Phase field feature of inclined hydraulic fracture propagation in naturally-layered rocks under stress boundaries

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Abstract. The phase field feature of inclined hydraulic fracture propagation in naturally-layered rocks under stress boundaries is investigated by using a numerical method. Based on a phase field fracture model, the coupled governing equations of displacement field, phase field, and flow field concerning hydraulic fracturing in naturally-layered rocks were established. The equations were solved using the finite element method and the influences of initial stress field and stiffness contrast on the fracture patterns were deeply studied. The numerical simulations indicate that: 1) The ratio of vertical in-situ stress to horizontal in-situ stress (\(S_v/S_h\)) has a significant effect on propagation of the hydraulic fracture. With the increase in \(S_v/S_h\), the hydraulic fracture deflects and propagates along the direction of \(S_v\), which is also the maximum in-situ stress; 2) The stiffness contrast of the two layers (\(E_1/E_2\)) has great influence on fracture penetration into the adjacent layer. For a low \(E_1/E_2\), a singly-deflected scenario is observed because the fracture propagation is depressed by the stiff rock. With the increase in \(E_1/E_2\), the hydraulic fracture more tends to penetrate into the adjacent layer.

1. Introduction

Hydraulic fracturing is a kind of technology which injects high-pressure fluid into the target reservoir to form a complex fracture network, thereby increasing the permeability of the reservoir and achieving the purpose of increasing the production of oil and gas resources [1]. Therefore, in recent decades, hydraulic fracturing has been widely used in the exploitation and extraction of oil, shale gas and other resources. However, hydraulic fracturing may have potential adverse effects on the engineering geological environment. For example, unexpected hydraulic fractures may propagate to adjacent rock layers, causing water contamination. In addition, uncontrolled hydraulic fractures along dominant faults or unknown faults may lead to increased seismic activity. In order to increase the production of oil and gas resources while avoiding adverse effects on the geological environment as much as possible, it is necessary to accurately predict the propagation path of hydraulic fractures. Therefore, the numerical methods for hydraulic fracturing have gained more and more attention in recent years.

Various numerical methods have been proposed with a lot of achievements in the field of hydraulic fracturing. Among all the numerical methods, the phase field method (PFM) [2-6] has received extensive attention due to its immense potential. The main idea of the phase field method is to use a diffusive scalar field to illustrate the sharp cracks (see figure 1). By introducing a phase field variable \(\phi\) (\(\phi=0\) and \(\phi=1\) means the material is intact and fully broken, respectively), the fracture model can be described by continuous functions. Besides, the governing equations for the displacement and the
phase field (the temperature field and the fluid field can also be included) can be obtained by using the variational approach \[7\]. Therefore, the fracture propagation problem is transformed into a multi-field problem, the key to which is to solve the partial differential equation system. Compared with other numerical methods, the advantages of the phase field method are as follows \[1\]: 1) simulations can be conducted on a fixed mesh without any remeshing or adaptive technique; 2) complex fracture propagation processes such as merging and branching can be obtained automatically; 3) external fracture criteria or fracture surface tracking algorithms are not required; 4) PFM can easily handle the problems related to heterogeneous media; 5) additional penetration criteria are not needed when the hydraulic fracture encounters the layer interface.

With the development of the phase field method, many achievements have been made in the field of hydraulic fracturing. However, current researches of the phase field method for simulating hydraulic fractures seldom consider the influence of initial stress field on the fracture propagation. Instead, most of the researches establish the homogeneous Dirichlet boundary conditions for the displacement field to solve the problem. In the phase field model, when the remote stress is applied on the boundaries of the calculation domain, it will cause a large deformation on these stress boundaries, which is inconsistent with the engineering observations \[1\]. On the other hand, naturally-layered rock strata are common phenomena in the engineering geological environment, so it is necessary to study the features of the hydraulic fracture crossing different rock layers. Existing researches \[8-9\] indicate that when the hydraulic fracture propagates to the layer interface, there are mainly three types of fracture patterns: penetration, singly-deflected and doubly-deflected scenarios (see figure 2). Nevertheless, predicting fracture propagation processes in naturally-layered rocks still remains a challenge.

**Figure 1.** Phase field representation of the sharp fracture in a layered porous medium (a) Sharp fracture, (b) Diffusive fracture.

**Figure 2.** Three hydraulic fracture patterns at the layer interface (a) Penetration, (b) Singly-deflected, (c) Doubly-deflected.
Therefore, to overcome the shortcomings discussed above, the phase field feature of inclined hydraulic fracture propagation in naturally-layered rocks under stress boundaries is investigated by using a numerical method. In addition, the influences of initial stress field and stiffness contrast on the fracture patterns were deeply studied.

2. Mathematical model of PFM

2.1. Energy functional

As shown in figure 1, the calculation domain is a two-dimensional or three-dimensional rock with internal fractures, which are denoted as $\Omega$ and $\Gamma$, respectively. $\Omega_j$ and $\Omega_s$ are two different layers of the rock with full bond between them. The boundary of the calculation domain is marked as $\partial \Omega$, where the time-dependent Dirichlet boundary conditions $\mathbf{u}(\mathbf{x}, t)$ and the Neumann condition $\mathbf{t}(\mathbf{x}, t)$ must be satisfied on $\partial \Omega_d$ and $\partial \Omega_n$, respectively. Note that $\mathbf{x}$ is the position vector. Besides, the whole domain $\Omega$ is subjected to a body force $\mathbf{b}(\mathbf{x}, t)$ and the outward unit normal vector is defined as $\mathbf{n}$. In addition, it is assumed that the rock is saturated with compressible viscous liquid. According to the variational approach proposed by Francfort and Marigo [7], fracture initiation and propagation at time $t$ is a process to minimize the total energy functional. The energy functional established in this paper is mainly composed of the contributions of the elastic energy, fracture energy, external work, and the energy contribution of fluid pressure. Moreover, in order to fully consider the impact of the initial stress field, the energy functional also adds an additional item of energy induced by it $[1]$

$$L(u, p, \Gamma) = \int_{\Omega} \psi'(\mathbf{e})d\Omega + \int_{\Omega_j} \mathbf{e} : \mathbf{g}d\Omega + \int_{\Omega} \alpha p \mathbf{t} \cdot (\nabla \mathbf{u})d\Omega + \int_{\Omega_s} G_\gamma d\Gamma - \int_{\partial \Omega_n} t \cdot \mathbf{u} dS - \int_{\partial \Omega} \mathbf{b} \cdot \mathbf{u} d\Omega \quad \text{MERGEFORMAT (1)}$$

where $\psi'(\mathbf{e})$ is the elastic energy density, $\alpha$ is the Biot coefficient, $G_\gamma$ is the critical energy release rate, and $\mathbf{e}$ is the initial stress.

For 2D and 3D problems, the fracture energy in equation is difficult to solve because the real shape of the sharp fracture is hard to obtain. Therefore, to simplify the numerical problem, a phase field $\phi(\mathbf{x}, t) \in [0, 1]$ is introduced to smear the sharp fracture. Note that $\phi = 0$ and $\phi = 1$ represent intact and fully broken material, respectively. Based on the research of Miehe et al. [3], the crack surface density per unit volume is utilized

$$\gamma(\phi, \nabla \phi) = \frac{\phi^2}{2l_0} + \frac{l_0}{2} \nabla \phi \cdot \nabla \phi \quad \text{MERGEFORMAT (2)}$$

and thereby the boundary integration in the fracture energy is converted into volume integration

$$\int_{\Omega} G_\gamma d\Omega \approx \int_{\Omega} G_\gamma d\Omega = \int_{\Omega} \left( \frac{\phi^2}{2l_0} + \frac{l_0}{2} \nabla \phi \cdot \nabla \phi \right) d\Omega \quad \text{MERGEFORMAT (3)}$$

Therefore, seeking for the real shape of the fracture is avoided and the numerical solution is simplified. This approximation method is widely used in PFM and it can be well applied in underground engineering and hydraulic fracturing under the assumption of linear elasticity and small deformation. In equation and , $l_0$ is the length scalar, which controls the width of the transition zone between the completed broken area and the intact area. A larger $l_0$ corresponds to a wider fracture in PFM.

The research of Bourdin et al. [10] showed that the fracture energy is transformed from the elastic energy, as a result, the phase field evolution is driven by the elastic energy. However, if the whole elastic energy is adopted as the driving part without being decomposed, unrealistic fracture patterns will be attained. Hence, according to the method proposed by Miehe et al. [3], the elastic energy is divided into the tensile and compressive parts, which are denoted as $\psi'_t(\mathbf{e})$ and $\psi'_c(\mathbf{e})$, respectively. In addition, it is assumed that the fracture evolution is only affected by the tensile part. Therefore, the elastic energy can be rewritten as
\[ \int_{\Omega} \psi_{\varepsilon}(e) \, d\Omega = \int_{\Omega} \left[ g(\phi)\psi_{\varepsilon}(e) + \psi_{\varepsilon}^e(\varepsilon) \right] \, d\Omega \]

\[ g(\phi) = (1 - k)(1 - \phi)^2 + k \]

where \( g(\phi) \) is a degradation function and \( k \) is a stability parameter to prevent numerical singularity when \( \phi = 1 \) [1]. In addition, due to the fact that the initial stress field has no effect on fracture propagation in a fully broken area, the degradation function is also applied to the part of the initial stress field in the energy functional. In summary, the energy functional is obtained by

\[ L(u, p, \Gamma) = \int_{\Omega} \left[ g(\phi)\psi_{\varepsilon}(e) + \psi_{\varepsilon}^e(\varepsilon) \right] \, d\Omega + \int_{\Omega} g(\phi)\sigma_0 : \varepsilon \, d\Omega - \int_{\Omega} \alpha p \cdot (\nabla \cdot \varepsilon) \, d\Omega + \int_{\Gamma} \frac{\partial^2 \phi}{\partial \tau^2} \, d\tau - \int_{\Omega} 2l_0 (1 - \phi)H \, d\Omega \]

2.2. Governing equations for the displacement and the phase field

The variational approach [7] is utilized to get the governing equations for the displacement and the phase field by means of setting the first variation of the energy functional \( L \) zero

\[ \frac{\partial \sigma_{\varepsilon}}{\partial x_j} + b_i = 0 \]

\[ \left[ \frac{2l_0 (1 - k)H}{G} + 1 \right] \phi - \frac{l_0^2}{2} \phi^2 \frac{\partial^2 \phi}{\partial x_i^2} = \frac{2l_0 (1 - k)H}{G} \]

In equation \( \sigma_{\varepsilon} \) is the component of the total stress tensor, which can be obtained from equation \( \varepsilon \), where \( I \) is a unit tensor, and \( \sigma^e \) is the effective stress tensor induced by the displacement field, which is calculated by equation [1]. In equation \( \phi \) is the reference variable (see equation ) which is used to satisfy the irreversibility condition: a crack cannot heal after initiation [5].

\[ \sigma(e) = \sigma^e(e) + g(\phi)\sigma_0 - \alpha p I \]

\[ \sigma^e = g(\phi) \frac{\partial \psi_{\varepsilon}^e(e)}{\partial e} + \frac{\partial \psi_{\varepsilon}^e(e)}{\partial e} \]

\[ H(x, t) = \max_{\varepsilon \in [\varepsilon]} \left[ \psi_{\varepsilon}(e(s, x)) + \sigma_0 : e(s, x) \right] \]

2.3. Governing equations for the flow field

The hydraulic fracture has an impact on the flow field, so the phase field is introduced into the flow field parameters. In terms of the method proposed by Lee et al. [11], the calculation domain is divided into three parts: unbroken domain, fracture domain and transition domain, which are distinguished according to the phase field variable \( \phi \) and two thresholds \( c_1 \) and \( c_2 \). The subdomain is an unbroken domain if \( \phi \leq c_1 \), but a fracture domain if \( \phi \geq c_2 \). In the case of \( c_1 \leq \phi \leq c_2 \), the subdomain is a transition domain. The parameters of the fluid in the unbroken domain and fracture domain are prescribed while those in the transition domain are obtained by linear interpolation of the unbroken domain and fracture domain. We following Lee et al. [11] and define two linear indicator functions \( \chi_r \) and \( \chi_f \) with the expressions as follows

\[ \chi_r(\phi, \phi) = \begin{cases} 1 & \phi \leq c_1 \\ \frac{c_2 - \phi}{c_2 - c_1} & c_1 < \phi < c_2 \\ 0 & \phi \geq c_2 \end{cases} \]
\[\chi_f(\phi) = \begin{cases} 
0 & \phi \leq c_1 \\
c_1 & c_1 < \phi < c_2 \\
c_2 - c_1 & \phi \geq c_2 \end{cases} \quad \text{\footnotesize{MERGEFORMAT (13)}}
\]

Darcy’s law and the storage model [12] are used to describe the flow field and the governing equation of the whole calculation domain is given by
\[\rho S \frac{\partial p}{\partial t} - \nabla \cdot \left( \frac{\rho K}{\mu} \nabla p \right) = q_m - \rho \alpha \chi_e \frac{\partial \varepsilon_{vol}}{\partial t} \quad \text{\footnotesize{MERGEFORMAT (14)}}\]
\[S = \varepsilon_r + \frac{(\alpha - \varepsilon_r)(1 - \alpha)}{K_r} \quad \text{\footnotesize{MERGEFORMAT (15)}}\]

where \( \rho \), \( S \), \( K \), \( \mu \), \( q_m \), \( \varepsilon_{vol} = \nabla \cdot \mathbf{u} \), \( \varepsilon_r \), \( c \), and \( K_r \) are the fluid density, storage coefficient, permeability, fluid viscosity, source term, volumetric strain, initial porosity, fluid compressibility and bulk modulus of the reservoir domain, respectively. In order to describe the fluid properties in different domains, parameters such as \( \rho \), \( K \), \( \mu \), \( c \), and \( \alpha \) are all interpolated from the corresponding parameters of the unbroken domain and fracture domain. Therefore, according to the linear indicator functions \( \chi_r \) and \( \chi_f \), equation can describe the evolution of the flow field in the entire calculation domain.

On the basis of the previous subsection and this subsection, combining equations , and , the governing equations of the phase field method for simulating hydraulic fracturing are as follows
\[\frac{\partial \sigma_{ij}}{\partial t} + b_j = 0 \]
\[\frac{2l_0(1-k)H}{G_c} + 1 \phi - l_0^2 \frac{\partial^2 \phi}{\partial \chi^2} - \frac{2l_0(1-k)H}{G_c} = 2l_0(1-k)H \quad \text{\footnotesize{MERGEFORMAT (16)}}\]
\[\rho S \frac{\partial p}{\partial t} - \nabla \cdot \left( \frac{\rho K}{\mu} \nabla p \right) = q_m - \rho \alpha \chi_e \frac{\partial \varepsilon_{vol}}{\partial t} \]

The phase field variable \( \phi \) solved from equation is a function of \( x \) and \( t \), as a result, the phase field distribution at any time in the calculation domain can be obtained. Since PFM uses the phase field distribution to describe the sharp fractures, solving equation can get the entire process of the hydraulic fracture propagation in the calculation domain.

3. Numerical implementation of PFM
The finite element method is adopted to solve the partial differential equations of the hydraulic fracturing problem and it is implemented in the commercial software, COMSOL Multiphysics [5]. We establish four modules: Solid mechanics Module, Darcy’s Law Module, History-strain Module and Phase Field Module, which correspond to the solution of four variables: \( \mathbf{u} \), \( p \), \( H \) and \( \phi \), respectively. The implicit Generalized-\( \alpha \) method is used for time-domain discretization while a segregated scheme is applied to solve the three unknown variables in each time step (where the displacement \( \mathbf{u} \) and the fluid pressure \( p \) are coupled together). The Newton-Raphson method is used to solve each segregated step. The numerical scheme for hydraulic fracturing in PFM is shown in figure 3.

4. Numerical results
4.1. Geometry and boundary conditions
The geometry and boundary conditions of the numerical simulation are shown in figure 4, where the origin of the coordinate coincides with the center of the calculation domain, and \( S_h \) and \( S_v \) are the vertical in-situ stress and horizontal in-situ stress, respectively. Two rock layers marked as ① and ②
are included in the calculation domain with a 0.4m long pre-existing notch set in the layer ① by the method proposed by Borden et al. [13]. In addition, the source term in the notch is set as \( q_m = 2 \text{kg/(m}^3\text{s)} \). Except the boundary conditions shown in figure 4, the left boundary of the calculation domain is impermeable while the other three boundaries are permeable boundaries with the pressure head set as zero. Moreover, in order to avoid rigid body displacement, the tangential displacement of the midpoint of the right boundary is constrained. The calculation domain is discretized with linear triangular elements with the maximum element size \( h = 0.025 \text{m} \).

### Table 1. Parameters of the numerical simulation.

| Parameter | Value | Unit | Parameter | Value | Unit |
|-----------|-------|------|-----------|-------|------|
| \( l_0 \) | 0.05 m | m | \( \alpha \) | 0.002 | – |
| \( c_1 \) | 0.4 | – | \( c_2 \) | 1.0 | – |
| \( c_3 \) | \( 1 \times 10^{-8} \) | 1/\text{Pa} | \( c_f \) | \( 1 \times 10^{-8} \) | 1/\text{Pa} |
| \( \rho_r \) | \( 1.0 \times 10^3 \) | \text{kg/m}^3 | \( k \) | \( 1 \times 10^{-9} \) | – |
| \( \varepsilon_p \) | 0.002 | – | \( \rho_f \) | \( 1.0 \times 10^3 \) | \text{kg/m}^3 |
| \( K_r \) | \( 1 \times 10^{-15} \) | \text{m}^2 | \( K_f \) | \( 2.083 \times 10^{-4} \) | \text{m}^2 |
| \( \mu_i \) | 1 \times 10^{-3} | \text{Pa} \cdot \text{s} | \( \mu_f \) | \( 1 \times 10^{-3} \) | \text{Pa} \cdot \text{s} |

### 4.2. Effect of initial stress field

To study the effect of initial stress field on propagation of the hydraulic fracture, three cases presented in table 2 are simulated and analyzed. In this table, \( E_1 \), \( E_2 \), \( G_{11} \), and \( G_{22} \) are the Young's modulus and critical energy release rate of the two layers, respectively while the Poisson's ratios of the two layers are both 0.3. Figure 5 shows the corresponding fracture patterns of these three cases. When \( S_v/S_h = 1/5 \) or \( S_v/S_h = 1 \), the fracture pattern is penetration while the fracture propagates down along the interface for a short distance before penetrating into the layer ②; while \( S_v/S_h = 10 \), the hydraulic fracture deflects and propagates along the direction of \( S_v \), which is also the maximum in-situ stress. In addition, the fracture does not encounter the layer interface.
Table 2. Numerical simulation cases.

| Initial stress field $S_v/\text{MPa}$ | $S_h/\text{MPa}$ | Stiffness contrast $E_1/\text{GPa}$ | $E_2/\text{GPa}$ | $G_{c1}/(\text{N/m})$ | $G_{c2}/(\text{N/m})$ |
|----------------------------------------|----------------|---------------------------------|----------------|---------------------|---------------------|
| $S_v/S_h=1/5$                          | 0.1            | 0.5                             |                |                     |                     |
| $S_v/S_h=1$                            | 0.5            | 0.5                             | $E_1/E_2=1/2$  | 60                  | 120                 | 450                 | 900                 |
| $S_v/S_h=10$                           | 5              | 0.5                             |                |                     |                     |

It can be seen from figure 5 that the initial stress field has a significant effect on propagation of the hydraulic fracture. For a low $S_v/S_h$, the hydraulic fracture propagates along the initial direction before reaching the layer interface. When $S_v/S_h$ increases to a certain degree, the hydraulic fracture deflects and propagates along the direction of the maximum in-situ stress, which is consistent with engineering observations and the numerical results shown in Zhou et al. [1].

![Figure 5](image)

Figure 5. Hydraulic fracture patterns under different initial stress conditions (a) $S_v/S_h=1/5$, (b) $S_v/S_h=1$, (c) $S_v/S_h=10$.

4.3. Effect of stiffness contrast

To study the effect of the stiffness contrast of the two layers on hydraulic fracture propagation, three cases in table 3 are simulated with the rest of the calculation parameters remained unchanged. Figure 6 presents the fracture patterns under different stiffness contrast conditions. When $E_1/E_2=1/3$, a singly-deflected scenario is observed; when $E_1/E_2=1/2$, the hydraulic fracture penetrates into the layer (2) and when $E_1/E_2=2$, the fracture pattern is also penetration except that the width of the fracture suddenly increases after the fracture crosses the layer interface, but decreases as the fracture propagates forward.

Table 3. Numerical simulation cases.

| Initial stress field $S_v/\text{MPa}$ | $S_h/\text{MPa}$ | Stiffness contrast $E_1/\text{GPa}$ | $E_2/\text{GPa}$ | $G_{c1}/(\text{N/m})$ | $G_{c2}/(\text{N/m})$ |
|----------------------------------------|----------------|---------------------------------|----------------|---------------------|---------------------|
| $S_v/S_h=1$                            | 0.5            | 0.5                             |                |                     |                     |
| $E_1/E_2=1/3$                          | 40             | 120                             | 300            |                     |                     |
| $E_1/E_2=1/2$                          | 60             | 120                             | 450            |                     |                     |
| $E_1/E_2=2$                            | 120            | 60                              | 900            |                     |                     |

According to figure 6, with the increase in $E_1/E_2$, the hydraulic fracture pattern gradually changes from no propagation across the interface to penetration into the adjacent layer. For a low $E_1/E_2$, a singly-deflected scenario is obtained because the fracture propagation is depressed by the stiff rock. However, as $E_1/E_2$ increases, the stiffness of layer (2) is not enough to prevent propagation of the hydraulic fracture, so penetration scenario occurs.
Figure 6. Hydraulic fracture patterns under different stiffness contrast conditions (a) $E_1/E_2=1/3$, (b) $E_1/E_2=1/2$, (c) $E_1/E_2=2$.

Note that because of page limit, the representative simulation cases shown above are only part of all simulation cases we have conducted. For other $E_1/E_2$ and $S_v/S_h$, the effects of $S_v/S_h$ and $E_1/E_2$ on the fracture pattern have the same trend, and the main conclusions still hold.

5. Conclusions

In this paper, the phase field feature of inclined hydraulic fracture propagation in naturally-layered rocks under stress boundaries is investigated by using a numerical method. Based on a phase field fracture model, the coupled governing equations of displacement field, phase field, and flow field concerning hydraulic fracturing in naturally-layered rocks were established. The equations were solved using the finite element method and the influences of initial stress field and stiffness contrast on the fracture patterns were deeply studied. The numerical simulations indicate that:

1) The ratio of vertical in-situ stress to horizontal in-situ stress ($S_v/S_h$) has a significant effect on propagation of the hydraulic fracture. With the increase in $S_v/S_h$, the hydraulic fracture deflects and propagates along the direction of $S_v$, which is also the maximum in-situ stress;

2) The stiffness contrast of the two layers ($E_1/E_2$) has great influence on fracture penetration into the adjacent layer. For a low $E_1/E_2$, a singly-deflected scenario is observed because the fracture propagation is depressed by the stiff rock. With the increase in $E_1/E_2$, the hydraulic fracture more tends to penetrate into the adjacent layer.

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