Nonlinear stability of the slowly-rotating Kerr-de Sitter family

Allen Juntao Fang

Abstract

In this paper, we provide a new proof of nonlinear stability of the slowly-rotating Kerr-de Sitter family of black holes as a family of solutions to the Einstein vacuum equations with cosmological constant $\Lambda > 0$, originally established by Hintz and Vasy in their seminal work [10]. Using the linear theory developed in the companion paper [8], we prove the nonlinear stability of slowly-rotating Kerr-de Sitter using a bootstrap argument, avoiding the need for a Nash-Moser argument, and requiring initial data small only in the $H^6$ norm.

1 Introduction

The aim of this paper is to use the linear theory developed in [8] in order to provide a new proof of the nonlinear stability of the slowly-rotating Kerr-de Sitter family of black hole solutions to Einstein’s vacuum equations.

1.1 The black hole stability problem

In this paper, we consider the Einstein vacuum equations (EVE) which govern Einstein’s theory of relativity under the assumption of vacuum and are given by

$$\text{Ric}(g) - \Lambda g = 0,$$

where $g$ is a Lorentzian metric with signature $(-, +, +, +)$ on a $3 + 1$ dimensional manifold $\mathcal{M}$, $\text{Ric}$ denotes its Ricci tensor, and $\Lambda$ is the cosmological constant.

Of particular interest are black hole solutions to (1.1). An explicit two-parameter family of black hole solutions to EVE with $\Lambda = 0$ is given by the asymptotically flat Kerr black hole solutions. The Kerr family represents a family of rotating, uncharged black hole solution, and is of particular interest due to the final state conjecture, which states that all rotating non-charged black holes are asymptotically Kerr.

A central conjecture in the field is the stability of Kerr black holes, which surmises that small perturbations of a Kerr spacetime asymptote to a nearby member of the Kerr family. There has been extensive work done towards addressing this question. The most simple member of the Kerr family is in fact the Minkowski spacetime, which does not itself contain a black hole. The stability of Minkowski was first shown in the breakthrough result of Christodoulou-Klainerman in [5]. Following substantial developments in the field, nonlinear stability of Schwarzschild was then shown by Klainerman-Szeftel in [15], and by Dafermos-Holzegel-Rodnianski in [6]. Most recently, Klainerman-Szeftel were able to obtain a nonlinear stability result for the slowly-rotating Kerr family [16]. Behind these nonlinear results, there was of course substantial work done on the stability of linear problems on black hole backgrounds. For an overview of the literature on

---

*Sorbonne Université, CNRS, Laboratoire Jacques-Louis Lions (LJLL), F-75005 Paris, France, fanga@ljll.math.upmc.fr

1 For black hole solutions, the question of stability is posed at the level of a given family of black hole solutions rather than an individual solution. This is in line with the physical expectation that nontrivial perturbations of a black hole alter its mass and angular momentum.
linear waves on black hole backgrounds and linearized stability of Einstein’s equations, we refer the reader to the introduction of [8].

A close relative of the Kerr family of solutions are the Kerr-de Sitter family of black hole solutions, which is the main focus of the present paper. Like their Kerr cousins, the Kerr-de Sitter black hole spacetimes are a 2-parameter family of black hole solutions to EVE representing rotating, uncharged black holes. The key difference between the two is that members of the Kerr-de Sitter family solve EVE with $\Lambda > 0$, while members of the Kerr family solve EVE with $\Lambda = 0$.

The Kerr-de Sitter family features many of the same inherent geometric obstacles to stability as the Kerr family. Both families feature trapped null geodesics which do not escape to either the black hole horizon or null infinity (the cosmological horizon in the case of Kerr-de Sitter). Ergoregions are also present in members of both families, reflecting a lack of a global timelike Killing vectorfield in both Kerr and Kerr-de Sitter metrics. However, it is generally expected that proving stability of Kerr-de Sitter is substantially easier than proving stability for Kerr, owing to the spatially compact nature of the domain of outer communication of members of the Kerr-de Sitter family. For this reason, studying the stability of Kerr-de Sitter is both an interesting question in and of itself, and a potential way of gaining insight to further study of the stability of the Kerr family.

1.2 Statement of the main theorem

We now state the main theorem of this paper.

**Theorem 1.1** (Nonlinear stability of the slowly-rotating Kerr-de Sitter family, first version). Fix $\Lambda > 0$. Assume that the initial data of a solution $g$ of (1.1) are sufficiently close to a slowly-rotating Kerr-de Sitter metric $g_{b^0}$ with black hole parameters $b^0 = (M^0, a^0)$. Then $g$ exists globally and moreover, there exist black hole parameters $b_\infty = (M_\infty, a_\infty)$ close to $(M^0, a^0)$ such that

$$g - g_{b_\infty} = O(e^{-\alpha t}), \quad t_\infty \to \infty,$$

for some $\alpha > 0$ constant that is independent of the initial data.

We refer the reader to Theorem 4.2 for the precise statement of the main theorem. This result was first proven by Hintz and Vasy in their seminal work [10] using harmonic coordinates to treat EVE as a system of quasilinear wave equations in terms of the metric coefficients. Their proof relies on the framework of the method of scattering resonances, using microlocal analysis and the $b$-calculus to prove extremely strong linear stability results, which they then used to solve the nonlinear problem with a Nash-Moser scheme.

As expected, the exponential decay we proved in [8] at the linear level allows for a relatively short nonlinear analysis in the present paper. We provide some comparisons between the proof presented here and the original proof of Hintz and Vasy in [10].

1. Like in the original work of Hintz and Vasy [10], we rely on wave coordinates, and the main estimates used are linear estimates on exact Kerr-de Sitter (see [8], or Chapter 5 of [10]).

2. Instead of employing an iterative Nash-Moser scheme to prove nonlinear stability, we use a standard bootstrap argument. In particular, we solve the true nonlinear Einstein equations up to a finite time, rather than solving a sequence of linearized Einstein equations as in

---

2The part of Kerr-de Sitter treated both in the current paper and in [10] is called the stationary region. One can also look at the stability of the complementary causal region, which is often referred to as the cosmological (or expanding) region. The cosmological region is itself not spatially compact, and features rather different behavior compared to the stationary region cf. [17].
[10]. Doing so preserves the structure of EVE and in particular allows us to avoid the need for constraint damping in contrast with the approach taken by Hintz and Vasy.

3. Using a bootstrap argument instead of Nash-Moser, we close the argument with a much lower level of regularity imposed on the initial data. Using our method, we expect to actually be able to improve the regularity requirements to require only $2 + \delta, \delta > 0$ derivatives (see Remark 6.7). However, for the sake of avoiding fractional Sobolev spaces, we prove the theorem here with initial data in $H^6$ instead. Regardless, this represents an improvement over the 21 derivatives used in [10].

1.3 Strategy of the proof of Theorem 1.1

We make the following three bootstrap assumptions (see Section 6.2):

1. low-regularity exponential decay;

2. high-regularity integrated slow exponential growth; and,

3. that there exists a suitably small gauge choice such that we have a certain orthogonality condition. This argument is similar in spirit to the last slice argument in other black hole stability proofs. See for example [14, 16].

Each of these bootstrap assumptions will be improved in Section 7.

1. Improving the low-regularity exponential decay bootstrap assumption is done in Section 7.1, and is a direct consequence of the linear theory in [8].

2. Improving the high-regularity exponential growth bootstrap assumption is done in Section 7.2. This is done using a weak Morawetz estimate on a perturbation of Kerr-de Sitter which only yields exponential growth, but does not lose derivatives.

3. Improving the gauge bootstrap assumption is done in Section 7.3 by relying on the implicit function theorem.

These improvements of the bootstrap assumptions allow us to extend the bootstrap time (see Section 7.4). This then allows us to use a continuity argument in 7.5 to conclude the proof of Theorem 1.1.

The previously mentioned weak Morawetz estimate needed to improve the high-regularity exponential growth bootstrap assumption is a high-frequency Morawetz estimate on exponentially decaying perturbations of a slowly-rotating Kerr-de Sitter background. This Morawetz estimate will be proven by perturbing the high-frequency Morawetz estimate proven in the linear theory (Theorem 8.2 in [8]). The perturbed high-frequency Morawetz estimate will feature arbitrarily slow exponential growth rather than small exponential decay but will no longer lose derivatives. This is critical in improving the high regularity bootstrap assumptions.

1.4 Outline of the paper

In Section 2 we set up the main geometric framework. In particular, we construct regular coordinate systems on the Kerr-de Sitter family and introduce some crucial vectorfields. In Section 3, we define the choice of wave gauge for Einstein’s vacuum equations. In Section 4, we state the precise form of the main theorem and provide an outline of the proof. In Section 5 we review the main points of the linear theory in [8] which will be needed in the nonlinear scheme.

In Section 6, we construct the semi-global extension of EVE, and state out bootstrap assumptions for solutions of the aforementioned semi-global extension. Bootstrap assumptions will be made at the level of high-regularity growth, low-regularity decay, and gauge. Each of
these will in turn be improved and extended in Section 7, leading to a proof of Theorem 1.1 using a standard continuity argument.

Finally, in Section 8, we prove a series of energy estimates on exponentially decaying perturbations of a slowly-rotating Kerr-de Sitter background. These estimates are direct perturbations of estimates in [8] and are needed to improve the high-regularity exponential growth bootstrap assumption.

1.5 Acknowledgments

The author would like to acknowledge Jérémie Szeftel for his support, encouragement, and many helpful discussions. This work is supported by the ERC grant ERC-2016 CoG 725589 EPGR.

2 Geometric set up

In this section, we define key geometric objects that we will make use of later.

2.1 Notational conventions

Throughout the paper, Greek indices will be used to indicate the spacetime indices \{0,1,2,3\}, lower-case Latin indices will be used to represent the spatial indices \{1,2,3\}, and upper-case Latin indices will be used to represent angular indices. We also use \(X^\alpha\) and \(\omega^a\) to denote the canonical one-form associated to the vectorfield \(X\) and the canonical vectorfield associated to the one-form \(\omega\) respectively.

We will use \(\cdot\) to indicate the natural contraction between two objects. For two vectors, \(v_1,v_2\), \(v_1\cdot v_2\) will indicate their dot product, while for tensors \(u,w\), \(u\cdot w\) will indicate tensor contraction.

We will use \(\nabla\) to indicate the full space-time derivative.

2.2 The Kerr-de Sitter family

The Kerr-de Sitter family of black holes, which will be presented explicitly in what follows, are a family of stationary black hole solutions to the Einstein vacuum equations (EVE) under the assumption of a positive cosmological constant \(\Lambda > 0\). The two-parameter family is parameterized by

1. the mass of the black hole \(M\) and,

2. the angular momentum of the black hole \(a = |\mathbf{a}|\), where \(\frac{\mathbf{a}}{|a|}\) is the axis of symmetry of the black hole.

We will denote by \(B\) the set of black-hole parameters \(b = (M,a)\).

In this paper, we will not deal with the full Kerr-de Sitter family, but instead are primarily concerned only a subfamily characterized by two features: first, that the mass of the black hole is subextremal and satisfies \(1 - 9\Lambda M^2 > 0\); and that the black hole is slowly rotating, \(|a| \ll M, \Lambda\). The subextremality of the mass ensures that the event horizon and the cosmological horizon remain physically separated, and the slow rotation ensures that the trapped set remains physically separated from both the cosmological and the black-hole ergoregions.

2.2.1 The Schwarzschild-de Sitter metric

Given a cosmological constant \(\Lambda\), and a black hole mass \(M > 0\) such that

\[
1 - 9\Lambda M^2 > 0,
\] (2.1)
we denote by
\[ b_0 = (M, 0) \]
the black hole parameters for a subextremal Schwarzschild-de Sitter black hole. The Schwarzschild-de Sitter family represents a family of spherically symmetric, non-rotating black hole solutions to Einstein’s equations with positive cosmological constant. On the domain of outer communication (also known in the literature as the static region), \( \mathcal{M}^0 = \mathbb{R}^4 \times (r_{b_0, H^+}, r_{b_0, \Pi^+}) \times S^2 \), with \( r_{b_0, H^+}, r_{b_0, \Pi^+} \) defined below, the Schwarzschild-de Sitter metric \( a \) can be expressed in standard Boyer-Lindquist coordinates by
\[ g_{b_0} = -\mu_{b_0} dt^2 + \mu_{b_0}^{-1} dr^2 + r^2 \hat{g}, \quad (2.2) \]
where
\[ \mu_{b_0}(r) = 1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}, \]
and \( \hat{g} \) denotes the standard metric on \( S^2 \). The subextremal mass restriction in (2.1) guarantees that \( \mu_{b_0}(r) \) has three roots: two positive simple roots, and one negative simple root,
\[ r_- < 0 < r_{b_0, H^+} < r_{b_0, \Pi^+} < \infty. \]
The hypersurfaces defined by \( \{ r = r_{b_0, H^+} \} \), \( \{ r = r_{b_0, \Pi^+} \} \) are the (future) event horizon and the (future) cosmological horizon, respectively, and bound the domain of outer communications, on which the form of the metric in (2.2) is valid. Because we will need to prove estimates which extend slightly into the black hole interior region and the cosmological region, we define, fixing \( b_0 \) and \( 0 < \varepsilon_M < 1 \),
\[ \mathcal{M} := \mathbb{R}^4 \times (r_{b_0, H^+} - \varepsilon_M, r_{b_0, \Pi^+} + \varepsilon_M) \times S^2, \quad (2.3) \]
\[ \Sigma := (r_{b_0, H^+} - \varepsilon_M, r_{b_0, \Pi^+} + \varepsilon_M) \]. \( (2.4) \)
We also define
\[ H_+^\prime := \{ r_{b_0, H^+} - \varepsilon_M \}, \quad \Pi_+^\prime := \{ r_{b_0, \Pi^+} + \varepsilon_M \}. \quad (2.5) \]

### 2.2.2 The Kerr-de Sitter metric

More general than the Schwarzschild-de Sitter family, the Kerr-de Sitter family represents a stationary, axi-symmetric, family of black-hole solutions to Einstein’s equations, of which the Schwarzschild-de Sitter family is clearly a sub-family. In this section, we will detail various useful coordinate systems that we will use subsequently. We remark that throughout the paper, we are mainly interested in Kerr-de Sitter metrics that are slowly-rotating, i.e \( a = |a| \) is small relative to \( M, \Lambda \), and thus, close to a Schwarzschild-de Sitter relative.

**Definition 2.1.** In the Boyer-Lindquist \((t, r, \theta, \varphi) \in \mathbb{R} \times (r_{b, H^+}, r_{b, \Pi^+}) \times (0, \pi) \times S^1_\varphi \) coordinates (with \( r_{b, H^+}, \) and \( r_{b, \Pi^+} \) defined below), the Kerr-de Sitter metric \( g_b := g(M, a) \), and inverse metric \( G_b := G(M, a) \) take the form:
\[ g_b = \rho_b^2 \left( \frac{dt^2}{\Delta_b} + \frac{d\theta^2}{\kappa_b \sin^2 \theta} \right) + \frac{\kappa_b \sin^2 \theta}{(1 + \lambda_b)^2 \rho_b^2} \left( a \, dt - (r^2 + a^2) \, d\varphi \right)^2 - \frac{\Delta_b}{(1 + \lambda_b)^2 \rho_b^2} \left( dt - a \sin^2 \theta \, d\varphi \right)^2, \]
\[ G_b = \frac{1}{\rho_b^2} \left( \Delta_b \rho_b^2 + \kappa_b \rho_b^2 \right) + \frac{(1 + \lambda_b)^2}{\rho_b^2} \left( a \sin^2 \theta \, \partial_t + \partial_\varphi \right)^2 - \frac{(1 + \lambda_b)^2}{\rho_b^2 \sin^2 \theta} \left( (r^2 + a^2) \, \partial_t + a \partial_\varphi \right)^2, \quad (2.6) \]
Remark 2.2. It is easy to observe that the metric reduces to the Schwarzschild-de Sitter metric $g_{b_0}$ expressed in Boyer-Lindquist coordinates in (2.2) when $b = b_0 (a = 0)$. When $a \neq 0$, the spherical coordinates $(\theta, \varphi)$ are chosen so that $\frac{\partial}{\partial \varphi} \in S^2$ is defined by $\theta = 0$, and $\partial_\varphi$ generates counter-clockwise rotation around the axis of rotation.

Definition 2.3. As in the Schwarzschild-de Sitter case, we define the event horizon and the cosmological horizon of $g_b$, denoted by $\mathcal{H}^+, \overline{\mathcal{H}}^+$ to be the $r$-constant hypersurfaces

$$\mathcal{H}^+ := \{ r = r_{b, \mathcal{H}^+} \}, \quad \overline{\mathcal{H}}^+ := \{ r = r_{b, \overline{\mathcal{H}}^+} \}$$

respectively, where $r_{b, \mathcal{H}^+} < r_{b, \overline{\mathcal{H}}^+}$ are the two largest distinct positive roots of $\Delta_b$.

A consequence of the implicit function theorem is that these roots depend smoothly on the black hole parameters $b = (M, a)$, and that for any $\varepsilon_M > 0$, there exists $a$ sufficiently small such that $|r_{b, \mathcal{H}^+} - r_{b_0, \mathcal{H}^+}| + |r_{b, \overline{\mathcal{H}}^+} - r_{b_0, \overline{\mathcal{H}}^+}| < \varepsilon_M$. Moreover, since these two horizons are null, the domain of outer communications, bounded by $\mathcal{H}^+, \overline{\mathcal{H}}^+$ is a causal domain that is foliated by compact space-like hypersurfaces.

Both the Schwarzschild-de Sitter metric (2.2) and the Kerr-de Sitter metric (2.6) in the Boyer-Lindquist coordinates have singularities at the event horizon and the cosmological horizon. Fortunately, this is only a coordinate singularity and can be smoothed out by an appropriate change of coordinates.

### 2.2.3 Regular coordinates on Kerr-de Sitter spacetimes

It is clear upon inspection that the Schwarzschild-de Sitter and Kerr-de Sitter metrics in Boyer-Lindquist coordinates as expressed in (2.2) and (2.6) have degeneracies at the event and cosmological horizons. However, these are merely coordinate singularities, and we can construct a smooth coordinate system that extends beyond both the event and the cosmological horizon.

On Schwarzschild-de Sitter, we introduce the function

$$t_\ast := t - F_{b_0}(r).$$

In the coordinates $(t_\ast, r, \omega)$, the Schwarzschild-de Sitter metric, $g_{b_0}$, and the inverse metric, $G_{b_0}$, take the form

$$g_{b_0} = -\mu_{b_0} dt_\ast^2 + 2F'_{b_0}(r) \mu dt_\ast dr + (\mu_{b_0} F'_{b_0}(r)^2 + \mu_{b_0}^{-1}) dr^2 + r^2 \hat{g},$$

$$G_{b_0} = - \left( \mu_{b_0}^{-1} + \mu_{b_0} F'_{b_0}(r)^2 \right) \hat{g}_{t_\ast t_\ast} + 2F'_{b_0}(r) \mu_{b_0} \hat{g}_{t_\ast r} + \mu_{b_0} \hat{g}_{\omega \omega} + r^2 \hat{G},$$

which can be required to fulfill certain useful properties.

Lemma 2.4. Fix some interval $I_{b_0} \subset (r_{b_0, \mathcal{H}^+}, r_{b_0, \overline{\mathcal{H}}^+})$. Then there exists a choice of $F_{b_0}(r)$ such that

1. the $t_\ast$-constant hypersurfaces are space-like. That is, that

$$- \frac{1}{\mu_{b_0}} + F'_{b_0}(r)^2 \mu_{b_0} < 0;$$
2. $F_{b_0}(r)$ satisfies that

$$F_{b_0}(r) \geq 0, \quad r \in (r_{b_0, \mathcal{H}^+}, r_{b_0, \mathcal{H}^+}),$$

with equality for $r \in \mathcal{I}_{b_0}$.

**Proof.** See Lemma 2.1 in [8].

We construct such a new, Kerr-star, coordinate system explicitly (see similar constructions in Section 5.5 of [7], Section 4 of [18], and Section 3.2 of [10]). First define the new variables

$$t_* = t - F_b(r), \quad \varphi_* = \varphi - \Phi_b(r), \quad (2.8)$$

where $F_b$ and $\Phi_b$ are smooth functions on $(r_{b_0, \mathcal{H}^+}, r_{b_0, \mathcal{H}^+} - \varepsilon_M)$. We can then compute that the metric takes the form

$$g_b = \frac{\sin^2 \theta}{(1 + \lambda_b)^2 \rho_b^2} \left( a(dt_* + F'_b dr) - (r^2 + a^2)(d\varphi_* + \Phi'_b dr) \right)^2$$

$$- \frac{\Delta_b}{(1 + \lambda_b)\rho_b^2} \left( dt_* + F'_b dr - a \sin^2 \theta (d\varphi_* + \Phi'_b dr) - \frac{(1 + \lambda_b)\rho_b^2}{\Delta_b} dr \right)$$

$$+ \frac{2}{1 + \lambda_b} \left( dt_* + F'_b dr - a \sin^2 \theta (d\varphi_* + \Phi'_b dr) - \frac{(1 + \lambda_b)\rho_b^2}{\Delta_b} dr \right) dr + \frac{\rho_b^2}{\Delta_b} d\theta^2. \quad (2.9)$$

We pick the $F_b, \Phi_b$ so that the $(t_*, r, \theta, \varphi_*)$ coordinate system extends smoothly beyond the horizons, is identical to the Boyer-Lindquist coordinates on a small neighborhood of $r = 3M$, and such that the $t_*$-constant hypersurfaces are spacelike.

**Lemma 2.5.** Fix an interval $\mathcal{I}_b := (r_1, r_2)$ such that $r_{b_0, \mathcal{H}^+} < r_1 < r_2 < r_{b_0, \mathcal{H}^+} - \varepsilon_M$. Then we can pick $F_b, \Psi_b$ so that

1. the choice extends the choice of regular coordinates for Schwarzschild-de Sitter in (2.7) in the sense that when $b = b_0$,

$$F_b = F_{b_0}, \quad \Phi_b = 0,$$

where $F_{b_0}$ is that of Lemma 2.4;

2. $F_b(r) \geq 0$ for $r \in (r_{b, \mathcal{H}^+}, r_{b, \mathcal{H}^+})$ with equality for $r \in \mathcal{I}_b$;

3. the $t_*$-constant hypersurfaces are space-like, and in particular, defining

$$A_b := -\frac{1}{G_b(dt_*^2, dt_*^2)}, \quad (2.10)$$

we have that

$$1 \leq A_b \leq 1$$

uniformly on $\mathcal{M}$;

4. the metric $g_b$ is smooth on $\mathcal{M}$.

**Proof.** See Lemma 2.5 in [8].
2.3 The Killing vectorfields $T$, $Φ$

**Definition 2.6.** Define the vectorfields $T = \overset{\circ}{\partial}_t$, $Φ = \overset{\circ}{\partial}_\varphi$ using the Kerr-star coordinates in (2.9). From the fact that the expression of $g_{ab}$ in the Kerr-star coordinates is independent of $t_*$ and $\varphi_*$, we immediately have that $T, Φ$ are Killing vectorfields.

**Definition 2.7.** On Kerr-de Sitter spacetimes, the ergoregion is defined by $E := \{(t, x) : g_b(T, T)(t, x) > 0\}$.

We define its boundary, the set of points where $T$ is null, by the ergosphere.

**Remark 2.8.** Observe that for the Schwarzschild-de Sitter sub-family, the ergosphere is exactly the event horizon and the cosmological horizon, and $T$ is timelike on the whole of the interior of the domain of outer communication.

2.4 Energy momentum tensor and divergence formulas

In this section, we define the energy momentum tensor and some basic divergence properties. Given a complex matrix function $h$, let us denote its complex conjugate by $\overline{h}$. Moreover, for any 2-tensor $h_{\mu\nu}$, we denote its symmetrization $h_{\mu\nu} = \frac{1}{2}(h_{\mu\nu} + h_{\nu\mu})$.

**Definition 2.9.** Let $h : \mathcal{M} \rightarrow \mathbb{C}^D$ be a complex-valued matrix function. We define the energy-momentum tensor to be the symmetric 2-tensor:

$$T_{\mu\nu}[h] = \nabla_\mu \overline{h} \cdot \nabla_\nu h = \frac{1}{2}g_{\mu\nu} \nabla_\alpha \overline{h} \cdot \nabla^\alpha h,$$

where $\cdot$ here is the dot product between matrices.

The energy-momentum tensor satisfies the following divergence property:

$$\nabla_\mu T^\mu_\nu[h] = \Re(\nabla_\nu \overline{h} \cdot \Box g h),$$ (2.11)

where we denote by

$$\Box g = \nabla^\alpha \nabla_\alpha$$

the scalar wave operator.

This property will be the key to producing the various divergence equations we use to derive the relevant energy estimates in the subsequent sections.

**Definition 2.10.** Let $X$ be a smooth vectorfield on $\mathcal{M}$, $m$ be a smooth one-form on $\mathcal{M}$, and $q$ be a smooth function on $\mathcal{M}$. We will refer to $X$ as the (vectorfield) multiplier, to $m$ as the auxiliary zero-order corrector, and to $q$ as the Lagrangian corrector. Then define

$$J_\mu^{X,q,m}[h] = X^\nu T_{\mu\nu}[h] + \frac{1}{2}q \nabla_\mu (|h|^2) - \frac{1}{2} \nabla_\mu q |h|^2 + \frac{1}{2}m_\mu |h|^2,$$

$$K_{\mu\nu}^{X,q,m}[h] = (^{(X)}\pi) \cdot T[h] + q \nabla^\alpha h \cdot \nabla_\alpha \overline{h} + \frac{1}{2} \nabla^\alpha m_\alpha |h|^2 + \frac{1}{2} \nabla^\alpha (m_\alpha - \partial_\alpha q)|h|^2,$$ (2.12)

where

$$(^{(X)}\pi)_{\mu\nu} := \frac{1}{2}(\nabla_\mu X_\nu + \nabla_\nu X_\mu)$$

denotes the deformation tensor of $X$.

---

3We will use $\nabla^\alpha \nabla_\alpha$ to denote the vectorial or tensorial wave operator.
Proposition 2.11. Let \(X\) denote a sufficiently regular vectorfield on \(\mathcal{M}\), a Kerr-de Sitter black hole spacetime, and \(D\) denote the spacetime region bounded by \(\Sigma_{t_1}, \Sigma_{t_2}, \mathcal{H}^+\), and \(\overline{\mathcal{H}}^+\). Moreover, denote
\[
\mathcal{H}^+_{t_1,t_2} := \mathcal{H}^+ \cap \{t_1 \leq t \leq t_2\}, \quad \overline{\mathcal{H}}^+_{t_1,t_2} := \overline{\mathcal{H}}^+ \cap \{t_1 \leq t \leq t_2\},
\]
where we recall the definitions of \(\mathcal{H}^+, \overline{\mathcal{H}}^+\) in (2.5).

Then we have the following divergence property:
\[
-\int_D \nabla g \cdot X = \int_{\Sigma_{t_2}} X \cdot n_{\Sigma_{t_2}} - \int_{\Sigma_{t_1}} X \cdot n_{\Sigma_{t_1}} + \int_{\mathcal{H}^+_{t_1,t_2}} X \cdot n_{\mathcal{H}^+} + \int_{\overline{\mathcal{H}}^+_{t_1,t_2}} X \cdot n_{\overline{\mathcal{H}}^+}.
\]
Here, \(n_{\Sigma_i}\) is the future-directed unit normal on \(\Sigma_i\), and \(n_{\mathcal{H}}\) denotes the future-directed timelike normal of \(\mathcal{H}\).

In proving the relevant energy estimates, we will typically apply the divergence formulas in Proposition 2.11 using the divergence property
\[
\nabla g \cdot J^{X,q,m}[h] = \Re \left[ (X + q) \overline{\mathcal{H}} \cdot \Box_g h \right] + K^{X,q,m}[h]. \tag{2.13}
\]
We will refer to \(X\) and \(q\) as the vectorfield (multiplier), and the Lagrangian correction respectively. When we want to emphasize the background on which \(J^{X,q,m}\) or \(K^{X,q,m}\) are defined, we will use the notation \(J^{X,q,m}_g\) or \(K^{X,q,m}_g\), where \(g\) is the background metric.

2.5 The redshift vectorfields \(N\)

In this subsection, we recall the construction of the redshift vectorfield \(N\).

Proposition 2.12. Let \(b = (M, a), |a| \ll M, \Lambda\) be the black hole parameters for a Kerr-de Sitter black hole \(g_b\), and let \(\Sigma_{t_2}\) be a \(t_2\)-constant uniformly spacelike hypersurface. There exist positive constants \(c_N\) and \(C_N\), parameters \(r_\mathcal{H} < r_0 < r_1 < R_0 < R_1 < r_{\overline{\mathcal{H}}^+}\), and a stationary uniformly time-like vectorfield \(N\) such that

1. \(K^{N,0,0}_g[h] \geq c_N J^{N,0,0}_g[h] \cdot n_{\Sigma_{t_2}}\) for \(r_\mathcal{H} < r \leq r_0\), and for \(R_0 < r \leq r_{\overline{\mathcal{H}}^+}\).
2. \(-K^{N,0,0}_g[h] \leq C_N J^{N,0,0}_g[h] \cdot n_{\Sigma_{t_2}}\) for \(r_0 < r \leq R_0\).
3. \(N = T\) for \(r_1 \leq r \leq R_1\).
4. There exists some \(\delta > 0\) such that
\[
g_b(N,N) < -\delta. \tag{2.14}
\]

Proof. See Proposition 2.20 in [8].

We use the redshift vectorfield \(N\) defined above to construct higher-order Sobolev spaces. To do so, we will require the following technical lemma (see Lemma 3.11 of [20] for the anti-de Sitter equivalent) which constructs the \(K_i\) vectorfields.

Lemma 2.13. There exists a finite collection of vectorfields \(\{K_i\}_{i=1}^N\) with the following properties:

1. \(K_i\) are stationary, smooth vectorfields on \(\mathcal{M}\).
2. Near \(\mathcal{H}^+, K_i\) is future-oriented null with \(g(K_i, K_{\mathcal{H}^+}) = -1\), and near \(\overline{\mathcal{H}}^+, K_i\) is future-oriented null with \(g(K_i, K_{\overline{\mathcal{H}}^+}) = -1\).
3. $K_i$ are tangent to both $\mathcal{H}^+$ and $\mathcal{H}_r^+$ for $2 \leq i \leq N$.

4. If $X$ is any vectorfield supported in $\mathcal{M}$, then there exist smooth functions $x^i$, not necessarily unique, such that

$$X = \sum_i x^i K_i.$$ 

5. We have the following decomposition of the deformation tensor of $K_i$,

$$(K_i)_\pi = \sum_{j,k} f_{ijk}^k K_j \otimes K_k^i,$$  

(2.15)

for stationary functions $f_{ijk}^k = f_{kji} \in C_c^\infty(\mathcal{M})$, where $X^\phi$ is the canonical one-form associated to the vectorfield $X$, and on $\mathcal{H} \in \{ \mathcal{H}^+, \mathcal{H}_r^+ \}$,

$$f_{11}^1 = \kappa_\mathcal{H},$$

$$f_{i1}^1 = 0, \quad i \neq 1.$$  

Proof. See Lemma 2.22 in [8].

2.6 Sobolev spaces

In this section, we define the Sobolev spaces that we make use of throughout the paper when discussing the Cauchy problem or establishing energy estimates.

Definition 2.14. Let $h : \mathcal{M} \rightarrow \mathbb{C}^D$. Then denoting by $D$ a subset of $\mathcal{M}$, we define the regularity spaces

$$L^2(D) := \left\{ h : \int_D |h|^2 < \infty \right\}, \quad H^k(D) := \left\{ h : \mathcal{K}^\alpha h \in L^2(D), |\alpha| \leq k \right\}.$$

Let us also define two $L^2$ inner products on spacelike slices.

Definition 2.15. We define the $L^2$ and $L^2$ inner products on the spacelike slice $\Sigma$ by

$$\langle h_1, h_2 \rangle_{L^2(\Sigma)} = \int_\Sigma h_1 \cdot \overline{h_2} \sqrt{A},$$

$$\langle h_1, h_2 \rangle_{L^2(\Sigma)} = \int_\Sigma h_1 \cdot \overline{h_2}$$

Observe that due to the construction of the $(t, r, \theta, \varphi)$ Kerr-star coordinate system, the two norms are equivalent.

We likewise have the following higher regularity Sobolev norms.

Definition 2.16. For any $u : \Sigma \rightarrow \mathbb{C}^D$, let $v : \mathcal{M} \rightarrow \mathbb{C}^D$ be the unique lifting satisfying

$$v|_{\Sigma} = u, \quad T v = 0.$$  

(2.16)

Then define the regularity spaces $H^k(\Sigma), \overline{H}^k(\Sigma)$ by:

$$H^k(\Sigma) := \left\{ u : \mathcal{K}^\alpha v|_{\Sigma} \in L^2(\Sigma), |\alpha| \leq k \right\},$$

$$\overline{H}^k(\Sigma) := \left\{ u : \mathcal{K}^\alpha v|_{\Sigma} \in \overline{L}^2(\Sigma), |\alpha| \leq k \right\},$$

where $\alpha$ is a multi-index and $\mathcal{K}_i$ are vectorfields satisfying the requirements of Lemma 2.13.

With the same $\mathcal{K}_i$, we define the regularity space $\overline{H}^k(\Sigma)$ by:

$$\overline{H}^k(\Sigma_{t_*}) := \left\{ h : \mathcal{K}^\alpha h(t_*, \cdot)|_{\Sigma_{t_*}} \in \overline{L}^2(\Sigma), |\alpha| \leq k \right\}.$$
Remark 2.17. Observe that different choices of the family \( \{K_i\}_{i=1}^N \) will result in different, though equivalent, \( H^k(\Sigma) \), \( \overline{H}^k(\Sigma) \), \( \overline{H}^k(\Sigma) \) norms.

We also use the vectorfields defined in Theorem 2.13 to define the following higher regularity Sobolev spaces.

**Definition 2.18.** Given some \( (\psi_0, \psi_1) \in H^k_{loc}(\Sigma, \mathbb{C}^D) \times H^{k-1}_{loc}(\Sigma, \mathbb{C}^D) \), letting \( \{K_i\}_{i=1}^N \) be as constructed in Lemma 2.13. We define \( \mathcal{H}^k(\Sigma) \) to be the space consisting of \( (\psi_0, \psi_1) \) such that

\[
\|(\psi_0, \psi_1)\|_{\mathcal{H}^k(\Sigma)}^2 := \|\psi_0\|_{H^k(\Sigma)}^2 + \|\psi_1\|_{H^{k-1}(\Sigma)}^2.
\]

We also define the following exponentially weighted Sobolev spaces.

**Definition 2.19.** For \( \alpha \in \mathbb{R}, h : \mathcal{M} \to \mathbb{C}^D \), define the weighted Sobolev norm with regularity \( k \) and decay \( \alpha \),

\[
\|h\|_{H^{k,\alpha}(\mathcal{M})}^2 := \int_0^\infty e^{2\alpha t_s} \|h\|_{H^k(\Sigma^s)}^2 \, dt_s.
\]

Then define the forcing parameter space of regularity \( k \), decay \( \alpha \), and loss \( m \) by

\[
\|(f, h_0, h_1)\|_{D^{k,\alpha,m}(\mathcal{M})} := \|f\|_{H^{k+m-1,\alpha}(\mathcal{M})} + \|(h_0, h_1)\|_{H^{k+m}(\Sigma_0)}.
\]

(2.17)

2.7 Implicit function theorem

We provide the explicit statement of the implicit function theorem that we use in the present paper.

**Theorem 2.20** (Theorem A.1.2 in [1]). Let \( X, Y, Z \) be Hilbert spaces. Let \( U \subset X, V \subset Y \) be open subsets and \( \mathcal{F} : U \times V \to Z \) be a \( C^1 \) mapping. If for some \( (x_0, y_0) \in U \times V \), the linearization in the first argument

\[
D_1\mathcal{F}|_{(x_0, y_0)} : X \to Z
\]

is an isomorphism, then there exist open neighborhoods \( U_0, V_0 \) such that \( x_0 \in U_0 \subset U \), \( y_0 \in V_0 \subset V \), and a unique \( C^1 \)-mapping \( \mathcal{G} : V_0 \to U_0 \) such that for all \( y \in V_0 \),

\[
\mathcal{F}(\mathcal{G}(y), y) = \mathcal{F}(x_0, y_0),
\]

and \( \mathcal{F}(x, y) = \mathcal{F}(x_0, y_0) \) is equivalent to \( x = \mathcal{G}(y) \) for \( (x, y) \in U_0 \times V_0 \).

3 Einstein’s equations

In this section, we review the formulation of Einstein’s equations in harmonic gauge and the nonlinear hyperbolic Cauchy problem associated to Einstein’s equations in harmonic gauge. For an in-depth reference on harmonic coordinates, we refer the reader to Chapter 7 of [2].

3.1 Harmonic gauge and the hyperbolic Cauchy problem

Recall that Einstein’s vacuum equations with a cosmological constant \( \Lambda \) for \( g \), a \((- , + , + , +)\) Lorentzian metric on a smooth manifold \( \mathcal{M} \), are

\[
\text{Ric}(g) - \Lambda g = 0.
\]

(3.1)
For any globally hyperbolic \((\mathcal{M}, g)\) solution to (3.1), and spacelike hypersurface \(\Sigma_0 \subset \mathcal{M}\), the induced Riemannian metric \(\bar{g}\) on \(\Sigma_0\) and the second fundamental form \(k(X,Y)\) of \(\Sigma_0\) satisfy the constraint equations

\[
R(\bar{g}) + (\text{tr}_{\bar{g}} k)^2 - |k|^2_{\bar{g}} = -2\Lambda, \tag{3.2}
\]

where \(R(\bar{g})\) is the scalar curvature of \(\bar{g}\). The Cauchy problem for Einstein’s equations then asks, given an initial data set consisting of the triple \((\Sigma_0, \bar{g}, k)\), where \(\bar{g}\) is a Riemannian metric on the smooth 3-manifold \(\Sigma_0\), and \(k\) is a symmetric 2-tensor on \(\Sigma_0\), for a Lorentzian 4-manifold \((\mathcal{M}, g)\) and an embedding \(\Sigma_0 \hookrightarrow \mathcal{M}\) such that \(\bar{g}\) is the induced metric on \(\Sigma_0\), and \(k\) is the second fundamental form of \(\Sigma_0\) in \(\mathcal{M}\). We denote initial data triplets with \((\bar{g}, k)\) satisfying the constraint equations (3.2) to be admissible initial data triplets.

It is well-known that (3.1) is a quasilinear second-order partial differential system of equations for the metric coefficients \(g_{\mu\nu}\). In local coordinates, we can write (3.1) as

\[
\text{Ric}(g)_{\mu\nu} = -\frac{1}{2} g^{\alpha\beta} \partial_{\alpha} \partial_{\beta} g_{\mu\nu} + \partial_{(\mu} \Gamma_{\nu\lambda)}^g + \mathcal{N}(g, \partial g), \quad \Gamma^g_{\mu} := g^{\alpha\beta} \Gamma^g_{\mu}{}^{\alpha\beta}, \tag{3.3}
\]

where the nonlinear term \(\mathcal{N}(g, \partial g)\) involves at most one derivative of \(g\). As a result of the presence of the \(\nabla_{(\mu} \Gamma_{\nu\lambda)}^g\) term EVE lacks any useful structure. However, as was first demonstrated by Choquet-Bruhat, this problem can be overcome by using the general covariance of Einstein’s equations and choosing wave coordinates, also referred to as harmonic coordinates. With this choice Einstein’s equations become a quasilinear hyperbolic system of equations [3, 4].

We first introduce the harmonic coordinate condition.

**Definition 3.1.** Define the (gauge) constraint operator

\[
\mathcal{C}(g, g^0)_{\mu} = g_{\mu\chi} g^{\nu\lambda} \left( \Gamma_g^{\nu\chi} - \Gamma^{g^0}_{\nu\chi} \right),
\]

where \(g^0\) is a fixed background metric which solves Einstein’s equations. Then we say that a Lorentzian metric \(g\) satisfies the harmonic coordinate condition (with respect to \(g^0\)) if

\[
\mathcal{C}(g, g^0) = 0. \tag{3.4}
\]

**Remark 3.2.** Instead of vanishing right-hand side, if we instead set the right-hand side equal to some one-form \(\Omega(g)\) depending on \(g\) but not its derivatives, we would obtain the generalized harmonic gauge.\(^4\)

**Remark 3.3.** For any given Lorentzian metric \(g\), finding the appropriate diffeomorphism \(\phi\) such that \(g^\phi := \phi^* g\) satisfies the harmonic coordinate condition with respect to \(g^0\) reduces to solving a semilinear wave equation, and thus the problem of finding such a \(\phi\) is locally well-posed.

Throughout the remainder of the paper, it will be convenient to pick \(g^0\) to be the Kerr-de Sitter background metric around which we linearize.

**Definition 3.4.** Define the linearized constraint

\[
\mathcal{C}_{g^0} h := D_{g^0} \mathcal{C}(g + h, g^0)(h) = -\nabla_{g^0} \cdot \mathcal{K}_{g^0} h, \tag{3.5}
\]

where

\[
\mathcal{K} h := h - \frac{1}{2} (\text{tr}_g h) g,
\]

denotes the trace reversal operator of \(g\). The linearized (wave coordinate) constraint condition is

\[
\mathcal{C}_{g^0} h = 0. \tag{3.6}
\]

\(^4\)For more applications of generalized harmonic coordinates we refer the reader to [12, 13]
Lemma 3.5. Let $g$ be a Lorentzian metric satisfying the harmonic coordinate condition in (3.4) with respect to a fixed background metric $g^0$. Then, there exists some nonlinear function $Q(\cdot, \cdot)$ such that
\[
\text{Ric}(g) - \Lambda g = -\frac{1}{2} \Box_g g + Q(g, \partial g),
\]
where we emphasize that $Q$ is independent of the second derivatives of $g$.

Proof. If $g$ satisfies $C^\mu g, g^0 \chi = 0$, then using (3.3), we can write that
\[
\text{Ric}(g)_{\mu \nu} = -\frac{1}{2} g^{\alpha \beta} \partial_\alpha \partial_\beta g_{\mu \nu} + \partial_{(\mu} \left( g_{\nu)}^{\alpha} g^{\alpha \beta} \Gamma(0)^{\chi}_{\alpha \beta} \right) + N(g, \partial g).
\]
Since $g^0$ is simply some prescribed background metric, we observe that the second term on the right hand side is itself now semilinear, and we can write in local coordinates
\[
\text{Ric}(g)_{\mu \nu} = -\frac{1}{2} g^{\alpha \beta} \partial_\alpha \partial_\beta g_{\mu \nu} + Q(g, \partial g),
\]
where
\[
Q(g, \partial g) := \partial_{(\mu} \left( g_{\nu)}^{\alpha} g^{\alpha \beta} \Gamma(0)^{\chi}_{\alpha \beta} \right) + N(g, \partial g).
\]
From (3.8), it is clear that Ric is a quasilinear hyperbolic operator. Observing that the scalar wave operator for a metric $g$ satisfying the harmonic gauge condition (3.4) can be expressed as
\[
\Box_g := \nabla^\alpha \partial_\alpha = g^{\alpha \beta} \partial_\alpha \partial_\beta - g^{\alpha \beta} \Gamma(0)^{\sigma}_{\alpha \beta} \partial_\sigma,
\]
we can then conclude.

Crucial to the utility of harmonic coordinates is that they are propagated by a hyperbolic operator.

Definition 3.6. Given a smooth one-form $\psi$, we define the constraint propagation operator,
\[
\Box_g^{CP} \psi := -2 \nabla_g \cdot \mathcal{M}_g \nabla_g \otimes \psi = \Box_g^{(1)} \psi - \text{Ric}(g)(\psi, \cdot),
\]
where $\nabla_g \otimes \psi = -\frac{1}{2} \mathcal{L}_{\psi^g} g$ is the symmetric gradient of $g$, $\psi^g$ is the canonical vectorfield associated to the one-form $\psi$, and $\Box_g^{(1)} = \nabla^\alpha \nabla_\alpha$ denotes the wave operator acting on one-tensors. Observe that $\Box_g^{CP}$ is a manifestly hyperbolic operator.

Lemma 3.7. Any solution $g$ to
\[
\text{Ric}(g) - \Lambda g - \nabla_g \otimes C(g, g^0) = 0
\]
must also satisfy
\[
\Box_g^{CP} C(g, g^0) = 0.
\]

Proof. The conclusion follows directly by applying the twice contracted second Bianchi identity to the equation.

The fact that the gauge constraints are propagated by a hyperbolic operator allows us to conclude that solving EVE in harmonic gauge is equivalent to solving the ungauged EVE provided that the initial data satisfies the gauge constraint.
Proposition 3.8. Let $g$ be a solution to the Cauchy problem
\begin{equation}
\frac{1}{2} g^{\alpha\beta} \partial_{\alpha} \tilde{c}_\beta g_{\mu\nu} + \Lambda g_{\mu\nu} = N_{g^0}(g, \tilde{c}g)_{\mu\nu}
\end{equation}
where
\begin{equation}
\gamma_0(g) = (g_0, g_1).
\end{equation}
If $g$ satisfies
\begin{equation}
C(g, g^0)|_{\Sigma_0} = 0;
\end{equation}
and $(g, k)$, the induced metric and second fundamental form respectively by $g$ on $\Sigma_0$, satisfy the constraint conditions (3.2); then in fact
\begin{equation}
C(g, g^0) = 0
\end{equation}
uniformly, and $g$ is a solution to (3.1) with admissible initial data posed by $(\Sigma_0, g, k)$.

Proof. This follows directly from the semilinear nature of the constraint propagation operator and uniqueness of solutions for semilinear hyperbolic PDEs.

It will be convenient to rewrite (3.10) as a quasilinear system in terms of the metric perturbation.

Lemma 3.9. Let $g = g_b + h$ be a solution to the Einstein vacuum equations in harmonic gauge with respect to a background metric $g^0$. If we choose $g^0 = g$, then linearizing around $g$, we find that $h$ solves
\begin{equation}
L_{g_b} h = N_{g_b}(h, \partial h, \partial^2 h),
\end{equation}
where $N_{g_b}(h, \partial h, \partial^2 h)$ is a quasilinear term in $h$, and
\begin{equation}
L_{g_b} h = \Box_{g_b} h + R_{g_b}(h),
\end{equation}
where $\Box_{g_b}^{(2)} = \nabla^\alpha \nabla_\alpha$ denotes the wave operator on $g_b$ acting on symmetric two-tensors. We will often refer to $L_{g_b}$ as the gauged linearized Einstein operator.

In particular, $h$ also solves the system
\begin{equation}
L_g h = Q_g(h, \partial h),
\end{equation}
where $Q_g(h, \partial h)$ is a semilinear term in $h$ at least quadratic in its arguments, and
\begin{equation}
L_g h = \Box_g h + S_g[h] + V_g[h],
\end{equation}
where $S_g$ is a vectorfield-valued matrix, $V_g$ is a function-valued matrix and both $S_g$ and $V_g$ have coefficients depending on at most one derivative of $g$, and $L_{g_b} = L_{g_b}$.

Proof. The linearization in (3.12) follows from equation (2.4) in [9]. Equation (3.14) then follows from Lemma 3.5, observing that the only quasilinear terms in (3.7) are the quasilinear terms involved in the scalar wave.
3.2 Initial data

In this section, we will construct the mapping $i_{b,\phi}$ between admissible initial data triplets $(\Sigma_0, g_{b0}, k_0)$ for the ungauged Einstein equations and the admissible initial data $(g_0, g_1)$ for the Cauchy problem for the gauged Einstein equations. An important property of this mapping is that a metric $g$ such that $(g|_{\Sigma_0}, \mathcal{L}_T g|_{\Sigma_0}) = (g_0, g_1)$ satisfies the gauge constraint $\mathcal{C}(g, g^0)|_{\Sigma_0} = 0$. To construct $i_{b,\phi}$ we need to specify a choice of $g^0$. As previously mentioned, it will be convenient to choose $g^0 = g_b$.

Consider the Kerr-de Sitter initial data triplet $(\Sigma_0, g_{b0}, k_0)$ that launches $g_b$. That is, let $g_{b0}$ and $k_0$ denote the induced metric and second fundamental form on $\Sigma_0$ by $g_b$. We will construct $i_{b,\phi}$ mapping $(\Sigma_0, g_{b0}, k_0)$ into Cauchy data for the gauged Einstein vacuum equations launching the Kerr-de Sitter solution $\phi^* g_b$. That is, we will have that

$$i_{b,\phi}(\phi^* g_{b0}, \phi^* k_0) = (0, 0).$$

The linearization of this mapping will also produce the correctly gauged initial data for the gauged linearized Einstein equation.

**Proposition 3.10.** Fix a diffeomorphism $\phi$. Then for each $b \in \mathcal{B}$, there exists a map

$$i_{b,\phi} : H^{N_e}(\Sigma_0; S^2 T^* \Sigma_0) \times H^{N_e-1}(\Sigma_0; S^2 T^* \Sigma_0) \to H^{N_e}(\Sigma_0; S^2 T^* \Sigma_0) \times H^{N_e-1}(\Sigma_0; S^2 T^* \Sigma_0),$$

that is smooth for $N_e \geq 1$ depending smoothly on $b$, such that

1. if $h$ is some symmetric two-tensor such that $i_{b,\phi}(g_{b0}, k_0) = \gamma_0(h)$, then

$$g_{b0} = (g_{b0}, k_0),$$

where $(g, k)$ are the induced metric and the second fundamental form respectively of $\phi(\Sigma_0)$ induced by $g = \phi^*(g_b + h)$. Moreover, $g_b + h$ satisfies the gauge constraint

$$\mathcal{C}(g_b + h, g_b)|_{\Sigma_0} = 0;$$

2. if $(g_{b0}, k_0)$ is the admissible initial data launching the Kerr-de Sitter metric $g_b$, then

$$i_{b,\phi}(\phi^* g_{b0}, \phi^* k_0) = (0, 0);$$

3. if $\phi = e^\theta$, for some smooth vectorfield $\theta$, then $(g_0, g_1) = i_{b,\phi}(g, k)$ satisfies the condition

$$\|(g_0, g_1)\|_{H^k(\Sigma_0)} \leq \sum_{0 \leq |I| \leq k} \|\partial_x^I (g - g_0)\|_{L^2(\Sigma_0)} + \sum_{0 \leq |I| \leq k+1} \|\partial_x^I \theta\|_{L^2(\Sigma_0)} + \sum_{0 \leq |I| \leq k-1} \|\partial_x^I (k - k_0)\|_{L^2(\Sigma_0)},$$

where $I, J$ are multi-indexes, and $k \leq N_e$.

**Proof.** See Proposition 3.12 of [8].

4 Main theorem

In this section, we present the convention regarding smallness constants, the statement of the main theorem, as well as a basic outline of the strategy.
4.1 Smallness constants

We introduce the following constants involved in the main theorem and its ensuing proof.

- The black hole parameters \( b^0 = (M^0, a^0) \) represents the mass and angular momentum respectively of the initial Kerr-de Sitter spacetime which we perturb.
- The integers \( N_e \) represents the maximum number of derivatives for our high regularity exponential growth bootstrap assumption of the solution, and \( N_d \) represents the maximum number of derivatives for the low-regularity exponential decay bootstrap assumption.
- The smallness of the perturbation of the initial data norm from \( g_{b^0} \) is measured by \( \varepsilon_0 \).
- \( \delta_d \) and \( \delta_e \) measure the bootstrap rate of low-regularity exponential decay and high-regularity exponential growth respectively.
- The size of the bootstrap norms are controlled by \( \Delta_0 \).

In what follows, \( b^0 \) is fixed, \( \delta_e, \delta_d \) will be chosen such that

\[
0 \leq |a^0| \ll \min \delta_d, \delta_e, M^0, \Lambda, 1.
\]

We will choose \( \delta_e, \delta_d, N_e, N_d, \) subject to the condition that

\[
2\delta_e < (N_e - N_d - 2)\delta_d, \quad N_e - N_d > 2, \quad N_d > \frac{5}{2}.
\]

We will also choose \( \varepsilon_0, \Delta_0 \) such that

\[
\varepsilon_0, \Delta_0 \ll \min M^0, \Lambda, 1, \quad \varepsilon_0, \Delta_0 \ll |a^0|, \quad \text{if } a^0 \neq 0, \tag{4.1}
\]

and

\[
\Delta_0 = \varepsilon_0^2.
\]

**Remark 4.1.** Observe that we may always assume (4.1), even if \( 0 < |a^0| \leq \varepsilon_0 \). In that case, the initial data continues to satisfy a smallness estimate of the form

\[
\left\| (g^0, k_0) - (g_{b^0}, k_{b^0}) \right\|_{H^{N_e}(\Sigma_0) \times H^{N_e-1}(\Sigma_0)} \lesssim \varepsilon_0
\]

by setting \( b^0 = (M^0, 0) \). See Remark 3.4.1 in [16] for a similar observation in the \( \Lambda = 0 \) case.

In what follows, we will use \( \lesssim \) to denote a quantity bounded by a constant depending only on universal geometric constants as well as the constants \( M^0, a^0, \Lambda \) but not on \( \varepsilon_0 \) or \( \Delta_0 \).

4.2 Statement of the main theorem

We are now ready to state the main theorem.

**Theorem 4.2.** Fix slowly-rotating Kerr-de Sitter parameters \( b^0 = (M^0, a^0) \). Let \( (\Sigma_0, g_{b^0}, k_0) \) be an admissible initial data triplet for EVE with cosmological constant \( \Lambda > 0 \), such that

\[
\left\| (g^0, k_0) - (g_{b^0}, k_{b^0}) \right\|_{H^{N_e}(\Sigma_0) \times H^{N_e-1}(\Sigma_0)} \lesssim \varepsilon_0 \tag{4.2}
\]

where \( (g_{b^0}, k_{b^0}) \) is the initial data induced by \( g_{b^0} \) on \( \Sigma_0 \), and \( N_e = 6 \). Then there exist slowly-rotating Kerr-de Sitter black hole parameters \( b_X = (M_X, a_X) \in B \), a diffeomorphism \( \phi_X = e^{i\sigma_0}X \), and a metric perturbation \( h \in S^2T^*\mathcal{M} \) such that the 2-tensor \( g \) satisfying

\[
g = \phi_X^*(g_{b_X} + h),
\]

[16]
is a solution of the Einstein vacuum equations

\[ \text{Ric}(g) - \Lambda g = 0, \quad (g, k) = (g_0, k_0), \]

where \( g, k \) are the induced metric and second fundamental form of \( \Sigma_0 \subset M \) by \( g \). Moreover, \( g_{b,c} + h \) is in harmonic gauge with respect to \( g_{b,c} \), so that

\[ \mathcal{C}(g_{b,c} + h, g_{b,c}) = 0 \]

uniformly on \( M \). Furthermore, there exists some \( \delta_d > 0 \) such that \( h \) satisfies the pointwise estimates

\[ \sup_{(t_*, x) \in M} e^{\delta_d t_*} |h(t_*, x)| \lesssim \varepsilon_0, \]

where \( t_* \) takes values in \([0, +\infty)\) in \( M \).

### 4.3 Strategy of proof

The proof of Theorem 4.2 will be provided in Section 7. We will employ a bootstrap scheme with three bootstrap assumptions (see Section 6.2):

1. A bootstrap assumption of integrated (small) exponential growth at the level of high regularity norms of the solution, which we call the energy bootstrap assumption.

2. A bootstrap assumption of exponential decay at the level of low regularity norms of the solution, which we call the decay bootstrap assumption.

3. A bootstrap assumption on the existence of a small gauge choice such that an orthogonality condition is satisfied which avoids certain linear obstacles to decay.

**Remark 4.3.** These bootstrap assumptions are imposed on solutions of a semi-global extension of EVE with a gauge condition enforcing a certain orthogonality condition that avoids finitely many linear obstacles to decay (see Section 5 for a more in-depth discussion). The introduction of a semi-global extension of EVE is done so that we are allowed to use the linear theory developed in [8], where certain global assumptions are needed.

To prove Theorem 4.2, it then suffices to improve each of the bootstrap assumptions, and show that the bootstrap time can be extended.

1. We improve the low-regularity exponential decay bootstrap assumption, we directly apply the linear theory of [8] on a fixed Kerr-de Sitter background, recalled in Section 5.1, by treating all the nonlinear terms, including the quasilinear terms, as forcing terms on the right-hand side. This results in estimates which lose derivatives. This is handled by an interpolation argument, using the fact that the high-regularity bootstrap assumption assumes only very slow exponential growth. This is done in Proposition 7.1.

2. To improve the high-regularity integrated exponential growth bootstrap assumptions, we use a weak Morawetz estimate which does not lose derivatives. This estimate is stated in Proposition 5.8, and is a perturbation of the high-frequency Morawetz estimate proven in [8]. It in turn is proven in Section 8. The actual improvement of the high-regularity exponential growth bootstrap assumption is then carried out in Proposition 7.3. Here, we no longer treat the quasilinear terms as forcing terms since we can not lose derivatives in this step.

3. To improve the bootstrap assumption on the gauge, we use the implicit function theorem. This is done in Proposition 7.6.
4. To extend the gauge, we use the continuity of the flow and again rely on the implicit function theorem. This is done in Proposition 7.8.

The proof of Theorem 4.2 is then finished in Section 7.5.

5 Linear Theory

In this section, we state the results from the linear theory needed for the proof of Theorem 4.2.

5.1 Exponential decay on exact Kerr-de Sitter spacetimes

We introduce some of the key conclusions of the linear theory in the companion paper in [8].

One of the main conclusions of the linear theory was that solutions to the linearized Einstein equations decay exponentially, up to a finite number of obstacles. These finite obstacles are $\Lambda_{\text{QNM}}^k(\mathbb{L}_{g_b},\mathbb{H}^+)$, the non-decaying $H^k$-quasinormal modes of $\mathbb{L}_{g_b}$.

**Corollary 5.1** (See Corollary 4.25 of [8]). Let $k \geq 3$, and let $Z \subset D^{k,\alpha}(M)$ be a finite-dimensional linear subspace such that the map

$$Z \ni (f, \tilde{h}_0, \tilde{h}_1) \mapsto \left( \frac{1}{i} \begin{pmatrix} \delta_0 h_0 \\ -A_b f + \delta_0 \tilde{h}_1 \end{pmatrix}, \right) \in \mathcal{L}(\Lambda_{\text{QNM}}^k(\mathbb{L}_{g_b},\mathbb{H}^+),\mathbb{C}),$$

(5.1)

mapping from $Z$ to the space of linear functionals on the dual $H^k$-quasinormal modes is bijective. Then there exists a continuous linear map

$$\lambda_Z : D^{k,\alpha,1}(M) \to Z$$

(5.2)

such that if $D^{k,\alpha,1} \ni \lambda_Z(f, h_0, h_1) = z = (\tilde{f}, \tilde{h}_0, \tilde{h}_1) \in Z$, then the initial value problem

$$Lh = f + \tilde{f},$$

$$\gamma_0(h) = (h_0 + \tilde{h}_0, h_1 + \tilde{h}_1)$$

has an exponentially decaying solution $h$ that satisfies the estimate

$$\|h\|_{H^1(\Sigma_\tau)} \leq e^{-\alpha \tau} \|(f, h_0, h_1)\|_{D^{k+1,\alpha}(M)}.$$  

(5.3)

If moreover, $\lambda_Z$ is bijective, then $z$ is unique and the map $(f, h_0, h_1) \to z$ is continuous.

The following proposition corresponds to Corollary 4.25 of [8] applied to the case where $L = \mathbb{L}_{g_b}$ (see also Proposition 5.7 in [10]).

**Proposition 5.2.** Fix $g_b$ a slowly-rotating Kerr-de Sitter black hole. Then there exist some $\alpha > 0$ and linear maps

$$\lambda_M[g_b] : D^{k,\alpha,1}(M) \to \mathbb{R}^N, \quad \lambda_C[g_b] : D^{k,\alpha,1}(M) \to \mathbb{R}^N,$$

such that for $(f, h_0, h_1) \in D^{k,\alpha,1}(M)$, where $k \geq 3$, if $\lambda_M[g_b](f, h_0, h_1) = \lambda_C[g_b](f, h_0, h_1) = 0$, then for any $0 < \delta_d < \alpha$, a solution $h$ to the Cauchy problem

$$\mathbb{L}_{g_b} h = f,$$

$$\gamma_0(h) = (h_0, h_1),$$

(5.4)

where $\mathbb{L}_{g_b}$ is as defined in (3.13), satisfies the decay estimate

$$\sup_{t \geq 0} e^{\delta_d t} \|h\|_{H^1(\Sigma_\tau)} \leq \|(f, h_0, h_1)\|_{D^{k,\alpha,1}(M)}.$$  

(5.5)

$^5$In the context of [8], Corollary 4.25 is shown for a strongly hyperbolic operator $\mathbb{L}$ with a discrete quasinormal spectrum satisfying certain high frequency resolvent estimates. That $\mathbb{L}_{g_b}$ satisfies these conditions is shown in Lemma 3.9, Theorems 5.4 and 5.6 in [8].
Remark 5.3. The appearance of the requirement that \((f, h_0, h_1) \in D^{k, \alpha, 1}(M)\) for \(k \geq 3\) has to do with the threshold regularity level in [8]. For slowly-rotating Kerr-de Sitter black holes, it is sufficient to take \(k > 2\). Since we are only working with integer Sobolev spaces, and since we will in any case always work with \((f, h_0, h_1) \in D^{3, \alpha, 1}(M)\) in this paper, we make no further attempts to optimize this.

The finite number of obstacles to exponential decay of the Cauchy problem defined by \((f, h_0, h_1)\) for the linearized Einstein operator are characterized by the condition in Proposition 5.2 that \((f, h_0, h_1) \in \ker \lambda_\Sigma(g_0) \bigcap \ker \lambda_\Sigma(g_b)\). A priori, these finite obstacles could actually grow exponentially, and thus, for general solutions to the Cauchy problem generated by \((f, h_0, h_1)\) in (5.4), it is not possible to enforce exponential decay. What makes the result of Proposition 5.2 useful then is that in considering Einstein’s equations, we do not consider a general Cauchy problem but one satisfying certain constraints. These constraints manifest themselves in the nature of the mappings \(\lambda_\Sigma\) and \(\lambda_\Sigma\).

In particular, the following proposition states that the kernel of \(\lambda_\Sigma(g_b)\) consists only of solutions that do not satisfy the linearized wave coordinate constraint condition in (3.6). This is the content of Proposition 10.9 in [8].

**Proposition 5.4.** Fix \(g_b\) a slowly-rotating Kerr-de Sitter metric, and let \(h\) be the solution to the Cauchy problem in (5.4), for some \((f, h_0, h_1) \in D^{k, \alpha, 1}(M), k > 0\).

If there exists some \(T_*\) such that

\[
f(t_*, \cdot) = C_{g_b}(h)(t_*, \cdot) = 0, \quad t_* > T_*,
\]

where \(C_{g_b}(h)\) is as defined in (3.5), then

\[
(f, h_0, h_1) \in \ker(\lambda_\Sigma(g_b)).
\]

Proposition 5.4 implies that some of the finite number of linear obstacles to decay present in Proposition 5.2 are solutions to the linearized Einstein equations in harmonic gauge which themselves do not satisfy the linearized harmonic gauge condition. These solutions are unphysical, and are avoided in when considering the linearized Einstein equations in linearized harmonic gauge with data respecting the linearized gauge constraint.

To handle the remaining obstacles to decay, we have the following proposition, which amounts to a mode stability statement. This is the content of Proposition 10.10 in [8].

**Proposition 5.5.** There exists an \(N_\Theta\)-dimensional family of non-exponentially-decaying vectorfields \(\Theta\), parametrized by

\[
i_\Theta : \mathbb{R}^{N_\Theta} \to \Theta
\]

such that for any \(i_\vartheta \vartheta, \vartheta \in \mathbb{R}^{N_\Theta}\), for any slowly-rotating Kerr-de Sitter metric \(g_b\),

\[
\mathbb{L}_{g_b}(\nabla_{g_b} \otimes i_\vartheta \vartheta) = 0,
\]

so that \(\nabla_{g_b} \otimes i_\vartheta \vartheta\) is a non-decaying mode solution of \(\mathbb{L}_{g_b}\).

Moreover, with \(\lambda_\Sigma[g_b]\) as in Proposition 5.2, \(\lambda_\Sigma\) is an isomorphism

\[
\lambda_\Sigma[g_b] : \{(0, \gamma_0 ((g_b')^T(b') + \nabla_{g_b} \otimes i_\vartheta \vartheta)) : (b', \vartheta') \in \mathbb{R}^4 \times \mathbb{R}^{N_\Theta}\} \to \mathbb{R}^{N_\Sigma},
\]

where \(N_\Sigma = 4 + N_\Theta\), and

\[
(g_b')^T(b') := \frac{\hat{g}_b}{\hat{\vartheta}_b} b' + \nabla_{g_b} \otimes \omega^\Sigma_b(b'),
\]

(5.7)

and \(\omega^\Sigma_b(b')\) denotes the solution of the Cauchy problem

\[
\left\{
\begin{aligned}
C_{g_b} \circ \nabla_{g_b} \otimes \omega^\Sigma_b(b') &= -C_{g_b}(g_b'(b')) & \text{in } M, \\
\gamma_0(\omega^\Sigma_b(b')) &= (0, 0) & \text{on } \Sigma_0,
\end{aligned}
\right.
\]

(5.8)
Observe that \((g'_b)(b') + \nabla_{g_b} \otimes \gamma \partial'\) are infinitesimal diffeomorphisms and infinitesimal changes in black hole parameters, and are solutions satisfying the linearized harmonic gauge condition. These are precisely the solutions to the linearized Einstein equations in harmonic gauge which correspond to general covariance and changes in the black hole parameters \(b\). Thus, Proposition 5.4 and Proposition 5.5 together state that the finite linear obstacles to exponential decay in Proposition 5.2 are unphysical in that they either

1. do not satisfy the linearized harmonic gauge constraint, or
2. are an infinitesimal diffeomorphism or an infinitesimal change of the black hole parameters.

It will be convenient to observe that \(\lambda_Y\) and \(\lambda_C\) are invariant under time translation.

**Proposition 5.6.** Fix \(g_b\) a slowly-rotating Kerr-de Sitter metric, and let \(h\) be the solution to the Cauchy problem in (5.4), for some \((f, h_0, h_1) \in D^{k,\alpha,1}(M), k > 0\). Then in fact

\[
\lambda_Y[g_b](f(t_* + T_*), h(T_*), \hat{c}_{t_*} h(T_*)) = \lambda_Y[g_b](f(t_*), h_0, h_1),
\]

\[
\lambda_C[g_b](f(t_* + T_*), h(T_*), \hat{c}_{t_*} h(T_*)) = \lambda_C[g_b](f(t_*), h_0, h_1).
\]

(5.9)

Finally, we will need the following lemma controlling the dependence of \(\lambda_Y[g_b]\) on the black hole parameters \(b\). This is the content of Corollary 10.10 in [8].

**Lemma 5.7.** Let \(g_b\) be a slowly-rotating Kerr-de Sitter metric. If \(f \in L^2(M)\) is supported on \(\mathcal{D} = [0, T_*] \times \Sigma\), then

\[
|D_b \lambda_Y[g_b](f, h_0, h_1)| \leq \|f\|_{L^2(\mathcal{D})} + \|(h_0, h_1)\|_{H^1(\Sigma_0)}.
\]

5.2 Exponential growth on perturbation of Kerr-de Sitter

In this section we state the main energy estimate that we need on a perturbation of Kerr-de Sitter. The proof of the following proposition is postponed to Section 8.1.

**Proposition 5.8.** Fix \(\delta_* > 0\) and let \(\mathcal{D} = [0, T_*] \times \Sigma\). Then fix \(g = g_b + \tilde{g}\), where \(g_b\) is a slowly-rotating Kerr-de Sitter metric, and

\[
\sup_{t_* > 0} e^{-\delta_* t_*} \|\tilde{g}\|_{H^1(\Sigma_{t_*})} \leq \Delta_0
\]

and let \(h\) solve the Cauchy problem

\[
L_g h = f,
\]

\[
\gamma_0(h) = (h_0, h_1),
\]

where \(L_g\) is as in (3.15) and \((f, h_0, h_1) \in D^{k,\alpha,0}(M), k \geq 3\). Then for \(\Delta_0\) sufficiently small, \(h\) satisfies the estimate

\[
\left\| e^{-\delta_* t_*} h \right\|_{H^1(\mathcal{D})} \leq \|(h_0, h_1)\|_{H^1(\Sigma_0)} + \left\| e^{-\delta_* t_*} h \right\|_{H^1(\Sigma_{t_*})} + \int_0^{T_*} e^{-\delta_* t_*} \left( \left\| f \right\|_{L^2(\Sigma_{t_*})} + \|h\|_{L^2(\Sigma_{t_*})} \right) dt_*.
\]

(5.10)

We make a few remarks regarding Proposition 5.8. Proposition 5.8 is a perturbation of a weak Morawetz estimate on exact Kerr-de Sitter spacetimes with two key properties:

1. first, the estimate in (5.10) does not lose derivatives. This is critical to improve the bootstrap assumptions at the highest level of regularity;
2. second, the estimate in (5.10) is proven for arbitrarily small \(\delta_* > 0\). To improve the low-regularity exponential-decaying bootstrap assumption, we use an interpolation argument. Having (5.10) for arbitrarily small \(\delta_* > 0\) then allows us to drastically reduce the number of derivatives needed for the aforementioned interpolation argument.

\textsuperscript{6}The requirement that \(k \geq 3\) again has to do with the threshold regularity level in [8], see Remark 5.3.
6 Bootstrap assumptions

In this section, we set up the bootstrap assumptions that we will use to prove Theorem 4.2.

6.1 The semi-global extension

Instead of making bootstrap assumptions on solutions of EVE up to a finite time, we make bootstrap assumptions on a semi-global extension of EVE.

We first introduce an auxiliary proposition that generates admissible gauged initial data for the linearized gauged Einstein equations from a solution to the nonlinear Einstein equations in harmonic gauge that is quadratically close to the nonlinear solution.

Proposition 6.1. Fix $T_* > 0$. Then there exists a map
\[
\iota_{b,T_*} : H^k(\Sigma_{T_*}) \to H^k(\Sigma_{T_*})
\]
\[(h_0, h_1) \mapsto (\tilde{h}_0, \tilde{h}_1),
\]
satisfying the following properties.

1. If $\tilde{h} \in S^2T^*\mathcal{M}$ is a two-tensor inducing $\iota_{b,T_*}(h_0, h_1)$ on $\Sigma_{T_*}$ in the sense that
\[
\left(\tilde{h}(T_*, \cdot), \mathcal{L}_T \tilde{h}(T_*, \cdot)\right)\big|_{\Sigma_{T_*}} = \iota_{b,T_*}(h_0, h_1),
\]
then $\tilde{h}$ satisfies the linearized harmonic gauge constraint on $\Sigma_{T_*}$ given by
\[
\mathcal{C}_{g_0}(\tilde{h})\big|_{\Sigma_{T_*}} = 0.
\]

2. If $\left(h\big|_{\Sigma_{T_*}}, \mathcal{L}_T h\big|_{\Sigma_{T_*}}\right)$ satisfies the nonlinear harmonic gauge constraint on $\Sigma_{T_*}$ with respect to $g_0$, and
\[
\left(\tilde{h}\big|_{\Sigma_{T_*}}, \mathcal{L}_T \tilde{h}\big|_{\Sigma_{T_*}}\right) = \iota_{b,T_*}\left(h\big|_{\Sigma_{T_*}}, \mathcal{L}_T h\big|_{\Sigma_{T_*}}\right),
\]
then there exist $q_{T_*}(\cdot, \cdot), \tilde{q}_{T_*}(\cdot, \cdot)$ such that
\[
\left(h - \tilde{h}\right)\big|_{\Sigma_{T_*}} = 0, \quad \partial_t\left(h - \tilde{h}\right)\big|_{\Sigma_{T_*}} = q_{T_*}(h|_{\Sigma_{T_*}}, \partial h|_{\Sigma_{T_*}}) + \tilde{q}_{T_*}(h|_{\Sigma_{T_*}}, \partial h|_{\Sigma_{T_*}}), \quad (6.1)
\]
where $q_{T_*}$ is quadratic in its arguments, and $\tilde{q}_{T_*}$ is at least cubic in its arguments.

Proof. Let $h \in S^2T^*\mathcal{M}$ be a symmetric two-tensor inducing $(h_0, h_1)$ on $\Sigma_{T_*}$. Then, let $(q_{T_*}, \tilde{q}_{T_*})$ denote the induced metric by $g_0 + h$, $\mathcal{L}_T(g_0 + h)$ respectively on $\Sigma_{T_*}$. Now we will construct some $\tilde{g} = g_0 + \tilde{h}$ such that $(\tilde{h}, \partial_t\tilde{h})\big|_{\Sigma_{T_*}} = \iota_{b,T_*}(h_0, h_1)$ explicitly satisfying the conditions in the lemma.

First, we define $\tilde{h} = q_{T_*} - g_0$, and $\tilde{h}_{ij} = (q'_{T_*})_{ij}$. It remains to define $\tilde{h}'_{\alpha\mu}$ using the linearized gauge condition. To this end, we recall the gauge condition on $\Sigma_{T_*}$ linearized around $g_0$.

\[
0 = -\frac{1}{2} \tilde{h}^{\alpha\beta} \tilde{c}_{\mu}(g_0)_{\alpha\beta} + \frac{1}{2} \tilde{g}^{\alpha\beta} \tilde{c}_{\mu}(g_0)_{\alpha\beta} + \tilde{h}_{\mu\nu}(g_0)_{\alpha\beta} \Gamma(g_0)^{\gamma}_{\alpha\beta} - \tilde{h}_{\mu\nu}(g_0)_{\alpha\beta} \Gamma(g_0)^{\gamma}_{\alpha\beta}. \quad (6.2)
\]

Contracting (6.2) with $\mathbf{T}$, we have that
\[
\frac{1}{2} (g_0)^{\alpha\beta} \tilde{c}_{\mu}(g_0)_{\alpha\beta} + \frac{1}{2} \tilde{h}^{\alpha\beta} \tilde{c}_{\mu}(g_0)_{\alpha\beta} - \tilde{h}^{\alpha\beta} \tilde{c}_{\alpha}(g_0)_{\alpha\beta} - \tilde{h}_{\mu\nu}^{\gamma}(g_0)_{\alpha\beta} \Gamma(g_0)^{\gamma}_{\alpha\beta} + (g_0)^{\alpha\beta} \Gamma(g_0)^{\gamma}_{\alpha\beta}. \quad (6.3)
\]
As we already have \( \tilde{h} \big|_{\Sigma_T} \), \( \partial_t \tilde{h} \big|_{\Sigma_T} \), and \( \partial_{tt} \tilde{h} \big|_{\Sigma_T} \), solving equation (6.3) uniquely determines \( \tilde{h}'_{t*} = \partial_{tt} \tilde{h} \big|_{\Sigma_T} \).

Now consider contracting (6.2) with \( \partial_t \). In this case we have that

\[
(g_b)^{t*} \partial_t \tilde{h}_{t*} = - (g_b)^{t*} \partial_t \tilde{h}_{t*} + \frac{1}{2} (g_b)^{\alpha \beta} \partial_t \tilde{h}_{t*} \partial_{tt} \tilde{h}_{t*} \partial_{tt} \tilde{h}_{t*} - \tilde{h}^{\alpha \beta} \partial_t \tilde{h}_{t*} (g_b)_{\alpha \beta} + \frac{1}{2} \tilde{h}^{\alpha \beta} \partial_t \tilde{h}_{t*} (g_b)_{\alpha \beta} \\
+ \tilde{h}_{t*} (g_b)^{\alpha \beta} \Gamma (g_b)^{\gamma} \partial_{tt} \tilde{h}_{t*} (g_b)_{\alpha \beta}.
\]  

(6.4)

As before, (6.4) uniquely determines \( \tilde{h}'_{t*} = \partial_{tt} \tilde{h} \big|_{\Sigma_T} \).

By construction, we have defined \((\tilde{h}, \partial_{tt} \tilde{h})\) satisfying the linearized gauge constraint on \(\Sigma_T\).

Moreover, by construction

\[
\left( h - \tilde{h} \right) \big|_{\Sigma_T} = 0, \quad \partial_{tt} \left( h_{ij} - \tilde{h}_{ij} \right) \big|_{\Sigma_T} = 0.
\]

If \(g_b + h\) solves the nonlinear gauge constraint on \(\Sigma_T\), and \(g_b + \tilde{h}\) solves the linearized gauge constraint on \(\Sigma_T\), then it is clear that \(\partial_{tt} \left( h - \tilde{h} \right) \big|_{\Sigma_T} \) can be decomposed into a nonlinearity

\[
\partial_{tt} \left( h - \tilde{h} \right) \big|_{\Sigma_T} = \mathcal{L}_{T*} (h, \partial h) + \mathcal{L}_{T*} (h, \partial h),
\]

consisting of a quadratic nonlinearity and a higher-order nonlinearity respectively as desired. \(\square\)

We can now explicitly construct the desired semi-global extension of the EVE. First, let \(\chi(t_a)\) be a smooth, compactly supported function such that

\[
\chi(t_a) = \begin{cases} 
1 & t_a \leq 0 \\
0 & t_a \geq 1
\end{cases}, \quad \chi_{T*}(t_a) = \chi(t_a - T_*). \quad (6.5)
\]

**Definition 6.2.** From this point forward in the paper, we denote

\[
h_{T*} := \chi_{T*} h + (1 - \chi_{T*}) \tilde{h}_{T*}, \quad (6.6)
\]

where \(\phi^*(g_b + h)\) is a solution to EVE with initial data \((g, k)\), and where \(\tilde{h}_{T*}\) is the solution to

\[
\mathcal{L}_{g_b} \tilde{h}_{T*} = 0 \\
\gamma_{T*} (\tilde{h}_{T*}) = \iota_{h_{T*}} (h | \Sigma_T), \quad \partial_{tt} h | \Sigma_T = 0.
\]

(6.7)

To emphasize their dependence on the choice of parameters \((b, \vartheta, g, k, T_*)\), we will write

\[
h_{T*} = h_{T*}(b, \vartheta, g, k), \\
h = h(b, \vartheta, g, k), \\
\tilde{h}_{T*} = \tilde{h}_{T*}(b, \vartheta, g, k). \quad (6.8)
\]

**Definition 6.3.** Fix some \(T_* > 0\). Then we define

\[
\mathcal{N}_{g_b}^{T*}(h_{T*}, \partial h_{T*}, \partial^2 h_{T*}) := \chi_{T*} \mathcal{N}_{g_b}(h, \partial h, \partial^2 h) + \mathcal{L}_{g_b} \chi_{T*} (h - \tilde{h}_{T*}),
\]

(6.9)

where we recall \(\tilde{h}_{T*}\) is the solution to the Cauchy problem in (6.7), \(\mathcal{L}_{g_b}\) is as defined in (3.13), and \(\mathcal{N}_{g_b}\) is as defined in (3.12).
We can now define the relevant semi-global extension of EVE.

**Proposition 6.4.** Fix \( T_* > 0 \), some diffeomorphism \( \phi = e^{i s \vartheta} \) where \( \vartheta \in \mathbb{R}^{N_o} \), a slowly rotating Kerr-de Sitter metric \( g_0 \), and \((\Sigma_0, g_0, k_0)\) an admissible initial data triplet. Then, for \( \phi^*(g_0 + h) \) a solution to EVE launched by the initial data \((\Sigma_0, g_0, k_0)\), \( h_{T_*} \) as defined in Definition 6.2 solves the semi-global extension of EVE generated by \((b, \vartheta, T_*)\), given by

\[
\begin{align*}
\mathbb{L}_{g_0} h_{T_*} &= \mathcal{N}_{g_0}^T (h_{T_*}, \partial h_{T_*}, \partial^2 h_{T_*}), \\
\gamma_0 (h_{T_*}) &= i_{b, \phi}(g_0, k_0),
\end{align*}
\]

where \( \mathcal{N}_{g_0}^T \) is as given in (6.9). Moreover,

1. \( h_{T_*} = h \) for \( t_* \leq T_* \),
2. \( \mathcal{C}_{g_0}(h_{T_*}) = 0 \) for \( t_* \geq T_* + 1 \).

**Proof.** The proof is straightforward from the definitions of \( h_{T_*} \) and \( \mathcal{N}_{g_0}^T \) in Definitions 6.2 and 6.3 respectively. \( \square \)

To improve the bootstrap assumptions, we need certain control on \( \mathcal{N}_{g_0}^T \).

**Proposition 6.5.** Fix some \( T_* > 0 \). Then, there exist nonlinear functions \( q(\cdot, \cdot, \cdot), \tilde{q}(\cdot, \cdot, \cdot), q_{T_*}(\cdot, \cdot), \) and \( \tilde{q}_{T_*}(\cdot, \cdot) \) such that for \( t_* \leq T_* \),

\[
\mathcal{N}_{g_0}^T (h_{T_*}, \partial h_{T_*}, \partial^2 h_{T_*}) = q(h_{T_*}, \partial h_{T_*}, \partial^2 h_{T_*}) + \tilde{q}(h_{T_*}, \partial h_{T_*}, \partial^2 h_{T_*})
\]

and

\[
\sup_{t_*=T_*} \left\| \mathcal{N}_{g_0}^T (h_{T_*}, \partial h_{T_*}, \partial^2 h_{T_*}) \right\|_{L^2(\Sigma_*)} \leq \sup_{t_*=T_*} \left\| q(h_{T_*}, \partial h_{T_*}, \partial^2 h_{T_*}) \right\|_{L^2(\Sigma_*)} + \sup_{t_*=T_*} \left\| \tilde{q}(h_{T_*}, \partial h_{T_*}, \partial^2 h_{T_*}) \right\|_{L^2(\Sigma_*)} \sup_{t_*=T_*} \left\| q_{T_*}(h_{T_*}, \partial h_{T_*}) \right\|_{L^2(\Sigma_*)} + \sup_{t_*=T_*} \left\| \tilde{q}_{T_*}(h_{T_*}, \partial h_{T_*}) \right\|_{L^2(\Sigma_*)},
\]

where

\[
q(h_{T_*}, \partial h_{T_*}, \partial^2 h_{T_*}) = \sum_{|\alpha|+|\beta| \leq 2} q_{\alpha \beta} \partial^\alpha h_{T_*} \partial^\beta h_{T_*},
\]

\[
\tilde{q}(h_{T_*}, \partial h_{T_*}, \partial^2 h_{T_*}) = \sum_{|\alpha|+|\beta| \leq 2} \tilde{q}_{\alpha \beta} \partial^\alpha h_{T_*} \partial^\beta h_{T_*},
\]

and \( \tilde{q}, \tilde{q}_{T_*} \) are at least cubic in their arguments, and \( q(h_{T_*}, \partial h_{T_*}, \partial^2 h_{T_*}) \) and \( \tilde{q}(h_{T_*}, \partial h_{T_*}, \partial^2 h_{T_*}) \) are quasilinear in \( h_{T_*} \), and \( q_{T_*}(h_{T_*}, \partial h_{T_*}) \) and \( \tilde{q}_{T_*}(h_{T_*}, \partial h_{T_*}) \) are semilinear in \( h_{T_*} \).

**Proof.** The first conclusion in (6.11) follows directly from the definition of \( \mathcal{N}_{g_0}^T \) and \( \mathcal{N}_{g_0} \).

For the second conclusion, observe that \( h - \tilde{h}_{T_*} \) satisfies the Cauchy problem given by

\[
\begin{align*}
\mathbb{L}_{g_0} (h - \tilde{h}_{T_*}) &= \mathcal{N}_{g_0} (h, \partial h, \partial^2 h), \\
\gamma_{T_*} (h - \tilde{h}_{T_*}) &= \left( 0, q_{T_*} (h_{T_*}, \partial h_{T_*}) + \tilde{q}_{T_*} (h_{T_*}, \partial h_{T_*}) \right),
\end{align*}
\]

where \( q_{T_*} (h_{T_*}, \partial h_{T_*}), \tilde{q}_{T_*} (h_{T_*}, \partial h_{T_*}) \) are as defined in (6.1). Then, since \([T_*, T_* + 1]\) is a time interval of bounded length, an energy estimate with a timelike vectorfield (see for example Lemma 8.14) and Gronwall allows us to conclude using the definition of \( h_{T_*} \) in (6.6). \( \square \)
The following corollary follows immediately from the construction of \( N^T_{g_0} \).

**Corollary 6.6.** Fix a slowly-rotating Kerr-de Sitter metric \( g_0 \), and some \( T_*>0 \). Then, let \( h_{T_*} \) denote the solution to the semi-global extension of EVE in (6.10). Then,

\[
\lambda_C[g_0] \left( \mathcal{N}^T_{g_0}(h_{T_*}, \hat{\partial}h_{T_*}, \hat{\partial}^2h_{T_*}), ib_\phi(g_0, k_0) \right) = 0,
\]

where \( \phi = e^{i\omega \theta} \). Moreover, if

\[
\lambda_Y[g_0] \left( \mathcal{N}^T_{g_0}(h_{T_*}, \hat{\partial}h_{T_*}, \hat{\partial}^2h_{T_*}), ib_\phi(g_0, k_0) \right) = 0,
\]

then for \( k \geq 3, h_{T_*} \) satisfies the decay estimate

\[
\sup_{t_\ast \geq 0} e^{\delta_3 t_*} \| h_{T_*} \|_{\mathcal{T}^k(\Sigma_{t_\ast})} \lesssim \left\| \left( \mathcal{N}^T_{g_0}(h_{T_*}, \hat{\partial}h_{T_*}, \hat{\partial}^2h_{T_*}), ib_\phi(g_0, k_0) \right) \right\|_{D^{k,\lambda,1}(M)}.
\]

**Proof.** Observe that by construction, \( \mathcal{N}^T_{g_0}(h_{T_*}, \hat{\partial}h_{T_*}, \hat{\partial}^2h_{T_*}) = 0 \) for \( t_\ast > T_* + 1 \), and that a solution \( h_{T_*} \) to the semi-global Cauchy problem in (6.10) eventually satisfies the linearized gauge constraint. That is, for \( t_\ast > T_* + 1 \),

\[
C_{g_0}(h_{T_*}) = 0.
\]

The conclusion is then an immediate consequence of Propositions 5.2 and 5.4.

\[ \square \]

### 6.2 The bootstrap assumptions

We are now ready to specify our bootstrap assumptions. Fix a bootstrap constant \( \Delta_0 = \varepsilon_0^\frac{2}{3} \). Then let \( T_* > 0 \) be a bootstrap time for which there exist Kerr-de Sitter black hole parameters \( b_{T_*} \), and a vector \( \partial_{T_*} \in \mathbb{R}^{N_{\alpha}} \) satisfying

\[
|\partial_{T_*}| + |b_{T_*} - b^0| \leq \Delta_0, \quad \text{(BA-G)}
\]

such that, recalling \( g = \phi_{T_*}(b)^*(g_{b_{T_*}} + h) \), the metric perturbation \( h_{T_*} \) solving (6.10) satisfies the orthogonality condition

\[
\lambda_Y[g_{b_{T_*}}] \left( \mathcal{N}^T_{b_{T_*}}(h_{T_*}, \hat{\partial}h_{T_*}, \hat{\partial}^2h_{T_*}), ib_{T_*}(g_{b_{T_*}}, k_0) \right) = 0, \quad \text{(ORT)}
\]

where \( \lambda_T \) is as constructed in Proposition 5.2. Moreover, assume that \( h_{T_*} \) satisfies in addition the bootstrap estimates

\[
\sup_{t_\ast \geq 0} e^{-\delta_{3} t_*} \| h_{T_*} \|_{\mathcal{T}^j(\Sigma_{t_\ast})} \leq \Delta_0, \quad j \leq N_e, \quad \text{(BA-E)}
\]

\[
\| e^{\delta_{3} t_*} h_{T_*} \|_{H^j(M)} \leq \Delta_0, \quad j \leq N_d, \quad \text{(BA-D)}
\]

where we will take \( N_e = 6 \), and \( N_d = 3 \). We pick \( \delta_3 > 0 \) and \( \delta_d > 0 \) such that \( \delta_d < \alpha \), where \( \alpha \) is as in Proposition 5.2, and moreover, \( \delta_3, \delta_d, N_e, \) and \( N_d \) satisfy the conditions

\[
2\delta_3 < (N_e - N_d - 2)\delta_d, \quad \text{(6.14)}
\]

\[
2 < N_e - N_d. \quad \text{(6.15)}
\]

**Remark 6.7.** The conditions on \( \delta_3, \delta_d, N_e, \) and \( N_d \) in (6.14) and (6.15) are made so that we can use an interpolation argument to improve the bootstrap assumptions in what follows. We also require that \( N_d > \frac{5}{2} \) in order to use Sobolev embeddings. As a result, it should be possible to sharpen the regularity requirements in Theorem 4.2 to require only \( N_e > \frac{5}{2} \) derivatives. However, for the sake of simplicity and to avoid fractional Sobolev spaces, we proceed here with the choice \( N_e = 6 \), and \( N_d = 3 \).
6.3 Initialization of the bootstrap argument

In this section, we will prove that there exists some sufficiently small \( T_0 > 0 \) such that the bootstrap assumptions detailed in Section 6.2 hold.

The main proposition of this section is the following.

**Proposition 6.8.** There exists some \( \delta_0 > 0 \) such that for any \( (g_j, k_0) \in H^{N_d}(\Sigma_0) \times H^{N_d-1}(\Sigma_0) \), and \( T_0 > 0 \) satisfying

\[
\left\| (g_j, k_0) - (g_{j0}, k_{00}) \right\|_{H^{N_d}(\Sigma_0) \times H^{N_d-1}(\Sigma_0)} + T_0 < \delta_0,
\]

there exists a choice of \( b \in \mathbb{R}^4 \) and \( \vartheta \in \mathbb{R}^{N_{e0}} \) such that

\[
|\vartheta| + |b - b^0| \leq \delta_0,
\]

and such that the solution \( h \) of (6.10) satisfies the orthogonality condition (ORT), as well as the bootstrap estimates (BA-E) and (BA-D).

From Proposition 6.8, it is straightforward to see that we can define \( \Delta_0 \) and \( \varepsilon_0 \) to initialize the bootstrap argument.

**Corollary 6.9.** There exists a time \( T_0 > 0 \) such that for initial data \( (g_j, k_0) \) satisfying (4.2), there exist \( (b_{T_0}, \vartheta_{T_0}) \) in \( \mathbb{R}^4 \times \mathbb{R}^{N_{e0}} \) such that

\[
\lambda_T[g_j] \left( N_{g_{T_0}}^{T_0}(h_{T_0}(b_{T_0}, \vartheta_{T_0}, g_j, k_0), i_{b_{T_0}} \vartheta_{T_0}(g_j, k_0)) \right) = 0,
\]

\[
|\vartheta_{T_0}| + |b_{T_0} - b^0| \leq \Delta_0,
\]

\[
\sup_{t_0 \geq 0} e^{-\delta_0 t} \left\| h_{T_0}(b_{T_0}, \vartheta_{T_0}, g_j, k_0) \right\|_{H^{N_{e}}(\Sigma_{T_0})} \leq \Delta_0, \quad j \leq N_e,
\]

\[
\sup_{t_0 \geq 0} e^{\delta_0 t} \left\| h_{T_0}(b_{T_0}, \vartheta_{T_0}, g_j, k_0) \right\|_{H^{N_{d}}(\Sigma_{T_0})} \leq \Delta_0, \quad j \leq N_d.
\]

**Proof.** Setting \( \Delta_0 = \delta \frac{2}{3} \), and taking \( \delta \) sufficiently small we have that

\[
\varepsilon_0 < \frac{\delta}{2} < \Delta_0 < 1.
\]

Moreover, taking \( T_0 = \frac{\delta}{2} \),

\[
\left\| (g_j, k_0) - (g_{j0}, k_{00}) \right\|_{H^{N_{e}}(\Sigma_0) \times H^{N_{e}-1}(\Sigma_0)} \leq \varepsilon_0 \leq \delta \frac{2}{3}.
\]

Then, applying Proposition 6.8 and taking \( \delta \) sufficiently small concludes the proof of Corollary 6.9.

The remainder of this section is focused on the proof of Proposition 6.8. We first prove a lemma utilizing the implicit function theorem that we will make repeated use of in the remainder of the proof.

**Lemma 6.10.** Define

\[
\mathcal{F} : \mathbb{R}^4 \times \mathbb{R}^{N_{e0}} \times H^{N_d}(\Sigma_0) \times H^{N_d-1}(\Sigma_0) \times \mathbb{R}^+ \to \mathbb{R}^{4+N_{e0}},
\]

\[
\mathcal{F}(b, \vartheta, g, k, T_*) = \lambda_T[g_j] \left( N_{g_{T_*}}^{T_*}(h_{T_*}(b_{T_*}, \vartheta_{T_*}, \partial h_{T_*}, \partial h_{T_*}, i_{b_{T_*}}(g, k)) \right),
\]

(6.17)
where $\phi = e^{i\omega \vartheta}$, and $h_{T_*}$ is the solution to the Cauchy problem in (6.10). Also let $(\tilde{b}, \tilde{\vartheta}, \tilde{\varphi}, \tilde{T}) \in \mathbb{R}^4 \times \mathbb{R}^N \times H^{N_d}(\Sigma_0) \times H^{N_d-1}(\Sigma_0) \times \mathbb{R}^+$ such that
\[
\mathcal{F}(b^\dagger, \vartheta^\dagger, \varphi^\dagger, T^\dagger) = 0,
\] and moreover,
\[
\begin{align*}
D_{b, \vartheta}N^T_{b^\dagger}(h_{T_*}, \partial h_{T_*}, \partial^2 h_{T_*}) \big|_{(b^\dagger, \vartheta^\dagger, \varphi^\dagger, T^\dagger)} &= 0, \\
D_{b, \vartheta} \lambda(g_b) \left(\lambda_{N_{b^\dagger}}(h_{T_*}, \partial h_{T_*}, \partial^2 h_{T_*}), i_{b, \varphi}(g^1, k)\right) \big|_{(b^\dagger, \vartheta^\dagger, \varphi^\dagger, T^\dagger)} &= 0. 
\end{align*}
\] Then there exists some $\delta > 0$ such that defining
\[
\mathcal{X}_{\delta, \tilde{b}, \tilde{k}, \tilde{T}} \subset H^{N_d}(\Sigma_0) \times H^{N_d-1}(\Sigma_0) \times \mathbb{R}^+,
\] such that $(g, k, T_*) \in \mathcal{X}_{\delta, \tilde{b}, \tilde{k}, \tilde{T}}$ if and only if
\[
\|
\begin{bmatrix} q \\ k - \tilde{k} \end{bmatrix}
\|_{H^{N_d}(\Sigma_0)} + \|
\begin{bmatrix} q \\ k - \tilde{k} \end{bmatrix}
\|_{H^{N_d-1}(\Sigma_0)} + \|T_* - \tilde{T}\| < \delta,
\]
there exists a function $\mathcal{G}(g, k, T_*)$ depending on $(\tilde{b}, \tilde{\vartheta}, \tilde{\varphi}, \tilde{T})$ such that
\[
\mathcal{G}(g, k, T_*) : \mathcal{X}_{\delta, \tilde{b}, \tilde{k}, \tilde{T}} \to \mathbb{R}^4 \times \mathbb{R}^N,
\]
which is well-defined and $C^1$ in its arguments on $\mathcal{X}_{\delta, \tilde{b}, \tilde{k}, \tilde{T}}$. Moreover, for $(g, k, T_*) \in \mathcal{X}_{\delta, \tilde{b}, \tilde{k}, \tilde{T}}$, \[
\mathcal{F}(b, \vartheta, g, k, T_*) = 0, \quad \iff \quad (b, \vartheta) = \mathcal{G}(g, k, T_*).
\] Proof. Recalling the definition of $(g^0_\gamma)^\gamma(b')$ in (5.7), observe that we can calculate
\[
D_{b, \vartheta} i_{b, \varphi}(g^\dagger, k) \big|_{(b, \vartheta, \varphi, T)}(b', \vartheta') = \gamma_0 \left( (g^0_\gamma)^\gamma(b') + \nabla_{g_b} \otimes i_\vartheta \vartheta' \right),
\]
as a result, using (6.19),
\[
D_{b, \vartheta} \mathcal{F} = \lambda_{\gamma} \left( 0, \gamma_0 \left( (g^0_\gamma)^\gamma(b') + \nabla_{g_b} \otimes i_\vartheta \vartheta' \right) \right).
\]
Moreover, this is an isomorphism by Proposition 5.5. The existence of $\mathcal{X}_{\delta, \tilde{b}, \tilde{k}, \tilde{T}}$ as in (6.20), and of $\mathcal{G}$ as in (6.21) satisfying (6.22) is then a direct result of the implicit function theorem in Theorem 2.20.

Next we prove two auxiliary lemmas that will be used in conjunction with the application of the implicit function theorem in Lemma 6.10. The first, Lemma 6.11, locates the neighborhood on which we will perform the implicit function theorem, and the second, Lemma 6.12, gives control over the derivatives of $\mathcal{F}$.

Lemma 6.11. There exists some $\delta_{\text{init}} > 0$ sufficiently small, and a neighborhood $(b_0, 0) \in \mathcal{B}_{\text{init}} \subset \mathbb{R}^4 \times \mathbb{R}^N$ such that for any $(b, \vartheta) \in \mathcal{B}_{\text{init}}$, the solution
\[
h_{\delta_{\text{init}}}(b, \vartheta, g_0, k_0) := \chi_{\delta_{\text{init}}} h(b, \vartheta, g_0, k_0) + (1 - \chi_{\delta_{\text{init}}}) h_{\delta_{\text{init}}}(b, \vartheta, g_0, k_0),
\]
using the definitions of $h_{T_*}(b, \vartheta, g, k)$, $h(b, \vartheta, g, k)$, and $h_{T_*}(b, \vartheta, g, k)$ in (6.8), satisfies the following estimates
\[
\begin{align*}
\sup_{t_* < \delta_{\text{init}} + 1} e^{\delta t_*} \left\| h_{\delta_{\text{init}}}(b, \vartheta, g_0, k_0) \right\|_{\mathbb{P}(\Sigma_{t_*})} &\leq 1, \quad k \leq N_d, \\
\sup_{t_* < \delta_{\text{init}} + 1} e^{-\delta t_*} \left\| h_{\delta_{\text{init}}}(b, \vartheta, g_0, k_0) \right\|_{\mathbb{P}(\Sigma_{t_*})} &\leq 1, \quad k \leq N_v.
\end{align*}
\]
Proof. We consider the neighborhood
\[ B_{\text{init}} := \{(b, \vartheta) : |b - b^0| + |\vartheta| < \delta_{\text{init}}\}, \]
and show that for a sufficiently small choice of \( \delta_{\text{init}} \), \( B_{\text{init}} \) satisfies the conditions in the lemma.

Observe that by local existence, if \( \delta_{\text{init}} \) is sufficiently small, we have that
\[ \sup_{t_\ast \in \delta_{\text{init}} + 1} \left\| h(b, \vartheta, g, k) \right\|_{\mathcal{H}^{N_d}(\Sigma_{t_\ast})} \lesssim \varepsilon_0 + \delta_{\text{init}}. \]

The estimates in (6.23) are then satisfied by taking \( \varepsilon_0, \delta_{\text{init}} \) sufficiently small. \( \Box \)

**Lemma 6.12.** Let \( \mathcal{F} \) be as defined in (6.17). Then, let
\[ (b^\dagger, \vartheta^\dagger, q^\dagger, k^\dagger, T^\dagger) \in \mathbb{R}^4 \times \mathbb{R}^{N_d} \times H^{N_d}(\Sigma_0) \times H^{N_d-1}(\Sigma_0) \times \mathbb{R}^+, \]
and let us denote \( h^\dagger_{T^\dagger} := h_{T^\dagger}(b^\dagger, \vartheta^\dagger, q^\dagger, k^\dagger) \) using the definition of \( h_{T^\dagger}(b, \vartheta, q, k) \) given in (6.8).

Moreover assume that \( (b^\dagger, \vartheta^\dagger, q^\dagger, k^\dagger, T^\dagger) \) was chosen so that \( h^\dagger_{T^\dagger} \) satisfies the control
\[ \sup_{t_\ast < T^\dagger + 1} e^{\delta_{T^\dagger}} \left\| h^\dagger_{T^\dagger} \right\|_{\mathcal{H}^{N_d}(\Sigma_{t_\ast})} \lesssim 1, \]
\[ \sup_{t_\ast < T^\dagger + 1} e^{-\delta_{T^\dagger}} \left\| h^\dagger_{T^\dagger} \right\|_{\mathcal{H}^{N_d}(\Sigma_{t_\ast})} \lesssim 1, \]
and the orthogonality condition (ORT).

Then for \( T^\dagger \) sufficiently small,
\[ \left\| D_{b, \vartheta, q, k, T^\dagger} \mathcal{F}(b, \vartheta, q, k, T^\dagger) \right\|_{(b^\dagger, \vartheta^\dagger, q^\dagger, k^\dagger, T^\dagger)} \leq 1. \]

**Proof.** We first consider
\[ D_{b, \vartheta} \mathcal{F}_{(b^\dagger, \vartheta^\dagger, q^\dagger, k^\dagger, T^\dagger)} = D_{b, \vartheta} \lambda_{b, \vartheta}(\vartheta, h_{T^\dagger}(b, \vartheta, q, k)) \left( (b^\dagger, \vartheta^\dagger, q^\dagger, k^\dagger, T^\dagger) \right). \]

Since \( \lambda_{b, \vartheta} \) is supported on \([0, T^\dagger + 1]\), we have from Lemma 5.7 and (6.24) that
\[ \left\| D_{b, \vartheta} \lambda_{b, \vartheta}(\vartheta, h_{T^\dagger}(b, \vartheta, q, k)) \right\|_{(b^\dagger, \vartheta^\dagger, q^\dagger, k^\dagger, T^\dagger)} \lesssim 1. \]

Observe that by construction,
\[ \left\| D_{b, \vartheta, q, k, T^\dagger} b, \vartheta, q, k \right\|_{(b^\dagger, \vartheta^\dagger, q^\dagger, k^\dagger, T^\dagger)} \lesssim 1. \]

We now consider \( D_{b, \vartheta, q, k, T^\dagger} \lambda_{b, \vartheta} \). Recall that up to \( T^\dagger \), the nonlinearity \( \lambda_{b, \vartheta} \) can be decomposed as
\[ \lambda_{b, \vartheta}(h_{T^\dagger}, \vartheta, q, k) = q(h_{T^\dagger}, \vartheta, q, k) + \tilde{q}(h_{T^\dagger}, \vartheta, q, k), \]
where \( q(\cdot, \cdot, \cdot, \cdot) \) and \( \tilde{q}(\cdot, \cdot, \cdot, \cdot) \) are as defined in (6.13), and that on \([T^\dagger, T^\dagger + 1]\), we have the control given in (6.12).

We show how to deal with \( q(\cdot, \cdot, \cdot, \cdot) \) since this is the primary difficulty. First observe that
\[ \left\| D_{b, \vartheta, q, k, T^\dagger} q(h_{T^\dagger}, \vartheta, q, k) \right\|_{(b^\dagger, \vartheta^\dagger, q^\dagger, k^\dagger, T^\dagger)} \lesssim 1. \]

Next, we consider \( D_{b, \vartheta, q, k, T^\dagger} \tilde{q}(h_{T^\dagger}, \vartheta, q, k) \). Recall that up to \( T^\dagger \), the nonlinearity \( \tilde{q}(\cdot, \cdot, \cdot, \cdot) \) can be decomposed as
\[ \tilde{q}(h_{T^\dagger}, \vartheta, q, k) = \tilde{q}(h_{T^\dagger}, \vartheta, q, k) + \tilde{\tilde{q}}(h_{T^\dagger}, \vartheta, q, k), \]
where \( \tilde{q}(\cdot, \cdot, \cdot, \cdot) \) and \( \tilde{\tilde{q}}(\cdot, \cdot, \cdot, \cdot) \) are as defined in (6.13), and that on \([T^\dagger, T^\dagger + 1]\), we have the control given in (6.12).
Thus we see that in order to prove the desired bound in (6.25), it suffices to prove that

$$\sup_{t_s \leq T_* + 1} \left\| D_{b, \vartheta, g, k, T_*} h_{T_*} \right\|_{(b', \vartheta', g', k', T')} \leq 1. \tag{6.26}$$

To prove (6.26), we first observe that

$$D_{b, \vartheta, g, k, T_*} h_{T_*} = \chi_{T_*} D_{b, \vartheta, g, k} h + (1 - \chi_{T_*}) D_{b, \vartheta, g, k, T_*} \tilde{h}_{T_*} + D_{T_*} \chi_{T_*} (h - \tilde{h}_{T_*}),$$

where $h, \tilde{h}_{T_*}$ are as defined in (6.8). For shorthand, we denote

$$h^\dagger := h \left( b^\dagger, \vartheta^\dagger, g^\dagger, k^\dagger \right), \quad \tilde{h}^\dagger := \tilde{h}_{T_*} \left( b^\dagger, \vartheta^\dagger, g^\dagger, k^\dagger \right).$$

Using the controls in (6.24), it suffices to show that

$$\sup_{t_s \leq T_* + 1} \left\| D_{b, \vartheta, g, k, T_*} h_{T_*} \right\|_{(b', \vartheta', g', k', T')} \leq 1, \tag{6.27}$$

$$\sup_{t_s \leq T_* + 1} \left\| D_{b, \vartheta, g, k, T_*} \tilde{h}_{T_*} \right\|_{(b', \vartheta', g', k', T')} \leq 1 \tag{6.28}$$

to prove (6.26). We first consider (6.28) and show that in fact it suffices to show (6.27). Observe that we can write the following Cauchy problem for $D_{b, \vartheta, g, k, T_*} \tilde{h}_{T_*} \left( b', \vartheta', g', k', T' \right)$,

$$\nabla_{g_s} D_{b, \vartheta, g, k, T_*} \tilde{h}_{T_*} \left( b', \vartheta', g', k', T' \right) = 0$$

$$\gamma \left( \nabla_{T} \left( D_{b, \vartheta, g, k, T_*} \tilde{h}_{T_*} \left( b', \vartheta', g', k', T' \right) \right) \right) = D_{b, \vartheta, g, k, T_*} h_{T_*} \left( \nabla_{T} \right) \left( b', \vartheta', g', k', T' \right).$$

From the construction of $t_b, T_*$ in Proposition 6.1,

$$\left\| D_{b, \vartheta, g, k, T_*} h_{T_*} \left( \nabla_{T} \right) \left( b', \vartheta', g', k', T' \right) \right\|_{H^{n+2, \delta}(\Sigma_{T_2})} \leq \left\| h^\dagger \right\|_{H^{n+2, \delta}(\Sigma_{T_2})} + \left\| D_{b, \vartheta, g, k} h \right\|_{(b', \vartheta', g', k', T')} \leq 1. \tag{6.29}$$

Thus it is clear that in order to prove (6.26), it suffices to prove (6.27).

To prove (6.27), we consider separately the derivatives involved. We will obtain the bound in (6.25) separately for each of the derivatives desired. First observe that by construction of the semi-global extension Cauchy problem, we have that for $t_s \leq T_*$

$$h(b, \vartheta, g, k) = (\phi^{-1})^* g - g_b, \quad g = \phi^* (g_b + h_{T_1}(b, \vartheta, g, k))$$

where $\phi = e^{i \omega \vartheta}$. Then, for $t_s \leq T_* + 1$

$$D_{b, \vartheta} h(b, \vartheta, g, k) \left( b', \vartheta', g', k' \right) = (g_b)^\dagger (b') + \nabla g_b \otimes \vartheta', \tag{6.29}$$

where $(g_b)^\dagger (b')$ is as defined in (5.7). We briefly note that the appearance of the $\omega^{-1}(b')$ term, reflected in the appearance of $(g_b)^\dagger (b')$ rather than $(g_b)(b')$ in (6.29) is necessary in light of the fact that the specific form of the Kerr-de Sitter metrics defined in (2.9) did not take any gauge considerations into account.

Since we know that there exists some fixed $\mathbf{M} > 0$ such that

$$\left\| (g_b)^\dagger (b') + \nabla g_b \otimes \vartheta' \right\|_{H^{n+2, \delta}(\Sigma_{T_2})} \leq e^{\mathbf{M} t_*},$$

28
we have that for $T^+ < 1$,

$$\sup_{t_* \leq T^+} \left\| D_{g,k} h_{T_*} \left( b', \vartheta', \varrho_1, k^1, T^+ \right) \right\|_{H^{N_d+1}(\Sigma_{t_*})} \lesssim 1. \quad (6.30)$$

Now let us consider $D_{g,k} h_{T_*} \left( b', \vartheta', \varrho_1, k^1 \right)$. Observe that $D_{g,k} h_{T_*} \left( b', \vartheta', \varrho_1, k^1 \right)$ solves the equation

$$\mathcal{L}_{\vartheta} D_{g,k} h_{T_*} \left( b', \vartheta', \varrho_1, k^1 \right) = \mathcal{P} \left( h_{T_*}, D_{g,k} h_{T_*} \right) \left( b', \vartheta', \varrho_1, k^1 \right),$$

where

$$\mathcal{P} \left( h_{T_*}, D_{g,k} h_{T_*} \right) := D_{g,k} Q(h, \partial h) - D_{g,k} S_g[h] - D_{g,k} V_g[h] + D_{g,k} \Box_g h.$$  

Then directly using local existence, we have that

$$\sup_{t_* \leq T^+} \left\| D_{g,k} h_{T_*} \left( b', \vartheta', \varrho_1, k^1 \right) \right\|_{H^{N_d+1}(\Sigma_{t_*})} \lesssim \left\| D_{g,k} z h_{T_*} \left( b', \vartheta', \varrho_1, k^1 \right) \right\|_{H^{N_d+1}(\Sigma_0)} \lesssim 1. \quad (6.31)$$

This proves (6.27) and consequently concludes the proof of Lemma 6.12.

We are now ready to prove Proposition 6.8.

**Proof of Proposition 6.8.** We first show that if $\tilde{T}$ is sufficiently small and there exists a choice of $(\tilde{b}, \tilde{\vartheta})$ such that (ORT) is satisfied for $(\tilde{b}, \tilde{\vartheta}, g_0, k_0, \tilde{T})$, and in addition we have that

$$\left\| g_0 - \tilde{\varrho}^* g_0 \right\|_{H^N(\Sigma_0)} + \left\| k_0 - \tilde{\varrho}^* k_0 \right\|_{H^{N-1}(\Sigma_0)} \lesssim \Delta_0^{\frac{3}{4}}, \quad \tilde{\varrho} = e^{i\varrho \vartheta}, \quad (6.32)$$

then for sufficiently small $\Delta_0$, we also have that (BA-D) and (BA-E) are satisfied. To begin, observe that using local existence theory, we have if (6.32) holds, then for $\delta_0$ sufficiently small,

$$\sup_{0 \leq t_* \leq 1+\delta_0} e^{-\delta_0 t_*} \left\| \tilde{h}_{\tilde{T}} \right\|_{H^N(\Sigma_{t_*})} + \sup_{0 \leq t_* \leq 1+\delta_0} e^{\delta_0 t_*} \left\| \tilde{h}_{\tilde{T}} \right\|_{H^N(\Sigma_{t_*})} \lesssim \Delta_0^{\frac{4}{5}}. \quad (6.33)$$

Since (ORT) is satisfied for $(\tilde{b}, \tilde{\vartheta}, g_0, k_0, \tilde{T})$, we can apply Propositions 5.2 and 5.8 to deduce that on $t_* > 1 + \delta_0$,

$$\sup_{t_* > 1+\delta_0} e^{-\delta_0 t_*} \left\| \tilde{h}_{\tilde{T}} \right\|_{H^N(\Sigma_{t_*})} \lesssim e^{-\delta_0(1+\delta_0)} \left\| \tilde{h}_{\tilde{T}} \right\|_{H^N(\Sigma_{t_*})},$$

and

$$\sup_{t_* > 1+\delta_0} e^{\delta_0 t_*} \left\| \tilde{h}_{\tilde{T}} \right\|_{H^N(\Sigma_{t_*})} \lesssim e^{\delta_0(1+\delta_0)} \left\| \tilde{h}_{\tilde{T}} \right\|_{H^N(\Sigma_{t_*})}.$$  

Then using (6.33), we have that for $\varepsilon_0$, and consequently $\Delta_0$ sufficiently small, (BA-E) and (BA-D) are satisfied as desired.

We now prove that we can indeed find some choice of $(\tilde{b}, \tilde{\vartheta}, \tilde{T})$ such that the orthogonality condition (ORT), the smallness condition (6.32), and the bootstrap assumption (BA-G) are satisfied. The main idea will be to apply Lemma 6.10. To this end, let us first define

$$F_{\text{init}} : \mathbb{R}^4 \times \mathbb{R}^{N_0} \times H^{N_d}(\Sigma_0) \times H^{N_d-1}(\Sigma_0) \times \mathbb{R}_+ \rightarrow \mathbb{R}^{4+N_0},$$

$$F_{\text{init}}(b, \vartheta, g_0, k, T_* = \lambda_T[g_0]) = \lambda_T[g_0] \left( N_{g_0}^* \left( h_{T*}, \partial h_{T*}, \partial^2 h_{T*}, \partial h_{T*} \right), i_{b, \vartheta}(g_0, k) \right).$$  


where \( \phi = e^{i\vartheta \theta} \), and \( h_{T_*} = h_{T_*}(b, \vartheta, g, k) \) is as defined in (6.8). We will show that there exists some \( \delta_0 > 0 \) sufficiently small such that for

\[
\left\| g - g_\rho \right\|_{H^{N_d}(\Sigma_0)} + \left\| k - k_\rho \right\|_{H^{N_d-1}(\Sigma_0)} + T_* < \delta_0,
\]  

(6.34)

there exists a \( C^1 \) mapping \( G_{\text{init}} : H^{N_d}(\Sigma_0) \times H^{N_d-1}(\Sigma_0) \times \mathbb{R}^+ \rightarrow \mathbb{R}^{4+N_\theta} \) such that

\[
F_{\text{init}} (G_{\text{init}}(g, k, T_*), g, k, T_*) = 0
\]

for \((g, k, T_*)\) satisfying (6.34). Observe that using the construction and estimates in Lemma 6.11, it is clear that \( F_{\text{init}} \) is \( C^1 \) in the arguments for \((b, \vartheta, T_*) \in \mathcal{B}_{\text{init}} \times (0, \delta_{\text{init}}) \), and \((g, k)\) satisfying (4.2). Moreover, observe that if \((b, \vartheta, g, k) = (b^0, 0, g_\rho, k_\rho)\), then for any \( T_* > 0 \), \( h_{T_*}(b^0, 0, g_\rho, k_\rho) = 0 \) is the solution of (6.10). Keeping this in mind, we can calculate

\[
D_{b,\vartheta}N_{T_0}(h_{T_*}, \partial h_{T_*}, \partial^2 h_{T_*})|_{(\vartheta^0, 0, g_\rho, k_\rho, 0)} (b', \vartheta') = 0.
\]

Applying Lemma 6.10, with the choice

\[
(b^1, \vartheta^1, g^1, k^1, T^1) = (b^0, 0, g_\rho, k_\rho, 0)
\]

we see that there exists some \( \delta_0 \) sufficiently small such that defining

\[
X_{\text{init}} \subset H^{N_d}(\Sigma_0) \times H^{N_d-1}(\Sigma_0) \times \mathbb{R}^+,
\]

where

\[
X_{\text{init}} := \left\{ (g, k, T_*) : \left\| g - g_\rho \right\|_{H^{N_d}(\Sigma_0)} + \left\| k - k_\rho \right\|_{H^{N_d-1}(\Sigma_0)} + |T_*| < \delta_0 \right\},
\]

there exists a \( C^1 \) mapping

\[
G_{\text{init}} : X_{\text{init}} \rightarrow \mathcal{B}_{\text{init}}
\]

such that for all \((g, k, T_*) \in X_{\text{init}},

\[
F_{\text{init}}(G_{\text{init}}(g, k, T_*), g, k, T_*) = 0.
\]

As a result, for all \((g, k, T_*) \in X_{\text{init}},\) we have that \( h_{T_*}(G_{\text{init}}(g, k, T_*), g, k) \) satisfies (ORT).

It remains to control \(|b - b^0| + |\vartheta|\) so that we can guarantee (6.32), and (BA-G). To this end, we apply Lemma 6.12. From Lemma 6.11, we have that for \((b, \vartheta, g, k, T_*) \in \mathcal{B}_{\text{init}} \times X_{\text{init}},\) the controls in (6.24) are satisfied. As a result, for

\[
(G_{\text{init}}(\tilde{g}, \tilde{k}, \tilde{T}), \tilde{g}, \tilde{k}, \tilde{T}) \in \mathcal{B}_{\text{init}} \times X_{\text{init}},
\]

we have from Lemma 6.12 that for \( \tilde{T} \ll 1 \) sufficiently small,

\[
\left| D_{g,k,T_*} F_{\text{init}} \left( G_{\text{init}}(\tilde{g}, \tilde{k}, \tilde{T}), \tilde{g}, \tilde{k}, \tilde{T} \right) \right| \leq 1,
\]

\[
\left| D_{b,\vartheta} F_{\text{init}} \left( G_{\text{init}}(\tilde{g}, \tilde{k}, \tilde{T}), \tilde{g}, \tilde{k}, \tilde{T} \right) \right| \leq 1.
\]

(6.35)

From the chain rule, we have that

\[
D_{g,k,T_*} G_{\text{init}}(\tilde{g}, \tilde{k}, \tilde{T}) = D_{b,\vartheta} F_{\text{init}} \left( G_{\text{init}}(\tilde{g}, \tilde{k}, \tilde{T}), \tilde{g}, \tilde{k}, \tilde{T} \right)^{-1} D_{g,k,T_*} F_{\text{init}} \left( G_{\text{init}}(\tilde{g}, \tilde{k}, \tilde{T}), \tilde{g}, \tilde{k}, \tilde{T} \right).
\]
Then for \( (\bar{\bar{b}}, \bar{k}, \bar{T}) \in \mathcal{X}_{init} \), denote \( (\bar{b}, \bar{\vartheta}) = \mathcal{G}(\bar{\bar{b}}, \bar{k}, \bar{T}) \). From the \( C^1 \) nature of \( \mathcal{G}_{init} \) and (6.35),

\[
|\bar{b} - \bar{b}^0| + |\bar{\vartheta}| \leq \sup_{\mathcal{X}_{init}} D_{\bar{\bar{b}}, \bar{k}, \bar{T}} \mathcal{G}_{init} \left( \left( \|\bar{\bar{b}} - \bar{b}^0\|_{H^{N_d}(\Sigma_0)} + \|\bar{k} - k\bar{\vartheta}\|_{H^{N_d-1}(\Sigma_0)} + |\bar{T}| \right) \right)
\]

\[
\leq C \left( \|\bar{\bar{b}} - \bar{b}^0\|_{H^{N_e}(\Sigma_0)} + \|\bar{k} - k\bar{\vartheta}\|_{H^{N_e-1}(\Sigma_0)} + |\bar{T}| \right),
\]

(6.36)

where \( C \) is an implicit constant independent of \( \delta_0, \Delta_0, \) and \( \varepsilon_0 \). Now choosing \( \Delta_0^4 = \delta < \delta_0 \), and making \( \delta \) smaller as necessary, we can in fact guarantee that

\[
2C\varepsilon_0 \leq \delta_0.
\]

Thus, choosing \( T_0 = \frac{1}{2C}\varepsilon_0 \), we have in fact that

\( (\bar{\bar{b}}, k_0, T_0) \in \mathcal{X}_{init} \).

As a result, denoting

\( (b_{T_0}, \vartheta_{T_0}) := \mathcal{G}_{init}(\bar{\bar{b}}, k_0, T_0) \),

it is clear that (6.36) implies that

\[
|b_{T_0} - b_0| + |\vartheta_{T_0}| \leq \Delta_0^4.
\]

This shows that (BA-G) holds. Finally, it remains to show that we can fulfill (6.32). Denoting \( \phi_{T_0} = e^{i\omega \vartheta_{T_0}} \), observe that

\[
\left\| b_{T_0} - \phi_{T_0}^* g_{b_{T_0}} \right\|_{H^{N_e}(\Sigma_0)} + \left\| k_0 - \phi_{T_0}^* k_{b_{T_0}} \right\|_{H^{N_e-1}(\Sigma_0)}
\]

\[
\leq \left\| b_0 - \bar{b}^0 \right\|_{H^{N_e}(\Sigma_0)} + \left\| k_0 - k\bar{\vartheta} \right\|_{H^{N_e-1}(\Sigma_0)}
\]

\[
+ \left\| g_{\vartheta_0} - \phi_{T_0}^* g_{\vartheta_{T_0}} \right\|_{H^{N_e}(\Sigma_0)} + \left\| k\vartheta_0 - \phi_{T_0}^* k_{\vartheta_{T_0}} \right\|_{H^{N_e-1}(\Sigma_0)}
\]

\[
\leq \Delta_0^4.
\]

Thus (6.32) holds, and so do (BA-D), (BA-E) by the argument made at the beginning of the proof. This concludes the proof of Proposition 6.8. \( \square \)

7 Proof of Theorem 4.2

This section is devoted to the proof of Theorem 4.2. Recall from Proposition 6.8 that the bootstrap assumptions (BA-D), (BA-E), (BA-G), and the orthogonality condition (ORT) hold for some \( T_* > 0 \) sufficiently small. We now consider any bootstrap time \( T_* > 0 \) and show that (BA-D), (BA-E), (BA-G) can be improved, and thus, that the bootstrap time \( T_* \) can be extended slightly. We will then conclude the continuity argument in Section 7.5.

7.1 Improving the bootstrap assumption on decay

We first improve the bootstrap assumption on decay (BA-D). The main goal of this section is the following proposition.
Proposition 7.1. Let $T_*>0$ be the bootstrap time, for which there exist slowly-rotating Kerr-de Sitter black hole parameters $b_{T*}$ and a diffeomorphism $\varphi_{T*} = e^{i\theta_{T*}}$ such that the solution $h_{T*} = h_{T*}(b, \varphi_{T*}, k_0)$ to (6.10) satisfies the bootstrap assumptions given in (BA-E), (BA-D), (BA-G), and (ORT). Then in fact, $h_{T*}$ satisfies the improved estimate

$$\sup_{t_* \geq 0} e^{\delta_* t_*} \|h_{T*}\|_{\mathcal{H}(\Sigma_{t_*})} \leq \varepsilon_0 + \Delta_d^2, \quad j \leq N_d. \tag{7.1}$$

**Proof.** Recall that by assumption, we have $b_{T*}, \varphi_{T*}, h_{T*}, T_*$ such that (ORT) is satisfied. As a result, using Proposition 5.2 we have that for any $k \leq N_d$,

$$e^{\delta_* t_*} \|h_{T*}(t_*, \cdot)\|_{\mathcal{H}(\Sigma_{t_*})} \leq \left\| \left( N_{g_{\varphi_{T*}}} (h_{T*}, \partial h_{T*}, \partial \partial h_{T*}), h_0, h_1 \right) \right\|_{D_{\varepsilon_0}(\mathcal{M})};$$

where $(h_0, h_1) = i_{b_{T*}, \varphi_{T*}}(g_0, k_0)$. Then, using the control that we have for $N_{g_{\varphi_{T*}}}$ in Proposition 6.5, we have that

$$\int_{\mathbb{R}^+} e^{\delta_* t_*} \left\| N_{g_{\varphi_{T*}}} (h_{T*}, \partial h_{T*}, \partial \partial h_{T*}) \right\|_{\mathcal{H}(\Sigma_{t_*})} dt_* \leq \int_{0}^{T_*} e^{\delta_* t_*} \left\| q(h_{T*}, \partial h_{T*}, \partial \partial h_{T*}) \right\|_{\mathcal{H}(\Sigma_{t_*})} dt_* + \int_{T_*}^{T_* + 1} e^{\delta_* t_*} \left\| q(h_{T*}, \partial h_{T*}, \partial \partial h_{T*}) \right\|_{\mathcal{H}(\Sigma_{t_*})} dt_* + \int_{0}^{T_*} \left\| \tilde{q}(h_{T*}, \partial h_{T*}, \partial \partial h_{T*}) \right\|_{\mathcal{H}(\Sigma_{t_*})} dt_* + \int_{T_*}^{T_* + 1} \left\| \tilde{q}(h_{T*}, \partial h_{T*}, \partial \partial h_{T*}) \right\|_{\mathcal{H}(\Sigma_{t_*})} dt_*.
$$

We show how to control the quadratic terms $q, q_{T*}$ first since they are the main difficulty. For $k \leq N_d$, using that $\mathcal{H}^N(\Sigma)$ is an algebra since $N_d \geq 2$, we have that

$$\int_{\mathbb{R}^+} e^{\delta_* t_*} \left\| N_{g_{\varphi_{T*}}} (h_{T*}, \partial h_{T*}, \partial \partial h_{T*}) \right\|_{\mathcal{H}(\Sigma_{t_*})} dt_* \leq \int_{0}^{T_*} e^{\delta_* t_*} \left\| q(h_{T*}, \partial h_{T*}, \partial \partial h_{T*}) \right\|_{\mathcal{H}(\Sigma_{t_*})} dt_* + \int_{T_*}^{T_* + 1} e^{\delta_* t_*} \left\| q(h_{T*}, \partial h_{T*}, \partial \partial h_{T*}) \right\|_{\mathcal{H}(\Sigma_{t_*})} dt_* + \int_{0}^{T_*} \left\| \tilde{q}(h_{T*}, \partial h_{T*}, \partial \partial h_{T*}) \right\|_{\mathcal{H}(\Sigma_{t_*})} dt_* + \int_{T_*}^{T_* + 1} \left\| \tilde{q}(h_{T*}, \partial h_{T*}, \partial \partial h_{T*}) \right\|_{\mathcal{H}(\Sigma_{t_*})} dt_*.
$$

Then, we control $e^{\delta_* t_*} \|h_{T*}\|_{\mathcal{H}^{N_d}(\Sigma_{t_*})}$ using the bootstrap assumption (BA-D). To handle the $\|h_{T*}\|_{\mathcal{H}^{N_d+2}(\Sigma_{t_*})}$ term, we interpolate between the $\mathcal{H}^N(\Sigma)$ and $\mathcal{H}^{N_d}(\Sigma)$ norms, observing that

$$\|h_{T*}\|_{\mathcal{H}^{N_d+2}(\Sigma_{t_*})} \lesssim \|h_{T*}\|_{\mathcal{H}^{N_d}(\Sigma_{t_*})}^{1-\theta} \|h_{T*}\|_{\mathcal{H}^{N}(\Sigma_{t_*})}^\theta, \quad \theta = \frac{2}{N_c - N_d}. \tag{7.3}$$

As a result, we have that

$$\int_{\mathbb{R}^+} \|h_{T*}\|_{\mathcal{H}^{N_d+2}(\Sigma_{t_*})} dt_* \lesssim \int_{\mathbb{R}^+} e^{(\delta_* \theta - (1-\theta)\delta_d) t_*} e^{-\delta_* \theta t_*} \|h_{T*}\|_{\mathcal{H}^{N_d}(\Sigma_{t_*})} \|h_{T*}\|_{\mathcal{H}^N(\Sigma_{t_*})} \|h_{T*}\|_{\mathcal{H}^{N_d}(\Sigma_{t_*})} dt_* \lesssim \Delta_0 \left\| e^{(\delta_* \theta - (1-\theta)\delta_d) t_*} \right\|_{L^2(\mathcal{M})}.
$$

where the last line follows by applying the bootstrap assumptions in (BA-E), and (BA-D). Moreover, recall that we chose $N_c - N_d > 2$, and $2\delta_c < (N_c - N_d - 2)\delta_d$. Thus,

$$\delta_* \theta - (1-\theta)\delta_d < 0, \tag{7.5}$$

32
and plugging (7.4) and (BA-D) in to (7.2) and taking into account (7.5), we have that for \( k \leq N_{d} \),
\[
\int_{0}^{T_{*}} e^{\delta t_{*}} \| q(h_{T_{*}}, \partial h_{T_{*}}, \partial^{2} h_{T_{*}}) \|_{\Pi^{k}(\Sigma_{t})} dt_{*} + \int_{T_{*}}^{T_{*} + 1} e^{\delta t_{*}} \| q(\partial h_{T_{*}}, \partial^{2} h_{T_{*}}) \|_{\Pi^{k}(\Sigma_{t})} dt_{*} \leq \Delta_{0}^{2},
\]
(7.6)

The higher-order nonlinear terms can be handled similarly. Recall from the definition of \( D^{k,\delta_{d},1}(\mathcal{M}) \) in (2.17) that
\[
\begin{align*}
\left\| \left( \Lambda_{g_{T_{*}}}^{T_{*}} (h_{T_{*}}, \partial h_{T_{*}}, \partial^{2} h_{T_{*}}), h_{0}, h_{1} \right) \right\|_{D^{k,\delta_{d},1}(\mathcal{M})} & \leq \left\| \Lambda_{g_{T_{*}}}^{T_{*}} (h_{T_{*}}, \partial h_{T_{*}}, \partial^{2} h_{T_{*}}) \right\|_{H^{k,\delta_{d}(\mathcal{M})}} + \left\| i_{g_{T_{*}}} \phi_{T_{*}} (g_{0}, k_{0}) \right\|_{H^{k+1}(\Sigma_{0})}.
\end{align*}
\]

Using (7.6), we then have that for \( k \leq N_{d} \),
\[
e^{\delta t_{*}} \| h_{T_{*}} \|_{\Pi^{k}(\Sigma_{t})} \leq \varepsilon_{0} + \Delta_{0}^{2}.
\]

improving the bootstrap assumption (BA-D) as desired and concluding the proof of Proposition 7.1.

**Remark 7.2.** The fact that we have to estimate \( \| h_{T_{*}} \|_{\Pi^{N_{d} + 2}(\Sigma)} \) terms in (7.4) reflects two different losses of derivatives. The first is due to derivative loss in the Morawetz estimate in the underlying estimate, which is due to the presence of the trapped set in Kerr-de Sitter. The second derivative loss comes from treating the quasilinear terms as a forcing term on the right-hand side.

### 7.2 Improving the bootstrap assumption on energy

The goal of this section is to improve the high-regularity exponential-growing bootstrap assumption in (BA-E). The main proposition is as follows.

**Proposition 7.3.** Fix some \( T_{*} > 0 \), for which there exist slowly-rotating Kerr-de Sitter black hole parameters \( b_{T_{*}} \) and a diffeomorphism \( \phi_{T_{*}} = e^{i\vartheta_{0} \partial \varrho_{0}} \) such that letting \( h_{T_{*}} = h_{T_{*}}(b, \vartheta, g_{0}, k_{0}) \) be the solution to the Cauchy problem in (6.10), the bootstrap assumptions given in (BA-E), (BA-D), (BA-G), and the orthogonality condition (ORT) are satisfied. Then in fact, \( h_{T_{*}} \) satisfies the improved estimate
\[
\left\| e^{-\delta t_{*}} h_{T_{*}} \right\|_{H^{j}(\mathcal{M})} \leq \varepsilon_{0} + \Delta_{0}^{2}, \quad j \leq N_{e}.
\]
(7.7)

In order to prove (7.7), we need an estimate that does not lose derivatives. To this end, we cannot rely on an estimate on exact Kerr-de Sitter, and must use an estimate on a perturbation of Kerr-de Sitter. This is where Proposition 5.8 comes in.

First, we write down the commuted system of equations.

**Lemma 7.4.** Let \( g = g_{b} + \tilde{g} \), where \( g_{b} \) is a slowly-rotating Kerr-de Sitter metric. Moreover, let \( h \) be a solution to the Cauchy problem given by
\[
\begin{align*}
\mathcal{L}_{g} h &= f, \\
\gamma_{0}(h) &= (h_{0}, h_{1}).
\end{align*}
\]

Then for any multi-index \(|\alpha|=k\),
\[
\mathcal{L}_{g} \mathcal{K}^{\alpha} h = \mathcal{K}^{\alpha} f + \tilde{f},
\]
(7.8)

where
\[
\| \tilde{f} \|_{L^{2}(\Sigma)} \leq \| \tilde{g} \|_{H^{3}(\Sigma)} \| h \|_{\Pi^{k+1}(\Sigma)} + \| h \|_{H^{3}(\Sigma)} \| \tilde{g} \|_{\Pi^{k+1}(\Sigma)}.
\]
(7.9)
Proof. Recall from the definition of $L_g$ in (3.15) that

$$L_g = \Box_g + S_g + V_g,$$

where the coefficients of both $S_g$ and $V_g$ depend on at most one derivative of $g$. Then, commuting $L_g$ and $K^{\alpha}$, we have that

$$[L_g, K^{\alpha}] = \sum_{|\beta| + |\gamma| \leq k + 2, 1 \leq |\beta| \leq k + 1} \tilde{k}_{\beta \gamma} (t, x) \partial^{\beta} \bar{g} \partial^{\gamma},$$

where $\tilde{k}_{\beta \gamma}$ are smooth bounded functions. The conclusion then follows from the fact that $\Sigma$ is 3-dimensional and an application of standard Sobolev embedding.

Since at the level of improving the high-regularity bootstrap, we are forced to work with estimates that do not lose derivatives, it will be convenient to consider a slightly different semi-global extension.

**Lemma 7.5.** Let $h_{T_*}$ be the solution to the Cauchy problem in (6.10), and assume that the bootstrap assumptions given in (BA-E), (BA-D), (BA-G), and the orthogonality condition (ORT) are satisfied.

Then there exists some $\bar{z}$ such that the solution $h_{T_*}$ to the Cauchy problem

$$L_{\chi T_* - 1 g_{T_*} + (1 - \chi T_* - 1) g_{T_*}} h_{T_*} = \chi T_* - 1 Q_{g T_*} (h_{T_*}, \partial h_{T_*}) + \bar{z}$$

$$\left. (h_{T_*}(t, \cdot), \partial_{t} h_{T_*}(t, \cdot)) \right|_{t = 0} = (h_{0}, h_{1}),$$

(7.10)

where $\chi T_* - 1(t_*)$ is a smooth cutoff function as specified in (6.5) and $g_{T_*} := g_{T_*} + h_{T_*}$, satisfies

$$\left\| e^{\delta_{t_*} h_{T_*}} \right\|_{H^{N_c}(\mathcal{M})} \leq \varepsilon_0 + \Delta_0^2,$$

and

$$\sup_{t_* > 0} e^{\delta_{t_*}} \left\| h_{T_*} \right\|_{\mathcal{P}^{n_d}(\Sigma_{t_*})} \leq \varepsilon_0 + \Delta_0^2.$$  (7.12)

Proof. The main idea will be to construct some $\bar{z}$ supported on $[T_*, T_* + 1]$ which depends nonlinearly on $h_{T_*}$ in order to enforce exponential decay of $h_{T_*}$ after time $T_*$. We will then use the bootstrap assumptions to prove the desired control.

We start with the construction of $\bar{z}$. Let $h$ be the solution to

$$L_{\chi T_* - 1 g_{T_*} + (1 - \chi T_* - 1) g_{T_*}} h = \chi T_* - 1 Q_{g_{T_*}} (h, \partial h)$$

$$\left. (h(t, \cdot), \partial_{t} h(t, \cdot)) \right|_{t = 0} = (h_{0}, h_{1}).$$

We define $\mathcal{N}(h, \partial h, \partial^2 h)$ such that

$$\mathcal{N}(h, \partial h, \partial^2 h) = L_{\chi T_* - 1 g_{T_*} + (1 - \chi T_* - 1) g_{T_*}} h - \| g_{T_*} \|_{T^1(\Sigma_{t_*})} h - \chi T_* - 1 Q_{g_{T_*}} (h, \partial h).$$

(7.13)

Observe that by construction,

$$\left\| L_{\chi T_* - 1 g_{T_*} + (1 - \chi T_* - 1) g_{T_*}} h - \| g_{T_*} \|_{T^1(\Sigma_{t_*})} h \right\|_{T^1(\Sigma_{t_*})} \leq \left\| h_{T_*} \right\|_{T^1(\Sigma_{t_*})} \left\| h \right\|_{T^1(\Sigma_{t_*})} + \| h_{T_*} \|_{T^1(\Sigma_{t_*})} \left\| h_{T_*} \right\|_{T^1(\Sigma_{t_*})}.$$

(7.14)

Since $h = h_{T_*}$ up until time $T_* - 1$, we observe that $h_{T_*} - h_{T_*}$ solves the system

$$\mathcal{N}_{g_{T_*}} (h - h_{T_*}) = \mathcal{N}(h, \partial h, \partial^2 h) - \mathcal{N}_{g_{T_*}} (h_{T_*}, \partial h h_{T_*}, \partial^2 h h_{T_*}),$$

$$\left. (h_{T_*}(t, \cdot), \partial_{t} h_{T_*}(t, \cdot)) \right|_{t = T_* - 1} = (0, 0).$$

(34)
Now, let us consider the finite-dimensional family of compactly-supported\(^7\) functions
\[
\mathcal{Z} := \left\{ \mathbb{L}_{g_{\gamma}T_*} (\chi T_*(t_*) \phi) : \phi \in \Lambda^k_{QNM}(\mathbb{L}_{g_0}, \mathbb{H}^+) \right\}.
\]
The map in (5.1) is bijective just by dimension counting. All non-decaying asymptotic behavior can be eliminated by adding some forcing term in \(\mathcal{Z}\) to the right hand side, while at the same time, \(\mathcal{Z}\) is at most as large as the space of non-decaying \(H^k\)-quasinormal mode solutions. Then from Lemma 5.1, we know that for\(^8\) \(k \geq 3\) there exists a continuous linear map
\[
\lambda_{\mathcal{Z}} : D^{k, \alpha, 1}(\mathcal{M}) \to \mathcal{Z},
\]
\[
\lambda_{\mathcal{Z}} (\mathcal{R}(h_0, \partial h_0, \partial^2 h_0)) = \mathcal{N}(h_{T_*}, \partial h_{T_*}, \partial^2 h_{T_*}, 0, 0) \to \tilde{z}
\]
such that \(h_{T_*}\) solving (7.10) is exponentially decaying.

Moreover, since \(\mathcal{Z}\) is a finite-dimensional family of compactly supported functions, we have that for \(4 \leq k \leq 6\),
\[
\left\| e^{-\delta t_{avg} T_*} \right\|_{H^k(\mathcal{M})} \lesssim \left\| \mathcal{R}(h_0, \partial h_0, \partial^2 h_0) \right\|_{H^3, \alpha}(\mathcal{M}) \left\| \mathcal{N}^{T_*(h_{T_*}, \partial h_{T_*}, \partial^2 h_{T_*})} \right\|_{H^3, \alpha}(\mathcal{M}) \lesssim \varepsilon_0 + \Delta_0^2.
\]
To bound the other term, we use the relationship in (7.14), to write that
\[
\int_{T_*-1}^{T_*} \mathcal{R}(h_0, \partial h_0, \partial^2 h_0) \mathcal{R}(\Sigma_{t_*}) dt_* \lesssim \int_{T_*-1}^{T_*} e^{\delta t_*} \left( \left\| h_{T_*} \right\|_{\mathcal{P}^3(\Sigma_{t_*})} \left\| \mathcal{P}^3(\Sigma_{t_*}) \right\| + \left\| h_{T_*} \right\|_{\mathcal{P}^3(\Sigma_{t_*})} \left\| \mathcal{P}^3(\Sigma_{t_*}) \right\| \right) dt_*.
\]
Now using the mean value theorem, there exists some \(t_{avg} \in [T_* - 2, T_* - 1]\) such that
\[
\left\| h_0 \right\|_{\mathcal{P}^3(\Sigma_{t_{avg}})} = \int_{T_* - 2}^{T_* - 1} \left\| h_0 \right\|_{\mathcal{P}^3(\Sigma_{t_*})} dt_*.
\]
Recall from local existence that
\[
\sup_{t_* \in [T_* - 1, T_*]} e^{-\delta t_*} \left\| h_0 \right\|_{\mathcal{P}^3(\Sigma_{t_*})} \lesssim e^{-\delta t_{avg}} \left\| h_0 \right\|_{\mathcal{P}^3(\Sigma_{t_{avg}})} \lesssim \left\| e^{-\delta t_*} h \right\|_{H^b(\mathcal{M})}.
\]
Using the interpolation argument in (7.4), we have that
\[
\int_{T_* - 1}^{T_* + 1} \left\| h_0 \right\|_{\mathcal{P}^3(\Sigma_{t_*})} dt_* \lesssim \sup_{t_* \in [T_* - 1, T_* + 1]} e^{\delta t_*} \left\| h_0 \right\|_{\mathcal{P}^3(\Sigma_{t_*})} \left\| e^{-\delta t_*} h \right\|_{H^b((T_* - 1, T_* + 1) \times \Sigma)}.
\]
Then the fact that \(h_{T_*}(t_{avg}, \cdot) = h(t_{avg}, \cdot)\), and the bootstrap assumptions for \(h_{T_*}\), we have that
\[
\int_{T_* - 1}^{T_*} e^{\delta t_*} \left\| \mathcal{R}(h_0, \partial h_0, \partial^2 h_0) \right\|_{\mathcal{R}(\Sigma_{t_*})} dt_* \lesssim \varepsilon_0 + \Delta_0^2.
\]
\(^1\)In particular, compactly supported in the \(t_*\) direction.
\(^2\)The \(H^3\) regularity required has to do with the threshold regularity level required in the \(H^k\)-quasinormal spectrum in order to pick up exponential decay.
Plugging this back into (7.15), we have that,

$$\| e^{-\delta_t \cdot \bar{z}} \|_{H^0(M)} \lesssim \varepsilon_0 + \Delta_0^2, \tag{7.18}$$

as desired.

We now prove (7.12). To this end, we rewrite (7.10) using (7.13) as

$$\mathbb{L}_{g_{\nu} T_a} \mathbf{h}_T = \mathcal{N}(\mathbf{h}_T, \partial \mathbf{h}_T, \partial \partial \mathbf{h}_T) + \bar{z},$$

where \( \mathcal{N} \) vanishes after time \( T_a \). Then, we can use (5.5) on \( \mathbf{h}_T \) to deduce that

$$\sup_{t_a > 0} e^{\delta_t t_a} \| \mathbf{h}_T \|_{\mathcal{P}^1(\Sigma_{t_a})} \lesssim \int_{\mathbb{R}^+} e^{\delta_t t_a} \| \mathcal{N}(\mathbf{h}_T, \partial \mathbf{h}_T, \partial \partial \mathbf{h}_T) \|_{\mathcal{P}^1(\Sigma_{t_a})} \, dt_a + \int_{\mathbb{R}^+} e^{\delta_t t_a} \| \bar{z} \|_{\mathcal{P}^1(\Sigma_{t_a})} \, dt_a + \varepsilon_0.$$

Until time \( t_a = T_a - 1 \),

$$\mathcal{N}(\mathbf{h}_T, \partial \mathbf{h}_T, \partial \partial \mathbf{h}_T) = \mathcal{N}_{g_{\nu} T_a} (h, \partial h, \partial \partial h), \quad h_{T_a} = \mathbf{h}_T,$$

so from Proposition 7.1, we have that

$$\int_{\mathbb{R}^+} e^{\delta_t t_a} \| \mathcal{N}(\mathbf{h}_T, \partial \mathbf{h}_T, \partial \partial \mathbf{h}_T) \|_{\mathcal{P}^1(\Sigma_{t_a})} \, dt_a = \int_0^{T_a-1} e^{\delta_t t_a} \| \mathcal{N}_{g_{\nu} T_a} (h_{T_a}, \partial h_{T_a}, \partial \partial h_{T_a}) \|_{\mathcal{P}^1(\Sigma_{t_a})} \, dt_a$$

$$+ \int_{T_a-1}^{T_a} e^{\delta_t t_a} \| \mathcal{N}(\mathbf{h}_T, \partial \mathbf{h}_T, \partial \partial \mathbf{h}_T) \|_{\mathcal{P}^1(\Sigma_{t_a})} \, dt_a.$$

The first term we already know from Proposition 7.1 is controlled by \( \varepsilon_0 + \Delta_0^2 \). To control the second term, we use (7.17), observing that \( \mathbf{h}_{T_a} = \mathbf{h} \) for \( t_a \leq T_a \).

We now move onto proving the control in (7.11), we apply Proposition 5.8 commuted equation for \( \mathcal{K}^\alpha \mathbf{h}_T \) with \( |\alpha| = k, k \leq N_e - 1 \) in (7.8) to \( \mathbf{h}_T \) using \( \mathcal{D} = \mathcal{M} \). Since we constructed \( \bar{z} \) such that \( \mathbf{h}_{T_a} \) is exponentially decaying, this yields

$$\| e^{-\delta_t t_a} \mathcal{K}^\alpha \mathbf{h}_T \|_{H^1(M)} \lesssim \varepsilon_0 + \int_0^{T_a} e^{-\delta_t t_a} \| \mathcal{K}^\alpha Q_{g_{\nu} T_a} (\mathbf{h}_T, \partial \mathbf{h}_T) \|_{L^2(\Sigma_{t_a})} \, dt_a$$

$$+ \| e^{-\delta_t t_a} \bar{z} \|_{H^N(M)} + \| e^{-\delta_t t_a} \bar{f} \|_{L^2(M)} + \| e^{-\delta_t t_a} \mathcal{K}^\alpha \mathbf{h}_T \|_{L^2(M)}.$$

We consider each of the terms on the right-hand side in turn.

Recall from the definition of \( Q_{g_{\nu} T_a} \) in (3.14) that \( Q_{g_{\nu} T_a} (\mathbf{h}_T, \partial \mathbf{h}_T) \) is at least quadratic. Thus, there exist nonlinear functions \( q(\cdot, \cdot) \) and \( \tilde{q}(\cdot, \cdot) \) such that

$$Q_{g_{\nu} T_a} (\mathbf{h}_T, \partial \mathbf{h}_T) = q(\mathbf{h}_T, \partial \mathbf{h}_T) + \tilde{q}(\mathbf{h}_T, \partial \mathbf{h}_T),$$

where \( q(\cdot, \cdot) \) is quadratic in its arguments, and \( \tilde{q}(\cdot, \cdot) \) is at least cubic. We first address the quadratic terms as they are the main difficulty. Recalling that \( H^k(\Sigma_{t_a}) \) is a calculus for \( k > \frac{3}{2} \), we have that

$$\| q(\mathbf{h}_T, \partial \mathbf{h}_T) \|_{\mathcal{P}^1(\Sigma_{t_a})} \lesssim \| \mathbf{h}_T \|_{\mathcal{P}^N(\Sigma_{t_a})} \| \mathbf{h}_T \|_{\mathcal{P}^1(\Sigma_{t_a})},$$

where the appearance of \( \mathcal{P}^1(\Sigma_{t_a}) \) comes from the Sobolev inequalities.

Recalling the bootstrap assumptions (BA-E) and (BA-D), and the fact that \( h_{T_a} = \mathbf{h}_T \) up until time \( T_a - 1 \), we then have that

$$\int_0^{T_a-1} e^{-\delta_t t_a} \| q(\mathbf{h}_T, \partial \mathbf{h}_T) \|_{\mathcal{P}^1(\Sigma_{t_a})} \, dt_a \lesssim \Delta_0^2.$$
On the other hand, we can control
\[ \int_{T_{s-1}}^{T_s} e^{-\delta_s t_*} \| q(h_{T_s}, \partial h_{T_s}) \|_{\mathcal{P}^k(\Sigma_{t_*})} \, dt_* \]
\[ \lesssim \sup_{t_* \in [T_{s-1}, T_s]} e^{\delta_s t_*} \| h_{T_s} \|_{\mathcal{P}^k(\Sigma_{t_*})} \int_{T_{s-1}}^{T_s} e^{-\delta_s t_*} e^{-\delta_s t_*} \| h_{T_s} \|_{\mathcal{P}^k(\Sigma_{t_*})} \, dt_* \]
\[ \lesssim \Delta^2_0 \left\| e^{-\delta_s t_*} h_{T_s} \right\|_{H^k(D)}. \] (7.19)

At this point, we again use the mean value theorem and the local existence theory to bound the remaining term by an integral of \( h_{T_s} \). Plugging (7.16) into (7.19) yields that
\[ \int_{T_{s-1}}^{T_s} e^{-\delta_s t_*} \| q(h_{T_s}, \partial h_{T_s}) \|_{\mathcal{P}^k(\Sigma_{t_*})} \, dt_* \lesssim \Delta^2_0. \]

The higher-order nonlinear terms in \( \tilde{q}(h, \partial h) \) can be handled similarly using the calculus inequality and the bootstrap assumptions (BA-E) and (BA-D) to show that
\[ \int_0^{T_s} e^{-\delta_s t_*} \| \tilde{q}(h_{T_s}, \partial h_{T_s}) \|_{\mathcal{P}^k(\Sigma_{t_*})} \, dt_* \lesssim \Delta^2_0, \quad k \leq N_e - 1. \]

Again using the the bootstrap assumptions (BA-E) and (BA-D), the fact that \( h_{T_s} = h_{T_s} \) until time \( T_s - 1 \), and (7.16), we also have from (7.9) that
\[ \left\| e^{-\delta_s t_*} f \right\|_{L^2(M)} \lesssim \Delta^2_0. \]

Finally, to handle the lower-order control, we observe that we can argue iteratively from \( k = 4 \), since we have already proven the stronger lower-order control in (7.12).

We now move onto the proof of the main proposition.

**Proof of Proposition 7.3.** Recall from (6.6) that we can decompose \( h_{T_s} = \chi_{T_s}(t_*) (h - \tilde{h}_{T_s}) + \tilde{h}_{T_s} \) where \( h \) solves the full nonlinear Einstein vacuum equations in harmonic gauge, and \( \tilde{h}_{T_s} \) solves the Cauchy problem in (6.7).

To improve (BA-E), we observe that
\[ \| h_{T_s} \|_{H^k(M)} \lesssim \| h_{T_s} \|_{H^k(M)} + \| h_{T_s} \|_{H^k(D_{t > T_s - 1})}, \]
where
\[ D_{t > T_s - 1} := [T_s - 1, +\infty) \times \Sigma, \]
and \( h_{T_s} \) is as constructed in Lemma 7.5. But from Lemma 7.5 we already have that
\[ \| h_{T_s} \|_{H^k(M)} \lesssim \varepsilon_0 + \Delta^2_0. \] (7.20)

Thus, it suffices to control \( \| h_{T_s} \|_{H^k(D_{t > T_s - 1})} \). To this end, we improve (BA-E) for \( t_* > T_s - 1 \) by controlling \( h \) on \([T_s - 1, T_s + 1]\), and controlling \( \tilde{h}_{T_s} \) on \([T_s, +\infty)\). We begin with estimating \( h \) on \([T_s - 1, T_s + 1]\). Using the mean value theorem, we define \( t'_{\text{avg}} \) such that
\[ h_{T_s} \|_{\mathcal{P}^k(\Sigma_{t'_{\text{avg}}})} = \int_{T_s - 1}^{T_s - 1} h_{T_s} \|_{\mathcal{P}^k(\Sigma_{t_*})} \, dt_* .\]

Then we have from local existence theory that for \( 3 < k \leq N_e \),
\[ \sup_{t_* \in [T_{s-1}, T_{s+1}]} e^{-\delta_s t_*} \| h \|_{\mathcal{P}^k(\Sigma_{t_*})} \lesssim e^{-\delta_s t'_{\text{avg}}} \| h_{T_s} \|_{\mathcal{P}^k(\Sigma_{t'_{\text{avg}}})} \lesssim \varepsilon_0 + \Delta^2_0, \] (7.21)

37
where the last inequality follows from (7.11) using the fact that on \( t_\ast \leq T_\ast - 1 \), \( b_{T_\ast} = h_{T_\ast} \).

Furthermore, using Proposition 5.8, we have that

\[
\sup_{t_\ast > T_\ast} e^{-\delta_\ast (t_\ast - T_\ast)} \left\| \tilde{h}_{T_\ast} \right\|_{H^k(\Sigma_{t_\ast})} \lesssim \left\| h_{T_\ast} \right\|_{H^k(\Sigma_{T_\ast})} + \left\| h_{T_\ast} - h_{T_\ast} \right\|_{H^k(\Sigma_{T_\ast})}.
\]

By the construction of \( t_b, t_\ast \) in Proposition 6.1, we also have that \( \tilde{h}_{T_\ast} - h_{T_\ast} \) vanishes quadratically at \( h = 0 \). Thus, following similar reasoning as above and using the bootstrap assumptions (BA-E) and (BA-D) we have that for \( k \leq N_c, t_\ast \geq T_\ast \),

\[
\sup_{t_\ast > T_\ast} e^{-\delta_\ast t_\ast} \left\| \tilde{h}_{T_\ast} \right\|_{H^k(\Sigma_{t_\ast})} \lesssim e^{-\delta_\ast T_\ast} \left( \left\| h_{T_\ast} \right\|_{H^k(\Sigma_{T_\ast})} + \left\| h_{T_\ast} - h_{T_\ast} \right\|_{H^k(\Sigma_{T_\ast})} \right)
\]

\[
\lesssim e^{-\delta_\ast T_\ast} \left\| h_{T_\ast} \right\|_{H^k(\Sigma_{T_\ast})} + \Delta_0^2 e^{-\delta_\ast T_\ast}
\]

\[
\lesssim \varepsilon_0 + \Delta_0^2,
\]

where the last inequality follows from (7.21).

Combining (7.20), (7.21), and (7.22) concludes the proof of Proposition 7.3.

7.3 Improving bootstrap assumption on gauge

In this section, we improve the bootstrap assumption on the gauge, (BA-G). The main proposition is as follows.

**Proposition 7.6.** Let \( T_\ast > 0 \) be some bootstrap time for which there exist \( b_{T_\ast}, \vartheta_{T_\ast}, h_{T_\ast}, T_\ast \) such that the bootstrap assumptions (BA-G), (ORT), (BA-E), (BA-D) are satisfied. Then in fact \( b_{T_\ast}, \vartheta_{T_\ast} \) satisfy the improved estimate

\[
\left| \vartheta_{T_\ast} \right| + \left| b_{T_\ast} - b_0 \right| \lesssim \varepsilon_0 + \Delta_0^2.
\]

To prove Proposition 7.6, we use the following application of the implicit function theorem.

**Lemma 7.7.** Define

\[
\mathcal{F} : \mathbb{R}^4 \times \mathbb{R}^{N_\varphi} \times H^{N_\varphi}(\Sigma_0) \times H^{N_\varphi - 1}(\Sigma_0) \times H^{N_\delta}(M) \rightarrow \mathbb{R}^{4 + N_\varphi}
\]

\[
\mathcal{F}(b, \vartheta, g, k, f) = \lambda_T (g_0)(f, i_{b, \vartheta}(g, k)),
\]

where \( \phi = e^{i\varphi_{\vartheta}} \), and \( h_{T_\ast} \) is the solution to the Cauchy problem in (6.10). Then, if \( (b_\ast, \vartheta_\ast, g_\ast, k_\ast) \in \mathbb{R}^4 \times \mathbb{R}^{N_\varphi} \times H^{N_\varphi}(\Sigma_0) \times H^{N_\varphi - 1}(\Sigma_0) \) is chosen such that

\[
\mathcal{F}(b_\ast, \vartheta_\ast, g_\ast, k_\ast, 0) = 0
\]

and moreover,

\[
D_b \mathcal{F}(b_\ast, \vartheta_\ast, g_\ast, k_\ast, 0) = 0, \quad D_\vartheta \mathcal{F}(b_\ast, \vartheta_\ast, g_\ast, k_\ast, 0) = 0,
\]

then there exists some \( \delta > 0 \) such that defining

\[
\mathcal{X}_\delta \subset H^{N_\varphi}(\Sigma_0) \times H^{N_\varphi - 1}(\Sigma_0) \times H^{N_\delta}(M)
\]

such that \( (g, k, T_\ast) \in \mathcal{X}_\delta \) if and only if

\[
\left\| g - g_\ast \right\|_{H^{N_\varphi}(\Sigma_0)} + \left\| k - k_\ast \right\|_{H^{N_\varphi - 1}(\Sigma_0)} + \left\| f \right\|_{H^{N_\delta}(M)} < \delta,
\]

there exists a function

\[
\mathcal{G}(g, k, f) : \mathcal{X}_\delta \rightarrow \mathbb{R}^4 \times \mathbb{R}^{N_\varphi},
\]

which is well-defined and \( C^1 \) in its arguments on \( \mathcal{X}_\delta \), and moreover, for \( (g, k, f) \in \mathcal{X}_\delta \),

\[
\mathcal{F}(b, \vartheta, g, k, f) = 0, \quad \iff (b, \vartheta) = \mathcal{G}(g, k, f).
\]
Proof. Recalling the definition of \((g'_0)^\gamma(b')\) in (5.7), observe that we can calculate
\[
D_{b_0}\lambda(g_0, k)(b_0, 0, g_0, k, 0) = \gamma_0 \left( (g'_0)^\gamma(b') \right),
\]
\[
D_{\theta_0}\lambda(g_0, k)(b_0, 0, g_0, k, 0) = \gamma_0 \left( \nabla g_0 \otimes \theta \theta' \right).
\]
As a result,
\[
D_{b_0}\lambda = \lambda [g_0] \left( 0, \gamma_0 \left( (g'_0)^\gamma(b') + \nabla g_0 \otimes \theta \theta' \right) \right),
\]
is an isomorphism by Proposition 5.5. The conclusion then follows from a direct application of the implicit function theorem in Theorem 2.20.

We are now ready to move onto the proof of Proposition 7.6.

Proof of Proposition 7.6. We will apply Lemma 7.7 with the map
\[
\mathcal{F}_{T} : \mathbb{R}^4 \times \mathbb{R}^{N_{\omega}} \times H^{N_{\omega}}(\Sigma_0) \times H^{N_{\omega}-1}(\Sigma_0) \times H^{N_{\omega}, \delta}(\mathcal{M}) \rightarrow \mathbb{R}^{4+N_{\omega}},
\]
\[
\mathcal{F}_{T}(b, \theta, g, k, f) = \lambda T[g_0](f, i_b, \phi(g, k)),
\]
where \(\phi = e^{i\omega \theta}\), around the choice
\[
(b, \theta, g, k, f) = (b_0, 0, g_0, k_0, 0).
\]
Observe that
\[
D_{b_0}f |_{(b_0, 0, g_0, k_0, 0)} = 0, \quad D_{\theta_0}f |_{(b_0, 0, g_0, k_0, 0)} = 0.
\]
Then, we can apply Lemma 7.7 to see that there exists some \(C^1\) function
\[
\mathcal{G}_{T} : H^{N_{\omega}}(\Sigma_0) \times H^{N_{\omega}-1}(\Sigma_0) \times H^{N_{\omega}, \delta}(\mathcal{M}) \rightarrow \mathbb{R}^4 \times \mathbb{R}^{N_{\omega}},
\]
and some \(\delta_{T_0} > 0\) such that for
\[
\|g - g_0\|_{H^{N_{\omega}}(\Sigma_0)} + \|k - k_0\|_{H^{N_{\omega}-1}(\Sigma_0)} + \|f\|_{H^{N_{\omega}, \delta}(\mathcal{M})} < \delta_{T_0},
\]
we have that
\[
\mathcal{F}_{T}(b, \theta, g, k, f) = 0 \iff (b, \theta) = \mathcal{G}_{T}(g, k, f).
\]
Then, we can pick \((g, k, f) = (g_0, k_0, N_{T_0}^{T_0}(h_{T_0}, \partial h_{T_0}, \partial \partial h_{T_0}))\). Recall that we have already shown in (7.6) that
\[
\left\| N_{T_0}^{T_0}(h_{T_0}, \partial h_{T_0}, \partial \partial h_{T_0}) \right\|_{H^{N_{\omega}, \delta}(\mathcal{M})} \lesssim \Delta^2_0.
\]
Thus, taking \(\varepsilon_0, \Delta_0\) sufficiently small, we can ensure that (7.28) is satisfied for \((g, k, f) = (g_0, k_0, N_{T_0}^{T_0}(h_{T_0}, \partial h_{T_0}, \partial \partial h_{T_0}))\).

Since we already know that
\[
\mathcal{F}_{T_0}\left(b_{T_0}, \theta_{T_0}, g_0, k_0, N_{T_0}^{T_0}(h_{T_0}, \partial h_{T_0}, \partial \partial h_{T_0})\right) = 0,
\]
we have in fact that
\[
(b_{T_0}, \theta_{T_0}) = \mathcal{G}_{T_0}(g_0, k_0, N_{T_0}^{T_0}(h_{T_0}, \partial h_{T_0}, \partial \partial h_{T_0})\).
\]
We then have the following estimate
\[
|b_{T_0} - b_0| + |\theta_{T_0}|
\]
\[
\lesssim \left\|g_0 - g_0\right\|_{H^{N_{\omega}}(\Sigma_0)} + \|k_0 - k_0\|_{H^{N_{\omega}-1}(\Sigma_0)} + \left\|N_{T_0}^{T_0}(h_{T_0}, \partial h_{T_0}, \partial \partial h_{T_0})\right\|_{H^{N_{\omega}, \delta}(\mathcal{M})}
\]
\[
\lesssim \varepsilon_0 + \Delta^2_0,
\]
where the first inequality follows from the fact that \(\mathcal{G}_{T_0}\) is \(C^1\) in its arguments and the second follows from the smallness of the initial data in (4.2) and from the improvements to the bootstrap assumptions in (7.1) and (7.7).
7.4 Extending the bootstrap time

We will also need to extend the bootstrap gauge. We formalize this in the following proposition.

**Proposition 7.8.** Assume that the orthogonality condition in (ORT), bootstrap assumptions (BA-E), (BA-D) and (BA-G) all hold with bootstrap time $T_\ast > 0$. Then there exists some $\delta_{ext} > 0$ sufficiently small, slowly rotating Kerr-de Sitter black-hole parameters $b_{T_\ast + \delta_{ext}}$ and a diffeomorphism $\phi_{T_\ast + \delta_{ext}} = e^{i\Theta_{T_\ast + \delta_{ext}}}$ such that

$$h_{T_\ast + \delta_{ext}} := h_{T_\ast + \delta_{ext}}(b_{T_\ast + \delta_{ext}}, \vartheta_{T_\ast + \delta_{ext}}, g_{0}, k_0)$$

satisfies the orthogonality condition

$$\lambda_T \left[ g_{b_{T_\ast + \delta_{ext}}}, \left( N^{T_{T_\ast + \delta_{ext}}} h_{T_\ast + \delta_{ext}}, i_{b_{T_\ast + \delta_{ext}}}(g_{0}, k_0) \right) \right] = 0. \quad (7.29)$$

Moreover, $h_{T_\ast + \delta_{ext}}$ satisfies the bootstrap assumptions (BA-E) and (BA-D), and $\vartheta_{T_\ast + \delta_{ext}}$ and $b_{T_\ast + \delta_{ext}}$ satisfy (BA-G).

We first prove the following auxiliary lemma.

**Lemma 7.9.** Under the same assumptions as in Proposition 7.8, there exists $\delta_\ast > 0$ sufficiently small, and a neighborhood $(b_{T_\ast}, \vartheta_{T_\ast}) \in B_{ext} \subset \mathbb{R}^4 \times \mathbb{R}^{N_\ast}$ such that for any $(b, \vartheta) \in B_{ext}$, and $\delta < \delta_\ast$ the solution

$$h_{T_\ast + \delta}(b, \vartheta, g_{0}, k_0) := \chi_{T_\ast + \delta} h(b, \vartheta, g_{0}, k_0) + (1 - \chi_{T_\ast + \delta}) \bar{h}_{T_\ast + \delta}(b, \vartheta, g_{0}, k_0)$$

satisfies the following estimates

$$\sup_{t_\ast < T_\ast + \delta + 1} e^{\delta_{ext} t_\ast} \left\| h_{T_\ast + \delta}(b, \vartheta, g_{0}, k_0) \right\|_{\mathcal{H}^T(\Sigma_{t_\ast})} \leq \Delta_0, \quad k \leq N_d,$$

$$\sup_{t_\ast < T_\ast + \delta + 1} e^{-\delta_{ext} t_\ast} \left\| h_{T_\ast + \delta}(b, \vartheta, g_{0}, k_0) \right\|_{\mathcal{H}^T(\Sigma_{t_\ast})} \leq \Delta_0, \quad k \leq N_c, \quad (7.30)$$

$$|b - b^0| + |\vartheta| \leq \Delta_0.$$

If in addition, for some $(b, \vartheta) \in B_{ext}, \delta < \delta_\ast$, we have that

$$\lambda_T [g_b] \left( N^{T_{T_\ast + \delta}} \left( h_{T_\ast + \delta}(b, \vartheta, g_{0}, k_0) \right), i_{b, \vartheta}(g_{0}, k_0) \right), \quad (7.31)$$

then in fact for $\Delta_0$ sufficiently small, we have the global estimates

$$\sup_{t_\ast > 0} e^{\delta_d t_\ast} \left\| h_{T_\ast + \delta}(b, \vartheta, g_{0}, k_0) \right\|_{\mathcal{H}^T(\Sigma_{t_\ast})} \leq \Delta_0, \quad k \leq N_d,$$

$$\sup_{t_\ast > 0} e^{-\delta_d t_\ast} \left\| h_{T_\ast + \delta}(b, \vartheta, g_{0}, k_0) \right\|_{\mathcal{H}^T(\Sigma_{t_\ast})} \leq \Delta_0, \quad k \leq N_c. \quad (7.32)$$

**Proof.** We consider the neighborhood

$$B_{ext} := \{(b, \vartheta) : |b - b_{T_\ast}| + |\vartheta - \vartheta_{T_\ast}| < \delta \},$$

and show that for a sufficiently small choice of $\delta_\ast$, $B_{ext}$ satisfies the conditions in the lemma.

Observe that on $[0, T_\ast]$,

$$\phi_{T_\ast}^b (g_{b_{T_\ast}} + h_{T_\ast}) = \phi_{T_\ast + \delta_{ext}}^b (g_{b_{T_\ast + \delta_{ext}}} + h_{T_\ast + \delta_{ext}}),$$

40
so for some $M > 0$ sufficiently large,

$$
\sup_{t_* \in T_*} e^{\delta t_*} \| h_{T_* + \delta_*} - h_{T_*} \|_{\mathcal{P}_{\delta}} (\Sigma_{T_*}) \lesssim e^{MT_* \delta_*}, \quad k \leq N_d,
\quad \sup_{t_* \in T_*} e^{-\delta t_*} \| h_{T_* + \delta_*} - h_{T_*} \|_{\mathcal{P}_{\delta}} (\Sigma_{T_*}) \lesssim e^{MT_* \delta_*}, \quad k \leq N_c.
$$

(7.33)

On the interval $[T_*, T_* + 1 + \delta_*]$, for $\delta_*$ sufficiently small, a local existence result and (7.33) yield that

$$
\sup_{t_* \in [T_* + 1 + \delta_*]} e^{\delta t_*} \| h_{T_* + \delta_*} - h_{T_*} \|_{\mathcal{P}_{\delta}} (\Sigma_{T_*}) \lesssim e^{\Delta_*} + e^{MT_* \delta_*}, \quad k \leq N_d,
\quad \sup_{t_* \in [T_* + 1 + \delta_*]} e^{-\delta t_*} \| h_{T_* + \delta_*} - h_{T_*} \|_{\mathcal{P}_{\delta}} (\Sigma_{T_*}) \lesssim e^{\Delta_*} + e^{MT_* \delta_*}, \quad k \leq N_c.
$$

(7.34)

Combining (7.33) and (7.34), and the improvements to the bootstrap assumptions we have already proven in Propositions 7.1, 7.3, and 7.6 respectively, we have that

$$
\sup_{t_* < T_* + 1 + \delta_*} e^{\delta t_*} \| h_{T_* + \delta_*} - h_{T_*} \|_{\mathcal{P}_{\delta}} (\Sigma_{T_*}) \lesssim e_0 + \Delta_* + e^{MT_* \delta_*}, \quad k \leq N_d,
\quad \sup_{t_* < T_* + 1 + \delta_*} e^{-\delta t_*} \| h_{T_* + \delta_*} - h_{T_*} \|_{\mathcal{P}_{\delta}} (\Sigma_{T_*}) \lesssim e_0 + \Delta_* + e^{MT_* \delta_*}, \quad k \leq N_c,
$$

$$
|b - b^0| + |\vartheta| \lesssim e_0 + \Delta_* + \delta_*.
$$

The estimates in (7.30) the follow by taking $e_0$, $\Delta_0$, and $\delta_*$ sufficiently small.

If in addition, the orthogonality condition in (7.31) is satisfied, then the global estimates in (7.32) follow directly from an application of Propositions 5.2 and 5.8.

We are now ready to prove Proposition 7.8.

**Proof of Proposition 7.8.** The main goal will be to apply Lemma 6.10 to bifurcate around the point $(b_{T_*}, \vartheta_{T_*}, q_{0T_*}, k_0, T_*)$ in order to find an appropriate $(b_{T_* + \delta_{ext}}, \vartheta_{T_* + \delta_{ext}}, q_{0T_* + \delta_{ext}}, k_0, T_*)$ satisfying the conditions in the proposition.

First, observe that using the interpolation inequality in (7.3), we know that (7.32) follow directly from an application of Propositions 5.2 and 5.8.

Now, to apply Lemma 6.10, we define

$$
\mathcal{F}_{ext} : \mathbb{R}^4 \times \mathbb{R}^{N_0} \times H^{N_0 + 1} (\Sigma_{T_*}) \times \mathbb{R} \to \mathbb{R}^4 \times \mathbb{R}^{N_0},
(b, \vartheta, g, k, \delta) \mapsto \lambda_T [g_b] \left( \mathcal{N}^{T_* + \delta}_{g_b} (h(b, \vartheta, g, k, \delta)(t_* + T_* + \delta)) \right).
$$

Observe from Proposition 5.6 that

$$
\lambda_T [g_b] \left( \mathcal{N}^{T_* + \delta}_{g_b} (h(b, \vartheta, g, k, \delta)(t_* + T_*)) \right) = \lambda_T [g_b] \left( \mathcal{N}^{T_* + \delta}_{g_b} (h(b, \vartheta, g, k, \delta)(t_*)) \right).
$$

The main step will be to show that $D_{b, \vartheta} \mathcal{F}_{ext}(b_{T_*}, \vartheta_{T_*}, q_{0T_*}, k_{T_*}, 0)$ is invertible, where $(g_{T_*}, k_{T_*})$ denotes the induced metric and second fundamental form respectively by $g = \phi_{T_*}^* (g_{T_*} + h_{T_*})$ on $\phi^* (\Sigma_{T_*}).$
To this end, observe that
\[
D_b \partial_i b, i(b, g, k) (b_T, \partial_T, g_T, k_T) (b', \vartheta') = D_b \partial_i \gamma_T (h_T(b, \partial, g, k)) (b_T, \partial_T, g_T, k_T) (b', \vartheta') = \gamma_T (g'_{bT})^Y (b') + \nabla g_{bT} \otimes i_{\vartheta} \vartheta' + \gamma_T (h_T b, \partial_T, g_T, k_T) (b') + \nabla h_T \otimes i_{\vartheta} \vartheta').
\] (7.36)

From Propositions 5.5 and 5.6, \( \lambda_T [g_{bT}] (0, \gamma_T (g'_{bT})^Y (b') + \nabla g_{bT} \otimes i_{\vartheta} \vartheta') \) is an isomorphism of \( \mathbb{R}^4 \times \mathbb{R}^{N_0} \) to itself. Furthermore, since \( h_T \) satisfies the improved bootstrap estimates as proven in Propositions 7.1 and 7.3,
\[
\left\| \gamma_T (h_T b, \partial_T, g_T, k_T) (b') + \nabla h_T \otimes i_{\vartheta} \vartheta') \right\|_{H^N (\Sigma)} \leq \varepsilon_0 + \Delta_0^2.
\] (7.37)

Likewise, we have using local existence that
\[
\sup_{T \in \Sigma} \left\| D_b \partial_i \gamma_T (h_T(b, \partial, g, k)) (b_T, \partial_T, g_T, k_T, 0) \right\|_{H^N (\Sigma)} \leq \sup_{T \in \Sigma} \left\| h_T(b_T, \partial_T, g_T, k_T, 0) \right\|_{H^N (\Sigma)} \left\| D_b \partial_i h_T (b_T, \partial_T, g_T, k_T, 0) \right\|_{H^N (\Sigma)} \leq \Delta_0 \left\| D_b \partial_i h_T (b_T, \partial_T, g_T, k_T, 0) \right\|_{H^N (\Sigma)}.
\] (7.38)

where the last inequality follows from the observation made in (7.35).

Moreover, from Lemma 5.7, we have that
\[
\left\| D_b \partial \lambda_T [g_{bT}] (b_T, \partial_T, g_T, k_T) \right\| \leq \Delta_0^2.
\] (7.39)

Combining (7.36), (7.38), and (7.39) and taking \( \Delta_0 \) sufficiently small then yields the invertibility of \( D_b \partial \mathcal{F}_{ext}(b_T, \partial_T, g_T, k_T, 0) \), as desired.

We can now apply Lemma 6.10, bifurcating around \( (b_T, \partial_T, g_T, k_T, 0) \) to see that there exists a neighborhood \( \mathcal{X}_{ext} \subset H^N (\Sigma) \times H^N (\Sigma) \times [0, 1] \) on which there exists a mapping
\[
\mathcal{G}_{ext} : \mathcal{X}_{ext} \to \mathcal{B}_{ext}
\]
such that
\[
\mathcal{F}_{ext}(b, \partial, g, k, \delta) = 0 \iff (b, \partial) = \mathcal{G}(g, k, \delta).
\]

Thus, for \( \delta_{ext} \) sufficiently small, it is clear that \( (g_{T*}, k_{T*}, \delta_{ext}) \in \mathcal{X}_{ext} \), and thus, there exists some
\[
(b_{T*} + \delta_{ext}, \partial_{T*} + \delta_{ext}) := \mathcal{G}(g_{T*}, k_{T*}, \delta_{ext})
\]
such that
\[
\mathcal{F}_{ext}(b_{T*} + \delta_{ext}, \partial_{T*} + \delta_{ext}, g_{T*}, k_{T*}, \delta_{ext}) = 0,
\]
and moreover, \( (b_{T*} + \delta_{ext}, \partial_{T*} + \delta_{ext}) \in \mathcal{B}_{ext} \). Then, using Lemma 7.9 concludes the proof of Proposition 7.8. □
7.5 Closing the proof of Theorem 4.2

We are now ready to prove Theorem 4.2 via a standard continuity argument. Recall from Proposition 6.8 that there exists some $T_0 > 0$ sufficiently small so that (BA-G), (BA-D), and (BA-E) are satisfied. We now let $T^*$ be the supremum of all such $T_0$, and assume for the sake of contradiction that $T^* < +\infty$. By the continuity of the flow, there exists $(b_{T^*}, \vartheta_{T^*})$ such that (BA-G), (BA-D), and (BA-E) hold at $T = T^*$.

But then, Proposition 7.8 states that there exist choice of $(b_{T^* + \delta_{ext}}, \vartheta_{T^* + \delta_{ext}})$ for which in fact (BA-G), (BA-D), and (BA-E) continue to hold. Thus, we have a contradiction and in fact, $T^* = +\infty$. We thus have a family of black hole parameters and diffeomorphism parameterizations $(b_{t_k}, \vartheta_{t_k}, h_{t_k})$, parametrized by $t_k \to +\infty$ such that $h_{t_k} = h_{t_k}(b_{t_k}, \vartheta_{t_k}, g_0, k_0)$ solves (6.10) with $(b_{t_k}, \vartheta_{t_k})$ satisfying (BA-G), (BA-D), and (BA-E). In particular, from (BA-G), $(b_{t_k}, \vartheta_{t_k})$ is a bounded, finite-dimensional family. Thus, it must possess a convergent subsequence

$$\lim_{k \to +\infty} (b_{t_k}, \vartheta_{t_k}) \to (b_\infty, \vartheta_\infty).$$

Furthermore, $(b_\infty, \vartheta_\infty)$ is the unique limit since otherwise, distinct Kerr-de Sitter metrics would be diffeomorphic to each other.

Thus, the solution $g$ to EVE is global and writing $g = \phi^*_\infty(g_{b_\infty} + h)$, where $\phi_\infty = e^{i\omega \vartheta_\infty}$, we have that

$$\sup_{t_k > 0} e^{-\delta t_k} \|h\|_{H^1(\Sigma_{t_k})} \lesssim \varepsilon_0.$$

This concludes the proof of Theorem 4.2.

8 Energy estimates on perturbations of Kerr-de Sitter

This section contains all the necessary analysis on perturbations of Kerr-de Sitter for the rest of the paper.

8.1 Proof of Proposition 5.8

We prove Proposition 5.8 by proving the following, stronger perturbation of the high-frequency Morawetz estimate in Proposition 8.2 of [8].

**Proposition 8.1.** Let $g = g_0 + \tilde{g}$, where $g_0$ is a fixed, slowly-rotating Kerr-de Sitter metric, and

$$\sup_{t > 0} e^{\delta t} \|\tilde{g}\|_{H^1(\Sigma_{t_k})} \lesssim \Delta_0. \quad (8.1)$$

Moreover, consider some solution $h$ to the Cauchy problem given by

$$L_y h = f = \mathcal{H}_0(h) = (h_0, h_1),$$

where $L_y$ is defined as in (3.15), and the initial data $(h_0, h_1) \in H^4(\Sigma_0)$. Also, let $D = [0, T_0] \times \Sigma$, $T_0 > 0$. Then for any fixed $\delta_0 > 0$, there exists a choice of $\Delta_0$ sufficiently small such that $h$ satisfies the estimate

$$\left\|e^{-\delta_0 t} h\right\|_{H^1(D)} \lesssim \|(h_0, h_1)\|_{H^1(\Sigma_0)} + \left\|e^{-\delta_0 t} f\right\|_{H^1(\Sigma_{T_0})} + \left\|e^{-\delta_0 t} f\right\|_{L^2(D)} + \left\|e^{-\delta_0 t} h\right\|_{L^2(D)}. \quad (8.2)$$

The proof of Proposition 8.1 is postponed to Section 8.4.
Remark 8.2. When compared to the Morawetz estimate in Theorem 8.2 in [8] in the linear theory, the main estimate (8.2) is weaker in the sense that it is consistent with exponential growth (albeit slow exponential growth) instead of exponential decay, but is stronger in the sense that it only involves non-degenerate norms, and moreover, does not lose any derivatives.

Remark 8.3. The appearance of \( H^3_p \Sigma^q \) in (8.1) is chosen so that we have the control

\[
\sup_{t \geq 0} \left( \| \tilde{g} \|_{W^{1,\infty}(\Sigma_t)} + \| T \tilde{g} \|_{L^\infty(\Sigma_t)} \right) \lesssim \Delta_0
\]

by Sobolev inequalities. Observe that again, we only need \( H^s_p \Sigma^q \), but since we choose \( N_d = 3 \) anyways, we do not make attempts to optimize this further.

Given Proposition 8.1, Proposition 5.8 follows directly.

Proof of Proposition 5.8. The estimate in (5.10) follows immediately taking \( \delta_0 = \delta_c \).

8.2 Basic pseudo-differential theory

Before we move onto the remainder of the section, we first introduce the basics of the pseudo-differential analysis we will be using. For an in-depth reference, we refer the reader to Chapter 1 of [1], Chapter 18 of [11], or Chapters 1-4 of [19].

Definition 8.4. For \( m \in \mathbb{R} \), we define \( \Psi^m \) to be the class of order-\( m \) symbols on \( \mathbb{R}^d \), consisting of \( C^\infty \) functions \( a(x, \zeta) \) such that

\[
|D_\zeta^{\alpha}D_x^{\beta}a(x, \zeta)| \leq C_{\alpha \beta} \langle \zeta \rangle^{m-|\alpha|}
\]

for all multi-indexes \( \alpha \), where \( \langle \zeta \rangle = (1 + |\zeta|^2)^{\frac{1}{2}} \). We also define the symbol class

\[
\Psi^{-\infty} := \bigcap_{m \in \mathbb{Z}} \Psi^m.
\]

To each symbol is its associated pseudo-differential operator acting on Schwartz functions \( \phi \),

\[
a(x, D)(\phi) = Op(a) \phi(x) := (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix \cdot \zeta} a(x, \zeta) \hat{\phi}(\zeta) \, d\zeta,
\]

where \( \hat{\phi} \) is the Fourier transform of \( \phi \).

These pseudo-differential operators have well-behaved mapping properties based on their symbol.

Proposition 8.5. If \( a \in \Psi^m(\mathbb{R}^d) \), then the operator \( a(x, D) \) is a well-defined mapping from \( H^s(\mathbb{R}^d) \) to \( H^{s-m}(\mathbb{R}^d) \) for any \( s \in \mathbb{R} \).

Definition 8.6. We call a symbol \( a(x, \zeta) \in \Psi^m(\mathbb{R}^d) \) and its corresponding operator \( a(x, D) \) an elliptic symbol and an elliptic operator respectively if there exists some \( c, C \) such that for \( \langle \zeta \rangle > C \),

\[
|a(x, \zeta)| \geq c \langle \zeta \rangle^m.
\]

Proposition 8.7. If \( a(x, \zeta) \in \Psi^m \) is elliptic, then it has a parametrix \( b(x, \zeta) \in \Psi^{-m} \) such that

\[
a(x, D)b(x, D) - Id \in Op(\Psi^{-\infty}), \quad b(x, D)a(x, D) - Id \in Op(\Psi^{-\infty}).
\]

A crucial property of pseudo-differential operators is the following commutation property.
Proposition 8.8 (Coifman-Meyer commutator estimate, see Proposition 4.1A in [19]). For \( f \in C^\infty, P \in OP \Psi^1 \),
\[
\| [P,f] u \|_{L^2} \leq C \| f \|_{C^1} \| u \|_{L^2}.
\]

Finally, we can also define pseudo-differential operators on a manifold.

Proposition 8.9 (See Proposition 7.1 in Chapter 1 of [1]). Let \( \phi : \Omega \to \Omega' \) be a smooth diffeomorphism between two open subsets of \( \mathbb{R}^d \). Moreover, let \( a \in \Psi^m \) be an order \( m \) symbol such that the operator \( a(x,D) \) has kernel with compact support in \( \Omega \times \Omega \).

Then the following hold.

1. The function \( a'(y, \zeta) \) defined by
\[
a'(\phi(x), \zeta) = e^{-i\phi(x)\zeta} a(x,D) e^{i\phi(x)\zeta}, \quad a' = 0 \text{ for } y \notin \Omega',
\]
is also a member of \( \Psi^m \).

2. The kernel of \( a'(x,D) \) has compact support in \( \Omega' \times \Omega' \).

3. For \( u \in S'(\mathbb{R}^d) \),
\[
a(x,D)(u \circ \phi) = (a'(x,D)u) \circ \phi.
\]

4. If \( a \) has the form
\[
a = a_m \mod \Psi^{m-1}(\mathbb{R}^d),
\]
where \( a_m \) is a homogeneous symbol of order \( m \), then the same is true for \( a' \). That is, there is a homogeneous symbol \( a'_m \) of order \( m \) such that
\[
a' = a'_m \mod \Psi^{m-1}(\mathbb{R}^d),
\]
and in fact
\[
a'_m(\phi(x), \zeta) = a_m(x, \chi'(x)\zeta).
\]

Definition 8.10 (See Definition 7.1 in Chapter 1 of [1]). An operator \( A : C^\infty_0(M) \to C^\infty_0(M) \) is a pseudo-differential operator of order \( m \) is for any coordinate system \( \kappa : V \to \tilde{V} \subset \mathbb{R}^n \), the transported operator
\[
\tilde{A} : C^\infty_0(\tilde{V}) \to C^\infty_0(\tilde{V}),
\]
\[
u \mapsto A(\nu \circ \kappa) \circ \kappa^{-1}
\]
is a pseudo-differential of operator of order \( m \) in \( \tilde{V} \). In other words, \( A \) is a pseudo-differential operator of order \( m \) if for all \( \phi, \psi \in C^\infty_0(\tilde{V}) \), \( \hat{\phi} \hat{\tilde{A}} \hat{\psi} \in OP(\Psi^m) \).

8.3 Perturbations of linear estimates

In this section, we prove a series of estimates for \( L_g \) where \( g = g_b + \tilde{g} \), \( g_b \) is a slowly-rotating Kerr-de Sitter metric, and \( \tilde{g} \) is an exponentially decaying perturbation.

We first introduce a series of basic results that stem from the fact that \( \tilde{g} \) is an exponentially decaying perturbation.

Lemma 8.11. Let \( g = g_b + \tilde{g} \), where \( g_b \) is a fixed, slowly-rotating Kerr-de Sitter metric, and \( \tilde{g} \) satisfies the bootstrap assumptions (BA-E), (BA-D). Then for \( X \) a smooth vectorfield with deformation tensor \( ^{(X)}\pi \),
\[
^{(X)}\pi_g = ^{(X)}\pi_{g_b} + ^{(X)}\pi_{\tilde{g}},
\]
where
\[
\left| ^{(X)}\pi_{\tilde{g}} \right| \leq \Delta_0 e^{-\delta_* t_*}.
\]

45
Proof. The inequality in (8.3) follows immediately from observing that in local coordinates,
\[
(\langle X \rangle_t g)_{\mu\nu} = \frac{1}{2} \left( X(g_{\mu\nu}) + g_{\gamma\nu} \frac{\partial X^\gamma}{\partial x^\mu} + g_{\gamma\mu} \frac{\partial X^\gamma}{\partial x^\nu} \right)
\]
and that \( g = g_b + \tilde{g} \).

As a result, we immediately have the following corollary.

**Corollary 8.12.** For \( g \) as in Lemma 8.11, if \( X \) is a smooth vectorfield, and \( q \) is a smooth function, then
\[
K_g^{X,q,m}[h] = K_{g_b}^{X,q,m}[h] + \tilde{K}^{X,q,m}[h],
\]
where
\[
\tilde{K}^{X,q,m}[h] \leq \Delta_0 e^{-\delta_0 t} (|\nabla h|^2 + |h|^2).
\]
Moreover,
\[
(S_g h \cdot X \tilde{h}) = (S_{g_b} h \cdot X \tilde{h}) + \tilde{S}[X][h],
\]
where
\[
\tilde{S}[X][h] \leq \Delta_0 e^{-\delta_0 t} |\nabla h|^2.
\]

**Proof.** The conclusion in (8.4) follows directly from Lemma 8.11 and the definition of \( K_g^{X,q,m}[h] \) in (2.12).

The second conclusion follows from the observation that \( S_g \) is a linear combination of terms of the form
\[
S(g, \partial g) \partial \mu,
\]
where \( S(g, \partial g) \) is linear in its arguments. The conclusion then follows directly from the Sobolev inequality.

The redshift vector \( \mathbf{N} \) of Proposition 2.12 is uniformly timelike on \( \mathcal{M} \) with respect to slowly-rotating Kerr-de Sitter metrics \( g_b \). Since we are only considering metrics \( g = g_b + \tilde{g} \) which are small (and exponentially decaying) perturbation of \( g_b \), we must have that \( \mathbf{N} \) remains globally timelike on \( g \) as well.

**Lemma 8.13.** Fix \( g_b \) a slowly-rotating Kerr-de Sitter metric, and let \( \mathbf{N} \) be as constructed in Proposition 2.12. Then for \( \Delta_0 \) sufficiently small, if \( \tilde{g} \in S^2 T^* \mathcal{M} \) is a metric perturbation such that
\[
\sup_{t_\# > 0} e^{\delta_0 t_\#} \| \tilde{g} \|_{L^2(\Sigma_{t_\#})} \leq \Delta_0,
\]
then \( \mathbf{N} \) is uniformly timelike with respect to the perturbed Kerr-de Sitter metric \( g = g_b + \tilde{g} \),
\[
g(\mathbf{N}, \mathbf{N}) < 0.
\]

**Proof.** The result of the lemma then follows from (2.14) by choosing sufficiently small \( \Delta_0 \).

The globally timelike nature of \( \mathbf{N} \) immediately gives the following Gronwall-type estimate.

**Lemma 8.14.** Let \( g = g_b + \tilde{g} \), where \( g_b \) is a sufficiently slowly-rotating Kerr-de Sitter metric, and \( \sup_{t_\#} e^{\delta_0 t_\#} \| \tilde{g} \|_{L^2(\Sigma_{t_\#})} \leq \Delta_0 \), where \( \Delta_0 \) is sufficiently small. Furthermore, let \( h \) be a solution to the Cauchy problem
\[
\mathbf{L}_g h = f,
\]
\[
\gamma^0(h) = (h_0, h_1),
\]
with \( \mathbf{L}_g \) as in (3.15). Then, there exists some \( \mathbf{M} \) such that the solution \( h \) to (8.5) satisfies the following energy estimate
\[
\sup_{t_\# \leq T_\#} e^{-\mathbf{M} t_\#} \| h \|_{\mathbf{L}^2(\Sigma_{t_\#})} \leq \| (h_0, h_1) \|_{\mathbf{H}^1(\Sigma_0)} + \int_0^{T_\#} e^{-\mathbf{M} t_\#} \| f \|_{L^2(\Sigma_{t_\#})} dt_\#.
\]

46
Thus, using the divergence theorem, there exists a constant $\Delta_0$ such that
\[
E(t_*)[h] \leq \| \nabla h \|_{L^2(D)}^2 \leq E(t_*)[h].
\] (8.7)

Now we apply the divergence theorem in Proposition 2.11 with $X = J^N_{\delta}[h]$ on the region
\[D := [0, T_*) \times \Sigma,\]
where we recall the definition of $\Sigma$ in (2.4).

Due to the timelike future-directed nature of $N$,
\[
\int_{\Sigma(t_*) \cap \mathcal{H}^-} J^N_{\delta}[h] \cdot n_{\mathcal{H}^-} \geq 0, \quad \int_{\Sigma(t_*) \cap \mathcal{H}^+} J^N_{\delta}[h] \cdot n_{\mathcal{H}^+} \geq 0.
\]
To estimate the divergence term, we use (2.13) with $X = N$, $q = 0$, i.e.
\[
\nabla_{\delta} \cdot J^N_{\delta}[h] = \Re \left[ N \nabla \cdot \Box_{\delta} h \right] + K^N_{\delta}[h].
\]
Using Corollary 8.12, we can write
\[
K^N_{\delta}[h] = K^N_{\delta}[h] + \tilde{K}^N_{\delta}[h],
\]
where
\[
\left| \tilde{K}^N_{\delta}[h] \right| \leq \Delta_0 e^{-\delta t_*} (\| \nabla h \|^2 + | h |^2).
\]
Since $N$ is uniformly timelike with respect to $\delta$ on $\mathcal{M}$, we have that
\[
\int_{D} \left| K^N_{\delta}[h] \right| + \left| \Re \langle Nh, S_{\delta} h \rangle_{L^2(D)} \right| + \left| \Re \langle Nh, V_{\delta} h \rangle_{L^2(D)} \right| \leq \int_{0}^{T_*} \mathcal{E}(t_*)[h] \, dt_* + \| h \|_{L^2(D)}^2.
\]
Thus, using the divergence theorem, there exists a constant $\mathcal{M}$ sufficiently large, such that
\[
\sup_{0 \leq t_* \leq T_*} \mathcal{E}(t_*)[h] \leq \mathcal{E}(0)[h] + C \int_{0}^{T_*} \| L_{\delta} h \|_{L^2(\Sigma(t_*))}^2 \, dt_* + \mathcal{M} \int_{0}^{T_*} \mathcal{E}(t_*)[h] \, dt_*.
\] (8.8)

Then, directly via Gronwall’s Lemma, we have that
\[
e^{-\mathcal{M} t_*} \mathcal{E}(t_*)[h] \leq \mathcal{E}(0)[h] + \int_{0}^{T_*} e^{-\mathcal{M} t_*} \| L_{\delta} h \|_{L^2(\Sigma(t_*))}^2 \, dt_*,
\]
from which the conclusion follows immediately given (8.7). \qed

We also have the following analogue of Lemma 8.38 in [8], which follows from a simple integration-by-parts argument.

**Lemma 8.15.** Let $g = g_b + \tilde{g}$, where $g_b$ is a sufficiently slowly-rotating Kerr-de Sitter metric, and $\sup_{t_*} e^{\delta t_*} \| \tilde{g} \|_{H^2(\Sigma(t_*))} \leq \Delta_0$, where $\Delta_0$ is sufficiently small, and let
\[D := [0, T_*) \times \Sigma,\]
and let $h$ be a sufficiently regular function on $D$ such that $h(t_*, \cdot)$ is compactly supported on $\Sigma$ for all $t_* > 0$. Then
\[
\| \langle D_x \rangle^{-1} D_{t_*} h \|_{L^2(D)} \leq \| L_{\delta} h \|_{H^{-1}(D)} + \| h \|_{L^2(D)} + \| h \|_{H^1(\Sigma(t_*))} + \| h \|_{H^1(\Sigma_0)}.
\]
Proof. Let $R_{-1} \in \text{Op} S^{-1}(\Sigma)$ be a compactly supported self-adjoint operator. We use $R_{-1}$ as a Lagrangian multiplier. Integrating by parts, we have that

\[
\begin{align*}
2\langle (g^{\alpha\beta} + \delta g_{\alpha\beta})^{-1} \Box_{gh} h, R^{\alpha}_{\beta} h \rangle_{L^2(D)} \\
= \|R_{-1} D_{t*} h\|_{L^2(D)}^2 + O \left( 1 + \|\tilde{g}\|_{W^{1,\infty}(D)} \right) \left( \|R_{-1} D_{t*} h\|_{L^2(D)} \|h\|_{L^2(D)} + \|h\|^2_{L^2(D)} \right) \\
+ \langle n_{\Sigma}, R_{-1} h \rangle_{L^2(\Sigma)} \big|_{t_*=T_*}.
\end{align*}
\]

Thus, applying Cauchy-Schwarz,

\[
\|R_{-1} D_{t*} h\|^2_{L^2(D)} \leq \|L_{gh} h\|^2_{L^2(D)} + \|h\|^2_{L^2(D)} + \|h\|^2_{\mathcal{F}^2(\Sigma)} + \|h\|^2_{\mathcal{F}^2(\Sigma_0)},
\]

as desired. \(\square\)

We now recall the following result from the linear theory:

**Proposition 8.16.** Fix $\delta > 0$. Let $g_0$ be a fixed, sufficiently slowly-rotating Kerr-de Sitter metric, and let $D = [0, T_*] \times \Sigma$.

Then, there exist some vectorfield multipliers $\tilde{X}$ and $\tilde{X}$, smooth function $\tilde{q}$ and $\tilde{q}$, one-form $\tilde{m}$, elliptic zero-order operator $Q$, and cutoff functions

\[
\tilde{\chi}, \tilde{\chi}, \tilde{\chi} \in \text{Op} S^0,
\]

such that the following properties hold.

1. $\tilde{\chi}$ and $\tilde{\chi}$ satisfy the conditions that

\[
\Sigma \subset \text{supp} \tilde{\chi} \bigcup \text{supp} \tilde{\chi} \bigcup \text{supp} \tilde{\chi}, \quad \text{supp} \tilde{\chi}_1 \bigcap \text{supp} \tilde{\chi}_2 = \emptyset, \quad \chi_1, \chi_2 \in \{ \tilde{\chi}, \tilde{\chi}, \tilde{\chi} \}.
\]

2. The following inequalities hold,

\[
\begin{align*}
\int_D K_{g_0}^{\tilde{X}, \tilde{m}} [\chi h] - \langle S_{g_0} [\chi h], \tilde{X} (\chi h) \rangle_{L^2(D)} + \int_D K_{g_0}^{\tilde{X}, \tilde{m}, 0} [\chi h] - \langle S_{g_0} [\chi h], \tilde{X} (\chi h) \rangle_{L^2(D)} \\
\gtrless \|\chi h\|^2_{H^1(D)} + \|\chi h\|^2_{H^1(D)} + \|C \nabla (\chi h)\|_{L^2(D)} \|\tilde{X} (\chi h)\|_{L^2(D)} \\
+ \|C \nabla (\chi h)\|_{L^2(D)} \|\tilde{X} (\chi h)\|_{L^2(D)} - C_1 \|h\|^2_{L^2(D)},
\end{align*}
\]

and

\[
\langle S_{g_0} [\chi h], T (\chi h) \rangle_{L^2(D)} \leq \delta \|\chi h\|^2_{H^1(D)} + C_1 \|h\|^2_{L^2(D)},
\]

where

\[
\overline{S}_{g_0} = QS_{g_0} Q^* + Q \left[ Q_{g_0}, Q^* \right],
\]

where we denote by $Q^*$ the parametrix of $Q$, and

\[
\|h\|^2_{H^1(D)} \leq \|h\|^2_{L^2(D)} + \int_D K_{g_0}^{\tilde{X}, \tilde{m}, 0} [\chi h] - \langle S_{g_0} [\chi h], \tilde{X} (\chi h) \rangle_{L^2(D)} + \int_D K_{g_0}^{\tilde{X}, \tilde{m}, 0} [\chi h] \\
- \langle S_{g_0} [\chi h], \tilde{X} (\chi h) \rangle_{L^2(D)} + \int_D K_{g_0}^{\tilde{T}, 0, 0} [\chi h] - \langle \overline{S}_{g_0} [\chi h], T (\chi h) \rangle_{L^2(D)} \\
+ \langle [Q_{g_0}], \chi h, \tilde{X} h \rangle_{L^2(D)} + \langle [Q_{g_0}], \chi h, \tilde{X} h \rangle_{L^2(D)} + \langle [Q_{g_0}], \chi h, \tilde{X} h \rangle_{L^2(D)}.
\]
3. $T$ is uniformly timelike with respect to $g_b$ on the support of $\hat{\chi}$,

$$g_b(T, T) < 0, \quad \text{on supp } \hat{\chi}. \quad (8.14)$$

4. $J_{g_b}^{X, q, m}[\chi h]$ satisfies the following properties along $\mathcal{H}_+^+$ and $\mathcal{H}_-^+$,

$$\int_{\mathcal{H}_+^+} J_{g_b}^{X, q, m}[\chi h] \cdot n_{\mathcal{H}_+^+} \geq 0, \quad \int_{\mathcal{H}_-^+} J_{g_b}^{X, q, m}[\chi h] \cdot n_{\mathcal{H}_-^+} \geq 0. \quad (8.15)$$

On the other hand, $\hat{\chi} h, \hat{\chi} h$ vanish along $\mathcal{H}_-^+, \mathcal{H}_+^+$.

**Remark 8.17.** We remark that $Q, \hat{\chi}, \hat{\chi}$ are all pseudodifferential operators, although their exact construction is irrelevant here.

**Proof.** Let

$$\hat{\chi} = \hat{\chi} \hat{\chi}_\zeta, \quad \hat{\chi} = \hat{\chi} \hat{\chi}_\xi, \quad \hat{\chi} = \hat{\chi},$$

as in Section 8 of [8], and

$$(\hat{X}, \hat{q}, \hat{m}) := (X + \hat{\chi}^2 N, \hat{q}, \hat{m}), \quad (\hat{X}, \hat{q}, 0) := (X, \hat{q}, 0)$$

where the right-hand sides of the definitions are as defined in (8.12) and Lemma (8.46) of [8], and in (8.27) of [8] respectively.

The bulk estimate in (8.11) is contained in Theorem 8.6, Lemma 8.8, and Lemma 8.11 in [8].

The combined bulk estimate containing the commutation error terms in (8.13) is directly implied by Lemma 8.43 in [8].

The flux estimate in (8.15) is contained in (8.105) in [8].

The remaining statements follow by construction of $\hat{\chi}, \hat{\chi}, \hat{\chi}$, and the definition of $g_b$. \qed

These properties are in fact preserved under perturbation by an exponentially decaying perturbation $\hat{g}$.

**Lemma 8.18.** Fix some $C > 0$, $\delta > 0$. Let $g_b$ be a fixed, sufficiently slowly-rotating Kerr-de Sitter metric. Then let the spacetime region $\mathcal{D}$, vectorfield multipliers $\hat{X}$ and $\hat{X}$, smooth function $\hat{q}$ and $\hat{q}$, one-form $\hat{m}$, elliptic zero-order operator $Q$, and cutoffs $\hat{\chi}, \hat{\chi}$ and $\hat{\chi}$ be those of Proposition 8.16. Furthermore, let $h$ be a function compactly supported on $\mathcal{D}$. If

$$g = g_b + \hat{g}, \quad \sup_{t_* > 0} e^{\delta t_*} \|\hat{g}\|_{L^2(\Sigma_{t_*})} \leq \Delta_0,$$

then for sufficiently small $\Delta_0$, the following properties hold.

1. There exists some $C_1 > 0$ sufficiently large so that

$$\int_{\mathcal{D}} K_g^{X, q, m}[\chi h] - \left< S_g[\hat{\chi} h], \hat{X}(\hat{\chi} h) \right>_{L^2(\mathcal{D})} + \int_{\mathcal{D}} K_g^{X, q, m}[\chi h] - \left< S_g[\hat{\chi} h], \hat{X}(\hat{\chi} h) \right>_{L^2(\mathcal{D})} \geq \|\hat{\chi} h\|_{H^1(\mathcal{D})} + \|\hat{\chi} h\|_{H^1(\mathcal{D})} + \|C \nabla (\hat{\chi} h)\|_{L^2(\mathcal{D})} \|\hat{X}(\hat{\chi} h)\|_{L^2(\mathcal{D})}$$

$$+ \|C \nabla (\hat{\chi} h)\|_{L^2(\mathcal{D})} \|\hat{X}(\hat{\chi} h)\|_{L^2(\mathcal{D})} - C_1 \|\hat{h}\|_{L^2(\mathcal{D})}^2, \quad (8.16)$$

and

$$\int_{\mathcal{D}} K_g^{T, q, m}[\chi h] - \left< S_g[\hat{\chi} h], T(\hat{\chi} h) \right>_{L^2(\mathcal{D})} \leq \delta \|\hat{\chi} h\|_{H^1(\mathcal{D})} + C_1 \|\hat{h}\|_{L^2(\mathcal{D})}^2, \quad (8.17)$$

49
where
\[ \mathcal{S}_g = \mathcal{S}_{g_0} + \mathcal{S}_{g_0 g_0}, \quad \mathcal{S}_{g_0} := Q \left[ \Box_g - \Box_{g_0}, Q^- \right] + Q \left( S_g - S_{g_0} \right) Q^- , \tag{8.18} \]
and
\[
\left\| h \right\|_{L^1(D)} \lesssim \left\| h \right\|_{L^2(D)}^2 + \int_D K_g \tilde{X}_{\tilde{g}, \tilde{m}} \left[ \tilde{\chi} h \right] - \left\langle \mathcal{S}_g \left[ \tilde{\chi} h \right], \tilde{X} \left( \tilde{\chi} h \right) \right\|_{L^2(D)} + \int_D K_g^{X, 0} \left[ \tilde{\chi} h \right] \\
- \left\langle \mathcal{S}_g \left[ \tilde{\chi} h \right], \tilde{X} \left( \tilde{\chi} h \right) \right\|_{L^2(D)} + \int_D K_g^{T, 0} \left[ \tilde{\chi} h \right] - \left\langle \mathcal{S}_g \left[ \tilde{\chi} h \right], T \left( \tilde{\chi} h \right) \right\|_{L^2(D)} \\
+ \left\langle \left[ \Box_g, \tilde{\chi} h \right], \tilde{X} \tilde{\chi} h \right\|_{L^2(D)} \tag{8.19} + \left\langle \left[ \Box_g, \tilde{\chi} h \right], \tilde{X} \tilde{\chi} h \right\|_{L^2(D)} + \left\langle \left[ \Box_g, \tilde{\chi} h \right], \tilde{X} \tilde{\chi} h \right\|_{L^2(D)}.
\]

2. **\( T \) is uniformly timelike with respect to \( g \) on the support of \( \tilde{\chi} \), i.e.\]
\[ g(T, T) < 0, \quad \text{on supp} \tilde{\chi}. \tag{8.20} \]

3. \( J_g \tilde{X}_{\tilde{g}, \tilde{m}} \chi h \) has the following properties along \( \mathcal{H}_+^t \) and \( \mathcal{H}_+^t \),
\[
\int_{\mathcal{H}_+^t} J_g \tilde{X}_{\tilde{g}, \tilde{m}} \left[ \chi h \right] \cdot n_{\mathcal{H}_+^t} \geq 0, \quad \int_{\mathcal{H}_+^t} J_g \tilde{X}_{\tilde{g}, \tilde{m}} \left[ \chi h \right] \cdot \tilde{m}_{\mathcal{H}_+^t} \geq 0. \tag{8.21} \]

On the other hand, \( \tilde{\chi} h \) and \( \chi h \) vanish along \( \mathcal{H}_+^t, \mathcal{H}_+^t \).

**Proof.** Observe that in light of Proposition 8.16 and 8.11, the first conclusion in (8.16) follows immediately.

For the second and third conclusion in (8.19) and (8.17) respectively, we use following observation. For any zero-order pseudo-differential operator \( P_0 \in \text{Op} S^0(\Sigma) \),
\[ \left\{ \Box_g - \Box_{g_0}, P_0 \right\} = \tilde{g}^{\alpha \beta} \left[ \tilde{\alpha}_\alpha \tilde{\beta}_\beta, P_0 \right] + \left[ \tilde{g}^{\alpha \beta}, P_0 \right] \tilde{\alpha}_\alpha \tilde{\beta}_\beta + \left\| \tilde{g} \right\|_{W^{1, \infty}(D)} \left\| \text{Op} S^0(\Sigma) \right\|.
\]

Then since \( \tilde{g} = g - g_0 \) is a scalar function, we have by the classical Coifman-Meyer commutator estimates in Proposition 8.8
\[
\left\| \tilde{g}^{\alpha \beta} \left[ \tilde{\alpha}_\alpha \tilde{\beta}_\beta, P_0 \right] h + \left[ \tilde{g}^{\alpha \beta}, P_0 \right] \tilde{\alpha}_\alpha \tilde{\beta}_\beta h \right\|_{L^2(D)} \lesssim \left\| \tilde{g} \right\|_{W^{1, \infty}(D)} \left\| h \right\|_{H^1(D)}. \tag{8.22}
\]

To prove the second conclusion in (8.17), we first observe that since \( T \) is Killing on \( g_0 \), applying Lemma 8.11 immediately yields that
\[
\left\| \int_D K_g^{T, 0} \left[ \tilde{\chi} h \right] \right\|_{L^2(D)} \lesssim \left\| \tilde{g} \right\|_{W^{1, \infty}(D)} \left\| \tilde{h} \right\|_{H^1(D)}. \tag{8.23}
\]
Applying, (8.22) with \( P_0 = Q^- \), and using Proposition 8.16 we observe that there exists some \( C_1 \) such that
\[
\left\langle \mathcal{S}_{g, g_0} \left( \tilde{\chi} h \right), T \left( \tilde{\chi} h \right) \right\|_{L^2(D)} \lesssim \left\| \tilde{g} \right\|_{W^{1, \infty}(D)} \left\| h \right\|_{H^1(D)}^2 + C_1 \left\| h \right\|_{L^2(D)}. \tag{8.24}
\]
The inequality in (8.16) then follows by using Proposition 8.16 and combining with (8.23), taking \( \Delta_0 \) sufficiently small.

To prove the third conclusion in (8.19), it suffices to show that
\[
\left\| \left[ \Box_g - \Box_{g_0}, \tilde{\chi} h \right], \tilde{X} \left( \tilde{\chi} h \right) \right\|_{L^2(D)} + \left\| \left[ \Box_g - \Box_{g_0}, \tilde{\chi} h \right], T \left( \tilde{\chi} h \right) \right\|_{L^2(D)} \lesssim \left\| \tilde{g} \right\|_{W^{1, \infty}(D)} \left\| h \right\|_{H^1(D)}. \tag{8.24}
\]
But (8.24) follows directly from (8.22) with \( P_0 = \tilde{\chi} \), and \( P_0 = \tilde{\chi} \). Then (8.19) follows from (8.11) (8.15), (8.13), (8.12), where we take \( \Delta_0 \) to be sufficiently small to conclude.

The remaining properties (8.20), (8.21), follow directly from (8.14), (8.15), by taking \( \Delta_0 \) sufficiently small.
Lemma 8.19. Let $g_0$ be a fixed, slowly-rotating Kerr-de Sitter metric, and fix $\delta_0 > 0$, $T_\ast > 0$, and denote

$$D = [0, T_\ast] \times \Sigma.$$  

Let $\phi = e^{-\delta_0 t_\ast}h$. Then there exists a choice of elliptic zero-order operator $Q$, and cutoff operators $\tilde{\chi}$ and $\bar{\chi}$ satisfying (8.10) such that for

$$\|\tilde{\chi}\phi\|_{H^1(D)} + \|\bar{\chi}\phi\|_{H^1(D)} \lesssim \|e^{-\delta_0 t_\ast}L_g(\tilde{\chi}h)\|_{L^2(D)} + \|e^{-\delta_0 t_\ast}L_g(\bar{\chi}h)\|_{L^2(D)}$$

$$+ \|\phi\|_{L^2(D)} + \|\phi\|_{\mathcal{P}(\Sigma_{T_\ast})} + \|\phi\|_{\mathcal{P}(\Sigma_0)},$$

(8.25)

$$\|\tilde{\chi}\phi\|_{H^1(D)} \lesssim \|e^{-\delta_0 t_\ast}L_g(\tilde{\chi}h)\|_{L^2(D)} + \|\phi\|_{L^2(D)} + \|\phi\|_{\mathcal{P}(\Sigma_{T_\ast})} + \|\phi\|_{\mathcal{P}(\Sigma_0)},$$

(8.26)

where

$$\mathbf{L}_g := QL_gQ^-$$

Proof. Observe that for $\phi = e^{-\delta_0 t_\ast}h$, we have that

$$L_g\phi + A^{-1}_g (2\delta_0 T + \delta_0^2) \phi = e^{-\delta_0 t_\ast}L_g h.$$  

Both of the conclusions in (8.25) and (8.26) are direct applications of the vectorfield multipliers, although the proof of (8.26) is slightly more technical due to the inclusion of the pseudo-differential conjugation by $Q$.

We begin with the first conclusion, (8.25). Applying the divergence theorem in Proposition 2.11 with $J_g \tilde{\chi}^0.\tilde{m}[\tilde{\chi}h]$, we have that

$$- \mathbb{R}\left\langle e^{-\delta_0 t_\ast}L_g(\tilde{\chi}h), (\tilde{X} + \tilde{q})(\bar{\chi}\phi) \right\rangle_{L^2(D)}$$

$$= \int_D K_g \tilde{\chi}^0.\tilde{m}[\tilde{\chi}\phi] - \mathbb{R}\left\langle S_g(\tilde{\chi}\phi), (\tilde{X} + \tilde{q})(\bar{\chi}\phi) \right\rangle_{L^2(D)} + \mathbb{R}\left\langle V_g(\bar{\chi}\phi), (\tilde{X} + \tilde{q})(\bar{\chi}\phi) \right\rangle_{L^2(D)}$$

$$+ \mathbb{R}\left\langle S_g(\bar{\chi}\phi), \tilde{q}\bar{\chi}\phi \right\rangle_{L^2(D)} + \left\langle (\delta_0 S_0 + A^{-1}_g (2\delta_0 T + \delta_0^2)) \bar{\chi}\phi, (\tilde{X} + \tilde{q})(\bar{\chi}\phi) \right\rangle_{L^2(D)}$$

$$+ \int_{H^+} J_g \tilde{\chi}^0.\tilde{m}[\tilde{\chi}\phi] \cdot n_{H^+} + \int_{\mathcal{P}_+} J_g \tilde{\chi}^0.\tilde{m}[\tilde{\chi}\phi] \cdot n_{\mathcal{P}_+},$$

where

$$S = S_0 \tilde{\partial}_t + S_1,$$

where $S_1$ is an $i$-th order differential operator that does not involve $\partial_t$. Similarly, applying the divergence theorem in Proposition 2.11 with $J_g \tilde{\chi}^0.0[\tilde{\chi}h]$, we have that

$$- \mathbb{R}\left\langle e^{-\delta_0 t_\ast}L_g(\tilde{\chi}h), (\tilde{X} + \tilde{q})(\bar{\chi}\phi) \right\rangle_{L^2(D)}$$

$$= \int_D K_g \tilde{\chi}^0.0[\tilde{\chi}\phi] - \mathbb{R}\left\langle S_g(\tilde{\chi}\phi), (\tilde{X} + \tilde{q})(\bar{\chi}\phi) \right\rangle_{L^2(D)} + \mathbb{R}\left\langle V_g(\bar{\chi}\phi), (\tilde{X} + \tilde{q})(\bar{\chi}\phi) \right\rangle_{L^2(D)}$$

$$+ \mathbb{R}\left\langle S_g(\bar{\chi}\phi), \tilde{q}\bar{\chi}\phi \right\rangle_{L^2(D)} + \left\langle (\delta_0 S_0 + A^{-1}_g (2\delta_0 T + \delta_0^2)) \bar{\chi}\phi, (\tilde{X} + \tilde{q})(\bar{\chi}\phi) \right\rangle_{L^2(D)}$$

$$+ \int_{H^+} J_g \tilde{\chi}^0.0[\tilde{\chi}\phi] \cdot n_{H^+} + \int_{\mathcal{P}_+} J_g \tilde{\chi}^0.0[\tilde{\chi}\phi] \cdot n_{\mathcal{P}_+}.$$  

Choosing $C$ sufficiently large, we have from Lemma 8.18, we then have that up to lower-order terms,

$$\|\tilde{\chi}\phi\|^2_{H^1(D)} + \|\bar{\chi}\phi\|^2_{H^1(D)}$$

$$\lesssim \int_D K_g \tilde{\chi}^0.\tilde{m}[\tilde{\chi}\phi] - \mathbb{R}\left\langle S_g(\tilde{\chi}\phi), (\tilde{X} + \tilde{q})(\bar{\chi}\phi) \right\rangle_{L^2(D)} + \left\langle A^{-1}_g (2\delta_0 T + \delta_0^2) \bar{\chi}\phi, (\tilde{X} + \tilde{q})(\bar{\chi}\phi) \right\rangle_{L^2(D)}$$

$$+ \int_D K_g \tilde{\chi}^0.0[\tilde{\chi}\phi] - \mathbb{R}\left\langle S_g(\tilde{\chi}\phi), (\tilde{X} + \tilde{q})(\bar{\chi}\phi) \right\rangle_{L^2(D)} + \left\langle A^{-1}_g (2\delta_0 T + \delta_0^2) \bar{\chi}\phi, (\tilde{X} + \tilde{q})(\bar{\chi}\phi) \right\rangle_{L^2(D)}.$$  

51
Then, we conclude using the control of the boundary fluxes in (8.21) and applying Cauchy-Schwarz, to control the lower-order terms, and using the bulk control in (8.16).

We now prove the second conclusion, (8.26). In this case, we will first use the multiplier argument with the conjugated operator $\mathbf{L}_g$, and then show that this is sufficient to conclude. First observe that we can write
\[
\mathbf{L}_g = \Box_g + \mathbf{S}_g + \nabla_g,
\]
where $\mathbf{S}_g$ is as defined in (8.18) and
\[
\nabla_g = QVgQ^-.
\]

Then, we can apply the divergence theorem in Proposition 2.11 with the vectorfield multiplier $J_g^{T,\delta_0}[\chi \phi]$, on the region $\mathcal{D}$, to yield that
\[
\begin{align*}
\tilde{E}[T_*](\chi \phi) + \int_{\mathcal{D}} K_g^{T,\delta_0}[\chi \phi] + 2\delta_0 \|T_\chi \phi\|_{L^2(\mathcal{D})}^2 + \delta_1 \int_{\mathcal{D}} g^{\alpha\beta} \partial_\alpha(\chi \phi) \cdot \partial_\beta(\chi \phi) \\
= \tilde{E}[0](\chi \phi) - \left\langle e^{-\delta_0 t_*} \mathbf{L}_g(\chi h), (T + \delta_1)(\chi \phi) \right\rangle_{L^2(\mathcal{D})} + \langle \mathbf{S}_g(\chi \phi), (T + \delta_1)(\chi \phi) \rangle_{L^2(\mathcal{D})} \\
+ \langle \nabla_g(\chi \phi), (T + \delta_1)(\chi \phi) \rangle_{L^2(\mathcal{D})} - \delta_0^2 \langle \chi \phi, (T + \delta_1)(\chi \phi) \rangle_{L^2(\mathcal{D})} + \langle \delta_0 \mathbf{S}_0(\chi \phi), (T + \delta_1)(\chi \phi) \rangle_{L^2(\mathcal{D})},
\end{align*}
\]
where the boundary fluxes vanish since $\chi h$ vanishes on the boundaries, and
\[
\tilde{E}[t_*](\chi \phi) := \int_{\Sigma_{t_*}} J_g^{T,\delta_0}[\chi \phi] \cdot n_{\Sigma_{t_*}},
\]
which we emphasize may not be positive.

First observe that for $\delta_0$ appropriately small with respect to $\delta_0$,\[
2\delta_0 \sqrt{\mathcal{A}_g}^{-1} T(\chi \phi) \parallel_{L^2(\mathcal{D})}^2 + \delta_1 \int_{\mathcal{D}} (g_0)^{\alpha\beta} \partial_\alpha(\chi \phi) \cdot \partial_\beta(\chi \phi) \geq \delta_0 \|\chi \phi\|^2_{H^1(\mathcal{D})}.
\]
Then, using Lemma 8.18 and (8.12) we see that for $\Delta_0$ sufficiently small and $\delta$ sufficiently small, we can guarantee that\[
\|\chi \phi\|^2_{H^1(\mathcal{D})} \leq 2\delta_0 \sqrt{\mathcal{A}_g}^{-1} T(\chi \phi) \parallel_{L^2(\mathcal{D})}^2 + \int_{\mathcal{D}} K_g^{T,\delta_0}[\chi \phi] \\
+ \delta_1 \int_{\mathcal{D}} g^{\alpha\beta} \partial_\alpha(\chi \phi) \cdot \partial_\beta(\chi \phi) - \langle \mathbf{S}_g(\chi \phi), T(\chi \phi) \rangle_{L^2(\mathcal{D})},
\]
Then, by a Cauchy-Schwarz argument we have that
\[
\|\chi \phi\|^2_{H^1(\mathcal{D})} \leq \left\| e^{-\delta_0 t_*} \mathbf{L}_g h \right\|^2_{L^2(\mathcal{D};\Sigma_{t_*})} + C_1 \|\phi\|^2_{L^2(\mathcal{D};\Sigma_{t_*})} + \|\phi\|^2_{\mathcal{H}^1(\Sigma_{t_*})} + \|\phi\|^2_{\mathcal{H}^1(\Sigma_{t_*})}, \tag{8.27}
\]
as desired.

We now show that we can recover (8.26) from (8.27). We first observe that since $Q, Q^- \in \text{OpS}^0(\Sigma)$, we have by combining standard elliptic estimates with Lemma 8.15 that
\[
\|\chi \phi\|_{H^1(\mathcal{D})} \leq \|Q(\chi \phi)\|_{H^1(\mathcal{D})} + \left\| e^{-\delta_0 t_*} \mathbf{L}_g(\chi h) \right\|_{L^2(\mathcal{D})} + \|\chi \phi\|_{\mathcal{H}^1(\Sigma_{t_*})} + \|\chi \phi\|_{\mathcal{H}^1(\Sigma_{t_*})}.
\]
Moreover, using (8.27), we have that
\[
\|Q(\chi \phi)\|_{H^1(\mathcal{D})} \leq \left\| e^{-\delta_0 t_*} Q \mathbf{L}_g(\chi h) \right\|_{L^2(\mathcal{D})} + \left\| e^{-\delta_0 t_*} \mathbf{L}_g \circ R_{-\infty}(\chi h) \right\|_{L^2(\mathcal{D})};
\]
where $R_{-\infty} = 1 - Q^- Q \in \text{OpS}^{0,-\infty}(\Sigma)$. Then,
\[
\left\| e^{-\delta_0 t_*} \mathbf{L}_g \circ R_{-\infty}(\chi h) \right\|_{L^2(\mathcal{D})} \leq \left\| e^{-\delta_0 t_*} \mathbf{L}_g(\chi h) \right\|_{L^2(\mathcal{D})} + \left\| e^{-\delta_0 t_*} \mathbf{L}_g(\chi h) \right\|_{H^{-1}(\mathcal{D})}.\]
We conclude the proof of (8.26), and with it the proof of Lemma 8.19, using the control of $\|D_{t_*}(\chi h)\|_{H^{-1}(\mathcal{D})}$ from Lemma 8.15. □
8.4 Proof of Proposition 8.1

We are now ready to prove Proposition 8.1.

Proof of Proposition 8.1. We observe that the estimates in (8.25) and (8.26) are almost enough to conclude immediately. The only complication is the presence of the cutoff functions inside of $L_\rho$. To commute the cutoffs, we use the fact that up to lower-order terms,

$$
\langle L_\rho (\chi h), (X + q)(\chi h) \rangle_{L^2(D)} + \langle L_\rho (\chi h), T(\chi h) \rangle_{L^2(D)}
$$

$$
= \langle \chi L_\rho h, (X + q)(\chi h) \rangle_{L^2(D)} + \langle \chi L_\rho h, T(\chi h) \rangle_{L^2(D)} + \langle [\Box_g, \chi] h, \chi (X + q) h \rangle_{L^2(D)}
$$

$$
+ \langle [\Box_g, \chi] h, \chi Th \rangle_{L^2(D)}.
$$

Then, using (8.19), we see that the extra commutation terms can be absorbed into the positive bulk, and the rest of the argument proceeds exactly as above for Lemma 8.19.

References

[1] Alinhac, Serge and Gérard, Patrick. *Pseudo-Differential Operators and the Nash–Moser Theorem*. Trans. by Stephen Wilson. Vol. 82. Graduate Studies in Mathematics. Providence, Rhode Island: American Mathematical Society, 2007.

[2] Choquet-Bruhat, Yvonne. *General Relativity and the Einstein Equations*. Oxford Mathematical Monographs. Oxford ; New York: Oxford University Press, 2009. 785 pp.

[3] Choquet-Bruhat, Yvonne. *Théorème d’existence pour certains systèmes d’équations aux dérivées partielles non linéaires*. Acta Mathematica 88 (1952), pp. 141–225.

[4] Choquet-Bruhat, Yvonne and Geroch, Robert. *Global Aspects of the Cauchy Problem in General Relativity*. Communications in Mathematical Physics 14 (4) (1969), pp. 329–335.

[5] Christodoulou, Demetrios and Klainerman, Sergiu. *The Global Nonlinear Stability of the Minkowski Space*. 1993.

[6] Dafermos, Mihalis, Holzegel, Gustav, Rodnianski, Igor, and Taylor, Martin. *The Non-Linear Stability of the Schwarzschild Family of Black Holes*. 2021.

[7] Dafermos, Mihalis and Rodnianski, Igor. *Lectures on Black Holes and Linear Waves*. 2008.

[8] Fang, Allen Juntao. *Linear Stability of Slowly-Rotating Kerr-de Sitter Black Hole Space-times*. preprint (2021), p. 140.

[9] Graham, C. Robin and Lee, John M. *Einstein Metrics with Prescribed Conformal Infinity on the Ball*. Advances in Mathematics 87 (2) (1991), pp. 186–225.

[10] Hintz, Peter and Vasy, András. *The Global Non-Linear Stability of the Kerr–de Sitter Family of Black Holes*. Acta Mathematica 220 (1) (2018), pp. 1–206.

[11] Hörmander, Lars. *The Analysis of Linear Partial Differential Operators III: Pseudo-Differential Operators*. Classics in Mathematics. Berlin, Heidelberg: Springer Berlin Heidelberg, 2007.

[12] Huneau, Cécile. *Stability of Minkowski Space-Time with a Translation Space-Like Killing Field*. Annals of PDE 4 (1) (2018).

[13] Johnson, Thomas. *On the Linear Stability of the Schwarzschild Solution to Gravitational Perturbations in the Generalised Wave Gauge*. 2018.

[14] Klainerman, Sergiu and Szefter, Jérémie. *Effective Results on Uniformization and Intrinsic GCM Spheres in Perturbations of Kerr*. 2019.
[15] Klainerman, Sergiu and Szeftel, Jérémie. *Global Nonlinear Stability of Schwarzschild Spacetime under Polarized Perturbations*. Annals of Mathematics Studies 210. Princeton University Press, 2020.

[16] Klainerman, Sergiu and Szeftel, Jérémie. *Kerr Stability for Small Angular Momentum*. 2021.

[17] Schlue, Volker. *Decay of the Weyl Curvature in Expanding Black Hole Cosmologies*. 2021.

[18] Tataru, Daniel and Tohaneanu, Mihai. *A Local Energy Estimate on Kerr Black Hole Backgrounds*. International Mathematics Research Notices 2011 (2) (2010), pp. 248–292.

[19] Taylor, Michael E. *Pseudodifferential Operators and Nonlinear PDE*. Boston, MA: Birkhäuser Boston, 1991.

[20] Warnick, Claude M. *On Quasinormal Modes of Asymptotically Anti-de Sitter Black Holes*. Communications in Mathematical Physics 333 (2) (2015), pp. 959–1035.