Can Magnetic Monopoles And Massive Photons Coexist In The Framework Of The Same Classical Theory?

C. Cafaro and S. A. Ali

Department of Physics, State University of New York at Albany-SUNY, 1400 Washington Avenue, Albany, NY 12222, USA

S. Capozziello

Dipartimento di Scienze Fisiche, Università di Napoli "Federico II", Via Cinthia, 80126 Napoli, Italy

Ch. Corda

INFN Sezione di Pisa and University of Pisa, Via F. Buonarroti 2, 56127 Pisa, Italy; European Gravitational Observatory (EGO), Via E. Amaldi, 56021 Cascina (PI), Italy

It is well known that one cannot construct a self-consistent quantum field theory describing the non-relativistic electromagnetic interaction mediated by massive photons between a point-like electric charge and a magnetic monopole. We show that, indeed, this inconsistency arises in the classical theory itself. No semi-classic approximation or limiting procedure for $\hbar \to 0$ is used. As a result, the string attached to the monopole emerges as visible also if finite-range electromagnetic interactions are considered in classical framework.

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In his classical works, Dirac showed that the existence of a magnetic monopole would explain the electric charge quantization \[1\]. This is known as the Dirac quantization rule. There exist various arguments based on quantum mechanics, theory of representations, topology and differential geometry on behalf of Dirac’s rule \[2, 3\]. Dirac’s formulation of magnetic monopoles takes into account a singular vector potential. Other approaches exist where two non-singular vector potentials, related through a gauge transformation, are used \[4, 5\]. Finite-range electrodynamics is a theory with non-zero photon mass. It is an extension of the standard theory and is fully compatible with experiments.

The existence of Dirac’s monopole in massless electrodynamics is compatible with the above quantization condition if the string attached to the monopole is invisible. The quantization condition can be obtained either with the help of gauge invariance or angular momentum quantization. In massive electrodynamics, both these approaches are no longer applicable \[6\]. These conclusions are formulated in a quantum framework which is a quantized version of the classical one. The Hamiltonian formulation and the problems involved in quantization of Dirac’s theory of monopoles have been extensively discussed in the past and is still an active field of research \[7, 8\]. Major work on the quantum field theory of magnetic charges has been developed by Schwinger \[9, 10, 11\] and Zwanziger \[12\]. Recent work on constructing a satisfactory classical relativistic framework for massive electrodynamics and magnetic monopoles from a geometrical point of view has been considered in \[13, 14\]. A complete update on the experimental and theoretical status of monopoles is presented in \[15\].

In this letter we consider the problem of constructing the static limit of a consistent classical, non-relativistic electromagnetic theory describing a point-like electric particle with charge $e$ and mass $m$ moving in the field of a fixed composite monopole of charge $e_m$, where their mutual interaction is mediated by massive carrier gauge fields. The total magnetic field $\vec{B}$ is comprised of point-like magnetic charge, a semi-infinite string along the negative $z$-axis and diffuse magnetic field contributions. We impose that the electrically charged particle must never pass through the string (“Dirac-veto”) \[16\] and therefore the motion of the test charged particle is constrained to region of motion $\mathbb{R}^+ := \{ (r, \theta, \varphi) : r \in \mathbb{R}^+_0, \theta \in [0, \pi), \varphi \in [0, 2\pi] \}$. It is known that no spherically symmetric diffuse magnetic field solutions are allowed in Maxwell’s classical electrodynamics with massive photons and magnetic monopoles \[6\]. Requiring the theory presented here be endowed with a well-defined canonical Poisson bracket structure, it is shown that the total angular momentum is the generator of rotations. Furthermore, by demanding proper transformation rules under spatial rotations for the allowed magnetic vector field solutions, it is shown that only spherically symmetric diffuse magnetic fields satisfy the Lie algebra of the system. This leads to conclude that the permitted solutions to the

*Electronic address: carlocafaro2000@yahoo.it
†Electronic address: alis@alum.rpi.edu
‡Electronic address: capozziello@na.infn.it (Corresponding Author)
§Electronic address: christian.corda@ego-gw.it
generalized Maxwell theory are incompatible with the Lie algebra of the Hamiltonian formulation. As a consequence, any quantization procedure applied to this classical theory would lead to an inconsistent quantum counterpart.

Maxwell’s equations with non-zero photon mass and magnetic charge follow from a standard variational calculus \[17\] \[18\] of the Maxwell-Proca-Monopole action functional. The field equations for the electromagnetic 4-vector potential \( A_\mu \) together with the Bianchi identities and Lorenz gauge condition \( \partial_\mu A^\mu = 0 \), lead to the generalized Maxwell equations in three-dimensions:

\[
\nabla \cdot \vec{E} = 4\pi \rho_e - m_\gamma^2 \vec{A}_0, \quad \nabla \times \vec{E} = -e^{-1} \partial_t \vec{B} - 4\pi e^{-1} j_m, \quad (1)
\]

\[
\nabla \cdot \vec{B} = 4\pi \rho_m, \quad \nabla \times \vec{B} = 4\pi e^{-1} j_e + e^{-1} \partial_t \vec{E} - m_\gamma^2 \vec{A}, \quad (2)
\]

where \( m_\gamma = \frac{\gamma}{\omega} \) and \( \omega \) is the frequency of the photon. In absence of electric fields, charges and currents, as well as the absence of magnetic current, the static monopole-like solution of this system is,

\[
\vec{B} = \vec{B}^{(Dirac)} + \vec{B}_\gamma, \quad (3)
\]

where \( \vec{B}^{(Dirac)} \) is the standard Dirac magnetic field,

\[
\vec{B}^{(Dirac)} = \frac{e_m}{r^2} \hat{r} \quad (4)
\]

whose divergence and curl are given by,

\[
\nabla \cdot \vec{B}^{(Dirac)} = 4\pi e_m \delta^{(3)}(r) \quad \text{and} \quad \nabla \times \vec{B}^{(Dirac)} = 0. \quad (5)
\]

The diffuse magnetic field \( \vec{B}_\gamma(r) \) is given by the following general expression,

\[
\vec{B}_\gamma(r) = b_\gamma^{(1)}(r, \hat{n} \cdot \hat{r}) + b_\gamma^{(2)}(r, \hat{n} \cdot \hat{r})\hat{n} \quad (6)
\]

where \( b_\gamma^{(1)} \) and \( b_\gamma^{(2)} \) are general scalar field functions and \( \hat{n} \) is a unitary vector along the monopole string. The magnetic field \( \vec{B}_\gamma(\hat{r}) \) is such that,

\[
\nabla \cdot \vec{B}_\gamma = 0 \quad \text{and} \quad \nabla \times \vec{B}_\gamma = -m_\gamma^2 (\vec{A}^{(Dirac)} + \vec{A}_\gamma). \quad (7)
\]

The vector \( \vec{A}^{(Dirac)} \) is the standard singular vector potential representing the field of a fixed monopole,

\[
\vec{A}^{(Dirac)}(\hat{r}) = \frac{e_m}{r^2} \frac{\sin(\theta)}{1 + \cos(\theta)} (\hat{n} \times \vec{r}), \quad \theta \neq \pi, \quad (8)
\]

with semi-infinite singularity line oriented along the negative z-axis, where \( e_m \) is the magnetic charge. The vector potential \( \vec{A}_\gamma(\hat{r}) \) is given by the following general expression,

\[
\vec{A}_\gamma(\hat{r}) = e_m m_\gamma^2 f_\gamma(m_\gamma r, m_\gamma \hat{r} \cdot \hat{n})(\hat{n} \times \vec{r}) \quad (9)
\]

where \( f_\gamma \) is a generic scalar field function. Because of the second equation in (7), it is clear that no spherically symmetric diffuse magnetic field solutions are allowed, that is to say, solutions like

\[
\vec{B}_\gamma(\hat{r}) = B_\gamma(r) \hat{r} \quad (10)
\]

are not allowed.

On the other hand, it is known that the classical non-relativistic theory describing the massless electromagnetic scattering of an electric charge from a fixed magnetic monopole does have a Hamiltonian formulation \[18\]. With this result in mind, let us consider the classical non-relativistic theory describing a point-like electric particle with charge \( e \) and mass \( m \) moving in the field of a fixed monopole of charge \( e_m \), but let us suppose that the electromagnetic interaction is mediated by massive photons. The total magnetic field \( \vec{B} \) is comprised of the point-like magnetic charge, string and diffuse magnetic field contributions

\[
\vec{B} = B^{(Dirac)} + \vec{B}_\gamma = \left[ \nabla \times \vec{A}^{(Dirac)} + e_m \vec{f}(r) \right] + \nabla \times \vec{A}_\gamma = \nabla \times \vec{A} + e_m \vec{f}(r), \quad \vec{A} = \vec{A}^{(Dirac)} + \vec{A}_\gamma \quad (11)
\]
where
\[ \left\| \vec{f}(\vec{r}) \right\| = 4\pi \delta(x) \delta(y) \Theta(-z) = \frac{4\pi}{r^2} \delta(\vartheta) \delta(\varphi) \Theta(-\cos \vartheta) \] (12)
is the string function having support only along the line $\vec{n} = -\hat{z}$ and passing through the origin while $\Theta$ is the Heaviside step function.

The classical Newtonian equation of motion describing this system is
\[ m \frac{d^2 \vec{r}}{dt^2} - \frac{e}{c} \frac{d\vec{r}}{dt} \times (\vec{\nabla} \times \vec{A}) - \frac{ee_m}{c} \frac{d\vec{r}}{dt} \times \vec{f}(\vec{r}) = 0. \] (13)
The restricted Hamiltonian associated with (15) is given by
\[ H_{\text{total}}(\vec{p}, \vec{r}) = \left( \vec{p} - \frac{e}{c} \vec{A} \right)^2 + H_{\text{string}}, \] (14)
where $\vec{p} = m \frac{d\vec{r}}{dt} + \frac{e}{c} \vec{A}$ is the canonical momentum vector, $\vec{P} = \vec{p} - \frac{e}{c} \vec{A} = m \frac{d\vec{r}}{dt}$ is the kinetic momentum vector, $\vec{L} = \vec{r} \times \vec{p}$ is the orbital angular momentum of the system and $\vec{J} = \vec{L} + \vec{s}$ is the total angular momentum such that $\vec{J} \cdot \vec{s} = 0$ where
\[ \vec{s} = (4\pi c)^{-1} \int (\vec{r} \times (\vec{E} \times \vec{B})) \, d^3\vec{r} = \vec{s}_{\text{massless}} + \frac{e}{4\pi c} \int d\vec{r} \times \left[ \vec{r} \times \vec{B} \right] \left( \vec{r} + \vec{R} \right), \] (17)
with $\vec{s}_{\text{massless}} = \frac{ee_m}{c} \vec{R}$ and $\vec{R}$ is the relative vector position between the monopole and the electric charge. The vector $\vec{s}$ is taken as an angular momentum with independent degrees of freedom and must obey the following classical Poisson-bracket relation
\[ \{s_i, s_j\} = -\varepsilon_{ijk} s_k. \] (18)

Observe that $H_{\text{total}}(\vec{p}, \vec{r})$ is not spherically symmetric due to the occurrence of $H_{\text{string}}$ and even in the restricted case of $H(\vec{p}, \vec{r})$, the term $\vec{\nabla} \times \vec{A}_\gamma$ breaks rotational invariance since
\[ \frac{d\vec{r}}{dt} \times (\vec{\nabla} \times \vec{A}_\gamma) = \frac{d\vec{r}}{dt} \cdot (\vec{\nabla} \times \vec{A}_\gamma) - \left( \frac{d\vec{r}}{dt} \cdot \vec{\nabla} \right) \vec{A}_\gamma = -\left( \frac{d\vec{r}}{dt} \cdot \vec{\nabla} \right) \vec{A}_\gamma \neq 0 \text{ in general.} \] (19)
We made use of the transversality condition $\vec{\nabla} \cdot \vec{A}_\gamma = 0$ in computing $\frac{d\vec{r}}{dt} \times (\vec{\nabla} \times \vec{A}_\gamma)$ [6]. Furthermore, we emphasize that we may obtain a spherically symmetric Hamiltonian provided the auxiliary condition $\left( \frac{d\vec{r}}{dt} \right)_k \partial_k (A_\gamma) = 0 \forall j = 1, 2, 3$ is satisfied. Such condition is however unnecessary for our present analysis.

The Poisson brackets between two generic functions $f(\vec{p}, \vec{r}, t)$ and $g(\vec{p}, \vec{r}, t)$ of the dynamical variables $\vec{p}$ and $\vec{r}$, are defined as,
\[ \{f(\vec{p}, \vec{r}, t), g(\vec{p}, \vec{r}, t)\} \overset{\text{def}}{=} \sum_i \left( \partial_{p_i} f \partial_{r_i} g - \partial_{r_i} f \partial_{p_i} g \right) \] (20)
and the basic canonical Poisson bracket structure for the conjugate variables is given by,
\[ \{r_i, r_j\} = 0, \quad \{r_i, p_j\} = -\delta_{ij}, \quad \{p_i, p_j\} = 0. \] (21)
Let us show explicitly that \( \vec{J} \) is the generator of spatial rotations so that we can safely define the rank of a tensor by studying its transformation rules under such rotations. Let us prove,

\[
\{ J_i, J_j \} = -\varepsilon_{ijk} J_k. \tag{22}
\]

Using the tensorial notation for the cross product appearing in the definition of \( \vec{J} \), and using the standard properties of a well-defined Poisson bracket structure, the brackets in equation (22) become,

\[
\{ J_i, J_j \} = \{ \varepsilon_{ijk} r_j p_k, \varepsilon_{lmp} r_m p_n \} - \{ \varepsilon_{ijk} r_j p_k, \varepsilon_{lmn} r_m A_n \} + \{ \varepsilon_{ijk} r_j A_k, \varepsilon_{lmn} r_m p_n \} + \{ \varepsilon_{ijk} r_j A_k, \varepsilon_{lmn} r_m A_n \} + \{ s_i, s_l \}. \tag{23}
\]

Using the basic canonical Poisson bracket structure expressed in (21) and the standard properties of Poisson brackets together with the following identity,

\[
\varepsilon_{ijk} \varepsilon_{mlk} = \delta_{im} \delta_{jl} - \delta_{il} \delta_{jm} \tag{24}
\]

the first bracket on the rhs of (23) becomes,

\[
\{ \varepsilon_{ijk} r_j p_k, \varepsilon_{lmn} r_m p_n \} = r_i p_i - r_l p_l. \tag{25}
\]

Similarly, the second, the third and the fourth brackets on the rhs of (23) become,

\[
- \{ \varepsilon_{ijk} r_j p_k, \varepsilon_{lmn} r_m A_n \} = \delta_{il} r_n A_n - r_l A_i + \varepsilon_{ijk} \varepsilon_{lmp} p_k \{ A_n, r_j \} \tag{26}
\]

\[
- \{ \varepsilon_{ijk} r_j A_k, \varepsilon_{lmn} r_m p_n \} = -\delta_{il} r_k A_k + r_i A_l + \varepsilon_{ijk} \varepsilon_{lmp} r_j p_n \{ r_m, A_k \} \tag{27}
\]

\[
\varepsilon_{ijk} \varepsilon_{lmp} r_m A_n \{ A_n, r_j \} - \varepsilon_{ijk} \varepsilon_{lmp} r_j A_k \{ A_k, r_m \} - \varepsilon_{ijk} \varepsilon_{lmp} r_m A_k \{ A_n, r_j \}. \tag{28}
\]

The last bracket on the rhs of (23) is given by (18). Finally, substituting these five brackets in the rhs of (23) and ordering them properly, the Poisson brackets of \( \vec{J} \) become,

\[
\{ J_i, J_j \} = (r_i p_i - r_l p_l - r_l A_i + r_i A_l - \varepsilon_{ilm} s_m) + \varepsilon_{ijk} \varepsilon_{lmp} [r_m p_k \{ A_n, r_j \} - r_j p_n \{ r_m, A_k \}] + \varepsilon_{ijk} \varepsilon_{lmp} [r_j A_n \{ A_k, r_m \} - r_m A_k \{ A_n, r_j \}]. \tag{29}
\]

Because of the full antisymmetry of the Levi-Civita tensor,

\[
\varepsilon_{ijk} \varepsilon_{lmp} r_m p_k \{ A_n, r_j \} - \varepsilon_{ijk} \varepsilon_{lmp} r_j p_n \{ A_n, r_m \} = 0. \tag{30}
\]

Therefore, equation (29) becomes,

\[
\{ J_i, J_j \} = r_i p_i - r_l p_l - r_l A_i + r_i A_l - \varepsilon_{ilm} s_m = -\varepsilon_{ilm} [\varepsilon_{mnk} r_n (p_k - A_k) + s_m]. \tag{31}
\]

Using equation (24), we obtain

\[
- \varepsilon_{ilm} \varepsilon_{mnk} r_n p_k = (r_l p_i - r_i p_l) \text{ and } \varepsilon_{ilm} \varepsilon_{mnk} r_n A_k = -(r_l A_i - r_i A_l) \tag{32}
\]

and finally,

\[
\{ J_i, J_j \} = -\varepsilon_{ilm} J_m. \tag{33}
\]

At this point, we have all the elements to show the classical inconsistency of the problem. Recall the kinetic momentum vector is defined as,

\[
\vec{P} \overset{\text{def}}{=} \vec{p} - \frac{e}{c} \vec{A}, \quad \vec{A} = \vec{A}_g + \vec{A}^{(\text{Dirac})}. \tag{34}
\]


Let us assume that there exist a well-defined Poisson bracket structure in the classical theoretical setting in consideration. In particular, let us assume a well-defined classical Poisson bracket structure among the vector fields $\vec{J}$, $\vec{P}$, and $\vec{r}$, that is,
\begin{equation}
\{J_i, J_j\} = -\varepsilon_{ijk} J_k, \quad \{J_i, r_j\} = -\varepsilon_{ijk} r_k, \quad \{J_i, P_j\} = -\varepsilon_{ijk} P_k.
\end{equation}

Being $\vec{J}$ the generator of rotations, it is required that any arbitrary vector $\vec{v}$ must satisfy the following classical commutation rules,
\begin{equation}
\{J_i, v_j\} = -\varepsilon_{ijk} v_k.
\end{equation}

Therefore, let us study the transformation properties of the magnetic field under spatial rotations. It must be,
\begin{equation}
\{J_i, B_j\} = -\varepsilon_{ijk} B_k.
\end{equation}

In terms of the magnetic field decomposition, equation (37) is equivalent to,
\begin{equation}
\{J_i, B_j^{\text{(Dirac)}}\} = -\varepsilon_{ijk} B_k^{\text{(Dirac)}} \text{ and } \{J_i, (B_\gamma)_j\} = -\varepsilon_{ijk} (B_\gamma)_k.
\end{equation}

It is quite straightforward to check the validity of the first equation in (38), as a matter of fact,
\begin{equation}
\{J_i, B_j^{\text{(Dirac)}}\} = \left\{ J_i, \frac{e_m}{p^3} r_j \right\} = \frac{e_m}{p^3} \{ J_i, r_j \} + \left\{ J_i, \frac{e_m}{p^3} \right\} r_j
\end{equation}
\begin{equation}
= -\varepsilon_{ijk} \frac{e_m}{p^3} r_k \equiv -\varepsilon_{ijk} B_k^{\text{(Dirac)}}.
\end{equation}

Let us consider the validity of equation (37), where the total magnetic field $\vec{B}$ is given by
\begin{equation}
B_j (r, \theta, \varphi) = \varepsilon_{jm} \partial_t A_m (r, \theta, \varphi) + e_m f_j (\vec{r}).
\end{equation}

By virtue of the "Dirac-veto", the magnetic field $B_j (r, \theta, \varphi)$ "felt" by the electric charge reduces to
\begin{equation}
B_j (r, \theta, \varphi) = \varepsilon_{jm} \partial_t A_m (r, \theta, \varphi).
\end{equation}

Fixing the constants $c$ and $e$ equal to one for the sake of convenience, let us consider first the Poisson brackets of the kinetic momentum vector components. Using (21), the standard properties of Poisson brackets together with equations (24) and (41), we obtain,
\begin{equation}
\{P_i, P_j\} = -\varepsilon_{ijk} B_k.
\end{equation}

Multiplying both sides of (42) by $\varepsilon_{ijn}$, we obtain
\begin{equation}
\varepsilon_{ijn} \{P_i, P_j\} = -\varepsilon_{ijn} \varepsilon_{ijk} B_k = -2 \delta_{nk} B_k = -2B_n
\end{equation}

and therefore,
\begin{equation}
B_k = \frac{1}{2} \varepsilon_{ijk} \{P_i, P_j\}.
\end{equation}

Therefore, substituting $B_k$ of equation (44) into (37), we obtain
\begin{equation}
\{J_i, B_j\} = -\frac{1}{2} \varepsilon_{lmj} \{J_i, \{P_l, P_m\}\}.
\end{equation}

The double commutator in equation (45) cannot be calculated in a direct way. However, because we are assuming the existence of a well-defined Poisson bracket structure among the vectors $\vec{J}$, $\vec{B}$ and $\vec{r}$, this double commutator can be evaluated by using the following Jacobi identity,
\begin{equation}
\{J_i, \{P_l, P_m\}\} + \{P_m, \{J_i, P_l\}\} + \{P_l, \{P_m, J_i\}\} = 0.
\end{equation}

Thus, using the fact that $\vec{J}$ is the generator of rotations, that $\vec{B}$ transforms as a vector quantity under rotations, and using equation (24), we obtain
\begin{equation}
\{J_i, \{P_l, P_m\}\} = -\delta_{il} B_m + \delta_{lm} B_l.
\end{equation}
Substituting equations (44) into (47), we obtain
\[ \{ J_i, B_j \} = -\varepsilon_{ijm} B_m. \] (48)

Therefore, we have shown that in a pure classical theoretical framework given by the Poisson brackets formalism, the commutation rule between the generator of spatial rotations and the total magnetic field is expressed in (48). Our last step is to calculate the Poisson brackets between \( \vec{J} \) and the magnetic field \( \vec{B} \). Using equation (6), standard Poisson brackets properties and the fact that \( \vec{J} \) is the generator of rotations, these brackets become,
\[ \{ J_i, (\vec{B} \gamma)_j \}_{\text{Poisson}} = -\varepsilon_{ijk} (\vec{B} \gamma)_k + \{ J_i, b^{(1)}_\gamma \} r_j + \{ J_i, b^{(2)}_\gamma \} n_j. \] (49)

In order to have proper Poisson brackets, for each vectors \( \hat{n} \) and \( \hat{r} \), the following relation must hold
\[ \{ J_i, b^{(1)}_\gamma \} r_j + \{ J_i, b^{(2)}_\gamma \} n_j = 0. \] (50)

Observe that the second Poisson bracket in the rhs of (49) contains a term quadratic in \( n_k \),
\[ \{ J_i, b^{(2)}_\gamma \} n_j = (\partial_{p_k} J_i)(\partial_{r_k} b^{(2)}_\gamma) n_j = (\partial_{p_k} J_i) \left[ \partial_{r_k} b^{(2)}_\gamma \frac{r_k}{r} + \partial_{(\vec{r} \widehat{\gamma})} b^{(2)}_\gamma n_k \right] n_j \]
\[ = \frac{1}{r} \partial_{p_k} J_i \partial_{r} b^{(2)}_\gamma r_{k} n_j + \partial_{p_k} J_i \partial_{(\vec{r} \widehat{\gamma})} b^{(2)}_\gamma n_k n_j. \] (51)

Since, the proper Poisson brackets should be linear in \( n_k \), we require
\[ \partial_{(\vec{r} \widehat{\gamma})} b^{(2)}_\gamma = 0. \] (52)

There is no way to cancel out this term in (49), then it must be,
\[ b^{(2)}_\gamma = 0. \] (53)

We now consider the first Poisson bracket on the rhs of (49). Because of the anti-symmetry in the indices \( i \) and \( j \) of the term \( \varepsilon_{ijk}(\vec{B} \gamma)_k \), it must be
\[ \{ J_i, b^{(1)}_\gamma \} r_j + \{ J_j, b^{(1)}_\gamma \} r_i = 0 \] (54)

that is,
\[ \{ J_i, b^{(1)}_\gamma \} r_i = 0. \] (55)

Explicitly, equation (55) becomes,
\[ 0 = (\partial_{p_k} J_i)(\partial_{r_k} b^{(1)}_\gamma) r_i = (\partial_{p_k} J_i) \left[ \partial_{r_k} b^{(1)}_\gamma \frac{r_k}{r} + \partial_{(\vec{r} \widehat{\gamma})} b^{(1)}_\gamma n_k \right] r_i = \]
\[ = \frac{1}{r} (\partial_{p_k} J_i)(\partial_{r} b^{(1)}_\gamma) r_k r_i + (\partial_{p_k} J_i)(\partial_{(\vec{r} \widehat{\gamma})} b^{(1)}_\gamma) n_k r_i. \] (56)

We neglect the quadratic term in \( r_k \) in equation (56) since this term has no analog in the proper Poisson brackets. Then, we have
\[ \partial_{(\vec{r} \widehat{\gamma})} b^{(1)}_\gamma = 0. \] (57)

Recalling that
\[ \hat{n} = -\hat{z} = - \left\{ \cos(\theta)\hat{r} - \sin(\theta)\hat{\theta} \right\} = - \cos(\theta)\hat{r} + \sin(\theta)\hat{\theta} \] (58)

then,
\[ \hat{n} \cdot \hat{r} = -\cos(\theta) = \theta - \text{dependence}. \] (59)
Therefore, equation (57) is satisfied by an arbitrary scalar function \( b_{\gamma}(r) \). As a consequence, the magnetic field \( \vec{B}_{\gamma} \) is not \( \theta \)-dependent (in a more general situation in which \( \vec{n} \) is not along the z-axis, we would conclude that the magnetic field is not \( (\theta, \varphi) \)-dependent). \( \vec{B}_{\gamma} \) must be a spherically symmetric field whose general expression is the following,

\[
\vec{B}_{\gamma}(\vec{r}) = B_{\gamma}(r)\hat{r}.
\]

In conclusion, in order to have a well-defined classical Poisson bracket structure in the problem under investigation, one must deal with diffuse magnetic field solutions exhibiting spherical symmetry. However, those very same solutions are not compatible with massive classical electrodynamics with magnetic monopoles. This result means that it is not possible to formulate a consistent non-relativistic classical theory of "true" Dirac monopoles (invisible string, "monopole without a string") and massive photons unless the string attached to the monopole is treated as an independent dynamical quantity. An important feature of our approach is that we do not use any kind of semiclassical approximation or limiting procedure for \( \hbar \rightarrow 0 \).

**APPENDIX A: THE GENERATOR OF SPATIAL ROTATIONS**

We show that \( \vec{J} \) is the generator of spatial rotations, that is,

\[
\{ J_i, J_j \} = -\varepsilon_{ijk} J_k.
\]  

(A1)

Notice that,

\[
\{ J_i, J_j \} = \{ \varepsilon_{ijk} r_j (p_k - A_k) + s_i, \varepsilon_{lmn} r_m (p_n - A_n) + s_l \} = \{ \varepsilon_{ijk} r_j p_k - \varepsilon_{ijk} r_j A_k + s_i, \varepsilon_{lmn} r_m p_n - \varepsilon_{lmn} r_m A_n + s_l \}
\]

\[
= \{ \varepsilon_{ijk} r_j p_k, \varepsilon_{lmn} r_m p_n \} - \{ \varepsilon_{ijk} r_j A_k, \varepsilon_{lmn} r_m A_n \} + \{ \varepsilon_{ijk} r_j A_k, \varepsilon_{lmn} r_m p_n \} + \{ \varepsilon_{ijk} r_j A_k, \varepsilon_{lmn} r_m A_n + \{ s_i, s_l \} \}.
\]

(A2)

Therefore there are five Poisson brackets to be calculated. Consider the first one,

\[
\{ \varepsilon_{ijk} r_j p_k, \varepsilon_{lmn} r_m p_n \} = \varepsilon_{ijk} \varepsilon_{lmn} \{ r_j p_k, r_m p_n \} = \varepsilon_{ijk} \varepsilon_{lmn} \{ r_j \{ p_k, r_m p_n \} + \{ r_j, r_m p_n \} p_k \}
\]

\[
= \varepsilon_{ijk} \varepsilon_{lmn} \{ -r_j \{ r_m p_n, p_k \} - \{ r_m p_n, r_j \} p_k \}
\]

\[
= \varepsilon_{ijk} \varepsilon_{lmn} \{ -r_j \{ r_m \{ p_k, p_n \} + \{ r_m, p_k \} p_n \} \} + \varepsilon_{ijk} \varepsilon_{lmn} \{ -\{ r_m \{ p_n, r_j \} + \{ r_m, r_j \} p_n \} p_k \}
\]

\[
= \varepsilon_{ijk} \varepsilon_{lmn} \{ \delta_{mk} r_j p_k - \delta_{mj} r_m p_k \} = \varepsilon_{ijk} \varepsilon_{lmn} \delta_{mk} r_j p_k - \varepsilon_{ijk} \varepsilon_{lmn} \delta_{mj} r_m p_k
\]

\[
= \varepsilon_{ijk} \varepsilon_{lmn} \delta_{mk} r_j p_k - \varepsilon_{ink} \varepsilon_{lmn} r_m p_k = -\varepsilon_{ijk} \varepsilon_{ink} r_j p_n + \varepsilon_{ink} \varepsilon_{lmn} r_m p_k
\]

\[
= -\{ \delta_{il} \delta_{jn} - \delta_{in} \delta_{jl} \} r_j p_n + \{ \delta_{il} \delta_{km} - \delta_{im} \delta_{lk} \} r_m p_k
\]

\[
= -\delta_{il} \delta_{jn} r_j p_n + \delta_{in} \delta_{jl} r_j p_n + \delta_{il} \delta_{km} r_m p_k - \delta_{im} \delta_{lk} r_m p_k
\]

\[
= -\delta_{il} r_j p_n + r_l p_i + \delta_{il} r_k p_k - r_i p_l = r_l p_i - r_i p_l
\]

(A3)

thus,

\[
\{ \varepsilon_{ijk} r_j p_k, \varepsilon_{lmn} r_m p_n \} = r_l p_i - r_i p_l.
\]  

(A4)
Consider the second bracket,

\[- \{ \varepsilon_{ijk} r_j p_k, \varepsilon_{lmn} r_m A_n \} = -\varepsilon_{ijk} \varepsilon_{lmn} \{ r_j p_k, r_m A_n \} \]

\[= -\varepsilon_{ijk} \varepsilon_{lmn} \{ r_j \{ p_k, r_m A_n \} + \{ r_j, r_m A_n \} p_k \} \]

\[= -\varepsilon_{ijk} \varepsilon_{lmn} \{ -r_j \{ r_m A_n, p_k \} - \{ r_m A_n, r_j \} p_k \} \]

\[= -\varepsilon_{ijk} \varepsilon_{lmn} \{ -r_j r_m \{ A_n, p_k \} - r_j \{ r_m, p_k \} A_n \} + \]

\[= -\varepsilon_{ijk} \varepsilon_{lmn} \{ -r_m \{ A_n, r_j \} p_k - \{ r_m, r_j \} A_n p_k \} \]

\[= -\varepsilon_{ijk} \varepsilon_{lmn} \{ \delta_{mk} r_j A_n - r_m p_k \{ A_n, r_j \} \} \]

\[= -\varepsilon_{ijk} \varepsilon_{lmn} \{ r_j \{ A_n, p_k \} \} + \varepsilon_{ijk} \varepsilon_{lmn} \{ p_k, r_m A_n \} \]

\[= \varepsilon_{ijk} \varepsilon_{lmn} r_j A_n + \varepsilon_{ijk} \varepsilon_{lmn} r_m p_k \{ A_n, r_j \} \]

\[= \delta_{jl} \delta_{jn} - \delta_{in} \delta_{jl} r_j A_n + \varepsilon_{ijk} \varepsilon_{lmn} r_m p_k \{ A_n, r_j \} \]

\[= \delta_{il} \delta_{jn} r_j A_n - \delta_{in} \delta_{jl} r_j A_n + \varepsilon_{ijk} \varepsilon_{lmn} r_m p_k \{ A_n, r_j \} \]

\[= \delta_{il} r_n A_n - r_i A_i + \varepsilon_{ijk} \varepsilon_{lmn} r_m p_k \{ A_n, r_j \} \]  \hspace{1cm} (A5)

thus,

\[- \{ \varepsilon_{ijk} r_j p_k, \varepsilon_{lmn} r_m A_n \} = \delta_{il} r_n A_n - r_i A_i + \varepsilon_{ijk} \varepsilon_{lmn} r_m p_k \{ A_n, r_j \}. \]  \hspace{1cm} (A6)

Using the standard canonical algebra, the third bracket becomes,

\[- \{ \varepsilon_{ijk} r_j A_k, \varepsilon_{lmn} r_m p_n \} = -\delta_{il} r_k A_k + r_i A_i + \varepsilon_{ijk} \varepsilon_{lmn} r_j p_n \{ r_m, A_k \}. \]  \hspace{1cm} (A7)

For the fourth bracket, we obtain

\[\{ \varepsilon_{ijk} r_j A_k, \varepsilon_{lmn} r_m A_n \} = \varepsilon_{ijk} \varepsilon_{lmn} \{ r_j A_k, r_m A_n \} \]

\[= \varepsilon_{ijk} \varepsilon_{lmn} \{ r_j \{ A_k, r_m A_n \} + \{ r_j, r_m A_n \} A_k \} \]

\[= \varepsilon_{ijk} \varepsilon_{lmn} \{ -r_j \{ r_m A_n, A_k \} - \{ r_m A_n, r_j \} A_k \} \]

\[= \varepsilon_{ijk} \varepsilon_{lmn} \{ -r_j r_m \{ A_k, A_n \} - r_m \{ A_n, r_j \} A_k \} \]

\[= -\varepsilon_{ijk} \varepsilon_{lmn} r_j A_k \{ r_m, A_k \} - \varepsilon_{ijk} \varepsilon_{lmn} r_m A_k \{ A_n, r_j \}. \]  \hspace{1cm} (A8)

For the last bracket, let us remind that the vector s is such the Poisson brackets of its components satisfy equation (18). In conclusion, using equations (A4), (A6), (A7), (A8) and using the commutation rules of the classical spin, equation (A2) becomes,

\[\{ J_i, J_j \} = r_i p_j - r_j p_i + \delta_{il} r_n A_n - r_i A_i + \varepsilon_{ijk} \varepsilon_{lmn} r_m p_k \{ A_n, r_j \} - \delta_{il} r_k A_k + \]

\[+ r_i A_i + \varepsilon_{ijk} \varepsilon_{lmn} r_j p_n \{ r_m, A_k \} - \varepsilon_{ijk} \varepsilon_{lmn} r_j A_n \{ r_m, A_k \} + \]

\[-\varepsilon_{ijk} \varepsilon_{lmn} r_j A_k \{ A_n, r_j \} - \varepsilon_{ilm} s_m + \]

\[= (r_i p_j - r_j p_i - r_i A_i + r_i A_i - \varepsilon_{ilm} s_m) + \]

\[+ (\varepsilon_{ijk} \varepsilon_{lmn} r_m p_k \{ A_n, r_j \} - r_j p_n \{ r_m, A_k \} + r_j A_n \{ A_k, r_m \} - r_m A_k \{ A_n, r_j \})]. \]  \hspace{1cm} (A9)

Notice that,

\[\varepsilon_{ijl} \varepsilon_{lmn} r_m p_k \{ A_n, r_j \} - \varepsilon_{ijk} \varepsilon_{lmn} r_j p_n \{ A_n, r_m \} = (\varepsilon_{ijl} \varepsilon_{lmn} - \varepsilon_{ijm} \varepsilon_{ljk}) r_m p_k \{ A_n, r_j \} = 0. \]  \hspace{1cm} (A10)

If \( i = l \), then,

\[\varepsilon_{ijl} \varepsilon_{lmn} - \varepsilon_{ijm} \varepsilon_{ljk} = \varepsilon_{ijl} \varepsilon_{lmn} - \varepsilon_{ijm} \varepsilon_{ijk} \equiv 0. \]  \hspace{1cm} (A11)

If \( i \neq l \), let us say \( i = 1 \) and \( l = 2 \), then

\[\varepsilon_{ijl} \varepsilon_{lmn} - \varepsilon_{ijm} \varepsilon_{ljk} = \varepsilon_{12l} \varepsilon_{2mn} - \varepsilon_{12m} \varepsilon_{2lk}. \]  \hspace{1cm} (A12)

Therefore, the possible non-vanishing pieces are:

\[\varepsilon_{123} \varepsilon_{213} - \varepsilon_{213} \varepsilon_{123} \equiv 0, \varepsilon_{132} \varepsilon_{231} - \varepsilon_{231} \varepsilon_{132} \equiv 0, \varepsilon_{132} \varepsilon_{213} - \varepsilon_{213} \varepsilon_{132} \equiv 0, \text{ etc. etc.} \]  \hspace{1cm} (A13)
Therefore, equation \((9)\) becomes,

\[
\{ J_i, J_j \} = (r_ip_i - r_ip_j - r_iA_i + r_iA_j - \varepsilon_{ilm}s_m)
= -\varepsilon_{ilm} [\varepsilon_{mnk}r_n(p_k - A_k) + s_m]
= -\varepsilon_{ilm}J_m. \tag{A14}
\]

Indeed,

\[
-\varepsilon_{ilm}\varepsilon_{mnk}r_n p_k = -\varepsilon_{ilm}\varepsilon_{kmn}r_n p_k = -\varepsilon_{ilm}\varepsilon_{nkm}r_n p_k =
= (\delta_{in}\delta_{lk} - \delta_{ik}\delta_{ln}) r_n p_k = -\delta_{in}\delta_{lk} r_n p_k + \delta_{ik}\delta_{ln} r_n p_k
= -r_ip_i + r_ip_i = (r_ip_i - r_ip_i) \tag{A15}
\]

and,

\[
\varepsilon_{ilm}\varepsilon_{mnk}r_n A_k = r_iA_l + r_iA_i = -(r_iA_i - r_iA_l). \tag{A16}
\]

This concludes our proof.

**APPENDIX B: THE JACOBI IDENTITY**

Consider the kinetic momentum vector,

\[
\vec{P} \equiv \vec{p} - \frac{\varepsilon}{c} \vec{A}, \vec{A} = \vec{A}_{\gamma} + \vec{A}^{(Dirac)}. \tag{B1}
\]

Consider the Poisson bracket of the kinetic momentum vector components,

\[
\{ P_i, P_j \} = \{ p_i - A_i, p_j - A_j \} =
= \{ p_i, p_j \} - \{ p_i, A_j \} - \{ A_i, p_j \} + \{ A_i, A_j \} = \{ A_j, p_i \} - \{ A_i, p_j \}
= \{ A_j, p_i \} - \{ A_i, p_j \} = -\partial_i A_j + \partial_j A_i = - (\partial_i A_j - \partial_j A_i)
= -\varepsilon_{ijk}B_k \tag{B2}
\]

where

\[
B_j = \varepsilon_{jlm}\partial_l A_m. \tag{B3}
\]

Using the fact that \(\{ J_i, B_j \} = -\varepsilon_{ijk}B_k\) and the identity \(\varepsilon_{ijk}\varepsilon_{mlk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\), it follows that,

\[
\varepsilon_{ijk}B_k = \varepsilon_{ijk}\varepsilon_{klm}\partial_l A_m = \varepsilon_{ijk}\varepsilon_{mlk}\partial_l A_m = -\varepsilon_{ijk}\varepsilon_{mlk}\partial_l A_m
= - (\delta_{im}\delta_{jl} - \delta_{il}\delta_{jm})\partial_m A_m = -\delta_{im}\delta_{jl}\partial_m A_m + \delta_{il}\delta_{jm}\partial_m A_m
= -\delta_{im}\partial_j A_m + \delta_{il}\partial_i A_j = \partial_i A_j - \partial_j A_i. \tag{B4}
\]

Using equation \((B2)\), we obtain

\[
\varepsilon_{ijn} \{ P_i, P_j \} = -\varepsilon_{ijn}\varepsilon_{ijk}B_k = -2\delta_{nk}B_k = -2B_n. \tag{B5}
\]

Thus,

\[
B_k = -\frac{1}{2}\varepsilon_{ijk} \{ P_i, P_j \}. \tag{B6}
\]

Finally, let us focus on the following Poisson bracket,

\[
\{ J_i, B_j \} = \left\{ J_i, -\frac{1}{2}\varepsilon_{lmj} \{ P_l, P_m \} \right\} = -\frac{1}{2}\varepsilon_{lmj} \{ J_i, \{ P_l, P_m \} \}. \tag{B7}
\]

Using the Jacobi identity,

\[
\{ J_i, \{ P_l, P_m \} \} + \{ P_m, \{ J_i, P_l \} \} + \{ P_l, \{ P_m, J_i \} \} = 0 \tag{B8}
\]
we obtain
\[
\{J_i, \{P_l, P_m\}\} = -\{P_m, \{J_i, P_l\}\} - \{P_l, \{P_m, J_i\}\} = \{P_l, \{J_i, P_m\}\} - \{P_m, \{J_i, P_l\}\}
\]
\[
= \{P_l, -\varepsilon_{imk} P_k\} - \{P_m, -\varepsilon_{ilk} P_k\} = -\varepsilon_{imk} \{P_l, P_k\} + \varepsilon_{ilk} \{P_m, P_k\}
\]
\[
= -\varepsilon_{imk}(\varepsilon_{ikq} B_q) + \varepsilon_{ilk}(\varepsilon_{mkq} B_q) = \varepsilon_{imk}\varepsilon_{lkq} B_q - \varepsilon_{ilk}\varepsilon_{mkq} B_q
\]
\[
= -\varepsilon_{imk}\varepsilon_{lkq} B_q + \varepsilon_{ilk}\varepsilon_{mkq} B_q = -(\delta_{im}\delta_{lq} - \delta_{iq}\delta_{ml}) B_q + (\delta_{im}\delta_{lq} - \delta_{iq}\delta_{lm}) B_q
\]
\[
= -\delta_{il}\delta_{mq} B_q + \delta_{iq}\delta_{ml} B_q + \delta_{im}\delta_{lq} B_q - \delta_{iq}\delta_{lm} B_q
\]
\[
= -\delta_{il} B_m + \delta_{ml} B_l + \delta_{im} B_l - \delta_{im} B_l = -\delta_{il} B_m + \delta_{im} B_l.
\]
Then, using equations (B6) and (B9), we obtain
\[
\{J_i, B_j\} = -\frac{1}{2}\varepsilon_{ilm}(-\delta_{il} B_m + \delta_{im} B_l) = \frac{1}{2}\varepsilon_{ilm}\delta_{il} B_m - \frac{1}{2}\varepsilon_{ilm}\delta_{im} B_l
\]
\[
= -\frac{1}{2}\varepsilon_{ilm} B_m - \frac{1}{2}\varepsilon_{ilm} B_m = -\frac{1}{2}\varepsilon_{ilm} B_m - \frac{1}{2}\varepsilon_{ilm} B_m = -\frac{1}{2}\varepsilon_{ilm} B_m.
\]
We have shown that in a pure classical theoretical framework given by the Poisson brackets formalism, the commutation rule between the generator of spatial rotations and the total magnetic field is,
\[
\{J_i, B_j\} = \iota\varepsilon_{ijk} B_k.
\]

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