Flows of $G_2$-Structures, I.

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Abstract

This is a foundational paper on flows of $G_2$-structures. We use local coordinates to describe the four torsion forms of a $G_2$-structure and derive the evolution equations for a general flow of a $G_2$-structure $\varphi$ on a 7-manifold $M$. Specifically, we compute the evolution of the metric $g$, the dual 4-form $\psi$, and the four independent torsion forms. In the process we obtain a simple new proof of a theorem of Fernández-Gray.

As an application of our evolution equations, we derive an analogue of the second Bianchi identity in $G_2$-geometry which appears to be new, at least in this form. We use this result to derive explicit formulas for the Ricci tensor and part of the Riemann curvature tensor in terms of the torsion. These in turn lead to new proofs of several known results in $G_2$-geometry.
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1 Introduction

1.1 Overview

This work can be considered as a foundational paper on geometric flows on manifolds with G\(_2\)-structure. This article also serves a second purpose, which is to collect together many useful identities, relations, and computational techniques for manifolds with G\(_2\)-structure, some of which have never before appeared in the literature, although they may be known to experts in the field.

Most research in geometric flows, such as Ricci flow or mean curvature flow, is done using local coordinates, and this paper describes such an approach for flows of G\(_2\)-structures. One advantage to such an approach is that we can appeal to known results on the evolution of quantities which depend only on the variation of the metric, such as the Christoffel symbols and the Riemann, Ricci, and scalar curvatures, as can be found, for example, in [5]. The author hopes that the foundational results collected here will be useful to others, particularly researchers working on geometric flows who may not yet be familiar with G\(_2\)-structures.

A general evolution of a G\(_2\)-structure is described by a symmetric 2-tensor \(h\) and a vector field \(X\), and it is only \(h\) which affects the evolution of the associated Riemannian metric. On the other hand, for the various geometric quantities unique to G\(_2\)-structures, such as the four torsion forms, their evolution is determined by both \(h\) and \(X\).

To date the only attempt at considering a flow of G\(_2\)-structures has been the Laplacian flow discussed in [2]. When restricted to closed G\(_2\)-structures, this flow is actually the gradient flow (with respect to an appropriately chosen inner product) for the Hitchin functional introduced in [10] (arXiv version.) This flow is still not very well understood.

It is not clear if the Laplacian flow is really the ‘natural’ flow to consider for G\(_2\)-structures, or indeed if there even exists a natural flow at all in this context. The advantage of developing the general theory of flows of G\(_2\)-structures is that it allows us to examine the evolution equations for the torsion under a general flow in the hopes that one can find obvious choices for specific flows which might have nice properties. This is clearly more efficient than trying to guess the right flow and computing all the associated evolution equations from scratch each time.

The main results of this paper are Theorem 3.8 on the evolution of the full torsion tensor under a general flow of G\(_2\)-structures, and Theorem 4.2 which proves a Bianchi-like identity relating the Riemann curvature and intrinsic torsion of a G\(_2\)-structure.

In the rest of Section 1 we review our notation and conventions. In Section 2 we review G\(_2\)-structures, the decomposition of the space of forms, and the intrinsic torsion forms of a G\(_2\)-structure. Along the way we provide a simple new computational proof of a Theorem of Fernández-Gray [8]. Section 3 is the heart of the paper, where we compute the evolution equations for the metric, dual 4-form, and torsion forms for a general flow of G\(_2\)-structures. In Section 4 we apply our evolution equations to derive Bianchi-type identities in G\(_2\)-geometry, and use these to produce new proofs of several known results. We also obtain an explicit formula for the Ricci tensor of a general G\(_2\)-structure in terms of the torsion. The paper closes with two appendices; the first of which discusses various identities in G\(_2\)-geometry, some of which are new and which should be useful in other contexts as well, and the second collecting some standard facts about flows of metrics.

The author is currently preparing a sequel [16] to this paper in collaboration with S.T. Yau, in which we present a detailed analysis of several specific flows, including the Laplacian flow and the Ricci flow in the context of G\(_2\)-geometry. We also discuss short-time existence and soliton solutions.

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1.2 Notation and Conventions

In this section we set up our notation and conventions. Throughout this paper, $M$ is a (not necessarily compact) smooth manifold of dimension 7 which admits a $G_2$-structure. (See Section 2.1 for a review of $G_2$-structures.) The Einstein summation convention is employed throughout: an index which appears both as a subscript and as a superscript in the same term is summed from 1 to 7.

We use $S_7$ to denote the group of permutations of seven letters, and $\text{sgn}(\sigma)$ denotes the sign ($\pm 1$) of an element $\sigma$ of $S_7$.

The space of $k$-forms on $M$ will be denoted by $\Omega^k$. It is the space of sections of the bundle $\Lambda^k(T^*M)$. A differential $k$-form $\alpha$ on $M$ will be written as

$$\alpha = \frac{1}{k!} \alpha_{i_1i_2\ldots i_k} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}$$

in local coordinates $(x^1, \ldots, x^7)$, where the sums are all from 1 to 7, and $\alpha_{i_1i_2\ldots i_k}$ is completely skew-symmetric in its indices. With this convention $\alpha$ can also be written as

$$\alpha = \sum_{i_1 < i_2 < \cdots < i_k} \alpha_{i_1i_2\ldots i_k} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}$$

but we will not have need to do so. The advantage of this approach is that if we take the interior product $\frac{\partial}{\partial x^m} \lhd \alpha$ of the $k$-form $\alpha$ with a vector field $\frac{\partial}{\partial x^m}$, we obtain the $(k-1)$-form

$$\frac{\partial}{\partial x^m} \lhd \alpha = \frac{1}{(k-1)!} \delta_{m i_1i_2\ldots i_{k-1}} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_{k-1}}$$

Given a Riemannian metric $g$ on $M$, it induces a metric on $k$-forms which is defined on decomposable elements to be

$$g(dx^{i_1} \wedge \cdots \wedge dx^{i_k}, dx^{j_1} \wedge \cdots \wedge dx^{j_k}) = \det_{a,b=1, \ldots, k} (g(dx^a, dx^b)) = \det(g^{ij})$$

$$= \sum_{\sigma \in S_7} \text{sgn}(\sigma) g^{i_1j_{\sigma(1)}} g^{i_2j_{\sigma(2)}} \cdots g^{i_kj_{\sigma(k)}}$$

where $g^{ij} = g(dx^i, dx^j)$ is the induced metric on the cotangent bundle and $g^{ij}$ is the inverse matrix of the matrix $g_{ij}$. In this convention, one can check that the inner product of two $k$-forms $\alpha = \frac{1}{k!} \alpha_{i_1\ldots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ and $\beta = \frac{1}{k!} \beta_{j_1\ldots j_k} dx^{j_1} \wedge \cdots \wedge dx^{j_k}$ is

$$g(\alpha, \beta) = \frac{1}{k!} \alpha_{i_1\ldots i_k} \beta_{j_1\ldots j_k} g^{i_1j_1} \cdots g^{i_kj_k}$$

which differs from other conventions by the factor of $k!$.

The Levi-Civita covariant derivative associated to $g$ is denoted by $\nabla$, and its associated Christoffel symbols by $\Gamma^k_{ij}$, where $\Gamma^k_{ij} = \Gamma^k_{ji}$. We write $\nabla_{\xi}$ for covariant differentiation in the $\frac{\partial}{\partial x^\xi}$ direction. In local coordinates, we have $\nabla_{\xi}(dx^k) = -\Gamma^k_{\xi j} dx^j$. If $T_{i_1\ldots i_k}$ is a tensor of type $(0,k)$, then $\nabla_{\xi} T_{i_1\ldots i_k}$ always means $(\nabla_{\xi} T)_{i_1\ldots i_k}$, which is

$$\nabla_{\xi} T_{i_1\ldots i_k} = \frac{\partial}{\partial x^m} T_{i_1\ldots i_k} - \Gamma^l_{m i_1} T_{li_2\ldots i_k} - \Gamma^l_{m i_2} T_{li_1\ldots i_k} - \cdots - \Gamma^l_{m i_k} T_{li_1\ldots i_{k-1}}$$

(1.2)
and is a tensor of type $(0, k + 1)$. Because the metric $g$ is parallel with respect to $\nabla$, covariant differentiation commutes with contractions. Explicitly, we have

$$
\nabla_m (T_{a_1 \cdots a_k} S_{b_1 \cdots b_j} g^{ab}) = (\nabla_m T_{a_1 \cdots a_k}) S_{b_1 \cdots b_j} g^{ab} + T_{a_1 \cdots a_k} (\nabla_m S_{b_1 \cdots b_j}) g^{ab}
$$

where $T$ is a $(0, k + 1)$ tensor and $S$ is a $(0, l + 1)$ tensor.

The exterior derivative $d\alpha$ of a $k$-form $\alpha$ can be written in terms of the covariant derivative as

$$
d\alpha = \frac{1}{k!} (\nabla_m \alpha_{i_1 \cdots i_k} + \Gamma^l_{mi_k} \alpha_{i_1 \cdots i_{k-1} \cdots i_l \cdots i_k} + \cdots + \Gamma^l_{m_{k-1}i_k} \alpha_{i_1 \cdots i_{k-2} \cdots i_l \cdots i_{k-1} \cdots i_k}) \, dx^m \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}
$$

(1.3)

since the $\Gamma^k_{ij}$’s are symmetric in $i, j$. We will always write exterior derivatives of forms in this way.

The metric $g$ determines a ‘musical’ isomorphism between the tangent and cotangent bundles of $M$. If $v$ is a vector field, then the metric dual 1-form $v^\flat$ is defined by $v^\flat(w) = g(v, w)$ for all vector fields $w$. In coordinates, $(\frac{\partial}{\partial x^i})^\flat = g_{ik} dx^k$. Similarly a 1-form $\alpha$ has a metric dual vector field $\alpha^\sharp$ defined by $\beta(\alpha^\sharp) = g(\alpha, \beta)$ for all 1-forms $\beta$, and $(dx^i)^\sharp = g^{ik} \frac{\partial}{\partial x^k}$. This isomorphism is an isometry: $g(v^\flat, w^\flat) = g(v, w)$.

We use ‘vol’ to denote the volume form on $M$ associated to a metric $g$ and an orientation, rather than something like $d\text{vol}$, to avoid confusion, since the volume form is never exact on a compact manifold. In local coordinates the volume form is

$$
\text{vol} = \sqrt{\det(g)} \, dx^1 \wedge \cdots \wedge dx^7
$$

where $\det(g)$ is the determinant of the matrix $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$.

The metric and orientation together determine the Hodge star operator $*$ taking $k$-forms to $(7 - k)$-forms, characterized by the relation

$$
\alpha \wedge \star \beta = g(\alpha, \beta) \text{vol}
$$

on two $k$-forms $\alpha$ and $\beta$. We also have $*^2 = 1$. Suppose $v$ is a vector field and $\alpha$ is a $k$-form. The interior product, wedge product, and Hodge star operator are all related by the following identities (the signs here are all specific to the odd dimension 7):

$$
\begin{align*}
*(v \lrcorner \alpha) &= (-1)^{k+1} (v^\flat \wedge \star \alpha) \\
*(v \lrcorner \star \alpha) &= (-1)^{k} (v^\flat \wedge \alpha) \\
v \lrcorner \text{vol} &= \star v^\flat
\end{align*}
$$

(1.4)

More details can be found, for example, in [13].

We also have need to consider the adjoint $\delta = d^* : \Omega^k \to \Omega^{k-1}$ of the exterior derivative, with respect to the metric. This operator is called the coderivative and it satisfies

$$
\int_M g(d\alpha, \beta) \text{vol} = \int_M g(\alpha, \delta \beta) \text{vol}
$$

whenever $M$ is compact without boundary. The operator $\delta$ can be written in terms of $d$ and $*$ as

$$
\delta = (-1)^k \star d \star \quad \text{on } \Omega^k
$$

(1.5)
where again, the signs are specific to the odd dimension 7 which we consider exclusively throughout this paper. The coderivative \( \delta \) can be written in terms of the metric \( g \) and the covariant derivative \( \nabla \) as follows:

\[
\delta \alpha = \frac{1}{(k-1)!} (\delta \alpha)_{i_1 i_2 \cdots i_{k-1}} \, dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_{k-1}}
\]

where \( (\delta \alpha)_{i_1 i_2 \cdots i_{k-1}} = -g^{lm} \alpha_{mi_1 \cdots i_{k-1}} \) \( \tag{1.6} \)

We need the expression for the Lie derivative \( L_Y(\alpha) \) of a tensor \( \alpha \) in the direction of a vector field \( Y \) in terms of the covariant derivative. Suppose \( \alpha = \alpha_{i_1 \cdots i_k} \) is a tensor, where \( \alpha_{i_1 \cdots i_k} = \alpha(\frac{\partial}{\partial x^{i_1}}, \ldots, \frac{\partial}{\partial x^{i_k}}) \). Then by the Liebnitz rule for both \( L_Y \) and \( \nabla_Y \), we have

\[
(L_Y (\alpha))_{i_1 \cdots i_k} = Y(\alpha_{i_1 \cdots i_k}) - \sum_{j=1}^{k} \alpha(\frac{\partial}{\partial x^{i_j}}, \ldots, L_Y(\frac{\partial}{\partial x^{i_j}}), \ldots, \frac{\partial}{\partial x^{i_k}})
\]

\[
= Y(\alpha_{i_1 \cdots i_k}) - \sum_{j=1}^{k} \alpha(\frac{\partial}{\partial x^{i_j}}, \ldots, \nabla_Y(\frac{\partial}{\partial x^{i_j}}) - \nabla_j Y, \ldots, \frac{\partial}{\partial x^{i_k}})
\]

\[
= (\nabla_Y \alpha)_{i_1 \cdots i_k} + (\nabla_{i_1} Y)\alpha_{i_2 \cdots i_k} + \cdots + (\nabla_{i_k} Y')\alpha_{i_1 \cdots i_{k-1} l} \quad (1.7)
\]

Finally, we discuss our conventions for labelling the Riemann curvature tensor. We define

\[
R^m_{ijkl} \frac{\partial}{\partial x^m} = (\nabla \nabla - \nabla \nabla)(\frac{\partial}{\partial x^k})
\]

in terms of coordinate vector fields (so the usual Lie bracket term vanishes), and we choose to lower indices by

\[
R_{ijkl} = R^m_{ijkl} g_{ml}
\]

With this convention, the Ricci tensor must be defined as \( R_{jk} = R_{ijkl} g^{il} \) to ensure that the round sphere has positive Ricci curvature. Recall that we have

\[
R_{ijkl} = -R_{ijlk} = -R_{ijlk} = R_{klji}
\]

and the first Bianchi identity

\[
R_{ijkl} + R_{iklj} + R_{ijlk} = 0 \quad (1.8)
\]

We will also need the Ricci identity

\[
\nabla_i \nabla_j X_l - \nabla_i \nabla_k X_l = -R_{klm} X^m \quad (1.9)
\]

Two possible references for this section (although the conventions do not always agree with ours) are [6, 11].

1.3 Acknowledgements

The bulk of this paper was completed while the author was a Postdoctoral Fellow at the Mathematical Sciences Research Institute in 2006-2007 as part of the program on ‘Geometric Evolution Equations and Related Topics.’ The inspiration for this work are the papers [2] and [10] by Robert
Bryant and Nigel Hitchin, respectively. These were the first papers (as far as the author knows) to discuss flows in $G_2$-geometry, and served to motivate the author to consider the general situation of $G_2$-flows.

The author is indebted to Robert Bryant, Hsiao-Bing Cheng, Ben Chow, Dominic Joyce, Naichung Conan Leung, Lei Ni, and Andrejs Treibergs for useful discussions. The author also benefited from a brief talk with Richard Hamilton, from whom the author learned valuable intuition. Finally, the author would like to acknowledge his former thesis advisor Shing-Tung Yau, without whose constant encouragement and advice, none of this work would have been possible.

2 Manifolds with $G_2$-structure

In this section we review the concept of a $G_2$-structure on a manifold $M$ and the associated decompositions of the space of forms. More details about $G_2$-structures can be found, for example, in [2, 13, 14, 18]. We also describe explicitly the four torsion tensors associated to a $G_2$-structure and compute their expressions in local coordinates. These results are needed to determine the evolution equations of the torsion tensors in Section 3.3.

2.1 Review of $G_2$-structures

Consider a 7-manifold $M$ with a $G_2$ structure $\varphi$. Such a structure exists if and only if $M$ is orientable and spin, which is equivalent to the vanishing of the first and second Stiefel-Whitney classes $w_1(M) = w_2(M) = 0$. In fact the space of 3-forms $\varphi$ on $M$ which determine a $G_2$-structure is an open subbundle $\Omega^3_{\text{pos}}$ of the bundle $\Omega^3$ of 3-forms on $M$, sometimes called the bundle of positive 3-forms. Such a structure determines a Riemannian metric and an orientation in a non-linear fashion which we now describe. Given local coordinates $x^1, \ldots, x^7$, we define

$$B_{ij} dx^1 \wedge \ldots \wedge dx^7 = \left( \frac{\partial}{\partial x^i} \varphi \right) \wedge \left( \frac{\partial}{\partial x^j} \varphi \right) \wedge \varphi \tag{2.1}$$

It is clear that $B_{ij} = B_{ji}$. This top form can be shown to be equal to

$$B_{ij} dx^1 \wedge \ldots \wedge dx^7 = -6g_{ij} \text{vol} = -6g_{ij} \sqrt{\det(g)} dx^1 \wedge \ldots \wedge dx^7 \tag{2.2}$$

In other words, the 3-form $\varphi$ naturally determines the tensor product of the metric $g_{ij}$ with the volume form $\text{vol}$. From this we can extract the metric $g_{ij}$ and subsequently the volume form $\text{vol}$ as follows.

$$B_{ij} = \left( \frac{\partial}{\partial x^i} \varphi \right) \wedge \left( \frac{\partial}{\partial x^j} \varphi \right) \wedge \varphi$$

$$B_{ij} = -6g_{ij} \sqrt{\det(g)}$$

$$\det(B) = (-6)^7 \det(g) \det(g)^{\frac{7}{2}} = -6^7 \det(g)^{\frac{9}{2}}$$

$$\sqrt{\det(g)} = \frac{1}{6^\frac{7}{2}} \det(B)^{\frac{1}{2}}$$

$$g_{ij} = \frac{1}{6} \frac{B_{ij}}{\sqrt{\det(g)}} = \frac{1}{6^\frac{7}{2}} \frac{B_{ij}}{\det(B)^{\frac{1}{2}}}$$
Some authors chose a different convention in which the \((-6)\) factor above is actually a \((+6)\). Since this cancels out in the above calculation, the equation

$$g_{ij} = \frac{1}{6^2} \frac{B_{ij}}{\det(B)^{\frac{1}{2}}}$$  \hspace{1cm} (2.3)$$

is true for both sign conventions (see [15].)

We stress that the tensor \(B\) is actually a section of \(\text{Sym}^2(T^*M) \otimes \wedge^7(T^*M)\). Therefore, if we change to a new basis \(dx^i = P^i_l d\tilde{x}^l\), then \(\tilde{B}_{ij} = P^k_i P^l_j \det(P) B_{kl}\) from which it follows from (2.3) that \(\tilde{g}_{ij} = P^k_i P^l_j g_{kl}\) as expected.

The metric \(g\) and orientation (determined by the volume form) determine a Hodge star operator \(*\), and we therefore have the associated dual 4-form \(\psi = *\varphi\). The metric also determines the Levi-Civita connection \(\nabla\), and the manifold \((M, \varphi)\) is called a G\(_2\) manifold if \(\nabla \varphi = 0\). Note that this is a nonlinear partial differential equation for \(\varphi\), since \(\nabla\) depends on \(g\) which depends non-linearly on \(\varphi\). Such manifolds (where \(\varphi\) is parallel) have Riemannian holonomy \(\text{Hol}_g(M)\) contained in the exceptional Lie group \(G_2 \subset \text{SO}(7)\).

The \(G_2\)-structure corresponding to the 3-form \(\varphi\) is also called \textit{torsion-free} if \(\varphi\) is parallel with respect to the metric \(g_\varphi\) induced by \(\varphi\). Torsion-free \(G_2\)-structures have sometimes been called ‘integrable’ by some authors, but since that term has also been used in different contexts as well, we prefer to stick to the unambiguous ‘torsion-free’ at all times. See also [2] for more discussion about the sometimes confusing terminology about \(G_2\)-structures.

In [8], the following theorem is proved.

\textbf{Theorem 2.1} (Fernández-Gray, 1982). \textit{The \(G_2\)-structure corresponding to \(\varphi\) is torsion-free if and only if \(\varphi\) is both closed and co-closed:}

$$d\varphi = 0 \quad d*\varphi = d\psi = 0$$

More recent proofs of this theorem can be found in [4, 13, 2].

\textbf{Remark 2.2.} A differential form \(\varphi\) is harmonic if \((dd^* + d^*d)\varphi = 0\). On a compact manifold, this is equivalent to \(d\varphi = 0\) and \(d*\varphi = 0\). Since a parallel differential form is always closed and co-closed, Theorem 2.1 says that for a compact manifold with \(G_2\) structure \(\varphi\), the 3-form \(\varphi\) being parallel is equivalent to it being harmonic (with respect to its induced metric.)

\textbf{Remark 2.3.} It is well known (see [13], for example), that for a \(G_2\)-manifold the holonomy must be one of the following four possibilities:

\begin{align*}
\text{Hol}_g(M) &= \{1\} & \iff b_1(M) &= 7 \\
\text{Hol}_g(M) &= \text{SU}(2) & \iff b_1(M) &= 3 \\
\text{Hol}_g(M) &= \text{SU}(3) & \iff b_1(M) &= 1 \\
\text{Hol}_g(M) &= G_2 & \iff b_1(M) &= 0
\end{align*}

We are interested in constructing manifolds with full holonomy \(G_2\), and not a strictly smaller subgroup. Therefore we can assume that the fundamental group \(\pi_1(M)\) is finite.

\textbf{Remark 2.4.} There are some topological obstructions (in the compact case) to the existence of a torsion-free \(G_2\)-structure which are known. First, we need \(b_3(M) \geq 1\), since the 3-form \(\varphi\) is a non-zero harmonic 3-form, and hence represents a non-trivial cohomology class by the Hodge theorem.
Additionally, if we insist on full holonomy $G_2$ rather than a strictly smaller subgroup, then we must have $b_1(M) = 0$, which was already mentioned in Remark 2.3 and also that the first Pontryagin class $p_1(M)$ of the manifold must be non-zero. There are also some conditions on the cohomology ring structure. A detailed discussion can be found in [13]. Of course, sufficient conditions for the existence of a torsion-free $G_2$-structure (analogous to the Calabi conjecture in Kähler geometry) are far from being known.

2.2 Decomposition of the space of forms

The existence of a $G_2$-structure $\varphi$ on $M$ (with no condition on $\nabla \varphi$) determines a decomposition of the spaces of differential forms on $M$ into irreducible $G_2$ representations. This is analogous to the decomposition of complex-valued differential forms on an almost complex manifold into forms of type $(p,q)$. We will see explicitly that the spaces $\Omega^2$ and $\Omega^3$ of 2-forms and 3-forms decompose as

$$\Omega^2 = \Omega^2_3 \oplus \Omega^3_7$$

$$\Omega^3 = \Omega^3_1 \oplus \Omega^2_3 \oplus \Omega^3_{27}$$

where $\Omega^k_l$ has (pointwise) dimension $l$ and this decomposition is orthogonal with respect to the metric $g$. The spaces $\Omega^2_3$ and $\Omega^3_7$ are both isomorphic to the cotangent bundle $\Omega^1_7 = T^*M$ (and hence also to the tangent bundle $TM$). We show below that the space $\Omega^3_{14}$ is isomorphic to the Lie algebra $\mathfrak{g}_2$, and $\Omega^3_{27}$ is isomorphic to the traceless symmetric 2-tensors $\text{Sym}^2_0(T^*M)$ on $M$. The explicit descriptions are as follows:

$$\Omega^2_3 = \{ X \cdot \varphi; \ X \in \Gamma(TM) \} = \{ \beta \in \Omega^2; \ast (\varphi \wedge \beta) = -2\beta \}$$ (2.4)

$$\Omega^3_7 = \{ \beta \in \Omega^3; \beta \wedge \psi = 0 \} = \{ \beta \in \Omega^3; \ast (\varphi \wedge \beta) = \beta \}$$ (2.5)

$$\Omega^2_3 = \{ f \varphi; \ f \in C^\infty(M) \}$$ (2.6)

$$\Omega^3_7 = \{ X \cdot \psi; \ X \in \Gamma(TM) \}$$ (2.7)

$$\Omega^3_{27} = \{ h_{ij} g^{il} dx^i \wedge (\frac{\partial}{\partial x^j} \varphi) \}; \ h_{ij} = h_{ji}, \ Tr_g(h_{ij}) = g^{ij}h_{ij} = 0 \}$$ (2.8)

The decompositions $\Omega^4 = \Omega^4_1 \oplus \Omega^4_2 \oplus \Omega^4_7$ and $\Omega^5 = \Omega^5_1 \oplus \Omega^5_7$ are obtained by taking the Hodge star of the decompositions of $\Omega^3$ and $\Omega^2$, respectively.

Remark 2.5. There is another orientation convention for $G_2$-structures which differs from this one. In the other convention, the eigenvalues of the operator $\beta \mapsto \ast (\varphi \wedge \beta)$ are $+2$ and $-1$ instead of $-2$ and $+1$, respectively. See [15] for more on sign and orientation conventions in $G_2$ geometry.

We now establish the decompositions of the spaces $\Omega^2$ and $\Omega^3$ in detail. Let $\beta = \frac{1}{2} \beta_{ij} dx^i \wedge dx^j$ be an arbitrary 2-form. Then we have

$$\ast (\varphi \wedge \beta) = \frac{1}{2} \beta_{ij} \ast (dx^i \wedge dx^j \wedge \varphi) = \frac{1}{2} \beta_{ij} g^{il} \frac{\partial}{\partial x^j} \ast (dx^l \wedge \varphi)$$

$$= -\frac{1}{2} \beta_{ij} g^{il} g^{jm} \frac{\partial}{\partial x^j} \frac{\partial}{\partial x^m} \ast \varphi = -\frac{1}{2} \beta_{ij} g^{il} g^{jm} \left( \frac{1}{2} \psi_{mab} dx^a \wedge dx^b \right)$$

$$= \frac{1}{4} \beta_{ij} \psi_{mab} g^{il} g^{jm} \left( dx^a \wedge dx^b \right)$$

9
where we have used (1.4) twice. This computation allows us to express the projection operators \( \pi_7 \) and \( \pi_{14} \) from \( \Omega^2 \) to \( \Omega^2_7 \) and \( \Omega^2_{14} \), respectively, as follows. From (2.4) and (2.5), we see that

\[
\pi_7(\beta) = \frac{1}{3} \beta - \frac{1}{3} \ast (\varphi \wedge \beta) \quad \pi_{14}(\beta) = \frac{2}{3} \beta + \frac{1}{3} \ast (\varphi \wedge \beta)
\]

Hence for \( \beta = \frac{1}{2} \beta_{ij} dx^i \wedge dx^j \), we have

\[
\pi_7(\beta) = \frac{1}{2} \left( \frac{1}{3} \beta_{ab} - \frac{1}{6} \beta_{ij} g^{il} g^{jm} \varphi_{lma} \right) dx^a \wedge dx^b \tag{2.9}
\]

\[
\pi_{14}(\beta) = \frac{1}{2} \left( \frac{2}{3} \beta_{ab} + \frac{1}{6} \beta_{ij} g^{il} g^{jm} \varphi_{lma} \right) dx^a \wedge dx^b \tag{2.10}
\]

The alternative characterizations of \( \Omega^2_7 \) and \( \Omega^2_{14} \) given in (2.4) and (2.5) can also be expressed in local coordinates: \( \beta \in \Omega^2_7 \) if and only if \( \beta = X \varphi \) for some vector field \( X \), which is equivalent to \( \beta_{ij} = X^k \varphi_{ijk} \), and \( \beta \in \Omega^2_{14} \) if and only if \( \varphi \wedge \beta = 0 \), which, using (1.4), becomes \( \beta_{ij} g^{il} g^{jm} \varphi_{lma} = 0 \).

We summarize the descriptions of \( \Omega^2 \) in the following:

**Proposition 2.6.** Let \( \beta = \frac{1}{2} \beta_{ij} dx^i \wedge dx^j \) be in \( \Omega^2 \). Then

\[
\beta \in \Omega^2_7 \iff \beta_{ij} g^{il} g^{jm} \varphi_{lma} = -4 \beta_{ab} \iff \beta_{ij} = X^k \varphi_{ijk} \tag{2.12}
\]

\[
\beta \in \Omega^2_{14} \iff \beta_{ij} g^{il} g^{jm} \varphi_{lma} = 2 \beta_{ab} \iff \beta_{ij} g^{il} g^{jm} \varphi_{lma} = 0 \tag{2.13}
\]

**Remark 2.7.** It is an enlightening exercise to establish the above equivalences (in index notation) directly using the identities of Lemmas A.12, A.13, and A.14.

**Remark 2.8.** Because the space \( \Omega^2_7 \) is isomorphic to the space of vector fields (and also to the space of 1-forms), it is convenient to be able to go back and forth between the two descriptions. If \( X = X^k \frac{\partial}{\partial x^k} \) is a vector field, the associated \( \Omega^2_7 \) form, by Proposition 2.6, is \( \frac{1}{2} X_{ab} dx^a \wedge dx^b \), where \( X_{ab} = X^k \varphi_{lab} \). It is easy to check using Lemma A.12 that we can solve for \( X^k \) in this expression as

\[
X^k = \frac{1}{6} X_{ab} \varphi_{mpq} g^{ap} g^{bq} g^{mk} \tag{2.11}
\]

which we will have occasion to use frequently.

We need a few more relations involving \( \Omega^2_7 \) and \( \Omega^2_{14} \) which will be used to simplify the evolution equations for the torsion forms in Section 3.3

**Lemma 2.9.** Suppose \( \beta_{ij} \) is a 2-form. Then if \( \beta_{ij} \in \Omega^2_{14} \),

\[
\beta_{ab} g^{bl} \varphi_{lpq} = \beta_{pl} g^{lm} \varphi_{maq} - \beta_{ql} g^{lm} \varphi_{map} \tag{2.12}
\]

whereas if \( \beta \in \Omega^2_7 \), then

\[
\beta_{ab} g^{bl} \varphi_{lpq} = -\frac{1}{2} \beta_{pl} g^{lm} \varphi_{maq} + \frac{1}{2} \beta_{ql} g^{lm} \varphi_{map} - \frac{3}{2} g_{pa} \beta_q + \frac{3}{2} g_{qa} \beta_p \tag{2.13}
\]

where in this case \( \beta_k = \frac{1}{2} \beta_{ij} g^{ia} g^{bk} \varphi_{abk} \) as given by (2.11).
Proof. We prove both statements at once. By Proposition \ref{g2prop}, we can write $\beta_{ij}g^{im}g^{jn}\lambda_{mnab}$ where $\lambda = \frac{1}{2}$ if $\beta \in \Omega_{14}^2$ and $\lambda = -\frac{1}{4}$ if $\beta \in \Omega_{14}^4$. Now we use Lemma \ref{2.13} and compute

$$\beta_{ab}g^{ij}\varphi_{pq} = \lambda\beta_{ij}g^{im}g^{jn}(\varphi_{pq}\lambda_{mnab}g^{kb})$$

$$= \lambda\beta_{ij}g^{im}g^{jn}(g_{pq}\varphi_{qna} + g_{pn}\varphi_{mpa} + g_{pa}\varphi_{mnq} - g_{qm}\varphi_{mpa} - g_{qn}\varphi_{mpa} - g_{qna}\varphi_{mpa})$$

$$= \lambda(\beta_{ij}g^{im}g^{jn}\varphi_{qna} + \beta_{ip}g^{im}\varphi_{mpa} + \beta_{iq}g^{im}\varphi_{mpa} - \beta_{pj}g^{im}\varphi_{mpa} - \beta_{iq}g^{im}\varphi_{mpa} - 6g_{qna}\varphi_{mpa})$$

where $\lambda_k = 0$ if $\beta \in \Omega_{14}^2$ by Proposition \ref{g2prop}. This can be simplified to

$$\beta_{ab}g^{ij}\varphi_{pq} = \lambda(2\beta_{pl}g^{im}\varphi_{maq} - 2\beta_{ql}g^{im}\varphi_{map} + 6g_{pa}\varphi_{q} - 6g_{qa}\varphi_{p})$$

and the statements now follow by substituting the value of $\lambda$ in each case.

\[\Box\]

Corollary 2.10. The space $\Omega_{14}^2$ is a Lie algebra with respect to the commutator of matrices:

$$[\beta, \mu]_{ij} = \beta_{ij}g^{im}\mu_{mj} - \mu_{ij}g^{im}\beta_{mj}$$

Proof. We know that $\Omega^2 \cong so(7)$, and that the matrix commutator is a Lie algebra bracket on $\Omega^2$ which satisfies the Jacobi identity. We need to show is that the bracket of two elements of $\Omega_{14}^2$ is again in $\Omega_{14}^2$. Suppose $\beta$ and $\mu$ are in $\Omega_{14}^2$. Then $[\beta, \mu] \in \Omega_{14}^2$ if and only if

$$[\beta, \mu]_{ij}g^{ia}g^{jb}\varphi_{abc} = 0$$

We have

$$[\beta, \mu]_{ij}g^{ia}g^{jb}\varphi_{abc} = \beta_{it}g^{im}\mu_{mj}g^{ia}g^{jb}\varphi_{abc} - \mu_{it}g^{im}\beta_{mj}g^{ia}g^{jb}\varphi_{abc}$$

$$= \beta_{it}g^{im}\mu_{mj}g^{ia}g^{jb}\varphi_{abc} + \mu_{it}g^{im}g^{ia}(\beta_{nk}g^{kn}\varphi_{mc} - \beta_{ck}g^{kn}\varphi_{ma})$$

where we have used \ref{2.12} in the last line above since $\beta \in \Omega_{14}^2$. The first two terms cancel each other, and the last term vanishes by Proposition \ref{g2prop} since $\mu \in \Omega_{14}^2$.

\[\Box\]

Remark 2.11. Of course, $\Omega_{14}^2 \cong g_2$, the Lie algebra of $G_2$.

We can regard a vector field $X^k$ as a $\Omega_{14}^2$ form $X_{ab}$ by \ref{2.11}. Given a 2-form $\beta_{ij}$, it acts on $X^k$ by matrix multiplication: $(\beta(X))^m = \beta_{ij}X^jg^{im}$ to give another vector field $\beta(X)$, which we can turn into a $\Omega_{14}^4$-form by $(\beta(X))_{ab} = (\beta(X))^m\varphi_{mab}$.

Proposition 2.12. If $\beta \in \Omega_{14}^2$, then

$$(\beta(X))_{ab} = [\beta, X]_{ab}$$

whereas if $\beta \in \Omega_{14}^4$, then

$$(\beta(X))_{ab} = -\frac{1}{2}[\beta, X]_{ab} - \frac{3}{2}\beta_aX_b + \frac{3}{2}\beta_bX_a$$

where $[\beta, X]$ is the matrix commutator of $\beta$ and $X$ regarded as elements of $\Omega^2$.
Proof. We use (2.12) and compute, if \( \beta \in \Omega_2^7 \):

\[
(\beta(X))_{ab} = \beta_{ij} X^j g^{il} \varphi_{lab} = -X^j (\beta_{al} g^{lm} \varphi_{jm} - \beta_{bl} g^{lm} \varphi_{mj})
\]

and similarly using (2.13) in the case \( \beta \in \Omega_2^7 \) to establish (2.15).

To conclude our discussion of \( \Omega_2^7 \), we note that if \( \beta \in \Omega_2^7 \), we can also write \( \varphi_{ij} X^j \):

\[
\beta_{aX_b} = -X^j (\beta_{al} g^{lm} \varphi_{mjb} - \beta_{bl} g^{lm} \varphi_{mja}) \]

and therefore, considering the \( \Omega_2^7 \) parts of (2.15) as vectors via (2.11), and using (2.17) gives

\[
-(\beta \times X) = -\frac{1}{2} \pi_7([\beta, X]) - \frac{3}{2} (\frac{1}{3} \beta \times X)
\]

from which it follows that

\[
\pi_7([\beta, X]) = \beta \times X
\]

for \( \beta \in \Omega_2^7 \), which will be used in Section 3.3.

The decomposition of the space \( \Omega^3 \) of 3-forms can be understood by considering the infinitesimal action of \( GL(7, \mathbb{R}) \) on \( \varphi \). Let \( A = A^i_j \in \mathfrak{gl}(7, \mathbb{R}) \). Hence \( e^{At} \in GL(7, \mathbb{R}) \), and we have

\[
e^{At} \cdot \varphi = \frac{1}{6} \varphi_{ijk} (e^{At} dx^i) \wedge (e^{At} dx^j) \wedge (e^{At} dx^k)
\]

Differentiating with respect to \( t \) and setting \( t = 0 \), we obtain:

\[
\frac{d}{dt} \bigg|_{t=0} (e^{At} \cdot \varphi) = \frac{1}{6} (A^j_i \varphi_{ijkl} + A^j_i \varphi_{iklj} + A^k_i \varphi_{ijlk}) dx^i \wedge dx^j \wedge dx^k
\]

Now let \( A^i_j = g^{ij} A_{ij} \), and decompose \( A_{ij} = S_{ij} + C_{ij} \) into symmetric and skew-symmetric parts, where \( S_{ij} = \frac{1}{2}(A_{ij} + A_{ji}) \) and \( C_{ij} = \frac{1}{2}(A_{ij} - A_{ji}) \). We have a map

\[
D : GL(7, \mathbb{R}) \to \Omega^3
\]

\[
D : A \mapsto \frac{d}{dt} \bigg|_{t=0} (e^{At} \cdot \varphi)
\]

\[
= S_{ij} g^{kl} dx^i \wedge \left( \frac{\partial}{\partial x^l} \varphi \right) + C_{ij} g^{kl} dx^i \wedge \left( \frac{\partial}{\partial x^l} \varphi \right)
\]
Proposition 2.13. The kernel of $D$ is isomorphic to the subspace $\Omega^2_{14}$. It is also isomorphic to the Lie algebra $\mathfrak{g}_2$ of the Lie group $G_2$ which is the subgroup of $GL(7,\mathbb{R})$ which preserves $\varphi$.

Proof. Since we are defining $G_2$ to be the group preserving $\varphi$, the kernel of $D$ is isomorphic to $\mathfrak{g}_2$ by definition. To show explicitly that this is isomorphic to $\Omega^2_{14}$, decompose $C_{ij} = (C_7)_{ij} + (C_{14})_{ij}$, where $C_7 \in \Omega^2_7$ and $C_{14} \in \Omega^2_{14}$. We have

$$
(C_{14})_{ij} g^{jl} \, dx^l \wedge \left( \frac{\partial}{\partial x^j} \varphi \right) = \frac{1}{6} \left( (C_{14})_i^{lj} \varphi_{lj} + (C_{14})_j^{lk} \varphi_{ik} + (C_{14})_k^{lj} \varphi_{ij} \right) \, dx^i \wedge dx^j \wedge dx^k
$$

From Proposition 2.9 we have $(C_{14})_{ij} = \frac{1}{2} (C_{14})_{ab} g^{ap} g^{bq} \psi_{pqij}$. Using this together with the final equation of Lemma A.13, one can compute easily that

$$
(C_{14})_i^{lj} \varphi_{lj} + (C_{14})_j^{lk} \varphi_{ik} + (C_{14})_k^{lj} \varphi_{ij} = \frac{1}{2} \left( (C_{14})_i^{lj} \varphi_{lj} + (C_{14})_j^{lk} \varphi_{ik} + (C_{14})_k^{lj} \varphi_{ij} \right)
$$

and hence that $(C_{14})_{ij} g^{jl} \, dx^l \wedge \left( \frac{\partial}{\partial x^j} \varphi \right) = 0$. Therefore $\Omega^2_{14}$ is in the kernel of $D$. We will see below that $D$ restricted to $\Omega^2_7$ or the symmetric tensors $S^3(T)$ is injective. This completes the proof. \qed

By counting dimensions, we must have $\Omega^2_7 = \{(C_7)_{ij} g^{jl} \, dx^l \wedge \left( \frac{\partial}{\partial x^j} \varphi \right) \}$ and also $\Omega^2_1 \oplus \Omega^3_7 = \{ S_{ij} g^{jl} \, dx^l \wedge \left( \frac{\partial}{\partial x^j} \varphi \right) \}$. We now proceed to establish these explicitly.

To show $(C_7)_{ij} g^{jl} \, dx^l \wedge \left( \frac{\partial}{\partial x^j} \varphi \right)$ is $X \cdot \psi$ for some vector field $X$, we use Proposition 2.9 to write $(C_7)_{ij} = (C_7)^k \varphi_{kij}$, where $(C_7)^k = \frac{1}{6} (C_7)_{ij} g^{ia} g^{jb} \varphi_{abc} g^{kc}$. By (2.11), in fact since $(C_{14})_{ij} g^{ia} g^{jb} \varphi_{abc} = 0$, we actually have $(C_7)^k = \frac{1}{6} C_{ij} g^{ia} g^{jb} \varphi_{abc} g^{kc}$. Now we use Lemma A.12 and compute:

$$
\frac{1}{6} \left( (C_7)_{il} g^{lm} \varphi_{mjk} + (C_7)_{jl} g^{lm} \varphi_{imk} + (C_7)_{kl} g^{lm} \varphi_{ijm} \right)
$$

$$
= \frac{1}{6} \left( (C_7)^n \varphi_{nij} g^{lm} \varphi_{mjk} + (C_7)^n \varphi_{njk} g^{lm} \varphi_{imk} + (C_7)^n \varphi_{nkj} g^{lm} \varphi_{ijm} \right)
$$

$$
= \frac{1}{6} (C_7)^n (g_{nj} g_{ik} - g_{nk} g_{ij} + g_{nj} g_{ji} - g_{ni} g_{jk} - g_{ni} g_{jk} + g_{njk} - g_{njk} - g_{njk} - g_{njk})
$$

$$
= \frac{1}{6} (-3(C_7)^n) \psi_{nijk} = \frac{1}{6} X^n \psi_{nijk}
$$

and hence we have shown that for $C_{ij}$ skew-symmetric,

$$
C_{ij} g^{ij} \varphi_{mjk} + C_{ji} g^{ij} \varphi_{imk} + C_{kl} g^{ij} \varphi_{ijm} = X^n \psi_{nijk}
$$

(2.19)

where

$$
X^n = -\frac{1}{2} C_{ij} g^{ia} g^{jb} \varphi_{abc} g^{cn}
$$

Following the notation of Bryant [2], we define maps $i : S^2(T) \to \Omega^3$ and $j : \Omega^3 \to S^2(T)$ as follows (our definition of the map $i$ differs from Bryant’s by a factor of 2):

$$
i(h_{ij}) = h_{ij} g^{jl} \, dx^l \wedge \left( \frac{\partial}{\partial x^j} \varphi \right) = \frac{1}{2} h_{ij} \varphi_{lj} dx^l \wedge dx^j \wedge dx^k
$$

(2.20)

$$
(j(\eta))(v,w) = * ((v \cdot \varphi) \wedge (w \cdot \varphi) \wedge \eta)
$$

(2.21)

We will have several occasions to use the following.
Proposition 2.14. Suppose that $h_{ij}$ is a symmetric tensor. It corresponds to the form $\eta = i(h_{ij})$ in $\Omega^3$ given by

$$\eta = h_{ij} g^{il} dx^i \wedge \left( \frac{\partial}{\partial x^l} \psi \right) = \frac{1}{2} h^i_{ij} \phi_{ijk} dx^i \wedge dx^j \wedge dx^k$$

Then the Hodge star $\ast \eta$ of $\eta$ is

$$\ast \eta = \frac{1}{4} \text{Tr}_g(h) g_{ij} - h_{ij} \right) g^{il} dx^i \wedge \left( \frac{\partial}{\partial x^l} \psi \right)$$

where $\text{Tr}_g(h) = g^{ij} h_{ij}$.

Proof. We compute

$$\ast \eta = h^i_{ij} \ast \left( dx^i \wedge \left( \frac{\partial}{\partial x^l} \psi \right) \right) = h^i_{ij} g^{im} \frac{\partial}{\partial x^m} \ast \left( \frac{\partial}{\partial x^n} \psi \right)$$

$$= h^m_{ij} \frac{\partial}{\partial x^m} \left( g^{lk} dx^k \wedge \psi \right) = h^m_{ij} \left( \delta^k_m \psi - dx^k \wedge \left( \frac{\partial}{\partial x^l} \psi \right) \right)$$

$$= \frac{1}{4} \text{Tr}_g(h) \delta^i_j dx^i \wedge \left( \frac{\partial}{\partial x^j} \psi \right) - h^i_{ij} \wedge \left( \frac{\partial}{\partial x^i} \psi \right)$$

where we have used (1.4) several times.

Proposition 2.15. Suppose $f_{ij}$ and $h_{ij}$ are two symmetric tensors. Let $i(f)$ and $i(h)$ be their corresponding forms in $\Omega^3$. Then we have

$$i(f) \wedge \ast (i(h)) = g(i(f), i(h)) \text{vol} = \left( \text{Tr}_g(f) \text{Tr}_g(h) + 2 f^k_i h^i_k \right) \text{vol}$$

Proof. Using Proposition 2.14 we compute $i(f) \wedge \ast (i(h)) =$

$$f^i_a dx^i \wedge \left( \frac{\partial}{\partial x^a} \psi \right) \wedge \left( \frac{1}{4} \text{Tr}_g(h) \delta^m_l \delta^n_k - h^m_k \right) dx^k \wedge \left( \frac{\partial}{\partial x^n} \psi \right)$$

$$= f^{ia} \frac{1}{4} \text{Tr}_g(h) (g^{mb} - h^{mb}) \left( \frac{\partial}{\partial x^a} \right)^b \wedge \left( \frac{\partial}{\partial x^b} \psi \right) \wedge \left( \frac{\partial}{\partial x^n} \psi \right)$$

$$= f^{ia} \left( \frac{1}{4} \text{Tr}_g(h) g^{mb} - h^{mb} \right) \left( 2 g_{ab} g_{km} - 2 g_{am} g_{bl} + \psi_{ablm} \right) \text{vol}$$

$$= \left( \frac{14}{4} \text{Tr}_g(f) \text{Tr}_g(h) - 2 \text{Tr}_g(f) \text{Tr}_g(h) - \frac{2}{4} \text{Tr}_g(f) \text{Tr}_g(h) + 2 f^m_i h^i_m \right) \text{vol} + 0$$

$$= \left( \text{Tr}_g(f) \text{Tr}_g(h) + 2 f^i_i h^i_i \right) \text{vol}$$

using Proposition A.11 and the symmetry of $f_{ij}$ and $h_{ij}$.

Corollary 2.16. The map $i : S^2(T) \to \Omega^3$ is injective. It is therefore an isomorphism onto its image, $\Omega^1 \oplus \Omega^3_{\Sigma^7}$.

Proof. Suppose $i(h) = 0$. Then Proposition 2.15 gives $\text{Tr}_g(h)^2 + g^{ab} g^{ij} h_{ij} h_{ab} = 0$. The second term is just $g(h, h)$, the natural inner product on $S^2(T)$. Thus both terms are non-negative and hence vanish. Therefore $g(h, h) = 0$, so $h_{ij} = 0$. 

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Proof. The map \( j : \Omega^3 \rightarrow S^2(T) \) is an isomorphism between \( \Omega_1^3 \oplus \Omega_2^3 \) and \( S^2(T) \). Consequently, \( \Omega_1^3 \) is the kernel of \( j \). Explicitly, we have

\[
\text{If } \eta = h_{ij} g^{il} dx^i \wedge \left( \frac{\partial}{\partial x^l} \psi \right) + (X \psi) = i(h) + (X \psi)
\]

then \( j(\eta) = -2 \text{Tr}_g(h) g_{ij} - 4 h_{ij} \)

by Theorem 2.4.7 of \[14\]. Thus \( \Omega_3^3 \) is in the kernel of the map \( j \). Now suppose that \( \eta = h_{ij} g^{il} dx^i \wedge (\nabla_{\partial x^l} \psi) = \frac{1}{2} h_{ij} \varphi_{ijk} dx^i \wedge dx^j \wedge dx^k \). Then we have

\[
(j(\eta))_{ab} = \frac{1}{2} h_{ij} \varphi_{ijk} * \left( \left( \frac{\partial}{\partial x^a} \psi \right) \wedge \left( \frac{\partial}{\partial x^b} \psi \right) \wedge dx^i \wedge dx^j \wedge dx^k \right)
\]

\[
= \frac{1}{8} h_{ij} \varphi_{ijk} \varphi_{apq} \varphi_{bmn} * (dx^p \wedge dx^q \wedge dx^m \wedge dx^n \wedge dx^j \wedge dx^k)
\]

\[
= \frac{1}{8} \sum_{\alpha \in S_7} \text{sgn}(\alpha) \varphi_{a(1)\sigma(2)\varphi_{b(3)\sigma(4)} \varphi_{c(5)\sigma(6)} h_{ij}^{l(7)}} * (dx^1 \wedge \cdots \wedge dx^7)
\]

\[
= \frac{1}{8} \left( \frac{8}{3} (B_{ij} h_i^l + B_{il} h_i^j + B_{jl} h_i^l) dx^1 \wedge \cdots \wedge dx^7 \right)
\]

\[
= \frac{1}{3} \left( -6 \text{Tr}_g(h) g_{ij} \text{vol} - 6 h_{ij} \text{vol} - 6 h_{ij} \text{vol} \right)
\]

\[
= -2 \text{Tr}_g(h) g_{ij} - 4 h_{ij}
\]

where we have used \([A.4]\) and \( B_{ij} = -6 g_{ij} \sqrt{\text{det}(g)} \). From this it follows immediately that \( j \) is injective on \( \Omega_1^3 \oplus \Omega_2^3 \), for if \( j(i(h)) = -2 \text{Tr}_g(h) g_{ij} - 4 h_{ij} = 0 \), taking the trace gives \(-18 \text{Tr}_g(h) = 0 \) and hence \( h_{ij} = 0 \).

To summarize, we have seen that an arbitrary 3-form \( \eta \) on a manifold \( M \) with G2-structure \( \varphi \) consists of the data of a vector field \( X \) and a symmetric 2-tensor \( h \). Explicitly, we have

\[
\eta = h_{ij} g^{il} dx^i \wedge \left( \frac{\partial}{\partial x^l} \psi \right) + X^l \frac{\partial}{\partial x^l} \psi
\]

\[
= \frac{1}{2} h_{ij} \varphi_{ijk} dx^i \wedge dx^j \wedge dx^k + \frac{1}{6} X^l \psi_{ijk} dx^i \wedge dx^j \wedge dx^k
\]

**Remark 2.18.** Note that the symmetric 2-tensor \( h_{ij} \) decomposes as \( h_{ij} = \frac{1}{7} \text{Tr}_g(h) g_{ij} + h_{ij}^0 \) where \( h_{ij}^0 \) is the trace-free part of \( h_{ij} \). Hence the first term in the above expression can be written as

\[
\frac{3}{7} h_{ij} \varphi + \frac{1}{2} (h_{ij}^0 \varphi_{ijk}) dx^i \wedge dx^j \wedge dx^k
\]

which shows explicitly the \( \Omega_1^3 \) and \( \Omega_2^3 \) components. However, it is more convenient to consider the tensor \( h_{ij} \) directly, using the isomorphism \( \Omega_1^3 \oplus \Omega_2^3 \cong \text{Sym}^2(T^*M) \).
2.3 The intrinsic torsion forms of a $G_2$-structure

Using the decomposition of the spaces of forms on $M$ determined by $\varphi$, given in Section 2.2, we can decompose $d\varphi$ and $d\psi$ into types. This defines the torsion forms of the $G_2$-structure.

**Definition 2.19.** There are four independent torsion forms corresponding to a $G_2$-structure $\varphi$. Following the notation introduced in [2], we denote them by

\[
\begin{align*}
\tau_0 &\in \Omega^0_1 & \tau_1 &\in \Omega^1_7 \\
\tau_2 &\in \Omega^2_{14} & \tau_3 &\in \Omega^3_{27}
\end{align*}
\]

They are defined via the equations

\[
\begin{align*}
d\varphi &= \tau_0 \psi + 3\tau_1 \wedge \varphi + \ast \tau_3 \\
d\psi &= 4\tau_1 \wedge \psi + \ast \tau_2
\end{align*}
\] (2.22)

The constants are chosen for convenience. The fact that the $\Omega^1_7$ component of $d\varphi$ and the $\Omega^2_{14}$ component of $d\psi$ are the same, up to a constant (when projected back to $\Omega^1_7$ via the $G_2$-invariant isomorphisms), is something that needs to be proved. A nice representation-theoretic proof is described in [4, 2]. Below we will give a brute force computational proof of this fact, which is useful as it is an exercise in the type of manipulations that will be used frequently in Section 3.3 when we compute the evolution equations for the torsion forms.

**Remark 2.20.** We call $\tau_0$ the scalar torsion, $\tau_1$ the vector torsion, $\tau_2$ the Lie algebra torsion, and $\tau_3$ the symmetric traceless torsion. These names are non-standard, but are clearly reasonable.

**Remark 2.21.** In [2], the second defining equation is given as $d\psi = 4\tau_1 \wedge \psi + \tau_2 \wedge \varphi$. This is the same thing, since in our chosen orientation convention, $\tau_2 \wedge \varphi = \ast \tau_2$ for $\tau_2 \in \Omega^2_{14}$. We prefer this way of writing the equation to make it more symmetric with the $d\varphi$ equation.

**Remark 2.22.** In [14], a one-form $\theta$ is discussed, sometimes called the Lee form. It is easy to check that $\theta = -12\tau_1$.

**Theorem 2.23.** The appearance of the same one-form $\tau_1$ in the expressions for both $d\varphi$ and $d\psi$ in (2.22) above is justified.

**Proof.** We begin by not assuming that the two $\tau_1$’s are the same. Let $d\varphi = \tau_0 \psi + 3\tau_1 \wedge \varphi + \ast \tau_3$ and $d\psi = 4\tilde{\tau}_1 \wedge \psi + \ast \tau_2$. We must show that $\tilde{\tau}_1 = \tau_1$. We manipulate these relations as follows

\[
\begin{align*}
d\varphi &= \tau_0 \psi + 3\tau_1 \wedge \varphi + \ast \tau_3 \\
* (d\varphi) &= \tau_0 \varphi + 3 \ast (\tau_1 \wedge \varphi) + \tau_3 \\
\varphi \wedge *(d\varphi) &= 0 - 3\varphi \ast (\varphi \wedge \tau_1) + 0 \\
\varphi \wedge *(d\varphi) &= 12 \ast \tilde{\tau}_1
\end{align*}
\]

where we have used Proposition [A.3]. Therefore we see that

\[
\begin{align*}
\tau_1 = \tilde{\tau}_1 \iff \varphi \wedge *(d\varphi) = \psi \wedge *(d\psi) \\
\iff d\varphi \wedge \varphi \wedge *(d\varphi) = d\psi \wedge \psi \wedge *(d\psi) \quad \text{for all } p \\
\iff g(d\varphi, d\varphi \wedge \varphi) = g(d\psi, d\psi \wedge \psi) \quad \text{for all } p
\end{align*}
\]

and it is the last equality above that we will now establish.
Let $X = X_i dx^i$ be an arbitrary one-form. Then we have

$$X \wedge \varphi = \frac{1}{6} X_q \varphi_{ijk} \, dx^i \wedge dx^j \wedge dx^k$$

$$= \frac{1}{24} (X_q \varphi_{ij} - X_i \varphi_{qjk} - X_j \varphi_{iqk} - X_k \varphi_{ijq}) \, dx^i \wedge dx^j \wedge dx^k$$

$$= \frac{1}{24} A_{gijk} \, dx^j \wedge dx^i \wedge dx^k \wedge dx^k$$

where we have skew-symmetrized the coefficients. Similarly we have

$$d \varphi = \frac{1}{6} (\nabla_m \varphi_{abc} - \nabla_a \varphi_{mbc} - \nabla_b \varphi_{amic} - \nabla_c \varphi_{abm}) \, dx^m \wedge dx^a \wedge dx^b \wedge dx^c$$

$$= \frac{1}{24} B_{mabc} \, dx^m \wedge dx^a \wedge dx^b \wedge dx^c$$

Now using (1.1), we have

$$g(X \wedge \varphi, d\varphi) = \frac{1}{24} A_{qijk} B_{mabc} g^{qm} g^{ia} g^{jb} g^{kc}$$

$$= \frac{1}{6} (X_q \varphi_{ij} - X_i \varphi_{qjk} - X_j \varphi_{iqk} - X_k \varphi_{ijq}) (\nabla_m \varphi_{abc}) g^{qm} g^{ia} g^{jb} g^{kc}$$

Let $X = dx^p$, so that $X_i = \delta_i^p$, and this expression becomes

$$g(dx^p \wedge \varphi, d\varphi) = \frac{1}{6} (\delta_q^p \varphi_{ij} - \delta_i^p \varphi_{qjk} - \delta_j^p \varphi_{iqk} - \delta_k^p \varphi_{ijq}) (\nabla_m \varphi_{abc}) g^{qm} g^{ia} g^{jb} g^{kc}$$

$$= \frac{1}{6} \varphi_{ijk} (\nabla_m \varphi_{abc}) g^{pm} g^{ia} g^{jb} g^{kc} - \frac{1}{2} \varphi_{qjk} (\nabla_m \varphi_{abc}) g^{qm} g^{pa} g^{jb} g^{kc}$$

By Proposition A.16, the first term vanishes, and the second term becomes

$$g(dx^p \wedge \varphi, d\varphi) = \frac{1}{2} (\nabla_m \varphi_{ijk}) \varphi_{abc} g^{im} g^{pa} g^{jb} g^{kc} \quad (2.23)$$

An exactly analogous calculation, which we omit, yields the expression

$$g(dx^p \wedge \psi, d\psi) = \frac{1}{6} (\nabla_m \psi_{ijkl}) \varphi_{abcd} g^{im} g^{pa} g^{jb} g^{kc} g^{ld} \quad (2.24)$$

Combining the two expressions, $g(dx^p \wedge \varphi, d\varphi) = g(dx^p \wedge \psi, d\psi)$ if and only if

$$(\nabla_m \varphi_{ijk}) \varphi_{abc} g^{im} g^{ja} g^{kc} g^{ld} = 3 (\nabla_m \varphi_{ijk}) \varphi_{abc} g^{im} g^{jb} g^{kc}$$

But this is precisely the content of Proposition A.17 after contracting with $g^{im}$. \qed

The reason to consider the torsion forms of a $G_2$-structure $\varphi$ is because $\varphi$ is torsion-free if and only if all four torsion forms vanish, and these forms are independent. This is clear since the decomposition of $\Omega^k$ into $G_2$-representations is orthogonal, and because the maps $\alpha \mapsto \varphi \wedge \alpha$ from $\Omega^1 \to \Omega^5_2$ and $\alpha \mapsto \psi \wedge \alpha$ from $\Omega^1 \to \Omega^5_2$ are isomorphisms.

Having defined the four torsion forms, we need to derive their expressions in local coordinates, in order to be able to compute their evolutions under a flow. To do this most efficiently, we will first determine how to write $\nabla \varphi$ in terms of the four torsion tensors. In the process we will obtain a new computational proof of the Fernández-Gray Theorem 2.1. We begin with the following important observation.
Lemma 2.24. For any vector field $X$, the 3-form $\nabla_X \varphi$ lies in the subspace $\Omega^3_l$ of $\Omega^3$. Therefore, the covariant derivative $\nabla \varphi$ lies in the space $\Omega^1_l \otimes \Omega^3_l$, a 49-dimensional space (pointwise.)

Proof. Let $X = \frac{\partial}{\partial x^i}$, and consider the 3-form $\nabla \varphi$. An arbitrary element $\eta$ of $\Omega^3_l \oplus \Omega^3_{27}$ can be written (for some symmetric tensor $h_{ij}$) as

$$\eta = \frac{1}{2} h_{lm} \varphi_{mjk} \, dx^i \wedge dx^j \wedge dx^k = \frac{1}{6} \left( h^m_{i} \varphi_{mjk} + h^m_{j} \varphi_{imk} + h^m_{k} \varphi_{ijm} \right) dx^i \wedge dx^j \wedge dx^k$$

Using (1.1), the inner product of $\eta$ with $\nabla = \frac{1}{6} \nabla \varphi_{abc} \, dx^a \wedge dx^b \wedge dx^c$ is

$$g(\nabla \varphi, \eta) = \frac{1}{6} \left( \nabla \varphi_{abc} \right) \left( h^m_{i} \varphi_{mjk} + h^m_{j} \varphi_{imk} + h^m_{k} \varphi_{ijm} \right) g^{ai} g^{bj} g^{ck}$$

$$= \frac{1}{2} \left( \nabla \varphi_{abc} \right) h^m_{i} \varphi_{mjk} g^{ai} g^{bj} g^{ck} = \frac{1}{2} \left( \nabla \varphi_{abc} \right) h^m_{i} \varphi_{mjk} g^{ai} g^{bj} g^{ck}$$

which vanishes since the third equation of Proposition A.16 says $(\nabla \varphi_{abc}) \varphi_{mjk} g^{bj} g^{ck}$ is skew-symmetric in $a$ and $m$. Since $g(\nabla \varphi, \eta) = 0$ for all $\eta \in \Omega^3_l \oplus \Omega^3_{27}$, we have that $\nabla \varphi \in \Omega^3_l$ for all $l = 1, \ldots, 7$ as claimed. \hfill \Box

Remark 2.25. In [8], Fernández and Gray study the symmetries of $\nabla \varphi$, and deduce that $\nabla \varphi \in \Omega^3_l \otimes \Omega^3_l$ as we have just shown. Their arguments involve the complicated analysis of the extension of the cross product operation to the full exterior bundle. The above proof is, in our opinion, much more transparent.

We pause here to consider Lemma 2.24 in more detail. The reason why $\nabla \varphi \in \Omega^3_l$ is essentially because of the way that the 3-form $\varphi$ of a G2-structure determines a metric $g$. We have seen that

$$(v \cdot \varphi) \wedge (w \cdot \varphi) \wedge \varphi = -6g(v, w)\text{vol}$$

Since the metric and volume form are parallel, the covariant derivative of this gives

$$(v \cdot \nabla \varphi) \wedge (w \cdot \varphi) \wedge \varphi + (v \cdot \varphi) \wedge (w \cdot \nabla \varphi) \wedge \varphi + (v \cdot \varphi) \wedge (w \cdot \varphi) \wedge \nabla \varphi = 0$$

Using the identity $\varphi \wedge (v \cdot \varphi) = -2 \ast (v \cdot \varphi)$ from Proposition A.3, this becomes

$$-2 \left( v \cdot \nabla \varphi \right) \wedge \ast (w \cdot \varphi) - 2 \left( v \cdot \varphi \right) \wedge \ast (w \cdot \nabla \varphi) + (v \cdot \varphi) \wedge (w \cdot \varphi) \wedge \nabla \varphi = 0$$

Finally, the covariant differentiation of the identity $v \cdot \varphi \wedge \ast (w \cdot \varphi) = 4g(v, w)\text{vol}$ (which is also implicit in Proposition A.3) gives

$$(v \cdot \nabla \varphi) \wedge \ast (w \cdot \varphi) + (v \cdot \varphi) \wedge \ast (w \cdot \nabla \varphi) = 0$$

Combining this with the previous equation yields the important relation

$$(v \cdot \varphi) \wedge (w \cdot \varphi) \wedge \nabla \varphi = 0$$

But by the definition of the map $j$ in (2.21), this precisely says that the $\Omega^3_l \oplus \Omega^3_{27}$ component of $\nabla_X \varphi$ is zero.

Definition 2.26. Since $\nabla \varphi \in \Omega^3_l$, by (2.7) we can write $\nabla \varphi_{abc} = T_{lm} g_{3m} \psi_{nabc}$ for some 2-tensor $T_{lm}$, which we shall call the full torsion tensor.
In coordinates, the four independent torsion forms are the following: \( \tau_0 \) (a function); \( \tau_1 = (\tau_1)_i \, dx^i \), a 1-form, which by Remark 2.8 can also be written as an \( \Omega_3 \)-form \( \tau_1 = \frac{1}{2} (\tau_1)_{ab} \, dx^a \wedge dx^b \) where \((\tau_1)_{ab} = (\tau_1)_{g^{lk} \varphi_{kab}}\); \( \tau_2 = \frac{1}{2} (\tau_2)_{ab} \, dx^a \wedge dx^b \), an \( \Omega_4 \)-form; and \( \tau_3 = \frac{1}{2} (\tau_3)_{lm} g^{mn} \varphi_{ijk} dx^i \wedge dx^j \wedge dx^k \), where \((\tau_3)_{lm} \) is traceless symmetric. The relationship between the four torsion forms and the full torsion tensor \( T_{lm} \) is given by the following theorem.

**Theorem 2.27.** The covariant derivative \( \nabla \varphi \) of the 3-form \( \varphi \) can be written as

\[
\nabla_\varphi \equiv T_{lm} g^{mn} \psi_{nabc}
\]

where the full torsion tensor \( T_{lm} \) is

\[
T_{lm} = \frac{\tau_0}{4} g_{lm} - (\tau_3)_{lm} + (\tau_1)_{lm} - \frac{1}{2} (\tau_2)_{lm}
\]

**Proof.** Write the full torsion tensor as \( T_{lm} = S_{lm} + C_{lm} \), where \( S_{lm} = \frac{1}{2} (T_{lm} + T_{ml}) \) and \( C_{lm} = \frac{1}{2} (T_{lm} - T_{ml}) \) are the symmetric and skew-symmetric parts of \( T_{lm} \). Thus we have

\[
\nabla_\varphi \equiv (S_{lm} + C_{lm}) g^{mn} \psi_{nabc} \tag{2.25}
\]

Since \( d\varphi = \tau_0 \psi + 3 \varphi \wedge \varphi \wedge \ast \varphi_3 \), the \( \Omega^3_3 \) \( \Omega^3_2 \) component of \( \ast d\varphi \) is \( \tau_0 \varphi_3 \), which we write as \( \frac{3}{4} (\frac{1}{7} \tau_0) \varphi_3 \). By Remark 2.18 this is \( f_{ij} g^{jl} \wedge (\frac{\partial}{\partial x^l} \varphi) \), where \( f_{ij} = \frac{1}{4} (\frac{3}{7} \tau_0) g_{ij} + (\tau_3)_{ij} \). Therefore, by Proposition 2.14 we have that the \( \Omega^3_3 \) \( \Omega^3_2 \) component of \( d\varphi = \ast (d\varphi) \) is \( \frac{1}{4} \text{Tr}_g f_{ij} - f_{ij} \) \( g^{jl} \wedge \left( \frac{\partial}{\partial x^l} \varphi \right) \). But \( \text{Tr}_g(f) = \frac{3}{4} \tau_0 \), so

\[
\frac{1}{4} \text{Tr}_g(f) g_{ij} - f_{ij} = \frac{7}{12} \tau_0 g_{ij} - \frac{4}{12} \tau_0 g_{ij} = \frac{1}{4} \tau_0 g_{ij} - (\tau_3)_{ij} \tag{2.26}
\]

Now we can also write \( d\varphi = \frac{1}{6} \nabla_\varphi \equiv dx^i \wedge dx^a \wedge dx^b \wedge dx^c \), which by (2.25) is

\[
d\varphi = \frac{1}{6} (S_{lm} + C_{lm}) g^{mn} \psi_{nabc} \wedge dx^a \wedge dx^b \wedge dx^c
\]

\[
\quad \quad = S_{lm} g^{mn} dx^l \wedge \left( \frac{\partial}{\partial x^m} \varphi \right) + C_{lm} g^{mn} dx^l \wedge \left( \frac{\partial}{\partial x^m} \varphi \right)
\]

The second term above is in \( \Omega^3_3 \). Therefore, comparing the \( \Omega^3_3 \) \( \Omega^3_2 \) term of \( d\varphi \) given above and by (2.26), we see that

\[
S_{lm} = \frac{\tau_0}{4} g_{lm} - (\tau_3)_{lm}
\]

This proves the first half of the theorem.

To establish the second half, we write \( \delta \varphi = - \ast d \ast \varphi \) in two ways. First, since \( d\psi = 4 \varphi \wedge \varphi \wedge \ast \varphi_2 \), we have \( \delta \varphi = - * d \psi = - 4(\tau_1 \diamond \varphi) - \tau_2 \), using (1.4). Therefore

\[
\delta \varphi = - 4 \frac{1}{2} (\tau_1)_{ab} dx^a \wedge dx^b \tag{2.27}
\]

From (1.0), we also have \( \delta \varphi = - \frac{1}{2} g^{lk} \nabla \varphi_{kab} dx^a \wedge dx^b \). Using (2.25), this is

\[
\delta \varphi = - \frac{1}{2} g^{lk} (S_{lm} + C_{lm}) g^{mn} \psi_{nabc} \wedge dx^a \wedge dx^b
\]

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The first term vanished by the symmetry of $S_{lm}$ and skew-symmetry of $\psi_{nkab}$. Now we decompose $C_{lm} = (C_7)_{lm} + (C_{14})_{lm}$ into $\Omega_2^7 \oplus \Omega_2^{14}$ components. Using Proposition 2.6 and interchanging $n$ and $k$, we see

$$\delta \varphi = \frac{1}{2} g^{lk} ((C_7)_{lm} + (C_{14})_{lm}) g^{mn} \psi_{knab} \, dx^a \wedge dx^b$$

$$= \frac{1}{2} (-4(C_7)_{ab} + 2(C_{14})_{ab}) \, dx^a \wedge dx^b$$

Comparing this to (2.24), we see that $(C_7)_{ab} = (\tau_1)_{ab}$ and $(C_{14})_{ab} = -\frac{1}{2}(\tau_2)_{ab}$, hence

$$C_{lm} = (\tau_1)_{lm} - \frac{1}{2}(\tau_2)_{lm}$$

and the proof is complete. \[\square\]

**Corollary 2.28.** The 3-form $\varphi$ is parallel if and only if it is both closed and co-closed. (This is Theorem 2.1.)

**Proof.** A parallel form is always automatically closed and co-closed, since the exterior derivative $d$ and the coderivative $\delta$ can both be written using the covariant derivative $\nabla$. The converse follows from Theorem 2.27 since $d\varphi = 0$ and $\delta \varphi = 0$ are equivalent to the vanishing of all four torsion tensors, so $T_{lm} = 0$ and hence $\nabla \varphi_{abc} = 0$. \[\square\]

**Remark 2.29.** Starting from $\nabla l \varphi_{abc} = T_{lm} g^{mn} \psi_{nabc}$ and using (A.11) and Lemma A.13, it is an easy computation to show that

$$\nabla m \psi_{ijkl} = -T_{mi} \varphi_{jkl} - T_{mj} \varphi_{ikl} - T_{mk} \varphi_{ijl} + T_{ml} \varphi_{ijk}$$

(2.28)

which will be used in Section 4.

The following lemma gives an explicit formula for $T_{lm}$ in terms of $\nabla \varphi$. This will be used in Section 3.3 to derive the evolution equations for the torsion tensors.

**Lemma 2.30.** The full torsion tensor $T_{lm}$ is equal to

$$T_{lm} = \frac{1}{24} (\nabla \varphi_{abc}) \psi_{mijk} g^{ia} g^{jb} g^{kc}$$

(2.29)

**Proof.** We begin with $\nabla l \varphi_{abc} = T_{lk} g^{kn} \psi_{nabc}$ and use Lemma A.14 to compute:

$$\nabla l \varphi_{abc} \psi_{mijk} g^{ia} g^{jb} g^{kc} = T_{lk} g^{kn} \psi_{nabc} \psi_{mijk} g^{ia} g^{jb} g^{kc}$$

$$= T_{lk} g^{kn} (24 \, g_{nm}) = 24 \, T_{lm}$$

as claimed. \[\square\]

Next we present expressions for the four torsion tensors from which we will calculate their evolution equations.
Proposition 2.31. The four torsion forms can be written in terms of $T_{pq} = S_{pq} + C_{pq}$ as follows:

$$
\begin{align*}
\tau_0 &= \frac{4}{7} g^{pq} S_{pq} \\
(\tau_3)_{pq} &= \frac{1}{4} \tau_0 g_{pq} - S_{pq} \\
(\tau_1)_{pq} &= \frac{1}{3} C_{pq} - \frac{1}{6} C_{ij} g^{ia} g^{jb} \psi_{abpq} \\
(\tau_2)_{pq} &= -\frac{4}{3} C_{pq} - \frac{1}{3} C_{ij} g^{ia} g^{jb} \psi_{abpq}
\end{align*}
$$

Proof. This is immediate from Theorem 2.27 and equations (2.9) and (2.10).

We close this section with an important observation about the vector torsion $\tau_1$. The effect of a conformal scaling on $G_2$-structures is well understood, and a detailed discussion can be found in Section 3.1 of [14]. In particular, we have the following result.

Theorem 2.32. Let $\tilde{\varphi} = f^3 \varphi$ be a new $G_2$-form, where $f$ is any nowhere vanishing smooth function, then the metric scales as $\tilde{g} = f^2 g$ and the 4-form as $\tilde{*\varphi} = f^4 * \varphi$. Furthermore, the torsion forms transform as follows:

$$
\begin{align*}
\tilde{\tau}_0 &= f^{-1} \tau_0 \\
\tilde{\tau}_1 &= \tau_1 + d \log(f) \\
\tilde{\tau}_2 &= f \tau_2 \\
\tilde{\tau}_3 &= f^2 \tau_3
\end{align*}
$$

Proof. This is essentially Theorem 3.1.4 in [14]. In that paper, $\theta = -12 \tau_1$.

The relevance of this result is evident. If we can construct a $G_2$-structure $\varphi$ for which the three torsion forms $\tau_0$, $\tau_2$, and $\tau_3$ all vanish, then any conformal scaling of such a structure remains of such type. It is easy to check that in this case that vector torsion $\tau_1$ is a closed 1-form. If we are interested in manifolds with full holonomy $G_2$, then by Remark 2.33 we must necessarily start with a manifold $M$ with finite fundamental group, and hence $H^1(M) = 0$, so the form $\tau_1$ is exact. Then we see from Theorem 2.32 that we can always find a conformal scaling factor $f$ (unique up to a multiplicative constant) to make the vector torsion $\tau_1$ vanish as well. Such $G_2$-structures are called conformally parallel. Hence we can restrict attention to constructing $G_2$-structures where $\tau_0$, $\tau_2$, and $\tau_3$ all vanish, since we can then always make $\tau_1$ vanish as well, provided the manifold is topologically able to admit a $G_2$-structure with full holonomy $G_2$.

3 General flows of $G_2$-structures

In this section we derive the evolution equations for a general flow $\frac{\partial}{\partial t} \varphi$ of a $G_2$-structure $\varphi$. Let $X$ be a vector field and $h$ a symmetric 2-tensor on $M$. Then a general variation of the $G_2$-structure $\varphi$ can be written as

$$
\frac{\partial}{\partial t} \varphi = \frac{1}{2} h^{ij}_{i} \varphi_{ijk} \, dx^i \wedge dx^j \wedge dx^k + \frac{1}{6} X^i \psi_{i i j k} \, dx^i \wedge dx^j \wedge dx^k
$$
We write the left hand side in coordinates, and skew-symmetrize the first term on the right hand side to obtain
\[
\frac{1}{6} \frac{\partial}{\partial t} \varphi_{ijk} \, dx^i \wedge dx^j \wedge dx^k = \frac{1}{6} \left( h^l_{ij} \varphi_{lik} - h^l_{lj} \varphi_{ilk} - h^l_{lk} \varphi_{ijl} \right) \, dx^i \wedge dx^j \wedge dx^k \\
+ \frac{1}{6} X^l \psi_{lik} \, dx^i \wedge dx^j \wedge dx^k
\]
We can now equate the (totally skew-symmetric) coefficients on both sides to obtain the general flow equation:
\[
\frac{\partial}{\partial t} \varphi_{ijk} = \frac{1}{6} \left( h^l_{ij} \varphi_{lik} + h^l_{lj} \varphi_{ilk} + h^l_{lk} \varphi_{ijl} + X^l \psi_{lik} \right) \, dx^i \wedge dx^j \wedge dx^k
\] (3.1)

3.1 Evolution of the metric \( g_{ij} \) and related objects

We now proceed to derive the evolution equations for the metric \( g_{ij} \) and objects related to the metric, specifically the volume form \( \text{vol} \) and the Christoffel symbols \( \Gamma^k_{ij} \). The first step is to compute the evolution of the tensor \( B_{ij} \), which was defined as
\[
B_{ij} \, dx^1 \wedge \ldots \wedge dx^7 = \left( \frac{\partial}{\partial x^i} \varphi \right) \wedge \left( \frac{\partial}{\partial x^j} \varphi \right) \wedge \varphi
\]
which in coordinates becomes
\[
B_{ij} \, dx^1 \wedge \ldots \wedge dx^7 = \frac{1}{24} \varphi_{ik_1 k_2} \varphi_{j k_3 k_4} \varphi_{k_5 k_6 k_7} \, dx^{k_1} \wedge \ldots \wedge dx^{k_7}
= \frac{1}{24} \sum_{\sigma \in S_7} \text{sgn}(\sigma) \, \varphi_{i \sigma(1) \sigma(2)} \varphi_{j \sigma(3) \sigma(4)} \varphi_{\sigma(5) \sigma(6) \sigma(7)} \, dx^1 \wedge \ldots \wedge dx^7
\]
and hence
\[
B_{ij} = \frac{1}{24} \sum_{\sigma \in S_7} \text{sgn}(\sigma) \, \varphi_{i \sigma(1) \sigma(2)} \varphi_{j \sigma(3) \sigma(4)} \varphi_{\sigma(5) \sigma(6) \sigma(7)}
\] (3.2)
where the sum is taken over all permutations \( \sigma \) of the group \( S_7 \) of seven letters.

**Theorem 3.1.** The evolution of \( B_{ij} \) under the flow (3.1) is given by
\[
\frac{\partial}{\partial t} B_{ij} = \text{Tr}_g(h) B_{ij} + h^l_{ij} B_{jl} + h^l_{lj} B_{il}
\] (3.3)

Note that the evolution of \( B_{ij} \) depends only on the symmetric 2-tensor \( h_{ij} \) and not on the vector field \( X^k \).

**Proof.** We will need to appeal to various identities for \( G_2 \)-structures that are proved in Section A.

We start by computing
\[
\frac{\partial}{\partial t} B_{ij} = \frac{1}{24} \sum_{\sigma \in S_7} \text{sgn}(\sigma) \left( \frac{\partial}{\partial t} \varphi_{i \sigma(1) \sigma(2)} \right) \varphi_{j \sigma(3) \sigma(4)} \varphi_{\sigma(5) \sigma(6) \sigma(7)}
+ \frac{1}{24} \sum_{\sigma \in S_7} \text{sgn}(\sigma) \varphi_{i \sigma(1) \sigma(2)} \left( \frac{\partial}{\partial t} \varphi_{j \sigma(3) \sigma(4)} \right) \varphi_{\sigma(5) \sigma(6) \sigma(7)}
+ \frac{1}{24} \sum_{\sigma \in S_7} \text{sgn}(\sigma) \varphi_{i \sigma(1) \sigma(2)} \varphi_{j \sigma(3) \sigma(4)} \left( \frac{\partial}{\partial t} \varphi_{\sigma(5) \sigma(6) \sigma(7)} \right)
\]
We substitute the evolution equation (3.1) into this expression:

\[
24 \frac{\partial}{\partial t} B_{ij} = \sum_{\sigma \in S_7} \text{sgn}(\sigma) \left( h^l_{\sigma} \varphi_{l \sigma (1) \sigma (2)} + h^l_{\sigma (1) j} \varphi_{l \sigma (2)} + h^l_{\sigma (2) j} \varphi_{l \sigma (1)} \right) \varphi_{\sigma (3) \sigma (4) \varphi_{\sigma (5) \sigma (6) \sigma (7)}}
\]

\[
+ \sum_{\sigma \in S_7} \text{sgn}(\sigma) \varphi_{i \sigma (1) \sigma (2)} \left( h^l_{\sigma (3) j} \varphi_{l \sigma (4)} + h^l_{\sigma (4) j} \varphi_{l \sigma (3)} \right) \varphi_{\sigma (5) \sigma (6) \sigma (7)}
\]

\[
+ \sum_{\sigma \in S_7} \text{sgn}(\sigma) \varphi_{i \sigma (1) \sigma (2)} \varphi_{j \sigma (3) \sigma (4)} \left( h^l_{\sigma (5) j} \varphi_{l \sigma (6) \sigma (7)} + h^l_{\sigma (6) j} \varphi_{l \sigma (5) \sigma (7)} + h^l_{\sigma (7) j} \varphi_{l \sigma (5) \sigma (6) \sigma (7)} \right)
\]

\[
+ \sum_{\sigma \in S_7} \text{sgn}(\sigma) X^l \psi_{l \sigma (1) \sigma (2)} \varphi_{j \sigma (3) \sigma (4)} \varphi_{\sigma (5) \sigma (6) \sigma (7)}
\]

\[
+ \sum_{\sigma \in S_7} \text{sgn}(\sigma) \varphi_{i \sigma (1) \sigma (2)} \varphi_{j \sigma (3) \sigma (4)} X^l \psi_{l \sigma (5) \sigma (6) \sigma (7)}
\]

(3.4)

where we have separated the expression into three terms involving \( h_{ij} \) and three terms involving \( X^l \).

Let us consider the \( X^l \) terms first. Equation (A.7) tells us immediately that the last term is zero. Let \( \tau \) be the permutation in \( S_7 \) which sends \( (1, 2, 3, 4, 5, 6, 7) \) \( \rightarrow \) \( (3, 4, 1, 2, 5, 6, 7) \). This permutation is even, so \( \text{sgn}(\sigma \circ \tau) = \text{sgn}(\sigma) \). Therefore if we write \( \sigma' = \sigma \circ \tau \), then the first two terms involving \( X^l \) can be written as

\[
\sum_{\sigma \in S_7} \text{sgn}(\sigma) X^l \psi_{l \sigma (1) \sigma (2)} \varphi_{j \sigma (3) \sigma (4)} \varphi_{\sigma (5) \sigma (6) \sigma (7)}
\]

\[
+ \sum_{\sigma' \in S_7} \text{sgn}(\sigma') \varphi_{i \sigma' (1) \sigma' (2)} X^l \psi_{l \sigma' (3) \sigma' (4)} \varphi_{\sigma' (5) \sigma' (6) \sigma' (7)}
\]

which also vanishes because of (A.6). Hence all the \( X^l \) terms are zero and thus the vector field \( X^l \) does not affect the evolution of \( B_{ij} \). Therefore it does not affect the evolution of the metric \( g_{ij} \) or the volume form \( \text{vol} \). It is well known that infinitesimal variations in the \( \Omega^2 \) direction do not affect the metric. See [4, 14] for other demonstrations of this fact.

We now return to the terms in (3.4) involving \( h_{ij} \). We start with the first term on the first line, together with the first term on the second line. These are:

\[
\sum_{\sigma \in S_7} \text{sgn}(\sigma) h^l_{\sigma} \varphi_{i \sigma (1) \sigma (2)} \varphi_{j \sigma (3) \sigma (4)} \varphi_{\sigma (5) \sigma (6) \sigma (7)}
\]

\[
+ \sum_{\sigma \in S_7} \text{sgn}(\sigma) \varphi_{i \sigma (1) \sigma (2)} h^l_{\sigma (3) j} \varphi_{l \sigma (4)} \varphi_{\sigma (5) \sigma (6) \sigma (7)}
\]

which, using (3.2), are seen to be equal to

\[
24 \left( h^l_i B_{ij} + h^l_j B_{il} \right)
\]

Next, we consider in (3.4) the remaining two terms from the first line, together with the remaining
two terms from the second line. These are:

\[
\sum_{\sigma \in S_7} \text{sgn}(\sigma) \left( h_{\sigma(1)}^l \varphi_{1\sigma(2)} + h_{\sigma(2)}^l \varphi_{\sigma(1)} \right) \varphi_{\sigma(3)\sigma(4)} \varphi_{\sigma(5)\sigma(6)\sigma(7)} + \sum_{\sigma \in S_7} \text{sgn}(\sigma) \varphi_{\sigma(1)\sigma(2)} \left( h_{\sigma(3)}^l \varphi_{j\sigma(4)} + h_{\sigma(4)}^l \varphi_{j\sigma(3)} \right) \varphi_{\sigma(5)\sigma(6)\sigma(7)}
\]

If we consider the permutation \(\tau\) which interchanges 1 and 2, and denote \(\sigma' = \sigma \circ \tau\), then the first line can be rewritten as

\[
\sum_{\sigma \in S_7} \text{sgn}(\sigma) h_{\sigma(1)}^l \varphi_{il\sigma(2)} \varphi_{j\sigma(3)\sigma(4)} \varphi_{\sigma(5)\sigma(6)\sigma(7)} - \sum_{\sigma' \in S_7} \text{sgn}(\sigma') h_{\sigma'(1)}^l \varphi_{i\sigma'(2)} \varphi_{j\sigma'(3)\sigma'(4)} \varphi_{\sigma'(5)\sigma'(6)\sigma'(7)} = 2 \sum_{\sigma \in S_7} \text{sgn}(\sigma) h_{\sigma(1)}^l \varphi_{il\sigma(2)} \varphi_{j\sigma(3)\sigma(4)} \varphi_{\sigma(5)\sigma(6)\sigma(7)}
\]

by the skew-symmetry of \(\varphi_{ijk}\). The two terms on the second line can be similarly combined as

\[
2 \sum_{\sigma \in S_7} \text{sgn}(\sigma) h_{\sigma(3)}^l \varphi_{j\sigma(4)} \varphi_{i\sigma(1)\sigma(2)} \varphi_{\sigma(5)\sigma(6)\sigma(7)} = 2 \sum_{\sigma \in S_7} \text{sgn}(\sigma) h_{\sigma(1)}^l \varphi_{j\sigma(2)} \varphi_{i\sigma(3)\sigma(4)} \varphi_{\sigma(5)\sigma(6)\sigma(7)}
\]

by interchanging 1 with 2 and 3 with 4. We can now apply [A.4] to these two expressions (with \(\alpha_k = h_{\sigma}^l\)). These terms then become

\[
2 \left( -4 (B_{ij} h_{i}^l - B_{ji} h_{j}^l) + 24 (\psi_{ilm} g^{km} h_{k}^l \sqrt{\det(g)}) \right) + 2 \left( -4 (B_{il} h_{i}^l - B_{ili} h_{i}^l) + 24 (\psi_{lim} g^{km} h_{k}^l \sqrt{\det(g)}) \right) = 16 \text{Tr}_g(h) B_{ij} - 8 h_{i}^l B_{ij} - 8 h_{j}^l B_{il}
\]

using the symmetry of \(B_{ij}\) and the skew-symmetry of \(\psi_{ijk}\).

All that remains now in (3.4) are the final three terms involving \(h_{ij}\). These are:

\[
\sum_{\sigma \in S_7} \text{sgn}(\sigma) \varphi_{\sigma(1)\sigma(2)} \varphi_{\sigma(3)\sigma(4)} \left( h_{\sigma(5)}^l \varphi_{\sigma(6)\sigma(7)} + h_{\sigma(6)}^l \varphi_{\sigma(5)\sigma(7)} + h_{\sigma(7)}^l \varphi_{\sigma(5)\sigma(6)} \right)
\]

If we consider the cyclic permutation 5 \(\mapsto\) 6 \(\mapsto\) 7 \(\mapsto\) 5, which is even, we see that the three terms above are actually the same, so we can combine them into

\[
3 \sum_{\sigma \in S_7} \text{sgn}(\sigma) \varphi_{\sigma(1)\sigma(2)} \varphi_{\sigma(3)\sigma(4)} \varphi_{\sigma(5)\sigma(6)} h_{\sigma(7)}^l
\]

Now we can use [A.4], (again with \(\alpha_k = h_{\sigma}^l\)), to write this expression as

\[
3 \left( \frac{8}{3} \right) (B_{ij} h_{i}^l + B_{il} h_{i}^l + B_{ji} h_{j}^l) = 8 \text{Tr}_g(h) B_{ij} + 8 h_{i}^l B_{ij} + 8 h_{j}^l B_{il}
\]

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Finally, we combine the results of these calculations of the terms in (3.4) to obtain
\[
24 \frac{\partial}{\partial t} B_{ij} = 24 \left( h_i^l B_{lj} + h_j^l B_{il} \right) + \left( 16 \text{Tr}_g(h) B_{ij} - 8 h_i^l B_{lj} - 8 h_j^l B_{il} \right) \\
+ \left( 8 \text{Tr}_g(h) B_{ij} + 8 h_i^l B_{lj} + 8 h_j^l B_{il} \right)
\]
and the proof is complete. □

**Corollary 3.2.** The evolution of the metric $g_{ij}$ under the flow (3.1) is given by
\[
\frac{\partial}{\partial t} g_{ij} = 2 h_{ij} \tag{3.5}
\]

*Proof.* We begin by computing the evolution of $\det(B)$ using Theorem 3.1 and Lemma B.1 (which applies to any symmetric matrix $g_{ij}$ with inverse $g^{ij}$.)
\[
\frac{\partial}{\partial t} \det(B) = \left( \frac{\partial}{\partial t} B_{ij} \right) B^{ij} \det(B)
\]
\[
= \left( \text{Tr}_g(h) B_{ij} + h_i^l B_{lj} + h_j^l B_{il} \right) B^{ij} \det(B)
\]
\[
= \left( \text{Tr}_g(h) \delta_i^i + h_i^l \delta_i^l + h_j^l \delta_j^l \right) \det(B)
\]
\[
= 9 \text{Tr}_g(h) \det(B)
\]
We now use this result to differentiate (2.3):
\[
\frac{\partial}{\partial t} g_{ij} = \frac{\partial}{\partial t} \left( \frac{1}{6^2} \frac{B_{ij}}{\det(B)^{\frac{1}{2}}} \right)
\]
\[
= \frac{1}{6^2} \left( \frac{\partial}{\partial t} B_{ij} \frac{1}{\det(B)^{\frac{1}{2}}} - 9 \frac{B_{ij} \frac{\partial}{\partial t} \det(B)}{\det(B)^{\frac{1}{2}}} \right)
\]
\[
= \frac{1}{6^2} \left( \text{Tr}_g(h) B_{ij} + h_i^l B_{lj} + h_j^l B_{il} - 9 \frac{B_{ij} \text{Tr}_g(h) \det(B)}{\det(B)^{\frac{1}{2}}} \right)
\]
\[
= \text{Tr}_g(h) g_{ij} + h_i^l g_{lj} + h_j^l g_{il} - \text{Tr}_g(h) g_{ij} = 2 h_{ij}
\]
as claimed. □

**Corollary 3.3.** The evolution of the inverse $g^{ij}$ of the metric and the evolution of the volume form $\text{vol}$ under the flow (3.1) are given by
\[
\frac{\partial}{\partial t} g^{ij} = -2 h^{ij} \tag{3.6}
\]
\[
\frac{\partial}{\partial t} \text{vol} = \text{Tr}_g(h) \text{vol} \tag{3.7}
\]

*Proof.* These follows directly from Lemmas B.1 and B.2 □

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Finally we consider the evolution of the Christoffel symbols $\Gamma^k_{ij}$.

**Proposition 3.4.** The evolution of the Christoffel symbols $\Gamma^k_{ij}$ under the flow (3.1) is given by

$$\frac{\partial}{\partial t} \Gamma^k_{ij} = g^{kl} (\nabla_i h_{jl} + \nabla_j h_{il} - \nabla_l h_{ij})$$ (3.8)

**Proof.** This result is standard and can be found, for example, in [5], Lemma 3.2. (Note that by our Corollary 3.2 above, their $h_{ij}$ in [5] is replaced by our $2h_{ij}$.) \qed

### 3.2 Evolution of the dual 4-form $\psi_{ijkl}$

We proceed now to the computation of the evolution equation for the dual 4-form $\psi = \ast \phi$. To do this directly would be quite complicated, because we would need to compute the evolution of the Hodge star operator $\ast$. We can avoid this by using the final equation from Lemma A.12, which can be rearranged into the form

$$\psi_{ijkl} = g_{ik} g_{jl} - g_{il} g_{jk} - \varphi_{ija} \varphi_{kl\beta} g^{\alpha\beta}$$ (3.9)

This essentially says that, up to some correction terms involving only the metric, the form $\psi$ is the contraction of $\varphi$ with itself.

**Theorem 3.5.** The evolution of the 4-form $\psi_{ijkl}$ under the flow (3.1) is given by

$$\frac{\partial}{\partial t} \psi_{ijkl} = h^m_i \psi_{mjkl} + h^m_j \psi_{imkl} + h^m_k \psi_{ijkm}$$

$$- X^i \varphi_{ijkl} + X^j \varphi_{iklj} - X^k \varphi_{ijlk} + X^l \varphi_{ijkl}$$ (3.10)

where $X_k = g_{kl} X^l$.

**Proof.** We differentiate (3.9) with respect to time:

$$\frac{\partial}{\partial t} \psi_{ijkl} = \left( \frac{\partial}{\partial t} g_{ik} \right) g_{jl} + \left( \frac{\partial}{\partial t} g_{jl} \right) g_{ik} - \left( \frac{\partial}{\partial t} g_{il} \right) g_{jk} - \left( \frac{\partial}{\partial t} g_{jl} \right) g_{ik}$$

$$- \varphi_{ija} \varphi_{kl\beta} \left( \frac{\partial}{\partial t} g^{\alpha\beta} \right) - \left( \frac{\partial}{\partial t} \varphi_{ija} \right) \varphi_{kl\beta} g^{\alpha\beta} - \varphi_{ija} \left( \frac{\partial}{\partial t} \varphi_{kl\beta} \right) g^{\alpha\beta}$$ (3.11)

Using Corollary 3.2 and equation (3.6), the first five terms in (3.11) are

$$2 h_{ik} g_{jl} + 2 h_{jl} g_{ik} - 2 h_{il} g_{jk} - 2 h_{jk} g_{il} + 2 \varphi_{ija} \varphi_{kl\beta} h^{\alpha\beta}$$

Meanwhile the next to last term in (3.11), using (3.1) and Lemma A.12 is

$$- h^m_i \varphi_{mj\alpha} + h^m_j \varphi_{im\alpha} + h^m_i \varphi_{jm\alpha} + X^m \psi_{mj\alpha} \varphi_{kl\beta} g^{\alpha\beta}$$

$$= - h^m_i \left( g_{mk} g_{jl} - g_{ml} g_{jk} - \psi_{mjkl} \right) - h^m_j \left( g_{ik} g_{ml} - g_{il} g_{km} - \psi_{imkl} \right)$$

$$- h^{m\beta} \varphi_{ij\alpha} \varphi_{kl\beta} - X^m \psi_{mj\alpha} \varphi_{kl\beta} g^{\alpha\beta}$$

$$= - h_{ik} g_{jl} + h_{jl} g_{ik} - h_{il} g_{jk} + h_{jk} g_{il} + h^m_i \psi_{mjkl} + h^m_j \psi_{imkl}$$

$$- h^{m\beta} \varphi_{ij\alpha} \varphi_{kl\beta} - X^m \psi_{mj\alpha} \varphi_{kl\beta} g^{\alpha\beta}$$
Finally, the last term in (3.11) is the same as the next to last term if we interchange \( i \) with \( k \) and \( j \) with \( l \), so it equals

\[
-h_{ik}g_{jl} + h_{jk}g_{il} - h_{it}g_{jk} + h_{tk}^m \psi_ml_{ij} + h_{tl}^m \psi_{km_{ij}} \\
h^{m\beta} \varphi_{mkl} \varphi_{ij\beta} - X^m \psi_{mkl} \varphi_{ij\beta} g^{\alpha\beta}
\]

When we add up these three sets of terms, all the ‘\( g \cdot h \)’ terms and the ‘\( \varphi \cdot \varphi \)’ terms cancel, and we are left with

\[
\frac{\partial}{\partial t} \psi_{ijkl} = h_{ik}^m \psi_{mjkl} + h_{jk}^m \psi_{imkl} + h_{tk}^m \psi_{ijml} + h_{tl}^m \psi_{ijkm} \\
- X^m \psi_{mij\alpha} \varphi_{kl\beta} g^{\alpha\beta} - X^m \psi_{mkl} \varphi_{ij\beta} g^{\alpha\beta}
\]

(3.12)

We deal with the \( X^m \) terms above using the final equation of Lemma A.13. They become

\[
- X^m \left( g_{km} \varphi_{lij} + g_{ki} \varphi_{mjl} + g_{kj} \varphi_{mil} - g_{li} \varphi_{mkj} - g_{lj} \varphi_{mki} \right) \\
- X^m \left( g_{im} \varphi_{jkl} + g_{ik} \varphi_{mjl} + g_{il} \varphi_{mki} - g_{jm} \varphi_{ikl} - g_{jk} \varphi_{mil} - g_{jl} \varphi_{mki} \right) \\
- X^k \varphi_{lij} + X_i \varphi_{kj} - X_i \varphi_{jkl} + X_j \varphi_{ikl}
\]

because the other terms cancel in pairs. Substituting this expression into (3.12) above completes the proof.

Remark 3.6. Let the infinitesimal deformation of a G2 3-form \( \varphi(t) \) be given by

\[
\frac{\partial}{\partial t} \varphi(t) = \eta_1 + \eta_7 + \eta_{27}
\]

where \( \eta_k \) belongs to the subspace \( \Omega^3_k \) associated to \( \varphi(t) \). Then one can show that the infinitesimal deformation of the associated 4-form \( \psi(t) \) is

\[
\frac{\partial}{\partial t} \psi(t) = \frac{4}{3} \left( * \varphi(t) \eta_1 \right) + \left( * \varphi(t) \eta_7 \right) - \left( * \varphi(t) \eta_{27} \right)
\]

This is mentioned in [2, 12, 13] and an explicit proof is given in [10]. The purpose of this remark is to clarify that Theorem 3.5 agrees with this result. Let \( \frac{\partial}{\partial t} \varphi = \eta \) be an arbitrary 3-form. It can be written uniquely as

\[
\eta = \frac{1}{2} h^i_{lj} \varphi_{ijk} \, dx^i \wedge dx^j \wedge dx^k + \frac{1}{6} X^i \psi_{lijk} \, dx^i \wedge dx^j \wedge dx^k
\]

for some vector field \( X^i \) and some symmetric 2-tensor \( h_{ij} \). We can write \( h_{ij} = \frac{1}{2} \text{Tr}_g(h) g_{ij} + h^0_{ij} \), where \( h^0_{ij} \) is the trace free part, and \( \text{Tr}_g(h) = g^{ij} h_{ij} \). Taking the Hodge star, and using (1.4) and Proposition 2.14 gives

\[
* \eta = f^m_k \, dx^k \wedge \left( \frac{\partial}{\partial x^m} \psi \right) - X^b \wedge \varphi
\]
where \( f_{ij} = \frac{1}{4} \text{Tr}_g(h) g_{ij} - h_{ij} = \frac{1}{4} \text{Tr}_g(h) g_{ij} - \frac{1}{7} \text{Tr}_g(h) g_{ij} - h_{ij}^0 = \frac{3}{4} \left( \frac{1}{7} \text{Tr}_g(h) g_{ij} \right) - h_{ij}^0 \). In coordinates, we have

\[
* \eta = f^m_k \, dx^k \wedge \left( \frac{\partial}{\partial x^m} \eta \right) - (X \wedge \varphi)
\]

\[
= \frac{1}{6} f^m_k \psi_m jkl \, dx^i \wedge dx^j \wedge dx^k \wedge dx^l - \frac{1}{6} X_i \psi_{jkl} \, dx^i \wedge dx^j \wedge dx^k \wedge dx^l
\]

\[
= \frac{1}{24} \left( f^m_i \psi_{m jkl} + f^m_j \psi_{imkl} + f^m_k \psi_{ijml} \right) \, dx^i \wedge dx^j \wedge dx^k \wedge dx^l
\]

\[
+ \frac{1}{24} \left( -X_i \varphi_{jkl} + X_j \varphi_{ikl} - X_k \varphi_{ijl} + X_l \varphi_{ijk} \right) \, dx^i \wedge dx^j \wedge dx^k \wedge dx^l
\]

Comparing with (3.10), we see that

\[
\left( \frac{\partial}{\partial t} \varphi \right)_1 = 4 \left( \frac{\partial}{\partial t} \psi \right)_1 ; \quad \left( \frac{\partial}{\partial t} \varphi \right)_7 = * \left( \frac{\partial}{\partial t} \psi \right)_7 ; \quad \left( \frac{\partial}{\partial t} \varphi \right)_{27} = - * \left( \frac{\partial}{\partial t} \psi \right)_{27}
\]

as expected. The approach we have adopted is advantageous, because in this setup

\[
\text{If } \frac{\partial}{\partial t} \varphi = h^k_m \, dx^k \wedge \left( \frac{\partial}{\partial x^m} \varphi \right) + (X \wedge \varphi)
\]

then

\[
\frac{\partial}{\partial t} \psi = h^k_m \, dx^k \wedge \left( \frac{\partial}{\partial x^m} \psi \right) + *(X \wedge \psi)
\]

so the two equations look much more symmetric this way.

### 3.3 Evolution of the torsion forms

In this section we derive the evolution equations for the four torsion tensors of a G\(_2\)-structure under a general flow described by a symmetric tensor \( h_{ij} \) and a vector field \( X^k \). We begin with the evolution of \( \n_i \varphi_{ijk} \).

**Lemma 3.7.** The evolution of \( \n_i \varphi_{ijk} \) under the flow (3.1) is given by

\[
\frac{\partial}{\partial t} (\n_i \varphi_{ijk}) = h^m_i (\n_i \varphi_{mjk}) + h^m_j (\n_j \varphi_{imk}) + h^m_k (\n_k \varphi_{ijm}) + X^m (\n_i \psi_{mijk})
\]

\[
+ (\n_i h_{ij}) g^{ms} \varphi_{mjk} + (\n_k h_{ij}) g^{ms} \varphi_{imk} + (\n_j h_{ij}) g^{ms} \varphi_{ijm}
\]

\[
- (\n_i h_{ik}) g^{ms} \varphi_{mjk} - (\n_j h_{ik}) g^{ms} \varphi_{imk} - (\n_k h_{ik}) g^{ms} \varphi_{ijm}
\]

\[
+ (\n_i X^m) \psi_{mijk}
\]

(3.13)

in terms of \( h_{ij} \) and \( X^k \).

**Proof.** Recall that

\[
\n_i \varphi_{ijk} = \frac{\partial}{\partial x^i} \varphi_{ijk} - \Gamma^m_i \varphi_{mjk} - \Gamma^m_j \varphi_{imk} - \Gamma^m_k \varphi_{ijm}
\]
We differentiate this equation with respect to $t$ to obtain

$$
\frac{\partial}{\partial t} (\nabla \varphi_{ijk}) = \frac{\partial}{\partial t} \left( \frac{\partial}{\partial t} \varphi_{ijk} \right) - \left( \frac{\partial}{\partial t} \Gamma^m_{li} \right) \varphi_{mjk} - \left( \frac{\partial}{\partial t} \Gamma^m_{lj} \right) \varphi_{imk} - \left( \frac{\partial}{\partial t} \Gamma^m_{lk} \right) \varphi_{ijm}
$$

$$
- \Gamma^m_{li} \left( \frac{\partial}{\partial t} \varphi_{mjk} \right) - \Gamma^m_{lj} \left( \frac{\partial}{\partial t} \varphi_{imk} \right) - \Gamma^m_{lk} \left( \frac{\partial}{\partial t} \varphi_{ijm} \right)
$$

\[= \nabla \left( \frac{\partial}{\partial t} \varphi_{ijk} \right) - \left( \frac{\partial}{\partial t} \Gamma^m_{li} \right) \varphi_{mjk} - \left( \frac{\partial}{\partial t} \Gamma^m_{lj} \right) \varphi_{imk} - \left( \frac{\partial}{\partial t} \Gamma^m_{lk} \right) \varphi_{ijm} \]

Now we substitute (3.1) and (3.8) to get

$$
\frac{\partial}{\partial t} (\nabla \varphi_{ijk}) = \nabla \left( h^m_i \varphi_{mjk} + h^m_j \varphi_{imk} + h^m_k \varphi_{ijm} + X^m \psi_{ijk} \right)
$$

$$
- g^{ms} (\nabla h_{is} + \nabla h_{is} - \nabla h_{il}) \varphi_{mjk}
$$

$$
- g^{ms} (\nabla h_{js} + \nabla h_{is} - \nabla h_{ij}) \varphi_{imk}
$$

$$
- g^{ms} (\nabla h_{ks} + \nabla h_{is} - \nabla h_{kl}) \varphi_{ijm}
$$

We use the product rule on the first line, and see that all the terms involving $\nabla h$ cancel in pairs. The result now follows. \[\square\]

We are now ready to compute the evolution equation of the full torsion tensor $T_{pq}$ for a general flow of $G_2$-structures.

**Theorem 3.8.** The evolution of the full torsion tensor $T_{pq}$ under the flow (3.1) is given by

$$
\frac{\partial}{\partial t} T_{pq} = T_{pl} g^{lm} h_{mq} + T_{pl} g^{lm} X_{mq} + (\nabla_p h_{ip}) g^{ka} g^{ib} \varphi_{abq} + \nabla_p X_q
$$

(3.14)

where $X_j = g_{jk} X^k$ and $X_{ij} = X^k \varphi_{kij}$ is the element of $\Omega^2_7$ corresponding to $X$.

**Proof.** We start with Lemma 2.30 and differentiate:

$$
\frac{\partial}{\partial t} T_{pq} = \frac{1}{24} \frac{\partial}{\partial t} \left( \nabla_p \varphi_{ijk} \right) \psi_{abcd} g^{ib} g^{jc} g^{kd} + \frac{1}{24} \left( \nabla_p \varphi_{ijk} \right) \left( \frac{\partial}{\partial t} \nabla_p \varphi_{ijk} \right) g^{ib} g^{jc} g^{kd}
$$

$$
+ \frac{3}{24} \left( \nabla_p \varphi_{ijk} \right) \psi_{abcd} \left( \frac{\partial}{\partial t} g^{ib} \right) g^{jc} g^{kd}
$$

(3.15)

where we have relabelled indices and used the skew-symmetry of $\varphi$ and $\psi$ to combine the three terms involving derivatives of $g$. By Lemma 3.7, the first term in (3.15) is

$$
\frac{1}{24} \left( h^m_i \left( \nabla_p \varphi_{mjk} \right) + h^m_j \left( \nabla_p \varphi_{imk} \right) + h^m_k \left( \nabla_p \varphi_{ijm} \right) + X^m \left( \nabla_p \psi_{mijk} \right) \right) \psi_{abcd} g^{ib} g^{jc} g^{kd}
$$

$$
+ \frac{1}{24} \left( (\nabla_p h_{ip}) g^{ms} \varphi_{mjk} + (\nabla_p h_{jp}) g^{ms} \varphi_{imk} + (\nabla_p h_{kp}) g^{ms} \varphi_{ijm} \right) \psi_{abcd} g^{ib} g^{jc} g^{kd}
$$

$$
- \frac{1}{24} \left( (\nabla_p h_{ps}) g^{ms} \varphi_{mjk} + (\nabla_p h_{ps}) g^{ms} \varphi_{imk} + (\nabla_p h_{ps}) g^{ms} \varphi_{ijm} \right) \psi_{abcd} g^{ib} g^{jc} g^{kd}
$$

$$
+ \frac{1}{24} \left( \nabla_p X^m \right) \psi_{mijk} \psi_{abcd} g^{ib} g^{jc} g^{kd}
$$
Again exploiting the skew-symmetry of $\varphi$ and $\psi$ and relabelling indices, the above expression simplifies to

\[
\frac{3}{24} \left( \nabla_p \varphi_{mjk} \right) \psi_{qbed} g^{ib} g^{jc} g^{kd} + \frac{1}{24} X^m \left( \nabla_p \psi_{mijk} \right) \psi_{qbed} g^{ib} g^{jc} g^{kd} \\
+ \frac{3}{24} \left( \nabla h_{ip} \right) g^{ms} \varphi_{mjk} \psi_{qbed} g^{ib} g^{jc} g^{kd} - \frac{3}{24} \left( \nabla h_{ps} \right) g^{ms} \varphi_{mjk} \psi_{qbed} g^{ib} g^{jc} g^{kd} \\
+ \frac{1}{24} \left( \nabla_p X^m \right) \psi_{mijk} \psi_{qbed} g^{ib} g^{jc} g^{kd}
\]

Now we use Proposition A.17 on the second term, Lemma A.13 on the third and fourth terms, and Lemma A.14 on the fifth term above to obtain

\[
\frac{3}{24} \left( \nabla_p \varphi_{mjk} \right) \psi_{qbed} h^{mb} g^{jc} g^{kd} + \frac{3}{24} X^m \left( \nabla_p \varphi_{mjk} \right) \varphi_{qcd} g^{jc} g^{kd} + \left( \nabla h_{ps} \right) g^{ms} \varphi_{mjk} g^{ib} + \nabla_p X_q \quad (3.15)
\]

We now consider the second term of (3.15). Using Theorem 3.5. It is

\[
\frac{1}{24} \left( \nabla_p \varphi_{ijk} \right) \left( h^m \psi_{mbed} + h^m \psi_{qmed} + h^m \psi_{qbdm} + h^m \psi_{qbecm} \right) g^{ib} g^{jc} g^{kd} \\
+ \frac{1}{24} \left( \nabla_p \varphi_{ijk} \right) \left( -X_q \varphi_{bed} + X_b \varphi_{qcd} - X_{c} \varphi_{qbd} + X_{d} \varphi_{qbc} \right) g^{ib} g^{jc} g^{kd}
\]

As usual, by exploiting symmetries, this simplifies to

\[
\frac{1}{24} \left( \nabla_p \varphi_{ijk} \right) h^m \psi_{mbed} g^{ib} g^{jc} g^{kd} + \frac{3}{24} \left( \nabla_p \varphi_{ijk} \right) h^m \psi_{qmed} g^{ib} g^{jc} g^{kd} \\
- \frac{1}{24} X_q \left( \nabla_p \varphi_{ijk} \right) \varphi_{qbed} g^{ib} g^{jc} g^{kd} + \frac{3}{24} X_b \left( \nabla_p \varphi_{ijk} \right) \varphi_{qcd} g^{ib} g^{jc} g^{kd}
\]

The third term above vanishes by Proposition A.10. Therefore we see that the second term of (3.15) can be written as

\[
T_{pm} h^m_q + \frac{3}{24} \left( \nabla_p \varphi_{ijk} \right) \psi_{qmed} h^{im} g^{jc} g^{kd} + \frac{3}{24} X^i \left( \nabla_p \varphi_{ijk} \right) \varphi_{qcd} g^{jc} g^{kd} \quad (3.17)
\]

Finally, using equation (3.6), the third term of (3.15) is

\[
- \frac{6}{24} \left( \nabla_p \varphi_{ijk} \right) \psi_{qbed} h^{ib} g^{jc} g^{kd} \quad (3.18)
\]

Adding the expressions (3.16), (3.17), and (3.18) gives

\[
\frac{\partial}{\partial t} T_{pq} = T_{pm} h^m_q + \frac{1}{4} X^i \left( \nabla_p \varphi_{ijk} \right) \varphi_{qcd} g^{jc} g^{kd} + \left( \nabla h_{ps} \right) g^{ms} \varphi_{mjk} g^{ib} + \nabla_p X_q \quad (3.19)
\]

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Furthermore, the case of skew-symmetric matrices, this differs from their norm as 2-forms by a factor of 2.

Substituting this into (3.19) and relabelling some indices gives

\[ \frac{\partial}{\partial t} T_{pq} = T_{pl} g^{lm} h_{mq} + T_{pl} g^{lm} X_{mq} + (\nabla_k h_{ip}) g^{ka} g^{lb} \varphi_{abq} + \nabla_p X_q \]

which is what we wanted to prove.

We can now derive the evolution equations of the four independent torsion forms using Proposition 2.31. We use \( \langle \cdot, \cdot \rangle \) to denote the matrix norm: \( \langle A, B \rangle = A_{ij} B_{kl} g^{ik} g^{jl} \). Note that in the case of skew-symmetric matrices, this differs from their norm as 2-forms by a factor of 2. Furthermore, \[ [A, B]_{pq} = A_{pl} g^{lm} B_{mq} - B_{pl} g^{lm} A_{mq} \] is the matrix commutator while \( \{ A, B \}_{pq} = A_{pl} g^{lm} B_{mq} + B_{pl} g^{lm} A_{mq} \) is the anti-commutator.

**Proposition 3.9.** The evolution equations of the scalar torsion \( \tau_0 \) and the symmetric traceless torsion \( \tau_3 \) under the flow (3.1) are given by

\[ \frac{\partial}{\partial t} \tau_0 = -\frac{1}{7} \text{Tr}_g(h) \tau_0 + \frac{4}{7} \langle h, \tau_3 \rangle - \frac{4}{7} \langle X, \tau_1 \rangle + \frac{4}{7} g^{pq} (\nabla_p X_q) \]  

and

\[ \frac{\partial}{\partial t} (\tau_3)_{ij} = \left( -\frac{1}{28} \text{Tr}_g(h) \tau_0 + \frac{1}{7} \langle h, \tau_3 \rangle - \frac{1}{7} \langle X, \tau_1 \rangle + \frac{1}{7} g^{pq} (\nabla_p X_q) \right) g_{ij} \]

\[ + \frac{1}{4} \tau_0 h_{ij} + \frac{1}{2} \{ h, \tau_3 \}_{ij} + \frac{1}{2} \{ h, \tau_1 \}_{ij} - \frac{1}{4} [h, \tau_2]_{ij} \]

\[ - \frac{1}{2} [X, \tau_3]_{ij} - \frac{1}{2} \{ X, \tau_1 \}_{ij} + \frac{1}{4} \{ X, \tau_2 \}_{ij} \]

\[ - \frac{1}{2} (\nabla_k h_{ij}) g^{ka} g^{lb} \varphi_{abq} - \frac{1}{2} (\nabla_k h_{ij}) g^{ka} g^{lb} \varphi_{abi} \]

\[ - \frac{1}{2} (\nabla_j X_j + \nabla_i X_i) \]

**Proof.** We have \( \tau_0 = \frac{4}{7} g^{pq} T_{pq} \). Differentiating and using Theorem 3.8 and 3.9,

\[ \frac{\partial}{\partial t} \tau_0 = \frac{4}{7} \left( \frac{\partial}{\partial t} g^{pq} \right) T_{pq} + \frac{4}{7} g^{pq} \left( \frac{\partial}{\partial t} T_{pq} \right) \]

\[ = -\frac{8}{7} h^{pq} T_{pq} + \frac{4}{7} g^{pq} \left( T_{pl} g^{lm} h_{mq} + T_{pl} g^{lm} X_{mq} + (\nabla_k h_{ip}) g^{ka} g^{lb} \varphi_{abq} + \nabla_p X_q \right) \]

\[ = -\frac{4}{7} \langle T, h \rangle - \frac{4}{7} \langle T, X \rangle + \frac{4}{7} (\nabla_k h^{pq}) \varphi_{abq} + \frac{4}{7} g^{pq} (\nabla_p X_q) \]

where we have used the symmetry of \( h \) and the skew-symmetry of \( X \). The third term vanishes by the skew-symmetry of \( \varphi \). The result now follows by recalling that \( T_{lm} = \frac{1}{4} \tau_0 g_{lm} - (\tau_3)_{lm} + \)
\[
(\tau_1)_{lm} - \frac{1}{4} (\tau_2)_{lm} \text{ and that this decomposition is orthogonal with respect to the matrix inner product. Equation (3.21) is proved similarly using } (\tau_3)_{ij} = \frac{1}{4} \tau_0 g_{ij} - \frac{1}{2} (T_{ij} + T_{ji}) \text{ and substituting (3.20) into the computation. We omit the details.}
\]

Before we can compute the evolution equations for \( \tau_1 \) and \( \tau_2 \), we need the following preliminary result. Define a linear map \( P \) which takes 2-tensors to skew-symmetric 2-tensors by \( (P(A))_{pq} = A_{ij} g^{ia} g^{jb} \psi_{abpq} \). Clearly the symmetric 2-tensors are in the kernel of \( P \), and by Proposition 2.40 we have \( P(C_7 + C_{14}) = -4 C_7 + 2 C_{14} \) where \( C_7 \in \Omega^2_7 \) and \( C_{14} \in \Omega^2_{14} \).

**Lemma 3.10.** If \( C = C_7 + C_{14} \) is a skew-symmetric tensor, then the evolution of the skew-symmetric tensor \( P(C) \) under the flow (3.1) is given by

\[
\frac{\partial}{\partial t} (P(C))_{ij} \equiv (P \frac{\partial}{\partial t} C)_{ij} + 6 \pi_7 ([h, C_{14}])_{ij} - 6 \pi_{14} ([h, C_7])_{ij} - 2 \pi_7 ([X, C_{14}])_{ij} + 2 \pi_{14} ([X, C_7])_{ij}
\]

where \( \pi_7 \) and \( \pi_{14} \) denote the projections onto \( \Omega^2_7 \) and \( \Omega^2_{14} \), respectively.

**Proof.** Using (3.5) and (3.10), we see that \( \frac{\partial}{\partial t} \left( C_{ab} g^{ap} g^{bq} \psi_{pqij} \right) \) equals

\[
\left( \frac{\partial}{\partial t} C_{ab} \right) g^{ap} g^{bq} \psi_{pqij} + 2 C_{ab} \left( \frac{\partial}{\partial t} g^{ap} \right) g^{bq} \psi_{pqij} + C_{ab} g^{ap} g^{bq} \left( \frac{\partial}{\partial t} \psi_{pqij} \right)
\]

\[
= (P \frac{\partial}{\partial t} C)_{ij} - 4 C_{ab} h^{ap} g^{bq} \psi_{pqij} + C_{ab} g^{ap} g^{bq} (h^i_j \psi_{pqij} + h^i_j \psi_{pqij})
\]

\[
+ C_{ab} g^{ap} g^{bq} (h^i_j \psi_{pqij} + h^i_j \psi_{pqij} - X_i \psi_{pqij} + X_i \psi_{pqij} - X_j \psi_{pqij})
\]

\[
= (P \frac{\partial}{\partial t} C)_{ij} - 2 C_{ab} h^{ap} g^{bq} \psi_{pqij} + h^i_j (P(C))_{ij} + (P(C))_{ij} h^i_j
\]

\[
+ 2 (C_{ab} X^b g^{ap}) \psi_{pqij} - 6 (C_7)_i X_i + 6 (C_7)_j X_j
\]

where we have used the skew-symmetry of \( C \) and \( \phi \) and relabeled indices to combine terms. The second term above can be written as

\[
- 2 h_{al} g^{lm} C_{mb} g^{ap} g^{bq} \psi_{pqij} = - (h_{al} g^{lm} C_{mb} + C_{al} g^{lm} h_{mb}) g^{ap} g^{bq} \psi_{pqij}
\]

\[
= - \{h, C\}_{ab} g^{ap} g^{bq} \psi_{pqij} = - P \{h, C\}_{ij} = 4 \pi_7 \{h, C\}_{ij} - 2 \pi_{14} \{h, C\}_{ij}
\]

\[
= 4 \pi_7 \{h, C_7\}_{ij} + 4 \pi_7 \{h, C_{14}\}_{ij} - 2 \pi_{14} \{h, C_7\}_{ij} - 2 \pi_{14} \{h, C_{14}\}_{ij}
\]

Meanwhile the third and fourth terms of (3.23) become

\[
\{h, P(C)\}_{ij} = \{h, -4 C_7 + 2 C_{14}\}_{ij}
\]

\[
= - 4 \pi_7 \{h, C_7\}_{ij} + 2 \pi_7 \{h, C_{14}\}_{ij} - 4 \pi_{14} \{h, C_7\}_{ij} + 2 \pi_{14} \{h, C_{14}\}_{ij}
\]

Combining these expressions, after some cancellation we see that

\[
\frac{\partial}{\partial t} \left( C_{ab} g^{ap} g^{bq} \psi_{pqij} \right) = (P \frac{\partial}{\partial t} C)_{ij} + 6 \pi_7 ([h, C_{14}])_{ij} - 6 \pi_{14} ([h, C_7])_{ij}
\]

\[
+ 2 (C_{ab} X^b g^{ap}) \psi_{pqij} - 6 (C_7)_i X_i + 6 (C_7)_j X_j
\]
Consider now the third to last term above. In the notation of Proposition 2.12, this is $2C(X)_{ij}$, which by (2.15) and (2.14) is

$$2(C_7)(X)_{ij} + C_{14}(X)_{ij} = 2(-\frac{1}{2}[C_7, X]_{ij} - \frac{3}{2}(C_7)_iX_j + \frac{3}{2}(C_7)_jX_i + [C_{14}, X]_{ij})$$

$$= -[C_7, X]_{ij} + 2[C_{14}, X]_{ij} - 3(C_7)_iX_j + 3(C_7)_jX_i$$

Hence the final three terms of (3.24) are

$$- [C_7, X]_{ij} + 2[C_{14}, X]_{ij} + 3(C_7)_iX_j - 3(C_7)_jX_i$$

using (2.16). By equation (2.18), the first and third terms above combine to give $-2\pi_{14}([C_7, X]_{ij})$. Also, equation (2.14) says $[C_{14}, X] = \pi([C_{14}, X])$. Equation (3.22) now follows since the commutator $[\cdot, \cdot]$ is skew-symmetric in its arguments. □

**Proposition 3.11.** The evolution equations of the vector torsion $\tau_1$ (as a $\Omega_T^2$-form) and the Lie algebra torsion $\tau_2$ under the flow (3.1) are given by

$$\frac{\partial}{\partial t}(\tau_1)_{ij} = \pi_7(\frac{1}{2}[h, \tau_3]_{ij} + \frac{1}{2}[h, \tau_1]_{ij} + \frac{1}{4}\{h, \tau_2\}_{ij}) + \pi_{14}([h, \tau_1]_{ij})$$

$$+ \frac{1}{4}\tau_0X_{ij} + \pi_7(-\frac{1}{2}[X, \tau_3]_{ij} - \frac{1}{2}[X, \tau_1]_{ij} + \frac{1}{12}[X, \tau_2]_{ij}) + \pi_{14}(-\frac{1}{3}[X, \tau_1]_{ij})$$

$$- \frac{1}{6}g^{pq}(\nabla_p h_{qk})g^{kl}\varphi_{lij} + \frac{1}{6}(\nabla_k \text{Tr}_g(h))g^{kl}\varphi_{lij}$$

$$+ \frac{1}{6}(\nabla_p X_q)g^{pq}g^{gh}\varphi_{abk}g^{kl}\varphi_{lij} \tag{3.25}$$

and

$$\frac{\partial}{\partial t}(\tau_2)_{ij} = \pi_7([h, \tau_2]_{ij} + \pi_{14}(-[h, \tau_3]_{ij} + [h, \tau_1]_{ij} + \frac{1}{2}\{h, \tau_2\}_{ij})$$

$$+ \pi_7(-\frac{1}{3}[X, \tau_2]_{ij} + \pi_{14}([X, \tau_3]_{ij} + \frac{1}{3}[X, \tau_1]_{ij})$$

$$- (\nabla_k h_{li})g^{ka}g^{ib}\varphi_{abi} + (\nabla_k h_{li})g^{ka}g^{ib}\varphi_{abi}$$

$$- \frac{1}{3}g^{pq}(\nabla_p h_{qk})g^{kl}\varphi_{lij} + \frac{1}{3}(\nabla_k \text{Tr}_g(h))g^{kl}\varphi_{lij}$$

$$- \nabla_i X_j + \nabla_j X_i + \frac{1}{3}(\nabla_p X_q)g^{pq}g^{gh}\varphi_{abk}g^{kl}\varphi_{lij} \tag{3.26}$$

**Proof.** This is a tedious but straightforward computation. By Proposition 2.31

$$\frac{\partial}{\partial t}(\tau_1)_{ij} = \frac{1}{3}\left(\frac{\partial}{\partial t}C_{ij}\right) - \frac{1}{6}\left(\frac{\partial}{\partial t}(P(C))_{ij}\right)$$

$$\frac{\partial}{\partial t}(\tau_2)_{ij} = -\frac{4}{3}\left(\frac{\partial}{\partial t}C_{ij}\right) - \frac{1}{3}\left(\frac{\partial}{\partial t}(P(C))_{ij}\right)$$

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where \( C_{ij} = \frac{1}{2}(T_{ij} - T_{ji}) = (\tau_1)_{ij} - \frac{1}{2}(\tau_2)_{ij}. \) Now we use Theorem 3.8 together with Lemma 3.10 and repeated applications of Proposition 2.6. We also need Lemmas A.12 and A.13 to put the terms involving \( \nabla h \) and \( \nabla X \) in the form given in (3.25) and (3.26). We omit the details.

**Remark 3.12.** The evolution equations given in Propositions 3.9 and 3.11 are clearly quite complicated. Ideally one could find an \( h_{ij} \) and \( X^k \) depending on the four torsion tensors (and their covariant derivatives) for which these evolution equations become simpler. The sequel [16] to this paper will discuss several specific flows of \( G_2 \)-structures.

### 4 Bianchi-type identities in \( G_2 \)-geometry

In this section, we apply the evolution equations from Section 3 to derive Bianchi-type identities for manifolds with \( G_2 \)-structure. As a consequence, we obtain explicit formulas for the Ricci tensor and part of the Riemann curvature tensor in terms of the full torsion tensor. This leads to new simple proofs of some known results in \( G_2 \)-geometry.

#### 4.1 Identities via diffeomorphism invariance

Let \( \alpha \) be a smooth tensor on a manifold \( M \), defined entirely in terms of some other tensor \( \beta \). We write \( \alpha = \alpha[\beta] \). Suppose \( f_t \) is a one-parameter family of diffeomorphisms of \( M \) such that \( f_0 \) is the identity, and \( \frac{d}{dt} f_t = Y \) for some smooth vector field \( Y \) on \( M \). Then, by diffeomorphism invariance, we have

\[
f_t^*(\alpha[\beta]) = \alpha[f_t^*(\beta)]
\]

where \( f_t^* \) denotes the pullback by \( f_t \). Differentiating with respect to \( t \) and setting \( t = 0 \) gives

\[
\mathcal{L}_Y \alpha = (D\alpha)[\mathcal{L}_Y \beta] \tag{4.1}
\]

where \( D\alpha \) is the linearization of the tensor \( \alpha \) as a function of \( \beta \). Explicitly, if \( \beta(t) \) satisfies \( \frac{d}{dt}\beta(t) = \beta_1 \), then \( (D\alpha)(\beta_1) = \frac{d}{dt}(\alpha[\beta(t)]) \). Since (4.1) holds for any vector field \( Y \), this yields identities involving \( \alpha \). This idea was exploited by Kazdan in [17] to show that the first and second Bianchi identities of Riemannian geometry were consequences of the diffeomorphism invariance of the Riemann curvature tensor \( R_{ijkl} \) as a function of the metric \( g_{ij} \). A discussion can be found in [5].

We will apply this idea to the setting of \( G_2 \)-geometry, to derive ‘Bianchi-type’ identities. First, we note that from (4.1), we have

\[
(\mathcal{L}_Y \varphi)_{ijk} = (\nabla_Y \varphi)_{ijk} + (\nabla_i Y^l)\varphi_{jk} + (\nabla_j Y^l)\varphi_{ik} + (\nabla_k Y^l)\varphi_{ij} \tag{4.2}
\]

This should be compared to the well-known analogous expression for the Riemannian metric:

\[
(\mathcal{L}_Y g)_{ij} = (\nabla_i Y^l)g_{lj} + (\nabla_j Y^l)g_{il} = \nabla_i Y_j + \nabla_j Y_i \tag{4.3}
\]

which also follows from (1.7) since \( \nabla g = 0 \).

**Proposition 4.1.** The diffeomorphism invariance of the metric \( g \) as a function of the 3-form \( \varphi \) is equivalent to the vanishing of the \( \Omega^3_1 \oplus \Omega^3_{17} \) component of \( \nabla_Y \varphi \) for any vector field \( Y \). This is the fact which was proved earlier in Lemma 2.24. See also the discussion following Remark 2.22.
Proof. Equation (4.1) for $g[\varphi]$ says

$$(\mathcal{L}_Y g)_{ij} = \nabla_i Y_j + \nabla_j Y_i = (Dg)[\mathcal{L}_Y \varphi]$$

(4.4)

By equation (3.5), the right hand side above equals $2h_{ij}$ where the symmetric 2-tensor $h_{ij}$ and the vector field $X^k$ are defined by the decomposition of the 3-form $\mathcal{L}_Y \varphi$ as

$$(\mathcal{L}_Y \varphi)_{ijk} = \frac{1}{2} h_{im} g^{ml} \varphi_{ljk} + X^m \psi_{mijk}$$

(4.5)

Consider the last three terms of (4.2). They can be written as

$$A_{im} g^{ml} \varphi_{ljk} + A_{jm} g^{ml} \varphi_{ilk} + A_{km} g^{ml} \varphi_{ijl}$$

(4.6)

where $A_{ij} = \nabla_i Y_j$. From the discussion following Remark 2.8, the $\Omega_7^2 \oplus \Omega_{27}^3$ component of this expression is $\frac{1}{2} S_{im} g^{ml} \varphi_{ljk}$ where $S_{ij} = \frac{1}{2}(A_{ij} + A_{ji}) = \frac{1}{2}(\nabla_i Y_j + \nabla_j Y_i)$. Since $2S_{ij}$ is already equal to the left hand side of (4.4), we see that equation (4.4) holds for all $Y$ if and only if the $\Omega_7^2 \oplus \Omega_{27}^3$ component of $\nabla_Y \varphi$ vanishes for all $Y$.

We pause here to note that this result enables us to explicitly describe the decomposition of $\mathcal{L}_Y \varphi$ into a symmetric tensor $h_{ij}$ and a vector field $X^k$ as given in (4.5). We already remarked in the proof of Proposition 4.1 that

$$h_{ij} = \frac{1}{2} (\nabla_i Y_j + \nabla_j Y_i)$$

(4.7)

Also, from (2.19) and (4.6), we see that the $\Omega_7$ component of the last three terms of (4.2) is

$$Z^n \psi_{nijk}$$

where

$$Z^n = -\frac{1}{2} \left( \frac{1}{2} (\nabla_b Y_a - \nabla_a Y_b) \right) g^{ai} g^{bj} \varphi_{ijk} g^{kn}$$

$$= -\frac{1}{2} (\nabla_b Y_a) g^{ai} g^{bj} \varphi_{ijk} g^{kn}$$

using the skew-symmetry of $\varphi_{ijk}$. Now combining this with $\nabla_i \varphi_{ijk} = T_{lm} g^{mn} \psi_{nijk}$, we see that $X^k$ is

$$X^k = Y^i T_{lm} g^{mk} - \frac{1}{2} (\nabla_b Y_a) g^{ai} g^{bj} \varphi_{ijm} g^{mk}$$

(4.8)

in equation (4.16).

Using (2.28) and Theorem 3.5 one can consider the analogous calculation of diffeomorphism invariance for the 4-form $\psi$ as a function of $\varphi$. It is straightforward to check that in this case no new information is obtained. Therefore we turn our attention to the full torsion tensor $T_{lm}$.

**Theorem 4.2.** The diffeomorphism invariance of the full torsion tensor $T$ as a function of the 3-form $\varphi$ is equivalent to the following identity:

$$\nabla_l T_{jl} - \nabla_j T_{ll} = T_{la} T_{jb} g^{am} g^{bn} \varphi_{mla} + \frac{1}{2} R_{ijab} g^{am} g^{bn} \varphi_{mnl}$$

(4.9)

Proof. Equation (4.1) for $T[\varphi]$ says

$$(\mathcal{L}_Y T)_{ij} = Y^l (\nabla_l T_{ij}) + (\nabla_i Y^l) T_{lj} + (\nabla_j Y^l) T_{il} = (DT)[\mathcal{L}_Y \varphi]$$

(4.10)
By Theorem 3.8, the right hand side above equals

\[ T_{il}g^{lm}h_{mj} + T_{il}g^{lm}X_{mj} + (\nabla_k h_{li})g^{ka}g^{lb}\varphi_{abj} + \nabla_l X_j \]  (4.11)

where the symmetric 2-tensor \( h_{ij} \) and the vector field \( X^k \) are given by (4.7) and (4.8), respectively. Remark 2.8 shows that

\[ X_{mj} = X^k\varphi_{kmj} = Y^l T_{ln}g^{nk}\varphi_{kmj} - \frac{1}{2} (\nabla_a Y_b)g^{ap}g^{aq}\varphi_{pqmn}g^{nk}\varphi_{kmj} \]

\[ = Y^l T_{ln}g^{nk}\varphi_{kmj} - \frac{1}{2} (\nabla_a Y_b)g^{ap}g^{aq}(g_{pm}g_{qj} - g_{pj}g_{qm} - \psi_{pqmj}) \]

\[ = Y^l T_{ln}g^{nk}\varphi_{kmj} - \frac{1}{2} \nabla_m Y_j + \frac{1}{2} \nabla_j Y_m + \frac{1}{2} (\nabla_a Y_b)g^{ap}g^{aq}\psi_{pqmj} \]  (4.12)

Substitute (4.7), (4.8), and (4.12) into (4.11) to obtain

\[ T_{il}g^{lm}\left(\frac{1}{2} (\nabla_m Y_j + \nabla_j Y_m)\right) + T_{il}g^{lm}\left(Y^b T_{bn}g^{nk}\varphi_{kmj} - \frac{1}{2} \nabla_m Y_j + \frac{1}{2} \nabla_j Y_m + \frac{1}{2} (\nabla_a Y_b)g^{ap}g^{aq}\psi_{pqmj}\right) \]

\[ + \nabla_k \left(\frac{1}{2} (\nabla_l Y_i + \nabla_l Y_i)\right)g^{ka}g^{lb}\varphi_{abj} + \nabla_l \left(Y^b T_{bj} - \frac{1}{2} (\nabla_a Y_b)g^{ap}g^{aq}\psi_{pqj}\right) \]

Now we expand the above expression using \( \nabla_i \varphi_{abc} = T_{im}g^{mn}\psi_{nabc} \) and the product rule, and collect terms. After some cancellation, we are left with

\[ (\nabla_l Y^l)T_{il} + (\nabla_l Y^b)T_{bj} + Y^b (\nabla_l T_{bj}) + Y^b T_{bn}T_{il}g^{nk}\varphi_{kmj} \]

\[ + \frac{1}{2} (\nabla_l \nabla Y_i + \nabla_l \nabla Y_i)g^{ka}g^{lb}\varphi_{abj} - \frac{1}{2} (\nabla_l \nabla Y_b)g^{ap}g^{aq}\varphi_{pqj} \]

for the right hand side of (4.10). Substituting this expression into (4.10), cancelling terms, rearranging, and relabeling indices gives

\[ Y^l (\nabla_l T_{ij}) - Y^l (\nabla_l T_{ij}) = Y^l T_{la}T_{lb}g^{am}g^{bn}\varphi_{mnj} \]

\[ + \frac{1}{2} (\nabla_k \nabla Y_i + \nabla_k \nabla Y_i - \nabla \nabla Y_i)g^{ka}g^{lb}\varphi_{abj} \]  (4.13)

Consider the last term in (4.13). Using the Ricci identity (1.9) and the skew-symmetry of \( \varphi \), we have

\[ \frac{1}{2} (\nabla_k \nabla Y_i + \nabla_k \nabla Y_i)g^{ka}g^{lb}\varphi_{abj} = \frac{1}{4} (\nabla_k \nabla Y_i - \nabla \nabla Y_i)g^{ka}g^{lb}\varphi_{abj} \]

\[ = -\frac{1}{4} R_{klim}Y^{m}g^{ka}g^{lb}\varphi_{abj} \]

and also

\[ \frac{1}{2} (\nabla_k \nabla Y_i - \nabla \nabla Y_i)g^{ka}g^{lb}\varphi_{abj} = \frac{1}{2} R_{kilm}Y^{m}g^{ka}g^{lb}\varphi_{abj} \]

\[ = -\frac{1}{4} (R_{kilm} - R_{kilm})Y^{m}g^{ka}g^{lb}\varphi_{abj} \]

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Putting these two expressions together, the last term of \((4.13)\) becomes

\[
-\frac{1}{4}(R_{klm} + R_{kilm} - R_{lkm})Y^m g^{ka} g^{lb} \varphi_{abj} = -\frac{1}{4}(-R_{mikl} + R_{mlik} + R_{mkli})Y^m g^{ka} g^{lb} \varphi_{abj} = -\frac{1}{4}(-R_{mikl} - R_{mlik})Y^m g^{ka} g^{lb} \varphi_{abj} = \frac{1}{2}R_{mikl}Y^m g^{ka} g^{lb} \varphi_{abj}
\]

where we have used the symmetries of \(R_{ijkl}\) in the first line, and the Riemannian first Bianchi identity \((1.8)\) in the second line. Hence \((4.13)\) becomes

\[
Y^l(\nabla_l T_{ij}) - Y^l(\nabla_i T_{lj}) = Y^l T_{la} T_{ib} g^{am} g^{bn} \varphi_{mnj} + \frac{1}{2} Y^m R_{mikl} g^{ka} g^{lb} \varphi_{abj}
\]

Since this must hold for any \(Y\), after relabelling indices this is exactly \((4.9)\).

Remark 4.3. The identity \((4.9)\) can also be established directly by using \((2.29), (2.28), \) Lemma \(A.13\), and the Ricci identities. However the proof above shows that \((4.9)\) is equivalent to the diffeomorphism invariance of the full torsion tensor \(T_{lm}\), and as such it deserves to be called a Bianchi-type identity for \(G_2\)-geometry.

Corollary 4.4. The following identity holds

\[
\frac{7}{4} \nabla_i \tau_0 - g^{jl} \nabla_j T_{il} = -6 T_{ia} \tau_a - \frac{1}{2} Y_{ab} R_{ijkl} g^{ka} g^{lb} \varphi_{mnj}
\]

We call it the contracted Bianchi-type identity for \(G_2\)-geometry.

Proof. This follows easily by contracting \((4.9)\) on \(j\) and \(l\), and using the fact that \(T_{lm} = \frac{1}{4} \tau_0 g_{lm} - (\tau_3)_{lm} + (\tau_1)_{lm} - \frac{1}{2} (\tau_2)_{lm}\) and equation \((2.11)\) for \(\tau_1\).

Remark 4.5. The usefulness of \((4.14)\) is that it shows that the ‘divergence-like’ expression \(g^{jl} \nabla_j T_{il}\) is equal to a gradient plus zero order terms in the torsion and curvature. This can potentially be used to simplify expressions, in the same way that the contracted second Bianchi identity of Riemannian geometry is used in Ricci flow.

4.2 Curvature formulas in terms of the torsion tensor

We now examine some consequences of Theorem 4.2. For \(i\) and \(j\) fixed, the Riemann curvature tensor \(R_{ijkl}\) is skew-symmetric in \(k\) and \(l\). Hence we can use \((2.9)\) and \((2.10)\) to decompose it as

\[
R_{ijkl} = (\pi_7(\text{Riem}))_{ijkl} + (\pi_{14}(\text{Riem}))_{ijkl}
\]

where

\[
(\pi_7(\text{Riem}))_{ijkl} = \frac{1}{3} R_{ijkl} - \frac{1}{6} R_{ijab} g^{ap} g^{bq} \psi_{pqkl} \quad (4.15)
\]

\[
(\pi_{14}(\text{Riem}))_{ijkl} = \frac{2}{3} R_{ijkl} + \frac{1}{6} R_{ijab} g^{ap} g^{bq} \psi_{pqkl} \quad (4.16)
\]

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Furthermore, by Remark 2.8 we also know that

\[(\pi_7(\text{Riem}))_{ijkl} = (\pi_7(\text{Riem}))_{ij}^m \varphi_{mkl}\]  

(4.17)

where

\[(\pi_7(\text{Riem}))_{ij}^m = \frac{1}{6} R_{ijkl} g^{ka} g^{lb} \varphi_{abc} g^{cm}\]

Thus the identity (4.18) can be expressed as

\[3 (\pi_7(\text{Riem}))_{ij} g_{ml} = \nabla_i \nabla_j T_{jl} - \nabla_j \nabla_i T_{il} - T_{ia} T_{jb} g^{am} g^{bn} \varphi_{mnl}\]  

(4.18)

Remark 4.6. The fact that the \(\Omega_2^7\) part of the Riemann curvature tensor can be expressed entirely in terms of the full torsion tensor \(T_{lm}\) was first demonstrated using frame bundle calculations in section [4.5] of [2].

Corollary 4.7. If the \(G_2\)-structure \(\varphi\) is torsion-free, then the Riemann curvature tensor \(R_{ijkl} \in \text{Sym}^2(\Omega_{14}^2)\) actually takes values \(\text{Sym}^2(\Omega_{14}^2)\), where \(\Omega_{14}^2 \cong \mathfrak{g}_2\), the Lie algebra of \(G_2\).

Proof. Setting \(T = 0\) in (4.18) shows the for fixed \(i, j\), we have \(R_{ijkl} \in \Omega_{14}^2\) as a skew-symmetric tensor in \(k, l\). The result now follows from the symmetry \(R_{ijkl} = R_{klij}\).

Remark 4.8. This result is well-known. When \(T = 0\), the three form \(\varphi\) is parallel and the Riemannian holonomy of the associated metric is contained in the group \(G_2\). It is a general fact that the Riemann curvature tensor of a metric with holonomy contained in a group \(G\) is an element of \(\text{Sym}^2(g)\), where \(g\) is the Lie algebra of \(G\). Here we have a direct proof of this fact in the \(G_2\) case.

Corollary 4.9. Let \(Q_{ijkl} = R_{ijab} g^{ap} g^{bq} \psi_{pqkl}\). Then we have \(Q_{ijkl} g^{il} = 0\).

Proof. We use the Riemannian Bianchi identity (4.8) and compute:

\[
Q_{ijkl} g^{il} = R_{ijab} g^{ap} g^{bq} \psi_{pqkl} g^{il} \\
= -(R_{jaib} + R_{aijb}) g^{ap} g^{bq} \psi_{pqkl} g^{il} \\
= R_{aijk} g^{ap} g^{bq} \psi_{pqkl} g^{il} + R_{bija} g^{ap} g^{bq} \psi_{pqkl} g^{il} \\
= -R_{aijk} g^{il} g^{bq} \psi_{lqkp} g^{ap} - R_{bija} g^{ap} g^{il} \psi_{pilkq} g^{bq} \\
= -Q_{ajaq} g^{ap} - Q_{bjaq} g^{bq}
\]

and thus \(3 Q_{ijkl} g^{il} = 0\).

Remark 4.10. This is essentially Proposition [3.2] in Cleyton-Ivanov [7].

Corollary 4.11. The Ricci tensor \(R_{jk}\) can be expressed as

\[
R_{jk} = R_{ijkl} g^{il} = 3 (\pi_7(\text{Riem}))_{ijkl} g^{il} = \frac{3}{2} (\pi_{14}(\text{Riem}))_{ijkl} g^{il}
\]

Proof. This follows immediately from Lemma 4.9 and equations (4.15) and (4.16).

Corollary 4.12. The metric of a torsion-free \(G_2\)-structure is necessarily Ricci-flat.

Proof. From Corollary 4.7 we have \(\pi_7(\text{Riem}) = 0\). The result then follows from Corollary 4.11.
Proposition 4.15. Given a $G_2$-structure $\varphi$ with full torsion tensor $T_{im}$, its associated metric $g$ has Ricci curvature $R_{jk}$ given by

$$R_{jk} = (\nabla_j T_{jm} - \nabla_j T_{im}) \varphi_{nkl} g^{mn} g^{il} - T_{jl} g^{il} T_{ik} + \text{Tr}_g(T) T_{jk} - T_{jb} T_{ia} g^{il} g^{ap} \psi_{lpqk} g^{bq}$$  \hspace{1cm} (4.19)$$

This can also be written in the form

$$R_{jk} = (\nabla_j (T_{jm} \varphi_{nkl} g^{mn} g^{il}) - \nabla_j (T_{im} \varphi_{nkl} g^{mn} g^{il}) - T_{jl} g^{il} T_{ik} + \text{Tr}_g(T) T_{jk} + T_{jb} T_{ia} g^{il} g^{ap} \psi_{lpqk} g^{bq}$$  \hspace{1cm} (4.20)$$

(note the change of sign on the last term.)

Proof. We use Corollary [4.11] and equations (4.17) and (4.18) to compute:

$$R_{jk} = (\nabla_j T_{jm} - \nabla_j T_{im} - T_{ia} T_{jb} g^{ap} g^{bq} \varphi_{pqm}) g^{mn} \varphi_{nkl} g^{il}$$

$$= (\nabla_j T_{jm} - \nabla_j T_{im}) \varphi_{nkl} g^{mn} g^{il} - T_{ia} T_{jb} g^{ap} g^{bq} g^{il} (g_{pqk} g_{ql} - g_{plqk} - \psi_{pqkl})$$

$$= (\nabla_j T_{jm} - \nabla_j T_{im}) \varphi_{nkl} g^{mn} g^{il} - T_{ik} T_{jl} g^{il} + T_{il} T_{jk} g^{il} - T_{jb} T_{ia} g^{il} g^{ap} \psi_{lpqk} g^{bq}$$

using Lemma [A.12]. To obtain (4.20), we write

$$\nabla_j (T_{jm} \varphi_{nkl} g^{mn} g^{il}) = \nabla_j (T_{jm} \varphi_{nkl} g^{mn} g^{il}) - T_{jm} (\nabla_i \varphi_{nkl}) g^{mn} g^{il}$$

$$= \nabla_j (T_{jm} \varphi_{nkl} g^{mn} g^{il}) - T_{jm} T_{ip} g^{pq} \psi_{lpqk} g^{mn} g^{il}$$

and similarly for the other term involving $\nabla T$. The two extra terms add up and are negatives of the last term in (4.19). The details are left to the reader. \hfill \Box
Remark 4.16. Robert Bryant derived a formula, equation [4.27] in [2], for the Ricci tensor in terms of the intrinsic torsion in the language of frame bundles and canonical $G_2$-connections. Our formula (4.19) is an equivalent expression in the language of local coordinates. Additionally, equation [4.30] in [2] expresseses the Ricci tensor in terms of the four independent torsion forms and the representation-theoretic operators ‘$Q$’ and ‘$j$.’ Our expression for $R_{jk}$ in terms of $T_{lm}$ can also be combined with Theorem 2.27 to obtain an expression for $R_{jk}$ in terms of $\tau_0$, $\tau_1$, $\tau_2$, and $\tau_3$.

Remark 4.17. Cleyton and Ivanov [7] also derived a formula for the Ricci tensor for closed $G_2$-structures in terms of $\delta \varphi$.

**Corollary 4.18.** Given a $G_2$-structure $\varphi$ with full torsion tensor $T_{lm}$, its associated metric $g$ has scalar curvature $R$ given by

$$R = -12 g^{il} \nabla_l (\tau_j) + \frac{21}{8} \tau_0^2 - |\tau_3|^2 + 5 |\tau_1|^2 - \frac{1}{4} |\tau_2|^2$$  \hspace{1cm} (4.21)

where $|A|^2 = A_{ij} A_{kl} g^{ik} g^{jl}$ is the matrix norm.

**Proof.** We use (4.20) and trace $R = g^{jk} R_{jk}$ to obtain $R = -2 g^{il} \nabla_l (T_{jm} \varphi_{kn} g^{jk} g^{mn}) - T_{jl} T_{ik} g^{il} g^{jk} + (Tr_g(T))^2 + T_{jb} (T_{ai} g^{il} g^{aq} \psi_{lpqk}) g^{bk} g^{jk}$  \hspace{1cm} (4.22)

Now recall that $T_{lm} = \frac{1}{4} \tau_0 g_{lm} - (\tau_3)_{lm} + (\tau_1)_{lm} - \frac{1}{2} (\tau_2)_{lm}$ from Theorem 2.27. Using the fact that the four components of $T_{lm}$ are mutually orthogonal, and that $\tau_3$ is symmetric while $\tau_1$ and $\tau_2$ are skew-symmetric, the second term in (4.22) becomes

$$- \left( \frac{1}{4} \tau_0 g_{jl} - (\tau_3)_{jl} + (\tau_1)_{jl} - \frac{1}{2} (\tau_2)_{jl} \right) \left( \frac{1}{4} \tau_0 g_{ik} - (\tau_3)_{ik} + (\tau_1)_{ik} - \frac{1}{2} (\tau_2)_{ik} \right) g^{li} g^{jk}$$

$$= - \frac{7}{16} \tau_0^2 - |\tau_3|^2 + |\tau_1|^2 + \frac{1}{4} |\tau_2|^2$$

The third term is $\frac{7}{8} \tau_0^2$. We use Proposition 2.6 on the third term to write it as

$$T_{jb} \left( \left( \frac{1}{4} \tau_0 g_{ia} - (\tau_3)_{ia} + (\tau_1)_{ia} - \frac{1}{2} (\tau_2)_{ia} \right) g^{il} g^{ap} \psi_{lpqk} \right) g^{bk} g^{jk}$$

$$= T_{jb} (0 + 0 - 4 (\tau_1)_{qk} - (\tau_2)_{qk}) g^{bk} g^{jk} = 4 |\tau_1|^2 - \frac{1}{2} |\tau_2|^2$$

Combining these three expressions gives the last four terms of (4.21). Finally, for the first term of (4.22) we note that by Proposition 2.0 and (2.11), we have

$$T_{jm} \varphi_{kn} g^{jk} g^{mn} = (\tau_1)_{jm} \varphi_{kn} g^{jk} g^{mn} = 6 (\tau_1)_{j}$$

and the proof is complete. \hfill $\square$

Remark 4.19. The equation (4.21) agrees exactly with equation [4.28] of [2]. It looks different because Bryant is using the differential form norms, while we use the matrix norms in our equation (4.21).
We close this section by considering a specific subclass of $G_2$-structures, namely those for which
the only nonvanishing component of the torsion tensor is $\tau_0$, the scalar torsion. These are sometimes
called nearly $G_2$ manifolds. In this case we have $d\psi = 0$ and $d\phi = \rho_0 \psi$, so differentiating the second
equation gives $d\tau_0 \wedge \psi = 0$, and hence $d\tau_0 = 0$, since wedge product with $\psi$ is an isomorphism from
$\Omega^1$ to $\Omega^2$. Thus $\tau_0$ is constant and hence $\nabla T_{jl} = 0$ in this case. Substituting into (4.20) gives

$$R_{jk} = \frac{3}{8} \tau_0^2 g_{jk}$$

and we recover the known fact that nearly $G_2$ manifolds are always Einstein with positive Einstein constant. The squashed 7-sphere $S^7$ is an example of such a manifold.

A Identities involving $\phi$ and $\psi$

In this section we derive several useful identities involving the 3-form $\phi$ and the 4-form $\psi$ for a
manifold with a $G_2$-structure. Some of these have appeared previously in [7] and in [2]. Here we
give proofs of them as well as some new identities. Some identities are described invariantly and
others in terms of arbitrary local coordinates.

A.1 Basic relations of $G_2$-geometry

The main ingredients for deriving the identities of $G_2$-geometry are the following:

**Lemma A.1.** The metric $g$, cross product $\times$, and 3-form $\phi$ satisfy the following relations:

\[
\begin{align*}
g(u \times v, w) &= \phi(u, v, w) \quad \text{(A.1)} \\
(u \times v)^\flat &= v \cdot u \cdot \phi = * (u^\flat \wedge v^\flat \wedge \psi) \quad \text{(A.2)} \\
u \times (v \times w) &= -g(u, v)w + g(u, w)v - (u \cdot v \cdot w \cdot \psi)^\sharp \quad \text{(A.3)}
\end{align*}
\]

where $u, v, w$ are vector fields and $v^\flat$ denotes the 1-form which is metric dual to $v$.

**Proof.** See, for example, [13] for a proof. Note that in [13] there is a sign error in [A.3]. With the
sign convention used in that paper, the last term should have a plus sign instead of a minus sign.
In this paper we use the opposite orientation convention, and so [A.3] has a minus sign in front of
the last term. See also [15] for more about sign conventions and orientations. \[\square\]

From Lemma [A.1] we get:

**Corollary A.2.** Let $a, b, c, d$ be vector fields. Then

$$g(a \times b, c \times d) = g(a \wedge b, c \wedge d) - \psi(a, b, c, d)$$

**Proof.** We compute

\[
\begin{align*}
g(a \times b, c \times d) &= \phi(a, b, c \times d) \\
&= -\phi(a, c \times d, b) \\
&= -g(a \times (c \times d), b) \\
&= -g(-g(a, c)d + g(a, d)c - (a \cdot c \cdot d \cdot \psi)^\sharp, b) \\
&= g(a, c)g(b, d) - g(a, d)g(b, c) + \psi(d, c, a, b)
\end{align*}
\]
which is what we wanted to prove.

We will also have occasion to use the following relations, whose proofs can be found in [14].

**Proposition A.3.** Let \( \alpha \) be a 1-form on \( M \), let \( w \) be a vector field on \( M \), and \( w^b \) be the 1-form dual to \( w \). Then the following relations hold:

\[
\begin{align*}
|\varphi|^2 &= 7 \\
|\varphi \wedge \alpha|^2 &= 4|\alpha|^2 \\
*(\varphi \wedge (\varphi \wedge \alpha)) &= -4\alpha \\
\psi \wedge *(\varphi \wedge \alpha) &= 0 \\
*(\varphi \wedge w^b) &= \underline{w \wedge \varphi} \\
\varphi \wedge (w \wedge \varphi) &= -2*(w \wedge \varphi) \\
\varphi \wedge (w \wedge \psi) &= -4*w^b
\end{align*}
\]

\[
\begin{align*}
|\psi|^2 &= 7 \\
|\psi \wedge \alpha|^2 &= 3|\alpha|^2 \\
*(\psi \wedge (\psi \wedge \alpha)) &= 3\alpha \\
\varphi \wedge *(\psi \wedge \alpha) &= -2\psi \wedge \alpha \\
*(\psi \wedge w^b) &= w \wedge \varphi \\
\psi \wedge (w \wedge \varphi) &= 3*w^b \\
\psi \wedge (w \wedge \psi) &= 0
\end{align*}
\]

A.2 Some coordinate-free identities of \( G_2 \)-geometry

We now derive some identities which are useful for describing the decomposition of the space of forms in Section 2.2 and for computing the evolution of the metric \( g \) from the 3-form \( \varphi \) in Section 3.1.

**Lemma A.4.** Let \( u, v, \) and \( w \) be vector fields on \( M \). Let \( u^b, v^b, \) and \( w^b \) denote their dual 1-forms with respect to the metric \( g \). Then the following identity holds:

\[
*((u \wedge \varphi) \wedge (v \wedge \varphi) \wedge (w \wedge \varphi)) = -2g(u, v)w^b - 2g(u, w)v^b - 2g(v, w)u^b
\]

**Proof.** We begin with the general relation between \( \varphi, g \), and the volume form:

\[
(u \wedge \varphi) \wedge (v \wedge \varphi) \wedge \varphi = -6g(u, v)\text{vol}
\]

Taking the interior product of this equation with \( w \) and using the fact that \( w \text{vol} = *w^b \), we obtain

\[
-6g(u, v) * w^b = (w \wedge u \wedge \varphi) \wedge (v \wedge \varphi) \wedge \varphi + (w \wedge \varphi) \wedge (u \wedge \varphi) \wedge (v \wedge \varphi) + (u \wedge \varphi) \wedge (v \wedge \varphi) \wedge (w \wedge \varphi) = -2(u \wedge w)^b \wedge *(v \wedge \varphi) - 2(v \wedge w)^b \wedge *(u \wedge \varphi) + (u \wedge \varphi) \wedge (v \wedge \varphi) \wedge (w \wedge \varphi)
\]

where we have used equation (A.2) and the relation \( (v \wedge \varphi) \wedge \varphi = -2*(v \wedge \varphi) \) from Proposition A.3. We rearrange this equation and use \( *(v \wedge \varphi) = v^b \wedge \psi \) to obtain

\[
(u \wedge \varphi) \wedge (v \wedge \varphi) \wedge (w \wedge \varphi) = -6g(u, v) * w^b - 2v^b \wedge (u \wedge w) \wedge \psi - 2u^b \wedge (v \wedge w) \wedge \psi
\]

We now use (A.2) and (A.3), and take * of both sides to get

\[
*(((u \wedge \varphi) \wedge (v \wedge \varphi) \wedge (w \wedge \varphi)) = -6g(u, v)w^b - 2(v \wedge (u \wedge w))^b - 2(u \wedge (v \wedge w))^b
\]

\[
= -6g(u, v)w^b - 2\left(-g(u, v)w^b + g(v, w)u^b - (u \wedge v \wedge w \wedge \psi)^b\right) - 2\left(-g(u, v)w^b + g(v, w)u^b - (u \wedge v \wedge w \wedge \psi)^b\right)
\]

\[
= -2g(u, v)w^b - 2g(u, w)v^b - 2g(v, w)u^b
\]

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and the proof is complete. □

Remark A.5. If we put \( u = \frac{\partial}{\partial x^1}, \ v = \frac{\partial}{\partial x^2}, \ \text{and} \ w = \frac{\partial}{\partial x^3} \) this identity takes the form

\[ \ast \left( \left( \frac{\partial}{\partial x^1} \varphi \right) \land \left( \frac{\partial}{\partial x^2} \varphi \right) \land \left( \frac{\partial}{\partial x^3} \varphi \right) \right) = -2 \left( g_{ij} g_{lm} + g_{il} g_{jm} + g_{ij} g_{lm} \right) dx^m \]

If we take \( \ast \) of both sides of this equation, and wedge both sides with an arbitrary 1-form \( \alpha = \alpha_k dx^k \), we get

\[ \alpha \land \left( \left( \frac{\partial}{\partial x^1} \varphi \right) \land \left( \frac{\partial}{\partial x^2} \varphi \right) \land \left( \frac{\partial}{\partial x^3} \varphi \right) \right) = -2 \left( g_{ij} g_{lm} + g_{il} g_{jm} + g_{ij} g_{lm} \right) \alpha_k dx^k \land \ast dx^m \]

\[ \frac{1}{8} \alpha_s \varphi_{is_1 s_2} \varphi_{js_3 s_4} \varphi_{ls_5 s_6} dx^{s_1} \land \ldots \land dx^{s_7} = -2 \left( g_{ij} g_{lm} + g_{il} g_{jm} + g_{ij} g_{lm} \right) \alpha_k g^{km} \text{vol} \]

and hence

\[ \frac{1}{8} \sum_{\sigma \in S_7} \text{sgn}(\sigma) \varphi_{is_1 s_2} \varphi_{js_3 s_4} \varphi_{ls_5 s_6} \alpha_{\sigma(1)} \alpha_{\sigma(2)} \alpha_{\sigma(3)} \alpha_{\sigma(4)} \alpha_{\sigma(5)} \alpha_{\sigma(6)} \alpha_{\sigma(7)} dx^1 \land \ldots \land dx^7 = -2 \left( g_{ij} \alpha_i + g_{il} \alpha_j + g_{ij} \alpha_i \right) \sqrt{\det(g)} dx^1 \land \ldots \land dx^7 \]

which yields the useful relation

\[ \sum_{\sigma \in S_7} \text{sgn}(\sigma) \varphi_{is_1 s_2} \varphi_{js_3 s_4} \varphi_{ls_5 s_6} \alpha_{\sigma(1)} \alpha_{\sigma(2)} \alpha_{\sigma(3)} \alpha_{\sigma(4)} \alpha_{\sigma(5)} \alpha_{\sigma(6)} \alpha_{\sigma(7)} = -16 \left( g_{ij} \alpha_i + g_{il} \alpha_j + g_{ij} \alpha_i \right) \sqrt{\det(g)} \]

\[ = \frac{8}{3} \left( B_{ij} \alpha_i + B_{il} \alpha_j + B_{ij} \alpha_i \right) \quad \text{(A.4)} \]

using the fact that \( B_{ij} = -6g_{ij} \sqrt{\det(g)} \).

Corollary A.6. Let \( u, \ v, \ \text{and} \ w \) be vector fields on \( M \). Then the following holds:

\[ \ast ((u \land \varphi) \land (v \land \varphi) \land \varphi) = 2g(u, v)w^b - 2g(u, w)v^b + 2 \ast (u \land v \land w \land \psi)^4 \]

Proof. This can be seen in the proof of Lemma [A.4] above, by observing what happens to the appropriate term. □

Remark A.7. If we put \( v = \frac{\partial}{\partial x^1}, \ w = \frac{\partial}{\partial x^2}, \ \text{and} \ u = \frac{\partial}{\partial x^3} \) this identity takes the form

\[ \ast \left( \left( \frac{\partial}{\partial x^1} \varphi \right) \land \left( \frac{\partial}{\partial x^2} \varphi \right) \land \varphi \right) = 2(g_{ij} g_{im} - g_{ij} g_{jm} + \psi_{ijjm})dx^m \]

If we take \( \ast \) of both sides of this equation, and wedge both sides with an arbitrary 1-form \( \alpha = \alpha_k dx^k \), we get

\[ \alpha \land \left( \left( \frac{\partial}{\partial x^1} \varphi \right) \land \left( \frac{\partial}{\partial x^2} \varphi \right) \land \varphi \right) = 2(g_{ij} g_{im} - g_{ij} g_{jm} + \psi_{ijjm})\alpha_k dx^k \land \ast dx^m \]

\[ \frac{1}{12} \alpha_s \varphi_{is_1 s_2} \varphi_{js_3 s_4} \varphi_{ls_5 s_6} dx^{s_1} \land \ldots \land dx^{s_7} = 2(g_{ij} g_{im} - g_{ij} g_{jm} + \psi_{ijjm})\alpha_k g^{km} \text{vol} \]
and hence

\[ \frac{1}{12} \sum_{\sigma \in S_7} \text{sgn}(\sigma) \alpha_{\sigma(1)} \varphi_{\sigma(2)} \varphi_{\sigma(3)} \varphi_{\sigma(4)} \varphi_{\sigma(5)} \varphi_{\sigma(6)} \varphi_{\sigma(7)} \, dx^1 \wedge \ldots \wedge dx^7 \]

\[ = 2 \left( g_{ij} \alpha_i - g_{ji} \alpha_l + \psi_{ijkm} g^{km} \alpha_k \right) \sqrt{\det(g)} \, dx^1 \wedge \ldots \wedge dx^7 \]

which yields the useful relation

\[ \sum_{\sigma \in S_7} \text{sgn}(\sigma) \alpha_{\sigma(1)} \varphi_{\sigma(2)} \varphi_{\sigma(3)} \varphi_{\sigma(4)} \varphi_{\sigma(5)} \varphi_{\sigma(6)} \varphi_{\sigma(7)} = 24 \left( g_{lj} \alpha_i - g_{ji} \alpha_l + \psi_{ijkm} g^{km} \alpha_k \right) \sqrt{\det(g)} \]

\[ = -4 \left( B_{lj} \alpha_i - B_{ji} \alpha_l \right) + 24 \psi_{ijkm} g^{km} \alpha_k \sqrt{\det(g)} \]  \hspace{1cm} (A.5)

where we have again used \( B_{ij} = -6 g_{ij} \sqrt{\det(g)} \).

**Proposition A.8.** Let \( u, v, \) and \( w \) be vector fields on \( M \). Then the following holds:

\[ (u \cdot v \cdot \psi) \wedge (w \cdot \varphi) \wedge \varphi = (v \cdot \psi) \wedge (u \cdot w \cdot \varphi) \wedge \varphi \]

**Proof.** We start with the (necessarily zero) 8-form

\[ (v \cdot \psi) \wedge (w \cdot \varphi) \wedge \varphi = 0 \]

and take the interior product with \( u \). This gives

\[ (u \cdot v \cdot \psi) \wedge (w \cdot \varphi) \wedge \varphi = (v \cdot \psi) \wedge (u \cdot w \cdot \varphi) \wedge \varphi + (v \cdot \psi) \wedge (w \cdot \varphi) \wedge (u \cdot \varphi) \]

from which the result follows, using the fact that \((v \cdot \psi) \wedge (w \cdot \varphi) \wedge (u \cdot \varphi) = 0\) for any \( u, v, w \) which is proved in [14], Theorem 2.4.7. \( \square \)

**Remark A.9.** If we put \( u = \frac{\partial}{\partial x^i} \), \( w = \frac{\partial}{\partial x^j} \), and \( v = \frac{\partial}{\partial x^l} \) this identity takes the form

\[ \left( \frac{\partial}{\partial x^i} \left( \frac{\partial}{\partial x^l} \psi \right) \right) \wedge \left( \frac{\partial}{\partial x^j} \left( \frac{\partial}{\partial x^l} \varphi \right) \right) \wedge \varphi = \left( \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} \psi \right) \wedge \left( \frac{\partial}{\partial x^l} \frac{\partial}{\partial x^j} \varphi \right) \wedge \varphi \]

In coordinates this becomes

\[ \frac{1}{24} \psi_{ijs} \varphi_{js} \varphi_{js} \varphi_{s} \, dx^i \wedge \ldots \wedge dx^7 = \left( \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} \psi \right) \wedge \left( \frac{\partial}{\partial x^l} \frac{\partial}{\partial x^j} \varphi \right) \wedge \varphi \]

and hence

\[ \sum_{\sigma \in S_7} \text{sgn}(\sigma) \psi_{i \sigma(1) \sigma(2)} \varphi_{\sigma(3) \sigma(4)} \varphi_{\sigma(5) \sigma(6) \sigma(7)} \]  \hspace{1cm} is skew-symmetric in \( i, j \).  \hspace{1cm} (A.6)

We also remark that the identity \((v \cdot \psi) \wedge (w \cdot \varphi) \wedge (u \cdot \varphi) = 0\) in local coordinates becomes

\[ \sum_{\sigma \in S_7} \text{sgn}(\sigma) \psi_{i \sigma(1) \sigma(2)} \varphi_{\sigma(3) \sigma(4)} \psi_{\sigma(5) \sigma(6) \sigma(7)} = 0 \]  \hspace{1cm} (A.7)

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Proposition A.10. Let \( v, w, a, b, c, d \) be vector fields on \( M \). The following identities hold:

\[
\begin{align*}
& a^b \land b^c \land c^d \land \psi = \phi(a, b, c) \text{vol} \\
& a^b \land b^c \land d^e \land \varphi = \psi(a, b, c, d) \text{vol} \\
& a^b \land b^c \land w^d \land (v \lrcorner \psi) = (g(v, w)\varphi(a, b, c) - g(a, v)\varphi(w, b, c) \\
& \quad - g(b, v)\psi(a, w, c) - g(c, v)\psi(a, b, w)) \text{vol} \\
& a^b \land b^c \land d^e \land w^f \land (v \lrcorner \varphi) = (g(v, w)\psi(a, b, c, d) - g(a, v)\psi(w, b, c, d) \\
& \quad - g(b, v)\psi(a, w, c, d) - g(c, v)\psi(a, b, w, d) \\
& \quad - g(d, v)\psi(a, b, c, w)) \text{vol}.
\end{align*}
\]

Proof. The first two equations follow from repeated application of \( (1.4) \). To prove the third, start with the (necessarily zero) 8-form

\[ a^b \land b^c \land \psi = 0 \]

and take the interior product with \( v \), and rearrange terms. The fourth equation is proved similarly. \( \square \)

Proposition A.11. Let \( a, b, c, d \) be vector fields on \( M \). The following relation holds:

\[
\begin{align*}
& a^b \land b^c \land (c \lrcorner \varphi) \land (d \lrcorner \psi) = (2g(a \land b, c \land d) + \psi(a, b, c, d)) \text{vol}.
\end{align*}
\]

Proof. We start with the relation \( \psi \land (c \lrcorner \varphi) = 3 \ast c^e \) from Proposition A.3. Taking the interior product with \( d \), using Lemma A.1, and rearranging, we obtain

\[(c \lrcorner \varphi) \land (d \lrcorner \psi) = 3 \ast (c^e \land d^f) - (c \times d)^h \land \psi \]

Taking the wedge product with \( a^b \land b^c \),

\[
\begin{align*}
& a^b \land b^c \land (c \lrcorner \varphi) \land (d \lrcorner \psi) = 3 (a^b \land b^c) \land (c \times d)^h \land \psi \\
& \quad - (a^b \land b^c \land (c \times d)^h \land \psi \\
& \quad - 3 g(a \land b, c \land d) \text{vol} - \varphi(a, b, c \times d) \text{vol} \\
& \quad = 3 g(a \land b, c \land d) \text{vol} - g(a \times b, c \times d) \text{vol} \\
& \quad = (2g(a \land b, c \land d) + \psi(a, b, c, d)) \text{vol}.
\end{align*}
\]

In the second equality, we used \( u^b \land v^b \land w^b \land \psi = \phi(u, v, w) \text{vol} \) from Proposition A.10 in the third equality we used \( \varphi(u, v, w) = g(u \times v, w) \), and in the final equality we used Corollary A.2. \( \square \)

A.3 Contraction of \( \varphi \) and \( \psi \)

The next set of identities are various contractions of \( \varphi \), \( \psi \), and their derivatives, in index notation. They are used repeatedly in calculations throughout this paper.

In local coordinates \( x^1, x^2, \ldots, x^7 \), the 3-form \( \varphi \) and the dual 4-form \( \psi \) are

\[
\begin{align*}
& \varphi = \frac{1}{6} \varphi_{ijk} \, dx^i \land dx^j \land dx^k \\
& \psi = \frac{1}{24} \psi_{ijkl} \, dx^i \land dx^j \land dx^k \land dx^l
\end{align*}
\]
where $\varphi_{ijk}$ and $\psi_{ijkl}$ are totally skew-symmetric in their indices. The metric is given by $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$. The cross product is a $(2,1)$ tensor which we write as

$$\frac{\partial}{\partial x^i} \times \frac{\partial}{\partial x^j} = P^k_{ij} \frac{\partial}{\partial x^k}$$  \hspace{1cm} (A.8)

where $P^k_{ij} = -P^k_{ji}$. From (A.1) it follows that

$$\varphi_{ijk} = g_{kl} P^l_{ij} \quad \quad P^l_{ij} = g^{kl} \varphi_{ijk}$$  \hspace{1cm} (A.9)

Setting $u = \frac{\partial}{\partial x^i}$, $v = \frac{\partial}{\partial x^j}$, and $w = \frac{\partial}{\partial x^k}$ in (A.3), we obtain

$$\frac{\partial}{\partial x^i} \times \left( \frac{\partial}{\partial x^j} \times \frac{\partial}{\partial x^k} \right) = -g_{ij} \frac{\partial}{\partial x^k} + g_{ik} \frac{\partial}{\partial x^j} + \psi_{ijk}(dx^l)^2$$

$$P^m_{il} P^l_{jk} \frac{\partial}{\partial x^m} = -g_{ij} \frac{\partial}{\partial x^k} + g_{ik} \frac{\partial}{\partial x^j} + \psi_{ijk} g^{lm} \frac{\partial}{\partial x^m}$$  \hspace{1cm} (A.10)

The first set of identities is the following.

**Lemma A.12.** Let the tensors $g$, $\varphi$, $\psi$, and $P$ be as given above. Then the following identities hold:

$$P^k_{il} P^l_{jk} = -6g_{ij}$$

$$\varphi_{ijk} \varphi_{abc} g^{ia} g^{jb} g^{kc} = 42$$

$$\varphi_{ijk} \varphi_{abc} g^{jb} g^{kc} = 6g_{ia}$$

$$\varphi_{ijk} \varphi_{abc} g^{kc} = g_{ia} g_{jb} - g_{ib} g_{ja} - \psi_{ijab}$$

**Proof.** We prove the fourth equation. The other three follow by contraction with $g^{ij}$ and using (A.9). To obtain the fourth equation, take the inner product of (A.10) with $\frac{\partial}{\partial x^i}$ and simplify:

$$P^m_{il} P^l_{jk} g_{mn} = -g_{ij} g_{kn} + g_{ik} g_{jn} + \psi_{ijk}$$

$$\varphi_{da} g^{am} \varphi_{jkb} g^{bl} g_{mn} = -g_{ij} g_{kn} + g_{ik} g_{jn} + \psi_{ijnk}$$

$$\varphi_{iln} \varphi_{jkb} g^{bl} = -g_{ij} g_{kn} + g_{ik} g_{jn} + \psi_{ijnk}$$

which is what we wanted to show, since $\varphi_{iln} = -\varphi_{inl}$. We also note that the second identity above is just a restatement in terms of indices of the fact that the pointwise norm $|\varphi|^2$ of $\varphi$ is 7.

The next set of identities involves contractions of $\varphi$ with $\psi$.

**Lemma A.13.** Let the tensors $g$, $\varphi$, and $\psi$ be as given above. Then the following identities hold:

$$\varphi_{ijk} \psi_{abc} g^{ib} g^{jc} g^{kd} = 0$$

$$\varphi_{ijk} \psi_{abc} g^{jc} g^{kd} = -4\varphi_{lab}$$

$$\varphi_{ijk} \psi_{abc} g^{kd} = g_{ia} \varphi_{jbc} + g_{ib} \varphi_{ajc} + g_{ic} \varphi_{abj}$$

$$-g_{aj} \varphi_{ibe} - g_{bj} \varphi_{aic} - g_{cj} \varphi_{abi}$$

**Proof.** Again, the first two follow from the third. To prove the third, we take the inner product of (A.8) with (A.3) and use Corollary A.2

$$g \left( \frac{\partial}{\partial x^a} \times \frac{\partial}{\partial x^b} \times \frac{\partial}{\partial x^c} \right) \times \left( \frac{\partial}{\partial x^i} \times \frac{\partial}{\partial x^j} \times \frac{\partial}{\partial x^k} \right) = g_{ai} \varphi_{jkb} - g_{ib} \varphi_{jka} - \psi_{abij} P^l_{jk}$$
But this also equals
\[
\begin{align*}
= & \ g \left( P_{ab}^l \frac{\partial}{\partial x^l} - g_{ij} \frac{\partial}{\partial x^j} + g_{jk} \frac{\partial}{\partial x^k} + \psi_{ijkn} g_{nm} \frac{\partial}{\partial x^m} \right) \\
= & \ -g_{ij}g_{lk} P_{ab}^l + g_{lk}g_{ij} P_{ab}^l + P_{ab}^l g_{nm} g_{lm} \psi_{ijkn} \\
= & \ -g_{ij} \varphi_{abk} + g_{ik} \varphi_{abj} + P_{ab}^l \psi_{ijkl}
\end{align*}
\]

Combining the two expressions and rearranging, we obtain
\[
g_{ia} \varphi_{jkb} - g_{ib} \varphi_{jka} + g_{ij} \varphi_{abk} - g_{ik} \varphi_{abj} - \varphi_{jkc} \psi_{abcd} g^{cd} - \varphi_{abc} \psi_{ijkl} g^{kl} = 0
\]

Denote the above expression by \( A_{ijkab} \). Then it is tedious but straightforward to check that
\[
A_{ijkab} + A_{ajkbi} + A_{bijka} - A_{kjabi} - A_{jikab} = 0
\]
yields the desired identity. \( \Box \)

Finally we can contract \( \psi \) with itself.

**Lemma A.14.** Let the tensors \( g, \varphi, \) and \( \psi \) be as given above. Then the following identities hold:
\[
\begin{align*}
\psi_{ijkl} \psi_{abcd} g^{ia} g^{jb} g^{kc} g^{ld} &= 168 \\
\psi_{ijkl} \psi_{abcd} g^{ia} g^{kbc} g^{ld} &= 24g_{ia} \\
\psi_{ijkl} \psi_{abcd} g^{kbc} g^{ld} &= 4g_{ia} g_{jb} - 4g_{ib} g_{ja} - 2\psi_{ijab} \\
\psi_{ijkl} \psi_{abcd} g^{ld} &= -\varphi_{ajk} \varphi_{bhc} - \varphi_{akj} \varphi_{bhc} - \varphi_{ija} \varphi_{kbc} + g_{ia} g_{jb} g_{kc} + g_{ib} g_{jc} g_{ka} + g_{ic} g_{ja} g_{kb} - g_{ia} g_{jb} g_{kc} - g_{ic} g_{ja} g_{kb} - g_{ia} g_{jkc} - g_{ja} g_{kbc} - g_{ka} \psi_{ijbc} + \psi_{ijkc} - g_{ac} \psi_{ijkb}
\end{align*}
\]

**Proof.** As usual, all the identities follow from the last one. (However, this time establishing the third from the fourth also requires using Lemma A.12.) To prove the last identity, we take the inner product of \( A_{10} \) with itself. The calculation involves the use of Corollary A.2 twice as well as Lemmas A.12 and A.13. The details are tedious and not enlightening, hence they are left to the reader. As in the case of \( \varphi \), the first identity is a restatement of the fact that \( |\psi|^2 = 7 \) pointwise. \( \Box \)

**Remark A.15.** The particular expression for \( \psi_{ijkl} \psi_{abcd} g^{ld} \) above is not manifestly skew-symmetric in \( a, b, c \). This is due to the particular way in which it was derived. Since it of course is skew-symmetric in \( a, b, c \), one can skew-symmetrize to obtain the more natural expression \( \psi_{ijkl} \psi_{abcd} g^{ld} =
\]
\[
\begin{align*}
g_{ia} g_{jb} g_{kc} + g_{ib} g_{jc} g_{ka} + g_{ic} g_{ja} g_{kb} - g_{ia} g_{jc} g_{kb} - g_{ib} g_{ja} g_{kc} - g_{ic} g_{jk} g_{ka} \\
- \frac{1}{3} (\varphi_{bck} \varphi_{ajk} + \varphi_{aic} \varphi_{bjk} + \varphi_{abi} \varphi_{cjk}) - \frac{1}{3} (\varphi_{ajc} \varphi_{ibk} + \varphi_{aib} \varphi_{cj} + \varphi_{abi} \varphi_{ck}) \\
- \frac{1}{3} (\varphi_{jbc} \varphi_{ij} + \varphi_{ajc} \varphi_{ibj} + \varphi_{abc} \varphi_{ijc}) - \frac{1}{3} (\varphi_{akc} \varphi_{jib} + g_{ia} \psi_{jkbc} + g_{ib} \psi_{jka} + g_{ic} \psi_{jkc}) \\
- \frac{1}{3} (g_{ja} \psi_{kbc} + g_{jb} \psi_{kca} + g_{jc} \psi_{kab}) + \frac{1}{3} (g_{ka} \psi_{jbc} + g_{kb} \psi_{jca} + g_{kc} \psi_{jab})
\end{align*}
\]

but since this is much more unwieldy and contains non-integer coefficients, we prefer to use the more manageable expression given in Lemma A.14.
Next we consider contractions involving the covariant derivatives of $\varphi$ and $\psi$.

**Proposition A.16.** Let $g$, $\varphi$, and $\psi$ be as before. The following identities hold:

\[

\begin{align*}
(\nabla_m \varphi_{ijk}) \varphi_{abc} g^{ia} g^{jb} g^{kc} &= 0 \\
(\nabla_m \psi_{ijkl}) \psi_{abcd} g^{ja} g^{jb} g^{kc} g^{ld} &= 0 \\
(\nabla_m \varphi_{ijk}) \psi_{abcd} g^{ib} g^{jc} g^{kd} &= - \varphi_{ijk} (\nabla_m \psi_{abcd}) g^{ib} g^{jc} g^{kd} \\
(\nabla_m \psi_{ijkl}) \varphi_{abc} g^{ib} g^{kc} &= - \varphi_{ijk} (\nabla_m \varphi_{abc}) g^{ib} g^{kc} \\
(\nabla_m \varphi_{ijkl}) \psi_{abcd} g^{jc} g^{kd} &= - \varphi_{ijk} (\nabla_m \psi_{abcd}) g^{jc} g^{kd} - 4 (\nabla_m \varphi_{abcd}) g^{jc} g^{kd}
\end{align*}
\]

and finally also

\[
(\nabla_m \psi_{abcd}) = - (\nabla_m \varphi_{abc}) g^{ib} g^{jc} - \varphi_{abc} (\nabla_m \varphi_{abcd}) g^{ib} g^{jc} \tag{A.11}
\]

**Proof.** The method for establishing all of these identities should be clear. Simply take the covariant derivative $\nabla_m$ of the various equations in Lemmas A.12, A.13, and A.14, and in some cases relabel indices to obtain the desired result. We omit the details. \qed

We also have the following important relation.

**Proposition A.17.** The following relation holds between $\nabla \varphi$ and $\nabla \psi$:

\[
(\nabla_m \psi_{ijkl}) \psi_{abcd} g^{ib} g^{jc} g^{kd} = 3 (\nabla_m \varphi_{ijk}) \varphi_{abc} g^{ib} g^{jc} g^{kd}
\]

**Proof.** We substitute (A.11) into the left hand side above, and use Lemma A.13 to obtain:

\[
\begin{align*}
- ((\nabla_m \varphi_{ijk}) \varphi_{klq} g^{pq} + \varphi_{ijk} (\nabla_m \varphi_{klq}) g^{pq}) \psi_{abcd} g^{ib} g^{jc} g^{kd} &= - (\nabla_m \varphi_{ijk}) g^{pq} g^{ib} (\varphi_{klq} \psi_{abcd} g^{jc} g^{kd}) - (\nabla_m \varphi_{klq}) g^{pq} g^{jc} g^{kd} (\varphi_{ijk} \psi_{abcd} g^{ib}) \\
&= - (\nabla_m \varphi_{ijk}) g^{pq} g^{ib} (-4 \varphi_{abc}) \\
&= - (\nabla_m \varphi_{klq}) (g^{pq} g^{jc} g^{kd} (g_{pc} \varphi_{ida} + g_{pd} \varphi_{cia} + g_{pa} \varphi_{cdi} - g_{ic} \varphi_{pda} - g_{id} \varphi_{cpa} - g_{ia} \varphi_{cdp}) \\
&= 4 (\nabla_m \varphi_{ijk}) \varphi_{abc} g^{pq} g^{ib} + 0 + 0 \\
&= 4 (\nabla_m \varphi_{ijkl}) (g^{pq} g^{jc} g^{kd} (g_{pc} \varphi_{ida} + g_{pd} \varphi_{cia} + g_{pa} \varphi_{cdi} - g_{ic} \varphi_{pda} - g_{id} \varphi_{cpa} - g_{ia} \varphi_{cdp}) \\
&= 4 (\nabla_m \varphi_{ijkl}) \varphi_{abc} g^{ib} g^{jc} g^{kd} - (\nabla_m \varphi_{klq}) \varphi_{cda} g^{ib} g^{jc} g^{kd} + (\nabla_m \varphi_{klq}) \varphi_{pda} g^{ib} g^{jc} g^{kd} \\
&+ (\nabla_m \varphi_{klq}) \varphi_{red} g^{ib} g^{jc} g^{kd} + (\nabla_m \varphi_{klq}) \varphi_{cda} g^{ib} g^{jc} g^{kd} g_{ia}
\end{align*}
\]

Using Proposition A.16 on the second and final terms, the final term vanishes and the remaining terms all combine (after relabelling some indices) to yield

\[
3 (\nabla_m \varphi_{ijk}) \varphi_{abc} g^{ib} g^{jc} g^{kd}
\]

and the proof is complete. \qed

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B  Review of Basic Flow Formulas

In this section we briefly review several formulas that are used frequently when studying geometric flows. Let \((M, g)\) be an oriented Riemannian manifold. We choose local coordinates \(x^1, \ldots, x^n\) and denote the Riemannian metric by \(g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})\). The associated volume form is then \(\text{vol} = \sqrt{\det(g)} \, dx^1 \wedge \ldots \wedge dx^n\). We use \(g^{ij}\) to denote the inverse matrix to \(g_{ij}\), and \(g_{ij}\) is the induced metric on 1-forms: \(g^{ij} = g(dx^i, dx^j)\).

**Lemma B.1.** Suppose \(g_{ij}\) depends smoothly on some parameter \(t\). Then

\[
\frac{\partial}{\partial t} g^{ij} = -g^{ik} \left( \frac{\partial}{\partial t} g_{kl} \right) g^{lj}
\]

**Proof.** We differentiate the relation \(g^{kl} g^{lj} = \delta^j_k\), to obtain

\[
\left( \frac{\partial}{\partial t} g^{kl} \right) g^{lj} + g^{kl} \left( \frac{\partial}{\partial t} g^{lj} \right) = 0
\]

\[
g^{ik} g^{kl} \left( \frac{\partial}{\partial t} g^{lj} \right) = \frac{\partial}{\partial t} g^{ij} = -g^{ik} \left( \frac{\partial}{\partial t} g_{kl} \right) g^{lj}
\]

as claimed. \(\Box\)

**Lemma B.2.** Suppose \(g_{ij}\) depends smoothly on some parameter \(t\). Then

\[
\frac{\partial}{\partial t} \det(g) = \left( \frac{\partial}{\partial t} g_{ij} \right) g^{ij} \det(g)
\]

\[
\frac{\partial}{\partial t} \text{vol} = \frac{1}{2} \left( \frac{\partial}{\partial t} g_{ij} \right) g^{ij} \text{vol}
\]

**Proof.** The second formula follows easily from the first, using the local coordinate expression for the volume form. Cramer’s rule from linear algebra says

\[
g_{ik} G^{kj} = \det(g) \delta^j_i
\]

where \(G^{kj}\) is the classical adjoint of \(g_{ij}\), the transpose of the matrix of cofactors. Note that \(g^{ij} = \frac{1}{\det(g)} G^{ij}\). The determinant \(\det(g)\) is a linear function \(F(g_1, \ldots, g_n)\) of the columns \(g_1, \ldots, g_n\) of the matrix \(g_{ij}\). Therefore

\[
\frac{\partial}{\partial t} \det(g) = F\left( \frac{\partial}{\partial t} g_1, g_2, \ldots, g_n \right) + F\left( g_1, \frac{\partial}{\partial t} g_2, \ldots, g_n \right) + \ldots + F\left( g_1, g_2, \ldots, \frac{\partial}{\partial t} g_n \right)
\]

where, for example, \(F\left( \frac{\partial}{\partial t} g_1, g_2, \ldots, g_n \right)\) is the determinant of the matrix \(g_{ij}\) with the first column \(g_{i1}\) replaced with \(\frac{\partial}{\partial t} g_{i1}\). Now we expand the first of these determinants along the first column, the second along the second column, and so on. We obtain

\[
\frac{\partial}{\partial t} \det(g) = \left( \frac{\partial}{\partial t} g_{i1} \right) G^{i1} + \left( \frac{\partial}{\partial t} g_{i2} \right) G^{i2} + \ldots + \left( \frac{\partial}{\partial t} g_{in} \right) G^{in}
\]

\[
= \left( \frac{\partial}{\partial t} g_{ij} \right) G^{ij} = \left( \frac{\partial}{\partial t} g_{ij} \right) g^{ij} \det(g)
\]

which completes the proof. \(\Box\)
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