ARITHMETIC HYPERBOLICITY: ENDMORPHISMS, AUTOMORPHISMS, HYPERKÄHLER VARIETIES, GEOMETRICITY

ARIYAN JAVANPEYKAR

Abstract. We verify some “arithmetic” predictions made by conjectures of Campana, Hassett–Tscheppel, Green–Griffiths, Lang, and Vojta. Firstly, we prove that every dominant endomorphism of an arithmetically hyperbolic variety over an algebraically closed field of characteristic zero is in fact an automorphism of finite order, and that the automorphism group of an arithmetically hyperbolic variety is a locally finite group. To prove these two statements we use (a mild generalization of) a theorem of Amerik on dynamical systems which in turn builds on work of Bell–Ghioca–Tucker, and combine this with a classical result of Bass–Lubotzky. Furthermore, we show that if the automorphism group of a projective variety is torsion, then it is finite. In particular, we obtain that the automorphism group of a projective arithmetically hyperbolic variety is finite, as predicted by Lang’s conjectures. Next, we apply this result to verify that projective hyperkähler varieties with Picard rank at least three are not arithmetically hyperbolic. Finally, we show that arithmetically hyperbolicity is a “geometric” notion, as predicted by Green–Griffiths–Lang’s conjecture, under suitable assumptions related to Demailly’s notion of algebraic hyperbolicity. For instance, if \( k \subset \mathbb{C} \) is an algebraically closed subfield and \( X \) is an arithmetically hyperbolic variety over \( k \) such that \( X_{\mathbb{C}} \) is Brody hyperbolic, then \( X \) remains arithmetically hyperbolic after any algebraically closed field extension of \( k \).

Contents

1. Introduction 1
2. Groupless varieties 7
3. Arithmetic hyperbolicity 8
4. Geometricity of arithmetic hyperbolicity 12
5. Torsion automorphism groups are finite 17
6. Endomorphisms of arithmetically hyperbolic varieties 19
7. Application to hyperkähler varieties 22
References 23

1. INTRODUCTION

A conjecture of Lang says that a projective variety over \( \mathbb{C} \) is Brody hyperbolic if and only if it is arithmetically hyperbolic. In particular, arithmetically hyperbolic projective varieties should have the same properties as Brody hyperbolic varieties. For instance, as a Brody hyperbolic projective variety has only finitely many automorphisms, Lang's
conjecture predicts that the same should hold for arithmetically hyperbolic projective varieties. In this paper we verify this prediction, and also establish several other results predicted by related conjectures of Campana, Green–Griffiths, Hassett–Tschinkel, and Vojta.

1.1. **Arithmetic hyperbolicity.** In studying the conjectural arithmetic properties of hyperbolic varieties, one is naturally led to the notion of arithmetic hyperbolicity. We start with a precise definition.

Let \( k \) be an algebraically closed field of characteristic zero. A finite type separated scheme \( X \) over \( k \) is **arithmetically hyperbolic over** \( k \) if, for all \( \mathbb{Z} \)-finitely generated subrings \( A \subset k \) and all finite type separated schemes \( \mathcal{X} \) over \( A \) with \( \mathcal{X}_k \cong X \), the set \( \mathcal{X}(A) \) is finite. We refer the reader to Section 3 for a discussion of basic results on arithmetically hyperbolic varieties. Note that arithmetically hyperbolic varieties are also referred to as “Mordellic” or “Siegelsch” (see for instance [63]).

We unravel what the notion of arithmetic hyperbolicity entails for affine varieties. To do so, let \( X \) be an affine variety over \( k \). Choose integers \( n \geq 1 \) and \( m \geq 1 \), choose polynomials \( f_1, \ldots, f_n \in k[x_1, \ldots, x_m] \), and choose an isomorphism

\[
X \cong \text{Spec}(k[x_1, \ldots, x_m]/(f_1, \ldots, f_n)).
\]

Let \( A_0 \) be the subring of \( k \) generated by the (finitely many) coefficients of the polynomials \( f_1, \ldots, f_n \). Note that \( A_0 \subset k \) is a \( \mathbb{Z} \)-finitely generated subring. Define

\[
\mathcal{X} := \text{Spec}(A_0[x_1, \ldots, x_m]/(f_1, \ldots, f_n))
\]

and note that \( \mathcal{X}_k \cong X \). Now, the following statements are equivalent.

1. The variety \( X \) is arithmetically hyperbolic over \( k \).
2. For every \( \mathbb{Z} \)-finitely generated subring \( A \subset k \) containing \( A_0 \), the set

\[
\{(a_1, \ldots, a_m) \in A^m \mid f_1(a_1, \ldots, a_m) = \ldots = f_n(a_1, \ldots, a_m) = 0\}
\]

is finite.

Thus, roughly speaking, one could say that an algebraic variety over \( k \) is arithmetically hyperbolic over \( k \) if it has only finitely many “\( A \)-valued points”, for any choice of finitely generated subring \( A \subset k \).

1.2. **Lang’s conjecture on integral points.** Conjecturally, a projective (integral) variety \( X \) is arithmetically hyperbolic over \( \mathbb{C} \) if and only if it is Brody hyperbolic, i.e., every holomorphic map \( \mathbb{C} \to X^{\text{an}} \) is constant. More generally, we have the following conjecture that relates all “notions of hyperbolicity”.

**Conjecture 1.1** (Consequence of conjectures of Green–Griffiths and Lang). Let \( X \) be a projective variety over \( k \). The following are equivalent.

1. The projective variety \( X \) is arithmetically hyperbolic over \( k \).
2. Every integral closed subvariety of \( X \) is of general type.
3. For every subfield \( k_0 \subset \mathbb{C} \), every embedding \( k_0 \to k \), and every variety \( X_0 \) over \( k_0 \) with \( X \cong X_0 \otimes_{k_0} k \), we have that \( X_{0, \mathbb{C}} \) is Brody hyperbolic.
4. For every abelian variety \( A \) over \( k \), every morphism of varieties \( A \to X \) over \( k \) is constant.
The original versions of this conjecture appeared in [59] and later in [25, Conjecture XV.4.3]; see also [11, §0.3] for a version over finitely generated subrings. Also, in [24] Conjecture 4.3 Vojta extended this conjecture to quasi-projective varieties. The first striking consequence of Lang’s conjecture was obtained by Caporaso-Harris-Mazur [21]; we refer the reader to [51, 54] for other consequences of Lang’s conjecture.

To the extent of our knowledge, Conjecture 1.1 does not imply the more general conjecture that varieties of general type have no dense set of rational points. Indeed, a variety of general type could have a dense set of rational points, even if the above conjecture is true.

In general, one can show that (1) \(\implies\) (4), (2) \(\implies\) (4), and (3) \(\implies\) (4). We refer the reader to Section 3.2 for details.

By Faltings’s Big Theorem [35, 36], the above conjecture is known for projective curves and (more generally) closed subvarieties of abelian varieties. By Faltings’s earlier work on the moduli space of abelian varieties [33, 34] and Zuo’s theorem for period maps [79], Lang’s conjecture is also known to hold for projective varieties over \(k\) with a finite morphism to the moduli stack of principally polarized abelian varieties over \(k\). The more general version of this conjecture for quasi-projective varieties is known in several other cases, by work of Autissier, Corvaja-Zannier, Levin, and Vojta; see Example 3.15.

1.3. Verifying Green–Griffiths–Lang’s predictions. The aim of this paper is to verify certain “arithmetic” predictions made by the Green–Griffiths–Lang conjecture (Conjecture 1.1). In this section we state our main results.

1.3.1. Finiteness of automorphism groups. Our first result says that the automorphism group of a projective arithmetically hyperbolic variety over \(k\) is finite.

**Theorem 1.2.** Let \(k\) be an algebraically closed field of characteristic zero. If \(X\) is a projective arithmetically hyperbolic variety over \(k\), then \(\text{Aut}_k(X)\) is finite.

Note that a Brody hyperbolic projective variety has only finitely many automorphisms [56, Theorem 5.4.4], and that a projective variety over \(k\) of general type has only finitely many automorphisms [38, §11]. Thus, Theorem 1.2 is in accordance with the Green–Griffiths–Lang conjecture.

1.3.2. Geometricity of arithmetic hyperbolicity. Note that “being of general type” is a geometric notion, i.e., persists over any field extension. Similarly, as is shown in [49, Lemma 2.3] and also [52], admitting no non-trivial maps from an abelian variety persists over field extension of \(k\). Thus, the Green–Griffiths–Lang’s conjecture (Conjecture 1.1) predicts that an arithmetically hyperbolic projective variety over \(k\) remains arithmetically hyperbolic after any algebraically field extension \(L\) of \(k\). That is, the notion “being arithmetically hyperbolic” should be a “geometric” notion for projective varieties. In the direction of this “reasonable” expectation, we prove the following result.

**Theorem 1.3.** Let \(k \subset \mathbb{C}\) be an algebraically closed subfield. Let \(X\) be a projective variety over \(k\). Assume that \(X_{\mathbb{C}}\) is Brody hyperbolic. Then \(X\) is arithmetically hyperbolic over \(k\) if and only if \(X_{\mathbb{C}}\) is arithmetically hyperbolic over \(\mathbb{C}\).

A projective variety satisfying either of the properties in the Green–Griffiths–Lang conjecture (Conjecture 1.1) is also conjectured to be algebraically hyperbolic over \(k\) (see [49,
Definition 1.1] for a precise definition). In Section 4.2 we collect some properties of algebraically hyperbolic varieties following [15, 28, 49], and we use these properties to prove Theorem 1.3. We stress that the “geometricity” of Demailly’s notion of algebraic hyperbolicity, as proven in [49, Theorem 7.1], is used to prove our result on the “geometricity” of arithmetic hyperbolicity.

We deduce Theorem 1.3 from the following more general result.

**Theorem 1.4.** Let \( k \subset L \) be an extension of algebraically closed fields of characteristic zero, and let \( X \) be a projective algebraically hyperbolic variety over \( k \). Then \( X \) is arithmetically hyperbolic over \( k \) if and only if \( X_L \) is arithmetically hyperbolic over \( L \).

More generally, it seems reasonable to suspect that the arithmetic hyperbolicity of a (not necessarily projective) algebraic variety over \( k \) persists over any algebraically closed field extension of \( L \). To explain what this means, let \( n \geq 1 \) be an integer, let \( r \geq 1 \) be an integer, and let \( f_1, \ldots, f_r \in \mathbb{Z}[x_1, \ldots, x_n] \) be polynomials with the property that, for any number field \( K \) and any finite set of finite places \( S \) of \( K \) with ring of \( S \)-integers \( \mathcal{O}_{K,S} \), the set of solutions

\[
\{(a_1, \ldots, a_n) \in \mathcal{O}_{K,S}^n \mid f_1(a_1, \ldots, a_n) = \ldots = f_r(a_1, \ldots, a_n) = 0\}
\]

is finite. That is, suppose that the finite type affine scheme

\[
X := \text{Spec} \mathbb{Q}[x_1, \ldots, x_n]/(f_1, \ldots, f_r)
\]

over \( \mathbb{Q} \) is arithmetically hyperbolic over \( \mathbb{Q} \). The persistence of the arithmetic hyperbolicity of \( X \) over an algebraically closed field extension \( k \) of \( \mathbb{Q} \) would entail that, for any finitely generated subring \( A \subset k \), the set

\[
\{(a_1, \ldots, a_n) \in A^n \mid f_1(a_1, \ldots, a_n) = \ldots = f_r(a_1, \ldots, a_n) = 0\}
\]

is finite. In the direction of this “reasonable” expectation, we show that arithmetic hyperbolicity persists, under a “mild boundedness” assumption; see Theorem 4.4.

**1.3.3. Endomorphisms and arithmetic hyperbolicity.** In Theorem 1.2 we verified the finiteness of automorphism groups of projective arithmetically hyperbolic varieties. In our next result, we show that every dominant endomorphism of an arithmetically hyperbolic variety is in fact an automorphism of finite order.

More precisely, it seems reasonable to suspect that the automorphism group of any arithmetically hyperbolic variety is finite. Instead of the finiteness, our next result verifies the local finiteness. Here, we follow standard terminology and say that a group \( G \) is **locally finite** if every finitely generated subgroup of \( G \) is finite.

**Theorem 1.5.** Let \( k \) be an algebraically closed field of characteristic zero and let \( X \) be an arithmetically hyperbolic variety over \( k \). Then, \( \text{Aut}_k(X) \) is a locally finite group and every dominant endomorphism of \( X \) is an automorphism of finite order.

To prove the second statement of Theorem 1.5 we use properties of dynamical systems of arithmetically hyperbolic varieties and Amerik’s theorem on existence of points with infinite orbit. The first statement then follows from combining the first statement with a well-known theorem of Bass–Lubotzky [5, Corollary 1.2]: if \( X \) is a variety over \( k \) and \( \Gamma \subset \text{Aut}_k(X) \) is a finitely generated torsion subgroup, then \( \Gamma \) is finite.

The “analytic” analogue of the second statement in Theorem 1.5 for Brody hyperbolic projective varieties is [56, Theorem 6.6.20]. The “algebraic” analogue of Theorem 1.5
for projective varieties of general type is [8, Proposition 2.2]. Thus, needless to stress, Theorem 1.3 is in accordance with the Green–Griffiths–Lang conjecture.

1.4. **Torsion automorphism groups are finite.** In our proof of the fact that projective arithmetically hyperbolic varieties have a finite automorphism group (Theorem 1.2), we will require a criterion for finiteness of the automorphism group of a projective variety. This criterion is provided by the following result.

**Theorem 1.6.** Let $k$ be an algebraically closed field of characteristic zero, and let $X$ be a projective variety over $k$. If $\text{Aut}_k(X)$ is a torsion group, then $\text{Aut}_k(X)$ is a finite group.

Our proof of Theorem 1.6 uses the theorem of the base, the existence of elements of infinite order in positive-dimensional algebraic groups over $k$, and the well-known fact that automorphisms preserving a fixed ample class in the Néron-Severi group form a finite type group scheme. We refer the reader to Section 5 for the proof of Theorem 1.6 and a discussion of several intermediate results.

We stress that, if $k = \mathbb{C}$ and $X$ is smooth, the proof of Theorem 1.6 is much simpler. Indeed, in this case, one could for instance appeal to [30, Theorem 2.1] to see that $\text{Aut}_\mathbb{C}(X)$ is finitely generated, so that the desired finiteness result follows directly from Bass–Lubotzky’s theorem [5, Corollary 1.2].

1.5. **Application to hyperkähler varieties.** As before, let $k$ be an algebraically closed field of characteristic zero. A smooth projective variety over $k$ is a hyperkähler variety over $k$ if $\pi_1^\text{et}(X)$ is trivial (i.e., $X$ is algebraically simply connected), and $H^{2,0}(X) = H^0(X, \Omega^2_X)$ is a one-dimensional $k$-vector space which can be generated by a non-degenerate form.

Hyperkähler varieties are a part of the building blocks of smooth projective varieties. In fact, by Beauville–Bogomolov’s decomposition theorem [7], every smooth projective variety of Kodaira dimension zero over $k$ admits a finite étale cover by a finite product of hyperkähler varieties, Calabi–Yau varieties, and abelian varieties.

Standard conjectures in the MMP suggest that hyperkähler varieties are not arithmetically hyperbolic. This brings us to our next result. To state this result, we define the Picard rank $\rho(X)$ of a smooth projective variety $X$ over $k$ to be the rank of the Néron-Severi group $\text{NS}(X)$ of $X$.

**Theorem 1.7.** Let $X$ be a smooth projective hyperkähler variety over $k$. If $\rho(X) \geq 3$, then $X$ is not arithmetically hyperbolic over $k$.

Note that a conjecture of Campana [19, Conjecture 13.23] predicts that, if $X$ is a smooth projective variety over a number field $K$ with $\omega_X$ numerically trivial, then there is a finite field extension $L/K$ such that $X(L)$ is Zariski dense in $X$; for results related to Campana’s conjecture we refer to [12, 13, 14, 40, 41, 42, 43]. Note that Theorem 1.7 is a modest contribution to these conjectures.

We note that Theorem 1.7 implies that K3 surfaces of Picard rank at least three are not arithmetically hyperbolic. However, since a K3 surface $X$ over $k$ contains a rational curve [46], it is clear that $X$ is not arithmetically hyperbolic over $k$.

Our proof of Theorem 1.7 uses our result that projective varieties with an infinite automorphism group are not arithmetically hyperbolic (Theorem 1.2). In fact, if $X$ is a hyperkähler variety with Picard rank at least three and finite automorphism group, then $X$ admits a rational curve (Theorem 7.2) in which case it is clear that $X$ is not arithmetically hyperbolic.
1.6. **Outline of paper.** We start with recalling basic properties of “groupless” varieties in Section 2. The main result of this section is Lemma 2.2 and its proof is given in [49]. After this brief discussion of groupless varieties, we move our attention to arithmetic hyperbolicity in Section 3. We start with the definition of arithmetic hyperbolicity (as in [53]) and then show the folklore fact that arithmetically hyperbolic varieties are groupless in Proposition 3.11. The proof of the latter result uses Hassett–Tschinkel’s work on integral points of abelian varieties; we recall their results in Section 3.1. As an application of our results so far, we prove that arithmetically hyperbolic projective varieties are hyperbolic in a “non-archimedean” sense; see Corollary 3.18.

In Section 4 we study the “geometricity” of arithmetic hyperbolicity. That is, we investigate whether the arithmetic hyperbolicity of a variety $X$ over $k$ persists over field extensions. We prove this under the additional assumption that $X$ is “mildly bounded” in Theorem 4.4. As an application, we obtain that arithmetic hyperbolicity is a geometric notion for algebraically hyperbolic projective varieties (Theorem 4.12). In particular, as Brody hyperbolic projective varieties over $\mathbb{C}$ are algebraically hyperbolic over $\mathbb{C}$, this achieves the proof of Theorem 1.3.

In Section 5 we briefly focus on properties of automorphism groups of projective varieties and prove Theorem 1.6. That is, if $X$ is a projective variety over $k$ with $\text{Aut}_k(X)$ a torsion group, then $\text{Aut}_k(X)$ is finite. We isolate several preliminary results in Section 5 and discuss at each stage where our assumption that $k$ is of characteristic zero is used.

In Section 6 we first show that endomorphisms of arithmetically hyperbolic varieties have only finite orbits; this is a fairly easy consequence of the definition of arithmetic hyperbolicity. Then, we state a (mild generalization of a) theorem of Amerik and prove that dominant endomorphisms with finite orbits are automorphisms of finite order (Corollary 6.5). We combine Amerik’s theorem with a well-known structure theorem of Bass–Lubotzky to prove Theorem 1.5. Then, we conclude Section 6 by combining Theorem 1.5 with our earlier established result on automorphism groups of projective varieties (Theorem 1.6) to prove Theorem 1.2.

We conclude this paper by proving that hyperkähler varieties with Picard rank at least three are not arithmetically hyperbolic (Theorem 1.7) in Section 7.

**Acknowledgements.** We are most grateful to Michel Brion for pointing us to the work of Hu–Meng–Zhang [44] and Falcone–Plaumann–Strambach [32], and for several helpful discussions and comments. We thank Antoine Chambert-Loir for many helpful comments on an earlier version of our paper. We thank Jason Starr for discussions on automorphisms of groupless varieties and Daniel Loughran for discussions on arithmetically hyperbolic varieties. We thank Ljudmila Kamenova for helpful discussions on algebraic hyperbolicity and hyperkähler varieties. We thank Ekaterina Amerik and Ronan Terpereau for helpful discussions on dynamical systems. This research was supported through the programme “Oberwolfach Leibniz Fellows” by the Mathematisches Forschungsinstitut Oberwolfach in 2018. We gratefully acknowledge support from SFB/Transregio 45.

**Conventions.** Throughout this paper $k$ will be an algebraically closed field of characteristic zero. A variety over $k$ is a finite type separated integral $k$-scheme.
2. Groupless varieties

In this section we introduce the notion of grouplessness for a variety over \( k \). Roughly speaking, an algebraic variety is groupless if it admits no non-trivial morphisms from an algebraic group. The precise definition reads as follows (see also [49, 52]).

**Definition 2.1** (Grouplessness). A variety \( X \) over a field \( k \) is **groupless** if, for every finite type connected group scheme \( G \) over \( k \), every morphism \( G \to X \) is constant.

Grouplessness is a well-known notion, and sometimes referred to as “algebraic hyperbolicity” or “algebraic Lang hyperbolicity”; see [44], [56, Remark 3.2.24], or [57, Definition 3.4]. To avoid confusion, we will only use the term “algebraically hyperbolic” for the notion defined by Demailly [28, 49].

**Lemma 2.2.** The following statements hold.

1. A finite type scheme \( X \) over \( k \) is groupless over \( k \) if and only if every morphism \( \mathbb{G}_{m,k} \to X \) is constant and, for every abelian variety \( A \) over \( k \), every morphism \( A \to X \) is constant.
2. A proper scheme \( X \) over \( k \) is groupless over \( k \) if and only if, for every abelian variety \( A \) over \( k \), every morphism of varieties \( A \to X \) is constant.
3. Let \( X \) be a proper groupless scheme over \( k \). Then, for every smooth variety \( S \) over \( k \) and every dense open \( U \subset S \), we have that any morphism \( U \to X \) extends uniquely to a morphism \( S \to X \).

**Proof.** This is proven in [49]. (Since a proper groupless variety over \( k \) has no rational curves, the third statement also follows from [27, Corollary 1.44].)

**Lemma 2.3.** If \( X \) is a proper groupless two-dimensional variety over \( k \), then every integral subvariety of \( X \) is of general type and \( \text{Aut}_k(X) \) is finite.

**Proof.** (This follows from [44, Corollary 1.6.2]). For the reader’s convenience, we include a simple proof.) As \( X \) clearly admits no morphisms from a rational or elliptic curve, it suffices to show that \( X \) is of general type.

Let \( X' \to X \) be the minimal resolution of singularities, and let \( X_{\text{min}} \) be the minimal regular model of \( X \); see [61, Ch. 8]. Note that \( X_{\text{min}} \) is a smooth projective surface. Since \( X \) is groupless, the varieties \( X' \) and \( X_{\text{min}} \) have only finitely many rational curves and genus one curves. In particular, by the classification of smooth projective surfaces, the Kodaira dimension of \( X \) is 0, 1, or 2. As smooth proper surfaces of Kodaira dimension 1 admit an elliptic fibration (over a curve) and \( X_{\text{min}} \) admits only finitely many genus one curves, we see that the Kodaira dimension of \( X \) is 0 or 2. To conclude the proof, it suffices to show that the Kodaira dimension of \( X \) is not 0.

Suppose that \( X_{\text{min}} \) admits a finite étale cover \( Y \to X_{\text{min}} \) with \( Y \) an abelian surface. Then, by the third part of Lemma 2.2, the dominant rational map \( Y \to X \) is in fact a well-defined (surjective) morphism. The existence of such a map contradicts the fact that \( X \) is groupless (as \( Y \) is a connected positive-dimensional algebraic group). Thus, \( X_{\text{min}} \) is not an abelian surface up to a finite étale cover.

Therefore, if \( X_{\text{min}} \) is of Kodaira dimension zero, by the classification of surfaces, it is a K3 surface up to a finite étale cover. However, a K3 surface contains infinitely many pairwise distinct connected genus one curves [61, Cor.2.2]. But any finite cover of \( X_{\text{min}} \) admits only finitely many curves of genus one. We conclude that \( X_{\text{min}} \) is not covered by a K3 surface, so that \( X_{\text{min}} \) is of Kodaira dimension two as required.
The fact that $\text{Aut}_k(X)$ is finite for $X$ of general type over $k$ is well-known; see for instance [48, Theorem. 11.12]. □

**Remark 2.4.** Combining the second part of Lemma 2.2 with Lemma 2.3, we see that, for any proper surface $X$ over $k$ which is not of general type, there is an abelian surface $A$ over $k$ and a non-constant morphism $A \to X$. In particular, if $X$ is a proper surface over $k$ with infinite automorphism group, then there is an abelian surface $A$ over $k$ and a non-constant morphism $A \to X$.

### 3. Arithmetic Hyperbolicity

Central to the theme of this paper is the notion of being arithmetically hyperbolic. This notion was introduced by Lang over $\mathbb{Q}$; see also [72, §2], and [3, 4]. Roughly speaking, to test the arithmetic hyperbolicity of a variety, one has to choose a model and “check” the finiteness of integral points on every $\mathbb{Z}$-finitely generated subring of $k$. The precise definition reads as follows.

**Definition 3.1 (Arithmetic hyperbolicity).** Let $X$ be variety over $k$. Then $X$ is **arithmetically hyperbolic** over $k$ (or: **has-only-finitely-many-integral-points over $k$**) if there is a $\mathbb{Z}$-finitely generated subring $A \subset k$ and a finite type separated $A$-scheme $\mathcal{X}$ with $\mathcal{X}_k \cong X$ over $k$ such that, for all $\mathbb{Z}$-finitely generated subrings $A \subset A' \subset k$, the set of $A'$-points $\mathcal{X}(A')$ on $\mathcal{X}$ is finite.

We refer the reader to [53, §4] for basic properties of arithmetically hyperbolic varieties. For example, a variety $X$ over $k$ is arithmetically hyperbolic over $k$ if, for all $\mathbb{Z}$-finitely generated subrings $A \subset k$ and all finite type separated schemes $\mathcal{X}$ over $A$ with $\mathcal{X}_k \cong X$, the set $\mathcal{X}(A)$ is finite. [53, Lemma 4.7].

**Remark 3.2.** If $X$ is a variety over a finitely generated field $K$ of characteristic zero such that, for every finite field extension $L/K$, the set $X(L)$ is finite, then $X_K$ is arithmetically hyperbolic over $K$.

#### 3.1. Integral points on abelian varieties

In the next section we will show that arithmetically hyperbolic varieties are groupless. To prove this result, we will collect some preliminary results on integral points of abelian varieties in this section; these results are due to Hassett–Tschinkel, Lang, and Néron. We start with the following generalization of Mordell–Weil’s theorem for abelian varieties.

**Lemma 3.3** (Lang–Néron’s Mordell–Weil). Let $K$ be a finitely generated field of characteristic zero and let $A$ be an abelian variety over $K$. Then, the abelian group $A(K)$ is finitely generated.

**Proof.** This is [24, Cor. 7.2] (see also the original paper of Lang–Néron [60]). □

We now show that the rank of an abelian variety $A$ over a finitely generated field $K$ of characteristic zero can grow arbitrarily large over finite extensions of $K$. We stress that Lemma 3.4, Lemma 3.5, Lemma 3.6, and Lemma 3.7 below are due to Hassett–Tschinkel [40].

**Lemma 3.4** (Rank jumping). Let $K$ be a finitely generated field of characteristic zero and let $A$ be an abelian variety over $K$. Then there is a finite field extension $L/K$ such that the rank of $A(L)$ is strictly bigger than the rank of $A(K)$. 

Proof. If $K$ is a number field, then this is due to Hassett–Tschinkel [40, Lemma 3.2]. To prove the lemma, we follow the arguments in loc. cit.. Thus, let $\Gamma$ be the saturation of $A(K)$ in $A(\overline{K})$, and note that $\Gamma$ contains all the torsion points of $A(\overline{K})$. Choose a finite extension $K_1 \subset \overline{K}$ of $K$ and a smooth curve $C$ of genus at least two contained in $A$. By [67, Theorem 1] (formerly the Manin-Mumford conjecture, proven in [69]), the intersection $C(\overline{K}) \cap \Gamma$ is finite. Thus, there is a finite field extension $L$ of $K_1$ such that $C(L) \setminus \Gamma$ is non-empty, so that the rank of $A(L)$ is strictly bigger than the rank of $A(K)$. \hfill $\square$

Lemma 3.5. Let $K$ be a finitely generated field of characteristic zero and let $A$ be a geometrically simple abelian variety over $K$. Then, there is a finite field extension $L/K$ and a point $P$ in $A(L)$ such that the subgroup generated by $P$ in $A(L)$ is Zariski dense in $A$.

Proof. We follow Hassett–Tschinkel’s proof of [40, Proposition 3.1]. Thus, let $L/K$ be a finite field extension and let $P$ be a point of infinite order in $A(L)$; such data exists by Lemma 3.4. The connected component of the closure of the subgroup generated by $P$ is a non-trivial abelian subvariety of $A$. Since $A$ is geometrically simple, we conclude that the closure of the subgroup generated by $P$ is equal to $A$. \hfill $\square$

Lemma 3.6. Let $K$ be a finitely generated field of characteristic zero and let $A_1$ be a geometrically simple abelian variety over $K$. Let $B$ be an abelian variety over $K$ and let $Q \in B(K)$ be a point such that the subgroup generated by $Q$ is Zariski dense in $B$. Then, there is a finite field extension $L/K$ such that $A_1(L) \times B(L)$ contains a point $P$ such that the subgroup generated by $P$ is Zariski dense in $A_1 \times B$.

Proof. When $K$ is a number field, this is proven in [40, Lemma 3.3]. The proof given there works over finitely generated fields of characteristic zero, as we explain now.

By Lemma 3.5, replacing $K$ by a finite field extension if necessary, there is a point $P \in A_1(K)$ such that the subgroup generated by $P_1$ is Zariski dense in $A_1$. Moreover, replacing $K$ by a finite field extension if necessary, we may and do assume that $\text{Hom}(A_1, B) = \text{Hom}_K(A_1, B)$. Let $Z_1, \ldots, Z_\ell$ be a basis of $\text{Hom}(A_1, B)$. Considering $B$ as an element $\beta$ of $\text{Hom}^0(A_1, B)$ (with $\beta(P_1) = Q$), we see that there exists integers $b_1, \ldots, b_\ell$ and $d \neq 0$ with $dQ = (b_1Z_1 + \ldots + b_\ell Z_\ell)(P_1)$. Thus, $Q$ is contained in the saturation $\Gamma$ of the subgroup of $B(K)$ generated by the images of $P_1$ under the $Z_\i$. By Lemma 3.4, there is a finite field extension $L/K$ and point $Q'$ in $B(K)$ such that $q$ is not contained in $\Gamma$. Then, it is clear that the point $P := (P_1, Q')$ in $A_1(L) \times B(L)$ has the required property. \hfill $\square$

Proposition 3.7. Let $K$ be a finitely generated field of characteristic zero and let $A$ be an abelian variety over $K$. Then, there is a finite field extension $L/K$ and a point $P$ in $A(L)$ such that the subgroup generated by $P$ in $A(L)$ is Zariski dense in $A$.

Proof. As in Hassett–Tschinkel’s proof of [40, Prop. 3.1], replacing $K$ by a finite field extension and $A$ by an isogenous abelian variety if necessary, we may and do assume that $A$ is a product of $n$ geometrically simple abelian varieties with $n \geq 1$. We proceed by induction on $n$ (as in loc. cit.). If $n = 1$, then the statement follows from Lemma 3.5. If $n > 1$, then we write $A = A_1 \times B$ with $A_1$ a geometrically simple abelian variety over $k$ and $B$ the product of precisely $n - 1$ geometrically simple abelian varieties. Now, the induction hypothesis implies that there is a finite field extension $K_1/K$ and a point $Q$ in $B(K_1)$ such that the subgroup generated by $Q$ is Zariski dense in $B$. It follows from Lemma 3.6 that
there is a finite field extension $L/K_1$ and a point $P$ in $A(L) = A_1(L) \times B(L)$ such that the subgroup generated by $P$ is Zariski dense in $A = A_1 \times B$. This concludes the proof.

**Corollary 3.8.** Let $k$ be an algebraically closed field of characteristic zero and let $G$ be an abelian variety over $k$. Then there is a finitely generated subfield $L \subset k$ and an abelian variety $\mathcal{G}$ over $L$ with $\mathcal{G}_k \cong G$ over $k$ such that $\mathcal{G}(L)$ is Zariski-dense in $G$.

**Proof.** We first “descend” the abelian variety $G$ over $k$ to a finitely generated subfield. Thus, choose a finitely generated subfield $K \subset k$ and an abelian variety $G'$ over $K$ such that $G'_k \cong G$ over $K$. By Lemma 3.7 there is a finite field extension $L$ of $K$ contained in $k$ such that $G'(L)$ is Zariski dense in $G$. Thus, the corollary holds with $\mathcal{G} := G'_L$. □

**Lemma 3.9.** Let $S$ be an integral regular noetherian scheme. Let $\mathcal{G}$ be an abelian scheme over $S$. Then $\mathcal{G}(S) = \mathcal{G}(K(S))$.

**Proof.** Note that the geometric fibres of $\mathcal{G} \to S$ do not contain any rational curves. Therefore, the result follows from [37, Proposition 6.2]. □

**Corollary 3.10.** Let $k$ be an algebraically closed field of characteristic zero and let $G$ be an abelian variety over $k$. Then there is a smooth $\mathbb{Z}$-finitely generated subring $A \subset k$ and an abelian scheme $\mathcal{G} \to \text{Spec } A$ with $\mathcal{G}_k \cong G$ such that $\mathcal{G}(A)$ is Zariski-dense in $G$.

**Proof.** Choose a smooth $\mathbb{Z}$-finitely generated subring $A \subset k$ and an abelian scheme $\mathcal{G} \to \text{Spec } A$ such that $\mathcal{G}_k \cong G$ and such that $\mathcal{G}(\text{Frac}(A))$ is Zariski dense in $G$; such data exists by Corollary 3.8. To conclude the proof, note that $\mathcal{G}(A) = \mathcal{G}(\text{Frac}(A))$ by Lemma 3.9. □

### 3.2. Arithmetically hyperbolic varieties are groupless.

Lang conjectured that a projective groupless variety over $k$ is arithmetically hyperbolic over $k$. In other words, if a projective variety $X$ over $k$ has infinitely many “integral points”, then Lang’s conjecture implies that there should be an abelian variety $A$ and a non-constant morphism $A \to X$ (by Lemma 2.2 (2)). In this section we prove the converse of this statement.

**Proposition 3.11.** If $X$ is an arithmetically hyperbolic variety over $k$, then $X$ is groupless over $k$.

**Proof.** We first show that every morphism $f : \mathbb{G}_{m,k} \to X$ is constant. To do so, choose a $\mathbb{Z}$-finitely generated subring $A \subset k$, a model $\mathcal{X}$ for $X$ over $A$ and a morphism $F : \mathbb{G}_{m,A} \to \mathcal{X}$ with $F_k \cong f$ such that $\mathbb{G}_m(A)$ is infinite. It follows that $\mathcal{X}(A)$ is infinite, unless $f$ is constant.

Now, let $G$ be an abelian variety over $k$, and let $G \to X$ be a morphism. To show that $G \to X$ is constant, choose a smooth $\mathbb{Z}$-finitely generated subring $A \subset k$, a model $\mathcal{X}$ for $X$ over $A$, an abelian scheme $\mathcal{G}$ over $A$ with $\mathcal{G}_k \cong G$, and a morphism $\mathcal{G} \to \mathcal{X}$ such that $\mathcal{G}(A)$ is Zariski dense in $G$; such data exists by Corollary 3.10. Note that the set $\mathcal{G}(A)$ maps to the set $\mathcal{X}(A)$ via $\mathcal{G} \to \mathcal{X}$. Therefore, since $\mathcal{G}(A)$ is Zariski dense in $G$, the image of the finite set $\mathcal{G}(A)$ in $\mathcal{X}(A)$ is Zariski dense in the (closed, scheme-theoretic) image of $G \to X$. However, since $X$ is arithmetically hyperbolic over $k$, any closed subscheme of $X$ is arithmetically hyperbolic over $k$. Thus, the image of $G \to X$ is an arithmetically hyperbolic connected variety whose set of $k$-points contains a finite and dense subset. This implies that the image of $G \to X$ is finite, so that $G \to X$ is constant.

To conclude the proof, apply the first part of Lemma 2.2. □
Remark 3.12. Let $X$ be a projective variety over $k$. If every integral closed subvariety of $X$ is of general type, then it is not hard to see that $X$ is groupless. Moreover, if $k = \mathbb{C}$ and $X$ is Brody hyperbolic, then $X$ is groupless (cf. the proof of [52, Lemma 2.14]). Thus, needless to stress, Proposition 3.11 is in accordance with the Green–Griffiths–Lang conjecture (Conjecture 1.1).

The grouplessness of arithmetically hyperbolic varieties implies that rational maps from a normal variety are defined everywhere, as we show now.

Corollary 3.13. Let $X$ be a proper arithmetically hyperbolic variety over $k$. Let $Y$ be a normal variety over $k$. Let $f : Y \rightarrow X$ be a rational map. Then $f$ extends uniquely to a morphism $Y \rightarrow X$.

Proof. Since $X$ is arithmetically hyperbolic, it follows that $X$ is groupless. Therefore, the lemma follows from the third part of Lemma 2.2. \qed

Remark 3.14. Let $X$ be a proper surface over $k$. If $X$ is arithmetically hyperbolic over $k$, then every integral subvariety of $X$ is of general type. To prove this, note that $X$ is groupless (3.11), so that the claim follows from Lemma 2.3. In particular, a K3 surface over $k$ is not arithmetically hyperbolic over $k$.

We conclude this section with examples of arithmetically hyperbolic varieties.

Example 3.15. It follows from Faltings’s theorem [33, 34] that a one-dimensional variety $X$ over $k$ is arithmetically hyperbolic over $k$ if and only if it is groupless over $k$. Moreover, it follows from Faltings’s theorem [35] that a closed subvariety $X$ of an abelian variety $A$ over $k$ is arithmetically hyperbolic over $k$ if and only if $X$ is groupless. We refer to [3, 4, 6, 26, 35, 63, 66, 70, 77, 75] for more examples of arithmetically hyperbolic varieties.

Example 3.16. It follows from Faltings’s theorem [35] that a smooth projective groupless surface over $k$ with $\text{h}^1(X, \mathcal{O}_X) > 2$ is arithmetically hyperbolic over $k$. Moreover, if $k = \mathbb{C}$, then such a surface is Brody hyperbolic. Thus, as every subvariety of such a surface is of general type (Lemma 2.3), we conclude that the Green–Griffiths–Lang conjecture (Conjecture 1.1) is known for smooth projective surfaces with $\text{h}^1(X, \mathcal{O}_X) > 2$.

Example 3.17. Let $X$ be a smooth projective curve over $k$ and let $n \geq 1$ be an integer. Then, it follows from Faltings’s theorem that the Green–Griffiths–Lang conjecture holds for the $n$-fold symmetric product $\text{Sym}^n_X$ of $X$. Namely, $\text{Sym}^n_X$ is groupless over $k$ if and only if it is arithmetically hyperbolic over $k$ if and only if it is every subvariety of $\text{Sym}^n_X$ is of general type. Moreover, if $k = \mathbb{C}$, then $\text{Sym}^n_X$ is groupless if and only if it is Brody hyperbolic.

We conclude this section with the following “non-archimedean” application of the grouplessness of an arithmetically hyperbolic variety. In the following statement, we follow the notation and conventions of [52].

Corollary 3.18. Let $k$ be an algebraically closed field of characteristic zero and let $X$ be an arithmetically hyperbolic projective variety over $k$. Let $K$ be the completion of an algebraic closure of the field $k((t))$ with its natural $t$-adic valuation. Then, for every finite type connected group scheme $G$ over $K$, every morphism of adic spaces $G^\text{an} \rightarrow X_K^\text{an}$ is constant.
Proof. Since $X$ is arithmetically hyperbolic over $k$, it follows that $X$ is groupless over $k$ (Proposition 3.11). Since $X$ is groupless over $k$, it follows from [52] Theorem 1.3 that $X_{an}^K$ is $K$-analytically Brody hyperbolic [52] Definition 2.3], i.e., for every finite type connected group scheme $G$ over $K$, every morphism of adic spaces $G_{an} \to X_{an}^K$ is constant. \hfill \Box

4. Geometricity of arithmetic hyperbolicity

In this section we study the persistence of arithmetic hyperbolicity over field extensions, under suitable “boundedness” assumptions related to Demailly’s notion of algebraic hyperbolicity [28]. Our main result (Theorem 1.3) says that arithmetic hyperbolicity of a projective variety persists over field extensions provided that the variety is Brody hyperbolic; see Section 4.3 for the proof.

4.1. Mild boundedness. With the aim of isolating the weakest property we require for proving the persistence of arithmetic hyperbolicity along field extensions, we start with the notion of “mild boundedness”.

Definition 4.1. A finite type scheme $X$ over $k$ is mildly bounded if, for every smooth quasi-projective curve $C$ over $k$, there exists an integer $m \geq 1$ and points $c_1, \ldots, c_m \in C(k)$ such that, for every $x_1, \ldots, x_m \in X(k)$ the set

\[ \text{Hom}_k((C, c_1, \ldots, c_m), (X, x_1, \ldots, x_m)) := \{ f : C \to X \mid f(c_1) = x_1, \ldots, f(c_m) = x_m \} \]

is finite.

The precise interplay between arithmetic hyperbolicity and mild boundedness should become clear in Theorem 4.4. We start with a preliminary result.

Lemma 4.2. Let $k \subset L$ be an extension of algebraically closed fields of characteristic zero such that $L$ is of transcendence degree 1 over $k$. If $X$ is an arithmetically hyperbolic mildly bounded variety over $k$, then $X_L$ is arithmetically hyperbolic over $L$.

Proof. Let $A \subset k$ be a $Z$-finitely generated subring and let $\mathcal{X} \to \text{Spec} A$ be a finite type separated model for $X$ over $A$ (so that $\mathcal{X}_k \cong X$). Note that $\mathcal{X}$ is also a finite type model for $X_L$ over $A \subset L$.

Let $A \subset B \subset L$ be a $Z$-finitely generated subring with $B$ a smooth $Z$-algebra. To prove the lemma, it suffices to show that $\mathcal{X}(B)$ is finite. We define $C := \text{Spec} B$, and note that $\mathcal{X}(C) = \mathcal{X}(B)$.

Since $X$ is arithmetically hyperbolic over $k$, for any intermediate subring $A \subset A' \subset k$ with $A'$ finitely generated over $Z$, the set $\mathcal{X}(A')$ is finite. Therefore, to prove that $\mathcal{X}(C)$ is finite, we may and do assume that the subring $B \subset L$ is not contained in $k$.

Let $K$ be the fraction field of $A$, and note that $K \subset k \subset L$. Then, as $B$ is not contained in $k$ and $L$ is of transcendence degree one over $k$, we have that $\mathcal{C}_K \to \text{Spec} K$ is a smooth affine connected one-dimensional scheme over $K$.

Define $C := \mathcal{C}_k$. Note that $C$ is a smooth affine curve over $k$, and that there is an inclusion of sets

\[ \mathcal{X}(C) = \text{Hom}_A(C, \mathcal{X}) \subset \text{Hom}_k(C, X). \]

We now use that $X$ is mildly bounded over $k$ to show that $\mathcal{X}(C)$ is finite. Indeed, since $X$ is mildly bounded over $k$, there exists an integer $m \geq 1$ and points $c_1, \ldots, c_m \in C(k)$ such that, for all $x_1, \ldots, x_m \in X(k)$, the set

\[ \text{Hom}_k((C, c_1, \ldots, c_m), (X, x_1, \ldots, x_m)) \]
is finite. We choose $m$ and $c_1, \ldots, c_m \in C(k)$ with this property.

Let $A \subset A' \subset k$ be a smooth $\mathbb{Z}$-finitely generated subring and let $\overline{c}_1, \ldots, \overline{c}_m \in C(A')$ be such that $\overline{c}_{1,k} = c_1, \ldots, \overline{c}_{m,k} = c_m$ in $C(k)$. (In other words, extend the base ring $A$ in such a way that the points $c_1, \ldots, c_m$ become sections of $C$.)

Now, define $\mathcal{D} := C \times_{A'} A'$. Note that $\mathcal{X}(C) \subset \mathcal{X}(\mathcal{D})$. Thus, it suffices to show that $\mathcal{X}(\mathcal{D})$ is finite. Note that $\overline{c}_1, \ldots, \overline{c}_m$ are sections of $\mathcal{D}$, so $\overline{c}_i$ is an element of $\mathcal{X}(\mathcal{D})$. Therefore, we have an inclusion of sets

$$\mathcal{X}(\mathcal{D}) \subset \bigcup_{(\overline{c}_1, \ldots, \overline{c}_m) \in \mathcal{X}(A')^m} \text{Hom}((\mathcal{D}, \overline{c}_1, \ldots, \overline{c}_m), (\mathcal{X}, \overline{c}_1, \ldots, \overline{c}_m)),$$

where $\mathcal{X}(A')^m$ denotes the product of sets $\mathcal{X}(A') \times \ldots \times \mathcal{X}(A')$.

Note that $\mathcal{D}_k = C$. Thus, for any $(\overline{c}_1, \ldots, \overline{c}_m) \in \mathcal{X}(A')^m$, we have an inclusion of sets

$$\text{Hom}_C((\mathcal{D}, \overline{c}_1, \ldots, \overline{c}_m), (\mathcal{X}, \overline{c}_1, \ldots, \overline{c}_m)) \subset \text{Hom}_k((C, c_1, \ldots, c_m), (X, \overline{x}_1, \ldots, \overline{x}_m)).$$

Thus, we conclude that $\mathcal{X}(\mathcal{D})$ is finite from the finiteness of $\mathcal{X}(A')^m$ and the finiteness of the set $\text{Hom}_k((C, c_1, \ldots, c_m), (X, \overline{x}_1, \ldots, \overline{x}_m))$. This concludes the proof. \qed

Lemma 4.3. Let $k \subset L$ be an extension of algebraically closed fields of characteristic zero such that $L$ has finite transcendence degree over $k$. Let $X$ be an arithmetically hyperbolic variety over $k$. If $X_L$ is mildly bounded over $L$, then $X_L$ is arithmetically hyperbolic over $L$.

Proof. We proceed by induction on the transcendence degree of $L$ over $k$. Let $n := \text{trdeg}_k L$. If $n = 0$, we are done by our assumption that $X$ is arithmetically hyperbolic over $k$ (as $k = L$ in this case). Thus, let us assume that the statement of the lemma has been proven for extensions of transcendence degree $\leq n - 1$. Let $k \subset K \subset L$ be an algebraically closed subfield with $\text{trdeg}_k K = n - 1$. Then $X_K$ is arithmetically hyperbolic over $K$ (by the induction hypothesis and the fact that $X_K$ is mildly bounded over $K$). Thus, as $K \subset L$ has transcendence degree 1 and $X_K$ is a mildly bounded arithmetically hyperbolic variety over $K$, the result follows from Lemma 4.2. \qed

Theorem 4.4. Let $k \subset L$ be an extension of algebraically closed fields of characteristic zero. Let $X$ be an arithmetically hyperbolic variety over $k$. If $X_L$ is mildly bounded over $L$, then $X_L$ is arithmetically hyperbolic over $L$.

Proof. To say that $X_L$ is arithmetically hyperbolic over $L$ is equivalent to saying that, for every algebraically closed subfield $k \subset K \subset L$ of finite transcendence degree over $k$, the variety $X_K$ is arithmetically hyperbolic over $K$. Therefore, the result follows from Lemma 4.3. \qed

4.1.1. A conjecture on mildly bounded varieties. It would be interesting to show that arithmetically hyperbolic varieties are actually mildly bounded, as this would imply that arithmetically hyperbolic varieties remain arithmetically hyperbolic over any extension of the base field.

Conjecture 4.5. If $X$ is an arithmetically hyperbolic variety over $k$, then $X$ is mildly bounded over $k$.

Mild boundedness is essentially the “weakest” notion of boundedness required for arithmetical hyperbolicity to persist over a field extension, and Conjecture 4.5 predicts that this “weak” notion on boundedness holds for all arithmetically hyperbolic varieties.
In the next section we use our earlier work with Ljudmila Kamenova on Demailly’s notion of algebraic hyperbolicity to show that arithmetic hyperbolicity persists over field extensions, under assumptions related to the notion of mild boundedness.

4.2. Demailly’s notion of algebraic hyperbolicity. In this section we combine our results from our earlier work with Kamenova [49] with Theorem 4.4, and provide new results on the geometricity of arithmetic hyperbolicity for projective varieties.

Before gathering the relevant statements from [49], we recall the definitions (see also [49, §4]).

**Definition 4.6 (Algebraic hyperbolicity).** A projective variety $X$ over $k$ is algebraically hyperbolic over $k$ if, for every ample line bundle $L$ on $X$, there is a real number $\alpha(L)$ such that, for every smooth projective connected curve $C$ and every morphism $f : C \to X$, the inequality
\[
\deg_C f^*L \leq \alpha(L) \cdot \text{genus}(C)
\]
holds.

In [49] several properties of of algebraically hyperbolic projective varieties are proven using earlier work of Kovács-Lieblich [58] and Hwang–Kebekus–Peternell [47].

In [49] the reader will also find notions of “boundedness” related to algebraic hyperbolicity (Definition 4.6). The definitions of these notions of boundedness are as follows.

**Definition 4.7 (Boundedness).** Let $n \geq 1$ be an integer. A projective variety $X$ over $k$ is $n$-bounded over $k$ (or: $(n,0)$-bounded over $k$) if, for every normal projective variety $Y$ over $k$ of dimension at most $n$, the scheme $\text{Hom}_k(Y,X)$ is of finite type over $k$.

**Definition 4.8 (Pointed boundedness).** Let $n \geq 1$ and $m \geq 1$ be integers. A projective variety $X$ over $k$ is $(n,m)$-bounded over $k$ if, for every normal projective variety $Y$ over $k$ of dimension at most $n$, every set of pairwise distinct points $y_1, \ldots, y_m$ in $Y(k)$, and every $x_1, \ldots, x_m \in X(k)$, the scheme
\[
\text{Hom}_k((Y,y_1,\ldots,y_m),(X,x_1,\ldots,x_m))
\]
parametrizing morphisms $f : Y \to X$ with $f(y_1) = x_1, \ldots, f(y_m) = x_m$ is of finite type over $k$.

It is clear that for $n \geq 1$ and $m \geq 0$, if $X$ is $(n,m)$-bounded over $k$, then $X$ is $(n,m+1)$-bounded. We now recall the results of [49].

**Remark 4.9.** Motivated by the fact that one can “test” hyperbolicity on curves (see for instance [50, Theorem 1.5]), we established that boundedness can be “tested” on maps from curves. Let $X$ be a projective variety over $k$. Then it follows from [49, Theorem 9.2] that the following are equivalent.

1. There is an integer $n \geq 1$ such that $X$ is $n$-bounded over $k$.
2. For all integers $n \geq 1$, the projective variety $X$ is $n$-bounded over $k$.

Moreover, it follows from [49, Theorem 8.4] that the following are equivalent.

1. There are integers $n \geq 1$ and $m \geq 1$ such that $X$ is $(n,m)$-bounded over $k$.
2. For all integers $n \geq 1$ and $m \geq 1$, the projective variety $X$ is $(n,m)$-bounded over $k$.

**Theorem 4.10 (Relations between notions of boundedness).** Let $X$ be a projective variety over $k$. 
(1) Let \( m \geq 1 \) be an integer. The projective variety \( X \) is \((1, m)\)-bounded over \( k \) if and only if, for every smooth projective connected curve \( C \) over \( k \), every set of pairwise distinct \( c_1, \ldots, c_m \) in \( C(k) \), and every \( x_1, \ldots, x_m \in X(k) \), the set
\[
\text{Hom}_k((C, c_1, \ldots, c_m), (X, x_1, \ldots, x_m)) := \{ f : C \to X \mid f(c_1) = x_1, \ldots, f(c_m) = x_m \}
\]
is finite.

(2) Let \( m \geq 0 \) be an integer. If \( X \) is \((1, m)\)-bounded over \( k \), then \( X \) is mildly bounded.

(3) Assume \( X \) is algebraically hyperbolic over \( k \). Then, for all integers \( n \geq 1 \) and \( m \geq 0 \), the projective variety \( X \) is \((n, m)\)-bounded and mildly bounded over \( k \).

**Proof.** Note that (1) is precisely [49, Lemma 4.6]. Now, to prove (2), we may and do assume that \( m = 1 \); see [49, Proposition 4.8]. Then, by (1), for every smooth projective connected curve \( C \) over \( k \), every \( c \) in \( C(k) \), and every \( x \) in \( X(k) \), the set \( \text{Hom}_k((C, c), (X, x)) \) is finite. Thus, since \( X \) is proper over \( k \), by the valuative criterion of properness, it follows that, for every smooth quasi-projective connected curve \( C \) over \( k \), every \( c \) in \( C(k) \), and every \( x \) in \( X(k) \), the set \( \text{Hom}_k((C, c), (X, x)) \) is finite. This clearly implies that \( X \) is mildly bounded over \( k \). Finally, note that (3) follows from [49, Theorem 9.4] and (2). This concludes the proof. 

The following result says that algebraic hyperbolicity and boundedness are “geometric” properties, i.e., persist over any field extension of \( k \).

**Theorem 4.11 (Geometricity of boundedness).** Let \( k \subset L \) be an extension of algebraically closed fields of characteristic zero. Let \( X \) be a projective variety over \( k \).

1. If \( X \) is algebraically hyperbolic over \( k \), then \( X_L \) is algebraically hyperbolic over \( L \).
2. Let \( m \geq 1 \) be an integer. If \( X \) is \((1, m)\)-bounded over \( k \), then \( X_L \) is \((1, m)\)-bounded over \( L \).

**Proof.** We note that (1) is [49, Theorem 7.1], and that (2) is [49, Corollary 9.3].

We can now combine the above results from [49] with the main result of the previous section (Theorem 4.4) to get a useful criterion for geometricity.

**Theorem 4.12.** Let \( k \subset L \) be an extension of algebraically closed fields of characteristic zero. Let \( X \) be a projective arithmetically hyperbolic variety over \( k \).

1. If \( m \geq 0 \) and \( X \) is \((1, m)\)-bounded over \( k \), then \( X_L \) is arithmetically hyperbolic over \( L \).
2. If \( X \) is algebraically hyperbolic over \( k \), then \( X_L \) is arithmetically hyperbolic over \( L \).

**Proof.** To prove (1), assume that \( X \) is \((1, m)\)-bounded over \( k \). Note that \( X_L \) is \((1, m)\)-bounded over \( L \) by the geometricity of \((1, m)\)-boundedness (Theorem 4.11). In particular, the projective variety \( X_L \) is mildly bounded over \( L \) (Theorem 4.10 (2)). Since \( X \) is an arithmetically hyperbolic variety over \( k \) and \( X_L \) is mildly bounded over \( L \), by Theorem 4.10 (3), we may conclude that \( X_L \) is arithmetically hyperbolic over \( L \). This proves (1).

If \( X \) is algebraically hyperbolic over \( k \), then \( X \) is \((1, m)\)-bounded over \( k \) by Theorem 4.10 (3). Thus, (2) follows from (1).
Proof of Theorem 4.4. If $X$ is arithmetically hyperbolic over $k$, then it follows from Theorem 4.12(2) that $X_L$ is arithmetically hyperbolic over $L$. Conversely, if $X_L$ is arithmetically hyperbolic over $L$, then it is clear that $X$ is arithmetically hyperbolic over $k$. This concludes the proof. □

4.3. Integral points on analytically hyperbolic varieties. In this section we prove Theorem 1.3. Our proof uses that Brody hyperbolic projective varieties over $\mathbb{C}$ are algebraically hyperbolic; a result due to Demailly [28, Theorem 2.1]. With the aim at greater clarity, we first state and prove a slightly more general result.

Proposition 4.13. Let $k \subset L$ be an extension of algebraically closed fields of characteristic zero. Let $X$ be a projective arithmetically hyperbolic variety over $k$. Suppose that there is a subfield $k_0 \subset k$, a projective variety $X_0$ over $k_0$, an isomorphism $X_0, k_0 \cong X$ over $k$, and an embedding $k_0 \subset \mathbb{C}$ such that $X_0, \mathbb{C}$ is Brody hyperbolic. Then $X_L$ is arithmetically hyperbolic over $L$.

Proof. Since $X_0, \mathbb{C}$ is a projective Brody hyperbolic, it follows from Brody’s lemma that $X_0, \mathbb{C}$ is a Kobayashi hyperbolic variety [56, Theorem 3.6.3]. In particular, as $X_0, \mathbb{C}$ is a Kobayashi hyperbolic projective variety, it follows that $X_0, \mathbb{C}$ is algebraically hyperbolic over $\mathbb{C}$; see [49, Theorem 1.2]. It follows that $X_0$ is algebraically hyperbolic over $k_0$. However, by the geometricity of algebraic hyperbolicity [49, Theorem 7.1], we conclude that $X_0, k_0 \cong X$ is algebraically hyperbolic over $k$. Now, as $X$ is an algebraically hyperbolic and arithmetically hyperbolic projective variety over $k$, we conclude that $X_L$ is arithmetically hyperbolic (Theorem 4.12(2)). □

Proof of Theorem 1.3. Let $X$ be a projective arithmetically hyperbolic variety over $k$ such that $X, \mathbb{C}$ is Brody hyperbolic. It follows that $X, \mathbb{C}$ is arithmetically hyperbolic over $\mathbb{C}$ from Proposition 4.13 with $k_0 := k$ and $L := \mathbb{C}$. □

It might be useful to include an overview of the relations (currently known to us) between the notions of hyperbolicity appearing in this paper. We do so in the following remark.

Remark 4.14. Let $X$ be a projective variety over $k$. We briefly say that $X$ is conjugate-Brody hyperbolic over $k$ if there is a subfield $k_0 \subset k$, an embedding $k_0 \to \mathbb{C}$, a projective variety $X_0$ over $k_0$, and an isomorphism $X_0, k_0 \cong X$ over $k$ such that $X_0, \mathbb{C}$ is Brody hyperbolic. With this definition, the following implications hold for $X$ over $k$: 

- $X$ is algebraically hyperbolic over $k$ $\implies$ $X$ is bounded over $k$
- $X$ is conjugate–Brody hyperbolic over $k$ $\implies$ $X$ is $1$–bounded over $k$
- $X$ is mildly bounded over $k$ $\iff$ $X$ is $(1, 1)$–bounded over $k$
- $X$ is arithmetically hyperbolic over $k$ $\implies$ $X$ is groupless over $k$
5. TORSION AUTOMORPHISM GROUPS ARE FINITE

The main result of this section says that, for a projective variety \( X \) over the field \( k \) (of characteristic zero), the automorphism group of \( X \) is infinite if and only if it is non-torsion; see Theorem \[\text{1.6}\]. To prove this result, we will use basic facts about Néron-Severi groups, automorphisms preserving some fixed ample class, and \( k \)-points of positive-dimensional finite type group schemes over \( k \). Presumably this result is “well-known” to experts, as the arguments we use already appear in the literature (in some form or another); see for instance [20], [61, Corollary 6.1.7], or [78].

Let \( S \) be a scheme and let \( X \to S \) be a morphism. The functor \( \text{Aut}_{X/S} \) on the category of schemes over \( S \) is defined by \( \text{Aut}_{X/S}(T) = \text{Aut}_T(X_T) \). We first use basic representability results for this functor to prove the following result.

**Lemma 5.1.** Let \( k \subset L \) be an extension of algebraically closed fields. Let \( X \) be a projective variety over \( k \). If \( \text{Aut}_k(X) \) is finite, then \( \text{Aut}_L(X_L) \) is finite.

**Proof.** The group scheme \( \text{Aut}_{X/k} \) is locally of finite type over \( k \) and has only finitely many \( k \)-points. This implies that \( \text{Aut}_{X/k} \) is finite over \( k \). Thus, \( \text{Aut}_{X/L} = \text{Aut}_{X/k} \otimes_k L \) is finite over \( L \), so that \( \text{Aut}(X_L) = \text{Aut}_{X/L}(L) \) is finite. This proves the lemma. \( \square \)

For \( X \) a proper scheme over a field \( k \), we let \( \text{NS}(X) \) be the Néron-Severi group of \( X \), and we define \( \text{NS}(X)_\mathbb{Q} := \text{NS}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \). If \( L \) is a line bundle on \( X \), we let \([L] \) denote the class of \( L \) in \( \text{NS}(X)_\mathbb{Q} \). The following well-known proposition says that, for \( L \) an ample line bundle on a projective variety \( X \) over \( k \), the group of automorphisms of \( X \) over \( k \) which fix the class of \( L \) in \( \text{NS}(X)_\mathbb{Q} \) is the group of \( k \)-points on a finite type group scheme over \( k \).

**Proposition 5.2.** Let \( X \) be a projective scheme over \( k \), and let \( L \) be an ample line bundle on \( X \). The functor defined by

\[
(S\text{ch}/k)^{\text{op}} \to \text{Groups},
\]

\[
S \mapsto \{ g \in \text{Aut}_S(X_S) \mid \text{for all geometric points } \overline{\pi} \to S, \ g_{\overline{\pi}}[L] = [L] \text{ in } \text{NS}(X_{\overline{\pi}})_\mathbb{Q} \}
\]

is representable by a finite type group scheme \( \text{Aut}_{X/k,[L]} \) over \( k \).

**Proof.** The proof of this is given in [65, Remark 2.6]. (Note that we do not need that \( k \) is of characteristic zero.) \( \square \)

We will also require the following simple group-theoretic lemma. It is essentially a consequence of the fact that a homomorphism of groups \( G \to H \) is trivial, provided \( G \) is torsion and \( H \) is torsion free.

**Lemma 5.3.** Let \( G \) be a torsion group. Let \( \Gamma \) be a finitely generated abelian group. Then, any morphism of groups \( G \to \text{Aut}(\Gamma) \) has finite image.

**Proof.** Let \( G \to \text{Aut}(\Gamma) \) be a morphism and let \( G' \) be its image. Note that \( G' \) is a torsion subgroup of \( \text{Aut}(\Gamma) \).

Minkowski showed that, for all positive integers, the group \( \text{GL}_n(\mathbb{Z}) \) has a torsion-free normal finite index subgroup (cf. Selberg’s lemma [22, Theorem 2]). Therefore, by [39, Corollary 6], the group \( \text{Aut}(\Gamma) \) has a torsion-free normal finite index subgroup, say \( H \).

Consider the morphism \( G' \subset \text{Aut}(\Gamma) \to \text{Aut}(\Gamma)/H \). The kernel of this morphism is \( G' \cap H \). Since \( H \) is torsion-free and \( G' \) is torsion, we see that \( G' \cap H \) is trivial. Since the index of \( G' \cap H \) in \( G \) is bounded by the index of \( H \) in \( \text{Aut}(\Gamma) \), we see that \( G' \cap H \) has...
finite index in $G'$. Thus, as $G' \cap H$ is trivial and of finite index in $G'$, we conclude that $G' = \text{Im}[G \to \text{Aut}(\Gamma)]$ is finite.

We now show that positive-dimensional algebraic groups over algebraically closed fields of characteristic zero have elements of infinite order. This is a non-trivial fact when $k$ is countable.

Lemma 5.4. Let $k$ be an algebraically closed field of characteristic zero. Let $G$ be a finite type group scheme over $k$ such that $G(k)$ is torsion. Then $G$ is finite.

Proof. We may and do assume that $G$ is connected. Since $k$ is of characteristic zero, by Cartier's theorem [71, Tag 047N], the group scheme $G$ is smooth. Thus, $G$ is an “algebraic group over $k$” in the sense of [23]. By Chevalley’s theorem [23, Theorem 1.1], there is a unique normal affine connected linear algebraic subgroup $H$ in $G$ such that $G/H$ is an abelian variety. If $H$ is non-trivial, then $H$ contains either $\mathbb{G}_a(k)$ or $\mathbb{G}_m(k)$ as a subgroup. Since $k$ is of characteristic zero, the group $\mathbb{G}_a(k)$ is not torsion, and the group $\mathbb{G}_m(k)$ is not torsion. Thus, if $H$ is non-trivial, then $H(k)$ contains non-torsion elements. Therefore, as $G(k)$ is torsion, it follows that $H$ is trivial, so that $G$ is an abelian variety (by the defining property of $H$). However, as $k$ is of characteristic zero, if $G$ is a positive-dimensional abelian variety over $k$, then $G(k)$ contains a point of infinite order [32, Theorem 9]. Therefore, we conclude that $G$ is the trivial group, as required. □

Remark 5.5. Note that, if $L$ is a field of characteristic $p > 0$, then $\mathbb{G}_a(L)$ is a positive-dimensional (non-finite) group scheme over $L$, and $\mathbb{G}_a(L) = (L,+)$ is an abelian $p$-torsion group. Thus, Lemma 5.4 is false over any algebraically closed field $L$ of positive characteristic.

We are now ready to prove our “criterion” for finiteness of the automorphism group of a projective variety over an algebraically closed field of characteristic zero.

Proof of Theorem 1.6. Let $X$ be a projective variety over $k$ such that $\text{Aut}_k(X)$ is a torsion group. Since $\Gamma := \text{NS}(X)$ is a finitely generated abelian group (see for instance [16, Theorem 8.4.7]), it follows from Lemma 5.3 that there is a finite index subgroup $H \subset \text{Aut}_k(X)$ which acts trivially on $\Gamma := \text{NS}(X)$. Let $L$ be an ample line bundle on $X$, and note that $H$ fixes the class of $L$ in $\text{NS}(X)_\mathbb{Q}$. By Proposition 5.2, the group of automorphisms which leave the class of $L$ fixed in $\text{NS}(X)_\mathbb{Q}$ is representable by a finite type group scheme $G := \text{Aut}_{X/k,[L]}$. Note that

$$H \subset G(k) \subset \text{Aut}_k(X).$$

In particular, as $\text{Aut}_k(X)$ is torsion (by assumption), it follows that $G(k)$ is torsion. Thus, by Lemma 5.4 the finite type group scheme $G$ is finite over $k$. As $H \subset G(k)$, we see that $H$ is finite. Since $H$ is of finite index in $\text{Aut}_k(X)$, we conclude that $\text{Aut}_k(X)$ is finite. □

Remark 5.6. The analogue of Theorem 1.6 is false for projective varieties over $\mathbb{F}_p$. Indeed, let $X$ be a smooth proper connected curve of genus one over $K = \mathbb{F}_p$. Then, the automorphism group $\text{Aut}_K(X)$ is torsion and infinite.

Remark 5.7. It seems reasonable to suspect that the analogue of Theorem 1.6 fails over any algebraically closed field of positive characteristic. Indeed, at the bottom of page 10 in [18], Brion constructs a smooth projective surface $S$ over $k$ such that $\text{Aut}_S^0 = \mathbb{G}_a$. In particular, for this surface $S$, the group $\text{Aut}_S^0(k) = \mathbb{G}_a(k)$ is infinite and torsion.
Remark 5.8. Theorem 1.6 confirms that, if \( X \) is a projective variety over \( k \) and \( \text{Aut}_k(X) \) is torsion, then \( \text{Aut}_k(X) \) is finite. We stress that this is \textit{not} a consequence of Bass–Lubotzky’s theorem which in this case “only” says that every finitely generated subgroup of \( \text{Aut}_k(X) \) is finite (i.e., \( \text{Aut}_k(X) \) is locally finite); see [5] and [10, Theorem 1.2]. (It also seems worthwhile stressing that there are smooth projective varieties over \( \mathbb{C} \) such that \( \text{Aut}_{\mathbb{C}}(X) \) is a discrete non-finitely generated group; see Remark 5.12.)

As an application of Theorem 1.6 we now prove the following more general result.

Corollary 5.9. Let \( k \subset L \) be an extension of algebraically closed fields of characteristic zero. Let \( X \) be a projective variety over \( k \). If \( \text{Aut}_k(X) \) is torsion, then \( \text{Aut}_L(X_L) \) is finite.

\textbf{Proof.} By Theorem 1.6 the group \( \text{Aut}_k(X) \) is finite. Thus, by Lemma 5.1 the group \( \text{Aut}_L(X_L) \) is finite. This proves the corollary. \( \square \)

Corollary 5.10. Let \( k \subset L \) be an extension of algebraically closed fields of characteristic zero. Let \( X \) be a projective variety over \( k \). Then \( X \) has an automorphism of infinite order if and only if \( X_L \) has an automorphism of infinite order.

\textbf{Proof.} If \( X \) has an automorphism of infinite order, then \( X_L \) has an automorphism of infinite order. Thus, to prove the corollary, suppose that \( X \) has no automorphism of infinite order. Then \( \text{Aut}_k(X) \) is torsion, so that \( \text{Aut}_L(X_L) \) is finite (Corollary 5.9). Therefore, the group \( \text{Aut}_L(X_L) \) has no element of infinite order. This proves the corollary. \( \square \)

Remark 5.11. Corollary 5.10 is false in positive characteristic. Indeed, let \( E \) be a smooth proper connected genus one curve over \( k := \mathbb{F}_p \). Then \( E \) has no automorphisms of infinite order over \( k \). Let 0 be an element of \( E(k) \), and let \( L \) be an uncountable algebraically closed field containing \( k \). Then the elliptic curve \( (E, 0) \) over \( k \) has an \( L \)-point of infinite order, say \( x \). Translation by \( x \) is an infinite order automorphism of the \( L \)-scheme \( E_L \).

Remark 5.12. For all \( n \geq 2 \), there exists a smooth projective simply connected \( n \)-dimensional variety \( X \) over \( \mathbb{C} \) such that \( \text{Aut}^0_{X/\mathbb{C}} \) is trivial and \( \text{Aut}_{\mathbb{C}}(X) \) is a non-finitely generated (infinite, non-torsion) group; see [29] [62]. Note that the arguments and ideas in \textit{loc. cit.} are different from those used in our proof of Theorem 1.6.

6. Endomorphisms of arithmetically hyperbolic varieties

In this section we study orbits of endomorphisms of arithmetically hyperbolic varieties. Obviously, if \( f : X \to X \) is an endomorphism of an arithmetically hyperbolic variety \( X \), then the \( f \)-orbit of each point is finite. This basic observation is the starting point of our proofs of Theorems 1.2 and 1.5.

6.1. Orbits of endomorphisms of arithmetically hyperbolic varieties. Let \( k \) be an algebraically closed field of characteristic zero. If \( f : X \to X \) is an endomorphism of a variety over \( k \) and \( i \geq 1 \), we let \( f^i : X \to X \) be the composition of \( f \) with itself \( i \)-times. We let \( f^0 := \text{id}_X \). Also, for \( f : X \to X \) an endomorphism and \( x \in X(k) \), we define \( O_f(x) := \{ f^k(x) \}_{k \geq 0} \). We will refer to \( O_f(x) \) as the \textit{(forward) \( f \)-orbit of \( x \).}

The crucial “arithmetic” observation is the following (very) simple lemma.

\textbf{Lemma 6.1.} Let \( X \) be an arithmetically hyperbolic variety over \( k \). Let \( E \subset \text{End}(X) \) be a subset. Assume that there is a \( \mathbb{Z} \)-finitely generated subring \( A \subset k \) and a finite type
separated model $X$ for $X$ over $A$ such that every $f \in E$ descends to an endomorphism of $X$ over $A$. Then, for every $x$ in $X(k)$, the set
$$\bigcup_{f \in E} O_f(x)$$
is finite.

Proof. For every $f : X \to X$ in $E$, let $F : X \to X$ be a model for $f$ over $A$, and let $F(A) : \mathcal{X}(A) \to \mathcal{X}(A)$ be the induced map of sets. Note that, for $i \geq 1$, the morphism $F^i : X \to X$ is a model for $f^i$ over $A$. Since $X$ is arithmetically hyperbolic over $k$, the set $\mathcal{X}(A)$ is finite.

Let $x$ be in $X(k)$. Note that $\mathcal{X}(A) \subset X(k)$. Let $A \subset B \subset k$ be a $\mathbb{Z}$-finitely generated subring such that $x$ lies in $\mathcal{X}(B)$. Now, for all $f$ in $E$, the $f$-orbit $O_f(x)$ of $x$ lies in the finite set $\mathcal{X}(B)$. Thus, we see that $O_f(x)$ is finite. This concludes the proof of the lemma.

Lemma 6.2. Let $X$ be an arithmetically hyperbolic variety over $k$. Let $f \in \text{End}(X)$ and let $x \in X(k)$. Then the $f$-orbit $O_f(X)$ of $x$ is finite.

Proof. Define $E := \{f\} \subset \text{End}(X)$. Note that there is a $\mathbb{Z}$-finitely generated subring $A \subset k$ and a finite type model $\mathcal{X}$ over $A$ such that $f$ descends to an endomorphism $F : \mathcal{X} \to \mathcal{X}$ of $\mathcal{X}$ over $A$. Thus, the lemma follows from applying Lemma 6.1 to $E$.

6.2. Dynamical systems of infinite order. Let $k$ be an algebraically closed field of characteristic zero, and let $X$ be a variety over $k$. Recall that a dominant endomorphism $f : X \to X$ has finite order if there exist pairwise distinct positive integers $n$ and $m$ such that $f^n = f^m$.

In [2] Amerik proved that dominant endomorphisms which are not of finite order have points of infinite order. The methods of Amerik are inspired by the work of many authors on dynamical systems of varieties over number fields; see for instance [9, 11, 31, 38, 68, 70]. We will require a mild generalization of Amerik’s theorem in which we allow the base field to be an algebraically closed field of characteristic zero (which is not necessarily $\overline{\mathbb{Q}}$).

Theorem 6.3 (Amerik + $\epsilon$). Let $X$ be a variety over $k$, and let $f : X \to X$ be a dominant morphism. If the orbit of every point $x$ in $X(k)$ is finite, then $f$ has finite order.

Proof. If $k$ is uncountable, this is “obvious”. If $k = \overline{\mathbb{Q}}$ this is proven by Amerik [2, Corollary 9]. The arguments in Amerik can be used (with minor modifications) to prove the theorem, as we explain now.

Firstly, by standard “spreading out” arguments, we may and do choose the following data.

1. A $\mathbb{Z}$-finitely generated subring $A \subset k$;
2. A finite type separated model $\mathcal{X}$ for $X$ over $A$;
3. A morphism of schemes $\tilde{f} : \mathcal{X} \to \mathcal{X}$ with $\tilde{f}_k = f$;
4. A prime number $p$, a finite extension $K$ of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_K$, and an embedding $A \subset \mathcal{O}_K$;
5. A maximal ideal $p \subset \mathcal{O}_K$;
6. A section $x \in \mathcal{X}(A) \subset \mathcal{X}(\mathcal{O}_K) \subset \mathcal{X}(K)$;
7. A dense open affine subscheme $U \subset \mathcal{X}$ containing $x$ and a finite surjective morphism of schemes $U \to \mathbb{A}^n_A$ with $U = \text{Spec} A[x_1, \ldots, x_n, x_{n+1}, \ldots, x_m]/I$. 20
Replacing $A$ by a $\mathbb{Z}$-finitely generated, $p$ by a larger prime number if necessary and $K$ by a finite field extension if necessary, we may and do assume that the above data satisfies the following properties.

1. every point in the orbit of $x_p$ is smooth on $X_p$ and the orbit of $x_p$ is disjoint from the ramification locus of $\tilde{f}_p$.
2. The coefficients of the power series $x_{n+1}, \ldots, x_m, \tilde{f}^s x_1, \ldots, \tilde{f}^s x_m$ lie in $A$ (when considered as power series in $x_1, \ldots, x_n$).
3. For all $n+1 \leq i \leq m$, the (monic) minimal polynomial $P_i$ of $x_i$ over $K[x_1, \ldots, x_n]$ has coefficients in $A$ and the derivative $P'_i$ of $P_i$ is not identically zero modulo $p$.

To construct this data, one can use Cassels’ embedding theorem (which is arguably the only “additional” ingredient necessary to adapt Amerik’s arguments) and (following Amerik) Hrushovski’s theorem on intersections of graphs with Frobenius [2, Corollary 2] (which relies on [43, 73]).

In the rest of the proof we follow Amerik. Thus, define $\mathcal{N}_{p,x}$ to be

$$\mathcal{N}_{p,x} := \{ t \in U(K) \mid x_i(t) \equiv x_i(x) \text{ for } 1 \leq i \leq m \}. $$

Now, the $p$-adic “uniformisation” theorem of Bell–Ghioca–Tucker as proven by Amerik (see [2, Proposition 3]) implies that the following holds. There is an integer $\ell \geq 1$ and an integer $N \geq 1$ such that $f^\ell$ maps $\mathcal{N}_{p,x}$ into itself, every preperiodic point in $\mathcal{N}_{p,x}$ has order at most $N$, and the subset $X(k) \cap \mathcal{N}_{p,x}$ is dense in $\mathcal{N}_{p,x}$. This implies the result by [2, Corollary 8].

To prove the main result of this section (Corollary 6.5), we will require the following simple lemma.

**Lemma 6.4.** Let $X$ be a finite type separated scheme over $k$. Let $f : X \to X$ be a dominant endomorphism. Suppose that there are distinct positive integers $n$ and $m$ such that $f^n = f^m$. Then $f^{n-m} = \text{id}_X$.

**Proof.** We may and do assume that $n > m$. Write $g := f^{n-m}$. Let $k \geq 2$ be an integer such that $(k-1)n - km \geq 0$. Define $G := (f^{n-m})^{k-1} = f^{n(k-1)-m(k-1)}$. Let $P \in X$ be in the image of $G$. Let $Q$ be a point such that $G(Q) = P$. Then,

$$f^{n-m}(P) = f^{n-m}(Q) = f^{n(k-1)-m(k-1)}(Q) = f^{n(k-1)-mk}(Q) \equiv f^n(Q) \equiv f^{n-m}(Q).$$

Thus, $P$ is a fixed point of $f^{n-m}$. We now use this observation to show that $f^{n-m} = \text{id}_X$.

Let $X^g := \{ P \in X \mid g(P) = P \}$ be the fixed locus of $X$. Since $X$ is separated, $X^g$ is a closed subscheme of $X$. Indeed, $X^g$ is the intersection of $\Delta$ and the graph of $g$ in $X \times X$. However, since every point in the image of $G$ is a fixed point of $f^{n-m}$, we see that $X^g$ contains the image of $G$. Moreover, as $f$ is dominant, the morphism $G := (f^{n-m})^{k-1}$ is dominant. Thus, the closed subscheme $X^g$ contains the dense subset $G(X)$ of $X$. Therefore, since $X^g$ is closed and dense, it follows that $X^g = X$. We conclude that $g = \text{id}_X$, as required. □

**Corollary 6.5.** Let $X$ be a finite type separated integral scheme over $k$, and let $f : X \to X$ be a dominant endomorphism. If the orbit of every point $x$ in $X(k)$ is finite, then $f$ is an automorphism of finite order.
Proof. By Amerik’s theorem (Theorem 6.3), the morphism \( f \) has finite order. Therefore, by Lemma 6.4 we conclude that \( f \) is an automorphism of finite order. □

6.3. Proof of Theorems 1.2 and 1.5. We first combine the theorem of Amerik [2] with the basic observation that endomorphisms of arithmetically hyperbolic varieties have finite orbits to prove Theorem 1.5.

Proof of Theorem 1.5. Let \( X \) be an arithmetically hyperbolic variety over \( k \). We first show that every dominant endomorphism is an automorphism of finite order. Thus, let \( f : X \to X \) be a dominant endomorphism of \( X \) over \( k \). Since \( X \) is arithmetically hyperbolic, we see that \( f \) has finite orbits (Lemma 6.2). Thus, it follows from Corollary 6.5 that \( f \) is an automorphism of finite order. This proves the second statement of the theorem. To prove the first statement, we argue as follows. As every automorphism of \( X \) has finite order, the group \( \text{Aut}_k(X) \) is a torsion group. Let \( \Gamma \subset \text{Aut}_k(X) \) be a finitely generated subgroup. Then, as \( \text{Aut}_k(X) \) is torsion, the group \( \Gamma \) is a finitely generated torsion subgroup of \( \text{Aut}_k(X) \). Therefore, by the theorem of Bass–Lubotzky [5, Corollary 1.3] (see also [10, Theorem 1.2]), the group \( \Gamma \) is finite. This shows that \( \text{Aut}_k(X) \) is a locally finite group, and concludes the proof. □

It follows from Theorem 1.5 that the automorphism group of an arithmetically hyperbolic variety is torsion. Therefore, automorphism group of a projective arithmetically hyperbolic must be finite by Theorem 1.6. We now make this more precise.

Corollary 6.6. Let \( k \subset L \) be an extension of algebraically closed fields of characteristic zero. Let \( X \) be an arithmetically hyperbolic projective variety over \( k \). Then \( \text{Aut}(X_L) \) is finite.

Proof. Let \( X \) be a projective arithmetically hyperbolic variety over \( k \). Since \( X \) is arithmetically hyperbolic over \( k \), it follows that \( \text{Aut}_k(X) \) is a torsion group (Theorem 1.5). Thus, it follows from Corollary 5.9 that \( \text{Aut}_L(X_L) \) is finite. □

Proof of Theorem 1.2. This follows from Corollary 6.6 (with \( k = L \)). □

7. Application to hyperkähler varieties

In this section we prove Theorems 1.7 and 7.2. As usual, let \( k \) be an algebraically closed field of characteristic zero. We start with a well-known property of complex algebraic hyperkähler varieties.

Lemma 7.1. If \( Y \) is a hyperkähler variety over \( \mathbb{C} \) with Picard rank at least three and only finitely many automorphisms over \( \mathbb{C} \), then \( Y \) admits a rational curve.

Proof. This is shown by Kamenova–Verbitsky in the proof of [55, Theorem 3.7], and relies on earlier work of Boucksom and Huybrechts; see [17, 45]. Indeed, under our assumptions, the Kähler cone does not coincide with the positive cone, so that the result follows from [55, Lemma 3.6]. □

Theorem 7.2. Let \( X \) be a hyperkähler variety over \( k \). If \( \rho(X) \geq 3 \) and \( \text{Aut}_k(X) \) is finite, then \( X \) admits a rational curve over \( k \) and \( X \) is not groupless over \( k \).

Proof. Clearly, it suffices to prove that \( X \) admits a rational curve over \( k \).

We first descend “everything” to a subfield of \( \mathbb{C} \). Thus, let \( k_0 \subset k \) be an algebraically closed subfield, let \( k_0 \to \mathbb{C} \) be an embedding, and let \( X_0 \) be a smooth projective variety
with $X_{0,k} \cong X$ over $k$. Note that $X_0$ (resp. $X_{0,C}$) is a hyperkähler variety over $k_0$ (resp. $\mathbb{C}$) with Picard rank at least three. Moreover, since $\text{Aut}_k(X)$ is finite, it follows from Lemma 5.1 that $\text{Aut}_{k_0}(X_0)$ and $\text{Aut}_{\mathbb{C}}(X_{0,C})$ are finite.

Since $X_{0,C}$ is a hyperkähler variety over $\mathbb{C}$ with Picard rank at least three and only finitely many automorphisms, the hyperkähler variety $X_{0,C}$ admits a rational curve (Lemma 7.1). Therefore, by a standard argument (see for instance [49, Lemma 2.3]), the hyperkähler variety $X_0$ also admits a rational curve over $k_0$. We conclude that $X$ admits a rational curve over $k$. 

We are now ready to apply our result on automorphism groups of arithmetically hyperbolic varieties (Theorem 1.2) to prove that certain hyperkähler varieties are not arithmetically hyperbolic.

Proof of Theorem 7.7. Let $X$ be a hyperkähler variety over $k$ with Picard rank $\rho(X)$ at least three. If $\text{Aut}_k(X)$ is infinite, it follows from our main result (Theorem 1.2) that $X$ is not arithmetically hyperbolic over $k$. If $\text{Aut}_k(X)$ is finite, then $X$ is groupless by Theorem 7.2 and therefore not arithmetically hyperbolic (Lemma 3.11). We conclude that in either case $X$ is not arithmetically hyperbolic over $k$. 

Remark 7.3. Assume the SYZ conjecture holds, and let $X$ be a hyperkähler variety over $k$ with $\rho(X) = 2$. Following the arguments at the end of the proof of [55, Theorem 3.7], we see that $X$ is not arithmetically hyperbolic over $k$.

Remark 7.4. The infinitude of the group of automorphisms on a projective variety $X$ over $k$ implies that $X$ is not arithmetically hyperbolic over $k$ (Theorem 1.2). However, it is not true that every such variety has a dense set of “integral points” (on some model). For instance, if $C$ is a smooth projective connected curve over $k$ of genus two, then the smooth projective surface $X := C \times \mathbb{P}^1_k$ has an infinite automorphism group. However, since $C$ is arithmetically hyperbolic over $k$ (Example 3.15), for any $\mathbb{Z}$-finitely generated subring $A \subset k$ and any model $X'$ for $X$ over $A$, the set of $A$-points $X'(A)$ is not Zariski dense in $X$.

References

[1] D. Abramovich. Uniformity of stably integral points on elliptic curves. Invent. Math., 127(2):307–317, 1997.
[2] Ekaterina Amerik. Existence of non-preperiodic algebraic points for a rational self-map of infinite order. Math. Res. Lett., 18(2):251–256, 2011.
[3] P. Autissier. Géométries, points entiers et courbes entières. Ann. Sci. Éc. Norm. Supér. (4), 42(2):221–239, 2009.
[4] P. Autissier. Sur la non-densité des points entiers. Duke Math. J., 158(1):13–27, 2011.
[5] Hyman Bass and Alexander Lubotzky. Automorphisms of groups and of schemes of finite type. Israel J. Math., 44(1):1–22, 1983.
[6] Ingrid Bauer and Michael Stoll. Geometry and arithmetic of primary Burniat surfaces. Math. Nachr., 290(14-15):2132–2153, 2017.
[7] Arnaud Beauville. Variétés Kähleriennes dont la première classe de Chern est nulle. J. Differential Geom., 18(4):755–782 (1984), 1983.
[8] Arnaud Beauville. Endomorphisms of hypersurfaces and other manifolds. Internat. Math. Res. Notices, (1):53–58, 2001.
[9] Jason P. Bell, Dragos Ghioca, and Thomas J. Tucker. The dynamical Mordell-Lang conjecture, volume 210 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2016.
[10] Jason Pierre Bell, Dragos Ghioca, and Thomas John Tucker. Applications of $p$-adic analysis for bounding periods for subvarieties under étale maps. *Int. Math. Res. Not. IMRN*, (11):3576–3597, 2015.

[11] Robert L. Benedetto, Dragos Ghioca, Pär Kurlberg, and Thomas J. Tucker. A case of the dynamical Mordell-Lang conjecture. *Math. Ann.*, 352(1):1–26, 2012. With an appendix by Umberto Zannier.

[12] F. A. Bogomolov and Yu. Tschinkel. Density of rational points on Enriques surfaces. *Math. Res. Lett.*, 5(5):623–628, 1998.

[13] F. A. Bogomolov and Yu. Tschinkel. On the density of rational points on elliptic fibrations. *J. Reine Angew. Math.*, 511:87–93, 1999.

[14] F. A. Bogomolov and Yu. Tschinkel. Density of rational points on elliptic $K3$ surfaces. *Asian J. Math.*, 4(2):351–368, 2000.

[15] Fedor Bogomolov, Ljudmila Kamenova, and Misha Verbitsky. Algebraically hyperbolic manifolds have finite automorphism groups. *ArXiv:1709.09774*, 2017.

[16] S. Bosch, W. Lütkebohmert, and M. Raynaud. *Néron models*, volume 21 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3)*. Springer-Verlag, Berlin, 1990.

[17] Sébastien Boucksom. Le cône kählérien d’une variété hyperkählérienne. *C. R. Acad. Sci. Paris Sér. I Math.*, 333(10):935–938, 2001.

[18] Michel Brion. On automorphisms and endomorphisms of projective varieties. In *Automorphisms in birational and affine geometry*, volume 79 of *Springer Proc. Math. Stat.*, pages 59–81. Springer, Cham, 2014.

[19] F. Campa. Orbifolds géométriques spéciales et classification biméromorphe des variétés kählériennes compactes. *J. Inst. Math. Jussieu*, 10(4):809–934, 2011.

[20] Serge Cantat. Sur les groupes de transformations birationnelles des surfaces. *Ann. of Math. (2)*, 174(1):299–340, 2011.

[21] L. Caporaso, J. Harris, and B. Mazur. Uniformity of rational points. *J. Amer. Math. Soc.*, 10(1):1–35, 1997.

[22] J. W. S. Cassels. An embedding theorem for fields. *Bull. Austral. Math. Soc.*, 14(2):193–198, 1976.

[23] B. Conrad. A modern proof of Chevalley’s theorem on algebraic groups. *J. Ramanujan Math. Soc.*, 17(1):1–18, 2002.

[24] B. Conrad. Chow’s $K/k$-image and $K/k$-trace, and the Lang-Néron theorem. *Enseign. Math. (2)*, 52(1-2):37–108, 2006.

[25] G. Cornell and J. H. Silverman, editors. *Arithmetic geometry*. Springer-Verlag, New York, 1986. Papers from the conference held at the University of Connecticut, Storrs, Connecticut, July 30–August 10, 1984.

[26] P. Corvaja, A. Levin, and U. Zannier. Integral points on threefolds and other varieties. *Tohoku Math. J. (2)*, 61(4):598–601, 2009.

[27] O. Debarre. *Higher-dimensional algebraic geometry*. Universitext. Springer-Verlag, New York, 2001.

[28] Jean-Pierre Demaillay. Algebraic criteria for kobayashi hyperbolic projective varieties. *Proc. Symp. Pure Math.*, 62(2):285–360, 1997.

[29] T.-C. Dinh and K. Oguiso. A surface with discrete and non-finitely generated automorphism group. *arXiv:1710.07019*.

[30] Tien-Cuong Dinh, Fei Hu, and De-Qi Zhang. Compact Kähler manifolds admitting large solvable groups of automorphisms. *Adv. Math.*, 281:333–352, 2015.

[31] Najmuddin Fakhruddin. The algebraic dynamics of generic endomorphisms of $\mathbb{P}^n$. *Algebra Number Theory*, 8(3):587–608, 2014.

[32] Giovanni Falcone, Peter Plaumann, and Karl Strambach. Monothetic algebraic groups. *J. Aust. Math. Soc.*, 82(3):315–324, 2007.

[33] G. Faltings. Endlichkeitssätze für abelsche Varietäten über Zahlkörpfern. *Invent. Math.*, 73(3):349–366, 1983.

[34] G. Faltings. Complements to Mordell. In *Rational points (Bonn, 1983/1984)*, Aspects Math., E6, pages 203–227. Vieweg, Braunschweig, 1984.

[35] G. Faltings. The general case of S. Lang’s conjecture. In *Barsotti Symposium in Algebraic Geometry (Abano Terme, 1991)*, volume 15 of *Perspect. Math.*, pages 175–182. Academic Press, San Diego, CA, 1994.
[36] Gerd Faltings. Diophantine approximation on abelian varieties. *Ann. of Math. (2)*, 133(3):549–576, 1991.

[37] O. Gabber, Q. Liu, and D. Lorenzini. Hypersurfaces in projective schemes and a moving lemma. *Duke Math. J.*, 164(7):1187–1270, 2015.

[38] D. Ghioca and T. J. Tucker. Periodic points, linearizing maps, and the dynamical Mordell-Lang problem. *J. Number Theory*, 129(6):1392–1403, 2009.

[39] Marek Golasiński and Daicíber Lima Gonçalves. On automorphisms of finite abelian $p$-groups. *Math. Slovaca*, 58(4):405–412, 2008.

[40] B. Hassett and Y. Tschinkel. Abelian fibrations and rational points on symmetric products. *Internat. J. Math.*, 11(9):1163–1176, 2000.

[41] Brendan Hassett and Yuri Tschinkel. Density of integral points on algebraic varieties. In *Rational points on algebraic varieties*, volume 199 of *Progr. Math.*, pages 169–197. Birkhäuser, Basel, 2001.

[42] Brendan Hassett and Yuri Tschinkel. Potential density of rational points for K3 surfaces over function fields. *Amer. J. Math.*, 130(5):1263–1278, 2008.

[43] E. Hrushovski. The Elementary Theory of the Frobenius Automorphisms. *arXiv:math/0406514*.

[44] Fei Hu, Sheng Meng, and De-Qi Zhang. Ampleness of canonical divisors of hyperbolic normal projective varieties. *Math. Z.*, 278(3-4):1179–1193, 2014.

[45] Daniel Huybrechts. The Kähler cone of a compact hyperkähler manifold. *Math. Ann.*, 326(3):499–513, 2003.

[46] Daniel Huybrechts. *Lectures on K3 surfaces*, volume 158 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2016.

[47] Jun-Muk Hwang, Stefan Kebekus, and Thomas Peternell. Holomorphic maps onto varieties of non-negative Kodaira dimension. *J. Algebraic Geom.*, 15(3):551–561, 2006.

[48] Shigeru Iitaka. *Algebraic geometry*, volume 76 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1982. An introduction to birational geometry of algebraic varieties, North-Holland Mathematical Library, 24.

[49] A. Javanpeykar and L. Kamenova. Demailly’s notion of algebraic hyperbolicity: geometricity, boundedness, moduli of maps. *arXiv:1807.03665*.

[50] A. Javanpeykar and A. Robert Kucharczyk. Algebraicity of analytic maps to a hyperbolic variety. *arXiv:1806.09880*.

[51] A. Javanpeykar and D. Loughran. Complete intersections: moduli, Torelli, and good reduction. *Math. Ann.*, 368(3-4):1191–1225, 2017.

[52] A. Javanpeykar and A. Vezzani. Non-archimedean hyperbolicity and applications. *arXiv:1808.09880*.

[53] A. Javanpeykar and Daniel Loughran. Arithmetic hyperbolicity and a stacky Chevalley-Weil theorem. *arXiv:1808.09876*.

[54] A. Javanpeykar and Daniel Loughran. Good reduction of Fano threefolds and sextic surfaces. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 18(2):509–535, 2018.

[55] Ljudmila Kamenova and Misha Verbitsky. Algebraic nonhyperbolicity of hyperkähler manifolds with Picard rank greater than one. *New York J. Math.*, 23:489–495, 2017.

[56] Shosshichi Kobayashi. *Hyperbolic complex spaces*, volume 318 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1998.

[57] Sándor J. Kovács. Subvarieties of moduli stacks of canonically polarized varieties: generalizations of Shafarevich’s conjecture. In *Algebraic geometry—Seattle 2005. Part 2*, volume 80 of *Proc. Sympos. Pure Math.*, pages 685–709. Amer. Math. Soc., Providence, RI, 2009.

[58] S.J. Kovács and M. Lieblich. Erratum for Boundedness of families of canonically polarized manifolds: a higher dimensional analogue of Shafarevich’s conjecture. *Ann. of Math. (2)*, 174(1):585–617, 2011.

[59] S. Lang. Hyperbolic and Diophantine analysis. *Bull. Amer. Math. Soc. (N.S.)*, 14(2):159–205, 1986.

[60] S. Lang and A. Néron. Rational points of abelian varieties over function fields. *Amer. J. Math.*, 81:95–118, 1959.

[61] J. Lesieutre and D. Litt. Dynamical mordell-lang and automorphisms of blow-ups. *arXiv:1604.08216*.

[62] John Lesieutre. A projective variety with discrete, non-finitely generated automorphism group. *Invent. Math.*, 212(1):189–211, 2018.

[63] Aaron Levin. Generalizations of Siegel’s and Picard’s theorems. *Ann. of Math. (2)*, 170(2):609–655, 2009.
[64] Q. Liu. *Algebraic geometry and arithmetic curves*, volume 6 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, 2006. Translated from the French by Reinie Erné, Oxford Science Publications.

[65] Sheng Meng and De-Qi Zhang. Jordan property for non-linear algebraic groups and projective varieties. *Amer. J. Math.*, 140(4):1133–1145, 2018.

[66] Atsushi Moriwaki. Remarks on rational points of varieties whose cotangent bundles are generated by global sections. *Math. Res. Lett.*, 2(1):113–118, 1995.

[67] J. Oesterlé. Courbes sur une variété abélienne (d’après M. Raynaud). *Astérisque*, (121-122):213–224, 1985. Seminar Bourbaki, Vol. 1983/84.

[68] Bjorn Poonen. $p$-adic interpolation of iterates. *Bull. Lond. Math. Soc.*, 46(3):525–527, 2014.

[69] M. Raynaud. Courbes sur une variété abélienne et points de torsion. *Invent. Math.*, 71(1):207–233, 1983.

[70] Joseph H. Silverman. *The arithmetic of elliptic curves*, volume 106 of *Graduate Texts in Mathematics*. Springer, Dordrecht, second edition, 2009.

[71] The Stacks Project Authors. *Stacks Project*. http://stacks.math.columbia.edu, 2015.

[72] E. Ullmo. Points rationnels des variétés de Shimura. *Int. Math. Res. Not.*, (76):4109–4125, 2004.

[73] Yakov Varshavsky. Intersection of a correspondence with a graph of Frobenius. *J. Algebraic Geom.*, 27(1):1–20, 2018.

[74] P. Vojta. A higher-dimensional Mordell conjecture. In *Arithmetic geometry (Storrs, Conn., 1984)*, pages 341–353. Springer, New York, 1986.

[75] P. Vojta. A refinement of Schmidt’s subspace theorem. *Amer. J. Math.*, 111(3):489–518, 1989.

[76] P. Vojta. Integral points on subvarieties of semiabelian varieties. I. *Invent. Math.*, 126(1):133–181, 1996.

[77] P. Vojta. Integral points on subvarieties of semiabelian varieties. II. *Amer. J. Math.*, 121(2):283–313, 1999.

[78] De-Qi Zhang. A theorem of Tits type for compact Kähler manifolds. *Invent. Math.*, 176(3):449–459, 2009.

[79] K. Zuo. On the negativity of kernels of Kodaira-Spencer maps on Hodge bundles and applications. *Asian J. Math.*, 4(1):279–301, 2000.

Ariyan Javanpeykar, Institut fü r Mathematik, Johannes Gutenberg-Universität Mainz, Staudingerweg 9, 55099 Mainz, Germany.

E-mail address: peykar@uni-mainz.de