Maximal automatic complexity and context-free languages

Bjørn Kjos-Hanssen*

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Abstract

Let $A_N$ denote nondeterministic automatic complexity and

$$L_{k,c} = \{ x \in [k]^* : A_N(x) > \frac{|x|}{c} \}.$$  

In particular, $L_{2,2}$ is the language of all $k$-ary words for which $A_N$ is maximal, while $L_{3,2}$ gives a rough dividing line between complex and simple. Let $\text{CFL}$ denote the complexity class consisting of all context-free languages. While it is not known that $L_{2,2}$ is infinite, Kjos-Hanssen (2017) showed that $L_{3,2}$ is $\text{CFL}$-immune but not $\text{coCFL}$-immune. We complete the picture by showing that $L_{3,2} \notin \text{coCFL}$.

Turning to Boolean circuit complexity, we show that $L_{2,3}$ is $\text{SAC}^0$-immune and $\text{SAC}^0$-coimmune. Here $\text{SAC}^0$ denotes the complexity class consisting of all languages computed by (non-uniform) constant-depth circuits with semi-unbounded fanin.

As for arithmetic circuits, we show that $\{ x : A_N(x) > 1 \} \notin \oplus \text{SAC}^0$. In particular, $\text{SAC}^0 \not\subseteq \oplus \text{SAC}^0$, which resolves an open implication from the Complexity Zoo.

1 Introduction

Automatic complexity is a computable form of Kolmogorov complexity which was introduced by Shallit and Wang in 2001 [17]. It was studied by Jordan and Moser in 2021 [10], and in a series of papers by the author and his coauthors. Roughly speaking, the automatic complexity $A(x)$ of a string is the minimal number of states of an automaton accepting only $x$ among its equal-length peers. The nondeterministic version $A_N$ was introduced by Hyde [9]:

**Definition 1.** Let $\Sigma$ be finite a set called the alphabet and let $Q$ be a finite set whose elements are called states. A nondeterministic finite automaton (NFA) is a 5-tuple $M = (Q, \Sigma, \delta, q_0, F)$. The transition function $\delta : Q \times \Sigma \rightarrow \mathcal{P}(Q)$

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maps each \((q, b) \in Q \times \Sigma\) to a subset of \(Q\). Within \(Q\) we find the initial state \(q_0 \in Q\) and the set of final states \(F \subseteq Q\). The function \(\delta\) is extended to a function \(\delta^* : Q \times \Sigma^* \to \mathcal{P}(Q)\) by

\[
\delta^*(q, \sigma i) = \bigcup_{s \in \delta^*(q, \sigma)} \delta(s, i).
\]

Overloading notation we also write \(\delta = \delta^*\). The language accepted by \(M\) is

\[
L(M) = \{x \in \Sigma^* : \delta(q, x) \cap F \neq \emptyset\}.
\]

A deterministic finite automaton (DFA) is also a 5-tuple \(M = (Q, \Sigma, \delta, q_0, F)\). In this case, \(\delta : Q \times \Sigma \to Q\) is a total function and is extended to \(\delta^*\) by \(\delta^*(q, \sigma i) = \delta(\delta^*(q, \sigma), i)\). If the domain of \(\delta\) is a subset of \(Q \times \Sigma\), \(M\) is a partial DFA. Finally, the set of words accepted by \(M\) is

\[
L(M) = \{x \in \Sigma^* : \delta(q, x) \in F\}.
\]

**Definition 2** (Hyde [9, 17]). Let \(L(M)\) be the language recognized by the automaton \(M\). The nondeterministic automatic complexity \(A_N(w)\) of a word \(w\) is the minimum number of states of an NFA \(M\) such that \(M\) accepts \(w\) and the number of paths along which \(M\) accepts words of length \(|w|\) is 1.

The fundamental upper bound on \(A_N\) was discovered by Kayleigh Hyde in 2013:

**Theorem 3** (Hyde [9]). Let \(x\) be any word of length \(n \in \mathbb{N}\). Then

\[
A_N(x) \leq \lfloor n/2 \rfloor + 1.
\]

The sharpness of Theorem 3 over a ternary alphabet is also known [9]. Therefore, if \(A_N(x) < \lfloor n/2 \rfloor + 1\), we say that \(x\) is \(A_N\)-simple (see Lemma 8).

The question whether the computation of automatic complexity is itself computationally tractable was raised by Shallit and Wang and also by Allender [2]. Here we show that a natural decision problem associated with automatic complexity does not belong to the complexity class \(\text{SAC}^0\) consisting of languages accepted by constant-depth Boolean circuits with semi-unbounded fan-in. We also confirm [11, Conjecture 11] by showing that the set of maximally complex words over a ternary alphabet is not co-context free.

## 2 Context-free languages

A word is called *squarefree* if it cannot be written as \(xyyz\) where \(y\) is nonempty. In ruling out context-freeness of the set of words of nonmaximal nondeterministic automatic complexity, our method of proof goes back to the 1980s. It consists in analyzing the proof in [14] in the form given in Shallit 2008 [16], to

\[^1\text{We denote concatenation by juxtaposition, } \sigma \tau.\]
conclude that it pertains to any language \( L \) with \( \text{SQ}_k \subseteq L \subseteq \text{REP}_k \). Here \( \text{SQ}_k \) is the set of square words over the alphabet \([k]\) and \( \text{REP}_k \) is the set of words over \([k]\) that are repetitive, i.e., not squarefree.

Our key tool will be the Interchange Lemma 4, originally due to Ehrenfeucht and Rozenberg [6]; see also Berstel and Boasson [3]. We denote the length of a word \( x \) by \(|x|\), and let \([k] = \{0, 1, \ldots, k-1\}\) for a nonnegative integer \( k \).

**Lemma 4** (Interchange lemma [14]). For each \( L \in \text{CFL} \) there is an integer \( c > 0 \) such that for all integers \( n \geq 2 \), all subsets \( R \subseteq L \cap \Sigma^n \), and all integers with \( 2 \leq m \leq n \), there exists a subset \( Z \subseteq R \), \( Z = \{z_1, z_2, \ldots, z_k\} \) such that \( k \geq \frac{|R|}{c(n+1)^2} \) and such that there exist decompositions \( z_i = w_i x_i y_i \), \( 1 \leq i \leq k \), such that for all \( 1 \leq i, j \leq k \),

(a) \(|w_i| = |w_j|\),
(b) \(|y_i| = |y_j|\),
(c) \( \frac{n}{2} < |x_i| = |x_j| \leq m \); and
(d) \( w_i x_i y_i \in L \).

**Remark 5.** When establishing Lemma 6 and Lemma 8 we needed a word with a certain property. We first found an example over a 6-letter alphabet, \((123)^20(12345)^2\). We were able to reduce the alphabet size from 6 to 5 by using the slightly modified word \((123)^20(12341)^2\). We expect that the alphabet size, and word length, can be reduced further, but we do not pursue it here.

**Lemma 6.** Let \( w = (123)^20(12341)^2 \in \{0, 1, 2, 3, 4\}^{17} \). Then \( A_N(w) = 8 \).

**Proof.** Let \( M \) be the following NFA.

\[
\begin{align*}
\text{start} & \rightarrow q_0 \\
q_0 & \rightarrow q_1 \quad (0, 3) \\
q_1 & \rightarrow q_2 \quad (1, 2) \\
q_2 & \rightarrow q_3 \quad (2, 3) \\
q_3 & \rightarrow q_4 \quad (1, 4) \\
q_4 & \rightarrow q_5 \\
q_5 & \rightarrow q_6 \\
q_6 & \rightarrow q_7 \\
q_7 & \rightarrow q_8 \quad (3, 1) \\
\end{align*}
\]

Since the equation \( 3x + 1 + 5y = 17 \) has the unique solution \((x, y) = (2, 2)\) over \( \mathbb{N} \), we see that \( M \) accepts \( w \) uniquely via the sequence of states \((q_0, q_1, q_2, q_0, q_1, q_2, q_0, q_0, q_0, q_1, q_2, q_3, q_3, q_3, q_3, q_3, q_3, q_3, q_3)\).

Since \( M \) has only 8 states, it follows that \( A_N(w) \leq 8 \). Conversely, the inequality \( A_N(w) \geq 8 \) was verified with a brute force computerized search in Python 3.7 on July 7, 2021. \( \square \)
**Definition 7.** For a word $x$ and positive integers $p, q$, we define the power $x^\alpha$, \( \alpha = p/q \), to be the prefix of length $n$ of $x^n = xxx \cdots$, where $n = p|x|/q$ is an integer, if it exists. The word $x^\alpha$ is called an $\alpha$-power; if $\alpha = 2$ this is read as squarefree, if $\alpha = 3$ cubefree. If a word $w$ has a (consecutive) subword of the form $x^\alpha$ then $x$ is said to occur with exponent $\alpha$ in $w$. The word $w$ is $\alpha$-power-free if it contains no subwords which are $\alpha$-powers. Finally, $w$ is overlap-free if it is $\alpha$-power-free for all $\alpha > 2$.

As an example of Definition 7 we have $(0110)^{3/2} = 011001$.

**Lemma 8.** There exists an $A_N$-simple overlap-free word of odd length.

*Proof.* Consider the word $w$ of Lemma 6 of length 17.

**Lemma 9.** Let $\Sigma$ be a finite alphabet. Let $\alpha \in \mathbb{Q}$, $\alpha \geq 1$ and let $x \in \Sigma^*$ such that $x$ is an $\alpha$-power. Then $A_N(x) \leq |x|/\alpha$.

*Proof.* Let $x = (x_1 \cdots x_v)^\alpha$ where $m, v \in \mathbb{N}$ such that the fractional part of $\alpha$ is $m/v$ and $x_1, \ldots, x_v \in \Sigma$. Let $M$ be the following NFA, whose digraph is a cycle:

![Diagram](image)

The number of states is $v$, so $A_N(x) \leq v = |x|/\alpha$.

**Lemma 10.** Let $\Sigma$ be a finite alphabet. Consider the following implication for a word $x \in \Sigma^*$.

\[
A_N(x) \leq |x|/\alpha \implies x \text{ contains an } \alpha\text{-power.} \tag{1}
\]

(i) Let $\alpha \geq 1$ be an integer. Then (1) holds.

(ii) (1) fails for $\alpha = 2 + \frac{1}{8}$.

*Proof.* Proof of (i): Suppose that $A_N(x) \leq |x|/k$ as witnessed by an NFA $M$. Then the $|x| + 1$ visiting times of $M$ during its computation on input $x$ are distributed among the at most $|x|/k$ states. Consequently, some state is visited at least \( \left\lfloor \frac{|x|+1}{k} \right\rfloor = k(1 + \frac{1}{|x|}) > k \), and hence at least $k + 1$, times. However, [9] Theorem 16] says: If an NFA $M$ uniquely accepts $x$ of length $n$, and visits a state $p$ at least $k + 1$ times, where $k \geq 2$, then $x$ contains a $k$th power.

Proof of (ii): consider the word $x$ of Lemma 6 of length 17 and let $\alpha = 2 + \frac{1}{8}$. Note that $|x|/\alpha = 17/(2 + \frac{1}{8}) = 8$. 

\[\square\]
Let $\Sigma, \Delta$ be finite alphabets. A morphism $\varphi : \Sigma^* \rightarrow \Delta^*$ is a function satisfying $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in \Sigma^*$. Note that in order to define a particular morphism $\varphi$ it suffices to define $\varphi(a)$ for each $a \in \Sigma$.

A function $f$ is called squarefree-preserving if for each squarefree $x$, $f(x)$ is squarefree.

**Theorem 11** (Brandenburg [5]). The following morphism $h$ is squarefree-preserving:

1. $0 \rightarrow 010201201202101202120120212021012021202102120210212$
2. $1 \rightarrow 01020120212012101202120120212$
3. $2 \rightarrow 0102012101202101210212$
4. $3 \rightarrow 0102012101202120121012$
5. $4 \rightarrow 0102012102010210120212$
6. $5 \rightarrow 0102012102120210120212$

**Definition 12.** The perfect shuffle of two words $x, y$ of the same length $a$ is

$$X(x, y) = x_1y_1 \cdots x_{a}y_{a}.$$ 

We follow Shallit’s presentation [16, Theorem 4.5.4] in Theorem 13, which is a slight strengthening thereof.

**Theorem 13.** For each language $L$,

$$\text{SQ}_6 \subseteq L \subseteq \text{REP}_6 \implies L \not\in \text{CFL}.$$ 

**Proof.** Assume that $L \in \text{CFL}$. Let $c$ be the constant in the Interchange Lemma and choose $n$ divisible by 8 and sufficiently large so that

$$\frac{2^{n/4}}{c(n + 1)^2} > 2^{n/8}.$$ 

Let $r'$ be a squarefree word over the alphabet $\{0, 1, 2\}$ of length $\frac{n}{4} - 1$ and let $r = 3^{r'}$, so that $|r| = \frac{n}{4}$. Let

$$A_n = \{\text{III}(rr, s) : s \in \{4, 5\}^{n/2}\} \subseteq \{0, \ldots, 5\}^n.$$ 

As shown in [16],

1. if $z_i = w_i x_i y_i \in A_n$ for $i \in \{1, 2\}$ with $|w_1| = |w_2|$, $|x_1| = |x_2|$, and $|y_1| = |y_2|$, then $w_1 x_2 y_1 \in A_n$ and $w_2 x_1 y_2 \in A_n$, too; and
2. if $z \in A_n$, then $z$ contains a square if and only if $z$ is a square.

Now let $B_n = A_n \cap \text{SQ}_6 = A_n \cap \text{REP}_6 = \{\text{III}(rr, ss) : s \in \{4, 5\}^{n/4}\}$ and note that $|B_n| = 2^{n/4}$. By the Interchange Lemma with $m = n/2$ and $R = B_n$, there is a subset $Z \subseteq B_n$, $Z = \{z_1, \ldots, z_k\}$ with $z_i = w_i x_i y_i$ satisfying the conclusions of that lemma. In particular,

$$k = |Z| \geq \frac{|R|}{c(n + 1)^2} = \frac{2^{n/4}}{c(n + 1)^2} > 2^{n/8}.$$ 

5
Case 1: There exist indices \( g, h \) such that \( x_g \neq x_h \). By the Interchange Lemma, \( w_gx_hy_g \in L \). Moreover, one of the 4 or 5's was changed going from \( x_g \) to \( x_h \). Since \( |x| \leq m = n/2 \), the corresponding 4 or 5 in the other half was not changed. So \( w_gx_hy_g \) is not a square and hence does not contain a square. So \( w_gx_hy_g \notin L \), contradiction.

Case 2: Case 1 fails. Then all the \( x_i \) are the same and have length at least \( m/2 = n/4 \). Therefore there are at least \( n/4 \) positions in which all the \( z_i \) are the same. Because of the shuffle and since \( n/4 \) is even, at least half of these, i.e., at least \( n/8 \) positions, contain 4's and 5's, which means that these \( n/8 \) positions have constant values within \( Z \). This leaves at most at most \( n/8 \) positions where both choices 4, 5 are available within \( Z \). Thus \( |Z| \leq 2n/8 \), which is a contradiction.

**Theorem 14** ([10, Theorem 4.1.4]). Let \( h \) be a morphism and \( L \in \text{CFL} \). Then \( h^{-1}(L) = \{ x \mid h(x) \in L \} \in \text{CFL} \).

**Theorem 15.** For each language \( L \),
\[
\text{SQ}_3 \subseteq L \subseteq \text{REP}_3 \implies L \notin \text{CFL}.
\]

*Proof.* Assume that \( \text{SQ}_3 \subseteq L \subseteq \text{REP}_3 \). Let \( h \) be the squarefree-preserving morphism of Theorem 11. Since \( h \) is a morphism, for all \( x, w \) we have
\[
w = xx \implies h(w) = h(x)h(x) \in \text{SQ}_3.
\]
Thus \( \text{SQ}_6 \subseteq h^{-1}(\text{SQ}_3) \). Also, the statement that \( h \) is squarefree-preserving is equivalent to: \( h^{-1}(\text{REP}_3) \subseteq \text{REP}_6 \). Thus
\[
\text{SQ}_6 \subseteq h^{-1}(\text{SQ}_3) \subseteq h^{-1}(L) \subseteq h^{-1}(\text{REP}_3) \subseteq \text{REP}_6.
\]
By Theorem 13, \( h^{-1}(L) \notin \text{CFL} \). Hence by Theorem 14, \( L \notin \text{CFL} \).  

**Corollary 16.** \( \{ x \in \{0, 1, 2\}^* : A_N(x) \leq |x|/2 \} \notin \text{CFL} \).

*Proof.* Let \( L = \{ x \in \{0, 1, 2\}^* : A_N(x) \leq |x|/2 \} \). By Lemma 9 and Lemma 10 with \( \alpha = 2 \), \( \text{SQ}_3 \subseteq L \subseteq \text{REP}_3 \). By Theorem 15, \( L \notin \text{CFL} \).

**3 Sensitivity of automatic complexity**

A natural decision problem associated to automatic complexity can be defined as follows.

**Definition 17.** Let \( k, c \geq 1 \) be integers, and let \( A_N \) denote nondeterministic automatic complexity. We define the language \( L_{k,c} \) by
\[
L_{k,c} = \{ x \in [k]^* : A_N(x) > |x|/c \}.
\]
Figure 1: Known implications (⊆) denoted by solid arrow,containments (∈) by dashed arrow, non-containments (∉) by dotted arrow. The non-containments are demonstrated in this paper for the first time except for the one from $L_{3,2}$ to CFL.

The intent is that the decision problem $L_{2,3}$ captures the “random or not” characteristic of a word $x$. Shallit and Wang [17] asked whether automatic complexity can be computed in polynomial time. This remains open, in particular so does the question whether $L_{2,3} \in P$. It is easy to see that $L_{2,3} \in \text{coNP}$ [9].

Our state of knowledge is shown in Figure 1.

In order to prove the remaining main result of this paper, Theorem 33, we shall use Theorem 27 on the abundance of high-complexity words, and a new result Theorem 25. The latter states that for any $x \in \{0,1\}^n$, and $a_1 < \cdots < a_c < n$, there exists $y \in \{0,1\}^n$ which agrees with $x$ on the $a_i$, and has $A_N(y) = O(c^2 \log n)$ where the constant $O$ is independent of $c$ and $n$. Before starting the proof, the reader is invited to consider Table 1.

We now prove some lemmas that will culminate in Theorem 25. Lemma 18 provides a small but valuable refinement of what we would otherwise get more crudely in Lemma 20.

**Lemma 18 (Max Alekseyev [8]).** Let $c \geq 2$ be an integer and let $a_1 < a_2 < \cdots < a_c$ be real numbers. The average of the values $a_j - a_i$, $i < j$, is at most $\frac{(c-1)c}{2}$, where $\ell = a_c - a_1$. 
Table 1: The best bound on $A_N(y)$ as a function of $c$ and $n$ for some small values of $c$ and $n$.

Proof. The average of the values $a_j - a_i$, $i < j$ is

$$
\left(\frac{c}{2}\right)^{-1} \sum_{i<j} |a_j - a_i| = \frac{2}{c^2 - c} \sum_{i<j} a_j - a_i
$$

$$
= \frac{2}{c(c-1)} \sum_{i=0}^{(c-1)/2} (c-1-2i)(a_{c-i} - a_{i+1})
$$

(2)

$$
\leq \frac{2\ell}{c(c-1)} \sum_{i=0}^{(c-1)/2} (c-1-2i)
$$

(3)

$$
\leq \frac{c}{2(c-1)^\ell}.
$$

(4)

For (4), the “large” values $a_c, a_{c-1}, \ldots, a_{c-i}, \ldots, a_{c-(c-1)/2}$ are the larger of the two numbers in a difference $c - 1 - i$ times and the smaller of the two $i$ times. The “small” values $a_{i+1}$ are the smaller $c - 1 - i$ times and the larger $i$ times.

For (1), if $c \in \{2d, 2d+1\}$, the sum is

$$
\sum_{i=0}^{d-1} (c - (2i + 1)) = cd - d^2 = \begin{cases} 
  d^2, & c = 2d \\
  d^2 + d, & c = 2d + 1 
\end{cases}
$$

$$
\leq (c/2)^2 = \begin{cases} 
  d^2, & c = 2d \\
  (d + \frac{1}{2})^2, & c = 2d + 1.
\end{cases}
$$

Definition 19. Let $p_n$ denote the $n$th prime number: $p_1 = 2, p_2 = 3$, etc. The primorial function $x#$ is defined by $x# = \prod_{i=1}^q p_i$, where $q$ is maximal such that $p_q \leq x$.

Lemma 20. Let $0 \leq a_1 < \cdots < a_c < n$, all integers. Let $q \in \mathbb{N}$ be such that $n \leq \frac{2(c-1)}{(p_q#)^{1/2}}$. There exists a modulus $m$ and an integer $j \leq q$ such that $m = p_j \leq p_q$ and all the $a_i$ are distinct mod $m$.
Proof. Suppose that for each $1 \leq m = p_j \leq p_q$, there exist $i \neq j$ with $a_i \equiv a_j \pmod{m}$. Then $\prod_{i=1}^q p_j$ divides $\prod_{i<j} (a_j - a_i)$. Since $i \neq j \implies a_i \neq a_j$ this gives, using Lemma 18 and the AM-GM inequality,
\[ p_q\# = \prod_{i=1}^q p_j \leq \prod_{1 \leq i < j \leq c} a_j - a_i \leq \left( \text{average}(a_j - a_i) \right)^\binom{q}{2} \leq \left( \frac{c}{2(c-1)} \ell \right)^\binom{q}{2}, \]
where $\ell = a_c - a_1 < n$. So
\[ \frac{2(c-1)}{c} (p_q\#)^{1/\binom{q}{2}} < n. \]
Contrapositively, if all the $a_i < n \leq \frac{2(c-1)}{c} (p_q\#)^{1/\binom{q}{2}}$ then there exists a prime $p_j \leq p_q$ such that all the $a_i$ are non-congruent mod $p_j$. \qed

Let log with no subscript denote the natural logarithm.

**Definition 21.** The first Chebyshev function $\vartheta : \mathbb{N} \to \mathbb{R}$ is defined by $\vartheta(x) = \sum_{p \leq x < p+1} \log p = \log(x\#)$ for $p_q \leq x < p_{q+1}$.

**Theorem 22** ([15, equation (3.16)]).
\[ x(1 - 1/\log x) < \vartheta(x), \quad \text{for all } x \geq 41. \]

**Corollary 23.** For each $\varepsilon > 0$, and all sufficiently large $x$,
\[ (1 - \varepsilon) \exp(x) \leq x\#. \]

**Proof.** Theorem 22 is equivalent to
\[ \frac{\exp(x)}{\exp(x/\log x)} < x\#, \quad \text{for } 41 \leq x. \]
\[ \square \]

**Corollary 24.** Let $\varepsilon > 0$ and $c \in \mathbb{N}$. Let $0 \leq a_1 < \cdots < a_c < n$, all integers, with $n$ sufficiently large. Let $q \in \mathbb{N}$ be such that $n \leq \frac{2(c-1)}{c} ((1-\varepsilon) \exp(p_q))^{1/\binom{q}{2}}$. There exists a modulus $m$ and an integer $j \leq q$ such that $m = p_j \leq p_q$ and all the $a_i$ are distinct mod $m$.

**Proof.** From Corollary 23 and Lemma 20 \[ \square \]

**Theorem 25.** Let $c \in \mathbb{N}$ and let $n$ be sufficiently large. For any $a_1 < \cdots < a_c < n$, for each $y \in \{0,1\}^n$, there is an $x \in \{0,1\}^n$ which agrees with $y$ on the $a_i$, and satisfies $A_N(x) \leq \binom{c}{2} \log n < n/3$. \[ \square \]
Proof. Notice that if we take \( n = \frac{2(c-1)}{c} (1 - \varepsilon) \exp(p_q) \) (within \( \pm 1 \)) in Corollary \( 24 \) then solving for \( p_q \), if \( c \geq 2 \) we have

\[
p_q = \log \left\{ \left( \frac{c}{2(c-1)n} \right)^{\frac{n}{2}} \cdot \frac{1}{1 - \varepsilon} \right\}
\]

\[
= \left( \frac{c}{2} \right) \left( \log \frac{n}{2} + \log \frac{c}{(c-1)} \right) - \log(1 - \varepsilon)
\]

\[
\leq \left( \frac{c}{2} \right) \log n,
\]

and so

\[
m \leq p_q = \left( \frac{c}{2} \right) \log n. \tag{5}
\]

Take \( x \) to be a (non-integral, in general) power of \( z \), i.e., \( x = z^{n/m} \), \( |z| = m \) as in \( \text{[15]} \), where \( z(a_i \mod m) = y(a_i) \) for each \( i \). In other words, given \( y \in \{0, 1\}^n \), consider \( y \) on \( \{a_1, \ldots, a_c\} \), copy those values \( \{y(a_1), \ldots, y(a_c)\} \) to \( z \in \{0, 1\}^m \), then extend that \( z \) to \( x \in \{0, 1\}^n \) by repetition. \hfill \( \square \)

**Remark 26.** Let us consider an example of the construction in Theorem \( 25 \).

Let \( c = 12, m = 14 \) (admittedly not a prime, but optimal in this example), and let \( a_i \) and \( b_i = a_i \mod m \) for each \( i \) be as follows.

\[
(a_1, \ldots, a_{12}) = (3, 4, 5, 7, 8, 11, 20, 23, 24, 26, 27, 28)
\]

\[
(b_1, \ldots, b_{12}) = (3, 4, 5, 7, 8, 11, 6, 9, 10, 12, 13, 0)
\]

We let \( y \) be an interesting word from \( \text{[13]} \) Theorem 20\) and let \( z \) be obtained as in Theorem \( 25 \). We underline the positions \( a_i \) in \( x \) and \( y \).

\[
y = 00010101101000011001200111110111
\]

\[
x = 12210101111010122101201111010122 = z^{31/14}
\]

\[
z = 122101011111010
\]

The occurrences of \( 2 \) in \( z \) can be replaced by either \( 0 \) or \( 1 \) as they are unconstrained. The NFA showing that \( A_N(x) \leq m \) is as follows

\[
\begin{align*}
\text{start} & \quad \rightarrow q_0 \quad 1 \quad q_1 \quad 2 \quad q_2 \quad 2 \quad q_3 \quad 1 \quad q_4 \quad 0 \quad q_5 \quad 1 \quad q_6 \\
q_0 & \quad \rightarrow q_1 \quad 0 \quad q_2 \quad 2 \quad q_3 \quad 1 \quad q_4 \quad 0 \quad q_5 \quad 1 \quad q_6
\end{align*}
\]

where the final state, \( q_3 \), is indicated by a double circle.

Theorem \( 25 \) showed that that it is possible for a low-complexity word to agree with a given high-complexity word in given positions. Theorem \( 27 \) will be used for a converse problem: showing that a high-complexity word can agree with a given low-complexity word in given positions.
Theorem 27. Let \( \mathbb{P}_n \) denote the uniform probability measure on words \( x \in \Gamma^n \), where \( \Gamma \) is a finite alphabet of cardinality at least 2. For all \( \epsilon > 0 \),

\[
\lim_{n \to \infty} \mathbb{P}_n \left( \left| \frac{A_N(x)}{n/2} - 1 \right| < \epsilon \right) = 1.
\]

3.1 Defining the class \( \text{SAC}^0 \)

The main fact about \( \text{SAC}^0 \) that we shall use is Theorem 28.

Theorem 28. If \( A \subseteq \Sigma^\ast \) is in \( \text{SAC}^0 \) then there is a constant \( c \) such that for each \( n \), there is a formula \( \psi = \bigvee_{i=1}^k \phi_i \) such that each \( \phi_i \) mentions at most \( c \) many variables, and for \( x = (x_1, \ldots, x_n) \in \Sigma^n \), \( x \in A \) iff \( \psi(x) \) holds.

Remark 29. The literature is not consistent on whether \( \text{SAC}^0 \) is by definition uniform, i.e., uniformly efficiently computable. Aaronson [1] states that “a uniformity condition may also be imposed”. Our result does not require the circuit families to be uniform.

Remark 30. The requirement of polynomial size circuits, while presumably important for \( \text{SAC}^k \) with \( k \geq 1 \), is redundant for \( \text{SAC}^0 \). When the depth is bounded by \( c \), the number of variables in any \( \phi_i \) is bounded by another constant \( d = d(c) \), and there are only \( e = e(d) \) many formulas in \( d \) variables, so that the \( \phi_i \) are chosen from a set of size at most \( \binom{n}{d} e(d) \), which is a polynomial in \( n \) of degree \( d \).

Theorem 28 can be considered already well known. Borodin et al. [4, page 560], referring back to Venkateswaran [18, page 383] describe the semi-unbounded fan-in circuit model as follows.

[...] in this model, we allow OR gates with arbitrary fan-in, whereas all AND gates have bounded fan-in. Input variables and their negations are supplied, but negations are prohibited elsewhere.

An example of a circuit computing the function \( f(x_1, x_2) = x_2 \), with some redundant gates:

```
\begin{center}
\begin{tikzpicture}[node distance = 2cm, auto]
  \node (A) at (0,0) [circle,draw] {\text{A}};
  \node (B) at (0,2) [circle,draw] {\text{A}};
  \node (X1) at (-1,1) [circle,draw] {x_1};
  \node (Y) at (0,1) [circle,draw] {x_2};
  \node (X2) at (1,1) [circle,draw] {x_2};

  \draw[->] (X1) -- (A);
  \draw[->] (Y) -- (B);
  \draw[->] (X2) -- (B);
  \draw[->] (B) -- (Y);
\end{tikzpicture}
\end{center}
```

It is important for us to note that for the described model, all unbounded fan-in OR gates may be assumed to be at the maximum depth, i.e., the OR gates is the output gate and is the furthest removed from the input variables and their negations. This is shown by repeated use of the distributive law for \( \land \) and \( \lor \).
3.2 Immunity to SAC\(^0\)

**Definition 31.** Let \( L \subseteq \Sigma^* \) and \( C \subseteq \mathcal{P}(\Sigma^*) \) where \( \mathcal{P} \) denotes power set. Let \( \text{Inf} = \{ L : L \text{ is an infinite set} \} \).

- The set \( L \) is \( C \)-immune if for all \( C \in \text{Inf} \), \( C \in C \Rightarrow C \not\subseteq L \).
- The set \( L \) is \( C \)-coimmune if \( \Sigma^* \setminus L \) is \( C \)-immune.

Writing \( \text{coC} = \{ L \subseteq \Sigma^* : L \not\in C \} \), we then also have:
- The set \( L \) is \( \text{coC} \)-immune if for all \( C \in \text{Inf} \), \( C \not\in C \Rightarrow C \not\subseteq L \).
- The set \( L \) is \( \text{coC} \)-coimmune if for all \( C \in \text{Inf} \), \( C \not\in C \Rightarrow C \not\subseteq \Sigma^* \setminus L \).

**Remark 32.** We can think of immune sets as “small” and coimmune sets as “big” in some sense. Conversely, if a set \( L \) is both \( C \)-immune and \( C \)-coimmune then it points to an ability to avoid having subsets in \( C \), i.e., sets in \( C \) may tend to be “big” in some sense.

From this point of view, Theorem 33 says that \( L_{2,3} \) is neither big nor small (or if the reader prefers, both big and small), and many of the sets in the class SAC\(^0\) are rather large.

**Theorem 33.** \( L_{2,3} \) is SAC\(^0\)-immune and SAC\(^0\)-coimmune.

**Proof.** Let \( \varphi_n \) be a SAC\(^0\) formula, representing a circuit family. Then \( \varphi_n = \bigvee_i \varphi_{i,n} \) where \( \varphi_{i,n} \) depends only on \( c \) bits of the input, where \( c \) does not depend on \( n \) but the choice of \( c \) bits can depend on both \( n \) and \( i \).

Suppose some \( \varphi_{i,n} \) determines membership in \( L_{2,3} \) based on the bits \( p_1 < \cdots < p_c \) of the input.

To show \( L_{2,3} \) is SAC\(^0\)-coimmune, we show that the bits \( p_i \) cannot guarantee low complexity, i.e., \( x \not\in L_{2,3} \). Let \( n \) be sufficiently large. Fix values \( x(p_i) \) and select the values \( x(k), k \not\in \{p_1, \ldots, p_c\}, 1 \leq k \leq n \), randomly (uniformly and independently). The probability of the values \( x(p_i) \) is \( 2^{-c} \). Let \( \epsilon = 1/3 \). Then

\[
\frac{A_N(x)}{n/2} - 1 < \epsilon \implies A_N(x) \geq \frac{n/2 + 1}{2} > n/3 \implies x \in L_{2,3},
\]

By Theorem 27 for large enough \( n \),

\[
P_n \left( \frac{A_N(x)}{n/2} - 1 < \epsilon \right) \geq 1 - 2^{-(c+1)}.
\]

Therefore, by the union bound, the probability of either disagreeing with some \( x(p_i) \), or having low complexity, is at most

\[ (1 - 2^{-c}) + 2^{-(c+1)} < 1,\]

so the probability of agreeing with the \( x(p_i) \) and also having high complexity is positive.
Since this occurs with positive probability, in particular it occurs for at least one \( x \). For that \( x, x \in L_{2,3} \), as desired. This shows that no single \( \varphi_{i_0,n} \) can imply that \( x \not\in L_{2,3} \). Since \( \varphi_{i_0,n} \) implies \( \bigvee_i \varphi_{i,n} \), neither can \( \bigvee_i \varphi_{i,n} \) imply that \( x \not\in L_{2,3} \). So \( L_{2,3} \) is \( \text{coSAC}_0 \)-immune.

To show \( L_{2,3} \) is \( \text{SAC}_0 \)-immune, we show that the bits \( p_i \) cannot guarantee high complexity, i.e., \( x \in L_{2,3} \).

This follows from Theorem 25, which shows how in the limit we can force \( x \not\in L_{2,3} \).

\[ \square \]

4 The class \( \oplus\text{SAC}^0 \)

Gál and Wigderson [7] considered arithmetic circuits with gates from the basis \{+,-,\times\} over fields such as \( \mathbb{Z}/2\mathbb{Z} \), i.e., \( GF(2) \). All constants of the field may be used. Boolean circuits have the standard Boolean basis \{\&,\lor,\neg\}. Semi-unbounded fan-in circuits have constant fan-in \times\ (resp. \&) gates and unbounded fan-in + (resp. \lor) gates. Semi-unbounded fan-in Boolean circuits may have negations only at the input level. \( \text{SAC}^k \) denotes the class of languages accepted by polynomial size, depth \( O((\log n)^k) \) semi-unbounded fan-in Boolean circuits. \( \oplus\text{SAC}^k \) denotes the class of languages accepted by polynomial size, depth \( O((\log n)^k) \) semi-unbounded fan-in arithmetic circuits over \( GF(2) \).

Gál and Wigderson showed that \( \text{SAC}^1 \subseteq \oplus\text{SAC}^1 \). We note that \( \text{SAC}_0 \neq \text{coSAC}_0 \) was shown in [4]. However, \( \text{co}\oplus\text{SAC}_0 = \oplus\text{SAC}_0 \) since if \( \varphi \) is an multilinear polynomial representing \( L \) then \( \varphi + 1 \) represents \( \Sigma^* \setminus L \). Analogously to Theorem 25 for \( \text{SAC}_0 \), the main fact about \( \oplus\text{SAC}_0 \) that we shall use is Theorem 34.

**Theorem 34.** If \( A \subseteq \Sigma^* \) is in \( \oplus\text{SAC}_0 \) then there is a constant \( c \) such that for each \( n \), there is a formula \( \psi = \bigoplus_{i=1}^k \varphi_i \) such that each \( \varphi \) mentions at most \( c \) many variables, and for \( x = (x_1,\ldots,x_n) \in \Sigma^n \), \( x \in A \) iff \( \psi(x) \) holds.

**Proof.** A formula of bounded depth with bounded fan-in can only mention a bounded number of variables. All the unbounded fan-in occurrences of \( \oplus \) can be folded into one using the distributive law of \( \cdot \) and +.

\[ \square \]

**Remark 35.** In Theorem 34 it follows that the formulas \( \psi \) have size polynomial in \( n \), much as in Remark 30.

We can show that \( \{ x : A_N(x) > 1 \} \in \text{SAC}_0 \setminus \oplus\text{SAC}_0 \) and hence \( \text{SAC}_0 \not\subseteq \oplus\text{SAC}_0 \). Theorem 39 resolves an open implication from the Complexity Zoo [1].

**Definition 36.** We define the degree of a multilinear polynomial by

\[ \deg \left( \sum_{F \in \mathcal{F}} \prod_{i \in F} x_i \right) = \max \{|F| : F \in \mathcal{F} \}. \]

**Lemma 37.** A multilinear polynomial over the ring \( (\mathbb{Z}/2\mathbb{Z}, +) \) is identically 0 as a function over \( \mathbb{Z}/2\mathbb{Z} \) only if all the coefficients are 0.
Proof. Let $P$ be the set of (formal) multilinear polynomials in $n$ variables with coefficients in $\{0, 1\}$. Let $F_P$ be the set of Boolean functions computed by elements of $F$. It suffices to show that two multilinear polynomials are equal as functions only if they are equal as (formal) polynomials, i.e., the map sending a formal polynomial to its function is one-to-one. Thus, it suffices to show that $|F_P| \leq |P|$.

Let $B$ be the set of all Boolean functions in $n$ variables. Each Boolean function may be expressed as a multilinear monomial over $\mathbb{Z}/2\mathbb{Z}$. To wit, $\bot = 0$, $\top = 1$, $\neg a = a + 1$, and $a \land b = a \cdot b$. Thus $B = F_P$.

For each set $S \subseteq \{1, \ldots, n\}$ there is a multilinear monomial $\prod_{i \in S} x_i$. Thus there are $2^n$ multilinear monomials in $n$ variables. Any subset $S$ of these may be included in a multilinear polynomial

$$
\sum_{S \subseteq S} \prod_{i \in S} x_i.
$$

Thus $|P| = 2^{2^n}$. As it is well known that $|B| = 2^{2^n}$, we conclude $|F_P| = |B| = 2^{2^n} = |P|$. \qed

Remark 38. Lemma 37 is a known result, but it should not be confused with the similar representation of Boolean functions as multilinear polynomials over $\mathbb{R}$. For instance, XOR is $x + y - xy$ over $\mathbb{R}$, but $x + y$ over $\mathbb{GF}(2)$. Also, Lemma 37 is somewhat sharp in that if we go beyond linear polynomials we quickly get a counterexample: $x^2 + x$ is identically 0 as a function.

Theorem 39. $\text{SAC}^0 \not\subseteq \oplus \text{SAC}^0$.

Proof. We will show that the family of disjunction functions

$$
\bigvee_n (x_1, \ldots, x_n) = x_1 \lor \cdots \lor x_n
$$

is in $\text{SAC}^0 \setminus \oplus \text{SAC}^0$. To see that $\bigvee_n$ is in $\text{SAC}^0$ is trivial: it is computed by the following constant-depth circuit.

It remains to show that the function $\bigvee_n$ is not in $\oplus \text{SAC}^0$. By de Morgan’s law, $\bigvee_n (x_1, \ldots, x_n) = 1 + \prod_{i=1}^n (1 + x_i) = 1 + \sum_{F \subseteq [n]} x_F = \sum_{\emptyset \neq F \subseteq [n]} x_F$ (6)
where $x_F = \prod_{i \in F} x_i$, $x_\emptyset = 1$. For instance, $V_2(x, y) = 1 + (xy + x + y + 1) = xy + x + y$.

By Lemma 37 if $V_n$ is equal to the function expressed by a $\oplus\text{SAC}^0$ formula (circuit) then that formula expands to the polynomial in (6). By Theorem 34 an $\oplus\text{SAC}^0$ formula is a sum of terms $\sum_i \varphi_i \mod 2$ where each $\varphi_i$ depends on a bounded number of variables. So when we express $\varphi_i$ as a multilinear polynomial, it will have bounded degree since

$$\deg \left( \sum_i \varphi_i \right) \leq \max_i \deg(\varphi_i).$$

Since $V_n$ has degree $n$ by (6), the result follows. \qed

**Corollary 40.** \{x \in \{0, 1\}^n : A_N(x) = 1\} \not\in \oplus\text{SAC}^0.

**Proof.** We have $A_N(x) = 1$ iff $x \in \{0^n, 1^n\}$ (where $n = |x|$), iff

$$\prod_{i \in [n]} x_i + \prod_{i \in [n]} (x_i + 1) = \sum_{\emptyset \neq F \subseteq [n]} \prod_{i \in F} x_i.$$

Since $\sum_{\emptyset \neq F \subseteq [n]} \prod_{i \in F} x_i$ has degree $n - 1$, which is unbounded as $n \to \infty$, the result follows as in Theorem 39. \qed

While we have not settled whether $L_{2,3} \in \oplus\text{SAC}^0$, Corollary 40 does show that $\oplus\text{SAC}^0$ is too limited to capture the computational complexity of $A_N$.

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