ON THE $E_\alpha$-ENVELOPES OF HYPERCENTRAL SUBGROUPS

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Abstract. The $E_k$ envelopes that generalize the double centralizers form a descending chain. In this paper we show that this descending chain stops after finitely many steps for hypercentral subgroups by defining the transfinite forms of some basic descriptions. In particular, we prove that the $E_\alpha$-envelopes of hypercentral subgroups are solvable in the class of groups satisfying chain condition on centralizers. These extend previous results on $E_k$ envelopes.

1. Introduction

This paper continues a line of research in the footsteps of [1] and [3] and analyzes the properties of a technical tool, namely the $E_k$-envelopes introduced in [1] to prove some definability properties (in the sense of the first-order logic) in the class of $M_C$-groups, the groups that satisfy the descending chain condition on centralizers, i.e. that do not have infinite descending chains of centralizers of subsets. Several important classes of groups, of which stable groups in model theory are a notable example, satisfy the descending chain condition on centralizers. The introduction of the paper by Roger Bryant ([2]) contains a detailed description of the basic properties of $M_C$-groups.

In [1], Altınel and Baginski showed that in an $M_C$-group, every nilpotent subgroup is contained in a definable subgroup of the same nilpotency class. In doing this, they introduced special enveloping subgroups of an arbitrary subgroup $H$, denoted $E_k(H)$ ($k \in \mathbb{N}$). If $G$ is an arbitrary group, $H$ a subgroup of $G$, then $E_k(H)$ is a double centralizer of $H$ in a special section of $G$. For every subgroup $H \leq G$, the $E_k(H)$ form a descending chain. In [3], group theoretic and topological properties of $E_k$ chains were analyzed. It was shown that if $G$ is an arbitrary group and $H \leq G$ is nilpotent the descending chain $(E_k(H))$ stabilizes after finitely many steps. This conclusion was based on another that showed if $H$ is $k$-nilpotent then so is $E_k(H)$.

In this paper we continue our investigation of these envelopes in a broader context. We analyze the envelopes of hypercentral subgroups of arbitrary groups and also of $M_C$-groups. This broader analysis necessitates an ordinal-indexed version of our envelopes that we denote using greek letters, the $E_\alpha$-envelopes. Their definition is the natural continuation of the integer-indexed envelopes. In Theorem 4.3, we obtain a new finiteness condition.

This work was supported by TÜBİTAK, the Scientific and Technological Research Council of Turkey, through its programs 2214A and 2211E.
The organization of the paper is as follows. In section 2, we revise various tools. Section 3 is devoted to the $E_n$-envelopes. In section 4, we use the technical bases set up in section 3 to prove the main results of the paper.

2. Preliminaries

In this section we will review the main facts required for the present paper. Our notation is standard for basic group-theoretic notions: the normalizer of any subset $H$ in $G$ is $N_G(H) = \{g \in G \mid \forall h \in H \quad g^{-1}hg \in H\}$; the centralizer of $H$ in $G$ is $C_G(H) = \{g \in G \mid \forall h \in H \quad gh = hg\}$; $[g, h] := g^{-1}h^{-1}gh$ is the commutator of $g, h \in G$ elements; when $A, B \subseteq G$ we write $[A, B] := \{[a, b] \mid a \in A, b \in B\}$. Also, we write $H \leq G$ to denote that $H$ is a subgroup of $G$ and $H \leq G$ to denote $H$ is normal in $G$. In particular, $\mathcal{M}_C$ denotes the class of groups satisfying the minimal condition on centralizers.

We recall the definition of $E_k$ envelopes, introduced in [1].

Definition 2.1. Let $G$ be a group and $H$ a subgroup. For $k \in \mathbb{N}$, a sequence of subgroups $E_k(H)$ of $G$ is defined
\[
E_k(H) = \left\{g \in E_{k-1}(H) \mid \left[g, C_{E_{k-1}(H)}^k(H)\right] \leq C_{E_{k-1}(H)}^{k-1}(H)\right\}
\]
if $k > 0$ and $E_0(H) = G$.

It is clear that $E_1(H) = C_G(C_G(H))$. We remind a simple fact from [3]:

Fact 2.2. Let $G$ be a group and $H$ an abelian subgroup of $G$. Then $C_G(C_G(H))$ is abelian.

Definition 2.3 ([2]). Let $A$ be any subset of the group $G$. Set $C^0_G(A) = 1$ and for $k \geq 1$, the $k$th iterated centralizer of $A$ in $G$ is
\[
C^k_G(A) = \left\{x \in \bigcap_{n<k} N_G(C^n_G(A)) \mid [x, A] \subseteq C^{k-1}_G(A)\right\}.
\]

One can show by induction that the iterated centralizers form an ascending sequence: $1 = C^0_G(H) \leq C^1_G(H) \leq \ldots \leq G$. In contrast with iterated centralizers $E_k$ envelopes form a descending sequence such as $G = E_0(H) \geq E_1(H) \geq \ldots \geq H$.

When $A = G$, the $k$th iterated centralizer of $G$ is more commonly known as $Z_k(G)$ and defined as follows:

Definition 2.4. Let $G$ be a group. Setting $Z_0(G) = \{1\}$, the $k$th center of $G$ is
\[
Z_k(G) = \{g \in G \mid [g, G] \subseteq Z_{k-1}(G)\}
\]
for all $k \geq 1$.

Some of the basic relations between the iterated centralizers and iterated centers are stated below:

Fact 2.5 ([1, Lemma 2.5]). Let $A \leq B \leq C$ be groups and suppose that for all $j \leq k$ we have $C_j^C(A) = Z_j(C)$. Then
(i) $C_j^C(A) = C_j^C(B) = Z_j(C), \forall j \leq k$.
(ii) $C^j_B(A) = Z_j(B) = Z_j(C) \cap B, \forall j \leq k$
(iii) $C^{k+1}_B(A) = C^{k+1}_C(A) \cap B, \forall j \leq k$.

**Fact 2.6** ([1, Theorem 3.7, (2)]). Let $G$ be an arbitrary group and $H$ a subgroup of $G$. Then

$$C^j_{E_k(H)}(H) = Z_j(E_k(H))$$

for all $j \leq k$.

In [3], the relation between any nilpotent subgroup of an arbitrary group and its $E_k$ envelope is given.

**Fact 2.7** ([3]). Let $G$ be a group and $H \leq G$. If $H$ is a $k$-nilpotent subgroup, then the envelope $E_k(H)$ is also $k$-nilpotent.

Hypercentral groups generalize nilpotent groups. These groups can be characterized in terms of the transfinitely extended upper central series, which is defined in the following manner.

**Definition 2.8.** Let $G$ be a group and $\alpha$ an ordinal, the terms $Z_\alpha(G)$ of the upper central series of $G$ are defined by the usual rules

$$Z_0(G) = \{1\} \quad \text{and} \quad Z_{\alpha+1}(G) / Z_\alpha(G) = Z(G / Z_\alpha(G))$$

together with the completeness condition

$$Z_\lambda(G) = \bigcup_{\alpha < \lambda} Z_\alpha(G)$$

where $\lambda$ is a limit ordinal. The transfinite upper central series terminate with a subgroup called the hypercenter of $G$. It sometimes convenient to call $Z_\alpha(G)$ the $\alpha$–hypercenter of $G$. A group $G$ is called hypercentral if $G = Z_\alpha(G)$ for some ordinal $\alpha$. The smallest such $\alpha$ is called the degree of hypercentrality of $G$.

The following is a formal definition of a group satisfying the descending chain condition on centralizers.

**Definition 2.9.** Let $G$ be a group. If there exist no infinite sequence of subsets $A_n \subseteq G$ such that $C_G(A_n) \supseteq C_G(A_{n+1})$ for all $n \in \mathbb{N}$, then $G$ has the chain condition on centralizers and denoted as $M_C$. By elementary properties of centralizers the descending chain condition on centralizers is equivalent to the ascending chain condition on centralizers.

The following property of $M_C$-groups will be useful in the paper.

**Fact 2.10** ([2, Corollary 2.3]). Let $G$ be a locally nilpotent $M_C$-group. Then $G$ is solvable.
3. Technical Definitions and Facts

In this section, we will introduce several technical notions and prove their properties needed for our main result, Theorem 4.3. For this purpose, we shall define the transfinite forms of $E_k$ definable envelopes and iterated centralizers. After that, we will prove the transfinite forms of the Facts 2.5 and 2.6. For simplicity, we will denote $E_\alpha (H)$ by $E_\alpha$ when the context is clear.

**Definition 3.1.** Let $H$ be any subgroup of the group $G$. Set $C_0^G (H) = 1$ and for a ordinal number, the $C_\alpha^G (H)$ iterated centralizers are defined as follow:

(i) If $\alpha$ is a successor ordinal

$$C_\alpha^G (H) = \left\{ x \in \cap_{\beta < \alpha} N_G \left( C_\beta^G (H) \right) \mid [x, H] \subseteq C_\alpha^{-1} (H) \right\},$$

(ii) If $\alpha$ is a limit ordinal

$$C_\alpha^G (H) = \cup_{\beta < \alpha} C_\beta^G (H).$$

**Definition 3.2.** Let $H$ be any subgroup of the group $G$. Set $E_0 (H) = G$ and for a ordinal number, the $E_\alpha (H)$ envelopes are defined as follow:

(i) If $\alpha$ is a successor ordinal

$$E_\alpha (H) = \left\{ g \in E_{\alpha-1} (H) \mid \left[ g, C_\alpha^{E_{\alpha-1}(H)}(H) \right] \leq C_{E_{\alpha-1}(H)}^{\alpha-1}(H) \right\},$$

(ii) If $\alpha$ is a limit ordinal

$$E_\alpha (H) = \cap_{\beta < \alpha} E_\beta (H).$$

When the subgroup is clear, we will shorten $E_\alpha (H)$ to $E_\alpha$.

Now we will give a technical lemma that generalizes Fact 2.5.

**Lemma 3.3.** Let $A \leq B \leq C$ be groups and $\lambda$ an ordinal such that

$$C_\alpha^C (A) = Z_\alpha (C)$$

for all $\alpha \leq \lambda$ ordinal numbers. Then the following equalities hold:

(i) $C_\alpha^C (A) = C_\alpha^C (B) = Z_\alpha (C)$

(ii) $C_\beta^B (A) = Z_\alpha (B) = Z_\alpha (C) \cap B$

(iii) $C_\beta^{\lambda + 1} (A) = C_{C_{\beta}^{\lambda + 1}} (A) \cap B$.

**Proof.** We proceed by transfinite induction. For $\alpha = 0$, our claims are trivial. Now we will show that the claims (i) and (ii) are true both of the successor and limit ordinals. Suppose that the claims (i) and (ii) hold for all $\beta < \lambda$ ordinals.

(i)

- Let $\alpha$ be a successor ordinal, namely $\alpha = \beta + 1$. By the hypothesis of theorem, it is known that $C_{\beta + 1}^C (A) = Z_{\beta + 1} (C)$ since $\beta + 1 \leq \lambda$ when
\( \beta < \lambda \). From this fact and the induction hypothesis we have

\[
C_{C}^{\beta + 1} (B) = \left\{ x \in \bigcap_{\gamma < \beta + 1} N_C (C_{C}^{\gamma} (B)) \mid [x, B] \subseteq C_{C}^{\beta} (B) \right\}
\]

\[
= \left\{ x \in \bigcap_{\gamma < \beta + 1} N_C (Z_{\gamma} (C)) \mid [x, B] \subseteq Z_{\beta} (C) \right\}
\]

\[
= \left\{ x \in C \mid [x, B] \subseteq C_{C}^{\beta} (A) \right\} \subseteq C_{C}^{\beta + 1} (A)
\].

Then

\[
C_{C}^{\beta + 1} (A) = Z_{\beta + 1} (C) = C_{C}^{\beta + 1} (C) \subseteq C_{C}^{\beta + 1} (B) \subseteq C_{C}^{\beta + 1} (A)
\],

so \( C_{C}^{\beta + 1} (A) = C_{C}^{\beta + 1} (B) \) is obtained.

- If \( \alpha \) is a limit ordinal; by using Definition \ref{d:limit} and the previous step, the following equations are obtained:

\[
C_{C}^{\alpha} (A) = \bigcup_{\beta < \alpha} C_{C}^{\beta} (A) = \bigcup_{\beta < \alpha} C_{C}^{\beta} (B) = C_{C}^{\alpha} (B),
\]

\[
C_{C}^{\alpha} (A) = \bigcup_{\beta < \alpha} Z_{\beta} (C) = Z_{\alpha} (C).
\]

Thus claim (i) holds for all ordinal numbers.

(ii)

- If \( \alpha \) is a successor ordinal, namely \( \alpha = \beta + 1 \); from Definition \ref{d:limit} and the induction hypothesis,

\[
C_{B}^{\beta + 1} (A) = \left\{ x \in \bigcap_{\delta < \beta + 1} N_B (C_{B}^{\delta} (A)) \mid [x, A] \subseteq C_{B}^{\beta} (A) \right\}
\]

\[
= \left\{ x \in \bigcap_{\delta < \beta + 1} N_B (Z_{\delta} (B)) \mid [x, A] \subseteq Z_{\beta} (B) \right\}
\]

\[
= \left\{ x \in B \mid [x, A] \subseteq Z_{\beta} (B) \right\}
\]

and also

\[
C_{C}^{\beta + 1} (A) \cap B = \left\{ x \in C \mid [x, A] \subseteq C_{C}^{\beta} (A) \right\} \cap B
\]

\[
= \left\{ x \in C \mid [x, A] \subseteq Z_{\beta} (C) \right\} \cap B
\]

\[
= \left\{ x \in B \mid [x, A] \subseteq Z_{\beta} (B) \right\} = C_{B}^{\beta + 1} (A).\]

is written. So, we get

\[
C_{B}^{\beta + 1} (A) = C_{C}^{\beta + 1} (A) \cap B = Z_{\beta + 1} (C) \cap B.
\]

From the first and last terms, one of the equation of claim (ii) is obtained.

On the other hand, it can be written \( Z_{\beta + 1} (C) \cap B = C_{C}^{\beta + 1} (B) \cap B \) by using claim (i). Then we have

\[
Z_{\beta + 1} (C) \cap B = C_{C}^{\beta + 1} (B) \cap B = \left\{ x \in C \mid [x, B] \subseteq C_{C}^{\beta} (B) \right\} \cap B
\]

\[
= \left\{ x \in B \mid [x, B] \subseteq Z_{\beta} (C) \cap B \right\}
\]

\[
= \left\{ x \in B \mid [x, B] \subseteq Z_{\beta} (B) \right\} = Z_{\beta + 1} (B).\]

Thus claim (ii) holds for successor ordinals.
If \( \alpha \) is a limit ordinal; by Definition 3.1 and induction hypothesis for claim \((ii)\) we have
\[
C_\alpha^B(A) = \bigcup_{\beta < \alpha} C_\beta^B(A) = \bigcup_{\beta < \alpha} Z_\beta(B) = Z_\alpha(B)
\]
and
\[
C_\beta^B(A) = \bigcup_{\beta < \alpha} C_\beta^B(A) = C_\beta^B(A) = \bigcup_{\beta < \alpha} (Z_\beta(C) \cap B)
\]
\[
= \left( \bigcup_{\beta < \alpha} Z_\beta(C) \right) \cap B = Z_\alpha(C) \cap B.
\]
Considering these equalities claim \((ii)\) follows for limit ordinals. Thus claim \((ii)\) holds for all ordinal numbers.

We now shall prove claim \((iii)\). By using Definition 3.1 and the truth of claim \((ii)\) we write
\[
C^{\lambda+1}_A \cap B = \{ x \in \bigcap_{\gamma < \lambda+1} N_B(Z_{\gamma}(B)) \mid [x, A] \subseteq Z_{\lambda}(B) \}
\]
(3.3.1)
\[
= \{ x \in B \mid [x, A] \subseteq Z_{\lambda}(B) \}.
\]
On the other hand, for the iterated centralizer \(C^{\lambda+1}_C(A)\)
\[
C^{\lambda+1}_C(A) = \left\{ x \in \bigcap_{\delta \leq \lambda} N_C(C^\delta(A)) \mid [x, A] \subseteq C^\lambda_C(A) \right\}
\]
\[
= \left\{ x \in \bigcap_{\delta \leq \lambda} N_C(Z_{\delta}(C)) \mid [x, A] \subseteq Z_{\lambda}(C) \right\}
\]
\[
= \{ x \in C \mid [x, A] \subseteq Z_{\lambda}(C) \}.
\]
is obtained from Definition 3.1 and claim \((i)\). If the intersection of the iterated centralizer \(C^{\lambda+1}_C(A)\) with group \(B\) is taken and claim \((ii)\) is used, the following equation is obtained:
\[
(3.3.2)
\]
\[
C^{\lambda+1}_C(A) \cap B = \{ x \in C \mid [x, A] \subseteq Z_{\lambda}(C) \} \cap B
\]
\[
= \{ x \in B \mid [x, A] \subseteq Z_{\lambda}(B) \}.
\]
Thus the result follows from the equations \((3.3.1)\) and \((3.3.2)\) \(\square\)

The following lemma is of general interest.

**Lemma 3.4.** Let \((C_\alpha)_{\alpha < \lambda}\) be an ascending sequence of nonempty subsets of a set \(E\). If \(x \in \bigcup_{\alpha < \lambda} C_\alpha\), then there exists a minimal \(\beta\) such that \(x \in C_\beta\) and \(\beta < \lambda\) and \(\beta\) is a successor ordinal.

**Proof.** If \(x \in \bigcup_{\alpha < \lambda} C_\alpha\), then \(x \in C_\beta\), for at least one \(\beta < \lambda\). If \(\beta\) is a limit ordinal, then the set \(\{ \beta \leq \lambda, \beta\ \text{limit ordinal} \mid x \in C_\beta \}\) has a minimal element, \(\beta_0\). Since \(C_{\beta_0} = \bigcup_{\delta < \beta_0} C_\delta\), there exists \(\delta_0 < \beta_0\) such that \(x \in C_{\delta_0}\). By the choice of \(\beta_0\), \(\delta_0\) is successor. \(\square\)

We now prove the iterated centralizers in transfinite form also compose an ascending chain.

**Lemma 3.5.** Let \(\alpha\) be an ordinal number. Then \(C^\alpha_G(H) \leq C^\lambda_G(H)\) for \(\lambda \geq \alpha\).
Lemma 3.6. Let $\lambda$ be an ordinal number. Then $C_{E_\lambda(H)}^\alpha(H) = Z_\alpha(E_\lambda(H))$ for all ordinals such that $\alpha \leq \lambda$.

Proof. We proceed by transfinite induction on $\lambda$. When $\lambda = 0$, $\alpha = 0$. So, our claim is trivial for $\lambda = 0$.
Let $\lambda$ be a successor ordinal, i.e $\lambda = \beta + 1$. In particular $\alpha \leq \beta$. It is known that $C^\alpha_{E_\lambda}(H) = Z_\alpha(E_\beta)$ by induction. So, it will suffice to show the equality

$$C^\alpha_{E_{\beta+1}}(H) = Z_\alpha(E_{\beta+1})$$

for $\alpha = \beta + 1$. Applying Lemma 3.3 (ii) to the $H \leq E_{\beta+1} \leq E_\beta$ triple we have

$$C^\alpha_{E_{\beta+1}}(H) = Z_\alpha(E_{\beta+1}) \subseteq E_{\beta+1} \Rightarrow C^\alpha_{E_{\beta+1}}(H) \leq E_{\beta+1}.$$ 

On the other hand from Definition 3.1 and Lemma 3.3 (ii)

$$[Z_{\beta+1}(E_{\beta+1}), H] \leq Z_\beta(E_{\beta+1}) = C^\beta_{E_{\beta+1}}(H)$$

is obtained. Now we shall show bidual inclusion by using Definition 3.1 and Lemma 3.3

$$C^\beta_{E_{\beta+1}}(H) = \left\{ x \in \bigcap_{\gamma < \beta+1} N_{E_{\beta+1}} \left( C^\gamma_{E_{\beta+1}}(H) \right) \mid [x, H] \subseteq C^\beta_{E_{\beta+1}}(H) \right\} = \left\{ x \in \bigcap_{\gamma < \beta+1} N_{E_{\beta+1}} \left( Z_\gamma(E_{\beta+1}) \right) \mid [x, H] \subseteq Z_\beta(E_{\beta+1}) \right\} = \left\{ x \in E_{\beta+1} \mid [x, H] \subseteq Z_\beta(E_{\beta+1}) \right\}.$$ 

So, we get

$$Z_{\beta+1}(E_{\beta+1}) \leq C^\beta_{E_{\beta+1}}(H) \leq C^\beta_{E_{\beta+1}}(H).$$

Thus the inclusion $Z_{\beta+1}(E_{\beta+1}) \leq C^\beta_{E_{\beta+1}}(H)$ is verified. We now will prove the reverse inclusion. Considering Lemma 3.3 (ii)

$$[C^\beta_{E_{\beta+1}}(H), E_{\beta+1}] = [C^\beta_{E_{\beta}}(H) \cap E_{\beta+1}, E_{\beta+1}]$$

is written. We will show this commutator is in $Z_\beta(E_{\beta+1})$ to verify the inclusion $C^\beta_{E_{\beta+1}}(H) \leq Z_{\beta+1}(E_{\beta+1})$. Considering the definition of $E_{\beta+1}$ and Lemma 3.3 (i)

$$E_{\beta+1} = \left\{ x \in E_\beta \mid x.C^\beta_{E_\beta}(H) \leq C^\beta_{E_\beta}(H) \right\} = \left\{ x \in E_\beta \mid [x, C^\beta_{E_\beta}(H)] \leq Z_\beta(E_{\beta+1}) \right\}.$$ 

Since $E_{\beta+1} \subseteq E_\beta$, the commutator of $x \in E_{\beta+1}$ and $C^\beta_{E_{\beta+1}}(H)$ is in $Z_\beta(E_{\beta+1})$. So, by using this fact and Lemma 3.3

$$[C^\beta_{E_{\beta+1}}(H), E_{\beta+1}] = [C^\beta_{E_{\beta}}(H) \cap E_{\beta+1}, E_{\beta+1}]$$

$$\leq Z_\beta(E_{\beta+1}) = Z_\beta(E_{\beta+1})$$

$$\Rightarrow C^\beta_{E_{\beta+1}}(H) \leq Z_\beta(E_{\beta+1}) \leq Z_{\beta+1}(E_{\beta+1})$$

is obtained. So we are done. Then, the claim holds for $\lambda = \beta + 1$ successor ordinal.

Finally let $\lambda$ be a limit ordinal. While $\alpha < \lambda$, we have $C^\alpha_{E_\lambda}(H) = Z_\alpha(E_\alpha)$ for all $\beta < \alpha$ from inductive hypothesis. Then we can apply Lemma 3.3 to $H \leq E_\lambda \leq E_\alpha$ subgroups for $\alpha \leq \lambda$. So, we get

$$(3.6.1) \quad C^\beta_{E_\lambda}(H) = Z_\beta(E_\lambda)$$

for $\beta < \alpha < \lambda$. When $\beta = \lambda$, consideringly Definition 3.1 and the equality 3.6.1 the following result is obtained:

$$C^\lambda_{E_\lambda}(H) = \bigcup_{\beta < \lambda} C^\beta_{E_\lambda}(H) = \bigcup_{\beta < \lambda} Z_\beta(E_\lambda) = Z_\lambda(E_\lambda).$$

Thus, our claim follows for ordinal numbers. □
In the rest of this section, we will prove a special ascendance property of the $E_\alpha$-envelopes.

**Lemma 3.7.** Let $\alpha$ be an ordinal number. Then $Z_\alpha (E_\alpha) \leq E_\lambda$ for $\alpha \leq \lambda$.

**Proof.** For $\alpha = \lambda$, since $Z_\alpha (E_\alpha) \leq E_\alpha$, our claim is clear. Suppose that $\alpha < \lambda$. Let $\lambda$ be a successor ordinal, i.e. $\lambda = \beta + 1$. By induction, for $\alpha \leq \beta$ we have $Z_\alpha (E_\alpha) \leq E_\beta$. It is known that $E_\beta \leq E_\alpha$ for $\alpha \leq \beta$ by Definition 3.2. Applying Lemma 3.3 (ii) to the $H \leq E_\beta \leq E_\alpha$ groups,

$$Z_\alpha (E_\beta) = Z_\alpha (E_\alpha) \cap E_\beta$$

is obtained. By using the induction hypothesis we have

$$Z_\alpha (E_\beta) = Z_\alpha (E_\alpha) \cap E_\beta = Z_\alpha (E_\alpha).$$

On the other hand considering the facts that $Z_\alpha (E_\beta) \leq Z_\beta (E_\beta)$ for $\alpha \leq \beta$ and $Z_\beta (E_\beta) \leq E_\beta$ we get

$$Z_\alpha (E_\alpha) = Z_\alpha (E_\alpha) \cap E_\beta = Z_\alpha (E_\beta) \leq Z_\beta (E_\beta).$$

It remains to show that

$$\left[ Z_\alpha (E_\alpha (H)), C_{E_\beta (H)}^{\beta+1} \right] \leq Z_\beta (E_\beta (H))$$

to prove the claim for successor ordinals. Since $C_{E_\beta (H)}^{\beta+1} (H) \leq E_\beta (H)$ and $Z_\alpha (E_\alpha) \leq E_\beta$ by induction, we find the following inclusion

$$\left[ Z_\alpha (E_\alpha (H)), C_{E_\beta (H)}^{\beta+1} \right] \subseteq Z_\alpha (E_\alpha) \subseteq Z_\beta (E_\beta).$$

According to this $Z_\alpha (E_\alpha) \leq E_\beta$. Then our claim holds for $\lambda = \beta + 1$ successor ordinal.

For $\lambda$ limit ordinal, let $Z_\alpha (E_\alpha)$ be a subgroup of $E_\beta$ for $\alpha \leq \beta < \lambda$. From Definition 3.2

$$E_\lambda = \bigcap_{\beta < \lambda} E_\beta = \bigcap_{\alpha \leq \beta < \lambda} E_\beta \geq Z_\alpha (E_\alpha)$$

is written. Thus, the result follows from the first and last terms. \hfill \Box

**Corollary 3.8.** Let $\alpha$ be an ordinal number. Then

$$Z_\alpha (E_\alpha) \leq Z_\lambda (E_\lambda)$$

for $\alpha \leq \lambda$.

**Proof.** When $\alpha \leq \lambda$, $E_\lambda \leq E_\alpha$. By Lemma 3.3 and 3.7 we have

$$Z_\lambda (E_\lambda) \geq Z_\alpha (E_\lambda) = Z_\alpha (E_\alpha) \cap E_\lambda = Z_\alpha (E_\alpha).$$

So, we are done. \hfill \Box
4. Hypercentral Subgroups

In this section, we apply the technical tools developed in the previous sections to the analysis of the $E_{\alpha}$-envelopes of hypercentral subgroups of various classes of groups. This allows us to draw conclusions on hypercentral subgroups of $\mathfrak{M}_{\alpha}$-groups (Corollary 4.2) and prove a general finiteness result (Theorem 4.3) which is the main conclusion of the paper.

Proposition 4.1. Let $G$ be a group and $H \leq G$. Let $\alpha$ be an ordinal. Then:

(i) If $H$ is an $(\alpha + 1)$-hypercentral subgroup, $E_{\alpha+1} (H)$ is also $(\alpha + 1)$-hypercentral.

(ii) If $H$ is an $\alpha$-hypercentral subgroup, then $E_{\alpha+1} (H)$ is at most $\alpha$-hypercentral.

Proof. (i) Let $H$ be an $(\alpha + 1)$-hypercentral subgroup. By using the second isomorphism theorem and Lemma 3.3 (ii) for the triple $H \leq H \leq E_{\alpha}$

$$HZ_{\alpha} (E_{\alpha}) / Z_{\alpha} (E_{\alpha}) \cong H / H \cap Z_{\alpha} (E_{\alpha}) = H / Z_{\alpha} (H)$$

is written. Since $H$ is $(\alpha + 1)$-hypercentral, $H / Z_{\alpha} (H)$ is abelian. By the second isomorphism theorem, Lemma 3.7 and applying Lemma 3.3 (ii) to $H \leq E_{\alpha+1} (H) \leq E_{\alpha} (H)$ triple

$$E_{\alpha+1} / Z_{\alpha} (E_{\alpha}) \cong E_{\alpha+1} / E_{\alpha+1} \cap Z_{\alpha} (E_{\alpha}) = E_{\alpha+1} / Z_{\alpha} (E_{\alpha+1})$$

is obtained. On the other hand, by the definition of $E_{\alpha+1}$ and $C_{E_{\alpha}}^{\alpha+1} (H)$, we have

$$E_{\alpha+1} / Z_{\alpha} (E_{\alpha}) = C_{E_{\alpha}} / Z_{\alpha} (E_{\alpha}) \left( C_{E_{\alpha}} / Z_{\alpha} (E_{\alpha}) (HZ_{\alpha} (E_{\alpha}) / Z_{\alpha} (E_{\alpha})) \right).$$

Since $H / Z_{\alpha} (H)$ is abelian and $HZ_{\alpha} (E_{\alpha}) / Z_{\alpha} (E_{\alpha}) \cong H / Z_{\alpha} (H)$, the following group

$$E_{\alpha+1} / Z_{\alpha} (E_{\alpha}) \cong E_{\alpha+1} / Z_{\alpha} (E_{\alpha+1}).$$

is also abelian from Fact 2.2. Thus the subgroup $E_{\alpha+1}$ is at most $(\alpha + 1)$-hypercentral. But at the same time $E_{\alpha+1}$ is exactly $(\alpha + 1)$-hypercentral since the hypercentrality class of a group can not be smaller than the hypercentrality class of subgroup.

(ii) When $\alpha$ is a limit ordinal $H / Z_{\alpha} (H) = 1$ since $Z_{\alpha} (H) = H$. So, we have

$$HZ_{\alpha} (E_{\alpha}) / Z_{\alpha} (E_{\alpha}) \cong H / Z_{\alpha} (H) = 1.$$
is written for $\beta < \alpha$. When $\beta = \alpha$, from the hypothesis of proposition and the fact that $E_{\alpha+1}$ is at most $(\alpha + 1)$-hypercentral, we have

\[(4.1.1) \quad Z_\alpha(H) = Z_{\alpha+1}(H) \Rightarrow Z_\alpha(E_\alpha) = Z_{\alpha+1}(E_{\alpha+1}) = E_{\alpha+1}.\]

Considering the equality (4.1.1) we get

\[(4.1.2) \quad Z_\alpha(E_{\alpha+1}) = Z_\alpha(E_\alpha) \cap E_{\alpha+1} = Z_\alpha(E_\alpha) = Z_{\alpha+1}(E_{\alpha+1}) = E_{\alpha+1}.\]

Here Lemma 3.3 (ii) was applied to $H \leq E_{\alpha+1} \leq E_\alpha$ subgroups for the first equation, while Lemma 3.7 and the fact that at most $(\alpha + 1)$-hypercentrality of $E_{\alpha+1}(H)$ were used respectively for the second and third equations. So $Z_\alpha(E_{\alpha+1}) = E_{\alpha+1}$.

\[\square\]

We now prove a corollary of Proposition 4.1 for an $M_c$-group.

**Corollary 4.2.** Let $G$ be an $M_c$-group and $H$ be an $\alpha$-hypercentral subgroup of $G$. Then $E_{\alpha+1}(H)$ is solvable.

**Proof.** By Proposition 4.1 (ii) $E_{\alpha+1}(H)$ is hypercentral. Since any hypercentral group is locally nilpotent and the class $M_c$ is closed under the formation of subgroups, $E_{\alpha+1}(H)$ is a locally nilpotent $M_c$-group. By Fact 2.10 $E_{\alpha+1}(H)$ envelope is solvable.

\[\square\]

The conclusion of this corollary is the best possible in this direction. Indeed, the following example shows that the envelope of an hypercentral subgroup of an $M_c$-group be non nilpotent:

*Let $G$ denote $GL_2(\mathbb{C})$ that has $M_c$-property and the hypercentral subgroup from the successor ordinal degree of $G$*

\[H_\infty = \langle \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} | \lambda_i^{2^n} = 1, \lambda_i \in \mathbb{C}, i, n \in \mathbb{N} \rangle \leq G\]

The envelopes of $H_\infty$ subgroup are determined as follows:

- $E_0(H_\infty) = E_1(H_\infty) = G$,
- For $n \in \mathbb{N}, n \neq 0, 1; E_\alpha(H_\infty) = \langle \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} | x, y \in \mathbb{C}^* \rangle \rtimes \langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle$,
- $E_\omega(H_\infty) = \langle \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} | x, y \in \mathbb{C}^* \rangle \rtimes \langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle$,
- $E_{\omega+1}(H_\infty) = \langle \begin{pmatrix} x & 0 \\ 0 & x(2^n) \end{pmatrix} | x \in \mathbb{C}^*, n \in \mathbb{N}^* \rangle \rtimes \langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle$.  

where \(1^{(\frac{1}{2})^n}\) denotes the \(2^n - \text{th}\) roots of 1. \(E_{\omega+1}(H_\infty)\) is a hypercentral subgroup from \((\omega + 1) - \text{th}\) degree by Proposition 4.7 and so it is locally nilpotent. At the same time \(E_{\omega+1}(H_\infty)\) is a \(M_\omega\)-group since \(G\) is an \(M_\omega\)-group. By Fact 2.11 \(E_{\omega+1}(H_\infty)\) is solvable. But it is not nilpotent.

Now we shall prove the main result of the paper, a finiteness condition that extends Corollary 3.1.2 of [3]:

**Theorem 4.3.** Let \(\alpha\) be a limit ordinal, \(G\) a group and \(H\) an \(\alpha\)-hypercentral subgroup of \(G\). Then \(E_{\alpha+1}(H) = E_\lambda(H)\) for all ordinals \(\lambda\) such that \(\alpha + 1 \leq \lambda\).

**Proof.** We proceed by transfinite induction on \(\lambda\). For \(\lambda = \alpha + 1\), the claim is trivially satisfied. If \(\lambda\) is successor ordinal strictly bigger than \(\alpha + 1\), i.e \(\lambda = \beta + 1\), it is known that the claim is true for all \(\alpha + 1 \leq \beta\) ordinals by induction. So, the following sequence of equalities holds:

\[
E_{\beta+1} = \left\{ x \in E_\beta \mid x, C_{E_\beta}^{\beta+1}(H) \leq C_{E_\beta}^\beta(H) \right\} = \left\{ x \in E_\beta \mid x, C_{E_\beta}^{\beta+1}(H) \leq Z_\beta(E_\beta) \right\} = \left\{ x \in E_\beta \mid x, C_{E_\beta}^{\beta+1}(H) \leq E_\beta \right\} = E_\beta = E_{\alpha+1},
\]

using also Lemma 3.6 and Proposition 4.1 (ii). If \(\lambda\) limit is limit ordinal, by using the facts that the \(E_\alpha\) form a descending chain and \(E_{\alpha+1} = E_\beta\) for all ordinals such that \(\alpha + 1 \leq \beta\), we get

\[
E_\lambda = \bigcap_{\beta<\lambda} E_\beta = \bigcap_{\alpha+1\leq\beta<\lambda} E_{\alpha+1} = E_{\alpha+1}.
\]

Thus the result follows for all ordinals \(\lambda\) such that \(\alpha + 1 \leq \lambda\). \(\square\)

5. **Acknowledgements**

The author is deeply grateful to her Ph.D. supervisors Tuna Altunel and Erdal Karaduman for their valuable comments and suggestions all along the way. She would also like to acknowledge the warm hospitality of Université Claude Bernard Lyon-1 where the results of the article were obtained.

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