Active Contour Models for Manifold Valued Image Segmentation

Sumukh Bansal · Aditya Tatu

Received: 13 June 2013 / Accepted: 16 January 2015 / Published online: 14 February 2015
© Springer Science+Business Media New York 2015

Abstract Image segmentation is the process of partitioning an image into different regions or groups based on some characteristics like color, texture, motion, or shape etc. Active contours are a popular variational method for object segmentation in images, in which the user initializes a contour which evolves in order to optimize an objective function designed such that the desired object boundary is the optimal solution. Recently, imaging modalities that produce Manifold-valued images are frequently used, for example, DT-MRI images, vector fields. The traditional active contour model does not work on such images. In this paper, we generalize the active contour model to work on Manifold-valued images. As expected, our algorithm detects regions with similar Manifold values in the image. Our algorithm also produces expected results on usual gray-scale images, since these are nothing but trivial examples of Manifold-valued images. As another application of our general active contour model, we perform texture segmentation on gray-scale images by first creating an appropriate Manifold-valued image. We demonstrate segmentation results for manifold-valued images and texture images.

Keywords Active contours · Manifold-valued images · Segmentation · Texture segmentation

1 Introduction

Image segmentation approaches are based on a characteristic property that defines the region of interest to be segmented out of the image, for example color, texture, motion, shape, and/or others.

Image segmentation approaches can be broadly categorized into edge-based or region-based ones. There are various approaches in both categories based on intensity, color, texture, and motion using statistical and geometrical framework [5]. Active contour model is a popular segmentation approach, which has both edge-based [3] and region-based [4] versions.

Active contours, also known as ‘snakes’, are based on evolving an initial contour toward the boundary of an object to be detected [15]. Usually, this evolution equation is a gradient descent for minimizing an appropriate energy functional.

In the Geodesic active contours model [3], the energy functional is written as a length functional with a modified metric such that the minimum, i.e., curve of minimum length corresponds to the object’s boundary. A region-based approach was proposed by Chan and Vese [4] where the energy functional was not based on edges. While the traditional approach works with parametric representation of curve, a simpler and efficient representation—the level set approach was introduced by Osher and Sethian [20].

Gray-scale images can be modeled as functions $I : \Omega \rightarrow \mathbb{R}$ where $\Omega \subseteq \mathbb{R}^2$ is the image domain. There are images and other data that can be modeled as a function $I : \Omega \rightarrow M$, where $M$ is a Riemannian manifold. We call such images as Manifold-valued images (henceforth written as MVI). Examples are DT-MRI images where $M = PD(3)$, the set of $3 \times 3$ symmetric positive definite matrices, optical flow field images $M = \mathbb{R}^2$, or wind velocity data where $M = \mathbb{R}^3$. The standard active contour methods will not work on MVIs.

S. Bansal · A. Tatu
Dhirubhai Ambani - Institute of Information & Communication Technology, Gandhinagar 382007, Gujarat, India
e-mail: aditya.tatu@gmail.com
S. Bansal
e-mail: sumukhbansal@gmail.com
this paper, we generalize the active contour model to work on MVI. We adapt both, Geodesic active contours and Chan–Vese active contour models to work on MVI’s. Since $\mathbb{R}$ is also an example of a manifold, our model also works on gray-scale images (modeled as $f : \Omega \to \mathbb{R}$).

As another application, we pose the texture segmentation problem as a MVI segmentation problem, as follows. Since covariance matrices are often used to characterize textures, we define a covariance matrix of gray-valued images in a neighborhood on every pixel of the given texture image. Covariance matrices are symmetric positive definite (actually semi-definite) matrices of size $n \times n$, the set of these matrices forms a manifold, denoted by $PD(n)$. Our algorithm is then able to identify regions of similar covariance matrices, thus segmenting textures in the image. In the rest of the paper, $|| \cdot ||$ denotes the usual Euclidean norm, $|\cdot |$ denoted absolute value of a real number, and symbols for other norms used have been defined where used for the first time.

1.1 Related Work

There have been related works on DT-MRI image segmentation using active contours, for example work by Jonasson [13,14], where the surface $S$ is evolved according to

$$\frac{\partial S}{\partial t} = (F + H)\hat{n},$$

(1)

where $F$ is a speed term proportional to the similarity of the Diffusion tensors of adjacent pixels along the normal direction to the surface, $H$ is a curvature-based regularization term and $\hat{n}$ is the unit normal to the surface $S$ being evolved. The similarity measure between two tensors $T_1$ and $T_2$ used in the above mentioned paper is the normalized tensor scalar product (NTSP):

$$NTSP(T_1, T_2) = \frac{\text{Trace}(T_1 T_2)}{\text{Trace}(T_1) \text{Trace}(T_2)}.$$

An active contour model that tries to obtain a binary approximation (similar to the Chan–Vese model for gray-scale images) was proposed by Wang and Vemuri [29] for DT-MRI segmentation, where they define the following energy function on the set of curves $C$:

$$E(C, T_1, T_2) = \int_{\text{int}(C)} ||T(x) - T_1||^2_F \, dx$$

$$+ \int_{\text{ext}(C)} ||T(x) - T_2||^2_F \, dx,$$

where $T(x)$ is the tensor defined at $x$, $\text{int}(C)$ ($\text{ext}(C)$) is the interior (exterior) of the curve $C$, and

$$T_1 = \arg \min_{\mu} \left( \int_R ||\mu - T(x)||^2_F \, dx \right),$$

$$T_2 = \arg \min_{\mu} \left( \int_{R_0} ||\mu - T(x)||^2_F \, dx \right)$$

are the mean tensors in the interior and exterior of the curve $C$. They use the Frobenius norm ($||\cdot||_F$) in their computations. In Ref. [17], the authors use Geodesic active contour to segment DT-MRI images. They use the Euclidean, KL-divergence based and the geodesic distance metric to induce an edge-stopping term. They conclude that the geodesic distance-based metric yields the best segmentation results.

In Ref. [28], the authors use a front propagation scheme to segment DT-MRI images.

Sagiv et al. [25] propose a Chan–Vese type active contour model for texture segmentation. Instead of defining the model on intensity-valued texture image, they first compute responses to a manually selected set of Gabor filters $h_{mn}$:

$$W_{mn}(x, y) = I(x, y) * h_{mn}(x, y),$$

$$m = 1, \ldots, M, \ n = 1, \ldots, N.$$

The scale, orientation, and the response which are maximal are included into the feature at every point of the image. This feature space is a 2D manifold embedded in $\mathbb{R}^N$ (they use $N = 7$). Chan–Vese active contour is then run on an image where every pixel maps to a point in this feature space. These features can also be used to induce a metric on the 2D manifold. Lee et al. [16] use this metric to define a stopping term for Geodesic active contours that segment out texture. One of the other commonly used feature for characterizing textures is the covariance matrix. In [31], the authors use covariance matrices of the intensity and first order Gaussian derivatives at every pixel over a $5 \times 5$ neighborhood as a feature and segment textures using multi-scale Graph cuts. Donoser and Bischof [8] use covariance matrices of intensity and first and second derivatives of intensity values as features to characterize textures. They use the manifold distance on the set of positive semi-definite matrices to extend the ROI-SEG [7] clustering algorithm to segment textures.

Our contribution is twofold: Our adaptation of traditional active contour models allows them to be used for segmentation of manifold-valued images and as a computer vision application we show how this algorithm can be used to segment textures.

The paper is organized as follows. In the next section we provide a brief background on active contours, manifold-valued images and cite other work on manifold-valued data. In Sect. 3, we explain our proposed active contour model for MVI segmentation, followed by implementation details and results.
2 Background

2.1 Active Contours and Level Sets

In classical active contours [15], the user initializes a curve $C: q : [0, 1] \rightarrow \Omega \subseteq \mathbb{R}^2$ on an intensity image $I : \Omega \rightarrow \mathbb{R}$ which evolves and hopefully converges to the object boundary. The gradient descent of an energy functional, $E(C)$, is given by

$$E(C) = \alpha \int_0^1 \| C(q) \|^2 dq + \beta \int_0^1 \| C''(q) \|^2 dq - \lambda \int_0^1 \| \nabla I(C(q)) \| dq \quad (4)$$

where $\alpha$, $\beta$, and $\lambda$ are real positive constants, $C'$ and $C''$ are first and second derivatives of $C$ and $\nabla I$ is the image gradient, gives us a curve evolution equation. The first two terms are regularizers, while the third term pushes the curve toward the object boundary.

Geodesic active contours [3] are an active contour model where the objective function can be interpreted as the length of a curve $C : [0, 1] \rightarrow \mathbb{R}^2$ in a Riemannian space with metric induced by image intensity. The energy functional for geodesic active contour is given by

$$E = \int_0^1 g(\| \nabla I(C(q)) \|) \| C'(q) \| dq,$$

where $g : \mathbb{R}^+ \rightarrow \mathbb{R}$ is a positive monotonically decreasing function of the image gradient. One such choice is $g(s) = \exp(-s)$. We set $g := g(\| \nabla I \|)$ to make notations simpler. The curve evolution equation that minimizes this energy is given by

$$\frac{\partial C}{\partial t} = (g\kappa - \langle \nabla g, \hat{n} \rangle) \hat{n}, \quad (5)$$

where $\hat{n}$ is the inward unit normal and $\kappa$ is the curvature of the curve $C$.

A convenient computational procedure for curve evolution is the level set formulation [18,20]. Here the curve is embedded in the zero level set of a function $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}$ which evolves so that the corresponding zero level set evolves according to the desired curve evolution equation. For a curve evolution equation of the form $\frac{\partial C}{\partial t} = v \hat{n}$, the corresponding level set evolution is $\frac{\partial \phi}{\partial t} = v \| \nabla \phi \|$. See Appendix in [3]. In particular the level set evolution for geodesic active contours is given by

$$\frac{\partial \phi}{\partial t} = g \| \nabla \phi \| dv \left( \frac{\nabla \phi}{\| \nabla \phi \|} \right) + \langle \nabla g, \nabla \phi \rangle. \quad (6)$$

Another active contour approach was introduced by Chan et al. [4] where the energy function was based on regional similarity properties of an object, rather than its edges (image gradient). With the assumption that the object and the background are approximately piecewise constant, the object boundary $C$ is an obvious minimizer of the function

$$F_1(C) + F_2(C) = \int_{int(C)} (I(x, y) - \mu_1)^2 dx dy + \int_{ext(C)} (I(x, y) - \mu_2)^2 dx dy, \quad (7)$$

where the object lies in the interior of $C (int(C))$ and background in the exterior of $C (ext(C))$. After adding some regularizing terms, the segmentation problem can be solved by minimizing the energy functional $F(\mu_1, \mu_2, C)$, given as

$$F(\mu_1, \mu_2, C) = \mu_1 \text{Length}(C) + v \text{Area}(int(C)) + \lambda_1 \int_{int(C)} (I(x, y) - \mu_1)^2 dx dy + \lambda_2 \int_{ext(C)} (I(x, y) - \mu_2)^2 dx dy, \quad (8)$$

where $\mu \geq 0, v \geq 0, \lambda_1, \lambda_2 > 0$ are fixed scalar parameters. Re-writing the above functional using a level set function $\phi$ with the following convention:

$$int(C) = \{ (x, y) \in \Omega : \phi(x, y) < 0 \}$$
$$ext(C) = \{ (x, y) \in \Omega : \phi(x, y) > 0 \},$$

the Heaviside function $H$, and the Dirac Delta function $\delta$ defined by

$$H(z) = \begin{cases} 1, & \text{if } z \geq 0 \\ 0, & \text{if } z < 0 \end{cases}, \quad \delta(z) = \frac{d}{dz} H(z),$$

we obtain the following energy functional:

$$F(\mu_1, \mu_2, \phi) = \int_{\Omega} \mu \delta(\phi(x, y)) \| \nabla \phi(x, y) \| dx dy + v(1 - H(\phi(x, y))) + \lambda_1 (I(x, y) - \mu_1)^2 (1 - H(\phi(x, y))) + \lambda_2 (I(x, y) - \mu_2)^2 H(\phi(x, y)) dx dy. \quad (9)$$

For a fixed $C$ (and therefore a fixed $\phi$), the minimizers for $\mu_1, \mu_2$ can easily be seen to be

$$\mu_1(\phi) = \frac{\int_{\Omega} I(x, y) (1 - H(\phi(x, y))) dx dy}{\int_{\Omega} (1 - H(\phi(x, y))) dx dy}. \quad (10)$$
Fig. 1 Example for different type of images. a Intensity image. b DT-MRI. c Vector field with vectors from $S^1$. d Texture image with leopard

\[
\mu_2(\phi) = \frac{\int_{\Omega} I(x, y) H(\phi(x, y)) \, dx \, dy}{\int_{\Omega} H(\phi(x, y)) \, dx \, dy}. \tag{11}
\]

The minimization in terms of $\phi$, keeping $\mu_1, \mu_2$ fixed (and using smooth approximations $H_\epsilon, \delta_\epsilon$ of $H, \delta$ as given in [4]) is given by the following gradient descent based level set evolution equation:

\[
\frac{\partial \phi}{\partial t} = \delta_\epsilon(\phi) \left[ \mu \, div \left( \frac{\nabla \phi}{\|\nabla \phi\|} \right) + v + \lambda_1 (I - \mu_1)^2 - \lambda_2 (I - \mu_2)^2 \right]. \tag{12}
\]

A nice survey on active contours and level set implementation can be found in [1]. In this paper, we generalize the two active contour models, the Geodesic active contour and Chan–Vese active contours model to segment objects in MVIs. Before describing our active contour model, we will now introduce and define the kind of images we focus on in this paper.

2.2 Manifold-Valued Images (MVI)

In this paper, instead of working with intensity images represented as functions $I : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$, we work on images represented as $I : \Omega \to M$, where $M$ is a Riemannian manifold. In what follows we assume the familiarity with basic concepts from differential geometry like geodesics, Exp map, Log map, etc. For the sake of completeness, we define these terms in the Appendix. A thorough introduction and discussion can be found in the textbooks [2,6].

We present some practical instances of MVI on which we propose to segment “objects” using active contours. Refer to Fig. 1 to see examples of the following MVIs.

---

1 A Manifold is a topological space which is locally Euclidean and a smooth Manifold equipped with a smooth inner product (called Riemannian inner product) on tangent space at every point is called a Riemannian manifold.
a. Diffusion tensor magnetic resonance imaging (DT-MRI) produces diffusion tensors corresponding to diffusion of white matter. The diffusion tensors produced are found to be symmetric and positive definite matrices of size $3 \times 3$, which forms a Riemannian manifold. A detailed analysis of diffusion tensor data from DT-MRI can be found in the work by Fletcher and Joshi [9]. In order to display such images in this paper, we represent the positive definite matrices which form a manifold. We mention a few here. Computing nonlinear statistics on such data was proposed by Fletcher et al. [11] and by Pennec [22]. Weickert and Brox [30], Tschumperlé and Deriche[26] and Rosman et al. [24] have all proposed regularization schemes for vector-valued and/or matrix-valued images using PDE’s. Tuzel et al. [27] presented methods using Lie group modeling for tracking objects. In the next section, we describe our active contour model for MVI segmentation.

3 Adapting Active Contour Model for MVI Segmentation

In what follows, we adapt the Geodesic active contours and Chan–Vese active contours for MVI’s.

3.1 Geodesic Active Contours (GAC)

In edge-dependent active contour models, for example GAC, the curve evolution is made to stop at the object boundary by defining a speed function (or a metric) $g$ that is inversely proportional to the image gradient magnitude (Refer to Sect. 2). The gradient of a differentiable intensity function $I : \Omega \rightarrow \mathbb{R}$ is a vector $v \in \mathbb{R}^2$ along which the function $I$ increases the most and whose length $||v||$ is that amount of increase. With Euclidean inner product on $\mathbb{R}^2$, the gradient at a point $(p, q) \in \mathbb{R}^2$ can be computed as $\nabla I(p, q) = (I_x(p, q), I_y(p, q)) \in \mathbb{R}^2$. Note that in geodesic active contours, the image gradient magnitude (not the image gradient vector itself) plays a significant role: the edge detector function $g$ is a function of the image gradient magnitude. Image gradient magnitude can be interpreted as the maximum rate of change in the value of the function at a point $(p, q) \in \mathbb{R}^2$.

$$||\nabla I(p, q)|| = \max_{v \in \mathbb{R}^2, ||v||=1} ||DI(p,q)(v)|| = \max_{v \in \mathbb{R}^2, ||v||=1} \left|\left| \begin{bmatrix} I_x & I_y \end{bmatrix} v \right|\right| = ||(I_x, I_y)||,$$  (13)

where $DI(p,q) : \mathbb{R}^2 \rightarrow \mathbb{R}$ is the differential of the function $I$ at point $(p, q)$, which in this case is the $1 \times 2$ Jacobian matrix $\begin{bmatrix} I_x & I_y \end{bmatrix}$, and $I_x, I_y$ are evaluated at $(p, q)$. Since this map is linear, the magnitude of maximum increase is the same as the magnitude of maximum decrease in the function value. The function increases the maximum along $\nabla I(p, q)$ and decreases the maximum along $-\nabla I(p, q)$ while the magnitude of rate of change in both cases is equal to $||\nabla I(p, q)||$. We generalize this concept for active contours to work on MVIs. Note that MVI’s are maps of the kind $I : \Omega \subset \mathbb{R}^2 \rightarrow M$. It does not make sense to talk about gradient of such maps, but we define an analogous concept to the gradient magnitude of intensity images for MVIs. The differential of the MVI function $I$ at point $(p, q) \in \mathbb{R}^2$ is given as a linear map:

$$DI(p,q) : T_{(p,q)}\mathbb{R}^2 (\simeq \mathbb{R}^2) \rightarrow T_I(p,q)M,$$  (14)

where $T_I(p,q)M$ is the tangent space to $M$ at the point $I(p,q)$, and is defined using

$I_x := DI(p,q)(1, 0)$
$I_y := DI(p,q)(0, 1)$

as

$$DI(p,q)(a, b) = aI_x + bI_y.$$  (15)

In what follows, we assume that $I$ is differentiable in the sense described in the book [2]. The definition of Differential of a smooth map between manifolds is given in the Appendix 1.
Moreover,
\[ ||D_I(p,q)(a,b)||_{I(p,q)} = ||aI_x + bI_y||_{I(p,q)}, \]  
(16)

where \( ||\cdot||_{I(p,q)} \) is the Riemannian norm on the tangent space \( T_{I(p,q)}M \). We define the gradient magnitude of \( I \) at \( (p,q) \) (denoted by \( \delta_M I(p,q) \)) as the maximum possible value of the norm of the differential:
\[
\delta_M I(p,q) = \max_{||u||_2=1} ||aI_x + bI_y||_{I(p,q)},
\]  
(17)

Let the inner product on \( T_{I(p,q)}M \) be defined as
\[
\langle u, w \rangle_{I(p,q)} = u^T G_{I(p,q)} w,
\]  
(18)

where \( G_{I(p,q)} \) is a symmetric positive definite matrix that varies smoothly over \( M \). Using Eqs. (17) and (18), we get
\[
\delta_M I(p,q) = \max_{||u||_2=1} \sqrt{\langle aI_x + bI_y, aI_x + bI_y \rangle_{I(p,q)}}
\]
\[
= \max_{||u||_2=1} \sqrt{\langle aI_x + bI_y, G_{I(p,q)}(aI_x + bI_y) \rangle}
\]
\[
= \max_{||u||_2=1} \sqrt{\begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} I_x^I(p,q) & I_x^I(p,q) I_y^I(p,q) \\ I_y^I(p,q) & I_y^I(p,q) I_y^I(p,q) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}}.
\]  
(19)

Let
\[
A_{pq} = \begin{bmatrix} I_x^I(p,q) & I_x^I(p,q) I_y^I(p,q) \\ I_y^I(p,q) & I_y^I(p,q) I_y^I(p,q) \end{bmatrix}
\]  
(20)

and \( v = (a, b) \in \mathbb{R}^2 \). \( \delta_M I(p,q) \) can then be computed via a constrained maximization problem:
\[
\delta_M I(p,q) = \max_{||v||=1} \sqrt{v^T A_{pq} v}.
\]  
(21)

Using Lagrange multipliers, it can be shown that
\[
\delta_M I(p,q) = \sqrt{\lambda_{\max}(A_{pq})},
\]  
(22)

where \( \lambda_{\max}(A_{pq}) \) is the maximum eigenvalue\(^4\) of the matrix \( A_{pq} \). Finally, we explain how we compute \( I_x(p,q) \) and \( I_y(p,q) \) on discretized MVIs. In case of intensity images, the following approximation is frequently used:
\[
I_x(p,q) \approx I(p+1,q) - I(p,q)
\]  
(23)
\[
I_y(p,q) \approx I(p,q+1) - I(p,q).
\]  
(24)

Subtraction in vector space can be re-interpreted on Riemannian manifolds as the Riemannian Log map\([21]\). Using this, the analogous approximations to \( I_x, I_y \) for MVIs \( I: \Omega \to M \), are given by
\[
I_x(p,q) \approx I(p+1,q) - I(p,q) := \text{Log}_{I(p,q)} (I(p+1,q))
\]  
(25)
\[
I_y(p,q) \approx I(p,q+1) - I(p,q) := \text{Log}_{I(p,q)} (I(p,q+1)).
\]  
(26)

where \( \text{Log}_{a}(b) \in T_a M \) is the Riemannian Log map at point \( a \in M \) for point \( b \in M \). Having defined the required gradient magnitude for MVIs, the level set evolution equation corresponding to geodesic active contours is given as
\[
\frac{\partial \phi}{\partial t} = \|\nabla \phi\| div \left( g \cdot \frac{\nabla \phi}{\|\nabla \phi\|} \right),
\]  
(27)

where
\[
g(\delta_M I(p,q)) = \frac{1}{1 + \delta_M I(p,q)}.
\]  
(28)

3.2 Chan–Vese Active Contour Model

In Chan–Vese active contour model the energy function is
\[
F(\mu_1, \mu_2, C) = \mu_1 \cdot \text{Length}(C) + v \cdot \text{Area}(\text{int}(C))
\]
\[
+ \lambda_1 \int_{\text{int}(C)} (I(x,y) - \mu_1)^2 \, dx \, dy
\]
\[
+ \lambda_2 \int_{\text{ext}(C)} (I(x,y) - \mu_2)^2 \, dx \, dy.
\]  
(29)

The corresponding energy function for Chan–Vese active contours on MVI is given by
\[
F(\mu_1, \mu_2, C) = \mu_1 \cdot \text{Length}(C) + v \cdot \text{Area}(\text{int}(C))
\]
\[
+ \lambda_1 \int_{\text{int}(C)} d(I(x,y), \mu_1)^2 \, dx \, dy
\]
\[
+ \lambda_2 \int_{\text{ext}(C)} d(I(x,y), \mu_2)^2 \, dx \, dy.
\]  
(30)

As stated earlier in Eqs. (10) and (11) for scalar images \( I \), the minimizers of \( F \) for a fixed \( C \) are simply the mean over the interior and exterior of the contour \( C \), respectively. Similarly, for MVIs, \( \mu_i, i = 1, 2 \) are the intrinsic means of the manifold data in the interior and exterior of the curve \( C \). For a fixed \( \phi \), the functional \( F \) reduces to
\[
F(\mu_1, \mu_2) = \lambda_1 \int_{\text{int}(C)} d(I(x,y), \mu_1)^2 \, dx \, dy
\]
\[
+ \lambda_2 \int_{\text{ext}(C)} d(I(x,y), \mu_2)^2 \, dx \, dy.
\]  
(31)

\(^4\) With \( u = aI_x + bI_y \), one can see that \( v^T A_{pq} v = u^T G_{pq} u \geq 0, \forall v \in \mathbb{R}^2 \), therefore \( A_{pq} \) is positive semi-definite and all its eigen-values are non-negative.
Therefore, the minimizers $\mu_1, \mu_2$ of $F$ (in a discrete form) are simply

$$
\mu_1 = \arg\min_{p \in M} \sum_{(x, y) \in \text{int}(C)} d(I(x, y), p)^2
$$

$$
\mu_2 = \arg\min_{p \in M} \sum_{(x, y) \in \text{ext}(C)} d(I(x, y), p)^2
$$

which by definition are the intrinsic means\(^5\) of manifold values inside and outside the curve $C$. The intrinsic mean is computed using a gradient descent approach. For a detailed explanation refer to [11], and refer to Sect. 4 for an algorithm to compute the intrinsic mean.

Using the same convention for the level set function $\phi$ as in Sect. 2, the energy functional can be re-written in terms of a level set function $\phi$:

$$
F(\mu_1, \mu_2, \phi) = \int \mu \delta_\epsilon(\phi(x, y)) ||\nabla\phi(x, y)||
\quad + v(1 - H_\epsilon(\phi(x, y)))
\quad + \lambda_1 d(I(x, y), \mu_1)^2 (1 - H_\epsilon(\phi(x, y)))
\quad + \lambda_2 d(I(x, y), \mu_2)^2 H_\epsilon(\phi(x, y))
\quad \cdot \mathrm{d}x \mathrm{d}y.
$$

The gradient descent equation that minimizes $F$ with respect to $\phi$, keeping $\mu_1, \mu_2$ fixed comes out to be

$$
\frac{\partial \phi}{\partial t} = \delta_\epsilon(\phi) \left[ \mu \cdot \nabla \phi \left( \frac{\nabla \phi}{||\nabla \phi||} \right) + v + \lambda_1 d(I(x, y), \mu_1)^2
\quad - \lambda_2 d(I(x, y), \mu_2)^2 H_\epsilon(\phi(x, y)) \right].
$$

In the next section, we implement both GAC and Chan–Vese active contour model on several examples of MVIs. We also provide computational algorithms for computing Riemannian Exp map, Log map, and intrinsic mean on the corresponding manifolds. The Riemannian Exp map is the inverse of Riemannian Log map, and intuitively can be interpreted as vector addition in Riemannian manifolds [21].

\(^5\) Intrinsic mean of a collection of points $x_1, \ldots, x_n \in M$ is the minimizer of the sum of squared Riemannian distances from each of the given points:

$$
\mu = \arg\min_{x \in M} \sum_{i=1}^n d(x, x_i)^2,
$$

where $d(\cdot, \cdot)$ is the Riemannian distance on $M$ and is a generalization of the Euclidean distance $d(x, y) = ||x - y||$ to Riemannian manifolds [21].

### 4 Experiments

Our Active contour model requires computing the Exp map, Log map, and intrinsic mean for each manifold. For short definitions of these terms refer to the Appendix. A summary of algorithms to compute these maps on $S^2$ (2D sphere), $SO(3)$ (3D rotation matrices) and $PD(3)$ manifolds are provided next, while detailed derivations can be found in one or more of these papers: [10,11,19,23].

#### 4.1 Computation on Different Manifolds

1. $S^2$

   (a) Exp map

   $$
   \exp_p(v) = \cos(||v||) p + \sin(||v||) \frac{v}{||v||},
   $$

   where $v \in T_p(S^2)$ and $p \in S^2$.

   (b) Log map

   $$
   \log p(q) = \frac{\theta}{\sin\theta}(q - p \cos(\theta)), \quad \theta = \cos^{-1}(p, q),
   $$

   where $p, q \in S^2$.

2. $SO(3)$

   Let us first describe the tangent space of $SO(3)$. Given a curve $R(t) \in SO(3)$, one can differentiate the constraints $R(t)' R(t) = R(t) R(t)' = 1d$ to obtain the tangent space at every point of the curve $R(t)$ in $SO(3)$. With $R(0) = 1d$, we get $R'(0) + (R'(0))' = 0$. Thus, tangent space of $SO(3)$ at $1d$, denoted by $T_{1d}(SO(3))$, (also called the Lie algebra $so(3)$) consists of all $3 \times 3$ real skew-symmetric matrices of the form

   $\begin{bmatrix}
   0 & -v_3 & v_2 \\
   v_3 & 0 & -v_1 \\
   -v_2 & v_1 & 0
   \end{bmatrix}$

   There is an obvious isomorphism between $so(3)$ and $\mathbb{R}^3$, $\phi : so(3) \rightarrow \mathbb{R}^3$ given by $\phi(v) = \hat{v} = (v_1, v_2, v_3)$. The tangent space at any other point $p \in SO(3)$ is simply $T_p(SO(3)) = \{p \cdot v | v \in so(3)\}$.

   (a) Exp map

   $$
   \exp_p : T_p(SO(3)) \rightarrow SO(3)
   $$

   $$
   \exp_p(p \cdot v) = p \exp(v)
   $$

   $$
   = p \left( 1d + \frac{\sin(||\hat{v}||)}{||\hat{v}||} \hat{v} + \frac{1 - \cos(||\hat{v}||)}{||\hat{v}||^2} \hat{v}^2 \right),
   $$

   (40)
Fig. 2 Segmentation results on different types of synthetic MVIs. (left column) Original images with manifolds $S^1$, $S^2$, $SO(3)$, and $PD(3)$ in top to bottom order, with initial contour. (center column) Corresponding geodesic active contour segmentation results. (right column) Corresponding Chan–Vese active contour segmentation results.
Fig. 3 Results on Real DT-MRI data. (left column) Image with initial contour, (right column) segmentation output

where $v \in so(3)$, $\hat{v} = \phi(v)$ and $\exp(\cdot)$ is the matrix exponential.

(b) Log map

$Log_p : SO(3) \rightarrow so(3)$

$Log_p(q) = \phi^{-1}\left( \frac{\theta}{2\sin\theta} \begin{bmatrix} -R(1,2) + R(2,1) \\ R(3,1) - R(1,3) \\ -R(2,3) + R(3,2) \end{bmatrix} \right)$

where $p, q \in SO(3)$, $R = p^T q$ and $\theta = \cos^{-1}\left( \frac{\text{Trace}(R)-1}{2} \right)$.

3. $PD(n)$

(a) Exp map

Input: Initial point $p \in PD(n)$
Tangent vector $X \in Sym(n)$

Output: $Exp_p(X)$
Let $p = u\Lambda u^T$ ($u \in SO(n)$, $\Lambda$ diagonal)
$g = u\sqrt{\Lambda}$
$Y = g^{-1}X(g^{-1})^T$
Let $Y = v\Sigma v^T$ ($v \in SO(n)$, $\Sigma$ diagonal)
$Exp_p(X) = (gv)exp(\Sigma)(gv)^T$

(b) Log map

Input: Initial point $p \in PD(n)$
End point $q \in PD(n)$

Output: $Log_p(q)$
Let $p = u\Lambda u^T$ ($u \in SO(n)$, $\Lambda$ diagonal)
$g = u\sqrt{\Lambda}$
$y = g^{-1}q(g^{-1})^T$
Let $y = v\Sigma v^T$ ($v \in SO(n)$, $\Sigma$ diagonal)
$Log_p(q) = (gv)log(\Sigma)(gv)^T$

We have used the following algorithm to find the intrinsic mean on all manifolds, details of which can be found in...
Fig. 4 Segmentation results on different texture images. The initial contour is shown in yellow, while the final contour is shown in red. a and b show the results on small-scale texture images, c shows the result on a leopard image, while d shows result on a texture image where the mean intensity of the two regions is the same (Color figure online)

[11]. With all the required computational machinery set, we next demonstrate our segmentation results for both Geodesic active contours and Chan–Vese active contours on various manifold-valued images.

Algorithm 1: Intrinsic Mean on a manifold

| Input: | $x_1, x_2, \ldots, x_n \in M$ |
|---------|--------------------------------|
| Output: | $\mu \in M$, the intrinsic mean |
| $\mu_0 = x_1$ |
| Do |
| $\Delta \mu = \frac{1}{N} \sum_{i=1}^{N} \log_{\mu_j} x_i$ |
| $\mu_{j+1} = \exp_{\mu_j}(\Delta \mu)$ |
| While $\|\Delta \mu\| > \epsilon$ |

4.2 Results

Results on synthetic $S^1$, $S^2$, $SO(3)$ and $PD(M^2)$-valued images for Geodesic active contours as well as Chan–Vese active contours are shown in Fig. 2. Results on Real DT-MRI data$^6$ are shown in Fig. 3. Texture segmentation problem can be posed as an MVI segmentation problem, as already explained in Sect. 2. We obtain a $PD(M^2)$-valued image from a texture image over which our algorithm successfully segments different textures$^7$ as shown in Fig. 4. Since texture boundaries are not defined based on gray-value image edges, the geodesic active contour model does not yield appropriate segmentation results. The results shown are for our version of the Chan–Vese active contour model. We have used a $5 \times 5$ covariance matrix to characterize a texture and it is computed over a neighborhood of $13 \times 13$ pixels.

$^6$ Images taken from (top) http://www.cise.ufl.edu/~abarmpou/lab/fandTasia/ and (bottom) http://cmic.cs.ucl.ac.uk/camino/uploads/Tutorials/example_dwi.zip

$^7$ From http://sipi.usc.edu/database/database.php?volume=textures and http://www.nada.kth.se/cvap/databases/kth-tips
5 Conclusion and Future Scope

In this paper, we have generalized the active contours for MVI segmentation. We provide several such examples which can be dealt under our general framework. The drawbacks and benefits are the same as those of the usual active contour models. As a computer vision application, we pose the texture segmentation problem as an MVI segmentation problem and demonstrate some texture segmentation results using our algorithm. We take a neighborhood of size $5 \times 5$ pixels in an intensity image to form covariance matrices over a larger neighborhood of size $13 \times 13$ at every pixel to get a $PD(25)$-valued image. This fixes the scale of features in which we are interested. Some textures may need multi-scale information to be properly characterized. Simply increasing the neighborhood size is not going to help in dealing with large scale texture as it may prevent proper localization of the texture boundary. One needs to incorporate a mechanism that automatically detects the scale of the given texture.

The computational cost for the model extended from Chan–Vese active contour is high since one needs to compute the Riemannian Exp and Log map for every manifold value in the image. This cost will of course vary from manifold to manifold. The texture segmentation algorithm is computationally very expensive: To generate results on images of size $150 \times 150$ pixels, it took about 30 min on a 2 GHz Intel laptop. This is primarily due to repeated Exp and Log map computation on $PD(25)$ manifold. One may use multi-grid approaches to speed up the evolution.

Appendix 1

We give some basic definitions from Differential geometry required for our paper. For a thorough explanation, we refer the reader to the books [2, 6].

1. Differentiable manifolds:

A differentiable manifold of dimension $n$ is a set $M$ and a family of injective mappings $\mathcal{T} = \{x_i : U_i \subset \mathbb{R}^n \rightarrow M\}$ of open sets $U_i$ of $\mathbb{R}^n$ into $M$ such that

- $\bigcup x_i(U_i) = M$, i.e. the open sets cover $M$.
- for any pair $i, j$ with $x_i(U_i) \cap x_j(U_j) = W \neq \phi$, the mapping $x_j^{-1} \circ x_i$ is differentiable.
- The family $\mathcal{T}$ is maximal, which means that if $(y, V)$, $y : V \subset \mathbb{R}^n \rightarrow M$ is such that: for each element of $\mathcal{T}$, $(x_i, U_i)$ with $x_i(U_i) \cap y(V) \neq 0$ implies that $y^{-1} \circ x_i$ is a diffeomorphism, then in fact $(y, V) \in \mathcal{T}$.

2. Differential of a smooth map between differentiable manifolds:

Let $F : M \rightarrow N$ be a smooth map between two differentiable manifolds $M$ and $N$. Given a point $p \in M$, the differential of $F$ at $p$ is a linear map $DF_p : T_p M \rightarrow T_{F(p)} N$ from the tangent space of $M$ at $p$ to the tangent space of $N$ at $F(p)$.

3. Riemannian metric:

A Riemannian metric on a manifold $M$ is a correspondence which associates to each point $p \in M$ an inner product $\langle \cdot, \cdot \rangle_p$ on the tangent space $T_p M$, which varies smoothly. In terms of local coordinates, the metric at each point $x$ is given by a matrix, $g_{ij} = \langle X_i, X_j \rangle_x$, where $X_i, X_j$ are tangent vectors to $M$ at $x$, and it varies smoothly with $x$. A Geodesic curve is a local minimizer of arc-length computed with a Riemannian metric.

4. Geodesics:

A parameterized curve $\gamma : I \rightarrow M$ is a geodesic if $\frac{d}{dt} \left( \frac{d \gamma}{dt} \right) = 0, \forall t \in I$, where $\frac{d}{dt}$ is called the Covariant derivative and intuitively is the orthogonal projection of the usual derivative to the tangent space. Geodesics are also local minimizers of arc-length.

5. Exponential map:

The exponential map is a map $Exp : TM \rightarrow M$, that maps $v \in T_q M$ for $q \in M$, to a point on $M$ obtained by going out the length equal to $||v||_M$, starting from $q$, along a geodesic which passes through $q$ with velocity equal to $\frac{v}{||v||_M}$. The geodesic starting at $q$ with initial velocity $t$ can thus be parametrized as

$t \mapsto Exp_q(tv)$.

6. Log map: For $\tilde{q}$ in a sufficiently small neighborhood of $q$, the length minimizing curve joining $q$ and $\tilde{q}$ is unique as well. Given $q$ and $\tilde{q}$, the direction in which to travel geodesically from $q$ in order to reach $\tilde{q}$ is given by the result of the logarithm map $Log_q(\tilde{q})$. We get the corresponding geodesic as the curve $t \mapsto Exp_q(t Log_q \tilde{q})$. In other words, $Log$ is the inverse of $Exp$ in the neighborhood.

References

1. Aubert, G., Kornprobst, P.: Mathematical Problems in Image Processing: Partial Differential Equations and the Calculus of
Variations (Applied Mathematical Sciences). Springer, Secaucus (2006).

2. Boothby, W.M.: An Introduction to Differentiable Manifolds and Riemannian Geometry. Academic Press, New York (1975).

3. Caselles, V., Kimmel, R., Sapiro, G.: Geodesic active contours. Int. J. Comput. Vis. 22 (1), 61–79 (1997).

4. Chan, T.F., Sandberg, B.Y., Vese, L.A.: Active contours without edges for vector-valued images. J. Vis. Commun. Image Represent. 11, 130–141 (2000).

5. Cremers, D., Rousson, M., Deriche, R.: A review of statistical approaches to level set segmentation: Integrating color, texture, motion and shape. Int. J. Comput. Vis. 72, 215 (2007).

6. do Carmo, M.P.: Differential Geometry of Curves and Surfaces. Prentice-Hall, Englewood Cliffs (1976).

7. Donoser, M., Bischof, H.: Roi-seg: Unsupervised color segmentation by combining differently focused sub results. In: CVPR, (2007).

8. Donoser, M., Bischof, H.: Using covariance matrices for unsupervised texture segmentation. In: ICCP, pp. 1–4, (2008).

9. Thomas, P.T., Joshi, S.: Riemannian geometry for the statistical analysis of diffusion tensor data. Signal Process. 87(2), 250–262 (2007).

10. Fletcher, P.T., Joshi, S.C.: Principal geodesic analysis on symmetric spaces: Statistics of diffusion tensors. In ECCV Workshops CVAMIA and MMBIA, pp. 1–15, (2004).

11. Thomas Fletcher, P., Lu, C., Pizer, S.M., Joshi, S.: Principal geodesic analysis for the study of nonlinear statistics of shape. IEEE Trans. Med. Imaging 23(8), 995–1005 (2004).

12. Berthold, K.P., Schunck, B.G.: Determining optical flow. Artif. Intell. 17, 185–203 (1981).

13. Jonasson, L., Bresson, X., Hagmann, P., Cuisenaire, O., Meuli, Reto, Thiran, Jean-Philippe: White matter fiber tract segmentation in DT-MRI using geometric flows. Med. Image Anal. 9(3), 223–236 (2005).

14. Jonasson, L., Hagmann, P., Pollo, C., Bresson, X., Wilson, Cecilia Richero, Meuli, Reto, Thiran, Jean-Philippe: A level set method for segmentation of the thalamus and its nuclei in DT-MRI. Signal Process. 87(2), 309–321 (2007).

15. Kass, M., Witkin, A., Terzopoulos, D.: Snakes: active contour models. Int. J. Comput. Vis. 1(4), 321–331 (1988).

16. Lee, S.-M., Abbott, A.L., Clark, N.A., Araman, P.A.: Active contours on statistical manifolds and texture segmentation. In ICIP (3), pp. 828–831, (2005).

17. Lenglet, C., Rousson, M., Deriche, R.: DTI segmentation by statistical surface evolution. IEEE Trans. Med. Imaging 25(6), 685–700 (2006).

18. Malladi, R., Sethian, J.A., Vemuri, B.C.: Shape modeling with front propagation: a level set approach. IEEE Trans. Pattern Anal. Mach. Intell. 17, 158–175 (1995).

19. Moakher, M., Zéraï, M.: The riemannian geometry of the space of positive-definite matrices and its application to the regularization of positive-definite matrix-valued data. J. Math. Imaging Vis. 40(2), 171–187 (2011).

20. Osher, S., Sethian, J.A.: Fronts propagating with curvature dependent speed: algorithms based on Hamilton–Jacobi formulations. J. Comput. Phys. 79(1), 12–49 (1988).

21. Pennec, X.: Statistical computing on manifolds: From riemannian geometry to computational anatomy. In: Frank Nielsen, editor, Emerging Trends in Visual Computing, volume 5416 of LNCS, pp. 347–366, (2008).

22. Pennec, X.: Intrinsic statistics on Riemannian manifolds: Basic tools for geometric measurements. J. Math. Imaging Vis., 25(1), 127–154, (July 2006). A preliminary appeared as INRIA RR-5093, January 2004.

23. Rentmeesters, Q., Absil, P.-A.: Algorithm comparison for karcher mean computation of rotation matrices and diffusion tensors. In: Proceedings of the 19th European Signal Processing Conference (EUSIPCO 2011). EURASIP, pp. 2229–2233, (2011).

24. Guy Rosman, Y., Wang, X.-C.T.: Fast regularization of matrix-valued images. ECCV 3, 173–186 (2012).

25. Sagiv, C., Sochen, N.A., Zeevi, Y.Y.: Integrated active contours for texture segmentation. IEEE Trans. Image Process. 15(6), 1633–1646 (2006).

26. Tschumperlé, D., Deriche, R.: Vector-valued image regularization with pdes: a common framework for different applications. In: IEEE Transactions on Pattern Analysis and Machine Intelligence, pp. 506–517, (2003).

27. Tuzel, O., Porikli, F., Meer, P.: Learning on lie groups for invariant detection and tracking. In: IEEE Conference on Computer Vision and Pattern Recognition, CVPR 2008, pp. 1–8, (2008).

28. Guo, W., Chen, Y., Zeng, Q.: DTI segmentation using an information theoretic tensor dissimilarity measure. Philos. Trans. A Math. Phys. Eng. Sci. 366(10), 2279–2292 (2008).

29. Wang, Z., Vemuri, B.C.: DTI segmentation using an information theoretic tensor dissimilarity measure. IEEE Trans. Med. Imaging 24(10), 1267–1277 (2005).

30. Weickert, J., Brox, T.: Diffusion and regularization of vector- and matrix-valued images. Contemp. Math. 313, 251–268 (2002).

31. Wildenauer, H., Mičuššk, B., Vincze, M.: Efficient texture representation using multi-scale regions. In: Proceedings of the 8th Asian conference on Computer vision - Volume Part I, ACCV’07, Berlin, Heidelberg, Springer, pp. 65–74, (2007).

Sumukh Bansal received an M.Tech (Information & Communication Technology) from the Dhirubhai Ambani - Institute of Information and Communication Technology (DA-IICT), India, in 2013. His M.Tech Thesis was on Manifold-Valued Image Segmentation. His research interests include Differential geometry, Manifold theory, and mathematical aspects of image processing and computer Vision.

Aditya Tatu did his Bachelor of Engineering (Electronics & Communication) from Sardar Patel University, India in 2003. He received his M.Tech (Information & Communication Technology) from Dhirubhai Ambani Institute of Information & Communication Technology (DA-IICT), Gandhinagar, Gujarat, India in 2005, followed by an M.Sc. and Ph.D. from the Department of Computer Science (DIKU), University of Copenhagen, Denmark in 2007 and 2010 (April), respectively. He is currently an Assistant Professor at DA-IICT, India. His topics of interest include applications of differential geometry and calculus of variations in image analysis (segmentation, shapes, and image features).