On Einstein’s Equations for Spacetimes
Admitting a Non–Null Killing Field

István Ráczi
ON EINSTEIN’S EQUATIONS FOR SPACETIMES ADMITTING
A NON-NULL KILLING FIELD

István Rácz†

MTA KFKI Research Institute for Particle and Nuclear Physics
H-1525 Budapest 114, P.O.B. 49, Hungary

Abstract: We consider the 3-dimensional formulation of Einstein’s theory for spacetimes possessing a non-null Killing field \( \xi^a \). It is known that for the vacuum case some of the basic field equations are deducible from the others. It will be shown here how this result can be generalized for the case of essentially arbitrary matter fields. The systematic study of the structure of the fundamental field equations is carried out. In particular, the existence of geometrically preferred reference systems is shown. Using local coordinates of this type two approaches are presented resulting resolvent systems of partial differential equations for the basic field variables. Finally, the above results are applied for perfect fluid spacetimes describing possible equilibrium configurations of relativistic dissipative fluids.

PACS number: 05.20.Cv, 05.40.+c

1 Introduction

If one is given a complicated system of nonlinear partial differential equations to solve – as it happens frequently, for instance, in Einstein’s gravitational theory – it is hard to see whether there exists any relationship between the equations or not. Sometimes the realization of certain type of connection might induce the introduction of an entirely new technique in solving the selected problem. This was the case, for example, when Cosgrove [1] gave a new formulation of field equations for stationary axisymmetric vacuum gravitational fields, or when, by a generalization of Cosgrove’s approach, Fackerell and Kerr [2] derived a resolvent system of differential equations for the vacuum field equation of Einstein’s theory for spacetimes with a single non-null Killing vector field.

† Present address: Enrico Fermi Institute, University of Chicago, 5640 S. Ellis Ave., Chicago, IL 60637
Email: istvan@rmkthe.rmki.kfki.hu
In the first part of this paper we are going to show that the fundamental results the introduction of the new approach was based on in Refs. [1,2] can be generalized for spacetimes possessing a non-null Killing field with essentially arbitrary matter fields. Subsequently, the properties of the basic field equations are studied in the situation where the gradient of the norm of the Killing field and the twist of the Killing field are linearly independent. It is shown, for instance, that there exists a geometrically preferred vector field on the space of Killing orbits so that the basic field equations possess – in local coordinate systems adopted to this vector field – very simple form. In particular, a number of the relevant field variables and/or their partial derivatives with respect to the coordinate associated with the preferred vector field are found to be identically zero. By the application of the associated simplifications, two different approaches in deriving resolvent systems of partial differential equations for the basic field variables are presented. The first is a general approach while the second one is a generalization, for particular matter fields, of the techniques applied for the study of stationary axisymmetric vacuum fields by Cosgrove [1]. In the last part of this paper the application of both of these techniques for the case of perfect fluids possessing 4-velocity parallel to a timelike Killing field will be presented.

2 The field equations

In this section, first, we shall recall some of the notions and techniques of the formalism of general relativity developed for spacetimes possessing a non-null Killing vector field. Then, it will be shown that some of the field equations involved are always deducible from the others. Finally, as a direct application of this result the basic field equations will be reformulated – displaying the simplest form of the relevant equations – corresponding to the possible subcases.

It is well known that for a spacetime, \((M,g_{ab})\), with a non-null Killing vector field, \(\xi^a\), the formulation of the Einstein’s theory can be simplified considerably by using the 3-dimensional formulation of these spacetimes [3,4]. In particular, this is done as follows: Let \(\mathcal{S}\) denote the space of Killing orbits of \(\xi^a\). It is assumed here that \(\mathcal{S}\) can be given the structure of a 3-dimensional differentiable manifold so that the projection map, \(\phi : M \to \mathcal{S}\), from \(M\) onto \(\mathcal{S}\) is a smooth mapping [3]. This condition always holds locally, and for the case of a timelike Killing field in a chronological spacetime is shown to be satisfied [5]. Consider, now, the following three fields on \(M\): the norm of the Killing field

\[ v = \xi^a \xi_a, \quad (2.1) \]
the twist of the Killing field
\[ \omega_a = \epsilon_{abcd} \xi^b \nabla^c \xi^d, \tag{2.2} \]
and the symmetric tensor field
\[ h_{ab} = g_{ab} - v^{-1} \xi_a \xi_b. \tag{2.3} \]
These three fields give rise to tensor fields on the 3-space \( S \). The image of \( h_{ab} \) by the map \( \phi \), is, for instance, the natural induced Lorentzian or Riemannian metric on \( S \) depending on whether the Killing field, \( \xi^a \), is spacelike or timelike. (Hereafter we restrict our considerations to the 3-space \( S \) so it should not cause a big confuse that the same notation will be used for the tensor fields living on \( S \) and for their natural ‘pull backs’ onto \( M \).)

Then the basic field equations are, for a spacetime possessing a non-null Killing field, \([4,5]\)
\[ R^{(3)}_{ab} = \frac{1}{2} v^{-1} D_a D_b v - \frac{1}{4} v^{-2} (D_a v)(D_b v) + \frac{1}{2} v^{-2} \{\omega_a \omega_b - h_{ab} (\omega_m \omega^m)\} + h^m_{\ a} h^b_{\ m} R^{(4)}_{mn}, \tag{2.4} \]
\[ D_a [\omega_b] = -\epsilon_{abmn} \xi^m h^n_q R^{(4)}_{pq}, \tag{2.5} \]
\[ D^a D_a v = \frac{1}{2} v^{-1} (D_m v)(D^n v) - v^{-1} \omega_m \omega^m - 2 R^{(4)}_{mn} \xi^m \xi^n, \tag{2.6} \]
\[ D^a \omega_a = \frac{3}{2} v^{-1} \omega_m D^m v, \tag{2.7} \]
where \( R^{(3)}_{ab} \) and \( D_a \) denote the Ricci tensor and the covariant derivative operator associated with \( h_{ab} \), while, \( R^{(4)}_{ab} \) is supposed to be given in terms of the energy-momentum tensor, \( T_{ab} \), of the matter fields by virtue of Einstein’s equations
\[ R^{(4)}_{ab} = 8\pi (T_{ab} - \frac{1}{2} g_{ab} T). \tag{2.8} \]
Equations (2.4) - (2.7) relate different types of projections of the 4-Ricci tensor to tensor fields and their covariant derivatives living on the 3-space \( S \). It is important that the entire geometrical content of Einstein’s theory for a spacetime, \((M, g_{ab})\), with a non-null Killing vector field, \( \xi^a \), can be uniquely represented by a 3-dimensional metric space, \((S, h_{ab})\), along with the fields \( v \) and \( \omega_a \) satisfying the above set of field equations. Even more important that, to any 3-dimensional formulation, \( \{(S, h_{ab}); v, \omega_a\} \) of this type – up to gauge transformations – there exists a unique 4-dimensional spacetime, \((M, g_{ab})\), with a Killing field, \( \xi^a \), so that the projection map \( \phi : M \rightarrow S \) reproduces the 3-dimensional formulation we started with. In fact, (2.7) is just the integrability condition ensuring that the 4-geometry can be recovered from the 3-dimensional formulation \([3,4]\).
Note that no restrictions have been raised concerning the matter fields. In fact, what we really need throughout this paper is that the matter fields be represented by tensor fields, \( \Psi^{a_1 \ldots a_k}_{b_1 \ldots b_l} (i \in I) \), on \( M \), and, a diffeomorphism invariant action be associated with them so that the energy-momentum tensor, \( T_{ab} \), and the Euler-Lagrange equations can be expressed in terms of appropriate variations of this action.

It is important to note that the invariance of \( T_{ab} \) under the action of the isometry group associated with \( \xi^a \) do not imply that the fields \( \Psi^{a_1 \ldots a_k}_{b_1 \ldots b_l} \) are invariant. There are, for instance, exact solutions of the stationary vacuum Einstein-Maxwell field equations so that the electromagnetic fields are non-stationary [6]. On the other hand, whenever \( L_{\xi} \Psi^{a_1 \ldots a_k}_{b_1 \ldots b_l} = 0 \) for each value of \( i \) one might consider the unique decomposition of the fields \( \Psi^{a_1 \ldots a_k}_{b_1 \ldots b_l} \) into tensor fields which possess definite ‘tangential’ or ‘perpendicular’ character with regard to their free indices. These fields can be built up from tensorial products of \( \xi^a \), \( \xi_a \) and the pull backs of tensor fields \( \psi^{a_1 \ldots a_m}_{b_1 \ldots b_n} \) living on \( S \).

It is well known that equations (2.4) - (2.7) can be simplified by the introduction of the conformal metric \( \hat{h}_{ab} \) defined as

\[ \hat{h}_{ab} = \varepsilon v h_{ab}, \]  

where \( \varepsilon \) takes the value +1 (resp. −1) for spacelike (resp. timelike) Killing fields. Then (2.4) - (2.7) take the form

\[ \hat{R}_{ab} = \frac{1}{2} v^{-2} \{(\hat{D}_a v)(\hat{D}_b v) + \omega_a \omega_b\} + \{h^m_a h^n_b + \varepsilon v^{-2} \hat{h}_{ab} \xi^m \xi^n\} R_m^{(4)}_{\kappa \kappa}, \]  

\[ \hat{D} [\omega_b] = -\epsilon_{abmn} \xi^m R_p^{(4)} \xi^p, \]  

\[ \hat{D}^a \hat{D}_a v = v^{-1} \{(\hat{D}_m v)(\hat{D}^m v) - \omega_m \omega^m\} - 2 \varepsilon v^{-1} R_m^{(4)} \xi^m \xi^n, \]  

\[ \hat{D}^a \omega_a = 2 v^{-1} \omega_m \hat{D}^m v, \]  

where \( \hat{D}_a \) and \( \hat{R}_{ab} \) are the covariant derivative operator and the Ricci tensor associated with \( \hat{h}_{ab} \).

Although we are considering the set of basic field equations for spacetimes with a Killing vector field – in which case some simplification arise compared to the general case – the whole set of field equations is still rather complicated. For instance, equations (2.10) - (2.13) give rise – in local coordinates – to a system of coupled non-linear second order partial differential equations for the function \( v \) and the components of the tensor fields \( \omega_a, \hat{h}_{ab}, \Psi^{a_1 \ldots a_k}_{b_1 \ldots b_l} \).

In fact, the situation is, in general, even worse because, in addition to (2.10) - (2.13), we have
to solve simultaneously the Euler-Lagrange equations which govern the evolution of matter fields in the spacetime. Note that these equations of motion, in general, couple to the above set of field equations increasing thereby the complexity of the whole problem. Therefore it is important to know whether there exists any relationship between these equations or not.

Now, we are going to show that in the formulation of Einstein’s theory for spacetimes possessing a non-null Killing vector field the same type of simplification arises as for the case of stationary axisymmetric vacuum case realized by Cosgrove [1]. In particular, it can be shown that equations (2.10) and (2.11) are actually far more fundamental than (2.12) and (2.13). More precisely, by using (2.10) and (2.11) one can derive the following algebraic relationship

\[(\bar{D}_b v) [\bar{D}^a \bar{D}_a v - v^{-1} ((\bar{D}_m v) (\bar{D}^m v) - \omega_m \omega^m) + 2 \varepsilon v^{-1} R^{(4)}_{mn} \xi^m \xi^n] + \omega_b [\bar{D}^a \omega_a - 2 v^{-1} \omega_m \bar{D}^m v] + \varepsilon v^{-1} h_{b}^m [\nabla^m R_{mn} - \frac{1}{2} \nabla_m \bar{R}^{(4)}] = 0.\]  

(2.14)

The way one could get this result is the following: Substitute the right hand side of (2.10) for \(\bar{R}_{ab}\) into the following expression

\[\bar{D}^a \bar{R}_{ab} = \frac{1}{2} \bar{D}_b \bar{R}.\]  

(2.15)

Then by using (2.11) a straightforward calculation yields that

\[\bar{D}^a \bar{R}_{ab} - \frac{1}{2} \bar{D}_b \bar{R} = \frac{1}{2} v^{-2} \left\{ (\bar{D}_b v) \left[ \bar{D}^a \bar{D}_a v - v^{-1} ((\bar{D}_m v) (\bar{D}^m v) - \omega_m \omega^m) + 2 \varepsilon v^{-1} R^{(4)}_{mn} \xi^m \xi^n \right] + \omega_b [\bar{D}^a \omega_a - 2 v^{-1} \omega_m \bar{D}^m v] \right\} \]

\[\hspace{1cm} - \varepsilon v^{-3} (\bar{D}_b v) (R^{(4)}_{mn} \xi^m \xi^n) + v^{-2} \omega^a \bar{D}_a \omega_b + \left[ \bar{D}^a \rho_{ab} - \frac{1}{2} \bar{D}_b (\bar{h}^m \rho_{mn}) \right],\]

(2.16)

where \(\rho_{ab} = \{ h^m_a h^b_v + \varepsilon v^{-2} \bar{h}_{ab} \xi^m \xi^n \} R^{(4)}_{mn}.\) Since \(\xi^a\) is a Killing field on \(M\) we get by (2.13)

\[v^{-2} \omega^a \bar{D}_a \omega_b = 2 \varepsilon v^{-2} h_{b}^m (\nabla_m \omega_a) R^{(4)}_{ap} \xi^p + \varepsilon v^{-3} (\bar{D}_b v) (R^{(4)}_{mn} \xi^m \xi^n).\]

(2.17)

We also have, for instance, \(\bar{L}_x \bar{R}^{(4)}_{ab} = 0.\) Moreover, it can be shown by using the relationship between the covariant derivative operators \(D_a\) and \(\bar{D}_a\) with a tedious but straightforward calculation – that

\[\bar{D}^a \rho_{ab} - \frac{1}{2} \bar{D}_b (\bar{h}^m \rho_{mn}) = \varepsilon v^{-1} h_{b}^m [\nabla^m R^{(4)}_{mn} - \frac{1}{2} \nabla_m \bar{R}^{(4)}] - 2 \varepsilon v^{-2} h_{b}^m (\nabla_m \omega_a) R^{(4)}_{ap} \xi^p.\]  

(2.18)

Now using (2.16),(2.17) and (2.18) we obtain

\[\bar{D}^a \bar{R}_{ab} - \frac{1}{2} \bar{D}_b \bar{R} = \frac{1}{2} v^{-2} \left\{ (\bar{D}_b v) \left[ \bar{D}^a \bar{D}_a v - v^{-1} ((\bar{D}_m v) (\bar{D}^m v) - \omega_m \omega^m) + 2 \varepsilon v^{-1} R^{(4)}_{mn} \xi^m \xi^n \right] + \omega_b [\bar{D}^a \omega_a - 2 v^{-1} \omega_m \bar{D}^m v] + \varepsilon v^{-1} h_{b}^m [\nabla^m R^{(4)}_{mn} - \frac{1}{2} \nabla_m \bar{R}^{(4)}] \right\}.\]  

(2.19)

5
Since the tensor field $\bar{R}_{ab}$ is just the Ricci tensor associated with the three metric, $\hat{h}_{ab}$, – in virtue of the twice contracted Bianchi identity – we have that the left hand side of the previous equation is identically zero. This proves then that (2.14) holds identically.

In the remaining part of this section we are going to study the consequences of the algebraic relation (2.14). We shall use the assumption that the Euler-Lagrange equations are satisfied for matter fields which implies in the case when they are derived from a diffeomorphism invariant action that

$$\nabla^a T_{ab} = 0$$  \hspace{1cm} (2.20)

so the third term of (2.14) is zero. Therefore, we have that the relevant form of (2.14) says then that the above particular linear combination of the form fields, $\bar{D}_a v$ and $\omega_a$, must vanish identically. Correspondingly, there are two subcases which have to be treated separately, namely, $\bar{D}_a v$ and $\omega_a$ might be either linearly independent or not.

Whenever the two form fields $\bar{D}_a v$ and $\omega_a$ are linearly independent only the trivial combinations of them can vanish identically. In this case (2.12) and (2.13) can be deduced from (2.10), (2.11) and the Euler-Lagrange equations. It is then sufficient to solve (2.10) and (2.11) along with the relevant equations of motion for matter fields since any solution of these equations will automatically satisfy (2.12) and (2.13) as well.

Suppose now that the two form fields, $\bar{D}_a v$ and $\omega_a$, are linearly dependent. This might happen whenever one of them vanishes throughout or there exists a function, $f$, such that

$$\omega_a = f \cdot (\bar{D}_a v).$$  \hspace{1cm} (2.21)

$\alpha$: Consider first the case of vanishing $\bar{D}_a v$, i.e., we suppose that $v$ is constant throughout. Since we can introduce then a new Killing field instead of $\xi^a$ by rescaling $\xi^a$ with an arbitrarily chosen constant factor we may assume here, without loss of generality, that $v = \varepsilon$. Furthermore, for this case (2.14) implies that the relevant form of (2.13) is a consequence of (2.10), (2.11) and (2.20). Hence, the whole content of the basic field equations reduce to

$$\bar{R}_{ab} = \frac{1}{2} \omega_a \omega_b + \{ h^{m}_{a} h^{n}_{b} + \varepsilon h_{ab} \xi^{m} \xi^{n} \} R^{(4)}_{mn},$$ \hspace{1cm} (2.\alpha.1)

$$\omega_m \omega^m = -2 \varepsilon R^{(4)}_{mn} \xi^m \xi^n, \hspace{1cm} (2.\alpha.2)$$

$$\bar{D}_{(a} \omega_{b)} = -\varepsilon_{abcd} \xi^{m} R^{(4)}_{np} \xi^{p}. \hspace{1cm} (2.\alpha.3)$$
Suppose now that $\omega_a = 0$, i.e., $\xi^a$ is hypersurface orthogonal. Then (2.11) and (2.13) are expected to hold, furthermore, the relevant form of (2.12) is simply a consequence of (2.14). Hence, the basic equations for the case under consideration simplify to

$$\tilde{R}_{ab} = \frac{1}{2} v^{-2} (\tilde{D}_a v)(\tilde{D}_b v) + \{ h^{m}_{a} h^{n}_{b} + \varepsilon v^{-2} h_{ab} \xi^{m} \xi^{n} \} \bar{R}^{(4)}_{mn},$$

(2.\beta)

Finally, suppose that neither $\tilde{D}_a v$ nor $\omega_a$ vanishes, and, there exists a function, $f$, such that (2.21) holds. Then the elimination of $\omega_a$ from (2.10) - (2.13) yields by using the above relationship

$$\tilde{R}_{ab} = \frac{1}{2} v^{-2} (1 + f^2)(\tilde{D}_a v)(\tilde{D}_b v) + \{ h^{m}_{a} h^{n}_{b} + \varepsilon v^{-2} h_{ab} \xi^{m} \xi^{n} \} \bar{R}^{(4)}_{mn},$$

(2.\gamma.1)

$$\tilde{D}^a \tilde{D}_a v = v^{-1} (1 - f^2)(\tilde{D}_m v)(\tilde{D}^m v) - 2 \varepsilon v^{-1} \bar{R}^{(4)}_{mn} \xi^{m} \xi^{n},$$

(2.\gamma.2)

$$(\tilde{D}_a f)(\tilde{D}^a v) = v^{-1} [(1 + f^2)(\tilde{D}_m v)(\tilde{D}^m v) + 2 \varepsilon \bar{R}^{(4)}_{mn} \xi^{m} \xi^{n}],$$

(2.\gamma.3)

$$(\tilde{D}_{[a} f)(\tilde{D}_{b]} v) = -\epsilon_{abmn} \xi^{n} \bar{R}^{(4)}_{mp} \xi^{p}.$$

(2.\gamma.4)

It can easily be checked that the relevant form of (2.14) implies that (2.\gamma.2) is deducible from the Euler-Lagrange equations and (2.\gamma.1), (2.\gamma.3) and (2.\gamma.4). Hence, for this last case, equations (2.21), (2.\gamma.1), (2.\gamma.3) and (2.\gamma.4) display the entire content of the basic field equations.

3. Geometrically preferred local coordinates

In the remaining part of this paper we shall restrict our consideration to the case of independent form fields, i.e., we suppose that $(\tilde{D}_{[a} v) \omega_{b]} \neq 0$ on a subset $\tilde{S}$ of $S$. (The other possibility, when $\tilde{D}_a v$ and $\omega_b$ are linearly dependent, will be examined elsewhere [7]). According to the results of the previous section to get a solution of the basic field equations, (2.10) - (2.13), it is sufficient to solve (2.10) and (2.11) along with the relevant set of Euler-Lagrange equations. In this section we are going to examine the properties of the fundamental equations (2.10) and (2.11). In particular, it will be shown that there exist geometrically preferred local coordinate systems in which these equations possess very simple form.

We shall use the following shortened form of (2.10) and (2.11)

$$\tilde{R}_{ab} = \frac{1}{2} v^{-2} \{(\tilde{D}_a v)(\tilde{D}_b v) + \omega_a \omega_b \} + \rho_{ab},$$

(3.1)

$$\tilde{D}_{[a} \omega_{b]} = \sigma_{ab},$$

(3.2)
where
\[ \rho_{ab} = (\hat{h}^m_a h^m_b + \varepsilon v^{-2} \hat{h}_{ab} k^m \xi^n) R^{(4)}_{mn}, \] (3.3)
and
\[ \sigma_{ab} = -\varepsilon_{abnm} k^m R^{(4)}_{npq} \xi^p. \] (3.4)

Note that \( \rho_{ab} \) is a symmetric while \( \sigma_{ab} \) an antisymmetric tensor field on \( S \), both depending on the fields \( v, \hat{h}_{ab}, \Psi, \xi \).

Since \( (\hat{D}_a v) \omega_{b} \neq 0 \) on \( S \) there exists a preferred vector field, \( k^a \), there defined as
\[ k^a = \epsilon^{abc} (\hat{D}_b v) \omega_c, \] (3.5)
where \( \epsilon^{abc} \) denotes the 3-dimensional volume element associated with \( \hat{h}_{ab} \), i.e., \( \epsilon^{abc} = \epsilon_{abcd} \xi^d \).

Then the following hold
\[ \mathcal{L}_k v = 0, \] (3.6)
\[ k^a \omega_a = 0, \] (3.7)
\[ k^a (\hat{R}_{ab} - \rho_{ab}) = 0, \] (3.8)
and
\[ \mathcal{L}_k (\hat{R}_{ab} - \rho_{ab}) = v^{-2} k^e \{ \sigma_{ea} \omega_b + \sigma_{eb} \omega_a \}. \] (3.9)

Equations (3.6) - (3.8) are direct consequences of the definition of \( k^a \). For (3.9) note that
\[ \mathcal{L}_k (\hat{R}_{ab} - \rho_{ab}) = k^e \hat{D}_c (\hat{R}_{ab} - \rho_{ab}) + (\hat{R}_{cd} - \rho_{cd}) \hat{D}_a k^e + (\hat{R}_{ae} - \rho_{ae}) \hat{D}_b k^e. \] (3.10)

However, according to (3.8) we have \( (\hat{R}_{cd} - \rho_{cd}) \hat{D}_a k^e = -k^e \hat{D}_a (\hat{R}_{ce} - \rho_{ce}) \), and so
\[ \mathcal{L}_k (\hat{R}_{ab} - \rho_{ab}) = k^e \{ \hat{D}_c (\hat{R}_{ab} - \rho_{ab}) - \hat{D}_a (\hat{R}_{cb} - \rho_{cb}) - \hat{D}_b (\hat{R}_{ae} - \rho_{ae}) \}. \] (3.11)

Now, using (3.1),(3.6) and (3.7) we get
\[ \mathcal{L}_k (\hat{R}_{ab} - \rho_{ab}) = v^{-2} k^e \{ (\hat{D}_e \hat{D}_a v) (\hat{D}_b v) + (\hat{D}_a v) (\hat{D}_e \hat{D}_b v) + \hat{D}_e \omega_a \omega_b + \omega_a \hat{D}_e \omega_b \}, \] (3.12)
which imply, along with (3.2) and the fact that \( \hat{D}_a \) is torsion free, that (3.9) holds.

Note that whenever \( k^e \sigma_{ea} \) is vanishing on \( \tilde{S} \), we have
\[ \mathcal{L}_k (\hat{R}_{ab} - \rho_{ab}) = 0, \] (3.13)
i.e., $k^a$ is a collineation vector field of $\hat{R}_{ab} - \rho_{ab}$. According to the definition of $\sigma_{ab}$ the contraction $k^c \sigma_{ca}$ is identically zero whenever there exist functions $\alpha, \beta$ such that $\hat{R}^p_{\mu} \xi^p = \alpha \xi^a + \beta k^a$. For the vacuum case, $\rho_{ab} = \sigma_{ab} = 0$ (or $\alpha = \beta = 0$). Then (3.9) reduces to the well-known result that $k^a$ is a Ricci collineation vector [8].

Just like in the vacuum case (see Ref. [2]) there are geometrically preferred local coordinate systems associated with the vector field $k^a$. In this subsection, we shall restrict our attention to adopted local coordinates, $(x^1, x^2, x^3)$, defined so that

$$k^a = \left( \frac{\partial}{\partial x^1} \right)^a, \quad i.e., \quad k^a = \delta_3^a,$$  \hspace{1cm} (3.14)

holds. Such coordinates (at least locally) can always be introduced on $\tilde{S}$.

In such a local coordinate system, $(x^1, x^2, x^3)$, equations (3.6) - (3.9) take the form

$$\frac{\partial v}{\partial x^3} = 0,$$  \hspace{1cm} (3.15)

$$\omega_3 = 0,$$  \hspace{1cm} (3.16)

$$\hat{R}_{33} - \rho_{33} = 0,$$  \hspace{1cm} (3.17)

and

$$\frac{\partial}{\partial x^3} (\hat{R}_{a3} - \rho_{a3}) = v^{-2} \{ \sigma_{3a} \omega_3 + \sigma_{33} \omega_a \}$$  \hspace{1cm} (3.18)

where $\beta$ takes the values 1, 2, 3. Note that whenever one of the functions, $\rho_{3\beta}$ ($\beta = 1, 2, 3$), does not vanish identically (3.17) gives algebraical relationship(s) between the variables $v$, $\hat{h}_{ab}$, $\Psi_{\cdots b_1 \cdots b_k}^{\cdots a_1 \cdots a_k}$ and the derivatives of $\hat{h}_{ab}$ and $\Psi_{\cdots b_1 \cdots b_k}^{\cdots a_1 \cdots a_k}$. Then we get that in such an adopted local coordinate system, $(x^1, x^2, x^3)$, (3.1) is equivalent to (3.17) and

$$\hat{R}_{AB} = \frac{1}{2} v^{-2} \{ (\partial_A v)(\partial_B v) + \omega_A \omega_B \} + \rho_{AB},$$  \hspace{1cm} (3.19)

where $\partial_A v$ denotes the partial derivative of $v$ with respect to the variable $x^A$, and the capital Latin indices take the values 1, 2.

It can be easily checked that in such a coordinate system (3.2) takes the form

$$\partial_1 \omega_2 - \partial_2 \omega_1 = \sigma_{12}$$

$$\partial_3 \omega_A = \sigma_{3A}.$$  \hspace{1cm} (3.20)

Observe that, whenever $\sigma_{ab}$ vanishes identically these equations imply that (at least locally) there exists a function, $\omega = \omega(x^1, x^2)$, so that

$$\omega_A = \partial_A \omega.$$  \hspace{1cm} (3.21)
Using these simplifications, in the next two sections two different methods in establishing resolvent systems of partial differential equations for the basic field variables will be presented.

4. General method

This section is devoted to the introduction of a general approach to get a resolvent system of differential equations for the basic field variables. This approach is based on the following observation: It seems to be a general feature of the present formulation of Einstein’s theory that (3.17) can be solved for the function $v$ in many cases. Hereafter, we shall assume that the fields $\psi_{(i)}^{a_1 \cdots a_k}{}_{b_1 \cdots b_l}$ are invariant under the action of the isometry group associated with $\xi^a$ and we use the fields $\psi_{(i)}^{a_1 \cdots a_m}{}_{b_1 \cdots b_n}$ to represent the matter content, instead of them. Combining these two facts, hereafter we shall assume that the norm of the Killing field, $v$, can be given in terms of quantities derived from the induced 3-geometry, $\tilde{h}_{ab}$, and, possibly, from the tensor fields $\psi_{(j)}^{a_1 \cdots a_m}{}_{b_1 \cdots b_n}$, representing the matter fields. Correspondingly, we shall assume that there exists a function

$$v = v(\tilde{h}_{\alpha \beta}, \partial_\alpha \tilde{h}_{\alpha \beta}, \partial_\beta \tilde{h}_{\alpha \beta}; \psi_{(i)}^{a_1 \cdots a_m}{}_{\beta_1 \cdots \beta_n}, \partial_\beta \psi_{(i)}^{a_1 \cdots a_m}{}_{\beta_1 \cdots \beta_n}, \partial_\beta \partial_\gamma \psi_{(i)}^{a_1 \cdots a_m}{}_{\beta_1 \cdots \beta_n}),$$

(4.1)

where the presence of second order partial derivatives of the fields $\psi_{(i)}^{a_1 \cdots a_m}{}_{\beta_1 \cdots \beta_n}$ indicates that the matter Lagrangian is supposed to contain at most second order partial derivatives of these fields and the Greek indices refer to components of tensor fields in geometrically preferred adopted local coordinates.

To start off note that (3.19) can be recast into the form

$$H_{AB} = v^{-2}\{ (\partial_A v)(\partial_B v) + \omega_A \omega_B \},$$

(4.2)

where

$$H_{AB} = 2(\tilde{R}_{AB} - \rho_{AB}).$$

(4.3)

It is important to emphasize that at each (explicit or implicit) appearance of the function $v$ in (4.2) the substitution of the right hand side of (4.1) is understood. Since we are dealing with the case of linearly independent form fields, i.e., $(\tilde{D}_v v)_{\omega \beta} \neq 0$ on $\tilde{\mathcal{S}}$, (4.2) can be shown to be equivalent to the following set of equations

$$\omega_A = \epsilon \frac{H_{A2}(\partial_1 v) - H_{1A}(\partial_2 v)}{[\det(H_{AB})]^\frac{3}{2}},$$

(4.4)

and

$$H_{11}(\partial_2 v)^2 - 2H_{12}(\partial_1 v)(\partial_2 v) + H_{22}(\partial_1 v)^2 - v^2 \det(H_{AB}) = 0,$$

(4.5)
where the sign ambiguity of $\omega_A$ is indicated by the factor $\epsilon$ (i.e., $\epsilon = \pm 1$) in (4.4).

Using the definition, (3.3), of $\rho_{ab}$ and (3.17) it can be checked easily that $v$ depends on at most second order derivatives of the metric functions, $h_{\alpha\beta}$, since only the terms $\tilde{R}_{\alpha\beta}$ enter (3.17). Therefore, with the assumption that at most second order covariant derivatives of the fields $\psi_{(i)}^{\alpha_1...\alpha_m b_1...b_n}$ are involved in the matter Lagrangian, we can conclude that (4.5) is at most a third order partial differential equation for the fields $h_{\alpha\beta}$ and $\psi_{(i)}^{\alpha_1...\alpha_m b_1...b_n}$. Three additional partial differential equations restricting these fields have to be taken into consideration. These are derived by substituting the right hand side of (4.4) for $v$ and $\omega_A$ into (3.20) and can be given as follows:

\[
\begin{align*}
(\partial_1 H_{22})(\partial_1 v) + H_{22}(\partial_1 \partial_1 v) - (\partial_1 H_{12})(\partial_2 v) - 2H_{12}(\partial_2 \partial_1 v) - (\partial_2 H_{12})(\partial_1 v) \\
+ (\partial_2 H_{11})(\partial_2 v) + H_{11}(\partial_2 \partial_2 v) + \partial_1 \left( \ln |\det (H_{AB})|^{-\frac{1}{2}} \right) \left( H_{22}(\partial_1 v) - H_{12}(\partial_2 v) \right) \\
- \partial_2 \left( \ln |\det (H_{AB})|^{-\frac{1}{2}} \right) \left( H_{12}(\partial_1 v) - H_{11}(\partial_2 v) \right) = \epsilon |\det (H_{AB})|^{\frac{1}{2}} \sigma_{12},
\end{align*}
\]

(4.6)

\[
\begin{align*}
(\partial_3 H_{12})(\partial_1 v) - (\partial_3 H_{11})(\partial_2 v) + \partial_3 \left( \ln |\det (H_{AB})|^{-\frac{1}{2}} \right) \left( H_{12}(\partial_1 v) - H_{11}(\partial_2 v) \right) \\
= \epsilon |\det (H_{AB})|^{\frac{1}{2}} \sigma_{31},
\end{align*}
\]

(4.7)

\[
\begin{align*}
(\partial_3 H_{22})(\partial_1 v) - (\partial_3 H_{21})(\partial_2 v) + \partial_3 \left( \ln |\det (H_{AB})|^{-\frac{1}{2}} \right) \left( H_{22}(\partial_1 v) - H_{12}(\partial_2 v) \right) \\
= \epsilon |\det (H_{AB})|^{\frac{1}{2}} \sigma_{32}.
\end{align*}
\]

(4.8)

The first equation, (4.6), is a fourth order while the last two ones are third order non-linear partial differential equations. These equations along with (3.17), (4.5) and the relevant set of Euler-Lagrange equations give rise to a resolvent system of field equations for the variables $h_{\alpha\beta}$ and $\psi_{(i)}^{\alpha_1...\alpha_m b_1...b_n}$. Once one could get a solution of these field equations one can determine $v$ via (4.1), moreover, $\omega_A$ can be given in virtue of (4.4).

Clearly, the applicability of this approach strongly depends on the detailed functional form of $v$ which was implicitly used throughout this section. For instance, the explicit form of the basic field equations, (3.17), (4.5)-(4.8), for the variables $h_{\alpha\beta}$ and $\psi_{(i)}^{\alpha_1...\alpha_m b_1...b_n}$ can be examined only for particular matter fields separately. In section 7 we are going to give the functional form of $v$ for perfect fluid matter sources possessing 4-velocity parallel to a timelike Killing field and for particular equations of state.

5. Generalization of Cosgrove’s method

In this section we generalize the techniques developed originally for stationary axisymmetric vacuum fields for spacetimes possessing a single non-null Killing field with matter fields satisfying the additional conditions given below. More precisely, a slightly modified version of Cosgrove’s
approach will be established so as to derive from the basic set of field equation a resolvent system of differential equations for the basic variables.

The two conditions are the following:

**Condition 5.1:** The tensor field $\sigma_{ab}$ vanishes throughout, i.e., $\xi_a R^{(4)}_{b\epsilon} \xi^\epsilon = 0$.

**Condition 5.2:** The tensor field $\rho_{ab}$ has the property that, in a geometrically preferred local coordinate system, its components, $\rho_{AB}$ ($A, B = 1, 2$), can be given exclusively in terms of the induced 3-geometry, $\hat{h}_{ab}$.

In particular, **Condition 5.1** implies that (at least locally) there exists such a function $\omega$ that $\omega_a = \hat{D}_a \omega$. Since we are dealing with the case of independent form fields – i.e., $\hat{D}_a \omega \neq 0$ – the functions $v$ and $\omega$ are then functionally independent. **Condition 5.2** might be satisfied when (3.17) can be solved for $v$, moreover, (by using the relevant expression for $v$) one can eliminate thereby $v$ and $\psi^{a_1 \ldots a_m}_{b_1 \ldots b_n}$ from $\rho_{AB}$. Whenever both of the above conditions hold (3.19) can be recast into the form

$$H_{AB} = v^{-2} [(\partial_A v)(\partial_B v) + (\partial_A \omega)(\partial_B \omega)],$$

(5.1)

where we used the expression (4.3) for $H_{AB}$. Furthermore, due to **Condition 5.2** the left hand side of (5.1) depends exclusively on the induced 3-metric while the right hand side of it depends merely on the functions $v$ and $\omega$. Since $v$ and $\omega$ are functions of $x^1$ and $x^2$, equation (5.1) shows that the same is true for the functions $H_{AB}$ even if, for instance, some of the components of $\hat{h}_{ab}$ may depend on $x^3$. (Note that this property of the functions, $H_{AB}$, is in fact a simple consequence of the general result (3.13).) Since $v$ and $\omega$ are functionally independent we have that the functions $H_{AB}$ can be considered as the components of a non-singular Riemannian metric on a 2-dimensional manifold. Note that the right hand side of (5.1) is just the well-known representation of a Riemannian 2-metric in local coordinates $(v, \omega)$ with Gaussian curvature $-1$.

Hence, for the Gaussian curvature, $K_H$, of the metric, $H_{AB}$,

$$K_H = -1$$

(5.2)

has to hold. This equation is, in fact, a fourth-order partial differential equation for the components of the tensor fields $\hat{h}_{ab}$. For the case of linearly independent form fields under consideration (5.2) is the necessary and sufficient condition for the existence of functions $v$ and $\omega$ satisfying (5.1).
The outline of the proof of the above statement can be given as follows: Since we are considering the case of linearly independent form fields, (3.21) and (4.4) yield

\[ \partial_A \omega = \epsilon_1 \frac{H_{A2}(\partial_1 v) - H_{1A}(\partial_2 v)}{\det(H_{AB})^{\frac{1}{2}}} , \quad (5.3) \]

where the ambiguity in sign of \( \omega \) is indicated by \( \epsilon_1 \) (i.e., \( \epsilon_1 = \pm 1 \)). Substituting (5.3) into (5.1) with setting \( A, B = 2 \) and solving for \( \partial_1 v \) we obtain

\[ \partial_1 v = \frac{H_{12}(\partial_2 v) + \epsilon_2 \det(H_{AB})^{\frac{1}{2}}(v^2 H_{22} - (\partial_2 v)^2)^{\frac{1}{2}}}{H_{22}} , \quad (5.4) \]

where \( H_{22} \neq 0 \) since otherwise \( (\hat{D}_v)\omega_0 \) should vanish and \( \epsilon_2 = \pm 1 \). Furthermore, the substitution of (5.4) into (5.3) yields

\[ \partial_2 \omega = \epsilon_1 \epsilon_2 [v^2 H_{22} - (\partial_2 v)^2]^{\frac{1}{2}} , \quad (5.5) \]

and

\[ \partial_1 \omega = -\epsilon_1 \frac{\det(H_{AB})^{\frac{1}{2}}(\partial_2 v) - \epsilon_2 H_{12}(v^2 H_{22} - (\partial_2 v)^2)^{\frac{1}{2}}}{H_{22}} . \quad (5.6) \]

Equations (5.4) - (5.6) are equivalent to (5.1). The integrability condition, \( \partial_2 \partial_1 \omega = \partial_1 \partial_2 \omega \), for the function \( \omega \) can be shown [1] to give rise to the following Appel equation for the function \( U = \frac{1}{2} \ln(\varphi v) \)

\[ 2H_{22}(\partial_2 \partial_1 U) - 4H_{22}(\partial_2 U)^2 - (\partial_2 H_{22})(\partial_2 U) + H_{22}^2 + \Phi[H_{22} - 4(\partial_2 U)^2]^{\frac{1}{2}} = 0 , \quad (5.7) \]

where

\[ \Phi = \frac{1}{4} \epsilon_2 \cdot \det(H_{AB})^{-\frac{1}{2}} \left\{ -2H_{22}(\partial_2 H_{21}) + H_{21}(\partial_2 H_{22}) + H_{22}(\partial_1 H_{22}) \right\} . \quad (5.8) \]

Utilizing Cosgrove's substitution (see Ref. [1,2])

\[ \partial_2 U = -(H_{22})^{\frac{1}{2}} M (1 + M^2)^{-1} , \quad (5.9) \]

we obtain from (5.4) and (5.7) the following pair of Riccati equations

\[ \partial_A M = X_A + 2Y_A M + Z_A M^2 . \quad (5.10) \]

Here the functions \( X_A, Y_A \) and \( Z_A \) are defined as

\[ X_A = \frac{\epsilon_2}{4\det(H_{AB})^{\frac{1}{2}} H_{22}} \left[ H_{12}(\partial_1 H_{22}) - H_{22}(\partial_A H_{12} + \partial_2 H_{A1} - \partial_1 H_{A2}) \right] + \frac{1}{2} H_{A2}(H_{22})^{-\frac{1}{2}} , \quad (5.11) \]
\[ Y_A = \frac{1}{2} \epsilon_2 \cdot \delta A_1 \cdot \text{det}(H_{AB})^{\frac{1}{2}} (H_{22})^{-\frac{1}{2}}, \quad (5.12) \]
and
\[ Z_A = X_A - H_{A2}(H_{22})^{-\frac{1}{2}}. \quad (5.13) \]

The integrability conditions for the simultaneous set of Riccati equations, (5.10), reduce to a single condition [1,2], which, not unexpectedly, may be put into the form of (5.2).

Summarizing the results of this section we can say the following: To get a resolution of the basic field variables we have to solve first (5.2) for \( \hat{h}_{\alpha\beta} \). Then the solutions of the simultaneous Riccati equations, (5.10), can be used to determine the function \( v \) via (5.4) and (5.9). Afterwards, (5.3) can be applied to construct the function \( \omega \). A detailed discussion about the resolution of the corresponding problems for the vacuum case – particularly, about the solutions of Riccati equations of the above type – can be found in Ref. [1]. Finally, using these functions \( v, \omega \) and \( \hat{h}_{\alpha\beta} \) – the Euler-Lagrange equations have to be solved for the components of tensor fields representing the matter content.

It is worth mentioning that (5.1) inherits a remarkable feature of the corresponding equation given for the vacuum case noticed by Geroch [3]. Namely, by starting with a particular solution, \( (v_0, \omega_0) \), of (5.1) associated with a fixed set of functions \( H_{AB} \) one can generate a one-parameter family of solutions, \( (v_\tau, \omega_\tau) \). More precisely, one can show by a straightforward modification of the proof of Theorem 1. of Ref. [1] that for fixed functions \( H_{AB} \) satisfying (5.3) the full set of solutions of (5.1) – apart from those related to gauge transformations of the spacetime, \( (M, g_{ab}) \) – is generated from the particular solution, \( (v_0, \omega_0) \), by the transformation

\[ v_\tau = \frac{v_0}{(\cos \tau - \omega_0 \sin \tau)^2 + v_0^2 \sin^2 \tau}, \quad (5.14) \]
\[ \omega_\tau = \frac{(\sin \tau + \omega_0 \cos \tau)(\cos \tau - \omega_0 \sin \tau) - v_0^2 \sin \tau \cos \tau}{(\cos \tau - \omega_0 \sin \tau)^2 + v_0^2 \sin^2 \tau}. \quad (5.15) \]

There is, however, a significant difference between the vacuum case and the case under consideration. Namely, for the case of vacuum the relevant form of (5.1) is the only field equation to be solved while for the general case with matter the basic field variables have to satisfy, beside (5.1), both (3.17) and the relevant set of Euler-Lagrange equations, as well. Therefore, one would expect that there is no matter field so that the above transformation can be applied. Nevertheless, there exists such a matter field (see section 7) where certain restrictions on the basic field variables (associated with the matter content) can ensure the applicability of the transformation (5.14)-(5.15), and, consequently, one may generate new solutions of Einstein’s equations.
6. Perfect fluids

In this section some of the basic notions and results in connection with perfect fluids will be recalled and some of the consequences of the presence of Killing fields in the spacetimes will be discussed.

Consider a perfect fluid with mass density, \( \rho \), and pressure, \( P \), (both quantity measured in the rest frame of the fluid), furthermore, with 4-velocity \( u^a \), where \( u^a u_a = -1 \). (Note that the tensor fields \( \Psi_{(i)}^{a_1 \ldots a_k} b_1 \ldots b_l \) on \( M \) for the present case are the fields \( \rho, P \) and \( u^a \).) The energy-momentum tensor is given as

\[
T_{ab} = \rho u_a u_b + P (g_{ab} + u_a u_b),
\]

(6.1)

furthermore, the Euler - Lagrange equations are

\[
\begin{align*}
\rho u^a \nabla_a \rho + (\rho + P) \nabla^a u_a &= 0, \\
(\rho + P) u^a \nabla_a u_b + (g_{ab} + u_a u_b) \nabla^a P &= 0.
\end{align*}
\]

(6.2)

It is known that for perfect fluid sources these equations are equivalent to the ‘integrability’ condition of Einstein’s equation

\[
\nabla^a T_{ab} = 0.
\]

(6.4)

In particular, (6.2) and (6.3) are equivalent to the ‘parallel to \( u^a \)’ and the ‘orthogonal to \( u^a \)’ projections of (6.4), respectively. Thereby, it is usual in the formulation of Einstein’s theory for spacetimes with perfect fluids to postulate merely the form of the energy-momentum tensor, \( T_{ab} \), and solve Einstein’s equations since the equations of motion for the fluid then are automatically satisfied. We have chosen, however, a somewhat reversed approach here. In section 2 it was assumed that Euler-Lagrange equations are satisfied (which implies for the present case that \( \nabla^a T_{ab} = 0 \)) and this condition was used to show that some of basic field equations are deducible from the others. It is important to emphasize that we earn much than we loss by replacing the two basic field equations, (2.11) and (2.12), by Euler-Lagrange equations. Equations (2.11) and (2.12) are second order partial differential equations while above the Euler-Lagrange equations are first order one for perfect fluid.

Consider now the consequences of the presence of a Killing field, \( \xi^a \), for perfect fluid matter sources. First of all,

\[
\mathcal{L}_\xi T_{ab} = 0.
\]

(6.5)

15
Again, by the presence of a preferred vector field, \( u^a \), one might consider the unique decomposition of \( \mathcal{L}_\xi T_{ab} \) into symmetric tensor fields so that each of these tensor fields has definite ‘tangential’ or ‘perpendicular’ character with regard to their free indices. Since \( \mathcal{L}_\xi T_{ab} \) vanishes all of these projections must vanish, as well. Thereby \( (\mathcal{L}_\xi T_{ab})u^a u^b = 0 \) which gives that

\[
\mathcal{L}_\xi \rho = 0, \tag{6.6}
\]

Then \( (\mathcal{L}_\xi T_{ab})u^a \pi^b c = 0 \) yields that

\[
\mathcal{L}_\xi u^a = 0, \tag{6.7}
\]

(or \( \rho + P = 0 \) but this case has been excluded earlier), and, finally, from \( (\mathcal{L}_\xi T_{ab})\pi^a c \pi^b f = 0 \) we get

\[
\mathcal{L}_\xi \Pi = 0 \tag{6.8}
\]

throughout, where the projector, \( \pi^a_b \), is defined to be \( \pi^a_b = \delta^a_b - u^a u_b \). All in all, each of the physical quantities related to the perfect fluid are invariant under the action of the isometry group associated with \( \xi^a \). Consequently, for a general perfect fluid spacetime possessing a non-null Killing field we can use, without loss of generality, instead of the fields \( \rho, P, u^a \) (“\( \Psi^a \cdots b_1 \cdots b_n \)”) given on \( M \) the fields \( \rho, P, u^a \Pi = u^a \xi_a, u^a \Pi = \eta^a_b u^b \Psi^a \cdots b_1 \cdots b_n \) defined on \( S \).

Determine now the relevant form of \( \rho_{ab} \) and \( \sigma_{ab} \). According to (2.8) and (6.1) we have

\[
R^{(4)}_{ab} = 8\pi\left[ (\rho + P)u_a u_b + \frac{1}{2}(\rho - P)g_{ab} \right], \tag{6.9}
\]

furthermore, by the definition of \( \rho_{ab} \) and \( \sigma_{ab} \)

\[
\rho_{ab} = 8\pi \left[ (\rho + P)\left\{ (h_u^m u_m)(h_v^n u_n) + \varepsilon v^{-2}(u^a \xi_a)^2 \hat{h}_{ab} \right\} + \varepsilon v^{-1}(\rho - P)\hat{h}_{ab} \right], \tag{6.10}
\]

and

\[
\sigma_{ab} = -8\pi\varepsilon_{abmn} \xi^m u^n (\rho + P)(u^a \xi_a), \tag{6.11}
\]

hold.

For simplicity, one may restrict ones considerations to the case of vanishing \( \sigma_{ab} \). Equation (6.11) implies that \( \sigma_{ab} = 0 \) whenever either of the followings hold: \( \rho + P = 0 \), or \( u_a \xi^a = 0 \), or \( \xi^a u^b = 0 \). Note that when the equation of state is chosen to be \( \rho + P = 0 \) then the energy-momentum tensor is of the form \( T_{ab} = P g_{ab} \), and (6.4) implies that \( P \) is constant throughout.

This is precisely the case of vacuum fields with non-zero cosmological constant so it is reasonable
to assume that \( \rho + P \) is not identically zero. Thereby, we can say that \( \sigma_{ab} = 0 \) throughout if and only if either the 4-velocity of the fluid, \( u^a \), is parallel to the Killing field, \( \xi^a \), which means that the spacetime is stationary and

\[
u^a = (-v)^{-\frac{1}{2}} \xi^a, \quad (6.12)
\]
or

\[
\nu^a \xi_a = 0, \quad (6.13)
\]

which might be the case whenever the Killing field, \( \xi^a \), is spacelike. For both of these cases \textit{Condition 5.1} holds which implies that there exists (at least locally) a function \( \omega \) such that

\[
\omega_\alpha = \bar{D}_\alpha \omega. \quad (6.14)
\]

Let us consider the following particular case of perfect fluid sources: There are two commuting Killing fields, \( \xi^a_{(A)} \) (\( A = 1, 2 \)), on the spacetime and the 4-velocity of the fluid, \( u^a \), can be given as a linear combination of these Killing fields

\[
u^a = \mathcal{A}\left(\xi^a_{(1)} + \mathcal{B} \xi^a_{(2)}\right), \quad (6.15)
\]

Then with linearly independent Killing fields (6.7) and (6.15) yield that the functions \( \mathcal{A} \) and \( \mathcal{B} \) satisfy

\[
\mathcal{L} \xi_{(A)} \mathcal{A} = \mathcal{L} \xi_{(A)} \mathcal{B} = 0 \quad (A = 1, 2). \quad (6.16)
\]

Now, applying (6.7) and (6.8) for the Killing fields, \( \xi^a_{(A)} \), and using (6.15) we get that

\[
\mathcal{L}_u \rho = \mathcal{L}_u P = 0, \quad (6.17)
\]

and, the equations of motion, (6.2) and (6.3), reduce to

\[
(\rho + P) \nabla^a u_a = 0, \quad (6.18)
\]

\[
(\rho + P) u^a \nabla_a u_b + \nabla_b P = 0. \quad (6.19)
\]

One extracts from (6.15) - (6.16) that the fluid is expansion free, i.e., \( \nabla_a u^a = 0 \) throughout. Thereby, (6.18) holds identically. Furthermore, a straightforward calculation yields that

\[
u^a \nabla_a u_b = -\frac{1}{2} \mathcal{A}^2 \left\{ \nabla_b (-\mathcal{A}^{-2}) - \frac{\partial (-\mathcal{A}^{-2})}{\partial B} \nabla_b B \right\}, \quad (6.20)
\]

which along with (6.19) and (6.20) gives that

\[
\nabla_a P + \frac{1}{2} (\rho + P) \left[ \nabla_a (\ln \mathcal{A}^{-2}) - \frac{\partial (\ln \mathcal{A}^{-2})}{\partial B} \nabla_a B \right] = 0. \quad (6.21)
\]
As it was argued in Ref. [9], (6.21) implies that \( P = P(A, B) \) and \( \rho = \rho(A, B) \) even if \( A \) and \( B \) are functionally dependent or constant. Furthermore, since the 4-velocity – given by (6.15) – is a unite timelike vector we have

\[
A^{-2} = -\left\{ \left( \xi^a_{(1)}, \xi^b_{(1)} \right) + 2B \left( \xi^a_{(1)}, \xi^b_{(2)} \right) + B^2 \left( \xi^a_{(2)}, \xi^b_{(2)} \right) \right\},
\]

(6.22)

which along with (6.21) (and the above conclusion) gives that the equation of state must be of the form

\[
\rho = \rho(P).
\]

(6.23)

The remained Euler - Lagrange equation, (6.21), simplifies further whenever \( \frac{\partial (\ln A^{-2})}{\partial B} \nabla_a B = 0 \), i.e.,

\[
\nabla_a B = 0 \quad \text{or} \quad \frac{\partial (\ln A^{-2})}{\partial B} = 0.
\]

(6.24)

The case \( \nabla_a B = 0 \) is that of a ‘rigid fluid’, i.e., the 4-velocity, \( u^a \), is parallel to the timelike Killing field \( \xi^a = \xi^a_{(1)} + B \xi^a_{(2)} \). It is important to emphasize that equations (6.15) - (6.23) along with their consequences hold (with \( B = 0 \)) without any alteration even if the spacetime admits only a single timelike Killing field, \( \xi^a = \xi^a_{(1)} \), parallel to the 4-velocity of the fluid, \( u^a \).

The other possibility, \( \frac{\partial (\ln A^{-2})}{\partial B} = 0 \), along with (6.22) gives that \( B = -\frac{\xi^a_{(1)} \xi^b_{(2)}}{\xi^a_{(2)} \xi^b_{(2)}} \), i.e., the 4-velocity of the fluid, \( u^a \), is orthogonal to \( \xi^a_{(2)} \) which, therefore, must be a spacelike Killing field.

Note that for both of these cases not merely the Euler-Lagrange equations are simplified but, in accordance with this fact, the potential space associated with the Lagrangian of this particular case of ‘gravity plus perfect fluid’ system admits a symmetry [9].

Moreover, equations (6.21), (6.23) and (6.24) yield then that

\[
A^{-2}(P) = A_0^{-2} \cdot \exp \left[ -2 \int_{P_0}^{P} \frac{dP'}{\rho(P') + P'} \right],
\]

(6.25)

where \( A_0^{-2} \) and \( P_0 \) are constants of the integration. Consequently, whenever the 4-velocity of the fluid, \( u^a \), is either parallel to a Killing field, \( \xi^a \), or spanned by two commuting Killing fields, as in (6.15), with \( B = -\frac{\xi^a_{(1)} \xi^b_{(2)}}{\xi^a_{(2)} \xi^b_{(2)}} \), and, the equation of state, \( \rho = \rho(P) \), is known then the function \( A^{-2} = A^{-2}(P) \) or \( P = P(A^{-2}) \) can be determined via (6.25). Note that the function \( A^{-2} \) possesses the form

\[
A^{-2} = \left\{ \begin{array}{ll}
-v_x, & \text{if } u^a \xi^b = 0; \\
W^2 v^{-1}, & \text{if } u^a \xi^b_{(a)} = 0,
\end{array} \right.
\]

(6.26)

where

\[
W^2 = -\left( \xi^a_{(1)} \xi^b_{(1)} \right) \left( \xi^c_{(2)} \xi^b_{(2)} \right) + \left( \xi^d_{(1)} \xi^b_{(2)} \right)^2.
\]

(6.27)
Note that the function \( W \) has the following simple geometrical meaning. In canonical Weyl coordinates, \((\rho, z, \phi)\), the 3-metric, \( \hat{h}_{\alpha\beta} \), can be given as

\[
\hat{h}_{\alpha\beta} = \text{diag}\{\text{Exp}(2\gamma), \text{Exp}(2\gamma), -W^2\},
\]

where \( \gamma \) and \( W \) are functions of the coordinates \((\rho, z)\) [4].

7. Perfect fluids with 4-velocity parallel to a Killing field

In this section we shall apply the results of the previous sections for perfect fluid spacetimes possessing a timelike Killing field, \( \xi^a \), parallel to the 4-velocity of the fluid, \( u^a \). Such a fluid has expansion- and shear-free flow, i.e., it is ‘rigid’. Thereby one might ask whether there exists any physically realistic situation in which such a model can be applied. However, it was shown by Geroch and Lindblom [10] that in a generic theory of relativistic dissipative fluids the equilibrium states of are perfect fluid states. Furthermore, they showed that for these perfect fluids – which represent the equilibrium configurations of dissipative relativistic fluids – the 4-velocity is parallel to a Killing field [10]. Therefore, the model we are dealing with in this section has to have physical relevance, and, in fact, it is the needed one as long as we are looking for faithful description of possible equilibrium configurations of relativistic dissipative fluids.

First the applicability of the generalization of Cosgrove’s method then the general approach will be considered. Clearly, for this type of perfect fluids, \( \text{Condition 5.1} \) is satisfied and we show that \( \text{Condition 5.2} \) holds, as well. Now, since \( u^a \xi^b = 0 \), the Killing field, \( \xi^a \), is timelike so \( \varepsilon \) takes the value \(-1\). Furthermore, (6.6) yields that

\[
\rho_{ab} = 16\pi v^{-1} P \hat{h}_{ab}.
\]

The relevant form of (3.17)

\[
\hat{R}_{33} = 16\pi v^{-1} P \hat{h}_{33},
\]

can then be solved for \( v \). Since we have a non-vanishing spacelike vector field, \( k^a \), and the 3-metric, \( \hat{h}_{\alpha\beta} \), is non-singular, \( \hat{h}_{33} \) cannot vanish. Whenever there is another non-vanishing one among the functions, \( \hat{h}_{33}, \) then (7.2) gives rise to an algebraical restriction on the components \( \hat{R}_{33} \) of the Ricci tensor associated with \( \hat{h}_{ab} \). We obtain from (7.1) and (7.2)

\[
\rho_{AB} = \frac{\hat{R}_{33}}{\hat{h}_{33}} \hat{h}_{AB},
\]

which means that \( \text{Condition 5.2} \) is satisfied. Furthermore, this equation yields, along with (4.3),

\[
H_{AB} = 2(\hat{R}_{AB} - \frac{\hat{R}_{33}}{\hat{h}_{33}} \hat{h}_{AB}).
\]
For the particular case under consideration the functions $H_{AB}$ depend exclusively on the induced 3-metric, $\hat{h}_{ab}$, and the relevant form of (5.2) is, in fact, a fourth order partial differential equation for the components of $\hat{h}_{ab}$. It is striking to what an extent the corresponding basic field equations are similar in structure to the vacuum counterparts. Turning back to the main issue, note that the functions $v$ and $\omega$ can be determined by virtue of (5.3)-(5.13). Finally, the pressure, $P$, can be determined by (7.2) and the mass density, $\rho$, by the Euler-Lagrange equations, (6.21).

After solving (5.2) and fixing the functions $H_{AB}$ we may ask for the conditions under which the transformation (5.14)-(5.15) yields new solutions of the basic field equations. The two equations to be solved are, for the present case, (7.2) and (6.21). It is straightforward to check that by choosing $P_\tau$ to satisfy the equation $P_\tau v_\tau^{-1} = P_0 v_0^{-1}$ and deriving $\rho_\tau$ — for each pair of the functions $v_\tau$ and $P_\tau$ — in virtue of (6.21) we get a one-parameter family of solutions of the basic field equations. This property of the basic field equations was noticed earlier by Kramer, Neugebauer and Stephani [4,11] who analyzed the invariance properties of the associated Lagrangian of this system.

An interesting subcase of these perfect fluid spacetimes, discovered many years ago by Ehlers [12,4], is the case of vanishing pressure, i.e., of a stationary spacetime with dust possessing 4-velocity, $u^a$, parallel to a timelike Killing field, $\xi^a$. According to (3.17), (3.18) and (7.1) the field equations are then the same as for the vacuum case. Note, however, that (6.21) implies then $v = const$. The value of $v$ may be chosen throughout to be $-1$, so the field equations simplify to

\[ \hat{R}_{33} = 0, \]  
\[ \hat{R}_{AB} = \frac{1}{2} (\partial_A \omega)(\partial_B \omega). \]

For the energy density, $\rho$, we obtain from (2.0.2), (6.9) and (6.12) the constraint

\[ \rho = \frac{1}{8\pi} (\partial^A \omega)(\partial_A \omega). \]

Accordingly, a stationary dust (sd) solution — represented by $v^{(sd)} = -1$, $\omega^{(sd)} = \ln[v^{(sv)}]$ and $\hat{h}_{ab}^{(sd)} = \hat{h}_{ab}^{(sv)}$ — can be assigned to every static vacuum (sv) solution — given in terms of $v^{(sv)}$ and $\hat{h}_{ab}^{(sv)}$ — where the energy density of the dust satisfies the constraint (7.7) [12,4].

Although the above method can be applied in a straightforward way to get solutions of Einstein’s equations for the selected perfect fluid source there is an unfavourable aspect of this method. Namely, the most significant physical quantities — the mass density, $\rho$, and the
pressure, $P$, characterizing the possible physical states of the fluid – can be determined only at the very end of the entire process in terms of the function $v$ and the 3-geometry, $h_{ab}$. Therefore the equation of state, $\rho = \rho(P)$, has to be in accordance of the corresponding constraints, i.e., it cannot be chosen freely. The general approach, introduced in section 4, can be used to cure this problem but the price we have to pay is the appearance of extra non-linearities.

The general method was developed to ensure more control on the physical properties of matter fields in solving the relevant set of field equations. The significance of the seemingly technical differences between the two methods can be transparented for the examined perfect fluid sources as follows: Remember that the basic point in the general approach was the specification of the functional form of $v$ in terms of the 3-geometry and tensor fields representing the matter content. Moreover, for the present situation the basic field equations are (5.1) – where $H_{AB}$ is given by (7.4), (7.2) and (6.25). Note that (6.25) – which is, indeed, the integrated form of the Euler-Lagrange equation – gives now the functional relation between the functions $v, \rho$ and $P$ as

$$v(P) = v_0 \cdot \exp\left[-2 \int_{P_0}^{P} \frac{dP'}{\rho(P') + P'}\right].$$

We show, with the help of (7.2) and (7.8), that the function $v$ can be given in terms of the 3-geometry exclusively. Note that (7.8) is independent of the 3-geometry and, indeed, it is the only equation where one can control the physical properties of the solution describing the fluid by the substitution of a physically realistic equation of state. Consider, for instance, the case of a ‘stiff matter’ [4], i.e., choose the equation of state to be

$$\rho = P + \rho_0 \quad (\rho_0 \geq 0).$$

Using (7.8) and (7.9) one gets

$$v = v_0 \left(\frac{2P_0 + \rho_0}{2P + \rho_0}\right) \quad \text{or} \quad P = -\frac{\rho_0}{2} + \left(P_0 + \frac{\rho_0}{2}\right)v_0v^{-1},$$

The substitution of this function $P = P(v)$ into (7.2) yields

$$v = \frac{-4\pi\rho_0 + \left((4\pi\rho_0)^2 + 8\pi(2P_0 + \rho_0)v_0 \frac{R_{23}}{h_{33}}\right)^{\frac{1}{2}}}{\frac{R_{23}}{h_{33}}}.$$
3-metric and the 3-Ricci tensor. Although the derivation of the equations is straightforward the appearance of extra non-linearities – related to the functional form of \( v \) given by (7.11) – might be frightening. Note, however, that whenever one is able to find solutions of these equations the physical relevance of the solutions is automatically assured.

8. Final remarks

A new formulation of Einstein’s equations for spacetimes admitting a non-null Killing vector field and arbitrary matter field was given in this paper. First it was shown that some of the basic field equations are always deducible from the others. Then the existence of a geometrically preferred vector field and related coordinate systems were shown. Based on the related simplifications, two methods were presented obtaining systems of partial differential equations for the basic variables associated with the spacetime geometry and with the matter content. Both of the developed approaches were applied for perfect fluid spacetimes which describe equilibrium configurations of relativistic dissipative fluids. The symmetry properties of the relevant equations and differences of the two approaches were studied. It was shown, furthermore, that the techniques which were developed by Cosgrove [1] for the vacuum stationary axisymmetric problem can be generalized straightforwardly for these perfect fluid spacetimes despite the fact that in our examinations we assumed merely the existence of a single timelike Killing field.

It worth emphasizing that the general results of this paper, given in details in sections 2 – 5, are valid for any spacetime in Einstein’s theory which possesses a non-null Killing field and essentially arbitrary matter fields. Thereby, it would deserve further studies to find out how to apply these results for even more interesting situations in which time dependence my occur and/or different type of matter fields are present.

Acknowledgements

This research was supported in parts by the OTKA grant No. F14196. I would like to say thank the Ervin Schrödinger Institute in Vienna for its hospitality during work on this paper.

References

[1] C.M. Cosgrove: J. Phys. A 11, 2389 (1978)
[2] E.D. Fackerell and R.P. Kerr: Gen. Rel. Grav. 23, 861 (1991)
[3] R. Geroch: J. Math. Phys. 12, 918 (1971)
[4] D. Kramer, H. Stephani, M. MacCallum and E. Herlt: Exact solutions of Einstein’s equations (Cambridge; Cambridge University Press, 1980)
[5] S. Harris: Class. Quant. Grav. 9, 1823 (1992)
[6] B. Lukács and Z. Perjés: in Proc. 1st. Marcel Grossmann Meeting, Ed. R. Ruffini, (North-Holland 1976) p. 281
[7] I. Rácz and Á. Sebestyén: in preparation
[8] C. Hoenselaers: Prog. Theor. Phys. 57, 1223 (1977)
[9] H. Stephani and R. Grosso: Class. Quant. Grav. 6, 1673 (1989)
[10] R. Geroch and L. Limdblom: Ann. Phys. 207, 394-416 (1991)
[11] D. Kramer, G. Neugebauer and H. Stephani: Fortschr, Physik. 20, 1 (1972)
[12] J. Ehlers: in Colloques Internationaux C.N.R.S. No. 91 (Les théories relativistes de la gravitation, 275, 1962)