Integer Powers of Certain Complex Pentadiagonal 2—Toeplitz Matrices

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Abstract
In this study, we get a general expression for the entries of the sth power of even order pentadiagonal 2-Toeplitz matrices.

1 Introduction
Gover [1] gained eigenvalues and eigenvectors of a tridiagonal 2-Toeplitz matrix in terms of the Chebyshev polynomials. Hadj and Elouafi [2] obtained the general expression of the characteristic polynomial, determinant and eigenvectors for pentadiagonal matrices. Rimas [3] offered general expression for the entries of the power of tridiagonal 2-Toeplitz matrix, in terms of the Chebyshev polynomials of the second kind. Obvious formulas for the determinants of a band symmetric Toeplitz matrix is restated in [4]. Álvarez-Nodarse et al. [5] gained the general expressions for the eigenvalues, eigenvectors and the spectral measure of 2 and 3-Toeplitz matrices. The powers of even order symmetric pentadiagonal matrices are calculated in [6]. Öteleş and Akbulak [7] viewed powers of tridiagonal matrices. Wu [8] calculated the powers of Toeplitz Matrices. The powers of complex pentadiagonal Toeplitz matrices are computed in [9].

This paper is organized as follows: the first section, motivated by [2], we apply $K_n$ pentadiagonal 2-Toeplitz matrix to the characteristic polynomial and eigenvectors of this matrix in given [2]. In Section 2, we obtain the eigenvalues and eigenvectors of $K_n$ pentadiagonal 2-Toeplitz matrix. In Section 3, the sth power of pentadiagonal 2-Toeplitz matrix we will get by using the expression $K_n^s = L_n J_n^s L_n^{-1}$ [11], where $J_n$ is the Jordan’s form of $K_n$ and $L_n$ is the transforming matrix. In Section 4, some numerical examples are given.

Consider the polynomial sequence $\{A_i\}_{i \geq 0}$ and $\{B_i\}_{i \geq 0}$ characterized by a three-term recurrence relation

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\[ \begin{align*}
x A_0(x) &= a_1 A_0(x) + b_1 A_2(x) \\
x A_1(x) &= a_2 A_1(x) + b_2 A_3(x) \\
x A_{i-1}(x) &= a_1 A_{i-3}(x) + a_1 A_{i-1}(x) + b_1 A_{i+1}(x) \quad \text{for } i \geq 3 \text{ and } i = 2t + 1 \quad (t \in \mathbb{N}) \\
x A_{i-1}(x) &= c_2 A_{i-3}(x) + a_2 A_{i-1}(x) + b_2 A_{i+1}(x) \quad \text{for } i \geq 3 \text{ and } i = 2t \quad (t \in \mathbb{N})
\end{align*} \]

with initial conditions \( A_0(x) = 0 \) and \( A_1(x) = 1 \), and

\[ \begin{align*}
x B_0(x) &= a_1 B_0(x) + b_1 B_2(x) \\
x B_1(x) &= a_2 B_1(x) + b_2 B_3(x) \\
x B_{i-1}(x) &= a_1 B_{i-3}(x) + a_1 B_{i-1}(x) + b_1 B_{i+1}(x) \quad \text{for } i \geq 3 \text{ and } i = 2t + 1 \quad (t \in \mathbb{N}) \\
x B_{i-1}(x) &= c_2 B_{i-3}(x) + a_2 B_{i-1}(x) + b_2 B_{i+1}(x) \quad \text{for } i \geq 3 \text{ and } i = 2t \quad (t \in \mathbb{N})
\end{align*} \]

with initial conditions \( B_0(x) = 1 \) and \( B_1(x) = 0 \), here \( a_1, a_2 \in \mathbb{C} \) and \( b_1, b_2, c_1, c_2 \in \mathbb{C} \setminus \{0\} \). We can write a matrix form to this three-term recurrence relations

\[ \begin{align*}
x A_{n-1}(x) &= K_n A_{n-1}(x) + A_n(x) d_{n-1} + A_{n+1}(x) d_n \\
x B_{n-1}(x) &= K_n B_{n-1}(x) + B_n(x) d_{n-1} + B_{n+1}(x) d_n
\end{align*} \]

where \( A_{n-1}(x) = [A_0(x), A_1(x), A_2(x), \ldots, A_{n-1}(x)]^T \),
\( B_{n-1}(x) = [B_0(x), B_1(x), B_2(x), \ldots, B_{n-1}(x)]^T \),
\( d_{n-1} = [0, 0, 0, \ldots, 0, b_2, 0]^T \),
\( d_n = [0, 0, 0, \ldots, 0, 0, b_1]^T \quad (n = 2t + 1, \quad t \in \mathbb{N}) \)

and
\[ \begin{align*}
d_{n-1} &= [0, 0, 0, \ldots, 0, b_1, 0]^T \\
d_n &= [0, 0, 0, \ldots, 0, 0, b_2]^T \quad (n = 2t, \quad t \in \mathbb{N}).
\end{align*} \]

Let
\[
K_n = \begin{bmatrix}
a_1 & 0 & b_1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & a_2 & 0 & b_2 & 0 & 0 & \cdots & 0 & 0 & 0 \\
c_1 & 0 & a_1 & 0 & b_1 & 0 & \cdots & 0 & 0 & 0 \\
0 & c_2 & 0 & a_2 & 0 & b_2 & \cdots & 0 & 0 & 0 \\
0 & 0 & c_1 & 0 & a_1 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & b_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & a_1 & 0 & b_1 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & a_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & c_1 & 0 & a_1
\end{bmatrix}
\]

for \( n = 2t + 1 \quad (t \in \mathbb{N}) \)
and
\[
K_n = \begin{bmatrix}
0 & a_1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & a_2 & 0 & b_1 & 0 & \cdots & 0 & 0 & 0 \\
c_1 & 0 & a_1 & 0 & b_1 & 0 & \cdots & 0 & 0 \\
0 & c_2 & 0 & a_2 & 0 & b_2 & \cdots & 0 & 0 \\
0 & 0 & c_1 & 0 & a_1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & b_1 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & a_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & a_3 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & c_2 & 0 \\
\end{bmatrix}
\]
for \(n = 2t (t \in \mathbb{N})\) (5)

here \(a_1, a_2 \in \mathbb{C}\) and \(b_1, b_2, c_1, c_2 \in \mathbb{C} \setminus \{0\}\).

**Lemma 1**  The polynomial sequences \(\{A_i\}_{i \geq 0}\) and \(\{B_i\}_{i \geq 0}\) confirm

\[
\deg(A_{2i+1}) = i \text{ and the leading coefficient of } A_{2i+1} \text{ is equal to } \frac{1}{b_2},
\]
\[
\deg(A_{2i}) = 0,
\]
\[
\deg(B_{2i+1}) = 0,
\]
\[
\deg(B_{2i}) = i \text{ and the leading coefficient of } B_{2i} \text{ is equal to } \frac{1}{b_1}.
\]

**Proof.** Let us prove by the inductive method. For the basis step, we possess

For \(i = 0\) : \(A_0 (x) = 0\), \(B_0 (x) = 1\).

For \(i = 1\) : \(A_1 (x) = 1\), \(B_1 (x) = 0\).

For \(i = 2\) : \(A_2 (x) = 0\), \(B_2 (x) = \frac{a_1 a_2}{b_1}\).

For \(i = 3\) : \(A_3 (x) = \frac{a_1 a_2}{b_2} , B_3 (x) = 0\).

For \(i = 4\) : \(A_4 (x) = 0\), \(B_4 (x) = \frac{(x-a_1)^2 - b_1 c_1}{b_1^2}\).

Assuming that (1) and (2) are correct for \(p = k \geq 3\). We will prove it for \(k = p + 1\), we obtain

\[
xA_{p-1}(x) = c_1 A_{p-3}(x) + a_1 A_{p-1}(x) + b_1 A_{p+1}(x) \text{ for } p \geq 3 \text{ and } p = 2t + 1 (t \in \mathbb{N})
\]
\[
xA_{p-1}(x) = c_2 A_{p-3}(x) + a_2 A_{p-1}(x) + b_2 A_{p+1}(x) \text{ for } p \geq 3 \text{ and } p = 2t (t \in \mathbb{N})
\]
\[
xB_{p-1}(x) = c_1 B_{p-3}(x) + a_1 B_{p-1}(x) + b_1 B_{p+1}(x) \text{ for } p \geq 3 \text{ and } p = 2t + 1 (t \in \mathbb{N})
\]
\[
xB_{p-1}(x) = c_2 B_{p-3}(x) + a_2 B_{p-1}(x) + b_2 B_{p+1}(x) \text{ for } p \geq 3 \text{ and } p = 2t (t \in \mathbb{N}).
\]

(6)

If \(p := 2i\), then we have

\[
xA_{2i-1}(x) = c_2 A_{2i-3}(x) + a_2 A_{2i-1}(x) + b_2 A_{2i+1}(x)
\]
\[
xB_{2i-1}(x) = c_2 B_{2i-3}(x) + a_2 B_{2i-1}(x) + b_2 B_{2i+1}(x).
\]

Accordingly

\[
\deg(A_{2i+1}(x)) = \deg(xA_{2i-1}(x))
\]
\[
\deg(B_{2i+1}(x)) = 0
\]
and the leading coefficient of

$$A_{2i+1} = \frac{1}{b_2^i} \text{(leading coefficient of } A_{2i-1}(x)),$$

$$= \frac{1}{b_2^i (b_2^i - 1)} = \frac{1}{b_2^i}. \quad (7)$$

If $p := 2i - 1$, then we write

$$xA_{2i-2}(x) = c_1A_{2i-4}(x) + a_1A_{2i-2}(x) + b_1A_{2i}(x),$$

$$xB_{2i-2}(x) = c_1B_{2i-4}(x) + a_1B_{2i-2}(x) + b_1B_{2i}(x).$$

So,

$$\deg(A_{2i}(x)) = 0$$

$$\deg(B_{2i}(x)) = \deg(xB_{2i-2}(x))$$

and the leading coefficient of

$$B_{2i} = \frac{1}{d_1^i} \text{(leading coefficient of } B_{2i-2}(x)),$$

$$= \frac{1}{d_1^i (d_1^i - 1)} = \frac{1}{d_1^i}. \quad (8)$$

**Definition 2** $K_n$ be $n$-square pentadiagonal 2-Toeplitz matrix, one correlates the sequence polynomial $P_i$ described by

$$P_i = \det \begin{bmatrix} A_i & A_i \\ B_i & B_i \end{bmatrix}. \quad (9)$$

**Lemma 3** Due to (4) and (3), we own

$$xP_{n-1}(x) = K_n P_{n-1}(x) + P_n(x)d_{n-1} + P_{n+1}(x)d_n$$

$$= K_n P_{n-1}(x) + P_{n+1}(x)d_n \quad (10)$$

where $P_{n-1}(x) = [P_0(x), P_1(x), P_2(x), \ldots, P_{n-1}(x)]^T$,

$$d_{n-1} = [0, 0, 0, \ldots, 0, b_2, 0]^T \quad (n = 2t + 1, \ t \in \mathbb{N}),$$

$$d_n = [0, 0, 0, \ldots, 0, 0, b_1]^T \quad (n = 2t, \ t \in \mathbb{N}),$$

$$d_{n-1} = [0, 0, 0, \ldots, 0, b_1, 0]^T \quad (n = 2t, \ t \in \mathbb{N}).$$

**Lemma 4** The polynomial $P_{n+1}$ is degree $n$ and the leading coefficient of $P_{n+1}$ is

$$\frac{1}{b_1 b_2^{n-1}}, \quad \text{if } n = 2t (t \in \mathbb{N}),$$

$$\frac{1}{b_1 b_2}, \quad \text{if } n = 2t + 1 (t \in \mathbb{N}). \quad (11)$$
Proof. If \( n = 2i \)

\[
P_{2i+1} = \det \begin{bmatrix} A_{2i} & A_{2i+1} \\ B_{2i} & B_{2i+1} \end{bmatrix} = A_{2i}B_{2i+1} - A_{2i+1}B_{2i}
\]

and using Lemma 1

\[
\deg(P_{2i+1}) = \deg(A_{2i+1}B_{2i})
\]

and the leading coefficient of

\[
P_{2i+1} = -\frac{1}{b_2} \frac{1}{b_1} = -\frac{1}{(b_1b_2)^i}.
\]

If \( n = 2i + 1 \)

\[
P_{2i+1} = \det \begin{bmatrix} A_{2i+1} & A_{2i+2} \\ B_{2i+1} & B_{2i+2} \end{bmatrix} = A_{2i+1}B_{2i+2} - A_{2i+2}B_{2i+1}
\]

and using Lemma 1

\[
\deg(P_{2i+2}) = \deg(A_{2i+1}B_{2i+2})
\]

and the leading coefficient of

\[
P_{2i+2} = \frac{1}{b_2} \frac{1}{b_1^{i+1}}.
\]

\[\blacksquare\]

Lemma 5 If \( \alpha \) is a zero of the polynomial \( P_{n+1} \) then \( \alpha \) is an eigenvalues of the matrix \( K_n \).

Proof. Let \( \alpha \) is a zero of the polynomial \( P_{n+1} \), from equation (10), we acquire

\[
K_nP_{n-1}(\alpha) = \alpha P_{n-1}(\alpha).
\]

There are four cases to be noted.

Case I. Suppose for \( n = 2t \ (t \in \mathbb{N}) \) either \( A_{n+1}(\alpha) \neq 0 \) or \( B_n(\alpha) \neq 0 \). In that case \( P_0(\alpha) = A_{n+1}(\alpha) \) and \( P_1(\alpha) = -B_n(\alpha) \), then \( P_{n-1}(\alpha) \) is a corresponding non-null eigenvector of \( K_n \), we acquire that \( \alpha \) is an eigenvalue of the matrix \( K_n \).

Case II. Suppose for \( n = 2t-1 \ (t \in \mathbb{N}) \) either \( A_{n}(\alpha) \neq 0 \) or \( B_{n+1}(\alpha) \neq 0 \). In that case \( P_0(\alpha) = A_{n}(\alpha) \) and \( P_1(\alpha) = -B_{n+1}(\alpha) \), then \( P_{n-1}(\alpha) \) is a corresponding non-null eigenvector of \( K_n \), we acquire that \( \alpha \) is an eigenvalue of the matrix \( K_n \).

Case III. Suppose for \( n = 2t \ (t \in \mathbb{N}) \), \( A_n(\alpha) = B_{n+1}(\alpha) = 0 \) and \( B_n(\alpha) \neq 0 \). Let \( F_{n-1}(\alpha) = -A_{n-1}(\alpha)B_n(\alpha) \), we own \( K_nF_{n-1}(\alpha) = \alpha F_{n-1}(\alpha) \), then \( F_{n-1}(\alpha) \) is a corresponding non-null eigenvector of \( K_n \), we acquire that \( \alpha \) is an
The eigenvalue of the matrix $K_n$.

Case IV. Suppose for $n = 2t + 1 \ (t \in \mathbb{N})$, $A_{n-1}(\alpha) = B_n(\alpha) = 0$ and $A_n(\alpha) \neq 0$. Let $F_{n-1}(\alpha) = A_n(\alpha)B_{n-1}(\alpha)$, we own $K_nF_{n-1}(\alpha) = \alpha F_{n-1}(\alpha)$, then $F_{n-1}(\alpha)$ is a corresponding non-null eigenvector of $K_n$, we acquire that $\alpha$ is an eigenvalue of the matrix $K_n$.

Theorem 6 Let $K_n$ be $n$-square pentadiagonal 2-Toeplitz matrix and the corresponding polynomial $P_{n+1}$ in the eq. $\mathbf{(10)}$. Suppose that $P_{n+1}$ has simple zeros, the characteristic polynomial of $K_n$ is completely

$$
|xI_n - K_n| = (b_1b_2)^\frac{n}{2}A_{n+1}B_n, \quad {\text{if}} \ n = 2t \ (t \in \mathbb{N})
$$

$$
|xI_n - K_n| = b_1^{\frac{n+1}{2}}b_2^{\frac{n-1}{2}}A_nB_{n+1}, \quad {\text{if}} \ n = 2t + 1 \ (t \in \mathbb{N})
$$

(12)

here $I_n$ is the $n$-square identity matrix.

Proof. Let $\alpha_1, \ldots, \alpha_n$ the zeros of characteristic polynomial of $K_n$. Lemma 5, $\alpha_1, \ldots, \alpha_n$ are the eigenvalues of the matrix $K_n$.

2 Eigenvalues and eigenvectors of $K_n$

Theorem 7 Let $K_n$ be $n$-square ($n = 2t, \ t \in \mathbb{N}$) pentadiagonal 2-Toeplitz matrix as in $\mathbf{(5)}$. Then the eigenvalues and eigenvectors of the matrix $K_n$ are

$$
\alpha_k = \begin{cases} 
  a_1 - 2\sqrt{b_1}c_1 \cos \left( \frac{(k+1)\pi}{n+2} \right), & (k = 2t + 1, \ t \in \mathbb{N}) \\
  a_2 - 2\sqrt{b_2}c_2 \cos \left( \frac{k\pi}{n+2} \right), & (k = 2t, \ t \in \mathbb{N})
\end{cases}
$$

(13)

and

$$
\begin{bmatrix}
  B_0(\alpha_j) \\
  B_1(\alpha_j) \\
  B_2(\alpha_j) \\
  \vdots \\
  B_{n-2}(\alpha_j) \\
  B_{n-1}(\alpha_j)
\end{bmatrix} \quad (j = 1, 3, 5, \ldots, n-3, n-1); \quad (14)
$$

and

$$
\begin{bmatrix}
  A_0(\alpha_j) \\
  A_1(\alpha_j) \\
  A_2(\alpha_j) \\
  \vdots \\
  A_{n-2}(\alpha_j) \\
  A_{n-1}(\alpha_j)
\end{bmatrix} \quad (j = 2, 4, 6, \ldots, n-2, n). \quad (15)
$$

6
Proof. We obtain for
\[
b_2^n A_{n+1}(x) = (b_2 c_2)^{\frac{n}{2}} U_{\frac{n}{2}} \left( \frac{x - a_2}{2\sqrt{b_2 c_2}} \right)
\]  (16)
and
\[
b_1^n B_n(x) = (b_1 c_1)^{\frac{n}{2}} U_{\frac{n}{2}} \left( \frac{x - a_1}{2\sqrt{b_1 c_1}} \right)
\]  (17)
from the recurrence relations (11) and (12), here \( n = 2t \ (t \in \mathbb{N}) \) and \( U_n(.) \) is the \( n \)th degree Chebyshev polynomial of the second kind [10]:
\[
U_n(x) = \frac{\sin((n + 1) \text{arccos}x)}{\sin(\text{arccos}x)}
\]
All the roots of \( U_n(x) \) are included in the interval \([-1, 1]\). Due to (12), (16) and (17), we have
\[
|x I_n - K_n| = (b_1 c_1)^{\frac{n}{2}} U_{\frac{n}{2}} \left( \frac{x - a_1}{2\sqrt{b_1 c_1}} \right) (b_2 c_2)^{\frac{n}{2}} U_{\frac{n}{2}} \left( \frac{x - a_2}{2\sqrt{b_2 c_2}} \right)
= (b_1 c_1 b_2 c_2)^{\frac{n}{2}} U_{\frac{n}{2}} \left( \frac{x - a_1}{2\sqrt{b_1 c_1}} \right) U_{\frac{n}{2}} \left( \frac{x - a_2}{2\sqrt{b_2 c_2}} \right).
\]  (18)
The eigenvalues of \( K_n \) obtained as
\[
\alpha_k = \begin{cases} 
  a_1 - 2\sqrt{b_1 c_1} \cos \left( \frac{(k + 1)\pi}{n + 2} \right), & (k = 2t + 1, \ t \in \mathbb{N}) \\
  a_2 - 2\sqrt{b_2 c_2} \cos \left( \frac{k\pi}{n + 2} \right), & (k = 2t, \ t \in \mathbb{N})
\end{cases}
\]
from (18). In [2] Hadj and Elouafi proved eigenvectors of a pentadiagonal matrix. Based on Theorem 6 and [2], we can calculate eigenvectors of the matrix \( K_n \) ■

3 The integer powers of the matrix \( K_n \)
Considering (14) and (15), we write down the transforming matrices \( L_n \) \((n = 2t, \ t \in \mathbb{N})\) as following:

\[
L_n = \begin{bmatrix}
B_0 (\alpha_1) & A_0 (\alpha_2) & B_0 (\alpha_3) & A_0 (\alpha_4) \\
B_1 (\alpha_1) & A_1 (\alpha_2) & B_1 (\alpha_3) & A_1 (\alpha_4) \\
B_2 (\alpha_1) & A_2 (\alpha_2) & B_2 (\alpha_3) & A_2 (\alpha_4) \\
\vdots & \vdots & \vdots & \vdots \\
B_n-3 (\alpha_1) & A_{n-3} (\alpha_2) & B_{n-3} (\alpha_3) & A_{n-3} (\alpha_4) \\
B_n-2 (\alpha_1) & A_{n-2} (\alpha_2) & B_{n-2} (\alpha_3) & A_{n-2} (\alpha_4) \\
B_n-1 (\alpha_1) & A_{n-1} (\alpha_2) & B_{n-1} (\alpha_3) & A_{n-1} (\alpha_4)
\end{bmatrix}
\]
Now, let us find the inverse matrix $L_n^{-1}$ of the matrix $L_n$. If we denote $i$-th row of the inverse matrix $L_n^{-1}$ by $\mu_i$, then we obtain

$\begin{bmatrix}
q_i r_1^l B_0 (\alpha_i) \\
q_i r_1^l B_1 (\alpha_i) \\
q_i r_1^l B_2 (\alpha_i) \\
q_i r_1^l B_3 (\alpha_i) \\
q_i r_1^l B_4 (\alpha_i) \\
\vdots \\
q_i r_1^l B_{n-4} (\alpha_i) \\
q_i r_1^l B_{n-3} (\alpha_i) \\
q_i r_1^l B_{n-2} (\alpha_i) \\
q_i r_1^l B_{n-1} (\alpha_i)
\end{bmatrix}^T$ \hspace{1cm} (i = 1, 3, 5, \ldots, n - 3, n - 1) \hspace{1cm} (20)

and

$\begin{bmatrix}
q_i r_2^l A_0 (\alpha_i) \\
q_i r_2^l A_1 (\alpha_i) \\
q_i r_2^l A_2 (\alpha_i) \\
q_i r_2^l A_3 (\alpha_i) \\
q_i r_2^l A_4 (\alpha_i) \\
\vdots \\
q_i r_2^l A_{n-4} (\alpha_i) \\
q_i r_2^l A_{n-3} (\alpha_i) \\
q_i r_2^l A_{n-2} (\alpha_i) \\
q_i r_2^l A_{n-1} (\alpha_i)
\end{bmatrix}^T$ \hspace{1cm} (i = 2, 4, 6, \ldots, n - 2, n) \hspace{1cm} (21)

where $r_1 = \sqrt{\frac{2a}{c_1}}$, $r_2 = \sqrt{\frac{2a}{c_2}}$,

$l = \left\{ \begin{array}{ll}
\{ j - 1, & j = 1, 3, 5, \ldots, n - 3, n - 1 \\
\{ j - 2, & j = 2, 4, 6, \ldots, n - 2, n \\
\end{array} \right.$

and

$q_i = \begin{cases} 
4 - \left( \frac{a - \sqrt{a^2 - 4b}}{n+2} \right)^2, & i = 2t + 1 \ (t \in \mathbb{N}) \\
4 - \left( \frac{a + \sqrt{a^2 - 4b}}{n+2} \right)^2, & i = 2t \ (t \in \mathbb{N})
\end{cases}$
Thus, we obtain

\[
L_n^{-1} = \begin{bmatrix}
q_1 B_0 (\alpha_1) & q_1 B_1 (\alpha_1) & q_1 r_1^2 B_2 (\alpha_1) & q_1 r_1^2 B_3 (\alpha_1) \\
q_2 A_0 (\alpha_2) & q_2 A_1 (\alpha_2) & q_2 r_2^2 A_2 (\alpha_2) & q_2 r_2^2 A_3 (\alpha_2) \\
q_3 B_0 (\alpha_3) & q_3 B_1 (\alpha_3) & q_3 r_3^2 B_2 (\alpha_3) & q_3 r_3^2 B_3 (\alpha_3) \\
\vdots & \vdots & \vdots & \vdots \\
q_{n-1} B_0 (\alpha_{n-1}) & q_{n-1} B_1 (\alpha_{n-1}) & q_{n-1} r_{n-1}^2 B_2 (\alpha_{n-1}) & q_{n-1} r_{n-1}^2 B_3 (\alpha_{n-1}) \\
q_n A_0 (\alpha_n) & q_n A_1 (\alpha_n) & q_n r_n^2 A_2 (\alpha_n) & q_n r_n^2 A_3 (\alpha_n)
\end{bmatrix}
\]

We write the \(s\)th powers of the matrix \(K_n\) as

\[
K_n^s = L_n J_n^s L_n^{-1} = W (s) = (w_{ij} (s)).
\]

Then

\[
w_{ij} (s) = \begin{cases} 
0, & \text{if } (-1)^{i+j} = -1 \\
\sum_{j=1}^{n} q_{2j-1} r_{2j-1} \alpha_{2j-1} B_{j-1} (\alpha_{2j-1}) B_{j-1} (\alpha_{2j-1}) & j = 1, 3, \ldots, n-1 \\
\sum_{j=1}^{n} q_{2j} r_{2j} \alpha_{2j} A_{j-1} (\alpha_{2j}) A_{j-1} (\alpha_{2j}) & j = 2, 4, \ldots, n
\end{cases}
\]

(24)

here \(r_1 = \sqrt{\frac{b_1}{c_1}}, r_2 = \sqrt{\frac{b_2}{c_2}}\),

\[
l = \begin{cases} 
j - 1, & j = 1, 3, 5, \ldots, n - 3, n - 1 \\
j - 2, & j = 2, 4, 6, \ldots, n - 2, n
\end{cases}
\]

\[
q_i = \begin{cases} 
\frac{4 - (\alpha_i - \alpha_{i+2})^2}{n+2}, & i = 2t + 1 (t \in \mathbb{N}) \\
\frac{4 - (\alpha_i - \alpha_{i+2})^2}{n+2}, & i = 2t (t \in \mathbb{N})
\end{cases}
\]

and \(\alpha_k\) are the eigenvalues of the matrix \(K_n\) \((n = 2t, t \in \mathbb{N})\).

**Corollary 8** Let \(K_n\) be \(n\)-square \((n = 2t, t \in \mathbb{N}; \, a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{C} \setminus \{0\})\) pentadiagonal 2-Toeplitz matrix as in (7), from Theorem 7

\[
a_1 \neq 2\sqrt{b_1 c_1} \cos \left( \frac{(k+1)\pi}{n+2} \right)
\]

(25)

\((k = 2t + 1, t \in \mathbb{N})\) and

\[
a_2 \neq 2\sqrt{b_2 c_2} \cos \left( \frac{k\pi}{n+2} \right).
\]

(26)
(k = 2t, t ∈ N). In that case, there exists the inverse and negative integer powers of the matrix $K_n$.

4 Numerical examples

Example 9 Taking $n = 6$ in Theorem 7, we obtain

$$J_6 = \text{diag}(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6)$$
$$= \text{diag}(a_1 - \sqrt{2b_1c_1}, a_2 - \sqrt{2b_2c_2}, a_1, a_2, a_1 + \sqrt{2b_1c_1}, a_2 + \sqrt{2b_2c_2})$$
and

$$K^*_6 = L_6 J^*_6 L^{-1}_6 = W(s)$$

$$= (w_{ij}(s)) = \begin{bmatrix}
    x_1 & 0 & x_7 & 0 & x_{11} & 0 \\
    0 & x_2 & 0 & x_8 & 0 & x_{12} \\
    x_3 & 0 & x_9 & 0 & x_7 & 0 \\
    0 & x_4 & 0 & x_{10} & 0 & x_8 \\
    x_5 & 0 & x_3 & 0 & x_1 & 0 \\
    0 & x_6 & 0 & x_4 & 0 & x_2
\end{bmatrix}$$

$$x_1 = \frac{1}{t^2} \left( (a_1 + \sqrt{2b_1c_1})^s - 2a_1^s + (a_1 - \sqrt{2b_1c_1})^s \right)$$
$$x_2 = \frac{1}{t^2} \left( (a_2 + \sqrt{2b_2c_2})^s - 2a_2^s + (a_2 - \sqrt{2b_2c_2})^s \right)$$
$$x_3 = \frac{1}{t^2} \left( (a_1 + \sqrt{2b_1c_1})^s - 2a_1^s + (a_1 - \sqrt{2b_1c_1})^s \right)$$
$$x_4 = \frac{1}{t^2} \left( (a_2 + \sqrt{2b_2c_2})^s - 2a_2^s + (a_2 - \sqrt{2b_2c_2})^s \right)$$
$$x_5 = \frac{1}{t^2} \left( (a_1 + \sqrt{2b_1c_1})^s - 2a_1^s + (a_1 - \sqrt{2b_1c_1})^s \right)$$
$$x_6 = \frac{1}{t^2} \left( (a_2 + \sqrt{2b_2c_2})^s - 2a_2^s + (a_2 - \sqrt{2b_2c_2})^s \right)$$

Example 10 Taking $s = 3, n = 8, a_1 = 1, a_2 = i + 1, b_1 = 3, b_2 = i + 3, c_1 = 5$ and $c_2 = i + 5$ in Theorem 7, we obtain

$$J_8 = \text{diag}(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)$$
$$= \text{diag}(-5.267, -5.280 - 0.668i, -1.394, -1.399 - 0.363i, 3.3394, 3.399 + 1.637i, 7.267, 7.280 + 2.668i)$$
and

\[ K_8^3 = L_8 J_8^3 L_8^{-1} = W(3) \]

\[ = (w_{ij}(3)) = \begin{bmatrix}
46 & 0 & 99 & 0 & 27 \\
0 & 16 + 68i & 0 & 62 + 94i & 0 \\
165 & 0 & 91 & 0 & 144 \\
0 & 118 + 138i & 0 & 34 + 134i & 0 \\
75 & 0 & 240 & 0 & 91 \\
0 & 42 + 102i & 0 & 180 + 192i & 0 \\
125 & 0 & 75 & 0 & 165 \\
0 & 110 + 74i & 0 & 42 + 102i & 0
\end{bmatrix} \]

\[ = \begin{bmatrix}
0 & 27 & 0 \\
6 + 42i & 0 & 18 + 26i \\
0 & 27 & 0 \\
96 + 12i & 0 & 6 + 42i \\
0 & 99 & 0 \\
34 + 134i & 0 & 62 + 94i \\
0 & 46 & 0 \\
118 + 138i & 0 & 16 + 68i
\end{bmatrix}. \]

Example 11 Taking \( s = -4, n = 10, a_1 = 1, a_2 = 2, b_1 = 3, b_2 = 4, c_1 = 5 \) and \( c_2 = 6 \) in Theorem 7, we get

\[ J_{10} = \text{diag}(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9, \alpha_{10}) \]

\[ = \text{diag}(-5.7082, -6.4853, -2.8730, -2.8990, 1, 2, 4.8730, 6.8990, 7.7082, 10.4853) \]

and

\[ K_{10}^{-4} = L_{10} J_{10}^{-4} L_{10}^{-1} = W(-4) \]

\[ = (w_{ij}(-4)) = \begin{bmatrix}
0.3375 & 0 & -0.0026 & 0 & -0.1999 \\
0 & 0.0245 & 0 & -0.0029 & 0 \\
-0.0043 & 0 & 0.0044 & 0 & -0.0001 \\
0 & -0.0043 & 0 & 0.0038 & 0 \\
-0.5552 & 0 & -0.0002 & 0 & 0.3337 \\
0 & -0.0311 & 0 & -0.0002 & 0 \\
0.0067 & 0 & -0.0063 & 0 & -0.0002 \\
0 & 0.0062 & 0 & -0.0052 & 0 \\
0.9148 & 0 & 0.0067 & 0 & -0.5552 \\
0 & 0.0388 & 0 & 0.0062 & 0
\end{bmatrix} \]
\[
\begin{bmatrix}
0 & 0.0015 & 0 & 0.1186 & 0 \\
-0.0138 & 0 & 0.0018 & 0 & 0.0077 \\
0 & -0.0023 & 0 & 0.0015 & 0 \\
-0.0001 & 0 & 0.0023 & 0 & 0.0018 \\
0 & -0.0001 & 0 & -0.1999 & 0 \\
0.0210 & 0 & -0.0001 & 0 & -0.0138 \\
0 & 0.0044 & 0 & -0.0026 & 0 \\
-0.0001 & 0 & 0.0038 & 0 & -0.0029 \\
0 & -0.0043 & 0 & 0.3375 & 0 \\
0.0311 & 0 & -0.0043 & 0 & 0.0245
\end{bmatrix}
\]

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