THE CASIMIR OPERATOR OF A METRIC CONNECTION WITH SKEW-SYMMETRIC TORSION

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Abstract. For any triple \((M^n, g, \nabla)\) consisting of a Riemannian manifold and a metric connection with skew-symmetric torsion we introduce an elliptic, second order operator \(\Omega\) acting on spinor fields. In case of a naturally reductive space and its canonical connection, our construction yields the Casimir operator of the isometry group. Several non-homogeneous geometries (Sasakian, nearly Kähler, co calibrated \(G_2\)-structures) admit unique connections with skew-symmetric torsion. We study the corresponding Casimir operator and compare its kernel with the space of \(\nabla\)-parallel spinors.

1. Introduction

Consider a Riemannian manifold \((M^n, g, \nabla)\) equipped with a metric connection with skew-symmetric torsion \(T\), and denote by \((D^{1/3})^2\) the square of the Dirac operator corresponding to the connection with torsion form \(T/3\). We introduce a second order differential operator \(\Omega\) that differs from \((D^{1/3})^2\) by a zero order term. This parameter shift has been already used by Bismut in the proof of the local index theorem for hermitian manifolds. Later, generalizing the well-known Parthasarathy formula for the square of the Dirac operator of a symmetric space, Kostant noticed a simple algebraic formula for some element in the tensor product of the universal enveloping algebra by the Clifford algebra of a naturally reductive space. The geometric interpretation of Kostant’s “cubic Dirac operator” as a \(1/3\)-parameter shifted Dirac operator for such a space endowed with its canonical connection as well as the formula for the square of any operator \(D^s\) in the family have been discussed in the paper [1]. Our operator \(\Omega\) is constructed in such a way to coincide with the Casimir operator of the naturally reductive space in the homogenous situation, hence motivating its name. The integral formulas for \((D^{1/3})^2\) are then used in order to study the new operator \(\Omega\) in greater detail. In general, the kernel of the operator \(\Omega\) contains all \(\nabla\)-parallel spinors. If the torsion form \(T\) is \(\nabla\)-parallel, the formula for \(\Omega\) simplifies to

\[\Omega = (D^{1/3})^2 - \frac{1}{16} (2 \text{Scal}^g + ||T||^2),\]

and the operators \(\Omega\) and \((D^{1/3})^2\) commute with the action of the torsion form on spinors. Triples \((M^n, g, \nabla)\) occur in the study of non integrable special Riemannian manifolds in a natural way. For example, any Sasakian manifold in odd dimensions, any hermitian manifold with skew-symmetric Nijenhuis tensor in even dimensions, any

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cocalibrated $G_2$-manifold in dimension seven and any Spin(7)-manifold in dimension eight admit a unique metric connection with skew-symmetric torsion and preserving the additional geometric structure (see [10] and [9]). The torsion forms of these connections are models for the $B$-field in the string equations and their parallel spinor fields are the supersymmetries of the models. From the mathematical point of view, the basic role of these connections is closely related to the fact that many of the geometric data of the non integrable geometric structure can be read of its unique torsion form.

We study the Casimir operator of a Riemannian manifold equipped with a metric connection. In particular, we compare its kernel with the space of $\nabla$-parallel or with the space of Riemannian Killing spinors. The low dimensions are specially interesting. Therefor we investigate Sasakian manifolds in dimension five, nearly Kähler manifolds in dimension six, and cocalibrated $G_2$-manifolds in dimension seven in detail. In case that a non integrable geometric structure admits a transitive automorphism group and the space is naturally reductive, its unique geometric connection coincides with the canonical connection of the reductive space. Henceforth, our geometric Casimir operator is the group-theoretical Casimir operator acting on spinors and we can study some of its properties in a purely geometric way, for example through vanishing theorems.

2. An overview of Schrödinger-Lichnerowicz type formulas for Dirac operators

Consider a Riemannian spin manifold $(M^n, g, T)$ with a 3-form $T$. Then we may define a metric connection with torsion $T$ by the formula

$$\nabla_X Y := \nabla^g_X Y + \frac{1}{2} T(X, Y, -),$$

where we denoted by $\nabla^g$ the Levi-Civita connection of $M$. The connection $\nabla$ can be lifted to a connection on the spinor bundle $S$ of $M$, where it takes the expression

$$\nabla_X \psi := \nabla^g_X \psi + \frac{1}{4} (X \lrcorner T) \cdot \psi.$$

We shall write $D$ for the Dirac operator associated with the connection $\nabla$, and $D^g$ for the classical Riemannian Dirac operator, the two being related by $D = D^g + (3/4)T$. In this section, we review the known Weitzenböck formulas for the square of $D$ and its relatives which will be needed in all subsequent sections. First, let us introduce the first order differential operator

$$D \psi := \sum_{k=1}^{n} (e_k \lrcorner T) \cdot \nabla e_k \psi = D^g \psi + \frac{1}{4} \sum_{k=1}^{n} (e_k \lrcorner T) \cdot (e_k \lrcorner T) \cdot \psi,$$

where $e_1, \ldots, e_n$ denotes an orthonormal basis and $D^g$ the part of the operator $D$ coming from the Levi-Civita connection. It will be convenient to introduce a 4-form derived from $T$,

$$\sigma_T := \frac{1}{2} \sum_{k=1}^{n} (e_k \lrcorner T) \wedge (e_k \lrcorner T).$$

By [2] Prop. 6.1, $\sigma_T$ is linked to the square of $T$ inside the Clifford algebra by $T^2 = -2 \sigma_T + ||T||^2$. On spinors, the difference between the endomorphisms $\sigma_T$ and $(D - D^g)$
is given by the formula
\[
\sum_{k=1}^{n} (e_k \cdot T)(e_k \cdot T) = 2 \sigma_T - 3 \|T\|^2.
\]

**Theorem 2.1** ([10, Thm 3.1, 3.3]). Let \((M^n, g, \nabla)\) be an \(n\)-dimensional Riemannian manifold with a metric connection \(\nabla\) of skew-symmetric torsion \(T\). Then, the square of the Dirac operator \(D\) associated with \(\nabla\) acts on an arbitrary spinor field \(\psi\) as
\[
D^2 \psi = \Delta_T(\psi) + \frac{3}{4} dT \cdot \psi - \frac{1}{2} \sigma_T \cdot \psi + \frac{1}{2} \delta T \cdot D \psi + \frac{1}{4} \text{Scal} \cdot \psi,
\]
where \(\Delta_T\) is the spinor Laplacian of \(\nabla\),
\[
\Delta_T(\psi) = (\nabla)^* \nabla \psi = -\sum_{k=1}^{n} \nabla_{e_k} \nabla e_k \psi + \nabla \nabla e_k e_k \psi,
\]
and \(\text{Scal}\) is the scalar curvature of the connection \(\nabla\). It is related to the Riemannian scalar curvature \(\text{Scal}^g\) by \(\text{Scal} = \text{Scal}^g - (3/2)\|T\|^2\). Furthermore, the anti-commutator of \(D\) and \(T\) is
\[
D \circ T + T \circ D = dT + \delta T - 2 \sigma_T - 2 D.
\]
This formula for \(D^2\) has the disadvantage of still containing a first order differential operator. By shifting the parameter in the torsion of the connection \(\nabla\), we can state a more useful Schrödinger-Lichnerowicz type formula. It links the Dirac operator \(D_{1/3} = D^g + T/4\) of the connection with torsion \(T/3\) and the Laplacian of the connection with torsion \(T\). The remainder is a zero order operator. Details on this parameter shift and its history are given in [2].

**Theorem 2.2** ([2, Thm 6.2]). The spinor Laplacian \(\Delta_T\) and the square of the Dirac operator \(D_{1/3}\) are related by
\[
(D_{1/3})^2 = \Delta_T + \frac{1}{4} dT + \frac{1}{4} \text{Scal}^g - \frac{1}{8} \|T\|^2.
\]
Integrating the latter formula on a compact manifold \(M^n\), we obtain
\[
\int_{M^n} \|D_{1/3} \psi\|^2 = \int_{M^n} \left[ \|\nabla \psi\|^2 + \frac{1}{8} (dT \cdot \psi, \psi) + \frac{1}{4} \text{Scal}^g \|\psi\|^2 - \frac{1}{8} \|T\|^2 \|\psi\|^2 \right].
\]
Finally, we state the Kostant-Parthasarathy formula for \((D_{1/3})^2\) in the homogeneous case, as it is the main motivation for what follows.

**Theorem 2.3** ([1, Thm 3.3]). Let \(M = G/H\) be a naturally reductive homogeneous space, and \(\mathfrak{g} = \mathfrak{h} + \mathfrak{m}\). Then its canonical connection \(\nabla\) has skew-symmetric torsion
\[
T(X,Y,Z) = -g([X,Y]_m, Z) \quad (X,Y,Z \in \mathfrak{m}),
\]
\(T\) is \(\nabla\)-parallel and \(D_{1/3}\) satisfies the identity
\[
(D_{1/3})^2 = \Omega_\mathfrak{g} + \frac{1}{8} \text{Scal}^g + \frac{1}{16} \|T\|^2,
\]
where \(\Omega_\mathfrak{g}\) denotes the Casimir operator of \(\mathfrak{g}\).

Typically, the canonical connection of a naturally reductive homogeneous space \(M\) can be given an alternative geometric characterization—for example, as the unique metric connection with skew-symmetric torsion preserving a given \(G\)-structure (see [1] or [10] for examples and details). Once this is done, \(D_{1/3}, \text{Scal}^g\) and \(\|T\|^2\) are geometrically invariant objects, whereas \(\Omega_\mathfrak{g}\) still heavily relies on the concrete realization of the homogeneous space \(M\) as a quotient. At the same time, the same interesting
3. The Casimir operator of a triple \((M^n, g, \nabla)\)

We consider a Riemannian spin manifold \((M^n, g, \nabla)\) with a metric connection \(\nabla\) and skew-symmetric torsion \(T\). Denote by \(\Delta_T\) the spinor Laplacian of the connection.

**Definition 3.1.** The Casimir operator of the triple \((M^n, g, \nabla)\) is the differential operator acting on spinor fields by

\[
\Omega := (D^{1/3})^2 + \frac{1}{8} (dT - 2\sigma_T) + \frac{1}{4} \delta(T) - \frac{1}{8} \text{Scal}_g - \frac{1}{16} ||T||^2
\]

\[
= \Delta_T + \frac{1}{8} (3dT - 2\sigma_T + 2\delta(T) + \text{Scal}).
\]

**Remark 3.1.** A naturally reductive space \(M^n = G/H\) endowed with its canonical connection satisfies \(dT = 2\sigma_T\) and \(\delta T = 0\), hence \(\Omega = \Omega_g\) by Theorem 2.3. For connections with \(dT \neq 2\sigma_T\) and \(\delta T \neq 0\), the numerical factors are chosen in such a way to yield an overall expression proportional to the scalar part of the right hand side of equation (1).

**Example 3.1.** For the Levi-Civita connection (\(T = 0\)) of an arbitrary Riemannian manifold, we obtain

\[
\Omega = (D^g)^2 - \frac{1}{8} \text{Scal}_g = \Delta^g + \frac{1}{8} \text{Scal}_g.
\]

The second equality is just the classical Schrödinger-Lichnerowicz formula for the Riemannian Dirac operator, whereas the first one is—in case of a symmetric space—the classical Parthasarathy formula.

**Example 3.2.** Consider a 3-dimensional manifold of constant scalar curvature, a constant \(a \in \mathbb{R}\) and the 3-form \(T = 2a dM^3\). Then

\[
\Omega = (D^g)^2 - a D^g - \frac{1}{8} \text{Scal}_g.
\]

The kernel of the Casimir operator corresponds to eigenvalues \(\lambda \in \text{Spec}(D^g)\) of the Riemannian Dirac operator such that

\[
8(\lambda^2 - a\lambda) - \text{Scal}_g = 0.
\]

In particular, the kernel of \(\Omega\) is in general larger than the space of \(\nabla\)-parallel spinors. Indeed, such spinors exist only on space forms. More generally, fix a real-valued smooth function \(f\) and consider the 3-form \(T := f \cdot dM^3\). If there exists a \(\nabla\)-parallel spinor

\[
\nabla^g_X\psi + (X \, \nabla T) \cdot \psi = \nabla^g_X\psi + f \cdot X \cdot \psi = 0,
\]

then, by a theorem of A. Lichnerowicz (see [18]), \(f\) is constant and \((M^3, g)\) is a space form.

Let us collect some elementary properties of the Casimir operator of a triple \((M^n, g, \nabla)\).

**Proposition 3.1.** The kernel of the Casimir operator contains all \(\nabla\)-parallel spinors.

**Proof.** By Theorem 2.1, one of the integrability conditions for a \(\nabla\)-parallel spinor field \(\psi\) is

\[
(3dT - 2\sigma_T + 2\delta(T) + \text{Scal}) \cdot \psi = 0.
\]

\(\square\)
If the torsion form $T$ is $\nabla$-parallel, the formulas for the Casimir operator simplify. Indeed, in this case we have (see [10])

$$dT = 2\sigma_T, \quad \delta(T) = 0,$$

and the Ricci tensor $\text{Ric}$ of $\nabla$ is symmetric. Using the formulas of Section 2 (in particular, Theorems 2.1 and 2.2), we obtain a simpler expression for the Casimir operator.

**Proposition 3.2.** The Casimir operator of a triple $(M^n, g, \nabla)$ with $\nabla T = 0$ can equivalently be written as:

$$\Omega = (D^{1/3})^2 - \frac{1}{16} \left( 2\text{Scal}^g + ||T||^2 \right) = \Delta_T + \frac{1}{16} \left( 2\text{Scal}^g + ||T||^2 \right) - \frac{1}{4} T^2 = \Delta_T + \frac{1}{8} \left( 2dT + \text{Scal} \right).$$

Integrating these formulas, we obtain a vanishing theorem for the kernel of the Casimir operator.

**Proposition 3.3.** Let $(M^n, g, \nabla)$ be a compact triple such that the torsion form is $\nabla$-parallel. If one of the conditions

$$2\text{Scal}^g \leq -||T||^2 \quad \text{or} \quad 2\text{Scal}^g \geq 4T^2 - ||T||^2,$$

holds, the Casimir operator is non-negative in $L^2(S)$.

**Example 3.3.** For a naturally reductive space $M = G/H$, the first condition can never hold, since a representation theoretic argument [1, Lemma 3.6] shows that $2\text{Scal}^g + ||T||^2$ is strictly positive. In concrete examples, the second condition typically singles out the normal homogeneous metrics among the naturally reductive ones. Notice a small mistake in Lemma 3.5 of [1]: in general, the fact that the negative definite contribution of the scalar product comes from an abelian summand in $g$ is not enough to conclude that $\Omega_g$ is non-negative.

Two further consequences of Proposition 3.2 are:

**Proposition 3.4.** If the torsion form is $\nabla$-parallel, the Casimir operator $\Omega$ and the square of the Dirac operator $(D^{1/3})^2$ commute with the endomorphism $T$,

$$\Omega \circ T = T \circ \Omega, \quad (D^{1/3})^2 \circ T = T \circ (D^{1/3})^2.$$

The endomorphism $T$ acts on the spinor bundle as a symmetric endomorphism with constant eigenvalues.

**Theorem 3.1.** Let $(M^n, g, \nabla)$ be a compact Riemannian spin manifold equipped with a metric connection $\nabla$ with parallel, skew-symmetric torsion, $\nabla T = 0$. The endomorphism $T$ and the Riemannian Dirac operator $D^g$ act in the kernel of the Dirac operator $D^{1/3}$. In particular, if, for all $\mu \in \text{Spec}(T)$, the number $-\mu/4$ is not an eigenvalue of the Riemannian Dirac operator, then the kernel of $D^{1/3}$ is trivial.

**Proof.** On a compact manifold, the kernels of $D^{1/3}$ and $(D^{1/3})^2$ coincide. \qed

If $\psi$ belongs to the kernel of $D^{1/3}$ and is an eigenspinor of the endomorphism $T$, we have $4 \cdot D^g \psi = -\mu \cdot \psi$, $\mu \in \text{Spec}(T)$. Using the estimate of the eigenvalues of the Riemannian Dirac operator (see [8]) we obtain an upper bound for the minimum $\text{Scal}^g_{\text{min}}$ Riemannian scalar curvature in case that the kernel of the operator $D^{1/3}$ is non-trivial.
Proposition 3.5. Let \((M^n, g, \nabla)\) be a compact Riemannian spin manifold equipped with a metric connection \(\nabla\) with parallel, skew-symmetric torsion, \(\nabla T = 0\). If the kernel of the Dirac operator \(D^{1/3}\) is non trivial, then the minimum of the Riemannian scalar curvature is bounded by

\[
\max \{ \mu^2 : \mu \in \text{Spec}(T) \} \geq \frac{4n}{n-1} \text{Scal}_g^{\text{min}}.
\]

Remark 3.2. If \((n-1) \mu^2 = 4n \text{Scal}_g^{\text{min}}\) is in the spectrum of \(T\) and there exists a spinor field \(\psi\) in the kernel of \(D^{1/3}\) such that \(T \cdot \psi = \mu \cdot \psi\), then we are in the limiting case of the inequality in [8]. Consequently, \(M^n\) is an Einstein manifold of non-negative Riemannian scalar curvature and \(\psi\) is a Riemannian Killing spinor,

\[

\nabla^g_X \psi - \frac{\mu}{4n} \cdot X \cdot \psi = 0.
\]

Examples of this type are 7-dimensional 3-Sasakian manifolds. The possible torsion form has been discussed in [2], Section 9.

4. The Casimir operator of a 5-dimensional Sasakian manifold

Let \((M^5, g, \xi, \eta, \varphi)\) be a compact 5-dimensional Sasakian spin manifold (with a fixed spin structure) and denote by \(\nabla\) its unique connection with skew-symmetric torsion and preserving the contact structure. We orient \(M^5\) by the condition that the differential of the contact form is given by \(d\eta = 2(e_1 \wedge e_2 + e_3 \wedge e_4)\), and write henceforth \(e_{ij}...\) for \(e_i \wedge e_j \wedge ...\). Then we have (see [10])

\[
\nabla T = 0, \quad T = \eta \wedge d\eta = 2(e_{12} + e_{34}) \wedge e_5, \quad T^2 = 8 - 8e_{1234}
\]

and

\[
\Omega = (D^{1/3})^2 - \frac{1}{8} \text{Scal}_g - \frac{1}{2} = \Delta_T + \frac{1}{8} \text{Scal}_g - \frac{3}{2} + 2e_{1234}.
\]

We study the kernel of the Dirac operator \(D^{1/3}\). The endomorphism \(T\) acts in the 5-dimensional spin representation with eigenvalues \((-4, 0, 0, 4)\) and, according to Theorem 3.1, we have to distinguish two cases. If \(D^{1/3} \psi = 0\) and \(T \cdot \psi = 0\), the spinor field is harmonic and the formulas of Proposition 3.2 yield in the compact case the condition

\[
\int_{M^5} (2 \text{Scal}_g^{\text{min}} + 8) \||\psi||^2 \leq 0.
\]

Examples of that type are the 5-dimensional Heisenberg group with its left invariant Sasakian structure or certain \(S^1\)-bundles over a flat torus. On these spaces, there exist \(\nabla\)-parallel spinors \(\psi_0\) satisfying the algebraic equation \(T \cdot \psi_0 = 0\) (see [10], [11]). Their scalar curvature equals \(\text{Scal}_g = -4\). Let us describe the 5-dimensional Heisenberg group. Its Sasakian structure is given on \(\mathbb{R}^5\) with coordinates \((x_1, x_2, y_1, y_2, z)\) by the 1-forms

\[
e_1 := \frac{1}{2} dx_1, \quad e_2 := \frac{1}{2} dy_1, \quad e_3 := \frac{1}{2} dx_2, \quad e_4 := \frac{1}{2} dy_2,
\]

\[
e_5 = \eta := \frac{1}{2} (dz - y_1 \cdot dx_1 - y_2 \cdot dx_2).
\]

The space of all \(\nabla\)-parallel spinors satisfying \(T \cdot \psi_0 = 0\) is a 2-dimensional subspace of the kernel of the operator \(D^{1/3}\). In a left-invariant frame of \(M^5\), spinors are simply functions \(\psi : M^5 \rightarrow \Delta_5\) with values in the 5-dimensional spin representation. It turns out that the spinors \(\psi_0\) are constant. Consequently, for any discrete subgroup \(\Gamma\) of the Heisenberg group, the manifold \(M^5/\Gamma\) equipped with its trivial spin structure is
a Sasakian manifold admitting spinors in $\text{Ker}(D^{1/3})$. The second case for spinors in the kernel is given by $D^{1/3}\psi = 0$ and $T \cdot \psi = \pm 4\psi$. The spinor field is an eigenspinor for the Riemannian Dirac operator, $D^g \psi = \mp 4\psi$. The formulas of Proposition 3.2 and Proposition 3.5 yield in the compact case two conditions:

$$\int_M (\text{Scal} - 12)||\psi||^2 \leq 0 \quad \text{and} \quad 5 \text{Scal}_{\text{min}}^g \leq 16.$$  

The paper [15] contains a construction of Sasakian manifolds admitting a spinor field of that algebraic type in the kernel of $D^{1/3}$. We describe the construction explicitly. Suppose that the Riemannian Ricci tensor $\text{Ric}^g$ of a simply-connected, 5-dimensional Sasakian manifold is given by the formula

$$\text{Ric}^g = -2g + 6\eta \otimes \eta.$$  

Its scalar curvature equals $\text{Scal}^g = -4$. In the simply-connected and compact case, they are total spaces of $S^1$ principal bundles over 4-dimensional Calabi-Yau orbifolds (see [5]). There exist (see [15], Theorem 6.3) two spinor fields $\psi_1, \psi_2$ such that

$$\nabla^g X \psi_1 = -\frac{1}{2}X \cdot \psi_1 + \frac{3}{2}\eta(X) \cdot \xi \cdot \psi_1, \quad T \cdot \psi_1 = -4\psi_1,$$

$$\nabla^g X \psi_2 = \frac{1}{2}X \cdot \psi_2 - \frac{3}{2}\eta(X) \cdot \xi \cdot \psi_2, \quad T \cdot \psi_2 = 4\psi_2.$$  

In particular, we obtain

$$D^g \psi_1 = \psi_1, \quad T \cdot \psi_1 = -4\psi_1, \quad \text{and} \quad D^g \psi_2 = -\psi_2, \quad T \cdot \psi_2 = 4\psi_2,$$

and therefore the spinor fields $\psi_1$ and $\psi_2$ belong to the kernel of the operator $D^{1/3}$.

Next, we investigate the kernel of the Casimir operator. Under the action of the torsion form, the spinor bundle $S$ splits into three subbundles $S = S_0 \oplus S_4 \oplus S_{-4}$ corresponding to the eigenvalues of $T$. Since $\nabla T = 0$, the connection $\nabla$ preserves the splitting. The endomorphism $e_{1234}$ acts by the formulas

$$e_{1234} = 1 \quad \text{on} \quad S_0, \quad e_{1234} = -1 \quad \text{on} \quad S_4 \oplus S_{-4}.$$  

Consequently, the formula

$$\Omega = \Delta_T + \frac{1}{8}\text{Scal}^g - \frac{3}{2} + 2e_{1234}$$

shows that the Casimir operator splits into the sum $\Omega = \Omega_0 \oplus \Omega_4 \oplus \Omega_{-4}$ of three operators acting on sections in $S_0$, $S_4$ and $S_{-4}$. On $S_0$, we have

$$\Omega_0 = \Delta_T + \frac{1}{8}\text{Scal}^g + \frac{1}{2} = (D^{1/3})^2 - \frac{1}{8}\text{Scal}^g - \frac{1}{2}.$$  

In particular, the kernel of $\Omega_0$ is trivial if $\text{Scal}^g \neq -4$. The Casimir operator on $S_4 \oplus S_{-4}$ is given by

$$\Omega_{\pm 4} = \Delta_T + \frac{1}{8}\text{Scal}^g - \frac{7}{2} = (D^{1/3})^2 - \frac{1}{8}\text{Scal}^g - \frac{1}{2}$$

and a non trivial kernel can only occur if $-4 \leq \text{Scal}^g \leq 28$. A spinor field $\psi$ in the kernel of the Casimir operator $\Omega$ satisfies the equations

$$(D^{1/3})^2 \cdot \psi = \frac{1}{8}(4 + \text{Scal}^g)\psi, \quad T \cdot \psi = \pm 4\psi.$$
In particular, we obtain
\[
\int_{\mathcal{M}} \langle (D^g + 1)^2 \psi, \psi \rangle = \frac{1}{8} \int_{\mathcal{M}} (4 + \text{Scal}^g) ||\psi||^2,
\]
and the first eigenvalue of the operator \((D^g + 1)^2\) is bounded by the scalar curvature,
\[
\lambda_1(D^g + 1)^2 \leq \frac{1}{8} (4 + \text{Scal}^g_{\text{max}}).
\]

Let us consider special classes of Sasakian manifolds. A first case is \(\text{Scal}^g = -4\). Then the formula for the Casimir operator simplifies,
\[
\Omega_0 = \Delta_T = (D^{1/3})^2, \quad \Omega_{\pm 4} = \Delta_T - 4 = (D^{1/3})^2.
\]

If \(M^5\) is compact, the kernel of the operator \(\Omega_0\) coincides with the space of \(\nabla\)-parallel spinors in the bundle \(S_0\). A spinor field \(\psi\) in the kernel the operator \(\Omega_{\pm 4}\) is an eigen-spinor of the Riemannian Dirac operator,
\[
D^g(\psi) = \mp \psi, \quad T \cdot \psi = \pm 4 \psi.
\]

Compact Sasakian manifolds admitting spinor fields in the kernel of \(\Omega_0\) are quotients of the 5-dimensional Heisenberg group (see [11], Theorem 4.1). Moreover, the 5-dimensional Heisenberg group and its compact quotients admit spinor fields in the kernel of \(\Omega_{\pm 4}\), too. Indeed, the non trivial connection forms of the Levi-Civita connection are
\[
\omega_{12} = e_5 = \omega_{34}, \quad \omega_{15} = e_2, \quad \omega_{25} = -e_2, \quad \omega_{35} = e_4, \quad \omega_{45} = -e_2,
\]
and a computation of the Riemannian Dirac operator yields the formula
\[
D^g(\psi) = \sum_{k=1}^5 e_k \cdot e_k(\psi) \text{ on } S_0, \quad D^g(\psi) = \sum_{k=1}^5 e_k \cdot e_k(\psi) \mp \psi \text{ on } S_{\pm 4}.
\]

Spinors in the kernel of \(\Omega_{\pm 4}\) occur on Sasakian \(\eta\)-Einstein manifolds of type \(\text{Ric}^g = -2 \cdot g + 6 \cdot \eta \otimes \eta\), too. This example has been discussed above.

A second case is \(\text{Scal}^g = 28\). Then
\[
\Omega_0 = \Delta_T + 4 = (D^{1/3})^2 - 4, \quad \Omega_{\pm 4} = \Delta_T = (D^{1/3})^2 - 4.
\]

The kernel of \(\Omega_0\) is trivial and the kernel of \(\Omega_{\pm 4}\) coincides with the space of \(\nabla\)-parallel spinors in the bundle \(S_{\pm 4}\). Sasakian manifolds admitting spinor fields of that type have been described in [10], Theorem 7.3 and Example 7.4.

If \(-4 < \text{Scal}^g < 28\), the kernel of the operator \(\Omega_0\) is trivial and the kernel of \(\Omega_{\pm 4}\) depends on the geometry of the Sasakian structure. Let us discuss Einstein-Sasakian manifolds. Their scalar curvature equals \(\text{Scal}^g = 20\) and the Casimir operators are
\[
\Omega_0 = \Delta_T + 3, \quad \Omega_{\pm 4} = \Delta_T - 1 = (D^{1/3})^2 - 3.
\]

If \(M^5\) is simply-connected, there exist two Riemannian Killing spinors (see [13])
\[
\nabla_X^g \psi_1 = \frac{1}{2} X \cdot \psi_1, \quad D^g(\psi_1) = -\frac{5}{2} \psi_1, \quad T \cdot \psi_1 = 4 \psi_1,
\]
\[
\nabla_X^g \psi_2 = -\frac{1}{2} X \cdot \psi_2, \quad D^g(\psi_2) = \frac{5}{2} \psi_2, \quad T \cdot \psi_2 = -4 \psi_2.
\]
We compute the Casimir operator
\[ \Omega(\psi_1) = -\frac{3}{4} \psi_1, \quad \Omega(\psi_2) = -\frac{3}{4} \psi_2. \]

In particular, the Casimir operator of a Einstein-Sasakian manifold has negative eigenvalues. The Riemannian Killing spinors are parallel sections in the bundles \( S_{\pm 4} \) with respect to the flat connections \( \nabla^{\pm} \)
\[ \nabla^{\pm}_{X} \psi := \nabla_{X} \psi - \frac{1}{2} X \cdot \psi \quad \text{in} \quad S_{4}, \quad \nabla^{\mp}_{X} \psi := \nabla^{g}_{X} \psi + \frac{1}{2} X \cdot \psi \quad \text{in} \quad S_{-4}. \]

We compare these connections with our canonical connection \( \nabla \):
\[ (\nabla^{\pm}_{X} - \nabla_{X}) \cdot \psi^{\pm} = \pm \frac{i}{2} g(X, \xi) \cdot \psi^{\pm}, \quad \psi^{\pm} \in S_{\pm 4}. \]
The latter equation means that the bundle \( S_{4} \oplus S_{-4} \) equipped with the connection \( \nabla \)
is equivalent to the 2-dimensional trivial bundle with the connection form
\[ \mathcal{A} = \frac{i}{2} \eta \cdot \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}. \]
The curvature of \( \nabla \) on these bundles is given by the formula
\[ R^{\nabla} = \frac{i}{2} d\eta \cdot \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = i (e_1 \wedge e_2 + e_3 \wedge e_4) \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \]
Since the divergence \( \text{div}(\xi) = 0 \) of the Killing vector field vanishes, the Casimir operator on \( S_{4} \oplus S_{-4} \) is the following operator acting on pairs of functions:
\[ \Omega_{4} \oplus \Omega_{-4} = \Delta_{T} - 1 = \Delta - \frac{3}{4} + \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \xi. \]
Here \( \Delta \) means the usual Laplacian of \( M^{5} \) acting on functions and \( \xi \) is the differentiation in direction of the vector field \( \xi \). In particular, the kernel of \( \Omega \) coincides with solutions \( f : M^{5} \to \mathbb{C} \) of the equation
\[ \Delta(f) - \frac{3}{4} f \pm i \xi(f) = 0. \]
The \( L^{2} \)-symmetric differential operators \( \Delta \) and \( i \xi \) commute. Therefore, we can diagonalize them simultaneously. The latter equation is solvable if and only if there exists a common eigenfunction
\[ \Delta(f) = \mu f, \quad i \xi(f) = \lambda f, \quad 4(\mu + \lambda) - 3 = 0. \]
The Laplacian \( \Delta \) is the sum of the non-negative horizontal Laplacian and the operator \((i \xi)^{2}\). Now, the conditions
\[ \lambda^{2} \leq \mu, \quad 4(\mu + \lambda) - 3 = 0 \]
restrict the eigenvalue of the Laplacian, \( 0 \leq \mu \leq 3 \). On the other side, by the Lichnerowicz-Obata Theorem (see [3]) we have \( 5 \leq \mu \), a contradiction. In particular, we proved

**Theorem 4.1.** *The Casimir operator of a compact 5-dimensional Sasakian-Einstein manifold has trivial kernel.*
The same argument estimates the eigenvalues of the Casimir operator. It turns out that the smallest eigenvalues of $\Omega$ is negative and equals $-3/4$. The eigenspinors are the Riemannian Killing spinors. The next eigenvalue of the Casimir operator is at least
\[
\lambda_2(\Omega) \geq \frac{17}{4} - \sqrt{5} \approx 2.014.
\]

5. An explicit example: The 5-dimensional Stiefel manifold

The 5-dimensional Stiefel manifold $V_{4,2} = SO(4)/SO(2)$ admits a homogeneous Einstein-Sasakian metric. This metric can be constructed via the Kaluza-Klein approach, observing that $V_{4,2}$ is a principal $SO(2)$-bundle over the 4-dimensional Einstein-Kähler manifold $G_{4,2}$ of all oriented 2-planes in $\mathbb{R}^4$. As a homogeneous space, the geometry and the Dirac operator of $V_{4,2}$ have been described in [8]. We will use these formulas in our computation, with a slight change in normalization: we set the scalar curvature of a 5-dimensional Einstein-Sasakian manifold equal to 20, whereas the metric as described in the latter paper has scalar curvature $20/3$. The manifold $V_{4,2}$ can be discussed as a naturally reductive space by writing it as $SO(4) \times SO(2)/SO(2) \times SO(2)$, and its canonical connection does then coincide with the unique metric connection $\nabla$ with skew-symmetric torsion preserving the Sasakian structure as discussed in the previous section (see also [1]). In this discussion, we concentrate on its contact structure and show that many properties can be derived from it alone. In order to fix the notation, let $E_{ij}$ be the standard basis of the Lie algebra $\mathfrak{so}(4)$. The subalgebra $\mathfrak{so}(2)$ is generated by the matrix $E_{34}$ and
\[
X_1 := \sqrt{3}E_{13}, \quad X_2 := \sqrt{3}E_{14}, \quad X_3 := \sqrt{3}E_{23}, \quad X_4 := \sqrt{3}E_{24}, \quad \xi = X_5 := \frac{3}{2}E_{12}
\]
constitute an orthonormal basis defining the metric of $V_{4,2}$. The formula for the Riemannian Dirac operator has been computed in [8]:
\[
D^9(\psi) = \sqrt{3} \sum_{i=1}^{5} X_i \cdot X_i(\psi) + S(\psi), \quad S := \frac{5i}{2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}
\]
Using the commutator relations for $[X_i,X_j]$ as well as the matrix of the endomorphism $T = \eta \wedge d\eta$, we compute the square of the operator $D^{1/3}$,
\[
(D^{1/3})^2(\psi) = -3 \sum_{i=1}^{5} X_i^2(\psi) + M_1 \cdot \psi + M_2 \cdot E_{34}(\psi) + M_3 \cdot X_5(\psi).
\]
Here the matrices $M_1$, $M_2$ and $M_3$ are given by
\[
M_1 := \frac{9}{4} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad M_2 := 6i \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad M_3 := \sqrt{3} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}
\]
According to the lift of the isotropy representation into the spin module (see [8]), a spinor field is a triple $\psi = (\psi_+,\psi_-)$ of maps $\psi_{\pm} : SO(4) \to \mathbb{C}$ and $\psi_+ : SO(4) \to \mathbb{C}^2$ such that $E_{34}(\psi_{\pm}) = \pm i \psi_{\pm}$ and $E_{34}(\psi_+) = 0$. The map $\psi_+$ is a section in the bundle $S_4 \oplus S_{-4}$ and $(\psi_+,\psi_-)$ are sections in $S_0$. Specially over $V_{4,2}$ the latter bundle splits.
into the sum of two line bundles. The Casimir operator \( \Omega = \Omega_0 \oplus \Omega_4 \oplus \Omega_{-4} \) is equivalent to the operators

\[
\Omega_0 = -3 \sum_{\alpha=1}^{5} X_\alpha^2 + 3, \quad \Omega_4 \oplus \Omega_{-4} = -3 \sum_{\alpha=1}^{5} X_\alpha^2 - \frac{3}{4} \pm \sqrt{3} i \cdot X_5
\]

acting on functions \( f: \text{SO}(4) \to \mathbb{C} \) satisfying the quasi-periodicity conditions \( E_{34}(f) = \pm i f \) and \( E_{34}(f) = 0 \), respectively.

6. The Casimir operator of 6-dimensional nearly Kähler manifolds

Let \((M^6, g, J)\) be a 6-dimensional nearly Kähler manifold. Then \(M^6\) is an Einstein manifold of positive scalar curvature,

\[
\text{Ric}^g = \frac{5}{2} a g, \quad \text{Scal}^g = 15 a > 0.
\]

The Nijenhuis tensor \( N \) does not vanish. There exists a unique connection \( \nabla \) with skew-symmetric torsion \( T \). This connection is Gray’s characteristic connection (see [16]) and its geometric data are given by

\[
\nabla T = 0, \quad 4 T = N, \quad \text{Ric} = 2 a g.
\]

Moreover, we have

\[
2 \sigma_T = dT = a (\omega \wedge \omega) = 2 a (e_{1234} + e_{1256} + e_{3456}), \quad ||T||^2 = 2 a,
\]

where \( \omega \) denotes the fundamental form of the nearly Kähler structure. A general reference for all these formulas is the paper [10]. We compute the symmetric endomorphism \( dT \) in the spinor bundle:

\[
2 dT + \text{Scal} = 16 a \cdot \text{diag}(0, 0, 1, 1, 1, 1).
\]

Consequently, the Casimir operator

\[
\Omega = \Delta_T + \frac{1}{8} (2 dT + \text{Scal}) = (D^{1/3})^2 - 2 a
\]

is non-negative. Its kernel coincides with the two-dimensional space of all \( \nabla \)-parallel spinors. These spinor fields are the Riemannian Killing spinors on \( M^6 \). The Dirac operator \( (D^{1/3})^2 \) is bounded from below by

\[
(D^{1/3})^2 \geq \frac{2}{15} \text{Scal}^g > 0.
\]

7. The Casimir operator of 7-dimensional G2-manifolds

Let \((M^7, g, \omega^3)\) be a 7-dimensional cocalibrated G2-manifold \((d \ast \omega^3 = 0)\) such that the scalar product \((d \omega^3, \ast \omega^3)\) is constant. There exists a unique connection \( \nabla \) preserving the G2-structure with skew-symmetric torsion

\[
T = - \ast d\omega^3 + \frac{1}{6} (d\omega^3, \ast \omega^3) \cdot \omega^3, \quad \delta(T) = 0.
\]

The Riemannian scalar curvature is given by the formula

\[
\text{Scal}^g = \frac{1}{18} (d\omega^3, \ast \omega^3)^2 - \frac{1}{2} ||T||^2 = 2 (T, \omega^3)^2 - \frac{1}{2} ||T||^2.
\]

Moreover, there exists a parallel spinor field \( \psi_0 \) such that

\[
\nabla \psi_0 = 0, \quad T \cdot \psi_0 = -\frac{1}{6} (d\omega^3, \ast \omega^3) \cdot \psi_0.
\]
A general reference for these facts are the papers [10] and [12]. The Casimir operator is given by the formula
\[ \Omega = \left( D^{1/3} \right)^2 - \frac{1}{4} (T, \omega^3)^2 + \frac{1}{8} (dT - 2 \sigma_T) \]
\[ = \Delta_T + \frac{1}{4} (T, \omega^3)^2 + \frac{1}{8} (3dT - 2 \sigma_T - 2 ||T||^2). \]

There are two special types of cocalibrated G\(_2\)-structures. A nearly parallel G\(_2\)-manifold is characterized by the equation \( d\omega^3 = -a (\ast \omega^3) \). The paper [14] contains examples of compact nearly parallel G\(_2\)-manifolds and their relation to Riemannian Killing spinors. The torsion form as well as the Riemannian Ricci tensor are given by the formulas
\[ T = -\frac{a}{6} \omega^3, \quad \text{Ric}^g = \frac{3}{8} a^2 \cdot g, \quad \text{Scal}^g = \frac{21}{8} a^2, \quad ||T||^2 = \frac{7}{36} a^2. \]

The torsion form of a nearly parallel G\(_2\)-manifold is \( \nabla \)-parallel (see [10], Corollary 4.9) and \( dT = 2 \sigma_T \). The Casimir operator is given by
\[ \Omega = \left( D^{1/3} \right)^2 - \frac{49}{144} a^2. \]

The \( \nabla \)-parallel spinor \( \psi_0 \) is the Riemannian Killing spinor and satisfies the equations (see [10])
\[ D^g \psi_0 = -\frac{7}{8} a \psi_0, \quad T \cdot \psi_0 = \frac{7}{6} a \psi_0. \]

In particular, \( \psi_0 \) belongs to the kernel of the Casimir operator. Consider now an arbitrary spinor field \( \psi \) in its kernel. Since the 3-form \( \omega^3 \) acts in the spinor bundle with two eigenvalues \(-7\) and \(+1\), there are two possibilities. If
\[ \Omega(\psi) = 0, \quad T \cdot \psi = \frac{7}{6} a \psi, \]
we obtain in the compact case the equation
\[ \frac{49}{144} a^2 \int_{M^7} ||\psi||^2 = \int_{M^7} ||(D^g + \frac{7}{24} a) \psi||^2. \]
Consequently, there exists an eigenvalue \( \lambda \in \text{Spec}(D^g) \) of the Riemannian Dirac operator such that
\[ \left( \lambda + \frac{7}{24} a \right)^2 \leq \frac{49}{144} a^2, \quad \frac{7}{8} a \leq |\lambda|. \]
The latter conditions imply that
\[ \lambda = -\frac{7}{8} a \]
and we are in the limiting case of the well-known estimate for the eigenvalues of the Riemannian Dirac operator (see [8]). The spinor field \( \psi \) is a Riemannian Killing spinor, i.e., \( \psi \) is \( \nabla \)-parallel. In a similar way, we discuss the second possibility
\[ \Omega(\psi) = 0, \quad T \cdot \psi = -\frac{1}{6} a \psi. \]
Then we obtain the inequalities
\[ \left( \lambda - \frac{1}{24} a \right)^2 \leq \frac{49}{144} a^2, \quad \frac{7}{8} a \leq |\lambda|. \]
and a solution \( \lambda \) does not exist. Let us summarize the result:
Theorem 7.1. Let \((M^7, g, \omega^3)\) be a compact, nearly parallel \(G_2\)-manifold \((d\omega^3 = -a\cdot(*\omega^3))\) and denote by \(\nabla\) its unique connection with skew-symmetric torsion. The kernel of the Casimir operator of the triple \((M^7, g, \nabla)\) coincides with the space of \(\nabla\)-parallel spinors,

\[
\text{Ker}(\Omega) = \{ \psi : \nabla \psi = 0, \quad T \cdot \psi = \frac{7}{6} a \cdot \psi \} = \text{Ker}(\nabla).
\]

A cocalibrated \(G_2\)-structure of type \(W_3\) in the Fernandez/Gray classification is characterized by the equations \(d \ast \omega^3 = 0\) and \((d\omega^3, \ast\omega^3) = 0\) (see \([7], [9]\)). The geometric data are \([10], [12]\)

\[
T = -d\omega^3, \quad \text{Scal}^g = -\frac{1}{2} ||T||^2, \quad \nabla \psi_0 = 0, \quad T \cdot \psi_0 = 0.
\]

In contrast to the nearly parallel case, cocalibrated \(G_2\)-manifolds of type \(W_3\) do not satisfy the condition \(d\Gamma = 2 \sigma_T\). The Casimir operator is given by the formula

\[
\Omega = (D^{1/3})^2 + \frac{1}{8} (dT - 2 \sigma_T) = \Delta_T + \frac{1}{8} (3dT - 2 \sigma_T - 2 ||T||^2).
\]

Examples of \(G_2\)-structures of type \(W_3\) on nilpotent Lie groups are discussed in the paper \([10]\) on the Aloff-Wallach space \(N(1,1)\) in \([2]\). We recall these examples and compute the relevant endomorphisms; they show that no general pattern is to be expected for this class of manifolds.

Example 7.1. There exists a \(G_2\)-structure of type \(W_3\) on the product of \(\mathbb{R}^1\) by the Heisenberg group. In this case, we have \(||T||^2 = 4\) and

\[
3dT - 2 \sigma_T = \text{diag}(8,0,8,-16,8,-16,8,0), \quad dT - 2 \sigma_T = \text{diag}(0,8,0,-8,0,-8,0,8).
\]

A second example on the product of \(\mathbb{R}^1\) by a 3-dimensional complex, solvable Lie group has been described in \([10]\), too. In both examples, \(3dT - 2 \sigma_T - 2 ||T||^2\) is a non-positive endomorphism acting on spinors. Consequently, the Casimir operator is dominated by the spinorial Laplacian,

\[
\int_{M^7} \langle \Omega(\psi), \psi \rangle \leq \int_{M^7} \langle \Delta_T(\psi), \psi \rangle.
\]

Example 7.2. In \([2]\), we constructed on the Aloff-Wallach space \(N(1,1) = SU(3)/S^1\) a family of metrics depending on a parameter \(0 < y < 1\) as well as \(G_2\)-structures of type \(W_3\) (see Proposition 8.8). In the notation of that paper, the spinor \(\psi_5\) is the \(\nabla\)-parallel spinor and algebraically the torsion form is given by \(4 \cdot T_5\) with

\[
T_5 = -y + \frac{2}{4} \left[ X_{135} + X_{146} + X_{245} - X_{236} \right] + \frac{3y}{y-1} X_{127} + \frac{2 + 2y - y^2}{2y - 2} \left[ X_{347} - X_{567} \right].
\]

Using the structure equations of the underlying geometry, we compute the exterior derivative,

\[
dT_5 = (2 + 4y) \left[ X_{2357} + X_{2467} - X_{1457} + X_{1367} \right] + \frac{3y(-2 - 2y + y^3)}{(y-1)^2} X_{3456}
\]

\[
+ \frac{10 + 9y + 12y^2 + 5y^3}{(y-1)^2} \left[ X_{1234} - X_{1256} \right].
\]

Inserting the matrices of the 7-dimensional spin representation, we compute the endomorphism \(3 \cdot (d\Gamma)_5 + (4T_5)^2 - 3 \cdot ||4T_5||^2\). It turns out that this endomorphism has the
eigenvalues \( \text{diag}(a, a, b, b, 0, c, a, a) \), where \( c := 64(7 + 10y + y^2) > 0 \) and

\[
a := -\frac{72(2 + y + y^2 - y^3 + y^4)}{(y-1)^2} < 0, \quad b := \frac{16(20 + 7y + 33y^2 + 13y^3 - y^4)}{(y-1)^2} > 0.
\]

The endomorphism \( 4dT_5 - 2\sigma_4 T_5 = 4dT_5 + (4T_5)^2 - ||4T_5||^2 \) has the eigenvalues \( \text{diag}(a^*, a^*, b^*, b^*, 0, c^*, a^*, a^*) \), where \( c^* := 64(5 + 6y + y^2) > 0 \) and

\[
a^* := \frac{24(-2 + y)(1 + y)^2}{1 - y} < 0, \quad b^* := \frac{16(4 - 7y - 10y^2 + y^3)}{(y-1)}.
\]

Hence, \( \Omega \) does not compare in any way to \( (D^{1/3})^2 \) or \( \Delta_T \); in particular, no statement about its kernel or positivity properties is possible.

Let us finally consider arbitrary cocalibrated \( G_2 \)-structures. The following example on \( N(1,1) \) is described in the paper [2], including the computation of the canonical connection and its geometric data. Surprisingly, its behaviour is almost the opposite of that of Example 7.1.

**Example 7.3.** In [2], Proposition 8.5, we constructed on \( N(1,1) \) a cocalibrated \( G_2 \)-structure with some special symmetry property. Its torsion form is given by \( 4 \cdot T \) with

\[
T = \frac{\sqrt{3}}{6} [X_{135} + X_{146} - X_{245} + X_{236}].
\]

Using the structure equations of the underlying geometry we compute the exterior derivative,

\[
dT = -X_{2357} - X_{2467} - X_{1457} + X_{1367},
\]

and finally the endomorphism

\[
\frac{1}{4}(4T, \omega^3)^2 + \frac{1}{8}(12dT - 2\sigma_4 T - 2||4T||^2) = \text{diag}\left(\frac{10}{3}, \frac{10}{3}, 0, 12, \frac{10}{3}, \frac{10}{3}, \frac{10}{3}, \frac{10}{3}\right).
\]

In particular, the Casimir operator of this \( G_2 \)-structure is non-negative,

\[
\int_{N(1,1)} \langle \Omega(\psi), \psi \rangle \geq \int_{N(1,1)} \langle \Delta_T(\psi), \psi \rangle \geq 0.
\]

and its kernel coincides with the space of \( \nabla \)-parallel spinors.

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