Dislocation model of an asymmetric weak zone for problems of interaction between crack-like defects

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An adhesively bonded asymmetric weak zone is proposed as a model for studying the problem of interaction between crack-like defects in an elastic medium. The opening of the weak zone is prescribed by a two-parameter basis function, i.e. by a special dislocation which automatically accounts for the asymmetry and other expected physical features of the stress–strain field near the tips of the weak zone. The adhesive forces corresponding to the prescribed opening are then calculated from the solution of the particular problem. The application of the model is demonstrated on the problem of a long interface crack subjected to wedge opening forces which is separated from a short collinear interface weak zone by a small unbroken strong microstructural feature (a small obstacle). Two key questions pertaining to limiting situations are addressed: (i) when does the weak zone become the nucleus of a cohesive crack on its own without linking with the pre-existing long crack; and (ii) when does it force the rupture of the obstacle and coalesce with the long crack.

1. Introduction

Crack-like defect models range from the well-known traction-free Griffith–Irwin cracks, Barenblatt cuts containing small process zones at tips, where the cohesive forces are distributed in a simple or complex manner, to cuts subjected to cohesive forces over their entire length. This last type of defect may be regarded as a weak zone (WZ) in the solid which is normally closed but which can open progressively under sufficiently large remote external tensile forces. This defect serves as a forerunner of a cohesive crack in the Barenblatt sense: the WZ transforms into a cohesive crack when the remote tensile stresses are sufficiently large to overcome the adhesive forces. The fact that the WZ begins to open only when the remote force has reached a certain threshold level distinguishes the WZ from a real cohesive crack. The adhesive forces can be of very different physical origin—atomic, dislocational, localized porosity, etc. Healed cracks in glaciers and in the earth’s crust are also WZs. The length of WZs can thus range from a few nanometers to hundreds of kilometers. It is interesting to investigate the fundamental behaviour of the WZs.

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during their interaction with other defects, notwithstanding their size and physical origin. Note that the direct approach using some cohesive force-opening relationship (see, for example, [1], where an arbitrary nonlinear relationship was assumed), leads to a complicated problem which is reduced to the solution of a system of nonlinear singular integral equations in [1]. Another approach was suggested in [2], whereby the cohesive forces were expressed in a series containing \( N \) terms with \( N \) free parameters. Without going into the details of the physical origin of these forces, the free parameters were determined by imposing physically consistent conditions on the solution. The opening of WZ faces was obtained in an analytical form. It was revealed, in particular, that if the cohesive forces are constant over the length of the WZ, as in a Dugdale-type cohesive crack, the WZ will never open.

This paper focuses on two topics. Firstly, guided by the results obtained in [2], we shall develop a general approach to examining the behaviour of a WZ embedded in a very asymmetric external tensile stress field. This approach consists of prescribing \textit{a priori} the WZ asymmetric opening by a two-parameter basis function which meets all physical constraints, i.e. by a special dislocation. Secondly, this approach is exemplified by the problem of interaction between a long crack subjected to wedge opening forces and a short collinear WZ. This collinear crack-like defect system is assumed to be situated along the interface between two dissimilar elastic half-planes. The short WZ is separated from the long main crack by a small strong microstructural feature of the material (i.e. by a small obstacle). This situation can easily arise in a structured material, for example in a composite, or in materials in which special localized thermal or other strengthening techniques have been used to arrest cracks or to deflect them along curved high-energy trajectories. The key questions that will be addressed below are: (i) when does the WZ become the nucleus of a cohesive crack on its own without linking with the pre-existing long crack; and (ii) when does the WZ force the obstacle to rupture allowing the pre-existing crack to link with it. The critical applied load levels corresponding to these limiting situations will be determined. These depend on the interaction of several scales: the pre-existing crack is much longer than the WZ which, in turn, is much longer than the size of the obstacle and the size of the cohesive zone at the tip of the pre-existing crack (figure 1). A method based on a combination of theory and experiment is suggested for the determination of the distribution of cohesive forces over the WZ.

2. An asymmetric weak zone modelled as a special dislocation

For future use, let us recapitulate the result for a symmetric WZ obtained in [2] under the action of a remote tensile stress \( \sigma_0 \). If the cohesive forces, \( \sigma_c(X) \), are assumed to be distributed according to a quadratic relation along the WZ faces \( |X| < 1 \), then the corresponding WZ opening, \( w(X) \), is given by

\[
\sigma_c(X) = \sigma_\infty + \sigma_0 (1 - 2t), \quad w(X) = w(0) t^{3/2}, \quad t = 1 - X^2,
\]

\[
w(0) = \frac{2}{3} q \sigma_0 = w_* \quad \iff \quad \sigma(0) = 0, \quad \sigma_\infty = \sigma_0 = \sigma_* = \frac{(3w_*)}{(2q)},
\]
where $w_c$ is the critical opening displacement at which the WZ begins to open, $\sigma_0$ is a free parameter in the range $\sigma_\text{th} < \sigma_\infty < \sigma_\ast$; $\sigma_0 = 0$ at $\sigma_\infty = \sigma_\text{th}$, and $\sigma_0 = \sigma_\ast$ when $\sigma_\infty = \sigma_\ast$. Here, $\sigma_\text{th}$ is the threshold value of $\sigma_\infty$ at which the WZ begins to open. This indeterminacy in the value of $\sigma_0$ is a result of the arbitrariness in the description of the cohesive forces of different origin.

Expressions (2) are the conditions for the transition of a WZ to a cohesive crack; when $w(0)$ becomes equal to or greater than $w_c$, traction-free domains will appear in the WZ, beginning at its centre (in a symmetrically loaded WZ).

In contrast to [2], we shall take an inverse approach when the WZ is located in an asymmetric tensile stress field. We shall search for a basic function of asymmetric opening displacement of WZ, which reduces to (1) in the case of symmetry. Let this basic function be

$$w = w_0 T^{3/2}, \quad T = tG(X, \eta), \quad |\eta| < 1. \quad (3)$$

The first multiplier of $T(X, \eta)$ here is the same as in the symmetric case (see (1) above). The function $G(X, \eta)$ also depends on the parameter of asymmetry, $\eta$, i.e. the distance by which the maximum of $w(X)$ is displaced from the centre of WZ, $X = 0$. It is assumed that $\eta > 0$, if the maximum is displaced to the left of centre.

We shall seek the unknown function $G(X, \eta)$ under the following mathematical constraints which result from obvious physical and geometrical considerations.

1. It is even with respect to a simultaneous change of sign of both variables: $G(X, \eta) = G(-X, -\eta)$.
2. It is normalized such that: $T = (\eta, \eta) = 1 \iff G(-\eta, \eta) = (1 - \eta^2)^{-1}$; $G(X, 0) = 1$.
3. It has a maximum at point $X = -\eta$: $\partial T/\partial X(-\eta, \eta) = 0$.
4. It is bounded at the right tip $X = 1$ as $\eta \to -1 + 0$. 

Figure 1. A semi-infinite crack subjected to wedge forces $\Sigma$ at the interface between two dissimilar elastic half-planes with a small cohesive zone of length $\ell$ separated from an adhesively bonded collinear weak zone (WZ) of length $2L = (b - a)$ by a microstructural obstacle of length $a$. All lengths $\ell$, $a$, $b$, $L$ are $\ll 1$. 

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(5) $0 < G(1, \eta) < G(-1, \eta)$, when $\eta > 0$. This means that the maximum of $w(X)$ moves to the left and we require that the intensity of the opening displacement be greater near the left tip.

We shall search for a rational function of parameter $\eta$ as a possible form of $G(X, \eta)$, taking into account immediately the first of the five constraints above

$$G(X, \eta) = \sum_{n=0}^{\infty} \frac{A_n \eta^{2n}}{\eta X B_n \eta^{2n} + C_n \eta^{2n}}.$$  

By subjecting this rational function to the remaining four conditions, we obtain first a discrete recursive system of equations for the coefficients $A_n$, $B_n$, and $C_n$, and then, rather unexpectedly, the following simple result:

$$T(X, \eta) = \frac{1 - X^2}{(1 + \eta^2)(1 + \alpha X)}, \quad \alpha = \frac{2\eta}{1 + \eta^2},$$  

$$w_0 = w_s \iff \sigma(-\eta) = 0.$$  

Note that when $\eta = 0$, expression (4) reduces to (1). Moreover, in the general case $\eta \neq 0$, it appears that the asymptotic behaviour of (4) over a ‘large’ neighbourhood of the WZ tips obtained theoretically resembles a cusp, which is similar to the curve obtained by fitting experimental data for the crack opening displacement inside the process zone [3]. In fact, the normalization of $G(X, \eta)$ (condition 2 above) was chosen with this requirement in mind.

By prescribing the displacement jump in the form (3) we are actually considering a kind of special dislocation. This simplifies the solution of the WZ behaviour considerably in comparison with the direct method adopted in [2].

The parameter $\eta$ is determined from the additional condition that the minimum of the cohesive force occurs at the location of the maximum of the opening displacement $w(X)$, i.e. at $X = -\eta$. We also require that $\sigma(X)$ and $w(X)$ be positive right up to the instant when the WZ becomes a cohesive crack, i.e. up to the instant when the maximum opening displacement reaches the critical value $w_s$, a material constant, as in the symmetric case (2). The critical value $w_s$ establishes a limit on the external applied stress $\sigma_s$.

Generally speaking, to complete the statement of the model defect system it is necessary to stipulate another material constant, namely the threshold value of the nominal (i.e. when WZ is absent) external applied stress, as in (2) above, below which the WZ cannot open, $\sigma_{th}$. In practice, we only consider situations where $\sigma_{th} < \sigma_{\infty} < \sigma_s$. In this range, $w_0$ in (5) remains a free parameter. This is a consequence of a certain indeterminacy inherent to the present model.

3. Interaction between a long interface crack and a collinear short weak zone

The phenomenon of intermittent crack growth has often been observed in structured materials, e.g. in toughened ceramics [4]. In all likelihood, Broberg [5] was the first to propose a possible explanation for this phenomenon—a zone of micro-defects forms ahead of the growing crack which subsequently coalesces with the crack. The formation of micro-defects and their coalescence with the propagating crack causes the
latter to grow intermittently. This explanation has been confirmed experimentally on toughened ceramics by Mueller and Karihaloo [6]. Morozov [7] has proposed a series of discrete analytical fracture models and has shown that intermittent growth of a crack is only possible in structured materials.

Thus, one of the possible explanations for intermittent growth of a crack is the following. The crack front periodically meets a microstructural feature (an obstacle) of enhanced strength and is arrested by it if its driving force is insufficient to overcome the obstacle. Ahead of this obstacle, in the region of normal strength, micro-defects (micro-voids, micro-cracks) are formed in time. These micro-defects alter the local stress field in such a way as to cause a stress concentration on the obstacle which ruptures as a result, allowing the crack front to advance until it meets another obstacle. Many such situations have been reported in the literature. It is therefore important to investigate how a crack overcomes an obstacle when there is a weak zone (WZ) or a micro-crack ahead of the obstacle.

Below we shall study how a long crack with a small cohesive zone at its tip situated at the interface between two dissimilar elastic half-planes interacts with a WZ which is situated very close to, and is collinear with, it. The long crack is subjected to point loads (figure 1). As the WZ is located in a highly asymmetric external stress field generated by the long crack, we will use the basic model of the asymmetric opening of a WZ developed in the preceding section.

3.1. Problem formulation and general solution

Consider the plane elastic problem of a semi-infinite crack with a small cohesive zone at its tip and a WZ along the interface between two dissimilar linear isotropic half-planes, \( y > 0 \) (material 1) and \( y < 0 \) (material 2). The semi-infinite crack and the WZ are located at \( -\infty < x < 0 \) and \( 0 < a < x < b \), respectively (figure 1). The crack faces are subjected to wedge forces at unit distance \( x = -1 \) from the tip. The cohesive forces over the small process zone near the tip of the crack, \( -\ell < x < 0 \), are distributed quadratically [2]

\[
\sigma_c = \sigma(0)(1 - R^2), \quad R = x/\ell. \tag{6}
\]

The unit distance from the crack tip to the points of application of the wedge forces sets the largest length scale of the problem. All other distances are much smaller than unity. On the intervals of complete contact \( 0 < x < a \) and \( x > b \) the stresses and displacements are continuous, but in the interval \( a < x < b \), unknown cohesive forces prevent the opening \( w(x) \) of the WZ. The latter is prescribed through the special dislocation described by (3) and (4) with the following co-ordinate transformation:

\[
X = \frac{x}{L} - D, \quad D = \frac{b + a}{b - a} > 1, \quad L = \frac{b - a}{2},
\]

where \( L \) is the half-length of WZ.

The stress fields, \( \sigma_{km} \), and derivatives with respect to \( x \) of the displacement vector, \( \mathbf{u} = (u, v) \), can be expressed through the complex functions (potentials), \( \chi_j(z) \), \( z = x + iy \) [8, 9], subject to the following conditions of continuation across the interface:
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\[
\chi_1(z) = -\overline{\chi_1(z)} , \quad \chi_2(z) = \overline{\chi_2(z)} , \quad \Im z > 0.
\] (7)

The potentials \(\chi_j(z) (j = 1, 2)\) are linear transforms of the well-known Kolosov–Muskhelishvili complex potentials [8] so that the stress–strain fields can be expressed through these functions in the planes \(y > 0\) and \(y < 0\) [9]. They are holomorphic in the whole \(z\) plane except along the axis \(y = 0\) and almost everywhere have finite limits as \(y \to \pm 0\). At \(y = \pm 0\), the normal and shear stresses and the derivatives of the displacement vector with respect to \(x\) are expressed as follows:

\[
\begin{align*}
\sigma_{12} &= \Im \chi_1 , \quad \sigma_{22} = \Re \chi_2 , \\
U^\pm &= -\Re \{b_j \chi_1^\pm + a_j \chi_2^\pm\} , \quad V^\pm = \Im \{a_j \chi_1^\pm + b_j \chi_2^\pm\} , \\
s(x) &= [U] = U^+ - U^- = -q \Re \{\chi_1 + \beta \chi_2\} , \quad w(x) = [V] = q \Im \{\beta \chi_1 + \chi_2\} , \\
U &= du/dx , \quad q = b_1 + b_2 , \quad \beta = (a_1 - a_2)/ q , \\
4\mu_j a_j &= 1 - \kappa_j , \quad 4\mu_j b_j = 1 + \kappa_j , \quad j = 1, 2 , \\
\kappa_j &= 3 - 4\nu_j \quad \text{(plane strain)} , \\
\kappa_j &= (3 - \nu_j)(1 + \nu_j)^{-1} \quad \text{(generalized plane stress)}.
\end{align*}
\] (8)

Here, \(w(x)\) and \(s(x)\) are the normal and horizontal displacement discontinuities.

The continuations (7) allow us to reduce the initial problem in the theory of complex functions in the whole \(z\) plane to a boundary-valued one for the upper half-plane. Besides the usual boundary conditions on \(y = +0\) resulting from physical assumptions and relationships (8), the desired functions must satisfy the continuity condition at the special points, \(x = 0, a, b; y = +0\), have the proper asymptotic behaviour at infinity, and must ensure that the crack faces do not interpenetrate

\[
\begin{align*}
\Re \chi_2 &= -\Sigma \delta(x + 1) + \sigma_0(x)(x + \ell) \equiv \sigma_0(x) , \quad \Im \chi_1 = 0 , \quad x < 0 , \\
\Re \{q \chi_1 + d \chi_2\} &= s'(x)H(x - a)H(b - x) , \quad (') = d/dx , \\
\Im \{d \chi_1 + q \chi_2\} &= w'(x)H(x - a)H(b - x) , \quad x > 0 , \\
|\chi_j(z)| &= O(1) , \quad z \to z_* = 0 , \quad a, b , \quad \chi_j(z) = o(z^{-1}) , \quad z \to \infty , \\
w(x) &= [v(x)] = \int_0^x [V(\xi)] d\xi = \int_0^x \Im \{d \chi_1 + q \chi_2\}(\xi) d\xi \geq 0 ,
\end{align*}
\] (9)

where \(\delta(\bullet)\) and \(H(\bullet)\) are the Dirac and Heaviside functions.

This problem belongs to the class of the generalized coupled boundary-value Riemann–Hilbert problem for the two complex functions [8]. The solution will contain rapidly oscillating cofactors. But as shown in [9, 10], they influence only a fairly small (less than \(10^{-4}\)) zone around the crack tips. Besides, relatively minor shear stresses appear along the sections with full contact. These effects do not play an important role in the problem at hand and can be neglected for simplicity. If we ignore terms of \(O(\beta)\), \(\beta = dq\), and the jump in the horizontal displacement \(s(x)\) in the boundary conditions (9), the coupled boundary-value problem is uncoupled with \(\chi_1(z) \equiv 0\). Instead of conditions (9) and (11), we now have the following two
conditions for just one scalar function $\chi \equiv \chi_2$ [8]:

$$\forall \chi = \sigma_0(x), \quad x < 0, \quad q\chi = w(x)H(x-a)H(b-x), \quad x > 0, \quad (12)$$

$$w(x) = q \int_0^x \chi(\xi) \, d\xi \geq 0. \quad (13)$$

The problem described by (10), (12), and (13) coincides with that described by (9)–(11) if Dundurs’ parameter $\beta$ determining the degree of mismatch in the elastic properties of the materials 1 and 2 vanishes. Note, however, that the materials still need not be identical (see (8)). When $\beta \neq 0$, the solution of (10), (12), (13) is only an approximation which is the better the smaller the value of $\beta$. The solution is given by Cauchy integrals

$$\chi(z) = \frac{1}{\pi} \int_{-\infty}^0 \sqrt{\frac{\xi}{z}} \sigma_0(\xi) \, d\xi + \frac{1}{\pi q} \int_a^b \sqrt{\frac{\xi}{z}} w'(\xi) \, d\xi, \quad (14)$$

$$\chi^+(x) = \pm \frac{i}{q} w'(x) + \sigma_1(x) + \sigma_2(x), \quad x > 0,$n

$$\chi^-(x) = \pm \sigma_0(x) + i\sigma_1(x) + i\sigma_2(x), \quad x < 0,$n

$$\sigma_1(x) = \frac{1}{\pi q} \int_a^b \frac{w'(\xi) \, d\xi}{|x| - \xi - x}, \quad \sigma_2(x) = \frac{1}{\pi} \int_{-\infty}^0 \frac{\sigma_0(\xi) \, d\xi}{|x| - \xi - x},$$

where the cut for the isolation of the single-valued branch of $\sqrt{z}$ is constructed along half $x$ axis $y=0$, $x<0$, implying that $\sqrt{1} = 1$. Here and below, the singular integrals are understood in the sense of principal value. The normal stress distribution non-dimensionalized by $\Sigma$ follows from (8) and (14):

$$\sigma(x) = \sigma_1(x) + \sigma_2(x), \quad x > 0, \quad \sigma(0) = \frac{5}{8\sqrt{\ell}} (1 + I), \quad (15)$$

$$\sigma_2(x) = \frac{\Theta(x, \ell)}{\pi \sqrt{x}} = \frac{1}{\pi \sqrt{x}} \left\{ \frac{1}{1 + x} + O\left(\frac{1}{R}\right) \right\}, \quad R \gg 1$$

$$\Theta(x, \ell) = (1 + I)S(R) - I - \frac{x}{1 + x}, \quad I = \int_a^b \frac{w(\xi) \, d\xi}{2q \Sigma \xi^{3/2}}$$

$$S = \frac{5}{4} \left\{ \sqrt{R(1 - R^2)} \arctan \frac{1}{\sqrt{R}} - \frac{R}{3 + R^2} \right\} \approx 1 - \frac{4}{21R} + O\left(\frac{1}{R^2}\right), \quad R \gg 1.$$n

Here, $\sigma_1(x)$ is the contribution of WZ, $\sigma_2(x)$ is the nominal stress in the absence of WZ, and $\sigma(0)$ is calculated from the finiteness condition (10) by eliminating the square-root singularity. Note that if the opening $w(x)$ is such that $w'(x) = O(|x - x_0|^\nu)$, $x \to x_0 = a, b; \nu > 0$ then the integral in (14) exists everywhere as $x \to a + 0, b - 0$, the function $\sigma(x)$ is bounded, and, moreover, the condition (10) will be satisfied.

The normal stress $\sigma_2(x)$ varies rapidly over the WZ, if the zone $a < x < b$ is situated close to the long crack tip $x=0$. Thus, the WZ is embedded in a significantly asymmetric external stress field leading to its asymmetric opening. It was this fact that motivated the construction of the asymmetric WZ model described by (3) and (4).
On the other hand, the stress field in the vicinity of the long crack is only altered significantly by the presence of the WZ when the latter is very close to it \((a \ll b)\); otherwise the long crack is practically unaffected by the presence of the WZ.

Let us now calculate the necessary expressions starting from (3) and (4)

\[
w'(x) = \frac{dw}{dX} dX = -\frac{3w_0}{L} g(X), \quad a < x < b,
\]

\[
g(X) = \frac{(1 + \eta X)(\eta + X)\sqrt{1 - X^2}}{(1 + \eta^2 + 2\eta X)^{5/2}}, \quad X = \frac{x}{L} - D.
\]

Substituting this expression into the integral for \(\sigma_1(x)\) from (14), we get

\[
\sigma_1(x) = \frac{3W_0}{\pi L} \int_{-1}^{1} \sqrt{\frac{\xi + D}{X + D}} \frac{g(\xi) d\xi}{\xi - X}, \quad W_0 = \frac{w_0}{q\Sigma}.
\]

Let us also transform the integral I in (15)

\[
I = W_0 I_0, \quad I_0 = \int_{-1}^{1} \left( \frac{T(\xi)}{\xi + D} \right)^{3/2} \frac{d\xi}{2\sqrt{L}}.
\]

When \(w_0 = w_*\) we can relate the normalized limit value \(W_0 = W_*\) to other parameters using (15)–(17)

\[
W_* = \frac{S(R_m) - x_m/(1 + x_m)}{(1 - S(R_m)) I_0 + 3 I_1 L^{-1/2}} \approx \frac{\sqrt{L}}{3(1 + x_m) I_1}, \quad R_m = \frac{x_m}{\ell} \gg 1,
\]

\[
I_1 = \int_{-1}^{1} \frac{\sqrt{T(\xi)(\xi + D)}}{(1 + \eta^2 + 2\eta \xi)} (1 + \eta \xi) d\xi, \quad x_m = L(D - \eta).
\]

### 3.2. Determination of free geometrical parameters \(\eta\) and \(\ell\)

Differentiating the boundary conditions (12) with respect to \(x\) and solving the boundary-value problem for the complex function \(d\chi/dz\), which is similar to the problem described by (10)–(13), we get an expression for \(\sigma'(x)\). Then equating \(\sigma'(x_m) = 0\) leads to the following equation in \(\eta\):

\[
2L^{3/2}[2\Theta'(x_m, \ell) - \Theta(x_m, \ell)/x_m] + 3W_0J(\eta) = 0,
\]

\[
J(\eta) = \int_{-1}^{1} \sqrt{D + \xi} \frac{h(\xi, \eta) d\xi}{\xi + \eta},
\]

\[
h(\xi, \eta) = \frac{\alpha^2 \xi^4 + 4\alpha^3 \xi^3 + (8 - 2\alpha^2)\xi^2 + 4\alpha \xi + 5\alpha^2 - 4}{(1 + \eta^2)^{3/2}(1 + \alpha \xi)^{7/2}(1 - \xi^2)^{1/2}}.
\]

The uniqueness of the solution of (19) for fixed \(\ell\) is guaranteed by the continuous, strictly-monotonic dependence of the singular integral \(J\) on \(\eta\); it assumes all values along with the whole \(x\) axis for any fixed \(|\eta| < 1\).
In the limiting situation \( W_0 = W_* \), substituting expression (18) into (19) and noting that \( R_m \gg 1 \), (19) is simplified to

\[
\frac{J}{I_1} = \frac{2}{D - \eta} \left\{ 1 + 3x_m - \frac{5}{7R_m} \left( 1 + \frac{\sqrt{L} I_0}{3(1 + x_m) I_1} \right) \right\} + O \left( \frac{1}{R_m^2} \right).
\] (20)

If the first term only is retained in (20), then it becomes independent of \( \ell \), and the determination of \( \eta \) is no longer coupled with that of \( \ell \).

The process zone length \( \ell \) at the tip of the long crack is found from the condition \( w( -\ell ) = w_c \), where \( w_c \) is the critical crack opening at which the cohesive forces vanish. For this it is necessary to integrate the expression \( w_0 = q\sqrt{\chi} \) obtained from (14) from zero to \( -\ell \) to get

\[
\sqrt{\ell} = \frac{\pi W_c}{25\sqrt{L}/18 - W_0 I_2} \approx \frac{\pi w_c/\mathbf{w}_*}{25(1 + x_m) I_1/6 - I_2}, \quad w_0 = w_*, \quad R_m \gg 1,
\] (21)

The right-hand side of the integro-algebraic equation (21) depends weakly on \( \ell \) so that the system of equations (19) or (20), and (21) can be solved by successive approximation.

### 3.3. Limit and critical loads

It is expedient to use a Neuber–Novozhilov [11, 12] type failure criterion to establish the force at which the obstacle \( 0 < x < a \) will rupture; it will rupture when the average normal stress over it will attain the critical value \( \sigma_{cr} \), a material constant to be determined experimentally,

\[
\frac{1}{a} \int_0^a \sigma(x) \, dx = \frac{\sigma_{cr}}{\Sigma_{cr}}.
\] (22)

Then, after explicitly eliminating the singularity in \( \sigma(x) \) at \( x = 0 \), we get the critical applied force

\[
\Sigma_{cr} = \frac{\pi \sqrt{\ell} \sigma_{cr} - (w_0/q)(I_3 + I_0 I_d)}{I_d - 2a - 1 / \sqrt{\ell} (\sqrt{\ell} - \arctg \sqrt{\ell})}, \quad I_d = \frac{1}{\hbar} \int_0^\rho S(R) \, dR / \sqrt{R},
\] (23)

\[
I_3 \approx \frac{5}{4} \arctg \frac{1}{\sqrt{h}}, \quad h \ll 1, \quad I_3 = \frac{3}{2a} \int_0^{r_0} \sqrt{r} \, i_0(r) \, dr \approx \sqrt{\frac{a\ell}{L^2}} i_0(0), \quad r_0 \ll 1,
\]

\[
i_0 = \int_{-1}^1 \left[ \frac{T(\xi)}{D + \xi} \right]^{3/2} \frac{D + \xi - r/3}{(D + \xi - r)^2} \, d\xi, \quad h = \frac{a}{\ell}, \quad r = \frac{x}{L}, \quad r_0 = \frac{a}{L}.
\]

It should be noted that \( \Sigma_{cr} \) decreases when \( w_0 \) increases from nothing to \( w_* \), because the WZ increases the stress level in the vicinity of the obstacle. At \( w_0 = w_* \) the force reaches the minimum \( \Sigma_{cr, min} \) which will be needed below for comparison with its other limit value.
On the other hand, the limit load \( \Sigma_{a} \) at which the WZ becomes the nucleus of a cohesive crack is governed by the criterion (5). It can be found from expressions (18)

\[
\Sigma_{a} = \frac{w_{a}}{qW_{a}} \approx \frac{3w_{a}(1 + x_{m})I_{1}}{q\sqrt{L}}, \quad R_{m} \gg 1.
\]

(24)

It is clear from physical considerations that different situations are possible during a continuous increase in the applied force. Thus, if \( \Sigma_{c_{a}}^{\text{min}} < \Sigma_{a} \), then the obstacle will fracture first and the long crack will jump over it and ‘engulf’ the WZ. If, on the other hand, \( \Sigma_{c_{a}}^{\text{min}} > \Sigma_{a} \), then the WZ will become the nucleus for a cohesive crack. As mentioned in [2], this transition can be unstable and accompanied by a sudden loss of cohesive forces, thus increasing the mean stress over the obstacle. In this case, the defect system takes the classic crack–micro-crack configuration.

3.4. Computational results

Table 1 gives the results for the parameter \( \eta \), the non-dimensional process zone length \( \ell^{*} = \ell/L \), the peak stress \( \sigma(0) \), the force coefficient \( f = P/\Sigma = a(\sigma) \), where \( P \) is the force acting on the obstacle, and the mean stress \( \langle \sigma \rangle/a \), as a function of the geometrical parameters \( a \) and \( L \). The value \( \zeta = a/b \) is shown with the aim of comparison with the results for crack–micro-crack interaction given in section 4. All calculations refer only to the limiting situation when \( w_{0} = w_{a} = w_{c} \).

The parameter \( \eta \) increases sharply with increasing \( L \) when \( L \) is small but less so when \( L > 0.1 \), and it decreases significantly when the width of the obstacle \( a \) increases. It attains the maximum value \( \approx 0.31 \) at \( a = 0.1, L = 0.5 \). Thus, it is neither too small for the effect of WZ asymmetry to be neglected, nor too large for the model of asymmetric WZ used here to be physically unrealistic.

The ratio \( \ell^{*} = \ell/L \) is important to control the accuracy of computations in view of the use of the approximate expression (20) for the determination of \( \eta \) which is only valid for \( \ell^{*} \ll 1 \). Estimates show that the error in \( \eta \) in table 1 is less than 10%. Note that there is little point in demanding higher accuracy as the method itself is approximate. The process zone length \( \ell \) is smaller than 0.14 in the given ranges of values of \( a \) and \( L \) and varies rapidly with these parameters because of the strong influence of the WZ on the process zone of the long crack. In the absence of WZ, this length is directly related to \( w_{c} \)

\[
\ell = \left[ \frac{18\pi w_{c}}{25q\Sigma} \right]^{2},
\]

(25)

whereas the values \( \ell \) in table 1 relate to the limiting situation \( w_{c} = w_{a} \), so that the material constant does not appear explicitly.

One of the many possible criteria for assessing the stability of a long crack with a cohesive zone near the tip in which the stress varies according to (6) requires two material constants \( w_{c} \) and \( \ell_{c} \), where \( w(\ell_{c}) = w_{c} \). If \( \ell > \ell_{c} \), the crack remains stationary, whereas if \( \ell < \ell_{c} \), it grows. The critical state is defined by \( \ell = \ell_{c} \), and the corresponding critical force and the peak stress \( \sigma_{c}(0) \) are determined uniquely by expressions similar to (15) and (25). For this reason, any pair of the three quantities \( \sigma_{c}(0), w_{c}, \) and \( \ell_{c} \) may be chosen as the independent fracture parameters.
The peak stress $\sigma(0)$ which was calculated independently from (15) decreases markedly when $L$ increases, according to $\sigma(0) \sim 1/\sqrt{L}$, $a/L < 0.5$. The same asymptotic behaviour is exhibited by the limit force $\Sigma_a$. The Cauchy integrals in (15)–(17) were calculated with the use of regularization. The accuracy of these calculations was judged by how closely $\sigma(x)$ approached the independently calculated $\sigma(0)$ as $x \to +0$ and how close to zero it was at $x = x_m$ where it must vanish.

When a WZ of constant $L$ approaches the crack tip, i.e. when the width of obstacle $a$ tends to zero, the force coefficient $f$ decreases, but mean stress value $\langle \sigma \rangle$ grows logarithmically. Thus, when the width of the obstacle is reduced 20-fold, the mean stress increases by a factor of 3 for $L = 0.005$, a factor of 1.7 for $L = 0.5$, and in the interval $0.01 \leq a \leq 0.1$ (with $L = 0.005$) it follows the
law $\langle \sigma \rangle \approx 2.28 + 1.61 \ln(10a)$. On the other hand, when $a$ is held constant and $L$ is increased, both $f$ and $\langle \sigma \rangle$ decrease logarithmically. The reason for this apparent contradiction follows.

The distribution of stress $\sigma(x)$ ahead of the long crack is shown in figures 2 and 3 for $a = 0.01, L = 0.01, 0.02, 0.05$, and $a = 0.1, L = 0.01$, respectively. From figure 2 we notice that the peak stress at the left tip of WZ depends weakly on its half-length $L$; calculations not reported here show that this value is primarily determined by the width of the obstacle $a$. It is also clear from this figure that the stress distribution over the obstacle $0 < x < a$ is greatly influenced by a closely located WZ. The influence is localized only when the WZ is much farther removed from the long crack tip ($a \gg L$), as can be clearly seen in figure 3.

Recall that all the above conclusions apply to the critical situation $w_0 = w_\ast$, and that the applied force $\Sigma$ varies nonlinearly with $a$ and $L$ which, in turn, affects $\sigma(0), \ell, f$, etc.

From the solution (3)–(5), one can calculate the $\sigma$–$w$ deformation relationship over the WZ domain. This relationship appears to be close to a linear function with a small distortion in the vicinity of the WZ tips.

The theory developed above suggests a way for measuring experimentally the constitutive parameters and functions appearing in it. Thus, for example, laser or X-ray techniques can be employed to measure the WZ opening $w_c(x)$ in plates with high accuracy [3]. How well the experimental data $w_c(x)$ agrees with the theoretical prediction (3) and (4) will give an indication of how well the idealized model represents reality. If the measured opening deviates substantially from the
predicted one, then the true $\sigma$–$w$ relationship (which is difficult to measure directly) can be obtained by solving a problem similar to that considered above after replacing (3) by the measured function $w_c(x)$. This represents an indirect theoretico-experimental method for the determination of the true relationship between the cohesive forces and the opening in a WZ.

4. Interaction between a long crack and a micro-crack

We now turn our attention to the defect system consisting of a semi-infinite crack and a micro-crack (MC). Both defects are assumed to be traction free. Such a defect system may arise as a result of the transition of a WZ first into the nucleus of a cohesive crack when the load exceeds $\Sigma_{cr}$, followed by rapid loss of cohesive forces, as mentioned above. The MC can, of course, also form immediately after the long crack meets an obstacle and stops, as explained in section 2 in connection with WZ.

Obviously, the influence of two traction-free cracks will be much stronger on the obstacle than was the case with the crack–WZ system. It is therefore of interest to compare the solution of the two-crack problem with the results of section 3 for the crack–WZ problem. The crack–MC interaction problem has been extensively studied and reported in the literature (see, for example, [13]), but here we shall approach it as a special case of the problem solved in section 3.
The boundary conditions for the problem of finding a complex function \( \chi(z) \), simplified mathematically as in section 3, are now

\begin{align}
\Re \chi &= -\Sigma \delta(x+1), \quad -\infty < x < 0 \cup a < x < b, \\
\Im \chi &= 0, \quad 0 < x < a \cup x > b, \\
w(b) \equiv [v(b)] &= q \int_a^b \Im \chi(x) \, dx = 0.
\end{align}

The additional condition (27) necessary for obtaining a unique solution ensures a single-valued displacement field. Taking into account the fact that the stresses at the tips of both cracks have square-root singularities, the solution of the boundary-value problem (26) and (27) can be obtained following the procedure described in [8]. Note that the same procedure can also be applied if there are multiple micro-cracks present.

Without going into details, let us give the expressions for the stress \( \sigma(x) \) in the regions of full contact, the force coefficient \( f \), the relative crack opening \( W(x) \), and asymptotics of the stress intensity factors (SIFs), \( K_i \), at the tips \( x = 0, a, b \) normalized by nominal SIF \( K_0 = \Sigma \sqrt{2/\pi} \), ignoring quantities of \( O(b) \), \( b \ll 1 \) and of \( O(\zeta^2 \ln \zeta) \), \( \zeta = a/b \ll 1 \)

\[
\sigma(x) = \frac{1}{\pi} G(x) \text{sgn}(x-b), \quad x > b \cup 0 < x < a,
\]

\[
f = \int_a^b \sigma(x) \, dx = \frac{2}{\pi} \sqrt{(1+a)(1+b)} \left\{ \Pi(-a, \sqrt{\zeta}) - pK(\sqrt{\zeta}) \right\}
\]

\[
\sim \frac{\sqrt{b(1+b)}}{(4+\zeta)^{-1}} \left\{ \frac{1}{4 \ln 16} \right\} \left\{ \frac{2 + 2\zeta(A - B \ln \zeta)}{4 \ln 16 - (4 + \zeta) \ln \zeta} - \frac{x}{8} \right\}, \quad G(x) = \frac{P(x)}{(1+x)\sqrt{|Q(x)|}},
\]

\[
K_i(0) \sim \frac{1}{\sqrt{\Pi \ln 16 - (1 + \zeta/4) \ln \zeta}}, \quad K_i(b) \sim \frac{1 - 2\ln (16/\zeta)}{\sqrt{1 - \zeta}},
\]

\[
K_i(a) \sim \frac{2 + (2A - \ln 16)\zeta + (1 - 2B + \zeta/4) \ln \zeta}{\sqrt{(1 - \zeta) \ln 16 - (1 + \zeta/4) \ln \zeta}},
\]

\[
W(x) \equiv \frac{w(x)}{q \Sigma} = \frac{1}{\pi} \int_0^x G(x) \, dx = \frac{2}{\pi} \sqrt{\frac{1+b}{b(1+a)}}
\]

\[
\times \left\{ (p_0 - ap)F(\varphi, n) + \Pi(\varphi, 1+a, n) \right\} = \sqrt{b(1+b)}
\]

\[
\times \frac{2}{\pi} \left\{ (e_0(\zeta) - \zeta)F(\varphi, n) + \ln \left[ \frac{1 + \sqrt{\zeta}}{1 - \sqrt{\zeta}} \right] \right\} [1 - O(b)], \quad -1 < x < 0,
\]
where $A = 0.463$, $B = 0.245$, $K(n)$ and $\Pi(m, n)$ are the complete elliptic integrals, and $F(\varphi, k)$, $E(\varphi, k)$, and $\Pi(\varphi, m, n)$ are the elliptic integrals of the first, second and third kinds, respectively. The SIFs depend only on the small parameter $\zeta$, and as $\zeta \to 0$, they approach the following limits:

$$1 > K_f(b) \to 1 - 0, \quad K_f(0) > K_f(a) \to \infty, \quad \frac{K_f(0)}{K_f(a)} \to 1 + 0. \quad (29)$$

However, as is well known [13], the rate of convergence to these limits is logarithmically slow, that is why the numerical values shown in table 2 are important in practice. Besides the SIFs, the force coefficient $f$ and the mean stress $\langle \sigma \rangle$ over the obstacle width are presented for the same geometrical parameters as in table 1. However, as the simplified formulae (28) used are only valid for $\zeta \ll 1$ (i.e. $L = [a(1 - \zeta)]/[2\zeta]$) the values are only given for $\zeta \leq 0.5$. The SIFs depending on $\zeta$ are shown only for the first block of parameters in table 2.

The SIF at the right tip of MC, $K_f(b)$, increases slightly as $\zeta$ decreases, but remains noticeably less than the nominal SIF $K_0$. This means that there is less likelihood of MC growth or of the appearance of new MCs ahead of it. On the other hand, at even the relatively large value of the obstacle width, $\zeta = 0.25, 0.5$, the SIF $K_f(0)$ is only slightly larger than $K_0$, and $K_f(a)$ is significantly smaller than it. Consequently, as expected, the effect of MC is only felt when the obstacle is narrow. Thus, as in the case of WZ, the assumption that $a \ll b$ in the study of this effect and the use of the Neuber–Novozhilov criterion (22) for the rupture of the obstacle are justified. The expression for the critical force follows from (22) and is $\Sigma_{cr} = a\sigma_{cr}/f$.

For a given geometry of the crack-like defect system, a comparison of the results in tables 1 and 2 shows, as expected, that the crack–micro-crack system has a much stronger influence on the obstacle than the crack–WZ system, i.e. the integral characteristic quantities $f$ and $\langle \sigma \rangle$ in table 2 are significantly larger. In order to better
understand this difference in \( f \) and \( \langle \sigma \rangle \), let us write the statical equilibrium equation of the upper half-plane in the case of the crack–WZ defect system

\[
\Sigma + \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4 = 0, \quad \Sigma_2 = P,
\]

where \( \Sigma_k \) are the integrals of the stress \(-\sigma(x)\) over the intervals \((-\ell, 0)\), \((0, a)\), \((a, b)\), and \((b, \infty)\), respectively. In the case of the crack–MC system, \( \Sigma_1 = \Sigma_3 = 0 \), so that the applied load \( \Sigma \) is resisted only by two forces, \( \Sigma_2 \) and \( \Sigma_4 \), thus increasing the load carried by the obstacle. Note that the statical equation of equilibrium was also used to check the accuracy of computations.

The force coefficient \( f \) decreases (table 2), but the mean stress value \( \langle \sigma \rangle \) increases logarithmically as the tip of the MC of constant length approaches the tip of the main crack. Both quantities increase logarithmically with increasing MC length for constant \( a \) (roughly five-fold when \( L \) increases 100-fold for \( a = 0.01 \)). It is worth recalling that both these quantities decreased (table 1) in the crack–WZ system. Figure 4 shows this decrease in more detail for \( L = 0.02, 0.04, 0.1, 0.2; a = 0.02 \). However, direct comparison of these conclusions based on the parametric analysis would be incorrect because, in the case of the crack–WZ system, the geometrical

| \( a = 0.01 \) | \( \xi \) | 0.500 | 0.250 | 0.100 | 0.050 | 0.025 | 0.010 |
| \( L \) | 0.005 | 0.015 | 0.045 | 0.095 | 0.195 | 0.495 |
| \( K_f(0) \) | 1.048 | 1.131 | 1.359 | 1.633 | 2.017 | 2.750 |
| \( K_f(a) \) | 0.482 | 0.729 | 1.099 | 1.446 | 1.883 | 2.663 |
| \( f \) | 0.079 | 0.095 | 0.129 | 0.169 | 0.231 | 0.383 |
| \( \langle \sigma \rangle \) | 7.89 | 9.55 | 12.91 | 16.87 | 23.08 | 38.27 |
| \( a = 0.02 \) | \( \xi \) | 0.500 | 0.250 | 0.100 | 0.050 | 0.025 | 0.010 |
| \( L \) | 0.010 | 0.030 | 0.090 | 0.190 | 0.390 | 0.990 |
| \( f \) | 0.113 | 0.138 | 0.191 | 0.258 | 0.370 | 0.663 |
| \( \langle \sigma \rangle \) | 5.63 | 6.88 | 9.53 | 12.88 | 18.50 | 33.15 |
| \( a = 0.05 \) | \( \xi \) | 0.500 | 0.250 | 0.100 | 0.050 | 0.025 | 0.010 |
| \( L \) | 0.025 | 0.075 | 0.225 | 0.475 | 0.975 | 2.475 |
| \( f \) | 0.183 | 0.229 | 0.337 | 0.487 | 0.755 | 1.482 |
| \( \langle \sigma \rangle \) | 3.67 | 4.59 | 6.74 | 9.74 | 15.11 | 29.65 |
| \( a = 0.10 \) | \( \xi \) | 0.500 | 0.250 | 0.100 | 0.050 | 0.025 | 0.010 |
| \( L \) | 0.050 | 0.150 | 0.450 | 0.950 | 1.950 | 4.950 |
| \( f \) | 0.271 | 0.350 | 0.550 | 0.843 | 1.379 | 2.839 |
| \( \langle \sigma \rangle \) | 2.71 | 3.50 | 5.50 | 8.43 | 13.79 | 28.39 |
| \( a = 0.20 \) | \( \xi \) | 0.500 | 0.250 | 0.100 | 0.050 | 0.025 | 0.010 |
| \( L \) | 0.100 | 0.300 | 0.900 | 1.900 | 3.900 | 9.900 |
| \( f \) | 0.413 | 0.562 | 0.953 | 1.540 | 2.617 | 5.547 |
| \( \langle \sigma \rangle \) | 2.07 | 2.81 | 4.77 | 7.70 | 13.08 | 27.73 |
parameters are varied at the limit situation \( w_0 = w_a \) (and, correspondingly, \( \Sigma = \Sigma_a \)), so that the load would have to change significantly as \( a \) and \( L \) are varied, which, in turn, would have a nonlinear effect on the stress state at the crack tip.

In view of the well-known \( \sigma-\tau \) dualism in the theory of elasticity, a model of the shear WZ and solution of the corresponding problem for the interaction between the crack under shear forces and the shear WZ follow automatically from the above results. The principle of superposition can then be used to solve the problem of crack–WZ interaction under an inclined force. However, the ratio of shear to normal force components must be sufficiently small for the approximations of section 3 to be valid when \( \beta \neq 0 \). If this condition is not satisfied, it is necessary to revert to the exact formulation (9)–(11) of the problem.

5. Conclusions

A semi-inverse method has been proposed for solving the problems of elastic bodies containing adhesively bonded weak zones (WZs) situated in highly asymmetric stress fields. It is based on prescribing the displacement jumps in the WZ in the form of a two-parameter basis function (i.e. a special dislocation) that is asymmetric and has the expected behaviour of the stress and displacement fields near the ends of the WZ. The advantages of this method are that it is independent of the physical origin of the adhesive forces and is simple to use for the solution of crack–WZ interaction problems. It is an approximate method which can, however, be improved by including higher-order terms in the basis function.
The method was demonstrated on the problem of interaction between a long crack and a short WZ separated by an obstacle. Two limit states exist in this problem: (i) the moment at which the WZ becomes the nucleus of a cohesive crack; and (ii) the moment at which the barrier ruptures allowing coalescence of the crack with the WZ. The analysis of these two limit states is important for understanding the phenomenon of intermittent crack growth in structured materials. Explicit expressions have been obtained for limit loads in terms of the given or calculable parameters of the problem. A procedure has been developed for determining the unknown parameters and the stress distribution ahead of the crack front.

Computational results did not reveal any physical inconsistency in the proposed dislocation model of the WZ, thus providing a justification for the semi-inverse method.

An indirect mixed theoretico-experimental route to determining the distribution of cohesive forces in the WZ has also been indicated. For this it is necessary to measure accurately the opening of a crack-like defect using laser or X-ray techniques and then to follow the semi-inverse method developed in this paper.

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