Crystal bases and \( q \)-identities

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Abstract. The relation of crystal bases with \( q \)-identities is discussed, and some new results on crystals and \( q \)-identities associated with the affine Lie algebra \( C_n^{(1)} \) are presented.

Contents

1. Introduction 1
2. The Rogers–Ramanujan identities and the Hard Hexagon model 3
3. Crystal bases 7
4. Bosonic evaluation 13
5. Fermionic evaluation 16
6. Summary and open problems 23
References 23

1. Introduction

The purpose of this paper is two-fold. First, we would like to advocate the importance of crystal theory to the theory of \( q \)-series. In particular crystal base theory provides a unifying and general setting for a large class of \( q \)-identities. Second, as evidence, some new identities associated to the affine Lie algebra \( C_n^{(1)} \) are presented. The emphasis here will not be on the completeness of the results since the field is evolving quite rapidly, but rather on the presentation of the main ideas and techniques used.

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The Rogers–Ramanujan identities are undoubtably the most famous \(q\)-series identities. They are given by

\[
\sum_{n=0}^{\infty} q^{n^2} = \prod_{j=1}^{\infty} \frac{1}{(1-q^{5j-4})(1-q^{5j-1})} \tag{1.1}
\]

\[
\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q)_n} = \prod_{j=1}^{\infty} \frac{1}{(1-q^{5j-3})(1-q^{5j-2})} \tag{1.2}
\]

where \((q)_n = (1-q)(1-q^2)\cdots(1-q^n)\). We will view them here as identities of formal power series, meaning that in the expansion as series in the formal variable \(q\) the coefficients of \(q^N\) match on both sides for all \(N \geq 0\). What contributes to their beauty is that these coefficients can be interpreted combinatorially. The coefficient of \(q^N\) on the left-hand side of (1.1) is the number of partitions of \(N\) for which the difference between any two parts is at least two. The coefficient of \(q^N\) of the right-hand side of (1.1) on the other hand is the number of partitions of \(N\) with parts congruent to 1 or 4 modulo 5. Similarly, the coefficients of \(q^N\) on the left and right side of (1.2) can be interpreted as the number of partitions of \(N\) for which the difference between any two parts is at least two and the smallest part is greater than one, and the number of partitions of \(N\) with parts congruent to 2 and 3 modulo 5, respectively.

Many of the ideas regarding crystals and \(q\)-identities can already be demonstrated in terms of the Rogers–Ramanujan identities. The point of focus here shall be the debut of the Rogers–Ramanujan identities on the mathematical physics stage, in particular their appearance in the hard hexagon model in a paper by Baxter \[3\] in 1981. In this setting the Rogers–Ramanujan identities can be viewed as two different evaluations of the generating function of certain paths which are coined bosonic and fermionic evaluations. Details are discussed in section 2. As it turns out the relation between the Rogers–Ramanujan identities and the hard hexagon model is only part of a much bigger picture. In terms of representation theory, the paths that occur in the hard hexagon model are elements of tensor products of crystals associated with the affine Lie algebra \(\hat{\mathfrak{sl}}_2\). Crystal bases were introduced by Kashiwara \[17, 18\] and roughly speaking are bases of representations of quantum universal enveloping algebras \(U_q(\mathfrak{g})\) as the parameter \(q\) (not to be confused with the \(q\) in the \(q\)-series!) tends to zero. Here \(\mathfrak{g}\) is any symmetrizable Kac–Moody algebra. As in the Rogers–Ramanujan case, there are two different ways to evaluate generating functions of tensor products of crystals, thereby giving rise to \(q\)-identities. Hence crystal base theory provides a natural framework for \(q\)-identities. Crystal bases, path spaces and their generating functions are discussed in section 3. The two different ways to evaluate the paths generating functions are subject of sections 4 and 5, respectively. In particular in section 5.5 we present new fermionic formulas for level-restricted paths. We close in section 6 with some outstanding open problems.

Before indulging in the fascinating theory of crystal bases there is one important point that needs to be addressed. Applying Jacobi’s triple product identity

\[
\sum_{n=-\infty}^{\infty} z^n q^{n^2} = \prod_{n=0}^{\infty} (1-q^{2n+2})(1+q^{2n+1})(1+zq^{2n+1})
\]
Figure 1. An allowed configuration of particles in the hard hexagon model. The hexagons around the particles are drawn in different colours.

The right-hand sides of (1.1) and (1.2) can be rewritten as alternating sums yielding the identities

\[
\sum_{n=0}^{\infty} \frac{q^n}{(q)_n} = \frac{1}{(q)_\infty} \sum_{j=-\infty}^{\infty} (-1)^j q^{\frac{1}{2}(5j+1)}
\]

(1.3)

\[
\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q)_n} = \frac{1}{(q)_\infty} \sum_{j=-\infty}^{\infty} (-1)^j q^{\frac{1}{2}(5j+3)}.
\]

(1.4)

The two different evaluations of generating functions of crystal paths that were mentioned above really yield (polynomial) analogues of (1.3) and (1.4) rather than (1.1) and (1.2). Identities relating sums to alternating sums are more general than identities relating sums to products. Only in special cases can the alternating sums be evaluated as products, namely when Jacobi’s triple product identity or more generally the Macdonald identities \(^{30}\) can be applied.

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2. The Rogers–Ramanujan identities and the Hard Hexagon model

The hard hexagon model is a two-dimensional lattice model of a gas of hard or non-overlapping particles. The particles are placed on a triangular lattice such that no two particles can occupy two adjacent sites. If one views each particle as the center of a hexagon, then the condition that no two particles can be adjacent translates into the condition that the hexagons cannot overlap. This explains the name of the model. An example of an allowed particle configuration is shown in Figure 1.

To pack the lattice densely all particles must lie on one of the three sublattices corresponding to the three corners of the triangles of the lattice. Hence one of the sublattices is distinguished from the others. At low particle density the probability
for a particle to be on a particular site is equal for sites on all three sublattices (assuming either an infinite or sufficiently large lattice so that boundary effects can be neglected). Let \( \rho_a \) for \( a = 1, 2, 3 \) be the probability that there is a particle at a fixed site on sublattice 1, 2, 3, respectively. If the boundary conditions of the model are such that at close packing all particles are on sublattice 1 then it is intuitively clear that the order parameter defined as \( R = \rho_1 - \rho_2 \) must undergo a phase transition. At low densities \( R \) is zero, but at some critical density \( R \) will become positive until at high densities it is one. Baxter \cite{3} managed to determine the precise point at which the phase transition occurs exactly by using corner transfer matrices. In essence, the corner transfer matrix method reduces the two-dimensional problem to a one-dimensional problem. The precise details are beyond the scope of this paper and can be found in section 14 of Baxter’s book \cite{4}.

The one-dimensional problem that Baxter encountered and which turns out to be of importance to the Rogers–Ramanujan identities is the following. Consider \( L + 1 \) points on a line labeled by \( i = 0, 1, 2, \ldots, L \). Assign to each point a height variable \( \sigma_i \) which takes on the values 0 or 1. In addition the height variables satisfy the restrictions \( \sigma_0 = \sigma_L = 0 \) and \( \sigma_i\sigma_{i+1} = 0 \). An allowed configuration of height variables for a given length \( L \) is called a path of length \( L \), and the set of all paths of length \( L \) is denoted by \( \mathcal{D}_L \). One can illustrate a path graphically by drawing all points \((i, \sigma_i)\) and connecting adjacent points by straight lines. An example for a path with \( L = 9 \) is given in Figure 2. The condition \( \sigma_i\sigma_{i+1} = 0 \) requires that the paths consist of a certain number of non-overlapping triangles (this condition comes directly from the condition in the two-dimensional hard hexagon model requiring that no two particles can be on adjacent sites). To each path \( \sigma = (\sigma_0, \ldots, \sigma_L) \) one may assign an energy \( E(\sigma) \) by summing up the positions of the peaks, that is

\[
E(\sigma) = \sum_{j=1}^{L} j\sigma_j.
\]

The energy of the path in Figure 2 is \( E(\sigma) = 1 + 5 + 8 = 14 \). The generating function of paths of length \( L \) which is also called a one dimensional configuration sum is defined as

\[
X(L) = \sum_{\sigma \in \mathcal{D}_L} q^{E(\sigma)}.
\]

The path picture immediately implies that \( X(L) \) satisfies the following initial conditions and recurrence which completely specify it

\[
X(0) = X(1) = 1 \\
X(L) = X(L-1) + q^{L-1}X(L-2).
\]

The aim is to find explicit expressions for \( X(L) \). We will describe two ways to obtain such an expression which will be related to the two sides of \[13\].
2.1. Fermionic formula. Interpret each peak in a path as a particle, that is, there is a particle at site \( i \) if \( \sigma_i = 1 \). Fix the number of particles to be \( n \). The ground state path \( \sigma_G \) with minimal energy is the path with particles at positions 1, 3, \ldots, 2n − 1. The energy of \( \sigma_G \) is \( E(\sigma_G) = 1 + 3 + \cdots + 2n − 1 = n^2 \). An example of the ground state path with 3 particles is shown in Figure 3. All other paths with \( n \) particles can be obtained from \( \sigma_G \) by moving particles to the right in such a way that the particles never overtake each other. If the length of the path is \( L \), the rightmost particle can move at most \( L - 2n \) positions to the right. Hence the paths of length \( L \) with \( n \) particles are in one-to-one correspondence with partitions with at most \( n \) parts not exceeding \( L - 2n \). If a path \( \sigma \) corresponds to partition \( \lambda \) then its energy is \( E(\sigma) = E(\sigma_G) + |\lambda| = n^2 + |\lambda| \). The generating function of partitions with at most \( n \) parts not exceeding \( m \) is the \( q \)-binomial coefficient defined as

\[
\binom{m+n}{n} = \frac{(q)_{m+n}}{(q)_m(q)_n}
\]

for \( m, n \in \mathbb{N} \) and zero otherwise. This implies the following explicit expression for \( X(L) \)

\[
X(L) = \sum_{n=0}^{\infty} q^{n^2} \binom{L-n}{n}.
\]  

(2.3)

Because of its interpretation in terms of non-overlapping particles this expression is called a quasiparticle or fermionic formula.

2.2. Bosonic formula. As opposed to (2.3) there is another expression for \( X(L) \) given by

\[
X(L) = \sum_{j=-\infty}^{\infty} (-1)^j q^{\frac{1}{2} (5j+1)} \left[ \frac{L}{\frac{L-5j}{2}} \right]
\]

(2.4)

where \( \lfloor x \rfloor \) is the largest integer not exceeding \( x \). It can be proven by showing that the right-hand side satisfies the recurrence and initial condition (2.2).

However, formula (2.4) can also be interpreted in terms of paths. For this purpose the above path picture needs to be slightly reformulated. The state diagram of the paths in the hard hexagon model is

This diagram is to be understood as follows. If the system is in the bottom state at position \( i \), that is \( \sigma_i = 0 \), then at position \( i + 1 \) it can either be in the bottom state again so that \( \sigma_{i+1} = 0 \) as indicated by the loop or it can be in the top state which means \( \sigma_{i+1} = 1 \) as indicated by the line. On the other hand, if the system is in the
top state at position $i$ then at the next position it has to be in the bottom state, meaning that $\sigma_i = 1$ implies $\sigma_{i+1} = 0$. This corresponds to condition on paths that $\sigma_i\sigma_{i+1} = 0$ for all $i$. Up to a $\mathbb{Z}_2$ symmetry this state diagram is isomorphic to

\[(2.5) \quad \bigcirc \approx \bigg/ \mathbb{Z}_2.\]

Under this correspondence the paths of the hard hexagon model become paths in a strip of height 4 with steps going up or down at each position. For example, with the convention that the paths start at height three, the path in Figure 2 becomes the path in Figure 4.

Let us now consider the following set of paths. Let $p = (p_1, p_2, \ldots, p_L)$ be a sequence of 1's and 2's and let $\lambda = (\lambda_1, \lambda_2)$ be the content of $p$ where $\lambda_i$ is the number of $i$'s in $p$. Denote by $P_{L, \lambda}$ the set of all $p = (p_1, \ldots, p_L)$ with content $\lambda$. Another way to represent $p$ is by height variables $\sigma = (\sigma_0, \sigma_1, \ldots, \sigma_L)$ where by convention $\sigma_0 = 3$ and $\sigma_i = \sigma_{i-1} + 1$ if $p_i = 1$ and $\sigma_i = \sigma_{i-1} - 1$ if $p_i = 2$ for $1 \leq i \leq L$. We will use $\sigma$ and $p$ interchangeably. We are actually mostly concerned here with the set $P_L := P_{L, \lambda}$ where $\lambda = (\lfloor L/2 \rfloor, \lfloor L/2 + 1 \rfloor)$.

Consider the following subsets of $P_L$. Let $\mathcal{P}_L$ be the subset of $P_L$ consisting of all paths in the strip of height four, that is all $\sigma \in P_L$ such that $1 \leq \sigma_i \leq 4$ for all $1 \leq i \leq L$. Furthermore, let $P_{L, \lambda}^{i,j}$ be the set of all paths $\sigma \in P_L$ such that there exist indices $1 \leq i_1 < i_2 < \cdots < i_j \leq L$ so that $\sigma_{i_1}, \sigma_{i_3}, \sigma_{i_5}, \ldots$ are greater than four and $\sigma_{i_2}, \sigma_{i_4}, \sigma_{i_6}, \ldots$ are less than one. Similarly let $P_{L, \lambda}^{i,j}$ be the set of all paths $\sigma \in P_L$ such that there exist indices $1 \leq i_1 < i_2 < \cdots < i_j \leq L$ such that $\sigma_{i_1}, \sigma_{i_3}, \sigma_{i_5}, \ldots$ are less than one and $\sigma_{i_2}, \sigma_{i_4}, \sigma_{i_6}, \ldots$ are greater than four. By inclusion-exclusion we have

$$\mathcal{P}_L = \left(P_L \cup \bigcup_{j \geq 1} (P_{L, \lambda}^{i,2j} \cup P_{L, \lambda}^{i,2j+1}) \right) \setminus \bigcup_{j \geq 1} (P_{L, \lambda}^{i,2j-1} \cup P_{L, \lambda}^{i,2j-2}).$$

Now define the energy function $E$ for $\sigma \in \mathcal{P}_L$ as

$$E(\sigma) = \sum_{i=1}^{L-1} i \cdot h(\sigma_{i-1}, \sigma_i, \sigma_{i+1})$$
where \( h \) is the local energy function given by

\[
h(\sigma_{i-1}, \sigma_i, \sigma_{i+1}) = \begin{cases} 
1 & \text{if } \sigma_{i-1} = \sigma_i - 1 = \sigma_{i+1} \text{ and } \sigma_i > 3, \\
0 & \text{or } \sigma_{i-1} = \sigma_i + 1 = \sigma_{i+1} \text{ and } \sigma_i < 2,
\end{cases}
\]

The term \( q^{\frac{1}{2}(5j+1)} \left( \frac{L}{2} \right) \) in (2.4) can then be interpreted as the generating function of \( P_{L}^{\downarrow,j} \) given by \( \sum_{\sigma \in P_{L}^{\downarrow,j}} q^{E(\sigma)} \) if \( j > 0 \) and the generating function of \( P_{L}^{\uparrow,j} \) given by \( \sum_{\sigma \in P_{L}^{\uparrow,j}} q^{E(\sigma)} \) if \( j < 0 \). The term \( j = 0 \) in (2.4) is the generating function of \( P_{L} \).

In section 4 we will encounter more general inclusion-exclusion arguments in terms of operations on crystals.

### 2.3. Identities.

Equations (2.3) and (2.4) yield the following polynomial identity

\[
\sum_{n=0}^{\infty} q^{n^2} \left( \frac{L-n}{n} \right) = \sum_{j=-\infty}^{\infty} (-1)^j q^{\frac{1}{2}(5j+1)} \left( \frac{n}{L} \right).
\]

Using \( \lim_{L \to \infty} \left( \frac{L}{n} \right) = \frac{1}{(q)_n} \) this identity implies (1.3) in the limit \( L \to \infty \).

All discussion so far focused on (1.3). The second Rogers–Ramanujan identity (1.4) can in fact be treated in a very similar fashion. Define the set \( D_{L}' \) analogous to \( D_{L} \) with the only difference that now \( \sigma_0 = 1 \) instead of 0. The generating function is defined to be \( X'(L) = \sum_{\sigma \in D_{L}'} q^{E(\sigma)} \). Analogous arguments to those in sections 2.1 and 2.2 yield the following two expressions for \( X'(L) \)

\[
X'(L) = \sum_{n=0}^{\infty} q^{n(n+1)} \left( \frac{L-n}{n} \right) = \sum_{j=-\infty}^{\infty} (-1)^j q^{\frac{1}{2}(5j+3)} \left( \frac{n}{L} \right).
\]

In the limit \( L \to \infty \) this polynomial identity implies (1.4).

### 3. Crystal bases

In this section we review the main results about crystal bases and explain how they provide a general setting for the definition of one-dimensional configuration sums.

The quantized universal enveloping algebra \( U_q(\mathfrak{g}) \) associated with a symmetrizable Kac–Moody Lie algebra \( \mathfrak{g} \) was introduced independently by Drinfeld [7] and Jimbo [14] in their study of two-dimensional solvable lattice models in statistical mechanics. The parameter \( q \) corresponds to the temperature of the underlying model. Kashiwara [16] showed that at zero temperature or \( q = 0 \) the representations of \( U_q(\mathfrak{g}) \) have bases, which he coined crystal bases, with a beautiful combinatorial structure and favorable properties such as uniqueness and stability under tensor products. The existence and uniqueness of crystal bases for integrable highest weight modules for an arbitrary symmetrizable Kac–Moody algebra was given in [17].
3.1. Axiomatic definition of crystals. Let $\mathfrak{g}$ be a symmetrizable Kac-Moody algebra, $P$ the weight lattice, $I$ the index set for the vertices of the Dynkin diagram of $\mathfrak{g}$, $\{\alpha_i \in P \mid i \in I\}$ the simple roots, and $\{h_i \in P^* = \text{Hom}_\mathbb{Z}(P, \mathbb{Z}) \mid i \in I\}$ the simple coroots. Let $U_q(\mathfrak{g})$ be the quantized universal enveloping algebra of $\mathfrak{g}$. A $U_q(\mathfrak{g})$-crystal is a nonempty set $B$ equipped with maps $\text{wt} : B \to P$ and $\epsilon_i, f_i : B \to B \cup \{\emptyset\}$ for all $i \in I$, satisfying

\begin{align}
& (3.1) \quad f_i(b) = b' \iff \epsilon_i(b') = b \text{ if } b, b' \in B \\
& (3.2) \quad \text{wt}(f_i(b)) = \text{wt}(b) - \alpha_i \text{ if } f_i(b) \in B \\
& (3.3) \quad \langle h_i, \text{wt}(b) \rangle = \varphi_i(b) - \epsilon_i(b)
\end{align}

where $\langle \cdot, \cdot \rangle$ is the natural pairing. Here for $b \in B$,

\begin{align}
& (3.4) \quad \epsilon_i(b) = \max\{n \geq 0 \mid e_i^n(b) \neq \emptyset\} \\
& \quad \varphi_i(b) = \max\{n \geq 0 \mid f_i^n(b) \neq \emptyset\}.
\end{align}

(It is assumed that $\varphi_i(b), \epsilon_i(b) < \infty$ for all $i \in I$ and $b \in B$.)

A $U_q(\mathfrak{g})$-crystal $B$ can be viewed as a directed edge-colored graph (the crystal graph) whose vertices are the elements of $B$, with a directed edge from $b$ to $b'$ labeled $i \in I$, if and only if $f_i(b) = b'$. The element $b$ is said to be highest weight if $\epsilon_i(b) = \emptyset$ for all $i \in I$. Let $K$ be a subset of $I$. Then the $K$-component of the crystal $B$ is the graph obtained by only considering edges colored by $i \in K$.

We also define the crystal reflection operator $s_i : B \to B$ by

\begin{align}
& s_i(b) = \\
& \quad \begin{cases} 
  f_i^{-\epsilon_i(b)}(b) & \text{if } \varphi_i(b) > \epsilon_i(b) \\
  b & \text{if } \varphi_i(b) = \epsilon_i(b) \\
  \epsilon_i(b)^{-\varphi_i(b)}(b) & \text{if } \varphi_i(b) < \epsilon_i(b).
\end{cases}
\end{align}

It is obvious that $s_i$ is an involution.

3.2. Tensor products of crystals. Given two crystals $B$ and $B'$, there is also a crystal obtained by taking the tensor product $B \otimes B'$. As a set $B \otimes B'$ is just given by the Cartesian product of the sets $B$ and $B'$. The weight function $\text{wt}$ for $b \otimes b' \in B \otimes B'$ is $\text{wt}(b \otimes b') = \text{wt}(b) + \text{wt}(b')$ and the raising and lowering operators $\epsilon_i$ and $f_i$ act as follows

\begin{align}
& (3.5) \quad \epsilon_i(b \otimes b') = \\
& \quad \begin{cases} 
  \epsilon_i(b) \otimes b' & \text{if } \epsilon_i(b) > \phi_i(b'), \\
  b \otimes \epsilon_i(b') & \text{otherwise},
\end{cases} \\
& f_i(b \otimes b') = \\
& \quad \begin{cases} 
  f_i(b) \otimes b' & \text{if } \epsilon_i(b) \geq \phi_i(b'), \\
  b \otimes f_i(b') & \text{otherwise}.
\end{cases}
\end{align}

The reader is warned that this convention is different from Kashiwara’s convention. The order of the tensor factors is interchanged.

3.3. Finite and infinite crystals. Let us fix some notation. From now on let $\mathfrak{g}$ be a simple complex Lie algebra and $\hat{\mathfrak{g}}$ be the associated untwisted affine algebra. That is, let $\hat{\mathfrak{g}}' = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}d$ be the central extension of the loop algebra of $\mathfrak{g}$ and $\hat{\mathfrak{g}} = \hat{\mathfrak{g}}' \oplus \mathbb{C}d$ where $d$ is the degree derivation. Let $J$ (resp. $I = J \cup \{0\}$) be a set indexing the vertices of the Dynkin diagram of $\mathfrak{g}$ (resp. $\hat{\mathfrak{g}}$ and $\hat{\mathfrak{g}}'$). A classical weight is a weight with respect to the algebra $\mathfrak{g}$. Denote by $U_q(\mathfrak{g})$, $U_q(\hat{\mathfrak{g}})$, and $U_q(\hat{\mathfrak{g}}')$ the quantized universal enveloping algebras of $\mathfrak{g}$, $\hat{\mathfrak{g}}'$, and $\hat{\mathfrak{g}}$ respectively.
There are two main categories of crystals. The first one contains the crystal bases of irreducible integrable $U_q(\mathfrak{g})$-modules $V(\Lambda)$ of highest weight $\Lambda \in P^+$ where $P^+$ denotes the set of dominant weights of $\mathfrak{g}$. These crystals are infinite-dimensional. The second one contains finite-dimensional crystals corresponding to $U_q'(\mathfrak{g})$. Since the set $B$ is finite in this case these crystals are called finite crystals. The level of a finite crystal $B$ is defined as

$$\text{lev}B = \min\{\langle c, e(b) \rangle \mid b \in B\}$$

where $e(b) = \sum_{i \in I} \epsilon_i(b) \Lambda_i$ and $\{\Lambda_i \mid i \in I\}$ is the $\mathbb{Z}$-basis of the weight lattice of $\mathfrak{g}'$. A $U_q'(\mathfrak{g})$-crystal can be viewed as a $U_q(\mathfrak{g})$-crystal by restricting to the $J$-component.

The crystal of each integrable highest weight $U_q(\mathfrak{g})$-module can be realized in terms of a semi-infinite tensor product of perfect crystals $\{22, 23, 21\}$. Perfect crystals are finite crystals with some additional properties (for details see [24, Definition 4.6.1]). At least one perfect crystal for each integrable $U_q(\mathfrak{g})$-module for $\hat{\mathfrak{g}} = A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}$, and $D_n^{(2)}$ was given in [23]. Kashiwara and Nakashima [25] constructed the finite $U_q(\mathfrak{g})$-crystals for $\mathfrak{g} = A_n$, $B_n$, $C_n$ and $D_n$ explicitly in terms of tableaux. The cases $\mathfrak{g} = A_n$ and $C_n$ are discussed in more detail in the next subsection.

### 3.4. Finite crystals of type $A_n$ and $C_n$.

The finite crystals associated with $\mathfrak{g} = A_n$ and $C_n$ are presented in more detail since they will be our main examples throughout the paper. Let $\Lambda_i$ for $i \in J$ be the fundamental weights of $\mathfrak{g}$. For later purposes it will be convenient to give the root and weight structure of $A_n$ and $C_n$ explicitly. Let $\langle \cdot, \cdot \rangle$ be the standard bilinear form normalized such that $\langle \alpha_i, \alpha_i \rangle = 2$ for the long roots $\alpha_i$.

**Example 3.1.** For either $\mathfrak{g} = A_n$ or $\mathfrak{g} = C_n$ the index set for the simple roots is $J = \{1, 2, \ldots, n\}$.

For $\mathfrak{g} = A_n$ the weight lattice is embedded in the subspace of $\mathbb{R}^{n+1}$ orthogonal to the vector $e = \sum_{j=1}^{n+1} \epsilon_j$, where $\epsilon_j$ is the $j$-th standard basis vector of $\mathbb{R}^{n+1}$. The simple roots are $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for all $i \in J$. The fundamental weights are $\Lambda_i = \sum_{j=1}^{i} \epsilon_j - \sum_{j=i+1}^{n} \epsilon_j$. The half-sum of positive roots is $\rho = (n, n-1, \ldots, 1, 0) - \frac{n}{2} e$. The scalar product is the standard one: $\langle \alpha_i, \beta \rangle = \alpha_i \cdot \beta$.

For $\mathfrak{g} = C_n$ the simple roots are given by the short roots $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $1 \leq i < n$ and the long root $\alpha_n = 2 \epsilon_n$ where $\epsilon_i$ is the $i$-th unit vector in $\mathbb{R}^n$. The fundamental weights are $\Lambda_i = \epsilon_1 + \cdots + \epsilon_i$ for $i \in J$. The sum of all positive roots is $\rho = (n, n-1, \ldots, 1)$. The bilinear form is given by $\langle \alpha_i, \beta \rangle = 2 \alpha_i \cdot \beta$.

For both type $A_n$ and $C_n$ we will identify dominant weights, that is weights of the form $\Lambda = \Lambda_{k_1} + \cdots + \Lambda_{k_m}$, with partitions. A partition $\lambda = (\lambda_1, \ldots, \lambda_{n+1})$ (with $\lambda_{n+1} = 0$ for type $C_n$) corresponds to the weight $\Lambda = \sum_{i \in J} (\lambda_i - \lambda_{i+1}) \Lambda_i$.

For each dominant weight $\Lambda$ there is a finite classical crystal $B(\Lambda)$ [23]. The crystals $B(\Lambda_i)$ for type $A_n$ and $C_n$ are given in Table 1 where the arrow $\overrightarrow{i}$ stands for $f_i$. The finite crystals $B(\Lambda_k)$ for $k \in J$ can be obtained in the following way. Let $u_{\Lambda_k} = \underbrace{[k] \otimes \cdots \otimes [2] \otimes [1]}$ be the unique highest weight vector of weight $\Lambda_k$ in $B(\Lambda_1)^{\otimes k}$. Then $B(\Lambda_k)$ is the connected component of $B(\Lambda_1)^{\otimes k}$ containing $u_{\Lambda_k}$. In [23] the elements in this connected component were identified with certain one-column tableaux. The finite crystal $B(\Lambda)$ for a dominant weight $\Lambda = \Lambda_{k_1} + \cdots + \Lambda_{k_m}$ with $k_1 \geq k_2 \geq \cdots \geq k_m$ is isomorphic to the connected component in $B(\Lambda_{k_1}) \otimes \cdots \otimes B(\Lambda_{k_m})$ which contains the highest weight element $u_{\Lambda_{k_1}} \otimes \cdots \otimes u_{\Lambda_{k_m}}$. 
Combining the two embeddings it follows that $B(\Lambda)$ is isomorphic to a certain connected component in $B(\Lambda_1)^\otimes M$ where $M = k_1 + \cdots + k_m$.

For type $A\,n$ the elements in $B(\Lambda)$ can be identified with semi-standard Young tableaux of shape $\lambda$ where $\lambda$ is the partition corresponding to the weight $\Lambda$. The paths encountered in section 2 are sequences of 1’s and 2’s. Viewed as single box Young tableaux over the alphabet $\{1, 2\}$, these are exactly the elements of the crystal $B(\Lambda_1)$ of type $A_1$.

3.5. Simple crystals. Simple crystals were introduced by Akasaka and Kashiwara [1]. As will be explained in the next section they have an isomorphism and energy function which are required for the definition of the one-dimensional configuration sums.

Let $\hat{W}$ be the Weyl group of the affine Kac–Moody algebra $\hat{\mathfrak{g}}$ generated by the simple reflections $r_i$ for $i \in I$ defined as

$$r_i(\beta) = \beta - \langle h_i, \beta \rangle \alpha_i$$

(3.6) where $\beta$ is a root. Let $B$ be the crystal graph of an integrable $U_q(\hat{\mathfrak{g}})$-module. Say that $b \in B$ is an extremal vector of weight $\Lambda \in P$ provided that $\text{wt}(b) = \Lambda$ and there exists a family of elements $\{b_w \mid w \in \hat{W}\} \subset B$ such that

1. $b_w = b$ for $w = e$.
2. If $\langle h_i, w\Lambda \rangle \geq 0$ then $e_i(b) = 0$ and $f_i^{(h_i, w\Lambda)}(b_w) = b_{r_iw}$.
3. If $\langle h_i, w\Lambda \rangle \leq 0$ then $f_i(b) = 0$ and $e_i^{(h_i, w\Lambda)}(b_w) = b_{r_iw}$.

DEFINITION 3.2. Say that a $U'_q(\hat{\mathfrak{g}})$-crystal $B$ is simple if

1. $B$ is the crystal base of a finite-dimensional integrable $U'_q(\hat{\mathfrak{g}})$-module.
2. There is a dominant weight $\Lambda$ with respect to the weight lattice of the classical algebra $\mathfrak{g}$ such that $B$ has a unique vector (denoted $u(B)$) of weight $\Lambda$, and the weight of any extremal vector of $B$ is contained in $W\Lambda$. Here $W$ is the Weyl group corresponding to the classical algebra $\mathfrak{g}$.

THEOREM 3.3. [1]

1. Simple crystals are connected as graphs.
2. The tensor product of simple crystals is simple.

Let $B$ be a simple $U'_q(\hat{\mathfrak{g}})$-crystal, equipped with a function $D = D_B : B \to \mathbb{Z}$, called its intrinsic energy, which is required to be constant on $J$-components and defined up to a global additive constant. Call the pair $(B, D)$ a graded simple $U'_q(\hat{\mathfrak{g}})$-crystal. We normalize the intrinsic energy function by the requirement that

$$D_B(u(B)) = 0.$$
3.6. Finite dimensional affine crystals. Recently, new families of crystals of the finite dimensional representations of $U_q(\hat{g})$ were conjectured \cite{13, 19} where $\hat{g}$ is an untwisted affine Lie algebra.

Conjecture 3.4. \cite{13, 19} For each $r \in J$ and $s \geq 1$, there exists an irreducible finite-dimensional integrable $U'_q(\hat{g})$-module $W_s^{(r)}$ with simple crystal base $B^{r,s}$ generated a unique extremal vector $u(B^{r,s})$ of weight $s\Lambda_r$, and a prescribed $U_q(\hat{g})$-crystal decomposition of the form $B^{r,s} \cong B(s\Lambda_r) \oplus B$, where $B$ is a direct sum of $U_q(\hat{g})$-crystals of the form $B(\Lambda)$ where $\Lambda$ is a classical dominant weight and $s\Lambda_r \triangleright \Lambda$. Here $\Lambda' \supseteq \Lambda$ if and only if $\Lambda' - \Lambda \in \bigoplus_{i \in J} \mathbb{N}\alpha_i$. Moreover there is a prescribed intrinsic energy function $D = D_{B^{r,s}} : B^{r,s} \to \mathbb{Z}$, that is constant on $J$-components, such that $0 = D(u) > D(b)$ where $u$ is the $J$-highest weight vector of weight $s\Lambda_r$ in $B^{r,s}$, and $b$ is any element not in the $J$-component of $u$.

3.7. Combinatorial $R$-matrix and energy function. Suppose $B_1$ and $B_2$ are simple $U_q(\hat{g})$-crystals. Then there is a unique isomorphism of $U'_q(\hat{g})$-crystals $\sigma : B_2 \otimes B_1 \to B_1 \otimes B_2$. There is also a function $H = H_{B_2,B_1} : B_2 \otimes B_1 \to \mathbb{Z}$, called local energy function which is unique up to global additive constant, such that for all $b_2 \otimes b_1 \in B_2 \otimes B_1$ with $b'_1 \otimes b'_2 = \sigma(b_2 \otimes b_1)$

\begin{equation}
H(e_i(b_2 \otimes b_1)) = H(b_2 \otimes b_1) + \begin{cases} -1 & \text{if } i = 0, \epsilon_0(b_2) > \varphi_0(b_1), \epsilon_0(b'_1) > \varphi_0(b'_2) \\ 1 & \text{if } i = 0, \epsilon_0(b_2) \leq \varphi_0(b_1), \epsilon_0(b'_1) \leq \varphi_0(b'_2) \\ 0 & \text{otherwise.} \end{cases}
\end{equation}

The pair $(\sigma, H)$ is called the combinatorial $R$-matrix. It is convenient to normalize the local energy function $H$ by requiring that

\begin{equation}
H(u(B_2) \otimes u(B_1)) = 0.
\end{equation}

With this convention it follows by definition that

\begin{equation}
H_{B_1,B_2} \circ \sigma_{B_2,B_1} = H_{B_2,B_1}
\end{equation}

as operators on $B_2 \otimes B_1$.

Let $(B_j, D_j)$ be graded simple $U_q(\hat{g})$-crystals for $1 \leq j \leq L$ and set $B = B_L \otimes \cdots \otimes B_1$. Following \cite{32} define the energy function $E_B : B \to \mathbb{Z}$ by

\begin{equation}
E_B = \sum_{1 \leq i < j \leq L} H_i \sigma_{i+1} \sigma_{i+2} \cdots \sigma_{j-1}
\end{equation}

where $H_i$ (resp. $\sigma_i$) is the local energy function (resp. isomorphism) acting on the $i$-th and $(i + 1)$-st tensor factor. By the normalization assumption (3.8) it follows that

\begin{equation}
E_B(u(B)) = 0.
\end{equation}

As shown in \cite{33}, the intrinsic energy $D_B$ for the $L$-fold tensor product $B = B_L \otimes \cdots \otimes B_1$ is given by

\begin{equation}
D_B = E_B + \sum_{j=1}^{L} D_j \sigma_1 \sigma_2 \cdots \sigma_{j-1}
\end{equation}

where $D_j$ acts on the rightmost tensor factor which is $B_j$. 

The coenergy and intrinsic coenergy are defined as
\[ E_B = -E_B \quad \text{and} \quad D_B = -D_B. \]

3.8. One-dimensional configuration sums. There are three different sets of paths that we consider. Let \( B = B_L \otimes \cdots \otimes B_1 \) where all \( B_i \) are simple crystals.

For a classical weight \( \Lambda \) the set of unrestricted paths is defined as
\[ P(B, \Lambda) = \{ b \in B \mid \text{wt}(b) = \Lambda \}. \]  
(3.13)

For a dominant classical weight \( \Lambda \) the set of classically restricted paths is
\[ P'(B, \Lambda) = \{ b \in B \mid \text{wt}(b) = \Lambda \quad \text{and} \quad e_i b = 0 \quad \text{for all} \quad i \in J \} \]
(3.14)

and the set of level-restricted paths for \( \ell \in \mathbb{N} \) is
\[ P^\ell(B, \Lambda) = \{ b \in B \mid \text{wt}(b) = \Lambda, \quad e_i b = 0 \quad \text{for all} \quad i \in J \quad \text{and} \quad e^{\ell+1} b = 0 \}. \]
(3.15)

The corresponding one-dimensional configuration sums are the generating functions of these sets of paths with energy/coenergy statistics. The one-dimensional sums
\[ S(B, \Lambda) = \sum_{b \in P(B, \Lambda)} q^{D_B(b)} \]
(3.16)
\[ \overline{S}(B, \Lambda) = \sum_{b \in P(B, \Lambda)} q^{D_B(b)} \]
are called supernomials, whereas
\[ X(B, \Lambda) = \sum_{b \in P'(B, \Lambda)} q^{D_B(b)} \]
(3.17)
\[ \overline{X}(B, \Lambda) = \sum_{b \in P'(B, \Lambda)} q^{D_B(b)} \]
are the classically restricted configuration sums or generalized Kostka polynomials, and
\[ X^\ell(B, \Lambda) = \sum_{b \in P^\ell(B, \Lambda)} q^{D_B(b)} \]
(3.18)
\[ \overline{X}^\ell(B, \Lambda) = \sum_{b \in P^\ell(B, \Lambda)} q^{D_B(b)} \]
are the level-\( \ell \) restricted configuration sums or \( \ell \)-generalized Kostka polynomials.

The classically restricted configurations sums \( P(B, \Lambda) \) are graded tensor product multiplicities. The level-restricted configuration sums \( P^\ell(B, \Lambda) \) are graded level \( \ell \) fusion coefficients. Let \( B_{\Lambda'} \) denote the crystal corresponding to the affine irreducible highest weight representation \( V(\Lambda') \). By the Verlinde formula \( [38] \), the fusion coefficient is the coefficient of \( B_{\Lambda'} \) of weight \( \Lambda' = \Lambda + \ell \Lambda_0 \) in the decomposition of \( B \otimes B_{\ell \Lambda_0} \). The affine highest weight vectors of weight \( \Lambda' \), whose number is the above multiplicity, are the summands of \( \overline{X}(B, \Lambda) \).

It will be shown in sections \( [8] \) and \( [9] \) that there are two different evaluations of \( \overline{X}(B, \Lambda) \) and \( \overline{X}^\ell(B, \Lambda) \) which give rise to \( q \)-identities.
4. Bosonic evaluation

Here we present the bosonic evaluation of $\Xi(B, \Lambda)$ and $\Xi'(B, \Lambda)$ as defined in (3.17) and (3.18). Similarly to the bosonic evaluation of $X(L)$ of section 2.2, the bosonic evaluation of $\Xi(B, \Lambda)$ and $\Xi'(B, \Lambda)$ can be obtained by inclusion-exclusion arguments as shown in [34]. We discuss the main ideas and techniques of sign-reversing involutions.

4.1. Classically-restricted case. Let $B = B_L \otimes \cdots \otimes B_1$ be a $U_q(\mathfrak{g})$-crystal and let $\Lambda$ be a classical weight. The Weyl group $W$ of $\mathfrak{g}$ which is generated by the simple reflections $r_i$ as in (3.6) with $i$ restricted to $i \in J$. The bosonic expression for the generating function of classically-restricted paths $\Xi(B, \Lambda)$ as defined in (4.1) is given by

$$\Xi(B, \Lambda) = \sum_{\omega \in W} (-1)^\omega \mathfrak{S}(B, \omega(\Lambda + \rho) - \rho)$$

where $(-1)^\omega$ is the sign of $\omega$ and $\rho$ is half the sum of all positive roots. This formula follows directly from Weyl’s character formula.

As a warm-up for the level-restricted case, we would like to briefly sketch for $\mathfrak{g} = A_n$ how (4.1) can be derived via a sign-reversing involution.

**Example 4.1.** For $\mathfrak{g} = A_n$ the Weyl group $W$ is generated by the reflections $r_1, \ldots, r_n$ which act on $\lambda \in \mathbb{Z}^{n+1}$ as follows

$$r_i(\lambda) = (\lambda_1, \ldots, \lambda_{i-1}, \lambda_i', \lambda_{i+1}, \ldots, \lambda_{n+1}).$$

Hence on $\mathbb{Z}^{n+1}$ the Weyl group acts as the symmetric group $S_{n+1}$.

For $\mathfrak{g} = C_n$ the Weyl group is generated by

$$r_i(\lambda) = (\lambda_1, \ldots, \lambda_{i-1}, \lambda_i, \lambda_{i+1}, \ldots, \lambda_n)$$

for $1 \leq i < n$ and

$$r_n(\lambda) = (\lambda_1, \ldots, \lambda_{n-1}, -\lambda_n)$$

so that $W$ on $\mathbb{Z}^n$ acts by all permutations and sign changes.

We define an involution $\Phi : S \to S$ with the properties that the fixed points are the pairs $(1, b)$ with $b \in \mathcal{P}'(B, \Lambda)$ which recall are the paths underlying $\Xi(B, \Lambda)$ (see (3.17)). Furthermore, if $\Phi(\omega, b) = (\omega', b')$ with $(\omega, b) \neq (\omega', b')$ then $\omega$ and $\omega'$ have opposite signs.

Let $S_i$ for $i \in J$ be the set of pairs $(\omega, b) \in S$ such that $\epsilon_i(b) > 0$. Define $\Phi_i : S_i \to S_i$ by $\Phi_i(\omega, b) = (\omega_r, s_i e_i(b))$. Define the set $S' = S - \{(1, b) \mid b \in \mathcal{P}'(B, \Lambda)\}$ so that $S' = \bigcup_{i \in J} S_i$. Then $\Phi$ is given by

$$\Phi(\omega, b) = \begin{cases} (\omega, b) & \text{if } (\omega, b) \in S \setminus S' \\ \Phi_i(\omega, b) & \text{if } (\omega, b) \in S' \text{ and } i = v(\omega, b) \end{cases}$$

where $v$ is some functions $v : S' \to J$. To show that $\Phi$ indeed exists and is an involution it needs to be shown that if $v(\omega, b) = i$ then

$$(\omega, b) \in S_i \quad \text{and} \quad v(\Phi_i(\omega, b)) = i.$$
b ∈ B under this embedding. Then v can be defined as follows. Let k be minimal such that ε_i(p_k ⊗ ⋅ ⋅ ⋅ ⊗ p_1) > 0 for some i ∈ I. Then it is clear from Table 3 that there is a unique i satisfying ε_i(p_k ⊗ ⋅ ⋅ ⋅ ⊗ p_1) > 0. Set v(ω, b) = i. It follows from equation (2.5) and Table 1 that the first k tensor factors of p stay invariant under Φ_i since there are no strings of length greater than one. This ensures (1.2). Hence inclusion-exclusion implies (1.1).

4.2. Level-restricted case. The bosonic expression for the level-restricted generating function X(B, Λ) defined in (1.18) can also be found by a sign-reversing involution. The difference is that one needs to consider elements ω in the affine Weyl group ˆW which is generated by r_i with i ∈ I (rather than i ∈ J).

Example 4.2. For ˆg = A_n(1) the affine Weyl group ˆW is generated by the reflections r_1, . . . , r_n as in example 4.1 and

\[ r_0(λ) = (λ_{n+1} + ℓ + n + 1, λ_2, . . . , λ_n, λ_1 - (ℓ + n + 1)). \]

For ˆg = C_n(1) the affine Weyl group ˆW is generated by the reflections r_1, . . . , r_n as in example 4.1 and

\[ r_0(λ) = (-λ_1 + 2(ℓ + n + 1), λ_2, . . . , λ_n). \]

The affine Weyl group is isomorphic to ˆW ≅ T × W where T is the set of certain translations t_α indexed by α ∈ M for a particular set M (for more details see [15], Section 6). Then it was shown in [14] that

\[ X^X(B, Λ) = \sum_{ω∈W} \sum_{α∈M} (-1)^{ω_α} q^{ω(α|α)(ℓ+|h^∨|α)} \times S(B, ω(Λ + ρ - (ℓ + |h^∨|α) - ρ) \]

where |h^∨| is the dual Coxeter number of ˆg and a_0 is the label of the zeroth node in the Dynkin diagram of ˆg.

Example 4.3. Let us give (4.3) more explicitly for ˆg = A_n(1) and C_n(1). In both cases the dual Coxeter number is |h^∨| = n + 1 and a_0 = 1. For type A_n(1) the set M is given by all β ∈ Z_n+1 such that |β| := β_1 + ⋅ ⋅ ⋅ + β_n+1 = 0 so that

\[ X^X(B, λ) = \sum_{ω∈W} \sum_{β∈Z_n+1} (-1)^{ω_β} q^{ω(β(ℓ+n+1)-ρ+λ)β} \times S(B, ω(λ + (ℓ + n + 1)β) - ρ) \]

where W = S_{n+1} is the set of permutations.

For type C_n(1) we have M = 2Z^n. Hence

\[ X^X(B, λ) = \sum_{ω∈W} \sum_{β∈2Z^n} (-1)^{ω_β} q^{ω(β(ℓ+n+1)-ρ+λ)β} \times S(B, ω(λ + (ℓ + n + 1)β) - ρ) \]

1The arguments in [14] require that B is a tensor product of almost perfect crystals and that the energy function obeys certain properties. For the examples of type A_n(1) with B_i = B^{r_i,1} and C_n(1) with B_i = B^{r_i,2}, for which we will consider fermionic formulas in the next section, these conditions are all satisfied.
where the Weyl group $W$ in this case is the set of all permutations and sign changes. The extra factor of $1/2$ in the exponent of $q$ comes from the normalization of $(\cdot|\cdot)$ as alluded to in example $[3,4].$

The arguments in $[3,4]$ involve a sign-reversing involution. Similarly to the classically restricted case set
\[ |(\omega, b) \in W \times B \mid \omega(wt(b) + \rho) = \Lambda + \rho |. \]

For $i \in J$ define $S_i$ and $\Phi_i$ as before. In addition, let $S_0$ be the subset of all pairs $(\omega, b)$ in $S$ such that $\ell_0(b) > \ell$ and define $\Phi_0(\omega, b) = (\omega r_0, s_0 e_0^{e_0 + 1} b)$. One can find a sign-reversing involution $\Phi$ with fixed point set being the set of level-$\ell$ restricted paths $\mathcal{P}^\ell(B, \Lambda)$ by again specifying a function $v : S - \{(1, b) \mid b \in \mathcal{P}^\ell(B, \Lambda)\} \rightarrow I$ satisfying (4.3). The existence of such a function $v$ was proven in $[3,4]$ for a large class of crystals.

### 4.3. Supernomial coefficients.

The bosonic formulas (4.1) and (4.3) involve the supernomial coefficients defined in (4.10). To obtain truly explicit expressions it is still necessary to give formulas for the supernomial coefficients. These are not yet known in general. Here we give a few examples for which formulas exist.

**Example 4.4.** The supernomial coefficients for type $A_n^{(1)}$ for single columns were given in $[12, 20]$. Let $B = B^{\mu_1, 1} \otimes \cdots \otimes B^{\mu_l, 1}$ be the tensor product of crystals of type $A_n^{(1)}$ corresponding to the partition $\mu = (\mu_1, \ldots, \mu_l)$. Furthermore, let $\lambda \in \mathbb{N}^n$ be a composition. Then

\[
\mathcal{S}(B, \lambda) = \sum_{\nu} \prod_{1 \leq a \leq n} \left[ \nu^{(a)} - \nu^{(a+1)} \right]
\]

where the sum is over all sequences of partitions $\nu = (\nu^{(1)}, \ldots, \nu^{(n)})$ such that $0 = \nu^{(0)} \subset \nu^{(1)} \subset \cdots \subset \nu^{(n+1)} = \mu^t$. $\nu^{(a)}/\nu^{(a-1)}$ is a horizontal $\lambda_a$-strip.

A horizontal $p$-strip is a skew shape with $p$ boxes such that each column contains at most one box. For $\mu = (1^{\lambda})$ equation (4.4) reduces to the $q$-multinomial coefficient

\[
\begin{bmatrix} L \\ \lambda_1, \ldots, \lambda_{n+1} \end{bmatrix} = \frac{(q)_L}{(q)_{\lambda_1} \cdots (q)_{\lambda_{n+1}}} \frac{1}{|\lambda| = L}
\]

and zero otherwise.

**Example 4.5.** The supernomials for type $A_n^{(1)}$ for single rows were also given in $[12, 20]$. Let $\mu = (\mu_1, \ldots, \mu_L)$ be a partition, $B = B^{1, \mu_L} \otimes \cdots \otimes B^{1, \mu_1}$ and $\lambda \in \mathbb{N}^{n+1}$ a composition. Then

\[
\mathcal{S}(B, \lambda) = \sum_{\nu} q^{\phi(\nu)} \prod_{1 \leq a \leq n} \left[ \frac{\nu^{(a+1)} - \nu^{(a)}}{\nu^{(a)} - \nu^{(a+1)}} \right]
\]

where the sum is over all sequences of partitions $\nu = (\nu^{(1)}, \ldots, \nu^{(n)})$ such that $0 = \nu^{(0)} \subset \nu^{(1)} \subset \cdots \subset \nu^{(n+1)} = \mu^t$. $|\nu^{(a)}| = \lambda_1 + \cdots + \lambda_a$ for all $1 \leq a \leq n$. 

Here \( \phi(\nu) \) is defined as
\[
\phi(\nu) = \sum_{1 \leq \alpha \leq n} \sum_{1 \leq i \leq \mu_1} \nu_1(\alpha) (\nu_i(\alpha+1) - \nu_i(\alpha)).
\]

As in the previous example this reduces to the \( q \)-multinomial coefficient \((4.3)\) if \( \mu = (1|\lambda|) \). For type \( A_1 \) this formula coincides with that in \(36\).

**Example 4.6.** The supernomials of type \( C_n \) for single boxes can be obtained by the following arguments. Let \( B = (B^{1,1}) \otimes L \) and \( \lambda \in \mathbb{Z}^n \) with \( ||\lambda|| := |\lambda_1| + \cdots + |\lambda_n| \leq L \). This means that there are at least \( \lambda_i \) letters \( i \) if \( \lambda_i \geq 0 \) and at least \( \lambda_i \) letter \( \mathfrak{t} \) if \( \lambda_i < 0 \). If \( ||\lambda|| < L \) then there have to be \( (L - ||\lambda||)/2 \) pairs of barred and unbarred letters in order to have weight \( \lambda \). Hence we have
\[
\mathcal{S}(B, \lambda) = \sum_{2|\mu = L - ||\lambda||} \frac{L}{||\lambda|| + \mu_1, \ldots, |\lambda_n| + \mu_n, \mu_1, \ldots, \mu_n}
\]
where \( \frac{L}{||\lambda|| + \mu_1, \ldots, |\lambda_n| + \mu_n, \mu_1, \ldots, \mu_n} \) is the \( q \)-multinomial as defined in \((4.5)\).

Explicit formulas also exist for level one cases for \( B_n^{(1)} \), \( D_n^{(1)} \) \(8\) and \( A_{2n}^{(2)}, D_{n+1}^{(2)} \) \(24\).

5. **Fermionic evaluation**

The derivation of fermionic evaluations of the classically- and level-restricted configuration sums \((3.17)\) and \((3.18)\) is in general much more intricate than for the hard-hexagon model. There exists a vast literature on conjectures and proofs of fermionic formulas, most of which deal with the case \( \mathfrak{g} = A_1^{(1)} \) or \( A_n^{(1)} \). A relatively complete list of references can be found in \(13\). Fermionic formulas for all untwisted quantum affine algebras were recently conjectured in \(13\). For type \( A_{2n}^{(1)} \) these are proven for the classically-restricted case in \(29\) and the level-restricted case in \(35\). For type \( C_n^{(1)} \) the classically-restricted formulas are proven in \(33\). We will present these results here and also derive the fermionic level-restricted formulas for type \( C_n^{(1)} \) in section \(5.3\).

Interestingly, Kirillov and Reshetikhin \(28\) conjectured that the coefficients of the decomposition of the representations of \( U_q(\mathfrak{g}) \) naturally associated with multiples of the fundamental weights into direct sums of irreducible representations of \( U_q(\mathfrak{g}) \) are given by the fermionic formulas at \( q = 1 \). Chari \(8\) proved this conjecture for a single tensor factor for \( \mathfrak{g} \) a simple Lie algebra of classical type and also for some exceptional cases.

5.1. **Classically-restricted case.** We will state here the fermionic formulas conjectured in \(13\). Let \( \mathfrak{g} = X_n^{(1)} \) with \( X = A, B, C, D \) or \( E \) for \( n = 6, 7, 8 \) or \( F \) for \( n = 4 \) or \( G \) for \( n = 2 \). Let \( B = \bigotimes_{a=1}^n \bigotimes_{i \geq 1} (B_n^{a, i}) \otimes L^{(a)}_i \) where \( L^{(a)}_i \in \mathbb{N} \) for all \( i \geq 1 \) and \( 1 \leq a \leq n \) and only finitely many \( L^{(a)}_i \) are nonzero. Define the following polynomial in \( q \) depending on \( B \) and a dominant weight \( \Lambda \)
\[
\mathcal{F}(B, \Lambda) = \sum_{\{m\}} q^{\varepsilon(\{m\})} \prod_{a=1}^n \prod_{i \geq 1} \left[ \frac{m_i^{(a)} + p_i^{(a)}}{m_i^{(a)}} \right]
\]
where the sum is over all \( \{ m_i^{(a)} \} \in \mathbb{N} \mid 1 \leq a \leq n, \ i \geq 1 \) subject to the constraints

(5.2) \[
\sum_{a=1}^{n} \sum_{i \geq 1} i m_i^{(a)} \alpha_a = \sum_{a=1}^{n} \sum_{i \geq 1} i L_i^{(a)} \Lambda_a - \Lambda.
\]

The variables \( p_i^{(a)} \) and the exponent \( cc(\{ m \}) \) are defined as

(5.3) \[
p_i^{(a)} = \sum_{j \geq 1} L_j^{(a)} \min(i, j) - \sum_{b=1}^{n} (\alpha_a | \alpha_b) \sum_{k \geq 1} \min(t_{b \cdot j}, t_{a \cdot k}) m_j^{(b)} m_k^{(b)}
\]

(5.4) \[
cc(\{ m \}) = \frac{1}{2} \sum_{a, b=1}^{n} (\alpha_a | \alpha_b) \sum_{j, k \geq 1} \min(t_{b \cdot j}, t_{a \cdot k}) m_j^{(a)} m_k^{(b)}
\]

where \( t_a = \frac{2}{(\alpha_a | \alpha_a)} \). Recall that \((\cdot | \cdot)\) is normalized such that \((\alpha_a | \alpha_a) = 2\) if \( \alpha_a \) is a long root. Then it was conjectured [13] that

\[
\Xi(B, \Lambda) = F(B, \Lambda).
\]

For type \( A_n^{(1)} \) and general \( B \) this is proven in [29] and for type \( C_n^{(1)} \) and \( B = \otimes_{a=1}^{n} (B^{(1)}) \otimes L_i^{(a)} \) a proof is given in [33]. Parts of the proofs given in [29, 33] are quite technical, but we would like to highlight the general ideas of the proof which also give more insight into the fermionic formulas.

### 5.2. Rigged configurations

Rigged configurations provide a combinatorial interpretation of the fermionic formula (5.1). We will focus first on type \( A_n^{(1)} \).

The sum over the variables \( \{ m_i^{(a)} \} \) in (5.1) subject to the restriction (5.2) can be interpreted as follows. Let \( \nu = (\nu^{(1)}, \ldots, \nu^{(n)}) \) be a sequence of partitions with constraints on their sizes given by

(5.5) \[
|\nu^{(a)}| = -\sum_{j=1}^{a} \lambda_j + \sum_{i \geq 1} \sum_{b=1}^{n} i L_i^{(b)} \min(a, b)
\]

where \( \lambda \) is the partition corresponding to the weight \( \Lambda \). Then (5.3) and (5.2) are equivalent provided that \( m_i^{(a)} \) is interpreted as the number of parts of \( \nu^{(a)} \) of size \( i \). In terms of \( \nu \) the definitions (5.3) and (5.4) read

\[
P_i^{(a)}(\nu) = Q_i(\nu^{(a-1)}) - 2Q_i(\nu^{(a)}) + Q_i(\nu^{(a+1)}) + \sum_{j \geq 1} L_j^{(a)} Q_i(j)
\]

\[
cc(\nu) = \sum_{a=1}^{n} \sum_{i \geq 1} (\alpha_i^{(a)} - \alpha_i^{(a+1)})
\]

where \( Q_i(\mu) \) is the number of boxes in the first \( i \) columns of the partition \( \mu \), \( \alpha_i^{(a)} \) is the size of the \( i \)-th column of \( \nu^{(a)} \) and \( p_i^{(a)} = \widetilde{P}_i^{(a)}(\nu) \).

To interpret (5.1) combinatorially one uses the fact the \( q \)-binomial coefficient \([m+p]_m \) is the generating function of partitions in a box of size \( m \times p \). More precisely, these are the partitions with at most \( m \) parts each not exceeding \( p \). Hence (5.1) can be restated as

\[
\Xi(B, \Lambda) = \sum_{(\nu, J) \in RC(B, \Lambda)} q^{cc(\nu, J)}
\]
where \( \text{RC}(B, A) \) is the set of all \((\nu, J)\) where \(\nu\) is a sequence of partitions satisfying (5.3) and \(J\) is a double sequence of partitions \(J = \{J^{(a,i)}\}_{1 \leq a \leq n, i \geq 1}\).

The partition \(J^{(a,i)}\) has to fit in a box of size \(m_i^{(a)}(\nu) \times P_i^{(a)}(\nu)\) where \(m_i^{(a)}(\nu) = m_i^{(a)}\) is the number of parts of size \(i\) in \(\nu^{(a)}\). In particular this requires that \(P_i^{(a)}(\nu) \geq 0\) for all \(i \geq 1\) and \(1 \leq a \leq n\). The exponent is defined as

\[
\text{cc}(\nu, J) = \text{cc}(\nu) + \sum_{a=1}^{n} \sum_{i \geq 1} |J^{(a,i)}|.
\]

The elements in \(\text{RC}(B, A)\) are called rigged configurations.

Originally, rigged configurations were introduced in papers by Kerov, Kirillov and Reshetikhin [26, 28] in their study of the XXX model using Bethe Ansatz techniques. Rigged configurations index the solutions to the Bethe Ansatz equations.

In [29] the fermionic formula was proven by showing that there is a statistic preserving bijection between classically restricted paths and rigged configurations. It should be noted that for type \(A_n^{(1)}\) the intrinsic energy is equal to the energy, \(\overline{D}_B = \overline{E}_B\).

**Theorem 5.1.** [29] For \(B = \bigotimes_{a=1}^{n} \bigotimes_{i \geq 1} (B_a^{n,i} \otimes L_i^{(a)})\) a crystal of type \(A_n^{(1)}\), there is a bijection \(\phi : \mathcal{P}'(B, A) \rightarrow \text{RC}(B, A)\) such that for \(b \in \mathcal{P}'(B, A)\) we have \(\overline{E}_B(b) = \text{cc}(\theta \circ \phi(b))\). Here \(\theta : \text{RC}(B, A) \rightarrow \text{RC}(B, A)\) maps \((\nu, J)\) to \((\nu, \tilde{J})\) where \(\tilde{J}\) is obtained from \(J\) by complementing each \(J^{(a,i)}\) in the box of dimensions \(m_i^{(a)}(\nu) \times P_i^{(a)}(\nu)\).

The bijection \(\phi\) is given explicitly in [29], [28], Section 5.4.

The fermionic formula for type \(C_n^{(1)}\) crystals of the form \(B_C = \bigotimes_{a=1}^{n} (B_C^{n,1} \otimes L_i^{(a)})\) was proven in [33] using an embedding of type \(C_n^{(1)}\) crystals into type \(A_n^{(1)}\) crystals. Let

\[
\Psi(B_C^{r,1}) = \begin{cases} 
B_A^{2n-r,1} \otimes B_A^{r,1} & \text{if } 1 \leq r < n \\
B_A^{n,2} & \text{if } r = n.
\end{cases}
\]

Baker [2] showed that there is an embedding \(\Psi^{r,1} : B_C^{r,1} \rightarrow \Psi(B_C^{r,1})\). It can be defined by requiring that the \(U_q(C_n)\)-highest weight vector \(u_A^{r,1}\) in \(B_C^{r,1}\) is mapped to \(u_A^{2n-r,1} \otimes u_A^{r,1}\) where \(u_A^{r,s}\) is the \(U_q(A_{2n-1})\)-highest weight vector in \(B_A^{r,s}\), and

\[
\begin{align*}
\Psi^{r,1} \circ f_i^C &= f_i^{2n-r,1} \circ f_i^A \circ \Psi^{r,1} \\
\Psi^{r,1} \circ e_i^C &= e_i^{2n-r,1} \circ e_i^A \circ \Psi^{r,1}.
\end{align*}
\]

For a tensor product, define \(\Psi_L : B_C^{r_L,1} \otimes \cdots \otimes B_C^{r_1,1} \rightarrow \Psi(B_C^{r_L,1}) \otimes \cdots \otimes \Psi(B_C^{r_1,1})\) by \(\Psi_L = \Psi^{r_L,1} \otimes \cdots \otimes \Psi^{r_1,1}\).

**Theorem 5.2.** [33] Let \(B_C = B_C^{r_L,1} \otimes \cdots \otimes B_C^{r_1,1}\). The image \(\text{Im}(\phi \circ \Psi_L)\) of \(\phi \circ \Psi_L : \mathcal{P}'(B_C, \cdot) \rightarrow \text{RC}(\Psi(B_C), \cdot)\) is characterized by the set of rigged configurations \((\nu, J)\) satisfying:

1. \((\nu, J)^{(k)} = (\nu, J)^{(2n-k)}\).
2. All parts of \(\nu^{(n)}\) are even.
3. All riggings in \((\nu, J)^{(n)}\) are even.
This characterization of the image of \( \phi \circ \Psi_L \) suggests the following definition of type \( C \) rigged configurations. Let \( \lambda \) be a partition and \( B_C = \bigotimes_{a=1}^n (B_{L_i}^{(a)}) \), and let \( \nu = (\nu^{(1)}, \ldots, \nu^{(n)}) \) be a sequence of partitions with the properties

\[
|\nu^{(a)}| = - \sum_{j=1}^a \lambda_j + \sum_{b=1}^n L_1^{(b)} \min(a, b) \quad \text{for } 1 \leq a \leq n
\]

where \( \nu^{(n)} \) has only even parts.

Define the vacancy numbers as

\[
P_i^{(a)}(\nu) = Q_i(\nu^{(a-1)}) - 2Q_i(\nu^{(a)}) + Q_i(\nu^{(a+1)}) + L_i^{(a)} \quad \text{for } 1 \leq a < n,
\]

\[
P_i^{(n)}(\nu) = Q_i(\nu^{(n-1)}) - Q_i(\nu^{(n)}) + \frac{1}{2} L_i^{(n)} Q_i(2).
\]

The set of rigged configurations of type \( C \) corresponding to a weight \( \Lambda \) with associated partition \( \lambda \) and crystal \( B_C \), denoted by \( RC_C(B_C, \Lambda) \), is given by \( (\nu, J) \) where \( \nu \) is a sequence of partitions satisfying (5.6) and \( J \) is a double sequence of partitions

\[J = \{J^{(a,i)}\}_{1 \leq a \leq n, i \geq 1}\]

where \( J^{(a,i)} \) is a partition in a box of size \( m_i^{(a)}(\nu) \times P_i^{(a)}(\nu) \) with \( P_i^{(a)}(\nu) \) as in (5.7) and \( m_i^{(a)}(\nu) \) the number of parts of \( \nu^{(a)} \).

It is shown in [33] that \( \bar{D}_A \circ \Psi_L = 2\bar{D}_C \). Hence using Theorem 5.2 the statistics of type \( C \) rigged configurations becomes

\[
ccc(\nu, J) = ccc(\nu) + \sum_{a=1}^n \sum_{i \geq 1} |J^{(a,i)}|
\]

where \( ccc(\nu) = \sum_{i \geq 1} \left( \sum_{a=1}^{n-1} \alpha_i^{(a)} \alpha_i^{(a)} - \alpha_i^{(a+1)} - \frac{1}{2} \alpha_i^{(n)} \right) \)

which implies that

\[
\mathbf{X}(B_C, \Lambda) = \sum_{(\nu, J) \in RC_C(B_C, \Lambda)} q^{ccc(\nu, J)}.
\]

It is also not so hard to show that

\[
\mathbf{F}(B_C, \Lambda) = \sum_{(\nu, J) \in RC_C(B_C, \Lambda)} q^{ccc(\nu, J)}
\]

by identifying

\[
P_i^{(a)} = \begin{cases} P_i^{(a)}(\nu) & \text{for } 1 \leq a < n \\ P_i^{(n)}(\nu) & \text{for } a = n \end{cases},
\]

\[
m_i^{(a)} = \begin{cases} m_i^{(a)}(\nu) & \text{for } 1 \leq a < n \\ m_i^{(n)}(\nu) & \text{for } a = n. \end{cases}
\]

This proves that \( \mathbf{X}(B_C, \Lambda) = \mathbf{F}(B_C, \Lambda) \).
5.3. Level-restricted case. Fermionic formulas for the level-restricted one-dimensional configuration sums $\mathbf{X}(B, \Lambda)$ were conjectured in [13] for all $\hat{g}$ as in section 5.1 and special weight $\Lambda = 0$. Let $\ell \in \mathbb{N}$ and define the following polynomial in $q$ depending on $B$

\[
(5.8) \quad \mathcal{F}^\ell(B) = \sum_{\{m\}} q^{cc^\ell(m)} \prod_{(a,i) \in H^\ell} \left[ \frac{m_i^{(a)} + p_i^{(a)}}{m_i^{(a)}} \right]
\]

where $H^\ell = \{(a,i) \mid 1 \leq a \leq n, 1 \leq j \leq t_a \ell \}$ and the sum is over all $\{m_i^{(a)} \in \mathbb{N} \mid (a,i) \in H^\ell \}$ subject to the constraints

\[
(5.9) \quad \sum_{(a,i) \in H^\ell} i m_i^{(a)} \alpha_a = \sum_{(a,i) \in H^\ell} i L_i^{(a)} \Lambda_a.
\]

The variables $p_i^{(a)}$ and the exponent $cc^\ell(m)$ are defined as

\[
(5.10) \quad p_i^{(a)} = \sum_{j=1}^{t_a \ell} j \min(i, j) - \sum_{(b,k) \in H^\ell} (\alpha_b | \alpha_k) \min(t_b, t_a) m_k^{(b)}
\]

\[
(5.11) \quad cc^\ell(m) = \frac{1}{2} \sum_{(a,j), (b,k) \in H^\ell} (\alpha_a | \alpha_b) \min(t_b, t_a) m_j^{(a)} m_k^{(b)}.
\]

Then it was conjectured [13] that

\[
(5.12) \quad \mathbf{X}^\ell(B, 0) = \mathcal{F}^\ell(B).
\]

5.4. Level-restricted case: type $A_n^{(1)}$. For type $A_n^{(1)}$ the conjecture [5,12] and its generalization to arbitrary weights $\Lambda$ was proven in [15]. These formulas can again be understood in terms of rigged configurations. We will explain this here since it will enable us to derive the level-restricted fermionic formulas for type $C_n^{(1)}$ in the next section.

Let $\lambda = (\lambda_1, \ldots, \lambda_{n+1})$ be the partition corresponding to the dominant integral weight $\Lambda$. A partition $\lambda$ is restricted of level $\ell$ if $\lambda_1 - \lambda_{n+1} \leq \ell$. Define $\tilde{\ell} = \ell - (\lambda_1 - \lambda_{n+1})$, which is nonnegative by assumption. Set $\lambda' = (\lambda_1 - \lambda_{n+1}, \ldots, \lambda_n - \lambda_{n+1})^t$ (where $^t$ stands for transpose) and denote the set of all column-strict tableaux of shape $\lambda'$ over the alphabet $\{1, 2, \ldots, \lambda_1 - \lambda_{n+1}\}$ by CST($\lambda'$). Define a table of modified vacancy numbers depending on a sequence of partitions $\nu$ and $t \in CST(\lambda')$ by

\[
(5.13) \quad P_i^{(a)}(\nu, t) = P_i^{(a)}(\nu) - \sum_{j=1}^{\lambda_a - \lambda_{n+1}} \chi(i \geq \tilde{\ell} + t_{j,a}) + \sum_{j=1}^{\lambda_{a+1} - \lambda_{n+1}} \chi(i \geq \tilde{\ell} + t_{j,a+1})
\]

where $t_{j,a}$ denotes the entry in $t$ in row $j$ and column $a$ and $\chi(S) = 1$ if the statement $S$ is true and $\chi(S) = 0$ otherwise.

**Definition 5.3.** Let $B = \bigotimes_{i=1}^{\ell} \bigotimes_{a=1}^{t_i} (B^{a,i}) \otimes L_i^{(a)}$ be a crystal of type $A_n^{(1)}$ and $\Lambda$ a dominant integral weight with corresponding partition $\lambda$. Say that $(\nu, J) \in RC(B, \Lambda)$ is restricted of level $\ell$ provided that

1. $\nu^{(a)} \leq \ell$ for all $a$.
2. There exists a tableau $t \in CST(\lambda')$, such that for every $i, a \geq 1$, the largest part of $J^{(a,i)}$ does not exceed $P_i^{(a)}(\nu, t)$. 

Denote by $RC_{\ell}(B, \Lambda)$ the set of $(\nu, J) \in RC(B, \Lambda)$ that are restricted of level $\ell$.

Note in particular that the second condition requires that $P^{(a)}_{\nu}(\nu, t) \geq 0$ for all $i, a \geq 1$.

**Example 5.4.** Let $\lambda = (m_n+1)$ be rectangular with $n + 1$ rows. Then $\lambda' = \emptyset$ and $P^{(a)}_{\nu}(\nu, \emptyset) = P^{(a)}_{\nu}(\nu)$ for all $i, a \geq 1$ so that the modified vacancy numbers are equal to the vacancy numbers.

**Theorem 5.5.** [35, Theorem 8.2] The bijection $\phi : P^{c*(B, \Lambda)} \to RC(B, \Lambda)$ restricts to a well-defined bijection $\phi : P^{c*(B, \Lambda)} \to RC^{c*(B, \Lambda)}$.

Since $E_B = cc \circ \theta \circ \phi$ by Theorem 5.1 it follows from Theorem 5.3 that for type $A^{(1)}_n$:

\[
\mathcal{X}^{c}(B, \Lambda) = \sum_{(\nu, J) \in RC^{c}(B, \Lambda)} q^{cc(\theta(\nu, J))}.
\]

(5.14)

Note that $cc(\theta(\nu, J)) = cc(\nu) + \sum_{a=1}^{n} \sum_{i=1}^{\ell} P^{(a)}_{i}(\nu) m^{(a)}_{i}(\nu) - \sum_{a=1}^{n} \sum_{i=1}^{\ell} |J^{(a, i)}|$.

Let $SCST(\lambda')$ be the set of all nonempty subsets of $CST(\lambda')$. Since the $q$-binomial $\binom{m+p}{m}$ is the generating function of partitions with at most $m$ parts each not exceeding $p$ it follows by inclusion-exclusion that

\[
\mathcal{X}^{c}(B, \Lambda) = \sum_{S \in SCST(\lambda')} (-1)^{|S|+1} \sum_{(a, i) \in H^\ell} q^{c^{\ell}(m)} \prod_{(a, i) \in H^\ell} \left[ \binom{m^{(a)}_{i} + p^{(a)}_{i}(S)}{m^{(a)}_{i}} \right]_{1/q}^{1/q}
\]

(5.15)

where the sum is over all $\{m^{(a)}_{i} \in \mathbb{N} \mid (a, i) \in H^\ell \}$ subject to the constraints

\[
\sum_{(a, i) \in H^\ell} i m^{(a)}_{i} \alpha_a = \sum_{(a, i) \in H^\ell} i L^{(a)}_{i} \Lambda_a - \Lambda.
\]

Also $p^{(a)}_{i}(S)$ and $c^{\ell}(\{m\})$ are defined as

\[
p^{(a)}_{i}(S) = p^{(a)}_{i} + \min_{t \in S} \left\{ - \sum_{j=1}^{\lambda_{a} - \lambda_{a+1}} \chi(i \geq \ell + t_{j,a}) + \sum_{j=1}^{\lambda_{a+1} - \lambda_{a+1}} \chi(i \geq \ell + t_{j,a+1}) \right\}
\]

\[
c^{\ell}(\{m\}) = cc^{\ell}(\{m\}) + \sum_{(a, i) \in H^\ell} p^{(a)}_{i} m^{(a)}_{i}
\]

with $p^{(a)}_{i}$ as in (5.10) and $cc^{\ell}(\{m\})$ as in (5.11). $\binom{m+p}{m}_{1/q}$ is the $q$-binomial with $q$ replaced by $1/q$. In particular if $\lambda = (m_n+1)$ as in Example 5.4 so that $\Lambda = 0$ and $p^{(a)}_{i}(\nu, \emptyset) = p^{(a)}_{i}(\nu)$ the fermionic form (5.13) reduces to (5.8) since $\binom{m+p}{m}_{1/q} = q^{-mp} \binom{m+p}{m}$.

Further details can be found in [35].

It should be remarked that even though (5.15) contains explicit signs, it is clear from the equivalent combinatorial formula (5.14) that it is a nonnegative polynomial in $q$.

**5.5. Level-restricted case: type $C^{(1)}_n$.** In this section we will show how the level-restricted fermionic formulas for type $C^{(1)}_n$ can be obtained from Theorems 5.1 and 5.2.
Under the embedding Ψ a dominant weight ΛC = \( \sum_{k=1}^{m} \Lambda_{ik}^C \) of type \( C_n^{(1)} \) becomes the weight \( \Lambda_A = \sum_{k=1}^{m} (\Lambda_{ik}^A + \Lambda_{2n-ik}^A) \) of type \( A_{2n-1}^{(1)} \) where all \( 1 \leq ik \leq n \).

In terms of the corresponding partitions \( \lambda^A \) and \( \lambda^C \) this implies that

\[
\begin{align*}
\lambda_A^a - \lambda_{2n}^A &= \lambda_1^C + \lambda_a^C & \text{for } 1 \leq a \leq n \\
\lambda_A^a - \lambda_{2n}^A &= \lambda_1^C - \lambda_{2n+1-a}^C & \text{for } n < a \leq 2n.
\end{align*}
\]

Let \( B_C = \bigotimes_{a=1}^{n} (B_{C,a}^{(1)} \otimes L_1^{(a)}) \). Hence, under the embedding \( \Psi_L : B_C \to \Psi(B_C) \), the conditions for level-restriction for rigged configurations of type \( A \) as given in Definition 5.3 become the following.

For a partition \( \lambda^C \) define \( (\lambda^C)^' = (2\lambda_1^C, \lambda_2^C + \lambda_2^C, \ldots, \lambda_n^C, \lambda_n^C - \lambda_{n-1}^C, \lambda_{n-1}^C, \ldots, \lambda_1^C - \lambda_2^C)^t \). Let \( \text{CST}((\lambda^C)^') \) be the set of all semi-standard tableaux of shape \( (\lambda^C)^' \) in the alphabet \( \{1, 2, \ldots, 2\lambda_1^C\} \). For \( t \in \text{CST}((\lambda^C)^') \) set

\[
f_i^{(a)}(t) = \begin{cases} 
- \sum_{j=1}^{\lambda_1^C + \lambda_a^C} \chi(i \geq 2\ell - 2\lambda_1^C + t_{j,a}) \\
+ \sum_{j=1}^{\lambda_1^C + \lambda_a^C} \chi(i \geq 2\ell - 2\lambda_1^C + t_{j,a+1}) & \text{for } 1 \leq a < n \\
- \sum_{j=1}^{\lambda_1^C + \lambda_a^C} \chi(i \geq 2\ell - 2\lambda_1^C + t_{j,n}) \\
+ \sum_{j=1}^{\lambda_1^C + \lambda_a^C} \chi(i \geq 2\ell - 2\lambda_1^C + t_{j,n+1}) & \text{for } a = n \\
- \sum_{j=1}^{\lambda_1^C + \lambda_a^C} \chi(i \geq 2\ell - 2\lambda_1^C + t_{j,a}) \\
+ \sum_{j=1}^{\lambda_1^C + \lambda_a^C} \chi(i \geq 2\ell - 2\lambda_1^C + t_{j,a+1}) & \text{for } n < a < 2n.
\end{cases}
\]

Define modified vacancy numbers as

\[
P_i^{(a)}(\nu, t) = P_i^{(a)}(\nu) + \left\{ \begin{array}{ll}
\min \{ f_i^{(a)}(t), f_i^{(2n-a)}(t) \} & \text{for } 1 \leq a < n \\
\frac{1}{2} f_i^{(n)}(t) & \text{for } a = n
\end{array} \right.
\]

with \( P_i^{(a)}(\nu) \) as defined in (5.7).

**Definition 5.6.** Let \( \Lambda_C \) be a dominant weight and \( \lambda^C \) the corresponding partition. Let \( B_C = \bigotimes (B_{C,a}^{(1)} \otimes L_1^{(a)}) \) be a crystal of type \( C_n^{(1)} \). Say that \( (\nu, J) \in \text{RC}_C(B_C, \Lambda_C) \) is restricted of level \( \ell \) provided that

1. \( \nu_i^{(a)} \leq 2\ell \) for all \( a \).
2. There exists a tableau \( t \in \text{CST}((\lambda^C)^') \), such that for every \( i, a \geq 1 \), the largest part of \( J^{(a,i)} \) does not exceed \( P_i^{(a)}(\nu, t) \) defined in (5.16).

Denote by \( \text{RC}_C^\ell(B_C, \Lambda_C) \) the set of \((\nu, J) \in \text{RC}_C(B_C, \Lambda_C)\) that are restricted of level \( \ell \).

It follows that

\[
\mathcal{X}^\ell(B_C, \Lambda_C) = \sum_{(\nu, J) \in \text{RC}_C^\ell(B_C, \Lambda_C)} q^{c^{C}(0(\nu, J))}.
\]

Let \( \text{SCST}((\lambda^C)^') \) be the set of all nonempty subsets of \( \text{CST}((\lambda^C)^') \). By the same arguments as in the type \( A \) we find

\[
\mathcal{X}^\ell(B_C, \Lambda_C) = \sum_{S \in \text{SCST}((\lambda^C)^')} (-1)^{|S|+1} \sum_{\{m\}} q^{|S|} \prod_{(a,i) \in H^\ell} \left[ m_i^{(a)} + P_i^{(a)}(S) \right]^{1/q}.
\]
where the sum is over all \( \{m_i^{(a)} \in \mathbb{N} \mid (a, i) \in H^\ell \} \) subject to the constraints

\[ \sum_{(a, i) \in H^\ell} i m_i^{(a)} \alpha_a = \sum_{(a, 1) \in H^\ell} L_1^{(a)} \Lambda_a^C - \Lambda_C. \]

The variable \( p_i^{(a)}(S) \) is defined as

\[ p_i^{(a)}(S) = p_i^{(a)} + \begin{cases} \text{min}_{t \in S} \{ f_i^{(a)}(t), f_i^{(2n-a)}(t) \} & \text{for } 1 \leq a < n \\ \frac{1}{2} \text{min}_{t \in S} \{ f_i^{(a)}(t) \} & \text{for } a = n \end{cases} \]

\[ c^\ell(\{m\}) = cc^\ell(\{m\}) + \sum_{(a, i) \in H^\ell} p_i^{(a)} m_i^{(a)} \]

with \( p_i^{(a)}(S) \) as in (5.3) and \( cc^\ell(\{m\}) \) as in (5.4). In particular if \( \lambda_C = \emptyset \) so that \( \Lambda_C = 0 \) the fermionic form (5.17) reduces to (5.8).

6. Summary and open problems

Equating the bosonic and fermionic evaluations for \( \overline{X}(B, \Lambda) \) and \( \overline{X}^\ell(B, \Lambda) \) as given in sections 4 and 5 yields polynomial identities in \( q \). One may take limits of these identities in various ways to obtain \( q \)-series identities. We refer the interested reader to [12, 13, 35, 36] for details.

It is still an outstanding problem to prove the conjectured fermionic formulas (5.1) and (5.8) for general \( \mathfrak{g} \). There is strong evidence that the rigged configuration approach will still be applicable in this case. Part of this program also requires the proof of the existence and structure of the crystals \( B^{r,s} \) for general \( \mathfrak{g} \) as mentioned in Conjecture 3.4.

The astute reader will have noticed that the energy function introduced in section 3 does not actually specialize to the statistics on path of section 2 which yield the Rogers–Ramanujan identities. This is due to the fact that the Rogers–Ramanujan identities correspond to a non-unitary physical model (more precisely the minimal model \( M(2, 5) \)) whereas the crystal base theory introduced in section 3 is related to unitary physical models. To really embed the Rogers–Ramanujan identities into the framework of crystal base theory it is hence necessary to generalize the definition of the (intrinsic) energy function of section 3 to include the non-unitary setting. For \( \mathfrak{g} = \mathfrak{sl}_2 \) results are available [9, 5, 10, 11]. However for general \( \mathfrak{g} \) details are not yet known in general. Some results in this direction can be found in [31].

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