Reflexive combinatory algebras

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Abstract

We introduce the notion of reflexivity for combinatory algebras. Reflexivity can be thought of as an equational counterpart of the Meyer–Scott axiom of combinatory models, which indeed allows us to characterise an equationally definable counterpart of combinatory models. This new structure, called strongly reflexive combinatory algebra, admits a finite axiomatisation with seven closed equations, and the structure is shown to be exactly the retract of combinatory models. Lambda algebras can be characterised as strongly reflexive combinatory algebras which are stable. Moreover, there is a canonical construction of a lambda algebra from a strongly reflexive combinatory algebra. The resulting axiomatisation of lambda algebras by the seven axioms for strong reflexivity together with those for stability is shown to correspond to the axiomatisation of lambda algebras due to Selinger [J. Funct. Programming, 12(6), 549–566, 2002].

Keywords: combinatory algebra; reflexivity; combinatory model; lambda algebra; lambda model

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1 Introduction

The paper is written in the hope that the study of structures more general than the established notion of models of the lambda calculus (i.e., lambda algebras) should lead to a better understanding of the properties of these models. To this end, we introduce the notion of reflexivity in a general setting for combinatory algebras, and show how this property relates to models of the lambda calculus.

Recall that a lambda algebra is a combinatory algebra which satisfies Curry’s five closed axioms (cf. Definition 6.11). Apart from this succinct axiomatisation, they have nice structural properties: lambda algebras are exactly the retracts of lambda models [1], the first-order models of lambda calculus characterised as lambda algebras satisfying the Meyer–Scott axiom (cf. Definition 4.6). Moreover, lambda algebras correspond to the reflexive objects in cartesian closed...
categories \[6, 12\]. However, the choice of axioms for lambda algebras by Curry is elusive and looks arbitrary \[1, 11, 13, 6\].

On the other hand, lambda models can be characterised independently from lambda algebras as combinatory models which are stable \[11\]. Combinatory models can be characterised simply as combinatory algebras satisfying the Meyer–Scott axiom:

\[
\forall c \in A (ac = bc) \implies ea = eb
\]

where \(e\) is a distinguished constant (often defined as \(s(\mathbf{ki})\)). Combinatory models are sufficient for interpreting lambda calculus. Moreover, there is a canonical way of stabilising a combinatory model to obtain a lambda model. In this sense, combinatory models encapsulate the essence of lambda models.

The observation above allows us to identify reflexivity as a fundamental property of combinatory algebras, which comes into play as follows: the combinatory completeness implies that there is a surjection \(\varphi: a \mapsto ax\) from a combinatory algebra \(A\) to its polynomial algebra \(A[x]\), whose section is provided by any choice of a defined \(\lambda\)-abstraction. The problem is that familiar abstraction mechanisms fail to respect the equality on the polynomials: the question is thus under what conditions this succeeds. Since \(\varphi\) is surjective, it induces an equivalence relation \(\sim_A\) on \(A\) which makes \(A/\sim_A\) a combinatory algebra isomorphic to \(A[x]\). The key observation is that the relation \(\sim_A\) is generated by finite schemas of equations on the elements of \(A\). The above problem is then reduced to the condition, called reflexivity, that the constant \(e\) preserves these equations, namely

\[
a \sim_A b \implies ea = eb,
\]

or in terms of polynomial algebra, \(ax = bx \implies ea = eb\). Reflexivity yields seven simple universal sentences on \(A\), which are equivalent to the requirement that an alternative choice of lambda abstraction, denoted \(\lambda^t\), is a well-defined operation on the polynomial algebra with one indeterminate. Then, taking \(\lambda^t\)-closures of these seven sentences yields seven closed equations; these equations correspond to the requirement that the polynomial algebra be reflexive, or equivalently, that \(\lambda^t\) be well-defined on polynomial algebras with any finite numbers of indeterminates.

We call a combinatory algebra satisfying the seven closed equations as mentioned above a strongly reflexive combinatory algebra. The class of strongly reflexive combinatory algebras provides the algebraic (i.e., equational) counterpart to the notion of combinatory models in that they are precisely the retracts of combinatory models. Indeed, the condition \((1.1)\) can be considered as an equational counterpart of the Meyer–Scott axiom. On the other hand, combinatory models can be characterised as strongly reflexive combinatory algebras satisfying the Meyer–Scott axiom. The relation between strongly reflexive combinatory algebras and lambda algebras can then be captured by stability \[11, 5.6.4 (ii)\]. Moreover, every strongly reflexive combinatory algebra can be made into a stable one (i.e., a lambda algebra) with an appropriate choice of constants. This passage to a lambda algebra also manifests itself in another form: we can associate a cartesian closed monoid (and thus a cartesian closed category

\[\text{See also Lambek \[9\], Freyd \[2\], Hindley and Seldin \[5\] Chapter 8, 8B} \] for discussions on the finite axiomatisation of Curry algebras (namely, the models of \(\lambda_{\beta\eta}\)).
with a reflexive object) to a strongly reflexive combinatory algebra, from which the above mentioned lambda algebra is obtained. Furthermore, the resulting axiomatisation of lambda algebras with the seven closed equations and the axiom of stability naturally corresponds to the axiomatisation of lambda algebras due to Selinger [13]. Thus, the notion of strongly reflexive combinatory algebras serves as a common generalisation of those of combinatory models and lambda algebras, which moreover is finitely axiomatisable (see Figure 1).

Throughout this paper, we work with combinatory pre-models, combinatory algebras extended with distinguished elements $i$ and $e$. Of course, these elements can be defined in terms of elements $k$ and $s$ as $i = skk$ and $e = s(ki)$. Nevertheless, the inclusion of $e$ as a primitive, in particular, may be justified given its fundamental role in reflexivity. Accordingly, most of the notions for combinatory algebras mentioned above (such as polynomial algebra, reflexivity, strong reflexivity, stability) will be introduced for combinatory pre-models. The exceptions are lambda algebras and lambda models, which are defined with respect to the conventional notion of combinatory algebras with $k$ and $s$ as the only primitive constants.

**Organisation**  In Section 2 we introduce the notion of combinatory pre-models and establish some elementary properties of polynomial algebras. We also introduce an alternative representation of a polynomial algebra without using indeterminates. In Section 3 we introduce the notion of reflexivity for com-
binatory pre-models, and show that this can be characterised by seven simple universal sentences. We also introduce a new abstraction mechanism, denoted $\lambda^\dagger$, in terms of which reflexivity can be rephrased. In Section 4, we introduce the notion of strong reflexivity for combinatory pre-models. We show that the class of strongly reflexive combinatory pre-models is axiomatisable with a finite set of equations which can be obtained by taking $\lambda^\dagger$-closures of seven axioms of reflexivity. We then show that strongly reflexive combinatory pre-models are exactly the retracts of combinatory models. In Section 5, we generalise the construction of a cartesian closed monoid from a lambda algebra to the setting of strongly reflexive combinatory pre-models. In Section 6, we introduce the notion of stability for strongly reflexive combinatory pre-models and characterise lambda algebras as stable strongly reflexive combinatory pre-models. We then clarify how this characterisation of lambda algebras corresponds to that of Selinger.

2 Combinatory pre-models

We begin with a preliminary on quotients of applicative structures.

Definition 2.1. An applicative structure is a pair $A = (A, \cdot)$ where $A$ is a set and $\cdot$ is a binary operation on $A$, called an application. The application $a \cdot b$ is often written as $(ab)$, and parentheses are omitted following the convention of association to the left.

A homomorphism between applicative structures $A = (A, \cdot_A)$ and $B = (B, \cdot_B)$ is a function $f: A \rightarrow B$ such that

$$f(a \cdot_A b) = f(a) \cdot_B f(b)$$

for each $a, b \in A$.

In what follows, we fix an applicative structure $A = (A, \cdot)$.

Definition 2.2. A congruence relation (or simply a congruence) on $A$ is an equivalence relation $\sim$ on $A$ such that

$$a \sim b \text{ and } c \sim d \implies a \cdot c \sim b \cdot d$$

for each $a, b, c, d \in A$.

Notation 2.3. If $\sim$ is a congruence on $A$, we often write $a$ for the equivalence class $[a]_\sim$ of $\sim$ whenever it is clear from the context. In this case, we write $a \sim b$ for $[a]_\sim = [b]_\sim$. The convention also applies to the other congruence relations in this paper.

Definition 2.4. Let $\sim$ be a congruence on $A$. The quotient of $A$ by $\sim$ is an applicative structure $A/{\sim} = (A/{\sim}, \cdot)$

where $a \cdot b = a \cdot_A b$. There is a natural homomorphism $\pi_{\sim}: A \rightarrow A/{\sim}$ defined by $\pi_{\sim}(a) = a$.

For any binary relation $R$ on $A$, there is a smallest congruence $\sim_R$ on $A$ containing $R$, which is inductively generated by the following rules:
1. if \( a R b \), then \( a \sim R b \),
2. \( a \sim R a \),
3. if \( a \sim R b \), then \( b \sim R a \),
4. if \( a \sim R b \) and \( b \sim R c \), then \( a \sim R c \),
5. if \( a \sim R b \) and \( c \sim R d \), then \( a \cdot c \sim R b \cdot d \),
where \( a, b, c, d \in A \).

Proposition 2.5. Let \( R \) be a binary relation on \( A \) and let \( f : A \to B \) be a homomorphism of applicative structures such that \( f(a) = f(b) \) for each \( a R b \). Then, there exists a unique homomorphism \( \tilde{f} : A/\sim_R \to B \) such that \( \tilde{f} \circ \pi \sim_R = f \).

Proof. Define \( \tilde{f} \) by \( \tilde{f}(a) = f(a) \). By induction on \( \sim_R \), one can show that \( a \sim R b \) implies \( f(a) = f(b) \) for each \( a, b \in A \). The fact that \( \tilde{f} \) is a homomorphism is clear. The uniqueness of \( \tilde{f} \) follows from the fact that \( \pi \sim_R \) is surjective.

Throughout the paper, we work with the following notion of combinatorial algebras where constants \( i \) and \( e \) are given as primitives.

Definition 2.6. A combinatorial pre-model is a structure \( A = (A, \cdot, k, s, i, e) \) where \( (A, \cdot) \) is an applicative structure and \( k, s, i, e \) are elements of \( A \) such that
1. \( kab = a \),
2. \( sabc = ac(bc) \),
3. \( ia = a \),
4. \( eab = ab \)
for each \( a, b, c \in A \). The reduct \( (A, \cdot, k, s) \) is called a combinatorial algebra.

A homomorphism between combinatorial pre-models \( A = (A, \cdot, k_A, s_A, i_A, e_A) \) and \( B = (B, \cdot, k_B, s_B, i_B, e_B) \) is a homomorphism between applicative structures \( (A, \cdot) \) and \( (B, \cdot) \) such that \( f(k_A) = k_B, f(s_A) = s_B, f(i_A) = i_B \), and \( f(e_A) = e_B \). Combinatory pre-models \( A \) and \( B \) are isomorphic if there exists a bijective homomorphism between \( A \) and \( B \). Homomorphisms and isomorphisms between combinatorial algebras are defined similarly.

In what follows, homomorphisms mean homomorphisms between combinatorial pre-models unless otherwise noted.

Notation 2.7. We often use a combinatorial pre-model denoted by the letter \( A \). Unless otherwise noted, we assume that \( A \) has the underlying structure \( A = (A, \cdot, k, s, i, e) \).

We recall the construction of a polynomial algebra, and establish its basic properties.

Definition 2.8. Let \( S \) be a set. The set \( T(S) \) of terms over \( S \) is inductively generated by the following rules:
1. \( a \in T(S) \) for each \( a \in S \),
2. if \(t, u \in T(S)\), then \((t, u) \in T(S)\).

Note that \(T(S)\) is a free applicative structure \(\langle T(S), \cdot \rangle\) over \(S\) where \(t \cdot u = (t, u)\).

In the rest of this section, we work over a fixed combinatory pre-model \(A\). We assume that a countably infinite set \(X = \{x_i \mid i \geq 1\}\) of distinct indeterminates is given. For each term \(t \in T(X + A)\), \(\text{FV}(t)\) denotes the set of indeterminates that occur in \(t\). As usual, \(t\) is said to be closed if \(\text{FV}(t) = \emptyset\).

**Definition 2.9.** Let \(\approx_x\) be the congruence relation on \(T(X + A)\) generated from the following basic relation:

1. \(((k, s), t) \approx_x s\),
2. \(((s, s), t, u) \approx_x ((s, u), (t, u))\),
3. \((i, s) \approx_x s\),
4. \(((e, s), t) \approx_x (s, t)\),
5. \((a, b) \approx_x ab\),

where \(a, b \in A\) and \(s, t, u \in T(X + A)\). The polynomial algebra \(A[X]\) over \(A\) is a combinatory pre-model

\[
A[X] = \langle A[X], *, k, s, i, e \rangle
\]

(2.1)

where \(\langle A[X], * \rangle\) is the quotient of \(T(\{X\} + A)\) with respect to \(\approx_x\). There is a homomorphism \(\sigma_A : A \to A[X]\) defined by \(\sigma_A(a) = a\).

Similarly, for each \(n \in \mathbb{N}\), the congruence relation \(\approx_{x_1, \ldots, x_n}\) on \(T(\{x_1, \ldots, x_n\} + A)\) is generated from the five basic equations above. The polynomial algebra \(A[x_1, \ldots, x_n]\) over \(A\) in indeterminates \(x_1, \ldots, x_n\) is then defined as in (2.1), whose underlying set is the quotient of \(T(\{x_1, \ldots, x_n\} + A)\) with respect to \(\approx_{x_1, \ldots, x_n}\). Let \(\eta_n : A \to A[x_1, \ldots, x_n]\) be the homomorphism defined by \(\eta_n(a) = a\).

**Notation 2.10.**

1. We sometimes write \(\eta_A : A \to A[x_1]\) for \(\eta_1 : A \to A[x_1]\).
2. We use \(x, y, z\) for \(x_1, x_2, x_3\), respectively; thus, \(A[x] = A[x_1]\), \(A[x, y] = A[x_1, x_2]\), and \(A[x, y, z] = A[x_1, x_2, x_3]\). However, we sometimes use \(x\) for an arbitrary element of \(X\) (cf. Definition 2.16 and Definition 3.7). The meaning of \(x\) should be clear from the context.
3. Terms are often written without parentheses and commas; e.g., \((t, s)\) will be written simply as \(ts\). In most cases, the reader should be able to reconstruct the original terms following the convention of association to the left. However, there still remain some ambiguities; e.g., it is not clear whether \(ab\) for \(a, b \in A\) denotes \((a, b)\) or \(ab \in A\). In practice, this kind of distinction does not matter as terms are usually considered up to equality of polynomial algebras.

We recall some standard properties of polynomial algebras.
Lemma 2.11. Let $B = (\mathcal{B}_*, \mathcal{B}, \mathcal{s}_B, \mathcal{i}_B, \mathcal{e}_B)$ be a combinatory pre-model, and let $n \geq 1$. For each homomorphism $f: A \to B$ and elements $b_1, \ldots, b_n \in \mathcal{B}$, there exists a unique homomorphism $\mathcal{J}: A[x_1, \ldots, x_n] \to B$ such that $\mathcal{J} \circ \eta_n = f$ and $\mathcal{J}(x_i) = b_i$ for each $i \leq n$.

Proof. Let $f: A \to B$ be a homomorphism and $b_1, \ldots, b_n \in \mathcal{B}$. By the freeness of $\mathcal{T}((x_1, \ldots, x_n) + \mathcal{A})$, $f$ uniquely extends to a homomorphism of applicative structures $\mathcal{J}: \mathcal{T}((x_1, \ldots, x_n) + \mathcal{A}) \to B$ such that $\mathcal{J}(x_i) = b_i$ for $i \leq n$, $\mathcal{J}(a) = f(a)$ for $a \in \mathcal{A}$, and $\mathcal{J}((t, u)) = f(t) \cdot_B \mathcal{J}(u)$. Since $f$ is a homomorphism of combinatory pre-models, $\mathcal{J}$ satisfies the assumption of Proposition 2.14 hence it uniquely extends to a homomorphism $\mathcal{F}: A[x_1, \ldots, x_n] \to B$. Then, we have $\mathcal{F}(x_i) = f(x_i) = b_i$ and $\mathcal{F}(\eta_A(a)) = f(a)$ for each $a \in \mathcal{A}$. \qed

Similarly, we have the following for $A[X]$.

Lemma 2.12. Let $B = (\mathcal{B}_*, \mathcal{B}, \mathcal{s}_B, \mathcal{i}_B, \mathcal{e}_B)$ be a combinatory pre-model. For any homomorphism $f: A \to B$ and a sequence $(b_n)_{n \geq 1}$ of elements of $\mathcal{B}$, there exists a unique homomorphism $\mathcal{J}: A[X] \to B$ such that $\mathcal{J} \circ \sigma_A = f$ and $\mathcal{J}(x_n) = b_n$ for each $n \geq 1$.

Remark 2.13. By Lemma 2.11, the construction $A \mapsto A[x]$ determines a functor $\mathcal{T}$ on the category of combinatory pre-models. The functor $\mathcal{T}$ sends each homomorphism $f: A \to B$ to the unique homomorphism $\mathcal{T}(f): A[x] \to B[x]$ such that $\mathcal{T}(f)(x) = x$ and $\mathcal{T}(f) \circ \eta_A = \eta_B \circ f$. By induction on $n \in \mathbb{N}$, one can show that $\mathcal{T}^n A \equiv (\cdots (\cdot (A[x]) \cdots )|x])$ is isomorphic to $A[x_1, \ldots, x_n]$ via the unique homomorphism

\[
\begin{align*}
h_n : A[x_1, \ldots, x_n] & \to \mathcal{T}^n A \\
h_n(x_i) & = \eta_{\mathcal{T}^{n-1} A} \circ \cdots \circ \eta_A(n) \\
\end{align*}
\]

(2.2)

such that $h_n \circ \eta_n = \eta_{\mathcal{T}^{n-1} A} \circ \cdots \circ \eta_A$ and $h_n(x_i) = (\eta_{\mathcal{T}^{n-1} A} \circ \cdots \circ \eta_A)(x)$ for each $i \leq n$.

Notation 2.14. As usual, the interpretation of a term $t \in \mathcal{T}(X + \mathcal{A})$ in $A$ under a valuation $\rho: \mathbb{N} \to \mathcal{A}$ is the unique homomorphism $\llbracket t \rrbracket_\rho: \mathcal{T}(X + \mathcal{A}) \to A$ that extends the identity function $\text{id}_A: A \to A$ with respect to the sequence $(\rho(n))_{n \in \mathbb{N}}$. When $t$ is a closed term, the interpretation $\llbracket t \rrbracket_\rho$ does not depend on $\rho$, and thus can be written as $\llbracket t \rrbracket$. Because of this, we often identify a closed term $t$ with its interpretation $\llbracket t \rrbracket \in \mathcal{A}$ and treat $t$ as if it is an element of $\mathcal{A}$. It should be clear from the context whether closed terms are treated as terms (or elements of $A[X]$) or elements of $\mathcal{A}$ via the interpretation. When a closed term $t$ is treated as an element of $A[X]$, however, this distinction is irrelevant since we have $t \equiv_X \llbracket t \rrbracket$. The similar notational convention applies to closed terms of $\mathcal{T}((x_1, \ldots, x_n) + \mathcal{A})$.

Next, recall that an object $X$ of a category is a retract of another object $Y$ if there exist morphisms $s: X \to Y$ and $r: Y \to X$ such that $r \circ s = \text{id}_X$. In the context of the combinatory pre-model $A$, we have the following.

Proposition 2.15.

1. $A[x_1, \ldots, x_n]$ is a retract of $A[X]$ for each $n \in \mathbb{N}$.

2. $(A[X])[x]$ is isomorphic to $A[X]$.

\[\text{Remark:}\] Here $h_n(x_i) = x$. We also define $\mathcal{T}^0 A = A$.

\[\text{Example:}\] term $(e, ((s, k), i))$ will be identified with $e(ski)$. 

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Lemma 2.17. For each $x \in \mathcal{X}$, there exists a unique homomorphism $f : A[x] \to (A[x])[x]$ such that $f \circ \sigma_A = \eta_n$ and $f(x_i) = x_{\min(n,i)}$ for each $i \geq 1$. In the other direction, by Lemma 2.11 there is a unique homomorphism $g : A[x] \to A[x]$ such that $g \circ \sigma_A = \sigma_A$ and $g(x_i) = x_{i+1}$ for each $i \geq 1$. By Lemma 2.11 $g$ extends uniquely to a homomorphism $h : (A[x])[x] \to A[x]$ such that $h(x) = x_1$ and $h \circ \eta_A = g$. Then, it is straightforward to show that $f$ and $h$ are mutual inverse.

As a corollary of Proposition 2.15[1], we have $t \approx x_n \cdots x_1 u \iff t \approx x u$ for each $t, u \in \mathcal{T}(\{x_1, \ldots, x_n\} + A)$.

Before proceeding further, we recall one of the standard abstraction mechanisms for combinatory algebras (cf. Barendregt [1, 7.3.4]).

**Definition 2.16.** For each $t \in \mathcal{T}(\mathbf{x} + A)$ and $x \in \mathbf{x}$, define $\lambda^x.t \in \mathcal{T}(\mathbf{x} + A)$ inductively by

1. $\lambda^x.x = \mathbf{i}$,
2. $\lambda^x.a = \mathbf{ka}$,
3. $\lambda^x.(t, u) = \mathbf{s}(\lambda^x.t)(\lambda^x.u)$,

where $a \in \mathbf{x} + A$ such that $a \neq x$. For each $n \geq 1$, we write $\lambda^x x_1 \cdots x_n.t$ for $\lambda^x x_1 \cdots \lambda^x x_n.t$.

**Lemma 2.17.** For each $t, u \in \mathcal{T}(\mathbf{x} + A)$ and $x \in \mathbf{x}$, we have $(\lambda^x.t)u \approx x t[x/u]$, where $t[x/u]$ denotes the substitution of $u$ for $x$ in $t$.

**Proof.** By induction on the complexity of $t$. \qed

**Proposition 2.18.** $A[x_1, \ldots, x_n]$ is a retract of $A[x]$ for each $n \in \mathbb{N}$.

**Proof.** By Lemma 2.11 the identity $\text{id}_A : A \to A$ extends to a homomorphism $\text{id}_A : A[x] \to A$ such that $\text{id}_A(x) = \mathbf{i}$ and $\text{id}_A \circ \eta_1 = \text{id}_A$. Thus, $A$ is a retract of $A[x]$. By the functoriality of $\mathcal{T}$, it remains to show that $A[x, y] (\approx T^2A)$ is a retract of $A[x] (\approx T^1A)$. To see this, define the following terms (cf. Barendregt [1, Section 6.2]):

$$t = k, \quad f = \lambda^xxy.y, \quad [\cdot, \cdot] = \lambda^xxyz.zxy. \quad (2.3)$$

By Lemma 2.11 there exists a unique homomorphism $f : A[x, y] \to A[x]$ such that $f \circ \eta_2 = \eta_1$, $f(x) \approx x t$, and $f(y) \approx x f$. In the other direction, there exists a unique homomorphism $g : A[x] \to A[x, y]$ such that $g \circ \eta_1 = \eta_2$ and $g(x) \approx xy [x, y]$. Since $g(f(x)) \approx xy g(xt) \approx xy [x, y] t \approx xy x$, $g(f(y)) \approx xy g(xy f) \approx xy [x, y] f \approx xy y$, and $g \circ f \circ \eta_2 = \eta_2$, we must have $g \circ f = \text{id}_{A[x, y]}$ by Lemma 2.11. \qed

\*When $n = 0$, we define $x_{\min(n, i)} = \mathbf{i}$. 

The rest of the section concerns an alternative representation of $A[x]$ without using indeterminates. We begin with the following observation: since $(\lambda^x.t)x \approx_k t$ for each $t \in \mathcal{T}(\{x\} + A)$, a function $f: A \rightarrow A[x]$ defined by $f(a) = ax$ is surjective. Let $\sim_A$ be the equivalence relation on $A$ generated by the kernel

$$\{(a, b) \in A \times A \mid f(a) = f(b)\}$$

of $f$, and let $\pi_{\sim_A}: A \rightarrow A/\sim_A$ be the natural map onto the (set theoretic) quotient of $A$ by $\sim_A$. Then, $f$ uniquely extends to a bijection $\gamma: A/\sim_A \rightarrow A[x]$ such that $\gamma \circ \pi_{\sim_A} = f$ with an inverse $\lambda: A[x] \rightarrow A/\sim_A$ defined by $\lambda(t) = \lambda^x[t]$. Thus, $A[x]$ induces the following combinatory pre-model structure on $A/\sim_A$:

$$\tilde{A} = (A/\sim_A, \bullet, kk, ks, ki, ke)$$

where $a \bullet b = sabc$ (note that $sabc \approx_k ax(bx)$). The point of the following is that the relation $\sim_A$ can be characterised directly without passing through $A[x]$.

**Definition 2.19.** Define an applicative structure $A_1 = (A, \cdot_1)$ on $A$ by

$$a \cdot_1 b = sabc.$$  

For each $a \in A$, define $a_1 \in A_1$ by $a_1 = ka$.

Let $\sim_1$ be the congruence relation on $A_1$ generated by the following relation:

1. $k_1 \cdot_1 a \cdot_1 b \sim_1 a$,
2. $s_1 \cdot_1 a \cdot_1 b \cdot_1 c \sim_1 a \cdot_1 c \cdot_1 (b \cdot_1 c)$,
3. $i_1 \cdot_1 a \sim_1 a$,
4. $e_1 \cdot_1 a \cdot_1 b \sim_1 a \cdot_1 b$,
5. $(ka) \cdot_1 (kb) \sim_1 k(ab)$,
6. $(ka) \cdot_1 i \sim_1 a$,

where $a, b, c \in A$. Then, the structure

$$\tilde{A}_1 = (A/\sim_1, \cdot_1, k_1, s_1, i_1, e_1),$$

where $(A/\sim_1, \cdot_1)$ is the quotient of $A_1$ by $\sim_1$, is a combinatory pre-model.

The crucial axiom of $\sim_1$ is Definition 2.19(6), as can be seen from the proof of the following theorem.

**Theorem 2.20.** Combinatory pre-models $\tilde{A}_1$ and $A[x]$ are isomorphic.

**Proof.** First, a function $f: \mathcal{T}(\{x\} + A) \rightarrow A/\sim_1$ defined by $f(t) = \lambda^x.t$ is a homomorphism of applicative structures $\mathcal{T}(\{x\} + A)$ and $\tilde{A}_1$. It is also straightforward to check that $f$ preserves (1)–(5) of Definition 2.19. Hence, by Proposition 2.23 $f$ extends uniquely to a homomorphism $\bar{f}: A[x] \rightarrow \tilde{A}_1$ of the underlying applicative structures. Then, it is easy to see that $\bar{f}$ is a homomorphism.

$^5$More precisely, $\lambda(t) = [\lambda^x.t]\sim_A$.  

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of combinatory pre-models. In the other direction, a function \( g: A \to A[x] \) defined by \( g(a) = ax \) is a homomorphism of applicative structures \( A_1 \) and \( A[x] \), and it is easy to see that \( g \) preserves (1)–(6) of Definition 2.19. By a similar argument as above, \( g \) extends uniquely to a homomorphism \( g: \bar{A} \to A[x] \).

Lastly, we have \((f \circ g)(a) \sim_1 \lambda^* . ax \sim_1 xa \sim_1 a \) for each \( a \in \bar{A} \), where the last equality follows from the equation (6). Thus \( f \) and \( g \) are inverse to each other.

From the proof of Theorem 2.20, we can derive the following correspondence.

**Proposition 2.21.** For any combinatory pre-model \( A \), we have

\[
a \sim_1 b \iff ax \approx_x bx
\]

for each \( a, b \in \bar{A} \).

As an immediate consequence, we have the following.

**Corollary 2.22.** For any combinatory pre-model \( A \), we have \( ea \sim_1 a \) for each \( a \in \bar{A} \).

### 3 Reflexivity

The notion of reflexivity introduced below can be understood as an algebraic (rather than first-order) analogue of the Meyer–Scott axiom for combinatory models (cf. Definition 4.6). We still follow the convention of Notation 2.7.

**Definition 3.1.** A combinatory pre-model \( A \) is **reflexive** if

\[
a \sim_1 b \implies ea = eb
\]

for each \( a, b \in \bar{A} \).

In terms of polynomial algebras, reflexivity can be stated as follows (cf. Proposition 2.21).

**Lemma 3.2.** A combinatory pre-model \( A \) is reflexive if and only if \( ax \approx_x bx \) implies \( ea = eb \) for each \( a, b \in \bar{A} \).

In the rest of the paper, we sometimes use Lemma 3.2 implicitly.

Since the relation \( \sim_1 \) is generated from the equations on the elements on \( A \) (cf. Definition 2.19), reflexivity can be characterised by a set of simple universal sentences on \( A \).

**Proposition 3.3.** A combinatory pre-model \( A \) is reflexive if and only if it satisfies the following equations:

1. \( e(k_1 \cdot 1 a \cdot 1 b) = ea \),
2. \( e(s_1 \cdot 1 a \cdot 1 b \cdot 1 c) = e(a \cdot 1 c \cdot 1 (b \cdot 1 c)) \),
3. \( e(i_1 \cdot 1 a) = ea \),
4. \( e(e_1 \cdot 1 a \cdot 1 b) = e(a \cdot 1 b) \),
5. \( e((ka) \cdot 1 \cdot (kb)) = e(k(ab)) \),

6. \( e((ka) \cdot 1) = ea \),

7. \( e((ea) \cdot 1 \cdot (eb)) = e(a \cdot 1 b) \)

for each \( a, b, c \in A \).

Proof. \((\Rightarrow)\) Suppose that \( A \) is reflexive. Then (1)-(6) hold by Definition 2.19. Moreover, we have \( ea \cdot 1 e b \sim 1 \cdot a \cdot b \) by Corollary 2.22 from which (7) follows.

\((\Leftarrow)\) Suppose that \( A \) satisfies (1)-(7) above. Let \( A_1 = (A, \cdot, 1) \) be the applicative structure defined as in Definition 2.19, and define another applicative structure \( A_e = (A^*, \cdot, 1) \) by \( e a \cdot 1 e b = e((ea) \cdot 1 (eb)) \). Then, a function \( f: A \to A^* \) given by \( f(a) = ea \) is a homomorphism from \( A_1 \) to \( A_e \) by (7), which moreover preserves relations (1)-(6) of Definition 2.19 by (1)-(6) above. Thus, \( f \) extends uniquely to a homomorphism from \( \overline{A}_1 \) to \( A_e \). In particular, \( a \sim 1 b \) implies \( ea = e b \). Hence \( A \) is reflexive.

Remark 3.4. The condition (7) in Proposition 3.3 does not have a counterpart in Definition 2.19, but it seems to be needed here. This is because \( e \) is required to preserve the congruence relation generated by (1)-(6) of Definition 2.19 and not the (weaker) equivalence relation.

Since the seven equations of Proposition 3.3 are simple universal sentences, we have the following corollary. Here, a substructure of a combinatory pre-model \( A \) is a subset \( B \subseteq A \) which contains \( k, s, i, e \), and is closed under the application of \( A \).

Corollary 3.5. Reflexivity is closed under substructures and homomorphic images. In particular, it is closed under retracts.

As a consequence, reflexivity is preserved under the addition of further indeterminates to a polynomial algebra.

Proposition 3.6. For any combinatory pre-model \( A \), if \( A[x] \) is reflexive, then \( A[x_1, \ldots, x_n] \) is reflexive for each \( n \in \mathbb{N} \).

Proof. Immediate from Corollary 3.5 and Proposition 2.18.

Note that we do not necessarily have that \( A[x] \) is reflexive when \( A \) is.

Next, we introduce an alternative abstraction mechanism for combinatory pre-models. This abstraction mechanism allows us to see reflexivity as a requirement that the equality of \( A[x] \) be preserved by the abstraction.

Definition 3.7. Let \( A \) be a combinatory pre-model. For each \( t \in T(X + A) \) and \( x \in X \), define \( \lambda^1 x.t \in T(X + A) \) inductively by

1. \( \lambda^1 x.x = e 1 \),

2. \( \lambda^1 x.a = e(ka) \),

3. \( \lambda^1 x.(a, x) = ea \),

4. \( \lambda^1 x.(t, u) = e(s(\lambda^1 x.t)(\lambda^1 x.u)) \) otherwise,

where \( a \in X + A \) such that \( a \neq x \). As in Lemma 2.17 we have \( (\lambda^1 x.t)u \sim_x t[x/u] \) for each \( t, u \in T(X + A) \).
First, we note that the choice of abstraction mechanisms does not affect the isomorphism between $A[x]$ and $A_1$ in Theorem 2.20.

**Proposition 3.8.** For any combinatory pre-model $A$, we have $\lambda^* x.t \sim_1 \lambda^1 x.t$ for each $t \in \mathcal{T}(\{x\} + A)$.

**Proof.** Immediate from Proposition 2.21.

Nevertheless, $\lambda^1$-abstraction enjoys some properties which $\lambda^*$-abstraction need not. The following is crucial for our development.

**Lemma 3.9.** If $A$ is a reflexive combinatory pre-model, then $e(ea) = ea$ for each $a \in A$. In particular, we have $e(\lambda^1 x.t) = \lambda^1 x.t$ for each $t \in \mathcal{T}(\{x\} + A)$.

**Proof.** By Corollary 2.22 and the definition of reflexivity.

We can now characterise reflexivity in terms of $\lambda^1$-abstraction. The proposition below says that reflexivity amounts to the requirement that the mapping $t \mapsto \lambda^1 x.t$ be well-defined on the polynomials with one indeterminate.

**Proposition 3.10.** A combinatory pre-model $A$ is reflexive if and only if 

$t \approx_\pi u \Rightarrow \lambda^1 x.t = \lambda^1 x.u$

for each $t, u \in \mathcal{T}(\{x\} + A)$.

**Proof.** Suppose that $A$ is reflexive, and let $t \approx_\pi u$. Then $\lambda^1 x.t \sim_1 \lambda^1 x.u$ by Proposition 3.8 and so $e(\lambda^1 x.t) = e(\lambda^1 x.u)$ by reflexivity. By Lemma 3.9, we obtain $\lambda^1 x.t = \lambda^1 x.u$. The converse is immediate from Definition 3.7(3).

The reflexivity of polynomial algebras admits similar characterisations.

**Corollary 3.11.** For each $n \in \mathbb{N}$, the following are equivalent:

1. $A[x_1, \ldots, x_n]$ is reflexive.
2. $t \approx_{x_1 \ldots x_{n+1}} u \Rightarrow \lambda^1 x_{n+1}.t \approx_{x_1 \ldots x_n} \lambda^1 x_{n+1}.u$ for each $t, u \in \mathcal{T}(\{x_1, \ldots, x_{n+1}\} + A)$.
3. $tx_{n+1} \approx_{x_1 \ldots x_{n+1}} ux_{n+1} \Rightarrow et \approx_{x_1 \ldots x_n} eu$ for each $t, u \in \mathcal{T}(\{x_1, \ldots, x_n\} + A)$.

**Proof.** By Remark 2.13, it suffices to show that items 2 and 3 are equivalent to the reflexivity of $T^n A$. To this end, consider the following commutative diagram:
Here

- $h_k : A[x_1, \ldots, x_k] \to T^k A$ is the isomorphism of (2.2) for each $k \in \mathbb{N}$;
- $\varphi : T(\{x_1, \ldots, x_n\} + A) \to T^n A$ is the unique extension of $\eta_{T^n-1 A} \circ \cdots \circ \eta_{T A}$ such that $\varphi(x_i) = (\eta_{T^n-1 A} \circ \cdots \circ \eta_{T A})(x)$ for each $i \leq n$;
- $\psi : T(\{x_1, \ldots, x_{n+1}\} + A) \to T(\{x\} + T^n A)$ is the unique extension of $\varphi$ such that $\psi(x_{n+1}) = x$;
- each $\pi$ is the quotient map with respect to the congruence relation of Definition 2.9;
- the other unnamed maps are the natural inclusions.

By induction on the complexity of terms, one can show that

$$\varphi(\lambda^t x_{n+1}.t) = \lambda^t x. \psi(t)$$

(3.1)

for each $t \in T(\{x_1, \ldots, x_{n+1}\} + A)$. Then, since $\psi$ is surjective, $T^n A$ is reflexive if and only if

$$\psi(t) \approx_x \psi(u) \Rightarrow \lambda^t x. \psi(t) = \lambda^t x. \psi(u)$$

(3.2)

for each $t, u \in T(\{x_1, \ldots, x_{n+1}\} + A)$ by Proposition 3.11. By equation (3.1) and the commutativity of the above diagram, (3.2) is equivalent to item 2.

By the similar argument using Lemma 3.9, $T^n A$ is reflexive if and only if

$$\varphi(t)x \approx_x \varphi(u)x \Rightarrow e \varphi(t) = e \varphi(u)$$

for each $t, u \in T(\{x_1, \ldots, x_n\} + A)$, which is equivalent to

$$\psi(tx_{n+1}) \approx_x \psi(ux_{n+1}) \Rightarrow \varphi(\epsilon t) = \varphi(\epsilon u)$$

(3.3)

for each $t, u \in T(\{x_1, \ldots, x_n\} + A)$. By the commutativity of the above diagram, (3.3) is equivalent to item 3.

To close this section, we show that the reflexivity of $A$ allows us to represent $A[x]$ by a structure on the fixed-points of $\epsilon$. The reader should recall some notations from Definition 2.19.

**Definition 3.12.** For a reflexive combinatory pre-model $A$, define

$$A^* = \{a \in A \mid \epsilon a = a\},$$

or equivalently $A^* = \{\epsilon a \mid a \in A\}$ (cf. Lemma 3.9).

Now, define a combinatory pre-model structure on $A^*$ by

$$A^* = (A^*, \circ, e k_1, e s_1, e i_1, e e_1),$$

where $a \circ b = \epsilon(a \cdot_1 b)$ for each $a, b \in A^*$.

**Proposition 3.13.** If $A$ is a reflexive combinatory pre-model, then $A^* \simeq \check{A}_1$, and hence $A^* \simeq A[x]$.
Proof. Let \( f : A \to A^* \) be a function defined by \( f(a) = ea \). As the proof of Proposition 3.3 shows, \( f \) extends uniquely to an isomorphism \( \bar{f} : \bar{A}_1 \to A^* \) of applicative structures. By the very definition of \( \bar{A}_1 \) and \( A^* \), \( \bar{f} \) is a homomorphism of combinatory pre-models.

Remark 3.14. Constructions similar to \( A^* \) have appeared in the literature [8, 2, 13]. For example, Krivine [8, Section 6.3] defined an applicative structure \( B = (A^*, \cdot, k k, k s) \) with \( a \cdot b = s a b \) for a combinatory algebra \( A = (A, \cdot, k, s) \) under the slightly stronger condition than reflexivity. His condition consists of (1), (2), and (5) of Proposition 3.3 without \( e \) in front of both sides of the equations, together with the following weak form of stability:

\[
\forall a, b \in A[e(ka) = ka \& e(sab) = sab]. \tag{3.4}
\]

It is clear that the structures \( A^* \) and \( B \) coincide in Krivine’s context when one ignores \( i \) and \( e \); in this case, \( B \) is isomorphic to \( A[x] \). The same representation of \( A[x] \) as \( B \) can be found in Selinger [13, Proposition 4] in the case where \( A \) is a lambda algebra. In view of this, Proposition 3.13 generalises the construction of the previous works.

4 Strong reflexivity

In this section, we introduce a stronger notion of reflexivity, which can be seen as an algebraic analogue of the Meyer–Scott axiom for combinatory models. In the following, the conventions of Notation 2.7 and Notation 2.14 still apply.

Definition 4.1. A combinatory pre-model \( A = (A, \cdot, k, s, i, e) \) is strongly reflexive if \( A[x] \) is reflexive.

We first note the following.

Lemma 4.2. A combinatory pre-model \( A \) is strongly reflexive if and only if \( A[x_1, \ldots, x_n] \) is reflexive for each \( n \in \mathbb{N} \).

Proof. By Proposition 3.6.

In particular, strong reflexivity implies reflexivity. From the above lemma, we obtain the following characterisation.

Proposition 4.3. The following are equivalent for a combinatory pre-model \( A \):

1. \( A \) is strongly reflexive.
2. \( tx \approx_t \bar{x} u \implies et \approx_t e u \) for each \( x \notin FV(tu) \) and \( t, u \in T(X + A) \).
3. \( t \approx_t \bar{x} u \implies \lambda x.t \approx_t \lambda x.u \) for each \( x \in X \) and \( t, u \in T(X + A) \).

Proof. (1 \( \iff \) 2) By Lemma 4.2, \( A \) is strongly reflexive if and only if Corollary 3.11(3) holds for each \( n \in \mathbb{N} \). This is equivalent to (2) by a suitable rearrangement of indeterminates \( FV(tu) \cup \{ x \} \) using Lemma 2.11.

(1 \( \iff \) 3) Similar.

Krivine defined \( e \) by \( e = \lambda x.y.x.y \), which makes (3) of Proposition 3.3 (without \( e \) in front of the left side) superfluous. Moreover, under Krivine’s condition, the equation (7) is derivable from the other equations. In short, Krivine’s condition consists of instantiations of three of Curry’s axioms for lambda algebras (cf. Definition 6.11) together with (3.4).
In particular, the polynomial algebra of a strongly reflexive combinatory pre-model is closed under the $\xi$-rule with respect to the abstraction mechanism $\lambda^t x$. In this way, each strongly reflexive combinatory pre-model gives rise to a model of the lambda calculus.

Next, we show that the class of strongly reflexive combinatory pre-models is axiomatisable with a finite set of closed equations which can be obtained by taking $\lambda^t$-closures of both sides of the equations (1)–(7) of Proposition 3.3.

**Theorem 4.4.** A combinatory pre-model $A$ is strongly reflexive if and only if it satisfies the following equations:

1. $\lambda^t xy. e(s(s(kk)x)y) = \lambda^t xy. ex$,
2. $\lambda^t xyz. e(s(s(s(ks)x)y)z) = \lambda^t xyz. e(s(sx)(sy))$,
3. $\lambda^t x.e(s(ki)x) = \lambda^t x.ex$,
4. $\lambda^t xy. e(s(s(ks)x)y) = \lambda^t xy. e(sx)$,
5. $\lambda^t xy. e(s(s(kk)(ky))) = \lambda^t xy. e(kxy)$,
6. $\lambda^t x.e(s(kx)i) = \lambda^t x.ex$,
7. $\lambda^t xy. e(s(ex)(ey)) = \lambda^t xy. e(sxy)$.

**Proof.** $(\Rightarrow)$ Suppose that $A$ is strongly reflexive. By Lemma 4.2, $A[x_1, \ldots, x_n]$ is reflexive for each $n \in \mathbb{N}$. Then, (1)–(7) follow from the reflexivity of $A[x_1, \ldots, x_n]$ ($n \leq 3$) together with Proposition 3.3 and Corollary 3.11(2).

$(\Leftarrow)$ Suppose that $A$ satisfies (1)–(7). Since these equations consist of constants $k, s, i, e$ only, $A[x]$ satisfies these equations as well. Then, $A[x]$ is reflexive by Proposition 3.3, i.e., $A$ is strongly reflexive.

Note that the equations (1)–(7) in Theorem 4.4 are closed in the sense that they correspond to terms built up from $k, s, i, e$ only. Thus, we have the following.

**Proposition 4.5.** If $A$ is strongly reflexive and $f: A \rightarrow B$ is a homomorphism of combinatory pre-models, then $B$ is strongly reflexive.

Next, we relate strongly reflexive combinatory pre-models and combinatory models.

**Definition 4.6 (Meyer [11]).** A combinatory model is a combinatory pre-model $A$ satisfying the Meyer–Scott axiom:

$$[\forall e \in A(ac = bc)] \Rightarrow e\!a = e\!b$$

for each $a, b \in A$.

Note that every combinatory model is reflexive by Lemma 3.2.

**Lemma 4.7.** Every combinatory model is strongly reflexive.
Proof. Let $A$ be a combinatory model. It suffices to show that Corollary 3.11 holds for $n = 1$. Let $t, u \in T(\{x\} + A)$ be such that $ty \approx xyuy$. Fix $c \in A$. Since $(\lambda^t x. t)xy \approx xy(\lambda^u x. u)xy$, we have $(\lambda^t x. t)c x \approx (\lambda^u x. u)c x$. This implies $e((\lambda^t x. t)c) = e((\lambda^u x. u)c)$ by the reflexivity of $A$, which is equivalent to $(\lambda^t x. e((\lambda^t x. t)x))c = (\lambda^u x. e((\lambda^t x. t)x))c$. Since $c$ was arbitrary and $A$ is a combinatory model, we have $e((\lambda^t x. t)c) = (\lambda^u x. e((\lambda^t x. t)x))c$.

Theorem 4.8. The following are equivalent for a combinatory pre-model $A$:

1. $A$ is strongly reflexive.
2. $A[X]$ is strongly reflexive.
3. $A[X]$ is reflexive.
4. $A[X]$ is a combinatory model.

Proof. (1 $\leftrightarrow$ 2) By Proposition 4.5 and Proposition 2.15 (1).
(2 $\leftrightarrow$ 3) By Corollary 3.5 and Proposition 2.15 (2).
(4 $\rightarrow$ 2) By Lemma 4.7.
(1 $\rightarrow$ 4) Suppose that $A$ is strongly reflexive. Let $t, u \in T(X + A)$, and suppose that $ts \approx xus$ for all $s \in T(X + A)$. Choose $n \in \mathbb{N}$ such that $t, u \in T(\{x_1, \ldots, x_n\} + A)$. Then $tx_{n+1} \approx x_1 \ldots x_{n+1}$, so by Lemma 4.7 and Corollary 3.11(3), we have $et \approx x_1 \ldots x_{n+1} eu$. Then $et \approx x eu$.

It is known that lambda algebras are exactly the retracts of lambda models. The following is its analogue for combinatory pre-models.

Theorem 4.9. A combinatory pre-model is strongly reflexive if and only if it is a retract of a combinatory model.

Proof. (⇒) By Proposition 2.15 (1) and Theorem 4.8
(⇐) By Lemma 4.7 and Proposition 4.5.

5 Cartesian closed monoids

In this section, we generalise the construction of a cartesian closed category with a reflexive object from a lambda algebra due to Scott [12] (see also Koymans [6]) to the setting of strongly reflexive combinatory pre-models. To this end, we construct a cartesian closed monoid from a strongly reflexive combinatory pre-model. The connection between cartesian closed monoids and cartesian closed categories with reflexive objects will be reviewed toward the end of this section. The reader is referred to Koymans [7, Chapter 2], Lambek and Scott [10, Part I, 15–17], and Hyland [5, 4] for a detailed account of cartesian closed monoids and their relation to untyped lambda calculus.

Throughout this section, we work over a fixed reflexive combinatory pre-model $A = (A, k, s, i, e)$. For each $a, b \in A$, define $a \circ b = \lambda^t x. a(bx)$.

Lemma 5.1. For each $a, b, c \in A$, we have
1. \( a \circ (b \circ c) = (a \circ b) \circ c, \)
2. \( i \circ a = a \circ i = e a. \)

**Proof.** We use Lemma 3.2 and Lemma 3.9.

1. Since \((a \circ (b \circ c))x \approx x a (b(c)x))\,x,\) we have
   \[ a \circ (b \circ c) = e(a \circ (b \circ c)) = e((a \circ b) \circ c) = (a \circ b) \circ c. \]

2. Since \((i \circ a) \approx x a x \approx x (a \circ i)x,\) we have
   \[ i \circ a = e(i \circ a) = e(a \circ i) = a \circ i. \]

We recall the following construction from Definition 3.12:

\[ A^* = \{ a \in A \mid e a = a \} = \{ e a \in A \mid a \in A \}. \]

Define \( I \in A^* \) by \( I = e i. \)

**Proposition 5.2.** The structure \((A^*, \circ, I)\) is a monoid with unit \( I.\)

**Proof.** By Lemma 5.1(1), it suffices to show that \( I \) is a unit of \( \circ. \) For any \( a \in A^*, \) we have \( I \circ a = i \circ a = e a = a \) by Lemma 5.1(2). Similarly, we have \( a \circ I = a. \)

**Definition 5.3** (Koymans [6, Definition 6.3]). Let \((M, \circ, I)\) be a monoid with unit \( I.\)

1. \( M \) has a paring if it is equipped with elements \( p,q \in M \) and an operation \( \langle \cdot, \cdot \rangle : M \times M \to M \) satisfying
   
   \[ \begin{align*}
   (a) & \quad p \circ \langle a, b \rangle = a \quad \text{and} \quad q \circ \langle a, b \rangle = b, \\
   (b) & \quad \langle a, b \rangle \circ c = \langle a \circ c, b \circ c \rangle
   \end{align*} \]
   
   for each \( a, b, c \in M. \)

2. \( M \) is cartesian closed if it has a paring together with an element \( \varepsilon \in M \) and an operation \( \Lambda(\cdot) : M \to M \) satisfying
   
   \[ \begin{align*}
   (c) & \quad \varepsilon \circ \langle p, q \rangle = \varepsilon, \\
   (d) & \quad \varepsilon \circ (\Lambda(a) \circ p, q) = a \circ \langle p, q \rangle, \\
   (e) & \quad \Lambda(\varepsilon) \circ \Lambda(a) = \Lambda(a), \\
   (f) & \quad \Lambda(\varepsilon \circ \langle a \circ p, q \rangle) = \Lambda(\varepsilon) \circ a
   \end{align*} \]
   
   for each \( a \in M. \)

Coming back to the context of the monoid \((A^*, \circ, I),\) define elements \( p, q, \varepsilon \in A^* \) and operations \( \langle \cdot, \cdot \rangle : A^* \times A^* \to A^* \) and \( \Lambda(\cdot) : A^* \to A^* \) by

\[ \begin{align*}
   p &= \lambda^1 x. x t, \\
   q &= \lambda^1 x. x f, \\
   \varepsilon &= \lambda^1 x. x t (x f), \\
   \langle a, b \rangle &= \lambda^1 x. [a x, b x], \\
   \Lambda(a) &= \lambda^1 x y. a[x, y].
\end{align*} \]

Here, \( t, f, \) and \( [\cdot, \cdot] \) are defined as in (2.3) using \( \lambda^1 \) instead of \( \lambda^*. \)
2. We verify (2c)–(2f) of Definition 5.3. Fix $A$, thus, the required equations follow from Lemma 3.2 and Lemma 3.9.

4. For a strongly reflexive combinatory pre-model $A$, when $A$ have an isomorphism between $A$ and $I$, this is the subject of the next section.

Theorem 5.4 (cf. Koymans [6] Lemma 7.2)]. Let $A$ be a combinatory pre-model.

1. If $A$ is reflexive, then the structure $(A^*, \circ, I, p, q, \langle \cdot, \cdot \rangle)$ is a monoid with paring.

2. If $A$ is strongly reflexive, then the structure $(A^*, \circ, I, p, q, \varepsilon, \langle \cdot, \cdot \rangle, \Lambda(\cdot))$ is a cartesian closed monoid.

Proof. We verify (1a)–(1b) of Definition 5.3. Fix $a, b, c \in A^*$:

$$(p \circ (a, b))x \approx_x p((a, b)x) \approx_x p(ax, bx) \approx_x (ax, bx)t \approx_x ax.$$ 

$$(q \circ (a, b))x \approx_x bx. \quad \text{(similar to the above)}$$

$$(a, b) \circ cx \approx_x (a, b)(cx) \approx_x [a(cx), b(cx)]$$

Thus, the required equations follow from Lemma 3.2 and Lemma 3.9.

We verify (2c)–(2f) of Definition 5.3. Fix $a \in A^*$:

$$(\varepsilon \circ (p, q)x) \approx_x \varepsilon(\langle p, q \rangle x) \approx_x [px, qx] \approx_x [px, qx]t([px, qx]f) \approx_x px(qx) \approx_x xt(xf) \approx_x e.$$ 

$$(\varepsilon \circ (\Lambda(a) \circ p, q)x) \approx_x \varepsilon((\Lambda(a) \circ p, q)x) \approx_x [\Lambda(a)(px), qx] \approx_x \varepsilon([\Lambda(a)(px), qx] \approx_x \varepsilon((\Lambda(a)(xt), xf) \approx_x \varepsilon([\Lambda(a)(xt), xf] \approx_x a[xt, xf] \approx_x a[px, qx] \approx_x a(p, q)x \approx_x (a \circ (p, q))x.$$ 

$$(\Lambda(\varepsilon) \circ \Lambda(a))xy \approx_{xy} \Lambda(\varepsilon)(\Lambda(a)xy) \approx_{xy} [\Lambda(a)xy]$$

$$\approx_{xy}[\Lambda(a)x, y]t([\Lambda(a)x, y]f) \approx_{xy} \Lambda(a)xy.$$ 

$$\Lambda(\varepsilon \circ (a \circ p, q))xy \approx_{xy} \varepsilon((a \circ p, q)(x, y) \approx_{xy} [\varepsilon([\Lambda(a)x, y], q[x, y])] \approx_{xy} \varepsilon([\Lambda(a)x, y], q[x, y]) \approx_{xy} \varepsilon([\Lambda(a)x, y], q[x, y]) \approx_{xy} \varepsilon([\Lambda(a)x, y], q[x, y]) \approx_{xy} \varepsilon([\Lambda(a)x, y], q[x, y]) \approx_{xy} \varepsilon([\Lambda(a)x, y], q[x, y]) \approx_{xy} \varepsilon([\Lambda(a)x, y], q[x, y])$$

Since $A$ is strongly reflexive (i.e., $A[x]$ is reflexive), we have the required equations as in the first case.

Now, let $\mathcal{C}_A$ be the Karoubi envelope of the monoid $(A^*, \circ, I)$ seen as a single-object category (see Koymans [6] Definition 4.1; Barendregt [11] 5.5.11]). If $A$ is strongly reflexive, then $\mathcal{C}_A$ is a cartesian closed category with a reflexive object $U = I$ by Theorem 5.4 (see Koymans [6] Section 7). Then, the homset $\mathcal{C}_A(T, U)$, where $T$ is a terminal object of $\mathcal{C}_A$, has a structure of a lambda algebra, and when $A$ is a lambda algebra, $\mathcal{C}_A(T, U)$ is isomorphic to $A$ [6] Sections 3 and 4. For a strongly reflexive combinatory pre-model $A$, in general, we would not have an isomorphism between $A$ and $\mathcal{C}_A(T, U)$. Instead, we obtain a certain lambda algebra structure on $(A, \cdot)$ induced by $\mathcal{C}_A(T, U)$, which is analogous to the construction of a lambda model from a combinatory model [11] Section 6]. This is the subject of the next section.
It is known that lambda models are combinatory models which are stable (Barendregt [11, 5.6.3, 5.6.6], Koymans [7, Section 1.4]). Having seen that strongly reflexive combinatory pre-models are the retracts of combinatory models in Section 4, it is now straightforward to establish an algebraic analogue of this fact for lambda algebras.

We begin with the following construction, which extends $e$ to finitely many arguments. Note that the conventions of Notation 2.7 and Notation 2.14 continue to apply.

**Definition 6.1** (Scott [12]). For a combinatory pre-model $A$, define $\varepsilon_n \in A$ for each $n \geq 1$ inductively by

$$
\varepsilon_1 = e, \quad \varepsilon_{n+1} = s(ke)(s(k\varepsilon_n)).
$$

One can observe the following by straightforward calculation.

**Lemma 6.2.** Let $A$ be a combinatory pre-model. For each $n \geq 1$, and for each $m \in \mathbb{N}$ and $s,t \in T(\{x_1, \ldots, x_m\} + A)$, we have

$$
\varepsilon_{n+1}st \approx x_1 \cdots x_m \varepsilon_n(st).
$$

In particular,

1. $\varepsilon_na x_1 \cdots x_n \approx x_1 \cdots x_n a x_1 \cdots x_n$,
2. $\varepsilon_na x_1 \cdots x_{n-1} \approx x_1 \cdots x_{n-1} e(ax_1 \cdots x_{n-1})$

for each $n \geq 1$ and $a \in A$.

If $A$ is a strongly reflexive combinatory pre-model, then the element $\varepsilon_na$ for each $a \in A$ admits a succinct characterisation.

**Lemma 6.3.** If $A$ is strongly reflexive, then

$$
\varepsilon_na = \lambda ^{\dagger}x_1 \cdots x_n ax_1 \cdots x_n
$$

for each $a \in A$ and $n \geq 1$.

**Proof.** By straightforward induction $n \in \mathbb{N}$. □

The construction $\varepsilon_n$ provides yet another characterisation of strong reflectivity.

**Proposition 6.4.** A combinatory pre-model $A$ is strongly reflexive if and only if

$$
ax_1 \cdots x_n \approx x_1 \cdots x_n bx_1 \cdots x_n \implies \varepsilon_na = \varepsilon_nb
$$

for each $n \geq 1$ and $a, b \in A$.

**Proof.** ($\Rightarrow$) By the $n$-time applications of Proposition 4.3 and Lemma 6.3 ($\Leftarrow$). We must show that $A[x]$ is reflexive. Let $t,u \in T(\{x\} + A)$ be such that $ty \approx xy$. Then, $(\lambda ^{\dagger}x.t)xy \approx xy$, and so $\varepsilon_2(\lambda ^{\dagger}x.t) = \varepsilon_2(\lambda ^{\dagger}x.u)$. By Lemma 6.2, we have $et \approx x e((\lambda ^{\dagger}x.t)x) \approx_x e((\lambda ^{\dagger}x.u)x) \approx_x eu$. □

---

7If $\varepsilon_n$ were defined by $\varepsilon_1 = e$ and $\varepsilon_{n+1} = e(s(ke)(s(k\varepsilon_n)))$, we could even show $\varepsilon_n = \lambda ^{\dagger}yx_1 \cdots x_n yx_1 \cdots x_n$. 

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The notion of stability for combinatory models also makes sense for strongly reflexive combinatory pre-models (cf. Barendregt [1, 5.6.4]; Meyer [11, Section 6]; Scott [12]).

**Definition 6.5.** Let \( A = (A, \cdot, k, s, i, e) \) be a strongly reflexive combinatory pre-model. Then, \( A \) is said to be **stable** if
\[
  k = \varepsilon_2 k, \quad s = \varepsilon_3 s, \quad i = \varepsilon_1 i, \quad e = \varepsilon_2 e.
\]

A combinatory model is **stable** if it is stable as a strongly reflexive combinatory pre-model (cf. Lemma 4.7).

In a stable strongly reflexive combinatory pre-model, the constants \( e \) and \( i \) coincide with the usual construction of these constants from \( k \) and \( s \).

**Lemma 6.6.** If \( A \) is strongly reflexive and stable, then \( i = skk \) and \( e = s(ki) \).

*Proof.* Suppose that \( A \) is strongly reflexive and stable. Then
\[
  ix \approx_s kx(kx) \approx_s skkx,
\]
\[
  exy \approx_{xy} xy \approx_{xy} kiy(xy) \approx_{xy} s(ki)xy.
\]

Since \( A \) is stable and reflexive, we have \( i = ei = e(skk) = skk \). Similarly, we have \( e = s(ki) \). \( \square \)

The following is immediate from Lemma 6.2.

**Lemma 6.7.** If \( A \) is a combinatory pre-model such that \( i = skk \) and \( e = s(ki) \), then \( s = \varepsilon_3 s \) implies \( i = \varepsilon_1 i \) and \( e = \varepsilon_2 e \).

**Remark 6.8.** By the above two lemmas, the notion of stable combinatory models in the sense of Definition 6.5 agrees with the corresponding notion in the literature which does not include \( i \) as a primitive [1, 5.6.4].

We recall the connection between stable combinatory models and lambda models. As the definition of the latter, we adopt the following characterisation.

**Definition 6.9** (Barendregt [1, 5.6.3]). A **lambda model** is a combinatory algebra \( A = (A, \cdot, k, s) \) such that the structure \( (A, \cdot, k, s, i, e) \), where \( i = skk \) and \( e = s(ki) \), is a combinatory model satisfying \( k = \varepsilon_2 k \) and \( s = \varepsilon_3 s \).

The following is immediate from Lemma 6.6 and Lemma 6.7.

**Proposition 6.10** (Meyer [11, Section 6]; Barendregt [1, 5.6.6(i)]). The following are equivalent for a combinatory pre-model \( A \):

1. \( A \) is a stable combinatory model.
2. \( (A, \cdot, k, s, i, e) \) is a lambda model, and \( i = skk \) and \( e = s(ki) \).

We establish an analogue of the above proposition for strongly reflexive combinatory pre-models and lambda algebras. The following characterisation of lambda algebras is often attributed to Curry.

---

The notion of stability also makes sense for combinatory pre-models in general. However, we have not found any significant consequence of the notion in that general setting.
Definition 6.11 (Barendregt [1, 5.2.5, 7.3.6]). A lambda algebra is a combinatorial algebra $A = (A, \cdot, k, s)$ satisfying the following equations:

1. $\lambda^*x.y.kxy = k$,
2. $\lambda^*x.y.z.xyz = s$,
3. $\lambda^*x.y.s(s(kk)x)y = \lambda^*x.y.z.xyz$,
4. $\lambda^*x.y.s(s(ks)x)y.z = \lambda^*x.y.s(s(xz)(syz))$,
5. $\lambda^*x.y.s(kx)(ky) = \lambda^*x.y.kxy$.

We recall the following fundamental result, which relates lambda algebras to lambda models (cf. Theorem 4.8 and Lemma 4.7).

Proposition 6.12 (Meyer [11, Section 7]).

1. A combinatory algebra $A$ is a lambda algebra if and only if $A[X]$ is a lambda model.
2. Every lambda model is a lambda algebra.

Proof. See Meyer [11, Section 7].

Remark 6.13. In Proposition 6.12, $A[X]$ denotes the polynomial algebra of $A = (A, \cdot, k, s)$ as a combinatory algebra. However, note that $A[X]$ coincides with the polynomial algebra of $A$ as a combinatory pre-model $(A, \cdot, k, s, i, e)$ where $i = skk$ and $e = s(ki)$. This is because the properties of $i$ and $e$ in $A[X]$ (as a combinatory pre-model) are derivable from those of $k$ and $s$.

We can now establish the following correspondence (cf. Proposition 6.10).

Theorem 6.14. The following are equivalent for a combinatory pre-model $A$:

1. $A$ is strongly reflexive and stable.
2. $(A, \cdot, k, s)$ is a lambda algebra, and $i = skk$ and $e = s(ki)$.

Proof. Since the axioms of stability 6.1 are closed equations, the following are equivalent by Theorem 4.8 Proposition 6.10 and Proposition 6.12:

1. $A$ is strongly reflexive and stable.
2. $A[X]$ is a stable combinatory model.
3. $(A[X], *, k, s)$ is a lambda model, and $i \approx_x skk$ and $e \approx_x s(ki)$.
4. $(A, \cdot, k, s)$ is a lambda algebra, and $i = skk$ and $e = s(ki)$.

Here, $*$ denotes the application of $A[X]$ (cf. Definition 2.9).

We can also establish an algebraic analogue of the following result.

Proposition 6.15 (Meyer [11, Section 6]; Barendregt [1, 5.6.6(ii)]). If $A$ is a combinatory model, then $(A, \cdot, \varepsilon_2k, \varepsilon_3s)$ is a lambda model.

Proof. See Barendregt [1 5.6.6(ii)].
Theorem 6.16. If \( A \) is a strongly reflexive combinatory pre-model, then \((A, \cdot, \varepsilon_2k, \varepsilon_3s)\) is a lambda algebra.

Proof. If \( A \) is strongly reflexive, then \( A[X] \) is a combinatory model by Theorem 4.8. Then, \((A[X], \cdot, \varepsilon_2k, \varepsilon_3s)\) is a lambda model by Proposition 6.15; hence it is a lambda algebra by Proposition 6.12. Since \( A \) is a retract of \( A[X] \), we see that \((A, \cdot, \varepsilon_2k, \varepsilon_3s)\) is a lambda algebra.

One can also check that the stable strongly reflexive combinatory pre-model determined by the lambda algebra \((A, \cdot, \varepsilon_2k, \varepsilon_3s)\) is of the form \((A, \cdot, \varepsilon_2k, \varepsilon_3s, \varepsilon_1i, \varepsilon_2e)\).

Remark 6.17. As we have noted at the end of Section 5, a strongly reflexive combinatory pre-model \( A \) determines a cartesian closed category with a reflexive object, which induces a lambda algebra structure on \((A, \cdot)\). One can verify that the constants \(k\) and \(s\) of the lambda algebra thus obtained coincide with \(\varepsilon_2k\) and \(\varepsilon_3s\), respectively.

We now understand that stable strongly reflexive combinatory pre-models and lambda algebras are equivalent. However, compared with the five axioms of lambda algebras (Definition 6.11), we have seven axioms of strong reflexivity (Definition 4.1) and four axioms of stability (6.1). Nevertheless, since \(i\) and \(e\) are definable from \(k\) and \(s\) in a stable strongly reflexive combinatory pre-model, some of the axioms of strong reflexivity and stability concerning the properties of \(i\) and \(e\) become redundant. Specifically, consider the following conditions for a combinatory pre-model \( A = (A, \cdot, k, s, i, e) \):

(CA) \( i = skk \) and \( e = s(ki) \);
(L1) \( k = \varepsilon_2k \) and \( s = \varepsilon_3s \);
(L2) The equations 1, 2, 5, and 6 of Definition 4.1:

1. \( \lambda^1xy.e(s(skk)x)y = \lambda^1xy.ex \),
2. \( \lambda^1xyz.e(s(s(sks)x)y)z = \lambda^1xyz.e(s(sxx)(syz)) \),
5. \( \lambda^1xy.e(s(kx)(ky)) = \lambda^1xy.e(k(xy)) \),
6. \( \lambda^1x.e(s(kx)i) = \lambda^1x.e \).

By Lemma 6.6, the conditions (CA), (L1), and (L2) hold if \( A \) is strongly reflexive and stable. Below, we show that these conditions are sufficient for \( A \) to be a stable strongly reflexive combinatory pre-model.

We fix a combinatory pre-model \( A \) satisfying (CA), (L1), and (L2). First, (CA) and (L1) imply that \( A \) is stable by Lemma 6.7. Thus, it remains to show that \( A \) is strongly reflexive. To this end, it suffices to show that \( A \) is reflexive: for then \( A[X] \) is reflexive because (CA), (L1), and (L2) are closed equations.

To see that \( A \) is reflexive, the following observation is useful.

Lemma 6.18. (CA), (L1), and (L2) together imply \( \lambda^1x.t = \lambda^1x.t \) for each \( t \in T(\{x\} + A) \).

Proof. By induction on the complexity of \( t \). The non-trivial case is \( t \equiv (a, x) \) for \( a \in A \); in this case, we can use (L2) (6). For the other cases, the results follow from (CA), (L1), and Lemma 6.2 (2).

Lemma 6.19. \( A \) is reflexive.
Proof. By Proposition 3.10 and Lemma 6.18, it suffices to show that 
\[ t \approx x u \] implies \[ \lambda^* x.t = \lambda^* x.u \] for each \( t, u \in T(\{x\} + A) \). The proof is by induction
on the derivation of \( t \approx x u \). Note that we only need to show that the following
defining equations of \( A[\{x\}] \) are preserved by \( \lambda^* x \):

1. \( ((k, s), t) \approx x s \),
2. \( (((s, s), t), u) \approx x ((s, u), (t, u)) \),
3. \( (i, s) \approx x s \),
4. \( ((e, s), t) \approx x (s, t) \),
5. \( (a, b) \approx x ab \),

where \( a, b \in A \) and \( s, t, u \in T(\{x\} + A) \). By the assumption \((CA)\), equations 3 and 4 are derivable from 1 and 2; thus we only need to deal with 1, 2, and 5.

We consider equation 1 as an example. First, note that \( e(\lambda^* x.t) = \lambda^* x.t \) for each \( t \in T(\{x\} + A) \) by (L1) and Lemma 6.2(2). Then, \( \lambda^* x.(k, s, t) = \lambda^* x.s \) follows from (L2)(1) by applying \( \lambda^* x.s \) and \( \lambda^* x.t \) to both sides and removing some \( e \) using Lemma 6.2(2). Similarly, one can show that equations 2 and 5 are preserved by \( \lambda^* x \) using (L2)(2) and (L2)(5).

From the above lemma, we obtain the desired conclusion.

**Theorem 6.20.** A combinatory pre-model is strongly reflexive and stable if and only if it satisfies \((CA)\), \((L1)\) and \((L2)\).

In the context of combinatory algebras, the conditions \((L1)\) and \((L2)\) provide an alternative characterisation of lambda algebras. These conditions are similar to Curry’s axiomatisation in Definition 6.11, but they do not directly correspond to each other. Instead, \((L1)\) and \((L2)\) naturally correspond to Selinger’s axiomatisation of lambda algebras [13], as we will see below.

The relation between the two axiomatisations can be clarified by the following characterisation of stable combinatory models [1, 5.6.5]. In the light of Lemma 6.6, the result can be stated as follows.

**Lemma 6.21.** Let \( A \) be a combinatory model satisfying \( i = skk \) and \( e = s(ki) \). Then, \( A \) is stable if and only if it satisfies the following equations:

\[ ek = k, \quad es = s, \quad e(ka) = ka, \quad e(sa) = sa, \quad e(sab) = sab \]
for each \( a, b \in A \).

**Proof.** See Barendregt [1, 5.6.5].

From the above lemma, we can derive the following algebraic analogue.

**Proposition 6.22.** Let \( A \) be a strongly reflexive combinatory pre-model satisfying \( i = skk \) and \( e = s(ki) \). Then, \( A \) is stable if and only if the following equations hold:

\[ ek = k, \quad es = s, \quad e(kx) \approx x kx, \quad e(sx) \approx x sx, \quad e(sxy) \approx xy sxy. \quad (6.2) \]
Proof. By Theorem 4.8, \( A \) is stable if and only if \( A[X] \) is stable as a combinatory model. By Lemma 6.21, the latter is equivalent to

\[
\begin{align*}
  e_k &\approx_{x} k, & e_s &\approx_{x} s, & e(kt) &\approx_{x} kt, & e(st) &\approx_{x} st, & e(stu) &\approx_{x} stu
\end{align*}
\]

for all \( t, u \in T(X + A) \), which in turn is equivalent to (6.2).

Proposition 6.22 allows us to see the connection between (L1) and (L2) and the following axiomatisation of lambda algebras by Selinger.

**Theorem 6.23** (Selinger [13, Theorem 3]). Let \( A = (A, \cdot, k, s) \) be a combinatory algebra. Then, \( A \) is a lambda algebra if and only if the following equations hold with \( i = skk \) and \( e = s(ki) \):

\[
\begin{align*}
  (a) \quad e_k &= k, &
  (f) \quad s(s(kk)x)y &\approx_{xy} ex, \\
  (b) \quad e_s &= s, &
  (g) \quad s(s(s(ks)x)y)z &\approx_{xyz} s(sxz)(sy), \\
  (c) \quad e(kx) &\approx_{x} kx, &
  (h) \quad s(kx)(ky) &\approx_{xy} k(xy), \\
  (d) \quad e(sx) &\approx_{x} sx, &
  (i) \quad s(kx)i &\approx_{x} ex. \\
  (e) \quad e(sxy) &\approx_{xy} sxy.
\end{align*}
\]

Proof. By Proposition 6.22, the conditions (L1) and (L2) imply the nine conditions above. Conversely, the proofs of Lemma 6.18 and Lemma 6.19 show that the conditions (c) and (e)–(i) imply that the structure \( (A, \cdot, k, s, i, e) \) with \( i = skk \) and \( e = s(ki) \) is a strongly reflexive combinatory pre-model; hence they imply (L2). Then, (L1) follows from (a)–(e) by Proposition 6.22.

The correspondence between (L1)–(L2) and (a)–(i) are evident: in (a)–(i) the conditions (L1) is broken down into an equivalent set of smaller pieces and some e’s are omitted from (L2) using (c) and (e).

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\[\text{As noted by Selinger [13 Section 2.4], condition (c) follows from (h) so it is redundant.}\]
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