A NEW PROOF OF A THOMAE-LIKE FORMULA FOR NON HYPERELLIPTIC GENUS 3 CURVES

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Abstract. We discuss Weber’s formula which gives the quotient of two Thetanullwerte for a plane smooth quartic in terms of the bitangents. In particular, we show how it can easily be derived from the Riemann-Jacobi formula.

1. Introduction

Let $g > 0$ be an integer and $M_g$ (resp. $A_g$) be the coarse moduli space of smooth, irreducible and projective curves of genus $g$ (resp. principally polarized abelian varieties of dimension $g$) over $\mathbb{C}$. These two important moduli spaces are related through the Torelli map $j$ which associates to the isomorphism class of a curve, the isomorphism class of its Jacobian with its canonical polarization. Thomae-like formulae can be seen as an explicit description of the Torelli map. Indeed, as Mumford showed in [21], a principally polarized abelian variety can be written down as intersection of explicit quadrics in a projective space. Now, the coefficients of these quadrics are determined by a certain projective vector of constants called Thetanullwerte (or Thetanulls) that we shall denote $\vartheta[q](\tau)$ (see Section 2.2) where $\tau$ is a Riemann matrix for a specific choice of bases of regular differentials and homology and $[q]$ is a characteristic. Thomae-like formulae express these constants (or quotients raised to a certain power) in terms of the geometry of the curve. In the case of a hyperelliptic curve $y^2 = \prod_{i=1}^{2g+2} (x - \alpha_i)$, Thomae himself [31, p.218] found that

$$\vartheta[q](\tau)^4 = (2i\pi)^{-2g} \cdot \det(\Omega_1)^2 \cdot \prod_{i,j \in U} (\alpha_i - \alpha_j)$$

where $\Omega_1$ is the first half of a period matrix and $U$ is a set of indices depending on the characteristic $[q]$. This formula, which we call the absolute Thomae formula has then been reproved by [10, 3, 9] using the variational method. A simpler version, which we call the relative Thomae formula, expressing the quotient $\vartheta[q](\tau)^8/\vartheta[q'](\tau)^8$ was then achieved in [34, 22, 7] using elementary arguments. Note that this formula, which involves only the roots $\alpha_i$ is generally sufficient to recover the Jacobian and can moreover be worked out over an arbitrary field [28]. The issue of finding the correct $8$th roots of the quotients is considered for $g = 1, 2$ in [5] and can be simply solved over $\mathbb{C}$ by computing the Thetanullwerte with a weak precision.

In the last 20 years, the subject came to a renaissance thanks to its applications, on one side, to theoretical physics ([29, 1] and the references of [8]) and on the other side to cryptography ([33, 26, 19, 20]). With the first applications in mind, the authors of [11, 2, 23, 8, 11, 15] have been able to find absolute or relative versions of Thomae-like formula in the case of $Z_N$-curves, i.e. curves of the form $y^N = f(x)$. As for cryptography,
the use of Thomae-like formula for the AGM point counting algorithm in the spirit of Mestre lead the second author to dig out a relative formula for non hyperelliptic genus 3 curves due to Weber [32]. This is the formula we will consider in this article (see Theorem 3.1). Note that Thomae-like formulae are also naturally connected to the Schottky problem of characterizing the locus of \( j(M_g) \) in \( \mathbb{A}_g \) and explicit solutions to reconstruct a curve from its Jacobian can be found in [27] for the genus 2 case, in [30, 18] for the general hyperelliptic case and in [32, 14] for the non hyperelliptic genus 3 case.

The combinatoric behind Weber’s formula for non hyperelliptic genus 3 curves is more involved than in the hyperelliptic case as there is no obvious ordering of the geometric data (the 28 bitangents) unlike the roots \( \alpha_i \) on the projective line. The Ancients solved this issue by the use of \textit{Aronhold bases}. We recall this theory and derive some useful lemmas in Section 2.1. In order to formulate a coordinate-free result, we consider these notions in the framework of quadratic forms over \( \mathbb{F}_2 \) as in [12]. In Section 3.1 we give an overview and a simplification of Weber’s original proof in order to compare it with ours. We want to point out (see Remark 2) that Weber’s proof may lead to an algorithm for computing Thetanullwerte in arbitrary genus in the spirit of [28].

In Section 3.2 we present our proof. It is shorter and based on \textit{Riemann-Jacobi formula} (see Corollary 2.1). This formula gives a link between \textit{Jacobian Nullwerte} (see Definition 2.5) and certain products of Thetanullwerte. Now, Jacobian Nullwerte are determinants of bitangents (Corollary 2.2) up to multiplicative constants. We use an elementary combinatoric operation to isolate one Thetanullwert, get rid of the multiplicative constants in the quotient and then get Weber’s formula up to a sign (which is left unspecified in the Riemann-Jacobi formula). In Section 3.3 we find the sign using a low precision computation and a transformation formula due to Igusa.

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2. Review on Aronhold sets, Fundamental systems and Riemann-Jacobi formula

We start with some general definitions and results on combinatorics of theta characteristics and Aronhold systems in the spirit of [12]. We then review some basic notions on theta functions (see for instance [25, Chap.I]) and the Riemann-Jacobi formula. We end up with some general results about the links between a curve and its Jacobian.

2.1. Quadratic forms over \( \mathbb{F}_2 \). Let \( g \geq 1 \) be an integer and \( V \) be a vector space of dimension \( 2g \) over \( \mathbb{F}_2 \). We fix a nondegenerate alternating form \( \langle \cdot, \cdot \rangle \) on \( V \) and we say that \( q: V \to \mathbb{F}_2 \) is a quadratic form on \( V \) if for all \( u, v \in V \)

\[
q(u + v) = q(u) + q(v) + \langle u, v \rangle.
\]

Fixing a symplectic basis \((e_1, \ldots, e_g, f_1, \ldots, f_g)\) of \((V, \langle \cdot, \cdot \rangle)\), we define the \textit{Arf invariant} \( a(q) \) of a quadratic form \( q \) by

\[
a(q) = \sum_{i=1}^{g} q(e_i)q(f_i).
\]

This invariant does not depend on the choice of the symplectic basis. One says that the form is even (resp. odd) if \( a(q) = 0 \) (resp. \( a(q) = 1 \)). The symplectic group \( \Gamma = \text{Sp}(V, \langle \cdot, \cdot \rangle) \simeq \text{Sp}_{2g}(\mathbb{F}_2) \) acts transitively on the sets of even and odd quadratic forms.
by \((\sigma \cdot q)(v) = q(\sigma^{-1}v)\). There are \(2^{g-1}(2^g + 1)\) (resp. \(2^{g-1}(2^g - 1)\)) forms with Arf invariants 0 (resp. 1).

The set \(QV\) of quadratic forms on \(V\) is a principal homogeneous space for \(V\): if \(q \in QV\) and \(v \in V\), we define \(q + v\) by \((q + v)(u) = q(u) + \langle v, u \rangle\). Similarly if \(q\) and \(q'\) are two quadratic forms, then we can define \(v = q + q' \in V\) as the unique vector such that \(\langle v, u \rangle = q(u) + q'(u)\) \(\forall u \in V\).

For any quadratic form \(q\) we compute \(q(w)\) in terms of the coordinates,

\[w = \lambda_1 e_1 + \cdots + \lambda_g e_g + \mu_1 f_1 + \cdots + \mu_g f_g\]

of any vector \(w \in V\). For simplicity, we shall write \(w = (\lambda, \mu)\), with \(\lambda = (\lambda_1, \ldots, \lambda_g)\) and \(\mu = (\mu_1, \ldots, \mu_g)\) in \(\mathbb{F}_2^g\). In coordinates, the most simple quadratic form is:

\[(1) \quad q_0(w) = \lambda \cdot \mu,\]

where \(\cdot\) denotes the usual dot product of \(g\)-tuples. Now, for any other vector \(v \in V\), with coordinates \(v = (\epsilon', \epsilon)\) (in this order), the form \(q = q_0 + v\) acts by:

\[q(w) = \epsilon \cdot \lambda + \epsilon' \cdot \mu + \lambda \cdot \mu.\]

We write \(q = \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}\). Note that

\[\epsilon = (q(e_1), \ldots, q(e_g)), \quad \epsilon' = (q(f_1), \ldots, q(f_g))\]

and therefore \(a(q) = \epsilon \cdot \epsilon'\). In coordinates we have:

\[\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} + (\lambda, \mu) = \begin{bmatrix} \epsilon + \mu \\ \epsilon' + \lambda \end{bmatrix},\]

\[\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon'_1 \\ \epsilon'_2 \\ \epsilon'_3 \end{bmatrix} + \begin{bmatrix} \epsilon_2 \\ \epsilon_3 \\ \epsilon'_2 \\ \epsilon'_3 \end{bmatrix} = \begin{bmatrix} \epsilon_1 + \epsilon_2 + \epsilon_3 \\ \epsilon'_1 + \epsilon'_2 + \epsilon'_3 \end{bmatrix}.\]

With respect to the symplectic basis \((e_i, f_j)\), the action of any \(\sigma \in \Gamma\) is represented by a matrix \(\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\) with \(a, b, c, d \in \mathbb{F}_2\) and \(a + d, b + c\) are symmetric. Then \(\sigma \cdot \begin{bmatrix} \epsilon \\ \nu \end{bmatrix} = \begin{bmatrix} \nu' \end{bmatrix}\) where

\[\begin{pmatrix} t \nu \\ t' \nu' \end{pmatrix} = \begin{pmatrix} d & c \\ b & a \end{pmatrix} \begin{pmatrix} t \epsilon \\ t' \epsilon' \end{pmatrix} + \begin{pmatrix} (c'd)_0 \\ (a'b)_0 \end{pmatrix}\]

and the 0 subscript means the column vector of the diagonal elements of the matrix.

The following lemma will be useful in computations and can be easily proven using a basis as above.

**Lemma 2.1.** Let \(q, q', q''\) be three quadratic forms. Then

\[a(q + q' + q'') = a(q) + a(q') + a(q'') + \langle q + q', q + q'' \rangle\]

and

\[q(q' + q'') = a(q + q' + q'') + a(q).\]
Definition 2.1. Let \( S = \{q_1, \ldots, q_{2g+1}\} \) be a set of quadratic forms such that any quadratic form \( q \) can be written \( q = \sum \alpha_i q_i \in QV \) with \( \alpha_i \in \{0, 1\} \subset \mathbb{Z} \). One says that \( S \) is an Aronhold set provided that the Arf invariant of any element satisfies
\[
a(q) = \frac{\#q - 1}{2} + \begin{cases} 0 & g \equiv 0, 1 \pmod{4}, \\ 1 & g \equiv 2, 3 \pmod{4}
\end{cases}
\]
where \( \#q \) is the odd integer \( \sum \alpha_i \).

There exist Aronhold sets and the symplectic group \( \Gamma \) acts transitively on them. We call an Aronhold basis an ordered Aronhold set and we denote it \((q_1, \ldots, q_{2g+1})\). Aronhold bases have a strong connection with the notion of azygetic bases.

Definition 2.2. An azygetic family of vectors is an ordered sequence \((v_1, \ldots, v_k)\) such that \( \langle v_i, v_j \rangle = 1 \) for all \( i \neq j \). An azygetic family of \( 2g \) vectors is necessarily a basis of \( V \); we say that it is an azygetic basis.

An azygetic family of quadratic forms is an ordered sequence \((q_1, q_2, \ldots, q_k)\) of quadratic forms, such that \( q_1 + q_2, \ldots, q_1 + q_k \) is an azygetic family of vectors. It is easy to check that this property is preserved under any reordering of the family.

Lemma 2.2. If \( \{q_1, \ldots, q_{2g+1}\} \) is an Aronhold set, then \((q_1, \ldots, q_{2g+1})\) is an azygetic family.

Proof. It suffices to check that any triple \( q_i, q_j, q_k \) of pair-wise different quadratic forms is azygetic. Since we have an Aronhold set the Arf invariants of \( q_i + q_j + q_k \) and of \( q_i \) are different, i.e. \( a(q_i + q_j + q_k) + a(q_i) = 1 \). Hence using Lemma 2.1
\[
1 = a(q_i + q_j + q_k) + a(q_i) = a(q_j) + a(q_k) + \langle v, w \rangle = \langle v, w \rangle,
\]
where \( v = q_i + q_j \) and \( w = q_i + q_k \).

This shows that one can associate to an Aronhold basis \((q_1, \ldots, q_{2g+1})\) an azygetic basis \((q_{2g+1} + q_1, \ldots, q_{2g+1} + q_{2g})\). This induces a bijection between Aronhold bases and azygetic bases.

Definition 2.3. A fundamental system is an azygetic family \((q_1, \ldots, q_{2g+2})\) of \( 2g + 2 \) quadratic forms such that \( q_1, \ldots, q_g \) are odd, \( q_{g+1}, \ldots, q_{2g+2} \) are even.

Let us show how to construct a fundamental system from an Aronhold basis when \( g \equiv 3 \pmod{4} \).

Proposition 2.1. Let \( g \equiv 3 \pmod{4} \), \( S = (q_1, \ldots, q_{2g+1}) \) be an Aronhold basis and denote \( q_S = \sum_{i=1}^{2g+1} q_i \). Let \( v = \sum_{i=g+1}^{2g+1} q_i \), then
\[
(p_1, \ldots, p_{2g+2}) = (q_1, \ldots, q_g, q_{g+1} + v, \ldots, q_{2g+1} + v, q_S)
\]
is a fundamental system.

Proof. Since \( g \equiv 3 \pmod{4} \), it is easy to check that the \( g \) first quadratic forms are odd and the \( g + 2 \) others are even using the formula \( a(q) = \frac{\#q - 1}{2} + 1 \) if \( q = \sum \alpha_i q_i \). So it remains to show the azygetic property. Let us denote \( v_1, v_2 \in \{v, 0\} \). Clearly,
\[
\langle p_{2g+2} + p_1, p_{2g+2} + p_j \rangle = \langle q_S + q_i + v_1, q_S + q_j + v_2 \rangle
\]
\[
= \langle q_S + q_i, q_S + q_j \rangle + \langle q_S + q_i, v_2 \rangle + \langle v_1, q_S + q_j \rangle + \langle v_1, v_2 \rangle.
\]
Since \( \#(q_S + q_i + q_j) = 2g - 1 \) we have \( a(q_S + q_i + q_j) = 1 \); hence the first term is 1 by Lemma 2.1. The last term is always 0. The second term is 1 if and only if \( i \in \{g + 1, \ldots, 2g + 1\} \) (and then \( v_1 = v \)) and \( v_2 = v \) (and then \( j \in \{g + 1, \ldots, 2g + 1\} \)). This is symmetric in \( i \) and \( j \). Hence the two central terms are always equal.  \( \square \)
Finally, we will need the following operation on fundamental systems for our proof of Weber’s formula. Let \( P = (p_1, \ldots, p_{2g+2}) \) be a fundamental system. For \( 1 \leq i \leq g \), let \( v_i = p_i + p_{2g+2} \). We will denote \( v_i + P \) the sequence of forms where we exchange the place of the \( i \)th and last element in the sequence \((p_1 + v_i, \ldots, p_{2g+2} + v_i)\).

**Proposition 2.2.** With the notation as above, \( v_i + P \) is a fundamental system.

**Proof.** Let us denote \( v_i + P = (p'_1, \ldots, p'_{2g+2}) \). First \( p'_i = p_i \) and \( p'_{2g+2} = p_{2g+2} \), so we need to check the Arf invariant of the other elements

\[
a(p'_j) = a(p_j + p_i + p_{2g+2}) = a(p_j) + a(p_i) + a(p_{2g+2}) + (p_{2g+2} + p_i, p_{2g+2} + p_j)
\]

Finally let us check the azytic condition for all triples \( p'_{2g+2}, p'_j, p'_k \). For \( j, k \neq i \) we have

\[
\langle p'_{2g+2} + p'_j, p'_{2g+2} + p'_k \rangle = (p_i + p_j, p_i + p_k) = 1
\]

and

\[
\langle p'_{2g+2} + p'_j, p'_{2g+2} + p'_k \rangle = (p_{2g+2} + p_i, p_i + p_k) = 1,
\]

because the fundamental system \( P \) is an azytic family. \( \square \)

2.2. **Riemann-Jacobi Formula.** For \( g \geq 1 \), let \( \mathbb{H}_g = \{ \tau \in \text{GL}_g(\mathbb{C}) \mid \tau = \tau, \text{Im} \tau > 0 \} \) be the Siegel upper half plane. For any \( x \in \mathbb{C} \), let \( e(x) = \exp(2i\pi x) \).

**Definition 2.4.** For \( \tau \in \mathbb{H}_g, z = (z_1, \ldots, z_g) \in \mathbb{C}^g \) and

\[
[q] = \left[ \begin{array}{c} \varepsilon \\ \varepsilon' \end{array} \right] \in \mathbb{Z}^g \oplus \mathbb{Z}^g,
\]

the function

\[
\vartheta[q](z, \tau) = \sum_{n \in \mathbb{Z}^g} e\left( \frac{1}{2}(n + \varepsilon/2)\tau^t(n + \varepsilon/2) + (n + \varepsilon/2)^t(z + \varepsilon'/2) \right),
\]

is well defined and is called the theta function with characteristic \([q]\).

Using the notation of Section 2.1, we can identify a characteristic \([q]\) modulo 2 with a quadratic form over \( \mathbb{F}_2^{2g} \), which we will still denote \( q \). The form corresponding to the characteristic \( \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] \) will be denote \( q_0 \) in the sequel. If starting with a quadratic form \( q \) (and a fixed symplectic basis), and if not mentioned otherwise, we choose for the characteristic \([q]\) a specific representative with coefficients in \( \{0, 1\} \). The choice of a representative has an impact on the sign of the theta function.

**Lemma 2.3** ([25, Th.I.3]). For any characteristic \( \left[ \begin{array}{c} \varepsilon \\ \varepsilon' \end{array} \right] \) and \( m, n \in \mathbb{Z}^g \), one has

\[
\vartheta \left[ \begin{array}{c} \varepsilon + 2m \\ \varepsilon' + 2n \end{array} \right] (z, \tau) = (-1)^{m \varepsilon} \cdot \vartheta \left[ \begin{array}{c} \varepsilon \\ \varepsilon' \end{array} \right] (z, \tau).
\]

The function \( z \mapsto \vartheta[q](z, \tau) \) is even (resp. odd) if \( a(q) \equiv \varepsilon_1 \varepsilon_2 \pmod{2} = 0 \) (resp. \( a(q) = 1 \)). When the function is even, its value at \( z = 0 \) is called a Theta:nullwert (with characteristic \([q]\)) and denoted \( \vartheta[q](\tau) \).

**Definition 2.5.** Let \([q_1], \ldots, [q_g]\) be \( g \) odd characteristics. We denote

\[
[q_1, \ldots, q_g](\tau) = \pi^{-g} \cdot \det \left( \frac{\partial \vartheta[q_j](z, \tau)}{\partial z_i} \right)_{1 \leq i, j \leq g}(0, \tau) \]

the Jacobian Nullwert with characteristics \([q_1], \ldots, [q_g]\).
There is a vast literature devoted to relations between Thetanullwerte and Jacobian Nullwerte, originating in the famous \textit{Jacobi identity}

\[
\vartheta \left[ \begin{array}{c} 1 \\ 1 \end{array} \right]' \left( 0, \tau \right) = -\pi \cdot \vartheta \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] \left( 0, \tau \right) \cdot \vartheta \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] \left( 0, \tau \right) \cdot \vartheta \left[ \begin{array}{c} 0 \\ 1 \end{array} \right] \left( 0, \tau \right).
\]

The formula has been generalized by Rosenhain, Frobenius, Weber and Riemann (see \cite{11, Th.3, 17, p.171, 9}) for precise references) up to genus 4 and in modern time by Fay \cite{9} for genus 5 (see also \cite{13} for higher derivative relations). Fay also proved that the Ancients’ conjectural formula does not hold for genus 6. All these results fit in the following general background.

\textbf{Theorem 2.1} \cite{(11, Th.3, 17, p.171, 9)}. Let \( q_1, \ldots, q_g \) be \( g \) odd characteristics such that the function \( [q_1, \ldots, q_g](\tau) \) is different from the constant 0 and is contained in the \( \mathbb{C} \)-algebra generated by the Thetanullwerte. Then

\[
[q_1, \ldots, q_g](\tau) = \sum_{\{q_{g+1}, \ldots, q_{g+2}\} \in \mathcal{S}} \pm \prod_{i=g+1}^{2g+2} \vartheta[q_i](\tau),
\]

where \( \mathcal{S} \) is the set of all sets of \( g+2 \) even forms \( \{q_{g+1}, \ldots, q_{g+2}\} \) such that \( (q_1, \ldots, q_{g+2}) \) is a fundamental system. The signs are independent of \( \tau \).

For \( g = 3 \), the result can be stated in the following simpler form.

\textbf{Corollary 2.1.} Let \( (q_1, \ldots, q_8) \) be a fundamental system, then

\[
[q_1, q_2, q_3](\tau) = \pm \prod_{i=4}^{8} \vartheta[q_i](\tau),
\]

and the sign does not depend on \( \tau \).

The sign can actually be determined by computing with a well chosen fundamental system and with a scalar matrix \( \tau \) in order to reduce the problem to a (non-zero) Jacobi identity. One then moves to a different fundamental system by the transitive action of \( \Gamma \) (see Section 3.3).

\textbf{2.3. Link between the curve and its Jacobian.} We follow here the presentation of \cite{14}. Let \( \mathcal{C} \) be a smooth, irreducible projective curve of genus \( g > 0 \) over \( \mathbb{C} \) and \( \omega = (\omega_1, \ldots, \omega_g) \) be a basis of regular differentials. Let \( \delta = (\delta_1, \ldots, \delta_{2g}) \) be a symplectic basis of \( H_1(\mathcal{C}, \mathbb{Z}) \) such that the intersection pairing has matrix \( \begin{bmatrix} 0 & \text{id} \\ -\text{id} & 0 \end{bmatrix} \). With respect to these choices, the period matrix of \( \mathcal{C} \) is \( \Omega = [\Omega_1, \Omega_2] \) where \( \Omega_1 = (\int_{\delta_i} \omega_j)_{1 \leq i \leq g, 1 \leq j \leq g} \) and \( \Omega_2 = (\int_{\delta_i} \omega_j)_{g+1 \leq i \leq 2g, 1 \leq j \leq g} \). We consider a second basis \( \eta \) of regular differentials obtained by \( \eta = \Omega_1^{-1} \omega \). The period matrix with respect to this new basis is \( [\text{id}, \tau] \) where \( \tau = \Omega_1^{-1} \Omega_2 \in \mathbb{H}_g \) and we let \( \text{Jac}(\mathcal{C}) = \mathbb{C}^g / (\mathbb{Z}^g + \tau \mathbb{Z}^g) \).

Let us denote for \( 1 \leq i \leq g \),

\[
e_i = \left( \frac{1}{2} \int_{\delta_i} \eta_j \right)_{1 \leq j \leq g} = (0, \ldots, 0, \frac{1}{2}, 0, \ldots, 0) \in \mathbb{C}^g, \quad f_i = \left( \frac{1}{2} \int_{\delta_{g+i}} \eta_j \right)_{1 \leq j \leq g} \in \mathbb{C}^g
\]

and \( v = \sum_{i=1}^{g} \lambda_i e_i + \mu_j f_j = (\lambda, \mu) \) with \( \lambda, \mu \in \mathbb{Z}^g \). We let \( W \) be the \( \mathbb{Z} \)-module generated by \( e_1, \ldots, e_g, f_1, \ldots, f_g \), so that \( \text{Jac}(\mathcal{C})[2] = W / \mathbb{Z}^g + \tau \mathbb{Z}^g \). An element \( v \in W \) also acts on a theta function as follows.
Lemma 2.4 ([25 Th.I.5]). Let \([q] = \left[ \frac{\epsilon}{\epsilon'} \right] \) be a characteristic and \(v = (\lambda, \mu) \in W\). Then
\[
(2) \quad \vartheta[q](z + v, \tau) = e \left( -\frac{1}{4} \mu^t (\epsilon' + \lambda) - \frac{1}{2} \mu^t z - \frac{1}{8} \mu^t \mu \right) \cdot \vartheta \left[ \frac{\epsilon + \mu}{\epsilon' + \lambda} \right](z, \tau).
\]

We will write \([q] + v = \left[ \frac{\epsilon + \mu}{\epsilon' + \lambda} \right]\) (the convention is different from [25 Def.I.6]). Using this notation, we can see the difference of two characteristics as an element of \(W\).

Thanks to the identifications of Section 2.1, the reduction modulo 2 of the characteristics and of \((\lambda, \mu)\) is coherent with the theory of quadratic forms on the \(\mathbb{F}_2\)-vector space \(V = \text{Jac}(C)[2]\), naturally equipped with the Weil pairing and for the choice of the symplectic basis induced by the \(e_i, f_i\) on \(V\). If we denote \(\bar{v} \in V\) the class of \(v, \bar{v}\) is identified with \((\lambda \mod 2, \mu \mod 2)\) in the isomorphism \(V \simeq \mathbb{F}_2^{2g}\) and we see that \(q + \bar{v}\) is the quadratic form associated to the characteristic \([q] + v\).

Let \(\Theta \subset \text{Jac}(C)\) be the zero divisor of the theta function \(\vartheta(z, \tau)\). The divisor \(\Theta\) can be interpreted in terms of the geometry of \(C\). For a divisor \(D \in \text{Pic}(C)\), we denote \(l(D)\) the dimension of the Riemann-Roch space associated to \(D\).

Proposition 2.3 (Riemann theorem). Let \(W_{g-1} = \{ D \in \text{Pic}^{g-1}(C), l(D) > 0 \}\) and \(\kappa\) the canonical divisor on \(C\). There exists a unique divisor class \(D_0\) of degree \(g - 1\) with \(2D_0 \sim \kappa\) and \(l(D_0)\) such that \(W_{g-1} = \Theta + D_0\). Moreover for any \(v \in V\), \(\text{mult}_v(\Theta) = l(D_0 + v)\).

A divisor (class) \(D\) such that \(2D \sim \kappa\) is called a theta characteristic divisor. Any theta characteristic divisor \(D\) is linearly equivalent to \(D_0 + v\) with \(v = (\lambda, \mu) \in V\). We can associate to \(D\) the quadratic form \(q = q_0 + v\) where \(q_0\) is the quadratic form defined in [1]. Note that
\[
a(q) = a(q_0 + v) \equiv \text{mult}_v(\Theta) \pmod{2}
\]
since \(\text{mult}_v(\Theta)\) is equal to the multiplicity at 0 of \(\vartheta[q](z, \tau)\) and the latter has the same parity as \(q\). Therefore, using Proposition 2.3, for any \(w \in V\), one has
\[
q(w) = a(q + w) + a(q) \equiv l(D + w) + l(D) \pmod{2}.
\]

Lemma 2.5. Any theta characteristic divisor \(D\) corresponds to a quadratic form \(q\) defined by
\[
q(v) = l(D + v) + l(D) \pmod{2}, \quad v \in V.
\]
It has Arf invariant \(a(q) \equiv l(D) \pmod{2}\). Note that the divisor \(D_0\) corresponds to the quadratic form \(q_0\).

Conversely, starting from a quadratic form \(q\), this correspondence defines a divisor class \(D_q = D_0 + q_0 + q\).

The basis of regular differentials \(\omega\) defines the canonical map
\[
\phi : \ C \to \mathbb{F}_2^{g-1}
\]
\[
P \mapsto (\omega_1(P) : \ldots : \omega_g(P)).
\]
If \(D \in \text{Pic}^{g-1}(C)\) is such that \(l(D) = 1\), then \(D \sim P_1 + \ldots + P_{g-1}\) with \(\phi(P_i) \in \phi(C)\) being the support of the intersection of \(\phi(C)\) with a unique hyperplane \(H_D\) of \(\mathbb{P}^{n-1}\). An equation of this hyperplane is given by the following proposition.
Proposition 2.4 ([14]). Let us denote \( \vartheta_i(z, \tau) = \frac{\partial \varphi}{\partial z_i}(z, \tau) \). Let \( D \in \text{Pic}^{g-1}(C) \) such that \( l(D) = 1 \) then

\[
(\vartheta_1(D - D_0, \tau), \ldots, \vartheta_g(D - D_0, \tau)) \Omega_1^{-1} \begin{pmatrix} X_1 \\ \vdots \\ X_g \end{pmatrix} = 0
\]

is an equation of \( H_D \).

Let \( q_1, \ldots, q_g \) be \( g \) odd quadratic forms and assume that the theta characteristic divisors \( D_{q_i} \) are such that \( l(D_{q_i}) = 1 \). Then \( H_{D_{q_i}} \) is tangent to the curve at each point \( \phi(P_i) \) such that \( D_{q_i} \sim P_1 + \ldots + P_{g-1} \). Let \( \beta_{q_i} \in \mathbb{C}[X_1, \ldots, X_g] \) be any linear polynomials such that \( H_{D_{q_i}} \) is the hyperplane with equation \( \beta_{q_i} = 0 \).

Corollary 2.2. With the notation above, there exist constants \( \eta_i = \eta_{[q_i], \beta_{q_i}} \) depending on \( [q_i], \beta_{q_i} \) (and the period matrix \( \Omega \)) such that

\[
[\beta_{q_1}, \ldots, \beta_{q_g}] = \left( \prod_{i=1}^{g} \eta_i \right) \cdot [q_1, \ldots, q_g]
\]

where \( [\beta_{q_1}, \ldots, \beta_{q_g}] \) is the determinant of the coefficients of the \( \beta_{q_i} \) in the basis \( X_1, \ldots, X_g \).

Proof. Let \( v_i = D_{q_i} - D_0 = [q_0] + [q_i] = (\lambda_i, \mu_i) \in W \) for \( 1 \leq i \leq g \). By (2) one has

\[
\vartheta(z + v_i, \tau) = e \left( -\frac{1}{4} \mu_i^t \lambda_i - \frac{1}{2} \mu_i^t z - \frac{1}{8} \mu_i \tau^t \mu_i \right) \cdot \vartheta \left( \frac{\mu_i}{\lambda_i} \right)(z, \tau)
\]

and for all \( 1 \leq j \leq g \), we have

\[
\vartheta_j(D_{q_i} - D_0, \tau) = \frac{\partial \vartheta(z + v_i, \tau)}{\partial z_j}(0, \tau) = c_i \cdot \frac{\partial \vartheta(q_i)(z, \tau)}{\partial z_j}(0, \tau)
\]

where \( c_i \) depends on \( \tau \) and \( [q_i] \). Proposition 2.4 shows that

\[
\beta_{q_i} = c_i \cdot \left( \frac{\partial \vartheta(q_i)(z, \tau)}{\partial z_1}(0, \tau), \ldots, \frac{\partial \vartheta(q_i)(z, \tau)}{\partial z_g}(0, \tau) \right) \Omega_1^{-1} \begin{pmatrix} X_1 \\ \vdots \\ X_g \end{pmatrix}
\]

for a constant \( c_i \) depending on \( \beta_{q_i}, [q_i] \) and \( \tau \). Taking the determinant, we get the result.

\[ \square \]

3. Proofs of Weber’s formula

We now restrict to \( g = 3 \) and we assume that \( C \) is a non hyperelliptic curve of genus \( 3 \) over \( \mathbb{C} \). Let \( (\omega_1, \omega_2, \omega_3) \) be a basis of regular differentials. The canonical embedding \( \phi : P \mapsto (\omega_1(P) : \omega_2(P) : \omega_3(P)) \in \mathbb{P}^2 \) identifies \( C \) with a smooth plane quartic. Let \( D \) be a theta characteristic divisor of \( C \). If \( l(D) > 0 \), then \( D \sim P + Q \), where \( P, Q \in C \). But then \( l(D) = 1 \), otherwise, there would be a non constant function of degree 2 with pole \( P + Q \) and \( C \) would be hyperelliptic. For the canonical embedding, the line \( H_D \) defined by \( P, Q \) (resp. the tangent to \( C \) if \( P = Q \)) is tangent to \( C \) at \( P \) and \( Q \) (resp. has intersection multiplicity 4 at \( P \)). Such a line is called a bitangent to \( C \). Using the bijection of Lemma 2.5 we see that such a \( D \) correspond to an odd quadratic form \( q \). Hence the number of bitangents in 28. To describe this set, we introduce an Aronhold set \( S = \{q_1, \ldots, q_7\} \) associated to a given even form \( q_S = \sum_{i=1}^{7} q_i \) (this is always possible by the transitive action of \( \Gamma \) on Aronhold sets). For all \( 1 \leq i \neq j \leq 7 \), we denote \( q_{ij} = q_S + q_i + q_j \) the sum of 5 distinct \( q_i \)s, hence this is an odd form. The 28 odd forms can all be written as
$q_i$ or $q_{ij}$ and we denote by $D_i = D_{q_i}$ or $D_{ij} = D_{q_{ij}}$ (resp. $\beta_i, \beta_{ij}$) the theta characteristic divisor (resp. an arbitrary fixed linear polynomial defining $H_{\alpha_i}$ or $H_{\alpha_{ij}}$) associated to them. Note also that any even form different from $q_S$ can be written $q_{ijk} = q_i + q_j + q_k$ with $i,j,k$ distinct. We can now state Weber’s formula.

**Theorem 3.1 (Weber’s formula [32, p.162]).** Let $q_S, q_T$ be two distinct even forms. Let $S = \{q_1, \ldots, q_7\}$ be an Aronhold set such that $q_S = \sum_{i=1}^7 q_i$ and assume that we have ordered $S$ so that $q_1 + q_2 + q_3 = q_T$. Define a Riemann matrix $\tau \in \mathbb{H}_3$ attached to $\text{Jac}(C)$ following the beginning of Section 2.3. Then

$$(3) \quad \left(\frac{\partial[q_S](\tau)}{\partial[q_T](\tau)}\right)^4 = (-1)^{a(q_0 + q_2 + q_T)} \cdot \left[\beta_1, \beta_2, \beta_3 \cdot [\beta_1, \beta_{12}, \beta_{13}] \cdot [\beta_{12}, \beta_2, \beta_{23}] \cdot [\beta_{13}, \beta_{23}, \beta_3] \cdot [\beta_{23}, \beta_{13}, \beta_{12}] \cdot [\beta_{23}, \beta_3, \beta_2] \cdot [\beta_3, \beta_{13}, \beta_1] \cdot [\beta_2, \beta_1, \beta_{12}]ight]$$

where $[\beta_i, \beta_j, \beta_k]$ is the determinant of the coefficients of $\beta_i, \beta_j$ and $\beta_k$.

Let us point out that each defining polynomial of a bitangent appears as many times on the numerator as on the denominator, so the quotient of the two expressions does not depend on the choice of a fixed polynomial. Similarly, as the characteristics $[q_S], [q_T]$ appear in Thetanullwerte raised to an even power, one can choose any representative for the characteristics associated to $q_S, q_T$. However, the dependence on the choices of symplectic basis and regular differentials appear on the left in the choice of $\tau$ and on the right side in the choice of $q_0$.

### 3.1 Sketch of Weber’s proof.

The original proof of Weber’s formula can be found in his book [32]. We want to give here an overview of his proof, formulated in a simpler and modern form. For symmetry, we denote $p_1 = q_S$ and $p_2 = q_T$ and then

$$p_1 + p_2 = q_1 + q_{23} = q_2 + q_{13} = q_3 + q_{12}.$$  

Let

$$D_1 \sim A + B, \quad D_{23} \sim G + H$$

be the two theta characteristics divisors associated to $q_1$ and $q_{23}$. The points $A, B$ (resp. $H, G$) are the support of the bitangents $\beta_1, (\text{resp. } \beta_{23})$. Let $S = S_1 + S_2 + S_3$ be an arbitrary generic effective divisor of degree 3 on $C$. We introduce

$$f_{i,S}(P) = \partial[p_1](P + S - \kappa)$$

with $\kappa = 2(A + B)$, so this fixes a precise value for $f_{i,S}(P)$ in $\mathbb{C}$ once paths have been chosen to each point. The $f_{i,S}(P)$ are regular sections of line bundles over $C$. According to Riemann theorem [25, V. Th.1], if $f_{i,S}$ is not identically zero then its zero divisor $(f_{i,S})_0$ has degree three and satisfies

$$(f_{i,S})_0 \sim D_0 + (p_i + q_0) + \kappa - S = D_{p_i} + \kappa - S.$$  

Since $l(\kappa + D_{p_i}) = 4$, we let $t_i, u_i, v_i, w_i$ be a basis of sections (called Wurzelfunctionen in Weber’s book). We then define

$$\chi_{i,S}(P) = \det \begin{pmatrix}
  t_i(P) & u_i(P) & v_i(P) & w_i(P) \\
  t_i(S_1) & u_i(S_1) & v_i(S_1) & w_i(S_1) \\
  t_i(S_2) & u_i(S_2) & v_i(S_2) & w_i(S_2) \\
  t_i(S_3) & u_i(S_3) & v_i(S_3) & w_i(S_3)
\end{pmatrix}.$$  

Since $\chi_{i,S}(S_j) = 0$ for $1 \leq j \leq 3$, we see that $(\chi_{1,S})_0 = S + R_i$ where $R_i$ is an effective divisor of degree 3, uniquely defined by $R_i + S \sim \kappa + D_{p_i}$. Now

$$(f_{i,S})_0 \sim D_{p_i} + \kappa - S \sim R_i$$  

so actually \((f_{1,S})_0 = R_i\). Therefore, \((f_{1,S})_0 - (f_{2,S})_0 = R_1 - R_2 = (\chi_{1,S})_0 - (\chi_{2,S})_0\) and there exists a constant \(\alpha_S\) such that
\[
\frac{f_{1,S}(P)}{f_{2,S}(P)} = \alpha_S \cdot \frac{\chi_{1,S}(P)}{\chi_{2,S}(P)}.
\]

**Lemma 3.1.** \(\alpha_S\) does not depend on \(S\).

**Proof.** One has
\[
\frac{f_{1,S}(P)}{f_{2,S}(P)} \cdot \frac{\chi_{2,S}(P)}{\chi_{1,S}(P)} = \alpha_S.
\]
We have to prove that the expression on the left side does not depend on the support of \(S = S_1 + S_2 + S_3\). It is enough to show that \(\alpha_S = \alpha_{S'_1 + S_2 + S_3}\) for another generic point \(S'_1\). Note that \(f_{1,S}(S'_1) = \vartheta[p_1](S'_1 + S_1 + S_2 + S_3 - \kappa) = f_{1,S'_1 + S_2 + S_3}(S_1)\) and \(\chi_{1,S}(S'_1) = -\chi_{1,S'_1 + S_2 + S_3}(S_1)\). Hence
\[
\alpha_S = \frac{f_{1,S}(S'_1)}{f_{2,S}(S'_1)} \cdot \frac{\chi_{2,S}(S'_1)}{\chi_{1,S}(S'_1)} = \frac{f_{1,S'_1 + S_2 + S_3}(S_1)}{f_{2,S'_1 + S_2 + S_3}(S_1)} \cdot \frac{\chi_{2,S'_1 + S_2 + S_3}(S_1)}{\chi_{1,S'_1 + S_2 + S_3}(S_1)} = \alpha_{S'_1 + S_2 + S_3}.
\]

In the sequel we are going to use two particular divisors \(S\).

**Lemma 3.2.** If \(S = B + A + B\) then
\[
\frac{f_{1,S}(A)^2}{f_{2,S}(A)^2} = \frac{\vartheta[p_1](0)^2}{\vartheta[p_2](0)^2}.
\]
If moreover \(S' = B + G + H\) then
\[
\frac{f_{1,S'}(P)^2}{f_{2,S'}(P)^2} = (-1)^{a(q_0 + p_1 + p_2)} \cdot \frac{f_{2,S}(P)^2}{f_{1,S}(P)^2}.
\]

**Proof.** The first equality is trivial. As for the second, let \([p_1] = \left[\begin{array}{c} \epsilon \\ \epsilon' \end{array}\right]\) and
\[
(G + H) - (A + B) \sim D_{23} - D_1 = [q_{23}] - [q_1] = (\lambda, \mu),
\]
so that \([p_2] = [p_1] + [q_{23}] - [q_1] = \left[\begin{array}{c} \epsilon + \mu \\ \epsilon' + \lambda \end{array}\right]\) (the choices for the lifts of the quadratic forms are irrelevant because we are going to take squares). Then using (2)
\[
f_{1,S'}(P)^2 = \vartheta[p_1](P + B + G + H - \kappa)^2
= \vartheta[p_1](P + B + A + B - \kappa + (G + H) - (A + B))^2
= (-1)^{\mu' + \epsilon - \lambda} \cdot c_{r,\mu,\varepsilon} \cdot f_{2,S}(P)^2
\]
where \(z = P + B + A + B - \kappa, c_{r,\mu,\varepsilon}\) is a constant depending on \(\tau, \mu, \varepsilon\) and \(f_{2,S}(P)^2 = (-1)^{\mu + \epsilon} \cdot c_{r,\mu,\varepsilon} \cdot f_{1,S}(P)^2\). Hence for the quotient we get
\[
\frac{f_{1,S'}(P)^2}{f_{2,S'}(P)^2} = (-1)^{\mu' + \epsilon - \lambda} \cdot \frac{f_{2,S}(P)^2}{f_{1,S}(P)^2}.
\]
From this we get that
\[
\frac{f_1 s(A)^2 \cdot f_2 s'(A)^2}{f_2 s(A)^2 \cdot f_1 s'(A)^2} = (-1)^{(q_0 + p_1 + p_2)} \cdot \frac{\vartheta [p_1](0)^4}{\vartheta [p_2](0)^4} = \frac{\chi_1 s(A)^2 \cdot \chi_2 s'(A)^2}{\chi_2 s(A)^2 \cdot \chi_1 s'(A)^2}.
\]

Note, however, that the expression \( \chi_1 s(A)/\chi_2 s(A) \) take the indeterminate form 0/0 so we need first to resolve this ambiguity and then we will express everything in terms of the bitangents.

We denote as Weber did \( \sqrt{\beta_1} \) (resp. \( \sqrt{\beta_2} \)) a (fixed) section (Abelsche Function) of the bundle associate to \( D_i \) (resp. to \( D_j \)). Since
\[
p_1 + q_1 = q_3 + q_{13} = q_2 + q_{12}, \quad p_1 + q_{23} = q_2 + q_3 = q_{13} + q_{12}
\]
and
\[
p_2 + q_1 = q_2 + q_3 = q_{13} + q_{12}, \quad p_2 + q_{23} = q_3 + q_{13} = q_2 + q_{12}
\]
We can then choose for \( t_1, u_i, v_i \) and \( w_i \) the following expressions
\[
t_1 = \sqrt{\beta_1 \beta_2 \beta_{13}}, \quad u_1 = \sqrt{\beta_1 \beta_2 \beta_{12}}, \quad v_1 = \sqrt{\beta_{23} \beta_2 \beta_3}, \quad w_1 = \sqrt{\beta_{23} \beta_{13} \beta_{12}}
\]
and
\[
t_2 = \sqrt{\beta_1 \beta_2 \beta_3}, \quad u_2 = \sqrt{\beta_1 \beta_{13} \beta_{12}}, \quad v_2 = \sqrt{\beta_{23} \beta_3 \beta_{13}}, \quad w_2 = \sqrt{\beta_{23} \beta_{23} \beta_{12}}.
\]
We start with a divisor \( S = S_1 + A + B \) and we will let \( S_1 = B \) and \( P = A \) once we have resolved the ambiguity 0/0. Note that \( \sqrt{\beta_1(A)} = \sqrt{\beta_1(B)} = 0 \). Hence the determinant
\[
\chi_{1, s}(P) = (t_i(P)u_i(S_1) - t_i(S_1)u_i(P)) \cdot (v_i(A)w_i(B) - v_i(B)w_i(A)).
\]
In the quotient \( \chi_{1, s}(P)/\chi_{2, s}(P) \) we see that \( \sqrt{\beta_1(P)} \sqrt{\beta_1(S_1)} \) and \( \sqrt{\beta_{23}(A)} \sqrt{\beta_{23}(B)} \) appear in the numerator and in the denominator, so after cancellation and taking \( S_1 = B \) and \( P = A \), we are left with (writting \( \sqrt{\beta_1^A} = \sqrt{\beta_1(A)} \) and \( \sqrt{\beta_1^B} = \sqrt{\beta_1(B)} \))
\[
\frac{\chi_1 s(A)}{\chi_2 s(A)} = \left( \frac{\sqrt{\beta_1^A \beta_1^A \beta_1^A \beta_2^B \beta_1^B}}{\sqrt{\beta_1^A \beta_1^A \beta_1^A \beta_2^B \beta_1^B}} \right) \cdot \left( \frac{\sqrt{\beta_1^A \beta_1^A \beta_1^A \beta_2^B \beta_1^B}}{\sqrt{\beta_1^A \beta_1^A \beta_1^A \beta_2^B \beta_1^B}} \right) = 1.
\]

**Remark 1.** Until this point, the proof could be easily generalized to a curve of arbitrary genus \( g \geq 3 \). Let us indicate the main modifications. One would consider an effective divisor \( S = S_1 + \ldots + S_{2g-3} \) of degree \( 2g - 3 \) and the section
\[
\chi_{1, s}(P) = \det \begin{pmatrix}
t_i^{(1)}(P) & \ldots & t_i^{(2g-2)}(P) \\
t_i^{(1)}(S_1) & \ldots & t_i^{(2g-2)}(S_1) \\
\vdots & \ddots & \vdots \\
t_i^{(1)}(S_{2g-3}) & \ldots & t_i^{(2g-2)}(S_{2g-3})
\end{pmatrix}, \quad 1 \leq i \leq 2
\]
for the bundle associated to the divisor \( \kappa + D_{p_i} \).

The previous decompositions of \( p_1 + p_2 \) as sum of two odd characteristics are special cases of Steiner systems [24][24]. In general there are \( 2^{g-2}(2g-1) - 1 \) pairs \( \{q_i, \bar{q}_i\} \) of odd characteristics such that \( p_1 + p_2 = q_i + \bar{q}_i \) (above we wrote only half of them). Among the characteristics \( q_i, \bar{q}_i \) consider the ones which also appears in the pairs of the Steiner system relative to \( p_1 + q_1 \). After ordering we can write \( p_1 + q_1 = p_2 + \bar{q}_1 \) in \( g + 1 \) ways \( q_i + q_j \) or \( \bar{q}_i + \bar{q}_j \). One has similarly \( p_1 + \bar{q}_1 = p_2 + q_1 \) in \( g + 1 \) ways as \( \bar{q}_i + q_j \) or \( q_i + \bar{q}_j \) for
the same indices. If we denote \((i)\) (resp. \(\bar{i}\)) a section relative to the bundle \(D_{q_1}\) (resp. \(D_{q_2}\)) we then choose to write for the \(g + 1\) choices of \(\{i, j\}\) above

\[
t_1^{(k)} = (1)(i)(j) \text{ or } (\bar{1})(\bar{i})(\bar{j}), \quad t_2^{(k)} = (\bar{1})(\bar{i})(\bar{j}), \quad 1 \leq k \leq g - 1
\]

and

\[
t_1^{(k)} = (\bar{1})(\bar{i})(\bar{j}), \quad t_2^{(k)} = (1)(i)(j) \text{ or } (1)(i)(j) \quad g \leq k \leq 2g - 2.
\]

The support of the theta-characteristic divisor \(D_{q_1}\) is a sum of \(g - 1\) points \(A_1, \ldots, A_{g-1}\). Letting first \((S_{g-1}, \ldots, S_{2g-3}) = (A_1, \ldots, A_{g-1})\) gives the sections \(\chi_{i,S}(P)\) as products of determinants of size \(g - 1\) from which we can simplify the sections \((1)\) and \((\bar{1})\) in the quotient \(\chi_{1,S}(P)/\chi_{2,S}(P)\). It is then enough to take \((P, S_1, \ldots, S_{g-2}) = (A_1, A_2, \ldots, A_{g-1})\) to obtain the same expression for the numerator and denominator and conclude that the quotient is 1.

We now deal with the divisor \(S' = B + G + H\). We now have \(\sqrt{\beta_{23}}(G) = \sqrt{\beta_{23}}(H) = 0\); hence

\[
\chi_{i,S'}(A) = -(v_i(A)w_i(B) - v_i(B)w_i(A)) \cdot (t_i(G)u_i(H) - t_i(H)u_i(G)).
\]

Again we can simplify a bit the quotient (writing \(\sqrt{\beta_i'^2} = \sqrt{\beta_i}(G)\) and \(\sqrt{\beta_i''} = \sqrt{\beta_i}(H)\))

\[
\frac{\chi_{1,S'}(A)}{\chi_{2,S'}(A)} = \frac{\begin{pmatrix} M_1 \\ \text{N}_1 \end{pmatrix}}{\begin{pmatrix} M_2 \\ \text{N}_2 \end{pmatrix}} = \frac{\begin{pmatrix} \sqrt{\beta_2^A \beta_3^A \beta_1^B \beta_2^B - \sqrt{\beta_2^B \beta_3^B \beta_1^A \beta_2^A}} \\ \sqrt{\beta_2^A \beta_3^A \beta_1^B \beta_2^B - \sqrt{\beta_2^B \beta_3^B \beta_1^A \beta_2^A}} \end{pmatrix}}{\begin{pmatrix} \sqrt{\beta_2^G \beta_3^G \beta_1^H \beta_2^H - \sqrt{\beta_2^H \beta_3^H \beta_1^G \beta_2^G}} \\ \sqrt{\beta_2^G \beta_3^G \beta_1^H \beta_2^H - \sqrt{\beta_2^H \beta_3^H \beta_1^G \beta_2^G}} \end{pmatrix}}.
\]

Using the fact that the space of regular sections of the bundle associated to the divisor \(\kappa + (p_1 + p_2)\) has dimension 2, we see that there is a linear relation of the form

\[
h_1\sqrt{\beta_1\beta_{23}} + h_2\sqrt{\beta_2\beta_{13}} + h_3\sqrt{\beta_3\beta_{12}} = 0.
\]

Changing the value of the \(\sqrt{\beta_i}\), we can even assume that \(h_1 = h_2 = 1\) and \(h_3 = -1\). Using the fact that \(\sqrt{\beta_i^2} = \sqrt{\beta_i'} = \sqrt{\beta_i''} = \sqrt{\beta_i} = 0\), we get that

\[
\sqrt{\beta_2^A \beta_3^A} = \sqrt{\beta_2^A \beta_3^A}, \quad \sqrt{\beta_2^B \beta_3^B} = \sqrt{\beta_2^B \beta_3^B}, \quad \sqrt{\beta_2^G \beta_3^H} = \sqrt{\beta_2^G \beta_3^H},
\]

and similarly for \(G, H\). We can now rewrite the \(M_1, N_1\) in the following way

\[
\sqrt{\beta_2^A \beta_3^B} \cdot M_1 = \sqrt{\beta_2^A \beta_2^B} \cdot (\beta_3^A \beta_{13}^B - \beta_3^B \beta_{13}^A), \quad \sqrt{\beta_3^A \beta_2^B} \cdot N_1 = \sqrt{\beta_3^A \beta_{13}^B} \cdot (\beta_3^A \beta_{13}^B - \beta_3^B \beta_{13}^A),
\]

\[
\sqrt{\beta_2^G \beta_3^H} \cdot M_2 = \sqrt{\beta_2^G \beta_2^H} \cdot (\beta_3^G \beta_{13}^H - \beta_3^H \beta_{13}^G), \quad \sqrt{\beta_3^G \beta_2^H} \cdot N_2 = \sqrt{\beta_3^G \beta_{13}^H} \cdot (\beta_3^G \beta_{13}^H - \beta_3^H \beta_{13}^G).
\]

Now, we write \(\beta_3\) as a linear combinaison of \(\beta_{13}, \beta_2, \beta_1\) (resp. \(\beta_{13}, \beta_2, \beta_{23}\))

\[
\beta_3 = a_1 \beta_{13} + b_1 \beta_2 + c_1 \beta_1 = a_2 \beta_{13} + b_2 \beta_2 + c_2 \beta_{23}.
\]

Using the first equality we get

\[
\begin{align*}
\beta_3^A &= a_1 \beta_{13}^A + b_1 \beta_2^A, \\
\beta_3^B &= a_1 \beta_{13}^B + b_1 \beta_2^B.
\end{align*}
\]
Hence using Cramer’s rule we get
\[
\frac{M_1}{N_1} = \frac{\sqrt{\beta^2_2 \beta^2_2}}{\sqrt{\beta^2_{13} \beta^2_{13}}} \cdot \frac{b_1}{a_1} \quad \text{and similarly} \quad \frac{M_2}{N_2} = \frac{\sqrt{\beta^2_{13} \beta^2_{13}}}{\sqrt{\beta^2_2 \beta^2_2}} \cdot \frac{a_2}{b_2}.
\]
It remains to deal with the quotient \(\sqrt{\beta^2_2 \beta^2_2} / \sqrt{\beta^2_{13} \beta^2_{13}}\) (and similarly with \(\sqrt{\beta^2_{13} \beta^2_{13}} / \sqrt{\beta^2_2 \beta^2_2}\)).

In order to do so, we introduce two other linear combinations

\[
\beta_{12} = a_1' \beta_{13} + b_1' \beta_2 + c_1' \beta_1 = a_2' \beta_{13} + b_2' \beta_2 + c_2' \beta_{23},
\]

Because \(\beta_{12}^A \beta_{13}^A = \beta_{13}^A \beta_{23}^A\) by (5), we can rewrite this equality using (6)

\[
\beta_{13}^A \beta_{12}^A = \beta_{13}^A \beta_{23}^A = (a_1' \beta_{13} + a_2' \beta_{23}) \cdot (a_1 \beta_{13} + a_2 \beta_{23}).
\]

Hence

\[
\frac{\beta_{12}^A}{\beta_{13}^A} = (a_1 + b_1 \beta_{13}^A) \cdot (a_1' + b_1' \beta_{13}^A)
\]

and we get the same expression replacing \(A\) by \(B\). Therefore, the quotients \(\frac{\beta_{13}^A}{\beta_{13}}\) and \(\frac{\beta_{13}^B}{\beta_{13}}\) can be seen as the two solutions of a quadratic equation and their product is equal to the constant term divided by the leading coefficients; hence

\[
\frac{\beta_{13}^A \beta_{13}^B}{\beta_{13}^A \beta_{13}^B} = \frac{a_1 a'_1 a_1' a'_1}{b_1 b'_1 b_1' b'_1}
\]

and similarly

\[
\frac{\beta_{13}^A \beta_{13}^B}{\beta_{13}^A \beta_{13}^B} = \frac{b_2 b'_2 a_1 a'_1}{a_2 a'_2 b_1 b'_1}.
\]

Putting everything together, we get

\[
\frac{\psi[p_1](0)^4}{\psi[p_2](0)^4} = (-1)^{a(q_0 + p_1 + p_2)} \cdot \frac{N_1^2 N_2^2}{M_1^2 M_2^2} = (-1)^{a(q_0 + p_1 + p_2)} \cdot \frac{N_1^2 N_2^2}{M_1^2 M_2^2} = (-1)^{a(q_0 + p_1 + p_2)} \cdot \frac{a_1 a'_1 a_1' a'_1}{b_1 b'_1 b_1' b'_1} \cdot \frac{b_2 b'_2 a_1 a'_1}{a_2 a'_2 b_1 b'_1}.
\]

To get the final expression in Weber’s formula, we now look for instance at the linear system (6). Using again Cramer’s rule, one finds for instance

\[
\frac{a_1}{b_1} = \frac{[\beta_3, \beta_2, \beta_1]}{[\beta_{13}, \beta_3, \beta_1]} \quad \text{and} \quad \frac{b_2}{a_2} = \frac{[\beta_{13}, \beta_3, \beta_{23}]}{[\beta_3, \beta_2, \beta_{23}]}
\]

and looking at (7)

\[
\frac{b'_1}{a'_1} = \frac{[\beta_{13}, \beta_{12}, \beta_1]}{[\beta_{12}, \beta_2, \beta_1]} \quad \text{and} \quad \frac{a'_2}{b'_2} = \frac{[\beta_{12}, \beta_2, \beta_{23}]}{[\beta_{13}, \beta_{12}, \beta_{23}]},
\]

Changing the order of the columns, one gets the result.

**Remark 2.** The complexity of the manipulations in this second part makes it difficult to work out a generalization of Weber’s formula for arbitrary genus. However, Remark 4 indicates that one should be able to design an algorithm to compute the quotients of two Thetanullwerte in terms of the equations of the hyperplanes supporting the odd theta characteristics divisors. Indeed, if we denote \(B_1, \ldots, B_{g-1}\) the support of \(D_{\tilde{q}_i}\) and let
$S' = A_2 + \ldots + A_{g-1} + B_1 + \ldots + B_{g-1}$, then with the choice of sections of Remark 2.1 we get that

$$\frac{\partial [p_1]}{\partial [p_2]}(0)^4 = (-1)^{\omega(p_1+p_2)} \cdot \chi_{2,s'}(A_1)^2 \cdot \chi_{1,s'}(A_1)^2.$$  

This should be compared to a similar algorithm suggested in [28]. As far as we know, this latter version has never been implemented.

3.2. A new proof. In order to prove Weber's formula, we need an extra combinatoric result which can be easily obtained using the results in Section 2.1.

**Lemma 3.3.** Let $q_S, q_T$ be two distinct even forms. Let $(q_1, \ldots, q_7)$ be an Aronhold basis attached to $q_S$ ordered such that $q_1 + q_2 + q_3 = q_T$. Then

$$S' = (q_1', \ldots, q_7') = (q_{23}, q_{13}, q_{12}, q_4, q_5, q_6, q_7)$$

is an Aronhold basis attached to $q_T$ such that $q_1' + q_2' + q_3' = q_S$.

By the relation between Aronhold basis and fundamental systems given in Proposition 2.1 and applying Lemma 3.3 we get

**Lemma 3.4.** Let $S = (q_1, \ldots, q_7)$ be an Aronhold basis attached to an even characteristic $q_S$ and $q_1 + q_2 + q_3 = q_T$. Then

$$P_0 = (p_i)_{i=1,\ldots,8} = (q_1, q_2, q_3, q_{567}, q_{467}, q_{457}, q_{456}, q_S)$$

and

$$P'_0 = (p'_i)_{i=1,\ldots,8} = (q_{23}, q_{13}, q_{12}, q_{567}, q_{467}, q_{457}, q_{456}, q_T)$$

are fundamental systems.

Using Corollary 2.2 for the fundamental systems $P_0$ and $P'_0$

$$\frac{[p_1, p_2, p_3]}{[p'_1, p'_2, p'_3]}(\tau) = \frac{[q_1, q_2, q_3]}{[q_{23}, q_{13}, q_{12}]}(\tau) = \pm \prod_{i=4}^{8} \frac{\partial [p_i]}{\partial [p'_i]}(\tau) = \pm \frac{\partial [q_S]}{\partial [q_T]}(\tau).$$

Then Corollary 2.2 shows that there exists constants $\eta_i, \eta_{ij}$ (depending on $\beta_i, [q_i]$ or $\beta_{ij}, [q_{ij}]$) such that

$$\frac{[\beta_1, \beta_2, \beta_3]}{[\beta_{23}, \beta_{13}, \beta_{12}]} = \pm \frac{\eta_1 \eta_2 \eta_3}{\eta_{23} \eta_{13} \eta_{12}} \cdot \frac{\partial [q_S]}{\partial [q_T]}(\tau).$$

In order to kill the constants $\eta_i, \eta_{ij}$, we need to make each $\beta_i$, $\beta_{ij}$ appears as many times in the numerator as in the denominator. In order to do this we use Proposition 2.2 to create new fundamental systems. To simplify the notation and by analogy with the $q_{ij}$ let us denote $p_{ij} = p_8 + p_i + p_j$ (for $1 \leq i < j \leq 3$ we have $p_{ij} = q_{ij}$). For $1 \leq i \leq 3$, let $v_i = p_8 + p_i, v'_i = p'_8 + p'_i, P_i = v_i + P_0$ and $P'_i = v'_i + P'_0$. Since $v_i = v'_i$, we get the following explicit forms.

$$P_0 = (p_1, p_2, p_3, p_4, p_5, p_6, p_7, q_S),$$
$$P'_0 = (p_{23}, p_{13}, p_{12}, p_4, p_5, p_6, p_7, q_T),$$
$$P_i = (p_1, p_{12}, p_{13}, p_{14}, p_{15}, p_{16}, p_7, q_S),$$
$$P'_i = (p_{23}, p_3, p_2, p_{14}, p_{15}, p_{16}, p_{17}, q_T),$$
$$P_2 = (p_{12}, p_{23}, p_{24}, p_{25}, p_{26}, p_{27}, q_S),$$
$$P'_2 = (p_3, p_{13}, p_1, p_{24}, p_{25}, p_{26}, p_{27}, q_T),$$
$$P_3 = (p_{13}, p_{23}, p_{34}, p_{35}, p_{36}, p_{37}, q_S),$$
$$P'_3 = (p_2, p_{13}, p_{12}, p_{34}, p_{35}, p_{36}, p_{37}, q_T).$$

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Hence

\begin{align}
(9) \quad \frac{[\beta_1, \beta_{12}, \beta_{13}]}{[\beta_{23}, \beta_3, \beta_2]}(\tau) &= \pm \frac{\eta_1 \eta_2 \eta_3}{\eta_{23} \eta_3 \eta_2} \vartheta[q_s](\tau), \\
(10) \quad \frac{[\beta_{12}, \beta_2, \beta_{23}]}{[\beta_3, \beta_{13}, \beta_1]}(\tau) &= \pm \frac{\eta_2 \eta_3 \eta_2}{\eta_{31} \eta_3 \eta_1} \vartheta[q_r](\tau), \\
(11) \quad \frac{[\beta_{13}, \beta_{23}, \beta_3]}{[\beta_2, \beta_{12}, \tau]}(\tau) &= \pm \frac{\eta_3 \eta_3 \eta_3}{\eta_{21} \eta_2 \eta_2} \vartheta[q_r](\tau).
\end{align}

Multiplying (9), (10), (11) and (11) gives Weber’s formula up to a sign which does not depend on \( \tau \).

3.3. The question of the sign. Following the different steps of the proof, we see that the sign in Weber’s formula only depends on the fundamental system \( P_0 \) and we will denote it \( \nu(P_0) \). Let us denote also for a list of characteristics \([P] = ([p_1], \ldots, [p_N])\) such that \( P = (p_1, \ldots, p_N) \) is a fundamental system and \( \tau \in \mathbb{H}_3 \)

\[ \mathcal{S}([P], \tau) = \frac{[p_1, p_2, \ldots, p_N](\tau)}{\prod_{i=1}^N \vartheta[p_i](\tau)} = \pm 1. \]

When starting with a fundamental system \( P \), we let \([P]\) be the associated list of characteristics with coefficients 0 and 1.

**Lemma 3.5 ([10] p.420).** The following list \( N_0 = (n_1, \ldots, n_N) \) is a fundamental system (of quadratic forms)

\[
\begin{pmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 1 & 0 \\
1 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
1 & 1 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
1 & 1 & 1
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 1 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 1 & 1
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
1 & 1 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 1 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 1 & 1
\end{pmatrix}.
\]

For \( 1 \leq i \leq 3 \), we can derive from \( N_0 \) the \( N_i, N_i' \) and the \( N_i'' \) as in Section 3.2. For instance, we have for \( N_0' = (n_1', \ldots, n_N') \) the following quadratic forms

\[
\begin{pmatrix}
0 & 1 & 1 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 1 \\
0 & 1 & 0
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
1 & 1 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 1 \\
1 & 1 & 1 \\
1 & 0 & 1
\end{pmatrix}.
\]

Using a computer algebra system like Magma\(^4\)[1], we see that

**Lemma 3.6.**

\[
\nu(N_0) = \prod_{i=0}^3 \frac{\mathcal{S}([N_i], \tau)}{\mathcal{S}([N_i''], \tau)} = 1.
\]

**Remark 3.** One would rather compute the sign using the classical trick to evaluate the expression with \( \tau \) a diagonal matrix. In this case one can reduce the formula to expressions involving only genus 1 Thetanullwerte and then use Jacobi identity. If this works well for \( \mathcal{S}([N_0], \tau) \), then for \( \mathcal{S}([N_0'], \tau) \) (for instance) the numerator and denominator are both zero. Actually, because of the geometric meaning of the problem –Jac(C) is an undecomposable principally polarized abelian variety−, it seems that this will happen for any choice of \( N_0 \), as soon as we consider a reducible \( \tau \). This is why we had to adopt the computational approach to get Lemma 3.6.

We now want to understand what happens when we move to the given fundamental system \( P_0 \) we are interested in. For this purpose, we will need a transformation formula

\[^4\text{see } \url{http://perso.univ-rennes1.fr/christophe.ritzenthaler/programme/theta-proof.magma}\]
which we give here for $g = 3$. Up to identifying a characteristic $[q] = \begin{pmatrix} \varepsilon' \\ \varepsilon \end{pmatrix}$ with the vector $(\varepsilon' \varepsilon)$, we let $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Sp}_6(\mathbb{Z})$ act by

$$\sigma \cdot [q] = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} \varepsilon' \\ \varepsilon \end{pmatrix} + \begin{pmatrix} (c'd)d_0 \\ (a'b)b_0 \end{pmatrix}.$$ 

Note that when we reduce modulo 2, this action coincides with the action of $\Gamma$ on quadratic forms as introduced in Section 2.1. Let us also denote $\phi_{[q]}(\sigma) = \frac{1}{8} \left( \varepsilon' b d' e - 2 \varepsilon' b c' \varepsilon' + \varepsilon'' a c' \varepsilon' - 2 \varepsilon (a'b)b_0 (\varepsilon' d' e - c' \varepsilon') \right)$.

For a list of characteristics $[P] = ([p_1], \ldots, [p_8])$ such that $P = (p_1, \ldots, p_8)$ is a fundamental system, $\tau \in \mathbb{H}_3$ and $\sigma \in \text{Sp}_6(\mathbb{Z})$, let us denote $\sigma \cdot [P] = (\sigma \cdot [p_1], \ldots, \sigma \cdot [p_8]).$

**Lemma 3.7** ([16, p.433]). With the notation above, we have

$$(12) \quad S(\sigma \cdot [P], \sigma \cdot \tau) = s([P], \sigma) \cdot S([P], \tau)$$

where $s([P], \sigma) = \kappa(\sigma)^{-2} \cdot e \left( \sum_{i=1}^{3} \phi_{[p_i]}(\sigma) - \sum_{i=4}^{8} \phi_{[p_i]}(\sigma) \right)$ and $\kappa(\sigma)$ is an 8-th root of unity.

Let $P_0 = (p_0, \ldots, p_8)$ and let $\tilde{\sigma} \in \Gamma$ be a matrix such that $\tilde{\sigma} \cdot n_i = p_i$ for $1 \leq i \leq 8$. Such a matrix always exists by the transitive action of $\Gamma$ on fundamental systems. Let $\sigma \in \text{Sp}_6(\mathbb{Z})$ be any lift of $\tilde{\sigma}$. If we apply $\sigma$ to the normalized characteristics coming from the $N_i$ and $N'_i$, we get characteristics for the forms in the $P_i$ and $P'_i$ because of the linearity of the transformations involved in the definition of these fundamental systems.

Note that since the value of a given quadratic form in the various fundamental systems $N_i, N'_i$ is fixed in the various list of characteristics $[N_i], [N'_i]$ the characteristics of the $[P_i] = \sigma \cdot [N_i], [P'_i] = \sigma \cdot [N'_i]$ have the same property. Moreover, even if the characteristics of the $[P_i]$ and $[P'_i]$ are not necessarily normalized, we have already noticed that the value of the global quotient does not change, since all of them appear (twice) in the numerator and denominator. Because of all these considerations, we get that

$$\ell(P_0) = \frac{\prod_{i=0}^{3} S([P_i], \sigma \cdot \tau)}{\prod_{i=0}^{3} S([P'_i], \sigma \cdot \tau)} = \frac{\prod_{i=0}^{3} s([N_i], \sigma) \cdot s([N'_i], \sigma)}{\prod_{i=0}^{3} s([N'_i], \sigma)} = \frac{\prod_{i=0}^{3} s([N_i], \sigma)}{\prod_{i=0}^{3} s([N'_i], \sigma)} = \frac{e(4 \cdot \phi_{[n'_i]}(\sigma))}{e(4 \cdot \phi_{[n_i]}(\sigma))} = (-1)^{s_{[n'_i]}(\sigma) - 8 \cdot s_{[n_i]}(\sigma)}$$

as all the characteristics apart from $[n_8]$ and $[n'_8]$ appear twice in the numerator and the denominator. To finish the proof we hence need the following lemma.

**Lemma 3.8.**

$8 \cdot \phi_{[n'_8]}(\sigma) - 8 \cdot \phi_{[n_8]}(\sigma) \equiv a(\sigma \cdot [n_8] + \sigma \cdot [n'_8] + q_0) \pmod{2}$.

**Proof.** Let $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Sp}_6(\mathbb{Z})$ and $[n'_8] = \begin{pmatrix} \varepsilon' \\ \varepsilon \end{pmatrix}$. The left hand side of the expression is equivalent modulo 2 to $r_1 = \varepsilon' b d' e + \varepsilon'' a c' \varepsilon'$. On the other hand

$$[p_8] = \sigma \cdot [n_8] = \sigma \cdot \begin{pmatrix} 000 \\ 000 \end{pmatrix} = \begin{pmatrix} (c'd)d_0 \\ (a'b)b_0 \end{pmatrix}$$

and

$$[p'_8] = \sigma \cdot [n'_8] = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} \varepsilon' \\ \varepsilon \end{pmatrix} + \begin{pmatrix} (c'd)d_0 \\ (a'b)b_0 \end{pmatrix}. $$
So

\[ q = [p] + [q'] + [q_0] = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} t' \\ t' \end{pmatrix} \equiv \begin{pmatrix} d't' - c't' \\ -b't' + a't' \end{pmatrix} \pmod{2}. \]

Finally

\[ a(q) \equiv t'(d't' - c't')(-b't' + a't') \equiv c't'bd't' + c't'ac't' + c'(t'bc + t'da)t' \equiv r_1 + e't' \equiv r_1 + a(n'_q) \equiv r_1 \pmod{2}. \]

\[ \square \]

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