Hermite-Padé approximation, isomonodromic deformation and hypergeometric integral

Toshiyuki Mano and Teruhisa Tsuda

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Abstract

We develop an underlying relationship between the theory of rational approximations and that of isomonodromic deformations. We show that a certain duality in Hermite’s two approximation problems for functions leads to the Schlesinger transformations, i.e. transformations of a linear differential equation shifting its characteristic exponents by integers while keeping its monodromy invariant. Since approximants and remainders are described by block-Toeplitz determinants, one can clearly understand the determinantal structure in isomonodromic deformations. We demonstrate our method in a certain family of Hamiltonian systems of isomonodromy type including the sixth Painlevé equation and Garnier systems; particularly, we present their solutions written in terms of iterated hypergeometric integrals. An algorithm for constructing the Schlesinger transformations is also discussed through vector continued fractions.

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Introduction

Let $L$ be an integer greater than one. For a given $L$-tuple of analytic functions (or formal power series) $f_0(w), f_1(w), \ldots, f_{L-1}(w)$ and of nonnegative integers $\mathbf{m} = (m_0, m_1, \ldots, m_{L-1})$, Hermite considered the following two rational approximation problems. The first is to find $L$ polynomials

$$q_i^m(w) \quad (0 \leq i \leq L - 1)$$

of degree $m_i - 1$ such that

$$\sum_{i=0}^{L-1} f_i q_i^m = O(w^{m_i-1}),$$

where $|\mathbf{m}| = \sum_{i=0}^{L-1} m_i$; i.e., the left-hand side has a zero of order at least $|\mathbf{m}| - 1$ at $w = 0$. The second is to find $L$ polynomials

$$p_i^m(w) \quad (0 \leq i \leq L - 1)$$

of degree $|\mathbf{m}| - m_i$ such that

$$f_i p_j^m - f_j p_i^m = O(w^{m_i+1}).$$

Each system of polynomials \{q_i^m\} and \{p_i^m\} generically turns out to be unique up to simultaneous multiplication by constants, as an elementary consequence of linear algebra. The above approximation problems come from Hermite’s study on arithmetic properties of the exponential function and are called collectively the Hermite-Padé approximations; often, the former is referred to as the ‘type I’ and the latter as the ‘type II’ or as the simultaneous Padé approximation. Note that if $L = 2$ then both of them reduce to the (usual) Padé approximations. Although these two types of approximations were seemingly unrelated, Mahler discovered that they were fundamentally connected to each other. Put $\mathbf{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^L$ for $0 \leq i \leq L - 1$.

**Theorem 0.1** (Mahler’s duality). It holds that

$$\begin{vmatrix}
q_0^{m+1} & q_1^{m+1} & \cdots & q_{L-1}^{m+1} \\
q_0^{m+1} & q_1^{m+1} & \cdots & q_{L-1}^{m+1} \\
\vdots & \vdots & \ddots & \vdots \\
q_0^{m+1} & q_1^{m+1} & \cdots & q_{L-1}^{m+1}
\end{vmatrix}
= w^{m_i} \cdot D$$

for $0 \leq i \leq L - 1$.
with \( D \) being a diagonal constant matrix. Moreover, if every diagonal part \( q_i^m e_i \) and \( p_i^{m-e_i} \) is chosen to be a monic polynomial then \( D \) becomes the identity matrix.

The aim of this paper is to develop an underlying relationship of Hermite’s two approximations with the theory of linear differential equations in the complex domain, especially with that of isomonodromic deformations. Interestingly enough, Mahler’s duality plays a crucial role in constructing a certain class of Schlesinger transformations, i.e. transformations of a linear differential equation shifting its characteristic exponents by integers while keeping its monodromy invariant.

In Sect. 1, we begin by introducing the two types of rational approximations for an \( L \)-tuple of functions, which are slightly modified (in order to fit the construction of Schlesinger transformations) from the original Hermite-Padé and simultaneous Padé approximations. We then prove a variation of Mahler’s duality between them (see Theorem 1.3). Applying the approximations for the solution of an \( L \times L \) Fuchsian system of linear differential equations yields its Schlesinger transformation (see Theorem 1.5); in fact, Mahler’s duality guarantees the absence of apparent singularities in the new Fuchsian system after the Schlesinger transformation. In Sect. 2 we deduce determinantal representations for the approximants and remainders from the approximation conditions (see Propositions 2.1 and 2.3 and also Remark 2.2). It should be noted that any key ingredient here is written in terms of block-Toeplitz determinants. In Sect. 3 we present an algorithm for constructing the Schlesinger transformation via vector continued fraction expansions, which is a variation of the Jacobi–Perron algorithm (i.e. a higher dimensional analogue of the Euclidean algorithm).

The last two sections are devoted to the study of isomonodromic deformations. In Sect. 4 we first review the Schlesinger system of nonlinear differential equations, which governs isomonodromic deformations of a Fuchsian system. Since a Schlesinger transformation preserves the monodromy of the Fuchsian system, it gives rise to a discrete symmetry of the corresponding Schlesinger system. We clarify, based on the above relationship with rational approximations, the determinantal structure in the general solutions of the Schlesinger systems. Next we concern a particular family of the Schlesinger systems, which possesses a unified description as a polynomial Hamiltonian system denoted by \( \mathcal{H}_{L,N} \) (\( L \geq 2, N \geq 1 \)); it includes various noteworthy examples of isomonodromic deformations such as the sixth Painlevé equation \( (\mathcal{H}_{2,1}) \) and the Garnier system in \( N \) variables \( (\mathcal{H}_{2,N}) \). In Sect. 5 we demonstrate Schlesinger transformations on the previously known hypergeometric solution of \( \mathcal{H}_{L,N} \) (see [Tsu14b]); as a result, we obtain solutions of \( \mathcal{H}_{L,N} \) written in terms of iterated hypergeometric integrals through Fubini’s theorem and the Vandermonde determinant (see Theorem 5.3 and its sequel).

1 From Hermite’s two approximation problems to Schlesinger transformations

In this section we show how rational approximations are useful for constructing Schlesinger transformations of linear differential equations. Fix an integer \( L \geq 2 \). We shall first introduce two different types of rational approximation problems for an \( L \)-tuple

\[
\begin{align*}
&f_0(w), f_1(w), \ldots, f_{L-1}(w) \in \mathbb{C}[w]
\end{align*}
\]

of formal power series, where we assume \( f_0(0) \neq 0 \) without loss of generality.
1.1 Hermite-Padé approximation (of type I)

Let $n$ be a positive integer. Consider for each $i$ ($0 \leq i \leq L - 1$) an $L$-tuple of polynomials $Q^{(i)}_j = Q^{(i)}_j(w)$ ($0 \leq j \leq L - 1$) of degree at most $n - 1 + \delta_{i,j}$, where

$$\delta_{i,j} = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases}$$

is the Kronecker delta. Suppose the approximation condition

$$Q^{(i)}_0 f_0 + \cdots + Q^{(i)}_j f_j + wQ^{(i)}_{i+1} f_{i+1} + \cdots + wQ^{(i)}_{L-1} f_{L-1} = O(w^{nL})$$

is fulfilled for each $i$. This condition amounts to a system of $nL$ homogeneous linear equations for the $nL + 1$ unknown coefficients of the polynomials $Q^{(i)}_j$ ($0 \leq j \leq L - 1$). Under a generic condition for the power series $f_0, \ldots, f_{L-1}$, these polynomials are uniquely determined up to simultaneous multiplication by constants; see Sect. 2.1. We will be concerned with the row vector

$$\overline{Q}^{(i)} = (Q^{(i)}_j)_{0 \leq j \leq L-1} = (Q^{(i)}_0, \ldots, Q^{(i)}_j, wQ^{(i)}_{i+1}, \ldots, wQ^{(i)}_{L-1}).$$

**Remark 1.1.** By construction, the polynomial $Q^{(0)}_0(w)$ has no constant term. Moreover, the degree of the diagonal part $Q^{(i)}_i(w)$ ($0 \leq i \leq L - 1$) turns out to be $n$ exactly; see (1.4) in Sect. 1.3.

1.2 Simultaneous Padé approximation

We treat another type of approximation problem for the same power series $f_0, \ldots, f_{L-1}$. Consider for each $j$ ($0 \leq j \leq L - 1$) an $L$-tuple of polynomials $P^{(j)}_i = P^{(j)}_i(w)$ ($0 \leq i \leq L - 1$) of degree at most $n(L-1) - 1 + \delta_{i,j}$ which satisfies the following approximation conditions:

- if $j = 0$ 
  $$f_0 P^{(j)}_i - f_i P^{(0)}_0 = O(w^{nL})$$
  for $1 \leq i \leq L - 1$;

- if $1 \leq j \leq L - 1$ 
  $$f_0 P^{(j)}_i - f_i P^{(0)}_0 = O(w^{nL})$$
  for $1 \leq i \leq j - 1$,

  $$f_0 P^{(j)}_i - f_i w P^{(j)}_0 = O(w^{nL})$$
  for $j \leq i \leq L - 1$.

These conditions are interpreted as a system of $nL(L - 1)$ homogeneous linear equations for the $nL(L - 1) + 1$ unknown coefficients of the polynomials $P^{(j)}_i$ ($0 \leq i \leq L - 1$). Hence the column vector

$$\overline{P}^{(j)} = \left( P^{(j)}_i \right)_{0 \leq i \leq L-1} = \left( wP^{(j)}_0, \ldots, wP^{(j)}_{j-1}, P^{(j)}_j, \ldots, P^{(j)}_{L-1} \right)$$

is generically unique up to multiplication by constants; see Sect. 2.2.

**Remark 1.2.** Let $a \neq b$. It is immediate from $1/f_0 \in \mathbb{C}[w]$ to verify that

- if $a, b < j$ 
  $$f_a \overline{P}^{(j)}_b - f_b \overline{P}^{(j)}_a = O(w^{nL+1});$$

- otherwise 
  $$f_a \overline{P}^{(j)}_b - f_b \overline{P}^{(j)}_a = O(w^{nL}).$$
1.3 Mahler’s duality in the two approximation problems

There is an interesting connection between the two approximation problems (1.1) and (1.2) although they are seemingly unrelated. The following theorem is thought of as a variation of Mahler’s duality; see Theorem 0.1 or [Mah68]. We will give a proof of it because our setup is slightly different from the original; cf. [BG-M96] Theorem 8.1.2 and [Coo66].

Theorem 1.3. It holds that

\[
\begin{bmatrix}
\tilde{Q}^{(0)} \\
\vdots \\
\tilde{Q}^{(L-1)}
\end{bmatrix} \begin{bmatrix}
P^{(0)}_0, \ldots, P^{(L-1)}_0 \\
P^{(0)}_1, \ldots, P^{(L-1)}_1 \\
\vdots \\
P^{(0)}_{L-1}, \ldots, P^{(L-1)}_{L-1}
\end{bmatrix} = w^{nL} \cdot D
\]

with \( D \) being a diagonal constant matrix.

Proof. Let us first estimate the degree of the \((i, j)\)-entry

\[
M_{ij} = \sum_{k=0}^{L-1} \tilde{Q}^{(i)}_k \tilde{P}^{(j)}_k
\]

of the left-hand side. The degree of each polynomials \( \tilde{Q}^{(i)}_k \) and \( \tilde{P}^{(j)}_k \) reads

\[
\begin{align*}
\deg \leq n - 1 & \quad \tilde{Q}^{(i)}_0, \ldots, \tilde{Q}^{(i)}_{j-1}, \tilde{Q}^{(i)}_j, w\tilde{Q}^{(j)}_{j+1}, \ldots, w\tilde{Q}^{(j)}_{L-1}, \\
\deg \leq n & \quad \tilde{Q}^{(i)}_{j+1}, \ldots, \tilde{Q}^{(i)}_{L-1}, \\
\deg \leq n(L-1) & \quad \tilde{P}^{(j)}_0, \ldots, \tilde{P}^{(j)}_{j-1}, \tilde{P}^{(j)}_j, \tilde{P}^{(j)}_{j+1}, \ldots, \tilde{P}^{(j)}_{L-1}.
\end{align*}
\]

Hence we have

\[
\deg_w M_{ij} \leq \begin{cases} 
nL & (i \leq j) \\
nL - 1 & (i > j).
\end{cases} \tag{1.3}
\]

Next we shall estimate the multiplicity of \( M_{ij} \) at \( w = 0 \) by means of the approximation conditions. Consider

\[
f_0 M_{ij} = f_0 \sum_{k=0}^{L-1} \tilde{Q}^{(i)}_k \tilde{P}^{(j)}_k \\
= f_0 \tilde{Q}^{(i)}_0 \tilde{P}^{(j)}_0 + \sum_{k \neq 0} \tilde{Q}^{(i)}_k f_0 \tilde{P}^{(j)}_k.
\]

(i) Case \( i < j \) (strictly upper triangular part)

\[
\sum_{k \neq 0} \tilde{Q}^{(i)}_k f_0 \tilde{P}^{(j)}_k = \sum_{0 < k < j} \tilde{Q}^{(i)}_k f_0 \tilde{P}^{(j)}_k + \sum_{j \leq k} \tilde{Q}^{(i)}_k f_0 \tilde{P}^{(j)}_k \\
= \sum_{0 < k < j} \tilde{Q}^{(i)}_k (f_0 \tilde{P}^{(j)}_0 + O(w^{nL+1})) + \sum_{j \leq k} \tilde{Q}^{(i)}_k (f_0 \tilde{P}^{(j)}_0 + O(w^{nL})), \quad \text{using (1.2)}
\]

\[
= \tilde{P}^{(j)}_0 \sum_{k \neq 0} \tilde{Q}^{(i)}_k f_k + O(w^{nL+1}), \quad \text{since } \tilde{Q}^{(i)}_k \text{ is divisible by } w \text{ if } i < (j \leq k).
\]
Therefore,

\[ f_0 M_{ij} = \overline{P}_0^{(j)} \sum_k Q_k^{(i)} f_k + O(w^{nL+1}) \]

\[ = \overline{P}_0^{(j)} \cdot O(w^{nL}) + O(w^{nL+1}), \quad \text{using (1.1)} \]

\[ = O(w^{nL+1}), \quad \text{since } \overline{P}_0^{(j)} \text{ is divisible by } w \text{ if } 0(\leq i) < j. \]

(ii) Case \( i \geq j \) (lower triangular part)

\[
\sum_{k \neq 0} Q_k^{(i)} f_0 \overline{P}_k^{(j)} = \sum_{0 < k < j} Q_k^{(i)} f_0 \overline{P}_k^{(j)} + \sum_{j \leq k} Q_k^{(i)} f_0 \overline{P}_k^{(j)}
\]

\[ = \sum_{0 < k < j} Q_k^{(i)} \left( f_k \overline{P}_0^{(j)} + O(w^{nL+1}) \right) + \sum_{j \leq k} Q_k^{(i)} \left( f_k \overline{P}_0^{(j)} + O(w^{nL}) \right), \quad \text{using (1.2)} \]

\[ = \overline{P}_0^{(j)} \sum_{k \neq 0} Q_k^{(i)} f_k + O(w^{nL}). \]

Therefore,

\[ f_0 M_{ij} = \overline{P}_0^{(j)} \sum_k Q_k^{(i)} f_k + O(w^{nL}) \]

\[ = \overline{P}_0^{(j)} \cdot O(w^{nL}) + O(w^{nL}), \quad \text{using (1.1)} \]

\[ = O(w^{nL}). \]

Noticing \( 1/f_0 \in \mathbb{C}[w] \) we verify

\[ M_{ij} = \begin{cases} O(w^{nL+1}) & (i < j) \\ O(w^{nL}) & (i \geq j). \end{cases} \]

Combining this with (1.3), we can conclude that \( (M_{ij}) = w^{nL} \cdot D. \) \( \square \)

Consequently, the diagonal entry \( M_{ii} \) coincides with the term of highest degree in \( Q_i^{(i)}(w)P_i^{(i)}(w) \)

and thus

\[ \deg_w Q_i^{(i)} = n, \quad \deg_w P_i^{(i)} = n(L - 1). \]  \( \text{(1.4)} \)

We henceforth normalize \( \overline{Q}^{(i)} \) and \( \overline{P}^{(j)} \) so that their diagonal parts \( Q_i^{(i)} \) and \( P_j^{(j)} \) become monic polynomials, i.e.

\[ z^n Q_i^{(i)}(z^{-1}) \big|_{z=0} = z^{n(L-1)} P_i^{(i)}(z^{-1}) \big|_{z=0} = 1 \]

and thereby \( D = I \) (the identity matrix).

**Corollary 1.4.** The polynomial matrix

\[ R(z) = z^n \begin{bmatrix} \overline{Q}^{(0)}(z^{-1}) \\ \vdots \\ \overline{Q}^{(l-1)}(z^{-1}) \end{bmatrix} \in \mathbb{C}^{L \times L}[z] \]

satisfies

(i) \( R(z)^{-1} = z^{n(L-1)} \begin{bmatrix} \overline{P}^{(0)}(z^{-1}), \ldots, \overline{P}^{(l-1)}(z^{-1}) \end{bmatrix}; \)

(ii) \( \det R(z) = 1, \quad i.e. \ R(z) \in \text{SL}(L, \mathbb{C}[z]). \)
Proof. Theorem 1.3 shows (i) immediately. Then, it holds that \( R \in \text{GL}(L, \mathbb{C}[z]) \) and thus \( \det R \in \mathbb{C} \setminus \{0\} \). By definition, \( R \) takes the form

\[
R = \begin{bmatrix}
1 & \cdots & * \\
& \ddots & \\
& & 1
\end{bmatrix} + O(z);
\]

namely, its constant term is an upper triangular matrix whose diagonal entries are all one. Therefore, we have \( \det R = 1 \).

### 1.4 Schlesinger transformations

Consider an \( L \times L \) Fuchsian system

\[
\frac{dY}{dz} = AY = \sum_{i=0}^{N+1} \frac{A_i}{z-u_i} Y \quad (A_i : \text{constant matrix})
\]

of linear ordinary differential equations with \( N+3 \) regular singularities

\[
S = \{ u_0 = 1, u_1, \ldots, u_N, u_{N+1} = 0, u_{N+2} = \infty \} \subset \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}
\]
on the Riemann sphere. Let \( A_{N+1} \) and \( A_{N+2} = -\sum_{i=0}^{N+1} A_i \) be upper and lower triangular matrices, respectively. Assume for simplicity there is no integer difference among the characteristic exponents \( \{ \varepsilon_{0,j} \}_{0 \leq j \leq L-1} \) at \( z = 0 \) (resp. \( \{ \varepsilon_{\infty,j} \}_{0 \leq j \leq L-1} \) at \( z = \infty \)), i.e. the eigenvalues of the residue matrix \( A_{N+1} \) (resp. \( A_{N+2} \)). Then we have a solution \( Y = Y(z) \) of (1.7) normalized as

\[
Y = \begin{bmatrix}
1 & \cdots & * \\
& \ddots & \\
& & 1
\end{bmatrix} + O(z) \cdot \text{diag} \left( z^{\varepsilon_{0,j}} \right)_{0 \leq j \leq L-1}
= \Phi(w) \cdot \text{diag} \left( w^{\varepsilon_{\infty,j}} \right)_{0 \leq j \leq L-1} \cdot C
\]

with \( w = 1/z \) and \( C \) being an invertible constant matrix (the \textit{connection matrix} between \( z = 0 \) and \( z = \infty \)). Here \( \Phi(w) \) is a matrix function holomorphic at \( w = 0 \) (\( z = \infty \)) and \( \Phi(0) \) is invertible and lower triangular, i.e.

\[
\Phi(w) = \begin{bmatrix}
* & & \\
& \ddots & \\
* & & *
\end{bmatrix} + O(w).
\]

An analytic continuation along a loop on \( \mathbb{P}^1 \setminus S \) based at some point \( z_0 \) induces a linear transformation of \( Y \) according to its multi-valuedness at the branch points \( S \). We thus obtain an \( L \)-dimensional representation of the fundamental group \( \pi_1(\mathbb{P}^1 \setminus S; z_0) \), which is called the \textit{monodromy} of \( Y \). A left multiplication \( Y \mapsto \hat{Y} = RY \) of a rational function matrix \( R = R(z) \) is said to be a \textit{Schlesinger transformation} if the new equation

\[
\frac{d\hat{Y}}{dz} = \hat{A}\hat{Y}
\]

for some \( \hat{A} \) and \( \hat{Y} \).
satisfied by \( \hat{Y} \) becomes the same form as the original (1.7). Because \( R(z) \) is rational, \( \hat{Y} \) and \( Y \) have the same monodromy though they have different characteristic exponents by integers. It is known that if we specify an admissible discrete change of the characteristic exponents then the corresponding rational function matrix \( R \) of the Schlesinger transformation is algebraically computable from \( Y \); see [JM81, Sch12]. In fact, the construction problem of Schlesinger transformations is naturally related to rational approximation problems; see also Remark 1.7.

In this paper we focus on a class of Schlesinger transformations, which is of particular interest from the viewpoint of Hermite’s two approximation problems and also of vector continued fractions (see Sect. 3). Note that, for a general Schlesinger transformation other than the present direction, though it can also be controlled by some rational approximation problems but it becomes much more complicated due to the absence of a duality like Mahler’s, e.g. \( R^{-1} \) seems not to have a concise determinantal representation; cf. [MT15].

Let us define the \( L \)-tuple \( f = T(f_0, \ldots, f_{L-1}) \) of power series in \( w \) as the first column of \( \Phi(w) \), which is the power series part of the solution \( Y \) of (1.7) near \( z = \infty \). Notice that \( f_0(0) \neq 0 \) certainly holds. Therefore, all the general arguments in Sects. 1.1–1.3 are still valid for this specific case, and we are led to the Schlesinger transformation through the two approximation problems for \( f \).

Now we state the result.

**Theorem 1.5.** The polynomial matrix \( R(z) \) given by (1.5) realizes the Schlesinger transformation \( Y \mapsto \hat{Y} = RY \) shifting the characteristic exponents at \( z = \infty \) by \( (n(L-1), -n, \ldots, -n) \).

**Proof.** It follows from the Hermite-Padé approximation condition (1.1) that \( Rf = O(w^n(L-1)) \). By definition, \( R \) takes the form

\[
R = w^{-n} \begin{pmatrix} 0 & * & * & \cdots & * \\ * & * & \vdots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & * \\ * & \cdots & * & * & * \end{pmatrix} + O(w). \tag{1.9}
\]

Hence, if we write as

\[
R\Phi(w) = \hat{\Phi}(w) \cdot \text{diag} (w^n(L-1), w^{-n}, \ldots, w^{-n}),
\]

then \( \hat{\Phi} \) becomes the same form as \( \Phi \). Also, we verify from (1.6) that \( R \) does not change the form of the power series expansion of \( Y \) near \( z = 0 \).

On the other hand, the coefficient

\[
A(z) = \sum_{i=0}^{N+1} \frac{A_i}{z - u_i}
\]

of the Fuchsian system (1.7) is transformed as

\[
A \mapsto \hat{A} = RAR^{-1} + \frac{dR}{dz} R^{-1}. \tag{1.10}
\]

If we remember both \( R \) and \( R^{-1} \) being polynomials (see Corollary 1.4), then \( \hat{A} \) turns out to be a rational function matrix having only simple poles at \( S \) as well as the original \( A \). In this sense Mahler’s duality guarantees the absence of apparent singularities in the new equation \( d\hat{Y}/dz = \hat{A}\hat{Y} \) satisfied by \( \hat{Y} = RY \). \( \square \)
Remark 1.6. In the rank two case \((L = 2)\) a similar construction of Schlesinger transformations as Theorem 1.5 has been established in [Man12] based on (usual) Padé approximations.

Remark 1.7. A series of pioneering works was done by D. Chudnovsky and G. Chudnovsky on the close connection between rational approximation problems and Riemann’s monodromy problem, involving (semi-classical) orthogonal polynomials; see [CC82, CC94] and references therein. The “Padé method” recently proposed by Yamada [Yam09] is a recipe for Lax formalism of Painlevé equations and, at the same time, for their special solutions, which is based on Padé approximations (or interpolations) of elementary functions; interestingly enough, it is applicable also for various discrete analogues of Painlevé equations beyond the originals; see [Ika13, Nag14, NTY12].

The essential idea of the above works could be exemplified by the following: let us consider a function \(\varphi(z) = z^a(z - 1)^b(z - u)^c\) with four branch points \(S = \{0, 1, u, \infty\} \subset \mathbb{P}^1\). The remainder \(\rho := P\varphi - Q = O(z^{-n-1})\) of its Padé approximation then satisfies a second-order linear differential equation denoted by \(E\), which may have an apparent singularity besides the four regular singularities at \(S\). However, the two functions \(\varphi\) and \(\rho\) share the same multi-valuedness since they are rationally related; thus, the monodromy of \(E\) is obviously constant with respect to \(u\). This fact leads to special solutions of the sixth Painlevé equation \(P_{VI}\), i.e. the isomonodromic deformation (cf. Sect. 4) of a second-order linear differential equation with four regular singularities.

It is interesting to note that such an idea had been recognized implicitly by Laguerre (before the discovery of Painlevé equations); see [Lag80] and also [Mag95].

Remark 1.8. The approximation conditions (1.1) and (1.2) can be interpreted as certain multi-orthogonality relations among the \(L\)-tuples of polynomials \(\tilde{Q}^{(i)}\) and \(\tilde{P}^{(j)}\), respectively; i.e., these polynomials can constitute multi-orthogonal polynomial systems. In this paper, although we do not enter into details on such aspects, we present below the determinantal representations for them, which will crucially work in the last two sections.

2 Determinantal representations for approximation polynomials and remainders

In this section we derive determinantal representations for the approximation polynomials \(Q_j^{(i)}(w)\) and \(P_j^{(i)}(w)\). We write the power series as

\[
f_i(w) = \sum_{j=0}^{\infty} a_j w^j \in \mathbb{C}[\llbracket w \rrbracket] \quad (2.1)
\]
The solution of (2.4) is unique up to multiplication by constants if and only if the rank of the homogeneous linear equations for the coefficients of the approximation polynomial equals 1 blocks. Consequently, we have a system for the sequence \( \{a_j^i\}_{j=0}^{\infty} \), where \( a_j^i = 0 \) if \( j < 0 \). It holds that
\[
(A_n^k(m,n))_{i,j} = (A_m^k(n,m))_{n-j+1,m-i+1}
\]
by definition.

### 2.1 Hermite-Padé polynomials

We can calculate separately for each \( i \) (\( 0 \leq i \leq L - 1 \)). Therefore, for brevity, we shall express the coefficients of the approximation polynomial \( Q_j^{(i)}(w) \) as
\[
Q_j^{(i)}(w) = \sum_{k=0}^{n-1+\delta_{i,j}} b_{j,k} w^k
\]
with omitting the index \( i \). The condition (1.1) implies vanishing of the coefficients of \( 1, w, w^2, \ldots, w^{nL-1} \) in the left-hand side. Consequently, we have a system
\[
\mathcal{A}^{(i)} \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{nL-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}
\]
(2.4)
of homogeneous linear equations for the \( nL + 1 \) unknowns
\[
b_j = ^t(b_{j,0}, \ldots, b_{j,nL-1+\delta_{i,j}}) \quad (0 \leq j \leq L - 1)
\]
where
\[
\mathcal{A}^{(i)} = \begin{bmatrix} A_0^0(nL,n) & \cdots & A_0^{i-1}(nL,n) & A_0^i(nL,n+1) & A_1^{i+1}(nL,n) & \cdots & A_{L-1}^{L-1}(nL,n) \end{bmatrix}_{i+1 \text{ blocks}} \begin{bmatrix} \begin{bmatrix} 0 \cdots 0 \end{bmatrix}_{1 \text{ blocks}} \end{bmatrix}_{L-i-1 \text{ blocks}}
\]
The solution of (2.4) is unique up to multiplication by constants if and only if the rank of the \( nL \times (nL + 1) \) matrix \( \mathcal{A}^{(i)} \) equals \( nL \) (which we will always assume).

Interestingly enough, we have the following determinantal representation of \( Q_j^{(i)}(w) \).
Proposition 2.1. It holds that

\[ Q^{(i)}_j(w) = \frac{1}{NQ^{(i)}} \det \left[ \begin{array}{c|c|c|c|c} \hline \text{0th block} & \text{1st block} & \ldots & \text{(i-1)th block} & \text{i-th block} \\ \hline 0 & 1, w, \ldots, w^{n-1}+\delta_{ij} & 0 & \ldots & 0 \\ \hline & & & & \end{array} \right], \]  

(2.5)

where \(NQ^{(i)}\) are some normalizing constants.

Proof. Consider

\[ \rho_i(w) = Q^{(i)}_0 f_0 + \cdots + Q^{(i)}_i f_i + wQ^{(i)}_{i+1} f_{i+1} + \cdots + wQ^{(i)}_{L-1} f_{L-1}, \]

which is the remainder of the approximation condition (1.1). Substituting (2.5) shows that

\[ \rho_i(w) = \frac{1}{NQ^{(i)}} \det \left[ \begin{array}{c|c|c|c|c} \text{0th block} & \text{1st block} & \ldots & \text{(i-1)th block} & \text{i-th block} \\ \hline f_0, f_0^2, \ldots, f_0^{n-1} & f_1, f_1^2, \ldots, f_1^{n-1} & \ldots & f_{i-1}, f_{i-1}^2, \ldots, f_{i-1}^{n-1} & f_i, f_i^2, \ldots, f_i^{n-1} \\ \hline A_0(nL, n) & A_1(nL, n) & \ldots & A_{i-1}(nL, n) & A_i(nL, n + 1) \\ \hline & & & & \end{array} \right]. \]

Therefore, if we put \( \rho_i(w) = \sum_{k=0}^{\infty} \rho_k^{(i)} w^k \), then the coefficients read

\[ \rho_k^{(i)} = \frac{1}{NQ^{(i)}} \det \left[ \begin{array}{c|c|c|c|c} A_0(1, n) & \cdots & A_{i-1}(1, n) & A_i(1, n + 1) & A_{i+1}(1, n) & \ldots & A_{L-1}(1, n) \\ A_0(nL, n) & \ldots & A_{i-1}(nL, n) & A_i(nL, n + 1) & A_{i+1}(nL, n) & \ldots & A_{L-1}(nL, n) \end{array} \right]. \]

It is immediate from a property of determinants to verify \( \rho_k^{(i)} = 0 \) for any \( k \) less than \( nL \); thus, we have \( \rho_i(w) = O(w^{nL}) \) indeed. \( \square \)

We will normalize the polynomials so that its diagonal part \( Q^{(i)}_i(w) \) becomes monic as well as in Sect. 1.3. Accordingly, the normalizing constant \( NQ^{(i)} \) should be

\[ NQ^{(i)} = \det \left[ \begin{array}{c|c|c|c|c} \hline \text{0th block} & \text{1st block} & \ldots & \text{(i-1)th block} & \text{i-th block} \\ \hline 0 & 0, \ldots, 0, 1 & 0 & \ldots & 0 \\ \hline & & & & \end{array} \right] \]

\[ = (-1)^{n+i} \det \left[ \begin{array}{c|c|c|c|c} A_0(nL, n) & \ldots & A_{i-1}(nL, n) & A_i(nL, n) & \ldots & A_{L-1}(nL, n) \end{array} \right]. \]

Thus, the leading coefficient \( \rho_{nL}^{(i)} \) of the remainder is given by

\[ \rho_{nL}^{(i)} = \frac{(-1)^{nL}}{NQ^{(i)}} \det \left[ \begin{array}{c|c|c|c|c} A_0(nL + 1, n) & \ldots & A_{i-1}(nL + 1, n) & A_i(nL + 1, n + 1) & A_{i+1}(nL + 1, n) & \ldots & A_{L-1}(nL + 1, n) \end{array} \right]; \]

the constant term of the polynomial \( Q^{(i)}_i(w) \) \( (i \neq 0) \) is given by

\[ Q^{(i)}_i(0) = \frac{1}{NQ^{(i)}} \det \left[ \begin{array}{c|c|c|c|c} \hline \text{0th block} & \text{1st block} & \ldots & \text{(i-1)th block} & \text{i-th block} \\ \hline 0 & 1, 0, \ldots, 0 & 0 & \ldots & 0 \\ \hline & & & & \end{array} \right] \]

\[ = \frac{(-1)^{ni}}{NQ^{(i)}} \det \left[ \begin{array}{c|c|c|c|c} A_0(nL, n) & \ldots & A_{i-1}(nL, n) & A_i(nL, n) & \ldots & A_{L-1}(nL, n) \end{array} \right]. \]
Remark 2.2. We here restrict ourselves to the case where \( f_0(w) = 1 \). Let us introduce the block-Toeplitz determinant

\[
\Delta^{(i)} = \det \begin{bmatrix}
A^1_n(m,n) & \cdots & A^{i-1}_n(m,n) \\
A^1_{n-1}(m,n) & \cdots & A^{L-1}_{n-1}(m,n)
\end{bmatrix}
\]

of size \( m = n(L - 1) \) for each \( i (1 \leq i \leq L) \); e.g. \( NQ^{(i)} = (-1)^{n+1} \Delta^{(i+1)} \). We see in particular that

\[
\rho^{0}_{nl} = (-1)^{n} \frac{\Delta^{(L)}}{\Delta^{(1)}},
\]

\[
Q^{(i)}(0) = (-1)^{n} \frac{\Delta^{(i)}}{\Delta^{(i+1)}} \quad (1 \leq i \leq L - 1).
\]

These simple formulae will be used later in Sect. 5.

2.2 Simultaneous Padé polynomials

Suppose \( f_0(w) = 1 \) for simplicity. Or, equivalently, we may understand that we have renamed \( f_i/f_0 \) \((i \neq 0)\) as \( f_i \). For a given power series \( F(w) = \sum_{k=0}^{\infty} F_k w^k \), we employ the notation

\[
[F(w)]_a^b = \sum_{k=a}^{b} F_k w^k
\]

denoting its section between \( w^a \) and \( w^b \) if \( a \leq b \). From now on, we set

\[
m = n(L - 1)
\]

as well as in Remark 2.2.

First we shall construct the formulae for \( P^{(j)}_0 \) \((0 \leq j \leq L - 1)\).

(i) Case \( j = 0 \). The approximation condition (1.2) requires that

\[
\left[ f_i P^{(0)}_0 \right]_m^{m+n-1} = 0 \quad (1 \leq i \leq L - 1)
\]

since \( P^{(0)}_i \) \((i \neq 0)\) is a polynomial of degree at most \( m - 1 \). If we write

\[
P^{(0)}_0(w) = \sum_{k=0}^{m} b_k w^k,
\]

then we find a system

\[
\begin{bmatrix}
A^1_m(n,m+1) & \cdots & A^{i-1}_m(n,m+1) \\
A^2_m(n,m+1) & \cdots & A^{L-1}_m(n,m+1)
\end{bmatrix}
\begin{bmatrix}
b_0 \\
b_1 \\
\vdots \\
b_m
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
\vdots \\
0
\end{bmatrix}
\]

of homogeneous linear equations for the \( m+1 \) unknowns \( b_0, \ldots, b_m \). The solution of (2.6) is unique up to multiplication by constants if and only if the rank of the \( m \times (m + 1) \) matrix in the left-hand side equals \( m \).
Similarly, it follows from (1.2) that
\[
\begin{align*}
[f_iP^{(j)}_0]_{m}^{m+n-1} &= 0 \quad (1 \leq i \leq j - 1) \\
[f_jP^{(j)}_0]_{m}^{m+n-2} &= 0 \quad (i = j) \\
[f_iP^{(j)}_0]_{m-1}^{m+n-2} &= 0 \quad (j + 1 \leq i \leq L - 1).
\end{align*}
\]

These amount to the simultaneous linear equation
\[
\begin{bmatrix}
A_{m}^{1}(n, m) \\
\vdots \\
A_{m}^{j-1}(n, m) \\
A_{m}^{n}(n - 1, m) \\
A_{m}^{j+1}(n, m) \\
\vdots \\
A_{m}^{L-1}(n, m)
\end{bmatrix}
\begin{bmatrix}
b_0 \\
b_1 \\
\vdots \\
b_{m-1}
\end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}_{m-1}
\]

for the \(m\) unknown coefficients \(b_0, \ldots, b_{m-1}\) of the polynomial
\[
P^{(j)}_0(w) = \sum_{k=0}^{m-1} b_k w^k.
\]

**Proposition 2.3.** The polynomials \(P^{(j)}_0\) admit the following determinantal representations:
\[
P^{(0)}_0(w) = \frac{1}{\text{NP}^{(0)}} \det \begin{bmatrix} 1, w, w^2, \ldots, w^n \\ A_{m}^{1}(n, m) \\ \vdots \\ A_{m}^{L-1}(n, m) + 1 \end{bmatrix}
\]

and
\[
P^{(j)}_0(w) = \frac{1}{\text{NP}^{(j)}} \det \begin{bmatrix} 1, w, w^2, \ldots, w^{m-1} \\ A_{m}^{1}(n, m) \\ \vdots \\ A_{m}^{j-1}(n, m) \\ A_{m}^{j}(n - 1, m) \\ A_{m}^{j+1}(n, m) \\ \vdots \\ A_{m}^{L-1}(n, m) \end{bmatrix}
\]

for \(1 \leq j \leq L - 1\), where \(\text{NP}^{(j)}\) are some normalizing constants.

Next the other \(P^{(j)}_i (i \neq 0)\) can be written as follows:

- if \(j = 0\)
  \[
P^{(0)}_i = [f_iP^{(0)}_0]_{0}^{m-1};
  \]
- if \(1 \leq j \leq L - 1\)
  \[
P^{(j)}_i = [f_iP^{(j)}_0]_{0}^{m-1} \quad \text{for } 1 \leq i \leq j - 1,
  \]
  \[
P^{(j)}_j = w [f_iP^{(j)}_0]_{0}^{m-1} \quad \text{for } i = j,
  \]
  \[
P^{(j)}_i = w [f_iP^{(j)}_0]_{0}^{m-2} \quad \text{for } j + 1 \leq i \leq L - 1.
  \]
We will choose the normalization so that the diagonal part \( P_i^j(w) \) becomes monic. Accordingly, we obtain

\[
NP^{(0)} = \det \begin{bmatrix} 0, \ldots, 0, 1 \\ A_m^1(n, m + 1) \\ \vdots \\ A_m^{L-1}(n, m + 1) \end{bmatrix} = (-1)^m \det \begin{bmatrix} A_m^1(n, m) \\ \vdots \\ A_m^{L-1}(n, m) \end{bmatrix} = (-1)^{\frac{m(n-1)+n(L-1)}{2}} \Delta^{(L)}
\]  

and

\[
NP^{(j)} = \det \begin{bmatrix} a_{m-1}^j, a_{m-2}^j, \ldots, a_0^j \\ A_m^1(n, m) \\ \vdots \\ A_m^{j-1}(n, m) \\ A_m^j(n-1, m) \\ A_{m-1}^j(n, m) \\ \vdots \\ A_{m-1}^{L-1}(n, m) \end{bmatrix} = (-1)^{n(j-1)} \det \begin{bmatrix} A_m^1(n, m) \\ \vdots \\ A_m^{j-1}(n, m) \\ A_m^j(n-1, m) \\ A_{m-1}^j(n, m) \\ \vdots \\ A_{m-1}^{L-1}(n, m) \end{bmatrix} = (-1)^{\frac{m(n-1)+n(j-1)}{2}} \Delta^{(j)}
\]  

for \( 1 \leq j \leq L - 1 \). Here we have employed the notation of the block-Toeplitz determinant (see Remark 2.2) in view of (2.3).

**Remark 2.4.** If \( i \geq j \geq 1 \), \( P_i^j(w) \) is divisible by \( w \). By construction (see Corollary 1.4), we observe that \( R^{-1} \) takes the form (cf. (1.9))

\[
R^{-1} = w^{-m} \begin{bmatrix} * & 0 \\ * & \end{bmatrix} + O(w).
\]

### 3 From vector continued fractions to Schlesinger transformations

In this section we present an alternative construction of the same Schlesinger transformation (considered in Sect. 1) through an algorithm for expanding a vector-valued function into a vector continued fraction.

#### 3.1 Algorithm for vector continued fraction expansion

Let \( f = (f_0, \ldots, f_{L-1}) \) be the \( L \)-tuple of formal power series (2.1). For simplicity, we assume tentatively that \( f_i(0) \neq 0 \) for all \( i \). We abbreviate the constant term \( d_i^0 \) of \( f_i(w) = \sum_{j=0}^{\infty} d_i^j w^j \) as \( d_i^j \).

First we apply a left multiplication of a permutation matrix to \( f \):

\[
\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_{L-1} \end{bmatrix} = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_{L-1} \end{bmatrix}.
\]
Next we eliminate the constant term of \( f_i \) by a subtraction of constant multiple of \( f_{i+1} \) for each \( 0 \leq i \leq L - 2 \) and by a multiplication by \( w \) for \( i = L - 1 \):

\[
\begin{bmatrix}
    w \\
    1 - \frac{a^0}{a^1} \\
    1 - \frac{a^1}{a^2} \\
    \ddots \\
    1 - \frac{a^{L-2}}{a^{L-1}} \\
    1
\end{bmatrix}
\begin{bmatrix}
    f_{L-1} \\
    f_0 \\
    f_1 \\
    \vdots \\
    f_{L-3} \\
    f_{L-2}
\end{bmatrix}
= O(w) = \begin{bmatrix}
    f'_0 \\
    f'_1 \\
    \vdots \\
    f'_{L-2}
\end{bmatrix}.
\]

Eventually we obtain a new \( L \)-tuple \( f' = T(f'_0, \ldots, f'_{L-1}) \) of power series from the original \( f \).

The above procedure is summarized as a left multiplication

\[
f' = T f
\]

of an invertible matrix

\[
T = \frac{1}{w} \begin{bmatrix}
    0 \\
    1 - \frac{a^0}{a^1} \\
    1 - \frac{a^1}{a^2} \\
    \ddots \\
    1 - \frac{a^{L-2}}{a^{L-1}} \\
    1
\end{bmatrix}
\]

with \( \det T = w^{1-L} \). This is an analogue of the Euclidean algorithm and can be repeated generically. Let \( f[k] = T(f_0[k], \ldots, f_{L-1}[k]) \) denote the corresponding vector of power series at the \( k \)th step and let \( a'[k] \) denote their constant terms. Hence, we have

\[
f[k + 1] = T[k] f[k] \quad \text{and} \quad f[0] = f,
\]

where

\[
T[k] = \frac{1}{w} \begin{bmatrix}
    0 \\
    1 - \frac{a^0[k]}{a'[k]} \\
    1 - \frac{a^1[k]}{a'[k]} \\
    \ddots \\
    1 - \frac{a^{L-2}[k]}{a'[k]} \\
    1
\end{bmatrix}.
\]

On the other hand, solving (3.1) for \( f \) yields

\[
f_i = w \sum_{j=i}^{L-1} \frac{a^i}{a^j} f'_{j+1} \quad (0 \leq i \leq L - 1)
\]

where \( f'_L = f'_0/w \). Let us introduce the inhomogeneous coordinates \( \varphi = T(\varphi_1, \ldots, \varphi_{L-1}) \) by \( \varphi_i = f_i/f_0 \). Therefore, we have

\[
\frac{\varphi_i}{\varphi_{L-1}} = \frac{a^i}{a^{L-1}} + w \sum_{j=i}^{L-2} \frac{a^i}{a^j} \varphi'_{j+1} \quad (0 \leq i \leq L - 2)
\]

where \( \varphi_0 = 1 \).
Definition 3.1 (cf. [Par81, NS91]). Let \( \varphi = (\varphi_1, \ldots, \varphi_{L-1}) \) be an \((L - 1)\)-vector such that \( \varphi_1 \neq 0 \). Then the vector

\[
\iota(\varphi) = \frac{1}{\varphi} = \left( \frac{\varphi_2}{\varphi_1}, \frac{\varphi_3}{\varphi_1}, \ldots, \frac{\varphi_{L-1}}{\varphi_1}, 1 \right)
\]

is called the reciprocal of \( \varphi \). Note that \( \iota^L = \text{id} \).

Under this notation, the correspondence (3.4) can be translated into

\[
T \left( \frac{1}{\varphi_{L-1}}, \frac{\varphi_1}{\varphi_{L-1}}, \ldots, \frac{\varphi_{L-2}}{\varphi_{L-1}} \right) = \iota^{-1}(\varphi) = a + wB\varphi
\]

or equivalently into

\[
\varphi = \frac{1}{a + wB\varphi},
\]

where

\[
a = \frac{1}{a^{L-1}} \begin{bmatrix}
a^0 \\
a^1 \\
\vdots \\
a^{L-2}
\end{bmatrix}
\quad \text{and} \quad
B = \begin{bmatrix}
1 & \frac{a^0}{a^1} & \frac{a^0}{a^2} & \cdots & \frac{a^0}{a^{L-2}} \\
\frac{a^1}{a^0} & 1 & \frac{a^1}{a^2} & \cdots & \frac{a^1}{a^{L-2}} \\
\vdots & \vdots & \ddots & \vdots & \ddots \\
\frac{a^{L-3}}{a^{L-2}} & \frac{a^{L-3}}{a^{L-1}} & \frac{a^{L-3}}{a^2} & \frac{a^{L-3}}{a^1} & 1
\end{bmatrix}
\]

Namely, the vector \( a \) is determined as the constant term of \( \iota^{-1}(\varphi) \) and the matrix \( B \) is then specified by \( a \). Let \( \varphi[k] \) denote the inhomogeneous coordinates of the vector \( f[k] \in \mathbb{C}^L \langle w \rangle \) at the \( k \)th step. Taking the reciprocal repeatedly in this way, we obtain formally the vector continued fraction

\[
\varphi = \frac{1}{a[0] + \frac{wB[0]}{a[1] + \frac{wB[1]}{a[2] + \frac{wB[2]}{a[3] + \cdots}}}}
\]

(3.5)

which is regarded as an \((L - 1)\)-dimensional generalization of the Stieltjes-type continued fraction. Refer to [JT09, Appendix A] for a classification of continued fractions. Our algorithm differs from the other known examples such as the Jacobi–Perron algorithm; cf. [Par81, Per07]. Note also that some dynamical system, like the Toda lattice, has been studied based on the connection among the Jacobi–Perron algorithm, rational approximations and bi-orthogonal polynomials; see [KA84].

The following theorem can be verified straightforwardly through the above algorithm, as well as the case of a Stieltjes-type continued fraction (i.e., \( L = 2 \) case).
**Theorem 3.2.** The kth convergents (rational functions)

\[
\Pi_i = \frac{1}{a[0]},
\]

\[
\Pi_k = \frac{1}{a[0] + \frac{wB[0]}{a[1] + \frac{wB[1]}{a[2] + \cdots + \frac{wB[k-2]}{a[k-1]}}}} \in \mathbb{C}^{L-1}(w) \quad (k \geq 2)
\]

of the vector continued fraction (3.5) provide approximants of the vector \( \varphi \in \mathbb{C}^{L-1}[w] \) of power series in the sense that \( \varphi - \Pi_k = O(w^k) \).

In calculating \( \Pi_k \), it is convenient to apply the projective transformations (3.3) successively as follows:

\[
\begin{bmatrix}
\sigma_0 \\
\sigma_1 \\
\vdots \\
\sigma_{L-1}
\end{bmatrix} = T[0]^{-1}T[1]^{-1} \cdots T[k-1]^{-1} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}
\quad \text{and} \quad \Pi_k = \frac{1}{\sigma_0} 
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\vdots \\
\sigma_{L-1}
\end{bmatrix}
\]

**3.2 Schlesinger transformations, revisited**

Let \( Y = Y(z) \) be a solution (1.8) of the Fuchsian system (1.7) having the local behaviors

\[
Y = \Psi(z) \cdot \text{diag} \left( z^\nu \right)_{0 \leq \nu \leq L-1} \quad \text{(near } z = 0) \\
= \Phi(w) \cdot \text{diag} \left( w^\nu \right)_{0 \leq \nu \leq L-1} \cdot C \quad \text{(near } z = 1/w = \infty)
\]

where the power series parts are normalized by

\[
\Psi(z) = \begin{bmatrix} 1 & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & 1 \end{bmatrix} + O(z), \quad \Phi(w) = \begin{bmatrix} * \\ \vdots & \ddots & \vdots \\ * & \cdots & * \end{bmatrix} + O(w)
\]

and \( C \) is the connection matrix. Let \( f = T(f_0, \ldots, f_{L-1}) \) denote the first column of \( \Phi(w) \).

It is clear from the construction of the matrix \( T = T[0] \) that

\[
T\Phi = \left( \begin{bmatrix} * \\ \vdots & \ddots & \vdots \\ * & \cdots & * \end{bmatrix} + O(w) \right) \cdot \text{diag} \left( 1, w^{-1}, \ldots, w^{-1} \right)
\]
near $w = 0 (z = \infty)$. On the other hand, it holds that

$$T\Psi = z \begin{pmatrix} \ast & \cdots & \ast & 1 \\ 1 & \cdots & \ast & 0 \\ & \ddots & \vdots & \vdots \\ & & \ast & 0 \\ & & & 1 \end{pmatrix} + O(z) \cdot \text{diag} (1, \ldots, 1, z^{-1})$$

near $z = 0$. After we repeat the same procedure $L$ times, the power series part thus recovers its original form:

$$T[L - 1] \cdots T[1]T[0]\Psi = z^{L-1} \begin{pmatrix} 1 & \cdots & \ast \\ & \ddots & \vdots \\ & & \ast \\ & & & 1 \end{pmatrix} + O(z).$$

In conclusion, the matrix $z^{1-L}T[L - 1] \cdots T[1]T[0]$ turns out to be a polynomial in $z$ and to be the multiplier of the Schlesinger transformation shifting the characteristic exponents at $z = \infty$ by $(L - 1, -1, \ldots, -1)$; cf. Theorem 1.5.

**Remark 3.3.** In fact, the same approximation problem considered in Sect. 1 appears in the following manner. Concerning the polynomial matrix $w^L T[L - 1] \cdots T[1]T[0]$, we observe from the form of $T[k]$ that the diagonal entries are all monic linear functions, the strictly upper triangular part is linear and divisible by $w$, and the strictly lower triangular part is a constant. Moreover, in view of (3.2) we have

$$w^L T[L - 1] \cdots T[1]T[0] f = O(w^L),$$

which coincides with the approximation condition (1.1), and thus

$$w^L T[L - 1] \cdots T[1]T[0] = \begin{pmatrix} \bar{Q}^{(0)}(w) \\ \vdots \\ \bar{Q}^{(L-1)}(w) \end{pmatrix}$$

under $n = 1$.

### 4 Application to isomonodromic deformations

In this section we first review some basic results on the *Schlesinger system*, which governs isomonodromic deformations of a Fuchsian system of linear ordinary differential equations. As explained in Sect. 1.4 a Schlesinger transformation preserves the monodromy of the Fuchsian system under consideration and, thereby, leads to a discrete symmetry of the associated Schlesinger system. Combining this fact with the result in Sect. 2 reveals a determinantal nature of isomonodromic deformations. Next we treat a particular case of the Schlesinger systems unifying various Painlevé-type differential equations and show its relationship with certain hypergeometric functions, which will be needed later.
4.1 Schlesinger systems and their symmetries

Let us consider again the \( L \times L \) Fuchsian system (1.7): 
\[
\frac{dY}{dz} = AY = \sum_{i=0}^{N+1} \frac{A_i}{z-u_i} Y,
\]
where \( u_0 = 1 \) and \( u_{N+1} = 0 \). We start with a well-known result on isomonodromic deformations of (4.1).

**Theorem 4.1.** The monodromy of a fundamental solution \( Y \), i.e. \( \det Y \neq 0 \), does not depend on \( u = (u_1, \ldots, u_N) \) if and only if
\[
B_i = \frac{\partial Y}{\partial u_i} Y^{-1} \quad (1 \leq i \leq N)
\]
are rational functions in \( z \).

We henceforth impose on our Fuchsian system (4.1) the following assumptions:

(i) all the residue matrices \( A_i \) are semi-simple, i.e. diagonalizable;
(ii) there is no integer difference other than zero among the eigenvalues of each \( A_i \).

Let us choose a normalization as before such that \( A_{N+1} \) and \( A_{N+2} = -\sum_{i=0}^{N+1} A_i \) are upper and lower triangular matrices, respectively. Then we can take a fundamental solution \( Y = Y(z) \) of the form
\[
Y = \Psi(z) \cdot \text{diag} \left( z^{\nu_j} \right)_{0 \leq j \leq L-1} \quad \text{(near } z = 0) \\
= \Phi(w) \cdot \text{diag} \left( w^{\nu_j} \right)_{0 \leq j \leq L-1} \cdot C \quad \text{(near } z = 1/w = \infty)
\]
with
\[
\Psi(z) = \begin{bmatrix} 1 & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & 1 \end{bmatrix} + O(z),
\]
\[
\Phi(w) = \begin{bmatrix} * & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & * \end{bmatrix} + O(w) = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ \phi & \cdots & 1 \end{bmatrix} \Xi
\]
and \( \Xi = \text{diag} \left( \xi_0, \ldots, \xi_{L-1} \right) \), where each \( \xi_j \neq 0 \) may depend on \( u = (u_1, \ldots, u_N) \). Moreover, we have
\[
Y = G_i(I + O(z-u_i))(z-u_i)^{\Lambda_i}
\]
(4.3) near each of the other regular singularities \( z = u_i \) \((0 \leq i \leq N)\), where \( G_i \) and \( \Lambda_i \) are certain constant matrices satisfying \( G_i \Lambda_i G_i^{-1} = A_i \). Therefore, the monodromy matrices of \( Y \) attached to loops around \( z = u_i \) \((0 \leq i \leq N)\) and \( z = 0, \infty \) read
\[
e^{2\pi \sqrt{-1} \Lambda_i}, \quad e^{2\pi \sqrt{-1} \text{diag} \left( \nu_{0,j} \right)_{0 \leq j \leq L-1}}, \quad C^{-1} e^{2\pi \sqrt{-1} \text{diag} \left( \nu_{\infty,j} \right)_{0 \leq j \leq L-1}} C.
\]

Suppose now that every monodromy matrix of \( Y \) is constant with respect to \( u \) and, additionally, so is the connection matrix \( C \). Then the rational functions \( B_i = B_i(z) \) can be explicitly written as
\[
B_i = \frac{A_i}{u_i - z} - \frac{1}{u_i}(A_i)_{LT},
\]
(4.4)
where \((A_i)_{LT}\) denotes the lower triangular part of \(A_i\); see Appendix for details. Note in particular that the diagonal part \((A_i)_D\) of \(A_i\) is expressible in terms of \(\Xi\) as

\[
(A_i)_D = -u_i \frac{\partial}{\partial u_i} \log \Xi. \tag{4.5}
\]

The compatibility condition

\[
\frac{\partial A}{\partial u_i} - \frac{\partial B_i}{\partial z} + [A, B_i] = 0
\]

of (4.1) and (4.2) is equivalent to a set of nonlinear differential equations for the matrices \(A_i\) with respect to \(u\), which is called the Schlesinger system \([Sch12]\). If \((L, N) = (2, 1)\), then the Schlesinger system reduces to the sixth Painlevé equation \(P_{VI}\).

Next we shall investigate how the solution \(Y\) and the coefficient \(A = A(z)\) of the Fuchsian system (4.1) are connected with each other. Concerning the power series expansion

\[
Y = \Phi(w) \cdot \text{diag} (w^{\infty,j}) \cdot C, \quad \Phi(w) = \sum_{k=0}^{\infty} \Phi_k w^k
\]

at the point of infinity \((z = 1/w = \infty)\), the coefficients \(\Phi_k\) turn out to be polynomials in the entries of \(A_{N+2} = -\sum_{i=0}^{N+1} A_i\) through Frobenius’ method. Conversely, substituting this solution \(Y\) in (4.1), we find that

\[
z \frac{dY}{dz} = -w \frac{dY}{dw}
\]

\[
= -w \frac{d}{dw} \left( \Phi_0 + \Phi_1 w + \Phi_2 w^2 + \cdots \right) \text{diag} (w^{\infty,j}) \cdot C
\]

\[
= - \left( [\Phi_0 + \Phi_1 w + \Phi_2 w^2 + \cdots] \text{diag} (w^{\infty,j}) + \Phi_1 w + 2\Phi_2 w^2 + 3\Phi_3 w^3 + \cdots \right) \text{diag} (w^{\infty,j}) \cdot C
\]

and

\[
z \sum_{i=0}^{N+1} \frac{A_i}{z - u_i} Y = \sum_{i=0}^{N+1} \frac{A_i}{1 - u_i w} Y
\]

\[
= \left[ A_{N+1} + \sum_{i=0}^{N} \left( 1 + u_i w + u_i^2 w^2 + \cdots \right) A_i \right] Y.
\]

Recall \(u_{N+1} = 0\) here. Equating the coefficients of \(w^k\) in these power series yields

\[
A_{N+2} = \Phi_0 \cdot \text{diag} (\infty,j) \cdot \Phi_0^{-1} \quad \text{for} \ k = 0,
\]

thus \(A_{N+2}\) becomes a polynomial in the entries of the leading coefficient \(\Phi_0\) of the solution, and also

\[
\sum_{i=0}^{N} u_i A_i = \left( -\Phi_1 \cdot \text{diag} (\infty,j + 1) + \Phi_0 \cdot \text{diag} (\infty,j) \cdot \Phi_0^{-1} \Phi_1 \right) \Phi_0^{-1} \quad \text{for} \ k = 1.
\]
Moreover, if one needs similar expressions for all other residue matrices $A_i$ besides $A_{N+2}$, it is convenient to use the deformation equation (4.2); one can verify in fact

$$A_i = \left( \frac{\partial \Phi_0}{\partial u_i} \Phi_0^{-1} \Phi_1 - \frac{\partial \Phi_1}{\partial u_i} \right) \Phi_0^{-1} \quad (1 \leq i \leq N).$$

In summary, each residue matrix $A_i$ of $A(z)$ is expressible as a polynomial in the entries of the coefficients of $Y$ (and their derivatives with respect to $u$), and vice versa.

A Schlesinger transformation keeps the monodromy of the Fuchsian system invariant but shifts its characteristic exponents by integers; recall Sect. 1.4. Consequently, it gives rise to a discrete symmetry of the Schlesinger system via the above correspondence between the solutions and coefficients of the Fuchsian system. On the other hand, any ingredient of Schlesinger transformations or of the associated rational approximations is described in terms of block-Toeplitz determinants; recall Sect. 2. This fact thus provides a natural explanation for the determinantal structure appearing in solutions of isomonodromic deformations, e.g. Painlevé equations. Refer also to [IMT15] for a detailed investigation on the determinantal structure in Jimbo–Miwa–Ueno’s $\tau$-functions (see [JMU81]).

### 4.2 Polynomial Hamiltonian system $\mathcal{H}_{L,N}$ of isomonodromy type

We turn now to a particular case of the Schlesinger systems, which will be the main subject in the rest of this paper.

Consider an $L \times L$ Fuchsian system of the form (4.1) whose spectral type is given by the partitions of $L$:

- \(1, L-1\) at $z = u_i$ ($0 \leq i \leq N$) and
- \(1, 1, \ldots, 1\) at $z = 0, \infty$,

which indicate how the characteristic exponents overlap at each of the $N + 3$ singularities. Fix the characteristic exponents as listed in the following table (Riemann scheme):

| Singularity | Characteristic exponents |
|-------------|--------------------------|
| $u_i$ ($0 \leq i \leq N$) | $(-\theta_i, 0, \ldots, 0)$ |
| $u_{N+1} = 0$ | $(e_0, e_1, \ldots, e_{L-1})$ |
| $u_{N+2} = \infty$ | $(\kappa_0 - e_0, \kappa_1 - e_1, \ldots, \kappa_{L-1} - e_{L-1})$ |

Assume the sum of all the characteristic exponents equals zero (Fuchs relation), i.e.

$$\sum_{i=0}^{L-1} \kappa_i = \sum_{i=0}^{N} \theta_i. \quad (4.7)$$

Let $A_{N+1}$ and $A_{N+2} = - \sum_{i=0}^{N+1} A_i$ be upper and lower triangular matrices, respectively. Then such a Fuchsian system, denoted by $\mathcal{L}_{L,N}$, can be parametrized as follows:

$$A_i = \tau \begin{pmatrix} b_i^{(i)} & b_i^{(i)} & \ldots & b_i^{(i)} \\ b_i^{(i)} & b_i^{(i)} & \ldots & b_i^{(i)} \\ \vdots & \vdots & \ddots & \vdots \\ b_i^{(i)} & b_i^{(i)} & \ldots & b_i^{(i)} \end{pmatrix} \cdot \begin{pmatrix} c_i^{(i)} & c_i^{(i)} & \ldots & c_i^{(i)} \\ c_i^{(i)} & c_i^{(i)} & \ldots & c_i^{(i)} \\ \vdots & \vdots & \ddots & \vdots \\ c_i^{(i)} & c_i^{(i)} & \ldots & c_i^{(i)} \end{pmatrix} \quad \text{with} \quad c_i^{(i)} = 1 \quad (0 \leq i \leq N),$$

$$A_{N+1} = \begin{pmatrix} e_0 & w_{0,1} & \cdots & w_{0,L-1} \\ e_1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ e_{L-1} & \cdots & w_{L-2,L-1} & e_{L-1} \end{pmatrix}. \quad (4.8)$$
under the relations

\[
(tr A_i) - \theta_i = \sum_{k=0}^{L-1} b_k^{(i)} c_k^{(i)}, \quad \kappa_i = -\sum_{j=0}^{N} b_j^{(i)} c_j^{(i)} \quad \text{and} \quad w_{k,l} = -\sum_{i=0}^{N} b_k^{(i)} c_l^{(i)} \quad (k < l);
\]

the last two of which come from the triangularity of \(A_{N+2}\). Also, we can and will normalize the characteristic exponents at \(z = 0\) by

\[
tr A_{N+1} = \sum_{k=0}^{L-1} e_k = \frac{L - 1}{2}
\]

without loss of generality.

As shown in [Tsu14a], the Schlesinger system governing isomonodromic deformations of \(L_{L,N}\) reduces to the multi-time Hamiltonian system \(H_{L,N}\):

\[
\frac{\partial q_k^{(i)}}{\partial x_j} = \frac{\partial H_j}{\partial p_k^{(i)}}, \quad \frac{\partial p_k^{(i)}}{\partial x_j} = -\frac{\partial H_j}{\partial q_k^{(i)}} \quad \left(1 \leq i, j \leq N, 1 \leq k \leq L - 1\right).
\]

Here we let \(x_i = 1/u_i\) and define the Hamiltonian function \(H_i\) by

\[
x_i H_i = \sum_{k=0}^{L-1} e_k q_k^{(i)} p_k^{(i)} + \sum_{j=0}^{N} \sum_{0 \leq k < j \leq L-1} q_k^{(j)} p_k^{(j)} q_j^{(i)} p_j^{(i)} + \sum_{j=0}^{N} \frac{x_j}{x_l} \sum_{k,l=0}^{L-1} q_k^{(i)} p_k^{(i)} q_l^{(i)} p_l^{(i)}
\]

with \(x_0 = q_0^{(0)} = q_0^{(i)} = 1, p_0^{(0)} = \kappa_i - \sum_{j=1}^{N} q_j^{(0)} p_j^{(0)}\) and \(p_0^{(i)} = \theta_i - \sum_{k=1}^{L-1} q_k^{(i)} p_k^{(i)}\). Therefore, \(H_i\) is a polynomial in the unknowns (canonical variables)

\[
q_k^{(i)} = \frac{c_k^{(i)}}{c_k^{(0)}} \quad \text{and} \quad p_k^{(i)} = -b_k^{(i)} c_k^{(0)} \quad \left(1 \leq i \leq N, 1 \leq k \leq L - 1\right).
\]

The number of the constant parameters

\[
(e, \kappa, \theta) = (e_0, \ldots, e_{L-1}, \kappa_0, \ldots, \kappa_{L-1}, \theta_0, \ldots, \theta_N)
\]

contained in \(H_{L,N}\) is essentially \(2L + N - 1\) in view of (4.7) and (4.9). For example, the case where \(L = 2\) and any \(N \geq 1\) coincides with the Garnier system in \(N\)-variables and, thereby, the first nontrivial case \(H_{2,1}\) does with the Hamiltonian form of \(P_{V1}\).

**Remark 4.2.** We have a priori known from their spectral type that the Fuchsian systems equipped with the Riemann scheme (4.6) constitute a \(2N(L-1)\)-dimensional family. The coordinates of such a family are called accessory parameters, which are realized by the \(2N(L - 1)\) canonical variables (4.10) in this instance.

### 4.3 Solution of \(H_{L,N}\) in terms of hypergeometric function \(F_{L,N}\)

Although the phase space of \(H_{L,N}\) is quite-complicated algebraic variety in general, there exists a family of solutions parametrized by a point in the projective space \(\mathbb{P}^{N(L-1)}\) when the constants

\[
\text{...}
\]
$(e, \kappa, \theta)$ take certain special values. In fact, these solutions are written in terms of the hypergeometric function

$$F_{L,N} \left[ \begin{array}{c} \alpha, \beta \\ \gamma \end{array} ; x \right] = \sum_{m_i \geq 0} \frac{(\alpha_1)_{|m_1|} \cdots (\alpha_{L-1})_{|m_1|}(\beta_1)_{m_1} \cdots (\beta_N)_{m_N} x_1^{m_1} \cdots x_N^{m_N}}{(\gamma_1)_{|m_1|} \cdots (\gamma_{L-1})_{|m_1|} m_1! \cdots m_N!}.$$  

(4.12)

where $|m| = m_1 + \cdots + m_N$ and $(a)_k = \Gamma(a+k)/\Gamma(a)$. If $(L, N) = (2, 1)$, then (4.12) is exactly Gauß’s hypergeometric function.

To state the result precisely, we introduce the integral representation

$$y_0 = \int_c \frac{U(t) \, dt_1 \cdots dt_{L-1}}{\prod_{l=1}^{L-1} (t_{l-1} - t_l)} = \prod_{l=1}^{L-1} \frac{\Gamma(\alpha_l)\Gamma(\gamma_l - \alpha_l)}{\Gamma(\gamma_l)} \times F_{L,N} \left[ \begin{array}{c} \alpha, \beta \\ \gamma \end{array} ; x \right]$$

(4.13)

of $F_{L,N}$, where the multi-valued function $U = U(t)$ in $t = (t_1, t_2, \ldots, t_{L-1})$ is given by

$$U(t) = \prod_{l=1}^{L-1} t_l^{\alpha_l - \gamma_l} (t_{l-1} - t_l)^{\gamma_l - \alpha_l} \prod_{l=1}^{N} (1 - x_l t_{L-1})^{-\beta_l} \quad \text{and} \quad t_0 = \gamma_L = 1,$$

and the cycle $c$ is chosen to be an $(L - 1)$-simplex

$$\{0 \leq t_{L-1} \leq \cdots \leq t_2 \leq t_1 \leq 1 \} \subset \mathbb{R}^{L-1}.$$  

(4.14)

Also, we introduce supplementarily the integrals

$$y_k^{(i)} = \int_c \frac{U(t) \, dt_1 \cdots dt_{L-1}}{x_i t_{L-1} - 1 \prod_{l=1}^{L-1} (t_{l-1} - t_l)} \left( \begin{array}{c} 1 \leq i \leq N \\ 1 \leq k \leq L - 1 \end{array} \right).$$

We are now ready to state the hypergeometric solution of $\mathcal{H}_{L,N}$; see [Tsu12b Theorem 3.2].

**Theorem 4.3.** If $\kappa_0 - \sum_{i=1}^{N} \theta_i = 0$ then the Hamiltonian system $\mathcal{H}_{L,N}$ possesses a solution

$$q_k^{(i)} = 0 \quad \text{and} \quad p_k^{(i)} = \theta_i y_k^{(i)}$$

under the correspondence

$$\alpha_k = e_k - e_0, \quad \beta_i = -\theta_i, \quad \gamma_k = e_k - e_0 - \kappa_k$$

of constant parameters.

The vector-valued function

$$y = \left[ y_0, y^{(1)}_1, \ldots, y^{(1)}_{L-1}, y^{(2)}_1, \ldots, y^{(2)}_{L-1}, \ldots, y^{(N)}_1, \ldots, y^{(N)}_{L-1} \right]$$

satisfies a certain linear Pfaffian system $\mathcal{P}_{L,N}$ of rank $N(L - 1) + 1$, whose fundamental solution is prepared by collecting admissible cycles along with the foregoing $(L - 1)$-simplex (4.14). Note that the linear space of these cycles, i.e. *twisted de Rham homology group*, is generated by the
chambers framed by the real section of branch locus of \( U = U(t) \); see [AK11, Tsu12b]. Of course, Theorem 4.3 is valid for any solution \( \{y_0, y^{(i)}_k\} \) of \( \mathcal{P}_{L,N} \).

The Fuchsian system \( \mathcal{L}_{L,N} \) is specialized as \( \kappa_0 - \sum_{i=1}^N \theta_i = 0 \) and

\[
\begin{align*}
  b_0^{(0)} &= 0, & b_k^{(0)} &= \kappa_k y_0, & b_0^{(i)} &= -\theta_i, & b_k^{(i)} &= \theta_i y_k^{(i)}, & c_k^{(0)} &= \frac{-1}{y_0}, & c_k^{(i)} &= 0
\end{align*}
\]

along the above hypergeometric solution of \( \mathcal{H}_{L,N} \); it thus becomes reducible. In fact, via the gauge transformation

\[
Y = \left( w^{s_0 - \kappa_0} \prod_{i=1}^N (1 - u_i w)^{-\theta_i} u_i^{\theta_i} \right) Y',
\]

we have a solution of the form

\[
Y' = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ f_1 & \cdots & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ f_{L-1} & \cdots & \cdots & 1 \end{bmatrix} + O(w),
\]

where \( f_k = f_k(w) (1 \leq k \leq L - 1) \) are holomorphic functions at \( w = 0 \) defined by the integrals

\[
f_k = \int_c \frac{U(t) \ dt_1 \cdots dt_{L-1}}{(1 - wt_{L-1}) \prod_{i=1}^{L-1} (t_{L-1} - t_i)}.
\]

Note that an \((L - 1) \times (L - 1)\) matrix \( W \) can be described by Thomae’s hypergeometric function \( \Gamma_{L}F_{L-2} \). For details we refer to [Tsu14b], in which a curious coincidence between \( \mathcal{P}_{L,N+1} \) and the Lax pair of \( \mathcal{H}_{L,N} \), i.e. the pair of the original Fuchsian system (4.1) and its deformation equation (4.2), is also discussed.

Our aim here is to generalize Theorem 4.3 by application of Schlesinger transformations starting from this hypergeometric solution at \( \kappa_0 - \sum_{i=1}^N \theta_i = 0 \). Notice that the Schlesinger transformation established in Theorem 4.5 shifts the constant parameters (4.11) as

\[
(\kappa_0, \kappa_1, \ldots, \kappa_{L-1}) \mapsto (\kappa_0 + n(L - 1), \kappa_1 - n, \ldots, \kappa_{L-1} - n)
\]

while all the others are unchanged; cf. the Riemann scheme (4.6). Hence, by virtue of the algebraic relation between the solution of \( \mathcal{L}_{L,N} \) and the canonical variables of \( \mathcal{H}_{L,N} \) (recall Sect. 4.1 and also (4.8) and (4.10)), we know in principle how to derive a solution \( (q_k^{(i)}, p_k^{(i)}) \) of \( \mathcal{H}_{L,N} \) at \( \kappa_0 - \sum_{i=1}^N \theta_i = n(L - 1) \) for any positive integer \( n \) even though the resulting expression in this way will be terribly complicated. In the next section we explore this problem to achieve much simpler formulae for these special solutions.

### 5 Solutions of \( \mathcal{H}_{L,N} \) in terms of iterated hypergeometric integrals

This section is concerned with the Schlesinger transform of the hypergeometric solution of \( \mathcal{H}_{L,N} \). We present its explicit formula by using the block-Toeplitz determinant whose entries are given
by the hypergeometric functions. Key ingredients of the argument are the determinantal representations for the approximation polynomials; see Sect. 2. Moreover, we prove through Fubini’s theorem and the Vandermonde determinant that these block-Toeplitz determinants can be written in the form of iterated hypergeometric integrals. Our result will be summarized in Theorem 5.3, which is regarded as a generalization of Theorem 4.3, i.e. the previously known hypergeometric solution of $H_{L,N}$; cf [Tsu12b].

5.1 Preliminaries

Let $f_0(w), f_1(w), \ldots, f_{L-1}(w)$ be the functions defined by

$$f_0(w) \equiv 1,$$

$$f_k(w) = \int_c \frac{U(t) \, dt_1 \cdots dt_{L-1}}{(1 - wt_{L-1}) \prod_{l \neq k}^{L-1} (t_l - t_i)} \quad (1 \leq k \leq L - 1).$$

If the cycle $c$ is chosen such that $|t_{L-1}| < \infty$, then $f_k(w)$ is holomorphic at $w = 0$. For instance, it is enough to choose a bounded cycle as $c$. Accordingly, we have a power series expansion

$$f_k(w) = \sum_{j=0}^{\infty} h^k_j w^j = h^k_0 + h^k_1 w + h^k_2 w^2 + \cdots$$

with the coefficients

$$h^k_j = \int_c t_{L-1}^j U(t) \, dt_1 \cdots dt_{L-1} \prod_{l \neq k}^{L-1} (t_l - t_i)$$

(5.1)

for $1 \leq k \leq L - 1$. Observe that each $h^k_j$ can be regarded as a moment

$$h^k_j = \int_{pr(c)} s^j \, d\mu_k(s)$$

of the ‘measure’

$$d\mu_k(t_{L-1}) = \left( \int_{c[t_{L-1}]} \frac{U(t) \, dt_1 \cdots dt_{L-2}}{\prod_{l \neq k}^{L-1} (t_l - t_i)} \right) \, dt_{L-1}$$

upon the following notations:

$$pr(c) = \{ t_{L-1} \mid (t_1, \ldots, t_{L-2}, t_{L-1}) \in c \},$$

$$c[t_{L-1}] = \{ (t_1, \ldots, t_{L-2}) \mid (t_1, \ldots, t_{L-2}, t_{L-1}) \in c \}.$$

Namely $f_k(w) \ (1 \leq k \leq L - 1)$ is written as the Stieltjes transform

$$f_k(w) = \int_{pr(c)} \frac{d\mu_k(s)}{1 - ws}$$

of a function $\mu_k = \mu_k(s)$. 25
In parallel, we introduce the functions

\[ h^0_j = \int_c t_{L-1}U(t) \prod_{i=1}^{L-1} (t_i - t) \]  

also. We thus see that

\[ h^0_j = \int_{c(t_{L-1})} s^j \, d\mu_0(s), \]

where

\[ d\mu_0(t_{L-1}) = \left( \int_{c(t_{L-1})} \frac{U(t) \, dt \prod_{i=1}^{L-2} (t_i - t)}{\prod_{i=1}^{L-1}(t_i - t)} \right) dt_{L-1}. \]

**Lemma 5.1.** The following linear relations (contiguity relations) hold:

\[ h^1_j + h^2_j + \cdots + h^{L-1}_j = h^0_j - h^0_{j+1}, \]  

\[ h^k_j - \ell_i h^1_{j+1} = \ell_i(h^k_j), \]

where \( \ell_i \) denotes the down-shift operator with respect to \( \beta_i \) defined by

\[ \ell_i(\beta_i) = \beta_i - 1 \quad \text{and} \quad \ell_i(\beta_j) = \beta_j \quad (i \neq j). \]

**Proof.** By definitions (5.1) and (5.2) it is immediate to verify these formulae. \( \square \)

**Remark 5.2.** If \( c \) is chosen to be the \((L - 1)\)-simplex (4.14):

\[ \{0 \leq t_{L-1} \leq \cdots \leq t_2 \leq t_1 \leq 1\} \subset \mathbb{R}^{L-1}, \]

both (5.1) and (5.2) are written by the hypergeometric function \( F_{L,N} \). Let us introduce the function

\[ h = h \left[ \begin{array}{c} \alpha, \beta \\ \gamma \end{array} ; x \right] = \prod_{l=1}^{L-1} \frac{\Gamma(\alpha_l)\Gamma(\gamma_l - \alpha_l)}{\Gamma(\gamma_l)} \times F_{L,N} \left[ \begin{array}{c} \alpha, \beta \\ \gamma \end{array} ; x \right]. \]

Then it holds by definition that

\[ h^k_j = h \left[ \begin{array}{c} \alpha_1 + j + 1, \ldots, \alpha_{k-1} + j + 1, \alpha_k + j, \ldots, \alpha_{L-1} + j, \beta \\ \gamma_1 + j + 1, \ldots, \gamma_{k-1} + j + 1, \gamma_k + j, \ldots, \gamma_{L-1} + j \end{array} ; x \right] \]  

\[(0 \leq k \leq L - 1).\]

For example we have

\[ h^0_j = h \left[ \begin{array}{c} \alpha_1 + j, \ldots, \alpha_{L-1} + j, \beta \\ \gamma_1 + j, \ldots, \gamma_{L-1} + j \end{array} ; x \right], \]

\[ h^1_j = h \left[ \begin{array}{c} \alpha_1 + j, \ldots, \alpha_{L-1} + j, \beta \\ \gamma_1 + j + 1, \gamma_2 + j, \ldots, \gamma_{L-1} + j \end{array} ; x \right], \]

\[ h^2_j = h \left[ \begin{array}{c} \alpha_1 + j + 1, \alpha_2 + j, \ldots, \alpha_{L-1} + j, \beta \\ \gamma_1 + j + 1, \gamma_2 + j + 1, \gamma_3 + j, \ldots, \gamma_{L-1} + j \end{array} ; x \right] \]

and so forth.
It is convenient to prepare the notation of the block-Toeplitz determinant (cf. Remark 2.2) for any $L$-tuple of nonnegative integers:

$$n = (n_0, n_1, \ldots, n_{L-1}) \in (\mathbb{Z}_{\geq 0})^L.$$  

Let $|n| = \sum_{i=0}^{L-1} n_i$. We set

$$\Delta^{(k)}(n) := \det \begin{pmatrix} A^0_{[n]}(n_0, n_0) & \cdots & A^{k-1}_{[n]}(n_0, n_{k-1}) & A^k_{[n]}(n, n_k) & \cdots & A^{L-1}_{[n]}(n, n_{L-1}) \\ \vdots & & & & & \vdots \\ A_{[n]}^0(n_0, |n|) & \cdots & A_{[n]}^{k-1}(n_{k-1}, |n|) & A_{[n]}^k(n_k, |n|) & \cdots & A_{[n]}^{L-1}(n_{L-1}, |n|) \end{pmatrix}$$

for each $k$ ($0 \leq k \leq L$). If $n = 0 = (0, \ldots, 0)$, we fix $\Delta^{(k)}(0) = 1$. As well as (2.2), the symbol $A_j^i(k, l)$ denotes the $k \times l$ rectangular Toeplitz matrix for the sequence $\{h_j^i\}_{j=0}^\infty$ whose top left corner is $h_j^i$, and $h_j^i = 0$ if $j < 0$. The second equality in (5.5) can be verified easily from (2.3). Henceforth we suppose

$$n = (0, n, \ldots, n)$$

unless expressly stated otherwise. We will often abbreviate $\Delta^{(k)}(n)$ for $n = (0, n, \ldots, n)$ as $\Delta^{(k)}$. Note that this convention is consistent with the description in Remark 2.2. We also prepare the canonical basis $\{e_0, e_1, \ldots, e_{L-1}\}$ of $\mathbb{Z}^L$, i.e.

$$e_k = (0, \ldots, 0, 1, 0, \ldots, 0).$$

As seen in (4.15), the coefficient

$$A(z) = \sum_{i=0}^{N+1} \frac{A_i}{z - u_i} \quad (u_0 = 1, u_{N+1} = 0)$$

of the Fuchsian system $\mathcal{L}_{L,N}$ attached to the hypergeometric solution of $\mathcal{H}_{L,N}$ with $\kappa_0 - \sum_{i=1}^N \theta_i = 0$ (see Theorem 4.3) is expressed as $A_i = b^{(i)} c^{(i)} (0 \leq i \leq N)$, where

$$b^{(0)} = h_0^0 \cdot \tau (0, \kappa_1, \kappa_2, \ldots, \kappa_{L-1}), \quad c^{(0)} = \left(1, \frac{-1}{h_0^0}, \ldots, \frac{-1}{h_0^0}\right), \quad \text{and}$$

$$b^{(i)} = -\tau \left(1, \ell_i^{-1}(h_0^1), \ell_i^{-1}(h_0^2), \ldots, \ell_i^{-1}(h_0^{L-1})\right), \quad c^{(i)} = (1, 0, \ldots, 0) \quad \text{for} \ 1 \leq i \leq N. \quad (5.6)$$

Cf. (4.8). Our next task is applying the Schlesinger transformation to this Fuchsian system.
5.2 Calculation of the Schlesinger transform (I)

To derive the action of the Schlesinger transformation, we need basically to deal with (1.10):

\[ A \mapsto \hat{A} = \sum_{i=0}^{N+1} \frac{\hat{A}_i}{z - u_i} = RAR^{-1} + \frac{dR}{dz}R^{-1}. \]

Namely, since both \( R = R(z) \) and \( R^{-1} \) are polynomials in \( z \), each residue matrix \( \hat{A}_i \) can be calculated by

\[ \hat{A}_i = R(u_i)A_i R^{-1}(u_i) \quad \text{for} \quad 0 \leq i \leq N + 1. \]

(5.7)

However, thanks to (4.5), it is rather easy to calculate the diagonal parts even in the general case. First we will demonstrate it.

The multiplier \( R = R(z) \) of the Schlesinger transformation (see Theorem 1.5) can be written, a little more specifically than (1.9), as

\[ R = w^{-n} \begin{bmatrix} 0 & * & Q^{(1)}_1(0) & * & Q^{(2)}_2(0) & * & \cdots & \cdots & \cdots & * & Q^{(L-1)}_{L-1}(0) \\ * & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} + O(w) \]

Recall (1.5). Multiplying

\[ \Phi = \begin{bmatrix} 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & \ddots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ * & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} + O(w) \]

by \( R \) from the left yields \( R\Phi = \hat{\Phi} \cdot \text{diag}(w^{n(L-1)}, w^{-n}, \ldots, w^{-n}) \) with

\[ \hat{\Phi} = \begin{bmatrix} 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & \ddots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ * & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} + O(w) \]

as shown in Theorem 1.5, where both \( \Xi \) and \( \hat{\Xi} \) are diagonal matrices independent of \( w = 1/z \). We mention, without fear of repetition, that the Hermite-Padé approximation condition (1.1) assures

\[ Rf = O(w^{n(L-1)}) = \begin{bmatrix} \rho_{nL}^0 \\ \rho_{nL}^1 \\ \vdots \\ \rho_{nL}^{L-1} \end{bmatrix} w^{n(L-1)} + (\text{terms of higher order}) \]

with \( f = [f_0, \ldots, f_{L-1}] \) denoting the first column of \( \Phi \). We thus find the formula

\[ \hat{\Xi}\Xi^{-1} = \text{diag} \left( \rho_{nL}^0, Q^{(1)}_1(0), Q^{(2)}_2(0), \ldots, Q^{(L-1)}_{L-1}(0) \right) \]

\[ = (-1)^n \cdot \text{diag} \left( (-1)^n \mu L \frac{\Delta^{(L)}}{\Delta^{(1)}}, \frac{\Delta^{(1)}}{\Delta^{(2)}}, \frac{\Delta^{(2)}}{\Delta^{(3)}}, \ldots, \frac{\Delta^{(L-1)}}{\Delta^{(L)}} \right) \]
by virtue of Remark 2.2. Combining this with (4.5) under \( x_i = 1/u_i \):

\[
(A_i)_D = x_i \frac{\partial}{\partial x_i} \log \Xi,
\]

we arrive at the formulae

\[
(\hat{A}_i)_D - (A_i)_D = x_i \frac{\partial}{\partial x_i} \log \text{diag} \left( \frac{\Delta^{(L)}}{\Delta^{(1)}}, \frac{\Delta^{(1)}}{\Delta^{(2)}}, \ldots, \frac{\Delta^{(L-1)}}{\Delta^{(L)}} \right)
\]

\[
= x_i \text{ diag} \left( \frac{\mathcal{D}_i A^{(L)}}{\Delta^{(L)}} \Delta^{(1)}, \frac{\mathcal{D}_i A^{(1)}}{\Delta^{(1)}} \Delta^{(2)}, \ldots, \frac{\mathcal{D}_i A^{(L-1)}}{\Delta^{(L-1)}} \Delta^{(L)} \right)
\]

(5.8)

for \( 1 \leq i \leq N \), where \( \mathcal{D}_i \) denotes the Hirota differential with respect to \( \partial/\partial x_i \).

Next we turn to the particular case, i.e. the Fuchsian system \( \mathcal{L}_{L,N} \) upon the substitution (4.15) corresponding to the hypergeometric solution (see Theorem 4.3). Write the residue matrices as \( \hat{A}_i = \hat{b}^{(i)} \hat{c}^{(i)} \) \((0 \leq i \leq N)\). In order to reconstruct the canonical variables \((\hat{a}_k^{(i)}, \hat{p}_k^{(i)})\) of the Schlesinger transform of \( \mathcal{H}_{L,N} \), it is only necessary to know the quantities

\[
\hat{c}^{(i)} = (\hat{c}_k^{(i)})_{0 \leq k \leq L-1} = (1, \hat{c}_1^{(i)}, \ldots, \hat{c}_{L-1}^{(i)}) \quad \text{for} \quad 0 \leq i \leq N
\]

(5.9)

because we have already known from (5.8) the diagonal entries \((\hat{A}_i)_{k,k} = \hat{b}_k^{(i)} \hat{c}_k^{(i)} = -\hat{q}_k^{(i)} \hat{p}_k^{(i)} \) for \( 1 \leq i \leq N \); cf. (4.10). Calculations of (5.9) are a little complicated; therefore, we will separately carry out the cases \( i = 0 \) and \( i \neq 0 \) in Sects. 5.3 and 5.4 respectively. The result can be found in Sect. 5.5

### 5.3 Calculation of the Schlesinger transform (II)

Consider the row vector \( c^{(0)} R^{-1}(1) \) in view of (5.7) and \( u_0 = 1 \), where \( c^{(0)} = (1, -1/h_0^0, \ldots, -1/h_0^0) \); recall (5.6).

(i) The 0th component \[ \text{It follows from the determinantal representation of the polynomial} \]

\[
P^{(0)}_0(w) \text{ and } P^{(0)}_l(w) = \left[ f_l P^{(0)}_m \right]_{0}^{m-1} \text{ for } l > 0 \text{ (see Proposition 2.3 and its sequel) that}
\]

\[
P^{(0)}_l(1) = \frac{1}{NP^{(0)}} \det \begin{bmatrix}
h_0^0 \cdots h_{m-1}^0, h_0^l \cdots h_{m-2}^l, \ldots, h_0^l, h_0^l, 0 \\
A^1_m(n, m + 1) \\
\vdots \\
A^{L-1}_m(n, m + 1)
\end{bmatrix}.
\]

Summation over \( l = 1, 2, \ldots, L - 1 \) of this formula entails

\[
\sum_{l=1}^{L-1} P^{(0)}_l(1) = \frac{1}{NP^{(0)}} \det \begin{bmatrix}
h_0^0 - h_{m}^0, h_0^0 - h_{m-1}^0, \ldots, h_0^0 - h_{m-2}^0, h_0^0 - h_{m-1}^0, 0 \\
A^1_m(n, m + 1) \\
\vdots \\
A^{L-1}_m(n, m + 1)
\end{bmatrix}
\]


via the contiguity relation (5.3). Hence the 0th component of $e^{(0)} R^{-1}(1)$ reads

$$P_0^{(0)}(1) - \frac{1}{\hbar_0} \sum_{i=1}^{L-1} P_i^{(0)}(1) = \frac{1}{\hbar_0^0 \mathbb{P}^{(0)}} \det \begin{bmatrix} h_0^0, h_0^0, \ldots, h_0^0 \\ A_m^1(n, m+1) \\ \vdots \\ A_m^{L-1}(n, m+1) \end{bmatrix} = \frac{\Delta^{(0)}(n + e_0)}{\hbar_0^0 \Delta^{(0)}(n)}. \quad (5.10)$$

Here we have used (2.7).

(ii) The $k(>0)$th component Similarly it holds for $l > 0$ that

$$P_{l}^{(k)}(1) = \frac{1}{\mathbb{P}^{(k)}} \det \begin{bmatrix} h_0^l + \cdots + h_{m-1}^l, h_0^l + \cdots + h_{m-2}^l, \ldots, h_0^l + h_1^l, h_0^0 \\ A_m^1(n, m) \\ \vdots \\ A_m^{k-1}(n, m) \\ A_m^k(n - 1, m) \\ \vdots \\ A_m^{k+1}(n, m) \\ \vdots \\ A_m^{L-1}(n, m) \end{bmatrix}.$$ 

Taking a sum and using (5.3) thus yield

$$P_0^{(k)}(1) - \frac{1}{\hbar_0} \sum_{i=1}^{L-1} P_i^{(k)}(1) = \frac{1}{\hbar_0^0 \mathbb{P}^{(k)}} \det \begin{bmatrix} h_0^0, h_0^0, \ldots, h_1^0 \\ A_m^1(n, m) \\ \vdots \\ A_m^{k-1}(n, m) \\ A_m^k(n - 1, m) \\ \vdots \\ A_m^{k+1}(n, m) \\ \vdots \\ A_m^{L-1}(n, m) \end{bmatrix} = \frac{(-1)^{nk+1} \Delta^{(k+1)}(n + e_0 - e_k)}{\hbar_0^0 \Delta^{(k)}(n)}. \quad (5.11)$$

Here we have used (2.8).

The ratio of (5.10) and (5.11) leads to the formula

$$\hat{c}_k^{(0)} = (-1)^{nk+1} \frac{\Delta^{(k)}(n)\Delta^{(k+1)}(n + e_0 - e_k)}{\Delta^{(k)}(n)\Delta^{(0)}(n + e_0)}.$$ 

### 5.4 Calculation of the Schlesinger transform (III)

Consider for $1 \leq i \leq N$ the row vector $e^{(i)} R^{-1}(u_i)$ in view of (5.7) and $u_i = 1/x_i$, which is nothing but the top row of the matrix $R^{-1}(u_i)$ due to $e^{(i)} = (1, 0, \ldots, 0)$; recall (5.6).
(i) The 0th component

We are interested in the following determinant

\[ x_i^{-m} P_0^{(0)}(x_i) = \frac{x_i^{-m}}{N P^{(0)}} \det \begin{bmatrix} 1, x_i, \ldots, x_i^m \\ A_m^{(n, m)} \\ \vdots \\ A_m^{L-1}(n, m + 1) \end{bmatrix} \]

Subtracting the \((j - 1)\)th column from the \(j\)th column for \(1 \leq j \leq L - 1\) and using the contiguity relation \(5.4\), we thus obtain

\[ x_i^{-m} P_0^{(0)}(x_i) = \frac{x_i^{-m}}{N P^{(0)}} \ell_i \left( A_m^{(n, m)} \right) = \frac{x_i^{-m}}{N P^{(0)}} \ell_i \left( A_m^{(n, m)} \right) \]

\[ = \frac{(-1)^m x_i^{-m} \ell_i \left( \Delta^{(0)}(n) \right)}{\Delta^{(L)}(n)}. \tag{5.12} \]

(ii) The \(k(>0)\)th component

In the same way as above, we find

\[ x_i^{-m+1} P_0^{(k)}(x_i) = \frac{x_i^{-m+1}}{N P^{(k)}} \det \begin{bmatrix} 1, x_i, \ldots, x_i^{m-1} \\ A_m^{(n, m)} \end{bmatrix} = \frac{x_i^{-m+1}}{N P^{(k)}} \ell_i \left( A_m^{(n, m-1)} \right) \]

\[ = \frac{(-1)^{m(L-k-1)} x_i^{-m+1} \ell_i \left( \Delta^{(k+1)}(n - e_k) \right)}{\Delta^{(k)}(n)}. \tag{5.13} \]

Hence we verify from the ratio of \(5.12\) and \(5.13\) that

\[ c^{(i)}_k = (-1)^k x_i \frac{\Delta^{(L)}(n) \ell_i \left( \Delta^{(k+1)}(n - e_k) \right)}{\Delta^{(k)}(n) \ell_i \left( \Delta^{(0)}(n) \right)}. \]

### 5.5 Result

Summarizing the above we are led to the following result.
Remark 5.4. Under the correspondence of constant parameters, where

\[ H = \Delta \]

Theorem 5.3. Calculating \( \hat{q}_k^{(i)} \), \( \hat{p}_k^{(i)} \) given by

\[
\hat{q}_k^{(i)} = -x_i \frac{\Delta^{(0)}(n + e_0)\ell_i(\Delta^{(k+1)}(n - e_k))}{\Delta^{(k+1)}(n + e_0 - e_k)\ell_i(\Delta^{(0)}(n))},
\]

\[ \hat{q}_k^{(i)} \hat{p}_k^{(i)} = -x_i \frac{\mathcal{D}_i\Delta^{(k)}(n) \cdot \Delta^{(k+1)}(n)}{\Delta^{(k)}(n) \Delta^{(k+1)}(n)} \]

under the correspondence

\[ \alpha_k = e_k - e_0, \quad \beta_i = -\theta_i, \quad \gamma_k = e_k - e_0 - \kappa_k + n \]

of constant parameters, where \( n = (0, n, \ldots, n) \in (\mathbb{Z}_{\geq 0})^L \).

Remark 5.4. We have in fact an alternative expression

\[
\hat{q}_k^{(i)} \hat{p}_k^{(i)} = \theta_i x_i \frac{\ell_i^{-1}(\Delta^{(k)}(n + e_k)) \ell_i(\Delta^{(k+1)}(n - e_k))}{\Delta^{(k)}(n) \Delta^{(k+1)}(n)}
\]

of the above solution with no use of the Hirota differentials. This formula can be verified by calculating \( \hat{b}^{(i)} \) in the same manner as Sects. 5.3 and 5.4 further details might be left to the reader.

As seen in Theorem 5.3, we have constructed a particular solution of \( \mathcal{H}_{L,N} \) expressed in terms of the block-Toeplitz determinant \( \Delta^{(k)}(n) \) whose entries are given by the hypergeometric functions. Finally we shall rewrite \( \Delta^{(k)}(n) \) as an iterated hypergeometric integral: if we remember that \( h_j^i \) is defined as a moment (see Sect. 5.1), then we observe that

\[
\Delta^{(k)}(n) = \int \det \left[ \mathbb{I}_{l_0, \ldots, l_m} \right]_{l_j \leq n_j} \times \prod_{a=0}^{L-1} \prod_{j=1}^{n_a} d\mu_a(s_{a,j})
\]

\[
= \int \det \left[ \mathbb{I}_{l_0, \ldots, l_m} \right]_{l_j \leq n_j} \times \prod_{a=0}^{L-1} \prod_{j=1}^{n_a} s_{a,j}^{-1} d\mu_a(s_{a,j})
\]

through Fubini’s theorem. The Vandermonde determinant shows that

\[
\det \left[ \mathbb{I}_{l_0, \ldots, l_m} \right]_{l_j \leq n_j} = \prod_{(a,b) > (c,d)} (s_{a,b} - s_{c,d}),
\]

where \((a, b) > (c, d)\) is defined to mean

\[ “a > c” \quad \text{or} \quad “a = c \text{ and } b > d”. \]
Also it holds that
\[ \sum_{\sigma \in \text{Sym}(n_a)} \text{sgn} \sigma \prod_{j=1}^{n_a} s_{a,\sigma(j)}^{n_a-j} = \prod_{b \leq c} (s_{a,b} - s_{a,c}), \]
where \( \text{Sym}(n_a) \) denotes the symmetric group on \( \{1, 2, \ldots, n_a\} \). Therefore, because the value of integral (5.14) is invariant under a permutation of the variables \( \{s_{a,b}\}_{1 \leq b \leq n_a} \) for each \( 0 \leq a \leq L - 1 \), we conclude that
\[ \Delta^{(k)}(n) = \prod_{a=0}^{L-1} \frac{1}{n_a!} \int \left( \prod_{a=0}^{n_a} s_{a,j} \right) \left( \prod_{(a,b) \neq (c,d)} (s_{a,b} - s_{c,d}) \right) \prod_{a=0}^{L-1} \left( \prod_{b \leq c} (s_{a,b} - s_{a,c}) \right) \prod_{j=1}^{n_a} d\mu_a(s_{a,j}). \]

A Verification of (4.4)

Usually, we often normalize the Fuchsian system (4.1) so that \( A_{N+2} = -\sum_{i=0}^{N+1} A_i \) (the residue matrix at \( z = \infty \)) is diagonal when considering its isomonodromic deformation; see e.g. [MU81]. But, in this paper, we adopt a different normalization treating the two points \( z = 0 \) and \( z = \infty \) equally; i.e. we choose \( A_{N+1} \) (the residue matrix at \( z = 0 \)) and \( A_{N+2} \) to be upper and lower triangular, respectively. Note that the latter normalization is more versatile in a general setting than the former.

In this appendix, for a supplement to Sect. 4.1, we demonstrate how to determine the coefficient \( B_i = B_i(z) \) of the deformation equation (4.2):
\[ \frac{\partial Y}{\partial u_i} = B_i Y \]
of (4.1).

Differentiating (4.3) with respect to \( u_j \) tells us that
- if \( i \neq j \) then \( \frac{\partial Y}{\partial u_j} Y^{-1} \) is holomorphic at \( z = u_i \);
- if \( i = j \) then \( \frac{\partial Y}{\partial u_i} Y^{-1} = -\frac{A_i}{z - u_i} + \text{(holomorphic at } z = u_i). \)

Similarly, we observe that
\[ \frac{\partial Y}{\partial u_i} Y^{-1} = \frac{\partial \Psi}{\partial u_i} \Psi^{-1} = \begin{bmatrix} 0 & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & 0 \end{bmatrix} + O(z) \text{ near } z = 0 \]
(A.1)

and
\[ \frac{\partial Y}{\partial u_i} Y^{-1} = \frac{\partial \Phi}{\partial u_i} \Phi^{-1} = \begin{bmatrix} * & \cdots \\ \vdots & \ddots \\ * & \cdots \end{bmatrix} + O(w) \text{ near } z = 1/w = \infty. \]
(A.2)

Here we have used the assumption that the connection matrix \( C \) does not depend on \( u_i \). Consequently, \( B_i = \frac{\partial Y}{\partial u_i} Y^{-1} \) is a rational function matrix in \( z \) that has only a simple pole at \( z = u_i \) with residue \(-A_i\) and thus
\[ B_i = \frac{A_i}{u_i - z} + K_i, \]
}\]
where $K_i$ is a constant matrix. Substituting $z = \infty$ ($w = 0$) in (A.2) shows that $K_i$ is a lower triangular matrix. Therefore, substituting $z = 0$ in (A.1), we conclude that $K_i = -(A_i)_{1T}/u_i$.

**Remark A.1.** Such a normalization as above emerges naturally from the similarity reduction of the UC hierarchy, which is a context of infinite-dimensional integrable systems; see [Tsu04, Tsu12a, Tsu14a]. Furthermore, as clarified by Haraoka [Har12], this normalization is effective to find a ‘good’ coordinate of the space of Fuchsian systems having a given Riemann scheme.

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