Zero-density estimates for Epstein zeta functions *

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Abstract

We investigate the zeros of Epstein zeta functions associated with a positive definite quadratic form with rational coefficients in the vertical strip $\frac{1}{2} < \Re s < \sigma_2$, where $1/2 < \sigma_1 < \sigma_2 < 1$. When the class number of the quadratic form is bigger than 1, Voronin gives a lower bound and Lee gives an asymptotic formula for the number of zeros. In this paper, we improve their results by providing a new upper bound for the error term.

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1 Introduction and statement of results

Let $Q(m, n) = am^2 + bmn + cn^2$ be a positive definite quadratic form with $a, b, c \in \mathbb{Z}$ and $D = b^2 - 4ac < 0$. Let $s = \sigma + it$ be a complex variable. The Epstein zeta function associated with $Q$ is defined by

$$E(s, Q) = \sum_{m,n}^\prime \frac{1}{Q(m,n)^s}$$

for $\sigma > 1$, where the sum is over all integers $m, n$ not both zero. It has a meromorphic continuation to $\mathbb{C}$ with a simple pole at $s = 1$ and it satisfies the functional equation

$$\Psi(s, Q) = \Psi(1-s, Q),$$

(1.1)

where

$$\Psi(s, Q) := \left(\frac{\sqrt{-D}}{2\pi}\right)^s \Gamma(s)E(s, Q).$$

In this paper we study the distribution of the zeros of $E(s, Q)$ in the right half of the critical strip, $1/2 < \sigma < 1$. This distribution is different depending on whether the class number $h(D)$ of $\mathbb{Q}(\sqrt{D})$ is 1 or is greater than 1. If $h(D) = 1$, then

$$E(s, Q) = w_D \zeta_K(s),$$

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where $w_D$ is the number of roots of unity in $K = \mathbb{Q}(\sqrt{D})$ and $\zeta_K$ is the Dedekind zeta function of $K$. Hence, in this case we expect $E(s, Q)$ to satisfy the analogue of the Riemann hypothesis. However, if $h(D) > 1$, Davenport and Heilbronn [2] proved that $E(s, Q)$ has infinitely many zeros on $\sigma > 1$. For $1/2 < \sigma_1 < \sigma_2$ set

$$N_E(\sigma_1, \sigma_2; T) = \sum_{T \leq \gamma \leq 2T, \sigma_1 < \beta \leq \sigma_2} 1,$$

where $\rho = \beta + i\gamma$ denotes a generic zero of $E(s, Q)$. Voronin [11] proved the following theorem.

**Theorem A** (Voronin). Let $Q$ be a quadratic form with integer coefficients whose discriminant is $D < 0$. Suppose that the class number $h(D)$ is greater than 1. Then for any $\sigma_1$ and $\sigma_2$ with $1/2 < \sigma_1 < \sigma_2 < 1$, $N_E(\sigma_1, \sigma_2; T) \geq cT$, where $c = c(\sigma_1, \sigma_2, Q) > 0$ is independent of $T$.

Recently, the second author [7] improved this to an asymptotic formula.

**Theorem B** (Lee). Assume the same hypothesis as in Theorem A. Then for $1/2 < \sigma_1 < \sigma_2$ we have

$$N_E(\sigma_1, \sigma_2; T) = cT + o(T),$$

where $c = c(\sigma_1, \sigma_2, Q) \geq 0$. If $\sigma_1 \leq 1$, then $c > 0$.

Our main theorem provides an improvement of the error term.

**Theorem 1.** Assume the same hypothesis as in Theorem A. If $1/2 < \sigma_1 < \sigma_2$, then there exists an absolute constant $b > 0$ such that

$$N_E(\sigma_1, \sigma_2; T) = cT + O\left(T \exp\left(-b\sqrt{\log \log T}\right)\right),$$

where $c = c(\sigma_1, \sigma_2, Q) \geq 0$. If $\sigma_1 \leq 1$, then $c > 0$.

The proofs of the above theorems begin with the well-known identity

$$E(s, Q) = \frac{w_D}{h(D)} \sum_{\chi} \chi(a_Q)L(s, \chi),$$

where the sum is over all characters of the class group, $a_Q$ is any integer ideal in the ideal class corresponding to the equivalence class of $Q$, and $L(s, \chi)$ is the Hecke $L$-function defined by

$$L(s, \chi) = \sum_{n} \frac{\chi(n)}{\mathfrak{N}(n)^s} = \prod_{p} \left(1 - \frac{\chi(p)}{\mathfrak{N}(p)^s}\right)^{-1}$$

for $\sigma > 1$. Here $\mathfrak{N}$ is the norm. Each Hecke $L$-function has a meromorphic continuation to $\mathbb{C}$, and it has a simple pole at $s = 1$ only when the character $\chi$ is trivial. It also satisfies the functional equation (1.1) except that this time

$$\Psi(s, \chi) := \left(\frac{\sqrt{-D}}{2\pi}\right)^{s} \Gamma(s)L(s, \chi).$$
The $L$-functions in the sum (1.2) are not distinct. For each rational prime $p$, a principal ideal $(p)$ is a prime ideal $p$ or a product of two prime ideals $p_1p_2$. If $(p) = p$, then $\chi(p) = 1$ and
\[
\prod_{p | p} \left( 1 - \frac{\chi(p)}{\mathcal{N}(p)^s} \right)^{-1} = \left( 1 - \frac{1}{p^{2s}} \right)^{-1}.
\]
If $(p) = p_1p_2$, then $\chi(p_1)\chi(p_2) = 1$. Thus $\chi(p_1) = \overline{\chi(p_2)}$ and
\[
\prod_{p | p} \left( 1 - \frac{\chi(p)}{\mathcal{N}(p)^s} \right)^{-1} = \left( 1 - \frac{\chi(p_1) + \chi(p_2)}{p^s} + \frac{1}{p^{2s}} \right)^{-1} = \left( 1 - \frac{2\Re\chi(p_1)}{p^s} + \frac{1}{p^{2s}} \right)^{-1}.
\]
It follows that $L(s, \chi) = L(s, \overline{\chi})$. We now let $J$ be the number of real characters plus one-half the number of complex characters, and list the characters as $\chi_1, \ldots, \chi_J$ in such a way that $\chi_j \neq \chi_k$ and $\chi_j \neq \overline{\chi_k}$ for $j \neq k$. Then, writing
\[
L_j(s) = L(s, \chi_j),
\]
we may rewrite (1.2) as
\[
E(s, Q) = \sum_{j=1}^{J} c_j L_j(s), \quad (1.3)
\]
where
\[
c_j = \frac{w_D}{h(D)} \chi_j(a_Q)
\]
for real characters $\chi_j$, and
\[
c_j = \frac{w_D}{h(D)} 2\Re \chi_j(a_Q)
\]
for complex characters $\chi_j$. Note that $J > 1$ if and only if $h(D) > 1$.

Voronin [11] deduced Theorem 1 from a joint distribution result for the inequivalent Hecke $L$-functions $L_1(s), \ldots, L_J(s)$ in (1.3). On the other hand, Lee’s proof of Theorem 1 in [7] proceeded via a study of the Jensen function
\[
\varphi(\sigma) = \lim_{T \to \infty} \frac{1}{T} \int_1^T \log |E(\sigma + it, Q)| dt.
\]
Lee showed that $\varphi(\sigma)$ is twice differentiable and that the density of zeros of $E(s)$ in the strip $\sigma_1 < \sigma < \sigma_2$ equals
\[
\varphi'(\sigma_2) - \varphi'(\sigma_1) = \int_{\sigma_1}^{\sigma_2} \varphi''(\sigma) d\sigma.
\]
Our proof of Theorem 1 also proceeds through the estimation of the integral
\[
\frac{1}{T} \int_T^{2T} \log |E(\sigma + it, Q)| dt = \frac{1}{T} \int_T^{2T} \log \left| \sum_{j \leq J} c_j L_j(\sigma + it) \right| dt
\]
but, in addition, incorporates recent ideas of Lamzouri, Lester, and Radziwiłł [8] in their study of the distribution of $\alpha$-points of the Riemann zeta function.
It is worth noting that when $J = 2$,
\[
\frac{1}{T} \int_T^{2T} \log |E(\sigma + it, Q)| dt = \frac{1}{T} \int_T^{2T} \log |c_1 L_1(\sigma + it)| dt + \frac{1}{T} \int_T^{2T} \log \left| \frac{c_2 L_2(\sigma + it)}{c_1 L_1(\sigma + it)} + 1 \right| dt.
\]
It is not difficult to show that the first term here equals $\log |c_1|$ plus a small error term. We can also estimate the second term by a straightforward adaptation of the method in [8]. However, when $J > 2$ this approach no longer works. In what follows, therefore, we are mostly interested in the case $J \geq 3$.

Corresponding to the Hecke $L$-functions
\[
L_j(s) = L(s, \chi_j) = \prod_p \left( 1 - \frac{\chi_j(p)}{\mathfrak{N}(p)^s} \right)^{-1} \quad (1 \leq j \leq J)
\]
we define the random models
\[
L_j(\sigma, X) := \prod_p \left( 1 - \frac{\chi_j(p)X(p)}{\mathfrak{N}(p)^s} \right)^{-1} \quad (1 \leq j \leq J),
\]
where $p$ is the unique rational prime dividing $\mathfrak{N}(p)$, and the $X(p)$ are uniformly and independently distributed on the unit circle $\mathbb{T}$. Note that these products converge almost surely on $\mathbb{T}^\infty$ for $\sigma > 1/2$. We define
\[
\log L_j(\sigma, X) := \sum_p \sum_{k=1}^{\infty} \frac{\chi_j(p^k)X(p)^k}{k \mathfrak{N}(p)^{k\sigma}}
\]
and define $\log |L_j(\sigma, X)|$ and $\arg L_j(\sigma, X)$ as its real and imaginary parts, respectively. These too converge almost surely on $\mathbb{T}^\infty$ for $\sigma > 1/2$.

Let
\[
\mathbf{L}(s) := (\log |c_1 L_1/c_j L_j(s)|, \ldots, \log |c_{J-1} L_{J-1}/c_j L_j(s)|, \log |c_j L_j(s)|; \quad \arg c_1 L_1(s) - \arg c_j L_j(s), \ldots, \arg c_{J-1} L_{J-1}(s) - \arg c_j L_j(s), \arg c_j L_j(s))
\]
and
\[
\mathbf{L}(\sigma, X) := (\log |c_1 L_1/c_j L_j(\sigma, X)|, \ldots, \log |c_{J-1} L_{J-1}/c_j L_j(\sigma, X)|, \log |c_j L_j(\sigma, X)|; \quad \arg c_1 L_1(\sigma, X) - \arg c_j L_j(\sigma, X), \ldots, \arg c_{J-1} L_{J-1}(\sigma, X) - \arg c_j L_j(\sigma, X), \arg c_j L_j(\sigma, X)).
\]
For a Borel set $\mathcal{B}$ in $\mathbb{R}^{2J}$ and for $1/2 < \sigma < 1$ fixed, we define
\[
\Psi_T(\mathcal{B}) := \frac{1}{T} \text{meas}\{t \in [T, 2T] : \mathbf{L}(s) \in \mathcal{B}\} \quad (1.4)
\]
and
\[
\Psi(\mathcal{B}) := \mathbb{P}(\mathbf{L}(\sigma, X) \in \mathcal{B}) = \text{meas}\{X \in \mathbb{T}^\infty : \mathbf{L}(\sigma, X) \in \mathcal{B}\}. \quad (1.5)
\]
We define the discrepancy between these two distributions as
\[ D_T(B) := \Psi_T(B) - \Psi(B). \]

The key ingredient in our proof of Theorem 1 is the following bound for \( D_T(B) \), which is an analogue of Theorem 1.1 of [8] and is interesting in its own right.

**Theorem 2.** Let \( \frac{1}{2} < \sigma < 1 \) be fixed. Then
\[ \sup_{\mathcal{R}} |D_T(\mathcal{R})| \ll \frac{1}{(\log T)^\sigma}, \]
where \( \mathcal{R} \) runs over all rectangular regions (possibly unbounded) with sides parallel to the coordinate axes.

The letters \( A, B \) and \( C \) denote positive constants throughout that are not necessarily the same at each occurrence. Boldfaced letters denote vectors whose components may be functions. We also write \( \mathcal{L} = \log \log T \).

## 2 Basic lemmas

In this section we provide several of the technical lemmas we shall need later.

**Lemma 1.** Let \( a \geq b > 0 \). There exists an absolute positive constant \( C \) such that for any positive integer \( k \) we have
\[ \frac{1}{2\pi} \int_0^{2\pi} (\log |a - be^{iv}|)^{2k} dv \ll (C |\log a|)^{2k} + (Ck)^{2k}. \]

**Proof.** First assume that \( a > b \). Then writing \( z \) for \( e^{iv} \), we find that
\[ \frac{1}{2\pi} \int_0^{2\pi} (\log |a - be^{iv}|)^{2k} dv \]
\[ = \frac{1}{(2\pi i)^4} \int_{|z|=1} \left( -\log(a - bz) - \log(a - bz^{-1}) \right)^{2k} \frac{dz}{z} \quad (2.1) \]
\[ = \frac{1}{(2\pi i)^4} \int_{|z|=1} \left( -2 \log a + \sum_{n=1}^{\infty} \frac{(bz/a)^n}{n} + \sum_{m=1}^{\infty} \frac{(b/az)^m}{m} \right)^{2k} \frac{dz}{z} \]
\[ = \frac{1}{4^k} \sum_{k_1+k_2+k_3=2k} \left( \frac{2k}{k_1, k_2, k_3} \right) (-2 \log a)^{k_1} \left\{ \frac{1}{(2\pi i)^4} \int_{|z|=1} \left( \sum_{n=1}^{\infty} \frac{(bz/a)^n}{n} \right)^{k_2} \left( \sum_{m=1}^{\infty} \frac{(b/az)^m}{m} \right)^{k_3} \frac{dz}{z} \right\}. \]
We calculate the expression in braces by the calculus of residues. If \( k_2 = k_3 = 0 \), it equals 1. If one of \( k_2, k_3 \) is 0 but the other is not, it equals 0. In all other cases, since \( a > b > 0 \), it equals

\[
\sum_{n_1 + \cdots + n_{k_2} = m_1 + \cdots + m_{k_3}} \frac{(b/a)^{n_1 + \cdots + n_{k_2} + m_1 + \cdots + m_{k_3}}}{n_1 \cdots n_{k_2} m_1 \cdots m_{k_3}} < \sum_{n_1 + \cdots + n_{k_2} = m_1 + \cdots + m_{k_3}} \frac{1}{n_1 \cdots n_{k_2} m_1 \cdots m_{k_3}} = \sum_{\ell=1}^{\infty} \left( \sum_{n_1 + \cdots + n_{k_2} = \ell} \frac{1}{n_1 \cdots n_{k_2}} \right) \left( \sum_{m_1 + \cdots + m_{k_3} = \ell} \frac{1}{m_1 \cdots m_{k_3}} \right).
\]

In the first sum at least one of the \( n_i \) is maximal, and therefore \( \geq \ell / k_2 \). There are \( k_2 \) choices for the maximal term, so

\[
\sum_{n_1 + \cdots + n_{k_2} = \ell} \frac{1}{n_1 \cdots n_{k_2}} \leq k_2 \left( \frac{\ell}{k_2} \right) \sum_{n_1, \ldots, n_{k_2-1} \leq \ell} \frac{1}{n_1 \cdots n_{k_2-1}} \ll \frac{k_2^2}{\ell} \left( \log \ell \right)^{k_2-1}.
\]

Thus the above is

\[
(k_2 k_3)^2 \sum_{\ell=1}^{\infty} \frac{(\log \ell)^{k_2+k_3-2}}{\ell^2}.
\]

Combining our estimates, we find that (2.1) is

\[
\ll \frac{1}{4k} (2 \log a)^{2k} + \frac{k^4}{4k} \sum_{k_1+k_2+k_3=2k} \left( \frac{2k}{k_1, k_2, k_3} \right) (2 \log a)^k \left( \sum_{\ell=1}^{\infty} \frac{(\log \ell)^{k_2+k_3-2}}{\ell^2} \right).
\]

The function \( f(x) = (\log x)^{k_2+k_3-2} x^{-2} \) has a maximum at \( x = x_0 = e^{(k_2+k_3-2)/2} \) and it is increasing for \( 0 < x < x_0 \) and decreasing for \( x > x_0 \). Thus

\[
\sum_{\ell=1}^{\infty} \frac{(\log \ell)^{k_2+k_3-2}}{\ell^2} \ll x_0 (\log x_0)^{k_2+k_3-2} x_0^2 + \int_{x_0}^{\infty} \frac{(\log u)^{k_2+k_3-2}}{u^2} du
\ll (Ck)^{k_2+k_3-2} + \int_{x_0}^{\infty} v^{k_2+k_3-2} e^{-v} dv
\ll (Ck)^{k_2+k_3-2} + \Gamma(k_2 + k_3 - 1) \ll (Ck)^{k_2+k_3-\frac{3}{2}}.
\]

Hence,

\[
\frac{1}{2\pi} \int_0^{2\pi} (\log |a - be^{i\theta}|)^{2k} d\theta \ll |\log a|^{2k} + k^{7/2} \sum_{k_1+k_2+k_3=2k} \left( \frac{2k}{k_1, k_2, k_3} \right) |2 \log a|^{k_1} (Ck)^{k_2+k_3}
\ll |\log a|^{2k} + k^{7/2} (|2 \log a| + Ck)^{2k}
\ll (C|\log a|)^{2k} + (Ck)^{2k}.
\]
Now consider the case $a = b > 0$. We have
\[
\frac{1}{2\pi} \int_0^{2\pi} (\log |a - ae^{iv}|)^{2k} dv = \frac{1}{2\pi} \int_0^{2\pi} (\log |2a| + \log |\sin v/2|)^{2k} dv \\
\leq 2^{2k-1} \left( (\log |2a|)^{2k} + \frac{1}{\pi} \int_0^{\pi} (\log |\sin v/2|)^{2k} dv \right).
\]
Note that for $0 \leq v \leq \pi$ we have $v/4 \leq |\sin(v/2)| \leq 1$. Thus, the last line is
\[
\leq 2^{2k-1} \left( (\log |2a|)^{2k} + \frac{1}{\pi} \int_0^{\pi} (\log(v/4))^{2k} dv \right) \\
\ll C^{2k} \left( |\log a|^{2k} + \int_0^1 (\log x)^{2k} dx \right) \ll C^{2k} \left( |\log a|^{2k} + \Gamma(2k + 1) \right) \\
\ll (C|\log a|)^{2k} + (Ck)^{2k}.
\]

\[\square\]

Lemma 2. Let $L(s) = L(s, \chi)$ be a Hecke $L$-function attached to an ideal class character of the quadratic field $\mathbb{Q}(\sqrt{D})$. For $\sigma > 1$ write
\[
\log L(s) = \sum_{p,n} \frac{a(p^n)}{p^{ns}},
\]
and for $Y \geq 2$ and any $s$ let
\[
R_Y(s) = \sum_{p^n \leq Y} \frac{a(p^n)}{p^{ns}}.
\]
Suppose that $1/2 < \sigma < 1$ and $B_1 > 0$ are fixed, and that $Y = (\log T)^{B_2}$ with $B_2 > 2(B_1 + 1)/(\sigma - 1/2)$. Then
\[
\log L(s) = R_Y(s) + O((\log T)^{-B_1})
\]
for all $t \in [T, 2T]$ except on a set of measure $\ll T^{1-d(\sigma)}$, where $d(\sigma) > 0$.

Proof. Using an approximate functional equation for $L(s)$ (for example, see Section A.12 of [4]) in a standard way, we find that
\[
\int_T^{2T} |L(1/2 + it)|^2 dt \ll T(\log T)^4.
\]
From this and Theorem 1 of [6] we obtain the zero-density estimate
\[
N_L(\sigma, T, 2T) := \sum_{T < \gamma \leq 2T} \frac{1}{\beta \geq \sigma} \ll T^{1-a_1(\sigma-1/2)}(\log T)^{12}
\]
uniformly for $\sigma \geq 1/2$, where $\rho = \beta + i\gamma$ denotes a generic nontrivial zero of $L(s)$ and $a_1 > 0$ is a constant independent of $\sigma$. 

Now let $s = \sigma + it$ with $1/2 < \sigma < 1$ and $T \leq t \leq 2T$. By Perron’s formula (see Titchmarsh [9], pp.60–61)

$$R_Y(s) = \sum_{p^n \leq Y} \frac{a(p^n)}{p^{nss}} = \frac{1}{2\pi i} \int_{c-iY}^{c+iY} \log L(s+w) \frac{Y^w}{w} dw + O(Y^{-\sigma+\epsilon}),$$  \hspace{1cm} (2.2)

where $c = 1 - \sigma + \epsilon$ with $0 < \epsilon < 1/4$. Let $w_0 = (1/2 - \sigma)/2$ and assume that $L(s+w)$ has no zeros in the half-strip given by $\Re w \geq \frac{3}{4}(1/2 - \sigma), |\Im w| \leq Y + 1$. Then in the slightly smaller half-strip $\Re w \geq w_0, |\Im w| \leq Y$ we have

$$L'(s+w) \ll \log T$$  \hspace{1cm} (2.3)

(see Iwaniec and Kowalski [3], Proposition 5.7). Observe that this holds for all $t \in [T, 2T]$ except for $t$ in a set of measure

$$\ll (2Y + 2) \cdot N(\frac{1}{2} + \frac{1}{4}(\sigma - \frac{1}{2}), T, 2T) \ll T^{1-a_1(\sigma-1/2)/4} (\log T)^{B_2+12}.$$

Now, integrating (2.3) along the horizontal segment from $w$ to $w + 2$, we see that

$$\log L(s+w) = O(\log T).$$

Using this and shifting the contour to the left in (2.2), we obtain

$$R_Y(s) = \log L(s) + \frac{1}{2\pi i} \int_{w_0-iY}^{w_0+iY} \log L(s+w) \frac{Y^w}{w} dw + O(Y^{-\sigma+\epsilon})$$

$$= \log L(s) + O((\log T)^{1-B_2(\sigma-1/2)/2} + (\log T)^{(-\sigma+\epsilon)B_2}).$$

Therefore,

$$\log L(s) = R_Y(s) + O((\log T)^{1-B_2(\sigma-1/2)/2} + (\log T)^{-(\sigma-\epsilon)B_2})$$  \hspace{1cm} (2.4)

holds for all $t \in [T, 2T]$ except for a set of measure

$$\ll T^{1-c_1(\sigma-1/2)/4} (\log T)^{B_2+12}.$$ 

Given $B_1 > 0$, if we take $B_2 > 2(B_1+1)/\sigma-1/2)$, both error terms in (2.4) are $O((\log T)^{-B_1})$.

This proves the lemma.

**Lemma 3.** Let $2 \leq y \leq z$ and let $k$ be a positive integer $\leq \log T/(3\log z)$. Suppose that $|a(p)| \leq 2$. Then

$$\frac{1}{T} \int_T^{2T} \left| \sum_{y < p \leq z} \frac{a(p)}{p^{\sigma+n}} \right|^{2k} dt \ll 2^{2k} k! \left( \sum_{y < p \leq z} \frac{1}{p^{2\sigma}} \right)^k + 2^{2k} T^{-1/3}$$

and

$$\mathbb{E} \left( \left| \sum_{y < p \leq z} \frac{a(p)X(p)}{p^{\sigma}} \right|^{2k} \right) \ll 2^{2k} k! \left( \sum_{y < p \leq z} \frac{1}{p^{2\sigma}} \right)^k.$$
This is a simple modification of Lemma 3.2 of [8] so we omit the proof.

**Lemma 4.** Let \( R_{j,Y}(s) \) be the Dirichlet polynomial approximation corresponding to \( \log L_j(s) \) in Lemma 2 and let \( R_{j,Y}(\sigma, X) \) be the analogous expression for \( \log L_j(\sigma, X) \). Let \( 1/2 < \sigma < 1 \) and \( Y = (\log T)^{B_2} \), where \( B_2 \) is as in Lemma 2. Then for any positive integers \( k \leq \log T/(3 \log Y) \) and \( j \leq J \), we have

\[
\frac{1}{T} \int_T^{2T} |R_{j,Y}(\sigma + it)|^{2k} \, dt \ll \left( \frac{C k^{1-\sigma}}{(\log k)^\sigma} \right)^{2k}
\]

and

\[
\mathbb{E}(|R_{j,Y}(\sigma, X)|^{2k}) \ll \left( \frac{C k^{1-\sigma}}{(\log k)^\sigma} \right)^{2k}.
\]

Here \( C \) is a constant depending only on \( \sigma \).

**Proof.** To prove the first estimate it is enough to show that

\[
\frac{1}{T} \int_T^{2T} \left| \sum_{p \leq Y} \frac{a_j(p)}{p^{\sigma + it}} \right|^{2k} \, dt \ll \left( \frac{C k^{1-\sigma}}{(\log k)^\sigma} \right)^{2k}.
\]

By Lemma 3 and the prime number theorem

\[
\frac{1}{T} \int_T^{2T} \left| \sum_{p \leq Y} \frac{a_j(p)}{p^{\sigma + it}} \right|^{2k} \, dt \leq \frac{2^{2k-1}}{T} \int_T^{2T} \left| \sum_{p \leq k \log k} \frac{a_j(p)}{p^{\sigma + it}} \right|^{2k} \, dt + \frac{2^{2k-1}}{T} \int_T^{2T} \left| \sum_{k \log k < p \leq Y} \frac{a_j(p)}{p^{\sigma + it}} \right|^{2k} \, dt
\]

\[
\ll 2^{2k-1} \left( \sum_{p \leq k \log k} \frac{2}{p^\sigma} \right)^{2k} + 2^{2k-1} k! \left( \sum_{k \log k < p \leq Y} \frac{1}{p^{2\sigma}} \right)^k + 2^{4k-1} T^{-1/3}
\]

\[
\ll \left( \frac{C k^{1-\sigma}}{(\log k)^\sigma} \right)^{2k}.
\]

The estimate for the expectation may be treated similarly. 

\[\Box\]

**Lemma 5.** Let

\[
Q_{j,Y}(s) = \sum_{n \leq Y} \frac{b_j(n)}{n^s} \quad \text{and} \quad Q_{j,Y}(\sigma, X) = \sum_{n \leq Y} \frac{b_j(n)X(n)}{n^\sigma},
\]

where for each \( j \leq J \), the \( b_j(n) \) are bounded complex numbers, and, if \( n = \prod_p p^{\alpha} \), then \( X(n) = \prod_p X(p)^{\alpha} \) with the \( X(p) \) uniformly and independently distributed on \( \mathbb{T} \). Let \( k_j, k_j' \) for \( j \leq J \) be positive integers and \( k = \sum_{j \leq J} k_j \), \( k' = \sum_{j \leq J} k_j' \). Then

\[
\frac{1}{T} \int_T^{2T} \prod_{j \leq J} \left( Q_{j,Y}(\sigma + it)^{k_j} \overline{Q_{j,Y}(\sigma + it)^{k_j'}} \right) \, dt = \mathbb{E} \left( \prod_{j \leq J} \left( Q_{j,Y}(\sigma, X)^{k_j} \overline{Q_{j,Y}(\sigma, X)^{k_j'}} \right) \right) + O\left( \frac{(CY^{2-\sigma})^{k+k'}}{T} \right).
\]
Proof.

\[
\frac{1}{T} \int_T^{2T} \prod_{j \leq J} \left( Q_{j,Y}(\sigma + it)^{k_j} Q_{j,Y}(\sigma + it)^{k_j'} \right) dt \\
= \frac{1}{T} \int_T^{2T} \left( \sum_{n_{i,j} \leq Y} \frac{b_1(n_{1,1}) \cdots b_1(n_{k_1,1})}{(1_1 \cdots n_{k_1,1})^{\sigma + it}} \cdots \frac{b_f(n_{1,J}) \cdots b_f(n_{k_f,J})}{(1_J \cdots n_{k_f,J})^{\sigma + it}} \right) \\
\cdot \left( \sum_{m_{i,j} \leq Y} \frac{b_1(m_{1,1}) \cdots b_1(m_{k_1',1})}{(m_{1,1} \cdots m_{k_1',1})^{\sigma - it}} \cdots \frac{b_f(m_{1,J}) \cdots b_f(m_{k_f',J})}{(m_{1,J} \cdots m_{k_f',J})^{\sigma - it}} \right) \, dt.
\]

The contribution of the diagonal terms to this is

\[
\mathcal{D} = \sum_{\substack{n_{i,j},m_{i,j} \leq Y \\cap n_{i,j} = \prod m_{i,j}}} \frac{b_1(n_{1,1}) \cdots b_1(n_{k_1,1})}{(n_{1,1} \cdots n_{k_1,1})^{\sigma}} \cdots \frac{b_f(n_{1,J}) \cdots b_f(n_{k_f,J})}{(n_{1,J} \cdots n_{k_f,J})^{\sigma}} \\
\cdot \frac{b_1(m_{1,1}) \cdots b_1(m_{k_1',1})}{(m_{1,1} \cdots m_{k_1',1})^{\sigma}} \cdots \frac{b_f(m_{1,J}) \cdots b_f(m_{k_f',J})}{(m_{1,J} \cdots m_{k_f',J})^{\sigma}} \\
= \mathbb{E} \left( \prod_{j \leq J} \left( Q_{j,Y}(\sigma, X)^{k_j} Q_{j,Y}(\sigma, X)^{k_j'} \right) \right).
\]

The off-diagonal contribution is

\[
\mathcal{O} = \sum_{\substack{n_{i,j},m_{i,j} \leq Y \\cap n_{i,j} \neq \prod m_{i,j}}} \frac{b_1(n_{1,1}) \cdots b_1(n_{k_1,1})}{(n_{1,1} \cdots n_{k_1,1})^{\sigma}} \cdots \frac{b_f(n_{1,J}) \cdots b_f(n_{k_f,J})}{(n_{1,J} \cdots n_{k_f,J})^{\sigma}} \\
\cdot \frac{b_1(m_{1,1}) \cdots b_1(m_{k_1',1})}{(m_{1,1} \cdots m_{k_1',1})^{\sigma}} \cdots \frac{b_f(m_{1,J}) \cdots b_f(m_{k_f',J})}{(m_{1,J} \cdots m_{k_f',J})^{\sigma}} \\
\cdot \left( \frac{(m/n)^{2iT} - (m/n)^{iT}}{iT \log(m/n)} \right),
\]

where \( n = \prod n_{i,j} \) and \( m = \prod m_{i,j} \). Since \( n, m \leq Y^{k+k'} \) and \( n \neq m \),

\[
\frac{1}{|\log(m/n)|} \ll Y^{k+k'}.
\]

Hence,

\[
\mathcal{O} \ll \frac{Y^{k+k'}}{T} \sum_{\substack{n_{i,j},m_{i,j} \leq Y \\cap n_{i,j} \neq \prod m_{i,j}}} \frac{|b_1(n_{1,1}) \cdots b_1(n_{k_1,1})|}{(n_{1,1} \cdots n_{k_1,1})^{\sigma}} \cdots \frac{|b_f(n_{1,J}) \cdots b_f(n_{k_f,J})|}{(n_{1,J} \cdots n_{k_f,J})^{\sigma}} \\
\cdot \frac{|b_1(m_{1,1}) \cdots b_1(m_{k_1',1})|}{(m_{1,1} \cdots m_{k_1',1})^{\sigma}} \cdots \frac{|b_f(m_{1,J}) \cdots b_f(m_{k_f',J})|}{(m_{1,J} \cdots m_{k_f',J})^{\sigma}} \\
\ll \frac{(CY)^{k+k'}}{T} \sum_{n_{i,j},m_{i,j} \leq Y} \frac{1}{(n_{1,1} \cdots n_{k_1,1})^{\sigma}} \cdots \frac{1}{(n_{1,J} \cdots n_{k_f,J})^{\sigma}} \\
\cdot \frac{1}{(m_{1,1} \cdots m_{k_1',1})^{\sigma}} \cdots \frac{1}{(m_{1,J} \cdots m_{k_f',J})^{\sigma}} \\
\ll \frac{(CY^{2-\sigma})^{k+k'}}{T}.
\]
The lemma now follows on combining our estimates for the diagonal and off-diagonal terms $D$ and $O$.

\[ \square \]

## 3 Lemmas on moments of logarithms of $L$-functions

We will frequently appeal to the following three lemmas.

**Lemma 6.** Let $1/2 < \sigma \leq 2$ be fixed. There exists a constant $C > 0$ depending at most on $J$, such that for every positive integer $k$ we have

\[
\frac{1}{T} \int_{T}^{2T} \left| \log \sum_{j \leq J} c_j L_j(\sigma + it) \right|^{2k} dt \ll (Ck)^{4k}.
\]

**Lemma 7.** Let $1/2 < \sigma \leq 2$ be fixed. There exist an absolute constant $C_1 > 0$ and a constant $C_2 > 0$ depending on $\sigma$ such that for every positive integer $k \leq \log T / (C_2 \log \log T)$, we have

\[
\frac{1}{T} \int_{T}^{2T} |\log c_j L_j(s)|^{2k} dt \ll (C_1 k)^{2k}
\]

and

\[
\frac{1}{T} \int_{T}^{2T} |\log c_i L_i(s) - \log c_j L_j(s)|^{2k} dt \ll (C_1 k)^{2k}.
\]

**Lemma 8.** Let $1/2 < \sigma \leq 2$ be fixed. For every integer $k > 0$ we have

\[
\mathbb{E} \left( \left| \log \sum_{j \leq J} c_j L_j(\sigma, X) \right|^{2k} \right) \ll (Ck)^{2k}
\]

and

\[
\mathbb{E} \left( \left| \log c_j L_j(\sigma, X) \right|^{2k} \right) \ll (Ck)^{k},
\]

where, in either case, $C > 0$ is a constant depending at most on $J$.

We will sometimes use Lemmas 6 and 7 to show that we may restrict certain sets to lie within intervals of length $\approx L = \log \log T$ at the cost of a small error. Here is a typical example. Let $\mathcal{B}$ be a Borel set in $\mathbb{R}$ and let $A$ be a positive constant. Set

\[ I_T = (-AL, AL). \]

Then

\[
\text{meas}\{ t \in [T, 2T] : |\log |c_i L_i(s)|| \notin I_T \} \leq \int_{T}^{2T} \left( \frac{|\log |c_i L_i(s)||}{AL} \right)^{2k} dt \ll T \left( \frac{Ck}{AL} \right)^{2k}.
\]

Taking $k = \mathcal{L}$ and $A$ sufficiently large relative to $C$, we see that this is $\ll T(\log T)^{-B}$, where $B > 0$ is an arbitrarily large constant. Thus,

\[
\text{meas}\{ t \in [T, 2T] : |\log |c_i L_i(s)|| \in \mathcal{B} \} = \text{meas}\{ t \in [T, 2T] : |\log |c_i L_i(s)|| \in \mathcal{B} \cap I_T \} + O(T(\log T)^{-B}).
\]
When required, we will restrict sets in this way by simply writing “by Lemma 6 (or 7)”.

The proof of Lemma 6 is a straightforward modification of the proof of Proposition 2.5 in [8] (one just replaces \( \zeta(s) - a \) by \( E(s, Q) \) throughout), so we do not include it.

### 3.1 Proof of Lemma 7

A little thought shows that it is enough to prove that
\[
\frac{1}{T} \int_{T}^{2T} |\log L_j(\sigma + it)|^{2k} dt \ll (Ck)^2k.
\]

Let \( \mathcal{A}(T) = \mathcal{A}_\sigma(T) \) be the set of \( t \in [T, 2T] \) such that
\[
\log L(\sigma + it) = R_Y(\sigma + it) + O((\log T)^{−B_1})
\]
and let \( \mathcal{A}'(T) = [T, 2T] \setminus \mathcal{A}(T) \). Then by Lemma 2, \( \text{meas}(\mathcal{A}'(T)) \ll T^{1−d(\sigma)} \). (3.1)

Splitting the integral as
\[
\frac{1}{T} \int_{T}^{2T} |\log L(\sigma + it)|^{2k} dt = \left\{ \int_{\mathcal{A}(T)}^{\mathcal{A}'(T)} \right\} |\log L(\sigma + it)|^{2k} dt,
\]
we first estimate the integral over \( \mathcal{A}'(T) \). We have
\[
\frac{1}{T} \int_{\mathcal{A}'(T)} |\log L(\sigma + it)|^{2k} dt
\leq 2^{k−1} \left( \frac{1}{T} \int_{\mathcal{A}'(T)} |\log |L(\sigma + it)||^{2k} dt + \frac{1}{T} \int_{\mathcal{A}'(T)} |\arg L(\sigma + it)|^{2k} dt \right).
\]

By Lemma 9.4 in [9]
\[
\arg L(\sigma + it) = O(\log T),
\]
so
\[
\frac{1}{T} \int_{\mathcal{A}'(T)} |\arg L(\sigma + it)|^{2k} dt \ll T^{−d(\sigma)} (C\log T)^{2k}.
\]

By the Cauchy-Schwartz inequality, Lemma 6, and (3.1), the second integral in (3.3) is
\[
\frac{1}{T} \int_{\mathcal{A}'(T)} |\log |L(\sigma + it)||^{2k} dt \ll T^{−d(\sigma)/2} \left( \frac{1}{T} \int_{T}^{2T} |\log |L(\sigma + it)||^{4k} dt \right)^{1/2}
\ll T^{−d(\sigma)/2} (Ck)^{4k}.
\]

Thus,
\[
\frac{1}{T} \int_{\mathcal{A}'(T)} |\log L(\sigma + it)|^{2k} dt \ll T^{−d(\sigma)} (C\log T)^{2k} + T^{−d(\sigma)/2} (Ck)^{4k}.
\] (3.4)
Next we estimate the integral over $\mathcal{A}(T)$. By the definition of $\mathcal{A}(T)$

$$\frac{1}{T} \int_{\mathcal{A}(T)} |\log L(\sigma + it)|^{2k} dt = \frac{1}{T} \int_{\mathcal{A}(T)} |R_Y(\sigma + it) + O((\log T)^{-B_1})|^{2k} dt$$

$$\leq \frac{1}{T} \int_T^{2T} \left| \sum_{p \leq Y} \frac{a(p)}{p^\sigma + it} + O(1) \right|^{2k} dt$$

$$\leq 2^{2k-1} \left( \frac{1}{T} \int_T^{2T} \left| \sum_{p \leq Y} \frac{a(p)}{p^\sigma + it} \right|^{2k} dt + C^k \right).$$

Defining

$$a_k(n) = \sum_{p_1 \cdots p_k = n} a(p_1) \cdots a(p_k),$$

we find that since $|a(p)| \leq 2$,

$$|a_k(n)| \leq \sum_{p_1 \cdots p_k = n} 2^k \leq 2^k k! \ll (Ck)^k.$$

Thus, for $Y^k \ll T$ and $1/2 < \sigma \leq 2$ fixed,

$$\frac{1}{T} \int_T^{2T} \left| \sum_{p \leq Y} \frac{a(p)}{p^\sigma + it} \right|^{2k} dt = \frac{1}{T} \int_T^{2T} \left| \sum_{n \leq Y^k} \frac{a_k(n)}{n^\sigma + it} \right|^{2k} dt$$

$$= \frac{1}{T} (T + O(Y^k)) \sum_{n \leq Y^k} \frac{|a_k(n)|^2}{n^{2\sigma}}$$

$$\ll \sum_{n=1}^{\infty} \frac{(Ck)^{2k}}{n^{2\sigma}} \ll (Ck)^{2k}.$$

Since $Y = (\log T)^{B_2}$, this inequality holds for $k \leq \log T/(B_2 \log \log T)$.

Combining this with (3.2) and (3.4), we obtain

$$\frac{1}{T} \int_T^{2T} |\log L(\sigma + it)|^{2k} dt \ll (Ck)^{2k} + T^{-d(\sigma)} (C \log T)^{2k} + T^{-d(\sigma)/2} (Ck)^{4k}$$

for $k \leq \log T/(B_2 \log \log T)$. In the second and third error terms we have

$$(C \log T)^{2k} \leq C^{2k}T^{2/B_2} \quad \text{and} \quad (Ck)^{4k} \leq T^{4/B_2}.$$

Thus, choosing $B_2$ large enough, we find that

$$\frac{1}{T} \int_T^{2T} |\log L(\sigma + it)|^{2k} dt \ll (Ck)^{2k}$$

for $k \leq \log T/(B_2 \log \log T)$, as required.
3.2 Proof of Lemma

We define a measure on Borel sets $\mathcal{B} \in \mathbb{R}^{2J}$ by

$$\Phi(\mathcal{B}) := \mathbb{P}(L_0(\sigma, X) \in \mathcal{B}),$$

where

$$L_0(\sigma, X) := (\log |L_1(\sigma, X)|, \ldots, \log |L_J(\sigma, X)|, \arg L_1(\sigma, X), \ldots, \arg L_J(\sigma, X)).$$

By a straightforward generalization of the proofs of Theorems 5 and 6 in Borchsenius and Jessen [11], one can show that $\Phi$ is absolutely continuous and that its density function $F(u, v)$ satisfies

$$F(u, v) \ll \exp \left( -A(u_1^2 + \cdots + u_J^2 + v_1^2 + \cdots + v_J^2) \right)$$

for some constant $A > 0$, where $u = (u_1, \ldots, u_J)$ and $v = (v_1, \ldots, v_J)$. Hence

$$E\left( \left| \log \left| \sum_{j=1}^{J} c_j L_j(\sigma, X) \right|^2 \right|^{2k} \right) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \log \left| \sum_{j=1}^{J} c_j e^{u_j + iv_j} \right|^{2k} F(u, v) du dv \ll \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \log \left| \sum_{j=1}^{J} c_j e^{u_j + iv_j} \right|^{2k} e^{-A(u_1^2 + \cdots + u_J^2 + v_1^2 + \cdots + v_J^2)} du dv.$$

Since $e^{iv_j}$ is periodic, the integral with respect to $v_j$ is of the form

$$\sum_{m=-\infty}^{\infty} \int_{0}^{2\pi} \log \left| \sum_{j=1}^{J} c_j e^{u_j + iv_j} \right|^{2k} e^{-A(v_j + 2\pi m)^2} dv_j = \int_{0}^{2\pi} \log \left| \sum_{j=1}^{J} c_j e^{u_j + iv_j} \right|^{2k} e^{-A(v_j)^2} dv_j \ll \int_{0}^{2\pi} \log \left| \sum_{j=1}^{J} c_j e^{u_j + iv_j} \right|^{2k} dv_j.$$

Thus,

$$E\left( \left| \log \left| \sum_{j=1}^{J} c_j L_j(\sigma, X) \right|^2 \right|^{2k} \right) \ll \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left( \int_{0}^{2\pi} \log \left| \sum_{j=1}^{J} c_j e^{u_j + iv_j} \right|^{2k} dv \right) e^{-A(u_1^2 + \cdots + u_J^2)} du.$$

We first integrate with respect to $u_1$ and $v_1$ and see that

$$\int_{-\infty}^{\infty} \int_{0}^{2\pi} \log \left| \sum_{j=1}^{J} c_j e^{u_j + iv_j} \right|^{2k} e^{-A u_1^2} dv_1 du_1 = \int_{-\infty}^{\infty} \int_{0}^{2\pi} \log |B + c_1 e^{u_1 + iv_1}|^{2k} e^{-A u_1^2} dv_1 du_1,$$
where $B = \sum_{j=2}^J c_j e^{u_j + iv_j}$. Dividing the $u_1$-integral into two pieces, we see that this equals

$$\left(\int_{|B|\geq |c_1|} + \int_{|B|< |c_1|}\right) \left(\int_0^{2\pi} \log |B + c_1 e^{u_1 + iv_1}|^{2k}dv_1\right) e^{-Au_1^2}du_1. \quad (3.6)$$

By Lemma [1], the first integral is

$$\ll \int_{|B|\geq |c_1|} ((C \log |B|)^{2k} + (Ck)^{2k}) e^{-Au_1^2}du_1 \ll (C \log |B|)^{2k} + (Ck)^{2k}.$$

Also by Lemma [1], the second integral is

$$\ll \int_{|B|< |c_1|} ((C \log |c_1| e^{u_1})^{2k} + (Ck)^{2k}) e^{-Au_1^2}du_1$$

$$\ll \int_{-\infty}^{\infty} (C + |u_1|)^{2k} e^{-Au_1^2}du_1 + (Ck)^{2k}$$

$$\ll C^{2k} \int_{-\infty}^{\infty} u_1^{2k} e^{-Au_1^2}du_1 + (Ck)^{2k} \ll C^{2k} \Gamma(k + \frac{1}{2}) + (Ck)^{2k}$$

$$\ll (Ck)^{2k}.$$

Hence

$$\int_{-\infty}^{\infty} \int_0^{2\pi} \log \left| \sum_{j=1}^J c_j e^{u_j + iv_j} \right|^{2k} e^{-Au_1^2}du_1dv_1 \ll \left( C \log \left| \sum_{j=2}^J c_j e^{u_j + iv_j} \right| \right)^{2k} + (Ck)^{2k}$$

Proceeding inductively, we find that

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left( \int_0^{2\pi} \cdots \int_0^{2\pi} \log \left| \sum_{j=1}^J c_j e^{u_j + iv_j} \right|^{2k} du \right) e^{-A(u_1^2 + \cdots + u_j^2)}dv \ll (Ck)^{2k}.$$

Thus, by (3.5)

$$E \left( \log \left| \sum_{j=1}^J c_j L_j(\sigma, X) \right|^{2k} \right) \ll (Ck)^{2k}.$$

We now prove the second inequality of the lemma, namely,

$$E \left( \log c_j L_j(\sigma, X) \right)^{2k} \ll (Ck)^{k}.$$
as in the proof for the first inequality, we have

\[
E \left( \left| \log L_j(\sigma, X) \right|^2 \right) \ll \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |u + iv|^{2k} e^{-A(u^2 + v^2)} \, du \, dv = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (u^2 + v^2)^{k/2} e^{-A(u^2 + v^2)} \, du \, dv = \int_{0}^{2\pi} \int_{0}^{\infty} r^{2k} e^{-Ar^2} r \, dr \, d\theta \ll (Ck)^k.
\]

4 Proof of Theorem

4.1 Two lemmas pertaining to the proof of Theorem

To prove Theorem we also require the following two lemmas.

Lemma 9. Let \( u = (u_1, \ldots, u_J) \) and similarly \( v, w, \) and \( z \) be vectors in \( \mathbb{R}^J \). Let

\[
(w, z) \cdot (u, v) = \sum_{j \leq J} (w_j u_j + z_j v_j),
\]

and define

\[
\hat{\Psi}_T(w, z) := \int_{\mathbb{R}^{2J}} e^{i(w, z) \cdot (u, v)} d\Psi_T(u, v)
\]

and

\[
\hat{\Psi}(w, z) := \int_{\mathbb{R}^{2J}} e^{i(w, z) \cdot (u, v)} d\Psi(u, v).
\]

If \( A > 1 \), there exists a constant \( b = b(\sigma, A) > 0 \) such that for all \( w \) and \( z \) with \( |w_j|, |z_j| \leq b(\log T)^\sigma \), we have

\[
\hat{\Psi}_T(w, z) - \hat{\Psi}(w, z) = O((\log T)^{-A}).
\]

Proof. By the definition of \( \hat{\Psi}_T \) and \( \hat{\Psi} \) (see (1.4) and (1.5)), we may write

\[
\hat{\Psi}_T(w, z) = \frac{1}{T} \int_{-T}^{T} \exp \left[ i \left( \sum_{j < J} w_j \log \left| \frac{c_j L_j}{c_J L_J}(\sigma + it) \right| + w_J \log |c_J L_J(\sigma + it)| \right. \right.

+ \left. \left. \sum_{j < J} z_j \arg \frac{c_j L_j}{c_J L_J}(\sigma + it) + z_J \arg c_J L_J(\sigma + it) \right) \right] \, dt
\]

and

\[
\hat{\Psi}(w, z) = \mathbb{E} \left( \exp \left[ i \left( \sum_{j < J} w_j \log \left| \frac{c_j L_j}{c_J L_J}(\sigma, X) \right| + w_J \log |c_J L_J(\sigma, X)| \right. \right.

+ \left. \left. \sum_{j < J} z_j \arg \frac{c_j L_j}{c_J L_J}(\sigma, X) + z_J \arg c_J L_J(\sigma, X) \right) \right] \right).
\]
Now we use Lemma 2 to replace the logarithms of the $L$-functions by short Dirichlet polynomials. Let

$$Q_{j,Y}(\sigma + it) = \log \frac{c_j}{c_j} L_j(\sigma + it) + \sum_{p^n \leq Y} \frac{a_j(p^n) - a_j(p^n)}{p^n(\sigma + it)}$$

for $j < J$, and

$$Q_{j,Y}(\sigma + it) = \log c_j + \sum_{p^n \leq Y} \frac{a_j(p^n)}{p^n(\sigma + it)}.$$

By Lemma 2 for any fixed $B_1 > 0$ we have

$$\log \frac{c_j}{c_j} L_j(\sigma + it) = Q_{j,Y}(\sigma + it) + O((\log T)^{-B_1}) \quad (j < J)$$

and

$$\log c_j L_j(\sigma + it) = Q_{j,Y}(\sigma + it) + O((\log T)^{-B_1})$$

for all $t \in [T, 2T]$ except for a set of measure $T^{1-d(\sigma)}$, where $d(\sigma) > 0$ is a constant. Here $Y = (\log T)^{B_2}$ and $B_2 > 2(B_1 + 1)/(\sigma - 1/2)$. Letting $B_1(T)$ be the set of $t \in [T, 2T]$ such that (4.1) and (4.2) hold, we then see that

$$\hat{\Psi}_T(w, z) = \frac{1}{T} \int_{B_1(T)} \exp \left[ i \left( \sum_{j < J} w_j \log \frac{c_j}{c_j} L_j(\sigma + it) + w_j \log |c_j L_j(\sigma + it)| \right. \right.$$  

$$\left. + \sum_{j < J} z_j \arg \frac{c_j}{c_j} L_j(\sigma + it) + z_j \arg c_j L_j(\sigma + it) \right] dt + O(T^{-d(\sigma)})$$

$$= \frac{1}{T} \int_{B_1(T)} \exp \left[ i \sum_{j < J} (w_j \Re Q_{j,Y}(\sigma + it) + z_j \Im Q_{j,Y}(\sigma + it)) \right] dt + O((\log T)^{-B_1})$$

$$= \frac{1}{T} \int_T^{2T} \exp \left[ i \sum_{j \leq J} \Re ((w_j - iz_j)Q_{j,Y}(\sigma + it)) \right] dt + O((\log T)^{-B_1}).$$

To estimate this we define

$$B_2(T) := \{ t \in [T, 2T] : |Q_{j,Y}(\sigma + it)| \leq (\log T)^{1-\sigma}/\log \log T, \ j \leq J \}.$$

Then by Lemma 3 and a Chebyshev inequality-type argument, we find that

$$\text{meas} \left( [T, 2T] \setminus B_2(T) \right) \ll T e^{-B_3 \log T / \log \log T},$$

where $B_3 = \frac{2}{3B_2} \log \frac{(3B_2)^{1-\sigma}}{C}$. Clearly $B_3$ will be positive if we choose $B_2$ sufficiently large. Assuming this is the case, we have

$$\frac{1}{T} \int_T^{2T} \exp \left[ i \sum_{j < J} \Re ((w_j - iz_j)Q_{j,Y}(\sigma + it)) \right] dt$$

$$= \frac{1}{T} \int_{B_2(T)} \exp \left[ i \sum_{j \leq J} \Re ((w_j - iz_j)Q_{j,Y}(\sigma + it)) \right] dt + O(e^{-B_3 \log T / \log \log T}).$$
Now for $|w_j|, |z_j| \leq b(\log T)^\sigma$ and $t \in B_2(T)$, we have

$$\left| \sum_{j \leq J} \Re \((w_j - iz_j)Q_{j,Y}(\sigma + it)\) \right| \leq \sqrt{2} b J \frac{\log T}{\log \log T}.$$ 

Taking $N = [e^2 \sqrt{2} b J \log T / \log \log T]$, we see that

$$\frac{1}{T} \int_{B_2(T)} \exp \left[ i \sum_{j \leq J} \Re \((w_j - iz_j)Q_{j,Y}(\sigma + it)\) \right] dt = \frac{1}{T} \int_{B_2(T)} \sum_{n \leq N} \frac{i^n}{n!} \left( \sum_{j \leq J} \Re \((w_j - iz_j)Q_{j,Y}(\sigma + it)\) \right)^n dt + O(e^{-N})$$

$$= \frac{1}{T} \int_{B_2(T)} \sum_{n \leq N} \frac{i^n}{n!} \left( \sum_{j \leq J} ((w_j - iz_j)Q_{j,Y}(\sigma + it) + (w_j + iz_j \overline{Q_{j,Y}(\sigma + it)}) \right)^n dt + O(e^{-N})$$

$$\sum_{n \leq N} \frac{i^n}{n!} \sum_{k \cdot \mathbf{e} = n} \left( \prod_{j \leq J} (w_j - iz_j)^{k_j} (w_j + iz_j)^{k'_j} \right) \cdot \left( \frac{1}{T} \int_{B_2(T)} \prod_{j \leq J} Q_{j,Y}(\sigma + it)^{k_j} \overline{Q_{j,Y}(\sigma + it)^{k'_j}} dt \right) + O(e^{-N}),$$

(4.3)

where $\mathbf{e} = (1, 1, \ldots, 1)$, $k \cdot \mathbf{e} = k_1 + \cdots + k_j$, $k' \cdot \mathbf{e} = k'_1 + \cdots + k'_j$, and

$$\left( \binom{n}{k, k'} \right) = \left( \begin{array}{c} n \\ k_1, \ldots, k_j, k'_1, \ldots, k'_j \end{array} \right).$$

We write the last integral as

$$\left\{ \int_{[T,2T]} - \int_{[T,2T] \setminus B_2(T)} \right\} \prod_{j \leq J} Q_{j,Y}(\sigma + it)^{k_j} \overline{Q_{j,Y}(\sigma + it)^{k'_j}} dt$$

By Lemma[4] if $0 < b < (6B_2 e^2 \sqrt{2} J)^{-1}$, then

$$\int_{[T,2T] \setminus B_2(T)} \prod_{j \leq J} Q_{j,Y}(\sigma + it)^{k_j} \overline{Q_{j,Y}(\sigma + it)^{k'_j}} dt$$

$$\leq \left( \text{meas}([T,2T] \setminus B_2(T)) \right)^{1/2} \left( \int_T^{2T} \prod_{j \leq J} |Q_{j,Y}(\sigma + it)|^{2(k_j + k'_j)} dt \right)^{1/2}$$

$$\ll T^{1/2} e^{-B_3 \log T / (2 \log \log T)} \prod_{j \leq J} \left( \int_T^{2T} |Q_{j,Y}(\sigma + it)|^{2n} dt \right)^{(k_j + k'_j)/(2n)}$$

$$\ll T e^{-B_3 \log T / (2 \log \log T)} \left( \frac{C n^{1-\sigma}}{(\log n)^\sigma} \right)^n.$$
The contribution of this integral to (4.3) is therefore
\[
\ll e^{-B_3 \log T/(2 \log T)} \sum_{n \leq N} \frac{1}{n!} \sum_{k+e+k' = n} \binom{n}{k, k'} (\sqrt{2}b(\log T)^\sigma)^n \left( \frac{Cn^{1-\sigma}}{(\log n)^\sigma} \right)^n 
\]
\[
= e^{-B_3 \log T/(2 \log T)} \sum_{n \leq N} \frac{1}{n!(2J)^n} (\sqrt{2}b(\log T)^\sigma)^n \left( \frac{Cn^{1-\sigma}}{(\log n)^\sigma} \right)^n 
\]
\[
\ll e^{-B_3 \log T/(2 \log T)} \sum_{n \leq N} \frac{1}{n!} \left( 2J \sqrt{2}C(\log T)^\sigma \frac{N^{1-\sigma}}{(\log N)^\sigma} \right)^n 
\]
\[
\ll e^{-B_3 \log T/(2 \log T)} \exp \left( 2J \sqrt{2}C(\log T)^\sigma \frac{N^{1-\sigma}}{(\log N)^\sigma} \right) 
\]
\[
\ll \exp \left( - \frac{B_3}{2} + 2J \sqrt{2}C(e^{2} \sqrt{2})^{1-\sigma}b^{2-\sigma} \frac{\log T}{\log \log T} \right) 
\]
\[
\leq \exp \left( - \frac{B_3 \log T}{4 \log \log T} \right), 
\]
provided that \( b \) also satisfies \( 0 < b < B_3^{1/(2-\sigma)}(8 \sqrt{2}JC(e^{2} \sqrt{2})^{1-\sigma})^{-1/(2-\sigma)} \). Thus,
\[
\frac{1}{T} \int_{T}^{2T} \exp \left[ i \sum_{j \leq J} \Re \left( (w_j - iz_j)Q_{j,Y}(\sigma + it) \right) \right] dt 
\]
\[
= \sum_{n \leq N} \frac{i^n}{n!} \sum_{k+e+k' = n} \binom{n}{k, k'} \prod_{j \leq J} (w_j - iz_j)^{k_j} (w_j + iz_j)^{k'_j} 
\]
\[
\cdot \left( \frac{1}{T} \int_{T}^{2T} \prod_{j \leq J} Q_{j,Y}(\sigma + it)^{k_j} \overline{Q_{j,Y}(\sigma + it)^{k'_j}} dt \right) 
\]
\[
+ O(e^{-N}) + O(e^{-(B_3/4) \log T/\log \log T}). 
\]
Note that if we take \( b > 0 \) sufficiently small, the first of the two \( O \)-terms will be the largest. Assuming this to be the case, we see by Lemmas 4 and 5 that the above is
\[
= \sum_{n \leq N} \frac{i^n}{n!} \sum_{k+e+k' = n} \binom{n}{k, k'} \prod_{j \leq J} (w_j - iz_j)^{k_j} (w_j + iz_j)^{k'_j} 
\]
\[
\cdot \Exp \left( \prod_{j \leq J} Q_{j,Y}(\sigma, X)^{k_j} \overline{Q_{j,Y}(\sigma, X)^{k'_j}} \right) + O(e^{-N}) 
\]
\[
= \sum_{n \leq N} \frac{i^n}{n!} \Exp \left( \left( \sum_{j \leq J} \Re \left( (w_j - iz_j)Q_{j,Y}(\sigma, X) \right) \right)^n \right) + O(e^{-N}) 
\]
\[
= \Exp \left( \exp \left[ i \sum_{j \leq J} \Re \left( (w_j - iz_j)Q_{j,Y}(\sigma, X) \right) \right] \right) + O(e^{-N}). 
\]
We have now shown that with appropriate choices of the parameters \( B_2, B_3, \) and \( b \)
\[
\hat{\Psi}_T(w, z) = \Exp \left( \exp \left[ i \sum_{j \leq J} \Re \left( (w_j - iz_j)Q_{j,Y}(\sigma, X) \right) \right] \right) + O(e^{-N}), 
\] (4.4)
where \( N = [e^2 \sqrt{2b} J \log T / \log \log T] \).

Now, by direct calculation,
\[
\mathbb{E}(\| \log c_j L_j(\sigma, X) - Q_j, Y(\sigma, X) \|^2) = \sum_{p^n \geq Y} \frac{|a_j(p^n)|^2}{p^{2n\sigma}} \ll Y^{(1-2\sigma)+\epsilon} \ll (\log T)^{(1-2\sigma)B_2+\epsilon}.
\]

From this and Chebyshev’s inequality we easily see that
\[
\log c_j L_j(\sigma, X) - Q_j, Y(\sigma, X) = O((\log T)^{-B_1}),
\]
except for a set of \( X \in \mathbb{T}^\infty \) of measure \( O((\log T)^{(1-2\sigma)B_2+2B_1+\epsilon}) \). Similarly,
\[
\log \frac{c_j}{c_j} L_j(\sigma, X) = Q_j, Y(\sigma, X) + O((\log T)^{-B_1}) \quad (j < J)
\]
holds except for a set of \( X \in \mathbb{T}^\infty \) of measure \( O((\log T)^{(1-2\sigma)B_2+2B_1+\epsilon}) \). Thus,
\[
\mathbb{E} \left( \exp \left[ i \sum_{j \leq J} \Re((w_j - iz_j)Q_j, Y(\sigma, X)) \right] \right)
= \mathbb{E} \left( \exp \left[ i \sum_{j < J} \Re((w_j - iz_j) \log \frac{c_j}{c_j} L_j(\sigma, X)) + i\Re((w_j - iz_j) \log c_j L_j(\sigma, X)) \right] \right)
+ O((\log T)^{(1-2\sigma)B_2+2B_1+\epsilon}) + O((\log T)^{-B_1})
= \tilde{\Psi}(w, z) + O((\log T)^{(1-2\sigma)B_2+2B_1+\epsilon}) + O((\log T)^{-B_1}).
\]

Choosing first \( B_1 \) and then \( B_2 \) sufficiently large as a function of \( B_1 \), we can ensure that for any given \( A > 1 \), the last line equals
\[
\tilde{\Psi}(w, z) + O((\log T)^{-A}).
\]

Combining this with (4.4), we see that
\[
\tilde{\Psi}_T(w, z) = \tilde{\Psi}(w, z) + O((\log T)^{-A}).
\]

This completes the proof of the Lemma. \( \square \)

**Lemma 10.** There is a positive constant \( C_\sigma \) such that
\[
|\tilde{\Psi}(y, 0, \ldots, 0)| \leq \exp \left( - \frac{C_\sigma y^{1/\sigma}}{4 \log y} \right)
\]
as \( y \to \infty \).

**Proof.**
\[
\tilde{\Psi}(y, 0, \ldots, 0) = \mathbb{E} \left( \exp \left[ iy \log \frac{c_1 L_1}{c_j L_j}(\sigma, X) \right] \right)
= \mathbb{E} \left( \exp \left[ iy \left( \log \frac{c_1}{c_j} + \Re \sum_{p^n} \left( \frac{a_1(p^n) - a_j(p^n)}{p^n \sigma} \right) X(p)^n \right) \right] \right)
= \left\{ \frac{c_1}{c_j} \right\}^iy \prod_p \mathbb{E} \left( \exp \left[ iy \left( \Re \sum_n \left( \frac{a_1(p^n) - a_j(p^n)}{p^n \sigma} \right) X(p)^n \right) \right] \right).
\]
It is easy to see that for each \( p \)

\[
\left| \mathbb{E} \left( \exp \left( iy \left( \Re \sum_n \frac{(a_1(p^n) - a_J(p^n))X(p)^n}{p^{n\sigma}} \right) \right) \right) \right| \leq 1.
\]

We next show that there are a number of \( p \) for which

\[
\left| \mathbb{E} \left( \exp \left( iy \left( \Re \sum_n \frac{(a_1(p^n) - a_J(p^n))X(p)^n}{p^{n\sigma}} \right) \right) \right) \right| \leq e^{-1/3}.
\]

Since \( a_j(p) \) is a real number for every prime \( p \) and for each \( j \leq J \), we have for \( y \leq p^{2\sigma}/2 \) that

\[
\mathbb{E} \left( \exp \left( iy \left( \Re \sum_n \frac{(a_1(p^n) - a_J(p^n))X(p)^n}{p^{n\sigma}} \right) \right) \right)
= \mathbb{E} \left( \exp \left( iy \left( \frac{a_1(p) - a_J(p)}{p^\sigma} \Re X(p) \right) + O \left( \frac{y}{p^{2\sigma}} \right) \right) \right)
= \mathbb{E} \left( \exp \left( iy \left( \frac{a_1(p) - a_J(p)}{p^\sigma} \Re X(p) \right) \right) + O \left( \frac{y}{p^{2\sigma}} \right) \right)
= J_0 \left( \frac{a_1(p) - a_J(p)}{p^\sigma} \right) + O \left( \frac{y}{p^{2\sigma}} \right),
\]

where \( J_0 \) is the Bessel function of order 0.

Recall that \( a_j(p) = 2\Re \chi_j(p) \) if \( p \) splits and \( p | p \). Since \( \chi_1 \neq \chi_J \) and \( \chi_1 \neq \overline{\chi_J} \), there is an ideal class \( C \) such that \( \chi_1(p) \neq \chi_J(p) \) and \( \chi_1(p) \neq \overline{\chi_J(p)} \) for all \( p \in C \). Since \( |\chi_j(p)| = 1 \), we see that

\[
a_1(p) - a_J(p) = 2\Re (\chi_1(p) - \chi_J(p)) = a \neq 0
\]

for all \( p \in C \) and \( p | p \). Using the crude inequality

\[
|J_0(x)| \leq e^{-1/2} \quad (x \geq 2),
\]

we find that

\[
\left| J_0 \left( \frac{a_1(p) - a_J(p)}{p^\sigma} \right) \right| + O \left( \frac{y}{p^{2\sigma}} \right) \leq e^{-1/2} + O \left( \frac{y}{p^{2\sigma}} \right) \leq e^{-1/3},
\]

provided that \( p \), with \( p \in C \) and \( p | p \), satisfies the conditions

\[
\frac{ay}{p^\sigma} \geq 2 \quad \text{and} \quad \frac{y}{p^{2\sigma}} \leq c
\]

for some small constant \( c > 0 \). Therefore

\[
|\hat{\Psi}(y, 0, \ldots, 0)| \leq \prod_{p \in \mathcal{P}} e^{-1/3},
\]

where \( \mathcal{P} \) is the set of all \( p \) with \( p \in C \), \( p | p \), and

\[
(y/c)^{1/(2\sigma)} \leq p \leq (ay/2)^{1/\sigma}.
\]
By Lemma 2.6 of [7], for example, there are
\[
\frac{1}{h(D)} \frac{(ay/2)^{1/\sigma}}{\log(ay/2)^{1/\sigma}} (1 + o(1)) \sim C_\sigma \frac{y^{1/\sigma}}{\log y}
\]
such \( p \) as \( y \to \infty \). Thus,
\[
|\hat{\Psi}(y, 0, \ldots, 0)| \leq \exp \left( -\frac{C_\sigma y^{1/\sigma}}{4 \log y} \right)
\]
as \( y \to \infty \).

\[
\square
\]

4.2 Completion of the proof of Theorem 2

We can now prove Theorem 2. For a sufficiently large constant \( A > 0 \) the set
\[
\{ t \in [T, 2T] : \frac{L(\sigma + it)}{\sigma} \notin [-A\Delta, A\Delta] \}
\]
has small measure by Lemma 7. Similarly, by Lemma 8
\[
\mathbb{P}(L(\sigma, X) \notin [-A\Delta, A\Delta]^{2J})
\]
is small. Thus it is enough to consider rectangular regions \( R \) contained in \([-A\Delta, A\Delta]^{2J}\).

Let \( \eta = b_1(\log T)^\sigma \), where \( b_1 \) is a positive constant such that \( b = 2\pi b_1 \) satisfies Lemma 9, and define
\[
G(u) = \frac{2u}{\pi} + 2(1 - u)u \cot(\pi u)
\]
for \( 0 \leq u \leq 1 \). By Lemma 4.1 of [10], the characteristic function of the interval \([\alpha, \beta]\) is
\[
1_{[\alpha, \beta]}(x) = 3 \int_0^\eta G(u/\eta)e^{2\pi iux} f_{\alpha,\beta}(u) du
\]
+ \( O \left\{ \left( \frac{\sin \pi \eta(x - \alpha)}{\pi \eta(x - \alpha)} \right)^2 + \left( \frac{\sin \pi \eta(x - \beta)}{\pi \eta(x - \beta)} \right)^2 \right\} \),
where \( f_{\alpha,\beta}(u) = (e^{-2\pi i \alpha u} - e^{-2\pi i \beta u})/2 \). Thus the characteristic function of the rectangular region \( R = \prod_{j \leq J} [\alpha_j, \beta_j] \times \prod_{j \leq J} [\alpha_j', \beta_j'] \) is
\[
1_R(u, v) = W_{\eta, R}(u, v) + \sum_{j \leq J} O \left\{ \left( \frac{\sin \pi \eta(u_j - \alpha_j)}{\pi \eta(u_j - \alpha_j)} \right)^2 + \left( \frac{\sin \pi \eta(u_j - \beta_j)}{\pi \eta(u_j - \beta_j)} \right)^2 \right\}
\]
+ \( O \left\{ \left( \frac{\sin \pi \eta(v_j - \alpha'_j)}{\pi \eta(v_j - \alpha'_j)} \right)^2 + \left( \frac{\sin \pi \eta(v_j - \beta'_j)}{\pi \eta(v_j - \beta'_j)} \right)^2 \right\} \),

where
\[
W_{\eta, R}(u, v) := \prod_{j \leq J} \left( 3 \int_0^\eta G(u/\eta)e^{2\pi i u_j} f_{\alpha_j,\beta_j}(u) du \right) \times \prod_{j \leq J} \left( 3 \int_0^\eta G(u/\eta)e^{2\pi i u_j} f_{\alpha'_j,\beta'_j}(u) du \right). \tag{4.5}
\]

(4.5)
Assuming that \( R = \prod_{j \leq J}[\alpha_j, \beta_j] \times \prod_{j \leq J}[\alpha'_j, \beta'_j] \subset [-A, L, A, L]^2J \), we see that
\[
\Psi_T(R) = \frac{1}{T} \int_T^{2T} 1_R(L(\sigma + it)) dt = \frac{1}{T} \int_T^{2T} W_{\eta, R}(L(\sigma + it)) dt + \epsilon_1
\]
and
\[
\Psi(R) = \mathbb{E}(W_{\eta, R}(L(\sigma, X))) + \epsilon_2,
\]
where \( \epsilon_1 \) and \( \epsilon_2 \) are the error terms arising from the \( O \)-terms in (4.3).

To estimate these error terms we begin with the identity
\[
\sin^2(\frac{\pi x}{y}) = \Re \left[ \frac{1}{2\pi^2} \int_0^{2\pi} (2\pi y - y) e^{ixy} dy \right].
\]
A typical term in \( \epsilon_1 \) is
\[
\frac{1}{T} \int_T^{2T} \sin^2(\frac{\pi(\log |c_1L_1/c_1L_2(s)| - \alpha_1)}{\log |c_1L_1/c_1L_2(s)| - \alpha_1}) dt
= \Re \left( \frac{1}{2\pi^2} \int_0^{2\pi} (2\pi y - y) \left( \frac{1}{T} \int_T^{2T} e^{iy(\log |c_1L_1/c_1L_2(s)| - \alpha_1)} dt dy \right) \right)
= \Re \left( \frac{1}{2\pi^2} \int_0^{2\pi} (2\pi y - y)e^{-iy\alpha_1} \tilde{\Psi}(y, 0, \ldots, 0) dy \right).
\]
By Lemma 9 this equals
\[
\Re \left( \frac{1}{2\pi^2} \int_0^{2\pi} (2\pi y - y)e^{-iy\alpha_1} \tilde{\Psi}(y, 0, \ldots, 0) dy \right) + O((\log T)^{-A}).
\]
Note that this is also a typical term in \( \epsilon_2 \). By Lemma 10 this is
\[
\ll \frac{1}{\eta^2} \int_0^{2\pi} (2\pi y - y) dy + \frac{1}{\eta^2} \int_2^{2\pi} (2\pi y - y) \exp \left( -\frac{C_\sigma y^{1/\sigma}}{4 \log y} \right) dy + O((\log T)^{-A})
\ll \frac{1}{\eta}.
\]
All the other terms in \( \epsilon_1 \) and \( \epsilon_2 \) are estimated similarly. Thus, it is enough to show that
\[
\frac{1}{T} \int_T^{2T} W_{\eta, R}(L(\sigma + it)) dt = \mathbb{E}(W_{\eta, R}(L(\sigma, X))) + O(1/\eta).
\]

Using \( \Im z = (z - \bar{z})/2i \) to rewrite the imaginary parts in (4.6), we see that
\[
W_{\eta, R}(u, v) = (-4)^{-J} \sum_{\epsilon_j, \epsilon'_j = \pm 1} \left\{ \prod_{j \leq J} \left( \int_0^{\eta} G(w_j/\eta) \epsilon_j e^{\epsilon_j 2\pi i w_j u_j} f_{\epsilon_j, \alpha_j, \epsilon'_j, \beta_j}(w_j) dw_j/w_j \right) \right\}
\times \prod_{j \leq J} \left( \int_0^{\eta} G(z_j/\eta) \epsilon'_j e^{\epsilon'_j 2\pi i z_j v_j} f_{\epsilon'_j, \alpha'_j, \epsilon'_j, \beta'_j}(z_j) dz_j/z_j \right)
= (-4)^{-J} \sum_{\epsilon_j, \epsilon'_j = \pm 1} \int_{[0, \eta]^{2J}} \prod_{j \leq J} \left[ G(w_j/\eta)G(z_j/\eta) \epsilon_j \epsilon'_j f_{\epsilon_j, \alpha_j, \epsilon'_j, \beta_j}(w_j) f_{\epsilon'_j, \alpha'_j, \epsilon'_j, \beta'_j}(z_j) \right]
\times \exp \left( \sum_{j \leq J} \epsilon_j 2\pi i w_j u_j + \sum_{j \leq J} \epsilon'_j 2\pi i z_j v_j \right) dw_1/w_1 \cdots dw_J/w_J \frac{dz_1}{z_1} \cdots \frac{dz_J}{z_J}.
\]
Thus
\[
\frac{1}{T} \int_T^{2T} W_{\eta,R}(L(\sigma + it)) \, dt
\]
\[= (-4)^{-J} \sum_{\epsilon_j, \epsilon_j' = \pm 1} \int_{[0, \eta]^{2J}} \prod_{j \leq J} (G(w_j/\eta)G(z_j/\eta)\epsilon_j \epsilon_j' f_{\epsilon_j \alpha_j, \epsilon_j' \beta_j}(w_j)f_{\epsilon_j' \alpha_j' \epsilon_j' \beta_j'}(z_j)) \hat{\Psi}_T(2\pi \epsilon_1 w_1, \ldots, 2\pi \epsilon_J w_J, 2\pi \epsilon_1 z_1, \ldots, 2\pi \epsilon_J z_J) \frac{dw_1}{w_1} \ldots \frac{dw_J}{w_J} \frac{dz_1}{z_1} \ldots \frac{dz_J}{z_J}
\]
and
\[
\mathbb{E}(W_{\eta,R}(L(\sigma, X)))
\]
\[= (-4)^{-J} \sum_{\epsilon_j, \epsilon_j' = \pm 1} \int_{[0, \eta]^{2J}} \prod_{j \leq J} (G(w_j/\eta)G(z_j/\eta)\epsilon_j \epsilon_j' f_{\epsilon_j \alpha_j, \epsilon_j' \beta_j}(w_j)f_{\epsilon_j' \alpha_j' \epsilon_j' \beta_j'}(z_j)) \hat{\Psi}(2\pi \epsilon_1 w_1, \ldots, 2\pi \epsilon_J w_J, 2\pi \epsilon_1 z_1, \ldots, 2\pi \epsilon_J z_J) \frac{dw_1}{w_1} \ldots \frac{dw_J}{w_J} \frac{dz_1}{z_1} \ldots \frac{dz_J}{z_J}.
\]
Since
\[
\hat{\Psi}_T(w, z) - \hat{\Psi}(w, z) = O((\log T)^{-A})
\]
for all \(|w_j|, |z_j| \leq \eta\) by Lemma 9 we have
\[
\frac{1}{T} \int_T^{2T} W_{\eta,R}(L(\sigma + it)) \, dt - \mathbb{E}(W_{\eta,R}(L(\sigma, X)))
\]
\[\ll \sum_{\epsilon_j, \epsilon_j' = \pm 1} \int_{[0, \eta]^{2J}} \prod_{j \leq J} |G(w_j/\eta)G(z_j/\eta)\epsilon_j \epsilon_j' f_{\epsilon_j \alpha_j, \epsilon_j' \beta_j}(w_j)f_{\epsilon_j' \alpha_j' \epsilon_j' \beta_j'}(z_j)| \frac{1}{(\log T)^A} \frac{dw_1}{w_1} \ldots \frac{dw_J}{w_J} \frac{dz_1}{z_1} \ldots \frac{dz_J}{z_J}
\]
\[\ll \frac{\eta^{2J} \log^{2J}}{(\log T)^A}
\]
\[\ll 1/\eta,
\]
provided we choose \(A > 0\) sufficiently large. Here, we have used the inequalities
\[
0 \leq G(x) \leq 2/\pi
\]
for \(x \in [0, 1]\) and
\[
f_{\alpha, \beta}(u) \ll |(\beta - \alpha) u| \ll L|u|.
\]
This completes the proof of Theorem 2.

5 Proof of Theorem 1

Let \(J \geq 2\) and for each \(j \leq J\) define
\[
S_j(T) = \{ t \in [T, 2T] : |c_j L_j(\sigma + it)| \geq |c_i L_i(\sigma + it)| \text{ for all } i \leq J \}.
\]
By the inclusion-exclusion principle we see that

\[
\frac{1}{T} \int_T^{2T} \log |E(\sigma + it, Q)| \, dt = \sum_{1 \leq j \leq J} \frac{1}{T} \int_{S_j(T)} \log |E(\sigma + it, Q)| \, dt
\]

\[
- \sum_{1 \leq j_1 < j_2 \leq J} \frac{1}{T} \int_{S_{j_1}(T) \cap S_{j_2}(T)} \log |E(\sigma + it, Q)| \, dt 
\]

\[
+ \sum_{1 \leq j_1 < j_2 < j_3 \leq J} \frac{1}{T} \int_{S_{j_1}(T) \cap S_{j_2}(T) \cap S_{j_3}(T)} \log |E(\sigma + it, Q)| \, dt
\]

\[
+ \cdots
\]

\[
+ (-1)^{j-1} \frac{1}{T} \int_{S_1(T) \cap S_2(T) \cap \ldots \cap S_J(T)} \log |E(\sigma + it, Q)| \, dt.
\]

We first show that for \(1 \leq i \neq j \leq J\) we have

\[
\int_{S_i(T) \cap S_j(T)} \log |E(\sigma + it, Q)| \, dt \ll T \frac{\mathcal{L}^2}{(\log T)^\sigma}.
\]

Without loss of generality, we may assume that \(i = 1\) and \(j = J\). By Hölder’s inequality and Lemma 6

\[
\int_{S_1(T) \cap S_J(T)} \log |E(\sigma + it, Q)| \, dt 
\]

\[
\ll \left( \int_T^{2T} |\log |E(\sigma + it, Q)||^{2k} \, dt \right)^{\frac{1}{2k}} \times \text{meas} (S_1(T) \cap S_J(T))^{1 - \frac{1}{2k}} \quad (5.1)
\]

\[
\ll k^2 T^{\frac{1}{2k}} \left( \text{meas} (S_1(T) \cap S_J(T)) \right)^{1 - \frac{1}{2k}}. \quad (5.2)
\]

Now

\[
S_1(T) \cap S_J(T) \subseteq \{ t \in [T, 2T] : |c_1 L_1(s)| = |c_J L_J(s)| \}
\]

\[
= \{ t \in [T, 2T] : \log |c_1 L_1(s)/c_J L_J(s)| = 0 \}
\]

\[
= \{ t \in [T, 2T] : L(s) \in \{0\} \times \mathbb{R}^{2J-1} \}.
\]

By Lemma 2 and since \(\Psi\) is absolutely continuous, the measure of this set equals

\[
T \Psi(\{0\} \times \mathbb{R}^{2J-1}) + O(T (\log T)^{-\sigma}) = O(T (\log T)^{-\sigma}).
\]

Hence

\[
\text{meas} (S_1(T) \cap S_J(T)) \ll T (\log T)^{-\sigma}.
\]

Thus, by (5.2) we see that

\[
\int_{S_1(T) \cap S_J(T)} |\log |E(\sigma + it, Q)|| \, dt \ll k^2 T \left( \frac{1}{(\log T)^\sigma} \right)^{1 - \frac{1}{2k}}.
\]

Taking \(k = \mathcal{L}\), we obtain

\[
\int_{S_1(T) \cap S_J(T)} |\log |E(\sigma + it, Q)|| \, dt \ll T \frac{\mathcal{L}^2}{(\log T)^\sigma},
\]
as claimed.

Clearly this bound applies to all the integrals on the right-hand side of (5.1) that involve two or more $S_i(T)$’s. Thus we see that

\[
\frac{1}{T} \int_T^{2T} \log |E(\sigma + it, Q)| dt = \sum_{j \leq J} \frac{1}{T} \int_{S_j(T)} \log |E(\sigma + it, Q)| dt + O\left( \frac{2^J L^2 }{(\log T)^\sigma} \right).
\]

Our next task is to estimate the individual terms here and, without loss of generality, we consider only the case $j = J$. We write

\[
\frac{1}{T} \int_{S_J(T)} \log |E(\sigma + it, Q)| dt = \frac{1}{T} \int_{S_J(T)} \log |c_J L_J(\sigma + it)| dt + \frac{1}{T} \int_{S_J(T)} \log \left| 1 + \sum_{j < J} \frac{c_j L_j}{c_J L_J}(\sigma + it) \right| dt,
\]

and calculate the integrals on the right in the next two subsections.

5.1 The first integral in (5.3)

Recall that

\[ S_J(T) = \{ t \in [T, 2T] : |c_J L_J(\sigma + it)| \geq |c_j L_j(\sigma + it)|, j < J \}. \]

and

\[ I_T = (-A \mathcal{L}, A \mathcal{L}). \]

We also define

\[ I_T^- = (-A \mathcal{L}, 0]. \]

By Lemma 6 we may restrict the range of integration to the set

\[ S_{J,1}(T) = \left\{ t \in S_J(T) : \log |c_J L_J(\sigma + it)| \in I_T, \arg c_J L_J(\sigma + it) \in I_T, \log \left| \frac{c_j L_j}{c_J L_J}(\sigma + it) \right| \in I_T^-, \arg c_J L_J(\sigma + it) - \arg c_J L_J(\sigma + it) \in I_T, j < J \right\} \]

at the cost of an error term of size $O(T(\log T)^{-B})$. That is,

\[
\frac{1}{T} \int_{S_J(T)} \log |c_J L_J(\sigma + it)| dt = \frac{1}{T} \int_{S_{J,1}(T)} \log |c_J L_J(\sigma + it)| dt + O((\log T)^{-B}). \quad (5.4)
\]
Letting \( u = (u_1, \ldots, u_J) \) and \( v = (v_1, \ldots, v_J) \), we see that

\[
\frac{1}{T} \int_{S_{J,1}(T)} \log |c_J L_J(\sigma + it)| \, dt = \int_{(I_T^J)^{-1} \times I_T \times (I_T)^J} u_J \, d\Psi_T(u, v)
\]

\[
= \int_{(I_T^J)^{-1} \times I_T \times (I_T)^J} \left( \int_{-A}^{u_J} du - A \mathcal{L} \right) d\Psi_T(u, v)
\]

\[
= \int_{(I_T^J)^{-1} \times I_T \times (I_T)^J} \left( \int_{-A}^{u_J} du \right) d\Psi_T(u, v) - A \mathcal{L} \Psi_T((I_T^J)^{-1} \times I_T \times (I_T)^J)
\]

\[
= \int_{I_T} \Psi_T((I_T^J)^{-1} \times (u, A \mathcal{L}) \times (I_T)^J) du - A \mathcal{L} \Psi_T((I_T^J)^{-1} \times I_T \times (I_T)^J).
\]

Thus, by Lemma 2, the last line equals

\[
\int_{I_T} \Psi((I_T^J)^{-1} \times (u, A \mathcal{L}) \times (I_T)^J) du - A \mathcal{L} \Psi((I_T^J)^{-1} \times I_T \times (I_T)^J) + O\left( \frac{\mathcal{L}}{(\log T)^\sigma} \right).
\]

Now simply reverse all the steps leading to the last line of (5.5) to see that

\[
\frac{1}{T} \int_{S_{J,1}(T)} \log |c_J L_J(\sigma + it)| \, dt = \int_{(I_T^J)^{-1} \times I_T \times (I_T)^J} u_J \, d\Psi_T(u, v) + O\left( \frac{\mathcal{L}}{(\log T)^\sigma} \right).
\]

Therefore, by (5.4)

\[
\frac{1}{T} \int_{S_J(T)} \log |c_J L_J(\sigma + it)| \, dt = \int_{(I_T^J)^{-1} \times I_T \times (I_T)^J} u_J \, d\Psi_T(u, v) + O\left( \frac{\mathcal{L}}{(\log T)^\sigma} \right). \tag{5.6}
\]

5.2 The second integral in (5.3)

As with the first integral in (5.3), we begin by limiting the range of integration. Let

\[
S_{J,2}(T) := \{ t \in [T, 2T] : \log |c_J L_J(\sigma + it)| \in I_T, \arg c_J L_J(\sigma + it) \in I_T, \log |c_J L_j / c_J L_J(\sigma + it)| \in I_T, \arg c_J L_j(\sigma + it) - c_J L_J(\sigma + it) \in I_T \text{ for all } j < J \}
\]

\[
\subseteq S_J(T) = \{ t \in [T, 2T] : |c_J L_J(\sigma + it)| \geq |c_J L_j(\sigma + it)|, \, j < J \}.
\]

Then

\[
\frac{1}{T} \int_{S_J(T)} \log \left( 1 + \sum_{j < J} c_J L_j(\sigma + it) \right) dt = \frac{1}{T} \int_{S_{J,2}(T)} \log \left( 1 + \sum_{j < J} c_J L_j(\sigma + it) \right) dt + O((\log T)^{-B})
\]

by Lemmas 5 and 7.
Let \( u = (u_1, \ldots, u_J) \) and \( v = (v_1, \ldots, v_J) \) be two vectors in \( \mathbb{R}^J \). We would like to show that
\[
\frac{1}{T} \int_{S_{J,2}(T)} \log \left| 1 + \sum_{j<J} \frac{c_j L_j}{L_j} (\sigma + it) \right| dt = \int_{(I_T)^{J-1} \times I_T \times (I_T)^J} \log \left| 1 + \sum_{j\leq J-1} e^{u_j+iv_j} \right| d\Psi_T(u, v),
\]
but this is not straightforward because the integrand has logarithmic singularities. To handle this we split the integral into small pieces by dividing the set \( (I_T)^{J-1} \times I_T \times (I_T)^J \) into at most \( O(\mathcal{L}) \) rectangular regions.

Let \( \delta = (\log T)^{-c_0} \) with a small constant \( c_0 > 0 \). Let \( m_{J-2} = (m_1, \ldots, m_{J-2}) \) and \( n_{J-2} = (n_1, \ldots, n_{J-2}) \) be two vectors in \( \mathbb{Z}^{J-2} \) and \( u_{J-2} = (u_1, \ldots, u_{J-2}) \) and \( v_{J-2} = (v_1, \ldots, v_{J-2}) \) be projections of the vectors \( u = (u_1, \ldots, u_J) \) and \( v = (v_1, \ldots, v_J) \) into \( \mathbb{R}^J \). Define a \((2J-4)\)-dimensional rectangular region
\[
\text{Rect}(m, n) := \left\{ (u_{J-2}, v_{J-2}) \in (I_T)^{J-2} \times (I_T)^{J-2} : m_j \delta < u_j \leq (m_j + 1) \delta, \right. \\
\left. n_j \delta < v_j \leq (n_j + 1) \delta \text{ for all } j \leq J-2 \right\}
\]
for \( 0 \leq m_j \leq 1/\delta - 1, |n_j| \leq A\mathcal{L}/\delta, \) and \( j \leq J-2 \). Then
\[
(I_T)^{J-2} \times (I_T)^{J-2} = \bigcup_{m, n} \text{Rect}(m, n).
\]
If \( m < 1/\delta \), the set
\[
\left\{ e^{u+iv} \in \mathbb{C} : m \delta < u \leq (m + 1) \delta, \ n \delta < v \leq (n + 1) \delta \right\}
\]
has diameter \( \leq \delta \). The set
\[
\left\{ 1 + \sum_{j\leq J-2} e^{u_j+iv_j} \in \mathbb{C} : (u_{J-2}, v_{J-2}) \in \text{Rect}(m, n) \right\}
\]
is therefore contained in a circle in \( \mathbb{C} \) of radius at most \( J\delta \). Let \( s_0 \) be the center of this circle. Then since each \( u_j \leq 0 \), we have \( |s_0| \leq J - 1 \). We consider four cases depending on the size of \( |s_0| \).

**Case 1:** \( |s_0| \leq 10J\delta \). Define
\[
R_{\text{main}}(m, n) = \{(u, v) \in (I_T)^{J-1} \times I_T \times (I_T)^J : (u_{J-2}, v_{J-2}) \in \text{Rect}(m, n), \ 12J\delta < e^{u_{J-1}} \leq 1 \}
\]
and
\[
R_{\text{error}}(m, n) = \{(u, v) \in (I_T)^{J-1} \times I_T \times (I_T)^J : (u_{J-2}, v_{J-2}) \in \text{Rect}(m, n), \ e^{-A\mathcal{L}} < e^{u_{J-1}} \leq 12J\delta \},
\]
so that
\[
(I_T)^{J-1} \times I_T \times (I_T)^J = \bigcup_{m, n} \left( R_{\text{main}}(m, n) \bigcup R_{\text{error}}(m, n) \right).
\]
Then
\[
\left| 1 + \sum_{j\leq J-1} e^{u_j+iv_j} \right| \geq e^{u_{J-1}} - \left| 1 + \sum_{j\leq J-2} e^{u_j+iv_j} \right| \geq J\delta \quad (5.7)
\]
for \((u, v) \in R_{main}(m, n)\), and
\[
1 + \sum_{j \leq J-1} e^{uj+ivj} \leq 1 + \sum_{j \leq J-2} e^{uj+ivj} + e^{uj-1} \leq 23J\delta
\]
for \((u, v) \in R_{error}(m, n)\). Observe that
\[
\log \left| 1 + \sum_{j \leq J-1} e^{uj+ivj} \right|
\]
may have singularities on \(R_{error}(m, n)\), but not on \(R_{main}(m, n)\) because of (5.7).

**Case 2:** \(10J\delta < |s_0| \leq 1 - 2J\delta\). In this case the inequality
\[
\left| 1 + \sum_{j \leq J-1} e^{uj+ivj} \right| \geq J\delta
\]
holds if \(|e^{uj-1+ivj-1} + s_0| \geq 2J\delta\). We define
\[
R_{main}(m, n) = R_1(m, n) \cup R_2(m, n) \cup \left( \bigcup_{l \in \mathbb{Z}} R_{3,l}(m, n) \right),
\]
where
\[
R_1(m, n) = \{(u, v) \in (I_T)^{J-1} \times I_T \times (I_T)^{J} : (u_{J-2}, v_{J-2}) \in \text{Rect}(m, n), e^{-A/2} < e^{uj-1} \leq |s_0| - 2J\delta\},
\]
\[
R_2(m, n) = \{(u, v) \in (I_T)^{J-1} \times I_T \times (I_T)^{J} : (u_{J-2}, v_{J-2}) \in \text{Rect}(m, n), |s_0| + 2J\delta < e^{uj-1} \leq 1\},
\]
and
\[
R_{3,l}(m, n) = \{(u, v) \in (I_T)^{J-1} \times I_T \times (I_T)^{J} : (u_{J-2}, v_{J-2}) \in \text{Rect}(m, n), |s_0| - 2J\delta < e^{uj-1} \leq |s_0| + 2J\delta,
\]
\[
- (\pi - \arcsin(2J\delta/|s_0|)) < v_{J-1} - \arg s_0 - 2\pi l \leq \pi - \arcsin(2J\delta/|s_0|)\}.
\]

Also define
\[
R_{error}(m, n) = \bigcup_{l \in \mathbb{Z}} \left( \bigcup_{|l| \leq A/2\pi} \{(u, v) \in (I_T)^{J-1} \times I_T \times (I_T)^{J} : (u_{J-2}, v_{J-2}) \in \text{Rect}(m, n),
\]
\[
|s_0| - 2J\delta < e^{uj-1} \leq |s_0| + 2J\delta,
\]
\[
- \arcsin(2J\delta/|s_0|) < v_{J-1} - \arg s_0 - (2l + 1)\pi \leq \arcsin(2J\delta/|s_0|)\}.
\]

Then we see that the sets \(R_{main}(m, n)\) and \(R_{error}(m, n)\) are each unions of roughly \(A/\pi\) rectangular regions, and that
\[
\left| 1 + \sum_{j \leq J-1} e^{uj+ivj} \right| \geq J\delta
\]
for \((u, v) \in R_{main}(m, n)\), and
\[
\left| 1 + \sum_{j \leq J-1} e^{uj+ivj} \right| \leq \left| 1 + \sum_{j \leq J-2} e^{uj+ivj} - s_0 \right| + \left| s_0 + e^{uj-1+ivj-1} \right| \leq 10J\delta
\]
for \((u, v) \in R_{\text{error}}(m, n)\).

**Case 3:** \(1 - 2J\delta < |s_0| \leq 1 + 2J\delta\). Similarly to Case 2, we define

\[
R_{\text{main}}(m, n) = R_1(m, n) \bigcup \left( \bigcup_{l \in \mathbb{Z}} R_{4,l}(m, n) \right),
\]

where

\[
R_{4,l}(m, n) = \{ (u, v) \in (I_T)^{J-1} \times I_T \times (I_T)^J : (u_{J-2}, v_{J-2}) \in \text{Rect}(m, n), |s_0| - 2J\delta < e^{u_{J-1}}, (\pi - \arcsin(2J\delta/|s_0|)) < v_{J-1} - \arg s_0 - 2\pi l \leq \pi - \arcsin(2J\delta/|s_0|) \}.
\]

Also define

\[
R_{\text{error}}(m, n) = \bigcup_{l \in \mathbb{Z}} \bigcup_{|l| \leq AJ/2\pi} \{ (u, v) \in (I_T)^{J-1} \times I_T \times (I_T)^J : (u_{J-2}, v_{J-2}) \in \text{Rect}(m, n), \quad |s_0| - 2J\delta < e^{u_{J-1}}, (\pi - \arcsin(2J\delta/|s_0|)) < v_{J-1} - \arg s_0 - (2l + 1)\pi \leq \pi - \arcsin(2J\delta/|s_0|) \}.
\]

Then we see that the sets \(R_{\text{main}}(m, n)\) and \(R_{\text{error}}(m, n)\) are each unions of roughly \(AJ/\pi\) rectangular regions, and that

\[
\left| 1 + \sum_{j \leq J-1} e^{u_j + iv_j} \right| \geq J\delta
\]

for \((u, v) \in R_{\text{main}}(m, n)\), and

\[
\left| 1 + \sum_{j \leq J-1} e^{u_j + iv_j} \right| \leq \left| 1 + \sum_{j \leq J-2} e^{u_j + iv_j} - s_0 \right| + \left| s_0 + e^{u_{J-1} + iv_{J-1}} \right| \leq 10J\delta
\]

for \((u, v) \in R_{\text{error}}(m, n)\).

**Case 4:** \(1 + 2J\delta < |s_0| \leq J - 1\). We define

\[
R_{\text{main}}(m, n) = \{ (u, v) \in (I_T)^{J-1} \times I_T \times (I_T)^J : (u_{J-2}, v_{J-2}) \in \text{Rect}(m, n) \}
\]

and

\[
R_{\text{error}}(m, n) = \emptyset.
\]

Then

\[
\left| 1 + \sum_{j \leq J-1} e^{u_j + iv_j} \right| \geq \left| 1 + \sum_{j \leq J-1} e^{u_j + iv_j} \right| - e^{u_{J-1}} \geq (|s_0| - J\delta) - 1 \geq J\delta
\]

for \((u, v) \in R_{\text{main}}(m, n)\).

Summarizing, we note that in each case, we have

\[
\left| 1 + \sum_{j \leq J-1} e^{u_j + iv_j} \right| \geq J\delta
\]  \hspace{1cm} (5.8)
if \((u, v) \in R_{\text{main}}(m, n)\),

\[
1 + \sum_{j \leq J-1} e^{u_j + iv_j} \leq 23J\delta
\]  

(5.9)

if \((u, v) \in R_{\text{error}}(m, n)\), and

\[
(I_T)^{J-1} \times I_T \times (I_T)^J = \bigcup_{m,n} \left( R_{\text{main}}(m, n) \cup R_{\text{error}}(m, n) \right). 
\]  

(5.10)

We also note that for each \(u_j\) with \(j < J\) we have

\[
e^{u_j} \leq 1,
\]  

(5.11)

since \(u \in (I_T)^{J-1} \times I_T\) and \(v \in (I_T)^J\).

We now write

\[
1 \frac{1}{T} \int_{S_{J,2}(T)} \log \left| 1 + \sum_{j < J} \frac{c_j}{L_j} (\sigma + it) \right| dt = \frac{1}{T} \int_{S_{\text{main}}(T)} \log \left| 1 + \sum_{j < J} \frac{c_j}{L_j} (\sigma + it) \right| dt
\]

\[
+ \frac{1}{T} \int_{S_{\text{error}}(T)} \log \left| 1 + \sum_{j < J} \frac{c_j}{L_j} (\sigma + it) \right| dt,
\]

(5.12)

where

\[
S_{\text{main}}(T) = \bigcup_{m,n} S_{\text{main}}(m, n, T),
\]

\[
S_{\text{error}}(T) = \bigcup_{m,n} S_{\text{error}}(m, n, T),
\]

and

\[
S_{\text{main}}(m, n, T) = \{t \in [T, 2T] : L(\sigma + it) \in R_{\text{main}}(m, n)\},
\]

\[
S_{\text{error}}(m, n, T) = \{t \in [T, 2T] : L(\sigma + it) \in R_{\text{error}}(m, n)\}.
\]

Here, the \((m, n)\)-sum and the \((m, n)\)-union are over \(0 \leq m_j \leq 1/\delta - 1\) and \(|n_j| \leq A\mathcal{L}/\delta\) for \(j \leq J - 2\).

The main term: For each \(m, n\), we have

\[
1 \frac{1}{T} \int_{S_{\text{main}}(m, n, T)} \log \left| 1 + \sum_{j < J} \frac{c_j}{L_j} (\sigma + it) \right| dt = \int_{R_{\text{main}}(m, n)} \log \left| 1 + \sum_{j < J} e^{u_j + iv_j} \right| d\Psi_T(u, v),
\]

and we wish to express this in terms of the distribution function \(\Psi(u, v)\). Since each \(R_{\text{main}}(m, n)\) is a union of rectangular regions, we let

\[
R = \prod_{j \leq 2J} (a_j, b_j)
\]
be one of them and consider

\[
\int_{R} \log \left| 1 + \sum_{j<J} e^{u_{j} + iv_{j}} \right| d\Psi_{T}(u, v).
\]

Our argument is similar to that in Subsection 5.1, but slightly more complicated. For \( w = (w_{1}, \ldots, w_{2J}) \) define

\[
\begin{align*}
    h_{0}(w) &= \log \left| 1 + \sum_{j<J} e^{w_{j} + iw_{j+J}} \right|, \\
    h_{1}(w) &= h_{0}(a_{1}, w_{2}, \ldots, w_{2J}), \\
    h_{2}(w) &= h_{0}(a_{1}, a_{2}, w_{3}, \ldots, w_{2J}), \\
    & \vdots \\
    h_{2J}(w) &= h_{0}(a_{1}, a_{2}, \ldots, a_{2J}).
\end{align*}
\]

(5.13)

Notice that \( h_{2J} \) is a constant function and

\[
    h_{0}(w) = \sum_{j=0}^{2J-1} (h_{j}(w) - h_{j+1}(w)) + h_{2J}(w).
\]

Thus we have

\[
\begin{align*}
    \int_{R} \log \left| 1 + \sum_{j<J} e^{u_{j} + iv_{j}} \right| d\Psi_{T}(u, v) &= \int_{R} h_{0}(w) d\Psi_{T}(w) \\
    &= \int_{R} \left( \sum_{j=0}^{2J-1} (h_{j}(w) - h_{j+1}(w)) + h_{2J}(w) \right) d\Psi_{T}(w) \\
    &= \sum_{j=0}^{2J-1} \int_{R} (h_{j}(w) - h_{j+1}(w)) d\Psi_{T}(w) + h_{2J}(w) \Psi_{T}(R),
\end{align*}
\]

where \( a = (a_{1}, \ldots, a_{2J}) \). Now, letting \( \bar{w}_{j+1} = (w_{1}, \ldots, w_{j}, \bar{w}_{j+1}, w_{j+2}, \ldots, w_{2J}) \), we have

\[
    h_{j}(w) - h_{j+1}(w) = \int_{a_{j+1}}^{w_{j+1}} \frac{\partial h_{j}(\bar{w}_{j+1})}{\partial \bar{w}_{j+1}} d\bar{w}_{j+1}.
\]

By (5.8), (5.11), and (5.13), we have

\[
    \frac{\partial h_{j}(w)}{\partial \bar{w}_{j+1}} \ll \left| 1 + \sum_{1 \leq j \leq J-1} e^{w_{j} + iw_{j+j}} \right|^{-1} \ll \delta^{-1}
\]

on \( R = \Pi_{j \leq 2J} (a_{j}, b_{j}) \). Thus, if

\[
    R_{j+1}(\bar{w}_{j+1}) := (a_{1}, b_{1}) \times \cdots \times (a_{j}, b_{j}) \times (\bar{w}_{j+1}, b_{j+1}) \times (a_{j+2}, b_{j+2}) \times \cdots \times (a_{2J-2}, b_{2J-2}),
\]
we see that
\[
\int_R \left( h_j(w) - h_{j+1}(w) \right) d\Psi_T(w) \\
= \int_R \left( \int_{a_{j+1}}^{b_{j+1}} \frac{\partial h_j(\bar{w}_{j+1})}{\partial \bar{w}_{j+1}} d\bar{w}_{j+1} \right) d\Psi_T(w) \\
= \int_{a_{j+1}}^{b_{j+1}} \left( \int_{R_{j+1}(\bar{w}_{j+1})} \frac{\partial h_j(\bar{w}_{j+1})}{\partial \bar{w}_{j+1}} d\bar{w}_{j+1} \right) d\Psi_T(w) \\
= \int_{a_{j+1}}^{b_{j+1}} \Psi_T(R_{j+1}(\bar{w}_{j+1})) \frac{\partial h_j(\bar{w}_{j+1})}{\partial \bar{w}_{j+1}} d\bar{w}_{j+1}. \tag{5.14}
\]

Now by Theorem 2
\[
\Psi_T(R_{j+1}(\bar{w}_{j+1})) - \Psi(R_{j+1}(\bar{w}_{j+1})) = O((\log T)^{-\sigma}).
\]

Reversing our steps, we see that the last line in (5.14) equals
\[
\int_{a_{j+1}}^{b_{j+1}} \Psi(R_{j+1}(\bar{w}_{j+1})) \frac{\partial h_j(\bar{w}_{j+1})}{\partial \bar{w}_{j+1}} d\bar{w}_{j+1} + O \left( \frac{|b_{j+1} - a_{j+1}|}{\delta(\log T)^\sigma} \right)
\]
\[
= \int_{a_{j+1}}^{b_{j+1}} \left( \int_{\bar{R}_{j+1}(\bar{w}_{j+1})} \frac{\partial h_j(\bar{w}_{j+1})}{\partial \bar{w}_{j+1}} d\bar{w}_{j+1} \right) d\Psi_T(w) + O \left( \frac{L}{\delta(\log T)^\sigma} \right)
\]
\[
= \int_R \left( h_j(w) - h_{j+1}(w) \right) d\Psi_T(w) + O \left( \frac{L}{\delta(\log T)^\sigma} \right).
\]

Hence, we obtain
\[
\int_R \log \left| 1 + \sum_{j<J} e^{\nu_j + iv_j} \right| d\Psi_T(u, v) = \int_R h_0(u, v) d\Psi_T(u, v) + O \left( \frac{L}{\delta(\log T)^\sigma} \right).
\]

Since \(R_{\text{main}}(m, n)\) is a union of at most \(O(L)\) rectangles \(R\), we have
\[
\int_{R_{\text{main}}(m, n)} \log \left| 1 + \sum_{j \leq J-1} e^{\nu_j + iv_j} \right| d\Psi_T(u, v) = \int_{R_{\text{main}}(m, n)} h_0(u, v) d\Psi_T(u, v) + O \left( \frac{L^2}{\delta(\log T)^\sigma} \right). \tag{5.15}
\]

**The error term:** We bound
\[
\frac{1}{T} \int_{S_{\text{error}}(T)} \log \left| 1 + \sum_{j<J} \frac{c_j L_j}{c_j L_j} (\sigma + it) \right| dt
\]
by using the Cauchy-Schwarz inequality and showing that \(\text{meas}(S_{\text{error}}(T))\) is small.

If \(t \in S_{\text{error}}(T)\), then \(L(\sigma + it) \in R_{\text{error}}(m, n)\) for some \(m, n\). Thus, by (5.9)
\[
\log \left| 1 + \sum_{j<J} \frac{c_j L_j}{c_j L_j} (\sigma + it) \right| \leq \log(23J) = -c_0L + \log(23J).
\]
By Lemma 6,
\[
\frac{1}{T} \int_T^{2T} \left| \log \left( 1 + \sum_{j<J} \frac{c_j L_j}{c_J L_J} (\sigma + it) \right) \right|^{2k} dt 
\leq 4^k \frac{1}{T} \int_T^{2T} \left| \log \left( \sum_{j \leq J} c_j L_j (\sigma + it) \right) \right|^{2k} dt + 4^k \frac{1}{T} \int_T^{2T} \left| c_J L_J (\sigma + it) \right|^{2k} dt 
\ll (Ck)^{4k}.
\]

Hence,
\[
\text{meas}(S_{\text{error}}(T)) \leq \int_{S_{\text{error}}(T)} \left( \frac{\log \left( 1 + \sum_{j<J} \frac{c_j L_j}{c_J L_J} (\sigma + it) \right)}{c_0 L} \right)^{2k} dt 
\ll T \left( \frac{C^2 k^2}{c_0 L} \right)^2.
\]

Taking \(k = a \sqrt{L}\) for any \(0 < a < \sqrt{c_0}/C\), we see that
\[
\frac{1}{T} |S_{\text{error}}(T)| \ll e^{-2b \sqrt{L}} \tag{5.16}
\]
with \(b = 2a \log(\sqrt{c_0}/aC) > 0\). It now follows by the Cauchy-Schwarz inequality and Lemma 6 (with \(k = 1\)) that
\[
\frac{1}{T} \left| \int_{S_{\text{error}}(T)} \log \left( 1 + \sum_{j<J} \frac{c_j L_j}{c_J L_J} (\sigma + it) \right) dt \right| 
\leq \left( \frac{1}{T} \text{meas}(S_{\text{error}}(T)) \right)^{\frac{1}{2}} \left( \frac{1}{T} \int_T^{2T} \left( \log \left( 1 + \sum_{j<J} \frac{c_j L_j}{c_J L_J} (\sigma + it) \right) \right)^2 dt \right)^{\frac{1}{2}} 
\ll e^{-b \sqrt{L}}.
\]

By (5.12), (5.15), and (5.16) we now see that
\[
\frac{1}{T} \int_{S_J(T)} \log \left| 1 + \sum_{j<J} \frac{c_j L_j}{c_J L_J} (\sigma + it) \right| dt 
= \sum_{m,n} \int_{R_{\text{main}}(m,n)} h_0(u, v) d\Psi(u, v) + O\left( \frac{L^{J+1}}{(\log T)^{\sigma-(2J-3)c_0}} \right) + O(e^{-b \sqrt{L}}) 
= \sum_{m,n} \int_{R_{\text{main}}(m,n)} h_0(u, v) d\Psi(u, v) + O(e^{-b \sqrt{L}}). \tag{5.17}
\]

Next we show that
\[
\int_{R_{\text{error}}} h_0(u, v) d\Psi(u, v) = O(e^{-b \sqrt{L}}), \tag{5.18}
\]
where
\[ R_{error} = \bigcup_{m,n} R_{error}(m, n). \] (5.19)

It will then follow from (5.10) and (5.17) – (5.19) that
\[ \frac{1}{T} \int_{S,T(T)} \log \left| 1 + \sum_{j<J} \frac{c_j L_j}{L_j} (\sigma + i t) \right| dt = \int_{(IT')^{J-1} \times IT \times (IT')^J} h_0(u, v) d\Psi(u, v) \]
\[ + O(e^{-b\sqrt{\mathcal{L}}}). \] (5.20)

To prove (5.18), we first note that by Hölder’s inequality
\[ \left| \int_{R_{error}} h_0(u, v) d\Psi(u, v) \right| \leq \left( \int_{R_{error}} |h_0(u, v)|^{2k} d\Psi(u, v) \right)^{1/2k} \left\| \int_{R_{error}} d\Psi(u, v) \right\|^{1-1/2k} \]
\[ \leq 4^k \left( E \left[ \log \left| \sum_{j<J} c_j L_j (\sigma, X) \right|^{2k} \right] \right) \]
\[ \ll (Ck)^2. \] (5.21)

Hence
\[ \left| \int_{R_{error}} h_0(u, v) d\Psi(u, v) \right| \ll k \left\| \int_{R_{error}} d\Psi(u, v) \right\|^{1-1/2k}. \] (5.22)

By (5.9) we see that for \((u, v) \in R_{error}\),
\[ h_0(u, v) = \log \left| 1 + f \sum_{j<J} e^{u_j + iv_j} \right| \leq \log(23J\delta) \leq -c_0 \mathcal{L} + \log(23J). \]
Thus
\[ |h_0(u, v)| \geq c_0 \mathcal{L} - \log(23J), \]
and we obtain
\[ 0 \leq \Psi(R_{error}) = \int_{R_{error}} d\Psi(u, v) \leq \int_{R_{error}} \frac{|h_0(u, v)|^{2k}}{(c_0 \mathcal{L} - \log(23J))^{2k}} d\Psi(u, v) \]
\[ \ll \left( \frac{Ck}{c_0 \mathcal{L} - \log(23J)} \right)^{2k}, \]
by (5.21). Using this in (5.22), we obtain
\[ \left| \int_{R_{error}} h_0(u, v) d\Psi(u, v) \right| \ll \mathcal{L} \left( \frac{Ck}{c_0 \mathcal{L} - \log(23J)} \right)^{2k}. \]
We now take \( k = (c_0 \mathcal{L} - \log(23J))/(eC) \) and find that
\[ \left| \int_{R_{error}} h_0(u, v) d\Psi(u, v) \right| \ll e^{-2(c_0 \mathcal{L})} \ll e^{-b\sqrt{\mathcal{L}}}. \]
This proves (5.18) and thus (5.20).
5.3 Completion of the proof of Theorem \[5.1\]

By (5.3), (5.6), and (5.20),

\[
\frac{1}{T} \int_{S_J(T)} \log |E(\sigma + it, Q)| dt = \int_{(I_T)^J \times I_T \times (I_T)^J} \left( u_J + \log \left| 1 + \sum_{j < J} e^{u_j + iv_j} \right| \right) d\Psi(u, v)
\]

\[
+ O(e^{-b\sqrt{L}})
\]

\[
= \mathbb{E} \left[ 1_{\mathcal{J}_{J,T}} \cdot \left( \log |c_J L_J(\sigma, X)| + \log \left| 1 + \sum_{j < J} \frac{c_j L_j(\sigma, X)}{c_J L_J(\sigma, X)} \right| \right) \right]
\]

\[
+ O(e^{-b\sqrt{L}})
\]

\[
= \mathbb{E} \left[ 1_{\mathcal{J}_{J,T}} \cdot \log \left| \sum_{j \leq J} c_j L_j(\sigma, X) \right| \right] + O(e^{-b\sqrt{L}}),
\]

where \(\mathcal{J}_{J,T}\) is the event

\[
L(\sigma, X) \in (I_T)^J \times I_T \times (I_T)^J,
\]

and \(1_{\mathcal{J}_{J,T}}\) is its characteristic function. By Lemma \[8\]

\[
\mathbb{E} \left[ 1_{\mathcal{J}_{J,T}} \cdot \log \left| \sum_{j \leq J} c_j L_j(\sigma, X) \right| \right] + O(e^{-b\sqrt{L}}) = \mathbb{E} \left[ 1_{\mathcal{J}_J} \cdot \log \left| \sum_{j \leq J} c_j L_j(\sigma, X) \right| \right] + O(e^{-b\sqrt{L}}),
\]

where \(\mathcal{J}_J\) is the event

\[
L(\sigma, X) \in (\infty, 0)^J \times \mathbb{R}^{J+1},
\]

and \(1_{\mathcal{J}_J}\) is its characteristic function. Hence \(\mathcal{J}_J\) is the event

\[
|L_J(\sigma, X)| = \max_j |L_j(\sigma, X)|.
\]

Reversing all the steps in (5.1), we therefore obtain

\[
\frac{1}{T} \int_T^{2T} \log |E(\sigma + it, Q)| dt = \sum_{j \leq J} \frac{1}{T} \int_{S_J(T)} \log |E(\sigma + it, Q)| dt + O\left( \frac{L^2}{(\log T)^\sigma} \right)
\]

\[
= \sum_{j \leq J} \mathbb{E} \left[ 1_{\mathcal{J}_J} \cdot \log \left| \sum_{j \leq J} c_j L_j(\sigma, X) \right| \right] + O(e^{-b\sqrt{L}}) \quad (5.23)
\]

where

\[
\mathcal{M}(\sigma) = \mathbb{E} \left[ \log \left| \sum_{j \leq J} c_j L_j(\sigma, X) \right| \right].
\]

In [7] the second author proved that \(\mathcal{M}(\sigma)\) is twice differentiable. We use this to show that

\[
N_E(\sigma_1, \sigma_2; T) = -\frac{T}{2\pi} (\mathcal{M}'(\sigma_1) - \mathcal{M}'(\sigma_2)) + O(e^{-b/2\sqrt{L}}).
\]
Applying Littlewood’s lemma in a standard way, we find that
\[ \int_{\sigma_0}^{\sigma} \left( \sum_{\beta > u \atop T \leq \gamma \leq 2T} 1 \right) du = \frac{1}{2\pi} \int_T^{2T} \log |E(\sigma + it, Q)| dt - \frac{1}{2\pi} \int_T^{2T} \log |E(\sigma_0 + it, Q)| dt + O(\log T), \]
where \( \sigma_0 \) is a large but fixed real number such that \( E(s, Q) \) has no zero in \( \Re s \geq \sigma_0 \). By (5.23)
\[ \int_{\sigma}^{\sigma_0} \left( \sum_{\beta > u \atop T \leq \gamma \leq 2T} 1 \right) du = \frac{T}{2\pi} M(\sigma) - \frac{1}{2\pi} \int_T^{2T} \log |E(\sigma_0 + it, Q)| dt + O(T e^{-bv\sqrt{L}}). \]

Differencing this at \( \sigma \) and \( \sigma + h \), with \( h > 0 \) small, we deduce that
\[ \frac{1}{h} \int_{\sigma}^{\sigma + h} \left( \sum_{\beta > u \atop T \leq \gamma \leq 2T} 1 \right) du = \frac{T}{2\pi} \frac{M(\sigma) - M(\sigma + h)}{h} + O\left(\frac{T}{h} e^{-bv\sqrt{L}}\right). \]
Since \( M(\sigma) \) is twice differentiable, we may write this as
\[ \frac{1}{h} \int_{\sigma}^{\sigma + h} \left( \sum_{\beta > u \atop T \leq \gamma \leq 2T} 1 \right) du = -\frac{T}{2\pi} M'(\sigma) + O\left(\frac{T}{h} e^{-bv\sqrt{L}}\right). \]
The integrand is a non increasing function of \( u \), so
\[ \sum_{\beta > \sigma + h \atop T \leq \gamma \leq 2T} 1 \leq -\frac{T}{2\pi} M'(\sigma) + O\left(\frac{T}{h} e^{-bv\sqrt{L}}\right) \leq \sum_{\beta > \sigma \atop T \leq \gamma \leq 2T} 1. \]
On the left-hand side we replace \( \sigma \) by \( \sigma - h \) and use \( M'(\sigma - h) = M'(\sigma) + O(h) \). We then find that
\[ \sum_{\beta > \sigma \atop T \leq \gamma \leq 2T} 1 = -\frac{T}{2\pi} M'(\sigma) + O\left(\frac{T}{h} e^{-bv\sqrt{L}}\right). \]
Taking \( h = e^{-\frac{b}{2}\sqrt{L}} \), we obtain
\[ \sum_{\beta > \sigma \atop T \leq \gamma \leq 2T} 1 = -\frac{T}{2\pi} M'(\sigma) + O\left(T e^{-\frac{b}{2}\sqrt{L}}\right). \]

Equation (5.24) follows easily from this.

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