Influence of Implementation on the Properties of Pseudorandom Number Generators with a Carry Bit

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Abstract

We present results of extensive statistical and bit level tests on three implementations of a pseudorandom number generator algorithm using the lagged Fibonacci method with an occasional addition of an extra bit. First implementation is the RCARRY generator of James, which uses subtraction. The second is a modified version of it, where a suggested error present in the original implementation has been corrected. The third is our modification of RCARRY such that it utilizes addition of the carry bit. Our results show that there are no significant differences between the performance of these three generators.

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Key words: Randomness, lagged Fibonacci random number generators, Monte Carlo simulations.
1 Introduction

Random numbers are needed in various applications, including cryptography [8], stochastic optimization [5], and Monte Carlo methods [4]. Because of practical reasons random numbers are usually produced by deterministic rules, implemented as pseudorandom number generators. In spite of their fully deterministic origin the quality of pseudorandom numbers may often be good enough for many applications.

To confirm the suitability of a given pseudorandom number generator for practical use, it should be subjected to a rigorous test program which reveals the strengths and weaknesses of the algorithm and, in particular, its implementation. Recently, such an extensive test program has been carried out by the present authors [18]. By performing a comparative evaluation using statistical, bit level and visual tests we were able to assess the quality of a group of random number generators which are commonly used in the applications of physics.

One of the generators included in Ref. [18] was RCARRY, which uses the so called “subtract-and-borrow” algorithm which has been implemented by James [8]. In the tests, RCARRY clearly displayed the poorest statistical properties of the generators tested, suggesting possible problems in the implementation. Supporting this, James has recently reported [9] the observation of M. Lüscher that the original implementation of RCARRY may contain a small error, which may adversely affect the quality of the random number sequence. The purpose of the present work is to address this issue. To this end, we present results of extensive statistical and bit level tests on the corrected version of RCARRY, and compare the results to those of Ref. [18]. In addition, we test a slightly different version of the RCARRY generator, which uses an “add-and-carry” algorithm based on the addition of a carry bit. We call this generator ADCARRY. Our results reveal that there is very little difference between the statistical properties of the original RCARRY and its corrected version, as well as the ADCARRY generator. All these generators display a relatively poor performance in two of the gap tests presented here.

2 Implementation of the Generators

The three pseudorandom number generators tested in this work are based on a lagged Fibonacci algorithm, which is augmented by an occasional addition of a carry bit.
The basic formula is:

\[ X_i = (X_{i-24} \pm X_{i-10} \pm c) \mod b. \]  

(1)

The carry bit \( c \) is zero if the sum is less than or equal to \( b \), and otherwise “\( c = 1 \) in the least significant bit position” \[8\]. The choice for \( b \) is \( 2^{24} \). The period of the generator is about \( 2^{1407} \) \[8\] and it produces random numbers distributed between \([0,1)\). Only the 24 most significant bits are guaranteed to be good.

The inclusion of the carry bit \( c \) in the lagged Fibonacci algorithm was done in order to improve its properties \[16\]. Recently, however, it has been shown \[5, 17\] that this type of algorithms are in fact equivalent to linear congruential generators with very large prime moduli. Consequently, they inherit unfavourable lattice structures in higher dimensions.

The original implementation of Eq. (1) was done by James \[8\], based on the ideas of Marsaglia et al. \[16\]. It uses the subtraction contained in Eq. (1). In this work, we shall denote it by I1. The second generator I2 includes the suggested correction of Lüscher and James, who recommend replacing line 13 of the code of Ref. \[8\]

\[
\text{uni} = \text{seeds(i24)} - \text{seeds(j24)} - \text{carry},
\]

by

\[
\text{uni} = \text{seeds(j24)} - \text{seeds(i24)} - \text{carry}.
\]

The third generator ADCARRY (I3) uses the operation known as “add-and-carry”, in which subtraction in Eq. (1) has been replaced by addition. In this version the lines 13 - 15 of \[8\] are rewritten as:

\[
\text{uni} = \text{seeds(j24)} + \text{seeds(i24)} + \text{carry}
\]

\[
\text{if}(\text{uni} \geq 1.) \text{ then}
\]

\[
\text{uni} = \text{uni} - 1.
\]

Otherwise, the implementation is identical to that of RCARRY \[12, 13\].

3 Test methods

Tests scrutinizing the quality of random numbers can be divided into three main categories: statistical tests \[10\], bit level tests \[3, 15, 18\] for testing the properties of random numbers on binary level, and visual tests \[10\] which may give some further qualitative information on the statistical properties of random numbers. A number
of these tests were implemented and employed extensively in Ref. [18]. In this work, we have repeated the same statistical tests for I2 and I3. They are listed in Table 1, where the numbering refers to the parameters of Ref. [18]. From bit level tests, only the $d$-tuple test [3, 15] was done since it was shown to be sufficient. Finally, the random numbers were plotted in two dimensions for purposes of visual inspection.

The test bench is described in detail in Ref. [18]. Description of the statistical tests can also be found in Ref. [10]. In brief, the statistical accuracy of all the tests was improved by utilizing a one way Kolmogorov - Smirnov test [10] to a large number (1000 or more) of test statistics. This approach has been realized earlier by L’Ecuyer [11]. The final test variables are therefore the values $K^+$ and $K^-$ of a Kolmogorov - Smirnov test statistic $K$ [10]. In each test the generator was considered to fail the test if the observed descriptive level $\delta = P(K \leq \{K^+, K^-\} | H_0)$ was less than 0.05 or larger than 0.95.

4 Results

Results of the statistical tests for the descriptive levels $\delta^+$ and $\delta^-$ are summarized in Table 2, where the numbering refers to Table 1. In each test the chosen generator was initialized with the seed 667790. In case a failure occurred, the generator was subjected to another test starting from the final state of the first test. If another immediate failure occurred, the generator was tested for the third time starting from a new state with an initial seed 14159 (from the decimals of $\pi$).

In Table 2, frames with thin lines indicate a single failure, frames with double single lines two consecutive failures, and frames with bold lines three consecutive failures in the corresponding tests. The results of the original RCARRY (I1) by James [8] are shown on the left (from Ref. [18]), whereas the results of the corrected version (I2) and ADCARRY (I3) are at the center and on the right, respectively.

Based on the results, it is clear that the corrected version of RCARRY (I2) using arithmetic subtraction performs no significantly better than the original RCARRY (I1). The main malady of RCARRY, namely the clear failing of the gap tests 6 and 8 with parameters $\alpha = 0, \beta = 0.05$ and $\alpha = 0.95, \beta = 1$ [10], respectively, is still characteristic of I2. This signifies the existence of local correlations in the vicinity of zero and one. The same conclusion applies to ADCARRY as well, signaling basic problems with these algorithms.
In the $d$-tuple test each implementation was tested two times and the bits considered failed had two consecutive failures. The results are shown in Table 3. In our notation, bit number one is the most significant bit (excluding the sign bit). For the original implementation of RCARRY (I1) only the 24 most significant bits are guaranteed to be good, which the tests confirm [18]. The implementation I2 yields identical results, whereas ADCARRY (I3) gives only 22 good bits (see Fig. 3).

Finally, visual tests on bit level support the results above. In Figs. 1, 2 and 3 we show subsequent random numbers for I1, I2 and I3 in binary form on a 120 × 120 matrix, when only 24 most significant bits are included. No clear correlations are visible, except for the last two bits of ADCARRY where strong correlations are apparent. No visual indications of the suggested [5, 17] lattice structure were found in these generators.

5 Summary and conclusions

In this work, we have compared the results of detailed statistical and bit level tests for three implementations of random number algorithms using a lagged Fibonacci sum with the addition of a carry bit. Results for RCARRY and its corrected version show very little difference. Also, a new generator ADCARRY using purely additive arithmetics fares no better statistically, and has two good bits less than RCARRY. Fortunately enough, these bits are not among the most significant ones. Overall, our results suggest that the basic algorithm of Eq. (1) on which these generators are based seems to lead to observable correlations. The persistent failure of this class of generators in the gap tests may lead to problems in some applications, e.g. in lattice simulations [14].

Finally, we would like to emphasize the importance of extensive testing such as presented here before using any new pseudorandom number generator. Even a good algorithm can be corrupted by a poor implementation, as we have previously demonstrated [18]. Hence, a good amount of scepticism towards pseudorandom number generators without extensive test results seems prudent. It should also be noted that even when no statistical or bit level correlations are found, direct physical tests of random number generators should be used to reveal possible “hidden” correlations [4, 7].
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[12] In a private communication, M. Lüscher has pointed out that the period for ADCARRY is unknown and it is difficult to prove analytically that it is always large.

[13] In a private communication, M. Lüscher has suggested a more faithful implementation of the add-and-carry algorithm by replacing lines 13-19 of RCARRY by the following:
uni = seeds(i24) - 1.
uni = uni + seeds(j24) + carry
if(uni.lt.0.) then
  uni = uni + 1.0
  carry = 0.
else
  carry = twom24
endif

This implementation should avoid possible rounding errors. We have tested it also, but find no difference in the results of the statistical tests here. However, in bit level tests all 24 bits pass as in RCARRY.

[14] More recently, M. Lüscher and F. James (unpublished) have suggested using RCARRY with a “luxury parameter” which means skipping numbers from the sequence.

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**Table captions**

**Table 1.** List of the statistical tests. Numbers refer to the choice of parameters in Ref. [18].

**Table 2.** Results of the statistical tests. I1, I2 and I3 refer to RCARRY, its corrected version, and ADCARRY, respectively. Results for RCARRY are from Ref. [18]. Depicted numbers are the observed descriptive levels $\delta^+$ and $\delta^-$ of the test variables $K^+$ and $K^-$, respectively. A generator was considered to fail the test if the descriptive level was less than 0.05 or more than 0.95. Single, double and triple consecutive failures are indicated by single, double, and bold lines, respectively. The numbers shown are from the first run only.

**Table 3.** Results of the bit level $d$-tuple test. The bits marked failed have failed the test twice. See text for details.

**Figure captions**

**Figure 1.** 24 bit binary representation of random numbers produced by implementation RCARRY (I1) on a 120 $\times$ 120 matrix.

**Figure 2.** 24 bit binary representation of random numbers produced by implementation I2 of RCARRY on a 120 $\times$ 120 matrix.

**Figure 3.** 24 bit binary representation of random numbers produced by ADCARRY on a 120 $\times$ 120 matrix.
| Test number | Test method                          |
|-------------|-------------------------------------|
| 1           | Equidistribution test (1)           |
| 2           | Equidistribution test (2)           |
| 3           | Serial test in 2 dimensions         |
| 4           | Serial test in 3 dimensions         |
| 5           | Serial test in 4 dimensions         |
| 6           | Gap test (1)                        |
| 7           | Gap test (2)                        |
| 8           | Gap test (3)                        |
| 9           | Maximum of $t$ test, $t = 5$        |
| 10          | Maximum of $t$ test, $t = 3$        |
| 11          | Collision test (1)                  |
| 12          | Collision test (2)                  |
| 13          | Collision test (3)                  |
| 14          | Runs-up test                        |
Table III

| Implementation | Failing bits | Number of “good” bits |
|----------------|--------------|----------------------|
| I1             | 25 – 31      | 24                   |
| I2             | 25 – 31      | 24                   |
| I3             | 23 – 31      | 22                   |