Influence of contact area coefficient on the results of $u$-$p$ formation dynamic equations of fluid-saturated porous media

Chaoqun Feng$^1$, Pei Zhang$^{2,3}$, Chengshun Xu$^1$$^*$, Xiuli Du$^1$, Chong Yue$^1$

$^1$The Key Laboratory of Urban Security and Disaster Engineering of Ministry of Education, Beijing University of Technology, Beijing 100124, China
$^2$School of Civil and Transportation Engineering, Beijing University of Civil Engineering and Architecture, Beijing 102616, China
$^3$Department of Civil and Environmental Engineering, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong, China

First author’s e-mail: fengchaoqun@emails.bjut.edu.cn
$^*$Corresponding author’s e-mail: xuchengshun@bjut.edu.cn

Abstract. The principle of effective stress is the keystone of modern physics, which becomes the basic difference between soil mechanics and general solid mechanics. Terzaghi’s effective stress principle has been controversial since it was put forward, especially for the calculation of effective stress, many scholars have revised it from different aspects. The pore pressure is reduced in all different modified formulas, that is, the pore pressure need to be multiplied by a pore pressure factor. However, the variation of pore pressure factor has always been ignored because the reduction of pore pressure is very small. In this paper, the effective stress expression considering the contact area coefficient is modified to the $u$-$p$ formation dynamic equation, and the influence of the change of contact area coefficient on the dynamic characteristics of soil is studied.

1. Introduction
Saturated soil widely exists in nature and it is usually simplified as a loose particle aggregate material composed of soil skeleton and fluid filled in pores. The effective stress principle was proposed by Terzaghi[1] and it illustrated the difference between saturated soil and continuous solid media in mechanical properties. In addition, the calculation formula of effective stress $\sigma'$ was given by Terzaghi (as shown in Equation (1), where $\sigma$ is the total stress, $p$ is the pore pressure), so that the mechanical properties of saturated soil can be analyzed in the framework of continuous mechanics.

$$\sigma' = \sigma - p$$ (1)

For the expression of effective stress proposed by Terzaghi, many scholars[2] have proposed different correction formulas of effective stress from the different understanding of the physical meaning of effective stress. These correction formulas can be unified as the following expression:

$$\sigma' = \sigma - \eta p$$ (2)

Where, $\eta$ is the pore pressure factor. On the one hand, several scholars believe that the effective stress principle of Terzaghi does not need to be modified, that is, $\eta$ =1; On the other hand, several scholars...
think that the $\eta$ is affected by the soil material parameters, such as, porosity, the ratio of soil particle contact area, the compression coefficient of soil skeleton and soil particles, the internal friction angle, etc. For example, the effective stress calculation formulas related to strength and deformation was given by Du et al.[2], and they were built according to the stress balance principle and the interaction between fluid pressure and soil particles. The pore pressure factor in the effective stress formula related to strength is related to the contact area ratio $\alpha_c$ and porosity. The modified effective stress calculation expression based on the contact area coefficient $\alpha$ is shown as following:

$$\sigma' = \sigma - p + \alpha p', \text{ where } (\alpha = \alpha_c (1 - n))$$

(3)

For Equation (3), $\eta = (1 - \alpha)$. In general, $\eta$ is usually regarded as 1 in application. However, some researchers[3] found that the initial hydrostatic pressure affects the liquefaction resistance of saturated sand, which was also a challenge to the calculation formula of Terzaghi effective stress from another point of view. It also shows that the influence of influencing factors on $\eta$ can not be ignored. Especially with the development of underground space and marine resources towards the super-deep, it is necessary to explore the influence of contact area coefficient more accurately.

In the study of sand liquefaction and strength, the $u$-$p$ formed equations for the fluid-saturated porous media are usually used to analyze the development of soil displacement and pore pressure. The dynamic equation of saturated two-phase media is established based on the constitutive relationship, mass conservation equation, dynamic equilibrium equation, liquid dynamic equilibrium equation and effective stress principle. The factor $\eta$ has so far been equal to 1 when using the $u$-$p$ formation.

In this paper, the $u$-$p$ formation is taken as the object. By changing the contact area coefficient in the effective stress, the influence of the contact area coefficient on the soil displacement and pore pressure are analyzed. Because there are still many difficulties in the accurate measurement of contact area coefficient, the contact area coefficients $\alpha$ of 0.01, 0.02, 0.03 and 0.04 are selected to make the law more obvious.

2. $u$-$p$ formation of fluid-saturated porous media

Based on variable substitution and simplifying assumptions, various approximations to Biot’s model were obtained by Zienkiewicz[4], in which $u$-$p$ form is more commonly used in practice, where $u$ is the soil skeleton displacement and $p$ symbolizes the pore pressure. The $u$-$p$ formation was obtained by neglecting the pore fluid acceleration with respect to the soil skeleton and it is the simplest form for problems in which high-frequency components are absent[6-7]. The simplified $u$-$p$ equations was given in Equation (4). It should be noted that, The total stress and effective stress are positive with pressure, while the pore pressure is positive with tension[8].

$$\nabla (\sigma' - mp) + \rho b = \rho \ddot{u}$$

$$-\nabla^T k \nabla p + \frac{1}{Q_h} \dot{p} + \eta \nabla^T u + \bar{k} \rho \gamma \nabla^T b = \bar{k} \rho \gamma \nabla^T \ddot{u}$$

(4)

The effective stress calculation formula considering the contact area correction were derived by Du et al.[2]. ‘$\alpha p$’ was added to the Terzaghi effective stress calculation formula and $\alpha = \alpha_c (1 - n)$. When the contact area is considered in the $u$-$p$ formed equations, the Equation (4) can be modified as:

$$\nabla (\sigma' - mp + \alpha mp) + \rho b = \rho \ddot{u}$$

$$-\nabla^T k \nabla p + \frac{1}{Q_h} \dot{p} + \eta \nabla^T u + \bar{k} \rho \gamma \nabla^T b = \bar{k} \rho \gamma \nabla^T \ddot{u}$$

(5)

The finite element method (FEM) was adopted to solve the Equation (5), and the entire analysis process can be summarized by two steps: discretizing the infinite continuous media in spatial domain and solving the dynamic equations of the discrete system in discrete time domain subsequently. After discretizing the Equation (5) in spatial and time domain successively, the Galerkin weak form of the $u$-
formed equations for fluid-saturated porous media can be expressed as Equation (6). An explicit-explicit method[9] was proposed to solve it and this method is used in the subsequent analysis of this paper.

\[ M_i \ddot{u}_i + K u_i - Q_p = f_{p_k} \quad \text{a)} \\
S_j \ddot{p}_j + J p_j + Q^T \dot{u}_j = f_{q_k} \quad \text{b)} \]

\[ M_i = \sum \int N_i^T \rho N_i d\Omega \\
K = \sum \int N_i^T N_i^T d\Omega \\
Q = \sum \int N_i^T \nabla N_i d\Omega \\
J = \sum \int \left( \nabla N_i \right)^T \kappa \left( \nabla N_i \right) d\Omega \\
S_j = \sum \int N_j^T \frac{1}{Q} N_j d\Omega \\
f_u = \sum \int N_j^T \{ f_u \} d\Gamma \\
f_p = -\sum \int N_j^T \{ f_q \} d\Gamma \]

Where, \( M_i \) is the mass matrix of two-phase media after diagonalization, \( K \) is the stiffness matrix of soil skeleton, \( Q \) is the coupling matrix, \( S_i \) is the compressibility matrix of the pore fluid after diagonalization, \( J \) is the permeability matrix of the pore fluid, and \( f_u, f_q \) are respectively the external mechanical loadings. Other symbols are explained in table 1.

| Symbols | Definitions |
|--------|-------------|
| \( u \) | Soil skeleton displacement |
| \( p \) | Pore pressure |
| \( \sigma' \) | Effective stress |
| \( n \) | Porosity |
| \( K_f \) | Bulk modulus of pore fluid |
| \( K_s \) | Bulk modulus of solid |
| \( b \) | Body force |
| \( \rho \) | Density of mixture |
| \( \rho_f \) | Density of pore fluid |
| \( k \) | Permeability |
| \( \bar{k} \) | Dynamic permeability |
| \( Q_b \) | Compressibility coefficient of fluid |
| \( \lambda \) and \( G \) | Lame constants |

3. Numerical model and result analysis

In Equation (6), \( D \) is the stiffness matrix. In this section, according to whether \( D \) is a constant, the examples are divided into elastic example and nonlinear example. In the elastic example, \( D \) remains unchanged during the loading process. Although this situation is usually unreasonable, the elastic example is usually used as a way to verify the correctness of one method. In the nonlinear model, the \( D \) changes with loading and soil state. In this section, the PM4Sand constitutive model are applied to simulate the stress-strain development of sand.

3.1. Elastic example

The calculation model and the saturated soil parameters are shown in figure 1. A uniform loading is applied to the top free drained boundary. The bottom of the model is fixed and impermeable boundary. In addition, only horizontal displacement is allowed on the lateral impermeable boundary. The size of soil column is 3m×30m. Four node element is used to discretize the model and the size of each element is 3m × 3m. The initial pore pressure is 1.0Pa. The results are shown in figure 2.
The comparison results show that the contact area coefficient has little influence on the elastic calculation results of $u-p$ dynamic equation, but it still presents a certain regularity. With the increase of the depth of the target point, the greater the influence of the contact area coefficient on the vertical displacement, the smaller the influence on the pore pressure value. For one node, the larger the contact area coefficient is, the larger the vertical displacement is, and the faster the pore pressure drops. It means that the contact area coefficient affects the strength of soil: the increase of contact area coefficient leads to the increase of effective stress, however, the contact area coefficient affects the development of pore pressure, that is, pore pressure decreases more (see figure 2(b)), therefore, the part of shear strength
increase caused by the increase of effective stress is offset, which resulting in the decrease of shear strength of soil. Eventually, the larger the contact area coefficient is, the larger the displacement is and the more the pore pressure decreases.

3.2. Nonlinear example

In this section, the sand liquefaction under cyclic loading is analyzed based on OpenSees platform, and the influence of the contact area coefficient on the results of u-p model when D is nonlinear is studied. Four node plane strain quadUP element is used in the model. The bottom boundary is fixed and the water line is set at the top of the soil to ensure the sand is fully saturated. In addition, equal DOF is applied to 3 and 4 nodes to make the model in pure shear state. The sand model is shown in figure 3. The mechanical parameters of saturated soil are shown in table 2.

![Figure 3. Sand model under cyclic loading.](image)

| Parameters     | Values | Parameters     | Values | Parameters     | Values |
|----------------|--------|----------------|--------|----------------|--------|
| Dr/%           | 50     | G₀/kPa         | 677    | hₙ₀            | 0.4    |
| hₚ₀            | 0.4    | p₀/kPa         | 101.3  | eₘₐₓ/eₘᵟᵣₜₜ | 0.8/0.5|
| nₜ₀/nₜ₁       | 0.5/0.1| C₉G₁D         | 2.0    | cₑ            | -1.0   |
| Aₐ₀            | -1.0   | zₘₐₓ          | -1.0   | Cₒₐᵢƒ        | -1.0   |
| φₛᵥ           | 33.0   | v              | 0.3    | Q              | 10.0   |
| R              | 1.5    | C₆r           | -1.0   | m              | 0.01   |
| Fsₑₜₙₜₜ,min   | -1.0   | p₀ₜₜₑ          | -1.0   |                 |        |

The results are shown in figure 4. The contact area coefficient has little influence on the development of stress-strain relationship and pore pressure before initial liquefaction. After initial liquefaction, the larger the contact area coefficient is, the slower the shear strain develops. The contact area coefficient has no effect on the cumulative rate of pore pressure, but it affects the peak value of pore pressure development.

It can be seen from equation (3) that the effective stress includes two parts in fact: the effective stress proposed by Terzaghi (σ₋ₚ) and that provided by the contact area coefficient term αp. Under the same shear stress state, the larger the α is, the larger the effective stress and the shear modulus are, and the smaller the shear strain is.

4. Conclusion

The modified effective stress expression considering the contact area coefficient is applied to the u-p form fluid-saturated porous media. The influence of the contact area coefficient on the results of u-p formation under linear and nonlinear situations is analyzed in this paper. The results shows that the contact area coefficient have effects on the development of soil displacement, the relationship of strain-stress and pore pressure. The larger the contact area coefficient is, the faster the soil deforms. At present, it is difficult to measure the contact area coefficient, the influence of contact area coefficient on soil
dynamic characteristics will be further studied, especially for the saturated soil under the deep sea or super deep underground.

Figure 4. Influence of contact area coefficient on the development of stress strain and pore pressure

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