On φ-w-Flat modules and Their Homological Dimensions

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Abstract

In this paper, we introduce and study the class of φ-w-flat modules which are generalizations of both φ-flat modules and w-flat modules. The φ-w-weak global dimension φ-w-gl.dim(R) of a strongly φ-ring R is also introduced and studied. We show that, for a strongly φ-ring R, φ-w-gl.dim(R) = 0 if and only if w.dim(R) = 0 if and only if R is a φ-von Neumann ring. It is also proved that, for a strongly φ-ring R, φ-w-gl.dim(R) ≤ 1 if and only if each nonnil ideal of R is φ-w-flat, if and only if R is a φ-PvMR, if and only if R is a PvMR.

Key Words: φ-w-flat module; φ-w-weak global dimension; φ-von Neumann ring; φ-PvMR.

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Throughout this paper, R denotes a commutative ring with 1 ≠ 0 and all modules are unitary. We denote by Nil(R) the nilpotent radical of R, Z(R) the set of all zero-divisors of R and T(R) the localization of R at the set of all regular elements. The R-submodules I of T(R) such that sI ⊆ R for some regular element s are said to be fractional ideals. Recall from [3] that a ring R is an NP-ring if Nil(R) is a prime ideal, and a ZN-ring if Z(R) = Nil(R). A prime ideal P is said to be divided prime if P ⊆ (x), for every x ∈ R − P. Set \( \mathcal{H} = \{ R | R \text{ is a commutative ring and } \text{Nil}(R) \text{ is a divided prime ideal of } R \} \). A ring R is a φ-ring if R ∈ \( \mathcal{H} \). Moreover, a ZN φ-ring is said to be a strongly φ-ring. For a φ-ring R, there is a ring homomorphism \( \phi : T(R) \to R_{\text{Nil}(R)} \) such that \( \phi(a/b) = a/b \) where \( a \in R \) and \( b \) is a regular element. Denote by the ring \( \phi(R) \) the image of \( \phi \) restricted to \( R \). In 2001, Badawi [4] investigated φ-chain rings (φ-CRs for short) and φ-pseudo-valuation rings as a φ-version of chain rings and pseudo-valuation rings. In 2004, Anderson and Badawi [1] introduced the concept of φ-Prüfer rings and showed that a φ-ring R is φ-Prüfer if and only if \( R_m \) is a φ-chain ring for any maximal ideal \( m \) of \( R \) if and only if \( R/\text{Nil}(R) \) is a Prüfer domain if and only if \( \phi(R) \) is Prüfer. Later, the authors in [2, 5] generalized the concepts of Dedekind domains, Krull domains and
Mori domains to the context of rings that are in the class $\mathcal{H}$. In 2013, Zhao et al. \cite{22} introduced and studied the conceptions of $\phi$-flat modules and $\phi$-von Neumann rings and obtained that a $\phi$-ring is $\phi$-von Neumann if and only if its Krull dimension is 0. Recently, Zhao \cite{21} gave a homological characterization of $\phi$-Prüfer rings as follows: a strongly $\phi$-ring $R$ is $\phi$-Prüfer, if and only if each submodule of a $\phi$-flat module is $\phi$-flat, if and only if each nonnil ideal of $R$ is $\phi$-flat.

Some other important generalizations of classical notions are their $w$-versions. In 1997, Wang and McCasland \cite{16} introduced the $w$-modules over strong Mori domains (SM domains for short) which can be seen as a $w$-version of Noetherian domains. In 2011, Yin et al. \cite{19} extended $w$-theories to commutative rings containing zero divisors. The notion of $w$-flat modules appeared first in \cite{11} for integral domains and was extended to arbitrary commutative rings in \cite{14}. In 2012, Kim and Wang \cite{13} introduced $\phi$-SM rings which can be seen as both a $\phi$-version and a $w$-version of Noetherian domains and obtained that a $\phi$-ring $R$ is $\phi$-SM if and only if $R/\text{Nil}(R)$ is an SM domain if and only if $\phi(R)$ is an SM ring. In 2014, Wang and Kim \cite{13} introduced $w$-$\text{w.gl.dim}(R)$ as a generalization of the classical weak global dimension and obtained that a ring $R$ is a von Neumann ring if and only if each $R$-module is $w$-flat, i.e., $w$-$\text{w.gl.dim}(R) = 0$. In 2015, Wang and Qiao \cite{17} studied several properties of the $w$-weak global dimension, and proved that an integral domain $R$ is a Prüfer $v$-multiplication domain (PvMD for short) if and only if $w$-$\text{w.gl.dim}(R) \leq 1$ if and only if $R_m$ is a valuation domain for any maximal $w$-ideal $m$ of $R$. As $\phi$-rings are natural extensions of integral domains, we introduce and study the $\phi$-versions of $w$-flat modules, von Neumann rings and PvMDs in this article. As our work involves $w$-theories, we give a review as below.

Let $R$ be a commutative ring and $J$ a finitely generated ideal of $R$. Then $J$ is called a GV-ideal if the natural homomorphism $R \to \text{Hom}_R(J, R)$ is an isomorphism. The set of all GV-ideals is denoted by $\text{GV}(R)$. An $R$-module $M$ is said to be GV-torsion if for any $x \in M$ there is a GV-ideal $J$ such that $Jx = 0$; an $R$-module $M$ is said to be GV-torsion free if $Jx = 0$, then $x = 0$ for any $J \in \text{GV}(R)$ and $x \in M$. A GV-torsion free module $M$ is said to be a $w$-module if for any $x \in E(M)$ there is a GV-ideal $J$ such that $Jx \subseteq M$ where $E(M)$ is the injective envelope of $M$. The $w$-envelope $M_w$ of a GV-torsion free module $M$ is defined by the minimal $w$-module that contains $M$. Therefore, a GV-torsion free module $M$ is a $w$-module if and only if $M_w = M$. A maximal $w$-ideal for which is maximal among the $w$-submodules of $R$ is proved to be prime (see \cite{19} Proposition 3.8)). The set of all maximal $w$-ideals is denoted by $w$-$\text{Max}(R)$. The $w$-dimension $w$-$\text{dim}(R)$ of a ring $R$ is defined to be the supremum of the heights of all maximal $w$-ideals.
An $R$-homomorphism $f : M \to N$ is said to be a $w$-monomorphism (resp., $w$-epimorphism, $w$-isomorphism) if for any $p \in w\text{-Max}(R)$, $f_p : M_p \to N_p$ is a monomorphism (resp., an epimorphism, an isomorphism). Note that $f$ is a $w$-monomorphism (resp., $w$-epimorphism) if and only if $\ker(f)$ (resp., $\coker(f)$) is GV-torsion. A sequence $A \to B \to C$ is said to be $w$-exact if for any $p \in w\text{-Max}(R)$, $A_p \to B_p \to C_p$ is exact. A class $C$ of $R$-modules is said to be closed under $w$-isomorphisms provided that for any $w$-isomorphism $f : M \to N$, if one of the modules $M$ and $N$ is in $C$, so is the other. An $R$-module $M$ is said to be of finite type if there exist a finitely generated free module $F$ and a $w$-epimorphism $g : F \to M$, or equivalently, if there exists a finitely generated $R$-submodule $N$ of $M$ such that $N_w = M_w$. Certainly, the class of finite type modules is closed under $w$-isomorphisms. Now we proceed to introduce the notion of $\phi$-$w$-flat modules.

1. $\phi$-$w$-flat modules

We say an ideal $I$ of $R$ is nonnil provided that there is a non-nilpotent element in $I$. Denote by $\text{NN}(R)$ the set of all nonnil ideals of $R$. Certainly, GV-ideals are nonnil. Let $R$ be an NP-ring. It is easy to verify that $\text{NN}(R)$ is a multiplicative system of ideals. That is $R \in \text{NN}(R)$ and for any $I \in \text{NN}(R)$, $J \in \text{NN}(R)$, we have $IJ \in \text{NN}(R)$. Let $M$ be an $R$-module. Define

$$\phi\text{-tor}(M) = \{x \in M |Ix = 0 \text{ for some } I \in \text{NN}(R)\}.$$  

An $R$-module $M$ is said to be $\phi$-torsion (resp., $\phi$-torsion free) provided that $\phi\text{-tor}(M) = M$ (resp., $\phi\text{-tor}(M) = 0$). Clearly, if $R$ is an NP-ring, the class of $\phi$-torsion modules is closed under submodules, quotients, direct sums and direct limits. Thus an NP-ring $R$ is $\phi$-torsion free if and only if every flat module is $\phi$-torsion free if and only if $R$ is a ZN-ring (see [21, Proposition 2.2]). The classes of $\phi$-torsion modules and $\phi$-torsion free modules constitute a hereditary torsion theory of finite type. For more details, refer to [10].

**Lemma 1.1.** Let $R$ be an NP-ring, $m$ a maximal $w$-ideal of $R$ and $I$ an ideal of $R$. Then $I \in \text{NN}(R)$ if and only if $I_m \in \text{NN}(R_m)$.

**Proof.** Let $I \in \text{NN}(R)$ and $x$ a non-nilpotent element in $I$. We will show the element $x/1$ in $I_m$ is a non-nilpotent element of $R_m$. If $(x/1)^n = x^n/1 = 0$ in $R_m$ for some positive integer $n$, there is an $s \in R - m$ such that $sx^n = 0$ in $R$. Since $R$ is an NP-ring, $\text{Nil}(R)$ is the minimal prime $w$-ideal of $R$. In the integral domain $R/\text{Nil}(R)$, we have $\overline{sx^n} = \overline{0}$, thus $\overline{x^n} = \overline{0}$ since $s \notin \text{Nil}(R)$. So $x \in \text{Nil}(R)$, a contradiction.
Let \( x/s \) be a non-nilpotent element in \( I_m \) where \( x \in I \) and \( s \in R - m \). Clearly, \( x \) is non-nilpotent and thus \( I \in \text{NN}(R) \).

\[ \square \]

**Proposition 1.2.** Let \( R \) be an NP-ring, \( m \) a maximal \( w \)-ideal of \( R \) and \( M \) an \( R \)-module. Then \( M \) is \( \phi \)-torsion over \( R \) if and only \( M_m \) is \( \phi \)-torsion over \( R_m \).

**Proof.** Let \( M \) be an \( R \)-module and \( x \in M \). If \( M_m \) is \( \phi \)-torsion over \( R_m \), there is an ideal \( I_m \in \text{NN}(R_m) \) such that \( I_m x/1 = 0 \) in \( R_m \). Let \( I \) be the preimage of \( I_m \) in \( R \). Then \( I \) is nonnil by Lemma [1.1]. Thus there is a non-nilpotent element \( t \in I \) such that \( tkx = 0 \) for some \( k \notin m \). Let \( s = tk \). Then we have \((s) \in \text{NN}(R)\) and \((s)x = 0\). Thus \( M \) is \( \phi \)-torsion. Suppose \( M \) is \( \phi \)-torsion over \( R \). Let \( x/s \) be an element in \( M_m \). Then there is an ideal \( I \in \text{NN}(R) \) such that \( Ix = 0 \), and thus \( I_m x/s = 0 \) with \( I_m \in \text{Nil}(R_m) \) by Lemma [1.1]. It follows that \( M_m \) is \( \phi \)-torsion over \( R_m \).

Recall from [14] that an \( R \)-module \( M \) is said to be \( w \)-flat if for any \( w \)-monomorphism \( f : A \to B \), the induced sequence \( f \otimes_R 1 : A \otimes_R M \to B \otimes_R M \) is also a \( w \)-monomorphism. Obviously, GV-torsion modules and flat modules are all \( w \)-flat. It was proved that the class of \( w \)-flat modules is closed under \( w \)-isomorphisms (see [15 Corollary 6.7.4]). Following [22 Definition 3.1], an \( R \)-module \( M \) is said to be \( \phi \)-flat if for every monomorphism \( f : A \to B \) with \( \text{Coker}(f) \phi \)-torsion, \( f \otimes_R 1 : A \otimes_R M \to B \otimes_R M \) is a monomorphism. Obviously flat modules are both \( \phi \)-flat and \( w \)-flat. Now we give a generalization of both \( \phi \)-flat modules and \( w \)-flat modules.

**Definition 1.3.** Let \( R \) be a ring. An \( R \)-module \( M \) is said to be \( \phi \)-\( w \)-flat if for every monomorphism \( f : A \to B \) with \( \text{Coker}(f) \phi \)-torsion, \( f \otimes_R 1 : A \otimes_R M \to B \otimes_R M \) is a \( w \)-monomorphism; equivalently, if \( 0 \to A \to B \to C \to 0 \) is an exact exact sequence with \( C \phi \)-torsion, then \( 0 \to A \otimes_R M \to B \otimes_R M \to C \otimes_R M \to 0 \) is \( w \)-exact.

Clearly \( \phi \)-flat modules and \( w \)-flat modules are \( \phi \)-\( w \)-flat. It is well known that an \( R \)-module \( M \) is flat if and only if the induced homomorphism \( 1 \otimes_R f : M \otimes_R I \to M \otimes_R R \) is exact for any (finitely generated) ideal \( I \), if and only if the multiplication homomorphism \( i : I \otimes_R M \to IM \) is an isomorphism for any (finitely generated) ideal \( I \) of \( R \). Some similar characterizations of \( w \)-flat modules and \( \phi \)-flat modules are given in [13 Proposition 1.1] and [22 Theorem 3.2], respectively. We can also obtain some similar characterizations of \( \phi \)-\( w \)-flat modules.

**Theorem 1.4.** Let \( R \) be an NP-ring. The following statements are equivalent for an \( R \)-module \( M \):

1. \( M \) is \( \phi \)-\( w \)-flat;
(2) $M_m$ is $\phi$-flat over $R_m$ for all $m \in w\text{-}Max(R)$;

(3) $\text{Tor}_R^1(T, M)$ is $\text{GV}$-torsion for all (finite type) $\phi$-torsion $R$-modules $T$;

(4) $\text{Tor}_R^1(R/I, M)$ is $\text{GV}$-torsion for all (finite type) non-nil ideals $I$ of $R$;

(5) $f \otimes_R 1 : I \otimes_R M \to R \otimes_R M$ is $w$-exact for all (finite type) non-nil ideals $I$ of $R$;

(6) the multiplication homomorphism $i : I \otimes_R M \to IM$ is a $w$-isomorphism for all (finite type) ideals $I$;

(7) let $0 \to K \to F \to M \to 0$ be an exact sequence of $R$-modules, where $F$ is free. Then $(K \cap FI)_w = (IK)_w$ for all (finite type) non-nil ideals $I$ of $R$.

Proof. (1) $\Rightarrow$ (2): Let $m$ be a maximal $w$-ideal of $R$, $f : A_m \to B_m$ an $R_m$-homomorphism with $\text{Coker}(f) \phi$-torsion over $R_m$. Then $\text{Coker}(f)$ is $\phi$-torsion over $R$ by Proposition 1.2. It follows that $f \otimes_R M : A_m \otimes_R M \to B_m \otimes_R M$ is a $w$-monomorphism over $R$. Localizing at $m$, we have $f \otimes_{R_m} M : A_m \otimes_{R_m} M_m \to B_m \otimes_{R_m} M_m$ is a monomorphism over $R_m$ since $N_m \otimes_R M_m \cong N_m \otimes_{R_m} M_m$ for any $R$-module $N$. It follows that $M_m$ is $\phi$-flat over $R_m$.

(2) $\Rightarrow$ (1): Let $f : A \to B$ be a monomorphism with $\text{Coker}(f) \phi$-torsion. For any $m \in w\text{-}Max(R)$, we have $f_m : A_m \to B_m$ is a monomorphism with $\text{coker}(f_m) \phi$-torsion over $R_m$ by Proposition 1.2. Since $M_m$ is $\phi$-flat over $R_m$, $f_m \otimes_{R_m} M_m : A_m \otimes_{R_m} M_m \to B_m \otimes_{R_m} M_m$ is a monomorphism. Thus $f \otimes_R M : A \otimes_R M \to B \otimes_R M$ is a $w$-monomorphism. Consequently, $M$ is $\phi$-flat.

The equivalences of (2) – (7) hold from [22, Theorem 3.2] by localizing at all maximal $w$-ideals.

Corollary 1.5. Let $R$ be an NP-ring. The class of $\phi$-$w$-flat modules is closed under $w$-isomorphisms.

Proof. Let $f : M \to N$ be a $w$-isomorphism and $T$ a $\phi$-torsion module. There exist two exact sequences $0 \to T_1 \to M \to L \to 0$ and $0 \to L \to N \to T_2 \to 0$ with $T_1$ and $T_2$ $\text{GV}$-torsion. Considering the induced two long exact sequences $\text{Tor}_R^1(T, T_1) \to \text{Tor}_R^1(T, M) \to \text{Tor}_R^1(T, L) \to T \otimes T_1$ and $\text{Tor}_R^2(T, T_2) \to \text{Tor}_R^2(T, T_2) \to \text{Tor}_R^2(T, T_2) \to \text{Tor}_R^2(T, N) \to \text{Tor}_R^2(T, T_2)$, we have $M$ is $\phi$-$w$-flat if and only if $N$ is $\phi$-$w$-flat by Theorem 1.4.

Lemma 1.6. Let $R$ be a $\phi$-ring and $I$ a non-nil ideal of $R$. Then $\text{Nil}(R) = I\text{Nil}(R)$.

Proof. Let $I$ be a non-nil ideal of $R$ with a non-nilpotent element $s \in I$. Then $\text{Nil}(R) \subseteq (s)$. Thus for any $a \in \text{Nil}(R)$, there exists $b \in R$ such that $a = sb$. Thus $\bar{a} = \bar{s}\bar{b}$ in the integral domain $R/\text{Nil}(R)$. Since $\bar{a} = 0$ and $\bar{s} \neq 0$, we have $\bar{b} = 0$. So $b \in \text{Nil}(R)$ and then $\text{Nil}(R) \subseteq s\text{Nil}(R) \subseteq I\text{Nil}(R) \subseteq \text{Nil}(R)$. It follows that $\text{Nil}(R) = I\text{Nil}(R)$. 

\[\square\]
Proposition 1.7. Let $R$ be a $\phi$-ring and $M$ an $R$-module. Then $M/\text{Nil}(R)M$ is $\phi$-flat over $R$ if and only if $M/\text{Nil}(R)M$ is flat over $R/\text{Nil}(R)$. Consequently, $R/\text{Nil}(R)$ is always $\phi$-flat over $R$.

Proof. For the “only if” part, let $\overline{T} = I/\text{Nil}(R)$ be an ideal of $\overline{R} = R/\text{Nil}(R)$. If $\overline{T}$ is zero, certainly $\text{Tor}_1^R(\overline{R}/T, M/\text{Nil}(R)M) = 0$. Let $\overline{T}$ be a non-zero ideal of $\overline{R}$ with $I \in \text{NN}(R)$. Since $M/\text{Nil}(R)M$ is $\phi$-flat over $R$, $\text{Tor}_1^R(R/I, M/\text{Nil}(R)M) = 0$. By Lemma 1.6, $\text{Tor}_1^R(R/\text{Nil}(R), R/I) \cong I \cap \text{Nil}(R)/I\text{Nil}(R) = \text{Nil}(R)/I\text{Nil}(R) = 0$. We have $\text{Tor}_1^R(\overline{R}/T, M/\text{Nil}(R)M) \cong \text{Tor}_1^R(R/I, M/\text{Nil}(R)M) = 0$ by change of rings.

For the “if” part, let $I$ be a nonnil ideal of $R$. Similarly to the proof of “only if” part, since $\text{Tor}_1^R(R/\text{Nil}(R), R/I) = 0$, we have $\text{Tor}_1^R(R/I, M/\text{Nil}(R)M) \cong \text{Tor}_1^\overline{R}(\overline{R}/\overline{T}, M/\text{Nil}(R)M) = 0$. It follows that $M/\text{Nil}(R)M$ is $\phi$-flat over $R$.

By localizing at all maximal $w$-ideals, we obtain the following corollary.

Corollary 1.8. Let $R$ be a $\phi$-ring and $M$ an $R$-module. Then $M/\text{Nil}(R)M$ is $\phi$-$w$-flat over $R$ if and only if $M/\text{Nil}(R)M$ is $w$-flat over $R/\text{Nil}(R)$.

Proof. See Proposition 1.7, Theorem 1.4 and [8, Theorem 3.3].

Certainly if $R$ is an integral domain, every $\phi$-$w$-flat module is $w$-flat. Conversely, this property characterizes integral domains.

Theorem 1.9. The following statements are equivalent for a $\phi$-ring $R$:

1. $R$ is an integral domain;
2. every $\phi$-$w$-flat module is $w$-flat;
3. every $\phi$-flat module is $w$-flat.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3): Trivial.

(3) $\Rightarrow$ (1): Let $s$ be a nilpotent element of $R$. Then

$$\text{Tor}_1^R(R/(s), R/\text{Nil}(R)) \cong (s) \cap \text{Nil}(R)/s\text{Nil}(R) = (s)/s\text{Nil}(R)$$

is GV-torsion since $R/\text{Nil}(R)$ is $w$-flat by (3) and Proposition 1.7. Thus there is a GV-ideal $J$ such that $sJ \subseteq s\text{Nil}(R)$. Since $J$ is a nonnil ideal, $\text{Nil}(R) = J\text{Nil}(R)$ by Lemma 1.6. Thus $sJ \subseteq s\text{Nil}(R) = sJ\text{Nil}(R) \subseteq sJ$. That is, $sJ = sJ\text{Nil}(R)$. Since $sJ$ is finitely generated, $sJ = 0$ by Nakayama’s lemma. Since $J \in \text{GV}(R)$, $s \in R$ is GV-torsion free, then $s = 0$. Consequently, $\text{Nil}(R) = 0$ and $R$ is an integral domain.

Recall from [12] that a ring $R$ is said to be a DW ring if every ideal of $R$ is a $w$-ideal. Then a ring $R$ is a DW ring if and only if every $R$-module is a $w$-module, if and only if $\text{GV}(R) = \{R\}$ (see [12] Theorem 3.8). Certainly if $R$ is a DW ring, every $\phi$-$w$-flat module is $\phi$-flat. Conversely, this property characterizes DW rings.
Theorem 1.10. The following statements are equivalent for an NP-ring $R$:

1. $R$ is a DW ring;
2. every $\phi$-$w$-flat module is $\phi$-flat;
3. every $w$-flat module is $\phi$-flat.

Proof. $(1) \Rightarrow (2) \Rightarrow (3)$: Trivial.

$(3) \Rightarrow (1)$: For any $J \in \text{GV}(R)$, $R/J$ is GV-torsion, and thus $w$-flat. By $(3)$, $R/J$ is $\phi$-flat. Since every GV-ideal $J$ is a nonnil ideal of $R$, we have Tor$_R^1(R/J, R/J) \cong J/J^2 = 0$. It follows that $J$ is a finitely generated idempotent ideal, and thus $J$ is projective. So $J = J_w = R$ by [15, Exercise 6.10(1)] and thus $R$ is a DW ring by [15, Theorem 6.3.12]. □

Some non-integral domain examples are provided by the idealization construction $R(+)M$ where $M$ is an $R$-module (see [6]). We recall this construction. Let

$R(+)M = R \oplus M$ as an $R$-module, and define

1. $(r, m) + (s, n) = (r + s, m + n)$.
2. $(r, m)(s, n) = (rs, sm + rn)$.

Under these definitions, $R(+)M$ becomes a commutative ring with identity. Denote by $(0 :_RM)$ the set $\{r \in R | rM = 0\}$. Now we compute some examples of GV-ideals of $R(+)M$.

Proposition 1.11. Let $T$ be a commutative ring and $E$ a $w$-module over $T$ such that $(0 :_TE) = 0$. Set $R = T(+)E$. Then $J(+)E$ is a GV-ideal of $R$ for any $J \in \text{GV}(T)$.

Proof. Let $J$ be a GV-ideal of $T$. Then we claim that $J(+)E \in \text{GV}(R)$. Indeed, since $T(+)E/J(+)E \cong T/J$, for any $i = 0, 1$, we have

$$\text{Ext}_R^i(T(+)E/J(+)E, R) \cong \text{Ext}_T^i(T/J, \text{Hom}_R(T, R)).$$

Note that

$$\text{Hom}_R(T, R) = \text{Hom}_R(R/0(+)E, R) \cong 0(+)E \cong E$$

since $(0 :_TE) = 0$. Thus $\text{Ext}_R^i(T(+)E/J(+)E, R) \cong \text{Ext}_T^i(T/J, E)$ for any $i = 0, 1$. If $J \in \text{GV}(T)$ then $J(+)E \in \text{GV}(R)$ since $E$ is a $w$-module over $T$. □

Now we give an example to show the notion of $\phi$-$w$-flat modules is a strict generalization of $\phi$-flat modules and $w$-flat modules.

Example 1.12. Let $D$ be a non-DW integral domain and $K$ its quotient field. Then $R = D(+)K$ is a $\phi$-ring (see [2, Remark 1]). However, by Proposition 1.11, $R$ is neither an integral domain nor a DW ring. Consequently, there is a $\phi$-$w$-flat module over $R$ which is neither $\phi$-flat nor $w$-flat by Theorem 1.9 and Theorem 1.10.
2. HOMOLOGICAL PROPERTIES OF $\phi$-$w$-FLAT MODULES

Let $R$ be a ring. It is well known that the flat dimension of an $R$-module $M$ is defined as the shortest flat resolution of $M$ and the weak global dimension of $R$ is the supremum of the flat dimensions of all $R$-modules. The $w$-flat dimension $w$-$\text{fd}_R(M)$ of an $R$-module $M$ and $w$-weak global dimension $w$-$\text{w.gl.dim}(R)$ of a ring $R$ were introduced and studied in [17]. The following result shows that $\phi$-$w$-flat modules can be characterized by “higher Tor-funcotrs” when the base ring is a strongly $\phi$-ring, and so we can investigate homological dimensions in terms of $\phi$-$w$-flat modules over strongly $\phi$-rings.

Lemma 2.1. [7, Theorem 1.6] Let the $R$ be a strongly $\phi$-ring. Then an $R$-module $M$ is $\phi$-$w$-flat if and only if $\text{Tor}^R_k(R/I, M)$ is $\text{GV}$-torsion for all nonnil ideal $I$ of $R$ and all $k > 0$

Proof. Let $I$ be a nonnil ideal of $R$. Since $R$ is a strongly $\phi$-ring, then $I$ contains a nonzero-divisor $a$. Suppose $M$ is a $\phi$-$w$-flat $R$-module. Since $a$ is a non-zero-divisor of $R$, $\text{Tor}^R_n(R/I, M) = 0$ for any positive integer $n$. Then

$$\text{Tor}^R_1(R/I, M/aM) \cong \text{Tor}^R_1(R/I, M \otimes_R R/a) \cong \text{Tor}^R_1(R/I, M)$$

which is $\text{GV}$-torsion. Hence $M/Ma$ is a $w$-flat $R/a$-module. Consequently, for any $n \geq 1$ we have

$$\text{Tor}^R_n(R/I, M) \cong \text{Tor}^R_n(R/I, M \otimes_R R/a) \cong \text{Tor}^R_n(R/I, M/aM)$$

is $\text{GV}$-torsion by [13, Theorem 6.7.2].

In this section, we always assume $R$ is a strongly $\phi$-ring. We first introduce the notion of $\phi$-$w$-flat dimension of an $R$-module over strongly $\phi$-ring $R$ as follows.

Definition 2.2. Let $R$ be a strongly $\phi$-ring and $M$ an $R$-module. We write $\phi$-$w$-$\text{fd}_R(M) \leq n$ ($\phi$-$w$-$\text{fd}$ abbreviates $\phi$-$w$-flat dimension) if there is a $w$-exact sequence of $R$-modules

$$0 \rightarrow F_n \rightarrow ... \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0 \quad (\diamond)$$

where each $F_i$ is $w$-flat for $i = 0, ..., n - 1$ and $F_n$ is $\phi$-$w$-flat. The $w$-exact sequence $(\diamond)$ is said to be a $\phi$-$w$-flat $w$-resolution of length $n$ of $M$. If such finite $w$-resolution does not exist, then we say $\phi$-$w$-$\text{fd}_R(M) = \infty$; otherwise, define $\phi$-$w$-$\text{fd}_R(M) = n$ if $n$ is the length of the shortest $\phi$-$w$-flat $w$-resolution of $M$.

It is obvious that an $R$-module $M$ is $\phi$-$w$-flat if and only if $\phi$-$w$-$\text{fd}_R(M) = 0$. Certainly, $\phi$-$w$-$\text{fd}_R(M) \leq w$-$\text{fd}_R(M)$. If $R$ is an integral domain, then $\phi$-$w$-$\text{fd}_R(M) = w$-$\text{fd}_R(M)$.
Lemma 2.3. [17, Lemma 2.2] Let $N$ be an $R$-module and $0 \to A \to F \to C \to 0$ a $w$-exact sequence of $R$-modules with $F$ a $w$-flat module. Then for any $n > 0$, the induced map $\text{Tor}^R_{n+1}(C, N) \to \text{Tor}^R_n(A, N)$ is a $w$-isomorphism. Hence, $\text{Tor}^R_{n+1}(C, N)$ is GV-torsion if and only if so is $\text{Tor}^R_n(A, N)$.

Proposition 2.4. Let $R$ be a strongly $\phi$-ring. The following statements are equivalent for an $R$-module $M$:

1. $\phi$-$w$-$\text{fd}_R(M) \leq n$;
2. $\text{Tor}^R_{n+k}(M, N)$ is GV-torsion for all $\phi$-torsion $R$-modules $N$ and all $k > 0$;
3. $\text{Tor}^R_{n+1}(M, N)$ is GV-torsion for all $\phi$-torsion $R$-modules $N$;
4. $\text{Tor}^R_{n+1}(M, R/I)$ is GV-torsion for all non-nil ideals $I$;
5. $\text{Tor}^R_{n+1}(M, R/I)$ is GV-torsion for all finite type non-nil ideals $I$;
6. if $0 \to F_0 \to \ldots \to F_1 \to F_0 \to M \to 0$ is an exact sequence, where $F_0, F_1, \ldots, F_{n-1}$ are flat $R$-modules, then $F_n$ is $\phi$-$w$-flat;
7. if $0 \to F_n \to \ldots \to F_1 \to F_0 \to M \to 0$ is an $w$-exact sequence, where $F_0, F_1, \ldots, F_{n-1}$ are $w$-flat $R$-modules, then $F_n$ is $\phi$-$w$-flat;
8. if $0 \to F_n \to \ldots \to F_1 \to F_0 \to M \to 0$ is an exact sequence, where $F_0, F_1, \ldots, F_{n-1}$ are $w$-flat $R$-modules, then $F_n$ is $\phi$-$w$-flat;
9. if $0 \to F_n \to \ldots \to F_1 \to F_0 \to M \to 0$ is an $w$-exact sequence, where $F_0, F_1, \ldots, F_{n-1}$ are flat $R$-modules, then $F_n$ is $\phi$-$w$-flat.

Proof. (1) $\Rightarrow$ (2): We prove (2) by induction on $n$. For the case $n = 0$, (2) holds by Theorem 1.3 and Lemma 2.1 as $M$ is $\phi$-$w$-flat. If $n > 0$, then there is a $w$-exact sequence $0 \to F_n \to \ldots \to F_1 \to F_0 \to M \to 0$, where each $F_i$ is $w$-flat for $i = 0, \ldots, n-1$ and $F_n$ is $\phi$-$w$-flat. Set $K_0 = \ker(F_0 \to M)$. Then both $0 \to K_0 \to F_0 \to M \to 0$ and $0 \to F_n \to F_{n-1} \to \ldots \to F_1 \to K_0 \to 0$ are $w$-exact, and $\phi$-$w$-$\text{fd}_R(K_0) \leq n-1$. By induction, $\text{Tor}^R_{n-1+k}(K_0, N)$ is GV-torsion for all $\phi$-torsion $R$-modules $N$ and all $k > 0$. Thus, it follows from Lemma 2.3 that $\text{Tor}^R_{n+k}(M, N)$ is GV-torsion.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5): Trivial.

(5) $\Rightarrow$ (6): Let $K_0 = \ker(F_0 \to M)$ and $K_i = \ker(F_i \to F_{i-1})$, where $i = 1, \ldots, n-1$. Then $K_{n-1} = F_n$. Since all $F_0, F_1, \ldots, F_{n-1}$ are flat, $\text{Tor}^R_1(F_n, R/I) \cong \text{Tor}^R_{n+1}(M, R/I)$ is GV-torsion for all finite type non-nil ideal $I$. Hence $F_n$ is a $\phi$-$w$-flat module by Theorem 1.4.

(6) $\Rightarrow$ (1): Obvious.

(3) $\Rightarrow$ (7): Set $L_n = F_n$ and $L_i = \text{Im}(F_i \to F_{i-1})$, where $i = 1, \ldots, n-1$. Then both $0 \to L_{i+1} \to F_i \to L_i \to 0$ and $0 \to L_1 \to F_0 \to M \to 0$ are $w$-exact sequences.
By using Lemma 2.3 repeatedly, we can obtain that $\text{Tor}_1^R(F_n, N)$ is GV-torsion for all $\phi$-torsion $R$-modules $N$. Thus $F_n$ is $\phi$-$w$-flat.

$(7) \Rightarrow (8) \Rightarrow (6), (7) \Rightarrow (9)$ and $(9) \Rightarrow (6)$: Trivial. \hfill $\square$

**Definition 2.5.** The $\phi$-$w$-weak global dimension of a strongly $\phi$-ring $R$ is defined by

$$\phi$-$w$-w.gl.dim($R$) = sup\{ $\phi$-$w$-fd$_R$(M) | M is an $R$-module \}.

Obviously, by definition, $\phi$-$w$-w.gl.dim($R$) $\leq$ $w$-w.gl.dim($R$). Notice that if $R$ is an integral domain, then $\phi$-$w$-w.gl.dim($R$) = $w$-w.gl.dim($R$).

**Proposition 2.6.** Let $R$ be a strongly $\phi$-ring. The following statements are equivalent for $R$.

1. $\phi$-$w$-fd$_R$(M) $\leq$ $n$ for all $R$-modules $M$.
2. $\text{Tor}_{n+k}^R(M, N)$ is GV-torsion for all $R$-modules $M$ and $\phi$-torsion $N$ and all $k > 0$.
3. $\text{Tor}_{n+1}^R(M, N)$ is GV-torsion for all $R$-modules $M$ and $\phi$-torsion $N$.
4. $\text{Tor}_{n+1}^R(M, R/I)$ is GV-torsion for all $R$-modules $M$ and nonnil ideals $I$ of $R$.
5. $\text{Tor}_{n+1}^R(M, R/I)$ is GV-torsion for all $R$-modules $M$ and finite type nonnil ideals $I$ of $R$.
6. $\phi$-$w$-fd$_R$(R/I) $\leq$ $n$ for all nonnil ideals $I$ of $R$.
7. $\phi$-$w$-fd$_R$(R/I) $\leq$ $n$ for all finite type nonnil ideals $I$ of $R$.
8. $\phi$-$w$-w.gl.dim($R$) $\leq$ $n$.

Consequently, the $\phi$-$w$-weak global dimension of a strongly $\phi$-ring $R$ is determined by the formulas:

$$\phi$-$w$-w.gl.dim($R$) = sup\{ $w$-fd$_R$(R/I) | I is a nonnil ideal of $R$ \} = sup\{ $w$-fd$_R$(R/I) | I is a finite type nonnil ideal of $R$ \}.

**Proof.** (1) $\Leftrightarrow$ (8) and (1) $\Rightarrow$ (6) $\Rightarrow$ (7) $\Rightarrow$ (8): Trivial.

(1) $\Rightarrow$ (2) and (5) $\Rightarrow$ (1): Follows from Proposition 2.4.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5): Trivial.

(8) $\Rightarrow$ (1): Let $M$ be an $R$-module and $0 \rightarrow F_n \rightarrow ... \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$ an exact sequence, where $F_0, F_1, ..., F_{n-1}$ are flat $R$-modules. To complete the proof, it suffices, by Proposition 2.4, to prove that $F_n$ is $\phi$-$w$-flat. Let $I$ be a finite type nonnil ideal of $R$. Thus $\phi$-$w$-fd$_R$(R/I) $\leq$ $n$ by (8). It follows from Lemma 2.3 that $\text{Tor}_1^R(R/I, F_n) \cong \text{Tor}_{n+1}^R(R/I, M)$ is GV-torsion. \hfill $\square$
3. Rings with \(\phi\)-w-weak global dimension at most one

It is well known that a commutative ring \(R\) with weak global dimension 0 is exactly a von Neumann regular ring, equivalently \(a \in (a^2)\) for any \(a \in R\). It was proved in [13, Theorem 4.4] that a commutative ring \(R\) has \(w\)-weak global dimension 0, if and only if \(a \in (a^2)_w\) for any \(a \in R\), if and only if \(R_m\) is a field for any maximal \(w\)-ideal \(m\) of \(R\), if and only if \(R\) is a von Neumann regular ring. Recall from [22] that a \(\phi\)-ring \(R\) is said to be \(\phi\)-von Neumann regular provided that every \(R\)-module is \(\phi\)-flat. A \(\phi\)-ring \(R\) is \(\phi\)-von Neumann regular, if and only if there is an element \(x \in R\) such that \(a = xa^2\) for any non-nilpotent element \(a \in R\), if and only if \(R/\text{Nil}(R)\) is a von Neumann regular ring, if and only if \(R\) is zero-dimensional (see [22, Theorem 4.1]).

Now, we give some more characterizations of \(\phi\)-von Neumann regular rings.

**Theorem 3.1.** Let \(R\) be a strongly \(\phi\)-ring. The following statements are equivalent for \(R\):

1. \(\phi\)-w.-w.gl.dim\((R) = 0\);
2. every \(R\)-module is \(\phi\)-w-flat;
3. \(a \in (a^2)_w\) for any non-nilpotent element \(a \in R\);
4. \(w\)-dim\((R) = 0\);
5. \(\text{dim}(R) = 0\);
6. \(R\) is \(\phi\)-von Neumann regular.

**Proof.** (1) \(\Leftrightarrow\) (2) By definition.

(2) \(\Rightarrow\) (3): Let \(a\) be a non-nilpotent element in \(R\). Then \(Ra\) is a nonnil ideal of \(R\). It follows that \(\text{Tor}_1^R(R/Ra, R/Ra)\) is GV-torsion since \(R/Ra\) is \(\phi\)-torsion and \(\phi\)-w-flat. That is, \(Ra/Ra^2\) is GV-torsion, and thus \(a \in Ra \subseteq (Ra)_w = (Ra^2)_w\).

(3) \(\Rightarrow\) (4): Since \(R\) is a \(\phi\)-ring, \(\text{Nil}(R)\) is the minimal prime \(w\)-ideal of \(R\). We claim that the ring \(\overline{R}_m := (R/\text{Nil}(R))_{m/\text{Nil}(R)}\) is a field for any \(m \in w\text{-Max}(R)\). Indeed, let \(a\) be a non-nilpotent element in \(R\). By (3), \((a)_w = (a^2)_w\). Thus \((a)_m = (a^2)_m\). We have \((a)_m = (a^2)_m\) as an ideal of \(\overline{R}_m\). So \(\overline{R}_m\) is a local von Neumann regular ring, and thus a field. Note that \(\overline{R}_m = R_m/\text{Nil}(R_m)\). It follows that \(R_m\) is 0-dimensional (see [13, Theorem 3.1]). Thus \(w\)-dim\((R) = 0\).

(4) \(\Rightarrow\) (1): By Theorem 1.4 we just need to show \(\text{Tor}_1^R(R/I, R/J)\) is GV-torsion for all nonnil ideals \(I\) and all ideals \(J\) of \(R\). Since \(R\) is a \(\phi\)-ring with \(w\)-dim\((R) = 0\), \(\text{Nil}(R)\) is the unique maximal \(w\)-ideal of \(R\). We just need to show \(\text{Tor}_1^R(R/I, R/J)_{\text{Nil}(R)} = 0\). That is, \((I \cap J/IJ)_{\text{Nil}(R)} = 0\).

If \(J\) is a nonnil ideal of \(R\), there is non-nilpotent elements \(s \in I\) and \(t \in J\) such that \(st \in IJ\). Since \(st \notin \text{Nil}(R)\), \((I \cap J/IJ)_{\text{Nil}(R)} = 0\). If \(J\) is a nilpotent ideal of \(R\),
I \cap J = J. Thus \( \text{Tor}_1^R(R/I, R/J)^{\text{Nil}(R)} = (I \cap J/IJ)^{\text{Nil}(R)} = (J/IJ)^{\text{Nil}(R)}. \) Let \( s \) be a non-nilpotent element in \( I \). We have \( s(j + IJ) = 0 + (IJ) \) in \( J/IJ \) for any \( j \in J \). Thus \( (I \cap J/IJ)^{\text{Nil}(R)} = 0. \)

(4) \( \Rightarrow \) (5): By (4), \( \text{Nil}(R) \) is the unique \( w \)-maximal ideal of \( R \). If \( \text{Nil}(R) \) is a maximal ideal of \( R \), (6) holds obviously. Otherwise, there is a non-unit element \( a \) which is not nilpotent. Since \( (a) \) is not a GV-ideal, there is maximal \( w \)-ideal \( m \) such that \( \text{Nil}(R) \subset (a) \subset (a)_w \subset m \), Thus \( w \)-dim\((R) \geq 1 \), which is a contradiction.

(5) \( \Rightarrow \) (4): Trivial.

(5) \( \Leftrightarrow \) (6): See [22] Theorem 4.1. \( \square \)

Recall from [9] that a ring \( R \) is said to be a Prüfer ring provided that every finitely generated regular ideal \( I \) is invertible, i.e., \( II^{-1} = R \) where \( I^{-1} = \{ x \in T(R) | IX \subseteq R \} \), or equivalently, there is a fractional ideal \( J \) of \( R \) such that \( IJ = R \). It is well known that an integral domain is a Prüfer domain if and only if the weak global dimension of \( R \leq 1 \). Recall that a ring \( R \) is said to be a PvMR if every finitely generated regular ideal \( I \) is \( w \)-invertible, i.e., \( (II^{-1})_w = R \), or equivalently, there is a fractional ideal \( J \) of \( R \) such that \( (IJ)_w = R \). PvMDs are exactly integral domains which are PvMRs. It is known that an integral domain \( R \) is a PvMD if and only if \( R_m \) is a valuation domain for each \( m \in w\text{-Max}(R) \) if and only if \( w \text{-gl.dim}(R) \leq 1 \) (see [13] [17]).

Following [4], a \( \phi \)-ring \( R \) is said to be a \( \phi \)-chain ring (\( \phi \)-CR for short) if for any \( a, b \in R - \text{Nil}(R) \), either \( a|b \) or \( b|a \) in \( R \). A \( \phi \)-ring \( R \) is said to be a \( \phi \text{-Prüfer ring} \) if every finitely generated nonnil ideal \( I \) is \( \phi \)-invertible, i.e., \( \phi(I)\phi(I^{-1}) = \phi(R) \). It follows from [11] Corollary 2.10] that a \( \phi \)-ring \( R \) is \( \phi \)-Prüfer, if and only if \( R_m \) is a \( \phi \)-CR for any maximal ideal \( m \) of \( R \), if and only if \( R/\text{Nil}(R) \) is a Prüfer domain, if and only if \( \phi(R) \) is Prüfer. For a strongly \( \phi \)-ring \( R \), Zhao [21] Theorem 4.3] showed that \( R \) is a \( \phi \)-Prüfer ring if and only if all \( \phi \)-torsion free \( R \)-modules are \( \phi \)-flat, if and only if each submodule of a \( \phi \)-flat \( R \)-module is \( \phi \)-flat, if and only if each nonnil ideal of \( R \) is \( \phi \)-flat.

Let \( R \) be a \( \phi \)-ring. Recall from [9] that a nonnil ideal \( J \) of \( R \) is said to be a \( \phi \text{-GV-ideal} \) (resp., \( \phi \text{-w-ideal} \)) of \( R \) if \( \phi(J) \) is a GV-ideal (resp., \( w \)-ideal) of \( \phi(R) \). A \( \phi \)-ring \( R \) is called a \( \phi \text{-SM ring} \) if it satisfies the ACC on \( \phi \text{-w-ideals} \). An ideal \( I \) of \( R \) is \( \phi \text{-w-invertible} \) if \( (\phi(I)\phi(I^{-1}))_W = \phi(R) \) where \( W \) is the \( w \)-operation of \( \phi(R) \). A \( \phi \)-ring is \( \phi \text{-Krull} \) provided that any nonnil ideal is \( \phi \text{-w-invertible} \) (see [9] Theorem 2.23]). By extending \( \phi \)-Krull rings and PvMDs, we give the definition of \( \phi \)-Prüfer \( v \)-multiplication rings.
Definition 3.2. Let \( R \) be a \( \phi \)-ring. \( R \) is said to be a \( \phi \)-Prüfer \( v \)-multiplication ring (\( \phi \)-PvMR for short) provided that any finitely generated nonnil ideal is \( \phi \)-\( w \)-invertible.

Now we characterize \( \phi \)-Prüfer multiplication rings in terms of \( \phi \)-\( w \)-flat modules.

Theorem 3.3. Let \( R \) be a \( \phi \)-ring. The following statements are equivalent for \( R \):

1. \( R \) is a \( \phi \)-PvMR;
2. \( R_m \) is a \( \phi \)-CR for any \( m \in w\text{-}\text{Max}(R) \);
3. \( R/\text{Nil}(R) \) is a PvMD;
4. \( \phi(R) \) is a PvMR.

Moreover, if \( R \) is a strongly \( \phi \)-ring, all above are equivalent to

5. \( R \) is a \( \phi \)-\( w \)-\( w \text{-gl.dim}(R) \) \( \leq 1 \);
6. every submodule of a \( w \)-flat module is \( \phi \)-\( w \)-flat;
7. every submodule of a flat module is \( \phi \)-\( w \)-flat;
8. every ideal of \( R \) is \( \phi \)-\( w \)-flat;
9. every nonnil ideal of \( R \) is \( \phi \)-\( w \)-flat;
10. every finite type nonnil ideal of \( R \) is \( \phi \)-\( w \)-flat.

Proof. Let \( R \) be a \( \phi \)-ring. Denote by \( W, w \) and \( \overline{w} \) the \( w \)-operations of \( \phi(R) \), \( R \) and \( R/\text{Nil}(R) \) respectively. We will prove the equivalences of (1) – (4) and (5) – (10).

(1) \( \Rightarrow \) (4): Let \( K \) be a finitely generated regular ideal of \( \phi(R) \). Then \( K = \phi(I) \) for some finitely generated nonnil ideal \( I \) of \( R \) by [1, Lemma 2.1]. Since \( R \) is a \( \phi \)-PvMR, \( (KK^{-1})_w = (\phi(I)\phi(I)^{-1})_w = \phi(R) \). Thus \( \phi(R) \) is a PvMR.

(4) \( \Rightarrow \) (1): Let \( I \) be a finitely generated nonnil ideal of \( R \). We will show \( I \) is \( \phi \)-\( w \)-invertible. By [1, Lemma 2.1], \( \phi(I) \) is a finitely generated regular ideal of \( \phi(R) \). Thus \( (\phi(I)\phi(I)^{-1})_w = \phi(R) \) since \( \phi(R) \) is a PvMR.

(2) \( \Leftrightarrow \) (3): By [1, Theorem 3.7, Corollary 2.10], \( R_m \) is a \( \phi \)-CR for any \( m \in w\text{-}\text{Max}(R) \) if and only if \( R_m/\text{Nil}(R_m) = (R/\text{Nil}(R))_m \) is a valuation domain for any \( m \in w\text{-}\text{Max}(R) \) if and only if \( R/\text{Nil}(R) \) is a PvMD (see [13, Theorem 4.9]).

(3) \( \Rightarrow \) (4): Note that \( \phi(R)/\text{Nil}(\phi(R)) \cong R/\text{Nil}(R) \) is a PvMD (see [1, Lemma 2.4]). Let \( \phi(I) \) be a finitely generated regular ideal of \( \phi(R) \). Then, by [1, Lemma 2.1], \( I \) is a nonnil ideal of \( R \). Then \( I = I/\text{Nil}(R) \) is \( w \)-invertible over \( \overline{R} = R/\text{Nil}(R) \) by (3). That is, \( (I/I^{-1})_w = \overline{R} \). There is a GV ideal \( J \) of \( \overline{R} \) such \( J \subset T^{-1} \) (see [13, Exercise 6.10(2)]). So \( J \subset II^{-1} \) where \( J \) is a \( \phi \)-GV ideal of \( R \) by [9, Lemma 2.3]. Thus \( \phi(J) \subset \phi(I)\phi(I)^{-1} \). Since \( \phi(J) \in \text{GV}(\phi(R)) \), \( (\phi(I)\phi(I)^{-1})_w = \phi(R) \).

(4) \( \Rightarrow \) (3): Suppose \( \phi(R) \) is a PvMR. Let \( \overline{I} \) be a finitely generated nonzero ideal of \( \overline{R} \). Then \( I \) is a nonnil ideal of \( R \). Thus \( \phi(I) \) is a finitely generated regular ideal of
\[ \phi(R) \text{ by [1, Lemma 2.1]. So } (\phi(I)\phi(I)^{-1})_w = \phi(R) \text{ by (4). Hence } J \subseteq II^{-1} \text{ in } R \text{ for some } \phi-GV \text{ ideal } J \text{ of } R \text{ and thus } \mathcal{J} \subseteq \mathcal{II}^{-1} \text{ in } \mathcal{R}. \] By [9, Lemma 2.3], \( \mathcal{J} \in GV(R) \), and thus \( (\mathcal{I}^{-1})_{\mathfrak{m}} = \mathcal{R} \). So \( R/\text{Nil}(R) \) is a PvMD.

(5) \( \Rightarrow \) (6): Let \( K \) be a submodule of a \( w \)-flat module \( F \). Then \( \phi-w\text{-fd}_R(F/K) \leq 1 \) by (5). Thus \( K \) is \( \phi-w \)-flat by Proposition 2.4.

(6) \( \Rightarrow \) (7) \( \Rightarrow \) (9) \( \Rightarrow \) (10): Trivial.

(10) \( \Rightarrow \) (5): Let \( I \) be a finite type nonnil ideal of \( R \). Then \( \phi-w\text{-fd}_R(R/I) \leq 1 \) by Proposition 2.4. It follows from Proposition 2.6 that \( \phi-w\text{-gl.dim}(R) \leq 1 \).

Now, let \( R \) be a strongly \( \phi \)-ring.

(2) \( \Rightarrow \) (9): Let \( \mathfrak{m} \) be a maximal \( w \)-ideal of \( R \) and \( I \) a nonnil ideal of \( R \). Then \( I_{\mathfrak{m}} \) is a nonnil ideal of \( R_{\mathfrak{m}} \) by Lemma 1.1 and thus is \( \phi \)-flat by [21, Theorem 4.3]. So \( I \) is \( \phi-w \)-flat by Theorem 1.4.

(9) \( \Rightarrow \) (2): Let \( \mathfrak{m} \) be a maximal \( w \)-ideal of \( R \), \( I_{\mathfrak{m}} \) a nonnil ideal of \( R_{\mathfrak{m}} \). Then \( I \) is a nonnil ideal of \( R \) by Lemma 1.1. By (9), \( I \) is \( \phi-w \)-flat and so \( I_{\mathfrak{m}} \) is \( \phi \)-flat by Theorem 1.4. Thus \( R_{\mathfrak{m}} \) is a \( \phi \)-CR by [21, Theorem 4.3].

**Corollary 3.4.** Suppose \( R \) is a \( \phi \)-ring. Then \( R \) is a \( \phi \)-Krull ring if and only if \( R \) is both a \( \phi \)-PvMR and a \( \phi \)-SM ring.

**Proof.** By [9, Theorem 2.4] a \( \phi \)-ring \( R \) is a \( \phi \)-SM ring if and only if \( R/\text{Nil}(R) \) is an SM domain. A \( \phi \)-ring \( R \) is a \( \phi \)-Krull ring if and only if \( R/\text{Nil}(R) \) is a Krull domain (see [2, Theorem 3.1]). Since \( R \) is a Krull domain if and only if \( R \) is an SM PvMD (see [8, Theorem 7.9.3]), the equivalence holds by Theorem 3.3.

**Corollary 3.5.** Suppose \( R \) is a strongly \( \phi \)-ring. Then \( R \) is a \( \phi \)-PvMR if and only if \( R \) is a PvMR.

**Proof.** Suppose \( R \) is a \( \phi \)-PvMR and let \( I \) be a finitely generated regular ideal of \( R \). Then \( \mathcal{I} \) is a finitely generated regular ideal of \( \mathcal{R} \). By Theorem 3.3, \( \mathcal{R} \) is a PvMD. Then \( (\mathcal{I}^{-1})_{\mathfrak{m}} = \mathcal{R} \). Thus there is a GV-ideal \( \mathcal{J} \) of \( \mathcal{R} \) with \( J \) finitely generated over \( R \) such that \( \mathcal{J} \subseteq \mathcal{II}^{-1} \). Since \( R \) is a strongly \( \phi \)-ring, \( J \) is a GV-ideal of \( R \) by [9, Lemma 2.11]. Since \( J \subseteq II^{-1} \) in \( R \), \( (II^{-1})_w = R \). Assume \( R \) is a PvMR. Since \( R \) is a strongly \( \phi \)-ring, \( \phi(R) = R \) is a PvMR. Thus \( R \) is a \( \phi \)-PvMR by Theorem 3.3.

The condition that \( R \) is a strongly \( \phi \)-ring in Corollary 3.5 can’t be removed by the following example.

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
Example 3.6. Let $D$ be an integral domain which is not a PvMD and $K$ its quotient field. Since $K/D$ is a divisible $D$-module, the ring $R = D(+K/D$ is a $\phi$-ring but not a strongly $\phi$-ring (see [2, Remark 1]). Since $\text{Nil}(R) = 0(+K/D$, we have $R/\text{Nil}(R) \cong D$ is not a PvMD. Thus $R$ is not a $\phi$-PvMR by Theorem 3.3. Denote by $U(R)$ and $U(D)$ the sets of unit elements of $R$ and $D$ respectively. Since $\mathcal{Z}(R) = \{(r,m)|r \in Z(D) \cup Z(K/D)\} = R - U(D)(+K/D = R - U(R)$, $R$ is a PvMR obviously.

References

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