ORTHOGONAL POLYNOMIALS, LAGUERRE FOCK SPACE
AND QUASI-CLASSICAL ASYMPTOTICS

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Abstract. Continuing our earlier investigation of the Hermite case [J. Math.
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Toeplitz quantization scheme associated with Laguerre polynomials. In partic-
ular, we describe a "Laguerre analogue" of the classical Fock (Segal-Bargmann)
space and the relevant semi-classical asymptotics of its Toeplitz operators;
the former actually turns out to coincide with the Hilbert space appearing in
the construction of the well-known Barut-Girardello coherent states. Further
extension to the case of Legendre polynomials is likewise discussed.

1. Introduction

One of the very well studied methods of quantizing Kähler manifolds is the
Berezin-Toeplitz quantization [9, 10]. In the simplest case of a phase space Ω
admitting a global real-valued potential Ψ (so that the Kähler form is given by
ω = ∂Ψ), one considers the $L^2$ space

$$ L^2_h = \{ f \text{ measurable on } \Omega : \int_{\Omega} |f|^2 e^{-\Psi/\hbar} \omega^n < \infty \} \quad (h > 0), $$

its subspace $L^2_{\text{hol},h}$ of functions holomorphic on Ω (the weighted Bergman space),
and the orthogonal projection $P_h : L^2_h \to L^2_{\text{hol},h}$. For a bounded measurable function
$f$ on Ω, the Toeplitz operator $T_f$ on $L^2_{\text{hol},h}$ with symbol $f$ is then defined by

$$ T_f u(x) = \int_{\Omega} u(y) f(y) K_h(x,y) e^{-\Psi(y)/\hbar} \omega(y)^n. $$

When the manifold Ω is not simply connected, one has to assume that the coho-
mology class of $\omega$ is integral, so that there exists a Hermitian line bundle $L$
with the canonical connection whose curvature form coincides with $\omega$; and the spaces
$L^2_{\text{hol},h}$ (and $L^2_h$) get replaced by the space of all holomorphic (or all measurable,
respectively) square-integrable sections of $L^{\otimes k}$, $k = \frac{1}{\hbar} = 1, 2, 3, \ldots$. In any case,
under reasonable technical assumptions on Ω and $\omega$, the Toeplitz operators satisfy

$$ T_f T_g \approx T_{fg} + h T_{C_1(f,g)} + h^2 T_{C_2(f,g)} + \ldots \quad \text{as } h \to 0, $$

with some bidifferential operators $C_j$ such that $C_1(f,g) - C_1(g,f) = \frac{i}{2\pi} \{f,g\}$,
implying in particular that the "correspondence principle"

$$ T_f T_g - T_g T_f \approx \frac{i\hbar}{2\pi} T_{\{f,g\}} $$
and, moreover, of a surprise that actually yields, for \( f \) it immediately follows that even to expect (7) to be defined, not to say bounded, on some space (whereas with original quantization setting).

The purpose of the present paper, which is a sequel to [3], is to highlight an operator calculus of a completely different flavour, which nonetheless bears certain resemblance to [5] and [3], and arises in a quite unexpected setting — namely, in connection with orthogonal polynomials. To be more specific, let \( H_n(x) \) stand for the standard Hermite polynomials, and, for \( 0 < \epsilon < 1 \), set

\[
K_\epsilon(x, y) = \sum_{n=0}^{\infty} \epsilon^n \|H_n\|^{-2} H_n(x) \overline{H_n(y)}, \quad x, y \in \mathbb{R}.
\]

Here \( \|H_n\| \) denotes the norm in \( L^2(\mathbb{R}, e^{-x^2} \, dx) \), where the \( \{H_n\} \) form an orthogonal basis. Then \( K_\epsilon \) is a positive-definite function, and, hence, determines uniquely a Hilbert space \( \mathcal{H}_\epsilon \) of functions on \( \mathbb{R} \) for which \( K_\epsilon \) is the reproducing kernel [6]; this space has been studied in [19] and was also encountered in [4] when studying "squeezed" coherent states and their representations in terms of Hermite polynomials of a complex variable. (The definition of this kernel may perhaps seem a bit artificial at first glance, but so must have seemed (1) when it first came around in Berezin’s papers!)

For a (reasonable) function \( f \) on \( \mathbb{R} \), set

\[
T_f u(x) := \int_{\mathbb{R}} u(y) f(y) K_\epsilon(x, y) e^{-y^2} \, dy.
\]

This certainly resembles the expression [3] for Toeplitz operators, however, note that this time there is no \( L^2 \) space around like [11] which would contain \( \mathcal{H}_\epsilon \) as a closed subspace (in fact, the set \( \{f(x)e^{-x^2/2} : f \in \mathcal{H}_\epsilon\} \) is a dense, rather than proper closed, subset of \( L^2(\mathbb{R}) \)), so there is no projection like \( P_\hbar \) around and the original definition [2] makes no sense. In particular, there is no reason a priori even to expect (7) to be defined, not to say bounded, on some space (whereas with [2] it immediately follows that \( \|T_f\| \) is not greater than the norm of the operator of “multiplication by \( f \)” on \( L^2 \), hence \( \|T_f\| \leq \|f\|_{\infty} \)). It may therefore come as a bit of a surprise that (7) actually yields, for \( f \in L^\infty(\mathbb{R}) \), a bounded operator on \( L^2(\mathbb{R}) \), and, moreover, \( T_f \) enjoys a nice asymptotic behaviour as \( \epsilon \to 1 \), which we saw in [3] to correspond, in a very natural sense, to the semiclassical limit \( \hbar \searrow 0 \) in the original quantization setting.

Furthermore, it turns out that the space \( \mathcal{H}_\epsilon \) actually consists, up to a trivial equivalence, precisely of restrictions to \( \mathbb{R} \) of holomorphic functions forming a very standard reproducing kernel space on the entire complex plane \( \mathbb{C} \). Namely, in addition to being an orthogonal basis in \( L^2(\mathbb{R}, e^{-|z|^2} \, dx) \), the Hermite polynomials also satisfy an orthogonality relation over \( \mathbb{C} \) [19, 24]:

\[
\int_{\mathbb{C}} H_n(z) \overline{H_m(z)} e^{-\frac{\epsilon}{1+\epsilon} x^2 - \frac{\epsilon^2}{4 \pi n} y^2} \, dx \, dy = \frac{\sqrt{1-\epsilon^2}}{2\epsilon} n! 2^n n^2 \pi e^{-n} \delta_{mn}, \quad z = x + yi,
\]

It follows that the multiplication operator

\[
M : f(z) \mapsto \frac{\sqrt{2\epsilon}}{(1-\epsilon^2)^{1/4} \pi^{1/4}} e^{\frac{x^2}{1+\epsilon}} f(z)
\]
maps the space $H_\epsilon$ onto the space of holomorphic functions on $\mathbb{C}$ with reproducing kernel
\[
F_\epsilon(z, w) := \frac{2\epsilon}{(1-\epsilon^2)\pi} K_\epsilon(z, w) = \frac{2\epsilon}{(1-\epsilon^2)\pi} e^{\frac{2\epsilon}{1-\epsilon^2} z \bar{w},}
\]
that is, onto the standard Fock (Segal-Bargmann) space
\[
\mathcal{F}_\epsilon = L^2_{\text{hol}}(\mathbb{C}, d\mu_\epsilon)
\]
of all entire functions on $\mathbb{C}$ square-integrable with respect to the Gaussian measure
\[
d\mu_\epsilon(z) := e^{-2\epsilon|z|^2/(1-\epsilon)} \, dz,
\]
where $dz$ stands for the Lebesgue area measure on $\mathbb{C}$. Now $\mathcal{F}_\epsilon$ is precisely the space $L^2_{\text{hol},h}$ as in (1) for $\Omega = \mathbb{C}$ equipped with the standard (i.e. Euclidean) Kähler structure. Using the above correspondence between $\mathcal{F}_\epsilon$ and $L^2(\mathbb{R}, e^{-|x|^2} \, dx)$, one can thus transfer the Toeplitz operators (3) on $\mathcal{F}_\epsilon$ into operators on $L^2(\mathbb{R}, e^{-|x|^2} \, dx)$ and, via another multiplication operator, on $L^2(\mathbb{R})$. The latter turn out to belong to the standard Weyl calculus, and it was shown in [3] that in this way one can actually recover, from this seemingly totally unrelated Ansatz involving Hermite polynomials, the whole Berezin-Toeplitz quantization (on $\mathbb{C}$) reviewed in the beginning.

In the present paper, we show that all the above, in some sense, remains in force also for the Laguerre polynomials $L_n$ in the place of $H_n$. In particular, we establish the existence of a certain analogue, associated to the Laguerre polynomials, of the Fock spaces $\mathcal{F}_\epsilon$, and study the semi-classical asymptotics of the Toeplitz operators there. Surprisingly, this “Laguerre Fock space” turns out to coincide with the space of entire functions discovered by Barut and Girardello [7] in the construction of coherent states that nowadays bear their name. (Similar spaces were also obtained in [1] while working with ensembles of non-Hermitian matrices and in [17, 19, 23].) The associated Toeplitz operators and their asymptotics just mentioned, however, up to the authors’ knowledge seem not to have previously appeared in the literature: it turns out that they again satisfy the correspondence principle [5], but with the Poisson bracket coming from the flat metric on the punctured complex plane $\mathbb{C} \setminus \{0\}$ (which is somewhat surprising). We also discuss the case of Legendre polynomials, where things turn to work out somewhat differently.

The necessary standard material on Laguerre polynomials is reviewed in Section 2, and the associated reproducing kernel Hilbert spaces are introduced there as well. The Laguerre Fock space is discussed in Section 3, and its Toeplitz operators in Section 5. A result exhibiting the Laguerre polynomials as a certain “squeezed” basis of the Laguerre Fock space is discussed in Section 4. The case of Legendre polynomials is analyzed in Section 6, and some concluding remarks and speculations are collected in the final Sections 7 and 8.

2. Laguerre Polynomials

The Laguerre polynomials $L_n(x), n = 0, 1, 2, \ldots$, are defined by the formula
\[
L_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n} x^n e^{-x}.
\]
They are orthonormal on the half-line $\mathbb{R}_+ = (0, +\infty)$ with respect to the weight $e^{-x}$; thus the functions
\[
l_n(x) := e^{-x/2} L_n(x), \quad n = 0, 1, 2, \ldots,
\]
form an orthonormal basis of $L^2(\mathbb{R}_+)$. They can also be obtained from the generating function

$$
\sum_{n=0}^{\infty} L_n(x)z^n = \frac{1}{1-z}e^{xz}, \quad x \in \mathbb{C}, |z| < 1.
$$

The series

$$
L_\epsilon(x, y) = \sum_{n=0}^{\infty} \epsilon^n L_n(x)L_n(y), \quad \tilde{L}_\epsilon(x, y) = \sum_{n=0}^{\infty} \epsilon^n l_n(x)l_n(y) = \frac{L_\epsilon(x, y)}{e^{(x+y)/2}}
$$

converge for all $x, y > 0$, and

$$
L_\epsilon(x, y) = \frac{1}{1-\epsilon}e^{-\frac{\epsilon}{1-\epsilon}(x+y)}I_0\left(\frac{2\sqrt{xy\epsilon}}{1-\epsilon}\right),
$$

where $I_0(z) = \sum_{k=0}^{\infty} \left(\frac{z^k}{k!2^k}\right)^2$ is the modified Bessel function; see e.g. [5, §6.2]. The differential equation for Laguerre polynomials

$$
xL_n''(x) + (1-x)L_n'(x) + nL_n(x) = 0
$$

is equivalent to the equation

$$
Al_n = nl_n, \quad Au(x) := -xu''(x) - u'(x) + \frac{x-2}{4}u(x),
$$

for the functions $l_n(x)$.

Drawing inspiration from (3), we may define, for a function (“symbol”) $f$ on $\mathbb{R}_+$, the corresponding “Toeplitz operator” $\tilde{T}_f^{(\epsilon)}$, $0 < \epsilon < 1$, on $L^2(\mathbb{R}_+)$ by

$$
\tilde{T}_f^{(\epsilon)}u(x) := \int_{0}^{\infty} u(y)f(y)L_\epsilon(x, y)\, dy.
$$

As in [3], these turn out to be actually bounded operators, and possess a kind of “semi-classical” asymptotic expansion as $\epsilon \to 1$.

**Theorem 1.** For $f \in L^\infty(\mathbb{R}_+)$ the operator $\tilde{T}_f^{(\epsilon)}$ is bounded on $L^2(\mathbb{R}_+)$.

**Proof.** By (10)

$$
\tilde{T}_f^{(\epsilon)}u = \sum_{n} \epsilon^n \langle fu, l_n \rangle l_n.
$$

Thus, for any $0 < \epsilon < 1$,

$$
\|\tilde{T}_f^{(\epsilon)}u\|^2 = \sum_{n} \epsilon^{2n} |\langle fu, l_n \rangle|^2 \leq \sum_{n} |\langle fu, l_n \rangle|^2 = \|fu\|^2 \leq \|f\|_{L^\infty}^2 \|u\|^2,
$$

so $\|\tilde{T}_f^{(\epsilon)}\| \leq \|f\|_{L^\infty}$.

**Theorem 2.** We have

$$
\tilde{T}_f^{(\epsilon)} = \epsilon^A M_f,
$$

where $A$ is as in (12), $\epsilon^A$ is understood in the sense of the spectral theorem, and $M_f$ stands for the operator of “multiplication by $f$”. Consequently, as $\epsilon \to 1$,

$$
\tilde{T}_f^{(\epsilon)}u \approx \sum_{k=0}^{\infty} \frac{(\log \epsilon)^k}{k!}A^k(fu).
$$
Proof: We have

$$\tilde{T}_f^{(\epsilon)} u = \int_0^{\infty} u(y) f(y) \sum_n \epsilon^n l_n(y) l_n \, dy$$

$$= \sum_n \epsilon^n \langle uf, l_n \rangle l_n$$

$$= \sum_n \langle uf, l_n \rangle \epsilon A l_n$$

$$= \epsilon A(uf) = \sum_k (\log \epsilon)^k k! A^k(uf).$$

□

Of course, using the familiar series

$$\log \epsilon = -\sum_{j=1}^{\infty} \frac{(1-\epsilon)^j}{j}$$

one could easily pass in (13) from powers of $\log \epsilon$ to powers of $(1-\epsilon)$.

The beginning of the asymptotic expansion (13) reads

$$\tilde{T}_f^{(\epsilon)} u = fu + (1-\epsilon)A(uf) + O((1-\epsilon)^2),$$

or

$$\tilde{T}_f^{(\epsilon)} = M_f + (1-\epsilon)AM_f + O((1-\epsilon)^2).$$

Using the similar formulas for $g$ and $fg$ and subtracting, we arrive at

$$\tilde{T}_f^{(\epsilon)} \tilde{T}_g^{(\epsilon)} - \tilde{T}_f^{(\epsilon)} \tilde{T}_g = (1-\epsilon)\left[(x-1)(g(xf')'-f(xg')')I + 2x(fg'-gf')D\right] + O((1-\epsilon)^2),$$

where we introduced the notation

$$D u(x) := \frac{d u(x)}{dx}$$

for the differentiation operator on $\mathbb{R}$. Comparing these formulas with (5) and (4) — the role of the Planck constant being now played by the quantity $1-\epsilon$ — we see that, first of all, the role of the Poisson bracket is now played by the (second-order) expression $g(x f')' - f(x g')'$; and, secondly, that in addition to the "Toeplitz" operators $\tilde{T}^{(\epsilon)}$, the differentiation operator $D$ appears too.

As with Hermite polynomials, one also again has Hilbert spaces for which $L_\epsilon$ and $\tilde{L}_\epsilon$ are the reproducing kernels:

$$L_\epsilon := \{ f = \sum_n f_n L_n : \sum_n \epsilon^{-n} \| f_n \|^2 < \infty \},$$

$$\tilde{L}_\epsilon := \{ f = \sum_n f_n \tilde{l}_n : \sum_n \epsilon^{-n} \| f_n \|^2 < \infty \}.$$

Here $\tilde{L}_\epsilon$ is a space of functions on $\mathbb{R}_+$, dense in $L^2(\mathbb{R}_+)$. It turns out that just as for the Hermite polynomials in [3], $L_\epsilon$ again extends to a space of holomorphic functions on all of $\mathbb{C}$. 

Theorem 3. Each \( f \in \mathcal{L}_\epsilon \) extends to an entire function on \( \mathbb{C} \), and \( \mathcal{L}_\epsilon \) is the space of (the restrictions to \( \mathbb{R}_+ \) of) holomorphic functions on \( \mathbb{C} \) with reproducing kernel

\[
L_\epsilon(x, y) = \sum_{n=0}^{\infty} e^n L_n(x) L_n(y) = \frac{1}{1 - \epsilon} e^{-(x+y)} I_0\left(\frac{2\sqrt{\epsilon xy}}{1 - \epsilon}\right), \quad x, y \in \mathbb{C}.
\]

Proof. By (9) and Cauchy estimates, we have for each \( 0 < r < 1 \) and \( x \in \mathbb{C} \)

\[
(17) \quad r^n |L_n(x)| \leq \sup_{|z|=r} \left| \frac{1}{1 - z} e^{\frac{z}{1-\epsilon}} \right| \leq \frac{1}{1 - r} e^{\frac{|x|}{1-\epsilon}}.
\]

Thus

\[
\sum_n |f_n L_n(x)| \leq e^{\frac{|x|}{r(1-r)}} \sum_n |f_n r^{-n}|
\]

\[
\leq e^{\frac{|x|}{r(1-r)}} \|f\| \left( \sum_n e^{n r^{-2n}} \right)^{1/2}
\]

\[
\leq e^{\frac{|x|}{r(1-r)}} \|f\| \frac{r}{\sqrt{r^2 - \epsilon}}
\]

whenever \( r \in (\sqrt{\epsilon}, 1) \). Thus the series

\[
f(x) = \sum_n f_n L_n(x)
\]

converges for any \( x \in \mathbb{C} \), and uniformly on compact subsets. The rest follows as in the proof of Theorem 2 in [3]. \( \square \)

3. THE LAGUERRE FOCK SPACE

It turns out that the Laguerre polynomials also satisfy an orthogonality relation over the complex plane, similarly to \( [8, 7.12(19)] \) for the Hermite polynomials.

Recall that the modified Bessel function of the third kind \( K_0 \) is defined by

\[
(18) \quad K_0(t) = \int_{1}^{\infty} \frac{e^{-tx}}{\sqrt{x^2 - 1}} \, dx, \quad \text{Re} \ t > 0
\]

(see \([8\] 7.12(19)]\). One has \( K_0(t) \sim \log \frac{1}{t} \) as \( t \searrow 0 \), while

\[
(19) \quad K_0(t) \sim \sqrt{\frac{\pi}{2t}} e^{-t} \quad \text{as} \ t \to +\infty.
\]

Lemma 4. For any \( k = 0, 1, 2, \ldots, \)

\[
(20) \quad \int_{0}^{\infty} 2^kt K_0(2\sqrt{t}) \, dt = k!^2.
\]
Proof. Making the indicated changes of variables and using Fubini,
\[ \int_0^\infty 2t^k K_0(2\sqrt{t}) \, dt = \left( t \to \frac{t^2}{4} \right) \]
\[ = \int_0^\infty t^{2k+1} 2^{-2k} K_0(t) \, dt \]
\[ = \int_1^\infty \int_0^\infty t^{2k+1} 2^{-2k} \frac{e^{-tx}}{\sqrt{x^2 - 1}} \, dt \, dx \quad \left( t \to \frac{t}{x} \right) \]
\[ = \int_1^\infty 2^{-2k} x^{-2k-2} \frac{\Gamma(2k + 2)}{\sqrt{x^2 - 1}} \, dx \quad \left( x \to s^{-1/2} \right) \]
\[ = \int_0^1 2^{-2k} s^{k+1} \frac{\Gamma(2k + 2)}{\Gamma(k + \frac{3}{2})} \frac{ds}{2 \sqrt{s}} \]
\[ = 2^{-2k-1} \frac{\Gamma(1/2)k!}{\Gamma(k + \frac{3}{2})} = k!^2 \]
by the doubling formula for the Gamma function. \qed

Remark. Another way to arrive at (20) is the following: starting with the double integral
\[ k!^2 = \int_0^\infty \int_0^\infty e^{-x} e^{-y} x^k y^k \, dx \, dy, \]
we make the change of variable \( s = x + y, \ p = xy, \) so \( ds \, dp = |x - y| \, dx \, dy = \sqrt{s^2 - 4p} \, dx \, dy. \) This yields
\[ k!^2 = 2 \int_0^\infty p^k \int_0^\infty e^{-s} \frac{ds}{\sqrt{s^2 - 4p}} \, dp \]
\[ = 2 \int_0^\infty p^k K_0\left(2\sqrt{\frac{p}{s}}\right) \, dp \]
by the change of variable \( s \to 2s \sqrt{p}. \) Note that \( 2K_0(2\sqrt{t}) \) is the unique function whose moments are given by (20), in view of Carleman’s criterion \( [2, p. 85], \) since \( \sum_k k!^{-1/k} = \infty \) by Stirling’s formula.

Iterating the above argument, it is also clear how to construct functions on \( \mathbb{R}_+ \) whose moments will be \( k!^3, \) or \( k!^4, \) and so forth. \qed

Our main result in this section is the following.

**Theorem 5.** Let \( 0 < \epsilon < 1 \) and denote for brevity
\[ c = \frac{\epsilon}{1 - \epsilon}. \]
Then for \( m, n = 0, 1, 2, \ldots, \)
\[ \int_{\mathbb{C}} L_n(z) L_m(z) e^{cz + c\bar{z}} K_0\left(\frac{2\sqrt{\epsilon}}{1 - \epsilon} |z|\right) \, dz = \frac{\pi}{2c} \epsilon^{-n} \delta_{mn}. \]

*Proof.** By virtue of the last lemma,
\[ \int_0^\infty k^k K_0\left(2\sqrt{\frac{\epsilon}{1 - \epsilon} \sqrt{t}}\right) \, dt = \frac{1}{2} \left(1 - \frac{\epsilon}{\sqrt{\epsilon}}\right)^{2k+2} k!^2. \]
Using the generating function (9), we thus have
\[
\sum_{m,n=0}^{\infty} t^m s^n \int_{\mathbb{C}} L_n(z) L_m(z) e^{cz+\sigma} K_0(\frac{2\sqrt{\sigma}}{1-\tau^2} |z|) \, dz
\]
\[
= \frac{1}{(1-t)(1-s)} \int_{\mathbb{C}} e^{\frac{zt}{1-\tau^2} + \frac{zs}{1-\tau^2} + cz + \sigma} K_0(\frac{2\sqrt{\sigma}}{1-\tau^2} |z|) \, dz
\]
\[
= \frac{1}{(1-t)(1-s)} \sum_{j,k=0}^{\infty} \left( \frac{t}{t-1} + c \right)^j \left( \frac{s}{s-1} + c \right)^k \int_{\mathbb{C}} \frac{z^j}{j!} \frac{\pi}{k!} K_0(\frac{2\sqrt{\sigma}}{1-\tau^2} |z|) \, dz
\]
\[
= \frac{1}{(1-t)(1-s)} \sum_{j,k=0}^{\infty} \left( \frac{t}{t-1} + c \right)^j \left( \frac{s}{s-1} + c \right)^k \pi \delta_{jk} \left( \frac{1-\epsilon}{\sqrt{\epsilon}} \right)^{2k+2}
\]
\[
= \frac{(1-\epsilon)^2 \pi/(2\epsilon)}{(1-t)(1-s)} \left[ 1 - \frac{(1-\epsilon)^2}{\epsilon} \left( \frac{t}{t-1} + c \right) \left( \frac{s}{s-1} + c \right) \right]^{-1},
\]
where the interchange of the integration and summation in the first equality is legitimate for
\[
\left| \frac{t}{t-1} \right| + c < \frac{\sqrt{\epsilon}}{1-\sqrt{\epsilon}}, \quad \left| \frac{s}{s-1} \right| + c < \frac{\sqrt{\epsilon}}{1-\sqrt{\epsilon}}
\]
— hence, for \( t, s \) in some neighbourhood of zero — thanks to (17) and (19). Now

\[
(1-t)(1-s) \left[ 1 - \frac{(1-\epsilon)^2}{\epsilon} \left( \frac{t}{t-1} + c \right) \left( \frac{s}{s-1} + c \right) \right] =
\]
\[
= (t-1)(s-1) - \frac{(1-\epsilon)^2}{\epsilon} (t+s-1) + \frac{(1-\epsilon)^2}{\epsilon} (t+s-1)
\]
\[
= \left[ 1 - \epsilon^2 \frac{(1-\epsilon)^2}{\epsilon} \right] + \left[ (1+c)(1-\epsilon)^2 \right] (t+s) + \left[ 1 - (1-\epsilon)^2 \frac{(1-\epsilon)^2}{\epsilon} \right] st
\]
\[
= (1-\epsilon) \left( 1 - \frac{st}{\epsilon} \right)
\]
by (21). Thus (23) equals
\[
\frac{(1-\epsilon)^2 \pi/(2\epsilon)}{1-\epsilon} = \frac{(1-\epsilon)^2 \pi}{2\epsilon} \sum_{k=0}^{\infty} \left( \frac{ts}{\epsilon} \right)^k
\]
and (22) follows. \(\square\)

**Corollary 6.** The multiplication operator

\[ M_L : f(z) \mapsto e^{\frac{t}{1-\epsilon^2} z} f(z) \]

maps the space \( \mathcal{L}_\epsilon \) unitarily onto the space

\[ \mathcal{L}_\epsilon := L^2_{hol}(\mathbb{C}, d\nu_\epsilon) \]
of entire functions on \( \mathbb{C} \) square-integrable with respect to the measure
\[
(24) \quad d\nu_\epsilon(z) := \frac{2\epsilon}{(1-\epsilon)^\pi} K_0\left(\frac{2\sqrt{\epsilon}}{1-\epsilon}|z|\right) dz.
\]
where \( dz \) stands for the Lebesgue area measure.

The space \( L_\epsilon = L^2_{\text{hol}}(\mathbb{C}, d\nu_\epsilon) \) thus plays an analogous role for the Laguerre polynomials as the Fock space \( F_\epsilon \) played for the Hermite polynomials; we will call \( L_\epsilon \) the Laguerre Fock space.

The last corollary and (11) imply that the reproducing kernel of \( L_\epsilon \) is equal to
\[
\frac{1}{1-\epsilon} I_0\left(\frac{2\sqrt{\epsilon}}{1-\epsilon}\right),
\]
which can be verified also directly using the monomial basis. (Namely, quite generally, if a multiplication operator \( M_\phi : f \mapsto \phi f \) is unitary from a reproducing kernel Hilbert space \( H_1 \) into another reproducing kernel Hilbert space \( H_2 \), then the corresponding reproducing kernels are related by \( K_2(x, y) = \phi(x) K_1(x, y) \phi(y) \); this is immediate e.g. from the standard formula \( K(x, y) = \sum j e_j(x) e_j(y) \) for reproducing kernel in terms of an arbitrary orthonormal basis \( \{e_j\} \). As for the second claim, Lemma 4 shows that \( \{\frac{\epsilon^{j/2}}{(1-\epsilon)^{j/2}} z^j\}_{j=0}^\infty \) is an orthonormal basis in \( L_\epsilon \), and the claim follows again by the formula just mentioned.)

So far we have worked with the ordinary Laguerre polynomials \( L_n(x) \); it should be noted, however, that everything we did in this section extends in a routine manner also to the generalized Laguerre polynomials \( L_\alpha^n(x) \), \( \alpha > -1 \), defined by
\[
L_\alpha^n(x) = e^{x} e^{-\alpha} \frac{d^n}{dx^n} x^{n+\alpha} e^{-x}.
\]
They are orthogonal on the half-line \( \mathbb{R}_+ = (0, +\infty) \) with respect to the weight \( e^{-x} x^\alpha \)
\[
\int_0^\infty L_\alpha^n(x) L_\alpha^m(x) e^{-x} x^\alpha dx = \frac{\Gamma(n + \alpha + 1)}{n!} \delta_{mn},
\]
and can also be obtained from the generating function
\[
\sum_{n=0}^\infty L_\alpha^n(x) z^n = \frac{1}{(1-z)^{\alpha+1}} e^{z-x}, \quad x \in \mathbb{C}, |z| < 1.
\]
The ordinary Laguerre polynomials correspond to \( \alpha = 0 \). Similarly to our Lemma 4, one checks that the modified Bessel functions of the third kind
\[
K_\alpha(t) = \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2} + \alpha)} \left(\frac{t}{2}\right)^\alpha \int_1^\infty e^{-tx} (x^2 - 1)^{\alpha - \frac{1}{2}} dx, \quad \text{Re} \ t > 0, \ \alpha > -\frac{1}{2}
\]
satisfies
\[
(25) \quad \int_0^\infty 2t^{k+\frac{\alpha}{2}} K_\alpha(2\sqrt{t}) dt = k! \Gamma(k + \alpha + 1).
\]
With the $c$ from (21), the computation
\[
\sum_{m,n=0} t^m s^n \int_{\mathbb{C}} L_n^\alpha(z) \overline{L_m^\alpha(z)} e^{cz+\overline{c}\overline{z}} |z|^\alpha K_\alpha(\frac{2\sqrt{c}}{1-\epsilon} |z|) \, dz
\]
\[
= \frac{1}{(1-t)^{\alpha+1}(1-s)^{\alpha+1}} \int_{\mathbb{C}} e^{\frac{t}{t-1} + \frac{s}{s-1} + c} |z|^\alpha K_\alpha(\frac{2\sqrt{c}}{1-\epsilon} |z|) \, dz
\]
\[
= \frac{1}{(1-t)^{\alpha+1}(1-s)^{\alpha+1}} \sum_{j,k=0} \left( \frac{t}{t-1} + c \right)^j \left( \frac{s}{s-1} + c \right)^k \frac{1}{j!k!} \int_0^\infty t^{k+\frac{1}{2}} K_\alpha(\frac{2\sqrt{c}}{1-\epsilon} \sqrt{t}) \, dt
\]
\[
= \frac{1}{(1-t)^{\alpha+1}(1-s)^{\alpha+1}} \sum_{j,k=0} \left( \frac{t}{t-1} + c \right)^j \left( \frac{s}{s-1} + c \right)^k \frac{\pi}{2} \delta_{jk} \frac{\Gamma(k+\alpha+1)}{k!} \left( 1 - \frac{1}{\epsilon} \right)^{2k+2+\alpha}
\]
\[
= \frac{1-\epsilon}{\sqrt{\epsilon}} e^{2+\alpha} \Gamma(\alpha+1) \frac{\pi}{2} (1-\epsilon)^{2+\alpha} \left( \frac{1-\epsilon}{\epsilon} \right)^{1-\alpha} \Gamma(\alpha+1) \left( 1 - \frac{1}{\epsilon} \right)^{2+\alpha}
\]
\[
= \frac{1-\epsilon}{\sqrt{\epsilon}} e^{2+\alpha} \Gamma(\alpha+1) \frac{\pi}{2} \left( 1 - \frac{1}{\epsilon} \right)^{2+\alpha}
\]
\[
= \frac{1-\epsilon}{\sqrt{\epsilon}} e^{2+\alpha} \Gamma(\alpha+1) \frac{\pi}{2} \left( 1 - \frac{1}{\epsilon} \right)^{2+\alpha}
\]
shows as before that
\[
\int_{\mathbb{C}} L_n^\alpha(z) \overline{L_m^\alpha(z)} e^{cz+\overline{c}\overline{z}} |z|^\alpha K_\alpha(\frac{2\sqrt{c}}{1-\epsilon} |z|) \, dz = \frac{(1-\epsilon)^{\alpha+1}}{2\epsilon^{1-\alpha}} \delta_{mn} \epsilon^{-n},
\]
generalizing (22). The same multiplication operator as before
\[
M_L : f(z) \mapsto e^{\frac{1}{2}+\epsilon^2} f(z)
\]
thus maps the space
\[
L^\epsilon_{(\alpha)} = \{ f = \sum_n f_n L_n : \sum_n \frac{\Gamma(n+\alpha+1)}{n!} \epsilon^{-n} |f_n|^2 = \|f\|^2_{L^\epsilon_{(\alpha)}} < \infty \}
\]
unitarily onto the space
\[
L^\epsilon_{(\alpha)} := L^2_{hol}(\mathbb{C}, \mathcal{D}V^\epsilon_{(\alpha)})
\]
of entire functions on $\mathbb{C}$ square-integrable with respect to the measure
\[
d\mathcal{D}V^\epsilon_{(\alpha)}(z) := \frac{2^{1+\frac{1}{2}}}{(1-\epsilon)^{\alpha}} |z|^\alpha K_\alpha(\frac{2\sqrt{c}}{1-\epsilon} |z|) \, dz.
\]
The reproducing kernel of $L^\epsilon_{(\alpha)}$ equals (cf. [8, §10.12(20)])
\[
\frac{1}{1-\epsilon} \left( c \overline{c} \right)^{-\alpha/2} L_\alpha \left( \frac{2\sqrt{c}}{1-\epsilon} \right)
\]
where $L_\alpha$ again denotes the modified Bessel function of the first kind. In particular, each $f \in L^\epsilon_{(\alpha)}$ again actually extends to an entire function on $\mathbb{C}$, and $L^\epsilon_{(\alpha)}$ is the space of (the restrictions to $\mathbb{R}_+$ of) holomorphic functions on $\mathbb{C}$ with reproducing kernel
\[
\frac{1}{1-\epsilon} e^{-\frac{1}{2}(x+y)(x+y)} \left( c \overline{c} \right)^{-\alpha/2} L_\alpha \left( \frac{2\sqrt{c}}{1-\epsilon} \right).
\]
Remarkably, the space $L^{(\alpha)}$ is a very well-known object, which first appeared in Section VI of the paper by Barut and Girardello [7] on coherent states associated with $SU(1, 1)$; cf. the formulas (6.2) and (6.3) there. (Our $\alpha$ corresponds to $-2\Phi-1$ in the notation of [7], recall that the Bessel function satisfies $K_{\nu} = K_{-\nu}$ for any $\nu$. Also we note that our Lemma 4 and (25) are just a special case of the formula (3.26) there, however we have included the simple direct verification here for convenience.) It is noteworthy that Laguerre polynomials turn out to be related to this space of Barut and Girardello in the same way as Hermite polynomials were shown in [4] to be related to the standard Fock-Segal-Bargmann space. More recently, the space $L^{(\alpha)}$ has been studied in some detail in [19]. Another interesting point in this connection is the existence of families of complex orthogonal polynomials in $z, \overline{z}$, with real coefficients, which span, for example, the space $L^2(\mathbb{C}, d\mu_{\epsilon})$, of which $F_{\epsilon} = L^2_{\text{hol}}(\mathbb{C}, d\mu_{\epsilon})$ is a subspace. These polynomials are also known as complex Hermite polynomials (see, e.g., [14]), and are determined completely by the measure $d\mu_{\epsilon}$. A general procedure for constructing such a family of polynomials, starting with a measure, has been developed in [15, 16]. It would be interesting to work out the analogous complex orthogonal polynomials starting with the measure $d\nu^{(\alpha)}$.

4. The Laguerre “squeeze” operator

In [4], it was shown that the Hermite polynomial basis in the Fock space actually arises as a “squeezed” variant of the standard monomial basis, namely, the former is obtained from the latter by a certain “squeezing” unitary operator. We show that all this persists also in the context of Laguerre polynomials and the Laguerre Fock space of Barut and Girardello described in the preceding section.

For simplicity, we treat only the case $\alpha = 0$, leaving the extension to the generalized Laguerre polynomials $L^{(\alpha)}$ to the interested reader.

It is immediate from Lemma 4 that

$$\left\{\frac{(-z)^n}{n!2^n\sqrt{2\pi}}\right\}^\infty_{n=0} =: \{e_n(z)\}^\infty_{n=0}$$

is an orthonormal basis of $L^2_{\text{hol}}(\mathbb{C}, K_0(|z|) dz)$.

On the other hand from Theorem 5, by the simple change of variable $z \mapsto \frac{1-\epsilon}{\sqrt{\epsilon}} z$, it transpires that

$$\left\{\sqrt{\frac{1-\epsilon}{2\pi}} e^{n/2\sqrt{\epsilon}z/2} L_n(\frac{1-\epsilon}{2\sqrt{\epsilon} z})\right\}^\infty_{n=0} =: \{E_{\epsilon,n}(z)\}^\infty_{n=0}$$

is another orthonormal basis of the same space.

**Theorem 7.** Denote

$$Qf(z) := \frac{z}{2} f(z) - 2 \frac{d}{dz} z \frac{d}{dz} f(z).$$

Then the operator

$$U_{\epsilon} := \left(\sqrt{\frac{1+\sqrt{\epsilon}}{1-\sqrt{\epsilon}}} \right)^Q = \exp \left[\frac{Q}{2} \log \frac{1+\sqrt{\epsilon}}{1-\sqrt{\epsilon}} \right]$$

satisfies

$$U_{\epsilon}e_n = E_{\epsilon,n}, \quad \forall n = 0, 1, 2, \ldots,$$

i.e. maps the basis $\{e_n\}^\infty_{n=0}$ into the basis $\{E_{\epsilon,n}\}^\infty_{n=0}$.
Note that, by a simple computation, 
\[ \frac{z}{2} e_n = -(n+1)e_{n+1}, \quad 2 \frac{d}{dz} \frac{z}{2} e_n = -ne_{n-1}, \]
from which one easily checks that \( Q = T - T^* \) where \( T f(z) = \frac{\hat{\phi}}{z} f(z) \) is the operator of multiplication by \( \frac{\hat{\phi}}{z} \) on \( L^2_{\text{hol}}(\mathbb{C}, K_0(|z|) dz) \). Thus \( iQ \) is self-adjoint, and \( U_\epsilon \) is unitary. However, to see that \( \mathcal{U}_\epsilon e_n = E_{\epsilon,n} \) requires more work.

**Proof.** Recall once again the generating function for Legendre polynomials
\[ (1 - a) \sum_{n=0}^{\infty} a^n L_n(z) = e^{\frac{z}{1-\epsilon}}, \quad |a| < 1, z \in \mathbb{C}. \]
Taking in particular \( a = \frac{w-1}{w+1} \) (so \( |a| < 1 \) corresponds to \( \text{Re} w > 0 \)), we have
\[ \frac{a}{a-1} = \frac{1-w}{2} \]
and
\[ \frac{2}{w+1} \sum_{n=0}^{\infty} \left( \frac{w-1}{w+1} \right)^n L_n(z) = e^{\frac{1-w}{2} z}, \]
or
\[ \frac{2}{w+1} \sum_{n=0}^{\infty} \left( \frac{w-1}{w+1} \right)^n e^{-z/2} L_n(z) = e^{-wz/2}. \]
On the other hand, taking \( a = \frac{w-1}{w+1} \frac{1-\epsilon}{1+\epsilon} \) (so \( |a| < 1 \) now corresponds to \( w \) in the disc with diameter \( (\epsilon, \frac{1}{\epsilon}) \) in the right half-plane), we similarly get
\[ \frac{2(1-\epsilon w)}{(w+1)(1-\epsilon)} \sum_{n=0}^{\infty} \left( \frac{w-1+\epsilon}{w+1-\epsilon} \right)^n e^{-z/2} L_n(z) = e^{\frac{z-w}{2} z}. \]
Now from the differential equation for Legendre polynomials
\[ z L_n''(z) + (1-z) L_n'(z) = -n L_n(z) \]
we obtain upon a simple computation using just the Leibniz rule
\[ \left( \frac{z}{2} - 2 \frac{d}{dz} \frac{1}{z} \right) e^{-z/2} L_n(z) = (2n+1)e^{-z/2} L_n(z), \]
i.e. \( Q(e^{-z/2} L_n(z)) = (2n+1) e^{-z/2} L_n(z) \). Hence
\[ U_\epsilon e^{z/2} L_n(z) = \left( \frac{1+\epsilon}{1-\epsilon} \right) e^{-z/2} L_n(z). \]
Substituting this into (26) yields
\[ U_\epsilon e^{-wz/2} = \frac{2}{w+1} \sqrt{1+\epsilon} \sum_{n=0}^{\infty} \left( \frac{w-1+\epsilon}{w+1-\epsilon} \right)^n e^{-z/2} L_n(z) \]
\[ = \frac{2}{w+1} \sqrt{1+\epsilon} (w+1)(1-\epsilon) e^{\frac{-w}{2(1-\epsilon)}} \]
\[ = \sqrt{1-\epsilon^2} \frac{e^{-w/2}}{1-\epsilon w} \]
by (27). Expanding the exponential on the left-hand side shows that it equals
\[ U_\epsilon \sum_{n=0}^{\infty} \frac{(-z)^n}{n!2^n} w^n = \sqrt{2\pi} \sum_{n=0}^{\infty} w^n U_\epsilon e_n(z). \]
On the other hand, using one more time the generating function for Legendre polynomials, this time with $a = \epsilon w$, shows that

$$
\sqrt{2\pi} \sum_{n=0}^{\infty} w^n E_{\epsilon z,n}(z) = \sum_{n=0}^{\infty} \sqrt{1 - \epsilon^2} e^n \epsilon \sqrt{\frac{z}{2}} L_n \left( \frac{1 - \epsilon^2}{2\epsilon} z \right) 
= \sqrt{1 - \epsilon^2} e^{z/2} \frac{1}{1 - \epsilon w} e^{-\epsilon w} \left( 1 - \epsilon^2 \right) 
= \frac{\sqrt{1 - \epsilon^2}}{1 - \epsilon w} e^{\frac{z-\epsilon w}{1-\epsilon^2}}.
$$

Consequently,

$$
\sum_{n=0}^{\infty} w^n U_{\epsilon z} \epsilon_n(z) = \sum_{n=0}^{\infty} w^n E_{\epsilon z,n}(z).
$$

Comparing coefficients at like powers of $w$ and replacing $\epsilon$ by $\sqrt{\epsilon}$, the theorem follows. \qed

Note that the operator $T$ and its adjoint $T^*$ mentioned before the last proof coincide (up to a different normalization) with the generators $L_+$ and $L_-$, respectively, of the action of the Lie algebra $\mathfrak{su}(1, 1)$ on $\mathfrak{L}_\epsilon$ defined in (6.19) in [7]. The reader is referred to Section V in [6] for further discussion and physical interpretation of the “squeezing” procedure in the Hermite case. By analogy we shall refer to $U_\epsilon$ as the Laguerre squeeze operator, although at this point we do not have a physical meaning for this squeezing. Furthermore, using the squeeze operator, we could also derive a family of squeezed Barut-Girardello coherent states, or express the Barut-Girardello states themselves in terms of the squeezed basis, just as was done for the canonical coherent states in [3].

5. Toeplitz operators on the Laguerre Fock space

For a “symbol” $f \in L^\infty(\mathbb{C})$, the associated Toeplitz operator $T_f(\epsilon) = T_f$ on the Laguerre Fock space $\mathfrak{L}_\epsilon$ is again given by

$$
T_f(\epsilon) u = P_\epsilon(fu), \quad u \in \mathfrak{L}_\epsilon,
$$

where $P_\epsilon : L^2(\mathbb{C}, d\nu_\epsilon) \to \mathfrak{L}_\epsilon$ is the orthogonal projection. Our aim in this section is to find the “semi-classical” asymptotics like (5) of these operators (with $h = 1 - \epsilon$). There are well-established methods to handle this for measures $d\nu_\epsilon$ with power-like dependence on $\epsilon$, that is, of the form $d\nu_\epsilon(z) = F(z)^c(z) G(z) dz$ with some fixed positive weights $F, G$ and some real-valued function $c(\epsilon)$ of $\epsilon$, $c(\epsilon) \to +\infty$ as $\epsilon \nearrow 1$; however, our $d\nu_\epsilon$ in (24) are plainly not quite of this type, so we need to work from scratch.

Recall that, quite generally, on a family of reproducing kernel Hilbert spaces $L^2_{\text{hol}}(\Omega, d\rho_\epsilon)$ of holomorphic functions with some measures $d\rho_\epsilon$, $0 < \epsilon < 1$, on a domain $\Omega \subset \mathbb{C}^n$, establishing an asymptotic expansion like (4) for $T_f T_g$ is actually tantamount to establishing the asymptotic behaviour of the Berezin transform

$$
B_\epsilon f(z) := \int_{\Omega} f(w) |K_\epsilon(z, w)|^2 d\rho_\epsilon(w),
$$

where $K_\epsilon(z, w)$ is the reproducing kernel of $L^2_{\text{hol}}(\Omega, d\rho_\epsilon)$. Indeed, from the definition (28) it is immediate that $T_f T_g = T_{fg}$ whenever $g$ is holomorphic or (upon taking adjoints) $f$ is anti-holomorphic. Thus the bidifferential operators $C_j(f, g)$ in (4) involve only holomorphic derivatives of $f$ and anti-holomorphic derivatives.
of \( g \). It is therefore enough to determine \( C_j(f, g) \) for holomorphic \( f \) and anti-holomorphic \( g \). For such \( f, g \), let us apply both sides of (4) to the reproducing kernel \( K_{e,w} = K_e(\cdot, w) \) at \( w \in \Omega \), and evaluate at \( w \). Since \( T_g K_{e,w} = g(w)K_e(w, w) \) for anti-holomorphic \( g \) by the reproducing property of \( K_e \), the left-hand side of (4) gives just \( f(w)g(w)K_e(w, w) \); while the right-hand side, in view of (28), becomes
\[
\sum_{j=0}^{\infty} h^j \int_{\Omega} C_j(f, g)(z)|K_e(z, w)|^2 d\rho_e(z) = \sum_{j=0}^{\infty} h^j B_e[C_j(f, g)](w)K_e(w, w).
\]
(Remember that \( h = 1 - \epsilon \).) Consequently, we get, at least formally,
\[
\sum_{j=0}^{\infty} h^j C_j(f, g) = B_e^{-1}(fg),
\]
with the inverse being understood in the sense of formal power series in \( \epsilon = 1 - \epsilon \). In other words, if \( B_e \) has an asymptotic expansion
\[(29) \quad B_e \approx \sum_{j=0}^{\infty} (1 - \epsilon)^j Q_j\]
with some differential operators \( Q_j \), and
\[
B_e^{-1} \approx \sum_{j=0}^{\infty} (1 - \epsilon)^j R_j, \quad R_j = \sum_{\alpha, \beta} R_{j\alpha\beta} \partial^\alpha \overline{\partial}^\beta,
\]
is the inverse of (29) (as a formal power series in \( \epsilon = 1 - \epsilon \)), then
\[(30) \quad C_j(f, g) = \sum_{\alpha, \beta} R_{j\alpha\beta}(\partial^\alpha f)(\overline{\partial}^\beta g).
\]
(Here the summations extend over all multiindices \( \alpha, \beta \).)

See [12] for more details of the above argument.

**Example.** For the ordinary Fock space \( \mathcal{F}_h = L^2_{\text{hol}}(\mathbb{C}, e^{-|z|^2/h \frac{dz}{\pi}}), \; h > 0 \), the reproducing kernel is given by \( K_h(z, w) = e^{z\overline{w}/h} \), so
\[
B_h f(z) = \frac{1}{\pi h} \int_{\mathbb{C}} f(w)e^{-|z-w|^2/h} dw = e^{h\Delta/4} f(z)
\]
is just the heat solution operator at time \( t = \frac{h}{4} \). Its formal inverse \( B_h^{-1} \) is thus \( e^{-h\Delta/4} \), and
\[
C_j(f, g) = (-1)^j j! (\partial^j f)(\overline{\partial}^j g),
\]
recovering the well-known formula for the Berezin-Toeplitz quantization on \( \mathbb{C} \).

Returning to our Laguerre Fock space, we are thus confronted with finding the asymptotics as \( \epsilon \nearrow 1 \) of the associated Berezin transform
\[(31) \quad B_\epsilon f(z) = I_0(\frac{2\sqrt{\epsilon} |z|}{1 - \epsilon})^{-1} \int_{\mathbb{C}} f(w) \left| I_0\left(\frac{2\sqrt{\epsilon} |z|}{1 - \epsilon}\right)\right|^2 dw_e(w),
\]
where we have used the formula for the reproducing kernel of \( \mathcal{L}_\epsilon \) from the end of Section 3.

It turns out to be more convenient, instead of \( \epsilon \in (0, 1) \), to use the parameter
\[
\alpha := \frac{2\sqrt{\epsilon}}{1 - \epsilon}.
\]
Thus \( \epsilon \nearrow 1 \) corresponds to \( \alpha \to +\infty \). We will write \( B_\alpha \) instead of \( B_\epsilon \) from now on.
Theorem 8. Let $z \in \mathbb{C}$, $z \neq 0$. For any $f \in L^\infty(\mathbb{C})$ which is $C^\infty$ in a neighbourhood of $z$, we have

$$B_\alpha f(z) \approx \sum_{j=0}^{\infty} \alpha^{-j} Q_j f(z) \quad \text{as } \alpha \to +\infty,$$

for some differential operators $Q_j$ on $\mathbb{C} \setminus \{0\}$, $j = 0, 1, 2, \ldots$ (not depending on $f$ and $z$). Explicitly,

$$Q_0 = I, \quad Q_1 = |z| \Delta, \quad Q_1 = \frac{1}{2} \Delta + (z \partial + \overline{z} \partial) \Delta + \frac{1}{2} \Delta^2.$$

For $z = 0$ and $f \in L^\infty(\mathbb{C})$ smooth near the origin, we have

$$B_\alpha f(0) \approx \sum_{j=0}^{\infty} \alpha^{-2j} \Delta^j f(0) \quad \text{as } \alpha \to +\infty.$$

Note that the asymptotics are thus discontinuous at $z = 0$; this can be viewed as an analogue of the familiar Stokes phenomenon in complex analysis.

Proof. For $z = 0$, (31) becomes simply

$$B_\alpha f(z) = \frac{1}{1 - \epsilon} \int_{\mathbb{C}} f(w) dw = \frac{2\epsilon}{(1 - \epsilon)^2} \int_{\mathbb{C}} f(w) K_0 \left( \frac{2\sqrt{\epsilon}}{1 - \epsilon} |w| \right) dw = \frac{\alpha^2}{2\pi} \int_{\mathbb{C}} f(w) K_0(\alpha |w|) dw.$$

For any $\delta > 0$ and $\alpha \geq 1$, we have

$$\left| \int_{|w| > \delta} f(w) K_0(\alpha |w|) dw \right| \leq \|f\|_\infty 2\pi \int_{\delta}^{\infty} K_0(\alpha r) r dr \leq \|f\|_\infty C_\delta \int_{\delta}^{\infty} e^{-\alpha r/2} dr = \|f\|_\infty \frac{2C_\delta}{\alpha} e^{-\alpha \delta/2}$$

for some finite $C_\delta$, thanks to (19). This decays faster than any negative power of $\alpha$ as $\alpha \to +\infty$.

On the other hand, for $|x| < \delta$ with $\delta$ small enough we may replace $f$ by its Taylor expansion at the origin, giving

$$\int_{|w| < \delta} f(w) K_0(\alpha |w|) dw \approx \frac{1}{1 - \epsilon} \sum_{j,k=0}^{\infty} \frac{\partial^j \overline{\partial}^k f(0)}{j! k!} \int_{|w| < \delta} w^j \overline{w}^k dw = \frac{1}{1 - \epsilon} \sum_{j,k=0}^{\infty} \frac{\partial^j \overline{\partial}^k f(0)}{j! k!} \delta_{jk} \frac{(1 - \epsilon)^{2j}}{\epsilon^j} j!^2.$$

Again, modulo an exponentially small error, the last integral equals, as we have seen in Section 3,

$$\int_{\mathbb{C}} w^j \overline{w}^k dw = \delta_{jk} \frac{(1 - \epsilon)^{2j}}{\epsilon^j} j!^2.$$

Hence

$$B_\alpha f(0) \approx \sum_{j,k=0}^{\infty} \frac{\partial^j \overline{\partial}^k f(0)}{j! k!} \delta_{jk} \frac{(1 - \epsilon)^{2j}}{\epsilon^j} j!^2 \approx \sum_{k=0}^{\infty} \frac{(1 - \epsilon)^{2k} \Delta^k f(0)}{4^k \epsilon^k} = \sum_{k=0}^{\infty} \frac{\Delta^k f(0)}{\alpha^{2k}}$$

as $\alpha \to +\infty$, establishing (34).
For the rest of the proof, we thus assume \( z \neq 0 \). The change of variable
\[
w = (1 + y)^2 z, \quad \text{Re} \, y > -1,
\]
then transforms (31) into
\[
B_\alpha f(z) = \frac{1}{I_0(\alpha |z|)} \int_{\text{Re} y > -1} f((1 + y)^2 z) I_0((1 + y)\alpha |z|) I_0((1 + y)\alpha |z|) \times \frac{\alpha^2}{2\pi} K_0(\alpha |1 + y|^2 |z|) |2z(1 + y)|^2 \, dy,
\]
or, introducing
\[
\lambda := \alpha |z|
\]
for convenience,
\[
(35) \quad B_\alpha f(z) = \frac{2\lambda^2}{\pi I_0(\lambda)} \int_{\text{Re} y > -1} f((1 + y)^2 z) |I_0((1 + y)\lambda)|^2 K_0(|1 + y|^2 \lambda) |1 + y|^2 \, dy.
\]
Using the integral representation
\[
I_0(z) = \frac{1}{\pi} \int_{-1}^{1} e^{-tz} \, dt
\]
for \( I_0 \) and the formula (18) for \( K_0 \), this can be rewritten as
\[
\frac{\pi^3 I_0(\lambda)}{2\lambda^2} B_\alpha f(z) = \int_{\text{Re} y > -1} \int_{-1}^{1} \int_{-1}^{1} \int_{1}^{\infty} f((1 + y)^2 z) \frac{e^{-t\lambda(1+y)\alpha - s\lambda(1+y) - x\lambda|1+y|^2}}{\sqrt{(1-t^2)(1-s^2)(x^2-1)}} |1 + y|^2 \, dx \, ds \, dt \, dy.
\]
Making one more change of variables
\[
t = T^2 - 1, \quad s = S^2 - 1, \quad x = X^2 + 1,
\]
the right-hand side becomes
\[
\int_{\text{Re} y > -1} \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2}}^{\sqrt{2}} \int_{\mathbb{R}} \frac{f((1 + y)^2 z)|1 + y|^2}{\sqrt{(2 - T^2)(2 - S^2)(2 + X^2)}} \times e^{(1-|y|^2+T^2+S^2+X^2+\mathcal{T}(y,T,S,X))\lambda} \, dX \, dS \, dT \, dy
\]
\[
\equiv e^\lambda \int_{\text{Re} y > -1} \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2}}^{\sqrt{2}} \int_{\mathbb{R}} F(y, T, S, X) \times e^{-(|y|^2+T^2+S^2+X^2)\lambda} \, dX \, dS \, dT \, dy,
\]
where
\[
(36) \quad F(y, T, S, X) := \frac{f((1 + y)^2 z)|1 + y|^2}{\sqrt{(2 - T^2)(2 - S^2)(2 + X^2)}}
\]
\[
(37) \quad \mathcal{T}(y, T, S, X) := T^2\mathcal{Y} + S^2y + X^2(y + \mathcal{Y} + |y|^2).
\]
Note that the factor at \( \lambda \) in the exponent in the integrand in (36) has a global maximum at the origin \( T = S = X = y = 0 \), and vanishes there precisely to the second order. Asymptotics as \( \lambda \to +\infty \) of such integrals is obtained by the standard stationary phase (WJKB) method; in the present case, this can be made
quite explicit as follows. Recall first of all that by the formula for the solution of the heat equation,
\[
\left(\frac{\lambda}{\pi}\right)^{5/2} \int_{\mathbb{C} \times \mathbb{R}^3} G(y, T, X) e^{-\left(|y|^2 + T^2 + X^2\right)\lambda} \, dX \, dS \, dT \, dy = e^{\Delta/(4\lambda)} G(0)
\]
\[
\approx \sum_{j=0}^{\infty} \frac{1}{j!(4\lambda)^j} \left[ 4\frac{\partial^2}{\partial y \partial y} + \frac{\partial^2}{\partial T^2} + \frac{\partial^2}{\partial S^2} + \frac{\partial^2}{\partial x^2} \right]^j G(0).
\]

Arguing as in the beginning of this proof, one sees that this holds also for integration over any open subset containing the origin, instead of the whole \(\mathbb{C} \times \mathbb{R}^3\); in particular, we can apply it to the integral \(36\), with
\[
G := F e^{-\lambda T}
\]
(note that this depends also on \(\lambda\)). We thus obtain, at least formally,
\[
\int \int \int \int F e^{-\left(|y|^2 + T^2 + X^2 + \frac{1}{2}T\right)\lambda} \, dX \, dS \, dT \, dy
\]
\[
\approx \left(\frac{\pi}{\lambda}\right)^{5/2} \frac{\lambda^k}{2 \lambda^2 e^{\lambda}} B_0 f(z) \approx
\sum_{m=0}^{\infty} \lambda^{-m} (R_m f)(z),
\]
\[
\left(\frac{\lambda}{\pi}\right)^{5/2} \frac{\pi^3 I_0(\lambda)}{2 \lambda^2 e^{\lambda}} B_0 f(z) \approx
\sum_{m=0}^{\infty} \lambda^{-m} (R_m f)(z),
\]
where \(R_m\) are some differential operators on \(\mathbb{C} \setminus \{0\}\) with \(C^\infty\) coefficients (in fact, \(R_m\) is of order \(2m\)). Explicit calculations (using computer for \(m = 2\)) yield
\[
R_0 f = 2^{-3/2} f, \quad R_1 f = 2^{-3/2} \left(\frac{1}{2} f + |z| \Delta f\right), \quad R_2 f = 2^{-3/2} \left(\frac{9}{128} f + \frac{5}{8} |z|^2 \Delta f + |z|^2 (z \partial + \bar{z} \partial) \Delta f + \frac{1}{2} |z|^2 \Delta^2 f\right).
\]
Observe that in view of the reproducing property of the reproducing kernel, one has \(B_0 1 = 1\) for all \(\alpha\) (where 1 denotes the function constant one). Consequently, taking \(f = 1\) in \(38\),
\[
\left(\frac{\lambda}{\pi}\right)^{5/2} \frac{\pi^3 I_0(\lambda)}{2 \lambda^2 e^{\lambda}} \approx \sum_{m=0}^{\infty} \lambda^{-m} (R_m 1)(z).
\]
Dividing \(38\) by \(41\), we finally obtain
\[
B_\alpha f(z) \approx \frac{2\sqrt{2}}{2\sqrt{2}} \sum_{m=0}^{\infty} \lambda^{-m} (R_m f)(z) =: \sum_{m=0}^{\infty} \lambda^{-m} (Q_m f)(z)
\]
with some differential operators \( Q_j \) on \( \mathbb{C} \setminus \{0\} \), proving (32). (Note that the division of formal power series above makes sense, since \( 2\sqrt{2}R_01 = 1 \).) Finally, lengthy but routine calculation using (40) yields (33).

\[ \square \]

Remark. A somewhat simpler way (which would require some justification however to make it completely rigorous) to get explicit expressions for the \( R_m \) and \( Q_m \) above is as follows. Recall that as \( z \to \infty \), the functions \( I_0 \) and \( K_0 \) possess the asymptotic expansions

\[ I_0(z) \approx \frac{e^z}{\sqrt{2\pi z}} \sum_{m=0}^{\infty} \frac{c_m}{z^m}, \quad \text{(42)} \]

\[ K_0(z) \approx \sqrt{\pi} e^{-z} \sum_{m=0}^{\infty} \frac{(-1)^m c_m}{z^m}, \quad \text{(43)} \]

where

\[ c_m = \frac{(\frac{1}{2} z)^m}{m!^{2^{m/2}}} = \frac{\Gamma(m + \frac{1}{2})^2}{m!2^{m}}; \]

see [8, vol. II, §7.13.1, (5) and (7)]. Substituting these into (35) yields

\[ \frac{\pi I_0(\lambda)}{2\lambda^2} B_n(z) = \sum_{j,k,l=0}^{\infty} c_j c_k (-1)^j c_l \lambda^{j+k+l+3/2} \left[ \int_{Re y > 1} \frac{f((1+y)^2 z)}{(1+y)^j(1-y)^k|1+y|^{2l}} e^{-\lambda |y|^2} dy \right] \]

\[ \approx \frac{\sqrt{\pi} e^{-\lambda}}{\sqrt{8\lambda^{5/2}}} \sum_{j,k,l,n=0}^{\infty} c_j c_k (-1)^j c_l \frac{\Gamma(1+n)}{n!\lambda^{j+k+l+n}} \partial^n \overline{\partial}^l \left[ \frac{f((1+y)^2 z)}{(1+y)^j(1-y)^k|1+y|^{2l}} \right]_{y=0}. \]

Comparing this with (39) yields immediately

\[ R_m f(z) = \sum_{j+k+l+n=m} c_j c_k (-1)^j c_l \frac{\Gamma(1+n)}{n!\lambda^{j+k+l+n}} \partial^n \overline{\partial}^l \left[ \frac{f((1+y)^2 z)}{(1+y)^j(1-y)^k|1+y|^{2l}} \right]_{y=0}, \]

which is a much simpler expression than in (38).

We pause to note that it is amusing to check that the asymptotics of \( I_0(\lambda) \) implicit from (11) coincide with (12).

Returning to our Toeplitz operator asymptotics on the Laguerre Fock space, we see from (33) and (30) that

\[ T_f T_g - T_g T_f \approx \frac{i\hbar}{2\pi} T_{\{f,g\}}, \]

where

\[ \frac{i\hbar}{2\pi} \{f, g\} = 2\frac{1 - \epsilon}{\sqrt{\epsilon}} |z| (\overline{\partial} f \partial g - \overline{\partial} f \partial g) + O(h^2) \]

(recall that \( h = 1 - \epsilon \)). Thus what we have is a quantization of the Kähler metric

\[ \frac{dz d\overline{z}}{2|z|^2}. \]

(44)

In view of the singularity at \( z = 0 \), we are in effect quantizing not \( \mathbb{C} \) but \( \mathbb{C} \setminus \{0\} \), where (11) is just the pullback of the (appropriately rescaled) Euclidean metric on the universal cover \( \mathbb{C} \) \setminus \{0\}. (This accounts for the discontinuity of the asymptotics at \( z = 0 \); physically, the origin does not belong to our phase space and the asymptotics there have no physical relevance.) A potential for this metric is given by \( \Psi(z) = 2|z| \), so the traditional Berezin-Toeplitz quantization would be using the spaces

\[ L^2_{\text{hol}}(\mathbb{C} \setminus \{0\}, e^{-2|z|/\hbar} dz) = L^2_{\text{hol}}(\mathbb{C}, e^{-2|z|/\hbar} dz), \quad \hbar > 0, \]
as described in the Introduction. (The equality of the last two spaces follows from the well-known fact — easily checked using the Laurent expansion and polar co-
ordinates — that any holomorphic and square-integrable function in a punctured neighbourhood of the origin has a removable singularity there.) The latter can be
carried out as in Example 2.16 in [11], and we leave to the reader the (amus-
ing) comparison of the outcomes of the two approaches.

We conclude this section by mentioning that, analogously as in Section 6 in [3],
one can in principle derive the asymptotics of the Toeplitz operators on \( L^2(\mathbb{R}) \), also by using the standard Weyl calculus on \( L^2(\mathbb{R}) \). Namely, the integral operator

\[
V_L f(z) := \int_0^\infty f(x) \beta_L(z, x) \, dx
\]

where

\[
\beta_L(z, x) = \sum_n I_n(x) e^{n/2} L_n(z) e^{\epsilon^z/(1-\epsilon)}
\]

\[
= \frac{1}{1 - \sqrt{1 - \epsilon}} e^{-\frac{\epsilon^z}{\sqrt{1 - \epsilon}}} + \frac{1}{1 + \sqrt{1 - \epsilon}} e^{\frac{\epsilon^z}{\sqrt{1 - \epsilon}}}
\]

is a unitary isomorphism of \( L^2(\mathbb{R}_+) \) onto \( L^2_\text{hol}(C, d\nu) \), where \( L^2_\text{hol}(C, d\nu) \) is the space of holomorphic functions that are square-integrable on the punctured disk around the origin. The latter can be
handled as described in the Introduction. (The equality of the last two spaces follows from the well-known fact — easily checked using the Laurent expansion and polar co-
ordinates — that any holomorphic and square-integrable function in a punctured neighbourhood of the origin has a removable singularity there.) The latter can be
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\[
= \frac{1}{1 - \sqrt{1 - \epsilon}} e^{-\frac{\epsilon^z}{\sqrt{1 - \epsilon}}} + \frac{1}{1 + \sqrt{1 - \epsilon}} e^{\frac{\epsilon^z}{\sqrt{1 - \epsilon}}}
\]

is a unitary isomorphism of \( L^2(\mathbb{R}_+) \) onto \( L^2_\text{hol}(C, d\nu) \) (taking the orthonormal basis \( \{I_n\}_n \) of \( L^2(\mathbb{R}_+) \) into the orthonormal basis \( \{e^{n/2}e^{\epsilon^z/(1-\epsilon)}L_n(z)\}_n \) of the latter). Composing it with the obvious unitary isomorphism

\[
Q: f(x) \mapsto x^{-1/2} f(\log x)
\]

of \( L^2(\mathbb{R}) \) onto \( L^2(\mathbb{R}_+) \), we thus obtain the operator

\[
V_L Q f(z) = \int_\mathbb{R} f(x) \beta_L(z, e^x) e^{x/2} \, dx
\]

sending \( L^2(\mathbb{R}) \) unitarily onto \( L^2(\mathbb{R}_+) \).

One can now, in principle, again consider the Toeplitz operators \( T_{\phi, \theta} \), \( \phi \in L^\infty(C) \),
on \( L^2_\text{hol}(C, d\nu) \) and try to identify the transferred operator \( Q^*V_L^* T_{\phi, \theta} V_L Q \) with some Weyl operator on \( L^2(\mathbb{R}) \). Proceeding as in Section 6 in [3],
we find that \( Q^*V_L^* T_{\phi, \theta} V_L Q = W_a \) with \( a \) given by

\[
a \left( \frac{x + y}{2}, x - y \right) = \int_C \phi(z) \beta_L(z, e^y) \beta_L(z, e^x) e^{\frac{x+y}{2}} \, d\nu(z)
\]

\[
= \frac{2\epsilon}{(1 - \epsilon)(1 - \sqrt{1 - \epsilon})^2 \pi} e^{\frac{1 - \epsilon + \sqrt{1 - \epsilon}}{2} e^{\frac{x+y}{2}}} e^{\frac{1}{1 - \sqrt{1 - \epsilon}} e^{\frac{x+y}{2}}}
\]

\[
\int_C \phi(z) e^{-\frac{\epsilon^z}{\sqrt{1 - \epsilon}}(z + \pi)} I_0 \left( \frac{2\sqrt{\epsilon} e^{y/2} e^{1/4}}{1 - \sqrt{1 - \epsilon}} \right) I_0 \left( \frac{2\sqrt{\epsilon} e^{x/2} e^{1/4}}{1 - \sqrt{1 - \epsilon}} \right) K_0 \left( \frac{2\sqrt{\epsilon}}{1 - \epsilon} |z| \right) \, dz,
\]

whence

\[
a(s, \eta) = \frac{2\epsilon}{(1 - \epsilon)(1 - \sqrt{1 - \epsilon})^2 \pi} \int_C \phi(z) K_0 \left( \frac{2\sqrt{\epsilon}}{1 - \epsilon} |z| \right) e^{-\frac{\epsilon^z}{\sqrt{1 - \epsilon}}(z + \pi) + s}
\]

\[
\times \int_R e^{-ir\eta} e^{\frac{1 - \epsilon}{1 - \sqrt{1 - \epsilon}} \cosh \frac{r}{2}} I_0 \left( \frac{2\sqrt{\epsilon} e^{y/2} e^{s}}{1 - \sqrt{1 - \epsilon}} e^{r/2} \right) I_0 \left( \frac{2\sqrt{\epsilon} e^{x/2} e^{s}}{1 - \sqrt{1 - \epsilon}} e^{r/2} \right) \, dr \, dz.
\]

One can now again replace \( I_0 \) and \( K_0 \) by their integral representations (or, at least on a heuristic level, by their asymptotic expansions (12) and (13)) and proceed as before to obtain an asymptotic expansion for \( a(s, \eta) \) as \( \epsilon \to 1 \). Invoking the standard
composition rules for the Weyl calculus would then lead to the asymptotics of the Toeplitz product $T_f T_g$. We omit the details.

6. LEGENDRE POLYNOMIALS

Another family of orthogonal polynomials susceptible to a similar treatment as with $H_n(x)$ and $L_n(x)$ are the Legendre polynomials $P_n(x)$, $n = 0, 1, 2, \ldots$, defined by

$$P_n(x) := \frac{(-1)^n}{n!} \frac{d^n}{dx^n} (1 - x^2)^n.$$  

These polynomials form an orthogonal basis on $L^2(-1, 1)$:

$$\int_{-1}^{1} P_n(x) P_m(x) \, dx = \frac{\delta_{mn}}{m + \frac{1}{2}}.$$  

The corresponding series

$$\tilde{P}_\epsilon(x, y) = \sum_{n=0}^{\infty} \epsilon^n \left( n + \frac{1}{2} \right) P_n(x) P_n(y), \quad x, y \in (-1, 1),$$  

can be summed to the rather complicated expression

$$\tilde{P}_\epsilon(\cos 2\phi, \cos 2\theta) = \frac{1 - \epsilon}{2(1 + \epsilon)^2} \sum_{m, n=0}^{\infty} \frac{(m + n)! \left( \frac{3}{2} \right)_{m+n} (4\epsilon \sin^2 \phi \sin^2 \theta)^m (4\epsilon \cos^2 \phi \cos^2 \theta)^n}{(m!n!)^2 (1 + \epsilon)^{2m+2n}},$$  

see [5, (7.5.6)]. (Incidentally, the series like $\tilde{K}_\epsilon, \tilde{L}_\epsilon$ and $\tilde{P}_\epsilon$ are called the “Poisson kernels” for the corresponding orthogonal polynomials in [5]. The series on the right-hand side in the last formula is Appell’s hypergeometric function $F_4$ in Horn’s notation [8, §5.7.1].)

The differential equation is

$$(1 - x^2) P_n''(x) - 2x P_n'(x) + n(n + 1) P_n(x) = 0,$$

implying that

$$A P_n = n P_n$$

for

$$A = \sqrt{-D(1 - x^2)D + \frac{1}{4} I - \frac{1}{2} I}$$

($D = d/dx$, and the square root is understood in the spectral-theoretic sense). As before, it follows that the corresponding “Toeplitz” operators

$$T_f^{(\epsilon)} u(x) = \int_{-1}^{1} u(y) f(y) \tilde{P}_\epsilon(x, y) \, dy, \quad 0 < \epsilon < 1,$$

satisfy (43), with the operator $A$ from (45). There are also the corresponding Hilbert spaces $\mathcal{P}_\epsilon$ of functions on $(-1, 1)$ having $\tilde{P}_\epsilon$ for their reproducing kernel; however, unlike the situation for Hermite and Laguerre polynomials, in this case $\mathcal{P}_\epsilon$ no longer extend to a reproducing kernel Hilbert space on a larger set.

**Proposition 9.** There exists no domain $\Omega$ in $\mathbb{C}$ containing the interval $(-1, 1)$ and such that for each $0 < \epsilon < 1$, $\mathcal{P}_\epsilon$ would consist of restrictions to $(-1, 1)$ of functions in some reproducing kernel Hilbert space of holomorphic functions on $\Omega$.  

Proof. Assume, to the contrary, that such a domain $\Omega$ and reproducing kernel Hilbert spaces $\mathcal{P}_\epsilon^\Omega$, $0 < \epsilon < 1$, exist. For each $\epsilon$, the function $\tilde{P}_\epsilon(x,y)$ then extends to a function (still denoted $\tilde{P}_\epsilon$) on all of $\Omega \times \Omega$, holomorphic in $x, y$, which is the reproducing kernel of $\mathcal{P}_\epsilon^\Omega$; furthermore, by the standard formula for the reproducing kernel in terms of an orthonormal basis, the series (45) converges for any $x, y \in \Omega$. Thus, in particular, the series
\[
\sum_{n=0}^\infty (n + \frac{1}{2}) \epsilon^n |P_n(x)|^2
\]
converges for any $\epsilon \in (0,1)$ and $x \in \Omega$. By the Cauchy-Schwarz inequality
\[
\left( \sum_n |z^n P_n(x)| \right)^2 \leq \left( \sum_n |z|^n \right) \left( \sum_n |z|^n |P_n(x)|^2 \right) \leq \frac{2}{1-|z|} \sum_n (n + \frac{1}{2}) |z|^n |P_n(x)|^2
\]
it thus follows that for any $x \in \Omega$, the series $\sum_n z^n P_n(x)$ converges for any $|z| < 1$. However, using the familiar generating function for Legendre polynomials
\[
\sum_{n=0}^\infty z^n P_n(x) = (1 - 2xz + z^2)^{-1/2}, \quad |z| < 1, -1 \leq x \leq 1,
\]
we quickly see that the series on the left-hand side converges precisely for
\[
|z| < \min (|x + \sqrt{x^2 - 1}|, |x - \sqrt{x^2 - 1}|).
\]
Since
\[
|x + \sqrt{x^2 - 1}| \cdot |x - \sqrt{x^2 - 1}| = 1,
\]
the series can thus converge for all $|z| < 1$ only if both $x + \sqrt{x^2 - 1}$ and $x - \sqrt{x^2 - 1}$ lie on the unit circle, that is, if and only if $x \in [-1,1]$. □

Remark. For a given $\epsilon \in (0,1)$, the domain of convergence of the series (45) is given by (cf. [8, §5.7])
\[
|(1-x)(1-y)|^{1/2} + |(1+x)(1+y)|^{1/2} < \epsilon^{1/2} + \epsilon^{-1/2}.
\]
Thus $\mathcal{P}_\epsilon$ actually extends to a reproducing kernel Hilbert space of holomorphic functions on the ellipse
\[
\Omega_\epsilon := \{ x \in \mathbb{C} : |1-x| + |1+x| < (1+\epsilon)/\sqrt{\epsilon} \}
\]
which however shrinks to the interval $[-1,1]$ as $\epsilon \nearrow 1$.

A more explicit description of the space $\mathcal{P}_\epsilon$ for a given $\epsilon$ was given in [19]. One can treat in the same way also the Jacobi polynomials $P_n^{(\alpha, \beta)}$, $\alpha, \beta > -1$ (of which $P_n$ are the special case $\alpha = \beta = 0$); we omit the details.

7. Final remarks: other sequences

The choice of the powers $\epsilon^n$ in (6), (10) and (45) may admittedly seem rather haphazard. It is in fact possible to give a fairly complete picture of what happens, from the point of view of existence of the reproducing kernel Hilbert spaces like $\mathcal{H}_\epsilon$, $\mathcal{L}_\epsilon$ and $\mathcal{P}_\epsilon$, when it is replaced by other sequences of positive coefficients.

Theorem 10. Let $c_n$ be a sequence of positive numbers. Then the following are equivalent:
(a) The series \( P_c(x, y) := \sum c_n P_n(x)P_n(y) \) converges for all \( x, y \in [-1, 1] \) and 
\( P_c(x, y) \) is the reproducing kernel of the Hilbert space 
\[
\mathcal{P}_c := \{ f = \sum f_n P_n : \sum c_n^{-1} |f_n|^2 =: \|f\|_{\mathcal{P}_c}^2 < \infty \}
\]

of functions on \([-1, 1]\).

(b) \( \sum c_n < \infty \).

Proof. (a) \( \Longrightarrow \) (b) This is immediate upon taking \( x = y = 1 \), since \( P_n(1) = 1 \) \( \forall n \).

(b) \( \Longrightarrow \) (a) Since

\[
\|P_n(x)\| \leq 1 \quad \forall n \forall x \in [-1, 1]
\]

(this follows e.g. from the first formula in [8, 10.10(42)]), clearly \( P_c(x, y) \) converges for all \( x, y \in [-1, 1] \), and thus \( P_{c, y} := P_c(y, \cdot) \) belongs to \( \mathcal{P}_c \) for each \( y \in [-1, 1] \).

The rest of the argument is the same as in the proof of Proposition 1 in [3], namely, 
for any \( f \in \mathcal{P}_c \), we have using again (47)

\[
\sum_n |f_n P_n(y)| = \left( \sum_n c_n^{-1} |f_n|^2 \right)^{1/2} \left( \sum_n c_n |P_n(y)|^2 \right)^{1/2} = \|f\|_{\mathcal{P}_c} \|P_c(y, y)\|^{1/2} < \infty
\]

showing that the series \( \sum_n f_n P_n(y) := \langle f, P_{c, y} \rangle_{\mathcal{P}_c} \) is a bounded linear functional on \( \mathcal{P}_c \). Thus \( \mathcal{P}_c \) is a reproducing kernel Hilbert space 
with reproducing kernel \( P_c(x, y) \), as asserted.

Theorem 11. Let \( c_n \) be a sequence of positive numbers. Then the following are equivalent:

(a) The series \( L_c(x, y) := \sum c_n L_n(x)L_n(y) \) converges for all \( x, y \geq 0 \) and 
\( L_c(x, y) \) is the reproducing kernel of the Hilbert space 
\[
\mathcal{L}_c := \{ f = \sum f_n L_n : \sum c_n^{-1} |f_n|^2 =: \|f\|_{\mathcal{L}_c}^2 < \infty \}
\]

of functions on \([0, \infty)\).

(b) \( \sum c_n < \infty \).

Proof. (a) \( \Longrightarrow \) (b) Immediate upon taking \( x = y = 0 \), since \( L_n(0) = 1 \) \( \forall n \).

(b) \( \Longrightarrow \) (a) Recall that the Legendre polynomials are related by the formula 

\[
L_n(x) = \frac{(-1)^n}{n!} \Psi(-n, 1, x), \quad x \neq 0,
\]

to the confluent hypergeometric function \( \Psi [8, 10.12(14)] \). The latter possesses the asymptotic behaviour 
[8, 6.13(8)]

\[
\Psi(a, c, x) = e^{-x} x^{\frac{1}{2}} e^{-\frac{1}{2} c} \sqrt{2} \cos(\sqrt{\kappa} x) \log(1 + O(\kappa^{-1/2}))
\]

as \( \kappa := \frac{a}{x} - 1 \rightarrow +\infty \). For \( x > 0 \), the cosine is bounded by 1 in modulus, thus by Stirling’s formula 

\[
|L_n(x)| \leq C_n x^{-1/4}
\]

for all \( n \) large enough, and the convergence of \( L_c(x, y) \) follows. The rest of the proof is the same as for the preceding theorem.

Remark. Proceeding as in the proof of Theorem 3 one can show that \( L_c(x, y) \) in fact converges for all \( x, y \in \mathbb{C} \), and \( \mathcal{L}_c \) extends to a space of holomorphic functions.
on all of \( \mathbb{C} \), as soon as \( \sum_n c_n r^{-2n} < \infty \) for some \( r \in (0, 1) \). The latter condition can in fact be relaxed to

\[
\sum_n c_n e^{a\sqrt{n}} < \infty \quad \forall a > 0
\]

(or, equivalently, \( c_n^{1/\sqrt{n}} \to 0 \)) by using \( [18] \).

For the spaces of Hermite polynomials, it is easy to see that \( \sum_n c_n \frac{H_n(x)H_n(y)}{n!2^n \sqrt{n}} \) converges for \( x = y = 0 \) if and only if \( \sum_{n=1}^{\infty} c_{2n}/\sqrt{n} < \infty \); unfortunately, handling the \( c_n \) with odd \( n \) seems more difficult. We can however offer the following result.

**Theorem 12.** Let \( c_n \) be a sequence of positive numbers. Then the following are equivalent:

(a) The two series \( H_c(x, y) := \sum_n c_n H_n(x)H_n(y)(n!2^n \sqrt{n})^{-1} \) and \( H^\#_c(x, y) := \sum_{n=1}^{\infty} c_{n+1} H_n(x)H_n(y)(n!2^n \sqrt{n})^{-1} \) converge for all \( x, y \in \mathbb{R} \) and \( H_c(x, y) \) is the reproducing kernel of the Hilbert space

\[
H_c := \{ f = \sum_n f_n(n!2^n \sqrt{n})^{-1/2} H_n : \sum_n c_n^{-1} |f_n|^2 := \| f \|_{H_c}^2 < \infty \}
\]

of functions on \( \mathbb{R} \).

(b) \( \sum_{n=1}^{\infty} n^{-1/2} c_n < \infty \).

**Proof.** (a) \( \Rightarrow \) (b) As already mentioned, taking \( x = y = 0 \), \( H_{2n+1}(0) = 0 \) and \( H_{2n}(0) = \frac{(2n)!}{n!} \) imply that

\[
\lim_{n \to \infty} \sum_n c_{2n} \frac{(2n)!}{n!2^{2n}} = \sum_n c_{2n} \frac{\left(\frac{1}{2}\right)_n}{n!} \sim \sum_n c_{2n} n^{-1/2};
\]

the same argument for \( H^\#_c \) gives \( \sum_n c_{2n+1} n^{-1/2} < \infty \), and (b) follows.

(b) \( \Rightarrow \) (a) According to a result of Schwid \[22\] Theorem VIII(a)] and Stirling’s formula,

\[
\frac{H_n(z)}{\sqrt{n!2^n\pi^{1/2}}} = 2^{1/4} \pi^{-1/2} e^{-3/2n^{1/4}} [1 + O(n^{-1})] \left[ \cos \left( \frac{\pi n}{2} - z \sqrt{2n+1} \right) + O(n^{-1/2}) \right]
\]

as \( n \to +\infty \). For \( z \) real, the cosine is bounded, so

\[
\left| \frac{H_n(z)}{\sqrt{n!2^n\pi^{1/2}}} \right| \leq C_n n^{-1/4} \quad \text{for } n \text{ large enough},
\]

and the convergence of \( H_c(x, y) \) for any \( x, y \in \mathbb{R} \) follows. The assertion for \( H^\#_c(x, y) \) is obtained upon replacing \( \{c_n\} \) by \( \{c_{n+1}\} \).

**Remark.** It follows from the proof of Theorem 2 in \[3\] that, again, \( H_c(x, y) \) in fact converges for all \( x, y \in \mathbb{C} \), and \( H_c \) extends to a space of holomorphic functions on all of \( \mathbb{C} \), as soon as the sequence \( \{c_n\} \) satisfies \( [13] \).

For Legendre polynomials, the condition for \( P_c(x, y) \) to converge for all \( x, y \in \mathbb{C} \), and for \( P_c \) to extend to a reproducing kernel Hilbert space of holomorphic functions on all of \( \mathbb{C} \), can similarly be shown to be \( c_n^{1/n} \to 0 \).
8. Conclusion

Generally, if we start with a family of real polynomials \( p_n(x), \quad x \in \mathbb{R}, \quad n = 0, 1, 2, \ldots, \infty \), which are orthonormal with respect to a measure \( d\mu \) over \( \mathbb{R} \), the sum \( \sum_{n=0}^{\infty} p_n(x)p_n(y) \) is usually divergent. However, there exist families of polynomials, such as the ones considered in this paper, for which the sum \( K_\epsilon(x, y) = \sum_{n=0}^{\infty} \epsilon^n p_n(x)p_n(y), \quad 0 < \epsilon < 1 \), converges for all \( x, y \). In that case \( K_\epsilon(x, y) \) defines a reproducing kernel and the polynomials \( \epsilon^n p_n(x) \) constitute an orthonormal basis for the corresponding Hilbert space \( \mathcal{H}_\epsilon \). However, although \( \mathcal{H}_\epsilon \subset L^2(\mathbb{R}, d\mu) \), it is in general not a Hilbert subspace. On the other hand, if we write the same polynomials in a complex variable, \( \epsilon^n p_n(z) \), \( z \in \mathbb{C} \), it often turns out that the sum \( K_\epsilon(z, z') = \sum_{n=0}^{\infty} \epsilon^n p_n(z)p_n(z') \), \( z, z' \in \mathbb{C} \), is convergent in some domain of the complex plane, in which it defines a reproducing kernel. Moreover, the corresponding reproducing kernel Hilbert space turns out to be a (holomorphic) subspace of an \( L^2 \)-space over this domain. This is the general situation which is known to happen, for example, for the Hermite, Laguerre, and Jacobi polynomials. Additionally, a large number of other interesting questions emerge, related to such families of polynomials and reproducing kernel Hilbert spaces. In this paper and in [3], we have looked at the questions of Berezin-Toeplitz quantization using the real kernel \( K_\epsilon(x, y) \) and its semi-classical approximation, and to certain physical questions related to “squeezing” of coherent states. In a future publication we plan to look at the problems of the associated non-linear coherent states and complex orthogonal polynomials related to such systems.

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