Analytical solutions for the internal forces of lining structure in shallow double-arched tunnel subject to unsymmetrical loads

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Abstract. This paper presents analytical solutions for the internal forces of lining structure in shallow double-arched tunnel subjected to unsymmetrical loads, mainly for the section of arch rings. The force method is employed during the derivation. The internal forces obtained from the analytical solutions are compared with that from the numerical simulation by FEM model utilizing the load-structure-method. It is found that the largest positive bending moment appears at the arch crown, while larger negative bending moment is located at both haunches of arch rings. The axial forces of both ends of arch rings are larger than that at the arch crown, which indicates that the side and middle walls are in larger compression than the arch crown. The larger shear forces of arch rings appear at the cross section approaching the top of side walls, which may be caused by the application of elastic resistance to the side wall. The comparison results and variation trend verify the applicability of presented analytical solutions for the double-arched tunnel subjected to unsymmetrical loads.

1. Introduction

The double-arched tunnel is a special tunnel type for shallow highway tunnels, which may usually be employed in the sloping terrain and subjected to the unsymmetrical loads. The loads or pressures exerted on the double-arched tunnel are not easy to be determined because of the special geometry. Some loading modes have been documented for double-arched tunnel in the existing researches[1-5]. Most of these studies focused on the symmetric loads, while few of them considered the unsymmetrical loading conditions on the double-arched tunnel. The latter is more common in practice.

The distribution of internal forces is an important issue that should be determined for the construction and design of tunnels, especially for double-arched tunnels. Researchers have conducted numerical analyses[6, 7], field monitoring[8, 9] and model tests[10, 11] to investigate the mechanical behavior of tunnel structure subjected to different types of loads. But these methods are always costly, time-consuming or lagged behind the actual construction schedule, and the unsymmetrical loading conditions were not considered. Analytical method is a conventional, fast and precise method for analyzing the mechanical behavior of both surrounding rock and tunnel lining[12, 13]. Because of the regularity in
geometry and simplicity in mathematics, most of the analytical analysis about internal forces or stresses of tunnel structure focus on the single circular tunnel[13-17], but few of them concerned about the non-circular tunnel lining[18]. Hu et al. [12] presented an analytical solution for determining the internal forces of a jointed segmental precast DOT (Double-O-tube) lining, which is the only analytical work available for the double tunnel lining. Furthermore, most of the existing researches mainly assumed that the tunnels experience symmetrical loads. No analytical solutions have been developed for determining the internal forces of double-arched tunnel subjected to unsymmetrical loads.

In this paper, analytical solutions for determining the internal forces of lining structure in shallow double-arched tunnels subjected to unsymmetrical loads are presented. Due to the space limit, only the analytical solutions for the arch ring are presented. The validity of the proposed analytical solutions is assessed by comparing the analytical values with the numerical simulation results.

2. The double-arched tunnel model

2.1 Model assumptions

The present study focuses on the shallow double-arched tunnel subjected to unsymmetrical loads, as shown in Figure 1. Some simplifications/assumptions are applied before the derivation, which include: the force method is used for the derivation of internal forces along the tunnel arch rings; the middle wall is considered as a cantilever beam while the two side walls are considered as elastic foundation beams; the bottom of the middle wall is fixed; the bottoms of both side walls are fixed in both horizontal and vertical directions, while rotation is allowed under the restriction by the elastic resistance at bottom; the thicknesses of the tunnel lining and the middle wall are not considered during the derivation and the size of lining is represented by its external dimensions.

![Diagram of double-arched tunnel](image)

Figure 1. The mechanical model for the double-arched tunnel subjected to unsymmetrical loads.

2.2 The surrounding rock mass loads

The method determining the unsymmetrical load of a single-arch tunnel follows the Code for Design of Road Tunnel[19], while Ding et al [1] gave a defining method for designing load of shallow double-arched tunnel subjected to symmetrical loads. Combining the two methods, the loads exerted on shallow double-arched tunnel subjected to unsymmetrical loads can be determined as shown in Figure 1.

The vertical pressure distribution pattern is consistent with surface slope, which is given by

\[ Q = \frac{\gamma}{2} \left( (h + h')B - \left( \lambda h^2 + \lambda' h'^2 \right) \tan \theta \right), \]

where \( h \) and \( h' \) are heights from the crown of the arch to the toe and crest of the surface slope, respectively; \( B \) is the total width of the double-arch tunnel; \( \gamma \) is the unit weight of the surrounding rock; \( \theta \) is the frictional angle of the two sides of the rock column over the crown of the arch; \( \lambda \) and \( \lambda' \) are the lateral pressure coefficients for shallowly-buried side and deeply-buried side[19].
The triangular vertical pressure \( q' \) is the weight of rock wedge between the middle wall and the arches of both sides, which is given by \( q' = \gamma (h_2 - h) \) \[^1\], where \( h_2 \) is the height from the top of the middle wall to the sloping surface.

The horizontal pressure outside the tunnel are \( e_i = \lambda \gamma h_i \) and \( e_c = \lambda' \gamma h'_i \), where \( h_i \) and \( h'_i \) are the heights from any point \( i \) on the left and right sides of the double-arched tunnel to the ground surface, respectively.

The horizontal pressures on top of middle wall exerting on the right/left of the left/right arch rings are \( e'_1 = q \tan^2 (45^\circ - \frac{\phi}{2}) \) and \( e'_2 = (q + q') \tan^2 (45^\circ - \frac{\phi}{2}) \), respectively \[^1\].

### 2.3 Adopted models in derivation

According to the assumptions in Section 2.1, some simplifications are made during the derivation. As Figure 2 shows, the tunnel arch feet are connected with the middle wall with the application of elastic resistance coefficient \( k_e \) of the lining structure. The tunnel lining is considered as linear elastic beam. The analytical solutions of internal forces along the tunnel arch, the side walls and the middle wall are derived separately. Due to the limiting space, only the solutions for tunnel arch are presented.

**Figure 2.** The simplified double-arched tunnel model in derivation.

**Figure 3.** The calculation model for the left arch ring.

In Figure 2, the symbols for the left tunnel are defined as following: \( R \) is the outer radius of the arch ring; \( a \) represents the bottom point of side wall; \( h \) is the left arch foot on top of the side wall; \( d \) is the right arch foot on top of the middle wall; \( l \) is the height of side wall; \( l_1 \) and \( l_2 \) are the widths of left and right half arch rings, respectively; \( f_1 \) and \( f_2 \) are the heights of left and right half arch rings. A prime is added to the same set of symbols to denote the corresponding quantities of the right tunnel.

### 3. Analytical solutions for the internal forces of arch rings of double-arched lining structure

During the derivation of analytical solutions for arch rings, the force method is adopted. The adopted calculation models for left arch ring is shown in Figure 3.

As shown in Figure 3, the pressures acting on the arch rings are active pressures, which are all in the positive directions. The redundant forces at the crown of the arch ring induced by the active pressures are defined as \( X_{1p}, X_{2p} \) and \( X_{3p} \), respectively, which are all in the positive directions.

The \( \beta_{pl}, u_{pl}, \) and \( v_{pl} \) represent the angular, horizontal and vertical displacements of the left arch foot, respectively. The \( \beta_{pr}, u_{pr}, v_{pr} \) denote the same quantities of the right arch foot. The angular displacement turning outwards is positive. The outwards horizontal displacement is positive. The upward vertical displacement of the right arch foot is positive, while the downward vertical displacement of the left arch foot is positive. The force method equations can be established at the crown of arch ring as:
where $\delta_{ij}$ ($i,j=1,2,3$) are the flexibility coefficients of the arch ring, $\Delta_{ip}(i=1,2,3)$ are the displacements under the active pressures in the direction of $X_{ip}$ at the crown of arch ring.

Using the superposition principle, the displacements of the left and right arch feet can be obtained as:

\[
\begin{align*}
\beta_{pl} &= X_{1p}\beta_{il} + X_{2p}(\beta_{2i} + f_{i}\beta_{il}) + X_{3p}(\beta_{3i} - l_{i}\beta_{il}) + \beta_{pl}\delta_{i}
\end{align*}
\]

(2)

where $\beta_{il}$, $u_{il}$ and $v_{il}$ represent the angular, horizontal and vertical displacements for the left arch foot under the unit loads $X_{i} = 1$ ($i=1,2,3$), respectively. The $\beta_{ir}$, $u_{ir}$ and $v_{ir}$ are the same quantities for the right arch foot. The $\beta_{pl}$, $u_{pl}$ and $v_{pl}$ are the angular, horizontal and vertical displacements for the left arch foot under the active pressures. The $\beta_{pr}$, $u_{pr}$ and $v_{pr}$ are the same quantities for the right arch foot.

The following relationships can be obtained according to the reciprocal theorem of displacement:

\[
\begin{align*}
\beta_{2i} &= u_{il}, \beta_{3i} = v_{il}, u_{3i} = u_{2i}, \beta_{2i} = u_{ir}, \beta_{3i} = v_{ir}. u_{3i} = v_{2r}
\end{align*}
\]

(3)

Substituting Equation (2) into Equation (1), the displacement equations containing the unknown redundant forces $X_{1p}$, $X_{2p}$ and $X_{3p}$ obey the following equations:

\[
\begin{align*}
a_{11}X_{1p} + a_{12}X_{2p} + a_{13}X_{3p} + a_{10} &= 0 \\
a_{21}X_{1p} + a_{22}X_{2p} + a_{23}X_{3p} + a_{20} &= 0 \\
a_{31}X_{1p} + a_{32}X_{2p} + a_{33}X_{3p} + a_{30} &= 0
\end{align*}
\]

(4)

where the coefficients $a_{ij}$ are:

\[
\begin{align*}
a_{11} &= \delta_{1i} + \beta_{il} + \beta_{ir} \\
a_{12} &= \delta_{2i} + \beta_{2i} + \beta_{2r} + f_{i}\beta_{il} + f_{2}\beta_{ir} \\
a_{13} &= \delta_{3i} + \beta_{3i} + \beta_{3r} + l_{2}\beta_{3l} - l_{1}\beta_{il} \\
a_{10} &= \Delta_{1p} + \beta_{pl}\delta_{i} + \beta_{pr}\delta_{i} \\
a_{21} &= \delta_{2i} + u_{il} + u_{2r} + f_{i}\beta_{il} + f_{3}\beta_{ir} \\
a_{22} &= \delta_{2i} + u_{il} + u_{2r} + f_{i}\beta_{il} + f_{3}\beta_{ir} + f_{2}\beta_{2l} + f_{2}\beta_{2r} + f_{2}\beta_{2l} \\
a_{23} &= \delta_{2i} + u_{il} + u_{2r} + l_{1}\beta_{3l} - l_{1}\beta_{il} + f_{2}\beta_{2l} + f_{2}\beta_{3r} + f_{2}\beta_{3l} \\
a_{20} &= \Delta_{2p} + u_{pl}\delta_{i} + u_{pr}\delta_{i} + f_{i}\beta_{pl}\delta_{i} + f_{2}\beta_{pr}\delta_{i} \\
a_{31} &= \delta_{3i} + v_{il} + v_{3r} + l_{2}\beta_{3l} - l_{1}\beta_{il} \\
a_{32} &= \delta_{3i} + v_{il} + v_{2r} + f_{i}\beta_{il} + f_{2}\beta_{ir} + l_{2}\beta_{2l} + l_{2}\beta_{2r} - l_{1}\beta_{il} - l_{1}\beta_{il} \\
a_{33} &= \delta_{3i} + v_{il} + v_{3r} - l_{1}\beta_{il} + l_{2}\beta_{3l} + l_{2}\beta_{3r} - l_{1}\beta_{il} + l_{1}\beta_{il} \\
a_{30} &= \Delta_{3p} + v_{pl}\delta_{i} + v_{pr}\delta_{i} + l_{2}\beta_{pl}\delta_{i} - l_{1}\beta_{pl}\delta_{i}
\end{align*}
\]

(5)

According to Equation (4), $X_{1p}$, $X_{2p}$ and $X_{3p}$ can be obtained as:
\[ X_{1p} = \frac{\Delta_1}{\Delta} \]
\[ X_{2p} = \frac{\Delta_2}{\Delta} \]
\[ X_{3p} = \frac{\Delta_3}{\Delta} \]

where \( \Delta = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad \Delta_1 = \begin{bmatrix} a_{10} & a_{12} & a_{13} \\ a_{20} & a_{22} & a_{23} \\ a_{30} & a_{32} & a_{33} \end{bmatrix}, \quad \Delta_2 = \begin{bmatrix} a_{11} & a_{10} & a_{13} \\ a_{21} & a_{20} & a_{23} \\ a_{31} & a_{30} & a_{33} \end{bmatrix} \) and \( \Delta_3 = -\begin{bmatrix} a_{11} & a_{12} & a_{10} \\ a_{21} & a_{22} & a_{20} \\ a_{31} & a_{32} & a_{30} \end{bmatrix}. \)

Based on the knowledge of structural mechanics, the unknown coefficients for the arch ring during the derivation of analytical solutions can be obtained. The basic structure of arch ring under the unit loads \( X_i = 1 \) \( (i=1,2,3) \) is presented in Figure 4, where the internal forces of the arch ring corresponding to the unit load \( X_i = 1 \), \( X_2 = 1 \) and \( X_3 = 1 \) can be obtained respectively. For \( X_1 = 1 \), \( M_1 = 1 \), \( N_1 = 1 \) and \( Q_1 = 1 \). For \( X_2 = 1 \), \( M_2 = R(1-\cos \phi_i) \), \( N_2 = \cos \phi_i \), \( Q_2 = -\sin \phi_i \) for left half arch ring and \( M_2 = R(1-\cos \phi_i) \), \( N_2 = \cos \phi_i \), \( Q_2 = \sin \phi_i \) for right half arch ring. For \( X_3 = 1 \), \( M_3 = -R\sin \phi_i \), \( N_3 = \sin \phi_i \), \( Q_3 = \cos \phi_i \) for left half arch ring and \( M_3 = R\sin \phi_i \), \( N_3 = -\sin \phi_i \), \( Q_3 = \cos \phi_i \) for right half arch ring.

![Figure 4](image-url)

**Figure 4.** The basic structure of left arch ring under the unit load.

Usually, only the effect of the bending moments is considered during the derivation of arch ring coefficients because the effect of axial forces and shear forces on displacement is relatively small\[15\]. However, this paper considers all the internal forces for the sake of precision. The flexibility coefficients \( \delta_{ij} \) for the arch ring can be determined by:

\[ \delta_{ij} = \sum \frac{M_i M_j}{EI} ds + \sum \frac{N_i N_j}{EA} ds + \sum \frac{kQ_i Q_j}{GA} ds \]

where, \( M_i (M_j) \), \( N_i (N_j) \) and \( Q_i (Q_j) \) represent the internal forces of the arch ring under the unit loads \( X_i(X) = 1 \) \((i,j = 1,2,3)\) at the crown of the arch ring; \( E \) and \( G \) are the elastic modulus and shear modulus of the arch ring; \( I \) is the polar moment of inertia of the arch ring’s cross-section with respect to the longitudinal direction; \( A \) is the area of the cross section; \( k \) is the nonuniform coefficient of the shear stress, which equals to 1.2 for the rectangular cross-section in this paper; \( s \) is the arch length.
Substituting the corresponding internal forces obtained according to Figure 4 into Equation (7) for \(i, j=1,2,3\), respectively, the flexibility coefficients for the left tunnel are obtained as:

\[
\delta_{11} = \frac{R}{EI} (\phi_a + \phi_c) \\
\delta_{22} = \frac{R^3}{EI} \left( \frac{3}{2} (\phi_a + \phi_c) - 2(\sin \phi_a + \sin \phi_c) + \frac{1}{4} (\sin 2\phi_a + \sin 2\phi_c) \right) \\
+ \left( \frac{R}{EA} + \frac{kr}{GA} \right) \left( \frac{1}{2} (\phi_a + \phi_c) + \frac{kr}{GA} \right) (\sin 2\phi_a + \sin 2\phi_c) \\
\delta_{33} = \frac{R^3}{EI} \left[ \frac{1}{2} (\phi_a + \phi_c) - \frac{1}{4} (\sin 2\phi_a + \sin 2\phi_c) \right] + \left( \frac{R}{EA} + \frac{kr}{GA} \right) \left[ \frac{1}{2} (\phi_a + \phi_c) + \frac{1}{4} (\sin 2\phi_a + \sin 2\phi_c) \right]
\]

The \(\phi_i\) in Equation (8) ranges from \(0~\phi_h\) for the left half arch ring and \(0~\phi_d\) for the right half arch ring. The flexibility coefficients for the right tunnel could be determined by replacing \(\phi_h, \phi_d\) by \(\phi'_h, \phi'_d\) in the same formulae of the left tunnel.

The displacement coefficients \(\Delta_p\) for the arch ring can be determined by:

\[
\Delta_p = \sum \left[ \frac{M_p^o}{EI} ds + \frac{N_p^o}{EA} ds + \sum \frac{kQ_p^o ds}{GA} \right]
\]

where \(M_p^o, N_p^o\), and \(Q_p^o\) denote the internal forces under the active pressures.

For the unsymmetrical pressure shown in Figure 5, \(M_p^o = \left[ -\frac{1}{2} q_o R^2 \sin^2 \phi_i - \frac{1}{6} R^3 \sin^3 \phi_i \cdot \tan \alpha \right] \), \(N_p^o = R \sin^2 \phi_i \cdot (q_o - \frac{1}{2} R \tan \alpha \cdot \sin \phi_i) \), \(Q_p^o = R \sin \phi_i \cdot \cos \phi_i \cdot (q_o - \frac{1}{2} R \tan \alpha \cdot \sin \phi_i) \) for left half arch ring and \(M_p^o = \left[ -\frac{1}{2} q_o R^2 \sin^2 \phi_i + \frac{1}{6} R^3 \sin^3 \phi_i \cdot \tan \alpha \right] \), \(N_p^o = R \sin^2 \phi_i \cdot \cos \phi_i \cdot (q_o + \frac{1}{2} R \tan \alpha \cdot \sin \phi_i) \), \(Q_p^o = -[R \sin \phi_i \cdot \cos \phi_i \cdot (q_o + \frac{1}{2} R \tan \alpha \cdot \sin \phi_i)] \) for right half arch ring.
Substituting the corresponding internal forces obtained according to Figure 4 and Figure 5 into Equation (9) for $i, j = 1, 2, 3$, respectively, the displacement coefficients for the arch ring under the vertical unsymmetrical pressure could be obtained as:

$$
\Delta_{\phi_{1}} = -\frac{R^{3}q_{o}}{4EI}(\phi_{h} + \phi_{d}) + \frac{R^{3}q_{o}}{8EI}(\sin 2\phi_{h} + \sin 2\phi_{d}) + \frac{R^{4}\tan \alpha}{6EI}(\cos \phi_{d} - \cos \phi_{h})
$$

$$
+ \frac{R^{4}\tan \alpha}{18EI}(\cos^{3} \phi_{h} - \cos^{3} \phi_{d})
$$

$$
\Delta_{\phi_{2}} = -\frac{1}{6} \frac{1}{q_{o}R^{4}(\phi_{h} + \phi_{d}) - \frac{1}{8} q_{o}R^{4}(\sin 2\phi_{h} + \sin 2\phi_{d}) - \frac{1}{6} q_{o}R^{4}(\sin^{3} \phi_{h} + \sin^{3} \phi_{d})
$$

$$
- \frac{1}{24} R^{3}\tan \alpha(\cos \phi_{d} - \cos \phi_{h}) + \frac{1}{3} (\cos^{3} \phi_{h} - \cos^{3} \phi_{d}) - \frac{1}{24} R^{3}\tan \alpha(\sin^{3} \phi_{d} - \sin^{3} \phi_{h})
$$

$$
+ \frac{1}{8} R^{3}\tan \alpha(\sin 2\phi_{h} + \sin 2\phi_{d}) - \frac{1}{16} R^{3}\tan \alpha(\sin 4\phi_{h} + \sin 4\phi_{d})
$$

$$
\Delta_{\phi_{3}} = \frac{1}{6} \frac{1}{EI} \frac{1}{2} q_{o}R^{4}(\cos \phi_{h} - \cos \phi_{d}) + \frac{1}{8} q_{o}R^{4}(\cos^{3} \phi_{h} - \cos^{3} \phi_{d}) - \frac{1}{16} R^{3}\tan \alpha(\phi_{h} + \phi_{d})
$$

$$
+ \frac{1}{24} R^{3}\tan \alpha(\sin 2\phi_{h} + \sin 2\phi_{d}) - \frac{1}{192} R^{3}\tan \alpha(\sin 4\phi_{h} + \sin 4\phi_{d})
$$

$$
+ \frac{1}{64} R^{3}\tan \alpha(\sin 4\phi_{h} + \sin 4\phi_{d})
$$

$$
+ \frac{1}{3} q_{o}R^{4}(\cos^{3} \phi_{d} - \cos^{3} \phi_{h}) + \frac{1}{16} R^{3}\tan \alpha(\phi_{h} + \phi_{d}) + \frac{1}{64} R^{3}\tan \alpha(\sin 4\phi_{h} + \sin 4\phi_{d})
$$

Using the same method, the internal forces and displacement coefficients of the arch ring under the other kind of pressures namely $q^{*}$, $\varepsilon^{*}$ and $\varepsilon'^{*}$ shown in Figure 3 could be obtained, which can be found in [20]. The displacement coefficients under the active pressures are the superposition of the obtained coefficients. The remaining unknown coefficients on top of the side and middle walls could be calculated by the elastic beam model and the force analysis on the top of the middle wall, which can be found in [20] and are not shown in the paper for simplicity.

The Equation (6) gives the redundant forces $X_{1p}$, $X_{2p}$ and $X_{3p}$, thus the internal forces along the arch ring can be determined according to the static equilibrium condition, as presented in Equation (11) for left half arch ring and Equation (12) for right half arch ring, respectively, where $M_{ip}$, $N_{ip}$ and $Q_{ip}$ are the internal forces along arch ring under all the active pressures. The bending moment $M_{ia}$ is positive when the intrados of the arch ring is in tension, the axial force $N_{ia}$ is positive when the arch ring is under compression, while the shear force $Q_{ia}$ leading to the clockwise rotation of the arch section is positive.

$$
\begin{align*}
M_{ia} &= X_{1p} + X_{2p}y_{1} - X_{3p}x_{1} + M_{ip}^{o} \\
N_{ia} &= X_{2p} \cos \phi_{1} + X_{3p} \sin \phi_{1} + N_{ip}^{o} \\
Q_{ia} &= -X_{2p} \sin \phi_{1} + X_{3p} \cos \phi_{1} + Q_{ip}^{o} \\
\end{align*}
$$

$$
\begin{align*}
M_{ia} &= X_{1p} + X_{2p}y_{1} + X_{3p}x_{1} + M_{ip}^{o} \\
N_{ia} &= X_{2p} \cos \phi_{1} - X_{3p} \sin \phi_{1} + N_{ip}^{o} \\
Q_{ia} &= X_{2p} \sin \phi_{1} + X_{3p} \cos \phi_{1} + Q_{ip}^{o} \\
\end{align*}
$$

By using the same method presented above, the analytical solutions of internal forces for the right tunnel arch ring can be determined accordingly.
4. Comparing the analytical solutions with numerical simulations

4.1 Model overview

The tunnel selected in this section is the Nandao river tunnel in the Sixiao freeway in Yunnan province in China. The cross section of the tunnel is shown in Figure 6.

The external diameter of the second lining is \( R = 6 \) m, the thickness of which is \( H = 0.6 \) m. The total width of the tunnel is \( B = 24.75 \) m. The heights of the side wall and middle wall are \( l = 3.29 \) m and \( h = 5.25 \) m, respectively. The angle of the joint of side wall and arch foot and the symmetrical middle line of the arch ring is \( \varphi_h = 90^\circ \), and the angle of the joint of middle wall and arch foot and the symmetrical middle line of the arch ring is \( \varphi_d = 71^\circ 38' 10'' \). The elastic modulus of the lining is \( E = 29.5 \) GPa, the shear modulus is \( G = 12.685 \) GPa, and the Poisson ratio is \( \mu = 0.2 \). The unit weight of the lining is \( \gamma = 25 \) kN/m\(^3\).

The geological conditions around the tunnel are as following. The grade of the surrounding rock is V. The unit weight of the surrounding rock is \( \gamma' = 19 \) kN/m\(^3\). The elastic modulus is \( E' = 1 \sim 2 \) GPa, and the Poisson ratio is \( \mu' = 0.35 \sim 0.45 \). The elastic resistance coefficient is \( K = 100 \sim 200 \) MPa/m. The calculating friction angle of surrounding rock is \( \phi = 40^\circ \sim 50^\circ \). The frictional angle of the two sides of the rock column over the crown of the arch is \( \theta = (0.5 \sim 0.7)\phi \). The height from arch crown to the surface slope for the shallowly-buried side is \( h = 4 \) m, and \( h' = 17.7 \) m for the deeply-buried side.

4.2 FEM numerical modelling

The 2D FEM model utilizing load-structure-method is adopted. The active pressures exerted on the double-arched tunnel are determined based on the method in Section 2.2 using the parameters in Section 4.1. The FEM analyses are carried out under the plane-strain condition and the model is shown in Figure 7, where only the secondary lining is simulated by the beam elements according to the dimensions in Figure 6. The extrados and bottom of both side walls are restricted by unidirectional compression springs with a stiffness \( k = 200 \) MPa/m modelling the elastic resistances. The bottoms of the side walls are fixed in both horizontal and vertical directions and the bottom of middle wall is fixed.

4.3 Comparison of analytical solutions and numerical simulations

The internal forces for both arch rings derived from analytical solutions in comparison with those from the numerical simulations are presented in Figure 8 and Figure 9, respectively.

It can be found that the variation trend and the distribution laws of the internal forces derived from both analytical solutions and numerical solutions are very similar to each other. The discrepancies are acceptable, except that the bending moment obtained from analytical solutions deviates increasingly from the numerical simulation results when approaching the middle wall. This may be caused by the selection of elastic resistance coefficient \( k_d \) between the top of the middle wall and the arch ring, which is equal to the elastic modulus \( E \) of the lining in the analytical solutions. Therefore, additional researches may be needed to investigate the real contact characteristics of the top of the middle wall and arch rings. Despite the discrepancies for the bending moment near the top of middle wall, the analytical solutions
overall have a good agreement with the numerical solutions, which indicates that the analytical solutions for the arch rings are reliable.

The results in Figure 8 and Figure 9 show that the largest positive bending moment appears at the arch crown, while larger negative bending moment is located at both haunches of arch rings, which means that the crown intrados and haunch extrados are in larger tension and should be paid more attention during design and construction. The axial forces of both ends of arch rings are larger than that at the arch crown, which indicates that the side and middle walls are in larger compression than the arch crown. The larger shear forces of arch rings appear at the cross section approaching the top of side walls, which may be caused by the application of elastic resistance to the side wall.

![Figure 8. Comparison of the internal forces of left arch ring obtained from the analytical solutions and numerical simulations: (a) bending moment; (b) axial force; (c) shear force.](image)

![Figure 9. Comparison of the internal forces of right arch ring obtained from the analytical solutions and numerical simulations: (a) bending moment; (b) axial force; (c) shear force.](image)

The results from the analytical solutions also show that a larger magnitude of bending moment and shear force are present in the left arch ring than that in the right arch ring, which implies that the left
tunnel is in the shallowly-buried side and suffers more thrust and deformation. The axial force is in the opposite case, meaning that the right tunnel is in the deeply-buried side and suffers larger vertical pressures. This tendency further confirms the validity of the analytical solutions for the double-arched tunnel subjected to the unsymmetrical loads.

5 Conclusions
This paper derived the analytical solutions for determining the internal forces of arch rings in shallow double-arched tunnel lining structure subjected to unsymmetrical loads. The proposed analytical solutions were compared with the results obtained from the numerical simulations, and a reasonable agreement were presented. A further analysis of the variation trend of analytical solutions confirms the validity of the derived analytical solutions.

The obtained analytical solutions mainly focus on the double-arched tunnel in shallow depth subjected to unsymmetrical pressures, while in practical engineering the arch ring may also be affected by the elastic resistance, also the analytical solutions for the side and middle walls need to be present, which are not considered in this paper and deserve further investigations.

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