Virtual Gromov-Witten Invariants and the Quantum Cohomology Rings of General Type Projective Hypersurfaces

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Abstract

In this paper, we propose another characterization of the generalized mirror transformation on the quantum cohomology rings of general type projective hypersurfaces. This characterization is useful for explicit determination of the form of the generalized mirror transformation. As applications, we rederive the generalized mirror transformation up to $d = 3$ rational Gromov-Witten invariants obtained in our previous article, and determine explicitly the generalized mirror transformation for the $d = 4, 5$ rational Gromov-Witten invariants in the case when the first Chern class of the hypersurface equals $-H$ (i.e., $k = N = 1$).

1 Introduction and Statement of the Main Results

Recently, some works on the quantum cohomology ring of the general type projective hypersurface have appeared [13], [5], [10]. In [13], Lian, Liu and Yau generalized their mirror principle to the case of the general type projective hypersurface and proposed a theoretical recipe to construct the generating function of a certain type of Gromov-Witten invariants including gravitational descendants from the hypergeometric data. In [5], Gathmann considered the relative Gromov-Witten invariants of the projective space with various tangency conditions on the ample hypersurface in the projective space. He verified the recursive formulas which increase the tangency condition like the results of Caporaso and Harris [3], and proposed an algorithm of computing the Gromov-Witten invariants of the hypersurface as a limit of iterative application of the recursive formulas.

In this paper, we continue the analysis of [10] on the quantum Kähler sub-ring $QH^*_e(M^k_N)$, where $M^k_N$ is the degree $k$ hypersurface in $CP^{N-1}$, especially in the case when its first Chern class is negative. Our approach is different from the ones of [13] and [5], mainly because we don’t use Mumford-Morita class (gravitational descendants). In [10], we proposed the generalized mirror transformation on the quantum cohomology rings of $M^k_N$, ($N - k \leq 1$) that represents the structural constant $L_{n,k,d}^N$ of $QH^*_e(M^k_N)$ in terms of the virtual structural constants $\tilde{L}_{n,k,d}^N$, which is the analogue of the hypergeometric series used in the mirror calculation in our context.

Now we restate the main conjecture in [10]. Let $P_m$ be the set of partitions of $m$ into positive integers and $\sigma_m$ be an element of $P_m$. We also denote the length of a partition $\sigma_m$ by $l(\sigma_m)$ (i.e.,

\[ l(\sigma_m) = \text{number of parts in } \sigma_m \]
σ_m : m = d_1 + d_2 + \cdots + d_(σ_m), \quad d_1 \geq d_2 \geq \cdots \geq d_(σ_m) \geq 1). We denote by \mul(i, σ_m), \ (1 \leq i \leq m) the multiplicity of i in σ_m. Our previous conjecture is the following:

Conjecture 1
The generalized mirror transformation takes the form
\[
L_n^{N,k,d} = \sum_{m=0}^{d-1} \sum_{σ_m \in P_m} (-1)^l(σ_m) \frac{d!(σ_m)}{\prod_{j=1}^{l(σ_m)} \prod_{i=1}^{d}(σ_m)!} \prod_{j=1}^{l(σ_m)} L_{1+(k-N)d}^{N,k,d} \cdot G_{d-m}^{N,k,d}(n; σ_m),
\]
(1.1)

where \(G_{d-m}^{N,k,d}(n; σ_m)\) is a polynomial of \(\hat{L}_n^{N,k,d}\) with weighted degree \(d\).

Now, we propose a conjecture on the explicit form of \(G_{d-m}^{N,k,d}(n; σ_m)\).

Conjecture 2
\[
V_{d-m}^{N,k,d}(n; σ_m) := \frac{1}{K} v(\mathcal{O}_{c^{N-2-n}} \mathcal{O}_{c^{n-1-(k-N)d}}) \prod_{i=1}^{l(σ_m)} \mathcal{O}_{c^{1+(k-N)d_i}} \cdot \frac{1}{(d-m)^l(σ_m) - 1}
\]
\[\approx G_{d-m}^{N,k,d}(n; σ_m)
\]
(1.2)

where \(v(\mathcal{O}_{c^{N-2-n}} \mathcal{O}_{c^{n-1-(k-N)d}}) \prod_{i=1}^{l(σ_m)} \mathcal{O}_{c^{1+(k-N)d_i}} \cdot \frac{1}{(d-m)^l(σ_m) - 1}) \) is the virtual Gromov-Witten invariant defined below. If \(l(σ_m) \leq 1\) or \(d - m = 1\), (1.2) becomes an exact equality.

We have to make some remarks on the meaning of \(\approx\) in (1.2). In the case when \(l(σ_m) \leq 1\) or \(d - m = 1\), we can see the coincidence between \(G_{d-m}^{N,k,d}(n; σ_m)\) and \(V_{d-m}^{N,k,d}(n; σ_m)\), but when \(l(σ_m) \geq 2\) and \(d - m \geq 2\), we have to modify slightly the coefficients of some polynomials in \(\hat{L}_n^{N,k,d}\) that appear in \(V_{d-m}^{N,k,d}(n; σ_m)\). These cases indeed appear when \(d \geq 4\). Even in such situations, \(V_{d-m}^{N,k,d}(n; σ_m)\) strongly predicts the form of \(G_{d-m}^{N,k,d}(n; σ_m)\).

Now we turn into the definition of the virtual Gromov-Witten invariants, that are the main ingredients of this paper.

Definition 1
The virtual Gromov-Witten invariant \(v(\prod_{j=1}^{n} \mathcal{O}_{e^j})\) on \(\overline{M}_{k,N}^k\) is the rational number that satisfy the condition:
(i) initial condition
\[
v(\mathcal{O}_{e^0} \mathcal{O}_{e^b} \mathcal{O}_{e^c})_0 = k \cdot \delta_{a+b+c,N-2},
\]
\[
v(\prod_{j=1}^{n} \mathcal{O}_{e^j})_0 = 0, \quad (n \neq 3),
\]
\[
\frac{1}{k} v(\mathcal{O}_{c^{N-2-n}} \mathcal{O}_{c^{n-1-(k-N)d}} \mathcal{O}_{e^d}) = \hat{L}_n^{N,k,d} - \hat{L}_1^{N,k,d} \cdot (d \geq 1),
\]
(1.3)

(ii) flat metric condition
\[
v(\mathcal{O}_{e^0} \mathcal{O}_{e^a} \mathcal{O}_{e^b})_0 = k \cdot \delta_{a+b,N-2},
\]
\[
v(\mathcal{O}_{e^0} \prod_{j=1}^{n} \mathcal{O}_{e^j})_d = 0, \quad (d \geq 1, \quad or \quad d = 0, \quad n \neq 2),
\]
(1.4)

(iii) topological selection rule
\[
v(\prod_{j=1}^{n} \mathcal{O}_{e^j})_d \neq 0 \implies (N - 5) + (N - k)d = \sum_{j=1}^{n} (a_j - 1),
\]
(1.5)
(iv) Kähler equation

\[ v(\mathcal{O}_e \prod_{j=1}^n \mathcal{O}_{e_j})_d = d \cdot v(\prod_{j=1}^n \mathcal{O}_{e_j})_d, \quad (1.6) \]

(v) associativity equation \([3], [4]\)

\[
\sum_{d_1=0}^d \sum_{\{\alpha_s\} \prod (\beta_s) = \{n_s\}} \sum_{i=0}^{N-2} v(\mathcal{O}_{e_i} \mathcal{O}_{e_b} \left( \prod_{\alpha_j \in \{\alpha_s\}} \mathcal{O}_{e_j} \right) \mathcal{O}_{e_c} \mathcal{O}_{e_d})_d \cdot v(\mathcal{O}_{e_{N-2-i}} \left( \prod_{\beta_j \in (\beta_s)} \mathcal{O}_{e_j} \right) \mathcal{O}_{e_{N-2-i}} \mathcal{O}_{e_{N-1}})_d - d_1,
\]

\[
= \sum_{d_1=0}^d \sum_{\{\alpha_s\} \prod (\beta_s) = \{n_s\}} \sum_{i=0}^{N-2} v(\mathcal{O}_{e_i} \mathcal{O}_{e_b} \left( \prod_{\alpha_j \in \{\alpha_s\}} \mathcal{O}_{e_j} \right) \mathcal{O}_{e_c} \mathcal{O}_{e_d})_d \cdot v(\mathcal{O}_{e_{N-2-i}} \left( \prod_{\beta_j \in (\beta_s)} \mathcal{O}_{e_j} \right) \mathcal{O}_{e_{N-2-i}} \mathcal{O}_{e_{N-1}})_d - d_1,
\]

\[
(a + b + c + d + \sum_{j=1}^m (n_j - 1) = N - 2 + (N - k)d).
\quad (1.7)
\]

The difference between the virtual Gromov-Witten invariants and the ordinary Gromov-Witten invariants comes from the initial condition \([3]\). In ordinary cases, we set

\[
\frac{1}{k} \langle \mathcal{O}_{e_{N-2-a}} \mathcal{O}_{e_{-1-(k-N)d}} \mathcal{O}_e \rangle_d = L_n^{N,k,d}, \quad (d \geq 1),
\quad (1.8)
\]

\([3]\) and \((1.8)\) indeed differ when \(d > 1\). With the above conditions, we can completely determine the virtual Gromov-Witten invariants like the ordinary Gromov-Witten invariants of genus 0. Using the generating function of the virtual Gromov-Witten invariants:

\[
F_v(z, t_i) = \frac{1}{3!} \sum_{i,j,k=0}^{N-2} k \cdot \delta_{i+j+k,N-2} \cdot t_i t_j t_k + f_v(z, t_i),
\quad (1.9)
\]

we can state the main conjecture in a more compact form,

\[
L_n^{N,k,d} \sim \frac{1}{k} \int_{C_0} dz \exp(-d \sum_{j=1}^{\infty} L_{1+(k-N)j} \partial t_j) \partial z^{j} \cdot \partial_{N-2-n \partial_{h-1-d(k-N)}} \partial f_v(z, t_i) |_{t_i=0},
\quad (1.10)
\]

where \(t_j (j = 0, \cdots, N - 2)\) is the variable that couples to the element \(\mathcal{O}_{e_j}\) of \(QH_\ast (M_k^N)\) and \(z\) is the formal degree counting variable.

This paper is organized as follows. In Section 2, we introduce the notation of the quantum Kähler subring of \(M_k^N\), and review the results obtained in \([1]\), \([8]\) and \([10]\). In Section 3, we reproduce the formulas obtained in \([10]\) under the assumption of Conjecture 2. In Section 4, we construct the explicit form of the generalized mirror transformation for degree 4, 5 rational Gromov -Witten invariants of \(M_k^N\) using the Conjecture 2 and some numerical data obtained from the fixed point computation in \([1]\).

2 Quantum Kähler Sub-ring of Projective Hypersurfaces

2.1 Notation

In this section, we introduce the quantum Kähler sub-ring of the quantum cohomology ring of a degree \(k\) hypersurface in \(CP^{N-1}\). Let \(M_k^N\) be a hypersurface of degree \(k\) in \(CP^{N-1}\). We denote by \(QH_\ast (M_k^N)\) the subring of the quantum cohomology ring \(QH_\ast (M_k^N)\) generated by \(\mathcal{O}_e\) induced from
the Kähler form $e$ (or, equivalently the intersection $H \cap M_N^k$ between a hyperplane class $H$ of $\mathbb{CP}^{N-1}$ and $M_N^k$). Additive basis of $QH^*(M_N^k)$ is given by $O_{e,j}$ $(j = 0, 1, \cdots, N - 2)$, which is induced from $e^j \in H^{j,j}(M_N^k)$. The multiplication rule of $QH^*(M_N^k)$ is determined by the Gromov-Witten invariant of genus 0 $(O_e O_{e^{N-2-m}} O_{e^{m-1-(k-N)d}})_{d,M_N^k}$ and it is given as follows:

$$
L_{m}^{N,k,d} := \frac{1}{k} \langle O_e O_{e^{N-2-m}} O_{e^{m-1-(k-N)d}} \rangle_d,
$$

\[ O_e \cdot 1 = O_e, \]

\[ O_e \cdot O_{e^{N-2-m}} = O_{e^{N-1-m}} + \sum_{d=1}^{\infty} L_{m}^{N,k,d} q^d O_{e^{N-1-m+(k-N)d}}, \]

\[ q := \exp(t), \quad (2.11) \]

where the subscript $d$ counts the degree of the rational curves measured by $e$. Therefore, $q = \exp(t)$ is the degree counting parameter.

**Definition 2** We call $L_{m}^{N,k,d}$ the structural constant of weighted degree $d$.

Since $M_N^k$ is a complex $(N - 2)$ dimensional manifold, we see that a structure constant $L_{m}^{N,k,d}$ is non-zero only if the following condition is satisfied:

$$
1 \leq N - 2 - m \leq N - 2, 1 \leq m - 1 + (N - k)d \leq N - 2, \quad \iff \quad \max\{0, 2 - (N - k)d\} \leq m \leq \min\{N - 3, N - 1 - (N - k)d\}. \quad (2.12)
$$

We rewrite $(2.12)$ into

$$
(N - k \geq 2) \implies 0 \leq m \leq (N - 1) - (N - k)d
$$

$$
(N - k = 1, d = 1) \implies 1 \leq m \leq N - 3
$$

$$
(N - k = 1, d \geq 2) \implies 0 \leq m \leq N - 1 - (N - k)d
$$

$$
(N - k \leq 0) \implies 2 + (k - N)d \leq m \leq N - 3. \quad (2.13)
$$

From $(2.13)$, we easily see that the number of the non-zero structure constants $L_{m}^{N,k,d}$ is finite except for the case of $N = k$. Moreover, if $N \geq 2k$, the non-zero structure constants come only from the $d = 1$ part and the non-vanishing $L_{m}^{N,k,1}$ is determined by $k$ and independent of $N$. The $N \geq 2k$ region is studied by Beauville [1], and his result plays the role of an initial condition of our discussion later. Explicitly, they are given by the formula :

$$
\sum_{n=0}^{k-1} L_{m}^{N,k,1} w^n = k \prod_{j=1}^{k-1} (jw + (k - j)), \quad (2.14)
$$

and the other $L_{m}^{N,k,d}$'s all vanishes. In the case of $N = k$, the multiplication rule of $QH^*(M^k)$ is given as follows:

$$
O_e \cdot 1 = O_e, \quad (2.15)
$$

\[ O_e \cdot O_{e^{k-2-m}} = (1 + \sum_{d=1}^{\infty} q^d L_{m}^{k,k,d}) O_{e^{(k-1)-m}} \quad (m = 2, 3, \cdots, k - 3), \]

\[ O_e \cdot O_{e^{k-3}} = O_{e^{k-2}}. \quad (2.16) \]

We introduce here the generating function of the structure constants of the Calabi-Yau hypersurface $M^k$:

$$
L_{m}^{k,k}(e^t) := 1 + \sum_{d=1}^{\infty} L_{m}^{k,k,d} e^{dt} \quad (m = 2, 3, \cdots, k - 3). \quad (2.16)
$$
2.2 Review of Results for Fano and Calabi-Yau Hypersurfaces and Virtual Structure Constants

Let us summarize the results of [4, 8]. In [4], we showed that the structure constants $L_{m}^{N,k,d}$ of $QH^{*}(M_{K}^{d})$ for $(N-k \geq 2)$ can be obtained by applying the recursive formulas which describe $L_{m}^{N,k,d}$ in terms of $L_{m'}^{N+1,k,d'}$ $(d' \leq d)$, with the initial conditions of $L_{m}^{N,k,1}$ given by (2.14) and $L_{m}^{N,k,d} = 0$ $(d \geq 2)$ in the $N \geq 2k$ region. Let us introduce the construction of the recursive formulas given in [8]. First, we introduce the polynomial $Poly_{d}$ in $x, y, z_{1}, z_{2}, \cdots, z_{d-1}$ defined by the formula:

$$Poly_{d}(x, y, z_{1}, z_{2}, \cdots, z_{d-1}) = \frac{1}{(2\pi i)^{d-1}} \int_{C_{t}} dt_{1} \cdots \int_{C_{d-1}} dt_{d-1} \prod_{j=1}^{d-1} \left( (d-j)x + jy \right) + \sum_{i=1}^{d-1} \left( \frac{d-j}{d-i} \right) t_{i} + \sum_{i=j+1}^{d} \left( \frac{d-j}{d-i} \right) t_{i} + z_{j} \left( \frac{(d-j)x + jy}{d} \right) + \sum_{i=1}^{d-1} \left( \frac{d-j}{d-i} \right) t_{i} + \sum_{i=j+1}^{d} \left( \frac{d-j}{d-i} \right) t_{i} + \sum_{i=j+1}^{d} \left( \frac{j}{d-i} \right) t_{i} + (d-1) \int \right).$$

(2.17)

In (2.17), we have to choose the path $C_{t}$ carefully to obtain the correct answer. See [8] for details. Consider the monomial $x^{d_{0}}z_{i_{1}}^{d_{1}} \cdots z_{i_{m}}^{d_{m}}y^{d_{m+1}}$ ($\sum_{j=0}^{m+1} d_{i_{j}} = d-1$), that appear in $Poly_{d}$, associated with the following ordered partition of a positive integer $d$ [8]:

$$0 = i_{0} < i_{1} < i_{2} < \cdots < i_{m} < i_{m+1} = d. \quad (2.18)$$

Next, we prepare some elements in (a free abelian group) $\mathbb{Z}^{m+1}$, which are determined for each monomial $x^{d_{0}}z_{i_{1}}^{d_{1}} \cdots z_{i_{m}}^{d_{m}}y^{d_{m+1}}$, as follows:

$$\alpha := (m+1-d, m+1-d, \cdots, m+1-d),$$
$$\beta := (0, i_{1}-1, i_{2}-2, \cdots, i_{m}-m),$$
$$\gamma := (0, i_{1}(N-k), i_{2}(N-k), \cdots, i_{m}(N-k)), $$
$$\epsilon_{1} := (1, 0, 0, 0, \cdots, 0),$$
$$\epsilon_{2} := (1, 1, 0, 0, \cdots, 0),$$
$$\epsilon_{3} := (1, 1, 1, 0, \cdots, 0),$$
$$\cdots$$
$$\epsilon_{m+1} := (1, 1, 1, 1, \cdots, 1). \quad (2.19)$$

Now we define $\delta = (\delta_{1}, \cdots, \delta_{m+1}) \in \mathbb{Z}^{m+1}$ by the formula:

$$\delta := \alpha + \beta + \gamma + \sum_{j=1}^{m} (d_{i_{j}} - 1) \epsilon_{j} + d_{m+1} \epsilon_{m+1}. \quad (2.20)$$

Then the recursive formulas are given as follows:

$$L_{n}^{N,k,d} = \phi(Poly_{d}), \quad (2.21)$$

where $\phi$ is a $\mathbb{Q}$-linear map from the $\mathbb{Q}$-vector space of the homogeneous polynomials of degree $d-1$ in $x, y, z_{1}, z_{2}, \cdots, z_{d-1}$ to the $\mathbb{Q}$-vector space of the weighted homogeneous polynomials of degree $d$ in $L_{m+1,k,d'}$. And it is defined on the basis by:

$$\phi(x^{d_{0}}y^{d_{1}}z_{i_{1}}^{d_{1}} \cdots z_{i_{m}}^{d_{m}}) = \prod_{j=1}^{m+1} L_{n+\delta_{j}}^{N+1,k,ij_{j}-i_{j-1}}. \quad (2.22)$$
In the $d \leq 5$ cases, we examined that these recursive formulas naturally lead us to the relation:

$$ (O_c)^{N-1} - k^k(O_c)^{k-1}q = 0, \quad (2.23) $$

of $QH^*_c(M_k^N)$ $(N - k \geq 2)$ by descending induction using Beauville’s result \[1\], \[9\], \[6\]. In the $N - k = 1$ case, the recursive formulas receive modification only in the $d = 1$ part:

$$ L_{m+1}^{k+1,k,1} = L_{m}^{k+2,k,1} - L_{0}^{k+2,k,1} = L_{m}^{k+2,k,1} - k!. \quad (2.24) $$

This leads us to the following relation of $QH^*_c(M_{k+1}^N)$:

$$ (O_c + k!q)^{N-1} - k^k(O_c + k!q)^{k-1}q = 0. \quad (2.25) $$

The structural constant $L_{m}^{k,k,d}$ for a Calabi-Yau hypersurface does not obey the recursive formulas. Instead, we introduce here the virtual structure constants $\tilde{L}_{m}^{N,k,d}$ as follows.

**Definition 3** Let $\tilde{L}_{m}^{N,k,d}$ be the rational number obtained by applying the recursion relations of Fano hypersurfaces for arbitrary $N$ and $k$ with the initial condition $\tilde{L}_{m}^{N,k,1} (N \geq 2k)$ and $\tilde{L}_{m}^{N,k,d} = 0$ ($d \geq 2, \ N \geq 2k$).

**Remark 1** In the $N - k \geq 2$ region, $\tilde{L}_{m}^{N,k,d} = L_{m}^{N,k,d}$.

**Definition 4** We call $\tilde{L}_{m}^{N,k,d}$ the virtual structural constant of weighted degree $d$.

We define the generating function of the virtual structural constants of the Calabi-Yau hypersurface $M_k^N$ as follows:

$$ \tilde{L}_{m}^{k,k,e^x} := 1 + \sum_{d=1}^{\infty} \tilde{L}_{m}^{k,k,d,e^x} dx, \quad (n = 0, 1, \ldots, k - 1). \quad (2.26) $$

In \[4\], we observed that $\tilde{L}_{m}^{k,k,e^x}$ gives us the information of the B-model of the mirror manifold of $M_k^N$. More explicitly, we conjectured

$$ \tilde{L}_{0}^{k,k} = \sum_{d=0}^{\infty} \frac{(kd)!}{(d!)^k} e^x, $$

$$ \tilde{L}_{1}^{k,k} = - \frac{dt(x)}{dx} := \frac{d}{dx} \left( x + \sum_{d=1}^{\infty} \frac{(kd)!}{(d!)^k} \sum_{i=1}^{d-1} \sum_{m=1}^{k-1} \frac{m}{i(ki - m)} e^x \right) \quad (2.27) $$

where the r.h.s. of (2.27) is derived from the solutions of the ODE for the period integral of the mirror manifold of $M_k^N$,

$$ ((\frac{d}{dx})^{k-1} - k e^x (\frac{d}{dx} + 1)(\frac{d}{dx} + 2) \cdots (\frac{d}{dx} + k - 1))w(x) = 0, \quad (2.28) $$

that was used in the computation based on the mirror symmetry, \[5\]. Of course, we can extend the conjecture (2.27) to the general $\tilde{L}_{m}^{k,k,e^x}$ if we compare the $\tilde{L}_{m}^{k,k,e^x}$ with the B-model three point functions in \[8\]. Hence we obtain the mirror map $t = t(x)$ without using the mirror conjecture:

$$ t(x) = x + \int_{-\infty}^{x} dx' (\tilde{L}_{1}^{k,k}(e^{x'}) - 1) = x + \sum_{d=1}^{\infty} \frac{\tilde{L}_{1}^{k,k,d}}{d} e^x. \quad (2.29) $$
With the conjecture given by (2.27), we can construct the mirror transformation that transforms the virtual structural constants of the Calabi-Yau hypersurface into the real ones as follows:

\[
L^{k,k}_m(e^t) = \frac{\tilde{L}^{k,k}_m(e^x(t))}{L^{k,k}_1(e^x(t))}, \quad (m = 2, \ldots, k - 3)
\]  

(3.30)

The above formula is further rewritten as follows:

\[
L^{k,k}_n = \sum_{m=0}^{d-1} \frac{1}{\text{Res} z = 0}(z^{-m-1} \exp(-d \sum_{j=1}^{\infty} \frac{\tilde{L}^{k,k,j}_1}{j} z^j)) \cdot (\tilde{L}^{k,k,d-m}_n - \tilde{L}^{k,k,d-m}_1).
\]

(3.31)

This formula motivated us to propose the conjecture described in [11] or in [10] [10].

3 Derivation of the Previous Results

In this section, we first show that \(V^{N,k,d}_{d-m}(n; \sigma_m)\) satisfy the ansatz of \(G^{N,k,d}_{d-m}(n; \sigma_m)\) proposed in [10]. Then, we show that \(V^{N,k,d}_{d-m}(n; \sigma_m)\) coincides with \(G^{N,k,d}_{d-m}(n; \sigma_m)\) in the \(d \leq 3\) cases.

Proposition 1

(i) flat metric condition

\[
V^{N,k,d}_{d-m}(1 + (k - N)d; \sigma_m) = V^{N,k,d}_{d-m}(N - 2; \sigma_m) = 0.
\]

(3.32)

(ii) symmetry

\[
V^{N,k,d}_{d-m}(n; \sigma_m) = V^{N,k,d}_{d-m}(N - 1 + (k - N)d - n; \sigma_m).
\]

(3.33)

(iii) flat metric condition

\[
V^{N,k,d}_{d-m}(n; (0)) = \tilde{L}^{N,k,d}_n - \tilde{L}^{N,k,d}_{1+k+d(k-N)}.
\]

(3.34)

(iv) flat metric condition

\[
V^{N,k,d}_{d-m}(2 + (k - N)(d + f); \sigma_m) = V^{N,k,d+f}_{d-m}(2 + (k - N)(d + f); \sigma_m \cup (f)).
\]

(3.35)

Proof

(i), (ii) and (iii) are obvious by definition. Here, we give a proof of (iv). By the definition of \(V^{N,k,d+f}_{d-m}(n; \sigma_m \cup (f))\) and by Kähler equation, we have

\[
V^{N,k,d+f}_{d-m}(2 + (k - N)(d + f); \sigma_m \cup (f))
= \frac{1}{k(d-m)^{(l(\sigma_m))}} \nu(O_{e_{N-2-2-(k-N)(d+f)}} O_{e^{l(\sigma_m) + (k-N)d_j}} \prod_{j=1}^{l(\sigma_m)} O_{e^{2+(k-N)d_j})}) d_{m-n}
= \frac{1}{k(d-m)^{(l(\sigma_m))}} \nu(O_{e_{N-2-2-(k-N)(d+f)}} O_{e^{2+(k-N)(d+f)-1-(k-N)d_j}} \prod_{j=1}^{l(\sigma_m)} O_{e^{2+(k-N)d_j})}) d_{m-n}.
\]

(3.36)

The last line of (3.36) is nothing but \(V^{N,k,d}_{d-m}(2 + (k - N)(d + f); \sigma_m)\). Q.E.D.

Then we introduce the definition:

Definition 5

\[
\tilde{V}^{N,k,d+f}_{d-m}(n; \sigma_m \cup (f)) := \sum_{j=0}^{(k-N)/f} V^{N,k,d}_{d-m}(n-j; \sigma_m) - \sum_{j=0}^{(k-N)/f} V^{N,k,d}_{d-m}(1+(k-N)(d+f)-j; \sigma_m).
\]

(3.37)

We denote by \(\pi_f\) the map which maps the function \(g(n)\) on \(\mathbb{Z}\) to \(\sum_{j=0}^{(k-N)/f} g(n-j) - \sum_{j=0}^{(k-N)/f} g(1+(k-N)(d+f)-j)\). Then we have

\[
\pi_f(V^{N,k,d}_{d-m}(n; \sigma_m)) = \tilde{V}^{N,k,d+f}_{d-m}(n; \sigma_m \cup (f)).
\]

(3.38)
Remark 2 The above decomposition depends on the order of \( m \). Hence it is not unique.
In (3.44), we omit the subscripts of $h_i^{N,k,d+f}(n;\sigma_m \cup (f))$. We can easily see that $h_i^{N,k,d+f}(n;\sigma_m \cup (f))$ consists of monomials of degree $(d - m)$ of $\tilde{L}_j^{N,k,m'}$ ($m' < d - m$) only, because the linear dependence on $\tilde{L}_j^{N,k,d-m}$ cannot satisfy the condition (3.43). Thus we are led to the proposition by picking up the top term of the decomposition in (3.44):

**Proposition 4** The linear part of $V_{d-m}^{N,k,d}(n;\sigma_m)$ is given by the formula:

$$
\sum_{j_1=0}^{d_1(k-N)} \sum_{j_2=0}^{d_2(k-N)} \cdots \sum_{j_l(k-N)}^{d_l(k-N)} \left( \tilde{L}_{N,k,d-m}^{N,k,d-m} - \tilde{L}_{N,k,d-m}^{N,k,d-m} \right).
$$

(3.45)

Especially in the $d - m = 1$ case, we obtain the following equality:

$$
V_1^{N,k,d}(n;\sigma_{d-1}) = \sum_{j_1=0}^{d_1(k-N)} \sum_{j_2=0}^{d_2(k-N)} \cdots \sum_{j_l(k-N)}^{d_l(k-N)} \left( \tilde{L}_{N,k,d}^{N,k,d} - \tilde{L}_{N,k,d}^{N,k,d} \right).
$$

(3.46)

**Remark 3** By introducing the polynomial in $x$:

$$
\prod_{j=1}^{l(\sigma_m)} \frac{1 - x^{d_j(k-N)+1}}{1 - x} = \sum_{j=0}^{(k-N)m} A_j^{N,k}(\sigma_m)x^j,
$$

(3.47)

we can write the linear part (3.44) in a more compact form,

$$
\sum_{j_1=0}^{d_1(k-N)} \sum_{j_2=0}^{d_2(k-N)} \cdots \sum_{j_l(k-N)}^{d_l(k-N)} \left( \tilde{L}_{N,k,d-m}^{N,k,d-m} - \tilde{L}_{N,k,d-m}^{N,k,d-m} \right)
$$

= \sum_{j=0}^{(k-N)m} A_j^{N,k}(\sigma_m)\left( \tilde{L}_{n-j}^{N,k,d} - \tilde{L}_{n-j}^{N,k,d} \right).
$$

(3.48)

These are the results directly applying the constraints obtained in Proposition 1. On the other hand, we can explicitly express $V_{d-m}^{N,k,d}(n;\sigma_m)$ in terms of $\tilde{L}_n^{N,k,d}$, because the virtual Gromov-Witten invariants satisfy the Kähler equation and the associativity equation. As examples, we compute $V_{d-m}^{N,k,d}(n;\sigma_m)$ ($d \leq 3$).

**Proposition 5** $V_{d-m}^{N,k,d}(n;\sigma_m)$ ($d \leq 3$) can be written in terms of $\tilde{L}_n^{N,k,d}$ ($d \leq 3$) as follows.

$$
V_1^{N,k,1}(n;0) = \tilde{L}_n^{N,k,1} - \tilde{L}_1^{N,k,1},
$$

$$
V_2^{N,k,2}(n;0) = \tilde{L}_n^{N,k,2} - \tilde{L}_2^{N,k,2},
$$

$$
V_1^{N,k,2}(n;1) = \sum_{j=0}^{k-N} (\tilde{L}_{n-j}^{N,k,1} - \tilde{L}_{1+2(k-N)-j}^{N,k,1}),
$$

$$
V_3^{N,k,3}(n;0) = \tilde{L}_n^{N,k,3} - \tilde{L}_3^{N,k,3},
$$

$$
V_2^{N,k,3}(n;1) = \sum_{j=0}^{k-N} (\tilde{L}_{n-j}^{N,k,2} - \tilde{L}_{1+3(k-N)-j}) + h_i^{N,k,3}(n;1)
$$
where \( h_2^{N,k,3}(n;1) \) is given by the formula:

\[
\begin{align*}
\left(3.50\right) \quad h_2^{N,k,3}(n;1) &= (k-N-1) \sum_{j=0}^{2(k-N)-1} \left( \sum_{m=0}^{j} \tilde{L}_{n-m}^{N,k,1} \tilde{L}_{n-2(k-N)+j-m}^{N,k,1} - \tilde{L}_{n-m}^{N,k,1} \tilde{L}_{n-2(k-N)+j-m}^{N,k,1} \right) \\
+ \tilde{L}_{1+(k-N)}^{N,k,1} \sum_{m=j+1}^{2(k-N)-j-1} \tilde{L}_{n-m}^{N,k,1} \\
- \sum_{j=0}^{(k-N)-1} \left( \sum_{m=0}^{j} \tilde{L}_{1+3(k-N)-m}^{N,k,1} \tilde{L}_{1+(k-N)+j-m}^{N,k,1} - \tilde{L}_{1+3(k-N)-m}^{N,k,1} \tilde{L}_{1+(k-N)+j-m}^{N,k,1} \right) \\
+ \tilde{L}_{1+(k-N)}^{N,k,1} \sum_{m=j+1}^{2(k-N)-j-1} \tilde{L}_{1+3(k-N)-m}^{N,k,1}.
\end{align*}
\]

**proof**

We give a proof of the formula for \( V_2^{N,k,3}(n;1) \), because the other formulas follow obviously from the preceding discussions. First, we introduce the following virtual G-W invariant:

\[
V_1^{N,k,3}(n;2) = \sum_{j=0}^{2(k-N)} (\tilde{L}_{n-j}^{N,k,1} - \tilde{L}_{1+3(k-N)-j}^{N,k,1}),
\]

\[
V_1^{N,k,3}(n;1 + 1) = \sum_{j=0}^{2(k-N)} A_j^{N,k,1}((1) + (1)) (\tilde{L}_{n-j}^{N,k,1} - \tilde{L}_{1+3(k-N)-j}^{N,k,1}).
\]

\[
\left(3.49\right)
\]

Using the associativity equation, we obtain the equality,

\[
\begin{align*}
\frac{1}{k} \cdot v(O_{e^n-2-n} O_{e^n-1-2(k-N)-m} O_{e^n+m})_2 - \frac{1}{k} \cdot v(O_{e^n-2-n} O_{e^n-2(k-N)-m} O_{e^n})_2 \\
= \tilde{L}_{n-m}^{N,k,2} - \tilde{L}_{1+m+2(k-N)}^{N,k,2} \\
+ \sum_{j=0}^{m-1} \left( \tilde{L}_{n-j}^{N,k,1} - \tilde{L}_{1+(k-N)+j}^{N,k,1} \right) \cdot \left( \tilde{L}_{n-m-(k-N)}^{N,k,1} - \tilde{L}_{1+(k-N)}^{N,k,1} \right) \\
- \sum_{j=0}^{m+(k-N)} \left( \tilde{L}_{n-j}^{N,k,1} - \tilde{L}_{1+(k-N)+j}^{N,k,1} \right) \cdot \left( \tilde{L}_{1+m+(k-N)}^{N,k,1} - \tilde{L}_{1+(k-N)}^{N,k,1} \right).
\end{align*}
\]

\[
\left(3.52\right)
\]

Adding up \( \left(3.52\right) \) with \( m = 0, 1, \ldots, (k-N) \) and using some algebras, we can reach the desired formula in the proposition. Q.E.D.

From this proposition and the conjectural form of the generalized mirror transformation for the \( d \leq 3 \) rational G-W invariants in [11], we can see the equality \( G_{d-m}^{N,k,d}(n;\sigma_m) = V_{d-m}^{N,k,3}(n;\sigma_m) \) \((d \leq 3)\). Thus, we have derived the generalized mirror transformation up to \( d \leq 3 \) cases in [10] under the assumption of Conjecture 2.

### 4 Explicit Determination in the \( k - N = 1, \ d = 4, 5 \) cases

In this section, we restrict \( k - N \) to 1 to avoid the complication of the formulas. In this setting, we compute \( V_{d-m}^{k-1,k,d}(n;\sigma_m) \) in the \( d = 4, 5 \) cases and discuss the modification of \( V_{d-m}^{k-1,k,d}(n;\sigma_m) \) into \( G_{d-m}^{k-1,k,d}(n;\sigma_m) \). With the aid of some numerical data, we fix the modification and derive the
generalized mirror transformation in these cases. Generalization to the general \( k - N \) cases is rather straightforward.

First, we repeatedly use the associativity equation and obtain the following formula that represent \( V_{n-m}^{k-1,k,d}(n;\sigma_m) \) in terms of \( \tilde{F}_{n-1,k,d} \).

**Proposition 6** \( V_{n-m}^{k-1,k,d}(n;\sigma_m) \)'s are inductively determined as follows:

\[
\begin{align*}
V_{4-m}^{k-1,k,4}(n;0)) &= \tilde{F}_{n-1,k,4} - \tilde{F}_{5-1,k,4}, \\
V_{4-m}^{k-1,k,4}(n;1)) &= \tilde{F}_{n-1,k,3} + \tilde{L}_{n-1}^{k-1,k,3} - \tilde{L}_{4}^{k-1,k,3} - \tilde{F}_{4}^{k-1,k,3} \\
&+ (\tilde{F}_{n-1,k,2} - \tilde{L}_{3}^{k-1,k,2}) \cdot (\tilde{F}_{n-3}^{k-1,k,1} - \tilde{F}_{2}^{k-1,k,1}) \\
&+ (\tilde{F}_{n-1,k,1} - \tilde{L}_{2}^{k-1,k,1}) \cdot (\tilde{F}_{n-2}^{k-1,k,2} - \tilde{L}_{3}^{k-1,k,2}) \\
&- (\tilde{L}_{3}^{k-1,k,1} - \tilde{L}_{2}^{k-1,k,1}) \cdot V_{2}^{k-1,k,4}(n;2)) \\
&- V_{1}^{k-1,k,4}(n;3)) \cdot (\tilde{F}_{4}^{k-1,k,2} - \tilde{L}_{2}^{k-1,k,2}), \\
V_{2}^{k-1,k,4}(n;2)) &= V_{2}^{k-1,k,3}(n;1)) + \tilde{F}_{n-2}^{k-1,k,2} - \tilde{L}_{5}^{k-1,k,2} \\
&+ V_{1}^{k-1,k,3}(n;2)) \cdot (\tilde{F}_{n-3}^{k-1,k,1} - \tilde{L}_{2}^{k-1,k,1}) \\
&- V_{1}^{k-1,k,4}(n;3)) \cdot (\tilde{L}_{4}^{k-1,k,1} - \tilde{L}_{2}^{k-1,k,1}), \\
V_{1}^{k-1,k,4}(n;3)) &= \tilde{F}_{n-1,k,1} + \tilde{F}_{n-1}^{k-1,k,1} + \tilde{L}_{n-2}^{k-1,k,1} + \tilde{L}_{n-3}^{k-1,k,1} \\
&- (\tilde{L}_{5}^{k-1,k,1} + \tilde{L}_{3}^{k-1,k,2} + \tilde{L}_{4}^{k-1,k,2} + \tilde{L}_{2}^{k-1,k,1}), \\
V_{1}^{k-1,k,4}(n;1)+ (1)) &= \tilde{F}_{n-1,k,1} + 2\tilde{L}_{n-1}^{k-1,k,1} + 2\tilde{L}_{n-2}^{k-1,k,1} + \tilde{L}_{n-3}^{k-1,k,1} \\
&- (\tilde{L}_{5}^{k-1,k,1} + 2\tilde{L}_{4}^{k-1,k,2} + 2\tilde{L}_{3}^{k-1,k,2} + \tilde{L}_{2}^{k-1,k,1}), \\
V_{2}^{k-1,k,4}(n;1)+ (1)) &= V_{2}^{k-1,k,3}(n;1)) + V_{2}^{k-1,k,3}(n-1;1)) - (\tilde{L}_{5}^{k-1,k,2} - \tilde{L}_{3}^{k-1,k,2}) \\
&+ \left( V_{1}^{k-1,k,1}(n;1)) \cdot (\tilde{F}_{n-3}^{k-1,k,1} - \tilde{L}_{2}^{k-1,k,1}) \\
&+ (\tilde{L}_{n-1,k,1} - \tilde{L}_{2}^{k-1,k,1}) \cdot V_{1}^{k-1,k,1}(n-2;1)) \\
&- V_{1}^{k-1,k,4}(n;1)+ (2)) \cdot (\tilde{F}_{3}^{k-1,k,1} - \tilde{L}_{2}^{k-1,k,1}) \\
&- V_{1}^{k-1,k,4}(n;3)) \cdot (\tilde{L}_{4}^{k-1,k,1} - \tilde{L}_{2}^{k-1,k,1})). \tag{4.53}
\end{align*}
\]

**Conjecture 3** In the \( d = 4 \) case, \( V_{n-m}^{k-1,k,d}(n;\sigma_m) \) equals \( G_{n-m}^{k-1,k,d}(n;\sigma_m) \) except for \( V_{2}^{k-1,k,4}(n;1)+ (1)) \). \( G_{2}^{k-1,k,d}(n;1)+ (1)) \) is given by the formula:

\[
\begin{align*}
G_{2}^{k-1,k,4}(n;1)+ (1)) &= V_{2}^{k-1,k,3}(n;1)) + V_{2}^{k-1,k,3}(n-1;1)) - (\tilde{L}_{5}^{k-1,k,2} - \tilde{L}_{3}^{k-1,k,2}) \\
&+ \left( V_{1}^{k-1,k,1}(n;1)) \cdot (\tilde{F}_{n-3}^{k-1,k,1} - \tilde{L}_{2}^{k-1,k,1}) \\
&+ (\tilde{L}_{n-1,k,1} - \tilde{L}_{2}^{k-1,k,1}) \cdot V_{1}^{k-1,k,1}(n-2;1)) \\
&- V_{1}^{k-1,k,4}(n;1)+ (2)) \cdot (\tilde{F}_{3}^{k-1,k,1} - \tilde{L}_{2}^{k-1,k,1}) \\
&- V_{1}^{k-1,k,4}(n;3)) \cdot (\tilde{L}_{4}^{k-1,k,1} - \tilde{L}_{2}^{k-1,k,1})). \tag{4.54}
\end{align*}
\]

The modification of the factor \( \frac{1}{2} \) in \( \frac{1}{\tilde{n}} \) into the factor \( \frac{1}{4} \) in \( \frac{1}{\tilde{L}_{n}} \) is determined by one numerical data:

\[
L_{7}^{11,12,4} = 1324882975682876246483412831870565329165165953902032, \tag{4.55}
\]
and the generalized mirror transformation obtained by Conjecture 3 and (1.1) correctly predicts $L_{n}^{k-1,k,4}$ up to $k \leq 18$.

In the $d = 5$ case, we can state the following proposition under the assumption of Conjecture 2.

**Proposition 7**

\[
G_{5}^{k-1,k,5}(n; (0)) = V_{5}^{k-1,k,5}(n; (0)),
\]
\[
G_{4}^{k-1,k,5}(n; (1)) = V_{4}^{k-1,k,5}(n; (1)),
\]
\[
G_{3}^{k-1,k,5}(n; (2)) = V_{3}^{k-1,k,5}(n; (2)),
\]
\[
G_{2}^{k-1,k,5}(n; (3)) = V_{2}^{k-1,k,5}(n; (3)),
\]
\[
G_{1}^{k-1,k,5}(n; (4)) = V_{1}^{k-1,k,5}(n; (4)),
\]
\[
G_{1}^{k-1,k,5}(n; (3) + (1)) = V_{1}^{k-1,k,5}(n; (3) + (1)),
\]
\[
G_{1}^{k-1,k,5}(n; (2) + (2)) = V_{1}^{k-1,k,5}(n; (2) + (2)),
\]
\[
G_{1}^{k-1,k,5}(n; (2) + (1) + (1)) = V_{1}^{k-1,k,5}(n; (2) + (1) + (1)),
\]
\[
G_{1}^{k-1,k,5}(n; (1) + (1) + (1) + (1)) = V_{1}^{k-1,k,5}(n; (1) + (1) + (1) + (1)).
\]  \hspace{1cm} (4.56)

On the other hand, we find that there exist some non-trivial modifications to obtain $G_{3}^{k-1,k,5}(n; (1) + (1))$, $G_{2}^{k-1,k,5}(n; (1) + (2))$ and $G_{2}^{k-1,k,5}(n; (1) + (1) + (1))$ from the corresponding virtual Gromov-Witten invariants.

**Conjecture 4**

\[
G_{2}^{k-1,k,5}(n; (1) + (2)) = V_{2}^{k-1,k,4}(n; (2)) + V_{2}^{k-1,k,4}(n - 1; (2)) - \tilde{L}_{6}^{k-1,k,2} + \tilde{L}_{3}^{k-1,k,2}
\]
\[
+ \frac{8}{5} h_{i_{1}}(n) + h_{i_{2}}(n) + \frac{4}{5} h_{i_{3}}(n) - \frac{3}{5} h_{i_{4}}(n), \hspace{1cm} (4.57)
\]
\[
G_{2}^{k-1,k,5}(n; (1) + (1)) = V_{2}^{k-1,k,3}(n; (1)) + V_{2}^{k-1,k,3}(n - 1; (1)) - (\tilde{L}_{5}^{k-1,k,2} - \tilde{L}_{3}^{k-1,k,2})
\]
\[
+ \frac{4}{5} \left( V_{1}^{k-1,k,1}(n; (1)) \cdot (\tilde{L}_{n-3}^{k-1,k,1} - \tilde{L}_{2}^{k-1,k,1})
\right.
\]
\[
+ \left. (\tilde{L}_{n-1}^{k-1,k,1} - \tilde{L}_{2}^{k-1,k,1}) \cdot V_{1}^{k-1,k,1}(n - 2; (1)) \right)
\]
\[
- V_{1}^{k-1,k,4}(n; (1) + (2)) \cdot (\tilde{L}_{3}^{k-1,k,1} - \tilde{L}_{2}^{k-1,k,1})
\]
\[
- V_{1}^{k-1,k,4}(n; (3)) \cdot (\tilde{L}_{4}^{k-1,k,1} - \tilde{L}_{2}^{k-1,k,1})
\]
\[
+ V_{2}^{k-1,k,3}(n - 1; (1)) + V_{2}^{k-1,k,3}(n - 2; (1)) - (\tilde{L}_{5}^{k-1,k,2} - \tilde{L}_{3}^{k-1,k,2})
\]
\[
+ \frac{4}{5} \left( V_{1}^{k-1,k,1}(n - 1; (1)) \cdot (\tilde{L}_{n-4}^{k-1,k,1} - \tilde{L}_{2}^{k-1,k,1})
\right.
\]
\[
+ \left. (\tilde{L}_{n-1}^{k-1,k,1} - \tilde{L}_{2}^{k-1,k,1}) \cdot V_{1}^{k-1,k,1}(n - 3; (1)) \right)
\]
\[
- V_{1}^{k-1,k,4}(n - 1; (1) + (2)) \cdot (\tilde{L}_{3}^{k-1,k,1} - \tilde{L}_{2}^{k-1,k,1})
\]
\[
- V_{1}^{k-1,k,4}(n - 1; (3)) \cdot (\tilde{L}_{4}^{k-1,k,1} - \tilde{L}_{2}^{k-1,k,1}) - V_{2}^{k-1,k,3}(6; (1))
\]
\[
+ \frac{46}{25} h_{i_{1}}(n) + \frac{46}{25} h_{i_{2}}(n) + \frac{16}{25} h_{i_{3}}(n) - \frac{2}{25} h_{i_{4}}(n), \hspace{1cm} (4.58)
\]

\[
G_{3}^{k-1,k,5}(n; (1) + (1)) = V_{3}^{k-1,k,4}(n; (1)) + V_{3}^{k-1,k,4}(n - 1; (1)) - (\tilde{L}_{6}^{k-1,k,3} - \tilde{L}_{4}^{k-1,k,3})
\]
where $h_i(n)$ is a degree 2 homogeneous polynomial of $L_{m-1,k,1}$ satisfying $h_i(6) = h_i(7) = 0$ and is given by,

\[
\begin{align*}
hi_1(n) &= L_{k-1,k,1} L_{k-1,k,1} - L_{k-1,k,1} (L_{n-1,k,1} + L_{n-2,k,1} + \tilde{L}_{k-1,k,1} + \tilde{L}_{k-1,k,1}) \\
&\quad + \tilde{L}_{k-1,k,1} (L_{n-1,k,1} + L_{n-2,k,1} + \tilde{L}_{k-1,k,1}) \\
&\quad - (L_{6,k-1,k,1} L_{k-1,k,1} - L_{k-1,k,1} (L_{6,k-1,k,1} + L_{5,k-1,k,1} + \tilde{L}_{3,k-1,k,1} + \tilde{L}_{2,k-1,k,1}) \\
&\quad + \tilde{L}_{k-1,k,1} (L_{5,k-1,k,1} + L_{4,k-1,k,1} + \tilde{L}_{3,k-1,k,1})),

hi_2(n) &= L_{k-1,k,1} L_{n-3,k,1} + L_{n-1,k,1} L_{n-4,k,1} \\
&\quad - L_{4,k-1,k,1} (L_{n,k-1,k,1} + L_{n-1,k-1,k,1} + L_{n-2,k-1,k,1} + L_{n-4,k-1,k,1}) + L_{2,k-1,k,1} L_{n-2,k-1,k,1} \\
&\quad - (L_{6,k-1,k,1} L_{n,k-1,k,1} + L_{5,k-1,k,1} L_{n-3,k-1,k,1} + L_{k-1,k,1} L_{n-4,k-1,k,1}), \\

hi_3(n) &= L_{k-1,k,1} L_{n-1,k-1,k,1} + L_{k-1,k,1} L_{n-2,k-1,k,1} + L_{k-1,k,1} L_{n-4,k-1,k,1} \\
&\quad - F_{k-1,k,1} (L_{n,k-1,k,1} + L_{n-1,k-1,k,1} + L_{n-2,k-1,k,1} + L_{n-4,k-1,k,1}) - L_{2,k-1,k,1} L_{n-2,k-1,k,1} \\
&\quad - (L_{6,k-1,k,1} L_{n,k-1,k,1} + L_{5,k-1,k,1} L_{n-3,k-1,k,1} + L_{k-1,k,1} L_{n-4,k-1,k,1}), \\

hi_4(n) &= L_{k-1,k,1} L_{n-3,k-1,k,1} - L_{4,k-1,k,1} (L_{n-1,k-1,k,1} + L_{n-2,k-1,k,1} + L_{n-3,k-1,k,1}) + L_{3,k-1,k,1} L_{n-2,k-1,k,1} \\
&\quad - (L_{6,k-1,k,1} L_{n-1,k-1,k,1} + L_{5,k-1,k,1} L_{n-2,k-1,k,1} + L_{3,k-1,k,1} L_{n-3,k-1,k,1} + L_{3,k-1,k,1} L_{n-2,k-1,k,1})).
\end{align*}
\]

As in the $d = 4$ case, we have fixed the modification of the rational factors by one numerical data:

\[
L_{8}^{12,13,5} = 100355724573836807695163109854598526931747042477505803923089934593470758513921 / 180000.
\]

Check of the prediction formula obtained by Proposition 7 and Conjecture 4 takes a lot of time due to numerical computation of fixed point formulas. We checked that the formula correctly
predicts $L_{3}^{13,14,5}$. Now, we discuss the rules of modification of $V_{d-m}^{k-1,k,d}(n;\sigma_m)$ into $G_{d-m}^{k-1,k,d}(n;\sigma_m)$. First, the corresponding $V_{5-m}^{k-1,k,5}(n;\sigma_m)$ is given as follows.

$$
V_{2}^{k-1,k,5}(n; (1) + (2)) = V_{2}^{k-1,k,4}(n; (2)) + V_{2}^{k-1,k,4}(n - 1; (2)) - \tilde{L}_{6}^{k-1,k,2} + \tilde{L}_{3}^{k-1,k,2}
+ \frac{1}{2}(2h_1(n) + h_2(n) + h_3(n) - h_4(n)), \tag{4.62}
$$

$$
V_{2}^{k-1,k,5}(n; (1) + (1)) = V_{2}^{k-1,k,3}(n; (1)) + V_{2}^{k-1,k,3}(n - 1; (1)) - (\tilde{L}_{5}^{k-1,k,2} - \tilde{L}_{3}^{k-1,k,2})
+ \frac{1}{2}(V_{1}^{k-1,k,1}(n; (1)) \cdot (\tilde{L}_{n-3}^{k-1,k,1} - \tilde{L}_{2}^{k-1,k,1})
+ (\tilde{L}_{n-1}^{k-1,k,1} - \tilde{L}_{2}^{k-1,k,1}) \cdot V_{1}^{k-1,k,1}(n - 2; (1))
- V_{1}^{k-1,k,4}(n; (1) + (2)) \cdot (\tilde{L}_{3}^{k-1,k,1} - \tilde{L}_{2}^{k-1,k,1})
- V_{1}^{k-1,k,4}(n; (3)) \cdot (\tilde{L}_{4}^{k-1,k,1} - \tilde{L}_{2}^{k-1,k,1})
+ V_{2}^{k-1,k,3}(n - 1; (1)) + V_{2}^{k-1,k,3}(n - 2; (1)) - (\tilde{L}_{5}^{k-1,k,2} - \tilde{L}_{3}^{k-1,k,2})
+ \frac{1}{2}(V_{1}^{k-1,k,1}(n - 1; (1)) \cdot (\tilde{L}_{n-4}^{k-1,k,1} - \tilde{L}_{2}^{k-1,k,1})
+ (\tilde{L}_{n-1}^{k-1,k,1} - \tilde{L}_{2}^{k-1,k,1}) \cdot V_{1}^{k-1,k,1}(n - 3; (1))
- V_{1}^{k-1,k,4}(n - 1; (1) + (2)) \cdot (\tilde{L}_{3}^{k-1,k,1} - \tilde{L}_{2}^{k-1,k,1})
- V_{1}^{k-1,k,4}(n - 1; (3)) \cdot (\tilde{L}_{4}^{k-1,k,1} - \tilde{L}_{2}^{k-1,k,1})
- V_{2}^{k-1,k,3}(6; (1))
+ \frac{1}{3}(3h_1(n) + 3h_2(n) + h_3(n)), \tag{4.63}
$$

$$
V_{3}^{k-1,k,5}(n; (1) + (1)) = V_{3}^{k-1,k,4}(n; (1)) + V_{3}^{k-1,k,4}(n - 1; (1)) - (\tilde{L}_{6}^{k-1,k,3} - \tilde{L}_{4}^{k-1,k,3})
+ \frac{2}{3}(V_{2}^{k-1,k,3}(n; (1)) \cdot (\tilde{L}_{n-4}^{k-1,k,1} - \tilde{L}_{2}^{k-1,k,1})
+ (\tilde{L}_{n-1}^{k-1,k,1} - \tilde{L}_{2}^{k-1,k,1}) \cdot V_{2}^{k-1,k,3}(n - 2; (1))
- V_{2}^{k-1,k,4}(n; (2)) + V_{2}^{k-1,k,4}(n - 1; (2))
- (\tilde{L}_{6}^{k-1,k,2} - \tilde{L}_{3}^{k-1,k,2}) \cdot (\tilde{L}_{3}^{k-1,k,1} - \tilde{L}_{2}^{k-1,k,1})
- V_{1}^{k-1,k,5}(n; (4)) \cdot (\tilde{L}_{5}^{k-1,k,2} - \tilde{L}_{3}^{k-1,k,2})
+ \frac{1}{3}(V_{1}^{k-1,k,2}(n; (1)) \cdot (\tilde{L}_{n-3}^{k-1,k,2} - \tilde{L}_{3}^{k-1,k,2})
+ (\tilde{L}_{n-1}^{k-1,k,2} - \tilde{L}_{3}^{k-1,k,2}) \cdot V_{1}^{k-1,k,2}(n - 3; (1))
- V_{1}^{k-1,k,5}(n; (1) + (3)) \cdot (\tilde{L}_{4}^{k-1,k,2} - \tilde{L}_{3}^{k-1,k,2})
- V_{2}^{k-1,k,5}(n; (3)) \cdot (\tilde{L}_{4}^{k-1,k,1} - \tilde{L}_{2}^{k-1,k,1})
- \frac{1}{3}(\tilde{L}_{3}^{k-1,k,1} - \tilde{L}_{2}^{k-1,k,1}) \cdot (2h_1(n) + h_2(n) + h_3(n) - h_4(n)), \tag{4.64}
$$

The non-trivial rational coefficients of $V_{d-m}^{k-1,k,d}(n;\sigma_m)$ come from the rational factor,

$$
\prod_{j=1}^{l(\sigma_m)-1} \left(1 - \frac{n_j}{d-m}\right), \quad (0 \leq n_j < d - m), \tag{4.65}
$$
whose origin is the factor \( \frac{d_m}{(d-m)!\sigma_m} \) in the definition of \( V_{d-m}^{k-1,k,d}(n;\sigma_m) \). Then looking at the above formulas of \( V_{d-m}^{k-1,k,5}(n;\sigma_m) \) and \( G_{d-m}^{k-1,k,5}(n;\sigma_m) \), we can speculate that the rational factor in (4.63) is modified into,

\[
\prod_{j=1}^{l(\sigma_m)-1} \left( 1 - \frac{n_j}{d} \right).
\] (4.66)

Let us take \( V_2^{k-1,k,5}(n;1+(1)+(1)) \) as an example. We decompose \( V_2^{k-1,k,5}(n;1+(1)+(1)) \) according to the decomposition in (4.44):

\[
V_2^{k-1,k,5}(n;1+(1)+(1)) = \pi_1 \circ \pi_1 \circ \pi_1(L_n^{k-1,k,2} - \tilde{L}_3^{k-1,k,2}) + \pi_1 \circ \pi_1(h_i^{k-1,k,3}(n;1))) + \pi_1(h_{i2}^{k-1,k,4}(n;1+(1))) + hi_2^{k-1,k,5}(n;1+(1)+(1)).
\] (4.67)

More explicitly, each decomposed part corresponds to,

\[
\pi_1(h_{i2}^{k-1,k,4}(n;1+(1))) = \frac{1}{2} \left( L_{n-3}^{k-1,k,1} - \tilde{L}_2^{k-1,k,1} \right) \cdot V_1^{k-1,k,1}(n;1) + \pi_1(h_i^{k-1,k,3}(n;1))) + V_2^{k-1,k,3}(n;1+(1) - (\tilde{L}_5^{k-1,k,2} - \tilde{L}_3^{k-1,k,2})) + V_2^{k-1,k,3}(n-1;1) - (\tilde{L}_5^{k-1,k,2} - \tilde{L}_3^{k-1,k,2}) - V_2^{k-1,k,3}(6;1),
\] (4.68)

\[
\pi_1(h_{i2}^{k-1,k,5}(n;1+(1)+(1))) = \frac{1}{4} \left( hi_2(n) + hi_2(n) + hi_2(n) \right).
\] (4.70)

According to (4.66), the modification is given by,

\[
\frac{1}{2} \to 4, \quad \frac{1}{4} = \frac{1}{2}^2 \to \frac{4}{5}^2 = 16/25.
\] (4.71)

This modification is almost correct, but there exist some errors in the modification of (4.70), which are given by,

\[
- \frac{2}{25} (hi_1(n) + hi_2(n) + hi_3(n)).
\] (4.72)

Similar errors also occur in the cases of \( V_2^{k-1,k,5}(n;1+(2)) \) and \( V_3^{k-1,k,5}(n;1+(1)) \). One of the reasons of such errors comes from the fact that the decomposition of (4.44) is not unique, as was suggested in the remark of (3.44), but there must be other reasons because in the case of \( V_2^{k-1,k,5}(n;1+(1)+(1)) \), the decomposition is unique. Therefore, further consideration is needed.

**Question 1** Fix the general rule of the modification of \( V_{d-m}^{N,k,d}(n; \sigma_m) \) into \( G_{d-m}^{N,k,d}(n; \sigma_m) \).
Remark 4 \( \pi_1(h_{k-1,k,4}^{N,k,d}(n; (1) + (1))) \) does not vanish when \( n = 7 \), but Conjecture 4 tells us that it receives modification. Thus, the condition (iv) of Proposition 1 does not hold true for \( G_{d-m}^{N,k,d}(n; \sigma_m) \) in the \( d \geq 5 \) cases.

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