LOG CALABI-YAU STRUCTURE OF PROJECTIVE THREEFOLDS ADMITTING POLARIZED ENDOMORPHISMS

SHENG MENG

Abstract. Let $X$ be a normal projective variety admitting a polarized endomorphism $f$, i.e., $f^*H \sim qH$ for some ample divisor $H$ and integer $q > 1$. It was conjectured by Broustet and Gongyo that $X$ is of Calabi-Yau type, i.e., $(X, \Delta)$ is lc for some effective $\mathbb{Q}$-divisor such that $K_X + \Delta \sim_{\mathbb{Q}} 0$. In this paper, we establish a general guideline based on the equivariant minimal model program and the canonical bundle formula. In this way, we prove the conjecture when $X$ is a smooth projective threefold.

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1. Introduction

We work over an algebraically closed field of characteristic 0. Let $f : X \to X$ be a surjective endomorphism of a normal projective variety $X$. In the curve case, it is well known by the Hurwitz formula that $X$ is either a rational curve or an elliptic curve when $f$ is non-isomorphic. This is equivalent to saying that the anticanonical divisor $-K_X$ is effective. In higher dimensional case, to easily eliminate the distraction of automorphism, one focuses on the polarized endomorphism $f$, i.e., $f^*H \sim qH$ for some ample divisor $H$ and integer $q > 1$. Then by making use of the ramification divisor formula, Zhang and the author showed that $-K_X$ is effective when $X$ is $\mathbb{Q}$-Gorenstein. However, the

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effectivity of the anticanonical divisor, though important, can say very few on the detailed characterization of higher dimensional varieties.

A delicate operation is running the $f$-equivariant (after iteration) minimal model program. The smooth surface case is settled by Nakayama and the general higher dimensional situation is settled by Zhang and the author (cf. [Nak02],[MZ18],[MZ19],[MZ20],[CMZ20]). In this way, Broustet and Gongyo [BG17] proposed the following conjecture and proved the surface case. Recall that a normal projective variety $X$ is of Calabi-Yau type if $(X, \Delta)$ is an lc pair for some effective Weil-$\mathbb{Q}$-divisor $\Delta$ such that $K_X + \Delta \sim_{\mathbb{Q}} 0$. By the abundance (cf. [Gon13, Theorem 1.2]), the latter condition is equivalent to $K_X + \Delta \equiv 0$. We also call the pair $(X, \Delta)$ log Calabi-Yau.

**Conjecture 1.1.** Let $X$ be a normal projective variety admitting a polarized endomorphism. Then $X$ is of Calabi-Yau type.

In higher dimensional cases, Conjecture 1.1 has been recently verified for rationally connected smooth projective varieties by Yoshikawa; see [Yos20] or Theorem 2.8. The main purpose of this paper is to give a partial guideline on Conjecture 1.1 and to provide a full solution for the case of smooth projective threefolds. The following is our main result.

**Theorem 1.2.** Let $X$ be a smooth projective threefold admitting a polarized endomorphism $f$. Then $X$ is of Calabi-Yau type, i.e., $(X, \Delta)$ is an lc pair for some effective $\mathbb{Q}$-divisor $\Delta$ with $K_X + \Delta \sim_{\mathbb{Q}} 0$.

We briefly explain the strategy and difficulty in our proof. We first run the $f$-equivariant minimal model program which ends up with a $\mathbb{Q}$-abelian variety $Y$, i.e., a quasi-étale quotient of an abelian variety. Since the smooth rationally connected case has been verified by Yoshikawa, we may assume $\dim(Y) > 0$. Then we observe the fibration $\pi : X \to Y$ and its $f$-periodic general fibre. By applying the canonical bundle formula and ramification divisor formula, the natural idea is to reduce the problem to the following Conjecture 1.3 proposed by Yoshinori Gongyo.

**Conjecture 1.3** (Gongyo). Let $f : X \to X$ be a $q$-polarized endomorphism of a smooth projective variety $X$. Then $(X, \frac{K_X}{q-1})$ is an lc pair after iteration.

However, for the surface case, we cannot fully prove Conjecture 1.3 for $\mathbb{P}^2$, which is the only left case. So the main difficulty of proving Theorem 1.2 remains in the case when $-K_X$ is big; see Section 5. This new condition allows us to reduce the problem to the only very concrete case...
when $X \cong \mathbb{P}_Y (\mathcal{F}_2 \oplus \mathcal{L})$ where $\mathcal{F}_2$ is the unique indecomposable rank 2 vector bundle with non-trivial global sections and $\mathcal{L}$ is a line bundle of negative degree.

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2. Preliminaries

We use the following notation throughout this paper.

**Notation 2.1.** Let $X$ be a projective variety.

- The symbols $\sim$ (resp. $\sim_\mathbb{Q}$, $\equiv$) denote the linear equivalence (resp. $\mathbb{Q}$-linear equivalence, numerical equivalence) on $\mathbb{Q}$- (or $\mathbb{R}$-) Cartier divisors. We also use $\equiv$ to denote the numerical equivalence of 1-cycles on $X$.
- Denote by $\text{NS}(X) = \text{Pic}(X)/\text{Pic}^0(X)$ the Néron-Severi group of $X$. Let $N^1(X) := \text{NS}(X) \otimes \mathbb{R}$ the space of $\mathbb{R}$-Cartier divisors modulo numerical equivalence and $\rho(X) := \dim_{\mathbb{R}} N^1(X)$ the Picard number of $X$. Let $N_1(X)$ be the dual space of $N^1(X)$ consisting of 1-cycles. Denote by $\text{Nef}(X)$ the cone of nef divisors in $N^1(X)$ and $\overline{\text{NE}}(X)$ the dual cone consisting of pseudo-effective 1-cycles in $N_1(X)$.
- Let $f : X \to X$ be a surjective endomorphism. A subset $Y \subseteq X$ is $f^{-1}$-invariant (resp. $f^{-1}$-periodic) if $f^{-1}(Y) = Y$ (resp. $f^{-s}(Y) = Y$ for some $s > 0$).
- A surjective endomorphism $f : X \to X$ is $q$-polarized if $f^*H \sim qH$ for some ample Cartier divisor $H$ and integer $q > 1$; see [MZ18, Propositions 1.1] for the equivalent definitions.
- A smooth projective variety $X$ is rationally connected if any two general points of $X$ can be connected by a chain of rational curves.
- A normal projective variety $X$ is of Fano type, if there is an effective Weil $\mathbb{Q}$-divisor $\Delta$ on $X$ such that the pair $(X, \Delta)$ has at worst klt singularities and $-(K_X + \Delta)$ is ample and $\mathbb{Q}$-Cartier. If $\Delta = 0$, we say that $X$ is a (klt) Fano variety.
- Let $Y$ be a projective variety and $\mathcal{E}$ a vector bundle of rank $n$. Denote by $\pi : \mathbb{P}_Y(\mathcal{E}) \to Y$ the projective bundle of hyperplanes in $\mathcal{E}$ (not lines in $\mathcal{E}$), so that $\pi_*\mathcal{O}_{\mathbb{P}_Y(\mathcal{E})}(1) = \mathcal{E}$.

The following lemma is well-known and useful.

**Lemma 2.2.** Let $\pi : X \to Y$ be a birational morphism of two normal projective varieties. Then $Y$ is of Calabi-Yau type if $X$ is of Calabi-Yau type.
Proof. Suppose the pair \((X, \Delta_X)\) is log Calabi-Yau. Let \(\Delta_Y := \pi_* \Delta_X\). Then \(K_Y + \Delta_Y = \pi_*(K_X + \Delta_X) \sim_Q 0\). Note that \(K_X + \Delta_X = \pi^*(K_Y + \Delta_Y)\). By [KM98, Lemma 3.38], \((Y, \Delta_Y)\) has singularities not worse than \((X, \Delta_X)\). So \((Y, \Delta_Y)\) is lc. \(\square\)

Lemma 2.3. Let \(\pi : X \to Y\) be a quasi-étale finite surjective morphism of normal projective varieties. Then \(X\) is of Calabi-Yau type if and only if so is \(Y\).

Proof. Suppose \((Y, \Delta_Y)\) is log Calabi-Yau. Let \(\Delta_X = \pi_* \Delta_Y\). Since \(\pi\) is quasi-étale, \(K_X = \pi^* K_Y\) and hence \((X, \Delta_X)\) is log Calabi-Yau by [KM98, Proposition 5.20].

Conversely, assume that \((X, \Delta_X)\) is log Calabi-Yau. Note that the Galois closure of \(\pi\) is still quasi-étale by [GKP16, Theorem 3.7]. So we may assume that \(\pi\) is the quotient map of \(X\) by a finite group \(G\). Note that \((X, g^* \Delta_X)\) is log Calabi-Yau for any \(g \in G\).

Let \(\Delta = 1/|G| \sum_{g \in G} g^* \Delta_X\) and \(\Delta_Y = 1/\deg \pi \pi_* \Delta\). Then \(\Delta = \pi^* \Delta_Y\) Note that \((X, \Delta)\) is log Calabi-Yau and hence \((Y, \Delta_Y)\) is log Calabi-Yau. \(\square\)

Proposition 2.4. Let \(f : X \to X\) be a polarized endomorphism of a projective variety. Let \(D\) be an effective \(\mathbb{Q}\)-Cartier divisor on \(X\) with \(\kappa(X, D) = 0\). Then \(\text{Supp} D\) is \(f^{-1}\)-periodic.

Proof. By [MZ19, Proposition 3.7], we have

\[
 f^* f_* D \sim_Q f_* f^* D = (\deg f) D.
\]

Since \(\kappa(X, D) = 0\), we have \(f^* f_* D = (\deg f) D\). In particular, \(f^{-i} f^i(\text{Supp} D) = \text{Supp} D\) for all \(i \geq 0\). By [Men20, Lemma 8.1], \(\text{Supp} D\) is \(f^{-1}\)-periodic. \(\square\)

A \(\mathbb{Q}\)-divisor \(D\) is said to be \(\mathbb{Q}\)-movable if for any prime divisor \(\Gamma\), one has \(\Gamma \not\subseteq \text{Supp} D'\) for some effective \(\mathbb{Q}\)-divisor \(D' \sim_Q D\).

Let \(f : X \to X\) be a polarized endomorphism of a normal projective variety. Denote by \(T_f\) the finite union of \(f^{-1}\)-periodic prime divisors (cf. [MZ20, Corollary 3.8]). Denote by \(P_f := -(K_X + T_f)\).

Proposition 2.5. Let \(X\) be a \(\mathbb{Q}\)-Gorenstein normal projective variety admitting a polarized endomorphism \(f\). Assume further \(\kappa(X, -K_X) = 0\). Then \(\text{Supp} R_f = T_f, -K_X \sim_Q T_f\) and \((X, T_f)\) is lc.

Proof. This follows from [MZ19, Theorem 6.2] and [Zha13, Theorem 1.3]. \(\square\)

We generalize [MZ19, Theorem 1.5].

Theorem 2.6. Let \(f : X \to X\) be a polarized endomorphism of a \(\mathbb{Q}\)-factorial normal projective variety. Then \(-(K_X + T_f)\) is \(\mathbb{Q}\)-movable.
Proof. After iteration, we may assume each component of $T_f$ is $f^{-1}$-invariant. Consider the log ramification divisor formula

$$f^*(-(K_X + T_f)) = -(K_X + T_f) + \Delta_f$$

where $\Delta_f = R_f - (q - 1)T_f$ is effective and contains no common component of $T_f$. Replacing $-K_X$ by $-(K_X + T_f)$ in the proof of [MZ19, Theorem 6.2], we have that $-(K_X + T_f)$ is effective. Consider the $\sigma$-decomposition

$$-(K_X + T_f) \sim_{\mathbb{Q}} P + N$$

where $\kappa(X, N) = 0$ and $P$ is $\mathbb{Q}$-movable. By Proposition 2.4, we may assume each component of $N$ is $f^{-1}$-invariant after iteration. So $\text{Supp} N \subseteq T$. Let $\Gamma$ be a prime divisor contained in $\text{Supp} N$. By the triangle inequality,

$$\sigma_{\Gamma}(f^*(-(K_X + T_f))) \leq \sigma_{\Gamma}(-(K_X + T_f)) + \sigma_{\Gamma}(\Delta_f) = \sigma_{\Gamma}(-(K_X + T_f))$$

On the other hand,

$$\sigma_{\Gamma}(f^*(-(K_X + T_f))) = q\sigma_{\Gamma}(-(K_X + T_f)).$$

So $\sigma_{\Gamma}(-(K_X + T_f)) = 0$ and hence $N = 0$. Since $P$ is $\mathbb{Q}$-movable, we may further assume that $P$ has coefficients $\leq 1$. □

We recall the following result by Yoshikawa [Yos20, Proposition 6.2] that Fano type can be preserved by (polarized) equivariant birational map.

**Proposition 2.7.** Let $f : X \to X$ be a polarized endomorphism of a normal projective variety $X$. Let $\pi : X -\to Y$ be an $f$-equivariant birational map. Then $X$ is of Fano type if and only if so is $Y$.

**Proof.** We may simply take $\Delta = \Gamma = 0$ in [Yos20, Proposition 6.2]. □

We recall the nice result of Yoshikawa [Yos20, Corollary 1.4].

**Theorem 2.8.** Let $X$ be a rationally connected smooth projective variety admitting a polarized endomorphism. Then $X$ is of Fano type.

3. **Singularities of ramification divisor**

In this section, we give some general results concerning Conjecture 1.3. First, we consider the coefficients of the ramification divisor.

**Theorem 3.1.** Let $f : X \to X$ be a $q$-polarized endomorphism of a projective variety. Then after iteration, the coefficient $r_P$ of $R_f$ on each prime divisor $P$ has $r_P \leq q - 1$ and the equality holds if and only if $P$ is $f^{-1}$-periodic.
Proof. Let \( b \) be the number of prime divisors contained in the branch locus of \( f \). Let \( c \) be the maximal coefficient appeared in \( f^*P \) with \( P \) being a prime divisor. Choose some \( a > 0 \) such that \( c^b < q^{a/2} \). Choose some \( s > 0 \) such that \( (\frac{q^s}{q-1})^{s/t} > c^b \) for any \( 1 \leq t < a \). Let \( N = \max\{a, s\} \) and take \( n > N \). Let \( P \) be a prime divisor which is not \( f^{-1}\)-periodic. Let \( Q \) be a prime divisor in \( f^{-n}(P) \). Let \( r \) be the coefficient of \( (f^n)^*P \) on \( Q \). We shall show that \( r < q^n \).

If \( P \) is not \( f\)-periodic, then all irreducible components of \( \{f^{-i}(P)\}_{i>0} \) are different with each other, and hence \( r \leq c^b < q^{a/2} < q^n \). In the following, we consider the case when \( P \) is \( f\)-periodic with period \( t \geq 1 \).

Let \( r_1 \) be the coefficient of \( (f^t)^*P \) on \( P \). Since all irreducible components of \( \{f^{-i}(P)\}_{0<i\leq t} \) are different with each other, we have \( r_1 < c^b < q^{a/2} \). Moreover, we claim that \( r_1 < q^t \).

Suppose the contrary that we can write \( (f^t)^*P = q^tP + E \) for some effective divisor \( E \). By the projection formula,

\[
\deg f^t = (f^t)_*(f^t)^*P = (f^t)_*(q^tP + E) = q^t(\deg f^t|_P)P + (f^t)_*E.
\]

Note that \( \deg f^t = q^t\dim(X) \) and \( \deg f^t|_P = q^t(\dim(X)-1) \). Therefore, we have \( E = 0 \) and hence \( P \) is \( f^{-1}\)-periodic, a contradiction. So the claim is proved.

Let \( e \) be the minimal non-negative integer such that \( f^e(Q) = P \). Let \( r_2 \) be the coefficient of \( (f^e)^*P \) on \( Q \). Then \( r = r_2 \cdot r_1^{(n-e)/t} \). If \( t \geq a \), then \( r < c^b \cdot q^{a/2n/a} < q^n \). If \( t < a \), then \( r < c^b \cdot (q^t - 1)^{n/t} < q^n \). □

Remark 3.2. The proof of Theorem 3.1 can be easily applied to consider general surjective endomorphisms, though the statement could be a bit wordy. We leave the readers the pleasure of the coefficient estimate.

We observe the behaviour of ramification divisors between equivariant birational morphisms.

Proposition 3.3. Let \( \pi : X \to Y \) be a birational morphism of \( \mathbb{Q}\)-factorial normal projective varieties. Let \( f : X \to X \) and \( g : Y \to Y \) be two \( q\)-polarized endomorphisms such that \( g \circ \pi = \pi \circ f \). Then

\[
K_X + \frac{R_f}{q-1} = \pi^*(K_Y + \frac{R_g}{q-1}).
\]

In particular, \( (X, \frac{R_f}{q-1}) \) is lc if \( (Y, \frac{R_g}{q-1}) \) is lc.

Proof. Since \( K_X \) and \( K_Y \) are \( \mathbb{Q}\)-Cartier, we may write

\[
K_X = \pi^*K_Y + E
\]
where $E$ has supports contained in the exceptional divisor of $\pi$. By the ramification divisor formula, we have

$$K_X = f^*K_X + R_f \text{ and } K_Y = g^*K_Y + R_g.$$ 

By the above three equations, we have

$$\pi^*K_Y + E = f^*(\pi^*K_Y + E) + R_f = \pi^*g^*K_Y + f^*E + R_f = \pi^*(K_Y - R_g) + f^*E + R_f.$$ 

Note that $f^*E = qE$. So we have

$$\pi^* \frac{R_g}{q-1} - \frac{R_f}{q-1} = E.$$ 

Then we have

$$K_X + \frac{R_f}{q-1} = \pi^*(K_Y + \frac{R_g}{q-1})$$ 

as desired. \qed

The canonical bundle formula plays a key role in our reduction.

**Proposition 3.4.** Let $\pi : X \to Y$ be an algebraic fibration where $X$ is an $n$-dimensional smooth projective variety and $Y$ is normal projective with $K_Y \equiv 0$. Let $f : X \to X$ and $g : Y \to Y$ be $q$-polarized endomorphisms such that $g \circ \pi = \pi \circ f$ and $f^*K_X \equiv qK_X$. Then $(X, \frac{R_f}{q-1})$ is log Calabi-Yau after iteration if Conjecture 1.3 holds true for $f^*|_F : F \to F$ where $F$ is a $f$-periodic (of period $s$) general fibre of $\pi$.

**Proof.** By the ramification divisor formula, we have

$$K_X + \frac{R_f}{q-1} \equiv 0$$ 

which holds true after arbitrary iteration.

Let $F$ be a general $f$-invariant smooth fibre of $\pi$ after iteration (cf. [Fak03, Theorem 5.1]). Note that $f|_F$ is $q$-polarized and $R_{f|_F} = R_f|_F$. By the assumption, $(F, \frac{R_f}{q-1})$ is lc after iteration. So $(X, \frac{R_f}{q-1})$ is lc over the generic point of $\pi$. By the lc canonical bundle formula [Fuj04, Theorem 4.1.1], we have

$$0 \equiv K_X + \frac{R_f}{q-1} = \pi^*(K_Y + B + M)$$ 

where $B$ is effective and $M$ is pseudo-effective. Note that $K_Y \equiv 0$. So $B = 0$ and hence $(X, \frac{R_f}{q-1})$ is lc. \qed
4. Surface case

We first prove Conjecture 1.3 for surfaces except $\mathbb{P}^2$.

**Theorem 4.1.** Let $f : X \to X$ be a $q$-polarized endomorphism of a smooth projective surface with $\rho(X) > 1$. Then $(X, \frac{R_f}{q-1})$ is lc after iteration.

**Proof.** We apply [MZ18, Theorem 1.8]. If $K_X$ is pseudo-effective, then $R_f = 0$ and we are done. In the following, we may assume $K_X$ is not pseudo-effective. After iteration, we may run an $f$-equivariant minimal model program of $X$ which ends up with a Fano contraction $\pi : X' \to Y$ with $Y$ being a curve of genus $\leq 1$. Then $f^*|_{N^1(X)} = q \text{id}$ and hence $K_X + \frac{R_f}{q-1} \equiv 0$ by the ramification divisor formula. By Proposition 3.3, we may assume $X = X'$. Note that $X$ is a ruled surface over $Y$. We finish the proof with the following 2 cases.

**Case 1.** Suppose $Y$ is elliptic. Note that Conjecture 1.3 holds for curves and $f^*|_{N^1(X)} = q \text{id}$. So we are done by Proposition 3.4.

**Case 2.** Suppose $Y = \mathbb{P}^1$. Then $Y \cong F_d$ with $d \geq 0$. If $d = 0$, then $R_f$ is simply a sum of fibres and horizontal sections with coefficients $\leq q - 1$ and hence this case is simple. We may assume now that $d > 0$. Then $\pi$ admits a unique negative section curve $C$. Note that $f^{-1}(C) = C$. Then we have the decomposition by effective $\mathbb{Q}$-divisors

$$\frac{R_f}{q-1} = C + V + H$$

where $V$ has support in the fibres of $\pi$ and each component of $H$ dominates $Y$. Clearly, $C$ is not a component of $H$ each component of $V$ has the coefficient $\leq 1$. Fix a point $y \in Y$. Let $F := \pi^{-1}(y)$. For the restriction

$$f|_F : F \cong \mathbb{P}^1 \to f(F) \cong \mathbb{P}^1,$$

we have $\frac{R_f}{q-1}|_F = C|_F + H|_F$. Note that $\frac{R_f}{q-1} \cdot F = 2$, each component of $\frac{R_f}{q-1}$ has coefficient $\leq 1$, and $C|_F$ is a reduced point. Consider the local intersection number at $x \in F$. If $x = C \cap F$, then $(H \cdot F)_x = 0$ and hence

$$((C + H) \cdot F)_x = 1.$$ 

If $x \not= C \cap F$, then

$$((C + H) \cdot F)_x \leq (C + H) \cdot F - (C \cdot F)_{C \cap F} = \frac{R_f}{q-1} \cdot F - 1 = 1.$$ 

In particular, we have $(X, F + C + H)$ is lc near $F$ by [KM98, Corollary 5.57]. Note that $\frac{R_f}{q-1} \leq F + C + H$ near $F$. So $(X, \frac{R_f}{q-1})$ is lc.

We give some partial results on the $\mathbb{P}^2$ case which is enough for the proof of Theorem 1.2.
Theorem 4.2. Let \( f : X \cong \mathbb{P}^2 \to X \) be a \( q \)-polarized endomorphism. Suppose \( T_f \neq \emptyset \). Then \( (X, \frac{R_f}{q-1}) \) is log Calabi-Yau after iteration.

Proof. Write \( \frac{R_f}{q-1} = T_f + \Delta_f \). Note that \( \Delta_f \) and \( T_f \) have no common components. By Theorem 3.1, we may assume that \( \Delta_f \) has coefficient \( < 1 \) after sufficient iteration of \( f \). Then the non-klt locus \( \text{Nklt}(X, \frac{R_f}{q-1}) = T_f \cup S \) where \( S \) is a finite set of points outside \( T_f \).

We first show that the non-lc locus \( \text{Nlc}(X, \frac{R_f}{q-1}) \cap T_f = \emptyset \). In particular, if \( S = \emptyset \), then \( (X, \frac{R_f}{q-1}) \) is lc. Suppose the contrary and let \( x \in \text{Nlc}(X, \frac{R_f}{q-1}) \cap T_f \). Let \( C \) be an irreducible component of \( T_f \) containing \( x \). By [Gur03], \( C \) is a line. After iteration, we may assume \( f^{-1}(C) = C \). Consider the log ramification divisor formula,

\[
(K_X + C) = f^*(K_X + C) + R_f - (q - 1)C.
\]

Apply adjunction on \( C \), we have

\[
K_C = (f|_C)^*K_C + (R_f - (q - 1)C)|_C = (f|_C)^*K_C + R_{f|_C}.
\]

So \( (R_f - (q - 1)C)|_C = R_{f|_C} \). Note that \( \frac{R_{f|_C}}{q-1} \) has coefficient \( \leq 1 \). Then \(((\frac{R_f}{q-1} - C) \cdot C)_x \leq 1 \). By inverse of adjunction (cf. [KM98, Corrolary 5.57]), \( (X, \frac{R_f}{q-1}) \) is lc near \( x \), a contradiction.

Assume now that \( S \neq \emptyset \). Then \( \text{Nklt}(X, \frac{R_f}{q-1}) \) is not connected. By [HHI19, Theorem 1.2], \( (X, \frac{R_f}{q-1}) \) is plt, a contradiction. \( \square \)

Corollary 4.3. Let \( f : X \cong \mathbb{P}^2 \to X \) be a \( q \)-polarized endomorphism. Suppose there is a birational morphism \( \pi : W \to X \) from a normal projective surface \( W \) such that \( \pi \circ h = f \circ \pi \) for some surjective endomorphism \( h : W \to W \). Suppose further either the exceptional locus of \( W \) is reducible or \( W \) has more than one negative curves. Then \( T_f \neq \emptyset \) and hence \( (X, \frac{R_f}{q-1}) \) is log Calabi-Yau after iteration.

Proof. By Proposition 2.7, \( W \) is of Fano type and hence \( \mathbb{Q} \)-factorial (cf. [KM98, Proposition 4.11]). By [MZ19, Lemma 4.3], the negative curves on \( W \) are \( h^{-1} \)-periodic. Note that the existence of negative curve on \( W \) not contracted by \( \pi \) will cause \( T_f \neq \emptyset \). So we may always assume that the exceptional locus of \( W \) is reducible.

Let \( E = \sum_{i=1}^{n} E_i \) be the reduced \( \pi \)-exceptional divisor with \( n > 1 \). Write \( K_W = \pi^*K_X + \sum_{i=1}^{n} a_i E_i \) where \( a_i > 0 \) since \( X \) is smooth. By the negativity lemma, \( K_W \) is not \( \pi \)-nef. So we can run \( h \)-equivariant (after iteration) relative minimal model program of \( W \) over \( X \) which finally ends up with \( X \). Denote by \( \tau : W_1 \to X \) the last step and \( \sigma : W \to W_1 \) the composition of the previous steps. Note that \( W_1 \) is of Fano type and \( \rho(W_1) = 2 \). Denote by \( p : W_1 \to Y \) be the contraction induced by the extremal ray different with \( \tau \). If \( p \) is birational, then the exceptional curve of \( p \) is not contracted by \( \tau \) and hence
So we may assume that $p : W_1 \to Y \cong \mathbb{P}^1$ is a $\mathbb{P}^1$-fibration. Suppose that $E_1$ is contracted by $\sigma$. Then $p(\sigma(E_1))$ is an $(h|_Y)^{-1}$-invariant point on $Y$ (cf. [CMZ20, Lemma 7.5]). Therefore, $p^{-1}(p(\sigma(E_1)))$ is an $(h|_{W_1})^{-1}$-invariant curve which is not contracted by $\tau$ and hence $T_f \neq \emptyset$. Finally, we apply Theorem 4.2. □

5. Anti-canonical divisor

Throughout this section, we use the following setting:

5.1. Let $f : X \to X$ be a $q$-polarized endomorphism of a smooth projective threefold $X$ admitting an $(f$-equivariant) Fano contraction $\pi : X \to Y$ with $Y$ being an elliptic curve and general fibres being $\mathbb{P}^2$. In this setting, $\rho(X) = 2$ and we may further assume $\rho|_{X} = q$ id. Denote by $\phi : X / \text{axisshort/axisshort/arrowaxisright} Z$ the Chow reduction of the Iitaka fibration of $-K_X$ which is $f$-equivariant by [MZ19, Theorem 7.8].

Proposition 5.2. Suppose there is another fibration $\tau : X \to V$ different with $\pi$ with $0 < \dim(V) < 3$. Then $-K_X$ is semi-ample. In particular, $X$ is of Calabi-Yau type.

Proof. Note that $\rho(V) = 1$ because $\rho(X) = 2$. Since $\pi$ and $\tau$ are two different fibrations to positive lower dimensional varieties, the two extremal rays of Nef($X$) are generated by the pullbacks of ample divisors on $Y$ and $V$ which are not big. Hence, Nef($X$) = PE$^1$($X$). Note that $-K_X$ is not ample but effective and $\pi$-ample. So $-K_X \equiv \phi^*H$ for some ample $\mathbb{Q}$-divisor $H$ on $V$. By Bertini’s theorem, we may assume $(X, \phi^*H)$ is lc (even terminal) after a suitable choice of $H$. By the abundance, $-K_X \sim_\mathbb{Q} \phi^*H$ and we are done. □

Theorem 5.3. Assume further $\kappa(X, -K_X) < 3$, i.e., $-K_X$ is not big. Then $X$ is of Calabi-Yau type.

Proof. By Proposition 2.5, we may assume $\kappa(X, -K_X) > 0$. By Proposition 5.2, we may assume $\phi$ is not well-defined. Let $W$ be the normalization of the graph of $\phi$. Denote by $h : W \to W$ the lifting of $f$ and $\sigma : W \to X$ the induced birational morphism. Since $X$ is smooth, the exceptional locus of $\sigma$ is of pure codimension one in $W$ (cf. [KM98, Corollary 2.63]). We denote it by $E := \sum_{i=1}^n E_i$ with $n > 0$. After iteration, we may assume $E_i$ is $h^{-1}$-invariant. Note that the elliptic curve $Y$ admits no $g^{-1}$-periodic points. So $\pi(\sigma(E_i)) = Y$.

Let $W_y := (\pi \circ \sigma)^{-1}(y)$ be an $h$-invariant general fibre after iteration (cf. [Fak03, Theorem 5.1]). Denote by $X_y := \pi^{-1}(y) \cong \mathbb{P}^2$. If $E$ is reducible, i.e., $n > 1$, then $E \cap W_y$ is also reducible. By Corollary 4.3, $(X_y, \frac{R_y}{q-1})$ is log Calabi-Yau after iteration of $f$. By
Proposition 3.4, \((X, \frac{R_Y}{Y_1})\) is log Calabi-Yau after iteration of \(f\). So we may assume \(E\) is irreducible. Note that \(E\) is \(\mathbb{Q}\)-Cartier (cf. [KM98, Lemma 2.62]). Then \(W\) is \(\mathbb{Q}\)-factorial. By [Zha13, Theorem 1.3], \((W, E)\) is lc. Since \(W \setminus E\) is smooth, \(W\) is klt.

Write \(K_W = \sigma^*K_X + aE\) with \(a > 0\) since \(X\) is smooth. Then

\[
0 \leq \kappa(W, -K_W) \leq \kappa(W, -\sigma^*K_X) = \kappa(X, -K_X)
\]

where the first inequality follows from Theorem 2.6 and the last equality follows from [Uen75, Theorem 5.13]. Let \(\phi_W : W \dashrightarrow V\) be the Chow reduction of the Iitaka fibration of \(-K_W\). So \(\dim(V) < 3\). Suppose \(\phi_W\) is not well-defined. Let \(\tilde{W}\) be the normalization of the graph of \(\phi_W\) and \(\tilde{\sigma} : \tilde{W} \to W\) the induced birational morphism. Since \(W\) is \(\mathbb{Q}\)-factorial, the exceptional locus of \(\tilde{\sigma}\) is of pure codimension one in \(\tilde{W}\) (cf. [KM98, Corollary 2.63]) and each irreducible component dominates \(Y\) by a similar argument. In particular, the restricted birational morphism \(\tilde{W}_y \to X_y\) has reducible exceptional locus. Then we are done by Corollary 4.3 and Proposition 3.4. So we may assume that \(\phi_W\) is well-defined, \(\rho(W_y) = 2\) and \(W_y\) only one negative curve \(C = E \cap W_y\), i.e., the exceptional curve of \(W_y \to X_y\). By Proposition 2.7 and [KM98, Proposition 4.11], \(W_y\) is of Fano type and \(\mathbb{Q}\)-factorial. Since \(W_y\) has only one negative curve (which is also \(K_{W_y}\)-negative since \(X\) is smooth), another contraction of \(W_y\) is a \(\mathbb{P}^1\)-fibration over \(\mathbb{P}^1\). In particular, \(-K_{W_y}\) is ample.

If \(\dim(V) = 0\), then \((W, T_h)\) is log Calabi-Yau by Proposition 2.5. If \(\dim(V) = 1\), then \(\phi_W\) is just the Iitaka fibration of \(-K_W\) and hence \(-K_W\) is semi-ample. In both cases, \(W\) and hence \(X\) are of Calabi-Yau type by Lemma 2.2.

We are left with \(\dim(V) = 2\). Note that \(-K_W|_{W_y} = -K_{W_y}\) is ample. So the restriction \(\phi_W|_{W_y} : W_y \to V\) is surjective. We first assume that the induced map \(X_y \dashrightarrow V\) is not well-defined. By the rigidity lemma (cf. [Deb01, Lemma 1.15]), \(\phi_W|_{W_y}\) contracts no curves. Then \(\phi_W|_{W_y} : W_y \to V\) is finite surjective. By [KM98, Lemma 5.16 and Proposition 5.20], \(V\) is klt and \(\mathbb{Q}\)-factorial. Let \(C_V := \phi_W|_{W_y}(C)\). By the projection formula, \(C_V\) is also a negative curve. Then, we have an \(h|_V\)-equivariant divisorial contraction \(V \to V_1\) which contracts \(C_V\). By the rigidity lemma again, the induced rational map \(X_y \dashrightarrow V_1\) is then well-defined. If the map \(X_y \to V\) is already well-defined, then we simply identify \(V_1\) with \(V\). Consider the induced \(f\)-equivariant dominant rational map \(\phi' : X \dashrightarrow V_1\). Note that indeterminant locus of \(\phi'\) is \(f^{-1}\)-invariant which does not dominate \(Y\) since \(\phi'\) is well-defined near \(X_y\). By [CMZ20, Lemma 7.5], \(\phi'\) is well-defined and we are done by Proposition 5.2. \(\square\)
6. Proof of Theorem 1.2

We refer to [Ati57] for well-known facts on vector bundles over elliptic curves. By \( \mathcal{F}_n \) we mean the unique indecomposable rank \( n \) vector bundle with non-trivial global sections over an elliptic curve. The following lemmas will be used later.

**Lemma 6.1.** Let \( S \cong \mathbb{P}_Y(\mathcal{F}_2) \) where \( Y \) is an elliptic curve and \( \mathcal{F}_2 \) is the unique indecomposable rank 2 vector bundle with non-trivial global sections. Then \( S \) admits no polarized endomorphism.

**Proof.** Suppose the contrary that there is a \( q \)-polarized endomorphism \( f : S \to S \). We may assume \( f^*|_{N^1(S)} = q \mathrm{id} \). Note that \( \pi : S \to Y \) has a unique section \( C \) with \( C^2 = 0 \) and \( -K_S \sim 2C \). Then \( \mathcal{O}_S(1) \cong \mathcal{O}_S(C) \) and \( \pi_\ast \mathcal{O}_S(n) \cong \text{Sym}^n(\mathcal{F}_2) \cong \mathcal{F}_{n+1} \) for \( n > 0 \) (cf. [Ati57, Theorem 9]). In particular, \( h^0(S, nC) = h^0(Y, \mathcal{F}_{n+1}) = 1 \) and \( \kappa(S, -K_S) = 0 \). By Proposition 2.5, we have \( \text{Supp } R_f = T_f \) and \( -K_S \sim_q T_f \). Note that \( T_f \) is reduced. So \( T_f \neq 2C \) and hence \( \kappa(S, C) > 0 \), a contradiction. \( \square \)

**Proof of Theorem 1.2. Step 1.** In this step, we reduce our situation to the case when \( T_f = \emptyset \), \( f^*|_{N^1(X)} = q \mathrm{id} \) and \( X \cong \mathbb{P}_Y(\mathcal{E}) \) for some rank 3 vector bundle \( \mathcal{E} \) on an elliptic curve \( Y \).

By [MZ18, Theorem 1.8], we run \( f \)-equivariant minimal model program (after iteration) and let \( Y \) be the end product: a \( \mathbb{Q} \)-abelian variety. Denote by \( \pi : X \to Y \) be the induced composition. If \( \dim(Y) = 3 \), then \( X = Y \) is \( \mathbb{Q} \)-abelian and hence \( (X, R_{\frac{T_f}{q-1}}) = (X, 0) \) is log Calabi-Yau. If \( \dim(Y) = 0 \), then \( X \) is of Fano type by Theorem 2.8.

Suppose \( \dim(Y) = 2 \). By Theorem 2.6, we have \( -(K_X + T_f) \sim_q P \) for some effective \( \mathbb{Q} \)-divisor \( P \) with coefficients \( \leq 1 \) and containing no component of \( T_f \). Let \( F \) be a general fibre of \( \pi \). Since \( F \) is general, we may assume that \( \text{Supp } P \cap T_f \) intersects with \( F \) transversally. In particular, \( (F, (T_f + P)|_F) \) is lc and \( (X, T_f + P) \) is lc over the generic point of \( \pi \). By the canonical bundle formula [Fuj04, Theorem 4.1.1], we have

\[ 0 \equiv K_X + T_f + P = \pi^\ast(K_Y + B + M) \]

Note that \( B \) is effective and \( M \) is pseudo-effective and \( K_Y \equiv 0 \). So \( B = 0 \) and hence \( (X, T_f + P) \) is lc and \( X \) is of Calabi-Yau type.

Now we may assume that \( \dim(Y) = 1 \). Then \( f^*|_{N^1(X)} = q \mathrm{id} \) after iteration by [MZ18, Theorem 1.8]. By Theorems 4.1 and Proposition 3.4, the only left case is when the general fibre of \( \pi \) is \( \mathbb{P}^2 \). In particular, \( \rho(X) = 2 \) and \( \pi \) is a Fano contraction. By [Mor82, Theorem 3.5] and since the Brauer group of the elliptic curve \( Y \) is trivial, \( X \cong \mathbb{P}_Y(\mathcal{E}) \) for some rank 3 vector bundle \( \mathcal{E} \) on \( Y \).

Suppose \( f^{-1}(P) = P \) for some prime divisor \( P \). Then \( P \) dominates \( Y \) since \( f|_Y \) is étale and polarized. Let \( F \) be an \( f \)-invariant (after iteration) fibre of \( \pi \). Then \( D \cap F \subseteq T_{f|_F} \).
By Theorem 4.2 and Proposition 3.4, \((X, \frac{R}{r-1})\) is log Calabi-Yau. So we may assume \(T_f = \emptyset\).

**Step 2.** In this step, we reduce our situation to the case when \(E \cong F_2 \oplus \mathcal{L}\) for some line bundle \(\mathcal{L}\).

Note that \(f|_Y\) is polarized and hence Zariski dense periodic points. So after iteration and choosing some fixed point as the identity element of \(Y\), we may assume \(f|_Y\) is an isogeny. In particular, \(f|_Y\) commutes with any multiplication map. By Lemma 2.3, we can always replace \(X\) by base change of multiplication map (by 3) on \(Y\). So we may assume \(3|\deg E\). By Theorem 5.3, we may further assume \(-K_X\) is big. Consider the following two cases.

Suppose \(E \cong L_1 \oplus L_2 \oplus L_3\) where \(L_1 \cong \mathcal{O}_E\). Let \(D_1\) be the divisor determined by the projection \(E \to L_2 \oplus L_3\) and define \(D_2\) and \(D_3\) similarly. Consider the exact sequence

\[
0 \to \mathcal{O}_X(1) \otimes \mathcal{O}_X(-D_1) \to \mathcal{O}_X(1) \to \mathcal{O}_X(1) \otimes \mathcal{O}_{D_1} \to 0.
\]

Taking \(\pi_*\), we have

\[
0 \to \pi_*(\mathcal{O}_X(1) \otimes \mathcal{O}_X(-D_1)) \to \mathcal{E} \to \pi_*\mathcal{O}_{D_1}(1) = \mathcal{L}_2 \oplus \mathcal{L}_3 \to 0
\]

with 0 on the right because \(R^1\pi_*(\mathcal{O}_X(1) \otimes \mathcal{O}_X(-D_1)) = 0\) since all the fibres are \(\mathbb{P}^2\). By the projection formula, we have \(\mathcal{O}_X(1) \cong \mathcal{O}_X(D_1)\) and similarly, \(D_1 \sim D_2 + \pi^*c_1(\mathcal{L}_2)\) and \(D_1 \sim D_3 + \pi^*c_1(\mathcal{L}_3)\). So by the relative Euler sequence, we have

\[
-K_X \sim 3D_1 - \pi^*(c_1(\mathcal{E})) = D_1 + D_2 + D_3.
\]

It is easy to see that \(D_1 + D_2 + D_3\) has simple normal crossing. In particular, \((X, D_1 + D_2 + D_3)\) is a log Calabi-Yau pair.

Suppose \(\mathcal{E}\) is indecomposable. Since \(3|\deg \mathcal{E}\), we have \(\mathcal{E} \cong \mathcal{F}_3 \otimes \mathcal{L} = 0\) for some line bundle \(\mathcal{L}\) (cf. [Ati57, Theorem 5]). Therefore, we may assume \(\mathcal{E} \cong \mathcal{F}_3\). However, \(-K_X\) is then nef but not big, a contradiction with our assumption.

Now we may assume \(\mathcal{E} \cong \mathcal{F} \oplus \mathcal{L}\) where \(\mathcal{F}\) is an indecomposable rank 2 vector bundle and \(\mathcal{L}\) is a line bundle. Note that we may also assume that \(\mathcal{F}\) does not split after taking pullback of the multiplication map (by 2) of \(Y\). So we may assume that \(\mathcal{F} \cong \mathcal{F}_2\).

**Step 3.** In this step, we show that \(\deg \mathcal{L} < 0\). Note that there is a non-splitting exact sequence

\[
0 \to \mathcal{O}_Y \to \mathcal{E} \to \mathcal{O}_Y \oplus \mathcal{L} \to 0
\]

which is induced by the non-splitting exact sequence of \(\mathcal{F}_2\). Let \(D\) be the divisor determined by \(\mathcal{E} \to \mathcal{O}_Y \oplus \mathcal{L}\). Then \(\mathcal{O}_X(1) \cong \mathcal{O}_X(D)\). Let \(D'\) be the divisor determined by the natural projection \(\mathcal{E} \to \mathcal{F}_2\). Then \(D \sim D' + \pi^*c_1(\mathcal{L})\) and we have

\[
-K_X \sim 3D - \pi^*c_1(\mathcal{L}) \sim 2D + D'.
\]
Note that $D|_D \sim C$ where $C$ is the section curve determined by $\mathcal{O}_E \oplus \mathcal{L} \to \mathcal{L}$. Then we see that $-K_X$ is big if and only if $\deg \mathcal{L} \neq 0$. By Lemma 6.1, $D'$ is not $f$-periodic. Note that $f^*D' \equiv qD'$ and $f^{-1}(D') \neq D'$. So $D'|_{D'}$ is pseudo-effective and hence nef on $D'$. Then $D'$ is nef. Note that $-K_X$ is not nef because otherwise $X$ is of Fano type. So $D$ is not nef and hence $\deg \mathcal{L} < 0$.

**Step 4.** In this step, we show that $\kappa(X, D) = 0$. Note that

$$\pi_*(\mathcal{O}_X(n)) \cong \text{Sym}^n(\mathcal{E}) \cong \bigoplus_{i=0}^n \mathcal{F}_{i+1} \otimes \mathcal{L}^{\otimes (n-i)}$$

for $n \geq 0$. Then

$$h^0(X, nD) = \sum_{i=0}^n h^0(Y, \text{Sym}^i(\mathcal{F}_2) \otimes \mathcal{L}^{\otimes (n-i)}) = h^0(Y, \mathcal{F}_{n+1}) = 1$$

by noticing that $h^0(Y, \mathcal{F}_{i+1} \otimes \mathcal{L}^{\otimes (n-i)}) = 0$ for any $i < n$. So $\kappa(X, D) = 0$.

**End of the proof.** By Proposition 2.4, $D$ is $f^{-1}$-periodic. However, this contradicts the reduction we obtained in Step 1. □

**Remark 6.2.** When $X \cong \mathbb{P}_Y(\mathcal{F}_2 \oplus \mathcal{L})$ for some line bundle $\mathcal{L}$ of negative degree, we do not know whether $X$ is of Calabi-Yau type or not. What we proved above is that either $X$ is of Calabi-Yau type or $X$ admits no polarized endomorphism. Therefore, it will be very interesting to study the following question.

**Question 6.3.** Let $X \cong \mathbb{P}_Y(\mathcal{E})$ be a projective bundle over an elliptic curve $Y$.

(1) When will $X$ be of Calabi-Yau type?

(2) When will $X$ admit a polarized endomorphism?

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Korea Institute For Advanced Study, Seoul 02455, Republic of Korea

Email address: ms@u.nus.edu, shengmeng@kias.re.kr