The Josephson current in Luttinger liquid-superconductor junctions

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\textbf{Abstract}

We study the Josephson current through a Luttinger liquid in contact with two superconductors. We show that it can be deduced from the Casimir energy in a two-boundary version of the sine-Gordon model. We develop a new thermodynamic Bethe Ansatz, which, combined with a subtle analytic continuation procedure, allows us to calculate this energy in closed form, and obtain the complete current-crossover function from the case of complete normal to complete Andreev reflection.

Low-dimensional condensed matter systems exhibit an exciting set of unusual properties, which have been actively studied theoretically for some time, and have recently become a topic of explosive experimental interest. The physics of these systems is crucially related to interactions, which in low dimensions behave nonperturbatively, giving rise to such nonintuitive phenomena as spin and charge separation, or charge fractionalization.

Dealing with these interactions theoretically is a challenge, especially as far as transport properties are concerned. These are of course the most relevant from an experimental point of view. New methods have had to be developed over the years to tackle these issues, like bosonization, conformal field theory and integrability. The area where most progress has been made is probably that of single impurity problems, including the Kondo effect, edge states tunneling, and quantum dots.

One of the paradigms of low-dimensional electronic systems is the Luttinger liquid, and it can only be expected that fascinating properties should appear when combined with another great paradigm of solid state physics: superconductivity. Indeed, it was recently shown that, as a result of the low-dimensional interactions, junctions between superconductors and Luttinger liquids behaved very differently from the usual ones: in particular, there is now an RG flow as the temperature or the length of the junction is changed, leading, for repulsive (resp. attractive) bulk interactions to perfect normal (resp. Andreev) reflection at low energies. In the non-interacting case there is no flow, and the relative amounts of normal and Andreev reflections are set by parameters like the superconducting gaps only.

The formalism developed so far allows one to predict global features of the RG flow, but not to exactly compute physical quantities like the current. This is due to the presence of Luttinger interactions, which require a non perturbative approach. In this letter, we present a new formalism to deal with this difficulty, and show how it gives rise to closed form expressions for the current, all the way from the UV to the IR fixed points.

We use here the formulation developed in \cite{composite}. Figure 1 illustrates our considerations. A quantum wire is suspended between two bulk superconductors $S_l,r$. We describe $S_l$ and $S_r$ with BCS theory, using gaps $\Delta_{l,r}^{(bulk)}$ and $\Delta_{l,r}^{(bulk)} e^{i\chi}$, with $\Delta_{l,r}^{(bulk)}$ real numbers. The phase difference between the gaps, $\chi$, is crucial in driving the Josephson current. Since the electronic states in $S_l$ and $S_r$ are suppressed by the bulk BCS gaps, an effective theory at low energies (compared to the gaps) can be obtained by integrating out all states in $S_l$ and $S_r$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.pdf}
\caption{The effective theory for a Josephson junction with a quantum wire between two superconductors $S_1$ and $S_2$, with bulk BCS gaps $\Delta_{1,2}^{(bulk)}$: a bounded Luttinger liquid, with BCS-like couplings $\Delta_{1,2}$ living on the boundaries only.}
\end{figure}

The only contributions left from the original superconductors are two boundary BCS terms modifying the Luttinger liquid Hamiltonian: in terms of fermionic left- and right-movers, we end up with the boundary contributions

$$H_B \propto \Delta_{1}^{(bulk)} \Psi_{L_1}^\dagger (0) \Psi_{L_1}^\dagger (0) + \Delta_{r}^{(bulk)} e^{i\chi} \Psi_{R_1}^\dagger (R) \Psi_{R_1}^\dagger (R) + ...$$

(1)

in which $\Psi_{L,R}$ means $h.c.$ $+ L \leftrightarrow R$. The values of the boundary parameters $\Delta_{L,R}$ are related to the original bulk gaps $\Delta_{l,r}^{(bulk)}$, and the physical processes at the boundaries depend on them. Namely, two things can happen when an electron hits one of the boundaries: it can be normal-reflected as an electron, or Andreev-reflected as the time-conjugate excitation (a hole), thereby creating an additional Cooper pair in the condensate. The relative amounts of normal versus Andreev reflections at each boundary is set by $\Delta$: for $\Delta = 0$, only normal reflection occurs, while for $\Delta \to \infty$, only Andreev reflection occurs. The Josephson current can thus be physically understood as excitations being Andreev-reflected back and forth between the two boundaries, thereby transferring charge from one end to the other.
Upon bosonization, the theory becomes $H = H_0 + H_B$ where $H_0$ is the charge part of the Luttinger liquid Hamiltonian, and the boundary Hamiltonian involves the dual charge boson (since we don’t consider any nontrivial spin-scattering processes at the boundaries, the spin and charge sectors decouple. We thus consider the charge sector only, and drop the spin boson in all further formulas):

$$H_B \propto \Delta_l \cos \frac{\beta}{2} \phi_c(0) + \Delta_r \cos \left( \frac{\beta}{2} \phi_c(R) - \chi \right).$$  \hspace{1cm} (2)

In the above notation, the interaction parameter is related to standard bosonization conventions according to $\beta^2 = (4\pi)^2 R_c^2 = 4/g \mu = 2/K \rho$. The current through the system is then easy to calculate, being given by the difference between right- and left-mover densities, which under bosonization becomes $I(x) = -e \sum_s (\Psi_{R,s}^\dagger \Psi_{R,s} - \Psi_{L,s}^\dagger \Psi_{L,s}) \propto -e \partial_x \phi_c$. Reabsorbing this linear derivative in the quadratic form of the Hamiltonian then shows that the zero-temperature current is given by the simple relation $I(\chi) = 2e \frac{\beta}{2} \chi E_0(\Delta_l, \Delta_r, \chi)$, where $E_0$ is the ground-state energy of the system.

We show in this letter how the Josephson current can be obtained exactly using the effective two-boundary model described above. The first step, generalizing the proposal of [3], is to think of the bulk theory as the massless limit of a sine-Gordon theory for the dual charge boson $\phi_c$, and to use the fact that the boundary sine-Gordon terms in [3] do not destroy the integrability of the sine-Gordon model [3]. Thus, we can derive our results from a study of the massless limit of the double-boundary sine-Gordon model

$$S = \int dt \int_0^R dx \frac{1}{2} \left[ (\partial_t \phi)^2 - (\partial_x \phi)^2 - m \cos \beta \phi \right] -$$

$$- \int dt \left[ \Delta_l \cos \frac{\beta}{2} \phi(0) + \Delta_r \cos \left( \frac{\beta}{2} \phi(R) - \chi \right) \right]$$  \hspace{1cm} (3)

As proposed in [3], the key role of integrability is to provide one with a basis of the free boson Hilbert space made of excitations that scatter in a simple, factorized way, with no particle production, both in the bulk and at the boundaries. In contrast with [3] and most other work on boundary integrable field theories, we have however to deal with a problem involving two boundaries. This gives rise to unexpected subtleties.

In imaginary time, two quantization schemes are possible [14]. In the first one, called the $L-$channel, we consider an inverse temperature $L \to \infty$ and write the partition function as

$$Z = Tr \ e^{-LH_{l,r}}$$  \hspace{1cm} (4)

where the trace is over eigenstates of the Hamiltonian with appropriate boundary conditions at the left and right ends. Alternately, we can permute the definitions of space and imaginary time and move to the $R-$channel picture, where time flows from one “boundary” state to the other. In this case, the partition function is given by

$$Z = \langle B_r | e^{-R \hat{H}} | B_l \rangle$$  \hspace{1cm} (5)

where the Hamiltonian is quantized along length $L$ with simple periodic boundary conditions. Thus, the states $|B_{l,r}\rangle$ become initial and final states, linked through time evolution along a slice of length $R$. The advantage of this approach is that the interpretation of the boundary states is physically clear: as no momentum can flow through the boundaries, only states of zero total momentum can be emitted or absorbed. As the boundary sine-Gordon is an integrable theory [3], factorizability of the scattering imposes very strong constraints on the shape of the boundary states, constraints which are in fact sufficient to determine them completely.

We specialize in this paper to the free fermion point $\beta^2 = 4\pi$, which corresponds to $R_c^2 = 1/4\pi$, in the bulk attractive regime of the Luttinger liquid. All difficulties due to the presence of two boundaries appear in this case, while the bulk scattering is trivial and does not obscure the issues. There are two fundamental excitations in the bulk: the soliton and the antisoliton, carrying rapidities $\theta$ (their creation operators being denoted $A_{l,r}^\dagger(\theta)$), with bulk scattering matrix $S = -1$. The boundary scattering however is nontrivial, and involves nondiagonal reflection processes. This affects the boundary states, whose general form is that of a coherent state of excitations with opposite rapidities, i.e. [3]

$$|B_{l,r} \rangle \propto \exp \left\{ \sum_0^\infty d \theta K_{l,r}^{ab}(\theta) A_{l,r}^\dagger(-\theta) A_{l,r}^\dagger(\theta) \right\} |0\rangle$$  \hspace{1cm} (6)

with amplitudes $K_{l,r}^{ab}(\theta)$ explicitly given in [12].

Given the two boundary states, the partition function can be calculated in the following way. For a complete set of states $|\alpha\rangle$, we have

$$Z = \sum_\alpha \frac{\langle B_{l} | \alpha \rangle \langle \alpha | B_{r} \rangle}{\langle \alpha | \alpha \rangle} e^{-R E_{\alpha}}.$$  \hspace{1cm} (7)

Now it is clear that the only states having nonzero internal product with the boundary states are those involving pairs of excitations: if rapidity $\theta$ is occupied, so must rapidity $-\theta$. As we are going to do thermodynamics using these states as a basis, it is important to classify them properly. Using only positive rapidities for the labeling, we can construct four different pairs, which we label 11, 22, 12, 21. A pair 12 has for example the meaning of a soliton at rapidity $-\theta$ and an antisoliton at rapidity $\theta$. A difficulty then occurs when one implements fermionic statistics. Whereas fermions of the same type cannot occupy the same rapidity state, we can in fact superimpose some pairs, e.g. pairs 11 and 22 but not 12 and 22. To do the thermodynamics, it is useful to consider that in fact none of these pairs can be superimposed, and then account for the possible superpositions.
by introducing one additional species, a quartet, obtained by superposition of 11 and 22, or, equivalently, 12 and 21. This appears in the boundary state with amplitude $K_{11}^{12}K_{12}^{22} - K_{11}^{12}K_{12}^{21} = \det(K_{11})$. The trace over states $\alpha$ now becomes a trace over configurations of these five excitations on the lattice of allowed rapidities. The fugacity of $\alpha$ as determined by the boundary states reads

$$\frac{\langle B_1|\alpha\rangle\langle\alpha|B_2\rangle}{\langle\alpha|\alpha\rangle} = \prod_{i=1}^{N} K_{1i}^{\alpha(i)} K_{2i}^{\alpha(i)} \prod_{i=1}^{N} \det(\tilde{K}_{1i}(\theta_j^1)K_{1i}(\theta_j^2))$$

(8)

where in this example we take $\alpha$ to have $N$ pairs of types $a_i, b_i$ at rapidities $\theta_i$, $i = 1, \ldots, N$, and $N_0$ quartets at rapidities $\theta_j$, $i = 1, \ldots, N_0$. The TBA can then be done according to standard procedures by looking for a saddle point in the sum $\tilde{F}$, yielding an expression for $\ln Z$ which we can interpret as the zero-temperature ground-state energy when $L \to \infty$. The end result is

$$E_0 = \lim_{m \to 0} -\frac{1}{2\pi} \int_0^\infty d\theta \ln \left[ 1 + tr[\tilde{K}_1K_r]e^{-2mR \cos \theta} + \det[\tilde{K}_1K_r]e^{-4mR \cos \theta} \right] \theta \cosh \theta.$$ 

(9)

Taking the limit, and substituting for the $K$ matrices from $\tilde{F}$, this formula becomes

$$E_0 = -\frac{1}{4\pi} \int_0^\infty d\kappa \ln \left[ 2\kappa^2 + 4\Delta_1^2 \Delta_2^2 \cos 2\chi \left( \frac{\kappa + 2\Delta_2^2}{\kappa + 2\Delta_1^2} \right) e^{-2\kappa R} + \left( \frac{\kappa - 2\Delta_1^2}{\kappa + 2\Delta_2^2} \right) e^{-2\kappa R} + 1 \right].$$

(10)

In fact, expression (10) hides unexpected subtleties. To understand the latter, and how to deal with them, it is useful to discuss for a while a simpler and closely related model, the critical Ising model with boundary magnetic fields $h_l, h_r$. As discussed in $\tilde{F}$, this theory is integrable, and the same kind of thermodynamical approach gives rise to the ground-state energy in a finite size $R$ $\tilde{F}$:

$$E_0^l(h, h') = -\frac{1}{4\pi R} \int_0^\infty d\kappa \times \ln \left[ 1 + \epsilon - 8\pi\hbar^2 R - \epsilon - 8\pi(h')^2 R \right].$$

(11)

This is in apparent contradiction with physical expectations. Indeed, it involves only the squares of the magnetic fields, while basic symmetry considerations lead one to expect an energy that depends on the fields $h_l, h_r$ themselves; lowest order perturbation theory in fact leads to

$$E_0^l(h, h') = -\frac{\pi}{48R} - h^2 R - f(h^2 R) - 2\pi h h' + \ldots$$

(12)

with $f(x) = 2x^2(\text{est.} + \ln x) / R$. The contradiction is partly resolved by realizing that expression (12) expands in fact in the variables $|h_l|, |h_r|$, and careful manipulations allow one to reproduce (12) in the case $h_l h_r > 0$. Expression (11) cannot be right however since it does not distinguish (even after the non-analyticity in $h_2^2, h_2^2$ has been taken into account) the cases $h_l h_r > 0$ and $h_l h_r < 0$. These cases should be different, and we think (11) applies to the former only. A quick assessment of the situation can be made in the limit of very large boundary fields, where $h_l h_r > 0$ corresponds to $++$ fixed boundary conditions, where based on conformal field theory we expect and do indeed get from (11) $E_0^l = -\frac{\pi}{48R}$, while the other case describes $+-$ fixed boundary conditions, for which we expect $E_0^l = -\frac{\pi}{48R} + \frac{\pi}{2R}$; the difference being the boundary dimension of the spin operator, $\Delta = 1/2$.

The correct expression for $h_l h_r < 0$ is obtained using the following physical considerations. The point is that, for a given boundary field $h$, the boundary state is fully defined only if one specifies the boundary conditions asymptotically far away from it. The results in $\tilde{F}$ assume implicitly that the spin at infinity is fixed at $+$ while the boundary field is positive. The case of a spin fixed at infinity should be obtained by considering an ‘excited’ boundary state. Following this procedure here leads to the result that, for $h_l h_r > 0$,

$$E_0^l(h, h') - E_0^l(h, -h') = -\frac{1}{2} k_+$$

(13)

where $k_+$ is a particular solution of the quantization condition $e^{iRk} \sinh^2 h - i k \sinh^2 h = -1$. Specifically, all solutions are analytic functions of $h_2^2$ and $h_2^2$, except for one pair going as $k_+ \approx \pm 8\pi |h_l h_r|$ for small fields. It can be proven that equation (13) does indeed provide the analytical continuation of (11) into the region $h_l h_r < 0$. For instance, the lowest term in the perturbative expansion reads now for $h_l h_r < 0$ $-2\pi |h_l h_r| + 4\pi |h_l h_r| = 2\pi |h_l h_r| = -2\pi h_l h_r$, as desired. We also observe that large magnetic fields, $k_+ \to \frac{\pi}{R}$, producing the expected gap in the conformal limit. Note that a similar analytical continuation procedure was proposed in $\tilde{F}$.

Expression (13) is extraordinarily simple; while each of the terms in the left-hand side requires calculation of a TBA integral, their difference is simply obtained by the solution of an elementary quantization condition. We have checked its validity by computing numerically the ground state energies of the critical Ising model with boundary magnetic fields, and studying their scaling limit. Results for $h_l = \pm h_r$ will be presented figure 2 and are in very good agreement with our formulas. We note that the lattice Ising model with boundary magnetic fields has never been solved to the best of our knowledge (the continuum version with $h_l = h_r$ has been treated in $\tilde{F}$), except in the case $h_l = 0$ $\tilde{F}$, where one can check that the results agree with those of our TBA approach.
In the sine-Gordon case, we have a similar situation. Since $E_0$ in $[10]$ is an even function of $\chi$, we can concentrate on the region $\chi \geq 0$. The crucial observation is that $[10]$ is valid only in the domain $\chi \in [0, \pi/2]$ since it exhibits a singularity at $\chi = \pi/2$, where the argument of the log has a zero that hits the real axis at $\kappa = 0$. Better intuition is gained by studying the limits $\Delta_{l,r} \to \infty$, where the integral reduces to $E_0 = -\frac{1}{24R} \int_0^\infty dx \ln \left[ 1 + 2 \cos \chi e^{-x} + e^{-2\pi} \right]$. This integral is tabulated, and we get $E_0 = -\frac{\pi}{24R} + \frac{\chi^2}{2\pi R}$ for $\chi \in [0, \pi/2]$, and $E_0 = -\frac{\pi}{24R} + \frac{(\pi-\chi)^2}{2\pi R}$ for $\chi \in [\pi/2, \pi]$. The current $I \propto \frac{\partial E}{\partial \chi}$ thus experiences a discontinuity at $\chi = \pi/2$, which contradicts perturbation theory: it should be odd in $\chi$, and a smooth function over the interval $[0, \pi]$. As in the Ising case, our procedure to repair this starts by identifying the root $k_0$ of the quantization condition $2 \left[ \frac{ik}{2\pi R} \right]^2 \cos 2\chi \kappa = 1$, which requires analytic continuation beyond $\chi = \pi/2$.

The limiting behaviours of the boundary contributions to the ground-state energy are $\delta E^{SG} \to \frac{\chi^2}{2\pi R}$ for $\Delta_{l,r}$ large, and $\delta E^{SG} \propto -\Delta_{l,r} \cos \chi$ for small $\Delta_{l,r}$.

With this formula, we can write down the Josephson current of our device at the free fermion point $\beta^2 = 4\pi$, in terms of the dimensionless parameters $\Delta^2_{L,R} = R\Delta^2_{l,r}$:

$$I(\chi) = \frac{8e\Delta^2_{L,R}}{\pi R} \sin 2\chi \cdot \int_0^\infty dk \times$$

$$\left[ \kappa + 2\Delta^2_{L,R} \right]^{-1} \left[ \kappa + 2\Delta^2_{R} \right]^{-1} e^{-\kappa} \left( \kappa + 2\Delta^2_{L} \right) \left( \kappa + 2\Delta^2_{R} \right)^{-1} e^{-2\kappa}. \quad (15)$$

with the understanding that this should be analytically continued beyond $\chi = \pi/2$ using the above procedure. This formula for the current provides complete interpolation between the limits of the sawtooth function for perfect Andreev boundary scattering ($\Delta_{l,r} \to \infty$), and the sin $\chi$ behaviour for small Andreev scattering. These and further results are discussed in more detail in $[11]$.

J.-S. C. would like to thank J. Cardy, P. Dorey, F. Essler and A. M. Tsvelik for useful discussions.