A Hilbert-type integral inequality in the whole plane related to the kernel of exponent function

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Abstract

By using real analysis and weight functions, we obtain a few equivalent statements of a Hilbert-type integral inequality in the whole plane related to the kernel of exponent function with intermediate variables. The constant factor related to the gamma function is proved to be the best possible. We also consider some particular cases and the operator expressions.

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1 Introduction

If \( 0 < \int_{0}^{\infty} f^2(x) \, dx < \infty \) and \( 0 < \int_{0}^{\infty} g^2(y) \, dy < \infty \), then we have the following well-known Hilbert integral inequality (see [1]):

\[
\int_{0}^{\infty} \int_{0}^{\infty} \frac{f(x)g(y)}{x+y} \, dx \, dy < \pi \left( \int_{0}^{\infty} f^2(x) \, dx \int_{0}^{\infty} g^2(y) \, dy \right)^{\frac{1}{2}},
\]

where the constant factor \( \pi \) is the best possible. In 1925, by introducing the pair of conjugate exponents \((p, q) \ (p > 1, \frac{1}{p} + \frac{1}{q} = 1)\), Hardy et al. gave an extension of (1) (see [1], Theorem 316). Recently, by means of weight functions, some new extensions of (1) and the Hardy’s work were given by Yang [2, 3] and in [4–9]. Most of them are built in the quarter plane of the first quadrant.

In 2007, Yang [10] provided a Hilbert-type integral inequality in the whole plane with the exponent function and intermediate variables as follows:

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f(x)g(y)}{1 + e^{\lambda x + \gamma y}} \, dx \, dy < B \left( \frac{\lambda}{2}, \frac{\gamma}{2} \right) \left( \int_{-\infty}^{\infty} e^{-\lambda x} f^2(x) \, dx \int_{-\infty}^{\infty} e^{-\gamma y} g^2(y) \, dy \right)^{\frac{1}{2}},
\]

where the constant factor \( B(\frac{\lambda}{2}, \frac{\gamma}{2}) \) is the best possible \((\lambda > 0, B(u, v) \) is the beta function). He et al. [11–19] proved some new Hilbert-type integral inequalities in the whole plane with the best possible constant factors.
In 2017, Hong [20] gave two equivalent statements between Hilbert-type inequalities with general homogenous kernel and a few parameters. A few authors continue to study this topic (see [21–25]).

In this paper, by using real analysis and weight functions we obtain a few equivalent statements of a Hilbert-type integral inequality in the whole plane related to the exponent function with intermediate variables. The constant factor related to the gamma function is proved to be the best possible. We also consider some particular cases and operator expressions.

2 Some lemmas

For \( \gamma, \rho, \sigma > 0 \), setting \( h(u) := e^{-\rho u^\gamma} \) \((u > 0)\), we find

\[
k'_\rho(\sigma) := \int_0^\infty h(u)u^{\sigma-1} \, du = \int_0^\infty e^{-\rho u^\gamma} u^{\sigma-1} \, du \quad (\nu = \rho u^\gamma)
\]

\[
= \frac{1}{\gamma \rho^{\sigma/\gamma}} \int_0^\infty e^{-\nu^{1/\gamma}} \frac{\Gamma(\sigma/\gamma)}{\nu^{\sigma/\gamma}} \, d\nu = \frac{\Gamma(\sigma/\gamma)}{\gamma \rho^{\sigma/\gamma}} \in \mathbb{R}_+ = (0, \infty),
\]

(3)

where \( \Gamma(s) := \int_0^\infty e^{-v v s} \, dv \) \((\text{Re} s > 0)\) is the gamma function (see [26]).

For \( \delta \in \{-1, 1\}, \alpha, \beta \in (-1, 1) \), we set

\[
x_\alpha := |x| + \alpha x, \quad y_\beta := |y| + \beta y \quad (x, y \in \mathbb{R} = (-\infty, \infty)),
\]

\[
E_\delta := \{t \in \mathbb{R}; |t|^\delta \geq 1\}, \quad E_{-\delta} := \{t \in \mathbb{R}; |t|^\delta \leq 1\}.
\]

**Lemma 1** For \( c > 0 \), \( \theta = \alpha, \beta \in (-1, 1) \), we have

\[
\int_{E_\delta} t_\theta^{c-1} \, dt = \frac{1}{c} \left[ \frac{1}{(1 + \theta)^{c+1}} + \frac{1}{(1 - \theta)^{c+1}} \right],
\]

(4)

\[
\int_{E_{-\delta}} t_\theta^{c-1} \, dt = \frac{1}{c} \left[ \frac{1}{(1 + \theta)^{-c+1}} + \frac{1}{(1 - \theta)^{-c+1}} \right];
\]

(5)

and for \( c \leq 0 \), we have

\[
\int_{E_\delta} t_\theta^{c-1} \, dt = \int_{E_{-\delta}} t_\theta^{c-1} \, dt = \infty.
\]

**Proof** Setting \( E_3^+ := \{t \in \mathbb{R}; t^\delta \geq 1\}, \ E_3^- := \{-t \in \mathbb{R}; (-t)^\delta \geq 1\} \), we find \( E_\delta = E_3^+ \cup E_3^- \) and

\[
\int_{E_\delta} t_\theta^{c-1} \, dt = \int_{E_3^+} \left[ (1 + \theta) t \right]^{c-1} \, dt + \int_{E_3^-} \left[ (1 - \theta)(-t) \right]^{c-1} \, dt
\]

\[
= \left[ \frac{1}{(1 + \theta)^{c+1}} + \frac{1}{(1 - \theta)^{-c+1}} \right] \int_{E_3^+} t^{c-1} \, dt.
\]

Setting \( t = u^{\frac{1}{\delta}} \), we find

\[
\int_{E_3^+} t^{c-1} \, dt = \frac{1}{|\delta|} \int_1^\infty u^{\frac{1}{\delta}(c-1)} u^{\frac{1}{\delta}-1} \, du = \int_1^\infty u^{c-1} \, du.
\]
Hence, for \( c > 0 \), (4) follows, and for \( c \leq 0, \int_{E_{\neg}} t_{\theta}^{c-1} \, dt = \infty \). Since, for \( c > 0, \)
\[
\int_{E_{\neg}} t_{\theta}^{c-1} \, dt = 2 \int_{E_{\neg}} t_{\theta}^{\alpha-1} \, dt = \left[ \frac{1}{(1 + \theta)^{c+1}} + \frac{1}{(1 - \theta)^{c+1}} \right] \int_{0}^{1} u^{c-1} \, du,
\]
we have (5), and for \( c \leq 0, \int_{E_{\neg}} t_{\theta}^{c-1} \, dt = \infty \).

The lemma is proved. \( \square \)

In the following, We further assume that \( p > 1, \frac{1}{p} + \frac{1}{q} = 1, \delta \in [-1, 1], \alpha, \beta \in (-1, 1), \gamma, \rho, \sigma > 0, \sigma_1 \in \mathbb{R}, K_{\sigma}^{(\gamma)}(\sigma) \) is given by (3), and

\[
K_{\sigma}^{(\gamma)}(\sigma) := \frac{2 k_{\sigma}^{(\gamma)}}{(1 - \alpha^2)^{q/2}(1 - \beta^2)^{1/p}}. \quad (6)
\]

For \( n \in \mathbb{N} = \{1, 2, \ldots\}, E_1 = [-1, 1], x \in E_3, \) we define:
\[
I^{(\gamma)}(x) := \int_{-1}^{0} e^{-\rho(\delta_{\gamma} \rho)^{\sigma}} y_{\beta}^{\sigma + \frac{1}{q} - 1} \, dy, \quad I^{(\gamma)}(x) := \int_{0}^{1} e^{-\rho(\delta_{\gamma} \rho)^{\sigma}} y_{\beta}^{\sigma + \frac{1}{q} - 1} \, dy,
\]
\[
I(x) := I^{(\gamma)}(x) + I^{(\gamma)}(x) = \int_{E_1} e^{-\rho(\delta_{\gamma} \rho)^{\sigma}} y_{\beta}^{\sigma + \frac{1}{q} - 1} \, dy.
\]

For \( y_{\beta} = (\text{sgn}(y) + \beta)y, \) where
\[
\text{sgn}(y) := \begin{cases} 
-1, & y < 0, \\
0, & y = 0, \\
1, & y > 0,
\end{cases}
\]
\[
x_{\alpha}^{\delta} = (1 + \alpha \text{sgn}(x))^{|\alpha|} |x|^{|\delta|} \geq \min_{\delta \in [-1, 1]} \left\{ (1 + |\alpha|)^{|\delta|} \right\} \quad (x \in E_3),
\]
and \( 1 - |\alpha| \leq (1 + |\alpha|)^{-1} \leq 1 + |\alpha| \leq (1 - |\alpha|)^{-1}, \) we have
\[
(1 + \beta)|x_{\alpha}^{\delta}| \geq m_{\alpha, \beta} := (1 - |\beta|)(1 - |\alpha|) > 0 \quad (x \in E_3). \quad (7)
\]

For fixed \( x \in E_3, \) setting \( u = x_{\alpha}^{\delta} y_{\beta}, \) we find
\[
I^{(\gamma)}(x) = \frac{x_{\alpha}^{\delta}}{1 - \beta} \int_{0}^{(1 - \beta) x_{\alpha}^{\delta}} e^{-\rho|u|^q} u^{\sigma + \frac{1}{q} - 1} \, du \geq \frac{x_{\alpha}^{\delta}}{1 - \beta} \int_{0}^{m_{\alpha, \beta}} e^{-\rho|u|^q} u^{\sigma + \frac{1}{q} - 1} \, du,
\]
\[
I^{(\gamma)}(x) = \frac{x_{\alpha}^{\delta}}{1 + \beta} \int_{0}^{(1 + \beta) x_{\alpha}^{\delta}} e^{-\rho|u|^q} u^{\sigma + \frac{1}{q} - 1} \, du \geq \frac{x_{\alpha}^{\delta}}{1 + \beta} \int_{0}^{m_{\alpha, \beta}} e^{-\rho|u|^q} u^{\sigma + \frac{1}{q} - 1} \, du,
\]
\[
I(x) = x_{\alpha}^{\delta} \left[ \frac{1}{1 - \beta} \int_{0}^{(1 - \beta) x_{\alpha}^{\delta}} e^{-\rho|u|^q} u^{\sigma + \frac{1}{q} - 1} \, du + \frac{1}{1 + \beta} \int_{0}^{(1 + \beta) x_{\alpha}^{\delta}} e^{-\rho|u|^q} u^{\sigma + \frac{1}{q} - 1} \, du \right]
\geq \frac{2 x_{\alpha}^{\delta}}{1 + \beta} \int_{0}^{m_{\alpha, \beta}} e^{-\rho|u|^q} u^{\sigma + \frac{1}{q} - 1} \, du. \quad (8)
\]
For \( n \in \mathbb{N} = \{1, 2, \ldots\}, x \in E_\delta, \) we define:

\[
f^{-}(x) := \int_{-\infty}^{x^{-1}} e^{-\rho(x_{\alpha} y_{\beta})^{\sigma}} y_{\beta}^{-\frac{1}{\eta_{\eta_{1}}}} dy, \\
\]

\[
f^{+}(x) := \int_{x^{+1}}^{\infty} e^{-\rho(x_{\alpha} y_{\beta})^{\sigma}} y_{\beta}^{-\frac{1}{\eta_{\eta_{1}}}} dy, \\
\]

\[
J(x) := f^{-}(x) + f^{+}(x) = \int_{E_{1}} e^{-\rho(x_{\alpha} y_{\beta})^{\sigma}} y_{\beta}^{-\frac{1}{\eta_{\eta_{1}}}} dy. \\
\]

Since, for \( x \in E_{\delta}, \)

\[
x_{\alpha}^{-} = (1 + \alpha \text{ sgn}(x)) \frac{\sigma}{|x|} \leq \max_{\delta \in [-1, 1]} \{ (1 \pm |\alpha|)^{\delta} \} = (1 - |\alpha|)^{-1},
\]

we have

\[
M_{\alpha, \beta} := (1 + |\beta|)(1 - |\alpha|)^{-1} \geq (1 \pm |\beta|) x_{\alpha}^{-} \quad (x \in E_{\delta}). \quad (9)
\]

For fixed \( x \in E_{\delta}, \) setting \( u = x_{\alpha}^{\delta} y_{\beta}, \) we find

\[
f^{-}(x) = \frac{x_{\alpha}^{- \delta(\sigma - \frac{1}{\eta})}}{1 - \beta} \int_{(1 - \beta)x_{\alpha}^{-}}^{\infty} e^{-\rho u^{\sigma}} u^{-\frac{1}{\eta_{1}}} du \geq \frac{x_{\alpha}^{- \delta(\sigma - \frac{1}{\eta})}}{1 - \beta} \int_{M_{\alpha, \beta}}^{\infty} e^{-\rho u^{\sigma}} u^{-\frac{1}{\eta_{1}}} du,
\]

\[
f^{+}(x) = \frac{x_{\alpha}^{\delta(\sigma - \frac{1}{\eta})}}{1 + \beta} \int_{(1 + \beta)x_{\alpha}^{\delta}}^{\infty} e^{-\rho u^{\sigma}} u^{-\frac{1}{\eta_{1}}} du \geq \frac{x_{\alpha}^{\delta(\sigma - \frac{1}{\eta})}}{1 + \beta} \int_{M_{\alpha, \beta}}^{\infty} e^{-\rho u^{\sigma}} u^{-\frac{1}{\eta_{1}}} du,
\]

\[
J(x) = x_{\alpha}^{- \delta(\sigma - \frac{1}{\eta})} \left[ \frac{1}{1 - \beta} \int_{(1 - \beta)x_{\alpha}^{-}}^{\infty} e^{-\rho u^{\sigma}} u^{-\frac{1}{\eta_{1}}} du + \frac{1}{1 + \beta} \int_{(1 + \beta)x_{\alpha}^{\delta}}^{\infty} e^{-\rho u^{\sigma}} u^{-\frac{1}{\eta_{1}}} du \right]
\]

\[
\geq \frac{2 x_{\alpha}^{- \delta(\sigma - \frac{1}{\eta})}}{1 + \beta} \int_{M_{\alpha, \beta}}^{\infty} e^{-\rho u^{\sigma}} u^{-\frac{1}{\eta_{1}}} du. \quad (10)
\]

In view of (8) and (10), we have the following:

**Lemma 2** We have the following inequalities:

\[
I_{1} := \int_{E_{\delta}} I(x) x_{\alpha}^{\delta(\alpha - \frac{1}{\eta_{1}}) - 1} dx \\
\geq \frac{2}{1 - \beta^{2}} \int_{E_{\delta}} x_{\alpha}^{- \delta(\sigma - \alpha + \frac{1}{\eta_{1}}) - 1} dx \int_{0}^{M_{\alpha, \beta}} e^{-\rho u^{\sigma}} u^{-\frac{1}{\eta_{1}}} du, \quad (11)
\]

\[
J_{1} := \int_{E_{\delta}} J(x) x_{\alpha}^{\delta(\alpha + \frac{1}{\eta_{1}}) - 1} dx \\
\geq \frac{2}{1 - \beta^{2}} \int_{E_{\delta}} x_{\alpha}^{- \delta(\sigma - \alpha - \frac{1}{\eta_{1}}) - 1} dx \int_{M_{\alpha, \beta}}^{\infty} e^{-\rho u^{\sigma}} u^{-\frac{1}{\eta_{1}}} du. \quad (12)
\]
Lemma 3 If there exists a constant $M$ such that, for any nonnegative measurable functions $f(x)$ and $g(y)$ in $\mathbb{R}$,

$$I := \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\rho(x^\alpha y^\beta)} f(x) g(y) \, dx \, dy$$

$$\leq M \left[ \int_{-\infty}^{\infty} x^{\alpha(1-\delta\sigma_1)} f_n(x) \, dx \right]^{\frac{1}{p}} \left[ \int_{-\infty}^{\infty} y^{\beta(1-\sigma)} g_n(y) \, dy \right]^{\frac{1}{q}}, \quad (13)$$

then we have $\sigma_1 = \sigma$.

Proof If $\sigma_1 > \sigma$, then for $n \geq \frac{1}{\sigma_1 - \sigma}$ ($n \in \mathbb{N}$), we define the functions:

$$f_n(x) := \begin{cases} x^{\alpha(1-\delta\sigma_1)} \cdot \frac{1}{x^\alpha}, & x \in E_\delta, \\ 0, & x \in \mathbb{R} \setminus E_\delta, \end{cases} \quad g_n(y) := \begin{cases} y^{\beta(1-\sigma)} \cdot \frac{1}{y^\beta}, & y \in E_1, \\ 0, & y \in \mathbb{R} \setminus E_1, \end{cases}$$

and by (4) and (5) it follows that

$$\tilde{J}_1 := \left[ \int_{-\infty}^{\infty} x^{\alpha(1-\delta\sigma_1)} f_n(x) \, dx \right]^{\frac{1}{p}} \left[ \int_{-\infty}^{\infty} y^{\beta(1-\sigma)} g_n(y) \, dy \right]^{\frac{1}{q}}$$

$$= \left( \int_{E_\delta} x^{\alpha-\frac{1}{p}} \, dx \right)^{\frac{1}{p}} \left( \int_{E_1} y^{\beta-\frac{1}{q}} \, dy \right)^{\frac{1}{q}}$$

$$= n \left[ \frac{1}{(1+\alpha)^{\frac{1}{p}+1}} + \frac{1}{(1-\alpha)^{\frac{1}{p}+1}} \right]^{\frac{1}{p}} \left[ \frac{1}{(1+\beta)^{\frac{1}{q}+1}} + \frac{1}{(1-\beta)^{\frac{1}{q}+1}} \right]^{\frac{1}{q}} < \infty.$$
and by (4) and (5) it follows that
\[
\tilde{J}_2 := \left[ \int_{-\infty}^{\infty} x_x^{(1-\sigma_1)-1} f^p(x) \, dx \right]^\frac{1}{\beta} \left[ \int_{-\infty}^{\infty} y^{(1-\sigma)-1} g^q(y) \, dy \right]^\frac{1}{\delta} \\
= \left( \int_{E_1} x_x^{\frac{1-\alpha}{2}+1} \, dx \right)^\frac{1}{\beta} \left( \int_{E_1} y_x^{\frac{1}{\beta}+1} \, dy \right)^\frac{1}{\delta} \\
= n \left[ \frac{1}{(1+\alpha)^{\frac{\beta}{2}+1}} + \frac{1}{(1-\alpha)^{\frac{\beta}{2}+1}} \right]^\frac{1}{\beta} \left[ \frac{1}{(1+\beta)^{\frac{1}{\delta}+1}} + \frac{1}{(1-\beta)^{\frac{1}{\delta}+1}} \right]^\frac{1}{\delta} < \infty.
\]

By (12) and (13) (for \( f = \tilde{f}_n, g = \tilde{g}_n \)) we have
\[
\frac{2}{1-\beta^2} \int_{E_1} x_x^{\delta(\sigma_1+\frac{1}{2})+1} \, dx \int_{E_1} e^{-\rho u' u^\alpha + \frac{1}{\rho \beta}} \, du \\
\leq J_1 = \int_{E_1} \int_{E_1} e^{-\rho u' u^\alpha + \frac{1}{\rho \beta}} \tilde{f}_n(x) \tilde{g}_n(y) \, dx \, dy \leq M \tilde{J}_2 < \infty.
\]

Since for any \( n \geq \frac{1}{\sigma_1} \), \( \sigma_1 - \sigma + \frac{1}{2} \leq 0 \), by Lemma 1 it follows that \( \int_{E_1} x_x^{\delta(\sigma_1+\frac{1}{2})+1} \, dx = \infty \). In view of \( \int_{E_1} e^{-\rho u' u^\alpha + \frac{1}{\rho \beta}} \, du > 0 \), we have \( \infty \leq M \tilde{J}_2 < \infty \), which is a contradiction.

Hence we conclude that \( \sigma_1 = \sigma \).

The lemma is proved. \( \square \)

**Lemma 4** If there exists a constant \( M \) such that, for any nonnegative measurable functions \( f(x) \) and \( g(y) \) in \( \mathbb{R} \),
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\rho |x|^\alpha} f(x) g(y) \, dx \, dy \\
\leq M \left[ \int_{-\infty}^{\infty} x_x^{(1-\sigma_1)-1} f^p(x) \, dx \right]^\frac{1}{\beta} \left[ \int_{-\infty}^{\infty} y^{(1-\sigma)-1} g^q(y) \, dy \right]^\frac{1}{\delta}, \tag{14}
\]
then we have \( K_{\alpha,\beta}(\sigma) \leq M \).

**Proof** For \( \sigma_1 = \sigma \), by (8) we have
\[
I_1 = \int_{E_1} I(x) x_x^{\delta(\sigma_1+\frac{1}{2})+1} \, dx = I_1^{(-)} + I_1^{(+)}, \\
I_1^{(-)} = \int_{E_1} I^{(-)}(x) x_x^{\delta(\sigma_1-\frac{1}{2})+1} \, dx, \\
I_1^{(+) :=} \int_{E_1} I^{(+)}(x) x_x^{\delta(\sigma_1-\frac{1}{2})+1} \, dx.
\]

In view of the presented results, we find
\[
I_1^{(-)} = \frac{1}{1-\beta} \int_{E_1} x_x^{\frac{1}{2}-1} \int_0^{(1-\beta)x_x^\delta} e^{-\rho u' u^\alpha + \frac{1}{\rho \beta} u^\alpha} \, du \, dx \\
= \frac{1}{1-\beta} \int_{E_1} x_x^{\frac{1}{2}-1} \left[ \int_0^{\infty} e^{-\rho u' u^\alpha + \frac{1}{\rho \beta} u^\alpha} \, du - \int_{(1-\beta)x_x^\delta}^{\infty} e^{-\rho u' u^\alpha + \frac{1}{\rho \beta} u^\alpha} \, du \right] \, dx
\]
\[
\begin{align*}
\int_{E}^{\infty} x_{\sigma}^{-\beta} \int_{E}^{\infty} e^{-\rho u} u^{\sigma+\frac{1}{\delta} - 1} du dx \\
\leq M_{1} \int_{E}^{\infty} x_{\sigma}^{-\beta} \int_{E}^{\infty} e^{-\rho u} u^{\sigma+\frac{1}{\delta} - 1} du dx = \frac{M_{1} \int_{E}^{\infty} x_{\sigma}^{-\beta} \int_{E}^{\infty} e^{-\rho u} u^{\sigma+\frac{1}{\delta} - 1} du dx}{(1-\beta)^{\frac{1}{\delta} - \frac{1}{\delta}}}
\end{align*}
\]

so that

\[
\begin{align*}
\frac{1}{1-\beta} \int_{E}^{\infty} x_{\sigma}^{-\beta} \int_{E}^{\infty} e^{-\rho u} u^{\sigma+\frac{1}{\delta} - 1} du dx = O(1).
\end{align*}
\]

By (15) it follows that

\[
\frac{1}{n} I_{1}^{(-)} = \frac{k^{(\gamma)}(\sigma + \frac{1}{\delta})}{1-\beta} \left[ \frac{1}{(1+\alpha)^{\frac{1}{\delta}+1}} + \frac{1}{(1-\alpha)^{\frac{1}{\delta}+1}} \right] - O(1) \frac{n}{n}.
\]

In the same way, we have

\[
\frac{1}{n} I_{1}^{(+)} = \frac{k^{(\gamma)}(\sigma + \frac{1}{\delta})}{1+\beta} \left[ \frac{1}{(1+\alpha)^{\frac{1}{\delta}+1}} + \frac{1}{(1-\alpha)^{\frac{1}{\delta}+1}} \right] - O(1) \frac{n}{n}.
\]

By (14) (for \(f = f_n, g = g_n\), we have

\[
\frac{1}{n} I_{1} = \frac{1}{n} (I_{1}^{(-)} + I_{1}^{(+)}) \leq \frac{1}{n} M_{1}.\]

For \(n \to \infty\), by Fatou lemma (see [27]), (16), and (17) we find

\[
\frac{2}{1-\beta^{2}} \leq M \left( \frac{2}{1-\alpha^{2}} \right)^{\frac{1}{2}} \left( \frac{2}{1-\beta^{2}} \right)^{\frac{1}{2}},
\]

so that \(K^{(\gamma)}(\sigma) = \frac{2k^{(\gamma)}(\sigma)}{(1-\alpha^{2})^{\frac{1}{2}}(1-\beta^{2})^{\frac{1}{2}}} \leq M\).

The lemma is proved.

\[\square\]

Lemma 5 We define the following weight functions:

\[
\omega_{\delta}(\sigma, y) := y_{\delta}^{\sigma} \int_{-\infty}^{\infty} e^{-\rho(x_{\sigma}^{\delta}y)^{\gamma}} x_{\sigma}^{-\delta - 1} dx \quad (y \in \mathbb{R}),
\]

\[\sigma_\delta(\sigma, x) := x_\alpha^\frac{\delta}{\sigma} \int_{-\infty}^{\infty} e^{-\rho(x^\gamma y)^\nu} y_\beta^{\sigma - 1} dy \quad (x \in \mathbb{R}). \quad (19)\]

Then we have
\[\frac{1 - \alpha^2}{2} \omega_\delta(\sigma, y) = \frac{1 - \beta^2}{2} \sigma_\delta(\sigma, x) = k^{(\nu)}(\sigma) \quad (x, y \in \mathbb{R} \setminus \{0\}). \quad (20)\]

**Proof** For fixed \(y \in \mathbb{R} \setminus \{0\}\), setting \(u = x_\alpha^\delta y_\beta\), we find
\[\omega_\delta(\sigma, y) = y_\beta^\sigma \int_{-\infty}^{0} e^{-\rho(x_\alpha^\gamma y_\beta)^\nu} [(1 - \alpha)(-x)]^{\sigma - 1} dx \]
\[+ y_\beta^\sigma \int_{0}^{\infty} e^{-\rho(x_\alpha^\gamma y_\beta)^\nu} [(1 + \alpha)x]^{\sigma - 1} dx \]
\[= \left( \frac{1}{1 - \alpha} + \frac{1}{1 + \alpha} \right) \int_{0}^{\infty} e^{-\rho u} u^{\sigma - 1} du = \frac{2}{1 - \alpha^2} k^{(\nu)}(\sigma);\]
for fixed \(x \in \mathbb{R} \setminus \{0\}\), setting \(u = x_\alpha^\delta y_\beta\), it follows that
\[\sigma_\delta(\sigma, x) = x_\alpha^\delta \int_{-\infty}^{0} e^{-\rho(x_\alpha^\gamma y_\beta)^\nu} y_\beta^{\sigma - 1} dy + x_\alpha^\delta \int_{0}^{\infty} e^{-\rho(x_\alpha^\gamma y_\beta)^\nu} y_\beta^{\sigma - 1} dy \]
\[= \frac{2}{1 - \beta^2} \int_{0}^{\infty} e^{-\rho u} u^{\sigma - 1} du = \frac{2}{1 - \beta^2} k^{(\nu)}(\sigma).\]

Hence we have (20).

The lemma is proved. \(\square\)

### 3 Main results

**Theorem 1** If \(M\) is a constant, then the following statements (i), (ii), and (iii) are equivalent:

(i) For any \(f(x) \geq 0\), we have:
\[J := \left\{ \int_{-\infty}^{\infty} y_\beta^{\sigma - 1} \left[ \int_{-\infty}^{\infty} e^{-\rho(x_\alpha^\gamma y_\beta)^\nu} f(x) dx \right]^p dy \right\}^{\frac{1}{p}} \leq M \left[ \int_{-\infty}^{\infty} x_\alpha^{\rho(1-\sigma_1)-1} f^p(x) dx \right]^{\frac{1}{p}}. \quad (21)\]

(ii) For any \(f(x), g(y) \geq 0\), we have:
\[I = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\rho(x_\alpha^\gamma y_\beta)^\nu} f(x)g(y) dx dy \leq M \left[ \int_{-\infty}^{\infty} x_\alpha^{\rho(1-\sigma_1)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[ \int_{-\infty}^{\infty} y_\beta^{\sigma - 1} g^q(y) dy \right]^{\frac{1}{q}}. \quad (22)\]

(iii) \(\sigma_1 = \sigma\), and \(K^{(\nu)}(\sigma) \leq M\).
Proof. (i) $\Rightarrow$ (ii). By Hölder’s inequality (see [28]) we have

$$I = \int_{-\infty}^{\infty} e^{-\rho(x_0^a y_0^b)^{\gamma}} f(x) \, dx \left[ g_{\beta}^{-\frac{\rho}{\beta}} g(y) \right] \, dy$$

$$\leq I \left[ \int_{-\infty}^{\infty} \left( e^{-\rho(x_0^a y_0^b)^{\gamma}} f(x) \, dx \right)^{\frac{\rho}{\beta}} \right]^{\frac{\beta}{\rho}}.$$  

(23)

Then by (21) we have (22).

(ii) $\Rightarrow$ (iii). By Lemma 1 we have $\sigma_1 = \sigma$. Then by Lemma 2 we have $K^{(\gamma)}_{a,\beta}(\sigma) \leq M$.

(iii) $\Rightarrow$ (i). For $\sigma_1 = \sigma$, by Hölder’s inequality with weight (see [28]) and (18) we have

$$\left[ \int_{-\infty}^{\infty} e^{-\rho(x_0^a y_0^b)^{\gamma}} f(x) \, dx \right]^{\frac{1}{p}}$$

$$= \left\{ \int_{-\infty}^{\infty} e^{-\rho(x_0^a y_0^b)^{\gamma}} \left[ \frac{y_0^{(\beta-1)/p} f(x)}{x_0^{(\delta-1)/q}} \right] \, dx \right\}^{\frac{1}{p}}$$

$$\leq \int_{-\infty}^{\infty} e^{-\rho(x_0^a y_0^b)^{\gamma}} \frac{y_0^{\beta-1} f(x)}{x_0^{(\delta-1)/q}} \, dx \left[ \int_{-\infty}^{\infty} e^{-\rho(x_0^a y_0^b)^{\gamma}} \frac{x_0^{\delta-1}}{y_0^{(\delta-1)/q}} \, dx \right]^{\frac{1}{p}}$$

$$= \int_{-\infty}^{\infty} e^{-\rho(x_0^a y_0^b)^{\gamma}} \frac{y_0^{\beta-1} f(x)}{x_0^{(\delta-1)/q}} \, dx \left[ \int_{-\infty}^{\infty} e^{-\rho(x_0^a y_0^b)^{\gamma}} \frac{x_0^{\delta-1}}{y_0^{(\delta-1)/q}} \, dx \right]^{\frac{1}{p}}.$$

(24)

By Fubini’s theorem (see [27]), (24), and (19) we have

$$J \leq \left( \frac{2k^{(\gamma)}_{a,\beta}(\sigma)}{1 - \alpha^2} \right)^{\frac{1}{3}} \left[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\rho(x_0^a y_0^b)^{\gamma}} \frac{y_0^{\beta-1} f(x)}{x_0^{(\delta-1)/q}} \, dx \, dy \right]^{\frac{1}{3}}$$

$$= \left( \frac{2k^{(\gamma)}_{a,\beta}(\sigma)}{1 - \alpha^2} \right)^{\frac{1}{3}} \left[ \int_{-\infty}^{\infty} \sigma_{\delta}(\sigma, x) x_0^{\beta(1-\delta)-1} f^\rho(x) \, dx \right]^{\frac{1}{3}}$$

$$= K^{(\gamma)}_{a,\beta}(\sigma) \left[ \int_{-\infty}^{\infty} x_0^{\beta(1-\delta)-1} f^\rho(x) \, dx \right]^{\frac{1}{3}}.$$  

For $K^{(\gamma)}_{a,\beta}(\sigma) \leq M$, we have (21) (when $\sigma_1 = \sigma$).

Therefore, statements (i), (ii), and (iii) are equivalent.

The theorem is proved. \qed

Theorem 2. If $M$ is a constant, then the following statements (i), (ii), and (iii) are equivalent:

(i) For any $f(x) \geq 0$ satisfying $0 < \int_{-\infty}^{\infty} x_0^{\beta(1-\delta)-1} f^\rho(x) \, dx < \infty$, we have:

$$\left\{ \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} e^{-\rho(x_0^a y_0^b)^{\gamma}} f(x) \, dx \right]^{\frac{1}{p}} \, dy \right\}^{\frac{1}{\rho}}$$

$$< M \left[ \int_{-\infty}^{\infty} x_0^{\beta(1-\delta)-1} f^\rho(x) \, dx \right]^{\frac{1}{p}}.$$  

(25)
(ii) For any \( f(x) \geq 0 \) satisfying \( 0 < \int_{-\infty}^{\infty} x_\alpha^{\delta(1-\sigma)-1} f^p(x) \, dx < \infty \), and \( g(x) \geq 0 \) satisfying \( 0 < \int_{-\infty}^{\infty} y^{\delta(1-\sigma)-1} g^q(y) \, dy < \infty \), we have:

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\rho(x_\alpha y_\beta y^{\gamma})} f(x)g(y) \, dx \, dy < M \left[ \int_{-\infty}^{\infty} x_\alpha^{\delta(1-\sigma)-1} f^p(x) \, dx \right]^{\frac{1}{p}} \left[ \int_{-\infty}^{\infty} y^{\delta(1-\sigma)-1} g^q(y) \, dy \right]^{\frac{1}{q}},
\]

where \( M = K_{a,b}(\sigma) \) is the best possible constant factor.

(iii) \( K_{\alpha,b}(\sigma) \leq M \).

Moreover, if statement (iii) holds, then the constant factor \( M = K_{a,b}(\sigma) \) in (25) and (26) is the best possible.

In particular, (1) for \( \delta = 1 \), \( M = K_{a,b}(\sigma) \) in (25) and (26) with nonhomogeneous kernel:

\[
\left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\rho(x_\alpha y_\beta y^{\gamma})} f(x)g(y) \, dx \, dy \right\}^{\frac{1}{p}} < \left[ \int_{-\infty}^{\infty} x_\alpha^{\delta(1-\sigma)-1} f^p(x) \, dx \right]^{\frac{1}{p}} \left[ \int_{-\infty}^{\infty} y^{\delta(1-\sigma)-1} g^q(y) \, dy \right]^{\frac{1}{q}},
\]

(27)

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\rho(x_\alpha y_\beta y^{\gamma})} f(x)g(y) \, dx \, dy < \left[ \int_{-\infty}^{\infty} x_\alpha^{\delta(1-\sigma)-1} f^p(x) \, dx \right]^{\frac{1}{p}} \left[ \int_{-\infty}^{\infty} y^{\delta(1-\sigma)-1} g^q(y) \, dy \right]^{\frac{1}{q}},
\]

(28)

where \( K_{a,b}(\sigma) \) is the best possible constant factor.

(2) for \( \delta = -1 \), \( M = K_{a,b}(\sigma) \), we have the following equivalent inequalities with homogeneous kernel of degree 0:

\[
\left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\rho(x_\alpha y_\beta y^{\gamma})} f(x)g(y) \, dx \, dy \right\}^{\frac{1}{p}} < \left[ \int_{-\infty}^{\infty} x_\alpha^{\delta(1+\sigma)-1} f^p(x) \, dx \right]^{\frac{1}{p}} \left[ \int_{-\infty}^{\infty} y^{\delta(1-\sigma)-1} g^q(y) \, dy \right]^{\frac{1}{q}},
\]

(29)

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\rho(x_\alpha y_\beta y^{\gamma})} f(x)g(y) \, dx \, dy < \left[ \int_{-\infty}^{\infty} x_\alpha^{\delta(1+\sigma)-1} f^p(x) \, dx \right]^{\frac{1}{p}} \left[ \int_{-\infty}^{\infty} y^{\delta(1-\sigma)-1} g^q(y) \, dy \right]^{\frac{1}{q}},
\]

(30)

where \( K_{a,b}(\sigma) \) is the best possible constant factor.

Proof. For \( \sigma_1 = \sigma \), under and the assumption of statement (i), if (24) takes the form of equality for \( y \in \mathbb{R} \setminus \{0\} \), then there exist constants \( A \) and \( B \) such that they are not both zero and (see [28])

\[
A x_\alpha^{\delta(1+\sigma)-1} y_\beta^{\delta(1-\sigma)-1} f^p(x) = B y_\beta^{\delta(1-\sigma)-1} x_\alpha^{\delta(1+\sigma)-1} g^q(y) \quad \text{a.e. in } \mathbb{R}.
\]
We suppose that $A \neq 0$ (otherwise, $B = A = 0$). Then it follows that

$$x_a^{p(1-\sigma)-1}f^p(x) = y_\beta^{q(\sigma-1)} \frac{B}{Ax_a} \text{ a.e. in } \mathbb{R}.$$ 

Since $\int_{-\infty}^{\infty} x_a^{-1} \, dx = \infty$, this contradicts the fact that $0 < \int_{-\infty}^{\infty} x_a^{p(1-\sigma)-1} f^p(x) \, dx < \infty$. Hence (24) takes the form of strict inequality, and so does (21). Hence (25) and (26) are valid.

In view of Theorem 1, we still can conclude that statements (i), (ii), and (iii) in Theorem 2 are equivalent.

When statement (iii) holds, namely, $K_{a,\beta}(\sigma) \leq M$, if there exists a constant $M(\leq K_{a,\beta}(\sigma))$ such that (26) is valid, then $M = K_{a,\beta}(\sigma)$, and we can conclude that the constant factor $M = K_{a,\beta}(\sigma)$ in (26) is the best possible.

The constant factor $M = K_{a,\beta}(\sigma)$ in (25) is still the best possible. Otherwise, by (23) (for $\sigma_1 = \sigma$), we would get a contradiction that the constant factor $M = K_{a,\beta}(\sigma)$ in (26) is not the best possible.

The theorem is proved. □

4 Operator expressions

We set the following functions: $\psi(x) := x_a^{p(1-\sigma)-1} (x \in \mathbb{R})$, and $\psi(y) := y_\beta^{q(\sigma-1)-1}$, where from $\psi^{1-p}(y) := y_\beta^{q\sigma-1} (y \in \mathbb{R})$. Define the following real normed linear spaces:

$$L_{p,\psi}(\mathbb{R}) := \left\{ f; \|f\|_{p,\psi} := \left( \int_{-\infty}^{\infty} \psi(x)|f(x)|^p \, dx \right)^{\frac{1}{p}} < \infty \right\},$$

$$L_{q,\psi}(\mathbb{R}) := \left\{ g; \|g\|_{q,\psi} = \left( \int_{-\infty}^{\infty} \psi(y)|g(y)|^q \, dy \right)^{\frac{1}{q}} < \infty \right\},$$

$$L_{p,\psi^{1-p}}(\mathbb{R}) := \left\{ h; \|h\|_{p,\psi^{1-p}} = \left( \int_{-\infty}^{\infty} \psi^{1-p}(y)|h(y)|^p \, dy \right)^{\frac{1}{p}} < \infty \right\}.$$

In view of Theorem 2, for $f \in L_{p,\psi}(\mathbb{R})$, setting

$$h_1(y) := \int_{-\infty}^{\infty} e^{-\psi(y)x_a} f(x) \, dx \ (y \in \mathbb{R}),$$

by (25) we have

$$\|h_1\|_{p,\psi^{1-p}} = \left( \int_{-\infty}^{\infty} \psi^{1-p}(y)|h_1(y)|^p \, dy \right)^{\frac{1}{p}} \leq M\|f\|_{p,\psi} < \infty. \quad (31)$$

Definition 1 Define the Hilbert-type integral operator $T : L_{p,\psi}(\mathbb{R}) \to L_{p,\psi^{1-p}}(\mathbb{R})$ as follows: For any $f \in L_{p,\psi}(\mathbb{R})$, there exists a unique representation $Tf = h_1 \in L_{p,\psi^{1-p}}(\mathbb{R})$, satisfying for any $y \in \mathbb{R}$, $Tf(y) = h_1(y)$.

In view of (31), it follows that $\|Tf\|_{p,\psi^{1-p}} = \|h_1\|_{p,\psi^{1-p}} \leq M\|f\|_{p,\psi}$, and then the operator $T$ is bounded and satisfies

$$\|T\| = \sup_{f \in L_{p,\psi}} \frac{\|Tf\|_{p,\psi^{1-p}}}{\|f\|_{p,\psi}} \leq M.$$
If we define the formal inner product of $Tf$ and $g$ as
\[(Tf, g) := \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-p(y|x)} f(x) dx g(y) dy,\]
then we can rewrite Theorem 2 as follows.

**Theorem 3** If $M$ is a constant, then the following statements (i), (ii), and (iii) are equivalent:

(i) For any $f(x) \geq 0, f \in L_{p,q}(\mathbb{R}), \|f\|_{p,q} > 0$, we have:
\[
\|Tf\|_{p,q} < M\|f\|_{p,q}. \tag{32}
\]

(ii) For $f(x), g(y) \geq 0, f \in L_{p,q}(\mathbb{R}), g \in L_{q,p}(\mathbb{R}), \|f\|_{p,q}, \|g\|_{q,p} > 0$, we have:
\[
(Tf, g) < M\|f\|_{p,q} \|g\|_{q,p}. \tag{33}
\]

(iii) $K_{a,b}(\sigma) \leq M$.

Moreover, if statement (iii) holds, then the constant factor $M = K_{a,b}(\sigma)$ in (32) and (33) is the best possible, namely, $\|T\| = K_{a,b}^{(p)}(\sigma)$.

**Remark 1** (1) In particular, for $\alpha = \beta = 0$ in (27) and (28), we have the following equivalent inequalities:

\[
\left\{ \int_{-\infty}^{\infty} |y|^{p-1} \left[ \int_{-\infty}^{\infty} e^{-\rho |x|^\gamma} f(x) dx \right]^p dy \right\}^{\frac{1}{p}} \leq \frac{2 \Gamma(\sigma / \gamma)}{\gamma \rho^{\sigma / \gamma}} \left[ \int_{-\infty}^{\infty} |x|^{p(1-\sigma)} f^p(x) dx \right]^{\frac{1}{p}}, \tag{34}
\]
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\rho |x|^\gamma} f(x) g(y) dx dy \leq \frac{2 \Gamma(\sigma / \gamma)}{\gamma \rho^{\sigma / \gamma}} \left[ \int_{-\infty}^{\infty} |x|^{p(1-\sigma)} f^p(x) dx \right]^{\frac{1}{p}} \left[ \int_{-\infty}^{\infty} |y|^{q(1-\sigma)} g^q(y) dy \right]^{\frac{1}{q}}, \tag{35}
\]
where $\frac{2 \Gamma(\sigma / \gamma)}{\gamma \rho^{\sigma / \gamma}}$ is the best possible constant factor.

If $f(-x) = f(x), g(-y) = g(y) \ (x, y \in \mathbb{R},)$, then we have the following equivalent inequalities:

\[
\left\{ \int_{0}^{\infty} y^{p-1} \left[ \int_{0}^{\infty} e^{-\rho (y|x)} f(x) dx \right]^p dy \right\}^{\frac{1}{p}} \leq \frac{\Gamma(\sigma / \gamma)}{\gamma \rho^{\sigma / \gamma}} \left[ \int_{0}^{\infty} x^{p(1-\sigma)} f^p(x) dx \right]^{\frac{1}{p}}, \tag{36}
\]
\[
\int_{0}^{\infty} \int_{0}^{\infty} e^{-\rho (y|x)} f(x) g(y) dx dy \leq \frac{\Gamma(\sigma / \gamma)}{\gamma \rho^{\sigma / \gamma}} \left[ \int_{0}^{\infty} x^{p(1-\sigma)} f^p(x) dx \right]^{\frac{1}{p}} \left[ \int_{0}^{\infty} y^{q(1-\sigma)} g^q(y) dy \right]^{\frac{1}{q}}, \tag{37}
\]
where $\frac{\Gamma(\sigma / \gamma)}{\gamma \rho^{\sigma / \gamma}}$ is the best possible constant factor.
For $\alpha = \beta = 0$ in (29) and (30), we have the following equivalent inequalities:

\[
\left\{ \int_{-\infty}^{\infty} \left| y \right|^{\rho\sigma-1} \left[ \int_{-\infty}^{\infty} e^{-\rho \left| y/x \right|^{\gamma}} f(x) \, dx \right]^p \, dy \right\}^{\frac{1}{p}} < \frac{2 \Gamma(\sigma/\gamma) \Gamma(\rho/\gamma)}{\gamma^{\rho\sigma/\gamma}} \left[ \int_{-\infty}^{\infty} \left| x \right|^{p(1+\sigma)-1} f^p(x) \, dx \right]^{\frac{1}{p}},
\]

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\rho \left| y/x \right|^{\gamma}} f(x) g(y) \, dx \, dy < \frac{2 \Gamma(\sigma/\gamma)}{\gamma^{\rho\sigma/\gamma}} \left[ \int_{-\infty}^{\infty} \left| x \right|^{p(1+\sigma)-1} f^p(x) \, dx \right]^{\frac{1}{p}} \left[ \int_{-\infty}^{\infty} \left| y \right|^{q(1-\sigma)-1} g^q(y) \, dy \right]^{\frac{1}{q}},
\]

where $\frac{2 \Gamma(\sigma/\gamma)}{\gamma^{\rho\sigma/\gamma}}$ is the best possible constant factor.

If $f(-x) = f(x)$, $g(-y) = g(y)$ $(x, y \in \mathbb{R}_+)$, then we have the following equivalent inequalities:

\[
\left\{ \int_{0}^{\infty} y^{\rho\sigma-1} \left[ \int_{0}^{\infty} e^{-\rho \left| y/x \right|^{\gamma}} f(x) \, dx \right]^p \, dy \right\}^{\frac{1}{p}} < \frac{\Gamma(\sigma/\gamma)}{\gamma^{\rho\sigma/\gamma}} \left[ \int_{0}^{\infty} x^{p(1+\sigma)-1} f^p(x) \, dx \right]^{\frac{1}{p}},
\]

\[
\int_{0}^{\infty} \int_{0}^{\infty} e^{-\rho \left| y/x \right|^{\gamma}} f(x) g(y) \, dx \, dy < \frac{\Gamma(\sigma/\gamma)}{\gamma^{\rho\sigma/\gamma}} \left[ \int_{0}^{\infty} x^{p(1+\sigma)-1} f^p(x) \, dx \right]^{\frac{1}{p}} \left[ \int_{0}^{\infty} y^{q(1-\sigma)-1} g^q(y) \, dy \right]^{\frac{1}{q}},
\]

where $\frac{\Gamma(\sigma/\gamma)}{\gamma^{\rho\sigma/\gamma}}$ is the best possible constant factor.

5 Conclusions

In this paper, by using real analysis and weight functions we obtain a few equivalent statements of a Hilbert-type integral inequality in the whole plane related to the kernel of exponent function with the intermediate variables (Theorem 1). The constant factor related to the gamma function is proved to be the best possible in Theorem 2. We also consider some particular cases and the operator expressions in Remark 1 and Theorem 3. The lemmas and theorems provide an extensive account of this type of inequalities.

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Competing interests
The authors declare that they have no competing interests.

Authors’ contributions
BY carried out the mathematical studies, participated in the sequence alignment, and drafted the manuscript. YZ and MH participated in the design of the study and performed the numerical analysis. All authors read and approved the final manuscript.

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