ON BASES OF QUANTUM AFFINE ALGEBRAS

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1. INTRODUCTION

Let \( U^+ \) be the positive part of the quantum group \( U \) associated with a generalized Cartan matrix \( A \).

In the case of finite type, Lusztig constructed the canonical basis \( B \) of \( U^+ \) via two approaches (\([8]\)). The first one is an elementary algebraic construction via Ringel-Hall algebra realization of \( U^+ \). The isomorphism classes of representations of the corresponding Dynkin quiver form a PBW-type basis of \( U^+ \). By a lemma (Lemma 3.1) of Lusztig, one can construct a bar-invariant basis, which is the canonical basis \( B \). A remarkable characteristic in his construction is that Lusztig reveals the triangular relations among three kind of bases: PBW-basis, monomial basis and canonical basis.

The second one is a geometric construction. In \([8]\) and \([9]\), Lusztig gave a geometric realization of \( U^+ \) via the category of some semisimple complexes on the variety \( E_\nu \) consisting of representations with dimension vector \( \nu \in \mathbb{N}^I \) of the corresponding quiver \( Q \). The set of the isomorphism classes of simple perverse sheaves gives the canonical basis \( B \) of \( U^+ \).

The geometric construction of canonical basis was generalized to the cases of all types in \([9]\) (see also \([11]\)). Furthermore, Lusztig in \([10]\) gave the construction of affine canonical bases by the perverse sheaves associated with tame quivers.

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an important feature is that those perverse sheaves are indexed by the classes of
aperiodic modules of tame quivers and irreducible modules of symmetric groups.

The generalization of this elementary algebraic construction to affine type is an
important problem.

In [1] and [2], Beck-Chari-Pressley and Beck-Nakajima defined a PBW-type basis
and gave an algebraic construction of canonical basis. However, the meaning of this
construction is not clear in terms of representation theory of quivers, in particular,
it is not clear how to parametrize the canonical basis in terms of aperiodic modules
of tame quivers and irreducible modules of symmetric groups.

In the case of Kronecker quiver, Chen ([3]) and Zhang ([17]) defined a PBW-
type basis by using Ringel-Hall algebra realization of $U^+$, McGerty ([12]) gave the
interpretation of the PBW-type basis of Beck-Nakajima in terms of representation
theory of the Kronecker quiver. In the case of cyclic quiver, Deng-Du-Xiao ([4])
defined a PBW-type basis and gave a concrete algebraic construction of the canoni-
cal basis by using the triangular relations between the monomial basis and the
PBW-basis.

Lin-Xiao-Zhang in [7] provided a process to construct a PBW-type basis of
$U^+$ and the canonical basis $B$ by using Ringel-Hall algebra approach. Recently Xiao-
Xu-Zhao ([16]) provided a direct method to construct a PBW-type basis of
$U^+$ and the canonical basis $B$. Compared with the construction Lin-Xiao-Zhang in
[7], the PBW-type basis of Xiao-Xu-Zhao is a $\mathbb{Z}[v,v^{-1}]$-basis. Particularly the
parametrization of the basis by aperiodic modules of tame quivers and irreducible
characters of $S_n$ naturally arises in the construction.

In this paper, we shall review these constructions. Since we shall use represen-
tations of quivers, we mainly consider the quantum group corresponding to a
symmetric generalized Cartan matrix.

2. Preliminaries

Let $I$ be a finite index set, $A = (a_{ij})_{i,j \in I}$ be a symmetric generalized Cartan
matrix. Denote by $U$ the quantum group associated with the Cartan matrix $A$ and
$U^+$ the positive part of $U$ ([11]).

Fix an indeterminate $v$. For any $n \in \mathbb{Z}$, set

$$[n]_v = \frac{v^n - v^{-n}}{v - v^{-1}}.$$ 

Let $[0]_v! = 1$ and $[n]_v! = [n]_v[n-1]_v \cdots [1]_v$ for any $n \in \mathbb{Z}_{>0}$.

Note that $U^+$ is an associate algebra over $\mathbb{Q}(v)$ generated by the elements $E_i$
for various $i \in I$ subject to the quantum Serre relations

$$\sum_{k=0}^{1-a_{ij}} (-1)^k E_i^{(k)} E_j E_i^{(1-a_{ij}-k)} = 0$$

for all $i \neq j \in I$, where $E_i^{(n)} = E_i^n / [n]_v!$.

Let $\overline{\cdot} : U^+ \to U^+$ be the unique $\mathbb{Q}$-algebra involution such that

$$\overline{v^n} = v^{-n} \quad \text{and} \quad \overline{E_i} = E_i.$$

Let $Q = (I, H, s, t)$ be a quiver without loops, where $I$ is the set of vertices, $H$
is the set of arrows and $s, t : H \to I$ are two maps sending an arrow $h \in H$ to the
source \(s(h)\) and target \(t(h)\) respectively. Let

\[ a_{ij} = |\{ h \in H \mid s(h) = i \text{ and } t(h) = j \}| + |\{ h \in H \mid s(h) = j \text{ and } t(h) = i \}|. \]

Then \(A = (a_{ij})_{i,j \in I}\) is a symmetric generalized Cartan matrix and called the generalized Cartan matrix associated to the quiver \(Q\). The Euler form on \(ZI\) is defined as

\[ \langle \nu, \nu' \rangle = \sum_{i \in I} \nu_i \nu'_i - \sum_{h \in H} \nu_{s(h)} \nu'_{t(h)} \]

and the symmetric Euler form is defined as \(\langle \nu, \nu' \rangle = \langle \nu, \nu' \rangle + \langle \nu', \nu \rangle\) for any \(\nu = \sum_{i \in I} \nu_i^1 i\) and \(\nu' = \sum_{i \in I} \nu'_i i\).

Let \(k = \mathbb{F}_q\) be a finite field with \(q\) elements and \(kQ\) be the path algebra. Denote by \(\text{rep}_k Q\) the abelian category of finite dimensional representations of \(Q\) over \(k\). There is an isomorphism between \(\text{rep}_k Q\) and the category \(\text{mod}-kQ\) of finite dimensional \(kQ\)-modules.

Let \(\mathcal{P}_k\) be the set of isomorphism classes of objects in \(\text{rep}_k Q\). For any \(\alpha \in \mathcal{P}_k\), choose an object \(M_\alpha \in \text{rep}_k Q\) such that the isomorphism classes \([M_\alpha] = \alpha\). Define the dimension vector of \(\alpha \in \mathcal{P}_k\) by \(\dim \alpha = \dim M_\alpha\).

Given three elements \(\alpha_1, \alpha_2\) and \(\alpha\) in \(\mathcal{P}_k\), denote by \(g^\alpha_{\alpha_1 \alpha_2}\) the number of subrepresentations \(N\) of \(M_\alpha\) such that \(N \simeq M_{\alpha_1}\) and \(M_\alpha/N \simeq M_{\alpha_2}\) in \(\text{rep}_k Q\). Let \(\nu_q = \sqrt{q} \in \mathbb{C}\). The twisted Ringel-Hall algebra \(H^*(kQ)\) is the \(\mathbb{Q}(\nu_q)\)-space with basis

\[ \{ u_\alpha \mid \alpha \in \mathcal{P}_k \}, \]

whose multiplication is given by

\[ u_{\alpha_1} * u_{\alpha_2} = \sum_{a \in \mathcal{P}} \nu_q^{(\dim \alpha_1, \dim \alpha_2)} g^\alpha_{\alpha_1 \alpha_2} u_\alpha. \]

For any \(\alpha \in \mathcal{P}_k\), let

\[ \langle M_\alpha \rangle = \nu_q^{-\dim M_\alpha + \dim \text{End} M_\alpha} u_\alpha. \]

The set \(\{ \langle M_\alpha \rangle \mid \alpha \in \mathcal{P}_k\}\) is also a \(\mathbb{Q}(\nu_q)\)-basis of \(H^*(kQ)\). There is a bilinear form \((\cdot, \cdot)\) on \(H^*(kQ)\) defined in [5].

Denote by \(\mathcal{C}^*(kQ)\) the composition subalgebra of \(H^*(kQ)\) generated by \(u_i = u_{[S_i]}\) for all \(i \in I\), where \(S_i\) is the simple \(kQ\)-module corresponding to \(i \in I\).

Then we shall recall the generic form of \(\mathcal{C}^*(kQ)\). Let \(\mathcal{K}\) be a set of some finite fields \(k\) such that the set \(\{ g_k = |k| \mid k \in \mathcal{K}\}\) is an infinite set. Consider the direct product

\[ H^*(Q) = \prod_{k \in \mathcal{K}} H^*(kQ) \]

and the elements \(v = (v_k)_{k \in \mathcal{K}}, v^{-1} = (v_k^{-1})_{k \in \mathcal{K}}\) and \(u_i = (u_i(k))_{k \in \mathcal{K}}.\) By \(\mathcal{C}^*(Q)_{\mathbb{Q}[v, v^{-1}]}\) we denote the subalgebra of \(H^*(Q)\) generated by \(v, v^{-1}\) and \(u_i\) over \(\mathbb{Q}\). We may regard it as a \(\mathbb{Q}[v, v^{-1}]\)-algebra generated by \(u_i\), where \(v\) is viewed as an indeterminate. Finally, define \(\mathcal{C}^*(Q) = \mathbb{Q}(v) \otimes_{\mathbb{Q}[v, v^{-1}]} \mathcal{C}^*(Q)_{\mathbb{Q}[v, v^{-1}]}\), which is called the generic twisted composition algebra of \(Q\).

**Theorem 2.1** ([5][14]). Let \(Q\) be a quiver without loops, \(A\) the corresponding generalized Cartan matrix and \(U^+\) the positive part of quantum group of type \(A\). There is an isomorphism of \(\mathbb{Q}(v)\)-algebras:

\[ \mathcal{C}^*(Q) \cong U^+ \]

\[ u_i \mapsto E_i. \]
Under this isomorphism, the bar involution on $U^+$ induces a bar involution $\overline{\cdot} : C^*(Q) \to C^*(Q)$ such that $\overline{v^n} = v^{-n}$ and $\overline{u_i} = u_i$.

In [8, 9], Lusztig gave a geometric realization of $U^+$. Let $Q$ be a quiver without loops with associated generalized Cartan matrix $A$. Lusztig considered the variety $E_\nu$ consisting of representations with dimension vector $\nu \in \mathbb{N}I$ of the quiver $Q$ over algebraically closed field $\overline{k}$ and the category $\mathcal{Q}_\nu$ of some semisimple complexes of constructible sheaves on $E_\nu$. Let $K(\mathcal{Q}_\nu)$ be the Grothendieck group of $\mathcal{Q}_\nu$.

Considering all dimension vectors, let

\[ K(Q) = \bigoplus_\nu K(\mathcal{Q}_\nu). \]

Lusztig define induction functors on $\mathcal{Q}_\nu$ and get a $\mathbb{Z}[v, v^{-1}]$-algebra structure on $K(Q)$. This algebra is isomorphic to the integral form of $U^+$. The set $B$ of the isomorphism classes of simple perverse sheaves gives a basis of $U^+$, which is called the canonical basis.

3. Canonical bases of finite types

Assume that $Q$ is a Dynkin quiver. In this case, the algebraic construction of canonical basis was introduced by Lusztig in [8] (see also [15]).

Let $A = (a_{ij})_{i,j \in I}$ be the generalized Cartan matrix associated to the quiver $Q$ and denote by $\Delta^+$ the set of positive roots of the Lie algebra $g(A)$ with $i$ corresponding to simple roots. The dimension vector $\text{dim}$ induces a bijection between the set of isomorphism classes of indecomposable objects $\text{ind-P}$ and the set $\Delta^+$ by Gabriel theorem. Given a positive root $\alpha$, choose an indecomposable representation $M_\alpha$ of $Q$ such that $[M_\alpha] = \alpha$.

Denote by $\mathbb{N}^{\Delta^+}$ the set of all functions $\phi : \Delta^+ \to \mathbb{N}$. For each $\phi : \Delta^+ \to \mathbb{N}$, define a representation

\[ M_\phi = \bigoplus_{\alpha \in \Delta^+} M_{\alpha}^{\oplus \phi(\alpha)}. \]

Then $\mathcal{P} = \{ [M_\phi] \mid \phi \in \mathbb{N}^{\Delta^+} \}$.

Now $\mathcal{H}^*(kQ)$ is spanned by the set

\[ \{ u_\phi = u_{[M_\phi]} \mid \phi : \Delta^+ \to \mathbb{N} \} \]

as $\mathbb{Q}(v)$-vector space, which is called a PBW-type basis.

Since $Q$ is representation-directed, we can define a total order on the set $\Delta^+$ such that

\[ \text{Hom}(M_\alpha, M_\beta) \neq 0 \Rightarrow \alpha \leq \beta \]

for any $\alpha, \beta \in \Delta^+$. This total order induces an order on $\mathbb{N}^{\Delta^+}$. For any $\phi, \psi : \Delta^+ \to \mathbb{N}$, define $\phi < \psi$ if and only if there exists $\alpha \in \Delta^+$ such that $\phi(\alpha) > \psi(\alpha)$ and $\phi(\beta) = \psi(\beta)$ for all $\alpha > \beta \in \Delta^+$.

For each $\phi : \Delta^+ \to \mathbb{N}$, there exists a monomial $m_\phi$ on the divided powers of Chevalley generators $u_i$ satisfying

\[ m_\phi = \langle M_\phi \rangle + \sum_{\phi' < \phi} a_{\phi'}^\phi \langle M_{\phi'} \rangle, \]
with \( a_\phi^{\phi'} \in \mathbb{Z}[v,v^{-1}] \). Since \( m_\phi = m_\phi \), we have

\[
\overline{(M_\phi)} = \langle M_\phi \rangle + \sum_{\phi' < \phi} b_\phi^{\phi'} \langle M_{\phi'} \rangle,
\]

with \( b_\phi^{\phi'} \in \mathbb{Z}[v,v^{-1}] \) such that

1. \( b_\phi^{\phi} = 1 \) for all \( \phi \) in \( \mathbb{N}^\Delta^+ \);
2. for all \( \phi' \leq \phi \) in \( \mathbb{N}^\Delta^+ \),
   \[
   \sum_{\phi'' : \phi'' \leq \phi' \leq \phi} b_\phi^{\phi''} b_\phi^{\phi''} = \delta_{\phi,\phi'}.
   \]

Here we need a lemma by Lusztig, which can be obtained by an elementary linear algebra method.

**Lemma 3.1** ([11]). Let \( H \) be a set with a partial order \( \leq \) such that for any \( h' \leq h \) in \( H \), the set \( \{ h'' : h' \leq h'' \leq h \} \) is finite. Assume that for each \( h' \leq h \) in \( H \), we are given an element \( r_h^{h'} \in \mathbb{Z}[v,v^{-1}] \) such that

1. \( r_h^{h} = 1 \) for all \( h \) in \( H \);
2. for all \( h' \leq h \) in \( H \),
   \[
   \sum_{h'' : h'' \leq h' \leq h} r_h^{h''} r_h^{h'} = \delta_{h,h'}.
   \]

Then there is a unique family of elements \( p_h^{h'} \in \mathbb{Z}[v^{-1}] \) defined for all \( h' \leq h \) in \( H \) such that

1. \( p_h^{h} = 1 \) for all \( h \) in \( H \);
2. \( p_h^{h'} \in v^{-1}\mathbb{Z}[v^{-1}] \) for all \( h' \leq h \) in \( H \);
3. for all \( h' \leq h \) in \( H \),
   \[
   p_h^{h'} = \sum_{h'' : h'' \leq h' \leq h} \overline{r_h^{h''}} r_h^{h'}.
   \]

By Lemma 3.1, there exists a unique family of elements \( c_\phi^{\phi'} \in \mathbb{Z}[v^{-1}] \) defined for all \( \phi' \leq \phi \) in \( \mathbb{N}^\Delta^+ \) such that

1. \( c_\phi^{\phi} = 1 \) for all \( \phi \) in \( \mathbb{N}^\Delta^+ \);
2. \( c_\phi^{\phi'} \in v^{-1}\mathbb{Z}[v^{-1}] \) for all \( \phi' \leq \phi \) in \( \mathbb{N}^\Delta^+ \);
3. for all \( \phi' \leq \phi \) in \( \mathbb{N}^\Delta^+ \),
   \[
   c_\phi^{\phi'} = \sum_{\phi'' : \phi'' \leq \phi' \leq \phi} \overline{c_\phi^{\phi''}} b_\phi^{\phi''}.
   \]

For any \( \phi \in \mathbb{N}^\Delta^+ \), let

\[
\overline{C_\phi} = \langle M_\phi \rangle + \sum_{\phi' < \phi} c_\phi^{\phi'} \langle M_{\phi'} \rangle.
\]

These formulas hold for every finite fields and may be viewed as formulas in \( \mathcal{H}^*(Q) = \mathcal{C}^*(Q) \). Then we have the following theorem.

**Theorem 3.2.** The set \( \{ \overline{C_\phi} : \phi : \Delta^+ \to \mathbb{N} \} \) is a \( \mathbb{Z}[v,v^{-1}] \)-basis of \( \mathcal{H}^*(Q)_{\mathbb{Z}[v,v^{-1}]} \) satisfying the following conditions.

1. \( \overline{C_\phi} = C_\phi \);
(2) \((C_\phi, C_{\phi'}) \in \delta_\phi,\phi' + v^{-1}\mathbb{Z}[v^{-1}] \cap \mathbb{Q}(v)\).

Under the isomorphism between \(\mathcal{H}^+(Q)\) and \(U^+\), the set
\[\{C_\phi \mid \phi : \Delta^+ \rightarrow \mathbb{N}\}\]
induces a basis of \(U^+\). This basis is just the canonical basis \(B\) of \(U^+\), by Theorem 3.2 and the uniqueness of canonical basis of \(U^+\).

**Example 3.3.** Take the quiver \(Q\) of type \(A_3\) for example.

\[Q: 1 \longrightarrow 2 \longrightarrow 3.\]

The AR-quiver is as following.

\[
\begin{array}{cccc}
\text{M}_{(111)} & \text{M}_{(110)} & \text{M}_{(011)} & \text{M}_{(010)} & \text{M}_{(001)} \\
\text{M}_{(100)} & \text{M}_{(001)} \\
\end{array}
\]

For dimension vector \(\nu = (111)\), there exist isomorphism classes of the following modules
\[\{M_{(111)}, [M_{(110)} \oplus M_{(001)}], [M_{(100)} \oplus M_{(011)}], [M_{(100)} \oplus M_{(010)} \oplus M_{(001)}]\}.\]

Hence,
\[\{(M_{(111)}), (M_{(110)} \oplus M_{(001)}), (M_{(100)} \oplus M_{(011)}), (M_{(100)} \oplus M_{(010)} \oplus M_{(001)})\}\]
are the elements in the PBW-type basis with dimension vector \(\nu = (111)\).

By construction, the elements in the corresponding monomial basis are
\[u_1u_2u_3 = (M_{(111)}) + v^{-2}(M_{(110)} \oplus M_{(001)}) + v^{-2}(M_{(100)} \oplus M_{(011)}) + v^{-3}(M_{(100)} \oplus M_{(010)} \oplus M_{(001)}),\]
\[u_3u_1u_2 = (M_{(110)} \oplus M_{(001)}) + v^{-1}(M_{(100)} \oplus M_{(010)} \oplus M_{(001)}),\]
\[u_2u_3u_1 = (M_{(100)} \oplus M_{(011)}) + v^{-1}(M_{(100)} \oplus M_{(010)} \oplus M_{(001)}),\]
\[u_3u_2u_1 = (M_{(100)} \oplus M_{(010)} \oplus M_{(001)}),\]
which are also the elements with dimension vector \((111)\) in the canonical basis.

It is clear that these PBW-type basis elements are the leading terms of the corresponding canonical basis elements.

4. Beck-Nakajima’s construction

In this section, we shall recall the construction of canonical basis given by Beck-Nakajima ([2],[14]).

Let \(A = (a_{ij})_{i,j \in I}\) be a generalized Cartan matrix of affine type, where \(I = \{0, 1, \ldots, n\}\), \(0 \in I\) is the exceptional point and \(I_0 = I \setminus \{0\}\). Let
\[D = \text{diag}(d_0, d_1, \ldots, d_n)\]
be a diagonal matrix such that \(DA\) is symmetric. Let \(\Delta^+\) be the set of positive roots and \(R\) the set of all positive real roots. Let \(\{\alpha_i \mid i \in I\}\) be the set of simple roots. Let \(v_i = v^{d_i}\).
We follow the notations of [2]. Denote by $\hat{W}$ the affine Weyl group generated by simple reflections $s_i$ for $i \in I$. Let $\hat{W}$ be the extended affine Weyl group. Then $\hat{W} = J \rtimes \hat{W}$, where $J$ is a subgroup of the group of Dynkin diagram automorphism and $\tau s_i = s_{\tau(i)} \tau \in \hat{W}$ for $\tau \in J, s_i \in \hat{W}$. For any $i \in I_0$, there exists $t_{\tilde{w}_i} \in \hat{W}$ such that

$$t_{\tilde{w}_i}(\alpha_j) = \begin{cases} 
\alpha_j & \text{if } j \neq i, \\
\alpha_i - di\delta & \text{if } j = i, \\
\alpha_0 + a_id\delta & \text{if } j = 0,
\end{cases}$$

where the minimal imaginary positive root $\delta = \sum a_i\alpha_i$ and $a_0 = 1$.

Let $s_{i_1}s_{i_2}\cdots s_{i_N} \tau$ be a reduced expression of $t_{\tilde{w}_n}t_{\tilde{w}_{n-1}}\cdots t_{\tilde{w}_1}$. Define an infinite sequence

$$h = (\cdots, i_{-1}, i_0, i_1, \cdots)$$

in $I$ such that $i_{k+N} = \tau(i_k)$ for any $k \in \mathbb{Z}$. Let

$$R_\prec = \{ \beta_0 = \alpha_{i_0}, \beta_1 = s_{i_0}(\alpha_{i_1}), \beta_2 = s_{i_0}s_{i_1}(\alpha_{i_2}), \cdots \}$$

and

$$R_\succ = \{ \beta_1 = \alpha_{i_1}, \beta_2 = s_{i_1}(\alpha_{i_2}), \beta_3 = s_{i_1}s_{i_2}(\alpha_{i_2}), \cdots \}.$$  

It is well-known that

$$R = R_\succ \cup R_\prec.$$

For all $j \in I$, denote by $T_j$ the Lusztig’s symmetries $T_{j,1}$ in [2]. For any $k \in \mathbb{Z}_{>0}$, let

$$E_{\beta_k} = T_{i_1}T_{i_2}\cdots T_{i_{k-1}}(E_{i_k}).$$

For any $k \in \mathbb{Z}_{\leq 0}$, let

$$E_{\beta_k} = T_{i_1}^{-1}T_{i_2}^{-1}\cdots T_{i_{k+1}}^{-1}(E_{i_k}).$$

Then $E_{\beta_k}$ are the root vectors for the real roots $\beta_k \in R$.

Then we shall define imaginary root vectors. For $k > 0$ and $i \in I_0$, let

$$\tilde{\Psi}_{i,kd} = E_{kd,\delta-\alpha_i}\alpha_i - v_{\lambda_i}^{-2}E_{kd}\alpha_i, 
\tilde{P}_{i,0,0} = 1$$

and

$$\tilde{P}_{i,kd} = \begin{cases} 
\frac{1}{[2k]_{\lambda_i}} \sum_{s=1}^{k} v_{\lambda_i}^{2(s-k)}\tilde{\Psi}_{i,s,0} \tilde{P}_{i,s,0} & \text{if } i = n \text{ and } A_{2n} \text{ is of type } A_{2n}^{(2)} \\
\frac{1}{[2k]_{\lambda_i}} \sum_{s=1}^{k} v_{\lambda_i}^{s-k}\tilde{\Psi}_{i,0,s} \tilde{P}_{i,k,0} & \text{otherwise.}
\end{cases}$$

Let $\mathcal{P}$ be the set of all partitions and $c_0 : I_0 \to \mathcal{P}$ be a map. For $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots) \in \mathcal{P}$, define

$$S_\lambda = \det(\tilde{P}_{i,(\lambda_i-k+m)}d_i)_{1 \leq k, m \leq t},$$

where $t$ is the length of $\lambda$. Denote

$$S_{c_0} = \prod_{i=1}^{n} S_{c_0(i)}.$$ 

Let $\mathcal{E}$ be the set of all such $\tilde{c} = (c, c_0)$, where $c_0 : I_0 \to \mathcal{P}$ is a map and $c : \mathbb{Z} \to \mathbb{N}$ is a function with finite support. For any $\tilde{c} \in \mathcal{E}$ and $p \in \mathbb{Z}$, let

$$L(\tilde{c}, p) = \left( E_{\tilde{P}_{i_p,0}}^{(c(p))} T_{i_p}^{-1}(E_{\tilde{P}_{i_{p-1},0}}^{(c(p-1))}T_{i_p}^{-1}(E_{\tilde{P}_{i_{p-2},0}}^{(c(p-2))})\cdots) \right) \times \left( T_{i_{p+1}}T_{i_p+1}\cdots T_{i_0}(S_{c_0}) \right) \times \left( \cdots T_{i_{p+1}}T_{i_p+2}(E_{\tilde{P}_{i_{p+3},0}}^{(c(p+3))})T_{i_{p+1}}(E_{\tilde{P}_{i_{p+2},0}}^{(c(p+2))}E_{\tilde{P}_{i_{p+1},0}}^{(c(p+1))}) \right).$$
where \( t_p \) are from the sequence \( h \).

For any \( p \in \mathbb{Z} \), Beck-Nakajima defined a partial ordering \( \prec_p \) on \( E \) such that the following Theorem holds.

**Theorem 4.1** ([2]). The set \( \{ L(\bar{c}, p) \mid \bar{c} \in E, p \in \mathbb{Z} \} \) is a \( \mathbb{Z}[v, v^{-1}] \)-basis of \( U_+^{\mathbb{Z}[v, v^{-1}]} \) such that

1. \( (L(\bar{c}, p), L(\bar{c}', p)) \in \delta_{\bar{c}, \bar{c}'} + v^{-1}\mathbb{Z}[v^{-1}] \cap Q(v) \);
2. \( L(\bar{c}, p) = L(\bar{c}, p) + \sum_{\bar{c}' \prec_p \bar{c}} a_{\bar{c}\bar{c}'} L(\bar{c}', p) \)

with \( a_{\bar{c}\bar{c}'} \in Q(v) \).

The set \( \{ L(\bar{c}, p) \mid \bar{c} \in E, p \in \mathbb{Z} \} \) is called a PBW-type basis of \( U^+ \).

Beck-Nakajima also proved the following theorem.

**Theorem 4.2** ([2]). For any \( \bar{c} \in E \) and \( p \in \mathbb{Z} \), there exists a unique \( b(\bar{c}, p) \in U_+^{\mathbb{Z}[v, v^{-1}]} \) satisfying the following conditions

1. \( b(\bar{c}, p) = b(\bar{c}, p) \);
2. \( (b(\bar{c}, p), b(\bar{c}', p)) \in \delta_{\bar{c}, \bar{c}'} + v^{-1}\mathbb{Z}[v^{-1}] \cap Q(v) \);
3. \( b(\bar{c}, p) = L(\bar{c}, p) + \sum_{\bar{c}' \prec_p \bar{c}} \xi_{\bar{c}\bar{c}'} L(\bar{c}', p) \)

with \( \xi_{\bar{c}\bar{c}'} \in v^{-1}\mathbb{Z}[v^{-1}] \).

Moreover, the set \( \{ b(\bar{c}, p) \mid \bar{c} \in E, p \in \mathbb{Z} \} \) is the canonical basis of \( U^+ \).

5. **Kronecker quiver**

Let \( Q \) be the Kronecker quiver with \( I = \{0,1\} \) and \( H = \{\rho_1, \rho_2\} \):

\[
\begin{array}{c}
0 \\
\rho_2 \quad \rho_1 \\
1
\end{array}
\]

Let \( kQ \) be the path algebra of \( Q \) over finite field \( k \).

The set of dimension vectors of indecomposable \( kQ \)-modules is

\[
\Delta^+ = \{ (l + 1, l), (m, m), (n, n + 1) \mid l, m, n \in \mathbb{Z}, l \geq 0, m \geq 1, n \geq 0 \}.
\]

The dimension vectors \( (l + 1, l) \) and \( (n, n + 1) \) correspond to preprojective and preinjective indecomposable \( kQ \)-modules respectively. The subset consisting of \( (l + 1, l) \) (resp. \( (n, n + 1) \)) is denoted by \( \text{Prep} \) (resp. \( \text{Prei} \)).

For any \( n \in \mathbb{N} \), let \( M(n + 1, n) \) and \( M(n, n + 1) \) be the indecomposable \( kQ \)-module of dimension vectors \( (n + 1, n) \) and \( (n, n + 1) \) respectively. For real root vectors, define

\[
E_{(n+1,n)} = \langle M(n+1,n) \rangle
\]

and

\[
E_{(n,n+1)} = \langle M(n,n+1) \rangle.
\]

Then we shall define imaginary root vectors. Let \( \delta = (1,1) \). As a special case of the definition in Section 3 let

\[
\tilde{\Psi}_k = \tilde{\Psi}_{1,k} = E_{(k-1,k)} E_{(1,0)} - v^{-2} E_{(1,0)} E_{(k-1,k)},
\]
\[ \tilde{P}_0 = 1 \text{ and } \]
\[ \tilde{P}_k = \frac{1}{|k|} \sum_{s=1}^{k} v^{s-k} \tilde{\Psi}_s \tilde{P}_{k-s}, \]
for \( k > 0. \)

**Proposition 5.1** ([3][7][12]). It holds that
\[ \tilde{P}_k = \sum_{\{M\} \text{ dim} M = k \delta} v^{-\text{dim} M} u[M] \]
for \( k \in \mathbb{Z}_{>0}. \)

For any partition \( \lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_t), \) let
\[ \tilde{P}_\lambda = \tilde{P}_{\lambda_1 \delta} \star \tilde{P}_{\lambda_2 \delta} \star \cdots \star \tilde{P}_{\lambda_t \delta} \]
and
\[ S_\lambda = \det(P(\lambda_k - k + m)_{1 \leq k, m \leq t}). \]

The relation between \( \tilde{P}_\lambda \) and \( S_\lambda \) is
\[ \tilde{P}_\lambda = \sum_{\mu \in P} K_{\lambda \mu} S_\mu, \]
where \( K_{\lambda \mu} \) is the Kostka number associated to the partitions \( \lambda \) and \( \mu. \)

**Theorem 5.2** ([12]). The set \( \{ S_\lambda \mid \lambda \text{ is a partition} \} \) is a subset of the canonical basis \( B. \)

Let \( \mathcal{G} \) be the set of \( (c = (c_-, c_+), \lambda), \) where \( c_- : \text{Prep} \to \mathbb{N}, c_+ : \text{Prei} \to \mathbb{N} \) are functions with finite support and \( \lambda \) is a partition.

For any \( (c, \lambda) \in \mathcal{G}, \) consider
\[ N'(c, \lambda) = \langle M(c_-) \rangle \star \tilde{P}_\lambda \star \langle M(c_+) \rangle \]
and
\[ N(c, \lambda) = \langle M(c_-) \rangle \star S_\lambda \star \langle M(c_+) \rangle \]
where
\[ M(c_-) = \bigoplus_{\alpha \in \text{Prep}} M_{\alpha}^{\oplus c_-(\alpha)} \]
and
\[ M(c_+) = \bigoplus_{\alpha \in \text{Prei}} M_{\alpha}^{\oplus c_+(\alpha)}. \]

**Proposition 5.3** ([3][7]). The sets \( \{ N(c, \lambda) \mid (c, \lambda) \in \mathcal{G} \} \) and \( \{ N'(c, \lambda) \mid (c, \lambda) \in \mathcal{G} \} \) are two \( \mathbb{Z}[v, v^{-1}] \)-bases of \( C^*(Q) \mathbb{Z}[v, v^{-1}]. \)

The sets \( \{ N(c, \lambda) \mid (c, \lambda) \in \mathcal{G} \} \) and \( \{ N'(c, \lambda) \mid (c, \lambda) \in \mathcal{G} \} \) are called PBW-type bases of \( C^*(Q). \)
6. THE CONSTRUCTION FOR CYCLIC QUIVERS

The construction of various bases of affine $A$ type was obtained in [4] by considering the Hall algebra of the cyclic quiver. Let $Q$ be the following cyclic quiver whose vertex set is $I = \{0, 1, 2, \ldots, n\}$:

\[
\begin{array}{cccccccc}
0 & \rightarrow & 1 & \rightarrow & 2 & \rightarrow & 3 & \rightarrow & \ldots & \rightarrow & n
\end{array}
\]

Denote by $\mathcal{H}^*$ the twisted Ringel-Hall algebra of the category of nilpotent representations of $Q$ and $C^*$ the twisted composition subalgebra of $\mathcal{H}^*$.

A multisegment is a formal sum
\[
\pi = \sum_{i \in I, l \geq 1} \pi_{il}[i, l],
\]
where $\pi_{il} \in \mathbb{N}$ and \{\(i \in I, l \geq 1 \mid \pi_{il} \neq 0\}\} is a finite set. Let $\Pi$ be the set of multisegments.

There is a bijection between the set $\Pi$ and the isomorphism classes of nilpotent representations of $Q$. The isomorphism classes corresponding to $\pi$ is
\[
M(\pi) = \bigoplus_{i \in I, l \geq 1} S_i[l]^{\oplus \pi_{il}},
\]
where $S_i[l]$ is the unique indecomposable $kQ$-module with top $S_i$ and length $l$.

An element $\pi \in \Pi$ is called aperiodic, if
\[
\prod_{i \in I} \pi_{il} = 0
\]
for each $l \geq 1$. The set of all aperiodic multisegments is denoted by $\Pi^a$.

There is a partial order on $\Pi$ defined as follows: for $\pi', \pi \in \Pi$ with the same dimension vector, $\pi' < \pi$ if and only if $\dim \text{Hom}(M, M(\pi')) > \dim \text{Hom}(M, M(\pi))$ for all indecomposable nilpotent representations $M$ of $Q$.

**Proposition 6.1 ([4]).** For each $\pi \in \Pi^a$, there exists a monomial $m^{\omega^*}$ on the divided powers of $u_i$ such that
\[
m^{\omega^*} = \langle M(\pi) \rangle + \sum_{\pi' < \pi} \eta_{\pi'}^\pi \langle M(\pi') \rangle
\]
with $\eta_{\pi'}^\pi \in \mathbb{Z}[v, v^{-1}]$.

Every non-empty subset of $\Pi^a$ contains a minimal element. Define $E_\pi$ for all $\pi \in \Pi^a$ inductively by the following relations. If $\pi \in \Pi^a$ is minimal,
\[
E_\pi = m^{\omega^*} = \langle M(\pi) \rangle + \sum_{\pi' < \pi, \pi' \in \Pi^a \setminus \Pi^*} \eta_{\pi'}^\pi \langle M(\pi') \rangle,
\]
If $E_{\pi'}$ have been defined for all $\pi > \pi' \in \Pi^a$, then
\[
E_\pi = m^{\omega^*} - \sum_{\pi' < \pi, \pi' \in \Pi^a} \eta_{\pi'}^\pi E_{\pi'} = \langle M(\pi) \rangle + \sum_{\pi' < \pi, \pi' \in \Pi^a \setminus \Pi^*} \gamma_{\pi'}^\pi \langle M(\pi') \rangle.
\]
Proposition 6.2 ([H]). The set \( \{ E_\pi \mid \pi \in \Pi^0 \} \) is a \( \mathbb{Z}[v, v^{-1}] \) basis of \( C^*_\mathcal{Z}[v, v^{-1}] \), satisfying the following conditions:

1. \( \{ E_\pi \mid \pi \in \Pi^0 \} \) is independent of the choice of monomials \( m^{\pi \pi} \);
2. \( \overline{E_\pi} = E_\pi + \sum_{\pi' < \pi} r^{\pi'}_\pi E_{\pi'} \)

with \( r^{\pi'}_\pi \in \mathbb{Z}[v, v^{-1}] \).

The set \( \{ E_\pi \mid \pi \in \Pi^0 \} \) is called a PBW-type basis of \( C^* \).

By Lemma 3.1, there exists a unique family of elements \( p^{\pi\pi'}_\pi \in \mathbb{Z}[v^{-1}] \) defined for all \( \pi' \leq \pi \) in \( \Pi^0 \) such that

1. \( p^{\pi\pi}_\pi = 1 \) for all \( \pi \in \Pi^0 \);
2. \( p^{\pi\pi'}_\pi \in v^{-1}\mathbb{Z}[v^{-1}] \) for all \( \pi' \leq \pi \) in \( \Pi^0 \);
3. for all \( \pi' \leq \pi \) in \( \Pi^0 \),

\[
p^{\pi\pi'}_\pi = \sum_{\pi'' \leq \pi} p^{\pi\pi''}_\pi r^{\pi''\pi'}_\pi.
\]

For any \( \pi \in \Pi^0 \), let

\[
c_\pi = E_\pi + \sum_{\pi' < \pi} p^{\pi\pi'}_\pi E_{\pi'}.
\]

Theorem 6.3 ([H]). The set \( \{ c_\pi \mid \pi \in \Pi^0 \} \) is a \( \mathbb{Z}[v, v^{-1}] \)-basis of \( C^*_\mathcal{Z}[v, v^{-1}] \) satisfying the following conditions:

1. \( \overline{c_\pi} = c_\pi \);
2. \( (c_\pi, c_{\pi'}) \in \delta_{\pi, \pi'} + v^{-1}\mathbb{Z}[v^{-1}] \cap \mathbb{Q}(v) \).

Corollary 6.4 ([H]). The set \( \{ c_\pi \mid \pi \in \Pi^0 \} \) is the canonical basis of \( C^* \).

7. The construction for tame quivers I

This construction of bases of affine \( A, D, E \) type was obtained in [7] by using the Ringel-Hall algebra approach. Let \( Q \) be an acyclic quiver of affine type. Give an order on \( I = \{0, 1, 2, \ldots, n\} \) such that \( i > j \) implies that there doesn’t exist an arrow \( i \to j \). Define a double infinite sequence

\[
h = (\cdots, i_{-1}, i_0, i_1, \cdots)
\]

such that \( i_k = k \) for all \( k = 0, 1, 2, \ldots, n \) and \( i_{k+n+1} = i_k \) for all \( k \in \mathbb{Z} \). Then

\[
\{ \beta_0 = \alpha_{i_0}, \beta_{-1} = s_{i_0}(\alpha_{i_{-1}}), \beta_{-2} = s_{i_0}s_{i_{-1}}(\alpha_{i_{-2}}), \cdots \}
\]

is the set of dimension vectors of all indecomposable preprojective modules and

\[
\{ \beta_1 = \alpha_{i_1}, \beta_2 = s_{i_1}(\alpha_{i_2}), \beta_3 = s_{i_1}s_{i_2}(\alpha_{i_3}), \cdots \}
\]

is the set of dimension vectors of all indecomposable preinjective modules.

The category \( \text{rep}_k Q \) has direct sum decomposition

\[
\text{rep}_k Q = \text{Prep} \oplus \text{Reg} \oplus \text{Prei}
\]

and each component is closed on taking extensions in \( \text{rep}_k Q \) and direct summands. Thus the Hall algebras of these components are subalgebras of \( \mathcal{H}(kQ) \). Generic composition algebra \( C^*(Q) \) contains the Hall algebras of the components \( \text{Prep} \) and \( \text{Prei} \) as subalgebras. Under the isomorphism between \( C^*(Q) \) and \( U^+ \), we can view them as subalgebras of \( U^+ \).
Let
\[ \langle M(\beta_k) \rangle = \begin{cases} \prod_{i=0}^{k} E_{i-1}(E_{ik}) & \text{if } k \leq 0, \\ \prod_{i=k}^{\infty} E_{i-1}(E_{ik}) & \text{if } k > 0. \end{cases} \]

For a support finite function \( a : \mathbb{Z}^{\leq 0} \rightarrow \mathbb{N} \), define
\[ \langle M(a) \rangle = \langle \oplus_{k \leq 0} M(\beta_k)^{(a(k))} \rangle = E_{t_0}^{(a(0))} T_{t_0}^{-1} E_{t_1}^{(a(1))} T_{t_1}^{-1} \cdots \]

For a support finite function \( b : \mathbb{Z}^{> 0} \rightarrow \mathbb{N} \), define
\[ \langle M(b) \rangle = \langle \oplus_{k > 0} M(\beta_k)^{(b(k))} \rangle = T_{t_1} T_{t_2} E_{t_3}^{(b(3))} T_{t_3} E_{t_2}^{(b(2))} E_{t_1}^{(b(1))} \).

Note that \( \langle M(a) \rangle \) and \( \langle M(b) \rangle \) belong to the Hall algebras of the components \( \text{Prei} \) and \( \text{Prej} \) respectively.

For regular part, there exist \( s(s \leq 3) \) non-homogeneous tubes \( J_1, J_2, \ldots, J_s \). It is well-known that the full subcategory corresponding to the tube \( J_t \) is equivalent to the category of nilpotent representations of the cyclic quiver with \( r_t \) vertices, where \( r_t \) is the rank of \( J_t \). Choose such an equivalence for each \( J_t \), which induces an algebra isomorphism
\[ \varepsilon_t : \mathcal{H}_{\tau_t}^* \rightarrow \mathcal{H}^*(J_t), \]
where \( \mathcal{H}_{\tau_t}^* \) is the twisted Ringel-Hall algebra corresponding to the cyclic quiver with \( r_t \) vertices and \( \mathcal{H}^*(J_t) \) is the twisted Ringel-Hall algebra of the full subcategory corresponding to the tube \( J_t \).

In Section 6, a PBW-type basis \( \{ E_\pi \mid \pi \in \Pi_t^o \} \) for the composition subalgebra \( C_{\tau_t}^* \) of the twisted Ringel-Hall algebra \( \mathcal{H}_{\tau_t}^* \) has been constructed. For any \( \pi \in \Pi_t^o \), the image of \( E_\pi \) under \( \varepsilon_t \) is still denoted by \( E_\pi \).

Let \( K_2 \) be the path algebra of the Kronecker quiver and \( F : \text{mod} kK_2 \rightarrow \text{mod} kQ \) be the canonical embedding. This embedding induces a map \( F : \mathcal{H}^*(kK_2) \rightarrow \mathcal{H}^*(kQ) \).

In Section 5, \( \tilde{\Psi}_{n\delta} \) and \( \tilde{P}_{n\delta} \) have been defined. Let
\[ E_{n\delta} = F(\tilde{P}_{n\delta}) \]
for \( n \in \mathbb{Z}_{>0} \) and
\[ E_{p\delta} = E_{p_1,\delta} \ast \cdots \ast E_{p_s,\delta} \]
for a partition \( p = (p_1 \geq \cdots \geq p_s) \).

Let \( M \) be the set of quadruples \( c = (a_c, b_c, \pi_c, p_c) \), where \( a_c : \mathbb{Z}^{\leq 0} \rightarrow \mathbb{N} \) and \( b_c : \mathbb{Z}^{>0} \rightarrow \mathbb{N} \) are functions with finite support, \( \pi_c \in \Pi_1^o \times \cdots \times \Pi_s^o \) and \( p_c \) is a partition. For each \( c \in M \), define
\[ E^c = \langle M(a_c) \rangle \ast E_{\pi_1,c} \ast \cdots \ast E_{\pi_s,c} \ast E_{p_c,\delta} \ast \langle M(b_c) \rangle. \]

Recall that \( E_\nu \) is the variety consisting of representations with dimension vector \( \nu \in \mathbb{N} \) of the quiver \( Q \) over \( \mathbf{k} \). For subset \( A \subset E_\alpha \) and \( B \subset E_\beta \), define the extension set \( A \ast B \) of \( A \) by \( B \) to be the set of all \( z \in E_{\alpha+\beta} \) such that \( M(z) \) is an extension of \( M(x) \) by \( M(y) \) for some \( x \in A, y \in B \).

Define the subvariety of \( E_\nu \)
\[ \mathcal{O}_c = \mathcal{O}_{M(a_c)} \ast \mathcal{O}_{M(\pi_1,c)} \ast \mathcal{O}_{M(\pi_2,c)} \ast \cdots \ast \mathcal{O}_{M(\pi_s,c)} \ast \mathcal{N}_{p_c} \ast \mathcal{O}_{M(b_c)} \]
for any \( c \in \mathcal{M} \), where \( \mathcal{N}_p = \mathcal{N}_{p_1} \ast \mathcal{N}_{p_2} \ast \cdots \ast \mathcal{N}_{p_s} \) if \( p = (p_1 \geq \cdots \geq p_s) \) and \( \mathcal{N}_{p_i} \) are the union of orbits of images of all regular modules in \( kK_2 \) under \( F \) with dimension vector \( p_i \delta \).

**Proposition 7.1** ([7]). The set \( \{ E^c \mid c \in \mathcal{M} \} \) is a \( \mathbb{Q}(v) \)-basis of \( C^*(kQ) \).

The set \( \{ E^c \mid c \in \mathcal{M} \} \) is a PBW-type basis of \( C^*(kQ) \).

**Proposition 7.2** ([7]). For each \( c \in \mathcal{M} \), there exists a monomial \( m_c \) on the divided powers of \( u_i \) such that

\[
m_c = E^c + \sum_{\dim \mathcal{O}_c < \dim \mathcal{O}_{c'}} h_c' E^{c'}
\]

with \( h_c' \in \mathbb{Q}[v,v^{-1}] \).

Similarly to the case of finite type, from this basis we can get a bar-invariant basis. But it is not the canonical basis considered by Lusztig. Hence in [7], another PBW-type basis is constructed.

There is a bilinear form \((\cdot,\cdot)\) on \( H^*_q(kQ) \) defined in [5]. Consider the \( \mathbb{Q}(v) \)-basis \( \{ E^c \mid c \in \mathcal{M} \} \). Let \( R(C^*(kQ)) \) be the \( \mathbb{Q}(v) \)-subspace of \( C^*(kQ) \) with the basis \( \{ E_{\pi_{1c}} \ast E_{\pi_{2c}} \ast \cdots \ast E_{\pi_{sc}} \ast E_{P_{i\delta}} \} \), where \( \pi_c = (\pi_{1c}, \pi_{2c}, \ldots, \pi_{sc}) \in \Pi_1^x \times \Pi_2^x \times \cdots \times \Pi_s^x \), and \( P_i = (p_1 \geq p_2 \geq \cdots \geq p_t) \) is a partition. Note that \( R(C^*(kQ)) \) is a subalgebra of \( C^*(kQ) \).

Let \( R^a(C^*(kQ)) \) be the subalgebra of \( R(C^*(kQ)) \) with the basis \( \{ E_{\pi_{1c}} \ast E_{\pi_{2c}} \ast \cdots \ast E_{\pi_{sc}} \mid \pi_c = (\pi_{1c}, \pi_{2c}, \ldots, \pi_{sc}) \in \Pi_1^x \times \Pi_2^x \times \cdots \times \Pi_s^x \} \). For any \( \alpha, \beta \in N_I \), define \( \alpha \leq \beta \) if \( \beta - \alpha \in N_I \). If \( \beta < \delta \), \( R(C^*(kQ))_\beta = R^a(C^*(kQ))_\beta \). Define \( F_\delta = \{ x \in R(C^*(kQ))_\delta \mid (x, R^a(C^*(kQ))_\delta) = 0 \} \).

In [7], it is proved that

\[
R(C^*(kQ))_\delta = R^a(C^*(kQ))_\delta \oplus F_\delta
\]

and \( \dim F_\delta = 1 \). By the method of Schmidt orthogonalization, we may set

\[
E'_\delta = E_\delta - \sum_{M(\pi_{se}) \mid \dim M(\pi_{se}) = \delta, 1 \leq i \leq s} a_{\pi_{se}} E_{\pi_{se}}.
\]

Then \( F_\delta = \mathbb{Q}(v) E'_\delta \).

Now let \( R(C^*(kQ))_1 \) be the subalgebra of \( R(C^*(kQ)) \) generated by \( R^a(C^*(kQ)) \) and \( F_\delta \). If \( \beta < 2 \delta \), \( R(C^*(kQ))_1)_\beta = R(C^*(kQ))_\beta \). Define

\[
F_{2\delta} = \{ x \in R(C^*(kQ))_{2\delta} \mid (x, R(C^*(kQ))_1_{2\delta}) = 0 \}.
\]

Then \( \dim F_{2\delta} = 1 \) and \( R(C^*(kQ))_{2\delta} = R(C^*(kQ))_1_{2\delta} \oplus F_{2\delta} \).

Suppose \( R(C^*(kQ))_{n-1} \) has been defined, we define \( F_{n \delta} = \{ x \in R(C^*(kQ))_{n \delta} \mid (x, R(C^*(kQ))_{(n-1) \delta}) = 0 \} \).

Let \( R(C^*(kQ))_n \) be the subalgebra of \( R(C^*(kQ)) \) generated by \( R(C^*(kQ))_{n-1} \) and \( F_{n \delta} \). Then \( \dim F_{n \delta} = 1 \) and \( R(C^*(kQ))_{n \delta} = R(C^*(kQ))_{(n-1) \delta} \oplus F_{n \delta} \). Similarly, choose \( E'_n \) such that \( E_{n \delta} - E'_n \in R(C^*(kQ))_{(n-1) \delta} \) and \( F_{n \delta} = \mathbb{Q}(v) E'_n \) for all \( n \geq 1 \).

Let \( P_{n \delta} = n E'_n \) and

\[
S_{p,\delta} = \det(P_{(p_i)_{k-k+m}})_{1 \leq k, m \leq t}
\]

be the Schur functions corresponding to \( P_{n \delta} \).
For each $c \in \mathcal{M}$, define

$$e^c = \langle M(a_c) \rangle * E_{\pi,c} \ast \cdots \ast E_{\pi,c} * S_{p,c} \ast \langle M(b_c) \rangle.$$ 

The set \{e^c | c \in \mathcal{M}\} is another PBW-type basis of $\mathcal{C}^*(kQ)$.

For two $c, c' \in \mathcal{M}$, define $e^{c'} < e^c$ if either $\dim \mathcal{O}_{c'} < \dim \mathcal{O}_c$ or $\dim \mathcal{O}_{c'} = \dim \mathcal{O}_c$ but $p_c < p_{c'}$ under lexicographic order of partitions.

**Proposition 7.3 (7).** The set \{e^c | c \in \mathcal{M}\} is a $\mathbb{Q}[v,v^{-1}]$-basis of $\mathcal{C}^*(kQ)_{\mathbb{Q}[v,v^{-1}]}$ satisfying

1. $(e^c, e^{c'}) \in \delta_{c,c'} + v^{-1} \mathbb{Q}[[v^{-1}]] \cap \mathbb{Q}(v)$;
2. $m_c = e^c + \sum_{c'' < c} a_{c''} e^{c''}$

with $a_{c''} \in \mathbb{Q}[v,v^{-1}]$.

Similarly to the case of finite type, Lin-Xiao-Zhang proved the following Theorem by using Lemma 3.1.

**Theorem 7.4 (7).** For any $c \in \mathcal{M}$, there exists a unique $\mathcal{E}^c \in \mathcal{C}^*(Q)_{\mathbb{Q}[v,v^{-1}]}$ satisfying the following conditions

1. $\mathcal{E}^c = \mathcal{E}^c$;
2. $(\mathcal{E}^c, \mathcal{E}^{c'}) \in \delta_{c,c'} + v^{-1} \mathbb{Q}[[v^{-1}]] \cap \mathbb{Q}(v)$;
3. $\mathcal{E}^c = e^c + \sum_{c'' < c} b_{c''} e^{c''}$

with $b_{c''} \in \mathbb{Q}[v,v^{-1}]$.

Moreover, the set \{E^c | c \in \mathcal{M}\} is the canonical basis of $\mathcal{C}^*(Q)$.

8. **The construction for tame quivers II**

Let $Q$ be an acyclic quiver of affine type. Denoted by $J_1, J_2, \ldots, J_s$ the non-homogeneous tubes. Let $\mathcal{H}^0(Q)$ be the $\mathbb{Q}(v)$-subalgebra of $\mathcal{H}^*(Q)$ generated by $u_i$ for $i \in I$ and $u_{i|M}$ for $M \in J_i$. Note that $\mathcal{C}^*(Q) \subset \mathcal{H}^0(Q)$ and $\mathcal{H}^0(Q)$ is called the reductive extension of $\mathcal{C}^*(Q)$.

With the same notations in Section 7 there is a double infinite sequence

$h = (\cdots, i_{-1}, i_0, i_1, \cdots)$

such that

$$\{\beta_0 = \alpha_{i_0}, \beta_1 = s_{i_0}(\alpha_{i_1}), \beta_2 = s_{i_0}s_{i_1}(\alpha_{i_2}), \cdots \}$$

is the set of dimension vectors of all indecomposable preprojective modules and

$$\{\beta_1 = \alpha_{i_1}, \beta_2 = s_{i_1}(\alpha_{i_2}), \beta_3 = s_{i_1}s_{i_2}(\alpha_{i_3}), \cdots \}$$

is the set of dimension vectors of all indecomposable preinjective modules. We order these $\beta_t$ for various $t \in \mathbb{Z}$ by

$$(\beta_0 < \beta_1 < \beta_2 < \cdots) < (\cdots \beta_3 < \beta_2 < \beta_1).$$

For a support finite function $c_{-} : \mathbb{Z}^{\leq 0} \rightarrow \mathbb{N}$, define

$$\langle M(c_{-}) \rangle = \langle \oplus_{k \leq 0} M(\beta_k) \rangle^{c_{-}}(k) \rangle$$

$$= E_{s_{i_0}}^{c_{-}(0)}(t_{s_{i_0}})^{-1}(E_{s_{i_1}}^{c_{-}(1)}(t_{s_{i_1}})^{-1}(E_{s_{i_2}}^{c_{-}(2)}(t_{s_{i_2}})^{-1}(E_{s_{i_3}}^{c_{-}(3)}(t_{s_{i_3}})^{-1}(E_{s_{i_4}}^{c_{-}(4)}(t_{s_{i_4}})^{-1}(E_{s_{i_5}}^{c_{-}(5)}(t_{s_{i_5}})^{-1}(E_{s_{i_6}}^{c_{-}(6)}(t_{s_{i_6}})^{-1}(E_{s_{i_7}}^{c_{-}(7)}(t_{s_{i_7}})^{-1}(E_{s_{i_8}}^{c_{-}(8)}(t_{s_{i_8}})^{-1}(E_{s_{i_9}}^{c_{-}(9)}(t_{s_{i_9}})^{-1}(E_{s_{i_{10}}}^{c_{-}(10)}(t_{s_{i_{10}}})^\cdots).$$
For a support finite function $c_+: \mathbb{Z}^+ \to \mathbb{N}$, define
\[
\langle M(c_+) \rangle = \langle \oplus_{k>0} M(\beta_k)^{\otimes c_+(k)} \rangle = \cdots T_{i_1} T_{i_2} (E_{i_3}^{c_+(3)}) T_{i_4} (E_{i_5}^{c_+(2)}) E_{i_6}^{c_+(1)}.
\]
For any $c_0 = (\pi_1, \ldots, \pi_s) \in \Pi_1 \times \cdots \times \Pi_s$, let
\[
M(c_0) = \varepsilon_1(M(\pi_1)) \cdots \varepsilon_s(M(\pi_s)),
\]
where
\[
\varepsilon_t: \mathcal{H}_{r_t}^\ast \to \mathcal{H}^\ast(J_t)
\]
is the algebra isomorphism in Section 7.

Similarly to the case of Kronecker quiver, define
\[
\hat{P}_m = \sum_{\substack{\lambda \vdash m \delta \vdash \lambda M \text{ is homogeneous regular}}} A_\lambda^{[M]}(v) \langle M \rangle,
\]
for $m \in \mathbb{Z}^+$ and
\[
S_\lambda = \det(P_{\lambda_k - k + m})_{1 \leq k \leq t}
\]
for a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_t)$.

We can write in terms of modules
\[
\hat{P}_\lambda = \sum_{\substack{\lambda \vdash m \delta \vdash \lambda M \text{ is homogeneous regular}}} A_\lambda^{[M]}(v) \langle M \rangle,
\]
\[
S_\lambda = \sum_{\substack{\lambda \vdash m \delta \vdash \lambda M \text{ is homogeneous regular}}} B_\lambda^{[M]}(v) \langle M \rangle,
\]
where $A_\lambda^{[M]}(v), B_\lambda^{[M]}(v) \in \mathbb{Z}[v, v^{-1}]$.

It is interesting to compute $A_\lambda^{[M]}(v), B_\lambda^{[M]}(v)$ for some special homogeneous regular $M$.

Let $Z_k$ be the set of all homogeneous tubes of mod $kQ$ and by deg $z$ we denote the degree of the corresponding irreducible polynomial of $z \in Z_k$. We denote by $M(l, z)$ the indecomposable module in tube $z$ with dimension vector $ld_z \delta$ where $d_z = \deg z$, that is, $l$ means the "level" of the corresponding module.

For a partition $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_t)$ and $\underline{z} = (z_1, z_2, \cdots, z_t)$ such that $z_i \in Z_k$ and $\deg z_i = 1$ for all $i$, we denote
\[
M(\mu, \underline{z}) = M(\mu_1, z_1) \oplus M(\mu_2, z_2) \oplus \cdots \oplus M(\mu_t, z_t).
\]

For a partition $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_t)$ and $\underline{z}' = (z'_1, z'_2, \cdots, z'_t)$ such that $z'_i \in Z_k$ and $\deg z'_i = \mu_i$ for all $i$, we denote
\[
M(\mu, \underline{z}') = M(1, z'_1) \oplus M(1, z'_2) \oplus \cdots \oplus M(1, z'_t).
\]

Note that both $M(\mu, \underline{z})$ and $M(\mu, \underline{z}')$ have the dimension vector $|\mu| \delta$.

Let $K_{\mu \lambda} \in \mathbb{Z}$ be the Kostka numbers.

**Proposition 8.1** *(16)*. For partitions $\lambda, \mu$ with $|\lambda| = |\mu|$ and $\underline{z} = (z_1, z_2, \cdots, z_t)$ such that $z_i \in Z_k$ and $\deg z_i = 1$ for all $i$, $B_\lambda^{[M(\mu, \underline{z})]}(v) = v^{-|\lambda|} K_{\mu \lambda}$.
For a partition $\lambda$ of $m$, let $S^\lambda$ be the Specht module for $S_m$. Let $t_\lambda(\mu) = \chi_{S^\lambda}(g_\mu)$ be the complex character value of $S^\lambda$ at $g_\mu \in S_m$ of cycle type $\mu$. Then $(t_\lambda(\mu))_{\lambda, \mu}$ is the character table of $S_m$. Let $t'_\lambda$ be the character of the permutation module $M^\lambda$. It is known that $t'_\lambda = \sum \mu K_{\lambda\mu} t_\mu$.

**Proposition 8.2** ([16]). For partitions $\lambda, \mu$ with $|\lambda| = |\mu|$ and $\mathbf{z'} = (z'_1, z'_2, \cdots, z'_\ell)$ such that $z'_i \in Z_k$ and $\deg z'_i = \mu_i$ for all $i$, $A^\lambda_\mu(v) = v^{-|\lambda|}t'_\lambda(\mu)$.

**Corollary 8.3** ([16]). For partitions $\lambda, \mu$ with $|\lambda| = |\mu|$ and $\mathbf{z'} = (z'_1, z'_2, \cdots, z'_\ell)$ such that $z'_i \in Z_k$ and $\deg z'_i = \mu_i$ for all $i$, $B^\lambda_\mu(v) = v^{-|\lambda|}t'_\lambda(\mu)$.

Let $\mathcal{G}$ be the set of $(c = (c_-, c_0, c_+), t_\lambda)$, where $c_- : \mathbb{Z}^{\geq 0} \to \mathbb{N}$ (resp. $c_+ : \mathbb{Z}^{> 0} \to \mathbb{N}$) is function with finite support, $c_0 \in \Pi_1 \times \cdots \times \Pi_s$ and $t_\lambda$ is the character of a Specht module $S^\lambda$. Let $\mathcal{G}^a$ be the subset of $\mathcal{G}$ consisting of all such $(c, t_\lambda)$ such that $c_0 \in \Pi_0^{\ell_1} \times \cdots \times \Pi_0^{\ell_s}$.

For any $(c, t_\lambda) \in \mathcal{G}$, consider

$$
N(c, t_\lambda) = \langle M(c_-) \ast \langle M(c_0) \ast S_\lambda \ast \langle M(c_+) \rangle \rangle \rangle,
$$

where $c = (c_-, c_0, c_+)$. 

**Proposition 8.4** ([16]). The set $\{N(c, t_\lambda) \mid (c, t_\lambda) \in \mathcal{G}\}$ is an $\mathbb{Q}(v)$-basis of $\mathcal{H}^0$ such that

1. $(N(c, t_\lambda), N(c', t'_\lambda)) \in \delta_{(c, t_\lambda), (c', t'_\lambda)} + v^{-1}\mathbb{Q}[v^{-1}] \cap \mathbb{Q}(v)$;
2. $N(c, t_\lambda) \ast N(c', t'_\lambda) = \sum_{(c'', t''_\lambda) \in \mathcal{G}} P_{(c, t_\lambda), (c', t'_\lambda)}^{(c'', t''_\lambda)} N(c'', t''_\lambda)$

with $P_{(c, t_\lambda), (c', t'_\lambda)}^{(c'', t''_\lambda)} \in \mathbb{Z}[v, v^{-1}]$.

There is a "combinatorial" order $<$ on $\mathcal{G}$ defined as follows. For $c_-, c'_- : \mathbb{Z}^{\geq 0} \to \mathbb{N}$, define $c_- < c'_-$ if and only if there exists $j > 0$ such that $c_-(t) = c'_-(t)$ for all $j < t \leq 0$ and $c_-(j) > c'_-(j)$. For $c_+, c'_+ : \mathbb{Z}^{> 0} \to \mathbb{N}$, define $c_+ < c'_+$ if and only if there exists $j > 0$ such that $c_+(t) = c'_+(t)$ for all $j < t > 0$ and $c_+(j) > c'_+(j)$. The partial order on $\Pi$ is given in Section 6 for $t_\lambda$ and $t'_\lambda$. $t_\lambda < t'_\lambda$ means that $\lambda$ is less than $\lambda'$ under lexicographic order of partitions.

**Definition 8.5.** For $(c, t_\lambda), (c', t'_\lambda) \in \mathcal{G}$, let $c = (c_-, c_0, c_+), c' = (c'_-, c'_0, c'_+), c_0 = (\pi_1, \ldots, \pi_s)$ and $c'_0 = (\pi'_1, \ldots, \pi'_s)$. Define $(c', t'_\lambda) < (c, t_\lambda)$ if one of the following three conditions holds.

1. $c = c'$ and $t_\lambda > t'_\lambda$;
2. $c'_- \leq c_-, c'_+ < c_+$ but not all equalities hold;
3. $c_0 = (\pi_1, \ldots, \pi_s)$, $c'_0 = (\pi'_1, \ldots, \pi'_s)$ but not all equalities hold.

**Proposition 8.6** ([16]). For each $(c, t_\lambda) \in \mathcal{G}^a$, there exists a monomial $m^{\omega(c, t_\lambda)}$ on the divided powers of $u_i$ such that

$$
m^{\omega(c, t_\lambda)} = N(c, t_\lambda) + \sum_{(c', t'_\lambda) < (c, t_\lambda)} a_{c, t_\lambda}^{c', t'_\lambda} N(c', t'_\lambda)
$$

with $a_{c, t_\lambda}^{c', t'_\lambda} \in \mathbb{Z}[v, v^{-1}]$. 

Li gave the geometric construction of this monomial basis in [8].

With this partial order on \( \mathcal{G}^a \), every nonempty subset has a minimal element. Define \( E(c, t_\lambda) \) for all \((c, t_\lambda) \in \mathcal{G}^a\) inductively by the following relations. If \((c, t_\lambda) \in \mathcal{G}^a\) is minimal,

\[
E(c, t_\lambda) = m^{\omega(c, t_\lambda)} = N(c, t_\lambda) + \sum_{(c', t_{\lambda'}) < (c, t_\lambda)} a^{c', t_{\lambda'}}_{c, t_\lambda} N(c', t_{\lambda'}).
\]

If \( E(c', t_{\lambda'}) \) have been defined for all \((c', t_{\lambda'}) \in \mathcal{G}^a\), then

\[
E(c, t_\lambda) = m^{\omega(c, t_\lambda)} - \sum_{(c', t_{\lambda'}) < (c, t_\lambda)} a^{c', t_{\lambda'}}_{c, t_\lambda} E(c', t_{\lambda'}) = N(c, t_\lambda) + \sum_{(c', t_{\lambda'}) < (c, t_\lambda)} b^{c', t_{\lambda'}}_{c, t_\lambda} N(c', t_{\lambda'}).\]

**Proposition 8.7** ([16]). The set \( \{E(c, t_\lambda) \mid (c, t_\lambda) \in \mathcal{G}^a\} \) is a \( \mathbb{Z}[v, v^{-1}] \) basis of \( \mathcal{C}^*(Q) \), such that

1. \( \{E(c, t_\lambda) \mid (c, t_\lambda) \in \mathcal{G}^a\} \) is independent of the choice of monomials \( m^{\omega(c, t_\lambda)} \).
2. \( \overline{E(c, t_\lambda)} = E(c, t_\lambda) + \sum_{(c', t_{\lambda'}) < (c, t_\lambda)} \gamma^{c', t_{\lambda'}}_{c, t_\lambda} E(c', t_{\lambda'}) \)

with \( \gamma^{c', t_{\lambda'}}_{c, t_\lambda} \in \mathbb{Z}[v, v^{-1}] \).

The set \( \{E(c, t_\lambda) \mid (c, t_\lambda) \in \mathcal{G}^a\} \) is called a PBW-type basis of \( \mathcal{C}^*(Q) \).

By Lemma 3.1 there exists a unique family of elements \( \zeta^{c', t_{\lambda'}}_{c, t_\lambda} \in \mathbb{Z}[v^{-1}] \) defined for all \((c', t_{\lambda'}) \leq (c, t_\lambda) \in \mathcal{G}^a\) such that

1. \( \zeta^{c, t_\lambda}_{c, t_\lambda} = 1 \) for all \((c, t_\lambda) \in \mathcal{G}^a\);
2. \( \zeta^{c', t_{\lambda'}}_{c, t_\lambda} \in v^{-1}\mathbb{Z}[v^{-1}] \) for all \((c', t_{\lambda'}) \leq (c, t_\lambda) \in \mathcal{G}^a\);
3. \( \zeta^{c', t_{\lambda'}}_{c, t_\lambda} \) for all \((c', t_{\lambda'}) \leq (c, t_\lambda) \in \mathcal{G}^a\),

\[
\zeta^{c', t_{\lambda'}}_{c, t_\lambda} = \sum_{(c'', t_{\lambda''}) \leq (c', t_{\lambda'}) \leq (c, t_\lambda)} \gamma^{c''', t_{\lambda''}}_{c', t_{\lambda'}} \gamma^{c', t_{\lambda'}}_{c, t_\lambda}.
\]

For any \((c, t_\lambda) \in \mathcal{G}^a\), let

\[
C(c, t_\lambda) = E(c, t_\lambda) + \sum_{(c', t_{\lambda'}) < (c, t_\lambda)} \zeta^{c', t_{\lambda'}}_{c, t_\lambda} E(c', t_{\lambda'}).\]

**Theorem 8.8** ([16]). The set \( \{C(c, t_\lambda) \mid (c, t_\lambda) \in \mathcal{G}^a\} \) is a \( \mathbb{Z}[v, v^{-1}] \)-basis of \( \mathcal{C}^*(Q)_{\mathbb{Z}[v, v^{-1}]} \) satisfying the following conditions.

1. \( \overline{C(c, t_\lambda)} = C(c, t_\lambda) \);
2. \( (\overline{C(c, t_\lambda)}), C(c', t_{\lambda'}) \in \delta_{(c, t_\lambda), (c', t_{\lambda'})} + v^{-1}\mathbb{Z}[v^{-1}] \cap \mathbb{Q}(v) \).

**Corollary 8.9** ([16]). The set \( \{C(c, t_\lambda) \mid (c, t_\lambda) \in \mathcal{G}^a\} \) is the canonical basis of \( \mathcal{C}^*(Q) \).
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