Higher dimensional rotating black holes in Einstein-Maxwell theory with negative cosmological constant

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Abstract

We present arguments for the existence of charged, rotating black holes with equal-magnitude angular momenta in an odd number of dimensions \( D \geq 5 \). These solutions possess a regular horizon of spherical topology and approach asymptotically the Anti-de Sitter spacetime background. We analyze their global charges, their gyromagnetic ratio and their horizon properties.

1 Introduction

Recently a tremendous amount of interest has focused on Anti-de Sitter (AdS) spacetime. This interest is mainly motivated by the proposed correspondence between physical effects associated with gravitating fields propagating in AdS spacetime and those of a conformal field theory (CFT) on the boundary of AdS spacetime [1, 2].

In this context, the black holes with cosmological constant \( \Lambda < 0 \) are of special interest since they would offer the possibility of studying the nonperturbative structure of some CFTs. Higher dimensional rotating black holes with AdS asymptotics have been studied by various authors, starting with Hawking et al. [3] who generalized the \( D = 4 \) Kerr-AdS solution to five dimensions with arbitrary angular momenta, and to all dimensions with only a single nonzero angular momentum. The generalization of these solutions to the full set of independent angular momenta was given in [4].

It is of interest to generalize these higher dimensional Kerr-AdS metrics further, by including matter fields. Several exact solutions describing charged rotating black holes have been found recently in gauged supergravities in \( D = 5 \) [5], and \( D = 7 \) [6] dimensions. Apart from Abelian fields with a Chern-Simons term, these configurations usually contain scalar fields with a nontrivial scalar potential.

The main purpose of this paper is to report progress on this problem by presenting numerical evidence for the existence a set of asymptotically AdS charged rotating black holes. These configurations exist in an odd number of dimensions, \( D \geq 5 \). They possess a regular horizon of spherical topology, and their angular momenta are all of equal-magnitude, thus factorizing the angular dependence. The same approach was employed recently to construct asymptotically flat charged rotating black holes in higher dimensions [7, 8].

Also, instead of specializing to a particular supergravity model, we shall consider pure Einstein-Maxwell (EM) theory with negative cosmological constant. Although this theory is non-supersymmetric in itself for \( D > 4 \), it enters all gauged supergravities as the basic building block. Therefore one can expect the basic features of its solutions to be generic.

The paper is structured as follows: in Section 2 we present the general framework and analyze the field equations. The boundary conditions and the black hole properties are discussed in Section 3. We present the numerical results for \( D = 5 \) and \( D > 5 \) black holes in Section 4, and conclude with Section 5, where further applications are addressed.
2 Action and Ansatz

We consider the EM action with a negative cosmological constant $\Lambda$
\[ I = \frac{1}{16\pi G_D} \int_M d^Dx \sqrt{-g}(R - 2\Lambda - F_{\mu\nu}F^{\mu\nu}) - \frac{1}{8\pi G_D} \int_{\partial M} d^{D-1}x \sqrt{-h}K, \tag{2.1} \]
where $D = 2N + 1$ ($N \geq 2$), $G_D$ is the $D$-dimensional Newton constant, $R$ is the curvature scalar, and $F_{\mu\nu}$ is the gauge field strength tensor ($F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, with gauge potential $A_\mu$). The last term in (2.1) is the Hawking-Gibbons surface term [9], which is required in order to have a well-defined variational principle. $K$ is the trace of the extrinsic curvature for the boundary $\partial M$ and $h$ is the induced metric of the boundary. We denote $\Lambda = -(D-2)(D-1)/(2\ell^2)$.

The field equations associated with the action (2.1) are the Einstein equations
\[ G_{\mu\nu} + \Lambda g_{\mu\nu} = 2 \left( F_{\mu\rho}F^{\rho\sigma} - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma}F^{\rho\sigma} \right), \tag{2.2} \]
and the gauge field equations
\[ \nabla_\mu F^{\mu\nu} = 0. \tag{2.3} \]

We consider stationary black hole space-times with $N$ azimuthal symmetries, representing charged $U(1)$ generalizations of the corresponding set of vacuum solutions discussed in [4]. The symmetries imply the existence of $N + 1$ commuting Killing vectors, $\xi = \partial_t$, and $\eta_{(k)} = \partial_{\varphi_k}$, for $k = 1, \ldots, N$. While the general EM-AdS black holes then possess $N$ independent angular momenta, we here restrict to black holes with equal-magnitude angular momenta and with spherical horizon topology [4].

We employ a parametrization for the metric corresponding to a generalization of the Ansatz used previously for asymptotically flat solutions [7]. It has the general form
\[ ds^2 = -b(r)dt^2 + \frac{dr^2}{u(r)} + g(r) \sum_{i=1}^{N-1} \left( \prod_{j=0}^{i-1} \cos^2 \theta_j \right) d\theta_i^2 + p(r) \sum_{k=1}^{N-1} \left( \prod_{l=0}^{k-1} \cos^2 \theta_l \right) \sin^2 \theta_k \left( \varepsilon_k d\varphi_k - \frac{w(r)}{r} dt \right)^2 \]
\[ + (g(r) - p(r)) \left\{ \sum_{k=1}^{N-1} \left( \prod_{l=0}^{k-1} \cos^2 \theta_l \right) \sin^2 \theta_k \sin^2 \varphi_k - \left[ \sum_{k=1}^{N-1} \left( \prod_{l=0}^{k-1} \cos^2 \theta_l \right) \sin^2 \theta_k \varepsilon_k \sin \varphi_k \right] \right\}, \tag{2.4} \]
where $\theta_0 \equiv 0$, $\theta_i \in [0, \pi/2]$ for $i = 1, \ldots, N - 1$, $\theta_N \equiv \pi/2$, $\varphi_k \in [0, 2\pi]$ for $k = 1, \ldots, N$, and $\varepsilon_k = \pm 1$ denotes the sense of rotation in the $k$-th orthogonal plane of rotation. For such solutions, the isometry group is enhanced from $R \times U(1)^{N+1}$ to $R \times U(N + 1)$, where $R$ denotes the time translation. This symmetry enhancement allows us to deal only with ordinary differential equations (ODE’s).

The vacuum black holes discussed in [4] are recovered for vanishing gauge field and
\[ u(r) = 1 + \frac{r^2}{\ell^2} - \frac{2\dot{M}\Xi}{r^{D-3}} + \frac{2\dot{M}^2}{r^{D-1}}, \quad p(r) = r^2 \left( 1 + \frac{2\dot{M}^2}{r^{D-1}} \right), \quad w(r) = \frac{2\dot{M}}{r^{D-4}p(r)}, \quad g(r) = r^2, \quad b(r) = \frac{r^2 u(r)}{p(r)}, \tag{2.5} \]
where $\dot{M}$ and $\dot{a}$ are two constants related to the solutions’ mass and angular momentum, and $\Xi = 1 - \dot{a}^2/\ell^2$ (see [11] for a discussion of the basic features of these solutions). For the numerical calculations a convenient parametrization is given by
\[ u(r) = \frac{f(r)}{m(r)} \left( \frac{r^2}{\ell^2} + 1 \right), \quad b(r) = \left( \frac{r^2}{\ell^2} + 1 \right) f(r), \quad g(r) = \frac{m(r)}{f(r)} r^2, \quad p(r) = \frac{n(r)}{f(r)} r^2. \tag{2.6} \]

\[ ^1 \text{Asymptotically AdS rotating charged topological black hole solutions with zero scalar curvature of the event horizon are known in closed form [10], but they were found for a different metric ansatz, and they possess rather different properties.} \]
The Ansatz for the U(1) potential, consistent with the symmetries of the line element (2.3), is given by

$$A_\mu dx^\mu = a_0 dt + a_\varphi \sum_{k=1}^{N} \left( \prod_{l=0}^{k-1} \cos^2 \theta_l \right) \sin^2 \theta_k \epsilon_k d\varphi_k$$.

(2.7)

Thus, independent of the odd dimension \( D \geq 5 \), this parametrization involves only four functions \( f, m, n, \omega \) for the metric and two functions \( a_0, a_\varphi \) for the gauge field, which all depend only on the radial coordinate \( r \).

When (2.4) and (2.7) are substituted into (2.2) and (2.3), a coupled system of ODE’s is obtained (one first-order equation (for \( n \)) plus five second-order equations (for \( f, m, \omega, a_0, A_\varphi \))). However, taking advantage of the existence of a first integral of that system

$$\frac{r^{D-2} m(D-5)/2}{f(D-3)/2} \sqrt{\frac{m m}{f}} \left( \frac{da_0}{dr} + \omega \frac{d a_\varphi}{r dr} \right) = (D - 3)q \ , \ q = \text{constant}$$.

(2.8)

we may eliminate \( a_0 \) from the equations, leaving a system of one first order equation (for \( n \)) and four second order equations \([7]\).

3 Black Hole Properties

3.1 Asymptotic expansion and boundary conditions

In order to generate black hole solutions which are asymptotically AdS and possess a regular event horizon of spherical topology, we have to impose appropriate boundary conditions.

A straightforward computation gives the following asymptotic expansion for the metric and matter fields, involving five parameters \( \tilde{\alpha}, \tilde{\beta}, J, q \) and \( \tilde{\mu} \),

\[
\begin{align*}
 f &= 1 + \frac{\tilde{\alpha}}{r^{D-1}} + O \left( \frac{1}{r^{D+1}} \right) \quad m = 1 + \frac{\tilde{\beta}}{r^{D-1}} + O \left( \frac{1}{r^{D+1}} \right) \quad n = 1 + \frac{(D - 2)(\tilde{\alpha} - \tilde{\beta})}{r^{D-1}} + O \left( \frac{1}{r^{D+1}} \right), \\
 \omega &= \frac{J}{r^{D-2}} + O \left( \frac{1}{r^{D-5}} \right) \quad a_0 = -q + O \left( \frac{1}{r^{D-3}} \right) \quad a_\varphi = \frac{\tilde{\mu}}{r^{D-3}} + O \left( \frac{1}{r^{D-4}} \right).
\end{align*}
\]

(3.1)

Therefore, in the numerical procedure, we impose the following boundary conditions at infinity

$$f |_{r = \infty} = m |_{r = \infty} = n |_{r = \infty} = 1 \ , \ \omega |_{r = \infty} = 0 \ , \ a_0 |_{r = \infty} = a_\varphi |_{r = \infty} = 0$$.

(3.2)

The horizon of these black hole solutions is a squashed \( S^{D-2} \) sphere. It resides at the constant value of the radial coordinate \( r = r_H \), and is characterized by \( f(r_H) = 0 \). Expanding the metric and matter functions at the horizon yields

\[
\begin{align*}
 f &= f_2 \bar{x}^2 \left( 1 - \frac{2r_H^2}{r_H^2 + \ell^2} \bar{x}^2 \right) + O(\bar{x}^4) \quad m = m_2 \bar{x}^2 \left( 1 - \frac{4r_H^2}{r_H^2 + \ell^2} \bar{x}^2 \right) + O(\bar{x}^4), \notag \\
 n &= n_2 \bar{x}^2 \left( 1 - \frac{4r_H^2}{r_H^2 + \ell^2} \bar{x}^2 \right) + O(\bar{x}^4) \quad \omega = \Omega r_H(1 + \bar{x}) + O(\bar{x}^2), \\
 a_0 &= -(\Phi_H + \Omega a_\varphi_0) + O(\bar{x}^2) \quad a_\varphi = a_\varphi_0 + O(\bar{x}^2),
\end{align*}
\]

(3.3)

(with \( f_2, m_2, n_2 \) positive constants). Here \( \Omega \) is the (constant) horizon angular velocity defined in terms of the Killing vector

$$\chi = \partial_t + \Omega \sum_{k=1}^{N} \epsilon_k \partial \varphi_k$$.

(3.4)

which is null at the horizon. \( \Phi_H \) denotes the (constant) horizon electrostatic potential, and the compactified radial coordinate \( \bar{x} \) is given by \( \bar{x} = 1 - r_H/r \).
At the horizon, the solutions satisfy the boundary conditions

\[ f|_{r=r_H} = m|_{r=r_H} = n|_{r=r_H} = 0 \, , \quad \omega|_{r=r_H} = r_H\Omega \, , \]
\[ \Phi_H = - (a_0 + \Omega a_r)|_{r=r_H} \, , \quad \frac{da_r}{dr}|_{r=r_H} = 0 \, . \tag{3.5} \]

### 3.2 Global charges

In an asymptotically flat spacetime, the total mass \( M \) and the angular momenta \( J_{(k)} \) of the black holes are obtained from the Komar expressions associated with the respective Killing vector fields. For AdS asymptotics, the calculation of the total mass is more involved, mainly because the analogous Komar integral for the relevant time-like Killing vector diverges, which then requires a regularization. Various definitions of the conserved charges for an AdS background have been presented in the literature \[12\].

Employing first the Ashtekar-Magnon-Das conformal mass definition \[13\], we obtain for the mass of rotating EM black holes the expression

\[ M = - \frac{A(S^{D-2})}{16\pi G_D} \frac{\hat{\beta} + (D-2)\hat{\alpha}}{\ell^2} \, , \tag{3.6} \]

where \( A(S^{D-2}) \) denotes the area of the unit \((D-2)\)-sphere, and \( \hat{\alpha}, \hat{\beta} \) the constants in the asymptotic expansion \[\ref{3.1}\].

A different technique, proposed by Balasubramanian and Kraus \[14\], was inspired by the AdS/CFT correspondence and consists in adding suitable counterterms \( I_{ct} \) to the action of the theory in order to ensure the finiteness of the boundary stress tensor \( T_{ab} = \frac{1}{\sqrt{-h}} \frac{\delta M}{\delta h_{ab}} \) derived by the quasilocal energy definition \[15\]. The expression for the mass we obtain in this approach\[2\] contains besides the expression \[3.6\] for the mass an additional term \( E_c^{(D)} \), which depends only on \( \ell \) and \( D \) (e.g. \( E_c^{(5)} = 3A(S^3)\ell^2/64\pi G_5 \), \( E_c^{(7)} = -5A(S^5)\ell^4/128\pi G_7 \)). \( E_c^{(D)} \) corresponds to the mass of the pure global \( \text{AdS}_{2N+1} \) and is usually interpreted as the energy dual to the Casimir energy of the dual CFT defined on the \((D-1)\)-dimensional boundary metric \[14\].

The angular momenta \( J_{(k)} \) may be computed from the standard Komar integral, since the divergent term vanishes. The Komar integral reads

\[ J_{(k)} = \frac{1}{16\pi G_D} \int_{S^{D-2}} \beta_{(k)} \, , \tag{3.7} \]

with \( \beta_{(k)}^{\mu_1...\mu_{D-2}} \equiv \epsilon_{\mu_1...\mu_{D-3}\rho\sigma} \nabla^\rho \eta_{(k)}^\sigma \), and for equal-magnitude angular momenta \( J_{(k)} = \varepsilon_k J, k = 1, \ldots, N \). With \[3.7\] we then obtain for the angular momentum the expression

\[ J = \frac{A(S^{D-2})}{8\pi G_D} \tilde{j} \, , \tag{3.8} \]

which agrees with the value found within the counterterm approach\[3\].

The electric charge is related to the first integral \[2.8\] by

\[ Q = \frac{(D-3)A(S^{D-2})}{4\pi G_D} q \, , \tag{3.9} \]

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\[2\] We have computed the mass and angular momenta for solutions in \( D = 5, 7 \) and 9 dimensions by using the expressions of the counterterms and boundary stress-tensor given e.g. in Ref. \[10\]. If there are matter fields on \( M \), additional counterterms may be needed to regularize the action and global charges. However, we have found that for a purely Abelian matter content, the usual counterterm prescription removes all divergences.

\[3\] There is ostensibly a mismatch between the total number of parameters in the asymptotic expansion \[3.1\] and the number of conserved charges of the solution, which we could not clarify. For \( Q = 0 \), one finds \( \tilde{J} = \sqrt{(2\tilde{\alpha} - \tilde{\beta})(D-1)/2 - (D-2)/\ell} \), while \( \tilde{\beta} = \tilde{\alpha}(D-2)/(D-1) \) in the static limit. However, for charged rotating solutions, the numerical results do not indicate the existence of any simple correlation between the constants \( \tilde{\alpha}, \tilde{\beta}, \tilde{J} \) and \( q \) in \[3.1\].
and the magnetic moment $\mu_{\text{mag}}$ is given by

$$\mu_{\text{mag}} = \frac{(D - 3)A(S^{D-2})}{4\pi G_D} \bar{\mu} .$$  

(3.10)

A further quantity of physical interest is the gyromagnetic ratio $g$ of these charged rotating AdS black holes, defined as

$$g = \frac{2M\mu_{\text{mag}}}{QJ} .$$  

(3.11)

### 3.3 Horizon properties

Employing the expansion at the horizon (3.3), we obtain for the surface gravity $\kappa$, defined by

$$\kappa^2 = -\left.\frac{1}{2}(\nabla_\mu \chi_\nu)(\nabla^\mu \chi^\nu)\right|_{r_H} ,$$  

(3.12)

the expression

$$\kappa = \left(1 + \frac{r^2_H}{\ell^2}\right) \frac{f_2}{r_H \sqrt{m_2}} ;$$  

(3.13)

and for the horizon area

$$A_H = r_H^{D-2} A(S^{D-2}) \sqrt{\frac{m_2^{D-3} n_2}{f_2^{D-2}}} .$$  

(3.14)

To have a measure of the deformation of the horizon, we introduce a deformation parameter defined as the ratio of the equatorial circumference $L_e$ and the polar one $L_p$, which for these solutions we are considering takes the form

$$\frac{L_e}{L_p} = \sqrt{\frac{n_2}{m_2}} .$$  

(3.15)

The horizon mass $M_H$ and the horizon angular momentum $J_H$ can be computed by evaluating the corresponding Komar integrals at the horizon [7], leading to

$$M_H = \frac{1}{8\pi G_D} \frac{D - 2}{D - 3} A(S^{D-2}) r_H^{D-3} \left[ 1 + \frac{r^2_H}{\ell^2} - \frac{n_2}{2f_2^2} r_H \Omega \left( \frac{d^2 \omega}{dx^2} \right|_{\bar{\epsilon}=0} - 2r_H \Omega \right] ,$$  

$$J_H = \frac{1}{8\pi G_D (D - 1)} A(S^{D-2}) r_H^{D-2} \sqrt{\frac{m_2^{D-3} n_2^3}{f_2^{D-2}}} \left( 2r_H \Omega - \frac{d^2 \omega}{dx^2} \right|_{\bar{\epsilon}=0} .$$  

(3.16)

These horizon quantities are related by the horizon mass formula

$$\frac{D - 3}{D - 2} M_H = \frac{\kappa A_H}{8\pi G_D} + N \Omega J_H .$$  

(3.17)

To obtain the global counterpart of (3.17) remains a challenge. It would be interesting to extend the approach used in [17], to derive a Smarr-type formula for Kerr-AdS solutions, to the case discussed in this paper.

### 3.4 Ergosurface and closed causal curves

These rotating black holes possess an ergosurface, inside of which observers cannot remain stationary, but will move in the direction of the rotation. The ergosurface is located at a constant value of the radial coordinate $r = r_e$, with $g_{tt}(r_e) = 0$, i.e.

$$\frac{n(r_e)}{f(r_e)} u^a(r_e) - f(r_e) \left(1 + \frac{r^2}{\ell^2}\right) = 0 ,$$  

(3.18)

and does not intersect the horizon.

Also, as a result of the metric ansatz (2.4), $F(x^a) = t$ is a global time coordinate ($g^{\mu\nu} F_{\mu} F_{\nu} = -1/(f(r)(r^2/\ell^2 + 1)) < 0$), and no closed causal curves occur in the region outside the event horizon.
Figure 1: Left: Scaled angular momentum $J/M^{(D-2)/(D-3)}$ vs. scaled charge $Q/M$ for extremal black holes with equal-magnitude angular momenta in $D = 5$ dimensions. Right: Mass $M$ vs. horizon angular velocity $\Omega$ for equal-magnitude angular momenta black hole solutions in $D = 5$ dimensions for fixed isotropic horizon radius $r_H = 0.5$, and fixed charge $Q = 10$, for several values of $\ell$; the uncharged Kerr-AdS counterparts are also shown.

4 Results

The numerical methods employed here are analogous to those used to construct asymptotically flat charged rotating black holes [7]. Working with the compactified coordinate $\bar{x}$, we choose units such that $G_D = 1$, and apply a collocation method for boundary-value ordinary differential equations, equipped with an adaptive mesh selection procedure [18]. Typical mesh sizes include $10^3 - 10^4$ points. The solutions have a relative accuracy of $10^{-8}$.

In the following we focus on the properties of $D = 5$ black hole solutions. These results can be easily extended to odd dimensions $D > 5$, since the main features of the solutions are common for all odd dimensions. We note, that the computation of the mass involves $1/r^{D-1}$ terms in the asymptotic expansion, decreasing the numerical accuracy of the mass as $D$ increases.

4.1 Domain of existence

Let us first address the domain of existence of these black hole solutions. In order to do so, we note that the solutions have a scale symmetry, which is broken once the value of the cosmological constant is fixed: $\ell = \lambda \ell, M = \lambda^{D-3} M, \hat{J} = \lambda^{D-2} J, \hat{Q} = \lambda^{D-3} Q, \hat{r}_H = \lambda r_H, \hat{\Omega} = \Omega/\lambda, \hat{\kappa} = \kappa/\lambda$, etc. To classify solutions related by this symmetry, we introduce the scale invariant ratio

$$\ell_Q = \frac{\ell}{|Q|^{1/(D-3)}}.$$ (4.1)

In Fig. 1 (left) we show the domain of existence of $D = 5$ solutions for several values of $\ell_Q$. The domain of existence is bounded by extremal solutions, which are characterized by a vanishing surface gravity $\kappa$. For finite $\ell_Q$ the extremal curve that delimits the corresponding region of existence reaches the $|J|/M^{(D-2)/(D-3)} = 0$ axis at $|Q|/M = 0$ and at a finite value of $|Q|/M$, affiliated to the corresponding extremal higher dimensional Reissner-Nordström (RN)-AdS solutions [4]. As $\ell_Q$ increases, the origin is approached more and more steeply and, in the limit $\ell_Q = \infty$, the extremal curve consists of two pieces: the

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4 The analytical RN-AdS values $Q/M = 0.677, 0.897, 1.135, 2/\sqrt{3}$ for $\ell_Q = 0.316, 0.632, 3.162, \infty$, respectively, in Fig. 1 (left) agree accurately with the numerical values.
pure EM part (without cosmological constant) [7], plus a vertical line corresponding to uncharged Kerr-AdS solutions. Clearly, the domain of existence enlarges with increasing $\ell_Q$, and is symmetric with respect to $Q \to -Q$.

4.2 Global charges and $g$-factor

We now turn to non-extremal black holes, and analyze their asymptotic properties. We first consider the global charges of $D = 5$ black hole solutions, obtained by keeping the charge $Q$ and the isotropic horizon radius $r_H$ fixed, while varying the horizon angular velocity $\Omega$. The mass $M$ of these solutions as given by (3.6) is exhibited in Fig. 1 (right), together with the mass of their higher dimensional Kerr-AdS counterparts.

For each set of solutions we observe two branches, extending up to a maximal value of $\Omega$, where they merge and end. The lower branch emerges from the static solution in the limit $\Omega = 0$. On the upper branch the mass diverges in the limit $|\Omega| \to 1/\ell$. Thus a static horizon ($\Omega = 0$) cannot be reached along the upper branch for finite $\ell$. The maximal value of $\Omega$ depends on the horizon radius $r_H$, the charge $Q$, the length $\ell$, and the dimension $D$.

The angular momentum $J$ of the solutions exhibits a very similar dependence on $\Omega$ and $\ell$ as the mass. We note that, for a fixed value of the charge, its influence on the solutions, and thus their deviation from the corresponding higher dimensional Kerr-AdS solutions, decreases with increasing dimension $D$, as expected from the scaling properties of the solutions.

![Figure 2](image-url): Left: Gyromagnetic ratio $g$ vs. horizon angular velocity $\Omega$ for the same set of solutions as in Fig. 1 (right). Right: Mass $M$ vs. $\ell$ for equal-magnitude angular momenta black hole solutions in $D = 5$ dimensions for fixed horizon radius $r_H = 0.5$, angular velocity $\Omega = 1/3$, and varying $\ell$ for several values of the charge $Q = 0, 5, 10$.

The gyromagnetic ratio $g$ of these solutions is exhibited in Fig. 2 (left). It reveals clearly, that $1/\ell$ is the limiting value of $\Omega$ on the upper branch. We note that the $g$-factor ranges between $D - 2$ and $D - 1$ for finite values of $\ell$, the value $D - 1$ being reached along the upper branch, when $|\Omega| \to 1/\ell$.

In the limit $Q \to 0$, the gyromagnetic ratio can be obtained analytically by linearly perturbing the higher dimensional Kerr-AdS solutions. We obtain for weakly charged solutions (in the limit $Q \to 0$), possessing equal-magnitude angular momenta,

$$g = (D - 2) + \frac{\hat{a}^2}{\ell^2},$$  

(4.2)

with

$$M = \frac{A(S^{D-2})}{8\pi G_D} \left[ (D - 2) + \frac{\hat{a}^2}{\ell^2} \right] \hat{M}, \quad J = \frac{A(S^{D-2})}{4\pi G_D} \hat{M} \hat{a}.$$  

(4.3)
This shows clearly, that $g$ ranges between $D - 2$ and $D - 1$ also for weakly charged solutions, since $\hat{a}^2/\ell^2 \leq 1$. We note, that for weakly charged solutions with a single non-vanishing angular momentum the $g$-factor ranges between $D - 2$ and $2$ [19].

In Fig. 2 (right) we exhibit the dependence of the mass on $\ell$, for fixed $\Omega$ and $r_H$, and several values of $Q$. For fixed (but not too large) $\Omega$, there is just a single solution when $\ell < 1/|\Omega|$. When $\ell > 1/|\Omega|$, depending on the value of the charge $Q$, there may be a maximal value of $\ell$ beyond which no solutions exist. As $Q$ increases, this maximal value of $\ell$ decreases. We note that this pattern is in agreement with Fig. 1 (right).

4.3 Horizon properties

Let us now address the horizon properties of the solutions, beginning with the surface gravity $\kappa$ and the horizon area $A_H$. For most of the sets of solutions with fixed $r_H$, $Q$, and $\ell$ and varying $\Omega$, $\kappa$ decreases monotonically, reaching a vanishing value at $|\Omega| = 1/\ell$, on the upper branch. At the same time, the horizon area $A_H$ increases monotonically, diverging in the limit.

Complementary information on these solutions is obtained from Fig. 3 (left), where the dependence of the area on the isotropic horizon radius $r_H$ is exhibited, for fixed $\Omega$, $Q$, and $\ell$. Again we see, that $|\Omega| = 1/\ell$ marks the borderline, concerning the existence of one or two solutions. For $|\Omega| \leq 1/\ell$, there is just a single solution for each value of $r_H$, and $r_H$ can be increased without bound. In addition, as $r_H$ is increased, the difference between charged and uncharged solutions decreases. Beyond $|\Omega| = 1/\ell$ and below a maximal value of $\Omega$ (given by the corresponding maximal value of $\Omega$ for the extremal solution), two solutions exist.

Figure 3: Left: Horizon area $A_H$ vs. isotropic horizon radius $r_H$ for equal-magnitude angular momenta black hole solutions in $D = 5$ dimensions for fixed charge $Q = 10$, $\ell = 1$, and several values of the horizon angular velocity $\Omega = 0.5, 1.0, 1.09$; their uncharged counterparts are also shown. Right: Ratio of horizon circumferences $L_e/L_p$ vs. charge $Q$ for equal-magnitude angular momenta black hole solutions in $D = 5$ dimensions for fixed $r_H = 0.5$, $\ell = 5$, and several values of the angular momentum $J = 0, 2, 10, 100, 500$.

To have a measure for the deformation of the horizon, we consider the ratio of equatorial and polar circumferences $L_e/L_p$ of the horizon. (The equatorial circumference $L_e$ is obtained with $\theta = 0$ and constant $\varphi_1$, and the polar circumference $L_p$ with constant $\varphi_1$ and $\varphi_2$.) Whereas Kerr-AdS solutions with equal angular momenta always have oblate deformation with ratio $L_e/L_p \geq 1$, we observe in Fig. 3 (right), that the presence of charge allows for prolate deformation of the horizon in certain regions of parameter space.

We also exhibit the ergosurface (isotropic) radius $r_e$ versus the charge $Q$ in Fig. 4 (left). The horizon area and the surface gravity of the solutions are related to the entropy and the temperature, respectively, $S = A_H/4G_D$ and $T = \kappa/2\pi$. The dependence of the entropy on the temperature is shown in Fig. 4 (right) and discussed below with respect to the thermodynamical properties of the solutions.
Figure 4: Left: Ergosurface (isotropic) radius $r_e$ vs. the charge $Q$ for equal-magnitude angular momenta black hole solutions in $D = 5$ dimensions for fixed $r_H = 0.5$, $\ell = 5$, and several values of the angular momentum $J = 0, 1, 2, 5, 10$. Right: Entropy $S$ vs. temperature $T$ for equal-magnitude angular momenta black hole solutions in $D = 5$ dimensions for fixed charge $Q = 0, 2, 7, 10$, angular momentum $J = 0, 1, 2, 10$ and $\ell = 5$.

5 Further remarks

Despite the increasing number of new interesting solutions, the closed form expression for the higher dimensional electrically charged rotating solutions in EM theory has not yet been obtained. We have therefore applied a numerical approach to study these solutions and their properties.

Our numerical analysis indicates the existence of charged rotating black holes in $D = 2N + 1$ dimensions, with $D \geq 5$, possessing a regular horizon of spherical topology and $N$ equal-magnitude angular momenta. These solutions represent the AdS counterparts of asymptotically flat EM solutions, discussed recently [7, 8].

This class of solutions may provide a fertile ground for further study of charged rotating configurations in gauged supergravity models. For $D = 5$ there should be no difficulty in principle, using the techniques applied in [8], to extend these solutions to the general case with two distinct angular momenta. Higher dimensional rotating EM topological black holes with a horizon of negative curvature are also likely to exist for $\Lambda < 0$.

The study of the solutions discussed in this paper in an AdS/CFT context represents an interesting open question. Although this may require to embed them in a supergravity model, one should remark that for the ansatz considered here, the boundary metric is not rotating and corresponds to a static Einstein universe in $2N$ dimensions. The background metric upon which the dual field theory resides is $\gamma_{ab} = \lim_{r \to \infty} \ell^2 r h_{ab}$ and corresponds to a $(D - 1)$ static Einstein universe with line element

$$\gamma_{ab} dx^a dx^b = -dt^2 + \ell^2 d\Omega_{D-2}^2.$$  (5.1)

One can use the AdS/CFT “dictionary” to predict qualitative features of a quantum field theory in this background. For example, the expectation value of the dual CFT stress-tensor can be calculated using the relation [20]

$$\sqrt{-\gamma} \gamma^{ab} < \tau_{bc} > = \lim_{r \to \infty} \sqrt{-h} h^{ab} T_{bc}.$$  (5.2)

Restricting to the five-dimensional case, one finds (with $x^1 = \theta_1$, $x^2 = \varphi_1$, $x^3 = \varphi_2$, $x^4 = t$) the following
interesting form for the stress-tensor

\[
8\pi G_5 < \tau^a_b > = \frac{1}{2\ell} \left( \frac{A}{\ell^4} + \frac{1}{4} \right) \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -3
\end{pmatrix} + \frac{2B}{\ell^5} \begin{pmatrix}
0 & 0 & \sin^2 \theta_1 & \cos^2 \theta_1 \\
0 & \sin^2 \theta_1 & \cos^2 \theta_1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix} + \frac{2\hat{J}}{\ell^5} \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{\ell^2} \\
0 & \sin^2 \theta_1 & \cos^2 \theta_1 & 0 \\
0 & 0 & 0 & -\frac{1}{\ell^2}
\end{pmatrix},
\]

where \( A = 5(\tilde{\beta} - \tilde{\alpha}) \), \( B = 3\tilde{\alpha} - 4\tilde{\beta} \). As expected, this stress-tensor is finite and covariantly conserved. It is also traceless as expected from the absence of a conformal anomaly for the static boundary metric \((5.1)\) [21].

From the AdS/CFT correspondence, we expect these black holes to be described by some thermal states in a dual theory, formulated in a static Einstein universe. Therefore it appears to be interesting to study the thermodynamics of charged rotating EM solutions to see, how rotation affects the thermodynamic properties as compared to those of the static RN-AdS solutions [22]. This can already be attempted based on the numerical results presented in Section 4, valid for solutions with Lorentzian signature.

A black hole as a thermodynamic system is unstable if it has negative specific heat. In the canonical ensemble, the charge and angular momentum are fixed parameters, and the response function whose sign determines the thermodynamic stability is the heat capacity \( C = T \left( \frac{\partial S}{\partial T} \right)_{J,Q} \). As is known, small Schwarzschild-AdS black holes (i.e. \( J = Q = 0 \)) have negative specific heat, but large size black holes have positive specific heat [23]. There exists a discontinuity of the specific heat as a function of the temperature for some critical value of the horizon radius \( r_H \), and thus small and large black holes are found to be somewhat disjoint objects.

The results here indicate that this discontinuity persists for charged rotating black holes in EM theory, provided that the electric charge \( Q \) and the angular momentum \( J \) are not very large. This is seen in Fig. 4 (right), where the entropy \( S \) of a set of \( D = 5 \) EM black holes is shown versus the temperature \( T \). For small values of \( Q \) and \( J \), the entropy increases with increasing temperature up to a first critical point, from where it continues to increase with now decreasing temperature up to a second critical point, beyond which entropy and temperature both increase monotonically. For fixed \( J \), the region where \( \frac{\partial S}{\partial T} \big|_{J,Q} < 0 \) shrinks, as \( Q \) is increased, and there exists a critical value of \( Q \) above which \( \frac{\partial S}{\partial T} \big|_{J,Q} > 0 \). The analogous pattern is seen for fixed \( Q \) and varying \( J \). Thus the solutions become more thermally stable as \( Q \) and/or \( J \) increase.

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