DESSINS D’ENFANTS AND SOME HOLOMORPHIC STRUCTURES ON THE
LOCH NESS MONSTER

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ABSTRACT. The classical theory of dessin d’enfants, which are bipartite maps on compact
surfaces, are combinatorial objects used to study branched covers between compact Rie-
mann surfaces and the absolute Galois group of the field of rational numbers. In this paper,
we show how this theory is naturally extended to non-compact surfaces and, in particular,
we observe that Loch Ness Monster (that is the surface of infinite genus with exactly one
end) admits infinite many regular dessins d’enfants (either chiral or reflexive). In addition,
we study different holomorphic structures on the Loch Ness Monster, which come from
homology covers of closed Riemann surfaces, infinite hyperelliptic, infinite superelliptic
and infinite $n$-gonal curves.

1. INTRODUCTION

In this paper a surface will mean a (possible non-compact) second countable connected
and orientable 2-manifold without boundary. A Riemann surface structure on a surface cor-
responds to a maximal holomorphic atlas (its local charts take their values in the complex
plane $\mathbb{C}$ and have biholomorphic transition functions where their domains overlap).

A dessin d’enfant corresponds to a bipartite map on a compact surface. These objects
were studied as early as the nineteenth century and rediscovered by Grothendieck in the
twentieth century in his ambitious research outline [Gro97]. Each dessin d’enfant defines
(up to isomorphisms) a unique Riemann surface structure $S$ together a branched cover
$\beta : S \rightarrow \hat{\mathbb{C}}$ branched over the set \{0, 1, $\infty$\} (called a Belyi map on $S$). Conversely, every
Belyi map $\beta : S \rightarrow \hat{\mathbb{C}}$ comes from a suitable dessin d’enfant.

The goal of this paper is to focus on the generalization of the theory of dessins d’enfants
to non-compact surfaces, and we study the different holomorphic structures on the Loch
Ness Monster.

From the point of view of the Kerekjarto’s theorem of classification of non-compact
surfaces (see e.g., [Ker23], [Ric63]), the topological type of any surface $S$ (recall that
we are assuming no boundary) is given by its genus $g \in \mathbb{N} \cup \{\infty\}$ and a couple of nested,
compact, metrizable and totally disconnected space $\text{Ends}_n(S) \subset \text{Ends}(S)$, which are know,
respectively, as the non-planar ends space and the ends of $S$. Of all non-compact surfaces
we focus is the Loch Ness monster.

In Section 2 we collect the principal tools to understand the classification of non-
compact surfaces theorem and introduce the Loch Ness Monster. Also, we explore the
concept of ends on groups. In Sections 3 we give an introduction to the theory of dessins
d’enfants, Belyi pairs and we give a characterisation of dessins d’enfants in terms of Belyi

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pairs on Riemann surfaces. In Section 4 we give a precise description of an infinite set of generators of a Fuchsian group \( \Gamma \), such that the quotient \( \mathbb{H}/\Gamma \) is the Loch Ness Monster. We show that there are infinitely many regular dessins d’enfants (either chiral or reflexive) on the Loch Ness monster, observing that the homology cover of a compact Riemann surface of genus \( g \geq 2 \) is topologically equivalent to the Loch Ness monster. We finally discuss on the explicit representation of the Loch Ness monster as a smooth infinite hyperelliptic, superelliptic and \( p \)-gonal curves, and of its moduli space.

2. The space of ends

We shall introduce the space of ends of topological spaces, Riemann surfaces and groups.

2.1. The ends space of a topological space. Let \( X \) be a locally compact, locally connected, connected, and Hausdorff space, and let \((U_n)_{n \in \mathbb{N}}\) be an infinite nested sequence \( U_1 \supset U_2 \supset \ldots \) of non-empty connected open subsets of \( X \), such that:

1. for each \( n \in \mathbb{N} \) the boundary \( \partial U_n \) of \( U_n \) is compact,
2. the intersection \( \bigcap_{n \in \mathbb{N}} \overline{U}_n = \emptyset \), and
3. for each compact \( K \subset X \) there is \( m \in \mathbb{N} \) such that \( K \cap U_m = \emptyset \).

Two nested sequences \((U_n)_{n \in \mathbb{N}}\) and \((U'_n)_{n \in \mathbb{N}}\) are equivalent if for each \( n \in \mathbb{N} \) there exist \( j, k \in \mathbb{N} \) such that \( U_n \supset U'_j \) and \( U'_n \supset U_k \). The corresponding equivalence classes of these sequences are called the ends of \( X \), and the set of all ends of \( X \) is denoted by \( \text{Ends}(X) \).

The ends space of \( X \) is the topological space having the ends of \( X \) as elements, and endowed with the following topology: for every non-empty open subset \( U \) of \( X \) such that its boundary \( \partial U \) is compact, we define

\[
U^* := \{ [U_n]_{n \in \mathbb{N}} \in \text{Ends}(X) \mid U_j \subset U \text{ for some } j \in \mathbb{N} \}.
\]

Then we take the set of all such \( U^* \), with \( U \) open with compact boundary of \( X \), as a basis for the topology of \( \text{Ends}(X) \).

**Theorem 2.1** (Theorem 1.5, [Ric63]). The space \( \text{Ends}(X) \) is Hausdorff, totally disconnected, and compact.

2.2. Ends of a Riemann surface. In a Riemann surface \( S \), one can distinguish the ends with infinite genus among the ends of \( S \). Thus, the Riemann surface \( S \) has associated two spaces: its respective ends space \( \text{Ends}(S) \), and the space \( \text{Ends}_\infty(S) \) conformed by the ends of \( S \) with infinite genus. Together they determine the topological type of \( S \). We shall introduce the concept end having infinite genus, the Theorem of Classification of non-compact Riemann Surfaces, and the definition of Loch Ness monster.

A closed subset of a Riemann surface \( S \) is called subsurface if its boundary in \( S \) consists of a finite number of nonintersecting simple closed curves. The reduced genus of a compact Riemann surface \( S \) with \( q(S) \) boundary curves and Euler characteristic \( \chi(S) \) is the number \( g(S) = 1 - \frac{1}{2}(\chi(S) + q(S)) \). We say that \( S \) has genus whenever \( g(S) \geq 1 \). In addition, the surface \( S \) is said to be planar if every compact subsurface of \( S \) is of genus zero, and the genus of the surface \( S \) is the maximum of the genera of its compact subsurfaces. An end \([U_n]_{n \in \mathbb{N}}\) of a Riemann surface \( S \) is called planar if there is \( l \in \mathbb{N} \) such that the open subset \( U_l \subset S \) is planar. Hence, we define the subset \( \text{Ends}_\infty(S) \) of \( \text{Ends}(S) \) conformed by all ends of \( S \), which are not planar (ends having infinity genus). It follows directly from the definition that \( \text{Ends}_\infty(S) \) is a closed subset of \( \text{Ends}(S) \) (see [Ric63]).
Theorem 2.2 (Classification of non-compact Riemann surfaces, [Ker23]). Two Riemann surfaces $S_1$ and $S_2$ having the same genus are topological equivalent if and only if there exists a homeomorphism $f : \text{Ends}(S_1) \to \text{Ends}(S_2)$ such that $f(\text{Ends}_\omega(S_1)) = \text{Ends}_\omega(S_2)$.

2.3. Loch Ness monster ([Val09]). The Loch Ness monster is the unique, up to homeomorphisms, Riemann surface of infinite genus and exactly one end (see Figure 1).

![Figure 1. The Loch Ness monster](image)

Remark 2.1 ([Spe49]). The Riemann surface $S$ has one end if and only if for all compact subset $K \subset S$ there is a compact $K' \subset S$ such that $K \subset K'$ and $S \setminus K'$ is connected.

2.4. Ends of a group. Given a generating set $H$ of a group $G$, the Cayley graph of $G$ with respect to the generating set $H$ is the graph $\text{Cay}(G, H)$ whose vertices of set is composed by the elements of $G$ and, there is an edge with ends points $g_1$ and $g_2$ if and only if there is an element $h \in H$ such that $g_1 h = g_2$. When the set $H$ is finite, the Cayley graph $\text{Cay}(G, H)$ is locally compact, locally connected, connected Hausdorff space.

The ends space of ends of a finitely generated group $G$ is $\text{Ends}(G) := \text{Ends}(\text{Cay}(G, H))$, where $H$ is a finite generating set of $G$.

Proposition 2.1 ([Löh17]). Let $G$ be a finitely generated group. The ends space of the Cayley graph of $G$ does not depend on the choice of the finite generating set.

Theorem 2.3 ([SW79]). Let $G$ be a finitely generated group. Then

1. $G$ has either 0, 1, 2 or infinitely many ends.
2. If $K$ is a finite index subgroup of $G$, then both groups have the same number of ends.
3. If $K$ is a finite normal subgroup of $G$, then groups $G$ and $G/K$ have the same number of ends.

The following result asserts that $G$ has more than one end if and only if $G$ splits over a finite subgroup.

Theorem 2.4 ([Sta68], [Sta71], [SW79] Theorem 6.1). Let $G$ be a finitely generated group.

1. If $G$ has infinitely many ends, then one of the following hold.
   a. If $G$ is torsion-free, then $G$ is a non-trivial free product.
   b. If $G$ has torsion, then $G$ is a non-trivial free product with amalgamation, with finite amalgamated subgroup.
2. The following are equivalent.
   a. $G$ has two ends.
   b. $G$ has a copy of $\mathbb{Z}$ as a finite index subgroup.

1From the historical point of view as shown in [ARM17], this nomenclature is due to A. Phillips and D. Sullivan [PS81].

2For us $\text{Cay}(G, H)$ will be the geometric realization of an abstract graph (see [Die17, p.226]).
(c) $G$ has a finite normal subgroup $N$ with $G/N$ isomorphic to either $\mathbb{Z}$ or $\mathbb{Z}_2 * \mathbb{Z}_2$.

(d) Either $G = F *_F F$, $F$ a finite group, or $G = A *_F B$, where $F$ is finite group and $[A : F] = [B : F] = 2$.

3. Dessins d’enfants and Belyi pairs

The classical theory of (Grothendieck’s) dessins d’enfants corresponds to bipartite maps on compact connected orientable surfaces \cite{GGD12, JW16}. This theory can be carry out without problems to non-compact connected orientable surfaces and, in this section, we describe this in such a generality.

3.1. Dessins d’enfant. A dessin d’enfant is a tuple

$$\mathcal{D} = (X, \Gamma, \iota : \Gamma \hookrightarrow X),$$

where

1. $X$ is a connected (not necessarily compact) orientable surface;
2. $\Gamma$ is a connected bipartite graph (vertices are either black or white) such that every vertex has finite degree;
3. $\iota : \Gamma \hookrightarrow X$ is an embedding such that every connected component of $X \setminus \iota(\Gamma)$, called a face of the dessin d’enfant, is a polygon with a finite number (necessarily even) of sides. The degree of a face is half the number of its sides.

Let us observe, from the above definition, that every compact subset $K$ of $X$ intersects only a finite number of faces, edges and vertices, respectively.

We say that a dessin d’enfant $\mathcal{D} = (X, \Gamma, \iota : \Gamma \hookrightarrow X)$ is:

1. a Grothendieck’s dessin d’enfant if $X$ is compact. In this case, we also say that it is a dessin d’enfant of genus $g$, where $g$ is the genus of $X$;
2. an uniform dessin d’enfant if all the black vertices (respectively, the white vertices and the faces) have the same degree;
3. a bounded dessin d’enfant if there is an integer $M > 0$ such that the degrees of all vertices and faces is bounded above by $M$;
4. a clean dessin d’enfant if all white vertices have degree 2. (This corresponds to the classical theory of maps on surfaces.)

3.2. Passport (valence) of a dessin d’enfant. Let $\mathcal{D} = (X, \Gamma, \iota : \Gamma \hookrightarrow X)$ be a dessin d’enfant. The collections of vertices and faces of $\mathcal{D}$ are either finite or countable infinite.

Let $\{v_i\}_{i \geq 1}$ (respectively, $\{w_j\}_{j \geq 1}$) be the collection of black (respectively, the collection of white) vertices of $\Gamma$, and let $\{f_k\}_{k \geq 1}$ be the collection of faces of the dessin d’enfant. The passport (or valence) of $\mathcal{D}$ is the tuple

$$\text{Val}(\mathcal{D}) = (\alpha_1, \alpha_2, \ldots; \beta_1, \beta_2, \ldots; \gamma_1, \gamma_2, \ldots),$$

where $\alpha_i$, $\beta_j$ and $\gamma_k$ denote the degrees of $v_i$, $w_j$ and $f_k$, respectively, with $\alpha_i \leq \alpha_{i+1}$, $\beta_j \leq \beta_{j+1}$ and $\gamma_k \leq \gamma_{k+1}$.

3.3. Equivalence of dessins d’enfants. Two dessins d’enfants

$$\mathcal{D}_1 = (X_1, \Gamma_1, \iota_1 : \Gamma_1 \hookrightarrow X_1)$$
$$\mathcal{D}_2 = (X_2, \Gamma_2, \iota_2 : \Gamma_2 \hookrightarrow X_2)$$

are called equivalent if there are exist:

1. an orientation-preserving homeomorphism $\phi : X_1 \rightarrow X_2$, and
(2) an isomorphism of bipartite graphs (i.e., an isomorphism of graphs sending black vertices to black vertices) \( \rho : \Gamma_1 \to \Gamma_2 \),

such that

\[ \phi \circ \iota_1 = \iota_2 \circ \rho. \]

**Remark 3.1.** Two equivalent dessins d’enfants necessarily have the same passport, but the converse is in general false.

### 3.4. Automorphisms of dessins d’enfants.

Let \( \mathcal{D} = (X, \Gamma, \iota : \Gamma \hookrightarrow X) \) be a dessin d’enfant.

An orientation-preserving automorphism of \( \mathcal{D} \) is an automorphism \( \rho \) of \( \Gamma \), as a bipartite graph, such that there exists an orientation-preserving homeomorphism \( \phi : X \to X \) with \( \phi \circ \iota = \iota \circ \rho \).

An orientation-reserving automorphism of \( \mathcal{D} \) is an automorphism \( \rho \) of \( \Gamma \), as a bipartite graph, such that there exists an orientation-reserving homeomorphism \( \phi : X \to X \) with \( \phi \circ \iota = \iota \circ \rho \).

The group of orientation-preserving automorphisms of \( \mathcal{D} \) is denoted by \( \text{Aut}^+ (\mathcal{D}) \), and the group all automorphisms of \( \mathcal{D} \), both orientation-preserving and orientation-reserving, is denoted by \( \text{Aut}(\mathcal{D}) \).

The subgroup \( \text{Aut}^+ (\mathcal{D}) \) is a normal subgroup of \( \text{Aut}(\mathcal{D}) \) of index at most two. If the index is two, then we say that \( \mathcal{D} \) is **reflexive**; otherwise we say that it is **chiral**; and if \( \text{Aut}^+ (\mathcal{D}) \) acts transitively on the set of edges of \( \Gamma \), then we say that \( \mathcal{D} \) is **regular**.

### 3.5. Monodromy groups of dessins d’enfants.

Let \( \mathcal{D} = (X, \Gamma, \iota : \Gamma \hookrightarrow X) \) be a dessin d’enfant. Let \( E \) be the set of edges of \( \Gamma \) and let \( \Sigma_E \) be the permutation group of \( E \).

Let \( v_i \) be a black vertex of \( \Gamma \), which has degree \( \alpha_i \). If \( e_{i1}, \ldots, e_{i\alpha_i} \in E \) are the edges of \( \Gamma \) adjacent to \( v_i \), following the counterclockwise orientation of \( X \), then we can construct a permutation

\[ \sigma = \prod_i \sigma_i \in \Sigma_E, \]

where \( \sigma_i = (e_{i1}, \ldots, e_{i\alpha_i}) \in \Sigma_E. \)

We may proceed in a similar fashion for the white vertices \( w_j \) to construct a permutation

\[ \tau = \prod_j \tau_j \in \Sigma_E. \]

The **monodromy group** of \( \mathcal{D} \) is the subgroup \( M_\mathcal{D} = \langle \sigma, \tau \rangle \) generated by \( \sigma \) and \( \tau \) in the symmetric group \( \Sigma_E \).

**Remark 3.2.** The connectivity of \( \Gamma \) asserts that the monodromy group \( M_\mathcal{D} \) is a transitive subgroup of \( \Sigma_E \). The permutation \( \tau \sigma \) is again a product of disjoint finite cycle permutations

\[ \tau \sigma = \prod_k \eta_k, \]

where there is a bijection between these \( \eta_k \) and the faces \( f_k \) of \( \mathcal{D} \) (the length of \( \eta_k \) is equal to the degree of the correspondent face \( f_k \)).

**Remark 3.3 ([GGDT12]).** If \( \sigma, \tau \in \Sigma_E \) are two permutations, such that \( \sigma, \tau \) and \( \tau \sigma \) are each a product of disjoint finite cycle permutation and the group generated by them is transitive, then \( M = \langle \sigma, \tau \rangle \) is the monodromy group of some dessin d’enfant with \( E \) as its set of edges.
3.6. The automorphisms of the dessin in terms of the monodromy group. Let $\mathcal{D}$ be a dessin d’enfant with monodromy group $M_\mathcal{D} = \langle \sigma, \tau \rangle \lhd \mathbb{E}_F$. In terms of the monodromy group $M_\mathcal{D}$, the group $\text{Aut}^+(\mathcal{D})$ can be identified with the centralizer of $M_\mathcal{D}$, that is, with the subgroup formed of those $\eta \in \mathbb{E}_F$ such that $\eta \sigma \eta^{-1} = \sigma$ and $\eta \tau \eta^{-1} = \tau$, i.e., the group $\text{Aut}^+(\mathcal{D})$ can be generated by two elements. Moreover, each orientation-reversing automorphism of the dessin can be identified with those $\eta \in \mathbb{E}_F$ such that $\eta \sigma \eta^{-1} = \sigma^{-1}$ and $\eta \tau \eta^{-1} = \tau^{-1}$.

3.7. Locally finite holomorphic branched coverings. Let $S_1, S_2$ be connected Riemann surfaces and $\varphi : S_1 \to S_2$ be a surjective holomorphic map. We say that $\varphi$ is a locally finite holomorphic branched cover map if:

1. the locus of branched values $B_\varphi \subset S_2$ of $\varphi$ is a (which might be empty) discrete set;
2. $\varphi : S_1 \setminus \varphi^{-1}(B_\varphi) \to S_2 \setminus B_\varphi$ is a holomorphic covering map; and
3. each point $q \in B_\varphi$ has an open connected neighborhood $U$ such that $\varphi^{-1}(U)$ consists of a collection $\{V_i\}$ of pairwise disjoint connected open sets such that each of the restrictions $\varphi|_{V_i} : V_i \to U$ is a finite degree branched cover (i.e., is equivalent to a branched cover of the form $z \in \mathbb{D} \to z^{d_i} \in \mathbb{D}$, where $\mathbb{D}$ denotes the unit disc).

Observe that in the case the surface $S_1$ is compact, then any non-constant holomorphic map is a locally finite branched cover map. The above definition is needed for the non-compact situation.

3.8. Belyi pairs. The classical theory of Belyi pairs corresponds to certain meromorphic maps on compact orientable surfaces and that is also related to algebraic curves defined over the field $\overline{\mathbb{Q}}$ of algebraic numbers. We generalize such a concept to the case of connected (not necessarily compact) Riemann surfaces.

Let $S$ be a connected Riemann surface (not necessarily compact). A Belyi map of $S$ is a locally finite holomorphic branched cover map $\beta : S \to \overline{\mathbb{C}}$ whose branch values are contained inside the set $\{\infty, 0, 1\}$. In this case, we also say that $S$ is a Belyi surface and that $(S, \beta)$ is a Belyi pair.

A Belyi pair $(S, \beta)$ is called a bounded Belyi pair if the set of local degrees of the preimages of $0$, $1$ and $\infty$ is bounded; and is called uniform if the local degrees of $\beta$ at the preimages of each value is the same.

3.9. Automorphisms of Belyi pairs. Let $(S, \beta)$ be a Belyi pair.

A holomorphic automorphism of $(S, \beta)$ is a conformal automorphism $\phi$ of $S$ such that $\beta \circ \phi = \beta$. The group of holomorphic automorphisms of $(S, \beta)$ is denoted by $\text{Aut}^+(S, \beta)$.

An antiholomorphic automorphism of $(S, \beta)$ is an anticonformal automorphism $\psi$ of $S$ such that $\beta \circ \psi = J \circ \beta$, where $J(z) = \overline{z}$.

The group all automorphisms of $(S, \beta)$, both holomorphic and antiholomorphic, is denoted by $\text{Aut}(S, \beta)$.

The group $\text{Aut}^+(S, \beta)$ is a normal subgroup of $\text{Aut}(S, \beta)$ of index at most two.

The Belyi pair $(S, \beta)$ is called regular if $\beta$ is a regular branched covering whose deck group is $\text{Aut}^+(S, \beta)$.

**Theorem 3.1** (Belyi’s theorem [Bel80]). Every Belyi pair of finite degree (Grothendieck’s dessin d’enfant) is equivalent to one of the form $(S, \beta)$, where $S$ is a smooth algebraic curve defined over $\overline{\mathbb{Q}}$ and with $\beta$ a rational map also defined over it.
The above results asserts that there is an action of the absolute Galois group $Gal(\overline{Q}/Q)$ on Grothendieck’s dessins d’enfants.

**Theorem 3.2** ([GGD07], [Gro97], [Sch94]). The action of $Gal(\overline{Q}/Q)$ on Grothendieck’s dessins d’enfants is faithful.

3.10. **Equivalence of Belyi pairs.** Two Belyi pairs $(S_1, \beta_1)$ and $(S_2, \beta_2)$ are called equivalent if there is a biholomorphism $\phi : S_1 \to S_2$ such that $\beta_1 = \beta_2 \circ \phi$.

**Theorem 3.3** ([GGD12]). There is an one-to-one correspondence between the category of equivalence classes of (bounded, Grothendieck) dessins d’enfants and the equivalence classes of (bounded, finite degree) Belyi pairs. The correspondence preserves regularity and also uniformity.

**Proof.** (Idea of the proof) A dessin d’enfant $D = (X, \Gamma, \iota : \Gamma \to X)$ induces a surjective continuous map $\beta : X \to \hat{C}$ which sends the black vertices to 0, the white vertices to 1 and center of faces to $\infty$ defining a covering map $\beta : X \setminus \beta^{-1}(\{\infty, 0, 1\}) \to \hat{C} \setminus \{\infty, 0, 1\}$. We may pull-back, under $\beta$, the Riemann surface structure of $\hat{C} \setminus \{\infty, 0, 1\}$ to obtain a Riemann surface structure on $X \setminus \beta^{-1}(\{\infty, 0, 1\})$ making the restriction of $\beta$ holomorphic. Such a Riemann surface structure extends to a Riemann surface structure $S$ on $X$ making $\beta$ a Belyi map. Conversely, if $(S, \beta)$ is a Belyi pair, then $\hat{\Gamma} = \beta^{-1}(\{0, 1\})$ produces a dessin d’enfant on $S$. □

3.11. **Dessins subgroups of $\Gamma(2)$**. Let $S$ be a compact Riemann surface and $\beta : S \to \hat{C}$ be a surjective meromorphic map, with branch values contained in $\{0, 1\}$.

In this generality, it might happen that $\beta$ is not a locally finite branched cover map (so it might be not a Belyi map).

Consider the Fuchsian group

$$\Gamma(2) = \langle A(z) = z + 2, B(z) = z/(1-2z) \rangle \cong F_2,$$

which is a normal subgroup of $\text{PSL}_2(\mathbb{R})$ of index 6 such that $\text{PSL}_2(\mathbb{R})/\Gamma(2) \cong \mathbb{Z}_3$. The quotient $\mathbb{H}/\Gamma(2)$ is isomorphic to $\hat{C} \setminus \{\infty, 0, 1\}$.

By the covering maps theory, there is a subgroup $K$ of $\Gamma(2)$ such that the unbranched holomorphic cover $\beta : S \setminus \beta^{-1}(\{\infty, 0, 1\}) \to \hat{C} \setminus \{\infty, 0, 1\}$ is induced by the inclusion $K < \Gamma(2)$.

The condition for $\beta$ to define a Belyi map on $S$ (i.e., to be a locally finite branched covering map) is equivalent for $K$ to satisfy the following property: for every parabolic element $Z \in \Gamma(2)$ there is some positive integer $n_Z > 0$ such that $Z^{n_Z} \in K$. We say that such kind of subgroup of $\Gamma(2)$ is a dessin subgroup.

The following result (see [Wol97] for the case of Grothendieck’s dessins d’enfants) states an equivalence between dessin subgroups of $\Gamma(2)$ and dessins d’enfants (Belyi pairs).

**Theorem 3.4.** There is an one-to-one correspondence between the category of equivalence dessins d’enfants and congugacy classes of dessins subgroups of $\Gamma(2)$. In this equivalence, Grothendieck’s dessins d’enfants corresponds to finite index subgroups.

Let us consider a bounded Belyi pair $(S, \beta)$. The boundness condition permits to compute the least common multiple of all local degrees of the points in each fiber $\beta^{-1}(p)$, $p \in \hat{C}$.

Let $a, b, c \geq 1$ be the least common multiple of the local degrees of $\beta$ at the preimages of 0, 1 and $\infty$, respectively. In this case, the triple $(a, b, c)$ is called the type of $(S, \beta)$. By
the equivalence with bounded dessins d’enfants, the above is also the type of the associated dessin d’enfant.

Set $\mathcal{X}(a,b,c)$ equal to the hyperbolic plane $\mathbb{H}$, the complex plane $\mathbb{C}$ or the Riemann sphere $\hat{\mathbb{C}}$ if $a^{-1} + b^{-1} + c^{-1}$ is less than 1, equal to 1 or bigger than 1, respectively.

Let us consider a triangular group (unique up to conjugation by Möbius transformations) $\Delta(a,b,c) = (x,y : x^a = y^b = (yx)^c = 1)$, acting as a group of holomorphic automorphisms of $\mathcal{X}(a,b,c)$. The quotient complex orbifold $\mathcal{X}/\Delta(a,b,c)$ is the Riemann sphere whose cone points are 0 of cone order $a$, 1 of cone order $b$ and $\infty$ of cone order $c$.

By the uniformization theorem, there exists a subgroup $K$ of $\Delta(a,b,c)$ such that (i) the quotient orbifold $\mathcal{X}(a,b,c)/K$ has a Riemann surface structure biholomorphically equivalent to $S$ and (ii) the Belyi map $\beta$ is induced by the inclusion $K < \Delta(a,b,c)$ (finite index condition on $K$ is equivalent for the orbifold to be compact).

**Theorem 3.5 ([GGD12], [IW16]).** There is a natural one-to-one correspondence between the category of equivalence classes of bounded Belyi (bounded dessins d’enfants) of type $(a,b,c)$ and the category of conjugacy subgroups of bounded $\Delta(a,b,c)$. The finite degree Belyi pairs (Grothendieck’s dessins d’enfants) correspond to finite index subgroups.

The regular Belyi pairs (regular dessins d’enfants) correspond to the torsion-free normal subgroups. The uniform ones correspond to torsion-free subgroups.

There is a group

$$\Delta(a,b,c) = \langle \tau_1, \tau_2, \tau_3 : \tau_1^a = \tau_2^b = \tau_3^c = (\tau_2 \tau_1)^a = (\tau_1 \tau_3)^b = (\tau_3 \tau_2)^c = 1 \rangle,$$

where $\tau_1$, $\tau_2$ and $\tau_3$ are reflection on the three sides of a circular triangle with angles $\pi/a, \pi/b, \pi/c$, such that $\Delta(a,b,c)$ is its index two subgroup of orientation-preserving elements ($x = \tau_2 \tau_1, y = \tau_1 \tau_3$).

**Theorem 3.6 ([GGD12], [IW16]).** Let $K < \Delta(a,b,c)$ and $(S,\beta)$ a Belyi pair associated to $K$ as described above. The group $\text{Aut}^+(S,\beta)$ corresponds to $N_{\Delta(a,b,c)}(K)/K$ and $\text{Aut}(S,\beta)$ corresponds to $N_{\Delta(a,b,c)}(K)\backslash K$, where $N_{\Delta(a,b,c)}(K)$ and $N_{\Delta(a,b,c)}$ are the corresponding normalizers of $K$ in $\Delta(a,b,c)$ and $\Delta(a,b,c)$, respectively.

The following result is immediately of the Theorems 2.4, 3.5 and 3.6.

**Corollary 3.1.** The ends space of the triangular group $\Delta(a,b,c)$, where $a^{-1} + b^{-1} + c^{-1} \leq 1$, has 1 end. Finite triangular groups have no ends.

The ends space of the automorphisms group of a regular and chiral map has 0 or 1 end (see [ARMV17]).

**Remark 3.4.** Let $\Delta(a,b,c)$ be a Fuchsian triangle group and $K < \Delta(a,b,c)$ such that $[\Delta(a,b,c) : K] = \infty$. Then the group $\Delta(a,b,c)/K$ has only one end ([Ser03 Example 6.3.3, p. 60]). In particular, the surface $\mathbb{H}^2/K$ has one end.

## 4. On the Loch Ness Monster

In this section we show that the Loch Ness monster admits infinitely many Riemann surface structures.

### 4.1. Construction 1: subgroups of PSL$_2(\mathbb{Z})$

Let us consider the modular group PSL$_2(\mathbb{Z})$, which is generated by $F(z) = z + 1$ and $E(z) = -1/z$. Inside this group is the index tree non-normal subgroup $K_0 = \mathbb{Z} \ast \mathbb{Z}_2$, generated by $A = F^2$ and $E$. A fundamental region for $K_0$ is the geodesic triangle whose vertices are $-1, 1$ and $\infty$ (see Figure 2) and the quotient
orbifold $\mathbb{H}/K_0$ is the punctured plane $\mathbb{C} \setminus \{0\}$ with exactly one cone point (say 1) with cone order 2.

The Möbius transformation $A^2E$ sends the half-circle $C_1(0)$ onto $C_1(4)$ and the element $AEA^{-3}$ of $\text{PSL}_2(\mathbb{Z})$ maps the half-circle $C_1(6)$ onto $C_1(2)$.

Set $K_1 = \langle A^4, AEA^{-3}, A^2E \rangle \cong \mathbb{Z} \ast \mathbb{Z} \ast \mathbb{Z}$,

which is a subgroup of index four of $K_0$. A fundamental region $R_1$ for $K_1$ is the geodesic polygon with 6 sides whose vertices are $-1, 1, 3, 5, 7$ and $\infty$ (see Figure 3) and $\mathbb{H}/K_1$ is homeomorphic to the torus with two puncture (or two caps each one having order $\infty$). Observe that the group $K_1$ is normalized by $A$, so it induces a holomorphic automorphism of order 4 on the torus $\mathbb{H}/K_1$, each of the two punctures is fixed. It follows that the compactification of this is defined by the elliptic curve $y^2 = x^4 - 1$. For each $n$ we define the subgroup $K_n = \langle A^{4n}, A^4AEA^{-3}A^{-4l}, A^4A^2EA^{-4l} : l \in \{0, 1, \ldots, n-1\} \rangle < K_{n-1}$.

**Remark 4.1.** A fundamental region $R_n$ for $K_n$ is the geodesic polygon with $4n + 2$ sides whose vertices are $\infty$ and the $2l - 1$ for each $l \in \{0, \ldots, 4n\}$ (see Figure 4). It can be seen (from the identification of sides) that $K_n \cong \mathbb{Z} \ast \mathbb{Z} \ast \mathbb{Z}$ has index $n$ in $K_1$ and that $\mathbb{H}/K_n$ is homeomorphic to the $n$-torus with two puncture (or two caps each one having order $\infty$).

The group $K_\infty = \langle A^4A^2EA^{-4l}, A^4AEA^{-3}A^{-4l} : l \in \mathbb{Z} \rangle$

is an infinite index subgroup of $K_0$, which is normalized by $A^4$, and such that

(2) $S = \mathbb{H}/K_\infty$

is topologically the Loch Ness monster [ARM20, Theorem 1.3].
4.2. **Construction 2: hyperbolic structures from compact surfaces.** A direct consequence of Corollary 3.1 we obtain an infinity of hyperbolic structures on the Loch Ness monster.

**Proposition 4.1.** Let \( \Gamma \) be a co-compact Fuchsian group of genus \( g \geq 2 \) and let \( K \) be a characteristic subgroup of \( \Gamma \) such that \( [\Gamma : K] = \infty \). Then \( \mathbb{H}^2 / K \) is topologically the Loch Ness monster.

**Proof.** For each genus \( g \geq 2 \) there is a regular Belyi pair \((S, \beta)\), where \( S \) is a closed Riemann surface of genus \( g \). Let \( G \subset \text{Aut}(S) \) be the deck group of \( \beta \). Let \( \Delta(a,b,c) \) be a triangular Fuchsian group such that \( S / G = \mathbb{H}^2 / \Delta(a,b,c) \). There is a normal (torsion free) subgroup \( F \) of \( \Delta(a,b,c) \) such that \( S = \mathbb{H}^2 / F \) and \( G = \Delta(a,b,c) / F \). There is an orientation preserving homeomorphism \( h : \mathbb{H}^2 \to \mathbb{H}^2 \) conjugating \( \Gamma \) to \( F \). This homeomorphism induces an orientation preserving homeomorphism between \( \mathbb{H}^2 / K \) and \( \mathbb{H}^2 / hK^{-1} \). As \( hK^{-1} \) is characteristic subgroup of \( F \), it is a normal subgroup of \( \Delta(a,b,c) \), so the result follows from Corollary 3.1 and Remark 3.3. \( \square \)

### 4.2.1. Example 1: the homology cover.

Let \( K = \Gamma' \triangleleft \Gamma \) be the derived subgroup of a co-compact Fuchsian group of genus \( g \geq 2 \). Set \( S_\Gamma = \mathbb{H} / \Gamma \), a compact Riemann surface of genus \( g \), and \( S_K = \mathbb{H} / K \), a non-compact Riemann surface (this is called the homology cover of \( S_\Gamma \)). As a consequence of Corollary 4.1 we obtain the following.

**Proposition 4.2.** The homology cover of a compact Riemann surface of genus \( g \geq 2 \) is topologically equivalent to the Loch Ness monster.

The above result can also be obtained as follows. The group \( G = \Gamma / K \cong \mathbb{Z}^{2g} \), has exactly one end, and acts as a group of holomorphic automorphisms of \( S_K \) such that \( S_\Gamma = S_K / G \). Let \( P \subset \mathbb{H} \) be a canonical fundamental polygon for \( \Gamma \) (it has \( 4g \) sides and its pairing sides are a set \( A_1, \ldots, A_g, B_1, \ldots, B_g \in \Gamma \), of generators of \( \Gamma \) such that \( \prod_{j=1}^{g} [A_j, B_j] = 1 \)). The \( \Gamma \)-translates of \( P \) induces a tessellation on \( \mathbb{H} \) and it descends to a tessellation on \( S_K \). The dual graph of such an induced tessellation is the Cayley graph of \( G \), with respect to the induced \( 2g \) generators by the elements \( A_j, B_j \). As \( \mathbb{Z}^{2g} \), \( g \geq 2 \), has one end, it follows that the number of ends of \( S_K \) must be one.

It is well-known that the homology cover determines the surfaces, more precisely:

**Theorem 4.1 (Mas86).** Let \( \Gamma_1 \) and \( \Gamma_2 \) be two co-compact Fuchsian groups. If \( \Gamma_1 = \Gamma_2 \), then \( \Gamma_1 = \Gamma_2 \). In particular, two compact Riemann surfaces, both of genus at least two, are isomorphic if and only if their homology covers are isomorphic.

### 4.2.2. Example 2.

**Proposition 4.3.** Let \( \Gamma \) be a torsion-free co-compact Fuchsian group of genus \( g \geq 2 \). For each \( k \geq 3 \), let \( \Gamma^k \) be the characteristic subgroup generated by all the \( k \)-powers of the elements of \( \Gamma \). Then \( \mathbb{H} / \Gamma^k \) is topologically the Loch Ness monster.
In this case,

\[ G = \Gamma/\Gamma^k = \langle x_1, \ldots, x_g, y_1, \ldots, y_g : x_1^k = \cdots = x_g^k = y_1^k = \cdots = y_g^k = \prod_{j=1}^g [x_j, y_j] = 1 \rangle. \]

By Theorem 2.4 and the fact that \( G \) is not finite, the group \( G \) has exactly one end.

4.3. **DeSSins d’enfants on the Loch Ness monster.** Let \((S, \beta)\) be a hyperbolic regular Belyi pair of type \((a, b, c)\), and \(S\) be a compact of genus \(g \geq 2\), and let \(G = \text{Aut}^+\langle S, \beta \rangle\). This Belyi pair corresponds to a torsion-free normal subgroup \(\Gamma \) of \(\Delta(a, b, c)\), that is, \(S = S_\Gamma = \mathbb{H}/\Gamma\) and \(\beta\) is induced by the normal inclusion \(\Gamma \triangleleft \Delta(a, b, c)\). Let \(K = \Gamma'\) be the derived subgroup of \(\Gamma\). As seen above, the homology cover \(S_K = \mathbb{H}/K\) of \(S\) is topologically equivalent to the Loch Ness Monster. The inclusion \(K \triangleleft \Delta(a, b, c)\) induces a regular dessin d’enfant \((S_K, \tilde{\beta})\) of type \((a, b, c)\). There is a regular unbranched covering \(P : S_K \to S\), whose deck group is \(\Gamma/K \cong \mathbb{Z}^d\), such that \(\tilde{\beta} = \beta \circ P\). If, moreover, \(S\) admits no anticonformal automorphism, then neither does \(S_K\), in particular, if \((S, \beta)\) is chiral, then the same happens with \((S_K, \tilde{\beta})\). In particular, this shows infinitely many regular deSSins d’enfants (either chiral or reflexive) on the Loch Ness monster.

4.4. **Infinite Hyperelliptic and infinite \(n\)-gonal Riemann surface structures on the Loch Ness Monster.** Let us consider a non-constant holomorphic map \(F : U \subset \mathbb{C}^2 \to \mathbb{C}\). If \(p \in F(U)\), then we have associated the affine plane curve given by the set \(S_F(p) := \{(z, w) \in U : F(z, w) = p\}\). When \(U = \mathbb{C}^2\) and \(F\) is a polynomial map, then usually we talk of an algebraic set.

**Remark 4.2.** The affine plane curve \(S_F(p)\) is a closed subset of \(U\), this is the inverse image of \(p\) under the continuous map \(F\).

If \(p\) is a regular value for \(F\), then the Implicit Function Theorem (e.g., [KP02], [Mir95], p. 10, Theorem 2.1) asserts that \(S_F(p)\) is a Riemann surface.

**Example 4.1.** Let \(F : \mathbb{C}^2 \to \mathbb{C}\) be a function given by \(F(z, w) = ze^w\). As \(\frac{\partial F}{\partial z}(z, w) = e^w \neq 0\), we have that, for instance, \(S_F(1) = \{(z, w) : ze^w = 1\}\) is a Riemann surface.

Given a sequence \((z_n)_{n \in \mathbb{N}}\) conformed by complex numbers such that the limit \(\lim_{n \to \infty} |z_n| = \infty\), and for each \(n = 1, 2, 3, \ldots\) let \(m_n\) be a positive integer. Then the Weierstrass’s theorem assures that there exists an entire function \(f\) that has the points \(z_1, z_2, \ldots\) as its zeros, and the zero at \(z_n\) being one of order \(m_n\). Moreover, \(f\) is uniquely determined up multiplication by a zero-free entire map (for example \(e^z\)) (see e.g., [Pal90] p. 498).

The function \(f\) admits representations of the type

\[ f(z) = g(z)e^w \prod_{n=1}^{\infty} \left( 1 - \frac{e}{z_n} \right)^{m_n} E_n(z) \]

where \(g\) is a zero-free entire function and \(E_n(z)\) is a function of the form

\[ E_n(z) = \exp \left[ \sum_{k=1}^{d(n)} \frac{1}{k} \left( \frac{z}{z_n} \right)^k \right] \]

for a suitably large non-negative integer \(d(n)\).
Infinite Hyperelliptic curve. Let \((z_n)_{n \in \mathbb{N}}\) be a sequence of different complex numbers such that \(\lim_{n \to \infty} |z_n| = \infty\) and let \(f\) be an entire function having the points \(z_1, z_2, \ldots\) as its zeros and each one of them is simple (i.e. \(m_n = 1\) for each \(n \in \mathbb{N}\)). The affine plane curve
\[ S(f) = \{(z, w) \in \mathbb{C}^2 : w^2 = f(z)\} \]
is called an Infinite Hyperelliptic curve.

Remark 4.3. We will see in the Theorem 4.2 that \(S(f)\) is a connected Riemann surface. The projection map \(\pi : S(f) \to \mathbb{C}, (z, w) \mapsto z\) satisfies the following properties.

1. It is a branch covering map having branch (ramification) points in the elements of the sequence \((z_n)_{n \in \mathbb{N}}\). The fiber \(\pi^{-1}(z)\) consists by two elements, for each \(z \in \mathbb{C}\) disjoint to the sequence \((z_n)_{n \in \mathbb{N}}\).

2. It is a proper map, it means, the inverse image of any compact subset \(K\) of \(\mathbb{C}\) is also a compact subset of \(S(f)\), because \(\pi^{-1}(K)\) is a closed subset of the compact \(K \times \{h^{-1}(f(K))\}\), where \(h\) is the complex square map \(h(w) = w^2\).

Theorem 4.2. An infinite hyperelliptic curve \(S(f)\) is a connected Riemann surface homeomorphic to the Loch Ness monster.

Proof. We must prove that the subspace \(S(f) \subset \mathbb{C}^2\) has a Riemann surface structure. We take the function \(F : \mathbb{C}^2 \to \mathbb{C}\), such that \(F(z, w) = w^2 - f(z)\). We shall define a chart \((\varphi, U)\) around the point \((z, w) \in S(f)\).

Case 1. If \(z \neq z_n\), for all \(n \in \mathbb{N}\), then we hold \(\frac{\partial F}{\partial w}(p) = 2w \neq 0\). From the Implicit Function Theorem exists a function \(g(z)\) holomorphic such that in an open \(U\) of \(p\), \(X\) is (locally) the graph \(w = h(z) = \sqrt{f(z)}\). Thus the restriction of the projection \(\pi_{z|U} : U \to \mathbb{C}\), such that \((z, w) \mapsto z\), is a homeomorphism from \(U\) to its image \(V\), which is an open subset of \(\mathbb{C}\). The couple \((\pi_{z|U}, U)\) is a chart around of the point \((z, w)\). We note that \(\pi_{z|U}^{-1} : V \to U\) is define as follows \(z \mapsto (z, h(z) = \sqrt{f(z)})\).

Case 2. If \((z_n, 0)\), for all \(n \in \mathbb{N}\), then the map \(f\) can be written as
\[ f(z) = \left(1 - \frac{z}{z_m}\right) g(z) \prod_{n \neq m} \left(1 - \frac{z}{z_n}\right)^{E_n(z)}, \]
and its complex derivative at \(z_m\) is
\[ \frac{\partial F}{\partial z}(p) = \frac{\partial f}{\partial z}(z_m) = -\frac{1}{z_m} g(z_m) \prod_{n \neq m} \left(1 - \frac{z_m}{z_n}\right)^{E_n(z_m)} \neq 0. \]

From the Implicit Function Theorem exists a function \(g_n(w)\) holomorphic such that in an open \(U\) of \(p\), \(X\) is (locally) the graph \(z = g_n(w)\). Thus the restriction of the projection \(\pi_{w|U} : U \to \mathbb{C}\), such that \((z, w) \mapsto w\), is a homeomorphism from \(U\) to its image \(V\), which is an open subset of \(\mathbb{C}\) does not has points \(z_k, k \in \mathbb{N}\) and \(k \neq n\). The couple \((\pi_{w|U}, U)\) is a chart around of the point \((z_n, 0)\). We note that \(\pi_{w|U}^{-1} : V \to U\) is define as follows \(w \mapsto (g_n(w), w)\).

The collection of these charts defines an complex atlas \(\mathcal{A}\) on \(S(f)\), because if the couples \((\varphi, U)\) and \((\psi, W)\) are charts satisfying \(U \cap U \neq \emptyset\), then the map
\[ \psi \circ \varphi^{-1}|_{\varphi(U \cap W)} : \varphi(U \cap W) \subset \mathbb{C} \to \mathbb{C} \]
is locally a restriction of either: the identity, \(h(z)\), or \(g_n(z)\), for any \(n \in \mathbb{N}\). Hence, we consider the maximal complex atlas contained the complex atlas \(\mathcal{A}\) and endowed to \(S(f)\) with a complex structure.
The Riemann surface $S(f)$ is connected. We shall prove that $S(f)$ is path-connected. The sequence $(z_n)_{n \in \mathbb{N}}$ is ordered as follows $|z_n| < |z_{n+1}|$ and, if $|z_n| = |z_{n+1}|$ then $\arg(z_n) < \arg(z_{n+1})$. As the complex plane $\mathbb{C}$ is path-connected, we can consider an injective path

\begin{equation}
\gamma : [0, \infty) \to \mathbb{C},
\end{equation}

such that there are real numbers $0 = x_1 < x_2 < \ldots$ satisfying $\gamma(x_n) = z_n$, for each $n \in \mathbb{N}$. The inverse image $G(\gamma) := \pi^{-1}_z(\gamma(0, \infty))$ is called a spine of $S(f)$ associated to $\gamma$, and we claim that the spine $G(\gamma)$ is a closed and connected subset of $S(f)$. Given that $\pi_z : S(f) \to \mathbb{C}$ is an branched covering (Remark 4.3 (1)), if we take a point $(z_0, w_0) \in S(f)$ and a path $\beta$ in $\mathbb{C} \setminus \{z_n : n \in \mathbb{N}\}$, such that one of its end points is $z_0$ and the other one end point is belonged to $\gamma(0, \infty)$, then there exists $\tilde{\beta}$ a lifting path in $S(f)$ of $\beta$, such that one of its end points is $(z_0, w_0)$ and the other one is belonged to the spine $G(\gamma)$. This is shows that $S(f)$ is path-connected.

The surface $S(f)$ is homeomorphic to the Loch Ness monster. We must prove that $S(f)$ has only one end and infinite genus.

The unique end of $S(f)$. Given a compact subset $K \subset S(f)$ we shall prove that there is a compact subset $K' \subset S(f)$ such that $K \subset K'$ and $S(f) \setminus K'$ is connected. The image $\pi_z(K)$ is a compact subset of $\mathbb{C}$. As the complex plane has only one end, then there exists a real number $r > 0$, such that the closed ball $\overline{B}_r(0)$ contains to the compact $\pi_z(K)$ and $\mathbb{C} \setminus \overline{B}_r(0)$ is connected.

We can suppose without loss of generality that there exists $N \in \mathbb{N}$ such that the intersection $B_r(0) \cap \{z_n : n \in \mathbb{N}\} = \{z_1, \ldots, z_{N-1}\}$. Moreover, the intersection $\overline{B}_r(0) \cap \gamma(x_N, \infty) = \emptyset$ (see equation (5)).

The inverse image $G := \pi^{-1}_z(\gamma(x_N, \infty))$ is a closed connected subset of the spine $G(\gamma)$, and given that the projection map $\pi_z$ is a proper map (Remark 4.3 (2)), the inverse image $K' := \pi^{-1}_z(\overline{B}_r(0))$ is a compact subset of $S(f)$. In addition, $K \subset K'$ and $G \cap K' = \emptyset$. Using the same ideas described in the connected case of $S(f)$, for each point $(z_0, w_0) \in S(f) \setminus K'$ we can found a path $\beta$ in $S(f) \setminus K'$ having as one of its end points $(z_0, w_0)$ and the other end point belonged to $G$. Hence, $S(f) \setminus K'$ is path-connected.

The surface $S(f)$ has infinite genus. For each $g \in \mathbb{N}$ let $S_g$ be the compact Riemann surface with genus $g$ associated to the algebraic curve

\begin{equation}
v^2 = \prod_{k=1}^{2g+2}(u - z_k).
\end{equation}

As the complex plane $\mathbb{C}$ is an $\sigma$-compact space, we can take an increasing sequence of compact connected subsets $K_1 \subset K_2 \subset \ldots$ such that $\mathbb{C} = \bigcup_{g \in \mathbb{N}} K_g$ and each $K_g$ just contains the points $\{z_1, \ldots, z_{2g+2}\}$ of the sequence $(z_n)_{n \in \mathbb{N}}$. By the Riemann-Hurwitz formula the subsurface

\begin{equation}
S_g(K_g) := \left\{(u,v) \in \mathbb{C}^2 : v^2 = \prod_{k=1}^{2g+2}(u - z_k) \text{ and, } u \in K_g\right\} \subset S_g,
\end{equation}

is homeomorphic to a $g$ torus with open disk.

Now, we define an embedding map $\varphi$ from $S_g(K_g)$ to $S(f)$ as follows: we fix a point $\bar{z} \in K - \{z_1, \ldots, z_{2g+2}\}$, then there are points $p_1$ and $p_2$ in the inverse image $\pi^{-1}_z(\bar{z}) \subset S_g(K_g)$. Similarly, there are points $q_1$ and $q_2$ in the inverse image $\pi^{-1}_z(\bar{z})$. Then we define

\begin{equation}
\varphi(p_i) = q_i, \text{ for each } i \in \{1, 2\}.
\end{equation}
For each point \( s \) in \( S_g(K_g) \), we take a path \( \gamma \) in \( K - \{ z_1, \ldots, z_{2g+1} \} \) having ends points \( \bar{z} \) and \( \pi_\gamma(s) \). Then there exists a lifting \( \gamma_1 \) of \( \gamma \) in \( S_g(K_g) \) such that its ends point are \( s \) and \( p_i \), for any \( i \in \{ 1, 2 \} \). Similarly, there exists a lifting \( \gamma_2 \) of \( \gamma \) in \( S(f) \) such that its ends points are \( q_i \) and \( t \). Hence, we define \( \varphi(s) = t \). By construction \( \varphi \) is a well define injective map. We remark that \( \pi_\gamma \circ \varphi(s) = \pi_\varphi(s) \), for each \( s \in S_g(K_g) \).

Next, we must prove that the map \( \varphi \) is continuous. Let \( s \) be a point of \( S_g(K_g) \) and let \( U \) be an open subset of \( \varphi(s) \in S(f) \). As the projection map \( \pi_\gamma \) is an open map, then \( \pi_\gamma(U) \) is an open subset of \( K_g \) containing the point \( \pi_\varphi(s) \). By the continuously of \( \pi_\gamma \), there exists an open \( V \) of \( s \) such that \( \pi_\gamma(V) \subset \pi_\gamma(U) \). It is easily shown that \( \varphi(V) \subset U \). The map \( \varphi \) closed because is a continuous map from a compact space into a Hausdorff space (see [Dug78, Theorem 2.1]). Hence \( f \) is an embedding. This fact implies that for each \( g \in \mathbb{N} \), there exists a subsurface of \( S(f) \) having genus \( g \).

**Moduli space of the infinite hyperelliptic curves.** Let \( (z_n)_{n \in \mathbb{N}} \) and \( (z'_n)_{n \in \mathbb{N}} \) be sequences of complex number such that they are "ordered" and \( \lim_{n \to \infty} |z_n| = \lim_{n \to \infty} |z'_n| \). Let \( f \) and \( g \) be the entire maps from Weierstrass’s theorem having as simple zeros the points \( z_1, z_2, \ldots \) and \( z'_1, z'_2, \ldots \), respectively.

**Theorem 4.3.** The Loch Ness monster \( S(f) \) and \( S(g) \) are biholomorphic equivalent if and only if there exists a biholomorphic \( T \) from \( \mathbb{C} \) to itself such that maps the sequence \( (z_n)_{n \in \mathbb{N}} \) onto the sequence \( (z'_n)_{n \in \mathbb{N}} \).

**Proof.** We shall prove the sufficiency. We will define \( T \) a lifting map of \( T \), which will be a biholomorphic from \( S(f) \) to \( S(g) \). The map \( \tilde{T} : S(f) \to S(g) \) is defined as follows, \( \tilde{T} \) sends the point \( z_n \) to \( z'_n \), for each \( n \in \mathbb{N} \). Now, we fix a point \( z_0 \in \mathbb{C} \) disjoint to the sequence \( (z_n)_{n \in \mathbb{N}} \). Let \( p_1 \) and \( p_2 \) be points in \( S(f) \) and, let \( q_1 \) and \( q_2 \) be points in \( S(g) \) such that \( \pi_\gamma(p_i) = z_0 \) and \( \pi_\gamma(q_i) = T(z_0) \), for each \( i \in \{1, 2\} \). The map \( \tilde{T} \) maps \( p_i \) to \( q_i \), for each \( i \in \{1, 2\} \). Given an unbranched point \( s \) in \( S(f) \), we consider a path \( \gamma \) in \( \mathbb{C} \) having ends points \( z_0 \) and \( \pi_\gamma(s) \), which does not pass over the points of the sequence \( (z_n)_{n \in \mathbb{N}} \), then there exists a lifting path \( \gamma \) of \( \gamma \) in \( S(f) \) with ends points \( s \) and \( p_j \), for any \( j \in \{1, 2\} \). For the path \( \gamma \) there is a lifting path \( \gamma' \) in \( S(g) \) having ends points \( q_j \) and \( t \), where \( t \) is a point belong to the fiber \( \pi_\gamma^{-1}(z_0) \subset S(g) \). The map \( \tilde{T} \) sends the point \( s \) to \( t \). By construction \( \tilde{T} \) is a bijective continuous map. In addition, from the Monodromy Principle follows that the map \( \tilde{T} \) and its inverse \( \tilde{T}^{-1} \) have as locally presentation the restriction of a holomorphic map.

We will show the necessary. Given the charts \( (U, \varphi) \) and \( (V, \psi) \) around the unbranch point \( p \in S(f) \) and the point \( \tilde{T}(p) \in S(g) \), respectively, from the Monodromy Principle the biholomorphic map

\[
\phi \circ \tilde{T} \circ \varphi^{-1} : \varphi(U \cap \varphi^{-1}(V)) \subset \mathbb{C} \to \psi(V \cap \varphi(U)) \subset \mathbb{C}
\]

can be extended to a biholomorphic map \( T \) from \( \mathbb{C} \) to \( \mathbb{C} \), such that \( T \) maps the sequence \( (z_n)_{n \in \mathbb{N}} \) onto the sequence \( (z'_n)_{n \in \mathbb{N}} \). □

**Example 4.2.** Given that the zeros of the sine and the cosine complex maps,

\[
f(z) = \sin(\pi z) = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) \quad \text{and} \quad g(z) = \cos(\pi z) = \prod_{n=1}^{\infty} \left(1 - \frac{4z^2}{(2n-1)^2}\right)
\]

differ by a translation, then the infinite hyperelliptic curves

\[
S(f) = \{(z, w) \in \mathbb{C}^2 : w^2 = f(z)\} \quad \text{and} \quad S(g) = \{(z, w) \in \mathbb{C}^2 : w^2 = g(z)\}
\]

are topologically equivalent to the Loch Ness monster.
The space $\ell^\infty$ conformed by all bounded sequence of complex numbers $(z_n)_{n\in\mathbb{N}}$ (equipped with the norm $\|z_n\|_{\ell^\infty} = \sup\{|z_n| < \infty\}$ is completed, and the set $c_0$ conformed by all sequences of complex number with limit 0 is a closed subset of $\ell^\infty$. The set $c_\infty$ conformed by all sequence of complex numbers $(z_n)_{n\in\mathbb{N}}$ such that $\lim |z_n| = \infty$ can be identified with $c_0$ using the map $z \mapsto \frac{1}{z}$. So two infinite hyperelliptic curves $S(f)$ and $S(g)$, such that the sequences $(z_n)_{n\in\mathbb{N}}$ and $(z'_n)_{n\in\mathbb{N}}$ are the simple zeros of $f$ and $g$ respectively, are biholomorphic if there exists affine map $z \mapsto az + b$ with $a, b \in \mathbb{C}$ and $a \neq 0$ such that $z_n \mapsto z'_n$, for each $n \in \mathbb{N}$. Then, by then above, the group $G = \{w \mapsto \frac{w}{bw + a} : a, b \in \mathbb{C}, a \neq 0\}$ acts on $c_0$ by $(w_n)_{n\in\mathbb{N}} \mapsto (\frac{w_n}{bw_n + a})_{n\in\mathbb{N}}$ with $w_n = \frac{1}{z_n}$. Thus, we have obtained the following fact.

Theorem 4.4. The moduli space of infinite hyperelliptic curves is given by $c_0/G$.

Infinite $n$-gonal curves. Let $(z_i)_{i\in\mathbb{N}}$ be a sequence of different complex numbers such that $\lim |z_i| = \infty$, let $(m_i)_{i\in\mathbb{N}}$ be a sequence of positive integer and let $f$ be an entire function having the points $z_1, z_2, \ldots$ as their zeros of order $m_i$ at $z_i$, for each $i \in \mathbb{N}$. We suppose that there exists a positive integer $n$ satisfying $1 \leq m_i < n$, for each $i \in \mathbb{N}$, then the affine plane curve

$$S(f) = \{(z, w) \in \mathbb{C}^2 : w^n = f(z)\}$$

is called an Infinite $n$-gonal curve. If $m_i = 1$ for all $i$, the curve $S(f)$ is called infinite superelliptic curve.

Using the same ideas described in the proof of the Theorem 4.2, we obtain the following.

Theorem 4.5. An infinite superelliptic curve and an infinite $n$-gonal curve $S(f)$ for $n$ prime is a connected Riemann surface homeomorphic to the Loch Ness monster.
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