Tight Regret Bounds for Infinite-armed Linear Contextual Bandits

Yingkai Li
Department of Computer Science
Northwestern University

Yining Wang
Machine Learning Department
Carnegie Mellon University

Yuan Zhou
Computer Science Department, Indiana University at Bloomington
Department of IESE, University of Illinois at Urbana-Champaign

Abstract

Linear contextual bandit is a class of sequential decision making problems with important applications in recommendation systems, online advertising, healthcare, and other machine learning related tasks. While there is much prior research, tight regret bounds of linear contextual bandit with infinite action sets remain open. In this paper, we prove regret upper bound of \( O(\sqrt{d^2 T \log T}) \times \text{poly}(\log \log T) \) where \( d \) is the domain dimension and \( T \) is the time horizon. Our upper bound matches the previous lower bound of \( \Omega(\sqrt{d^2 T \log T}) \) in (Li et al., 2019) up to iterated logarithmic terms.

1 Introduction

Linear contextual bandit is a class of sequential decision making problems with an extensive history of research in both machine learning and operations research (Abbasi-Yadkori et al., 2011; Chu et al., 2011; Auer, 2002; Rusmevichientong & Tsitsiklis, 2010; Dani et al., 2008; Li et al., 2019). In the linear contextual bandit problem, a player makes sequential decisions over \( T \) time periods. At each time period \( t \), an action set \( D_t \subseteq \mathbb{R}^d \) is provided; the player would select an action \( x_t \in D_t \), and subsequently receive a reward \( r_t \) parameterized as

\[
    r_t = \langle x_t, \theta \rangle + \xi_t,
\]

where \( \theta \in \mathbb{R}^d \) is a fixed but unknown regression model, and \( \{\xi_t\} \) are independent centered sub-Gaussian noise variables with variance proxy 1. The performance is evaluated by the cumulative regret, defined as

\[
    R_T := \sum_{t=1}^{T} \sup_{x \in D_t} \langle x, \theta \rangle - \langle x_t, \theta \rangle.
\]

The objective of this paper is to design algorithms that achieve the optimal expected regret under the worst case, when the action sets \( \{D_t\} \) are infinite (i.e., \( |D_t| = \infty \)). Our main results and comparison with existing work are summarized in the next section.

1.1 Existing work and our results

A summarization of our results as well as existing results is given in Table 1. The regularity conditions that \( \|\theta\|_2 \leq 1 \) and \( D_t \subseteq \{x \in \mathbb{R}^d : \|x\|_2 \leq 1\} \) are imposed, so that \( \mathbb{E}[r_t] = |\langle x_t, \theta \rangle| \leq 1 \)

*Author names listed in alphabetical order.

Preprint. Under review.
holds for all \( x_t \in D_t \). Additionally, as suggested by the title, we consider the infinite-armed case in which \( |D_t| = \infty \) for all \( t \).

For ease of presentation, we additionally assume that the action sets \( D_t \) are closed, so that the supremum over the sets can always be achieved by an action. This assumption can be easily removed by adding a slack of \( O(\sqrt{1/T}) \) whenever we choose an arm to maximize the UCB or the real expected value.

[Dani et al. (2008)] derived an algorithm based on confidence balls of prediction errors of \( \theta \), achieving a worst-case expected regret of \( O(\sqrt{d^2 T \log^2 T}) \). [Abbasi-Yadkori et al. (2011)] further improved the analysis and obtained \( O(\sqrt{d^2 T \log^3 T}) \) regret. On the lower bound side, [Dani et al. (2008)] proved a regret lower bound of \( \Omega(\sqrt{d^2 T}) \) for all policies, which was later improved to \( \Omega(\sqrt{d^2 T \log T}) \) by [Li et al. (2019)] as a direct corollary of regret lower bounds for finite-armed linear contextual bandits. While [Li et al. (2019)] derived matching upper bounds for the finite-armed case, their results and techniques cannot be directly applied to the infinite-armed case even if computational issues are disregarded, as covering nets of \( \{D_t\} \) up to \( 1/poly(T) \) accuracy would incur additional logarithmic terms in \( T \).

In this paper, we prove the following main result:

**Theorem 1** (Informal). There is a policy whose worst-case expected regret is asymptotically upper bounded by \( O(\sqrt{d^2 T \log T}) \times poly(\log \log T) \).

Comparing with the lower bound \( \Omega(\sqrt{d^2 T \log T}) \), the upper bound in Theorem 1 is tight up to iterated logarithmic terms. Our results thus close the \( O(\sqrt{\log T}) \) gap between upper and lower bounds in infinite-armed linear contextual bandit.

### 1.2 Proof techniques

**Sharp tail bounds of self-normalized empirical processes.** Due to the inherent statistical dependency between the chosen actions \( \{a_t\} \) and noise variables \( \{\xi_t\} \), the estimation error of \( \theta \) at each time step cannot be analyzed using standard closed-forms of linear regression estimators. The work of [Abbasi-Yadkori et al. (2011)] pioneered the use of self-normalized empirical processes to understand the estimation and prediction errors at each time step.

In this paper, we make use of sharp tail bounds on the supremum of self-normalized empirical processes in the high-dimensional probability society. By exploiting such tail bounds we have a more refined command of failure probabilities at each time step, which lays the foundation of our improved regret analysis.

**Varying confidence levels in UCB-type algorithm.** Most existing methods on linear contextual bandit can be categorized as Upper-Confidence-Band (UCB) or Optimism-in-Face-of-Uncertainty (OFU) type algorithms, which builds confidence bands/balls around unknown models at each time step and then pick actions in the most optimistic way.

While most existing algorithms set constant confidence levels (corresponding to failure probabilities at each time), in this paper we consider varying confidence levels, with higher failure probabilities towards the end of the time horizon \( T \). The intuition is that later fails would incur much less regret. Similar ideas were also employed in previous works [Audibert & Bubeck, 2009; Li et al., 2019; Wang et al., 2018] to improve regret guarantees in bandit problems.
Then for all \( \| \alpha \| \leq 1 \).

Throughout the paper, we adopt the standard asymptotic notations. In particular, we use \( f(\cdot) \leq g(\cdot) \) to denote that \( f(\cdot) = O(g(\cdot)) \). Similarly, by \( f(\cdot) \geq g(\cdot) \), we denote \( f(\cdot) = \Omega(g(\cdot)) \). We also use \( f(\cdot) \sim g(\cdot) \) for \( f(\cdot) = \Theta(g(\cdot)) \). Throughout this paper, we will use \( C_0, C_1, C_2, \ldots \) to denote universal constants.

1.3 Asymptotic notations

Theorem 2. Suppose the universal constant \( C > 0 \) in the input of Algorithm 1 is sufficiently large. Then for all \( \| \theta \|_2 \leq 1 \) and \( \{ D_t \subseteq \{ x \in \mathbb{R}^d : \| x \|_2 \leq 1 \} \} \), the regret of Algorithm 1 satisfies

\[
\mathbb{E}[R_T] \lesssim \sqrt{d^2 T \log T} + \sqrt{d T \log T \log \log T}.
\]

Remark 1. The \( \lesssim \) notation in Eq. (1) only hides universal constants. Furthermore, the right-hand side of Eq. (1) can be asymptotically upper bounded by \( O(\sqrt{d^2 T \log T \log \log T}) \).

The proof of Theorem 2 is stated in the next section.

2 Algorithm design and main results

Algorithm 1 presents the pseudo-code of the main algorithm considered in this paper, which is built upon the UCB/OFU framework used as well in previous linear contextual bandit work (Dani et al., 2008; Abbasi-Yadkori et al., 2011). The basic idea of Algorithm 1 is to build uniform confidence bands for the predicted reward \( \langle x, \hat{\theta}_{t-1} \rangle \) and then commit to action \( x_t \in D_t \) with the largest upper confidence band. The major difference between Algorithm 1 and previous approaches is the varying confidence levels (reflected by the inclusion of \( \omega_{x,t} \) in \( \alpha_{x,t} \)), which allows for sharper regret bounds.

The following theorem is the main result of this paper:

**Theorem 2.** Suppose the universal constant \( C > 0 \) in the input of Algorithm 1 is sufficiently large. Then for all \( \| \theta \|_2 \leq 1 \) and \( \{ D_t \subseteq \{ x \in \mathbb{R}^d : \| x \|_2 \leq 1 \} \} \), the regret of Algorithm 1 satisfies

\[
\mathbb{E}[R_T] \lesssim \sqrt{d^2 T \log T} + \sqrt{d T \log T \log \log T}.
\]  

**Remark 1.** The \( \lesssim \) notation in Eq. (1) only hides universal constants. Furthermore, the right-hand side of Eq. (1) can be asymptotically upper bounded by \( O(\sqrt{d^2 T \log T \log \log T}) \).

The proof of Theorem 2 is stated in the next section.

2.1 Remarks on implementation

The implementation of Algorithm 1 is straightforward, except for the computation of \( x^\top \hat{\theta}_{t-1} + C(\sqrt{d} + \alpha_{x,t})\omega_{x,t} \). In this section we make several remarks on the practical implementation of this step. More specifically, we present an efficient computational routine when the action sets \( D_t \) are convex.

First, we suggest replacing \( \alpha_{x,t} \) with \( \bar{\alpha}_{x,t} = \sqrt{\ln(e + \ln((T \ln T)\omega_{x,t}^2 / d))} \). The removal of the \( \max \{ \cdot \} \) operator makes the objective smoother and hence easier to optimize. Additionally, for any \( x, t \) it holds that \( \alpha_{x,t} \leq \bar{\alpha}_{x,t} \leq 2\alpha_{x,t} \), and therefore replacing \( \alpha_{x,t} \) with \( \bar{\alpha}_{x,t} \) will not affect the regret claims in Theorem 2.

With \( \alpha_{x,t} \) replaced by \( \bar{\alpha}_{x,t} \), the optimization question of \( x_t \) can be formulated as

\[
\max_{x \in D_t} F(x) \quad \text{where} \quad F(x) = C(\sqrt{d} + \sqrt{\ln(e (T \ln T)/d) + \ln(x^\top \Lambda_{t-1}^{-1} x)}) \sqrt{x^\top \Lambda_{t-1}^{-1} x}.
\]

It is easy to verify that \( F(\cdot) \) in Eq. (2) is concave and differentiable. Therefore, when \( D_t \) is convex, Eq. (2) can be efficiently solved with conventional convex optimization methods such as the projected gradient descent.

\[
\begin{align*}
\text{Input:} & \quad \text{Time horizon } T, \text{ domain dimension } d, \text{ universal constant } C \geq 1; \\
1 & \quad \text{Initialization: } \Lambda_0 = I_{d \times d}, \lambda_0 = \bar{\theta}_0 = \bar{\theta}_0 = \bar{\theta}_0; \\
2 & \quad \text{for } t = 1, 2, \ldots, T \text{ do} \\
3 & \quad \quad \text{Compute } \omega_{x,t} = \sqrt{x^\top \Lambda_{t-1}^{-1} x}, \alpha_{x,t} = \max\{1, \ln((T \ln T)\omega_{x,t}^2 / d)\} \text{ for all } x \in D_t; \\
4 & \quad \quad \text{Find } x_t \in D_t \text{ that maximizes } x^\top \hat{\theta}_{t-1} + C(\sqrt{d} + \alpha_{x,t})\omega_{x,t}; \\
5 & \quad \quad \text{Commit to action } x_t \text{ and receive reward } r_t = \langle x_t, \theta \rangle + \xi_t; \\
6 & \quad \quad \text{Update: } \Lambda_t = \Lambda_{t-1} + x_t x_t^\top, \lambda_t = \lambda_t + r_t x_t \text{ and } \hat{\theta}_t = \Lambda_t^{-1} \lambda_t. \\
7 & \end{align*}
\]

**Algorithm 1:** The UCB/OFU algorithm with varying confidence levels.
3 Proof of Theorem \[2\]

3.1 Uniform confidence region for $\hat{\theta}_t$

We first present a lemma that upper bounds the errors $| \langle x, \hat{\theta}_{t-1} - \theta \rangle |$ with high probability.

**Lemma 3.** For any $t \in [T]$ and $\delta \in (0, 1/2]$, with probability $1 - \delta$ it holds that

$$\sup_{x \in \mathbb{R}^d} (\omega_{x,t}^{-1}) \left| x^T (\hat{\theta}_{t-1} - \theta) \right| \lesssim \sqrt{d} + \sqrt{\log(1/\delta)}.$$  

The proof of Lemma 3 can be roughly divided into three steps. First, the closed-form expression of Ridge regression to express $\hat{\theta}_{t-1}$ in terms of $\theta$ and $\xi$. At the second step, a self-normalized empirical process is derived by manipulating and normalizing the expression derived in the first step. Finally, sharp tail bounds of sub-Gaussian processes are invoked to prove Lemma 3.

**Proof of Lemma 3** Let $X_{t-1}$ be a $(t - 1) \times d$ matrix constructed by stacking all $\{x_{\tau}\}_{\tau < t}$ together (i.e., $\Lambda_{t-1} = X_{t-1}^T X_{t-1} + I$), and $r_{t-1} = X_{t-1} \theta + \Xi_{t-1}$ be the $(t - 1)$-dimensional vector by concatenating all rewards from time periods $1$ through $t - 1$. Define also $\|x\|_A := \sqrt{x^T A x}$ for $d$-dimensional vectors $x$ and $d \times d$ positive-semidefinite matrices $A$. Then

$$\hat{\theta}_{t-1} = (X_{t-1}^T X_{t-1} + I)^{-1} X_{t-1}^T (X_{t-1} \theta + \Xi_{t-1}) = (I - \Lambda_{t-1}^{-1}) \theta + \Lambda_{t-1}^{-1} X_{t-1}^T \Xi_{t-1}.$$  

Subtracting one $\theta$ and left multiplying with $(\hat{\theta}_{t-1} - \theta)^T \Lambda_{t-1}$ on both sides of the above identity, we obtain

$$\|\hat{\theta}_{t-1} - \theta\|_{\Lambda_{t-1}}^2 = - (\hat{\theta}_{t-1} - \theta)^T \theta + (\hat{\theta}_{t-1} - \theta)^T X_{t-1}^T \Xi_{t-1}. \quad (3)$$

Note that $| (\hat{\theta}_{t-1} - \theta)^T \theta | \leq \| \theta \|_2 \|\hat{\theta}_{t-1} - \theta\|_2 \leq \|\hat{\theta}_{t-1} - \theta\|_{\Lambda_{t-1}}$ because $\| \theta \|_2 \leq 1$ and $\Lambda_{t-1} \succeq I$. Dividing both sides of Eq. (4) by $\|\hat{\theta}_{t-1} - \theta\|_{\Lambda_{t-1}}$, we have

$$\|\hat{\theta}_{t-1} - \theta\|_{\Lambda_{t-1}} \leq 1 + \phi^T X_{t-1}^T \Xi_{t-1}, \quad \text{where} \quad \phi = (\hat{\theta}_{t-1} - \theta)/\|\hat{\theta}_{t-1} - \theta\|_{\Lambda_{t-1}}. \quad (4)$$

It is easy to verify that $\phi$ satisfies $\|\phi\|_{\Lambda_{t-1}} \leq 1$. Consider linear transforms $\bar{x}_{\tau} = \Lambda_{t-1}^{-1/2} x_{\tau}$ for all $\tau < t$ and $\bar{\phi} = \Lambda_{t-1}^{-1/2} \phi$. Then $\bar{\phi}$ satisfies $\|\phi\|_2 \leq 1$. Subsequently, Eq. (4) can be re-formulated as

$$\|\hat{\theta}_{t-1} - \theta\|_{\Lambda_{t-1}} \leq 1 + \sup_{\|\phi\|_2 \leq 1} G_{\bar{\phi}}, \quad \text{where} \quad G_{\bar{\phi}} = \sum_{\tau < t} \xi_{\tau} (\bar{x}_{\tau}, \bar{\phi}). \quad (5)$$

We next show that $G_{\bar{\phi}}$ is a sub-Gaussian process with respect to $\| \cdot \|_2$. Indeed, for any $\phi, \phi'$, \(G_{\phi} - G_{\phi'} = \sum_{\tau < t} \xi_{\tau} (\bar{x}_{\tau}, \phi - \phi')\) is a centered sub-Gaussian random variable with variance proxy

$$\sum_{\tau < t} (\bar{x}_{\tau}, \phi - \phi')^2 = (\phi - \phi')^T (\sum_{\tau < t} \bar{x}_{\tau} \bar{x}_{\tau}^T) (\phi - \phi') = \phi^T A_{t-1}^{-1/2} (\sum_{\tau < t} x_{\tau} x_{\tau}^T) A_{t-1}^{-1/2} (\phi - \phi') = \|\phi - \phi\|_2^2.$$  

Subsequently, invoking Lemma 8 we have with probability $1 - \delta$ that

$$\|\hat{\theta}_{t-1} - \theta\|_{\Lambda_{t-1}} \lesssim 1 + \int_0^\infty \sqrt{ \log N(\{x \in \mathbb{R}^d : \|x\|_2 \leq 1, \|\cdot,\|_2,\| \cdot,\|_2\|_2 \}} \, d\epsilon + \sqrt{\log(1/\delta)} \lesssim 1 + \int_0^2 \sqrt{d \log(1/\epsilon) d \epsilon + \sqrt{\log(1/\delta)}} \lesssim \sqrt{d} + \sqrt{\log(1/\delta)}.$$

Finally, Lemma 3 is proved by the Cauchy-Schwarz inequality:

$$| x^T (\hat{\theta}_{t-1} - \theta) | \leq \|x\|_{\Lambda_{t-1}} \|\hat{\theta}_{t-1} - \theta\|_{\Lambda_{t-1}} \leq \omega_{x,t} \|\hat{\theta}_{t-1} - \theta\|_{\Lambda_{t-1}}, \quad \forall x \in \mathbb{R}^d.$$

\[\square\]
3.2 Regret upper bound at a single time step

In this subsection, we shall derive the following upper bound on the expected regret incurred at every single time step.

**Lemma 4.** For each time $t$, and the parameter $C$ in Algorithm 4 being a large enough universal constant, the expected regret incurred at time $t$ is

$$\mathbb{E} \left[ \sup_{x \in D_t} \langle x, \theta \rangle - \langle x_t, \theta \rangle \right] \lesssim \mathbb{E} \left[ (\sqrt{d} + \alpha_{x_t,t}) \omega_{x_t,t} \right] + d \sqrt{\ln T / T}. \quad (6)$$

Note that instead of a high probability bound, which is usual in the previous analysis (e.g., Dani et al. (2008); Abbasi-Yadkori et al. (2011)), our upper bound is in an expectation form. This crucially helps us to avoid the extra $\log T$ factor due to the union bound argument.

The key to proving Lemma 4 is the following reward error bound for the selected arm at time $t$.

**Lemma 5.** For each time $t$, it holds that

$$\mathbb{E} \left[ x_t^\top (\hat{\theta}_t - \theta) \right] \lesssim \mathbb{E} \left[ (\sqrt{d} + \alpha_{x_t,t}) \omega_{x_t,t} \right] + d \sqrt{\ln T / T}. \quad (7)$$

To prove Lemma 5 we adopt a novel argument that partitions the action set according to the geometric scale of the confidence levels of the actions. Using Lemma 4 for each partition, we derive a uniform error bound that for the actions in the partition. Since we have no control on the index of the the partition that $x_t$ belongs to, we finally employ a union bound argument to combine the error bounds for every partition and complete the proof.

It is worthwhile to note that a similar partitioning approach has been used in the algorithms for linear contextual bandits with finite-sized action sets (e.g., Audibert (2008); Chu et al. (2011); Li et al. (2019)). However, previous algorithms explicitly partition the action set, while in our work, the partitioning idea only appears in the proof. Furthermore, our proof technique is very different from the previous works.

**Proof of Lemma 5.** At each time $t$, consider a partition of the context vectors $\{A^\kappa_t\}_{\kappa \in \{1,2,\ldots,K\}}$ where $K = \lceil \log_2(T^2) \rceil + 1$, and we define

$$A^\kappa_t = \begin{cases} \{x \in D_t : \omega_{x,t} \in (2^{-\kappa}, 2^{-\kappa} + 1)\} & \text{when } \kappa \in \{1,2,\ldots,K-1\} \\ \{x \in D_t : \omega_{x,t} \in (0, 2^{-K+1})\} & \text{when } \kappa = K \end{cases}.$$

For each $\kappa$, we let

$$m^\kappa_t = \sup_{x \in A^\kappa_t} \left\{ x^\top (\hat{\theta}_{t-1} - \theta) \right\}$$

be the maximum estimation error for the context vectors in $A^\kappa_t$. By Lemma 4, there exists a universal constant $\zeta$, such that for all $\beta \geq \sqrt{\ln 2}$, we have that

$$\Pr \left[ m^\kappa_t \geq \zeta \cdot 2^{-\kappa}(\sqrt{d} + \beta) \right] = \Pr \left[ m^\kappa_t \geq (\zeta/2) \cdot 2^{-\kappa+1}(\sqrt{d} + \beta) \right] \leq \Pr \left[ \exists x \in A^\kappa_t : x^\top (\hat{\theta}_{t-1} - \theta) \geq (\zeta/2) \cdot \omega_{x,t}(\sqrt{d} + \beta) \right] \leq e^{-\beta^2}.$$

Now we let $\alpha^\kappa_t = C \sqrt{\max\{1, \ln[T \ln T \cdot 2^{-2\kappa} / d]\}}$, and use $1[\cdot]$ to denote the indicator function. For each $\kappa$, it holds that

$$\mathbb{E} \left[ 1 \left[ m^\kappa_t \geq \zeta \cdot 2^{-\kappa}(\sqrt{d} + \alpha^\kappa_t) \right] \cdot m^\kappa_t \right] \leq \Pr \left[ m^\kappa_t \geq \zeta \cdot 2^{-\kappa}(\sqrt{d} + \alpha^\kappa_t) \right] \cdot \left( \zeta \cdot 2^{-\kappa}(\sqrt{d} + \alpha^\kappa_t) \right) + \int_{\zeta \cdot 2^{-\kappa}(\sqrt{d} + \alpha^\kappa_t)}^{+\infty} \Pr[m^\kappa_t \geq z] dz \leq \exp \left( - (\alpha^\kappa_t)^2 / 2 \right) \cdot \left( \zeta \cdot 2^{-\kappa}(\sqrt{d} + \alpha^\kappa_t) \right) + \zeta \cdot 2^{-\kappa} \int_{\alpha^\kappa_t}^{+\infty} e^{-\beta^2} d\beta \lesssim \exp \left( - (\alpha^\kappa_t)^2 \right) \cdot 2^{-\kappa}(\sqrt{d} + \alpha^\kappa_t). \quad (8)$$
We first focus on the first term in the Right-Hand Side of Eq. (10). When $T \exp((-e^{-1} \cdot 2^{-\kappa} \cdot \ln(1 + d)) \leq d/\sqrt{T \ln T}$, therefore, Eq. (8) is upper bounded by $e^{-1} \cdot 2^{-\kappa} \cdot \ln(1 + d) \leq d/\sqrt{T \ln T}$. In the second case, when $\alpha > 1$, we have that $T \ln T \cdot 2^{-\alpha} / d = \exp((\alpha^*)^2) \cdot 2^{-\kappa} = \exp((\alpha^*)^2) / 2$. We can upper bound Eq. (8) by $\exp(-((\alpha^*)^2(1 - 1/2))) \cdot d/\sqrt{T \ln T}) \cdot \sqrt{d + \alpha^*)^2} \leq d/\sqrt{T \ln T}$. Summarizing the two cases, we have

$$
\mathbb{E} \left[ 1 \left[ m_t^\kappa \geq \zeta \cdot 2^{-\kappa} (\sqrt{d + \alpha^*)^2}) \cdot m_t^\kappa \right] \leq d/\sqrt{T \ln T}. 
$$

(9)

We now work with the Left-Hand Side of Eq. (7). Let $\kappa_t$ be the index of the partition such that $x_t \in A_{\kappa_t}$. We have

$$
\mathbb{E} \left[ x_t^\top (\hat{\theta}_{t-1} - \theta) \right] \leq \mathbb{E} \left[ 1 [\kappa_t = K] \left[ x_t^\top (\hat{\theta}_{t-1} - \theta) \right] \right] + \mathbb{E} \left[ (\sqrt{d + \alpha_{x_t,t}}) \omega_{x_t,t} \right] 
$$

+ $\mathbb{E} \left[ 1 [\kappa_t < K] \cdot 1 \left[ x_t^\top (\hat{\theta}_{t-1} - \theta) \right] \geq \zeta (\sqrt{d + \alpha_{x_t,t}}) \omega_{x_t,t} \right] \cdot \left[ x_t^\top (\hat{\theta}_{t-1} - \theta) \right].$

(10)

We first focus on the first term in the Right-Hand Side of Eq. (10). When $\kappa_t = K$, we have $\omega_{x_t,t} \leq 2/T^2$. Therefore,

$$
\mathbb{P} \left[ 1 [\kappa_t = K] \left[ x_t^\top (\hat{\theta}_{t-1} - \theta) \right] > \sqrt{1/2} \right] \leq \mathbb{P} \left[ \omega_{x_t,t}^{-1} \left[ x_t^\top (\hat{\theta}_{t-1} - \theta) \right] > T^{-1.5} \right] \leq \exp(-T),
$$

where the last inequality is due to Lemma 3 and for $T \gtrsim \sqrt{d}$. Therefore, we have

$$
\mathbb{E} \left[ 1 [\kappa_t = K] \left[ x_t^\top (\hat{\theta}_{t-1} - \theta) \right] \right] \leq \sqrt{1/2} + \mathbb{P} \left[ 1 [\kappa_t = K] \left[ x_t^\top (\hat{\theta}_{t-1} - \theta) \right] > \sqrt{1/2} \right] \leq \sqrt{1/2}. 
$$

(11)

Now we work with the third term in the Right-Hand Side of Eq. (10). When $\kappa_t < K$, we have

$$
2^{-\kappa_t} \omega_{x_t,t} \leq \alpha_{x_t,t} \leq \alpha_{x_t,t}^\kappa_t, \text{ and therefore}
$$

$$
\mathbb{E} \left[ 1 [\kappa_t < K] \cdot 1 \left[ x_t^\top (\hat{\theta}_{t-1} - \theta) \right] \geq \zeta (\sqrt{d + \alpha_{x_t,t}}) \omega_{x_t,t} \right] \cdot \left[ x_t^\top (\hat{\theta}_{t-1} - \theta) \right] \leq 1 \left[ x_t^\top (\hat{\theta}_{t-1} - \theta) \right] \geq \zeta (\sqrt{d + \alpha_{x_t,t}}) \cdot m_t^\kappa \cdot \mathbb{E} \left[ 1 \left[ x_t^\top (\hat{\theta}_{t-1} - \theta) \right] \right] \leq \sum_{k=1}^{K-1} \mathbb{E} \left[ 1 \left[ m_t^\kappa \geq \zeta (\sqrt{d + \alpha_{x_t,t}}) \cdot m_t^\kappa \right] \right] \leq d/\ln T/T. 
$$

(12)

Taking expectation and invoking Eq. (9), we have

$$
\mathbb{E} \left[ 1 [\kappa_t < K] \cdot 1 \left[ x_t^\top (\hat{\theta}_{t-1} - \theta) \right] \geq \zeta (\sqrt{d + \alpha_{x_t,t}}) \omega_{x_t,t} \right] \cdot \left[ x_t^\top (\hat{\theta}_{t-1} - \theta) \right] \leq \sum_{k=1}^{K-1} \mathbb{E} \left[ 1 \left[ m_t^\kappa \geq \zeta (\sqrt{d + \alpha_{x_t,t}}) \cdot m_t^\kappa \right] \right] \leq d/\ln T/T. 
$$

(13)

Combining Eq. (10), Eq. (11), and Eq. (13), we have

$$
\mathbb{E} \left[ x_t^\top (\hat{\theta}_{t-1} - \theta) \right] \leq \sqrt{1/2} + \mathbb{E} \left[ (\sqrt{d + \alpha_{x_t,t}}) \omega_{x_t,t} \right] + d/\ln T/T, 
$$

proving the desired upper bound.

Let $x_t^* \in D_t$ be a candidate arm that maximizes the mean reward $\langle x_t^*, \theta \rangle$ at time $t$. It is straightforward to verify that the same proof for Lemma 6 yields the following statement.

**Lemma 6.** There exists a universal constant $\xi > 0$ such that for any time $t$, it holds that

$$
\mathbb{E} \left[ \langle x_t^* (\hat{\theta}_{t-1} - \theta) \rangle \right] < \xi \left( \mathbb{E} \left[ (\sqrt{d + \alpha_{x_t^*,t}}) \omega_{x_t^*,t} \right] + d/\ln T/T \right). 
$$
We are now ready to prove Theorem 2. Besides the elliptical potential lemma, the proof exploits the continuous function \( f \) as well as the commonly used vectors \( y \) and therefore, combining Eq. 14 we have

\[
\mathbb{E} \left[ \sup_{x \in D_t} (x, \theta) - (x_t, \theta) \right] \leq \mathbb{E} \left[ (x^*_t, \theta) - (x_t, \theta) \right]
\]

Continuing with Eq. 14, we have

\[
\sum_{t=1}^{T} \mathbb{E} \left[ (x^*_t, \theta_{t-1}) + (x^*_t, \theta_{t-1} - \theta) - (x_t, \theta) \right] \leq O \left( d \sqrt{\ln T} \right) + \mathbb{E} \left[ (x^*_t, \theta_{t-1}) + C(\sqrt{d} + \alpha_{x^*_t,t})\omega_{x^*_t,t} - (x_t, \theta) \right], \tag{14}
\]

where the last inequality is because of Lemma 6, so long as \( C \geq \xi \). By Line 4 of Algorithm 1 we have that \( (x^*_t, \theta_{t-1}) + C(\sqrt{d} + \alpha_{x^*_t,t})\omega_{x^*_t,t} \leq (x_t, \theta_{t-1}) + C(\sqrt{d} + \alpha_{x^*_t,t})\omega_{x^*_t,t} \), and therefore, continuing with Eq. 14 we have

\[
\mathbb{E} \left[ \sup_{x \in D_t} (x, \theta) - (x_t, \theta) \right] \leq O \left( d \sqrt{\ln T} \right) + \mathbb{E} \left[ (\sqrt{d} + \alpha_{x^*_t,t})\omega_{x^*_t,t} + (x_t, \theta_{t-1}) - (x_t, \theta) \right]
\]

\[
\leq d \sqrt{\ln T} + \mathbb{E} \left[ (\sqrt{d} + \alpha_{x^*_t,t})\omega_{x^*_t,t} \right],
\]

where the last inequality is because of Lemma 5.

\[
\square
\]

3.3 The elliptical potential lemma, and putting everything together

Below we state a version of the celebrated elliptical potential lemma, key to many existing analysis of linearly parameterized bandit problems [Auer, 2002; Filippi et al., 2010; Abbasi-Yadkori et al., 2011; Chu et al., 2011; Li et al., 2017].

Lemma 7 (Abbasi-Yadkori et al. (2011)). Let \( U_0 = I \) and \( U_t = U_{t-1} + y_t y_t^\top \) for \( t \geq 1 \). For any vectors \( y_1, y_2, \ldots, y_T \), it holds that \( \sum_{t=1}^{T} y_t U_{t-1} y_t \leq 2 \ln(\det(U_T)) \).

We are now ready to prove Theorem 2. Besides the elliptical potential lemma, the proof exploits the power of varieted confidence levels (i.e., the specially designed \( \alpha_{x,t} \) quantity in Algorithm 1) and relies on an application of Jensen’s inequality to the concave function \( f(\tau) = \sqrt{\tau \ln((T \ln T)\tau/d)} \), as well as the commonly used \( f(\tau) = \sqrt{\tau} \).

Proof of Theorem 2. By Lemma 4, we have

\[
\mathbb{E}[R_T] \leq \sum_{t=1}^{T} \left( \mathbb{E} \left[ (\sqrt{d} + \alpha_{x^*_t,t})\omega_{x^*_t,t} \right] + d \sqrt{\ln T} \right)
\]

\[
\leq d \sqrt{T \ln T} + d \sqrt{d} \sum_{t=1}^{T} \mathbb{E} \left[ \omega_{x^*_t,t} \right] + \sum_{t=1}^{T} \mathbb{E} \left[ \alpha_{x^*_t,t} \omega_{x^*_t,t} \right]. \tag{15}
\]

For the second term in Eq. (15), by Lemma 7 and Jensen’s inequality for \( f(\tau) = \sqrt{\tau} \) (where \( \tau = \omega_{x^*_t,t}^2 \)), we have

\[
\sum_{t=1}^{T} \mathbb{E} \left[ \omega_{x^*_t,t} \right] \leq \mathbb{E} \left[ \left( T \cdot \sum_{t=1}^{T} \omega_{x^*_t,t}^2 \right)^{1/2} \right] \leq \mathbb{E} \left[ \sqrt{2T \ln(\det(\Lambda_T))} \right] \leq \sqrt{d T \ln(T/d)}, \tag{16}
\]

where the last inequality is due to

\[
\det(\Lambda_T) \leq \text{tr}(\Lambda_T/d)^d \leq ((T + 1)/d)^d. \tag{17}
\]
For the third term in Eq. (15), let $T^+ = \{ t \in [T] : \omega_{x,t} \geq \sqrt{d/(T \ln T)} \}$ and let $T^- = \{ t \in [T] : \omega_{x,t} < \sqrt{d/(T \ln T)} \} = [T] \setminus T^+$. We have that

$$\sum_{t=1}^{T} \alpha_{x,t} \omega_{x,t} = \sum_{t \in T^+} \alpha_{x,t} \omega_{x,t} + \sum_{t \in T^-} \alpha_{x,t} \omega_{x,t}$$

$$= \sum_{t \in T^+} \sqrt{\ln((T \ln T) \omega_{x,t}^2/d)} \omega_{x,t} + \sum_{t \in T^-} \omega_{x,t}$$

$$\leq \sum_{t \in T^+} \sqrt{\ln((T \ln T) \omega_{x,t}^2/d)} \omega_{x,t} + \sqrt{dT}. \quad (18)$$

Note that the univariate function $f(\tau) = \sqrt{\tau \ln((T \ln T)/\tau/d}$ is concave for $\tau \geq d/(T \ln T)$. Applying Jensen’s inequality to $f(\tau)$ with $\tau = \omega_{x,t}^2$ (t ∈ $T^+$), we have

$$\sum_{t \in T^+} \sqrt{\ln((T \ln T) \omega_{x,t}^2/d)} \omega_{x,t} \leq |T^+| \cdot \sqrt{\frac{\sum_{t \in T^+} \omega_{x,t}^2}{|T^+|}} \cdot \ln \left( \frac{T \ln T}{d} \cdot \frac{\sum_{t \in T^+} \omega_{x,t}^2}{|T^+|} \right)$$

$$\leq \sqrt{|T^+| \ln(T/d)} \ln \left( \frac{T \ln T}{d} \cdot \frac{\ln(T/d)}{|T^+|} \right)$$

$$\leq \sqrt{T d \ln(T/d)} \ln \ln T,$$

where the second inequality is due to Lemma[2] and Eq. (17), and the third inequality is due to the monotonicity of the function $g(x) = \sqrt{xd \ln(T/d) \ln(T \ln T) / d \cdot (d \ln(T/d)/x)}$ for large enough $x$. Combining Eq. (18), and Eq. (19), we have

$$\sum_{t=1}^{T} \alpha_{x,t} \omega_{x,t} \lesssim \sqrt{T d \ln(T/d)} \ln \ln T. \quad (20)$$

Finally, we combine Eq. (15), Eq. (16), and Eq. (20), and have that

$$\mathbb{E}[R_T] \lesssim d \sqrt{T \ln T} + d \sqrt{T \ln(T/d)} + \sqrt{T d \ln(T/d)} \ln \ln T \lesssim d \sqrt{T \ln T} + \sqrt{dT \ln T \ln \ln T},$$

which is the desired regret upper bound for Algorithm[1] □

4 Conclusion

In this paper we study the linearly parameterized contextual bandit problem and develop algorithms that achieve minimax-optimal regret up to iterated logarithmic terms. Future directions include generalizing the proposed approach to contextual bandits with generalized linear models, as well as other variants of contextual bandit problems.

Appendix: useful probability tools

A separable process $\{G_{\phi}\}_{\phi \in \Theta}$ with respect to a metric space $(\Theta, d)$ is sub-Gaussian if for any $\lambda \in \mathbb{R}$ and $\phi, \phi' \in \Theta$, $\mathbb{E}[e^{\lambda(X_{\phi} - X_{\phi'})}] \leq e^{\lambda^2 d^2(\phi, \phi')/2}$. Let also $\text{diam}(\Theta) = \sup_{\phi, \phi' \in \Theta} d(\phi, \phi')$ be the diameter of the metric space $(\Theta, d)$. The following result is cited from [van Handel, 2014, Theorem 5.29].

**Lemma 8.** There exists a universal constant $C_0 < \infty$ such that for all $z > 0$ and $\phi_0 \in \Theta$,

$$\Pr\left[ \sup_{\phi \in \Theta} G_{\phi} - G_{\phi_0} \geq C_0 \int_0^{\infty} \sqrt{\log N(\Theta; d, \epsilon) d\epsilon} + z \right] \leq C_0 e^{-z^2/C_0 \text{diam}(\Theta)},$$

where $N(\Theta; d, \epsilon)$ is the covering number of the metric space $(\Theta, d)$ up to precision $\epsilon$.

---

2See Definition 5.22 in [van Handel, 2014] for a technical definition of separable stochastic processes.
References

Abbasi-Yadkori, Y., Pál, D., & Szepesvári, C. (2011). Improved algorithms for linear stochastic bandits. In Proceedings of the Advances in Neural Information Processing Systems (NIPS).

Abbasi-Yadkori, Y., Pal, D., & Szepesvari, C. (2012). Online-to-confidence-set conversions and application to sparse stochastic bandits. In Proceedings of the International Conference on Artificial Intelligence and Statistics (AISTATS).

Audibert, J.-Y., & Bubeck, S. (2009). Minimax policies for adversarial and stochastic bandits. In Proceedings of the Conference on Learning Theory (COLT).

Auer, P. (2002). Using confidence bounds for exploitation-exploration trade-offs. Journal of Machine Learning Research, 3(Nov), 397–422.

Chu, W., Li, L., Reyzin, L., & Schapire, R. (2011). Contextual bandits with linear payoff functions. In Proceedings of the International Conference on Artificial Intelligence and Statistics (AISTATS).

Dani, V., Hayes, T. P., & Kakade, S. M. (2008). Stochastic linear optimization under bandit feedback. In Proceedings of the Conference on Learning Theory (COLT).

Filippi, S., Cappe, O., Garivier, A., & Szepesvári, C. (2010). Parametric bandits: The generalized linear case. In Proceedings of the Advances in Neural Information Processing Systems (NIPS).

Li, L., Lu, Y., & Zhou, D. (2017). Provable optimal algorithms for generalized linear contextual bandits. In Proceedings of the International Conference of Machine Learning (ICML).

Li, Y., Wang, Y., & Zhou, Y. (2019). Nearly minimax-optimal regret for linearly parameterized bandits. In Proceedings of the annual Conference on Learning Theory (COLT).

Rusmevichientong, P., & Tsitsiklis, J. N. (2010). Linearly parameterized bandits. Mathematics of Operations Research, 35(2), 395–411.

van Handel, R. (2014). Probability in high dimension. Tech. rep., Princeton University.

Wang, Y., Chen, X., & Zhou, Y. (2018). Near-optimal policies for dynamic multinomial logit assortment selection models. In Proceedings of the advances in Neural Information Processing Systems (NeurIPS).