Mixing Implicit and Explicit Deep Learning with Skip DEQs and Infinite Time Neural ODEs (Continuous DEQs)

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Abstract
Implicit deep learning architectures, like Neural ODEs and Deep Equilibrium Models (DEQs), separate the definition of a layer from the description of its solution process. While implicit layers allow features such as depth to adapt to new scenarios and inputs automatically, this adaptivity makes its computational expense challenging to predict. Numerous authors have noted that implicit layer techniques can be more computationally intensive than explicit layer methods. In this manuscript, we address the question: is there a way to simultaneously achieve the robustness of implicit layers while allowing the reduced computational expense of an explicit layer? To solve this we develop Skip DEQ, an implicit-explicit (IMEX) layer that simultaneously trains an explicit prediction followed by an implicit correction. We show that training this explicit layer is free and even decreases the training time by 2.5x and prediction time by 3.4x. We then further increase the “implicitness” of the DEQ by redefining the method in terms of an infinite time neural ODE which paradoxically decreases the training cost over a standard neural ODE by not requiring back-propagation through time. We demonstrate how the resulting Continuous Skip DEQ architecture trains more robustly than the original DEQ while achieving faster training and prediction times. Together, this manuscript shows how bridging the dichotomy of implicit and explicit deep learning can combine the advantages of both techniques.

1. Introduction
Implicit layer methods, such as neural ODEs and Deep Equilibrium models (Chen et al., 2018; Bai et al., 2019; Ghaoui et al., 2020), have gained popularity due to their ability to automatically adapt model depth based on the “complexity” of new problems and inputs. The forward pass of these methods involves solving steady-state problems, convex optimization problems, differential equations, etc., all defined by neural networks, which can be an expensive processes. However, training these more generalized models has empirically been shown to take significantly more time than traditional explicit models such as recurrent neural networks and transformers. Nothing within the structure of the model implicitly guarantees that it finds a fast solving forward process, so we asked, does that need to be the case?

Grathwohl et al. (2018); Dupont et al. (2019); Kelly et al. (2020); Finlay et al. (2020) have identified several problems with training implicit networks. These models grow in complexity over training, and a single forward pass can take over 100 iterations (Kelly et al., 2020) even for simple problems like MNIST. Deep Equilibrium Models (Bai et al., 2019; 2020) have better scaling in the backward pass, but are still bottle necked by slow steady-state convergence. Bai et al. (2021) quantified several convergence and stability problems with DEQs. They proposed a regularization technique by exploiting the “implicitness” of DEQs to stabilize their training. However, such a regularization process is expensive as it requires higher order automatic differentiation. Their results demonstrated faster prediction times at the expense of greatly increased training times. This leaves an open question as to whether such regularization could be imposed in a cheap or free enough manner to reduce the training cost.

\[\text{Figure 1. Training and Prediction Speedups for Skip DEQ: The best Skip DEQ model is on average 2.55x faster to train and 3.349x faster during prediction time.}\]
Our main contributions include:

1. An improved DEQ architecture (Skip-DEQ) with an additional neural network predicting better initial conditions, resulting in significantly reduced training and prediction times.

2. A regularization scheme (Skip DEQ V2) which incentivizes the DEQ to learn simpler dynamics and leads to faster training and prediction. Notably, this does not require nested automatic differentiation and thus is considerably less computationally-expensive than other published techniques.

3. A continuous formulation for DEQs as an infinite time ODE, which enables adaptivity to further decrease the number of iterations required for steady-state convergence and decreases the time cost over standard neural ODEs by not requiring backpropagation through time.

4. We release our code publicly (https://github.com/SciML/FastDEQ.jl) with the intention of wider adoption of the proposed methods in the community.

2. Deep Implicit Layers

Explicit Deep Learning Architectures specify a projection \( f : \mathcal{X} \rightarrow \mathcal{Z} \) by stacking multiple “layers”. Implicit models, however, define a solution process instead of directly specifying the projection. These methods enforce a constraint on the output space \( \mathcal{Z} \) by learning \( g : \mathcal{X} \times \mathcal{Z} \rightarrow \mathbb{R}^n \). By specifying a solution process, implicit models can effectively vary features like depth to automatically adapt to new scenarios and inputs. Some prominent implicit models include Neural ODEs (Chen et al., 2018), where the output \( z \) is defined by the ODE \( \frac{dz}{dt} = g_\theta(x, t) \), i.e., \( z(t) = z_0 + \int_{t_0}^{t} g_\theta(x, t) \, dt \). Liu et al. (2019) generalized this framework to Stochastic Differential Equations (SDEs) by stochastic noise injection, which regularizes the training of Neural ODEs allowing them to be more robust and achieve better generalization. Bai et al. (2019) designed equilibrium models (See Section 3) where the output \( z \) was constrained to be a steady state, \( z^* = f_\theta(z^*, x) \). Another example of implicit layer architectures is seen in Amos & Kolter (2017); Agrawal et al. (2019) set \( z \) to be the solution of convex optimization problems.

Deep Implicit Models essentially removed the design bottleneck of choosing the “depth” of neural networks. Instead, these models use a “tolerance” to determine the accuracy to which the constraint needs to be satisfied. Additionally, many of these models (Bai et al., 2019; 2020) only require \( O(1) \) memory for backpropagation, thus alluding to potential increased efficiency over their explicit layer counterparts. However, evaluating these models require solving differential equations (Chen et al., 2018; Liu et al., 2019), non-linear equations (Bai et al., 2019), convex optimization problems (Amos & Kolter, 2017; Agrawal et al., 2019), etc. Numerous authors (Dupont et al., 2019; Grathwohl et al., 2018; Finlay et al., 2020; Kelly et al., 2020; Ghosh et al., 2020; Bai et al., 2021) have noted that this solution process makes implicit models significantly slower in practice during training and prediction compared to explicit networks achieving similar accuracy.

3. Deep Equilibrium Network

Deep Equilibrium Networks (DEQs) (Bai et al., 2019) are implicit models where the output space represents a steady-state solution. Intuitively, this represents infinitely deep neural networks, i.e., an infinite composition of explicit layers \( z = (f_\theta \circ f_\theta \circ \cdots \circ f_\theta)(x) \). In practice, it is equivalent to evaluating a dynamical system until it reaches steady state \( z^* = f_\theta(z^*, x) \). Bai et al. (2019; 2020) perform nonlinear fixed point iterations of the discrete dynamical system using Broyden’s method (Broyden, 1965; Bai et al., 2020) to reach this steady-state solution.

Evaluating DEQs requires solving a steady-state equation involving multiple evaluations of the explicit layer slowing down the forward pass. However, driving the solution to steady-state makes the backward pass very efficient (Johnson, 2006). Despite a potentially infinite number of evaluations of \( f_\theta \) in the forward pass, backpropagation only requires solving a linear equation.

\[
\frac{\partial z^*}{\partial \theta} = f_\theta(z^*, x) \cdot \frac{\partial z^*}{\partial \theta} + \frac{\partial f_\theta(z^*, x)}{\partial \theta} \\
\Rightarrow \left( I - \frac{\partial f_\theta(z^*, x)}{\partial z^*} \right) \frac{\partial z^*}{\partial \theta} = \frac{\partial f_\theta(z^*, x)}{\partial \theta}
\]

For backpropagation, we need the Vector-Jacobian Prod-
Figure 3. Polynomial Regression: All the models converge to a similar final loss. Skip DEQ and Skip DEQ V2 require \( \sim 2.8x \) and \( \sim 1.5x \) fewer function evaluations compared to Vanilla DEQ.

where \( v \) is the gradients from layers after the DEQ module. Computing \( (I - \frac{\partial f_\theta(z^*, x)}{\partial z^*})^{-T} \) is expensive and makes DEQs non-scalable to high-dimensional problems. Instead, we solve the linear equation \( g = (\frac{\partial f_\theta(z^*, x)}{\partial z^*})^T g + v \) using Newton-Krylov Methods like GMRES (Saad & Schultz, 1986). To compute the final VJP, we need to compute \( (\frac{\partial f_\theta(z^*, x)}{\partial \theta})^T g \), which allows us to efficiently perform the backpropagation without explicitly computing the Jacobian.

3.1. Multiscale Deep Equilibrium Network

Multiscale modeling (Burt & Adelson, 1987) has been the central theme for several deep computer vision applications (Farabet et al., 2012; Yu & Koltun, 2015; Chen et al., 2016; 2017). The standard DEQ formulation drives a single feature vector to steady-state. Bai et al. (2020) proposed Multiscale DEQ (MDEQ) to simultaneously learn coarse and fine grained feature representations. MDEQs operate at multiple feature scales \( z = \{z_1, z_2, \ldots, z_n\} \), with the new equilibrium state \( z^* = f_\theta(z_1^*, z_2^*, \ldots, z_n^*, x) \). All the feature vectors in an MDEQ are interdependent and are simultaneously driven to steady-state. Bai et al. (2020) used a Limited-Memory Broyden Solver (Broyden, 1965) to solve these large scale computer vision problems. We use this MDEQ formulation for all our classification experiments.

3.2. Jacobian Stabilization

Infinite composition of a function \( f_\theta \) does not necessarily lead to a steady-state – chaos, periodicity, divergence, etc., are other possible asymptotic behaviors. The Jacobian Matrix \( J_{f_\theta}(z^*) \) controls the existence of a stable steady-state and also influences the convergence of DEQs in the forward and backward passes. Bai et al. (2021) describes how controlling the spectral radius of \( J_{f_\theta}(z^*) \) would prevent simpler iterative solvers from diverging or oscillating. Bai et al. (2021) introduce a Jacobian term to the training objective to regularize the model training. The authors use the Hutchinson estimator (Hutchinson, 1989) to compute and regularize the Frobenius norm of the Jacobian.

\[
L_{jac} = \lambda_{jac} \frac{\|z^T J_{f_\theta}(z^*)\|^2}{d}, \quad \epsilon \sim \mathcal{N}(0, I_d)
\]

While well-motivated, the disadvantage of this method is that the Hutchinson trace estimator requires using automatic differentiation in the loss function, thus requiring higher order differentiation in the training process and greatly increasing the training costs. However, in return for the increased cost it was demonstrated that increased robustness followed, along with faster forward passes in the trained results. Thus we looked into techniques which could achieve similar results without the increased training costs. Note that our methods are orthogonal to the Jacobian stabilization process and in Section 6, we demonstrate that Skip DEQs (See Section 4) are composable with Jacobian Stabilization to achieve even more robust results.

4. Skip Deep Equilibrium Network (Skip DEQ)

When solving a DEQ, previous authors typically set the initial condition \( u_0 = 0 \). Assuming that a steady-state exists, our solvers should be able to converge to that given enough iterations. However, each iteration is expensive, and a poor guess of the initial condition makes the convergence slower. To counteract these issues, we introduce an alternate architecture for DEQ (Skip DEQ), where we use an explicit model \( g_\phi \) to predict the initial condition for the steady-state problem \( u_0 = g_\phi(x) \) (See Figure 2). We jointly optimize for \( \{\theta, \phi\} \) by adding an auxiliary loss function:

\[
L_{skip} = \lambda_{skip} \|f_\theta(z^*, x) - g_\phi(x)\|_1
\]

Intuitively, our explicit model \( g_\phi \) better predicts a value closer to the steady-state (over the training iterations), and hence we need to perform fewer iterations during the forward pass. Given that its prediction is relatively free in comparison to the cost of the DEQ, this technique could decrease the total cost of the DEQ by reducing the total number of iterations required. However, this prediction-correction approach still uses the result of the DEQ for its
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final predictions and thus should achieve robustness properties equal.

Indeed, the robustness of the DEQ is even improved through this process because in practice the maximum iterations of the DEQ is bound (Bai et al., 2019; 2020; 2021). DEQs often do not converge to a steady-state due to this bound. We perform empirical studies in Section 6 to show that Skip DEQ can converge to a steady-state even where a DEQ fails due to its increased convergence speed. This ability allows Skip DEQs to be trained with a more stable gradient signal since the adjoint equation assumes steady-state convergence during the forward pass.

4.1. Skip DEQ V2: No Extra Parameters

One of the primary benefits of DEQs is the low memory footprint of these models (See Section 2). Introducing an explicit model $g_\phi$ increases the memory requirements for training. To alleviate this problem, we propose a regularization term to minimize the L1 distance between the first prediction of $f_\theta$ and the steady-state solution:

$$L_{skip} = \lambda_{skip} \| f_\theta(z^*, x) - u_0 \|_1 \text{ where } u_0 = f_\theta(0, x)$$

This technique follows the same principle as the Skip DEQ where the DEQ’s internal neural network is now treated as the prediction model. We hypothesize that this introduces an inductive bias in the model to learn simpler training dynamics which allows our models to generalize better at test time (See Section 6).

5. Continuous DEQs

The aforementioned DEQ formulations were all described using the steady states of discrete dynamical systems. However, there are many downsides this approach can have. To illustrate this, take the linear discrete dynamical system $u_{n+1} = \alpha u_n$ with $u_0 = 1$. When $|\alpha| < 1$, this system converges to a steady state $u_\infty = 0$. However, in many cases this convergence can be rather slow. If $\alpha = 0.9$, then after 10 steps the value is $u_{10} = 0.35$ because each successive step is only reducing by a small amount and thus convergence could only be accelerated by taking many steps together. Even further if $\alpha = -0.9$, the value ping-pongs over the steady state $u_1 = -0.9$, meaning that if we could take some fractional step $u_{\delta t}$ then it would be possible to approach the steady state much faster.

This thought experiment, along with many other issues with discrete steady-state dynamics (Rico-Martinez et al., 1992; Bulsari, 1995), motivates changing from a discrete description of the system (the fixed point or Broyden’s method approach) to a continuous description of the system that allows adaptivity to change the stepping behavior and accelerate convergence. To this end we propose an alternate formulation for DEQs by modeling continuous dynamical system (Continuous DEQ) where the forward pass is represented as an ODE which is solved from $t_0 = 0$ to $t_1 = \infty$:

$$\frac{dz}{dt} = f_\theta(z, x) - z$$

Continuous DEQs leverage fast adaptive ODE solvers, which terminate automatically once the solution is close to a steady-state, i.e., $\frac{dz}{dt} = 0$, which then satisfies $f_\theta(z^*, x) = z^*$ and is thus the solution to the same implicit system as before. In this form, the Continuous DEQ can be thought of as an infinite time neural ODE. However, almost paradoxically the infinite time version is cheaper to train than the finite time version as its solution is the solution to the nonlinear system, meaning the same implicit differentiation formula of the original DEQ holds for the derivative. This means that no backpropagation through the steps is required for the Continuous DEQ and only a linear system must be solved. In Section 6, we show that Continuous DEQs converge more consistently within a fixed number of iterations to the steady-state compared to Discrete DEQs.

6. Experiments

This section shows the effectiveness of Skip DEQs and Continuous DEQs on three image classification problems – MNIST (LeCun et al., 1998), CIFAR-10 (Krizhevsky et al., 2009) & SVHN (Netzer et al., 2011). We use perform our experiments in Julia (Bezanson et al., 2017) using Flux (Innes et al., 2018; Innes, 2018) and DifferentialEquations (Rackauckas & Nie, 2017; Rackauckas et al., 2018; 2020). Our primary metrics are classification accuracy, number of function evaluations (NFEs), convergence depth\(^1\), total training time, and prediction time per batch.\(^2\)

\(^1\)The continuous DEQ formulation uses ODE Solvers, which evaluate the function $f_\theta$ multiple times before taking one step. For LBroyden, one step corresponds to one function evaluation since we do not perform line search as per standard practice. Hence, we use convergence depth, which corresponds to the total steps taken, as an uniform metric to compare different solvers.

\(^2\)We note that due to limitations of our Automatic Differentiation system, we are unable to perform Jacobian Regularization for Convolutional Models. However, our preliminary analysis
Table 1. MNIST Classification with Fully Connected Layers: Continuous Skip DEQ V2 without Jacobian Regularization takes 4x less training time and speeds up prediction time by 4x compared to Continuous DEQ. Continuous DEQ with Jacobian Regularization has a similar prediction time but takes 6x more training time than Continuous Skip DEQ V2.

To motivate the use of Skip DEQs, we use a synthetic training dataset of 512 datapoints and a testing dataset of 128 data points generated from $h(x) = \frac{3}{2}x^3 - x^2 - 5x + 2\sin(x) - 3 + \epsilon$ where $x \in [-1, +1]$, $\epsilon \sim \mathcal{N}(0, 0.005)$. We use the continuous DEQ formulation and use BS3 (Bogacki & Shampine, 1989) to find the steady-state. Figure 3 & 4 show that Skip DEQ and Skip DEQ V2 are fastest to train while adding Jacobian Stabilization has minimal effect on NFEs.

6.1. MNIST Image Classification

6.1.1. Fully Connected Layers

Training Details: Following Kelly et al. (2020), our Fully Connected Model consists of 3 layers – a downsampling layer $\mathbb{R}^{784} \rightarrow \mathbb{R}^{128}$, continuous DEQ layer $f_\theta: \mathbb{R}^{128} \rightarrow \mathbb{R}^{128}$, and a linear classifier $\mathbb{R}^{128} \rightarrow \mathbb{R}^{10}$. For regularization we use $\lambda_{\text{skip}} = 0.01$ and train the models for 25 epochs with a batchsize of 32. We use Tsit5 (Tsitouras, 2011) with a relative tolerance for convergence of 0.005. For optimization, we use ADAM (Kingma & Ba, 2014) with a constant learning rate of 0.001. The training was distributed across 6 processes and 2 32 GB NVIDIA V100 GPUs.

Baselines: We use continuous DEQ and continuous DEQ with Jacobian Stabilization as our baselines. We additionally compose Skip DEQs with Jacobian Stabilization in our benchmarks. For all experiments, we keep $\lambda_{\text{jac}} = 1.0$.

Results: We summarize our results in Table 1 & Figure 5. Without Jacobian Stabilization, Skip DEQ V2 has the highest testing accuracy of 97.973% and has the lowest training and prediction timings overall. Using Jacobian Regularization, DEQ outperforms Skip DEQ models by $< 0.4\%$, however Jacobian regularization increases training time by 1.4 - 4x. Overall, Skip DEQ models are able to obtain the lowest prediction time per batch of $\sim 0.01$s.

6.1.2. Convolutional Layers

Training Details: We use the Multiscale DEQ formulation with two feature scales. We slightly modify the MDEQ-small CIFAR-10 architecture used in Bai et al. (2020) for MNIST. The two feature scales operate at a resolution of $28 \times 28$ and $14 \times 14$ and are directly fused by addition after a downsampling operation on the $28 \times 28$ resolution.

We use a fixed regularization weightage of $\lambda_{\text{skip}} = 0.01$ and the models are trained for 25 epochs with a batchsize of 64. For the continuous DEQ formulation, we use Tsit5 (Tsitouras, 2011) with a relative tolerance for convergence of 0.01. We use ADAMW (Loshchilov & Hutter, 2017) optimizer with a cosine scheduling on the learning rate – starting from $10^{-3}$ and terminating at $10^{-6}$ – and a weight decay of $2.5 \times 10^{-6}$. The training was distributed across 6 processes and 2 32 GB NVIDIA V-100 GPUs.

Baselines: Our Baseline Model is the Vanilla DEQ, trained with a similar setup as the Skip DEQ models. Additionally, we perform a comparison with the discrete DEQ formulation using LBroyden Solver (Bai et al., 2020) with a relative tolerance of 0.01.
Figure 6. MNIST Classification with Convolutional Layers: Accuracy and Convergence Depth of DEQ and Skip DEQ over the training epochs. Continuous forms exhibit a very stable convergence depth over the epochs while, Discrete forms are comparatively unstable for the Skip DEQ models, even though they attain similar testing accuracies.

Results: We summarize our results in Table 2 & Figure 6. For the continuous variant, Skip DEQ V2 outperforms all the other models in all metrics. Skip DEQ models speed up training by 1.7 - 3.2x and reduce prediction time by 4.49 - 5.5x compared against Vanilla DEQ. For the discrete variant, DEQ only outperforms the Skip DEQs by atmost 0.095% testing accuracy, while Skip DEQ V2 reduces training time by 1.9x and improves prediction time by 4.8x. Overall, continuous DEQs converge at the least “depth”, but discrete DEQs are fastest during training and prediction.

6.2. CIFAR-10 Image Classification

Training Details: Our Multiscale DEQ architecture is the same as MDEQ-small architecture used in Bai et al. (2020). For the explicit network in Skip DEQ, we use the residual block and downsampling blocks from Bai et al. (2020) which account for the additional 58K trainable parameters. We use a fixed regularization weightage of $\lambda_{skip} = 0.01$ and the models are trained for 50 epochs. We use a batch size of 64 for the continuous form, and we use Tsit5 (Tsitouras, 2011) with a relative tolerance for convergence of 0.05. We use ADAMW (Loshchilov & Hutter, 2017) optimizer with a cosine scheduling on the learning rate – starting from $10^{-3}$ and terminating at $10^{-6}$ – and a weight decay of $2.5 \times 10^{-6}$. The training was distributed across 6 processes and 2 32 GB NVIDIA V-100 GPUs.

Baselines: Vanilla DEQ is trained with the exact same training hyperparameters (taken from (Bai et al., 2020)). For the discrete forms, we used a relative tolerance for convergence of 0.05 and reduced the batch size to 32 since it utilizes more memory.

Results: Our results are summarized in Table 3 & Figure 7. Vanilla DEQ and Discrete forms are unable to converge to the steady-state. Skip DEQ V2 generalizes significantly better than DEQs to the testing dataset with an accuracy boost of 1.6% – 2.5%. Skip DEQ has a marginally lower convergence depth compared to Skip DEQ V2. Additionally, Skip DEQs improve the training time by 1.39 – 1.47x and the continuous form reduces the prediction time per batch by 1.2x. Using discrete Skip DEQ is detrimental.
Table 3. CIFAR-10 Classification: Skip DEQs generalize better to the testing data outperforming DEQ by 0.2% - 2.52%. Continuous Skip DEQs converge to the steady state and reduce the training time by 1.39 - 1.47x and prediction time by 1.2x. Discrete forms are unable to converge to the steady state and hence using Skip DEQ has a detrimental effect on the training and prediction timings.

Figure 7. CIFAR-10 Classification: Skip DEQs outperform DEQ on testing accuracy. None of the discrete models converge to the steady-state. Continuous Skip-DEQ models exhibit stable convergence to steady-state over the epochs.

to the performance since it is unable to converge to the steady-state.

6.3. SVHN Image Classification

Training Details & Baselines: We use the exact same setup as in CIFAR-10 Classification (See Section 6.2). However, we distribute the training across 4 processes and 2 32 GB NVIDIA V-100 GPUs.

Results: Our results are summarized in Table 4 & Figure 8. Vanilla DEQs are unable to converge to the steady-state. Skip DEQ V2 generalizes significantly better to the testing dataset with an accuracy boost of 8.2% – 14.6%. Discrete Skip DEQ V2 converges to the steady-state at the least “depth”. Additionally, Skip DEQs improve the training time by 1.07 – 1.5x and reduces the prediction time per batch by 1.32 – 2.8x.

7. Related Works

Implicit Models: Implicit Models have obtained competitive results in image processing (Bai et al., 2020), generative modeling (Grathwohl et al., 2018), time-series prediction (Rubanova et al., 2019), etc, at a fraction of memory requirements for explicit models. Additionally, Kawaguchi (2021) show that for a certain class of DEQs convergence to global optima is guaranteed at a linear rate. However, the slow training and prediction timings (Dupont et al., 2019; Kelly et al., 2020; Finlay et al., 2020; Ghosh et al., 2020; Pal et al., 2021; Bai et al., 2021) often overshadow these benefits.

Accelerating Neural ODEs: Finlay et al. (2020); Kelly et al. (2020) used higher-order regularization terms to constrain the space of learnable dynamics for Neural ODEs. Despite speeding up predictions, these models often increase the training time by 7x (Pal et al., 2021). Alternatively, Ghosh et al. (2020) randomized the endpoint of Neural ODEs to incentivize simpler dynamics. Pal et al. (2021) used internal solver heuristics – local error and stiffness estimates – to control the learned dynamics in a way that decreased both prediction and training time. Xia et al. (2021) rewrite Neural ODEs as heavy ball ODEs to accelerate both forward and backward passes. Djeumou et al. (2022) replace ODE solvers in the forward with a Taylor-Lagrange expansion and report significantly better training and prediction times.
Table 4. SVHN Classification: Skip DEQs generalize better to the SVHN testing data with an accuracy boost of 5.4% - 14.7%. Vanilla DEQs are unable to converge to the steady-state, while Skip DEQ V2 always converges to the steady-state at the least “depth”. Skip DEQs improve the training time by 1.07 - 1.5x and reduces the prediction time per batch by 1.32 - 2.8x.

Figure 8. SVHN Classification: Skip DEQs outperform Vanilla DEQ in terms of testing accuracy. The convergence depth stabilizes after a few epochs for continuous Skip DEQ while vanilla DEQ is unable to converge to steady-state. Among the discrete variants, only Skip DEQ V2 converges to the steady-state.

Regularized Neural ODEs can not be directly extended to discrete DEQs (Bai et al., 2019; 2020). Our continuous formulation introduces the potential to extend Xia et al. (2021); Djeumou et al. (2022) to DEQs. However, these methods benefit from the structure in the backward pass, which does not apply to DEQs. Additionally, relying on discrete sensitivity analysis (Pal et al., 2021) nullifies the benefit of a cost-effective backward pass.

Accelerating DEQs: Bai et al. (2021) uses second-order derivatives to regularize the Jacobian, stabilizing the training and prediction timings of DEQs. Fung et al. (2022) proposes a Jacobian-Free Backpropagation Model, which accelerates solving the Linear Equation in the backward pass. Our work complements these models and can be freely composed with them. We have shown that a poor initial condition harms convergence, and a better estimate for the same leads to faster training and prediction. We hypothesize that combining these methods would lead to more stable and faster convergence and demonstrated this possibility with the Jacobian regularization Skip DEQ.

8. Conclusion

We have empirically shown the effectiveness of Skip DEQs and Continuous DEQs. The Skip DEQ showcases how a cheap learning for predicting an improved initial condition allows for faster convergence to steady-state and increased stability. For memory constrained applications, our Skip DEQ V2 regularization learns a simpler dynamics without requiring additional trainable parameters. The continuous formulation of the Continuous DEQ often takes slightly longer during training and prediction than discrete DEQs, but our empirical results show that the continuous form more consistently converge to the steady-state which enables a stable gradient signal. We demonstrated applications where the combined Skip and Continuous DEQ approach is required to achieve robust training. We observe that our best models have a 3.74x reduction in convergence depth which amortizes the added optimization cost over the training epochs. Additionally, our models show an average reduction of 2.5x training time and 3.4x prediction time per batch. These results show that our improved DEQ and neural ODE architectures should be employed in any scenario.
where the previous implicit layer methods are used.

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