Geometric construction of Gelfand–Tsetlin modules over simple Lie algebras

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Abstract

In the present paper we describe a new class of Gelfand–Tsetl in modules for an arbitrary complex simple finite-dimensional Lie algebra \( g \) and give their geometric realization as the space of 'period functions' on the flag manifold \( G/B \) supported at the 1-dimensional submanifold. When \( g = \mathfrak{sl}(n) \) (or \( \mathfrak{gl}(n) \)) these modules form a subclass of Gelfand-Tsetlin modules with infinite dimensional weight subspaces. We discuss their properties and describe the simplicity criterion for these modules in the case of the Lie algebra \( \mathfrak{sl}(3, \mathbb{C}) \).

Keywords: Verma module, Gelfand–Tsetlin module, tensor category.

2010 Mathematics Subject Classification: 17B10, 17B20.

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Introduction

A study of simple weight modules for a complex reductive finite-dimensional Lie algebra \( g \) is a classical problem in representation theory. Such modules have the diagonalizable action of a certain Cartan subalgebra \( \mathfrak{h} \) of \( g \). However, a complete classification of simple weight modules is only known for the Lie algebra \( \mathfrak{sl}(2, \mathbb{C}) \) when the result is obvious. A classification of simple \( \mathfrak{sl}(2, \mathbb{C}) \)-modules was obtained in [1] and a classification of simple weight modules with finite-dimensional weight spaces for any \( g \) was described in [5] and [15]. A larger class of weight modules containing all modules with finite-dimensional weight spaces was extensively studied by many authors during several decades for the Lie algebra \( g \) of type \( A \). The corresponding modules are called Gelfand–Tsetlin modules. They are defined in a similar manner as the weight modules but instead of a Cartan subalgebra \( \mathfrak{h} \) of \( g \) one uses a certain maximal commutative subalgebra \( \Gamma \) of the universal enveloping algebra \( U(g) \), called the Gelfand–Tsetlin subalgebra, see [5]. Let us recall that Gelfand–Tsetlin modules are connected with the study of Gelfand–Tsetlin integrable systems [14], [17], [13], [2]. Properties and explicit constructions of Gelfand–Tsetlin modules for the Lie algebra \( g \) of type \( A \) was studied in [4], [3], [19], [20], [12], [6], [7], [9], [8], [11], [24], [22], [23], [21] among the others.
Allowing only some generators of the Gelfand–Tsetlin subalgebra to have torsion on simple \( \mathfrak{gl}(n, \mathbb{C}) \)-modules leads to partial Gelfand–Tsetlin modules for \( \mathfrak{gl}(n, \mathbb{C}) \) which were studied in [10]. These modules decompose into the direct sum of \( \Gamma \)-submodules parameterized by prime ideals of the Gelfand–Tsetlin subalgebra \( \Gamma \).

In the present paper, we study a special class of Gelfand–Tsetlin modules for \( \mathfrak{gl}(n) \) and their analogues for an arbitrary complex simple finite-dimensional Lie algebra \( \mathfrak{g} \). Let \( \mathfrak{b} \) be a Cartan subalgebra of \( \mathfrak{g} \) and \( \mathfrak{h} \) a Borel subalgebra of \( \mathfrak{g} \) containing \( \mathfrak{h} \). The Beilinson–Bernstein correspondence provides a geometric realization of Verma modules \( M^\mathfrak{g}_\mathfrak{b}(\lambda) \) for \( \lambda \in \mathfrak{h}^* \). By this correspondence, to the Verma module \( M^\mathfrak{g}_\mathfrak{b}(\lambda) \) we associate the vector space of \( \cdot \)-functions’ on the flag manifold \( G/B \) supported at the point \( eB \). From this point of view, we introduce a family of Gelfand–Tsetlin modules \( W^\mathfrak{g}_\mathfrak{b}(\lambda) \) for \( \lambda \in \mathfrak{h}^* \) as the vector space of \( \cdot \)-functions’ on the flag manifold \( G/B \) supported at the 1-dimensional submanifold going through the point \( eB \). Modules \( W^\mathfrak{g}_\mathfrak{b}(\lambda) \) provide first examples of Gelfand–Tsetlin modules for an arbitrary simple finite-dimensional \( \mathfrak{g} \) with respect to certain commutative subalgebra \( \Gamma \) of the universal enveloping algebra \( U(\mathfrak{g}) \).

We will not discuss this geometric approach in details, instead we describe everything from the algebraic point of view. However, geometry serves as a motivation for the purely algebraic construction of \( \mathfrak{g} \)-modules \( W^\mathfrak{g}_\mathfrak{b}(\lambda) \) considered in this paper.

| Verma module | Gelfand–Tsetlin module |
|--------------|------------------------|
| \( M^\mathfrak{g}_\mathfrak{b}(\lambda) \) | \( W^\mathfrak{g}_\mathfrak{b}(\lambda) \) |
| weights \( \lambda - Q_+ \) | weights \( \lambda - Q \) |
| infinitesimal character \( \chi_{\lambda+\rho} \) | infinitesimal character \( \chi_{\lambda+\rho} \) |
| cyclic module | cyclic module |
| weight \( \mathfrak{h} \)-module with finite-dimensional weight spaces | weight \( \mathfrak{h} \)-module with finite-dimensional weight spaces |
| \( \theta \)-Gelfand-Tsetlin module with finite \( \Gamma_\theta \)-multiplicities | \( \theta \)-Gelfand-Tsetlin module with finite \( \Gamma_\theta \)-multiplicities |

Table 1: Comparison of Verma modules and Gelfand–Tsetlin modules

The content of our article goes as follows. In Section [11] we recall a general notion of Harish-Chandra modules and define a class of partial Gelfand–Tsetlin modules for an arbitrary complex simple finite-dimensional Lie algebra \( \mathfrak{g} \). We also introduce various tensor categories of partial Gelfand–Tsetlin modules. In Section [2] we use the geometric realization of Verma modules \( M^\mathfrak{g}_\mathfrak{b}(\lambda) \) for \( \lambda \in \mathfrak{h}^* \) described in [16] to obtain a geometric realization of a new family of \( \Gamma_\theta \)-Gelfand–Tsetlin modules \( W^\mathfrak{g}_\mathfrak{b}(\lambda) \) for \( \lambda \in \mathfrak{h}^* \) called \( \theta \)-Gelfand–Tsetlin modules. We also discuss their properties, which are summarized in Table [1]. Our main result is the following theorem.

**Theorem.** Let \( \mathfrak{g} \) be a complex simple finite-dimensional Lie algebra with a Cartan subalgebra \( \mathfrak{h} \) and the corresponding triangular decomposition \( \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n} \). Let \( \theta \) be the maximal root in \( \Delta^+ \) and \( \Delta^+_\theta = \Delta^+ \setminus \{\theta\} \). If \( \lambda \in \mathfrak{h}^* \), then the vector space \( \mathbb{C}[\partial_{x_\alpha}, \alpha \in \Delta^+_\theta, x_\theta] \) has the structure of a cyclic \( \theta \)-Gelfand–Tsetlin \( \mathfrak{g} \)-module and every element \( a \in \mathfrak{g} \) acts as the following differential operator

\[
\alpha \mapsto - \sum_{\alpha \in \Delta^+} \left[ \frac{\text{ad}(u(x))e^{\text{ad}(u(x))}}{e^{\text{ad}(u(x))} - \text{id}} \right] \partial_{x_\alpha} + (\lambda + 2\rho)((e^{-\text{ad}(u(x))}a)_\mathfrak{h}),
\]

where \( [a]_\alpha \) denotes the \( \alpha \)-th coordinate of \( a \in \mathfrak{h} \) with respect to a root basis \( \{f_\alpha; \alpha \in \Delta^+\} \) of \( \mathfrak{n} \), \( x = \{x_\alpha; \alpha \in \Delta^+\} \) are the corresponding linear coordinate functions on \( \mathfrak{h} \) and \( u(x) = \sum_{\alpha \in \Delta^+} x_\alpha f_\alpha \).
Further $a_\mathfrak{h}$ and $a_{\mathfrak{n}}$ are $\mathfrak{h}$-part and $\mathfrak{n}$-part of $a \in \mathfrak{g}$ with respect to the triangular decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}$.

Furthermore, we establish the simplicity criterion for $W_{\mathfrak{g}}^\Theta(\lambda)$ in the case of the Lie algebra $\mathfrak{sl}(3, \mathbb{C})$ (Theorem 2.16). Let us note that the notions of Gelfand–Tsetlin modules and $\Theta$-Gelfand–
Tsetlin modules coincide for $\mathfrak{sl}(3, \mathbb{C})$. All such simple modules were classified in [1] (see also [2]). We also describe a relation of Gelfand–Tsetlin modules $W_{\mathfrak{g}}^\Theta(\lambda)$ to twisted Verma modules $M_{\mathfrak{g}}^\Theta(\lambda)$. For the reader’s convenience, we remind in Appendix A several important facts concerning the generalized eigenspace decomposition.

Throughout the article, we use the standard notation $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{N}_0$ for the set of integers, the set of natural numbers and the set of natural numbers together with zero, respectively.

1 Gelfand–Tsetlin modules

In this section we introduce a general notion of Harish-Chandra modules and then focus on a particular class of partial Gelfand–Tsetlin modules and their analogues for an arbitrary complex simple finite-dimensional Lie algebra.

1.1 Harish-Chandra modules and Gelfand–Tsetlin modules

For a commutative $\mathbb{C}$-algebra $\Gamma$ we denote by $\text{Hom}(\Gamma, \mathbb{C})$ the set of all characters of $\Gamma$, i.e. $\mathbb{C}$-algebra homomorphisms from $\Gamma$ to $\mathbb{C}$. Let us note that if $\Gamma$ is finitely generated, then there is a natural identification between the set $\text{Hom}(\Gamma, \mathbb{C})$ of all characters of $\Gamma$ and the set $\text{Specm} \Gamma$ of all maximal ideals of $\Gamma$, which corresponds to a complex algebraic variety.

Let $M$ be a $\Gamma$-module. For each $\chi \in \text{Hom}(\Gamma, \mathbb{C})$ we set

$$M_\chi = \{ v \in M; (\exists k \in \mathbb{N}) (\forall a \in \Gamma) (a - \chi(a))^k v = 0 \} \quad (1.1)$$

and call it the $\Gamma$-weight space of $M$ with weight $\chi$. When $M_\chi \neq \{0\}$, we say that $\chi$ is a $\Gamma$-weight of $M$ and the elements of $M_\chi$ are called $\Gamma$-weight vectors with weight $\chi$. If a $\Gamma$-module $M$ satisfies

$$M = \bigoplus_{\chi \in \text{Hom}(\Gamma, \mathbb{C})} M_\chi, \quad (1.2)$$

then we call $M$ a $\Gamma$-weight module. The dimension of the vector space $M_\chi$ will be called the $\Gamma$-multiplicity of $\chi$ in $M$. We say that a left $A$-module $M$ is a Harish-Chandra module with respect to $\Gamma$ if $M$ is a $\Gamma$-weight $A$-module.

Let $A$ be a $\mathbb{C}$-algebra and let $\Gamma$ be a commutative $\mathbb{C}$-subalgebra of $A$. Following [3] we denote by $\mathcal{H}(A, \Gamma)$ the category of all $\Gamma$-weight $A$-modules and by $\mathcal{H}_{\text{fin}}(A, \Gamma)$ the full subcategory of $\mathcal{H}(A, \Gamma)$ consisting of those $\Gamma$-weight $A$-modules $M$ satisfying $\dim M_\chi < \infty$ for all $\chi \in \text{Hom}(\Gamma, \mathbb{C})$. In the case when the $\mathbb{C}$-algebra $A$ is the universal enveloping algebra $U(\mathfrak{g})$ of a complex Lie algebra $\mathfrak{g}$, we will use the notation $\mathcal{H}(\mathfrak{g}, \Gamma)$ and $\mathcal{H}_{\text{fin}}(\mathfrak{g}, \Gamma)$ instead of $\mathcal{H}(U(\mathfrak{g}), \Gamma)$ and $\mathcal{H}_{\text{fin}}(U(\mathfrak{g}), \Gamma)$, respectively.

If $\Gamma$ and $\tilde{\Gamma}$ are commutative $\mathbb{C}$-subalgebras of $A$ satisfying $\Gamma \subset \tilde{\Gamma}$, then we have the following obvious inclusions

$$\mathcal{H}_{\text{fin}}(A, \Gamma) \subset \mathcal{H}_{\text{fin}}(A, \tilde{\Gamma}) \subset \mathcal{H}(A, \tilde{\Gamma}) \subset \mathcal{H}(A, \Gamma),$$

which are strict in general.

Finally, let us recall that a commutative $\mathbb{C}$-subalgebra $\Gamma$ of $A$ is a Harish-Chandra subalgebra if the $\Gamma$-bimodule $\Gamma a \Gamma$ is finitely generated both as left and as right $\Gamma$-module for all $a \in A$.

**Example.** If $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ and $\Gamma$ is the Gelfand–Tsetlin subalgebra of $U(\mathfrak{g})$ then objects of $\mathcal{H}(\mathfrak{g}, \Gamma)$ are the Gelfand–Tsetlin modules studied by many authors.

Let us note that the $\Gamma$-weight spaces in the case of the Gelfand–Tsetlin subalgebra $\Gamma$ of $U(\mathfrak{gl}(n, \mathbb{C}))$ are always finite-dimensional in simple $\mathfrak{gl}(n, \mathbb{C})$-modules. This need not be the case for
other commutative $\mathbb{C}$-subalgebras of $U(\mathfrak{gl}(n, \mathbb{C}))$. However, in all known to us cases the following is true. We state it as a conjecture in general.

**Conjecture.** Let $\mathfrak{g}$ be a complex simple finite-dimensional Lie algebra. If $M \in \mathcal{H}(\mathfrak{g}, \Gamma)$ is a simple $\mathfrak{g}$-module and $\dim M_\chi < \infty$ for some $\Gamma$-weight $\chi$ of $M$, then $M \in \mathcal{H}_{fn}(\mathfrak{g}, \Gamma)$.

**Remark.** If $M$ is a simple $\mathfrak{g}$-module such that $M_\chi \neq \{0\}$ for some $\chi \in \text{Hom}(\Gamma, \mathbb{C})$, then it is not clear (and probably not true in general) whether $M \in \mathcal{H}(\mathfrak{g}, \Gamma)$.

### 1.2 $\theta$-Gelfand–Tsetlin modules

In this section we introduce a class of partial Gelfand–Tsetlin modules for an arbitrary complex simple finite-dimensional Lie algebra.

Let us consider a complex simple finite-dimensional Lie algebra $\mathfrak{g}$ and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. We denote by $\Delta$ the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$, by $\Delta^+$ a positive root system in $\Delta$ and by $\Pi \subset \Delta$ the set of simple roots. Further, we denote by $\theta$ the maximal root of $\mathfrak{g}$ (by definition, the highest weight of its adjoint representation) and by $h_\alpha \in \mathfrak{h}$ the coroot corresponding to a root $\alpha \in \Delta$.

**Definition 1.1.** Let $\mathfrak{s}_\theta$ be the Lie subalgebra of $\mathfrak{g}$ generated by an $\mathfrak{sl}(2, \mathbb{C})$-triple $(e_\theta, h_\theta, f_\theta)$ associated to the maximal root $\theta$. Then we define the commutative $\mathbb{C}$-subalgebra $\Gamma_\theta$ of $U(\mathfrak{g})$ as a $\mathbb{C}$-subalgebra of $U(\mathfrak{g})$ generated by the Cartan subalgebra $\mathfrak{h}$ and by the center $\mathfrak{Z}(\mathfrak{s}_\theta)$ of $U(\mathfrak{s}_\theta)$.

The center $\mathfrak{Z}(\mathfrak{s}_\theta)$ of $U(\mathfrak{s}_\theta)$ is freely generated by the quadratic Casimir element $\text{Cas}_\mathfrak{s}_\theta$ defined by

$$\text{Cas}_\mathfrak{s}_\theta = e_\theta f_\theta + f_\theta e_\theta + \frac{1}{2} h_\theta^2.$$  

(1.3)

Hence, the $\mathbb{C}$-algebra $\Gamma_\theta$ is freely generated by the coroots $h_\alpha$, $\alpha \in \Pi$, and by the Casimir element $\text{Cas}_\mathfrak{s}_\theta$. As a consequence, we obtain that the complex algebraic variety corresponding to $\text{Hom}(\Gamma_\theta, \mathbb{C})$ and $\text{Specm}\Gamma_\theta$ is isomorphic to $\mathbb{C}^{\text{rank}(\mathfrak{g})+1}$. In the next, we will focus on the category $\mathcal{H}(\mathfrak{g}, \Gamma_\theta)$. The objects of this category will be called $\theta$-Gelfand–Tsetlin modules.

**Proposition 1.2.**

(i) The commutative $\mathbb{C}$-subalgebra $\Gamma_\theta$ of $U(\mathfrak{g})$ is a Harish-Chandra subalgebra of $U(\mathfrak{g})$.

(ii) Let $M$ be a simple $\mathfrak{g}$-module such that $M_\chi \neq \{0\}$ for some $\chi \in \text{Hom}(\Gamma_\theta, \mathbb{C})$. Then we have $M \in \mathcal{H}(\mathfrak{g}, \Gamma_\theta)$.

**Proof.** To prove the first statement it is sufficient to check it for the generators of the Lie algebra $\mathfrak{g}$. Let $a \in \mathfrak{g}$ be such a generator. We may assume that $a \in \mathfrak{g}_\alpha$ for some $\alpha \in \Delta$. If $\alpha = \pm \theta$, then we have $[\text{Cas}_\mathfrak{s}_\theta, a] = 0$. For $\alpha \neq \pm \theta$, let us consider the $\mathfrak{s}_\theta$-module

$$\mathfrak{g}_{\alpha, \theta} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_{\alpha + j \theta}.$$  

Since $\mathfrak{g}_{\alpha, \theta}$ is a simple finite-dimensional $\mathfrak{s}_\theta$-module, we obtain that $[\text{Cas}_\mathfrak{s}_\theta, a] = c_\alpha a$, where $c_\alpha \in \mathbb{C}$, which together with $[h, a] = \alpha(h) a$ implies the first and also the second statement. \hfill $\square$

We define the full subcategories $\mathcal{I}_{\theta, e}$ and $\mathcal{I}_{\theta, f}$ of the category of weight (with respect to $\mathfrak{h}$) $\mathfrak{g}$-modules consisting of those $\mathfrak{g}$-modules on which $e_\theta$ and $f_\theta$ is locally nilpotent, respectively. Also we set $\mathcal{I}_\theta = \mathcal{I}_{\theta, e} \cap \mathcal{I}_{\theta, f}$.

**Proposition 1.3.**

(i) The categories $\mathcal{I}_{\theta, e}$, $\mathcal{I}_{\theta, f}$ and $\mathcal{I}_\theta$ are full subcategories of $\mathcal{H}(\mathfrak{g}, \Gamma_\theta)$.

(ii) Let $M$ be a simple weight $\mathfrak{g}$-module such that there exists an integer $n \in \mathbb{N}$ and a nonzero vector $v \in M$ satisfying $e_\theta^n v = 0$ (respectively, $f_\theta^n v = 0$). Then we have $M \in \mathcal{I}_{\theta, e}$ (respectively, $M \in \mathcal{I}_{\theta, f}$), and hence $M \in \mathcal{H}(\mathfrak{g}, \Gamma_\theta)$. 

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Let us consider a complex simple Lie algebra \( g \). The standard Borel subalgebra \( b \) (respectively, lowest) weight \( I \) consists of sums of finite-dimensional \( g \)-modules; \( \mathcal{I}_{\theta,e,\text{fin}} \) contains all finite-dimensional \( g \)-modules when the Levi factor \( I \) of the parabolic subalgebra \( p = l \oplus u \) of \( g \) does not contain \( e_{\theta}, f_{\theta} \) and \( c_{\theta} = u \). Moreover, it was shown in [6] that certain simple non-parabolically induced Gelfand–Tsetlin \( s(\mathbb{C}, \mathbb{C}) \)-modules belong to the categories \( \mathcal{I}_{\theta,e,\text{fin}} \), \( \mathcal{I}_{\theta,f,\text{fin}} \) and \( \mathcal{I}_{\theta,\text{fin}} \). Such modules have infinite-dimensional weight spaces. Furthermore, using the parabolic induction from such simple Gelfand–Tsetlin \( s(\mathbb{C}, \mathbb{C}) \)-modules one can construct simple \( g \)-modules respectively in categories \( \mathcal{I}_{\theta,e} \), \( \mathcal{I}_{\theta,f} \) and \( \mathcal{I}_{\theta} \) for any \( g \) of rank \( \text{rank}(g) \geq 3 \) (in order to have \( s(\mathbb{C}, \mathbb{C}) \) as a Levi factor of some parabolic subalgebra).

Let us note that \( \mathcal{H}(g, \Gamma_{\theta}), \mathcal{H}_{\text{fin}}(g, \Gamma_{\theta}), \mathcal{I}_{\theta,e,\text{fin}}, \mathcal{I}_{\theta,f,\text{fin}} \) and \( \mathcal{I}_{\theta,\text{fin}} \) are not tensor categories, except the case \( g = s(2, \mathbb{C}) \) when \( \mathcal{I}_{\theta,e,\text{fin}} \) (respectively, \( \mathcal{I}_{\theta,f,\text{fin}} \)) consists of extensions of highest (respectively, lowest) weight \( g \)-modules and their direct sums and \( \mathcal{I}_{\theta,\text{fin}} \) consists of sums of finite-dimensional \( g \)-modules. Nevertheless, we have the following statement as a consequence of the definitions.

**Theorem 1.5.** The categories \( \mathcal{I}_{\theta,e}, \mathcal{I}_{\theta,f} \) and \( \mathcal{I}_{\theta} \) are tensor categories.

### 2 Geometric construction of \( \theta \)-Gelfand–Tsetlin modules

#### 2.1 Geometric realization of Verma modules

Let us consider a complex simple Lie algebra \( g \) and let \( h \) be a Cartan subalgebra of \( g \). We denote by \( \Delta \) the root system of \( g \) with respect to \( h \), by \( \Delta^+ \) a positive root system in \( \Delta \) and by \( \Pi \subset \Delta \) the set of simple roots. The standard Borel subalgebra \( b \) of \( g \) is defined through \( b = h \oplus n \) with the nilradical \( n \) and the opposite nilradical \( \Pi \) given by

\[
n = \bigoplus_{\alpha \in \Delta^+} g_{\alpha} \quad \text{and} \quad \Pi = \bigoplus_{\alpha \in \Delta^+} g_{-\alpha}.
\]

Moreover, we have a triangular decomposition

\[
g = \Pi \oplus h \oplus n
\]

of the Lie algebra \( g \). Furthermore, we define the height \( \text{ht}(\alpha) \) of \( \alpha \in \Delta \) by

\[
\text{ht}(\sum_{i=1}^{r} a_{i} \alpha_{i}) = \sum_{i=1}^{r} a_{i},
\]

where \( r = \text{rank}(g) \) and \( \Pi = \{ \alpha_{1}, \alpha_{2}, \ldots, \alpha_{r} \} \). If we denote \( k = \text{ht}(\theta) \), where \( \theta \) is the maximal root of \( g \) (by definition, the highest weight of its adjoint representation), then \( g \) is a \( \{ k \} \)-graded Lie algebra with respect to the grading given by \( g_{i} = \bigoplus_{\alpha \in \Delta, \text{ht}(\alpha) = i} g_{\alpha} \) for \( 0 \neq i \in \mathbb{Z} \) and \( g_{0} = h \). Moreover, we have

\[
\Pi = g_{-k} \oplus \cdots \oplus g_{-1}, \quad h = g_{0}, \quad n = g_{1} \oplus \cdots \oplus g_{k}
\]

(2.4)
together with \( g_{-k} = g_{-\theta} \) and \( g_{k} = g_{\theta} \). We also set

\[
Q = \sum_{\alpha \in \Delta^+} \mathbb{Z} \alpha = \bigoplus_{i=1}^{r} \mathbb{Z} \alpha_{i} \quad \text{and} \quad Q_{+} = \sum_{\alpha \in \Delta^+} \mathbb{N}_{0} \alpha = \bigoplus_{i=1}^{r} \mathbb{N}_{0} \alpha_{i}
\]

(2.5)
and call $Q$ the root lattice. Let us note that for any positive root $\alpha \in \Delta^+$, we have $\theta - \alpha \in Q_+$. Furthermore, we introduce the notation $\Delta^+_g$ to stand for $\Delta^+ \setminus \{\theta\}$.

For $\lambda \in \mathfrak{h}^*$ we denote by $\mathbb{C}_\lambda$ the 1-dimensional representation of $\mathfrak{b}$ defined by

$$av = \lambda(a)v$$

for $a \in \mathfrak{h}$, $v \in \mathbb{C}_\lambda \simeq \mathbb{C}$ (as vector spaces) and trivially extended to a representation of $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$.

**Definition 2.1.** Let $\lambda \in \mathfrak{h}^*$. Then the Verma module $M^\mathfrak{g}_\mathfrak{b}(\lambda)$ is the induced module

$$M^\mathfrak{g}_\mathfrak{b}(\lambda) = \text{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_\lambda \equiv U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda \simeq U(\mathfrak{p}) \otimes_{\mathbb{C}} \mathbb{C}_\lambda, \quad (2.7)$$

where the last isomorphism of $U(\mathfrak{p})$-modules follows from Poincaré–Birkhoff–Witt theorem.

In the next, we shall need the action of the center $\mathfrak{z}(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$ on Verma modules $M^\mathfrak{g}_\mathfrak{b}(\lambda)$ for $\lambda \in \mathfrak{h}^*$. Using (2.7), together with Poincaré–Birkhoff–Witt theorem we obtain

$$U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (\mathfrak{p} U(\mathfrak{g}) + U(\mathfrak{g}) \mathfrak{n}). \quad (2.8)$$

Let us consider the projection $p: U(\mathfrak{g}) \to U(\mathfrak{h})$ with respect to the direct sum decomposition (2.8) and the $\mathbb{C}$-algebra automorphism $f: U(\mathfrak{h}) \to U(\mathfrak{h})$ defined by

$$f(h) = h - \rho(h)1 \quad (2.9)$$

for all $h \in \mathfrak{h}$, where $\rho \in \mathfrak{h}^*$ is the Weyl vector given through

$$\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha. \quad (2.10)$$

We denote by $\gamma: \mathfrak{z}(\mathfrak{g}) \to U(\mathfrak{h})$ the restriction of $f \circ p$ to the center $\mathfrak{z}(\mathfrak{g})$ of $U(\mathfrak{g})$. The mapping $\gamma: \mathfrak{z}(\mathfrak{g}) \to U(\mathfrak{h})$ is a homomorphism of associative $\mathbb{C}$-algebras and does not depend on the choice of the positive root system $\Delta^+$. It is called the Harish-Chandra homomorphism.

A $\mathbb{C}$-algebra homomorphism from $\mathfrak{z}(\mathfrak{g})$ to $\mathbb{C}$ is called a central character. For each $\lambda \in \mathfrak{h}^*$ we define a central character $\chi_\lambda: \mathfrak{z}(\mathfrak{g}) \to \mathbb{C}$ by

$$\chi_\lambda(z) = (\gamma(z)) (\lambda) \quad (2.11)$$

for $z \in \mathfrak{z}(\mathfrak{g})$, where we identify $U(\mathfrak{h}) \simeq S(\mathfrak{h})$ with the $\mathbb{C}$-algebra of polynomial functions on $\mathfrak{h}^*$. The action of the center $\mathfrak{z}(\mathfrak{g})$ of $U(\mathfrak{g})$ on Verma modules $M^\mathfrak{g}_\mathfrak{b}(\lambda)$ for $\lambda \in \mathfrak{h}^*$ is then given by

$$zv = \chi_{\lambda + \rho}(z)v \quad (2.12)$$

for all $z \in \mathfrak{z}(\mathfrak{g})$ and $v \in M^\mathfrak{g}_\mathfrak{b}(\lambda)$.

We briefly describe the geometric realization of Verma modules given in [16]. This nice geometric realization enables us to construct new interesting simple modules for the Lie algebra $\mathfrak{g}$.

Let $\{f_\alpha: \alpha \in \Delta^+\}$ be a basis of the opposite nilradical $\mathfrak{p}$. We denote by $x = \{x_\alpha: \alpha \in \Delta^+\}$ the corresponding linear coordinate functions on $\mathfrak{p}$ with respect to the basis $\{f_\alpha: \alpha \in \Delta^+\}$ of $\mathfrak{p}$. Then we have a canonical isomorphism $\mathbb{C}[\mathfrak{p}] \simeq \mathbb{C}[x]$ of $\mathbb{C}$-algebras, and the Weyl algebra $\mathcal{A}_\mathfrak{p}$ of the vector space $\mathfrak{p}$ is generated by $\{x_\alpha, \partial x_\alpha: \alpha \in \Delta^+\}$ together with the canonical commutation relations. For $\lambda \in \mathfrak{h}^*$ there is a homomorphism

$$\pi_\lambda: \mathfrak{g} \to \mathcal{A}_\mathfrak{p} \quad (2.13)$$

of Lie algebras given through

$$\pi_\lambda(a) = - \sum_{\alpha \in \Delta^+} \left[ \frac{\text{ad}(u(x)) e^{\text{ad}(u(x))}}{e^{\text{ad}(u(x))}} - \text{id} \right] (e^{-\text{ad}(u(x))}a)_{\mathfrak{p}} \partial x_\alpha + (\lambda + \rho)((e^{-\text{ad}(u(x))}a)_{\mathfrak{h}}) \quad (2.14)$$

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for all \( a \in \mathfrak{g} \), where \([a]_\alpha\) denotes the \( \alpha \)-th coordinate of \( a \in \mathfrak{h} \) with respect to the basis \( \{f_\alpha; \alpha \in \Delta^+ \} \) of \( \mathfrak{h} \), further \( a_\mathfrak{n} \) and \( a_\mathfrak{h} \) are \( \mathfrak{h} \)-part and \( \mathfrak{h} \)-part of \( a \in \mathfrak{g} \) with respect to the trinangular decomposition \( \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n} \), and finally the element \( u(x) \in \mathbb{C}[\mathfrak{h}] \otimes \mathfrak{g} \) is given by
\[
u(x) = \sum_{\alpha \in \Delta^+} x_\alpha f_\alpha. \tag{2.15}\]

Let us note that \( \mathbb{C}[\mathfrak{h}] \otimes \mathfrak{g} \) has the natural structure of a Lie algebra. Hence, we have a well-defined \( \mathbb{C} \)-linear mapping \( \text{ad}(u(x)) : \mathbb{C}[\mathfrak{h}] \otimes \mathfrak{g} \to \mathbb{C}[\mathfrak{h}] \otimes \mathfrak{g} \).

In particular, we have
\[
\pi_\lambda(a) = -\sum_{\alpha \in \Delta^+} \left[ \frac{\text{ad}(u(x))}{\epsilon^{\text{ad}(u(x))} - \text{id}} \right]_\alpha \partial_\alpha \tag{2.16}
\]
for \( a \in \mathfrak{h} \) and
\[
\pi_\lambda(a) = \sum_{\alpha \in \Delta^+} [\text{ad}(u(x))a]_\alpha \partial_\alpha + (\lambda + \rho)(a) \tag{2.17}
\]
for \( a \in \mathfrak{g} \). The Verma module \( M_{\mathfrak{g}}^\lambda(\lambda) \) for \( \lambda \in \mathfrak{h}^* \) is then realized as \( \mathcal{A}_g^\mathfrak{h} / \mathcal{J}_V \), where \( \mathcal{J}_V \) is the left ideal of \( \mathcal{A}_g^\mathfrak{h} \) defined by \( \mathcal{J}_V = \{x_\alpha, \alpha \in \Delta^+ \} \), see e.g. [10]. The \( \mathfrak{g} \)-module structure on \( \mathcal{A}_g^\mathfrak{h} / \mathcal{J}_V \) is given through the homomorphism \( \pi_{\lambda + \rho} : \mathfrak{g} \to \mathcal{A}_g^\mathfrak{h} \) of Lie algebras.

In the next, we change the left ideal \( \mathcal{J}_V \) of the Weyl algebra \( \mathcal{A}_g^\mathfrak{h} \) to produce a class of \( \mathfrak{b} \)-Gelfand–Tsetlin modules. These \( \mathfrak{g} \)-modules will be the subject of our interest.

### 2.2 Geometric realization of \( \mathfrak{b} \)-Gelfand–Tsetlin modules

We need to introduce a suitable basis of the Lie algebra \( \mathfrak{g} \). Let \( \{e_\alpha; \alpha \in \Delta^+ \} \) be a root basis of the nilractal \( \mathfrak{n} \) and let \( \{f_\alpha; \alpha \in \Delta^+ \} \) be a root basis of the opposite nilractal \( \mathfrak{h} \). If we denote by \( h_\alpha \in \mathfrak{h} \) the coroot corresponding to a root \( \alpha \in \Delta^+ \), then \( \{h_\alpha; \alpha \in \Pi \} \) is a basis of the Cartan subalgebra \( \mathfrak{h} \).

Let us note that the Lie subalgebra of \( \mathfrak{g} \) generated by \( \{e_\alpha, h_\alpha, f_\alpha\} \) for \( \alpha \in \Delta^+ \) is isomorphic to \( \mathfrak{sl}(2, \mathbb{C}) \).

**Definition 2.2.** Let \( \lambda \in \mathfrak{h}^* \). Then the \( \mathfrak{g} \)-module \( W_{\mathfrak{g}}^\lambda(\lambda) \) is defined by
\[
W_{\mathfrak{g}}^\lambda(\lambda) \simeq \mathcal{A}_g^\mathfrak{h} / \mathcal{J}_{GT} \simeq \mathbb{C}[\partial_\alpha, \alpha \in \Delta^+_{\mathfrak{g}}, \partial_{h_\alpha}], \quad \mathcal{J}_{GT} = \{x_\alpha, \alpha \in \Delta^+_{\mathfrak{g}}, \partial_{h_\alpha}\}, \tag{2.18}
\]
where the \( \mathfrak{g} \)-module structure on \( \mathcal{A}_g^\mathfrak{h} / \mathcal{J}_{GT} \) is given through the homomorphism \( \pi_{\lambda + \rho} : \mathfrak{g} \to \mathcal{A}_g^\mathfrak{h} \) of Lie algebras.

**Remark.** Although, the definition of \( W_{\mathfrak{g}}^\lambda(\lambda) \) for \( \lambda \in \mathfrak{h}^* \) makes sense for any complex simple finite-dimensional Lie algebra \( \mathfrak{g} \), we exclude the case \( \mathfrak{g} = \mathfrak{sl}(2, \mathbb{C}) \). For \( \mathfrak{g} = \mathfrak{sl}(2, \mathbb{C}) \) the \( \mathfrak{g} \)-module \( W_{\mathfrak{g}}^\lambda(\lambda) \) is isomorphic to the contragredient Verma module \( M_{\mathfrak{g}}^\lambda(\lambda + 2\rho)^* \) for the opposite standard Borel subalgebra \( \mathfrak{b} = \mathfrak{h} \oplus \mathfrak{h} \).

Let \( M \) be an \( \mathfrak{h} \)-module. For each \( \mu \in \mathfrak{h}^* \) we set
\[
M_\mu = \{v \in M; (\forall h \in \mathfrak{h}) hv = \mu(h)v\} \tag{2.19}
\]
and call it the weight space of \( M \) with weight \( \mu \). When \( M_\mu \neq \{0\} \), we say that \( \mu \) is a weight of \( M \) and the elements of \( M_\mu \) are called weight vectors with weight \( \mu \). If an \( \mathfrak{h} \)-module \( M \) satisfies
\[
M = \bigoplus_{\mu \in \mathfrak{h}^*} M_\mu, \tag{2.20}
\]
then we call \( M \) a weight module.
Therefore, the first natural question is whether \( W_0^\vartheta(\lambda) \) is a weight module and how the action of the center \( Z(\mathfrak{g}) \) of \( U(\mathfrak{g}) \) on \( W_0^\vartheta(\lambda) \) looks.

**Lemma 2.3.** Let \( \lambda \in \mathfrak{h}^* \). Then we have

\[
\pi_\lambda(h) = \sum_{\alpha \in \Delta^+} \alpha(h) x_\alpha \partial x_\alpha + (\lambda + \rho)(h)
\]

\[
= \sum_{\alpha \in \Delta^+_g} \alpha(h) x_\alpha \partial x_\alpha + \theta(h) x_\theta \partial x_\theta + (\lambda - \rho + \theta)(h)
\]  \hspace{1cm} (2.21)

for all \( h \in \mathfrak{h} \).

**Proof.** For \( h \in \mathfrak{h} \) we have

\[
[u(x), h] = \sum_{\alpha \in \Delta^+} x_\alpha [f_\alpha, h] = \sum_{\alpha \in \Delta^+} \alpha(h) x_\alpha f_\alpha,
\]

which together with (2.17) enables us to write

\[
\pi_\lambda(h) = \sum_{\alpha \in \Delta^+} \alpha(h) x_\alpha \partial x_\alpha + (\lambda + \rho)(h)
\]

\[
= \sum_{\alpha \in \Delta^+_g} \alpha(h) x_\alpha \partial x_\alpha + \theta(h) x_\theta \partial x_\theta + (\lambda - \rho + \theta)(h)
\]

for all \( h \in \mathfrak{h} \), where we used (2.10) in the second equality. \( \square \)

We denote by \( \mathbb{Z}^{\Delta^+} \) and \( \mathbb{Z}^\Pi \) the set of all functions from \( \Delta^+ \) to \( \mathbb{Z} \) and from \( \Pi \) to \( \mathbb{Z} \), respectively. Since we have \( \Pi \subset \Delta^+ \), an element of \( \mathbb{Z}^\Pi \) will be also regarded as an element of \( \mathbb{Z}^{\Delta^+} \) extended by 0 on \( \Delta^+ \setminus \Pi \). A similar notation is introduced for \( \mathbb{N}_0^{\Delta^+} \) and \( \mathbb{N}_0^\Pi \).

For \( \alpha \in \mathbb{N}_0^{\Delta^+} \) we define by

\[
w_{\lambda, \alpha} = \prod_{\alpha \in \Delta^+_g} \partial x_\alpha ^{a_\alpha}
\]  \hspace{1cm} (2.22)

a vector in \( W_0^\vartheta(\lambda) \). The subset \( \{ w_{\lambda, \alpha}; \alpha \in \mathbb{N}_0^{\Delta^+} \} \) of \( W_0^\vartheta(\lambda) \) forms a basis of \( W_0^\vartheta(\lambda) \). As \( w_{\lambda, \alpha} \) is a weight vector with weight

\[
\mu_{\lambda, \alpha} = -\sum_{\alpha \in \Delta^+_g} a_\alpha \alpha + a_\theta \theta + \lambda + \theta
\]  \hspace{1cm} (2.23)

for all \( \alpha \in \mathbb{N}_0^{\Delta^+} \), as easily follows from (2.21), we obtain that \( W_0^\vartheta(\lambda) \) is a weight module. However, we need a basis of weight spaces. Since any positive root \( \alpha \in \Delta^+ \) can be expressed as

\[
\alpha = \sum_{i=1}^r m_{\alpha, i} \alpha_i
\]  \hspace{1cm} (2.24)

where \( m_{\alpha, i} \in \mathbb{N}_0 \) for all \( i = 1, \ldots, r \) and \( \Pi = \{ \alpha_1, \alpha_2, \ldots, \alpha_r \} \), we define \( t_\alpha \in \mathbb{Z}^{\Delta^+} \) for \( \alpha \in \Delta^+ \setminus \Pi \) by

\[
t_\theta(\beta) = \begin{cases} m_{\theta, j} & \text{for } \beta = \alpha_j, \\ 1 & \text{for } \beta = \theta, \\ 0 & \text{for } \beta \neq \theta \text{ and } \beta \notin \Pi \end{cases}
\]  \hspace{1cm} (2.25)

and

\[
t_\alpha(\beta) = \begin{cases} -m_{\alpha, i} & \text{for } \beta = \alpha_i, \\ 1 & \text{for } \beta = \alpha, \\ 0 & \text{for } \beta \neq \alpha \text{ and } \beta \notin \Pi \end{cases}
\]  \hspace{1cm} (2.26)
for $\alpha \neq \theta$. Furthermore, we define subsets $\Lambda_+$ and $\Lambda_0^+$ of $\mathbb{Z}^+$ by

$$\Lambda_+ = \{ \sum_{\alpha \in \Delta^+ \setminus \Pi} n_{\alpha} t_{\alpha}; n_{\alpha} \in \mathbb{N}_0 \text{ for all } \alpha \in \Delta^+ \setminus \Pi \}$$

and

$$\Lambda_0^+ = \{ \sum_{\alpha \in \Delta_0^+ \setminus \Pi} n_{\alpha} t_{\alpha}; n_{\alpha} \in \mathbb{N}_0 \text{ for all } \alpha \in \Delta_0^+ \setminus \Pi \},$$

respectively.

**Lemma 2.4.** Let $\lambda \in \mathfrak{h}^*$ and $a, b \in \mathbb{Z}^+$. Then $\mu_{\lambda,a} = \mu_{\lambda,b}$ if and only if

$$b = a + \sum_{\alpha \in \Delta^+ \setminus \Pi} n_{\alpha} t_{\alpha},$$

where $n_{\alpha} \in \mathbb{Z}$ for all $\alpha \in \Delta^+ \setminus \Pi$.

**Proof.** Let $\lambda \in \mathfrak{h}^*$ and $a, b \in \mathbb{Z}^+$. If $b = a + \sum_{\alpha \in \Delta^+ \setminus \Pi} n_{\alpha} t_{\alpha}$, where $n_{\alpha} \in \mathbb{Z}$ for $\alpha \in \Delta^+ \setminus \Pi$, then we have $\mu_{\lambda,a} = \mu_{\lambda,b}$. On the other hand, let us assume that $\mu_{\lambda,a} = \mu_{\lambda,b}$. Then we set $n_{\alpha} = b_{\alpha} - a_{\alpha}$ for $\alpha \in \Delta^+ \setminus \Pi$ and define $c = a + \sum_{\alpha \in \Delta^+ \setminus \Pi} n_{\alpha} t_{\alpha}$. Hence, we have $c_{\alpha} = a_{\alpha} + n_{\alpha} = b_{\alpha}$ for $\alpha \in \Delta^+ \setminus \Pi$ and $c_{\alpha} = a_{\alpha} - \sum_{\alpha \in \Delta^+ \setminus \Pi} n_{\alpha} m_{\alpha,i} + n_{\alpha} m_{\theta,i}$ for $i = 1, 2, \ldots, r$. Further, since $\mu_{\lambda,a} = \mu_{\lambda,b}$, we may write

$$\mu_{\lambda,b} - \mu_{\lambda,a} = \sum_{\alpha \in \Delta_+^*} (a_{\alpha} - b_{\alpha}) \alpha + (a_{\theta} - b_{\theta}) \theta = \sum_{i=1}^{r} (a_{\alpha,i} - b_{\alpha,i}) \alpha_i + \sum_{i=1}^{r} (a_{\alpha} - b_{\alpha}) m_{\alpha,i} \alpha_i = \sum_{i=1}^{r} (a_{\alpha} - b_{\alpha}) m_{\alpha,i} \alpha_i = 0,$$

where we used (2.23). As the set $\{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ forms a basis of $\mathfrak{h}^*$, we get

$$a_{\alpha,i} - b_{\alpha,i} - \sum_{\alpha \in \Delta_+^*} n_{\alpha} m_{\alpha,i} + n_{\alpha} m_{\theta,i} = 0,$$

which implies that $c_{\alpha,i} = b_{\alpha,i}$ for $i = 1, 2, \ldots, r$. Hence, we have $c_{\alpha} = b_{\alpha}$ for all $\alpha \in \Delta^+$ and we are done.

**Proposition 2.5.** Let $\lambda \in \mathfrak{h}^*$. Then the set of all weights of $W_{\theta}^\mathfrak{h}(\lambda)$ is $\{\mu_{\lambda,a}; a \in \mathbb{Z}^{\Pi}\}$. Moreover, we have that $W_{\theta}^\mathfrak{h}(\lambda)$ is a weight $\mathfrak{h}$-module with infinite dimensional weight spaces and the subset

$$\{w_{\lambda,a+n_{\alpha} t_{\alpha} + t}; t \in \Lambda_+; a + n_{\alpha} t_{\theta} + t \in \mathbb{N}_0^{\Delta^+}\}$$

of $W_{\theta}^\mathfrak{h}(\lambda)_{\mu_{\lambda,a}}$ forms a basis of $W_{\theta}^\mathfrak{h}(\lambda)_{\mu_{\lambda,a}}$ for all $a \in \mathbb{Z}^{\Pi}$, where $n_{\alpha} \in \mathbb{N}_0$ is the smallest non-negative integer satisfying $a + n_{\alpha} t_{\theta} \in \mathbb{N}_0^{\Delta^+}$.

**Proof.** From the previous considerations we know that $W_{\theta}^\mathfrak{h}(\lambda)$ is a weight $\mathfrak{h}$-module and that all weights are of the form $\mu_{\lambda,a}$ for some $a \in \mathbb{Z}^{\Pi}$ as follows from (2.23). Now, let us consider $a \in \mathbb{Z}^{\Pi}$. Since $\theta - \alpha \in Q_+$ for all $\alpha \in \Delta^+$, i.e. $m_{\theta,i} \neq 0$ for all $i = 1, 2, \ldots, r$, there exists the smallest non-negative integer $n_{\alpha} \in \mathbb{N}_0$ such that $a + n_{\alpha} t_{\theta} \in \mathbb{N}_0^{\Delta^+}$. Moreover, we have $\mu_{\lambda,a} = \mu_{\lambda,a+n_{\alpha} t_{\theta}}$, which implies that $\mu_{\lambda,a}$ is a weight of $W_{\theta}^\mathfrak{h}(\lambda)$, since $w_{\lambda,a+n_{\alpha} t_{\theta}}$ is a weight vector with weight $\mu_{\lambda,a}$. Further, since $a + (n_{\alpha} + n) t_{\theta} \in \mathbb{N}_0^{\Delta^+}$ and $\mu_{\lambda,a} = \mu_{\lambda,a+(n_{\alpha}+n) t_{\theta}}$ for all $n \in \mathbb{N}_0$, we obtain
that \( W^g_b(\lambda)_{\mu_{\lambda,a}} \) contains the subset \( \{ w_{\lambda,a+(n_a+n)t_a} : n \in \mathbb{N}_0 \} \), which implies that the weight space \( W^g_b(\lambda)_{\mu_{\lambda,a}} \) is infinite dimensional.

Let us assume that \( w_{\lambda,b} \in W^g_b(\lambda)_{\mu_{\lambda,a}} \) for \( a \in \mathbb{Z}^\Pi \) and \( b \in \mathbb{N}_0^{\Delta^+} \). Then we have \( \mu_{\lambda,a} = \mu_{\lambda,b} \). By Lemma 2.4 we get

\[
b = a + \sum_{\alpha \in \Delta^+ \setminus \Pi} n_\alpha t_\alpha,
\]

where \( n_\alpha \in \mathbb{Z} \) for all \( \alpha \in \Delta^+ \setminus \Pi \). Therefore, we have \( b_\alpha = a_\alpha + n_\alpha = n_\alpha \) for \( \alpha \in \Delta^+ \setminus \Pi \) and \( b_\alpha = a_\alpha + n_\alpha m_\theta, i = \sum_{\alpha \in \Delta^+ \setminus \Pi} n_\alpha m_\alpha, i \) for \( i = 1, 2, \ldots, r \). Since \( b_\alpha \in \mathbb{N}_0 \) for \( \alpha \in \Delta^+ \), we get \( n_\alpha \in \mathbb{N}_0 \) for \( \alpha \in \Delta^+ \setminus \Pi \) and

\[
0 \leq a_\alpha + n_\alpha m_\theta, i = \sum_{\alpha \in \Delta^+ \setminus \Pi} n_\alpha m_\alpha, i \leq a_\alpha + n_\theta m_\theta, i
\]

for \( i = 1, 2, \ldots, r \). Hence, we obtain that \( n_\theta \in \mathbb{N}_0 \) and \( a + n_\theta \theta, \theta, \theta \in \mathbb{N}_0^{\Delta^+} \). If we denote by \( n_\theta \in \mathbb{N}_0 \) the smallest non-negative integer satisfying \( a + n_\theta t_\theta \in \mathbb{N}_0^{\Delta^+} \), then we may write \( n_\theta \) in the form \( n_\theta = n_\alpha + n_\theta^\prime \) for \( n_\theta^\prime \in \mathbb{N}_0 \). This finishes the proof.

\textbf{Theorem 2.6.} Let \( \lambda \in \mathfrak{h}^* \). Then we have

\[
zv = \chi_{\lambda+\rho}(z)v \tag{2.31}
\]

for all \( z \in \mathfrak{z}(g) \) and \( v \in W^g_b(\lambda) \).

\textit{Proof.} For \( \lambda \in \mathfrak{h}^* \) we show that

\[
\pi_{\lambda+\rho}(z) = \chi_{\lambda+\rho}(z)
\]

for all \( z \in \mathfrak{z}(g) \), which implies the required statement. For that reason, let us consider the Verma module \( M^g_b(\lambda) \) for \( \lambda \in \mathfrak{h}^* \). Then we have \( zv = \chi_{\lambda+\rho}(z)v \) for all \( z \in \mathfrak{z}(g) \) and \( v \in M^g_b(\lambda) \), which follows from (2.12). As the Verma module \( M^g_b(\lambda) \) is realized as \( A^g_b/V \simeq C[\partial_{x_\alpha}, \alpha \in \Delta^+] \), where the \( g \)-module structure on \( A^g_b/V \) is given through the homomorphism \( \pi_{\lambda+\rho} : g \to A^g_b \) of Lie algebras, we have

\[
\pi_{\lambda+\rho}(z)v = \chi_{\lambda+\rho}(z)v
\]

for all \( v \in C[\partial_{x_\alpha}, \alpha \in \Delta^+] \). Therefore, we have \( \pi_{\lambda+\rho}(z) = \chi_{\lambda+\rho}(z) \) for all \( z \in \mathfrak{z}(g) \). \( \Box \)

In the following, we give an explicit form of \( \pi_{\lambda}(C_{sg}) \) for all \( \lambda \in \mathfrak{h}^* \). Let us recall that the Lie subalgebra of \( g \) generated by \( \{ e_\theta, h_\theta, f_\theta \} \) is denoted by \( s_\theta \).

For \( \lambda \in \mathfrak{h}^* \) we introduce a linear mapping

\[
\sigma_{\lambda} : \mathfrak{g} \to A^g_b \tag{2.32}
\]

by

\[
\sigma_{\lambda}(a) = -\sum_{\alpha \in \Delta^+} \left[ \frac{\text{ad}(v(x)) e^{\text{ad}(v(x))}}{e^{\text{ad}(v(x))} - \text{id}} (e^{-\text{ad}(v(x))} a) \right]_\alpha \partial_{x_\alpha} + (\lambda + \rho)((e^{-\text{ad}(v(x))} a) h) \tag{2.33}
\]

for all \( a \in \mathfrak{g} \), where

\[
v(x) = \sum_{\alpha \in \Delta^+} x_\alpha f_\alpha \tag{2.34}
\]

and we use the same convention as in (2.14). In particular, we have

\[
\sigma_{\lambda}(a) = -\sum_{\alpha \in \Delta^+} \left[ \frac{\text{ad}(v(x))}{e^{\text{ad}(v(x))} - \text{id}} a \right]_\alpha \partial_{x_\alpha} \tag{2.35}
\]
for $a \in \mathfrak{P}$ and
\[
\sigma_\lambda(a) = \sum_{\alpha \in \Delta^+} [\text{ad}(v(x))a]_\alpha \partial_{x_\alpha} + (\lambda + \rho)(a)
\]  
(2.36)
for $a \in \mathfrak{h}$.

**Proposition 2.7.** Let $\lambda \in \mathfrak{h}^*$. Then we have
\[
\pi_\lambda(e_\alpha) = x_\theta \pi_\lambda([e_\alpha, f_\theta]) + \sigma_\lambda(e_\alpha)
\]  
(2.37)
for $\alpha \in \Delta^+$.

**Proof.** Let us assume that $\alpha \in \Delta^+$. Since $u(x) = v(x) + x_\theta f_\theta$ and $[v(x), x_\theta f_\theta] = 0$, we may write
\[
e^{-\text{ad}(u(x))}e_\alpha = e^{-\text{ad}(v(x)) - \text{ad}(x_\theta f_\theta)}e_\alpha = e^{-\text{ad}(v(x))}e^{-\text{ad}(x_\theta f_\theta)}e_\alpha = e^{-\text{ad}(v(x))}(e_\alpha - \text{ad}(x_\theta f_\theta)e_\alpha) = e^{-\text{ad}(v(x))}(e_\alpha + x_\theta[e_\alpha, f_\theta]),
\]
which implies
\[
(e^{-\text{ad}(u(x))}e_\alpha)_\mathfrak{h} = (e^{-\text{ad}(v(x))}e_\alpha)_\mathfrak{h}
\]  
(2.38)
and
\[
(e^{-\text{ad}(u(x))}e_\alpha)_\mathfrak{P} = (e^{-\text{ad}(v(x))}e_\alpha)_\mathfrak{P} + x_\theta e^{-\text{ad}(v(x))}[e_\alpha, f_\theta].
\]  
(2.39)
Finally, using (2.14) we may write
\[
\pi_\lambda(e_\alpha) = -\sum_{\alpha \in \Delta^+} \left[\frac{\text{ad}(v(x))e^{\text{ad}(u(x))}}{e^{\text{ad}(u(x))} - \text{id}} (e^{-\text{ad}(u(x))}e_\alpha)_\mathfrak{P}\right] \partial_{x_\alpha} + (\lambda + \rho)((e^{-\text{ad}(u(x))}e_\alpha)_\mathfrak{h})
\]
\[
= -\sum_{\alpha \in \Delta^+} \left[\frac{\text{ad}(v(x))e^{\text{ad}(v(x))}}{e^{\text{ad}(v(x))} - \text{id}} (e^{-\text{ad}(u(x))}e_\alpha)_\mathfrak{P}\right] \partial_{x_\alpha} + (\lambda + \rho)((e^{-\text{ad}(u(x))}e_\alpha)_\mathfrak{h})
\]
\[
= -\sum_{\alpha \in \Delta^+} \left[\frac{\text{ad}(v(x))e^{\text{ad}(v(x))}}{e^{\text{ad}(v(x))} - \text{id}} (e^{-\text{ad}(v(x))}e_\alpha)_\mathfrak{P}\right] \partial_{x_\alpha} + (\lambda + \rho)((e^{-\text{ad}(v(x))}e_\alpha)_\mathfrak{h})
\]
\[
- \sum_{\alpha \in \Delta^+} x_\theta \left[\frac{\text{ad}(v(x))e^{\text{ad}(v(x))}}{e^{\text{ad}(v(x))} - \text{id}} e^{-\text{ad}(v(x))}[e_\alpha, f_\theta]\right] \partial_{x_\alpha},
\]
were we used $[u(x), a] = [v(x), a]$ for all $a \in \mathfrak{P}$ in the second equality and the formulas (2.38) and (2.39) in the last equality. Therefore, we get
\[
\pi_\lambda(e_\alpha) = -\sum_{\alpha \in \Delta^+} x_\theta \left[\frac{\text{ad}(v(x))}{e^{\text{ad}(v(x))} - \text{id}} [e_\alpha, f_\theta]\right] \partial_{x_\alpha} + \sigma_\lambda(e_\alpha) = x_\theta \pi_\lambda([e_\alpha, f_\theta]) + x_\theta \sigma_\lambda(e_\alpha).
\]
This finishes the proof.

**Proposition 2.8.** We have
\[
\pi_\lambda(f_\theta) = -\partial_{x_\theta},
\]
\[
\pi_\lambda(h_\theta) = \sum_{\alpha \in \Delta^+} \alpha(h_\theta)x_\alpha \partial_{x_\alpha} + (\lambda + \rho)(h_\theta),
\]  
(2.40)
\[
\pi_\lambda(e_\theta) = x_\theta (\pi_\lambda(h_\theta) - x_\theta \partial_{x_\theta}) + \sigma_\lambda(e_\theta),
\]
for $\lambda \in \mathfrak{h}^*$.
Proof. Since \( f_\theta \in \mathfrak{g}(\mathfrak{h}) \), we have \([u(x), f_\theta] = 0\). Hence, from (2.16) we obtain
\[
\pi_\lambda(f_\theta) = - \sum_{\alpha \in \Delta^+} [f_\theta]_{\alpha} \partial_{x_\alpha} = - \partial x_\theta.
\]
Further, we have
\[
[u(x), h_\theta] = \sum_{\alpha \in \Delta^+} x_\alpha [f_\alpha, h_\theta] = \sum_{\alpha \in \Delta^+} \alpha(h_\theta) x_\alpha f_\alpha,
\]
which together with (2.17) gives us
\[
\pi_\lambda(h_\theta) = \sum_{\alpha \in \Delta^+} [u(x), h_\theta]_{\alpha} \partial_{x_\alpha} + (\lambda + \rho)(h_\theta) = \sum_{\alpha \in \Delta^+} \alpha(h_\theta) x_\alpha \partial x_\alpha + (\lambda + \rho)(h_\theta).
\]
Since \( u(x) = v(x) + x_\theta f_\theta \) and \([v(x), x_\theta f_\theta] = 0\), we may write
\[
e^{-\text{ad}(u(x))} e_\theta = e^{-\text{ad}(v(x)) - \text{ad}(x_\theta f_\theta) - e^{-\text{ad}(x_\theta f_\theta) e_\theta}
= e^{-\text{ad}(v(x))} e_\theta - \text{ad}(x_\theta f_\theta) e_\theta + \frac{1}{2} \text{ad}(x_\theta f_\theta)^2 e_\theta = e^{-\text{ad}(v(x))} (e_\theta + x_\theta h_\theta - x_\theta^2 f_\theta),
\]
which implies
\[
(e^{-\text{ad}(u(x))} e_\theta)_h = (e^{-\text{ad}(v(x))} e_\theta)_h + x_\theta h_\theta \tag{2.41}
\]
and
\[
(e^{-\text{ad}(u(x))} e_\theta)_\pi = (e^{-\text{ad}(v(x))} e_\theta)_\pi + x_\theta (e^{-\text{ad}(v(x))} h_\theta)_\pi - x_\theta^2 f_\theta. \tag{2.42}
\]
Finally, using (2.14) we may write
\[
\pi_\lambda(e_\theta) = - \sum_{\alpha \in \Delta^+} \left[ \frac{\text{ad}(u(x)) e^{-\text{ad}(v(x))}}{\text{e}^{\text{ad}(u(x))} - \text{id}} \right] (e^{-\text{ad}(u(x))} e_\theta)_\pi \partial_{x_\alpha} + (\lambda + \rho)((e^{-\text{ad}(u(x))} e_\theta)_h)
- \sum_{\alpha \in \Delta^+} \left[ \frac{\text{ad}(v(x)) e^{-\text{ad}(v(x))}}{\text{e}^{\text{ad}(v(x))} - \text{id}} \right] (e^{-\text{ad}(u(x))} e_\theta)_\pi \partial_{x_\alpha} + (\lambda + \rho)((e^{-\text{ad}(u(x))} e_\theta)_h)
- \sum_{\alpha \in \Delta^+} x_\theta \left[ \frac{\text{ad}(v(x)) e^{-\text{ad}(v(x))}}{\text{e}^{\text{ad}(v(x))} - \text{id}} \right] (e^{-\text{ad}(v(x))} h_\theta)_\pi \partial_{x_\alpha} + \sum_{\alpha \in \Delta^+} x_\theta^2 \left[ \frac{\text{ad}(v(x)) e^{-\text{ad}(v(x))}}{\text{e}^{\text{ad}(v(x))} - \text{id}} - f_\theta \right] \partial_{x_\alpha}
+ x_\theta (\lambda + \rho)(h_\theta),
\]
were we used \([u(x), a] = [v(x), a]\) for all \( a \in \mathfrak{h} \) in the second equality and the formulas (2.41) and (2.42) in the last equality. Therefore, we get
\[
\pi_\lambda(e_\theta) = - \sum_{\alpha \in \Delta^+} x_\theta \left[ \frac{\text{ad}(v(x)) e^{-\text{ad}(v(x))}}{\text{e}^{\text{ad}(v(x))} - \text{id}} \right] (e^{-\text{ad}(v(x))} h_\theta)_\pi \partial_{x_\alpha} + \sum_{\alpha \in \Delta^+} x_\theta^2 [f_\theta]_{\alpha} \partial_{x_\alpha}
+ x_\theta (\lambda + \rho)(h_\theta) + \sigma_\lambda(e_\theta)
- \sum_{\alpha \in \Delta^+} x_\theta \left[ \frac{\text{ad}(v(x)) e^{-\text{ad}(v(x))}}{\text{e}^{\text{ad}(v(x))} - \text{id}} \right] (e^{-\text{ad}(v(x))} h_\theta)_\pi \partial_{x_\alpha} + x_\theta^2 \partial_{x_\theta}
+ x_\theta (\lambda + \rho)(h_\theta) + \sigma_\lambda(e_\theta).
\]
Further, since we have \((e^{-\text{ad}(v(x))} h_\theta)_\pi = (e^{-\text{ad}(v(x))} - \text{id}) h_\theta\), we may write
\[
\pi_\lambda(e_\theta) = - \sum_{\alpha \in \Delta^+} x_\theta \left[ \frac{\text{ad}(v(x)) e^{-\text{ad}(v(x))}}{\text{e}^{\text{ad}(v(x))} - \text{id}} \right] (e^{-\text{ad}(v(x))} - \text{id}) h_\theta \partial_{x_\alpha} + x_\theta^2 \partial_{x_\theta}
\]
12
+ x_\theta (\lambda + \rho)(h_\theta) + \sigma_\lambda (e_\theta) \\
= \sum_{\alpha \in \Delta^+} x_\theta [\mathrm{ad}(v(x))h_\theta]_\alpha \partial_x_\alpha + x_\theta^2 \partial_x + x_\theta (\lambda + \rho)(h_\theta) + \sigma_\lambda (e_\theta) \\
= \sum_{\alpha \in \Delta^+} x_\theta [u(x), h_\theta]_\alpha \partial_x_\alpha - x_\theta^2 \partial_x + x_\theta (\lambda + \rho)(h_\theta) + \sigma_\lambda (e_\theta) \\
= x_\theta (\pi_\lambda (h_\theta) - x_\theta \partial_x) + \sigma_\lambda (e_\theta).
This finishes the proof. \qed

**Theorem 2.9.** Let $\lambda \in \mathfrak{h}^*$. Then $W^g_\theta (\lambda)$ is a cyclic $\mathfrak{g}$-module.

**Proof.** Since the polynomials $[(e^{-\mathrm{ad}(v(x))}e_\theta)]_\alpha$ and $[(e^{-\mathrm{ad}(v(x))}e_\theta)]_\pi$ do not contain the constant and linear term for all $\alpha \in \Delta^+$, we obtain, using (2.33), that $\sigma_{\lambda+\rho}(e_\theta)x^n_\theta = 0$ for $n \in \mathbb{N}_0$. Hence, we have

$$
\pi_{\lambda+\rho}(e_\theta)x^n_\theta = x_\theta (\pi_{\lambda+\rho}(h_\theta) - x_\theta \partial_x_\alpha)x^n_\theta = x_\theta (x_\theta \partial_x_\alpha + (\lambda + \theta)(h_\theta))x^n_\theta = (\lambda (h_\theta) + n + 2)x^n_\theta + \pi_{\lambda+\rho}(h_\theta)x^n_\theta = (2x_\theta^{n+1}) + x_\theta (\lambda (h_\theta) + 2 + 2n)x^n_\theta,
$$

for $n \in \mathbb{N}_0$, where we used (2.21), which gives us $C[x_\theta] = U(\mathfrak{g}_\theta)1$ provided $\lambda(h_\theta) + 2 \notin \mathbb{N}_0$ and $C[x_\theta] = U(\mathfrak{g}_\theta)x_\theta^{\lambda(h_\theta)} - 1$ if $\lambda(h_\theta) + 2 \in \mathbb{N}_0$.

From (2.13) and the identity

$$
\frac{\mathrm{ad}(u(x))}{e^{\mathrm{ad}(u(x))} - \mathrm{id}} a = a + \frac{\mathrm{ad}(u(x)) - e^{\mathrm{ad}(u(x))} + \mathrm{id}}{e^{\mathrm{ad}(u(x))} - \mathrm{id}} a
$$

for all $a \in \mathfrak{g}$, we obtain

$$
\pi_{\lambda+\rho}(f_\alpha) = -\partial_x_\alpha + \pi'_{\lambda+\rho}(f_\alpha)
$$

for $\alpha \in \Delta^+$, where

$$
\pi'_{\lambda+\rho}(a) = -\sum_{\alpha \in \Delta^+} \left[ \frac{\mathrm{ad}(u(x)) - e^{\mathrm{ad}(u(x))} + \mathrm{id}}{e^{\mathrm{ad}(u(x))} - \mathrm{id}} a \right]_\alpha \partial_x_\alpha.
$$

Further, for $k \in \mathbb{N}_0$ we denote by $F_k W^g_\theta (\lambda)$ the vector subspace of $W^g_\theta (\lambda)$ consisting of all polynomials of degree at most $k$ in the variables $\partial_x_\alpha$, $\alpha \in \Delta^+$. Hence, we have an increasing filtration $\{F_k W^g_\theta (\lambda)\}_{k \in \mathbb{N}_0}$ on $W^g_\theta (\lambda)$. As the polynomial

$$
\left[ \frac{\mathrm{ad}(u(x)) - e^{\mathrm{ad}(u(x))} + \mathrm{id}}{e^{\mathrm{ad}(u(x))} - \mathrm{id}} a \right]_\alpha
$$

does not contain the constant term and does not depend on $x_\theta$ for all $\alpha \in \Delta^+$ and $a \in \mathfrak{g}$, we get

$$
\pi'_{\lambda+\rho}(f_\alpha)(F_k W^g_\theta (\lambda)) \subset F_k W^g_\theta (\lambda)
$$

for $k \in \mathbb{N}_0$ and $\alpha \in \Delta^+$. Therefore, we have

$$
\pi_{\lambda+\rho}(f_\alpha)(F_k W^g_\theta (\lambda))/F_k W^g_\theta (\lambda) = \partial_x_\alpha F_k W^g_\theta (\lambda)/F_k W^g_\theta (\lambda)
$$

for $k \in \mathbb{N}_0$ and $\alpha \in \Delta^+$, which together with $F_0 W^g_\theta (\lambda) = C[x_\theta]$ gives us that $W^g_\theta (\lambda)$ is generated by $C[x_\theta]$. As $C[x_\theta]$ is generated by one vector, we get that $W^g_\theta (\lambda)$ is a cyclic $\mathfrak{g}$-module. \qed

**Proposition 2.10.** Let $\lambda \in \mathfrak{h}^*$. Then the Casimir operator $\pi_\lambda (\mathrm{Cas}_\theta)$ has the form

$$
\pi_\lambda (\mathrm{Cas}_\theta) = \pi_\lambda (\mathrm{Cas}_\theta)_s + \pi_\lambda (\mathrm{Cas}_\theta)_n, \quad (2.43)
$$
where
\[ \pi_\lambda(Cas_\theta)_s = \frac{1}{2} \sigma_\lambda(h_\theta) (\sigma_\lambda(h_\theta) - 2) \] (2.44)
is a semisimple operator and
\[ \pi_\lambda(Cas_\theta)_n = 2 \sigma_\lambda(e_\theta) \sigma_\lambda(f_\theta) \] (2.45)
is a locally nilpotent operator. Moreover, the weight vector \( w_{\lambda,\mu+(n_\alpha+n_\beta)t_\theta+t} \) with weight \( \mu_\lambda,\mu \) for \( a \in \mathbb{Z}^n \), \( n_\theta \in \mathbb{N}_0 \) and \( t \in \Lambda^\theta_+ \) is an eigenvector of \( \pi_{\lambda+\mu}(Cas_\theta)_s \) with eigenvalue
\[ c_{\lambda,\mu,n_\theta} = \frac{1}{2} (\pi_{\lambda,\mu}(h_\theta) - 2(n_\alpha + n_\beta)) (\mu_{\lambda,\mu}(h_\theta) - 2(n_\alpha + n_\beta + 1)) \] (2.46)
provided \( a + (n_\alpha + n_\beta)t_\theta + t \in \mathbb{N}_0^{\Lambda^\theta_+} \), where \( n_\theta \in \mathbb{N}_0 \) is the smallest non-negative integer satisfying \( a + n_\theta t_\theta \in \mathbb{N}_0^{\Lambda^\theta_+} \).

**Proof.** As \( Cas_\theta = e_\theta f_\theta + f_\theta e_\theta + \frac{1}{2} h_\theta^2 = 2 f_\theta e_\theta + \frac{1}{2} h_\theta (h_\theta + 2) \), we may write
\[
\pi_\lambda(Cas_\theta)_s = -2(x_\theta f_\theta x_\theta + 1) (\pi_\lambda(h_\theta) - x_\theta f_\theta x_\theta + \frac{1}{2} \pi_\lambda(h_\theta) (\pi_\lambda(h_\theta) + 2)
\]
and
\[
\pi_\lambda(Cas_\theta)_s = -2 \partial_\theta x_\theta \sigma_\lambda(e_\theta),
\]
we obtain \( \pi_\lambda(Cas_\theta)_s = \pi_\lambda(Cas_\theta)_s + \pi_\lambda(Cas_\theta)_n \). Further, we have
\[
\pi_\lambda(Cas_\theta)_n = -2 \partial_\theta x_\theta \sigma_\lambda(e_\theta) = -2 \sigma_\lambda(e_\theta) \partial_\theta x_\theta = 2 \sigma_\lambda(e_\theta) \sigma_\lambda(f_\theta)
\]
and
\[
\pi_\lambda(Cas_\theta)_n = -2(x_\theta f_\theta x_\theta + 1) (\pi_\lambda(h_\theta) - x_\theta f_\theta x_\theta + \frac{1}{2} \pi_\lambda(h_\theta) (\pi_\lambda(h_\theta) + 2)
\]
where we used Proposition 2.8. If we denote
\[
\pi_\lambda(Cas_\theta)_s = -2(x_\theta f_\theta x_\theta + 1) (\sigma_\lambda(h_\theta) - x_\theta f_\theta x_\theta + \frac{1}{2} \sigma_\lambda(h_\theta) (\sigma_\lambda(h_\theta) + 2)
\]
and
\[
\pi_\lambda(Cas_\theta)_n = -2 \partial_\theta x_\theta \sigma_\lambda(e_\theta),
\]
we obtain \( \pi_\lambda(Cas_\theta)_n = \pi_\lambda(Cas_\theta)_s + \pi_\lambda(Cas_\theta)_n \). Further, we have
\[
\pi_\lambda(Cas_\theta)_n = -2 \partial_\theta x_\theta \sigma_\lambda(e_\theta) = -2 \sigma_\lambda(e_\theta) \partial_\theta x_\theta = 2 \sigma_\lambda(e_\theta) \sigma_\lambda(f_\theta)
\]
and
\[
\pi_\lambda(Cas_\theta)_n = -2(x_\theta f_\theta x_\theta + 1) (\pi_\lambda(h_\theta) - x_\theta f_\theta x_\theta + \frac{1}{2} \pi_\lambda(h_\theta) (\pi_\lambda(h_\theta) + 2)
\]
where we used \( \pi_\lambda(h_\theta) = \sigma_\lambda(h_\theta) + 2 x_\theta f_\theta x_\theta \), which easily follows from (2.40) and Proposition 2.8.

Now, let \( a \in \mathbb{Z}^n \). Then the weight vector \( w_{\lambda-\rho,a+(n_\alpha+n_\beta)t_\theta+t} \) with weight \( \mu_{\lambda-\rho,a} \) for \( n_\theta \in \mathbb{N}_0 \) and \( t \in \Lambda^\theta_+ \) is an eigenvector of \( \sigma_\lambda(h_\theta) \) and therefore also of \( \pi_\lambda(Cas_\theta)_s \) if \( a + (n_\alpha + n_\beta)t_\theta + t \in \mathbb{N}_0^{\Lambda^\theta_+} \), since we have
\[
\sigma_\lambda(h_\theta) w_{\lambda-\rho,a+(n_\alpha+n_\beta)t_\theta+t} = (\pi_\lambda(h_\theta) - 2 x_\theta f_\theta x_\theta) w_{\lambda-\rho,a+(n_\alpha+n_\beta)t_\theta+t}
\]
As \( \{ w_{\lambda-\rho,a+(n_\alpha+n_\beta)t_\theta+t}; a \in \mathbb{Z}^n, n_\theta \in \mathbb{N}_0, t \in \Lambda^\theta_+, a + (n_\alpha + n_\beta)t_\theta + t \in \mathbb{N}_0^{\Lambda^\theta_+} \} \) forms a basis of \( W^\theta_\mathfrak{h}(\lambda-\rho) \), we get that \( \pi_\lambda(Cas_\theta)_s \) is a semisimple operator.

Since \( \sigma_\lambda(f_\theta) \) is obviously a locally nilpotent operator and \( \sigma_\lambda(f_\theta) \sigma_\lambda(e_\theta) = \sigma_\lambda(e_\theta) \sigma_\lambda(f_\theta) \), we obtain that \( \pi_\lambda(Cas_\theta)_n \) is also a locally nilpotent operator.

□

The main disadvantage of modules \( W^\theta_\mathfrak{g}(\lambda) \) for \( \lambda \in \mathfrak{h}^* \), in comparison with objects of the category \( \mathcal{O} \), is infinite dimensional weight spaces. However, we can enlarge the Cartan subalgebra \( \mathfrak{h} \) of \( \mathfrak{g} \) to a commutative subalgebra \( \Gamma^\theta \) of \( U(\mathfrak{g}) \) in such a way that \( W^\theta_\mathfrak{h}(\lambda) \) for \( \lambda \in \mathfrak{h}^* \) is a \( \theta \)-Gelfand–Tsetlin module, i.e. \( W^\theta_\mathfrak{h}(\lambda) \in \mathcal{H}(\mathfrak{g}, \Gamma^\theta) \).
A comparison of the basic characteristics of Verma modules and Gelfand–Tsetlin modules is given.

For an arbitrary simple finite-dimensional $W$, we obtain, using (2.33), that

$$\sigma_{\lambda+\rho}(e_\theta)(F_r W^g_\theta(\lambda)) \subset F_{r-1} W^g_\theta(\lambda)$$

for $r \in \mathbb{N}$, which together with

$$\sigma_{\lambda+\rho}(\mathcal{G}_\theta)(W^g_\theta(\lambda)_r) \subset W^g_\theta(\lambda)_r$$

and

$$\sigma_{\lambda+\rho}(\mathcal{G}_\theta)(F_r W^g_\theta(\lambda)_r) \subset W^g_\theta(\lambda)_{r-1}$$

for $r \in \mathbb{N}$, according to Proposition 2.10, gives us

$$\pi_{\lambda+\rho}(\mathcal{G}_\theta)_r(W^g_\theta(\lambda)_r) \subset W^g_\theta(\lambda)_r$$

and

$$\pi_{\lambda+\rho}(\mathcal{G}_\theta)_r(F_r W^g_\theta(\lambda)_r) \subset F_{r-1} W^g_\theta(\lambda)$$

for all $r \in \mathbb{N}$. Therefore, the assumptions of Theorem A.2 are satisfied.

Let $\chi \in \text{Hom}(\Gamma_\theta, \mathbb{C})$ be a $\Gamma_\theta$-weight of $W^g_\theta(\lambda)$. Let us denote $\mu = \chi|_\mathfrak{g}$ and $c = \chi(\mathcal{G}_\theta)$. Then $\mu \in \mathfrak{h}^*$ is an $\mathfrak{h}$-weight of $W^g_\theta(\lambda)$ and we have $W^g_\theta(\lambda)_\chi \subset W^g_\theta(\lambda)_\mu$. By Proposition 2.5 we obtain that there exists $a \in \mathbb{Z}^1$ such that $\mu = \mu_{a,a}$. The $\mathbb{N}$-grading on $W^g_\theta(\lambda)$ induces the $\mathbb{N}$-grading on $W^g_\theta(\lambda)_{\mu_\alpha}$ defined by $W^g_\theta(\lambda)_{\mu_\alpha,r} = W^g_\theta(\lambda)_{\mu_\alpha} \cap W^g_\theta(\lambda)_r$ for all $r \in \mathbb{N}$. Hence, by using Theorem A.2 for the linear mapping $\pi_{\lambda+\rho}(\mathcal{G}_\theta)_r$ on the vector space $W^g_\theta(\lambda)_{\mu_\alpha}$, we get that eigenvalues of $\pi_{\lambda+\rho}(\mathcal{G}_\theta)_r$ on $W^g_\theta(\lambda)_{\mu_\alpha}$ are the same as eigenvalues of $\pi_{\lambda+\rho}(\mathcal{G}_\theta)_r$ on $W^g_\theta(\lambda)_{\mu_\alpha}$, and moreover the dimensions of generalized eigenspaces corresponding to the same eigenvalue are equal. Hence, by Proposition 2.10 there exists $n_\theta \in \mathbb{N}_0$ such that $c = c_{\lambda,a,n_\theta}$. Further, from Proposition 2.5 and Proposition 2.10 follows that the set

$$\{w_{\lambda,a+(n_a+n_\theta)t_\theta+t}; n_\theta \in \mathbb{N}_0, t \in \Lambda^\theta_+, a + (n_a+n_\theta)t_\theta + t \in \mathbb{N}^{\Delta^+}\}$$

forms a basis of $W^g_\theta(\lambda)_{\mu_\alpha}$ and $w_{\lambda,a+(n_a+n_\theta)t_\theta+t}$ is an eigenvectors of $\pi_{\lambda+\rho}(\mathcal{G}_\theta)_r$, with eigenvalue $c_{\lambda,a,n_\theta}$. As we have $c_{\lambda,a,n_\theta} = c_{\lambda,a,n_\theta}'$ for $n_\theta, n_\theta' \in \mathbb{N}_0$ if and only if $n_\theta = n_\theta$ or $n_\theta' = \mu_{a,a}(h_\theta) - 2n_a - n_\theta - 1$ provided $\mu_{a,a}(h_\theta) - 2n_a - n_\theta - 1 \in \mathbb{N}_0$, we obtain

$$\dim W^g_\theta(\lambda)_{\chi} = \frac{1}{2}\{a + (n_a+n_\theta)t_\theta + t; t \in \Lambda^\theta_+, a + (n_a+n_\theta)t_\theta + t \in \mathbb{N}_0^{\Delta^+}\}$$

$$+ \frac{1}{2}\{a + (\mu_{a,a}(h_\theta) - n_a - n_\theta - 1)t_\theta + t; t \in \Lambda^\theta_+, a + (\mu_{a,a}(h_\theta) - n_a - n_\theta - 1)t_\theta + t \in \mathbb{N}_0^{\Delta^+}\}$$

which is obviously a finite number. This completes the proof.

**Corollary 2.12.** The $\mathfrak{g}$-module $W^g_\theta(\lambda)$ belongs to the category $\mathcal{I}_{\theta,f,\tilde{\mathfrak{m}}}$ for all $\lambda \in \mathfrak{h}^*$.

**Corollary 2.13.** Let $\Gamma$ be any commutative subalgebra of $U(\mathfrak{g})$ containing $\Gamma_\theta$. Then the $\mathfrak{g}$-module $W^g_\theta(\lambda)$ belongs to the category $\mathcal{H}_\mathfrak{m}(U(\mathfrak{g}), \Gamma)$ for all $\lambda \in \mathfrak{h}^*$.

Hence, the $\theta$-Gelfand–Tsetlin $\mathfrak{g}$-modules $W^g_\theta(\lambda)$ is a $\Gamma$-weight module for any commutative $\Gamma$ containing $\Gamma_\theta$. In particular, if $\mathfrak{g} = \mathfrak{sl}(n)$ (or $\mathfrak{g} = \mathfrak{gl}(n)$) and $\Gamma$ is a Gelfand-Tsetlin subalgebra of $U(\mathfrak{g})$ containing $\mathcal{G}_\theta$, then $W^g_\theta(\lambda)$ is a Gelfand-Tsetlin module (with respect to a particular choice of a Gelfand-Tsetlin subalgebra) (in this case the chain of $\mathfrak{sl}(n)$-subalgebras contains $\mathfrak{sl}(2)$ associated to the root $\theta$). Therefore, the $\mathfrak{g}$-modules $W^g_\theta(\lambda)$ can be viewed as examples of Gelfand-Tsetlin modules for an arbitrary simple finite-dimensional $\mathfrak{g}$ with respect to any $\Gamma$ containing $\Gamma_\theta$.

By duality we can similarly construct certain $\theta$-Gelfand–Tsetlin modules in the category $\mathcal{I}_{\theta,e,\tilde{\mathfrak{m}}}$. A comparison of the basic characteristics of Verma modules and Gelfand–Tsetlin modules is given in Table 4. Their realization is given by the following theorem.
The coroots corresponding to the positive roots are given by

\[ \lambda \in \mathfrak{h}^* \quad \text{Then the g-module} \]

\[ \mathcal{A}_1^0/\mathcal{J}_{\mathrm{GT}} \simeq \mathbb{C}[\partial_{x_\alpha}, \alpha \in \Delta^+_0, x_\alpha], \quad \mathcal{J}_{\mathrm{GT}} = (x_\alpha, \alpha \in \Delta^+_0, \partial_{x_\alpha}), \quad (2.47) \]

where the g-module structure on \( \mathcal{A}_1^0/\mathcal{J}_{\mathrm{GT}} \) is given through the homomorphism \( \pi_{\lambda + \rho}: \mathfrak{g} \to \mathcal{A}_1^0 \) of Lie algebras with

\[ \pi_\lambda(a) = - \sum_{\alpha \in \Delta^+} \left[ \frac{\text{ad}(u(x))e^{\text{ad}(u(x))}}{e^{\text{ad}(u(x))} - 1} \right] \partial_{x_\alpha} + (\lambda + \rho)((e^{\text{ad}(u(x))}a)\mathfrak{h}) \quad (2.48) \]

for all \( a \in \mathfrak{g} \), is a weight, cyclic \( \theta \)-Gelfand–Tsetlin module with finite \( \Gamma_\theta \)-multiplicities.

2.3 Gelfand–Tsetlin modules for \( \mathfrak{sl}(3, \mathbb{C}) \)

In this section, we will focus more closely on the Gelfand–Tsetlin modules \( W_\theta\mathfrak{g}(\lambda) \) for \( \lambda \in \mathfrak{h}^* \) over the Lie algebra \( \mathfrak{sl}(3, \mathbb{C}) \). In particular, we describe the irreducibility condition for these modules. We use the notation introduced in the previous section.

Let us consider the complex simple Lie algebra \( \mathfrak{g} = \mathfrak{sl}(3, \mathbb{C}) \). A Cartan subalgebra \( \mathfrak{h} \) of \( \mathfrak{g} \) is given by diagonal matrices

\[ \mathfrak{h} = \{ \text{diag}(a_1, a_2, a_3); a_1, a_2, a_3 \in \mathbb{C}, a_1 + a_2 + a_3 = 0 \}. \quad (2.49) \]

For \( i = 1, 2, 3 \) we define \( \varepsilon_i \in \mathfrak{h}^* \) by \( \varepsilon_i(\text{diag}(a_1, a_2, a_3)) = a_i \). The root system of \( \mathfrak{g} \) with respect to \( \mathfrak{h} \) is \( \Delta = \{ \varepsilon_i - \varepsilon_j; 1 \leq i \neq j \leq 3 \} \). A positive root system in \( \Delta \) is \( \Delta^+ = \{ \varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \varepsilon_1 - \varepsilon_3 \} \) and the set of simple roots is \( \Pi = \{ \alpha_1, \alpha_2 \} \) with \( \alpha_1 = \varepsilon_1 - \varepsilon_2 \) and \( \alpha_2 = \varepsilon_2 - \varepsilon_3 \). The remaining positive root \( \alpha_1 + \alpha_2 = \varepsilon_1 - \varepsilon_3 \) is the maximal root \( \Theta \) of \( \mathfrak{g} \).

Let us introduce a root basis of the nilradical \( \mathfrak{n} \) by

\[ e_{\alpha_1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_\theta = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_{\alpha_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \]

and a root basis of the opposite nilradical \( \overline{\mathfrak{n}} \) through

\[ f_{\alpha_1} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad f_\theta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad f_{\alpha_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \]

The coroots corresponding to the positive roots are given by

\[ h_{\alpha_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad h_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad h_{\alpha_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \]

These generators fulfill, among others, the commutation relations \([e_{\alpha_1}, e_{\alpha_2}] = e_\theta \) and \([f_{\alpha_2}, f_{\alpha_1}] = f_\theta \). Moreover, we have that \([e_\theta, h_\theta, f_\theta] \) is an \( \mathfrak{sl}(2, \mathbb{C}) \)-triple.

Let us recall that \{\( x_{\alpha_1}, x_{\alpha_2}, x_\theta \)\} are the linear coordinate functions on \( \overline{\mathfrak{n}} \) with respect to the basis \{\( f_{\alpha_1}, f_{\alpha_2}, f_\theta \)\} of \( \overline{\mathfrak{n}} \) and that the Weyl algebra \( \mathcal{A}_1^0 \) of \( \overline{\mathfrak{n}} \) is generated by \{\( x_{\alpha_1}, \partial_{x_{\alpha_1}}, x_{\alpha_2}, \partial_{x_{\alpha_2}}, x_\theta, \partial_{x_\theta} \)\} together with the canonical commutation relations. For clarity, we denote \( x = x_{\alpha_1}, y = x_{\alpha_2} \) and \( z = x_\theta \).

Theorem 2.15. Let \( \lambda \in \mathfrak{h}^* \). Then the homomorphism \( \pi_\lambda: \mathfrak{g} \to \mathcal{A}_1^0 \) of Lie algebras is given by

\[ \pi_\lambda(f_{\alpha_1}) = -\partial_x + \frac{1}{2} y \partial_z, \]
\[ \pi_\lambda(f_{\alpha_2}) = -\partial_\theta - \frac{1}{2} x \partial_z, \]
\[ \pi_\lambda(f_\theta) = -\partial_z, \]

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Proof. It follows immediately by a straightforward computation from (2.14).

The homomorphism \( \pi_\lambda : \mathfrak{g} \to \mathcal{A}_\mathfrak{g}^w \) of Lie algebras enables us to construct a wide class of interesting \( \mathfrak{g} \)-modules. The easiest way is to take a left ideal \( I \) of \( \mathcal{A}_\mathfrak{g}^w \). Then \( \mathcal{A}_\mathfrak{g}^w/I \) has the \( \mathfrak{g} \)-module structure through the homomorphism \( \pi_\lambda : \mathfrak{g} \to \mathcal{A}_\mathfrak{g}^w \).

In the previous section, we considered only the two simplest possible left ideals of the Weyl algebra \( \mathcal{A}_\mathfrak{g}^w \). The first one \( J_V = \langle x, y, z \rangle \) leads to the Verma modules \( M_\mathfrak{g}^w(\lambda) \cong \mathcal{A}_\mathfrak{g}^w/J \), while the second one \( J_{\mathcal{G}T} = \langle x, y, z \rangle \) leads to the Gelfand–Tsetlin modules \( W_\mathfrak{g}^w(\lambda) \cong \mathcal{A}_\mathfrak{g}^w/J_{\mathcal{G}T} \). However, we have much more options, where some of them lead to the so called twisted Verma modules \( M_\mathfrak{g}^w(\lambda)^w \) labeled by the elements \( w \in W \) of the Weyl group \( W \) of \( \mathfrak{g} \), see [15].

The twisted Verma modules \( M_\mathfrak{g}^w(\lambda)^w \) for \( \lambda \in \mathfrak{h}^* \) and \( w \in W \) are realized as \( \mathcal{A}_\mathfrak{g}^w/J_w \), where \( J_w \) is the left ideal of \( \mathcal{A}_\mathfrak{g}^w \) defined by

\[
J_w = \langle x, y, z \rangle, \quad \alpha \in w^{-1}(\Delta^+) \cap \Delta^+, \quad \partial_{x,\alpha}, \alpha \in w^{-1}(-\Delta^+) \cap \Delta^+) \quad (2.50)
\]

and the \( \mathfrak{g} \)-module structure on \( \mathcal{A}_\mathfrak{g}^w/J_w \) is given through the homomorphism \( \pi_{\lambda+\rho}^w : \mathfrak{g} \to \mathcal{A}_\mathfrak{g}^w \) of Lie algebras defined by

\[
\pi_{\lambda+\rho}^w = \pi_w^{-1}(\lambda+\rho) \circ \text{Ad}(w^{-1}), \quad (2.51)
\]

where \( w^{-1} \) acts in the standard way and \( \check{w} \) is a representative of \( w \) in the Lie group \( G = \text{SL}(3, \mathbb{C}) \).

The list of all possibilities looks as follows:

1) \( w = e, J_e = \langle x, y, z \rangle, \mathcal{A}_\mathfrak{g}^w/I_e \cong \mathbb{C}[\partial_x, \partial_y, \partial_z]; \)

2) \( w = s_1, J_{s_1} = \langle \partial, y, z \rangle, \mathcal{A}_\mathfrak{g}^w/J_{s_1} \cong \mathbb{C}[\partial, \partial_y, \partial_z]; \)

3) \( w = s_2, J_{s_2} = \langle x, \partial, y \rangle, \mathcal{A}_\mathfrak{g}^w/J_{s_2} \cong \mathbb{C}[\partial, \partial_y, \partial_z]; \)

4) \( w = s_1 s_2, J_{s_1 s_2} = \langle x, y, \partial \rangle, \mathcal{A}_\mathfrak{g}^w/J_{s_1 s_2} \cong \mathbb{C}[\partial, y, z]; \)

5) \( w = s_2 s_1, J_{s_2 s_1} = \langle \partial, x, \partial \rangle, \mathcal{A}_\mathfrak{g}^w/J_{s_2 s_1} \cong \mathbb{C}[\partial, x, \partial_y]; \)

6) \( w = s_1 s_2 s_1, J_{s_1 s_2 s_1} = \langle \partial, \partial_y, \partial_z \rangle, \mathcal{A}_\mathfrak{g}^w/J_{s_1 s_2 s_1} \cong \mathbb{C}[x, y, z]. \)

The elements \( s_1 \) and \( s_2 \) of the Weyl group \( W \) denote the reflections about the hyperplanes perpendicular to \( \alpha_1 \) and \( \alpha_2 \), respectively.

In fact, we have another two possible left ideals of \( \mathcal{A}_\mathfrak{g}^w \), which look as follows:

7) \( J_{\mathcal{G}T} = \langle x, y, \partial_z \rangle, \mathcal{A}_\mathfrak{g}^w/J_{\mathcal{G}T} \cong \mathbb{C}[\partial_x, \partial_y, z]; \)

8) \( J_{\mathcal{G}T^w} = \langle \partial_x, \partial_y, z \rangle, \mathcal{A}_\mathfrak{g}^w/J_{\mathcal{G}T^w} \cong \mathbb{C}[x, y, \partial_z]. \)

Hence, we have the complete list of left ideals of the Weyl algebra \( \mathcal{A}_\mathfrak{g}^w \) of the given type, where we consider either \( x, \alpha \) or \( z, \alpha \), for each \( \alpha \in \Delta^+. \)

**Theorem 2.16.** Let \( \lambda \in \mathfrak{h}^* \). Then \( W_\mathfrak{g}^w(\lambda) \) is a simple \( \mathfrak{g} \)-module if and only if \( \lambda(h_{\alpha_1}), \lambda(h_{\alpha_2}) \notin \mathbb{N}_0 \) and \( \lambda(h_\rho) \notin \mathbb{Z}. \)

**Proof.** Let \( M \) be a nonzero \( \mathfrak{g} \)-submodule of \( W_\mathfrak{g}^w(\lambda) \) for \( \lambda \in \mathfrak{h}^* \). Then \( M \) is a weight \( \mathfrak{h} \)-module, since \( W_\mathfrak{g}^w(\lambda - \rho) \) is a weight \( \mathfrak{h} \)-module. Moreover, we have \( \pi\lambda(f_\theta) = -\partial_z \), which implies that \( M \)
contains a nonzero weight vector $v$ satisfying $\partial_z v = 0$. Hence, the weight vector $v$ is of the form $v = \partial_x^2 \partial_y^b$ for some $a, b \in \mathbb{N}_0$. Further, by Proposition 2.6 and Proposition 2.7 we have

$$\pi \lambda(e_\theta) = z(\pi \lambda(h_\theta) - z\partial_z) + \sigma \lambda(e_\theta)$$

and

$$\pi \lambda(e_\alpha) = z\pi \lambda([e_\alpha, f_\theta]) + \sigma \lambda(e_\alpha)$$

for $\alpha \in \Delta^+$. As we have $[\pi \lambda(h_\theta), z] = 2z$, we may write

$$(\pi \lambda(h_\theta) - 2)\pi \lambda(e_\alpha) = z\pi \lambda(h_\theta)\pi \lambda([e_\alpha, f_\theta]) + (\pi \lambda(h_\theta) - 2)\sigma \lambda(e_\alpha)$$

and

$$\pi \lambda(e_\theta)\pi \lambda([e_\alpha, f_\theta]) = z(\pi \lambda(h_\theta) - z\partial_z)\pi \lambda([e_\alpha, f_\theta]) + \sigma \lambda(e_\theta)\pi \lambda([e_\alpha, f_\theta])$$

for $\alpha \in \Delta^+$. Since $[\pi \lambda(f_\theta), \pi \lambda([e_\alpha, f_\theta])] = 0$ for $\alpha \in \Delta^+$ and $\partial_z v = 0$, we get

$$(\pi \lambda(h_\theta) - 2)\pi \lambda(e_\alpha) v = z\pi \lambda(h_\theta)\pi \lambda([e_\alpha, f_\theta]) v + (\pi \lambda(h_\theta) - 2)\sigma \lambda(e_\alpha) v;$$

$$\pi \lambda(e_\theta)\pi \lambda([e_\alpha, f_\theta]) v = z\pi \lambda(h_\theta)\pi \lambda([e_\alpha, f_\theta]) v + \sigma \lambda(e_\theta)\pi \lambda([e_\alpha, f_\theta]) v,$$

which implies that

$$((\pi \lambda(h_\theta) - 2)\pi \lambda(e_\alpha) - \pi \lambda(e_\theta)\pi \lambda([e_\alpha, f_\theta]))) v = ((\pi \lambda(h_\theta) - 2)\sigma \lambda(e_\alpha) - \pi \lambda(e_\theta)\pi \lambda([e_\alpha, f_\theta]))) v$$

for $\alpha \in \Delta^+$. If we denote

$$w_1 = ((\pi \lambda(h_\theta) - 2)\sigma \lambda(e_\alpha) - \sigma \lambda(e_\theta)\pi \lambda([e_\alpha, f_\theta])) v,$$

$$w_2 = ((\pi \lambda(h_\theta) - 2)\sigma \lambda(e_\alpha) - \sigma \lambda(e_\theta)\pi \lambda([e_\alpha, f_\theta])) v,$$

then we have $w_1, w_2 \in M$ and $\pi \lambda(f_\theta) w_1 = 0, \pi \lambda(f_\theta) w_2 = 0$. We may write

$$w_1 = ((\pi \lambda(h_\theta) - 2)\sigma \lambda(e_\alpha) + \sigma \lambda(e_\theta)\pi \lambda(f_\alpha)) v$$

$$= ((x\partial_x + y\partial_y + \lambda_1 + \lambda_2)(x\partial_x - y\partial_y + \lambda_1 + 1) - \frac{1}{2}xy(x\partial_x - y\partial_y + \lambda_1 - \lambda_2)\partial_y) v$$

$$= x(x\partial_x + \lambda_1 + \lambda_2 + 1)(x\partial_x + \lambda_1 + 1)v = x(\partial_x x + \lambda_1 + \lambda_2)(\partial_x x + \lambda_1)v$$

and similarly

$$w_2 = ((\pi \lambda(h_\theta) - 2)\sigma \lambda(e_\alpha) - \sigma \lambda(e_\theta)\pi \lambda(f_\alpha)) v$$

$$= (y\partial_y + \lambda_1 + \lambda_2 + 1)(y\partial_y + \lambda_1 + 1)v = y(\partial_y y + \lambda_1 + \lambda_2)(\partial_y y + \lambda_1)v,$$

where we used Theorem 2.15 and notation $\lambda_1 = \lambda(h_{\alpha_1})$ and $\lambda_2 = \lambda(h_{\alpha_2})$. Since $v = \partial_x^a \partial_y^b$ for some $a, b \in \mathbb{N}_0$, we obtain

$$w_1 = -a(\lambda_1 + \lambda_2 - a)(\lambda_1 - a)\partial_x^{a-1}\partial_y^b$$

and

$$w_2 = -b(\lambda_1 + \lambda_2 - b)(\lambda_2 - b)\partial_x^a\partial_y^{b-1},$$

which implies that $a\partial_x^{a-1}\partial_y^b, b\partial_x^a\partial_y^{b-1} \in M$ provided $\lambda_1, \lambda_2, \lambda_1 + \lambda_2 \notin \mathbb{N}$. As a consequence, we get that $1 \in M$ if $\lambda_1, \lambda_2, \lambda_1 + \lambda_2 \notin \mathbb{N}$. Since $W^g_B(\lambda)$ is generated by the vector $1 \in W^g_B(\lambda)$ for $\lambda_1 + \lambda_2 \notin -\mathbb{N}_0$, as follows from the proof of Theorem 2.9, we obtain that $W^g_B(\lambda)$ is a simple $g$-module if $\lambda_1, \lambda_2 \notin \mathbb{N}$ and $\lambda_1 + \lambda_2 \notin \mathbb{Z}$.

Now, let us assume that $\lambda_1 + \lambda_2 \in -\mathbb{N}_0$. Let $M$ be the $g$-submodule of $W^g_B(\lambda)$ generated by the vector $1 \in W^g_B(\lambda)$. For a vector $v \in M$ we may write $v = \pi \lambda(u)v_1$, where $u \in U(g)$, which gives us $\pi \lambda(e_\theta) v = \pi \lambda([e_\theta, u]) v + \pi \lambda(u)\pi \lambda(e_\theta) v$. Since $ad(e_\theta)$ is locally nilpotent on $U(g)$ and $\pi \lambda(e_\theta)^{1-\lambda_1-\lambda_2+1} = 0$, we obtain that $\pi \lambda(e_\theta)$ is locally nilpotent on $M$. However, we have
\( \pi_\lambda (e_\theta) z^{-\lambda_1 - \lambda_2 + 1} = k! z^{-\lambda_1 - \lambda_2 + k + 1} \) for \( k \in \mathbb{N}_0 \), which implies that \( z^{-\lambda_1 - \lambda_2 + 1} \notin M \). Hence, we get that \( W^g_\lambda (\lambda) \) is not a simple \( g \)-module.

Further, let us assume that \( \lambda_1 \in \mathbb{N} \). Then there exists a homomorphism

\[
\varphi_\lambda : M^g_\lambda (s_1 (\lambda + \rho) - \rho) \to M^g_\lambda (\lambda)
\]

of Verma modules. As the Verma module \( M^g_\lambda (\lambda) \) for \( \lambda \in \mathfrak{h}^* \) is realized as \( A^g_{\mathfrak{h}^*} / \mathcal{J}_\mathfrak{h} \simeq \mathbb{C}[\partial_x, \partial_y, \partial_z] \), where the \( g \)-module structure on \( A^g_{\mathfrak{h}^*} / \mathcal{J}_\mathfrak{h} \) is given through the homomorphism \( \pi_{\lambda+\rho} : g \to A^g_{\mathfrak{h}^*} \) of Lie algebras, we have

\[
\pi_\lambda (a) \varphi_\lambda (v) = \varphi_\lambda (\pi_{s_1 \lambda} (a)v)
\]

for all \( v \in \mathbb{C}[\partial_x, \partial_y, \partial_z] \) and \( a \in g \). Moreover, we have \( \varphi_\lambda = (\partial_x + \frac{1}{2} y \partial_z)^{\lambda_1} \), as follows immediately from [15, Theorem 2.1], which gives us \( \pi_\lambda (a) \circ \varphi_\lambda = \varphi_\lambda \circ \pi_{s_1 \lambda} (a) \) in the Weyl algebra \( A^g_{\mathfrak{h}^*} \) for all \( a \in g \). Therefore, we obtain a homomorphism

\[
\psi_\lambda : W^g_\lambda (s_1 (\lambda + \rho) - \rho) \to W^g_\lambda (\lambda)
\]

of Gelfand–Tsetlin modules, such that \( \psi_\lambda = \varphi_\lambda \) if we realize the Gelfand–Tsetlin module \( W^g_\lambda (\lambda) \) for \( \lambda \in \mathfrak{h}^* \) as \( A^g_{\mathfrak{h}^*} / \mathcal{J}_{GT} \simeq \mathbb{C}[\partial_x, \partial_y, \partial_z] \). As \( \psi_\lambda \) is not surjective, we obtain that \( W^g_\lambda (\lambda) \) is not a simple \( g \)-module. By a similar argument, we prove it in the case \( \lambda_2 \in \mathbb{N} \) or \( \lambda_1 + \lambda_2 \in \mathbb{N} \). If \( \lambda_2 \in \mathbb{N} \), then there exists a homomorphism

\[
\varphi_\lambda : M^g_\lambda (s_2 (\lambda + \rho) - \rho) \to M^g_\lambda (\lambda)
\]

of Verma modules with \( \varphi_\lambda = (\partial_y + \frac{1}{2} x \partial_z)^{\lambda_2} \). Finally, if \( \lambda_1 + \lambda_2 \in \mathbb{N} \), then there exists a homomorphism

\[
\varphi_\lambda : M^g_\lambda (s_1 s_2 s_1 (\lambda + \rho) - \rho) \to M^g_\lambda (\lambda)
\]

of Verma modules with \( \varphi_\lambda = \prod_{j=0}^{k-1} ((\partial_x + \frac{1}{2} y \partial_z)(\partial_y - \frac{1}{2} x \partial_z) - \frac{1}{2}(\lambda_1 - \lambda_2 - k + 2j)\partial_z) \), where \( k = -\lambda_1 - \lambda_2 \).

Therefore, the Gelfand–Tsetlin module \( W^g_\lambda (\lambda) \) is a simple \( g \)-module if and only if \( \lambda_1, \lambda_2 \notin \mathbb{N} \) and \( \lambda_1 + \lambda_2 \notin \mathbb{Z} \).

\[\square\]

### Appendix A  Generalized eigenspace decomposition

For the reader’s convenience, we summarize and prove several important facts concerning the generalized eigenspace decomposition.

Let \( V \) be a complex vector space and let \( T : V \to V \) be a linear mapping. For each \( \lambda \in \mathbb{C} \) we set

\[
V_\lambda = \{ v \in V; (\exists k \in \mathbb{N}) (T - \lambda \text{id})^k v = 0 \}
\]

(A.1)

and call it the generalized eigenspace of \( T \) with eigenvalue \( \lambda \). When \( V_\lambda \neq \{0\} \), we say that \( \lambda \) is an eigenvalue of \( T \) and the elements of \( V_\lambda \) are called generalized eigenvectors with eigenvalue \( \lambda \). Further, let us note that

\[
\sum_{\lambda \in \mathbb{C}} V_\lambda = \bigoplus_{\lambda \in \mathbb{C}} V_\lambda,
\]

(A.2)

however it does not hold, in general, that \( V = \bigoplus_{\lambda \in \mathbb{C}} V_\lambda \).

Let \( V = \bigoplus_{r \in \mathbb{N}_0} V_r \) be an \( \mathbb{N}_0 \)-graded complex vector space. Then we denote by \( p_r : V \to V_r \) for \( r \in \mathbb{N}_0 \) the canonical projection with respect to the direct sum decomposition. Furthermore, we define

\[
F_k V = \bigoplus_{r=0}^{k} V_r
\]

(A.3)
for all $k \in \mathbb{N}_0$, which gives us an increasing filtration $\{F_k V\}_{k \in \mathbb{N}_0}$ on $V$.

**Lemma A.1.** Let $V = \bigoplus_{r \in \mathbb{N}_0} V_r$ be an $\mathbb{N}_0$-graded complex vector space and let $\dim V_r < \infty$ for all $r \in \mathbb{N}_0$. Further, let $T : V \rightarrow V$ be a linear mapping satisfying

$$T(F_k V) \subset F_k V$$

(A.4)

for all $k \in \mathbb{N}_0$. Then we have a decomposition

$$V = \bigoplus_{\lambda \in \mathbb{C}} V_\lambda$$

(A.5)

called the generalized eigenspace decomposition of $V$ with respect to $T$.

**Proof.** We already know that $\bigoplus_{\lambda \in \mathbb{C}} V_\lambda \subset V$. Let us assume that $v \in V$. Then there exists $k \in \mathbb{N}_0$ such that $v \in F_k V$. Since $T(F_k V) \subset F_k V$ and $\dim F_k V < \infty$, we have the generalized eigenspace decomposition of $F_k V$ with respect to $T$. Therefore, we obtain a decomposition $v = \sum_{\lambda \in \mathbb{C}} v_\lambda$, were $v_\lambda \in V_\lambda \cap F_k V$, which gives us the required statement.

**Theorem A.2.** Let $V = \bigoplus_{r \in \mathbb{N}_0} V_r$ be an $\mathbb{N}_0$-graded complex vector space and let $\dim V_r < \infty$ for all $r \in \mathbb{N}_0$. Further, let $T_s, T_n : V \rightarrow V$ be linear mappings satisfying

$$T_s(V_r) \subset V_r \quad \text{and} \quad T_n(F_r V) \subset F_{r-1} V$$

(A.6)

for all $r \in \mathbb{N}_0$. Then there exists an isomorphism

$$\varphi : V \rightarrow V$$

(A.7)

of vector spaces such that

$$\varphi(V_s^r) = V_s^{r+n}$$

(A.8)

for all $\lambda \in \mathbb{C}$, where

$$V = \bigoplus_{\lambda \in \mathbb{C}} V_\lambda^s$$

and

$$V = \bigoplus_{\lambda \in \mathbb{C}} V_\lambda^{s+n}$$

(A.9)

called the generalized eigenspace decompositions of $V$ with respect to $T_s$ and $T_s + T_n$, respectively.

**Proof.** Since $T_s(F_k V) \subset F_k V$ and $(T_s + T_n)(F_k V) \subset F_k V$ for all $k \in \mathbb{N}_0$, as follows immediately from (A.6), we obtain by Lemma A.1 the generalized eigenspace decompositions

$$V = \bigoplus_{\lambda \in \mathbb{C}} V_\lambda^s$$

and

$$V = \bigoplus_{\lambda \in \mathbb{C}} V_\lambda^{s+n}$$

(A.9)

of $V$ with respect to $T_s$ and $T_s + T_n$, respectively.

Now, we construct a linear mapping $\varphi : V \rightarrow V$. Since we have the direct sum decomposition

$$V = \bigoplus_{r \in \mathbb{N}_0} \bigoplus_{\lambda \in \mathbb{C}} (V_r \cap V_\lambda^s),$$

it is enough to define a linear mapping $\varphi$ on $V_r \cap V_\lambda^s$, which we shall denote by $\varphi_{r,\lambda}$, for all $r \in \mathbb{N}_0$, $\lambda \in \mathbb{C}$ and then extend linearly to $V$. To define $\varphi_{r,\lambda}$, let $w \in V_r$ be a generalized eigenvector of $T_s$ with eigenvalue $\lambda \in \mathbb{C}$. As $w \in F_r V$ and $(T_s + T_n)(F_r V) \subset F_r V$, we get a decomposition

$$w = \sum_{\mu \in \mathbb{C}} w_\mu^{s+n},$$

were $w_\mu^{s+n} \in F_r V \cap V_\mu^{s+n}$ for all $\mu \in \mathbb{C}$. Applying the projection $p_r : V \rightarrow V_r$ on the previous equation, we obtain

$$w = \sum_{\mu \in \mathbb{C}} p_r(w_\mu^{s+n}),$$

where
since $p_r(w) = w$. As $w^{r+n}_\mu \in F_r V \cap V^{s+n}_\mu$, there exists $k \in \mathbb{N}$ such that $(T_s + T_n - \mu)^k w^{r+n}_\mu = 0$ and therefore we may write
\[ 0 = p_r((T_s + T_n - \mu)^k w^{r+n}_\mu) = p_r((T_s - \mu)^k w^{r+n}_\mu) = (T_s - \lambda)^k p_r(w^{r+n}_\mu), \]
where we used (A.6), which gives us that $p_r(w^{r+n}_\mu)$ is a generalized eigenvector of $T_s$ with eigenvalue $\mu$ for all $\mu \in \mathbb{C}$. Hence, the decomposition $w = \sum_{\mu \in \mathbb{C}} p_r(w^{r+n}_\mu)$ implies that $p_r(w^{r+n}_\mu) = 0$ for $\mu \neq \lambda$ and $p_r(w^{r+n}_\lambda) = w$. Therefore, we may set
\[ \varphi_{r,\lambda}(w) = w^{r+n}_\lambda \]
for all $w \in V_r \cap V^{s}_\lambda$. The previous construction ensures that $\varphi_{r,\lambda}: V_r \cap V^{s}_\lambda \to F_r V \cap V^{s+n}_\lambda$ is a linear mapping. Moreover, we have $\varphi(V^s_\lambda) \subset V^{r+n}_\lambda$ for all $\lambda \in \mathbb{C}$.

Further, since $\varphi(F_r V) \subset F_r V$ for all $r \in \mathbb{N}_0$, we obtain a linear mapping
\[ \text{gr}_r \varphi: F_r V/F_{r-1} V \to F_r V/F_{r-1} V \]
for all $r \in \mathbb{N}_0$. As $\text{gr}_r \varphi = \text{id}_{F_r V/F_{r-1} V}$ by construction, we get that $\varphi: V \to V$ is an isomorphism of vector spaces. \hfill \Box

**Acknowledgments**

V.F. is supported in part by CNPq (304467/2017-0) and by Fapesp (2014/09310-5); L.K. is supported by Capes (88887.137839/2017-00).

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