NETWORK DATA ENVELOPMENT ANALYSIS WITH FUZZY NON-DISCRETIONARY FACTORS

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ABSTRACT. Network data envelopment analysis (DEA) concerns using the DEA technique to measure the relative efficiency of a system, taking into account its internal structure. The results are more meaningful and informative than those obtained from the conventional DEA models. This work proposed a new network DEA model based on the fuzzy concept even though the inputs and outputs data are crisp numbers. The model is then extended to investigate the network DEA with fuzzy non-discretionary variables. An illustrative application assessing the impact of information technology (IT) on firm performance is included. The results reveal that modeling the IT budget as a fuzzy non-discretionary factor improves the system performance of firms in a banking industry.

1. Introduction. Data envelopment analysis (DEA) has been widely recognized as an effective technique for evaluating relative efficiency of decision making units (DMUs). In classical DEA models, only the inputs consumed by and the outputs produced from the system are considered, and internal interactions are not taken into account in measuring efficiencies. When a system is composed of several components operating interdependently, it has been found that ignoring the operations of the components may produce efficiency measures that are misleading [4]. Network DEA is extended from conventional DEA, which considers the relation and

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dependencies between internal links so that the efficiencies are measured more appropriately. It was suggested, for the first time, by Füre & Grosskopf [8]. Then, the multi-stage structures either with serial or parallel or mixed structure for network DEA model were proposed [2]. Hundreds of works that discuss network DEA have been published. Some develop models to measure efficiencies under specified conditions, some examine the properties possessed by certain models, and others apply existing models to solve real world problems [11]. Overviews of the related network DEA models and applications can be found in [5, 11].

Network DEA assumes that all inputs and outputs are controlled by DMUs. Inputs and/or outputs that do not conform to this assumption are denoted in DEA as non-discretionary (ND) factors. There have been a number of models proposed for analyzing performance in the presence of non-discretionary factors. Banker and Morey [1] provided the first DEA model to include non-discretionary variables for measuring technical efficiency. Muniz et al. [15] reviewed various non-discretionary approaches and compared their performance by simulation analysis. Taleb et al. [17] proposed a two-stage approach of super efficiency slack-based measure in non-discretionary factors and mixed integer requirements. Galagedera [9] developed a two-stage DEA model with non-discretionary first stage output for modeling social responsibility in mutual fund performance appraisal.

In the DEA applications dealing with non-discretionary factors, a factor is considered either fully controllable or totally uncontrollable. Golany and Roll [10] pointed out that in many real-life applications of efficiency studies, some factor might be neither fully controllable nor totally uncontrollable. In other words, the factor is partially controllable. Such a factor is called an almost discretionary variable or fuzzy non-discretionary (FND) variable [19]. Models allowing varying degrees of control (0-100%) over factors may be more suitable to many cases [10]. To formulate these cases in a way which will generalize the previous developments, the discretionary index concept was introduced to incorporate the FND variables into a DEA model, where an index taking on the values between 0 and 1 is utilized to represent the degree of discretion that the DMU has on the factor. However, the discretionary indexes are usually not known and are arbitrarily determined by decision makers in real-life applications [19].

To address this limitation, an alternative interpretation of efficiency in the network DEA is introduced in our work. A new model based on the fuzzy concept even though the inputs and outputs data are crisp numbers is developed. An equivalence between the proposed model and network DEA model is established. The approach is then extended to study the network DEA with fuzzy non-discretionary variables. To incorporate the fuzzy non-discretionary factor into a network DEA model, the membership function taking on the values between 0 and 1 is used to replace the need to determine the discretionary indexes of fuzzy non-discretionary and represent the degree of discretion that the DMU has on the fuzzy non-discretionary factor. An illustrative application assessing the impact of information technology on the firm performance in a banking industry is included.

The rest of the paper is organized as follows. A new network DEA model based on the fuzzy concept is provided in Section 2. An extension of the model
incorporating situations whereby some inputs or outputs are fuzzy non-discretionary variables is developed in Section 3. An illustrative application is included in Section 4. The paper is concluded in Section 5.

2. A new version of the general network DEA model. To investigate internal interactions of a system and examine the interrelationship of the components within a system, a variety of network DEA models have been studied. Figure 1 depicts a general network system proposed by Kao [12], where the operations of component processes are taken into consideration in measuring efficiency. Denote \( Z_{dj}^{(a,b)} \) as the \( d \)th intermediate product of DMU \( j \) produced by division \( a \) for division \( b \) to use. \( \sum_{a=1}^{q} Z_{fj}^{(a,k)} \) is then the total amount of the \( f \)th intermediate product, \( f \in M^{(k)} \), of DMU \( j \) produced by other divisions for division \( k \) to use, and \( \sum_{b=1}^{q} Z_{gj}^{(k,b)} \) is then the total amount of the \( g \)th intermediate product, \( g \in N^{(k)} \), of DMU \( j \) produced by division \( k \) for other divisions to use, where \( M^{(k)} \subset \{1, 2, \cdots, h\} \) and \( N^{(k)} \subset \{1, 2, \cdots, h\} \) are the index sets for the intermediate products to be consumed and produced by division \( k \), \( k = 1, 2, \cdots, q \), respectively. Any process \( k, k = 1, 2, \cdots, q \), utilizes exogenous inputs \( X_{ij}^{(k)} \), \( i \in I^{(k)} \), of DMU \( j \) supplied from outside and endogenous inputs \( \sum_{a=1}^{q} Z_{fj}^{(a,k)} \), \( f \in M^{(k)} \), of DMU \( j \) produced by other processes to produce exogenous outputs \( Y_{rj}^{(k)} \), \( r \in O^{(k)} \), of DMU \( j \) as final outputs of the system and endogenous output \( \sum_{a=1}^{q} Z_{gj}^{(k,b)} \), \( g \in N^{(k)} \), of DMU \( j \) to be utilized by other processes, where \( I^{(k)} \subset \{1, 2, \cdots, m\} \) and \( O^{(k)} \subset \{1, 2, \cdots, s\} \) are the index sets for the inputs and outputs of division \( k \), \( k = 1, 2, \cdots, q \), respectively.

![Figure 1. General network systems [12].](image)

Assume that the most general case where the technologies of all processes are allowed to be different, the production possibility set of this network system is defined by Färe & Grosskopf [7, 8] as

\[
T = \{ (x, y, z) \mid \sum_{j=1}^{n} \lambda_{j}^{(k)} X_{ij}^{(k)} \leq x_{i}, \ i \in I^{(k)}, \sum_{j=1}^{n} \lambda_{j}^{(k)} Y_{rj}^{(k)} \geq y_{r}, \ r \in O^{(k)}, \sum_{j=1}^{n} \lambda_{j}^{(k)} Z_{fj}^{(k)} \leq z_{f}, \ f \in M^{(k)}, \sum_{j=1}^{n} \lambda_{j}^{(k)} Z_{gj}^{(k)} \geq z_{g}, \ g \in N^{(k)}, \lambda_{j}^{(k)} \geq 0, j = 1, 2, \cdots, n, k = 1, 2, \cdots, q \}, \text{where } (\lambda_{1}^{k}, \lambda_{2}^{k}, \cdots, \lambda_{n}^{k}), k =
Consider the fuzzy decision \( \tilde{\theta} \) where \( \bar{\theta} \) or \( \theta \) mediate products \( z \). Assume that \( z \) has been omitted for simplicity of expression.

A virtual fuzzy variable with its corresponding membership function defined as obtained by solving the following max-min optimization problem:

\[
\max \{ \min_{i \in I(k)} \{ \mu_{\tilde{X}_{ip}}(\tilde{X}_{ip}^{(k)}) \} \} \quad \text{s.t.} \quad (\tilde{\theta}_p, \tilde{y}_p, \tilde{z}_p) \in T,
\]

where \( \theta_p \) is the efficiency evaluation of \( DMU_p \) with inputs \( x_p \), outputs \( y_p \), and intermediate products \( z_p \), and the non-Archimedean number \( \varepsilon \) in the objective function has been omitted for simplicity of expression.

Assume that \( X_{ip}^{(k)}, i \in I(k), k = 1, 2, \ldots, q \), are inputs of \( DMU_p \). Let \( \tilde{X}_{ip}^{(k)} \) be a virtual fuzzy variable with its corresponding membership function defined as

\[
\mu_{\tilde{X}_{ip}^{(k)}}(\tilde{X}_{ip}^{(k)}) = \frac{X_{ip}^{(k)} - \tilde{X}_{ip}^{(k)}}{X_{ip}^{(k)}},
\]

where \( \tilde{X}_{ip}^{(k)} \leq X_{ip}^{(k)}, i \in I(k), k = 1, 2, \ldots, q \), is the virtual fuzzy input of \( DMU_p \).

Consider the fuzzy decision \( \tilde{\theta} \) of model (1) with virtual fuzzy variables \( \tilde{X}_{ip}^{(k)}, i \in I(k), k = 1, 2, \ldots, q \), which can be defined as the fuzzy set resulting from the intersection of \( \tilde{X}_{ip}^{(k)}, i \in I(k), k = 1, 2, \ldots, q \), with the corresponding membership function

\[
\mu_{\tilde{\theta}}(\tilde{X}_{ip}^{(k)}) = \min_{k = 1, 2, \ldots, q} \{ \mu_{\tilde{X}_{ip}^{(k)}}(\tilde{X}_{ip}^{(k)}) \}.
\]

According to [3], a solution, say \( \tilde{X}_{ip}^{(k)*} \), of model (1) with virtual fuzzy variables \( \tilde{X}_{ip}^{(k)} \) can be taken as the solution with the highest membership in the fuzzy decision set \( \tilde{\theta} \). Therefore, a solution of model (1) with virtual fuzzy variables \( \tilde{X}_{ip}^{(k)} \) can be obtained by solving the following max-min optimization problem:

\[
\max \{ \min_{i \in I(k)} \{ \mu_{\tilde{X}_{ip}^{(k)}}(\tilde{X}_{ip}^{(k)}) \} \} \quad \text{s.t.} \quad (\tilde{\theta}_p, \tilde{y}_p, \tilde{z}_p) \in T,
\]
or equivalently

\[
\max \left\{ \min_{i \in I^{(k)}} \left\{ \mu_{\tilde{X}_{ip}^{(k)}} \left( \tilde{X}_{ip}^{(k)} \right) \right\} \right\} \tag{5a}
\]

\[
s.t. \sum_{j=1}^{n} \lambda_j^{(k)} X_{ij}^{(k)} \leq \tilde{X}_{ip}^{(k)}, \ i \in I^{(k)}, k = 1, 2, \cdots, q, \tag{5b}
\]

\[
\sum_{j=1}^{n} \lambda_j^{(k)} Y_{rj}^{(k)} \geq Y_{rp}^{(k)}, \ r \in O^{(k)}, k = 1, 2, \cdots, q, \tag{5c}
\]

\[
\sum_{j=1}^{n} \lambda_j^{(k)} Z_{fj}^{(k)} \leq Z_{fp}^{(k)}, \ f \in M^{(k)}, k = 1, 2, \cdots, q, \tag{5d}
\]

\[
\sum_{j=1}^{n} \lambda_j^{(k)} Z_{gj}^{(k)} \geq Z_{gp}^{(k)}, \ g \in N^{(k)}, k = 1, 2, \cdots, q, \tag{5e}
\]

\[
0 \leq \tilde{X}_{ip}^{(k)} \leq X_{ip}^{(k)}, \ i \in I^{(k)}, k = 1, 2, \cdots, q, \tag{5f}
\]

\[
\lambda_j^{(k)} \geq 0, \ j = 1, 2, \cdots, n, k = 1, 2, \cdots, q, \tag{5g}
\]

where \(\bar{x}_p, y_p, z_p\) is the fuzzy virtual input, output and the intermediate product of \(DMU_p\), respectively. Introducing one new variable \(\alpha_p = \min_{i \in I^{(k)}} \left\{ \mu_{\tilde{X}_{ip}^{(k)}} \left( \tilde{X}_{ip}^{(k)} \right) \right\}\) in (5a) results in the following problem:

\[
\max \alpha_p \tag{6a}
\]

\[
s.t. \sum_{j=1}^{n} \lambda_j^{(k)} X_{ij}^{(k)} \leq \tilde{X}_{ip}^{(k)}, \ i \in I^{(k)}, k = 1, 2, \cdots, q, \tag{6b}
\]

\[
\sum_{j=1}^{n} \lambda_j^{(k)} Y_{rj}^{(k)} \geq Y_{rp}^{(k)}, \ r \in O^{(k)}, k = 1, 2, \cdots, q, \tag{6c}
\]

\[
\sum_{j=1}^{n} \lambda_j^{(k)} Z_{fj}^{(k)} \leq Z_{fp}^{(k)}, \ f \in M^{(k)}, k = 1, 2, \cdots, q, \tag{6d}
\]

\[
\sum_{j=1}^{n} \lambda_j^{(k)} Z_{gj}^{(k)} \geq Z_{gp}^{(k)}, \ g \in N^{(k)}, k = 1, 2, \cdots, q, \tag{6e}
\]

\[
\alpha_p \leq \frac{X_{ip}^{(k)} - \tilde{X}_{ip}^{(k)}}{X_{ip}^{(k)}}, \ i \in I^{(k)}, k = 1, 2, \cdots, q, \tag{6f}
\]

\[
0 \leq \tilde{X}_{ip}^{(k)} \leq X_{ip}^{(k)}, \ i \in I^{(k)}, k = 1, 2, \cdots, q, \tag{6g}
\]

\[
\lambda_j^{(k)} \geq 0, \ j = 1, 2, \cdots, n, k = 1, 2, \cdots, q, \tag{6h}
\]

\[
\alpha_p \geq 0. \tag{6i}
\]

To illustrate the relationship between the system efficiency, \(\theta_p\), of model (2) and the membership degree, \(\alpha_p\), of model (6), we have the following result.

**Theorem 1.** Assume that \(\theta^*_p\) and \(\alpha^*_p\) are the optimal solutions of model (2) and model (6), respectively. Then, \(\theta^*_p = 1 - \alpha^*_p\). In other words, solving model (6) is equivalent to solving model (2).
Proof. Since \( \alpha_p = \min_{i \in I^{(k)}, k = 1, \ldots, q} \{ \mu_{\tilde{X}_i}^{(k)}(\tilde{X}_i^{(k)}) \} \), we have \( \alpha_p \leq \frac{X_i^{(k)} - \tilde{X}_i^{(k)}}{X_i^{(k)}}, i \in I^{(k)}, k = 1, \ldots, q \). It implies that \( \tilde{X}_i^{(k)} \leq X_i^{(k)} - \alpha_p X_i^{(k)} = (1 - \alpha_p)X_i^{(k)}, i \in I^{(k)}, k = 1, \ldots, q \). Replacing (6b) in model (6) by taking into the above consideration results in the following problem:

\[
\max \alpha_p \\
\text{s.t.} \sum_{j=1}^{n} \lambda_j^{(k)} X_{ij}^{(k)} \leq \tilde{X}_{ip}^{(k)} \leq (1 - \alpha_p)X_{ip}^{(k)}, i \in I^{(k)}, k = 1, \ldots, q,
\]

(7a)

\[
\sum_{j=1}^{n} \lambda_j^{(k)} Y_{ij}^{(k)} \geq Y_{ip}, r \in O^{(k)}, k = 1, \ldots, q,
\]

(7b)

\[
\sum_{j=1}^{n} \lambda_j^{(k)} Z_{ij}^{(k)} \leq Z_{ip}, f \in M^{(k)}, k = 1, \ldots, q,
\]

(7c)

\[
\sum_{j=1}^{n} \lambda_j^{(k)} Z_{ij}^{(k)} \geq Z_{ip}, g \in N^{(k)}, k = 1, \ldots, q,
\]

(7d)

\[
\alpha_p \leq \frac{X_{ip} - \tilde{X}_{ip}}{X_{ip}}, i = 1, \ldots, m,
\]

(7e)

\[
0 \leq \tilde{X}_{ip} \leq X_{ip}, i = 1, \ldots, m,
\]

(7f)

\[
\lambda_j^{(k)} \geq 0, j = 1, \ldots, n, k = 1, \ldots, q,
\]

(7g)

\[
\alpha_p \geq 0.
\]

(7h)

Let \( \theta = 1 - \alpha_p \). Then, model (7) can be converted to:

\[
\max 1 - \theta
\]

s.t. \( \sum_{j=1}^{n} \lambda_j^{(k)} X_{ij}^{(k)} \leq \tilde{X}_{ip}^{(k)} \leq \theta_p X_{ip}^{(k)}, i \in I^{(k)}, k = 1, \ldots, q \),

(8a)

\[
\sum_{j=1}^{n} \lambda_j^{(k)} Y_{ij}^{(k)} \geq Y_{ip}, r \in O^{(k)}, k = 1, \ldots, q,
\]

(8b)

\[
\sum_{j=1}^{n} \lambda_j^{(k)} Z_{ij}^{(k)} \leq Z_{ip}, f \in M^{(k)}, k = 1, \ldots, q,
\]

(8c)

\[
\sum_{j=1}^{n} \lambda_j^{(k)} Z_{ij}^{(k)} \geq Z_{ip}, g \in N^{(k)}, k = 1, \ldots, q,
\]

(8d)

\[
1 - \theta_p \leq \frac{X_{ip}^{(k)} - \tilde{X}_{ip}^{(k)}}{X_{ip}^{(k)}}, i \in I^{(k)}, k = 1, \ldots, q,
\]

(8e)

\[
0 \leq \tilde{X}_{ip}^{(k)} \leq X_{ip}^{(k)}, i \in I^{(k)}, k = 1, \ldots, q,
\]

(8f)

\[
\lambda_j^{(k)} \geq 0, j = 1, \ldots, n, k = 1, \ldots, q,
\]

(8g)

\[
\theta_p \geq 0.
\]

(8h)
It should be noted that the constraint (8f) in model (8) is implied by \( \bar{X}_{ip}^{(k)} \leq \theta_p X_{ip}^{(k)} \), the constraint (8b) in model (8). In addition, since \( 0 \leq \frac{X_{ip}^{(k)} - X_{ip}^{(k)}}{X_{ip}^{(k)}} \leq 1 \), we have \( 0 \leq \theta_p \leq 1 \). Model (2) is then resulted by considering \( \max (1 - \theta_p) \) as \( \min \theta_p \). This means that solving model (6) is equivalently solving model (2) and \( \theta_p^* = 1 - \alpha_p^* \).

Theorem 1 indicates the equivalency between the system efficiency, \( \theta_p \), and the membership degree of model (6). In other words, model (6) represents a new version of model (2).

3. Network DEA with fuzzy non-discretionary factors. In this section, the fuzzy concept of discretionary factors is extended to describe the fuzzy non-discretionary factors in network DEA. To incorporate the fuzzy non-discretionary factor into a network DEA model, the membership function, \( \mu_{\bar{X}_{ip}} \), taking on the values between 0 and 1 is used to represent the degree of discretion that the \( DMU \) has on the fuzzy non-discretionary factor \( \bar{X}_{ip}^{(k)} \), \( i \in I_{FND}^{(k)}, k = 1, 2, \cdots, q \). The membership function of a fuzzy non-discretionary factor can be specified by introducing the risk-free and impossible bounds. The risk-free bounds are interpreted as the conservation values of the fuzzy non-discretionary factors that are most realistically found. Whereas, the impossible bounds are associated with the values of the fuzzy non-discretionary factors that are the least realistic. For each fuzzy non-discretionary factor, the change from its risk-free to impossible bound is represented by its membership function. It is assumed that membership functions are monotonically linearly and are equal to zero, if the fuzzy non-discretionary factor bounds are impossible, and are equal to one if they are risk free. Suppose that \( X_{ip}^{(k)L} \) and \( X_{ip}^{(k)U} \) represent the impossible and risk-free bounds of the fuzzy non-discretionary factor \( \bar{X}_{ip}^{(k)} \), \( i \in I_{FND}^{(k)}, k = 1, 2, \cdots, q \), respectively. A possible linear membership functions associated with the fuzzy non-discretionary variable \( \bar{X}_{ip}^{(k)} \) is given by

\[
\mu_{\bar{X}_{ip}}(\bar{X}_{ip}^{(k)}) = \frac{\bar{X}_{ip}^{(k)} - X_{ip}^{(k)L}}{X_{ip}^{(k)U} - X_{ip}^{(k)L}},
\]

where \( X_{ip}^{(k)L} \leq \bar{X}_{ip}^{(k)} \leq X_{ip}^{(k)U} \), \( i \in I_{FND}^{(k)}, k = 1, 2, \cdots, q \). The inclusion of the fuzzy non-discretionary variables into (6) results in the following linear problem:

\[
\begin{align*}
\max & \quad \sum_{n} \alpha_p^n \\
\text{s.t.} & \quad \sum_{j=1}^{n} \lambda_{ij}^{(k)} X_{ij}^{(k)} \leq \bar{X}_{ip}^{(k)}, \quad i \in I^{(k)}, k = 1, 2, \cdots, q, \\
& \quad \sum_{j=1}^{n} \lambda_{ij}^{(k)} Y_{ij}^{(k)} \geq Y_{ip}^{(k)}, \quad r \in O^{(k)}, k = 1, 2, \cdots, q, \\
& \quad \sum_{j=1}^{n} \lambda_{ij}^{(k)} Z_{ij}^{(k)} \leq Z_{ip}^{(k)}, \quad f \in M^{(k)}, k = 1, 2, \cdots, q, \\
& \quad \sum_{j=1}^{n} \lambda_{ij}^{(k)} Z_{ij}^{(k)} \geq Z_{ip}^{(k)}, \quad g \in N^{(k)}, k = 1, 2, \cdots, q,
\end{align*}
\]
If other types of fuzzy membership functions are employed, the linearity of model (10) could not be guaranteed for efficient computation. As similar to the definition of the membership function in (3), the input of the $DMU_p^{(k)}, i \in I_{FND}^{(k)}, k = 1, 2, \cdots, q$, can be used to represent the risk-free bound, $X_{ip}^{(k)}$, for the fuzzy non-discretionary factor $\tilde{X}_{ip}^{(k)}$, $i \in I_{FND}^{(k)}, k = 1, 2, \cdots, q$. Moreover, let $\theta_p^{*}$ be the optimal solution of (2) with all factors of $DMU_p$ being treated as discretionary variables. The optimal system efficiency of (2) is used to determine variables $X_{ip}^{(k)L}$ with $X_{ip}^{(k)L} = \theta_p^{*}X_{ip}^{(k)}$. Model (10) is then transformed into the following problem:

$$\max_{\alpha_p} \quad \alpha_p \ni \quad \sum_{j=1}^{n} \lambda_{ij}^{(k)} X_{ij}^{(k)} \leq X_{ip}^{(k)}, i \in I^{(k)}, k = 1, 2, \cdots, q,$$

$$\quad \sum_{j=1}^{n} \lambda_{rj}^{(k)} Y_{rj}^{(k)} \geq Y_{rp}^{(k)}, r \in O^{(k)}, k = 1, 2, \cdots, q,$$

$$\quad \sum_{j=1}^{n} \lambda_{fj}^{(k)} Z_{fj}^{(k)} \leq Z_{fp}^{(k)}, f \in M^{(k)}, k = 1, 2, \cdots, q,$$

$$\quad \sum_{j=1}^{n} \lambda_{gj}^{(k)} Z_{gj}^{(k)} \geq Z_{gp}^{(k)}, g \in N^{(k)}, k = 1, 2, \cdots, q,$$

$$\quad \alpha_p \leq \frac{X_{ip}^{(k)} - X_{ip}^{(k)}}{X_{ip}^{(k)}}, i \in I_D^{(k)}, k = 1, 2, \cdots, q,$$

$$\quad \alpha_p \leq \frac{X_{ip}^{(k)} - \theta_p^{*}X_{ip}^{(k)}}{X_{ip}^{(k)}}, i \in I_{FND}^{(k)}, k = 1, 2, \cdots, q,$$

$$\quad 0 \leq X_{ip}^{(k)} \leq \tilde{X}_{ip}^{(k)}, i \in I_D^{(k)}, k = 1, 2, \cdots, q,$$

$$\quad \theta_p^{*}X_{ip}^{(k)} \leq X_{ip}^{(k)} \leq \tilde{X}_{ip}^{(k)}, i \in I_{FND}^{(k)}, k = 1, 2, \cdots, q,$$

$$\quad \alpha_p \geq 0, \lambda_{ij}^{(k)} \geq 0, j = 1, 2, \cdots, n, k = 1, 2, \cdots, q.$$

4. **An illustrative application.** In this section, the proposed model is applied to assess the impact of information technology (IT) on firm performance in a banking industry. According to [18], the problem of assessing the impact of IT on bank performance can be separated into two stages connected in series as depicted in Figure 2. The first stage is a fund collection stage, where IT is applied to collect funds from bank customers in the form of deposits. The second stage is a profit generation stage, where deposits collected in the first stage are invested in securities and provided for loans. Three factors were selected as the inputs for the first stage: IT budget ($X_1$), fixed assets ($X_2$), and the number of employees ($X_3$); and
one factor as the output: the dollar value of deposits ($Z$), which was in turn used as the input for the second stage. The outputs considered from the second stage were: profits earned ($Y_1$) and the percentage of loans recovered ($Y_2$). The data set for assessing IT impact on firm performance in [18] is utilized and shown in Table 1.

![Figure 2. Network system discussed in [18].](image)

### Table 1. Data set for assessing IT impact on firm performance

| DMU | IT budget ($b$ billions) | Fixed assets ($b$ billions) | No. of employees ($b$ billions) | Deposits ($b$ billions) | Z ($b$ billions) | Profit ($b$ billions) | Fraction of loans recovered |
|-----|--------------------------|-----------------------------|---------------------------------|-------------------------|-----------------|----------------------|----------------------------|
| 1   | 0.150                    | 0.713                       | 13.3                            | 14.478                  | 0.232           | 0.986                |
| 2   | 0.170                    | 1.071                       | 16.9                            | 19.502                  | 0.340           | 0.986                |
| 3   | 0.235                    | 1.224                       | 24.0                            | 20.952                  | 0.363           | 0.986                |
| 4   | 0.211                    | 0.363                       | 15.6                            | 13.902                  | 0.211           | 0.982                |
| 5   | 0.133                    | 0.409                       | 18.485                          | 15.206                  | 0.237           | 0.984                |
| 6   | 0.497                    | 5.846                       | 56.42                           | 81.186                  | 1.103           | 0.955                |
| 7   | 0.060                    | 0.918                       | 56.42                           | 81.186                  | 1.103           | 0.986                |
| 8   | 0.071                    | 1.235                       | 12.0                            | 11.441                  | 0.199           | 0.985                |
| 9   | 1.500                    | 18.120                      | 89.51                           | 124.072                 | 1.858           | 0.972                |
| 10  | 0.120                    | 1.821                       | 19.8                            | 17.425                  | 0.274           | 0.983                |
| 11  | 0.120                    | 1.915                       | 19.8                            | 17.425                  | 0.274           | 0.983                |
| 12  | 0.050                    | 0.874                       | 13.1                            | 14.342                  | 0.177           | 0.985                |
| 13  | 0.370                    | 6.918                       | 12.5                            | 32.491                  | 0.648           | 0.945                |
| 14  | 0.440                    | 4.432                       | 41.9                            | 47.653                  | 0.639           | 0.979                |
| 15  | 0.431                    | 4.504                       | 41.1                            | 52.63                   | 0.741           | 0.981                |
| 16  | 0.110                    | 1.241                       | 14.4                            | 17.493                  | 0.243           | 0.988                |
| 17  | 0.053                    | 0.450                       | 7.6                             | 9.512                   | 0.067           | 0.980                |
| 18  | 0.345                    | 5.892                       | 15.5                            | 42.469                  | 1.002           | 0.948                |
| 19  | 0.128                    | 0.973                       | 12.6                            | 18.987                  | 0.243           | 0.985                |
| 20  | 0.055                    | 0.444                       | 5.9                             | 7.546                   | 0.153           | 0.987                |
| 21  | 0.057                    | 0.508                       | 5.7                             | 7.595                   | 0.123           | 0.987                |
| 22  | 0.098                    | 0.370                       | 14.1                            | 16.906                  | 0.233           | 0.981                |
| 23  | 0.104                    | 0.395                       | 14.6                            | 17.264                  | 0.263           | 0.983                |
| 24  | 0.206                    | 2.680                       | 19.6                            | 36.430                  | 0.601           | 0.982                |
| 25  | 0.067                    | 0.781                       | 10.5                            | 11.581                  | 0.120           | 0.987                |
| 26  | 0.109                    | 0.872                       | 12.1                            | 22.207                  | 0.248           | 0.972                |
| 27  | 0.0106                   | 1.757                       | 12.7                            | 20.670                  | 0.253           | 0.988                |

Due to the lack of information on how IT budgets were spent, the IT budget ($X_1$) is assumed to be a fuzzy non-discretionary input. To define the membership function
of the fuzzy non-discretionary input, we consider solving the following problem:

$$\begin{align*}
\min & \quad \alpha_p \\
\text{s.t.} & \quad \sum_{j=1}^{27} \lambda_j^{(1)} X_{1j} \leq X_{1p}, \sum_{j=1}^{27} \lambda_j^{(1)} X_{2j} \leq X_{2p}, \sum_{j=1}^{27} \lambda_j^{(1)} X_{3j} \leq X_{3p}, \\
& \quad \sum_{j=1}^{27} \lambda_j^{(2)} Y_{1j} \geq Y_{1p}, \sum_{j=1}^{27} \lambda_j^{(2)} Y_{2j} \geq Y_{2p}, \\
& \quad \sum_{j=1}^{27} \lambda_j^{(1)} Z_j \geq Z_p, \sum_{j=1}^{27} \lambda_j^{(2)} Z_j \leq Z_p, \\
& \quad \lambda_j^{(1)}, \lambda_j^{(2)}, j = 1, 2, \ldots, 27.
\end{align*}$$

(13)

Table 2 shows the system efficiency, $\theta_p$, $p = 1, 2, \ldots, 27$, of the problem (12), which can be used to define the impossible-free bound of the fuzzy non-discretionary input. According to Theorem 1, the system efficiency of the problem (12) can also be obtained by solving the new network DEA model:

$$\begin{align*}
\min & \quad \alpha_p \\
\text{s.t.} & \quad \sum_{j=1}^{27} \lambda_j^{(1)} X_{1j} \leq \bar{X}_{1p}, \sum_{j=1}^{27} \lambda_j^{(1)} X_{2j} \leq \bar{X}_{2p}, \sum_{j=1}^{27} \lambda_j^{(1)} X_{3j} \leq \bar{X}_{3p}, \\
& \quad \sum_{j=1}^{27} \lambda_j^{(2)} Y_{1j} \geq \bar{Y}_{1p}, \sum_{j=1}^{27} \lambda_j^{(2)} Y_{2j} \geq \bar{Y}_{2p}, \\
& \quad \sum_{j=1}^{27} \lambda_j^{(1)} Z_j \geq \bar{Z}_p, \sum_{j=1}^{27} \lambda_j^{(2)} Z_j \leq \bar{Z}_p \\
& \quad \alpha_p \leq \frac{X_{1p} - \bar{X}_{1p}}{X_{1p}}, \alpha_p \leq \frac{X_{2p} - \bar{X}_{2p}}{X_{2p}}, \alpha_p \leq \frac{X_{3p} - \bar{X}_{3p}}{X_{3p}}, \\
& \quad \bar{X}_{1p} \leq X_{1p}, \bar{X}_{2p} \leq X_{2p}, \bar{X}_{3p} \leq X_{3p}, \\
& \quad \lambda_j^{(1)}, \lambda_j^{(2)}, j = 1, 2, \ldots, 27, \\
& \quad \bar{X}_{1p}, \bar{X}_{2p}, \bar{X}_{3p}, \alpha_p \geq 0.
\end{align*}$$

The comparison of the system efficiency, $\theta_p$, of the problem (12), and the membership degree, $\alpha$, of the problem (13) is also provided in Table 2. With the information in Table 2, the membership function of the fuzzy non-discretionary factor can
be determined. For example, to define the membership function of the fuzzy non-
discretionary factor for DMU_4, the impossible bound of the fuzzy non-discretionary
factor is defined as \( \bar{X}_{14} = \theta^*_p \times X_{14} = 0.5986 \times 0.211 = 0.1263 \). The membership
function of the fuzzy non-discretionary factor for DMU_4 can then be described as
\[
\mu_{\bar{X}_{14}}(\bar{X}_{14}) = \frac{\bar{X}_{14} - 0.1263}{0.211 - 0.1263}.
\]

The inclusion of the fuzzy non-discretionary variable into the problem (13) results
in the following optimization problem:
\[
\begin{align*}
\max & \quad \alpha_p \\
\text{s.t.} & \quad \sum_{j=1}^{27} \lambda_{j}^{(1)} X_{1j} \leq \bar{X}_{1p}, \sum_{j=1}^{27} \lambda_{j}^{(1)} X_{2j} \leq \bar{X}_{2p}, \sum_{j=1}^{27} \lambda_{j}^{(1)} X_{3j} \leq \bar{X}_{3p}, \\
& \quad \sum_{j=1}^{27} \lambda_{j}^{(2)} Y_{1j} \geq Y_{1p}, \sum_{j=1}^{27} \lambda_{j}^{(2)} Y_{2j} \geq Y_{2p}, \\
& \quad \sum_{j=1}^{27} \lambda_{j}^{(1)} Z_{j} \geq Z_{p}, \sum_{j=1}^{27} \lambda_{j}^{(2)} Z_{j} \leq Z_{p}, \\
& \quad \alpha_p \leq \frac{\bar{X}_{1p} - \theta^*_p X_{1p}}{X_{1p} - \theta^*_p X_{1p}}, \\
& \quad \alpha_p \leq \frac{\bar{X}_{2p} - \theta^*_p X_{2p}}{X_{2p} - \theta^*_p X_{2p}}, \\
& \quad \alpha_p \leq \frac{\bar{X}_{3p} - \theta^*_p X_{3p}}{X_{3p} - \theta^*_p X_{3p}}, \\
& \quad \theta^*_p X_{1p} \leq X_{1p} \leq \bar{X}_{1p}, 0 \leq \bar{X}_{2p} \leq X_{2p} \leq \bar{X}_{2p}, 0 \leq \bar{X}_{3p} \leq X_{3p}, \\
& \quad \lambda_{j}^{(1)}, \lambda_{j}^{(2)}, \alpha_p \geq 0, \quad j = 1, 2, \ldots, 27.
\end{align*}
\]

The results of the efficiency analysis when the IT budget is fuzzy non-discretionary
is shown in Table 3. It reveals that modeling the IT budget as a fuzzy non-
discretionary factor improves the system performance of a firm. For example, the
system efficiency of DMU_4 obtained by solving (14) is 0.7136 and it is larger than
the system efficiency obtained by solving (13) at 0.5986. Moreover, there are some
suggestions for improvement of the input variables. For instance, reducing the in-
put variable \( X_1 \) of DMU_4 from 0.211 to \( \bar{X}^*_1 = 0.1506 \), the input variable \( X_2 \) of
DMU_4 from 0.363 to \( \bar{X}^*_2 = 0.2564 \), and the input variable \( X_3 \) of DMU_4 from 15.6
to \( \bar{X}^*_3 = 10.9013 \) is suggested for the improvement of the firm’s system performance.

The extension of the network DEA model incorporating situations whereby some
inputs or outputs are partially controlled can be viewed as one which introduces
appropriate changes in the relevant data, rather than the model itself.

The proposed model can also be employed to applications relating to the service
industry where managers can exercise only a limited control over some variables, for
example, they might have the right to authorize a limited amount of over-time but
they have to follow general guidelines of their organizations in many other aspects
involving the use of human resources [10].

5. Conclusion. This work presents a new model for ranking DMUs in network
DEA based on fuzzy concept. An equivalence between the conventional network
DEA model and the proposed model is established. An extension of the new model
The results of solving the proposed fuzzy non-discretionary Model (14) are provided in Table 3.

Table 3. The results of solving the proposed fuzzy non-discretionary Model (14)

| DMU | Fuzzy non-discretionary input | $X_{1j}$ | $X_{2j}$ | $X_{3j}$ | $\alpha*$ | $1 - \alpha*$ | Rank |
|-----|-------------------------------|---------|---------|---------|-----------|-------------|------|
| 1   | 0.1102 0.5236 9.6335          | 0.2654  | 0.7346  | 18      |
| 2   | 0.1260 0.7723 12.4259         | 0.2589  | 0.7411  | 17      |
| 3   | 0.1586 0.8079 16.1328         | 0.3253  | 0.6747  | 25      |
| 4   | 0.1506 0.2564 10.9013         | 0.2864  | 0.7136  | 21      |
| 5   | 0.0921 0.2793 11.2165         | 0.3077  | 0.6923  | 23      |
| 6   | 0.4008 4.6342 45.4289         | 0.1936  | 0.8064  | 10      |
| 7   | 0.0600 0.9180 56.4200         | 0.0000  | 1.0000  | 1       |
| 8   | 0.0485 0.7677 8.0988          | 0.3173  | 0.6827  | 24      |
| 9   | 1.0908 13.1471 64.8529        | 0.2728  | 0.7272  | 20      |
| 10  | 0.0798 1.0715 12.8295         | 0.3351  | 0.6649  | 26      |
| 11  | 0.0797 1.0997 12.7416         | 0.3358  | 0.6642  | 27      |
| 12  | 0.0376 0.6544 9.7883          | 0.2490  | 0.7510  | 14      |
| 13  | 0.3519 5.3291 11.8900         | 0.0488  | 0.9512  | 5       |
| 14  | 0.3116 3.1047 29.5929         | 0.2918  | 0.7082  | 22      |
| 15  | 0.3212 3.2772 30.4969         | 0.2547  | 0.7453  | 16      |
| 16  | 0.0824 0.8943 10.7729         | 0.2512  | 0.7488  | 15      |
| 17  | 0.0413 0.3500 5.8871          | 0.2200  | 0.7798  | 11      |
| 18  | 0.3450 5.8920 15.5000         | 0.0000  | 1.0000  | 1       |
| 19  | 0.1080 0.8154 10.6151         | 0.1565  | 0.8435  | 7       |
| 20  | 0.0421 0.3349 4.4948          | 0.2343  | 0.7657  | 13      |
| 21  | 0.0441 0.3904 4.3686          | 0.2268  | 0.7732  | 12      |
| 22  | 0.0813 0.3043 11.6775         | 0.1707  | 0.8293  | 8       |
| 23  | 0.0853 0.3216 11.9554         | 0.1802  | 0.8198  | 9       |
| 24  | 0.1925 2.4125 18.3176         | 0.0654  | 0.9346  | 6       |
| 25  | 0.0488 0.5448 7.5342          | 0.2717  | 0.7283  | 19      |
| 26  | 0.1000 0.8720 12.1000         | 0.0000  | 1.0000  | 1       |
| 27  | 0.0106 1.7570 12.7000         | 0.0000  | 1.0000  | 1       |

Incorporating situations whereby some inputs are fuzzy non-discretionary variables is developed. In many situations the discretionary indexes which describe the degree of discretion over factors in the previous developments are usually not known and are arbitrarily determined by decision makers. Membership functions are employed to replace the need to determine the discretionary indexes of fuzzy non-discretionary variables in the proposed model. An illustrative application assessing the impact of IT on firm performance in a banking industry is included. Our results show that modeling the IT budget as a fuzzy non-discretionary factor improves the system performance of a firm. Some suggestions for the improvement of input variables in the illustrative application are also provided. The proposed network DEA model allowing varying degrees of control over factors can be viewed as one which introduces appropriate changes in the relevant data, which is more suitable to many real-life applications.

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