Casimir and Vacuum Energy of 5D Warped System and Sphere Lattice Regularization

Shoichi Ichinose

Laboratory of Physics, School of Food and Nutritional Sciences, University of Shizuoka
Yada 52-1, Shizuoka 422-8526, Japan

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We examine the Casimir energy of 5D electro-magnetism in the recent standpoint. $Z_2$ symmetry is taken into account. After confirming the consistency with the past result, we do new things based on a new regularization. The regularization is based on the minimal area principle and the regularized configuration is the sphere lattice. We do it not in the Kaluza-Klein expanded form but in the closed form. The formalism is based on the heat-kernel approach using the position/momentum propagator. A useful expression of the Casimir energy, in terms of the P/M propagator, is obtained. Renormalization flow is realized as the change along the extra-axis.

The present common image about the compactification of the higher-dimensional model is strongly based on the work by Appelquist and Chodos. They considered the Kaluza-Klein model, which is the 5D unified model of the graviton, the photon and the dilaton. The starting Lagrangian is the pure 5D Einstein gravity on $S^1 \times \mathcal{M}_4$. They took the standard approach of the quantum field theory, the background field method, and calculated the Casimir energy (taking the flat vacuum). After the appropriate regularization for the KK-expansion series expression, they obtain (for one scalar mode with the even parity)

$$V(l) = \frac{1}{5} A^5 - \frac{3}{4} \zeta(5) \frac{l^4}{l^5}, \quad F(l) = -\frac{\partial V}{\partial l} = -\frac{1}{5} A^5 - \frac{3}{4} \zeta(5) \frac{l^4}{l^5},$$

where $l$ is the periodicity parameter ($y \rightarrow y + 2l$), and $A$ is the 4D momentum cut-off. The first term of $V(l)$ is quintically divergent. This quantity comes from the UV-divergences of 5D quantum fluctuation. Dropping the (divergent) constant term, the Casimir force is finitely obtained as $-3\zeta(5)/l^5$.

In the closed form, $E_{\text{Cas}}$ of 5D electro-magnetism is expressed as

$$E_{\text{Cas}}(l) = \int \frac{d^4 p}{(2\pi)^4} \int_0^\infty \frac{1}{2} \text{Tr} G^\pm_k(y, y') + 2\text{Tr} G^+_k(y, y') \, dk^2 = \frac{2\pi^2}{(2\pi)^4} \int_0^\infty d\tilde{p} \int_0^l dy \tilde{p}^3 F(\tilde{p}, y),$$

where the P/M propagators are $G^\pm_k(y, y') = \pm\{\cosh \tilde{k}(|y + y'| - l) \mp \cosh \tilde{k}(|y - y'| - l)\}/4\tilde{k} \sinh \tilde{k} l$, $-l \leq y \leq l$, $-l \leq y' \leq l$, $\tilde{k} \equiv \sqrt{k_\mu k^\mu}$, $k_\mu k^\mu > 0$ (space-like). $F(\tilde{p}, y)$ is expressed by the Gauss’s hypergeometric function. The integral region of the equation (2) is displayed in Fig.1. In the figure, we introduce the UV and IR regularization cut-offs, $\mu = 1/l \leq \tilde{p} \leq A$, $\epsilon = 1/A \leq y \leq l$. From

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$^*)$ E-mail: ichinose@u-shizuoka-ken.ac.jp
a close numerical analysis of \((\tilde{p}, y)\)-integral (2), we have confirmed \(E_{\text{Cas}}(\Lambda, l) = (2\pi^2/(2\pi)^4) \{-0.1247l{\Lambda^5} - 1.773 \times 10^{-16}(1/l^4) - 1.253 \times 10^{-15}l^{-4}\ln(l/A)\}\). This is the (almost) same result of the previous one (1). The \(\Lambda^5\)-divergence shows the notorious problem of the higher dimensional theories. In spite of all efforts of the past literature, we have not succeeded in defining the higher-dimensional theories.

The divergences cause problems. The famous example is the divergent cosmological constant in the gravity-involving theories.\(^1\) We notice here we can avoid the divergence problem if we find a way to **legitimately restrict the integral region in \((\tilde{p}, y)\)-space.** One proposal of this was presented by Randall and Schwartz\(^2\). They introduced the position-dependent cut-off, \(\mu < \tilde{p} < 1/u, \ u \in [\epsilon, l]\), for the 4D-momentum integral in the "brane" located at \(y = u\). (See Fig.1) The total integral region is the lower part of the hyperbolic curve \(\tilde{p} = 1/y\). (They succeeded in obtaining the finite \(\beta\)-function in the 5D warped vector model.) Although they claim the holography is behind the procedure, the legitimacy of the restriction looks less obvious. We propose an alternate one\(^3\) and give a legitimate explanation within the 5D QFT.

On the "3-brane" at \(y = \epsilon\), we introduce the IR-cutoff \(\mu\) and the UV-cutoff \(\Lambda\) (\(\mu \ll \Lambda\)). See Fig.2. This is legitimate in the sense that we usually do this procedure in the 4D renormalizable theories. On the "3-brane" at \(y = l\), we have another set of IR and UV-cutoffs, \(\mu'\) and \(\Lambda'\). We consider the case: \(\mu' \leq \Lambda', \ \mu \sim \mu', \Lambda' \ll \Lambda\). This case leads to allow us to introduce the renormalization flow. (See the later explanation of Fig.3.) We claim here, as for the "3-brane" located at each point \(y\) (\(\epsilon < y < l\)), the regularization parameters are determined by the **minimal area principle.** To explain it, we depict the regularization configuration(Fig.2) in the 5D coordinate space \((x^\mu, y)\) in Fig.3. The 5D volume region bounded by \(B_{UV}\) and \(B_{IR}\) is the integral region of the Casimir energy \(E_{\text{Cas}}\). The forms of \(r_{UV}(y)\) and \(r_{IR}(y)\)
can be determined by the minimal area principle.

\[ \delta \text{(Surface Area)} = 0 \quad , \quad 3 - \frac{r \frac{d^2 r}{dy^2}}{1 + (\frac{dr}{dy})^2} = 0 \quad , \quad 0 \leq y \leq l \quad (3) \]

We have confirmed that there exist the solutions (geodesic curves) with the properties shown in Fig.2 or Fig.3 when we take appropriate boundary conditions: \( r(y = \epsilon), r(y = l) \).

Instead of restricting the integral region, we have another approach to suppress UV and IR divergences. We introduce a weight function \( W(\tilde{p}, y) \).

\[ E_{\text{Cas}}^W(l) \equiv \int \frac{d^4 p}{(2\pi)^4} \int_0^l dy \ W(\tilde{p}, y) F(\tilde{p}, y) \quad , \quad (4) \]

As the examples of \( W \), we present \( e^{-\tilde{p}^2/2 - (y^2/2\ell^2)} \equiv W_1(\tilde{p}, y) \) (elliptic suppression) and \( e^{-\tilde{p}y} \equiv W_2(\tilde{p}, y) \) (hyperbolic suppression). We have evaluated the divergence behaviour of \( E_{\text{Cas}}^W \) by numerically performing the \((\tilde{p}, y)\)-integral (4) for the rectangle region of Fig.1.

\[ E_{\text{Cas}}^W = \begin{cases} \frac{c_{10}}{\ell^4} - 21.4 A^4 + c_{11} A^4 \ln A & \text{for } W_1(\tilde{p}, y) \\ -\frac{c_{20}}{\ell^4} - 0.216 A^4 + c_{21} A^4 \ln A & \text{for } W_2(\tilde{p}, y) \end{cases} \quad (5) \]

where \( c's \) are unstable for the range \( l = (10, 20, 40), A = 10 \sim 10^2 \) and are given by \( c_{10} = (26.3, 18.2, 10.0), c_{11} = (5.52, 2.78, 1.39) \times 10^{-3}, c_{20} = (6.47, 55.6, 446) \times 10^8, c_{21} = (4.73, 2.35, 1.18) \times 10^{-5} \). In particular \( c_{11} \) and \( c_{21} \) changes as \( 1/l \) (decreases as \( l \) increases).

\( W_2 \) corresponds to the restriction approach by Randall-Schwartz and the above result is consistent with theirs. Its suppression, however, is insufficient. \( W_1 \) gives, after normalization by the factor \( \ell \Lambda \), the desired log-divergence. In this case, the Casimir energy is finitely obtained by the renormalization of the periodicity \( l \).

\[ -\frac{3 \zeta(5)}{4} \frac{1}{l^4} (1 - 4 \text{ln}(l \Lambda)) = \frac{3 \zeta(5)}{4} \frac{1}{l^4}, \]

\[ \frac{\partial}{\partial (\text{ln} \Lambda)} \ln \frac{\ell}{l} = c \text{ (anom. dim.)} \quad (6) \]

Fig.3 shows the renormalization flow. For interacting theories, such as 5D YM theories, the scaling of the renormalized coupling \( g(y) \) is given by

\[ \beta = -\frac{1}{3 \frac{\partial}{\partial y} \ln r(y)} \frac{1}{g \frac{\partial g}{\partial y}} \quad (7) \]
where \( g(y) \) is a renormalized coupling at \( y \) and \( r(y) \) is an appropriate geodesic.

Finally we comment on the meaning of the weight function. First we can define it by requiring that the dominant contribution to \( E_{\text{Cas}} \) [4], which is obtained by the steepest-descend method to (4), coincides with the geodesic curve, which is obtained by the minimal area principle for the surface in the bulk.

\[
d\tilde{p} = \frac{-\partial \ln(WF)}{\partial y} + \frac{3}{\tilde{p}} \frac{\partial \ln(WF)}{\partial \tilde{p}}. \tag{8}
\]

Secondly, we notice the \( W_1 \) is the harmonic oscillator Hamiltonian, hence the weighted system can be regarded as some quatum mechanical system of 5D-space coordinates where the extra coordinate \( y \) and the absolute value of the 4D momentum \( \tilde{p} \) are in the conjugate relation (new uncertainty principle).

So far the flat geometry is considered. For the warped case, the similar analysis can be done. One additional parameter (curvature parameter of AdS_5), besides \( l \), comes into the arguments. We are also examining the vacuum energy and the self energy using interacting theories such as 5D \( \Phi^4 \)-theory and 5D YM.

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References

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