A further multiplicity result for Lagrangian systems of relativistic oscillators

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Dedicated to Professor Adrian Petruşel on his 60th birthday

Abstract. This is our third paper, after [4] and [5], about a joint application of the theory developed by Brezis and Mawhin in [1] with our minimax theorems ([2], [3]) to get multiple solutions of problems of the type

\[
\begin{aligned}
(\phi(u'))' &= \nabla_x F(t,u) & \text{in } [0,T] \\
u(0) &= u(T), \ u'(0) = u'(T)
\end{aligned}
\]

which are global minima of a suitable functional over a set of Lipschitzian functions. A challenging conjecture is also formulated.

Key words: periodic solution; Lagrangian system of relativistic oscillators; minimax; multiplicity; global minimum; non-convex range.

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1. Introduction

In what follows, \(L, T\) are two fixed positive numbers. For each \(r > 0\), we set \(B_r = \{x \in \mathbb{R}^n : |x| < r\}\) (\(\cdot, \cdot\) being the Euclidean norm on \(\mathbb{R}^n\)) and \(\overline{B}_r\) is the closure of \(B_r\). The scalar product on \(\mathbb{R}^n\) is denoted by \((\cdot, \cdot)\).

We denote by \(A\) the family of all homeomorphisms \(\phi\) from \(B_L\) onto \(\mathbb{R}^n\) such that \(\phi(0) = 0\) and \(\phi = \nabla \Phi\), where the function \(\Phi : \overline{B}_L \to ]-\infty, 0]\) is continuous and strictly convex in \(\overline{B}_L\), and of class \(C^1\) in \(B_L\). Notice that \(0\) is the unique global minimum of \(\Phi\) in \(B_L\).

We denote by \(B\) the family of all functions \(F : [0,T] \times \mathbb{R}^n \to \mathbb{R}\) which are measurable in \([0,T]\), of class \(C^1\) in \(\mathbb{R}^n\) and such that \(\nabla_x F\) is measurable in \([0,T]\) and, for each \(r > 0\), one has \(\sup_{x \in B_r} |\nabla_x F(\cdot,x)| \in L^1([0,T])\), with \(F(\cdot,0) \in L^1([0,T])\). Clearly, \(B\) is a linear subspace of \(\mathbb{R}^{[0,T] \times \mathbb{R}^n}\).

Given \(\phi \in A\) and \(F \in B\), we consider the problem

\[
\begin{aligned}
(\phi(u'))' &= \nabla_x F(t,u) & \text{in } [0,T] \\
u(0) &= u(T), \ u'(0) = u'(T)
\end{aligned}
\]

\((P_{\phi,F})\)

A solution of this problem is any function \(u : [0,T] \to \mathbb{R}^n\) of class \(C^1\), with \(u'([0,T]) \subset B_L\), \(u(0) = u(T), u'(0) = u'(T)\), such that the composite function \(\phi \circ u'\) is absolutely continuous in \([0,T]\) and one has \((\phi \circ u')'(t) = \nabla_x F(t,u(t))\) for a.e. \(t \in [0,T]\).

Now, we set \(K = \{u \in \text{Lip}([0,T],\mathbb{R}^n) : |u'(t)| \leq L \ for \ a.e. \ t \in [0,T], u(0) = u(T)\}\), \(\text{Lip}([0,T],\mathbb{R}^n)\) being the space of all Lipschitzian functions from \([0,T]\) into \(\mathbb{R}^n\).

Clearly, one has

\[
\sup_{[0,T]} |u| \leq LT + \inf_{[0,T]} |u| \quad (1.1)
\]

for all \(u \in K\).
Next, consider the functional $I : K \rightarrow \mathbb{R}$ defined by

$$I(u) = \int_0^T (\Phi(u'(t)) + F(t, u(t)))dt$$

for all $u \in K$.

In [1], Brezis and Mawhin proved the following result:

**THEOREM 1.1** ([1], Theorem 5.2). - *Any global minimum of $I$ in $K$ is a solution of problem $(P_{\phi, F})$.*

On the other hand, very recently, in [6], we established the following:

**THEOREM 1.2** ([6], Theorem 2.2). - *Let $X$ be a topological space, let $E$ be a real normed space, let $I : X \rightarrow \mathbb{R}$, let $\psi : X \rightarrow E$ and let $S \subseteq \text{conv}$ be a convex set weakly-star dense in $E^*$. Assume that $\psi(X)$ is not convex and that $I + \eta \circ \psi$ is lower semicontinuous and inf-compact for all $\eta \in S$.

Then, there exists $\tilde{\eta} \in S$ such that the function $I + \tilde{\eta} \circ \psi$ has at least two global minima in $X$.

The aim of this paper is to establish a new multiplicity result for the solutions of problem $(P_{\phi, F})$ as a joint application of Theorems 1.1 and 1.2.

Notice that [4] and [5] are the only previous papers on multiple solutions for problem $(P_{\phi, F})$ which are global minima of $I$ in $K$.

2. The result

Here is our result:

**THEOREM 2.1.** - Let $\phi \in \mathcal{A}$, $F, G \in \mathcal{B}$ and $H \in C^1(\mathbb{R}^n)$. Assume that:

(a) there exists $q > 0$ such that

$$\lim_{|x| \rightarrow +\infty} \frac{\inf_{t \in [0, T]} F(t, x)}{|x|^q} = +\infty$$

and

$$\lim_{|x| \rightarrow +\infty} \frac{\sup_{t \in [0, T]} |G(t, x)| + |H(x)|}{|x|^q} < +\infty;$$

(b) there are $\gamma \in \{\inf_{R^n}, H, \sup_{R^n}, H\}$, with $H^{-1}(\gamma)$ at most countable, and $v, w \in H^{-1}(\gamma)$ such that

$$\int_0^T G(t, v)dt \neq \int_0^T G(t, w)dt.$$

Then, for each $\alpha \in L^\infty([0, T])$ having a constant sign and with $\text{meas}(\alpha^{-1}(0)) = 0$, there exists $(\lambda, \mu) \in \mathbb{R}^2$ such that the problem

$$\begin{cases}
(\phi(u'))' = \nabla_x \left( F(t, u) + \lambda G(t, u) + \mu \alpha(t) H(u) \right) & \text{in } [0, T] \\
u(0) = u(T), \quad u'(0) = u'(T)
\end{cases}$$

has at least two solutions which are global minima in $K$ of the functional

$$u \rightarrow \int_0^T (\Phi(u'(t)) + F(t, u(t)) + \lambda G(t, u(t)) + \mu \alpha(t) H(u(t)))dt.$$

**PROOF.** Fix $\alpha \in L^\infty([0, T])$ having a constant sign and with $\text{meas}(\alpha^{-1}(0)) = 0$. Let $C^0([0, T], \mathbb{R}^n)$ be the space of all continuous functions from $[0, T]$ into $\mathbb{R}^n$, with the norm $\sup_{[0, T]} |u|$. We are going to apply Theorem 1.2 taking $X = K$, regarded as a subset of $C^0([0, T], \mathbb{R}^n)$ with the relative topology, $E = \mathbb{R}^2$ and $I : K \rightarrow \mathbb{R}$, $\psi : K \rightarrow \mathbb{R}^2$ defined by

$$I(u) = \int_0^T (\Phi(u'(t)) + F(t, u(t)))dt.$$
\[
\psi(u) = \left( \int_0^T G(t, u(t)) dt, \int_0^T \alpha(t) H(u(t)) dt \right)
\]

for all \( u \in K \). Fix \((\lambda, \mu) \in \mathbb{R}^2\). By Lemma 4.1 of [1], the function \( I(\cdot) + \langle \psi(\cdot), (\lambda, \mu) \rangle \) is lower semicontinuous in \( K \). Let us show that it is inf-compact too. First, observe that if \( P \in \mathcal{B} \) then, for each \( r > 0 \), there is \( M \in L^1([0, T]) \) such that

\[
\sup_{x \in B_r} |P(t, x)| \leq M(t)
\]

for all \( t \in [0, T] \). Indeed, by the mean value theorem, we have

\[
P(t, x) - P(t, 0) = \langle \nabla_x P(t, \xi), x \rangle
\]

for some \( \xi \) in the segment joining \( 0 \) and \( x \). Consequently, for all \( t \in [0, T] \) and \( x \in B_r \), by the Cauchy-Schwarz inequality, we clearly have

\[
|P(t, x)| \leq r \sup_{y \in B_r} |\nabla_x P(t, y)| + |P(t, 0)|.
\]

So, to get (2.1), we can choose \( M(t) := r \sup_{y \in B_r} |\nabla_x P(t, y)| + |P(t, 0)| \) which is in \( L^1([0, T]) \) since \( P \in \mathcal{B} \).

Now, by \((a_1)\), we can fix \( c_1, \delta > 0 \) so that

\[
|G(t, x)| + |H(x)| \leq c_1|x|^q
\]

for all \((t, x) \in [0, T] \times (\mathbb{R}^n \setminus B_\delta)\). Then, set

\[
c_2 := c_1 \max \{ |\lambda|, |\mu||\alpha| \}_{L^\infty(0,T)} \}
\]

and, by \((a_1)\) again, fix \( c_3 > c_2 \) and \( \delta_1 > \delta \) so that

\[
F(t, x) \geq c_3|x|^q
\]

for all \((t, x) \in [0, T] \times (\mathbb{R}^n \setminus B_{\delta_1})\). On the other hand, for what remarked above, there is \( M \in L^1([0, T]) \) such that

\[
\sup_{x \in B_{\delta_1}} (|F(t, x)| + |\lambda G(t, x)| + |\mu \alpha(t) H(x)|) \leq M(t)
\]

for all \( t \in [0, T] \). Therefore, from (2.2), (2.3) and (2.4), we infer that

\[
F(t, x) \geq c_3|x|^q - M(t)
\]

and

\[
|\lambda G(t, x)| + |\mu \alpha(t) H(x)| \leq c_2|x|^q + M(t)
\]

for all \((t, x) \in [0, T] \times \mathbb{R}^n\). Set

\[
b := T\Phi(0) - 2 \int_0^T M(t)dt.
\]

For each \( u \in K \), with \( \sup_{[0,T]} |u| \geq LT \), taking (1.1), (2.5) and (2.6) into account, we have

\[
I(u) + \langle \psi(u), (\lambda, \mu) \rangle \geq T\Phi(0) + \int_0^T F(t, u(t)) dt - \int_0^T |\lambda G(t, u(t))| dt - \int_0^T |\mu \alpha(t) H(u(t))| dt
\]

\[
\geq T\Phi(0) + c_3 \int_0^T |u(t)|^q dt - \int_0^T M(t)dt - c_2 \int_0^T |u(t)|^q dt - \int_0^T M(t)dt
\]

\[
\geq (c_3 - c_2) T \inf_{[0,T]} |u|^q - b \geq (c_3 - c_2) T \left( \sup_{[0,T]} |u| - LT \right)^q + b.
\]
Consequently
\[
\sup_{[0,T]} |u| \leq \left( \frac{I(u) + \langle \psi(u), (\lambda, \mu) \rangle - b}{(c_3 - c_2)T} \right)^\frac{1}{p} + LT. \tag{2.7}
\]

Fix \( \rho \in \mathbb{R} \). By (2.7), the set
\[
C_\rho := \{ u \in K : I(u) + \langle \psi(u), (\lambda, \mu) \rangle \leq \rho \}
\]
turns out to be bounded. Moreover, the functions belonging to \( C_\rho \) are equi-continuous since they lie in \( K \). As a consequence, by the Ascoli-Arzelà theorem, \( C_\rho \) is relatively compact in \( C^0([0,T], \mathbb{R}^n) \). By lower semicontinuity, \( C_\rho \) is closed in \( K \). But \( K \) is closed in \( C^0([0,T], \mathbb{R}^n) \) and hence \( C_\rho \) is compact. The inf-compactness of \( I(\cdot) + \langle \psi(\cdot), (\lambda, \mu) \rangle \) is so shown. Now, we are going to prove that the set \( \psi(K) \) is not convex. By \((a_2), \) the set \( \{ \int_0^T G(t,x)dt : x \in H^{-1}(\gamma) \} \) is at most countable since \( H^{-1}(\gamma) \) is so. Hence, since \( \int_0^T G(t,v)dt \neq \int_0^T G(t,w)dt \), we can fix \( \lambda \in ]0,1[ \) so that
\[
\int_0^T G(t,x)dt \neq \int_0^T G(t,w)dt + \lambda \left( \int_0^T G(t,v)dt - \int_0^T G(t,w)dt \right) \tag{2.8}
\]
for all \( x \in H^{-1}(\gamma) \). Since \( K \) contains the constant functions, the points
\[
\left( \int_0^T G(t,v)dt, \gamma \int_0^T \alpha(t)dt \right)
\]
and
\[
\left( \int_0^T G(t,w)dt, \gamma \int_0^T \alpha(t)dt \right)
\]
belong to \( \psi(K) \). So, to show that \( \psi(K) \) is not convex, it is enough to check that the point
\[
\left( \int_0^T G(t,w)dt + \lambda \left( \int_0^T G(t,v)dt - \int_0^T G(t,w)dt \right), \gamma \int_0^T \alpha(t)dt \right)
\]
does not belong to \( \psi(K) \). Arguing by contradiction, suppose that there exists \( u \in K \) such that
\[
\int_0^T G(t,u(t))dt = \int_0^T G(t,w)dt + \lambda \left( \int_0^T G(t,v)dt - \int_0^T G(t,w)dt \right), \tag{2.9}
\]
\[
\int_0^T \alpha(t)H(u(t))dt = \gamma \int_0^T \alpha(t)dt. \tag{2.10}
\]
Since the functions \( \alpha \) and \( H \circ u - \gamma \) do not change sign, (2.10) implies that \( \alpha(t)(H(u(t)) - \gamma) = 0 \ a.e. \ in [0,T] \). Consequently, since \( \text{meas}(\alpha^{-1}(0)) = 0 \), we have \( H(u(t)) = \gamma \ a.e. \ in [0,T] \) and hence \( H(u(t)) = \gamma \) for all \( t \in [0,T] \) since \( H \circ u \) is continuous. In other words, the connected set \( u([0,T]) \) is contained in \( H^{-1}(\gamma) \) which is at most countable. This implies that the function \( u \) must be constant and so (2.9) contradicts (2.8). Therefore, \( I \) and \( \psi \) satisfy the assumptions of Theorem 1.2 and hence there exists \( (\hat{\lambda}, \hat{\mu}) \in \mathbb{R}^2 \) such that the function \( I(\cdot) + \langle \psi(\cdot), (\hat{\lambda}, \hat{\mu}) \rangle \) has at least two global minima in \( K \). Thanks to Theorem 1.1, they are solutions of Problem (P), and the proof is complete. \( \triangle \)

REMARK 1. Of course, \((a_2)\) is the leading assumption of Theorem 2.1. The request that \( H^{-1}(\gamma) \) must be at most countable cannot be removed. Indeed, if we remove such a request, we could take \( H = 0 \), \( G(t,x) = \langle x, \omega \rangle \), with \( \omega \in \mathbb{R}^n \setminus \{0\} \) and \( F(t,x) = \frac{2}{p}|x|^p \), with \( p > 1 \). Now, observe that, by Proposition 3.2 of [1], for all \( \lambda \in \mathbb{R} \), the problem

\[
\begin{cases}
(\phi(u'))' = |u|^{p-2}u + \lambda\omega & \text{in } [0,T] \\
u(0) = u(T), \ u'(0) = u'(T)
\end{cases}
\]
has a unique solution. To the contrary, the question of whether the condition $\int_0^T G(t, v) dt \neq \int_0^T G(t, w) dt$ (keeping $v \neq w$) can be dropped remains open at present. We feel, however, that it cannot be removed. In this connection, we propose the following

**CONJECTURE 2.1.** - There exist $\phi \in A$, $F \in B$, $H \in C^1(\mathbb{R}^n)$, $\alpha \in L^\infty([0, T])$, with $\alpha \geq 0$ and $\text{meas}(\alpha^{-1}(0)) = 0$, and $q > 0$ for which the following assertions hold:

(b1) \[ \lim_{|x| \to +\infty} \inf_{t \in [0, T]} \frac{F(t, x)}{|x|^q} = +\infty \]

and

(b2) the function $H$ has exactly two global minima;

(b3) for each $\mu \in \mathbb{R}$, the functional

\[ u \to \int_0^T (\Phi(u'(t)) + F(t, u(t)) + \mu \alpha(t)H(u(t))) dt \]

has a unique global minimum in $K$.

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