Abstract

We present a new family of sharp examples for the Szemerédi-Trotter theorem. These are the first examples not based on a rectangular lattice. We also include an application to the discrete inverse Loomis-Whitney problem.

1 Introduction

One formulation of the celebrated Szemerédi–Trotter theorem \[24\] provides a tight upper bound for the number of \(r\)-rich lines:

**Theorem 1.1. (Szemerédi and Trotter)** Let \(P\) be a set of \(n\) points and let \(L_r\) be a set of lines we call \(r\)-rich that contain at least \(r\) points in \(P\), both in \(\mathbb{R}^2\). Then

\[
|L_r| = O \left( \frac{n^2}{r^3} + \frac{n}{r} \right).
\]

This statement is equivalent to the statement in terms of point-line incidences, which goes as follows.

**Theorem 1.2. (Szemerédi and Trotter)** Let \(P\) be a set of \(n\) points and let \(L\) be a set of \(m\) lines, both in \(\mathbb{R}^2\). Then

\[
I(P, L) = O \left( \frac{m^{2/3}n^{2/3}}{n^2} + m + n \right).
\]

The many variants of this theorem constitute an entire discipline called incidence theory. The theorem has also proved useful in other domains: its numerous applications range from problems in additive number theory to harmonic analysis \[8, 9, 11, 17, 22\]. Despite the community’s interest \[2, 14, 15, 23\], the inverse problem: characterizing constructions that meet the Szemerédi-Trotter (mixed term) upper bound, remains widely open. Although there has been recent progress for lines in general position on Cartesian product point sets, \[12, 19\], not much else is known.

Up to recently, only two constructions were known to match the (non-linear) term in Theorem 1.1 and Theorem 1.2. The first example, given by Erdős in 1946, is based on a square lattice. The second example, given by Elekes \[15\] in 2001, is based on a rectangular lattice. Adam Sheffer and the second author recently introduced the first infinite family of...
sharp Szemerédi-Trotter examples, which has the Erdős and Elekes constructions as limits [21]. In all of these examples, the point set is a lattice: a Cartesian product of two arithmetic progressions.

Every previous sharp example for the Szemerédi-Trotter theorem was found by starting with a Cartesian product of two arithmetic progressions and then applying a projective transformation and/or point-line duality. In this paper, we give a new sharp example which does not have this structure.

Our family of constructions.

We present the first sharp Szemerédi-Trotter family of non-lattice point-line constructions in $\mathbb{R}^2$: the $x$ and $y$ coordinates of the point set are a generalized arithmetic progression and for any richness $r$ there is a maximal family of $r$-rich lines on the point set.

**Theorem 1.3.** For any non-square integer $k$, any large enough $N$ and $r \leq N$, let the point set $\mathcal{P} = A_N^2$ where $A_N = \left\{ x_1 + x_2\sqrt{k}; x_1, x_2 \in [-\sqrt{N}, \sqrt{N}] \right\}$. Then there exists a set of $r$-rich lines $|\mathcal{L}_r|$ such that

$$|\mathcal{L}_r| = \Theta\left(\frac{|\mathcal{P}|^2}{r^3} + \frac{|\mathcal{P}|}{r}\right).$$

See Section 3 for the proof and the explicit construction of the line set. Our point set $\mathcal{P} = A_N^2$ is not a product of arithmetic progressions. It is a product of generalized arithmetic progressions. However, not every product of generalized arithmetic progressions gives a sharp example for Szemerédi-Trotter. The algebraic structure coming from $\sqrt{k}$ is crucial. If we replace $\sqrt{k}$ by a transcendental number, then the construction would be far from sharp for Szemerédi-Trotter.

**Application to Inverse Discrete Loomis-Whitney.**

The Loomis-Whitney inequality [18] upperbounds the volume of an $n$ dimensional set by the product of the areas of its "shadows": the $(n-1)$ dimensional coordinate projections.

**Theorem 1.4.** Let $m$ be the measure of an open subset $O$ of the Euclidean $n$-space, and let $m_1, \ldots, m_n$ be the $(n-1)$-dimensional measures of the projections of $O$ on the coordinate hyperplanes. Then

$$m^{n-1} \leq \prod_{i=1}^{n} m_i.$$

The many variations of this theorem constitute a rich field of study [3, 7, 5, 13]. These results also find applications in other domains from group theory [16] to the Kakeya problem in harmonic analysis [6]. Recently there has been much interest in the inverse problem: characterizing sets that provide sharp examples of the Loomis-Whitney inequality [10, 1]. We focus on the discrete variant of the inverse problem in $\mathbb{R}^2$: characterizing point configurations in the plane whose 1 dimensional projections are minimal. Classical Loomis-Whitney tells us that in the case of a point set in $\mathbb{R}^2$ of size $n^2$ (using affine transformations to map 2 arbitrary projection directions to the coordinate projections) the product of the size of these two projections is greater or equal to $n^2$. Equivalently, it is not possible for both projections to have size less than $n$.

Thus the natural inverse discrete Loomis-Whitney problem in the plane asks under which structural conditions of the point set of size $n^2$, and for which set of projection directions, all the one-dimensional projections have size $\Theta(n)$. Elementary arguments yield the following necessary and sufficient condition for square lattices:
Lemma 1.5. Let the point set \( \mathcal{P} \) be a section of the integer lattice of size \( n \times n \). A one-dimensional projection of \( \mathcal{P} \) has size \( \Theta(n) \) if and only if the slope of the projection direction is an irreducible rational \( p/q \) such that \( p, q = O(1) \).

Note for any square lattice in the plane there exists an affine map which takes it to a square section of the integer square lattice. So up to affine transformation of the plane lemma 1.5 holds for any square lattice.

Obtaining sharp constructions for the discrete inverse Loomis-Whitney problem in the plane for an \( n \times n \) grid of points overlaps with finding sharp examples for Theorem 1.1 because finding a family of \( \Theta(n) \) parallel \( \Theta(n) \)-rich lines yields a projection direction along which a constant fraction of the points have minimal projection size. We obtain the following application of theorem 1.3:

Corollary 1.6. For any non-square integer \( k \), any large enough \( N \), let the point set \( \mathcal{P} = A_N^2 \) where \( A_N = \left\{ x_1 + x_2 \sqrt{k}; x_1, x_2 \in \left[ -\sqrt{N}, \sqrt{N} \right] \right\} \). Then for any constant \( p = O(1) \) there is a set of projections \( \{ \pi_i \}_{i=1}^{\Theta(p)} \) such that \( |\pi_i(\mathcal{P})| = \Theta\left( \sqrt{pn} \right) \).

Sharp example for Energy Bound.

Our constructions provide a new tight example for the following lemma which provide upperbounds for the additive energy of finite subsets of \( \mathbb{R} \):

Lemma 1.7. Let \( A, B \) and \( X \) be finite subsets of \( \mathbb{R} \) such that \( |X| \leq |A||B| \). Then

\[
\sum_{x \in X} E^+(A, xB) = O\left( |A|^{3/2} |B|^{3/2} |X|^{1/2} \right).
\]

Note there is an equivalent lemma for multiplicative energy. The proofs of this lemma relies on an application of Theorem 1.1 so all of our sharp examples from Theorem 1.3 are also sharp for this lemma.

Lemma 1.8. For any non-square constant \( k \), and constant \( M \leq N \) let

\( A_N = \left\{ x_1 + x_2 \sqrt{k}; x_1, x_2 \in \left[ -\sqrt{N}, \sqrt{N} \right] \right\} \) and let \( X = S \subset \left( 4N \right) \) be the slope set from the proof of Theorem 1.3. Then \( |X| \leq |A_N|^2 \) and

\[
\sum_{x \in X} E^+(A_N, xA_N) = O\left( |A_N|^3 |X|^{1/2} \right).
\]

Note we can construct an equivalent sharp example for the multiplicative energy version of the lemma.

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2 Background

This section presents tools that will be used in the proofs.

Asymptotic notation is used throughout. We say \( f(n) = O(g(n)) \) if there exist constants \( c, n_0 > 0 \) such that \( |f(n)| \leq c \cdot g(n) \) for all \( n \geq n_0 \). Likewise \( f(n) = \Omega(g(n)) \) if there exist constants \( c, n_0 > 0 \) such that \( |f(n)| \geq c \cdot g(n) \) for all \( n \geq n_0 \). We say \( f(n) = \Theta(g(n)) \) if and only if \( f(n) = O(g(n)) \) and \( f(n) = \Omega(g(n)) \).
We also use the stronger notation \( f(n) = o(g(n)) \) if for all \( \epsilon > 0 \) there exists \( n_\epsilon \) such that \( |f(n)| \leq \epsilon \cdot g(n) \) for all \( n \geq n_\epsilon \). Likewise we say \( f(n) = \omega(g(n)) \) if for all \( \epsilon > 0 \) there exists \( n_\epsilon \) such that \( |f(n)| \geq \epsilon \cdot g(n) \) for all \( n \geq n_\epsilon \).

The following classical result from Beck [4] is used in the proof of Theorem 1.3.

**Theorem 2.1. (Beck)** There exist constants \( c, k \) such that for any set of \( n \) points in \( \mathbb{R}^2 \)

- either there is a subset of \( n/c \) collinear points
- or there are \( \Omega(n^2/k) \) distinct lines containing at least two points of the point set. Such lines are said to be determined by the point set.

### 3 New constructions

In this section we prove Theorem 1.3 and provide an explicit description of the line set. We first recall the statement of the theorem.

**Theorem 1.3**

For any non-square integer \( k \), any large enough \( N \) and \( r \leq N \), let the point set \( \mathcal{P} = A_N^2 \) where \( A_N = \left\{x_1 + x_2\sqrt{k}; x_1, x_2 \in [-\sqrt{N}, \sqrt{N}]\right\} \). Then there exists a set of \( r \)-rich lines \( |\mathcal{L}_r| \) such that

\[
|\mathcal{L}_r| = \Omega\left(\frac{P^2}{r^3} + \frac{P}{r}\right).
\]

**Proof.** We double count the number of incidences to obtain a lower bound on the size of the line set. This involves proving each of the lines are \( r \)-rich. Let \( M = \frac{N}{r} \).

We define the point set, slope set and line set as follows:

\[
A_N = \left\{x_1 + x_2\sqrt{k} \mid x_1, x_2 \in [-\sqrt{N}, \sqrt{N}]\right\}
\]

There are \( 2\sqrt{N} \) choices of \( x_i \) and \( k \) is not a square so \( |A_N| = \left(2\sqrt{N}\right)^2 = 4N = \Theta(N) \). Letting \( P = A_N^2 \) we have \( |P| = \Theta(N^2) \). Next I define the slope set

\[
S = \left\{(p_1 + p_2 \sqrt{k}, q_1 + q_2 \sqrt{k}) \mid |p_i|, |q_i| \in \left[c\sqrt{M}, \sqrt{M}\right], \gcd(p_1^2 - kp_2^2, q_1^2 - kq_2^2) \leq 5, \gcd(p_1, p_2) \leq 5\right\}
\]

for some constant \( c < 1 \) sufficiently close to 1. Then let the line set

\[
L = \left\{y = s(x - a) + b; (a, b) \in A_N^{2/4}, s \in S\right\}
\]

Each point \((a, b) \in A_N^{2/4} \subset P\) has at least \( |S| \) lines of \( L \) so \( I(P, L) = \Omega \left(|A_N^{2/4}| |S|\right) = \Omega(N^2 |S|) \).

**Lemma 3.1.** \(|S| = \Theta(M^2)\)

**Proof.** We represent the slopes in \( S \) as points in the plane where the \( x \)-coordinate is the numerator and the \( y \)-coordinate is the denominator. Let \( S_P = \{(p_1 + p_2 \sqrt{k}, q_1 + q_2 \sqrt{k}) \in \).
$A^2_M$ where $|p_i|, |q_i| \in \left[ c\sqrt{M}, \sqrt{M} \right]$, $\gcd(p_i^2 - kp_2^2, q_i^2 - kq_2^2) \leq 5$, $\gcd(p_1, p_2) \leq 5$} $\subset \mathbb{R}^2$. Note the number of distinct elements in the slope set $S$ is equal to the number of lines determined by a point in $S_P$ and the origin. Let $S^+_P \subset S_P$ be the subset of elements where $p_i, q_i \geq 0$, then $|S_P| = 16|S^+_P|$

No line can contain more than $\sqrt{|S^+_P|}$ points of $S^+_P$ since each line contains at most one point per row/column. Thus we must be in the second case of Theorem 2.1: there exists a constant $K$ such that at least $|S^+_P|^2 / K$ distinct lines are determined by points in $S^+_P$. So there exists $\alpha = (\alpha_x, \alpha_y) \in S^+_P$ such that the set $L_\alpha$ of lines determined by $\alpha$ and another point in $S^+_P$ satisfies $|L_\alpha| \geq |S^+_P| / K$.

For all $l \in L_\alpha$ there exists a point $\beta = (\beta_x, \beta_y) \in S^+_P \setminus \{\alpha\}$ such that $\beta \in l$. Letting $l : y - \alpha_y = s(x - \alpha_x)$ we must have $(\beta_y - \alpha_y) = s(\beta_x - \alpha_x)$. $\alpha, \beta \in S^+_P$ so $\beta - \alpha \in S_p$. Thus the line of slope $s$ going through the origin also contains the point $\beta - \alpha \in S_p$. Finally all the lines in $L_\alpha$ are concurrent and distinct so none of them have the same slope. Thus $|S| \geq |L_\alpha| \geq |S^+_P| / K$.

To find a lower bound on $S^+_P$, we must remove the quadruples $(p_1, p_2, q_1, q_2)$ that do not satisfy the divisibility requirements. The number of quadruples $(p_1, p_2, q_1, q_2) \in \left[ c\sqrt{M}, \sqrt{M} \right]$ such that $\gcd(p_i^2 - kp_2^2, q_i^2 - kq_2^2) \leq 5$ and $\gcd(p_1, p_2) \leq 5$ is equal to $(1-c)^2 M^2$ minus the number of quadruples such that $d \mid p_1^2 - kp_2^2$ and $d \mid q_1^2 - kq_2^2$ for some odd prime $d > 5$ and the number of quadruples where $d \mid p_1$ and $d \mid p_2$ for some odd prime $d > 5$.

We first count the number of quadruples such that $d \mid p_1^2 - kp_2^2$ and $d \mid q_1^2 - kq_2^2$. If $k p_2^2$ is a quadratic residue mod $d$ (prime) then $\mathbb{F}_d$ is a field so the degree 2 equation for $p_1$ in $\mathbb{F}_d : p_1^2 = k p_2^2 \mod d$ has at most two solutions. Likewise for $q_1$, so we must remove at most $(1-c)^2 M^2 \cdot \phi(1-c)/2$ quadruples $(p_1, p_2, q_1, q_2)$. Furthermore the number of quadruples $(p_1, p_2, q_1, q_2)$ such that $d \mid p_1$ and $d \mid p_2$ for some odd prime $d$ which we must remove is upperbounded by $(1-c)^2 M^2 \cdot \phi(1-c)/2 = (1-c)^2 M^2 / d^2$.

So for each $d$ the number of quadruples we must remove is upperbounded by $\frac{4(1-c)^2 M^2}{d^2} + \frac{5(1-c)^2 M^2}{d^2}$. Summing over all odd primes $d > 5$ we must remove at most $5(1-c)^2 M^2 \sum_{d \geq 7 \text{ prime}} d^{-2} < 5(1-c)^2 M^2$.

So $|S^+_P| \geq (1-c)^2 M^2 - \frac{5}{6} (1-c)^2 M^2 = \Omega(M^2)$. Thus $|S| \geq |S^+_P| / K = \Omega(M^2)$. Furthermore, $|S| \leq M^2$ so $|S| = \Theta(M^2)$.

Therefore $I(P, L) = \Omega(N^2 M^2)$.

**Lemma 3.2.** Each line in $L$ has $\Theta\left(\frac{N^3}{M^3}\right)$ points in $P$.

**Proof.** Each line is $\Omega\left(\frac{N^3}{M^3}\right)$ rich: Let $l$ be an arbitrary line in $L$. There exist $s = \frac{m_1 + \sqrt{k}a_1}{q_1 + q_2 \sqrt{k}} \in S$ and $(a, b) \in A_{N/4}^2$ such that $l$ is the line $y = b = s(x - a)$. Then for all $x = a + (q_1 + \sqrt{k}a_2)(a_1 + \sqrt{k}a_2)$ where $a_i \in \left[ 0, \min_{i \in \{1, 2\}} \frac{\sqrt{N} - |b_i|}{(k+1)\sqrt{N}} \right] : s(x-a) = (p_1 + \sqrt{k}a_2)(a_1 + \sqrt{k}a_2) = (p_1 a_1 + kp_2 a_2) + (p_1 a_2 + p_2 a_1) \sqrt{k}$ where each of the linearly independent terms have integer coefficients in the range $\max_{i \in \{1, 2\}} \left[ -\sqrt{N} - b_i, \sqrt{N} - b_i \right]$. So for all $\Omega\left(\left(\frac{\sqrt{N}}{M^3}\right)^2 \right)$ choices of $a_1, a_2$ there exists $y \in A_N$ such that $y - b = s(x-a)$. Thus each line in $L$ has $\Omega\left(\frac{N^3}{M^3}\right)$ points in $P$.

Each line in $L$ has $O\left(\frac{N^3}{M^2}\right)$ points in $P$: Let $(x, y) \in A_N^2 = P$ such that $(x, y) \in l$ for
some line \( l : y = \frac{p_1 + p_2 \sqrt{k}}{q_1 + q_2 \sqrt{k}} (x - a) + b \in L \). Letting \( Y_1 + Y_2 \sqrt{k} = y - b \) this is equivalent to \( x = \frac{(Y_1 + Y_2 \sqrt{k})(q_1 + q_2 \sqrt{k})}{p_1 + p_2 \sqrt{k}} + a \).

\[ \Rightarrow x = a + \frac{(Y_1 q_1 p_1 - k Y_1 q_2 p_2 - k Y_2 q_2 p_1 + k Y_2 q_2 p_1) + (Y_2 q_1 p_1 + Y_1 q_2 p_1 - Y_1 q_1 p_2 - k Y_2 q_2 p_2) \sqrt{k}}{p_1^2 - k p_2^2} \]

\[ x \in A_N \Rightarrow \begin{cases} p_1^2 - k p_2^2 | q_1 (Y_1 p_1 - k Y_2 p_2) + k q_2 (Y_2 p_1 - Y_1 p_2) \\ p_1^2 - k p_2^2 | q_2 (Y_1 p_1 - k Y_2 p_2) + q_1 (Y_2 p_1 - Y_1 p_2) \end{cases} \]

\[ \Rightarrow p_1^2 - k p_2^2 | (q_1 - q_2)(Y_1 p_1 - k Y_2 p_2) \]

\[ \frac{p_1 + p_2 \sqrt{k}}{q_1 + q_2 \sqrt{k}} \in S \text{ so } \gcd(p_1^2 + k p_2^2, q_1^2 + k q_2^2) \leq 5 \] Thus \( p_1^2 - k p_2^2 | 30(Y_1 p_1 - k Y_2 p_2) \). \( Y_1, Y_2 \in [-\sqrt{N}, \sqrt{N}] \) and \( |p_1| \in [c \sqrt{M}, \sqrt{M}] \) where \( c < 1 \) is a constant that we choose to be sufficiently close to 1. Then \( |p_1^2 - k p_2^2| > |M - k(1 - c)^2 M| = \Omega(M) \). So \( \frac{|Y_1 p_1 - k Y_2 p_2|}{p_1^2 - k p_2^2} = O\left(\frac{\sqrt{N}}{M}\right) \). So \( p_1^2 - k p_2^2 | 30(Y_1 p_1 - k Y_2 p_2) \) if and only if there exists and integer \( |j| = O\left(\frac{\sqrt{N}}{M}\right) \) such that \( j(p_1^2 - k p_2^2) = 30(Y_1 p_1 - k Y_2 p_2) \)

\[ \Rightarrow \begin{cases} Y_1 = j p_1 / 30 + \frac{k p_2 (Y_2 - p_1)}{30 p_1} \\ Y_2 = j p_2 / 30 + \frac{p_1 (Y_2 - p_1)}{30 k p_2} \end{cases} \Rightarrow \begin{cases} p_1 | k (Y_2 - p_2) \\ k p_2 | Y_1 - p_1 \end{cases} \]

Since \( \gcd(p_1, p_2) \leq 5 \) and \( Y_1, p_1 \) are integers. Also \( |p_1| \geq c \sqrt{M} \) and \( Y_1 \equiv p_1 \mod k p_2 \) and \( Y_1 \in \left[-\sqrt{N}, \sqrt{N}\right] \) so there are \( O\left(\frac{\sqrt{N}}{M}\right) \) choices for \( Y_1 \). Plugging \( Y_1 \) into the system of equations above uniquely defines \( Y_2 \) so there are \( O\left(\frac{\sqrt{N}}{M}\right) \) choices of \( (Y_1, Y_2) \) for each \( j \). There are \( O\left(\frac{\sqrt{N}}{M}\right) \) choices of \( j \) so each line is \( O\left(\frac{\sqrt{N}}{M} \cdot \frac{\sqrt{N}}{M}\right) = O\left(\frac{N}{M^2}\right) \) rich.

Each line in \( L \) has \( \Theta\left(\frac{N}{M}\right) \) points in \( P \) so \( I(P, L) = \Theta\left(|L| \frac{N}{M}\right) \). Combining with \( I(P, L) = \Theta\left(N^2 M^2\right) \) we obtain \( |L| = \Theta\left(N M^3\right) \).

The Szemerédi-Trotter bound for point set \( P \) states that the number of \( \frac{N}{M} \) rich lines is \( O\left(\frac{N^2}{M^2}\right) + \frac{N^2}{M} = O\left(\frac{N}{M^2}\right) \). So we have achieved the Szemerédi-Trotter upper bound for any richness.

\[ \square \]

4 Applications

In this section we prove Lemma 1.5 and Corollary 1.6, two sharp examples of the inverse discrete Loomis-Whitney problem in the plane. We use the family of constructions from Theorem 1.3 to show two lemmas bounding additive and multiplicative energies are sharp. We first recall the statements.

**Corollary 1.6.**
For any non-square integer \( k \), any large enough \( N \), let the point set \( P = A_N^2 \) where \( A_N = \left\{ x_1 + x_2 \sqrt{k}; x_1, x_2 \in \left[-\sqrt{N}, \sqrt{N}\right]\right\} \). Then for any constant \( p = O(1) \) there is a set of projections \( \pi_i \) such that \( |\pi_i(P)| = \Theta\left(\sqrt{N}\right) \).
Proof. We see \( \mathcal{P} \) as embedded in the larger point set \( \mathcal{P}' = A_{4N}^2 \). Letting \( p = M^2 \), we construct the set of \( \frac{n}{\sqrt{p}} \)-rich lines on \( \mathcal{P}' \) from the proof of Theorem 1.3. These belong to \(|S|\) many families of parallel lines each of size \( \Theta(\sqrt{pm}) \), such that every point in \( \mathcal{P} \) is in exactly one line from every family. The size of the slope set is \(|S| = \Theta(M^2) = \Theta(p)\). Letting \( S \) be the projection directions, the size of each projection is equal to the number of lines in each family = \( \Theta(\sqrt{pm}) \).

Lemma 1.5
Let the point set \( \mathcal{P} \) be a section of the integer lattice of size \( n \times n \). A one-dimensional projection of \( \mathcal{P} \) has size \( \Theta(n) \) if and only if the slope of the projection direction is an irreducible rational \( \frac{p}{q} \) such that \( p, q = O(1) \).

Proof. Any line whose slope is non-rational will go through at most a single point, so the projection of the point set along this direction will have maximal size of \( n^2 \). So projections of size \( O(n) \) can only exist along rational projection directions.

Furthermore we know from the proof of 1.3, taking the case where \( k \) is a square, so the point set reduces to the case of a square lattice, that if \( p, q = O(1) \) then the projection along the slope \( \frac{p}{q} \) has size \( \Theta(n) \).

If \( p = \omega(1), y = \frac{x}{\omega(q)} \cdot x \in [0,n] \implies x/q = o(n) \). Similarly, if \( q = \omega(1), y = \frac{x}{\omega(q)} \cdot x \in [0,n] \implies y/q = o(n) \). In either case there are asymptotically less than \( n \) points of the lattice on each line of slope \( \frac{p}{q} \). So the projection along \( \frac{p}{q} \) has size \( \omega(n) \).

Lemma 1.8
For any non-square constant \( k \), and \( M \leq N \) let
\[ A_N = \left\{ x_1 + x_2 \sqrt{k}; x_1, x_2 \in \left[-\sqrt{N}, \sqrt{N}\right] \right\} \]
and let \( X = S \subset \frac{A_N}{N} \) be the slope set from the proof of Theorem 1.3. Then \(|X| \leq |A_N|^2\) and
\[ \sum_{x \in X} E^+(A_N, xA_N) = \Theta\left(|A_N|^3|X|^{1/2}\right). \]

Proof. We consider the dual situation of the proof of Lemma 2.3 [20]. I let the point set be \( \mathcal{P} = A_N^2 \) and the line set to be as in the construction of Theorem 1.3. Then \( \sum_{x \in X} E^+(A_N, xA_N) = \sum_{X \in S} \sum_y r_{A+Bx}(y) = \Theta\left(\sum_{\text{lines}}(N)^2\right) \) since each line in the construction is \( \frac{N}{M} \) rich. Furthermore there are \( \Theta(N \cdot M^2) \) lines in the construction so \( \sum_{x \in X} E^+(A_N, xA_N) = \Theta(N^3 \cdot M) \). Finally \( |A_N| = N \) and \(|X| = |S| = \Theta(M^2)\) [31] so I have shown \( \sum_{x \in X} E^+(A_N, xA_N) = \Theta\left(|A_N|^3|X|^{1/2}\right). \)

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