DOLBEAULT COHOMOLOGY OF A LOOP SPACE

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Abstract. The loop space $L\mathbb{P}_1$ of the Riemann sphere is an infinite dimensional complex manifold consisting of maps (loops) $S^1 \to \mathbb{P}_1$ in some fixed $C^k$ or Sobolev $W^{k,p}$ space. In this paper we compute the Dolbeault cohomology groups $H^{0,1}(L\mathbb{P}_1)$.

0. Introduction

Loop spaces $LM$ of compact complex manifolds $M$ promise to have rich analytic cohomology theories, and it is expected that sheaf and Dolbeault cohomology groups of $LM$ will shed new light on the complex geometry and analysis of $M$ itself. This idea first occurs in [W], in the context of the infinite dimensional Dirac operator, and then in [HBJ] that touches upon Dolbeault groups of loop spaces; but in all this both works stay heuristic. Our goal here is rigorously to compute the $H^{0,1}$ Dolbeault group of the first interesting loop space, that of the Riemann sphere $\mathbb{P}_1$. The consideration of $H^{0,1}(L\mathbb{P}_1)$ was directly motivated by [MZ], that among other things features a curious line bundle on $L\mathbb{P}_1$. More recently, the second named author in [Z] classified all holomorphic line bundles on $L\mathbb{P}_1$ that are invariant under a certain group of holomorphic automorphisms of $L\mathbb{P}_1$—a problem closely related to describing (a certain subspace of) $H^{0,1}(L\mathbb{P}_1)$. One noteworthy fact that emerges from the present research is that analytic cohomology of loop spaces, unlike topological cohomology (cf. [P, Theorem 13.14]), is rather sensitive to the regularity of loops admitted in the space. Another concerns local functionals, a notion from theoretical physics. Roughly, if $M$ is a manifold, a local functional on a space of loops $x: S^1 \to M$ is one of form

$$f(x) = \int_{S^1} \Phi(t, x(t), \dot{x}(t), \ddot{x}(t), \ldots) dt,$$

where $\Phi$ is a function on $S^1 \times \text{an appropriate jet bundle of } M$. It turns out that all cohomology classes in $H^{0,1}(L\mathbb{P}_1)$ are given by local functionals. Nonlocal cohomology classes exist only perturbatively, i.e., in a neighborhood of constant loops in $L\mathbb{P}_1$; but none of them extends to the whole of $L\mathbb{P}_1$.

We fix a smoothness class $C^k$, $k = 1, 2, \ldots, \infty$, or Sobolev $W^{k,p}$, $k = 1, 2, \ldots, 1 \leq p < \infty$. If $M$ is a finite dimensional complex manifold, consider the space $LM = L_k M$, resp. $L_{k,p} M$ of maps $S^1 = \mathbb{R}/\mathbb{Z} \to M$ of the given regularity. These spaces are complex manifolds modeled on a Banach space, except for $L_\infty M$, which is modeled on a Fréchet space. We shall focus on the loop space(s) $L\mathbb{P}_1$. As on any complex manifold, one can consider the

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space $C_{r,q}^\infty(L\mathbb{P}_1)$ of smooth $(r, q)$ forms, the operators $\bar{\partial} : C_{r,q}^\infty(L\mathbb{P}_1) \to C_{r,q+1}^\infty(L\mathbb{P}_1)$, and the associated Dolbeault groups $H^{r,q}(L\mathbb{P}_1)$; for all this, see e.g. [L1,2]. On the other hand, let $\mathfrak{F}$ be the space of holomorphic functions $F : \mathbb{C} \times L\mathbb{C} \to \mathbb{C}$ that have the following properties:

1. $F(\zeta/\lambda, \lambda^2 y) = O(\lambda^2)$, as $\mathbb{C} \ni \lambda \to 0$;
2. $F(\zeta, x + y) = F(\zeta, x) + F(\zeta, y)$, if $\text{supp } x \cap \text{supp } y = \emptyset$;
3. $F(\zeta, y + \text{const}) = F(\zeta, y)$.

As we shall see, the additivity property (2) implies $F(\zeta, y)$ is local in $y$.

**Theorem 0.1.** $H^{0,1}(L\mathbb{P}_1) \approx \mathbb{C} \oplus \mathfrak{F}$.

In the case of $L_\infty\mathbb{P}_1$, examples of $F \in \mathfrak{F}$ are

$$F(\zeta, y) = \zeta^\nu \left\langle \Phi, \prod_{j=0}^m y^{(d_j)} \right\rangle,$$

where $\Phi$ is a distribution on $S^1$, $y^{(d)}$ denotes $d$’th derivative, each $d_j \geq d_0 = 1$, and $0 \leq \nu \leq 2m$. A general function in $\mathfrak{F}$ can be approximated by linear combinations of functions of form (0.1), see Theorem 1.5.

On any, possibly infinite dimensional complex manifold $X$ the space $C_{r,q}^\infty(X)$ can be given the compact–$C^\infty$ topology as follows. First, the compact–open topology on $C_{0,0}^\infty(X) = C^\infty(X)$ is generated by $C^0$ seminorms $\|f\|_K = \sup_K |f|$ for all $K \subset X$ compact. The family of $C^\nu$ seminorms is defined inductively: each $C^{\nu-1}$ seminorm $\|\cdot\|$ on $C^\infty(TX)$ induces a $C^\nu$ seminorm $\|f\|' = \|df\|$ on $C^\infty(X)$. The collection of all $C^\nu$ seminorms, $\nu = 0, 1, \ldots$, defines the compact–$C^\infty$ topology on $C^\infty(X)$. The compact–$C^\infty$ topology on a general $C_{r,q}^\infty(X)$ is induced by the embedding $C_{r,q}^\infty(X) \subset C^\infty(X \oplus TX)$. With this topology $C_{r,q}^\infty(X)$ is a separated locally convex vector space, complete if $X$ is first countable. The quotient space $H^{r,q}(X)$ inherits a locally convex topology, not necessarily separated. We note that on the subspace $\mathcal{O}(X) \subset C^\infty(X)$ of holomorphic functions the compact–$C^\infty$ topology restricts to the compact–open topology. The isomorphism in Theorem 0.1 is topological; it is also equivariant with respect to the obvious actions of the group of $C^k$ diffeomorphisms of $S^1$.

There is another group, the group $G \approx \text{PSL}(2, \mathbb{C})$ of holomorphic automorphisms of $\mathbb{P}_1$, whose holomorphic action on $L\mathbb{P}_1$ (by post–composition) and on $H^{0,1}(L\mathbb{P}_1)$ will be of greater concern to us. Theorems 0.2, 0.3, 0.4 below will describe the structure of $H^{0,1}(L\mathbb{P}_1)$ as a $G$–module. Recall that any irreducible (always holomorphic) $G$–module is isomorphic, for some $n = 0, 1, \ldots$, to the space $\mathfrak{H}_n$ of holomorphic differentials $\psi(\zeta)(d\zeta)^{-n}$ of order $-n$ on $\mathbb{P}_1$; here $\psi$ is a polynomial, $\deg \psi \leq 2n$, and $G$ acts by pullback. (For this, see [BD, pp. 84-86], and note that the subgroup $\approx \text{SO}(3)$ formed by $g \in G$ that preserve the Fubini–Study metric is a maximally real submanifold; hence the holomorphic representation theory of $G$ agrees with the representation theory of $\text{SO}(3)$.) The $n$’th isotypical subspace of a $G$–module $V$ is the sum of all irreducible submodules isomorphic to $\mathfrak{H}_n$. In particular, the $0$’th isotypical subspace is the space $V^G$ of fixed vectors.
Theorem 0.2. If $n \geq 1$, the $n$‘th isotypical subspace of $H^{0,1}(L_\infty \mathbb{P}^1)$ is isomorphic to the space $\mathfrak{F}^n$ spanned by functions of form (0.1), with $m = n$.

The isomorphism above is that of locally convex spaces, as $\mathfrak{F}$ or $\mathfrak{F}^n$ have not been endowed with an action of $G$ yet. But in Section 2 they will be, and we shall see that the isomorphism in question is a $G$–morphism. — The fixed subspace of $H^{0,1}(L\mathbb{P}^1)$ can be described more explicitly, for any loop space:

Theorem 0.3. The space $H^{0,1}(L\mathbb{P}^1)^G$ is isomorphic to $C^{k-1}(S^1)^*$, resp. $W^{k-1,p}(S^1)^*$, if the dual spaces are endowed with the compact–open topology.

The isomorphisms in Theorem 0.3 are not Diff $S^1$ equivariant. To remedy this, one is led to introduce the spaces $C^l_r(S^1)$, resp. $W^{1,p}_r(S^1)$ of differentials $y(t)(dt)^r$ of order $r$ on $S^1$, of the corresponding regularity; $L^p_r = W^{0,p}_r$. Then $H^{0,1}(L\mathbb{P}^1)^G$ will be Diff $S^1$ equivariantly isomorphic to $C^{k-1}_{\infty}(S^1)^*$, resp. $W^{k-1,p}(S^1)^*$.

For low regularity loop spaces one can very concretely represent all of $H^{0,1}(L\mathbb{P}^1)$:

Theorem 0.4. (a) If $1 \leq p < 2$, all of $H^{0,1}(L_{1,p}\mathbb{P}^1)$ is fixed by $G$, hence it is isomorphic to $L^{p'}(S^1)$, with $p' = p/(p - 1)$.

(b) If $1 \leq p < \infty$ then $H^{0,1}(L_{1,p}\mathbb{P}^1)$ is isomorphic to

$$\bigoplus_{0 \leq n \leq p-1} \mathfrak{R}_n \otimes L^{p/(n+1)}_{n+1}(S^1)^* \approx \bigoplus_{0 \leq n \leq p-1} \mathfrak{R}_n \otimes L^{p_n}_{-n}(S^1), \quad p_n = \frac{p}{p - 1 - n},$$

and so it is the sum of its first $[p]$ isotypical subspaces. Indeed, the isomorphisms above are $G \times \text{Diff } S^1$ equivariant, $G$, resp. $\text{Diff } S^1$ acting on one of the factors $\mathfrak{R}_n$, $L^p_n$ naturally, and trivially on the other.

Again, the dual spaces are endowed with the compact–open topology.

It follows that the infinite dimensional space $H^{0,1}(L_{1,p}\mathbb{P}^1)$ can be understood in finite terms, if it is considered as a representation space of $S^1$. Here $S^1$ acts on itself (by translations), hence also on $L\mathbb{P}^1$ and on $H^{0,1}(L\mathbb{P}^1)$. One can read off from Theorem 0.4 that each irreducible representation of $S^1$ occurs in $H^{0,1}(L_{1,p}\mathbb{P}^1)$ with the same multiplicity $[p]^2$. On the other hand, for spaces of loops of regularity at least $C^1$, in $H^{0,1}(L\mathbb{P}^1)$ each irreducible representation of $S^1$ occurs with infinite multiplicity and, somewhat contrary to earlier expectations, it is not possible to associate with this cohomology space even a formal character of $S^1$. This indicates that Dolbeault groups of general loop spaces $LM$ should be studied as representations of Diff $S^1$ rather than $S^1$.

The structure of this paper is as follows. In Sections 1 and 2 we study the space $\mathfrak{F}$ as a $G$–module. We connect it with a similar but simpler space of functions that are required to satisfy only the first two of the three conditions defining $\mathfrak{F}$ (Theorem 1.1). Theorem 1.1 will be needed in proving the isomorphism $H^{0,1}(L\mathbb{P}^1) \approx \mathbb{C} \oplus \mathfrak{F}$, and also in concretely representing elements of $\mathfrak{F}$. Further, we shall rely on Theorem 1.1 in identifying isotypical
subspaces of $\mathfrak{F}$ (Theorems 2.1, 2.2). This will then prove Theorems 0.2, 0.3, and 0.4, modulo Theorem 0.1.

In Section 3 we introduce a $G$-module $\mathfrak{F}$ of holomorphic Čech cocycles of $L\mathbb{P}_1$, and prove $H^{0,1}(L\mathbb{P}_1) \approx \mathfrak{F}$ (Theorem 3.3). In Section 4 we construct a morphism $\alpha: \mathfrak{F} \to \mathfrak{F}$ that, in Section 5, is shown to induce an isomorphism $\mathfrak{F}/\text{Ker } \alpha \to \mathfrak{F}$. Also, $\dim \text{Ker } \alpha = 1$ (Theorem 5.1). Finally, in Section 6 we show how all this, put together, proves the results formulated in this introduction.

1. THE SPACE $\mathfrak{F}$

In this Section and the next we shall study the structure of the space $\mathfrak{F}$, independently of any cohomological content. It will be convenient to allow (but only in this Section!) $k$ to be any integer; when $k < 0$, elements of $C^k(S^1)$, $W^{k,p}(S^1)$ are distributions, locally equal to the $-k$th derivative of functions in $C(S^1)$, $L^p(S^1)$. Let $L^{-}\mathbb{C}$ denote the space $C^{k-1}(S^1)$, resp. $W^{k-1,p}(S^1)$. We shall write $L^{(-)}\mathbb{C}$ to mean either $L\mathbb{C}$ or $L^{-}\mathbb{C}$. Consider the space $\mathfrak{F}$ of those $F \in \mathcal{O}(\mathbb{C} \times L^{-}\mathbb{C})$ that have properties (1) and (2) of the Introduction. We shall refer to (2) as additivity. A function $F \in \mathcal{O}(\mathbb{C} \times L^{-}\mathbb{C})$ will be said to be posthomogeneous of degree $m$ if $F(\zeta, \cdot)$ is homogeneous of degree $m$ for all $\zeta \in \mathbb{C}$. Posthomogeneous degree endows the spaces $\mathfrak{F}$ and $\mathfrak{F}$ with a grading.—All maps below, unless otherwise mentioned, will be continuous and linear.

**Theorem 1.1.** The graded linear map $\tilde{\mathfrak{F}} \ni \tilde{F} \mapsto F \in \mathfrak{F}$ given by $F(\zeta, y) = \tilde{F}(\zeta, \dot{y})$ has a graded right inverse, and its kernel consists of functions $\tilde{F}(\zeta, x) = \text{const } \int_{S^1} x$.

First we shall consider functions $E \in \mathfrak{F}$, resp. $\mathfrak{F}$, that are independent of $\zeta$. We denote the space of these functions $\mathfrak{E} \subset \mathcal{O}(L\mathbb{C})$, resp. $\mathfrak{E} \subset \mathcal{O}(L^{-}\mathbb{C})$, graded by degree of homogeneity. Additivity of $E \in \mathcal{O}(L^{-}\mathbb{C})$ implies $E(0) = 0$, which in turn implies property (1) of the Introduction. Let

$$E = \sum_{1}^{\infty} E_{m}, \quad E_{m}(y) = \int_{0}^{1} E(e^{2\pi i \tau} y) e^{-2m\pi i \tau} d\tau$$

be the homogeneous expansion of a general $E \in \mathcal{O}(L^{-}\mathbb{C})$ vanishing at 0. Consider tensor powers $(L^{-}\mathbb{C})^\otimes_m$ of the vector spaces $L^{-}\mathbb{C}$ over $\mathbb{C}$. In particular, $C^{\infty}(S^1)^\otimes_m$ is an algebra, and a general $(L^{-}\mathbb{C})^\otimes_m$ is a module over it. Each $E_{m}$ in (1.1) induces a symmetric linear map $E_{m}: (L^{-}\mathbb{C})^\otimes_m \to \mathbb{C}$, called the polarization of $E_{m}$. On monomials $E_{m}$ is defined by

$$E_{m}(y_1 \otimes \ldots \otimes y_m) = \frac{1}{2^m m!} \sum_{\epsilon_{j}=\pm 1} \epsilon_1 \ldots \epsilon_mE_{m}(\epsilon_1 y_1 + \ldots + \epsilon_m y_m),$$

and then extended by linearity. Thus $E_{m}(y) = E_{m}(y^\otimes_m)$. We shall call $w \in (L^{-}\mathbb{C})^\otimes_m$ degenerate if it is a linear combination of monomials $y_1 \otimes \ldots \otimes y_m$ with one $y_j = 1$. 
Lemma 1.2. (a) $E$ is additive if and only if $E_m(y_1 \otimes \ldots \otimes y_m) = 0$ whenever $\bigcap_1^m \text{supp } y_j = \emptyset$.

(b) $E(y + \text{const}) = E(y)$ if and only if $E_m(w) = 0$ whenever $w$ is degenerate.

Proof. (a) Clearly $E$ is additive precisely when all the $E_m$ are, whence it suffices to prove the claim when $E$ itself is homogeneous, of degree $m$, say. In this case $E_n = 0$, $n \neq m$. Denoting $E_m$ by $E$, it is also clear that the condition on $E$ implies $E$ is additive. We show the converse by induction on $m$, the case $m = 1$ being obvious. Let $x, y \in L^{(-)} \mathbb{C}$ have disjoint support, so that

\[(1.3) \quad E((x + y)^{\otimes m}) = E(x^{\otimes m}) + E(y^{\otimes m}).\]

Write $\lambda x$ for $x$ and separate terms of different degrees in $\lambda$ to find $E(x \otimes \ldots \otimes y) = 0$, which settles the case $m = 2$. Now if the case $m - 1 \geq 2$ is already settled, take a $z \in L^{(-)} \mathbb{C}$ with $\text{supp } y \cap \text{supp } z = \emptyset$, and write $x + \lambda z$ for $x$ in (1.3). Considering the terms linear in $\lambda$ we obtain

\[(1.4) \quad E(z \otimes (x + y)^{\otimes m-1}) = E(z \otimes x^{m-1}) + E(z \otimes y^{m-1}),\]

the last term being 0. The same will hold if $\text{supp } x \cap \text{supp } z = \emptyset$. Since any $z \in L^{(-)} \mathbb{C}$ can be written $z' + z''$ with the support of $z'$ (resp. $z''$) disjoint from the support of $x$ (resp. $y$), (1.4) in fact holds for all $z$. By the induction hypothesis applied to $E(z \otimes \cdot)$

\[
E(z \otimes y_2 \otimes \ldots \otimes y_m) = 0, \quad \text{if } \bigcap_2^m \text{supp } y_j = \emptyset.
\]

Suppose now $\bigcap_1^m \text{supp } y_j = \emptyset$ and write $y_1 = y' + y''$ with $y' = 0$ near $\bigcap_{j \neq 2} \text{supp } y_j$ and $y'' = 0$ near $\bigcap_{j \neq 3} \text{supp } y_j$. Then

\[
E(y_1 \otimes \ldots \otimes y_m) = E(y' \otimes \ldots \otimes y_m) + E(y'' \otimes \ldots \otimes y_m) = 0.
\]

(b) Again we assume $E$ is $m$–homogeneous, and again one implication is trivial. So assume $E((y+1)^{\otimes m}) = E(y^{\otimes m})$, where $E = E_m$. Differentiating both sides in the directions $y_2, \ldots, y_m$ and setting $y = 0$ we obtain $E(1 \otimes y_2 \otimes \ldots \otimes y_m) = 0$, whence the claim follows.

Proposition 1.3. The graded map $\tilde{E} \ni \tilde{E} \mapsto E \in \mathfrak{E}$ given by $E(y) = \tilde{E}(y)$ has a graded right inverse, and its kernel is spanned by $\tilde{E}(x) = \int_{S^1} x$.

We shall write $\int x$ for $\int_{S^1} x$.

Proof. (a) To identify the kernel, because of homogeneous expansions, it will suffice to deal with homogeneous $\tilde{E}$. So assume $\tilde{E} \in \tilde{\mathfrak{E}}$ is homogeneous of degree $m$ and $\tilde{E}(\cdot)$ is homogeneous of degree $m$ and $\tilde{E}(\cdot) = 0$ for all $y \in L \mathbb{C}$. Its polarization $\tilde{E}$ satisfies $\tilde{E}(\cdot) = 0$. If $m = 1$, this implies
\[ \tilde{E}(x) = \text{const} \int x, \] so from now on we assume \( m \geq 2 \), and first we prove by induction that \( \tilde{E}(x_1 \otimes \ldots \otimes x_m) = \text{const} \prod \int x_j \). Suppose we already know this for \( m - 1 \). Then

\[ \tilde{E}(y \otimes x_2 \otimes \ldots \otimes x_m) = c(y) \prod_{2}^{m} \int x_j. \]

With arbitrary \( x_1 \in L^{-} \mathbb{C} \) the function \( x_1 - \int x_1 \) is of form \( \dot{y} \), so \( x_1 = \dot{y} + \int x_1 \), and

\[ \tilde{E}(x_1 \otimes \ldots \otimes x_m) = l(x_1) \prod_{1}^{m} \int x_j + \tilde{E}(1 \otimes x_2 \otimes \ldots \otimes x_m) \int x_1, \] \( (1.5) \)

where \( l(x_1) = c(x_1 - \int x_1) \) is linear in \( x_1 \). If \( \int x_1 = 0 \) and \( \text{supp } x_1 \neq S^1 \), then we can choose \( x_2, \ldots \) so that \( \bigcap_{1}^{m} \text{supp } x_j = \emptyset \) but \( \int x_j \neq 0, \ j \geq 2 \). This makes the left hand side of \( (1.5) \) vanish by Lemma 1.2a, and gives \( l(x_1) = 0 \). Since any \( x_1 \in L^{-} \mathbb{C} \) with \( \int x_1 = 0 \) can be written \( x_1 = x' + x'' \) with \( \int x' = \int x'' = 0 \) and \( \text{supp } x', \text{supp } x'' \neq S^1 \), it follows that \( l(x_1) = 0 \) whenever \( \int x_1 = 0 \). Hence \( l(x_1) = \text{const} \int x_1 \). In particular, the first term on the right of \( (1.5) \) is symmetric in \( x_j \). Therefore the second term must be symmetric too, which implies this term is \( \text{const} \prod_{1}^{m} \int x_j \). Thus \( \tilde{E}(x) = \text{const}(\int x)^m \).

Yet for \( m \geq 2 \) \( \tilde{E}(x) = \text{const}(\int x)^m \) is additive only if it is identically zero; so that indeed \( \tilde{E}(x) = \text{const} \int x \), as claimed.

(b) To construct the right inverse, consider \( E \in \mathfrak{E} \) with homogeneous expansion \( (1.1) \).

We shall construct \( m \)-homogeneous polynomials \( \tilde{E}_m \in \mathfrak{E} \) such that \( E_m(y) = \tilde{E}_m(\dot{y}) \); the case \( m = 1 \) being obvious, we assume \( m \geq 2 \). Let us say that an \( n \)-tuple of functions \( \rho_{\nu} : S^1 \rightarrow \mathbb{C} \) is centered if \( \bigcap_{1}^{n} \text{supp } \rho_{\nu} \neq \emptyset \). We start by fixing a \( C^\infty \) partition of unity \( \sum_{\rho \in \mathcal{P}} \rho = 1 \) on \( S^1 \) such that each \( \text{supp } \rho \) is an arc of length \( < 1/4 \). This implies that \( \bigcup_{\rho \in \mathcal{P}} \text{supp } \rho \) is an arc of length \( < 1/2 \) if \( \rho_1, \ldots, \rho_n \in \mathcal{P} \) are centered. Given \( x \in L^{-} \mathbb{C} \), for each centered \( R = (\rho_1, \ldots, \rho_n) \) in \( P \) construct \( y_R \in L\mathbb{C} \) so that \( \dot{y}_R = x \) on a neighborhood of \( \bigcup_{\rho \in \mathcal{P}} \text{supp } \rho \), making sure that \( y_R = y_Q \) if \( Q \) and \( R \) agree as sets. For noncentered \( n \)-tuples \( R \) in \( P \) let \( y_R \in L\mathbb{C} \) be arbitrary. We shall refer to the \( y_R \) as local integrals.

If \( Q, R \) are centered tuples in \( P \) then

\[ y_Q - y_R = c_{QR} = \text{constant} \quad \text{on } (\bigcup_{\rho \in Q} \text{supp } \rho) \cap (\bigcup_{\rho \in R} \text{supp } \rho). \] \( (1.6) \)

When the intersection in \( (1.6) \) is empty, or \( Q \) or \( R \) are not centered, fix \( c_{QR} \in \mathbb{C} \) arbitrarily. Define

\[ v_{QR} = m \int_{0}^{c_{QR}} (y_R + \tau)^{m-1} d\tau \in (L\mathbb{C})^{\otimes m-1}, \] \( (1.7) \)

and with the polarization \( \mathcal{E}_m \) of \( E_m \) from \( (1.2) \) consider

\[ \mathcal{E}_m \left( \sum_{R = (\rho_1, \ldots, \rho_m)} (\rho_1 \otimes \ldots \otimes \rho_m)(y_R^{m} + 1) \otimes \sum_{S = (\sigma_2, \ldots, \sigma_m)} (\sigma_2 \otimes \ldots \otimes \sigma_m)v_{SR} \right); \] \( (1.8) \)
we sum over all \(m\)-tuples \(R\) and \((m-1)\)-tuples \(S\) in \(P\). (We will not need it, but here is an explanation of (1.8). Say that tensors \(w, w' \in L(\mathbb{C}^m)\) are congruent, \(w \equiv w'\), if \(w - w'\) is the sum of a degenerate tensor and of monomials \(x_1 \otimes \ldots \otimes x_m\) with \(\bigcap \text{supp } x_j = \emptyset\). Denote by \(\partial^m\) the linear map \((L\mathbb{C})^m \to (L\mathbb{C})^m\) defined by \(\partial^m(y_1 \otimes \ldots \otimes y_m) = \dot{y}_1 \otimes \ldots \otimes \dot{y}_m\). Then the symmetrization of the argument of \(E_m\) in (1.8) is a solution \(w\) of the congruence \(\partial^m w = x^m\); in fact it is the unique symmetric solution, up to congruence. It follows that for the \(\dot{E}_m\) sought, \(\dot{E}_m(x)\) must be equal to \(E_m(w)\), which, in turn, equals (1.8).

We claim that the value in (1.8) depends only on \(x\) (and \(E_m\)), but not on the partition of unity \(P\) and the local integrals \(y_R\). Indeed, suppose first that the local integrals \(y_R\) are changed to \(\dot{y}_R\), so that the \(c_{QR}\) change to \(\dot{c}_{QR}\) and \(v_{QR}\) to \(\dot{v}_{QR}\); but we do not change \(P\). There are \(c_R \in \mathbb{C}\) such that for all centered \(R\)

\[
\dot{y}_R = y_R + c_R \quad \text{on } \bigcup_{\rho \in R} \text{supp } \rho.
\]

Let

\[
(1.9) \quad u_R = m \int_0^{c_R} (y_R + \tau)^{\otimes m-1} d\tau.
\]

Clearly \(\dot{c}_{QR} = c_{QR} + c_Q - c_R\) if \(Q \cup R\) is centered. In this case one computes also

\[
(1.10) \quad \frac{1}{m} \dot{v}_{QR} = \int_0^{c_{QR}} (\dot{y}_R + \tau)^{\otimes m-1} d\tau
\]

\[
= \int_0^{c_R} (\dot{y}_R - c_R + \tau)^{\otimes m-1} d\tau - \int_0^{c_R} (\dot{y}_R - c_R + \tau)^{\otimes m-1} d\tau
\]

\[
+ \int_0^{c_Q} (\dot{y}_R - c_R + c_{QR} + \tau)^{\otimes m-1} d\tau.
\]

Because of Lemma 1.2a, in (1.8) only centered \(R\), and such \(S\) that \(R \cup S\) is centered, will contribute. When \(y^\otimes_R\) is changed to \(\dot{y}^\otimes_R\), the corresponding contributions change by

\[
\sum_R E_m \left( \int_0^{c_R} (\rho_1 \otimes \ldots \otimes \rho_m) \frac{d}{d\tau}(y_R + \tau)^{\otimes m} d\tau \right)
\]

\[
= \sum_R E_m \left( \int_0^{c_R} (\rho_1 \otimes \ldots \otimes \rho_m)(m \otimes (y_R + \tau)^{\otimes m-1}) d\tau \right)
\]

\[
= \sum_R E_m ((\rho_1 \otimes \ldots \otimes \rho_m)(1 \otimes u_R)).
\]

When \(v_{QR}\) is changed to \(\dot{v}_{QR}\), in view of (1.10), (1.6), and (1.9), the contribution of the terms in the double sum in (1.8) changes by

\[
E_m \left( (m\rho_1 \otimes \rho_2 \sigma_2 \otimes \ldots \otimes \rho_m \sigma_m) \left( \int_0^{c_S} (y_S + \tau)^{\otimes m-1} d\tau - \int_0^{c_R} (y_R + \tau)^{\otimes m-1} d\tau \right) \right)
\]

\[
= E_m ((\rho_1 \otimes \rho_2 \sigma_2 \otimes \ldots \otimes \rho_m \sigma_m)(1 \otimes u_S - 1 \otimes u_R)).
\]
The net change in (1.8) is therefore

$$\mathcal{E}_m \left( \sum_{R,S} (\rho_1 \otimes \rho_2 \sigma_2 \otimes \ldots \otimes \rho_m \sigma_m)(1 \otimes u_S) \right) =$$

$$\mathcal{E}_m \left( \sum_S (1 \otimes \sigma_2 \otimes \ldots \otimes \sigma_m)(1 \otimes u_S) \right) = 0$$

by Lemma 1.2b, as needed.

Now to pass from $P$ to another partition of unity $P'$, introduce $\Pi = \{ \rho \rho' : p \in P, \rho' \in P' \}$. One easily shows that $P$ and $\Pi$ give rise to the same value in (1.8), hence so do $P$ and $P'$. Therefore (1.8) indeed depends only on $x$, and we define $\tilde{E}_m(x)$ by this value. We proceed to check that $\tilde{E}_m$ has the required properties.

If $x = \hat{y}$ then all $y_R$ can be chosen $y$, and (1.8) gives $\tilde{E}_m(\hat{y}) = E_m(y)$. Next suppose $x', x'' \in L^{-\mathbb{C}}$ have disjoint support, and $x = x' + x''$. If the supports of all $\rho \in P$ are sufficiently small, then the local integrals $y'_R$, $y''_R$ of $x', x''$ can be chosen so that for each $R$ one of them is 0. Hence the local integrals $y_R = y'_R + y''_R$ of $x$ will satisfy $y_R \otimes^m = y'_R \otimes^m + y''_R \otimes^m$, whence $\tilde{E}_m(x) = \tilde{E}_m(x') + \tilde{E}_m(x'')$ follows.

To show that $\sum \tilde{E}_m$ is convergent and represents a holomorphic function, note that $\tilde{E}_m(x)$ is the sum of terms

$$\mathcal{E}_m(\rho_1 y_R \otimes \ldots \otimes \rho_m y_R) \quad \text{and}$$

$$\int_0^1 \mathcal{E}_m(\rho_1 c_{SR} \otimes \rho_2 \sigma_2 (y_R + c_{SR} \tau) \otimes \ldots \otimes \rho_m \sigma_m (y_R + c_{SR} \tau)) d\tau$$

(we have substituted $c_{QR\tau}$ for $\tau$ in (1.7)). Since $y_R \in L^1 \mathbb{C}$ and $c_{QR} \in \mathbb{C}$ can be chosen to depend on $x$ in a continuous linear way, each $\tilde{E}_m$ is a homogeneous polynomial of degree $m$. Furthermore, let $K \subset L^{-\mathbb{C}}$ be compact. For each $x \in K$, $m \in \mathbb{N}$, and $m$--tuples $Q,R$ in $P$ we can choose $y_R$ and $c_{QR}$ so that all the functions

$$\rho c_{QR}, \rho' (y_R + c_{QR} \tau),$$

$\rho, \rho' \in P$, $0 \leq \tau \leq 1$, belong to some compact $H \subset L^1 \mathbb{C}$. By passing to the balanced hull, it can be assumed $H$ is balanced. If $\lambda > 0$, (1.1) implies

$$\max_H |E_m| = \lambda^{-m} \max_{\lambda H} |E_m| \leq \lambda^{-m} \max_{\lambda H} |E| = A \lambda^{-m},$$

so that by (1.2)

$$|\mathcal{E}_m(z_1 \otimes \ldots \otimes z_m)| \leq A(m^m/m!) \lambda^{-m} \leq A(e/\lambda)^m,$$

if each $z_\mu \in H$. Thus each term in (1.11) satisfies this estimate. If $|P|$ denotes the cardinality of $P$, we obtain, in view of (1.8)

$$\max_K |\tilde{E}_m| \leq (|P|^m + m|P|^{2m-1}) A(e/\lambda)^m.$$
Choosing $|\lambda| > e|P|^2$ we conclude that $\sum \tilde{E}_m$ uniformly converges on $K$, and, $K$ being arbitrary, $\tilde{E} = \sum \tilde{E}_m$ is holomorphic. By what we have already proved for $\tilde{E}_m$, $\tilde{E} \in \tilde{\mathcal{E}}$, and $\tilde{E}(\dot{y}) = E(y)$. The above estimates also show that the map $E \to \tilde{E}$ is continuous and linear, which completes the proof of Proposition 1.3.

Now consider an $F \in \mathcal{O}(\mathbb{C} \times L^{-} \mathbb{C})$ and its posthomogeneous expansion

\begin{equation}
F = \sum_{0}^{\infty} F_m, \quad F_m(\zeta, y) = \int_{0}^{1} F(\zeta, e^{2\pi i y}) e^{-2m\pi i \tau} d\tau.
\end{equation}

**Proposition 1.4.** The function $F$ satisfies condition (1) of the Introduction if and only if each $F_m$ is a polynomial in $\zeta$, of degree $\leq 2m - 2$ (in particular, $F_0 = 0$).

**Proof.** As $F$ satisfies (1) precisely when each $F_m$ does, the statement is obvious.

**Proof of Theorem 1.1.** Apply Proposition 1.3 on each slice $\{\zeta\} \times L^{-} \mathbb{C}$. Accordingly, an $\tilde{F}$ in the kernel is posthomogeneous of degree 1, hence, by Proposition 1.4, independent of $\zeta$. Thus indeed $\tilde{F}(\zeta, x) = \text{const} \int x$. Further, the slicewise right inverse applied to $F \in \mathfrak{F}$ clearly produces an additive $\tilde{F} \in \mathcal{O}(\mathbb{C} \times L \mathbb{C})$. To see that $\tilde{F}$ also verifies condition (1) of the Introduction, expand $F$ in a posthomogeneous series

\begin{equation}
F(\zeta, y) = \sum_{m=1}^{\infty} F_m(\zeta, y) = \sum_{m=1}^{\infty} \sum_{\nu=0}^{2m-2} \zeta^\nu E_{m\nu}(y),
\end{equation}

by Proposition 1.4, so that

$$\tilde{F}(\zeta, x) = \sum_{m=1}^{\infty} \sum_{\nu=0}^{2m-2} \zeta^\nu \tilde{E}_{m\nu}(x),$$

with $\tilde{E}_{m\nu}$ $m$–homogeneous. Again by Proposition 1.4, $\tilde{F}$ verifies condition (1), and so $\in \tilde{\mathfrak{F}}$.

Theorem 1.1 can be used effectively to describe elements of the space $\mathfrak{F}$. With ulterior motives we switch notation $m = n + 1$, and consider a homogeneous polynomial $\tilde{E} \in \mathcal{O}(L^{-} \mathbb{C})$ of degree $n + 1 \geq 1$. Its polarization $\mathcal{E}$ defines a distribution $D$ on the torus $(S^1)^{n+1} = T$. Indeed, denote the coordinates on $T$ by $t_j \in \mathbb{R}/\mathbb{Z}$ and set

\begin{equation}
\langle D, \prod_{j=0}^{n} e^{2\pi i \nu_j t_j} \rangle = \mathcal{E}(x_0 \otimes \ldots \otimes x_n), \quad x_j(\tau) = e^{2\pi i \nu_j \tau}, \ \nu_j \in \mathbb{Z}.
\end{equation}

Since $\tilde{E}$ is continuous,

$$|\mathcal{E}(x_0 \otimes \ldots \otimes x_n)| \leq c \prod \|x_j\|_{C^q(S^1)} \quad \text{with some } c > 0 \text{ and } q \in \mathbb{N}.$$
Hence (1.14) can be estimated, in absolute value, by $e^{\sum_{j} (1 + |\nu_j|)^q}$, and it follows by Fourier expansion that $D$ extends to a unique linear form on $C^\infty(T)$. Clearly, $D$ is symmetric, i.e., invariant under permutation of the factors $S^1$ of $T$. Also,

\begin{equation}
E(x_0 \otimes \ldots \otimes x_n) = \langle D, x_0 \otimes \ldots \otimes x_n \rangle,
\end{equation}

if on the right $x_0 \otimes \ldots \otimes x_n$ is identified with the function $\prod x_j(t_j)$.

Assume now $\tilde{E} \in \tilde{\mathcal{E}}$. Lemma 1.2a implies $D$ is supported on the diagonal of $T$. The form of distributions supported on submanifolds is in general well understood; in the case at hand, e.g. [H, Theorem 2.3.5] gives that $D$ is a finite sum of distributions of form

$$C^\infty(T) \ni \rho \mapsto \left\langle \Psi, \partial^{\alpha_1 + \ldots + \alpha_n} \rho \right|_{\text{diag}}, \quad \alpha_j \geq 0,$$

where $\Psi$ is a distribution on the diagonal of $T$. In view of Theorem 1.1 and (1.12)–(1.13) we therefore proved

**Theorem 1.5.** The restriction of an $(n+1)$–posthomogeneous $F \in \mathfrak{F}$, resp. $\tilde{\mathfrak{F}}$, to $\mathbb{C} \times C^\infty(S^1)$ is a finite sum of functions of form

$$f(\zeta, y) = \zeta^\nu \left\langle \Phi, \prod g^{(d_j)} \right\rangle, \quad \nu \leq 2n, \quad d_j \geq d_0 = 1, \quad \text{resp. } 0,$$

where $\Phi$ is a distribution on $S^1$. For a general $F \in \mathfrak{F}$, resp. $\tilde{\mathfrak{F}}$, the restriction $F|\mathbb{C} \times C^\infty(S^1)$ is the limit, in the topology of $\mathcal{O}(\mathbb{C} \times C^\infty(S^1))$, of finite sums of the above functions.

2. The $G$–action on $\mathfrak{F}$

For $g \in G$ let $J_g(\zeta) = d g \zeta / d \zeta$. By considering the posthomogeneous expansion (1.12)–(1.13) of $F \in \mathfrak{F}$, resp. $\tilde{\mathfrak{F}}$, one checks that the function $gF$ defined by

\begin{equation}
(gF)(\zeta, y) = F(g\zeta, y/J_g(\zeta)) J_g(\zeta)
\end{equation}

extends to all of $\mathbb{C} \times L^{(-)} \mathbb{C}$, and the extension (also denoted $gF$) belongs to $\mathfrak{F}$, resp. $\tilde{\mathfrak{F}}$. The action thus defined makes $\mathfrak{F}, \tilde{\mathfrak{F}}$ holomorphic $G$–modules. The $n'$th isotypical subspace $\mathfrak{F}^n$, resp. $\tilde{\mathfrak{F}}^n$ is the subspace of $(n+1)$–posthomogeneous functions. In this section we shall describe the space $\mathfrak{F}^0$, and, for $W^{1,p}$ loop spaces, the spaces $\mathfrak{F}^n$ as well, $n \geq 1$.

**Theorem 2.1.** $\mathfrak{F}^0 \approx (L^{-}\mathbb{C})^*/\mathbb{C}$, the dual endowed with the compact–open topology. If $L^{-}\mathbb{C}$ is interpreted as the space of one–forms on $S^1$ of the corresponding regularity, then the isomorphism is Diff $S^1$ equivariant.

**Proof.** Indeed, the map $(L^{-}\mathbb{C})^* = \tilde{\mathfrak{F}}^0 \to \mathfrak{F}^0$ associating with $\Phi \in (L^{-}\mathbb{C})^*$ the function $F(y) = \langle \Phi, dy \rangle$ (or $\langle \Phi, dy \rangle$) has one dimensional kernel and a right inverse by Theorem 1.1.
Theorem 2.2. In the case of $W^{1,p}$ loop spaces $\mathfrak{g} = \bigoplus_{n \leq p-1} \mathfrak{g}^n$. Furthermore

$$\mathfrak{g}_n \otimes L^{p/(n+1)}(S^1)^* \approx \mathfrak{g}^n, \quad 1 \leq n \leq p-1,$$

as $G$–modules, $G$ acting on $L^{p/(n+1)}(S^1)^*$ trivially. Indeed, the map $\varphi \otimes \Phi \mapsto F$ given by

$$F(\zeta, y) = \psi(\zeta)\langle \Phi, \dot{y}^{n+1} \rangle, \quad \varphi(\zeta) = \psi(\zeta)(d\zeta)^{-n},$$

induces the isomorphism above. (To achieve Diff $S^1$ equivariant isomorphism, replace $L^{p/(n+1)}(S^1)$ by the space $L^{p/(n+1)}_{n+1}(S^1)$ of $(n+1)$–differentials.)

We shall need a few auxiliary results to prove the theorem.

Lemma 2.3. Let $m \geq 2$ be an integer and $\Psi$ a distribution on $S^1$. If the function

$$C^\infty(S^1) \ni x \mapsto \langle \Psi, x^m \rangle \in \mathbb{C}$$

extends to a homogeneous polynomial $E$ on $L^p(S^1)$ then $\Psi \equiv 0$, or $m \leq p$ and $\Psi$ extends to a form $\Phi$ on $L^{p/m}(S^1)$. In the latter case the map $E \mapsto \Phi$ is continuous linear.

Proof. There is a constant $C$ such that

$$|\langle \Psi, x^m \rangle| = |E(x)| \leq C(\int |x|^p)^{m/p}, \quad x \in C^\infty(S^1).$$

Let $z \in C^\infty(S^1)$ be real valued and $x_\epsilon = (z + i\epsilon)^{1/m} \in C^\infty(S^1)$. By (2.4)

$$|\langle \Psi, z \rangle| = \lim_{\epsilon \to 0} |\langle \Psi, x_\epsilon^m \rangle| \leq C(\int |z|^{p/m})^{m/p}.$$ 

As the same estimate holds for imaginary $z$, it will hold for a general $z \in C^\infty(S^1)$ too, perhaps with a different $C$. Therefore $\Psi$ extends to a form $\Phi$ on $L^{p/m}(S^1)$. Unless $p \geq m$, $\Phi = 0$ by Day’s theorem [D]. With $z \in L^{p/m}(S^1)$, any choice of measurable $m$’th root $z^{1/m}$, and $y_\epsilon \in C^\infty(S^1)$ converging to $z^{1/m}$ in $L^p$,

$$\langle \Phi, z \rangle = \lim_{\epsilon \to 0} \langle \Phi, y_\epsilon^m \rangle = \lim_{\epsilon \to 0} E(y_\epsilon) = E(z^{1/m}).$$

This shows that $\Phi$ is uniquely determined by $E$, and depends continuously and linearly on $E$.

In the rest of this section we work with $W^{1,p}$ loop spaces. Write $\mathcal{C}^n \subset \mathcal{C}$, $\tilde{\mathcal{C}}^n \subset \tilde{\mathcal{C}}$ for the space of $(n+1)$–homogeneous functions.
Lemma 2.4. If \( m \geq 2 \) and \( E \in \check{C}^{m-1} \subset \mathcal{O}(L^p(S^1)) \), then \( E(x) = \langle \Phi, x^m \rangle \) with a unique \( \Phi \in L^{p/m}(S^1)^* \). In particular, \( E = 0 \) if \( m > p \). Also, the map \( E \mapsto \Phi \) is an isomorphism between \( \check{C}^{m-1} \) and \( L^{p/m}(S^1)^* \).

Proof. We shall prove by induction, first assuming \( m = 2 \). By Theorem 1.5 there are distributions \( \Phi_\alpha \) so that

\[
E(x) = \sum_{\alpha=0}^{d} \langle \Phi_\alpha, xx^{(\alpha)} \rangle, \quad x \in C^\infty(S^1).
\]

Now any \( x^{(\alpha)}x^{(\beta)} \) will be a linear combination of expressions \( (x^{(j)}x^{(j)})^{(h)} \), as one easily proves by induction of \( |\alpha - \beta| \). It follows that \( E \) can be written with distributions \( \Psi_j \) as

\[
E(x) = \sum_{j=0}^{d} \langle \Psi_j, (x^{(j)})^2 \rangle, \quad x \in C^\infty(S^1).
\]  

(2.5)

Next we show that \( d = 0 \).

Indeed, assuming \( d > 0 \), for fixed \( x \in C^\infty(S^1) \)

\[
E(\cos \lambda x) + E(\sin \lambda x) = \lambda^{2d} \langle \Psi_d, x^{2d} \rangle + \sum_{j=0}^{2d-1} c_j(x) \lambda^j
\]

(2.6)

is a polynomial in \( \lambda \). For fixed \( \lambda \in \mathbb{C} \) the maps \( x \mapsto \cos \lambda x \), \( x \mapsto \sin \lambda x \) map the Banach algebra \( W^{1,1}(S^1) \) holomorphically into itself, hence into \( L^p(S^1) \). Therefore the left hand side of (2.6) extends to \( W^{1,1}(S^1) \), and \( \langle \Psi_d, x^{2d} \rangle \) must also. The extension of this latter will be an additive, \( 2d \)-homogeneous polynomial \( E' \) on \( W^{1,1}(S^1) \), satisfying \( E'(x + \text{const}) = E'(x) \). By Proposition 1.3 there is therefore a unique additive \( 2d \)-homogeneous polynomial \( \tilde{E} \) on \( W^{0,1}(S^1) = L^1(S^1) \) such that \( E'(x) = \tilde{E}(\dot{x}) \). Since the restriction \( \tilde{E}|C^\infty(S^1) \) is also unique,

\[
\tilde{E}(x) = \langle \Psi_d, x^{2d} \rangle, \quad x \in C^\infty(S^1).
\]

In particular, the expression on the right continuously extends to \( L^1(S^1) \). By virtue of Lemma 2.3, \( \Psi_d \equiv 0 \). Thus (2.5) reduces to \( E(x) = \langle \Psi, x^2 \rangle, \ x \in C^\infty(S^1) \), and by another application of Lemma 2.3, \( \Psi \) extends to a form \( \Phi \) on \( L^{p/2}(S^1) \).

Now assume the Lemma is known for degree \( m - 1 \geq 2 \), and consider an \( E \in \check{C}^{m-1} \) and its polarization \( \mathcal{E} \). For fixed \( x_1 \in C^\infty(S^1) \) the inductive assumption implies that there is a distribution \( \Theta \) such that \( \mathcal{E}(x_1 \otimes \ldots \otimes x_m) = \langle \Theta, \prod_{j}^{m} x_j \rangle \); in particular,

\[
\mathcal{E}(x_1 \otimes \ldots \otimes x_m) = \mathcal{E}(x_1 \otimes \prod_{2}^{m} x_j \otimes 1 \otimes \ldots \otimes 1), \quad x \in C^\infty(S^1).
\]

The case \( m = 2 \) now gives a distribution \( \Psi \) such that \( \mathcal{E}(x_1 \otimes \ldots \otimes x_m) = \langle \Psi, \prod_{1}^{m} x_j \rangle \). We conclude by Lemma 2.3: \( \Psi \) extends to \( \Phi \in L^{p/m}(S^1)^* \), and \( \Phi = 0 \) unless \( m \leq p \). It is
clear that $\Phi$ is uniquely determined by $E$, and the map $\tilde{E}^{m-1} \ni E \mapsto \Phi \in L^{p/m}(S^1)^*$ is an isomorphism.

Proof of Theorem 2.2. To construct the inverse of the map defined by (2.2), write an arbitrary $F \in \mathfrak{F}^n$, $n \geq 1$, as

$$F(\zeta, y) = \sum_{\nu=0}^{2n} \zeta^\nu E_\nu(y), \quad E_\nu \in \mathfrak{E}^n,$$

cf. Proposition 1.4, and find the unique $\tilde{E}_\nu \in \tilde{E}^n$ so that $E_\nu(y) = \tilde{E}_\nu(\dot{y})$, see Proposition 1.3. By Lemma 2.4 there are unique $\Phi_\nu \in L^{p/((n+1))}(S^1)^*$ such that $\tilde{E}_\nu(x) = \langle \Phi_\nu, x^{n+1} \rangle$. If $p < n + 1$ then $\Phi_\nu = 0$ and so $\mathfrak{F}^n = (0)$. Otherwise the map

$$\mathfrak{F}^n \ni F \mapsto \sum_{\nu=0}^{2n} \zeta^\nu (d\zeta)^{-n} \otimes \Phi_\nu \in \mathfrak{F}_n \otimes L^{p/((n+1))}(S^1)^*$$

is the inverse of the map given in (2.2), so (2.2) indeed induces an isomorphism. Finally, the posthomogeneous expansion of an arbitrary $F \in \mathfrak{F}$ is

$$F = \sum_{0}^{\infty} F_n = \sum_{0}^{[p-1]} F_n,$$

which completes the proof.

3. Cuspidal cocycles

In this section we shall construct an isomorphism between $H^{0,1}(L\mathbb{P}_1)$ and a space of holomorphic Čech cocycles on $L\mathbb{P}_1$. We represent $\mathbb{P}_1$ as $\mathbb{C} \cup \{\infty\}$. Constant loops constitute a submanifold of $L\mathbb{P}_1$, that we identify with $\mathbb{P}_1$. If $a, b, \ldots \in \mathbb{P}_1$, set $U_{ab\ldots} = \mathbb{P}_1 \setminus \{a, b, \ldots\}$. Thus $LU_a$, $a \in \mathbb{P}_1$, form an open cover of $L\mathbb{P}_1$, with $LU_\infty = L\mathbb{C}$ a Fréchet algebra. If $g \in G$ then $g(LU_a) = LU_{ga}$.

Suppose we are given $v: \mathbb{P}_1 \to \mathbb{C}$, finitely many $a, b, \ldots \in \mathbb{P}_1$, and a function $u: LU_{ab\ldots} \to \mathbb{C}$. If $\infty$ is among $a, b, \ldots$, let us say that $u$ is $v$-cuspidal at $\infty$ if $u(x + \lambda) \to v(\infty)$ as $\mathbb{C} \ni \lambda \to \infty$, for all $x \in LU_{ab\ldots}$; and in general, that $u$ is $v$-cuspidal if $g^* u$ is $g^* v$-cuspidal at $\infty$ for all $g \in G$ that maps one of $a, b, \ldots$ to $\infty$. When $v \equiv 0$ we simply speak of cuspidal functions.

Proposition 3.1. Given a closed $f \in C_0^{\infty}(L\mathbb{P}_1)$ and $v \in C^{\infty}(\mathbb{P}_1)$ such that $\overline{\partial} v = f|\mathbb{P}_1$, for each $a \in \mathbb{P}_1$ there is a unique $v$-cuspidal $u_a \in C^{\infty}(LU_a)$ that solves $\overline{\partial} u_a = f|LU_a$. Furthermore, $u_a|U_a = v|U_a$, and $u_a(x) = u_a(x)$ is smooth in $(a, x)$, holomorphic in $a$.

Proof. Uniqueness follows since for fixed $g \in G$, $y \in L\mathbb{C}$, on the line $\{g(y + \lambda): \lambda \in \mathbb{P}_1\}$ the $\overline{\partial}$ equation is uniquely solvable up to an additive constant, which constant is determined
by the cuspidal condition. To construct \( u_a \), fix a \( g \in G \) with \( g\infty = a \), let \( Y = \{ y \in L\mathbb{C} : y(0) = 0 \} \) and
\[
P_g : \mathbb{P}_1 \times Y \ni (\lambda, y) \mapsto g(y + \lambda) \in L\mathbb{P}_1,
\]
a biholomorphism between \( \mathbb{C} \times Y \) and \( LU_a \). Setting \( f_g = P_g^* f \), by [L1, Theorem 5.4] on the \( \mathbb{P}_1 \) bundle \( \mathbb{P}_1 \times Y \) the equation \( \overline{\partial}u_g = f_g \) has a unique smooth solution satisfying \( u_g(\infty, x) = v(a) \). It follows that \( u_a = (P_g^{-1})^*(u_g|\mathbb{C} \times Y) \) solves \( \overline{\partial}u_a = f|LU_a \). Also, \( g^*u_a \) is \( g^*v \)-cuspidal at \( \infty \). On \( U_a \) both \( u_a \) and \( v \) solve the same \( \overline{\partial} \)-equation, and have the same limit at \( a \), hence \( u_a|U_a = v|U_a \).

One can also consider
\[
P : \mathbb{P}_1 \times G \times Y \ni (\lambda, g, y) \mapsto g(y + \lambda) \in L\mathbb{P}_1
\]
and \( f' = P^* f \). Again by [L1, Theorem 5.4], on the \( \mathbb{P}_1 \) bundle \( \mathbb{P}_1 \times G \times Y \) the equation \( \overline{\partial}u' = f' \) has a smooth solution satisfying \( u'(\infty, g, x) = v(g\infty) \). Uniqueness of \( u_g \) implies \( u'(\lambda, g, x) = u_g(\lambda, x) \), whence \( u_g(\lambda, x) \) depends smoothly on \( (\lambda, g, x) \), and \( u_a(x) \) on \( (a, x) \). Furthermore, \( u' \) is holomorphic on \( P^{-1}(x) \) for any \( x \). In particular, if \( g \in G \) with \( g\infty = a \) is chosen to depend holomorphically on \( a \) (which can be done locally), then it follows that \( u_a(x) = u'(g^{-1}x(0), g, g^{-1}x - g^{-1}x(0)) \) is holomorphic in \( a \).

Since \( f \) determines \( v \) up to an additive constant, we can uniquely associate with \( f \) the \( \check{\text{C}}ech \) cocycle \( \check{f} = (u_a - u_b : a, b \in \mathbb{P}_1) \). The components of \( \check{f} \) are cuspidal holomorphic functions on \( LU_a \). One easily verifies

**Proposition 3.2.** \( f \) is exact if and only if \( \check{f} = 0 \). Hence \( \check{f} \) depends only on the cohomology class \( [f] \in H^{0,1}(L\mathbb{P}_1) \). The components \( h_{ab}([f], x) \) of \( \check{f} \) depend holomorphically on \( a, b \in \mathbb{P}_1 \) and \( x \in LU_{ab} \), and satisfy the transformation formula
\[
(3.1) \quad h_{ga,gb}([f], gx) = h_{ab}(g*[f], x), \quad g \in G, \ x \in LU_{ab}.
\]

Set
\[
\Omega = \{(a, b, x) \in \mathbb{P}_1 \times \mathbb{P}_1 \times L\mathbb{P}_1 : a, b \notin x(S^1) \}.
\]
Let \( \mathcal{H} \) denote the space of those holomorphic cocycles \( \check{h} = (h_{ab})_{a, b \in \mathbb{P}_1} \) of the covering \( \{LU_a\} \), for which \( h_{ab}(x) \) depends holomorphically on \( a, b \), and \( x \in LU_{ab} \), and each \( h_{ab} \) is cuspidal. Then \( \mathcal{H} \subset \mathcal{O}(\Omega) \), with the compact open topology, is a complete, separated, locally convex space. The action of \( G \) on \( \Omega \) induces a \( G \)-module structure on \( \mathcal{H} \):
\[
(3.2) \quad (g^*h)_{ab}(x) = h_{ga,gb}(gx), \quad g \in G.
\]

Proposition 3.2 implies the map \( [f] \mapsto \check{f} \) is a monomorphism \( H^{0,1}(L\mathbb{P}_1) \rightarrow \mathcal{H} \) of \( G \)-modules.

**Theorem 3.3.** The map \( [f] \mapsto \check{f} \) is an isomorphism \( H^{0,1}(L\mathbb{P}_1) \rightarrow \mathcal{H} \).

The proof would be routine if the loop space \( L\mathbb{P}_1 \) admitted smooth partitions of unity; but a typical loop space does not, see [K]. The proof that we offer here will work only
when the loops in $L\mathbb{P}_1$ are of regularity $W^{1,3}$ at least, and we shall return to the case of $L_{1,p}(\mathbb{P}_1)$, $p < 3$, in Section 6.

Those $g \in G$ that preserve the Fubini–Study metric form a subgroup (isomorphic to) $\text{SO}(3)$. Denote the Haar probability measure on $\text{SO}(3)$ by $dg$.

**Lemma 3.4.** Unless $L\mathbb{P}_1 = L_{1,p}\mathbb{P}_1$, $p < 3$, there is a $\chi \in C^\infty(L\mathbb{P}_1)$ such that $\chi = 0$ in a neighborhood of $L\mathbb{P}_1 \setminus L\mathcal{C} = \{ x : x \in x(S^1) \}$, and $\int_{\text{SO}(3)} g^* \chi dg = 1$.

**Proof.** With $c_0 \in (0, \infty)$ to be specified later, fix a nonnegative $\rho \in C^\infty(\mathbb{R})$ such that $\rho(\tau) = 1$, resp. 0 when $|\tau| < c_0$, resp. $> 2c_0$. For $x \in L\mathcal{C}$ let

$$\psi(x) = \rho \left( \int_{S^1} (1 + |x|^2)^{3/4} \right),$$

and define $\psi(x) = 0$ if $x \in L\mathbb{P}_1 \setminus L\mathcal{C}$. We claim that $\psi$ vanishes in a neighborhood of an arbitrary $x \in L\mathbb{P}_1 \setminus L\mathcal{C}$. This will then also imply that $\psi \in C^\infty(L\mathbb{P}_1)$.

Indeed, suppose $x(t_0) = \infty$. In a neighborhood of $t_0 \in S^1$ the function $z = 1/x$ is $W^{1,3}$, hence Hölder continuous with exponent 2/3 by the Sobolev Embedding Theorem, [H, Theorem 4.5.12]. In this neighborhood therefore $|x(t)| \geq c|t-t_0|^{-2/3}$, and $\int_{S^1} (1 + |x|^2)^{3/4} = \infty$. When $y \in L\mathcal{C}$ is close to $x$, $\int_{S^1} (1 + |y|^2)^{3/4} > 2c_0$, i.e. $\psi(y) = 0$.

Next we show that for every $x \in L\mathbb{P}_1$ there is a $g \in \text{SO}(3)$ with $\psi(gx) > 0$. Let $d(a, b)$ denote the Fubini–Study distance between $a, b \in \mathbb{P}_1$; then with some $c > 0$

$$1 + |\zeta|^2 \leq \frac{c}{d(\zeta, \infty)^2}, \quad \text{and} \quad \int_{S^1} (1 + |x|^2)^{3/4} \leq c \int_{S^1} d(x, \infty)^{-3/2}.$$

Hence

$$\int_{\text{SO}(3)} \int_{S^1} (1 + |gx(t)|^2)^{3/4} dtdg \leq c \int_{S^1} \int_{\text{SO}(3)} d(gx(t), \infty)^{-3/2} dgd = cI,$$

where, for any $\zeta \in \mathbb{P}_1$

$$I = \int_{\text{SO}(3)} d(g\zeta, \infty)^{-3/2} dg = \int_{\mathbb{P}_1} d(\cdot, \infty)^{-3/2} < \infty,$$

the last integral with respect to the Fubini–Study area form. If $c_0$ is chosen $> cI$ then indeed $\int_{S^1} (1 + |gx|^2)^{3/4} < c_0$ and $\psi(gx) = 1$ for some $g \in \text{SO}(3)$.

It follows that $\int_{\text{SO}(3)} \psi(gx) dg > 0$, and we can take $\chi(x) = \psi(x)/\int_{\text{SO}(3)} \psi(gx) dg$.

**Proof of Theorem 3.3.** Given $\mathfrak{h} \in \mathfrak{h}_1$, extend the functions $(g^* \chi)\mathfrak{h}_{a, g\infty}$ from $LU_{a, g\infty}$ to $LU_a$ by zero, and define the cuspidal functions

$$u_a = \int_{\text{SO}(3)} (g^* \chi)\mathfrak{h}_{a, g\infty} dg, \quad a \in \mathbb{P}_1.$$

Then $u_a - u_b = \int_{\text{SO}(3)} (g^* \chi)\mathfrak{h}_{ab} dg = \mathfrak{h}_{ab}$, so that $f = \overline{\mathfrak{u}} u_a$ on $LU_a$ consistently defines a closed $f \in C^\infty_0(L\mathbb{P}_1)$. It is immediate that the map $\mathfrak{h} \mapsto [f] \in H^{0,1}(L\mathbb{P}_1)$ is left inverse to the monomorphism $[f] \mapsto f$, whence the theorem follows.
4. The map $\mathfrak{H} \to \mathfrak{F}$

Consider an $h = (h_{ab}) \in \mathfrak{H}$. The cocycle relation implies that $d_\zeta h_{a\zeta}(x)$ is independent of $a$; for $\zeta \in \mathbb{C}$ we can write it as

$$d_\zeta h_{a\zeta}(x) = F(\zeta, \frac{1}{\zeta - x}) d\zeta,$$

where $F \in \mathcal{O}(\mathbb{C} \times L\mathbb{C})$. Set $F = \alpha(h)$. Since $h_{aa} = 0$,

$$h_{ab}(x) = \int_a^b F\left(\zeta, \frac{1}{\zeta - x}\right) d\zeta,$$

provided $a, b$ are in the same component of $P_1 \setminus x(S^1)$—that we shall express by saying $x$ does not separate $a, b$—, and we integrate along a path within this component. The main result of this section is

**Theorem 4.1.** $\alpha(h) = F \in \mathfrak{F}$.

The heart of the matter will be the special case when $h$ is in an irreducible submodule $\cong \mathfrak{R}_n$. A vector that corresponds in this isomorphism to $\text{const}(d_\zeta)^{-n} \in \mathfrak{R}_n$ is said to be of lowest weight $-n$. Thus, if $l$ is of lowest weight $-n \leq 0$, then

$$g_\lambda^* l = \lambda^{-n} l, \quad \text{when} \quad g_\lambda \zeta = \lambda \zeta, \quad \lambda \in \mathbb{C},$$

and

$$g_\lambda^* l = l, \quad \text{when} \quad g_\lambda \zeta = \zeta + \lambda, \quad \lambda \in \mathbb{C}.$$

Conversely, an $l \neq 0$ satisfying (4.3), (4.4) is a lowest weight vector and spans an irreducible submodule, isomorphic to $\mathfrak{R}_n$, but we shall not need this fact.

If $l \in \mathfrak{H}$ satisfies (4.4) then $l_{=0}(x) = l_{=0}(x + \lambda)$ by (3.2), whence $d_\zeta l_{=0}(x)$ depends only on $\zeta - x$, and $\alpha(l)$ is of form $F(\zeta, y) = E(y)$. If, in addition, $l$ satisfies (4.3), then similarly it follows that $E \in \mathcal{O}(L\mathbb{C})$ is homogeneous of degree $n+1$. We now fix a nonzero lowest weight vector $l \in \mathfrak{H}$, the corresponding $(n+1)$–homogeneous polynomial $E$, and its polarization $\mathcal{E}$, cf. (1.2).

**Proposition 4.2.** $\mathcal{E}(1 \otimes y_1 \otimes \ldots \otimes y_n) = 0$, and so $E(y + \text{const}) = E(y)$.

**Proof.** Since $l_{=0} \in \mathcal{O}(LU_{=0})$ is cuspidal and homogeneous of order $-n$,

$$0 = \lim_{\lambda \to \infty} l_{=0}\left(\frac{1}{\lambda + x}\right) = \lim_{\lambda \to \infty} \lambda^n l_{=0}\left(\frac{1}{1 + x/\lambda}\right).$$

Thus $l_{=0}$ vanishes at 1 to order $\geq n + 1$. Hence

$$\left.\frac{\partial}{\partial \zeta}\right|_{\zeta=0} l_{=0}(x - \zeta) = \left.\frac{\partial}{\partial \zeta}\right|_{\zeta=0} l_{=0}(x) = E\left(\frac{1}{x}\right)$$

vanishes at $x = 1$ to order $\geq n$, and the same holds for $E(x)$. Differentiating $E$ in the directions $y_1, \ldots, y_n$, we obtain at $x = 1$, as needed, that $n!\mathcal{E}(1 \otimes y_1 \otimes \ldots \otimes y_n) = 0$.

Let $\mathfrak{R}_n \ni \varphi \mapsto h^\varphi \in \mathfrak{H}$ denote the homomorphism that maps $(d_\zeta)^{-n}$ to $l$. 


Proposition 4.3.

\[ d_{\zeta} h_{a,\zeta}^{\varphi}(x) = \psi(\zeta) E\left( \frac{1}{\zeta - x} \right) d\zeta, \quad \varphi(\zeta) = \psi(\zeta)(d\zeta)^{-n}. \]

By homogeneity, the right hand side can also be written \( \varphi(\zeta) E(d\zeta/(\zeta - x)) \).

**Proof.** Denote the form on the left hand side of (4.5) by \( \omega_{\varphi} \). In view of (3.2) it transforms under the action of \( G \) on \( P_1 \times L P_1 \) as

\[ g^* \omega_{\varphi} = \omega_{g \varphi}, \quad g \in G. \]

If we show that the right hand side of (4.5) transforms the same way, then (4.5) will follow, since it holds when \( \psi \equiv 1 \), see (4.1). In fact, it will suffice to check the transformation formula for \( g\zeta = \lambda \zeta \), \( g\zeta = \zeta + \lambda \) (\( \lambda \in \mathbb{C} \)), and \( g\zeta = 1/\zeta \), maps that generate \( G \). We shall do this for the last map, the most challenging of the three types. The pullback of the right hand side of (4.5) by \( g\zeta = 1/\zeta \) is

\[ (g\varphi)(\zeta) E\left( \frac{d(g\zeta)}{g\zeta - gx} \right) = (g\varphi)(\zeta) E\left( \frac{-d\zeta/\zeta^2}{(1/\zeta) - (1/x)} \right) \]
\[ = (g\varphi)(\zeta) E\left( \frac{d\zeta}{\zeta - x} - \frac{d\zeta}{\zeta} \right) = (g\varphi)(\zeta) E\left( \frac{d\zeta}{\zeta - x} \right), \]

by Proposition 4.2, which is what we needed.

The form \( \mathcal{E} \) defines a symmetric distribution \( D \) on the torus \( T = (S^1)^{n+1} \) as in Section 1, cf. (1.14). By (1.15), (4.2), and Proposition 4.3

\[ h_{ab}^{\varphi}(x) = \int_{a}^{b} \psi(\zeta) \left\langle D, \frac{1}{\zeta - x} \otimes \ldots \otimes \frac{1}{\zeta - x} \right\rangle d\zeta, \quad \varphi = \psi(\zeta)(d\zeta)^{-n}, \]

provided \( x \in L_\infty U_{ab} \) does not separate \( a, b \). To prove Theorem 4.1, we have to understand \( \text{supp } \mathcal{D} \). Let

\[ O = \{ x \in C^\infty(S^1): \pm i \not\in x(S^1) \}, \quad \text{and } O' = \{ x \in O: [-i, i] \cap x(S^1) = \emptyset \}, \]

where \([ -i, i ]\) stands for the segment joining \( \pm i \).

**Lemma 4.4.** With \( \Delta \) a symmetric distribution on \( T = (S^1)^{n+1} \) and \( \nu = 0, \ldots, 2n - 2 \), let

\[ I_\nu(x) = \int_{[-i,i]} \left\langle \Delta, \frac{1}{\zeta - x} \otimes \ldots \otimes \frac{1}{\zeta - x} \right\rangle \zeta^\nu d\zeta, \quad x \in O'. \]

If each \( I_\nu \) continues analytically to \( O \) then \( \Delta \) is supported on the diagonal of \( T \).
In preparation to the proof, consider a holomorphic vector field $V$ on $O$, and observe that $VI_\nu$ also continues analytically to $O$. Such vector fields can be thought of as holomorphic maps $V: O \to C^\infty(S^1)$. Using the symmetry of $\Delta$ we compute

\begin{equation}
(4.8) \quad (VI_\nu)(x) = (n + 1) \int_{[-i,i]} \left\langle \Delta, \frac{V(x)}{(\zeta - x)^2} \otimes \frac{1}{\zeta - x} \otimes \ldots \otimes \frac{1}{\zeta - x} \right\rangle c^{-\nu} d\zeta, \quad x \in O'.
\end{equation}

**Proof of Lemma 4.4, case $n = 1$.** Let $\bar{s}_0 \neq \bar{s}_1 \in S^1$. To show $\Delta$ vanishes near $\bar{s} = (\bar{s}_0, \bar{s}_1)$, construct a smooth family $x_{\varepsilon,s} \in O$ of loops, where $\varepsilon \in [0,1]$ and $s \in T$ is in a neighborhood of $\bar{s}$, so that

\begin{equation}
(4.9) \quad x_{\varepsilon,s}(\tau) = (-1)^j (\varepsilon^2 + (\tau - s_j)^2), \quad \text{when } \tau \in S^1 \text{ is near } \bar{s}_j, \ j = 0,1;
\end{equation}

here, perhaps abusively, $\tau - s_j$ denotes both a point in $S^1 = \mathbb{R}/\mathbb{Z}$ and its representative in $\mathbb{R}$ that is closest to 0. Make sure that $x_{\varepsilon,s} \in O'$ when $\varepsilon > 0$. Fix $y_0, y_1 \in C^\infty(S^1)$ so that $y_j \equiv 1$ near $\bar{s}_j$, and (4.9) holds when $\tau, s_j$ are in a neighborhood of $\text{supp} \ y_j$. This forces $y_0, y_1$ to have disjoint support. With constant vector fields $V_j = y_j$

\begin{equation}
(4.10) \quad (V_1 V_0 I_0)(x) = 2 \int_{[-i,i]} \left\langle \Delta, \frac{y_0}{(\zeta - x)^2} \otimes \frac{y_1}{(\zeta - x)^2} \right\rangle d\zeta, \quad x \in O',
\end{equation}

analytically continues to $O$. In particular, for $\varepsilon > 0$ and $t = (t_0, t_1) \in T$ setting

\begin{equation}
K_\varepsilon(t, s) = \int_{[-i,i]} \frac{y_0(t_0)y_1(t_1)d\zeta}{(\zeta - x_{\varepsilon,s}(t_0))^2(\zeta - x_{\varepsilon,s}(t_1))^2}, \quad s \text{ near } \bar{s},
\end{equation}

it follows that $\langle \Delta, K_\varepsilon(\cdot, s) \rangle$ stays bounded as $\varepsilon \to 0$. Therefore, if $\rho \in C^\infty(T)$ is supported in a sufficiently small neighborhood of $\bar{s}$,

\begin{equation}
(4.11) \quad \langle \Delta, \varepsilon^4 \int_T K_\varepsilon(\cdot, s) \rho(s)ds \rangle \to 0, \quad \varepsilon \to 0.
\end{equation}

On the other hand we shall show that for such $\rho$

\begin{equation}
(4.12) \quad \varepsilon^4 \int_T K_\varepsilon(\cdot, s) \rho(s)ds \to c\rho, \quad \varepsilon \to 0,
\end{equation}

in the topology of $C^\infty(T)$; here $c \neq 0$ is a constant.

It will suffice to verify (4.12) on $\text{supp} \ y_0 \otimes y_1$, since both sides vanish on the complement. Thus we shall work on small neighborhoods of $\bar{s}$; we can pretend $\bar{s} \in \mathbb{R}^2$, and work on $\mathbb{R}^2$ instead of $T$. When $s, t \in \mathbb{R}^2$ are close to $\bar{s}$, the left hand side of (4.12) becomes

\begin{equation}
(4.13) \quad \varepsilon^4 y_0(t_0)y_1(t_1) \int_{\mathbb{R}^2} \int_{[-i,i]} \frac{\rho(s)d\zeta ds}{(\zeta - \varepsilon^2 - (s_0 - t_0)^2)^2(\zeta + \varepsilon^2 + (s_1 - t_1)^2)^2}.
\end{equation}
Substituting $s = t + \varepsilon u$ and $\zeta = \varepsilon^2 \xi$, we compute the limit in (4.12) is

$$
\lim_{\varepsilon \to 0} y_0(t_0)y_1(t_1) \int_{\mathbb{R}^2} \int_{[0, i/\varepsilon^2, i/\varepsilon^2]} \frac{\rho(t + \varepsilon u)d\xi du}{(\xi - 1 - u_0^2)^2(\xi + 1 + u_1^2)^2} = 4\pi i y_0(t_0)y_1(t_1) \int_{\mathbb{R}^2} \frac{\rho(t)du}{(2 + u_0^2 + u_1^2)^3} = c\rho(t),
$$

if $y_0 \otimes y_1 = 1$ on supp $\rho$. This limit is first seen to hold uniformly. However, since the integral operator in (4.13) is a convolution, in (4.14) in fact all derivatives converge uniformly. Now (4.11) and (4.12) imply $(\Delta, \rho) = 0$, so that $\Delta$ vanishes close to $\mathfrak{R}$, q.e.d.

**Proof of Lemma 4.4, general $n$.** The base case $n = 1$ settled and the statement being vacuous when $n = 0$, we prove by induction. Assume the Lemma holds on the $n$–dimensional torus, and with $y \in C^\infty(S^1)$, consider holomorphic vector fields $V_\mu(x) = y x^\mu$, $\mu = 0, 1, 2$. (These vector fields continue to all of $L^P_1$, and generate the Lie algebra of the loop group $LG$.) In view of (4.8), for $x \in O'$

$$
\int_{[-i, i]} \left\langle \Delta, y \otimes \frac{1}{\zeta - x} \otimes \ldots \otimes \frac{1}{\zeta - x} \right\rangle \zeta^\nu d\zeta = \frac{1}{n + 1} (V_0 I_{\nu + 2} - 2V_1 I_{\nu + 1} + V_2 I_\nu).
$$

Therefore the left hand side continues analytically to $O$, provided $\nu = 0, \ldots, 2n - 4$. If $\Delta^y$ denotes the distribution on $(S^1)^n$ defined by $\langle \Delta^y, \rho \rangle = \langle \Delta, y \otimes \rho \rangle$, the left hand side of (4.15) is

$$
\int_{[-i, i]} \left\langle \Delta^y, \frac{1}{\zeta - x} \otimes \ldots \otimes \frac{1}{\zeta - x} \right\rangle \zeta^\nu d\zeta.
$$

The inductive hypothesis implies $\Delta^y$ is supported on the diagonal of $(S^1)^n$. This being true for all $y$, the symmetric distribution $\Delta$ itself must be supported on the diagonal.

**Corollary 4.5.** The distribution $D$ in (4.7) is supported on the diagonal of $T$.

**Proof of Theorem 4.1.** First assume that $h \in \mathfrak{H}$ is in an irreducible submodule $\approx \mathfrak{R}_n$, and $l \neq 0$ is a lowest weight vector in this submodule. Thus $h = h^\varphi$ with some $\varphi \in \mathfrak{R}_n$, $\varphi(\zeta) = \psi(\zeta)(d\zeta)^{-n}$. With $l$ we associated an $(n + 1)$–homogeneous polynomial $E$ on $L\mathbb{C}$ and a distribution $D$ on $(S^1)^{n + 1}$. By Proposition 4.3 $F(\zeta, y) = \psi(\zeta)E(y)$, and so $F(\zeta, y + \text{const}) = F(\zeta, y)$ by Proposition 4.2. Since $\deg \psi \leq 2n$, $F(\zeta / \lambda, \lambda^2 y) = O(\lambda^2)$ as $\lambda \to 0$. Finally, take $x, y \in L\mathbb{C}$ with disjoint support. If $x, y \in C^\infty(S^1)$,

$$
E(x + y) = \langle D, (x + y)^{\otimes n + 1} \rangle = \langle D, x^{\otimes n + 1} \rangle + \langle D, y^{\otimes n + 1} \rangle = E(x) + E(y),
$$

as supp $D$ is on the diagonal. By approximation $E(x + y) = E(x) + E(y)$ follows in general, whence $F$ itself is additive. We conclude $F \in \mathfrak{F}$ if $h$ is in an irreducible submodule.

By linearity it follows that $F \in \mathfrak{F}$ whenever $h$ is in the span of irreducible submodules. Since this span is dense in $\mathfrak{H}$ (cf. [BD, III.5.7] and the explanation in the Introduction connecting representations of $G$ with those of the compact group $SO(3)$), $\alpha(h) \in \mathfrak{F}$ for all $h \in \mathfrak{H}$.
Theorem 4.6. The map \( \alpha \) is a \( G \)-morphism.

Proof. It suffices to verify that the restriction of \( \alpha \) to an irreducible submodule of \( \mathfrak{H} \) is a \( G \)-morphism, which follows directly from Proposition 4.3.

5. The structure of \( \mathfrak{H} \)

The main result of this Section is

Theorem 5.1. The \( G \)-morphism \( \alpha: \mathfrak{H} \to \mathfrak{F} \) has a right inverse \( \beta \). Its kernel is one dimensional, spanned by the \( G \)-invariant cocycle

\[
\mathfrak{h}_{ab}(x) = \text{ind}_{ab}x
\]

(\( = \) the winding number of \( x: S^1 \to U_{ab} \)).

We shall need the following

Lemma 5.2. With notation as in Section 1, suppose \( z_1, \ldots, z_N \in L^- \subset \mathbb{C} \) are such that no point in \( S^1 \) is contained in the support of more than two \( z_j \). If \( \tilde{F} \in \tilde{\mathfrak{F}} \) then

\[
(5.2) \quad \tilde{F}(\zeta, \sum_{j=1}^{N} z_j) = \sum_{i<j} \tilde{F}(\zeta, z_i + z_j) - (N-2) \sum_{j=1}^{N} \tilde{F}(\zeta, z_j).
\]

In particular, if \( N \geq 3 \), and writing \( z_0 = z_N \) only consecutive \( \text{supp} z_j \)'s intersect each other, then

\[
\tilde{F}(\zeta, \sum_{j=1}^{N} z_j) = \sum_{j=1}^{N} \tilde{F}(\zeta, z_{j-1} + z_j) - \sum_{j=1}^{N} \tilde{F}(\zeta, z_j).
\]

Proof. It will suffice to verify (5.2) when \( \tilde{F}(\zeta, z) = \tilde{E}(z) \) is homogeneous, in which case it follows by expressing both sides in terms of the polarization of \( \tilde{E} \), and using Lemma 1.2a. The second formula follows from (5.2) by applying additivity to terms with non-consecutive \( i, j \).

Proof of Theorem 5.1. (a) Construction of the right inverse. By Theorem 1.1, for \( F \in \mathfrak{F} \) we can choose \( \tilde{F} \in \tilde{\mathfrak{F}} \), depending linearly on \( F \), so that \( F(\zeta, y) = \tilde{F}(\zeta, \dot{y}) \). With \( x \in L^\mathbb{P}_1 \) consider the differential form

\[
(5.3) \quad F\left(\zeta, \frac{1}{\zeta-x}\right) d\zeta = \tilde{F}\left(\zeta, \frac{\dot{x}}{(\zeta-x)^2}\right) d\zeta,
\]

holomorphic in \( \mathbb{C} \setminus x(S^1) \). In fact, it is holomorphic at \( \zeta = \infty \) as well, provided \( \infty \notin x(S^1) \), since the coefficient of \( d\zeta \) vanishes to second order at \( \zeta = \infty \). This latter is easily verified
when \( \tilde{F}(\zeta, z) = \zeta^\nu \tilde{E}(z) \) and \( \tilde{E} \) is \((n + 1)\)-homogeneous, \( \nu \leq 2n \); in general it follows from the posthomogeneous expansion

\[
\tilde{F}(\zeta, z) = \sum_{0}^{\infty} \tilde{F}_n(\zeta, z) = \sum_{\nu=0}^{\infty} \sum_{n=0}^{2n} \zeta^\nu \tilde{E}_{n\nu}(\zeta).
\]

Hence, if \( x \in L\mathbb{P}_1 \) does not separate \( a \) and \( b \), the integral

\[
(5.4) \quad h_{ab}(x) = \int_{a}^{b} \tilde{F} \left( \zeta, \frac{\dot{x}}{(\zeta - x)^2} \right) d\zeta,
\]

is independent of the path joining \( a, b \) within \( \mathbb{P}_1 \setminus x(S^1) \), and defines a holomorphic function of \( a, b, \) and \( x \).

We claim that \( h_{ab} \) can be continued to a cuspidal cocycle \( h = (h_{ab}) \in \mathcal{H} \). First we prove a variant. Let \( \sigma \in C^\infty(S^1) \) be supported in a closed arc \( I \neq S^1 \). Given finitely many \( a, b, \ldots \in \mathbb{P}_1 \), set

\[
W_{ab\ldots} = \{ x \in L\mathbb{P}_1 : a, b, \ldots \notin x(I) \} \supset LU_{ab\ldots}.
\]

We shall show that the integrals

\[
(5.5) \quad \int_{a}^{b} \tilde{F} \left( \zeta, \frac{\sigma \dot{x}}{(\zeta - x)^2} \right) d\zeta,
\]

can be continued to functions \( t_{ab}(x) \) depending holomorphically on \( a, b, \in \mathbb{P}_1 \), and \( x \in W_{ab} \). The main point will be that, unlike \( LU_{ab\ldots} \), the sets \( W_{ab\ldots} \) are connected.

If \( x_1 \in W_{ab} \), construct a continuous curve \([0, 1] \ni \tau \mapsto x_\tau \in W_{ab}, \) \( x_0 = \text{constant loop} \). Cover \( S^1 \) with open arcs \( J_1, \ldots, J_N = J_0, N \geq 3 \), so that only consecutive \( J_j \)'s intersect, and no \( x_\tau(\overline{J}_i \cup \overline{J}_j) \) separates \( a \) and \( b \). Choose a \( C^\infty \) partition of unity \( \{ \rho_j \} \) subordinate to \( \{ J_j \} \). For \( x \) in a connected neighborhood \( W \subset W_{ab} \) of \( \{ x_\tau : 0 \leq \tau \leq 1 \} \) define

\[
(5.6) \quad t_{ab}(x) = \sum_{1}^{N} \int_{a}^{b} \tilde{F} \left( \zeta, \frac{(\rho_{j-1} + \rho_j)\sigma \dot{x}}{(\zeta - x)^2} \right) d\zeta - \sum_{1}^{N} \int_{a}^{b} \tilde{F} \left( \zeta, \frac{\rho_j \sigma \dot{x}}{(\zeta - x)^2} \right) d\zeta.
\]

In the first sum we extend \( (\rho_{j-1} + \rho_j)\sigma \dot{x}/(\zeta - x)^2 \) to \( S^1 \setminus (J_{j-1} \cup J_j) \) by 0, and integrate along paths in \( \mathbb{P}_1 \setminus x(\overline{J}_{j-1} \cup \overline{J}_j) \); we interpret the second sum similarly. The neighborhood \( W \) is to be chosen so small that no \( x(\overline{J}_i \cup \overline{J}_j) \) separates \( a \) and \( b \) when \( x \in W \).

As above, the integrals in \( (5.6) \) are independent of the path, and define a holomorphic function in \( W \). By Lemma 5.2, \( t_{ab} \) agrees with \( (5.5) \) when \( x \) is near \( x_0 \). Furthermore, the germ of \( t_{ab} \) at \( x_1 \) depends on the curve \( x_\tau \) only through the choice of the \( \rho_j \). In fact, it does not even depend on \( \rho_j \); let \( t'_{ab} \) be the function obtained if in \( (5.6) \) the \( \rho_j \) are replaced by another partition of unity \( \rho'_h \). It will suffice to show that \( t_{ab} = t'_{ab} \) under the additional assumption that each \( \rho'_h \) is supported in some \( J_j \). In this case \( t'_{ab} \) is holomorphic in \( W \) and agrees with \( t_{ab} \) near \( x_0 \), hence on all of \( W \).
Therefore, by varying the partition of unity ρj, we can use (5.6) to define ι_{ab}(x) depending holomorphically on a, b ∈ P1, and x ∈ W_{ab}. Also, ι_{ab} + ι_{bc} = ι_{ac} on W_{abc}, since this is so in a neighborhood of constant loops, and W_{abc} is connected.

Now, to obtain a continuation of h_{ab} in (5.4), construct a partition of unity σ1, σ2, σ3 ∈ C^∞(S^1), so that supp(σi + σj) ≠ S^1 and \( \bigcap_1^3 \text{supp } σ_j = \emptyset \). Setting σ0 = σ3, in light of Lemma 5.2 we can rewrite (5.4)

\[
h_{ab}(x) = \sum_1^3 \int_a^b \tilde{F} \left( \zeta, \frac{(\sigma_j-1 + \sigma_j)\dot{x}}{(\zeta - x)^2} \right) d\zeta - \sum_1^3 \int_a^b \tilde{F} \left( \zeta, \frac{\sigma_j\dot{x}}{(\zeta - x)^2} \right) d\zeta,
\]

and continue each term to LU_{ab}, as above. We obtain a holomorphic cocycle β(F) = h = (h_{ab}), with h_{ab} depending holomorphically on a, b, and one easily checks that each h_{ab} is cuspidal. Therefore β(F) ∈ \( \mathcal{H} \). Finally, \( \alpha β(F) \) can be computed by considering \( d_\zeta h_{a\zeta}(x) \) with a in the same component of \( \mathbb{P}_1 \setminus x(S^1) \) as ζ, so that (5.4) gives

\[
d_\zeta h_{a\zeta}(x) = d_\zeta h_{a\zeta}(x) = \tilde{F} \left( \zeta, \frac{\dot{x}}{(\zeta - x)^2} \right) d\zeta = F \left( \zeta, \frac{1}{\zeta - x} \right) d\zeta.
\]

Thus \( \alpha β(F) = F \) as needed.

(b) The kernel of \( α \). Take an irreducible submodule of Ker \( α \), spanned by a vector l of lowest weight \( -n \leq 0 \). Since \( F = α(l) = 0 \), (4.2) implies \( ι_{ab}(x) = 0 \) if x does not separate a, b; hence, by analytic continuation, whenever \( \text{ind}_{ab}x = 0 \). By the cocycle relation \( ι_{ac}(x) = ι_{bc}(x) \) if \( \text{ind}_{ab}x = 0 \), i.e., if \( \text{ind}_{ac}x = \text{ind}_{bc}x \).

Consider the components of \( LU_{0\infty} \)

\[
X_r = \{ x ∈ LU_{0\infty}; \text{ind}_{0\infty}x = r \}, \quad r ∈ \mathbb{Z}.
\]

Let

\[
x_1(t) = e^{irt}, \quad y(t) = e^{2irt} + e^{3irt-4}.
\]

We shall shortly show that whenever \( x ∈ LU_{0\infty} \) is in a sufficiently small neighborhood of \( x_1 \), and \( (κ, \lambda) ∈ \mathbb{C}^2 \setminus (0,0) \), then \( z_{κλ} = κx + λy ∈ X_r + \mathbb{C} \). It follows that with such \( x, y \) we can define \( h(κ, λ) = ι_{a∞}(z_{κλ}) \), where a is chosen so that \( \text{ind}_{a∞}z_{κλ} = r \). Thus \( h ∈ \mathcal{O}(\mathbb{C}^2 \setminus (0,0)) \), and by Hartogs’ theorem it extends to all of \( \mathbb{C}^2 \); also, it is homogeneous of degree \( -n \). It follows that \( h \) is constant, indeed zero when \( n > 0 \). In all cases \( ι_{0∞}(x) = h(1,0) = h(0,1) \) is independent of \( x \). This being true for \( x \) in a nonempty open set, \( ι_{0∞} \) is constant on \( X_r \). It follows that \( ι_{a∞}(x) = ι_{0∞}(x - a) \) is locally constant, and so is \( ι_{ab} = ι_{a∞} - ι_{b∞} \). Moreover, \( ι_{ab} = 0 \) unless \( n = 0 \).

Suppose now \( n = 0 \), and let \( ι_{0∞}|X_1 = l ∈ \mathbb{C} \). We have \( ι_{a∞}(x) = ι_{0∞}(x - a) = l \) if \( \text{ind}_{a∞}x = 1 \). Choose a homeomorphic \( x ∈ L \mathcal{C} \) and \( a, b ∈ \mathbb{C} \setminus x(S^1) \) so that \( \text{ind}_{ab}x = 1 \); say \( b \) is in the unbounded component. Then \( ι_{ab}(x) = ι_{a∞}(x) - ι_{b∞}(x) = l \), and the same will hold if \( x \) is slightly perturbed. It follows that \( ι_{ab}(x) = l \) whenever \( \text{ind}_{ab}x = 1 \), and in
Following the proof of Theorem 5.1, to compute $\mathfrak{h}$, the substitution

$$F_{\mathfrak{h}, \mathfrak{z}} = \sum_{1}^{m} t_{a_{j-1}a_{j}}(y) = l \sum_{1}^{m} \text{ind}_{a_{j-1}a_{j}}y = l \text{ind}_{a_{j-1}a_{j}}y.$$ 

The upshot is that any irreducible submodule of $\text{Ker} \mathfrak{a}$ is spanned by $\mathfrak{h}$ in (5.1), whence $\text{Ker} \mathfrak{a}$ itself is spanned by $\mathfrak{h}$, as claimed.

We still owe the proof that $\kappa x + \lambda y \in X_{r} + \mathbb{C}$ unless $\kappa = \lambda = 0$, for $x$ near $x_{1}$ and $y$ given in (5.7). In fact, the general statement follows once we prove it for $r = 1$ and $x = x_{1}$, that we henceforward assume. If $|\kappa| \geq 2|\lambda|$ then $z_{\kappa \lambda} \in X_{1}$ by Rouché’s theorem. Otherwise consider the polynomial

$$P(\zeta) = \kappa \zeta + \lambda (\zeta^{2} + e^{-4} \zeta^{3}), \quad \zeta \in \mathbb{C}.$$ 

For fixed $|\zeta| < 2$ the equation $P(\eta) = P(\zeta)$ has two solutions with $|\eta| < 5$, again by Rouché’s theorem. One of the solutions is $\eta = \zeta$. Let $\eta = R(\zeta)$ be the other one, so that $R$ is holomorphic. There are only finitely many $\zeta$ with $|\zeta| = |R(\zeta)| = 1$. Indeed, otherwise $|R(\zeta)| = 1$ would hold for all unimodular $\zeta$, and by the reflection principle $R$ would be rational. However, $P(R(\zeta)) = P(\zeta)$ cannot hold with rational $R(\zeta) \neq \zeta$. We conclude that $z_{\kappa \lambda}(S^{1})$ has only finitely many self-intersection points.

Since $P(0) = 0$, $\text{ind}_{0, \infty} z_{\kappa \lambda} \geq 1$. Drag a point $a$ from $0$ to $\infty$ along a path that avoids multiple points of $z_{\kappa \lambda}(S^{1})$. Each time we cross $z_{\kappa \lambda}(S^{1})$, $\text{ind}_{a, \infty} z_{\kappa \lambda}$ changes by $\pm 1$. It follows that $\text{ind}_{a, \infty} z_{\kappa \lambda} = 1$ for some $a$, which completes the proof.

For the space $L_{1, P_{1}}$ Theorems 2.1, 2.2, and the construction in Theorem 5.1 lead to explicit representations of elements of $\mathfrak{h}$. First there are the multiples of the cocycle (5.1), and then there is the complementary subspace $\beta(\mathfrak{h}) = \bigoplus_{n \leq p-1} \beta(\mathfrak{h}^{n})$, see Theorem 2.2. According to Theorems 2.1, 2.2 elements of $\mathfrak{h}^{n}$ are of form

$$F(\zeta, y) = \sum_{\nu = 0}^{2n} \zeta^{\nu} \langle \Phi_{\nu}, \dot{y}^{n+1} \rangle, \quad \Phi_{\nu} \in L^{p/(n+1)}(S^{1})^{*}.$$ 

Following the proof of Theorem 5.1, to compute $\mathfrak{h} = \beta(F)$ we set $\tilde{F}(\zeta, z) = \sum_{\nu} \zeta^{\nu} \langle \Phi_{\nu}, z^{n+1} \rangle$. The substitution $\zeta = \xi + c$ shows that

$$R_{\nu}(a, b, c) = \int_{a}^{b} \frac{\zeta^{\nu} d\zeta}{(\zeta - c)^{2n+2}}, \quad 0 \leq \nu \leq 2n, \quad c \in P_{1} \setminus \{a, b\},$$ 

are rational functions with poles at $c = a, b$, so that

$$\mathfrak{h}_{ab}(x) = \int_{a}^{b} \tilde{F} \left( \zeta, \frac{\dot{x}}{(\zeta - x)^{2}} \right) d\zeta = \sum_{\nu = 0}^{2n} \langle \Phi_{\nu}, R_{\nu}(a, b, x) x^{n+1} \rangle,$$
when \( x \) does not separate \( a, b \). However, the right hand side makes sense for any \( x \in LU_{ab} \) and, as one checks, defines \( \mathfrak{h} = \beta(F) \). For example, if \( F \), hence \( \mathfrak{h} \) are of lowest weight, then \( \Phi_{\nu} = 0 \) for \( \nu \geq 1 \), and

\[
(5.8) \quad \mathfrak{h}_{ab}(x) = \left\langle \Phi_0, \frac{\dot{x}^{n+1}}{2n+1} \left( \frac{1}{(x-a)^{2n+1}} - \frac{1}{(x-b)^{2n+1}} \right) \right\rangle.
\]

Letting \( n = 0 \) and \( \langle \Phi_0, z \rangle = \int_{S^1} z/2\pi i \), formula (5.8) recovers the locally constant cocycle (5.1) as well. Thus we proved

**Theorem 5.3.** In the case of \( W^{1,p} \) loop spaces, any lowest weight cocycle in the \( n \)'th isotypical subspace \( \mathfrak{h}^n \subset \mathfrak{h} \) is of form (5.8) with (a unique) \( \Phi_0 \in L^{p/(n+1)}(S^1) \), \( 0 \leq n \leq p-1 \).

### 6. Synthesis

In this last section we show how the results obtained by now imply the theorems of the Introduction. Theorems 0.1 and 0.2 follow from the isomorphism \( H^{0,1}(L\mathbb{P}_1) \approx \mathfrak{h} \) of \( G \)-modules (Theorem 3.3) and from the isomorphism \( \mathfrak{h} \approx \mathbb{C} \oplus \mathfrak{g} \), a consequence of Theorem 5.1. In particular, \( H^{0,1}(L\mathbb{P}_1)^G \approx \mathbb{C} \oplus \mathfrak{g}^0 \). The latter being isomorphic to the dual of \( L^{-1} = C^{k-1}(S^1) \), resp. \( W^{k-1,p}(S^1) \) by Theorem 2.1, Theorem 0.3 also follows. Finally, Theorem 0.4 is a consequence of Theorems 2.2 and 2.1.

Seemingly we are done with all the proofs. However, Theorem 3.3 has not yet been proved for loop spaces \( L_{1,p}\mathbb{P}_1 \), \( p < 3 \), and we still have to revisit spaces of loops of low regularity. This will give us the opportunity to explicitly represent classes in \( H^{0,1}(L_{1,p}\mathbb{P}_1) \), in fact, for all \( p \in [1, \infty) \).

Generally, given a complex manifold \( M \), \( 1 \leq p < \infty \), and a natural number \( m \leq p \), consider the space \( C^\infty_{0,q}((T^*M)^\otimes m) \) of \( (T^*M)^\otimes m \) valued \((0, q)\) forms on \( M \). If \( \omega \) is such a form, \( v \in \bigoplus q T^{0,1}_s \), and \( w \in T^{1,0}_s M \), we can pair \( \omega(v) \in (T^*M)^\otimes m \) with \( w^\otimes m \), to obtain what we shall denote \( \omega(v, w^m) \in \mathbb{C} \). Write \( LM \) for the space of \( W^{1,p} \) loops in \( M \), and observe that the tangent space \( T^x_{LM} \) is naturally isomorphic to the space \( W^{1,p}(x^*T^{0,1}M) \) of \( W^{1,p} \) sections of the induced bundle \( x^*T^{0,1}M \to S^1 \) (see [L2, Proposition 2.2] in the case of \( C^k \) loops).

There is a bilinear map

\[
I = I_q: L^{p/m}(S^1)^* \times C^\infty_{0,q}((T^*M)^\otimes m) \to C^\infty_{0,q}(LM),
\]

obtained by the following Radon type transformation. If

\[
(\Phi, \omega) \in L^{p/m}(S^1)^* \times C^\infty_{0,q}((T^*M)^\otimes m),
\]

\( x \in LM \), and \( \xi \in \bigoplus q T^{0,1}_x LM \approx \bigoplus q W^{1,p}(x^*T^{0,1}M) \), then \( \omega(\xi, \dot{x}^m) \in L^{p/m}(S^1) \). Define \( I(\Phi, \omega) = f \) by

\[
f(\xi) = \langle \Phi, \omega(\xi, \dot{x}^m) \rangle.
\]
One verifies that $\bar{\partial}(\Phi, \omega) = I(\Phi, \bar{\partial}\omega)$, whence $I_q$ induces a bilinear map

$$L^{p/m}(S^1)^* \times H^{0,q}((T^*M)^{\otimes m}) \to H^{0,q}(LM).$$

Henceforward we take $M = \mathbb{P}_1$, $q = 1$, $m = n + 1$, and $\omega$ given on $\mathbb{C}$ by

$$\omega = \frac{-1}{2n+1} \frac{\zeta^{-2n} d\zeta \otimes (d\zeta)^{n+1}}{(1 + |\zeta|^{4n+2})^{(2n+2)/(2n+1)}}, \quad \zeta \in \mathbb{C},$$

so that $f = I_1(\Phi, \omega)$ is a closed form on $L\mathbb{P}_1$. Explicitly,

$$(6.1) \quad f(\xi) = \frac{-1}{2n+1} \left( \phi \left( \frac{\xi^{2n+1}}{(1 + |\xi|^{2n+2})^{(2n+2)/(2n+1)}} \right) \right), \quad \xi \in T_{x_1}^0 L\mathbb{P}_1.$$

To compute its image in $\mathfrak{H}$ under the map of Theorem 3.3, let

$$\theta_a = \frac{1}{2n+1} \left( \frac{\zeta^{-2n-1}}{(1 + |\zeta|^{4n+2})^{1/(2n+1)}} - \zeta^{-2n-1} + (\zeta - a)^{-2n-1} \right) (d\zeta)^n \quad \text{on } U_a.$$

Thus $\bar{\partial}\theta_a = \omega|_{U_a}$, and the cuspidal functions $u_a = I_0(\Phi, \theta_a) \in C^\infty(U_a)$ solve $\bar{\partial}u_a = f|_{LU_a}$. Hence the image of $f$ in $\mathfrak{H}$ is

$$\mathfrak{h}_{ab}(x) = u_a(x) - u_b(x) = \left( \phi \left( \frac{1}{(x-a)^{2n+1}} - \frac{1}{(x-b)^{2n+1}} \right) \right).$$

Comparing this with Theorem 5.3 we see that by associating with a lowest weight $\mathfrak{h} \in \mathfrak{H}^n$ the functional $\Phi = \Phi_0$ of (5.8), and then $f \in C^\infty_0(L\mathbb{P}_1)$ of (6.1), the image of $f$ in $\mathfrak{H}$ will be $\mathfrak{h}$. In particular, the class $[f] \in H^{0,1}(L\mathbb{P}_1)$ is also of lowest weight $-n$. Therefore the linear map $\mathfrak{h} \mapsto [f]$, defined for $\mathfrak{h} \in \mathfrak{H}^n$ of lowest weight, can be extended to a $G$–morphism $\mathfrak{H}^n \to H^{0,1}(L\mathbb{P}_1)$, and then to a $G$–morphism $\bigoplus_{n \leq p-1} \mathfrak{H}^n = \mathfrak{H} \to H^{0,1}(L\mathbb{P}_1)$, inverse to the morphism $H^{0,1}(L\mathbb{P}_1) \to \mathfrak{H}$ of Theorem 3.3. This completes the proof of Theorem 3.3, and now we are really done.

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