A split special Lagrangian calibration associated with helicity

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Abstract

Let $M$ be an oriented three dimensional Riemannian manifold. We define a notion of helicity of local sections of the bundle $SO(M) \to M$ of all its positively oriented orthonormal tangent frames. When $M$ is a space form, we relate the concept to a suitable invariant split pseudo-Riemannian metric on $\text{Iso}_o(M) \cong SO(M)$: A local section has positive helicity if and only if it determines a space-like submanifold. In the Euclidean case we find explicit homologically volume maximizing sections using a split special Lagrangian calibration. We introduce the concept of optimal helicity and give an optimal screwed global section for the three-sphere. We prove that it is also homologically volume maximizing (now using a common one-point split calibration). Besides, we show that no optimal section can exist in the Euclidean and hyperbolic cases.

Keywords: Special Lagrangian calibration, helicity, split bi-invariant metric

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1 Introduction

In this article we obtain several results on the screwedness of sections of the orthonormal frame bundle of a three dimensional space form. We highlight the particular one involving a split special Lagrangian calibration, due to the relevance of this technique. Particularly for curved spaces with signature, although there have been important applications (for instance, in relation with optimal transport), we feel that concrete, natural examples could be welcome.

Pseudo-Riemannian geometry is often the appropriate setting when dealing with manifolds possessing two qualitatively different types of tangent vectors (and a third borderline type), the paramount example being Lorentzian geometry in relativity, reflecting the distinction between space-like or time-like curves of events. On the manifold

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of all rigid transformations of a three dimensional space form $M$ we distinguish curves (that is, motions of $M$) that describe, at each instant, positive or negative screws. We consider on it a pseudo-Riemannian metric of signature $(3,3)$ which accounts for this dichotomy.

The importance of extrema in mathematics cannot be overemphasized. Calibrations detect submanifolds of minimum or maximum volume in a homology class. The method is powerful because of the global nature of its results. One can see the history of calibrations in the book [18]. They grew impressively in strength in the celebrated paper [9] by Harvey and Lawson of 1982; see also [8]. Calibrations provided substantial achievements in volume minimization of submanifolds of Euclidean space (see for instance [2, 13, 17, 23]), and also of several Riemannian manifolds, see for instance [3, 4, 6, 11, 15, 19, 20] (the first one inaugurated the much pursued search for the best among structures of a certain type on a Riemannian manifold).

In 1989 Mealy introduced in [16] calibrations on pseudo-Riemannian manifolds. The split special Lagrangian calibrations were rediscovered by Warren [22], with a new approach set up by Hitchin in [12]. Warren applied them to the volume maximization problem of the special Lagrangian submanifolds of a pseudo-Euclidean space (see also [10]). For curved manifolds, they were also useful for instance in relation to optimal transport [14] or foliations [5].

**Definition 1.** Let $N$ be a pseudo-Riemannian manifold of dimension $2m$ with signature $(m,m)$ and let $M$ be an oriented space-like submanifold of $N$ of dimension $m$. One says that $M$ is **volume maximizing** in $N$ if for any open subset $U$ of $M$ with compact closure and smooth border $\partial U$, one has that

$$\text{vol}_m(U) \geq \text{vol}_m(V)$$

for any space-like submanifold $V$ of $N$ of dimension $m$ with compact closure and $\partial V = \partial U$. Moreover, $M$ is said to be **homologically volume maximizing** in $N$ if in addition $V$ is required to be homologous to $U$.

Let $o_3 = \{ A \in \mathbb{R}^{3 \times 3} \mid A^T = -A \}$ be the Lie algebra of $SO_3$. There is a linear isomorphism

$$C : \mathbb{R}^3 \to o_3, \quad \xi \mapsto C_\xi, \quad \text{where} \quad C_\xi(y) = \xi \times y.$$ (1)

Note that for $x \neq 0$, $\exp(C_x)$ is the rotation through the angle $|x|$ around the oriented axis spanned by $x$.

Let $\mathbb{R}^3 \rtimes SO_3$ be the group of direct (that is, orientation preserving) isometries of Euclidean space $\mathbb{R}^3$, the multiplication being given by $(x, A) \cdot (y, B) = (x + Ay, AB)$. We endow it with the left invariant pseudo-Riemannian metric of signature $(3,3)$ given at the identity $(0,I_3)$ by

$$\langle (x,C_\xi), (y,C_\eta) \rangle = \frac{1}{4} (\langle x, \eta \rangle + \langle y, \xi \rangle)$$ (2)
\(I_k\) is the \((k \times k)\)-identity matrix. We show below, in Section 3, that the metric is actually bi-invariant.

For a vector \(X\) tangent to a pseudo-Riemannian manifold we denote \(\|X\| = \langle X, X \rangle\) and \(|X| = \sqrt{\langle X, X \rangle}\), provided that the integrand is nonnegative. A map \(F\) from \(\mathbb{R}^3\) to a group is said to be odd if \(F(-x) = F(x)^{-1}\) for all \(x \in \mathbb{R}^3\). By smooth we mean of class \(C^\infty\).

We can now state one of the main results of the article.

**Theorem 2.** Given \(c > 0\), let

\[
\ell : \mathbb{R} \to \mathbb{R}, \quad \ell(r) = c(r - \sin r)^{1/3},
\]

which is a strictly increasing odd smooth function with \(\ell'(0) > 0\). Then the map

\[
\Phi : \mathbb{R}^3 \to \mathbb{R}^3 \rtimes SO_3, \quad \Phi(ru) = (\ell(r)\ u, \exp(Cru)),
\]

with \(|u| = 1\) and \(r \in \mathbb{R}\), is well defined, smooth and odd. Its restrictions to the open ball \(\{v \in \mathbb{R}^3 \mid |v| < \pi\}\) and to the open spherical shells

\[
\{v \in \mathbb{R}^3 \mid 2k\pi < |v| < (2k + 1)\pi\},
\]

for \(k \in \mathbb{N}\), determine homologically volume maximizing space-like submanifolds of \(\mathbb{R}^3 \rtimes SO_3\).

In Section 2 we give the proof of this theorem, passing to the universal covering, using a split special Lagrangian calibration. The submanifolds of the statement, being calibrated, are in particular maximal, that is, they have zero mean curvature. We do not know properties of general solutions of the equation of Monge-Ampère type for maximal space-like submanifolds, e.g. whether they can project onto \(\mathbb{R}^3\); we refer to the recent survey [24].

In Section 3 we present the geometric significance of the theorem, namely, that the given submanifolds are maximally screwed, in a certain sense. We go beyond inquiring further about related subjects. Let \(M\) be an oriented three dimensional Riemannian manifold and let \(SO(M) \to M\) be the bundle of its positively oriented orthonormal frames. First, we introduce the concept of helicity of a section of this bundle. Then, when \(M\) is a space form, we relate its positivity with the fact that the submanifold of \(SO(M)\) determined by the section is space-like for a split pseudo-Riemannian metric on it.

In Section 4 we study the intrinsic geometry of (a lift of) the submanifold \(\Phi|_B\) in Theorem 2, where \(B\) is the ball of radius \(\pi\) in \(\mathbb{R}^3\) centered at the origin. It is not complete and its metric completion is homeomorphic to the three-sphere.

In Section 5 we define the notion of optimal helicity for sections of \(SO(M) \to M\). We give an example for \(M = S^3\) (which turns out to be also homologically volume maximizing, using a common one-point calibration) and show that no local section has that property for Euclidean or hyperbolic space.
2 Calibrations and the proof of the theorem

We recall some definitions and results from Mealy in [16] (see also [10]), which are analogous to those for Riemannian manifolds in [9]. Let \( N \) be a split pseudo-Riemannian manifold of dimension \( 2m \) (by split we mean that it has signature \( (m,m) \)). For \( q \in N \), an \( m \)-vector \( \xi \) in \( T_qN \) is said to be space-like if the subspace generated by \( \xi \) is space-like (we are not considering the induced indefinite inner product on \( \Lambda^m (T_qN) \)).

**Definition 3.** Let \( N \) be a split pseudo-Riemannian manifold of dimension \( 2m \). A closed \( m \)-form \( \psi \) on \( N \) is called a calibration if \( \psi_q(\xi) \geq \text{vol}_m(\xi) \) for any space-like \( m \)-vector \( \xi \) in \( T_qN \) with \( \psi_q(\xi) > 0 \), for all \( q \in N \).

**Theorem 4.** [16] Let \( N \) be a split pseudo-Riemannian manifold of dimension \( 2m \) and let \( \psi \) be a calibration on \( N \). If \( M \) is an oriented space-like submanifold of \( N \) of dimension \( m \) which is calibrated by \( \psi \), then \( M \) is homologically volume maximizing in \( N \).

Next we introduce the split special Lagrangian calibration on a split Euclidean space, which appeared first in the work of Mealy [16]. We consider the presentation of this calibration in null coordinates given in [22].

**Proposition 5.** [22] Let \( \mathbb{R}^{m,m} \) be \( \mathbb{R}^m \times \mathbb{R}^m \) endowed with the split inner product whose associated square norm is \( \| (x,y) \| = \langle x,y \rangle \), where \( \langle .,. \rangle \) is the canonical inner product on \( \mathbb{R}^m \). For any \( C > 0 \), the \( m \)-form

\[
\frac{1}{2} \left( C e^1 \wedge \cdots \wedge e^m + \frac{1}{C} e^1 \wedge \cdots \wedge e^m \right)
\]

is a calibration on \( \mathbb{R}^{m,m} \) (here \( \{e^1,\ldots,e^m,\bar{e}^1,\ldots,\bar{e}^m\} \) is the dual of the canonical bases of \( \mathbb{R}^{2m} \)). Moreover, a space-like \( m \)-vector \( \xi \) is calibrated by it if and only if

\[
\epsilon^1 \wedge \cdots \wedge e^m(\xi) = C^2 \epsilon^1 \wedge \cdots \wedge e^m(\xi),
\]

that is, \( \xi \) is special Lagrangian.

To prove Theorem 2 it will be convenient to pass to the universal covering of \( \mathbb{R}^3 \rtimes SO_3 \) and compute with quaternions. Let \( \mathbb{H} \) be the skew field of quaternions and let \( S^3 = \{ q \in \mathbb{H} \mid |q| = 1 \} \). We identify as usual \( \mathbb{R}^4 = \mathbb{R} + \text{Im}(\mathbb{H}) \) and consider the canonical basis \( \{1,i,j,k\} \). Let \( \mathbb{R}^3 \rtimes S^3 \) be the product \( \mathbb{R}^3 \times S^3 \) endowed with the multiplication

\[
(x,p)(y,q) = (x + py\bar{p}, pq),
\]
where $\bar{q} = q^{-1}$. For further reference we compute

$$
(dL_{(x,p)})_{(0,1)} (\xi, \eta) = \frac{d}{dt} \bigg|_0 (x + pt\bar{\xi}\bar{p}, pe^{\xi}) = (p\xi\bar{p}, p\eta). 
$$

(6)

We endow the Lie group $\mathbb{R}^3 \rtimes S^3$ with the left invariant split pseudo-Riemannian metric given at the identity $(0, 1)$ by

$$
\langle (x, \xi), (y, \eta) \rangle = \frac{1}{2} (\langle x, \eta \rangle + \langle y, \xi \rangle).
$$

(7)

for $x, y \in \text{Im}(H)$ and $\xi, \eta \in T_1S^3 = \text{Im}H$.

Let $I : S^3 \to SO_3$, $I(q) = I_q$, with $I_q(x) = qx\bar{q}$ for $x \in \text{Im}H \cong \mathbb{R}^3$, and let

$$
\Pi : \mathbb{R}^3 \rtimes S^3 \to \mathbb{R}^3 \rtimes SO_3, \quad \Pi(x, q) = (x, I_q).
$$

(8)

Both are smooth two-to-one surjective Lie group morphisms.

**Proposition 6.** The metric above on $\mathbb{R}^3 \rtimes S^3$ is bi-invariant. In particular, we have the following rotational symmetry: the left action of $S^3$ on $\mathbb{R}^3 \rtimes S^3$ given by

$$
(q, (x, p)) \mapsto L_{(0,q)}R_{(0,\bar{q})} (x, p) = (qx\bar{q}, qp\bar{q}) = (I_q(x), qp\bar{q})
$$

(9)

is by isometries. Also, $\Pi$ is a local isometry if $\mathbb{R}^3 \rtimes SO_3$ is endowed with the metric defined in (2).

**Proof.** We prove first the last assertion. We compute $(dI_1) : T_1S^3 = \text{Im}H \to \mathfrak{o}_3$: For $z \in \text{Im}H$ we have

$$
dI_1 (\xi) (z) = \frac{d}{dt} \bigg|_0 e^{t\xi}ze^{-t\xi} = \xi z - z\xi = 2\xi \times z = C_{2\xi} (z).
$$

Hence,

$$
\|dI_{(0,1)} (x, \xi)\| = \|(x, C_{2\xi})\| = \frac{1}{2} \langle 2\xi, \eta \rangle = \langle \xi, \eta \rangle = \|(x, \xi)\|.
$$

Then, using the left invariance we have that $\Pi$ is a local isometry.

In the paragraph containing expression (19) below we show in particular that the metric on $\mathbb{R}^3 \rtimes SO_3$ (called $G_0$ there) defined in (2) is bi-invariant. Now, the map $d\Pi_{(0,1)}$ is a linear isometric Lie algebra isomorphism. Since for connected Lie groups (as in our case) bi-invariance depends only on the inner product on the Lie algebra, the metric on $\mathbb{R}^3 \rtimes S^3$ is also bi-invariant. The second assertion is an immediate corollary of that property. \hfill $\square$

We consider maps of the form

$$
\varphi : \mathbb{R}^3 \to \mathbb{R}^3 \rtimes S^3, \quad \varphi (ru) = (l(r) u, \exp (\theta (r) u)),
$$

with $|u| = 1$ and $r \in \mathbb{R}$, where $l : \mathbb{R} \to \mathbb{R}$ and $\theta : \mathbb{R} \to \mathbb{R}$ are two smooth odd functions.

We call the map $\varphi$ a *screw-radial map* from $\mathbb{R}^3$ to $\mathbb{R}^3 \rtimes S^3$.

From the following lemma one deduces conditions on $l$ and $\theta$ for restrictions of $\varphi$ to open sets of $\mathbb{R}^3$ to be space-like submanifolds of $\mathbb{R}^3 \rtimes S^3$. 

5
Lemma 7. The map \( \varphi \) is well defined, smooth and odd. For \( r > 0 \) and \( |u| = 1 \), the image of \( (d\varphi)_u \) is a space-like three dimensional subspace of \( T_{\varphi(u)}(\mathbb{R}^3 \times S^3) \) if and only if

\[
l'(r) > 0, \quad \theta'(r) > 0 \quad \text{and} \quad k\pi < \theta(r) < k\pi + \frac{\pi}{2}
\]

for some \( k \in \mathbb{N} \cup \{0\} \). For \( r = 0 \), the same holds, but dropping the third condition.

Proof. We have \( \varphi(0) = (0,1) \). For \( 0 \neq v \in \mathbb{R}^3 \) we write \( v = r(v/r) \), with \( r = |v| \), and check that \( \varphi((-r)(-v/r)) = \varphi(r(v/r)) \), hence \( \varphi \) is well defined. The map \( \varphi \) is odd since \( e^{sw} \) commutes with \( w \) for all \( w \in \text{Im}(H) \) and \( s \in \mathbb{R} \). Since \( s \) is smooth and odd, the function \( \mathbb{R} \to \mathbb{R} \) given by \( s \mapsto l \Big( \sqrt{|s|}/\sqrt{|s|}, 0 \mapsto l'(0) \), is smooth; similarly for \( \theta \). Hence, \( \varphi \) is smooth.

Notice that \( \varphi \) is equivariant by the left actions of \( S^3 \) on \( \mathbb{R}^3 = \text{Im}(H) \) given by \( (q,x) \mapsto qxq^{-1} = I_q(x) \) and that on \( \mathbb{R}^3 \times S^3 \) in (9). The first action is clearly isometric and transitive on the two-sphere and the second one is isometric by Proposition\(^6\). Thus, it suffices to analyze when \( d\varphi_{ri} \) is one-to one and its image is space-like for \( r \geq 0 \). We consider first the case \( r > 0 \) and then study the limit for \( r \to 0^+ \). We compute

\[
d\varphi_{ri}(i) = \frac{d}{dt} \bigg|_0 \big( l(t)i, e^{\theta i}\big) = \left( l'(r)i, \theta'(r)i e^{\theta r}i\right) \tag{10}
\]

Now let \( z \) be a unit vector in \( \text{span}\{j,k\} \) and let \( \alpha(t) = \cos(t/r) i + \sin(t/r) z \). This curve satisfies

\[
r\alpha(0) = ri, \quad r\alpha'(0) = z \quad \text{and} \quad e^{\theta r \alpha(t)} = \cos(\theta(r)) + \sin(\theta(r)) \alpha(t).
\]

Hence,

\[
d\varphi_{ri}(z) = \frac{d}{dt} \bigg|_0 \varphi(r\alpha(t)) = \left( l(r)\alpha(t), e^{\theta r \alpha(t)}\right) \tag{11}
\]

Using (6), we have that \( (dL_{\varphi(r)})(0,1) (\xi,\eta) = (e^{\theta(r)i}\xi e^{-\theta(r)i}, e^{\theta(r)i}\eta) \). We apply the inverse of this linear map to (10) and (12), obtaining

\[
(l'(r)i, \theta'(r)i) \quad \text{and} \quad \frac{1}{r} \left( l(r) e^{-\theta(r)i} z e^{\theta(r)i}, \sin(\theta(r)) e^{-\theta(r)i} z \right), \tag{12}
\]

respectively. Since the metric is left invariant and \( \langle ze^{\theta(r)i}, z \rangle = \langle e^{\theta r}, 1 \rangle = \cos(\theta(r)) \), we have that

\[
\|d\varphi_{ri}(i)\| = \langle l'(r)i, \theta'(r)i \rangle = l'(r) \theta'(r), \tag{13}
\]

\[
\|d\varphi_{ri}(z)\| = \frac{1}{r^2} \langle l(r) e^{-\theta(r)i} z e^{\theta(r)i}, \sin(\theta(r)) e^{-\theta(r)i} z \rangle = \frac{1}{r^2} l(r) \sin(\theta(r)) \langle z e^{\theta(r)i}, z \rangle = \frac{1}{2r^2} l(r) \sin(2\theta(r)). \tag{14}
\]
For \( z \perp i, e^{si}z \) and \( e^{-si}ze^{si} \) are orthogonal to \( i \). Hence (7) yields \( d\varphi_{ri}(i) \perp d\varphi_{ri}(z) \).

Consequently, for \( r > 0 \), \( d\varphi_{ri} \) is injective and its image is space-like if and only if \( \theta, \theta' \) and \( l' \) evaluated at \( r \) are as stated. Taking the limit for \( r \to 0 \), we have that \( \langle d\varphi_0(i), d\varphi_0(z) \rangle = 0 \) and \( \|d\varphi_0(i)\| = l'(0) \theta'(0) = \|d\varphi_0(z)\| \). Thus, the last assertion follows.

The lemma shows that \( \theta' > 0 \) is necessary for the screw-radial map to be space-like.

So we can reparametrize and in the next theorem we consider \( \theta(r) = r/2 \).

Let \( \mathbb{R}^3 \rtimes S^3 \) be endowed with the left invariant split pseudo-Riemannian metric defined in (7) and let \( \ell : \mathbb{R} \to \mathbb{R} \) be as in (3). The following is the lifted version of Theorem 2.

**Theorem 8.** The map

\[
\phi : \mathbb{R}^3 \cong \text{Im} \mathbb{H} \to \mathbb{R}^3 \rtimes S^3, \quad \phi(ru) = (\ell(r)u, \exp(ru/2)),
\]

with \( |u| = 1 \) and \( r \in \mathbb{R} \), is well defined, smooth and odd. Its restrictions to the open ball \( \{v \in \mathbb{R}^3 \mid |v| < \pi\} \) and to the open spherical shells

\[
\{v \in \mathbb{R}^3 \mid 2k\pi < |v| < (2k + 1)\pi\},
\]

for \( k \in \mathbb{N} \), determine homologically volume maximizing space-like submanifolds of \( \mathbb{R}^3 \rtimes S^3 \).

**Proof.** Using Lemma 7 with \( l = \ell \) and \( \theta(r) = r/2 \), one shows that those restrictions of \( \phi \) are space-like submanifolds. Indeed, on the one hand, \( \theta' = 1/2 > 0 \) and \( 2k\pi < r < (2k + 1)\pi \) implies that \( k\pi < \frac{r}{2} < k\pi + \frac{\pi}{2} \) (for \( k \in \mathbb{N} \cup \{0\} \)); on the other hand,

\[
\ell'(r) = \frac{c}{3} \frac{1 - \cos r}{(r - \sin r)^{2/3}}
\]

is positive for \( 0 < r \neq 2k\pi \) (\( k \in \mathbb{N} \)) and also \( \ell'(0) > 0 \). In particular, \( \ell \) is strictly increasing and so \( \varphi \) is injective.

Now we prove that the given restrictions of \( \varphi \) are homologically volume maximizing.

Let \( \pi_1 \) and \( \pi_2 \) be the canonical projections of \( \mathbb{R}^3 \rtimes S^3 \) onto the first and second factor, and let \( \omega^1 \) and \( \omega^2 \) be the canonical volume forms of \( \mathbb{R}^3 \) and \( S^3 \), respectively. Define the 3-form \( \omega \) on \( \mathbb{R}^3 \rtimes S^3 \) by

\[
\omega = \frac{1}{2} \left( C\pi_1^*\omega^1 + \frac{1}{C} \pi_2^*\omega^2 \right),
\]

for some positive constant \( C \) to be determined later. Let us see that \( \omega \) is a calibration. Clearly, \( \omega \) is closed. We call \( \{e^1, e^2, e^3, e^1, e^2, e^3\} \) the dual basis of the juxtaposition of \( \{(i, 0), (j, 0), (k, 0)\} \) and \( \{(0, i), (0, j), (0, j)\} \) (sometimes, we will abuse the notation omitting the zeros). We have that

\[
\omega_{(0,1)} = \frac{1}{2} \left( Ce^1 \wedge e^2 \wedge e^3 + \frac{1}{C} e^1 \wedge e^2 \wedge e^3 \right),
\]
which is a calibration on $T_{(0,1)}(\mathbb{R}^3 \times S^3) = \mathbb{R}^3 \times \mathbb{R}^3$ by Proposition \ref{prop:calibration}. Since both $\omega$ and the split metric are left invariant, $\omega$ is a calibration on $\mathbb{R}^3 \times S^3$.

Next we verify that $\omega$ calibrates $\phi$. By the rotational symmetry we have already used in the proof of Lemma \ref{lem:rotation}, if suﬃces to show that the image of $d\phi_{ri}$ is calibrated by $\omega_{\phi_{ri}}$, or equivalently, by invariance, that $(L_{\phi_{ri}})^* d\phi_{ri}(\mathbb{R}^3)$ is calibrated by $\omega_{(0,1)}$. For any $r > 0$ and any unit vector $z \perp i$ in $\text{Im} \mathbb{H}$, by \eqref{eq:calibration} with $l = \ell$ and $\theta = r/2$, we have

\begin{align*}
(L_{\phi_{ri}})^* d\phi_{ri}(i) &= (\ell'(r)i, i/2) \\
(L_{\phi_{ri}})^* d\phi_{ri}(z) &= \frac{1}{r} (\ell(r) - ri/2, z e^{ri/2}, \sin(r/2) e^{-ri/2} z).
\end{align*}

We compute

\[
e^1 \land e^2 \land e^3 \left( \ell'(r)i, \frac{1}{r} \ell(r) e^{-ri/2} j e^{ri/2}, \frac{1}{r} \ell(r) e^{-ri/2} k e^{ri/2} \right) =
\]

\[
= \frac{1}{r^2} \ell(r)^2 \ell'(r) e^1 \land e^2 \land e^3 (i, j, k) = \frac{1}{3r^2} \frac{d}{dr} (\ell(r)^3),
\]

since $i = e^{-ri/2} j e^{ri/2}$ and conjugation by $e^{-ri/2}$ preserves $\omega_0^1$. Also,

\[
e^1 \land e^2 \land e^3 \left( \frac{1}{r} i, \frac{1}{r} \sin(r/2) e^{-ri/2} j, \frac{1}{r} \sin(r/2) e^{-ri/2} k \right) =
\]

\[
= \frac{\sin^2(r/2)}{2r^2} e^1 \land e^2 \land e^3 (i, j, k) = \frac{1 - \cos r}{4r^2}
\]

(on $i^1$, multiplication by $e^{is}$ is the rotation through the angle $s$). Then equation \eqref{eq:calibration} is satisfied in our case if and only if

\[
\frac{1 - \cos r}{4} = \frac{C^2}{3} \frac{d}{dr} (\ell(r)^3)
\]

(the case $r = 0$ is dealt with using continuity). Since $\ell(0) = 0$, this amounts to $\ell(r)^3 = \frac{3}{4c^2} (r - \sin r)$. So, we can choose $C > 0$ with $4c^2C^2 = 3$ and the theorem follows.

Now, Theorem \ref{thm:main} is a corollary of the previous theorem:

**Proof of Theorem** \ref{thm:main}. Let $\omega^1$ and $\omega^2$ be, as above, the canonical volume forms of $\mathbb{R}^3$ and $S^3$, respectively. Since odd dimensional real projective spaces are orientable, there exists a 3-form $\varpi^2$ on $SO_3$ such that $I^* \varpi^2 = \omega^2$.

Let $p_1, p_2$ be the canonical projections of $\mathbb{R}^3 \times SO_3$ onto the first and second factors, respectively, and let $\Omega = \frac{1}{2} (C p_1^* \omega^1 + C p_2^* \varpi^2)$ on $\mathbb{R}^3 \times SO_3$. Then, $\Pi^* \Omega = \omega$, where $\omega$ is the calibration form in \eqref{eq:calibration}, and thus $\Omega$ is a calibration. We also have that $\Pi \circ \phi = \Phi$, where $\phi$ is as in \eqref{eq:phi}. Since $\Pi$ is a local isometry and $\phi$ is calibrated by $\omega$, then $\Phi$ is calibrated by $\Omega$. 

\[\square\]
3 Geometric significance: helicity

Helicity of vector fields on a three dimensional Riemannian manifold is a central concept in some areas of mathematics. We introduce below the concept of helicity of sections of orthonormal frame bundles. As a motivation, we comment on the simplest screw-radial map from $\mathbb{R}^3$ to $\mathbb{R}^3 \times SO_3$.

Let $SO(\mathbb{R}^3) = \mathbb{R}^3 \times SO_3$ be the positively oriented orthonormal frame bundle of $\mathbb{R}^3$. We consider the section $b_0: \mathbb{R}^3 \to SO(\mathbb{R}^3)$, $b_0(x) = (x, R_x)$, where $R_x$ is the rotation through the angle $|x|$ around the line $\mathbb{R}x$, more precisely, $R_0 = \text{id}$ and for $x \neq 0$, $R_x$ is the linear map defined by $R_x(x) = x$ and $R_x(y) = \cos(|x|) y + \sin(|x|) (x/|x|) \times y$ for $y \perp x$. Notice that $\frac{d}{dt}|_0 R_{\alpha(t)} = C_x$, with $C$ as in (1), and so the tangent space at $(0, I_3)$ of the submanifold of $SO(\mathbb{R}^3)$ determined by $b_0$ is $\{(x, C_x) \mid x \in \mathbb{R}^3\}$.

The section $b_0$ may be described informally as follows: Moving away from the origin in one direction $x$ entails rotating through an angle $|x|$ around the oriented line determined by $x$. In broad terms, we want to discern to what extent a local section of $SO(\mathbb{R}^3) \to \mathbb{R}^3$, at each point of $\mathbb{R}^3$, resembles $b_0$ near the origin. For the sake of generality we study the problem in a wider context.

Let $M$ be an oriented three dimensional Riemannian manifold. For an orthonormal set $\{u, v\}$ in $T_pM$, let $u \times v$ be the unique $w \in T_pM$ such that $\{u, v, w\}$ is a positively orthonormal basis. The bilinear extension gives a well defined cross product $\times$ on $T_pM$ depending smoothly on $p$. For $x \in T_pM$ we denote $C_x(y) = x \times y$, which defines a skew symmetric operator on $T_pM$. Also, all such operators have that form.

Fix a point $o$ in $M$ and let $SO(M)$ be the bundle of positively oriented orthonormal frames of $M$, that is,

$$SO(M) = \{b: T_oM \to T_pM \mid b \text{ is a direct linear isometry, } p \in M\}.$$  

This is a principal fiber bundle over $M$ with typical fiber $SO_3$. The definition differs inessentially from the usual one, with $\mathbb{R}^3$ instead of $T_oM$. On the one hand, this will allow us to think of the elements of $SO(M)$ as positions of a body in $M$ with reference state at $o$. On the other hand, when $M$ is a space form, this will induce a simple identification between $SO(M)$ and the group of direct isometries of $M$.

Let $b$ be a smooth section of $SO(M) \to M$. If $x \in T_pM$, then $\nabla_x b: T_oM \to T_pM$ is well defined, as usual, by

$$\left(\nabla_x b\right)(y) = \left.\frac{D}{dt}\right|_0 b(\alpha(t))(y),$$

where $\alpha$ is any curve in $M$ with $\alpha(0) = p$ and $\alpha'(0) = x$. 
Proposition 9. Let $b$ be a smooth section of $SO(M) \to M$. For each vector field $X$ on $M$ there exists a unique vector field $X^b$ on $M$ such that $\nabla_X b = C_{X^b} \circ b$.

The map $X \mapsto X^b$ determines a $(1,1)$-tensor field on $M$ prompting the following definition ($T^1M$ denotes the unit tangent bundle of $M$).

**Definition 10.** The helicity of a smooth section $b$ of $SO(M) \to M$ is the function

$$h^b : T^1M \to \mathbb{R}, \quad h^b(x) = \langle X^b, X \rangle.$$ 

Positive helicity of $b$ means, informally, that at each point of $M$, when moving according to $b$ in any direction, this direction forms an angle smaller than $\pi/2$ with the axis of rotation.

**Proof of Proposition 9.** Let $p \in M$ and let $\alpha$ be a smooth curve on $M$ such that $\alpha(0) = p$ and $\alpha'(0) = X_p$. Given $y \in T_0 M$, $t \mapsto b(\alpha(t)) (y)$ is a unit vector field along $\alpha$ taking the value $b(p)(y)$ at $t = 0$. Hence,

$$\left( \nabla_{\alpha_t} b \right)(y) = \frac{D}{dt} \big|_0 b(\alpha(t))(y) \perp b(p)(y).$$

Putting $z = b(p)y$, we have that $\left\langle \left( \nabla_{\alpha_t} b \right)(b(p)^{-1}z), z \right\rangle$ for all $z \in T_p M$. Then, $(\nabla_{\alpha_t} b) \circ b(p)^{-1}$ is a skew symmetric operator on $T_p M$ and so it can be realized as $C_w$ for a unique $w \in T_p M$, which we call $X_p^b$. \qed

Now we turn our attention to the three dimensional space form $M_\kappa$ of constant sectional curvature $\kappa = 0, 1$ or $-1$, that is, $M_0 = \mathbb{R}^3$, $M_1$ is the sphere $S^3$ and $M_{-1}$ is hyperbolic space $H^3$.

Let $\{e_0, e_1, e_2, e_3\}$ be the canonical basis of $\mathbb{R}^4$ and consider the inner product given by $\langle x, y \rangle_\kappa = \kappa x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3$. For $\kappa = \pm 1$, $M_\kappa$ is the connected component of $e_0$ of $\{ x \in \mathbb{R}^4 \mid \langle x, x \rangle_\kappa = \kappa \}$, with the induced (Riemannian) metric. To handle the three cases simultaneously, sometimes it will be convenient to identify $\mathbb{R}^3 \cong \{ x \in \mathbb{R}^4 \mid x_0 = 1 \}$. For any $p \in M_\kappa$, the orientation on $T_p M_\kappa$ is given, as usual, by declaring an oriented basis $\{ u, v, w \}$ positive if $\{ p, u, v, w \}$ is a positive basis of $\mathbb{R}^4$. In particular, $e_1 \times e_2 = e_3$ at $e_0 \in M_\kappa$.

We denote by $G_\kappa = \text{Iso}_0(M_\kappa)$, the group of direct isometries of $M_\kappa$. For $\kappa = 1, -1$, it coincides with the identity component of the automorphism group of the inner product $\langle , \rangle_1$ or $\langle , \rangle_{-1}$ (that is, $SO_4$ or $O_\kappa(1, 3)$), respectively. With the identification $\mathbb{R}^3 \cong e_0 + \mathbb{R}^3$, we have

$$G_0 = \left\{ \begin{pmatrix} 1 & 0 \\ a & A \end{pmatrix} \mid a \in \mathbb{R}^3, A \in SO_3 \right\} \cong \mathbb{R}^3 \times SO_3.$$ 

The Lie algebra of $G_\kappa$ is $\mathfrak{g}_\kappa = \{ Z_\kappa(x, \xi) \mid x, \xi \in \mathbb{R}^3 \}$, where

$$Z_\kappa(x, \xi) = \begin{pmatrix} 0 & -\kappa x^T \\ x & C_\xi \end{pmatrix}$$
is a column vector and \( T \) denotes transpose). One has that \( g_1 = o(4) \) and \( g_{-1} = o(1, 3) \). We take \( o = e_0 \) and denote by \( K_\kappa \) the isotropy subgroup at this point and by \( \mathfrak{k}_\kappa \) the Lie algebra of \( K_\kappa \).

Let \( g_\kappa = p_\kappa \oplus \mathfrak{k}_\kappa \) be the Cartan decomposition of \( g_\kappa \) associated with the point \( o \), that is, \( p_\kappa \) and \( \mathfrak{k}_\kappa \) consist of the matrices \( Z_\kappa (x, 0) \) and \( Z_\kappa (0, \xi) \), with \( x, \xi \in \mathbb{R}^3 \), respectively. The corresponding Cartan decomposition of \( G_\kappa \) is given by \( G_\kappa = \exp(p_\kappa)K_\kappa \). Let \( \pi : G_\kappa \to M_\kappa, \pi (g) = g (o) \).

We consider on \( G_\kappa \) the left invariant pseudo-Riemannian structure given at the identity by
\[
\langle Z_\kappa (x, \xi), Z_\kappa (y, \eta) \rangle = \frac{1}{4} (\langle x, \eta \rangle + \langle y, \xi \rangle),
\]
which in the Euclidean case coincides with that in (2). It has signature \((3, 3)\) and is bi-invariant. Indeed, since \( G_\kappa \) is connected, it suffices to show that \( \text{ad} Z \) is skew symmetric for all \( Z \in g_\kappa \). This follows from this expression for the Lie bracket:
\[
[Z_\kappa (x, \xi), Z_\kappa (y, \eta)] = Z_\kappa (C_\xi y - C_\eta x, \kappa x \times y + \xi \times \eta),
\]
which can be checked using the identities \( yx^T - xy^T = C_{xy} = [C_x, C_x] \). Note that for \( \kappa = \pm 1 \), this metric is not the one corresponding to the Killing form on the simple Lie algebra \( g_\kappa \); the Killing form is not the only nondegenerate bi-invariant form (since \( g_1 \) and \( g_{-1} \) are the complexifications of \( o(3) \) and \( o(1, 2) \), respectively, they are not absolutely simple).

The group \( G_\kappa \) acts simply transitively on \( SO (M_\kappa) \). The action induces the bijection
\[
\mathcal{I} : G_\kappa \to SO (M_\kappa), \quad \mathcal{I} (g) = (dg)_o.
\]
For \( g, h \in G_\kappa \) we have \( (dg) (\mathcal{I} (h)) = (dg) (dh)_o = (dg \circ h)_o = d (L_g (h))_o = \mathcal{I} (L_g (h)) \), where \( L_g : G_\kappa \to G_\kappa \) denotes left multiplication by \( g \). Hence,
\[
dg = L_g = \text{def} \mathcal{I} \circ L_g \circ \mathcal{I}^{-1} : SO (M_\kappa) \to SO (M_\kappa).
\]

We consider on \( SO (M_\kappa) \) the pseudo-Riemannian metric induced from that on \( G_\kappa \) in (19) by the identification \( \mathcal{I} \) in (21).

The next proposition provides a geometrical meaning of the pseudo-Riemannian metric (2) on \( \mathbb{R}^3 \times SO_3 \) considered in the introduction and moreover for the metric (19) on \( G_\kappa \) above. Theorem 2 thereby acquires significance in relation to positive helicity.

**Proposition 11.** Let \( U \) be an open subset of \( M_\kappa \). A section \( b : U \to SO (U) \) has positive helicity if and only if it determines a space-like submanifold of \( SO (M_\kappa) \).

Before proving the proposition, we state the following lemma. We call \( \iota = \text{id}_{T_o M_\kappa} \) and identify \( T_o M_\kappa = e_0^\perp \cong \mathbb{R}^3 \). For simplicity, in the proofs we sometimes omit the subindex \( \kappa \).
Lemma 12. Let $b : U \to SO(M_\kappa)$ be a local section of $SO(M_\kappa) \to M_\kappa$.

a) If $o \in U$, $b(o) = i$ and $x \in T_oM_\kappa$, then $(db)_o(x) = dI_{I_o}(Z_\kappa(x, x^b))$.

b) Let $p \in U$ and suppose that $b(p) = I(p)$, with $g \in G$ (in particular, $g(o) = p$). Let $\bar{b} = (dg)^{-1} \circ b \circ g$, which is a section from $g^{-1}(U)$ to $SO(M_\kappa)$ with $\bar{b}(o) = i$. Then

$$(db)_p(y) = (dL_g)_o(\bar{db}_o(x))$$

and

$$y^b = (dg)_o(x^b)$$

for $x \in T_oM_\kappa$ and $y = dg_o(x)$.

Proof. We call $\sigma(t) = \exp(tZ(x, 0))$, a curve in $G$. Let $\gamma$ be the geodesic in $M$ with $\gamma(0) = o = e_o$ and $\gamma'(0) = x$. It is well known that $\gamma(t) = \pi(\sigma(t)) = \sigma(t)(o)$.

For some curve $g$ in $G$. Hence $(db)_o(x) = dI_{I_o}(g'(0))$.

Suppose that the Cartan factorization of $g(t)$ is given by $g(t) = \exp(X(t))k(t)$, with $X(t) \in \mathfrak{p}$ and $k(t) \in K$. Evaluating at $t = 0$ we get $g(0) = I_6$ and so $X(0) = 0$ and $k(0) = I_6$.

Differentiating $g$ at $t = 0$ we obtain that $g'(0) = X'(0) + k'(0)$, which gives the Cartan decomposition of $g'(0)$. Hence, it suffices to show that

$$X'(0) = Z(x, 0)$$

and

$$k'(0) = Z(0, x^b).$$

We have $\pi(\exp(X(t))) = \pi(g(t)) = \gamma(t)$. Since $\pi \circ \exp_{|\mathfrak{p}} : \mathfrak{p} \to M$ is a local diffeomorphism near $0 \in \mathfrak{p}$, we have that $X(t) = tZ(x, 0)$ and so the first identity in (21) holds. Also,

$$b(\gamma(t)) = \mathcal{I}(g(t)) = \mathcal{I}(\sigma(t)k(t)) = (d\sigma(t))_o(dk(t))_o.$$

It is well known that $(d\sigma(t))_o$ realizes the parallel transport along $\gamma$ between 0 and $t$. Thus, we have

$$C_{x^b}(v) = (\nabla_xb)(v) = D\left|_0 b(\gamma(t))(v) = D\left|_0 (dk(t))_o(v) = k'(0)(v)$$

for all $v \in T_oM \cong \mathbb{R}^3$. This implies the validity of the second identity in (21).

b) It is a direct consequence of (a) using the invariance by the action of $G$. For the sake of completeness we present the computations. By the hypothesis and (22) we have that

$$db \circ dg = d(dg) \circ \bar{db} = dL_g \circ \bar{db}.$$

Evaluating at $x$ we get the first expression in (23). The second one follows from the identities defining $y^b$ and $x^b$ and the standard facts that $(dg) \circ C_u = C_{dg(u)} \circ (dg)$ and $\nabla_v(dg \circ b \circ g^{-1}) = (dg) \circ (\nabla_u\bar{b}) \circ g^{-1}$ for $v = dg(u)$, since $g$ is a direct isometry of $M$. \qed
Proof of Proposition 11. Let \( b : U \to SO(U) \) be a section and let \( p \in U \). We will show that the tangent space of the submanifold determined by \( b \) at \( b(p) \) is space-like if and only if \( b \) has positive helicity at \( p \). It suffices to see that, given \( y \in T_p M_\kappa, \|(db)_p(y)\| > 0 \) if and only if \( \langle y^b, y \rangle > 0 \).

Suppose that \( b(p) = \mathcal{L}(g) \) with \( g \in G_\kappa \). Let \( \tilde{b} = (dg)^{-1} \circ b \circ g \), which is a section \( g^{-1}(U) \to SO(M_\kappa) \) with \( \tilde{b}(o) = \iota \). Using the notation and assertions of Lemma 12, we write \( y = (dg)_o(x) \) with \( x \in T_o M_\kappa \) and have that

\[
\|(db)_p(y)\| = \|db_o(x)\| = \|Z_\kappa(x, x^b)\| = \frac{1}{2} \langle x, x^b \rangle = \frac{1}{2} \langle y, y^b \rangle.
\]

The first and last equalities follow from part (b) of that lemma (since \( \mathcal{L}_g \) is an isometry of \( SO(M_\kappa) \) for the metric in (19)) and the second one follows from part (a), since \( \mathcal{L} \) is an isometry. \( \square \)

Remark 13. The local sections \( \varphi \) of \( \mathbb{R}^3 \rtimes SO_3 \) corresponding to the spherical shells \( \{ v \in \mathbb{R}^3 \mid |2k-1)\pi < |v| < 2k\pi \} \) with \( k \in \mathbb{N} \) in Theorem 2 have negative helicity.

4 The intrinsic geometry of the submanifold \( \phi(B) \)

We present some properties of the submanifold \( \phi : B = \{ v \in \mathbb{R}^3 \mid |v| < \pi \} \to \mathbb{R}^3 \rtimes S^3 \) of Theorem 8 endowed with the induced Riemannian metric.

Proposition 14. For any unit vector \( u \in \mathbb{R}^3 \), the radial curve \( r \mapsto \phi(ru) \) is the reparametrization of an inextendible geodesic in \( \phi(B) \) of finite length. In particular, \( \phi(B) \) is not complete.

The metric completion of \( \phi(B) \) is \( (\phi(B) \cup \{\ast\}, d) \), obtained by adding a point \( \ast \), and the metric topology is the one-point compactification of \( \phi(B) \), rendering it homeomorphic to \( S^3 \). The distance \( d \) is not Riemannian.

Proof. It is convenient to work with the Riemannian metric on \( B \) induced by \( \phi \), which we call \( g \). Notice that \( S^3 \) acts on \( (B, g) \) by isometries via \( (q, x) \mapsto I_q(x) \), by the rotational symmetry stated in Proposition 4.

Given a unit vector \( w \in \mathbb{R}^3 \), the intersection with \( B \) of the line \( \mathbb{R}w \) through the origin is the connected component the set of fixed points of an isometry of \( B \), for instance, any nontrivial rotation around \( \mathbb{R}w \). Then it is the image of a geodesic in \( (B, g) \) (see, for instance, Proposition 10.3.6 in [1]). Hence, the arc length reparametrization with respect to \( g \) of \( \alpha(r) = rw \) is a geodesic. We compute

\[
|\alpha'(r)|^2 = \|(\phi \circ \alpha)'(r)\| = \|d\phi_{rw}(w)\| = \frac{1}{2} \ell'(r)
\]

(we have used [13] with \( \theta(r) = r/2 \); the expression is valid with \( w \) instead of \( i \) due to the rotational symmetry). Now, let

\[
\sigma : (-\pi, \pi) \to \mathbb{R}, \quad \sigma(r) = \int_0^r \frac{1}{\sqrt{2}} (\ell'(t))^{1/2} \, dt
\]
be the signed arc length of \( \alpha \). We call \( L = \sigma (\pi) \), which is finite, since \( \ell' \) is bounded on \( (-\pi, \pi) \) (see (13)). Hence \( (-L, L) \to B, t \mapsto \sigma^{-1}(t) w \) is an inextendible geodesic of \( (B, g) \). Therefore, the geodesic exponential map of \( (B, g) \) at 0 is given by
\[
\text{Exp}_0 : \{ v \in \mathbb{R}^3 \mid |v| < L \} \to (B, g), \quad \text{Exp}_0(tw) = \sigma^{-1}(t) w
\]
and it is moreover a diffeomorphism. In particular, \( B \) is a normal ball centered at the origin and for any \( 0 < r < \pi \) the sphere \( S_r = \{ v \in B \mid |v| = r \} \) is a geodesic sphere, which is round, since it is preserved by the action of \( S^3 \).

Let \( (N, d) \) be the completion of \( (B, g) \). We consider equivalence classes of Cauchy sequences in \( (B, g) \). In order to show that \( N \) is the one-point compactification of \( B \), it suffices to verify that given sequences \( r_n u_n, s_n v_n \) in \( B \) with \( |u_n| = |v_n| = 1 \) and \( \lim_{n \to \infty} r_n = \lim_{n \to \infty} s_n = \pi \), then \( d_g(r_n u_n, s_n v_n) \to 0 \) as \( n \to \infty \).

Suppose that \( 0 \leq s \leq r < \pi \) and \( |u| = |v| = 1 \). Consider the piecewise smooth curve obtained by juxtaposing parametrizations of the radial segment joining \( su \) with an arc of circle joining \( ru \) with \( rv \). The segment has \( g \)-length
\[
\sigma^{-1}(r) - \sigma^{-1}(s) = \int_{s}^{r} \frac{1}{\sqrt{2}} (\ell'(t))^{1/2} \ dt.
\]  
(25)
Also, by (14) with \( l = \ell \) and \( \theta(r) = r/2 \), the \( g \)-length of any Euclidean great circle in \( S_r \) is
\[
2\pi r \| d\phi_{ri}(z) \|^{1/2} = \sqrt{2} \pi (\ell(r) \sin r)^{1/2}
\]  
(26)
(any unit \( z \perp i \)). Since both (25) and (26) tend to 0 as \( s \to \pi^- \), we have that \( d_g(r_n u_n, s_n v_n) \to 0 \) as \( n \to \infty \).

For \( 0 \leq r < \pi \), \( S_r \) defined above is the \( d \)-sphere in \( N \) of radius \( \sigma(\pi) - \sigma(r) \) centered at \( * \). By (14) with \( l = \ell \) and \( \theta(r) = r/2 \), its \( g \)-area is
\[
4\pi r^2 |d\phi_{ri}(z)|^2 = 4\pi r^2 \| d\phi_{ri}(z) \| = 2\pi \ell(r) \sin r.
\]

Now, for \( \rho > 0 \) let \( A(\rho) \) be the area of the distance sphere of radius \( \rho \) centered at the point \( * \). We compute
\[
\lim_{\rho \to 0} \frac{A(\rho)}{4\pi \rho^2} = \lim_{r \to \pi^-} \frac{\text{g-area} (S_r)}{4\pi (\sigma(\pi) - \sigma(r))^2 r} = \lim_{r \to \pi^-} \frac{\ell(r) \sin r}{\left( \int_{r}^{\pi} (\ell'(t))^{1/2} \ dt \right)^2} = \infty
\]
by L’Hospital’s rule, since the derivatives of the numerator and the denominator tend to \( -\ell(\pi) \neq 0 \) and 0 as \( r \to \pi^- \), respectively. Then the metric is not Riemannian, since otherwise, the limit would have been equal to 1 (see for instance [7]).

Let \( S \) be the boundary of \( \phi(B) \) in \( \mathbb{R}^3 \times S^3 \), which is a two-sphere. By the proposition above, one may think that it collapses to the point \( * \) in \( N \). The following proposition should confirm the insight we gained in Section 2: On the one hand, the metric on \( \mathbb{R}^3 \times S^3 \) degenerates completely on \( S \), and on the other hand, positive helicity fails for the section of \( SO(\mathbb{R}^3) \) associated to \( \Phi = \Pi \circ \phi \) for directions tangent to the two-sphere \( \pi(\Pi(S)) \) in \( \mathbb{R}^3 \).
Proposition 15. The boundary $S$ equals $\{(\ell (\pi) u, u) \in \text{Im} \, \mathbb{H} \times S^3 \mid |u| = 1\}$ and its projection to $\mathbb{R}^3 \times S^3$ is $\{(\ell (\pi) u, R_{\pi u}) \mid |u| = 1\}$. Both are totally null.

Let $b: \mathbb{R}^3 \to SO (\mathbb{R}^3)$ be the section associated with $\Phi$, that is, $b (\ell (r) u) = R_{\pi u}$. Let $\partial B = \{v \in \mathbb{R}^3 \mid |v| = \ell (\pi)\}$ be the boundary of $B$. Then $\langle X^b, X \rangle = 0$ for any vector field $X$ tangent to $\partial B$.

We recall an expression for the rotation $R_x$ in (18) in quaternionic terms that will be useful below: For any $0 \neq x \in \mathbb{R}^3 \cong \text{Im} \, \mathbb{H}$ one has

$$R_x = I \left( \cos \left( \frac{|x|}{2} \right) + \sin \left( \frac{|x|}{2} \right) x / |x| \right),$$

(27)

with $I$ as defined before (5).

Proof. The boundary of $\phi (B)$ in $\text{Im} \, \mathbb{H} \times S^3$ equals $\{ (\ell (\pi) u, u) \mid |u| = 1 \}$, since $e^{\pi u/2} = u$. By (14) with $r = \pi, l = \ell$ and $\theta (r) = r/2$, it is totally null (by the rotational symmetry we may consider $\pi i$ instead of an arbitrary element of $\partial B$). The projection of $S$ onto $\mathbb{R}^3 \times S^3$ is as stated, since for $|u| = 1$, $I_u = R_{\pi u}$ holds by (27). It is also totally null, since $\Pi$ is a local isometry.

Let $v = \ell (\pi) u \in \partial B$, with $|u| = 1$ and let $x \in T_v (\partial B) = v^\perp$. We may suppose that $x = \ell (\pi) y$ with $|y| = 1$, $y \perp u$. By definition of $x^b$,

$$(\nabla_x b)_v = C_{x^b} \circ b (v).$$

(28)

In order to compute the left hand side we call $\alpha (t) = \cos t \, u + \sin t \, y$. Note that $\beta = \ell (\pi) \alpha$ satisfies $\beta (0) = v$ and $\beta' (0) = x$. By (27),

$$b (\ell (\pi) \alpha (t)) (z) = R_{\pi \alpha (t)} (z) = \alpha (t) z \alpha (t),$$

(29)

for $z \in \mathbb{R}^3$. Then we have

$$(\nabla_x b)_v (z) = (\nabla_{\ell (\pi) y} b)_v (z) = \left. \frac{d}{dt} \right|_{t=0} b (\ell (\pi) \alpha (t)) (z)$$

$$= \left. \frac{d}{dt} \right|_{t=0} \alpha (t) z \alpha (t) = yz (-u) - uzy = -yzu - uzy.$$

Evaluating (29) at $t = 0$ yields $b (v) (z) = uz \bar{u} = -uzz$; in particular $(b (v))^{-1} = b (v)$ ($u^2 = -1$). Consequently, by (28),

$$C_{x^b} (w) = (\nabla_x b)_v ((b (v))^{-1} (w)) = (\nabla_x b)_v (-uwu).$$

(30)

Using (30) and properties of quaternions, such as $u^2 = -1$ and $-uy = uy = u \times y$ since $\{y, u\}$ is orthonormal, we have

$$C_{x^b} (w) = uwy - wuy = 2 (u \times y) \times w = C_{2(u \times y)} (w)$$

for all $w$. Hence, $x^b = 2 (u \times y)$, which is orthogonal to $x$. \qed
5 Optimal helicity

**Definition 16.** Let $M$ be an oriented three dimensional Riemannian manifold. A section $b$ of $SO(M) \to M$ is said to have **optimal helicity** if for any vector field $X$ on $M$, the associated vector field $X^b$ satisfies $X^b = X$.

We will show below that $SO(S^3) \to S^3$ admits a global section with optimal helicity.

Recall that $P : S^3 \times S^3 \to SO_4$, $P(p,q) \to L_p \circ R_q$, is a surjective two-to-one morphism, where $L_p, R_q : G \to G$ denote left and right multiplication by $p$, respectively.

**Lemma 17.** Let $G_1 = SO_4$ and $S^3 \times S^3$ be endowed with the split pseudo-Riemannian metrics given by (19) and

$$\langle (x,y), (x',y') \rangle = \frac{1}{2} (\langle x, x' \rangle - \langle y, y' \rangle),$$

(31)

for $x, x' \in T_pS^3$, $y, y' \in T_qS^3$, respectively. Then $P$ is a local isometry.

**Proof.** Since both metrics are invariant and $P$ is a surjective morphism, it suffices to show that $dP_{(1,1)} : T_1S^3 \times T_1S^3 = \text{Im} \mathbb{H} \times \text{Im} \mathbb{H} \to \mathfrak{o}_4$ is a linear isometry. Let $z \in \mathbb{R}^4 \cong \mathbb{H}$. For $v, w \in \text{Im} \mathbb{H}$ we compute

$$\left( dP_{(1,1)} (v, w) \right) (z) = \frac{d}{dt} \bigg|_0 e^{tv} e^{-tw} = vz - zw = (\ell_v - \rho_w)(z),$$

where $\ell_v, \rho_v$ denote left (respectively, right) multiplication by $v$ on $\mathbb{H}$. We verify that

$$\ell_v = Z_1 (v,v) \quad \text{and} \quad \rho_v = Z_1 (v,-v).$$

(32)

Indeed, if $a \in \mathbb{R}$ and $u \in \text{Im} \mathbb{H}$, we have

$$\left( \begin{array}{c} 0 \\ v \\ -v^T C_v \\ u \end{array} \right) \left( \begin{array}{c} a \\ \rho_v \end{array} \right) = \left( \begin{array}{c} -\langle v, u \rangle \\ av + C_v u \end{array} \right).$$

Since $v (a + u) = av + vu = av - \langle v,u \rangle + v \times u$, the first identity in (32) follows. The second one can be checked in a similar manner. Hence,

$$dP_{(1,1)} (v, w) = Z_1 (v,v) - Z_1 (w,-w) = Z (v-w, v+w).$$

Since $2 \frac{1}{2} (v - w, v + w) = \frac{1}{2} (|v|^2 - |w|^2)$, (19) implies that $P$ is a local isometry.

**Theorem 18.** The section $b$ of $SO(S^3) \to S^3$ given by $b(p) = (dL_p)_{(1)}$ has optimal helicity.

**Proof.** Let $X$ be a vector field on $S^3$ and $p \in S^3$. We want to see that $\nabla_X b = C_{X_p} \circ b$. We consider first the case $p = 1$ and call $x = X_1$. Let $p : \mathbb{H} \to T_1S^3$ be the orthogonal projection. For $z \in T_1S^3 = \text{Im} (\mathbb{H})$ we compute

$$\left( \nabla_x b \right) (z) = \nabla_x (q \mapsto (dL_q)_{(1)} (z)) = \frac{d}{dt} \bigg|_0 (dL_{e^{tx}})_{(1)} (z) = p \left( \frac{d}{dt} \bigg|_0 e^{tx} z \right) = xz - (xz, 1) = xz - \text{Re} (xz) = x \times z = C_x (z).$$

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Hence $x^b = x$. Now, we deal with the case when $p$ is arbitrary in $S^3$. Suppose $X_p = (dL_p)_1(x)$. Then $\gamma(t) = pe^{tx}$ has initial velocity $X_p$. For $z \in \text{Im} \mathbb{H}$ we compute

\[
(\nabla_{X_p} b)(z) = \left. \frac{D}{dt} \right|_0 (dL_{pe^{tx}})_1(z) = (dL_p)_1 \left. \frac{D}{dt} \right|_0 (dL_{e^{tx}})_1(z)
\]

\[
= (dL_p)_1 \left. \frac{D}{dt} \right|_0 b(e^{tx})(z) = (dL_p)_1(\nabla_x b)(z)
\]

\[
= (dL_p)_1 C_x(z) = C_{(dL_p)_1x}((dL_p)_1(z)) = C_{X_p}(b(z)).
\]

Consequently, $X^b = X$, as desired. \( \square \)

Not surprisingly, the section $b$ in the previous theorem is also distinguished concerning the volume:

**Theorem 19.** The submanifold of $SO(S^3)$ determined by the section $b$ of $SO(S^3) \to S^3$ given by $b(p) = (dL_p)_1$ is space-like and homologically volume maximizing.

**Remark 20.** Compare with the calibration on the double covering of $SO(S^3)$ in the Riemannian setting in [21].

**Proof.** By Lemma 17, $\mathcal{I} \circ P : S^3 \times S^3 \to SO(S^3)$ is a double covering and a local isometry with respect to the metrics (31) and the one induced on $SO(S^3)$ from (19) via $\mathcal{I}$ (with $\kappa = 0$). Since $\mathcal{I} \circ P$ sends $S^3 \times \{1\}$ to the image of the section $b$, it suffices to show that $S^3 \times \{1\}$ is homologically volume maximizing. Let $\theta$ be the bi-invariant 3-form on $S^3$ such that $\theta_1(i,j,k) = 2^{-3/2}$ and let $p : S^3 \times S^3 \to S^3$ be the projection onto the first factor.

Now, $\{ \sqrt{2}(i,0), \sqrt{2}(j,0), \sqrt{2}(k,0), \sqrt{2}(0,i), \sqrt{2}(0,j), \sqrt{2}(0,k) \}$ is a positively oriented orthonormal basis of $T_{(1,1)}(S^3 \times S^3)$ for the split metric (31). Let $\{\alpha_1, \ldots, \alpha_6\}$ be its dual basis. It is easy to see that $p^*\theta$ is a left invariant 3-form on $S^3 \times S^3$ such that $(p^*\theta)_{(1,0)} = \alpha_1 \wedge \alpha_2 \wedge \alpha_3$.

Since $p^*\theta$ is closed, it is a split one-point calibration by the remark after the Fundamental Lemma of Calibrations in [16]. The computations above show that it calibrates $S^3 \times \{1\}$, as desired. \( \square \)

**Proposition 21.** For $\kappa = 0, -1$, no local section of $SO(M_\kappa)$ has optimal helicity.

**Proof.** Suppose that $b : U \to SO(M_\kappa)$ is a local section of $SO(M_\kappa) \to M_\kappa$ with optimal helicity. Let $\mathcal{D}$ be the left invariant distribution on $G_\kappa$ defined at the identity by

\[
\mathcal{D}_{I_6} = \{ Z_\kappa(x,x) \mid x \in \mathbb{R}^3 \}
\]

and let $\mathcal{E}$ be the corresponding distribution on $SO(M_\kappa)$ via the identification $\mathcal{I}$ in (21), that is, $d\mathcal{I}_g(\mathcal{D}_g) = \mathcal{E}_{\mathcal{I}(g)}$ for $g \in G_\kappa$.

Let us see first that $b(U)$ is an integral submanifold of the distribution $\mathcal{E}$. Let $p \in U$ and $y \in T_pU$. Let $g, b$ and $x$ as in Lemma 12. By parts (b) and (a) of that lemma,

\[
(db)_p(y) = (d\mathcal{L}_g)_y(db_v(x)) = (d\mathcal{L}_g)_y d\mathcal{I}_{I_6}(Z_\kappa(x,x^b))
\]
which belongs to \((d\mathcal{L}_o), \mathcal{E} (\tilde{b} (o)) = \mathcal{E} (b (p))\), since \(x^b = x\) by hypothesis. Hence \(T_{b(p)}b(U) = \mathcal{E}_{b(p)}\). Then \(b(U)\) is an integral submanifold of the distribution \(\mathcal{E}\) (both of the same dimension). We have reached a contradiction, since the distribution \(\mathcal{D}\) is nowhere involutive for \(\kappa = 0, -1\). In fact, it is left invariant and by (20), at the identity one has

\[
[Z_\kappa (x, x), Z_\kappa (y, y)] = Z_\kappa (2x \times y, (\kappa + 1) x \times y),
\]

which does not belong to \(\mathcal{D}_{I_0}\) if \(\kappa \neq 1\). \(\square\)

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