Scale versus Conformal Invariance Revisited

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Abstract

For a scale invariant theory with gauge-invariant local virial current we argue that the existence of a well defined ground state implies the vanishing of all conformal dilaton scattering amplitudes.
A long standing problem is to understand under which conditions scale invariance implies conformal invariance for space-time dimension $d > 2$ (for early and recent attempts see [1, 2, 3, 4] and references therein). This is tantamount to the condition that the theory admits an improved, local, symmetric, conserved and traceless stress tensor. This, in turn, implies that for an arbitrary correlator we have, modulo boundary terms,

$$\int \omega(x) < \cdots T(x) \cdots > = \int \nabla^2 \omega(x) < \cdots L \cdots >$$

where $L$ is a local operator. In particular this implies

$$< \cdots \tilde{T}(p) \cdots > \propto p^2.$$

In a beautiful paper [3] Dymarsky et al. argued that scale invariance and unitarity imply the vanishing of the $n \geq 4$ conformal dilaton amplitude. Furthermore, assuming the property that "if the S-matrix of some particle scattering is trivial in a unitary theory, then the particle can be rendered free after some change of variables" they conclude that the on-shell dilaton action is trivial and thus the theory is conformal invariant. An exception are non-interacting scale invariant but not conformally invariant theories where their argument does not apply. Their key argument uses analyticity of the "S-matrix" to conclude that $<0| \prod_{i=1}^n T(p_i) anything >_{p^2=0} = 0$ for $n \geq 2$. Note, however, that (2) does not imply (1).

While the arguments reviewed above are open to debate we have nothing to add in support of their validity. The purpose of this note is to argue that the vanishing of the above correlators can be generalized straightforwardly to $n = 1$ and dimensions different from 4 assuming that the virial current, not used in the previous analysis, exists as a well defined operator in the quantum theory.

Consider a scale invariant action, $S[\Phi]$, that is, under $x^\mu \rightarrow e^\omega x^\mu$ together with $\Phi \rightarrow e^{d_\Phi \omega} \Phi$, ($\omega = const.$) the action is invariant. For a local transformation, $\omega(x)$, then have to first order in $\omega$ and its derivatives

$$\delta_\omega S[\Phi] = \int \partial_\mu \omega \ j^\mu d^d x,$$

where

$$j^\mu = x^\nu T_{\mu \nu} + V_\mu$$

is the improved Noether current associated with scale invariance and [7]

$$V_\mu = \pi_\Phi \Lambda_{\nu \mu} \Phi, \quad \Lambda_{\nu \mu} = \partial_\Phi g_{\mu \nu} + 2 \Sigma_{\mu \nu}$$

is the virial current. The first term in (4) implements the coordinate transformation on the fields (or operators) whereas the charge associated with $V_\mu$ implements the scale transformation.

\[3\] see also [5] and [6] for a scrutinization of this argument.
on the fields. In particular, if the fields appearing in $S$ are dimensionless then $V_\mu$ vanishes. For instance, for a free scalar field in $d$ dimension we have $V_\mu = \frac{d-2}{4} \partial_\mu \phi^2$ with the associated charge

$$Q(V) = \int d^d x \: \pi_\phi \phi.$$ \hfill (6)

In what follows we will assume that $V_\mu$ exists as a well defined, local gauge invariant operator in the quantum theory.

This property is not universally true. Well known counter examples are Maxwell theory in three dimensions or, more generally, $p$-forms in $d \neq 2(p+1)$ dimensions. Perhaps less well known, are other counter examples provided by higher spins theories. Indeed, symmetric massless higher spin fields described by the second order Fronsdal action

$$S = \frac{1}{2} \int d^d x (\partial_\mu \phi_{\nu_1 \ldots \nu_s} \partial^\mu \phi^{\nu_1 \ldots \nu_s} + \ldots) \quad \hfill (7)$$

are obviously scale invariant. The associated equations of motion define also unitary representations of the Poincare group. On the other hand, they are not conformally invariant (see, for example, [3, 9] and references therein). One can easily see that Fronsdal’s action (7) defines the virial current which is not gauge invariant. Indeed, the virial current carries one derivative of the field, while it is well-known, that the invariant of higher spin transformations with the least number of derivatives is the Fronsdal tensor, carrying two derivatives of the gauge field.

On the other hand, if $V_\mu$ exists then it is easy to see that the associated charge does not leave the vacuum invariant, $Q(V)|0 > \neq 0$. Indeed, we have from

$$[Q(V), \Phi(x)] = d_\Phi \Phi(x) \quad \hfill (8)$$

that

$$< 0|[Q(V), \Phi(x)\Phi(y)]|0 > = 2d_\Phi < 0|\Phi(x)\Phi(y)|0 > \neq 0,$$ \hfill (9)

where, for simplicity we have assumed that extra indices in the correlator are contracted. Repeating the arguments from spontaneous chiral symmetry breaking in QCD we then conclude that

$$\tilde{V}_\mu(p)|0 > = p_\mu |\pi(p) >,$$ \hfill (10)

where $p_\mu$ is the momentum of the created state $\pi$, which must be a scalar for Lorenz invariance, but need not be massless since $V_\mu$ is not conserved. On the other hand from $\partial_\mu V^\mu = T^\mu_\mu \equiv T$ we conclude from (10) that

$$< \ldots \tilde{T}(p)|0 > = p^2 < \ldots |\pi(p) >.$$ \hfill (11)

\footnote{However, one can write conformally invariant equations equivalent to those coming from (7). They are formulated in terms of the so-called higher spin curvatures, carrying $s$ derivatives of the spin $s$ gauge field. Nevertheless, reformulation of these equations in terms of gauge potentials breaks conformal symmetry. In this respect the higher spin counterexample is similar to the 2-form counterexample.}
Consequently, if we consider the coupling
\[ \int \omega(x)T(x) = \int \tilde{\omega}(-p)\tilde{T}(p) \] (12)
with \( \nabla^2 \omega = 0 \), i.e. \( \tilde{\omega}(-p) = \delta^d(p^2)\tilde{\omega}_0(-p) \) we then have
\[ \int \tilde{\omega}(-p) < 0 | \tilde{T}(p) \cdots > = 0 \] (13)
Note that if (13) is true then this trivially implies the vanishing of all higher on-shell dilaton scattering amplitudes since
\[ < 0 | \tilde{T}(p) \prod_{i=2}^n \tilde{T}(p_i) \cdots > = p^2 < \pi(p) | \prod_{i=2}^n \tilde{T}(p_i) \cdots > . \] (14)

We should emphasize that (11) or (14) does not necessarily imply \( \tilde{T}(p)|0> = p^2 \tilde{L}(p)|0> \) for some local operator \( L \) (see [5]) although fairly convincing arguments for this were given in [3].

Note that we haven’t used unitarity explicitly in the above arguments. However, unitarity, or rather reflection positivity appears to be necessary to prove (11) since only for theories with this property \( \tilde{T}(p)|0> = p^2 \tilde{L}(p)|0> \), \( \forall p \) implies \( \tilde{T}(p) = p^2 \tilde{L}(p) \) as an operator equation\( ^5 \). A class of models that do not satisfy the positivity property are given by generalizations of the Fronsdal Lagrangian without higher spin symmetry (See [9] for a detailed discussion and other examples). Another interesting class of counter examples are Lagrangians with non-canonical kinetic terms such as \( L = f(\theta)(\partial \varphi)^2 + \varphi^2(\partial \theta)^2 \) which have a well defined virial current at the classical level [9]. Such theories, however, do not allow for a perturbative quantization and thus we can, in particular not infer a well defined ground state which is key for the implication (10) to hold.

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\[ ^5 \] We like to thank S. Theisen for explanations on this point.
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