ i.
\end{align*}
$$

Alternately, we can say that a matrix $A$ is doubly-stochastic if and only if the corresponding linear operator $A$ is positivity-preserving, trace-preserving and unital.

Then, the results about the majorization process are as follows:

**Theorem-1:** A matrix $A$ is doubly-stochastic if and only if $Ax \prec x$, for all vector $x \in \mathbb{R}^n$.

**Theorem-2:** If $x, y \in \mathbb{R}^n$, then $x \prec y$ if and only if $x = Ay$, for some doubly-stochastic matrix $A$.

### 4.2.1 Matrix/Operator majorization

Quantum mechanics is a probabilistic theory where states of a physical system are represented by density matrices. Any density matrix is hermitian, positive semi-definite, trace preserving. The evolution of the system has some relation with majorization. For this reason, we require the idea of majorization of matrices, preferably for hermitian matrices.

If $\Omega_1$ and $\Omega_2$ are hermitian matrices and $\lambda_i$ be the vector of eigenvalues arranged in non-increasing order, corresponding to the matrix $\Omega_i$, for $i = 1, 2$, then we define $\Omega_1 \prec \Omega_2$ if and only if $\lambda_1 \prec \lambda_2$.

Now, quantum operators are largely connected with the notion of majorization. Briefly, they are as follows:

**Lemma-** A quantum operation is doubly stochastic if it is both trace preserving and unital.

Thus doubly stochastic operators are of great importance in quantum information theory. They have many properties including that, they will never increase the entropy of a system.

**Theorem-1:** Suppose $A$ is hermitian and $\xi$ is a doubly stochastic operator, then $\xi(A) \prec A$.

**Theorem-2:** If $\xi$ is a trace-preserving quantum operation that is not doubly stochastic, then there exists a hermitian operator $A$ such that $A \prec \xi(A)$ but $\xi(A) \not\prec A$. 

4.2. Doubly stochasticity and majorization

**Schur’s theorem** - Let \( A \) be a \( d \times d \) hermitian matrix. Let \( \text{diag}(A) \) denote the vector whose components are the diagonal entries of \( A \) and \( \lambda_A \) denote the vector whose components are the eigenvalues of the operator \( A \). Then \( \text{diag}(A) \preceq \lambda_A \).

The set of all doubly-stochastic matrices have a nice geometric configuration along with the permutation matrices.

**Birkhoff’s theorem**: The set of all \( n \times n \) doubly-stochastic matrices is a convex set whose extreme points are the \( n \times n \) permutation matrices.

As there are \( n! \) number of permutation matrices of size \( n \), so from Birkhoff’s theorem, every \( n \times n \) doubly stochastic matrix is a convex combination of these \( n! \) number of matrices.

### 4.2.2 Random unitary evolution

Unitary evolution is a very important and general kind of physical operation. To study the nature of any physical system it is necessary to explore its behavior under every possible unitary operation performed on it. Transformation of the state of a system under any unitary operation is termed as a unitary evolution of the system.

**Random unitary evolution**: Any random unitary evolution, of a physical system in an initial state \( \rho \), will be denoted by \( \xi(\rho) \) and is described as a set of unitary matrices \( U_i \) together with the probabilities \( p_i \) which evolve the physical system to the final state \( \rho_f \) such that

\[
\rho_f = \xi(\rho) = \sum_i p_i U_i \rho U_i^\dagger
\]

If we represent the evolution operator \( \xi \), as a set of operations \( \{\xi_i ; i = 1, 2, \cdots n\} \) then the elements of this set can be expressed as \( \xi_i = \sqrt{p_i} U_i \), where

\[
\sum_i \xi_i \xi_i^\dagger = \sum_i p_i U_i U_i^\dagger = \sum_i p_i = 1
\]

Thus we see that every random unitary evolution is unital in nature. Next, if we compute the trace of the final state, then we have,

\[
\text{Tr}(\rho_f) = \text{Tr}(\sum_i p_i U_i \rho U_i^\dagger) = \sum_i p_i \text{Tr}(U_i \rho U_i^\dagger) = \sum_i p_i \text{Tr}(\rho) = \text{Tr}(\rho).
\]

Thus any unitary evolution is trace-preserving in nature.

It is interesting to observe that every random unitary evolution is a doubly stochastic operation, as \( \xi(\rho) \) is both unital and trace-preserving. And we have the following theorem:

**Uhlmann’s theorem**: For any pair of hermitian matrices \( A \) and \( B \), the following conditions are equivalent:

1. \( A \preceq B \)
2. There exists a random unitary evolution characterized by a set of unitary matrices $U_i$ and probabilities $p_i$, connecting $A$ and $B$ such that $A = \sum_i p_i U_i B U_i^\dagger$.

3. There exists a doubly stochastic operation $\xi$, such that $A = \xi(B)$.

We previously seen that every random unitary evolution is a doubly stochastic matrix, but the converse is not true. For our purpose we confine ourselves with the evolution of hermitian matrices. Uhlmann’s theorem implies that in this case, every doubly stochastic matrix is also a random unitary evolution.

**Change in entropy of the system:** For a single quantum system in state $\rho$, we measure the disorder by the von-Neumann entropy of the system, i.e., $S(\rho) = -\text{Tr}(\rho \log_2 \rho)$. It is interesting to note the behavior of entropy of two systems, when one of them is majorized by the other. For any pair of density matrices $\rho$ and $\sigma$, the relation $\rho \prec \sigma$ will then imply $S(\rho) \geq S(\sigma)$. This may be also described as a natural equivalency between two processes of measuring disorder of a system.

### 4.3 Majorization in quantum mechanics

The algebraic relation via majorization between two vectors is connected very nicely with the evolution of quantum system. In different areas of quantum mechanics, idea of majorization is used very successfully. It reveals the characteristics of physical system and its possible physical evolutions. Majorization criteria also detects possible measurement outcomes and operational outputs of a physical system. We mention some of those results \[104\] here.

**Probability of any Measurement outcome:** *Probabilities of any measurement in an orthonormal basis are connected with the eigenvalues of the initial state by the majorization relation.*

Suppose the state corresponding to a physical system is represented in its spectral decomposition form as, $\rho = \sum_i \lambda_i |i\rangle \langle i|$, where $\{\lambda_i ; i = 1,2,\cdots d\}$ are the eigenvalues of the density matrix $\rho$ and $\{|i\rangle ; i = 1,2,\cdots d\}$ are the corresponding eigenvectors. If a measurement in an orthonormal basis $\{|\phi_k\rangle ; k = 1,2,\cdots d\}$ is performed on this system, then the $k^{th}$ result will occur with the probability,

$$p_k = \langle \phi_k | \rho | \phi_k \rangle = \sum_{i=1}^d \lambda_i |\langle \phi_k | i \rangle|^2$$ (4.11)

Now, for any density matrix $\rho$ characterizing the state of a system, the necessary and sufficient condition for the existence of any measurement basis of the system, for which a given set of values $\lambda_\rho = \{\lambda_i ; i = 1,2,\cdots d\}$ will serve as the probabilities of that measurement is described by;
Quantum Measurement without Post-Selection: If $\rho$ be the density matrix corresponding to the initial state of a physical system then measuring the system in any complete set of orthonormal projectors, the final state of the system will be $\rho^f$ if and only if $\rho^f \prec \rho$.

The measurement associated with any complete set of projection operators, represented by, $\{P_j; \ j = 1,2,\cdots,d\}$, is usually called von Neumann measurement. If such a measurement is performed on the physical system, then the measurement outcomes will be one of the state corresponding to a projector $P_j$. Without knowing the actual outcome obtained, the final state after measurement will be described by,

$$\rho^f = \sum_j P_j \rho P_j$$

And then we will find the evolution of the system, characterized by some unitary operations, as follows,

$$\rho^f = \sum_k U_k \rho U_k$$

Therefore, by Uhlmann’s theorem, $\rho^f \prec \rho$.

Possible Pure State Decomposition of any Density matrix: For any given density matrix $\rho$ with vector of eigenvalues $\lambda \equiv (\lambda_1,\lambda_2,\cdots,\lambda_d)$ there exists a pure state ensemble $\{p_j,|\phi_j\rangle\}$ corresponding to some fixed probability vector $p \equiv (p_1,p_2,\cdots,p_d)$ if and only if $p \prec \lambda$.

In many tasks of information processing it is often necessary to decompose some given density matrix in terms of different pure state ensembles $\{p_j,|\phi_j\rangle\}$. It is then important to decide whether for a given probability vector, there exists some set of pure states that may be represented by the density matrix,

$$\rho = \sum_j p_j |\phi_j\rangle \langle \phi_j|$$

Clearly, it is determined by the majorization criteria.

Measurement increases disorder of the system: If $\lambda \equiv (\lambda_1,\lambda_2,\cdots,\lambda_d)$ be the vector of eigenvalues of some physical system then there exists a measurement with possible set of outcomes $\{\rho_j; \ j = 1,2,\cdots,d\}$ occurring with probabilities $\{p_j; \ j = 1,2,\cdots,d\}$, if and only if

$$\lambda \prec \sum_j p_j \rho_j$$

This result implies, measurements acquire some information about the system being measured.
4.4 Bipartite state transformation by LOCC

The simplest kind of non-local states are the states shared between two distant parties. In such bipartite cases, many properties of the joint system is revealed by the states of the local subsystems, i.e., by the reduced density matrices. Bipartite entangled states are required for performing different kinds of computational tasks like, Teleportation, Dense Coding, etc. Now, sometimes for performing specific kind of information processing tasks, one requires some specific entangled states. But the specific state may not be available always. Then further task is that, given a known state shared between two parties, whether it is possible to convert the given state according to their own choice. For such purpose, it is interesting to find out the set of states that may obtained from a particular entangled state. And, it is much preferable to perform the transformation of entangled states by LOCC \[88, 94\]. This LOCC restriction is not only imposed due to practical usefulness, but also due to the cause that conversion of entangled states by global operations is physically equivalent of transforming states of a single system by valid physical operations. Thus properties of entangled systems may not be reflected if we allow global operations to perform on a physical system.

Various ideas are prescribed for making possible the transformation of pure bipartite states \(|\Psi\rangle_{AB}\) to \(|\Phi\rangle_{AB}\), shared between two parties A and B, under LOCC in both the deterministic and probabilistic manner. Main idea of most of those schemes can be described as a scheme for manipulating entangled states to enhance the required tasks. The most important result is due to M.A. Nielsen \[105, 137\], where we find a necessary and sufficient condition for deterministic conversion of pure bipartite entangled states under LOCC.

4.4.1 Nielsen’s criteria

Nielsen provided a criteria \[105\], related with majorization, for transforming one pure bipartite state to another by any sort of allowed local operations and classical communications with certainty.

**Nielsen’s majorization criteria:** Suppose the pure bipartite state \(|\Psi\rangle_{AB}\) is shared between two parties, say, Alice and Bob situated at two distant places. The task is to determine the possibility of converting the joint state \(|\Psi\rangle_{AB}\) with certainty, to another pure bipartite state \(|\Phi\rangle_{AB}\) under LOCC performed by Alice and Bob. First, let us represent \(|\Psi\rangle_{AB}, |\Phi\rangle_{AB}\) in the Schmidt form with decreasing order of Schmidt coefficients:

\[
|\Psi\rangle_{AB} = \sum_{i=1}^{d} \sqrt{\alpha_i} |\mu_i\rangle_A |\nu_i\rangle_B ; \quad \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_d \geq 0, \quad \sum_{k=1}^{d} \alpha_k = 1
\]

\[
|\Phi\rangle_{AB} = \sum_{j=1}^{d} \sqrt{\beta_j} |\eta_j\rangle_A |\zeta_j\rangle_B ; \quad \beta_1 \geq \beta_2 \geq \cdots \geq \beta_d \geq 0, \quad \sum_{k=1}^{d} \beta_k = 1
\]  

\[ (4.12) \]
4.5 Incomparable pair of states

Here \( \{|\mu_i\rangle_A |\nu_i\rangle_B \; ; \; i = 1,2,\ldots,d \} \) is the Schmidt basis for the state \(|\Psi\rangle_{AB}\) and \( \{|\eta_j\rangle_A |\zeta_j\rangle_B \; ; \; j = 1,2,\ldots,d \} \) is the same for state \(|\Phi\rangle_{AB}\). We denote, the Schmidt vectors corresponding to the states \(|\Psi\rangle_{AB}\) and \(|\Phi\rangle_{AB}\) as: \( \lambda_\Psi \equiv (\alpha_1, \alpha_2, \ldots, \alpha_d) \), \( \lambda_\Phi \equiv (\beta_1, \beta_2, \ldots, \beta_d) \). Then according to Nielsen’s criterion, the necessary and sufficient condition for the conversion of \(|\Psi\rangle_{AB}\) to \(|\Phi\rangle_{AB}\) (expressed as, \(|\Psi\rangle_{AB} \rightarrow |\Phi\rangle_{AB}\)) with certainty under LOCC is, \( \lambda_\Psi < \lambda_\Phi \), i.e.,

\[
\sum_{i=1}^{k} \alpha_i \leq \sum_{i=1}^{k} \beta_i, \; \forall \; k = 1,2,\ldots,d
\]  

(4.13)

**Change in entanglement:** Let us now recall the thermodynamical law of entanglement, i.e., the amount of entanglement shared between some spatially separated parties can not be increased by LOCC. Therefore, as a consequence of thermodynamical law of entanglement, if \(|\Psi\rangle_{AB} \rightarrow |\Phi\rangle_{AB}\) is possible under LOCC with certainty, then \( E(|\Psi\rangle_{AB}) \geq E(|\Phi\rangle_{AB}) \); where \( E(\cdot) \) denotes the entropy of entanglement. Now, from Nielsen’s criteria we have the following corollary.

**Corollary:** Let \(|\psi\rangle_{AB}\) be any pure entangled state of a \( m \times n \) system. The number of Schmidt coefficients of \(|\psi\rangle_{AB}\) is \( d \leq \min\{m,n\} \). Then, \(|\Psi\rangle_{AB} \rightarrow |\psi\rangle_{AB}\) is always possible, where \(|\Psi\rangle_{AB}\) is any \( d \times d \) maximally entangled state. This is obvious, as the Schmidt vector of \(|\Psi\rangle_{AB}\) is \((\frac{1}{d}, \frac{1}{d}, \ldots, \frac{1}{d})\). Also, for any pure bipartite product state \(|Y\rangle_{AB}\) we have, \(|\psi\rangle_{AB} \rightarrow |Y\rangle_{AB}\), as the Schmidt vector corresponding to any pure product state is \((1,0,\cdots,0)\).

### 4.5 Incomparable pair of states

If the above criterion for transforming \(|\Psi\rangle_{AB} \rightarrow |\Phi\rangle_{AB}\) with certainty under LOCC, is not satisfied, then that pair of states is usually denoted by, \(|\Psi\rangle_{AB} \not\leftrightarrow |\Phi\rangle_{AB}\). Though, in such cases, it may happen that \(|\Phi\rangle_{AB} \rightarrow |\Psi\rangle_{AB}\) under LOCC. If for some pair of pure bipartite states \((|\Psi\rangle, |\Phi\rangle)\) shared between, say, Alice and Bob, both \(|\Psi\rangle \not\leftrightarrow |\Phi\rangle\) and \(|\Phi\rangle \not\leftrightarrow |\Psi\rangle\) happen together, then we denote it by, \(|\Psi\rangle \not\leftrightarrow |\Phi\rangle\) and describe \((|\Psi\rangle, |\Phi\rangle)\), as a pair of incomparable states. Also, \( E(|\Psi\rangle) \geq E(|\Phi\rangle) \) does not guarantee \(|\Psi\rangle \rightarrow |\Phi\rangle\) under LOCC with certainty. One of the most interesting feature of the existence of an incomparable pair of states is that we are unable to say which state has a greater amount of entanglement than the other.

For every pair of pure states in \( 2 \times 2 \) system it is always the case that either \(|\Psi\rangle \rightarrow |\Phi\rangle\) (when \( E(|\Psi\rangle) \geq E(|\Phi\rangle) \)) or \(|\Phi\rangle \rightarrow |\Psi\rangle\) (when \( E(|\Psi\rangle) \leq E(|\Phi\rangle) \)). So, in \( 2 \times 2 \), there is no pair of pure entangled states which are incomparable as described above. Therefore, we look beyond the \( 2 \times 2 \) system of states.

In \( 3 \times 3 \) system, there exists incomparable pair of states. The incomparability of Schmidt rank 3 states has some strange character which we will discuss latter. Now, we explicitly mention the criterion of incomparability for a pair of pure entangled states \((|\Psi\rangle, |\Phi\rangle)\) of \( m \times n \) system, where \( \min\{m,n\} = 3 \).
4.5. Incomparable pair of states

Suppose, the Schmidt vectors corresponding to the states $|\Psi\rangle$, $|\Phi\rangle$ are $(a_1,a_2,a_3)$ and $(b_1,b_2,b_3)$ respectively, where $a_1 > a_2 > a_3$, $b_1 > b_2 > b_3$, $a_1 + a_2 + a_3 = b_1 + b_2 + b_3$. Then it follows from Nielsen’s criterion that $|\Psi\rangle$, $|\Phi\rangle$ are incomparable if and only if either of the two relations,

$$a_1 > b_1 > b_2 > a_2 > a_3 > b_3,$$

$$b_1 > a_1 > a_2 > b_2 > b_3 > a_3$$

(4.14)

hold.

For higher dimensional states, incomparable pair of states appear in several ways. However, it is interesting to note that for infinite dimensional states incomparability is generic [46, 108].

4.5.1 Catalysis

Now, it is natural to ask that what we could do with an incomparable pair of states, by means of LOCC? If we require a pure bipartite state $|\phi\rangle$ and we have a finite but sufficient number of copies of a pure bipartite state $|\psi\rangle$, then Vidal’s theorem [134] provided us the way of converting probabilistically $|\psi\rangle$ to $|\phi\rangle$ by LOCC. This is of course of no use for deterministic conversion. For this purpose, Jonathan and Plenio [89], found that sometimes collective operation may be useful to convert deterministically $|\psi\rangle$ to $|\phi\rangle$ by LOCC, where $|\psi\rangle \not\rightarrow |\phi\rangle$. They showed that if we assist the conversion by another pure bipartite entangled state $|\chi\rangle$, say, a catalyst, then the conversion $|\psi\rangle \otimes |\chi\rangle \rightarrow |\phi\rangle \otimes |\chi\rangle$ may be possible by collective LOCC deterministically. Let us illustrate it by an example in detail.

Example: Consider two pure bipartite entangled states $|\psi\rangle$, $|\phi\rangle$ of $4 \times 4$ system, having the following Schmidt vectors,

$$\lambda_{|\psi\rangle} = (.4,.4,.1,.1)$$

$$\lambda_{|\phi\rangle} = (.5,.25,.25,.0)$$

(4.15)

This two Schmidt vectors violates Nielsen’s criteria for transforming the state $|\psi\rangle$ to $|\phi\rangle$ by LOCC with certainty. Calculating entanglement of the two states we found that $E(|\psi\rangle) > E(|\phi\rangle)$. Thus there is no restriction for local conversion of $|\psi\rangle$ to $|\phi\rangle$ from the constraint of non-increase of entanglement by local manipulation. Therefore, one may ask for a scheme to implement the task of transforming the pure state shared among two parties by allowing some extra entanglement shared between them without exchange of quantum communication through some quantum channel, i.e., whether there exists any catalyst or not? Suppose, two parties share a pure bipartite state $|\chi\rangle$, having Schmidt vector $\lambda_{|\chi\rangle} = (.6,.4)$. In this new situation, the local system of each party has two parts corresponding to the two entangled states $|\psi\rangle$ and $|\chi\rangle$. The Schmidt vectors corresponding to the initial and final joint states $|\psi\rangle \otimes |\chi\rangle$ and $|\phi\rangle \otimes |\chi\rangle$ are respectively, $\lambda_{|\psi\rangle \otimes |\chi\rangle} = (.24,.24,.16,.16,.06,.06,.04,.04)$ and $\lambda_{|\phi\rangle \otimes |\chi\rangle} = (.3,.2,.15,.15,.1,.1,0,0)$. Then, it is easy to check from Nielsen’s criteria that by performing LOCC collectively on the joint entangled system it becomes possible to transform $|\psi\rangle \otimes |\chi\rangle \rightarrow |\phi\rangle \otimes |\chi\rangle$ with certainty. Here, it is interesting to note that the extra amount of entanglement shared between the parties, preserved
by the process. Also, the entangled state $|\chi\rangle$ is not used as a channel for transforming quantum information.

But what type of pairs are really catalyzable? It is really hard to categorize. Consider two pure bipartite states $|\psi\rangle, |\phi\rangle$ of Schmidt rank $d$ with Schmidt vectors,

$$
\lambda_{|\psi\rangle} = (\alpha_1, \alpha_2, \ldots, \alpha_d)
\lambda_{|\phi\rangle} = (\beta_1, \beta_2, \ldots, \beta_d)
$$

(4.16)

respectively. Let $|\chi\rangle$ be a pure bipartite state acting as a catalyst for the local conversion of $|\psi\rangle$ to $|\phi\rangle$ with certainty.

Jonathan and Plenio [89] showed that, if the form of Schmidt vector for the catalytic state $|\chi\rangle$ is assumed to be $\lambda_{|\chi\rangle} = (\gamma_1, \gamma_2, \ldots, \gamma_d)$, then from Nielsen’s criteria, the first and last conditions for $|\psi\rangle \otimes |\chi\rangle \rightarrow |\phi\rangle \otimes |\chi\rangle$ by LOCC with certainty are $\alpha_1 \gamma_1 \leq \beta_1 \gamma_1$ and $\alpha_d \gamma_d \geq \beta_d \gamma_d$. And it is equivalent with the conditions,

$$
\alpha_1 \leq \beta_1 \quad \text{and} \quad \alpha_d \geq \beta_d
$$

(4.17)

This is of course a necessary condition, but not sufficient, for catalytic transformation. Thus if it is violated for some pair of states $(|\psi\rangle, |\phi\rangle)$, then there does not exist any pure bipartite entangled state, that may be used as catalyst. Also, if $|\psi\rangle \rightarrow |\phi\rangle$ is possible by catalysis, then $E(|\psi\rangle) \geq E(|\phi\rangle)$. For $3 \times 3$ system of states, violation of Nielsen’s criteria for the pair $(|\psi\rangle, |\phi\rangle)$ implies either both the conditions $\alpha_1 > \beta_1$ and $\alpha_3 > \beta_3$ or the conditions $\beta_1 > \alpha_1$ and $\beta_3 > \alpha_3$ must hold simultaneously. This implies the violation of necessary condition for catalysis. So, for any incomparable pair of states of $3 \times 3$ system, the transformation by LOCC with certainty could not be possible by any catalytic state. The existence of catalytic state is first observed in $4 \times 4$ incomparable pairs and only in this level a necessary and sufficient condition for the existence of $2 \times 2$ system of catalytic state is found until now [4]. Therefore, except of some numerical evidences, it is really hard to determine whether an incomparable pair is catalyzable or not. Investigation in this direction is going on by several groups [48, 59, 135, 140].

### 4.5.2 Multi-copy transformation

Another interesting feature of local conversion of non-local states shown by Bandyopadhyay et.al. [8]. Sometimes if we increase the number of copies of the states, then deterministic conversion of incomparable states under LOCC may be possible. There exists some pair of bipartite entangled states $(|\psi\rangle, |\phi\rangle)$ such that, $|\psi\rangle \nleftrightarrow |\phi\rangle$ where $E(|\psi\rangle) > E(|\phi\rangle)$, but $|\psi\rangle^k \rightarrow |\phi\rangle^k$ is possible for some positive integer $k$. This phenomena is called multi-copy transformation. A sufficient condition for an incomparable pair to remain incomparable even if we increase the number of copies as large as possible is that, either $\alpha_1 < \beta_1$ and $\alpha_d < \beta_d$, or, $\alpha_1 > \beta_1$ and $\alpha_d > \beta_d$ must hold simultaneously [6, 7, 8]. Existence of this kind of incomparable pair of states are known as strongly incomparable.
4.5. Incomparable pair of states

Let us reconsider the example in catalytic transformation. It is shown by Duan et al. [53] that for the incomparable pair of states of $4 \times 4$ system represented by,

\[
\begin{align*}
\lambda_{\psi} &= (.4,.4,.1,.1), \\
\lambda_{\phi} &= (.5,.25,.25,0)
\end{align*}
\]

the conversion is possible if three copies of the initial state are available to produce the three copies of the final state under deterministic LOCC, i.e.,

\[
|\psi\rangle^\otimes 3 \rightarrow |\phi\rangle^\otimes 3
\]

is possible under deterministic LOCC. It must be noted here that the condition for strong incomparability implies a violation of the necessary condition for catalytic transformation. Thus when multi-copy transformation is not possible for a pair of pure bipartite states, then so also the catalytic transformation. Later it has been shown that catalytic transformation is asymptotically equivalent with multi-copy transformation [53]. This equivalency is established in two steps. Firstly, if for a pair of incomparable states $(|\psi\rangle, |\phi\rangle)$, multi-copy transformation is possible, then there exists some finite dimensional bipartite entangled state, which may assist this transformation catalytically. Conversely, if there exist some catalytic state $|\chi\rangle$, for which $|\psi\rangle \otimes |\chi\rangle \rightarrow |\phi\rangle \otimes |\chi\rangle$ by LOCC, then $|\psi\rangle^\otimes k \rightarrow |\phi\rangle^\otimes k$ is possible, if $k \rightarrow \infty$.

4.5.3 Mutual catalysis

Now, another interesting process is provided by Feng et al., and other groups [64, 6, 7, 101], known as mutual catalysis. The basic objective in this process is: given two pairs of incomparable states, say, $|\psi_1\rangle \not\rightarrow |\phi_1\rangle, |\psi_2\rangle \not\rightarrow |\phi_2\rangle$, whether $|\psi_1\rangle \otimes |\psi_2\rangle \rightarrow |\phi_1\rangle \otimes |\phi_2\rangle$ is possible under LOCC with certainty or not. Emphasis is given on the special kind of mutual catalysis (in [6], it is defined as super catalysis), where in the conversion process, we recover not only the entanglement assisted in the process but more than that, i.e., for some incomparable pair $|\psi\rangle \not\rightarrow |\phi\rangle$ there exists $(|\chi\rangle, |\eta\rangle)$ with $E(|\eta\rangle) \geq E(|\chi\rangle)$ and $|\eta\rangle \rightarrow |\chi\rangle$ such that by collective local operation $|\psi\rangle \otimes |\chi\rangle \rightarrow |\phi\rangle \otimes |\eta\rangle$ is possible deterministically. It is interesting that the necessary condition for the existence of such special kind of mutual catalytic pair $(|\chi\rangle, |\eta\rangle)$, is the same as that for catalyst. Hence this type of mutual catalysis is not possible for $3 \times 3$ system of incomparability. It is shown, not analytically, but by some numerical examples, that there are system of states for which catalyst does not exist but mutual catalysis works. Trivially it is always possible that $|\psi\rangle \otimes |\phi\rangle \rightarrow |\phi\rangle \otimes |\psi\rangle$ under LOCC with certainty. So, existence of mutual catalytic state is always possible. But it is not of use, as our target state $|\phi\rangle$ is not in our hand and in the process of trivial mutual catalysis we have to use it.

We have investigated possibility of deterministic conversion by LOCC on the ground of mutual catalysis. We have tried to give analytic method of searching out some optimal process of finding mutual catalytic states that may be suitable from some physical viewpoint. Our work
emphasis on the idea of using entanglement of a composite system as a resource and also on conservation and proper manipulation of this valuable resource.

### 4.5.4 Probabilistic Conversion by Bound entanglement

Here comes the question of proper use of entanglement to reach the target state. By the use of entanglement we mean, forget about recovering the entanglement used in the process, but to concentrate on converting the input state to the desired one. Use of bound entanglement solves the problem of conversion to the desired one probabilistically. Recently, Ishizaka \[87\] showed that using PPT-bound entanglement local conversion of any two pure entangled states (no matter what the Schmidt rank of the states are) is always possible, at least with some probability. So, the problem of finding target state of our interest now remains for the case of deterministic local conversion only.

### 4.5.5 Deterministically unresolved class

We have observed that the necessary condition for the existence of super-catalytic pair \((|\chi\rangle, |\eta\rangle)\) is the same as that for catalyst. All pure incomparable states in \(3 \times 3\) are strongly incomparable. So, all the processes of catalysis, mutual catalysis with some recovery and multi-copy transformation will fail for all \(3 \times 3\) pure incomparable pair of states and also for all pure strongly incomparable bipartite classes. Therefore one may ask, is it not possible to get the target state under collective LOCC with certainty for such incomparable pairs? The answer is yes. Two possible ways for resolving incomparability of such classes are discussed here \[34\]. In both the cases we use free unrecoverable entanglement for deterministic local transformation.

### 4.6 Assistance by entanglement

Suppose we have a pair of incomparable pure bipartite states \(|\psi\rangle_{AB}, |\phi\rangle_{AB}\) of \(d \times d\) system, where the source state \(|\psi\rangle_{AB}\) and the target state \(|\phi\rangle_{AB}\) are taken in the following form:

\[
|\psi\rangle_{AB} = \sum_{i=1}^{d} \sqrt{a_i} |i\rangle_A |i\rangle_B, \quad a_i \geq a_{i+1} \geq 0, \quad \forall \ i = 1, 2, \ldots, (d-1), \quad \sum_{i=1}^{d} a_i = 1
\]

\[
|\phi\rangle_{AB} = \sum_{i=1}^{d} \sqrt{b_i} |i\rangle_A |i\rangle_B, \quad b_i \geq b_{i+1} \geq 0, \quad \forall \ i = 1, 2, \ldots, (d-1), \quad \sum_{i=1}^{d} b_i = 1
\]

Now, consider the \((d-1) \times (d-1)\) maximally entangled state \(|\Psi_{\text{max}}^{d-1}\rangle_{AB} = \frac{1}{\sqrt{d-1}} \sum_{i=d+1}^{2d-1} |i\rangle_A |i\rangle_B\), and the product state \(|P\rangle_{AB} = |d+1\rangle_A |d+1\rangle_B\). We want to make possible the joint transformation \(|\psi\rangle_{AB} \otimes |\Psi_{\text{max}}^{d-1}\rangle_{AB} \rightarrow |\phi\rangle_{AB} \otimes |P\rangle_{AB}\) under LOCC with certainty. If the Schmidt coefficients of \(|\psi\rangle_{AB} \otimes |\Psi_{\text{max}}^{d-1}\rangle_{AB}\) is arranged in a decreasing order, then all the first \(d\) coefficients are equal to \(\frac{a_1}{d-1}\). Again the Schmidt vector of \(|\phi\rangle_{AB} \otimes |P\rangle_{AB}\) is \(\{b_1, b_2, \ldots, b_d, 0, 0, \ldots\}\). Thus by Nielsen’s
4.6. Assistance by entanglement

criteria, one has to check the first $d$ conditions. We may write those conditions in the simplified form as,

$$\frac{ka_1}{d-1} \leq \sum_{i=1}^{k} b_i \quad \forall k = 1, 2, \ldots, d-1. \quad (4.21)$$

To prove this, we first state a theorem which shows an intricate relation holds between first and last Schmidt coefficients of any pair of incomparable states.

**Theorem.** For any pair of incomparable states $|\psi\rangle_{AB} \not\leftrightarrow |\phi\rangle_{AB}$ in $d \times d$ system, where $|\psi\rangle_{AB} = \sum_{i=1}^{d} \sqrt{a_i} |i\rangle_A |i\rangle_B$ and $|\phi\rangle_{AB} = \sum_{i=1}^{d} \sqrt{b_i} |i\rangle_A |i\rangle_B$, with $a_i \geq a_{i+1} \geq 0$, $\forall$ $i$, $\sum_{i=1}^{d} a_i = 1$ and $b_i \geq b_{i+1} \geq 0$, $\forall$ $i$, $\sum_{i=1}^{d} b_i = 1$, the following always holds:

$$a_1 + b_d < 1, \ b_1 + a_d < 1. \quad (4.22)$$

Proof: From Nielsen’s criteria, $|\psi\rangle_{AB} \not\leftrightarrow |\phi\rangle_{AB}$ implies either of the following two cases must be hold.

Case-1. When, $a_1 \leq b_1$ and $\sum_{i=1}^{k} a_i \geq \sum_{i=1}^{k} b_i$, for some $k = 2, 3, \cdots, d-1$. Then

$$a_1 + b_d \leq b_1 + b_d < 1 \quad (4.23)$$

and

$$b_1 + a_d = b_1 + 1 - \sum_{i=1}^{d-1} a_i = b_1 + 1 - \sum_{i=1}^{d-1} a_i - \sum_{i=k+1}^{d} a_i < b_1 + 1 - \sum_{i=1}^{k} b_i - \sum_{i=k+1}^{d-1} a_i = 1 - \sum_{i=2}^{k} b_i - \sum_{i=k+1}^{d-1} a_i < 1 \quad (4.24)$$

Case-2. When, $a_1 \geq b_1$ and $\sum_{i=1}^{k} a_i < \sum_{i=1}^{k} b_i$, for some $k = 2, 3, \cdots, d-1$; the proof is similar to the previous case.

This proves the theorem.

So, the incomparability condition itself implies that $|\psi\rangle_{AB} \otimes |\Psi_{\text{max}}^{d-1}\rangle_{AB} \rightarrow |\phi\rangle_{AB} \otimes |P\rangle_{AB}$ is possible under LOCC with certainty. Therefore, for any pair of incomparable states with a given Schmidt rank the maximally entangled state of the next lower rank is sufficient to assist the joint transformation under LOCC.

Next, we show that instead of using lower rank maximally entangled state, the conversion may be possible under LOCC if we use lower rank non-maximally entangled states so that we need as minimum use of the resource as possible. In explicit form the minimum amount of entanglement that is required for the local transformation of any $3 \times 3$ incomparable pair is

$$\min E(|\chi\rangle) = -c \log_2 c - (1 - c) \log_2 (1 - c)$$

where $|\chi\rangle = \sqrt{c} |00\rangle + \sqrt{(1-c)} |11\rangle$. Let the Schmidt vectors of the pure bipartite states $|\psi\rangle$ and $|\phi\rangle$ are respectively, $\lambda_i |\psi\rangle = (a_1, a_2, a_3)$ and
4.6. Assistance by entanglement

\[ \lambda_{(\phi)} = (b_1, b_2, b_3). \]  
The joint transformation \( |\psi\rangle \otimes |\chi\rangle \rightarrow |\phi\rangle \otimes |P\rangle \) will be possible if the following conditions,

\[
\begin{align*}
& a_1 c \\
& a_1 c + \max\{a_2 c, a_1 (1-c)\} \leq b_1 + b_2
\end{align*}
\]

hold (all other conditions are obvious). Now, \( a_2 c \geq a_1 (1-c) \) if and only if \( c \geq \frac{a_1}{a_1 + a_2} \). Here we want to choose the maximum possible value of \( c \) for which the required transformation will occur. Thus, if we assume \( c > \frac{a_1}{a_1 + a_2} \), i.e., \( \max\{a_2 c, a_1 (1-c)\} = a_2 c \). Then, the second condition becomes \( c \leq \frac{b_1 + b_2}{a_1 + a_2} \). Again, if \( \max\{a_2 c, a_1 (1-c)\} = a_1 (1-c) \), then the second condition is obvious. So, we only need to check the possibility of the transformation when \( c > \frac{a_1}{a_1 + a_2} \). We discuss separately the cases of incomparability of the two states, as follows:

**Type-1:** When, \( a_1 < b_1, a_1 + a_2 > b_1 + b_2 \), for all values of \( c \in \left[ \frac{1}{2}, 1 \right] \), we have \( a_1 c < b_1 \). So, we have to select the state \( |\chi\rangle \) in such a way that the second condition \( c \leq \frac{b_1 + b_2}{a_1 + a_2} < 1 \), will be satisfied. Thus in this case the only condition to make possible the joint transformation \( |\psi\rangle \otimes |\chi\rangle \rightarrow |\phi\rangle \otimes |P\rangle \) will be,

\[ c \leq \frac{b_1 + b_2}{a_1 + a_2} < 1 \]

As the maximum possible value of \( c \) is, \( c_0 = \frac{b_1 + b_2}{a_1 + a_2} < 1 \), so the minimum amount of entanglement required in this process is, \( E_0 = -c_0 \log_2 c_0 - (1-c_0) \log_2 (1-c_0) \).

**Type-2:** If, \( a_1 > b_1, a_1 + a_2 < b_1 + b_2 \), the second condition is obvious, as for all \( c \in \left[ \frac{1}{2}, 1 \right] \), \( c(a_1 + a_2) < a_1 + a_2 < b_1 + b_2 \). The condition for the above transformation to be possible is given by \( c \leq \frac{b_1}{a_1} < 1 \). The minimum amount of entanglement required in the process is, \( E = E_0 \) corresponding to \( c = c_0 = \frac{b_1}{a_1} \).

We conclude this section with an interesting result that in any \( d \times d \) system, from a non-maximally pure entangled state \( |\psi^d\rangle_{AB} \) of \( d \times d \) system \( (d \geq 3) \), we are able to reach the maximally entangled state \( |\phi\rangle_{AB} = |\Psi^d_{\text{max}}\rangle_{AB} \) of the same \( d \times d \) system by the use of the next lower rank maximally entangled state \( |\Psi^{d-1}_{\text{max}}\rangle_{AB} \) through collective local operation with certainty.

**Corollary-1.** In any \( d \times d \) system for \( (d \geq 3) \), a non-maximally pure entangled state \( |\psi^d\rangle_{AB} = \sum_{i=1}^{d} \sqrt{a_i} |i\rangle_A |i\rangle_B \) can be transformed to the maximally entangled state \( |\Psi^d_{\text{max}}\rangle_{AB} = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle_A |i\rangle_B \) of same dimension by using the next lower rank maximally entangled state \( |\Psi^{d-1}_{\text{max}}\rangle_{AB} \) through collective local operations with certainty. i.e., \( |\psi^d\rangle_{AB} \otimes |\Psi^{d-1}_{\text{max}}\rangle_{AB} \rightarrow |\Psi^d_{\text{max}}\rangle_{AB} \otimes |P\rangle_{AB} \), where \( |P\rangle_{AB} \) is a product state, is possible under LOCC with certainty, if the largest Schmidt coefficient, \( a_1 \) of \( |\psi^d\rangle_{AB} \) satisfies the relation, \( a_1 \leq \frac{d-1}{d} \).
We have from Nielsen criteria,

\[
\frac{a_1}{d - 1} \leq \frac{1}{d} \\
\frac{a_1}{d - 1} \leq \frac{1}{d} \\
\vdots \\
(d - 1) \frac{a_1}{d - 1} \leq (d - 1) \frac{1}{d} \\
a_1 + \frac{a_2}{d - 1} \leq 1 \\
\vdots
\]

(4.26)

We observe that each of the first (d-1) conditions are equivalent with the relation, \( a_1 \leq \frac{d-1}{d} \) and all the next conditions are obvious, as the right-hand is always 1.

Therefore, the result of Corollary-1, follows directly from the theorem above. In fact, instead of using the state \(|\psi^d\rangle_{AB}\), the above transformation is possible by a \(2 \times 2\) state only.

**Corollary-2.** The transformation \(|\psi\rangle_{AB} \otimes |\Psi^d_{\text{max}}\rangle_{AB} \rightarrow |\Psi^d_{\text{max}}\rangle_{AB} \otimes |P\rangle_{AB}\), is possible under LOCC with certainty, if we take \(|\psi\rangle_{AB}\), as a \(2 \times 2\) state with Schmidt coefficients \((\frac{d}{d-1}, \frac{1}{d})\).

By a recursive way, Corollary-2 immediately suggests that it is possible to achieve a maximally entangled state of any arbitrary Schmidt rank \(d, d \geq 3\) by using a finite number (to be exact, \(d - 1\)) of suitably chosen \(2 \times 2\) states only. This will be presented in the next corollary, given below.

**Corollary-3.** The transformation, \(|\psi_1\rangle_{AB} \otimes |\psi_2\rangle_{AB} \otimes \cdots |\psi_{d-1}\rangle_{AB} \rightarrow |\Psi^d_{\text{max}}\rangle_{AB} \otimes |P\rangle_{AB}\), is possible under LOCC with certainty, where \(|\psi_i\rangle_{AB}, \forall i = 1, 2, \cdots, d - 1\) are \(2 \times 2\) states with Schmidt coefficients \((\frac{d-i}{d-i+1}, \frac{1}{d-i+1})\), respectively.

### 4.7 Mutual co-operation

In this section our main goal is to provide an auxiliary incomparable pair so that the collective operation enables us to find the desired states; i.e., given a pair \(|\psi\rangle \not\leftrightarrow |\phi\rangle\), we want to find an auxiliary pair \(|\chi\rangle \not\leftrightarrow |\eta\rangle\) such that \(|\psi\rangle \otimes |\chi\rangle \rightarrow |\phi\rangle \otimes |\eta\rangle\), is possible under LOCC with certainty. There are several ways to find nontrivial \((|\chi\rangle, |\eta\rangle)\). We explicitly provide the form of the auxiliary pair for all possible incomparable pair \((|\psi\rangle, |\phi\rangle)\) in \(3 \times 3\) system. One of the interesting feature of such incomparable pairs is that we are unable to say that which state has a greater amount of entanglement than the other. So in this way we may resolve the incomparability of \((|\psi\rangle, |\phi\rangle)\) with \(E(|\psi\rangle) < E(|\phi\rangle)\) by mutual co-operation which obviously claims that \(E(|\chi\rangle) > E(|\eta\rangle)\). To remove the trivial case of joint transformation through the existence of single crosswise transformations \(|\psi\rangle \rightarrow |\eta\rangle\) or \(|\chi\rangle \rightarrow |\phi\rangle\), we further impose the restriction that \(|\psi\rangle \not\leftrightarrow |\eta\rangle\) and \(|\chi\rangle \not\leftrightarrow |\phi\rangle\)
We have also studied analytically that in $3 \times 3$ system, from two copies of a pure entangled state we are able to find two different pure entangled states, both of which are incomparable with the source state.

We first provide some examples that will show such features and then in two subsections we shall give analytical results for $3 \times 3$ system of incomparable states. We explicitly provide the form of the auxiliary pair for all possible incomparable pairs $(|\psi\rangle,|\phi\rangle)$ in $3 \times 3$ system.

**Example 1.**– Consider a pair of pure entangled states of the form

$$|\psi\rangle = \sqrt{0.4}|00\rangle + \sqrt{0.4}|11\rangle + \sqrt{0.2}|22\rangle,$$

$$|\phi\rangle = \sqrt{0.48}|00\rangle + \sqrt{0.26}|11\rangle + \sqrt{0.26}|22\rangle,$$

$$|\chi\rangle = \sqrt{0.49}|33\rangle + \sqrt{0.255}|44\rangle + \sqrt{0.255}|55\rangle,$$

$$|\eta\rangle = \sqrt{0.41}|33\rangle + \sqrt{0.41}|44\rangle + \sqrt{0.18}|55\rangle.$$

It is easy to check that $|\psi\rangle \not\leftrightarrow |\phi\rangle$ and $|\chi\rangle \not\leftrightarrow |\eta\rangle$; whereas, $E(|\psi\rangle) \approx 1.5219 > E(|\phi\rangle) \approx 1.5188, E(|\chi\rangle) \approx 1.5097 > E(|\eta\rangle) \approx 1.5001$; and if we allow collective operations locally on the joint system, then the transformation $|\psi\rangle \otimes |\chi\rangle \rightarrow |\phi\rangle \otimes |\eta\rangle$ is possible with certainty, i.e., the two incomparable pairs, after mutual co-operation, are able to make the joint transformation possible.

If we review the whole process critically, then we find something more than what we have discussed earlier. Here we see that the comparability of the joint operation actually evolved through the co-operation with the comparable class of states, i.e., the four states chosen are related in such a way that $|\psi\rangle \rightarrow |\eta\rangle$ and $|\phi\rangle \rightarrow |\chi\rangle$. So here we reduce the incomparability of two states by choosing some class of states comparable with them. It is obvious that such a pair of states always exist for any incomparable pair, i.e., incomparable pairs can always be made to compare by collective LOCC. Without going into the details of the proof, we just mention here that this approach resolves the incomparability of the $3 \times 3$ states in one way. However, someone may think that only by the help of comparable classes we may able to reach the desired states which are incomparable initially. Obviously, the answer is in the negative. The next example is given in support of this last remark.

**Example 2.**– Consider two pairs of pure entangled states $(|\psi\rangle,|\phi\rangle)$ and $(|\chi\rangle,|\eta\rangle)$ of the form

$$|\psi\rangle = \sqrt{0.41}|00\rangle + \sqrt{0.38}|11\rangle + \sqrt{0.21}|22\rangle,$$

$$|\phi\rangle = \sqrt{0.4}|00\rangle + \sqrt{0.4}|11\rangle + \sqrt{0.2}|22\rangle,$$
|χ⟩ = \sqrt{0.45} |33⟩ + \sqrt{0.34} |44⟩ + \sqrt{0.21} |55⟩,

|η⟩ = \sqrt{0.48} |33⟩ + \sqrt{0.309} |44⟩ + \sqrt{0.211} |55⟩.

It is quite surprising to see that not only |ψ⟩ \not\leftrightarrow |φ⟩ and |χ⟩ \not\leftrightarrow |η⟩ but also |ψ⟩ \not\leftrightarrow |η⟩, |χ⟩ \not\leftrightarrow |φ⟩. Beside this we also get the extra facility to prepare |χ⟩ from |ψ⟩ as |ψ⟩ → |χ⟩. From the informative point of view the picture is although, \(E(|ψ⟩) \approx 1.5307 > E(|φ⟩) \approx 1.5219\), and \(E(|χ⟩) \approx 1.5204 > E(|η⟩) \approx 1.5054\), but still independently we can not convert |ψ⟩ to either one of |φ⟩ or |η⟩ and also |χ⟩ to either one of |φ⟩ or |η⟩ with certainty under LOCC. Therefore, although the resource states have greater information content in terms of pure state entanglement, the individual pairs aren’t convertible. But considering them together, we are able to break the overall incomparability. Thus, apart from the usual way in mutual catalysis, i.e., achieve transformation of the first pair and recover as much as possible the amount of entanglement from second pair, here, we have tried to overcome the incomparability of both the two pairs of incomparable states together, by collective LOCC with certainty.

To give rise of the fact that mutual co-operation also exists in other dimensions, we are now providing other two sets of incomparable pairs in \(4 \times 4\) system which are strongly incomparable so that deterministic local conversions are not possible by assistance of catalytic states and \(2 \times 2\) mutual catalytic states, but by mutual co-operation the transformation is possible.

**Example 3.**—Consider two pairs of pure entangled states \((|ψ⟩, |φ⟩)\) and \((|χ⟩, |η⟩)\) of the form:

\[|ψ⟩ = \sqrt{0.4} |00⟩ + \sqrt{0.3} |11⟩ + \sqrt{0.2} |22⟩ + \sqrt{0.1} |33⟩,\]

\[|φ⟩ = \sqrt{0.45} |00⟩ + \sqrt{0.29} |11⟩ + \sqrt{0.14} |22⟩ + \sqrt{0.12} |33⟩,\]

\[|χ⟩ = \sqrt{0.5} |44⟩ + \sqrt{0.25} |55⟩ + \sqrt{0.2} |66⟩ + \sqrt{0.05} |77⟩,\]

\[|η⟩ = \sqrt{0.48} |44⟩ + \sqrt{0.36} |55⟩ + \sqrt{0.12} |66⟩ + \sqrt{0.04} |77⟩.\]

It is easy to check that |ψ⟩ \not\leftrightarrow |φ⟩ and |χ⟩ \not\leftrightarrow |η⟩, and \(E(|ψ⟩) \approx 1.846 > E(|φ⟩) \approx 1.800, E(|χ⟩) \approx 1.680 > E(|η⟩) \approx 1.592\). However, one may check |ψ⟩ \otimes |χ⟩ \rightarrow |φ⟩ \otimes |η⟩, is possible under LOCC. From crosschecking, we find that |ψ⟩ \rightarrow |η⟩ but |χ⟩ \not\leftrightarrow |φ⟩, however, |φ⟩ \rightarrow |χ⟩.

**Example 4.**—Consider another two pairs of pure entangled states \((|ψ⟩, |φ⟩)\) and \((|χ⟩, |η⟩)\) of the form:

|ψ⟩ = \sqrt{0.4} |00⟩ + \sqrt{0.3} |11⟩ + \sqrt{0.2} |22⟩ + \sqrt{0.1} |33⟩,
\[ |\phi\rangle = \sqrt{0.45}|00\rangle + \sqrt{0.29}|11\rangle + \sqrt{0.14}|22\rangle + \sqrt{0.12}|33\rangle, \]

\[ |\chi\rangle = \sqrt{0.5}|44\rangle + \sqrt{0.23}|55\rangle + \sqrt{0.22}|66\rangle + \sqrt{0.05}|77\rangle, \]

\[ |\eta\rangle = \sqrt{0.48}|44\rangle + \sqrt{0.36}|55\rangle + \sqrt{0.12}|66\rangle + \sqrt{0.04}|77\rangle. \]

Here also it is easy to verify that \(|\psi\rangle \not\leftrightarrow |\phi\rangle, |\psi\rangle \not\leftrightarrow |\eta\rangle, |\chi\rangle \not\leftrightarrow |\phi\rangle\) and \(|\chi\rangle \not\leftrightarrow |\eta\rangle\). But surprisingly \(|\psi\rangle \rightarrow |\chi\rangle\). Now it is very interesting that we can prepare the state of co-operation from the state in our hand. The relations between the entanglement of those states are, \(E(|\psi\rangle) \approx 1.846 > E(|\phi\rangle) \approx 1.800, \) and \(E(|\chi\rangle) \approx 1.684 > E(|\eta\rangle) \approx 1.592, \) and \(|\psi\rangle \otimes |\chi\rangle \rightarrow |\phi\rangle \otimes |\eta\rangle\), is possible under LOCC with certainty. All the examples we have provided here, are non-trivial one. Next, we will show some analytical results for \(3 \times 3\) system of incomparable states.

### 4.7.1 Conversion by an auxiliary incomparable pair

Now we concentrate only to the case of incomparable pairs in \(3 \times 3\) system of states. We will show that for every pair of incomparable pure entangled states \((|\psi_1\rangle, |\phi_1\rangle)\), there is always a pair of incomparable pure entangled states \((|\psi_2\rangle, |\phi_2\rangle)\) such that \(|\psi_1\rangle \otimes |\psi_2\rangle \longrightarrow |\phi_1\rangle \otimes |\phi_2\rangle\), is possible under LOCC with certainty. The main idea of this portion is, assuming \(|\psi_1\rangle\) as the source state and \(|\phi_1\rangle\) as the target state, we choose the nontrivial auxiliary incomparable pair \((|\psi_2\rangle, |\phi_2\rangle)\) such that by collective LOCC the joint transformation of both pairs is possible with certainty.

Consider, \(|\psi_1\rangle \equiv (a_1, a_2, a_3), |\phi_1\rangle \equiv (b_1, b_2, b_3)\) where \(a_1 \geq a_2 \geq a_3 \geq 0, a_1 + a_2 + a_3 = 1, b_1 \geq b_2 \geq b_3 \geq 0, b_1 + b_2 + b_3 = 1\). There are two possible cases of incomparability that exist in this dimension, which are discussed and treated differently below. Let us first consider the most arbitrary form of the other pair of states as: \(|\psi_2\rangle \equiv (c_1, c_2, c_3)\) and \(|\phi_2\rangle \equiv (d_1, d_2, d_3)\), then a necessary condition for the existence of the joint transformation by LOCC with certainty will be

\[
\begin{align*}
 a_1 c_1 &< b_1 d_1 \\
 a_3 c_3 &> b_3 d_3
\end{align*}
\]

Thus, in the case \(a_1 > b_1\), we must have \(c_1 < d_1\) and in case of \(a_1 < b_1\), we must choose the other pair so that \(c_3 > d_3\) (as, for the incomparability of the states \(|\psi_1\rangle, |\phi_1\rangle\), we have \(a_3 < b_3\).

**Case-1:** \(a_1 > b_1, a_1 + a_2 < b_1 + b_2\). As described above, we have the restriction \(c_1 < d_1\), then imposing the condition that the pair of states \((|\psi_2\rangle, |\phi_2\rangle)\) are also incomparable in nature, we have \(d_1 \geq c_1 \geq c_2 \geq d_2 \geq d_3 \geq c_3\). Now, without disturbing our target pair, for simplicity, we consider, \(c_1 = c_2\) and \(d_2 = d_3\). We rename the Schmidt vectors of the second pair as \(|\psi_2\rangle \equiv (\beta_1, \beta_1, \beta_2), |\phi_2\rangle \equiv (\alpha_1, \alpha_2, \alpha_2)\) where \(\beta_1 > \beta_2 > 0, 2\beta_1 + \beta_2 = 1, \alpha_1 > \alpha_2 > 0, \alpha_1 + 2\alpha_2 = 1, \beta_1 < \)
\( \alpha_1, 2\beta_1 > \alpha_1 + \alpha_2 \). Therefore, with the choice of the state \( |\phi_2\rangle \) in such a manner that \( b_3\alpha_1 > b_1\alpha_2 \), we arrange the Schmidt coefficients of the final joint state in the decreasing order as:

\[
|\phi_1\rangle \otimes |\phi_2\rangle \equiv \{b_1\alpha_1, b_2\alpha_1, b_3\alpha_1, b_1\alpha_2, b_2\alpha_2, b_3\alpha_2, b_1\alpha_2, b_2\alpha_2, b_3\alpha_2\} \quad (4.28)
\]

We consider separately the different subcases.

If \( a_3\beta_1 > a_1\beta_2 \), then

\[
|\psi_1\rangle \otimes |\psi_2\rangle \equiv \{a_1\beta_1, a_1\beta_1, a_2\beta_1, a_2\beta_1, a_3\beta_1, a_1\beta_2, a_2\beta_2, a_3\beta_2\} \quad (4.29)
\]

Thus the joint transformation is possible by LOCC with certainty, if the following conditions, using Nielsen’s criteria, are satisfied.

\[
\begin{align*}
2a_1\beta_1 &< (b_1 + b_2)\alpha_1 \\
(2a_1 + a_2)\beta_1 &< \alpha_1 \\
2(a_1 + a_2)\beta_1 &< \alpha_1 + b_1\alpha_2 \\
(2a_1 + 2a_2 + a_3)\beta_1 &< \alpha_1 + 2b_1\alpha_2 \\
2\beta_1 &< \alpha_1 + (2b_1 + b_2)\alpha_2 \\
(a_2 + a_3)\beta_2 &> 2b_3\alpha_2 \\
a_3\beta_2 &> b_3\alpha_2
\end{align*}
\] (4.30)

Among the above conditions, the second one implies the first (as, \( 2a_1\beta_1 < (b_1 + b_2)\alpha_1 < 2b_1\alpha_1 \)) and the 8th condition would imply the 7th (as, \( (a_2 + a_3)\beta_2 > 2a_3\beta_2 > 2b_3\alpha_2 \)). Again by assuming the validity of the second and fourth conditions we found that the third condition is satisfied automatically (adding both sides of the second the fourth condition we have, \( 2(2a_1 + a_2)\beta_1 < 2(b_1 + b_2)\alpha_1 + b_3\alpha_1 + b_1\alpha_2 < 2\alpha_1 \)). In the same way the fifth condition follows from the fourth and the sixth conditions. Thus, we are left with four conditions. We transform them into conditions given on \( \beta_2 \), by using \( 2\beta_1 + \beta_2 = 1 \). We arrive at the final condition of transformation as:

\[
\beta_2 > \max\left\{ \frac{a_2b_3}{a_3}, \frac{a_2(b_2 + 2b_3)}{1 - a_3}, 1 - \frac{\alpha_1}{a_1} \right\} \quad (4.31)
\]

when \( a_3\beta_1 > a_1\beta_2 \). And if,

\[
\max\left\{ \frac{a_1}{a_2}, \frac{b_1}{b_3} \right\} < \frac{\alpha_1}{\alpha_2} \quad (4.32)
\]

and

\[
\frac{a_3}{(2a_1 + a_3)} > \beta_2 > \max\left\{ \frac{a_2b_3}{a_3}, \frac{a_2(b_2 + 2b_3)}{1 - a_3}, 1 - \frac{\alpha_1}{a_1} \right\} \quad (4.33)
\]
Under such a choice the required joint transformation is always possible.

In the above process there may arise a similar condition like our first example. For this type of choice we have always \(|\psi_1\rangle \rightarrow |\phi_2\rangle\). Except this choice we further require that those cross pairs \((|\psi_1\rangle, |\phi_2\rangle)\) or \((|\psi_2\rangle, |\phi_1\rangle)\), remain incomparable too. To fulfill this requirement the state \(|\psi_2\rangle\) is chosen slightly differently, as \(|\psi_2\rangle \equiv (\beta_1, \beta_2, \beta_3)\), where \(\beta_1 > \beta_2 > \beta_3 > 0, \beta_1 + \beta_2 + \beta_3 = 1\), such that

\[ \alpha_1 > a_1 > \beta_1 > b_1 > a_2 > a_2 > a_2 > a_3 > a_3 > \beta_3 > b_3. \]

We also impose the extra conditions, \(a_1 \beta_3 > \beta_1 a_3 > b_1 \alpha_2\), \(a_1 \beta_1 < b_1 \alpha_1, a_3 \beta_3 > b_3 \alpha_2\) and \(\{(\beta_1 a_3 - a_2 \beta_3) - (a_3 - a_3)\} < \min\{0, (\alpha_1 b_3 - \alpha_2 b_2), (a_2 \beta_2 - \alpha_2 b_2)\}\).

Therefore, from Nielsen’s criteria, \(|\psi_1\rangle \otimes |\psi_2\rangle \rightarrow |\phi_1\rangle \otimes |\phi_2\rangle\), if

\[
\begin{align*}
\alpha_1 \beta_1 + a_1 (\beta_1 + \beta_2) &< b_1 \alpha_1, \\
\alpha_1 (\beta_1 + \beta_2) + \max\{a_2 \beta_1, a_3 \beta_3\} a_1^2 + 2a_1 a_2 &< (2b_1 + b_2) \alpha_1, \\
(a_1 + a_2)^2 &< 2(b_1 + b_2) \alpha_1, \\
(a_1 + a_2)^2 + a_1 a_3 &< (2b_1 + 2b_2 + b_3) \alpha_1, \\
(a_1 + a_2)^2 + 2a_1 a_3 &< 2 \alpha_1, \\
(a_1 + a_2)^2 + (2a_1 + a_2) a_3 &< 2 \alpha_1 + b_1 \alpha_3, \\
\alpha_1^2 &< b_3 \alpha_3.
\end{align*}
\]

After such a choice, the pair \((|\psi_2\rangle, |\phi_1\rangle)\) became incomparable except when \(a_2 = a_3\). But, whenever we face the case \(b_1 = b_2\) and \(a_2 = a_3\), then correspondingly we see that, \(|\psi_1\rangle \rightarrow |\phi_2\rangle\) and \(|\psi_2\rangle \rightarrow |\phi_1\rangle\).

**Case-2:** \(a_1 < b_1, a_1 + a_2 > b_1 + b_2\). In this case we choose, \(|\psi_2\rangle \equiv (\beta_1, \beta_2, \beta_3), |\phi_2\rangle \equiv (\alpha_1, \alpha_1, \alpha_2)\) where \(\beta_1 > \beta_2 > \beta_3 > 0, \beta_1 + \beta_2 + \beta_3 = 1, \alpha_1 > \alpha_2 > 0, 2\alpha_1 + \alpha_2 = 1, \beta_1 > \alpha_1, \beta_1 + \beta_2 > 2\alpha_1\). Now there arises two different subcases, \(a_1 \geq \frac{1}{2}\) which must be considered separately as in the case of \(a_1 < \frac{1}{2}\) there is a possibility for \(a_1 = a_2\).

Firstly, when \(a_1 < \frac{1}{2}\), we choose the state \((|\psi_2\rangle, |\phi_2\rangle)\) in such a way that \(\alpha_1 b_3 > a_1 \beta_3 > \beta_1 a_3\) and

\[
\alpha_1 > \max\left\{ \frac{\beta_1 a_1}{b_1}, \frac{\beta_1 (a_1 + a_2) a_1 \beta_2}{2b_1 + b_2}, \frac{(1 - \beta_2)(1 - a_3)}{2(1 - b_3)} \right\} \quad (4.35)
\]

Secondly, when \(a_1 \geq \frac{1}{2}\), we choose the state \((|\psi_2\rangle, |\phi_2\rangle)\) in such a way that \(\beta_1 = \frac{1}{2}\) and \(\alpha_1 b_3 > a_1 \beta_3 > \beta_1 a_3\),

\[
\alpha_1 > \max\left\{ \frac{a_1}{2b_1}, \frac{a_1 + a_2 + 2a_1 \beta_2}{2(2b_1 + b_2)}, \frac{(0.5 + \beta_2)(1 - a_3)}{2(1 - b_3)}, \frac{2a_1 + a_2}{4(b_1 + b_2)}, \frac{a_1 + a_2 + a_2 \beta_3}{2 - b_3} \right\} \quad (4.36)
\]

It is interesting to note that in the first subcase when \(a_1 = a_2\) we have, \(|\psi_1\rangle \rightarrow |\phi_2\rangle\). Except this case, our choice maintains \(|\psi_i\rangle \not\rightarrow |\phi_j\rangle\), \(\forall i, j = 1, 2\).
4.7.2 Joint transformation of pairs with the same initial state

At the beginning of this subsection we want to present the special result for $3 \times 3$ system of pure entangled states as follows:

**Theorem:** For any source state $|\psi\rangle$ in $3 \times 3$ system, with distinct Schmidt coefficients there always exist two states $(|\chi\rangle, |\eta\rangle)$ such that both of them are incomparable with $|\psi\rangle$, but from two copy of $|\psi\rangle$ we are able to get them by collective LOCC with certainty.

Suppose the source state is $|\psi\rangle \equiv (a_1, a_2, a_3)$ with $a_1 > a_2 > a_3 > 0$, $a_1 + a_2 + a_3 = 1$ and the other states are expressed with the most arbitrary form as: $|\chi\rangle \equiv (b_1, b_2, b_3)$ and $|\eta\rangle \equiv (c_1, c_2, c_3)$ with $b_1 \geq b_2 \geq b_3 > 0$, $b_1 + b_2 + b_3 = 1$ and $c_1 \geq c_2 \geq c_3 > 0$, $c_1 + c_2 + c_3 = 1$. For incomparability of the two pair of states $(|\psi\rangle, |\chi\rangle)$ and $(|\psi\rangle, |\eta\rangle)$, we need to impose either of the two relations between their Schmidt coefficients, as specified in Eq.(4.14). Then it follows from Nielsen’s condition that there is always a possible range of $(|\chi\rangle, |\eta\rangle)$ such that $|\psi\rangle \otimes 2 \rightarrow |\chi\rangle \otimes |\eta\rangle$, under LOCC with certainty. It should be noted that the cases of failure of this general result is only the small number of cases where irrespective of the incomparability condition, the Schmidt coefficients of the source state are not all distinct, i.e., either $a_1 = a_2$ or $a_2 = a_3$.

This result is very important because we must keep in our mind the fact, that multiple copy transformation is not possible for states in $3 \times 3$ system. Thus, whenever we require the joint transformation to be possible under deterministic LOCC, while both of the final states are incomparable with the initial state, we are constrained to keep the pair of final states $(|\chi\rangle, |\eta\rangle)$, to be incomparable too. Now with this result in our hand, let us try to fix $|\chi\rangle$ as our target state and find the possible range (if exists at all) of $|\eta\rangle$; i.e., given two copies of the source state $|\psi\rangle$, our aim is to find a $|\eta\rangle$; incomparable with the source state, so that $|\psi\rangle \otimes 2 \rightarrow |\chi\rangle \otimes |\eta\rangle$, is possible under LOCC with certainty.

Like the previous section here also we have two cases of incomparability of the target pair $(|\psi\rangle, |\chi\rangle)$, as given in Eq.(4.14). From Nielsen’s criteria, to perform the joint transformation by deterministic LOCC, we necessarily require to satisfy the relation, $a_1^2 \leq b_1 c_1$ and $a_3^2 > b_3 c_3$. Thus, in case $a_1 > b_1$, we are bound to impose the condition $a_1 < c_1$. Otherwise, when $a_1 < b_1$, to retain the incomparability of $(|\psi\rangle, |\chi\rangle)$, we have $a_3 < b_3$. And, then we must choose $|\chi\rangle$ in such a manner that $a_3 > c_3$ and as, we have to satisfy $|\chi\rangle \leftrightarrow |\eta\rangle$, we also require $a_1 > c_1$.

**Case-1:** When, $a_1 < b_1, a_1 + a_2 > b_1 + b_2$. Here, without any loss of generality, we may assume, $c_1 = c_2$ and rename the Schmidt coefficients of the pure state $|\eta\rangle$ as, $|\eta\rangle \equiv (\alpha_1, \alpha_1, \alpha_2)$ where $\alpha_1 > \alpha_2 > 0, 2\alpha_1 + \alpha_2 = 1, a_1 > \alpha_1, a_1 + a_2 < 2\alpha_1$. Thus, we obtain the relation between the Schmidt coefficients as:

$$b_1 > a_1 > \alpha_1 > a_2 > b_2 \geq b_3 > a_3 > \alpha_2.$$

Consider now two subcases separately.
Firstly, we consider the case, when \(a_2^2 > a_1a_3\). We require the joint transformation \(|\psi\rangle \otimes^2 \rightarrow |\chi\rangle \otimes |\eta\rangle\), is possible under LOCC with certainty. As, \(a_2^2 > a_1a_3\), the Schmidt vector of \(|\psi\rangle \otimes^2\) is:
\[
(a_1^2, a_1a_2, a_1a_2, a_2^2, a_1a_3, a_2a_3, a_2a_3, a_2^2)
\]
and the Schmidt vector of \(|\chi\rangle \otimes |\eta\rangle\) is:
\[
(b_1\alpha_1, b_1\alpha_1, b_1\alpha_1, b_2\alpha_2, b_3\alpha_1, b_3\alpha_1, b_2\alpha_3, b_3\alpha_3).
\]
Choose, \(|\eta\rangle\) in such a way that \(\alpha_3 < \frac{b_3}{2b_1 + b_3}\).

Then, by Nielsen’s criteria, for the transformation \(|\psi\rangle \otimes^2 \rightarrow |\chi\rangle \otimes |\eta\rangle\), we must have:
\[
\begin{align*}
a_1^2 &< b_1\alpha_1 \\
\frac{a_1^2}{b_1\alpha_1} &< 2b_1\alpha_1 \\
a_1^2 + a_1a_2 &< (2b_1 + b_2)\alpha_1 \\
\frac{a_1^2 + a_1a_2}{(2b_1 + b_2)\alpha_1} &< 2(b_1 + b_2)\alpha_1 \\
(a_1 + a_2)^2 &< (2b_1 + 2b_2 + b_3)\alpha_1 \\
\frac{(a_1 + a_2)^2}{(2b_1 + 2b_2 + b_3)\alpha_1} &< 2\alpha_1 \\
\frac{(a_1 + a_2)^2 + 2a_1a_3}{2\alpha_1} &< 2\alpha_1 + b_1\alpha_3 \\
\frac{(a_1 + a_2)^2 + (2a_1 + a_2)a_3}{2\alpha_1 + b_1\alpha_3} &> b_3\alpha_3 \\
\frac{1}{b_2} &> 0.
\end{align*}
\]

Now, we investigate those conditions in detail. If \(a_1^2 < b_1\alpha_1\), then \(a_1^2 + a_1a_2 < 2\alpha_1^2 < 2b_1\alpha_1\).

Again, adding both sides of the 4th and 6th conditions, we obtain the 5th condition. Next, we try to prove the 7th condition:
\[
\{(a_1 + a_2)^2 + (2a_1 + a_2)a_3\} - (2\alpha_1 + b_1\alpha_3) = \{(a_1 + a_2 + a_3)^2 - (a_2a_3 + a_3^2)\} - (1 - \alpha_3 + b_1\alpha_3) = 1 - (a_2a_3 + a_3^2) - (1 - \alpha_3 + b_1\alpha_3) = (1 - b_1)\alpha_3 - (a_2 + a_3)a_3 = \{(b_2 + b_3)a_3 - (a_2 + a_3)b_3\} \frac{a_3}{b_3} = \{b_2a_3 - a_2b_3\} \frac{a_3}{b_3} < 0.
\]

Also, \(a_2^2 - b_3\alpha_3 = a_3^2 - b_3(1 - 2\alpha_1) = a_3^2 - b_3 + 2\alpha_1b_3 > a_3^2 - b_3 + \frac{(a_1 + a_2)^2}{b_1}\alpha_3 = a_3^2 - b_3 + \frac{(1 - \alpha_3)^2}{b_2}b_3 > 0.
\]

Then, such a joint transformation occurs if \(\frac{(a_1 + a_2)^2}{2(b_1 + b_2)} < a_1\) and the range of \(|\eta\rangle\) is specified by the relation:
\[
\alpha_1 > \max\left\{a_1 - \frac{a_1^2 - a_2^2}{2(b_1 + b_2)}, \frac{(a_1 + a_2)^2}{b_1}, \frac{a_1(a_1 + 2a_2)}{2(b_1 + b_2)}\right\}
\]
(4.38)

Next, when \(a_2^2 < a_1a_3\), then the condition for such transformation is; \(a_1 + 2a_2 < 2b_1 + b_2\). Under this condition range of \(|\eta\rangle\) is specified by the relation:
\[
\alpha_1 > \max\left\{a_1 - \frac{a_1^2 - a_2^2}{2(2 - b_3)}, \frac{a_1(2 - a_1)}{b_1}, \frac{a_1(a_1 + 2a_2)}{(2b_1 + b_2)}\right\}
\]
(4.39)

Case-2: When \(a_1 > b_1, a_1 + a_2 < b_1 + b_2\), we take \(|\eta\rangle \equiv (\alpha_1, \alpha_2, \alpha_2)\) where \(\alpha_1 > \alpha_2 > 0, \alpha_1 + 2\alpha_2 = 1, a_1 < \alpha_1, a_1 + a_2 > \alpha_1 + \alpha_2\). Then the condition for such transformation is, \(a_3 <
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\( \frac{1}{2} (1 - \frac{a_2^2}{a_1^2}) \). Under this condition, we have not only one \( |\eta\rangle \), but a range of it specified either by the relation:

\[
\alpha_2 < \min \left\{ \frac{a_1 a_3}{b_1}, \frac{a_3^2}{b_3}, \frac{a_3 (2a_2 + a_3)}{(b_2 + 2b_3)} \right\}, \text{ for } a_2^2 > a_1 a_3,
\]

(4.40)

or, by the relation:

\[
\alpha_2 < \min \left\{ a_3 + \frac{a_2^2 - a_3^2}{2}, \frac{a_3^2}{b_3}, \frac{a_3 (2a_2 + a_3)}{(b_2 + 2b_3)} \right\}, \text{ for } a_2^2 < a_1 a_3
\]

(4.41)

Finally, here we must mention that our process works for most of the cases of incomparability. But, it is not always successful; i.e., choosing any arbitrary incomparable pair, we might not be able to reach the target state by this method. This small range of failure of the process is possibly due to the fact that we didn’t ever bother about the amount of entanglement contained into the states. It is possible that \( E(|\psi\rangle) \ll E(|\chi\rangle) \); for which there doesn’t exist such a state \( |\eta\rangle \), incomparable with \( |\psi\rangle \) and \( E(|\psi\rangle \otimes |\eta\rangle) > E(|\chi\rangle \otimes |\eta\rangle) \).

In conclusion, with this method any incomparable pair of pure bipartite entangled states in any finite dimension, can be maid to compare (i.e., transform one to another), under LOCC with certainty, by providing some pure entanglement. We observe that mutual co-operation is an useful process to break the incomparability of two pairs under LOCC. This is not only discussed as an abstract or rather complicated theory, but we provide the algorithmic structure by which this goal can be really achieved.

4.8 The role of entanglement

The various methods of transforming a pure bipartite states to other described above are again examples of the role of entanglement in performing different information processing tasks. The amount of entanglement content determines the direction of local transformation for a particular pair of pure entangled states. The presence of unperturbed entangled state sometimes enhance the transformation. When the presence of entanglement is not sufficient to assist the transformation, then by exploiting entanglement, one can also able to perform the local conversion. Our schemes are proposed in this direction. Both of the schemes proposed here are elaborated analytically, starting from some numerical examples. They provide us in practical senses, the way of proper utilization of the resource, for the process of transforming an arbitrary incomparable pair of states, with extra input of entanglement, by deterministic LOCC. The results are obviously related with the manipulation and processing of pure bipartite entangled states.

In conclusion, in this chapter, we have studied incomparability as a peculiar feature of pure state entanglement that exists in quantum systems. Incomparability shows some constraint on local conversion of entangled states. The results also indicate that a kind of non-locality is associated with the system that may not be fully describable by the amount of entanglement,
but rather it may concerned with the construction or purification processes which are connected with the Schmidt vector of the states. It again brings us to explore a very fundamental area of the quantum information theory regarding the notion of quantification of pure bipartite entangled states that all are equal with entropy of entanglement. It is thus interesting to search for different non-local features of such systems of incomparable states through different views of quantum information theory. Apart from the manipulating entanglement, we will show in the next two chapters that the notion of incomparability also plays a role of detection for some classes of nonphysical operations.
Chapter 5

Spin-flipping and Incomparability

5.1 Spin-flipping of a qubit

In quantum information theory, one of the main objects is to encode some information in quantum states. In classical world, if the information of the space direction \( \overrightarrow{n} \) is represented by a classical spin-\( \frac{1}{2} \) particle polarized in the direction \( \overrightarrow{n} \), then it is as good as the spin-\( \frac{1}{2} \) particle polarized along \( -\overrightarrow{n} \). But, the situation in quantum systems is quite different. There is no single quantum operation that could reverse the spin direction in an arbitrary manner. Further, if we use composite quantum states to encode information, then the state \(|\overrightarrow{n}, -\overrightarrow{n}\rangle\) of two spin-\( \frac{1}{2} \) particles both polarized along \( \overrightarrow{n} \), is not equivalent with the state \(|\overrightarrow{n}, -\overrightarrow{n}\rangle\) corresponding to two spin-\( \frac{1}{2} \) particles polarized in opposite directions \( \overrightarrow{n} \) and \( -\overrightarrow{n} \). The root of this problem is not completely solved, however, many interesting results found in this direction \([65, 70, 97]\). Below, we have described one such situation.

There is an intimate relation between discriminating a set of quantum states and the process of estimating the set. It is shown by Gisin et.al. \([70]\), the set of anti-parallel spin states defined by,

\[
S_a = \{|\overrightarrow{n}, -\overrightarrow{n}\rangle; \overrightarrow{n} \in \mathbb{R}^3\}
\]

can be better distinguished than the corresponding set of states of parallel spins describes by,

\[
S_p = \{|\overrightarrow{n}, \overrightarrow{n}\rangle; \overrightarrow{n} \in \mathbb{R}^3\}
\]

Thus, two set of states are not equivalent while we want to discriminate its members. Also, it is found that the ensemble \(S_a\) of states with antiparallel spins, have a larger entropy. The physical reason behind such behavior of states of spin-\( \frac{1}{2} \) particles may be the non-universality of the operation that transforms an arbitrary spin-direction to the opposite spin direction. However, it is clear that the quantum spin-flipping operation has its own limitations which imposes some

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5.1. Spin-flipping of a qubit

Constraints over the system. In this chapter, we will show a connection between spin-flipping of qubits and the class of pure incomparable states.

5.1.1 Flipping operation

The flipping operation acts on a qubit to reverse the spin-polarization direction of the qubit. In other words, a flipping operation transforms a qubit to its orthogonal qubit. For example, the NOT gate represented by $2 \times 2$ matrix in $z$-basis

$$
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
$$

flips perfectly the states $|0\rangle$ and $|1\rangle$, i.e., the spin directions $Z^+$ and $Z^-$, but it cannot flip the spin direction $X^+$. Mathematically, we define the flipping operation $\Omega$ on a single arbitrary input qubit $|\varphi\rangle = \alpha|0\rangle + \beta|1\rangle$; $|\alpha|^2 + |\beta|^2 = 1$ as follows:

$$
|\varphi\rangle \rightarrow |\varphi\rangle = \Omega|\varphi\rangle = \beta^*|0\rangle - \alpha^*|1\rangle
$$

so that the inner product of the input qubit $|\varphi\rangle$ and the output qubit $|\varphi\rangle$ will vanish. Thus, if defined on a single qubit, it is a very natural operation achievable by a simple rotation of the Bloch sphere about its center. Now if one requires a single flipping operation defined to act perfectly on a large class of qubits, there arises some restrictions.

5.1.2 No-flipping principle

Gisin first showed that there is a restriction over general flipping operation which is similar in some sense with the 'No-Cloning' and 'No-Deleting' principles, but completely different in its operational status. Non-existence of universal exact flipping machine is a kind of constraint on the quantum systems that has been directly observed from the unitary dynamics of quantum evolutions. One could verify that the universal exact flipper, which if operated on an arbitrary qubit, will reverse the spin polarization direction, is not unitary but an anti-unitary operation and thus it is not in general a physical operation. No-flipping principle says that exact flipping of an unknown qubit state, is not possible. Like no-cloning, no-deleting principles, it is also a fundamental restriction to the allowable operations on quantum systems. However, instead of at least two subsystems to describe no-cloning or no-deleting principles, we require only one single system to describe flipping operation. Further, there always exist a quantum machine which can act as an exact flipper for any set of two qubit states, even if they are non-orthogonal in nature. One interesting observation is found by Gisin and Popescu and also by Massar that the two antiparallel-spin state $|\vec{n}, -\vec{n}\rangle$ contains more information than that of two-parallel-spin state $|\vec{n}, \vec{n}\rangle$. It is conjectured that the origin of this feature is the property that the anti parallel spin states span the whole four dimensional Hilbert space of two spin-$\frac{1}{2}$ system, while the parallel spin-$\frac{1}{2}$ system spans only a three dimensional subspace constituted...
by the symmetric states. Apart from such restrictions, there exists universal optimal flipping machine for qubits (Buzek et.al. [31]).

5.1.3 Exact flipping of states of one great circle

The flipping operation on qubit system can be visualized nicely by the Bloch sphere representation. It is easy to imagine a valid physical operation on the system which may flip a fixed spin polarization direction by a $180^\circ$ rotation of the Bloch sphere about its center. Obviously, all the vectors lying on the circle of rotation, will be flipped exactly by this operation. Thus, we see that the operation acts exactly on every qubit represented by a ray on a great circle. It is further proved that the largest set of states which can be exactly flipped by a single quantum machine, is a set of states lying on a great circle of the Bloch sphere [65]. Interestingly, one could find that any three states of the Bloch sphere, not lying in one great circle can not be flipped exactly by a single quantum machine. Thus, a stronger version of no-flipping principle can be expressed as,

Any three qubit states, with spin-polarization directions not lying in one great circle can not be flipped exactly by a single quantum machine.

5.1.4 Relation with other detectors

Any physical system would necessarily respect no-signalling condition of the special theory of relativity. Violation of this principle of nature is used as a detector for exploring the impossibilities of some processes to be a physical one [110, 113]. Recently, we proved [35] that, both the principles of No-Signaling and Non-increase of entanglement by LOCC, separately implies the exact flipping operation, defined on the minimum number (i.e., three) of qubits not lying in one great circle, is impossible in nature. The work also shows the importance of the underlying linearity assumption, as by allowing linearity in pure state superposition level, one could even create entanglement between separated subsystems. Research along the direction of finding inter-relations between No-flipping principle and other constrains of quantum mechanics [61], has created lots of interest.

5.2 No-flipping verses incomparability

In this section, we will show the inter-relation between the no-flipping principle and the notion of incomparable states in pure bipartite entangled states [36]. We have illustrated our results firstly with examples.

**Example-1**: Consider three states representing the three axes of the Bloch sphere in usual basis, as $|0_x\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$, $|0_y\rangle = \frac{|0\rangle + i|1\rangle}{\sqrt{2}}$, $|0_z\rangle = |0\rangle$. Beside of not lying in one great circle of the
5.2. No-flipping verses incomparability

Bloch sphere, this three states are mutually non-orthogonal and far separated from each other. We choose the particular setting of a pure bipartite state in the form:

$$|\Psi^i\rangle_{AB} = \frac{1}{\sqrt{3}} \{ |0\rangle_A |0_z 0_z\rangle_B + |1\rangle_A |0_x 0_y\rangle_B + |2\rangle_A |0_y 0_x\rangle_B \}$$  \hspace{1cm} (5.2)

This is a three particle state, shared between two space-like separated parties Alice and Bob, so that Alice has a 3-dimensional system with the basis \{0,1,2\} and Bob has two particles, each one belongs to a qubit system. As a bipartite state it represents a $3 \times 4$ system. We also impose the restriction that the reduced density matrix of Bob’s subsystem admits a representation in terms of the three input states $|0_x\rangle, |0_y\rangle, |0_z\rangle$. As the operation is performed on Bob’s system and it is defined on the above input states only, thus it is necessary that the reduced density matrix of Bob’s system must be represented by $|0_x\rangle, |0_y\rangle, |0_z\rangle$, on which our flipping machine is defined to act. Such as here we have $\rho_B = Tr_A (|\Psi^i\rangle_{AB}\langle\Psi^i|) = \frac{1}{3} \{ P[|0_z\rangle \otimes P[|0_z\rangle] + P[|0_y\rangle] \otimes P[|0_y\rangle] + P[|0_x\rangle] \otimes P[|0_x\rangle] \}$. The Schmidt vector corresponding to the initial joint state is $\lambda^i = \left( \frac{2}{3}, \frac{1}{6}, \frac{1}{6} \right)$. If the existence of exact flipping machine for the three states $|0_x\rangle, |0_y\rangle, |0_z\rangle$ is possible, then by applying this machine to one of the two particles on Bob’s side (say, on the last qubit system), the joint state between them exactly transforms to the state:

$$|\Psi^f\rangle_{AB} = \frac{1}{\sqrt{3}} \{ |0\rangle_A |0_z 0_z\rangle_B + e^{i\chi} |1\rangle_A |0_x 0_y\rangle_B + e^{i\eta} |2\rangle_A |0_y 0_x\rangle_B \}$$  \hspace{1cm} (5.3)

where $e^{i\chi}, e^{i\eta}$ are some arbitrary phase-factors. The Schmidt vector corresponding to the final joint state is $\lambda^f = \left( \frac{1}{3} + \frac{1}{2\sqrt{3}}, \frac{1}{3}, \frac{1}{3} - \frac{1}{2\sqrt{3}} \right)$. From Eq.4.14 it is easy to check that $|\Psi^i\rangle_{AB}, |\Psi^f\rangle_{AB}$ is a pair of incomparable states. Hence by Nielsen’s criterion it is impossible to locally transform $|\Psi^i\rangle_{AB}$ to $|\Psi^f\rangle_{AB}$ with certainty. In this example, we see that the impossibility of an operation in quantum mechanics can be established from the contradiction that it forces two incomparable states to become comparable by LOCC with certainty.

The result shows, how a local anti-unitary operation evolves the system in an unphysical way. To explore this unphysical nature of anti-unitary operators, we have considered in example-1, a joint system between some separated parties, because on a single system it is really difficult to distinguish unitary and anti-unitary operators. Now, applying an anti-unitary operator (say, L) locally on a joint system of $3 \times (2 \times 2)$ dimension, i.e., applying the operator $I_A \otimes (I \otimes L)_B$ on the joint system (where I indicates, identity operator), we find a case of incomparability. The bi-partite entanglement of the joint system changes and it changes in such a manner that there is no way to compare the initial and final joint states locally. Next, we provide another example that shows a reverse phenomena.

**Example-2:** Consider a pair of pure entangled states, shared between Alice and Bob, in the form:

$$|\Psi\rangle_{AB} = \sqrt{.51} |0\rangle_A |0\rangle_B + \sqrt{.30} |1\rangle_A |1\rangle_B + \sqrt{.19} |2\rangle_A |2\rangle_B,$$

$$|\Phi\rangle_{AB} = \sqrt{.49} |0\rangle_A |0\rangle_B + \sqrt{.36} |1\rangle_A |1\rangle_B + \sqrt{.15} |2\rangle_A |2\rangle_B.$$  \hspace{1cm} (5.4)
The Schmidt vectors corresponding to the states $|\Psi\rangle_{AB}$ and $|\Phi\rangle_{AB}$ are \( \lambda^\Psi = (0.51, 0.30, 0.19) \) and \( \lambda^\Phi = (0.49, 0.36, 0.15) \) respectively. Using Eq. (4.14) we find, \( (|\Psi\rangle_{AB}, |\Phi\rangle_{AB}) \) is a pair of incomparable states. Suppose Bob has a two qubit system on his side and the orthogonal states \( \{|0\rangle_B, |1\rangle_B, |2\rangle_B\} \) have the form,

\[
|0\rangle_B = |\psi\rangle_B |\psi\rangle_B, |1\rangle_B = |\Psi\rangle_{B_1} |\Psi\rangle_{B_2}, |2\rangle_B = |\Phi\rangle_{B_1} |\Phi\rangle_{B_2},
\]

where \( |\psi\rangle \) is an arbitrary qubit state with Bloch vector \( \vec{n}_\psi \), i.e., \( |\psi\rangle\langle\psi| = \frac{1}{2} [I + \vec{n}_\psi \cdot \vec{\sigma}] \) and \( |\Phi\rangle \) is orthogonal to \( |\psi\rangle \).

Now tracing out Alice’s system and the second qubit of Bob’s side (i.e., system \( B_2 \)), one reduced subsystem of Bob corresponding to the two joint systems in Eq. (5.4) will be of the form,

\[
\rho^\Psi_{\overline{B}_1} = Tr_{AB_2} \{|\Psi\rangle\langle\Psi|\} = \frac{1}{2} [I + 0.02 \vec{n}_\psi \cdot \vec{\sigma}]
\]

\[
\rho^\Phi_{\overline{B}_1} = Tr_{AB_2} \{|\Phi\rangle\langle\Phi|\} = \frac{1}{2} [I - 0.02 \vec{n}_\psi \cdot \vec{\sigma}].
\]

The states of the above equation are two mixed qubit states with spin-polarization directions along two exactly opposite vectors. Here, we have extended the idea of flipping operation from only pure qubit states to general qubit states. Thus, by applying any set of local operations, if it is possible to transform the joint state between Alice and Bob from \( |\Psi\rangle_{AB} \) to \( |\Phi\rangle_{AB} \), then the reduced state of one qubit subsystem on Bob’s side will be changed from \( \rho^\Psi_{\overline{B}_1} \) to \( \rho^\Phi_{\overline{B}_1} \) exactly, by LOCC with certainty. It is clear that the spin direction \( \vec{n}_\psi \) of the arbitrary qubit state \( |\psi\rangle \) is reversed after the operation, i.e., transformed to \( \vec{n}_{\overline{\psi}} \). So, if we extend the LOCC transformation criterion so that the states \( |\Psi\rangle_{AB}, |\Phi\rangle_{AB} \) are interconvertible by some operation then consequently on a subsystem, the spin-polarization of an arbitrary qubit state is being reversed. This is quite similar of preparing an arbitrary spin flipper machine and is an alternative way of establishing the incomparable nature of the pair of states in Eq. (5.4).

### 5.2.1 Spin-flipping for mixed qubit

As discussed in the second chapter, the general form of any qubit system in Bloch sphere representation is given by, \( \rho = \frac{1}{2} (I + \vec{n} \cdot \vec{\sigma}) \); \( |\vec{n}| \leq 1 \), where \( \vec{n} \) denotes the Bloch vector of the state or the direction of the spin polarization of the state. Bloch sphere is a three-dimensional geometrical visualization of the state of the system and it is reasonable to define the flipped state to be the state, corresponding to the direction that is exactly opposite to the direction \( \vec{n} \). The new direction representing the mixed qubit will obviously be denoted by \( -\vec{n} \), preserving the previous length of the vector \( |\vec{n}| \). Thus, mixed qubit flipping may be described as reversal of the direction of spin polarization of the qubit. The operation can be described as, \( \Gamma : \rho^i \longrightarrow \rho^f \) where the initial mixed state is \( \rho^i = \frac{1}{2} \{I + \vec{n} \cdot \vec{\sigma}\} \) and the flipped state will be of the form \( \rho^f = \frac{1}{2} \{I - \vec{n} \cdot \vec{\sigma}\} \), with \( |\vec{n}| \leq 1 \).
5.3 Non-existence of Universal exact flipper

To generalize the main result corresponding to the first example, we consider three arbitrary states not lying in one great circle in their simplest form,

\[
|0\rangle, \\
|\psi\rangle = a|0\rangle + b|1\rangle, \\
|\phi\rangle = c|0\rangle + d e^{i\theta}|1\rangle
\]

(5.6)

where a, b, c, d are real numbers satisfying the relation \(a^2 + b^2 = 1 = c^2 + d^2\) and \(0 < \theta < \pi\).

We call these three states as three arbitrary qubits, as by suitable change of basis it is possible to express any three qubits in this form.

Suppose two spatially separated parties Alice and Bob are sharing the entangled state,

\[
|\Omega\rangle_{AB} = \frac{1}{\sqrt{3}} \left( |0\rangle_A|00\rangle_B + |1\rangle_A|\psi\rangle_B + |2\rangle_A|\phi\rangle_B \right)
\]

(5.7)

where Alice has a 3-dimensional orthogonal local system, having the basis, \{|0\rangle, |1\rangle, |2\rangle\} and Bob has a two qubit system.

Using Eq. (5.6) one can rewrite the joint system shared between A and B in the usual basis as,

\[
|\Omega\rangle_{AB} = \frac{1}{\sqrt{3}} \left\{ \right. \\
\left. |0\rangle_A|00\rangle_B + |1\rangle_A((a|0\rangle + b|1\rangle)(c|0\rangle + d e^{i\theta}|1\rangle))B \right. \\
+ |2\rangle_A((c|0\rangle + d e^{i\theta}|1\rangle)(a|0\rangle + b|1\rangle))B \right\}
\]

(5.8)

Tracing out the part of the state on Bob’s side we focus on the reduced density matrix of Alice’s side, as follows:

\[
\rho^i_A = Tr_B(|\Omega\rangle_{AB}\langle\Omega|)
\]

\[
= \frac{1}{3} \left\{ P|0\rangle + ac|1\rangle + ac|2\rangle \right. \\
+ P|b|1\rangle + ad e^{i\theta}|2\rangle + b^2 d^2 P|1\rangle + |2\rangle \left. \right\}
\]

(5.9)

where to avoid notational complexity, we drop above the suffix \(A\) on the R.H.S. for the subsystem of Alice. In the sequel, sometimes we will also drop suffix \(A\) or \(B\) where the case may be.

Consider now that Bob has a exact flipping machine defined on just these three states \(|0\rangle, |\psi\rangle, |\phi\rangle\). The flipping operation can be described as:

\[
|0\rangle \longrightarrow |1\rangle \\
|\psi\rangle \longrightarrow e^{i\mu}|\psi\rangle = e^{i\mu}(b|0\rangle - a|1\rangle) \\
|\phi\rangle \longrightarrow e^{i\nu}|\phi\rangle = e^{i\nu}(d e^{-i\theta}|0\rangle - c|1\rangle)
\]

(5.10)
where \(|\psi\rangle, |\phi\rangle\) are the states orthogonal to \(|\psi\rangle, |\phi\rangle\) respectively and \(e^{i\mu}\) and \(e^{i\nu}\) are some arbitrary phase factors. Now assume that Bob applies the above mentioned flipping machine on any one of his two subsystems (say, on the second subsystem). After this local operation on Bob’s subsystem, the shared state between Alice and Bob takes the form,

\[
|\Omega^f_{AB}\rangle = \frac{1}{\sqrt{3}}\{|0\rangle_A|01\rangle_B + e^{i\nu}|1\rangle_A|\psi\rangle B + e^{i\mu}|2\rangle_A|\psi\rangle B\}
\]

\[
= \frac{1}{\sqrt{3}}\{|0\rangle_A|01\rangle_B + e^{i\nu}|1\rangle_A((a|0\rangle + b|1\rangle)(d e^{-i\theta}|0\rangle - c|1\rangle))B
+ e^{i\mu}|2\rangle_A((c|0\rangle + d e^{i\theta}|1\rangle)(b|0\rangle - a|1\rangle))B\}
\]

\[
= \frac{1}{\sqrt{3}}\{(ade^{i(v-\theta)}|1\rangle + bce^{i\mu}|2\rangle)A|00\rangle_B
+ (|0\rangle - ac e^{iv}|1\rangle - ace^{i\mu}|2\rangle)A|01\rangle_B
+(bde^{i(v-\theta)}|1\rangle + bde^{i(\mu+\theta)}|2\rangle)A|10\rangle_B
+(-bce^{iv}|1\rangle + ade^{i(\mu+\theta)}|2\rangle)A|11\rangle_B\} \tag{5.11}
\]

The final density matrix of Alice’s side is:

\[
\rho^f_A = \frac{1}{3}\{P[ade^{i(v-\theta)}|1\rangle + e^{i\mu}bc|2\rangle] + P[|0\rangle - ac e^{iv}|1\rangle - ace^{i\mu}|2\rangle]
+P[bde^{i(v-\theta)}|1\rangle + e^{i\mu}bde^{i\nu}|2\rangle] + P[e^{iv}bc|1\rangle - e^{i\mu}ade^{\theta}|2\rangle]\}
\]

\[
= \frac{1}{3}\{P[|0\rangle] + P[|1\rangle] + P[|2\rangle]
-ae^{iv}e^{-i\mu}|0\rangle\langle 0| + e^{iv}|1\rangle\langle 0| + e^{-i\mu}|0\rangle\langle 2| + e^{i\mu}|2\rangle\langle 0|
+ (|\phi\rangle|\psi\rangle)^2 e^{i(v-\mu)}|1\rangle\langle 2| + (|\phi\rangle|\psi\rangle)^2 e^{i(\mu+v)}|2\rangle\langle 1|\} \tag{5.12}
\]

The eigenvalue equation for the initial local density matrix \(\rho^f_A\) is,

\[
(1 - 3\lambda)^3 - 3(1 - 3\lambda)A + B = 0 \tag{5.13}
\]

and that of the final local density matrix \(\rho^f_A\) is,

\[
(1 - 3\lambda)^3 - 3(1 - 3\lambda)A + B' = 0 \tag{5.14}
\]

where \(A = \frac{1}{3}[2a^2c^2 + |\langle \psi |\phi \rangle|^4]\), \(B = 2a^2c^2|\langle \psi |\phi \rangle|^2\) and \(B' = 2a^2c^2 \text{Re}\{\langle \phi |\psi \rangle^2\}\). It is interesting to observe that the phase factors \(e^{i\mu}, e^{i\nu}\) vanishes in this stage. So the final result obtained, doesn’t care about the phase factors related with the operation.

We find the roots of the equations (5.13) and (5.14) (i.e., initial and final eigenvalues) and compare them by using Cardan’s method. This method is applied to find roots of the cubic equation of the form:

\[
x^3 - 3Gx + H = 0 \tag{5.15}
\]

with \(G \geq 0\). The roots of this equation will denoted by \(x_1, x_2, x_3\) and may be expressed in the following form:

\[
x_1 = 2\sqrt{G}\cos\left(\frac{2\pi}{3} + \alpha\right),
\]

\[
x_2 = 2\sqrt{G}\cos(\alpha),
\]

\[
x_3 = 2\sqrt{G}\cos\left(\frac{2\pi}{3} - \alpha\right) \tag{5.16}
\]
where \( \cos(3\alpha) = \frac{-H}{2\sqrt{\Omega}} \). By suitable change of the variables, our initial and final eigenvalue equations can be transformed to the above form of Eq. (5.15). If we denote the initial and final eigenvectors as \( \vec{\lambda} = (\alpha_1, \alpha_2, \alpha_3) \) and \( \vec{\lambda}' = (\beta_1, \beta_2, \beta_3) \), then we can express them as

\[
\begin{align*}
\alpha_1 &= \frac{1}{3} \{1 - 2\sqrt{A} \cos \left( \frac{2\pi}{3} + \theta^i \right) \}, \\
\alpha_2 &= \frac{1}{3} \{1 - 2\sqrt{A} \cos \theta^i \}, \\
\alpha_3 &= \frac{1}{3} \{1 - 2\sqrt{A} \cos \left( \frac{4\pi}{3} - \theta^i \right) \}
\end{align*}
\]

and

\[
\begin{align*}
\beta_1 &= \frac{1}{3} \{1 - 2\sqrt{A} \cos \left( \frac{2\pi}{3} + \theta^f \right) \}, \\
\beta_2 &= \frac{1}{3} \{1 - 2\sqrt{A} \cos \theta^f \}, \\
\beta_3 &= \frac{1}{3} \{1 - 2\sqrt{A} \cos \left( \frac{4\pi}{3} - \theta^f \right) \}
\end{align*}
\]

where, \( \cos(3\theta^i) = \frac{-B}{2\sqrt{A}} \), and \( \cos(3\theta^f) = \frac{-B'}{2\sqrt{A'}} \).

The eigenvectors \( \vec{\lambda} \) and \( \vec{\lambda}' \) are not the Schmidt vectors as the eigenvalues are not arranged in decreasing order. However, we will establish the relation of incomparability as given in Eq. (4.14), by rearranging the eigenvalues in decreasing order. Now to compare the states \( |\Omega\rangle_{AB} \) with \( |\Omega'\rangle_{AB} \) in terms of their local convertibility, we do not need the explicit values of the Schmidt coefficients but only we have to find the relations between them. To do this, we first compare the coefficients of the eigenvalue equations, \( B \) and \( B' \). Here \( B = B' + 4a^2b^2c^2d^2 \sin^2 \theta \), so \( B \geq 0, B \geq B' \). Therefore, we have two possibilities, either, \( 0 < B' < B \) or, \( B' < 0 < B \).

If \( 0 < B' < B \), we have \( 0 > \cos(3\theta^f) > \cos(3\theta^i) \). This imply, \( 3\theta^i, 3\theta^f \in \left( \frac{\pi}{2}, \frac{3\pi}{2} \right) \). We find four cases corresponding to the different regions of \( \theta^i \), and \( \theta^f \).

Firstly, we consider the case when \( 3\theta^i, 3\theta^f \in \left( \frac{\pi}{2}, \pi \right) \). We also know previously that \( \cos(3\theta^f) > \cos(3\theta^i) \). Now in the region \( \left( \frac{\pi}{2}, \pi \right) \), \( \cos(3\theta^f) > \cos(3\theta^i) : 3\theta^i > 3\theta^f \). So, we have,

\[
\begin{align*}
\frac{\pi}{6} < \theta^f < \theta^i < \frac{\pi}{3}, \\
\sqrt{3} > \cos(\theta^f) > \cos(\theta^i) > \frac{1}{2}, \\
\sqrt{3}A > 2\sqrt{A} \cos(\theta^f) > 2\sqrt{A} \cos(\theta^i) > \sqrt{A}, \\
\frac{1}{3}(1 - \sqrt{A}) > \frac{1}{3} \{1 - 2\sqrt{A} \cos(\theta^i)\} > \frac{1}{3} \{1 - 2\sqrt{A} \cos(\theta^f)\} > \frac{1}{3}(1 - \sqrt{3A})
\end{align*}
\]

i.e., \( \frac{1}{3}(1 - \sqrt{A}) > \alpha_2 > \beta_2 = \frac{1}{3}(1 - \sqrt{3A}) \).
5.3. Non-existence of Universal exact flipper

Proceeding in this way, we find,

$$\frac{\pi}{6} < \theta^f < \theta^i < \frac{\pi}{3},$$

$$\frac{2\pi}{3} + \frac{\pi}{6} < \frac{2\pi}{3} + \theta^f < \frac{2\pi}{3} + \theta^i < \frac{2\pi}{3} + \frac{\pi}{3},$$

i.e., $\frac{\pi}{3} < \theta^f < \theta^i < \frac{\pi}{2}$

$$\cos(\pi - \frac{\pi}{6}) > \cos(\frac{2\pi}{3} + \theta^f) > \cos(\frac{2\pi}{3} + \theta^i) > \cos \pi$$

$$-\frac{\sqrt{3}}{2} > \cos(\frac{2\pi}{3} + \theta^f) > \cos(\frac{2\pi}{3} + \theta^i) > -1$$

$$-\sqrt{3} > 2\sqrt{A} \cos(\frac{2\pi}{3} + \theta^f) > 2\sqrt{A} \cos(\frac{2\pi}{3} + \theta^i) > -2\sqrt{A}$$

$$\frac{1}{2}(1 + \sqrt{3}A) < \frac{1}{3}(1 - 2\sqrt{A} \cos(\frac{2\pi}{3} + \theta^f)) < \frac{1}{3}(1 - 2\sqrt{A} \cos(\frac{2\pi}{3} + \theta^i))$$

Thus, we have, $\frac{1}{3}(1 + \sqrt{3}A) < \beta_1 < \alpha_1 < \frac{1}{3}(1 + 2\sqrt{A}).$

Again,

$$\frac{\pi}{6} < \theta^f < \theta^i < \frac{\pi}{3},$$

$$\frac{2\pi}{3} - \frac{\pi}{6} > \frac{2\pi}{3} - \theta^f > \frac{2\pi}{3} - \theta^i > \frac{2\pi}{3} - \frac{\pi}{3},$$

i.e., $\frac{\pi}{3} > \theta^f > \theta^i > \frac{\pi}{2}$

$$\cos \frac{\pi}{2} < \cos(\frac{2\pi}{3} - \theta^f) < \cos(\frac{2\pi}{3} - \theta^i) < \cos \frac{\pi}{3}$$

$$0 < \cos(\frac{2\pi}{3} - \theta^f) < \cos(\frac{2\pi}{3} - \theta^i) < \frac{1}{2}$$

$$0 < 2\sqrt{A} \cos(\frac{2\pi}{3} - \theta^f) < 2\sqrt{A} \cos(\frac{2\pi}{3} - \theta^i) < \sqrt{A}$$

$$\frac{1}{3} > \frac{1}{3}(1 - 2\sqrt{A} \cos(\frac{2\pi}{3} - \theta^f)) > \frac{1}{3}(1 - 2\sqrt{A} \cos(\frac{2\pi}{3} - \theta^i))$$

Thus, the eigenvalues of $\rho^i_A, \rho^f_A$ are related as:

$$\alpha_1 > \beta_1 > \beta_2 > \alpha_3 > \alpha_2 > \beta_2.$$  (5.22)

So, by Eq.(4.14), the states $|\Omega\rangle_{AB}, |\Omega^f\rangle_{AB}$ are incomparable in this region.

In a similar manner we have investigated the other regions and we find incomparability between the initial and final bipartite states. Results of the other three possible regions for $0 < B' < B$, will be as follows:

When, $3\theta^i, 3\theta^f \in (\pi, \frac{3\pi}{2})$ then, $\alpha_1 > \beta_1 > \beta_2 > \alpha_2 > \alpha_3 > \beta_3$.

When, $3\theta^i \in (\frac{\pi}{2}, \pi)$ and $3\theta^f \in (\pi, \frac{3\pi}{2})$ then, $\alpha_1 > \beta_1 > \beta_2 > \alpha_3 > \alpha_2 > \beta_3$.

When, $3\theta^i \in (\pi, \frac{3\pi}{2})$ and $3\theta^f \in (\frac{\pi}{2}, \pi)$ then, $\alpha_1 > \beta_1 > \beta_3 > \alpha_2 > \alpha_3 > \beta_2$.

Otherwise, $B' < 0 < B$, and we have, $\cos(3\theta^f) > 0 > \cos(3\theta^i)$, which implies $3\theta^i \in (\frac{\pi}{2}, \frac{3\pi}{2})$ and $3\theta^f \in ((0, \frac{\pi}{2}) \cup (\frac{3\pi}{2}, 2\pi))$. The following subcases for different regions of $\theta^i, \theta^f$ are considered separately.

When, $3\theta^i \in (\frac{\pi}{2}, \pi)$ and $3\theta^f \in (0, \frac{\pi}{2})$ then, $\alpha_1 > \beta_1 > \beta_3 > \alpha_3 > \alpha_2 > \beta_2$.

When, $3\theta^i \in (\frac{\pi}{2}, \pi)$ and $3\theta^f \in (\frac{3\pi}{2}, 2\pi)$ then, $\alpha_1 > \beta_1 > \beta_2 > \alpha_3 > \alpha_2 > \beta_3$. 


5.4. Possible physical reason behind this connection

When, $3\theta^i \in (\pi, \frac{3\pi}{2})$ and $3\theta^f \in (0, \frac{\pi}{2})$ then, $\alpha_1 > \beta_1 > \beta_3 > \alpha_2 > \alpha_3 > \beta_2$.
When, $3\theta^i \in (\pi, \frac{3\pi}{2})$ and $3\theta^f \in (\frac{3\pi}{2}, 2\pi)$ then, $\alpha_1 > \beta_1 > \beta_2 > \alpha_2 > \alpha_3 > \beta_3$.
In all the above cases, the results show that the states $|\Omega\rangle_{AB}$ and $|\Omega^f\rangle_{AB}$ are incomparable in nature.

Equations (5.13) and (5.14) will be identical (and hence the eigen vectors of $\rho^i_A$ and $\rho^f_A$) only when $B = B'$, which imply $abcd \sin \theta = 0$, i.e., the three input states $|0\rangle, |\psi\rangle, |\phi\rangle$, on which the flipping machine is defined, will lie on one great circle of the Bloch sphere. This is clear from the fact that there exists exact flipping machine for the set of states taken from one great circle.

Thus, if the exact flipping machine does exist, and is applied locally on one subsystem of the initial pure bipartite state, then an impossible transformation is shown to occur. Obviously this impossibility comes through our assumption on the existence of universal exact flipping machine. It is interesting to observe that the arbitrary phase factor of the flipping operation we have considered, does not make a difference in the result obtained.

5.4 Possible physical reason behind this connection

In this chapter, we have considered first two examples by which we are able to show, how an impossible local operation is connected with the restrictions imposed on state transformations by LOCC. One of the root of this connection is the anti-unitary nature of the exact universal spin-flipping operation. In the case of state transformation criterion, the allowed local operations on each parties are such that as a whole it can be implemented by an unitary evolution and if we restrict further individual local operations as unitary then bipartite entanglement cannot be changed under such local unitary operations. We, however considered here a local operation which is anti-unitary and it is observed that anti-unitary operator acts in a nonphysical way.

Also, the above results show an interesting interplay between the notion of incomparability and no-flipping principle. It indicates no-flipping can be used to determine the interrelations between LOCC and entanglement behavior of the quantum system. We observe that the incomparability criterion of local state transformations is also capable of revealing some more fundamental properties of the quantum systems. It can detect operations which are nonphysical in nature, such as, here it is anti-unitary. Naturally one could conjecture that the two impossibilities are equivalent, as they both require anti-unitary operators. Our results support this conjecture. It also exhibit the impossibility of extending LOCC operations to incorporate anti-unitary operators which can create an increase of information content of the system, as anti-parallel spin states contain more information than that of the parallel ones. In the next chapter, we will find incomparability as a new detector of impossible operations.
Chapter 6

General Impossible Operations

6.1 Physical Operations and LOCC

Quantum systems allow physical operations to perform some tasks that seems to be impossible in classical domain. However, varying with the nature of the operations performed, there are several restrictions imposed on correctness or exact behavior of the operations to act for the whole class of states of the quantum system. Possibilities or impossibilities of various kind of such operations acting on some specified system is thus one of the basic tasks of quantum information processing. In case of cloning and deleting, the input states must be orthogonal to each other for the exactness of the operations performed. Rather, if the operation considered is spin-flipping or Hadamard type then the allowable set of input states enhanced to the infinite set of qubits defined by any great circle of the Bloch sphere. It indicates that any angle preserving operation has some restrictions imposed on the set of allowable input states. The unitary nature of all physical evolution raised the question that whether the non-physical nature of the anti-unitary operations is a natural constraint over the system or not. In other words, it is nice to show how an impossible operation like anti-unitary, evolve with the physical systems concerned.

First part of this chapter concerned with a connection between general anti-unitary operations and evolution of a joint system through local operations together with classical communications, in short, LOCC. Some constraint over the system are always imposed by the condition that the system is evolved under LOCC. Such as, performing any kind of LOCC, the amount of entanglement between some spatially separated subsystems can not be increased. If we further assume that the concerned system is pure bipartite, then by Nielsen’s criteria it is possible to determine whether a pure bipartite state can be transformed to another pure bipartite state with certainty by LOCC or not. Consequently, we find earlier that there are pairs of pure bipartite

\[ \text{References:} \]

\[ \text{Footnote:} \]

\[ \text{1Some portions of this chapter is published in Quantum Information and Computation, 7(4), 392 (2007).} \]
states, known as incomparable states which are not interconvertible by LOCC with certainty. The existence of such class of states prove that the amount of entanglement content does not always determine the possibility of exact transformation of a joint system by applying LOCC. Now, we first pose the problem of detecting the possibility of existence of some operations. In this chapter, we will able to detect some single qubit impossible operations.

Suppose, $\rho_{ABCD...}$ be a state shared between distinct parties situated at distant locations. They are allowed to do local operations on their subsystems and also they may communicate any amount of classical information among themselves. But they do not know whether their local operations are valid physical operations or not. By valid physical operation, we mean a completely positive map (may be trace-preserving or not) acting on the physical system. Sometimes an operation is confusing in the sense that it works as a valid physical operation for a certain class of states but not as a whole. Therefore, they want to judge their local operations using quantum formalisms or with other physical principles, may be along with quantum formalism or may not be. No-signalling, non-increase of entanglement by LOCC are some of the good detectors to detect nonphysical operations. In this chapter, we want to establish another good detector for a large number of nonphysical operations. The existence of incomparable states enables us to find this good detector. Suppose, $L_A \otimes L_B \otimes L_C \otimes L_D \otimes \cdots$ be an operation acting on the physical system described by the state $\rho_{ABCD...}$ and $\rho'_{ABCD...}$ be the transformed state. Now, if it is known that the states $\rho_{ABCD...}$ and $\rho'_{ABCD...}$ are incomparable by the action of any deterministic LOCC, then we can certainly say that at least one of the operations $L_A, L_B, L_C, L_D, \cdots$ are nonphysical. Therefore, if somehow we find two states that are incomparable and an operation acting on the local system of any party (or a number of parties) of one state, can transform it to another state, then we certainly claim that the operation is a nonphysical one.

6.1.1 Separable Superoperator, LOCC and impossible operations

In this section, we recall the notion of a physical operation in the sense of Kraus [91]. Suppose a physical system is described by a state $\rho$. By a physical operation on $\rho$, we mean a completely positive map $\mathcal{E}$ acting on the system and described by

$$\mathcal{E}(\rho) = \sum_k A_k \rho A_k^\dagger$$

where each $A_k$ is positive linear operator that satisfies the relation $\sum_k A_k^\dagger A_k \leq I$. If $\sum_k A_k^\dagger A_k = I$, then the operation is known trace preserving. When the state is shared between a number of parties, say, A, B, C, D, .... and each $A_k$ has the form $A_k = L_k^A \otimes L_k^B \otimes L_k^C \otimes L_k^D \otimes \cdots$ with all the $L_k^A, L_k^B, L_k^C, L_k^D, \cdots$ are linear positive operators, the operator is then called as a separable superoperator. In this context, we want to mention an interesting result concerned with LOCC. Every LOCC is a separable superoperator but it is not known whether the converse is also true or not. It is affirmed that there are separable superoperators which cannot be expressed by finite LOCC
Now, if a physical system evolved under LOCC (may be deterministic or stochastic) then quantum mechanics does not allow the system to behave arbitrarily. More precisely, under the action of any LOCC one fundamental constraint arises for any entangled system. The content of entanglement will not increase under LOCC. This is usually known as principle of non-increase of entanglement under LOCC. Further, for any closed system as unitarity is the only possible evolution, the constraint is then: the entanglement content will not change under LOCC. So, if we find some violation of these principles under the action of any local operation, then we certainly claim that the operation is not a physical one. No-flipping [35], no-cloning, no-deleting, all those theorems are already established with these principles, basically with the principles of non-increase of entanglement. These kind of proof for those important no-go theorems will always give us a more powerful physically intuitive approaches for quantum information processing, apart from the mathematical proofs that the dynamics should be linear as well as unitary. Linearity and unitarity are the building blocks of every physical operation. But within the quantum formalism we always search for more and more new physical situations that are much useful and intuitive for quantum information processing. Existence of incomparable states in pure bipartite entangled states allow us to use it as a new detector. In particular, we have proved three impossibilities, viz., exact cloning, deleting [24] and flipping operations [36] by the existence of incomparable states under LOCC.

### 6.1.2 Two Examples of Impossible operations

**Spin flipping:** Exact flipping operation $\Omega$ acting on an arbitrary qubit, $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$; $|\alpha|^2 + |\beta|^2 = 1$ is defined by,

$$|\phi\rangle = \Omega|\psi\rangle = |\overline{\psi}\rangle$$

where, $\langle\phi|\psi\rangle = 0$, i.e., $|\phi\rangle = \beta^*|0\rangle - \alpha^*|1\rangle$.

This operation is not in general a physical operation. The impossibility of the operation occurs due to arbitrariness of the input states. i.e., it works exactly for a class of states, but not for the whole class.

**Hadamard gate:** Universal Hadamard gate $\Lambda$ acting on the arbitrary qubit $|\psi\rangle$ defined by,

$$|\phi\rangle = \Lambda|\psi\rangle = \frac{1}{\sqrt{2}}(|\psi\rangle + i|\overline{\psi}\rangle)$$  \hspace{1cm} (6.2)

This is also an impossible operation for single-qubit system [112].

### 6.2 Anti-Unitary operators and Incomparability

**General class of anti-unitary operations:** A general class of anti-unitary operations can be defined in the form, $\Gamma = CU$; where $C$ is the conjugation operation and $U$ be the most general
6.2.2 Relation with incomparability

To prove that the operation $\Gamma$ is nonphysical and its existence leads to an impossibility, we choose particular pure bipartite state $|\chi^i\rangle_{AB}$ shared between two spatially separated parties Alice and Bob in the form,

$$|\chi^i\rangle_{AB} = \frac{1}{\sqrt{3}} \left[ |0\rangle_A |0\rangle_B + |0\rangle_A |1\rangle_B + |1\rangle_A |0\rangle_B \right]$$

(6.4)

The Schmidt vector corresponding to the initial state $|\chi^i\rangle_{AB}$ is $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Assuming that Bob operates $\Gamma$ on one of his two qubits, say, on the last qubit, the joint state shared between Alice and Bob will transform to:

$$|\chi^f\rangle_{AB} = \frac{1}{\sqrt{3}} \left[ |0\rangle_A |0\rangle_B \Gamma(|0\rangle_B) + |1\rangle_A |0\rangle_B \Gamma(|0\rangle_B) \right]$$

(6.6)

Tracing out Bob’s subsystem we again consider the reduced density matrix of Alice’s subsystem. The final reduced density matrix is

$$\rho^f_A = \frac{1}{3} \left[ P[|0\rangle] + P[|1\rangle] + \frac{1}{2} (|0\rangle \langle 1| + |1\rangle \langle 0|) + \frac{1}{2} (|0\rangle \langle 2| + |2\rangle \langle 0|) + |i\rangle \langle i| + |i\rangle \langle i|) \right]$$

(6.7)
The Schmidt vector corresponding to the final state $|\chi_f\rangle_{AB}$ is $(\frac{1}{3} + \frac{1}{2\sqrt{3}}, \frac{1}{3}, \frac{1}{3} - \frac{1}{2\sqrt{3}})$. Interestingly, the Schmidt vector of the final state does not contain the arbitrary parameters of anti-unitary operator $\Gamma$. It is now easy to check that the final and initial Schmidt vectors are incomparable as, $\frac{2}{3} > \frac{1}{3} + \frac{1}{2\sqrt{3}} > \frac{1}{3} > \frac{1}{6} > \frac{1}{3} - \frac{1}{2\sqrt{3}}$. Thus we have, $|\chi_i\rangle \not\leftrightarrow |\chi_f\rangle$ so that the transformation of the pure bipartite state $|\chi_i\rangle$ to $|\chi_f\rangle$ by LOCC with certainty is not possible following Nielsen’s criteria. Though by applying the anti-unitary operator $\Gamma$ on Bob’s local system the transformation $|\chi_i\rangle \rightarrow |\chi_f\rangle$ is performed exactly. This impossibility emerges out of the impossible operation $\Gamma$ which we have assumed to be exist and apply it to generate the impossible transformation.

If, instead of operating $\Gamma = CU$, we will operate only $U$, i.e., the general unitary operator, the initial and final density matrices of one side will be seen to be identical, implying that there is not even a violation of No-Signalling principle. This is true as we only operate the unitary operator on any qubit not restricting on any particular choices, such as they will act isotropically for all the qubits, etc. Thus it can not even used to send a signal here.

As a particular case, we have verified the non-existence of exact universal flipper by our above method, where we have chosen, $\theta = \pi/2, \alpha = 0, \beta = 0$.

If input states lying in one great circle:

Now, if the three input states are chosen from any great circle of the Bloch sphere, then the anti-unitary operator $\Gamma$ defined on them will exists uniquely, which is very natural, as it can be generated by a 180 degree rotation of the Bloch sphere about the axis perpendicular to that great circle.

Thus, we are now able to detect non-physical nature of anti-unitary operators through the existence of incomparable pairs of pure entangled states. In the next section, we will consider another class of operations, viz., inner product preserving operations.

### 6.3 Universal inner product preserving operations

There is an intrinsic relation between incomparability with the impossibility of some inner product preserving operations defined only on the minimum number of qubits $|0_x\rangle, |0_y\rangle, |0_z\rangle$. Here, we consider the existence of the operation defined on these three qubits in the following manner,

\[
\begin{align*}
|0_z\rangle &\quad \rightarrow \quad (\alpha|0_z\rangle + \beta|1_z\rangle), \\
|0_x\rangle &\quad \rightarrow \quad (\alpha|0_x\rangle + \beta|1_x\rangle), \\
|0_y\rangle &\quad \rightarrow \quad (\alpha|0_y\rangle + \beta|1_y\rangle),
\end{align*}
\]  

(6.8)

where $|\alpha|^2 + |\beta|^2 = 1$.

This operation exactly transforms the input qubit into an arbitrary superposition of the input qubit with its orthogonal one. To verify the possibility or impossibility of existence of this
operation as physical one, we consider a pure bipartite state shared between Alice and Bob as follows:

\[ |\Pi_i^{(f)}\rangle_{AB} = \frac{1}{\sqrt{3}} \{ |0\rangle_A (|0_z\rangle |0_z\rangle)_B + |1\rangle_A (|0_x\rangle |0_x\rangle)_B + |2\rangle_A (|0_y\rangle |0_y\rangle)_B \} \]  \hspace{1cm} (6.9)

Then, the reduced density matrix of Alice’s side will be of the form,

\[ \rho_A^i = \frac{1}{3} \left\{ P[|0\rangle] + P[|1\rangle] + P[|2\rangle] + \frac{1}{2}\langle 1|0\rangle + \frac{1}{2}\langle 1|0\rangle + \langle 1|0\rangle \right\} \]  \hspace{1cm} (6.10)

Therefore, the Schmidt vector corresponding to the initial joint state \( |\Pi_i^{(f)}\rangle_{AB} \) is, \( \left( \frac{1}{3} + \frac{1}{2\sqrt{3}}, \frac{1}{3}, \frac{1}{3} - \frac{1}{2\sqrt{3}} \right) \).

Suppose, Bob has now a machine that can operate on the three input qubits \( |0_x\rangle, |0_y\rangle, |0_z\rangle \) as defined in Eq. (6.8). He operates that machine on a part of his local subsystem (say, on the last qubit). Then, the joint state between Alice and Bob will be of the form:

\[ |\Pi_i^{(f)}\rangle_{AB} = \frac{1}{\sqrt{3}} \{ |0\rangle_A |0_z\rangle_B (|\alpha\rangle |\alpha\rangle + |\beta\rangle |1_z\rangle)_B + |1\rangle_A |0_x\rangle_B (|\alpha\rangle |\alpha\rangle + |\beta\rangle |1_x\rangle)_B + |2\rangle_A |0_y\rangle_B (|\alpha\rangle |\alpha\rangle + |\beta\rangle |1_y\rangle)_B \} \]  \hspace{1cm} (6.11)

And thus, the final reduced density matrix of Alice’s side will be of the form:

\[ \rho_A^f = \frac{1}{3} \left\{ P[|0\rangle] + P[|1\rangle] + P[|2\rangle] + p\langle 0|2\rangle + q\langle 1|0\rangle + r\langle 2|1\rangle \right\} \]  \hspace{1cm} (6.12)

where \( p = \frac{1}{2} \{ |\alpha|^2 - |\beta|^2 + \alpha \overline{\beta} + \beta \overline{\alpha} \}, q = \frac{1}{2} \{ |\alpha|^2 + i|\beta|^2 + \alpha \overline{\beta} - i\beta \overline{\alpha} \} \) and \( r = \frac{1}{2} \{ \alpha \overline{\beta} + \beta \overline{\alpha} - i \} \).

To compare the initial and final state, we have to check whether the initial and final eigenvalues will satisfy either of the relations of Eq. (4.14). We write, the final eigenvalue equation as:

\[ x^3 - 3Ax + B = 0 \]  \hspace{1cm} (6.13)

where, \( A = \frac{1}{4}(p\overline{\gamma} + q\overline{\gamma} + r\overline{\gamma}) \geq 0 \) and \( B = pr\overline{\gamma} + p\overline{\gamma}q \). The eigenvalues \( \{ \lambda_1, \lambda_2, \lambda_3 \} \) can then be expressed (by, Carden’s method) as:

\[ \{ \lambda_1 = \frac{1}{4} \{ 1 - 2\sqrt{A}\cos\left( \frac{2\pi}{3} + \theta \right) \}, \lambda_2 = \frac{1}{4} \{ 1 - 2\sqrt{A}\cos\left( \frac{2\pi}{3} - \theta \right) \}, \lambda_3 = \frac{1}{4} \{ 1 - 2\sqrt{A}\cos\left( \frac{2\pi}{3} - \theta \right) \} \} \]  \hspace{1cm} (6.14)

where, \( \cos(3\theta) = \frac{B}{2\sqrt{A}} \). We discuss the matter case by case.

**Case-1** : For \( B < 0 \), we observe an incomparability between the initial and final joint states if \( A = \frac{1}{4} \). In case of \( A < \frac{1}{4} \), we observe that either there is an incomparability between the initial and final states or the entanglement content of the final state is larger than that of the initial states. Lastly, if \( A > \frac{1}{4} \), we also see a case of incomparability if the condition \( 2\sqrt{A}\cos\left( \frac{2\pi}{3} + \theta \right) > -\frac{\sqrt{3}}{2} \) holds. Numerical searches also support that for real values of \( (\alpha, \beta) \) incomparability is observed almost everywhere in this region. Details are as follows:
Now, $B < 0$ implies $3\theta \in [0, \frac{\pi}{2}) \cup (\frac{3\pi}{2}, 2\pi]$. We analyze this in two parts.

If, $3\theta \in [0, \frac{\pi}{2})$ we have, $\frac{\sqrt{3}}{2} < \cos \theta \leq 1$; $\lambda_2 \in \left[\frac{1}{2}(1 - 2\sqrt{A}), \frac{1}{2}(1 - \sqrt{3A})\right]$. Again, $0 \leq \theta < \frac{\pi}{6}$; $-\frac{\sqrt{3}}{2} < \cos(\frac{\pi}{2} - \theta) \leq -\frac{1}{2}; \lambda_1 \in \left[\frac{1}{2}(1 + \sqrt{A}), \frac{1}{2}(1 + \sqrt{3A})\right]$. Finally, $0 \leq \theta < \frac{\pi}{6}$: $\cos(\frac{\pi}{2} - \theta) \in [-\frac{1}{2}, 0]; \lambda_3 \in \left(\frac{1}{2}, \frac{1}{2}(1 + \sqrt{A})\right]$.

Otherwise, $3\theta \in (\frac{3\pi}{2}, 2\pi]$, i.e., $\theta \in (\frac{\pi}{6}, \frac{\pi}{2}]$, we have, $\lambda_3 \in \left[\frac{1}{2}(1 - 2\sqrt{A}), \frac{1}{2}(1 - \sqrt{3A})\right)$, $\lambda_2 \in \left[\frac{1}{2}(1 + \sqrt{A}), \frac{1}{2}(1 + \sqrt{3A})\right)$ and $\lambda_1 \in \left(\frac{1}{2}, \frac{1}{2}(1 + \sqrt{A})\right]$.

So, in both the cases, $\lambda_{f_{\text{MIN}}}^f \in \left[\frac{1}{2}, \frac{1}{2}(1 + \sqrt{3A})\right)$ and $\lambda_{f_{\text{MAX}}}^f \in \left[\frac{1}{2}(1 - 2\sqrt{A}), \frac{1}{2}(1 - \sqrt{3A})\right)$.

Thus, for $A = \frac{1}{4}$, we observe that $\lambda_{f_{\text{MIN}}}^f \in [0, \frac{1}{4}(1 - \sqrt{3})]) < \lambda_{f_{\text{MIN}}}^f$ and $\lambda_{f_{\text{MAX}}}^f \in \left[\frac{1}{2}, \frac{1}{2}(1 + \sqrt{3})\right)$, which implies $|\Pi^i\rangle_{AB}, |\Pi^f\rangle_{AB}$ are incomparable.

If, $A < \frac{1}{4}$ then, $\lambda_{f_{\text{MAX}}}^f \leq \lambda_{f_{\text{MIN}}}^f$. So, in case of $\lambda_{f_{\text{MIN}}}^f \leq \lambda_{f_{\text{MIN}}}^f$, the states $|\Pi^i\rangle_{AB}, |\Pi^f\rangle_{AB}$ are incomparable, otherwise, we have $\lambda_{f_{\text{MIN}}}^f \geq \lambda_{f_{\text{MIN}}}^f$, and then $E(|\Pi^i\rangle_{AB}) < E(|\Pi^f\rangle_{AB})$. For real values of $\alpha, \beta$, we can express $A, B$ as:

$$\begin{align*}
A &= \frac{1}{4} + \frac{1}{6}[2\alpha^2\beta^2 + 3\alpha\beta(\alpha^2 - \beta^2)] \\
B &= \frac{\beta}{4}(\alpha^2 - \beta^2 + 2\alpha\beta)[\alpha(2\alpha^2 + 1) + \beta(\alpha^2 - \beta^2)]
\end{align*}$$

(6.15)

Numerical evidences support that for real $\alpha, \beta$ most of the cases show incomparability between $|\Pi^i\rangle_{AB}, |\Pi^f\rangle_{AB}$.

Lastly if $A \geq \frac{1}{4}$, then $\lambda_{f_{\text{MAX}}}^f < \lambda_{f_{\text{MAX}}}^f$. Thus incomparability between $|\Pi^i\rangle_{AB}, |\Pi^f\rangle_{AB}$ will hold if $\lambda_{f_{\text{MIN}}}^f < \lambda_{f_{\text{MIN}}}^f$. For this, we get the condition that $2\sqrt{A}\cos\phi < \frac{\sqrt{3}}{2}$, where $\phi = \min\{\theta, \frac{2\pi}{3} - \theta\} \in (\frac{\pi}{6}, \frac{\pi}{2})$. For real values of $\alpha, \beta$ from Eq. (6.15), we see that $A > \frac{1}{4}$ implies $\alpha\beta(\alpha^2 - \beta^2) > 0$ if $B > 0$. Thus, for real $\alpha, \beta$, this subcase does not arise.

**Case-2:** For $B = 0$, we found that there does not arise a case of incomparability. It is also seen that there is always an increase of entanglement by LOCC if $A < \frac{1}{4}$, which is the only possibility for real values of $\alpha, \beta$.

Here, the final eigenvalues are $\left\{\frac{1}{2}(1 + \sqrt{3A}), \frac{1}{2}, \frac{1}{2}(1 - \sqrt{3A})\right\}$. Thus, $E(|\Pi^i\rangle_{AB}) \geq E(|\Pi^f\rangle_{AB})$ for $A \geq \frac{1}{4}$. Incomparability between the initial and final joint states $|\Pi^i\rangle_{AB}, |\Pi^f\rangle_{AB}$ will not occur in this case.

Hence, for all values of $\alpha, \beta$ for which $A < \frac{1}{4}$, there is an increase of entanglement by applying the local operation defined in Eq. (6.8) in Bob’s system. This impossibility indicates the impossibility of the operation defined in Eq. (6.8) for those values of $\alpha, \beta$ which satisfy $A < \frac{1}{4}$.

And for real values of $\alpha, \beta$, in all possibilities for $B = 0$ we have $A < \frac{1}{4}$. This case also always shows an increase of entanglement.

**Case-3:** $B > 0$. In this case, we also find similar results like Case-1, only the condition for incomparability in case $A > \frac{1}{4}$ if changed to the form $2\sqrt{A}\cos\phi < \frac{\sqrt{3}}{2}$, where $\phi = \min\{\theta, \frac{2\pi}{3} - \theta\} \in (\frac{\pi}{6}, \frac{\pi}{2})$. It must be noted here that for real values of $\alpha, \beta$ this subcase do not arise at all. Now, $3\theta \in (\frac{\pi}{2}, \frac{3\pi}{2}); \theta \in (\frac{\pi}{6}, \frac{\pi}{2})$: $\cos \theta \in (\frac{\sqrt{3}}{2}, 0); \lambda_2 \in (\frac{1}{2}(1 - \sqrt{3A}), \frac{1}{2})$. 


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Again, \( \theta \in \left( \frac{\pi}{6}, \frac{\pi}{2} \right) : \cos \left( \frac{2\pi}{3} + \theta \right) \in (-1, -\sqrt{3}/2) : \lambda_1 \in \left( \frac{1}{3}(1 + \sqrt{3}A), \frac{1}{3}(1 + 2\sqrt{A}) \right) \). Lastly, \( \theta \in \left( \frac{\pi}{6}, \frac{\pi}{2} \right) : \cos \left( \frac{2\pi}{3} - \theta \right) \in (0, \sqrt{3}/2) : \lambda_3 \in \left( \frac{1}{3}(1 - \sqrt{3}A), \frac{1}{3} \right) \).

Hence, in this case, \( \lambda_{\text{MAX}}^f \in \left( \frac{1}{3}(1 + \sqrt{3}A), \frac{1}{3}(1 + 2\sqrt{A}) \right) \) and \( \lambda_{\text{MIN}}^f \in \left( \frac{1}{3}(1 - \sqrt{3}A), \frac{1}{3} \right) \).

Thus, for \( A = \frac{1}{4} \) we have, \( \lambda_{\text{MAX}}^i < \lambda_{\text{MAX}}^f \) and \( \lambda_{\text{MIN}}^i < \lambda_{\text{MIN}}^f \). These two relations together implies that \( |\Pi_i\rangle_{AB}, |\Pi_f\rangle_{AB} \) are incomparable.

Also, for \( A \leq \frac{1}{4} \) we see, \( \lambda_{\text{MIN}}^f > \lambda_{\text{MIN}}^i \). Thus, if \( \lambda_{\text{MAX}}^f > \lambda_{\text{MAX}}^i \), then the states \( |\Pi_i\rangle_{AB}, |\Pi_f\rangle_{AB} \) are incomparable or, if \( \lambda_{\text{MAX}}^f < \lambda_{\text{MAX}}^i \), then \( E(\Pi_{AB}) < E(\Pi_{AB}) \).

Lastly, if \( A \geq \frac{1}{4} \), then \( \lambda_{\text{MAX}}^i < \lambda_{\text{MAX}}^f \). Incomparability between the initial and final joint states \( |\Pi_i\rangle_{AB}, |\Pi_f\rangle_{AB} \) will hold if \( \lambda_{\text{MIN}}^i < \lambda_{\text{MIN}}^f \). For this we get the condition that \( 2\sqrt{A} \cos \left( \frac{2\pi}{3} + \theta \right) > -\sqrt{\frac{3}{2}} \). From Eq. (6.15) we find, for real values of \( \alpha \) and \( \beta \), numerical results also support that in most of the cases there is an incomparability between \( |\Pi_i\rangle_{AB}, |\Pi_f\rangle_{AB} \).

In particular, if we check the values of \( \alpha, \beta \) be such that they represent the operations flipping (i.e., \( \alpha = 0 \)) and Hadamard (i.e., \( \alpha = \beta = \frac{1}{\sqrt{2}} \)) respectively, then we find from the above that in both the cases the initial and final states are incomparable.

In almost all the cases above, we find some kind of violation of physical laws, which implies that in general, most of the inner product preserving operations defined only on three states is non-physical in nature and we observe for a large class of such inner-product-preserving operation incomparability senses efficiently as a good detector.

6.4 Incomparability as a detector of Impossible operations

Thus, in this chapter together with the previous chapter, we find how local state transformation criteria and correspondingly the existence of incomparable states provide a new direction of search for the possibility and impossibility of various local operations. This is sometimes more preferable than the constraint on local evolution of entangled state with the amount of entanglement content. Sometimes the non-increase of entanglement under LOCC, may become less powerful in showing such violations. Many known and unknown impossibilities may be easily detected by our above method [24].
Chapter 7

Bound entanglement-Strange but strong resource

7.1 Bound entanglement exists in nature

Bound entanglement is quite a similar notion with that of bound energy in classical physics. This is a peculiar kind of characteristic, observed in bi-partite as well as multi-partite systems. Pure state entanglement in bi-partite case, is reversible in nature. For any pure bi-partite entangled state $|\phi\rangle_{AB}$, it is physically possible to extract out all the entanglement required to prepare the state, i.e., $E_F(|\phi\rangle_{AB}) = E_D(|\phi\rangle_{AB}) = E(|\phi\rangle_{AB})$. This feature is remarkably absent in mixed state level.

There are strong physical restrictions on distillation process to recover all the entanglement for some mixtures. If a state is entangled, i.e., the entanglement cost of preparing such states is non-zero, but it is not possible to distill out any positive amount of entanglement from the state as long as the parties sharing the state will remain specially separated and are allowed to perform only LOCC, then the state is said to be Bound entangled. The notion of bound entanglement was first shown by Horodecki et.al. in system of $3 \times 3$ states [81].

We first discuss the example provided by P. Horodecki [80] that turns out to be a bound entangled state in $3 \times 3$ dimension. The state has the form,

$$\rho_a = \frac{8a}{8a+1} \rho_{ins} + \frac{1}{8a+1} |\phi_a\rangle\langle \phi_a|$$ (7.1)

where $\rho_{ins}$ is an inseparable state (as it has negative partial transpose) described by,

$$\rho_{ins} = \frac{1}{8} \{I + P[|00\rangle + |11\rangle + |22\rangle] - P[|00\rangle] - P[|11\rangle] - P[|22\rangle] - P[|20\rangle]\}$$ (7.2)

1Some portions of this chapter is published in Physical Review A 71, 062317 (2005).
and $|\phi_a\rangle$ be a pure product state defined as,

$$
|\phi_a\rangle = |2\rangle \otimes \left( \sqrt{\frac{1+a^2}{2}} |0\rangle + \sqrt{\frac{1-a^2}{2}} |2\rangle \right), \quad 0 \leq a \leq 1
$$

(7.3)

Now, the state $\rho_a$ is found to be PPT, but entangled. This also shows that PPT criteria is only sufficient but not a necessary condition for separability. Again, it is proved that every PPT state is undistillable [81]. Thus, existence of PPT entangled state immediately provides the example of states, that are enriched with some resources of entanglement, but it is truly impossible to extract out any non-negative amount of entanglement from those states. The states of this kind are certainly bound entangled. Next, we will show a constructive way to find bound entangled states, provided by Bennett et. al. [21]. Before going to discuss this matter, we first describe the aspect of local distinguishability of states which have a close connection with the bound entangled states.

### 7.1.1 Distinguishing entangled states by LOCC

In the first chapter, we have described some notions of non-orthogonal states and the aspect of discriminating set of states of a single quantum system. It is also discussed that the concept of non-orthogonality, exact distinguishability and exact cloning are physically equivalent. If we consider any composite system (in other words, a non-local system empowered with the source of quantum communication allowed between its subsystems) as a whole (i.e., globally), the situation appears to be similar with a single system. The situation is more complex in case of composite systems shared between different parties situated at distant locations where it becomes difficult to create a global set up to distinguish a set of states. Rather, it is much more desirable to discriminate a set of mutually orthogonal quantum states belonging to some composite systems, by allowing only local quantum operations on the subsystems together with the classical communications between different locations (i.e., by LOCC). Though performing a task via LOCC sometimes seen to be unsatisfactory. The physical reasons behind this notion are the constraints imposed on possible evolution of a physical system under local operations. For example, amount of entanglement can not be increased under LOCC. In other words, if we allow only some restricted set of operations, then it is reasonable to accomplish less satisfactory results for some specific tasks [143][144]. Naturally we expect that such differences in achievement by using global or local operations would only be reflected in some entangled system, but not in any separable set up. The first result we will discuss in this respect is quite satisfactory indeed [143][144]. It is proved that there are orthogonal composite quantum states for which the task of discrimination can be achieved locally, are as good as with global discrimination processes.

**Local distinguishability of Pure states:** Walgate et.al. [143] proved that any two pure orthogonal states $|\psi\rangle, |\phi\rangle$ bipartite or multipartite, whether entangled or separable, can always be distinguished with certainty by LOCC.
For simplicity, we only describe the bipartite case. Suppose, Alice and Bob are two persons situated at distant laboratories. Each of them share a part of some quantum system in one of the two pure quantum states $|\psi\rangle$, $|\phi\rangle$, which are orthogonal to each other. The task is to determine which state they share, by LOCC with certainty, if single copy of the states are given. Walgate et al. [143] showed that any two pure orthogonal bipartite states can always be represented by (ignoring the normalization factors),

$$
|\psi\rangle = |1\rangle_A |\eta_1\rangle_B + |2\rangle_A |\eta_2\rangle_B + \cdots + |n\rangle_A |\eta_n\rangle_B \\
|\phi\rangle = |1\rangle_A |\nu_1\rangle_B + |2\rangle_A |\nu_2\rangle_B + \cdots + |n\rangle_A |\nu_n\rangle_B
$$

(7.4)

where \{1_A, 2_A, \ldots, n_A\} form an orthonormal basis for Alice’s local system, chosen very specifically so that $\langle \eta_i | \nu_i \rangle = 0$, \forall \ i = 1, 2, \ldots, n. Either of the two sets \{|\eta_i\rangle_B; i = 1, 2, \ldots, n\}$, \{|\nu_i\rangle_B; i = 1, 2, \ldots, n\} may not necessarily constitute set of orthogonal vectors for Bob’s subsystem. Thus, the form of the two states $|\psi\rangle$, $|\phi\rangle$, immediately proves the local distinguishability of the states with certainty.

In multipartite case, the same notion could also be generalized for local distinguishability of any two pure orthogonal states. The situation drastically changes if we consider more than only two pure orthogonal states. In the simplest possible case of $2 \times 2$ system of states, there is an example of four Bell states $|\Phi^\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$, $|\Psi^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$, that are not locally distinguishable, if single copy of the states are given [66]. Actually, any three of them are locally indistinguishable. For pure $2 \times 2$ system of states, the analysis is almost complete [68] [144]. However, for higher dimensions, there is no local discriminating procedure by which we can certainly show that a set of orthogonal pure states of quantum system are locally distinguishable. For mixed states, the aspect of discrimination is much more difficult to deal with. Chefles provided a criteria for perfect discrimination of mixed states in finite case, incorporating some ideas of entanglement witness [41]. Thus, observing such kind of peculiarity in composite quantum systems, we may understood a form of non-locality plays in quantum systems [22] [145]. Now, we elaborate some cases of local indistinguishability further for higher dimensions.

Let us consider the Bennett set of pure orthogonal maximally entangled states in $3 \times 3$ system represented by,

$$
|\Psi_1\rangle = \frac{1}{\sqrt{3}} \{|0\rangle_A |0\rangle_B + |1\rangle_A |1\rangle_B + |2\rangle_A |2\rangle_B\} \\
|\Psi_2\rangle = \frac{1}{\sqrt{3}} \{|0\rangle_A |1\rangle_B + |1\rangle_A |2\rangle_B + |2\rangle_A |0\rangle_B\} \\
|\Psi_3\rangle = \frac{1}{\sqrt{3}} \{|0\rangle_A |2\rangle_B + |1\rangle_A |0\rangle_B + |2\rangle_A |1\rangle_B\} \\
|\Psi_4\rangle = \frac{1}{\sqrt{3}} \{|0\rangle_A |0\rangle_B + \omega |1\rangle_A |1\rangle_B + \omega^2 |2\rangle_A |2\rangle_B\} \\
|\Psi_5\rangle = \frac{1}{\sqrt{3}} \{|0\rangle_A |1\rangle_B + \omega |1\rangle_A |2\rangle_B + \omega^2 |2\rangle_A |0\rangle_B\} \\
|\Psi_6\rangle = \frac{1}{\sqrt{3}} \{|0\rangle_A |2\rangle_B + \omega |1\rangle_A |0\rangle_B + \omega^2 |2\rangle_A |1\rangle_B\} \\
|\Psi_7\rangle = \frac{1}{\sqrt{3}} \{|0\rangle_A |0\rangle_B + \omega^2 |1\rangle_A |1\rangle_B + \omega |2\rangle_A |2\rangle_B\} \\
|\Psi_8\rangle = \frac{1}{\sqrt{3}} \{|0\rangle_A |1\rangle_B + \omega^2 |1\rangle_A |2\rangle_B + \omega |2\rangle_A |0\rangle_B\} \\
|\Psi_9\rangle = \frac{1}{\sqrt{3}} \{|0\rangle_A |2\rangle_B + \omega^2 |1\rangle_A |0\rangle_B + \omega |2\rangle_A |1\rangle_B\}
$$

(7.5)
7.1. Bound entanglement exists in nature

where $\omega, \omega^2$ are the distinct cube roots of unity. In a similar manner, one can also describe the Bennett set of pure orthogonal maximally entangled states in $d \times d$ system. It is proved that any set of $d + 1$ number of states chosen arbitrarily from Bennett set of $d^2$ states in $d \times d$, $d \geq 2$ system, can not be distinguishable locally if single copy of the states are given [67]. Nathanson later extended this result using the idea of mutually unbiased bases for bipartite systems [103]. Fan showed that it is not possible to distinguish locally $l$ number of pure maximally entangled states if $l(l-1) \leq 2d$ [62]. However, it is interesting to note that if two copies of the set of maximally entangled states are given, then they are distinguishable by LOCC with certainty [32, 67]. Related with the local or global distinguishability of set of states, there is also an important issue of conclusive discrimination. Conclusive distinguishability relates the complete discrimination of the states with some probability. Inconclusive discrimination of a set of states can be described as partly discriminated set of states where recovery of only a part of the information about the states is possible. Unlike this case of distinguishability of states, discrimination of unitary operators is somehow a less constrained phenomena [1, 42]. Now, we will discuss a constructive approach to find bound entangled states through unextendible product bases, where local indistinguishability is inherent.

7.1.2 Unextendible product bases and bound entanglement

Naturally, the general notion of quantum non-locality was directly associated with the existence of entangled states. Surprisingly, with the discovery of Bennett group [22], this assumption was found to be wrong. They provide a complete orthonormal bipartite product basis in $3 \times 3$ system, (i.e., a set of orthogonal quantum states) that can not be perfectly distinguished by LOCC. This peculiar phenomena of local indistinguishability is termed as non-locality of a set of quantum states without entanglement. Another, very interesting class of local indistinguishable states is unextendible product basis that also generates bound entangled states [21, 51].

Unextendible product basis: An incomplete product basis of a multipartite quantum system described in the Hilbert space $H = \bigotimes_{i=1}^{m} H_i$, where the dimension of the $i^{th}$ Hilbert space is $d_i$; $i = 1, 2, \ldots, m$, is a set of pure orthogonal product states spanning a proper subspace $H_s$ of $H$. Unextendible Product Basis (in short, UPB) is a product basis whose complementary subspace $H - H_s$ contains no product state [21]. In other words, a set of product orthogonal vectors in $H$ is such that: (a) it has fewer elements than the dimension of the space, and (b) there does not exist any product vector orthogonal to all of them is called an unextendible product basis (UPB).
**Example:** Consider the five states of $3 \times 3$ system defined as,

\[
\begin{align*}
|\psi_0\rangle &= \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) \\
|\psi_1\rangle &= \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle)|2\rangle \\
|\psi_2\rangle &= \frac{1}{\sqrt{2}} |2\rangle(|1\rangle - |2\rangle) \\
|\psi_3\rangle &= \frac{1}{\sqrt{2}} (|1\rangle - |2\rangle)|0\rangle \\
|\psi_4\rangle &= \frac{1}{3} (|0\rangle + |1\rangle + |2\rangle)(|0\rangle + |1\rangle + |2\rangle)
\end{align*}
\]

This set of states forms an unextendible product basis of $3 \times 3$ system. The states are locally indistinguishable [21]. There arises a natural question that whether a complete product basis is always distinguishable by LOCC or not. The answer is in negative and the criteria determining this result is given below as a theorem.

**Theorem:** Any complete orthogonal product basis of a bipartite system $H = H_A \otimes H_B$ is distinguishable by LOCC if and only if it can not be decomposed into two disjoint sets of states spanning the subspaces $H_A \otimes H'_B$ and $H_A \otimes H''_B$ (or, the two subspaces $H'_A \otimes H_B$ and $H''_A \otimes H_B$), in any manner [122].

The following result is a constructive way of finding bound entangled states corresponding to any unextendible product basis.

**Theorem:** [21] Let $S = \{|\psi_j\rangle \equiv \bigotimes_{i=1}^m |\phi_j\rangle; \ j = 1, 2, \cdots, n\}$ be an unextendible product basis of the $m$-partite system $H = H_1 \otimes H_2 \otimes \cdots \otimes H_m$, such that $\dim(H) = d$. Then the state corresponding to the uniform mixture on the space complementary to the orthogonal UPB $S$ is a bound entangled state, described by,

\[
\rho = \frac{1}{d-n} \left(1 - \sum_{j=1}^n |\psi_j\rangle\langle\psi_j|\right)
\]

### 7.1.3 Notion of bound entanglement in general

Entanglement is generally defined as a resource of the system and studied through various protocols of computational tasks performed on entangled states. Protocols are usually given on Maximally entangled or other well known states. On other states they usually act as, first distill out some entanglement in terms of known states and implement the tasks on such known states. Bound entangled states are inactive in this sense. This notion is also supported by a work of Linden et.al., where it is shown that for teleportation some PPT bound entangled states can not be used to obtain a better result than classical case [93]. It is also demonstrated that presence of an infinite amount of bound entanglement can not enhance the amount of distillable entanglement of a state [132]. Quite opposite to all those results, it is also investigated that sometimes bound entanglement could be used even to perform otherwise impossible tasks [87]. For some special kind of bound entangled states, however, it is possible to perform some computational tasks by means of some global operations allowed by quantum mechanics. Such processes of performing
tasks on the states that are seemed to be inactive resource are designated as different activation processes.

Now, for bipartite systems, bound entanglement is defined clearly. But, in a multipartite setting, due to several distinct spatially separated configurations the definition of bound entanglement is not unique. A multipartite quantum state is said to be bound entangled if there is no distillable entanglement between any subset of parties as long as all the parties remain spatially separated from each other [56]. However, one may allow some of the parties to group together and perform local operations collectively. It leads us to two qualitatively different classes of bound entangled states. In 2001, Smolin found [127] a bound entangled state defined as Unlockable Bound Entangled State, that actually provides us the scheme for classifying bound entangled states into two different way. They are specified as activable and inactivable bound entangled states.

**Activable Bound Entanglement:** The states that are not distillable when every party is separated from each other, but becomes distillable if certain parties decide to group together. i.e., there is at least one bipartite cut where the state is negative under partial transposition (NPT).

**Non-activable Bound Entanglement:** Non-activable bound entangled states are not distillable under any modified configuration as long as there are at least two spatially separated groups. In other words, such states are always positive under partial transposition across any bipartite partition.

### 7.2 A class of activable bound entangled states

Recently, we discovered [9] that there is a large class of bound entangled states in multi-qubit systems. This is a generalization of the state given by Smolin, but besides of simple generalization we find some new results about the structure of the concerned Hilbert spaces. We will now analyze the matter step by step starting from the cases with lower number of qubits. The states are constructed by mixing up the four Bell states,

\[
|\Phi^\pm\rangle = \frac{|00\rangle \pm |11\rangle}{\sqrt{2}}, \quad |\Psi^\pm\rangle = \frac{|01\rangle \pm |10\rangle}{\sqrt{2}} \tag{7.8}
\]

which are pure maximally entangled states of two-qubit system.

#### 7.2.1 Bound entangled states in Hilbert space of four qubits

Consider the equiprobable Bell mixture,

\[
\rho_4^+ = \frac{1}{4} \{ P[|\Phi^+\rangle \otimes |\Phi^+\rangle] + P[|\Phi^-\rangle \otimes |\Phi^-\rangle] + P[|\Psi^+\rangle \otimes |\Psi^+\rangle] + P[|\Psi^-\rangle \otimes |\Psi^-\rangle] \} \tag{7.9}
\]

which we recognize as the unlockable bound entangled state, presented by Smolin, where for simplicity, we have used the notation \( P[\cdot] \) as projector corresponding to a pure state.
Applying Pauli operators $\sigma_z, \sigma_x, i\sigma_y$ on any one qubit of $\rho_4^+$, say, on the last qubit, three other four qubit states of this type will be obtained as follows,

$$\rho_4^- = \frac{1}{4} \{ P[[\Phi^+]] \otimes P[[\Phi^-]] + P[[\Phi^-]] \otimes P[[\Phi^+]] + P[[\Psi^+]] \otimes P[[\Psi^-]] + P[[\Psi^-]] \otimes P[[\Psi^+]] \}$$

$$\sigma_4^+ = \frac{1}{4} \{ P[[\Phi^+]] \otimes P[[\Psi^+]] + P[[\Phi^-]] \otimes P[[\Psi^-]] + P[[\Psi^+]] \otimes P[[\Phi^+]] + P[[\Psi^-]] \otimes P[[\Phi^-]] \}$$

$$\sigma_4^- = \frac{1}{4} \{ P[[\Phi^+]] \otimes P[[\Psi^-]] + P[[\Phi^-]] \otimes P[[\Psi^+]] + P[[\Psi^+]] \otimes P[[\Phi^-]] + P[[\Psi^-]] \otimes P[[\Phi^+]] \}$$

(7.10)

As the four Bell states are mutually orthogonal then so also the four states $\rho_4^\pm, \sigma_4^\pm$. The states can also be rewritten in the symmetric form,

$$\rho_4^\pm = \frac{1}{2} (P[[0000 \pm 1111]] + P[[0011 \pm 1100]] + P[[0101 \pm 1010]] + P[[0110 \pm 1001]])$$

$$\sigma_4^\pm = \frac{1}{2} (P[[0001 \pm 1110]] + P[[0010 \pm 1101]] + P[[0100 \pm 1011]] + P[[0111 \pm 1000]])$$

(7.11)

Following Smolin’s argument, it is easy to show that the states are activable bound entangled. Assume that any one of the four states $\rho_4^\pm, \sigma_4^\pm$ are shared among four distant parties Alice, Bob, Charlie and Daniel. We allow Charlie and Daniel to come to one lab. Then they will make a projective Bell measurement on their two qubit subsystem and communicates their results to Alice and Bob by any classical means. After this measurement Alice and Bob will able share one Bell state among them. Thus the state $\rho_4^\pm, \sigma_4^\pm$ are entangled. Now, to express our results in a more convenient way, we will also use the notations for the four states $\rho_4^\pm, \sigma_4^\pm$ as $\rho_{ABCD}^\pm, \sigma_{ABCD}^\pm$ and with these notations we again, express the four states as,

$$\rho_{ABCD}^+ = \frac{1}{4} \{ P[[\Phi^+]]_{AB} \otimes P[[\Phi^+]]_{CD} + P[[\Phi^-]]_{AB} \otimes P[[\Phi^-]]_{CD} + P[[\Psi^+]]_{AB} \otimes P[[\Psi^-]]_{CD} + P[[\Psi^-]]_{AB} \otimes P[[\Psi^+]]_{CD} \}$$

$$\rho_{ABCD}^- = \frac{1}{4} \{ P[[\Phi^-]]_{AB} \otimes P[[\Phi^-]]_{CD} + P[[\Phi^+]]_{AB} \otimes P[[\Phi^+]]_{CD} + P[[\Psi^-]]_{AB} \otimes P[[\Psi^-]]_{CD} + P[[\Psi^+]]_{AB} \otimes P[[\Psi^+]]_{CD} \}$$

$$\sigma_{ABCD}^+ = \frac{1}{4} \{ P[[\Phi^+]]_{AB} \otimes P[[\Phi^+]]_{CD} + P[[\Phi^-]]_{AB} \otimes P[[\Phi^-]]_{CD} + P[[\Psi^+]]_{AB} \otimes P[[\Psi^+]]_{CD} + P[[\Psi^-]]_{AB} \otimes P[[\Psi^-]]_{CD} \}$$

$$\sigma_{ABCD}^- = \frac{1}{4} \{ P[[\Phi^-]]_{AB} \otimes P[[\Phi^-]]_{CD} + P[[\Phi^+]]_{AB} \otimes P[[\Phi^+]]_{CD} + P[[\Psi^-]]_{AB} \otimes P[[\Psi^-]]_{CD} + P[[\Psi^+]]_{AB} \otimes P[[\Psi^+]]_{CD} \}$$

(7.12)

The above expressions immediately tell us, that the four states are separable in the bipartite cut $AB : CD$. Again, from the permutation symmetry of the states, the form of the states are identical if we express them in any combination of the parties like ABDC, ACBD, BCAD, etc. For example, if we exchange the positions of the parties $B$ and $C$, then the state $\rho_4^+$ will also be represented as,

$$\rho_{ABCD}^+ = \rho_{ACBD}^+ = \frac{1}{4} \{ P[[\Phi^+]]_{AC} \otimes P[[\Phi^+]]_{BD} + P[[\Phi^-]]_{AC} \otimes P[[\Phi^-]]_{BD} + P[[\Psi^+]]_{AC} \otimes P[[\Psi^+]]_{BD} + P[[\Psi^-]]_{AC} \otimes P[[\Psi^-]]_{BD} \}$$

(7.13)
This will imply that the state $\rho_4^+$ is also separable in the bipartite cut $AC : BD$. In a similar manner, we can show that each of the four states are separable in every possible $2 : 2$ bipartite cut. Now, if it could be possible to distill out any non-zero amount of entanglement from any of the four states by LOCC when the parties are far apart, then it would certainly violate the result that the states are separable across any $2 : 2$ bipartite cut. This shows that the states $\rho_4^+, \sigma_4^\pm$ are bound entangled, i.e., they are activable bound entangled.

Now, if we denote the support of the density matrices corresponding to the four states as the sets $S_1, S_2, S_3, S_4$, then,

$$
S_1 = \{ |0000 + 1111angle, |0011 + 1100angle, |0101 + 1010angle, |0110 + 1001\rangle \} \\
S_2 = \{ |0000 - 1111\rangle, |0011 - 1100\rangle, |0101 - 1010\rangle, |0110 - 1001\rangle \} \\
S_3 = \{ |0001 + 1110\rangle, |0001 + 1110\rangle, |0100 + 1011\rangle, |0110 + 1100\rangle \} \\
S_4 = \{ |0001 - 1110\rangle, |0001 - 1110\rangle, |0100 - 1011\rangle, |0110 - 1100\rangle \}
$$

(7.14)

The above form shows that the four sets $S_1, S_2, S_3, S_4$ together span the total Hilbert space of four qubits. Also, the sets are mutually exclusive and equal in size. Thus, the Hilbert space of the four qubit system has been divided into four disjoint sections, i.e., four subspaces mutually orthogonal to each other.

Having the result of four qubit system in our hand, we now gradually proceed to higher number of parties each holding a qubit system in a similar geometric way. For this, we try to split the total Hilbert space by dividing the usual basis in four parts. Unfortunately, such splitting that may construct bound entangled states will only seen in even number of qubit systems. For odd number of qubit systems such Hilbert space symmetry is absent.

### 7.2.2 Bound entangled states in Hilbert space of six qubits

Following the symmetric structure, it is now easy to construct similar states of six qubits system. Firstly, the state similar in structure to the Smolin state can be constructed as,

$$
\rho_6^+ = \frac{1}{32} \{ P[000000 + 111111] + P[000011 + 111100] + P[001100 + 110011] + P[001010 + 110101] + P[001010 + 110101] + P[001010 + 110101] + P[001100 + 100111] + P[001100 + 100111] + P[001100 + 100111] + P[001100 + 100111] \}
$$

(7.15)

If we express the last two qubits of this mixed state in terms of Bell states then this state can be expressed as Bell mixture of the four states of four qubit system, i.e.,

$$
\rho_6^+ = \frac{1}{4} \{ \rho_4^+ \otimes P[|\Phi^+\rangle] + \rho_4^- \otimes P[|\Phi^-\rangle] + \sigma_4^+ \otimes P[|\Psi^+\rangle] + \sigma_4^- \otimes P[|\Psi^-\rangle] \}
$$

(7.16)
One can now generate three other mutually orthogonal activable bound entangled states of six qubit by operating Pauli operators on any one qubit. Thus, the four states of six qubit system are expressed as,

\[
\rho_6^\pm = \frac{1}{4} \left\{ \rho_4^+ \otimes P[|\Phi^\pm\rangle] + \rho_4^- \otimes P[|\Phi^\mp\rangle] + \sigma_4^+ \otimes P[|\Psi^\pm\rangle] + \sigma_4^- \otimes P[|\Psi^\mp\rangle] \right\}
\]

(7.17)

\[
\sigma_6^\pm = \frac{1}{4} \left\{ \rho_4^+ \otimes P[|\Psi^\pm\rangle] + \rho_4^- \otimes P[|\Psi^\mp\rangle] + \sigma_4^+ \otimes P[|\Phi^\pm\rangle] + \sigma_4^- \otimes P[|\Phi^\mp\rangle] \right\}
\]

7.2.3 General class of bound entangled states in higher dimension

By mathematical induction formula, we find in any even number of qubit system starting from four, there are only four activable bound entangled states belonging to this class. We will show this in a constructive way.

**Constructive approach:** Consider the usual basis of the Hilbert space of $2N + 2$ qubits. First, we divide the basis set in two section according as the first qubit is in the state $|0\rangle$ or $|1\rangle$. Then we subdivide the two subsets according as the number of 0’s are even or odd. So, each of the four subsets have exactly $\frac{2^{2N+2}}{4} = 2^N$ number of elements. The explicit form of the sets are given below,

\[
\Gamma_{2N+2}^1 = \{|p_{2N+2}^i : |p_{2N+2}^i\rangle = |a_0^i\rangle |a_1^i\rangle \cdots |a_{2N+1}^i\rangle\}
\]

(7.18)

where $a_0^i = 0, \forall i = 1, 2, \cdots 2^N$ and $\sum_{j=0}^{2N+1} a_j^i = 0 \mod 2$, i.e., $|p_{2N+2}^i\rangle$ has a string of 0’s and 1’s of length $2N + 2$ such that the first element of the string is 0 and there is an even number of 0’s in the string.

Now, we consider the subset with elements $|\overline{p_{2N+2}^i}\rangle$ orthogonal to $|p_{2N+2}^i\rangle$ as,

\[
\Gamma_{2N+2}^2 = \{|\overline{p_{2N+2}^i} : |\overline{p_{2N+2}^i}\rangle = |\overline{a_0^i}\rangle |\overline{a_1^i}\rangle \cdots |\overline{a_{2N+1}^i}\rangle\}
\]

(7.19)

Thus, the form of elements of the second subset will be determined by

\[
\langle a_j^i | \overline{a_j^i} \rangle = 0 \forall i, j
\]

i.e., $|\overline{p_{2N+2}^i}\rangle$ has a string of 0’s and 1’s of length $2N + 2$ such that the $j$-th element of the $i$-th string of $|\overline{p_{2N+2}^i}\rangle$ is 0 or 1, according as the $j$-th element of the $i$-th string of $|p_{2N+2}^i\rangle$, will be 1 or 0, for all $i, j$. By construction, there is also an even number of 0’s in the strings of this set. Next, we consider other two class of sets.

\[
\Gamma_{2N+2}^3 = \{|q_{2N+2}^i : |q_{2N+2}^i\rangle = |b_0^i\rangle |b_1^i\rangle \cdots |b_{2N+1}^i\rangle\}
\]

(7.20)

such that $b_0^i = 0, \forall i = 1, 2, \cdots 2^N$ and $\sum_{j=0}^{2N+1} b_j^i = 1 \mod 2$, i.e., $|q_{2N+2}^i\rangle$ has a string of 0’s and 1’s of length $2N + 2$ such that the first element of the string is 0 and there is an odd number of 0’s in the string.
Lastly, we consider the set with elements orthogonal to the elements of the third set as,
\[ \Gamma_{2N+2}^4 = \{ |q_{2N+2}^i \rangle : |q_{2N+2}^i \rangle = |b_0^i \rangle |b_1^i \rangle \cdots |b_{2N+1}^i \rangle \} \] (7.21)
such that \( \langle b_j^i | b_j^i \rangle = 0, \forall i, j, \) i.e., \( |q_{2N+2}^i \rangle \) has a string of 0’s and 1’s of length \( 2N + 2 \) such that the \( j \)-th element of the \( i \)-th string of \( |q_{2N+2}^j \rangle \) is 0 or 1, according as the \( j \)-th element of the \( i \)-th string of \( |q_{2N+2}^i \rangle \), will be 1 or 0, for all \( i, j \). There will be an odd number of 0’s in each string of this set.

Now, we reconstruct four new sets with the elements of the above four sets as follows:
\[ S_{2N+2}^+ = \{ |s_{2N+2}^i \rangle : |s_{2N+2}^i \rangle = \frac{1}{\sqrt{2}} (|p_{2N+2}^i \rangle + |q_{2N+2}^i \rangle), \forall i = 1, 2, \ldots 2^N \} \]
\[ S_{2N+2}^- = \{ |s_{2N+2}^i \rangle : |s_{2N+2}^i \rangle = \frac{1}{\sqrt{2}} (|p_{2N+2}^i \rangle - |q_{2N+2}^i \rangle), \forall i = 1, 2, \ldots 2^N \} \] (7.22)
\[ R_{2N+2}^+ = \{ |r_{2N+2}^i \rangle : |r_{2N+2}^i \rangle = \frac{1}{\sqrt{2}} (|q_{2N+2}^i \rangle + |q_{2N+2}^i \rangle), \forall i = 1, 2, \ldots 2^N \} \]
\[ R_{2N+2}^- = \{ |r_{2N+2}^i \rangle : |r_{2N+2}^i \rangle = \frac{1}{\sqrt{2}} (|q_{2N+2}^i \rangle - |q_{2N+2}^i \rangle), \forall i = 1, 2, \ldots 2^N \} \]

Then, we prepare four mixed states by taking an equal mixture of all the states from each of the four sets. The four mixed states will be of the following form,
\[ \rho_{2N+2}^+ = \frac{1}{2^{2N}} \sum_{i=1}^{2^{2N}} P_i [|s_{2N+2}^i \rangle \langle s_{2N+2}^i |] \]
\[ \rho_{2N+2}^- = \frac{1}{2^{2N}} \sum_{i=1}^{2^{2N}} P_i [|s_{2N+2}^i \rangle \langle s_{2N+2}^i |] \]
\[ \sigma_{2N+2}^+ = \frac{1}{2^{2N}} \sum_{i=1}^{2^{2N}} P_i [|r_{2N+2}^i \rangle \langle r_{2N+2}^i |] \]
\[ \sigma_{2N+2}^- = \frac{1}{2^{2N}} \sum_{i=1}^{2^{2N}} P_i [|r_{2N+2}^i \rangle \langle r_{2N+2}^i |] \] (7.23)

So far we have described the process of constructing the four mixed states in any system consists of an even number of local qubit subsystem. We will now discuss the inter-relations of this class of states. To investigate the properties and uses of this class of states, we need a simple rule of connection between two successive systems. We obtain a nice correlation between two successive systems. This relation plays the most fundamental step of generalization of this class. We represent the result in the following theorem.

**Theorem:** The four states \( \rho_{2N+2}^\pm, \sigma_{2N+2}^\pm \) of \( 2N + 2 \) qubit system can be expressed as equal Bell-mixture of the four states of the previous states \( \rho_{2N}^\pm, \sigma_{2N}^\pm \) of \( 2N \) qubit system.

**Proof:** The states of the \( 2N + 2 \) qubit system \( |s_{2N+2}^i \rangle \), can be expressed as,
\[ |s_{2N+2}^i \rangle = \frac{1}{\sqrt{2}} (|p_{2N+2}^i \rangle + |q_{2N+2}^i \rangle) \] (7.24)

Now, the string of \( |p_{2N+2}^i \rangle \) has its first element as 0 and an even number of 0’s in total. Thus, it must be either in the form \( |00\rangle |\mu_{2N}^1 \rangle \) or \( |00\rangle |\nu_{2N}^1 \rangle \) or \( |01\rangle |\nu_{2N}^1 \rangle \) or \( |01\rangle |\nu_{2N}^1 \rangle \), where \( |\mu_{2N}^1 \rangle \) has a string of 0’s and 1’s of length \( 2N \) with the first element to be 0 and an even number of 0’s in it.
Similarly, $|\mu^2_{2N}\rangle$ has a string of 0’s and 1’s of length $2N$ with the first element to be 1 and an even number of 0’s in it, $|\nu^1_{2N}\rangle$ has a string of 0’s and 1’s of length $2N$ with the first element to be 0 and an odd number of 0’s in it, and lastly, $|\nu^2_{2N}\rangle$ has a string of 0’s and 1’s of length $2N$ with the first element to be 1 and an odd number of 0’s in it. It is obvious that $|\mu^1_{2N}\rangle \in \Gamma^1_{2N}$, $|\mu^2_{2N}\rangle \in \Gamma^2_{2N}$, $|\nu^1_{2N}\rangle \in \Gamma^3_{2N}$ and $|\nu^2_{2N}\rangle \in \Gamma^4_{2N}$. In other words, the set $S^+_{2N+2}$ can be divided into four disjoint parts as, $S^+_{2N+2} = Y^1_{2N+2} \cup Y^2_{2N+2} \cup Y^3_{2N+2} \cup Y^4_{2N+2}$ where,

$$
Y^1_{2N+2} = \{ |\psi^i\rangle : |\psi^i\rangle = \frac{1}{\sqrt{2}} (|00\rangle|p^i_{2N}\rangle + |11\rangle|p^i_{2N}\rangle), \forall |p^i_{2N}\rangle \in \Gamma^1_{2N} \},
$$

$$
Y^2_{2N+2} = \{ |\phi^i\rangle : |\phi^i\rangle = \frac{1}{\sqrt{2}} (|00\rangle|p^i_{2N}\rangle - |11\rangle|p^i_{2N}\rangle), \forall |p^i_{2N}\rangle \in \Gamma^1_{2N} \},
$$

$$
Y^3_{2N+2} = \{ |\chi^i\rangle : |\chi^i\rangle = \frac{1}{\sqrt{2}} (|01\rangle|q^i_{2N}\rangle + |10\rangle|q^i_{2N}\rangle), \forall |q^i_{2N}\rangle \in \Gamma^3_{2N} \},
$$

$$
Y^4_{2N+2} = \{ |\phi^i\rangle : |\phi^i\rangle = \frac{1}{\sqrt{2}} (|01\rangle|q^i_{2N}\rangle - |10\rangle|q^i_{2N}\rangle), \forall |q^i_{2N}\rangle \in \Gamma^3_{2N} \}.
$$

Number of elements of each of the four sets above will be $2^{2N-2}$. As the state $\rho^+_{2N+2}$ is formed as equal mixture of all possible states $|s^i_{2N+2}\rangle \in S^+_{2N+2}$, therefore, we can represent it as,

$$
\rho^+_{2N+2} = \frac{1}{2^{2N}} \sum_{i=1}^{2^{2N-2}} \{ |\psi^i\rangle \langle \psi^i | + |\phi^i\rangle \langle \phi^i | + |\chi^i\rangle \langle \chi^i | + |\phi^i\rangle \langle \phi^i | \} \tag{7.26}
$$

Now, using the form of Bell basis in terms of the usual product basis, we can write,

$$
|00\rangle = \frac{1}{\sqrt{2}} (|\Phi^+\rangle + |\Phi^-\rangle)
$$

$$
|11\rangle = \frac{1}{\sqrt{2}} (|\Phi^+\rangle - |\Phi^-\rangle)
$$

$$
|01\rangle = \frac{1}{\sqrt{2}} (|\Psi^+\rangle + |\Psi^-\rangle)
$$

$$
|10\rangle = \frac{1}{\sqrt{2}} (|\Psi^+\rangle - |\Psi^-\rangle) \tag{7.27}
$$

Thus, we may express,

$$
|\psi^i\rangle = \frac{1}{\sqrt{2}} (|00\rangle|p^i_{2N}\rangle + |11\rangle|p^i_{2N}\rangle) = \frac{1}{2} \{ (|\Phi^+\rangle + |\Phi^-\rangle)|p^i_{2N}\rangle + (|\Phi^+\rangle - |\Phi^-\rangle)|p^i_{2N}\rangle \}
$$

$$
= \frac{1}{2} \{ (|\Phi^+\rangle + |\Phi^-\rangle)|p^i_{2N}\rangle + (|\Phi^+\rangle - |\Phi^-\rangle)|p^i_{2N}\rangle \} \tag{7.28}
$$

Similarly,

$$
|\phi^i\rangle = \frac{1}{2} \{ (|\Phi^+\rangle + |\Phi^-\rangle)|p^i_{2N}\rangle - (|\Phi^+\rangle - |\Phi^-\rangle)|p^i_{2N}\rangle \},
$$

$$
|\chi^i\rangle = \frac{1}{2} \{ (|\Psi^+\rangle + |\Psi^-\rangle)|q^i_{2N}\rangle + (|\Psi^+\rangle - |\Psi^-\rangle)|q^i_{2N}\rangle \},
$$

$$
|\phi^i\rangle = \frac{1}{2} \{ (|\Psi^+\rangle + |\Psi^-\rangle)|q^i_{2N}\rangle - (|\Psi^+\rangle - |\Psi^-\rangle)|q^i_{2N}\rangle \} \tag{7.29}
$$
Inserting all these relations in Eq. (7.26), we obtain,

\[
\rho_{2N+2} = \frac{1}{2^{2N}} \sum_{i=1}^{2^{2N}-1} \left( \{ \Phi^+ \} \langle p'_{i2N} \rangle + \{ \Phi^- \} \langle p''_{i2N} \rangle + \{ \Phi^+ \} \langle p''_{i2N} \rangle + \{ \Phi^- \} \langle p'_{i2N} \rangle \right) \\
+ \{ \Psi^+ \} \langle q'_{i2N} \rangle + \{ \Psi^- \} \langle q'_{i2N} \rangle + \{ \Psi^- \} \langle q''_{i2N} \rangle + \{ \Psi^+ \} \langle q''_{i2N} \rangle \\
+ \{ \Psi^+ \} \langle q''_{i2N} \rangle + \{ \Psi^- \} \langle q''_{i2N} \rangle + \{ \Psi^- \} \langle q''_{i2N} \rangle + \{ \Psi^+ \} \langle q''_{i2N} \rangle \\
= \frac{1}{2^{2N+2}} \sum_{i=1}^{2^{2N}-1} 2 \{ \{ \Phi^+ \} \langle p_{i2N} \rangle + \{ \Phi^- \} \langle p_{i2N} \rangle \} + \{ \Psi^+ \} \langle q_{i2N} \rangle + \{ \Psi^- \} \langle q_{i2N} \rangle \\
= \frac{1}{4} \{ \{ \Phi^+ \} \langle p_{i2N} \rangle + \{ \Phi^- \} \langle p_{i2N} \rangle \} + \{ \Psi^+ \} \langle q_{i2N} \rangle + \{ \Psi^- \} \langle q_{i2N} \rangle \\
(7.30)
\]

Thus, we obtain a nice Bell-correlation between the states of two successive systems for the state \( \rho_{2N+2} \). In a similar way, we can obtain the Bell-correlated form of other three states \( \rho_{2N+2} \), \( \sigma_{2N+2} \). If, instead of considering the first two qubits in the above analysis, we proceed with the last two qubits, then the Bell-correlated form of the four states of \( 2N + 2 \) qubit system will be given by,

\[
\begin{align*}
\rho_{2N+2}^\pm &= \frac{1}{2} \left\{ \rho_{2N}^+ \otimes \{ \Phi^\pm \} + \rho_{2N}^- \otimes \{ \Phi^\mp \} + \sigma_{2N}^+ \otimes \{ \Psi^\pm \} \\
&+ \sigma_{2N}^- \otimes \{ \Psi^\mp \} \right\} \\
\sigma_{2N+2}^\pm &= \frac{1}{2} \left\{ \rho_{2N}^+ \otimes \{ \Psi^\pm \} + \rho_{2N}^- \otimes \{ \Psi^\mp \} + \sigma_{2N}^+ \otimes \{ \Phi^\pm \} \\
&+ \sigma_{2N}^- \otimes \{ \Phi^\mp \} \right\} \\
(7.31)
\end{align*}
\]

Now, by the above correlated formula, it is possible to prove that the four states of \( 2N + 2 \) qubit system, for any \( N \geq 2 \), are mutually orthogonal activable bound entangled states. This correlation also enables one to generate the whole class of states from the four qubit states mentioned earlier by a recursive process. Next, we will explain the various properties of this large class of states.

**Property-1:** The whole class of states are symmetric over all the parties concerned, i.e., the states remain invariant under the interchange of any two parties. The four states \( \rho_{2N+2}^\pm \), \( \sigma_{2N+2}^\pm \) can be expressed as,

\[
\begin{align*}
\rho_{2N+2}^\pm &= \frac{1}{2^{2N}} \left\{ \sum_{i=1}^{2^{2N}} \{ p'_{i2N+2} \} \pm \{ p''_{i2N+2} \} \right\} \\
\sigma_{2N+2}^\pm &= \frac{1}{2^{2N}} \left\{ \sum_{i=1}^{2^{2N}} \{ q'_{i2N+2} \} \pm \{ q''_{i2N+2} \} \right\} \\
\end{align*}
\]

where, \( |p_{i2N+2}^k\rangle \) \( |q_{i2N+2}^j\rangle \), for \( k, j = 1, 2, \ldots, 2^{2N} \), are given by,

\[
|p_{i2N+2}^k\rangle = |a_{i0}^k\rangle \otimes |a_{i1}^k\rangle \otimes \cdots \otimes |a_{i2N+1}^k\rangle \\
(7.32)
\]
with \(a_i^k \in \{0, 1\}, \ \forall \ i = 0, 1, \cdots, 2N + 1\) and \(a_0^k = 0\),

\[
|q_{2N+2}^i\rangle = |b_0^i\rangle \otimes |b_1^i\rangle \otimes \cdots \otimes |b_{2N+1}^i\rangle
\]

\[(7.33)\]

with \(b_i^j \in \{0, 1\}, \ \forall \ i = 0, 1, \cdots, 2N + 1\) and \(b_0^i = 0\), such that

\[
\sum_{i=0}^{2N+1} a_i^j = 0 \quad \text{(mod 2)}, \quad \sum_{i=0}^{2N+1} b_i^j = 1 \quad \text{(mod 2)}
\]

The states \(|p_{2N+2}^k\rangle\) and \(|q_{2N+2}^l\rangle\) are orthogonal to the states \(|p_{2N+2}^k\rangle\) and \(|q_{2N+2}^l\rangle\) respectively, for all possible values of \(k\) and \(j\). The set \(\{|p_{2N+2}^k\rangle : \forall k\}\) contains all possible permutations of strings of 0’s and 1’s with an even number of zeros. Thus, any permutation of the positions of different parties will interchange all the \(|p_{2N+2}^k\rangle\), in between themselves and their orthogonals \(|p_{2N+2}^k\rangle\). This proves the permutation symmetry of the all the four states \(p_{2N+2}^k, \ \sigma_{2N+2}^\pm\).

Explicitly, from the Eq. (7.11) of the four qubit systems, i.e.,

\[
\rho_4^\pm = \frac{1}{8}(P[|0000 \pm 1111\rangle] + P[|0011 \pm 1100\rangle] + P[|0101 \pm 1010\rangle] + P[|0110 \pm 1001\rangle])
\]

\[
\sigma_4^\pm = \frac{1}{8}(P[|0001 \pm 1110\rangle] + P[|0010 \pm 1101\rangle] + P[|0100 \pm 1011\rangle] + P[|0111 \pm 1000\rangle])
\]

we find clearly the permutation symmetry of the states over all the parties concerned.

**Property-2:** From Eq. (7.31), it is clear that the four states of \(2N + 2\) qubit system are orthogonal to each other if the \(2N\) qubit states are so. Also, from Eq. (7.9) and Eq. (7.10), we observe that the four states \(\rho_4^\pm, \ \sigma_4^\pm\) of four qubit system are mutually orthogonal. Thus, in a recursive way it provides us the mutual orthogonality of the four activable bound entangled states of any even qubit system, starting from four. This is also evident from the orthogonality of the support sets \(S_{2N+2}^+, S_{2N+2}^-, R_{2N+2}^+, R_{2N+2}^-\) corresponding to those four states of \(2N + 2\) qubit system.

**Property-3:** All the states above are mixed entangled states. Suppose, the four states of \(2N + 2\) qubit system are shared between the parties denoted by \(A_1, A_2, \cdots A_{2N}, B, C\) as,

\[
\rho_{2N+2}^\pm = \frac{1}{4} \left\{ (\rho_{2N}^+)_{A_1A_2\cdots A_{2N}} \otimes (P[|\Phi^\pm\rangle])_{BC} + (\rho_{2N}^-)_{A_1A_2\cdots A_{2N}} \otimes (P[|\Phi^\mp\rangle])_{BC} + (\sigma_{2N}^+)_{A_1A_2\cdots A_{2N}} \otimes (P[|\Psi^\pm\rangle])_{BC} + (\sigma_{2N}^-)_{A_1A_2\cdots A_{2N}} \otimes (P[|\Psi^\mp\rangle])_{BC} \right\}
\]

\[(7.34)\]

\[
\sigma_{2N+2}^\pm = \frac{1}{4} \left\{ (\rho_{2N}^+)_{A_1A_2\cdots A_{2N}} \otimes (P[|\Psi^\pm\rangle])_{BC} + (\rho_{2N}^-)_{A_1A_2\cdots A_{2N}} \otimes (P[|\Psi^\mp\rangle])_{BC} + (\sigma_{2N}^+)_{A_1A_2\cdots A_{2N}} \otimes (P[|\Phi^\pm\rangle])_{BC} + (\sigma_{2N}^-)_{A_1A_2\cdots A_{2N}} \otimes (P[|\Phi^\mp\rangle])_{BC} \right\}
\]

Now, the states \(\rho_{2N}^\pm, \ \sigma_{2N}^\pm\), are orthogonal to each other (follows from the above property-2). Then, the first \(2N\) parties \(A_1A_2\cdots A_{2N}\) can join together and perform a projective measurement...
on their system of $2N$ qubit in those four mixed orthogonal states $\rho_{2N+2}^\pm$, $\sigma_{2N+2}^\pm$. The result of their measurement is then communicated to $B$ and $C$ and consequently, the remaining state shared between $B$ and $C$ will be the corresponding Bell state determined by the above expression. For example, when the parties share the state $\rho_{2N+2}^+$, and if the result of the orthogonal measurement of the $2N$ parties is $\sigma_{2N}^+$, then the state shared between the other two parties $B$ and $C$, who are still remain specially separated is $|\Psi^+\rangle$. As all the four Bell states are maximally entangled state in $2\times 2$, it is always possible to extract non-zero amount of entanglement from the state $\rho_{2N+2}^+$ from the above procedure. Thus, the mixed state $\rho_{2N+2}$ is entangled and so also the others. Therefore, by induction method, the whole class of states are entangled.

**Property-4:** States are activable bound entangled. From the Eq.(7.31), we have obtained that for each $N \geq 2$, the $2N + 2$ qubit states $\rho_{2N+2}^\pm$, $\sigma_{2N+2}^\pm$, can be expressed in Bell-correlated form with the $2N$ qubit states $\rho_{2N}^\pm$, $\sigma_{2N}^\pm$. This form also tells us that they are separable across a $2N : 2$ cut. Now, proceeding in the same manner, we could further divide the $2N$ qubit states in terms of the $2N - 2$ qubit states and so on. Thus the states of $2N + 2$ qubit system could be represented as states separable across any $2K : 2N + 2 - 2K$ ($1 \leq K \leq N$) cut. Again, as the states are symmetric over all possible permutations of the concerned parties, therefore, it is clear that the states of this class are separable in any bipartite cut, if the number of parties in each of the two groups are even. Thus, in any $2K : 2N + 2 - 2K$ cut, i.e., when we join all the concerned parties in any two separate groups so that the number of parties in each group is even, the states are separable. So, it is not possible to distill out any entanglement from the states by LOCC, when all the $2N + 2$ parties remain spatially separated. It shows the bound entangled character of all the four states of the $2N + 2$ qubit systems, for all $N \geq 2$.

Also, if any of the $2N$ parties of $\rho_{2N+2}^\pm$, $\sigma_{2N+2}^\pm$, join together in a lab and make a projective measurement on the four orthogonal states $\rho_{2N}^\pm$, $\sigma_{2N}^\pm$, then this operation with classical communications results in sharing a Bell state between the other two separated parties. Hence, the states could be activated by collective operations on a subset of the concerned parties. So, the states are Activable, or in other words, Unlockable bound entangled states.

**Property-5:** The class of states are Bell-correlated. It is proved earlier before mentioning property-1. From Eq.(7.31), we find each of the four states $\rho_{2N+2}^\pm$, $\sigma_{2N+2}^\pm$ for $N \geq 2$, is expressed as equiprobable mixture of $\rho_{2N}^\pm$, $\sigma_{2N}^\pm$ with the four Bell states $|\Phi^\pm\rangle$, $|\Psi^\pm\rangle$, taken in a specific order. Explicitly, the four qubit states $\rho_{4}^\pm$, $\sigma_{4}^\pm$ are themselves written in terms of Bell states. In fact, the whole class of states can be written explicitly as equiprobable mixture of products of Bell states taken in some specific order.

**Property-6:** The four states $\rho_{2N+2}^\pm$, $\sigma_{2N+2}^\pm$ for $N \geq 1$, are locally Pauli-connected in one party, i.e., starting from any one of the four states and operating the Pauli matrices on the local subsystem of any one party of the $2N + 2$ qubit system, one would able to
get the other three states. Thus, denoting, \( \rho^1_{2N+2} = \rho^+_{2N+2}, \ \rho^2_{2N+2} = \rho^-_{2N+2}, \ \rho^3_{2N+2} = \sigma^i_{2N+2} \) and the Pauli matrices as, \( \sigma^1 = I, \sigma^2 = \sigma_z, \sigma^3 = \sigma_x, \sigma^4 = i\sigma_y \), for each \( i = 1, 2, 3, 4 \), we can express, \( \rho^i_{2N+2} = \Gamma_k \rho^i_{2N+2} \), for any \( k = 1, 2, \ldots, 2N + 2 \), where, \( \Gamma_k = I_1 \otimes I_2 \otimes \cdots \otimes I_{2N-2} \otimes \sigma^i \otimes I_{2N-k+1} \otimes \cdots \otimes I_{2N+2} \), is the operator whose action can be described as the operation by the Pauli operator \( \sigma^i \) on k-th qubit.

**Property-7:** The support of the density matrices corresponding to the four states \( \rho^\pm_{2N+2}, \ \sigma^\pm_{2N+2} \) for \( N \geq 1 \), are the four sets \( S^\pm_{2N+2}, S^-_{2N+2}, R^\pm_{2N+2}, R^-_{2N+2} \) respectively, described earlier. This four sets will together span the total Hilbert space of \( 2N + 2 \) qubit system. Also, the four sets are mutually disjoint and equal in size. Thus, any two states of the \( 2N + 2 \) qubit Hilbert space, belonging to the support of two different sets. Geometrically, the Hilbert spaces of any even qubit systems are divided into four equal and mutually orthogonal parts.

**Property-8:** The entanglement cost of preparing the \( 2N \) qubit states is \( N \) e-bits [10], shown by possing a lower bound on entanglement cost and then achieving the bound by an explicit protocol.

Now, we will establish an important property of the class of four activable bound entangled states in any \( 2N + 2 \) qubit system for \( N \geq 1 \). The four states are local indistinguishable. We will represent it as a theorem.

**Theorem:** For all \( N \geq 2 \), the probability for distinguishing the four states \( \rho^\pm_{2N}, \ \sigma^\pm_{2N} \) exactly, by LOCC is zero.

To prove it, let us assume that for some value of \( N \geq 2 \), the four states \( \rho^\pm_{2N}, \ \sigma^\pm_{2N} \) are locally distinguishable. Now, consider the state,

\[
\rho^+_{2N+2} = \frac{1}{4} \left( \rho^+_{2N} \otimes P[\Phi^+] + \rho^-_{2N} \otimes P[\Phi^-] + \sigma^+_2 \otimes P[\Psi^+] + \sigma^-_2 \otimes P[\Psi^-] \right)
\]

where the first \( 2N \) parties are \( A_1, A_2, \ldots, A_{2N-1}, B_1 \) and the last two parties are \( A_{2N}, B_2 \), i.e., the state is separable by construction in \( A_1 A_2 \ldots A_{2N-1} B_1 : A_{2N} B_2 \) cut. Again, the state is symmetric with respect to the interchange of any two parties, i.e., \( \rho^+_{2N+2} \) has the same form if the first \( 2N \) parties are \( A_1, A_2, \ldots, A_{2N} \) and the last two parties are \( B_1, B_2 \). If, the four states \( \rho^\pm_{2N}, \ \sigma^\pm_{2N} \) are locally distinguishable, then by LOCC only, \( A_1, A_2, \ldots, A_{2N} \) are able to share a Bell state among \( B_1 \) and \( B_2 \), which is impossible as initially there is no entanglement in between \( B_1 \) and \( B_2 \). So, all the four states \( \rho^\pm_{2N}, \ \sigma^\pm_{2N} \) are locally indistinguishable for any \( N \geq 2 \). Our protocol also suggest that the states are even probabilistically indistinguishable for any \( N \geq 2 \), as it is impossible to share any entanglement by LOCC between \( B_1 \) and \( B_2 \). Let us assume that the four states are locally indistinguishable with probability \( p > 0 \), then having the shared state \( \rho^+_{2N+2} \) among the \( 2N + 2 \)
parties, any set of $2N$ parties may able to distinguish their joint local system with probability $1 > p > 0$ and correspondingly share one Bell state among the other two parties. In this way, it is possible to extract on average non-zero amount of entanglement by performing LOCC only. This contradicts with the bound entangled nature of $\rho_{2N+2}^\pm$. Thus, the four states of $2N$ qubit system are even probabilistically locally indistinguishable. So, the probability for distinguishing the four states $\rho_{2N}^\pm, \sigma_{2N}^\pm$ by LOCC is zero, for any $N \geq 2$.

### 7.2.4 Conclusion

In this chapter, we find interesting nature of the general class of Activable Bound Entangled States. Besides of being activable bound entangled, which itself is a peculiar feature, they have some other special properties, never attainable by any other known state. We focussed our attention here centrally on the property of local indistinguishability. This is because it is a very special kind of non-locality observed in composite quantum systems. For multi-partite systems, it is rare to observe and this non-locality could be used for practical purposes. Previously, Smolin state is being used successfully for performing many computational tasks [102,126]. One may search whether those tasks can also be performed by employing our multipartite activable bound entangled states. Such generalizations are always important for practical implementation of any computational schemes. Our next aim is to build some hiding protocol with the class of states. In the next chapter, we will elaborate this concept in detail.
Chapter 8

An Application- Quantum Data Hiding

8.1 Introduction

Information theory concerns largely with encoding, securing and manipulating information in terms of physical states. Quantum information theory is a newly developed subject with a vast field of applicability known basically after the discovery of quantum teleportation, dense coding, etc., \[15, 17, 16, 29\]. Quantum non-locality or, more specifically the quantum entanglement is the key invention of quantum information theory that has been used for lots of information processing schemes, which could not be possible by any classical protocol. Protocols are now supported by practical laboratory experiments or on the verge of implementations. Information theory cares largely with the objects that maintains the secrecy of a data. Unlike teleportation, where the objective is to send information from one place to another so that the secrecy of the data is not explored by any eavesdropper, the aim in this area is to share some information among a number of different parties situated at distant laboratories, so that any one or some subset of the concerned parties cannot break the security or change the secret data \[45\]. Thus, protocols are proposed against the cheating attempts of some parties sharing the states. This completely changes the old scenario. Obviously, the investigations become more successful by using the nonlocal character of the quantum states. Entanglement is now used for posing different protocols on quantum secret sharing or on data hiding. Secret sharing is comparatively more known to us, whereas data hiding is a new branch of information theory. It is mainly concerned with the sharing of information to some parties situated at distant places, so that each of the associated parties have access of some part of the data. Though, there are strong restrictions on recovering the secret information. Now, the difficulty in data hiding by using entangled states is the very important requirement, i.e., local indistinguishability of the states. We have

\[\text{\footnotesize 1Some portions of this chapter is published in Physics Letter A, } \textbf{365}, \text{ 273 (2007).}\]
observed in the previous chapter that our activable bound entangled states have the property of
local indistinguishability. In this chapter, we will explore the possibility and limitations of that
class of activable bound entangled states to build a data hiding protocol. Before that, we first
describe some basic ideas of different hiding schemes.

8.2 Sharing information secretly among distant parties

Early ideas came with the hiding classical information in terms of classical bits (in short, cbits).
The process is known as classical secret sharing protocol [123]. Later, we find the concept of
keeping quantum information secret in various purposes. The protection of secret information
can be defined either against eavesdropper attack or, against the cheating of the concerned par-
ties. In the first case, it is sufficient to detect the presence of an eavesdropper having some
(obviously, unwanted and secret) access of the communication channels between the associated
parties [60]. The classical one-time-pad schemes are proposed for using a classical state as the
secret channel, where the information is processed only once and then discarded by the choice of
another classical state. This scheme is generalized to quantum case by sharing a private quantum
channel between some parties so that the hidden information remains secured against any eaves-
droppers attempt to explore the secret. A more complicated situation may arise when the data is
kept secret from the parties themselves. The security of the hidden data is built on the premises
that the data cannot be perfectly revealed by any kind of cheating attack (local or global) of the
associated parties. Altogether, we have two different branches of research in this area, known as,
Secret Sharing and Data Hiding.

Now, in a secret sharing scheme the basic task is to share some information classical or
quantum, among a number of parties situated at different labs so that each of the parties have
some access of a part of the information, though the information is not known to them. The
information is hidden from the parties and thus the protection of it is defined against various kinds
of cheating attempts maid by the associated parties. This is done by encoding the information
in some classical or quantum state and sharing the state among all the parties. The associated
parties (may be only some of them) are not allowed to know the secret. Thus, to reveal the
hidden information they have to retrieve the secret state shared by them altogether. Schemes are
proposed in this direction must have a properly defined set of allowable operations that the parties
can perform on their shared part and a properly chosen security parameter specifying the level
of perfection of security of the secret information. Thus any computational scheme proposed for
this purpose is characterized by the following properties.

(1) The amount of information (in terms of cbits or qubits or qudits, etc.) that can be hidden
in the system.

(2) The maximum number of parties among which the system is distributed.
(3) Maintaining the security criteria, what is the allowable set of operation that the associated parties can perform to reveal the data.

(4) The maximum number of unfaithful parties among all the associated parties. Here, it is to be noticed that the number of parties is important; i.e., a symmetric structure of the states is required.

(5) What is the amount of (zero for perfect security and otherwise an asymptotically small quantity) of secret information that the unfaithful parties can at most retrieve from the system by their joint attack.

Usually, a \((k, n)\) threshold scheme is proposed for sharing secret, where \(n\) is the total number of parties sharing the state and \(k\) of them can be used to reconstruct the state, but \(k - 1\) or fewer have absolutely no information about the state [45, 75].

### 8.3 Hiding classical data

A further restricted and secured scenario can be found in the schemes of data hiding where a very strong constraint is always imposed over the system. Here, classical information is kept secret in terms of orthogonal quantum states shared among some parties situated at distant locations. The involved parties know which quantum state is used to encode which classical bit, but do not know the actual state they are sharing. Even considering all of the concerned parties to be unfaithful, the security in such schemes must guarantee the requirement that the parties can not retrieve the secret by LOCC only. This requirement turns out to be the local indistinguishability of the hiding states [50, 143]. This imply, in a quantum data hiding scheme the hiding states must be locally indistinguishable. The aim of such a hiding scheme is to build a considerably high level of security and to minimize the number of faithful parties, required to maintain the secrecy. We can imagine a situation where an employer is sharing some data to a number of his/her employees, without knowing the actual data. The aim of the scheme is, according to his/her own convenience, the employer can reveal the data. However, the parties could not be able to know the hidden data by their cooperations. For example, \(\log_2 k\) number of classical bits are shared among \(n\) number of distant parties with \(k\) number of orthogonal quantum states, say, \(\{\rho_i; i = 1, \ldots, k\}\). The hiding states, \(\{\rho_i; i = 1, \ldots, k\}\) are chosen to be locally indistinguishable. Also, it is desirable that the hiding states must be symmetric over all possible interchanges of the positions of the parties. Then, the next step is to check whether the states can be distinguished locally with some non-zero probability or not. This probability requirement indicates the amount of data that can be recovered at most locally. Here, one should note that the data is known to all the concerned parties with the guessing probability. Therefore, it is expected that in a hiding protocol, the unfaithful parties or anyone who wants to know the hidden data, could not be able to retrieve the hidden data more than the guessing probability. Thus, data hiding is really a challenging area of research where one have to define the security of the protocol, first by
8.4 Hiding quantum data

Quantum data hiding is a subject where the basic tasks is to hide information in terms of quantum states, like qubits, qudits, etc. Protocols proposed for hiding qubits are almost similar with the hiding classical data. DiVincenzo et al. [52] showed that in bipartite case, there is an equivalence between hiding one qubit and two cbits hiding with a similar security criteria. It is further shown that any quantum data hiding scheme for hiding $2^k$ cbits with a security parameter $\epsilon$, can be converted into a $\delta$-secure $k$-qubit hiding scheme, for $\delta = 2^{k+1}\epsilon$.

In multipartite case, data hiding is rather complex to describe. As here quantum communications may include the joint operations on any subset of the all $n$ number of parties. Security is defined by allowing a fixed number of parties to communicate within themselves quantum mechanically. Hayden et al. [74] provided a general scheme for hiding qubit information in multipartite quantum state. The hiding states belongs to a class of nearly perfect locally indistinguishable states, rather than only perfectly indistinguishable class. The scheme is proposed with $(k,n)$ access structure through suitably defined encoding and decoding maps with the required level of security. The quantum information is hidden in multipartite quantum states by some completely positive trace-preserving maps. Corresponding to the hiding states, the encoding map is defined as,

$$E(\Phi) = \frac{1}{r} \sum_i \{U_i \Phi U_i^\dagger \otimes \rho^i\} \quad (8.1)$$

where, $\Phi$ is a multipartite state, $\rho^i$ are hiding states, $U_i$ are some unitary operations and $r$ is the normalizing factor. Then, the encoded state is almost indistinguishable from maximally mixed state by arbitrary LOCC and quantum communications among any $k-1$ number of parties.
The security of the protocol will be guaranteed by the condition,
\[ \| L\{\rho^i\} - L\{\rho^j\} \|_1 \leq \epsilon \]  
(8.2)
where \( L \) is any admissible local operation. Again, the correctness of the scheme is guaranteed by
the requirement that the data is correctly decoded,
\[ \| (D^{(X)} \otimes E)(\Phi) - \Phi \|_1 \leq \delta \]  
(8.3)
where \( D^{(X)} \) is the decoding map and in all the above two equations \( \| \cdot \|_1 \) represents trace norm.

Now, we shall discuss an important property of our activable bound entangled states defined
previously and explore the possibilities of hiding information.

### 8.5 Activable bound entangled states- a property

In the previous chapter, we found that activable bound entanglement is a very special kind of
entanglement. They have a highly symmetric structure with some other very special characteristics. There are very few classes of mixed entangled states in multi-qubit systems, which are
used in so many computational tasks like the Smolin state \([55, 126]\). Previously, we have dis-
cussed local indistinguishability of the four activable bound entangled states \( \rho_{2N+2}^\pm \) and \( \sigma_{2N+2}^\pm \)
of \( 2N+2 \) qubit system. It provides a suitable ground for proposing a natural multipartite data
hiding scheme. For this purpose, we investigate how much information of the global states could
be gathered by the co-operation of any proper subset of the the concerned parties. This is re-
lected from the nature of the reduced density matrices corresponding to the four states of the
same system. We first show that all the reduced density matrices of the \( 2N+1 \) qubit systems
from \( \rho_{2N+2}^\pm \) and \( \sigma_{2N+2}^\pm \) are maximally mixed state or in other words, the subsystems do not have
any short of information of the global system.

**Theorem:** For all \( N \geq 1 \) all the reduced density matrices corresponding to the four states
\( \rho_{2N+2}^\pm, \sigma_{2N+2}^\pm \) are maximally mixed state. Ignorance of any one party (i.e., by tracing out one
qubit system) from any of the four states \( \rho_{2N+2}^\pm, \sigma_{2N+2}^\pm \) would result in the state \( \frac{1}{2^{2N+1}} \mathbf{1}^{2N+1} \).

To establish this result, let us first consider the four states of four qubit system. The Smolin
state is,
\[
\rho_4^+ = \frac{1}{4} \left\{ P[\Phi^+] \otimes P[\Phi^+] + P[\Phi^-] \otimes P[\Phi^-] + P[\Psi^+] \otimes P[\Psi^+] \right. \\
+ P[\Psi^-] \otimes P[\Psi^-] \right\} \\
= \frac{1}{4} \left\{ P[\frac{100}{\sqrt{2}} + |11\rangle \otimes P[|\Phi^+\rangle] + P[\frac{100}{\sqrt{2}} - |11\rangle \otimes P[|\Phi^-\rangle] \\
+ P[\frac{101}{\sqrt{2}} \otimes P[|\Psi^+\rangle] + P[\frac{101}{\sqrt{2}} \otimes P[|\Psi^-\rangle] \right\} \\
= \frac{1}{8} \left\{ (P[00] + P[11]) \otimes (P[\Phi^+] + P[\Phi^-]) + (P[01]) \otimes (P[\Psi^+] + P[\Psi^-]) \\
+ P[10]) \otimes (P[\Psi^+] + P[\Psi^-]) + (|00\rangle <11| + |11\rangle <00| \otimes (P[\Phi^+] + P[\Phi^-]) \\
- P[\Phi^-]) + (|01\rangle <10| + |10\rangle <01| \otimes (P[\Psi^+] - P[\Psi^-])) \right\} 
\]  
(8.4)
Thus, tracing out the first party of the Smolin state we have,

\[
\rho'_3 = \frac{1}{8} \left\{ (P[0]) + 1 \right\} \otimes \left[ (P[\Phi^+] + P[\Phi^-]) + (P[0]) + P[1] \right\} \otimes \left( (P[\Psi^+] + P[\Psi^-]) \right) \\
= \frac{1}{8} \left( (P[0]) + (P[1]) \right) \otimes \left( (P[\Phi^+] + P[\Phi^-]) + (P[\Psi^+] + P[\Psi^-]) \right) \\
= \frac{1}{8} I \otimes I^2 \\
= \frac{1}{2} I^3 
\]

As the state is invariant under all possible permutations of the concerned parties, therefore, by tracing out any one qubit system of the Smolin state, we would obtain a maximally mixed state. The other three activable bound entangled states of the four qubit system are Pauli-connected with the Smolin state in any one qubit. For example, if we consider the state \( \sigma_4^- \), then we can express it in the form,

\[
\sigma_4^- = (I \otimes I \otimes i\sigma_y \otimes I) \rho_4^+ (I \otimes I \otimes i\sigma_y \otimes I) 
\]

This form reflects clearly that tracing out the third party, we would obtain just the same as that of \( \rho_4^+ \), i.e., \( \frac{1}{2} I^3 \). Following this argument, we conclude that all the four states \( \rho_4^\pm, \sigma_4^\pm \) have this property. The next step is to prove this property for the whole class of states. It would follow from the mathematical induction process prescribed in Eq.(7.31). The process certainly ensures that if the statement of the property is true for the 2\( N \) qubit states then so also for the 2\( N + 1 \) qubit states and thus proceeding from the four qubit states to the six qubit activable bound entangled states, then from six to eight and so on. To prove this, let us assume that for some integer \( N \), the four states \( \rho_{2N}^\pm, \sigma_{2N}^\pm \) have this property. Thus, tracing out the first qubit system of \( \rho_{2N}^\pm, \sigma_{2N}^\pm \), we have, \( \frac{1}{2^{2N+1}} I^{2N-1} \). Then applying the relation Eq.(7.31), we find the state \( \frac{1}{2^{2N+2}} I^{2N+1} \) by tracing out the first qubit system of the state \( \rho_{2N+2}^\pm \). As, tracing out the first qubit system of \( \rho_{2N+2}^\pm \) we have,

\[
\rho_{2N+1}' = \frac{1}{4} \cdot \frac{1}{2^{2N+1}} I^{2N-1} \otimes \left( (P[\Phi^+] + P[\Phi^-]) + (P[\Psi^+] + P[\Psi^-]) \right) \\
= \frac{1}{4} \cdot \frac{1}{2^{2N+1}} I^{2N-1} \otimes I^2 \\
= \frac{1}{2^{2N+2}} I^{2N+1} 
\]

Similarly, for the states \( \sigma_{2N+2}^\pm \). Therefore, by the mathematical induction formula we have the result. Thus, through a recursive method we obtain that the property is true for the whole class of activable bound entangled states. As the states are symmetric over permutations of all the parties concerned, thus tracing out anyone party we find the same result. It would also imply that the individual density matrices of each party is a maximally mixed state, i.e., \( \frac{1}{2} I \).

This property of the states of generalized class (Eq. (7.31)) is very important for implementing the data hiding protocol using these states as the hiding states.
8.6 Hiding of two classical bits

The properties of the above class of states enable us to construct a protocol \[38\] to hide two cbits of information among any even number of spatially separated parties starting from four. Here, we assume that two cbits of information is hidden in the shared state between \(2N + 2\) number of parties separated by distance, i.e., corresponding to the classical information \(b = 0, 1, 2, 3\), the four states \(\rho_{2N+2}^\pm, \sigma_{2N+2}^\pm\), for \(N \geq 1\) are shared among themsevces. The hidden data is secured against every possible LOCC among all the parties and against any sort of quantum communication among \(2N + 1\) parties as the hidden data cannot retrieved perfectly, until and unless all the parties remain separated or all of the \(2N + 2\) parties are dishonest.

**Secured against LOCC:** The protocol appears to be secured against any LOCC attack as the four states of same system are locally indistinguishable, so that even probabilistically no information can be extracted by all possible LOCC among all concerned parties and no security bound is required against LOCC. However, this conclusion is not perfectly correct as the states are inconclusively discriminated by LOCC. We discuss this limitation later on.

**Security against global operation:** The data remains secure under the action of any \(2N + 1\) number of dishonest distant parties, who are allowed to make global operations, by joining in some laboratories and make collective operations on their joint system. It follows directly from the maximal ignorance property of the activable bound entangled states, as ignorance of the system of the honest party (there should be at least one such or, otherwise the states are obviously globally distinguishable as being orthogonal to each other) gives the reduced density matrix of the others to be the maximally mixed state. Thus, the quantum communication is allowable among a maximum number of parties, i.e., \(2N + 1\). It is interesting to note that in the above protocol we need only one honest party, not allowed to communicate with the others through some quantum channel. The hider may not be a part of the system. It is also not necessary that the hider herself encrypt the bit in the quantum state and thus knows the hidden data.

8.6.1 Limitations regarding inconclusive distinguishability

Though the above situation appears to be quite nice to maintain the secrecy of hidden data in very stronger manner, but it has some limitations. So far we have only considered perfect distinguishability of the states. Precisely, it implies to discriminate which state is given from the whole set (here the set of four states from \(2N + 2\) qubit system). Though it may possible to determine whether the given state belongs to some particular subset of the whole set of states. Such as here, though it is impossible to distinguish perfectly the four states \(\rho_{2N+2}^\pm, \sigma_{2N+2}^\pm\), for \(N \geq 1\) even with an arbitrarily small probability by LOCC, but it is possible to determine by LOCC, either the given state belongs to the subset \(\rho_{2N+2}^\pm\) or, from the subset \(\sigma_{2N+2}^\pm\). It is possible simply by measuring on \(\sigma_z\) basis in each party and checking only the parity (even or odd number of zeros
or ones). The basic fact of this set discrimination, taken two together, is that the four states are locally Pauli connected.

### 8.7 Conclusive remarks

In conclusion our scheme is proposed to hide two bits of classical information or, equivalently one quantum bit among \(2N + 2\) number of parties, for any \(N \geq 1\). The advantage of our protocol is that the number of parties can be extended in pairs up to any desired level keeping the individual systems only with dimension two. The hidden information cannot be exactly revealed by any classical attack of the corresponding parties and also against every quantum attack, as long as one party remains honest. However, the hiding scheme has some limitations from the viewpoint of set distinguishability. The states are nice for practical preparation by sharing Bell mixtures among distant parties. This class of locally Pauli connected but locally indistinguishable states with the power of activable boundness has created a new direction to investigate the relation between non-locality and local distinguishability.
Conclusion

In conclusion, this work highlights on some peculiar features of some special classes of entangled systems. For bipartite systems, we explore the nature of incomparability regarding the state transformation by LOCC with certainty, proposed by M. Nielsen. Our basic object was to search for the origin of the existence of such pair. We have provided methods to resolve this non-transferability of incomparable pairs by LOCC. This feature is further connected with various impossible operations defined on single systems. We have considered here the connection of incomparability with universal anti-unitary operations and universal angle-preserving operations. As a particular case, both include the universal spin-flipping operation. The existence of incomparable states also provides us to consider some kind of irreversibility that plays in pure state level also. We have succeeded in posing incomparability as a good detector of impossible operations. We have also observed that the action of a local operation on multipartite systems is a very peculiar phenomena. It is not always predictable the behaviour of an operation in a composite system from its action on a single system. In mixed state level we have found a general class of Activable Bound Entangled states in multi-qubit systems. Apart from the nice Bell-correlation and other important properties, these classes of bound entangled states have the remarkable feature of local indistinguishability in quantum systems. The four states of $2N$ qubit system are even probabilistically locally indistinguishable for each $N \geq 2$. We explore this feature to prescribe a hiding protocol with these classes. Our protocol, though has its own limitations regarding inconclusive distinguishability, applicable to any even number of multi-partite systems. We expect in future those queers will gloom into some fundamental properties of entangled systems.
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