Multilevel Preconditioning of Discontinuous-Galerkin Spectral Element Methods
Part I: Geometrically Conforming Meshe

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Abstract This paper is concerned with the design, analysis and implementation of preconditioning concepts for spectral Discontinuous Galerkin discretizations of elliptic boundary value problems. While presently known techniques realize a growth of the condition numbers that is logarithmic in the polynomial degrees when all degrees are equal and quadratic otherwise, our main objective is to realize full robustness with respect to arbitrarily large locally varying polynomial degrees, i.e., under mild grading constraints condition numbers stay uniformly bounded with respect to the mesh size and variable degrees. The conceptual foundation of the envisaged preconditioners is the auxiliary space method. The main conceptual ingredients that will be shown in this framework to yield “optimal” preconditioners in the above sense are Legendre-Gauss-Lobatto grids in connection with certain associated anisotropic nested dyadic grids as well as specially adapted wavelet preconditioners for the resulting low order auxiliary problems. Moreover, the preconditioners have a modular form that facilitates somewhat simplified partial realizations. One of the components can, for instance, be conveniently combined with domain decomposition, at the expense though of a logarithmic growth of condition numbers. Our analysis is complemented by quantitative experimental studies of the main components.

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1 Introduction

Attractive features of Discontinuous Galerkin (DG) discretizations are on the one hand their versatility regarding a variety of different problem types, and on the other hand their flexibility regarding local mesh refinement and even locally variable polynomial order of the discretization. While initially the main focus has been on transport problems like hyperbolic conservation laws, an increased attention has recently been paid to diffusion problems, which naturally enter the picture in more complex applications like incompressible Navier-Stokes equations. Due to the possible occurrence of both singularities and regions of high regularity on the other hand, the use of variable and possibly

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arbitrarily high degrees is particularly attractive, see, e.g., \cite{32, 34}; see also \cite{15}. The central theme of this paper is to develop efficient solvers for the systems of equations resulting from \textit{Discontinuous Galerkin Spectral Element} (DG-SE) discretizations. By this we mean that nodal-based polynomials of \textit{arbitrarily high locally variable} polynomial degree on meshes with arbitrarily small mesh size are permitted. In this work we confine the discussion to \textit{geometrically conforming} meshes, i.e., the intersection of any two mesh elements is empty or a common facet so that hanging nodes are not permitted. 

To formulate our objectives in more specific terms and also to indicate some intrinsic obstructions we briefly recall the state of the art regarding preconditioners for the DG method applied to \textit{second order} elliptic boundary value problems.

The first group of results refers to \textit{uniformly bounded} polynomial degrees. The multigrid scheme proposed in \cite{23} gives rise to uniformly bounded condition numbers provided that (i) the underlying hierarchy of meshes is \textit{quasi-uniform} and (ii) the solution exhibits a certain (weak) regularity. This scheme has been extended in \cite{27} to locally refined meshes showing a similar performance without a theoretical underpinning though. Domain decomposition preconditioners investigated in \cite{1, 2} give rise to only logarithmically growing condition numbers when the mesh size decreases. A two-level scheme in the sense of the \textit{auxiliary space method} (see, e.g., \cite{7, 30, 36}) is proposed in \cite{20} and shown to exhibit mesh-independent convergence again on quasi-uniform geometrically conforming meshes with a \textit{fixed} uniformly equal polynomial degree. In the framework of the auxiliary space method, preconditioners providing uniformly bounded condition numbers for locally refined meshes have been developed in \cite{9, 10} under weak grading constraints and for variable but uniformly bounded polynomial degrees. In all those results the condition numbers \textit{depend on the bound} for the polynomial degrees.

The following second group of results \textit{concerns the quantitative dependence} of condition numbers on the polynomial degree aiming at the use of polynomial elements of \textit{arbitrarily high order}. These results draw primarily on domain decomposition concepts, see, e.g., \cite{33}. More precisely, two essentially distinct cases arise, namely

- (a) all polynomial degrees are equal,
- (b) the polynomial degrees may vary from element to element.

In the case (a) the condition numbers can be shown to exhibit only a logarithmic growth in the polynomial degree \(p\) (see \cite{33}). This may be perceived as quite satisfactory for practical purposes if one accepts unnecessarily large polynomial degrees even near singularities. In the case (b), however, when arbitrarily high polynomial degrees are to be used only in part of the domain, the best known bounds to us grow like \(p^2\) (see \cite{32}) which does call for improvements.

In summary, (1) none of the currently known preconditioners gives rise to uniformly bounded condition numbers \textit{independent of the polynomial degrees}, (2) a \textit{strong} growth of condition numbers may occur when non-uniform degree distributions are used, i.e., when the quotient of the largest and lowest degree is allowed to be unbounded.

To see why there is an essential difference between (1) and (2) it is instructive to consider first the extreme case of a \textit{spectral} trial space spanned by a high-order polynomial on a single quadrilateral element. All strategies for this case known to us can be interpreted as employing a low-order \textit{auxiliary} space to precondition the system for the high-order discretization. To distinguish the two types of spaces we sometimes refer to the original elements in the high-order finite element mesh, comprised in the extreme case under consideration of a single element, as \textit{macro-elements} while the grid inside each macro-element can be viewed as a \textit{subgrid}. In \cite{6}, at least for the Laplace operator, piecewise linear finite elements on a Cartesian equidistant subgrid in conjunction with wavelet bases for weighted spaces give rise to precondition numbers with logarithmic growth in the polynomial degree. An alternative, perhaps more versatile (with respect to problem specification) approach put forward in \cite{19} and theoretically supported by \cite{13, 31}, is to our knowledge the only way to obtain uniformly condition numbers. It uses a very special low-order space based on so called \textit{Legendre-Gauss-Lobatto} (LGL) grids associated with the high-order space. It is known that using the inverse
of the low-order discretization on the LGL subgrid as a preconditioner for the high-order trial space on the single element gives rise to uniformly bounded condition numbers \[14\].

When the finite element mesh is comprised of more than a single element it is shown in \[33\] how to still use this concept in conjunction with domain decomposition in a near optimal way, provided that the LGL subgrids on adjacent elements match at element interfaces, which means that all polynomial degrees are equal. The growth of the condition numbers can then be kept logarithmic while, however, such bounds no longer exist when the polynomial degrees vary locally in the above strong sense. In fact, apparently the non-nestedness of LGL grids is then the essential obstruction to contriving optimal preconditioners in the sense that:

(P1) the condition numbers remain uniformly bounded independent of the mesh size and locally variable arbitrarily high degrees;

(P2) the preconditioner can be applied at a computational cost that stays proportional to the problem size.

Here are a few indications why non-nestedness causes serious problems when the degrees vary locally. First, the jump terms at element interfaces corresponding to the high-order trial functions are not equivalent to those for the corresponding auxiliary low-order spaces which are conforming only inside each macro-element. In fact, in the high-order case a jump between two adjacent polynomial elements of formally different polynomial degrees may still vanish due to matching traces, while the jumps for corresponding low-order finite element traces would not vanish because the nodes of the LGL subgrids interlace at the macro-element interfaces. Second, the non-nestedness of LGL grids implies that, when the polynomial degrees on adjacent macro-elements disagree, the corresponding subgrids have only a trivial intersection at such an interface. Therefore, one fails to find sufficiently rich globally conforming auxiliary low-order spaces. As a consequence, one faces serious difficulties in verifying the relevant auxiliary space conditions. Third, the non-nestedness of LGL grids not only affects (P1) but also the application complexity (P2). In fact, when employing locally large polynomial degrees, the iterative solution of an auxiliary problem, even on only a single macro-element, becomes problematic because one cannot resort to efficient multilevel techniques and hence (P2) is not clear.

In summary, it seems that the currently known concepts are not sufficient to provide optimal preconditioners for DG-SE discretizations. The central objective of this paper is to construct optimal preconditioners for DG-SE discretizations in the sense of (P1), (P2). Specifically, to overcome the obstructions outlined above, we introduce the following new conceptual ingredients. The first one is the construction of certain dyadic grids that are associated in a strong sense with the LGL grids but are in addition nested. This association manifests itself through a number of stability estimates based on suitable comparison criteria for different grids. These findings allow us eventually to ensure (P1). Concerning (P2), the auxiliary grid hierarchies allow one, in principle, to employ multilevel techniques to solve the resulting low order problems. However, the strong anisotropies of the dyadic meshes, which are inherited from the LGL grids, appear to prevent standard techniques like BPX-preconditioners from working well. Therefore, as a second ingredient, we propose a specially tailored wavelet preconditioner using suitable piecewise polynomial \(L^2\)-orthogonal multi-wavelets.

As mentioned above, we choose the auxiliary space method as a conceptual platform for putting these tools to work, see \[30, 35, 36\]. The key issue is to construct a conforming auxiliary space comprised of globally continuous piecewise multi-linear functions on dyadic subgrids of the original macro mesh. It turns out though that the identification of suitable ingredients and their analysis is facilitated best by realizing the “final” auxiliary space through several stages. That is the auxiliary space method is iterated. This offers in our opinion at least two major benefits. First, it turns out that the identification of a “proper smoother” is not obvious in the “one-step-mode” while it is naturally obtained as a result of concatenating the intermediate stages. Second, the intermediate stages can be used as “stand-alone” results in several contexts. For instance, in the first stage we use a high-order conforming subspace as an auxiliary space (to get rid of the above mentioned “jump-problem” arising from non-matching LGL subgrids). The corresponding result can be used directly as an essential
tool for a domain decomposition preconditioner for the DG-SE discretization, see [16], accepting a logarithmic growth of condition numbers, but now for arbitrary locally variable polynomial degrees. Also, associated dyadic grid concepts as well as the associated wavelet preconditioner can be used for high-order conforming Galerkin discretizations or for a pure spectral discretization on a single element offering a performance that does not seem to be available yet for these scenarios either.

The paper is organized as follows. In Section 2 we formulate a simple model problem and describe the main ingredients of the DG-SE discretization. In Section 3, following essentially [30], we briefly recall the auxiliary space concept in a way that is most conveniently applied in the present context. In Section 4 we explain first, as indicated above, why we essentially split the construction of a suitable auxiliary space into several stages, and formulate a result for the first “intermediate” stage, see Theorem 4.1. Section 5 is devoted to the second and main stage of constructing globally conforming low order finite element spaces on certain strongly associated dyadic grids. As stated in Theorem 5.6 even uniformly bounded condition numbers (avoiding the logarithmic growth of the domain decomposition approach) can be obtained, once a corresponding preconditioner for the conforming low-order problem is available. Due to the fact that the partitions for the low-order auxiliary spaces involve highly anisotropic cells, this is not completely obvious and standard BPX-type techniques do not work well enough. In Section 5.4 we develop, so to speak as a third stage, a change-of-bases preconditioner based on specially tailored multi-wavelets that does give rise to uniformly bounded condition numbers, see Theorem 5.8. In Section 6 we present the resulting “composite preconditioner”, see Theorem 6.1. Each stage is concluded by some numerical experiments quantifying its performance.

Sections 7 and 8 are then devoted to the proofs of Theorems 4.1 and 5.6. These are necessarily rather technical but we have tried to organize them in a way that brings out the main mechanisms. From a bird’s view one could say that the robust treatment of varying polynomial degrees hinges on two main ingredients, namely first the fact that certain interpolation operators provide uniformly stable $L^2$- and $H^1$-isomorphisms between high-order polynomial spaces and low-order finite element spaces on LGL partitions, and second, a proper notion of uniformly equivalent grids that allows one to deal with different polynomial degrees and to switch to the associated dyadic grids which, in turn, allows one to take advantage of nestedness.

Throughout the paper we shall employ the following notational convention. By $a \lesssim b$ we mean that the quantity $a$ can be bounded by a constant multiple of $b$ uniformly in the parameters $a$ and $b$ may depend on. Likewise $a \simeq b$ means $a \lesssim b$ and $b \lesssim a$. For two vectors $\mathbf{p}, \mathbf{q} \in \mathbb{R}^n$ an inequality $\mathbf{q} \leq \mathbf{p}$ is to be understood componentwise, i.e., $q_k \leq p_k$ for $1 \leq k \leq n$.

## 2 Model problem and discretizations

The methods developed in the sequel apply to second order symmetric elliptic boundary value problems with variable coefficients. To keep the technical level of the exposition as low as possible we confine the discussion to the model problem

$$- \Delta u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega,$$

(2.1)

where $\Omega \subset \mathbb{R}^d$, is a bounded Lipschitz domain with piecewise smooth boundary and $f \in L_2(\Omega)$. Here one should think of $d \in \{1, 2, 3\}$. Such domains can be partitioned into images of (hyper-)rectangles through smooth mappings (such as iso/sub-parametric mappings, or Gordon-Hall transforms). Again, for the sake of technical simplicity, it suffices to treat unions of closed (hyper-)rectangles, i.e., we assume that the closure of $\Omega$ can be partitioned into the union of an essentially disjoint finite collection $\mathcal{R}$ of closed (hyper-)rectangles $R$. These (hyper-)rectangles are often simply referred to as elements. Moreover, we confine the discussion in this paper to geometrically conforming partitions $\mathcal{R}$, which means that any nonempty intersection between two elements $R$ and $R'$ in
\( \mathcal{R} \) is an \( l \)-facet \( F \) for both of them, for some \( 0 \leq l \leq d-1 \). It will be convenient to introduce the complex

\[
\mathcal{F}_l := \bigcup_{R \in \mathcal{R}} \mathcal{F}_l(R)
\]

of all \( l \)-dimensional closed facets associated with the macro-mesh consisting of the elements \( R \in \mathcal{R} \). We call a \((d-1)\)-dimensional facet \( F \in \mathcal{F}_{d-1}(R) \) a face or an interface. Moreover, for each \( F \in \mathcal{F}_l \), we define \( \mathcal{R}(F) := \{ R' \in \mathcal{R} : R' \cap F \neq \emptyset \} \).

To describe the eligible discretizations, \( \mathbf{H} = (H_1, \ldots, H_d) = \mathbf{H}(R) \in \mathbb{R}^d \) denotes the vector of the \( k \)-th side lengths of \( R \), \( k = 1, \ldots, d \). Likewise \( \mathbf{p} = (p_1, \ldots, p_d) = \mathbf{p}(R) \in \mathbb{N}^d \) denotes the vector of coordinate-wise polynomial degrees of an element of \( Q_p(R) := \bigotimes_{k=1}^d \mathbb{P}_{p_k}(I_k) \) It will be convenient to denote by \( \mathbf{H} := \sum_{R \in \mathcal{R}} \mathbf{H}(R) \chi_R, \mathbf{p} = \sum_{R \in \mathcal{R}} \mathbf{p}(R) \chi_R \) the corresponding (global) meshsize- and degree-functions and we use \( \delta := (\mathbf{H}, \mathbf{p}) \) as the corresponding discretization parameter.

The trial spaces for the Discontinuous Galerkin Spectral Element (DG-SE) method are then of the form

\[
V_\delta = \{ v \in L_2(\Omega) : v|_{R \in \mathcal{R}} = v \in Q_p(R), \forall R \in \mathcal{R} \},
\]

while \( V^c_\delta := V_\delta \cap H^1_0(\Omega) \) denotes the largest conforming subspace of \( V_\delta \) accommodating the prescribed boundary conditions. \( \mathbf{H}, \mathbf{p} \) will always be assumed to satisfy the grading conditions

\[
\max_k \frac{H_k(R)}{\min_k H_k(R)} \lesssim 1, \quad \max_k \frac{p_k(R)}{\min_k p_k(R)} \lesssim 1, \quad \max_k \frac{p_k(R^-)}{p_k(R^+)} \lesssim 1, \quad R \in \mathcal{R},
\]

uniformly in \( \mathcal{R} \), where \( R^+, R^- \) stand for two adjacent elements sharing an interface.

We impose one further assumption on \( \mathbf{p} \) which is needed only when \( d \geq 3 \). For \( 2 \leq l \leq d-1 \) and for each \( F \in \mathcal{F}_l \), there exists \( R \in \mathcal{R}(F) \) such that

\[
\mathbf{p}(F, R) \leq \mathbf{p}(F, R') \quad \forall R' \in \mathcal{R}(F),
\]

where \( \mathbf{p}(F, R) := \mathbf{p}(R) \mid_F \). Note that the property is trivially true for \( l = 1 \) since \( \mathbf{p}(F, R) \in \mathbb{N} \), and for \( l = d \) since \( \mathcal{R}(F) = \{ F \} \).

Denoting by \( \mathbf{n}_{R^+, F} = -\mathbf{n}_{R^-, F} \) the unit normal vectors on \( F = R^+ \cap R^- \) pointing to the exterior of \( R^\pm \), and for \( F \in \mathcal{F}_{d-1}(R) \) we define as usual the jumps \( [v]_F \) and averages \( \{v\}_F \) as follows: taking the homogeneous Dirichlet boundary condition into account, for \( F \subset \partial \Omega \) we set \( [v]|_F = \mathbf{n}_{R^-, F} v|_F - \mathbf{n}_{R^+, F} v|_F \) and \( \{v\}_F = \frac{1}{2} (v|_F - v|_F) \).

The Symmetric Interior-Penalty Discontinuous Galerkin Spectral-Element discretization of Problem (2.1) is defined as follows (see [3, 4]): find \( u \in V_\delta \) such that

\[
a_d(u, v) = (f, v)_{0, \Omega} \quad \forall v \in V_\delta,
\]

where the bilinear form \( a_d(\cdot, \cdot) : V_\delta \times V_\delta \to \mathbb{R} \) is given by

\[
a_d(u, v) = \sum_{R \in \mathcal{R}} (\nabla u, \nabla v)_{0,R} + \sum_{F \in \mathcal{F}_{d-1}} \left( -([\nabla u], [v])_{0,F} - ([\nabla v], [u])_{0,F} + \gamma \omega_F([u], [v])_{0,F} \right).
\]

As usual, we denote by \( \langle \cdot, \cdot \rangle_{0,G} \) the standard \( L_2 \)-inner product over the domain \( G \) and set \( \|v\|_{0,G} := (v, v)_{0,G}^{1/2} \). The weights \( \omega_F \) are defined as follows. When \( F \) is orthogonal to the \( k \)-th coordinate direction and \( \mathcal{R}(F) = \{ R^k \} \), we set

\[
\omega_F = \max \left( \frac{(p_k(R^-) + 1)^2}{H_k(R^-)}, \frac{(p_k(R^+) + 1)^2}{H_k(R^+)} \right).
\]
with the obvious modification when $F \subset \partial \Omega$. The definition is motivated by the inverse trace inequality $\|v\|_{0,F} \leq \frac{\min(R+1)}{\sqrt{\min(R)}} \|v\|_{0,R}$ for all $v \in Q_p(R)$, $R = R^k$, which allows one to prove the uniform coercivity and continuity of $a_\delta(\cdot, \cdot)$ provided the constant $\gamma > 0$ is properly chosen. Indeed, the following result holds (see, e.g., \cite{32}).

**Proposition 2.1.** There exists a constant $\gamma_0 > 0$ such that for all $\gamma > \gamma_0$ the bilinear form $a_\delta$, defined in (2.7), satisfies

$$a_\delta(v, v) \simeq \|v\|_{DG, \delta}^2 := \sum_{R \in \mathcal{R}} \|\nabla v\|_{0, R}^2 + \gamma \sum_{F \in \mathcal{F}_{\delta-1}} \omega_F \|v\|_{0, F}^2 \quad \forall v \in \mathcal{V}_\delta. \quad (2.9)$$

The constant $\gamma_0$ and the constants implied by the symbol $\simeq$ can be chosen independently of $\delta = (\mathbf{H}, \mathbf{p})$.

The central objective of this paper is to develop and analyze preconditioners for the linear systems arising from (2.6). Their concrete structure depends on the bases for the spaces $V_\delta$. Nodal basis functions for certain specific subgrids on the elements $R$ will be seen to have particularly favorable properties, among them uniformly equivalent but computationally more efficient quadrature formulations as discussed in [8] [12].

### 3 The auxiliary space method

The so called auxiliary space method (ASM) will serve as the conceptual platform for developing preconditioners for the linear system (2.6), (see, e.g., [7] [33] [35] [10]). Specifically, we adopt the abstract framework, see [30] [25] [29] because the specific conditions formulated there are best suited for the present application. In particular, the envisaged auxiliary spaces are not contained in the DG trial spaces $V_\delta$. For convenience of the reader we briefly recall the relevant ingredients in appropriate generality.

For a given finite-dimensional Hilbert space $V$ and a symmetric positive-definite bilinear form $a : V \times V \to \mathbb{R}$ we seek an auxiliary finite-dimensional Hilbert space $\tilde{V}$, endowed with a symmetric positive-definite bilinear form $\tilde{a} : \tilde{V} \times \tilde{V} \to \mathbb{R}$ so that the respective variational problems are spectrally equivalent. To ensure this when $\tilde{V} \nsubseteq V$ we consider the sum $\tilde{V} = V + \tilde{V}$ and two further symmetric positive-definite bilinear forms $\tilde{a}, b : \tilde{V} \times \tilde{V} \to \mathbb{R}$ which satisfy the following conditions:

**ASM1:** $\tilde{a}$ is a spectrally equivalent extension of both $a$ and $\tilde{a}$, i.e.,

$$a(v, v) \simeq \tilde{a}(v, v), \quad \forall v \in V, \quad \tilde{a}(\tilde{v}, \tilde{v}) \simeq \tilde{a}(\tilde{v}, \tilde{v}) \quad \forall \tilde{v} \in \tilde{V}. \quad (3.1)$$

**ASM2:** $b$ dominates $a$ on $V$, i.e., $a(v, v) \lesssim b(v, v)$, for all $v \in V$;

**ASM3:** there exist linear operators $Q : \tilde{V} \to V$ and $\tilde{Q} : V \to \tilde{V}$ such that

$$\tilde{a}(\tilde{Q}v, \tilde{Q}v) \lesssim a(v, v) \quad \forall v \in V, \quad a(\tilde{Q}v, \tilde{Q}v) \lesssim \tilde{a}(\tilde{v}, \tilde{v}) \quad \forall \tilde{v} \in \tilde{V}, \quad (3.2)$$

and

$$b(v - \tilde{Q}v, v - \tilde{Q}v) \lesssim a(v, v) \quad \forall v \in V, \quad b(\tilde{v} - Q\tilde{v}, \tilde{v} - Q\tilde{v}) \lesssim \tilde{a}(\tilde{v}, \tilde{v}) \quad \forall \tilde{v} \in \tilde{V}. \quad (3.3)$$

These conditions imply the following stable splitting.

**Proposition 3.1** (\cite{30}, Theorem 1 (2)). *The conditions ASM1-3 imply the following norm equivalence in $V$:

$$\hat{c} a(v, v) \leq \inf_{w \in V, \tilde{v} \in \tilde{V}} \{ b(w, w) + \tilde{a}(\tilde{v}, \tilde{v}) \} \leq \hat{C} a(v, v) \quad \forall v \in V, \quad (3.4)$$

where the constants $\hat{c}$ and $\hat{C}$ depend only on the constants implied by the assumptions (see \cite{3} for explicit expressions).
Proposition 3.4 has the following main consequence. Let $A$, $\hat{A}$, and $B$ denote the Gramian matrices for the bilinear forms $a$, $\hat{a}$ and $b$ (restricted to $V \times V$) with respect to suitable bases of the spaces $V$ and $\hat{V}$. Let $S$ be the matrix representation of $Q$ with respect to these bases.

**Corollary 3.2** (see [30], Theorem 2). Let $C_B$ and $C_{\hat{A}}$ be symmetric preconditioners for $B$ and $\hat{A}$, respectively, satisfying the following spectral bounds:

$$0 < \lambda_{\min} \leq \min\{\lambda_{\min}(C_{B}B), \lambda_{\min}(C_{\hat{A}}\hat{A})\} \leq \max\{\lambda_{\max}(C_{B}B), \lambda_{\max}(C_{\hat{A}}\hat{A})\} \leq \Lambda_{\max}.$$ 

Moreover, let $\hat{c}, \hat{C} > 0$ be the constants in the norm equivalence (3.3). Then, under the assumptions ASM1-3, $C_A = C_B + SC_{\hat{A}}S^T$ is a symmetric preconditioner for $A$, and

$$\kappa(C_A A) \leq \frac{\Lambda_{\max}}{\Lambda_{\min}} \frac{\hat{C}}{\hat{c}}.$$ 

Note that only the operator $Q$ enters the actual construction of the preconditioner while $\hat{Q}$ is only needed for its analysis. We proceed recalling the following convenient simplifications.

**Proposition 3.3.** If ASM2 is replaced by the stronger condition: ASM2': $b$ dominates $\hat{a}$ on $\hat{V}$, i.e., $\hat{a}(\hat{v}, \hat{v}) \leq b(\hat{v}, \hat{v})$, for all $\hat{v} \in \hat{V}$, then one can skip checking the inequalities (3.2) in ASM3.

The statement easily follows from the first part of the proof of Theorem 1 in [30].

**Proposition 3.4.** When $\hat{V} \subset V$, then $\hat{V} = V$ and one can take $\hat{a} = \hat{a} = a$, so that an obvious choice for $Q$ is the canonical injection. In this case, the only conditions that need be verified for Proposition 3.3 are ASM2 together with the existence of a linear operator $\hat{Q}: V \to \hat{V}$ such that

$$b(v - \hat{Q}v, v - \hat{Q}v) \lesssim a(v, v) \quad \forall v \in V.$$ 

The central remaining objective is now to identify for $V = V_0$ a suitable *conforming* auxiliary space $\hat{V} \subset H_0^1(\Omega)$ along with the operators $Q, \hat{Q}$. Since one cannot expect $\hat{V} \subset V_0$ the operator $Q$ will be non-trivial.

### 4 Reduction to a conforming problem

A common strategy for treating nonconforming discretizations is to look for a suitable conforming subspace in order to treat the corresponding auxiliary problem with the aid of efficient multilevel techniques while the remaining high-frequency, non-conforming part is efficiently treated by smoothing realized by relaxation sweeps. Eventually, we shall be looking for a low-order conforming finite element space as auxiliary space as well. However, as will be explained later in more detail, the principal tool that suggests itself, namely introducing for each high-order element a so called Legendre-Gauss-Lobatto (LGL) (sub-)grid, on which piecewise multilinear finite elements could be defined, does not comply with global conformity as soon as the polynomial degrees $\mathbf{p}(R)$ vary from element to element.

We will therefore split the construction of a DG-preconditioner in two main steps. The first one is to view the largest conforming subspace $V_0^p \subset V_0 \cap H_0^1(\Omega)$ as an auxiliary space. This comes with the additional benefit that such a step can be combined with domain decomposition techniques to obtain a DG-preconditioner with only mildly growing condition numbers [16], but now retaining this performance for varying polynomial degrees.

As indicated above, the first main conceptual tool revolves around LGL quadrature nodes $a = \xi_0 < \cdots < \xi_{j-1} < \xi_j < \cdots < \xi_p = b$ of order $p$ in the interval $I = [a, b]$ of positive length $H = b - a$. We refer to the collection of these nodes as the LGL grid $\mathcal{G}_p(I)$. It is well-known that there exist positive weights $w_0, \ldots, w_p \in \mathbb{R}$ such that

$$\sum_{j=0}^p w_j \xi_j = \int_I v(x) dx \quad \forall v \in \mathbb{P}_{2p-1}(I).$$

(4.1)
Obviously, any \( v \in \mathbb{P}_p(I) \) is uniquely determined by its values on \( \mathcal{G}_p(I) \). We recall that nodes and weights are classically defined on the reference interval \( I = [-1, 1] \), as \( \xi_j, \tilde{w}_j \), respectively. By affine transformation, one has \( \xi_j = a + H(\tilde{\xi}_j + 1)/2 \) and \( w_j = (H/2)\tilde{w}_j \). We also recall that weights with index \( j \) close to 0 or \( p \) satisfy \( w_j \approx H^{-2} \) (in particular, \( w_0 = w_p = H(p(p+1))^{-1} \)), whereas weights with index \( j \) close to \( p/2 \) satisfy \( w_j \approx Hp^{-1} \). Yet, the variation in the order of magnitude is smooth, as made precise by the estimates

\[
w_{j-1} \approx w_j \quad \text{for} \quad 1 \leq j \leq p,
\]

which hold uniformly in \( p \) and \( H \) (see, e.g., [13] [31]).

Moreover, given any element \( R = \times_{k=1}^d I_k \) of \( \mathcal{R} \) and denoting by \( \xi_{k,j_k} \) the corresponding LGL nodes in \( I_k \) the product set

\[
\mathcal{G}_p(R) = \bigtimes_{k=1}^d \mathcal{G}_{p_k}(I_k) = \{ \xi = (\xi_{1,j_1}, \xi_{2,j_2}, \ldots, \xi_{d,j_d}) \mid 0 \leq j_k \leq p_k, \ 1 \leq k \leq d \} \quad (4.3)
\]

may be viewed as a sub-grid for the element \( R \) and will be referred to as (tensorial) LGL grid on \( R \), obviously forming a unisolvent set for \( \mathbb{Q}_{p}(R) \). Such grids have been intensely used and studied in the context of spectral methods (see, e.g., [5] [14] [26]) and more detailed properties will be given later.

Regarding the ASM framework, setting \( \hat{V} = V_\delta^c \subset V_\delta = V \), we can take \( \hat{a}(\cdot, \cdot) = a_\delta(\cdot, \cdot) \) and \( \hat{a}(\cdot, \cdot) := a_\delta(\cdot, \cdot) |_{V_\delta^c \times V_\delta^c} \). Moreover, since \( \hat{V} \subset V \), the operator \( Q : \hat{V} \to V \) is the canonical injection. Thus, defining a preconditioner based on \( V_\delta^c \) as an auxiliary space it remains to specify the auxiliary bilinear form \( b(\cdot, \cdot) \). The following definition is inspired by a refined version of the classical inverse inequality \( \|v''\|_{0,I} \lesssim p^2 H^{-1} \|v\|_{0,I} \) for all \( v \in \mathbb{P}_p(I) \), and reads in terms of LGL nodes and weights:

\[
\|v''\|_{0,I} \lesssim \left( \sum_{j=0}^p v^2(\xi_j) w_j^{-1} \right)^{1/2} \quad \forall v \in \mathbb{P}_p(I) \quad (4.4)
\]

(see Section [8.1] for a proof). Introducing the product weights \( w_\xi = w_{1,j_1} w_{2,j_2} \cdots w_{d,j_d} \) associated to each node \( \xi \in \mathcal{G}_p(R) \), and denoting by \( w_{\xi,k} \) the factor \( w_{j_k} \) coming from the \( k \)-th direction, we introduce the weights

\[
W_\xi = \left( \sum_{k=1}^d w_{\xi,k}^{-2} \right)^{1/2} w_\xi = \sum_{k=1}^d w_{\xi,k}^{-1} \left( \prod_{j \neq k} w_{\xi,j} \right) . \quad (4.5)
\]

The bilinear form \( b_1(\cdot, \cdot) : V_\delta \times V_\delta \to \mathbb{R} \) is defined as

\[
b_1(u, v) = \sum_{R \in \mathcal{R}} b_R(u, v) , \quad \text{with} \quad b_R(u, v) := \sum_{\xi \in \mathcal{G}_p(R)} u(\xi) v(\xi) c_\xi W_\xi . \quad (4.6)
\]

Here the strictly positive coefficients \( c_\xi \approx 1 \) (meaning that they are bounded from above and from below independently of \( \xi, p \) and \( H \)) will be chosen in applications so as to enhance the effectivity of the ASM preconditioner, see Section [4.1] for the details.

Note that \( b_1(\cdot, \cdot) \) is defined strictly element-wise, i.e., it does not involve any coupling between different elements \( R \); in particular, for the Lagrange basis \( \{ \phi_{R,\xi} : \xi \in \mathcal{G}_p(R), R \in \mathcal{R} \} \) associated with the LGL (sub-)grids of the elements, the matrix \( B_1 := (b_1(\phi_{R,\xi}, \phi_{R',\xi'}))_{(R,\xi),(R',\xi')} \) is diagonal.

Since \( \hat{V} = V_\delta^c \subset V_\delta \), the operator \( Q \) is just the canonical injection whose matrix representation \( S_1 \) is, however, not the identity matrix. The main result of this section, whose proof is deferred to Section [7] reads as follows.
Figure 1: Grid and distribution of polynomial degrees for the test scenarios. Darker shading indicates higher polynomial degree in the patch.

Theorem 4.1. Let $\tilde{A}_1$ be the stiffness matrix for the conforming problem: find $u_{\delta} \in V_{\delta}^c$ such that

$$a(u_{\delta}, v) = (f, v)_{0, \Omega}, \quad v \in V_{\delta}^c,$$

(4.7)

where $a(v, w) = \sum_{R \in \mathcal{R}} (\nabla v, \nabla w)_{0, R}$ agrees with $a_{\delta}(\cdot, \cdot)$ on $V_{\delta}^c \times V_{\delta}^c$, and assume that $C_{\tilde{A}_1}$ is a symmetric preconditioner for $\tilde{A}_1$. Then, denoting by $A$ the stiffness matrix for (2.1), there exists a constant $C_1$ such that $\tilde{C}_A := B_1^{-1} + S_1 C_{\tilde{A}_1} S_1^T$ is a symmetric preconditioner for (2.1) satisfying

$$\kappa(\tilde{C}_A A) \leq C_1 \kappa(C_{\tilde{A}_1} \tilde{A}_1),$$

(4.8)

uniformly in $\delta = (\mathbf{H}, p)$ with $C_1$ depending only on the grading conditions (2.4).

LGL quadrature is not only essential for the analysis of the proposed schemes but allows one to formulate “equivalent discrete DG-bilinear forms” which enhance computational efficiency, see the discussion in [8].

4.1 Numerical experiments

We demonstrate next the performance of the preconditioner $\tilde{C}_A$ from Theorem 4.1 when $C_{\tilde{A}_1} = \tilde{A}_1^{-1}$, that is, the auxiliary problem is solved exactly. For more extensive tests we refer to [8] [12]. We consider two test scenarios as shown in Fig. 1, exhibiting a checkerboard distribution of polynomial degrees and a more monotonic polynomial degree distribution which one would expect for a mesh refinement. Concerning the values of $p, q \in \mathbb{N}$, we consider the following five cases in the first scenario: (i) we either use a constant polynomial degree $q = p$, (ii) we simulate a small variation of the polynomial degree by choosing $q = p + 2$, or a large variation of the polynomial degree represented by multiplicative relations (iii) $q = 3/2p$ for even $p$, (iv) $q = 7/4p$ for $p$ chosen as a multiple of 4 or (v) $q = 2p$.

The main issue at this stage is to calibrate the tuning parameters in the bilinear form $b_1(\cdot, \cdot)$. From (4.6)

$$b_0(u, v) = \beta_1 (c_1^2 \sum_{R \in \mathcal{R}} \sum_{\xi \in \mathcal{G}_p(R)} u(\xi) v(\xi) W_\xi + \gamma \rho_1 \sum_{F \in \mathcal{F}_{d-1}} \omega_F \sum_{\xi \in \mathcal{G}_p(F, R^\pm)} w_{F, R^\pm} u^\pm(\xi) v^\pm(\xi),$$
where $w_{F,R;\xi}$ is the LGL quadrature weight on $F$ seen as a face of $R^{\pm}$. Consequently, a reasonable ansatz for the constants $c_\xi$ in (4.5) is

$$c_\xi = \begin{cases} 
\beta_1(c_1^2 + \gamma \rho_1 \omega^w_{F,R}/W_\xi), & \text{for } \xi \in \mathcal{G}_p(F,R), \ F \in \mathcal{F}_{d-1}(R), \ R \in \mathcal{R}, \\
\beta_1 c_1^2, & \text{else.}
\end{cases}$$

The diagonal structure of $B_1$ is, of course, not affected by the choice of these parameters. Preliminary experiments concerning the constant arising in the inverse estimate (4.4) reveal that $c_1^2 = 10$ is a good choice which we fix in our subsequent tests.

The condition numbers $\kappa(\mathbf{C}_\Lambda \mathbf{A})$ are depicted as contour plots for $p = 8$ and for $p = 16$ in Fig. 2 as functions of $\beta_1 \in [0.05, 1.1]$ and $\rho_1 \in [0, 2]$. We observe that for both values of $p$ there is a very flat minimum of the condition number that is located near the parameter point $(\beta_1, \rho_1) = (0.15, 1.25)$. From now on we fix these parameter values for the rest of the paper.

Fig. 3(a) shows the condition numbers obtained in the first scenario for the relations (i)-(v) between $p$ and $q$. We observe that the condition numbers stay uniformly bounded as $p$ increases although the upper bound depends on the ratio $q/p$. In the case of additively increasing the polynomial degree $q = p + 2$, the quotient $q/p$ decreases, which is also visible in Fig. 3(a). The analogous plot for the second test scenario, representing a typical $p$-adaptation, is depicted in Fig. 3(b). In this case the condition numbers are almost constant and slightly smaller than 7.5.

## 5 The conforming problem

In view of Theorem 4.1 it remains to develop a preconditioner for the related conforming problem (4.7) over the high-order conforming subspace $V^d_\delta$. We emphasize that the results of this section are of interest in their own right because the proposed preconditioner for the conforming problem offers a solver performance that does not seem to be available yet so far.

In what follows one should keep in mind that the degree function $p$ defining $V^d_\delta$, determines for each $R \in \mathcal{R}$ a unique LGL (sub-)grid $\mathcal{G}_p(R)$ along with the corresponding (micro-)partition $\mathcal{T}_p(R) = \mathcal{T}(\mathcal{G}_p(R))$ of $R$ into (hyper-)rectangles $S = S_\ell = \times_{k=1}^d I_{k,\ell_k}$ for $\ell \in \times_{k=1}^d \{1, \ldots, p_k\}$. 

![Figure 2: Contour plots illustrating the dependence of the condition number $\kappa(\mathbf{C}_\Lambda \mathbf{A})$ on $\beta_1 > 0$ and $\rho_1 \geq 0$.](image)
where each $I_{k,t_k}$ is an interval in the univariate partition $\mathcal{T}_{p_k}(I_k)$ of $I_k$ (see Fig. 4(a)). With $G_p(R)$ we associate next the local piecewise multi-linear finite element space on $\mathcal{T}_p(R)$ given by

$$V_{h,p}(R) = \{ v \in C^0(R) : v |_S \in Q_1 \ \forall S \in \mathcal{T}_p(R) \} = \otimes_{k=1}^{d} V_{h,p_k}(I_k).$$  \hspace{1cm} (5.1)

To state now a property that is pivotal for our purposes, we introduce the following tensor product interpolation operators

$$I_p = I_p^R : C^0(R) \rightarrow Q_p(R) , \quad I_p^R = \otimes_{k=1}^{d} I_{p_k}^l,$$  \hspace{1cm} (5.2)

and

$$I_{h,p} = I_{h,p}^R : \mathbb{Q}_p(R) \rightarrow V_{h,p}(R) , \quad I_{h,p}^R = \otimes_{k=1}^{d} I_{h,p_k}^l.$$  \hspace{1cm} (5.3)

where $I_{p_k}, I_{h,p_k}$ are the corresponding univariate operators, satisfying $I_{p_k}^l v \in \mathbb{P}_{p_k}(I_k), I_{h,p_k}^l v \in V_{h,p_k}(I_k)$ and

$$(I_{p_k}^l v)(\xi_{k,j}) = v(\xi_{k,j}), \quad (I_{h,p_k}^l v)(\xi_{k,j}) = v(\xi_{k,j}), \quad 0 \leq j \leq p_k.$$

The crucial fact is that the operators $I_{h,p}^R$ induce uniformly bounded topological isomorphisms between $\mathbb{Q}_p(R)$ and $V_{h,p}(R)$ with respect to both the $L_2$ and $H^1$ norms, whose inverse is $I_{p}^R$.

**Property 5.1.** \hspace{1cm} \textbf{[I3]} For any $v \in \mathbb{Q}_p(R)$, set $v_h := I_{h,p}^R v$. Then,

$$\|v\|_{0,R} \simeq \|v_h\|_{0,R} \quad \text{and} \quad \|
abla v\|_{0,R} \simeq \|
abla v_h\|_{0,R}.$$  \hspace{1cm} (5.4)

The constants in both relations are independent of $p$ and $H$. \hspace{1cm} \textbf{D}

The relations \hspace{1cm} (5.4) \hspace{1cm} follow from the analogous univariate relations (see Property \hspace{1cm} 8.6 \hspace{1cm} combined with the following frequently used fact.
Proposition 5.2. For $1 \leq k \leq d$, let $V_k, W_k \subset H^1(I_k)$ be finite dimensional subspaces. Let $L_k : V_k \rightarrow W_k$ be linear operators satisfying, for $m = 0, 1$,

$$\|L_k v\|_{m, I_k} \lesssim \|v\|_{m, I_k} \quad \forall v \in V_k. \quad (5.5)$$

Then, setting $V := \bigotimes_{k=1}^d V_k$ and $W := \bigotimes_{k=1}^d W_k$, the operator $L = \bigotimes_{k=1}^d L_k : V \rightarrow W$ satisfies, for $m = 0, 1$,

$$\|Lv\|_{m, R} \lesssim \|v\|_{m, R} \quad \forall v \in V. \quad (5.6)$$

Moreover, the same relations hold for the corresponding seminorms.

Property 5.1 plays a pivotal role in the analysis of the preconditioners, see in particular Section 8.1. Its significance is indicated by the fact that $V_{h,p}(R)$ is an ideal auxiliary space for $Q_p(R)$ when $R$ consists of a single element. However, even ignoring the question of efficiently solving the obtained low-order problem, when dealing with the general case of several elements the corresponding LGL (sub-)grids do not match at the element interfaces when the degrees $p_i$ vary from element to element as shown by Fig. 4(a). Globally conforming piecewise multilinear finite elements on the corresponding composite partitions would therefore not form suitable auxiliary spaces since the traces at the element interfaces would have to be at best linear on the whole interface.

The next major conceptual tool is therefore the construction of suitable composite grids which, on the one hand are still sufficiently close to the composite LGL grids but, on the other hand, give rise to sufficiently rich globally conforming nested low-order finite element spaces, to be used as auxiliary spaces.

5.1 Dyadic meshes

As before we refer to an ordered collection $\mathcal{G} = \{\xi_j : 0 \leq j \leq p\} \subset I = [a, b]$ with $a = \xi_0 < \cdots < \xi_{j-1} < \xi_j < \cdots < \xi_p = b$, as a grid in $I$ and continue to denote by $\mathcal{T} = \mathcal{T}(\mathcal{G})$ the induced partition comprised of the intervals $I_j = [\xi_{j-1}, \xi_j]$. Such a grid is called locally $C_g$-quasiuniform if there exists a constant $C_g > 1$ such that

$$C_g^{-1} \leq \frac{|I_{j+1}|}{|I_j|} \leq C_g, \quad 1 \leq j \leq p - 1. \quad (5.7)$$

LGL grids are locally $C_g$-quasiuniform, see [12] for estimates of $C_g$. 

Figure 4: LGL and dyadic sub-grids in a patch of elements in $\mathbb{R}^2$
Moreover, to be able to “compare” different grids we call a grid $\mathcal{G}$ in $I$ locally $(A,B)$-uniformly equivalent to another ordered grid $\tilde{\mathcal{G}}$ if there exist constants $0 < A < B$ such that

$$\forall I_j \in \mathcal{T}(\mathcal{G}), \ \forall \tilde{I}_i \in \mathcal{T}(\tilde{\mathcal{G}}), \ \ I_j \cap \tilde{I}_i \neq \emptyset \implies A \leq \frac{|I_j|}{|\tilde{I}_i|} \leq B.$$ (5.8)

The central issue of this section is to construct for a given grid $\mathcal{G}$ in $I$ an equivalent (in the sense of (5.8)) dyadic grid $\mathcal{D}$. To this end, we first introduce an intermediate construction.

Given a real $\alpha > 0$ and an initial dyadic partition $\mathcal{D}_0$, a dyadic partition $\mathcal{D}$ (identified for brevity with the associated grid) can be constructed iteratively as follows:

**Dyadic** $[\mathcal{G}, \mathcal{D}_0, \alpha] \to \mathcal{D}$:

(i) Set $\mathcal{D} := \mathcal{D}_0$.

(ii) While there exists $D \in \mathcal{D}$ such that

$$|D| > \alpha |\mathcal{T}(\mathcal{D}, \mathcal{G})|,$$ (5.9)

where $\mathcal{T}(\mathcal{D}, \mathcal{G}) := \arg\max \{|I_j| : I_j \in \mathcal{T}(\mathcal{G}), D \cap I_j \neq \emptyset\}$, then split $D$ by halving it and replace it by its two children $D', D''$, $D = D' \cup D''$, i.e.

$$(\mathcal{D} \setminus \{D\}) \cup \{D', D''\} \to \mathcal{D}.$$  

Such a dyadic mesh generator, applied to the sequence of LGL grids $\mathcal{G}_p = \mathcal{G}_p(I)$, produces grids $\mathcal{D}_p := \text{Dyadic} [\mathcal{G}_p, \{I\}, \alpha]$ which enjoy a number of useful structural properties such as always containing the midpoint of the interval and being symmetric around the midpoint. However, the sequence of these grids still fails to be nested although exceptions seem to be very rare (see [8]).

Since nestedness is essential we remedy the deficiency of $\text{Dyadic} [\mathcal{G}_p, \{I\}, \alpha]$ by employing the following recursive definition:

**NestedDyadic** $[\mathcal{G}_p, \{I\}, \alpha] \to \mathcal{D}_p$:

(i) Given $\alpha > 0$, set $\mathcal{D}_1 := \text{Dyadic} [\mathcal{G}_1, \{I\}, \alpha]$.

(ii) For $p > 1$, given $\mathcal{D}_{p-1}$, set $\mathcal{D}_p := \text{Dyadic} [\mathcal{G}_p, \mathcal{D}_{p-1}, \alpha]$.

The following result from [12] will be essential for the realization of ASM-conditions.

**Theorem 5.3.** For all $p \geq 1$, the dyadic meshes $\mathcal{D}_p$ are nested, locally quasi-uniform, and locally $(A,B)$-uniformly equivalent to $\mathcal{G}_p$ with constants $A,B$ specified as follows:

$$\forall D \in \mathcal{D}_p, \ \forall I_j \in \mathcal{T}(\mathcal{G}_p), \ \ I_j \cap D \neq \emptyset \implies A := \alpha^{-1} \leq \frac{|I_j|}{|D|} \leq \frac{2C_p}{\min\{\alpha C_p, 1\}} =: B.$$ (5.10)

Furthermore,

$$\text{card} \mathcal{D}_p \simeq \text{card} \mathcal{G}_p.$$ (5.11)

In addition, if $1 \leq \alpha \leq 1.25$, for all values of $p$ such that the sequence of LGL interval lengths is such that the quotients $\frac{|I_{j+1}|}{|I_j|}$ are monotonically decreasing from the left-end to the center of the interval $I$, then the grids $\mathcal{D}_p$ are graded, i.e., any two adjacent intervals differ in subdivision generation by at most one.

In order to keep the cardinality of $\mathcal{D}_p$ as close as possible to that of $\mathcal{G}_p$, numerical evidence suggests to choose the parameter $\alpha$ close to 1, see [8] for more details. The above condition on $p$ implying gradedness has been numerically checked for $p$ up to 2000.
With the above construction of dyadic grids in intervals at hand, dyadic grids \( D_p(R), p = p(R) \), on hyper-rectangles \( R \) are obtained in a straightforward manner by tensorization. Just as the LGL grids \( \mathcal{G}_p(R) \), the dyadic grids \( D_p(R) \) may be viewed as sub-grids on the (macro-) elements \( R \in \mathcal{R} \). Let \( T_{D_p}(R) = T(D_p(R)) \) be the partition of \( R \) into dyadic (hyper-)rectangles \( E = E_m = \times_{k=1}^N D_{k,m_k} \), where each \( D_{k,m_k} \) is a dyadic interval in the partition \( T_{D_p}(I_k) = T(D_{p_k}(I_k)) \) of \( I_k \); let \( V_{h,D_p}(I_k) \) be the space of continuous piecewise linear functions subordinate to this partition.

We are now prepared to define the auxiliary space for \( V_\delta \) as follows:

\[
V_{h,D_p} := \{ v \in H^1_0(\Omega) : v|_R \in V_{h,D_p}(R), R \in \mathcal{R} \},
\]

where

\[
V_{h,D_p}(R) = \{ v \in C^0(\Omega) : v|_E \in \mathcal{Q}_1 \ \forall \ E \in T_{D_p}(R) \} = \bigotimes_{k=1}^d V_{h,D_{p_k}}(I_k).
\]

Thus, \( V_{h,D_p} \) consists of globally conforming piecewise multi-linear functions on a composite dyadic partition of \( \Omega \). In sharp contrast to the spaces on analogous composite LGL grids, due to the nestedness of the dyadic grid hierarchies, the traces of functions in \( V_{h,D_p} \) on an element interface are now arbitrary continuous piecewise linear functions on the coarser one of the two adjacent “glueable” dyadic partitions.

### 5.2 The auxiliary bilinear forms

Since in terms of the auxiliary space method \( V = V_{\delta} \) and \( \tilde{V} = V_{h,D_p} \) are both conforming spaces we can simply take \( a(\cdot,\cdot) = \tilde{a}(\cdot,\cdot) = \hat{a}(\cdot,\cdot) \). A little more care is required to identify a suitable form \( b(\cdot,\cdot) \). Unfortunately, it turns out that simply retaining the definition \([4,6]\) will not work. In fact, using an inverse estimate throughout an element is too strong to allow the direct estimates in \( \text{ASM3} \) to hold since \( \hat{a}(\cdot,\cdot) \) no longer involves jump terms which in the first stage of reduction to a conforming problem had to be controlled by the \( b_1(\cdot,\cdot) \) form in \( \text{ASM2} \).

**Remark 5.4.** The latter comments hint at the fact that when going directly from the high-order DG-discretization to a globally conforming auxiliary low-order discretization, it is difficult to identify a suitable auxiliary form \( b(\cdot,\cdot) \). Our approach of splitting the construction of an auxiliary space into two stages will eventually provide a suitable preconditioner \( C_B \) by composition.

As a consequence we will propose next a “tamed” version of \( b_1(\cdot,\cdot) \) from \([1,6]\) using the inverse inequality only in those subcells of an element \( R \) which are not “too anisotropic”.

Observe that \( a_R(u,v) = \sum_{k=1}^d a_{R,k}(u,v) = \sum_{k=1}^d \int_R \partial_{x_k} u \partial_{x_k} v \ dx \), we make the ansatz

\[
b_2(u,v) = \sum_{R \in \mathcal{R}} b_{R}(u,v), \quad b_R(u,v) = \sum_{k=1}^d b_{R,k}(u,v),
\]

and proceed to define \( b_{R,k}(u,v) \) in a cell-wise manner for any \( u,v \in V \cup \tilde{V} \) so as to ensure, in view of \( \text{ASM2} \), that still \( a_{R,k}(v,v) \lesssim b_{R,k}(v,v) \). To that end, we consider the partition \( T_p(R) \) of \( R \) into LGL subcells \( S_k = S_k(R) = \prod_{k=1}^d I_{k,l_k} \) already introduced at the beginning of Section [5]. Next, fixing a constant \( C_{\text{aspect}} > 0 \), for each \( 1 \leq k \leq d \), we decompose \( T_p(R) \) into two parts \( T_p^{(0)}(R), T_p^{(1)}(R) \) defined as follows. Setting \( h_{k,l_k} := |I_{k,l_k}| \), we let

\[
S_k \in T^{(0)}_p(R) \quad \text{if} \quad \max_{l_k \neq k} \frac{h_{l_k}}{h_{k,l_k}} > C_{\text{aspect}}, \quad S_k \in T^{(1)}_p(R) \quad \text{otherwise}.
\]

Thus \( T^{(0)}_p(R) \) and \( T^{(1)}_p(R) \) are comprised of “strongly anisotropic” and “sufficiently isotropic” cells, respectively. We now define

\[
b_{R,k}(u,v) = \sum_{S_k \in T^{(0)}_p(R)} b_{R,k,S_k}^{(0)}(u,v) + \sum_{S_k \in T^{(1)}_p(R)} b_{R,k,S_k}^{(1)}(u,v) =: b_{R,k}^{(0)}(u,v) + b_{R,k}^{(1)}(u,v),
\]
Note that for a given edge \( \xi \in F \cap S' \), instead of \( \mathcal{S}_k \), we employ an inverse inequality to obtain
\[
\Phi^* = \frac{1}{V} \sum_{\xi \in F} \int_{\xi} \partial x_k u_h(x_k, \xi') \partial x_k v_h(x_k, \xi') \, dx,
\]
where \( u_h := \mathcal{T}_h^R u, \) \( v_h := \mathcal{T}_h^R v \) as in Property 5.1, \( \mathcal{S}'_k := \prod_{l=1, l \neq k}^d I_{l,h} \), the weight
\[
\omega'_{k,l} := \prod_{l=1, l \neq k}^d h_{l,h} = \text{vol}_{d-1}(\mathcal{S}'_k)
\]
is equal to the \((d-1)\)-dimensional volume of \( \mathcal{S}'_k \), and as before, \( \mathcal{F}_0(\mathcal{S}'_k) \) is the set of all its vertices. Instead, on isotropic cells we employ an inverse inequality to obtain
\[
b_{R,k,S}(u,v) := \sum_{\xi \in F} \sum_{\xi' \in F} c_{\xi,\xi'} \omega'_{k,l} \frac{1}{h_{l,h}} \int_{\xi} u_h(x_k, \xi') v_h(x_k, \xi') \, dx,
\]
where as before the \( c_{\xi,\xi'} \) are tuning constants of order one which have to be chosen judiciously in practical applications. To simplify the exposition we shall suppress them in what follows.

In summary, we have
\[
b_{R,k,S}(u,v) := \sum_{R \in \mathcal{R}} \sum_{k=1}^d \left( \sum_{S_k \in \mathcal{T}^{(0)}_p(R)} b_{R,k,S}(u,v) + \sum_{S_k \in \mathcal{T}^{(1)}_p(R)} b_{R,k,S}(u,v) \right),
\]
where \( b_{R,k,S}(u,v), \) \( i = 0, 1, \) are defined by (5.17), (5.19), respectively.

### 5.3 The operator \( Q \)

It remains to construct a suitable operator \( Q : \tilde{V} = V_{h,D,p} \to V = V_0 \) which is now non-trivial because \( \tilde{V} \not\subseteq V \). Given any \( R \in \mathcal{R} \), it is clear that the trace of a high-order function from \( V_0 \) on any face \( F \) of \( R \) must have (componentwise) the minimal degree among all the \( p(F',R') \) with \( R' \in \mathcal{R}(F) \). To satisfy this condition, we proceed vertex-wise. Precisely, for any vertex \( z \in \mathcal{F}_0(R) \), let \( E_{k}, k = 1, \ldots, d \), be the edges emanating from \( z \) and let the minimal vertex degree be defined as
\[
p_k^* := (p_1^*, \ldots, p_d^*) \in \mathbb{N}^d, \quad \text{with} \quad p_k^* = p_k(E_{k}) := \min_{R' \supseteq E_{k}} p_k(R').
\]
Note that for a given edge \( E = [z,y] \in \mathcal{F}_1(R) \) the degrees \( p_k^* \) and \( p_k' \) may differ. A proper “fusion” can be realized with the aid of the the element shape functions \( \Phi_{z} = \Phi_{z}^R \in \mathcal{Q}_1(R), \) defined by \( \Phi_{z}(z') = \delta_{z,z'}, \) \( z, z' \in \mathcal{F}_0(R), \) that allow us to disentangle first the minimal vertex degrees. In fact, introducing the interpolation operators associated with the dyadic grids \( \mathcal{D}_p(I) \)
\[
\mathcal{T}_{h,D,p}^I : C^0(I) \to V_{h,D,p}(I), \quad \mathcal{T}_{h,D,p}^I(\zeta) = v(\zeta) \quad \forall \zeta \in \mathcal{D}_p(I), \quad \forall v \in C^0(I),
\]
along with their element-wise tensor product
\[
\mathcal{T}_{h,D,p} = \mathcal{T}_{h,D,p}^R : C^0(R) \to V_{h,D,p}(R), \quad \mathcal{T}_{h,D,p}^R := \bigotimes_{k=1}^d \mathcal{T}_{h,D,p_k}^I,
\]
we readily see that if \( \tilde{v}_R \in V_{h,D,p}(R), \) then
\[
\tilde{v}_z^* := \mathcal{T}_{h,D,p_{z}}(\Phi_{z} \tilde{v}_R) \in V_{h,D,p_{z}}(R) \quad \text{and} \quad \tilde{v}_R^* := \sum_{z \in \mathcal{F}_0(R)} \tilde{v}_z^* \in V_{h,D,p}(R).
\]
Similarly, 
\[ v^*_p = T^R_{p; v} \tilde{v}^*_p \in Q_{p; v}(R) \quad \text{and} \quad Q_R \tilde{v}^*_R := v^*_R := \sum_{z \in \mathcal{F}_0(R)} v^*_z \in Q_p (R) \]  
(5.24)

The desired global operator \( Q : \tilde{V} = V_{h,D,p} \rightarrow V = \tilde{V}_\delta \) can now be defined as 
\[ (Q \tilde{v}) |_{\mathcal{R}} := Q_R \tilde{v}^*_R = v^*_R \quad \forall R \in \mathcal{R} \quad \forall \tilde{v} \in \tilde{V}. \]  
(5.25)

**Proposition 5.5.** \( Q \tilde{v} \), defined by (5.25), belongs to \( V^e_\delta \).

**Proof.** First note that for any edge \( E \in \mathcal{F}_1(R) \), one has
\[ (\tilde{v}^*_R)_{|E} = (\tilde{v}_R)_{|E}. \]  
(5.26)

In fact, if \( E \) has vertices \( z_1 \) and \( z_2 \), then
\[ (\tilde{v}^*_z + \tilde{v}^*_w)_{|E} T^E_{h,D,p^*} ((\Phi_{z_1} + \Phi_{z_2}) \tilde{v}^*_R)_{|E} T^E_{h,D,p^*} (\tilde{v}_R)_{|E} = (\tilde{v}_R)_{|E} \]  
since by continuity \( (\tilde{v}^*_z)_{|E} \in V_{h,D,p^*}(E) \); on the other hand, \( (\tilde{v}^*_y)_{|E} = 0 \) for all \( y \in \mathcal{F}_0(R) \setminus \{z_1, z_2\} \). Thus, \( (\tilde{v}^*_R)_{|E} = (\tilde{v}^*_z + \tilde{v}^*_w)_{|E} \) which confirms (5.26).

Furthermore, for any interface \( F \in \mathcal{F}_{d-1} \), with \( \mathcal{R}(F) = \{R', R''\} \), one has
\[ (\tilde{v}^*_R)_{|F} = (\tilde{v}^*_R')_{|F} \quad \text{and} \quad (\tilde{v}^*_R)_{|F} = (\tilde{v}^*_R'')_{|F}. \]  
(5.27)

To see this, let \( \mathcal{F}_0(F) = \mathcal{F}_0(R') \cap \mathcal{F}_0(R'') \) be the set of vertices of \( F \). Since, by continuity of \( \tilde{v} \), \( \tilde{v}_F := (\tilde{v}_R)_{|F} = (\tilde{v}_R')_{|F}, \) one has \( (\Phi_{z} \tilde{v}_R)_{|F} = (\Phi_{z} \tilde{v}_R')_{|F}, \) which confirms (5.27).

Hence, denoting by \( q^*_z = (p^*_z)' \in \mathbb{N}^{d-1} \) the reduced degree vector obtained from \( p^*_z \) by dropping the component in the direction orthogonal to \( F \), one concludes that
\[ (\tilde{v}^*_R')_{|F} = \sum_{z \in \mathcal{F}_0(F)} T^F_{h,D,q^*_z} (\Phi_z \tilde{v}_R')_{|F} = \sum_{z \in \mathcal{F}_0(F)} T^F_{h,D,q^*_z} (\Phi_z \tilde{v}_R')_{|F} = \sum_{z \in \mathcal{F}_0(F)} T^F_{h,D,q^*_z} (\tilde{v}^*_R')_{|F}, \]
where we have used that \( q^*_z \) is the same for both \( R' \) and \( R'' \). This confirms the first part of the assertion. Abbreviating \( \tilde{v}^*_R := T^F_{h,D,q^*_z} (\Phi_z \tilde{v}_R') \), the second one follows from \( (\tilde{v}^*_R')_{|F} = \sum_{z \in \mathcal{F}_0(F)} T^F_{h,D,q^*_z} \tilde{v}^*_R = (\tilde{v}^*_R')_{|F}, \) which completes the proof of (5.27) and hence of the proposition.

The main result of this section, whose proof is given in Section 8, can be phrased as follows.

**Theorem 5.6.** Let \( \tilde{A}_2 \) be the stiffness matrix with respect to a basis for \( \tilde{V} = V_{h,D,p} \) and let \( B_2 \) be the matrix representation of the operator \( B_2 \) defined by \( (B_2 v, w) = b_2(v, w), \) \( v, w \in \tilde{V}^e_\delta + V_{h,D,p} \), with \( b_2(\cdot, \cdot) \) defined by (5.20). Finally let \( S_2 \) denote the matrix representation of the operator \( Q \), defined by (5.25). Then, there exists a constant \( C_2 \) depending only on the grading conditions (2.4), such that, whenever \( C_{A_2} \) is a symmetric preconditioner for \( A_2 \), the matrix \( C_{A_2} : = B_2^{-1} + S_2 C_{A_2} S_2^T \) is a symmetric preconditioner for (2.1) satisfying
\[ \kappa(C_{A_2} A_2) \leq C_2 \kappa(C_{A_2} A_2) \]  
(5.28)
uniformly in \( \delta = (H, p) \), subject to the grading conditions (2.4).

Note that, due to the global continuity of the functions in \( \tilde{V}^e_\delta + V_{h,D,p} \) the matrix \( B_2 \) is no longer diagonal but very sparse. More precisely the sparsity depends quantitatively on the control of the aspect ratios in (5.15). Therefore, \( B_2 \) is, of course, not inverted exactly. A more elaborate discussion of this issue can be found in [8] [12]. The numerical experiments later given in Section 5.5 indicate that the approximate inversion of \( B_2 \) is facilitated by efficient smoothing relaxations.
5.4 A multiwavelet preconditioner for $\tilde{A}_2$

It remains to specify the preconditioner $C_{\tilde{A}_2}$ in Theorem 5.6. Again we stress that such a preconditioner is already of interest for spectral discretization on a single element as well as for high-order conforming discretizations, independent of the DG context.

On the one hand, we are now in the comfortable situation of dealing with a multi-level hierarchy of nested partitions $D_p(\mathcal{R})$ of $\Omega$, obtained as the union of dyadic element partitions $D_p(R), R \in \mathcal{R}$. On the other hand, a straightforward application of standard preconditioning concepts is impeded by the fact that the dyadic grids are highly anisotropic. In fact, the currently known results do not seem to imply that BPX-type techniques would give rise to uniformly bounded condition numbers.

We therefore propose a strategy that again resorts to the auxiliary space method, this time for the scenario that $V \subset \tilde{V}$. The envisaged auxiliary space will be seen to have uniformly $H^1$-stable splittings in the sense of [24] which, in turn, are obtained in two steps:

(a) For each element $R \in \mathcal{R}$ construct a local $H^1$-stable splitting for the corresponding finite element space $V_{h,D,p}(R)$ subordinate to the dyadic subgrid $D_p(R)$.

(b) Glue the local $H^1$-stable splittings to a global $H^1$-stable splitting for the global conforming finite element space $V_{h,D,p}$.

For step (a) we exploit the fact that $V_{h,D,p}(R)$ is a tensor product space. For this to work we need a frame or basis for the one-dimensional factors with the property that properly scaled versions form a frame or basis for $L_2$ and for $H^1$. To obtain such systems we use certain piecewise linear $L_2$-orthonormal compactly supported multi-wavelets on an interval. In particular, the boundary conditions of those multi-wavelets allows us to glue them across end points of adjacent intervals in such a way that they still give rise to globally continuous $L_2$-orthonormal wavelets on the unions of such intervals. This is the essential prerequisite for step (b).

5.4.1 Piecewise affine orthogonal multiwavelets on an interval

We briefly recall first the piecewise affine orthogonal multiwavelets on all of $\mathbb{R}$ constructed in [21] Example in Section 3.1. For our purpose it is important to relate them to the standard piecewise linear multi-resolution spaces $(\tilde{S}_j)_{j \in \mathbb{Z}}$ on the dyadic grids $2^{-j} \mathbb{Z}$. Specifically, $\tilde{S}_j$ is obtained by dilating the elements of $\tilde{S}_0$ by $2^j$, which itself is generated by the integer translates of the standard piecewise linear hat function with integer knots. An intertwining technique yields then a new multiresolution analysis of subspaces of piecewise linear functions $S_j$ that are spanned by three orthogonal scaling functions $\phi^0, \phi^1$, and $\phi^2$ in such a way that

$$\tilde{S}_{j+1} \subset S_j \subset \tilde{S}_{j+2}. \quad (5.29)$$

The orthogonal complements $W_j := S_{j+1} \ominus S_j$ can be shown to be spanned by translated and dilated versions of three orthogonal wavelet functions $\psi^0, \psi^1$, and $\psi^2$, i.e.,

$$W_j := \text{span} \{ \psi^i_{j,k} : i \in \{0, 1, 2\}, k \in \mathbb{Z} \}.$$

Here we denote the translated and dilated version of a function $f \in L_2(\mathbb{R})$ by $f_{j,k} := 2^{j/2} f(2^j \cdot -k)$. For convenience we gather in the following the indices of a translated and dilated scaling function or wavelet in a single multilevel index $\lambda = (i, j, k)$, where $|\lambda| := j$ denotes the level of $\lambda$. With the convenient definition $W_{-1} := V_0$ we have the orthogonal decomposition $L_2(\mathbb{R}) = \bigoplus_{j=-1}^\infty W_j$. The wavelets have two vanishing moments which is optimal for piecewise affine orthogonal wavelets.

The scaling functions $\phi^i$ and wavelets $\psi^i$ in the above setting are supported on the interval $[-1, 1]$. Their graphs are depicted in Fig. 5.

The above construction is geared to facilitate the construction of an orthonormal wavelet basis $\Psi = (\psi_\lambda)_{\lambda \in \Delta}$ for $L_2([0,1])$, essentially by restriction to the interval $[0, 1] \subset \mathbb{R}$, see [22] for details.
The multilevel basis on $[0,1]$ consists essentially of all translated and dilated scaling functions and wavelets with support inside the interval complemented by appropriate boundary scaling functions and wavelets. These boundary functions are obtained as restrictions of linear combinations of the scaling functions and wavelets with support exceeding the interval boundaries to $[0,1]$. Moreover, for each end point of the interval there is exactly one boundary scaling function and one wavelet that vanishes at that end point, see Figures 6(a) and (b). This is important for efficiently “gluing” local scaling functions and wavelets on $R \in \mathcal{R}$ to globally conforming ones on $\Omega$. For those elements $R$ which intersect $\partial \Omega$ one has to incorporate Dirichlet boundary conditions. For the scaling functions this simply amounts to omitting the single scaling function that does not vanish at the respective end point. The boundary wavelet needs to be modified somewhat as shown in Fig. 6(c).

With the analogous definition of multiresolution spaces $V_j$ and corresponding orthogonal complement spaces $W_j := \text{span} \{ \psi_\lambda : \lambda \in \Delta_j \}$, this time on the interval $[0,1]$, we have the orthogonal decomposition $L_2([0,1]) = \bigoplus_{j=-1}^\infty W_j$. By a suitable affine change of variables and adjusted scaling one obtains orthogonal decompositions of $L_2([a,b])$ for any interval $[a,b] \subset \mathbb{R}$.

![Figure 5: Piecewise affine orthogonal multiwavelets (21), blue: scaling functions, red: wavelets)](image)

![Figure 6: Boundary scaling functions and wavelets (22); blue: scaling functions, red: wavelets; solid line: left boundary, dashed line: right boundary; (a) and (b): Neumann boundary conditions, (c): zero Dirichlet boundary conditions.)](image)
5.4.2 Constructing an auxiliary multilevel space

We construct next an auxiliary space for $V_{h,D,p}$ that permits $H^1$-stable multilevel decompositions. According to (a) it is based on local tensor product spaces. To define these local tensor product spaces consider first an arbitrary interval $I = [a,b] \subset \mathbb{R}$ along with a dyadic meshes $D_p = \text{NestedDyadic}[\mathcal{G}_{p}, \{I\}, \alpha]$, which are nested when $p$ increases and graded. Recall that $V_{h,D,p}(I)$ denotes the corresponding piecewise linear finite element space spanned by the piecewise affine hat functions $(\theta_{\xi})_{\xi \in D_p(I)}$, i.e., $\theta_{\xi}(\xi') = \delta_{\xi,\xi'}$, $\xi, \xi' \in D_p(I)$.

In view of (5.29), $V_{h,D,p}(I)$ does not equal exactly a span of scaling functions from the spaces $V_{\tilde{\xi}}$. Also the grid $D_p(I)$ is non-uniform. To obtain a possibly small space $W_{h,D,p}(I)$, spanned by wavelets in $\Psi$, that contains $V_{h,D,p}(I)$, we consider first the set

$$ \hat{\Lambda} := \{ \lambda \in \Delta : \text{there is a } \zeta \in D_p(I) \text{ such that } \langle \psi_\lambda, \theta_\zeta \rangle_{L^2(0,1)} \neq 0 \}, $$

(5.30)

that need to be activated in order to span $V_{h,D,p}(I)$. Note that because of (5.29) the set $\hat{\Lambda}$ is finite. Moreover, because all $\theta_\zeta$ are piecewise affine functions and the wavelets are compactly supported with two vanishing moments, $\hat{\Lambda}$ “adapts” the non-uniform nature of the grid $D_p(I)$. In particular, the maximal level occurring in $\hat{\Lambda}$ is finite, i.e., $|\lambda| \leq j_{\text{max}}$ for all $\lambda \in \hat{\Lambda}$, where $j_{\text{max}} = j_{\text{max}}(p,\alpha)$.

In order to facilitate an efficient transformation of linear combinations of the hat functions $(\theta_{\xi})_{\xi \in D_p(I)}$ in $V_{h,D,p}(I)$ into a linear combination of wavelets one may possibly have to augment $\hat{\Lambda}$ somewhat to a set $\Lambda$ which is the smallest “multilevel tree” that contains $\hat{\Lambda}$ and set $\Lambda_j := \{ \lambda \in \Lambda : |\lambda| = j \}$. By this we mean that, when representing all wavelets with indices in $\Lambda$ in terms of scaling functions, the encountered scaling functions on level $j$ all appear in the two-scale relation of the active scaling functions on level $j-1$. The corresponding wavelet-transform then exhibits linear complexity which is important for the efficiency of the preconditioner based on wavelet representation of the stiffness matrix.

Defining for $-1 \leq j \leq j_{\text{max}}$ the spaces $W_{h,D,p,j}(I) := \text{span} \{ \psi_\lambda : \lambda \in \Lambda_j \}$, we obtain the $L_2$-orthogonal decomposition

$$ W_{h,D,p}(I) := \text{span} \{ \psi_\lambda : \lambda \in \Lambda \} = \bigoplus_{j=-1}^{\infty} W_{h,D,p,j}(I). $$

Note that since $D_p$ is graded and because of (5.29), the grid underlying $W_{h,D,p}(I)$ is a local refinement of $D_p(I)$ by at most one level. Furthermore, nestedness is preserved, i.e., $W_{h,D,p}(I) \subseteq W_{h,D,p+1}(I)$ for all $p$.

Given any $R = \times_{k=1}^{d} I_k \in \mathcal{R}$, we define, according to step (a), the local auxiliary space as

$$ \tilde{V}(R) = W_{h,D,p}(R) := \bigotimes_{k=1}^{d} W_{h,D,p_k}(I_k), $$

(5.31)

which leads to

$$ \tilde{V} = W_{h,D,p} := \{ w \in H^1_0(\Omega) : w|_R \in W_{h,D,p}(R), R \in \mathcal{R} \}, $$

(5.32)

as the auxiliary space $\tilde{V} \supset V = V_{h,D,p}$ used in the ASM. By construction, the dimension of $\tilde{V}$ remains uniformly proportional to dim $V_{h,D,p}$.

For the construction of $H^1$-stable splittings for $\tilde{V}$ we wish to identify next an $L_2$-orthogonal wavelet basis along with the corresponding scaling function basis for $\tilde{V} = W_{h,D,p}$ composed of the local wavelet bases for each $W_{h,D,p}(R)$. The basic principle is to properly “glue” those scaling functions and wavelets across an element interface which do not vanish on that interface. Due to the properties of the interval-adapted scaling functions and wavelets, mentioned above, one can follow essentially the lines of [17] [18]. A small difference to be perhaps addressed is the fact that the dyadic grids may vary from element to element and hence the restrictions of dyadic grids at an (inner)
element interface \( F \subset \Omega \) do not agree. However, since grids are nested the common minimal grid – the intersection – \( D^*(F) \) belongs to all participating grids. Moreover, global continuity of the functions in \( V_{h.D,p} \) implies that the traces of the local functions from the adjacent elements agree on \( F \) and belong to the finite element space induced by \( D^*(F) \). Let us denote by \( W(D^*(F)) \) the trace space obtained by the intersection of the spaces \( W_{h.D,p}(R) \) such that \( F \in F(R) \). The effect of the gluing process is now as follows. Any global scaling basis function or wavelet in \( W_{h.D,p} \) which does not vanish on \( F \), and hence has part of its support in all adjacent elements, has the following property: its restriction to any of the adjacent elements, whenever being nontrivial, is a tensor product of a scaling function or wavelet belonging to the trace space \( W(D^*(F)) \) and a boundary scaling function or wavelet in the complementary variables which do not vanish on \( F \). In short, the global basis, which we refer to as composite basis, consists of basis functions supported in a single element and on basis functions that are glued across element interfaces with factors from the intersection spaces \( W(D^*(F)) \). Moreover, it is important to note that the restriction of the global wavelets to any element \( R \in \mathcal{R} \) is still an \( L_2 \)-orthogonal basis for \( L_2(R) \) where the basis functions affected by the restriction still have norms that are uniformly equivalent to one. Finally, the (local and global) transformation from elements in \( V_{h.D,p} \) into wavelet representation has linear complexity. It remains to confirm that \( \Psi_{\Omega} \) gives rise to uniformly \( H^1 \)-stable splittings for \( \tilde{V} \). On account of the above restriction properties and the set-additivity of the \( H^1 \)-norm it suffices to show that the local wavelet bases for the tensor-product spaces \( W_{h.D,p}(R) \) give rise to uniformly \( H^1(R) \)-stable splittings.

To that end, we can now apply the theory of tensorial subspace splittings in its anisotropic variant. In fact, suitably scaled versions of the univariate wavelet bases on an interval are uniformly \( L_2 \)-stable with respect to any element \( R \in \mathcal{R} \). Thus, by Theorem 1 in [25] we know that on any \( R = \times_{k=1}^d I_k \in \mathcal{R} \) we have an \( H^1 \)-stable splitting of \( W_{h.D,p}(R) \) into the components

\[
a_R(u,u) \simeq \inf \sum_{j_1,\ldots,j_d \geq -1} (2^{2j_1} + \cdots + 2^{2j_d})(u_{j_1,\ldots,j_d}, u_{j_1,\ldots,j_d})_R ,
\]

where the infimum is taken over all representations of \( u \) as \( u = \sum_{j_1,\ldots,j_d \geq -1} u_{j_1,\ldots,j_d} \) with \( u_{j_1,\ldots,j_d} \in W_{j_1,\ldots,j_d} \). Since the wavelets form a basis the representations in the splittings are actually unique therefore providing an \( H^1(R) \)-stable splitting on each element \( R \in \mathcal{R} \).

We again apply the auxiliary space method. In fact, in terms of the terminology of Section 3 we now have \( \tilde{V} = \tilde{V} \), \( a = \tilde{a} = \tilde{a} \) and \( \tilde{Q} : V \rightarrow \tilde{V} \) can be taken as the canonical embedding of \( V \) into \( \tilde{V} \). Then (5.1) and the inequalities in (5.2) and (5.3) concerning \( \tilde{Q} \) are obviously satisfied.

Since \( \tilde{V} = V_{h.D,p} \subset \tilde{V} = W_{h.D,p} \) we could look for an appropriate operator \( \tilde{Q} \) which possesses this right inverse. This would allow us to dispense with an additional smoothing bilinear form in this stage. For simplicity of exposition we instead apply again Proposition 3.1, i.e. we define the form \( b_\Omega(\cdot,\cdot) \) in analogy to (5.14)-(5.20), (i) replacing \( V_{p}(R) \) by \( V_{h.D,p}(R) \) and (ii) substituting the underlying LGL grid by the dyadic grid and introducing the corresponding intervals \( S_{D,\ell} \), subcells \( I_{D,\ell,k,\ell} \) and, in analogy to (5.15), sets \( T_{D,p}(i) \) for \( i \in \{0,1\} \).

Moreover, also \( Q \) can be defined in complete analogy to (5.25), (i) replacing \( V_{h.D,p}(R) \) in (5.27) by \( W_{h.D,p}(R) \) with the same meaning of the degree vectors \( p_\zeta \), \( \zeta \in \mathcal{F}_0(R) \); (ii) employing hat function representations of the elements of \( W_{h.D,p}(R) \); (iii) replacing the operator \( T_{p}(R) \) in (5.24) by the dyadic interpolation operator \( I_{p}(R) \). Since the elements of the composite space \( W_{h.D,p} \) are continuous over \( \Omega \) one shows in exactly the same way as in Proposition 5.5 that \( \tilde{Q} \tilde{v} \in H^1(\Omega) \) for any \( \tilde{v} \in \tilde{V} = W_{h.D,p} \).

The following lemma asserts the uniform \( H^1 \)-stability of \( Q \) and the corresponding direct estimate, hence the validity of the ASM conditions.

**Lemma 5.7.** Let \( Q : \tilde{V} \rightarrow V \) be the operator defined above. Then one has

\[
\|Q\tilde{v}\|_{H^1} \lesssim \|\tilde{v}\|_{H^1} \quad \text{for all} \quad \tilde{v} \in \tilde{V}
\]

(5.34)
and
\[ b(\tilde{v} - Q\tilde{v}, \tilde{v} - Q\tilde{v}) \lesssim \tilde{a}(\tilde{v}, \tilde{v}) \quad \text{for all} \quad \tilde{v} \in \tilde{V}, \]

where the constants are independent of the discretization parameters \( \delta = (H, p) \).

The proof is a consequence of various stability estimates provided in Section 8 and is therefore deferred to that section (see Remarks 8.15 and 8.18).

To build a preconditioner \( C_{\tilde{A}} \) for the conforming problem over \( V_{h,D,P} \) let \( S_3 \) and \( B_3 \) denote the matrix representations of the operator \( Q \) and the auxiliary bilinear form \( b_3 \), referred to in Lemma 5.7 respectively. In the iterative solver we need to efficiently apply the stiffness matrix with respect to a properly scaled wavelet basis for \( W_{h,D,P} \). Specifically, the wavelets are normalized so that their \( H^1 \)-norms are uniformly equivalent to one. However, this matrix is not sparse and is therefore never assembled. Instead, we only assemble the stiffness matrix with respect to the corresponding scaling function basis, which overall exhibits the linear complexity of the fast wavelet transform. Let us denote by \( \tilde{A}_2^\Psi \) the stiffness matrix with respect to the local scaling function basis of a space \( \tilde{W}_{h,D,P} \), that is slightly larger than \( W_{h,D,P} \). Note that due to the grading of \( D_p \), the dyadic grid underlying \( \tilde{W}_{h,D,P} \) is a local refinement of that of \( W_{h,D,P} \) by finitely many levels. Since the scaling functions are piecewise (multi-)affine functions on a uniform grid, where the nodal values are known, the sparse stiffness matrix \( \tilde{A}_2^\Psi \) can easily be assembled. Then the stiffness matrix \( \tilde{A}_2^\Phi \) with respect to the corresponding wavelet basis of \( W_{h,D,P} \) is given as \( \tilde{A}_2^\Phi = T^T \tilde{A}_2^\Psi T \) and can be applied in the way described above at the expense of \( \mathcal{O}(\dim(V_{h,D,P})) \) operations. Finally let \( CG \) stand for a fixed number of conjugate gradient iterations applied to \( \tilde{A}_2^\Psi \).

As before for \( B_2 \), the matrix \( B_3 \) of the smoothing operator is not inverted exactly, but we shall approximate the inverse of \( B_3 \) along the same lines as the inverse of \( B_2 \).

The above findings can now be summarized as follows.

**Theorem 5.8.** Adhering to the above notation the composite preconditioner
\[ C_{\tilde{A}} := B_3^{-1} + S_3 CG(T^T \tilde{A}_2^\Psi T) S_3^T \]

satisfies \( \kappa(C_{\tilde{A}} \tilde{A}_2) = \mathcal{O}(1) \), uniformly in the discretization parameters \( \delta = (H, p) \).

Employing the approximate inversion of \( B_3 \) in analogy to the treatment of \( B_2 \) the application of \( C_{\tilde{A}} \) requires \( \mathcal{O}(\dim(V_{h,D,P})) \) operations.

### 5.5 Numerical experiments concerning the preconditioner \( C_{\tilde{A}} \)

We discuss next efficient strategies for applying \( C_{\tilde{A}} \) with \( C_{\tilde{A}} = \tilde{A}_2^{-1} \). The matrix \( B_2 \) for the smoothing operator is very sparse but no longer diagonal. In Fig. 7(a) this is illustrated for \( d = 2 \) and a patch with polynomial degree \( p = 25 \) in both directions. For a detailed discussion of exploiting the sparsity we refer to [8, 12].
We compare below the exact inversion of $B_2$ and an alternating Gauss-Seidel relaxation over the skeleton of $R$ and a block elimination for each $R \in R$ in the spirit of substructuring methods. Before we need to fix the parameter $\alpha$ in the dyadic grid generation, the aspect ratio control $C_{\text{aspect}}$ in the definition of $b_2(\cdot, \cdot)$, monitoring the influence of the inverse estimates, and the tuning constant $c_{\xi, \xi'} \sim 1$. $\alpha = 1.2$ appears to be a good compromise yielding sufficiently rich auxiliary spaces $V_h, D_P$ while keeping their dimension close to that of $V_q$. In subsequent experiments we use $C_{\text{aspect}} = 2$ since larger values turn out to more oscillations in the estimates for the condition numbers. Due to a somewhat stronger variation in the condition numbers, the choice of $c_{\xi, \xi'}$ is less clear. Based on extensive experiments we set $c_{\xi, \xi'} = 0.6$.

Figures 8(a) and 8(b) show the condition numbers $\kappa(C_{\tilde{A}_2} \tilde{A}_1)$ for the first and second test scenario, respectively, when $\tilde{A}_2$ is inverted exactly. The plot marks indicate the condition numbers obtained by approximately solving the smoothing problem with the aid of 7 iterations of the substructuring method. In contrast, in both plots the solid lines represent the results when the smoothing problem is solved exactly using a direct method.

For the first test scenario the condition numbers are presented for $p$ ranging from 4 to 40. In the cases (i) $q = p$, (ii) $q = p + 2$ and (iii) $q = 3/2p$ the condition numbers vary only mildly, probably due
to the discrete process of dyadic grid generation. In contrast, we observe larger condition numbers for larger jumps between the polynomial degrees on adjacent elements, see the cases (iv) $q = 7/4p$ and, in particular, (v) $q = 2p$. For a more thorough analysis and an explanation for these effects we refer to [11].

One observes enhanced oscillations when specific polynomial degrees are present in the grid, e.g. for odd polynomial degrees close to $27$ which we attribute to a resonance effect between the LGL and dyadic grids. More detailed investigations reveal that similar oscillations are also observed for some even polynomial degrees, e.g. 50. Moreover, the resonance can be shifted to nearby polynomial degrees when the parameter $\alpha$, controlling the associated dyadic grids, is varied. Therefore, our tests should be viewed as a guide for favorable choices of the dyadic grid generation parameters $\alpha$ depending on the employed polynomial degrees.

For the second test scenario, which is perhaps more relevant for practical applications than the first one, we investigate $p$ in the range of 4 to 50. When the smoothing problem is solved exactly, we observe that, aside from some oscillations due to the grid resonance effect, the condition numbers grow in essence mildly for small polynomial degrees and quickly tend to a limit below $10^{-5}$ when solving the smoothing problem only approximately by a fixed number of substructuring iterations, we observe increasing condition numbers for polynomial degrees larger than 41. This issue will be examined more closely in forthcoming work.

5.5.2 The multi-wavelet preconditioner $C_{\tilde{A}_2}$

In the more relevant second test scenario we compare the effect of an exact solution of the smoothing problem with a substructuring method for $B_3$ performing alternating Gauss-Seidel relaxation over the skeleton of $R$ and a block elimination for each $R \in R$. We always apply 7 iterations in the substructuring scheme and set $C_{\text{aspect}} = 2.0$ and $c_{\xi,\xi'} = 0.6$.

Our first observation is that the condition number of the multi-wavelet stiffness matrix $\tilde{A}^\psi_2$ is approximately $\kappa(\tilde{A}^\psi_2) \approx 60$, independent of the underlying polynomial degree. We observe that after approximately 30 iterations of the CG algorithm a desired absolute residual tolerance of $10^{-6}$ is reached.

Fig. 5 shows that the condition numbers produced by $C_{\tilde{A}_2}$ change abruptly when the polynomial degree varies. This seems to be again due to the discontinuous nature of dyadic grid generation. Moreover, we see that for $4 \leq p \leq 50$ the condition numbers are below 12.5 in all cases. Finally, note that the substructuring method works better on $B_3$ than on $B_2$. Using the same number of iterations as before, there is no visible difference in the results between exact and approximate inversion of $B_3$, which is probably due to the much more local coupling of the macro elements in $B_3$.

6 The composite preconditioner

We adhere to the previous notation concerning the matrices $C_{\tilde{A}_1}, \tilde{A}_1, C_{\tilde{A}_2}, \tilde{A}_2, B_i,$ and $S_i$ for $i = 1, 2$. Combining Theorems 4.1, 5.6, and 5.8, we obtain the following result.

**Theorem 6.1.** Assume that the grading conditions (2.4) hold. Then the composite preconditioner

$$C_A := B_1^{-1} + S_1(B_2^{-1} + S_2C_{\tilde{A}_2}S_2^T)S_1^T$$

(6.1)

for the DG-system (2.6) satisfies $\kappa(C_A A) = O(1)$ uniformly in the discretization parameters $\delta = (H, p)$.

Moreover, safe for the exact inversion of $B_2$, each application requires a number of operations that stays uniformly proportional to $\dim V_\delta$ with respect to the discretization parameters $\delta = (H, p)$.
Figure 9: Condition numbers $\kappa(C_{\tilde{A}_2} \tilde{A}_2)$ produced by the multi-wavelet preconditioner $C_{\tilde{A}_2}$ in the second test scenario, where the smoothing problem is solved exactly (solid line) or approximately using the substructuring method (blue marks).

Figure 10: Condition numbers $\kappa(C_{A_{A}})$ obtained by (6.1) with $C_{\tilde{A}_2} = \tilde{A}_2^{-1}$. The quantitative behavior of the combined preconditioner is illustrated by the subsequent experiments. First, we set $C_{\tilde{A}_2} = \tilde{A}_2^{-1}$ in (6.1), i.e., we invert $\tilde{A}_2$ exactly. The numerical results for both test scenarios are depicted in Fig. 10.

For both scenarios, the numerical effects observed earlier for the single stages are still present. The oscillations that have been striking in the second stage for the first test scenario now only have a mild effect, but their dependence on the ratio $q/p$ is still clearly visible. In the second test scenario, for $p \leq 40$ the condition numbers for the approximate and exact solutions of the smoothing problem almost agree. For larger $p$ the insufficiently accurate solutions of the smoothing problem by the substructuring method in the second stage lead to a deterioration of the overall condition number. When $B_2$ is inverted exactly, the condition number is bounded by 17.

Fig. 11 shows the condition numbers $\kappa(C_{A_{A}})$ obtained for the second test scenario when $C_{\tilde{A}_2}$ is the multi-wavelet preconditioner defined in (5.36); the polynomial degree $p$ ranges from 4 to 50. In Fig. 11(a) these condition numbers (solid line) are compared to those obtained for $C_{\tilde{A}_2} = \tilde{A}_2^{-1}$ (dashed line), see also Fig. [8(b)].
When the multi-wavelet preconditioner is applied, we again observe that, besides the oscillations, the condition numbers essentially tend to a limit smaller than 18. In comparison with the condition numbers obtained for the variant where $\tilde{A}_2$ is inverted exactly, we observe that both curves are nearly parallel. The oscillations, which are due to the LGL/dyadic grid resonance, in both curves appear for the same polynomial degrees, but their amplitudes are multiplied by a factor in the range from 2 to 3 in the variant using the multi-wavelet preconditioner.

The effect of the substructuring method is visualized as in Fig. 8, namely the blue plot marks indicate the condition numbers obtained when $B_2$ is approximated by 7 iterations of the substructuring method while the solid line represents the condition numbers when the smoothing problem is solved exactly by a direct method.

The results obtained with the composite preconditioner are qualitatively comparable to the ones obtained in Subsection 5.5.1, see also Fig. 8(b). For $p \leq 41$ we observe that the substructuring method performs as well as exact solution of the smoothing problem. In comparison with the variant where $\tilde{A}_2$ is inverted exactly the condition numbers are larger by a small factor below 2.

Finally, we numerically investigate a variant of the concatenated preconditioner (6.1), where in the definition of the multi-wavelet preconditioner (5.36) the inverse of the smoother $B_3$ is omitted. This can be interpreted as a single auxiliary space method with the concatenated operators $Q_2 \circ Q_3$ and $\tilde{Q}_3 \circ \tilde{Q}_2$ and the auxiliary bilinear form $b_2$.

In the numerical results displayed in Fig. 11(b) we observe that the condition number is bounded independently of the discretization parameters $\delta = (H,p)$. Quantitatively, the condition numbers are somewhat lower than in the first variant, i.e., we can save computational cost and obtain a better preconditioner when the summand $B_3^{-1}$ is omitted. On the other hand, when an approximate inversion of $B_3$ is used the need for a more accurate inversion of $B_2$ seems to be reduced.
7 Proof of Theorem 4.1

The proof of Theorem 4.1 follows from Corollary 3.2 once we have verified the validity of the ASM-conditions in Section 3. Since here \( V = V_\xi \subseteq V = V_\delta \), Proposition 3.4 applies, i.e., we have to verify ASM2 and find a suitable \( Q : V \to V \), satisfying the direct estimate (3.5).

To this end, we collect first some simple preparatory facts about product LGL grids and the structure of corresponding quadrature weights. To organize this it is convenient to localize the facet complexes \( F_1 \). Given two integers \( 0 \leq l, m \leq d \) and a facet \( F \in F_1 \) (see (2.2)), we set

\[
F_m(F) = \{ G \in F_m : G \subset F \text{ if } m < l, \ G = F \text{ if } m = l, \ G \supset F \text{ if } m > l \}.
\]  

This new definition consistently extends the ones given in Section 2, where in particular \( F_1 \) already been defined as the set of all \( l \)-facets of an element \( R \in \mathcal{R} \), whereas \( \mathcal{R}(F) = F_d(F) \) is the set of all elements \( R \) containing the facet \( F \).

Given any \( R \in \mathcal{R} \) and the corresponding LGL grid \( G_p(R) \), if \( F \in F_l(R) \) for some \( 1 \leq l \leq d - 1 \), then \( G_p(R) \cap F \) is the LGL grid of the induced quadrature formula on \( F \), which will be denoted by \( G_p(F, R) \). The order of the formula, denoted by \( p_\ast = p(F, R) \in \mathbb{N} \), is obtained from the vector \( \mathbf{p} = p(R) \) by deleting the \( d - l \) components corresponding to the frozen coordinates of \( F \). The weights of the formula, denoted by \( w_\xi = \{ \omega \} \xi \), are connected to the weights \( w_\xi \) of the original formula on \( R \) by the relation

\[
w_\xi = w_{f_1} \cdots w_{f_{d-1}} w_{\xi}^{F,R} \quad \forall \xi \in G_p(F, R),
\]

where each \( w_{f_l} \) is a boundary weight of the univariate LGL formula along one of the frozen coordinates of \( F \). This implies that if the facet \( F' \) belongs to \( F_{l-1}(F) \subset F_{l-1}(R) \), then

\[
w_{\xi}^{F,R} = w_f w_{\xi}^{F,R} \quad \forall \xi \in G_p(F', R),
\]

for some boundary weight \( w_f \).

Next, let us draw some consequences from the assumptions (2.4) of quasi-uniformity of the mesh and polynomial degree grading. Under these assumptions, it is easily seen that given any \( R \in \mathcal{R} \), there exists a real \( H_R > 0 \) and an integer \( p_R > 0 \), such that for all \( R' \in \mathcal{R} \) satisfying \( R' \cap R \neq \emptyset \), it holds

\[
H_{R'} \simeq H_R, \quad p_{R'} \simeq p_R, \quad 1 \leq k \leq d,
\]

i.e., lengths, as well as polynomial degrees are locally comparable. More generally, for any facet \( F \in F_1 \), \( l < d \), one can define analogous “representative parameters” \( H_F, p_F \), for instance, by averaging corresponding parameters from intersecting elements. This implies

\[
H_F \simeq H_G, \quad p_F \simeq p_G, \quad \text{for any two faces } F, G \text{ such that } F \cap G \neq \emptyset.
\]

Furthermore, for each boundary weight \( w_j \) of any univariate LGL formula used in the definition of the tensorial LGL formula on \( R' \), and for any face \( F \in F_{d-1}(R') \), one has

\[
w_j \simeq \frac{H_R}{p_R} , \quad j \in \{ 0, p_k(R') \}, \quad \omega_F \simeq \frac{p_F^2}{H_F}, \quad F \in F_{d-1}(R').
\]

In particular, all the faces \( F \in F_{d-1} \) having a nonempty intersection with an element \( R \) carry weights \( \omega_F \) of comparable magnitude.

As a final prerequisite recall the fundamental property that the bilinear form

\[
(u, v)_{0,F} = \sum_{j=0}^{p} u(\xi_j)v(\xi_j) w_j
\]

(7.7)
is an inner product in $\mathbb{P}_p(I)$, which induces a norm that is uniformly equivalent to the $L^2$-norm, namely $\|v\|_{0,I} \leq \|v\|_{0,1,p} \leq \sqrt[3]{3}\|v\|_{0,I}$ for all $v \in \mathbb{P}_p(I)$, see e.g. [13] (5.3.2) on p. 280. Tensorization yields that

$$
(u, v)_{0,R,p} = \sum_{\xi \in \mathcal{G}_p(R)} u(\xi)v(\xi)w_\xi 
$$

(7.8)
is a discrete inner product in $\mathbb{Q}_p(R)$, defining a norm which is uniformly equivalent to the $L^2$-norm, i.e.,

$$
\|v\|_{0,R} \leq \|v\|_{0,R,p} \leq (\sqrt[3]{3})^d \|v\|_{0,R} \quad \forall v \in \mathbb{Q}_p(R) .
$$

(7.9)

LGL grids as well as discrete inner products and norms restrict to any facet of $\Omega$ yielding objects with analogous properties.

We shall now verify condition $\text{ASM}_2$, i.e., the form $b_1(\cdot, \cdot)$ dominates $a_3(\cdot, \cdot)$. To this end, the following immediate consequences of (7.6) concerning the weights $W_\xi$ introduced in (4.5) will be useful.

**Lemma 7.1.** Let $R \in \mathcal{R}$ be arbitrary. For any face $F \in \mathcal{F}_{d-1}$ and any $\xi \in \mathcal{G}_p(F,R)$, one has

$$W_\xi \simeq \omega_F w^{F,R}_\xi .
$$

**Proposition 7.2.** For $b_1$ given in Definition 4.6 one has

$$a_3(v,v) \lesssim b_1(v,v) \quad \forall v \in V_\delta .
$$

**Proof.** For any $R \in \mathcal{R}$, using (7.7), (7.9) and (4.4) together with tensorization, one obtains

$$\|\nabla v\|_{0,R}^2 = \sum_{k=1}^d \|\partial_k v\|_{0,R}^2 \lesssim \sum_{\xi \in \mathcal{G}_p(R)} v(\xi)^2 w_{\xi,k}^{-1} \left( \prod_{j \neq k} w_{\xi,j} \right) = \sum_{\xi \in \mathcal{G}_p(R)} v(\xi)^2 W_\xi ,
$$

whence

$$\sum_{R \in \mathcal{R}} \|\nabla v\|_{0,R}^2 \lesssim b_1(v,v) .
$$

On the other hand, let $F \in \mathcal{F}_{d-1}$ be a face shared by two elements, say $R^\pm$, and let $p^\pm = p(F,R^\pm)$. We have

$$\|v\|_{p,F}^2 \lesssim \sum_+ \|v^+\|_{0,F}^2 \lesssim \sum_\pm \|v^\pm\|_{0,F,p^\pm}^2 = \sum_\pm \sum_{\xi \in \mathcal{G}_p(F,R^\pm)} (v^\pm(\xi))^2 W_{\xi} \pm .
$$

Multiplying by $\omega_F$ and using Lemma 7.1 for both $R = R^\pm$, we obtain

$$\omega_F \|v\|_{p,F}^2 \lesssim \sum_{\xi \in \mathcal{G}_p(F,R^\pm)} (v^\pm(\xi))^2 W_{\xi} \pm \lesssim \sum_\pm \sum_{\xi \in \mathcal{G}_p(F,R^\pm)} (v^\pm(\xi))^2 W_{\xi} \pm .
$$

A similar result holds for the faces $F \in \mathcal{F}_{d-1}$ sitting on the boundary of $\Omega$. Hence,

$$\sum_{F \in \mathcal{F}} \omega_F \|v\|_{p,F}^2 \lesssim b_1(v,v) ,
$$

which proves our claim.

To verify the remaining $\text{ASM}$ conditions we need to identify a suitable operator $\tilde{Q} : V_\delta \rightarrow V_\delta^\epsilon$. First recall that, on account of Assumption 2.5 (which indeed poses a restriction only for $d \geq 3$), we are for each $F \in \mathcal{F}_l$ with $1 \leq l \leq d$, entitled to select, once and for all, an element $R^\epsilon(F)$ such that

$$R^\epsilon(F) := \arg\min_{R' \in \mathcal{R}(F)} p(F,R'), \quad p^\epsilon(F) := p(F,R^\epsilon(F)) .
$$

(7.10)
The LGL grid $\mathcal{G}_p(F, R^l(F))$ will be denoted by $\mathcal{G}_{p^l}(F)$. Finally, for each vertex $v \in \mathcal{F}_0$ we select an arbitrary element $R^l(v)$.

Given $v \in \mathcal{V}_d$, the construction of $\tilde{Q}v =: \tilde{v}$ is based on the following simple idea. Ascending from lower to higher dimensional facets, first at each vertex $v$ of $\mathcal{R}$ we assign the value $\tilde{v}(z) := v_{R^l(z)}(z)$.

We now turn to the verification of (3.5) in Proposition 3.4. This will be accomplished through a recursive procedure.

**Definition 7.3.** For $0 \leq l \leq d$, set $F_l = \bigcup_{F \in \mathcal{F}_l} F$. Given any $v \in \mathcal{V}_d$, we define the sequence of piecewise polynomial functions $\tilde{q}_l : F_l \to \mathbb{R}$ by the following recursion:

i) For any $x \in \mathcal{F}_0$, set $\tilde{q}_0(x) = 0$ if $x \in \partial \Omega$ and $\tilde{q}_0(x) = v_{R^l(y)}(x)$ if $x \in \Omega$.

ii) For $l = 1, \ldots, d$, define $\tilde{q}_l$ on $\mathcal{F}_l$ as follows: for any $F \in \mathcal{F}_l$, set $\tilde{q}_l|_{\partial F} = 0$ if $F \subset \partial \Omega$, otherwise define $\tilde{q}_l|_F \in \mathbb{Q}_{p^l}(F)$ by the conditions

$$
\tilde{q}_l|_F(\xi) = v_{R^l(F)}(\xi) \quad \forall \xi \in \mathcal{G}_{p^l}(F) \setminus \partial_i F,
$$

$$
\tilde{q}_l|_F(\xi) = \tilde{q}_{l-1}(\xi) \quad \forall \xi \in \mathcal{G}_{p^l}(F) \cap \partial_i F.
$$

Finally, we set

$$
\tilde{Q}v = \tilde{q}_d.
$$

**Remark 7.4.** The above recursion defines a linear operator $\tilde{Q}$ from $\mathcal{V}_d$ into $\mathcal{V}_d$.

In fact, the linearity of $\tilde{Q}$ is obvious. Moreover, note that $\tilde{Q}v \in \mathcal{V}_d$ since $p^l(R) = p(R)$ for all $R \in \mathcal{R}$. Furthermore, one has the property

$$
\tilde{q}_l|_{F_{l-1}} = \tilde{q}_{l-1} \quad 1 \leq l \leq d.
$$

Indeed, if $F \in \mathcal{F}_l$ and $F' \in \mathcal{F}_{l-1}(F)$, then $\tilde{q}_l|_{F'} \in \mathbb{Q}_{p^l(F', R^l(F))}(F')$ whereas

$$
\tilde{q}_{l-1}|_{F'} \in \mathbb{Q}_{p^l(F', R^l(F'))}(F').
$$

The minimality property (7.10) applied to $F'$, together with the last set of conditions in (7.11), imply that $\tilde{q}_l|_{F'} = \tilde{q}_{l-1}|_{F'}$, whence (7.13) follows. This property implies that $\tilde{Q}v$ is continuous throughout $\Omega$, and vanishes on $\partial \Omega$. We conclude that $\tilde{Q}v$ belongs to $\mathcal{V}_d$.

We now turn to the verification of (3.5) in Proposition 3.4. This will be accomplished through a sequence of intermediate results.

**Lemma 7.5.** For any $v \in \mathcal{V}_d$, let its restriction to $R \in \mathcal{R}$ be denoted by $v_R$. The following bound holds

$$
b_1(v - \tilde{Q}v, v - \tilde{Q}v) \lesssim \mathcal{L} \sum_{F \in \mathcal{F}_{d-1}} \sum_{R \in \mathcal{R}(F)} \|v_R - \tilde{q}_{d-1}\|^2_{0,F}, \quad v \in \mathcal{V}_d,
$$

where $\tilde{q}_{d-1}$ is constructed in Definition 7.3 in order to define $\tilde{Q}v$.

**Proof.** By definition of $b_1$, one has

$$
b_1(v - \tilde{Q}v, v - \tilde{Q}v) = \sum_{R \in \mathcal{R}} \sum_{\xi \in \mathbb{G}_{p^l}(R)} |(v_R - \tilde{Q}v)(&\xi)|^2 c_\xi W_\xi.
$$
As for the first quantity, there exists a sequence $F$ proving the assertion. Let $R$ in order to define $\tilde{\eta}_{d-1}$, the following bound

$$b_1(v - \tilde{Q}v, v - \tilde{Q}v) \leq \sum_{R \in \mathcal{R}} \sum_{\xi \in \mathcal{G}_p(R) \cap \partial R} |(v_R - \tilde{q}_{d-1})(\xi)|^2 W_\xi$$

where we have used Lemma [7.1] in the last inequality. Finally, we observe that $v_{R|F} \in Q_{p(F,R)}(F)$, whereas $\tilde{q}_{d-1}|F \in Q_{p(1)}(F)$. Thus, (7.9) yields

$$\sum_{\xi \in \mathcal{G}_p(F,R)} |(v_R - \tilde{q}_{d-1})(\xi)|^2 w_\xi^{F,R} \simeq \|v_R - \tilde{q}_{d-1}\|_{0,F}^2$$

denoting the Lagrangian function associated with the node $\xi$ of the LGL grid on $F$, and, for simplicity, let us set $q = \tilde{q}_{d-1}$, the following bound

$$\sum_{\xi \in \mathcal{G}_p(F) \cap \partial F} \left|q_{\xi}(\xi) - q_{\xi}(\xi)\right|^2 w_\xi^{F,R} =: A^2 + B^2.$$
Thus, on the other hand, we have observe that Lemma 7.7 with 
Lemma 7.7. \( \text{Proposition 7.8.} \)

Lemma 7.7. Let \( F \in \mathcal{F}_l \) for some \( 0 \leq l \leq d - 2 \), and let \( G \in \mathcal{F}_{d-1}(F) \). Define \( p^* = \max_{R \in \mathcal{R}(G)} p(R) \) (componentwise). Then,

\[
(\mathcal{H}_G \mathcal{P}_G^{-2})^{d-l-1} \left\| v \right\|_{0,F}^2 \lesssim \left\| v \right\|_{0,G}^2 \quad \forall v \in \mathcal{Q}_G(G) .
\]

Proof. Using several times (7.3) and (7.6), as well as (7.9), we have

\[
\left\| v \right\|_{0,G}^2 \approx \sum_{\xi \in \mathcal{V}_p(G) \cap F} |v(\xi)|^2 w^G_\xi \geq \sum_{\xi \in \mathcal{V}_p(G) \cap F} |v(\xi)|^2 w^G_\xi \\
\approx (\mathcal{H}_G \mathcal{P}_G^{-2})^{d-l-1} \sum_{\xi \in \mathcal{V}_p(G)} |v(\xi)|^2 w^G_\xi \approx (\mathcal{H}_G \mathcal{P}_G^{-2})^{d-l-1} \left\| v \right\|_{0,F}^2 . \quad \square
\]

We are now ready to prove condition (3.5).

Proposition 7.8. For \( b_1 \) given in Definition 4.6 and for \( \bar{Q} \) given in Definition 7.3 one has

\[
b_1(v - \bar{Q}v, v - \bar{Q}v) \lesssim a_8(v, v) \quad \forall v \in V_b .
\]

Proof. First, we observe that the cardinality of any set \( \mathcal{F}_m(F) \) defined in (7.1) is bounded by a quantity depending only on the dimension \( d \). We start from the bound given by Lemma 7.5, and focus on any face \( F \in \mathcal{F}_{d-1} \). Using inequality (7.14) of Lemma 7.6 with \( l = d - 1 \), we get

\[
\sum_{R \in \mathcal{R}(F)} \left\| v_R - \bar{q}_{d-1} \right\|_{0,F}^2 \lesssim \left\| v \right\|_{0,F}^2 + \mathcal{H}_F \mathcal{P}_F^{-2} \mathcal{P}_{F'} \mathcal{P}_{F'} \mathcal{P}_{F'} \mathcal{P}_{F'} \left\| v_{R(F')} - \bar{q}_{d-2} \right\|_{0,F'}^2 .
\]

A further application of Lemma 7.6 to each summand on the right hand side of the above inequality, now with \( l = d - 2 \), yields taking into account (7.5)

\[
\sum_{R \in \mathcal{R}(F)} \left\| v_R - \bar{q}_{d-1} \right\|_{0,F}^2 \lesssim \left\| v \right\|_{0,F}^2 + \mathcal{H}_F \mathcal{P}_F^{-2} \mathcal{P}_{F'} \mathcal{P}_{F'} \mathcal{P}_{F'} \mathcal{P}_{F'} \left\| v_{R(F')} - \bar{q}_{d-2} \right\|_{0,F'}^2 .
\]

Lemma 7.7 with \( l = d - 2 \) and (7.5) yields \( \mathcal{H}_F \mathcal{P}_F^{-2} \left\| v \right\|_{0,F}^2 \lesssim \left\| v \right\|_{0,G}^2 . \) At this point, let us introduce the set \( \mathcal{F}_{d-1}(F) = \{ G \in \mathcal{F}_{d-1} : G \cap F \neq \emptyset \} \) of all faces intersecting the face \( F \), and let us observe that

\[
\mathcal{H}_F \mathcal{P}_F^{-2} \sum_{F' \in \mathcal{F}_{d-2}(F)} \sum_{G \in \mathcal{F}_{d-1}(F)} \left\| v \right\|_{G}^2 \left\| v \right\|_{0,F'}^2 \lesssim \sum_{F' \in \mathcal{F}_{d-2}(F)} \left\| v \right\|_{0,F'}^2 .
\]

On the other hand, we have

\[
\sum_{F' \in \mathcal{F}_{d-2}(F)} \sum_{G' \in \mathcal{F}_{d-3}(F')} \left\| v_{R(F')} - \bar{q}_{d-3} \right\|_{0,F'}^2 \lesssim \sum_{F'' \in \mathcal{F}_{d-3}(F')} \sum_{R \in \mathcal{R}(F'')} \left\| v_R - \bar{q}_{d-3} \right\|_{0,F''}^2 .
\]

Thus,

\[
\sum_{R \in \mathcal{R}(F)} \left\| v_R - \bar{q}_{d-1} \right\|_{0,F}^2 \lesssim \sum_{G \in \mathcal{F}_{d-1}(F)} \left\| v \right\|_{0,G}^2 + \mathcal{H}_F \mathcal{P}_F^{-2} \sum_{F' \in \mathcal{F}_{d-2}(F)} \sum_{R \in \mathcal{R}(F')} \left\| v_R - \bar{q}_{d-3} \right\|_{0,F''}^2 .
\]
We now proceed recursively, using Lemmas 7.6 and 7.7 with \( l = d - 3, d - 4, \ldots \). At the \( j \)-th stage of recursion, we obtain

\[
\sum_{R \in \mathcal{R}(F)} \|v_R - \tilde{q}_{d-j}\|_{0,F}^2 \lesssim \sum_{G \in \mathcal{F}_{d-j}^1(F)} \|v\|_{0,G}^2 + (H_F p_F^2)^{j-1} \sum_{F' \in \mathcal{F}_{d-j}(F)} \sum_{R \in \mathcal{R}(F')} \|v_R - \tilde{q}_{d-j}\|_{0,F'}^2.
\]

When \( j = d \), we use (7.15) and (7.6) to finally get

\[
\sum_{R \in \mathcal{R}(F)} \|v_R - \tilde{q}_{d-1}\|_{0,F}^2 \lesssim \sum_{G \in \mathcal{F}_{d-1}^1(F)} \|v\|_{0,G}^2.
\]

At last, we observe that \( \omega_G \simeq \omega_F \) for all \( G \in \mathcal{F}_{d-1}^1(F) \) by (7.5), so that, going back to Lemma 7.5, we conclude that

\[
b_1(v - \tilde{Q}v, v - \tilde{Q}v) \lesssim \sum_{G \in \mathcal{F}_{d-1}} \omega_G \|v\|_{0,G}^2 \leq a_6(v, v).
\]

\section{Proof of Theorem 5.6}

The proof of Theorem 5.6 follows again from Corollary 5.2 once we have verified the ASM conditions for the respective ingredients given in Section 5. Since now \( \tilde{V} = V_{h,D,p} \not\subset V = V_d^\circ \) we need to address ASM2-3 in full while ASM1 is trivial since all spaces are conforming and the standard energy bilinear form can be used.

We verify first ASM2.

\textbf{Proposition 8.1.} One has \( a(v, v) \lesssim b_2(v, v) \) for all \( v \in V \).

\textbf{Proof.} We first observe that \( a(v, v) \simeq a(v_h, v_h) \) for all \( v \in V_d^\circ \) due to Property 5.1. With the notation of Section 5.2, the idea is to bound the terms

\[
a_{R,k,S}(v_h, v_h) := \int_{S_k} |\partial x_k v_h|^2 \, dx = \int_{S_k} \left( \int_{I_k} |\partial x_k v_h(x_k, x')|^2 \, dx_k \right) \, dx'
\]

by using quadrature in all but the \( k \)-th variable while keeping the integral with respect to the \( k \)-th variable for \( S_k \in \mathcal{T}_p^{(0)}(R) \), the “anisotropic sub-cells”, and applying an inverse estimate for \( S_k \in \mathcal{T}_p^{(1)}(R) \), the “isotropic sub-cells”. Specifically, using the tensorized trapezoidal rule for integration over \( S_k \), which is the finite-element lumped mass matrix approximation (4.4.44) on p. 220, yields the terms \( b_{R,k,S}(v, v) \) in (5.17). Hence we still have \( a_{R,k,S}(v_h, v_h) \lesssim b_{R,k,S}(v, v) \) for all \( S_k \in \mathcal{T}_p^{(1)}(R) \). The same local relations hold for the sub-cells in \( \mathcal{T}_p^{(1)}(R) \) since we have used an inverse estimate as in the original version of \( b_1(\cdot, \cdot) \).

The remaining ASM conditions involve the operators \( Q \), defined by (5.25), and \( \tilde{Q} \) which is yet to be defined and is only needed for the analysis.

The definition of the operator \( \tilde{Q} \) follows the same lines as the one of \( Q \). Given any \( v \in V \) and any \( R \in \mathcal{R} \), we set \( v_R = v_{|R} \in Q_p(R) \). Then, the chain (5.23)-(5.24) is replaced by the following one:

\[
v_R^* := I_{P_1} (\Phi_z v_R) \in Q_p(R) \quad \text{and} \quad \tilde{v}_R^* := I_{h,D,p_1} (\tilde{I}_{h,D,p_1} R_{v_R^*}) \in V_{h,D,p_1}(R).
\]

Summing over the vertices of \( R \), we define

\[
R v_R := \sum_{z \in \mathcal{F}_0(R)} v_R^* \in Q_p(R) \quad \text{and} \quad \tilde{R} v_R := \tilde{v}_R^* = \sum_{z \in \mathcal{F}_0(R)} \tilde{v}_R^* \in V_{h,D,p_1}(R).
\]
As above, one easily confirms interelement continuity, which suggests defining the operator $\tilde{Q}: V \rightarrow \tilde{V}$ by
\[(\tilde{Q}v)_R := \tilde{Q}_{RV} = \tilde{v}_R^* \quad \forall R \in R \quad \forall v \in V , \quad (8.3)\]
where $\tilde{v}_R^*$ is defined in (8.2).

To complete the proof of Theorem 5.6 it remains to verify ASM3 which, in turn, requires establishing the following two results.

**Proposition 8.2.** The operators $Q$ and $\tilde{Q}$ are linear and satisfy the continuity assumption (3.2).

**Proposition 8.3.** The operator $Q$ and $\tilde{Q}$ satisfy the Jackson conditions (3.3), i.e., one has
\[b_2(\tilde{v} - Q\tilde{v}, v - \tilde{Q}v) \lesssim |v|^2_{1,\Omega} \quad \forall v \in V , \quad (8.3)\]
where the multiplicative constant in the above estimate depends on the constant $C_{\text{aspect}}$ in (5.15).

The remainder of this section is devoted to the proofs of Propositions 8.2 and 8.3 which require several further technical prerequisites.

### 8.1 The role of Property 5.1 and related facts

The following stability estimates draw in essential way on Property 5.1 and its univariate counterpart. A major issue is to interrelate grids of different orders.

We begin with a relevant property that concerns the locally uniform equivalence of LGL grids of comparable order and refer to [12] for the proof.

**Property 8.4.** Assume that $cp \leq q \leq p$ for some fixed constant $c > 0$. Let $I = [a, b]$. Then, $G_q(I)$ and $G_p(I)$ are locally $(A, B)$-uniformly equivalent, with $A$ and $B$ depending on the proportionality factor $c$ but independent of $q$, $p$ and $H = b - a$.

As a consequence of Properties 8.4 and 5.3 one obtains the following immediate extension to the associated dyadic grids.

**Corollary 8.5.** Assume that $cp \leq q \leq p$ for some fixed constant $c > 0$. Then, $D_q(I)$ is locally $(A, B)$-uniformly equivalent to both $G_p(I)$ and $D_p(I)$, with $A$ and $B$ depending on the proportionality factor $c$ but not on $p$ and $H$.

Next, recall that Property 5.1 follows from its univariate counterpart.

**Property 8.6.** One has
\[\|v\|_{0,I} \simeq \|v_h\|_{0,I}, \quad \|v'\|_{0,I} \simeq \|v_h'\|_{0,I} \quad \forall v \in P_p(I) , \quad (8.4)\]
where the involved constants are independent of $p$ and $H$.

Note that taking as $v$ in (8.4) each Lagrange basis function at the LGL nodes and using (4.2) and (5.7), it is easily seen that the size of each interval $I_j$ is comparable to that of the LGL weight associated with any of its endpoints, in the sense that the following bounds hold, uniformly in $p$ and $H$:
\[1 \lesssim \min_{1 \leq j \leq p} \frac{h_j}{w_j} \leq \max_{1 \leq j \leq p} \frac{h_j}{w_j} \lesssim 1 . \quad (8.5)\]
As a consequence, the second relation in (8.4), together with (8.5) and (4.2), provide a simple proof of the inverse inequality (4.4). Indeed, one has

\[ \|v_h\|_{0,I}^2 = \sum_{j=1}^{p} \left( \frac{v(\xi_j) - v(\xi_{j-1})}{h_j} \right)^2 h_j \leq \frac{2}{h_0} v^2(\xi_0) + \sum_{j=1}^{p-1} \left( \frac{2}{h_j} + \frac{2}{h_{j+1}} \right) v^2(\xi_j) + \frac{2}{h_p} v^2(\xi_p) . \]

We address next the continuity of various univariate interpolation operators in certain Sobolev norms or seminorms. The first result is classical.

**Lemma 8.7.** Let \( \mathcal{G} \) be any ordered grid in \( I \), which defines a partition \( \mathcal{T} = \mathcal{T}(\mathcal{G}) \), and let \( \mathcal{I}_G : H^1(I) \to V_h(\mathcal{T}) \) be the associated piecewise linear interpolation operator. Then,

\[ |\mathcal{I}_G v|_{1,I} \leq \|v\|_{1,I} \quad \forall v \in H^1(I) . \]

**Lemma 8.8.** Let \( \mathcal{G} \) and \( \mathcal{\bar{G}} \) be ordered grids in \( I \), with associated partitions \( \mathcal{T} = \mathcal{T}(\mathcal{G}) \) and \( \mathcal{\bar{T}} = \mathcal{T}(\mathcal{\bar{G}}) \). Assume that \( \mathcal{G} \) and \( \mathcal{\bar{G}} \) are locally \((A,B)\)-uniformly equivalent. If \( \mathcal{I}_G : H^1(I) \to V_h(\mathcal{T}) \) is the piecewise linear interpolation operator associated with \( \mathcal{G} \), one has

\[ \|\mathcal{I}_G v\|_{0,I} \lesssim \|v\|_{0,I} \quad \forall v \in V_h(\mathcal{T}) , \]

where the constant in the inequality depends only on the parameters \( A \) and \( B \).

**Proof.** Let \( \mathcal{G} = \{\xi_j : 0 \leq j \leq p\} \) and \( \mathcal{T} = \{I_j : 1 \leq j \leq p\} \), with \( h_j = |I_j| = \xi_j - \xi_{j-1} \). For \( 0 \leq j \leq p \), let us define \( h(\xi_j) = h_j + h_{j+1} \), where we set \( h_0 = h_{p+1} = 0 \). Similarly, let \( \mathcal{\bar{G}} = \{\eta_i : 0 \leq i \leq q\} \) and \( \mathcal{\bar{T}} = \{\bar{I}_i : 1 \leq i \leq q\} \) with \( \bar{h}_i = |\bar{I}_i| = \eta_i - \eta_{i-1} \), and let \( \bar{h}(\eta_i) \) be defined in a manner similar to \( h(\xi_j) \). Given any \( v \in V_h(\mathcal{T}) \), we have

\[ \|\mathcal{I}_G v\|_{0,I}^2 = \sum_{j=1}^{p} \|\mathcal{I}_G v\|_{0,I_j}^2 \lesssim \sum_{j=1}^{p} \left( v^2(\xi_j) + v^2(\xi_{j-1}) \right) h_j = \sum_{j=0}^{p} v^2(\xi_j) h(\xi_j) , \]

and

\[ \|v\|_{0,I}^2 = \sum_{i=1}^{q} \|v\|_{0,\bar{I}_i}^2 \lesssim \sum_{i=1}^{q} \left( v^2(\eta_i) + v^2(\eta_{i-1}) \right) \bar{h}_i = \sum_{i=0}^{q} v^2(\eta_i) \bar{h}(\eta_i) . \]

Now, for any \( \xi_j \) there exist \( \eta_i \) and \( \theta \in [0,1] \) such that \( v(\xi_j) = (1-\theta)v(\eta_i) + \theta v(\eta_{i+1}) \), whence \( v^2(\xi_j) \leq v^2(\eta_i) + v^2(\eta_{i+1}) \). If \( \theta \in (0,1) \), then \( \xi_j \in (\eta_i,\eta_{i+1}) \), hence both \( I_j \) and \( I_{j+1} \) intersect \( \bar{I}_i \). By the assumption of locally uniform equivalence of the two grids, we obtain \( |I_j| \lesssim |\bar{I}_i| \) and \( |I_{j+1}| \lesssim |\bar{I}_i| \), whence \( h(\xi_j) \lesssim \bar{h}_i \leq \min(\bar{h}(\eta_i),\bar{h}(\eta_{i+1})) \). On the other hand, if \( \theta = 0 \), i.e., \( \xi_j = \eta_i \), then \( I_j \) intersects \( \bar{I}_i \) and \( I_{j+1} \) intersects \( \bar{I}_{i+1} \) (with the obvious adaptation if \( \xi_j \) is a boundary point), thus \( |I_j| \lesssim |\bar{I}_i| \) and \( |I_{j+1}| \lesssim |\bar{I}_i| \), which yields \( h(\xi_j) \lesssim \bar{h}(\eta_i) \). This completes the proof.

To proceed recall the definitions of the interpolation operators \( \mathcal{I}_p, \mathcal{I}_{h,p}, \mathcal{I}_{h,D,p} \), in (5.2), (5.3), and (5.21), respectively.

**Lemma 8.9.** For any \( p > 0 \), the operator \( \mathcal{I}_p \) satisfies

\[ |\mathcal{I}_p v|_{1,I} \lesssim \|v\|_{1,I} \quad \forall v \in H^1(I) , \]

with a constant that does not depend \( \bar{a} \) on \( p \).

**Proof.** The result is classical (see, e.g., [14]). It can be derived from Property 8.6 and Lemma 8.7 observing that \( \mathcal{I}_p v = \mathcal{I}_p(\mathcal{I}_{h,p} v) \). 

\[ \square \]
Property 8.10. ([5, Remark 13.5]) Assume that $c p \leq q \leq p$ for some fixed constant $c > 0$. Then
\[ \| I_q v \|_{0, I} \lesssim \| v \|_{0, I} \quad \forall v \in P_p(I), \]
with a constant depending on the proportionality factor $c$ but not on $p$.

Lemma 8.11. Assume that $c p \leq q \leq p$ for some fixed constant $c > 0$. Then,
\[ |I_{h,q} v|_{m,I} \lesssim |v|_{m,I} \quad \forall v \in P_p(I), \quad m = 0, 1, \]
with a constant depending on the proportionality factor $c$ but not on $p$.

Proof. For $m = 0$, we observe that $I_{h,q} v = I_{h,q} (I_q v)$, so that $\| I_{h,q} v \|_{0, I} \lesssim \| I_q v \|_{0, I} \lesssim \| v \|_{0, I}$ by Properties 8.6 and 8.10. The result for $m = 1$ is included in Lemma 8.7.

Lemma 8.12. Assume that $c p \leq q \leq p$ for some fixed constant $c > 0$. Then, for $m = 0, 1$ one has
\[ |I_q v|_{m,I} \lesssim |I_{h,q} v|_{m,I} \lesssim |v|_{m,I} \quad \forall v \in V_{h,D,p}(I), \]
\[ |I_{h,D,q} v|_{m,I} \lesssim |v|_{m,I} \quad \forall v \in V_{h,D,p}(I), \]
\[ |I_{h,D,q} v|_{m,I} \lesssim |v|_{m,I} \quad \forall v \in V_{h,p}(I). \]

Proof. The results for $m = 0$ follow from Lemma 8.8 applied in various combinations to the grids $G_q(I), G_p(I), D_q(I)$ and $D_p(I)$, that are locally uniformly equivalent to each other by Corollary 8.5. The results for $m = 1$ follow again from Lemma 8.7.

We turn now to the multivariate case. Using Lemmas 8.7 and 8.8, Proposition 5.2 yields the following general result.

Property 8.13. For $1 \leq k \leq d$, let $G_k$ and $\tilde{G}_k$ be ordered grids in $I_k$, with associated partitions $\mathcal{T}_k$ and $\tilde{\mathcal{T}}_k$, which are locally $(A,B)$-uniformly equivalent; let $I_{G_k} = I_{G_k}^b : H^1(I_k) \to V_h(\mathcal{T}_k)$ be the piecewise linear interpolation operator associated with $\mathcal{T}_k$. Consider the spaces $V_h(\mathcal{T}) := \bigotimes_{k=1}^d V_h(\mathcal{T}_k)$ and $V_h(\tilde{\mathcal{T}}) := \bigotimes_{k=1}^d V_h(\tilde{\mathcal{T}}_k)$ of piecewise multi-linear functions on the Cartesian partitions $\mathcal{T} := \bigotimes_{k=1}^d \mathcal{T}_k$ and $\tilde{\mathcal{T}} := \bigotimes_{k=1}^d \tilde{\mathcal{T}}_k$ of $R$. Then, the piecewise multilinear interpolation operator $I_V = I_V^b := \bigotimes_{k=1}^d I_{G_k}^b : C^0(R) \to V_h(\mathcal{T})$ satisfies
\[ \| I_V v \|_{m,R} \lesssim \| v \|_{m,R} \quad \forall v \in V_h(\tilde{\mathcal{T}}), \quad m = 0, 1, \]
where the constant in the inequality depends only on the parameters $A$ and $B$.

From Lemma 8.12 and Proposition 5.2, we immediately get the following multidimensional result.

Property 8.14. Assume that $c \mathbf{p} \leq \mathbf{q} \leq \mathbf{p}$ for some fixed constant $c > 0$. Then,
\[ |I_q v|_{m,R} \lesssim |v|_{m,R} \quad \forall v \in V_{h,D,p}(R), \quad m = 0, 1, \]
with a constant depending on the proportionality factor $c$ but not on $\mathbf{p}$.
We are now prepared to complete the

Proof of Proposition 8.2. We treat only the operator $Q$. The argument for $\tilde{Q}$ is analogous. We have to prove that $|Q\tilde{v}|_{1,R} \lesssim |\tilde{v}|_{1,R}$ for all $\tilde{v} \in V$. This follows if, for any $R \in \mathcal{R}$, we prove that

$$|v^*_R|_{1,R} \lesssim |\tilde{v}_R|_{1,R} \quad \forall \tilde{v}_R \in V_{h,D,p}(R).$$

A classical mapping-and-scaling argument in Finite Element analysis tells us that this result holds provided it holds when $R$ is the reference element $\tilde{R} = \times_{k=1}^d \tilde{I}$, with $\tilde{I} = [-1,1]$. For this element, it is enough to prove that

$$|v^*|_{1,\tilde{R}} \lesssim \|\tilde{v}\|_{1,\tilde{R}} \quad \forall \tilde{v} \in V_{h,D,p}(\tilde{R}).$$

Indeed, changing $\tilde{v}$ into $\tilde{v} + \lambda$, by any $\lambda \in \mathbb{R}$, does not change the left-hand side, whence $|v^*|_{1,\tilde{R}} \lesssim \inf_{\lambda \in \mathbb{R}} \|\tilde{v} + \lambda\|_{1,\tilde{R}} \lesssim |\tilde{v}|_{1,\tilde{R}}$. In order to establish (8.7), let us first consider a univariate function $\tilde{v} \in V_{h,D,p}(\tilde{I})$ and let $\Phi$ denote the affine function taking the value 1 at one endpoint of the interval and 0 at the other one. Let us prove that if $cp \leq q \leq p$ for some fixed constant $c > 0$, one has

$$\|I_{h,D,q}(\Phi \tilde{v})\|_{m,\tilde{I}} \lesssim \|\tilde{v}\|_{m,\tilde{I}}, \quad m = 0, 1,$$

where, of course, the involved constant depends on the proportionality factor $c$. For $m = 0$, we have

$$\|I_{h,D,q}(\Phi \tilde{v})\|_{0,\tilde{I}} \lesssim \sum_{\zeta \in \mathcal{G}_{D,q}(\tilde{I})} (\Phi(\zeta)\tilde{v}(\zeta))^2 h_{D,q}(\zeta) \lesssim \sum_{\eta \in \mathcal{G}_{D,p}(\tilde{I})} \tilde{v}(\eta)^2 h_{D,p}(\eta) \lesssim \|\tilde{v}\|_{0,\tilde{I}}^2,$$

since $\Phi^2 \leq 1$ and $h_{D,q}(\zeta) \leq h_{D,p}(\zeta)$ for all $\zeta \in \mathcal{G}_q(\tilde{I}) \subseteq \mathcal{G}_p(\tilde{I})$. For $m = 1$, we have by Lemma 8.7, $|I_{h,D,q}(\Phi \tilde{v})|_{1,\tilde{I}} \lesssim |\Phi \tilde{v}|_{1,\tilde{I}} \lesssim \|\tilde{v}\|_{1,\tilde{I}}$, where the last inequality holds since we are working on the reference element. Hence, (8.8) is proven. Using this result and Proposition 5.2, we obtain the bound $|v^*_1|_{1,\tilde{R}} \lesssim \|\tilde{v}\|_{1,\tilde{R}}$ for each function $\tilde{v}$ defined as in (5.23) on $\tilde{R}$. Then, Property 8.14 yields $|v^*_1|_{1,\tilde{R}} \lesssim \|\tilde{v}\|_{1,\tilde{R}}$, and (8.7) follows by the triangle inequality, since $|F_0(\tilde{R})| \simeq 1$. \hfill \Box

Remark 8.15. Consider the operator $Q : \tilde{V} = W_{h,D,p} \to V = V_{h,D,p}$ introduced in Section 5.4.2 whose $H^1$-stability is claimed in [5.34] in Lemma 5.7. Since this operator is defined in complete analogy to (5.25), the proof of its stability is similar to that of Proposition 8.2 given above. Indeed, the grids underlying the wavelets spanning the spaces $W_{h,D,p_k}(I_k)$ are locally $(A,B)$-uniformly equivalent to the dyadic grids $D_{p_k}(I_k)$. Thus the result is a consequence of Property 8.13.

8.2 A localized Jackson estimate for the interpolation error

To prove Proposition 8.3 requires the following different types of estimate which are not covered by the results of the preceding section. Since these results will be applied to both the LGL and the dyadic tensorized grids, with various choices of finite-dimensional function spaces (comprised of either piecewise multi-linear or global polynomial functions), we first establish the key estimates in suitable generality in order to specialize them later to the cases at hand.

Consider again the general piecewise multilinear interpolation operator $I_G = I^D_0 : C^0(\bar{R}) \to V_h(\bar{T})$ introduced in the statement of Property 8.13 above. In addition, assume that each grid $G_k$ is locally quasiuniform according to [5.7]. For $k = 1, \ldots, d$, let $W_k$ be a finite dimensional subspace of $H^1(I_k)$ to be specified later. Then, the univariate piecewise linear interpolation operator $I_{G_k} = I^1_{G_k}$ is well-defined on $W_k$ and we have

$$\|I_{G_k}v\|_{0,I_k} \leq \tilde{c}_k\|v\|_{0,I_k} \quad \forall v \in W_k,$$

for some constant $\tilde{c}_k > 0$ independent of the size $|I_k|$ but possibly depending on the dimension of $W_k$ (although this will not be the case in all our applications). Let us set $W = \bigotimes_{k=1}^d W_k$. 

Next, consider the cells $S_k$ forming the partition $\mathcal{T} = \times_{k=1}^d T_k$, i.e., $\mathcal{T} = \{S_k = \times_{k=1}^d I_{k,\ell_k} : I_{k,\ell_k} \in T_k\}$, and let $h_k := \max \{\ell_k, \ell_k\}$ be the largest one-dimensional size of the cell $S_k$. Let $h = \sum_{k} h_k \chi_S$ be the meshsize function defined in $R$.

The following localized Jackson estimate will be used several times in the sequel.

**Proposition 8.16.** The following estimate holds

$$\|h^{-1}(v - I_{\mathcal{G}}v)\|_{0,R} \lesssim |v|_{1,R} \quad \forall v \in W,$$

where the constant implied by the inequality is independent of the meshsize but depends on the constants $c_k$ introduced in (8.9).

**Proof.** The result will be obtained by assembling local estimates in each cell $S_k$, which in turn are derived by a scaling argument from corresponding estimates on the unit box $B^d = [0, 1]^d$, with $B = [0, 1]$. To this end, let $I_{B^k}$ denote the multilinear interpolation operator on $B^k$, i.e.,

$$I_{B^k}v = \sum_{\xi \in \mathcal{F}_0(B^k)} v(\xi)\Phi_{\xi},$$

where $\Phi_{\xi}$ denotes the multilinear Lagrange basis function satisfying $\Phi_{\xi}(\xi') = \delta_{\xi,\xi'}$ for $\xi, \xi' \in \mathcal{F}_0(B^k)$. We make heavy use of the fact that $I_{B^k}$ is a tensor product operator, i.e., we have $I_{B^k} = \otimes^d I_{B}$. Moreover, it will be convenient to employ the following convention to write, for any $k = 1, \ldots, d$,

$$B^d = A^k \times B^{d-k},$$

meaning that $A^k$ is the unit $k$-cube representing the first $k$ variables and $B^{d-k}$ is the unit $(d-k)$-cube for the coordinates $d-k+1, \ldots, d$.

**Lemma 8.17.** Let $W = \bigotimes_{k=1}^d W_k$, where $W_k$ are finite-dimensional subspaces of $H^1(B)$. Then, one has

$$\|v - I_{B^k}v\|_{0,B^d}^2 \lesssim \sum_{k=1}^d \sum_{\xi \in \mathcal{G}(A^{k-1})} \|\partial_{x_k} v(\xi, \cdot)\|_{0,B^{d-k+1}}^2, \quad \forall v \in W,$$

(with the first summand on the right-hand side to be understood as $\|\partial_{x_k} v\|_{0,B^d}^2$), where the constant depends only on $d$.

**Proof.** Denoting by $\text{id}$ the identity operator on $B$, and writing

$$v - I_{B^k}v = \sum_{k=1}^d \left( (I_{B^{k-1}} \otimes \text{id}^{d-k+1})v - (I_{B^k} \otimes \text{id}^{d-k})v \right) = \sum_{k=1}^d (\text{id}^{k-1} \otimes (\text{id} - I_B) \otimes \text{id}^{d-k})(I_{B^{k-1}} \otimes \text{id}^{d-k+1})v,$$

we note that, when abbreviating $w_k := (I_{B^{k-1}} \otimes \text{id}^{d-k+1})v$, for $x_k' := (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_d)$ one has

$$(\text{id}^{k-1} \otimes (\text{id} - I_B) \otimes \text{id}^{d-k})w_k(x_1, \ldots, x_{k-1}, \xi, x_{k+1}, \ldots, x_d) = 0, \quad \xi_k = 0, 1, \quad x_k' \in A^{k-1} \times B^{d-k}.$$ 

Hence, we can apply the classical univariate inequality $\|v - I_Bv\|_{0,B} \leq 2\|v\|_{1,B}$, for all $v \in H^1(B)$, to get

$$\|\text{id}^{k-1} \otimes (\text{id} - I_B) \otimes \text{id}^{d-k})w_k\|_{0,B^d} \leq 2\|\partial_{x_k} w_k\|_{0,B^d}.$$
For $k \geq 2$, notice that $w_k(\zeta, \cdot) = v(\xi, \cdot), \xi \in \mathcal{F}_0(A^{k-1})$ and that $w_k$ is affine in the first $k - 1$ variables. Hence, we conclude that
\[
\|\partial_{x_k} w_k\|_{0,B^d}^2 \lesssim \sum_{\xi \in \mathcal{F}_0(A^{k-1})} \|\partial_{x_k} v(\xi, \cdot)\|^2_{0,B^{d-k+1}},
\]
from which the assertion of Lemma 8.17 easily follows.

To complete the proof of Proposition 8.16, consider now a generic cell $S_\ell \in \mathcal{T}$, which we can write as
\[
S_\ell = (\bigotimes_{l=1}^k I_{l,\ell_l}) \times (\bigotimes_{l=k+1}^d I_{l,\ell_l}) =: A_\ell^k \times B_{\ell}^{d-k}.
\]
Given any $v \in \mathcal{W}$ and considering its restriction to $S_\ell$, we write $\hat{v}(\hat{x}) = v(x)$ with $\hat{x} \in B^d$, so that $(\mathcal{I}_{S_\ell} v)(x) = (\mathcal{I}_{B^d} \hat{v})(\hat{x})$; setting $h_{l,\ell_l} := |I_{l,\ell_l}|$, a standard affine change of variables yields in view of (8.11),
\[
\|v - \mathcal{I}_{S_\ell} v\|_{0,0,S_\ell}^2 \leq |S_\ell| \sum_{k=1}^d \sum_{\xi \in \mathcal{F}_0(A^{k-1})} \|\partial_{x_k} \hat{v}(\xi, \cdot)\|^2_{0,B^{d-k+1}} \lesssim |S_\ell| \sum_{k=1}^d \sum_{\xi \in \mathcal{F}_0(A^{k-1})} |B_{\ell}^{d-k+1}| |\partial_{x_k} v(\xi, \cdot)\|^2_{0,B^{d-k+1}} \lesssim h_\ell^2 \sum_{k=1}^d \sum_{\xi \in \mathcal{F}_0(A^{k-1})} |A_\ell^{k-1}| \|\partial_{x_k} v(\xi, \cdot)\|^2_{0,B^{d-k+1}}.
\]
Dividing both sides by $h_\ell^2$ and summing over $\ell$ provides
\[
\|h^{-1}(v - \mathcal{I}_{\mathcal{G}} v)\|_{0,R}^2 \lesssim \sum_{k=1}^d \sum_{S_{\ell_l} \in \mathcal{T}} \sum_{\xi \in \mathcal{F}_0(A^{k-1})} |A_\ell^{k-1}| \|\partial_{x_k} v(\xi, \cdot)\|^2_{0,B^{d-k+1}}.
\]
Now, $|A_\ell^{k-1}| = \prod_{l=1}^{k-1} h_{l,\ell_l} \simeq \prod_{l=1}^{k-1} h_{l,\ell_l}$ for each $\xi \in \mathcal{F}_0(A^{k-1})$, where the weights $w_{l,\ell_l}$ are defined by the conditions $\sum_{\xi \in \mathcal{F}_0} v^2(\xi) w_{l,\ell} = ||\mathcal{I}_{\mathcal{G}} v||_{0,R}^2$ for all $v \in C^0(I_l)$. Then, the assertion follows from (8.9).

8.3 Proof of Proposition 8.3

Again, we treat only the first relation. The second one is analogous. For each $R \in \mathcal{R}$ and each $k = 1, \ldots, d$, let us recall the definitions (5.16), (5.17) and (5.19) of the form $b_{R,k}(u,v)$ and its portions $b^{(0)}_{R,k}(u,v)$ and $b^{(1)}_{R,k}(u,v)$. Concerning the former portion, observe that
\[
b^{(0)}_{R,k}(v,v) = \sum_{S_{\ell_l} \in \mathcal{T}_{p,k}(R)} \omega_{\ell,k}^{\ell} \int_{I_{k,\ell_k}} \|\partial_{x_k} \mathcal{I}_R^{\ell} v(\xi',x_k)\|^2 \, dx_k \lesssim \sum_{S_{\ell_l} \in \mathcal{T}_{p,k}(R)} \int_{S_{\ell_l}} \|\partial_{x_k} ^{\mathcal{I}_R^{\ell}} v(x)\|^2 \, dx \lesssim |\mathcal{I}_R^{\ell} v|_{1,R}^2,
\]
where we have used the uniform equivalence of the weights $\omega_{\ell,k}^{\ell}$ (see (5.18)) with the integration weights for multi-linear functions on the cell $S_{\ell,k}$. Thus, for all $\tilde{v}_R \in V_{h,D,p}(R)$, we have
\[
b^{(0)}_{R,k}(\tilde{v}_R - Q_R \tilde{v}_R, \tilde{v}_R - Q_R \tilde{v}_R) \lesssim |\mathcal{I}_R^{\ell} v - Q_R \mathcal{I}_R^{\ell} v|_{1,R}^2 \lesssim |\mathcal{I}_R^{\ell} v|_{1,R}^2 + |\mathcal{I}_R^{\ell} Q_R \mathcal{I}_R^{\ell} v|_{1,R}^2 \lesssim |\mathcal{I}_R^{\ell} v|_{1,R}^2,
\]
for $R \in \mathcal{R}$ and each $k = 1, \ldots, d$. The proof of the latter relation is similar.
where the last bound follows immediately from Lemmas 8.8 and 8.13 Property 8.11 and Proposition 5.22.

Consider now the portion $b^{(1)}_{R,k}(u, v)$. Arguing as above, one has

$$b^{(1)}_{R,k}(v, v) = \sum_{s_t \in T^{(1)}_{p,k}(R)} \sum_{t_s} \omega_{\ell,k} h^{-1}_{k,\ell} |T^R_{h,p}v(\xi', \xi)|^2 \int_{s_t} h^{-2}_{k,\ell} |T^R_{h,p}v(x)|^2 dx .$$

Now observe that, by definition of $T^{(1)}_{p,k}(R)$ (see (5.15)), one has $h_{k,\ell} \geq \max(1, C_\text{aspect}) h_\ell$, with $h_\ell = \max_t h_{t,\ell}$. Hence,

$$b^{(1)}_{R,k}(v, v) \lesssim \sum_{s_t \in T^{(1)}_{p,k}(R)} \int_{s_t} h^{-2}_{k,\ell} |T^R_{h,p}v(x)|^2 dx \lesssim \sum_{s_t \in T^{(1)}_{p,k}(R)} \int_{s_t} h^{-2}_{k,\ell} |T^R_{h,p}v(x)|^2 dx \lesssim (\|h^{-1} T^R_{h,p}v\|_{0,R}^2 + \|h^{-1} v\|_{0,R}^2),$$

where $h = \sum_t h_{t,\ell} s_t$ is the LGL meshsize function in $R$. If $\bar{v}_R \in V_{h,D,p}(R)$, this yields

$$b^{(1)}_{R,k}(\bar{v}_R - Q_R \bar{v}_R, \bar{v}_R - Q_R \bar{v}_R) \lesssim \|h^{-1}(\bar{v}_R - T^R_{h,p} \bar{v}_R)\|_{0,R}^2 + \|h^{-1}(Q_R \bar{v}_R - T^R_{h,p} (Q_R \bar{v}_R))\|_{0,R}^2 \quad (8.14)$$

Now we invoke Proposition 8.16 with different choices of the grid $G$ and the space $W$, to bound each of the three summands on the right-hand side. For the first summand, we have $G = G_p(R)$ (the LGL grid of order $p$ in $R$) and $W = V_{h,D,p}(R)$. We note that, due to Lemma 8.8 the bounds 8.9 are satisfied with $c_k \lesssim 1$. Thus we get

$$\|h^{-1}(\bar{v}_R - T^R_{h,p} \bar{v}_R)\|_{0,R}^2 \lesssim \|\bar{v}_R\|_{1,R}^2. \quad (8.15)$$  

For the second summand, we recall that $Q_R \bar{v}_R \in Q_p(R)$, so that we have again $G = G_p(R)$, whereas now $W = Q_p(R)$. The bounds 8.9 are now satisfied with $c_k \lesssim 1$ because of Lemma 8.11. Thus, recalling 8.6, we obtain

$$\|h^{-1}(Q_R \bar{v}_R - T^R_{h,p} (Q_R \bar{v}_R))\|_{0,R}^2 \lesssim \|Q_R \bar{v}_R\|_{1,R}^2 \quad (8.16)$$

At last, we bound the third summand on the right-hand side of (8.14). To this end, recalling the definition (5.24) of $Q_R \bar{v}_R$, noting that $\bar{v}_R = \sum_{z \in \mathcal{F}_0(R)} \Phi_z \bar{v}_R$, and defining $\tilde{v}_z := \Phi_z \bar{v}_R$, it suffices to bound the quantity $C^2_z := \|h^{-1}(\bar{v}_z - T^R_{p} (T^R_{h,D,pz} \bar{v}_z))\|_{0,R}^2$ for each $z \in \mathcal{F}_0(R)$. Writing

$$\bar{v}_z - T^R_{p} (T^R_{h,D,pz} \bar{v}_z) = (\bar{v}_z - T^R_{p} (T^R_{h,D,pz} \bar{v}_z)) + (T^R_{h,D,pz} \tilde{v}_z - T^R_{p} (T^R_{h,D,pz} \tilde{v}_z)),$$

and setting for simplicity $\tilde{w}_z := T^R_{h,D,pz} \tilde{v}_z \in V_{h,D,p}(R)$, we thus have

$$C^2_z \lesssim \|h^{-1}(\bar{v}_z - T^R_{h,D,pz} \bar{v}_z))\|_{0,R}^2 + \|h^{-1}(\tilde{w}_z - T^R_{p} \tilde{w}_z))\|_{0,R}^2 \quad (8.17)$$

We now proceed as in the proof of Proposition 8.2, i.e., we work on the reference element $\hat{R}$ viewed as an affine image of $R$. This simplifies handling the factor $\Phi_z$ when eventually bounding the $H^1$-seminorm of $\tilde{v}_z = \Phi_z \bar{v}_R$ by that of $\bar{v}_R$ on the element $R$.

We want to apply Proposition 8.16 to the first summand on the right-hand side of (8.17), with $G = D_p^\ast(\hat{R})$ (the dyadic grid of order $p^\ast$ in $\hat{R}$) and $W = \Phi \in V_{h,D,p^\ast}(\hat{R})$. To this end, we observe that the meshsize function $h = h_0$ associated with the grid $G_p(\hat{R})$, as a consequence of Corollary 8.5 is uniformly comparable to the meshsize function $h_{D,p^\ast}$ associated with the grid $D_p^\ast(\hat{R})$, i.e., $h_0 \approx h_{D,p^\ast}$ pointwise in $\hat{R}$. On the other hand, the bounds 8.9 are satisfied with $c_k \lesssim 1$; this easily follows
from the fact that the restriction of $\tilde{v}_z$ to any cell of the grid $D_{p^*}(\hat{R})$ is a piecewise multi-quadratic function belonging to a finite dimensional space whose dimension is bounded independently of $p$. Thus, we obtain

$$\|h^{-1}(\tilde{v}_z - \mathcal{I}_{h,D,p^*}^R \tilde{v}_z)\|_{0,\hat{R}}^2 \lesssim |\tilde{v}_z|_{1,\hat{R}}^2 \lesssim \|\tilde{v}_R\|_{1,\hat{R}}^2 .$$

(8.18)

The second summand on the right-hand side of (8.17) can be bounded with the aid of Proposition 8.16 as well. Indeed, using the property that $\mathcal{I}_{h,q}(\mathcal{I}_q v) = \mathcal{I}_{h,q} v$ if $\mathcal{I}_{h,q} v$ and $\mathcal{I}_q v$ are the low- and high-order interpolants of a continuous function on the same grid, one has

$$v - \mathcal{I}_q v = v - \mathcal{I}_{h,q}(\mathcal{I}_q v) + \mathcal{I}_{h,q}(\mathcal{I}_q v) - \mathcal{I}_q v = (v - \mathcal{I}_{h,q} v) - (\mathcal{I}_q v - \mathcal{I}_{h,q}(\mathcal{I}_q v)) .$$

In our situation, this yields with $\tilde{u}_z = \mathcal{I}_{p^*}^R \tilde{w}_z \in \mathcal{Q}_{p^*}(\hat{R})$

$$\|h^{-1}(\tilde{w}_z - \mathcal{I}_{p^*}^R \tilde{w}_z)\|_{0,\hat{R}}^2 \lesssim \|h^{-1}(\tilde{w}_z - \mathcal{I}_{p^*}^R \tilde{w}_z)\|_{0,\hat{R}}^2 + \|h^{-1}(\tilde{u}_z - \mathcal{I}_{p^*}^R \tilde{u}_z)\|_{0,\hat{R}}^2 .$$

so that we can apply Proposition 8.16 with $\mathcal{G} = \mathcal{G}_{p^*}(\hat{R})$ and either $W = V_{h,D,p}(\hat{R})$ or $W = \mathcal{Q}_{p^*}(\hat{R})$. Again, the function $h_p$ is uniformly comparable to the function $h_{p^*}$ associated with the grid $\mathcal{G}_{p^*}(\hat{R})$, and one easily checks that the bounds (8.9) are satisfied with $\tilde{c}_k \lesssim 1$ with both choices of $W$. Thus,

$$\|h^{-1}(\tilde{w}_z - \mathcal{I}_{p^*}^R \tilde{w}_z)\|_{0,\hat{R}}^2 \lesssim |\tilde{w}_z|_{1,\hat{R}}^2 \lesssim |\tilde{v}_z|_{1,\hat{R}}^2 \lesssim \|\tilde{v}_R\|_{1,\hat{R}}^2 ,$$

(8.19)

where the second inequality follows from Lemma 8.8 and Property 8.13. Going back to the element $R$, the bounds (8.17), (8.18) and (8.19) imply $C_z \lesssim |\tilde{v}_R|_{1,\hat{R}}^2$. This, together with (8.15) and (8.16), allows us to obtain $v^{(1)}_{R,k}(\tilde{v}_R - Q_R \tilde{v}_R, \tilde{v}_R - Q_R \tilde{v}_R) \lesssim |\tilde{v}_R|_{1,\hat{R}}^2$ from (8.14). Thus, the proof of Proposition 8.3 is complete. \(\square\)

**Remark 8.18.** Consider the operator $Q : \tilde{V} = W_{h,D,p} \rightarrow V = V_{h,D,p}$ and the auxiliary bilinear form $b_3$ introduced in Section 5.4.3, where condition ASM 3 is claimed to hold in (5.35) of Lemma 5.7. Since both objects are defined in complete analogy to (5.25), the proof of this property is similar to that of Proposition 8.3 given above.

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