COMBINATORIAL SOLUTION OF
ONE–DIMENSIONAL QUANTUM SYSTEMS

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Abstract

We give a self–contained exposition of the combinatorial solution of quantum mechanical systems of coupled spins on a one–dimensional lattice. Using Trotter formula, we write the partition function as a generating function of a spanning subgraph of a two–dimensional lattice and solve the combinatorial problem by the method of Pfaffians provided the weights satisfy the so–called free fermion condition. The free energy $f = f(\beta, J, h)$ and the ground state energy $e_0 = e_0(J, h)$ as a function of the inverse temperature $\beta$, couplings $J$ and magnetic fields $h$, for the XY model in a transverse field with period $p = 1$ and $2$, is then obtained.

1 Introduction

A quantum mechanical system of coupled spins on a one-dimensional lattice $\Lambda \in \mathbb{Z}$ with $|\Lambda| = N$ sites is described by Hamiltonians of the form

$$H_N(\sigma) = -\sum_{i \in \Lambda} h_{i,i+1}$$

where

$$h_{i,j} = J^{x}_{i,j} \sigma^{x}_{i} \sigma^{x}_{j} + J^{y}_{i,j} \sigma^{y}_{i} \sigma^{y}_{j} + J^{z}_{i,j} \sigma^{z}_{i} \sigma^{z}_{j} + \frac{h_{i}}{2}(\sigma^{z}_{i} + \sigma^{z}_{j}).$$

For each $i \in \Lambda$, $\sigma^{\alpha}_{i}, \alpha = x, y, z$, are Pauli matrices and periodic boundary condition is imposed, $\sigma^{\alpha}_{i+N} = \sigma^{\alpha}_{i}$. The family of couplings $J = \{(J^{x}_{i,i+1}, J^{y}_{i,i+1}, J^{z}_{i,i+1})\}_{i \in \Lambda}$ and magnetic fields $h = \{h_{i}\}_{i \in \Lambda}$ we shall considered here are assumed to be periodic of period $p$ and, for simplicity, only $p = 1$, $J^{x}_{i,i+1} = J^{x}$, $h_{i} = h$ and $p = 2$ arrays, $J^{x}_{i,i+1} = J^{x}_{1}$, $h_{i} = h_{1}$ if $i$ is odd and $J^{x}_{i,i+1} = J^{x}_{2}$, $h_{i} = h_{2}$ if $i$ is even, will be treated. The $z$–component couplings $J^{z}_{i,i+1}$ will also be set equal to zero at certain moment.

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One-dimensional quantum systems have been extensively studied, as compared to the higher
dimensional counterparts, because some of their physical quantities can be explicitly calculated. In
the present work we consider the free energy function, \( f = f(\beta, J, h) \), and the ground state energy,
\( \epsilon_0 = \epsilon_0(J, h) = \lim_{\beta \to \infty} f \), where

\[
f = \lim_{N \to \infty} -\frac{1}{\beta N} \ln Z_N.
\]

The partition function

\[
Z_N = \text{Tr} e^{-\beta H_N} \equiv \text{Tr} \rho_N,
\]

is given by the trace over a density matrix \( \rho_N \), defined in the \( 2^N \)-dimensional vector space \( \bigotimes_{i \in \Lambda} \mathbb{C}^2 \),
and \( \beta \) is the inverse temperature.

Among the available solution methods, we distinguish two types depending on whether one
approaches the partition function \( Z_N \) directly or by means of the eigenvalue problem of \( H_N \).

The eigenvalues are more easily accessible when, by a canonical transformation, \( H_N \) can be
written as a free fermion system. If \( J \) and \( h \) have period one with \( J^z = 0 \), then (1.1) is unitarily
equivalent to \( K \)

\[
\mathcal{H}_N = -t \sum_{i \in \Lambda} \left\{ c_i^\dagger c_{i+1}^\dagger - c_i c_{i+1} + \Gamma \left( c_i^\dagger c_{i+1}^\dagger - c_i c_{i+1} \right) + H \left( 1 - 2 c_i^\dagger c_i \right) \right\},
\]

where \( c_i^\dagger \) and \( c_i \) are the creation and annihilation operators for electrons at the site \( i \in \Lambda \), \( t = J^x + J^y \),
\( \Gamma = (J^x - J^y) / t \), \( H = h / t \) and the boundary condition terms has been omitted. Lieb et al \( \text{LSM} \)
and Katsura \( \text{Ka} \) have calculated the spectrum of the \( XY \)-model (1.5) with \( H = 0 \) and \( XY \)-
model in a transverse field (1.5), respectively, obtaining the free energy function and the ground
state. Subsequently, the \( XY \)-model with alternating interactions and magnetic moments has been
solved by Perk et al \( \text{PCZ} \).

The partition function is said to be approached directly if it is mapped into a problem of counting
spanning subgraphs in a lattice. Direct approach does not require \( H_N \) to be diagonalizable but in
order to solve the underlying problem certain conditions have to be met. The basic ingredient for
this approach is the Trotter formula \( \text{T} \)

\[
e^{(K^1 + K^2)} = \lim_{m \to \infty} \left( e^{K^1/m} e^{K^2/m} \right)^m,
\]

valid for any finite-dimensional matrices \( K^1, K^2 \) (see e.g. \( \text{S} \) for extensions and applications). The
partition function (1.4) can thus be mapped into a classical two dimensional lattice spin system as
explained in detail in the following sections.

There is another relationship between chains of quantum spins and the transfer matrix of two–
dimensional classical systems which has been discussed in the excellent review by Kasteleyn \( \text{K1} \).

The partition function has been directly approached by Suzuki and Inoue \( \text{SuI} \) who, using
Trotter formula, have mapped the \textit{period–one} \( XY \)-model in a transverse field into a \textit{period–two}
8–vertex model with two sets of weights, \( \{ w_i \}_{i=1}^8 \) and \( \{ w'_i \}_{i=1}^8 \), satisfying the free–fermion condition

\[
w_1 w_2 + w_3 w_4 = w_5 w_6 + w_7 w_8.
\]

Equation (1.7) is the very fact that allowed Fan and Wu \( \text{FW} \) to solve the 8–vertex model by the
Pfaffian method (see \( \text{HLW} \) for the period 2 model).
The purpose of this paper is to give a step–by–step account of the combinatorial method of solving chains of interacting quantum spins. In particular, Katsura’s results are shown to be recovered by mapping the $XY$–model into a period–one 8–vertex model with weight function satisfying (1.7). For this we use the one–to–one mapping established by Barma and Shastri [BS] in the context of $XYZ$–model. Due to the careful treatment of our exposition the method is extended to any period $p$. To our knowledge, we calculate for the first time the free energy of a period–two $XY$–model in a transverse magnetic field by the method of Pfaffian.

This paper is part of a program aiming at alternative solutions to the ground state energy of Heisenberg (1.1) models. The method based in the spectral analysis of $H_N$ is amazingly far more developed than the direct approach. The ground state energy of Hubbard [LW] and Heisenberg models [XY] can be obtained by an Ansatz to the eigenfunctions of $H_N$ called Bethe–Ansatz. The need for alternative solutions is more demanding since the “string hypothesis” (completeness of the Bethe–Ansatz eigenstates) remains unresolved.

Last but not least, the interest on the Pfaffian method has been renewed in view of a recent result by Pinson and Spencer [PS] on the universality of critical $d = 2$ Ising model.

The layout of this paper is as follows. In Section 2 the Hamiltonian of the quantum spin system (1.1) is mapped into a two–dimensional classical spin system. This in turn is mapped, in Section 3, into a spanning subgraph problem which is solved in Section 4 by the Pfaffian method. The evaluation of the determinant is presented in Section 4.1 and free energy functions for periodic systems with periods $p = 1$ and 2 are determined in Sections 6 and 7, respectively.

2 Trotter Formula

In this section the Trotter formula will be used to write the partition function (1.4) as the partition function of a two-dimensional lattice $\mathbb{Z}^N \times \mathbb{Z}^m$ of a classical spin system in the limit $m \to \infty$.

Trotter’s product formula (1.6) is a mathematical tool which has elucidated the “path integration” of a quantum mechanical particle in a potential (see [S] and references therein for an accountable survey). The application of this formula for coupled quantum spin systems was originally carried out by Suzuki [Su] but we shall here follow a slightly modification due to Barma–Shastry [BS]. Before working it out, let us define the Hamiltonian system precisely.

To each site $i \in \Lambda$ it is associated a Hilbert space $\mathcal{H}_i = \mathbb{C}^2$. The vectors $e_+ = (1\ 0)$ and $e_- = (0\ 1)$ form a base of $\mathcal{H}_i$ and represent the two possible states of a spin $s_i$: $s_i = +$ if the spin at $i$ is up and $s_i = -$ if it is down.

The vector space $\mathcal{H} = \mathcal{H}_\Lambda$ where the Hamiltonian operator (1.1) acts is a tensor product space,

$$\mathcal{H} = \bigotimes_{i \in \Lambda} \mathcal{H}_i.$$  \hfill (2.1)

Under the above interpretation, each of the $2^N$ vectors of the form

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

represents a possible spin configuration $s = \{s_i\}_{i \in \Lambda}$ of the whole system. Note that the collection $B = B_\Lambda = \{e_s\}$ of vectors $e_s := e_{s_1} \otimes e_{s_2} \otimes \cdots \otimes e_{s_N}$ with $s \in \{+,-\}^N$ forms an orthonormal base
of $\mathcal{H}$. So, if $E_s = \epsilon_s (\epsilon_s)^T$ denotes the (orthogonal) projector of $\mathcal{H}$ into the $\epsilon_s$ direction, we have

$$
\sum_{s \in \{+, -\}^N} E_s = I, \quad (2.2)
$$

where $I = I \otimes I \otimes \cdots \otimes I$ is the identity matrix: $Iv = v$, for all $v \in \mathcal{H}$.

A base of operators in $\mathbb{C}^2$ can be chosen to be the Pauli matrices

$$
\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (2.3)
$$

and the identity $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. For later reference, we shall also consider the “raising” and “lowering” matrices,

$$
\sigma^+ = \frac{\sigma^x + i \sigma^y}{2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \sigma^- = \frac{\sigma^x - i \sigma^y}{2} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad (2.4)
$$

whose operations over the base vectors are $\sigma^+ e_- = e_+$, $\sigma^- e_+ = e_-$, and $\sigma^+ e_+ = \sigma^- e_- = 0$.

The spin operator $\sigma^\alpha_i$ in (1.2) acts on the Hilbert space $\mathcal{H}$ and has the form

$$
\sigma^\alpha_i = I \otimes \cdots \otimes I \otimes \sigma^\alpha \otimes I \otimes \cdots \otimes I \quad (2.5)
$$

with $\sigma^\alpha$ acting on $\mathcal{H}_i$ (i.e. it is located at $i$-th position). Here $\alpha$ runs over $x, y, z, 0, +$ and $-$ with $\sigma^0 \equiv I$. The Hamiltonian is composed of pair interactions between nearest neighbor sites,

$$
\sigma^\alpha_i \sigma^\beta_{i+1} = I \otimes \cdots \otimes I \otimes \sigma^\alpha \otimes \sigma^\beta \otimes I \otimes \cdots \otimes I,
$$

which acts as the identity operator in $\mathcal{H}_k$ for all $k$ except at the positions $k = i, i + 1$. The type of interactions considered in (1.2) have pairs $(\alpha, \beta)$ equal to $(x, x)$, $(y, y)$ and $(z, z)$ which will subsequently be written in terms of $(+, +)$, $(+, -)$, $(-, +)$ and $(-, -)$ interactions.

Note that two consecutive terms of the Hamiltonian (1.1) do not commute, $[h_{i-1,i}, h_{i,i+1}] \neq 0$, due to the fact that

$$
\left[\sigma^\alpha_{i-1} \sigma^\beta_i, \sigma^\gamma_{i+1} \sigma^\delta_i\right] = I \otimes \cdots \otimes \sigma^\alpha \otimes \left[\sigma^\beta, \sigma^\gamma\right] \otimes \sigma^\delta \otimes I \otimes \cdots \otimes I
$$

does not vanish if $\beta \neq \gamma$. This expresses the quantum mechanical nature of our spin system. We shall now explain how Trotter’s formula resolves the non–commutativity character by converting quantum spin variables $\{\sigma^\alpha_i\}$ into infinite many copies $s_1, s_2, \ldots$ of “classical” spin variables $s_j = \{s_{ij}\}_{i=1}^N$.

For $N = 2n$ let us decompose the set $\Lambda = \{1, 2, \ldots, 2n\} = \Lambda^{\text{odd}} \cup \Lambda^{\text{even}}$, according to whether the site $i$ is odd or even, and write

$$
H_{2n} = H_n^{\text{odd}} + H_n^{\text{even}}, \quad (2.6)
$$

where $H_n^{\text{odd(even)}}$ is given by (1.1) with $\Lambda$ replaced by $\Lambda^{\text{odd(even)}}$. Because we have eliminated consecutive pairings, the terms which compose $H_n^{\text{odd(even)}}$ commute with each other. Note, however, that $[H_n^{\text{odd}}, H_n^{\text{even}}] \neq 0$.  

\[4\]
Using Trotter’s formula (1.6) we have

$$\rho_{2n} = \lim_{m \to \infty} \left( e^{-\beta H_n^{\text{odd}}/m} e^{-\beta H_n^{\text{even}}/m} \right)^m \equiv \lim_{m \to \infty} \left( \rho_{n,m}^{\text{odd}} \rho_{n,m}^{\text{even}} \right)^m. \tag{2.7}$$

The partition function (1.4) can thus be written as

$$Z_{2n} = \lim_{m \to \infty} Z_{n,m},$$

where

$$Z_{n,m} = \text{Tr} \rho_{n,m}^{\text{odd}} \rho_{n,m}^{\text{even}} \ldots \rho_{n,m}^{\text{odd}} \rho_{n,m}^{\text{even}} \text{ m-times}. \tag{2.9}$$

We are going to make use of our base $\mathcal{B}$ by inserting the spectrum decomposition of the identity (2.2) between all pairs of density matrices $\rho_{n,m}^{\text{odd}} \rho_{n,m}^{\text{even}}$ and $\rho_{n,m}^{\text{even}} \rho_{n,m}^{\text{odd}}$. In other words, we want to represent the density matrix $\rho$ according to the base $\mathcal{B}$. To make this representation clearer we shall first consider the special case $Z_{1,1}$.

Dropping, for simplicity, the subindex of $\rho$, we have

$$Z_{1,1} = \text{Tr} \rho^{\text{odd}} \rho^{\text{even}} = \text{Tr} \rho^{\text{odd}} I \rho^{\text{even}} I = \sum_{s,s'} \rho_{s,s'}^{\text{odd}} \rho_{s',s}^{\text{even}} \tag{2.10}$$

where $\rho_{s,s'} = (e_s)^T \rho e_{s'}$. For $n = 1, \Lambda = \{1, 2\}$ has only two sites and $\mathcal{H}$ is a four dimensional space, $s \in \{++, +-, --, --, ---\}$, leading to a $4 \times 4$ density matrix $\rho$. In view of the periodic boundary condition, $\rho^{\text{odd}}$ differs from $\rho^{\text{even}}$ by the order of the spins at the sites $\{1, 2\}$. We shall keep it in mind and drop their super-index too. As we shall see, the alternating position of the odd–even density matrix in the product (2.9) will lead to a chess-board pattern.

To compute the $4 \times 4$ density matrix $\rho$, the algebraic properties of the matrices (2.3) and (2.4) have to be used. Let us first write $h_{1,2}$, with $J^z = 0$, in terms of $\sigma^+, \sigma^-$ and $\sigma^z$. From (1.2) and (2.10), we have

$$h_{1,2} = (J^x - J^y) \left( \sigma_1^+ \sigma_2^+ + \sigma_1^- \sigma_2^- \right) + (J^x + J^y) \left( \sigma_1^+ \sigma_2^- + \sigma_1^- \sigma_2^+ \right) + (h/2) \left( \sigma_1^z + \sigma_2^z \right),$$

where $J_{1,2}^z = J_{2,1}^z = J^z$ and $h_1 = h_2 = h$.

Writing $\eta = (J^x - J^y)$, $\kappa = (J^x + J^y)$ and $\zeta = \sqrt{\eta^2 + h^2}$, the following relations

$$\begin{align*}
(h_{1,2})^{2k} e_s &= \begin{cases} 
\zeta^{2k} e_{++}, & \text{if } s = ++, \\
\kappa^{2k} e_{+-}, & \text{if } s = +-, \\
\kappa^{2k} e_{-+}, & \text{if } s = -+, \\
\zeta^{2k} e_{--}, & \text{if } s = --,
\end{cases} \tag{2.12}
\end{align*}$$

and

$$\begin{align*}
(h_{1,2})^{2k-1} e_s &= \begin{cases} 
\zeta^{2k-2} \eta e_{++} + \eta e_{--}, & \text{if } s = ++, \\
\kappa^{2k-1} e_{+-}, & \text{if } s = +-, \\
\kappa^{2k-1} e_{-+}, & \text{if } s = -+, \\
\zeta^{2k-2} - \eta e_{--} + \eta e_{++}, & \text{if } s = --,
\end{cases} \tag{2.13}
\end{align*}$$

hold for any integer number $k \geq 1$. 

5
Taylor-expanding $\rho = e^{\beta h_{1,2}}$ and using equations (2.12) and (2.13), combined with the orthogonal relation $(e_a)^T e_{a'} = \delta_{s,s'}$, gives

$$
\rho = \begin{pmatrix}
\cosh \beta \zeta + (h/\zeta) \sinh \beta \zeta & 0 & 0 & (\eta/\zeta) \sinh \beta \zeta \\
0 & \cosh \beta \kappa & \sinh \beta \kappa & 0 \\
0 & \sinh \beta \kappa & \cosh \beta \kappa & 0 \\
(\eta/\zeta) \sinh \beta \zeta & 0 & 0 & \cosh \beta \zeta - (h/\zeta) \sinh \beta \zeta
\end{pmatrix}.
$$

(2.14)

Note that $\rho$ is a symmetric matrix and $Z_{1,1} = \text{Tr} \rho^2$.

Now, we want to include the $J^z$ term into the above calculation. The simplest way to deal with this case is to introduce a slightly modification of the Trotter formula [KM],

$$
e^{K_0+K_1+K_2} = \lim_{m \to \infty} (e^{K_0/m} e^{K_1/m} e^{K_2/m})^m.
$$

As in [BS] we separate the $J^z$-dependent terms of $H_{2n}$ and divide the remaining terms according to the even–odd partition of $\Lambda$,

$$
H_{2n} = H_{2n}^0 + H_n^{\text{odd}} + H_n^{\text{even}},
$$

(2.15)

where $H_n^{\text{odd(even)}}$ is as in (2.6). We repeat the steps (2.7) – (2.9) with the modified Trotter formula.

For the special case $Z_{1,1}$, we have

$$
Z_{1,1} = \text{Tr} \rho^0 \rho^{\text{odd}} \rho^{\text{even}} = \text{Tr} (\rho^0)^{1/2} \rho^{\text{odd}} \rho^{\text{even}} (\rho^0)^{1/2} = \sum_{s,s'} \rho_{s,s'}^{\text{odd}} \rho_{s,s'}^{\text{even}},
$$

(2.16)

where $\rho^{\text{odd}} = (\rho^0)^{1/2} \rho \equiv \hat{\rho}$ with $\rho$ given by (2.14), $\hat{\rho}^{\text{even}} = (\hat{\rho})^T$ and

$$
\rho^0 = e^{2\beta J^z} \sigma^\sigma \otimes \sigma^\sigma = \begin{pmatrix}
e^{2\beta J^z} & 0 & 0 & 0 \\
0 & e^{-2\beta J^z} & 0 & 0 \\
0 & 0 & e^{-2\beta J^z} & 0 \\
0 & 0 & 0 & e^{2\beta J^z}
\end{pmatrix}.
$$

(2.17)

Note that $\hat{\rho} = (\hat{\rho})^T$.

To extend the calculation of $Z_{1,1}$ to $Z_{n,m}$ with $n, m \geq 1$, we may think of a rectangular $2m \times 2n$ lattice as a “chess–board” with $n \cdot m$ black squares. To each of these black pieces labeled by $p$ (of plaquetes) we associate a matrix $\rho_p$ given by (2.14) with $\beta$ replaced by $\beta/m$; $\rho$ is a transfer matrix between the state of spins at the up corners of the black square $p$ and the state of the spins at the bottom corners of $p$. If the up and bottom corners’ labels of $p$ are respectively $v, w$ and $v', w'$, we denote the matrix elements of $\rho_p$ by $\rho(s_v, s_w; s_{v'}, s_{w'})$ and equation (2.8) represented in the base $\mathcal{B}$ can be written as

$$
Z_{n,m} = \sum_{\{s_i,j\}} \prod_{j=1}^m \left( \prod_{i=1}^n \rho(s_{2i-1,2j-1}, s_{2i,2j-1}; s_{2i-1,2j}, s_{2i,2j}) \rho(s_{2i,2j}, s_{2i+1,2j}; s_{2i,2j+1}, s_{2i+1,2j+1}) \right),
$$

(2.18)
with periodic conditions assumed in both directions. For (2.18), note that the matrix elements of $\rho^{\text{odd(even)}}$ between two successive spin configurations, $s_l = \{s_{k,l}\}_{k=1}^{2n}$ and $s_{l+1} = \{s_{k,l+1}\}_{k=1}^{2n}$, factor out

$$
(\rho^{\text{odd}})_{s_{2j-1},s_{2j}} = \prod_{i=1}^{n} \rho(s_{2i-1,2j-1}, s_{2i,2j-1}; s_{2i-1,2j}, s_{2i,2j})
$$

and analogously for $\rho^{\text{even}}$, due to the even–odd decomposition.

As a consequence, two matrices $\rho_p$ and $\rho_{p'}$ are coupled if $p$ and $p'$ share the same vertex $v$ ($p \cap p' = v$). Each matrix labeled by a plaquette $p$ has a pairwise interaction with four other matrices labeled by the nearest–neighbor black plaquetes of $p$. We shall use this fact in the following section to write (2.18) as a counting problem in a $n \times m$ rectangular lattice (wrapped into a torus and rotated $45^\circ$) whose vertices are centers of the black plaquetes of the original lattice.

### 3 The Vertex Model

In the previous section Trotter’s formula has been used to write the partition function $Z_{2\pi n}$ of a one–dimensional quantum spin system in $\Lambda_{2\pi n} = \{1, \ldots, 2n\}$ as the $m \to \infty$ limit of a sequence $\{Z_{n,m}\}_{m \geq 1}$ of partition functions of two–dimensional classical spin systems in $\Lambda_{n,m} \equiv \Lambda_{2\pi n} \times \Lambda_{2\pi m}$ with four spin interactions mediated by the matrix $\rho$ (see equations (2.8), (2.9) and (2.18)). The mapping we shall now discuss is due to Barma and Shastry [BS].

We consider $\Lambda_{n,m}$ wrapped in a torus as a “chess-board” and define a $n \times m$ regular lattice $\Lambda^*_{n,m}$ with vertices at the center of the black plaquetes of $\Lambda_{n,m}$ and edges connecting nearest neighbor vertices (see Figure 1). Note that the edges of $\Lambda^*_{n,m}$ are $45^\circ$ with respect to edges of $\Lambda_{n,m}$. The remarkable property in this construction is the one–to–one correspondence between vertices of $\Lambda_{n,m}$ and edges of $\Lambda^*_{n,m}$.

![Figure 1: The regular $\Lambda_{n,m}$ and dual $\Lambda^*_{n,m}$ lattices](image)

Let us define the “dual” relation $*: \Lambda_{n,m} \longrightarrow \Lambda^*_{n,m}$, defined by

$$(i, j)^* = e,$$

if the the edge $e \in \Lambda^*_{n,m}$ crosses the site with coordinates $(i, j) \in \Lambda_{n,m}$, and

$$p^* = v,$$

if
if the vertex $v \in \Lambda_{n,m}^*$ is the center of a black plaquette $p \in \Lambda_{n,m}$. This allows us to establish a correspondence between classical spin configurations \( \{ s_{i,j} : i = 1, \ldots, 2n; j = 1, \ldots, 2m \} \) and the two–state edge configurations in $\Lambda_{n,m}^*$. We say that the edge $e = (i, j)^*$ is occupied or vacant according to whether the value of $s_{i,j}$ is 1 or $-1$.

Now, the density matrix $\rho$ at the plaquette $p$, whose values depend on the spin configuration of its four corners, is mapped, by the “dual” transformation, into a vertex function $w$ at $v = p^*$ whose values depend on the configuration $\xi_v$ of the four incident edges. In the period 1 system there is only one vertex function $w$ for all $v \in \Lambda_{n,m}^*$. This leads us to the following definitions.

A vertex configuration $\xi$ is an assignment $\xi : \Lambda_{n,m}^* \rightarrow \{1, \ldots, 8\}$ which associates to each vertex $v$ one of the 4–incident–edges configurations $\xi_v$ shown in Figure 2. There are actually $2^4 = 16$ possible assignments of occupied–vacant incident edges but only 8 of them (shown in Figure 2) are relevant to the problem at hand.

A vertex configuration $\xi$ is said to be compatible if for each edge $e$ with end points $v$ and $v'$, $\xi_v$ and $\xi_{v'}$ have the same assignment for the common edge $e$. From here on, by vertex configuration $\xi$ we always means a compatible configuration. It is worth noting that a vertex configuration $\xi$ corresponds to a covering of $\Lambda_{n,m}^*$ by closed polygons (see Figure 3).

The weight of a vertex configuration $\xi$ is given by

$$w(\xi) = \prod_{v \in \Lambda_{n,m}^*} w_{\xi_v},$$

(3.1)
where, in view of (2.14), (2.16), the “dual” mapping and the label chosen in Figure 2,

\[ w_1 = e^{\beta J^z/m} \left( \cosh \frac{\beta \zeta}{m} + \frac{h}{\zeta} \sinh \frac{\beta \zeta}{m} \right), \]

\[ w_2 = e^{\beta J^z/m} \left( \cosh \frac{\beta \zeta}{m} - \frac{h}{\zeta} \sinh \frac{\beta \zeta}{m} \right), \]

\[ w_3 = w_4 = e^{-\beta J^z/m} \sinh \frac{\beta \kappa}{m}, \]

\[ w_5 = w_6 = e^{-\beta J^z/m} \cosh \frac{m}{\beta \kappa}, \]

\[ w_7 = w_8 = e^{-\beta J^z/m} \eta \sinh \frac{\beta \zeta}{m}, \]

where \( \zeta, \eta \) and \( \kappa \) are given following equation (2.11). Note that only the eight vertex configurations shown in Figure 2 have non–vanishing vertex function.

With these definitions, we conclude

**Proposition 3.1** The partition function (2.18) can be written as

\[ Z_{n,m} = \sum_{\xi} w(\xi), \quad (3.3) \]

where the sum runs over all compatible vertex configurations \( \xi \) with weights given by (3.1) and (3.2).

At this point, one can verify whether (3.2) satisfies the free fermion condition stated in (1.7). The equation

\[ w_1w_2 + w_3w_4 - w_5w_6 - w_7w_8 = (e^{2\beta J^z/m} - e^{-2\beta J^z/m}) \left( 1 + \frac{\eta^2}{\zeta^2} \sinh^2 \frac{\beta \zeta}{m} \right) \]

vanishes if and only if \( J^z = 0 \). This will be assumed in the next sections.

4 **Pfaffian Solution**

In this section the partition function \( Z_{n,m} \), written as the generating function of a spanning subgraph of \( \Lambda^* \equiv \Lambda_{n,m}^* \) in (3.3), will be solved by the Pfaffian technique. For this, we introduce some terminology.

Let \( G \) be a graph which we may think to be a “decoration” of \( \Lambda^* \). A dimer covering \( D \) of \( G \) is a spanning subgraph consisting of non–overlapping close–packed configurations of elements formed by an edge and its two terminal vertices (dimer). Note that no vertex is left vacant in \( D \).

In the early 60’s the problem of enumerating all dimer coverings has been solved for certain lattice graphs (see [K] for a review). Right after this, Green and Hurst [GH] showed that the counting dimer coverings can be useful to solve the partition function of two–dimensional Ising models (see also [GH]). Since the Ising model has to be written as the generating function of a
spanning subgraph of the form (3.3), the method we now start to explain can be applied directly to this equation.

Given a collection of positive numbers \( z = \{z_e\}_{e \in G} \) indexed by the edges of \( G \), let \( \Gamma(z) \) be the generating function for the dimer covering problem on \( G \),

\[
\Gamma(z) = \sum_D z(D),
\]

where

\[
z(D) = \prod_{e \in D} z_e
\]
is the weight of a dimer covering \( D \) with the product running over edges covered by the dimers of \( D \).

If \( \{1, 2, \ldots, k\} \) is an arbitrary enumeration of the vertices of \( G \) let \( A = \{a_{xy}\} \) be a skew-symmetric matrix of order \( k \) whose elements \( a_{xy} = -a_{yx} \) satisfy

\[
|a_{xy}| = \begin{cases} 
  z_e, & \text{if } e = \langle xy \rangle \in G, \\
  0, & \text{otherwise}. 
\end{cases}
\]

The sign of \( a_{xy} \) is chosen according to a given orientation of the edges of \( G \): \( a_{xy} > 0 \) if \( e \) is oriented from \( x \) to \( y \).

Our ability to calculate the partition function (3.3) depends on whether the graph \( G \) obtained by decorating \( \Lambda^* \), and the associate weight function, agree with the hypotheses of the following results.

**Proposition 4.1** Let \( A \) be a \( k \times k \) matrix defined as above with \( k \) even. Then

\[
\det A = (\text{Pf } A)^2,
\]

where the Pfaffian, \( \text{Pf } A \), of \( A \) is defined by

\[
\text{Pf } A = \sum_\pi \varepsilon_\pi a_{j_1,j_2} a_{j_3,j_4} \cdots a_{j_{k-1},j_k},
\]

with the sum over all partitions of \( \{1, 2, \ldots, k\} \) into pairs; \( P_\pi \) is a permutation \( j_1 j_2 \cdots j_k \) such that \( |j_1 j_2| |j_3 j_4| \cdots |j_{k-1} j_k| \) is a description of the partition \( \pi \) and \( \varepsilon_\pi \) is the signature of \( P_\pi \).

For a proof of Proposition 4.1 see [K]. To understand, however, equation (4.3) and get a feeling of the next statement, let us observe that if we superpose a dimer covering \( D \) on another dimer covering \( D' \) we obtain a circuit covering of \( G \). For, starting from any vertex of \( G \) we walk along the edge in \( D \) till we reach other extremity; then, walk along the edge of \( D' \) and along the edge of \( D \) after the extremity has been reached and so on. Continuing this process we eventually arrive at the starting vertex when we have to choose a different one and start again this process till all vertices of \( G \) have been covered. It is a matter of fact that \( \det A \) is the generating function of circuit covering of \( G \). Note that

\[
\det A = \sum_P \varepsilon_P a_{1,P(1)} \cdots a_{k,P(k)},
\]
and all permutation \( P = \{ P(1), \ldots, P(k) \} \) of \( \{1, \ldots, k\} \) can be decomposed into cycles \((C_1, \ldots, C_s)\). Since by the superposition of \( D \) on \( D' \) with \( D, D' \) running over all dimer covering one gets all circuit covering of \( \mathcal{G} \), it is tempting to identify \( \det A \) with \( \Gamma^2(z) \). As we shall see, this requires to choose the orientation of \( \mathcal{G} \) properly.

**Proposition 4.2** If the number of contra–oriented steps, with respect to the chosen orientation of \( \mathcal{G} \), performed along each circuit generated from the superposition of \( D \) on \( D' \) is odd, for any two dimer coverings \( D \) and \( D' \) of \( \mathcal{G} \), then

\[
\Gamma(z) = |\text{Pf } A|.
\] (4.4)

By imposing that the parity of all circuits to be odd \( \det A \) becomes a sum of positive terms\(^1\) and consequently the sign of every term in Pf \( A \) remains the same implying Proposition 4.2 (see ref. [K], pg. 89, for a proof). We shall call an orientation of \( \mathcal{G} \) with such property admissible.

Combining the two propositions we arrive at

\[
\Gamma(z) = \sqrt{\det A}.
\] (4.5)

It thus remains only to write (3.3) as a generating function of dimer covering of a properly oriented graph \( \mathcal{G} \) and evaluate the determinant of the associated skew-symmetric matrix \( A \).

We begin by choosing the appropriated “decorated” graph \( \mathcal{G} \). We take \( \mathcal{G} \) as being the terminal graph \((\Lambda^*)^T\) (see Figure 4) obtained by the replacement of all vertices \( v \in \Lambda^* \) by a cluster of 4 vertices all connected with each other by single edges (a complete graph \( \mathbb{K}_4 \)) denoted by \( \square_v \). Each vertex of \( \mathcal{G} \) is an extreme or terminal of an incident edge \( e \in \Lambda^* \) a property by which \( \mathcal{G} \) is named. The edge \( e \in \mathcal{G} \) will be called internal if it belongs to \( \square_v \) for some \( v \) and external otherwise. The external edges are identified with the edges of \( \Lambda^* \). Note that the graph \( \mathcal{G} \) is not planar due to the crossing of diagonal edges of \( \square_v \) which prevents, in general, that an admissible orientation can be found leading the Pfaffian method useless to compute \( \Gamma(z) \). As we shall see, it is exactly this problem that will help us to solve another one.

![Figure 4: Construction of the terminal graph \((\Lambda^*)^T\)](image)

What makes a terminal graph significant for our problem is that we can associate to each (compatible) vertex configuration \( \xi \) in \( \Lambda^* \) a dimer covering \( D \) in the terminal lattice \( \mathcal{G} \) as follows:

\(^1\)Note that the signature \( \varepsilon_P \) of a permutation \( P \) is given by \((-1)^s\) where \( s \) is the number of cycles \((C_1, \ldots, C_s)\) in \( P \). If the parity of all circuits is odd, we get \((-1)\) from the product of matrix elements along each cycle \( C_i \) compensating the signature.
an external edge \( e \in D \) if the edge \( e \in \Lambda^* \) is occupied in the configuration \( \xi \). A list of vertex configurations \( \xi_v \) at vertex \( v \) and its associated dimer covering at \( \Box_v \) is shown in Figure 5. The problem here is the degeneracy of dimer covering associated with \( \xi_v = 2 \).

![Figure 5: Vertex and their associated dimer configurations](image)

It turns out that there exist a (non-admissible) orientation of \( G \) such that, if a dimer configuration \( D' \) differ from \( D \) by the substitution of a crossing dimer covering at \( \Box_v \) by a non-crossing one, the superposition of \( D' \) on \( D \) yields a circuit in \( \Box_v \) with even parity and a minus sign is given to this contribution. In addition, this orientation is such that all circuits, excluding the ones with crossings and those winding around the torus, generated by the superposition of \( D \) on \( D' \), with \( D \) and \( D' \) two dimer coverings in \( G \), have odd parity. The orientation of \( G \) with these properties is shown in Figure 6 for a given \( \Box_v \).

![Figure 6: The orientation of a cell \( \Box_v \)](image)

As demonstrated in ref. \[K\], pages 98–99, it is the crossing dimer configuration which gives rise to the minus sign and Proposition 4.2 holds if \( z(D) \) is replaced by \( \tilde{z}(D) = (−1)^c z(D) \) in equation (4.1) where \( c \) is the number of \( v \)'s such that \( \Box_v \) is cross-covered in \( D \) and free boundary condition is imposed in \( G \).

As \( \Lambda_{n,m} \) is wrapped on a torus, \( \Lambda^*_{n,m} \) may have many different topologies depending on the values of \((n,m)\). The lattice \( \Lambda^*_{n,m} \) will be wrapped on a torus only if \( m \) is a multiple of \( n \). We shall in the following assume that \( m = kn + 1 \) for some integer \( k \) which makes \( \Lambda^*_{n,m} \) to be helically wound on a torus and avoids circuits winding around \( G = (\Lambda^*_{n,m})^T \) (see McCoy and Wu \[McW\] for the dimer covering problem on a torus).
Once the orientation of $G$ has been resolved, we are in position to write (3.3) as a generating function $\Gamma(z)$ of dimer covering of $G = (\Lambda^*)^T$. We observe that the degeneracy in the association of $\xi_v = 2$ results from the dimer covering of $\square_v$ using only internal edges. For all dimer covering associated with $\xi_v \neq 2$ the association is unique and there are at least two external edges covered.

Now, let $C_{\xi}$ denote the class of equivalence defined as follows. Two dimer coverings of $G$, $D$ and $D'$, are said to be equivalent, $D \sim D'$, if they give rise a configuration $\xi$ under the association made in Figure 5. Then equation (4.1) can be written as

$$\Gamma(z) = \sum_{\xi} u(\xi), \quad (4.6)$$

where

$$u(\xi) = \sum_{D \in C_{\xi}} z(D). \quad (4.7)$$

As Fan and Wu [FW] (see also [HLW]), if in addition we set $z_e = 1$ for all external edges $e \in G$, then $u(\xi)$ factors out,

$$u(\xi) = \prod_{v \in \Lambda^*} u_{\xi_v}, \quad (4.8)$$

where $u_{\xi_v}$ depends only on $\{z_e\}_{e \in \square_v}$ and is explicitly given by (using the label of Figure 5)

$$u_{\xi_v} = \begin{cases} 1, & \text{if } \xi = 1, \\ z_{(14)} z_{(23)} + z_{(34)} z_{(12)} - z_{(13)} z_{(24)}, & \text{if } \xi = 2, \\ z_{(24)}, & \text{if } \xi = 3, \\ z_{(13)}, & \text{if } \xi = 4, \\ z_{(34)}, & \text{if } \xi = 5, \\ z_{(12)}, & \text{if } \xi = 6, \\ z_{(23)}, & \text{if } \xi = 7, \\ z_{(14)}, & \text{if } \xi = 8. \end{cases} \quad (4.9)$$

In order to compare equation (3.3) with (4.6), we identify $u(\xi)$ with the normalized vertex function $w(\xi)/w_1$: $u_{\xi_v} = w_{\xi_v}/w_1$. Equation (4.1) then reads

$$Z_{n,m} = w_{1}^{n-m} \sum_{\xi} u(\xi) = w_{1}^{n-m} \cdot \Gamma(z).$$

Note that $z = \{z_e\}$ has to be specified only for the six internal edges of $\square_v$ for a given $v$ (recall $z_e = 1$ for all external edges). Moreover, note that the last six equations of (4.9) determine completely the correspondence between the vertex function $u(\xi) = w(\xi)/w_1$ and the weight $z$ since, by substituting those equations into the second equation, we have

$$u_2 = u_5 u_6 + u_7 u_8 - u_3 u_4 = \frac{w_5 w_6 + w_7 w_8 - w_3 w_4}{w_1^2} = \frac{w_2}{w_1}, \quad (4.10)$$

provided $w_1, \ldots, w_8$ satisfy the free–fermion condition (1.7).

In view of (4.5), the conclusion of this section may be summarize by the following result.
Proposition 4.3 Let $A$ be a $(4nm \times 4nm)$ skew–symmetric matrix defined as in (4.2) with the sign determined by the orientation of $\mathcal{G}$ as in Figure 6. If $z_e = 1$ for all external edges and the equations $u_\xi = w_\xi/w_1$, $\xi \in \{3, 4, \ldots, 8\}$, given by (4.9) define $z_e$ for all internal edges $e \in \mathcal{G}$, then

$$Z_{n,m} = w_1^{n-m} \sqrt{\det A}.$$  \hspace{1cm} (4.11)

5 Evaluation of the Determinant

In this section we evaluate the determinant of the $4nm \times 4nm$ matrix $A$, with entries given by (4.2), as described in Proposition 4.3.

We begin by block–diagonalizing $A$ by the Fourier method. Let us first remind some facts about Kronecker (or tensor) product. If $M = \{m_{ij}\}$ and $N = \{n_{ij}\}$ are $k \times k$ and $l \times l$ matrices, respectively, the Kronecker product of $M$ and $N$, $M \otimes N$, is a $kl \times kl$ matrix given by $M \otimes N = \{m_{ij} N\}$. If $M_1 \otimes N_1$ and $M_2 \otimes N_2$ are two matrices as above, their product is given by

$$(M_1 \otimes N_1) (M_2 \otimes N_2) = M_1 M_2 \otimes N_1 N_2 .$$  \hspace{1cm} (5.1)

The Kronecker products do not commute, $M \otimes N \neq N \otimes M$, but there exists a permutation matrix $P$ such that $P^T (M \otimes N) P = N \otimes M$. So

$$\det M \otimes N = \det N \otimes M = \det (I \otimes M) \det (I \otimes N) = (\det M)^l (\det N)^k .$$

We want to write $A$ as a $nm \times nm$ matrix whose elements are $4 \times 4$ matrices, called inner matrices. Since the entry $a_{xy}$ of $A$ is nonvanishing if $\langle xy \rangle = e$ is either an edge at $\square_v$ or connecting nearest neighbor ones, there are only three relevant inner matrices. The inner matrices are labeled by the enumeration of vertices of $\square_v$ and the sign of their entries determined according to the orientation of $\mathcal{G}$ (see Figure 6).

Let $U$ be the skew-symmetric “adjacency matrix” of $\square_v$ defined by the last six equations of (4.3),

$$U = \begin{pmatrix} 0 & u_6 & -u_4 & -u_8 \\ -u_6 & 0 & u_7 & u_3 \\ u_4 & -u_7 & 0 & -u_5 \\ u_8 & u_4 & u_5 & 0 \end{pmatrix},$$  \hspace{1cm} (5.2)

and let $X$ and $Y$ be defined by the external edges bridging $\square_v$ and $\square_{v'}$ in the southeast and southwest directions, respectively,

$$X = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} .$$  \hspace{1cm} (5.3)

The $nm \times nm$ matrix is labeled by the vertices $v$ of $\Lambda_{n,m}^*$ which is helically wound on the torus (recall $m = kn + 1$). The vertex enumeration $\{1, 2, \ldots, mn\}$ can be made starting from the upper left corner following the southeast direction. Note that all vertices are visited in this way with the last site $mn$ being neighbor to the first one. In addition, the four nearest neighbors of a vertex $j$
have the following coordinates (modulo \( mn \)): in the northeast and southwest directions, \( j^{ne} = j + m \) and \( j^{sw} = j - m \), respectively; in the northwest and southeast neighbors, we have \( j^{nw} = j - 1 \) and \( j^{se} = j + 1 \).

Here, the relevant \( N \times N \) matrices are the identity \( I = I_N \) and the “forward shift” permutation,

\[
\Pi = \Pi_N = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
-1 & 0 & 0 & \cdots & 0
\end{pmatrix},
\]

which satisfies the following properties \( \Pi^N = -I \) and \( \Pi^T = \Pi^{-1} = -\Pi'^{-1} \).

We have chosen the shift matrix \( \Pi \) with a phase \( e^{ix} = -1 \) in order to give odd parity to circuits winding around (see [HG]).

From these definitions with \( N = mn \), the matrix \( A \) can be written as

\[
A = I \otimes U + \Pi \otimes X - \Pi^T \otimes X^T + \Pi^m \otimes Y - (\Pi^T)^m \otimes Y^T.
\]  

To block–diagonalize (5.4) we introduce the \( N \times N \) Fourier matrix \( F_N = \{f_{ij}\} \) defined as follows. Let \( \omega = \omega_N = e^{-\pi i/N} \), \( i = \sqrt{-1} \), and note the following property

\[
1 + \omega^2 + \omega^4 + \cdots + \omega^{2(N-1)} = 0.
\]

We define

\[
f_{ij} := \frac{1}{\sqrt{N}} \omega^{(2i-1)j}.
\]

The Fourier matrix \( F \) is a unitary matrix, \( FF^\dagger = F^\dagger F = I_N \), which diagonalizes the so–called class of \( N \times N \) “circulant” matrices. In particular, using (5.1) and property (5.6), we have

\[
F^{-1} \Pi F = \text{diag} \{\omega, \omega^3, \ldots, \omega^{2N-1}\} \equiv D
\]

and \( F^{-1} \Pi^k F = D^k \). In addition, taking the Hermitian conjugate of (5.8), gives

\[
(F^{-1} \Pi F)^\dagger = F^{-1} \Pi^T F = \bar{D},
\]

where \( \bar{D} = \text{diag} \{\bar{\omega}, \bar{\omega^3}, \ldots, \bar{\omega}^{2N-1}\} \).

Now, we apply the Fourier matrix \( \mathcal{F} = F_{mn} \otimes I_4 \) (\( \mathcal{F}^{-1} = F_{mn}^{-1} \otimes I_4 \)) in (5.9) and use the multiplication rule (5.4) with (5.8) to obtain

\[
\mathcal{F}^{-1} A \mathcal{F} = I \otimes U + D \otimes X - \bar{D} \otimes X^T + D^m \otimes Y - \bar{D}^m \otimes Y^T = \bigoplus_j R_j,
\]

where

\[
R_j = \begin{pmatrix}
0 & u_6 & -u_4 - \bar{\omega}^j & -u_8 \\
-u_6 & 0 & u_7 & u_3 + \omega^{jm} \\
u_4 + \omega^j & -u_7 & 0 & -u_5 \\
u_8 & -u_3 - \bar{\omega}^jm & u_5 & 0
\end{pmatrix},
\]
and $j = 2(r-1)n + 2l - 1$ with $r \in \{1, \ldots, m\}$ and $l \in \{1, \ldots, n\}$.

As a consequence, the determinant of $A$ can be written as

$$\text{det} \, A = \prod_{l=1}^{n} \prod_{r=1}^{m} |R_j|, \tag{5.11}$$

and evaluated by the Laplace method,

$$|R_j| = \frac{1}{w_1^j} \left( a + 2b \cos (\phi_r + \psi_l^0) + 2c \cos \varphi_l + 2d \cos (\phi_r + \psi_l^+) + 2e \cos (\phi_r - \psi_l^-) \right), \tag{5.12}$$

with

$$\phi_r = \frac{(2r-1)\pi}{m}, \quad \varphi_l = \frac{(2l-1)\pi}{n}, \tag{5.13}$$

$$\psi_l^0 = \varphi_l/m + \pi/m, \quad \psi_l^\pm = (1 \pm m)\varphi_l/m + \pi/m,$$

and

$$a = w_1^2 + w_2^2 + w_3^2 + w_4^2,$$

$$b = w_4 - w_2 w_3,$$

$$c = w_3 - w_3 w_4,$$

$$d = w_3 w_4 - w_5 w_6,$$

$$e = w_3 w_4 - w_7 w_8. \tag{5.14}$$

Here, we have used $u_\xi = w_\xi/w_1$ and equation (1.7) to obtain this form.

## 6 The Free Energy

This section is concerned with the calculation of free energy $f = f(\beta)$, given by (1.3), of a class of quantum spin systems satisfying (1.7). Two limits are required to be taken: the Trotter limit $m \to \infty$ of the partition function $Z_{n,m}$ and the thermodynamic limit $n \to \infty$ of its logarithm. So far, we have obtained $Z_{n,m}$ exactly in Proposition 4.3 and equations (5.11) - (5.14), for any value $n,m \in \mathbb{Z}_+$. In the limit, these equations involve infinite products and the way we deal with them depends on whether $m$ or $n$ is taken to infinite. A third limit, $\lim_{\beta \to \infty} f(\beta)$, will also be taken in order to get the ground state energy.

We begin by studying the Trotter limit. Taylor expanding (3.2) (with $J^z = 0$) up to second order in $1/m$ and substituting the result into (5.14), gives

$$a = 2 + 2\beta^2 (h^2 + \zeta^2 + \kappa^2) m^2 + \mathcal{O} \left( \frac{1}{m^3} \right),$$

$$b = c = \frac{2\beta^2 h \kappa}{m^2} + \mathcal{O} \left( \frac{1}{m^3} \right),$$

$$d = -1,$$

$$e = \frac{\beta^2 (\kappa^2 - \eta^2)}{m^2} + \mathcal{O} \left( \frac{1}{m^3} \right), \tag{6.1}$$

where $\eta = J^x - J^y$, $\kappa = J^x + J^y$, and $\zeta^2 = h^2 + \eta^2$. The value of $d$ is exact in view of (5.14) and (3.2).
We introduce a real valued function
\[ G(z) := \frac{1}{2} (a + c(z + \bar{z})) , \]
and a complex valued function
\[ H(z) := z^{-(1-n+m)/m} (d + bz + ez^2) \]
defined on the unit circle \(|z|^2 = 1\), and set
\[ \Delta^2(z) = G^2 - H \bar{H} \]
\[ = \frac{4\beta^2}{m^2} \left( \frac{h^2 + \zeta^2 + \kappa^2}{2} + h\kappa(z + \bar{z}) + \frac{\kappa^2 - \eta^2}{4}(z^2 + \bar{z}^2) \right) + \mathcal{O} \left( \frac{1}{m^3} \right) , \]
in view of (6.1). Note that \((G + \Delta)(G - \Delta) = H \bar{H}\).

Using these definitions, equation (5.12) can be rewritten as
\[ w_1^2|R_j| = 2G_l + H_l \omega_r + \bar{H}_l \bar{\omega}_r \]
\[ = \frac{H_l}{\omega_r} \left( \frac{G_l + \Delta_l}{-H_l} - \omega_r \right) \left( \frac{G_l - \Delta_l}{-H_l} - \omega_r \right) , \]
where \(\omega_r = e^{i\phi_r}, G_l = G(e^{i\varphi_l})\) with \(\phi_r, \varphi_l\) as in (6.13) and similar expressions holding for \(H_l\) and \(\Delta_l\).

Now, since \(\{\omega_r; r = 1, \ldots, m\}\) are the roots of \(-1\), we have
\[ \prod_{r=1}^{m} (x - \omega_r) = x^m + 1 , \]
and the substitution of (6.5) into (5.11) yields
\[ w_1^{2nm} \det A = \prod_{l=1}^{n} (-H_l)^m \left[ \left( \frac{G_l + \Delta_l}{-H_l} \right)^m + 1 \right] \left[ \left( \frac{G_l - \Delta_l}{-H_l} \right)^m + 1 \right] \]
\[ = \prod_{l=1}^{n} \left\{ (-H_l)^m + (-\bar{H}_l)^m + (G_l + \Delta_l)^m + (G_l - \Delta_l)^m \right\} , \]
where we have used \(\prod R(-\omega_r) = 1\).

To evaluate (6.7), note that the asymptotic estimate
\[ \left( 1 + \frac{c}{m} + \mathcal{O} \left( \frac{1}{m^2} \right) \right)^m = e^c \left( 1 + \mathcal{O} \left( \frac{1}{m} \right) \right) \]
holds for all \(c \in \mathbb{R}\). From (6.3) and the fact that \(m = kn + 1\) (helical condition) for some positive integer \(k\), we have
\[ (-H_l)^m = e^{-(k+1)n\varphi_l} \left( 1 + \mathcal{O} \left( \frac{1}{m} \right) \right) , \]
which tends to 1 or $-1$ depending on whether $k$ is odd or even. We always take $k$ an odd number.

From equations (6.2) and (6.4) and asymptotic expansions (6.1), we have

$$(G_l \pm \Delta_l)^m = e^{\pm 2\delta_l} \left(1 + O\left(\frac{1}{m}\right)\right),$$

(6.9)

where $\delta_l = \delta(\varphi_l)$ with

$$\delta(\varphi) = \beta \left(h^2 + (J^x)^2 + (J^y)^2 + 2h(J^x - J^y)\cos \varphi + 2J^x J^y \cos 2\varphi\right)^{1/2}. \quad (6.10)$$

This gives

$$w_{1mm}^2 \det A = \prod_{l=1}^{n} \left(2 + e^{2\delta_l} + e^{-2\delta_l}\right) \left(1 + O\left(\frac{1}{m}\right)\right)$$

$$= \prod_{l=1}^{n} (2 \cosh \delta_l)^2 \left(1 + O\left(\frac{1}{m}\right)\right)$$

(6.11)

$$= \exp \left(2 \sum_{l=1}^{n} \ln (2 \cosh \delta_l)\right) \left(1 + O\left(\frac{1}{m}\right)\right),$$

(6.11)

and leads to the following expression for the partition function (see Proposition 4.3),

$$Z_n = \lim_{m \to \infty} Z_{n,m} = \exp \left\{\frac{n}{\pi} \int_{0}^{2\pi} d\varphi \ln [2 \cosh (\delta(\varphi))]\right\} \left(1 + o\left(\frac{1}{n}\right)\right), \quad (6.12)$$

where the limit is taken over the subsequence with $m = kn + 1$, $k = 1, 3, 5, \ldots$. We have verified that the limit agrees with (6.12) if taken over the subsequence with $m = kn$, $k \in \mathbb{N}_+$, corresponding to toroidal boundary condition. The remaining subsequences are expected to converge to the same limit.

In approximating the Riemann sum by the Riemann integral in equation (6.12) we have used the fact that the integrand $g = \ln (2 \cosh \delta)$ is a periodic and smooth function of $\varphi$ to get the sharper than midpoint rule error estimate

$$\left|\frac{1}{n} \sum_{l} g(\varphi_l) - \frac{1}{2\pi} \int_{0}^{2\pi} g(\varphi) d\varphi\right| = O\left(\frac{1}{nK}\right)$$

for any $K \in \mathbb{N}_+$ (see Proposition 5.2 of [MFH]).

From equations (2.8) and (6.12), the free energy function (1.3) is thus given by

$$f(\beta) = \frac{1}{\pi \beta} \int_{0}^{\pi} d\varphi \ln [2 \cosh (\delta(\varphi))]$$

(6.13)

and the ground state energy, $e_0 = \lim_{\beta \to \infty} f(\beta)$, by

$$e_0 = \frac{1}{\pi} \int_{0}^{\pi} d\varphi \sqrt{h^2 + (J^x)^2 + (J^y)^2 + 2h(J^x - J^y)\cos \varphi + 2J^x J^y \cos 2\varphi},$$

(6.14)

where we have used the fact that $\delta(\varphi)$ is symmetric about $\varphi = \pi$.

This concludes the analysis of the period one $XY$ model in a transverse magnetic field.
The previous analysis of a homogeneous quantum spin system with period \( p = 1 \) can be extended to arbitrary period. In this section we shall illustrate the procedure fully for the period \( p = 2 \) case.

For a periodic model with period \( p \), all steps of Section 2 can be repeated without modification. Since the purpose of that section was to separate consecutive terms of the Hamiltonian (1.1) we may adopt the same odd–even decomposition (2.6), provided \( p \) is an even number. We also take \( N = 2pn \) and impose periodic boundary condition. The only difference is that we now have \( p \) matrices \( \rho^{(1)}, \ldots, \rho^{(p)} \), of the form (2.14) with distinct values \((\eta_i, \kappa_i, \zeta_i), i = 1, \ldots, p\).

Section 3 is also maintained with a minor modification. A vertex \( v \) in the dual lattice \( \Lambda^*_{n,m} \) will be now distinguished according to its type labeled by \( \{1, \ldots, p\} \): the label increases sequentially (modulo \( p \)) if one follows the northeast or southeast directions as seen in Figure 7.

![Figure 7: Dual lattice with period \( p \)](image)

We associate to each vertex \( v^{(i)} \) of type \( i \) a set \( \{w_1^{(i)}, \ldots, w_8^{(i)}\} \) of weights given by (3.2) with parameters \((\eta_i, \kappa_i, \zeta_i)\) and to each vertex (compatible) configuration \( \xi = \{\xi_v\} \), a weight \( w(\xi) = \prod_v w_{\xi_v} \) with \( w_{\xi_v} = w^{(i)}_{\xi_v} \) if \( v \) is of the type \( i \).

This completes the set up which allows Propositions 3.1 and 4.3 to be holden for any period \( p \) and we are now ready to evaluate the determinant of the correspondent matrix \( A \).

Up to this point there were no extra difficulties. The matrix \( A \) is defined by equation (4.2) with the signal determined by the orientation of the terminal graph \( G = (\Lambda^*_{n,m})^T \) as in Figure 6. But now the unit cells is composed by \( p \) different types of vertices \( \Box_v \) which leads to \( 4p \times 4p \) internal matrices. We shall write explicitly these matrices, whose rows and columns are indexed by the label of Figure 8, only for \( p = 2 \).

Similarly to what have been done in Section 5, we set \( u_{\xi} \equiv u^{(1)}_{\xi} = w^{(1)}_{\xi}/w^{(1)}_1 \) and \( u^{(2)}_{\xi} = \)
Figure 8: Unit cell for period $p = 2$

$$w^{(2)}_\xi / w^{(2)}_1,$$ and let

$$V = \begin{pmatrix}
0 & u_6 & -u_4 & -u_8 & 0 & 0 & 0 & 0 \\
-u_6 & 0 & u_7 & u_3 & 0 & 0 & 0 & 0 \\
u_4 & -u_7 & 0 & -u_5 & 1 & 0 & 0 & 0 \\
u_8 & -u_3 & u_5 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & u'_6 & -u'_4 & -u'_8 \\
0 & 0 & 0 & 0 & -u'_6 & 0 & u'_7 & u'_3 \\
0 & 0 & 0 & 0 & u'_4 & -u'_7 & 0 & -u'_5 \\
0 & 0 & 0 & 0 & u'_8 & -u'_3 & u'_5 & 0
\end{pmatrix} = \begin{pmatrix}
U & X \\
-XT & U'
\end{pmatrix}, \quad (7.1)$$

and

$$Q = \begin{pmatrix}
0 & 0 \\
X & 0
\end{pmatrix}, \quad R = \begin{pmatrix}
0 & Y \\
0 & 0
\end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix}
0 & 0 \\
Y & 0
\end{pmatrix}, \quad (7.2)$$

where $U, X$ and $Y$ is as in (5.2) and (5.3). If $m = kn + 1$, with $k$ an odd number, we recall that $\Lambda^{*}_{n,m}$ becomes helically wound on a torus. The matrix $A$ can thus be written as (see Figure 9 for a self–explanation)

$$A = I \otimes V + \Pi \otimes Q - \Pi^T \otimes Q^T + \Pi^m \otimes R - (\Pi^T)^m \otimes R^T + \Pi^{m+1} \otimes S - (\Pi^T)^{m+1} \otimes S^T, \quad (7.3)$$

where $I$ and $\Pi$ are the $nm \times nm$ identity and the forward shift.

Figure 9: Numbering nearest neighbors unit cells

Conjugating $A$ by the Fourier matrix $\mathcal{F} = F_{mn} \otimes I_8$ gives

$$\mathcal{F}^{-1} A \mathcal{F} = \bigoplus_j S_j \equiv \bigoplus_j \begin{pmatrix}
U & -M_j^\dagger \\
M_j & U'
\end{pmatrix}, \quad (7.4)$$
where \( j = 2(r - 1)n + 2l - 1 \), with \( r \in \{1, \ldots, m\} \), \( l \in \{1, \ldots, n\} \) and

\[
M_j = -X^T + \omega^j X + \omega^{j(m+1)} Y - \omega^{jm} Y^T = \begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & 0 & 0 & \omega^{j(m+1)} \\
\omega^j & 0 & 0 & 0 \\
0 & -\omega^{jm} & 0 & 0
\end{pmatrix}, \tag{7.5}
\]

with \( \omega = e^{i\pi/(nm)} \). This implies

\[
\det A = \prod_{i=1}^n \prod_{r=1}^m |S_j|, \tag{7.6}
\]

where

\[
|S_j| = |U| \left| U' + M_j U^{-1} M_j^\dagger \right|. \tag{7.7}
\]

Here, we used the fact that (7.4) can be brought to block diagonal form by a non–unitary transformation

\[
\begin{pmatrix}
1 & 0 \\
-M_j U^{-1} & 1
\end{pmatrix}
\begin{pmatrix}
U & -M_j^\dagger \\
M_j U' & 0
\end{pmatrix}
\begin{pmatrix}
1 & U^{-1} M_j^\dagger \\
0 & 1
\end{pmatrix} = \begin{pmatrix}
U & 0 \\
0 & U' + M_j U^{-1} M_j^\dagger
\end{pmatrix}. \tag{7.8}
\]

Using Laplace method and equation (4.10), we have \( |U| = u_2^2 \) and

\[
U^{-1} = \frac{1}{u_2} \begin{pmatrix}
0 & -u_5 & -u_3 & u_7 \\
u_5 & 0 & -u_8 & u_4 \\
u_3 & u_8 & 0 & u_6 \\
-u_7 & -u_4 & -u_6 & 0
\end{pmatrix},
\]

which, in view of (7.5) and (7.7), gives

\[
|S_j| = \frac{1}{u_2^2} \begin{vmatrix}
0 & u_2 u'_6 - u_6 \omega^{-j(m+1)} & -u_2 u'_4 - u_3 \omega^{-j} & -u_2 u'_8 + u_8 \omega^{jm} \\
u_2 u'_4 + u_3 \omega^j & 0 & u_2 u'_7 - u_7 \omega^{jm} \\
u_2 u'_8 - u_6 \omega^{-jm} & -u_2 u'_3 - u_4 \omega^{-j(m+1)} & 0 \\
\end{vmatrix},
\]

\[
(\omega_1 \omega_4')^2 |S_j| = 2a + 2b \cos(\xi_j) + 2c \cos(m \xi_j) + 2d \cos((m+1) \xi_j) + 2e \cos(2m \xi_j) + 2f \cos((2m+1) \xi_j) + 2g \cos(2(m+1) \xi_j), \tag{7.9}
\]

where \( \xi_j = \pi j/nm = \phi_r + \varphi_l/m - \pi/m \). Defining

\[
\begin{align*}
\Omega_1 & = \omega_1 \omega_4' + \omega_2 \omega_5', \\
\Omega_2 & = \omega_3 \omega_4' + \omega_4 \omega_3', \\
\Omega_3 & = \omega_5 \omega_6' + \omega_6 \omega_5', \\
\Omega_4 & = \omega_7 \omega_5' + \omega_8 \omega_7', \\
\end{align*}
\tag{7.10}
\]

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the $a$--$g$ constants are given by

\begin{align*}
2a &= \Omega_1^2 + \Omega_2^2 + \Omega_3^2 + \Omega_4^2 + 2(w_5w_6w_3'w_4' + w_7w_8w_3'w_6' - w_1w_2w_1'w_2' - w_3w_4w_3'w_4'), \\
b &= w_5w_8w_3'w_7' + w_6w_7w_5'w_8' - w_1w_4w_1'w_4' - w_2w_3w_2'w_3', \\
c &= \Omega_2\Omega_3 - \Omega_1\Omega_4, \\
d &= \Omega_2\Omega_4 - \Omega_1\Omega_3, \\
e &= (w_3w_4 - w_7w_8)(w_4'w_1' - w_7'w_8'), \\
f &= w_6w_8w_3'w_7' + w_5w_7w_6'w_8' - w_1w_3w_1'w_3' - w_2w_4w_2'w_4', \\
g &= (w_3w_4 - w_5w_6)(w_3'w_4' - w_5'w_6').
\end{align*}

(7.11)

Note that, in view of equations in (3.2), $b = f$ and $g = 1$ hold exactly. This fact together with the trigonometric relations

\[ \cos(\xi_j) + \cos(2m\xi_j + \xi_j) = 2\cos((m + 1)\xi_j)\cos(m\xi_j), \]

and

\[ \cos((m + 1)\xi_j) = 2\cos((m + 1)\xi_j) - 1, \]

lead the right hand side of (7.8) to be written as

\begin{align*}
(w_1w_1')^2|S_j| &= 4(\cos^2((m + 1)\xi_j) - 2B_t\cos((m + 1)\xi_j) + C_t) \\
&= 4(\cos((m + 1)\xi_j) - x_i^+)(\cos((m + 1)\xi_j) - x_i^-),
\end{align*}

(7.12)

where

\[ B_t = -\left(\frac{d}{4} + \frac{b}{2}\cos(\varphi_t)\right), \]

\[ C_t = \frac{a - 1}{2} + \frac{c}{2}\cos(\varphi_t) + \frac{e}{2}\cos(2\varphi_t) \]

(note that $\cos(m\xi_j) = \cos(2(r + 1)\pi + (2l - 1)/n) = \cos(\varphi_i)$) and

\[ x_i^\pm = B_t \pm \sqrt{B_t^2 - C_t}, \]

(7.13)

where

\[ D_t^2 = \frac{\beta_4}{m^4} \left\{ (\kappa\eta - \kappa'\eta')\cos(\varphi_i) + (h + h')\cos(\varphi_i) \right\} \pm D_t + O\left(\frac{1}{m^4}\right), \]

(7.14)

Here a word about the $1/m$ expansion is in order. Differently from the period $p = 1$ case, the constants (7.11) of period $p = 2$ have to be computed up to order $1/m^4$ to pick up the relevant terms. This happens because $B^2 - C$, whose leading order term is written in equation (7.14), is $O(1/m^4)$. So the square root $D_t$ has the same order $1/m^2$ of $B_t - 1$. Equation (7.12) reduces the present problem of evaluating $\det A$ to the one already dealt with in the previous section.
Repeating the steps in (6.2) – (6.5), we have

\[
(w_1 w_1')^2 \mid S_j \mid = \frac{H_i^2}{\omega_i^2} \left( \frac{x_i^+ + \Delta_i^+}{-H_i} - \omega_r \right) \left( \frac{x_i^+ - \Delta_i^+}{-H_i} - \omega_r \right) \left( \frac{x_i^- + \Delta_i^-}{-H_i} - \omega_r \right) \left( \frac{x_i^- - \Delta_i^-}{-H_i} - \omega_r \right),
\]

(7.15)

where \( H_i = (\omega_i)^{-1} (1 - n + m/m) \) and \((\Delta_i^\pm)^2 = (x_i^\pm)^2 - H_i H_t = (x_i^\pm)^2 - 1 \) which, in view of (7.14), can be estimated by

\[
(\Delta_i^\pm)^2 = \frac{\beta^2}{m^2} \left\{ \kappa^2 + (\kappa')^2 + \eta^2 + (\eta')^2 + (h + h')^2 + (\kappa \kappa' - \eta \eta') \cos(\varphi_l) \right\} \pm 2 D_t + \mathcal{O} \left( \frac{1}{m^4} \right) .
\]

(7.16)

Note that \( \Delta_i^\pm = \mathcal{O} (1/m) \).

Now, repeating the steps from (6.6) to (6.11), leads (7.6) to be written as

\[
(w_1 w_1')^2 \det A = \exp \left( 2 \sum_{l=1}^{n} \left[ \ln \left( 2 \cosh \delta_l^+ \right) + \ln \left( 2 \cosh \delta_l^- \right) \right] \right) \left( 1 + \mathcal{O} \left( \frac{1}{m} \right) \right),
\]

(7.17)

where \( \delta_l^\pm \equiv \delta^\pm(\varphi_l) = \lim_{m \to \infty} m \Delta_i^\pm / 2 \) which, when expressed in terms of the couplings and magnetic fields, is given by

\[
(\delta^\pm(\varphi))^2 = \frac{\beta^2}{2} \left\{ (J_1^x)^2 + (J_2^x)^2 + (J_1^y)^2 + (J_2^y)^2 + 2(J_1^x J_2^y + J_1^y J_2^x) \cos \varphi \right\}
\]

\[
\pm \frac{\beta^2}{2} \left\{ (J_1^x)^2 - (J_2^x)^2 - (J_1^y)^2 + (J_2^y)^2 + 2(J_1^x J_2^y - J_1^y J_2^x) \cos \varphi \right\}^2
\]

\[
+(h_1 + h_2)^2 \left[ (J_1^x + J_1^y)^2 + (J_2^x + J_2^y)^2 + 2(J_1^x J_2^x + J_1^y J_2^y + J_1^x J_2^y + J_1^y J_2^x) \cos \varphi \right] \}^{1/2}.
\]

The free energy function (1.3) for \( p = 2 \) periodic systems is thus given by

\[
f_2(\beta) = f^+(\beta) + f^-(\beta),
\]

(7.18)

with

\[
f^\pm(\beta) = \frac{1}{2\pi \beta} \int_0^\pi d\varphi \ln \left[ 2 \cosh (\delta^\pm(\varphi)) \right] .
\]

(7.19)

Note that the total size \( N \) of the system is now equal to \( 4n \) and this explain why (7.19) is formally 1/2 of (6.13).

It is interesting to observe that, if we set \( J_1^x = J_2^x = J^x, J_1^y = J_2^y = J^y \) and \( h_1 = h_2 = h \), we have

\[
(\delta^\pm(\varphi))^2 = \beta \left( (J^x)^2 + (J^y)^2 + 2J^x J^y \cos \varphi \pm 2h(J^x + J^y) |\cos(\varphi/2)| \right),
\]

and this gives (since \( \delta^\pm(\varphi) \) is symmetric about \( \varphi = \pi \))

\[
f_2(\beta) = \frac{1}{\pi \beta} \left( \int_0^{\pi/2} d\tau \ln \left[ 2 \cosh (\delta^+(2\tau)) \right] + \int_0^{\pi/2} d\tau \ln \left[ 2 \cosh (\delta^-(2\tau)) \right] \right),
\]

(7.20)

whose combination of the two terms yields (6.13).
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