TWO-LIMIT PROBLEMS FOR ALMOST SEMICONTINUOUS PROCESSES DEFINED ON A MARKOV CHAIN

Ievgen Karnaukh

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We consider almost upper semi-continuous processes defined on a finite Markov chain. The distributions of the functionals associated with the exit from a finite interval are studied. We also consider some modification of these processes.

Problems related to the exit of a process with independent increments from an interval were investigated in many works (see, e.g. [1], [2]). Analogous problems were investigated for processes on a finite Markov chain under the semicontinuity condition [3, 4]. For walks on a countable Markov chain, a two-dimensional problem was studied in [5]. In the present paper, we consider distributions of some functionals associated with the exit from a bounded interval for a process with independent increments on a finite Markov chain under the assumption that this process crosses a positive level only by exponentially distributed jumps (an almost semicontinuous process [6]).

The distributions of overjump functionals described by integral equations on a semi-axes are defined by the projection factorization method using an infinitely divisible factorization (instead of canonical factorization). In the present paper, we investigate functionals described by integral equations on an interval that can be extended to a semi-axes. In the solution of the extended equation, we use the method developed by Krein in [7, 8] and probability factorization identities.

Consider a two-dimensional Markov process:

\[ Z(t) = \{ \xi(t), x(t) \}, \quad t \geq 0, \]

where \( x(t) \) is a finite irreducible nonperiodic Markov chain with the set of states \( E' = \{ 1 \ldots m \} \) and the matrix of transition probabilities

\[ P(t) = e^{tQ}, \quad t \geq 0, \quad Q = N(P - I), \]

where \( N = ||\delta_{kr} \nu_k||_r \), \( \nu_k \) are the parameters of exponentially distributed random variables \( \zeta_k \) (the sojourn time of \( x(t) \) in the state \( k \)), \( P = ||p_{kr}|| \) is the matrix of transition probabilities of the embedded chain; \( \pi = (\pi_1, \ldots, \pi_m) \) is the stationary distribution, and \( \xi(t) \) is a process with stationary conditionally independent increments for fixed values of \( x(t) \) (see [3, p.13]).

The evolution of the process \( Z(t) \) is described by the matrix characteristic function:

\[ \Psi(\alpha) = ||E[e^{t\alpha \xi(t)}, x(t) = r/x(0) = k]|| = E[e^{t\alpha \xi(t)}] = e^{t\Psi(\alpha)}, \quad \Psi(0) = Q. \]

In what follows, we consider processes that have cumulant

\[ \Psi(\alpha) = \Lambda \overline{F}_0(0) \left( C (C - \alpha I)^{-1} - I \right) + \int_{-\infty}^{0} (e^{\alpha x} - 1) dK_0(x) + Q, \tag{1} \]

where

\[ dK_0(x) = N \alpha F(x) + \Pi(dx), \quad F(x) = ||P\{ \chi_{kr} < x; x(\zeta_k) = r/x(0) = k \}||, \]

and the jumps of \( \xi(t) \) at the time of transition of \( x(t) \) from the state \( k \) to the state \( r \),

\[ \Pi(dx) = \Lambda dF_0(x), \quad F_0(x) = ||\delta_{kr} F_k^x(\zeta_k)||, \]

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Taras Shevchenko Kyiv National University, Kyiv. kveugene@mail.ru

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$F_k^0(x)$ are the distribution functions of the jumps of $\xi(t)$ if $x(t) = k$, $A = \|\delta_k, \lambda_k\|$, $\lambda_k$ are the parameters of exponentially distributed random variables $\zeta_k^r$ (the time interval between two neighboring jumps of $\xi(t)$ if $x(t) = k$), $C = \|\delta_k, c_k\|$, and $c_k$ are the parameters of exponentially distributed positive jumps of $\xi(t)$ if $x(t) = k$. The process $Z(t)$ with this cumulant is the almost upper-semicontinuous process defined in [6, p.43].

Let $\theta_\ast$ denote an exponentially distributed random variable with parameter $s > 0 \ (P\{\theta_\ast > t\} = e^{-st}, t \geq 0)$, independent of $Z(t)$. In this case, we rewrite the characteristic function of $\xi(\theta_\ast)$ as follows

$$
\Phi(s, \alpha) = E e^{i\alpha \xi(\theta_\ast)} = s \int_0^\infty e^{-st} \Phi_1(\alpha) dt = s(sI - \Psi(\alpha))^{-1},
$$

$$
P_\ast = s \int_0^\infty e^{-st} P(t) dt = \Phi(s, 0) = s(sI - Q)^{-1}.
$$

Denote the time of the first hit of a positive (negative) level by

$$
\tau^+(x) = \inf\{t > 0 : \xi(t) > x\}, x > 0,
$$

$$
\tau^-(x) = \inf\{t > 0 : \xi(t) < x\}, x < 0
$$

and the time of the first exit from the interval $(x - T, x)$, $0 < x < T$, $T > 0$ by:

$$
\tau(x, T) = \inf\{t > 0 : \xi(t) \notin (x - T, x)\}.
$$

We introduce the events

$$
A_+(x) = \{\omega : \xi(\tau(x, T)) \geq x\}, \ A_-(x) = \{\omega : \xi(\tau(x, T)) \leq x - T\}.
$$

Then, for $x > 0$, we can write:

$$
\tau(x, T) = \begin{cases} 
\tau^+(x, T) = \tau^+(x), & \omega \in A_+(x); \\
\tau^-(x, T) = \tau^-(x - T), & \omega \in A_-(x).
\end{cases}
$$

Denote the overjumps at the time of exit from the interval by:

$$
\gamma_\ast^-(x) = x - T - \xi(\tau^-(x, T)), \ \gamma_\ast^+(x) = \xi(\tau^+(x, T)) - x.
$$

In the first part of the paper, we obtain closed-form representation of the following moment generating functions:

$$
B_T(s, x) = \left\| E \left[ e^{-st\tau(x, T)}, \xi(\tau(x, T)) \geq x, x(\tau(x, T)) = \tau(x, T) = k \right] \right\| =
$$

$$
= E \left[ e^{-st\tau(x, T)}, A_+(x) \right],
$$

$$
B_T(s, x) = \left\| E \left[ e^{-st\tau(x, T)}, \xi(\tau(x, T)) \leq x - T, x(\tau(x, T)) = \tau(x, T) = k \right] \right\| =
$$

$$
= E \left[ e^{-st\tau(x, T)}, A_+(x) \right],
$$

$$
B(s, x, T) = E e^{-st\tau(x, T)}, \ V(s, \alpha, x, T) = E \left[ e^{i\alpha \xi(\theta_\ast)}, \tau(x, T) > \theta_\ast \right],
$$

$$
V^\pm(s, \alpha, x, T) = E \left[ e^{i\alpha \gamma_\ast^\pm(x) - st \tau^\pm(x, T)}, A_\pm(x) \right],
$$

$$
V^\pm(s, \alpha, x, T) = E \left[ e^{i\alpha \gamma_\ast^\pm(x) - st \tau^\pm(x, T)}, A_\pm(x) \right].
$$

Denote the set of bounded functions absolutely integrable on the interval $I \subseteq (-\infty, \infty)$ and the set of their integral transforms by

$$
L_m(I) = \left\{ G(x) = \|G_k\| : \int_I |G_k(x)| dx < \infty; k, r = \overline{1, m} \right\},
$$

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Lemma 1. We introduce the projection operation onto $R_m^0((-\infty,\infty))$

\[ [C + g(\alpha)]_1 = \int_I e^{i\alpha x} G(x)dx, \quad [C + g(\alpha)]_0 = C + \int_I e^{i\alpha x} G(x)dx, \]

\[ [C + g(\alpha)]_- = [C + g(\alpha)]_{(-\infty,0)}, \quad [C + g(\alpha)]_+ = [C + g(\alpha)]_{(0,\infty)}. \]

Note that $V(s,\alpha,x,T) \in R_m^0((x-T,x))$, $V^+(s,\alpha,x,T) \in R_m^0([x,\infty))$, $V^-(s,\alpha,x,T) \in R_m^0((-\infty,x-T))$.

Further, we introduce extrema of $(s,x) = (s,\alpha,x,T) = (s,\alpha,x,T)$.

\[ \Phi(s,\alpha) = E e^{i\alpha \xi_0} = \Phi_+(s,\alpha) \phi^{-1}(s,\alpha), \]

\[ \Phi_-(s,\alpha) = E e^{i\alpha \xi_0} = \Phi_-(s,\alpha) \phi^{-1}(s,\alpha). \]

Lemma 1. [3, p.49] For the two-dimensional Markov process $Z(t) = \{\xi(t), x(t)\}$, the following factorization identity is true

\[ \Phi(s,\alpha) = E e^{i\alpha \xi_0} \Phi_+(s,\alpha), \]

where

\[ \Phi_+(s,\alpha) = E e^{i\alpha \xi_0}, \Phi_-(s,\alpha) = E e^{i\alpha \xi_0}. \]

Theorem 1. For a process $Z(t)$ with cumulant $1$, $B^T(s,x)$ is determined by the relation

\[ sB^T(s,x) = s(I - p^+(s)) e^{-R^+(s)x} - p^+(s) \int^x_{-\infty} dP^-(s,y) e^{-C(x-y)} C^0_T(s) - \]

\[ - (I - p^+(s)) \int^x_{-\infty} e^{-R^+(s)x} R^+(s) \int^y_{-\infty} dP^-(s,y) e^{-C(x-y)} dz C^0_T(s), 0 < x < T, \]

\[ C^0_T(s) = A \bar{F}_0(0) \left( I + C \int^T_0 e^{C}\bar{B}^T(s,z)dz \right). \]

Proof. Using the stochastic relations for $\tau^+_k(x,T)$, namely,

\[ \tau^+_k(x,T) = \begin{cases} \zeta_k, & \zeta_k < \xi_k, \xi_k > x, \\
\zeta_k + \tau^+_k(x-\xi_k,T), & \zeta_k < \xi_k, x - T < \xi_k < x, \\
\zeta_k + \tau^+_k(x-\chi_k,T), & \zeta_k > \xi_k, x - T < \chi_k < x, \end{cases} \]

where the subscripts denote, respectively, the initial state and the state of the chain $x(t)$ at the time of exit form $(x-T,x)$ ($x(0) = k, x(\tau(x,T)) = r$), we get

\[ B^T_{kr}(s,x) = \lambda_k \int^\infty_0 e^{-(s+\lambda_k+\nu_k)y} dF^0_k(z) \left( \int^z_x dF^0_k(z) + \int^z_{x-T} dF^0_k(z) B^T_{kr}(s,x-z) \right) + \]

\[ + \sum_{j=1}^m \nu_k \int^\infty_0 e^{-(s+\lambda_k+\nu_k)y} dF^0_k(z) \int^z_{x-T} dF^0_k(z) B^T_{kj}(s,x-z), 0 < x < T. \]
We can rewrite these equations in the matrix form:

\[(sI + \Lambda + N)BT(s, x) = \Lambda F_0(x) + \int_{x-T}^{x} dK_0(z)BT(s, x-z), \quad 0 < x < T, \quad (5)\]

\[BT(s, x) = 0, \quad x \geq T, \quad B^T(s, x) = I, \quad x < 0.\]

Performing the substitution \(B^T(s, x) = I - BT(s, x)\), in (5), we derive the following equation for \(B^T(s, x)\) \((0 < x < T)\)

\[(sI + \Lambda + N)B^T(s, x) = (sI - Q) + \int_{-\infty}^{\infty} dK_0(z)B^T(s, x-z), \quad 0 < x < T.\]

Extending this equation to the semi-axis \(x > 0\), we obtain

\[(sI + \Lambda + N)B^T(s, x) = (sI - Q) + \int_{-\infty}^{\infty} dK_0(z)B^T(s, x-z) + e^{-Cz}C_0^T(s)I \{x > T\}. \quad (6)\]

We denote \(C_\epsilon(x) = e^{-Cz}I \{x > 0\}\) and consider the following equation for \(Y_\epsilon(T, s, x)\) \((x > 0, \epsilon > 0)\), instead of Eq. (6):

\[(sI + \Lambda + N)Y_\epsilon(s, x) = (sI - Q)C_\epsilon(x) + \int_{-\infty}^{\infty} dK_0(z)Y_\epsilon(T, s, x-z) + e^{-Cz}C_0^T(s)I \{x > T\}, \quad (7)\]

Applying the integral transform with respect to \(x\) to (7), we get

\[(sI - \Psi(\alpha))\tilde{Y}_\epsilon(T, s, \alpha) = (sI - Q)\int_{0}^{\infty} e^{iz}e^{-\epsilon z}dz + \int_{0}^{\infty} e^{iz}e^{-Cz}C_0^T(s)I \{z > T\} dz -\]

\[- \left[ K_0(\alpha)\tilde{Y}_\epsilon(T, s, \alpha) \right], \quad (8)\]

\[K_0(\alpha) = \int_{0}^{\infty} e^{iz}dK_0(z), \quad \tilde{Y}_\epsilon(T, s, \alpha) = \int_{0}^{\infty} e^{iz}Y_\epsilon(T, s, z)dz.\]

Using (2) and (8) and performing the projection \([\cdot]_+\), we obtain

\[s\tilde{Y}_\epsilon(T, s, \alpha) = \Phi_+(s, \alpha)P_s^{-1}\left[ \Phi^-(s, \alpha) \left( (sI - Q)\int_{0}^{\infty} e^{iz}e^{-\epsilon z}dz + \int_{0}^{\infty} e^{iz}e^{-Cz}C_0^T(s)I \{z > T\} dz \right) \right]_+. \]

Inverting this relation, we get

\[sY_\epsilon(T, s, x) = \int_{0}^{x} dP_+(s, z)P_s^{-1}\int_{0}^{x} dP^-(s, y)(sI - Q)\epsilon^{-\epsilon(x-y-z)} +\]

\[+ \int_{0}^{x} dP_+(s, z)P_s^{-1}\int_{-\infty}^{\min\{x-z-T, 0\}} dP^-(s, y)e^{-C(x-y-z)}C_0^T(s), \quad (9)\]

Since \(Y_\epsilon(T, s, x) \rightarrow \tilde{B}^T(s, x)\) as \(\epsilon \rightarrow 0, 0 < x < T\), relation (9) yields

\[s\tilde{B}^T(s, x) = \int_{0}^{x} dP_+(s, z)(sI - Q) + \int_{0}^{x} dP_+(s, z)P_s^{-1}\int_{-\infty}^{\min\{x-z-T, 0\}} dP^-(s, y)e^{-C(x-y-z)}C_0^T(s).\]

Taking into account that \([6, p.45]\)

\[P \{\xi^+(\theta_s) > x\} = (I - p_+^*(s)) e^{-R^*(s)x}P_s, \quad x > 0, \quad (10)\]

we get (3).

Below, we present (without proof) analogs of the Pecherskii identities (see [2, p.108]).
Lemma 2. The following identities are true for \( Z(t) \)
\[
V(s, \alpha, x, T) = \Phi(s, \alpha) \left( I - V_+(s, \alpha, x, T) - V_-(s, \alpha, x, T) \right),
\]
\[
V(s, \alpha, x, T) = \Phi_+(s, \alpha) P_s^{-1} \left[ \Phi^-(s, \alpha)(I - V_+(s, \alpha, x, T)) \right]_{x-T, \infty},
\]
\[
V(s, \alpha, x, T) = \Phi_-(s, \alpha) P_s^{-1} \left[ \Phi^+(s, \alpha)(I - V_-(s, \alpha, x, T)) \right]_{-\infty, x}.\]

Denote the joint distribution of \( \{\xi(\theta_s), \xi^+(\theta_s), \xi^-(\theta_s)\} \) by
\[
H_s(T, x, y) = \left| \mathbb{P} \{ \xi(\theta_s) < y, \xi^+(\theta_s) < x, \xi^-(\theta_s) > x - T, \tau(x) = r/x(0) = k \} \right|
= \mathbb{P} \{ \xi(\theta_s) < y, \tau(x, T) > \theta_s \}.
\]

Theorem 2. For a process \( Z(t) \) with cumulant \( \gamma \), the joint distributions of \( \{\tau^+(x, T), \gamma_T^+(x)\} \) and \( \{\tau^-(x, T), \xi(\tau^+(x, T))\} \) are defined by the relations
\[
\begin{align*}
\mathbf{V}(s, \alpha, x, T) &= \mathbf{B}_+^T(s, x) \mathbf{C}^{-1} - (s, \alpha, x, T),
\end{align*}
\]
\[
\mathbf{V}(s, \alpha, x, T) = e^{i\alpha x} \mathbf{V}_+^T(s, \alpha, x, T).
\]

The characteristic function of \( \xi(\theta_s) \) up to the time of exit from the interval has the following form
\[
V(s, \alpha, x, T) = \Phi_+(s, \alpha) P_s^{-1} \left[ \Phi^-(s, \alpha) \left( I - e^{i\alpha x} \mathbf{B}_+^T(s, x) \mathbf{C}^{-1} \right) \right]_{x-T, \infty}.
\]

The corresponding distribution has density \( x < T \) for \( y < x, y \neq 0 \)
\[
h_s(T, x, y) = \frac{\partial}{\partial y} H_s(T, x, y) = \]
\[
= p_+^*(s) \left( \mathbf{P}^{-}(s, y) \right)' I \{ y < 0 \} + (I - p_+^*(s)) \int_{x-T}^{\min\{0, y\}} e^{-R_+^*(s)(y-z)} R_+^*(s) d\mathbf{P}^-(s, z)
\]
\[
- p_+^*(s) \int_{-\infty}^{y-x} d\mathbf{P}^-(s, z) \mathbf{B}_+^T(s, x) \mathbf{C} e^{-C(y-x-z)} - (I - p_+^*(s)) \int_{0}^{y-(x-T)} e^{-R_+^*(s)v} \times
\]
\[
\times \int_{-\infty}^{y-v-x} R_+^*(s) d\mathbf{P}^-(s, z) \mathbf{B}_+^T(s, x) \mathbf{C} e^{-C(y-v-x-z)} dv,
\]
with atom at zero
\[
\mathbb{P} \{ \xi(\theta_s) = 0, \tau(x, T) > \theta_s \} = s(sI + \Lambda - N) (\mathbb{P} \{ \chi_{kr} = 0, x(\zeta_1) = r/x(0) = k \} I)^{-1}.
\]

The probability of nonexit from the interval \( x, T) \) is determined by the relation
\[
\mathbb{P} \{ \tau(x, T) > \theta_s \} = \int_{x-T}^{x} dH_s(T, x, y).
\]

For the moment generating functions of \( \tau(x, T) \) and \( \tau^-(x, T) \), the following relations are true:
\[
\begin{align*}
\mathbf{B}(s, x, T) &= I - \mathbb{P} \{ \tau(x, T) > \theta_s \} P_s^{-1}, \quad 0 < x < T,
\mathbf{B}_T(s, x) &= \mathbf{B}(s, x, T) - \mathbf{B}_+^T(s, x), \quad 0 < x < T.
\end{align*}
\]

Proof. Using stochastic relations (4) for \( \tau^+(x, T) \) and
\[
\gamma_T^+(x) = \left\{ \begin{array}{cc}
\xi_k, & \xi_k < \xi_k, \xi_k > x, \\
\gamma_T^+(x - \xi_k)_{kr}, & \xi_k < \xi_k, x - \xi_k < x, \\
\gamma_T^+(x - \chi_{kj})_{jr}, & \xi_k > \xi_k, x - \chi_{kj} < x,
\end{array} \right.
\]
\[\]
we obtain the following equation for $\gamma_T^+(x)$:

$$(sI + N + \Lambda) V^+ (s, \alpha, x, T) = \Lambda \mathbf{F}_0(0) e^{-CsT} C (C - i\alpha I)^{-1} + \int_{x-T}^x dK_0(z) V^+ (s, \alpha, x - z, T).$$

Using (5), we deduce the first relation in (14). The second relation follows from the definition of $\gamma_T^+(x)$. Relation (15) follows from (12). After the inversion with respect to $\alpha$, we obtain (16) and (17) from (15).

Using relations (16), (17) and an analog of the Bratiichuk formulas [1, p.187], we can obtain a matrix analog for the generatrices of the joint distributions of $\{\tau^+(x, T), \xi(\tau^+(x, T))\}$ and $\{\tau^-(x, T), \xi(\tau^-(x, T))\}$:

**Theorem 3.** For $Z(t)$, the following relations are true:

$$sE \left[ e^{-s\tau^+(x,T)} \xi(\tau^+(x,T)) > z, \right] = \int_{x-T}^x dH_s(T, x, y) \mathbf{K}_0(y), x > 0, \mathbf{K}_0(x) = \int_{-\infty}^x dK_0(y), x < 0.$$

**Proof.** According to [4, p.469], we have

$$E_i \left[ e^{-s\tau^+(x,t)} f(x - \xi(\tau(x,T)), x(\tau(x,T))) \right] - f(x, i) = E_i \int_{x-T}^\tau e^{-st} g(x - \xi(t), x(t)) dt,$$

where $f$ is a bounded function, $g = Af - sf$ and $A$ is a generator of the semigroup defined by the cumulant $\Psi(\alpha)$. For the right-hand side of the equation, we have

$$E_i \int_{x-T}^\tau e^{-st} g(x - \xi(t), x(t)) dt = \sum_{j=1}^m \int_0^\infty e^{-st} E_i \left[ g(x - \xi(t), j), \tau(x, T) > t, x(t) = j \right] dt =$$

$$= \sum_{j=1}^m s^{-1} \int_{x-T}^x g(x - y, j) d(H_s(T, x, y))_{ij}.$$ (23)

Assuming that $f(x, i) = I\{x \geq -z\} \delta_{ir}, z > 0, i, r \in E'$, we obtain

$$E_i \left[ e^{-s\tau^+(x,t)} f(x - \xi(\tau(x,T)), x(\tau(x,T))) \right] - f(x, i) =$$

$$= E_i \left[ e^{-s\tau^+(x,t)}, \gamma_T^+(x) \geq z, x(\tau(x,T)) = r \right],$$ (24)

$$g(x, j) = \int_{-\infty}^\infty I\{x - y \leq -z\} dK_0^{jr}(y) = \mathbf{K}_0^{jr}(x + z).$$ (25)

Substituting (24) and (25) into (22) and taking (23) into account, we get

$$E_i \left[ e^{-s\tau^+(x,t)}, \gamma_T^+(x) \geq z, x(\tau(x,T)) = r \right] = \sum_{j=1}^m s^{-1} \int_{x-T}^x \mathbf{K}_0^{jr}(x - y + z) d(H_s(T, x, y))_{ij}.$$ (26)

Using the definition of $\gamma_T^+(x)$, we derive (20) from (26). By analogy, we obtain relation (21).

Consider the behavior of the functions $B^T(s, x)$ and $H_s(T, x, y)$ as $s \to 0$. Denote

$$M(y) = \lim_{s \to 0} s^{-1} p^+_s(s) P^-(s, y).$$
The existence of this function follows from the reasoning presented below. According to [6, p.46], for the moment generating function of \( \overline{\xi}(\theta_s) \) we have
\[
\lim_{s \to 0} s^{-1} \mathbf{p}^*(s) \mathbb{E} e^{r \overline{\xi}(\theta_s)} = -(\mathbf{p}^*(0) \mathbf{C} - r \mathbf{I}) (\mathbf{C} - r \mathbf{I})^{-1} \Psi^{-1}(-ir).
\]
Then we can define \( \mathbf{M}(y) \) as a function for which
\[
\int_{-\infty}^{0} e^{ry} d\mathbf{M}(y) = -(\mathbf{p}^*(0) \mathbf{C} - r \mathbf{I}) (\mathbf{C} - r \mathbf{I})^{-1} \Psi^{-1}(-ir).
\]
Furthermore, it follows from [6, p.41] that
\[
\mathbf{p}^*(0) = (\mathbf{I} - \| \mathbb{P} \{ \tau^+(0) < \infty, \mathbf{r}(\tau^+(0)) = r/x(0) = k \} \|).
\]

**Corollary 1.** For the process \( Z(t) \) following relations are true:
\[
\lim_{s \to 0} s^{-1} \mathbf{h}_s(T, x, y) = \mathbf{M}(y) I \{ y < 0 \} + (\mathbf{I} - \mathbf{p}^*(0)) \int_{x-T}^{\min(0,y)} e^{-R^*_s(0)(y-z)} C d\mathbf{M}(z) -
\]
\[
- (\mathbf{I} - \mathbf{p}^*(0)) \int_{0}^{y-(x-T)} e^{-R^*_s(0)y} C \int_{-\infty}^{y-v-x} d\mathbf{M}(z) \mathbf{B}^T(x) C e^{-C(y-v-x-z)} dv -
\]
\[
- \int_{-\infty}^{y-x} d\mathbf{M}(z) \mathbf{B}^T(x) C e^{-C(y-x-z)}, \quad (27)
\]
\[
\mathbf{B}^T(x) = \lim_{s \to 0} \mathbf{B}^T(s, x) = (\mathbf{I} - \mathbf{p}^*(0)) e^{-R^*_s(0)x} - \int_{-\infty}^{x-T} d\mathbf{M}(y) e^{-C(x-y)} C^T_0(0) -
\]
\[
- (\mathbf{I} - \mathbf{p}^*(0)) \int_{0}^{x} e^{-R^*_s(0)y} C \int_{-\infty}^{x-z-T} d\mathbf{M}(y) e^{-C(x-y)} dz C^T_0(0), \quad (28)
\]
\[
C^T_0(0) = \mathbf{A} \mathbf{F}_0(0) \left( \mathbf{I} + C \int_{0}^{T} e^{Cz} \mathbf{B}^T(z) dz \right).
\]
Assume that \( \mathbf{R} \equiv 0 \). If \( x(t) = k, k = 1, \ldots, m \), then we set
\[
\xi(t) = \sum_{n \leq \nu_k(t)} \xi_n^k - \sum_{n \leq \nu^*_k(t)} \xi_n^k,
\]
where \( \nu_k(t) \) and \( \nu^*_k(t) \) are Poisson processes with the rates \( \lambda_k^1 \) and \( \lambda_k^2 \), respectively. \( \xi_n^k \) and \( \xi_n^k \) are independent positive random variables, \( \xi_n^k \) and \( \xi_n^k \) are exponentially distributed with the parameters \( c_k \), the variables \( \xi_n^k \) have an arbitrary distribution with bounded expectation \( m_k \). It is clear that, in this case, the process \( Z(t) = \{ \xi(t), x(t) \} \) is the almost upper-semicontinuous and has cumulant
\[
\Psi(\alpha) = \mathbf{A} \mathbf{F}_0(0) \left( \mathbf{C} (\mathbf{C} - \alpha \mathbf{I})^{-1} - \mathbf{I} \right) + \int_{-\infty}^{0} \left( e^{\alpha x} - \mathbf{I} \right) \mathbf{P}(dx) + \mathbf{Q},
\]
where
\[
\mathbf{A} = \| \delta_{kr} (\lambda_k^1 + \lambda_k^2) \|, \quad \mathbf{F}_0(0) = \| \delta_{kr} \lambda_k^1 (\lambda_k^1 + \lambda_k^2) \|, \quad \mathbf{P}(dx) = \mathbf{A} \mathbf{F}_0(0) d\mathbf{F}_0^t(x),
\]
\[
\mathbf{F}_0(0) = \mathbf{I} - \mathbf{F}_0(0), \quad C^T_0(x) = \| \delta_{kr} \mathbb{P} \{ -\xi_n^k < x \} \|, \quad x < 0.
\]
Consider the process \( \eta^{kr}_{B,u}(t) \) defined by the stochastic relations
\[
\eta^{kr}_{B,u}(t) =
\begin{cases}
  u + \xi_{kr}(t) & t < T_1, \\
  B & t \in (T_1, T_2), T_1 < \infty, \\
  \eta^{kr}_{B,u}(t-T_2) & t > T_2, x(T_2) = j, T_1 < \infty,
\end{cases}
\]
where the superscripts kr mean that \( x(t) = r, x(0) = k \). Note that, \( T_1 \sim \tau^+(v), v = B-u \) and \( T_2 \sim \tau^+(v) + \zeta \), \( \zeta \) is the time of the first negative jump independent of \( T_1 \).

The process \( \eta_{B,u}(t) \) is called the risk process in a Markov environment with stochastic premium function and bounded reserve (see [9] - [11]). We also consider the dividend process \( Y_{B,u}(t) \equiv u + \xi(t) - \eta_{B,u}(t) \) (see [9, p.169]).

**Theorem 4.** The distribution of \( \eta_{B,u}(\theta_s) \) is determined by the characteristic function

\[
\Phi_{B,u}(s, \alpha) = E e^{\alpha \eta_{B,u}(\theta_s)} = e^{\alpha B} \left( (C - i\alpha I) p^*_+(s) e^{-i\alpha v} - \left( I - p^*_+(s) \right) e^{-R^*_+(s)v} R^*_+(s) \right) (R^*_+(s) - i\alpha I)^{-1} \Phi^-(s, \alpha) + \left( I - p^*_+(s) \right) e^{-R^*_+(s)v} (sI + \Lambda F_0(0) - Q)^{-1} \left( s e^{i\alpha B} + \Lambda F_0(0) \Phi_B(s, \alpha) \right),
\]

(30)

\[
\tilde{\Phi}_B(s, \alpha) = \int_{-\infty}^{0} dF_0(z) \Phi_{B,B+\zeta}(s, \alpha) = e^{i\alpha B} (\Lambda F_0(0))^{-1} (sI + \Lambda F_0(0) - Q) (p^*_+(s))^{-1} \times \left( sI + \Lambda - Q \right)^{-1} \int_{-\infty}^{0} \Pi(dz) \left( (I - p^*_+(s)) e^{R^*_+(s)z} (sI + \Lambda F_0(0) - Q)^{-1} + e^{i\alpha z} (C - i\alpha I) - (I - p^*_+(s)) e^{R^*_+(s)z} C \right) p^*_+(s) (R^*_+(s) - i\alpha I)^{-1} \Phi^-(s, \alpha). \]

(31)

If \( m^0_1 = \sum_{k=1}^{m} \pi_k(\lambda_k/c_k - \lambda_k^2 m_k) > 0 \), then the following relations hold for \( \eta_{B,u} = \lim_{s \to 0} \eta_{B,u}(\theta_s) \)

\[
\Phi_{B,u}(\alpha) = E e^{\alpha \eta_{B,u}} = e^{i\alpha B} (I - p^*_+(0)) e^{\alpha R^*_+(0)v} p^*_+(\Lambda - Q)^{-1} \int_{-\infty}^{0} \Pi(dz) \times \left( e^{-i\alpha z} \Phi^-(0) + (I - p^*_+(0)) e^{\alpha R^*_+(0)v} \right) (sI + \Lambda F_0(0) - Q)^{-1} + C (C - i\alpha I)^{-1} \Phi^-(0) \bigg), \]

(32)

\[
P \{ \eta_{B,u} = B \} = (I - p^*_+(0)) e^{-\alpha R^*_+(0)v} p^*_+(\Lambda - Q)^{-1} \times \int_{-\infty}^{0} \Pi(dz) (I - p^*_+(0)) e^{\alpha R^*_+(0)v} (sI + \Lambda F_0(0) - Q)^{-1}. \]

(33)

For the dividend process \( Y_{B,u}(\theta_s) \), the following relations are true:

\[
E e^{-\mu Y_{B,u}(\theta_s)} = \left( I - (I - p^*_+(s)) e^{-\alpha R^*_+(s)v} \left( \mu I + R^*_+(s) \right)^{-1} \right) P_s, \]

(34)

\[
P \{ Y_{B,u}(\theta_s) = 0 \} = P \{ \xi^+(\theta_s) < v \} = P_s - (I - p^*_+(s)) e^{-\alpha R^*_+(s)v} P_s. \]

(35)

**Proof.** It follows from (29) that the characteristic function of \( \eta_{B,u}(t) \) satisfies an integral relation, which, after the Laplace–Karson transform, takes form

\[
\Phi_{B,u}(s, \alpha) = e^{i\alpha u} E \left[ e^{i\alpha \xi(\theta_s), \xi^+(\theta_s) < v} \right] + e^{i\alpha B} E \left[ e^{-sT_1, T_1 < \infty} (I - E e^{-s\xi^+}) P_s + E [ e^{-sT_1, T_1 < \infty} E e^{-s\xi^+} \tilde{\Phi}_B(s, \alpha) \right].
\]

(36)

According to [3, p.50] and [6, p.43], we get

\[
E \left[ e^{-sT_1, T_1 < \infty} \right] = (I - p^*_+(s)) e^{-\alpha R^*_+(s)v},
\]

\[
E e^{-s\xi^+} = (sI + \Lambda F_0(0) - Q)^{-1} \Lambda F_0(0),
\]

(37)

\[
E \left[ e^{i\alpha \xi(\theta_s), \xi^+(\theta_s) < v} \right] = E \left[ e^{i\alpha \xi^+(\theta_s), \xi^+(\theta_s) < v} \right] P_s^{-1} \Phi^-(s, \alpha) = (C - i\alpha I) p^*_+(s) - (I - p^*_+(s)) e^{(i\alpha I - R^*_+(s))v} R^*_+(s) (R^*_+(s) - i\alpha I)^{-1} \Phi^+(s, \alpha).
\]
Substituting these formulas into (36), we obtain (30) and the relation

\[ \tilde{\Phi}_B(s, \alpha) = e^{i\alpha B} \left( I - \int_{-\infty}^{0} dF_1^s(z) \left( I - p_+^s(s) \right) e^{R_1^s(z)z} \right) \times \]

\[ \times (sI + AF_0(0) - Q)^{-1} \left( \int_{-\infty}^{0} e^{i\alpha z} dF_1^s(z) (C - i\alpha I) - \right. \]

\[ - \int_{-\infty}^{0} dF_1^s(z) \left( I - p_+^s(s) \right) e^{R_1^s(s)z} C p_+^s(s) \left( R_1^s(s) - i\alpha I \right)^{-1} \Phi^-(s, \alpha) + \]

\[ + \int_{-\infty}^{0} dF_1^s(z) \left( I - p_+^s(s) \right) e^{R_1^s(s)z} s (sI + AF_0(0) - Q)^{-1} \right) . \] (38)

The stochastic relations for \( \tau^+ \) (see [3, p.62]) yield the following equation for \( p_+^s(s) \):

\[ (sI + \Lambda - Q) \left( I - p_+^s(s) \right) = \Delta F_0(0) + \Delta F_0(0) \int_{-\infty}^{0} dF_1^s(z) \left( I - p_+^s(s) \right) e^{Cp_+^s(s)z} . \] (39)

Substituting (39) into (38), we receive (31). Note that, for \( m_0^s > 0 \): \( \xi^+ \) has a degenerate distribution. Therefore, the first term in relation (36) tends to 0 as \( s \to 0 \). Since \( (I - E e^{-s\xi^+}) = s (sI + AF_0(0) - Q)^{-1} P_+^1 \), the second term in (36) tends to 0 too. For the third term, by virtue of Theorem 3 in [6] and the first relation in (2) we get

\[ \lim_{s \to 0} s^{-1} p_+^s(s) \left( R_1^s(s) - i\alpha I \right)^{-1} \Phi^-(s, \alpha) = \lim_{s \to 0} \left( C - i\alpha I \right)^{-1} (sI - \Psi(\alpha))^{-1} = \]

\[ = - \left( C - i\alpha I \right)^{-1} \Psi^{-1}(\alpha) . \]

Thus, passing to the limit as \( s \to 0 \) in (36) we obtain (32). Relation (34) and (35) follow from the representation: \( Y_{B,u}^R(t) = \max \left( 0, \xi_{B,u}^R(t) - v \right) . \)

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