Stochastic Global Optimization Algorithms: A Systematic Formal Approach

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Abstract
As we know, some global optimization problems cannot be solved using analytic methods, so numeric/algorithmic approaches are used to find near to the optimal solutions for them. A stochastic global optimization algorithm (SGoAL) is an iterative algorithm that generates a new population (a set of candidate solutions) from a previous population using stochastic operations. Although some research works have formalized SGoALs using Markov kernels, such formalization is not general and sometimes is blurred. In this paper, we propose a comprehensive and systematic formal approach for studying SGoALs. First, we present the required theory of probability ($\sigma$-algebras, measurable functions, kernel, markov chain, products, convergence and so on) and prove that some algorithmic functions like swapping and projection can be represented by kernels. Then, we introduce the notion of join-kernel as a way of characterizing the combination of stochastic methods. Next, we define the optimization space, a formal structure (a set with a $\sigma$-algebra that contains strict $\epsilon$-optimal states) for studying SGoALs, and we develop kernels, like sort and permutation, on such structure. Finally, we present some popular SGoALs in terms of the developed theory, we introduce sufficient conditions for convergence of a SGOAL, and we prove convergence of some popular SGoALs.

Keywords: Stochastic Global Optimization; Markov model; Markov Kernel; $\sigma$-algebra; Optimization space; Convergence

1. Stochastic Global Optimization
A global optimization problem is formulated in terms of finding a point $x$ in a subset $\Omega \subseteq \Phi$ where a certain function $f : \Phi \to \mathbb{R}$, attains is best/optimal value (minimum or maximum) [1]. In the optimization field, $\Omega$, $\Phi$, and $f$ are called the feasible region, the solution space, and the objective function, respectively. The optimal value for the objective function (denoted as $f^* \in \mathbb{R}$) is suppose to exist and it is unique ($\mathbb{R}$ is a total order). In this paper, the global
optimization problem will be considered as the minimization problem described by equation 1.

\[
\min (f : \Phi \to \mathbb{R}) = x \in \Omega \subseteq \Phi \mid (\forall y \in \Omega) (f(x) \leq f(y))
\] (1)

Since a global optimization problem cannot be solved, in general, using analytic methods, numeric methods are largely applied in this task [2, 3]. Some numeric methods are deterministic, like Cutting plane techniques [4], and Branch and Bound [5] approaches while others are stochastic like Hill Climbing and Simulated Annealing. Many stochastic methods, like Evolutionary Algorithms and Differential Evolution, are based on heuristics and metaheuristics [6, 7, 8, 9]. In this paper, we will concentrate on Stochastic Global Optimization Methods (algorithms). A Stochastic Global Optimization Algorithm (SGoAL) is an iterative algorithm that generates a new candidate set of solutions (called population) from a given population using a stochastic operation, see Algorithm 1.

**Algorithm 1** Stochastic Global Optimization Algorithm.

\begin{align*}
\text{SGoAL}(n) \\
1. \quad t_0 &= 0 \\
2. \quad P &= \text{InitPop}(n) \\
3. \quad \text{while } \neg \text{End}(P_t, t) \text{ do} \\
4. \quad P_{t+1} &= \text{NextPop}(P_t) \\
5. \quad t &= t + 1 \\
6. \quad \text{return } \text{Best}(P_t)
\end{align*}

In Algorithm 1, \(n\) is the number of individuals in the population (population’s size), \(P_t \in \Omega^n\) is the population at iteration \(t \geq 0\), \(\text{InitPop}: \mathbb{N} \to \Omega^n\) is a function that generates the initial population (according to some distribution), \(\text{NextPop}: \Omega^n \to \Omega^n\) is a stochastic method that generates the next population from the current one (the stochastic search), \(\text{End}: \Omega^n \times \mathbb{N} \to \text{Bool}\) is a predicate that defines when the SGoAL(n) process is stopped and \(\text{Best}: \Omega^n \to \Omega\) is a function that obtains the best candidate solution (individual) in the population according to the optimization problem under consideration, see equation 2.

\[
\text{Best}(x) = x_i \mid \forall_{k=1}^n f(x_i) \leq f(x_k) \land f(x_i) < \forall_{k=1}^{i-1} f(x_k)
\] (2)

Although there are several different SGoAL models, such models mainly vary on the definition of the NextPop function. Sections 1.1 to 1.3 present three popular SGoALs reported in the literature.

\[ \text{Algorithm 1} \text{ Stochastic Global Optimization Algorithm.} \]

\[ \text{SGoAL}(n) \]

1. \(t_0 = 0\)
2. \(P = \text{InitPop}(n)\)
3. \(\text{while } \neg \text{End}(P_t, t) \text{ do}\)
4. \(P_{t+1} = \text{NextPop}(P_t)\)
5. \(t = t + 1\)
6. \(\text{return } \text{Best}(P_t)\)

\[ \text{In Algorithm 1, } n \text{ is the number of individuals in the population (population’s size).} \]

\[ P_t \in \Omega^n \text{ is the population at iteration } t \geq 0, \text{InitPop}: \mathbb{N} \to \Omega^n \text{ is a function that generates the initial population (according to some distribution), NextPop}: \Omega^n \to \Omega^n \text{ is a stochastic method that generates the next population from the current one (the stochastic search), End}: \Omega^n \times \mathbb{N} \to \text{Bool} \text{ is a predicate that defines when the SGoAL(n) process is stopped and Best}: \Omega^n \to \Omega \text{ is a function that obtains the best candidate solution (individual) in the population according to the optimization problem under consideration, see equation 2}. \]

\[ \text{Best}(x) = x_i \mid \forall_{k=1}^n f(x_i) \leq f(x_k) \land f(x_i) < \forall_{k=1}^{i-1} f(x_k) \] (2)

\[ \text{Although there are several different SGoAL models, such models mainly vary on the definition of the NextPop function. Sections 1.1 to 1.3 present three popular SGoALs reported in the literature.} \]

\[ ^2 \text{Although, parameters that control the stochastic process, like the size of the population, can be adapted (adjusted) during the execution of a SGoAL, in this paper we just consider any SGoAL with fixed control parameters (including population’s size).} \]
1.1. Hill Climbing (HC)

The hill climbing algorithm (HC), see Algorithm 2, is a SGOAL that uses a single individual as population \( n = 1 \), generates a new individual from it (using the stochastic method \( \text{Variate}: \Omega \rightarrow \Omega \)), and maintains the best individual among them (line 2). Notice that HC allows to introduce neutral mutations if the greater or equal operator \( (\geq) \) is used in line 2. In order to maintain more than one individual in the population, the HC algorithm can be parallelized using Algorithm 3.

**Algorithm 2 Hill Climbing Algorithm - NextPop Method.**

\begin{verbatim}
NextPopHC(\{x\})
1. \( x' = \text{Variate}(x) \)
2. if \( f(x') \{>,\geq\} f(x) \) then \( x' = x \)
3. return \( \{x'\} \)
\end{verbatim}

**Algorithm 3 Parallel Hill Climbing Algorithm (PHC) - NextPop Method.**

\begin{verbatim}
NextPopPHC(P)
1. \( \{Q_i\} = \text{NextPopHC}\{P_i\} \) for all \( i = 1,2,\ldots,|P| \)
2. return \( Q \)
\end{verbatim}

1.2. Genetic Algorithms (GA)

Genetic algorithms (GA)s are optimization techniques based on the principles of natural evolution \([6]\). Although there are several different versions of GAs, such as Generational Genetic (GGA) algorithms and Steady State Genetic (SSGA) algorithms, in general, all GAs have the same structure. Major differences between them are in the encoding scheme, in the evolution mechanism, and in the replacement mechanism. Algorithms 4 and 5 present the GGA and SSGA, respectively. There, \( \text{PickParents}: \Omega^n \rightarrow \mathbb{N}^2 \) picks two individuals (indices) as parents, \( \text{XOver}: \Omega^2 \rightarrow \Omega^2 \) combines both of them and produces two new individuals, \( \text{Mutate}: \Omega^2 \rightarrow \Omega^2 \) produces two individuals (offspring) that are mutations of such two new individuals, \( \text{Best}_2: \Omega^4 \rightarrow \Omega^2 \) picks the best two individuals between parents and offspring, and \( \text{Bernoulli}(r) \) generates a true value following a Bernoulli distribution with probability \( CR \).

1.3. Differential Evolution (DE)

Differential Evolution (DE) algorithm is an optimization technique, for linear spaces, based on the idea of using vector differences for perturbing a candidate solution, see Algorithm 6. Here, \( \Omega \) is a \( d \)-dimensional linear search space, \( \text{PickDiffParents}: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}^3 \) gets three individuals (indices \( a, b, \) and \( c \)) that

\[\text{\footnotesize \text{\text{\footnotesize A neutral mutation is a variation in the individual that does not change the value of the objective function \[10\].}}}\]
### Algorithm 4 Generational Genetic Algorithm (GGA) - NextPop Method.

**NextPopGGA(P)**

1. for $i = 1$ to $\frac{n}{2}$
2. ${a, b} = \text{PickParents}(P)$
3. if Bernoulli($CR$) then $\{Q_{2i-1}, Q_{2i}\} = \text{Mutate}($XOver$(P_a, P_b))$
4. else $\{Q_{2i-1}, Q_{2i}\} = \{P_a, P_b\}$
5. return $Q$

### Algorithm 5 Steady State Genetic Algorithm (SSGA) - NextPop Method.

**NextPopSSGA(P)**

1. $\{a, b\} = \text{PickParents}(P)$
2. $Q_k = P_k$ for all $k = 1, 2, \ldots |P|$, $k \neq a, b$
3. if Bernoulli($CR$) then $\{c_1, c_2\} = \text{Mutate}($XOver$(P_a, P_b))$
4. else $\{c_1, c_2\} = \text{Mutate}(P_a, P_b)$
5. $\{Q_a, Q_b\} = \text{Best}_2(c_1, c_2, P_a, P_b)$
6. return $Q$

are different from each other and different from the individual under consideration $(i)$, $0 \leq CR \leq 1$ is a crossover rate, and $0 \leq F \leq 2$ is the difference weight.

### Algorithm 6 Differential Evolution Algorithm (DE) - NextPop Method.

**NextPopDE(P)**

1. $\{a, b, c\} = \text{PickDifParents}(|P|, i)$
2. $R \sim \mathcal{N}[1, d]$
3. for $k = 1$ to $d$
5. if Bernoulli($CR$) or $k = R$ then $q_k = P_{a,k} + F \cdot (P_{b,k} - P_{c,k})$
6. else $q_k = P_{i,k}$
7. return $q$

**NextPopDE(P)**

1. for $i = 1$ to $n$
2. $Q_i = \text{NextInd}_{DE}(P, i)$
3. return $Q$

### 2. Measure and Probability Theory

In this section, we introduce the basic measure and probability theory concepts that are required for a formal treatment of SGOALS. First, we will concentrate on the concept of family of sets, required for formalizing concepts like event and random observation (subsections 2.1 and 2.2). Next, we will cover the concepts of measurable, measure, and probability measure functions (subsection 2.3) that are required for defining and studying the notion of Kernel (subsection
2.4), notion that will be used as formal characterization of stochastic methods used by SGOALS. Then, we will present the concept of Markov chains (subsection 2.5), concept that is used for a formal treatment of SGOALS. Finally, we introduce two concepts of random sequence convergence (subsection 2.6) for studying the convergence properties of a SGOAL.

2.1. Family of Sets

Probability theory starts by defining the space of elementary events (a nonempty set \( \Omega \)) and the system of observable events (a family of subsets of \( \Omega \)). In the case of a formal treatment of a SGOAL, the space of elementary events is the set of possible populations while the system of observable events is defined by any subset of populations that can be generated, starting from a single population, through the set of stochastic methods used by the SGOAL. In the rest of this paper, let \( \Omega \neq \emptyset \) be a non-empty set (if no other assumption is considered).

**Definition 1. (Power Set)** Let \( \Omega \) be a set, the power set of \( \Omega \), denoted as \( 2^\Omega \), is the family of all subsets of \( \Omega \), i.e. \( 2^\Omega = \{A \mid A \subseteq \Omega \} \).

Clearly, the system of observable events is a subset \( A \) of \( 2^\Omega \) that satisfies some properties. Here, we introduce families of sets with some properties that are required for that purpose. Then, we establish a relation between two of them.

**Definition 2.** Let \( A \subseteq 2^\Omega \) be a family of subsets of \( \Omega \).

- **DF (disjoint family)** \( A \) is a disjoint family if \( A \cap B = \emptyset \) for any pair of \( A \neq B \in A \).
- **CF (countable family)** \( A \) is a countable family if \( A = \{A_i\}_{i \in I} \) for some countable set \( I \).
- **CDF (countable disjoint family)** \( A \) is a countable disjoint family if \( A \) is CF and.
- **CU (close under complements)** \( A \) is close under complements if \( A \in A \) then \( A^c = \Omega \setminus A \in A \).
- **CDU (close under countable disjoint unions)** \( A \) is close under countable disjoint unions if \( \bigcup_{i \in I} A_i \in A \) for all \( \{A_i \in A\}_{i \in I} \) CDF.
- **CI (close under countable intersections)** \( A \) is close under countable intersections if \( \bigcap_{i \in I} A_i \in A \) for all \( \{A_i \in A\}_{i \in I} \) CF.
- **\( \pi \) (\( \pi \)-system)** \( A \) is a \( \pi \)-system if it is close under finite intersections, i.e. if \( A, B \in A \) then \( A \cap B \in A \).
- **\( \lambda \) (\( \lambda \)-system)** \( A \) is called \( \lambda \)-system iff (\( \lambda.1 \) \( \emptyset \in A \), (\( \lambda.2 \) \( A \) is CDU and (\( \lambda.3 \) \( A \) is CI.

**Lemma 3.** Let \( A \subseteq 2^\Omega \)
1. \((\lambda \rightarrow \mathbb{PD})\) If \(\mathcal{A}\) is \(\lambda\)-system then \(\mathcal{A}\) is \(\mathbb{PD}\).
2. \((\mathcal{C} \rightarrow (\mathcal{C}\cup \mathcal{C}) \iff \mathcal{C})\) If \(\mathcal{A}\) is \(\mathcal{C}\) then \(\mathcal{A}\) is \(\mathcal{C}\cup \mathcal{C}\) iff \(\mathcal{A}\) is \(\mathcal{C}\).

Proof. 1. \((\lambda \rightarrow \mathbb{PD})\) If \(A, B \in \mathcal{A}\) then \(B^c \in \mathcal{A}\) (\(\lambda.2: A\) is \(\mathbb{C}\)). Clearly, \(A \cap B^c = \emptyset\) \((A \subset B)\) and \(A \cup B^c \in \mathcal{A}\) (\(\lambda.3: A\) is \(\mathbb{C}\)). So, \(A \cup B^c \in \mathcal{A}\) (Morgan’s law). Therefore, \(B \setminus A \in \mathcal{A}\) (def. proper difference).
3. \((\mathcal{C} \rightarrow (\mathcal{C}\cup \mathcal{C}) \iff \mathcal{C})\) If \(\{A_i\}_{i \in I}\) is cf in \(\mathcal{A}\), then \(\{A_i^c\}_{i \in I}\) is cf in \(\mathcal{A}\) (\(\mathcal{A}\) is \(\mathcal{C}\)).

Finally, if \(\mathcal{A}\) is \(\mathcal{C}\) then \(\bigcap_{i \in I} A_i = \left(\bigcup_{i \in I} A_i^c\right)^c \in \mathcal{A}\) (Morgan’s law and \(\mathcal{A}\) is \(\mathcal{C}\)), so \(\mathcal{A}\) is \(\mathbb{C}\).

2.2. \(\sigma\)-algebras

Although each family of sets, in definition 2, is very interesting on its own, none of them allows by itself to define, in a consistent manner, a notion of probability. As we will see, \(\sigma\)-algebras play this role in a natural way.

Definition 4. \((\sigma\)-algebra\) A family of sets \(\Sigma \subseteq 2^\Omega\) is called a \(\sigma\)-algebra over \(\Omega\), iff \((\sigma.1)\) \(\emptyset \in \Sigma\), \((\sigma.2)\) \(\Sigma\) is \(\mathcal{C}\), and \((\sigma.3)\) \(\Sigma\) is \(\mathbb{C}\).

Now, we can establish some relations between \(\sigma\)-algebras and some of the previously defined families of sets. These relations are very useful when dealing with notions like measure, measurable, and kernel.

Lemma 5. Let \(\Sigma\) be a \(\sigma\)-algebra over \(\Omega\):

1. \(\emptyset \in \Sigma\)
2. \(\Sigma\) is \(\mathcal{C}\).
3. \(\Sigma\) is a \(\lambda\)-system.
4. \(\Sigma\) is \(\mathbb{PD}\).

Proof. \([1]\) \(\emptyset \in \Sigma\) (\(\sigma.1\)) then \(\emptyset \in \Sigma\) (\(\sigma.2: \Sigma\) is \(\mathcal{C}\)). \([2]\) Follows from \(\sigma.2\), \(\sigma.3\) and lemma 3 \([3]\) \(\lambda.1\) follows from \((1)\), \(\lambda.2\) and \(\lambda.3\) follow from \(\sigma.2\) and \(\sigma.3\), respectively. \([4]\) Follows from \((3)\) and lemma 3.

Proposition 6. Let \(\Omega\) be a set

1. \(2^\Omega\) is a \(\sigma\)-algebra.
2. If \(\{\Sigma_i\}_{i \in I}\) is a family of \(\sigma\)-algebras over \(\Omega\) then \(\bigcap_{i \in I} \Sigma_i\) is a \(\sigma\)-algebra over \(\Omega\).
3. If \(\mathcal{A} \subseteq 2^\Omega\) is an arbitrary family of subsets of \(\Omega\) then the minimum \(\sigma\)-algebra generated by \(\mathcal{A}\) is \(\sigma(\mathcal{A}) = \bigcap\{\Sigma | \mathcal{A} \subseteq \Sigma \text{ and } \Sigma \text{ is } \sigma\text{-algebra}\}\).

Proof. \([1]\) Obvious. \([2]\) \(\Omega \in \Sigma_i\) for all \(i \in I\) (\(\sigma.1\)) then \(\Omega \in \bigcap_{i \in I} \Sigma_i\) (def. \(\bigcap\)). If \(A \in \bigcap_{i \in I} \Sigma_i\) then \(A \in \Sigma_i\) for all \(i \in I\), then \(A^c \in \Sigma_i\) for all \(i \in I\) (\(\sigma.2: \Sigma_i\) is \(\mathcal{C}\)), therefore \(A^c \in \bigcap_{i \in I} \Sigma_i\) (def. \(\bigcap\)). If \(\{A_j \in \bigcap_{i \in I} \Sigma_i\}_{j \in J}\) is cf then \(A_j \in \Sigma_i\) for all \(j \in J\) and \(i \in I\), then \(\{A_j \in \Sigma_i\}_{j \in J}\) is cf in \(\Sigma_i\) for all \(i \in I\), therefore \(\bigcup_{j \in J} A_j \in \Sigma_i\) for all \(i \in I\) (\(\sigma.3: \Sigma_i\) is \(\mathbb{C}\)). So, \(\bigcup_{j \in J} A_j \in \bigcap_{i \in I} \Sigma_i\) (def. \(\bigcap\)). \([3]\) Follows from \((1)\) and \((2)\).
Theorem 7. \textbf{(Dynkin \(\pi\)-\(\lambda\) theorem)} Let \(A\) be a \(\lambda\)-system and let \(E \subseteq A\) be a \(\pi\)-system then \(\sigma(E) \subseteq A\).

\textit{Proof.} A proof of this theorem can be found on page 6 of Kenkle’s book \cite{11} (Theorem 1.19).

Now, notions of measure and probability measure are defined on the real numbers (\(\mathbb{R}\)), usually equipped with the Euclidean distance, so we need to define an appropriated \(\sigma\)-algebra on it. Such appropriated \(\sigma\)-algebra can be defined as a special case of a \(\sigma\)-algebra for topological spaces.

\textbf{Definition 8. (Borel \(\sigma\)-algebra)} Let \((\Omega, \tau)\) be a topological space. The \(\sigma\)-algebra \(B(\Omega) = B(\Omega, \tau) = \sigma(\tau)\) is called the Borel \(\sigma\)-algebra on \(\Omega\) and every \(A \in B(\Omega, \tau)\) is called Borel (measurable) set.

\textbf{Proposition 9.} If \(B(\mathbb{R})\) is the Borel \(\sigma\)-algebra where \(\mathbb{R}\) is equipped with the Euclidean distance, then \(B(\mathbb{R}) = \sigma(E_7)\) with \(E_7 = \{ (\alpha, \beta) \mid \alpha, \beta \in \mathbb{Q}, \alpha < \beta \}\).

\textit{Proof.} A proof of this proposition can be found on page 9 of Kenkle’s book \cite{11} (Theorem 1.23).

Now, we are ready to define the basic mathematical structure used by probability theory.

\textbf{Definition 10. (measurable space)} If \(\Sigma\) is a \(\sigma\)-algebra over a set \(\Omega\) then the pair \((\Omega, \Sigma)\) is called a measurable space. Sets in \(\Sigma\) are called measurable sets on \(\Omega\).

2.3. Functions

Having the playground defined (space of elementary events and observable events), probability theory defines operations over them (functions). Such functions will allow us to characterize stochastic methods used by a SGOAL. First, we introduce the concepts of set function and inverse function, that are used when working on \(\sigma\)-algebras.

\textbf{Definition 11. (set functions)} Let \(f: \Omega_1 \rightarrow \Omega_2\) be a function,

1. The power set function of \(f\) is defined as

\[
f: \quad 2^{\Omega_1} \rightarrow 2^{\Omega_2}
\]

\[
A \mapsto \{ f(x) \mid \forall (x \in A) \}
\]

2. The inverse function of \(f\) is defined as

\[
f^{-1}: \quad 2^{\Omega_2} \rightarrow 2^{\Omega_1}
\]

\[
B \mapsto \{ x \in \Omega_1 \mid (\exists y \in B) (y = f(x)) \}
\]

Next, we study the measurable functions, structure-preserving maps (homomorphisms between measurable spaces). A measurable function guarantees that observable events are obtained by applying the function to observable events. For SGOALs, a measurable function (stochastic methods) basically means that any generated subset of populations must be obtained by applying the stochastic methods to some generated subset of populations.
**Definition 12.** (measurable function) Let \((\Omega_1, \Sigma_1)\) and \((\Omega_2, \Sigma_2)\) be measurable spaces and \(f: \Omega_1 \rightarrow \Omega_2\) be a function. Function \(f\) is called \(\Sigma_1 - \Sigma_2\) measurable if for every measurable set \(B \in \Sigma_2\), its inverse image is a measurable set in \((\Omega_1, \Sigma_1)\), i.e., \(f^{-1}(B) \in \Sigma_1\).

**Corollary 13.** Let \((\Omega_1, \Sigma_1)\), \((\Omega_2, \Sigma_2)\) and \((\Omega_3, \Sigma_3)\) be measurable spaces and \(f: \Omega_1 \rightarrow \Omega_2\) be \(\Sigma_1 - \Sigma_2\) measurable and \(g: \Omega_2 \rightarrow \Omega_3\) be \(\Sigma_2 - \Sigma_3\) measurable then \(g \circ f: \Omega_1 \rightarrow \Omega_3\) is \(\Sigma_1 - \Sigma_3\) measurable.

**Proof.** If \(A \in \Sigma_3\) then \(g^{-1}(A) \in \Sigma_2\) (\(g\) is \(\Sigma_2 - \Sigma_3\) measurable), therefore \(f^{-1}(g^{-1}(A)) \in \Sigma_1\) (\(f^{-1}\) is \(\Sigma_1 - \Sigma_2\) measurable). Clearly, \(f^{-1}(g^{-1}(A)) = (g \circ f)^{-1}(A) \in \Sigma_1\) (def inverse). In this way, \(g \circ f\) is \(\Sigma_1 - \Sigma_3\) measurable. 

**Definition 14.** (isomorphism of measurable spaces) Let \((\Omega_1, \Sigma_1)\) and \((\Omega_2, \Sigma_2)\) be measurable spaces and \(\varphi: \Omega_1 \rightarrow \Omega_2\) be a bijective function. \(\varphi\) is called \((\Omega_1, \Sigma_1)\)-(\(\Omega_2, \Sigma_2\)) isomorphism if \((\Omega_1, \Sigma_1)\) and \((\Omega_2, \Sigma_2)\) are called isomorphic if \(\varphi\) is \(\Sigma_1 - \Sigma_2\) measurable and \(\varphi^{-1}\) is \(\Sigma_2 - \Sigma_1\) measurable.

After that, we consider measure functions, functions that quantify, in some way, how much observable events are. This concept of measure function is the starting point on defining probability measure functions. In the following, we just write \(f\) is measurable instead of \(f\) is \(\Sigma_1 - \Sigma_2\) measurable if the associated \(\sigma\)-algebras can be inferred from the context.

**Definition 15.** (measure function) Let \((\Omega, \Sigma)\) be a measurable space and \(\mu: \Sigma \rightarrow \mathbb{R}\) be a function from \(\Sigma\) to the extended reals \((\mathbb{R} = \mathbb{R} \cup \{-\infty, \infty\})\). Function \(\mu\) is called measure if it satisfies the following three conditions:

\[
\begin{align*}
\mu_1 & \text{ (nullity) } \mu(\emptyset) = 0, \\
\mu_2 & \text{ (non-negativity) } \mu(A) \geq 0 \text{ for all } A \in \Sigma, \\
\mu_3 & \text{ (}\sigma\text{-additivity) } \mu \left( \bigcup_{i \in I} A_i \right) = \sum_{i \in I} \mu(A_i) \text{ for all } \{A_i \in \Sigma\}_{i \in I} \text{ cdf}.
\end{align*}
\]

Then, we consider probability measure functions, functions that quantify, how probable observable events are. For SGOALS, a probability function will quantify how probable a set of populations can be generated using stochastic methods.

**Definition 16.** Let \((\Omega, \Sigma)\) be a measurable space and \(\mu: \Sigma \rightarrow \mathbb{R}\) be a measure.

1. (finite measure) \(\mu\) is a finite measure if \(\mu(A) < \infty\) for all \(A \in \Sigma\).
2. (probability measure) \(\mu\) is a probability measure if \(\mu(\Omega) = 1\).

Now, we are ready to define the mathematical structure used by probability theory.

**Definition 17.** If \((\Omega, \Sigma)\) is a measurable space and \(\mu: \Sigma \rightarrow \mathbb{R}\) is a measure function

1. (measure space) \((\Omega, \Sigma, \mu)\) is called measure space.
2. (probability space) If \(\mu\) is a probability measure, \((\Omega, \Sigma, \mu)\) is called probability space.
Finally, we can define the concept of random variable, a function that preserves observable events and quantifies how probable an observable event is.

**Definition 18. (random variable)** Let \((\Omega_1, \Sigma_1, Pr)\) be a probability space and \((\Omega_2, \Sigma_2)\) be a measurable space. If \(X: \Omega_1 \to \Omega_2\) is a measurable function then \(X\) is called a random variable with values in \((\Omega_2, \Sigma_2)\).

For \(A \in \Sigma_2\), we denote \(\{X \in A\} \equiv X^{-1}(A)\) and \(Pr [X \in A] \equiv Pr \left[ X^{-1} (A) \right]\).

In particular, if \((\Omega_2, \Sigma_2) = (\mathbb{R}, B(\mathbb{R}))\), then \(X\) is called a real random variable and we let \(Pr [X \geq 0] \equiv X^{-1} ([0, \infty))\).

2.4. Kernel

As pointed by Breiman in Section 4.3 of [12], the kernel is a regular conditional probability \(K(x, A) = P(x, A) = Pr [X_t \in A \mid X_{t-1} = x]\). For SGOALS, a kernel will be used for characterizing the stochastic process carried on iteration by iteration (generating a population from a population).

**Definition 19. (Markov kernel)** Let \((\Omega_1, \Sigma_1)\) and \((\Omega_2, \Sigma_2)\) be measurable spaces. A function \(K : \Omega_1 \times \Sigma_2 \to [0, 1]\) is called a (Markov) kernel if the following two conditions hold:

1. Function \(K_{x, \bullet} : A \mapsto K(x, A)\) is a probability measure for each fixed \(x \in \Omega_1\) and
2. Function \(K_{\bullet, A} : x \mapsto K(x, A)\) is a measurable function for each fixed \(A \in \Sigma_2\).

**Remark 20.** As noticed by Kenkle in Remark 8.26, page 181 of [11], it is sufficient to check \(K_2\) in definition [19] for sets \(A\) from a \(\pi\)-system \(\mathcal{E}\) that generates \(\Sigma_2\) and that either contains \(A\) or a sequence \(A_n \uparrow A\). Indeed, in this case, \(\mathcal{D} = \{ A \in \Sigma_2 \mid K_{\bullet, A}^{-1} \in \Sigma_1 \}\) is a \(\lambda\)-system. Since \(\mathcal{E} \subset \mathcal{D}\), by the Dynkin \(\pi - \lambda\) theorem [17], \(\mathcal{D} = \sigma (\mathcal{E}) = \Sigma_2\).

If the transition density \(K : \Omega_1 \times \Omega_2 \to [0, 1]\) exists, then the transition kernel can be defined using equation [3] In the rest of this paper, we will consider kernels having transition densities.

\[
K(x, A) = \int_A K(x, y) \, dy
\]  

(3)

Kernels that will play a main role in a systematic development of a formal theory for SGOALS are those associated to deterministic methods that are used by a particular SGOAL, like selecting any/the best individual in a population, or sorting a population. Theorem [21] provides a sufficient condition for characterizing deterministic methods as kernels.

**Theorem 21. (deterministic kernel)** Let \((\Omega_1, \Sigma_1)\) and \((\Omega_2, \Sigma_2)\) be measurable spaces, and \(f : \Omega_1 \to \Omega_2\) be \(\Sigma_1 - \Sigma_2\) measurable. The function \(1_f : \Omega_1 \times \Sigma_2 \to [0, 1]\) defined as follow is a kernel.

\[
1_f (x, A) = \begin{cases} 
1 & \text{if } f(x) \in A \\
0 & \text{otherwise}
\end{cases}
\]
Proof. [well-defined] Obvious, it is defined using the membership predicate and takes only two values 0 and 1. [K.1] Let $x \in \Omega_1$, clearly $1_{f \cdot \bullet} (A) \geq 0$ for all $A \subseteq \Sigma_2$ so $1_{f \cdot \bullet}$ is non-negative. Now, $1_f (x, \emptyset) = 0$ so it satisfies property $f$. Let $\{ A_i \subset \Sigma_2 \}_i \subset I$ be a cdf. If $x \in \bigcup_{i \in I} A_i$ then $1_f (x, \bigcup_{i \in I} A_i) = 1$ (def $1_f (x, A)$), and $\exists k \in I$ such that $x \in A_k$ (def $\{ A \}$). Therefore, $x \notin A_i$ for all $i \neq k \in I$ so $1_f (x, A_k) = 1$ and $1_f (x, A_i) = 0$ for all $i \neq k \in I$ (def $1_f (x, A)$). Clearly, $1_{f \cdot \bullet} (\bigcup_{i \in I} A_i) = 1 = \sum_{i \in I} 1_{f \cdot \bullet} (A_i)$. A similar proof is carried on when $x \notin \bigcup_{i \in I} A_i$, in this case $1_{f \cdot \bullet} (\bigcup_{i \in I} A_i) = 0 = \sum_{i \in I} 1_{f \cdot \bullet} (A_i)$. Then, $1_{f \cdot \bullet}$ is $\sigma$-additive, so it is a measure. $1_f (x, \Omega_2) = 1$ (obvious) then $1_{f \cdot \bullet}$ is a probability measure for a fixed $x \in \Omega_1$. [K.2] Let $A \subseteq \Sigma_2$ and $\alpha \in \mathbb{Q}^+$. If $\alpha < 1$ then $1_{f \cdot \bullet, A} ((0, \alpha]) = \emptyset \ (1_f (x, A) = \{0, 1\} \notin (0, \alpha])$, so $1_{f \cdot \bullet, A} ((0, \alpha]) \subseteq \Sigma_1$ (lemma [5.1]). Now, if $\alpha \geq 1$ then $1_{f \cdot \bullet, A} ((0, \alpha]) = \{ x \in \Omega_1 \mid 1_f (x, A) = \alpha \subset (0, \alpha] \} (1_f (x, A) = 1$ is the only value in $(0, \alpha])$, i.e., $1_{f \cdot \bullet, A} ((0, \alpha]) = \{ x \in \Omega_1 \mid f (x) \in A \} = f^{-1} (A) (\text{def } f^{-1})$, so $1_{f \cdot \bullet, A} ((0, \alpha]) \subseteq \Sigma_1$ ($f$ is measurable). Therefore, $1_{f \cdot \bullet, A}$ is measurable. \qed

Corollary 22. (Indicator kernel) Let $(\Omega, \Sigma)$ be a measurable space. The indicator function $1 : \Omega \times \Sigma \to [0, 1]$ defined as $1 (x, A) = 1_{\text{id}(x)} (A)$, with $\text{id}(x) = x$ is a kernel.

Proof. According to theorem [21] it is sufficient to prove that $\text{id}$ is a measurable function. It is obvious, we have that $A = \text{id} (A) = \text{id}^{-1} (A) (\text{def } \text{id})$ then $\text{id}^{-1} (A) \in \Sigma$ if $A \in \Sigma$. \qed

Transition probabilities (kernels) also represent linear operators over infinite-dimensional vector spaces [13]. Therefore, operations like kernels multiplication, and kernels convex combinations can be used in order to preserve the Markovness property of the resulting transition kernel (sometimes called update mechanism).

2.4.1. Random Scan (Mixing)

The random scan (mixing) update mechanism follows the idea of picking one update mechanism (among a collection of predefined update mechanisms) and then applying it. Such update mechanism is picked according to some weight associated to each one of the update mechanism. Following this idea, the mixing update mechanism is built using kernels addition and kernel multiplication by a scalar.

In order to maintain the Markovness property (both operations, kernels addition and kernel multiplication by a scalar, in general, do not preserve such property), a convex combination of them is considered.

Definition 23. (mixing) The mixing update mechanism of a set of $n$ Markov transition kernels $K_1, ..., K_n$, each of them with a probability of being picked $p_1, p_2, ..., p_n$ ($\sum p_i = 1$), is defined by equation [4]

$$\left( \sum_{i=1}^{n} p_i K_i \right) (x, A) = \int_A \sum_{i=1}^{n} p_i K_i (x, y) dy \tag{4}$$

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Since the integral in equation 4 is a linear operator, the mixing operation can be defined by equation 5.

\[
\left( \sum_{i=1}^{n} p_i K_i \right) (x, A) = \sum_{i=1}^{n} p_i \int_A K_i (x, y) \, dy
\]  

(5)

2.4.2. Composition

The composition update mechanism follows the idea of applying an update mechanism (kernel) followed by other update mechanism and so on. Following this idea, the composition update mechanism is built using the kernel multiplication operator.

**Definition 24. (composition)** The composition of two kernels \( K_1, K_2 \) is defined by equation 6.

\[
(K_2 \circ K_1) (x, A) = \int K_2 (y, A) K_1 (x, dy)
\]  

(6)

Since the kernel multiplication is an associative operation (using the conditional Fubini theorem, see Theorem 2 of Chapter 22, page 431 of the book of Frisedit and Gray [12]), the composition of update mechanisms that corresponds to a set of \( n \) transition kernels \( K_1, \ldots, K_n \) is defined as the product kernel \( K_n \circ K_{n-1} \circ \ldots \circ K_1 \).

2.4.3. Transition’s Kernel Iteration

The transition probability \( t \)-th iteration (application) of a Markovian kernel \( K \), given by equation 7, describes the probability to transit to some set \( A \in \Sigma \) within \( t \) steps when starting at state \( x \in \Omega \).

\[
K^{(t)} (x, A) = \begin{cases} 
K (x, A) & , \ t = 1 \\
\int_{\Omega} K^{(t-1)} (y, A) K (x, dy) & , \ t > 1 
\end{cases}
\]  

(7)

If \( p : \Sigma \rightarrow [0, 1] \) is the initial distribution of subsets, then the probability that the Markov process is in set \( A \in \Sigma \) at step \( t \geq 0 \) is given by equation 8.

\[
Pr \{ X_t \in A \} = \begin{cases} 
p (A) & , \ t = 0 \\
\int_{\Omega} K^{(t)} (x, A) p (dx) & , \ t > 0 
\end{cases}
\]  

(8)

2.5. Markov Chains

**Definition 25. (Markov chain)** A discrete-time stochastic process \( X_0, X_1, X_2, \ldots \), taking values in an arbitrary state space \( \Omega \) is a Markov chain if it satisfies:
1. **(Markov property)** The conditional distribution of $X_t$ given $X_0, X_1, \ldots, X_{t-1}$ is the same as the conditional distribution of $X_t$ given only $X_{t-1}$.

2. **(stationarity property)** The conditional distribution of $X_t$ given $X_{t-1}$ does not depend on $t$.

Clearly, transition probabilities of the chain are specified by the conditional distribution of $X_t$ given $X_{t-1}$ (kernel), while the probability law of the chain is completely specified by the initial distribution $X_0$. Moreover, many SGOALS may be characterized by Markov chains.

### 2.6 Convergence

**Definition 26.** Let $(D_t)$ be a random sequence, i.e., a sequence of random variables defined on a probability space $(\Omega, \Sigma, P)$. Then $(D_t)$ is said to

1. **Converge completely to zero**, denoted as $D_t \xrightarrow{c} 0$, if equation 9 holds for every $\epsilon > 0$

$$\lim_{t \to \infty} \sum_{i=1}^{t} Pr \{|D_i| > \epsilon\} < \infty \quad (9)$$

2. **Converge in probability to zero**, denoted as $D_t \xrightarrow{p} 0$, if equation 10 holds for every $\epsilon > 0$.

$$\lim_{t \to \infty} Pr \{|D_t| > \epsilon\} = 0 \quad (10)$$

Notice that convergence in probability to zero (equation 10) is a necessary condition for convergence completely to zero (equation 9).

### 3. Probability Theory on Cartesian Products

Since we are working on populations (finite tuples of individuals in the search space), we need to consider probability theory on generalized cartesian products of the search space (subsection 3.1). By considering some mathematical properties of the generalized cartesian product (we will move from tuples of tuples to just a single tuple), some mathematical proofs can be simplified and an appropriated $\sigma$-algebra for populations can be defined (subsection 3.2). These accessory definitions, propositions and theorems will allow us to define a kernel by joining some simple kernels (section 3.3). Therefore, we will be able to work with stochastical methods in SGOALS that are defined as joins of stochastical methods that produce subpopulations (sub-tuples) of the newly generated complete population (single tuple).
3.1. Generalized Cartesian Product

**Definition 27.** (cartesian product) Let \( \mathcal{L} = \{\Omega_1, \Omega_2, \ldots, \Omega_n\} \) be an ordered list of \( n \in \mathbb{N} \) sets. The Cartesian product of \( \mathcal{L} \) is the set of ordered \( n \)-tuples: 
\[
\prod_{i=1}^{n} \Omega_i = \{(a_1, a_2, \ldots, a_n) | a_i \in \Omega_i \text{ for all } i = 1, 2, \ldots, n\}.
\]
If \( \Omega = \Omega_i \) for all \( i = 1, 2, \ldots, n \) then \( \prod_{i=1}^{n} \Omega_i \) is noted \( \Omega^n \) and it is called the \( n \)-fold cartesian product of set \( \Omega \).

**Lemma 28.** Let \( \Omega_1, \Omega_2 \) and \( \Omega_3 \) be sets, then

1. (associativity) \((\Omega_1 \times \Omega_2) \times \Omega_3 \equiv \Omega_1 \times (\Omega_2 \times \Omega_3) \equiv \Omega_1 \times \Omega_2 \times \Omega_3\).
2. (commutativity) \(\Omega_1 \times \Omega_2 \equiv \Omega_2 \times \Omega_1\).

**Proof.** [1] Functions \( h_L : (\Omega_1 \times \Omega_2) \times \Omega_3 \rightarrow \Omega_1 \times \Omega_2 \times \Omega_3 \) and \( h_R : \Omega_1 \times (\Omega_2 \times \Omega_3) \rightarrow \Omega_1 \times \Omega_2 \times \Omega_3 \) such that \( h_L ((a, b), c) = (a, b, c) \) and \( h_R (a, (b, c)) = (a, b, c) \) are equivalence functions. [2] Function \( r : A \times B \rightarrow B \times A \) for any \( A, B \) such that \( r(a, b) = (b, a) \) is a bijective function. \( \square \)

**Corollary 29.** Let \( \{n_i \in \mathbb{N}^+\}_{i=1,2,\ldots,m} \) be an ordered list of \( m \in \mathbb{N} \) positive natural numbers.

1. \( \prod_{i=1}^{m} \prod_{j=1}^{n_i} \Omega_{i,j} \equiv \Omega_{1,1} \times \ldots \times \Omega_{1,n_1} \times \ldots \times \Omega_{m,1} \times \ldots \times \Omega_{m,n_m} \) with \( \Omega_{i,j} \) a set for all \( i = 1, 2, \ldots, m \) and \( j = 1, 2, \ldots, n_i \).
2. \( \prod_{i=1}^{m} \Omega_{n_i} \equiv \Omega^n \) with \( n = \sum_{i=1}^{m} n_i \).

3.2. Product \( \sigma \)-algebra

Products \( \sigma \)-algebra allow us to define appropriated \( \sigma \)-algebra for generalized cartesian products. If we are provided with a \( \sigma \)-algebra associated to the feasible region of a SGOAL, then we can define a \( \sigma \)-algebra for populations of it.

**Definition 30.** Let \( \mathcal{L} = \{\Sigma_1, \Sigma_2, \ldots, \Sigma_n\} \) be a \( n \in \mathbb{N} \) ordered list of \( \Sigma_i \subseteq 2^{\Omega_i} \) family of sets.

1. (generalized family product) The generalized product of \( \mathcal{L} \) is \( \prod_{i=1}^{n} \Sigma_i = \{\prod_{i=1}^{n} a_i | \forall_{i=1}^{n} a_i \in \Sigma_i\} \).
2. (product \( \sigma \)-algebra) If \( \Sigma_i \) is a \( \sigma \)-algebra for all \( i = 1, 2, \ldots, n \), then the product \( \sigma \)-algebra of \( \mathcal{L} \) is the \( \sigma \)-algebra \( \bigotimes_{i=1}^{n} \Sigma_i = \sigma(\prod_{i=1}^{n} \Sigma_i) \) defined over the set \( \prod_{i=1}^{n} \Omega_i \).

**Lemma 31.** If \( \mathcal{L} = \{(\Sigma_1, \Omega_1), (\Sigma_2, \Omega_2), \ldots, (\Sigma_n, \Omega_n)\} \) is a finite \( (n \in \mathbb{N}) \) ordered list of measurable spaces then \( \prod_{i=1}^{n} \Sigma_i \) is a \( \pi \)-system.

**Proof.** If \( U, V \in \prod_{i=1}^{n} \Sigma_i \) then for all \( i = 1, 2, \ldots, n \) exist \( U_i, V_i \in \Sigma_i \) such that \( U = \prod_{i=1}^{n} U_i \) and \( V = \prod_{i=1}^{n} V_i \) (def. \( \prod_{i=1}^{n} \Sigma_i \)). Clearly, \( U_i \cap V_i \in \Sigma_i \) (lemma[4]). Therefore, \( \prod_{i=1}^{n} (U_i \cap V_i) \in \prod_{i=1}^{n} \Sigma_i \) (def. \( \prod_{i=1}^{n} \Sigma_i \)). Let \( z = (z_1, z_2, \ldots, z_n) \in \prod_{i=1}^{n} \Omega_i \). Clearly, \( z \in U \cap V \) iff \( z \in U \) and \( z \in V \) (def \( \cap \)) iff \( z_i \in U_i \) and \( z_i \in V_i \) for all \( i = 1, 2, \ldots, n \) (def \( \cap \)) iff \( z \in \prod_{i=1}^{n} (U_i \cap V_i) \) (def [\( \prod \)]). Therefore, \( U \cap V = \prod_{i=1}^{n} (U_i \cap V_i) \in \prod_{i=1}^{n} \Sigma_i \). \( \square \)
Proposition 32 will allow us to move from the product \( \sigma \)-algebra of products \( \sigma \)-algebras to a single product \( \sigma \)-algebra (as we move from tuples of tuples to just a single tuple).

**Proposition 32. (associativity of \( \sigma \)-algebra product)** Let \( \Sigma_i \) be a \( \sigma \)-algebra defined over a set \( \Omega_i \) for all \( i = 1, 2, 3 \), then \( (\Sigma_1 \otimes \Sigma_2) \otimes \Sigma_3 \equiv \Sigma_1 \otimes (\Sigma_2 \otimes \Sigma_3) \equiv \Sigma_1 \otimes \Sigma_2 \otimes \Sigma_3 \).

**Proof.** We will use the Dynkin \( \pi-\lambda \) theorem here. So, we need to find a \( \pi \)-system that is contained by a \( \lambda \) system.

Consider \( \Sigma_3 \in \Sigma_3 \) and define \( \mathcal{A}_{\Sigma_3} = \{ X \in \Sigma_1 \otimes \Sigma_2 \mid X \times A_3 \in \sigma (\Sigma_1 \times \Sigma_2 \times \Sigma_3) \} \). [\( \pi \)-system] \( A_1 \times A_2 \times \Sigma_3 \in \Sigma_1 \times \Sigma_2 \times \Sigma_3 \) for all \( A_1 \in \Sigma_1 \) and \( A_2 \in \Sigma_2 \) (\( A_3 \in \Sigma_3 \).

\( \Sigma_1 \times \Sigma_2 \subseteq \mathcal{A}_{\Sigma_3}, \Sigma_1 \otimes \Sigma_2 \subseteq \mathcal{A}_{\Sigma_3} \). [\( \lambda \)-system] \( \Omega_i \in \Sigma_i \) for \( i = 1, 2, 3 \) (\( \Sigma_i \) \( \sigma \)-algebra) then \( \Omega_1 \times \Omega_2 \times \Sigma_3 \subseteq \Sigma_1 \times \Sigma_2 \times \Sigma_3 \). Therefore, \( \Omega_1 \times \Omega_2 \in \mathcal{A}_{\Sigma_3} \) (\( \Omega_1 \times \Omega_2 \) \( \sigma \)-algebra). [\( \lambda \)-system] \( \Omega_i \in \Sigma_i \) for \( i = 1, 2, 3 \) (\( \mathcal{A}_{\Sigma_3} \)).

Let \( X \in \mathcal{A}_{\Sigma_3} \), then \( X^c \times A_3 = ((\Omega_1 \times \Omega_2) \setminus X) \times A_3 \) (def. complement). Clearly, \( X^c \times A_3 = ((\Omega_1 \times \Omega_2) \times A_3) \cap (X \times A_3)^c \) (Distribution and Morgan’s law). Now, \( (X \times A_3) \in \sigma (\Sigma_1 \times \Sigma_2 \times \Sigma_3) \) (def. \( \mathcal{A}_{\Sigma_3} \), then \( (X \times A_3)^c \in \sigma (\Sigma_1 \times \Sigma_2 \times \Sigma_3) \) (\( \sigma \)-algebra). Moreover, \( (\Omega_1 \times \Omega_2) \times A_3 \in \sigma (\Sigma_1 \times \Sigma_2 \times \Sigma_3) \) (part 1 and def. \( \mathcal{A}_{\Sigma_3} \)) therefore, \( X^c \times A_3 \in \sigma (\Sigma_1 \times \Sigma_2 \times \Sigma_3) \) (lemma 3).

\( \bigcup_{i \in I} (X_i \times A_3) = ( \bigcup_{i \in I} X_i ) \times A_3 \) (sets algebra), and \( (\bigcup_{i \in I} X_i ) \times A_3 \in \Sigma_1 \otimes \Sigma_2 \times \Sigma_3 \). Clearly, \( \bigcup_{i \in I} (X_i \times A_3) \subseteq \mathcal{A}_{\Sigma_3} \) (Dynkin \( \pi-\lambda \) theorem).

And \( \sigma (\Sigma_1 \times \Sigma_2) \times A_3 \subseteq \sigma (\Sigma_1 \otimes \Sigma_2 \times \Sigma_3) \) (def. \( \mathcal{A}_{\Sigma_3} \)), i.e., \( (\Sigma_1 \otimes \Sigma_2) \times A_3 \subseteq \Sigma_1 \otimes \Sigma_2 \otimes \Sigma_3 \). Because, \( (\Sigma_1 \otimes \Sigma_2) \times A_3 \subseteq \Sigma_1 \otimes \Sigma_2 \otimes \Sigma_3 \) for all \( A_3 \in \Sigma_3 \) then \( (\Sigma_1 \otimes \Sigma_2) \times \Sigma_3 \subseteq \Sigma_1 \otimes \Sigma_2 \otimes \Sigma_3 \) and \( \sigma (\Sigma_1 \otimes \Sigma_2) \otimes \Sigma_3 \) \( \sigma (\Sigma_1 \otimes \Sigma_2) \otimes \Sigma_3 \subseteq \Sigma_1 \otimes \Sigma_2 \otimes \Sigma_3 \) (def \( \sigma (\cdot) \)). \( \sqsubseteq \)

It is clear that \( \Sigma_1 \otimes \Sigma_2 \subseteq \Sigma_1 \otimes \Sigma_2 \) (def \( \otimes \)), so \( \Sigma_1 \otimes \Sigma_2 \otimes \Sigma_3 \subseteq \sigma (\Sigma_1 \otimes \Sigma_2) \otimes \Sigma_3 \) (def \( \sigma (\cdot) \)), i.e., \( (\Sigma_1 \otimes \Sigma_2) \otimes \Sigma_3 \subseteq \Sigma_1 \otimes \Sigma_2 \otimes \Sigma_3 \). \( \sqsubseteq \)

A similar proof to the \( (\Sigma_1 \otimes \Sigma_2) \otimes \Sigma_3 \subseteq \Sigma_1 \otimes \Sigma_2 \otimes \Sigma_3 \) is carried on. \( \square \)

**Corollary 33.** If \( \Sigma \) is a \( \sigma \)-algebra defined over set \( \Omega \) and \( \{ \bigotimes_{k=1}^{m} \Sigma \}_{i=1,2,...,m} \) is an ordered lists of the \( m \in \mathbb{N} \) given product \( \sigma \)-algebras \( (n_i \in \mathbb{N}^+ \text{ for all } i = 1, 2, \ldots, m) \) then \( \bigotimes_{i=1}^{m} \Sigma \equiv \bigotimes_{k=1}^{m} (\bigotimes_{k=1}^{n} \Sigma) \) with \( n = \sum_{i=1}^{m} n_i \).

In the rest of this paper, we will denote \( \prod_{i=1}^{m} A \equiv A^n \) for any \( A \subseteq 2^{\Omega} \), and \( \Sigma \otimes n \equiv \bigotimes_{i=1}^{n} \Sigma \) for any \( \sigma \)-algebra \( \Sigma \) on \( \Omega \).

### 3.3. Kernels on product \( \sigma \)-algebras

Now, we are in the position of defining a kernel that characterizes a deterministic method that is commonly used by SGOALs (as part of individual’s selection methods based on fitness): the SWAP method.

**Definition 34. (SWAP)** Let \( \Omega_1 \) and \( \Omega_2 \) be two sets. The swap function \( \sw_{\Omega} \) is defined as followed.

\(^4\)We will use the notation \( \sw_{\Omega} \equiv \sw_{\Omega} (z) \) for any \( z \in \Omega_1 \times \Omega_2 \) and \( \sw_{\Omega} \equiv \sw_{\Omega} (A) \) for any \( A \subseteq \Omega_1 \times \Omega_2 \).
Lemma 35. Let $\Omega_1$ and $\Omega_2$ be two sets.

1. $\emptyset = \emptyset$ and $\emptyset \times \Omega_2 = \Omega_2 \times \emptyset$
2. $z \in A$ iff $z \not\in A$
3. $A = A$ for all $A \subseteq \Omega_1 \times \Omega_2$
4. $B \setminus A = B \setminus A$ for all $A, B \subseteq \Omega_1 \times \Omega_2$
5. $\bigcup_{i \in I} A_i = \bigcup_{i \in I} A_i$ for any family $\{A_i \subseteq \Omega_1 \times \Omega_2\}_{i \in I}$.

Proof. [1, 2, 3] Are obvious (just applying def swap). [4] Let $A, B \subseteq \Omega_1 \times \Omega_2$. Now, $B \setminus A = \{ z \mid z \in B \setminus A \}$ (def swap), so $B \setminus A = \{ z \mid z \in B \land z \not\in A \}$ (def proper diff). Clearly, $B \setminus A = \{ z \mid z \in B \land z \not\in A \}$ (def proper diff). [5] Let $\{A_i \subseteq \Omega_1 \times \Omega_2\}_{i \in I}$ a family of sets, $z \in \bigcup_{i \in I} A_i$ iff $\exists i \in I$ such that $z \in A_i$, if and only if $z \in \bigcup_{i \in I} A_i$.

Proposition 36. Let $(\Omega_1, \Sigma_1)$ and $(\Omega_2, \Sigma_2)$ be measurable spaces then $A \in \Sigma_1 \otimes \Sigma_2$ iff $A \in \Sigma_2 \otimes \Sigma_1$.

Proof. $[\rightarrow]$ We will apply the Dynkin $\pi$-$\lambda$ theorem (theorem 7). Let $\mathcal{A} = \{ A \in \Sigma_1 \otimes \Sigma_2 \mid A \in \Sigma_2 \otimes \Sigma_1 \}$. [\lambda.1] Obvious, $\emptyset = \emptyset \in \Sigma_2 \otimes \Sigma_1$ (lemma 35.1 and lemma 35.4). [\lambda.2] Let $A, B \in \mathcal{A}$ such that $A \subseteq B$. Since $B \setminus A = B \setminus A$ (lemma 35.4), and $A, B \in \Sigma_2 \otimes \Sigma_1$ (def $\mathcal{A}$), then $B \setminus A = B \setminus A$ (def $\Sigma_2 \otimes \Sigma_1$) (lemma 35.2), i.e., $A$ is $\mathcal{B}$. [\lambda.3] Let $\{A_i \subseteq \Omega\}_{i \in I}$ a $\pi$-system, $\bigcup_{i \in I} A_i = \bigcup_{i \in I} A_i$ (lemma 35.5) and $A_i \in \Sigma_2 \otimes \Sigma_1$ (def $\mathcal{A}$) then $\bigcup_{i \in I} A_i \in \Sigma_2 \otimes \Sigma_1 \sigma$ (\sigma.3). Therefore, $\mathcal{A}$ is $\lambda$-system. Now, let $A \in \Sigma_1 \otimes \Sigma_2$, then there are $A_1 \in \Sigma_1$ and $A_2 \in \Sigma_2$ such that $A = A_1 \times A_2$ (def $\Sigma_1 \times \Sigma_2$). Clearly, $A = A_1 \times A_2 \in \Sigma_2 \otimes \Sigma_1$ (def swap and $\Sigma_2 \times \Sigma_1$), i.e., $\Sigma_1 \otimes \Sigma_2 \subseteq \mathcal{A}$ (def $\mathcal{A}$). Because $\Sigma_1 \otimes \Sigma_2 = \sigma(\Sigma_1 \times \Sigma_2)$ and $\Sigma_1 \times \Sigma_2 \subseteq \mathcal{A}$ then $A \in \Sigma_1 \otimes \Sigma_2$. $[\leftarrow]$ If $A \in \Sigma_2 \otimes \Sigma_1$, we have that $A \in \Sigma_1 \otimes \Sigma_2$ (\rightarrow), therefore, $A \in \Sigma_1 \otimes \Sigma_2$ ($A = A$).

Corollary 37. Let $(\Omega_1, \Sigma_1)$ and $(\Omega_2, \Sigma_2)$ be measurable spaces.

1. (commutativity of $\sigma$-algebra product) $\Sigma_1 \otimes \Sigma_2 \equiv \Sigma_2 \otimes \Sigma_1$.
2. (measurability of swap) The swap function $\leftarrow$ is measurable.
3. (swap kernel) The function $1_{\leftarrow}$ is a kernel$^5$

$^5$We will use the ambiguous notation $1_{\leftarrow} \equiv \leftarrow$ in the rest of this paper.
Proof. [1] The swap function is a bijective function. [2] Follows from (1) and proposition 36. [3] Follows from (2) and theorem 21.

Moreover, we can define kernels for deterministic methods that select a group of individuals from the population (projections).

Lemma 38. (projection) Let \( L = \{ (\Sigma_1, \Omega_1), (\Sigma_2, \Omega_2), \ldots, (\Sigma_n, \Omega_n) \} \) be a finite \( (n \in \mathbb{N}) \) ordered list of measurable spaces and \( I = \{ k_1, k_2, \ldots, k_m \} \subseteq \{ 1, \ldots, n \} \) be a set of indices, i.e., \( k_i < k_{i+1} \) and \( m \leq n \). The function \( \pi_I \) defined as follows is \( \bigotimes_{i=1}^n \Sigma_i - \bigotimes_{i=1}^m \Sigma_{k_i} \) measurable.

\[
\pi_I : \prod_{i=1}^n \Omega_i \rightarrow \prod_{i=1}^m \Omega_{k_i}
\]

Proof. Because \( \prod_{i=1}^1 \Omega_i = \prod_{i=1}^m \Omega_{k_i} \times \prod_{i=1}^{n-m} \Omega_i \) with \( I^c = \{ l_1, l_2, \ldots, l_{n-m} \} \) the complement set of indices of \( I \) (by applying many times lemma 23), we can "rewrite" (under equivalences) \( \pi_I \) as \( \pi_I(x, y) = x \) with \( y \in \prod_{i=1}^{n-m} \Omega_i \) and \( x \in \prod_{i=1}^m \Omega_{k_i} \). Now, \( \bigotimes_{i=1}^n \Sigma_i \equiv \bigotimes_{i=1}^m \Sigma_{k_i} \times \bigotimes_{i=1}^{n-m} \Sigma_i \) (by applying many times proposition 32 and corollary 37). Thus, for any \( A \in \bigotimes_{i=1}^m \Sigma_{k_i} \), we have that \( \pi_I^{-1} (A) = A \times \prod_{i=1}^{n-m} \Omega_i \). Clearly, \( \pi_I^{-1} (A) \in \bigotimes_{i=1}^m \Sigma_{k_i} \times \bigotimes_{i=1}^{n-m} \Sigma_i \) (def. product \( \sigma \)-algebra), therefore, \( \pi_I \) is measurable.

Corollary 39. The function \( 1_{\pi_I} \) as defined in theorem 27 is a kernel.\( ^{6} \)

Proof. Follows from lemma 38 and theorem 21.

Finally, we are able to define a kernel for a stochastic method that is the join of several stochastic methods (methods that generate a subpopulation of the next population).

Theorem 40. (product probability measure) Let \( \{ (\Omega_i, \Sigma_i, \mu_i) \mid i = 1, \ldots, n \} \) an ordered list of \( n \in \mathbb{N} \) probability spaces. There exist a unique probability measure \( \mu : \bigotimes_{i=1}^n \Sigma_i \rightarrow \mathbb{R} \) such that \( \mu \left( \prod_{i=1}^n A_i \right) = \prod_{i=1}^n \mu(A_i) \) for all \( A_i \in \Sigma_i \), \( i = 1, 2, \ldots, n \). In this case \( \mu \) is called the product probability measure of the \( \mu_i \) probability measures and is denoted \( \bigotimes_{i=1}^n \mu_i \).

Proof. This theorem is the version of theorem 14.14 in page 277 of the book of Kenkle [11], when considering \( \Sigma_i \) not just a ring but a sigma algebra. In this case, any probability measure is a finite measure and any finite measure is \( \sigma \)-finite measure \( (\Omega_i \in \Sigma_i) \).

Theorem 41. (join-kernel) Let \( (\Omega', \Sigma') \) be a measurable space and \( \{ (\Omega_i, \Sigma_i) \} \) and \( \{ K_i : \Omega' \times \Sigma_i \rightarrow [0,1] \} \) be ordered lists of \( n \in \mathbb{N} \) measurable spaces and kernels, respectively. The following function is a kernel.

\[
\otimes K : \Omega' \times \bigotimes_{i=1}^n \Sigma_i \rightarrow [0,1]
\]

\[
(x, A) \mapsto \bigotimes_{i=1}^n K_i(x, \cdot) (A)
\]

\( ^{6} \)We will use the ambiguous notation \( 1_{\pi_I} \equiv \pi_I \) in the rest of this paper.
Proof. [well-defined and K.1] Let \( x \in \Omega' \), since \( K_i(x, \cdot) \) is a probability measure for all \( i = 1, 2, \ldots, n \) (K.1 for \( K_i \) kernel) then \( \otimes K_{x, \cdot} = \bigotimes_{i=1}^{n} K_i(x, \cdot) \) is a probability measure \( \otimes K_{x, \cdot}: \bigotimes_{i=1}^{n} \Sigma_i \rightarrow [0, 1] \) (theorem 40), thus its is well defined for any \( A \in \bigotimes_{i=1}^{n} \Sigma_i \) [K.2] Using remark 20 we just need to prove that \( \otimes K_{x, \cdot} A \) is measurable for any \( A \in \mathcal{E} \) with \( \mathcal{E} \subseteq 2^{\bigotimes_{i=1}^{n}} \) a \( \pi \)-system that generates \( \bigotimes_{i=1}^{n} \Sigma_i \). Therefore, \( \otimes K_{x, \cdot} A \) is a measurable function. Because \( \prod_{i=1}^{n} \Sigma_i \) is a \( \pi \)-system (lemma 31) and \( \bigotimes_{i=1}^{n} \Sigma_i \) is the \( \sigma \)-algebra generated by \( \prod_{i=1}^{n} \Sigma_i \) (def. 30), then we just need to prove that \( \otimes K_{x, \cdot} A \) is measurable for any \( A = \prod_{i=1}^{n} A_i \) with \( A_i \in \Sigma_i \) for all \( i = 1, 2, \ldots, n \). By definition, \( \otimes K(x, \prod_{i=1}^{n} A_i) = \bigotimes_{i=1}^{n} K_i(x, \cdot) (\prod_{i=1}^{n} A_i) \) and according to theorem 40 \( K(x, \prod_{i=1}^{n} A_i) = \prod_{i=1}^{n} K_i(x, \cdot) (A_i) (A_i \in \Sigma_i) \). Clearly, \( \otimes K_{x, \cdot} A \) is a measurable function for all \( i = 1, 2, \ldots, n \) (K.2 for \( K_i \) kernel) then their product is a measurable function (see Theorem 1.91, page 37 in Kenkle’s book [11]). Therefore, \( \otimes K_{x, \cdot} A \) is a measurable function.

Corollary 42. If \( (\Omega, \Sigma) \) is a measurable space and \( \{n_i \in \mathbb{N}^+\}_{i=1,2,\ldots,m} \) such that \( \Omega_i = \Omega^{n_i} \) and \( \Sigma_i = \Sigma^{\otimes n_i} \) for all \( i = 1, 2, \ldots, m \) in theorem 42 then \( \otimes K: \Omega' \times \Sigma^{\otimes n} \rightarrow [0, 1] \), with \( n = \sum_{i=1}^{m} n_i \), is a kernel.

Proposition 43. (permutation) Let \( (\Sigma, \Omega) \) be a measurable space and \( I = [i_1, i_2, \ldots, i_n] \) be a fixed permutation of the set \( \{1, 2, \ldots, n\} \) then the function \( K_I: \Omega^n \times \Sigma^{\otimes n} \rightarrow [0, 1] \) defined as \( K_I = \otimes_{k=1}^{n} \pi_{i_k} \) is a kernel.

Proof. Follows from corollaries 42 and 39.

Corollary 44. Let \( (\Sigma, \Omega) \) be a measurable space and \( \mathcal{P} \) be the set of permutations of set \( \{1, 2, \ldots, n\} \). Function \( K_{\mathcal{P}}: \Omega^n \times \Sigma^{\otimes n} \rightarrow [0, 1] \) defined as \( K_{\mathcal{P}} = \frac{1}{|\mathcal{P}|} \sum_{I \in \mathcal{P}} \pi_I \) is a kernel.

Proof. \( K_{\mathcal{P}} \) is a mixing update mechanisms of \( |\mathcal{P}| \) kernels (subsection 2.4.1) and proposition 44.

4. Characterization of a SGaOL using Probability Theory

Following the description of a SGaOL (see Algorithm 1), the initial population \( P_0 \) is chosen according to some initial distribution \( p(\cdot) \) and the population \( P_t \) at step \( t > 0 \) is generated using a stochastic method (NextPop) on the previous population \( P_{t - 1} \). If such NextPop method can be characterized by a Markov kernel, the stochastic sequence \( (P_t: t \geq 0) \) becomes a Markov chain. In order to develop this characterization, first we define appropriated measurable spaces and Markov kernels of stochastic methods, and then we define some properties of stochastic methods that cover many popular SGaOLs reported in the literature.

Since a SGaOL consists of a population of \( n \) individuals on the feasible region \( \Omega \), it is clear that the state space is defined on \( \Omega^n \). Moreover, the initial population \( P_0 \in \Omega^n \) is chosen according to some initial distribution \( p(\cdot) \). Now,
the σ-algebra must allow us to determine convergence properties on the kernel. In this paper, we will extend the convergence approach proposed by Günter Rudolph in \cite{16} to SGOALS. In the following, we call \textit{objective function} to a function \( f : \Phi \rightarrow \mathbb{R} \) if its has an optimal value (denoted as \( f^* \in \mathbb{R} \)) in the feasible region.

4.1. \( \epsilon \)-optimal states

We define and study the set of strict \( \epsilon \)-optimal states (the optimal elements according to Rudolph’s notation), i.e., a set that includes any candidate population which best individual has a value of the objective function close (less than \( \epsilon \in \mathbb{R}^+ \)) to the optimum objective function value. We also introduce two new natural definitions that we will use in some proofs (the \( \epsilon \)-optimal states and \( \epsilon \)-states) and study some properties of sets defined upon these concepts.

\textbf{Definition 45.} Let \( \Omega \subseteq \Phi \) be a set, \( f : \Phi \rightarrow \mathbb{R} \) be an objective function, \( \epsilon > 0 \) be a real number and \( x \in \Omega^m \).

1. (\textbf{optimality}) \( d(x) = f(\text{Best}(x)) - f^* \) (here \( f^* \) is the optimal value of \( f \) in \( \Omega \)).
2. (\textbf{strict} \( \epsilon \)-\textbf{optimum state}) \( x \) is a strict \( \epsilon \)-optimum element if \( d(x) < \epsilon \),
3. (\textbf{\( \epsilon \)-optimum state}) \( x \) is an \( \epsilon \)-optimum element if \( d(x) \leq \epsilon \), and
4. (\textbf{\( \epsilon \)-state}) \( x \) is an \( \epsilon \)-element if \( d(x) = \epsilon \).

Sets \( \Omega^m_\epsilon = \{ x \in \Omega^m : d(x) < \epsilon \} \), \( \Omega^m_m = \{ x \in \Omega^m : d(x) \leq \epsilon \} \), and \( \Omega^m_m = \{ x \in \Omega^m : d(x) = \epsilon \} \) are called set of strict \( \epsilon \)-optimal states, \( \epsilon \)-optimal states, and \( \epsilon \)-states, respectively. We will denote \( \Omega_\epsilon = \Omega^m_\epsilon \), \( \Omega^m = \bigcap_{\epsilon > 0} \Omega^m_\epsilon \) and \( \Omega^m = \bigcap_{\epsilon > 0} \Omega^m_\epsilon \).

\textbf{Remark 46.} Notice that

1. \( (\Omega^m_\epsilon)^c = \{ x \in \Omega^m : \epsilon \leq d(x) \} \) and
2. \( (\Omega^m_m)^c = \{ x \in \Omega^m : \epsilon < d(x) \} \).

\textbf{Lemma 47.} \( \Omega^m_m = \bigcap_{n=1}^{\infty} \Omega^m_{\epsilon + \frac{1}{n}} \) for all \( \epsilon > 0 \) and \( m \in \mathbb{N}^+ \).

\textit{Proof.} [\( \subseteq \)] If \( x \in \Omega^m_\epsilon \) then \( d(x) \leq \epsilon \) (def \( \Omega^m_\epsilon \)). Now, \( \epsilon < \epsilon + \frac{1}{n} \) for all \( n > 0 \), clearly \( d(x) < \epsilon + \frac{1}{n} \) for all \( n > 0 \). Therefore, \( x \in \Omega^m_{\epsilon + \frac{1}{n}} \) for all \( n > 0 \) so \( x \in \bigcap_{n=1}^{\infty} \Omega^m_{\epsilon + \frac{1}{n}} \). [\( \supseteq \)] Let \( x \in \bigcap_{n=1}^{\infty} \Omega^m_{\epsilon + \frac{1}{n}} \), if \( x \notin \Omega^m_\epsilon \) then \( d(x) > \epsilon \) (def \( \Omega^m_\epsilon \)), therefore, \( \exists \delta > 0 \) such that \( d(x) = \epsilon + \delta \). We have that \( \exists n \in \mathbb{N} \) such that \( 0 < \frac{1}{n} < \delta \) (Archimedes theorem and real numbers dense theorem). Clearly, \( \epsilon + \frac{1}{n} < \epsilon + \delta = d(x) \), so \( x \notin \Omega^m_{\epsilon + \frac{1}{n}} \), then \( x \notin \bigcap_{n=1}^{\infty} \Omega^m_{\epsilon + \frac{1}{n}} \) (contradiction). Therefore, \( x \in \Omega^m_\epsilon \). \( \square \)
4.2. Optimization space

We define the optimization $\sigma$-algebra property (a $\sigma$-algebra containing the family of sets of strict $\epsilon$-optimal states) and show that such property is preserved by the product $\sigma$-algebra.

**Definition 48. ($f$-optimization $\sigma$-algebra)** Let $f: \Phi \to \mathbb{R}$ be an objective function, and $\Omega \subseteq \Phi$. A $\sigma$-algebra $\Sigma$ on $\Omega$ is called $f$-optimization $\sigma$-algebra iff $\{\Omega_\epsilon\}_{\epsilon > 0} \subseteq \Sigma$.

**Lemma 49.** Let $\Sigma$ be an $f$-optimization $\sigma$-algebra on $\Omega$ then $\{\Omega_\epsilon\}_{\epsilon > 0} \subseteq \Sigma$ and $\{\Omega_\epsilon\}_{\epsilon > 0} \subseteq \Sigma$.

**Proof.** $\{\Omega_\epsilon\}_{\epsilon > 0} \subseteq \Sigma$ It follows from the facts that $\{\Omega_{\epsilon+\frac{1}{n}}\}_{n \in \mathbb{N}^+} \subseteq \{\Omega_\epsilon\} \subseteq \Sigma$ (Sigma optimization $\sigma$-algebra), $\Omega_\epsilon = \bigcap_{n=1}^{\infty} \Omega_{\epsilon+\frac{1}{n}}$ (lemma 47) and $\Sigma$ close under countable intersections (part 2, lemma 5). $\{\Omega_\epsilon\}_{\epsilon > 0} \subseteq \Sigma$ It follows from the fact that $\Omega_\epsilon = \Omega - \Omega_\epsilon$ for all $\epsilon > 0$ and $\Sigma$ is close under proper differences (part 4, lemma 5).

**Proposition 50.** $\Omega^m_\epsilon = \bigcup_{i=1}^{m} \left[\Omega^{i-1}_\epsilon \times \Omega_\epsilon \times \Omega^{m-i}_\epsilon\right]$ for all $\epsilon > 0$ and $m \in \mathbb{N}^+.$

**Proof.** $\subseteq$ Let $x \in \Omega^m_\epsilon$, then $d(x) < \epsilon$ (def $\Omega^m_\epsilon$) and $d(Best(x)) - f^* < \epsilon$ (def $d(x)$). It is clear that $x = \prod_{k=1}^{i-1} x_k \times x_i \times \prod_{k=i+1}^{m} x_k f(x_i) - f^* < \epsilon$ for some $i = 1, 2, \ldots, m$ (def Best), so $d(x_i) < \epsilon$ (def $d(x)$). Therefore, $x_i \in \Omega_\epsilon$ (def $\Omega_\epsilon$) so $x \in \Omega^{i-1}_\epsilon \times \Omega_\epsilon \times \Omega^{m-i}_\epsilon$ and $x \in \bigcup_{i=1}^{m} \left[\Omega^{i-1}_\epsilon \times \Omega_\epsilon \times \Omega^{m-i}_\epsilon\right]$.

$\supseteq$ if $x \in \bigcup_{i=1}^{m} \left[\Omega^{i-1}_\epsilon \times \Omega_\epsilon \times \Omega^{m-i}_\epsilon\right] \text{ then } x \in \left[\Omega^{i-1}_\epsilon \times \Omega_\epsilon \times \Omega^{m-i}_\epsilon\right]$ for some $i = 1, 2, \ldots, m$. Clearly, $f(Best(x)) < f(x_i)$ (def Best) so $d(x) \leq d(x_i)$. Now, $d(x) \leq d(x_i) < \epsilon (x_i \in \Omega_\epsilon$) therefore $x \in \Omega^m_\epsilon$ (def $\Omega^m_\epsilon$).

**Corollary 51.** Let $\Sigma$ be an $f$-optimization $\sigma$-algebra on $\Omega$ then $\Sigma^\otimes m$ is an $f$-optimization $\sigma$-algebra on $\Omega^m$ for all $m \in \mathbb{N}^+.$

**Proof.** $\Omega_\epsilon \in \Sigma$ (optimization $\sigma$-algebra) and $\Omega^i \in \Sigma^\otimes i$ for all $i = 1, 2, \ldots, m$ (universality of $\sigma$-algebra) then $[\Omega^{i-1}_\epsilon \times \Omega_\epsilon \times \Omega^{m-i}_\epsilon] \in \Sigma^\otimes m$ (def product $\sigma$-algebra), so $\bigcup_{i=1}^{m} \left[\Omega^{i-1}_\epsilon \times \Omega_\epsilon \times \Omega^{m-i}_\epsilon\right] \in \Sigma^\otimes m$ (Sigma $\otimes m$ is cdm). Therefore, $\Omega^m_\epsilon \in \Sigma^\otimes m$ for all $\epsilon > 0$, i.e., $\Sigma^\otimes m$ is an optimization $\sigma$-algebra.

Now, we are ready to define the mathematical structure that we use for characterizing a SGOAL.
Definition 52. **(optimization space)** If \( f : \Phi \rightarrow \mathbb{R} \) is an objective function, \( n \in \mathbb{N} \) and \( \Sigma \) is an \( f \)-optimization \( \sigma \)-algebra over a set \( \Omega \) then the triple \((\Omega^n, \Sigma^{\otimes n}, f)\) is called optimization space.

4.3. **Kernels on optimization spaces**

If we are provided with an optimization space \((\Omega^n, \Sigma^{\otimes n}, f)\), we can represent the population used by the SGOAL, as an individual \( x \in \Omega^n \), while we can characterize the NextPop (a \( f : \Omega^n \rightarrow \Omega^n \) stochastic method), as a Markov kernel \( K : \Omega^n \times \Sigma^{\otimes n} \rightarrow [0, 1] \). Because such NextPop can be defined in terms of more general \( f : \Omega^\eta \rightarrow \Omega^\upsilon \) stochastic methods, we study such general kernels.

4.3.1. **Join Stochastic Methods**

**Definition 53. (Join)** A stochastic method \( f : \Omega^\eta \rightarrow \Omega^\upsilon \) is called join if it is defined as the join of \( m \in \mathbb{N} \) stochastic methods \((f_i : \Omega^n \rightarrow \Omega^\upsilon_i)\), each method generating a subpopulation of the population, i.e. \( f = \prod_{i=1}^{m} f_i \). Here, \( \upsilon_i \in \mathbb{N}^+ \) is the size of the \( i \)-th sub-population \( i = 1, 2, \ldots, m \), and \( s_m = v \).

**Algorithm 7** Joined Stochastic Method.

\[
\text{f}(P) =
\begin{align*}
1. & \quad s_i = \sum_{k=1}^{i-1} \upsilon_k \text{ for all } i = 1, 2, \ldots, m \\
2. & \quad Q_{1+s_i-1}, \ldots, s_i = f_i(P) \text{ for all } i = 1, 2, \ldots, m \\
3. & \quad \text{return } Q
\end{align*}
\]

**Example 54.** The NextPop method of a GGA (see Algorithm 4), is a joined stochastic method: a stochastic method \( \text{NextSubPop}_{GGA} : \Omega^n \rightarrow \Omega^2 \) that generates two new candidate solutions (by selecting two parents from the population, recombining them and mutating the offspring), is applied \( \frac{n}{2} \) times in order to generate the next population, see equation (11).

\[
\text{NextPop}_{GGA}(P) = \prod_{i=1}^{\frac{n}{2}} \text{NextSubPop}_{GGA}(P) \quad (11)
\]

**Example 55.** The NextPop method of a DE (see Algorithm 6), is a joined stochastic method: \( n \) stochastic methods \( \text{NextInd}_{DE,i} : \Omega^n \rightarrow \Omega \) each one generating the \( i \)-th individual of the new population (by selecting three extra parents from the population and recombining each dimension using differences if required), are applied, see equation (12).

\[
\text{NextPop}_{DE}(P) = \prod_{i=1}^{n} \text{NextInd}_{DE,i}(P) \quad (12)
\]
Example 56. The NextPop method of a PHC (see Algorithm 3), is a joined stochastic method: \( n \) stochastic methods \( \text{NextInd}_{\text{PHC},i} : \Omega^n \to \Omega \) each one generating the \( i \)th individual of the new population \( (\text{NextInd}_{\text{PHC},i}(P) = \text{NextPop}_{\text{PHC}}(P_i)) \), see equation 13.

\[
\text{NextPop}_{\text{DE}}(P) = \prod_{i=1}^{n} \text{NextPop}_{\text{PHC}}(P_i)
\]

We are now in the position of providing a sufficient condition for characterizing PHC, GGA, and DE algorithms.

Proposition 57. Let \( \{F_i : \Omega^n \to \Omega^n\}_{i=1}^{m} \) be a finite family of stochastic methods, each one characterized by a kernel \( K_i : \Omega^n \times \Sigma \to [0,1] \), then the join stochastic method \( F = \prod_{i=1}^{m} F_i \) is characterized by the kernel \( \mathcal{K} : \Omega^n \times \Sigma \to [0,1] \) with \( v = \sum_{k=1}^{m} v_k \).

Proof. Follows from corollary 42 of theorem 41.

Corollary 58. Each of the \( \text{NextPop}_{\text{PHC}}, \text{NextPop}_{\text{GGA}}, \) and \( \text{NextPop}_{\text{DE}} \) stochastic methods can be characterized by kernels if each of the stochastic methods \( \text{NextInd}_{\text{PHC}}, \text{NextSubPop}_{\text{GGA}}, \) and \( \text{NextInd}_{\text{DE}} \) can be characterized by a kernel.

4.3.2. Sorting Methods

Although the result of sorting a population is, in general, a non stochastic method, we can model it as a kernel. We start by modeling the sorting of two elements according to their fitness value.

Definition 59. (Sort-Two) Let \( d : \Omega \to \mathbb{R} \), the sort-two function \( s_2 : \Omega^2 \to \Omega^2 \) is defined as follows:

\[
s_2(z = (x, y)) = \begin{cases} 
  z & \text{if } d(x) < d(y) \\
  \bar{z} = (y, x) & \text{otherwise}
\end{cases}
\]

In order to model the \( S_2 \) method as a kernel, we need to define sets that capture some notions of sorted couples.

Definition 60. (sorted couples sets) Let \( x, y \in \Omega \).

- \( \text{mM} \) The set \( \text{mM} = \{(x, y) \mid d(x) < d(y)\} \) is called min-max sorted couples set.
- \( \text{MM} \) The set \( \text{MM} = \{(x, y) \mid d(y) < d(x)\} \) is called max-min sorted couples set.
- \( \text{m} \) The set \( \text{m} = \{(x, y) \mid d(y) = d(x)\} \) is called equivalent couples set.

Lemma 61. The following set of equations holds.

1. \( \text{mM} = \bigcup_{r \in Q} \Omega_r \times \Omega_r^{\complement} \)
2. \( MM = \bigcup_{r \in \mathbb{Q}} \mathcal{O}_r \) × \( \Omega_r \)

3. \( M = (MM \cup MM)^c \)

Proof. [1] \( \subseteq \) Let \( (x, y) \in MM \) then \( d(x) < d(y) \) (def. \( MM \)), therefore \( \exists r \in \mathbb{Q} \) such that \( d(x) < r < d(y) \) (Archimedes theorem and real numbers dense theorem). Clearly, \( x \in \Omega_r \) and \( y \in \mathcal{O}_r \) (def \( \Omega, \mathcal{O} \) and remark [10]), then \( (x, y) \in \Omega_r \times \mathcal{O}_r \).

Corollary 64. \( \exists MM, M \in \Sigma^{\otimes 2} \) if \( \Sigma \) is an optimization \( \sigma \)-algebra.

Proof. \( (\Omega_r \times \mathcal{O}_r) \in (\Sigma_r \times \mathcal{O}_r) = \Sigma^{\otimes 2} \) (def \( \Sigma, \mathcal{O} \), def product \( \sigma \)-algebra), and \( MM = \bigcup_{r \in \mathbb{Q}} \Omega_r \times \mathcal{O}_r \) and \( MM = \bigcup_{r \in \mathbb{Q}} (\Omega_r \times \mathcal{O}_r) \) (lemma [61]), then \( MM \), \( MM \in \Sigma^{\otimes 2} \) (\( \Sigma^{\otimes 2} \) is \( \subseteq \) \( \mathbb{Q} \)). Clearly, \( (MM \cup MM) \in \Sigma^{\otimes 2} \) (\( \Sigma^{\otimes 2} \) is \( \subseteq \) \( \mathbb{Q} \)). Finally, \( \Sigma^{\otimes 2} \), \( MM^c \), \( MM^c \vee \Sigma^{\otimes 2} \) (\( \sigma \)-2).

Proposition 63. \( s_2 : \Omega^2 \rightarrow \Omega^2 \) is measurable.

Proof. Let \( A \in \Sigma \) and \( z = (x, y) \in \Omega^2 \). \( z \in s_2^{-1} (A) \) iff \( z \in A \land d(x) < d(y) \) \( \lor \) \( z \in A \land d(y) \leq d(x) \) (def \( s_2 \)) iff \( z \in A \land z \in MM \) \( \lor \) \( z \in A \land z \in MM^c \) (def \( MM \)) iff \( z \in (A \cap MM) \cup (A \cap MM^c) \) (def \( \cup \) and \( \cap \)). Since \( A, \Omega \in \Sigma \) then \( A, \Omega, MM, MM^c \in \Sigma^{\otimes 2} \) (corollary [50] and lemma [62]). Therefore, \( s_2^{-1} (A) \in \Sigma^{\otimes 2} \) (\( \Sigma^{\otimes 2} \) is \( \subseteq \), \( \subseteq \), \( \subseteq \)) and \( s_2^c \). Clearly, \( s_2 \) is measurable.

Corollary 64. \( 1_{s_2} : \Omega^2 \times \Sigma^{\otimes 2} \rightarrow [0, 1] \) as defined in theorem [21] is a kernel.

Proof. Follows from proposition [63] and theorem [21].

Having defined the kernel for \( s_2 \), we define a kernel \( s_{n, n-1} : \Omega^n \times \Sigma^{\otimes n} \rightarrow [0, 1] \) for characterizing a n-tuple sorting method.

Proposition 65. The following functions are kernels

1. \( W_{n,k} : \Omega^n \times \Sigma^{\otimes n} \rightarrow [0, 1] \) defined as \( W_{n,k} = \pi_{\{1, \ldots, k-1\}} \circ \otimes (\pi_{\{k+2, \ldots, n\}} \circ \pi_{\{k, k+1\}}) \) for \( k = 1, \ldots, n-1 \).

2. \( T_{n,k} : \Omega^n \times \Sigma^{\otimes n} \rightarrow [0, 1] \) defined as \( T_{n,1} = W_{n,1} \), and \( T_{n,k} = W_{n,k} \circ T_{n,k-1} \) for \( k = 2, \ldots, n-1 \).

3. \( s_{n,k} : \Omega^n \times \Sigma^{\otimes n} \rightarrow [0, 1] \) defined as \( s_{n,1} = T_{n,1} \), and \( s_{n,k} = T_{n,k} \circ s_{n,k-1} \) for \( k = 2, \ldots, n-1 \).
\textbf{Proof.} Obvious, all functions are defined in terms of composition and/or join of kernels.

**Corollary 66.** The \textsc{Best2} function used by the \textsc{SSGA} (line 5, algorithm 2) can be characterized by the kernel \( B_{2,4} : \Omega^4 \times \Sigma^{\otimes 2} \rightarrow [0,1] \) defined as \( B_{2,4} = \pi_{\{1,2\}} \circ \psi_{n,2} \).

**Proposition 67.** If lines 3-4 in the \textsc{SSGA} (see algorithm 3) can be modeled by a kernel \( v : \Omega^2 \times \Sigma^{\otimes 2} \rightarrow [0,1] \), The stochastic method \textsc{NextPop\textsc{SSGA}} can be characterized by the following kernel.

\[
K_{\text{SSGA}} = [(B_{2,4} \circ \pi_{\{1,\ldots,4\}}) \odot \pi_{\{5,\ldots,n+2\}}] \circ [([v \circ \pi_{\{1,2\}}] \odot 1] \circ K \varphi
\]

4.3.3. Variation-Replacement Stochastic Methods

Many \textsc{SGOALs} are defined as two-steps stochastic processes: First by applying a stochastic method that generates \( \varpi \in \mathbb{N} \) new individuals, in order to \textit{“explore”} the search space, and then by applying a stochastic method that selects candidate solutions among the current individuals and the new individuals, in order to \textit{“improve”} the quality of candidate solutions.

**Definition 68.** (\textbf{Variation-Replacement}) A stochastic method \( f : \Omega^n \rightarrow \Omega^v \) is called Variation-Replacement (VR) if there are two stochastic methods, \( v : \Omega^n \rightarrow \Omega^\varpi \) and \( r : \Omega^{n+\varpi} \rightarrow \Omega^v \), \( (t) \) such that \( f(P) = r(P,v(P)) \) or \( f(P) = r(v(P),P) \) for all \( P \in \Omega^n \).

**Example 69.** The \textsc{NextPop} method of \textsc{HC} with neutral mutations (see Algorithm 2) is a VR stochastic method, see equations 14 and 15. The \textsc{HC} algorithm will not consider neutral mutations just by changing the order of the arguments in the replacement stochastic method \( R_{\text{HC}} \), i.e., \( R_{\text{HC}}(\text{Variate}(x),x) \).

\[
\text{NextPop}_{\text{HC}}(x) = R_{\text{HC}}(x,\text{Variate}(x)) \tag{14}
\]

\[
R_{\text{HC}}(x,y) = \begin{cases} 
  x & \text{if } f(x) < f(y) \\
  y & \text{otherwise}
\end{cases} \tag{15}
\]

**Proposition 70.** If \( v : \Omega^n \rightarrow \Omega^\varpi \) and \( r : \Omega^{n+\varpi} \rightarrow \Omega^v \) are stochastic methods characterized by kernels \( K_V : \Omega^n \times \Sigma^{\otimes \varpi} \rightarrow [0,1] \) and \( K_R : \Omega^{n+\varpi} \times \Sigma^{\otimes v} \rightarrow [0,1] \), respectively, then \( K_F = K_R \circ [1_{\Omega^n} \odot K_V] \) and \( K_F = K_R \circ [K_V \odot 1_{\Omega^n}] \) are kernels that characterize the VR stochastic method \( f(P) = r(P,v(P)) \) and \( f(P) = r(v(P),P) \) with \( P \in \Omega^n \), respectively.

\textbf{Proof.} Clearly, \([1_{\Omega^n} \odot K_V]\) and \([K_V \odot 1_{\Omega^n}]\) are kernels (theorem 11 and lemma 21). Therefore, \( K_F = K_R \circ [1_{\Omega^n} \odot K_V] \) and \( K_F = K_R \circ [K_V \odot 1_{\Omega^n}] \) are kernels by composition of kernels, see Section 2.4.2.

We are now in the position of defining a kernel that characterizes the replacement method of a HC algorithm. Before doing that, notice that \( R_{\text{HC}}(x,y) = \pi_1(S_2(x,y)) \).

\[\square\]
Lemma 71. The function \( r_{HC} = \pi_1 \circ s_2 \) is measurable and \( K_{r_{HC}} \equiv 1_{r_{HC}} \) as defined in theorem \( \text{21} \) is a kernel.

Proof. Follows from the fact \( r_{HC} = \pi_1 \circ s_2 \) is measurable (composition of measurable functions is measurable) and theorem \( \text{21} \).

Corollary 72. The Hill Climbing algorithm shown in Algorithm \( \text{2} \) can be characterized by a kernel if its \( \text{Variate}_{HC} \) stochastic method can be characterized by a kernel.

Proof. Follows from example \( \text{69} \), proposition \( \text{70} \) and lemma \( \text{71} \).

Corollary 73. The Parallel Hill Climbing algorithm shown in Algorithm \( \text{3} \) can be characterized by a kernel if the \( \text{Variate}_{HC} \) stochastic method of the parallelized HC can be characterized by a kernel.

Proof. Follows from example \( \text{56} \), corollary \( \text{72} \) and proposition \( \text{57} \).

Lemma 74. The \( \text{NextPop} \) stochastic method of the SSGA shown in Algorithm \( \text{5} \) can be characterized by the composition of two kernels \( v_{\text{SSGA}} = [\left( V \circ \pi_{\{1,2\}} \right) \otimes 1] \circ K_P \) and \( r_{\text{SSGA}} = [b_{2,4} \circ \pi_{\{1,\ldots,4\}}] \otimes \pi_{\{5,\ldots,n+2\}} \) if lines 3-4 can be characterized by a kernel \( V : \Omega^2 \times \Sigma^{\otimes 2} \to [0,1] \).

Proof. Follows from composition of kernels and proposition \( \text{67} \).

4.3.4. Elitist Stochastic Methods

Some SGOALS use elitist stochastic methods, i.e., if the best candidate solution obtained after applying the method is at least as good as the best candidate solution before applying it, in order to capture the notion of “improving” the solution.

Definition 75. (elitist method) A stochastic method \( f : \Omega^n \to \Omega^n \) is called elitist if \( f(\text{Best}(f(P))) \leq f(\text{Best}(P)) \).

Example 76. The \( \text{NextPop} \) methods of the following algorithms\( ^7 \), are elitist stochastic methods. Here, we will denote \( Q_A \equiv \text{NextPop}_A(P) \).

1. SSGA: \( \text{Best}(Q_{\text{SSGA}}(P)) = \text{Best}(c_1 \times c_2 \times P) \), (see Algorithm \( \text{5} \)). Then, \( f(\text{Best}(c_1 \times c_2 \times P)) \leq f(\text{Best}(P)) \).
2. HC: \( \text{Best}(Q_{\text{HC}}(x)) = \text{Best}(\text{Variate}_{HC}(x) \times x) \) (see Algorithm \( \text{2} \)). Then, \( f(\text{Best}(\text{Variate}_{HC}(x) \times x)) \leq f(x) = f(\text{Best}(x)) \).
3. PHC: Let \( k \in [1,n] \) the index of the best individual in population \( P \), then \( f(\text{Best}(P)) = f(P_k) \). Since \( Q_{\text{PHC}}(P)_i = Q_{\text{HC}}(P_i) \) for all \( i = 1,2,\ldots,n \) (see Algorithm \( \text{3} \)), it is clear that \( f(Q_{\text{PHC}}(P)_k) \leq f(\text{Best}(P)) \) (\( \text{HC} \) is elitist). Then, \( f(\text{Best}(Q_{\text{PHC}}(P))) \leq Q_{\text{PHC}}(P)_i = f(\text{Best}(P)) \).

\( ^7 \)Here we just present the examples when such algorithms consider neutral mutations, but it is also valid when those do not consider neutral mutations (we just need to reverse the product order).
Definition 77. (elitist kernel) A kernel $K : \Omega^n \times \Sigma^{\otimes v} \rightarrow [0, 1]$ is called elitist if $K(x, A) = 0$ for each $A \in \Sigma^{\otimes v}$ such that $d(x) < d(y)$ for all $y \in A$.

Proposition 78. Kernels $RHC$ and $RSSGA$ are elitist kernels.

Proof. Let $(x, y) \in \Sigma^{\otimes 2}$ and $A \in \Sigma$ such that $d(z) < d(x, y)$ for all $z \in A$. Now, $RHC(x, y) = \pi_1 \circ S_2(x, y)$ (def $RHC$), clearly, $d(RHC(x, y)) \leq d(x, y)$ (def $d$), therefore $d(RHC(x, y)) \notin A$ (def $A$). In this way, $RHC(x, A) = 0$ (def kernel $RHC$ and theorem 21). Therefore, $RHC$ is elitist (def elitist kernel). A similar proof is carried on for $RSSGA$.

Lemma 79. If $K : \Omega^n \times \Sigma^{\otimes v} \rightarrow [0, 1]$ is elitist then

1. $K\left(x, \left(\Omega^w_{\alpha(x)}\right)^c\right) = 0$ and $K\left(x, \Omega^w_{\alpha(x)}\right) = 1$.
2. Let $x \in \Omega^n$, if $d(x) < \alpha \in \mathbb{R}$ then $K\left(x, \left(\Omega^w_{\alpha(x)}\right)^c\right) = 0$ and $K\left(x, \Omega^w_{\alpha(x)}\right) = 1$

Proof. [1] Let $y \in \left(\Omega^w_{\alpha(x)}\right)^c$ then $\neg (d(y) \leq d(x))$ (def complement, $\Omega^w_{\alpha(x)}$), i.e., $d(x) < d(y)$. Therefore, $K\left(x, \left(\Omega^w_{\alpha(x)}\right)^c\right) = 0$ (K elitist) and $K\left(x, \Omega^w_{\alpha(x)}\right) = 1$ (K, probability measure). [2] if $d(x) < \alpha$ then $\Omega^w_{\alpha(x)} \subseteq \Omega^w_\alpha$ (def $\alpha$) and $(\Omega^w_\alpha)^c \subseteq \left(\Omega^w_{\alpha(x)}\right)^c$ (def $\neg$). Clearly, $K\left(x, \left(\Omega^w_\alpha\right)^c\right) \leq K\left(x, \left(\Omega^w_{\alpha(x)}\right)^c\right) = 0$ and $K\left(x, \Omega^w_{\alpha(x)}\right) = 1$ (K, measure).

Definition 80. (optimal strictly bounded from zero) A kernel $K : \Omega^n \times \Sigma^{\otimes v} \rightarrow [0, 1]$ is called optimal strictly bounded from zero if $K(x, \Omega_\epsilon) \geq \delta(\epsilon) > 0$ for all $\epsilon > 0$.

5. Convergence of a SGoal

We will follow the approach proposed by Günter Rudolph in [16], to determine the convergence properties of a SGoal. In the rest of this paper, $\Sigma$ is an optimization $\sigma$-algebra. First, Rudolph defines a convergence property for a SGoal in terms of the objective function.

Definition 81. (SGoal convergence). Let $P_t \in \Omega^n$ be the population maintained by a SGoal $\mathcal{A}$ at iteration $t$. Then $\mathcal{A}$ converges to the global optimum if the random sequence $(D_t = d(P_t) : t \geq 0)$ converges completely to zero.

Then, Rudolph proposes a sufficient condition on the kernel when applied to the set of strict $\epsilon$-optimal states in order to attain such convergence.

Lemma 82. (Lemma 1 in [16]) If $K(x, \Omega_\epsilon) \geq \delta > 0$ for all $x \in \Omega^c_\epsilon$ and $K(x, \Omega_\epsilon) = 1$ for all $x \in \Omega_\epsilon$ then, equation [14] holds for $t \geq 1$.

$$K^{(t)}(x, \Omega_\epsilon) \geq 1 - (1 - \delta)^t$$ (16)
Proof. In [16], Rudolph uses induction on \( t \) in order to demonstrate lemma 82. For \( t = 1 \) we have that \( K^{(t)}(x, \Omega_\varepsilon) = K(x, \Omega_\varepsilon) \) (equation 7), so \( K^{(t)}(x, \Omega_\varepsilon) \geq \delta \) (condition lemma), therefore \( K^{(t)}(x, \Omega_\varepsilon) \geq 1 - (1 - \delta)^t \) (\( t = 1 \) and numeric operations). Here, we will use the notations \( K^{(t)}(x, \Omega_\varepsilon) = K^{(t)}_x(y, \Omega_\varepsilon) \) to reduce the visual length of the equations.

\[
\begin{align*}
K^{(t+1)}_x(\Omega_\varepsilon) &= \int \limits_{\Omega} K^{(t)}_y(\Omega_\varepsilon) K(x, dy) \\
&= \int \limits_{\Omega} K^{(t)}_y(\Omega_\varepsilon) K(x, dy) + \int \limits_{\Omega \setminus \Omega_\varepsilon} K^{(t)}_y(\Omega_\varepsilon) K(x, dy) \quad (\Omega = \Omega_\varepsilon \cup \Omega_\varepsilon^c) \\
&= \int \limits_{\Omega_\varepsilon} K(x, dy) + \int \limits_{\Omega_\varepsilon^c} K^{(t)}_y(\Omega_\varepsilon) K(x, dy) \quad (\text{If } y \in \Omega_\varepsilon, K^{(t)}_y(\Omega_\varepsilon) = 1) \\
&= K(x, \Omega_\varepsilon) + \int \limits_{\Omega_\varepsilon^c} K^{(t)}_y(\Omega_\varepsilon) K(x, dy) \quad (\text{def kernel}) \\
\geq K(x, \Omega_\varepsilon) + \left[ 1 - (1 - \delta)^t \right] \int \limits_{\Omega_\varepsilon^c} K(x, dy) \quad (\text{Induction hypothesis}) \\
\geq K(x, \Omega_\varepsilon) + \left[ 1 - (1 - \delta)^t \right] K(x, \Omega_\varepsilon^c) \quad (\text{del kernel}) \\
\geq K(x, \Omega_\varepsilon) + K(x, \Omega_\varepsilon^c) - (1 - \delta)^t K(x, \Omega_\varepsilon^c) \\
\geq 1 - (1 - \delta)^t (1 - K(x, \Omega_\varepsilon)) \quad (\text{Probability}) \\
\geq 1 - (1 - \delta)^t (1 - \delta) \quad (\text{condition lemma}) \\
\geq 1 - (1 - \delta)^{t+1}
\end{align*}
\]

Using lemma 82, Rudolph is able to stay a theorem for convergence of evolutionary algorithms (we rewrite it in terms of SGOALs). However, Rudolph’s proof is not wright, since \( Pr\{d(P_t) < \varepsilon\} = Pr\{P_t \in \Omega_\varepsilon\} \) for \( t \geq 0 \) by definition of \( \Omega_\varepsilon \) and Rudolph wrongly assumed that \( Pr\{d(P_t) \leq \varepsilon\} = Pr\{P_t \in \Omega_\varepsilon\} \). Here, we correct the proof proposed by Rudolph (see step 7 in our demonstration).

**Theorem 83.** (Theorem 1 in Rudolph [16]) A SGOAL, whose stochastic kernel satisfies the precondition of lemma 82 will converge to the global optimum (\( f^* \)) of a real valued function \( f : \Phi \to \mathbb{R} \) with \( f > -\infty \), defined in an arbitrary space \( \Omega \subseteq \Phi \), regardless of the initial distribution \( p(\cdot) \).

**Proof.** The idea is to show that the random sequence \( \{d(P_t) : t \geq 0\} \) converges completely to zero under the pre-condition of lemma 82 [16].


\[ Pr \{ P_t \in \Omega_\epsilon \} = \int_{\Omega} K^{(t)}(y, \Omega_\epsilon) \, p(dx) \] (Kernel definition)
\[ \geq 1 - (1 - \delta)^t \int_{\Omega} p(dx) \] (Lemma 82)
\[ \geq 1 - (1 - \delta)^t \] (Reversing order)
\[ 1 - Pr \{ P_t \in \Omega_\epsilon \} \leq (1 - \delta)^t - 1 \] (Adding 1 to both sides)
\[ Pr \{ d(P_t) < \epsilon \} \leq Pr \{ d(P_t) \leq \epsilon \} \] (Probability)
\[ Pr \{ P_t \in \Omega_\epsilon \} \leq Pr \{ d(P_t) \leq \epsilon \} \] (Definition \( \Omega_\epsilon \))
\[ -Pr \{ d(P_t) \leq \epsilon \} \leq -Pr \{ P_t \in \Omega_\epsilon \} \] (Organizing)
\[ 1 - Pr \{ d(P_t) \leq \epsilon \} \leq 1 - Pr \{ P_t \in \Omega_\epsilon \} \] (Adding 1 to both sides)
\[ Pr \{ d(P_t) > \epsilon \} \leq 1 - Pr \{ P_t \in \Omega_\epsilon \} \] (Probability)
\[ \leq (1 - \delta)^t \] (Transitivity with line 5)

Since \( (1 - \delta)^t \to 0 \) as \( t \to \infty \) then \( Pr \{ d(P_t) > \epsilon \} \to 0 \) as \( t \to \infty \), so \( D_t \to 0 \).

Now, \[ \sum_{i=1}^{\infty} Pr \{ d(P_t) > \epsilon \} \leq \sum_{i=1}^{\infty} (1 - \delta)^t \] (line 11)
\[ \leq \frac{(1 - \delta)}{1 - \delta} \] (geometric serie)
\[ < \infty \]

Therefore, \( \{ d(P_t) : t \geq 0 \} \) converges completely to zero.

5.1. Convergence of a VR-SGoal

We follow the approach proposed by Günter Rudolph in [16], to determine the convergence properties of a VR-SGoals but we formalize it in terms of kernels (both variation and replacement).

**Theorem 84.** A VR-SGoal with \( K_V \) an optimal strictly bounded from zero variation kernel and \( K_R \) an elitist replacement kernel, will converge to the global optimum of the objective function.

**Proof.** If we prove that \( K = K_R \circ [K_V \otimes 1_{\Omega_\eta}] \) satisfies the precondition of lemma 82 then the VR-SGoal will converge to the global optimum of the objective function (theorem 83), we use the notation \( \omega = \eta + \upsilon \) in this proof.

\[ [1. \ K(x, A) = \int_{\Omega_\omega \times \{x\}} [K_V \otimes 1_{\Omega_\eta}] (x, dy) K_R(y, A)] \]
\[ K(x, A) = (K_R \circ [K_V \otimes 1_{\Omega^n}]) (x, A) \] (def \( K \))

\[ = \int_{\Omega^n} [K_V \otimes 1_{\Omega^n}] (x, dy) K_R (y, A) \] (def \( \circ \))

\[ = \int_{\Omega^n} K_V (x, \pi_{\{v, ..., v+1, \ldots, \omega}\}) (dy) 1_{\Omega^n} (x, \pi_{\{v+1, ..., \omega\}} (dy)) K_R (y, A) \] (def \( \otimes \))

\[ = \int_{\Omega^n \times \{x\}} K_V (x, \pi_{\{1, ..., v\}}) (dy) 1_{\Omega^n} (x, \pi_{\{v+1, ..., \omega\}} (dy)) K_R (y, A) \] (def \( 1_{\Omega^n} \))

\[ = \int_{\Omega^n \times \{x\}} [K_V \otimes 1_{\Omega^n}] (x, dy) K_R (y, A) \] (def \( K \))

Notice, if \( y \in \Omega^n \times \{x\} \) then \( d(y) \leq d(x) \) (def \( d() \)) and if \( y \in \Omega^n \) then \( d(y) < \epsilon \) (def \( \Omega^n \)) therefore \( K_R (y, \Omega^n) = 1 \) (lemma 79.2).

2. \( K(x, \Omega^n) \geq \delta(\epsilon) > 0 \) for all \( x \in \Omega^n \)

\[ K(x, \Omega^n) = \int_{\Omega^n \cup (\Omega^n)^c} [K_V \otimes 1_{\Omega^n}] (x, dy) K_R (y, \Omega^n) \] (obvious)

\[ \geq \int_{\Omega^n} [K_V \otimes 1_{\Omega^n}] (x, dy) \ast K_R (y, \Omega^n) \] (\( K, \bullet \) measure)

\[ \geq \int_{\Omega^n} [K_V \otimes 1_{\Omega^n}] (x, dy) \] (lemma 79.2)

\[ \geq \int_{\Omega^n} K_V (x, \pi_{\{1, ..., v\}}) (dy) \] (line 3)

\[ \geq \int_{\Omega^n} K_V (x, \pi_{\{1, ..., v\}}) (dy) \] (obvious)

\[ \geq \int_{\Omega^n} K_V (x, dz) \] (notation)

\[ \geq K^* (x, \Omega^n) \] (def kernel)

Clearly, \( K(x, \Omega^n) \geq \delta(\epsilon) > 0 \) for all \( x \in \Omega^n \) (\( K_V \) optimal strictly bounded from zero).

3. \( K(x, \Omega^n) = 1 \) if \( x \in \Omega^n \). If \( x \in \Omega^n \) then \( d(x) < \epsilon \) (def \( \Omega^n \)). Clearly, \( d(y) < \epsilon \) (transitivity), therefore \( K(x, (\Omega^n)^c) = 0 \) (lemma 79.2) and \( K(x, \Omega^n) = 1 \) (\( K, \bullet \) probability measure). \( \square \)

**Corollary 85.** Algorithms HC, and SSGA will converge to the global optimum of the objective function if kernels\( \forall_{\text{HC}}, \) and \( \forall_{\text{SSGA}} \) are optimal strictly bounded from zero kernels.

**Proof.** Follows from theorem 84 and proposition 78. \( \square \)
6. Conclusions and Future Work

Developing a comprehensive and formal approach to stochastic global optimization algorithms (SGOALs) is not an easy task due to the large number of different SGOALs reported in the literature (we just formalize and characterize three classic SGOALs in this paper!). However, such SGOALs are defined as joins, compositions and/or random scans of some common deterministic and stochastic methods that can be represented as kernels on an appropriated structure (measurable spaces with some special property and provided with additional structure). Such special structure is the optimization space (defined in this paper). On this structure, we are able to characterize several SGOALs as special cases of variation/replacement strategies, join strategies, elitist strategies and we are able to inherit some properties of their associated kernels. Moreover, we are able to prove convergence properties (following Rudolph approach [10]) of SGOALs. Since the optimization $\sigma$-algebra property of the structure is preserved by product $\sigma$-algebras, our formal approach can be applicable to both single point SGOALs and population based SGOALs.

Although the theory developed in this paper is comprehensive for just studying SGOALs with fixed parameters (like population size and variation rates), it is a good starting point for studying adapting SGOALs (SGOALs that adapt/vary some search parameters as they are iterating). The central concept for doing that will be the join of kernels (if we consider the space of the parameter values as part of the $\sigma$-algebra). However, such study is far from the scope of this paper.

Our future work will concentrate on including in this formalization, as many as possible, selection mechanisms that are used in SGOALs, and extending and developing the theory required for characterizing both adaptable and Mixing SGOALs.

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