Investigation of spin-0 massive charged particle subject to a homogeneous magnetic field with potentials in a topologically trivial flat class of Gödel-type space-time

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Abstract

In this paper, we investigate relativistic quantum dynamics of spin-0 massive charged particle subject to a homogeneous magnetic field in the Gödel-type space-time with potentials. We solve the Klein-Gordon equation subject to a homogeneous magnetic field in a topologically trivial flat class of Gödel-type space-time in the presence of a Cornell-type scalar and Coulomb-type vector potentials and analyze the effects on the energy eigenvalues and eigenfunctions.

Keywords: Gödel-type space-time, Klein-Gordon equation, electromagnetic field, energy spectrum, wave-functions

PACS Number: 03.65.Pm, 03.65.Ge

1 Introduction

The first solution to the Einstein’s field equations containing closed time-like curves is the cylindrical symmetry Gödel rotating Universe [1]. Reboucas et al. [2, 3, 4] investigated the Gödel-type solutions characterized by vorticity, which represents a generalization of the original Gödel metric with possible sources and analyzed the problem of causality. The line element of Gödel-type solution is given by

$$ds^2 = - (dt + A_i dx^i)^2 + \delta_{ij} dx^i dx^j,$$  \hspace{1cm} (1)
where the spatial coordinates of the space-time are represented by $x^i$ and $i, j = 1, 2, 3$. The different classes of Gödel-type solutions have discussed in [5].

Investigation of relativistic quantum dynamics of spin-zero and spin-half particles in Gödel Universe and Gödel-type space-times as well the Schwarzschild, the Kerr black holes solution has been addressed by several authors. The study of relativistic wave-equations, particularly, the Klein-Gordon and Dirac equations in the background of Gödel-type space-time was first studied in [6]. The close relation between the relativistic energy levels of a scalar particle in the Som-Raychaudhuri space-time with the Landau levels was studied in [7]. The same problem in the Som-Raychaudhuri space-time was investigated and compared with the Landau levels [8]. The Klein-Gordon equation in the background of Gödel-type space-times with a cosmic string was studied in [9], and analyzed the similarity of the energy levels with Landau levels in flat space. The Klein-Gordon oscillator in the background of Gödel-type space-time under the influence of topological defects was studied in [10]. The relativistic quantum dynamics of scalar particle in the presence of an external fields in the Som-Raychaudhuri space-time under the influence of topological defects was studied in [11]. The relativistic quantum motion of spin-0 particles in a flat class of Gödel-type space-time was studied in [12]. The study of spin-0 system of DKP equation in a flat class of Gödel-type space-time was studied in [13]. In all the above system, the influence of topological defects and vorticity parameter characterizing the space-time on the relativistic energy eigenvalues was analyzed. Linear confinement of a scalar particle in the Som-Raychaudhuri space-time with cosmic string was studied in [14] (see also, [15]). The behavior of scalar particle with Yukawa-like confining potential in the Som-Raychaudhuri space-time in the presence of topological defects was investigated in [16]. Ground state of a bosonic massive charged particle in the presence of an external fields in a Gödel-type space-time was investigated in [17] (see also, [18]). The relativis-
tic quantum dynamics of spin-zero particles in 4D curved space-time with the 
cosmic string subject to a homogeneous magnetic field was studied in [19].
In addition, the relativistic wave-equations in (1+2)-dimensional rotational 
symmetry space-time background was investigated in [20, 21, 22, 23, 24, 25].

Furthermore, Dirac and Weyl fermions in the background of the Som-
Raychaudhuri space-times in the presence of topological defects with 
torsion was studied in [26]. Weyl fermions in the background of the Som-
Raychaudhuri space-times in the presence of topological defects was studied 
in [27] (see, Refs. [28, 29]). The relativistic wave-equations for spin-half par-
ticles in the Melvin space-time, a space-time where the metric is determined 
by a magnetic field was studied in [30]. The fermi field and Dirac oscillator 
in the Som-Raychaudhuri space-time was studied in [31]. The Fermi field with 
scalar and vector potentials in the Som-Raychaudhuri space-time was inves-
tigated in [32]. The Dirac particles in a flat class of Gödel-type space-time 
was studied in [5]. Dirac fermions in (1+2)-dimensional rotational symmetry 
space-time background was investigated in [33].

The relativistic quantum dynamics of a scalar particle subject to dif-
ferent confining potentials have been studied in several areas of physics by 
various authors. The relativistic quantum dynamics of scalar particles sub-
ject to Coulomb-type potential was investigated in [34, 35, 36, 37]. It is 
worth mentioning studies that have dealt with Coulomb-type potential in the 
propagation of gravitational waves [38], quark models [39], and relativistic 
quantum mechanics [40, 41, 42, 43]. Linear confinement of scalar particles 
in a flat class of Gödel-type space-time, were studied in [44]. The Klein-
Gordon equation with vector and scalar potentials of Coulomb-type under 
the influence of non-inertial effect in cosmic string space-time was studied in 
[45]. The Klein-Gordon oscillator in the presence of Coulomb-type potential 
in the background space-time generated by a cosmic string was studied in 
[43, 46]. Other works on the relativistic quantum dynamics are the Klein-
Gordon scalar field subject to a Cornell-type potential [47], and survey on
the Klein-Gordon equation in a Gödel-type space-time [48].

Our aim in this paper is to investigate the quantum effects on bosonic massive charged particle by solving the Klein-Gordon equation subject to a homogeneous magnetic field in the presence of a Cornell-type scalar and Coulomb-type vector potentials in Gödel-type space-time. We see that the presence of magnetic field as well as various potential modifies the energy spectrum.

2 Bosonic charged particle : The KG-Equation

The relativistic quantum dynamics of a charged particle of modifying mass $m \rightarrow m + S$, where $S$ is the scalar potential is described by the following equation [42]

$$\left[ \frac{1}{\sqrt{-g}} D_\mu (\sqrt{-g} g^{\mu\nu} D_\nu) - (m + S)^2 - \xi R \right] \Psi = 0, \quad (2)$$

where $g$ is the determinant of metric tensor with $g^{\mu\nu}$ its inverse, $D_\mu = \partial_\mu - ie A_\mu$ is the minimal substitution, $e$ is the electric charge and $A_\mu$ is the electromagnetic four-vector potential, and $\xi$ is the non-minimal coupling constant with the background curvature.

We choose the electromagnetic four-vector potential $A_\mu = (-V, \vec{A})$ with [49]

$$A_y = -xB_0 \quad , \quad \vec{A} = (0, A_y, 0) \quad (3)$$

such that the constant magnetic field is along the axis $\vec{B} = \vec{\nabla} \times \vec{A} = -B_0 \hat{z}$.

Consider the following stationary space-time [50] (see, [3, 12, 13, 44, 51]) in the Cartesian coordinates $(x^0 = t, x^1 = x, x^2 = y, x^3 = z)$ is given by

$$ds^2 = - (dt + \alpha_0 x dy)^2 + \delta_{ij} dx^i dx^j, \quad (4)$$

where $\alpha_0 > 0$ is a real positive constant. In Ref. [5], we have discussed different classes of Gödel-type space-time. For the space-time geometry [4], it belongs to a linear or flat class of Gödel-type metrics. The parameter
\( \alpha_0 = 2\Omega \) where, \( \Omega \) characterize the vorticity parameter of the space-time. For \( \Omega \to 0 \), the study space-time reduces to four-dimensional flat Minkowski metric.

The determinant of the corresponding metric tensor \( g_{\mu\nu} \) is

\[
\text{det} \, g = -1. \tag{5}
\]

The scalar curvature of the metric is

\[
R = \frac{\alpha_0^2}{2} = 2\Omega^2. \tag{6}
\]

For the space-time geometry (4), the equation (2) becomes

\[
\left[-\left(\frac{\partial}{\partial t} + ieV\right)^2 + \left\{\frac{\partial}{\partial y} - ieA_y\right\} - 2\Omega \left(\frac{\partial}{\partial t} + ieV\right)\right]^2 + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} - (m + S)^2 - 2\xi\Omega^2\right] \Psi = 0. \tag{7}
\]

Since the metric is independent of \( t, y, z \). One can choose the following ansatz for the function \( \Psi \)

\[
\Psi(t, x, y, z) = e^{i(-E t + ly + kz)} \psi(x), \tag{8}
\]

where \( E \) is the total energy of the particle, \( l = 0, \pm 1, \pm 2, \ldots \) are the eigenvalues of the \( y \)-component operator, and \( -\infty < k < \infty \) is the eigenvalues of the \( z \)-component operator.

Substituting the ansatz Eq. (8) into the Eq. (7), we obtain the following differential equation for \( \psi(x) \):

\[
\left\{\frac{d^2}{dx^2} + (E - e V)^2 - k^2 - \{(l + eB_0) + 2\Omega x (E - eV)\}^2 - (m + S)^2 - 2\xi\Omega^2\right\} \psi(x) = 0. \tag{9}
\]

### 2.1 Interaction with Cornell-type and Coulomb-type potentials

Here we study a spin-0 massive charged particle by solving the Klein-Gordon equation in the presence of an external fields in a flat class of Gödel-type
space-time subject to a Cornell-type scalar and Coulomb-type vector potentials. We obtain the energy eigenvalues and eigenfunctions and analyze the effects due to various physical parameters.

The Cornell-type potential contains a confining (linear) term besides the Coulomb interaction and has been successfully accounted for the particle physics data [52]. This type of potential is a particular case of the quark-antiquark interaction, which has one more harmonic-type term [53]. The Coulomb potential is responsible by the interaction at small distances and the linear potential leads to the confinement. The quark-antiquark interaction potential has been studied in the ground state of three quarks [54], and systems of bound heavy quarks [55, 56, 57]. This type of interaction has been studied by several authors ([11, 23, 42, 58, 59, 60, 61, 62, 63, 64]).

We consider the scalar $S$ to be Cornell-type [42]

$$S = \frac{\eta_c}{x} + \eta_L x,$$

where $\eta_c, \eta_L$ are the Coulombic and confining potential constants, respectively.

Another potential that we are interest here is the Coulomb-type potential which we discussed in the introduction. Therefore, the Coulomb-type vector potential is given by

$$V = \frac{\xi_c}{x},$$

where $\xi_c$ is the Coulombic potential constants.

Substituting the potentials (10) and (11) into the Eq. (9), we obtain the following equation:

$$\left[ \frac{d^2}{dx^2} + \lambda - \omega^2 x^2 - \frac{j^2}{x^2} - \frac{a}{x} - bx \right] \psi(x) = 0,$$

(12)
where we have defined

\[
\lambda = E^2 - m^2 - k^2 - 2 \eta c \eta L - 2 \xi \Omega^2 - (l - 2 e \Omega \xi c)^2,
\]

\[
\omega = \sqrt{4 (\Omega E + m \omega c)^2 + \eta^2 L},
\]

\[
j = \sqrt{\eta^2 c - e^2 \xi^2},
\]

\[
a = 2 (e \xi c E + m \eta c),
\]

\[
b = 2 [m \eta L + \omega (l - 2 e \Omega \xi c)],
\]

\[
\omega_c = \frac{e B_0}{2 m}
\]  

(13)
is called the cyclotron frequency of the particle moving in the magnetic field.

Let us define a new variable \(r = \sqrt{\omega x}\), Eq. (12) becomes

\[
\left[ \frac{d^2}{dr^2} + \beta - r^2 - j^2 r^2 - \eta r - \theta r \right] \psi(r) = 0,
\]

(14)

where

\[
\beta = \frac{\lambda}{\omega}, \quad \eta = \frac{a}{\sqrt{\omega}}, \quad \theta = \frac{b}{\omega^2}.
\]

(15)

We now use the appropriate boundary conditions to investigate the bound states solution in this problem. It is known in relativistic quantum mechanics that the radial wave-functions must be regular both at \(r \to 0\) and \(r \to \infty\). Then we proceed with the analysis of the asymptotic behavior of the radial eigenfunctions at origin and in the infinite. These conditions are necessary since the wave-functions must be well-behaved in these limit, and thus, the bound states of energy eigenvalues for this system can be obtained. Suppose the possible solution to the Eq. (14) is

\[
\psi(r) = r^j e^{-\frac{1}{2} (\theta + r) r} H(r).
\]

(16)

Substituting the solution Eq. (16) into the Eq. (14), we obtain

\[
\frac{d^2 H}{dr^2} + \left[ \gamma \gamma - \theta - 2 r \right] \frac{dH}{dr} + \left[ - \chi \gamma + \Theta \right] H(r) = 0,
\]

(17)
where
\[
\gamma = 1 + 2 j,
\]
\[
\Theta = \beta + \frac{\theta^2}{4} - 2 (1 + j),
\]
\[
\chi = \eta + \frac{\theta}{2} (1 + 2 j).
\]  
(18)

Equation (17) is the biconfluent Heun’s differential equation \([42, 43, 46, 59, 60, 61, 62, 63, 65, 66]\) with \(H(r)\) is the Heun polynomials function.

Writing the function \(H(r)\) as a power series expansion around the origin \([49]\):
\[
H(r) = \sum_{i=0}^{\infty} c_i r^i.
\]  
(19)

Substituting the series solution into the Eq. (17), we obtain the following recurrence relation:
\[
c_{n+2} = \frac{1}{(n + 2)(n + 1 + \gamma)} \left[ (\chi + \theta (n + 1)) c_{n+1} - (\Theta - 2 n) c_n \right].
\]  
(20)

Few coefficients of the series solution are
\[
c_1 = \left( \frac{\eta}{\gamma} + \frac{\theta}{2} \right) c_0,
\]
\[
c_2 = \frac{1}{2(1 + \gamma)} \left[ (\chi + \theta) c_1 - \Theta c_0 \right].
\]  
(21)

As the function \(H(r)\) has a power series expansion around the origin in Eq. (19), then, the relativistic bound states solution can be achieved by imposing that the power series expansion becomes a polynomial of degree \(n\) and we obtain a finite degree polynomial for the biconfluent Heun series. Furthermore, the wave-function \(\psi\) must vanish at \(r \to \infty\) for this finite degree polynomial of power series otherwise the function diverge for large values of \(r\). Therefore, we must truncate the power series expansion \(H(r)\) a polynomial of degree \(n\) by imposing the following two conditions \([11, 14, 42, 43, 44, 46]\\n
\[ \Theta = 2n, \quad (n = 1, 2, \ldots) \]
\[ c_{n+1} = 0. \quad (22) \]

By analyzing the condition \( \Theta = 2n \), we have the second degree eigenvalues equation

\[ E_{n,l}^2 = m^2 + k^2 + 2 \eta_c \eta_L + 2 \xi \Omega^2 + 2 \omega (n + 1 + \sqrt{\eta_c^2 - e^2 \xi_c^2}) \]
\[ - \frac{m^2 \eta_L^2}{\omega^2} = \frac{2 m \eta_L (l - 2 e \xi_c \Omega)}{\omega}. \quad (23) \]

The corresponding eigenfunctions is given by

\[ \psi_{n,l}(r) = r^{\sqrt{\eta_c^2 - e^2 \xi_c^2}} e^{-\frac{1}{2}(r+\theta)r} H(r). \quad (24) \]

Note that Eq. (23) does not represent the general expression of the eigenvalue problem. One can obtain the individual energy eigenvalues one by one, that is, \( E_1, E_2, E_3, \ldots \) by imposing the additional recurrence condition \( c_{n+1} = 0 \) on the eigenvalues. The solution with Heun’s equation makes it possible to obtain the individual eigenvalues one by one as done in [11, 14, 42, 43, 44, 46, 59, 60, 61, 62, 63, 67, 68]. In order to analyze the above conditions, we must assign values to \( n \). In this case, consider \( n = 1 \), that means we want to construct a first degree polynomial to \( H(r) \). With \( n = 1 \), we have \( \Theta = 2 \) and \( c_2 = 0 \) which implies from Eq. (21)

\[ \frac{a_{1,l}}{1 + 2 j} + \frac{b_{1,l}}{2 \omega_{1,l}} (a_{1,l} + \frac{b_{1,l}}{\omega_{1,l}} (j + \frac{3}{2})) = 2 \omega_{1,l} \]
\[ \omega_{3,l} - \frac{a^2}{2 (1 + 2 j)} \omega_{1,l}^2 - ab \frac{1 + j}{1 + 2 j} \omega_{1,l} - \frac{b^2}{8} (3 + 2 j) = 0, \quad (25) \]

a constraint on the physical parameter \( \omega_{1,l} \). The relation given in Eq. (25) gives the possible values of the parameter \( \omega_{1,l} \) that permit us to construct
first degree polynomial to \( H(r) \) for \( n = 1 \). Note that its values changes for each quantum number \( n \) and \( l \), so we have labeled \( \omega \rightarrow \omega_{n,l} \). In this way, we obtain the following energy eigenvalue \( E_{1,l} \):

\[
E_{1,l} = \pm \{ m^2 + k^2 + 2 \eta_c \eta L + 2 \xi \Omega^2 + 2 \left( 2 + \sqrt{\eta_c^2 - \xi_c^2} \right) \omega_{1,l} \}
\]

\[
\frac{m^2 \eta_c^2}{\omega_{1,l}^2} - \frac{2 m \eta L (l - 2 e \xi_c \Omega)}{\omega_{1,l}} \}
\]

(26)

Then, by substituting the real solution \( \omega_{1,l} \) from Eq. (25) into the Eq. (26) it is possible to obtain the allowed values of the relativistic energy levels for the radial mode \( n = 1 \) of a position-dependent mass system. We can see that the lowest energy state is defined by the real solution of the algebraic equation Eq. (25) plus the expression given in Eq. (26) for the radial mode \( n = 1 \), instead of \( n = 0 \). This effect arises due to the presence of Cornell-type potential in the system. Note that, it is necessary physically that the lowest energy state is \( n = 1 \) and not \( n = 0 \), otherwise the opposite would imply that \( c_1 = 0 \) which is not possible.

The corresponding radial wave-function for \( n = 1 \) is given by

\[
\psi_{1,l}(r) = r^{\frac{l}{2}} e^{\frac{-1}{2} \left( r + \frac{b}{r} \right)} \frac{1}{\omega_{1,l} \sqrt{a + 2 \sqrt{\eta_c^2 - \xi_c^2}}} \left( c_0 + c_1 r \right),
\]

(27)

where

\[
c_1 = \frac{1}{\omega_{1,l} \sqrt{a + 2 \sqrt{\eta_c^2 - \xi_c^2}}} \left( c_0 + \frac{b}{2 \omega_{1,l}} \right).
\]

(28)

### 2.2 Interaction without potential

Here we study a spin-0 massive charged particle by solving the Klein-Gordon equation in the presence of an external fields in a Gödel-type space-time without potential and obtain the relativistic energy eigenvalue.

We choose here zero scalar and vector potentials, \( S = 0 = V \). In that case, Eq (9) becomes

\[
\left[ \frac{d^2}{dx^2} + E^2 - m^2 - k^2 - l^2 - 2 \xi \Omega^2 - \omega^2 x^2 - 2 \omega l x \right] \psi(x) = 0,
\]

(29)
The above equation can be expressed as

\[
\left[ \frac{d^2}{dx^2} + E^2 - m^2 - k^2 - 2 \xi \Omega^2 - \omega^2 (x + \frac{l}{\omega})^2 \right] \psi(x) = 0, \quad (30)
\]

Let us define a new variable \( r = (x + \frac{l}{\omega}) \), Eq. (30) becomes

\[
\psi''(r) + (\delta - \omega^2 r^2) \psi(r) = 0, \quad (31)
\]

where

\[
\delta = E^2 - m^2 - k^2 - 2 \xi \Omega^2. \quad (32)
\]

Again introducing a new variable \( \rho = \sqrt{\omega} r \) into the Eq. (31), we obtain

\[
\psi''(\rho) + (\frac{\delta}{\omega} - \rho^2) \psi(\rho) = 0 \quad (33)
\]

which is similar a harmonic-type oscillator equation. Therefore, the energy eigenvalues equation is

\[
\frac{\delta}{\omega} = 2 n + 1 \Rightarrow \delta = (2 n + 1) \omega \\
\Rightarrow \quad E_n^2 = 2 \omega (2 n + 1) E_n - m^2 - k^2 - 2 \xi \Omega^2 - 2 m \omega c (2 n + 1) = 0. \quad (34)
\]

The energy eigenvalues associated with \( n^{th} \) modes is

\[
E_n = (2 n + 1) \Omega + \sqrt{(2 n + 1)^2 \Omega^2 + m^2 + k^2 + 2 \xi \Omega^2 + 2 m \omega c (2 n + 1)} \\
= (2 n + 1) \Omega + \sqrt{(2 n + 1)^2 \Omega^2 + m^2 + k^2 + 2 \xi \Omega^2 + |e B_0| (2 n + 1)} \quad (35)
\]

where \( n = 0, 1, 2, .. \). We can see that the energy eigenvalues (35) depend on the parameter \( \Omega \) characterizing the vorticity parameter of the space-time geometry, and the external magnetic field \( B_0 \) as well the non-minimal coupling constant \( \xi \) with the background curvature.

In absence of external magnetic fields, \( B_0 \rightarrow 0 \), and without non-minimal coupling constant, \( \xi \rightarrow 0 \), the eigenvalue (35) becomes

\[
E_n = (2 n + 1) \Omega + \sqrt{(2 n + 1)^2 \Omega^2 + m^2 + k^2}. \quad (36)
\]
Equation (36) is the energy eigenvalues of spin-0 particle in the background of a flat class of Gödel-type space-time and consistent with the result in [12]. Thus we can see that the energy eigenvalues (35) in comparison to the result in [12] get modify due to the presence of external fields and the non-minimal coupling constant with the background curvature.

Therefore, the individual energy levels for $n = 0, 1$ using (35) are follows:

$$
n = 0 : E_0 = \Omega + \sqrt{\Omega^2 + 2 \xi \Omega^2 + 2 m \omega_c + m^2 + k^2},$$

$$
n = 1 : E_1 = 3 \Omega + \sqrt{9 \Omega^2 + 2 \xi \Omega^2 + 6 m \omega_c + m^2 + k^2}. \quad (37)$$

A special case corresponds to $m = 0 = k$, the energy eigenvalues (35) reduces to

$$
E_n = (2 n + 1) \Omega + \sqrt{(2 n + 1)^2 \Omega^2 + 2 \xi \Omega^2 + |e B_0| (2 n + 1)}. \quad (38)
$$

The individual energy levels for $n = 0, 1$ in that case are follows:

$$
n = 0 : E_0 = \Omega + \sqrt{\Omega^2 + 2 \xi \Omega^2 + |e B_0|},$$

$$
n = 1 : E_1 = 3 \Omega + \sqrt{9 \Omega^2 + 2 \xi \Omega^2 + 3 |e B_0|}. \quad (39)$$

And others are in the same way. We can see that the presence of external magnetic field $B_0$ as well as the non-minimal coupling constant $\xi$ causes asymmetry in the energy levels and hence, the energy levels are not equally spaced.

The eigenfunctions is given by

$$
\psi_n(\rho) = |N| \, H_n(\rho) \, e^{-\frac{\rho^2}{2}}, \quad (40)
$$

where $|N| = \frac{1}{2^n n! \sqrt{\pi}}$ is the normalization constant and $H_n(\rho)$ are the Hermite polynomials and define as

$$
H_n(\rho) = (-1)^n e^{\rho^2} \frac{d^n}{d\rho^n} (e^{-\rho^2}), \quad \int_{-\infty}^{\infty} e^{-\rho^2} H_n(\rho) H_m(\rho) \, d\rho = \sqrt{\pi} 2^n n! \delta_{nm}. \quad (41)
$$
3 The Klein-Gordon Oscillator

Here we study a spin-0 massive charged particle by solving the Klein-Gordon equation of the Klein-Gordon oscillator in the presence of external fields in a Gödel-type space-time subject to a Cornell-type scalar and Coulomb-type vector potentials. We analyze the effects on the relativistic energy eigenvalue and corresponding eigenfunctions due to various physical parameters.

To couple Klein-Gordon field with oscillator \[^{(69, 70)}\], the generalization of Mirza et al. prescription \[^{(71)}\], in which the following change in the momentum operator is taken:

\[
p_\mu \rightarrow p_\mu + im\omega_0 X_\mu,
\]

where \(m\) is the particle mass at rest, \(\omega_0\) is the frequency of the oscillator and \(X_\mu = (0, x, 0, 0)\), with \(x\) being the distance of the particle. In this way, the Klein-Gordon oscillator equation becomes

\[
\frac{1}{\sqrt{-g}} (D_\mu + m\omega_0 X_\mu) \sqrt{-g} \frac{\partial}{\partial x} (D_\nu - m\omega_0 X_\nu) \Psi = (m + S)^2 \Psi.
\]

Using the space-time \[^{(2)}\], we obtain the following equation

\[
\frac{\partial}{\partial t} + \frac{\partial}{\partial y} - \frac{i}{\omega_0} \frac{\partial}{\partial z} \frac{\partial}{\partial x} + \frac{\partial}{\partial x} + m \omega_0 x \left( \frac{\partial}{\partial x} - m \omega_0 x \right) = 0.
\]

Using the ansatz \[^{(8)}\] into the above Eq. \(^{(44)}\), we arrive at the following equation

\[
\frac{d^2 \psi}{dx^2} + \left( (E - eV)^2 - \left\{ \alpha_0 x (E - eV) + (l - eA_y) \right\}^2 - k^2 - m \omega_0 \right) \psi = 0.
\]

Substituting the potentials Eq. \(^{(3)}\), \(^{(10)}\) and \(^{(11)}\) into the equation \(^{(45)}\), we obtain the following equation:

\[
\psi''(x) + \left[ \tilde{\lambda} - \tilde{\omega}^2 x^2 - \frac{j^2}{x^2} - \frac{a}{x} - \tilde{b} x \right] \psi(x) = 0,
\]
where we have defined

\[ \tilde{\lambda} = E^2 - m^2 - k^2 - 2 \eta_c \eta_L - 2 \xi \Omega^2 - (l - 2 e \Omega \xi_c)^2 - m \omega_0, \]
\[ \tilde{\omega} = \sqrt{4 (\Omega E + m \omega_e)^2 + m^2 \omega_0^2 + \eta_c^2}, \]
\[ j = \sqrt{\eta_c^2 - e^2 \xi_c^2}, \]
\[ a = 2 (e \xi_c E + m \eta_c), \]
\[ \tilde{b} = 2 [m \eta_L + \tilde{\omega} (l - 2 e \Omega \xi_c)]. \]  

(47)

Let us define a new variable \( r = \sqrt{\omega} x \), Eq. (46) becomes

\[ \frac{d^2}{dr^2} + \tilde{\beta} - r^2 - \frac{j^2}{r^2} - \frac{\tilde{\eta}}{r} - \tilde{\theta} r \] \( \psi(r) = 0, \)  

(48)

where

\[ \tilde{\beta} = \frac{\lambda}{\tilde{\omega}}, \quad \tilde{\eta} = \frac{a}{\sqrt{\omega}}, \quad \tilde{\theta} = \frac{b}{\omega^2}. \]  

(49)

Suppose the possible solution to the Eq. (48) is

\[ \psi(r) = r^j e^{-\frac{1}{2} (\tilde{\theta} + r) r} H(r). \]  

(50)

Substituting the solution Eq. (50) into the Eq. (48), we obtain

\[ H''(r) + \left[ \frac{\gamma}{r} - \tilde{\theta} - 2 r \right] H'(r) + \left[ - \frac{\tilde{\chi}}{r} + \tilde{\Theta} \right] H(r) = 0, \]  

(51)

where \( \gamma \) is given earlier and

\[ \tilde{\Theta} = \tilde{\beta} + \frac{\tilde{\theta}^2}{4} - 2 (1 + j), \]
\[ \tilde{\chi} = \tilde{\eta} + \frac{\tilde{\theta}}{2} (1 + 2 j). \]  

(52)

Equation (51) is the biconfluent Heun’s differential equation \[42, 43, 46, 59, 60, 61, 62, 63, 65, 66\].

Substituting the series solution Eq. (19) into the Eq. (51), we obtain the following recurrence relation:

\[ c_{n+2} = \frac{1}{(n + 2)(n + 1 + \gamma)} \{ \tilde{\chi} + \tilde{\Theta} (n + 1) \} c_{n+1} - (\tilde{\Theta} - 2 n) c_n. \]  

(53)
Few coefficients of the series solution are
\[ c_1 = \left( \frac{\bar{\eta}}{\gamma} + \frac{\tilde{\theta}}{2} \right) c_0, \]
\[ c_2 = \frac{1}{2 (1 + \gamma)} [(\tilde{\chi} + \tilde{\theta}) c_1 - \tilde{\Theta} c_0]. \] (54)

The power series expansion becomes a polynomial of degree \( n \) by imposing following two conditions \([11, 14, 42, 43, 44, 46, 59, 60, 61, 62, 63, 67, 68]\):
\[ \tilde{\Theta} = 2 n \quad (n = 1, 2, ..) \]
\[ c_{n+1} = 0 \] (55)

Using the first condition we obtain the following energy eigenvalues :
\[ E_{n,l}^2 = m^2 + k^2 + 2 \eta_c \eta_L + 2 \xi \Omega^2 + m \omega_0 + 2 \left( n + 1 + \sqrt{\eta_c^2 - e^2 \xi_c^2} \right) \tilde{\omega} \]
\[ - \frac{m^2 \eta_L^2}{\tilde{\omega}^2} - \frac{2 m \eta_L (l - 2 e \xi_c \Omega)}{\tilde{\omega}} \] (56)

The corresponding eigenfunctions is given by
\[ \psi_{n,l}(r) = r \sqrt{m^2 - e^2 \xi_c^2} e^{-\frac{1}{2} (r + \tilde{\theta}) r} H(r). \] (57)

As done earlier, we obtain the individual energy levels by imposing the recurrence condition \( c_{n+1} = 0 \). For \( n = 1 \), we have \( c_2 = 0 \) which implies from Eq. [54]
\[ \tilde{\omega}_{1,l}^3 - \frac{a^2}{2 (1 + 2 j)} \tilde{\omega}_{1,l}^2 \tilde{\omega}_{1,l} - a b \left( \frac{1 + j}{1 + 2 j} \right) \tilde{\omega}_{1,l} - \frac{b^2}{8} (3 + 2 j) = 0, \] (58)
a constraint on the physical parameter \( \tilde{\omega}_{1,l} \). The relation given in Eq. [58] gives the possible values of the parameter \( \tilde{\omega}_{1,l} \) that permit us to construct first degree polynomial to \( H(r) \) for \( n = 1 \) [42, 59, 43, 46]. In this way, we obtain the following second degree algebraic equation for \( E_{1,l} \):
\[ E_{1,l} = \pm \left\{ m^2 + k^2 + 2 \eta_c \eta_L + 2 \xi \Omega^2 + 2 \left( 2 + \sqrt{\eta_c^2 - e^2 \xi_c^2} \right) \tilde{\omega}_{1,l} \right\} \]
\[ - \frac{m^2 \eta_L^2}{\tilde{\omega}_{1,l}^2} = \frac{2 m \eta_L (l - 2 e \xi_c \Omega)}{\tilde{\omega}_{1,l}} \}^{\frac{1}{2}}. \] (59)
Then, by substituting the real solution \( \tilde{\omega}_{1,l} \) from Eq. (58) into the Eq. (59) it is possible to obtain the allowed values of the relativistic energy levels for the radial mode \( n = 1 \) of a position-dependent mass system. We can see that the lowest energy state is defined by the real solution of algebraic equation Eq. (58) plus the expression given in Eq. (59) for the radial mode \( n = 1 \), instead of \( n = 0 \). This effect arises due to the presence of Cornell-type potential in the system. Note that, it is necessary physically that the lowest energy state is \( n = 1 \) and not \( n = 0 \), otherwise the opposite would imply that \( c_1 = 0 \) which is not possible.

The corresponding radial wave-function for \( n = 1 \) is given by

\[
\psi_{1,l}(r) = r \sqrt{\eta^2 - e^2 \xi^2} e^{-\frac{1}{2} \left( r + \frac{\eta}{r} \right)} \left( c_0 + c_1 r \right),
\]

where

\[
c_1 = \frac{1}{\sqrt{\tilde{\omega}_{1,l}}} \left[ \frac{2 (e \xi E_{1,l} + m \eta e)}{1 + 2 \sqrt{\eta^2 - e^2 \xi^2}} + \frac{2 (m \eta L + \tilde{\omega}_{1,l} (l - 2 e \Omega \xi))}{2 \tilde{\omega}_{1,l}} \right] c_0.
\]

4 Conclusions

The relativistic quantum system of scalar and spin-half particles in Gödel-type space-times was investigated by several authors (e.g., [7, 8, 9, 11, 12, 13, 14, 59]). They demonstrated that the energy eigenvalues of the relativistic quantum system get modified and depend on the global parameters characterizing the space-times.

In this work, we have investigated the influence of vorticity parameter on the relativistic energy eigenvalues of a relativistic scalar particle in a Gödel-type space-time subject to a homogeneous magnetic field with potentials. We have derived the radial wave-equation of the Klein-Gordon equation in a class of flat Gödel-type space-time in the presence of an external fields with(-out) potentials by choosing a suitable ansatz of the wave-function. In sub-section 2.1, we have introduced a Cornell-type scalar and Coulomb-type
vector potentials into the considered relativistic system and obtained the energy eigenvalue Eq. \( (23) \) and corresponding eigenfunctions Eq. \( (24) \). We have seen that the presence of a uniform magnetic field and potential parameters modifies the energy spectrum in comparison those result obtained in \[12\]. By imposing the additional recurrence condition \( c_{n+1} = 0 \), we have obtained the ground state energy levels Eq. \( (26) \) and wave-functions Eq. \( (27)-(28) \) for \( n = 1 \). In sub-section 2.2, we have considered zero potential into the relativistic system and solved the radial wave-equation of the Klein-Gordon equation in the presence of an external field. We obtained the energy eigenvalues Eq. \( (35) \) and compared with the results obtained in \[12\]. We have seen that the relativistic energy eigenvalues Eq. \( (35) \) get modify in comparison to those in \[12\] due to the presence of a homogeneous magnetic field. In section 3, we have solved the Klein-Gordon equation of the Klein-Gordon oscillator in a Gödel-type space-time subjected to a homogeneous magnetic field in the presence of a Cornell-type scalar and Coulomb-type vector potentials. We have obtained the energy eigenvalue Eq. \( (56) \) and corresponding eigenfunctions Eq. \( (57) \). We have seen that the presence of a uniform magnetic field and potential parameters modifies the energy spectrum in comparison to those in \[12\]. By imposing the additional recurrence condition \( c_{n+1} = 0 \), we have obtained the ground state energy levels Eq. \( (59) \) and wave-function Eq. \( (60) \) for \( n = 1 \) and others are in the same way.

So, in this paper we have some results which are in addition to the previous results obtained in \[7,8,9,11,12,13,14,51,59\] present may interesting effects. This is the fundamental subject in physics and the connection between these theories (quantum mechanics and gravitation) are not well understood.

**Data Availability**

No data has been used to prepare this paper.
Conflict of Interest

Author declares that there is no conflict of interest regarding publication this paper.

Acknowledgment

Author sincerely acknowledge the anonymous kind referee(s) for his/her valuable comments and suggestions.

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