Multiparticle channel assemblages

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Motivated by the recent studies on post-quantum steering, we generalize notion of bipartite channel steering by introducing the concept of multipartite no-signaling channel assemblages. We show that beyond the bipartite case, no-signaling and quantum description of such scenarios do not coincide. With a help of Choi-Jamiolkowski isomorphism we present full description of considered classes of assemblages and in particular, we use this characterization to provide sufficient conditions for extremality of quantum channel assemblages in the set of all no-signaling channel assemblages. Finally, in the tripartite case, we introduce and discuss a relaxed version of channel steering where only certain subsystems obey no-signaling constrains. In this asymmetric scenario we are able to provide exactly 1 bit of key that is secure against no-signaling eavesdropper.

Introduction.- Quantum mechanics provides a fundamental framework governing physical processes at the microscopic level. The central idea of entanglement presented within this framework, enable us to observe and utilize phenomena contradicting our macroscopic intuitions [14] - starting from presence of correlations stronger that classically predicted [4] and finishing at possibility of steering local states of a distant parties [18,35].

In order to better understand the quantum paradigm from the informational perspective and relate it to the no-signaling principle of special relativity, it is often convenient to consider some generalized post-quantum theories. In particular, this approach gave a birth to considerations regarding steering scenarios with less restricted constrains [20,29–31] and analysis of underlying convex structures with potential cryptographic applications [3,23,24,26].

In this paper, motivated by the recent interest in notions of multipartite and post-quantum steering, we generalized idea of (bipartite) channel steering [23]. We start by recalling basic definitions of no-signaling and quantum assemblages. Next, based on the above description, we introduce a notions of no-signaling, quantum and local channel assemblages and provide their characterization via related Choi matrices. With this background we discuss problem of quantum realization of extremality and related issue of security against eavesdropper holding post-quantum resources. Finally, we also introduce and discuss slightly different paradigm of post-quantum channel steering base on the idea of no-signaling conditions restricting only some parties.

Multiparticle steering and no-signaling assemblages.- Since its heuristic conception [18] and modern reformulation [33] the idea of steering scenario become more and more important quantum phenomenon emphasizing fundamentally non-classical character of nature [7–11,24,34]. Recently, the notion of steering has been expanded to include also situations described only by no-signaling principles [20,24,31] and consider theories of post-quantum resources [10].

Let us consider a typical multipartite scenario of \( n + 1 \) separated parties, where quantum subsystem \( C \) is fully trusted (i.e. described by a known Hilbert space \( H_C \) under full operational control) and remaining quantum subsystems \( A_1, \ldots, A_n \) are untrusted. Assume that such system is described by some joint quantum state and each untrusted party perform local measurements on respective subsystem choosing at random out of \( m_i \) measurement settings \( x_i = 0, \ldots, m_i - 1 \) and obtaining one of \( k_i \) outcomes \( a_i = 0, \ldots, k_i - 1 \). This scenario may be then fully described by the triple \( n, m, k \) where \( m = (m_1, \ldots, m_n), k = (k_1, \ldots, k_n) \) and dimension \( d_C \) of \( H_C \). In case like that, possible subnormalized states describing subsystem \( C \) conditioned upon choice of measurements \( x_n = (x_1, \ldots, x_n) \) and outcomes \( a_n = (a_1, \ldots, a_n) \) form a quantum assemblage \( \Sigma = \{ \sigma_{a_n|x_n} \}_{a_n, x_n} \) consisting of positive operators (acting on a trusted subsystem \( C \)) for which there exist a quantum state \( \rho_{A_1 \ldots A_n,C} \) of some composed system \( A_1 \ldots A_n C \) and elements of local POVMs \( M_{a_1|x_1}^{(A_1)}, \ldots, M_{a_n|x_n}^{(A_n)} \) such that \( \sigma_{a_n|x_n} = \text{Tr}_{A_1 \ldots A_n} \left( M_{a_1|x_1}^{(A_1)} \otimes \cdots \otimes M_{a_n|x_n}^{(A_n)} \otimes I_C \rho_{A_1 \ldots A_n,C} \right) \).

Among assemblages with quantum description we may distinguish specific subclass of local assemblages (assemblages with local hidden states model [20]) given by those assemblages admitting \( \sigma_{a_n|x_n} = \sum_j q_j p_j^{(i)}(a_1|x_1) \cdots p_j^{(n)}(a_n|x_n) \sigma_j \) where \( q_j \geq 0, \sum_j q_j = 1, \sigma_j \) are some states of trusted subsystem \( C \) and \( \{ p_j^{(i)}(a_i|x_i) \}_{a_i, x_i} \) denotes conditional probability distributions for untrusted subsystem \( A_i \) respectively.
Finally, one can discard the restriction of quantum description of the composed system, demanding that only trusted subsystem obey quantum mechanical characterization, while the joint system $A_1, \ldots, A_n C$ is governed by some possibly post-quantum theory (see for example [4]) equipped only with fundamental physical constrains of no-signaling. Probabilistic description of steering experiment in this generalized approach is given by the mathematical structure of no-signaling assemblage $\{\sigma_{a_n|x_n}\}_{a_n,x_n}$ of subnormalised states acting on $d_C$-dimensional Hilbert space of trusted subsystem $C$, such that

$$\forall_{x_n} \sum_{a_n} \sigma_{a_n|x_n} = \sigma, \quad (1)$$

where $\sigma$ is some state, and for any set of indexes $I = \{i_1, \ldots, i_s\} \subset \{1, \ldots, n\}$ with $1 \leq s < n$ there exist an operator $\sigma_{a_{i_1}\ldots a_{i_s}|x_{i_1}\ldots x_{i_s}}$ that fulfills

$$\forall_{a_k,k \in I} \forall_{x_n} \sum_{a_j \not\in I} \sigma_{a_n|x_n} = \sigma_{a_{i_1}\ldots a_{i_s}|x_{i_1}\ldots x_{i_s}} \quad (2)$$

It is easy to see that distinction between quantum and local assemblages certifies existence of quantum entanglement in composed systems. On the other hand, sets of no-signaling and quantum assemblages coincide for $n = 1$ [11, 22] and start to differ if $n > 1$ [23, 30], i.e. there is a possibility of post-quantum steering.  

Within that framework question regarding possibility of quantum realization of non-local yet extreme points in the set of all no-signaling assemblages become a non-trivial one. This question present important practical challenge. Namely, in a case when adversary attack of eavesdropper is modeled by possible convex decomposition of assemblage playing a role of resource shared by parties performing some cryptographic task, an affirmative answer to the above question certifies security (against post-quantum resources) of the protocol based on quantum description that is non-local.

The analogous question related to sets of correlations (respectively no-signaling, quantum and local) is known to have a negative answer, regardless of the number of parties, measurements and outcomes [27]. However, as it has been recently shown [24], the evoked no-go result is no longer true in the case of no-signaling assemblages, even in the simplest nontrivial case.

**Multipartite channel steering.** Starting from the intuition laying behind the notion of post-quantum steering it is natural to consider also a generalized scenario of channel steering introduced for bipartite case in [23] and discussed from resource theory perspective in [28, 33]. Indeed, let us firstly once more consider $n + 1$ quantum subsystem $A_1, \ldots, A_n$ and $C$ where only the last subsystem is fully trusted. Subsystem $C$ initially interacts with all subsystems $A_1, \ldots, A_n$ while such interaction is modeled by an action of some quantum channel (i.e. completely positive and trace preserving map). In next step all subsystems become separated and final description of characterized subsystem $C$ is conditioned upon random quantum measurements performed locally by parties at each subsystem $A_i$ with $x_i$ denoting measurements settings and $a_i$ denoting measurements outcomes respectively. Probabilistic description of evolution of subsystem $C$ (with respect to vectors of labels $x_n = (x_1, \ldots, x_n)$, $a_n = (a_1, \ldots, a_n)$) is in that case encapsulated by the notion of quantum channel assemblage.

**Definition 1.** Family $\mathcal{L} = \{A_{a_n|x_n}\}_{a_n,x_n}$ consisting of completely positive maps $A_{a_n|x_n} : B(H_C) \to B(H_C)$ defines a quantum channel assemblage if for any state $\rho_C \in B(H_C)$

$$A_{a_n|x_n}(\rho_C) = \text{Tr}_{A_1, \ldots, A_n}(M^{(i)}_{a_{i_1}|x_{i_1}} \otimes \ldots \otimes M^{(n)}_{a_n|x_n} \otimes 1_C (E(\rho_{A_1, \ldots, A_n} \otimes \rho_C))), \quad (3)$$

where $\rho_{A_1, \ldots, A_n} \in \otimes^i B(H_{A_i})$ is some state, $M^{(i)}_{a_{i_1}|x_{i_1}}$ are some POVM elements acting on subsystem $A_i$, and $E : \otimes^i B(H_{A_i}) \otimes B(H_C) \to \otimes^i B(H_{A_i}) \otimes B(H_C)$ is some completely positive and trace preserving map (i.e. channel).

In analogy with assemblages of states, one can in principle relax the requirement of a quantum description of untrusted subsystems and demand a channel steering scenario with only certain constrains of no-signaling type, when (trusted) quantum subsystem evolves under interaction and later measurements performed on some joint state described by some possibly post-quantum theory. This lead to the natural notion of no-signaling channel assemblages.

**Definition 2.** Family $\mathcal{L} = \{A_{a_n|x_n}\}_{a_n,x_n}$ consisting of completely positive maps $A_{a_n|x_n} : B(H_C) \to B(H_C)$ defines a no-signaling channel assemblage if

$$\forall_{x_n} \sum_{a_n} A_{a_n|x_n} = \Lambda, \quad (4)$$

where $\Lambda$ is some completely positive and trace preserving map, and for any subset of indexes $I = \{i_1, \ldots, i_s\} \subset \{1, \ldots, n\}$ with $1 \leq s < n$ there exists a map
\[ A_{i_1 \ldots i_n} | x_{i_1} \ldots x_{i_n} \ldots x_{i_n} \ldots = A_{i_1 \ldots i_n} | x_{i_1} \ldots x_{i_n} \ldots x_{i_n} \ldots, \] (5)

On the other hand, restriction of quantum description only to the classical cases leads to the definition of local channel assemblages (generalizing concept introduced in [23]).

**Definition 3.** Family \( \mathcal{L} = \{ A_{n,|x_n|} \}_{n,|x_n|} \) consisting of completely positive maps \( A_{n,|x_n|} : B(H_C) \to B(H_C) \) defines a local channel assemblage if and only if \( A_{n,|x_n|} = \sum_j \prod_i^n p_j^{(i)}(a_i | x_i) A_j \), where \( \{ p_j^{(i)}(a_i | x_i) \}_{a_i, x_i} \) stand for conditional probabilities related to subsystems \( A_i \), and \( A_j \) are completely positive maps such that \( A = \sum_j A_j \) is a channel.

For a particular, fixed scenario \( n, m, k \) with \( d_C, d_C' \), we will denote set of no-signaling channel assemblages by \( \mathbf{nsL}(n, m, k, d_C, d_C') \), set of quantum channel assemblages by \( \mathbf{QL}(n, m, k, d_C, d_C') \) and set of local channel assemblages \( \mathbf{LA}(n, m, k, d_C, d_C') \) (we omit \( d_C \) if \( d_C = d_C' \)).

We will show that the difference between quantum and no-signaling (or local) description of multipartite channel steering can be seen as a result of the difference between sets of quantum and no-signaling (or local) assemblages of Choi matrices [12–19] related to maps forming channel assemblages.

To characterize channels assemblages we recall the well-known notion of a Choi-Jamiolkowski isomorphism \( J : B(B(A_n)) \to B(H_B) \otimes B(H_A) \) given as \( J(A) = A \otimes \text{id}_A(\hat{\phi}^+_A | \hat{\phi}^+_A) \) where \( \hat{\phi}^+_A \) is a maximally entangled state.

With this one can show the following result (see Appendix A for detailed calculations).

**Theorem 4.** Family of linear maps \( \mathcal{L} = \{ A_{n,|x_n|} \}_{n,|x_n|} \) given by \( A_{n,|x_n|} : B(H_C) \to B(H_C) \) defines a no-signaling (respectively quantum and local) channel assemblage if and only if there exist a Hermitian operator \( W \in \bigotimes_{i=1}^n B(H_{A_i}) \otimes B(H_C) \) such that \( \text{Tr}_{A_1, \ldots, A_n, C}(W) = \frac{1}{d_C} \in B(H_C) \).

In particular, above theorems provide that all considered sets of channel assemblages are indeed convex (as linear images of convex sets). Moreover, the family of sets \( \mathbf{nsL}(n, m, k, d_C, d_C') \) and \( \mathbf{LA}(n, m, k, d_C, d_C') \) are compact (as continuous images of compact sets).

Let us emphasize that characterization of assemblages provided by causal, localizable and local channels (see [20]) together with presented theorems leads to yet another description of channel assemblages.

**Theorem 5.** Family \( \mathcal{L} = \{ A_{n,|x_n|} \}_{n,|x_n|} \) of completely positive maps \( A_{n,|x_n|} : B(H_C) \to B(H_C) \) defines a no-signaling (respectively quantum and local) channel assemblage of channel is defined by the set of Choi matrices \( \{ a \} \) such that \( A_{n,|x_n|} = J^{-1}(\text{Tr}_{A_1, \ldots, A_n, C}((\bigotimes_{i=1}^n |a_i\rangle \langle a_i|_{H_{A_i}}(\bigotimes_{i=1}^n x_i) \otimes 0_C)\langle 0_C|)) \) and \( \text{Tr}_{A_1, \ldots, A_n, C}(\Phi(\bigotimes_{i=1}^n |x_i\rangle \langle x_i| \otimes 0_C)\langle 0_C|) = \frac{1}{d_C} \).

Note that taking one-dimensional output Hilbert space (i.e. \( H_C = C \)) in the case of Definitions 1–2 and 3 one obtains sets consisting of families of maps with outputs in a form of conditional probabilities. To distinguish that case we will say about channel correlations and we will introduce a separate notation \( \mathcal{P} = \{ P_{n,|x_n|} \}_{n,|x_n|} \). Starting from a recent idea of intermediate set of correlations (i.e. hybrid no-signaling-quantum correlations [2]) one can provide further refinement. Indeed, \( \mathcal{P} = \{ P_{n,|x_n|} \}_{n,|x_n|} \) is of a hybrid form if there exist a no-signaling channel assemblage \( \mathcal{L} = \{ A_{n,|x_n|} \}_{n,|x_n|} \) (with arbitrary output dimension) and measurements \( M_{n,|x_n|} \) such that \( P_{n,|x_n|} = \text{Tr}(M_{n,|x_n|}) \).

**Quantum realization of extreme points.**– Choi-Jamiolkowski isomorphism enable us to identify appropriate classes of channel assemblages with appropriate subclasses of assemblages (acting on a higher dimensional Hilbert spaces), so not all no-signaling channel assemblages admit quantum realization, as long as a given scenario have at least two untrusted subsystems (i.e. \( n > 1 \)). In particular, existence of non-local and quantum extreme points within the set of no-signaling channels assemblages is a nontrivial question that can be resolve by the recent solution for assemblages of states [26].

For the sake of completeness we will recall [20] that assemblage of pure states \( \mathcal{L} = \{ A_{n,|x_n|} \} \) is called if any other assemblages of the same pure
states $\Sigma' = \{q_{a_i}|x_n|\psi_{a_i}|x_n\rangle\langle\psi_{a_i}|x_n|\}_{a_i,x_n}$ such that $q_{a_i}|x_n| = 0$ whenever $p_{a_i}|x_n| = 0$ satisfies $\Sigma' = \Sigma$. It can be seen that inflexible assemblages are extremal points of the set of all no-signaling assemblages and there is a sufficient condition for inflexibility [20].

**Theorem 6 [24].** Let $\Sigma \in \text{nsA}(2,2,2,d_C)$ be an assemblage of pure states. If there exists $y_1$ and $y_2$ such that each of the following sets $\{\sigma_{a_1\,|y_1\,x_2\rangle\langle\sigma_{a_2\,|y_2\,x_1\rangle}\}_{a_2,x_1}$, $\{\sigma_{a_1\,|y_1\,x_2\rangle\langle\sigma_{a_2\,|y_2\,x_1\rangle}\}_{a_2,x_1}$ consists of different rank one operator, then $\Sigma$ is inflexible and not local.

Based on the above theorem we may provide an example of quantum (yet non-local) channel assemblage that is an extreme point.

**Example 7.** Let consider $\mathcal{L} = \{\Lambda_{a_{b_{xy}}}\}_{a_{b_{x,y}}}$ where $\Lambda_{a_{b_{xy}}} : B(H_C) \rightarrow B(H_C)$ admit quantum realization $\Lambda_{a_{b_{xy}}} = \text{Tr}_{AB}(P_{a_{xy}} \otimes Q_{b_{xy}} \otimes \mathcal{I}_C(\delta_{a,b} \otimes \mathcal{E}(\phi_{AB}^{+} \otimes \cdot))$ with two-dimensional Hilbert spaces $H_A, H_B, H_C$, maximally entangled state $|\phi_{AB}^{+}\rangle = |0\rangle_0 \otimes |1\rangle_1$, and $F_{a_{b_{xy}}} = Q_{b_{xy}} = Q_{b_{xy}} = \mathcal{I}_{a_{b}}$. Using Choi–Jamiołkowski isomorphism we obtain $\sigma_{a_{b_{xy}}} = \mathcal{J}(\Lambda_{a_{b_{xy}}} = \text{Tr}_{AB}(P_{a_{xy}} \otimes Q_{b_{xy}} \otimes \mathcal{I}_C(\phi_{AB}^{+} \otimes \cdot))$, where $|\psi_{a_{b_{xy}}} = \frac{1}{2}((00|00 + 01|11 + 10|11 + 11|11))$. Assuming that pairs $a,b$ labels rows and pairs $x,y$ label columns, assemblage of Choi matrices $\Sigma = \{\sigma_{a_{b_{xy}}}\}_{a_{b_{x,y}}}$ is given in a convenient graphical representation.

$$\Sigma = \begin{pmatrix}
\begin{array}{cc}
|0\rangle\langle0| & 0 \\
0 & 2|\phi\rangle\langle\phi|
\end{array}
\end{pmatrix},
\begin{pmatrix}
|\phi\rangle\langle\phi| & |\phi\rangle\langle\phi| & |\phi\rangle\langle\phi| \\
|\phi\rangle\langle\phi| & |\phi\rangle\langle\phi| & |\phi\rangle\langle\phi|
\end{pmatrix}
$$

with $|\phi\rangle = \frac{1}{\sqrt{2}}(|00| + |11|)$, $|\phi\rangle = \frac{1}{\sqrt{2}}(|10| + |01|)$ and $|\phi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, $|\phi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.

The above assemblage is inflexible and non-local (Theorem 3). Due to the previous discussion initial quantum channel assemblage $\mathcal{L}$ is an extreme point of $\text{nsA}(2,2,2,2)$ while $\mathcal{L} \notin \text{IA}(2,2,2,2)$.

**Multiparticle channel steering in an asymmetric scenario.** Notion of multipartite channel steering can be further modified, when we allow for signaling between certain (but not all) subsystems. For the sake of simplicity, we will consider this asymmetric generalization only in the tripartite case, as this configuration captures all interesting features. Namely, it provide operationally bipartite setting, where subsystem $BC$ are treated as single one (with untrusted part $B$ described by possibly post-quantum theory).

Let us consider subsystem $A$ that will be eventually separated form subsystems $B$ and $C$ with only subsystem $C$ being fully trusted and described by the principles of quantum theory. Similarly to the description introduced in the previous sections, assume that subsystem $C$ initially interacts with subsystem $AB$ and after that subsystems $A$ and $B$ become separated. Let both untrusted subsystems $A$ and $B$ provide measurements with (chosen at random) settings $x, y$ and possible outcomes $a, b$ respectively. Probabilistic description of evolution of quantum system $C$ with assumption of no-signaling between $A$ and $B$ is provided by an asymmetric channel assemblage.

**Definition 8.** Family $\mathcal{L} = \{\Lambda_{a_{b_{xy}}}\}_{a_{b_{x,y}}}$ consisting of completely positive maps $\Lambda_{a_{b_{xy}}} : B(H_C) \rightarrow B(H_C)$ defines an asymmetric no-signaling channel assemblage if i) $\forall_{x,x',y,b} \sum_a \Lambda_{a_{b_{xy}}} = \sum_a \Lambda_{a_{b_{x'y}}}$, ii) $\forall_{y',x,a} \sum_b \Lambda_{a_{b_{xy}}} = \sum_b \Lambda_{a_{b_{yx'}}}$, and iii) $\forall_{x,y} \sum_{a,b} \Lambda_{a_{b_{xy}}} = \Lambda$ where $\Lambda$ is some fixed quantum channel (i.e. trace preserving map).

For a particular, fixed scenario $m, k$ with $d_C, d_C$ we will denote set of asymmetric no-signaling channel assemblages by $\text{nsA}(2, m, k, d_C, d_C)$.

By arguments similar to the proof of Theorem 4 we obtain the following characterization.

**Theorem 9.** Family of linear maps $\mathcal{L} = \{\Lambda_{a_{b_{xy}}}\}_{a_{b_{x,y}}}$ given by $\Lambda_{a_{b_{xy}}} : B(H_C) \rightarrow B(H_C)$ defines an asymmetric no-signaling channel assemblage if and only if a family of positive Choi matrices $\Sigma = \{\sigma_{a_{b_{xy}}} = \mathcal{J}(\Lambda_{a_{b_{xy}}})\}_{a_{b_{x,y}}}$ fulfill the following conditions i) $\forall_{x,x',y,b} \sum_a \sigma_{a_{b_{xy}}} = \sum_a \sigma_{a_{b_{x'y}}}$, ii) $\forall_{y',x,a} \sum_b \sigma_{a_{b_{xy}}} = \sum_b \sigma_{a_{b_{yx'}}}$, and iii) $\forall_{x,y} \sum_{a,b} \sigma_{a_{b_{xy}}} = \sigma$ where $\sigma \in B(H_C) \otimes B(H_C)$ is some fixed state such that $\text{Tr}_C(\sigma) = \frac{1}{d_C}$.

Note that under provided assumptions (i.e. Definition 8) not all examples of channel assemblages that are extremal in multipartite paradigm, remain that way in the asymmetric scenario (however, one can still construct such examples - see Appendix B).

Indeed, consider channel assemblage $\mathcal{L} = \{\Lambda_{a_{b_{xy}}}\}_{a_{b_{x,y}}}$ form Example 7 now as an element of $\text{nsA}(2,2,2,2)$. Assume that $\Sigma = \mathcal{J}(\Lambda)$ and that we have a convex decomposition $\Sigma = p\Sigma(1) + (1-p)\Sigma(2)$. Due to conditions i)-iii) in Definition 8 any $\Sigma(i)$ must be of the form

$$\Sigma(i) = \begin{pmatrix}
2\alpha|\phi\rangle\langle\phi| & 0 \\
0 & 2\beta|\phi\rangle\langle\phi|
\end{pmatrix},
\begin{pmatrix}
\eta|\phi\rangle\langle\phi| & \kappa|\phi\rangle\langle\phi| \\
\beta|\phi\rangle\langle\phi| & \epsilon|\phi\rangle\langle\phi|
\end{pmatrix}
$$

where in particular $\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta, \kappa \geq 0$, $2\alpha = \eta + \kappa$, $2\beta = \eta + \kappa$ and $\alpha + \delta = 2$ (one can consider only single coefficients $\eta, \kappa$ for the whole third and fourth column respectively since both columns consist of four different rank one operators). Note that this leads to $\alpha = \delta = 1$. 
Nevertheless, when $\beta = \frac{1}{2}$, $\gamma = \frac{1}{2}$, $\zeta = \frac{1}{2}$, $\epsilon = \frac{1}{2}$ for $\Sigma^{(1)}$ and $\beta = \frac{1}{2}$, $\gamma = \frac{1}{2}$, $\zeta = \frac{1}{2}$, $\epsilon = \frac{1}{2}$ for $\Sigma^{(2)}$ ($\alpha, \delta, \eta, k = 1$) we have $\Sigma = \frac{1}{2} \Sigma^{(1)} + \frac{1}{2} \Sigma^{(2)}$ while $\Sigma \neq \Sigma^{(1)} \neq \Sigma^{(2)}$. In the end, $\Sigma$ in not extremal and so is $\Lambda$.

However, the above reasoning shows that for any $a, b = 0, 1$ we get
\[
A_{ab|00} = A^{(i)}_{ab|00}
\] where $A^{(i)}$ stands for asymmetric channel assembly related to assembly of Choi matrices from considered decomposition.

Considered paradigm is an example of a post-quantum scenario in which quantum realizable elements (channel assemblages) could provide security against no-signaling eavesdropper in certain cryptographic tasks. Moreover, looking form operational perspective, as classical input and output of subsystem $B$ may be treated as a description of subsystem $C$, this scenario can be seen as a practically bipartite experimental setup, where such security is presented.

Indeed, putting initial state as $\rho = |0\rangle\langle 0|$ and given measurements as $M_{0|0} = |0\rangle\langle 0|$, $M_{1|1} = |1\rangle\langle 1|$ we obtain correlations (for fixed $x = 0$, $y = 0, z = 0$)
\[
p(000|000) = \text{Tr}(|0\rangle\langle 0|A_{00|00})(|0\rangle\langle 0|)) = \frac{1}{2}
\]
\[
p(111|000) = \text{Tr}(|1\rangle\langle 1|A_{11|00})(|0\rangle\langle 0|)) = \frac{1}{2}
\] that seen as a bipartite correlations (in a cut $A|BC$) provide a perfect key. Note that with channel assemblage as a resource sharing between $A$ and $BC$ attack of the eavesdropper may be simulated by a family of completely positive maps $\mathcal{L} = \{A_{abc|xyz}\}_{a,b,c,x,y}$ (extension of considered channel assemblage $\mathcal{L}$) such that i) $\forall_{x,y,z} \sum_{a,b} A_{abc|xyz} = \sum_{a} A_{a|xyz}$, ii) $\forall_{y,z} \sum_{a,b} \text{Tr}(A_{abc|xyz}) = \sum_{b} \text{Tr}(A_{abc|xyz})$ and $\forall_{x,y,z} \sum_{a,b} \text{Tr}(A_{abc|xyz}) = \Lambda$ and iv) $\forall_{x} \text{Tr}(A_{abc|xyz}) = \rho_{xyz} A^{(e)}_{xyz}$ for some $\rho_{xyz} \geq 0$ and no-signaling channel assemblage $\mathcal{L}^{(e)} = \{A^{(e)}_{xyz}\}_{a,b,c,x,y}$. In other words such attack is given by a convex combination $A_{abc|xyz} = \sum_{c} \rho_{xyz} A^{(e)}_{abc|xyz}$ where no-signaling channel assemblages $\mathcal{L}^{(e)}$ introduce biased (toward bit 0 or 1) versions of correlations (IX) and (X). However, this is impossible as (X) holds and attack (within proposed paradigm) cannot be successful. Observe that (X) remains valid even with vi) removed in the above description.

Similar observation can be made in a different paradigm, when quantum channel behavior (acting from subsystem $C$) is given, while possible attack is simulated by the convex decomposition into asymmetric hybrid channel behaviors $\mathcal{P}^{(e)} = \{P^{(e)}_{abc|xyz}\}_{a,b,c,x,y,z}$, where $P^{(e)}_{abc|xyz}(\cdot) = \text{Tr} \mathcal{C}(\tilde{M}^{(e)}_{c|z} A^{(e)}_{ab|xy}(\cdot))$ with some asymmetric channel assemblage $\mathcal{L}^{(e)} = \{A^{(e)}_{ab|xy}\}_{a,b,x,y}$ (acting between subsystem $C$ and arbitrary $\tilde{C}$) and some POVM elements $M^{(e)}_{c|z}$ (acting on subsystem $\tilde{C}$).

Indeed, one can show (see calculations in Appendix C) that for $\mathcal{P} = \{P_{abc|xyz}\}_{a,b,c,x,y,z}$ with $P_{abc|xyz}(\cdot) = \text{Tr}(M_{c|z} A_{ab|xy}(\cdot))$ where $\mathcal{L} = \{A_{ab|xy}\}_{a,b,x,y}$ is as in Example 7 and $M_{0|0} = |0\rangle\langle 0|$, $M_{0|1} = +\rangle\langle +|$, we have $p_{abc|000} = p^{(e)}_{abc|000}$ for all $a, b, c$ and any $P^{(e)}$ from such convex decomposition. Moreover, $p(abc|000) = P_{abc|000}(000)$ fulfill (IX) and (X).

It is finally worth noting, that the analogous construction of a perfect key obtained from bipartite quantum correlations with security against adversary equipped with resource in a form of no-signaling correlations remains an open problem.

Discussion.- We introduced and analyzed concept of multipartite no-signaling channel assemblages and related subclasses of channel assemblages also in the relaxed framework with no-signaling constrains binding only some parties.

Despite these results, there are still some open questions related to addressed topics. First of all, it would we interesting to provide an expansion of presented results in terms of not only sufficient but sufficient and necessary conditions for extremality among no-signaling channel assemblages (at least in the simplest nontrivial case) as well. Similarly, further characterization of extremality within paradigm of asymmetric channel steering is also needed. Finally, from operational perspective, it also would be important to propose concrete cryptographic protocols based on structure of considered convex sets.

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Appendix A: Proof of Theorem 4

In order to justify characterization given in the main text and provide a proof for Theorem 4 let us evoke the following description of no-signaling assemblages \[1, 29\] and simple auxiliary lemma.

**Theorem 10** ([1], [29]). Family of positive operators \[\{\sigma_{\alpha_n|x_n}\}_{\alpha_n,x_n}\] defines a no-signaling assemblage if and only if there exist a Hermitian operator \(W \in \bigotimes_{i=1}^{n} B(A_i) \otimes B(C)_i\) and POVM elements \(M^{(i)}_{a_i|x_i} \in B(A_i)\) for which \(\sigma_{a_n|x_n} = \text{Tr}_{A_1,...,A_n}(M^{(1)}_{a_1|x_1} \otimes ... \otimes M^{(n)}_{a_n|x_n}) \otimes 1_C W\).

**Lemma 11.** Let \(E : B(H_A) \to B(H_A) \otimes B(H_B)\) be a completely positive and trace preserving map. There exist some state \(\tilde{\rho}_B \in B(H_B)\) and some completely positive and trace preserving map \(\tilde{E} : B(H_A) \otimes B(H_B) \to B(H_A) \otimes B(H_B)\) such that

\[E(\rho_A) = \tilde{E}(\rho_A \otimes \tilde{\rho}_B)\] (A1)

for all states \(\rho_A \in B(H_A)\).
Let \( \mathcal{J} = \{ A_n \}_{n=1}^{\infty} \in \text{nsA}(n, m, k, d_C, d_{\mathcal{C}}) \). In that case \( \mathcal{J}(A_n) \in B(H_{\mathcal{C}}) \otimes B(H_C) \) are positive operators such that, according to the linearity of \( \mathcal{J} \), fulfill relations (1) and (2) in the main text. Moreover, \( \mathcal{J}(\sum_{n} A_n) = \mathcal{J}(A) \) and \( \text{Tr}_{\mathcal{C}}(\mathcal{J}(A)) = \frac{1}{d_C} \), as \( A \) is completely positive and trace preserving.

Conversely, assume that \( \Sigma = \{ \mathcal{J}(A_n) \}_{n=1}^{\infty} \in \text{nsA}(n, m, k, d_C, d_{\mathcal{C}}) \) and \( \text{Tr}_{\mathcal{C}}(\mathcal{J}(A)) = \frac{1}{d_C} \). If so then \( A \) is a completely positive and trace preserving map and according to the linearity of \( \mathcal{J}^{-1} \) completely positive maps \( A_n \in \mathcal{J}^{-1}(\mathcal{J}(A_n)) \) fulfill constraints stated in Definition 2 in the main text. This proves the first statement regarding no-signaling realization.

The second part is a direct consequence of the above reasoning, characterization of no-signaling assemblages presented in Theorem 10 and the fact that for any no-signaling assemblage \( \Sigma = \{ \sigma_{a_n|x_n} \}_{n=1}^{\infty} \in \text{nsA}(n, m, k, d_C, d_{\mathcal{C}}) \) of the form

\[
\sigma_{a_n|x_n} = \text{Tr}_{A_1,\ldots,A_\infty}(M_{a_1|x_1}^{(1)} \otimes \ldots \otimes M_{a_n|x_n}^{(n)} \otimes 1_{\mathcal{C}}),
\]

for a state \( \rho = \mathcal{E} \otimes \text{id}_{\mathcal{C}}(\rho_{A_1,\ldots,A_\infty} \otimes |\phi_{\mathcal{C}}^{+}\rangle \langle \phi_{\mathcal{C}}^{+}|) \in \bigotimes_{i=1}^{n} B(H_{A_i}) \otimes B(H_{\mathcal{C}}) \otimes B(H_C) \) that provides a quantum realization of \( \Sigma = \{ \mathcal{J}(A_n) \}_{n=1}^{\infty} \). Moreover,

\[
\text{Tr}_{\mathcal{C}}(\mathcal{J}(\sum_{n} A_n)) = \text{Tr}_{A_1,\ldots,A_\infty}(\rho) = \text{Tr}_{\mathcal{C}}(|\phi_{\mathcal{C}}^{+}\rangle \langle \phi_{\mathcal{C}}^{+}|) = \frac{1}{d_C}.
\]

Conversely, let \( \Sigma = \{ \mathcal{J}(A_n) \}_{n=1}^{\infty} \in \text{nsA}(n, m, k, d_C, d_{\mathcal{C}}) \) and \( \text{Tr}_{\mathcal{C}}(\mathcal{J}(A)) = \frac{1}{d_C} \). In this case, there exists a quantum realization

\[
\mathcal{J}(A_{n|x}) = \text{Tr}_{A_1,\ldots,A_n}(M_{a_1|x_1}^{(1)} \otimes \ldots \otimes M_{a_n|x_n}^{(n)} \otimes 1_{\mathcal{C}}),
\]

Because of Lemma 11 there exist a state \( \rho_{A_1,\ldots,A_n} \in \bigotimes_{i=1}^{n} B(H_{A_i}) \) and a completely positive and trace preserving map \( \mathcal{E} : \bigotimes_{i=1}^{n} B(H_{A_i}) \otimes B(H_C) \to \bigotimes_{i=1}^{\infty} B(H_{A_i}) \otimes B(H_{\mathcal{C}}) \) such that

\[
\mathcal{J}^{-1}(\rho_{A_1,\ldots,A_\infty}) = \mathcal{E}(\rho_{A_1,\ldots,A_\infty} \otimes \rho_C)
\]

for any state \( \rho_C \in B(H_C) \). If so, then

\[
\text{Tr}_{A_1,\ldots,A_n}(\rho_{A_1,\ldots,A_n} \otimes 1_{\mathcal{C}}) = \text{Tr}_{A_1,\ldots,A_n}(\mathcal{E}(\rho_{A_1,\ldots,A_\infty} \otimes \rho_C)).
\]
which ends the proof of the first part of the thesis regarding quantum channel assemblages.

The second part of the thesis concerning quantum description follows from the argument analogous to the one presented in the no-signaling case.

By a reasoning similar to the one above and the fact that any assemblage with local hidden state model admits quantum realization with measurements performed on some fully separable state we conclude the proof.

\[
A_{ab|xy}(\cdot) = \text{Tr}_{AB}(P_{a|x} \otimes Q_{b|y} \otimes I_C \otimes \mathcal{E}_{CNOT}(\langle \phi_{AB}^+ \rangle \otimes \cdot)) \tag{B1}
\]

with \( |\phi_{AB}^+\rangle\langle \phi_{AB}^+| \) being a maximally entangled state of two qubits, \( \mathcal{E}(\cdot) = U_{\text{CNOT}}(\cdot)U_{\text{CNOT}}^* \) and \( Q_{00} = |\theta_1\rangle\langle \theta_1|, \quad Q_{10} = |\theta_2\rangle\langle \theta_2|, \quad P_{00} = |\theta_3\rangle\langle \theta_3|, \quad P_{10} = |\theta_4\rangle\langle \theta_4|, \quad Q_{01} = P_{01} = |+\rangle\langle +|, \quad Q_{11} = P_{11} = |\cdot\rangle\langle \cdot| \) where

\[
|\theta_1\rangle = \frac{1}{\sqrt{5}} |0\rangle + \frac{2}{\sqrt{5}} |1\rangle, \quad |\theta_2\rangle = \frac{2}{\sqrt{5}} |0\rangle - \frac{1}{\sqrt{5}} |1\rangle,
\]

\[
|\theta_3\rangle = \frac{1}{\sqrt{3}} |0\rangle + \sqrt{\frac{2}{3}} |1\rangle, \quad |\theta_4\rangle = \frac{\sqrt{2}}{\sqrt{3}} |0\rangle - \frac{1}{\sqrt{3}} |1\rangle.
\]

Define related assemblages of Choi matrices by

\[
\mathcal{J}(A_{ab|xy}) = A_{ab|xy} \otimes I_{C} = \text{Tr}_{AB}(P_{a|x} \otimes Q_{b|y} \otimes I_C |\psi_{ABCC}\rangle\langle \psi_{ABCC}|)
\]

\[
|\psi_{ABCC}\rangle = \frac{1}{2} (|0000\rangle + |0011\rangle + |1110\rangle + |1101\rangle).
\]

Explicit form of this assemblage \( \Sigma = \{ \mathcal{J}(A_{ab|xy}) \}_{a,b,x,y} \) is represented as

\[
\Sigma = \left( \begin{array}{cccc}
|\phi_1\rangle\langle \phi_1| & |\phi_2\rangle\langle \phi_2| & |\phi_3\rangle\langle \phi_3| & |\phi_4\rangle\langle \phi_4|

|\phi_5\rangle\langle \phi_5| & |\phi_6\rangle\langle \phi_6| & |\phi_7\rangle\langle \phi_7| & |\phi_8\rangle\langle \phi_8|

|\phi_9\rangle\langle \phi_9| & |\phi_10\rangle\langle \phi_10| & |\phi_{13}\rangle\langle \phi_{13}| & |\phi_{14}\rangle\langle \phi_{14}|

|\phi_{11}\rangle\langle \phi_{11}| & |\phi_{12}\rangle\langle \phi_{12}| & |\phi_{15}\rangle\langle \phi_{15}| & |\phi_{16}\rangle\langle \phi_{16}|
\end{array} \right)
\]

where

\[
|\phi_1\rangle = \frac{1}{2\sqrt{15}} \left( |00\rangle + |11\rangle + 2\sqrt{2}|10\rangle + 2\sqrt{2}|01\rangle \right),
\]

\[
|\phi_2\rangle = \frac{1}{2\sqrt{15}} \left( 2|00\rangle + 2|11\rangle - \sqrt{2}|10\rangle - \sqrt{2}|01\rangle \right),
\]

\[
|\phi_3\rangle = \frac{1}{2\sqrt{15}} \left( \sqrt{2}|00\rangle + \sqrt{2}|11\rangle - 2|10\rangle - 2|01\rangle \right),
\]

\[
|\phi_4\rangle = \frac{1}{2\sqrt{15}} \left( 2\sqrt{2}|00\rangle + 2\sqrt{2}|11\rangle + |10\rangle + |01\rangle \right),
\]

Appendix B: Example of quantum extreme point in asymmetric scenario

Consider a asymmetric quantum channel assemblage \( \Lambda = \{ A_{ab|xy} \}_{a,b,x,y} \) of qubit maps \( A_{ab|xy} : B(H_C) \to B(H_C) \) with quantum realization given as

\[
|\phi_{\phi}\rangle = \frac{1}{\sqrt{2}} \left( |00\rangle + |11\rangle + \sqrt{2}|10\rangle + \sqrt{2}|01\rangle \right),
\]

\[
|\phi_{\phi}\rangle = \frac{1}{\sqrt{2}} \left( \sqrt{2}|00\rangle + \sqrt{2}|11\rangle - |10\rangle - |01\rangle \right),
\]

\[
|\phi_{\phi}\rangle = \frac{1}{\sqrt{2}} \left( \sqrt{2}|00\rangle + \sqrt{2}|11\rangle + |10\rangle + |01\rangle \right),
\]

\[
|\phi_{\phi}\rangle = \frac{1}{\sqrt{2}} \left( 2|00\rangle + 2|11\rangle + |10\rangle + |01\rangle \right),
\]

\[
|\phi_{\phi}\rangle = \frac{1}{\sqrt{2}} \left( |00\rangle + |11\rangle + 2|10\rangle + 2|01\rangle \right),
\]

\[
|\phi_{\phi}\rangle = \frac{1}{\sqrt{2}} \left( 2|00\rangle + 2|11\rangle - |10\rangle - |01\rangle \right),
\]

\[
|\phi_{\phi}\rangle = \frac{1}{\sqrt{2}} \left( |00\rangle + |11\rangle - 2|10\rangle - 2|01\rangle \right),
\]

\[
|\phi_{\phi}\rangle = \frac{1}{\sqrt{2}} \left( 2|00\rangle + 2|11\rangle + |10\rangle + |01\rangle \right),
\]

\[
|\phi_{\phi}\rangle = \frac{1}{\sqrt{2}} \left( |00\rangle + |11\rangle + 2|10\rangle + 2|01\rangle \right),
\]

\[
|\phi_{\phi}\rangle = \frac{1}{\sqrt{2}} \left( 2|00\rangle + 2|11\rangle - |10\rangle - |01\rangle \right),
\]

Assume that there is a convex decomposition \( \Sigma = p \Sigma_1 + (1 - p) \Sigma_2 \), where both \( \Sigma_i \) are assemblages described by conditions i)-iii) of Theorem\textsuperscript{[3]} in the main text. So then

\[
\Sigma = \left( \begin{array}{cccc}
\alpha|\phi_1\rangle\langle \phi_1| & \beta|\phi_2\rangle\langle \phi_2| & \gamma|\phi_3\rangle\langle \phi_3| & \delta|\phi_4\rangle\langle \phi_4|

\alpha|\phi_5\rangle\langle \phi_5| & \beta|\phi_6\rangle\langle \phi_6| & \gamma|\phi_7\rangle\langle \phi_7| & \delta|\phi_8\rangle\langle \phi_8|

\alpha|\phi_9\rangle\langle \phi_9| & \beta|\phi_{10}\rangle\langle \phi_{10}| & \gamma|\phi_{13}\rangle\langle \phi_{13}| & \delta|\phi_{14}\rangle\langle \phi_{14}|

\alpha|\phi_{11}\rangle\langle \phi_{11}| & \beta|\phi_{12}\rangle\langle \phi_{12}| & \gamma|\phi_{15}\rangle\langle \phi_{15}| & \delta|\phi_{16}\rangle\langle \phi_{16}|
\end{array} \right)
\]
for some non-negative coefficients \(\alpha, \beta, \gamma, \delta\) (since each initial column consists of different rank one operators and condition i) of Theorem 9 in the main text must be fulfilled).

Observe that due to condition ii) of Theorem 9 in particular we have

\[
\alpha \text{Tr}_C(\langle \phi_1 | \phi_1 \rangle) + \beta \text{Tr}_C(\langle \phi_2 | \phi_2 \rangle) = \gamma \text{Tr}_C(\langle \phi_5 | \phi_5 \rangle) + \delta \text{Tr}_C(\langle \phi_6 | \phi_6 \rangle)
\]
and
\[
\alpha \text{Tr}_C(\langle \phi_9 | \phi_9 \rangle) + \beta \text{Tr}_C(\langle \phi_{10} | \phi_{10} \rangle) = \gamma \text{Tr}_C(\langle \phi_{13} | \phi_{13} \rangle) + \delta \text{Tr}_C(\langle \phi_{14} | \phi_{14} \rangle).
\]

These equalities implies that coefficients \(\alpha, \beta, \gamma, \delta\) must fulfill

\[
\begin{align*}
& \frac{2}{3} \alpha + \frac{6}{5} \beta - \frac{2}{3} \gamma - \frac{2}{5} \delta = 0 \\
& 4\alpha - 4\beta - 5\gamma + 5\delta = 0
\end{align*}
\]

\[
\alpha + \beta - \gamma - \delta = 0
\]

(B5)

According to Kronecker-Capelli theorem space of solutions of linear system (B5) is one dimensional. As \(\alpha = \beta = \gamma = \delta = 1\) form a non-zero solution of (B5), normalization constrain implies that \(\alpha = \beta = \gamma = \delta = 1\) is the only possible choice of coefficients, so \(\Sigma = \Sigma_i\) for \(i = 1, 2\) and due to Theorem 9 in the main text considered quantum channel assemblages is an extreme point in the set of all asymmetric no-signaling channel assemblages \(\mathfrak{nSA}(2, 2, 2, 2)\).

**Appendix C: Proof that** \(P_{abc|000} = P'_{abc|000}\)

Let \(\hat{\mathcal{L}} = \{\hat{A}_{ab|x,y}\}_{a,b,x,y}\) be any asymmetric channel assemblage (acting between quantum subsystems \(C\) and \(\hat{C}\)) with some POVM elements \(M_{c|z}\) such that the following formula

\[
\hat{P}_{abc|xyz}(\cdot) = \text{Tr}(M_{c|z} \hat{A}_{ab|x,y}(\cdot))
\]

defines an asymmetric hybrid channel behavior \(\hat{\mathcal{P}} = \{\hat{P}_{abc|xyz}\}_{a,b,c,x,y,z}\). Observe that as \(\mathcal{J}(\hat{P}_{abc|xyz}) = \text{Tr}_C(M_{c|z} \otimes 1_C \mathcal{J}(\hat{A}_{ab|x,y}))\), due to Theorem 9 in the main text we obtain a no-signaling assemblage \(\Sigma = \{\mathcal{J}(\hat{P}_{abc|xyz})\}_{a,b,c,x,y,z}\), fulfilling \(\Sigma\) conditions. Moreover, for any fixed \(b, y\) the following subcollection of operators \(\Sigma^{(b|y)} = \{\mathcal{J}(\hat{P}_{abc|xyz})\}_{a,c,x,z}\) form no-signaling assemblage up to normalization (i.e. \(\text{Tr}(\Sigma^{(b|y)})\) may be lesser than 1).

Recall the quantum channel assemblage \(\mathcal{L} = \{\hat{A}_{ab|x,y}\}_{a,b,x,y}\) from Example 7 in the main text and related quantum channel behavior \(\mathcal{P} = \{P_{abc|xyz}\}_{a,b,c,x,y,z}\) defined by \(P_{abc|xyz}(\cdot) = \text{Tr}(M_{c|z} \hat{A}_{ab|x,y}(\cdot))\) with \(M_{0|0} = |0\rangle\langle 0|\) and \(M_{0|1} = |+\rangle\langle +|\) (where \(a, b, c, x, y, z \in \{0, 1\}\)). Observe that

\[
\Sigma^{(0|0)} = \frac{1}{8}
\begin{bmatrix}
2|0\rangle\langle 0| & 2|1\rangle\langle 1| & 2|+\rangle\langle +| & 2|-\rangle\langle -|
\end{bmatrix}
\]

\[
\Sigma^{(1|0)} = \frac{1}{8}
\begin{bmatrix}
2|0\rangle\langle 0| & 2|1\rangle\langle 1| & |+\rangle\langle +| & |\rangle\langle -| & |\rangle\langle -|
\end{bmatrix}
\]

\[
\Sigma^{(0|1)} = \frac{1}{8}
\begin{bmatrix}
|0\rangle\langle 0| & |1\rangle\langle 1| & |+\rangle\langle +| & |\rangle\langle -| & |\rangle\langle -|
\end{bmatrix}
\]

\[
\Sigma^{(1|1)} = \frac{1}{8}
\begin{bmatrix}
|0\rangle\langle 0| & |1\rangle\langle 1| & |+\rangle\langle +| & |\rangle\langle -| & |\rangle\langle -|
\end{bmatrix}
\]

where \(\Sigma^{(b|y)} = \{\sigma^{(b|y)}_{ac|xz}\}_{a,c,x,z} = \{\mathcal{J}(\hat{P}_{abc|xyz})\}_{a,c,x,z}\).

Assume that there exists a convex combination \(\mathcal{P} = p_1 \mathcal{P}_1 + p_2 \mathcal{P}_2\) such that both \(\mathcal{P}_i = \{P^{(i)}_{abc|xyz}\}_{a,b,c,x,y,z}\) are asymmetric hybrid channel behaviors. Note that all \(\Sigma^{(b|y)} = \{\sigma^{(b|y)|i}_{ac|xz}\}_{a,c,x,z} = \{\mathcal{J}(\hat{P}_{abc|xyz})\}_{a,c,x,z}\) are no-signaling assemblages up to normalization (due to the discussion in the above paragraph). Without the loss of generality fix \(i\). Assumption concerning convex combination together with explicit form of (C3) imply that \(\sigma^{(b|y)|i}_{ac|xz} = \alpha^{(b|y)\alpha}_{ac|xz} \sigma^{(b|y)\alpha}_{ac|xz}\) for any \(a, b, c, x, y, z\) and some non-negative coefficients \(\alpha^{(b|y)\alpha}_{ac|xz}\). Theorem 9 in the main text applied respectively to \(\Sigma^{(0|1)}\) and \(\Sigma^{(1|1)}\) provides that in fact \(\sigma^{(b|y)|i}_{ac|xz} = \alpha^{(b|y)|i}_{ac|xz}\) for any \(a, b, c, x, z\) and some
non-negative coefficients $\alpha_{(b|0)}^{(1),i}$. To simplify notation we put $\alpha = \alpha_{(0|0)}^{(1),i}$, $\beta = \alpha_{(1|0)}^{(1),i}$. Moreover, analysis of $\Sigma_{(0|0)}$ and $\Sigma_{(1|0)}$ shows that for all $c,z$ we have $\alpha = \alpha_{(0|0)}^{(0),i} = \gamma_1$, $\alpha_{(1|0)}^{(0),i} = \alpha_{(0|0)}^{(1),i} = \gamma_2$, $\alpha_{(0|0)}^{(1),i} = \delta_1$, $\alpha_{(0|1)}^{(1),i} = \delta_3$, $\alpha_{(1|1)}^{(1),i} = \delta_2$, and $\alpha_{(1|0)}^{(1),i} = \delta_4$ for some $\gamma_1, \gamma_2, \delta_1, \delta_2, \delta_3, \delta_4$.

With this knowledge let us finally consider the no-signaling assemblage $\Sigma_i = \sum_c \mathcal{J}(P_{abc|xyz}^{(i)})$, given explicitly as (compare with (C2-C5))

$$\Sigma_i = \frac{1}{8} \begin{pmatrix} 2\gamma_1 \mathbb{1} & 0 & \alpha \mathbb{1} & \beta \mathbb{1} \\ 0 & 2\gamma_2 \mathbb{1} & \alpha \mathbb{1} & \beta \mathbb{1} \\ \delta_1 \mathbb{1} & \delta_2 \mathbb{1} & 2\alpha|+\rangle\langle +| + 2\beta|\rangle\langle -| \\ \delta_3 \mathbb{1} & \delta_4 \mathbb{1} & 2\alpha|\rangle\langle -| + 2\beta|+\rangle\langle +| \end{pmatrix}. \quad (C6)$$

In particular, the following no-signaling equality

$$(\delta_1 + \delta_2) \mathbb{1} = 2\alpha|+\rangle\langle +| + 2\beta|\rangle\langle -| \quad (C7)$$

can be true if and only if $\alpha = \beta$. Since no-signaling conditions also imply $2\gamma_1 = \alpha + \beta$ and $2\gamma_2 = \alpha + \beta$ while $2\alpha + 2\beta = 1$ (normalization) we get $\gamma_1 = \gamma_2 = 1$. This concludes the argument as by applying inverse of the Choi-Jamiolkowski isomorphism we obtain $P_{abc|000} = P''_{abc|000}$ for all $a, b, c$. 
