LEVI-CIVITA CONNECTION ON HOM-$\rho$-COMMUTATIVE ALGEBRAS

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Abstract. Recently, some concepts such as Hom-algebras, Hom-Lie algebras, Hom-Lie admissible algebras, Hom-coalgebras are studied and some of classical properties of algebras and some geometric objects are extended on them. In this paper by recall the concept of Hom-$\rho$-commutative algebras, we intend to develop some of the most classical results in Riemannian geometry such as metric, connection, torsion tensor, curvature tensor on it and also we discuss about differential operators and get some results of differential calculus using them. The notions of symplectic structures and Poisson structures are included and an example of $\rho$-Poisson bracket is given.

1. Introduction

One branch of differential geometry is Riemannian geometry that studies Riemannian manifolds (a smooth manifold with a Riemannian metric) (see [24] for more details). Riemannian geometry at the first was bring up by Bernhard Riemann in the nineteenth century. The concept of linear connection is one of the main concepts of the Riemannian geometry, which arose by the idea of parallel transport along a path in a Riemannian manifold at the end of 19th century. [25], [9]. There is no direct and quick way to companion between distance points of a curve space, however the connection permits to contrast what is happening at these points. Earlier, in the 1910’s, Albert Einstein discovered that the Riemannian geometry is substantial to general relativity theory. It is also the foundational revelation for gauge theories. This division into two branches has led to many representations tending to either the specific (e.g. presented in tensor notation assuming a coordinate frame and zero torsion) or the abstract (e.g. using the language of fiber bundles). By worth of its applications the Riemannian geometry stands at the nucleus of modern mathematics.

Differential calculus is a branch of mathematics concerned with the determination, properties, and application of derivatives and differentials in study of functions. The development of differential calculus is closely dealing with the concept of integral calculus. In this approach the differential calculus on the manifold is deduced from the properties of the manifold and it involves functions on the manifold, differential operators, differential forms and derivatives. The most important property of this calculus is that the operator $d$ satisfies $d^2 = 0$.

Definition a multiplication over a vector space was the origin of the notion of Hom-algebra structure, where the structure twisting by a homomorphism. The structure of Hom-Lie algebra appeared first as a generalization of Lie algebra by Hartwig, Larsson and Silvestrov in [12]. Physics and deformations of Lie algebras, in particular Lie algebra of vector fields were the stimulants to study Hom-Lie structures. Lie algebras are special cases of Hom-Lie algebras in which $\phi$ is the identity map. Also, $q$-deformations of the Witt and the Virasoro algebras have the structure of a Hom-Lie algebra. Later Hom-Lie algebras were extended to Hom-associative algebras by Makhlouf and Silvestrov in [14] and to quasi-Hom Lie and
quasi-Lie algebras by Larsson and Silvestrov in [20], [21]. Leibniz and Hom-Lie admissible algebras, Hom-alternative algebras, Hom-Hopf algebras, Hom-coalgebras are other interesting Hom-algebraic structures were studied [15, 16, 26].

Non-commutative geometry is a branch of mathematics concerned with a geometric approach to non-commutative algebras, and with the construction of spaces that are locally presented by non-commutative algebras of functions. Extension of the concept of differential forms on manifolds plays the basic role in non-commutative geometry (see [8], [10], [13], [19], for instance). Important examples of non-commutative algebras are $\rho$-commutative algebras. They have a great ability to generalize geometric objects. Accordingly, Riemannian geometry and its objects such as metric, connection, curvature, torsion, differential form and also differential calculus and application to hyperplane are discussed on $\rho$-commutative algebra by Bongaarts, Ciupala and Ngakeu in [7], [5] and [18]. In this paper we recall and study Hom-$\rho$-commutative algebra and develop some of the most classical results in Riemannian geometry and differential calculus on it.

This paper is arranged as follows. In Section 2, we recall some necessary background knowledge including $\rho$-commutative and Hom-$\rho$-commutative algebras, Hom-associative and Hom-$\rho$-commutative Lie algebras, $p$-forms and derivations. In Section 3, the reader will get some important properties of $\rho$-tensor products, this section further develops the foundational topics for Riemannian manifolds, metric, connection, torsion tensor and curvature tensor are included. Also, we check some examples, properties and lemmas to obtain important results. Section 4 has been assigned to discuss about representations, cochain, Hom-cochain and some results will be derived of differential calculus. Also, symplectic structures and Poisson brackets are studied.

2. Hom-$\rho$-commutative algebra

In this section, we summarize some definitions concerning $\rho$-commutative and Hom-$\rho$-commutative algebras and related results.

Let $k$ be a field ($k = \mathbb{R}$ or $k = \mathbb{C}$), $(G, +)$ be an abelian group and $A$ be a $G$-graded associative and unital algebra over a field $k$. This implies that the vector space $A$ has a $G$-grading $A = \oplus_{a \in G} A_a$ such that $A_a A_b \subset A_{a+b}$. A two-cycle is a map $\rho: G \times G \rightarrow k^*$ which satisfies

\[
\rho(a, b) = \rho(b, a)^{-1}, \quad a, b \in G, 
\]

\[
\rho(a + b, c) = \rho(a, c)\rho(b, c), \quad a, b, c \in G. 
\]

This implies that $\rho(a, b) \neq 0$, $\rho(0, b) = 1$ and $\rho(c, c) = \pm 1$ for all $a, b, c \in A$, $c \neq 0$.

The $\rho$-commutator of two homogeneous elements $f$ and $g$ of $A$ is

\[
[f, g]_\rho = fg - \rho(|f|, |g|)gf. 
\]

A $\rho$-commutative algebra is a $G$-graded algebra $A$ with a given cocycle $\rho$ such that $fg = \rho(|f|, |g|)gf$ for all homogeneous elements $f$ and $g$ in $A$ (i.e., $[f, g]_\rho = 0$).

In the following, we have some preliminary definitions from [3]:

**Definition 2.1.** A Hom-$\rho$-commutative algebra is a quadruple $(A, \cdot, \rho, \phi)$ consisting of a $G$-graded vector space $A$ i.e., $A = \bigoplus_{a \in G} A_a$, an even bilinear map $\cdot: A \times A \rightarrow A$ i.e., $A_a \cdot A_b \subset A_{a+b}$, for all $a, b \in G$, a bicharacter $\rho: G \times G \rightarrow k^*$ and an even linear map $\phi: A \rightarrow A$. In addition $f \cdot g = \rho(|f|, |g|)g \cdot f$, for any $f, g \in Hg(A)$.
The G-degree of a (non-zero) homogeneous element \( f \in A \) and the set of homogeneous elements in \( A \), respectively denoted by \( |f| \) and \( Hg(A) \).

**Definition 2.2.** A Hom-\( \rho \)-commutative algebra \((A, \cdot, \rho, \phi)\) is said to be

a) multiplicative if \( \phi \) is a morphism for \( \cdot \),

b) regular if \( \phi \) is an automorphism for \( \cdot \),

c) involutive if \( \phi^2 = id_A \).

**Definition 2.3.** A Hom-associative \( \rho \)-commutative algebra is a Hom-\( \rho \)-commutative algebra \((A, \cdot, \rho, \phi)\) such that

\[
\phi(f)(g \cdot h) = (f \cdot g)\phi(h),
\]

for any \( f, g, h \in Hg(A) \).

**Example 2.4.** The quaternion algebra \( H \) is a \( \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \)-graded algebra in the following sense. Associate the "Triple degree" to the standard basis elements of \( H \)

\[
\varepsilon = (0, 0, 0), i = (0, 1, 1), j = (1, 0, 1), k = (1, 1, 0),
\]

where \( \varepsilon \) denotes the unit and the following multiplication conditions are imposed

\[
i) \ i^2 = j^2 = k^2 = -1,
\]

\[
ii) \ ij = k, ji = -k, jk = i, kj = -i, ki = j, ik = j, ik = -j.
\]

Also the two-cocycle \( \rho \) is defined by \( \rho(a, b) = (-1)^{(a,b)} \), where \( (a, b) \) is the usual scalar product of 3-vectors. Indeed \( (i, j) = 1 \) and similarly for \( k \), so that \( i, j \) and \( k \) \( \rho \)-commute with each other. But \( (i, i) = 2 \), so that \( i, j \) and \( k \) commute with themselves. Thus, quaternion algebra \( H \) is a \( \rho \)-commutative algebra. If we set linear map \( \phi_H(i) = ai, \phi_H(j) = bj, \phi_H(k) = ck, \ a, b, c \in \mathbb{C}, \) then we have a Hom-\( \rho \)-commutative quaternion algebra. But, \( H \) is a Hom-associative \( \rho \)-commutative algebra if \( a = b = c \). Let us to set \( a = b = c = 1 \).

**Definition 2.5.** A Hom-\( \rho \)-Lie algebra is a \( G \)-graded vector space \( A \) together with a bilinear map \([-,-]_\rho : A \times A \rightarrow A \), a two-cycle \( \rho \) and a linear map \( \phi : A \rightarrow A \) satisfying the following relations

- \([A_g, A_{g'}]_\rho \subset A_{g+g'},\)
- \([f, g]_\rho = -\rho(|f|, |g|)[g, f]_\rho, \) (\( \rho \)-antisymmetry),
- \(\phi(f, g)_\rho = [\phi(f), \phi(g)]_\rho,\)
- \(\rho(|h|, |f|)[\phi(f), [g, h]_\rho]_\rho + \rho(|g|, |h|)[\phi(h), [f, g]_\rho]_\rho + \rho(|f|, |g|)[\phi(g), [h, f]_\rho]_\rho = 0, \) (\( \rho \)-Hom-Jacobi identity).

The triple \((A, \phi, [.,.]_\rho)\) consisting of \( G \)-graded associative algebra \( A \), linear map \( \phi = I \) and \( \rho \)-commutator \([f, g]_\rho = fg - \rho(f, g)gf\) is a Hom-\( \rho \)-Lie algebra that is called Hom-\( \rho \)-commutative Lie algebra.

**Definition 2.6.** Let \((A, \cdot, \rho, \phi)\) be a multiplicative Hom-\( \rho \)-commutative algebra. A \( \rho \)-derivation of degree \(|X|\) on \( A \) is a linear map \( X : A \rightarrow A \) such that

\[
X(f \cdot g) = X(f) \cdot g + \rho(|X|, |f|)f \cdot X(g).
\]
If $\rho$-Der$A$ is denoted the space of all $\rho$-derivations of $A$, then for $X \in \rho$-Der$A$ and $Y \in \rho$-Der$A$ the $\rho$-commutator of $X, Y$, defined by $[X, Y]_\rho = XY - \rho(|X|, |Y|)YX$, is a $\rho$-derivation. Furthermore, when $A$ is Hom-$\rho$-commutative, $\rho$-Der$A$ is also a $A$-bimodule with actions $\triangleright, \triangleleft$ defined by
\begin{equation}
(f \triangleright X)g = f(Xg), X \triangleleft f = \rho(|X|, |f|)f \triangleright X.
\end{equation}
In fact any $G$-graded left module $M$ over a Hom-$\rho$-commutative algebra $A$ is a $A$-bimodule with
\begin{equation}
f \triangleright (X \triangleleft g) = (f \triangleright X) \triangleleft g, f, g \in A, X \in M.
\end{equation}
Moreover, $\rho$-Der$A$ equipped with the $\rho$-commutator and the following even map
\begin{equation}
\phi_A : \rho$-Der$A \rightarrow \rho$-Der$A : \phi_A(X) = X \circ \phi,
\end{equation}
and action
\begin{equation}
\phi(f \triangleright X) = \phi(f) \triangleright X, \phi(X \triangleleft f) = X \triangleleft \phi(f),
\end{equation}
is a Hom-$\rho$-Lie algebra.

We will simply denote the left action $f \triangleright X$ as $fX$ and $\rho(|X|, |Y|)$ by $\rho(X, Y)$.

**Definition 2.7.** A $p$-form on module $A$ is a $p$-linear map $\alpha_p : \times^p(\rho$-Der$A) \rightarrow A$ satisfying the following relations
\begin{itemize}
    \item $\alpha_p(fX_1, \cdots, X_p) = f\alpha_p(X_1, \cdots, X_p),$
    \item $\alpha_p(X_1, \cdots, X_j f, X_{j+1}, \cdots, X_p) = \alpha_p(X_1, \cdots, X_j, fX_{j+1}, \cdots, X_p),$
    \item $\alpha_p(X_1, \cdots, X_j, X_{j+1}, \cdots, X_p) = -\rho(X_j, X_{j+1})\alpha_p(X_1, \cdots, X_{j+1}, X_j, \cdots, X_p),$
\end{itemize}
where $j = 1, \cdots, p - 1$, $X_k \in \rho$-Der$\circ A$, $k = 1, \cdots, p$, $f \in A$ and $Xf$ is the right $A$-action on $\rho$-Der$A$ defined by (2.5). Let $\Omega^p(A)$ denotes the set of $p$-forms, then $\Omega^p(A)$ is a $G$-graded right $A$-module with
\begin{equation}
(\alpha_p f)(X_1, \cdots, X_p) = \alpha_p(X_1, \cdots, X_p)f,
\end{equation}
\begin{equation}
|\alpha_p| = |\alpha_p(X_1, \cdots, X_p)| - (|X_1| + \cdots + |X_p|).
\end{equation}
The direct sum $\Omega(A) = \bigoplus_{p \geq 0} \Omega^p(A)$ with $\Omega^0(A) = A$ is again a $G$-graded $A$-module.

**Definition 2.8.** The wedge product $\wedge$ in $\Omega^p(A)$ is the map
\begin{equation}
\wedge : \Omega^p(A) \times \Omega^q(A) \rightarrow \Omega^{p+q}(A),
\end{equation}
defined by
\begin{equation}
(\alpha \wedge \beta)(X_1, \cdots, X_p, \cdots, X_{p+q})
\end{equation}
\begin{equation}
= \sum_{\sigma \in S_{p+q}} \text{sign}(\sigma) \times \rho(\sum_{j=p+1}^{p+q} X_{\sigma(j)}, \alpha)(X_{\sigma(k)}, X_{\sigma(l)}) \times \alpha(X_{\sigma(1)}, \cdots, X_{\sigma(p)})\beta(X_{\sigma(p+1)}, \cdots, X_{\sigma(p+q)}),
\end{equation}
for $\alpha \in \text{Hg}(\Omega^p(A)$ and $\beta \in \text{Hg}(\Omega^q(A)$, where $S_{p+q}$ is the set of permutations $\sigma \in S_{p+q}$ such that $\sigma(1) < \sigma(2) < \cdots < \sigma(p)$ and $\sigma(p+1) < \sigma(k+2) < \cdots < \sigma(p+q)$ and $\sigma(l) > \sigma(k)$ if $l < k$. 
3. \(\rho\)-Tensor Product

In this section, at first we turn to a brief discussion of the concept of tensor products to develop Riemannian geometry on Hom-\(\rho\)-commutative algebras, then we concentrate on metrics to introduce the notion of linear connection and define the torsion and curvature associated to the connection. Later, we survey some properties and give to the reader some main points. Symbols \(Hg(\Omega^1(A))\) and \(Hg(\rho\text{-Der}\,A)\) apply respectively for homogeneous elements of \(\Omega^1(A)\) and \(\rho\text{-Der}\,A\).

For \(\alpha_1, \cdots, \alpha_p \in Hg(\Omega^1(A))\), we make a \(p\)-linear homogeneous map \(s\) of \(G\)-degree \(|s| = \sum_{i=1}^{p} |\alpha_i|\) as \(s = \alpha_1 \otimes_\rho \cdots \otimes_\rho \alpha_p\) and define it by

\[
\alpha_1 \otimes_\rho \cdots \otimes_\rho \alpha_p(X_1, \cdots, X_p) = \prod_{i=1}^{p} \alpha_i(X_i) \prod_{k=1}^{p-1} \rho(\sum_{j=k+1}^{p} X_j, \alpha_k),
\]

where \(X_1, \cdots, X_p \in Hg(\rho\text{-Der}_\rho A)\), that satisfy the following relations

\[
s(fX_1, X_2, \cdots, X_p) = fs(X_1, X_2, \cdots, X_p),
\]

\[
s(X_1, \cdots, X_i \times f, X_{i+1}, \cdots, X_p) = s(X_1, \cdots, X_i, fX_{i+1}, \cdots, X_p).
\]

Let \(T^\otimes_p\) denotes the linear space generated by elements \(s = \alpha_1 \otimes_\rho \cdots \otimes_\rho \alpha_p\). Then we have the exterior algebra \(T^\otimes_p = \bigoplus_{n \geq 0} T^\otimes_{n} p\) with \(T^\otimes_0 = A\) and natural algebra structure \(\otimes_\rho\) which is defined on homogeneous elements in \(T^\otimes_p\) by

\[
T_p \otimes_\rho T_q = \bigoplus_{i+j=k} T_{p+i} \otimes T_{q+j}.
\]

\(\forall T_p \in T^\otimes_p, T_q \in T^\otimes_q, X_1, \cdots, X_{p+q} \in Hg(\rho\text{-Der}\,A)\). Also, \(T_p \otimes_\rho T_q\) has the degree \(|T_p \otimes_\rho T_q| = |T_p| + |T_q|\).

Notice that \(T^\otimes_p\) coincide with the space of all \(\rho\text{-p}\)-linear maps on \(\times^p \rho\text{-Der}\,A\), if \(\Omega^1(A)\) and \(\rho\text{-Der}_\rho A\) are finitely generated. In this case, \(T \in T^\otimes_p\) is called covariant \(\rho\)-tensor.

**Definition 3.1.** Let \((A, \rho, \phi)\) be a multiplicative Hom-\(\rho\)-commutative algebra endowed with a bilinear symmetric non-degenerate form \(g(\cdot, \cdot)\) such that for any \(X, Y \in \rho\text{-Der}(A)\) the following equation is satisfied

\[g(Y, X) = g(\phi_A(Y), \phi_A(X)).\]

Then, we say that \(A\) admits a metric \(g(\cdot, \cdot)\). Metric \(g\) gives us a left \(A\)-module isomorphism \(\tilde{g} : \rho\text{-Der} A \rightarrow \Omega^1(A)\) defined by \(\tilde{g}(X)Y = g(Y, X)\).

**Isomorphism \(\tilde{g}\) satisfies** \(g(X, fY) = g(Xf, Y)\) and \(g(aX) = ag(X)\), \(\tilde{g}(X \times a) = \tilde{g}(X)ap(a, g), a \in Hg(A)\).

**Definition 3.2.** A linear connection on \(\rho\text{-Der} A\) is a linear map

\[
\begin{cases}
\nabla : \rho\text{-Der} A \rightarrow \text{End}(\rho\text{-Der}A), \\
X \rightarrow \nabla_X,
\end{cases}
\]

such that

\[
\nabla_{aX}Y = a\nabla_XY, \quad a \in A, X, Y \in \rho\text{-Der} A,
\]

\[
\nabla_X(aY) = (X \cdot a)Y + \rho(X, a)\phi(a)\nabla_XY, \quad a \in Hg(A), X \in Hg(\rho\text{-Der} A).
\]
Definition 3.3. Let $T \in T^p$, $X \in Hg(\rho-DerA)$ and $\nabla_X T$ denotes the covariant derivative of $T$ which is a $p$-linear map. This map is defined on $\rho-DerA$ by

$$\rho(X, \sum_{i=1}^{p} X_i)\nabla_X T(X_1, \cdots, X_p) = \phi_A(X) \cdot T(X_1, \cdots, X_p)$$

$$-\sum_{i=1}^{p} \rho(X, \sum_{i=1}^{p} X_i)T(\phi_A(X_1), \cdots, \nabla_X X_i, \cdots, \phi_A(X_p)).$$

$\rho$-tensor $T$ is said to be parallel or compatible with respect to a linear connection $\nabla$ if $\nabla T = 0$.

Connection $\nabla$ is called compatible with metric $g$ if

$$\phi_A(X).g(Y, Z) = g(\nabla_X Y, \phi_A(Z)) + \rho(X, Y)g(\phi_A(Y), \nabla_X Z), \quad \forall X, Y, Z \in Hg(\rho-DerA).$$

The curvature of a linear connection $\nabla$ is the map

$$R : \rho-DerA \times \rho-DerA \rightarrow End(\rho-DerA),$$

$$(X, Y) \rightarrow R(X, Y) := R_{XY},$$

defined by

$$R(X, Y)(Z) = \nabla_{\phi_A(X)} \nabla_Y Z - \rho(X, Y)\nabla_{\phi_A(Y)} \nabla_X Z - \nabla_{[X, Y]_{\rho}} \phi_A(Z).$$

Lemma 3.4. By the definition of curvature $R$, we get the following equalities

a) $R(X, Y) = -\rho(X, Y)R(Y, X),$

b) $R(aX, Y)(Z) = \phi(a)R(X, Y)Z - \rho(X + a, Y)(Y \cdot \phi(a))\nabla_X Z$

$$+ \rho(X + a, Y)(Y \cdot a)\nabla_X \phi_A(Z) + a\nabla_{\phi_A(X)} \nabla_Y Z - \phi(a)\nabla_{\phi_A(X)} \nabla_Y Z,$$

c) $R(X, aY)(Z) = \rho(X, a)\phi(a)R(X, Y)Z + (X \cdot \phi(a))\nabla_Y Z$

$$- (X \cdot a)\nabla_Y \phi_A(Z) - \rho(X, a + Y)a\nabla_{\phi_A(Y)} \nabla_X Z + \rho(X, a + Y)\phi(a)\nabla_{\phi_A(Y)} \nabla_X Z$$

$$- \phi(a)\nabla_{[X, Y]_{\rho}} \phi_A(Z) + \rho(X, a)\phi(a)\nabla_{[X, Y]_{\rho}} \phi_A(Z),$$

d) $R(X, Y)(aZ) = \rho(X + Y, a)\phi^2(a)(R(X, Y)Z) + (\phi_A(X) \cdot Y \cdot a)Z$

$$- \rho(X, Y)(\phi_A(Y) \cdot X \cdot a)Z + \rho(Y, a)(\phi_A(X) \cdot \phi(a))\nabla_Y Z$$

$$- \rho(X, Y)\rho(X, a)(\phi_A(Y) \cdot \phi(a))\nabla_X Z - \rho(X, Y)\rho(Y, X + a)\rho(X, a)\phi(a) \cdot X$$

$$- \rho(X, Y)\rho(Y, X + a)\phi(X \cdot a)\nabla_{\phi_A(Y)} Z + \rho(X, Y + a)\phi(Y \cdot a)\nabla_{\phi_A(Y)} Z$$

$$- (X \cdot Y \cdot a)\phi_A(Z) + \rho(X, Y)(Y \cdot X \cdot a)\phi_A(Z)$$

$$- \rho(X + Y, a)\phi(a)\nabla_{[X, Y]_{\rho}} \phi_A(Z) + \rho(X + Y, a)\phi^2(a)\nabla_{[X, Y]_{\rho}} \phi_A(Z).$$

Proof. For sake of the brevity, we proof the relation b). By the definitin of curvature $R$, we have

$$R(aX, Y)Z = \nabla_{\phi_A(aX)} \nabla_Y Z - \rho(X + a, Y)\nabla_{\phi_A(Y)} \nabla_{aX} Z - \nabla_{[aX, Y]_{\rho}} \phi_A(Z).$$
Now, by the properties of connection $\nabla$ and the relation $[aX, Y]_\rho = -\rho(X + a, Y)(Y \cdot a)X + \phi(a)[X, Y]_\rho$, we get

$$R(aX, Y)Z = a\nabla_{\phi_A(X)}Y\nabla_Y Z - \rho(X + a, Y)(\phi_A(Y) \cdot a)\nabla_X Z$$

$$- \rho(X + a, Y)\rho(Y, a)\phi(a)\nabla_{\phi_A(Y)}Y\nabla_X Z$$

$$- (Y \cdot a)\nabla_X \phi_A(Z) - \phi(a)\nabla_{[X, Y]_\rho}\phi_A(Z)$$

$$= \phi(a)R(X, Y)Z - \rho(X + a, Y)(Y \cdot \phi(a))\nabla_X Z$$

$$+ \rho(X + a, Y)(Y \cdot a)\nabla_X \phi_A(Z) + a\nabla_{\phi_A(X)}Y\nabla_Y Z - \phi(a)\nabla_{\phi_A(X)}Y\nabla_Y Z.$$ 

Also, the torsion of the connection $\nabla$ is defined by

$$\begin{cases} T_\nabla : \rho DerA \times \rho DerA \rightarrow \rho DerA, \\ T_\nabla(X, Y) = \nabla_X Y - \rho(X, Y)\nabla_Y X - [X, Y]_\rho. \end{cases}$$

Connection $\nabla$ called torsion-free if

$$T_\nabla = 0, \text{ i.e., } [X, Y]_\rho = \nabla_X Y - \rho(X, Y)\nabla_Y X, \text{ } \forall X, Y \in Hg(\rho DerA).$$

In the similar way of the curvature, it is easy to see that the torsion have the following properties

$$T(aX, Y) = T(X, Y) + a\nabla_X Y - \phi(a)\nabla_X Y;$$

$$T(X, aY) = \rho(X, a)\phi(a)T(X, Y) - \rho(X, a + Y)a\nabla_Y X$$

$$+ \rho(X + a, Y)\phi(a)\nabla_Y X - \rho(X, a)\phi(a)[X, Y]_\rho + \phi(a)[X, Y]_\rho.$$ 

**Theorem 3.5.** There exists a unique linear connection on every Hom-$\rho$-commutative algebra with metric $g$ which is torsion-free and compatible with metric $g$. This connection is called Levi-Civita connection associated to $g$.

**Proof.** By the compatibility condition of the connection $\nabla$ with the metric $g$, we can easily show that the Koszul equation is given by

$$2\rho(Z, Y)g(\phi_A(X), \nabla_Y Z) = \rho(X, Z)\phi_A(Z) \cdot g(X, Y) + \rho(X, Z)g(\phi_A(Z), [X, Y]_\rho)$$

$$- \phi_A(X) \cdot g(Z, Y) - \rho(X, Z)g([Z, X]_\rho, \phi_A(Y))$$

$$+ \rho(Z, Y)\rho(X, Y + Z)\phi_A(Y) \cdot g(Z, X) + \rho(Z, Y)\rho(X, Y + Z)g([Y, Z]_\rho, \phi_A(X)), $$

where $X, Y, Z \in Hg(\rho DerA)$. So, the proof is clear. \qed

Assuming that $[\partial_i, \partial_j] = 0$ and $\nabla_{\partial_i} \partial_j = \sum_k \phi_A(\partial_k) \wedge \Gamma_{ij}^k$, the Christoffel symbols are as follows

$$\Gamma_{ij}^k = \frac{1}{2} \rho(g, x_i - x_i - x_j) \times \sum_k \tilde{g}^{ik} \{-\phi_A(\partial_k) \cdot \tilde{g}_{ij} + \rho(x_i + x_k, x_j) \phi_A(\partial_j) \cdot \tilde{g}_{ki} + \rho(x_k, x_i + x_j) \phi_A(\partial_i) \cdot \tilde{g}_{jk}\}.$$ 

**Example 3.6.** Extended-hyperplane $A_q = \langle 1, x, y, x^{-1}, y^{-1} : x \cdot y = qy \cdot x \rangle$ is a $\mathbb{Z} \times \mathbb{Z}$-graded algebra. Considering the bicharacter

$$\rho(n, n') = q^{\sum_{j=1}^n n_j n'_j \alpha_j},$$
where $\alpha_{jk} = 1$ if $j < k$, 0 if $j = k$ and $-1$ if $j > k$, $A_q^2$ is a $\rho$-commutative algebra. If we define linear map $\phi: A_q^2 \to A_q^2$ by

$$
\phi(x) = ax, \phi(y) = ay, \phi(x^{-1}) = bx^{-1}, \phi(y^{-1}) = by^{-1}, \quad a, b \in k,
$$

then we have the Hom-$\rho$-commutative algebra $A_q^2$. Note that $(A_q^2, \rho, \phi)$ is a Hom-associative $\rho$-commutative algebra if $a = b$.

The set $\rho$-$\text{Der}A_q^2$ of all $\phi$-$\rho$-derivations on $A_q^2$ is a $A_q^2$-bimodule generated by $\partial_\rho x$ and $\partial_\rho y$, and $\Omega^1(A)$ is generated by $dx, dy$ such that $dx_j(\partial_\rho x_j) = \partial_\rho (x_j) = \delta_{ij}, |\partial_\rho x_i| = -|x_i|$ and $|dx_i| = |x_i|$ where $x_1 = x$ and $x_2 = y$. We define linear map $\phi_{A_q^2}: \rho$-$\text{Der}A_q^2 \to \rho$-$\text{Der}A_q^2$ by

$$
\phi_{A_q^2}(\frac{\partial}{\partial x}) = \lambda \frac{\partial}{\partial x} + \gamma \frac{\partial}{\partial y}, \quad \phi_{A_q^2}(\frac{\partial}{\partial y}) = (\lambda + \gamma) \frac{\partial}{\partial y}.
$$

All homogeneous metrics on $\rho$-$\text{Der}A_q^2$ were defined by [18]

$$
g = g_{11}dx \otimes_\rho dx + g_{12}dx \otimes_\rho dy + g_{21}dy \otimes_\rho dx + g_{22}dy \otimes_\rho dy.
$$

Now, by using metric $g$, we try to find the following relation

$$
g(X, Y) = g(\phi_{A_q^2}(X), \phi_{A_q^2}(Y)), \quad \forall X, Y \in \rho$-$\text{Der}A_q^2.
$$

**Case 1:** If $X = \frac{\partial}{\partial x}, Y = \frac{\partial}{\partial x}$, we have $g(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}) = g_{11}$. Also, we get

$$
g(\phi_{A_q^2}(\frac{\partial}{\partial x}), \phi_{A_q^2}(\frac{\partial}{\partial x})) = g(\lambda \frac{\partial}{\partial x} + \gamma \frac{\partial}{\partial y}, \lambda \frac{\partial}{\partial x} + \gamma \frac{\partial}{\partial y})
$$

$$
= \lambda^2 g(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}) + \lambda \gamma g(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) + \gamma \lambda g(\frac{\partial}{\partial y}, \frac{\partial}{\partial x})
$$

$$
+ \gamma^2 g(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}) = \lambda^2 g_{11} + \lambda \gamma (1 + q) g_{12} + \gamma^2 g_{22}.
$$

In this case, we find $\lambda^2 = 1, \lambda \gamma = 0, \gamma^2 = 0$, so $\lambda = \pm 1, \gamma = 0$.

**Case 2:** If $X = \frac{\partial}{\partial y}, Y = \frac{\partial}{\partial y}$, we have

$$
g(\phi_{A_q^2}(\frac{\partial}{\partial y}), \phi_{A_q^2}(\frac{\partial}{\partial y})) = g((\lambda + \gamma) \frac{\partial}{\partial y}, (\lambda + \gamma) \frac{\partial}{\partial y}) = (\lambda + \gamma)^2 g(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}) = (\lambda + \gamma)^2 g_{22},
$$

$$
g(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}) = g_{22}.
$$

In this case, we find $(\lambda + \gamma)^2 = 1$.

**Case 3:** If $X = \frac{\partial}{\partial x}, Y = \frac{\partial}{\partial y}$, we get $g(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) = g_{21} = qg_{12}$ and

$$
g(\phi_{A_q^2}(\frac{\partial}{\partial x}), \phi_{A_q^2}(\frac{\partial}{\partial y})) = g(\lambda \frac{\partial}{\partial x} + \gamma \frac{\partial}{\partial y}, (\lambda + \gamma) \frac{\partial}{\partial y}) = \lambda(\lambda + \gamma) g(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})
$$

$$
+ \gamma(\lambda + \gamma) g(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}) = q\lambda(\lambda + \gamma) g_{12} + \gamma(\lambda + \gamma) g_{22}.
$$

In this case, we find $\lambda(\lambda + \gamma) = 1, \gamma(\lambda + \gamma) = 0$. 

Case 4: If \( X = \frac{\partial}{\partial y}, Y = \frac{\partial}{\partial x} \), we have

\[
\begin{align*}
&g(\phi_{A_{q}^{2}}(\frac{\partial}{\partial x}), \phi_{A_{q}^{2}}(\frac{\partial}{\partial y})) = g((\lambda + \gamma)\frac{\partial}{\partial y}, \frac{\partial}{\partial x}, \lambda \frac{\partial}{\partial x} + \frac{\partial}{\partial y}) = \lambda(\lambda + \gamma)g(\frac{\partial}{\partial y}, \frac{\partial}{\partial x}) \\
&\quad + \gamma(\lambda + \gamma)g(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}) = \lambda(\lambda + \gamma)g_{12} + \gamma(\lambda + \gamma)g_{22},
\end{align*}
\]

In this case, we also find \( \lambda(\lambda + \gamma) = 1, \gamma(\lambda + \gamma) = 0 \).

By the above cases, If \( \gamma = 0 \) and \( \lambda = \pm 1 \), thus we obtain

\[
\phi_{A_{q}^{2}}(\frac{\partial}{\partial x}) = \pm \frac{\partial}{\partial x}, \quad \phi_{A_{q}^{2}}(\frac{\partial}{\partial y}) = \pm \frac{\partial}{\partial y},
\]

Therefore \( \phi_{A_{q}^{2}} \) can be write in the following four cases

\[
\begin{align*}
\begin{cases}
\phi_{A_{q}^{2}}(\frac{\partial}{\partial x}) &= \frac{\partial}{\partial x}, \quad \phi_{A_{q}^{2}}(\frac{\partial}{\partial y}) = -\frac{\partial}{\partial y}, \\
\phi_{A_{q}^{2}}(\frac{\partial}{\partial x}) &= -\frac{\partial}{\partial x}, \quad \phi_{A_{q}^{2}}(\frac{\partial}{\partial y}) = -\frac{\partial}{\partial y}, \\
\phi_{A_{q}^{2}}(\frac{\partial}{\partial x}) &= \frac{\partial}{\partial x}, \quad \phi_{A_{q}^{2}}(\frac{\partial}{\partial y}) = \frac{\partial}{\partial y}, \\
\phi_{A_{q}^{2}}(\frac{\partial}{\partial x}) &= -\frac{\partial}{\partial x}, \quad \phi_{A_{q}^{2}}(\frac{\partial}{\partial y}) = \frac{\partial}{\partial y}.
\end{cases}
\end{align*}
\]

Also, we know that \( \rho\)-Der\( A_{q}^{2} \) with \( \rho \)-commutator \([X, Y] = XY - \rho(X, Y)YX \) and linear map \( \phi_{A_{q}^{2}} \) is a Hom-\( \rho \)-commutative Lie algebra. But, note that for the elements of the basis of \( \rho\)-Der\( A_{q}^{2} \), we have

\[
(3.1)
\]

\[
\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\big|_{\rho} = \frac{\partial}{\partial x}, \frac{\partial}{\partial y} - \rho(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) = \frac{\partial}{\partial x}, \frac{\partial}{\partial y}
\]

On the other hand, since \([\frac{\partial}{\partial x}, \frac{\partial}{\partial y}]_{\rho} \in \rho\)-Der\( A_{q}^{2} \), then we can write

\[
\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\big|_{\rho} = p \frac{\partial}{\partial x} + q \frac{\partial}{\partial y},
\]

so \([\frac{\partial}{\partial x}, \frac{\partial}{\partial y}]_{\rho}(x) = p \) and \([\frac{\partial}{\partial x}, \frac{\partial}{\partial y}]_{\rho}(y) = q \). But, relation \((3.1)\) gives us \([\frac{\partial}{\partial x}, \frac{\partial}{\partial y}]_{\rho}(x) = [\frac{\partial}{\partial x}, \frac{\partial}{\partial y}]_{\rho}(y) = 0 \).

Let us setting \( \tilde{g}_{mk} := \rho(x_{m}, x_{k})g_{mk} \) and continue with \( g_{11} = x^{2}, g_{12} = x^{-1}y^{-1}, g_{22} = y^{-2} \). In this case, \( g \) is a homogeneous metric on \( A_{q}^{2} \) (of degree \((0, 0)\)) if and only if

\[
D = \frac{1 - q^{2}}{q^{2}}x^{-2}y^{-2},
\]

is invertible, in other words if and only if \( q \neq 0, 1, -1 \). So, by the definition of \( \tilde{g} \) we have

\[
(\tilde{g}_{mk}) = \begin{pmatrix}
  x^{-2} & qx^{-1}y^{-1} \\
  x^{-1}y^{-1} & y^{-2}
\end{pmatrix},
\]

and

\[
(\tilde{g}^{mk}) = (\tilde{g}_{mk})^{-1} = \frac{1}{1 - q^{2}} \begin{pmatrix}
  x^{2} & -qxy \\
  -xy & y^{2}
\end{pmatrix}.
\]
Here, it seems that, we are ready to find the $\rho$-Christoffel coefficients $\Gamma^1_{ij}$ corresponding to each $\phi_A^{2q}$. So, we need to consider the following cases:

If $\phi_A^{2q} = \frac{\partial}{\partial x^1}$, $\phi_A^{2q} = \frac{\partial}{\partial y^1}$, then the $\rho$-Christoffel coefficients are as follows

$$\Gamma^1_{11} = -x^{-1}, \Gamma^2_{22} = y^{-1}, \Gamma^1_{12} = \Gamma^2_{12} = \Gamma^1_{21} = \Gamma^2_{21} = \Gamma^1_{11} = \Gamma^2_{12} = 0.$$

Also, when $\phi_A^{2q} = \frac{\partial}{\partial x^2}$, $\phi_A^{2q} = \frac{\partial}{\partial y^2}$, we have

$$\Gamma^1_{11} = x^{-1}, \Gamma^2_{22} = y^{-1}, \Gamma^1_{12} = \Gamma^2_{12} = \Gamma^1_{21} = \Gamma^2_{21} = \Gamma^1_{11} = \Gamma^2_{12} = 0.$$

If $\phi_A^{2q} = \frac{\partial}{\partial x^2}$, $\phi_A^{2q} = \frac{\partial}{\partial y^2}$, we can find the following $\rho$-Christoffel coefficients

$$\Gamma^1_{11} = -x^{-1}, \Gamma^2_{22} = -y^{-1}, \Gamma^1_{12} = \Gamma^2_{12} = \Gamma^1_{21} = \Gamma^2_{21} = \Gamma^1_{11} = \Gamma^2_{12} = 0.$$

In the last case, if $\phi_A^{2q} = \frac{\partial}{\partial x^2}$, $\phi_A^{2q} = \frac{\partial}{\partial y^2}$, in the similar way of the above cases, we have

$$\Gamma^1_{11} = x^{-1}, \Gamma^2_{22} = -y^{-1}, \Gamma^1_{12} = \Gamma^2_{12} = \Gamma^1_{21} = \Gamma^2_{21} = \Gamma^1_{11} = \Gamma^2_{12} = 0.$$

For a Levi-Civita connection $\nabla$ and linear map $\phi_A : \rho$-Der$A \rightarrow \rho$-Der$A$ we can find the following relation

$$(3.2) \quad \phi_A[X, Y]_\rho = [\phi_A(X), \phi_A(Y)]_\rho = \nabla_{\phi_A(X)}\phi_A(Y) - \rho(X, Y)\nabla_{\phi_A(Y)}\phi_A(X).$$

On the other hand, we have

$$[X, Y]_\rho = \nabla_X Y - \rho(X, Y)\nabla_Y X,$$

and

$$(3.3) \quad \phi_A[X, Y]_\rho = \phi_A(\nabla_X Y) - \rho(X, Y)\phi_A(\nabla_Y X).$$

If we compare the relations (3.2) and (3.3), then we obtain

$$\nabla_{\phi_A(X)}\phi_A(Y) - \rho(X, Y)\nabla_{\phi_A(Y)}\phi_A(X) = \phi_A(\nabla_X Y) - \rho(X, Y)\phi_A(\nabla_Y X),$$

so

$$\nabla_{\phi_A(X)}\phi_A(Y) - \phi_A(\nabla_X Y) + \rho(X, Y)(-\nabla_{\phi_A(Y)}\phi_A(X) + \phi_A(\nabla_Y X)) = 0.$$

Consequently, we have

$$(3.4) \quad \nabla_{\phi_A(X)}\phi_A(Y) = \phi_A(\nabla_X Y).$$

Here, we define the covariant derivation of $R$ as follows

$$(3.5) \quad (\nabla Z R)(X, Y) = \nabla_{\phi_A(Z)}(\rho R(X, Y)(\cdot) - R(\rho X, Z, \phi_A(Y))\phi_A(\cdot) - \rho(|Z|, |X|) R(\rho(\phi_A(X), \nabla Z Y)\phi_A(\cdot) - \rho(|Z|, |X| + |Y|) R(\phi_A(X), \phi_A(Y))\nabla Z(\cdot).$$

**Lemma 3.7.** For $X, Y, Z, V, W \in \rho$-Der$A$, the curvature $R$ satisfies the following equalities

(a) $\rho(X, Y) R_{Y Z} X + \rho(Y, Z) R_{Z X} Y + \rho(Z, X) R_{X Y} Z = 0$ (Bianchi 1),

(b) Second Bianchi identity for a torsion-free connection

$$\rho(V, X) R(X, Y, V, W) + \rho(Y, V) R(V, X, Y, W) + \rho(X, Y) R(Y, V, X, W) = 0,$$

(c) $\rho(Y, Z)(\nabla_Z R)(X, Y) + \rho(X, Y)(\nabla_Y R)(Z, X) + \rho(Z, X)(\nabla_X R)(Y, Z) = 0,$

where $R(X, Y, V, W) := g(R_{X Y} V, W)$. 
Proof. a) By the definition of curvature $R$, we can find the following relations

$$
\rho(Y, Z)R_{XY}X = \rho(Y, Z)\{\nabla_{\phi_A(Z)}\nabla_X Y - \rho(Z, X)\nabla_{\phi_A(X)}\nabla_Z Y - \nabla_{[Z, X], \rho}\phi_A(Y)\}
$$

$$
= \rho(Y, Z)\nabla_{\phi_A(Z)}\nabla_X Y - \rho(Y, Z)\rho(Z, X)\nabla_{\phi_A(X)}\nabla_Y Z
$$

$$
- \rho(Y, Z)\rho(Z, X)\nabla_{\phi_A(X)}[Z, Y]_\rho - \rho(Y, Z)\rho(Z, Y)\rho(X, Y)\nabla_{\phi(Y)}[Z, X]_\rho
$$

(3.6)

$$
- \rho(Y, Z)[[Z, X]_\rho, \phi_A(Y)]_\rho,
$$

$$
\rho(X, Y)R_{YZ}X = \rho(X, Y)\{\nabla_{\phi_A(Y)}\nabla_Z X - \rho(Y, Z)\nabla_{\phi_A(Z)}\nabla_Y X - \nabla_{[Y, Z], \rho}\phi_A(X)\}
$$

$$
= \rho(X, Y)\nabla_{\phi_A(Y)}\nabla_Z X - \rho(X, Y)\rho(Y, Z)\nabla_{\phi_A(Z)}\nabla_X Y
$$

$$
- \rho(X, Y)\rho(Y, Z)\nabla_{\phi_A(Z)}[Y, Z]_\rho - \rho(X, Y)\rho(Y, X)\rho(Z, X)\nabla_{\phi_A(X)}[Y, Z]_\rho
$$

(3.7)

$$
- \rho(X, Y)[[Y, Z]_\rho, \phi_A(X)]_\rho,
$$

and

$$
\rho(Z, X)R_{XY}Z = \rho(Z, X)\{\nabla_{\phi_A(X)}\nabla_Y Z - \rho(X, Y)\nabla_{\phi_A(Y)}\nabla_X Z - \nabla_{[X, Y], \rho}\phi_A(Z)\}
$$

$$
= \rho(Z, X)\nabla_{\phi_A(X)}\nabla_Y Z - \rho(Z, X)\rho(X, Y)\nabla_{\phi_A(Y)}\nabla_Z X
$$

$$
- \rho(Z, X)\rho(X, Y)\nabla_{\phi_A(Y)}[X, Z]_\rho - \rho(Z, X)\rho(Y, Z)\nabla_{\phi_A(Z)}[X, Y]_\rho
$$

(3.8)

$$
- \rho(Z, X)[[X, Y]_\rho, \phi_A(Z)]_\rho.
$$

Summing the relations (3.6), (3.7) and (3.8), imply

$$
\rho(X, Y)R_{YZ}X + \rho(Y, Z)R_{ZX}Y + \rho(Z, X)R_{XY}Z = 0.
$$

The second Bianchi identity follows immediately from the first and the relation (c) follows from a direct calculation. \qed

Based on our knowledge of Riemannian geometry the following properties hold for Riemannian curvature tensor $R(X, Y, Z, W)$

1. $R$ is skew-symmetric in the first two and last two entries

$$
R(X, Y, Z, W) = -R(Y, X, Z, W) = -R(Y, X, W, Z).
$$

2. $R$ is symmetric between the first two and last two entries

$$
R(X, Y, Z, W) = R(Z, W, X, Y).
$$

Also, in [18], Ngakeu presented that the curvature tensor $R(X, Y, Z, W)$ on $\rho$-commutative algebras has the same properties of Riemannian geometry, that is

(i) $R(X, Y, V, W) = -\rho(X, Y)R(Y, X, V, W) = -\rho(Y, W)R(X, Y, V, W),$

(ii) $R(X, Y, V, W) = \rho(X + Y, V + W)R(V, W, X, Y).$

For the curvature tensor $R(X, Y, V, W)$ on Hom-$\rho$-commutative algebras necessarily, we can not have the following properties

$$
R(X, Y, V, W) = \rho(X + Y, V + W)R(V, W, X, Y),
$$

$$
R(X, Y, V, W) = -\rho(X, Y)R(Y, X, V, W) = -\rho(V, W)R(X, Y, V, W).
$$
4. REPRESENTATIONS, COCHAINS AND HOM-COCHAINS ON HOM-\(\rho\)-COMMUTATIVE ALGEBRAS

In this part, we recall the notions of representation, cochains and Hom-cochains on Hom-\(\rho\)-commutative Lie algebras. These notions are defined analogously of the classical case by K. Abdaoui, F. Ammar and A. Makhlouf in [2]. Then, we try to develop differential calculus by using them and recall the notion of Poisson bracket on Hom-\(\rho\)-commutative Lie algebras and investigate some examples (for the classical case see [23, 17]).

**Definition 4.1.** Let \((A, ., \rho)\) be a multiplicative Hom-\(\rho\)-commutative Lie algebra. For any non-negative integer \(k\), a \(\phi^k\)-derivation of degree \(|X|\) on \(A\) is a linear map \(X : A \rightarrow A\) such that

\[
X \circ \phi = \phi \circ X \quad \text{ i.e.,} \quad [X, \phi]_\rho = 0,
\]

and

\[
X[f, g]_\rho = [X(f), \phi^k(g)]_\rho + \rho(X, f)[\phi^k(f), X(g)]_\rho. \tag{4.1}
\]

We denote by \(\rho\)-\(\mathcal{D}er\) the space of all \(\phi^k\)-\(\rho\)-derivations of \(A\). It is easy to see that for \(X \in \rho\)-\(\mathcal{D}er\) and \(Y \in \rho\)-\(\mathcal{D}er\) the \(\rho\)-commutator of \(X, Y\), defined by \([X, Y]_\rho = XY - \rho(X, Y)YX\), is a \(\phi^{k+s}\)-\(\rho\)-derivation and

\[
\rho\)-\(\mathcal{D}er(A) = \bigoplus_{k \geq 0} \rho\)-\(\mathcal{D}er\); \quad \rho\)-\(\mathcal{D}er\)
\]

is a \(\rho\)-Lie algebra with above bracket.

In this paper, we concentrate on \(\phi\)-\(\rho\)-derivations and denote by \(\rho\)-\(\mathcal{D}er\) the space of \(\phi\)-\(\rho\)-derivations. It is easy to see that \(\rho\)-\(\mathcal{D}er\) with linear map \(\phi : \rho\)-\(\mathcal{D}er\) given by \(\phi_A(X) = X \circ \phi\) and action

\[
\phi(f \cdot X) = \phi(f) \cdot X, \quad \phi(X \cdot f) = \rho(X, f)\phi(f) \cdot X,
\]

is a Hom-\(\rho\)-Lie algebra.

**Definition 4.2.** Let \(V\) be a vector space. A linear map \(\mu : A \rightarrow \text{End}(V)\) is called a representation of the Hom-\(\rho\)-Lie algebra \((A, [\cdot, \cdot]_\rho, \phi)\) on \(V\) with respect to \(B \in \text{End}(V)\) if the following equality is satisfied

\[
\mu[f, g]_\rho \circ B = \mu(\phi(f)) \circ \mu(g) - \rho(f, g)\mu(\phi(g)) \circ \mu(f).
\]

Moreover, a representation \((V, \mu)\) is said to be graded if \(V = \bigoplus_{a \in G} V_a\) is a \(G\)-graded space such that

\[
\mu(f)(V_a) \subseteq V_{|f|+a}.
\]

for all the homogeneous elements \(f \in A\) and \(a \in G\).

**Definition 4.3.** A \(k\)-cochain on a Hom-\(\rho\)-Lie algebra \((A, [\cdot, \cdot]_\rho, \phi)\) is a \(\rho\)-skew-symmetric and \(k\)-linear map \(\alpha : A \times \cdots \times A \rightarrow V\) of \(G\)-degree \(|\alpha|\). We denote by \(C^k(A; V)\) the set of \(k\)-cochains on \(A\).

\(\alpha \in C^k(A; V)\) is called a \(k\)-Hom-cochain on \(A\) if for \(B \in \text{End}(V)\) and \(f_1, \cdots, f_k \in \text{Hom}(A)\), the following relation holds

\[
B(\alpha(f_1, \cdots, f_k)) = \alpha(\phi(f_1), \cdots, \phi(f_k)).
\]
Let $C^k_\phi(A, V)$ denotes the set of $k$-Hom-cochains on $A$, then $C^k_\phi(A, V)$ is a graded algebra with $C^0_\phi(A, V) = V$ and we have (see [2] for more details)

$$C_\phi(A, V) = \bigoplus_{k \geq 0} C^k_\phi(A, V).$$

Consider Hom-$\rho$-Lie algebra $(\rho\cdot Der_A, [\cdot, \cdot], \rho_A)$ equipped with representation $\mu_A$ on $A$ ($\mu_A : \rho\cdot Der_A \rightarrow End(A)$) with respect to $B = Id : A \rightarrow A$. In the next, we intend to define some operators on the set of $k$-Hom-cochains $C^k_{\phi_A}(\rho\cdot Der_A, A)$ on $\rho\cdot Der_A$. Note that in this case, $C^k(\rho\cdot Der_A, A) = \Omega^k(A)$ and

$$C^k_{\phi_A}(\rho\cdot Der_A, A) = \{ \alpha \in \Omega^k(A) : \alpha \circ \phi_A = \alpha \} = \Omega^k_{\phi_A}(A).$$

So,

$$|\alpha| = |\alpha(X_1, \cdots, X_k)| - |X_1| - \cdots - |X_k|.$$ 

Now, we define the co-boundary operator $d_\mu : \Omega^k_{\phi_A}(A) \rightarrow \Omega^{k+1}_{\phi_A}(A)$ by

$$d_\mu f(X) = \mu_A(\phi_A^{-1}(X)) \cdot f, \quad f \in A,$$

and

$$(4.2)$$

$$d_\mu \alpha(X_1, \cdots, X_{k+1}) =: \sum_{j=1}^{k+1} (-1)^{j-1} \rho(\sum_{i=1}^{j-1} X_i, X_j) \mu_A(\phi_A^{-1}(X_j)) \cdot \alpha(X_1, \cdots, \hat{X}_j, \cdots, X_{k+1})$$

$$+ \sum_{1 \leq j < l \leq k+1} (-1)^{j+l} \rho(\sum_{i=1}^{j-1} X_i, X_j) \rho(\sum_{i=1}^{l-1} X_i, X_l)$$

$$\times \rho(\sum_{i=j+1}^{l-1} X_i, X_l) \alpha([X_j, X_l]_\rho, \phi_A(X_1), \cdots, \phi_A(\hat{X}_j), \cdots, \phi_A(\hat{X}_l), \cdots, \phi_A(X_{k+1})), $$

for $k \geq 1$, and $\alpha \in \Omega^k_{\phi_A}(A)$, where $\hat{X}_j$ means that $X_j$ is omitted. Note that $|d_\mu \alpha| = |\alpha|$ and $d_\mu^2 = 0$ (the condition $d_\mu^2 = 0$ does not follow if the condition $\alpha \circ \phi_A = \alpha$ is omitted, so it is necessary to define the differential operators on $k$-Hom-cochains).

The inner and Lie derivation also are defined on $\Omega_{\phi_A}(A)$ by

$$i_X \alpha(X_1, \cdots, X_{k-1}) = \rho(\sum_{i=1}^{k-1} X_i, X) \alpha(X, X_1, \cdots, X_{k-1}), \quad i_X(f) = 0,$$

$$L_X \alpha(X_1, \cdots, X_k) = \rho(\sum_{i=1}^{k} X_i, X) \mu_A(\phi_A^{-1}(X)) \cdot \alpha(X_1, \cdots, X_k)$$

$$- \sum_{j=1}^{k} \rho(\sum_{i=1}^{j-1} X_i, X_j) \alpha(\phi_A(X_1), \cdots, [X, X_1]_\rho, \cdots, \phi_A(X_k)).$$

Note that $|i_X| = |L_X| = |X|$.

**Definition 4.4.** Let $\Omega \in C^2_{\phi_A}(\rho\cdot Der_A, A) = \Omega^2_{\phi_A}(A)$ be a 2-Hom-cochain. We set

$$\Omega : \rho\cdot Der_A \rightarrow C^1_{\phi_A}(\rho\cdot Der_A, A) = \Omega^1_{\phi_A}(A); \quad \hat{\Omega}(X) = \Omega(\cdot, \phi_A(X)).$$
Ω is called a symplectic structure on A if Ω is a closed \((d_\mu \Omega = 0)\) nondegenerate (\(\tilde{\Omega}\) is isomorphism) and we have

\[
\Omega(\phi_A(X), Y) = -\Omega(X, \phi_A(Y)).
\]

(4.3)

It is remarkable that Ω is homogeneous iff \(\tilde{\Omega}\) is homogeneous and we have \(|\Omega| = |\tilde{\Omega}|\).

Note that, if \((\rho-Der_\phi A, [\cdot, \cdot], \phi_A)\) is an involutive Hom-\(\rho\)-Lie algebra then the condition \((4.3)\), is equivalent to the following condition

\[
\Omega(\phi_A(X), \phi_A(Y)) = -\Omega(X, Y).
\]

**Definition 4.5.** \(X \in \rho-Der_\phi A\) is called a locally Hamiltonian \(\phi-\rho\)-derivation if \(L_{\phi_A(X)} \Omega = 0\).

**Lemma 4.6.** \(X \in \rho-Der_\phi A\) is locally Hamiltonian if and only if \(d_\mu (i_{\phi_A(X)} \Omega) = 0\).

**Proof.** By the Cartan identity \(L_{\phi_A(X)} = d_\mu \circ i_{\phi_A(X)} + i_{\phi_A(X)} \circ d_\mu\) the proof is clear. \(\Box\)

By the relation \(L_{\phi_A(X)} \phi_A = [L_{\phi_A(X)}, L_{\phi_A(Y)}]_\rho\), we can show that if \(X, Y \in \rho-Der_\phi A\) are locally Hamiltonian then \([X, Y]_\rho\) is also locally Hamiltonian.

**Definition 4.7.** For any \(f \in A\), the vector \(X := \tilde{\Omega}^{-1}(d_\mu f)\) is called the Hamiltonian \(\phi-\rho\)-derivation associated to \(f\).

Let us \(X_f\) denote the Hamiltonian \(\phi-\rho\)-derivation associated to \(f\) i.e., \(X = X_f\). So \(X_f\) is of \(G\)-degree \(|X_f| = |f| - |\Omega|\) and

\[
d_\mu f = \Omega(\cdot, \phi_A(X_f)) = -i_{\phi_A(X_f)} \Omega.
\]

In other words

\[
\Omega(\phi_A(X_g), \phi_A(X_f)) = \mu_A(X_g) \cdot f.
\]

We make the following two conditions on the maps \(\phi\) and \(\phi_A\) and \(\mu_A(X) \in End(A)\)

\[
\mu_A(\phi_A(X_f)) = \mu_A(X_{\phi(f)}),
\]

(4.4)

\[
\mu_A(\phi_A(X_f)) \cdot [g, h]_\rho = [\mu_A(X_f) \cdot g, \phi(h)]_\rho + \rho(X_f, g)[\phi(g), \mu_A(X_f) \cdot h]_\rho.
\]

(4.5)

**Lemma 4.8.** Let \((\rho-Der_\phi A, [\cdot, \cdot], \phi_A)\) be an involutive Hom-\(\rho\)-Lie algebra and \(X, Y \in \rho-Der_\phi A\) be locally Hamiltonians. Then the following relation holds

\[
[X, Y]_\rho = -X_{\Omega(X, Y)} = \rho(X, Y) X_{\Omega(Y, X)},
\]

i.e., \(\phi_A[X, Y]_\rho\) is the Hamiltonian \(\phi-\rho\)-derivation associated to \(\Omega(X, Y)\).

**Proof.** By the equality \(i_{[X, Y]_\rho} = [L_X, i_Y]_\rho\) and equivalently \(i_{\phi_A[X, Y]_\rho} = [L_{\phi_A(X)}, i_{\phi_A(Y)}]_\rho\), we have

\[
i_{\phi_A[X, Y]_\rho} \Omega = [L_{\phi_A(X)}, i_{\phi_A(Y)}]_\rho \Omega = L_{\phi_A(X)} (i_{\phi_A(Y)} \Omega) - \rho(X, Y) i_{\phi_A(Y)} (L_{\phi_A(X)} \Omega).
\]

Since \(X\) is a locally Hamiltonian \(\phi-\rho\)-derivation, so

\[
i_{\phi_A[X, Y]_\rho} \Omega = [L_{\phi_A(X)}, i_{\phi_A(Y)}]_\rho \Omega = L_{\phi_A(X)} (i_{\phi_A(Y)} \Omega).
\]

In the next, by the Cartan identity and given that the \(\phi-\rho\)-derivation \(Y\) is locally Hamiltonian, we easily obtain the following relation

\[
i_{\phi_A[X, Y]_\rho} \Omega = \rho(X, Y) d_\mu (\Omega(\phi_A(Y), \phi_A(X)) = d_\mu (\Omega(X, Y)),
\]
and so

\[-i_{\phi A(X,Y)}\Omega = -d_\rho(\Omega(X,Y)).\]

Thus, the conclusion is held, that is

\[[X,Y]_\rho = -X_{\Omega(X,Y)} = \rho(X,Y)X_{\Omega(Y,X)}\].

\[\square\]

The above lemma says that the $\rho$-commutator of two locally Hamiltonian $\phi$-$\rho$-derivation is a Hamiltonian $\phi$-$\rho$-derivation.

**Definition 4.9.** [4] A Poisson Hom-$\rho$-commutative algebra consists of a $G$-graded vector space $A$, bilinear maps $: A \times A \to A$ and $\{\cdot, \cdot\}_\rho: A \times A \to A$ of $G$-degree $|\{\cdot, \cdot\}_\rho| = P$, a linear map $\phi: A \to A$ and a two-cycle $\rho: G \times G \to k^*$ such that

1. $(A, \{\cdot, \cdot\}_\rho, \phi)$ is a Hom-associative $\rho$-commutative algebra.
2. $(A, \{\cdot, \cdot\}_\rho, \phi)$ is a Poisson Hom-$\rho$-Lie algebra, i.e.,
   1. $|\{f, g\}_\rho| = P + |f| + |g|$, 
   2. $\{f, g\}_\rho = -\rho(f, g)\{g, f\}_\rho$, 
   3. $\rho(h, f)\{\phi(f), \{g, h\}_\rho\} + \rho(g, h)\{\phi(h), \{f, g\}_\rho\} + \rho(f, g)\{\phi(g), \{h, f\}_\rho\} = 0$.
3. For all $f, g, h \in A$, $\{\{f, g\}_\rho, \phi(h)\}_\rho = \rho(g, h + P)\{\{f, h\}_\rho, \phi(g)\}_\rho + \rho(f, g)\{\phi(f), \{g, h\}_\rho\}_\rho$.

Equivalently, Poisson Hom-$\rho$-Lie algebra can be defined in the following expression

**Definition 4.10.** A Poisson Hom-$\rho$-Lie algebra is a multiplex $(A, \{\cdot, \cdot\}_\rho, \{\cdot, \cdot\}_\rho, \phi, \rho)$ consisting of a $G$-graded vector space $A$, bilinear maps $\{\cdot, \cdot\}_\rho: A \times A \to A$ and $\{\cdot, \cdot\}_\rho: A \times A \to A$ of $G$-degree $|\{\cdot, \cdot\}_\rho| = P$, a linear map $\phi: A \to A$ and a two-cycle $\rho: G \times G \to k^*$ satisfying

1. $(A, \{\cdot, \cdot\}_\rho, \phi)$ is a Hom-$\rho$-Lie algebra.
2. For all $f, g, h \in A$, $\{\{f, g\}_\rho, \phi(h)\}_\rho = \rho(g, h + P)\{\{f, h\}_\rho, \phi(g)\}_\rho + \rho(f, g)\{\phi(f), \{g, h\}_\rho\}_\rho$.

**Theorem 4.11.** Let $(A, \{\cdot, \cdot\}_\rho, \phi)$ be an involutive Hom-$\rho$-Lie algebra and $\Omega$ be homogeneous symplectic structure. Defining the $\rho$-Poisson bracket $\{\cdot, \cdot\}_\rho$ associated to $\Omega$ as

\[\{f, g\}_\rho := -\rho(\Omega, g)\mu_A(X_f) \cdot g = -\rho(\Omega, g)\Omega(\phi_A(X_f), \phi_A(X_g)) \quad f, g \in A,\]

$(A, \{\cdot, \cdot\}_\rho, \{\cdot, \cdot\}_\rho, \phi)$ is a Poisson Hom-$\rho$-Lie algebra.

**Proof.** Since, $A$ is an involutive Hom-$\rho$-Lie algebra, then we have

\[\{f, g\}_\rho := -\rho(\Omega, g)\mu_A(X_f) \cdot g = -\rho(\Omega, g)\Omega(X_f, X_g) \quad f, g \in A.\]

At first, we show that $(A, \{\cdot, \cdot\}_\rho, \phi)$ is a Hom-$\rho$-Lie algebra. we have

\[|\{f, g\}_\rho| = |\{\cdot, \cdot\}_\rho| + |f| + |g|.\]

On the other hand, we have

\[|\Omega(X_f, X_g)| = |\Omega| + |X_f| + |X_g| = P + |f| + |g|.\]

By the definition of $\rho$-Poisson bracket, since $|\{f, g\}_\rho| = |\Omega(X_f, X_g)|$, then we can find $|\{\cdot, \cdot\}_\rho| = P = -|\Omega|$. Now, we investigate the following relation

\[\{f, g\}_\rho = -\rho(f, g)\{g, f\}_\rho.\]
By the relation \[4.6\], we have
\[
\{f, g\}_\rho = -\rho(\Omega, g)\Omega(X_f, X_g) = \rho(\Omega, g)\rho(X_f, X_g)\Omega(X_g, X_f)
\]
\[
= \rho(\Omega, g)\rho(f, g)\rho(f, -\Omega)\rho(-\Omega, g)\Omega(X_g, X_f)
\]
\[
= \rho(f, g)\rho(\Omega, f)\Omega(X_g, X_f) = -\rho(f, g)\Omega(X_g, X_f)
\]
\[
= -\rho(f, g)\{g, f\}_\rho.
\]
Now, it’s time to complete the proof by showing that the \(\rho\)-Hom-jacobi identity holds. At first note that
\[
(4.7) \quad \Omega([X_f, X_g]_\rho, \phi_A(X_h)) = -\Omega(\phi_A[X_f, X_g]_\rho, X_h) = -\mu_A([\phi_A(X_f), \phi_A(X_g)]_\rho)h
\]
\[
= -\mu_A(X_f)\mu_A(\phi_A(X_g)) \cdot h + \rho(X_f, X_g)\mu_A(X_g)\mu_A(\phi_A(X_f)) \cdot h.
\]
Since \(\Omega\) is a close form, then by \[4.4\], we have
\[
0 = d_\mu(X_f, X_g, X_h) = \mu_A(\phi_A(X_f))\mu_A(X_g) \cdot h - \rho(X_f, X_g)\mu_A(\phi_A(X_g))\mu_A(X_f) \cdot h
\]
\[
+ \rho(X_f + X_g, X_h)\mu_A(\phi_A(X_h))\mu_A(X_f) \cdot g + \mu_A(X_f)\mu_A(\phi_A(X_g)) \cdot h
\]
\[
- \rho(X_f, X_g)\mu_A(X_g)\mu_A(\phi_A(X_f)) \cdot h - \rho(X_g, X_h)\mu_A(X_f)\mu_A(\phi_A(X_h)) \cdot g
\]
\[
+ \rho(X_g, X_h)\rho(X_f, X_h)\mu_A(X_g)\mu_A(\phi_A(X_f)) \cdot f
\]
\[
- \rho(X_f, X_g)\rho(X_g, X_h)\rho(X_f, X_h)\mu_A(X_g)\mu_A(\phi_A(X_f)) \cdot f.
\]
By sorting the terms in the above relation, we have
\[
0 = d_\mu(X_f, X_g, X_h) = \rho(X_g, X_h)\rho(X_f, X_h)\rho(X_h, X_f)\mu_A(\phi_A(X_f))\mu_A(X_g) \cdot g
\]
\[
+ \rho(X_f, X_g + X_h)\mu_A(\phi_A(X_g))\mu_A(X_f) \cdot f
\]
\[
- \rho(X_f, X_g)\rho(X_g, X_h)\rho(X_h, X_f)\mu_A(\phi_A(X_g))\mu_A(X_h) \cdot f.
\]
By \[4.6\] again, this time vice versa, we obtain
\[
0 = d_\mu(X_f, X_g, X_h) = -\rho(g, \Omega)\rho(h, 2\Omega)\rho(f, h, f)\rho(h, f, g)\{\phi(f), \{g, h\}_\rho\}_\rho
\]
\[
- \rho(f + g, h)\rho(g, \Omega)\rho(h, 2\Omega)\{\phi(h), \{f, g\}_\rho\}_\rho
\]
\[
- \rho(f, g + h)\rho(g, \Omega)\rho(h, 2\Omega)\{\phi(g), \{h, f\}_\rho\}_\rho.
\]
With this result, we can write
\[
\rho(h, f)\{\phi(f), \{g, h\}_\rho\}_\rho + \rho(g, h)\{\phi(h), \{f, g\}_\rho\}_\rho + \rho(f, g)\{\phi(g), \{h, f\}_\rho\}_\rho = 0.
\]
We continue with the checking of the following relation
\[
\{[f, g]_\rho, \phi(h)\}_\rho = \rho(g, h + P)[\{f, h\}_\rho, \phi(g)]_\rho + [\phi(f), \{h, g\}_\rho\}_\rho.
\]
Corollary 4.12. We have

\[
\{[f, g]_\rho, \phi(h)\}_\rho = -\rho(\Omega, h)\Omega(X_{[f, g]_\rho}, X_{\phi(h)}) = \rho(\Omega, h)\rho(f + g - \Omega, h - \Omega)\Omega(X_{\phi(h)}, X_{[f, g]_\rho}).
\]

Now, by using relation (4.4), we have

\[
\{[f, g]_\rho, \phi(h)\}_\rho = \rho(\Omega, h)\rho(f + g - \Omega, h - \Omega)\mu_A(X_{\phi(h)}) \cdot [f, g]_\rho.
\]

In the next, the relations (4.4) and (4.5) give us

\[
\{[f, g]_\rho, \phi(h)\}_\rho = \rho(f + g, h)\rho(f + g, -\Omega)[\mu_A(X_h) \cdot f, \phi(g)]_\rho
\]

\[
+ \rho(f + g, h)\rho(f + g, -\Omega)\rho(X_h, f)[\phi(f), \mu_A(X_h) \cdot g]_\rho
\]

\[
= \rho(g, h - \Omega)[\{f, h\}_\rho, \phi(g)]_\rho + [\phi(f), \{g, h\}_\rho].
\]

Lemma 4.8 implies the following

Corollary 4.12. We have

\[
[X_f, X_g]_\rho = -X_{\Omega(X_f, X_g)} = \rho(\Omega, X_f, X_g)_\rho.
\]

The above theorem is the key to constructing Hamiltonian derivation and Poisson bracket on the specific Hom-\(\rho\)-commutative algebras. To illustrate an application of this lemma, let us state the following example.

Let’s go back to the Example 3.6, in this example if we set \(a = b = 1\) then \((A_q^2, \rho, \phi = Id, [, , ]_\rho = 0)\) is a Hom-\(\rho\)-Lie algebra. For \(\phi = Id\) and \(k = 0\), \(\frac{\partial}{\partial x}\) and \(\frac{\partial}{\partial y}\) are \(\phi^0, \rho\)-derivation. Thus, for the extended hyperplane \(A_q^2\) and space \(\rho\)-Der\(A_q^2\), assuming that \(\Omega = dy \wedge dx\) and \(|f| = (f_1, f_2)\), the Hamiltonian \(\phi, \rho\)-derivation associated to \(f \in A_q^2\) has the following expression

\[
X_f = q^{1-f_1}\mu_{A_q^2}(\frac{\partial}{\partial y}) \cdot f \frac{\partial}{\partial x} - q^{f_2}\mu_{A_q^2}(\frac{\partial}{\partial x}) \cdot f \frac{\partial}{\partial y}.
\]

So, the Poisson bracket corresponding to \(\phi_{A_q^2}\) gives as follows

\[
\{f, g\}_\rho = -\rho(\Omega, g)\mu_{A_q^2}(X_f)g = -\rho(x + y, g)\{q^{1-f_1}(\mu_{A_q^2}(\frac{\partial}{\partial y}) \cdot f)(\mu_{A_q^2}(\frac{\partial}{\partial x}) \cdot g) - q^{f_2}(\mu_{A_q^2}(\frac{\partial}{\partial x}) \cdot f)(\mu_{A_q^2}(\frac{\partial}{\partial y}) \cdot g)\}.
\]

Example 4.13. This example is intended to give us a Poisson structure on quaternion algebra \(H\) (Example 2.4). Let us to define the Poisson bracket \(\{., .\}_\rho\) on \(H\) by the following structure

\[
\{i, i\}_\rho = 0, \{j, j\}_\rho = 0, \{k, k\}_\rho = 0, \{i, j\}_\rho = -\{j, i\}_\rho = k, \{k, i\}_\rho = -\{i, k\}_\rho = j, \{j, k\}_\rho = -\{k, j\}_\rho = i.
\]

So, \((H, \{., .\}_\rho, \phi_H)\) is Poisson Hom-\(\rho\)-commutative algebra.
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