Compressed Sensing and Matrix Completion with Constant Proportion of Corruptions

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Abstract

In this paper we improve existing results in the field of compressed sensing and matrix completion when sampled data may be grossly corrupted. We introduce three new theorems. 1) In compressed sensing, we show that if the $m \times n$ sensing matrix has independent Gaussian entries, then one can recover a sparse signal $x$ exactly by tractable $\ell_1$ minimization even if a positive fraction of the measurements are arbitrarily corrupted, provided the number of nonzero entries in $x$ is $O(m/(\log(n/m) + 1))$. 2) In the very general sensing model introduced in [7] and assuming a positive fraction of corrupted measurements, exact recovery still holds if the signal now has $O(m/(\log^2 n))$ nonzero entries. 3) Finally, we prove that one can recover an $n \times n$ low-rank matrix from $m$ corrupted sampled entries by tractable optimization provided the rank is on the order of $O(m/(n\log^2 n))$; again, this holds when there is a positive fraction of corrupted samples.

Keywords. Compressed Sensing, Matrix Completion, Robust PCA, Convex Optimization, Restricted Isometry Property, Golfing Scheme.

1 Introduction

1.1 Introduction on Compressed Sensing with Corruptions

Compressed sensing (CS) has been well-studied in recent years [9,19]. This novel theory asserts that a sparse or approximately sparse signal $x \in \mathbb{R}^n$ can be acquired by taking just a few non-adaptive linear measurements. This fact has numerous consequences which are being explored in a number of fields of applied science and engineering. In CS, the acquisition procedure is often represented as $y = Ax$, where $A \in \mathbb{R}^{m \times n}$ is called the sensing matrix and $y \in \mathbb{R}^m$ is the vector of measurements or observations. It is now well-established that the solution $\hat{x}$ to the optimization problem

$$\min_{\tilde{x}} \|\tilde{x}\|_1 \text{ such that } A\tilde{x} = y,$$

(1.1)

is guaranteed to be the original signal $x$ with high probability, provided $x$ is sufficiently sparse and $A$ obeys certain conditions. A typical result is this: if $A$ has iid Gaussian entries, then exact recovery occurs provided $\|x\|_0 \leq Cm/(\log(n/m) + 1)$ [10,18,37] for some positive numerical constant $C > 0$. Here is another example, if $A$ is a matrix with rows randomly selected from the DFT matrix, the condition becomes $\|x\|_0 \leq Cm/\log n$ [9].

This paper discusses a natural generalization of CS, which we shall refer to as compressed sensing...
with corruptions. We assume that some entries of the data vector $y$ are totally corrupted but we have absolutely no idea which entries are unreliable. We still want to recover the original signal efficiently and accurately. Formally, we have the mathematical model

$$y = Ax + f = [A, I] \begin{bmatrix} x \\ f \end{bmatrix},$$

(1.2)

where $x \in \mathbb{R}^n$ and $f \in \mathbb{R}^m$. The number of nonzero coefficients in $x$ is $\|x\|_0$ and similarly for $f$. As in the above model, $A$ is an $m \times n$ sensing matrix, usually sampled from a probability distribution.

The problem of recovering $x$ (and hence $f$) from $y$ has been recently studied in the literature in connection with some interesting applications. We discuss a few of them.

- **Clipping.** Signal clipping frequently appears because of nonlinearities in the acquisition device \[27,38\]. Here, one typically measures $g(Ax)$ rather than $Ax$, where $g$ is always a nonlinear map. Letting $f = g(Ax) - Ax$, we thus observe $y = Ax + f$. Nonlinearities usually occur at large amplitudes so that for those components with small amplitudes, we have $f = g(Ax) - Ax = 0$. This means that $f$ is sparse and, therefore, our model is appropriate. Just as before, locating the portion of the data vector that has been clipped may be difficult because of additional noise.

- **CS for networked data.** In a sensor network, different sensors will collect measurements of the same signal $x$ independently (they each measure $z_i = \langle a_i, x \rangle$) and send the outcome to a center hub for analysis \[23,30\]. By setting $a_i$ as the row vectors of $A$, this is just $z = Ax$. However, typically some sensors will fail to send the measurements correctly, and will sometimes report totally meaningless measurements. Therefore, we collect $y = Ax + f$, where $f$ models recording errors.

There have been several theoretical papers investigating the exact recovery method for CS with corruptions \[28–30,38,40\], and all of them consider the following recovery procedure in the noiseless case:

$$\min_{\tilde{x}, \tilde{f}} \|\tilde{x}\|_1 + \lambda(m, n)\|\tilde{f}\|_1 \text{ such that } A\tilde{x} + \tilde{f} = [A, I] \begin{bmatrix} \tilde{x} \\ \tilde{f} \end{bmatrix} = y.$$  

(1.3)

We will compare them with our results in Section 1.4.

1.2 Introduction on matrix completion with corruptions

Matrix completion (MC) bears some similarity with CS. Here, the goal is to recover a low-rank matrix $L \in \mathbb{R}^{n \times n}$ from a small fraction of linear measurements. For simplicity, we suppose the matrix is square as above (the general case is similar). The standard model is that we observe $P_O(L)$ where $O \subset [n] \times [n] := \{1, \ldots, n\} \times \{1, \ldots, n\}$ and

$$P_O(L)_{ij} = \begin{cases} L_{ij} & \text{if } (i, j) \in O; \\ 0 & \text{otherwise.} \end{cases}$$

The problem is to recover the original matrix $L$, and there have been many papers studying this problem in recent years, see \[8,12,21,26,33\], for example. Here one minimizes the nuclear norm —
the sum of all the singular values \( \sum_{i=1}^{r} \sigma_i \) — to recover the original low rank matrix. We discuss below an improved result due to Gross \( \text{[21]} \) (with a slight difference).

Define \( O \sim \text{Ber}(\rho) \) for some \( 0 < \rho < 1 \) by meaning that \( 1_{\{(i,j) \in O\}} \) are iid Bernoulli random variables with parameter \( \rho \). Then the solution to

\[
\min_{\tilde{L}} \| \tilde{L} \|_* \text{ such that } \mathcal{P}_O(\tilde{L}) = \mathcal{P}_O(L),
\]

is guaranteed to be exactly \( L \) with high probability, provided \( \rho \geq \frac{C \rho r \mu \log n}{n} \). Here, \( C \rho \) is a positive numerical constant, \( r \) is the rank of \( L \), and \( \mu \) is an incoherence parameter introduced in \( \text{[8]} \) which is only dependent of \( L \).

This paper is concerned with the situation in which some entries may have been corrupted. Therefore, our model is that we observe

\[
\mathcal{P}_O(L) + S, \quad (1.5)
\]

where \( O \) and \( L \) are the same as before and \( S \in \mathbb{R}^{n \times n} \) is supported on \( \Omega \subset O \). Just as in CS, this model has broad applicability. For example, Wu et al. used this model in photometric stereo \( \text{[42]} \).

This problem has also been introduced in \( \text{[4]} \) and is related to recent work in separating a low-rank from a sparse component \( \text{[4, 13, 14, 24, 43]} \). A typical result is that the solution \( (\hat{L}, \hat{S}) \) to

\[
\min_{\tilde{L}, \tilde{S}} \| \tilde{L} \|_* + \lambda \| \tilde{S} \|_1 \text{ such that } \mathcal{P}_O(\tilde{L}) + \tilde{S} = \mathcal{P}_O(L) + S, \quad (1.6)
\]

is guaranteed to be the true pair \( (L, S) \) with high probability under some assumptions about \( L, O, S \) \( \text{[4, 16]} \). We will compare them with our result in Section 1.4.

1.3 Main results

This section introduces three models and three corresponding recovery results. The proofs of these results are deferred to Section 2 for Theorem 1.1, Section 3 for Theorem 1.2 and Section 4 for Theorem 1.3.

1.3.1 CS with iid matrices [Model 1]

**Theorem 1.1** Suppose that \( A \) is an \( m \times n \) \( (m < n) \) random matrix whose entries are iid Gaussian variables with mean 0 and variance \( 1/m \), the signal to acquire is \( x \in \mathbb{R}^n \), and our observation is \( y = Ax + f + w \) where \( f, w \in \mathbb{R}^m \) and \( \| w \|_2 \leq \epsilon \). Then by choosing \( \lambda(n,m) = \frac{1}{\sqrt{\log(n/m) + 1}} \), the solution \( (\hat{x}, \hat{f}) \) to

\[
\min_{\tilde{x}, \tilde{f}} \| \tilde{x} \|_1 + \lambda \| \tilde{f} \|_1 \text{ such that } \| (A\tilde{x} + \tilde{f}) - y \|_2 \leq \epsilon \quad (1.7)
\]

satisfies \( \| \hat{x} - x \|_2 + \| \hat{f} - f \|_2 \leq K \epsilon \) with probability at least \( 1 - C \exp(-cm) \). This holds universally; that is to say, for all vectors \( x \) and \( f \) obeying \( \| x \|_0 \leq \alpha m/(\log(n/m) + 1) \) and \( \| f \|_0 \leq \alpha m \). Here \( \alpha, C, c \) and \( K \) are numerical constants.

In the above statement, the matrix \( A \) is random. Everything else is deterministic. The reader will notice that the number of nonzero entries is on the same order as that needed for recovery from
clean data \[3, 10, 19, 37\], while the condition of \( f \) implies that one can tolerate a constant fraction of possibly adversarial errors. Moreover, our convex optimization is related to LASSO \[35\] and Basis Pursuit \[15\].

### 1.3.2 CS with general sensing matrices [Model 2]

In this model, \( m < n \) and

\[
A = \frac{1}{\sqrt{m}} \begin{pmatrix}
    a_1^* \\
    \vdots \\
    a_m^*
\end{pmatrix},
\]

where \( a_1, \ldots, a_m \) are \( n \) iid copies of a random vector \( a \) whose distribution obeys the following two properties: 1) \( \mathbb{E} aa^* = I \); 2) \( \|a\|_{\infty} \leq \sqrt{\mu} \). This model has been introduced in \[7\] and includes a lot of the stochastic models used in the literature. Examples include partial DFT matrices, matrices with iid entries, certain random convolutions \[34\] and so on.

In this model, we assume that \( x \) and \( f \) in (1.2) have fixed support denoted by \( T \) and \( B \), and with cardinality \( |T| = s \) and \( |B| = m_b \). In the remainder of the paper, \( x_T \) is the restriction of \( x \) to indices in \( T \) and \( f_B \) is the restriction of \( f \) to \( B \). Our main assumption here concerns the sign sequences: the sign sequences of \( x_T \) and \( f_B \) are independent of each other, and each is a sequence of symmetric iid ±1 variables.

**Theorem 1.2** For the model above, the solution \((\hat{x}, \hat{f})\) to (1.3), with \( \lambda(n, m) = 1/\sqrt{\log n} \), is exact with probability at least \( 1 - Cn^{-3} \), provided that \( s \leq \alpha \frac{m}{\mu \log^2 n} \) and \( m_b \leq \beta \frac{m}{\mu} \). Here \( C, \alpha \) and \( \beta \) are some numerical constants.

Above, \( x \) and \( f \) have fixed supports and random signs. However, by a recent de-randomization technique first introduced in \[4\], exact recovery with random supports and fixed signs would also hold. We will explain this de-randomization technique in the proof of Theorem 1.3. In some specific models, such as independent rows from the DFT matrix, \( \mu \) could be a numerical constant, which implies the proportion of corruptions is also a constant. An open problem is whether Theorem 1.2 still holds in the case where \( x \) and \( f \) have both fixed supports and signs. Another open problem is to know whether the result would hold under more general conditions about \( A \) as in \[6\] in the case where \( x \) has both random support and random signs.

We emphasize that the sparsity condition \( \|x\|_0 \leq \frac{Cm}{\mu \log^2 n} \) is a little stronger than the optimal result available in the noise-free literature \[7, 9\], namely, \( \|x\|_0 \leq \frac{Cm}{\mu \log n} \). The extra logarithmic factor appears to be important in the proof which we will explain in Section 3, and a third open problem is whether or not it is possible to remove this factor.

Here we do not give a sensitivity analysis for the recovery procedure as in Model 1. Actually by applying a similar method introduced in \[7\] to our argument in Section 3, a very good error bound could be obtained in the noisy case. However, technically there is little novelty but it will make our paper very long. Therefore we decide to only discuss the noiseless case and focus on the sampling rate and corruption ratio.
1.3.3 MC from corrupted entries [Model 3]

We assume $L$ is of rank $r$ and write its reduced SVD as $L = U\Sigma V^*$, where $U, V \in \mathbb{R}^{n \times r}$ and $\Sigma \in \mathbb{R}^{r \times r}$. Let $\mu$ be the smallest quantity such that for all $1 \leq i \leq n$,

$$\|UU^*e_i\|^2_2 \leq \frac{\mu r}{n}, \quad \|VV^*e_i\|^2_2 \leq \frac{\mu r}{n}, \quad \text{and} \quad \|UV^*\|_\infty \leq \frac{\sqrt{\mu r}}{n}.$$

This model is the same as that originally introduced in [8], and later used in [4, 12, 16, 21, 32]. We observe $P_O(L) + S$, where $O \in [n] \times [n]$ and $S$ is supported on $\Omega \subset O$. Here we assume that $O, \Omega, S$ satisfy the following model:

**Model 3.1:**

1. Fix an $n$ by $n$ matrix $K$, whose entries are either 1 or $-1$.
2. Define $O \sim \text{Ber}(\rho)$ for a constant $\rho$ satisfying $0 < \rho < \frac{1}{2}$. Specifically speaking, $1_{\{(i,j) \in O\}}$ are iid Bernoulli random variables with parameter $\rho$.
3. Conditioning on $(i, j) \in O$, assume that $(i, j) \in \Omega$ are independent events with $\mathbb{P}((i, j) \in \Omega| (i, j) \in O) = s$. This implies that $\Omega \sim \text{Ber}(\rho s)$.
4. Define $\Gamma := O/\Omega$. Then we have $\Gamma \sim \text{Ber}(\rho(1 - s))$.
5. Let $S$ be supported on $\Omega$, and $\text{sgn}(S) := P_{\Omega}(K)$.

**Theorem 1.3** Under Model 3.1, suppose $\rho > C_\rho \frac{\mu r \log^2 n}{n}$ and $s \leq C_\rho$. Moreover, suppose $\lambda := \frac{1}{\sqrt{\rho n \log n}}$ and denote $(\hat{L}, \hat{S})$ as the optimal solution to the problem (1.6). Then we have $(\hat{L}, \hat{S}) = (L, S)$ with probability at least $1 - Cn^{-3}$ for some numerical constant $C$, provided the numerical constants $C_s$ is sufficiently small and $C_\rho$ is sufficiently large.

In this model $O$ is available while $\Omega, \Gamma$ and $S$ are not known explicitly from the observation $P_O(L) + S$. By the assumption $O \sim \text{Ber}(\rho)$, we can use $|O|/(n^2)$ to approximate $\rho$. From the following proof we can see that $\lambda$ is not required to be $\frac{1}{\sqrt{\rho n \log n}}$ exactly for the exact recovery. The power of our result is that one can recover a low-rank matrix from a nearly minimal number of samples even when a constant proportion of these samples has been corrupted.

We only discuss the noiseless case for this model. Actually by a method similar to [6], a suboptimal estimation error bound can be obtained by a slight modification of our argument. However, it is of little interest technically and beyond the optimal result when $n$ is large. There are other suboptimal results for matrix completion with noise, such as [4], but the error bound is not tight when the additional noise is small. We want to focus on the noiseless case in this paper and leave the problem with noise for future work.

The values of $\lambda$ are chosen for theoretical guarantee of exact recovery in Theorem 1.1, 1.2 and 1.3. In practice, $\lambda$ is usually taken by cross validation.

1.4 Comparison with existing results, relative works and our contribution

In this section we will compare Theorems 1.1, 1.2 and 1.3 with existing results in the literature.
We begin with Model 1. In [40], Wright and Ma discussed a model where the sensing matrix $A$ has independent columns with common mean $\mu$ and normal perturbations with variance $\sigma^2/m$. They chose $\lambda(m,n) = 1$, and proved that $(\hat{x}, \hat{f}) = (x, f)$ with high probability provided $\|x\|_0 \leq C_1(\sigma, n/m)m$, $\|f\|_0 \leq C_2(\sigma, n/m)m$ and $f$ has random signs. Here $C_1(\sigma, 1/m)$ is much smaller than $C/(\log(n/m) + 1)$. We notice that since the authors of [40] talked about a different model, which is motivated by [11], it may not be comparable with ours directly. However, for our motivation of CS with corruptions, we assume $A$ satisfy a symmetric distribution and get better sampling rate.

A bit later, Laska et al. [28] and Li et al. [29] also studied this problem. By setting $\lambda(m, n) = 1$, both papers establish that for Gaussian (or sub-Gaussian) sensing matrices $A$, if $m > C(\|x\|_0 + \|f\|_0)\log((n + m)/(|\|x\|_0 + \|f\|_0)))$, then the recovery is exact. This follows from the fact that $[A, I]$ obeys a restricted isometry property known to guarantee exact recovery of sparse vectors via $\ell_1$ minimization. Furthermore, the sparsity requirement about $x$ is the same as that found in the standard CS literature, namely, $\|x\|_0 \leq Cm/(\log(n/m) + 1)$. However, the result does not allow a positive fraction of corruptions. For example, if $m = \sqrt{n}$, we have $\|f\|_0/m \leq 2/\log n$, which will go to zero as $n$ goes to zero.

As for Model 2, an interesting piece of work [30] (and later [31] on the noisy case) appeared during the preparation of this paper. These papers discuss models in which $A$ is formed by selecting rows from an orthogonal matrix uniformly at random; 2) $x$ is a random signal with independent signs and equally likely to be either $\pm 1$; 3) the support of $f$ is chosen uniformly at random. (By the de-randomization technique introduced in [4] and used in [30], it would have been sufficient to assume that the signs of $f$ are independent and take on the values $\pm 1$ with equal probability). Finally, the sparsity conditions require $m \geq C\mu^2\|x\|_0(\log n)^2$ and $\|f\|_0 \leq Cm$, which are nearly optimal, for the best known sparsity condition when $f = 0$ is $m \geq C\mu\|x\|_0\log n$. In other words, the result is optimal up to an extra factor of $\mu\log n$; the sparsity condition about $f$ is of course nearly optimal.

However, the model for $A$ does not include some models frequently discussed in the literature such as subsampled tight or continuous frames. Against this background, a recent paper of Candes and Plan [7] considers a very general framework, which includes a lot of common models in the literature. Theorem 1.2 in our paper is similar to Theorem 1 in [30]. It assumes similar sparsity conditions, but is based on this much broader and more applicable model introduced in [7]. Notice that, we require $m \geq C\mu\|x\|_0(\log n)^2$ whereas [30] requires $m \geq C\mu^2\|x\|_0(\log n)^2$. Therefore, we improve the condition by a factor of $\mu$, which is always at least 1 and can be as large as $n$. However, our result imposes $\|f\|_0 \leq Cm/\mu$, which is worse than $\|f\|_0 \leq \gamma m$ by the same factor. In [30], the parameter $\lambda$ depends upon $\mu$, while our $\lambda$ is only a function of $m$ and $n$. This is why the results differ, and we prefer to use a value of $\lambda$ that does not depend on $\mu$ because in some applications, an accurate estimate of $\mu$ may be difficult to obtain. In addition, we use different techniques of proof which the clever golfing scheme of [21] is exploited.

Sparse approximation is another problem of underdetermined linear system where the dictionary
matrix $A$ is always assumed to be deterministic. Readers interested in this problem (which always requires stronger sparsity conditions) may also want to study the recent paper \cite{38} by Studer et al. There, the authors introduce a more general problem of the form $y = Ax + Bf$, and analyzed the performance of $\ell_1$-recovery techniques by using ideas which have been popularized under the name of generalized uncertainty principles in the basis pursuit and sparse approximation literature.

As for Model 3, Theorem 1.3 is a significant extension of the results presented in \cite{4}, in which the authors have a stringent requirement $\rho = 0.1$. In a very recent and independent work \cite{16}, the authors consider a model where both $O$ and $\Omega$ are unions of stochastic and deterministic subsets, while we only assume the stochastic model. We recommend interested readers to read the paper for the details. However, only considering their results on stochastic $O$ and $\Omega$, a direct comparison shows that the number of samples we need is less than that in this reference. The difference is several logarithmic factors. Actually, the requirement of $\rho$ in our paper is optimal even for clean data in the literature of MC. Finally, we want to emphasize that the random support assumption is essential in Theorem 1.3 when the rank is large. Examples can be found in \cite{24}.

We wish to close our introduction with a few words concerning the techniques of proof we shall use. The proof of Theorem 1.1 is based on the concept of restricted isometry, which is a standard technique in the literature of CS. However, our argument involves a generalization of the restricted isometry concept. The proofs of Theorems 1.2 and 1.3 are based on the golfing scheme, an elegant technique pioneered by David Gross \cite{21}, and later used in \cite{4, 7, 32} to construct dual certificates. Our proof leverages results from \cite{4}. However, we contribute novel elements by finding an appropriate way to phrase sufficient optimality conditions, which are amenable to the golfing scheme. Details are presented in the following sections.

### 2 A Proof of Theorem 1.1

In the proof of Theorem 1.1 we will see the notation $P_T x$. Here $x$ is a $k$-dimensional vector, $T$ is a subset of $\{1, \ldots, k\}$ and we also use $T$ to represent the subspace of all $k$-dimensional vectors supported on $T$. Then $P_T x$ is the projection of $x$ onto the subspace $T$, which is to keep the value of $x$ on the support $T$ and to change other elements into zeros. In this section we use the notation $\lfloor \cdot \rfloor$ of “floor function” to represent the integer part of any real number.

First we generalize the concept of the restricted isometry property (RIP) \cite{11} for the convenience to prove our theorem:

**Definition 2.1** For any matrix $\Phi \in \mathbb{R}^{l \times (n+m)}$, define the RIP-constant $\delta_{s_1, s_2}$ by the infimum value of $\delta$ such that

$$
(1 - \delta)(\|x\|_2^2 + \|f\|_2^2) \leq \left\| \Phi \begin{bmatrix} x \\ f \end{bmatrix} \right\|_2^2 \leq (1 + \delta)(\|x\|_2^2 + \|f\|_2^2)
$$

holds for any $x \in \mathbb{R}^n$ with $|\text{supp}(x)| \leq s_1$ and $f \in \mathbb{R}^m$ with $|\text{supp}(f)| \leq s_2$. 

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Lemma 2.2 For any \( x_1, x_2 \in \mathbb{R}^n \) and \( f_1, f_2 \in \mathbb{R}^m \) such that \( \text{supp}(x_1) \cap \text{supp}(x_2) = \emptyset, \) \( |\text{supp}(x_1)| + |\text{supp}(x_2)| \leq s_1 \) and \( \text{supp}(f_1) \cap \text{supp}(f_2) = \emptyset, \) \( |\text{supp}(f_1)| + |\text{supp}(f_2)| \leq s_2, \) we have

\[
\left\langle \Phi \begin{bmatrix} x_1 \\ f_1 
\end{bmatrix}, \Phi \begin{bmatrix} x_2 \\ f_2 
\end{bmatrix} \right\rangle \leq \delta_{s_1, s_2} \sqrt{\|x_1\|_2^2 + \|f_1\|_2^2} \sqrt{\|x_2\|_2^2 + \|f_2\|_2^2}
\]

Proof First, we suppose \( \|x_1\|_2^2 + \|f_1\|_2^2 = \|x_2\|_2^2 + \|f_2\|_2^2 = 1. \) By the definition of \( \delta_{s_1, s_2}, \) we have

\[
2(1 - \delta_{s_1, s_2}) \leq \left\langle \Phi \begin{bmatrix} x_1 + x_2 \\ f_1 + f_2 
\end{bmatrix}, \Phi \begin{bmatrix} x_1 + x_2 \\ f_1 + f_2 
\end{bmatrix} \right\rangle \leq 2(1 + \delta_{s_1, s_2}),
\]

and

\[
2(1 - \delta_{s_1, s_2}) \leq \left\langle \Phi \begin{bmatrix} x_1 - x_2 \\ f_1 - f_2 
\end{bmatrix}, \Phi \begin{bmatrix} x_1 - x_2 \\ f_1 - f_2 
\end{bmatrix} \right\rangle \leq 2(1 + \delta_{s_1, s_2}).
\]

By the above inequalities, we have \( \left\langle \Phi \begin{bmatrix} x_1 \\ f_1 
\end{bmatrix}, \Phi \begin{bmatrix} x_2 \\ f_2 
\end{bmatrix} \right\rangle \leq \delta_{s_1, s_2}, \) and hence by homogeneity, we have \( \left\langle \Phi \begin{bmatrix} x_1 \\ f_1 
\end{bmatrix}, \Phi \begin{bmatrix} x_2 \\ f_2 
\end{bmatrix} \right\rangle \leq \delta_{s_1, s_2} \sqrt{\|x_1\|_2^2 + \|f_1\|_2^2} \sqrt{\|x_2\|_2^2 + \|f_2\|_2^2} \) without the norm assumption. □

Lemma 2.3 Suppose \( \Phi \in \mathbb{R}^{(n+m) \times (n+m)} \) with RIP-constant \( \delta_{2s_1,2s_2} < \frac{1}{18} \) \((s_1, s_2 > 0)\) and \( \lambda \) is between \( \frac{1}{2} \sqrt{\frac{s_1}{s_2}} \) and \( 2 \sqrt{\frac{s_1}{s_2}}. \) Then for any \( x \in \mathbb{R}^n \) with \( |\text{supp}(x)| \leq s_1, \) any \( f \in \mathbb{R}^m \) with \( |\text{supp}(f)| \leq s_2, \) and any \( w \in \mathbb{R}^m \) with \( \|w\|_2 \leq \epsilon \) the solution \( (\hat{x}, \hat{f}) \) to the optimization problem \((1.7)\) satisfies \( \|\hat{x} - x\|_2 + \|\hat{f} - f\|_2 \leq \frac{4\sqrt{13 + 13\delta_{2s_1,2s_2}}}{1 - 9\delta_{2s_1,2s_2}} \epsilon. \)

Proof Suppose \( \Delta x = \hat{x} - x \) and \( \Delta f = \hat{f} - f. \) Then by \((1.7)\) we have

\[
\left\| \Phi \begin{bmatrix} \Delta x \\ \Delta f 
\end{bmatrix} \right\|_2 \leq \|w\|_2 + \left\| \Phi \begin{bmatrix} \hat{x} \\ \hat{f} 
\end{bmatrix} - \left( \Phi \begin{bmatrix} x \\ f 
\end{bmatrix} + w \right) \right\|_2 \leq 2\epsilon.
\]

It is easy to check that the original \((x,f)\) satisfies the inequality constraint in \((1.7),\) so we have

\[
\|x + \Delta x\|_1 + \lambda \|f + \Delta f\|_1 \leq \|x\|_1 + \lambda \|f\|_1.
\]

Then it suffices to show \( \|\Delta x\|_2 + \|\Delta f\|_2 \leq \frac{4\sqrt{13 + 13\delta_{2s_1,2s_2}}}{1 - 9\delta_{2s_1,2s_2}} \epsilon. \)

Suppose \( T_0 \) with \( |T_0| = s_1 \) such that \( \text{supp}(x) \subseteq T_0. \) Denote \( T_0^c = T_1 \cup \cdots \cup T_l \) where \( |T_1| = \cdots = |T_{l-1}| = s_1 \) and \( |T_l| \leq s_1. \) Moreover, suppose \( T_1 \) contains the indices of the \( s_1 \) largest (in the sense of absolute value) coefficients of \( P_{T_0^c} \Delta x, \) \( T_2 \) contains the indices of the \( s_1 \) largest coefficients of \( P_{T_0 \cup T_1^c} \Delta x, \) and so on. Similarly, define \( V_0 \) such that \( \text{supp}(f) \subseteq V_0 \) and \( |V_0| = s_2, \) and divide \( V_0^c = V_1 \cup \cdots \cup V_k \) in the same way. By this setup, we easily have

\[
\sum_{j \geq 2} \|P_{T_j} \Delta x\|_2 \leq s_1 \frac{1}{2} \|P_{T_0^c} \Delta x\|_1,
\]

and

\[
\sum_{j \geq 2} \|P_{V_j} \Delta f\|_2 \leq s_2 \frac{1}{2} \|P_{V_0^c} \Delta f\|_1.
\]
On the other hand, by the assumption supp($x$) $\subset T_0$ and supp($f$) $\subset V_0$, we have,
\[
\|x + \Delta x\|_1 = \|P_{T_0}x + P_{T_1}\Delta x\|_1 + \|P_{T_0}\Delta x\|_1 \geq \|x\|_1 - \|P_{T_0}\Delta x\|_1 + \|P_{T_0}\Delta x\|_1, \tag{2.4}
\]
and similarly,
\[
\|f + \Delta f\|_1 \geq \|f\|_1 - \|P_{V_0}\Delta f\|_1 + \|P_{V_0}\Delta f\|_1. \tag{2.5}
\]
By inequalities (2.1), (2.4) and (2.5), we have
\[
\|P_{T_0}\Delta x\|_1 + \lambda\|P_{V_0}\Delta f\|_1 \leq \|P_{T_0}\Delta x\|_1 + \lambda\|P_{V_0}\Delta f\|_1. \tag{2.6}
\]
By the definition of $\delta_{2s_1,2s_2}$, the fact $\|\Phi \frac{\Delta x}{\Delta f}\|_2 \leq 2\epsilon$ and Lemma 2.2 we have
\[
(1 - \delta_{2s_1,2s_2}) \left(\|P_{T_0}\Delta x + P_{T_1}\Delta x\|_2^2 + \|P_{V_0}\Delta f + P_{V_1}\Delta f\|_2^2\right)
\leq \left\| \Phi \left[\frac{P_{T_0}\Delta x + P_{T_1}\Delta x}{P_{V_0}\Delta f + P_{V_1}\Delta f}\right] \right\|_2^2
\leq \left\langle \Phi \left[\frac{P_{T_0}\Delta x + P_{T_1}\Delta x}{P_{V_0}\Delta f + P_{V_1}\Delta f}\right], \Phi \left[\frac{\Delta x}{\Delta f}\right] - \Phi \left[\frac{P_{T_2}\Delta x + \ldots + P_{T_1}\Delta x}{P_{V_2}\Delta f + \ldots + P_{V_1}\Delta f}\right] \right\rangle
\leq - \left\langle \Phi \left[\frac{P_{T_0}\Delta x + P_{T_1}\Delta x}{P_{V_0}\Delta f + P_{V_1}\Delta f}\right], \Phi \left[\frac{P_{T_2}\Delta x + \ldots + P_{T_1}\Delta x}{P_{V_2}\Delta f + \ldots + P_{V_1}\Delta f}\right] \right\rangle
\leq 2\epsilon \left\| \Phi \left[\frac{P_{T_0}\Delta x + P_{T_1}\Delta x}{P_{V_0}\Delta f + P_{V_1}\Delta f}\right] \right\|_2^2
\leq \delta_{2s_1,2s_2} \left(\left\| \left[\frac{P_{T_0}\Delta x}{P_{V_0}\Delta f}\right]\right\|_2 + \left\| \left[\frac{P_{T_1}\Delta x}{P_{V_1}\Delta f}\right]\right\|_2\right) \left(\sum_{j \geq 2} \left\| P_{T_j}\Delta x\right\|_2 + \sum_{j \geq 2} \left\| P_{V_j}\Delta f\right\|_2\right)
\leq 2\epsilon \sqrt{1 + \delta_{2s_1,2s_2}} \sqrt{\|P_{T_0}\Delta x\|_2^2 + \|P_{T_1}\Delta x\|_2^2 + \|P_{V_0}\Delta f\|_2^2 + \|P_{V_1}\Delta f\|_2^2}.
\]
Moreover, since
\[
\sum_{j \geq 2} \left\| P_{T_j}\Delta x\right\|_2 + \sum_{j \geq 2} \left\| P_{V_j}\Delta f\right\|_2
\leq s_1^{-\frac{1}{2}} \|P_{T_0}\Delta x\|_1 + s_2^{-\frac{1}{2}} \|P_{V_0}\Delta f\|_1 \leq 2s_1^{-\frac{1}{2}} (\|P_{T_0}\Delta x\|_1 + \lambda \|P_{V_0}\Delta f\|_1) \leq 2s_1^{-\frac{1}{2}} (\|P_{T_0}\Delta x\|_1 + \lambda \|P_{V_0}\Delta f\|_1) \leq 2s_1^{-\frac{1}{2}} (s_1^{\frac{1}{2}} \|P_{T_0}\Delta x\|_2 + \lambda s_2^{\frac{1}{2}} \|P_{V_0}\Delta f\|_2) \leq 4\|P_{T_0}\Delta x\|_2 + 4\|P_{V_0}\Delta f\|_2,
\]
we have
\[
\left(\left\| \left[\frac{P_{T_0}\Delta x}{P_{V_0}\Delta f}\right]\right\|_2 + \left\| \left[\frac{P_{T_1}\Delta x}{P_{V_1}\Delta f}\right]\right\|_2\right) \left(\sum_{j \geq 2} \left\| P_{T_j}\Delta x\right\|_2 + \sum_{j \geq 2} \left\| P_{V_j}\Delta f\right\|_2\right)
\leq 8(\|P_{T_0}\Delta x\|_2^2 + \|P_{T_1}\Delta x\|_2^2 + \|P_{V_0}\Delta f\|_2^2 + \|P_{V_1}\Delta f\|_2^2).
Therefore, by \( \delta_{2s_1,2s_2} < 1/9 \), we have

\[
\sqrt{\|P_{T_0}\Delta x\|_2^2 + \|P_{T_1}\Delta x\|_2^2 + \|P_{V_0}\Delta f\|_2^2 + \|P_{V_1}\Delta f\|_2^2} \leq \frac{2\epsilon \sqrt{1 + \delta_{2s_1,2s_2}}}{1 - 9\delta_{2s_1,2s_2}}.
\]

Since

\[
\sum_{j \geq 2} \|P_{T_j}\Delta x\|_2 + \sum_{j \geq 2} \|P_{V_j}\Delta f\|_2 \leq 4\|P_{T_0}\Delta x\|_2 + 4\|P_{V_0}\Delta f\|_2,
\]

we have

\[
\|\Delta x\|_2 + \|\Delta f\|_2 \leq 5(\|P_{T_0}\Delta x\|_2 + \|P_{V_0}\Delta f\|_2) + (\|P_{T_1}\Delta x\|_2 + \|P_{V_1}\Delta f\|_2)
\]

\[
\leq \sqrt{52}\sqrt{\|P_{T_0}\Delta x\|_2^2 + \|P_{T_1}\Delta x\|_2^2 + \|P_{V_0}\Delta f\|_2^2 + \|P_{V_1}\Delta f\|_2^2}
\]

\[
\leq \frac{4\sqrt{13}+13\delta_{2s_1,2s_2}}{1 - 9\delta_{2s_1,2s_2}}\epsilon.
\]

We now cite a well-known result in the literature of CS, e.g. Theorem 5.2 of [3].

**Lemma 2.4** Suppose \( A \) is a random matrix defined in model 1. Then for any \( 0 < \delta < 1 \), there exist \( c_1(\delta), c_2(\delta) > 0 \) such that with probability at least \( 1 - 2 \exp(-c_2(\delta)m) \),

\[
(1 - \delta)\|x\|_2^2 \leq \|Ax\|_2^2 \leq (1 + \delta)\|x\|_2^2
\]

holds universally for any \( x \) with \( \|\text{supp}(x)\| \leq c_1(\delta)\frac{m}{\log\frac{m}{\alpha}} + 1 \).

Also, we cite a well-known result which can give a bound for the biggest singular value of random matrix, e.g. [17] and [39].

**Lemma 2.5** Let \( B \) be an \( m \times n \) matrix whose entries are independent standard normal random variables. Then for every \( t \geq 0 \), with probability at least \( 1 - 2 \exp(-t^2/2) \), one has \( \|B\|_{2,2} \leq \sqrt{m} + \sqrt{n} + t \).

We now prove Theorem 2.4

**Proof** Suppose \( \alpha, \delta \) are two constants independent of \( m \) and \( n \), and their values will be specified later. Set \( s_1 = \lfloor \alpha m \log \frac{m}{\alpha} \rfloor \) and \( s_2 = \lfloor \alpha m \rfloor \). We want to bound the RIP-constant \( \delta_{2s_1,2s_2} \) for the \( (n + m) \times m \) matrix \( \Phi = [A, I] \) when \( \alpha \) is sufficiently small. For any \( T \) with \( |T| = 2s_1 \) and \( V \) with \( |V| = 2s_2 \), and any \( x \) with \( \|\text{supp}(x)\| \leq T \), any \( f \) with \( \|\text{supp}(f)\| \leq V \), we have

\[
\left\| [A, I] \begin{bmatrix} x \\ f \end{bmatrix} \right\|_2^2 = \|Ax + f\|_2^2 = \|Ax\|_2^2 + \|f\|_2^2 + 2\langle PVAP_Tx, f \rangle.
\]

By Lemma 2.4 assuming \( \alpha \leq c_1(\delta) \), with probability at least \( 1 - 2 \exp(-c_2(\delta)m) \) we have

\[
(1 - \delta)\|x\|_2^2 \leq \|Ax\|_2^2 \leq (1 + \delta)\|x\|_2^2
\]

(2.7)
holds universally for any such $T$ and $x$.

Now we we fix $T$ and $V$, and we want to bound $\|P_V A P_T\|_{2,2}$. By Lemma 2.8, we actually have

$$\|P_V A P_T\|_{2,2} \leq \frac{1}{\sqrt{m}} (\sqrt{2s_1} + \sqrt{2s_2} + \sqrt{\delta^2 m}) \leq (2\sqrt{2\alpha} + \delta)$$

(2.8)

with probability at least $1 - 2\exp(-\delta^2 m/2)$. Then with probability at least $1 - 2\exp(-\frac{\delta^2 m}{2}) \binom{n}{2s_1} \binom{m}{s_2}$, inequality (2.8) holds universally for any $V$ satisfying $|V| = 2s_1$ and $T$ satisfying $|V| = 2s_2$. By $2s_1 \leq 2\alpha \frac{m}{\log (\frac{n}{m})}$, we have $2s_1 \log (\frac{en}{2s_1}) \leq \alpha_1 m$, where $\alpha_1$ only depends on $\alpha$ and $\alpha_1 \to 0$ as $\alpha \to 0$, and hence $\binom{n}{2s_1} \leq (\frac{en}{2s_1})^{2s_1} \leq \exp(\alpha_1 m)$. Similarly, because $2s_2 \leq 2\alpha m$, we have $2s_2 \log (\frac{em}{2s_2}) \leq \alpha_2 m$, where $\alpha_2$ only depends on $\alpha$ and $\alpha_2 \to 0$ as $\alpha \to 0$, and hence $\binom{m}{2s_2} \leq (\frac{em}{2s_2})^{2s_2} \leq \exp(\alpha_2 m)$. Therefore, inequality (2.8) holds universally for any such $T$ and $V$ with probability at least $1 - 2\exp((\delta^2/2 - \alpha_1 - \alpha_2)m)$.

Combined with 2.7, we have

$$(1 - \delta)\|x\|_2^2 + \|f\|_2^2 - (2\sqrt{2\alpha} + \delta)\|x\|_2 \|f\|_2 \leq \left\| [A, I] \left[ \begin{array}{c} x \\ f \end{array} \right] \right\|_2^2 \leq (1 + \delta)\|x\|_2^2 + \|f\|_2^2 + (2\sqrt{2\alpha} + \delta)\|x\|_2 \|f\|_2$$

holds universally for any such $T$, $U$, $x$ and $f$ which probability at least $1 - 2\exp((-\epsilon_2(\delta)m) - 2\exp((\delta^2/2 - \alpha_1 - \alpha_2)m)$. By choosing an appropriate $\delta$ and letting $\alpha$ sufficiently small, we have $\delta_{2s_1, 2s_2} < 1/9$ with probability at least $1 - Ce^{-cm}$.

Moreover, under the assumption that $\alpha \left( \frac{m}{\log (n/m) + 1} \right) \geq 1$, we have $s_1 = \left\lfloor \alpha \left( \frac{m}{\log (n/m) + 1} \right) \right\rfloor > 0$, $s_2 = [\alpha m] > 0$ and $\frac{1}{2} \sqrt{\frac{s_1}{s_2}} < \frac{1}{\sqrt{\log (\frac{n}{m}) + 1}} < 2 \sqrt{\frac{s_1}{s_2}}$. Then Theorem 1.1 as a direct corollary of Lemma 2.3.

3 A Proof of Theorem 1.2

In this section, we will encounter several absolute constants. Instead of denoting them by $C_1$, $C_2$, ..., we just use $C$, i.e., the values of $C$ change from line to line. Also, we will use the phrase “with high probability” to mean with probability at least $1 - Cn^{-c}$, where $C > 0$ is a numerical constant and $c = 3, 4$, or 5 depending on the context.

Here we will use a lot of notations to represent sub-matrices and sub-vectors. Suppose $A \in \mathbb{R}^{m \times n}$, $P \subset [m] := \{1, ..., m\}$, $Q \subset [n]$ and $i \in [n]$. We denote by $A_P$, the sub-matrix of $A$ with row indices contained in $P$, by $A_{i,Q}$ the sub-matrix of $A$ with column indices contained in $Q$, and by $A_{P,Q}$ the sub-matrix of $A$ with row indices contained in $P$ and column indices contained in $Q$. Moreover, we denote by $A_{P,i}$ the sub-matrix of $A$ with row indices contained in $P$ and column $i$, which is actually a column vector.

The term “vector” means column vector in this section, and all row vectors are denoted by an adjoin of a vector, such as $a^*$ for a vector $a$. Suppose $a$ is a vector and $T$ a subset of indices. Then we denote by $a_T$ the restriction of $a$ on $T$, i.e., a vector with all elements of $a$ with indices in $T$. For any vector $v$, we use $v_{(i)}$ to denote the $i$-th element of $v$. 

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3.1 Supporting lemmas

To prove Theorem 1.2 we need some supporting lemmas. Because our model of sensing matrix $A$ is the same as in [7], we will cite some lemmas from it directly.

**Lemma 3.1** (Lemma 2.1 of [7]) Suppose $A$ is as defined in model 2. Let $T \subset [n]$ be a fixed set of cardinality $s$. Then for $\delta > 0$, $P(\|A^*_T A_{:,T} - I\|_{2,2} \geq \delta) \leq 2s \exp\left(-\frac{m}{\mu s} \cdot \frac{\delta^2}{2(1+\delta/3)}\right)$. In particular, $\|A^*_T A_{:,T} - I\|_{2,2} \leq \frac{1}{2}$ with high probability provided $s \leq \gamma \frac{m}{\mu \log n}$, where $\gamma$ is some absolute constant.

This Lemma was proved in [7] by matrix Bernstein’s inequality, which is first introduced by [2]. A deep generalization is given in [25].

**Lemma 3.2** (Lemma 2.4 of [7]) Suppose $A$ is as defined in model 2. Fix $T \subset [n]$ with $|T| = s$ and $v \in \mathbb{R}^s$. Then $\|A^*_T A_{:,T} v\|_\infty \leq \frac{1}{20s} \|v\|_2$ with high probability provided $s \leq \gamma \frac{m}{\mu \log n}$, where $\gamma$ is some absolute constant.

**Lemma 3.3** (Lemma 2.5 of [7]) Suppose $A$ is as defined in model 2. Fix $T \subset [n]$ with $|T| = s$. Then $\max_{i \in T^c} \|A^*_T A_{:,i}\|_2 \leq 1$ with high probability provided $s \leq \gamma \frac{m}{\mu \log n}$, where $\gamma$ is some absolute constant.

3.2 A proof of Theorem 1.2

In this part we will give a complete proof of Theorem 1.2 with a powerful technique called "golfing-scheme" introduced by David Gross in [21], and later in [4] and [7]. Under the assumption of model 2, we additionally assume $s \leq \alpha \frac{m}{\mu \log^2 n}$ and $m_b \leq \beta \frac{m}{\mu}$, where $\alpha$ and $\beta$ are numerical constants whose values will specified later.

First we give two useful inequalities. By replacing $A$ with $\sqrt{\frac{m}{m-m_b}} A_{B^c,T}$ in Lemma 3.1 and Lemma 3.2, we have

$$\frac{m}{m-m_b} A^*_{B^c,T} A_{B^c,T} - I \|_{2,2} \leq 1/2$$

(3.1)

and

$$\max_{i \in T^c} \|\frac{m}{m-m_b} A^*_{B^c,T} A_{B^c,i}\|_2 \leq 1$$

(3.2)

with high probability provided $s \leq \gamma \frac{m}{\mu \log n}$. Since $s \leq \alpha \frac{m}{\mu \log^2 n}$ and $m_b \leq \beta \frac{m}{\mu}$, both (3.1) and (3.2) hold with high probability provided $\alpha$ and $\beta$ are sufficiently small. We assume (3.1) and (3.2) hold throughout this section.

First we prove that the solution $(\hat{x}, \hat{f})$ of (1.3) equals $(x, f)$ if we can find an appropriate dual vector $q_{B^c}$ satisfying the following requirement. This is actually an “inexact dual vector” of the optimization problem (1.3). This idea was first given explicitly in [22] and [21], and related to [5]. We give a result similar to [7].
Lemma 3.4 (Inexact Duality) Suppose there exists a vector \( q_{B^c} \in \mathbb{R}^{m-m_b} \) satisfying
\[
\|v_T - \text{sgn}(x_T)\|_2 \leq \lambda/4, \quad \|v_T\|_\infty \leq 1/4 \quad \text{and} \quad \|q_{B^c}\|_\infty \leq \lambda/4, \quad (3.3)
\]
where
\[
v = A_{B^c}^*q_{B^c} + A_{B^c}^*\lambda \text{sgn}(f_B). \quad (3.4)
\]
Then the solution \((\hat{x}, \hat{f})\) of \((1.3)\) equals \((x, f)\) provided \(\beta\) is sufficiently small and \(\lambda < \frac{3}{2}\).

Proof Set \(h = \hat{x} - x\). By \(x_{T^c} = 0\) we have
\[
h_{T^c} = \hat{x}_{T^c}. \quad (3.5)
\]
By \(f_{B^c} = 0\), and \(Ax + f = A\hat{x} + \hat{f}\), we have \(Ah = f - \hat{f}\) and
\[
A_{B^c}h = (f - \hat{f})_{B^c} = -\hat{f}_{B^c}. \quad (3.6)
\]
Then we have the following inequality
\[
\|\hat{x}\|_1 + \lambda \|\hat{f}\|_1
\geq (\hat{x}_T, \text{sgn}(\hat{x}_T)) + \|\hat{x}_{T^c}\|_1 + \lambda (\langle \hat{f}_B, \text{sgn}(\hat{f}_B) \rangle + \|\hat{f}_{B^c}\|_1)
\geq (x_T + h_T, \text{sgn}(x_T)) + \|h_{T^c}\|_1 + \lambda (\langle f_B - A_{B^c}h, \text{sgn}(f_B) \rangle + \|A_{B^c}h\|_1)
\geq (x, h) + \lambda \|f\|_1 + \|h_{T^c}\|_1 + \lambda \|A_{B^c}h\|_1 + \langle h, \text{sgn}(x_T) \rangle - \lambda \langle A_{B^c}h, \text{sgn}(f_B) \rangle.
\]
Since \(\|\hat{x}\|_1 + \lambda \|\hat{f}\|_1 \leq \|x\|_1 + \lambda \|f\|_1\), we have
\[
\|h_{T^c}\|_1 + \lambda \|A_{B^c}h\|_1 + \langle h, \text{sgn}(x_T) \rangle - \lambda \langle A_{B^c}h, \text{sgn}(f_B) \rangle \leq 0. \quad (3.7)
\]
By \((3.4)\), we have
\[
\langle h_T, v_T \rangle + \langle h_{T^c}, v_{T^c} \rangle = \langle h, v \rangle = \langle h, A_{B^c}^*q_{B^c} + A_{B^c}^*\lambda \text{sgn}(f_B) \rangle = \langle A_{B^c}h, q_{B^c} \rangle + \lambda \langle A_{B^c}h, \text{sgn}(f_B) \rangle,
\]
and then by \((3.3)\),
\[
\langle h_T, \text{sgn}(x_T) \rangle - \lambda \langle A_{B^c}h, \text{sgn}(f_B) \rangle = \langle h_T, (\text{sgn}(x_T) - v_T) \rangle + \langle A_{B^c}h, q_{B^c} \rangle - \langle h_{T^c}, v_{T^c} \rangle
\geq -\frac{\lambda}{4} \|h_T\|_2 - \frac{1}{4} \|A_{B^c}h\|_1 - \frac{1}{4} \|h_{T^c}\|_1.
\]
Unite it with \((3.7)\), we have
\[
-\frac{\lambda}{4} \|h_T\|_2 + \frac{3}{4} \|A_{B^c}h\|_1 + \frac{3}{4} \|h_{T^c}\|_1 \leq 0. \quad (3.8)
\]
By (3.1), we have \( \| \sqrt{\frac{m}{m-m_b}} A_{B^c,T}^* \|_2 \leq \sqrt{\frac{2}{2}} \) and the smallest singular value of \( \frac{m}{m-m_b} A_{B^c,T}^* A_{B^c,T} \) is at least \( \frac{1}{2} \). Therefore,

\[
\| h_T \|_2 \leq 2 \left( \| \frac{m}{m-m_b} A_{B^c,T}^* A_{B^c,T} h_T \|_2 \right)
\leq 2 \left( \| \frac{m}{m-m_b} A_{B^c,T}^* A_{B^c,T} h_T \|_2 + \| \frac{m}{m-m_b} A_{B^c,T}^* A_{B^c,T} h \|_2 \right)
\leq 2 \left( \| \frac{m}{m-m_b} A_{B^c,T}^* A_{B^c,T} h_T \|_2 + \sqrt{6} \right) \| \frac{m}{m-m_b} A_{B^c,T}^* A_{B^c,T} h \|_2
\leq 2 \| h_T \|_1 + \sqrt{6} \| \frac{m}{m-m_b} A_{B^c,T}^* A_{B^c,T} h \|_1
\]

By the triangle inequality

Plugging this into (3.8), we have

\[
\frac{3}{4} - \sqrt{6} \sqrt{\frac{m}{m-m_b}} > 0 \quad \text{when } \lambda \text{ is sufficiently small. Moreover, by the assumption } \lambda < \frac{3}{2}, \text{ we have } h_T = 0 \text{ and } A_{B^c,T} h = 0. \quad \text{Since } A_{B^c,T} h = A_{B^c,T} h_T + A_{B^c,T} h_T, \text{ we have } A_{B^c,T} h_T = 0. \quad \text{The inequality (3.1) implies that } A_{B^c,T} \text{ is injective, so } h_T = 0 \text{ and } h = h_T + h_T = 0, \text{ which implies } (\hat{x}, \hat{f}) = (x, f).
\]

Now let’s construct a vector \( q_{B^c} \) satisfying the requirement (3.3) by choosing an appropriate \( \lambda \).

**Proof** (of Theorem 1.2) Set \( \lambda = \frac{1}{\sqrt{\log n}} \). It suffices to construct a \( q_{B^c} \) satisfying (3.3). Denoting \( u = A_{B^c,T}^* q_{B^c} \), we only need to construct a \( q_{B^c} \) satisfying

\[
\| u_T + \lambda A_{B^c,T}^* \text{sgn}(f_B) - \text{sgn}(x_T) \|_2 \leq \frac{\lambda}{4}, \quad \| u_T \|_\infty \leq \frac{1}{8}, \quad \| \lambda A_{B^c,T}^* \text{sgn}(f_B) \|_\infty \leq \frac{1}{8}, \quad \| q_{B^c} \|_\infty \leq \frac{\lambda}{4}.
\]

Now let’s construct our \( q_{B^c} \) by the golfing scheme. First we have to write \( A_{B^c,T}^* \) as a block matrix. We divide \( B^c \) into \( l = \lceil \log_2 n + 1 \rceil = \lceil \frac{\log n}{\log 2} + 1 \rceil \) disjoint subsets: \( B^c = G_1 \cup \ldots \cup G_l \) where \( |G_i| = m_i \). Then we have \( \sum_{i=1}^l m_i = m - m_b \) and

\[
A_{B^c,T}^* = \begin{bmatrix} A_{G_1,T}^* & \cdots & A_{G_l,T}^* \end{bmatrix}.
\]

We want to mention that the partition of \( B^c \) is deterministic, not depending on \( A \), so \( A_{G_1,T}^* , \ldots , A_{G_l,T}^* \) are independent. Noticing \( m_b \leq \beta \frac{m}{\mu} \leq \beta m \), by letting \( \beta \) sufficiently small, we can require

\[
\frac{m}{m_1} \leq C, \quad \frac{m}{m_2} \leq C, \quad \frac{m}{m_k} \leq C \log n \quad \text{for } k = 3, \ldots, l
\]

for some absolute constant \( C \). Since \( s \leq \alpha \frac{m}{\mu \log n} \), we have

\[
s \leq \alpha C \frac{m_1}{\mu \log^2 n}, \quad s \leq \alpha C \frac{m_2}{\mu \log^2 n}, \quad s \leq \alpha C \frac{m_k}{\mu \log n} \quad \text{for } k = 3, \ldots, l.
\]
Then by Lemma 3.1 replacing \( A \) with \( \sqrt{\frac{m}{m'}} A_{G_j,T} \), we have the following inequalities:

\[
\begin{align*}
\left\| \frac{m}{m'} A_{G_j,T} A_{G_j,T} - I \right\|_{2,2} & \leq \frac{1}{2 \sqrt{\log n}} \quad \text{for } j = 1, 2; \\
\left\| \frac{m}{m'} A_{G_j,T} A_{G_j,T} - I \right\|_{2,2} & \leq \frac{1}{2} \quad \text{for } j = 3, ..., l;
\end{align*}
\]

with high probability provided \( \alpha \) is sufficiently small.

Now let’s give an explicit construction of \( q_{B^c} \). Define

\[
p_0 = \text{sgn}(x_T) - \lambda A_{B,T} \text{sgn}(f_B)
\]

and

\[
p_i = \left( I - \frac{m}{m_i} A_{G_j,T} A_{G_j,T} \right) p_{i-1} = \left( I - \frac{m}{m_i} A_{G_j,T} A_{G_j,T} \right) \cdots \left( I - \frac{m}{m_1} A_{G_1,T} A_{G_1,T} \right) p_0
\]

for \( i = 1, ..., l \), and construct

\[
q_{B^c} = \begin{bmatrix}
\frac{m}{m_1} A_{G_1,T} p_0 \\
\vdots \\
\frac{m}{m_l} A_{G_l,T} p_{l-1}
\end{bmatrix}
\]

Then by \( u = A_{B^c}^* q_{B^c} \), we have

\[
u = A_{B^c}^* \begin{bmatrix}
\frac{m}{m_1} A_{G_1,T} p_0 \\
\vdots \\
\frac{m}{m_l} A_{G_l,T} p_{l-1}
\end{bmatrix} = \sum_{i=1}^{l} \frac{m}{m_i} A_{G_i,T} A_{G_i,T} p_{i-1}.
\]

We now bound the \( \ell_2 \) norm of \( p_i \). Actually, by (3.10), (3.11) and (3.13), we have

\[
\|p_1\|_2 \leq \frac{1}{2 \sqrt{\log n}} \|p_0\|_2;
\]

\[
\|p_2\|_2 \leq \frac{1}{4 \log n} \|p_0\|_2;
\]

\[
\|p_j\|_2 \leq \frac{1}{\log n} \left( \frac{1}{2} \right)^j \|p_0\|_2 \quad \text{for } j = 3, ..., l.
\]

Now we will prove our constructed \( q_{B^c} \) satisfies the desired requirements:

The proof of \( \|\lambda A_{B^c}^* \text{sgn}(f_B)\|_\infty \leq \frac{1}{8} \)

By Hoeffding’s inequality, for any \( i = 1, ..., n \), we have \( \mathbb{P} \left( \left| A_{B,i}^* \text{sgn}(f_B) \right| \geq t \right) \leq 2 \exp \left( -\frac{2t^2}{4 \| A_{B,i} \|_2^2} \right) \).

By choosing \( t = C \sqrt{\log n} \| A_{B,i} \|_2 \) (\( C \) is some absolute constant), with high probability, we have \( \|\lambda A_{B,i}^* \text{sgn}(f_B)\|_\infty \leq C \sqrt{\log n} \| A_{B,i} \|_2 \leq C \sqrt{\frac{m'}{m}} \leq \sqrt{\beta} \leq \frac{1}{8} \), provided \( \beta \) is sufficiently small, and this implies \( \|\lambda A_{B^c}^* \text{sgn}(f_B)\|_\infty \leq \frac{1}{8} \).
The proof of \( \| u_T + \lambda A_{B,T}^* \text{sgn}(f_B) - \text{sgn}(x_T) \|_2 \leq \frac{\lambda}{4} \)

By (3.15) and (3.13), we have \( u_T = \sum_{i=1}^{l} \frac{m}{m_i} A_{G_i,T}^* A_{G_i,T} p_{i-1} = \sum_{i=1}^{l} (p_{i-1} - p_i) = p_0 - p_l \). Then by (3.12), we have \( \| u_T + \lambda A_{B,T}^* \text{sgn}(f_B) - \text{sgn}(x_T) \|_2 = \| u_T - p_0 \|_2 = \| p_l \|_2 \). Since \( \| \lambda A_{B,T}^* \text{sgn}(f_B) \|_\infty \leq 1/8 \), we have \( \| A_{B,T}^* \text{sgn}(f_B) \|_2 \leq \frac{1}{8} \sqrt{s} \), which implies

\[
\| p_0 \|_2 = \| \lambda A_{B,T}^* \text{sgn}(f_B) - \text{sgn}(x_T) \|_2 \leq \frac{9}{8} \sqrt{s}.
\]

Then by (3.18) and \( l = \lfloor \log_2 n + 1 \rfloor \), we have \( \| p_l \|_2 \leq \frac{1}{\log n} (\frac{1}{2})^{\frac{9}{8} \sqrt{s}} \leq \left( \frac{1}{\log n} \right) \left( \frac{1}{8} \right) \sqrt{\frac{\alpha n}{\mu \log^2 n}} \leq \frac{1}{4 \sqrt{\log n}} = \frac{\lambda}{4} \), provided \( \alpha \) is sufficiently small.

The proof of \( \| u_{T^c} \|_\infty \leq 1/8 \)

By (3.15), we have \( u_{T^c} = \sum_{i=1}^{l} \frac{m}{m_i} A_{G_i,T^c}^* A_{G_i,T} p_{i-1} \). Recall that \( A_{G_1:T^c}, ..., A_{G_l:T^c} \) are independent, so by the construction of \( p_{i-1} \) we know \( A_{G_i:T^c} \) and \( p_{i-1} \) are independent. Replacing \( A \) with \( \sqrt{\frac{m_i}{m}} A_{G_i:T^c} \) in Lemma 3.2 and by the sparsity condition (3.9), we have \( \sum_{i=1}^{l} \| \frac{m}{m_i} A_{G_i,T^c}^* A_{G_i,T} p_{i-1} \|_\infty \leq \sum_{i=1}^{l} \frac{1}{20} \sqrt{s} \| p_{i-1} \|_2 \) with high probability, provided \( \alpha \) is sufficiently small. By (3.16), (3.17), (3.18) and (3.19), we have \( \| u_{T^c} \|_\infty \leq \sum_{i=1}^{l} \frac{1}{20} \sqrt{s} \| p_{i-1} \|_2 \leq \frac{1}{20} \sqrt{s} \| p_0 \|_2 < \frac{1}{8} \).

The proof of \( \| q_{B^c} \|_\infty \leq \frac{\lambda}{4} \)

For \( k = 1, ..., l \), we denote \( A_{G_k:T^c}^* = \frac{1}{\sqrt{m}} \begin{bmatrix} a_{k1}^* & ... & a_{km_k}^* \end{bmatrix} \) and \( A_{B:j} = \frac{1}{\sqrt{m}} \begin{bmatrix} \tilde{a}_{1}^* & ... & \tilde{a}_{m}^* \end{bmatrix} \). By (3.13), (3.14) and (3.12), it suffices to show that for any \( 1 \leq k \leq l \) and \( 1 \leq j \leq m_k \),

\[
\left| \sqrt{\frac{m}{m_k}} (a_{k})_T^* \left( I - \frac{m}{m_{k-1}} A_{G_{k-1:T} A_{G_{k-1:T}}} \right) \cdots \left( I - \frac{m}{m_1} A_{G_{1:T} A_{G_{1:T}}} \right) (\text{sgn}(x_T) - \lambda A_{B,T}^* \text{sgn}(f_B)) \right| \leq \frac{\lambda}{4}.
\]

Set

\[
w = \left( I - \frac{m}{m_1} A_{G_{1:T} A_{G_{1:T}}} \right) \cdots \left( I - \frac{m}{m_{k-1}} A_{G_{k-1:T} A_{G_{k-1:T}}} \right) (a_{k})_T.
\]

Then it suffices to prove

\[
\left| \sqrt{\frac{m}{m_k}} w^* (\text{sgn}(x_T) - \lambda A_{B,T}^* \text{sgn}(f_B)) \right| \leq \frac{\lambda}{4}.
\]

Since \( w \) and \( \text{sgn}(x_T) \) are independent, by Hoeffding’s inequality and conditioning on \( w \), we have

\[
\mathbb{P} \left( |w^* \text{sgn}(x_T)| \geq t \right) \leq 2 \exp \left( -\frac{2t^2}{4||w||_2^2} \right) \text{ for any } t > 0.
\]

Then with high probability we have

\[
|w^* \text{sgn}(x_T)| \leq C \sqrt{\log n} ||w||_2
\]

(3.21)
for some absolute constant $C$.

Setting $z = \text{sgn}(f_B)$, we have $w^*A_{B,T}^*\text{sgn}(f_B) = \frac{1}{\sqrt{m}} \sum_{i=1}^{m_b} [(\tilde{a}_i)^*_T w] z(i)$. Since $w$, $A_{B,T}$ and $z$ are independent, conditioning on $w$ we have

$$\mathbb{E}\{[(\tilde{a}_i)^*_T w] z(i)\} = \mathbb{E}\{(\tilde{a}_i)^*_T w\} \mathbb{E}\{z(i)\} = 0,$$

$$|[(\tilde{a}_i)^*_T w] z(i)| \leq \|w\|_2 \|(\tilde{a}_i)^*_T\|_2 \leq \sqrt{s_f} \|w\|_2 \leq \sqrt{\frac{\alpha m}{\log^2 n}} \|w\|_2,$$

and

$$\mathbb{E}\{|[(\tilde{a}_i)^*_T w] z(i)|^2\} = \mathbb{E}\{|w^*(\tilde{a}_i)^*_T [(\tilde{a}_i)^*_T w]\} = w^* \mathbb{E}\{(\tilde{a}_i)^*_T (\tilde{a}_i)^*_T\} w = \|w\|_2^2.$$

By Bernstein’s inequality, we have

$$\mathbb{P}\left(|w^*A_{B,T}^*\text{sgn}(f_B)| \geq \frac{t}{\sqrt{m}}\right) \leq 2 \exp\left(-\frac{t^2/2}{m_b \|w\|_2^2 + \sqrt{\frac{\alpha m}{\log^2 n}} \|w\|_2^2 t/3}\right).$$

By choosing some numerical constant $C$ and $t = C \sqrt{m \log n} \|w\|_2$, we have

$$|w^*A_{B,T}^*\text{sgn}(f_B)| \leq C \sqrt{\log n} \|w\|_2$$

(3.22)

with high probability, provided $\alpha$ is sufficiently small.

By (3.21) and (3.22), we have

$$\left|\frac{\sqrt{m}}{m_b} w^* \left(\text{sgn}(x^T) - \lambda A_{B,T}^*\text{sgn}(f_B)\right)\right| \leq \frac{\sqrt{m}}{m_b} C \sqrt{\log n} \|w\|_2,$$

(3.23)

for some numerical constant $C$.

When $k \geq 3$, by (3.20), (3.10) and (3.11), we have $\|w\|_2 \leq (\frac{1}{2})^{k-1} \frac{1}{\log n} \sqrt{m_b} \leq \frac{\sqrt{\alpha m}}{\log^2 n}$. Recalling $\frac{m}{m_b} \leq C \log n$, by (3.23), we have

$$\left|\frac{\sqrt{m}}{m_b} w^* \left(\text{sgn}(x^T) - \lambda A_{B,T}^*\text{sgn}(f_B)\right)\right| \leq C \left(\frac{m}{m_b}\right) \sqrt{\alpha (\log n)^{-3/2}} \leq \frac{1}{4}$$

provided $\alpha$ is sufficiently small.

When $k \leq 2$, by (3.20) and (3.10), we have $\|w\|_2 \leq \sqrt{\frac{m_b}{s_f}} \leq \frac{\sqrt{\alpha m}}{\log^2 n}$. Recalling $\frac{m}{m_b} \leq C$, by (3.23), we have

$$\left|\frac{\sqrt{m}}{m_b} w^* \left(\text{sgn}(x^T) - \lambda A_{B,T}^*\text{sgn}(f_B)\right)\right| \leq C \left(\frac{m}{m_b}\right) \sqrt{\alpha (\log n)^{-1/2}} \leq \frac{1}{4}$$

provided $\alpha$ is sufficiently small.

Here we would like to compare our golfing scheme with that in [7]. There are mainly two differences. One is that we have an extra term $\lambda A_{B,T}^*\text{sgn}(f_B)$ in the dual vector. To obtain the inequality $\|v_T\|_{\infty} \leq 1/4$, we propose to bound $\|u_T\|_{\infty}$ and $\|\lambda A_{B,T}^*\text{sgn}(f_B)\|_{\infty}$ respectively, and this will lead to the extra log factor compared with [7]. Moreover, by using the golfing scheme to construct the dual vector, we need to bound the term $\|q_B\|_{\infty}$, which is not necessary in [7]. This inevitably incurs the random signs assumptions of the signal.
4 A Proof of Theorem 1.3

In this section, the capital letters \( X, Y \) etc represent matrices, and the symbols in script font \( \mathcal{I}, \mathcal{P}_T, \) etc represent linear operators from a matrix space to a matrix space. Moreover, for any \( \Omega_0 \subset [n] \times [n] \) we have \( \mathcal{P}_{\Omega_0}M \) is to keep the entries of \( M \) on the support \( \Omega_0 \) and to change other entries into zeros. For any \( n \times n \) matrix \( A, \) denote by \( \|A\|_F, \|A\|, \|A\|_\infty \) and \( \|A\|_s \) respectively the Frobenius norm, operator norm (the largest singular value), the biggest magnitude of all elements, and the nuclear norm (the sum of all singular values).

Similarly to Section 3, instead of denoting them as \( C_1, C_2, \ldots, \) we just use \( C, \) whose values change from line to line. Also, we will use the phrase “with high probability” to mean with probability at least \( 1 - Cn^{-c} \), where \( C > 0 \) is a numerical constant and \( c = 3, 4, \) or \( 5 \) depending on the context.

4.1 A model equivalent to Model 3.1

Model 3.1 is natural and used in [1], but we will use the following equivalent model for the convenience of proof:

**Model 3.2:** 1. Fix an \( n \) by \( n \) matrix \( K, \) whose entries are either \( 1 \) or \( -1. \)
2. Define two independent random subsets of \([n] \times [n]:: \Gamma' \sim \text{Ber}((1 - 2s)\rho) \) and \( \Omega' \sim \text{Ber}(\frac{2s\rho}{1 - \rho + 2s\rho}). \) Moreover, let \( O := \Gamma' \cup \Omega', \) which thus satisfies \( O \sim \text{Ber}(\rho). \)
3. Define an \( n \times n \) random matrix \( W \) with independent entries \( W_{ij} \) satisfying \( \mathbb{P}(W_{ij} = 1) = \mathbb{P}(W_{ij} = -1) = \frac{1}{2}. \)
4. Define \( \Omega'' \subset \Omega': \Omega'' := \{(i, j) : (i, j) \in \Omega', W_{ij} = K_{ij}\}. \)
5. Define \( \Omega := \Omega''/\Gamma', \) and \( \Gamma := O/\Omega. \)
6. Let \( S \) satisfy \( \text{sgn}(S) := \mathcal{P}_\Omega(K). \)

Obviously, in both Model 3.1 and Model 3.2 the whole setting is deterministic if we fix \( (O, \Omega). \) Therefore, the probability of \((L, \hat{S}) = (L, S)\) is determined by the joint distribution of \((O, \Omega). \) It is not difficult to prove that the joint distributions of \((O, \Omega)\) in both models are the same. Indeed, in Model 3.1, we have that \( \{1_{\{(i,j)\in\Omega\}}, 1_{\{(i,j)\in\Omega\}}\} \) are iid random vectors with the probability distribution \( \mathbb{P}(1_{\{(i,j)\in\Omega\}} = 1) = \rho, \mathbb{P}(1_{\{(i,j)\in\Omega\}} = 1 \| 1_{\{(i,j)\in\Omega\}} = 1) = s \) and \( \mathbb{P}(1_{\{(i,j)\in\Omega\}} = 1 \| 1_{\{(i,j)\in\Omega\}} = 0) = 0. \) In Model 3.2, we have

\[
(1_{\{(i,j)\in\Omega\}}, 1_{\{(i,j)\in\Omega\}}) = (\max(1_{\{(i,j)\in\Gamma'\}}, 1_{\{(i,j)\in\Omega'\}}, 1_{\{(i,j)\in\Omega'\}}, 1_{\{W_{ij} = K_{ij}\}}, 1_{\{(i,j)\in\Gamma''\}}).
\]

This implies that \( \{1_{\{(i,j)\in\Omega\}}, 1_{\{(i,j)\in\Omega\}}\} \) are independent random vectors. Moreover, it is easy to calculate that \( \mathbb{P}(1_{\{(i,j)\in\Omega\}} = 1) = \rho, \mathbb{P}(1_{\{(i,j)\in\Omega\}} = 1) = sp \) and \( \mathbb{P}(1_{\{(i,j)\in\Omega\}} = 1, 1_{\{(i,j)\in\Omega\}} = 0) = 0. \) Then we have

\[
\mathbb{P}(1_{\{(i,j)\in\Omega\}} = 1 \| 1_{\{(i,j)\in\Omega\}} = 1) = \mathbb{P}(1_{\{(i,j)\in\Omega\}} = 1, 1_{\{(i,j)\in\Omega\}} = 1) / \mathbb{P}(1_{\{(i,j)\in\Omega\}} = 1) = s,
\]

and

\[
\mathbb{P}(1_{\{(i,j)\in\Omega\}} = 1 \| 1_{\{(i,j)\in\Omega\}} = 0) = \mathbb{P}(1_{\{(i,j)\in\Omega\}} = 1, 1_{\{(i,j)\in\Omega\}} = 0) / \mathbb{P}(1_{\{(i,j)\in\Omega\}} = 0) = 0.
\]
Notice that although \((1_{\{(i,j)\in O\}}, 1_{\{(i,j)\in \Omega\}})\) depends on \(K\), its distribution does not. By the above we know that \((O, \Omega)\) has the same distribution in both models. Therefore in the following we will use Model 3.2 instead. The advantage of using Model 3.2 is that we can utilize \(\Gamma', \Omega', W, \) etc. as auxiliaries.

In the next section we prove some supporting lemmas which are useful for the proof of the main theorem.

### 4.2 Supporting lemmas

Define \(T := \{UX^* + YV^*, X, Y \in \mathbb{R}^{n \times r}\}\) a subspace of \(\mathbb{R}^{n \times n}\). Then the orthogonal projectors \(P_T\) and \(P_{T^\perp}\) satisfy \(P_TX = UU^*X + XVV^* - UU^*XVV^*\) and \(P_{T^\perp}X = (I - UU^*)X(I - VV^*)\) for any \(X \in \mathbb{R}^{n \times n}\). This means \(\|P_{T^\perp}X\| \leq \|X\|\) for any \(X\). Recalling the incoherence conditions: for any \(i \in \{1, \ldots, n\}\), \(\|UU^*e_i\|^2 \leq \frac{\mu_r}{n}\) and \(\|VV^*e_i\|^2 \leq \frac{\mu_r}{n}\), we have \(\|P_T(e_i e_i^*)\|_\infty \leq \frac{2\mu_r}{n}\) and \(\|P_T(e_i e_i^*)\|_F \leq \sqrt{\frac{2\mu_r}{n}}\).

**Lemma 4.1** (Theorem 4.1 of [8]) Suppose \(\Omega_0 \sim \text{Ber}(\rho_0)\). Then with high probability, \(\|P_T - \rho_0^{-1}P_T P_{\Omega_0} P_T\| \leq \epsilon\), provided that \(\rho_0 \geq C_0 \epsilon^{-2} \frac{\mu_r \log n}{n}\) for some numerical constant \(C_0 > 0\).

The original idea of the proof of this theorem is due to [36].

**Lemma 4.2** (Theorem 3.1 of [4]) Suppose \(Z \in \text{Range}(P_T)\) is a fixed matrix, \(\Omega_0 \sim \text{Ber}(\rho_0)\), and \(\epsilon \leq 1\) is an arbitrary constant. Then with high probability \(\|(I - \rho_0^{-1}P_T P_{\Omega_0})Z\|_\infty \leq \epsilon\|Z\|_\infty\) provided that \(\rho_0 \geq C_0 \epsilon^{-2} \frac{\mu_r \log n}{n}\) for some numerical constant \(C_0 > 0\).

**Lemma 4.3** (Theorem 6.3 of [8]) Suppose \(Z\) is a fixed matrix, and \(\Omega_0 \sim \text{Ber}(\rho_0)\). Then with high probability, \(\|\rho_0 I - P_{\Omega_0}\| \leq C_0' \sqrt{n \rho \log n \|Z\|_\infty}\) provided that \(\rho_0 \leq p\) and \(p \geq C_0' \frac{\log n}{\epsilon}\) for some numerical constants \(C_0 > 0\) and \(C_0' > 0\).

Notice that we only have \(\rho_0 = p\) in Theorem 6.3 of [8]. By a very slight modification in the proof (specifically, the proof of Lemma 4.2) we can have \(\rho_0 \leq p\) as stated above.

### 4.3 A proof of Theorem 1.3

By Lemma 3.1, we have we have \(\|\frac{1}{(1 - 2s)p}P_T P_{T^\perp} P_T\| \leq \frac{1}{2}\) and \(\|\frac{1}{\sqrt{1 - 2s}} P_T P_{T^\perp}\| \leq \sqrt{3/2}\) with high probability provided \(C_s\) is sufficiently large and \(C_s\) is sufficiently small. We will assume both inequalities hold all through the paper.

**Theorem 4.4** If there exists an \(n \times n\) matrix \(Y\) obeying

\[
\begin{align*}
\|P_T Y + P_T (\lambda P_{O/T} W - UV^*)\|_F & \leq \frac{\lambda}{n^2}, \\
\|P_T Y + P_{T^\perp} (\lambda P_{O/T} W)\| & \leq \frac{1}{4}, \\
P_{T^\perp} Y & = 0, \\
\|P_{T^\perp} Y\|_\infty & \leq \frac{1}{4},
\end{align*}
\]

where \(\lambda = \frac{1}{\sqrt{n \rho \log n}}\). Then the solution \((\hat{L}, \hat{S})\) to (1.6) satisfies \((\hat{L}, \hat{S}) = (L, S)\).
Proof. Set $H = \hat{L} - L$. The condition $\mathcal{P}_O(L + S) = \mathcal{P}_O(\hat{L}) + \hat{S}$ implies that $\mathcal{P}_O(H) = S - \hat{S}$. Then $\hat{S}$ is supported on $O$ because $S$ is supported on $\Omega \subseteq O$. By considering the subgradient of the nuclear norm at $L$, we have

$$\|\hat{L}\|_* \geq \|L\|_* + \langle P_T H, UV^* \rangle + \|P_T \perp H\|_*.$$ 

By the definition of $(\hat{L}, \hat{S})$, we have

$$\|\hat{L}\|_* + \lambda \|\hat{S}\|_1 \leq \|L\|_* + \lambda \|S\|_1.$$ 

By the two inequalities above, we have

$$\lambda \|S\|_1 - \lambda \|\hat{S}\|_1 \geq \langle P_T(H), UV^* \rangle + \|P_T \perp H\|_*,$$

which implies

$$\lambda \|S\|_1 - \lambda \|\mathcal{P}_{O/\Gamma'}(\hat{S})\|_1 \geq \langle H, UV^* \rangle + \|P_T \perp (H)\|_* + \lambda \|P_{\Gamma'}(\hat{S})\|_1.$$ 

On the other hand,

$$\|\mathcal{P}_{O/\Gamma'} \hat{S}\|_1 = \|S + \mathcal{P}_{O/\Gamma'}(-H)\|_1 \\
\geq \|S\|_1 + \langle \text{sgn}(S), \mathcal{P}_{\Omega}(-H) \rangle + \|\mathcal{P}_{O/(\Gamma' \cup \Omega)}(-H)\|_1 \\
\geq \|S\|_1 + \langle \mathcal{P}_{O/\Gamma'}(W), -H \rangle.$$ 

By the two inequalities above and the fact $\mathcal{P}_{\Gamma'} \hat{S} = \mathcal{P}_{\Gamma'}(\hat{S} - S) = -\mathcal{P}_{\Gamma'}H$, we have

$$\|\mathcal{P}_{T \perp} (H)\|_* + \lambda \|\mathcal{P}_{\Gamma'}(H)\|_1 \leq \langle H, \lambda \mathcal{P}_{O/\Gamma'}(W) - UV^* \rangle.$$ 

(4.2) By the assumptions of $Y$, we have

$$\langle H, \lambda \mathcal{P}_{O/\Gamma'}(W) - UV^* \rangle \\
= \langle H, Y + \lambda \mathcal{P}_{O/\Gamma'}(W) - UV^* \rangle - \langle H, Y \rangle \\
= \langle \mathcal{P}_T(H), \mathcal{P}_T(Y + \lambda \mathcal{P}_{O/\Gamma'}(W) - UV^*) \rangle + \langle \mathcal{P}_{T \perp}(H), \mathcal{P}_{T \perp}(Y + \lambda \mathcal{P}_{O/\Gamma'}(W)) \rangle \\
- \langle \mathcal{P}_{\Gamma'}(H), \mathcal{P}_{\Gamma'}(Y) \rangle - \langle \mathcal{P}_{\Gamma'}(H), \mathcal{P}_{\Gamma'}(Y) \rangle \\
\leq \frac{\lambda}{n^2} \|\mathcal{P}_T(H)\|_F + \frac{1}{4} \|\mathcal{P}_{T \perp}(H)\|_* + \frac{\lambda}{4} \|\mathcal{P}_{\Gamma'}(H)\|_1.$$ 

By inequality 1.2

$$\frac{3}{4} \|\mathcal{P}_{T \perp}(H)\|_* + \frac{3\lambda}{4} \|\mathcal{P}_{\Gamma'}(H)\|_1 \leq \frac{\lambda}{n^2} \|\mathcal{P}_T(H)\|_F.$$ 

(4.3) Recall that we assume $\|\frac{1}{(1-2s)} \mathcal{P}_T \mathcal{P}_{\Gamma'} \mathcal{P}_T - \mathcal{P}_T\| \leq \frac{1}{2}$ and $\|\frac{1}{\sqrt{(1-2s)}} \mathcal{P}_T \mathcal{P}_{\Gamma'}\| \leq \sqrt{3}/2$ all through the paper. Then

$$\|\mathcal{P}_T(H)\|_F \leq 2 \|\frac{1}{(1-2s)} \mathcal{P}_T \mathcal{P}_{\Gamma'} \mathcal{P}_T(H)\|_F \\
\leq 2 \|\mathcal{P}_T \mathcal{P}_{\Gamma'} \mathcal{P}_T \mathcal{P}_{T \perp}(H)\|_F + 2 \|\frac{1}{(1-2s)} \mathcal{P}_T \mathcal{P}_{\Gamma'}(H)\|_F \\
\leq \sqrt{\frac{6}{(1-2s)}} \|\mathcal{P}_{T \perp} H\|_F + \sqrt{\frac{6}{(1-2s)}} \|\mathcal{P}_{\Gamma'} H\|_F.$$ 

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By inequality [4.3] we have

\[
\left(\frac{3}{4} - \frac{\lambda}{n^2} \sqrt{\frac{6}{(1 - 2s)\rho}}\right)\|P_{T^\perp}(H)\|_F + \left(\frac{3\lambda}{4} - \frac{\lambda}{n^2} \sqrt{\frac{6}{(1 - 2s)\rho}}\right)\|P_{T'\perp}H\|_F \leq 0.
\]

Then \(P_{T^\perp}(H) = P_{T'\perp}H = 0\), which implies \(P_{T'}P_T(H) = 0\). Since \(P_{T'}P_T\) is injective (\(\|\frac{1}{(1 - 2s)\rho}P_T P_{T'\perp} - P_T\| \leq \frac{2}{3}\)) on \(T\), we have \(P_T(H) = 0\). Then we have \(H = 0\).

Suppose we can construct \(Y\) and \(\tilde{Y}\) satisfying

\[
\begin{align*}
\|P_TY + P_T(\lambda P_{\Omega^1}W - UV^*)\|_F &\leq \frac{\lambda}{2n^2}, \\
\|P_{T^\perp}Y + P_{T^\perp}(\lambda P_{\Omega^1}W)\| &\leq \frac{1}{4}, \\
P_{T'^{\perp}}Y & = 0, \\
\|P_{T'}Y\|_\infty &\leq \frac{\lambda}{4}.
\end{align*}
\]

and

\[
\begin{align*}
\|P_T\tilde{Y} + P_T(\lambda(2P_{\Omega^1}W) - U V^*)\|_F &\leq \frac{\lambda}{2n^2}, \\
\|P_{T^\perp}\tilde{Y} + P_{T^\perp}(\lambda(2P_{\Omega^1}W) - P_{\Omega^1}W)\| &\leq \frac{1}{4}, \\
P_{T'^{\perp}}\tilde{Y} & = 0, \\
\|P_{T'}\tilde{Y}\|_\infty &\leq \frac{\lambda}{4}.
\end{align*}
\]

Then \(Y = (Y + \tilde{Y})/2\) will satisfy [4.1]. By the assumptions in Model 2, \((\Gamma', P_{\Omega^1}W)\) and \((\Gamma', 2P_{\Omega^1}W - P_{\Omega^1}W)\) have the same distribution. Therefore, if we can construct \(Y\) satisfying [4.4] with high probability, we can also construct \(\tilde{Y}\) satisfying [4.5] with high probability. Therefore to prove Theorem 1.3, we only need to prove that there exists \(Y\) satisfying [4.4] with high probability:

**Proof** (of Theorem 1.3) Notice that \(\Gamma' \sim \text{Ber}(1 - 2s)\rho\). Suppose that \(q\) satisfying \(1 - (1 - 2s)\rho = (1 - (\frac{1 - 2s}{6})^2(1 - q)^{l - 2}\), where \(l = [5\log n + 1]\). This implies that \(q \geq C_\rho/\log(n)\). Define \(q_1 = q_2 = (1 - 2s)\rho/6\) and \(q_3 = ... = q_l = q\). Then in distribution we can let \(\Gamma' = \Gamma_1 \cup ... \cup \Gamma_l\), where \(\Gamma_j \sim \text{Ber}(q_j)\) independently.

Construct

\[
\begin{align*}
Z_0 &= P_T(U V^* - \lambda P_{\Omega^1}W), \\
Z_j &= (P_T - \frac{1}{q_j}P_T P_{\Gamma_j} P_T)Z_{j - 1} \text{ for } j = 1, ..., l, \\
Y &= \sum_{j=1}^{l} \frac{1}{q_j} P_{\Gamma_j} Z_{j - 1},
\end{align*}
\]

Then by Lemma 4.1 we have

\[
\|Z_j\|_F \leq \frac{1}{2} \|Z_{j - 1}\|_F \text{ for } j = 1, ..., l,
\]

with high probability provided \(C_\rho\) is large enough and \(C_\chi\) is small enough. Then \(\|Z_j\|_F \leq (\frac{1}{2})^j\|Z_0\|_F\). By the construction of \(Z_j\), we know that \(Z_j \in \text{Range}(P_T)\) and \(Z_j = (I - \frac{1}{q_j}P_T P_{\Gamma_j})Z_{j - 1}\).

Then similarly, by Lemma 4.2 we have

\[
\|Z_1\|_\infty \leq \frac{1}{2\sqrt{\log n}}\|Z_0\|_\infty,
\]

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and
\[ \|Z_j\|_\infty \leq \frac{1}{2j \log n} \|Z_0\|_\infty \text{ for } j = 2, \ldots, l \]
with high probability provided \( C_\rho \) is large enough and \( C_s \) is small enough. Also, by Lemma 4.3, we have
\[ \| (I - \frac{1}{q} P_T) Z_{j-1} \| \leq C \sqrt{\frac{n \log n}{q}} \|Z_{j-1}\|_\infty \text{ for } j = 1, \ldots, l \]
with high probability provided \( C_\rho \) is large enough and \( C_s \) is small enough.

We first bound \( \|Z_0\|_F \) and \( \|Z_0\|_\infty \). Obviously \( \|Z_0\|_\infty \leq \|UV^*\|_\infty + \lambda \|P_T P_Y(W)\|_\infty \). Recall that for any \( i, j \in [n] \), we have \( \|P_T(e_i e_j^*)\|_\infty \leq 2 \mu r \) and \( \|P_T(e_i e_j^*)\|_F \leq \sqrt{2 \mu r} \). Moreover, \( P_Y(W) \) satisfies (4.3), are iid random variables with the distribution
\[
(P_Y(W))_{ij} = \begin{cases} 1 & \text{with probability } \frac{s \rho}{1 - \rho + 2s \rho} \\ 0 & \text{with probability } \frac{1 - s \rho}{1 - \rho + 2s \rho} \\ -1 & \text{with probability } \frac{s \rho}{1 - \rho + 2s \rho} \end{cases}
\]
Then by Bernstein's inequality, we have
\[
P\left( \left| \langle P_T(P_Y(W)), e_i e_j^* \rangle \right| \geq t \right) = \mathbb{P}\left( \left| \langle P_T(W), P_T(e_i e_j^*) \rangle \right| \geq t \right) \leq 2 \exp\left( -\frac{t^2}{2 \sum EX_j^2 + Mt/3} \right),
\]
where we have
\[
\sum EX_j^2 = \frac{2s \rho}{1 - \rho + 2s \rho} \|P_T e_i e_j^*\|_F^2 \leq C \rho s \mu r \frac{n}{n},
\]
and
\[
M = \|P_T e_i e_j^*\|_\infty \leq \frac{2 \mu r}{n}.
\]
Then with high probability we have \( \|P_T P_Y(W)\|_\infty \leq C \sqrt{\frac{\mu r log n n}{n}} \geq C \sqrt{C_\rho \mu r n \mu r log n} \geq C \sqrt{C_\rho \mu r log n} \). Then by \( \|UV^*\|_\infty \leq \sqrt{\mu r} \) we have \( \|Z_0\|_\infty \leq C \sqrt{\mu r \log n} \), which implies \( \|Z_0\|_F \leq n \|Z_0\|_\infty \leq C \sqrt{\mu r} \).

Now we want to prove \( Y \) satisfies (4.4) with high probability. Obviously \( P_T^\perp Y = 0 \). It suffices to prove
\[
\begin{cases}
\|P_T Y + P_T(\lambda P_Y(W) - UV^*)\|_F \leq \frac{\lambda}{2n^2}, \\
\|P_T^\perp Y\| \leq \frac{1}{8}, \\
\|P_T^\perp (\lambda P_Y(W))\| \leq \frac{1}{8}, \\
\|P_T Y\|_\infty \leq \frac{\lambda}{4}.
\end{cases}
\]
(4.6)
First,
\[
\| P_T Y + P_T(\lambda P_{\Omega'}(W) - UV^*) \|_F = \| Z_0 - \left( \sum_{j=1}^{l} \frac{1}{q_j} P_T P_{\Gamma_j} Z_{j-1} \right) \|_F \\
= \| P_T Z_0 - \left( \sum_{j=1}^{l} \frac{1}{q_j} P_T P_{\Gamma_j} P_T Z_{j-1} \right) \|_F \\
= \| (P_T - \frac{1}{q_1} P_T P_{\Gamma_1} P_T) Z_0 - \left( \sum_{j=2}^{l} \frac{1}{q_j} P_T P_{\Gamma_j} P_T Z_{j-1} \right) \|_F \\
= \| P_T Z_1 - \left( \sum_{j=2}^{l} \frac{1}{q_j} P_T P_{\Gamma_j} P_T Z_{j-1} \right) \|_F \\
= \ldots = \| Z_l \|_F \leq C \left( \frac{1}{2} \right) \sqrt{\mu r} \leq \frac{\lambda}{n^2}.
\]

Second,
\[
\| P_{T^\perp} Y \| = \| P_{T^\perp} \sum_{j=1}^{l} \frac{1}{q_j} P_{\Gamma_j} Z_{j-1} \|
\leq \sum_{j=1}^{l} \| \frac{1}{q_j} P_{T^\perp} P_{\Gamma_j} Z_{j-1} \|
= \sum_{j=1}^{l} \| P_{T^\perp} (\frac{1}{q_j} P_{\Gamma_j} Z_{j-1} - Z_{j-1}) \|
\leq \sum_{j=1}^{l} \frac{1}{q_j} P_{\Gamma_j} Z_{j-1} - Z_{j-1} \|
\leq \sum_{j=1}^{l} C \sqrt{\frac{n \log n}{q_j}} \| Z_{j-1} \|_\infty
\leq C \sqrt{n \log n (\sum_{j=3}^{l} \frac{1}{2^{j-1} \log n \sqrt{q_j}} + \frac{1}{2 \sqrt{\log n \sqrt{q_2}} + \frac{1}{\sqrt{q_1}}})} \| Z_0 \|_\infty
\leq C \sqrt{\frac{n \mu r \log n}{n \sqrt{\rho}}} \frac{1}{8 \sqrt{\log n}},
\]
provided \( C_\rho \) is sufficiently large.

Third, we have \( \| \lambda P_{T^\perp} P_{\Omega'}(W) \| \leq \lambda \| P_{\Omega'}(W) \| \). Notice that \( W_{ij} \) is an independent Rademacher sequence independent of \( \Omega' \). By Lemma 4.3, we have
\[
\| \frac{2s\rho}{1 - \rho + 2s\rho} W - P_{\Omega'}(W) \| \leq C_0 \sqrt{np \log n} \| W \|_\infty
\]

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with high probability provided $\frac{2s\rho_1}{1-\rho+2s\rho} \leq p$ and $p \geq C_0 \log \frac{n}{n}$. By Theorem 3.9 of [39], we have $\|W\|_\infty \leq C_1 \sqrt{n}$ with high probability. Therefore,

$$\|P_{\Omega'}(W)\| \leq C_0' \sqrt{np \log n} + C_1 \sqrt{n} \frac{2s\rho}{1-\rho+2s\rho}.$$ 

By choosing $p = \rho C_2$ for some appropriate $C_2$, we have $\|P_{\Omega'}(W)\| \leq \frac{\sqrt{np \log n}}{8}$, provided $C_\rho$ is large enough and $C_s$ is small enough.

Fourth,

$$\|P_{\Gamma}Y\|_\infty = \|P_{\Gamma} \sum_j \frac{1}{q_j} P_{\Gamma_j} Z_{j-1}\|_\infty$$

$$\leq \sum_j \frac{1}{q_j} \|Z_{j-1}\|_\infty$$

$$\leq \left( \sum_{j=3}^l \frac{1}{q_j} 2^{j-1} \frac{1}{\log n} + \frac{1}{q_2} \frac{1}{2\sqrt{\log n}} + \frac{1}{q_1} \right) \|Z_0\|_\infty$$

$$\leq \frac{C \sqrt{\mu r}}{n^\rho} \leq \frac{\lambda}{4\sqrt{\log n}},$$

provided $C_\rho$ is sufficiently large.

Notice that in [4] the authors used a very similar golfing scheme. To compare these two methods, we use here a non-uniform sizes golfing scheme to achieve a result with fewer log factors. Moreover, unlike in [4] the authors used both golfing scheme and least square method to construct two parts of the dual matrix, here we only use golfing scheme. Actually the method to construct the dual matrix in [4] cannot be applied directly to our problem when $\rho = O(r \log^2 n/n)$.

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