The shape of multidimensional gravity

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In the case of one extra dimension, well known Newton’s potential \( \varphi(r_3) = -G_N m/r_3 \) is generalized to compact and elegant formula \( \varphi(r_3, \xi) = -(G_N m/r_3) \sinh(2\pi r_3/a) \cosh(2\pi r_3/a) - \cos(2\pi \xi/a) \)^{-1} if four-dimensional space has topology \( \mathbb{R}^3 \times T \). Here, \( r_3 \) is magnitude of three-dimensional radius vector, \( \xi \) is extra dimension and \( a \) is a period of a torus \( T \). This formula is valid for full range of variables \( r_3 \in [0, +\infty) \) and \( \xi \in [0, a] \) and has known asymptotic behavior: \( \varphi \sim 1/r_3 \) for \( r_3 > a \) and \( \varphi \sim 1/r_3^2 \) for \( r_4 = \sqrt{r_3^2 + \xi^2} < a \). Obtained formula is applied to an infinitesimally thin shell, a shell, a sphere and two spheres to show deviations from the newtonian expressions. Usually, these corrections are very small to observe at experiments. Nevertheless, in the case of spatial topology \( \mathbb{R}^3 \times T^d \), experimental data can provide us with a limitation on maximal number of extra dimensions.

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\[ \Delta_D \varphi_D = S_D G_D \rho_D(r_D), \quad (1) \]

where \( S_D = 2\pi^{D/2}/\Gamma(D/2) \) is a total solid angle (square of \( (D - 1) \)-dimensional sphere of a unit radius), \( G_D \) is a gravitational constant in \( (D = D + 1) \)-dimensional spacetime and \( \rho_D(r_D) = m\delta(x_1)\delta(x_2)...\delta(x_D) \). In the case of topology \( \mathbb{R}^D \), Eq. (1) has the following solution:

\[ \varphi_D(r_D) = - \frac{G_D m}{(D - 2)r_D^{D-2}}, \quad D \geq 3. \quad (2) \]

This is the unique solution of Eq. (1) which satisfies the boundary condition: \( \lim_{r_D \to +\infty} \varphi_D(r_D) = 0 \). Gravitational constant \( G_D \) in (1) is normalized in such a way that the strength of gravitational field (acceleration of a test body) takes the form: \(-d\varphi_D/dr_D = -G_D m/r_D^{D-1}\).

If topology of space is \( \mathbb{R}^3 \times T^d \), then it is natural to impose periodic boundary conditions in the directions of the extra dimensions: \( \varphi_D(r_3, \xi_1, \xi_2, \ldots, \xi_i, \ldots, \xi_d) = \varphi_D(r_3, \xi_1, \xi_2, \ldots, \xi_i + a_i, \ldots, \xi_d) \), \( i = 1, \ldots, d \), where \( a_i \) denotes a period in the direction of the extra dimension \( \xi_i \). Then, Poisson equation has solution (cf. also with [1, 2]):

\[ \varphi_D(r_3, \xi_1, \ldots, \xi_d) = - \frac{G_N m}{r_3} \times \sum_{k_1=-\infty}^{+\infty} \ldots \sum_{k_d=-\infty}^{+\infty} \exp \left[ -\frac{2\pi}{a_1} \sum_{i=1}^{d} \left( \frac{k_i}{a_i} \right)^2 \right] \left( \frac{2\pi k_1}{a_1} \xi_1 \ldots \frac{2\pi k_d}{a_d} \xi_d \right). \quad (3) \]

To get this result we, first, use the formula \( \delta(\xi_i) = \frac{1}{a_i} \sum_{k=+\infty}^{+\infty} \cos \left( \frac{2\pi k}{a_i} \xi_i \right) \) and, second, put the following relation between gravitational constants in four- and \( D \)-dimensional spacetimes:

\[ \frac{S_D}{S_3} \cdot \frac{G_D}{\prod_{i=1}^{d} a_i} = G_N. \quad (4) \]

The letter relation provides correct limit when all \( a_i \to 0 \). In this limit zero modes \( k_i = 0 \) give the main contribution.
and we obtain $\varphi_D(r_3, \xi_1, \ldots, \xi_d) \to -G_N m/r_3$. Eq. (1) was widely used in the concept of large extra dimensions which gives possibility to solve the hierarchy problem [1, 3]. It is also convenient to rewrite (1) via fundamental scales:

$$S_D \cdot M_{Pl(4)}^2 = M_{Pl(4)}^{2+d} \prod_{i=1}^{d} a_i,$$

(5)

where $M_{Pl(4)} = G_N^{1/2} = 1.2 \cdot 10^{18}$GeV and $M_{Pl(D)} = G_D^{1/(2+d)}$ are fundamental Planck scales in four and $D$ spacetime dimensions, respectively.

In opposite limit when all $a_i \to +\infty$ the sums in Eq. (3) can be replaced by integrals. Using the standard integrals (e.g. from [4] and relation (4), we can easily show that, for example, in particular cases $d = 1,2$ we get desire result: $\varphi_D(r_3, \xi_1, \ldots, \xi_d) \to -G_D m/[(D-2) r_3^{1+d}]$.

**One extra dimension** In the case of one extra dimension $d = 1$ we can perform summation of series in Eq. (3). To do it, we can apply the Abel-Plana formula or simply use the tables of series [4]. As a result, we arrive at compact and nice expression:

$$\varphi_4(r_3, \xi) = -\frac{G_N m}{r_3} \frac{\sinh \left( \frac{2\pi \xi}{a} \right)}{\cosh \left( \frac{2\pi \xi}{a} \right) - \cos \left( \frac{2\pi \xi}{a} \right)},$$

(6)

where $r_3 \in [0, +\infty)$ and $\xi \in [0, a]$. It is not difficult to verify that this formula has correct asymptotes when $r_3 >> a$ and $r_3 << a$. Fig. 1 demonstrates the shape of this potential. Dimensionless variables $\eta_1 \equiv r_3/a \in [0, +\infty)$ and $\eta_2 \equiv \xi/a \in [0, 1]$. With respect to variable $\eta_2$, this potential has two minima at $\eta_2 = 0, 1$ and one maximum at $\eta_2 = 1/2$. We continue the graph to negative values of $\eta_2 \in [-1, 1]$ to show in more detail the form of minimum at $\eta_2 = 0$. The potential (6) is finite for any value of $r_3$ if $\xi \neq 0, a$ and goes to $-\infty$ as $-1/r_3^2$ if simultaneously $r_3 \to 0$ and $\xi \to 0, a$ (see Fig. 2).

![FIG. 1: Graph of function $\tilde{\varphi}(\eta_1, \eta_2) \equiv \varphi_4(r_3, \xi)/(G_N m/a) = -\sinh(2\pi \eta_1)/[\eta_1 (\cosh(2\pi \eta_1) - \cos(2\pi \eta_2))]$.](image1)

Having at hand formula (6), we can apply it for calculation of some elementary physical problems and compare obtained results with famous newtonian expressions. For our calculations we shall use the case of $\xi = 0$. It means that test bodies have the same coordinates in extra dimensions. It takes place e.g. when test bodies are on the same brane. Also, to get numerical results we should define the size $a$ of the extra dimension. If the standard model fields are not localized on the brane, then experiments give upper bound $a \lesssim 10^{-11}$cm [5]. For this value of $a$ the 5-dimensional fundamental Planck scale is $M_{Pl(5)} \gtrsim 10^{11}$TeV (see Eq. (5)). This value $a$ can be greatly increased if we suppose that the standard model fields are localized on the brane. The gravitational inverse-square law experiments show that there is no deviations from three-dimensional Newton’s law up to $2.18 \cdot 10^{-2}$cm [6]. Thus, we can take for this model $a \approx 10^{-2} \div 10^{-3}$cm which results in $M_{Pl(5)} \approx 10^6$TeV. However, it is necessary to keep in mind that this case can be constrained by observations from supernova cooling (see e.g. [7]).

It is worth of noting that all formulas in this letter are applied to the Coulomb’s law if electromagnetic field is not localized on the brane. It was not found deviations from three-dimensional Coulomb’s law up to $10^{-16}$cm [5]. Therefore, for models with non-localized electromagnetic field we should take $a \lesssim 10^{-17}$cm.

**Infinitesimally thin shell** Let us consider first an infinitesimally thin shell of mass $m = 4\pi R^2 \sigma$, where $R$ and $\sigma$ are radius and surface density of the shell. Then, gravitational potential of this shell in a point with radius vector $r_3$ (from the center of the shell) is

$$\varphi(r_3) = -\frac{G_N \sigma R a}{r_3} \ln \left( \frac{\cosh \left( \frac{2\pi (r_3 + R)}{a} \right)}{\cosh \left( \frac{2\pi (r_3 - R)}{a} \right)} - 1 \right).$$

(7)

This formula demonstrates two features of the considered models. Firstly, we see that inside $(r_3 < R)$ of the shell gravitational potential is not a constant. Thus, a test body undergoes an acceleration in contrast to the newtonian case (and Birkhoff’s theorem of general relativity in four-dimensional spacetime which states that the metric inside an empty spherical cavity in the center of a spherically symmetric system is the Minkowski metric). Secondly, this potential has a logarithmic divergence when $r_3 \to R$: $\varphi(r_3) \approx -\frac{G_N \sigma}{R} \left[ 1 - \ln \left( \frac{2\pi R}{2\pi r_3} \right) \right]$. 

![FIG. 2: Section $\xi = 0$ of potential (6). Solid line is $\tilde{\varphi}(\eta_1, 0) = -\sinh(2\pi \eta_1)/[\eta_1 (\cosh(2\pi \eta_1) - \cos(2\pi \eta_2))]$ which goes to $-1/\eta_1$ (dotted line) for $\eta_1 \to +\infty$ and to $-1/(\pi \eta_1^2)$ (dashed line) for $\eta_1 \to 0$.](image2)
where we took into account \( R >> a \) and \(|R - r_3| << a\). For example, in the case \( 2\pi R = 10\text{cm} \) and \( 2\pi|R - r_3| = 10^{-1}\text{a} \), the deviation constitutes \( 2.3 \times 10^{-4} \) and \( 2.3 \times 10^{-18} \) parts of the newtonian value \(-G_N m/R\) for \( a = 10^{-3}\text{cm} \) and \( a = 10^{-17}\text{cm} \), respectively. In principle, the former estimate is not very small. However, it is very difficult to set an experiment which satisfies the condition \(|R - r_3| << a\). If the shell has a finite thickness, then the divergence disappears.

**Spherical shell** The gravitational potential of a spherical shell of inner radius \( R_1 \) and outer radius \( R_2 \) can be written as

\[
\varphi(r_3) = \frac{G_N m}{r_3} \int_{R_1}^{R_2} \left( \cos \frac{2\pi(r_3 + R)}{a} - 1 \right) \ln \left( \cos \frac{2\pi(r_3 - R)}{a} - 1 \right) dR,
\]

(8)

where \( \rho \) is a constant volume density of the shell: \( \rho = m/\left[ \frac{4\pi}{3}(R_3^3 - R_1^3) \right] \). It is useful to present this potential in the form of series. For example, inside \((r_3 \leq R_1)\) of the shell it reads:

\[
\varphi(r_3) = 2G_N \rho \left\{ -\pi (R_2^2 - R_1^2) + \frac{a^2}{\pi r_3} \sum_{k=1}^{+\infty} \sinh \left( \frac{2\pi k}{a} r_3 \right) \right\}
\times \left[ \left( R + \frac{a}{2\pi k} \right) \exp \left( -\frac{2\pi k}{a} R \right) \right]_{R_1}^{R_2} \geq 0,
\]

(9)

where we singled out zero mode \( k = 0 \) which corresponds to the newtonian limit. It can be easily seen that this series does not diverge when \( r_3 \to R_1 \). However, acceleration of a test body diverges when it approaches boundaries \( R_1 \) and \( R_2 \) of the shell. It clearly follows e.g. from the form of the acceleration inside \((r_3 \leq R_1)\) of the shell:

\[
-\frac{d^2\varphi}{dr_3^2} = \frac{2G_N m a^2}{\pi^2 r_3^3} \sum_{k=1}^{+\infty} \frac{1}{k^2} \left( -\frac{2\pi k}{a} r_3 \cosh \left( \frac{2\pi k}{a} r_3 \right) \right)
+ \sinh \left( \frac{2\pi k}{a} r_3 \right) \left[ \left( R + \frac{a}{2\pi k} \right) \exp \left( -\frac{2\pi k}{a} R \right) \right]_{R_1}^{R_2} \geq 0,
\]

(10)

and outside \((r_3 \geq R_2)\) of the shell:

\[
-\frac{d^2\varphi}{dr_3^2} = -\frac{G_N m}{r_3^3} - \frac{G_N \rho a^3}{\pi^2 r_3^3} \sum_{k=1}^{+\infty} \frac{1 + (2\pi k r_3)}{k^3}
\times \exp \left( -\frac{2\pi k}{a} r_3 \right) [h_k(R_2) - h_k(R_1)] < 0,
\]

(11)

where \( h_k(R) = \frac{2\pi k}{a} R \cosh \left( \frac{2\pi k}{a} R \right) - \sinh \left( \frac{2\pi k}{a} R \right) \). Divergence of acceleration originates from the divergence of the series \( \sum_{k=1}^{+\infty} 1/k \) and has the form \( \pm 2G_N \rho a \ln 2\pi a \). Here, \( \varepsilon = |R_2 - R_1| \) and \(-, +\) corresponds to \( r_3 \to R_1 \) and \( r_3 \to R_2 \), respectively. In the case of a sphere \((R_1 = 0 \text{ and } R_2 = R)\) this divergence can be rewritten in the form \(-3G_N m/2\pi R \ln (2\pi R - r_3)/a\). Similar to the case of the infinitesimally thin shell, this deviation from the newtonian acceleration \(-G_N m/R^2\) is also difficult to observe at experiments for considered above parameters. Eqs. (10) and (11) show that acceleration changes the sign from negative outside of the shell to positive inside of the shell. This change happens within the shell. In the limit \( R_1 \to 0 \), some of obtained above formulas can be applied for a solid sphere.

**Gravitational self-energy of a sphere** For a sphere of constant volume density \( \rho \), mass \( m \) and radius \( R \), a gravitational self-energy is

\[
U = U_N \left\{ 1 + 15 \left( \frac{a}{2\pi R} \right)^3 \sum_{k=1}^{+\infty} \frac{1}{k^3} \left[ \frac{2\pi k R}{3a} + \left( \frac{a}{2\pi k R} \right)^2 F_k \right] \right\},
\]

where

\[
F_k = \left( 1 + \frac{2\pi k R}{a} \right) \exp \left( -\frac{2\pi k R}{a} \right) \left[ \sinh \left( \frac{2\pi k R}{a} \right) - \frac{2\pi k R}{a} \cosh \left( \frac{2\pi k R}{a} \right) \right]
\]

and \( U_N = -3G_N m^2/(5R) \) is gravitational self-energy of a sphere in the newtonian limit \([3]\). In the case \( R >> a \), the difference \( \Delta U \equiv U - U_N \) reads:

\[
\Delta U \approx U_N \frac{15}{8\pi^3} \left( \frac{a}{3R} \right)^2 \sum_{k=1}^{+\infty} \frac{1}{k^2} = U_N \frac{5}{24} \left( \frac{a}{R} \right)^2,
\]

where we took into account \( \sum_{k=1}^{+\infty} 1/k^2 = \pi^2/6 \). Therefore, this difference is suppressed by power law (with respect to ratio between \( a \) and \( R \)) but not exponentially as it is usually expected for Kaluza-Klein modes. Nevertheless, for Sun with \( R \approx 7 \times 10^{10}\text{cm} \) this value is a negligible part of \( U_N \) even if \( a \approx 10^{-3}\text{cm} \): \( \Delta U \approx 4 \times 10^{-29} U_N \) \( \text{where } U_N \approx -2 \times 10^{-48}\text{erg} \[3] \).

**Gravitational interaction of two spheres** Energy of gravitational interaction between two spheres of constant volume densities \( \rho, \rho' \), masses \( m, m' \) and radiiuses \( R, R' \) on a distance \( r_3 \geq R + R' \) reads

\[
U(r_3) = -G_N m m' \left\{ 1 + 18 \left( \frac{a}{2\pi R} \right)^3 \left( \frac{a}{2\pi R'} \right)^3 \right. 
\times \sum_{k=1}^{+\infty} \frac{1}{k^6} \exp \left( -\frac{2\pi k}{a} r_3 \right) h_k(R) h_k(R') \right\}.
\]

(13)

The member of the series with \( k = 1 \) (first Kaluza-Klein mode) gives the main correction to the newtonian expression and acquires the form of Yukawa potential. In this case, for the force of gravitational interaction between these two spheres we obtain:

\[
-\frac{dU}{dr_3} \approx -G_N m m' \left\{ 1 + \frac{9}{2} \left( \frac{a}{2\pi R} \right)^2 \left( \frac{a}{2\pi R'} \right)^2 \frac{2}{a} \frac{r_3}{R - R'} \right\}.
\]

(14)
where we made additional natural assumption $R, R' >> a$. If $r - R - R' \approx a$, then we get an estimate:

\[
-dU \approx -\frac{G_N m n}{r_3^2} \left\{ 1 + 0.0084 \left( \frac{a}{2\pi R} \right)^2 \left( \frac{a}{2\pi R'} \right)^2 \frac{2\pi r_3}{a} \right\} .
\]

(15)

For example, in the case $2\pi R = 2\pi R' = 10$cm and $a \approx 10^{-2}$cm the correction is $1.68 \cdot 10^{-11}$, which is difficult to observe at experiments. However, in the case of the internal space topology $T^d$ the correction term in (15) acquires a prefactor $s$ which satisfies the condition $1 \leq s \leq d$ and represents a number of extra dimensions with periods of the torus which are equal (or approximately equal) to $a = \max a_i$. If all of $a_i$ are equal to each other, $s = d$. Increasing the number of extra dimensions $d$, finally we arrive at the condition when the correction term becomes big enough to contradict with experimental data. Therefore, in this case we can get a limitation on a maximal number of extra dimensions for considered models. Certainly, the models with infinite number of extra dimensions with $a = \max a_i$ are forbidden.

**Conclusions** We have considered generalization of the Newton's potential to the case of extra dimensions where multidimensional space has topology $M_D = \mathbb{R}^3 \times T^d$. It was shown that for model with one extra dimensions in the case of massive point source, the gravitational potential can be expressed via compact and elegant formula (6). This formula is valid for full range of variables $r_3$ (magnitude of a radius vector in three dimensions) and $\xi$ (extra dimension) and has well known asymptotic behavior. Then, this formula was applied to an infinitesimally thin shell, a shell, a sphere and two spheres to get gravitational potentials and acceleration of a test body for these configurations and to compare obtained results with the known newtonian formulas. In some cases, obtained potentials and accelerations have logarithmic divergences near the boundaries of shells and spheres. Additionally, in contrast to the newtonian case, test bodies accelerate inside of shells. For each considered problems, we found deviations from the known newtonian expressions and show that for proposed parameters of the models it is difficult to observe these deviations at experiments. Nevertheless, if internal space has topology $T^d$ with approximately equal periods $a_i$ of the torus in different dimensions, then we can get a limitation on maximal number of extra dimensions from experimental data with the help of formulas (14), (15) for gravitational interaction between two massive spheres. To conclude this letter, we want to stress that with the help of exact formulas of the form of (6) we can solve multidimensional quantum Schrödinger equation. It opens a possibility to find a new gravitationally bound states: ”dark matter atoms”. We postpone this investigation to our forthcoming paper.

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