LIE ALGEBRAS OF DIFFERENTIAL OPERATORS III: CLASSIFICATION

HELGE ÖYSTEIN MAAKESTAD
EMAIL HMAAKESTAD@HOTMAIL.COM

Abstract. In a previous paper we introduced the notion of a D-Lie algebra \( \bar{L} \). A D-Lie algebra \( \bar{L} \) is an \( A/k \)-Lie-Rinehart algebra with a right \( A \)-module structure and a canonical central element \( D \) satisfying several conditions. We used this notion to define the universal enveloping algebra of the category of \( \bar{L} \)-connections and to define the cohomology and homology of an arbitrary connection. In this note we introduce the canonical quotient \( L \) of a D-Lie algebra \( \bar{L} \) and use this to classify D-Lie algebras where \( L \) is projective as \( A \)-module. We define for any 2-cocycle \( f \in Z^2(\text{Der}_k(A), A) \) a functor \( F_f(-) \) from the category of \( A/k \)-Lie-Rinehart algebras to the category of D-Lie algebras and classify D-Lie algebras with projective canonical quotient using the functor \( F_f(-) \). We prove a similar classification for non-abelian extensions of D-Lie algebras. We moreover classify maps of D-Lie algebras and \( \bar{L} \)-connections \( (E, \rho) \) in the case when the canonical quotient \( L \) of \( \bar{L} \) is projective as \( A \)-module.

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1. Introduction

A D-Lie algebra \( \bar{L} \) is a refinement of the notion of an \( A/k \)-Lie-Rinehart algebra: It is a \( k \)-Lie algebra and left \( P^1_{A/k} \)-module, where \( P^1_{A/k} \) is the first order module of principal parts of \( A/k \). The Lie-product on \( \bar{L} \) satisfies the anchor condition

\[
[u, cv] = c[u, v] + \bar{\pi}(u)(c)v
\]

for all \( u, v \in \bar{L} \) and \( c \in A \). Here \( \bar{\pi} : \bar{L} \to \text{Der}_k(A) \) is a map of \( k \)-Lie algebras and left \( P \)-modules. Hence if we view \( \bar{L} \) as a left \( A \)-module and \( k \)-Lie algebra, it follows

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the pair \((\tilde{L}, \tilde{\pi})\) is an \(A/k\)-Lie-Rinehart algebra. There is a canonical central element \(D \in \tilde{L}\) with the property that
\[
uc = cu + \tilde{\pi}(u)(c)D
\]
holds for \(u \in \tilde{L}\) and \(c \in A\). We may define the notion of a connection \(\rho : \tilde{L} \to \text{Diff}^1(E)\) where \(E\) is a left \(A\)-module. In the papers \([8]\) and \([9]\) general properties of the category of D-Lie algebras are introduced and studied: Universal enveloping algebras of D-Lie algebras, cohomology and homology of connections on D-Lie algebras. Let \(\text{D-Lie}\) denote the category of D-Lie algebras and morphisms. In \([8]\) Theorem 3.5 and 3.18 we prove the following result:

**Theorem 1.1.** There are covariant functors
\[
\begin{align*}
U^\otimes & : \text{D-Lie} \to \text{Rings} \\
U^\rho & : \text{D-Lie} \to \text{Rings}
\end{align*}
\]
with the following property: For any D-Lie algebra \(\tilde{L}\) there are exact equivalences of categories
\[
\begin{align*}
F_1 : \text{Mod}(\tilde{L}, \text{Id}) & \cong \text{Mod}(U^\otimes(\tilde{L})) \\
F_2 : \text{Conn}(\tilde{L}, \text{Id}) & \cong \text{Mod}(U^\rho(\tilde{L}))
\end{align*}
\]
with the property that \(F_1\) and \(F_2\) preserves injective and projective objects.

We used the associative unital rings \(U^\otimes(\tilde{L})\) and \(U^\rho(\tilde{L})\) to define the cohomology and homology of an arbitrary \(L\)-connection \((E, \rho)\). By Lemma 3.19 in \([8]\) there is for any 2-cocycle \(f \in H^2(\text{Der}_k(A), A)\) an exact equivalence of categories
\[
\psi_f : \text{Conn}(L) \cong \text{Mod}(U^\otimes(L(f^\alpha)))
\]
preserving injective and projective objects. Here \(L\) is an \(A/k\)-Lie-Rinehart algebra and \(\text{Conn}(L)\) is the category of \(L\)-connections and morphisms of connections. Hence we may use the associative unital ring \(U^\otimes(L(f^\alpha))\) to define the cohomology and homology of any \(L\)-connection, flat or non-flat. Using \(\psi_f\) we get for any pair of \(L\)-connections \(V, W\) and any integer \(i \geq 0\) isomorphisms
\[
\text{Ext}_{U^\otimes(L(f^\alpha))}^i(\psi_f(V), \psi_f(W)) \cong \text{Ext}_{\text{Conn}(L)}^i(V, W).
\]
Hence the associative ring \(U^\otimes(L(f^\alpha))\) may be used to calculate the "true" Ext-groups of \(V\) and \(W\). Theorem 1.1 was one of the main reasons for the introduction of the notion D-Lie algebra.

Note: The category \(\text{Conn}(L)\) of connections on an \(A/k\)-Lie-Rinehart algebra \(L\) is a small abelian category, hence the Freyd-Mitchell full embedding theorem gives an equivalence \(\phi\) of \(\text{Conn}(L)\) with a sub category of \(\text{Mod}(R)\) where \(R\) is an associative ring. The equivalence \(\phi\) does not preserve injective and projective objects, hence we cannot use \(\phi\) to define the cohomology and homology of a connection. Theorem 1.1 gives a geometric construction of cohomology and homology groups of any \(\tilde{L}\) and \(L\)-connection where \(\tilde{L}\) is any D-Lie algebra and \(L\) is any \(A/k\)-Lie-Rinehart algebra. The construction is functorial and can be done for any scheme and any sheaf of Lie-Rinehart algebras.

The aim of this paper is to classify maps of D-Lie algebras. We also classify D-Lie algebras and non-abelian extensions of D-Lie algebras with projective canonical quotient in terms of a functor \(F_{f^\alpha}\) introduced in \([8]\).
Let \((L, \alpha)\) and \((L', \alpha')\) be \(A/k\)-Lie-Rinehart algebras and let \(f, g \in \mathbb{Z}^2(\text{Der}_k(A), A)\) be two 2-cocycles. Let \(F_f(L) := L(g^\alpha)\) and \(F_f(L') := L'(f^\alpha)\) be the \(D\)-Lie algebras introduced in [8], Theorem 2.7. In Theorem 2.3 we prove the following:

\textbf{Theorem 1.2.} There is a map of \(D\)-Lie algebras \(\phi : L(g^\alpha) \to L'(f^\alpha)\) if and only if \(\overline{g}^\alpha = f^\alpha\) in \(\alpha^* H^2(\text{Der}_k(A), A)\). In this case there is an equality between the set of maps of \(D\)-Lie algebras \(\phi : L(g^\alpha) \to L'(f^\alpha)\) and the set of maps of \(A/k\)-Lie-Rinehart algebras \(\psi : L \to L'\).

In Theorem 3.9 we prove the following:

\textbf{Theorem 1.3.} Let \((L, \tilde{\alpha}, \tilde{\pi}, [\cdot, \cdot], D)\) be a \(D\)-Lie algebra and let \((L, \alpha)\) be the canonical quotient \(A/k\)-Lie-Rinehart algebra of \(\tilde{L}\). Assume \(L\) is projective as left \(A\)-module. There is an isomorphism \(\tilde{L} \cong L(f^\alpha)\) as \(D\)-Lie algebras where \(L(f^\alpha) := F_f(L)\) and \(F_f\) is the functor from Example 4.3. Hence \(\tilde{L}\) is uniquely determined up to isomorphism by the canonical quotient \((L, \alpha)\) and the 2-cocycle \(f \in H^2(\text{Der}_k(A), A)\).

In Corollary 4.10 we classify maps of \(D\)-Lie algebras between two \(D\)-Lie algebras \(\tilde{L}_1\) and \(\tilde{L}_2\).

In Corollary 4.20 we get a classification of non-abelian extensions of \(D\)-Lie algebras in terms of the \(D\)-Lie algebras using the module of principal parts. This gives an equivalent definition of a \(D\)-Lie algebra to the one introduced in [8]. We prove in Theorem 4.19 that for any \(\tilde{L}\)-connection \((E, \rho)\) there is an extension of \(D\)-Lie algebras

\[(1.3.1) \quad 0 \to \text{End}_A(E) \to \text{End}((\tilde{L}, E) \to \tilde{L} \to 0)\]

with the property that \((1.3.1)\) splits in the category of \(D\)-Lie algebras if and only if \(E\) has a flat \(\tilde{L}\)-connection \(\rho' : \tilde{L} \to \text{Diff}^1(E)\). Theorem 4.19 generalize a similar result proved in [11] for non-abelian extensions of \(A/k\)-Lie-Rinehart algebras.

Using the \(P\)-module structure on the \(D\)-Lie algebra \(\tilde{L}\) we define in Definition 4.13 the correspondence \(Z(\rho, \sigma_1, \sigma_2)\) associated to a connection and degeneracy locies \(\sigma_1\) and \(\sigma_2\). The correspondence \(Z(\rho, \sigma_1, \sigma_2)\) induce an endomorphism

\[
I(\rho, \sigma_1, \sigma_2) : \text{CH}^\ast(X) \to \text{CH}^\ast(X)
\]

of the Chow-group \(\text{CH}^\ast(X)\). The correspondence \(Z(\rho, \sigma_1, \sigma_2)\) and Chow-operator \(I(\rho, \sigma_1, \sigma_2)\) depend in a non-trivial way on the \(P\)-module structure of \(\tilde{L}\) and \(P\)-linearity of the \(\tilde{L}\)-connection \(\rho\) and cannot be defined for an ordinary \(L\)-connection where \((L, \alpha)\) is an \(A/k\)-Lie-Rinehart algebra (see Example 4.3). There is no non-trivial right \(A\)-module structure on \(L\).

We also prove in Theorem 4.13 the following result relating the category of connections on an \(A/k\)-Lie-Rinehart algebra \(L\) to the category of connections on the \(D\)-Lie algebra \(L(f^\alpha)\):

\textbf{Theorem 1.4.} Let \((L, \alpha)\) be an \(A/k\)-Lie-Rinehart algebra and let \(f \in \mathbb{Z}^2(\text{Der}_k(A), A)\) be a 2-cocycle. There is an equivalence of categories

\[
C_f : \text{Conn}(L, \text{End}) \to \text{Conn}(L(f^\alpha))
\]

from the category \(\text{Conn}(L, \text{End})\) of \((L, \psi)\)-connections \(\nabla\), to the category \(\text{Conn}(L(f^\alpha))\) of \(L(f^\alpha)\)-connections \(\rho\). Let \(u := az + x, v := bz + y \in L(f^\alpha)\) and let \(e \in E\). Let
Let \((E, \nabla)\) be an \(L\)-connection and let \(\rho_\nabla := C_f(\nabla)\). The following holds:
\[
R_{\rho_\nabla}(u, v)(e) = R_{\nabla \alpha}(x, y)(e) - f'^\alpha(u, v)e.
\]
For any \(L(f^\alpha)\)-connection \((E, \rho)\) there is an \((L, \psi)\)-connection \((E, \nabla)\) with \(C_f(E, \nabla) = (E, \rho)\).

Hence there is for any 2-cocycle \(f \in Z^2(Der_k(A), A)\) a functorial way to define an \(L(f^\alpha)\)-connection \(C_f(E, \nabla)\) from an \(L\)-connection \((L, \alpha)\). As a Corollary we are able to classify \(\hat{L}\)-connections on \(D\)-Lie algebras with projective canonical quotient. In Corollary 1.16 we prove the following:

**Corollary 1.5.** Let \((\hat{L}, \bar{\alpha}, \bar{\nabla}, [\cdot, \cdot], D)\) be a \(D\)-Lie algebra and let \((E, \rho)\) be an \(\hat{L}\)-connection. Assume the canonical quotient \(L\) of \(\hat{L}\) is a projective \(A\)-module. It follows any \(\hat{L}\)-connection \((E, \rho)\) is on the form \(C_f(E, \nabla)\) where \(f \in Z^2(Der_k(A), A)\) and \((E, \nabla)\) is an \((L, \psi)\)-connection for \(\psi \in \text{End}_A(E)\). If \(\rho(D) = I\) it follows we may choose \((E, \nabla)\) to be an \(L\)-connection.

### 2. Classification of morphisms of \(D\)-Lie algebras

Let in the following \(A\) be a fixed commutative \(k\)-algebra where \(k\) is a fixed commutative unital ring. Let \(f, g \in Z^2(Der_k(A), A)\) be two 2-cocycles and consider the two \(k\)-Lie algebras \(D^1(A, f)\) and \(D^1(A, g)\). Let \(P := A \otimes_k A/I^2\) be the first order module of principal parts of \(A/k\). The \(k\)-Lie algebra \(D^1(A, f)\) has the following left and right \(A\)-module structure:

\[
(c, x) := (ca, cx)
\]
and
\[
(a, x)c := (ac + x(c), cx)
\]
for \((a, x) \in D^1(A, f)\) and \(c \in A\). Let \(z_f := (1, 0) \in D^1(A, f)\). It follows \(u := (a, x)\) that \(uc = c(u)z_f = cu + \pi_f(u)(c)z_f\). Similarly for \(D^1(A, g)\). Let \(d : A \to A \otimes_k A\) be defined by \(dc := 1 \otimes c - c \otimes 1\). It follows \(dc(u) = uc - cu = \pi_f(u)(c)z\). One checks that \(z_c = cz\) hence \(dcbd(u) = db(\pi_f(u)(c))z = 0\) hence \(D^1(A, f)\) is annihilated by \(I^2 \subseteq A \otimes_k A\) and it follows \(D^1(A, f)\) is a left \(P\)-module. The two projection maps \(\pi_f : D^1(A, f) \to Der_k(A)\) and \(\pi_g : D^1(A, g) \to Der_k(A)\) and \(\pi_f, \pi_g\) are maps of \(P\)-modules and \(k\)-Lie algebras. We want to classify maps

\[
\phi : D^1(A, g) \to D^1(A, f)
\]
of \(P\)-modules and \(k\)-Lie algebras such that

\[
(2.0.1) \quad \pi_f \circ \phi = \pi_g \quad \text{and} \quad \phi(z_g) = z_f.
\]

**Lemma 2.1.** Let \(\phi : D^1(A, g) \to D^1(A, f)\) be a map of \(P\)-modules and \(k\)-Lie algebras satisfying condition \(2.0.1\). It follows \(\phi(a, x) = (a + \phi_1(x), x)\) where \(\phi_1 \in C^1(Der_k(A), A)\) and \(g = f + d^1(\phi_1)\) and \(d^1 : C^1(Der_k(A), A) \to C^2(Der_k(A), A)\) is the first differential in the Lie-Rinehart complex of \(Der_k(A)\). Hence the map \(\phi\) exists if and only if \(\overline{\phi} = \overline{f} \in H^2(Der_k(A), A)\). If the map \(\phi\) exists it is an isomorphism of \(P\)-modules and \(k\)-Lie algebras with inverse \(\psi : D^1(A, f) \to D^1(A, g)\) defined by \(\psi(a, x) := (a - \phi_1(x), x)\).
Proof. Let \( \phi : D^1(A, g) \rightarrow D^1(A, f) \) be a map of \( P \)-modules and \( k \)-Lie algebras satisfying (2.0.1). It follows for any \((a, x) \in D^1(A, g)\) we get

\[
\phi(a, x) = \phi(a, 0) + \phi(0, x) = a z_f + (\phi_1(x), x) = (a + \phi_1(x), x)
\]

with \( \phi_1 \in C^1(Der_k(A), A) \). One checks the map \( \phi \) is a map of \( k \)-Lie algebras if and only if \( g = f + d^1(\phi_1) \). The map \( \phi \) in (2.1.1) is always a map of \( P \)-modules. Hence the map \( \phi \) exists if and only if \( \overline{f} = \overline{g} \in H^2(Der_k(A), A) \). If \( \phi \) exists, an inverse \( \psi \) is given by \( \psi(a, x) := (a - \phi_1(x), x) \). The Lemma follows. \( \square \)

Assume \((L, \alpha)\) is an \( A/k \)-Lie-Rinehart algebra and let \( f, g \in Z^2(Der_k(A), A) \) be two 2-cocycles and let \( f^\alpha, g^\alpha \in Z^2(L, A) \) be the pull-back cocycles. Let \( F_g(L) := L(g^\alpha) := A z_g \oplus L \) be the \( A/k \)-Lie-Rinehart algebra defined in [8]. It has the following \( k \)-Lie product:

\[
[(a, x), (b, y)] := (\alpha(a)(b) - \alpha(y)(a) + g^\alpha(x, y), [x, y])
\]

for \((a, x), (b, y) \in L(g^\alpha)\). There is a map

\[
\pi_g : L(g^\alpha) \rightarrow \text{Der}_k(A)
\]

defined by

\[
\pi_g(a, x) := \alpha(x).
\]

The pair \((L(g^\alpha), \pi_g)\) is an \( A/k \)-Lie-Rinehart algebra.

Note: The map \( \alpha \) induce a map of cohomology groups

\[
\alpha^* : H^2(Der_k(A), A) \rightarrow H^2(L, A).
\]

Let \( \text{Im}(\alpha^*) := \alpha^* H^2(Der_k(A), A) \) denote the image of the map \( \alpha^* \) in \( H^2(L, A) \).

**Lemma 2.2.** There is a map \( \alpha_g : L(g^\alpha) \rightarrow D^1(A, f) \) of \( P \)-modules and \( k \)-Lie algebras with \( \pi \circ \alpha_g = \pi_g \) if and only if there is an element \( \alpha_1 \in C^1(L, A) \) with

\[
g^\alpha = f^\alpha + d^1(\alpha_1).
\]

The map \( \alpha_g \) is defined as follows: \( \alpha_g(a, x) := (a + \alpha_1(x), \alpha(x)) \in D^1(A, f) \). Hence there exists a 5-tuple \((L(g^\alpha), \alpha_g, \pi_g, [\cdot, \cdot], z_g)\) which is a \( D \)-Lie algebra if and only if there is an equality of cohomology classes

\[
\overline{g^\alpha} = \overline{f^\alpha} \in \alpha^* H^2(Der_k(A), A) \subseteq H^2(L, A).
\]

**Proof.** By definition the map \( \alpha_g \) must look as follows:

\[
\alpha_g(a, x) = (a + \alpha_1(x), \alpha(x))
\]

where \((a, x) \in L(g^\alpha)\) and \( \alpha_1 \in C^1(L, A) \). One checks the map \( \alpha_g \) is a map of left \( P \)-modules. The map \( \alpha_g \) is a map of \( k \)-Lie algebras if and only if \( g^\alpha = f^\alpha + d^1(\alpha_1) \). Hence the 5-tuple \((L(g^\alpha), \alpha_g, \pi_g, [\cdot, \cdot], z_g)\) is a \( D \)-Lie algebra over \( D^1(A, f) \) if and only if \( \overline{g^\alpha} = \overline{f^\alpha} \) in \( \alpha^* H^2(Der_k(A), A) \). The Lemma follows. \( \square \)

We may now classify maps between arbitrary \( D \)-Lie algebras. Let \((L, \alpha)\) and \((L', \alpha')\) be \( A/k \)-Lie-Rinehart algebras and let \( f, g \in Z^2(Der_k(A), A) \) with \( g^\alpha = f^\alpha + d^1(\alpha_1) \) for \( \alpha_1 \in C^1(L, A) \). Hence there is by Lemma 2.2 a structure as \( D \)-Lie algebra \( \alpha_g : L(g^\alpha) \rightarrow D^1(A, f) \) given by the map \( \alpha_g(a, x) := (a + \alpha_1(x), \alpha(x)) \). The 5-tuple \((L(g^\alpha), \alpha_g, \pi_g, [\cdot, \cdot], z_g)\) is a \( D \)-Lie algebra over \( D^1(A, f) \) with \( \pi_g(a, x) := \alpha(x) \in \text{Der}_k(A) \).

Let \((L, \alpha)\) and \((L', \alpha')\) be \( A/k \)-Lie-Rinehart algebras and let \( f, g \in Z^2(Der_k(A), A) \) be two 2-cocycles. Let \( F_g(L) := L(g^\alpha) \) and \( F_f(L') := L'(f^\alpha) \) be the \( D \)-Lie algebras introduced in [8], Theorem 2.7.
Theorem 2.3. There is a map of D-Lie algebras \( \phi : L(g^a) \to L'(f^a) \) if and only if \( \overline{g^a} = \overline{f^a} \) in \( \alpha^a H^2(\text{Der}_k(A), A) \). In this case there is an equality between the set of maps of D-Lie algebras \( \phi : L(g^a) \to L'(f^a) \) and the set of maps of \( A/k \)-Lie-Rinehart algebras \( \psi_2 : L \to L' \).

Proof. If there is an equality \( \overline{g^a} = \overline{f^a} \) it follows there is a map

\[
\alpha_g : L(g^a) \to D^1(A, f)
\]

defined by

\[
\alpha_g(a, x) = (a + \alpha_1(x), \alpha(x)).
\]

The map \( \phi : L(g^a) \to L'(f^a) \) must look as follows:

\[
\phi(a, x) = (a + \phi_1(x), \phi_2(x))
\]

where \( \phi_1 : L \to A \) and \( \phi_2 : L \to L' \). Since \( \alpha'_1 \circ \phi = \alpha_g \) it follows \( \phi_1 = \alpha_1 \) and \( \alpha' \circ \phi_2 = \alpha \). Hence \( \phi_2 \) is a map of \( A/k \)-Lie-Rinehart algebras. One checks the map \( \phi \) is \( P \)-linear and a map of \( k \)-Lie algebras. Conversely, for an arbitrary map \( \phi_2 : L \to L' \) of \( A/k \)-Lie-Rinehart algebras it follows the map \( \phi(a, x) := (a + \alpha_1(x), \phi_2(x)) \) is a map of D-Lie algebras. The Theorem follows. \( \square \)

3. Classification of D-Lie algebras with projective canonical quotient

In this section we define the notion of a D-Lie algebra, the category of D-Lie algebras and connections on D-Lie algebras.

The module \( \text{Diff}^1(A) \) of first order differential operators on a commutative \( k \)-algebra \( A \) is in a canonical way a \( k \)-Lie algebra and a left \( P \)-module where \( P := A \otimes_k A / I^2 \) is the module of first order principal parts on \( A/k \). By definition \( \text{Diff}^1(A) := \text{Hom}_A(P, A) \) and it follows \( \text{Diff}^1(A) \) is in a canonical way a left and right \( A \)-module. There is a canonical projection map \( \pi : \text{Diff}^1(A) \to \text{Der}_k(A) \) which is a map of \( k \)-Lie algebras and left \( P \)-modules. There is a canonical inclusion of left and right \( A \)-modules and \( k \)-Lie algebras \( \text{Diff}^1(A) \subseteq \text{End}_k(A) \) and the Lie product \( [,] \) on \( \text{Diff}^1(A) \) satisfies the following formula:

\[
[u, cv] = c[u, v] + \pi(u)(c)v
\]

for all \( u, v \in \text{Diff}^1(A) \) and \( c \in A \). There is a canonical element \( D \in \text{Diff}^1(A) \) with the property that

\[
uc = cu + \pi(u)(c)D.
\]

The element \( D \) is central in \( \text{Diff}^1(A) \) and \( \pi(D) = 0 \). A D-Lie algebra \( \tilde{L} \) is a generalization of \( \text{Diff}^1(A) \) and we may speak the category of D-Lie algebras, extensions and non-abelian extensions of D-Lie algebras, cohomology of D-Lie algebras, connections etc. Equation \( 3.0.1 \) is the equation defining an \( A/k \)-Lie-Rinehart algebra and any D-Lie algebra \( \tilde{L} \) has an underlying \( A/k \)-Lie-Rinehart algebra. Hence we may view a D-Lie algebra as a refinement of the notion of a \( A/k \)-Lie-Rinehart algebra. A D-Lie algebra is an \( A/k \)-Lie-Rinehart algebra with extra structure defined by the \( P \)-module structure, the canonical element \( D \) and Equations \( 3.0.1 \) and \( 3.0.2 \).

Note: We may consider the \( A/k \)-Lie-Rinehart algebra \( \text{Der}_k(A) \subseteq \text{Diff}^1(A) \) and \( \text{Der}_k(A) \) has a canonical structure as \( k \)-Lie algebra and left \( A \)-module. It has no non-trivial right \( A \)-module structure. To get a non-trivial right \( A \)-module structure, we must consider the abelian extension \( \text{Diff}^1(A) \) and this is one of the motivations.
The following holds for all \( u,v \), the map
\[
P := \text{End}(\tilde{A}/k)
\]
is projective as left \( \tilde{A} \)-module. Let \( P := 1 \in D^1(A,f) \) and define
\[
[u,v] := (x(a) - y(b) + f(x, y), [x, y]) \in D^1(A,f).
\]
Define for any element \( c \in A \)
\[
cu := (ca, cx)
\]
and
\[
uc := (ac + x(c), cx).
\]
It follows \( D^1(A,f) \) is a left \( A \otimes_k k \)-module. Define the map \( \pi : D^1(A,f) \to \text{Der}_k(A) \)
by \( \pi(u) := (a, x) = x \). Define for \( x \in \text{Der}_k(A) \) \( xc := cx \). It follows \( \text{Der}_k(A) \) is a left \( A \otimes_k k \)-module. Let \( P := A \otimes_k A/I^2 \) where \( I \) is the kernel of the multiplication map. It follows \( P \) is the first order module of principal parts of \( A/k \). Let \( d : A \to P \) be the universal derivation. Let \( p(a) := 1 \otimes a \) and \( q(a) := a \otimes 1 \). It follows \( d = p - q \).

Let \( D := (1,0) \in D^1(A,f) \).

**Lemma 3.1.** The \( k \)-Lie algebra \( D^1(A,f) \) is in a canonical way a left \( P \)-module.

The map \( \pi : D^1(A,f) \to \text{Der}_k(A) \) is a map of \( k \)-Lie algebras and left \( P \)-modules.

The following holds for all \( u, v \in D^1(A,f) \) and \( c \in A \):

\[
\begin{align*}
(u,v) &= c[u,v] + \pi(u)(c)v \\
dc.u &= \pi(u)(c)D
\end{align*}
\]

The element \( D \) is a central element with \( \pi(D) = 0 \).
Proof. Since $D^1(A,f)$ is a left and right $A$-module with $(au)b = a(ub)$ for all $u \in D^1(A,f)$ and $a,b \in A$ it follows $D^1(A,f)$ is a left $A \otimes_k A$-module. One checks that for any element $w \in I^2$ it follows $wu = 0$ hence $D^1(A,f)$ is a left $P$-module. The map $\pi$ is left and right $A$-linear. It follows $\pi$ is a map of $P$-modules and $k$-Lie algebras. One checks Equation 3.1.1 and 3.1.2 holds and the Lemma is proved. □

The notion of a $D$-Lie algebra is a generalization of the $k$-Lie algebra and left $P$-module $D^1(A,f)$ from Lemma 3.1.

Definition 3.2. Let $f \in Z^2(Der_k(A),A)$ be a 2-cocycle and let $D^1(A,f)$ be the $k$-Lie algebra and $P$-module defined in Lemma 3.1. A 5-tuple $(\tilde{L},\tilde{\alpha},\tilde{\pi},[\cdot,\cdot],D)$ is a $D$-Lie algebra if the following holds. $\tilde{L}$ is a $k$-Lie algebra and left $P$-module. The element $\tilde{\alpha}$ is a map

$$\tilde{\alpha} : \tilde{L} \rightarrow D^1(A,f)$$

of $P$-modules and $k$-Lie algebras. The element $\tilde{\pi}$ is a map

$$\tilde{\pi} : \tilde{L} \rightarrow \text{Der}_k(A)$$

of left $P$-modules and $k$-Lie algebras with $\tilde{\pi} = \pi \circ \tilde{\alpha}$. The $k$-Lie product satisfies

$$[u,av] = a[u,v] + \tilde{\pi}(a)v$$

for all $u,v \in \tilde{L}$ and $a \in A$. The following holds for all $a \in A$:

$$da.a.u = \tilde{\pi}(a)(a)D.$$ 

The element $D$ is a central element in $\tilde{L}$ with $\tilde{\pi}(D) = 0$. Given two $D$-Lie algebras $(\tilde{L},\tilde{\alpha},\tilde{\pi},[\cdot,\cdot],D)$ and $(\tilde{L}',\tilde{\alpha}',\tilde{\pi}',[\cdot,\cdot],D')$. A map of $D$-Lie algebras is a map

$$\phi : \tilde{L} \rightarrow \tilde{L}'$$

of left $P$-modules and $k$-Lie algebras with $\phi(D) = D'$ and $\tilde{\pi}' = \tilde{\pi} \circ \phi$. Let $D$-Lie denote the category of $D$-Lie algebras and maps. An ideal $I$ in a $D$-Lie algebra $\tilde{L}$ is a sub-$k$-Lie algebra and sub-$P$-module $I \subseteq \tilde{L}$ such that $[\tilde{L},I] \subseteq I$. A D-ideal $I$ is a sub-$k$-Lie algebra and sub-$P$-module $I \subseteq \tilde{L}$ with $[\tilde{L},I] \subseteq I$ and with $D \notin I$.

Note: One checks Definition 3.2 gives the same notion as Definition 2.3 in [5].

Example 3.3. Non-trivial examples: D-Lie algebras and $A/k$-Lie-Rinehart algebras.

It follows from Lemma 3.1 that the 5-tuple $(D^1(A,f),id,\pi,[\cdot,\cdot],D)$ is a $D$-Lie algebra for any 2-cocycle $f \in Z^2(Der_k(A),A)$.

Given a $D$-Lie algebra $(\tilde{L},\tilde{\alpha},\tilde{\pi},[\cdot,\cdot],D)$ it follows the left $A$-module $\tilde{L}$ and the map $\tilde{\pi} : \tilde{L} \rightarrow \text{Der}_k(A)$ is an $A/k$-Lie-Rinehart algebra.

Given any 2-cocycle $f \in Z^2(Der_k(A),A)$, let $(L,\alpha)$ be an $A/k$-Lie-Rinehart algebra and let $f^\alpha \in Z^2(L,A)$ be the pull back 2-cocycle. There is by Theorem 2.7 in [5] a functor

$$F_{f^\alpha} : LR(A/k) \rightarrow D \text{-Lie}$$

from the category $LR(A/k)$ of $A/k$-Lie-Rinehart algebras to the category $D$-Lie of $D$-Lie algebras, hence it is easy to give non-trivial examples of $D$-Lie algebras. When $H^2(Der_k(A),A) \neq 0$ we get for any $f \in Z^2(Der_k(A),A)$ a non-trivial functor $F_{f^\alpha}$. Since $F_{f^\alpha}(L)$ is independent of choice of representative for the class $f := f^\alpha \in H^2(L,A)$ the following holds: There is for any two representatives $f,f'$ for the class $c$ a canonical isomorphism $F_f(L) \cong F_{f'}(L)$ of extensions of $A/k$-Lie-Rinehart
algebras. Hence the two functors $F_I$ and $F_{I'}$ are equal up to isomorphism. Hence we get from Theorem 2.7 in [8] a well defined functor $F_c$ for any cohomology class $c := \overline{f} \in H^2(L, A)$:

$$F_c : LR(A/k) \rightarrow D\text{Lie}.$$ 

When the class $c$ is the zero class it follows the underlying $A/k$-Lie-Rinehart algebra of $F_c(L) := Az \oplus L$ is the trivial abelian extension of $L$ by $A$. Hence the cohomology group $H^2(L, A)$ parametrize a large class of non-trivial functors $F_c$.

**Lemma 3.4.** Let $\phi : \hat{L}_1 \rightarrow \hat{L}_2$ be a map of $D$-Lie algebras. It follows the kernel $ker(\phi)$ is a $D$-ideal in $\hat{L}_1$. Let $\hat{I} := I/I^2 \subseteq P$ be the module of differentials and let $\hat{L}$ be any $D$-Lie algebra. It follows $\hat{I}\hat{L} \subseteq \hat{L}$ is an ideal. Let $(\hat{L}, \hat{\alpha}, \hat{\pi}, [\,\cdot\,,\,\cdot\,], D)$ be a $D$-Lie algebra and let $J := \{aD : \text{ with } a \in A\} \subseteq \hat{L}$. It follows $J$ is an ideal. Let $L := \hat{L}/J$. There is an exact sequence of $P$-modules and $k$-Lie algebras

$$0 \rightarrow J \rightarrow \hat{L} \rightarrow L \rightarrow 0 \quad (3.4.1)$$

There is a canonical map of left $A$-modules and $k$-Lie algebras $\alpha_L : L \rightarrow \text{Der}_k(A)$ making $(L, \alpha_L)$ into an $A/k$-Lie-Rinehart algebra. The sequence (3.4.1) is an exact sequence of $A/k$-Lie-Rinehart algebras. The ideal $J$ is a free left $A$-module on the element $D$.

**Proof.** One checks that $ker(\phi)$ is a $P$-submodule of $\hat{L}_1$ and that $[\hat{L}_1, ker(\phi)] \subseteq \hat{I}$, since $D \notin ker(\phi)$ it follows $ker(\phi)$ is a $D$-ideal. The same holds for $\hat{I}$: $\hat{I}$ is a $P$-module and $[\hat{L}, \hat{I}] \subseteq \hat{I}$. For any element $u \in \hat{L}$ and $c \in A$ it follows $uc = cu + \hat{\pi}(u)(c)D$. It follows

$$[u, aD] = a[u, D] + \hat{\pi}(u)(a)D = \hat{\pi}(u)(a)D$$

since $D$ is central. It follows $[\hat{L}, J] \subseteq J$. Hence $[J, J] \subseteq J$ and $J$ is a $k$-Lie algebra. We get

$$dc. u = uc - cu = \hat{\pi}(u)(c)D$$

hence

$$dc. dc'. u = dc(\hat{\pi}(u)(c')D) = 0$$

since $Dc = cD$. Hence $J$ is a left $P$-module. It follows the sequence (3.4.1) is an exact sequence of $P$-modules and $k$-Lie algebras. If $u \in \hat{L}/J$ is an element with $u \in \hat{L}$ it follows $\overline{uc} - c\overline{u} = \hat{\pi}(u)(c)D := 0$. Hence $\overline{uc} = c\overline{u}$ in $\hat{L}/J$. Hence $\hat{L}/J$ is trivially a left $P$-module. Since $\hat{\pi}(aD) = a\hat{\pi}(D) = 0$ it follows we get a canonical map

$$\pi_L : \hat{L}/J \rightarrow \text{Der}_k(A)$$

and one checks $(\hat{L}/J, \pi_L)$ is an $A/k$-Lie-Rinehart algebra. The rest is clear and the proof is finished.

**Definition 3.5.** Let $(\hat{L}, \hat{\alpha}, \hat{\pi}, [\,\cdot\,,\,\cdot\,], D)$ be a $D$-Lie algebra. The quotient $A/k$-Lie-Rinehart algebra $(L, \pi_L)$ with $L := \hat{L}/J$ from Lemma 3.4 is the canonical quotient of $\hat{L}$.

**Example 3.6.** When the canonical quotient is projective.

Assume $L := \hat{L}/J$ where $J$ is the ideal defined in Lemma 3.4 and let $\pi_L : L \rightarrow \text{Der}_k(A)$ be the anchor map. Assume $L$ is projective as left $A$-module and let $s$
be a left $A$-linear section of the canonical projection map $p : \tilde{L} \to L$. Define for $u, v \in L$ the following map

$$\psi_a : \wedge^2 L \to J$$

by

(3.6.1) $$\psi(\overline{u}, \overline{v}) := [s(\overline{u}), s(\overline{v})] - s(\overline{u}, \overline{v}).$$

Define the map $\nabla_s : L \to \text{End}_k(J)$ by

$$\nabla_s(\overline{x})(x) := [s(\overline{x}), x].$$

**Lemma 3.7.** It follows $(J, \nabla_s)$ is a flat $L$-connection and $\psi_s \in Z^2(L, (J, \nabla_s))$ where $C^0(L, (J, \nabla_s))$ is the Lie-Rinehart complex of the flat connection $(J, \nabla_s)$. If $s'$ is another left $A$-linear splitting of $p$ it follows there is an equality of connections $\nabla_s = \nabla_{s'}$. If $\psi_{s'}$ is the $2$-cocycle associated to $s'$ it follows there is an element $\rho \in C^1(L, (J, \nabla_s))$ with $\psi_{s'} = \psi_s + d\psi_s(\rho)$. Hence there is an equality of cohomology classes

$$\overline{\psi_s} = \overline{\psi_{s'}} \in H^2(L, (J, \nabla_s)).$$

**Proof.** One checks that $\nabla_s$ is a flat $L$-connection on $J$ and that $\psi \in Z^2(L, (J, \nabla_s))$. Assume $s' = s + \rho$ where $\rho \in \text{Hom}_A(L, J)$. We get for any element $u \in L$ and $x \in J$ the following:

$$\nabla_{s'}(u)(x) := [s(u) + \rho(u), x] = \nabla_s(u)(x) + [\rho(u), x] = \nabla_s(u)(x)$$

since $J$ is an abelian $k$-Lie algebra. It moreover follows

$$\psi_{s'}(u, v) = [s(u) + \rho(u), s(v) + \rho(v)] - s([u, v]) = \rho([u, v]) = \psi_s(u, v) + d\psi_s(\rho)(u, v) + [\rho(u), \rho(v)]$$

since $J$ is an abelian Lie algebra and hence $[\rho(u), \rho(v)] = 0$ for all $u, v \in L$. The Lemma follows. \hfill $\square$

Define the map

$$\phi(-, -) : L \times A \to J$$

by

(3.7.1) $$\phi(\overline{u}, a) := s(\overline{u})a - s(\overline{a}).$$

Make the following definition: $J \oplus (\phi, \psi) L$ is the left $A$-module $J \oplus L$ with the following right $A$-module structure: Given $z := (u, x) \in J \oplus L$ and $c \in A$ define

$$zc := (x, \overline{c})c := (xc + \phi(\overline{u}, c), \overline{c}).$$

We get since $\overline{xc} = c\overline{x}$ the following:

$$\phi(\overline{u}, c) = s(\overline{u})c - s(\overline{xc}) = s(\overline{x})c - s(\overline{x})c = cs(\overline{x}) + \tilde{\pi}(s(\overline{u}))(c)D - cs(\overline{x}) = \tilde{\pi}(s(\overline{u}))(c)D \in J.$$

Hence

$$zc := cz + \tilde{\pi}(s(\overline{u}))(c)D \in J \oplus L.$$

Since $\tilde{\pi} = \pi_L \circ p$ and $p \circ s = Id$ it follows $\tilde{\pi}(s(\overline{u})) = \pi_L(p(s(\overline{u}))) = \pi_L(\overline{u})$. It follows

$$zc = cz + \pi_L(\overline{u}))(c)(D, 0) = cz + \pi_L(\overline{u}))(c)\overline{D}$$

where $\overline{D} := (D, 0)$.

Let $z := (x, \overline{u})$, $w := (y, \overline{v}) \in J \oplus L$ and define the following product:

(3.7.2) $$[z, w] := (\nabla_s(\overline{u})(y) - \nabla_s(\overline{v})(x) + \psi(\overline{u}, \overline{v}), [\overline{u}, \overline{v}]).$$
It follows the product $[\cdot,\cdot]$ is a $k$-Lie product on $J \oplus L$. Define the map

$$\rho : J \oplus (\nabla_s,\psi) \rightarrow \tilde{L}$$

by

$$(3.7.3) \quad \rho(x,\overline{v}) := x + s(\overline{v}).$$

It follows $\rho$ is an isomorphism of $P$-modules and $k$-Lie algebras. Define the map

$$\alpha_J : J \oplus L \rightarrow D^1(A,f)$$

by

$$\alpha_J := \tilde{\alpha} \circ \rho.$$ 

Define $\pi_J : J \oplus L \rightarrow \text{Der}_k(A)$ by $\pi_J(u,x) := \pi_L(x)$. Associated to a left $A$-linear splitting $s$ of $p : \tilde{L} \rightarrow L$ we get by Lemma 3.7 a unique flat connection $\nabla_s : L \rightarrow \text{End}_k(J)$, a unique cohomology class $\psi_s \in H^2(L,(J,\nabla_s))$ and a 5-tuple $(J \oplus L,\alpha_J,\pi_J,[\cdot,\cdot],\tilde{D})$.

**Proposition 3.8.** Let $(\tilde{L},\tilde{\alpha},\tilde{\pi},[,],D)$ be a $D$-Lie algebra and let $(L,\alpha)$ be the canonical quotient $A/k$-Lie-Rinehart algebra of $L$. Assume $L$ is projective as left $A$-module. The 5-tuple $(J \oplus L,\alpha_J,\pi_J,[\cdot,\cdot],\tilde{D})$ constructed above is a $D$-Lie algebra and there is an isomorphism $J \oplus L \cong \tilde{L}$ of $D$-Lie algebras.

**Proof.** The proof follows from the construction and calculations above. $\square$

**Theorem 3.9.** Let $(\tilde{L},\tilde{\alpha},\tilde{\pi},[,],D)$ be a $D$-Lie algebra and let $(L,\alpha)$ be the canonical quotient $A/k$-Lie-Rinehart algebra of $L$. Assume $L$ is projective as left $A$-module. There is an isomorphism $\tilde{L} \cong L(f^\alpha)$ as $D$-Lie algebras where $L(f^\alpha) := F_{f^\alpha}(L)$ and $F_{f^\alpha}$ is the functor from Example 3.3. Hence $\tilde{L}$ is uniquely determined by the canonical quotient $(L,\alpha)$ and the 2-cocycle $f \in H^2(\text{Der}_k(A),A)$.

**Proof.** Let $s$ be a left $A$-linear splitting of the canonical projection map $p : \tilde{L} \rightarrow L$. From Proposition 3.8 it follows the 5-tuple $(J \oplus L,\alpha_J,\pi_J,[\cdot,\cdot],\tilde{D})$ is a $D$-Lie algebra and there is by 3.7.3 construction an isomorphism of $D$-Lie algebras

$$\rho : J \oplus L \rightarrow \tilde{L}$$

defined by

$$\rho(aD,\overline{v}) := aD + s(\overline{v}) \in \tilde{L}.$$ 

Consider the map $\tilde{\alpha} : \tilde{L} \rightarrow D^1(A,f)$. It looks as follows: $\tilde{\alpha}(u) = \alpha_1(u)I + \tilde{\pi}(u) \in A \oplus \text{Der}_k(A)$ with $\alpha_1 \in \text{Hom}_A(\tilde{L},A)$

Let $z := (aD,\overline{v})$ and $z' := (bD,\overline{v})$. It follows

$$[\rho(z),\rho(z')] =$$

$$(\alpha(\overline{v})(b) - \alpha(\overline{v})(a) + f^\alpha(\overline{v},\overline{v}) + \alpha(\overline{v})(\alpha_1(s(\overline{v}))) - \alpha(\overline{v})(\alpha_1(s(\overline{v}))))I + [\alpha(\overline{v}),\alpha(\overline{v})].$$

The Lie product on $J \oplus L$ is defined as follows:

$$[(aD,\overline{v}),bD,\overline{v}]] := (\alpha(\overline{v})(b) - \alpha(\overline{v})(a) + g(\overline{v},\overline{v}))D + [\overline{v},\overline{v}] \in J \oplus L$$

with

$$\psi(\overline{v},\overline{v}) := [s(\overline{v}),s(\overline{v})] - s([\overline{v},\overline{v}]) = g(\overline{v},\overline{v})D \in J$$

from equation 3.6.1.

Here $g(\overline{v},\overline{v}) \in Z^2(L,A)$ is a 2-cocycle. We get

$$\rho([z,z']) =$$
Example 3.11. \( \square \)

Corollary is proved.

Associated to a 2-cocycle \( f \) \( G \) \( A/k \) central element, we get a canonical map of 2-Lie algebras from Theorem 3.9 that the set of maps of 2-Lie algebras between \( \tilde{D} \) of \( \phi \) map of 2-Lie algebras

Proof. By Theorem 3.9 there are isomorphisms \( \tilde{L}_i \cong L(f^\alpha) \) for \( i = 1, 2 \) with projective canonical quotients \( (L_i, \alpha_i) \) for \( i = 1, 2 \). There is an equality between the set of maps of 2-Lie algebras \( \phi : L_1 \to \tilde{L}_2 \) and the set of maps of \( A/k \)-Lie-Rinehart algebras \( \phi^* : L_1 \to L_2 \).

\[ (\alpha(\bar{v})(b) - \alpha(\bar{v})(a) + g(\bar{v}, \bar{v}) + \alpha_1(s(\bar{v}, \bar{v}))))I + \alpha(\bar{v}, \bar{v}) \in D^1(A, f). \]

It follows

\[ \rho([z, z']) = [\rho(z), \rho(z')] \]

if and only if

\[ f^\alpha(\bar{v}, \bar{v}) + \alpha(\bar{v})(\alpha_1 \circ s(\bar{v})) - \alpha(\bar{v})(\alpha_1 \circ s(\bar{v})) = g(\bar{v}, \bar{v}) + \alpha_1 \circ s(\bar{v}, \bar{v}) \]

hence \( \rho \) is a map of 2-Lie algebras if and only if

\[ g(\bar{v}, \bar{v}) = f^\alpha(\bar{v}, \bar{v}) + d^1_s(\alpha_1 \circ s)(\bar{v} \wedge \bar{v}) \]

and \( \alpha_1 \circ s \in \text{Hom}_A(L, \alpha) \). Hence \( \rho \) is a map of 2-Lie algebras if and only if there is an isomorphism of 2-Lie algebras

\[ L(f^\alpha) \cong J \oplus L. \]

It follows there is an isomorphism \( \tilde{L} \cong L(f^\alpha) \) of 2-Lie algebras and the Theorem follows. \( \square \)

Corollary 3.10. Let \( (\tilde{L}_i, \tilde{\alpha}_i, \tilde{\pi}_i, [\cdot, \cdot], D_i) \) be 2-Lie algebras for \( i = 1, 2 \) with projective canonical quotients \( (L_i, \alpha_i) \) for \( i = 1, 2 \). There is an equality between the set of maps of 2-Lie algebras \( \phi^* : L_1 \to \tilde{L}_2 \) and the set of maps of \( A/k \)-Lie-Rinehart algebras \( \phi^* : L_1 \to L_2 \).

Proof. By Theorem 3.9 there are isomorphisms \( \tilde{L}_i \cong L(f^\alpha) \) for \( i = 1, 2 \). It follows again from Theorem 3.9 that the set of maps of 2-Lie algebras between \( \tilde{L}_1 \) and \( \tilde{L}_2 \) equal the set of maps of \( A/k \)-Lie-Rinehart algebras between \( L_1 \) and \( L_2 \) and the Corollary is proved. \( \square \)

Example 3.11. The D-Lie algebra associated to an A/k-Lie-Rinehart algebra.

Let \( (L, \alpha) \) be an \( A/k \)-Lie-Rinehart algebra and let \( f^\alpha \in Z^2(L, \alpha) \) be the 2-cocycle associated to a 2-cocycle \( f \in Z^2(\text{Der}_k(A), \alpha) \). In S, Theorem 2.8 we constructed a functor

\[ F : LR(A/k) \to D^1(A, f) - \text{Lie} \]

by

\[ F(L, \alpha) := (L(f^\alpha), \alpha_f, \pi_f, [\cdot, \cdot], z) \]

where \( L(f^\alpha) := Az \oplus L \) is the abelian extension of \( L \) by the free rank one \( A \)-module on the symbol \( z \). Define for any D-Lie algebra \( (\tilde{L}, \tilde{\alpha}, \tilde{\pi}, [\cdot, \cdot], \tilde{D}) \) the following:

\[ G(\tilde{L}, \tilde{\pi}, [\cdot, \cdot], \tilde{D}) := (L, \pi_L) \]

where \( (L, \pi_L) \) is the canonical quotient of \( \tilde{L} \). Since any map of D-Lie algebras \( \phi : L \to \tilde{L} \) satisfies \( \phi(D) = D' \) where \( D' \in \tilde{L}' \) is the canonical central element, we get a canonical map of \( A/k \)-Lie-Rinehart algebras

\[ G(\phi) : (L, \pi_L) \to (L', \pi_{L'}). \]

where \( (L', \pi_{L'}) \) is the canonical quotient of \( \tilde{L}' \). Hence we get a functor

\[ G : D^1(A, f) - \text{Lie} \to LR(A/k). \]

Theorem 3.9 says that in the case when the canonical quotient \( L \) of \( \tilde{L} \) is a projective \( A \)-module, it follows \( G(F(L, \alpha)) \cong (L, \alpha) \) and \( F(G(L)) \cong L \) are isomorphisms.
4. Classification of connections on D-Lie algebras with projective canonical quotient

In this section we classify connections on a D-Lie algebra \( \hat{L} \) with projective canonical quotient \((L, \alpha)\). We prove in Theorem 4.15 there is a 2-cocycle \( f \in Z^2(\text{Der}_k(A), A) \) and an equivalence of categories

\[(4.0.1) \quad C_f : \text{Conn}(L, \text{End}) \cong \text{Conn}(\hat{L}). \]

We use the equivalence \( C_f \) in 4.0.1 to classify arbitrary \( \hat{L} \)-connections in Corollary 4.16.

We also introduce the correspondence and Chow-operator of an \( \hat{L} \)-connection \((E, \rho)\).

**Lemma 4.1.** Let \( E \) be a left \( A \)-module and let \( \text{Diff}^1(E) \) be the module of first order differential operators on \( E \). It follows \( \text{Diff}^1(E) \) is a left \( P \)-module and \( k \)-Lie algebra. There is a map \( \psi : \text{Diff}^1(A) \times A \to \text{End}_A(E) \) defined by

\[ \psi(\partial, a) := [\partial, aI] := \partial \circ aI - aI \circ \partial \]

where \( I \) is the identity operator. It follows \( \psi(\partial, ab) = a\psi(\partial, b) + \psi(\partial, a)b \) for all \( a,b \in A \) and \( \partial \in \text{Diff}^1(E) \).

**Proof.** A differential operator \( \partial \in \text{Diff}^1(E) \) is by definition an operator \( \partial \in \text{End}_k(E) \) with \([\partial, aI]bI = 0\) for all elements \( a, b \in A \) where \( I \) is the identity operator. The module of differential operators \( \text{Diff}^1(E) \) has a left \( A \otimes A \)-module structure defined by \((a \otimes b, \partial)(e) := a\partial(be)\). It follows

\[ da.\partial := \partial \circ aI - aI \circ \partial := [\partial, aI]. \]

It follows \( da.db.\partial := [[\partial, bI], aI] = 0 \) hence for any element \( w \in I^2 \) it follows \( w\partial = 0 \) and it follows \( \text{Diff}^1(E) \) is a left \( P \)-module. One checks the product

\[ [\partial, \partial'] := \partial \circ \partial' - \partial' \circ \partial \]

defines a \( k \)-Lie algebra structure on \( \text{Diff}^1(E) \). The rest is trivial and the Lemma follows. \( \square \)

**Definition 4.2.** Let \((\hat{L}, \hat{\alpha}, \hat{\pi}, [,], D)\) be a D-Lie algebra and let \( E \) be a left \( A \)-module. An \( \hat{L} \)-connection \( \rho \) is a map

\[ \rho : \hat{L} \to \text{Diff}^1(E) \]

of left \( P \)-modules. The **curvature** of a connection \((\rho, E)\) is the map

\[ R_{\rho} : \hat{L} \times \hat{L} \to \text{End}_k(E) \]

defined by

\[ R_{\rho}(u, v) := [\rho(u), \rho(v)] - \rho([u, v]). \]

Given two \( \hat{L} \)-connections \((E, \rho_E)\) and \((F, \rho_F)\), a **map of \( \hat{L} \)-connections** is a map of left \( A \)-modules

\[ \phi : E \to F \]

with \( \rho_F(u) \circ \phi = \phi \circ \rho_E(u) \) for all \( u \in \hat{L} \). Let \( \text{Conn}^{\hat{L}} \) denote the category of \( \hat{L} \)-connections and maps of connections. Let \( \text{Conn}(\hat{L}, \text{Id}) \) denote the category of \( \hat{L} \) connections \((\rho, E)\) with \( \rho(D) = \text{Id}_E \in \text{End}_A(E) \).
Note: It follows $\text{Conn}(\tilde{L}, \text{Id})$ is a full sub category of $\text{Conn}(\tilde{L})$.

Note: A connection $\rho : \tilde{L} \to \text{Diff}^1(E)$ in $\text{Conn}(\tilde{L}, \text{Id})$ is in particular an $A \otimes_k A$-linear map with $\rho(D) = \text{Id}_E$.

**Example 4.3.** The degeneracy loci and correspondence associated to a connection.

A connection in the sense of Definition 4.2 is a map of left and right $A$-modules $\rho : \tilde{L} \to \text{Diff}^1(E)$ and we may associate to $\rho$ several types of correspondences.

**Definition 4.4.** Given a connection $\rho \in \text{Hom}_P(\tilde{L}, \text{Diff}^1(E))$. Let $I(\rho) \subseteq P := A \otimes_k A/I^2$ be the annihilator ideal of the element $\rho$. Let $Z_P(\rho) := \text{V}(I(\rho)) \subseteq \text{Spec}(P)$ be the correspondence $\rho$.

By definition $I(\rho)$ is the set of elements in $x \in P$ with $x\rho = 0$. The ideal $I(\rho)$ gives rise to an ideal $J(\rho) \subseteq A \otimes_k A$ containing the square of the diagonal $I^2$. We get in a canonical way a correspondence $Z(\rho) := \text{V}(J(\rho)) \subseteq \text{Spec}(A \otimes_k A) := X \times X$ where $X := \text{Spec}(A)$. Hence the connection $\rho$ gives in a canonical way rise to a correspondence $Z(\rho)$ on $X$.

The left $P$-module $\text{Diff}^1(E)$ is projective as left and right $A$-module when $A$ is a regular ring of finite type over a field and $E$ a finite rank projective $A$-module. Hence when $\tilde{L}$ is projective as left and right $A$-module it follows a connection $\rho : \tilde{L} \to \text{Diff}^1(E)$ is a map of left and right $A$-modules that are projective as left and right $A$-modules. Hence a connection is a geometric object and we may use $\rho$ to define a correspondence on $X \times X$. For a classical connection

\begin{equation}
\nabla : E \to E \otimes_A \Omega^1_{A/k}
\end{equation}

it is not immediate how to do this, since $\nabla$ is a map of $k$-vector spaces and not $A$-modules. We may view an ordinary connection as an $A$-linear map

\begin{equation}
\nabla : L \to \text{End}_k(A)
\end{equation}

where $L$ is an $A/k$-Lie-Rinehart algebra satisfying $\nabla(x)(ae) = a
abla(x)(e) + x(a)e$ and it is not immediate how to define a correspondence from $\nabla$.

We may associate to $\rho$ the degeneracy loci $D^l(\rho)$ and $D^r(\rho)$ where $l$ and $r$ refer to the degeneracy loci of $\rho$ as a left and right $A$-linear map. Here $D^l(\rho)$ and $D^r(\rho)$ are defined using local trivializations of the map $\rho$. Locally the map $\rho$ is a matrix $M$ with coefficients in a commutative ring $B$ and we may use minors of $M$ of a given size to define an ideal in $B$ associated to $\sigma$ as done in [2]. It is possible to do this in a way that is intrinsic and does not depend of the choice of local trivialization of the map $\rho$. We get for any two $\sigma_1, \sigma_2$ a correspondence $Z(\rho, \sigma_1, \sigma_2) := D^l(\rho) \times D^r(\rho) \subseteq X \times X$. Hence we may associate different types of correspondences to the connection $\rho$. If $X$ is smooth over $k$ we get for each connection $\rho$ and each correspondence $Z(\rho, \sigma_1, \sigma_2)$ an endomorphism

$I(\rho, \sigma_1, \sigma_2) : \text{CH}^*(X) \to \text{CH}^*(X)$

defined by

$I(\rho, \sigma_1, \sigma_2)(\alpha) := p_*(Z(\rho, \sigma_1, \sigma_2) \cap q^*(\alpha))$

where $p, q : X \times X \to X$ are the projection maps, $\text{CH}^*(X)$ is the Chow group of $X$ and $Z(\rho, \sigma_1, \sigma_2) \cap \alpha$ is the intersection product. We get a similar construction for
any reasonable cohomology theory $H(-)$ equipped with a cycle map. One would like to relate the correspondence $Z(\rho, \sigma_1, \sigma_2)$ and operation $I(\rho, \sigma_1, \sigma_2)$ to the Chern classes of $(E, \rho)$. Assume $\gamma : CH^*(X \times_k X) \to H^*(X \times_k X)$ is a cycle map and let $\alpha \in H^*(X)$ be a cohomology class. we get an operator

$$I_H(\rho, \sigma_1, \sigma_2) : H^*(X) \to H^*(X)$$

defined by

$$I_H(\rho, \sigma_1, \sigma_2)(\alpha) := p_*((Z(\rho, \sigma_1, \sigma_2)) \cap q^*(\alpha)).$$

**Example 4.5.** *Algebraic cycles and the Gauss-Manin connection.*

If $H^*(-)$ is a Weil cohomology theory, there are *Lefschetz operators*

$$\Lambda := (L^{n-i+2})^{-1} \circ L \circ (L^{n-i}) : H^i(X) \to H^{i-2}(X)$$
defined for any $i = 0, \ldots, \dim(X)$ and the operator $\Lambda$ is conjectured to be induced by an algebraic cycle $Z \subseteq X \times_k X$. The cycle $[Z]$ of the sub-scheme $Z$ induce an operator

$$I_H(Z) : H^*(X) \to H^*(X)$$
defined by

$$I_H(Z)(\alpha) := p_*([Z]) \cap q^*(\alpha)).$$

It has been conjectured that when $X \subseteq \mathbb{P}^n_k$ is a smooth projective variety over an algebraically closed field, the Lefschetz operator $\Lambda$ is induced by an operator on the form $I_H(Z)$ for some closed sub-scheme $Z \subseteq X \times_k X$. One wants to construct non-trivial cycle classes $\beta \in CH^*(X \times_k X)$ and calculate the operator $I_H(\beta)$. The Chow group $CH^*(X \times_k X)$ is hard to calculate and there are no general formulas for it. One also wants a construction of all Weil-cohomology theories $H(-)$. If one could realize a Weil cohomology theory $H^*(-)$ as the cohomology $H^*(\tilde{L}, -)$ of a D-Lie algebra $\tilde{L}$ as defined in $[8]$, Definition 3.23, one could approach conjectures on algebraic cycles for smooth projective families of varieties. One has to develop the formalism of the Gauss-Manin connection for the cohomology theory $H^*(\tilde{L}, -)$ in the language of D-Lie algebras. In previous papers (see $[2]$) I have developed a formalism aimed at making explicit calculations of such connections. If one could realize the action of $\Lambda$ on a Weil cohomology $H^*(X)$ of the total space $X$ of a smooth projective family $\pi : X \to S$, as the action of $\nabla(x)$ where $x$ is a vector field on $S$ and $\nabla$ the Gauss-Manin connection, this could be a first step in the direction of determining if $\Lambda$ is induced by an algebraic cycle. The vector field $x$ and the Gauss-Manin connection $\nabla$ are algebraic objects, hence we would get an algebraic construction of $\Lambda$.

**Definition 4.6.** Let $Z(\rho, \sigma_1, \sigma_2)$ be the correspondence of $\rho$ of type $(\sigma_1, \sigma_2)$. Let $I(\rho, \sigma_1, \sigma_2)$ be the Chow-operator of $\rho$ of type $(\sigma_1, \sigma_2)$.

When using the notion of a D-Lie algebra $\tilde{L}$, the derivation property of the connection $\rho : \tilde{L} \to Diff^1(E)$ is encoded in the right $A$-linearity of the map $\rho$. Hence the correspondence $Z(\rho, \sigma_1, \sigma_2)$ encodes properties of the left $A$-linearity of the map $\rho$ and the derivation property (the right $A$-linearity) of the map $\rho$. Hence the correspondence $Z(\rho, \sigma_1, \sigma_2)$ and the endomorphism $I(\rho, \sigma_1, \sigma_2)$ depends on the connection $\rho$ is a non-trivial way. It is not clear how to make a similar definition with connections on the form $[L4.1]$ and $[L4.2]$ depending in a non-trivial way on the derivation property of the connection. Hence in the case of an ordinary connection or a connection $(E, \nabla)$ on an $A/k$-Lie-Rinehart algebra $L$ it is essential we work
with the associated D-Lie algebra $F_I(L) := L(f^\alpha)$ and $L(f^\alpha)$-connection $C_f(E, \nabla)$ for some 2-cocycle $f \in Z^2(\text{Der}_k(A), A)$ if we want to define the correspondence and Chow-operator of $(E, \nabla)$.

**Example 4.7.** Restricted Lie-Rinehart algebras, logarithmic derivatives and Picard groups.

Let $B$ be a commutative ring over a field $k$ characteristic $p > 0$ and let $(L, \alpha)$ be a restricted $B/k$-Lie-Rinehart algebra. Let $A := \ker(L)$ be the kernel of $L$ in $B$. The ring $B$ is a purely inseparable Galois extension of $A$ if $B$ is a finitely generated and projective $A$-module and $B[L] = \text{Hom}_A(B, B)$ in the sense of $[17]$. Yuan proves in $[17]$ the existence of an exact sequence

$$(4.7.1) \quad 0 \to \text{Log}(B/A) \to \text{Pic}(A) \to \text{Pic}(B) \to \text{H}^2_{\text{res}}(L, B) \to \text{Br}(B/A) \to 0$$

where $\text{Log}(B/A)$ is the logarithmic derivative group of $B/A$, $\text{H}^2_{\text{res}}(L, B)$ is the restricted Lie-Rinehart cohomology group of $L$ with values in $B$, $\text{Br}(B/A)$ the Brauer group of $B/A$ and $\text{Pic}(B), \text{Pic}(A)$ the Picard groups of $B$ and $A$. Hence in characteristic $p > 0$, the restricted version of classical $B/k$-Lie-Rinehart cohomology calculates Picard groups and Brauer groups. One wants to generalize the exact sequence $\text{(4.7.1)}$ to the case of restricted D-Lie algebras.

**Example 4.8.** D-Lie algebras, groupoid schemes and algebraic stacks.

Recall the following from $[1]$, page 140: Let $T := (X, Y, s_1, s_2, t, p, i)$ be a groupoid scheme with $X$ the scheme of arrows and $Y$ the scheme of objects. We say $T$ is a schematic equivalence relation if the map $(s_1, s_2) : X \times X \to Y \times Y$ is a monomorphism. There is the following result:

**Proposition 4.9.** Assume $Y$ is smooth over a field $k$ of characteristic zero. There is a one-to-one correspondence between locally trivial sheaves of $\mathcal{O}_Y/k$-Lie-Rinehart algebras $(\mathcal{L}, \alpha)$ and formal infinitesimal groupoid schemes with $Y$ as a scheme of objects. Under this correspondence schematic equivalence relations corresponds to sub-sheaves of $\mathcal{O}_Y/k$-Lie-Rinehart algebras of the tangent sheaf $T_{Y/k}$.

**Proof.** See $[3]$, VIII.1.1.5. $\square$

Hence there is a close connection between $A/k$-Lie-Rinehart algebras, moduli spaces and algebraic stacks. If $\mathcal{F} \subseteq T_{Y/k}$ is a locally trivial finite rank sub $\mathcal{O}_Y$-module and sheaf of $k$-Lie algebras, and the corresponding equivalence relation $R \subseteq Y \times_k Y$ is an etale equivalence relation, we may form the stack quotient $[Y/\mathcal{F}]$. The quotient $[Y/\mathcal{F}]$ is a Deligne-Mumford stack. The sheaf of universal enveloping algebras $\mathcal{U}(\mathcal{O}_Y, \mathcal{F})$ is in a natural way a sheaf of rings on $[Y/\mathcal{F}]$ - the structure sheaf $\mathcal{O}_{[Y/\mathcal{F}]} := \mathcal{U}(\mathcal{O}_Y, \mathcal{F})$ of the stack $[Y/\mathcal{F}]$. A flat $\mathcal{F}$-connection $(\mathcal{E}, \nabla)$ which is a quasi coherent sheaf of $\mathcal{O}_Y$-modules, is in a natural way a quasi coherent sheaf of $\mathcal{O}_{[Y/\mathcal{F}]}$-modules. Any Deligne-Mumford stack $[Y/R]$ arise by Proposition $4.9$ from an involutive sub-bundle $\mathcal{F} \subseteq T_{Y/k}$. Hence if we are given the scheme of objects $Y$ of a DM-stack we may construct the equivalence relation $R$ on $Y$ using a bundle on the form of $\mathcal{F}$. The structure sheaf $\mathcal{O}_{[Y/R]}$ is a sheaf of filtered almost commutative rings, hence the “ringed space” $([Y/R], \mathcal{O}_{[Y/R]})$ may be viewed as a non-commutative ringed space. Hence connections arise naturally when studying algebraic stacks, quasi coherent sheaves on algebraic stacks and non-commutative ringed spaces.
Note: Associated to any D-Lie algebra \( \tilde{L} \) which is of finite rank and projective as left \( A \)-module, we get a formal groupoid scheme over \( Y := \text{Spec}(A) \) using the underlying \( A/k \)-Lie-Rinehart algebra of \( \tilde{L} \) and Proposition 4.9.

**Example 4.10.** L-functions and cristalline cohomology.

Let \( S := \text{Spec}(\mathcal{O}_K) \) where \( K \) is an algebraic number field and let \( f : X \rightarrow S \) be a regular scheme of finite type over \( S \) of dimension \( d \). There is an equality of L-functions

\[
L(X, s) = \prod_{p \neq (0)} L(X(p), s)
\]

where \( L(X, s) \) is the global L-function of the scheme \( X \) and \( L(X(p), s) \) is the L-function of the fiber \( X(p) := f^{-1}(p) \) for a closed point \( p \in S \). The residue field \( \kappa(p) \) is a finite field of characteristic \( q > 0 \) and the fiber \( X(p) \) is a scheme of finite type over \( \kappa(p) \). Hence there is an equality

\[
L(X(p), s) = Z(X(p), (1/q)^s)
\]

where \( Z(X(p), t) \) is the Weil zeta function of \( X(p) \). By the work of Kedlaya (see [6]) it follows the Weil conjectures can be proved using a p-adic cohomology theory \( H^*(\_\_) \). In fact Kedlaya has proved the Weil conjectures for an arbitrary scheme \( X \) of finite type over a finite field with no condition on smoothness or projectivity on \( X \). If the Ext-group \( \text{Ext}^*(V, W) \) of two connections \( V, W \) calculate cristalline cohomology (see [5]), it may be we can use \( \text{Ext}^*(V, W) \) to prove the Weil conjectures for any scheme \( X \) of finite type over a finite field. If the Weil zeta function \( Z(X(p), t) \) can be calculated using the Ext group \( \text{Ext}^*(V, W) \), it may be we can use such a description in the study of the global L-function \( L(X, s) \) via the product formula in 4.10.1. The Ext-group is defined in complete generality and may be defined for connections on the family \( X \). One has to calculate explicit examples to check if this idea leads to interesting constructions and results. If \( X \) is a regular scheme the following is conjectured in [12]:

\[
\chi(X, j) = \text{ord}_{s=j}(L(X, s)).
\]

Here

\[
\chi(X, j) := \sum_{m \geq 0} (-1)^{m+1} \dim_{\mathbb{Q}}(K_m(X)^{(d-j)})
\]

is the K-theoretic Euler characteristic of \( X \) as defined in [12]. The conjecture in 4.10.3 is referred to as a conjecture due to Lichtenbaum, Deligne, Bloch, Beilinson and others in Wiles official problem description [16] for the BSD-conjecture. If it can be proved the Ext-group \( \text{Ext}^*(V, W) \) calculate cristalline cohomology of the fibers \( X(p) \) for all primes \( p \neq (0) \), it might be the group \( \text{Ext}^*(V, W) \) can be used in the study of Conjecture 4.10.3.

**Lemma 4.11.** The following holds for an \( \tilde{L} \)-connection \( \rho : \tilde{L} \rightarrow \text{Diff}^1(E) \):

\[
\rho(u)(ae) = a\rho(u)(e) + \tilde{\pi}(u)(a)\rho(D)(e).
\]

for all elements \( a \in A, u \in \tilde{L} \) and \( e \in E \). It follows \( \rho(D) \in \text{End}_A(E) \). The curvature \( R_\rho \) defines a map

\[
R_\rho : \tilde{L} \times \tilde{L} \rightarrow \text{Diff}^1(E).
\]
Assume $P : \tilde{L} \to \text{End}_A(E)$ is an $P$-linear map. It follows $\rho' := \rho + P$ is an $\tilde{L}$-connection. If $P(D) = 0$ it follows $\rho'(D) = \text{Id}_E$.

Proof. Since $\rho$ is $P$-linear it follows $\rho$ is $A \otimes_k A$-linear. It follows $\rho(au) = a\rho(u)$ and $\rho(ua) = \rho(u)a$. We get

$$\rho(u)(ae) = \rho(ua)(e) = \rho(au + \tilde{\pi}(u)(a)D)(e) = \rho(au)(e) + \tilde{\pi}(u)(a)\rho(D).$$

Since $Da = aD$ it follows $\rho(D) \in \text{End}_A(E)$. The second statement holds since the element $[\rho(u), \rho(v)] := \rho(u)\rho(v) - \rho(v)\rho(u) \in \text{Diff}^1(E)$. Since $\rho$ is $P$-linear it follows $\rho' := \rho + P$ is $P$-linear. If $P(D) = 0$ it follows $\rho'(D) = \rho(D) + P(D) = \text{Id}_E$. The Lemma follows.

Hence the notion of an $\tilde{L}$-connection introduced in Definition 4.2 agrees with the notion introduced in the paper [3].

**Example 4.12.** The $L(f^\alpha)$-connection associated to an $(L, \psi)$-connection.

**Definition 4.13.** Let $(L, \alpha)$ an $A/k$-Lie-Rinehart algebra and let $\nabla : L \to \text{End}_k(E)$ be an $(L, \psi)$-connection where $\psi \in \text{End}_A(E)$. This means

$$\nabla(x)(ae) = a\nabla(x)(e) + \alpha(x)(a)\psi(e)$$

for all $a \in A, e \in E$ and $x \in L$. Let $\text{Conn}(L, \text{End})$ denote the category of $(L, \psi)$-connections and morphisms. The endomorphism $\psi$ may vary. Let $\text{Conn}(L)$ denote the category of ordinary $L$-connections and morphisms of $L$-connections.

Note: If $\nabla : L \to \text{End}_k(E)$ is an ordinary connection and $\psi \in \text{End}_A(E)$ it follows $\nabla \circ \psi$ is an $(L, \psi)$-connection.

Recall the following construction: Let $f \in Z^2(\text{Der}_k(A), A)$ be a 2-cocycle and let $F_f(L) := (L(f^\alpha), \alpha_f, \pi_f, [\cdot, \cdot], z)$ be the $D$-Lie algebra associated to $L$ and $f$. Define the following map:

$$\rho := \rho_{f \psi} : L(f^\alpha) \to \text{End}_k(E)$$

by

$$\rho(az + x) := a\psi + \nabla(x) \in \text{End}_k(E).$$

Let $C_f(E, \nabla) := (E, \rho_{f \psi})$.

Let $(F, \nabla')$ be an $(L, \psi')$-connection with $\psi' \in \text{End}_A(E)$ and let $\phi : (E, \nabla) \to (F, \nabla')$ be a map of connections. Define the following map

$$C_f(\phi) := \phi : E \to F.$$
and

\[ R_f : \text{Conn}(L(f^a)) \to \text{Conn}(L, \text{End}) \]

by

\[ R_f(F, \rho) := (F, \nabla_\rho). \]

It follows \( C_f \circ R_L = \text{Id} \) and \( R_f \circ C_f = \text{Id} \) hence \( C_f \) and \( R_f \) are equivalences of categories for any 2-cocycle \( f \in \mathbb{Z}^2(\text{Der}_k(A), A) \)

**Proof.** The proof follows immediately from the constructions above. \( \square \)

**Theorem 4.15.** Let \( (L, \alpha) \) be an \( A/k \)-Lie-Rinehart algebra and let \( f \in \mathbb{Z}^2(\text{Der}_k(A), A) \) be a 2-cocycle. Lemma 4.14 gives an equivalence of categories

\[ C_f : \text{Conn}(L, \text{End}) \to \text{Conn}(L(f^a)) \]

from the category \( \text{Conn}(L, \text{End}) \) of \((L, \psi)\)-connections, to the category \( \text{Conn}(L(f^a)) \) of \( L(f^a) \)-connections \( \rho \) for any 2-cocycle \( f \). Let \( u := az + x, v := bz + y \in L(f^a) \) and let \( e \in E \). Let \( (E, \nabla) \) be an \( L \)-connection and let \( \rho \psi := C_f(\nabla) \). The following holds:

\[ R_{\rho \psi}(u, v)(e) = R_{\psi \nabla}(x, y)(e) - f^a(u, v)e. \]

For any \( L(f^a) \)-connection \((E, \rho)\) there is an \((L, \psi)\)-connection \((E, \nabla)\) with \( C_f(E, \nabla) = (E, \rho) \). If \( \rho(z) = \text{Id} \) we may choose \((E, \nabla) \in \text{Conn}(L, \text{End})\).

**Proof.** One checks for any \((L, \psi)\)-connection \((E, \nabla)\) the corresponding map \( \rho \psi \) is a map

\[ \rho \psi : L(f^a) \to \text{Diff}^1(E) \]

of left \( \text{P} \)-modules. Moreover for any map of \((L, \psi)\)-connections \( \phi : (E, \nabla) \to (F, \nabla') \) it follows the map

\[ C_f(\phi) : C_f(E, \nabla) \to C_f(F, \nabla') \]

is a map of \( L(f^a) \)-connections with \( C_f(\phi \circ \psi) = C_f(\phi) \circ C_f(\psi) \). By definition \( C_f \circ R_f = \text{Id} \) and \( R_f \circ C_f = \text{Id} \) and the first claim follows. The statement on the curvature follows from Lemma 2.20 in [8]. In particular given an \( L(f^a) \)-connection \((E, \rho)\) it follows \((E, \rho) \cong C_f(R_f(E, \rho)) \) and \( R_f(E, \rho) \in \text{Conn}(L, \text{End}) \). The theorem follows. \( \square \)

We may classify \( \tilde{L} \)-connections in terms of \((L, \psi)\)-connections in the case when the canonical quotient \( L \) of \( \tilde{L} \) is a projective \( A \)-module.

**Corollary 4.16.** Let \((\tilde{L}, \tilde{\alpha}, \tilde{\pi}, [\cdot, \cdot], D)\) be a \( D \)-Lie algebra and let \((E, \rho)\) be an \( \tilde{L} \)-connection. Assume the canonical quotient \( L \) of \( \tilde{L} \) is a projective \( A \)-module. It follows any \( \tilde{L} \)-connection \((E, \rho)\) is on the form \( C_f(E, \nabla) \) where \( f \in \mathbb{Z}^2(\text{Der}_k(A), A) \) and \((E, \nabla)\) is an \((L, \psi)\)-connection for some \( \psi \in \text{End}_A(E) \). If \( \rho(D) = I \) it follows there is an \( L \)-connection \((E, \nabla)\) with \( C_f(E, \nabla) = (E, \rho) \).

**Proof.** By Theorem 4.15 there is an isomorphism \( \tilde{L} \cong L(f^a) \) for \( f \in \mathbb{Z}^2(\text{Der}_k(A), A) \). The corollary now follows from Theorem 4.15. \( \square \)

**Example 4.17.** Non-abelian extensions of \( D \)-Lie algebras.

**Definition 4.18.** Let \((\tilde{L}, \tilde{\alpha}, \tilde{\pi}, [\cdot, \cdot], D)\) be a \( D \)-Lie algebra and let \((\rho, E)\) be an \( \tilde{L} \)-connection with \( \rho(D) = \text{Id}_E \). Let \((\text{End}(\tilde{L}, E), \alpha_E, \pi_E, [\cdot, \cdot], \tilde{D})\) be the \( D \)-Lie algebra constructed in Section 4 in [8].
The following Theorem generalizes Theorem 2.14 from [9] from a connection on an $A/k$-Lie-Rinehart algebra to the case of a connection on a D-Lie algebra:

**Theorem 4.19.** Let $(\tilde{L}, \tilde{\alpha}, \tilde{\pi}, [\cdot,\cdot], D)$ be a D-Lie algebra and let $(\rho, E)$ be an $\tilde{L}$-connection with $\rho(D) = \text{Id}_E$. Let $\text{End}(\tilde{L}, E)$ be the non-abelian extension of $\tilde{L}$ with $\rho$ from Definition 4.18. There is a canonical flat connection

$$\rho^1 : \text{End}(\tilde{L}, E) \to \text{Diff}^1(E)$$

defined by $\rho^1(\phi, u) := \phi + \rho(u)$. The exact sequence

$$0 \to \text{End}_A(E) \to \text{End}(\tilde{L}, E) \to \tilde{L} \to 0$$

is split in the category of D-Lie algebras if and only if $E$ has a flat $\tilde{L}$-connection.

**Proof.** By Proposition 4.9 in [8] we get an extension of D-Lie algebras

$$0 \to \text{End}_A(E) \to \text{End}(\tilde{L}, E) \to \tilde{L} \to 0$$

where $\text{End}(\tilde{L}, E) := \text{End}_A(E) \oplus \tilde{L}$ with $\tilde{D} := (0, D)$ and $p_E : \text{End}(\tilde{L}, E) \to \tilde{L}$ defined by $p_E(\phi, u) := u \in \tilde{L}$. Define $c(\phi, u) := (c\phi, cu)$ and $(\phi, u)c := (\phi, uc)$ and define

$$\alpha_E : \text{End}(\tilde{L}, E) \to \text{D}^1(A, f)$$

by

$$\alpha_E(\phi, u) := \tilde{\alpha}(u) \in \text{D}^1(A, f)$$

and

$$\pi_E : \text{End}(\tilde{L}, E) \to \text{Der}_k(A)$$

by

$$\pi_E(\phi, u) := \tilde{\pi}(u).$$

Define moreover for any $z := (\phi, u), w := (\psi, v) \in \text{End}(\tilde{L}, E)$

$$[z, w] := ([\phi, \psi] + [\rho(u), \psi] - [\rho(v), \phi] + R_{\rho}(u, v), [u, v]).$$

It follows from Proposition 4.9 in [8] that $\text{End}(\tilde{L}, E)$ is an extension of $\tilde{L}$ by the $A$-Lie algebra $\text{End}_A(E)$. Assume $s : \tilde{L} \to \text{End}(\tilde{L}, E)$ is a section of the map $p_E$. Hence $p_E \circ s = \text{Id}_{\tilde{L}}$ and $s$ is a map of D-Lie algebras. It follows $s$ is $P$-linear, a map of $k$-Lie algebras and $s(D) = (0, D)$. It follows $s(u) = (P(u), u)$ where $P : \tilde{L} \to \text{End}_A(E)$ is a $P$-linear map with $P(D) = 0$. It follows from Lemma 4.11 the map $\rho' := \rho + P$ is an $\tilde{L}$-connection $\rho' : \tilde{L} \to \text{Diff}^1(E)$. One checks that the map $s$ is a map of $k$-Lie algebras if and only if $R_{\rho'} = 0$, hence the sequence 4.19.1 is split in the category of D-Lie algebras if and only if $E$ has a flat $\tilde{L}$-connection. The Theorem is proved. \hfill \Box

Note: In [9], Theorem 2.4 a result similar to Theorem 4.19 is proved for $A/k$-Lie-Rinehart algebras. Note moreover that by Theorem 4.19 we may view any $\tilde{L}$-connection $(\rho, E)$ as a representation of the $k$-Lie algebra $\text{End}(\tilde{L}, E)$. Since the induced connection $\rho^1$ is flat, it follows $\rho^1$ is a map of $k$-Lie algebras.

**Corollary 4.20.** Let $(\tilde{L}, \tilde{\alpha}, \tilde{\pi}, [\cdot,\cdot], D)$ be a D-Lie algebra and let $(\rho, E)$ be an $\tilde{L}$-connection with $\rho(D) = \text{Id}_E$. It follows the canonical quotient of $\text{End}(\tilde{L}, E)$ is isomorphic to the $A/k$-Lie-Rinehart algebra $\text{End}(L, E)$ where $(L, \pi_L)$ is the canonical quotient of $\tilde{L}$. If $L$ is projective there is an isomorphism $\text{End}(\tilde{L}, E) \cong \text{End}(L(f^\alpha), E)$, where $L(f^\alpha) = F_{f^\alpha}(L)$ and $F_{f^\alpha}$ is the functor from Example 3.3.
Proof. The proof follows from Theorem 4.19 and Theorem 3.9 since \( \text{End}(\tilde{L}, E) := \text{End}_A(E) \oplus \tilde{L} \) and \( \tilde{D} := (0, D) \). \qed

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E-mail address: h_maakestad@hotmail.com