The Heat-Kernel in a Schwarzschild Geometry and the Casimir Energy

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Abstract

We obtain an hybrid expression for the heat-kernel, and from that the density of the free energy, for a minimally coupled scalar field in a Schwarzschild geometry at finite temperature. This gives us the zero-point energy density as a function of the distance from the massive object generating the gravitational field. The contribution to the zero-point energy due to the curvature is extracted too, in this way arriving at a renormalised expression for the energy density (the Casimir energy density). We use this to find an expression for other physical quantities: internal energy, pressure and entropy. It turns out that the disturbance of the surrounding vacuum generates entropy. For $\beta$ small the entropy is positive for $r > 2M$. We also find that the internal energy can be negative outside the horizon pointing to the existence of bound states. The total internal energy inside the horizon turns out to be finite but complex, the imaginary part to be interpreted as responsible for particle creation.

1 Introduction

One of the most important quantum physical quantities is probably the zero-point energy. It is needed, for instance, when we want to study back-reaction, i.e., the influence the matter fields moving in a curved back-ground assert
on the back-ground geometry itself. This would be done by solving the Einstein equations with the expectation value of the energy-momentum tensor as source:

\[ G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \langle T_{\mu\nu} \rangle, \]

where we have chosen units such that \( \kappa = \hbar = c = 1 \). It is also important for the study of renormalisation properties of a quantum field theory in a curved space-time.

In order to find this quantity, we evaluate the heat-kernel corresponding to the equations of motion for a scalar field. This calculation is carried out at finite temperature. The integral of this heat-kernel with respect to some fictitious fifth coordinate, \( \sigma \), will give the density of Helmholtz' free energy (its integral over the entire space-time will give the zeta-function, which is essentially just Helmholtz' free energy). The resulting expression is regularised and renormalised in the subsequent sections.

We do this by using the spherical symmetry of the spacetime in order to collect all the unknown bits of the heat kernel into a function \( g_{nl}(r, r'; \sigma) \) depending only upon the radial coordinates and \( \sigma \). A recursive relation for an asymptotic expansion of this unknown function can be found and solved, thereby allowing us to find the heat kernel.

From the free energy one can derive expressions for the entropy and the pressure by using standard thermodynamic relations. We find these and comment on their meaning. We also show that, to lowest order in the mass generating the Schwarzschild geometry, the free energy is that of an infinite family of particles moving in one dimension with an \( r \)-dependent mass.

### 2 Set-Up

We consider a minimally coupled scalar field \( \phi \) moving in a Schwarzschild back-ground, hence the action is

\[ S = \frac{1}{2} \int (\partial_{\mu} \phi \partial_{\nu} \phi g^{\mu\nu} - \mu^2 \phi^2) \sqrt{|g|} d^4 x, \tag{1} \]

where the metric is given by the standard expression

\[ ds^2 = \left( 1 - \frac{2M}{r} \right) dt^2 - \left( 1 - \frac{2M}{r} \right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2, \]
where $M$ is the mass of the (classical) massive object generating the Schwarzschild geometry, a black hole or a star, say. We will put the mass, $\mu$, of the scalar field equal to zero. We can later reinsert it if we find it desirable. The d’Alembertian becomes

\[ \square = \frac{1}{h} \frac{\partial^2}{\partial t^2} - \frac{1}{r^2} \frac{\partial^2}{\partial r^2} + \frac{L^2}{r^2}, \]

where $h(r) = (1 - 2M/r)$ and $L^2$ is the square of the angular momentum operator. A finite temperature can be included by complexifying the time coordinate $t$, $t \mapsto \tau$. As $\phi$ describes a Bose field, it becomes periodic in $\tau$, $\phi(\tau + \beta) = \phi(\tau)$. This implies that the time direction is not only complexified but also compactified – by appropriate scaling we can take $\tau$ to lie on the unit circle $S^1$ (see e.g. Ramond [1] or Itzykson and Zuber, [3], for further details on this). The action, furthermore, becomes “euclideanised”.

Now, the partition function

\[ Z = \int e^{-\mathcal{S}} D\phi = \int e^{-\int \frac{1}{2} \phi \square \phi \sqrt{|g|} d^3x d\tau} D\phi, \]

is simply a Gaussian and hence the functional integral can be carried out, the result being

\[ Z = (\det \square)^{-1/2}. \tag{2} \]

See for instance [3]. The quantity we are particularly interested in, is Helmholtz’ free energy, $F$, which is defined as

\[ F = -\frac{1}{\beta} \ln Z = \frac{1}{2\beta} \ln \det \square. \tag{3} \]

The internal energy and pressure, which appear in the energy momentum tensor, are related to $F$ by the usual thermodynamic relations, as is the entropy.

### 3 Functional Determinants

The major problem is apparently the calculation of $\det \square$. For consistency, and in order to establish notation, I will give a short introduction to the topic here. Descriptions can be found in e.g. Hawking [4] or Ramond [1].
A priori, the determinant of an operator $A$ must be given by the product of its eigenvalues $\lambda$

$$\det A = \prod \lambda.$$  (4)

Now, obviously this is not an easy thing to calculate. This is where the zeta function, $\zeta_A(s)$, and the heat-kernel, $G_A(x, x'; \sigma)$, comes into play. Define

$$\zeta_A(s) = \sum \lambda^{-s},$$  (5)

then

$$\det A = e^{-\zeta'_A(0)}.$$  (6)

The heat-kernel is defined through the differential equation

$$A_x G_A(x, x'; \sigma) = -\frac{\partial}{\partial \sigma} G_A(x, x'; \sigma),$$  (7)

subject to the boundary condition

$$\lim_{\sigma \to 0} G_A(x, x'; \sigma) = \delta(x - x').$$  (8)

The reason for the terminology is transparent: when $A$ is $\frac{d^2}{d\theta^2}$, the Laplacean on $S^1$, then $\zeta_A(s) \propto \zeta(s)$, where $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, is the Riemann zeta-function, and when $A = \nabla^2$ and $\sigma$ is the temperature, then the heat kernel satisfies the usual heat equation.

Denoting the eigenfunctions of $A$ by $\psi_\lambda$ we have

$$G_A(x, x'; \sigma) = \sum_\lambda \psi_\lambda(x) \psi_\lambda^*(x') e^{-\lambda \sigma},$$  (9)

and one easily proves the important relationship

$$\zeta_A(s) = \frac{1}{\Gamma(s)} \int_0^\infty d\sigma \sigma^{s-1} \int d^4 x G_A(x, x; \sigma).$$  (10)

Notice that the integral is only along the diagonal $x = x'$. We don’t have to know the eigenfunctions in order to solve the heat equation, in a $d$-dimensional Euclidean space, for instance, one can show

$$G_A(x, x'; \sigma) = (4\pi \sigma)^{-d/2} e^{-\frac{(x-x')^2}{4\sigma}},$$
in Cartesian coordinates. We will later use a mixture of techniques to arrive at a workable expression for the heat kernel. The angular and thermal part will be handled by a mode sum, whereas we will have to make do with an asymptotic expansion for the radial part in order to find the Casimir energy density.

We can arrive at an important interpretation of the value of the heat kernel along the diagonal by noting

\[
f(x) \equiv -\frac{1}{2\beta} \frac{\partial}{\partial s} \left|_{s=0} \frac{1}{\Gamma(s)} \int_0^\infty d\sigma \sigma^{s-1} G_A(x, x; \sigma),
\]

is the density of Helmholtz’ free energy

\[
F = \int f(x) \sqrt{|g|} d^4x,
\]

and is thus exactly the quantity we are interested in calculating.

## 4 Finding the Heat-Kernel

We now return to our particular problem, the minimally coupled scalar field in a Schwarzschild background, i.e. the calculation of the determinant of the d’Alembertian. For this particular operator we know that the eigenfunctions can be written

\[
\psi_\lambda(\tau, r, \Omega) = e^{-i\omega_n \tau} Y_{lm}(\Omega) g_\lambda(r),
\]

where \( \Omega \) denotes the angles, and where \( \omega_n \) is the Matsubara frequency

\[
\omega_n = \frac{2\pi n}{\beta} \quad n = 0, \pm 1, \pm 2, \ldots
\]

We have no intention of finding the eigenvalues \( \lambda \), nor of finding \( g_\lambda(r) \), instead we will rewrite the d’Alembertian as

\[
\Box = -\frac{\omega_n^2}{h} - \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \frac{l(l+1)}{r^2}.
\]

And the heat-kernel will be written as

\[
G(x, x'; \sigma) = \sum_{nlm} g_{nl}(r, r'; \sigma) Y_{lm}(\Omega) Y_{lm}^*(\Omega') e^{-\lambda_n(r)\sigma - i\omega_n(\tau - \tau')},
\]

4 Finding the Heat-Kernel
where we have defined
\[ \lambda_{nl}(r) \equiv \frac{l(l + 1)}{r^2} + \frac{\omega_n^2}{\hbar}. \] (17)

The reasoning behind this formula is as follows. The eigenfunctions can be written according to (13) as a product of a spherical harmonic, a plane wave involving the Matsubara frequency and the complexified time, and finally some unknown radial function. Clearly, we would expect the heat kernel, being a sum of products of eigenfunctions, to have a similar form, but since we cannot find the radial eigenvalues \( \lambda \) we cannot simply use (13) to find \( G \). On the other hand, it must be possible to expand it on the functions \( Y_{lm}(\Omega)Y_{lm}^*(\Omega') \) and \( e^{-i\omega_n(\tau - \tau')} \), hence the expression (16).

A mass, \( \mu \), of the scalar field can be reinserted simply by adding \( \mu^2 \) to \( \lambda_{nl}(r) \). The heat equation determines the unknown functions \( g_{nl} \). We will Taylor expand these in \( \sigma \), writing (asymptotic expansion)\(^1\)
\[ g_{nl}(r, r'; \sigma) = \frac{1}{\sqrt{4\pi\sigma}} e^{-\left(\frac{(r-r')^2}{4\sigma}\right)} \sum_{k=0}^{\infty} a_k(r, r')\sigma^k, \]
(18)

where the first term is the form \( g_{nl} \) would have in a flat space-time. The boundary condition implies \( a_0 \equiv 1 \). This asymptotic expansion is similar to the standard Schwinger-DeWitt expansion \([5, 6]\). The difference here lies in the fact that we only make an asymptotic expansion of the radial part of the heat kernel, whereas we use a mode sum for the remaining part. Our expansion is thus a hybrid, combining the asymptotic Schwinger-DeWitt expansion with the exact mode sum.

Inserting this expansion into the heat equation yields a recursion relation for the \( a_k \)’s. As we only need to know the value of the heat-kernel along the diagonal \( x = x' \), we will put \( r = r' \) to arrive at the following recursion relation by straightforward computation
\[ \left(1 + k + \frac{M}{r}\right) a_{k+1} = -L_1 a_k - L_2 a_{k-1}, \]
(19)

where we have defined the operators
\[ L_1 = \left(1 - \frac{2M}{r}\right) \frac{d^2}{dr^2} + \left(\frac{2}{r} - \frac{8M}{r^2}\right) \frac{d}{dr}. \]

\(^1\)For simplicity we will follow the common abuse of terminology and refer to this expression as “the heat kernel” although it strictly speaking is only an asymptotic formula valid for \( \sigma \) not too large.
\[
L_2 = \frac{d^2}{dr^2} + \frac{2}{r} \left( 2h - 1 \right) \frac{d}{dr},
\]

\[
h' \lambda_n' \frac{d}{dr} + 2h \lambda_n' r^{-1} + h' \lambda_n' + h \lambda_n'',
\]

Putting
\[
H = L_2 a_0 = 2h \lambda_n' r^{-1} + h' \lambda_n' + h \lambda_n'',
\]

the first few functions \( a_k \) becomes

\[
a_0 = 1,
\]

\[
a_1 = 0,
\]

\[
a_2 = \frac{-2H}{5 - h},
\]

\[
a_3 = \frac{(-1)^2 2^2 H}{7 - h} L_1 \frac{H}{5 - h},
\]

\[
a_4 = \frac{(-1)^3 2^3}{9 - h} L_1 \frac{1}{7 - h} L_1 \frac{H}{5 - h} + \frac{(-1)^2 2^2}{11 - h} L_2 \frac{H}{5 - h},
\]

\[
a_5 = \frac{(-1)^4 2^4}{11 - h} L_1 \frac{1}{7 - h} L_1 \frac{1}{5 - h} + \frac{(-1)^3 2^3}{11 - h} L_1 \frac{1}{9 - h} L_2 \frac{H}{5 - h} + \frac{(-1)^3 2^3}{11 - h} L_2 \frac{1}{7 - h} L_1 \frac{H}{5 - h}.
\]

There are singularities in these at \( r = 2M \), as one would expect. Noting that both \( H \) and \( L_2 \) are linear in \( \omega_n^2 \), we see that \( a_k \) can contain all even powers of the Matsubara frequencies up to and including \( \omega_n^{2[k/2]} \).

The method of summation of modes when calculating zeta functions have been applied in a number of other spacetimes by various authors, \cite{10}, and a number of explicit results exist for some rather simple cases.

### 5 The Free Energy Density

We have now found an asymptotic expansion for the heat-kernel and are ready to integrate it. The \( \sigma \) integration is trivial (it usually is), and will just give a sum of \( \Gamma \)-functions. Explicitly

\[
\tilde{f}(x; s) \equiv \frac{1}{\Gamma(s)} \int_0^\infty d\sigma \sum_{nlm} |Y_{lm}(\Omega)|^2 \frac{1}{\sqrt{4\pi}} \sum_{k=0}^\infty a_k e^{-\lambda_n(r)\sigma} \sigma^{s+k-3/2}
\]
\( \sum_{nlm} |Y_{lm}|^2 \frac{1}{\sqrt{4\pi}} \sum_{k=0}^{\infty} a_k(r) \frac{\Gamma(s + k - \frac{1}{2})}{\Gamma(s)} \lambda_{nl}^{\frac{1}{2} - s - k}. \)

(29)

One should note that even though the asymptotic series is only valid for \( \sigma \) small, this integral still makes sense since the factor \( \lambda_{nl} \) acts as a mass term regularisation. If we hadn’t used our hybrid method we would have had to assume a mass for the scalar field in order to obtain a meaningful integral. We can still do this, if we wish, by simply adding \( \mu^2 \) to \( \lambda_{nl} \). The addition of such a finite mass, however, will not modify the convergence properties of \( \tilde{f} \), hence the successive regularisations to be performed below are still needed. This is so because only the sum over \( n \) need to be regularised, and for fixed \( l \), \( \lambda_{nl} \) acts like a finite mass irrespective of whether \( \mu^2 \neq 0 \) or not.

For \( k \neq 0 \), differentiating this with respect to \( s \) at \( s = 0 \) simply amounts to removing the \( \Gamma \)-function in the denominator and putting \( s = 0 \) in the remaining terms. The \( k = 0 \) term is singular and has to be treated separately. We have

\[ \tilde{f}_{k=0}(x; s) = \sum_{nlm} |Y_{lm}|^2 \frac{1}{\sqrt{4\pi}} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} \lambda_{nl}^{\frac{1}{2} - s}, \]

so we have a divergent sum (for \( s = 0 \)) in the numerator, namely \( \sum_n \sqrt{z^2 + 4\pi^2 n^2} \), where \( z^2 = \frac{l(l+1)h(r)^2}{r^2} \). We can carry out the sum for an arbitrary \( s \), see e.g. Ramond [1]:

\[ \frac{1}{\Gamma(s)} \sum_{n=-\infty}^{\infty} \left( z^2 + 4\pi^2 n^2 \right)^{\frac{1}{2} - s} = \frac{1}{\Gamma(s)} \left[ z^{1-2s} + \frac{2\zeta(2s+1)}{(4\pi^2)^{s-\frac{1}{2}}} + \frac{2}{\Gamma(s + \frac{1}{2})} \sum_{\nu=1}^{\infty} \frac{x^{2\nu}}{\nu!} \frac{(-1)^{\nu}}{(4\pi^2)^{s+\nu-\frac{1}{2}}} \Gamma(s + \nu - \frac{1}{2}) \zeta(2s + 2\nu - 1) \right]. \]

Differentiating and putting \( s = 0 \) we obtain

\[ z + \frac{1}{2\pi^{3/2}}(\gamma - 2 \ln 2\pi)(4\pi - \frac{z^2}{\pi}) + \frac{2}{\sqrt{\pi}} \sum_{\nu=2}^{\infty} \frac{z^{2\nu}}{\nu!} \frac{(-1)^{\nu}}{(4\pi^2)^{\nu-\frac{1}{2}} \Gamma(\nu - \frac{1}{2}) \zeta(2\nu - 1)}, \]

which is finite. This expression is to be multiplied by \( \beta^{-1} h^{-1/2} \).

The free energy density \( f(x) \) is then

\[ f(x) = -\frac{1}{2\beta} \frac{\partial \tilde{f}}{\partial s} \bigg|_{s=0} \]
\[
\begin{aligned}
&= -\frac{1}{2\beta} \sum_{lm} |Y_{lm}|^2 (4\pi)^{-1/2} \left[-2\sqrt{\pi} \sqrt{\frac{l(l+1)}{r^2}} - \\
&\quad + 2\sqrt{\pi} \frac{1}{2\pi^{3/2}} (\gamma - 2 \ln 2\pi) \left( \frac{4\pi}{\sqrt{h\beta}} - \frac{l(l+1)\sqrt{h\beta}}{\pi r^2} \right) + \\
&\quad + 4\sqrt{\pi} \sqrt{\frac{\beta}{2\pi^{3/2}} h^{-1/2}} \sum_{\nu=2}^{\infty} \frac{(-1)^\nu}{\nu!} \left( \frac{l(l+1)h^2}{4\pi^2 r^2} \right)^\nu \Gamma(\nu - \frac{1}{2}) \zeta(2\nu) \right] - \\
&\quad - \frac{1}{2\beta} \sum_{nlm} |Y_{lm}|^2 (4\pi)^{-1/2} \sum_{k=2}^{\infty} a_k(r) \lambda_{nl}(r) x_{l-k} \Gamma(k - \frac{1}{2}).
\end{aligned}
\]

On the face of it, this function depends on all the coordinates, but looking closer we see that there is no explicit time-dependence, and using the sum rule

\[
\sum_m Y_{lm}(\Omega) Y^*_{lm}(\Omega') = \frac{2l+1}{4\pi} P_l(\mathbf{u} \cdot \mathbf{u'}),
\]

where \(\mathbf{u}, \mathbf{u'}\) are unit vectors given by the solid angles \(\Omega, \Omega'\), we see that

\[
\sum_m |Y_{lm}(\Omega)|^2 = \frac{2l+1}{4\pi} P_l(1)
\]

\[
= \frac{2l+1}{2l+2}\pi \sum_{\nu=0}^{\lfloor \frac{3l}{2} \rfloor} \frac{(-1)^\nu (2l-2\nu)!}{\nu!(l-\nu)!(l-2\nu)!}.
\]

So the angular dependency also disappears. In accordance with what we would expect, the energy density depends only on the radial coordinate. We will write the density as a mode sum

\[
f(r) = \sum_l f_l(r),
\]

where

\[
f_l(r) \equiv \begin{cases} 
\pi^{-1} \beta^{-2} \frac{1}{2\beta} (4\pi)^{-3/2} (2l+1)^{2-l} \left( \sqrt{\frac{l(l+1)\beta^2}{r^2}} + \\
\frac{1}{2\pi^{3/2}} (\gamma - 2 \ln 2\pi) \left( \frac{4\pi}{\sqrt{h\beta^2}} - \frac{l(l+1)\sqrt{h\beta^2}}{\pi r^2} \right) + \\
\end{cases}
\]
The major problem here is that the coefficients $a_k(r)$ depend on $n$ through the Matsubara frequencies. This expression needs regularisation. For instance, when $l = 0$ we encounter a singularity. Remembering that $a_k$ can include all even powers of $n$ up to and including $n^{2[k/2]}$ we write

$$a_k(r) = \sum_{t=0}^{[k/2]} b_k^{(t)}(r)n^{2t},$$

where $b_k^{(s)}$ is independent of $n$. Defining $\lambda = 4\pi^2/h\beta^2$ we then have

$$\sum_{n=-\infty}^{\infty} \sum_{k=2}^{\infty} a_k|_{l=0} \Gamma(k - \frac{1}{2}) \lambda^{\frac{1}{2} - k} = 2 \sum_{k=2}^{\infty} \Gamma(k - \frac{1}{2}) \lambda^{\frac{1}{2} - k} \sum_{t=0}^{[k/2]} b_k^{(t)}(r) \sum_{n=1}^{\infty} n^{2t+1-2k}$$

$$= 2 \sum_{k=2}^{\infty} \Gamma(k - \frac{1}{2}) \lambda^{\frac{1}{2} - k} \sum_{t=0}^{[k/2]} b_k^{(t)}(r) \zeta(2k - 2t - 1),$$

which is divergent whenever $2k - 2s - 1 = 1$, as $s \leq \frac{1}{2}k$ this can only happen when $k = 2, s = 1$.

6 The Minkowski Space Contribution and the Casimir Energy Density

To get a physical quantity we must subtract the Minkowski space contribution [5, 6], because the physical energy must be such that it vanishes in flat spacetime. Doing this we thereby get the “Casimir energy density”. The relationship between this Casimir energy density, $\zeta$-functions and cut-off regularisations has been studied in [9]. This subtracting off the flat spacetime
contribution is sometimes enough to cancel divergences, because any mani-
fold looks locally like Minkowski spacetime and hence will have the same
leading divergence as in flat spacetime. It will turn out, however, that the
Casimir energy thus obtained in this case is not finite and still needs some
regularisation. The subtraction off the flat contribution should therefore not
be seen as much as a regularisation/renormalisation as simply a normalisa-
tion. 

The heat kernel for the d’Alembertian is already known in flat spacetime
using Cartesian coordinates, but this will not be useful to us here, since we
need to act on the asymptotic expansion (17). Hence what we need, is to
find a similar expression for the flat spacetime heat kernel using a mixture of
mode sum (angular coordinates and Matsubara frequencies) and asymptotic
expansion (radial coordinate). We obtain this contribution by letting $M \to 0$
in the metric (and hence the d’Alembertian etc.) Denoting the resulting
coefficients by $\tilde{a}_k(r)$ we get the following much simpler recursion relation

$$
\tilde{a}_{k+1} = -\frac{1}{k+1} \tilde{L}_1 \tilde{a}_k - \frac{1}{k+1} \tilde{L}_2 \tilde{a}_{k-1} \\
= -\frac{1}{k+1} \left( \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} \right) \tilde{a}_k - \frac{2l(l+1)}{(k+1)r^3} \left( \frac{d}{dr} + \frac{2}{r} \right) \tilde{a}_{k-1}.
$$

(33)

The solutions go like

$$
\tilde{a}_k(r) = \alpha_k(l)r^{-2k},
$$

(34)

there is no dependency upon the frequency $\omega_n$. The coefficients $\alpha_k$ satisfy
(for $k \neq 0$)

$$
\alpha_{k+1} = \frac{2k(2k-1)}{k+1} \alpha_k + \frac{4kl(l+1)}{k+1} \alpha_{k-1},
$$

(35)

and $\alpha_0 = 1, \alpha_1 = 0$. Written down explicitly, the first few of the $\tilde{a}_k$ are

$$
\tilde{a}_0 = 1 = \alpha_0(l), \\
\tilde{a}_1 = 0 = \alpha_1(l)r^{-2},
$$

2This is not the same as saying our result for the heat kernel is smooth in $M$. We merely
use the $M \to 0$ as a short hand for saying that in that limit the metric tensor reduces to
the Minkowski spacetime one - with a compact time due to the finite temperature. The
coefficients $a_k$ will not a priori be smooth in this limit, however. Hence, new recursion
relations have to be found.

3If the mass, $\mu$, of the quanta of the scalar field is non-zero, then these would be com-
pletely different. They would still be independent of the Matsubara frequencies, though.
\[ \bar{a}_2 = 2l(l+1)r^{-4} = \alpha_2(l)r^{-4}, \]
\[ \bar{a}_3 = -8l(l+1)r^{-6} = \alpha_3(l)r^{-6}, \]
\[ \bar{a}_4 = 12l(l+1)(5-l(l+1))r^{-8} = \alpha_4(l)r^{-8}, \]
\[ \bar{a}_5 = -672l(l+1)\left(1 - \frac{17}{105}l(l+1)\right)r^{-10} = \alpha_5(l)r^{-10}. \]

Denoting the corresponding free energy density by \( f^{(0)} \), the Casimir density is defined to be
\[
\tilde{f}_{\text{Cas}}(r) = f(r) - f^{(0)}(r) = \sum_l f^\text{Cas}_l(r). \tag{36}
\]

The very simple form the functions \( \tilde{a}_k \) take on actually allow us to carry out the summation over the Matsubara frequencies, using essentially the same formulas as in the \( M \neq 0 \) case. Defining
\[
c_l \equiv -\frac{1}{2\beta}(4\pi)^{-3/2}(2l+1)2^{-l/2}\sum_{\nu=0}^{[l/2]} \frac{(-1)^\nu(2l-2\nu)!}{\nu!(l-\nu)!(l-2\nu)!}, \tag{37}
\]
the resulting free energy becomes
\[
f^\text{Cas}_l = 2\sqrt{\pi}c_l \left[ -\frac{\gamma - 2\ln 2\pi}{2\pi^{3/2}}(z^2h^{-1/2} - \tilde{z}^2 - \frac{4\pi^2}{\sqrt{h\beta^2}} + \frac{4\pi^2}{\beta^2}) + \right.
\]
\[
\frac{2}{\beta\sqrt{\pi}} \sum_{\nu=2}^{\infty} (z^{2\nu}h^{-1/2} - \tilde{z}^{2\nu}) \frac{(-1)^\nu}{\nu!(4\pi^2)^{\nu-\frac{1}{2}}} \Gamma(\nu - \frac{1}{2}) \zeta(2\nu - 1) + \sum_{k=2}^{\infty} \Gamma(k - \frac{1}{2}) \times
\]
\[
\left( \sum_{n=-\infty}^{\infty} a_k(r) \lambda_{nl}^k - \alpha_k(l)r^{-2k}\beta^{2k-1}(2\pi)^{1-2k} \left\{ \left( \frac{\tilde{z}}{2\pi} \right)^{1-2k} + 2\zeta(2k-1) \right\} \right), \tag{38}
\]
where
\[
z^2 \equiv \frac{l(l+1)h(r)\beta^2}{r^2}, \quad \tilde{z}^2 \equiv \frac{l(l+1)\beta^2}{r^2}, \tag{39}
\]
and where we have used \( zh^{-1/2} - \tilde{z} = 0 \). We should note that \( f_0^{(0)} \) contains a countable infinity of divergences, this time coming from the terms \( \tilde{z}^{1-2k}\alpha_k(l) \sim l(l+1)^{1-2k+k/2} \). We should furthermore notice that all the
terms $z^{2\nu}h^{-1/2} - \tilde{z}^{2\nu}$ are finite as $r \to 2M$, since $zh^{-1/2} \to 0$ in this limit. The only divergent parts as $r \to 2M$ are the $a_k(r)\lambda^2 - k$-terms. We can find an improved expression for these by writing once more

$$a_k(r) = \sum_{t=0}^{\left[\frac{k}{2}\right]} b_k^{(t)}(r)n^{2t},$$

we then have to perform sums of the form

$$\xi_2(s, t; a; z) \equiv \sum_{n=-\infty}^{\infty} n^{2t}(z^2 + an^2)^{-s},$$  \hspace{1cm} (40)

with $a = 2\pi$. For $t = 0$ we know the result; it is just the formula we have been using a number of times by now. Denote this sum by $\xi_1(s; a; z)$. Explicitly it is

$$\xi_1(s; a; z) = \sum_{n=-\infty}^{\infty} (z^2 + an^2)^{-s} \hspace{1cm} = \hspace{1cm} z^{-2s} + 2\frac{\Gamma(s)}{\Gamma(s-t)} \sum_{\nu=0}^{\infty} \frac{(-1)^\nu z^{2\nu}}{\nu!a^{s+\nu}} \Gamma(s + \nu) \zeta(2s + 2\nu).$$

For $t$ a positive integer (which is the only case we need to worry about) we notice the following relationship

$$\xi_2(s, t; a; z) = \frac{(-1)^t \Gamma(s)}{\Gamma(s-t)} \frac{\partial^t}{\partial a^t} \xi_1(s - t; a; z).$$  \hspace{1cm} (41)

Inserting the expression of $\xi_1$ we finally arrive at

$$\xi_2(s, t; a; z) = \frac{2\Gamma(s)}{\Gamma(s-t)} \sum_{\nu=0}^{\infty} (-1)^\nu \frac{z^{2\nu}}{\nu!} \Gamma(s + \nu) \zeta(2s - 2t + 2\nu)a^{s-t-\nu}. \hspace{1cm} (42)$$

We have to evaluate this at $s = k - \frac{1}{2}, a = 4\pi^2$ and multiply it by $(\beta^2 h)^{k-\frac{1}{2}} 2\sqrt{\pi} c_l$ in order to get the contribution to the free energy, which then reads

$$f^\text{Cas}_l(r) = 2\sqrt{\pi} c_l \left[ -\frac{\gamma - 2\ln 2\pi}{2\pi^{5/2}} (z^2 h^{-1/2} - \tilde{z}^2) - \frac{4\pi^2}{\sqrt{h\beta^2}} + \frac{4\pi^2}{\beta^2} \right] +$$
\[ \frac{2}{\beta \sqrt{\pi}} \sum_{\nu=2}^{\infty} \frac{(-1)^\nu}{\nu!(2\pi)^{2\nu-1}} \left( e^{2\nu h^{-1/2}} - \bar{z}^{2\nu} \right) \Gamma(\nu - \frac{1}{2}) \zeta(2\nu - 1) + \]

\[ \sum_{k=2}^{\infty} \frac{\Gamma(k - \frac{1}{2})(2\pi)^{1-2k}}{\Gamma(k - \frac{1}{2}- t)} \sum_{t=0}^{[k/2]} \frac{\Gamma(k - \frac{1}{2})}{\Gamma(k - \frac{1}{2}- t)} \sum_{\nu=0}^{\infty} \frac{(-1)^\nu \bar{z}^{2\nu}}{\nu!} \times \]

\[ \frac{\Gamma(k + \nu)}{\Gamma(k - \frac{1}{2} - t)} (2\pi)^{-2k-2\nu+1-2t} \zeta(2\nu + 2k - 2t - 1) - \]

\[ \alpha_k (l) r^{-2k} \beta^{2k-1} (2\pi)^{1-2k} \left\{ \left( \frac{\bar{z}}{2\pi} \right)^{1-2k} + 2 \zeta(2k - 1) + \right\} \]

\[ \frac{2}{\Gamma(k - \frac{1}{2})} \sum_{\nu=1}^{\infty} \frac{(-1)^\nu \bar{z}^{2\nu}}{\nu!(2\pi)^{2\nu}} \Gamma(k + \nu - \frac{1}{2}) \zeta(2k + 2\nu - 1) \} \].

We note the presence of a singularity when \( k = 2, t = 1 \). It is the same kind of singularity we encountered in the \( l = 0 \) case. The regularisation of such singularities is the subject of the next section.

It is easy matter to derive a recursion relation for the new coefficients \( b_k^{(t)} \); inserting \( a_k = \sum_t b_k^{(t)} n^{2t} \) into the recursion relations for the \( a_k \) yields

\[ \left( 1 + k + \frac{M}{r} \right) b_k^{(t+1)} = -L_1 b_k^{(t)} - L_2 (0) b_{k-1}^{(t)} - L_2 (1) b_{k-1}^{(t-1)}, \]

where we have written \( L_2 = L_2 (0) + n^2 L_1 (1). \) Thus

\[ L_2 (0) = -2 \frac{l(l + 1)}{r^3} \left( h \frac{d}{dr} + h' - h \frac{1}{r} \right), \]

\[ L_2 (1) = -\frac{4\pi^2}{h \beta^2} \left( h' \frac{d}{dr} + 2h' \frac{1}{r} - h'' + h'' \right). \]

The first few of these have been written out explicitly in the appendix. Remembering that \( b_k^{(t)} \) vanishes whenever \( t < 0 \) or \( t > \left[ \frac{k}{2} \right] \), we can calculate these coefficients systematically. We furthermore notice that \( b_k^{(t)} \) is \( \beta^{-2t} \) times some function which only depends upon the metric and its derivatives (i.e. on \( h, h', h'', \ldots \)), we can thus write

\[ b_k^{(t)} (r, \beta) = \beta^{-2t} c_k^{(t)} (r), \]

in this way separating the coefficients into a purely thermal and a purely geometric part.
Now, from Helmholtz’ free energy density $f^{\text{Cas}}$ we can calculate various interesting physical quantities, namely the renormalised expressions for the (modes of) the pressure, $p_l$, internal energy density, $u_l$, and finally the entropy density $s_l$ by

$$ p_l = - \left( \frac{\partial f^{\text{Cas}}}{\partial V} \right)_\beta = - \frac{1}{4\pi r^2} \left( \frac{\partial f^{\text{Cas}}(r)}{\partial r} \right)_\beta, \quad (46) $$

$$ u_l = \beta^2 f^{\text{Cas}}_l + \beta s_l, \quad (47) $$

$$ s_l = \beta^2 \left( \frac{\partial f^{\text{Cas}}}{\partial \beta} \right)_r. \quad (48) $$

However, if one attempts to calculate, say, the entropy density, one will find that it is divergent; thus there is still some regularisation to be done.

### 7 The Final Regularisations

As we saw, $f^{\text{Cas}}$, had a singularity from $l = 0$, stemming from two kinds of singularities in $f$ and $f^{(0)}$ which did not cancel each other. We furthermore noticed a similar singularity in the $k = 2, t = 1$ term. We will now return to this problem once more.

The singularities in the curved and flat space-time free energies came from two different sources, one was $\zeta(1)$, i.e. the pole of the Riemann $\zeta$-function, whereas the flat space-time contribution had a countable infinity of poles $(l(l + 1))^{-\nu}$. We thus need two different ways of dealing with the problems. It is well-known that singularities in the $\zeta$-function regularization are to be removed by taking the appropriate principal parts (see e.g. [8]), which for a meromorphic function simply amounts to extracting the finite part near a pole. Hence

$$ f_0(r) = -2\sqrt{\pi} c_0 \left[ \frac{1}{2} \sqrt{\pi} \left( \frac{4\pi^2}{h\beta^2} \right)^{3/2} \left( b_2^{(0)} \right|_{l=0} (r) + \zeta(3) + \gamma b_2^{(1)} \right|_{l=0} (r) + $$

$$ + \sum_{k=3}^{\infty} \frac{\Gamma(k - \frac{1}{2}) (4\pi^2)^{1-k} r^{\frac{3}{2} - k}}{k \pi} \sum_{t=0}^{[k/2]} b_k^{(t)} \left|_{l=0} (r) \zeta(2k - 2t - 1) \right] \right), \quad (49) $$

where we have written

$$ a_k(r) = \sum_{t=0}^{[k/2]} b_k^{(t)} (r) n^{2t}, \quad (50) $$

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as before. In obtaining this result we have used \[4, 7\]

\[
\zeta(s) = \frac{1}{s - 1} + \gamma + \sum_{n=1}^{\infty} (-1)^n \frac{\gamma_n}{n!} (s - 1)^n.
\]

Now

\[
a_2 = \frac{-2H}{5 - h} = \frac{8\pi^2\nu^2}{(5 - h)h\beta^2} \left(2r^{-1} - h^{-1}(h')^2 + h''\right) \quad \text{(for } l = 0),
\]

so \[b_2^{(0)}\big|_{l=0} = 0\], leading to

\[
f_0(r) = \frac{1}{16\pi^{3/2}} \sqrt{h\gamma(5 - h)}^{-1}(2r^{-1} - h^{-1}(h')^2 + h'') + \]

\[
\frac{1}{8\beta\sqrt{\pi}} \sum_{k=3}^{\infty} \Gamma(k - \frac{1}{2}) \left(\frac{\sqrt{h\beta}}{2\pi}\right)^{2k-1} \sum_{t=0}^{[k/2]} b_k^{(t)}\big|_{l=0} (r)\zeta(2k - 2t - 1), \quad (51)
\]

where we have inserted \(c_0 = -\frac{1}{23}(4\pi)^{-3/2}\). We notice the appearance of a temperature independent term, this will not give any contribution to the entropy then.

The remaining singularity in \(f_l\) for \(l \geq 1\) also comes from the pole in Riemann’s zeta-function and again appears only in the \(k = 2\) contribution. Here \(b_2^{(0)}\) gets multiplied by

\[
4\sqrt{\pi} c_1 \beta^3 h^{3/2} \sum_{\nu=0}^{\infty} (-1)^\nu \frac{2^\nu}{\nu!} \Gamma(\nu + \frac{3}{2}) \zeta(3 + 2\nu)(4\pi^2)^{-\nu - 3/2},
\]

which is non-singular, but \(b_2^{(1)}\) gets multiplied by

\[
4\sqrt{\pi} c_1 \frac{1}{4} \beta^3 h^{3/2} (\zeta(1)\Gamma(\frac{3}{2})(4\pi^2)^{-3/2} + ...),
\]

with the non-singular terms left out. In the spirit of \(\zeta\)-function regularization then, we must interpret this \(\zeta(1)\) as \(\gamma\), and the result is then well behaved.

\footnote{The \(n = 0\) contribution will always be proportional to \(l(l + 1)\), as this is the only \(\omega_{\nu}\)-independent term on the right hand side of the recursion relations. For \(k = 2\) we find explicitly \(b_2^{(0)} = 4r^{-4}l(l + 1)(5 - h)^{-1}(h' - h/r)\).}
For the Minkowski space contribution we have to return to the initial problem, the solution of the heat-equation, and we have to redo the calculation with $l = 0$. Hence

$$f_0^{(0)}(r) = -\frac{1}{2\beta} \frac{d}{ds} \left|_{s=0} \left( \frac{1}{\Gamma(s)} \sum_{n=-\infty}^{\infty} \int_{0}^{\infty} \sigma^{s-1} g_{n0}^{(0)}(r, \tau; \sigma) Y_{00}^2 e^{-\omega_n^{2} \sigma} d\sigma \right) \right.,$$

(52)

where $g_{n0}^{(0)}$ solves $\Box g_{n0}^{(0)} = \omega_n^{2} g_{n0}^{(0)} - \frac{\beta}{\sigma} g_{n0}^{(0)}$. So $g_{n0}^{(0)}$ is the Minkowski space analogue of the function $g_{nl}$ we introduced in order to solve the heat-equation in a Schwarzschild geometry. The solution is easily seen to be

$$g_{n0}^{(0)} = \frac{1}{\sqrt{4\pi \sigma}} e^{-\left(\frac{e-r'}{2}\right)^2} + \text{terms vanishing at } r = r'.$$

(53)

This leads to (standard calculation)

$$f_0^{(0)}(r) = -\frac{1}{91\pi \beta^2}.$$ (54)

We notice that this contribution is independent of $r$ and that it vanishes as $\beta \to \infty$, it is thus a pure effect of the finite temperature. This also implies that there is no Minkowski spacetime contribution, for $l = 0$, to the pressure, but only to the entropy and internal energy. It is interesting to point out that in fact the entire Minkowski space contribution is $\beta$-dependent, also for $l \neq 0$.

The $l = 0$ contribution to the Casimir energy density can now be written down:

$$f_0^{\text{Cas}} = \frac{\sqrt{h} \gamma}{16\pi^{3/2}(5-h)} (2r^{-1} - h^{-1} h'^2 + h'') + \frac{1}{91\pi \beta^2} +$$

$$+ \frac{1}{8\beta \sqrt{\pi}} \sum_{k=3}^{\infty} \Gamma(k - \frac{1}{2}) \left( \frac{\sqrt{h} \beta}{2\pi} \right)^{2k-1} \left[ \sum_{t=0}^{[k/2]} b_k^{(t)} \right]_{l=0} \zeta(2k - 2t - 1).$$

(55)

The temperature independent part of this is negative for $r$ not too far away from the Schwarzschild radius, and in fact looks pretty much like the usual effective potential (i.e., Coulomb plus angular momentum part) for the hydrogen atom, thereby suggesting the existence of bound states. We will later see more convincing arguments for this.
From $f_0^{\mathrm{Cas}}$ we can evaluate the $l = 0$ contribution to the entropy density, which turns out to be

$$s_0 = \frac{1}{4} \pi^{-1/2} \sum_{k=3}^{\infty} \Gamma(k - \frac{1}{2}) \left( \frac{\sqrt{h} \beta}{2 \pi} \right)^{2k-1} \sum_{t=0}^{[k/2]} b_k^{(t)} (2k - t)(k - 1 - t) - \frac{2}{91 \pi \beta}. \quad (56)$$

For $l \neq 0$ we still had a singularity, using the above prescription for its regularization we arrive at (including the $l = 0$ contribution)

$$f_l^{\mathrm{Cas}}(r) = 2 \sqrt{\pi} c_l \left[ -\frac{\gamma - 2 \ln 2\pi}{2 \pi^{5/2}} \left( z^2 h^{1/2} - \tilde{z}^2 - \frac{4\pi^2}{\sqrt{h} \beta^2} + \frac{4\pi^2}{\beta^2} \right) + \frac{2}{\beta \sqrt{\pi}} \sum_{\nu=2}^{\infty} \frac{(-1)^\nu}{\nu! (2\pi)^{2\nu-1}} (z^{2\nu} h^{-1/2} - \tilde{z}^{2\nu}) \Gamma(\nu - \frac{1}{2}) \zeta(2\nu - 1) + \beta^3 \frac{1}{2} \sqrt{\pi} \left( 4 b_2^{(0)} h^{3/2} \pi^{-1} \sum_{\nu=0}^{\infty} \frac{(-1)^\nu \tilde{z}^{2\nu}}{\nu! (2\pi)^{2\nu}} \Gamma(\nu + 3) + \zeta(2\nu + 3) \right) - \frac{\alpha_2(l) r^{-4} (2\pi)^{-3}}{\nu \Gamma(2\pi)^{2\nu}} \left( \pi \zeta(2\nu + 3) \right) + \sum_{k=3}^{\infty} \frac{\Gamma(k - \frac{1}{2}) \beta^{2k-1}}{\Gamma(k - \frac{1}{2} - t)} \left( 2 \sum_{t=0}^{[k/2]} \frac{\Gamma(k - \frac{1}{2})}{\Gamma(k - \frac{1}{2} - t)} b_k^{(t)} h^{k-\frac{1}{2}} \sum_{\nu=0}^{\infty} \frac{(-1)^\nu \tilde{z}^{2\nu}}{\nu! (2\pi)^{2k+2\nu+2t-1}} \times \Gamma(k - \frac{1}{2} + \nu) \Gamma(k - \frac{1}{2} - t) \zeta(2k - 1 + 2\nu - 2t) - \alpha_k(l) r^{-2k} (2\pi)^{1-2k} \left( \frac{\tilde{z}}{2\pi} \right)^{1-2k} + \frac{2}{\Gamma(k - \frac{1}{2})} \sum_{\nu=0}^{\infty} \frac{(-1)^\nu \tilde{z}^{2\nu}}{\nu! (2\pi)^{2\nu}} \Gamma(k + \nu - \frac{1}{2}) \zeta(2k + 2\nu - 1) \right) \right] + \frac{\sqrt{h} \gamma}{16 \pi^{3/2} (5 - h)} (2r^{-1} - h^{-1} (h')^2 + h'') + \frac{1}{91 \pi \beta^2}. \quad (57)$$

This leads to the following expression for the modes of the entropy density

$$s_l^{\mathrm{Cas}} = -\beta f_l^{\mathrm{Cas}} + 2 \sqrt{\pi} c_l \left[ -\frac{\gamma - 2 \ln 2\pi}{\pi^{5/2}} \left( \beta (z^2 h^{-1/2} - \tilde{z}^2) + \frac{4\pi^2}{\sqrt{h} \beta^2} - \frac{4\pi^2}{\beta^2} \right) +$$

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One should note that $s_l$ is negative in some regions, this need not indicate that the entropy as such is negative, but only that one can not really localise the entropy: the correct physical quantity is $S = \sum_l \int s_l r^2 dr$ and not $s_l(r)$, and this could very well be positive still. Furthermore, the entropy density is not uniquely defined as one could add a total derivative and still get the same over all entropy. This arbitrariness in the choice of $s_l$ will of course be fixed by defining a zero for the entropy proper. Furthermore, the calculation is only valid outside the horizon, $r \geq 2M$, so in principle one could have a negative entropy in this part of the system (here the universe), provided that a suitable positive entropy is present inside the horizon $r \leq 2M$. It does seem rather strange, however, to insist on this interpretation, as $r \geq 2M$ is the only observable part of the universe, and we would certainly expect physical quantities to “behave properly” in this region.

The entropy density we have found here is partly geometric in nature (induced by the curvature) and partly thermal (coming from $\beta \neq \infty$). The “geometric entropy” of black holes has been studied by Moretti in ([11], where he

\[
\frac{2}{\sqrt{\pi}} \sum_{\nu=2}^{\infty} \frac{(-1)^{\nu}(2\nu-1)}{\nu!(2\pi)^{2\nu-1}} (z^{2\nu} h^{-1/2} - \tilde{z}^{2\nu}) \Gamma(\nu - \frac{1}{2}) \zeta(2\nu - 1) + \\
\frac{1}{2} \frac{\sqrt{\pi} \beta^4}{\nu! (2\pi)^{2\nu}} \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu} z^{2\nu}(2\nu+3)}{\nu!(2\pi)^{2\nu}} \Gamma(\nu + \frac{3}{2}) \zeta(2\nu + 3) + \\
b_2^{(1)} h^{3/2} \nu^{-1/2} \left( \frac{5}{2} \gamma + \pi^{-1} \sum_{\nu=0}^{\infty} \frac{(1)^{\nu} z^{2\nu}(2\nu+5)}{\nu!(2\pi)^{2\nu}} \Gamma(\nu + \frac{3}{2}) \zeta(2\nu + 1) \right) - \\
\frac{1}{2} \alpha_2 (l) r^{-4} \pi^{-7/2} \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu} z^{2\nu}(2\nu+3)}{\nu!(2\pi)^{2\nu}} \Gamma(\nu + \frac{3}{2}) \zeta(2\nu + 3) + \\
\sum_{k=3}^{\infty} \frac{\beta^{2k}}{\Gamma(k - \frac{1}{2} + \nu)} \zeta(2\nu + 2k - 2t - 1) - \\
2 \alpha_k (l) r^{-2k} (2\pi)^{1-2k} \frac{1}{\Gamma(k - \frac{1}{2})} \sum_{t=0}^{[k/2]} \frac{(-1)^{\nu} z^{2\nu}}{\nu!(2\pi)^{2\nu}} \Gamma(\nu + \frac{1}{2}) \times \\
(2k + 2\nu - 1) \zeta(2k + 2\nu - 1)) - \frac{2}{91\pi \beta}.
\]
is also using \( \zeta \)-function techniques. He shows that the Bekenstein-Hawking entropy is purely geometrical. We refer to his paper for further details.

The \( \beta \)-independent part of \( f_{\text{Cas}}^{(t)} \) is seen to be simply (remember, \( c_l \sim \beta^{-1}, b_k^{(t)} = \beta^{-2}c_k^{(t)} \))

\[
\left. f_{\text{Cas}}^{(t)} \right|_{\text{no temp.}} = 2\sqrt{\frac{\pi}{6}} \frac{1}{2} \sqrt{\pi} c_2^{(1)} h^{3/2} (2\pi)^{-5/2} \frac{1}{2} \gamma + \frac{\sqrt{h\gamma}}{16\pi^{3/2} (5 - h)} (2r^{-1} - h^{-1}(h')^2 + h'')
\]

\[
= \left( \frac{2l + 1}{2^6 64\pi^{3/2}} \right) M^2 r^{-2} (r - 2M)^{-1/2} (2r + M)^{-1} \sum_{l=0}^{\lfloor l/2 \rfloor} \frac{(-1)^\nu (2l - 2\nu)!}{\nu!(l - \nu)!(l - 2\nu)!} + \frac{1}{16\pi^{3/2}} \sqrt{h\gamma} (5 - h)^{-1}(2r^{-1} - h^{-1}(h')^2 + h''),
\]  

(59)

(60)

It is also interesting to note that outside the Schwarzschild radius this is positive, whereas as inside it is negative imaginary. The integral, moreover, over \( r \) is finite (for \( l \neq 0 \)) in any case, even though the function is divergent for \( r = 2M \), and we get the following result

\[
\left. f_{\text{Cas}}^{(t)} \right|_{\text{no temp.}} = \frac{M^2 \gamma}{16\pi^{5/2}} r^{-2} (r - 2M)^{-1/2} (2r + M)^{-1} \frac{\sqrt{h\gamma}}{16\pi^{3/2} (5 - h)} (2r^{-1} - h^{-1}(h')^2 + h'').
\]  

It is also interesting to note that outside the Schwarzschild radius this is positive, whereas as inside it is negative imaginary. The integral, moreover, over \( r \) is finite (for \( l \neq 0 \)) in any case, even though the function is divergent for \( r = 2M \), and we get the following result

\[
\left. F \right|_{\text{no temp.}} = \frac{\gamma}{16\pi^{5/2}} \left[ \frac{\pi}{\sqrt{10M}} - i \sqrt{\frac{2}{5M}} \text{Artanh} \left( \frac{2}{\sqrt{5}} \right) \right] \quad l \geq 1,
\]  

(61)

the first term is the \( r > 2M \) contribution, whereas the second comes from \( 0 < r < 2M \). This result represents, then the energy coming from the removal of the point \( r = 0 \) from Minkowski space and placing a mass \( M \) in that singularity. It is, in other words, a proper Casimir energy similarly to the one one obtains in the original case of two plates in flat spacetime.

From (61) we can also find a temperature independent pressure by simply
taking the derivative with respect to \( r \). Doing this we find

\[
p |_{\text{no temp}} = -\frac{M^2 \gamma}{64\pi^{5/2}} \frac{8M^2 + 19Mr - 14r^2}{2r^4h^{3/2}(2r + M)^2} - \frac{\gamma}{64\pi^{5/2}} \frac{10M^4 + 16M^3r - 32M^2r^2 - 2M^3r^2 + 12Mr^3 - 11M^2r^3 + 10Mr^4 - 2r^5}{r^5h^{3/2}(2r + M)^2}
\]

This local pressure is negative outside the horizon, i.e., for \( r > 2M \), and negative imaginary for \( r < 2M \).

For small but non-vanishing \( \beta \) we get (writing \( c_l = \beta^{-1}c_1, z^2 = \beta^2Z^2, \bar{z}^2 = \beta^2\bar{Z}^2 \))

\[
f^\text{Cas}_l = 2\sqrt{\pi}c_l \left[ -\frac{\gamma - 2\ln 2\pi}{2\pi^{5/2}} \beta(Z^2h^{-1/2} - \bar{Z}^2 - \beta^{-4}4\pi^2(h^{-1/2} - 1)) + \frac{1}{8}\beta^{-7/2}Z^2 \right] + \frac{1}{2\pi} \left[ \frac{1}{2}\beta^3 - (2\pi)^{-2}Z^2 \beta^2 \Gamma(5/2) \zeta(3) \right] - \frac{\alpha_2(l)r^{-4}(2\pi)^{-3}(2\pi)^3\bar{Z}^{-3}\beta^{-4} + 4\sqrt{\pi} \beta^2 \Gamma(3/2) \zeta(3)}{\Gamma(k - \frac{1}{2})a_k(l)r^{-2k}\bar{Z}^{1-2k}\beta^{-1}} + \frac{\sqrt{h}\gamma}{16\pi^{3/2}(5 - h)}(2r^{-1} - h^{-1}(h')^2 + h'') + \frac{1}{91\pi\beta^2} + O(\beta^3).
\]

Note that in the summation over \( k \) contribution, the summation does not extend to a summation over powers of \( \beta \); \( \beta^{-1} \) is multiplied by a function of \( r, l \), \( A_l(r) := \sum_{k=3}^\infty \Gamma(k - 1/2)a_k(l)r^{-2k}\bar{Z}^{1-2k} \). The free energy resulting from this expression turns out to be everywhere negative for \( r > 2M \) and imaginary for \( r < 2M \).

The entropy density we derive from this is

\[
s^\text{Cas}_l = 2\sqrt{\pi}c_l \left[ -\beta^{-2}\gamma - 2\ln 2\pi \beta^{-2}Z^2 - 6\beta^{-2}\gamma - 2\ln 2\pi \beta^{-2}Z^2 - \beta^{-4}4\pi^2(h^{-1/2} - 1) + \frac{\alpha_2(l)r^{-4}(2\pi)^{-3}(2\pi)^3\bar{Z}^{-3}\beta^{-3} - 4\alpha_2(l)r^{-4}(2\pi)^{-3}\beta^{-3} - \zeta(3)(2\pi)^{-3}\beta^{-3}}{A_l(r)} \right] - \frac{3}{2}\sqrt{\pi}2\pi^{-7}Z^2h^{3/2}c_2(1)\zeta(3)\beta^3 + 4\alpha_2(l)r^{-4}(2\pi)^{-3}\beta^{-3} - \zeta(3)(2\pi)^{-3}\beta^{-3} + A_l(r) \right] - \frac{3}{2}\sqrt{\pi}2\pi^{-7}Z^2h^{3/2}c_2(1)\zeta(3)\beta^3 + 4\alpha_2(l)r^{-4}(2\pi)^{-3}\beta^{-3} - \zeta(3)(2\pi)^{-3}\beta^{-3} + A_l(r) \right] -
\[
\frac{2}{91\pi\beta} + O(\beta^3),
\]  
(64)

This is positive outside the Schwarzschild radius. We see that the temperature independent part of the entropy density is simply \(A_l(r)\) for \(l \geq 1\) which is independent of the mass of the black hole and comes from the renormalisation, and the \(l = 0\) contribution then has all the information about \(M\). As well the free energy as the entropy and internal energy densities are divergent on the horizon \(r = 2M\), although in a rather mild way.

The internal energy density is found to be

\[
U_{\text{int}}^{\text{Cas}} = 2\sqrt{\pi} \bar{c}_l \left[ -(\gamma - 2 \ln 2\pi)\pi^{-5/2} \beta^3 (Z^2 h^{-1/2} - \tilde{Z}^2) - 4\pi^{-1/2} (\gamma - 2 \ln 2\pi)\beta^{-1} (h^{-1/2} - 1) + \frac{3}{16\pi^3} \beta^4 \zeta(3) (Z^2 h^{-1/2} - \tilde{Z}^2) + \frac{3}{4\sqrt{\pi}} c_2^{(0)} \zeta(3) h^{3/2} \beta^4 + \frac{(2\pi)^{-5/2} c_2^{(1)} (1/2) \gamma \beta^2 + \frac{9}{16\pi^3} Z^2 \zeta(3) \beta^4}{\alpha_2(l) r^{-4} (-3 \tilde{Z}^{-3} \beta^{-2} + \frac{4}{3\pi^3} \zeta(3) \beta^4)} \right] + \frac{\sqrt{h} \gamma}{16\pi^{3/2} (5 - h)} (2r^{-1} - h^{-1} (h')^2 + h''\beta^2 - \frac{1}{91\pi}) + O(\beta^5),
\]  
(65)

which is seen to be singular at \(r = 2M\) as mentioned above. It is interesting to note that, contrary to the free energy and the entropy densities, the internal energy is integrable inside the horizon. In fact we get

\[
U_{\text{int}}^{\text{Cas}} = \int_0^{2M} dr r^2 U_{\text{int}}^{\text{Cas}} \]  
(66)

\[
= -12M^2 (l(l + 1))^{-1/2} \beta^{-2} + \frac{\gamma M \ln 5}{128\sqrt{5\pi}} (24\pi^{3/2} - 32 - 5M \pi^{3/2}) \beta^2 + \frac{8}{273\pi} M^3 \beta^{-1} (364\gamma \sqrt{\pi} - \beta - 728\sqrt{\pi} \ln 2\pi) - \frac{i\gamma M}{128\sqrt{\pi}} (1280M^2 \pi (\gamma - 2 \ln 2\pi) + \gamma \beta^3 (M - 8)) - (i\pi + \ln 5) \frac{\gamma M}{128\sqrt{5\pi}} \beta^2 (32 - 24\pi^{3/2} + 5M \pi^{3/2}),
\]  
(67)

which is clearly complex but finite. The imaginary part of this internal energy is interpreted as giving a continuous creation of particles. Notice, by
the way, that only one of the terms in $U_{\text{Cas}}^l$ is $l$-dependent. The temperature independent part, moreover, is seen to be $-\frac{8M^4}{27\pi^2}$, a negative constant. In any case, for all values of $\beta$ there is a region in which $U_{\text{Cas}}^l < 0$, the size of this $r$-interval, however, becomes rapidly smaller as $\beta$ increases. The actual depth of the resulting potential “well”, moreover, also rapidly decreases as $\beta$ grows. But for $\beta \approx 0.1$ a very large range of $r$ values exist for which the internal energy is negative, once more pointing towards the possible existence of bound states. This region only exist for $l < 10$, though. For higher values of angular momentum, no such region seems to exist, suggesting that only orbits with low angular momentum can be bound, which is what one would expect on physical grounds anyway.

Inside the Schwarzschild radius both the real and imaginary parts of the internal energy density seem to be negative for all values of $\beta$, with the energy density becoming almost entirely a large negative imaginary number near the horizon and a large negative real number near the origin. This apparently holds for all values of $l$.

8 Interpretation of $f_{\text{Cas}}^l$

We can find a closed expression for the summations over powers of $z$, by noting with Ramond [1] that

$$\sigma^{-d/2} = \frac{2^{d+1}}{\sqrt{\pi(d-2)!}} \int_0^\infty e^{-\sigma w^2} w^{d-1} dw.$$  

Inserting this at an earlier stage (i.e. before the $\sigma$-integration) we arrive at the following closed formula

$$\sum_{\nu=0}^{\infty} \frac{(-1)^\nu y^{2\nu}}{\nu!} \Gamma(\nu-\frac{1}{2}) \zeta(2\nu-1) \propto -\beta^{-1} \pi^{-3/2} \int_0^\infty \sqrt{u^2 + y^2 + 2 \ln \left(1 - e^{-\sqrt{u^2+y^2}}\right)} dw,$$

where

$$y^2 = \frac{z^2}{2\pi} = \frac{l(l+1)h(r)\beta^2}{2\pi r^2}.\quad (69)$$

Now, this is essentially the expression for the free energy of a scalar quantum field in $d = 1$ dimension (see e.g. [1]) and with the energy of the individual quanta being given by $\omega^2 = y^2/\beta^2 = l(l+1)h(r)/(2\pi r^2)$. Our Casimir energy density is hence the regularised expression for the energy of an infinite family
(labelled by their angular momentum quantum number $l$) of such quanta. This was to be expected, since the spherical symmetry of the system implies that the problem is essentially unidimensional (the radial coordinate). Moreover, the angular quantum number $l$ appears as defining an $r$-dependent mass.

Expanding to first order in $M$ we then arrive at expressions of the form (valid only for $l \neq 0$)

$$f_{l}^{\text{Cas}} = \beta^{-1} \pi^{-3/2} \int_{0}^{\infty} \left[ \sqrt{w^{2} + \frac{l(l+1)\beta^{2}}{2\pi r^{2}}} + 2 \ln \left( 1 - e^{\sqrt{w^{2} + \frac{l(l+1)\beta^{2}}{2\pi r^{2}}}} \right) \right] dw$$

$$-M \frac{\beta l(l+1)}{2\pi^{5/2} r^{4}} \int_{0}^{\infty} (w^{2}r^{2} + \beta^{2}l(l+1))^{-1/2} \times$$

$$\left( 1 + 2 \frac{e^{-\sqrt{w^{2} + l(l+1)\beta^{2}/r^{2}}}}{1 - e^{-\sqrt{w^{2} + l(l+1)\beta^{2}/r^{2}}}} \left( w^{2} + \frac{l(l+1)\beta^{2}}{2\pi r^{2}} \right)^{-1/2} \right) dw + O(M^{2}).$$

(70)

The first two terms are the contribution coming from a massless particle in a flat one dimensional space. While the second term shows how the presence of a gravitational field (here $M$) modifies the energy.

## 9 Discussion and Conclusion

We have obtained asymptotic expressions for the heat kernel and thereby for the free energy density in a Schwarzschild geometry by use of the $\zeta$-function technique. We also took the $M \to 0$ limit, i.e. the flat space limit, and we subtracted the two energies, to obtain what we called the Casimir energy density, this is the part of the zero point energy density due to the curvature (i.e. to the deviation from Minkowski spacetime) and is thus an intrinsically interesting quantity. As we would expect, these densities all turned out to depend only on the radial coordinate.

The major unanswered question, however, concerns the boundary conditions. The asymptotic expansion of the heat kernel is independent of the chosen boundary conditions, i.e. on the particular vacuum state. It is known, however, that the full renormalised energy-momentum tensor is sensitive to the particular vacuum state. Page, [12], has computed the energy momentum tensor in a Schwarzschild background for conformal coupling, and he got the
The 00 contribution to be 

\[ T^0_0 \approx (-9216M^{10} - 21504M^8 r - 18688M^6 r^2 + 7168M^4 r^3 - 2560M^2 r^4 + 4M^2 r^8 - 4Mr + r^{10}) / (122880M^2 r^{10} \pi^2) \]

Anderson et al. [13] have extended the original calculation by Page to arbitrary coupling to curvature in the Hartle-Hawking vacuum state, and they then find a finite energy density at the horizon \( \rho(r = 2M) = (15\xi - 4)/(15360M^4 \pi^2) \). They also find that \( \rho \) is positive for \( r > 2M \) only for \( \frac{4}{15} < \xi < 1.2575 \), where \( \xi \) is the non-minimal coupling. In our case we have \( \xi = 0 \), and we found the Helmholtz free energy to be positive for \( r > 2M \) in the limit of vanishing temperature. We also found, however, the internal energy to be divergent at the Schwarzschild radius, and, moreover, to possess regions in which it was negative, suggesting the existence of bound states. Our computation seems to have more in common with Boulware vacuum. Candelas, [15], has computed the renormalised value of \( \langle \phi^2 \rangle \) in the Boulware, Unruh and Hartle-Hawking vacuum states, and found that for \( r \to 2M \) the Boulware vacuum expression diverges, whereas the others are finite. Furthermore, as \( r \to \infty \), \( \langle \phi^2 \rangle \) vanishes in the Boulware vacuum and not in the other two. Both of these features are reproduced by the present calculation, which therefore seems to be closer related to the Boulware vacuum than any of the other two known vacuum states. It is impossible to tell precisely, however, since the Schwinger-DeWitt expansion which we used partially in obtaining our result is insensitive to the details of the vacuum state. This does not imply, though, that the present calculation is meaningless from a physical point of view, since Anderson and coworkers, [14], have shown that for \( (r - 2M)/M \) not too large, the Schwinger-DeWitt expansion expression for \( \langle T_{00} \rangle \) is valid, and only for \( (r - 2M)/M \gg 1 \) (i.e., \( r \gg 3M \)) does the state-dependence become noticeable. Hence, for \( (r - 2M)/M \lesssim 1 \) at least, the present calculation is valid. Moreover, since we use a hybrid technique mixing the exact mode sum with the asymptotic Schwinger-DeWitt expansion, one can expect the region of validity of this calculation to be larger than the pure Schwinger-Dewitt approach. It remains for future research to specify the precise range of validity, to see just how far away from \( r \approx 3M \) we can push the present computation.

All the calculations were carried out at finite temperature. The quantities we have found are therefore also the effective actions for a free (minimally coupled) scalar field in these backgrounds. The Casimir density then shows
how the presence of curvature modifies the effective action for a flat spacetime.

We noticed, for instance, the presence of a constant (with respect to $r$), positive contribution to the entropy, which we can interpret as generation of radiation, somewhat related to the Hawking radiation but not quite the same, as this time the radiation clearly came from the distortion of the surrounding vacuum and could not be attributed to the internal (quantum) structure of the massive object generating the Schwarzschild geometry. This contribution, moreover, depended upon the temperature (was, in fact, proportional to it) and can in this way be seen as a correction to the Hawking radiation, by letting the temperature be equal to the Hawking temperature $T_H$.

Especially interesting would be the coupling of the Casimir energy density to the curvature, i.e. plugging in the Casimir energy density into the right hand side of Einstein’s field equations and assume $M = M(r, t)$ is a sufficiently slowly varying function of $r, t$ (if not, one will have to redo the entire calculation of $f^{\text{Cas}}$ with a time and radial dependent mass). Will the singularity at $r = 0$, stemming from the point mass ($M(r, t) = M \delta(r)$ independent of $t$), become “dressed” in this way? If so, then quantum effects could be responsible for the removal of other singularities as well, most notably the singularity at the Big Bang (or the Big Crunch). Will this constitute a general “quantum removal of singularities”-mechanism? This back reaction would also give us a more precise picture of the internal structure of a black hole, as well as of its stability; we already know that black holes can evaporate due to Hawking radiation, but the calculations put forward here suggests the existence of even more such effects, this time coming directly from the disturbance of the surrounding vacuum, from the “dressing” of the black hole so to speak. This means that at least part of the Bekenstein-Hawking entropy must be of a geometrical nature, consequently supporting Moretti’s findings, [11]. The fact that we get negative values for the candidate entropy density shows that this entropy cannot be localised in one particular region (such as near the Schwarzschild radius, say) but has to be considered a global object intimately related to the global topological properties of the spacetime manifold.

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A The Explicit Form of the $b^{(t)}_k$

In this appendix we list, for the readers convenience, the explicit form of the first few $b^{(t)}_k$. For $k = 0$ we have simply $b^{(0)}_0 = 1$ while for $k = 1$ we have $b^{(0)}_1 = 0$. All of these coefficients can be written as rational functions of $r, l, \beta$ and $M$. 

27
A.0.1 The Case of \( k = 2 \)

Here we have two possible values of \( t \), namely \( t = 0, 1 \) and the functions are readily found to be

\[
\begin{align*}
  b_2^{(0)} &= -2l(l + 1) \frac{r - 4M}{2r^5 + Mr^4} \\
  b_2^{(1)} &= -\frac{16\pi^2 M^2}{\beta^2 r(r - 2M)^2(2r + M)}
\end{align*}
\]

We note the appearance of a zero at \( r = 4M \) in \( b_2^{(0)} \) and a pole of order two at \( r = 2M \) in \( b_2^{(1)} \).

A.0.2 The Case of \( k = 3 \)

Again we have only two coefficients, corresponding once more to \( t = 0, 1 \). The functional expressions, however, become somewhat more complicated.

\[
\begin{align*}
  b_3^{(0)} &= l(l + 1) \frac{96r^4 - 640Mr^3 + 156M^2r^2 + 288M^3r + 64M^4}{r^6(2r + M)^3(3r + M)} \\
  b_3^{(1)} &= \frac{\pi^2 M^2}{\beta^2 r^3(r - 2M)^2(2r + M)^3(3r + M)} \frac{768r^4 + 512Mr^3 + 244M^2r^2 - 576M^3r - 128M^4}{r^3(2r + M)^3(3r + M)^3(3r + M)}
\end{align*}
\]

Here there are no zeroes but only a pole of order three at \( r = 2M \).

A.0.3 The Case of \( k = 4 \)

For \( k = 4 \) we get very complicated expressions for all three possible values of \( t, t = 0, 1, 2 \).

The numerator for \( t = 0 \) is

\[
l(l + 1)\{(8640 - 103680l(l + 1))r^9 + (995328 - 37152l(l + 1))r^8M - (532800 + 53712l(l + 1))r^7M^2 - (1407552 - 79016l(l + 1))r^6M^3 + (71328 + 207464l(l + 1))r^5M^4 + (1035336 + 182480l(l + 1))r^4M^5 + (717648 + 85024l(l + 1))r^3M^6 + (228192 + 22504l(l + 1))r^2M^7 + (36096 + 3208l(l + 1))rM^8 + (2304 + 192l(l + 1))M^9\}
\]

while the denominator is

\[
r^8\{M^9 + 23M^8r + 233M^7r^2 + 1365M^6r^3 + 5098M^5r^4 + 12592M^4r^5 + 20576M^3r^6 + 21456M^2r^7 + 12960Mr^8 + 3456r^9\}
\]
For $t = 1$ the numerator becomes
\[
\pi^2 M \{768l(l + 1)M^{11} + (12032l(l + 1) + 1536)M^{10}r + (76160l(l + 1) + 24064)M^9r^2 + (240384l(l + 1) + 150272)M^8r^3 + (350256l(l + 1) + 142976)M^7r^4 + \\
(37968l(l + 1) - 717248)M^6r^5 - (469392l(l + 1) + 436992)M^5r^6 - \\
(265040l(l + 1) - 1506816)M^4r^7 + (310048l(l + 1) - 993792)M^3r^8 + \\
(124992l(l + 1) - 884736)M^2r^9 - (141696l(l + 1) + 829440)Mr^{10} + 27648r^{11}\}
\]
whereas the denominator reads
\[
\beta^2 r^5 \{16M^{13} + 336M^{12}r + 3016M^{11}r^2 + 14928M^{10}r^3 + \\
43297M^9r^4 + 69255M^8r^5 + 37937M^7r^6 - 52347M^6r^7 - 81046M^5r^8 + \\
3504M^4r^9 + 49376M^3r^{10} + 720M^2r^{11} - 14688Mr^{12} + 3456r^{13}\}\]
For the last case, $t = 2$, the numerator is simply
\[
\pi^4 M^3(1024r - 128M)
\]
while the denominator, a bit more complicated, reads
\[
\beta^4 r(r - 2M)^4(2r + M)^2(4r + M)
\]