Variance reduction for Langevin Monte Carlo in high dimensional sampling problems

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Abstract

Sampling from a log-concave distribution function is one core problem that has wide applications in Bayesian statistics and machine learning. While most gradient free methods have slow convergence rate, the Langevin Monte Carlo (LMC) that provides fast convergence requires the computation of gradients. In practice one uses finite-differencing approximations as surrogates, and the method is expensive in high-dimensions.

A natural strategy to reduce computational cost in each iteration is to utilize random gradient approximations, such as random coordinate descent (RCD) or simultaneous perturbation stochastic approximation (SPSA). We show by a counter-example that blindly applying RCD does not achieve the goal in the most general setting. The high variance induced by the randomness means a larger number of iterations are needed, and this balances out the saving in each iteration.

We then introduce a new variance reduction approach, termed Randomized Coordinates Averaging Descent (RCAD), and incorporate it with both overdamped and underdamped LMC. The methods are termed RCAD-O-LMC and RCAD-U-LMC respectively. The methods still sit in the random gradient approximation framework, and thus the computational cost in each iteration is low. However, by employing RCAD, the variance is reduced, so the methods converge within the same number of iterations as the classical overdamped and underdamped LMC [12, 10]. This leads to a computational saving overall.

1 Introduction

Sampling is one of the core problems in statistics, data assimilation [54], and machine learning [1], with wide applications in inverse problems [45], atmospheric science [23], petroleum engineering [50], remote sensing [36] and epidemiology [37] in the form of volume computation [64], and bandit optimization [60].

Let \( f(x) \) be a convex function that is \( L \)-gradient Lipschitz and \( M \)-strongly convex in \( \mathbb{R}^K \). Define the target probability density function \( p(x) \propto e^{-f} \), then \( p(x) \) is a log-concave function. To sample from the probability distribution induced by \( p(x) \) amounts to finding an \( x \in \mathbb{R}^K \) (or a list of \( \{x^i \in \mathbb{R}^K \} \)) that can be regarded as i.i.d. (independent and identically distributed) drawn from the distribution.

There is vast literature on sampling, and proposed methods include Markov chain Monte Carlo methods (MCMC) [56, 58] and Metropolis-Hasting based MCMC (MH-MCMC) [48, 29]. Langevin dynamics based methods (including both the overdamped Langevin [53, 58, 11, 12] and underdamped Langevin [9, 41, 10, 21] Monte Carlo), Hamiltonian Monte Carlo methods [51, 42, 43], their different levels of combination (such as MALA) [58, 57, 19, 7], and some ensemble Kalman filter type methods [32, 26, 54].

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One popular method is MCMC. The essence of the method is to develop a Markovian transition kernel whose invariant measure is the target distribution, so that after many rounds of iteration, the invariant measure is achieved. However, if the design of the transition kernel does not involve \( \nabla f \) or sense the local behavior of \( f \), the convergence is slow [31, 30, 59, 47].

The other end of the spectrum is to formulate overdamped or underdamped Langevin dynamics. This is to find stochastic differential equations (SDEs) whose equilibrium-in-time is the target distribution. These SDEs are typically driven by \( \nabla f \). The corresponding algorithms are Overdamped or Underdamped Langevin Monte Carlo (O/U-LMC) that can be viewed as the discrete-in-time (such as Euler-Maruyama discretization) version of the Langevin dynamics (SDEs). Since \( \nabla f \) leads the dynamics, fast converge is expected [12, 10].

However, \( \nabla f \) is typically not available. Indeed, when the explicit formula is unknown, one usually surrogates the gradients with their finite-difference approximations \( \partial_i f \approx [f(x + \eta e^i) - f(x - \eta e^i)]/2\eta \) for every direction \( e_i \). In high dimension, \( K \gg 1 \), this means at least \( K \)-times of differentiation of \( f \) are calculated, bringing up the numerical cost. Therefore, how to sample with a small number of finite differencing approximations becomes rather crucial.

There are methods proposed to achieve gradient-free property, such as Importance Sampling (IS), Ensemble Kalman methods, random walks methods, and various finite difference approximations to surrogate the gradient. However, IS [28, 16, 17] has high variance of the weight terms and it leads to wasteful sampling: ensemble Kalman methods [22, 4, 54, 26] usually require Gaussianity assumption [14, 15]; random walk methods such that Metropolized random walk (MRW) [47, 58, 59], Ball Walk [38, 20, 39] and the Hit-and-run algorithm [13, 34, 40] cannot guarantee fast convergence [63]; and to our best knowledge, modification of LMC with derivatives replaced by its finite difference approximation [46] or Kernel Hilbert space [62] are not yet equipped with theoretical non-asymptotic analysis.

1.1 Contribution

We work under the O/U-LMC framework, and we look for methods that produce i.i.d. samples with only a small number of gradient computation. To this end, the contribution of the paper is twofold.

We first examine a natural strategy to reduce the number of gradient computation by adopting RCD, a random directional gradient approximation. This method replaces \( K \) finite difference approximations in \( K \) directions, by 1 in a randomly selected direction. Presumably this reduces the cost in each iteration by \( K \) fold, and hopefully the total cost. However, in this article we will show that this is not the case in the general setting. We will provide a counter-example: the high variance induced by the random direction selection process brings up the numerical error, and thus more iterations are needed to achieve the preset error tolerance. This in the end leads to no improvement in terms of the computational cost.

We then propose a variance reduction method to improve RCD in the LMC framework. We call the method Randomized Coordinates Averaging Descent Overdamped/Underdamped LMC (or RCAD-O/U-LMC). The methods start with a fully accurate gradient in the first round of iteration, and in the subsequent iterations they only update the gradient evaluation in one randomly selected direction. Since the method preserve some information about the gradient along the evolution, the variance is reduced. We prove the new methods converge as fast as the classical O/U-LMC [12, 10], namely the preset error tolerance is achieved in the same number of iterations. But since they require only 1 directional derivative per iteration instead of \( K \), the overall cost is reduced. We summarize the advantage over the classical O-LMC and U-LMC in Table 1 (assuming the standard finite-differencing is performed in each direction). The dependence on the conditioning of \( f \) is omitted in the table, but will be discussed in detail in Section 5.

In some sense, the new methods share some similarity with SAGA [13], a modification of SAG (stochastic average gradient) [61]. These are two methods designed for reducing variance in the stochastic gradient descent (SGD) framework where the cost function \( f \) has the form of \( \sum_i f_i \). Similar approaches are also found in SG-MCMC (stochastic-gradient Markov chain Monte Carlo (SG-MCMC)) [41, 23, 3, 46, 8]. In their cases, variance reduction is introduced in the selection of \( \nabla f_i \). In our case, the cost function \( f \) is a simple convex function, but the gradient \( \nabla f \) can be viewed as \( \nabla f = \sum \partial_i f e^i \) and the variance reduction is introduced in the selection of \( \partial_i f e^i \).
There are other variance reduction methods, such as SVRG [33] and CV-ULD [2,8]. We leave the discussion to future research.

| Algorithm       | Number of iterations | Number of $f$ evaluations |
|-----------------|----------------------|---------------------------|
| O-LMC[12]       | $O(K/\epsilon)$      | $O(K^2/\epsilon)$        |
| U-LMC[10]       | $O(K^{3/2}/\epsilon)$ | $O(K^{3/2}/\epsilon)$    |
| RCAD-O-LMC      | $O(K^{3/2}/\epsilon)$ | $O(K^{3/2}/\epsilon)$    |
| RCAD-U-LMC      | $O(\max\{K^{4/3}/\epsilon^{2/3}, K^{1/2}/\epsilon\})$ | $O(\max\{K^{4/3}/\epsilon^{2/3}, K^{1/2}/\epsilon\})$ |

Table 1: Number of iterations and directional derivative evaluations of $f(x)$ to achieve $\epsilon$-accuracy. We assume finite difference type approximation is used for each direction. $K$ is the dimension. $\tilde{O}(f) = \tilde{O}(f \log f)$. For the overdamped cases, we assume the Lipschitz continuity for the hessian term. Without this assumption, RCAD-O-LMC still outperforms O-LMC, as will be discussed in Section 5.

## 1.2 Organization

In Section 2, we discuss the essential ingredients of our methods: the random coordinate descent (RCD) method, the overdamped and underdamped Langevin dynamics and the associated Monte Carlo methods (O-LMC and U-LMC). In Section 3, we unify the notations and assumptions used in our methods. In Section 4, we discuss the vanilla RCD applied to LMC and present a counter-example to show it is not effective if used blindly. In Section 5, we introduce our new methods RCAD-O/U-LMC and present the results on convergence and numerical cost.

## 2 Essential ingredients

### 2.1 Random coordinate descent (RCD)

The most straightforward approximation of gradients is to apply finite difference method. For approximating a directional derivative $\partial_i f$, one uses $\partial_i f(x) \approx \frac{f(x+\eta e^i) - f(x-\eta e^i)}{2\eta}$ where $e^i$ is the $i$-th unit direction. Given enough smoothness, the introduced error is $O(\eta^2)$. For approximating the entire $\nabla f$, $K$ such finite differencing is required, and it is expensive in the high dimensional setting when $K \gg 1$.

Ideally one can take one random direction and computes the derivative in that direction, and hopefully this random directional derivative reveals some information of the entire gradient $\nabla f$. This approach is used in both RCD [68, 55, 52] and SPSA [27, 35]. Both methods, instead of calculating the full gradient, randomly pick one direction and use the directional derivative to find an approximation to $\nabla f$. More specifically, RCD computes the derivative in one random unit direction $e^r$ and approximates:

$$\nabla f \approx K (\nabla f(x) \cdot e^r) e^r \approx K f(x + \eta e^r) - f(x - \eta e^r) \frac{2\eta}{e^r},$$

where $r$ is randomly drawn from $1, 2, \cdots, K$ (see the distribution of drawing in [55]). This approximations is consistent in the expectation sense because

$$E_r (K (\nabla f(x) \cdot e^r) e^r) = \nabla f(x).$$

Here $E$ is to take expectation.

### 2.2 Overdamped Langevin dynamics and O-LMC

The O-LMC method is derived from the Langevin equation:

$$dX_t = -\nabla f(X_t) dt + \sqrt{2} dB_t.$$

The SDE characterizes the trajectory of $X_t$. Two forcing terms $\nabla f(X_t)$ and $dB_t$ compete: the former drives $X_t$ to the minimum of $f$ and the latter provides small oscillations. The initial data $X_0$ is a random variable drawn from a given distribution induced by $q_0(x)$. Denote $q(x, t)$ the probability
The underdamped Langevin Monte Carlo algorithm, U-LMC, can be viewed as a numerical solver of the following Fokker-Planck equation:

\[ \partial_t q = \nabla \cdot (\nabla f q + \nabla q), \quad \text{with} \quad q(x, 0) = q_0, \]  

and furthermore, \( q(x, t) \) converges to the target density function \( p(x) = e^{-f} \) exponentially fast in time [44].

The overdamped Langevin Monte Carlo (O-LMC), as a sampling method, is simply a discrete-in-time version of the SDE (2). A standard Euler-Maruyama method applied on the equation gives:

\[ x^{m+1} = x^m - \nabla f(x^m)h + \sqrt{2h}\xi^m, \]  

where \( \xi^m \) is i.i.d. drawn from \( N(0, I_K) \) with \( I_K \) being the identity matrix of size \( K \). Since (4) approximates (2), the density of \( x^m \), denoted as \( p_m(x) \), converges to \( p(x) \) as \( m \to \infty \). It was proved in [12] that the convergence to \( \epsilon \) is achieved within \( O(K/\epsilon) \) iterations if hessian of \( f \) is Lipschitz. If hessian of \( f \) is not Lipschitz, the number of iterations increase to \( O(K/\epsilon^2) \). In many real applications, the gradient of \( f \) is not available and some approximation is used, introducing another layer of numerical error. In [12], the authors did discuss the effect of such error, but they assumed the error has bounded variance.

### 2.3 Underdamped Langevin dynamics and U-LMC

The underdamped Langevin dynamics [9] is characterized by the following SDE:

\[
\begin{cases}
  dX_t = V_t dt \\
  dV_t = -2V_t dt - \gamma \nabla f(X_t) dt + \sqrt{4\gamma} dB_t,
\end{cases}
\]  

where \( \gamma > 0 \) is a parameter to be tuned. Denote \( q(x, v, t) \) the probability density function of \( (X_t, V_t) \), then \( q \) satisfies the Fokker-Planck equation

\[ \partial_t q = \nabla \cdot \left( \begin{bmatrix} -v \\ 2v + \gamma \nabla f \end{bmatrix} q + \begin{bmatrix} 0 & 0 \\ 0 & 2\gamma \end{bmatrix} \nabla q \right), \]

and under mild conditions, it converges to \( p_2(x, v) = \exp\left(-\frac{(f(x) + |v|^2)}{2\gamma}\right) \), making the marginal density function of \( x \) the target \( p(x) \).

The underdamped Langevin Monte Carlo algorithm, U-LMC, can be viewed as a numerical solver to (5). In each step, we sample new particles \( (x^{m+1}, v^{m+1}) \sim (Z_{x}^{m+1}, Z_{v}^{m+1}) \in \mathbb{R}^{2K} \), where \( (Z_{x}^{m+1}, Z_{v}^{m+1}) \) is a Gaussian random vector determined by \( (x^m, v^m) \) with the following expectation and covariance:

\[
\begin{align*}
  \mathbb{E} Z_{x}^{m+1} &= x^m + \frac{1}{2} \left( 1 - e^{-2h} \right) v^m - \frac{\gamma}{2} \left( h - \frac{1}{2} \left( 1 - e^{-2h} \right) \right) \nabla f(x^m), \\
  \mathbb{E} Z_{v}^{m+1} &= v^m e^{-2h} - \frac{\gamma}{2} \left( 1 - e^{-2h} \right) \nabla f(x^m), \\
  \text{Cov} (Z_{x}^{m+1}) &= \gamma \left[ h - \frac{3}{4} - \frac{1}{4} e^{-4h} + \frac{e^{-2h}}{2} \right] \cdot I_K, \\
  \text{Cov} (Z_{v}^{m+1}) &= \frac{\gamma}{2} \left[ 1 + e^{-4h} - 2e^{-2h} \right] \cdot I_K.
\end{align*}
\]

We here used the notation \( \mathbb{E} \) to denote the expectation, and \( \text{Cov}(a, b) \) to denote the covariance of \( a \) and \( b \). If \( b = a \), we abbreviate it to \( \text{Cov}(a) \). The scheme can be interpreted as sampling from the following dynamics in each time interval:

\[
\begin{cases}
  X_t = x^m + \int_0^t V_s ds \\
  V_t = v^m e^{-2t} - \frac{\gamma}{2} (1 - e^{-2t}) \nabla f(x^m) + \sqrt{4\gamma} e^{-2t} \int_0^t e^{2s} dB_s.
\end{cases}
\]

The advantage of underdamped Langevin dynamics over the overdamped Langevin dynamics is unclear, but U-LMC does demonstrate faster convergence rate [10] than O-LMC. Without the assumption on the hessian of \( f \) being Lipschitz, the number of iteration is \( O(\sqrt{K}/\epsilon) \) to achieve \( \epsilon \).
accuracy. The faster convergence on the discrete level should be explained by the better discretization solver instead of faster convergence of the underlying SDEs. Indeed, without the Lipschitz continuity on the hessian term, the discretizing of (5) produces \( O(h^2) \) numerical error. In contrast, the discretization error of (4) is \( O(h^{3/2}) \). A third-order discretization was discussed for (5) in [49], further enhancing the numerical accuracy. Similar to O-LMC, the method here also uses finite differencing to approximate \( \nabla f(x_m) \). This induces another layer of error, and also requires \( K \) times of evaluation and differentiation of \( f \).

3 Notations

3.1 Assumption

We make some standard assumptions on \( f(x) \):

Assumption 3.1. The function \( f \) is \( M \)-strongly convex and has an \( L \)-Lipschitz gradient:

- Convex, meaning for any \( x, x' \in \mathbb{R}^K \):
  \[
  f(x) - f(x') - \nabla f(x')^\top (x - x') \geq (M/2)|x - x'|^2. \tag{7}
  \]

- Gradient is Lipschitz, meaning for any \( x, x' \in \mathbb{R}^K \):
  \[
  |\nabla f(x) - \nabla f(x')| \leq L|x - x'|. \tag{8}
  \]

If \( f \) is second-order differentiable, these assumptions together mean \( MI_K \preceq H(f) \preceq LI_K \) where \( H(f) \) is the hessian of \( f \). We also define condition number of \( f(x) \) as

\[
R = L/M \geq 1. \tag{9}
\]

We will express our results in terms of \( R \) and \( M \). Furthermore, for some results we assume Lipschitz condition of the hessian too:

Assumption 3.2. The function \( f \) is second-order differentiable and the hessian of \( f \) is \( H \)-Lipschitz, meaning for any \( x, x' \in \mathbb{R}^K \):

\[
\|H(f)(x) - H(f)(x')\|_2 \leq H|x - x'|. \tag{10}
\]

3.2 Wasserstein distance

The Wasserstein distance is a classical quantity that evaluates the distance between two probability measures:

\[
W_p(\mu, \nu) = \left( \inf_{(X,Y) \in C(\mu,\nu)} \mathbb{E}|X - Y|^p \right)^{1/p},
\]

where \( C(\mu, \nu) \) is the set of distribution of \( (X, Y) \in \mathbb{R}^{2K} \) whose marginal distributions, for \( X \) and \( Y \) respectively, are \( \mu \) and \( \nu \). These distributions are called the couplings of \( \mu \) and \( \nu \). Here \( \mu \) and \( \nu \) can be either probability measures themselves or the measures induced by probability density functions \( \mu \) and \( \nu \). In this paper we mainly study \( W_2 \).

4 Direct application of RCD in LMC, a negative result

We study if RCD can be blindly applied to U-LMC for reducing numerical complexity. This is to replace \( \nabla f \) in the updating formula (4) for U-LMC by the random directional derivative surrogates (1). The resulting algorithms are presented as Algorithm 2 in Appendix A.1.

RCD was introduced in optimization. In [53], the authors show that despite RCD computes only 1, instead of \( K \) directional derivatives in each iteration, the number of iteration needed for achieving \( \epsilon \)-accuracy is \( O(K/\epsilon) \), as compared to \( O(1/\epsilon) \) when the full-gradient is used (suppose Lipschitz coefficient in each direction is at the same order with the total Lipschitz constant). This means there are counter-examples for which RCD cannot save compared with ordinary gradient descent. We emphasize that there are of course also plenty examples for which RCD significantly outperforms
when the hessian has special structures. In this article we would like to investigate the general lower-bound situations.

The story is the same for sampling. There are examples that show directly applying the vanilla RCD to U-LMC fails to outperform the classical U-LMC. One example is the following: assume 

\[ p_2(x, v) = \frac{1}{(2\pi)^{K/2}} \exp(-|x|^2/2 - |v|^2/2), \]

Denote \(\{(x^m, v^m)\}\) the trajectory of the sample computed through Algorithm 2 (underdamped) using RCD with stepsize \(h\). Let \(\eta\) be extremely small and the finite differencing error is negligible, and denote \(q_m\) the probability density function of \((x^m, v^m)\), then we can show \(W_2(q_m, p_2)\) cannot converge too fast.

**Theorem 4.1.** For the example above, choose \(\gamma = 1\), there exists uniform nonzero constant \(C_1, C_2\) such that if \(K, h\) satisfies

\[ K > \max\{48, C_1\}, h < \min\left\{\frac{1}{(1 + 2C_2)}, \frac{1}{4000K^2}\right\}. \]

If \(q_m\) satisfies

\[ W_2(q_m, p_2) \leq \sqrt{K}/2 \]  

for any \(m\), then

\[ W_2(q_m, p_2) \geq \frac{1}{28\sqrt{K}} (1 - 2h + 3h^2 + Ch^3)^m E|w^0|^2 + \frac{K^{3/2}h}{28}. \]  

The proof is found in Appendix A.2. The condition (11) is easy to satisfied if we choose a good enough initial condition and use convergence of RCD-U-LMC, which we omitted in this paper but can be worked out using similar technics.

We note the second term in (12) is rather big. The smallness comes from \(h\), the stepsize, and it needs be small enough to balance out the influence from \(K^{3/2} \gg 1\). This puts strong restriction on \(h\). Indeed, to have \(\epsilon\)-accuracy, \(W(q_m, p_2) \leq \epsilon\), we need both terms smaller than \(\epsilon\), and this term suggests that \(h \leq \frac{28}{h^3}K\) at least. And when combined with restriction from the first term, we arrive at the conclusion that at least \(\widetilde{O}(K^{3/2}/\epsilon)\) iterations are needed, and thus \(\widetilde{O}(K^{3/2}/\epsilon)\) finite differencing approximation are required. The \(K\) dependence is \(K^{3/2}\), and is exactly the same as that in U-LMC, meaning RCD-U-LMC brings no computational advantage over U-LMC.

We emphasize that that large second term, as shown in the proof, especially in Appendix A.2 equation (27), is induced exactly due to the high variance in the gradient approximation. If the variance can be controlled by a smaller value, this term can be reduced, which would eventually lead to a smaller number of needed iteration, and thus a lower numerical cost. In this paper we do not discuss other random direction approximations, such as SPSA (simultaneous perturbation stochastic approximation). With some calculation, it can be shown SPSA applied blindly on LMC will also lead to high variance, bringing no numerical saving. For the brevity of the paper we omit the discussion.

**5 Random direction approximation with variance reduction on O/U-LMC, two positive results**

The direct application of RCD induces high variance and thus high error. It leads to many more rounds of iterations for convergence, gaining no numerical saving in the end. In this section we propose RCAD-O/U-LMC with RCAD reducing variance in the framework of RCD. We will prove that while the numerical cost per iteration is reduced by \(K\)-fold, the number of required iteration is mostly unchanged.

**5.1 Algorithm**

The key idea is to compute one accurate gradient at the very beginning in iteration No. 1, and to preserve this information along the iteration to prevent the high variance. The algorithms for RCAD-O-LMC and RCAD-U-LMC are both presented in Algorithm 1 based on overdamped and
underdamped Langevin dynamics. Potentially the same strategy can be combined with SPSA, we do not explore that direction in this article.

In the methods, an accurate gradient (up to a finite-differencing error) is used in the first step, denoted by \( g \approx \nabla f \), and in the subsequent iterations, only one directional derivative of \( f \) gets computed and updated in \( g \).

**Theorem 5.1.** Suppose \( C \) with the classical O-LMC and U-LMC methods \([12, 10]\). We emphasize that these two papers indeed discuss the numerical error in approximating the gradients, but they both require the variance of error being bounded, which is not the case here. One related work is \([8]\), where the authors construct a contraction map is used for U-LMC, but such map cannot be directly applied in our situation because the variance depends on the entire trajectory of samples. Furthermore, the history of the trajectory is reflected in each iteration, deeming the process to be non-Markovian. We need to re-engineer the iteration formula accordingly for tracing the error propagation.

**Algorithm 1 Randomized Coordinate Averaging Decent O/U-LMC (RCAD-O/U-LMC)**

**Preparation:**
1. Input: \( \eta \) (space stepsize); \( h \) (time stepsize); \( \gamma \) (parameter); \( K \) (dimension) and \( f(x) \).
2. Initial: (overdamped): \( x^0 \) i.i.d. sampled from a initial distribution induced by \( q_0(x) \) and calculate \( g^0 \in \mathbb{R}^K \):
   \[
g^0_i = \frac{f(x^0 + \eta \varepsilon^i) - f(x^0 - \eta \varepsilon^i)}{2\eta}, \quad 1 \leq i \leq K. \tag{13}
   \]
(underdamped): \( (x^0, v^0) \) i.i.d. sampled from a initial distribution induced by \( q_0(x, v) \) and calculate \( g^0 \in \mathbb{R}^K \) as in (13).

**Run:** For \( m = 0, 1, \cdots \)
1. Draw a random number \( r^m \) uniformly from 1, 2, \cdots, \( K \).
2. Calculate \( g^{m+1} \) and flux \( F^m \in \mathbb{R}^K \) by letting \( g^{m+1} = g^m \) for \( i \neq r_m \) and
   \[
g^{m+1}_{r_m} = \frac{f(x^m + \eta \varepsilon^{m+1}) - f(x^m - \eta \varepsilon^{m})}{2\eta}, \quad F^m = g^m + K (g^{m+1} - g^m). \tag{14}
   \]
3. (overdamped): Draw \( \xi^m \) from \( \mathcal{N}(0, I_K) \):
   \[
x^{m+1} = x^m - F^m h + \sqrt{2h} \xi^m. \tag{15}
   \]
(underdamped): Sample \( (x^{m+1}, v^{m+1}) \sim Z^{m+1} = (Z^x_{x+1}, Z^v_{v+1}) \) where \( Z^{m+1} \) is a Gaussian random variable with expectation and covariance defined in (6), replacing \( \nabla f(x^m) \) by \( F^m \).

**Output:** \( \{x^m\} \).

5.2 Convergence and numerical cost analysis

We now discuss the convergence of RCAD-O-LMC and RCAD-U-LMC, and compare the results with the classical O-LMC and U-LMC methods \([12, 10]\). We emphasize that these two papers indeed discuss the numerical error in approximating the gradients, but they both require the variance of error being bounded, which is not the case here. One related work is \([8]\), where the authors construct the Lyapunov function to study the convergence of SG-MCMC. Our proof for the convergence of RCAD-O-LMC is inspired by its technicalities. In \([10, 8]\), a contraction map is used for U-LMC, but such map cannot be directly applied in our situation because the variance depends on the entire trajectory of samples. Furthermore, the history of the trajectory is reflected in each iteration, deeming the process to be non-Markovian. We need to re-engineer the iteration formula accordingly for tracing the error propagation.

5.2.1 Convergence for RCAD-O-LMC

For RCAD-O-LMC, we have the following theorem:

**Theorem 5.1.** Suppose \( f \) satisfies Assumption \([3, 12]\) and \( h, \eta \) satisfy
   \[
h < \frac{1}{3(1 + 9K)R^2M}, \quad \eta < h. \tag{16}
   \]

The Wasserstein distance between \( q^O_m \), the probability density function of the sample \( x^m \) derived from Algorithm (overdamped), and \( p \), the target density function, satisfies
   \[
W_2(q^O_m, p) \leq \exp(-Mhm/4)\sqrt{1 + 1/2h^2} W_2(q^O_0, p) + 2h \sqrt{K^3 C_1 + K^2 C_2}. \tag{17}
   \]
Here \( C_1 = 77R^2M, C_2 = H^2/M^2 + 20R^2 + R^3 M/K \).
Theorem 5.2. Assume 

\[ \epsilon \]

The proof is included in Appendix B. The theorem gives us the strategy of designing stopping criterion: to achieve \( \epsilon \)-accuracy, meaning to have \( W_2(q^o_m, p) \leq \epsilon \), we can choose to set both terms in (17) less than \( \epsilon/2 \), and it leads to:

\[
\frac{1}{(1 + D)R}\left(\frac{1}{1648 R K}\right), \quad \eta < h^3,
\]

and

\[
m \geq \frac{4}{hM} \log \left( \frac{2\sqrt{1 + 1/R^2W_2(q_0, p)}}{\epsilon} \right).
\]

This means \( O(K^{3/2}/\epsilon) \) times of finite differencing.

Note that the theorem here requires both Assumptions 3.1 and 3.2. We can relax the second assumption. If so, the numerical cost of degrades to \( \tilde{O}(\max\{K^{3/2}/\epsilon, K/\epsilon^2\}) \), whereas O-LMC with standard finite-differencing requires \( O(K^2/\epsilon^2) \). Our strategy still outperforms. The proof is the same, and we omit it from the paper.

5.2.2 Convergence for RCAD-U-LMC

For RCAD-U-LMC, we have the following theorem.

Theorem 5.2. Assume \( f(x) \) satisfies Assumption 3.1 and set \( \gamma = 1/L \), then there exists a uniformly constant \( D > 0 \) such that if \( h, \eta \) satisfy

\[
h \leq \min \left\{ \frac{1}{(1 + D)R}, \frac{1}{1648 R K} \right\}, \quad \eta < h^3,
\]

then the Wasserstein distance between the distribution of the sample \( (x^m, v^m) \), derived from Algorithm 1 (underdamped), and distribution induced by \( p_2 \) (whose marginal density in \( x \) is \( p \)) decays as:

\[
W_2(q^U_m, p_2) \leq 4\sqrt{2}\exp(-hm/(8R))W_2(q^U_0, p_2) + 60\sqrt{h^3K^4/M} + 7\sqrt{Rh^2K/M} + 35\sqrt{Rh^5K^2}.
\]

The proof is included in Appendix C. To achieve \( \epsilon \)-accuracy, meaning to have \( W_2(q^U_m, p_2) \leq \epsilon \), we can choose all terms in (19) less than \( \epsilon/4 \). This gives:

\[
h \leq \min \left\{ \frac{\epsilon^{2/3}M^{1/3}}{(240)^{2/3}K^{4/3}}, \frac{\epsilon^{1/2}M^{1/2}}{28R^{1/2}K^{1/2}}, \frac{\epsilon^{2/5}}{(140)^{2/5}R^{1/5}K^{2/5}}, \frac{1}{(1 + D)R}, \frac{1}{1648 R K} \right\}
\]

and

\[
m \geq \frac{8R}{h} \log \left( \frac{16\sqrt{2W_2(q^U_0, p_2)}}{\epsilon} \right).
\]

This means \( \tilde{O}(\max\{K^{4/3}/\epsilon^{2/3}, K^{1/2}/\epsilon\}) \) evaluation and finite-differencing of \( f \).

6 Conclusion and future work

To our best knowledge, this is the first work that discusses both the negative and positive aspects of applying random gradient approximation, mainly RCD type, to LMC, in both over-damped and under-damped situations without and with variance reduction. Without the variance reduction we show the RCD-LMC has the same numerical cost as the classical LMC, and with variance reduction, the numerical cost is reduced in both overdamped and underdamped cases.

There are a few future directions that we would like to pursue. 1. Our method, in its current version, is blind to the structure of \( f \). The only assumptions are reflected on the Lipschitz bounds. In \([55, 52, 24]\) the authors, in studying optimization problems, propose to choose random directions according to the Lipschitz constant in each direction. The idea could potentially be incorporated in our framework to enhance the sampling strategy. 2. Our algorithms are designed based on reducing variance in the RCD framework. Potentially one can also apply variance reduction methods to improve SPSA-LMC. There are also other variance reduction methods that one could explore.
7 Broader Impact

This work does not present any foreseeable societal consequence.

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We apply RCD as surrogates of the gradient in O/U-LMC. This amounts to replacing the gradient terms in \( \text{(1)} \) using the approximation \( \text{(11)} \). The new methods are presented in Algorithm 2, termed RCD-O/U-LMC.

Algorithm 2 RCD-overdamped(underdamped) Langevin Monte Carlo

Preparation:
1. Input: \( \eta \) (space step); \( h \) (time step); \( \gamma \) (parameter); \( K \) (dimension); \( f(x) \).
2. Initial: \( \text{(overdamped):} \) \( x^0 \) i.i.d. sampled from a initial distribution induced by \( q_0(x) \).
   \( \text{(underdamped):} \) \( (x^0, v^0) \) i.i.d. sampled from the initial distribution induced by \( q_0(x, v) \).

Run: For \( m = 0, 1, \cdots \)
1. Finite difference: calculate flux approximation either by RCD:
\[
F^m = K \frac{f(x^m + \eta e^r) - f(x^m - \eta e^r)}{2\eta} e^r
\]
with \( r \) randomly drawn from \( 1, \cdots, K \).
2. \( \text{(overdamped):} \) Draw \( \xi^m \) from \( \mathcal{N}(0, I_K) \):
\[
x^{m+1} = x^m - F^m h + \sqrt{2h} \xi^m.
\]
\( \text{(underdamped):} \) Sample \( (x^{m+1}, v^{m+1}) \sim Z^{m+1} = (Z_x^{m+1}, Z_v^{m+1}) \) where \( Z^{m+1} \) is a Gaussian random variable with expectation and covariance defined in \( \text{(6)} \), replacing \( \nabla f(x^m) \) by \( F^m \).

Output: \( \{x^m\} \).

A.2 A counter-example

In this section, we prove Theorem 4.1.

First, we define \( w^m = x^m + v^m \), and denote \( u_m(x, w) \) the probability density of \( (x^m, w^m) \) and \( u^*(x, w) \) the probability density of \( (x, w) \) if \( (x, v = w - x) \) is distributed according to density function \( p_2 \). One main reason to change \( (x, v) \) to \( (x, w) \) is that in [10], the authors showed that the map \( (x_0, w_0) \rightarrow (x_t, w_t) \) induced from \( \text{(5)} \) is a contracting map for \( t \). From [10], we also have:
\[
| x^m - x |^2 + | v^m - v |^2 \leq 4( | x^m - x |^2 + | w^m - w |^2 ) \leq 16( | x^m - x |^2 + | v^m - v |^2 )
\]
and
\[
W_2^2(q_m, p_2) \leq 4W_2^2(u_m, u^*) \leq 16W_2^2(q_m, p_2).
\]

Proof of Theorem 4.1 According to \( \text{(23)} \), it suffices to find a lower bound for \( W_2^2(u_m, u^*) \). We first notice
\[
W_2(u_m, u^*) \geq \sqrt{ \int |w|^2 u_m(x, w) \, dw \, dx } - \sqrt{ \int |w|^2 u(x, w) \, dw \, dx }\]
\[
= \sqrt{ \int |w|^2 u_m(x, w) \, dw \, dx } - \sqrt{2K} = \sqrt{\mathbb{E}|w|^2} - \sqrt{2K},
\]

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where $E$ takes all randomness into account. This implies to prove (12), it suffices to find a lower bound for second moment of $w^m$. We divide the proof into several steps:

- **First step:** Lower bound for $\mathbb{E}|x^m|^2$ Similar to (24), we also have
  $$W_2(q_m, p) \geq |\sqrt{\mathbb{E}|x^m|^2} - \sqrt{K}|.$$  
  Use (11), we have
  $$\sqrt{\mathbb{E}|x^m|^2} \geq \sqrt{K}/2$$  
  for any $m \geq 0$.

- **Second step:** Iteration formula of $\mathbb{E}|w^m|^2$.

  By the special structure of $p$, we can calculate the second moment explicitly. Since $f(x)$ can be written as
  $$f(x) = \sum_{i=1}^{K} \frac{|x_i|^2}{2},$$
  in each step of RCD-U-LMC, according to (6), for each $m \geq 0$ and $1 \leq i \leq K$, we have

  \[
  \begin{align*}
  \mathbb{E} \left( x_i^{m+1} \mid x^m, v^m, r^m \right) &= x_i^m + \frac{1}{2} \left(1 - e^{-2h} \right) v_i^m - \frac{1}{2} \left(h - \frac{1}{2} \left(1 - e^{-2h} \right) \right) \left( x_i^m - E_i^m \right), \\
  \mathbb{E} \left( v_i^{m+1} \mid x^m, v^m, r^m \right) &= v_i^m e^{-2h} - \frac{1}{2} \left(h - \frac{1}{2} \left(1 - e^{-2h} \right) \right) \left( x_i^m - E_i^m \right), \\
  \mathbb{E} \left( w_i^{m+1} \mid x^m, v^m, r^m \right) &= \frac{1}{2} \left(1 + e^{-2h} \right) w_i^m + \frac{1}{2} \left(h - \frac{1}{2} \left(1 - e^{-2h} \right) \right) E_i^m, \\
  \text{Var} \left( x_i^{m+1} \mid x^m, v^m, r^m \right) &= h - \frac{3}{4} e^{-4h} + e^{-2h}, \\
  \text{Var} \left( v_i^{m+1} \mid x^m, v^m, r^m \right) &= 1 - e^{-4h}, \\
  \text{Cov} \left( x_i^{m+1}, v_i^{m+1} \mid x^m, v^m, r^m \right) &= \frac{1}{2} \left[1 + e^{-4h} - 2e^{-2h}\right],
  \end{align*}
  \]

  where $E^m \in \mathbb{R}^K$ is a random variable defined as  
  $$E_i^m = x_i^m - K x_i^m e_i^m$$
  and satisfies  
  $$\mathbb{E}_{r^m}(E_i^m) = 0, \quad \text{Var}_{r^m}(E_i^m)^2 = (K-1)|x_i^m|^2$$
  for each $1 \leq i \leq K$.

  Now, for simplicity, we replace $e^{-2h}$ and $e^{-4h}$ by their Taylor expansion:
  $$e^{-2h} = 1 - 2h + 2h^2 + C_1 h^3, \quad e^{-4h} = 1 - 4h + 8h^2 + C_2 h^3,$$
  where $C_1, C_2$ is a constant depends on $h$. Since $h < 1/4000$, we have
  $$|C_1| < 1000, |C_2| < 1000.$$  
  Plug (28) into (26), we have

  \[
  \begin{align*}
  \mathbb{E} \left( w_i^{m+1} \mid x^m, w^m, r^m \right) &= \left(1 - h + h^2 + \frac{C_1 h^3}{2} \right) w_i^m + \left(h - h^2 - \frac{C_1 h^3}{2} \right) E_i^m, \\
  \text{Var} \left( x_i^{m+1} \mid x^m, v^m, r^m \right) &= \left(C_1 - \frac{C_2}{4}\right) h^3, \\
  \text{Var} \left( v_i^{m+1} \mid x^m, v^m, r^m \right) &= 4h - 8h^2 - C_2 h^3, \\
  \text{Cov} \left( x_i^{m+1}, v_i^{m+1} \mid x^m, v^m, r^m \right) &= 2h^2 + \frac{C_2 - 2C_1}{2} h^3.
  \end{align*}
  \]
The last three equalities in (29) implies
\[ \text{Var} \left( w_i^{m+1} \mid (x^m, v^m, r^m) \right) = \text{Var} \left( x_i^{m+1} \mid (x^m, v^m, r^m) \right) + \text{Var} \left( v_i^{m+1} \mid (x^m, v^m, r^m) \right) + 2 \text{Cov} \left( x_i^{m+1}, v_i^{m+1} \mid (x^m, v^m, r^m) \right) \]
\[ = 4h - 4h^2 - \left( C_1 + \frac{C_2}{4} \right) h^3. \]

Then, we can calculate the iteration formula for \( E[x_i^{m+1}] \) and \( E[\omega_i^{m+1}] \):
\[
E[\omega_i^{m+1}] = E_{x^m, v^m, r^m} \left( \left| \omega_i^m \right|^2 (x^m, v^m, r^m) \right) \]
\[ = E_{x^m, v^m, r^m} \left( \left| E \left( w_i^{m+1} \mid (x^m, v^m, r^m) \right) \right|^2 + \text{Var} \left( w_i^{m+1} \mid (x^m, v^m, r^m) \right) \right) \]
\[ = \left( 1 - h + h^2 + \frac{C_1 h^3}{2} \right)^2 E[|w_i^m|^2] + (K - 1) \left( h - h^2 - \frac{C_1 h^3}{2} \right)^2 E[|x_i^m|^2] \]
\[ + 4h - 4h^2 - \left( C_1 + \frac{C_2}{4} \right) h^3. \]
where we use the cross term equals to zero:
\[ E_{x^m} \left( w_i^m, E_i^m \right) = 0 \]
and (27).

Sum them up with \( i \), we finally obtain an iteration formula for \( E[w^m]^2 \):
\[
E[\omega^{m+1}] = \left( 1 - h + h^2 + \frac{C_1 h^3}{2} \right)^2 E[|w^m|^2] + (K - 1) \left( h - h^2 - \frac{C_1 h^3}{2} \right)^2 E[|x^m|^2] \]
\[ + 4K - 4K h^2 - K \left( C_1 + \frac{C_2}{4} \right) h^3 \]
(30)

- **Third step: Lower bound for \( E[\omega^m]^2 \)**

Plug (25) into (30), we have
\[
E[\omega^{m+1}] \geq \left( 1 - h + h^2 + C' h^3 \right)^2 E[|w^m|^2] + K^2 h^2 + C' h^3 + 4K h - 4K h^2 + C' h^3
\]
where \( C' \) is a uniformly constant.

Choose \( C_1 \) such that \( C_1^2 = 8 |C'| \), then we have with
\[
K^2/3 + C' K h - 4K + C' K h > K^2/3 - 4K - 2 |C'| > K^2/4 + 2C' > 0,
\]
which implies
\[
K^2 (h^2 + C' h^3) + 4K h - 6K h^2 + C' K h^3 \geq 4K h + 2K^2 h^2 / 3.
\]

use the inequality iteratively, we finally have
\[
E[\omega^m] \geq \left( 1 - 2h + 3h^2 + Ch^3 \right)^m E[|w^0|^2] + \frac{4K + 2K^2 h/3}{2 - 3h + Ch^2},
\]
where \( C \in \mathbb{R} \) is a uniformly constant.

Choose \( C_2 = |C| \), then we have \( Ch^2 > -h \) and
\[
2 - 3h + Ch^2 > 2 - 4h > 4/3, \quad \frac{4K + 2K^2 h/3}{2 - 3h + Ch^2} \leq 3K + K^2 h/2 \leq 4K,
\]
which implies
\[
(1 - 2h + 3h^2 + Ch^3)^m E[|w^0|^2] + \frac{4K + 2K^2 h/3}{2 - 3h + Ch^2} \leq 20K, \quad (31)
\]
where we also use
\[
\mathbb{E}|w^0|^2 \leq \left( \sqrt{\mathbb{E}_t|w|^2} + 4W_2(q^m, p_2) \right)^2 \leq 16K
\]
Plug (31) into (24), we further have
\[
W_2(u_m, u^*) \geq \frac{(1 - 2h + 3h^2 + Ch^3)^m \mathbb{E}|w^0|^2 + \frac{4K + 2K^2}{h^2} - 2K}{\sqrt{(1 - 2h + 3h^2 + Ch^3)^m \mathbb{E}|w^0|^2 + \frac{4K + 2K^2}{h^2} + \sqrt{2K}}}
\]
\[
\geq \frac{1}{7\sqrt{K}} (1 - 2h + 3h^2 + Ch^3)^m \mathbb{E}|w^0|^2 + \frac{6K^{1/2}h + 2K^{3/2}h/3 - 2CK^{1/2}h^2}{14}
\]
\[
\geq \frac{1}{7\sqrt{K}} (1 - 2h + 3h^2 + Ch^3)^m \mathbb{E}|w^0|^2 + \frac{K^{3/2}h}{7},
\]
where we use \( Ch^2 < h \) in the last inequality. Use (23), we obtain (12).

\[\square\]

B Proof of convergence of RCAD-O-LMC (Theorem 5.1)

As presented in the main text, the first step of RCAD-O-LMC uses the finite differencing approximation for every direction, namely, setting \( g^0 \in \mathbb{R}^K \) to be:
\[
g^0_i = \frac{f(x^0 + \eta e^i) - f(x^0 - \eta e^i)}{2\eta}, \quad i = 1, 2, \ldots, K.
\]
In the following iterations, one random direction is selected for the updating,
\[
g_{m+1}^{r_m} = \frac{f(x^m + \eta e^{r_m}) - f(x^m - \eta e^{r_m})}{2\eta}
\]
with other directions untouched: \( g^{m+1}_i = g^m_i \) for all \( i \neq r_m \). Define:
\[
F^m = g^m + K (g^{m+1} - g^m),
\]
then the updating formula is:
\[
x^{m+1} = x^m - F^m h + \sqrt{2h} \xi^m, \tag{32}
\]
where \( h \) is the time stepsize, and \( \xi^m \) i.i.d. drawn from \( \mathcal{N}(0, I_K) \). Denote
\[
E^m = \nabla f(x^m) - F^m,
\]
then this updating formula (32) writes to:
\[
x^{m+1} = x^m - \nabla f(x^m) h + E^m h + \sqrt{2h} \xi^m. \tag{33}
\]
This is the formula we use for the analysis under Assumptions 3.1 and 3.2.

To show the theorem, we let \( y_0 \) be a random vector drawn from target distribution induced by \( p \) such that \( W_2^2(q_0^0, p) = \mathbb{E}|x^0 - y^0|^2 \), and set
\[
y_t = y_0 - \int_0^t \nabla f(y_s) \, ds + \sqrt{2} \int_0^t \, dB_s, \tag{34}
\]
where we construct the Brownian motion that always satisfies
\[
B_{h(m+1)} - B_{hm} = \sqrt{h} \xi^m. \tag{35}
\]
Then $y_t$ is drawn from target distribution as well. On the discrete level, let $y^m = y_{hm}$, then:

$$y^{m+1} = y^m - \int_{m}^{(m+1)h} \nabla f(y_s) \, ds + \sqrt{2h} \xi^m.$$  

Noting

$$W_2^2(q_m^p, p) \leq \mathbb{E}[x^m - y^m]^2,$$

where $\mathbb{E}$ takes all randomness into account. We now essentially need to show the difference between (32) and (34), also see [8].

As for a preparation, we now define an a set of auxiliary gradients.

- $g^0$ is the true derivative used at the initial step:

$$g^0 = \nabla f(x^0),$$  

(36)

- $g^{m+1}$ is the continuous version of $g^m$:

$$g^{m+1}_r = \partial_{r_m} f(x^m) \quad \text{and} \quad g^{m+1}_i = g^m_\ \text{if} \quad i \neq r_m,$$

(37)

- $F^m$ is the continuous version of $F^m$:

$$F^m = g^m + K (g^{m+1} - g^m).$$

- Define $\beta^m$ using (36) with the same $r_m$ but replacing $x^m$ with $y^m$:

$$\beta^0 = \nabla f(y^0)$$

and

$$\beta^{m+1}_r = \partial_{r_m} f(y^m) \quad \text{and} \quad \beta^{m+1}_i = \beta^m_\ \text{if} \quad i \neq r_m.$$  

Indeed in the later proof we will give an upper bound for the following Lyapunov function:

$$T^m = T_1^m + c_p T_2^m = \mathbb{E}[y^m - x^m]^2 + c_p \mathbb{E}[g^m - \beta^m]^2.$$  

(38)

where $c_p$ will be carefully chosen later.

We further define

$$E^m = \nabla f(x^m) - F^m = E^m + F^m - F^m,$$

this leads to $E^m = E^m - E^m + F^m$. The properties of $E^m$ will be discussed in Appendix [D]. To quantify $F^m - E^m$ straightforward: it can be bounded using mean-value theorem. Since:

$$|g^0_1 - g^0_1|^2 = \frac{|f(x^m + \eta e^i) - f(x^m - \eta e^i) - 2\eta \partial_x f(x^m)|^2}{2\eta} \leq \frac{(\partial_x f(z) - \partial_x f(x^m))^2}{2\eta} \leq L^2 \eta^2$$

where $z \in \mathbb{R}^K$ is a point between $x^m + \eta e^i$ and we use the fact that $\nabla f$ is $L$-Lipschitz. Similarly, for all $m$:

$$|g^m - g^m|^2 \leq L^2 \eta^2 K,$$

we have:

$$\left| F^m - E^m \right|^2 \leq 2 |g^m - g^m|^2 + 2K^2 |g^{m+1}_r - g^{m+1}_r|^2 < 2L^2 \eta^2 K + 8L^2 \eta^2 K^2.$$  

(39)

Now we present the iteration formula for $T_1^{m+1}$, $T_2^{m+1}$, in Lemma [B.1] and Lemma [B.2] respectively:

**Lemma B.1.** Under conditions of Theorem 5.4, for any $a > 0$, there are upper bounds of $T_1^m$ and $T_2^m$:

$$T_1^{m+1} \leq (1 + a) AT_1^m + (1 + a) BT_2^m + (1 + a) h^3 C + \left(1 + \frac{1}{a}\right) h^4 D,$$

(40)

where

$$A = 1 - 2 M h + 3(1 + 3 K) L^2 h^2, \quad B = 9 h^2 K,$$

$$C = 2L^2 K + 72L^2 K^3 \left[ \frac{h^2 K}{M} + 1 \right], \quad D = (H^2 + 16 L^2) K^2 + (L^3 + 4 L^2) K.$$  

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Note that for the proof to proceed, one at least needs the coefficient \((1 + a)A < 1\). This can be made possible only if \(a\) is small enough. For small \(a\), the \(h^4D\) term is magnified, but it may not matter as \(h^4\) serves as a high order error so the term is negligible so long as \(a \ll h^4\).

**Proof.** Define \(\Delta^m = y^m - x^m\), we first divide \(\Delta^{m+1}\) into several parts:

\[
\Delta^{m+1} = \Delta^m + (y^{m+1} - y^m) - (x^{m+1} - x^m)
\]

\[
= \Delta^m + \left( - \int_{mh}^{(m+1)h} \nabla f(y_s) \, ds + \sqrt{2h} \xi_m \right) - \left( - \int_{mh}^{(m+1)h} F^m ds + \sqrt{2h} \xi_m \right)
\]

\[
= \Delta^m - \left( \int_{mh}^{(m+1)h} (\nabla f(y_s) - F^m) \, ds \right)
\]

\[
= \Delta^m - \left( \int_{mh}^{(m+1)h} (\nabla f(y_s) - \nabla f(y^m) + \nabla f(y^m) - \nabla f(x^m) + \nabla f(x^m) - F^m) \, ds \right),
\]

\[
= \Delta^m - h(\nabla f(y^m) - \nabla f(x^m)) - \int_{mh}^{(m+1)h} (\nabla f(y_s) - \nabla f(y^m)) \, ds
\]

\[
- h(\nabla f(x^m) - F^m)
\]

\[
= \Delta^m - hU^m - (V^m + h\Phi^m) - hE^m
\]

\[
= \Delta^m - (V^m + h(F^m - F^m)) - h(U^m + \Phi^m + \tilde{E}^m)
\]  \hspace{1cm} (41)

where we set \(U^m = \nabla f(y^m) - \nabla f(x^m)\),

\[
V^m = \int_{mh}^{(m+1)h} \left( \nabla f(y_s) - \nabla f(y^m) - \sqrt{2} \int_{mh}^{s} \nabla^2 f(y_r) \, dB_r \right) \, ds,
\]

\[
\Phi^m = \frac{\sqrt{2}}{h} \int_{mh}^{(m+1)h} \int_{mh}^{s} \nabla^2 f(y_r) \, dB_r \, ds.
\]

Upon getting equation (41) it is time to analyze each term and hopefully derive an induction inequality that states \(\mathbb{E}|\Delta^{m+1}|^2 \approx c\mathbb{E}|\Delta^m|^2 + d\) with \(c < 1\) and \(d\) being of high order in \(\eta\) and \(h\), some parameters we can tune. Indeed the \(\Delta^m\) term is what we would like to preserve, and the \(U^m\) term depends on \(\Delta^m\) with a Lipschitz coefficient. The opposite signs of these two terms essentially indicate that \(\eta\) can be made small. The \(V^m + h\Phi^m\) completely depends on the one-time step error. In some sense, it is close to the forward Euler error obtained in one timestep. The \(E^m\) term is the most crucial term and the only term that reflects the error introduced by the algorithm in one time step. By choosing the right discretization in the algorithm to approximate \(\nabla f\), one could expect this term to be small. We leave the analysis of this term to Appendix [1] and focus on how the other terms interact here.

We first control last two terms in the last line of (41). According to Lemma 6 of [12], we first have

\[
\mathbb{E}|V^m|^2 \leq \frac{h^4}{2} (H^2 K^2 + L^3 K), \quad \mathbb{E}|\Phi^m|^2 \leq \frac{2L^2 hK}{3}, \quad (42)
\]

and thus:

\[
\mathbb{E}|V^m + h(F^m - F^m)|^2 \leq 2 \left( \mathbb{E}|V^m|^2 + h^2 \mathbb{E}|\tilde{E}^m - F^m|^2 \right)
\]

\[
\leq h^4 (H^2 K^2 + L^3 K) + 2h^2 (2L^3 \eta^2 K + 8L^3 \eta^2 K^2)
\]

\[
\leq (H^2 + 16L^2)h^4 K^2 + (L^3 + 4L^2)h^4 K = h^4 D, \quad (43)
\]

In particular, it is obvious that the square of all terms except \(\Delta^m\) contribute small values and will enter \(d\), and the cross terms would dominate.
where we use (39) and (42) in the second inequality and the condition of $h$ and $\eta$ in (16) in last inequality. We also have:

$$E|U^m + \Phi^n + \tilde{E}^m|^2$$

$$\leq 3E|U|^2 + 3E|\Phi|^2 + 3E|\tilde{E}^m|^2,$$

$$\leq 3L^2T^m_1 + 2L^2hK + 9KL^2T^m_1 + 9KT^m_2 + 72hL^2K\left[\frac{hL^2K}{M} + 1\right].$$

(44)

where we used the Lipschitz continuity of $f$ for controlling $U^m$, (42) for $\Phi^n$, and Appendix D for $\tilde{E}^m$.

We then handle the cross terms. For example, due to the independence, (77), and the convexity, we have

$$E \langle \Delta^m, \Phi^m \rangle = 0, \quad E \langle \Delta^m, \tilde{E}^m \rangle = 0, \quad \langle \Delta^m, U^m \rangle \geq M|\Delta^m|^2,$$

(45)

this means the cross term between first and the third term in the last line (44) leads to $-2MhE|\Delta^m|^2$. The cross term produced by the first and the last term, however can be hard to control, mostly because $E\langle \Delta^m, V^m \rangle$ is unknown. We now employ Young's inequality, meaning, for any $a > 0$:

$$T^m_1 = E|\Delta^m + 1|^2$$

$$\leq (1 + a)E|\Delta^m + 1|^2 + h(V^m + h(\tilde{E}^m - F^m))^2 + \left(1 + \frac{1}{a}\right)E|V^m + h(\tilde{E}^m - F^m)|^2.$$  

(46)

While the second term is already investigated in (43), the first term of (46), according to (41) becomes:

$$E|\Delta^m + 1|^2 + h(V^m + h(\tilde{E}^m - F^m))^2 = E|\Delta^m|^2 - 2hE\langle \Delta^m, U^m + \Phi^m + \tilde{E}^m \rangle$$

$$+ 2h^2E|U^m + \Phi^m + \tilde{E}^m|^2,$$

(47)

$$\leq (1 - 2Mh)E|\Delta^m|^2 + h^2E|U^m + \Phi^m + \tilde{E}^m|^2$$

where we used (45). Plug (44) into (47), we have have, using the definition of the coefficients $A, B, C$:

$$E|\Delta^m - h(U^m + \Phi^m + \tilde{E}^m)|^2 \leq AT^m_1 + Ch^3 + BT^m_2,$$

(48)

and plug it together with (43) in (46) to conclude (40).

**Lemma B.2.** Under conditions of Theorem 5.1, we have the upper bound for $T^m_2$:

$$T^m_2 = \hat{A}T^m_1 + \hat{B}T^m_2$$

(49)

where $\hat{A} = \frac{L^2}{K}$ and $\hat{B} = 1 - 1/K$.

Note that the coefficient $\hat{B}$ is automatically $< 1$ and the gap $1/K$ is independent of $h$ and $\eta$. This gives us some room to tune the parameters.

**Proof.** We now expand $E|\beta^m + 1 - g^m + 1|^2$:

$$E_{r_m}|\beta^m + 1 - g^m + 1|^2 = E_{r_m}\left[|\beta^m + 1 - g^m + 1|^2 - |\beta^m - g^m|^2\right] + |\beta^m - g^m|^2$$

$$= \frac{1}{K}\left[|\partial_t f(y^m) - \partial_t f(x^m)|^2 - |\beta^m - g^m|^2\right] + |\beta^m - g^m|^2$$

$$= \left(1 - \frac{1}{K}\right)|\beta^m - g^m|^2 + \frac{1}{K}|\partial_t f(y^m) - \partial_t f(x^m)|^2$$

Therefore, we have

$$E|\beta^m + 1 - g^m + 1|^2 = \left(1 - \frac{1}{K}\right)E\sum_{i=1}^{K}|\beta^m_i - g^m_i|^2 + \frac{1}{K}E|\nabla f(y^m) - \nabla f(x^m)|^2$$

$$\leq \left(1 - \frac{1}{K}\right)E|\beta^m - g^m|^2 + \frac{L^2}{K}E|\Delta^m|^2$$

(50)

$\square$
Now, we are ready to prove Theorem 5.1 by adjusting \( a \) and \( c_p \).

**Proof of Theorem 5.1**  Plug (40) and (49) into (38), we have

\[
T^{m+1} \leq \left( (1 + a)A + c_p\tilde{A} \right) T^m_1 + \left( \frac{(1 + a)B}{c_p} + \tilde{B} \right) c_p T^m_2 + (1 + a)h^3C + \left( 1 + \frac{1}{a} \right) h^4D. \tag{51}
\]

To show the proof amounts to choosing proper \( c_p \) and \( a \). Note that according to the definitions, \( A \sim 1 - Mh, \tilde{A} \sim h^2 \) and \( \tilde{B} \sim 1 - K \), this suggests \( c_p \sim h^2 \) to cancel out the order in \( B \), and in the end we have estimates of the form:

\[
(1 + a)A + c_p\tilde{A} = 1 - O(h), \quad \frac{(1 + a)B}{c_p} + \tilde{B} = 1 - O(h).
\]

Indeed, let us choose

\[
c_p = 18(1 + a)h^2K^2,
\]

so that

\[
(1 + a)A + c_p\tilde{A} = (1 + a)(1 - 2Mh + 3(1 + 9K)L^2h^2), \quad \text{and} \quad \frac{(1 + a)B}{c_p} + \tilde{B} = 1 - \frac{1}{2K}.
\]

Since \( h \) satisfies (16), this relaxes them to

\[
(1 + a)A + c_p\tilde{A} \leq (1 + a)(1 - Mh), \quad \text{and} \quad \frac{(1 + a)B}{c_p} + \tilde{B} = 1 - \frac{1}{2K} \leq 1 - \frac{Mh}{2}.
\]

Setting \( a = \frac{Mh/2}{1-Mh} \) so that

\[
(1 + a)(1 - Mh) = 1 - \frac{Mh}{2}, \quad \text{and} \quad 1 + 1/a \leq 2/Mh,
\]

and this finally leads to

\[
T^{m+1} \leq (1 - Mh/2)T^m_1 + (1 - Mh/2)c_p T^m_2 + 2h^3C + \frac{2}{Mh} h^4D \tag{52}
\]

Noting

\[
W^2_2(q^O_m, p) \leq T^m
\]

and

\[
T^0 = E|y^0 - x^0|^2 + c_p E|g^0 - \beta^0|^2 = E|y^0 - x^0|^2 + c_p E|\nabla f(x^0) - \nabla f(y^0)|^2 \leq (1 + c_pL^2)E|y^0 - x^0|^2 \leq (1 + M^2/L^2)W^2_2(q^O_0, p) \leq (1 + 1/R^2)W^2_2(q^O_0, p),
\]

where we use \( hL < M/(27L) \), by iteration, we finally have

\[
W^2_2(q^O_m, p) \leq \exp(-Mhm/2)(1 + 1/R^2)W^2_2(q^O_0, p) + 4(\hbar^2C/M + \hbar^2D/M^2). \tag{53}
\]

The proof is concluded considering

\[
C/M \leq K^3 \left( 2L^2/(K^2M) + 75L^2/M \right) \leq 77K^3R^2M,
\]

\[
D/M^2 \leq K^2(H^2/M^2 + 20R^2 + R^3M/K).
\]
C Proof of convergence of RCAD-U-LMC (Theorem 5.2)

Recall the definitions:

- \( E^m \): \( E^m = \nabla f(x^m) - F^m \)
- \( \tilde{g}^0 \): \( \tilde{g}^0 = \nabla f(x^0) \)
- \( \tilde{g}^{m+1} \): \( \tilde{g}^{m+1}_{r_m} = \partial_r f(x^m) \) and \( \tilde{g}^{m+1}_i = \tilde{g}^m_i \) if \( i \neq r_m \),
- \( \tilde{F}^m \): \( \tilde{F}^m = \tilde{g}^m + K (\tilde{g}^{m+1} - \tilde{g}^m) \)
- \( \tilde{E}^m \): \( \tilde{E}^m = \nabla f(x^m) - \tilde{F}^m = \tilde{E}^m + F^m - \tilde{F}^m \)

Similarly, we also have

\[
|\tilde{g}^m - \tilde{g}^m| \leq L^2 \eta_2 K, \quad |\tilde{F}^m - F^m|^2 = |\tilde{E}^m - E^m|^2 \leq 2L^2 \eta^2 K + 8L^2 \eta^2 K^2. \tag{54}
\]

According to the algorithm, RCAD-U-LMC can be seen as drawing \((x^0, v^0)\) from distribution induced by \(q^0\), and update \((x^m, v^m)\) using the following coupled SDEs:

\[
\left\{ \begin{array}{l}
V_t = v^m e^{-2(t-mh)} - \gamma \int_{mh}^{t} e^{-2(t-s)} ds \right. \left. F^m + \sqrt{4\gamma} e^{-2(t-mh)} \int_{mh}^{t} e^2 ds dB_s \right.
X_t = x^m + \int_{0}^{t} V_s ds
\end{array} \right., \tag{55}
\]

where \(B_s\) is the Brownian motion and \((x^{m+1}, v^{m+1}) = (X_{(m+1)h}, V_{(m+1)h})\).

We then define \( w^m = x^m + v^m \), and denote \( u_m(x, w) \) the probability density of \((x^m, w^m)\) and \( u^*(x, w) \) the probability density of \((x, w)\) if \((x, v = w - x)\) is distributed according to density function \(p_2\). One main reason to change \((x, v)\) to \((x, w)\) is that in [10], the authors showed that the map \((x_0, w_0) \rightarrow (x_t, w_t)\) induced from \(f\) is a contracting map for for \(t\). From [10], we also have:

\[
|x^m - x|^2 + |v^m - v|^2 \leq 4(|x^m - x|^2 + |w^m - w|^2) \leq 16(|x^m - x|^2 + |v^m - v|^2) \tag{56}
\]

and

\[
W_2^2(q^m, p_2) \leq 4W_2^2(u_m, u^*) \leq 16W_2^2(q^m, p_2). \tag{57}
\]

Similar to RCAD-O-LMC, define another trajectory of sampling by setting \((\tilde{x}^0, \tilde{v}^0)\) to be drawn from the distribution induced by \(p_2\), and that \(\tilde{x}^m = \tilde{X}_{hm}, \tilde{v}^m = \tilde{V}_{hm}, \tilde{u}^m = \tilde{x}^m + \tilde{v}^m\) are samples from \(\tilde{X}_t, \tilde{V}_t\) that satisfy

\[
\left\{ \begin{array}{l}
\tilde{V}_t = \tilde{v}_0 e^{-2t} - \gamma \int_{0}^{t} e^{-2(t-s)} \nabla f(\tilde{X}_s) ds + \sqrt{4\gamma} e^{-2t} \int_{0}^{t} e^2 ds dB_s \\
\tilde{X}_t = \tilde{x}_0 + \int_{0}^{t} \tilde{V}_s ds
\end{array} \right., \tag{58}
\]

with the same Brownian motion as before. This leads to

\[
\left\{ \begin{array}{l}
\tilde{x}^{m+1} = \tilde{x}^m e^{-2h} - \gamma \int_{mh}^{(m+1)h} e^{-2((m+1)h-s)} \nabla f(\tilde{X}_s) ds + \sqrt{4\gamma} e^{-2h} \int_{mh}^{(m+1)h} e^2 ds dB_s \\
\tilde{x}^{m+1} = \tilde{x}^m + \int_{mh}^{(m+1)h} \tilde{V}_s ds
\end{array} \right.,\tag{59}
\]

Clearly \(\tilde{X}_t, \tilde{V}_t\) can be seen as drawn from target distribution for all \(t\), and initially we can pick \((\tilde{x}^0, \tilde{v}^0)\) such that

\[
W_2^2(q^U, p_2) = \mathbb{E} \left( |x^0 - \tilde{x}^0|^2 + |v^0 - \tilde{v}^0|^2 \right), \quad \text{and} \quad W_2^2(u_0, u^*) = \mathbb{E} \left( |x^0 - \tilde{x}^0|^2 + |w^0 - \tilde{w}^0|^2 \right).
\]
We then also define \( \beta^m \)
\[
\beta^0 = \nabla f(x^0)
\]
and
\[
\beta_r^{m+1} = \partial_r m f(x^m)
\]
We will be showing the decay of the following Lyapunov function:
\[
T^m \triangleq T_1^m + c_p T_2^m = E \left( |x^m - x^m|^2 + |w^m - w^m|^2 \right) + c_p E |g^m - \beta^m|^2,
\]
where \( c_p \) will be carefully chosen later.

The following lemma gives bounds for \( T_1^{m+1}, T_2^{m+1} \) using \( T_1^m, T_2^m \), and the proof of the theorem amounts to selecting the correct \( c_p \).

**Lemma C.1.** Under conditions of Theorem 5.2 we have
\[
T_1^{m+1} < D_1 T_1^m + D_2 T_2^m + D_3, \tag{61}
\]
\[
T_2^{m+1} \leq \frac{L^2}{K} T_1^m + \left( 1 - \frac{1}{K} \right) T_2^m, \tag{62}
\]
where
\[
D_1 = 1 - h/(2R) + 244h^2 K, \quad D_2 = 84\gamma^2 h^2 K, \quad D_3 = 672\gamma h^4 K^3 + 10h^3 K/M + 260h^6 K^2.
\]

**Proof.** The proof for bounding \( T_2^m \) is the same as the one in Appendix B Lemma B.2 and is omitted from here. We only prove the first inequality.

**Step 1:** We firstly define \( |\Delta^m|^2 = |\tilde{w}^m - w^m|^2 + |\tilde{x}^m - x^m|^2 \), and compare (55) and (59) for:
\[
|\Delta^{m+1}|^2 = \left( |\tilde{w}^m - v^m|e^{-2h} + |\tilde{x}^m - x^m| + \int_{m}^{(m+1)h} \tilde{V}_s - V_s \, ds \right)
- \gamma \int_{m}^{(m+1)h} e^{-2((m+1)h-s)} \left| \nabla f \left( \tilde{X}_s \right) - \nabla f(x^m) \right| \, ds
+ \gamma \int_{m}^{(m+1)h} e^{-2((m+1)h-s)} E^m \, ds
+ \left( |\tilde{x}^m - x^m| + \int_{m}^{(m+1)h} \tilde{V}_s - V_s \, ds \right)^2

= |J_1^m|^2 + |J_2^m|^2 = |J_1^{r,m} + J_1^{E,m}|^2 + |J_2^{m}|^2,
\]
where we denote
\[
J_1^{r,m} = (\tilde{w}^m - v^m)e^{-2h} + (\tilde{x}^m - x^m) + \int_{m}^{(m+1)h} \tilde{V}_s - V_s \, ds
- \gamma \int_{m}^{(m+1)h} e^{-2((m+1)h-s)} \left| \nabla f \left( \tilde{X}_s \right) - \nabla f(x^m) \right| \, ds
\]
and
\[
J_1^{E,m} = \gamma \int_{m}^{(m+1)h} e^{-2((m+1)h-s)} E^m.
\]
To control \( J_1^m \), we realize that \( J_1^{E,m} \) term, produced by \( E^m \), is not perpendicular to the rest of the terms, namely \( J_1^{r,m} \), and it will lead to a lot of cross terms. We thus replace it by \( J_1^{E,m} \) induced by \( E^m \). This allows us to eliminate all cross terms. Since \( E^m - E^m \) is small, such replacement brings only small perturbation. In particular, with Young’s inequality:
\[
E |J_1^m|^2 \leq (1 + h^2) E |J_1^m| + J_1^{E,m} - J_1^{E,m}|^2 + (1 + 1/h^2) E |J_1^{E,m} - J_1^{E,m}|^2,
\]
\[
\leq (1 + h^2) E |J_1^m + J_1^{E,m} - J_1^{E,m}|^2 + \gamma^2 (h^2 + 1)(2L^2\eta^2 K + 8L^2\eta^2 K^2), \tag{63}
\]
21
where we use the smallness of $E^m - \tilde{E}^m$ in (54). The first term of (65) can be separated into three terms:

$$E \left| J_1^m + J_1^{\tilde{E},m} - J_1^{E,m} \right|^2 = E \left| J_1^{r,m} + J_1^{\tilde{E},m} \right|^2$$

$$= E \left| J_1^{r,m} \right|^2 + E \left| J_1^{\tilde{E},m} \right|^2 + 2E \left\langle J_1^{r,m}, J_1^{\tilde{E},m} \right\rangle.$$  

Firstly note that

$$E \left| J_1^{\tilde{E},m} \right|^2 \leq \gamma^2 h^2 E \left| \tilde{E}^m \right|^2.$$  

And to bound the third term, note that

$$E \left\langle J_1^{r,m}, J_1^{\tilde{E},m} \right\rangle = E \left( \int_{mh}^{(m+1)h} \begin{align*}
\bar{V}_s - V_s & \, ds, J_1^{\tilde{E},m} \end{align*} \right)$$

due to the fact that

$$E \langle A, \tilde{E}^m \rangle = E \langle A, E_m \tilde{E}^m \rangle = 0$$  

for all $A$ that has no $r_m$ dependence. To further bound this term, we plug in the definition and have:

$$2E \left( \int_{mh}^{(m+1)h} \bar{V}_s - V_s \, ds, \gamma \int_{mh}^{(m+1)h} e^{-2((m+1)h-s)} \, ds \tilde{E}^m \right)$$

$$= -2E \left( \int_{mh}^{(m+1)h} V_s \, ds, \gamma \int_{mh}^{(m+1)h} e^{-2((m+1)h-s)} \, ds \tilde{E}^m \right)$$

$$= 2E \left( \gamma \int_{mh}^{(m+1)h} \int_s^t e^{-2(s-t)} \, dt \, ds E^m, \gamma \int_{mh}^{(m+1)h} e^{-2((m+1)h-s)} \, ds \tilde{E}^m \right)$$

$$\leq \gamma^2 h^3 (3E \left| \tilde{E}^m \right|^2 + 4L^2 \eta^2 K + 16L^2 \eta^2 K^2)$$

where we used (64) again in the first and second equalities and

$$E \left( E^m, \tilde{E}^m \right) \leq 3E \left| \tilde{E}^m \right|^2 + 2E \left| E^m - \tilde{E}^m \right|^2$$

together with (54) in the last inequality.

In conclusion, we have

$$T_1^{m+1} = E \left| \Delta^{m+1} \right|^2 \leq (1 + h^2) E \left| J_1^{r,m} \right|^2 + \left| J_2^m \right|^2 + \gamma^2 h^2 \left( E \left| \tilde{E}^m \right|^2 + 3E \left| \tilde{E}^m \right|^2 + 4L^2 \eta^2 K + 16L^2 \eta^2 K^2 \right).$$

Using $\gamma L = 1$, $h < 1$, $\eta < h^3$, we have

$$T_1^{m+1} = E \left| \Delta^{m+1} \right|^2 \leq (1 + h^2) E \left| J_1^{r,m} \right|^2 + \left| J_2^m \right|^2 + 2\gamma^2 h^2 (3h^3) E \left| \tilde{E}^m \right|^2 + 60h^6 K^2.$$  

\textbf{Step 2:} Now, we study first two terms in (66). We try to bound $(1 + h^2) E \left| J_1^{r,m} \right|^2 + \left| J_2^m \right|^2$ using $T_1^m$ and $E \left| \tilde{E}^m \right|^2$. We first try to separate out $(x^m, \overline{x}^m, v^m, \overline{v}^m)$ from $J_1^{r,m}$ and $J_2^m$. Denote

$$A^m = (\overline{v}^m - v^m) (h + e^{-2h}) + (\overline{x}^m - x^m)$$

$$- \gamma \int_{mh}^{(m+1)h} e^{-2((m+1)h-s)} \left[ \nabla f(\overline{x}^m) - \nabla f(x^m) \right] \, ds,$$

$$B^m = \int_{mh}^{(m+1)h} \overline{V}_s - V_s - (\overline{v}^m - v^m) \, ds$$

$$- \gamma \int_{mh}^{(m+1)h} e^{-2((m+1)h-s)} \left[ \nabla f(\overline{x}_s) - \nabla f(\overline{x}^m) \right] \, ds,$$
\[
C^m = (\bar{x}^m - x^m) + \int_{m}^{(m+1)h} \bar{v}^m - v^m \, ds = (\bar{x}^m - x^m) + h(\bar{v}^m - v^m),
\]

(69)

\[
D^m = \int_{m}^{(m+1)h} \tilde{V}_s - V_s - (\bar{v}^m - v^m) \, ds,
\]

(70)

then we have

\[
J^m_1 = A^m + B^m, \quad J^m_2 = C^m + D^m.
\]

By Young’s inequality, we have

\[
(1 + h^2)\mathbb{E}|J^m_1|^2 + \mathbb{E}|J^m_2|^2 = (1 + h^2)\mathbb{E}|A^m + B^m|^2 + \mathbb{E}|C^m + D^m|^2 \\
\leq (1 + a) \left((1 + h^2)\mathbb{E}|A^m|^2 + \mathbb{E}|C^m|^2\right) \\
+ (1 + 1/a)((1 + h^2)\mathbb{E}|B^m|^2 + \mathbb{E}|D^m|^2),
\]

(71)

where \(a > 0\) will be carefully chosen later. Now, the first term of (71) only contains information from previous step, using \(f\) is strongly convex, we can bound it using \(|\Delta^m|^2\) (showed in Lemma E.3). To bound the second term, we need to consider difference between \(x, v\) at \(t_{m+1}\) and \(t_m\), which can be bounded by \(|\Delta^m|^2\) and \(|E^m|^2\) (showed in Lemma E.2).

According to Lemma E.2, E.3 we first have

\[
(1 + h^2)\mathbb{E}|J^m_1|^2 + \mathbb{E}|J^m_2|^2 \\
\leq (1 + a) \left((1 - h/R)^2 + Dh^2\right) T^m_1 \\
+ (1 + 1/a) \left[80h^2 T^m_1 + 5\gamma^2 h^4 \mathbb{E}|E^m|^2 + 5\gamma h^4 K\right] \\
= C_1 T^m_1 + (1 + 1/a)\gamma^2 h^4 \mathbb{E}|E^m|^2 + 5\gamma h^4 K,
\]

(72)

where in the first inequality we use \(1 + h^2 < 2\) and

\[
C_1 = (1 + a)((1 - h/R)^2 + Dh^2) + 80(1 + 1/a)h^4.
\]

Plug (72) in (66) and also replace \(\mathbb{E}(|E^m|^2)\) with Lemma E.4 equation (89), we have

\[
T^{m+1}_1 \leq C_1 T^m_1 + \gamma^2 \left[10(1 + 1/a)h^4 + 8h^2\right] \mathbb{E}\left|\bar{E}^m\right|^2 \\
+ 100(1 + 1/a)h^10 K^2 + 5\gamma h^4 K + 60h^6 K^2,
\]

(73)

where we use \(\gamma L = 1, \eta < h^3\) and \(h < 1\).

- **Step 3:** To ensure the decay of \(T^m_1\), we need to choose \(a\) such that the coefficient in front of \(T^m_1\) is strictly smaller than 1. Noting in

\[
C_1 = (1 + a)((1 - h/R)^2 + Dh^2) + 80(1 + 1/a)h^4
\]

the second term is of high order, while the first one is of \(1 - O(h)\) amplified by \(1 + a\), so it is possible to choose \(a\) small enough to make the entire term \(1 - O(h)\). Indeed, since \(h \leq \frac{1}{(1+D)R}\),

we have

\[
(1 - h/R)^2 + Dh^2 \leq 1 - h/R,
\]

and thus by setting \(a\) so that

\[
1 + a = \frac{1 - h/(2R)}{1 - h/R}.
\]

The entire coefficient is \(1 - h/2R + 160Rh^3\) and is smaller than 1 for moderately small \(h\). Moreover, due to the definition of \(a\), we have

\[
1 + 1/a \leq 2R/h,
\]

plugging the calculation in (73) we have

\[
T^{m+1}_1 \leq \left\{1 - h/(2R) + 160Rh^3\right\} T^m_1 \\
+ \gamma^2 \left[20Rh^3 + 8h^2\right] \mathbb{E}\left|\bar{E}^m\right|^2 \\
+ 200Rh^9 K^2 + 10\gamma Rh^3 K + 60h^6 K^2.
\]

(74)
We further bound $\mathbb{E}[\tilde{E}^m]^2$ by plugging in Lemma E.4 equation (88) and use $\gamma L = 1$, $Rh < 1 \leq K, \gamma R = 1/M$, we have
\[
T_1^{m+1} \leq \left\{ 1 - h/(2R) + 160Rh^3 \right\} T_1^m + 84h^2 K \mathbb{E}[\tilde{x}^m - x^m]^2 
+ 28\gamma^2 h^2 (24Lh^2 K^4 + 3K\mathbb{E} |\beta^m - \bar{\gamma}^m|^2) 
+ 200Rh^9 K^2 + 10\gamma Rh^3 K + 60h^6 K^2,
\]
\[
< \left\{ 1 - h/(2R) + 244h^2 K \right\} T_1^m 
+ 84\gamma^2 h^2 K \mathbb{E} |\beta^m - \bar{\gamma}^m|^2 
+ 672\gamma h^4 K^4 + 10h^3 K/M + 260h^6 K^2,
\]
where we use $\mathbb{E}[\tilde{x}^m - x^m]^2 \leq \mathbb{E} |\Delta^m|^2 = T_1^m$ and try to absorb small terms into large terms to simplify the formula:
\[
20Rh^3 + 8h^2 < 28h^2, \quad 200Rh^9 K^2 + 60h^6 K^2 < 260h^6 K^2,
\]
and
\[
160Rh^3 + 84h^2 K \leq 244h^2 K, \quad 10\gamma Rh^3 K = 10h^3 K/M.
\]
This proves (61). \(\Box\)

Now we are ready to prove Theorem 5.2 by adjusting $c_p$.

**Proof of Theorem 5.2** Plug (61) and (62) into (60):
\[
T^{m+1} \leq \left\{ D_1 + \frac{c_p L^2}{K} \right\} T_1^m + \left( 1 - \frac{1}{K} \right) \left( 1 - \frac{D_2}{c_p} \right) c_p T_2^m + D_3.
\]
Note that according to the definition $D_3$ is of $O(h^3)$, and $D_2$ is of $O(h^2)$ while $D_1 \sim 1 - O(h)$, so it makes sense to choose $c_p$ small enough so that the coefficient for $T_1^m$ keeps being of $1 - O(h)$. Indeed, we let
\[
c_p = 168\gamma^2 h^2 K^2,
\]
and will have
\[
T^{m+1} \leq \left\{ 1 - h/(2R) + 412h^2 K \right\} T_1^m + \left( 1 - \frac{1}{2K} \right) T_2^m 
+ 672\gamma h^4 K^4 + 10h^3 K/M + 260h^6 K^2,
\]
where we use $\gamma L = 1$.

Using (18), we can verify
\[
\max\{1 - h/(2R) + 412h^2 K, 1 - 1/2K\} \leq 1 - h/(4R).
\]
Plug into (76), we have
\[
T^{m+1} \leq (1 - h/(4R)) T^m + 672\gamma h^4 K^4 + 10h^3 K/M + 260h^6 K^2,
\]
by induction
\[
T^m \leq (1 - h/(4R))^m T^0 + 2688\gamma Rh^3 K^4 + 40Rh^2 K/M + 104Rh^5 K^2
\]
\[
\leq (1 - h/(4R))^m T^0 + 2688\gamma h^3 K^4/M + 40Rh^2 K/M + 104Rh^5 K^2.
\]
Finally, consider
\[
T^0 = \mathbb{E}|x^0 - x^0|^2 + \mathbb{E}|u^0 - \bar{u}^0|^2 + c_p \mathbb{E}|g^0 - \beta^0|^2
= \mathbb{E}|x^0 - x^0|^2 + \mathbb{E}|u^0 - \bar{u}^0|^2 + c_p \mathbb{E} |\nabla f(x^0) - \nabla f(y^0)|^2
\leq (1 + c_p L^2) (\mathbb{E}|x^0 - x^0|^2 + \mathbb{E}|u^0 - \bar{u}^0|^2) \leq 2W_2(Q_{h^0}, p),
\]
where we use $168\gamma^2 h^2 K^2 L^2 < 1$. Taking square root on each term and use (57), we finally obtain
\(\Box\)
D Calculation of $\mathbb{E} \left| \vec{E}^m \right|^2$ for RCAD-O-LMC

According to the definition of (36)-(37):

$$\mathbb{E}_{x_m} \vec{g}^{m+1} = \vec{g}^m + \frac{1}{K} (\nabla f(x^m) - \vec{g}^m), \quad \mathbb{E}_{x_m} (\vec{g}^{m+1} - \vec{g}^m) = \frac{1}{K} (\nabla f(x^m) - \vec{g}^m),$$

and

$$\mathbb{E}_{x_m} \left| \vec{g}^{m+1} - \vec{g}^m \right|^2 = \sum_i \mathbb{E}_{x_m} (\vec{g}_i^{m+1} - \vec{g}_i^m)^2 = \frac{1}{K} \sum_i \left| \partial_i f(x^m) - \vec{g}_i^m \right|^2.$$ Naturally

$$\mathbb{E}_{x_m} \tilde{g}^m = \vec{g}^m + (\nabla f(x^m) - \vec{g}^m) = \nabla f(x^m).$$ Accordingly,

$$\mathbb{E}_{x_m} \left( \vec{E}^m \right) = \nabla f(x^m) - \mathbb{E}_{x_m} (\vec{F}^m) = 0 \quad (77)$$

and

$$\mathbb{E}_{x_m} \left| \vec{E}^m \right|^2 = \sum_{i=1}^K \mathbb{E}_{x_m} |\vec{E}^m_i|^2 = \sum_{i=1}^K \mathbb{E}_{x_m} |\partial_i f(x^m) - \vec{g}_i^m - K (\vec{g}_i^{m+1} - \vec{g}_i^m)|^2. \quad (78)$$

Taking the expectation over the random trajectory:

$$\mathbb{E} \left| \vec{E}^m \right|^2 = \mathbb{E} \left( \mathbb{E}_{x_m} \left| \vec{E}^m \right|^2 \right) < K \mathbb{E} |\nabla f(x^m) - \vec{g}^m|^2.$$

To analyze each entry of $\partial_i f(x^m) - \vec{g}_i^m$, we note:

$$|\partial_i f(x^m) - \vec{g}_i^m|^2 \leq 3 |\partial_i f(x^m) - \partial_i f(y^m)|^2 + 3 |\partial_i f(y^m) - \beta_i^m|^2 + 3 |\beta_i^m - \vec{g}_i^m|^2. \quad (79)$$

The first term, after taking expectation and summing over $i$, becomes

$$3 \mathbb{E} |\nabla f(x^m) - \nabla f(y^m)|^2 \leq 3L^2 \mathbb{E} |\Delta^m|^2 = 3L^2 T_1^m. \quad (80)$$

The last term, with the same procedure, becomes $3T_2^m$. They both will be left in the estimate. We now focus on giving an upper bound of the second term. To do so we adopt a technique from [8, 18]. Define $p = 1/K$, for fixed $m \geq 1$ and $1 \leq i \leq K$, we have

$$\mathbb{P}(\beta_i^m = \partial_i f(y^0)) = (1-p)^m + (1-p)^{m-1} p$$

and

$$\mathbb{P}(\beta_i^m = \partial_i f(y^j)) = (1-p)^{m-1-j} p, \quad 1 \leq j \leq m-1.$$
$$\mathbb{E} \sum_{i=1}^{K} [\partial_i f(y^m) - \beta_i^m]^2 = \sum_{i=1}^{K} \sum_{j=0}^{m-1} \mathbb{E}(\mathbb{E}(\partial_i f(y^m) - \beta_i^m | \beta_i^m = \partial_i f(y^i))) \mathbb{P}(\beta_i^m = \partial_i f(y^i))$$

$$= \sum_{j=0}^{m-1} \sum_{i=1}^{K} \mathbb{E}((\partial_i f(y^m) - \partial_i f(y^j))^2) \mathbb{P}(\beta_i^m = \partial_i f(y^j))$$

$$\leq (I) \sum_{j=0}^{m-1} \mathbb{E}((\nabla f(y^m) - \nabla f(y^j))^2) \mathbb{P}(\beta_1^m = \partial_1 f(y^j))$$

$$\leq L^2 \sum_{j=0}^{m-1} \mathbb{E}(|y^m - y^j|^2) \mathbb{P}(\beta_1^m = \partial_1 f(y^j))$$

$$\leq L^2 \sum_{j=0}^{m-1} \mathbb{E}(|y^m - y^j|^2) (1 - p)^{m-1-j}p$$

$$+ L^2 \mathbb{E}(|y^m - y^0|^2) (1 - p)^m$$

$$\leq (II) L^2 \sum_{j=0}^{m-1} \mathbb{E} \left( \int_{jh}^{(j+1)h} \nabla f(y_s) ds - \sqrt{2h} \sum_{i=j}^{m-1} \xi_i \right)^2 (1 - p)^{m-1-j}p$$

$$+ L^2 \mathbb{E} \left( \int_{0}^{mh} \nabla f(y_s) ds - \sqrt{2h} \sum_{i=0}^{m-1} \xi_i \right)^2 (1 - p)^m$$

$$\leq (III) L^2 \sum_{j=0}^{m-1} \left[ 2h^2 (m-j)^2 \mathbb{E}_p |\nabla f(y)|^2 + 4hK(m-j) \right] (1 - p)^{m-1-j}p$$

$$+ L^2 \left[ 2h^2 m^2 \mathbb{E}_p |\nabla f(y)|^2 + 4hKm \right] (1 - p)^m$$

$$\leq (IV) 2phL^2 \mathbb{E}_p |\nabla f(y)|^2 \left[ \sum_{j=1}^{m} j^2 (1 - p)^{j-1} + m^2 (1 - p)^m / p \right]$$

$$+ 4phL^2 K \left[ \sum_{j=1}^{m} j(1 - p)^{j-1} + m(1 - p)^m / p \right]$$

$$\leq (V) 8h^2 L^2 \mathbb{E}_p |\nabla f(y)|^2 \left[ \frac{hL^2 K}{p^2} + \frac{8hL^2 K}{p} \right]$$

$$\leq (VI) 8hL^2 K^2 \left[ \frac{hL^2 K}{M} + 1 \right]$$

where in (I) we use \( \mathbb{P}(\beta_1^m = \partial_1 f(y^j)) \) are same for different \( i \), (II) comes from (34), (III) comes from (35),(36), (IV) comes from changing of variable, in (V) we use the bound for terms in the bracket and in (VI) we use \( \mathbb{E}_p |x - x^*|^2 \leq K/M \) according to Theorem D.1 in [8], where \( x^* \) is the maximum point of \( f \).

In conclusion, we have

$$\mathbb{E} \left| \tilde{E}^m \right| \leq 3KL^2 T_1^m + 3KT_2^m + 24hL^2 K^3 \left[ \frac{hL^2 K}{M} + 1 \right].$$
E  Key lemma in proof of RCAD-U-LMC

Lemma E.1. Under conditions of Theorem 5.2, $\hat{X}_t, \tilde{V}_t$ are defined in (58), we have
\[
\mathbb{E} \int_{m}^{(m+1)h} |\tilde{X}_t - \bar{x}^m|^2 dt \leq \frac{h^3 \gamma K}{3}
\]  (83)
and
\[
\mathbb{E} \int_{m}^{(m+1)h} \left( |\tilde{V}_t - V_t| - (\bar{v}^m - v^m) \right)^2 dt \leq 16h^3 \mathbb{E}[\Delta^m]^2 + \gamma^2 h^3 \mathbb{E}[E^m]^2 + 0.4 \gamma h^5 K,
\]  (84)

Lemma E.2. Under conditions of Theorem 5.2 and $B^m, D^m$ are defined in (68), (70), we have
\[
\mathbb{E}|B^m|^2 \leq 32 h^4 \mathbb{E}|\Delta^m|^2 + 2 \gamma h^4 \mathbb{E}|E^m|^2 + 2 \gamma h^4 K
\]  (85)
\[
\mathbb{E}|D^m|^2 \leq 16 h^4 \mathbb{E}|\Delta^m|^2 + 2 \gamma h^4 \mathbb{E}|E^m|^2 + 0.4 \gamma h^6 K
\]  (86)

Lemma E.3. Under conditions of Theorem 5.2 and $A^m, C^m$ defined in (67), (69), there exists a uniform constant $D$ such that
\[
\mathbb{E}((1 + h^2)|A^m|^2 + |C^m|^2) \leq [(1 - h/R)^2 + Dh^2] \mathbb{E}|\Delta^m|^2
\]  (87)
where $R = L/M$ is the condition number of $f$.

Lemma E.4. Under conditions of Theorem 5.2 we have estimation for approximation gradient
\[
\mathbb{E}|\tilde{E}^m|^2 \leq 3KL^2\mathbb{E}|x^m - x^m|^2 + 24Lh^2 K^4 + 3KE|\beta^m - \gamma^m|^2
\]  (88)
and
\[
\mathbb{E}|E^m|^2 \leq 2\mathbb{E}|\tilde{E}^m|^2 + 20L^2 h^6 K^2.
\]  (89)

We prove these four lemmas below.

Proof of Lemma E.1. First we prove (83). According to (58), we have
\[
\mathbb{E} \int_{m}^{(m+1)h} |\tilde{X}_t - \bar{x}^m|^2 dt = \mathbb{E} \int_{m}^{(m+1)h} \left| \int_{m}^{t} \tilde{V}_s ds \right|^2 dt
\]
\[
\leq \int_{m}^{(m+1)h} |t - mh| \int_{m}^{t} \mathbb{E} \left| \tilde{V}_s \right|^2 dsdt
\]
\[
= \int |v|^2 p_2(x,v) dx dv \int_{m}^{(m+1)h} (t - mh)^2 dt = \frac{h^3 \gamma K}{3},
\]  (90)
where in the first inequality we use Hölder’s inequality, and for the second equality we use $p_2$ is a stationary distribution so that $\tilde{X}_t, \tilde{V}_t \sim p_2$ and $\tilde{V}_t \sim \exp(-|v|^2/(2\gamma))$ for any $t$.

Second, to prove (84), using (55) (58), we first rewrite $\left( \tilde{V}_t - V_t \right) - (\bar{v}^m - v^m)$ as
\[
\left( \tilde{V}_t - V_t \right) - (\bar{v}^m - v^m) = (\bar{v}^m - v^m) (e^{-2(t-mh)} - 1)
\]
\[ - \gamma \int_{m}^{t} e^{-2(t-s)} \left[ \nabla f(\tilde{X}_s) - \nabla f(x^m) \right] ds
\]
\[ + \gamma \int_{m}^{t} e^{-2(t-s)} ds E^m
\]
\[ = I(t) + II(t) + III(t).
\]  (91)
for $mh \leq t \leq (m+1)h$. Then we bound each term seperately:
\[ \mathbb{E} \int_{mh}^{(m+1)h} |I(t)|^2 \, dt \leq h \mathbb{E} \int_{mh}^{(m+1)h} \left| (\tilde{v}^m - v^m) \left( e^{-2(t - mh)} - 1 \right) \right|^2 \, dt \\
\leq h \int_{mh}^{(m+1)h} (2(t - mh))^2 \mathbb{E} |\tilde{v}^m - v^m|^2 \, dt \\
\leq \frac{4h^3}{3} \mathbb{E} |\tilde{v}^m - v^m|^2 , \]

where we use Hölder’s inequality in the first inequality and \( 1 - e^{-x} < x \) in the second inequality.

\[ \mathbb{E} \int_{mh}^{(m+1)h} |\Pi(t)|^2 \, dt \leq \gamma^2 \mathbb{E} \int_{mh}^{(m+1)h} \left| \int_{mh}^{t} e^{-2(t-s)} \left[ \nabla f(\tilde{X}_s) - \nabla f(x^m) \right] \, ds \right|^2 \, dt \\
\leq 2\gamma^2 \mathbb{E} \int_{mh}^{(m+1)h} \left| \int_{mh}^{t} e^{-2(t-s)} \left[ \nabla f(\tilde{X}_s) - \nabla f(x^m) \right] \, ds \right|^2 \, dt \\
+ 2\gamma^2 \mathbb{E} \int_{mh}^{(m+1)h} \left| \int_{mh}^{t} e^{-2(t-s)} \left[ \nabla f(x^m) - \nabla f(x^m) \right] \, ds \right|^2 \, dt \\
\leq 2\gamma^2 \int_{mh}^{(m+1)h} \left. (t - mh) \mathbb{E} \int_{mh}^{t} \left| \nabla f(\tilde{X}_s) - \nabla f(x^m) \right|^2 \, ds \, dt \\
+ 2\gamma^2 \int_{mh}^{(m+1)h} \left. (t - mh) \mathbb{E} \int_{mh}^{t} \left| \nabla f(x^m) - \nabla f(x^m) \right|^2 \, ds \, dt \\
\leq 2\gamma^2 L^2 \mathbb{E} \int_{mh}^{(m+1)h} \left. (t - mh) \int_{mh}^{t} \tilde{X}_s - x^m \right|^2 \, ds \, dt \\
+ 2\gamma^2 L^2 \mathbb{E} \int_{mh}^{(m+1)h} \left. (t - mh) \int_{mh}^{t} x^m - x^m \right|^2 \, ds \, dt \\
\leq 2\gamma^3 L^2 K (t - mh)^3 dt + 2\gamma^2 L^2 \int_{mh}^{(m+1)h} (t - mh)^2 dt \mathbb{E} |x^m - x^m|^2 \\
\leq \frac{2\gamma^3 L^2 h^3 K}{15} + \frac{2\gamma^2 L^2 h^3}{3} \mathbb{E} |x^m - x^m|^2 , \]

where in the third inequality we use gradient of \( f \) is \( L \)-Lipschitz function and we use \(^{83}\) in the fourth inequality.

\[ \mathbb{E} \int_{mh}^{(m+1)h} |\Pi(t)|^2 \, dt = \gamma^2 \mathbb{E} \int_{mh}^{(m+1)h} \left| \int_{mh}^{t} e^{-2(t-s)} \, ds E^m \right|^2 \, dt \\
\leq \gamma^2 \int_{mh}^{(m+1)h} (t - mh)^2 \, dt \mathbb{E} (|E^m|^2) \\
\leq \frac{\gamma^2 h^3}{3} \mathbb{E} (|E^m|^2) , \]

Plug \(^{92},^{93},^{94}\) into \(^{91}\) and using \( \gamma L = 1 \) we have

\[ \mathbb{E} \int_{mh}^{(m+1)h} \left| \left( \bar{V}_t - V_t \right) - (\tilde{v}^m - v^m) \right|^2 \, dt \\
\leq 3 \left( \mathbb{E} \int_{mh}^{(m+1)h} |I(t)|^2 \, dt + \mathbb{E} \int_{mh}^{(m+1)h} |\Pi(t)|^2 \, dt + \mathbb{E} \int_{mh}^{(m+1)h} |\Pi(t)|^2 \, dt \right) \\
\leq 4h^3 \mathbb{E} |x^m - x^m|^2 + \mathbb{E} |\tilde{v}^m - v^m|^2 + \gamma^2 h^3 \mathbb{E} (|E^m|^2) + 0.4\gamma h^5 K , \]

using \(^{50}\), we get the desired result.
Proof of Lemma [E.2] First, we separate $B^m$ into two parts:

$$
\mathbb{E}|B^m|^2 \leq 2\mathbb{E} \left| \int_{m}^{(m+1)h} \left( \tilde{V}_t - V_t \right) - (\tilde{v}^m - v^m) \right|^2 dt + 2\mathbb{E} \left| \gamma \int_{m}^{(m+1)h} e^{-2((m+1)h-t)} \left[ \nabla f(\tilde{X}_t) - \nabla f(\tilde{x}^m) \right] dt \right|^2 .
$$

And each terms can be bounded:

- \begin{align*}
\mathbb{E} \left| \int_{m}^{(m+1)h} \left( \tilde{V}_t - V_t \right) - (\tilde{v}^m - v^m) \right|^2 dt & \leq h\mathbb{E} \int_{m}^{(m+1)h} \left| \tilde{V}_t - V_t \right|^2 dt \\
& \leq 16h^4\mathbb{E}|\Delta m|^2 + \gamma^2 h^4(\mathbb{E}(|E^m|^2) + 0.4\gamma h^6K ,
\end{align*}

where we use Lemma [E.1] (84) in the second inequality.

- \begin{align*}
\mathbb{E} \left| \gamma \int_{m}^{(m+1)h} e^{-2((m+1)h-t)} \left[ \nabla f(\tilde{X}_t) - \nabla f(\tilde{x}^m) \right] dt \right|^2 & \leq h\gamma^2\mathbb{E} \int_{m}^{(m+1)h} \left| e^{-2((m+1)h-t)} \right|^2 dt \\
& \leq h\gamma^2L^2\mathbb{E} \int_{m}^{(m+1)h} \left| \tilde{X}_t - \tilde{x}^m \right|^2 dt \\
& \leq \frac{h^4\gamma^3L^2K}{3} \leq \frac{h^4\gamma K}{3} .
\end{align*}

where we use Lemma [E.1] (83) and $\gamma L = 1$ in the last two inequalities.

Combine (95), (96) together, we finally have

$$
\mathbb{E}|B|^2 \leq 32h^4\mathbb{E}|\Delta m|^2 + 2\gamma^2 h^4(\mathbb{E}(|E^m|^2) + 0.8h^6\gamma K + 2h^4\gamma K/3 ,
$$

which implies (85) if we further use $h < 1$.

Next, estimation of $(\mathbb{E}|D|^2)^{1/2}$ is a direct result of (95).

Proof of Lemma [E.3] Let $\tilde{x}^m - x^m = a$ and $\tilde{w}^m - w^m = b$. First, by the mean-value theorem, there exists a matrix $H$ such that $MI_K \preceq H \preceq LI_K$ and

\[ \nabla f(\tilde{x}^m) - \nabla f(x^m) = Ha . \]

By calculation, \[ \int_{m}^{(m+1)h} e^{-2((m+1)h-t)} dt = \frac{1-e^{-2h}}{2} \]

\[ A^m = (h + e^{-2h})(\tilde{v}^m - v^m) + \left( I_K - \frac{(1-e^{-2h})}{2} \gamma H \right)(\tilde{x}^m - x^m) \]

\[ = \left( 1 - h \right) a + (h + e^{-2h})b . \]

$$
C^m = (1-h)a + hb .
$$

Since $\|\gamma H\|_2 \leq 1$ and we also have following calculation

\[ h + e^{-2h} = h + e^{-2h} - 1 + 1 = 1 - h + O(h^2) , \]

\[ 1 - h - e^{-2h} = h + O(h^2) , \]

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1 - e^{-2h} = 2h + O(h^2).

If we further define matrix $\mathcal{M}_A$ and $\mathcal{M}_C$ such that
\[
|A|^2 = (a, b)^\top \mathcal{M}_A (a, b), \quad |C|^2 = (a, b)^\top \mathcal{M}_C (a, b),
\]
then, we have
\[
\|\mathcal{M}_A - \begin{bmatrix} 0 & hI_K - \gamma hH \\ hI_K - \gamma hH & (1 - 2h)I_K \end{bmatrix}\|_2 \leq D_1 h^2,
\]
and
\[
\|\mathcal{M}_B - \begin{bmatrix} (1 - 2h)I_K & hI_K \\ hI_K & 0 \end{bmatrix}\|_2 \leq D_1 h^2,
\]
where $D_1$ is a uniform constant since $h < 1/1648$ by (18). This further implies
\[
(1 + h^2)|A|^2 + |C|^2 = (a, b)^\top \begin{bmatrix} (1 - 2h)I_K & 2hI_K - \gamma hH \\ 2hI_K - \gamma hH & (1 - 2h)I_K \end{bmatrix} (a, b) + h^2 (a, b)^\top Q (a, b)
\]
where $\|Q\|_2 \leq D_2$ and $D_2$ is a uniform constant. Calculate the eigenvalue of the dominating matrix (first term), we need to solve
\[
\det \{ (1 - 2h - \lambda)^2 I_K - (2hI_K - \gamma hH)^2 \} = 0,
\]
which implies eigenvalues $\{\lambda_j\}_{j=1}^K$ solve
\[
(1 - 2h - \lambda_j)^2 - (2h - \gamma h\Lambda_j)^2 = 0,
\]
where $\Lambda_j$ is $j$-th eigenvalue of $H$. Since $\gamma \Lambda_j \leq \gamma L = 1$ and $h < 1$, we have
\[
\lambda_j \leq 1 - \gamma \Lambda_j h \leq 1 - Mh\gamma = 1 - h/R
\]
for each $j = 1, \ldots, K$. This implies
\[
\left\| \begin{bmatrix} (1 - 2h)I_K & 2hI_K - \gamma hH \\ 2hI_K - \gamma hH & (1 - 2h)I_K \end{bmatrix} \right\|_2 \leq 1 - h/R,
\]
and
\[
(1 + h^2)|A|^2 + |C|^2 \leq ((1 - h/R)^2 + Dh^2)(|a|^2 + |b|^2),
\]
where $D$ is a uniform constant. Take expectation on both sides, we obtain (87).

Proof of Lemma E.4. The proof is mostly the same as that in the calculation in Appendix D. Inequality (79) still holds true except the second term needs to be treated differently. Following the step in Appendix D, we define $p = 1/K$, and then for fixed $m \geq 1$ and $1 \leq i \leq K$, we have
\[
\mathbb{P}(\beta_i^m = \partial_i f(\bar{x}^0)) = (1 - p)^m + (1 - p)^{m-1} p,
\]
and
\[
\mathbb{P}(\beta_i^m = \partial_i f(\bar{x}^j)) = (1 - p)^{m-j} p, \quad 1 \leq j \leq m - 1.
\]
where (II) comes from (58), (III) comes from (39) and \( \eta < h \).

Next, to prove (89), we only need to notice inequality differ from the derivation in Appendix D only through (II).

\[
\begin{align*}
\mathbb{E} \sum_{i=1}^{K} |\partial_i f(\bar{x}^m) - \beta_i^m|^2 & = \sum_{i=1}^{m-1} \sum_{j=1}^{K} \mathbb{E}(\mathbb{E}(|\partial_i f(\bar{x}^m) - \beta_i^m|^2 | \beta_i^m = \partial_i f(\bar{x}^j))) \mathbb{P}(\beta_i^m = \partial_i f(\bar{x}^j)) \\
& \leq \sum_{j=0}^{m-1} \sum_{i=1}^{K} \mathbb{E}(|\partial_i f(\bar{x}^m) - \partial_i f(\bar{x}^j)|^2) \mathbb{P}(\beta_i^m = \partial_i f(\bar{x}^j)) \\
& \leq \sum_{j=0}^{m-1} \mathbb{E}(|\nabla f(\bar{x}^m) - \nabla f(\bar{x}^j)|^2) \mathbb{P}(\beta_i^m = \partial_i f(\bar{x}^j)) \\
& \leq L^2 \sum_{j=0}^{m-1} \mathbb{E}(|\bar{x}^m - \bar{x}^j|^2) \mathbb{P}(\beta_i^m = \partial_i f(\bar{x}^j)) \\
& \leq L^2 \sum_{j=0}^{m-1} \mathbb{E}(|\bar{x}^m - \bar{x}^j|^2)(1 - p)^{m-1-j} p \\
& + L^2 \mathbb{E}(|\bar{x}^m - \bar{x}^j|^2)(1 - p)^{m_1} \\
& \leq (II) L^2 \sum_{j=0}^{m-1} \mathbb{E} \left( \left( \int_{j_0}^{m_j} \bar{V}_s \, ds \right)^2 \right) (1 - p)^{m-1-j} p \\
& + L^2 \mathbb{E} \left( \left( \int_{0}^{m_j} \bar{V}_s \, ds \right)^2 \right) (1 - p)^m \\
& \leq (III) L^2 \sum_{j=0}^{m-1} \left[ 2h^2 (m - j)^2 \mathbb{E}_{p_2} |\bar{V}|^2 \right] (1 - p)^{m-1-j} p \\
& + L^2 \left[ 2h^2 m^2 \mathbb{E}_{p_2} |\bar{V}|^2 \right] (1 - p)^m \\
& \leq (IV) 2ph^2 L^2 \mathbb{E}_{p_2} |\bar{V}|^2 \left[ \sum_{j=1}^{m} j^2 (1 - p)^{j-1} + m^2 (1 - p)^m / p \right] \\
& \leq (V) \frac{8h^2 L^2 \mathbb{E}_{p_2} |\bar{V}|^2}{p^2} \\
& \leq (VI) 8\gamma h^2 L^2 K^3 = 8h^2 LK^3, \quad (97)
\end{align*}
\]

where (II) comes from (58), (III) comes from \( \left( \bar{X}_t, \bar{V}_t \right) \sim p_2 \) for any \( t \), (IV) comes from changing of variable, in (V) we use the bound for terms in the bracket and in (VI) we use \( \mathbb{E}_{p_2} |v|^2 \leq \gamma K \). This inequality differ from the derivation in Appendix D only through (II).

Next, to prove (39), we only need to notice

\[
\mathbb{E}|E^m|^2 \leq 2\mathbb{E}|\bar{E}^m|^2 + 2\mathbb{E}|F^m - \bar{F}^m|^2,
\]

(39) and \( \eta < h^3 \) and follow the same calculation as in done in Appendix D. \( \square \)