ON THE ZEROS OF THE EPSTEIN ZETA FUNCTION

ANIRBAN MUKHOPADHYAY, KRISHNAN RAJKUMAR, KOTYADA SRINIVAS

Abstract. In this article, we count the number of consecutive zeros of the Epstein zeta-function, associated to a certain quadratic form, on the critical line with ordinates lying in $[0, T]$, $T$ sufficiently large and which are separated apart by a given positive number $V$.

1. Introduction

The study of the distribution of zeros of the zeta-functions on the critical line is a fundamental problem in analytic number theory. A related problem is to find a good upper bound for the difference between consecutive zeros on the critical line. Such a study naturally leads one to ask the following

Question 1. Given $T$ sufficiently large, find $H = H(T)$ such that the interval $[T, T + H]$ contains the ordinate of a zero on the critical line of the corresponding zeta function under consideration?

Question 2. Given $T$ sufficiently large and $V > 0$, how many consecutive zeros of the zeta-function under consideration are there on the critical line with ordinates in $[0, T]$ which are at least $V$ distance apart?

Question 1 has been studied in great detail from the time of Hardy and Littlewood. We shall briefly mention few known results below. The main object of this note is to study the second question with respect to the Epstein zeta function $\zeta_Q(s)$ (see (2.1)). We shall address this issue in section 2.

In a classical paper [7], Hardy and Littlewood gave an answer to Question 1 for the Riemann zeta function, showing that one can take $H = T^{1/4 + \varepsilon}$. This was subsequently improved to $H = T^{1/6} (\log T)^{5 + \varepsilon}$ by Moser [17] and to $H = T^{1/6 + \varepsilon}$ by Balasubramanian [1], and the latest result is by Karatsuba [15] who showed that $H = T^{5/32} (\log T)^2$ holds. It must be mentioned here that Ivić (page 261, [9]) improved Karatsuba’s exponent $5/32 = 0.15625$ to $0.1559458\ldots$. Following the method of Hardy and Littlewood, Potter and Titchmarsh obtained analogous result for the Epstein zeta-function, which we shall discuss in section 2. For the zeta function, $\zeta_K(s, C)$ of an ideal class $C$ in a quadratic number field $K$, Bruce Bernt [3] showed that $H = T^{1/2 + \varepsilon}$ holds. This was improved to $H = T^{1/2 \log T}$ by Sankaranarayanan [22]. In [11], Jutila developed a method of transforming certain exponential sums into another sum which is much easier to handle (see section 4) and thereby proved that $H = T^{1/3 + \varepsilon}$ holds for the $L$-function associated to a cusp form (1.3) for the full modular group.

A well-known theorem of Selberg [23] states that the function $\zeta(\frac{1}{2} + it)$ has at least $T \log T$ zeros in the interval $[0, T]$, and this is best possible up to a value

2000 Mathematics Subject Classification. 11E45 (primary); 11M41 (secondary).
of the implied constant. In fact, Heath-Brown [8] showed that the same estimate holds even if we count only simple zeros. Hence the average gap between critical zeros of $\zeta(s)$ is $(\log T)^{-1}$. It is in this context that the Question 2 becomes relevant!

Let $R_1 := R_1(V)$ be the number of gaps between consecutive zeros of $\zeta(\frac{1}{2} + it)$, with ordinate between 0 and $T$, which are larger than $V$. Then trivially $R_1 \ll TV^{-1}$. Karatsuba [16] showed that $R_1 \ll TV^{-3/2}$ for $V = T^\delta$ for any $\delta > 0$. Ivić and Jutila [10] made substantial improvement in the case of $\zeta(s)$ as well as the $L$-function associated to a certain cusp form. We state their results below as Theorem A and B:

**THEOREM A:** Let $R_1$ be as defined above. Then uniformly

\begin{equation}
R_1 \ll TV^{-2} \log T
\end{equation}

and

\begin{equation}
R_1 \ll TV^{-3} \log^5 T.
\end{equation}

For $V \ll \log T$ these results are trivial and also $R_1 = 0$ for $V = T^{1/6}$ which follows from known results mentioned earlier on the gap between critical zeros for $\zeta(s)$. Thus, the result is non-trivial in the range $log T \ll V \leq T^{1/6 - \epsilon}$. In this article we shall be always considering $V$ in the range $T^{\epsilon} \ll V \leq T^{1/6 - \epsilon}$.

Let $a(n)$’s denote the Fourier coefficients of a cusp form $f$ of weight $\kappa$ for the full modular group. The $L$-series associated with $f$ is given by the Dirichlet series

\begin{equation}
\varphi(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}, \quad \text{Re}(s) > (\kappa + 1)/2.
\end{equation}

Let $R_2 := R_2(V)$ denote the number of gaps of length at least $V$ between consecutive zeros of $\varphi(\kappa/2 + it)$ in the interval $[0, T]$.

**THEOREM B:** Suppose that the $a(n)$’s defined above are real and let $V \gg \log^5 T$. Then

\begin{equation}
R_2 \ll TV^{-2} \log T
\end{equation}

and

\begin{equation}
R_2 \ll T^2V^{-6} \log^6 T.
\end{equation}

In this note, we obtain an analogue of the results (1.1) and (1.4) for the Epstein zeta-function, $\zeta_Q(s)$. We state the main result in the next section after a brief survey of the known results on the zeros of $\zeta_Q(s)$. In section 3 and 4 we shall state some Lemmas, section 5 will deal with the proof of the main result.

2. **Main result**

Let $Q(x, y) = ax^2 + bxy + cy^2$ be a binary positive-definite integral quadratic form. Let $r_Q(n)$ be the number of solutions of $Q(x, y) = n$. The Epstein zeta-function associated to $Q(x, y)$ is defined as

\begin{equation}
\zeta_Q(s) = \sum_{(x,y)\neq(0,0)} \frac{1}{Q(x,y)^s} = \sum_{n=1}^{\infty} \frac{r_Q(n)}{n^s}, \quad \sigma > 1.
\end{equation}
It is well known that \( \zeta_Q(s) \) can be analytically continued to the entire complex plane except for \( s = 1 \) where it has a simple pole with residue \( \frac{2\pi}{\sqrt{\Delta}} \) where \( \Delta = 4ac - b^2 > 0 \). It also has the functional equation

\[
\left( \frac{\sqrt{\Delta}}{2\pi} \right)^s \Gamma(s) \zeta_Q(s) = \left( \frac{\sqrt{\Delta}}{2\pi} \right)^{1-s} \Gamma(1-s) \zeta_Q(1-s).
\]

(2.2)

We suppose throughout the paper that \( \Delta \) is not a square, so that \( \sqrt{\Delta} \) is irrational; other cases like \( \Delta = 4 \) related to the form \( Q(x,y) = x^2 + y^2 \) are either easier or well-known.

Let \( h(-\Delta) \) be the number of inequivalent forms with discriminant \(-\Delta\). It is well known that there is a one-one correspondence between the equivalence classes of quadratic forms of discriminant \(-\Delta\) and the ideal classes of \( K = \mathbb{Q}(\sqrt{-\Delta}) \). In fact, the Dedekind zeta-function \( \zeta_K(s) \) for the number field \( K \) can be written as

\[
\zeta_K(s) = \frac{1}{t} \sum_r \zeta_Q_r(s),
\]

where \( Q_r \) runs over the distinct equivalence classes and \( t \) is the number of units in \( K \).

The Dedekind zeta-function has an Euler product and it belongs to that class of Dirichlet series for which Riemann’s hypothesis can be reasonably expected. In contrast, if the class number \( h(-\Delta) \) of the quadratic forms with discriminant \(-\Delta\) exceeds one, then \( \zeta_Q(s) \) has no Euler product and the Riemann hypothesis fails to hold for it. Indeed, when \( a, b, c \) are integers, \( -\Delta \) is a fundamental discriminant and the class number \( h(-\Delta) > 1 \), Davenport and Heilbronn \([6]\) had shown that \( \zeta_Q(s) \) has infinitely many zeros in the half plane \( \sigma > 1 \) arbitrarily close to the line \( \sigma = 1 \). S. M. Voronin \([27]\) proved that if \( h(-\Delta) > 1 \), then the number of zeros of \( \zeta_Q(s) \) in the rectangle \( \sigma_1 < \sigma < \sigma_2, |t| < T \), with \( 1/2 < \sigma_1 < \sigma_2 < 1 \) and \( T \) sufficiently large, exceeds \( c(\sigma_1, \sigma_2)T \), where \( c(\sigma_1, \sigma_2) > 0 \). Chowla and Selberg \([5]\) showed the existence of a real zero \( s \) with \( \frac{1}{2} < s < 1 \) when \( a = 1, b = 0 \) and \( c \) is large enough. Bateman and Grosswald \([2]\) improved it by showing that \( \zeta_Q(s) \) has a real zero between \( \frac{1}{2} \) and 1 if \( k = (\sqrt{\Delta})/2a > 7.0556 \).

Stark \([24]\) gave a striking description about the distribution of zeros of \( \zeta_Q(s) \) in a bounded region. More precisely, he proved that there exists a number \( K \) such that if \( k > K \), then all the zeros of \( \zeta_Q(s) \) in the region \(-1 < \sigma < 2, -2k < t < 2k \) are simple zeros and with the exception of two real zeros between 0 and 1, all are on the line \( \sigma = \frac{1}{2} \). Moreover, in the same paper he proved the following zero-density result: Let \( N(T, Q) \) denote the number of zeros of \( \zeta_Q(s) \) in the region \(-1 < \sigma < 2, 0 \leq t \leq T \). If \( k > K \) and \( 0 < T \leq 2k \), then

\[
N(T, Q) = (T/\pi) \log(kT/\pi e) + O(h(T + 3)),
\]

where \( h(x) = (\log x)^{1/3}(\log \log x)^{1/6} \) and the constant in "O" does not depend on \( k \).

Though \( \zeta_Q(s) \), in general, has thus infinitely many zeros off the critical line \( \sigma = 1/2 \), it has nevertheless infinitely many zeros on the critical line, and in fact it was proved by Potter and Titchmarsh \([20]\) that there is a zero \( 1/2 + i\gamma \) of \( \zeta_Q \) with \( \gamma \in [T, T + T^{1/2 + \varepsilon}] \), for any fixed \( \varepsilon > 0 \) and \( T \geq T(Q, \varepsilon) \). Sankaranarayanan \([22]\) sharpened this by showing the same for intervals of the type \([T, T + cT^{1/2} \log T}\).

For a long time improving the exponent of \( T \) below \( 1/2 \) was considered to be a challenging problem. First result in this direction is by Jutila and Srinivas \([14]\),...
where they prove that for any positive definite binary integral quadratic form $Q$ and for any fixed $\varepsilon > 0$ and $T \geq T(\varepsilon, Q)$, there is a zero $1/2 + i\gamma$ of the corresponding Epstein zeta-function $\zeta_Q$ with

(2.3) \[ |\gamma - T| \leq T^{5/11 + \varepsilon}. \]

Thus the best answer to Question 1 for $\zeta_Q$ is $H = T^{5/11 + \varepsilon}$, at the time of writing this manuscript. However, in analogy with the $L$-functions associated to a cusp forms (mentioned in section 1), the next big challenge seems to be to reduce the exponent of $T$ from $5/11$ to $1/3$!

In this article, we prove the following

THEOREM. Suppose $V$ satisfies

$T^\varepsilon \ll V \ll T^{1/2 - \varepsilon}$

and $R := R(V)$ denote the number of gaps of length at least $V$ between consecutive zeros of $\zeta_Q(1/2 + it)$ in the interval $[0, T]$, then

(2.4) \[ R \ll T^{1+\varepsilon}V^{-2} \]

for sufficiently large $T$, where the constant in $\ll$ may depend only on $\Delta$ and $\varepsilon$.

REMARK. The method of the proof closely follows the paper by Ivić and Jutila [10]. Using their approach to find an analogue of (1.2) and (1.5) for $\zeta_Q(s)$ does not seem to yield anything better than (2.4). Note that an analogue of (1.2) or (1.5) would immediately improve the exponent of $T$ to $1/3$ in (2.3).

3. SOME PRELIMINARY LEMMAS

Ramachandra showed (see [21], Chapter II ) that the first power mean of a generalized Dirichlet series satisfying certain conditions can not be too small. The following Lemma (Theorem 3 of [1] ) is a particular case of this general theorem, which is quite useful in obtaining lower bounds of this type, even in short-intervals.

LEMMA 1. Let $B(s) = \sum_{n=1}^{\infty} b_n n^{-s}$ be any Dirichlet series satisfying the following conditions:

(i) not all $b_n$’s are zero;
(ii) the function can be continued analytically in $\sigma \geq a$, $|t| \geq t_0$, and in this region $B(s) = O((|t| + 10)^A)$.

Then for every $\varepsilon > 0$, we have

\[ \int_T^{T+H} |B(\sigma + it)|dt \gg H \]

for all $H \geq (\log T)^\varepsilon$, $T \geq T_0(\varepsilon)$, and $\sigma > a$.

A suitable Dirichlet polynomial approximation is required to replace the divergent series $\zeta_Q(1/2 + it)$ in the course of the proof. We state the following Lemma which is a direct adaptation of Lemma 3 in [11].
LEMMA 2. Let $t \geq 2$ and $t^2 \ll X \ll t^A$, where $A$ is an arbitrarily large positive constant. Then we have

$$
\zeta_Q(1/2 + it) = \sum_{n \leq X} r_Q(n)n^{-1/2-it}
$$

(3.1)

$$
+ (\log 2)^{-1} \sum_{X < n \leq 2X} r_Q(n) \log(2X/n)n^{-1/2-it}
$$

$$
+ (\log 2)^{-1} 2\pi \Delta^{-1/2} (1/2 - it)^{-2} ((2X)^{1/2-it} - X^{1/2-it}) + O(tX^{-1/2}).
$$

The following result is Lemma 4.3 of Titchmarsh [26].

LEMMA 3. Let $F(x)$ and $G(x)$ be real functions, $G(x)/F(x)$ monotonic, and $F''(x)/G(x) \geq m > 0$, or $\leq -m < 0$. Then

$$
\left| \int_a^b G(x)e^{iF(x)} dx \right| \leq \frac{4}{m}
$$

4. SUMMATION AND TRANSFORMATION FORMULAE

Let

$$
S = \sum_n \eta(n) r_Q(n)n^{-1/2-it}
$$

and $e_k(x) = \exp(2\pi x/k)$. Introducing an additive character (mod $k$), we may write it formally as

$$
S = \sum_n \eta(n) r_Q(n)n^{-1/2-it} e_k(nh) \cdot e_k(-nh);
$$

as in [12], the purpose of the extra exponential factor is to damp the oscillations of the original exponential sum before an application of the Voronoi summation. Thus $S$ is of the general form

$$
\sum_n r_Q(n)f(n)e_k(hn),
$$

and a Voronoi summation formula for such sums was given in [13], Eq. (28). To state it, we need some notation. The Gauss sums related to the form $Q$ and additive characters are

$$
G_Q(k, h) = \sum_{x,y \ (\text{mod } k)} e_k(hQ(x, y)),
$$

and it holds (see [24], Lemma 1)

$$
|G_Q(k, h)| \leq (\Delta, k)k.
$$

Further, the summation formula involves an integral positive definite quadratic form $Q^*(x, y)$ depending on $Q$ and $k$, and the discriminant of $Q^*$ is at most $\Delta$ in absolute value. Also, there occurs an arithmetic function (corresponding to $r_{Q^*}(n)$ in [13]) of the form

$$
\rho(n) = \rho(n; Q, h/k) = \sum_{Q^*(x, y) = n} \alpha(x, y),
$$

where $|\alpha(x, y)| \leq (\Delta, k) \leq \Delta$ and $\alpha(x, y)$ depends only on the classes of $x$ and $y$ (mod $\Delta$) for given $Q$ and $h/k$. In this notation, a slightly simplified version of the summation formula (28) in [13] can be stated as follows.
Let \( \chi(s) \) be as in the functional equation \( \zeta(s) = \chi(s)\zeta(1 - s) \), thus \( \chi(s) = 2^s\pi^{s-1}\sin(\frac{1}{2}\pi s)\Gamma(1-s) \). The following Lemma is due to Jutila and Srinivas (Lemma 3.2 [14]).

**Lemma 4.** Let \( t \) be a large positive number, \( r = h/k \) a positive rational number with \((h,k) = 1\), and suppose that the positive numbers \( M_1 \) and \( M_2 \) satisfy

\[
M_1 < \frac{t}{2\pi r} < M_2
\]

\[
M_j = \frac{t}{2\pi r} + (-1)^j m_j, \quad m_1 \asymp m_2,
\]

\[
1 \leq k \ll M_1^{1/2-\delta},
\]

\[
t^{\delta_2}\max(t^{1/2-r^{-1}},hk) \ll m_1 \ll M_1^{1-\delta_3}
\]

for some small positive constants \( \delta_j \). Further, let

\[
U \gg t^{1/2+\delta_4}
\]

and let \( J \) be a fixed positive integer exceeding a certain bound which depends on \( \delta_4 \). Write

\[
M'_j = M_j + (-1)^j M_j' = \frac{t}{2\pi r} + (-1)^j m'_j,
\]

supposing that \( m_j \asymp m'_j \), and let

\[
n_j = \triangle_0 h^2 m_j^{-2} M_j^{-1}, \quad n'_j = \triangle_0 h^2 (m'_j)^2 (M'_j)^{-1}.
\]

Then for a certain weight function \( \eta \in C^{J-1}(\mathbb{R}) \) with support \([M_1,M_2]\) and satisfying \( \eta(x) = 1 \) for \( x \in [M'_1,M'_2]\), we have

\[
\sum_{n=1}^{\infty} \eta(n)r_Q(n)n^{-1/2-it} = \left\{ \begin{array}{ll}
2\pi \Delta^{-1/2} k^{-2} G_Q(k,-h) r^{-1/2} \\
+ \pi^{1/4} i(2hkt\triangle_0)^{-1/4}(\triangle_0/\Delta)^{1/2} \sum_{j=1}^{2} (-1)^j \sum_{n < n_j} w_j(n)\rho(n) \\
x \exp\left( 2\pi\imath\left( \frac{\tilde{h}\Delta_0}{k} - \frac{1}{2hk\triangle_0} \right) \right) n^{-1/4} \left( 1 + \frac{\pi n}{2hk\triangle_0} \right)^{-1/4} \\
x \exp\left( i(-1)^{j-1}\left( 2t\phi\left( \frac{\pi n}{2hk\triangle_0} + \frac{\pi}{4} \right) \right) \right) r^{it} \chi\left( \frac{1}{2} + it \right) \\
+ O(h^2k^{-1}m_1^{1/2}t^{-3/2}U \log t),
\end{array} \right.
\]

where

\[
\phi(x) = \arcsinh(x^{1/2}) + (x + x^2)^{1/2},
\]

\( w_j(n) = 1 \) for \( n < n'_j \), \( w_j(n) \ll 1 \) for \( n \leq n_j \), \( w_j(y) \) and \( w'_j(y) \) are piecewise continuous in the interval \((n'_j,n_j)\) with at most \( J - 1 \) jumps, and

\[
w'_j(y) \ll (n_j - n'_j)^{-1} \quad \text{for} \quad n'_j < y < n_j
\]

whenever \( w'_j(y) \) exists.
5. Proof of the theorem

We define the functions $f(s), \gamma(s)$ and $W(t)$ as

$$f(s) = e^{\frac{i}{2} \pi i (\frac{1}{2} - s)} \left( \frac{\sqrt{s}}{2\pi} \right) \Gamma(s) \zeta_Q(s) = \gamma(s) \zeta_Q(s)$$

and

$$W(t) = f(1/2 + it);$$

the latter is an analogue of Hardy’s function $Z(t)$ in the theory of Riemann’s zeta-function. The functional equation for $\zeta_Q(s)$ implies that $W(t)$ is real for real $t$. Thus the zeros of $\zeta_Q(s)$ on the critical line correspond to the real zeros of $W(t)$. To prove the result we can obviously restrict ourselves to the zeros lying in $[T/2, T]$. From now onwards we shall dwell on the upper bound estimation of $W(t)$ such that

$$[\tau, \tau + U] \subset [T/2, T], \text{ and } U \geq V,$$

where $T^* \leq V \leq T^{5/11}$. Further, let $L = 8(\log T)^{1/2}$ and $G = VL^{-1}$. Then, for $t \in [\tau + V/4, \tau + 3V/4]$, $W(u)$ has no zero in the interval $[t - V/4, t + V/4]$ and so $W(u)$ does not change sign in this interval. Let us define

$$I_1(t) = \int_{t-V/4}^{t+V/4} |W(u)| e^{-(t-u)^2/G^2} du$$

and

$$I_2(t) = \int_{t-V/4}^{t+V/4} W(u) e^{-(t-u)^2/G^2} du$$

Then for $t \in [\tau + V/4, \tau + 3V/4]$, $I_1(t) = |I_2(t)|$. Let $S$ be the set of $t \in [T/2 + V/4, T - V/4]$ such that $I_1(t) = |I_2(t)|$. Since the number of pairs of consecutive zeros of $\zeta_Q(1/2 + it)$, $\tau$ and $\tau + U$, such that (5.1) holds is $R$, the above discussion implies that the Lebesgue measure of $S$, $m(S) \gg RV$. Thus

$$RV < m(S) = \int_{t \in S} dt = \int_{t \in S} \frac{|I_2(t)|^2}{I_1(t)^2} dt.$$

Therefore, to obtain an upper bound on $R$, we have to find a lower bound for $I_1(t)$ and an upper bound for $I_2(t)$.

The lower bound

$$I_1(t) \gg G.$$

follows from Lemma 1.

From now onwards we shall dwell on the upper bound estimation of $I_2(t)$. First we define some notations; let $P(u) = u \sqrt{A}/2\pi$, for $J$ sufficiently large, we choose a smoothing function $\eta \in C^{J-1}(\mathbb{R})$ satisfying the following

$$\eta(x) = \begin{cases} 
1 & \text{if } x \in [P(T) - Y, P(T) + Y] \\
0 & \text{if } x > P(T) + 2Y \text{ or } x < P(T) - 2Y
\end{cases}$$

The parameter $Y$ is defined by

$$VY = T^{1+\epsilon}.$$
We shall use \( e^x \) and \( \exp(x) \) to denote the same function. Now the integral \( I_2(t) \) is written as

\[
I_2(t) = \int_{-V/4}^{V/4} \gamma(1/2 + i(t + u)) \zeta_Q(1/2 + i(t + u)) e^{-(u/G)^2} du.
\]

We substitute the expression for \( \zeta_Q(1/2+i(t+u)) \) given by Lemma \( \ref{lem:zeta_Q} \) with \( X = t^3 \) in (5.6) to obtain

\[
I_2(t) = \sum_{|n-P(t)| > 2Y} r_Q(n)n^{-1/2} \int_{-V/4}^{V/4} \gamma(1/2 + i(t + u)) n^{-i(t+u)} e^{-(u/G)^2} du
\]

\[
+ \int_{-V/4}^{V/4} \gamma(1/2 + i(t + u)) \left( \sum_{|n-P(t)| \leq 2Y} r_Q(n)n^{-1/2-i(t+u)} \right) e^{-(u/G)^2} du
\]

\[
+ (\log 2)^{-1} \sum_{t^3 < n \leq 2t^3} r_Q(n) \log(2t^3/n) n^{-1/2}
\]

\[
\times \int_{-V/4}^{V/4} \gamma(1/2 + i(t + u)) n^{-i(t+u)} e^{-(u/G)^2} du + O(Gt^{-1/2})
\]

\[
= S_1 + S_2 + S_3 + o(G).
\]

We expand the sum within brackets in \( S_2 \) and use (5.4) to get

\[
S_2 = \int_{-V/4}^{V/4} \gamma(1/2 + i(t + u)) \left( \sum_{|n-P(t)| \leq 2Y} \eta(n)r_Q(n)n^{-1/2-i(t+u)} \right) e^{-(u/G)^2} du
\]

\[
+ \sum_{|n-P(t)| \leq 2Y} (1 - \eta(n)) r_Q(n)n^{-1/2-i(t+u)} e^{-(u/G)^2} du
\]

\[
= \int_{-V/4}^{V/4} \gamma(1/2 + i(t + u)) \sum_{n=1}^{\infty} \eta(n)r_Q(n)n^{-1/2-i(t+u)} e^{-(u/G)^2} du
\]

\[
+ \sum_{Y < |n-P(t)| \leq 2Y} (1 - \eta(n)) r_Q(n)n^{-1/2}
\]

\[
\times \int_{-V/4}^{V/4} \gamma(1/2 + i(t + u)) n^{-i(t+u)} e^{-(u/G)^2} du
\]

\[
= S_4 + S_5.
\]

We now estimate \( S_1, S_3 \) and \( S_5 \). Note that in all these cases \( |n - P(t)| > Y \) and \( n \ll t^3 \). First we consider \( n > P(t) + Y \). We view the integral

\[
\int_{-V/4}^{V/4} \gamma(1/2 + i(t + u)) n^{-i(t+u)} e^{-(u/G)^2} du
\]

appearing in \( S_1, S_3, S_5 \), as a complex integral over the rectangular contour with vertices \( \pm V/4, \pm V/4 - iG \). We then use the well known Stirling’s formula for the \( \Gamma \)-function which states that in any fixed vertical strip \( -\infty < \alpha < \beta < \infty \),

\[
\Gamma(\alpha + it) = (2\pi)^{1/2} t^{\alpha + i\alpha} e^{-(t/2) \frac{\alpha^2}{2}} \left( 1 + O\left( (t/\alpha) \right) \right)
\]

as \( t \to \infty \), in order to estimate \( \gamma(1/2 + i(t + u)) \) along this contour.
Therefore, on the vertical sides, where \( u = \pm V/4 - iw \) with \( w \in [0, G] \), we estimate the factors in the integrand as follows:

\[
\gamma(1/2 + i(t + u))n^{-i(t+u)} = \Delta^{1/4} \left( \frac{2\pi n}{\sqrt{\Delta(t + \frac{V}{4})}} \right)^{-w} \times \\
\times \exp \left\{ i \left( (t + \frac{V}{4}) \log \left( \frac{2\pi n}{\sqrt{\Delta(t + \frac{V}{4})}} \right) - (t + \frac{V}{4}) \right) \right\} (1 + O(1/T))
\]

which is bounded. As for the other factor, we have

\[
e^{-u/G)^2} \ll e^{-(L/4)^2} = e^{-4\log T}.
\]

Hence the value of the integral on the vertical sides becomes \( O(G/T^4) \).

On the horizontal side in the lower half plane, let \( u = v - iG \) with \( v \in [-\frac{V}{4}, \frac{V}{4}] \).

Here \( e^{-u/G)^2} \) is bounded and the other factor is estimated by

\[
\gamma(1/2 + i(t + u))n^{-i(t+u)} \ll \exp \left( -G \log \left( \frac{n}{P(t) + \frac{\sqrt{\Delta}}{8\pi} V} \right) \right) \\
\ll \exp \left( -\frac{GY}{P(T)} \right) = \exp(-cT/L) \ll \exp(-\log(T^4L))
\]

where the constant \( c = \sqrt{\Delta}/(2\pi) \). Hence the value of the integral on the horizontal side is also \( O(G/T^4) \).

For \( n < P(t) - Y \) we take the rectangular contour with vertices \( \pm V/4, \pm V/4 + iG \) in the upper half plane and argue in a similar way to estimate the value of the integral to be \( O(G/T^4) \).

Finally we use the fact that \( r_Q(n) \ll n^{\epsilon_1} \) for any \( \epsilon_1 > 0 \) (see the discussion preceding \((5.21)\)) and estimate \( S_1, S_3, \) and \( S_5 \) as follows

\[
(5.9) \quad S_j \ll GT^{-4} \sum_{n \leq 2^{2G}} r_Q(n)n^{-1/2} \ll GT^{-4+3/2+\epsilon_1} = o(G) \quad j = 1, 3, 5.
\]

From equations \((5.7), (5.8), \) and \((5.9)\), after a change of variable, we conclude that

\[
(5.10) \quad I_2(t) \ll \int_{t-V/4}^{t+V/4} \left| \sum_{n=1}^{\infty} \eta(n)r_Q(n)n^{-1/2+iu}\right| e^{-(t-u)^2/G^2} \, du + o(G).
\]

Now we need to use the transformation formula \((4.5)\) of Lemma \(4\) to transform the sum in \((5.10)\). For that we first choose a rational approximation \( r \) to \( 1/\sqrt{\Delta} \) as follows. Since \( \sqrt{\Delta} \) is a quadratic irrational, we can choose \( r = h/k \) with \( (h, k) = 1 \), such that \( h^{-2} \ll |\sqrt{\Delta} - k/h| \ll h^{-2} \) and \( h \asymp \sqrt{T/Y} \). We also observe that \( k \asymp k \) as \( h/k = r \asymp 1/\sqrt{\Delta} \), which is a constant. Let \( M_j = P(u) + (-1)^j2Y, j = 1, 2 \). Then

\[
m_j = (-1)^j \frac{u}{2\pi} \left( \sqrt{\Delta} - \frac{1}{r} \right) + 2Y \asymp \max(Th^{-2}, Y) \ll Y.
\]

Also we choose

\[
M_j' = P(u) + (-1)^jY, \quad m_j' = (-1)^j \frac{u}{2\pi} \left( \sqrt{\Delta} - \frac{1}{r} \right) + Y \ll Y.
\]
We observe that $M_j, M'_j \asymp T$. It is easy to check that all conditions of the Lemma \[\text{3}\] are satisfied when $T^{1/2+\epsilon} \ll Y \ll T^{1-\epsilon}$ or in other words when $T^\epsilon \ll V \ll T^{1/2-\epsilon}$. Hence by Lemma \[\text{4}\] there exists a smoothing function $\eta$ satisfying \[\text{5.4}\] for which \[\text{4.5}\] holds. Finally we also get the following estimate for $n_j(u)$,

\begin{equation}
 n_j(u) \ll \frac{h^2m^2}{M_j} \ll Y.
\end{equation}

Now we replace the sum in \[\text{5.10}\] by using \[\text{4.5}\]. Then the leading term in \[\text{5.9}\] is $\ll k^{-2}|G_Q(k,-h)| \ll k^{-1}$ which after integration in \[\text{5.10}\] becomes $o(G)$. Similarly, the error term in \[\text{4.5}\] is $\ll h^2k^{-1}Y^{1/2}T^{-3/2}U \log T \ll T^{-1/2+\epsilon}$ which, after integration in \[\text{5.10}\], also becomes $o(G)$.

Thus we obtain the following upper bound on $I_2(t)$,

\begin{equation}
 I_2(t) \ll \int_{t-V/4}^{t+V/4} |\Sigma(u)|e^{-(t-u)^2/G^2} du + o(G),
\end{equation}

where $\Sigma(u)$ is as follows

\begin{equation}
 \Sigma(u) = (hku)^{-1/4}r^u \chi(1/2+iu) \sum_{j=1}^2 (-1)^j \sum_{n<n_j(u)} w_j(n)\rho(n)n^{-1/4} \exp(2\pi i C_1 n) (1 + C_2 n/u)^{-1/4} \exp(i(-1)^j (2u\phi(C_2 n/u) + \pi/4)),
\end{equation}

where $C_1 = h\Delta_0/k - 1/(2hk\Delta_0)$, and $C_2 = \pi/(2hk\Delta_0)$.

We now claim that for $t \in \mathcal{S}$ we can ignore the error term in \[\text{5.12}\]. This is because for these $t$ we have (by \[\text{5.3}\]) $G \ll I_1(t) = |I_2(t)|$. Hence we actually have

\begin{equation}
 I_2(t) \ll \int_{t-V/4}^{t+V/4} |\Sigma(u)|e^{-(t-u)^2/G^2} du.
\end{equation}

Using the lower bound \[\text{5.3}\] of $I_1(t)$ and the upper bound \[\text{5.14}\] of $I_2(t)$ in \[\text{5.2}\] we get

\begin{equation}
 RV \ll G^{-2} \int_{t \in \mathcal{S}} \left( \int_{t-V/4}^{t+V/4} |\Sigma(u)|e^{-(t-u)^2/G^2} du \right)^2 dt.
\end{equation}

Applying the Cauchy-Schwarz inequality to the inner integral, \[\text{5.15}\] becomes

\begin{equation}
 RV \ll G^{-2} \int_{t \in \mathcal{S}} \left( \int (|\Sigma(u)|e^{-\frac{1}{4}(t-u)^2/G^2})^2 du \int (e^{-\frac{1}{4}(t-u)^2/G^2})^2 du \right) dt,
\end{equation}

where the inner integrals are over the range $[t-V/4, t+V/4]$. Note that

\begin{equation}
 \int_{t-V/4}^{t+V/4} e^{-(t-u)^2/G^2} du = \int_{-V/4}^{V/4} e^{-x^2/G^2} dx \leq \sqrt{\pi}G.
\end{equation}

Hence \[\text{5.16}\] and \[\text{5.17}\] yields

\begin{equation}
 RV \ll G^{-1} \int_{t \in \mathcal{S}} \int_{t-V/4}^{t+V/4} |\Sigma(u)|^2 e^{-(t-u)^2/G^2} du.
\end{equation}
By replacing $\mathcal{S}$ by the bigger set $[T/2 + V/4, T - V/4]$ and interchanging the order of integration we get
\begin{align*}
RV & \ll G^{-1} \int_{T/2}^{T} |\Sigma(u)|^2 \int_{u-V/4}^{u+V/4} e^{-(t-u)^2/G^2} dt \, du \ll \int_{T/2}^{T} |\Sigma(u)|^2 du. \tag{5.18}
\end{align*}
Using (5.13) in (5.18), we have the following bound
\begin{align*}
R & \ll V^{-1}(hkT)^{-1/2} \sum_{j=1}^{2} \sum_{n<n_j(T_0)} w_j(n)\rho(n)n^{-1/4} \exp(2\pi C_1 n) \\
& \quad \times (1 + C_2 n/u)^{-1/4} \exp \left( i(-1)^j 2\pi \phi(C_2 n/u) \right) \left| \frac{\phi(C_2 m/u)}{\phi(C_2 n/u)} \right|^2 du. \tag{5.19}
\end{align*}
Suppose that $n_j(T_0) = \max\{n_j(u) : u \in [T/2, T]\}$. By expanding the sum within the modulus in (5.19) we get
\begin{align*}
R & \ll V^{-1}(hk)^{-1/2} T^{1/2} \sum_{j=1}^{2} \sum_{n<n_j(T_0)} |w_j(n)\rho(n)|^2 n^{-1/2} \\
& \quad + V^{-1}(hkT)^{-1/2} \sum_{j=1}^{2} \sum_{m,n<n_j(T_0) \atop m \neq n} |w_j(m)\rho(m)w_j(n)\rho(n)|(mn)^{-1/4} \\
& \quad \times \int \left( (1 + C_2 m/u)^{-1/4} (1 + C_2 n/u)^{-1/4} \right. \\
& \quad \left. \times \exp \left( i(-1)^j 2\pi \left( \frac{\phi(C_2 m/u)}{\phi(C_2 n/u)} \right) \right) \right) \, du \\
& = \Sigma_1 + \Sigma_2. \tag{5.20}
\end{align*}
In $\Sigma_2$, the integral will be over an appropriate subinterval of $[T/2, T]$ depending on $m$ and $n$.

For estimating $\Sigma_1$ we will need an estimate on the mean-square of the coefficients $r_{Q^*}(n)$. Since $r_{Q^*}(n)$ is a complex linear combination of the Dirichlet coefficients of the Dedekind zeta-function $\zeta_k(s)$, we have, for any $\epsilon > 0$, $r_{Q^*}(n) \ll n^\epsilon$ for sufficiently large $n$ depending on $\epsilon$. Therefore the following estimate trivially holds.
\begin{align*}
\sum_{n \leq x} r_{Q^*}^2(n) & = O \left( x^{1+\epsilon} \right), \tag{5.21}
\end{align*}
for any $\epsilon > 0$. In fact, for certain quadratic forms much better results are known (see [18], [19]). Since $\rho(n) \ll r_{Q^*}(n)$ by (4.10), (5.21) implies that the function $m(x)$ defined by
\begin{align*}
m(x) = \sum_{n \leq x} |w_j(n)\rho(n)|^2 & \ll x^{1+\epsilon} \tag{5.22}
\end{align*}
for $x \leq n_j(T_0)$. Hence, by partial summation we get that the inner sum over $n$ in $\Sigma_1$ is
\begin{align*}
\sum_{n<n_j} |w_j(n)\rho(n)|^2 n^{-1/2} & = \int_{0}^{n_j} x^{-1/2} dm(x) = \\
& = m(n_j)n_j^{-1/2} + \frac{1}{2} \int_{0}^{n_j} m(x)x^{-3/2} \, dx \ll n_j^{1/2+\epsilon},
\end{align*}
where \( n_j = n_j(T_0) \). Hence,

\[
\Sigma_1 \ll V^{-1/(hk)}^{-1/2}T^{1/2}Y^{1/2 + \epsilon} \ll V^{-1}Y^{1+\epsilon} \ll T^{1+\epsilon}V^{-2}.
\]

For estimating \( \Sigma_2 \), we use Lemma 3 with \( G(x) = ((1 + C_2m/x)(1 + C_2n/x))^{-1/4} \) and \( F(x) = 2x(\phi(C_2m/x) - \phi(C_2n/x)). \) From the definition \( \phi \) we get

\[
F'(x) = 2 \arcsinh \left((C_2m/x)^{1/2}\right) - 2 \arcsinh \left((C_2n/x)^{1/2}\right) + 4 ((C_2m/x) (1 + C_2m/x))^{1/2} - 4 ((C_2n/x) (1 + C_2m/x))^{1/2}.
\]

Therefore \( G(x)/F'(x) \) is monotonic in the interval \( [T/2, T] \), after replacing \( F(x) \) by \(-F(x)\) if necessary. We also have \( |F'(x)/G(x)| \gg (C_2/T)^{1/2}|\sqrt{m} - \sqrt{n}| \). Hence, by Lemma 3 we get that the integral in \( \Sigma_2 \) is \( \ll (C_2)T^{1/2}|\sqrt{m} - \sqrt{n}|^{-1} \ll (hkT)^{1/2}|\sqrt{m} - \sqrt{n}|^{-1}. \) Thus we estimate \( \Sigma_2 \) as follows.

\[
\Sigma_2 \ll V^{-1} \sum_j \sum_{m, n < n_j} |w_j(m)\rho(m)w_j(n)\rho(n)|(mn)^{-1/4}|\sqrt{m} - \sqrt{n}|^{-1}
\]

\[
 \ll V^{-1} \sum_j \sum_{n < m < n_j} |w_j(n)\rho(n)w_j(m)\rho(m)|n^{-1/4}m^{1/4}(m - n)^{-1}
\]

\[
 \leq V^{-1} \sum_j \sum_{h < n_j} h^{-1} \sum_{n < n_j - h} |w_j(n)\rho(n)w_j(n + h)\rho(n + h)|n^{-1/4}(n + h)^{-1/4},
\]

where \( n_j = n_j(T_0) \). Using the Cauchy-Schwarz inequality, the inner sum over \( n \) is bounded by

\[
\ll \left( \sum_{n < n_j} |w_j(n)\rho(n)|^2 n^{-1/2} \right)^{1/2} \left( \sum_{n + h < n_j} |w_j(n + h)\rho(n + h)|^2(n + h)^{-1/2} \right)^{1/2}.
\]

By using (5.22) and partial summation as was done in estimating \( \Sigma_1 \), we conclude that the above is

\[
\ll \left( n_j^{1/2 + \epsilon}n_j^{3/2 + \epsilon} \right)^{1/2} = n_j^{1+\epsilon}.
\]

Hence, we combine this with \( \sum_{h < x} h^{-1} \ll \log x \) to finally get the following estimate

\[
\Sigma_2 \ll V^{-1} \sum_j n_j^{1+\epsilon} \log n_j \ll V^{-1} Y^{1+\epsilon} \ll T^{1+\epsilon}V^{-2}.
\]

Thus from (5.23), (5.24) and (5.20), we get

\[
R \ll T^{1+\epsilon}V^{-2}.
\]

This completes the proof of the theorem.

REFERENCES

[1] R. Balasubramanian, An improvement of a theorem of Titchmarsh on the mean square of \( |\zeta(1/2 + it)| \), Proc. London Math. Soc. 36, (1978) 540–576.
[2] P. T. Bateman and E. Grosswald, On Epstein’s zeta function. Acta Arith. 9 (1964), 365–373.
[3] Bruce Berndt, The number of zeros of the Dedekind zeta-function on the critical line. J. Number Theory 3 (1971), 1–6.
[4] K. Chandrasekharan and Raghavan Narasimhan, Zeta-functions of ideal classes in quadratic fields and their zeros on the critical line, Comment. Math. Helv. 43 (1968) 18–30.
[5] S. Chowla and A. Selberg, On Epstein’s zeta function. I. Proc. Nat. Acad. Sci. U. S. A. 35, (1949). 371–374.
H. Davenport and H. Heilbronn, *J. London Math. Soc.* 11 (1936), 181–185; ibid. 11 (1936), 307–312.

G. H. Hardy and J. E. Littlewood, Contributions to the theory of the Riemann zeta-function and the distribution of primes, *Acta Math.* Vol. 41, (1918) 119–196.

D. R. Heath-Brown, Simple zeros of the Riemann zeta-function on the critical line, *Bull. London Math. Soc.* 11, (1979) 17-18.

A. Ivić, The Riemann zeta-function, The theory of the Riemann zeta-function with applications. *A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York*, (1985). xvi+517.

A. Ivić and M. Jutila, Gaps between consecutive zeros of the Riemann zeta-function on the critical line. *Monatsb. Math.* Vol. 105, no. 1, (1988), 59–73.

M. Jutila, 'Exponential Sums Connected with Quadratic Forms', *Number Theory* (ed. Richard A. Mollin), 271-286, Walter de Gruyter & Co, Berlin-New York, 1990.

M. Jutila and K. Srinivas, Gaps between the zeros of Epstein’s zeta-functions on the critical line. *Bull. London Math. Soc.* 37 (2005), no. 1, 45–53.

A. A. Karatsuba, On the distance between consecutive zeros of the Riemann zeta-function on the critical line (Russian), *Trudy Mat. Inst. Steklova AN SSSR*, 157 (1981), 49–63.

A. A. Karatsuba, The distribution of zeros of $\zeta(\frac{1}{2} + it)$ (Russian), *Izv. Akad. Nauk SSSR ser. mat.* 48, (1984) 1214-1224.

J. Moser, On a theorem of Hardy-Littlewood in the theory of the Riemann zeta-function (Russian), *Acta Arith.* 31, 45–51 (1976b) and 35, 403–404 (1979).

M. Ram Murty and R. Osburn, Representations of integers by certain positive definite binary quadratic forms. *Ramanujan J.* 14 (2007), no. 3, 351–359.

R. Osburn, A remark on a conjecture of Borwein and Choi. *Proc. Amer. Math. Soc.* 133 (2005), no. 10, 2903–2909.

H. S. A. Potter and E. C. Titchmarsh, The zeros of Epstein’s zeta-functions, *Proc. London Math. Soc.* 39 (2) (1935) 372–384.

K. Ramachandra, On the Mean-Value and Omega-Theorems for the Riemann Zeta-Function, *Tata Inst. Fund. Res. Lect. Math.* 85 (Springer, Berlin/Heidelberg/New York/Tokyo, 1995).

A. Sankaranarayanan, Zeros of quadratic zeta-functions on the critical line, *Acta Arith.* 69 (1995), 21-37.

Atle Selberg, On the zeros of Riemann’s zeta-function, *Skr. Norske Vid. Akad. Oslo I.* 10(1942) 1–59.

R. A. Smith, 'The average order of a class of arithmetic functions over arithmetic progressions with applications to quadratic forms', *J. reine angew. Math.* 317 (1980), 74-87.

H. M. Stark, On the zeros of Epstein’s zeta function, *Mathematika*, 14 (1967), 47–55.

E. C. Titchmarsh, The theory of the Riemann zeta-function, Second edition. Edited and with a preface by D. R. Heath-Brown, *The Clarendon Press, Oxford University Press*, New York, 1986.

S. M. Voronin, 'On the zeros of zeta-functions of quadratic forms' (in Russian), *Trudy Mat. Inst. Steklo*, 142 (1976), 135-147.

**Institute of Mathematical Sciences, CIT Campus, Tharamani, Chennai 600 113, India**

E-mail address, Anirban Muckhopadhyay: anirban@imsc.res.in

E-mail address, Kotyada Srinivas: srini@imsc.res.in

E-mail address, Krishnan Rajkumar: kkrishnan@imsc.res.in