Brane Vector Dynamics from Embedding Geometry

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\textbf{Abstract}

A Kaluza-Klein decomposition of higher dimensional gravity is performed in the flexible brane world scenario and the properties of the extra vectors resulting from this decomposition are explored. These vectors become massive due to a gravitational Higgs mechanism in which the brane oscillation Nambu-Goldstone bosons become the longitudinal component of the vector fields. The vector mass is found to be proportional to the exponential of the vacuum expectation value of the radion (dilaton) field and as such its magnitude is model dependent. Using the structure of the embedding geometry, the couplings of these vectors to the Standard Model, including those resulting from the extrinsic curvature, are deduced. As an example, we show that for 5D space-time the geometry of the bulk-brane world, either intrinsic or extrinsic, only depends on the extra vector and the 4D graviton. The connection between the embedding geometry and coset construction by non-linear realization is also presented.

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1 Introduction

If our world is a four dimensional brane floating in a higher dimensional space-time, an important physical consequence is that the brane can fluctuate into the extra dimension(s). As such, some higher dimensional symmetries, such as translation(s) along the extra dimension(s), will be spontaneously broken\cite{1} and there will appear the corresponding Nambu-Goldstone bosons. In the flexible brane limit where the scale that sets the brane tension, $F_X$, is much smaller than the $D$-dimensional Planck scale, $M_D$, the Kaluza-Klein modes of higher dimensional gravity decouple from the Standard Model particles. When the broken higher dimensional symmetries are realized locally, a gravitational Higgs mechanism ensues and these Nambu-Goldstone modes become the longitudinal components of massive vector fields\cite{1,2}. The phenomenology of these (brane) vector fields has recently been considered\cite{3-5} and contrasted with that resulting from including only the longitudinal (branon) modes\cite{1-5}.

Within a Kaluza-Klein formalism\cite{6}, these extra vectors originate from the off diagonal components of the higher dimensional metric. Using this decomposition, it will be established that the vector mass depends exponentially on the vacuum expectation value of the radion (dilaton) field which is the scalar component in this decomposition. As such the value of the vector mass is model dependent. In particular, it could be in the TeV range and thus may be accessible to the LHC. The coupling of these vectors to the Standard Model and to gravity can be obtained either via the method of nonlinear realizations of the spontaneously broken symmetries of higher dimensional space-time, or by the embedding geometry of the bulk-brane world. This paper addresses the latter approach.

Section 2 provides a decomposition of higher dimensional gravity as in Kaluza-Klein theories resulting in an expression for the brane vector’s mass. The embedding frames and the embedding conditions are introduced in section 3 along with their integrability conditions which are described by Gauss-Weingarten equations and Gauss-Coddazi-Ricci equations respectively. It is then shown how the intrinsic and extrinsic geometry depend on the graviton and the brane vectors. In section 4, the connections between the embedding geometry and non-linear realization method is established. Finally, conclusions are presented in section 5.

2 Decomposition of Metric and Brane Vectors

The Kaluza-Klein formalism\cite{6} provides the decomposition of the gravitational metric in $d > 4$ dimensions into its various spin components in $d = 4$. In general, the $d = 4$ fields will consist of the spin-2 graviton, spin-1 vector fields and scalar (radion or dilaton) degrees of freedom. Traditionally, most applications of this formalism have attempted to unify gravity with the Standard Model and as such have identified the vector fields with the gauge bosons
of the Standard Model. However, this does not have to be the case and the vectors could correspond to novel degrees of freedom. In this paper, they are identified with the vectors which emerge in flexible brane world models as a consequence of the spontaneous breaking of local space translation symmetries.

We begin by considering the zero modes of the 5D Kaluza-Klein metric tensor

\[ G_{MN}(x) = \rho^{-\frac{1}{3}} \begin{pmatrix} g_{\mu\nu} + \rho A_\mu A_\nu & \rho A_\mu \\ \rho A_\nu & \rho \end{pmatrix}. \] (1)

Compactified on a circle with radius \( r \), the 4D effective action is

\[ S_G = -\frac{1}{2\kappa_5^2} \int d^4x dy \ e(5) R^{(5)} \]

\[ = -\frac{1}{2\kappa_5^2} \int d^4x \ e(4) \left[ R^{(4)} + \frac{1}{4} \rho F_{\mu\nu} F^{\mu\nu} + \frac{1}{6\rho^2} \partial^\rho \rho \partial_\rho \rho \right] \] (2)

where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) and \( \kappa_5, \kappa \) are the 5D and 4D gravitational constants respectively which are related via \( \kappa_5^2 = \frac{2\pi r \kappa^2}{\kappa^5} \). The indices are raised by \( g^{\mu\nu} \) which is the inverse of \( g_{\mu\nu} \). The presence of a 3-brane in the 5D bulk breaks the extra 5D translation and Lorentz symmetry spontaneously. The position of the brane is provided by the embedding function \( Y^M = Y^M(x^\mu) \) with \( x^\mu \) the coordinates on the brane. The brane action is of the Nambu-Goto form built from the induced metric tensor \[ h_{\mu\nu} = G_{MN} \partial_\mu Y^M \partial_\nu Y^N \] and given by

\[ S_{brane} = F_X^4 \int d^4x \sqrt{\det h_{\mu\nu}}. \] (3)

We employ the static gauge defined by \( Y^\mu(x) = x^\mu, Y^5(x) = \phi(x) \). The Nambu-Goldstone boson \( \phi \) describes the brane fluctuation for a 5D space-time with non-dynamical gravity. When we consider a curved 5D space-time with dynamical gravity as \( [1] \) and compactify the 5D theory on a circle, an extra vector field appears in the induced metric \( h_{\mu\nu} \) after the field \( \phi \) is absorbed as the longitudinal component by \( A_\mu \) \([1]-[3]\). Defining \( X_\mu \equiv A_\mu + \partial_\mu \phi \), the induced metric can be written as

\[ h_{\mu\nu} = \rho^{-\frac{1}{3}} g_{\mu\nu} + \rho \hat{\nabla}(A_\mu + \partial_\mu \phi)(A_\nu + \partial_\nu \phi) = \rho^{-\frac{1}{3}} (g_{\mu\nu} + \rho X_\mu X_\nu). \] (4)

Note that the field strength is \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \partial_\mu X_\nu - \partial_\nu X_\mu \) and hence the kinetic term of \( A_\mu \) simply becomes the kinetic term of \( X_\mu \). The global limit is restored by taking

\[ g_{\mu\nu} = \eta_{\mu\nu}, \quad A_\mu = 0, \quad \rho = 1. \] (5)

\[ ^6 \text{The (4+N) dimensional space-time metric tensor } \eta_{AB} \text{ has signature } (-,+,+,+), \text{ Curved indices are denoted } M, N, \ldots \text{ for the (4+N) dimensional space-time and } \mu, \nu \text{ for the 4D theory, while the local Lorentz indices are } A, B, \ldots \text{ for (4+N)-dimensions and } a, b, \ldots \text{ for 4D respectively. Finally, the indices } i, j, \ldots \text{ label the co-volume.} \]
Plugging the induced metric (1) into the brane action (3) yields

\[ S_{\text{brane}} = \int d^4x \, F_\Theta^4 \rho^{-\frac{2}{3}} \sqrt{\det g_{\mu\nu}} \sqrt{\det (\delta^\tau_\alpha + \rho X^\tau X_\lambda)} = \int d^4x \, F_\Theta^4 \rho^{-\frac{2}{3}} \sqrt{\det g_{\mu\nu}(1 + \kappa^2 X^\lambda X_\lambda + O(X^4))}. \tag{6} \]

To canonically normalize the Maxwell term in eq. (2), we rescaled the vector field as \( X_\mu = \kappa \sqrt{\frac{2}{\rho}} \, \tilde{X}_\mu \), while the dilaton kinetic term in eq. (2) is put into canonical form after the redefinition, \( \rho = e^{\mp \sigma} \). So doing, the resulting vector mass is then gleaned as

\[ m_\Sigma^2 \sim e^{\pm \frac{2}{3} \langle \sigma \rangle} \kappa^2 F_\Theta^4. \tag{7} \]

If one takes \( \rho > 1 \), i.e. \( \sigma > 0 \), the mass of \( \tilde{X}_\mu \) is found to be \( m_\Sigma^2 \sim \kappa^2 F_\Theta^4 \). For \( F_\Theta \sim \text{TeV} \), this mass is very small \cite{4, 5}. However, in general, the size of the dilaton vacuum value is model dependent and consequently so is the vector mass. Thus the “scaling factor”, may exponentially increase or suppress the mass of the vector depending on the form of the metric tensor in the extra dimensional space-time.

Now consider the more general case where there are \( N > 1 \) co-dimensions. The 3-brane is embedded in a \((4+N)\)-dimensional bulk space-time with topology \( M_4 \times B \) and coordinates \( Y^M = (x^\mu, y^i) \), where the co-volume \( B \) is a compact manifold with an isometry. The \((4+N)\) dimensional metric is

\[ G_{MN} = \begin{pmatrix} g_{\mu\nu}(x) + \rho(x) \gamma_{ij}(y) A^i_\mu(x, y) A^j_\nu(x, y) & \rho(x) \gamma_{ij}(y) A^j_\mu(x, y) \\ \rho(x) \gamma_{ij}(y) A^i_\mu(x, y) & \rho(x) \gamma_{ij}(y) \end{pmatrix} \]

where \( A^i_\mu(x, y) \equiv \xi^i_\alpha(y) A^\alpha_\mu(x) \) and \( \xi^i_\alpha(y) \) are Killing vectors for describing the isometry of the co-volume. If the co-volume \( B \) is homogeneous and isotropic, then its maximal isometry group can have \( \frac{1}{2} N(N + 1) = N + \frac{1}{2} N(N - 1) \) Killing vectors. The 4-dimensional brane breaks all the isometries except the ones that belong to the stability group. More precisely, we denote \( \xi_\alpha = (\xi_i, \xi_a) \) and there are \( N \) Killing vectors \( \xi_i, i = 1, 2, ..., N \) which correspond to \( N \) broken translations due to the existence of the brane, i.e. \( \xi_i = \partial_i \) and \( \frac{1}{2} N(N - 1) \) Killing vectors \( \xi_a, a = 1, 2, ..., \frac{1}{2} N(N - 1), \) which correspond to \( \frac{1}{2} N(N - 1) \) unbroken generators. These Killing vectors \( \xi_a \) may form an \( \text{SO}(N) \) Lie algebra as the cases considered in \cite{2}, i.e. \( \xi_{jk} = \frac{1}{2} (y_k \partial_j - y_j \partial_k) \). Then one can also decompose \( A^\alpha_\mu(x) = (A^i_\mu(x), A^k_\mu(x)) \).

\[ R^{4+N} = R^4 + \frac{1}{4} \rho \gamma_{ij} \xi^i \xi^j F^\alpha_{\mu\nu} F^{\beta\mu\nu} + L_{\text{scalar}} \tag{9} \]

where the scalar term \( L_{\text{scalar}} \) can be calculated from ref. \cite{8}. The 4D effective action is

\[ S_G = -\frac{1}{2\kappa^2} \int d^4x \, \sqrt{g} \, [R^4 \rho^{\frac{N}{2}} + \frac{1}{4} \rho^{\frac{N+2}{2}} F^\alpha_{\mu\nu} F^{\beta\mu\nu} + L_{\text{scalar}}] \tag{10} \]

where we have used that

\[ \sqrt{G} = \sqrt{\det G_{MN}} = \sqrt{\det g_{\mu\nu} \sqrt{\det \gamma_{ij} \rho^{\frac{N}{2}}}}, \]

\[ \kappa_D^2 = \kappa^2 \int_B d^N y \sqrt{\gamma} \tag{11} \]
with
\[
\frac{\int_B d^N y \sqrt{\gamma} \gamma_{ij} \xi_i^j}{\int_B d^N y \sqrt{\gamma}} = \delta_{\alpha \beta}.
\] (12)

Here \( \tilde{L}_{\text{scalar}} \) is obtained from \( L_{\text{scalar}} \) by integrating over the extra dimensions. The brane action has the Nambu-Goto form
\[
S_{\text{brane}} = F_X^4 \int d^4 x \sqrt{\det h_{\mu \nu}} = F_X^4 \int d^4 x \sqrt{\det(g_{\mu \nu} + \rho \gamma_{ij} X_i^\mu X_j^\nu)}
\] (13)

where \( X_i^\mu(x) = A_i^\mu(x) + \partial_\mu \phi^i(x) \).\(^7\) Analogously to the 5D case, we rescale the metric, \( g_{\mu \nu} = \bar{g}_{\mu \nu} \bar{\rho}^{-\frac{N}{N+2}} \), the vector field, \( X_\mu = \kappa \sqrt{\frac{2}{\bar{\rho}}} \bar{X}_\mu \) and scalar field \( \rho = \bar{\rho} \frac{1}{\sqrt{\bar{\rho}}} \) so that the higher dimensional metric (8) takes the form
\[
G_{MN} = \bar{\rho}^{-\frac{N}{N+2}} \begin{pmatrix} \bar{g}_{\mu \nu} + \bar{\rho} \gamma_{ij} A_i^\mu A_j^\nu & \bar{\rho} \gamma_{ij} A_i^\mu \\ \bar{\rho} \gamma_{ij} A_j^\mu & \bar{\rho} \gamma_{kl} \end{pmatrix}.
\] (14)

With these rescalings the Einstein-Hilbert and Yang-Mills terms in the 4D effective action (10) assume their canonical form. As in 5D case, we take \( \rho = e^{\pm <\sigma>} \) and the brane action becomes
\[
S_{\text{brane}} = e^{\pm \frac{N}{N+2} <\sigma>} F_X^4 \int d^4 x \sqrt{\det(\bar{g}_{\mu \nu} + 2 \kappa^2 \gamma_{ij} \bar{X}_i^\mu \bar{X}_j^\nu)}
\] (15)

from which we extract the vector mass term \( e^{\pm \frac{N}{N+2} <\sigma>} \kappa^2 X_i^\mu X_i^\nu \). Thus for any extra dimensional space-time, there can be an exponential enhancement (or suppression) for the vectors masses. Note that this exponential factor is reminiscent of that employed by the Randall-Sundrum model [9] in relating the weak scale to the Planck scale. The fact that the vacuum expectation value (vev) of the dilaton can control various coupling constants is well known in string theory where the vacuum expectation value of the dilaton is related to the string coupling.

When the 4D effective theory is constructed using the method of non-linear realizations [2], the vector kinetic terms and mass terms arise as completely independent invariant Lagrangian monomials with the mass parameters arbitrary. Consequently, we treat the masses of these vectors as free parameters to be constrained by experiment. The couplings of these massive vectors to gravity and the Standard Model fields will be addressed in the next section by applying the embedding geometry and deriving the Einstein equation on the brane. Included in these interactions will be derivative couplings of \( X_\mu \) to the Standard Model fields which are related to the extrinsic curvature of the brane.

\(^7\)In the brane action (13), any \( y^i \)-coordinate dependence of the metric (8) and \( A_i^\mu(x, y) \) is eliminated using the embedding functions \( y^i = y^i(x) = y_0^i + \xi_j(y_0) \phi^j(x) \), where \( y_0^i \) is a particular position of the brane.
3 Couplings of Brane Vectors to Gravity and Matter

3.1 Embedding geometry and Einstein equation on brane

In this section, the general couplings of $X_\mu$ to matter and gravity are deduced using the embedding geometry \[16\] of the bulk-brane world scenario. This approach has been previously used \[10\] - \[15\] in brane scenarios and now we apply it to the case of brane vectors. Introducing the embedding frame, $\hat{e}_\mu = Y^M Y_\mu Y^N$, $n_i = n_i^M \partial_M$, with $n_i, i = 1, 2, ..., N$ the normal vectors to the brane, the embedding conditions

\[
G_{MN} \partial_\mu Y^M \partial_\nu Y^N = h_{\mu\nu},
\]

\[
G_{MN} \partial_\mu Y^M n_i^N = 0,
\]

\[
G_{MN} n_i^M n_j^N = \delta_{ij}
\]

relate the higher dimensional metric and the induced metric on the brane as well as provide the orthogonality condition of $\hat{e}_\mu$ and $n_i$ and the normalization of $n_i$. Defining $\nabla_\mu \equiv \hat{e}_\mu^M \nabla_M$, where $\Gamma^K_{MN}$ are the higher dimensional Christoffel connections, the covariant derivatives of the embedding frame basis are given by the Gauss-Weingarten equations \[10\] - \[16\]

\[
\nabla_\mu \hat{e}_\nu = \Gamma^\lambda_{\mu\nu} \hat{e}_\lambda - K^i_{\mu\nu} n_i,
\]

\[
\nabla_\mu n_i = K^i_{\mu\nu} \hat{e}_\nu + B^i_{\mu} n_j
\]

which introduce the extrinsic curvature $K^i_{\mu\nu}$, the 4-dimensional connection $\Gamma^\lambda_{\mu\nu}$, and the twist potential $B^i_{\mu}$. These can be expressed in terms of the embedding frame basis and their covariant derivatives, using the embedding conditions \[16\] as

\[
K^i_{\mu\nu} = -n_i^M \nabla_\mu (Y^M_{\nu}), \quad \Gamma^\lambda_{\mu\nu} = h^{\lambda\rho} Y^M_{\rho} G_{MN} \nabla_\mu (Y^N_{\nu}), \quad B^i_{\mu} = n_M \nabla_\mu n_i^M.
\]

Since the covariant differentiation $\nabla_M$ is symmetric \[16\], $K^i_{\mu\nu} = K^i_{\nu\mu}$. Further note that the twist potential vanishes in the case of co-dimension one, $N = 1$. Using the Gauss-Weingarten equations, it is straightforward to deduce their integrability conditions, the Gauss-Codazzi-Ricci equations \[10\] - \[16\], which relate the higher dimensional Riemannian curvature tensor to the lower dimensional induced one, plus the extrinsic curvature and the twist potential as

\[
\hat{R}_{KLMN} e^K_{\rho} e^L_{\sigma} e^M_{\mu} e^N_{\nu} = R_{\rho\sigma\mu\nu} + K^i_{\mu\nu} K_{\nu\sigma i} - K^i_{\mu\sigma} K_{\nu\nu i}.
\]

\[
\hat{R}_{KLMN} n^K_i e^L_{\sigma} e^M_{\mu} e^N_{\nu} = \nabla_\mu K^i_{\nu\sigma} - \nabla_\nu K^i_{\mu\sigma}
\]

\[
\hat{R}_{KLMN} n^K_i n^L_{kj} e^M_{\mu} e^N_{\nu} = F^i_{\mu \nu} + K^i_{\mu\sigma} K_{\nu\nu j} - K^i_{\nu\nu} K_{\mu\nu j}
\]

where

\[
R^\lambda_{\mu\nu} = \partial_\nu \Gamma^\lambda_{\mu\sigma} - \partial_\mu \Gamma^\lambda_{\nu\sigma} + \Gamma^\rho_{\mu\nu} \Gamma^\lambda_{\rho\sigma} - \Gamma^\rho_{\mu\sigma} \Gamma^\lambda_{\nu\rho},
\]

\[
F^i_{\mu\nu} = \partial_\mu B^i_{\nu} - \partial_\nu B^i_{\mu} + B^j_{\nu} B^i_{\mu j} - B^j_{\mu} B^i_{\nu j},
\]

\[
\hat{\nabla}_\mu K^i_{\nu\sigma} = \nabla_\mu K^i_{\nu\sigma} - B^i_{\mu} K^j_{\nu\sigma j}.
\]
These are the basic ingredients of the embedding geometry and now we apply them to the study of brane vectors. For simplicity, we consider a 5-dimensional space-time so the bulk-brane world has co-dimension one and the twist potential $B^{ij}_\mu$ vanishes. In this case, we can remove all the $i, j$ indices and set $B^{ij}_\mu = 0$ in the embedding condition \( (16) \), the Gauss-Weingarten equations \( (17) \) and the expression of $K_{\mu\nu}$ in \( (18) \). The last equation of \( (19) \) becomes trivial and the first two equations are also simplified yielding,

\[
\begin{align*}
\hat{R}_{KLMN} \tilde{e}^K_{\rho} \tilde{e}^L_{\sigma} \tilde{e}^M_{\mu} \tilde{e}^N_{\nu} &= R_{\rho\sigma\mu\nu} + K_{\mu\rho}K_{\nu\sigma} - K_{\mu\nu}K_{\rho\sigma} , \\
\hat{R}_{KLMN} n^K \tilde{e}^L_{\sigma} \tilde{e}^M_{\mu} \tilde{e}^N_{\nu} &= \nabla_\mu K_{\nu\sigma} - \nabla_\nu K_{\mu\sigma} . \\
\end{align*}
\]

Using the 5D Einstein equation $\hat{R}_{MN} - \frac{1}{2} G_{MN} \hat{R} = -\kappa_5^2 \hat{T}_{MN}$ where $\hat{T}_{MN}$ is the 5D stress energy tensor of matter sources and following \( (10) \), the Einstein equations on the brane take the form

\[
\begin{align*}
R_{\mu\nu} - \frac{1}{2} R h_{\mu\nu} + E_{\mu\nu} + Q_{\mu\nu} &= -\frac{2}{3} \kappa_5^2 [\hat{T}_{MN} \tilde{e}^M_{\mu} \tilde{e}^N_{\nu} + (\hat{T}_{MN} n^M n^N - \frac{1}{4} \hat{T}) h_{\mu\nu}] , \\
\nabla_\tau K^\tau_{\mu} - \nabla_\mu K &= \kappa_5^2 n^M \tilde{e}^N_{\mu} \hat{T}_{MN} \\
\end{align*}
\]

where

\[
\begin{align*}
E_{\mu\nu} &= \hat{C}_{KLMN} n^K n^M \tilde{e}^L_{\sigma} \tilde{e}^N_{\nu} , \quad \hat{T} = G^{MN} \hat{T}_{MN} , \\
Q_{\mu\nu} &= (K_{\mu\nu} K - K_{\mu\tau} K^\tau_{\nu}) - \frac{1}{2} h_{\mu\nu} (K^2 - K_{\sigma\tau} K^{\sigma\tau}) , \quad K = h^\mu_{\rho} K_{\nu\mu} = \text{Tr} K
\end{align*}
\]

with $\hat{C}_{KLMN}$ the 5D Weyl tensor. We first address the case where the 5D space-time is flat, \(^8\) and work in static gauge defined as $Y^\mu = x^\mu$, $Y^5 = \phi(x)$ so that

\[
\begin{align*}
\tilde{e}^\nu_{\mu} &= \delta^\nu_{\mu} , \quad \tilde{e}^5_{\mu} = \partial_\mu \phi , \quad E_{\mu\nu} = 0 , \\
n_{\mu\nu} &= -\partial_\mu \phi \sqrt{1 + \partial_\tau \phi \partial^\tau \phi} , \quad n_5 = 1/\sqrt{1 + \partial_\tau \phi \partial^\tau \phi} , \\
h_{\mu\nu} &= \eta_{\mu\nu} + \partial_\mu \phi \partial_\nu \phi , \quad h^\mu_{\nu} = \eta^{\mu\nu} - \partial_\mu \phi \partial_\nu \phi/(1 + \partial_\tau \phi \partial^\tau \phi) , \\
K_{\mu\nu} &= -\partial_\mu \partial_\nu \phi \sqrt{1 + \partial_\tau \phi \partial^\tau \phi} , \\
R_{\rho\sigma\mu\nu} &= (\partial_\mu \partial_\sigma \phi \partial_\nu \partial_\rho \phi - \partial_\mu \partial_\rho \phi \partial_\nu \partial_\sigma \phi)/\left(1 + \partial_\tau \phi \partial^\tau \phi\right) .
\end{align*}
\]

It follows that the only physical degree of freedom is the Nambu-Goldstone boson, $\phi$, which describes the fluctuation of the brane. Equations \( (22) \) are consistency equations for $\phi$ and its derivatives. Since the extra dimensional translation is spontaneously broken, the dynamics of the Nambu-Goldstone field $\phi$ can be secured using the conservation of the broken symmetry current $\partial_\mu \tilde{T}^{\mu5} = F^\mu_5 \partial_\mu \phi + ... = 0$. Alternatively, the field equation follows from a minimization of the trace of extrinsic curvature as shown in \( (13) \). This is equivalent to the p-brane equations of

\(^8\)Strictly speaking the 5D space-time cannot be flat due to the presence of the brane as the matter source. However, we assume that it does not bend the 5D space-time much so the metric can be considered as an almost flat one.
motion which one obtains from the Nambu-Goto action. It corresponds geometrically to the minimal volume obtained from the embedding of the corresponding world volume into higher dimensional space-time. In this case, if the brane is the only matter source in 5D spacetime, then the vanishing condition of the trace of the extrinsic curvature \( K = h^{\mu\nu} K_{\mu\nu} = 0 \) leads to

\[
\partial^2 \phi = \frac{\partial^\mu \phi \partial^\nu \phi \partial_\mu \partial_\nu \phi}{1 + \partial_\nu \phi \partial^\nu \phi} + \partial_\tau \phi \partial^\tau \phi
\]

(25)

which is recognized as the same equation of motion of \( \phi \) as that obtained from the Nambu-Goto action [7, 13]. Also the extrinsic curvature can be related to the rigidity of strings or branes [17]. In general it is difficult to solve these equations and, moreover, the form of the stress energy tensor \( \hat{T}^{MN} \) must be specified. However, the equations for \( \phi \) can be converted to an action which includes the leading couplings like \( \partial_\mu \phi \partial_\nu \phi T_{\mu\nu}^{SM} \) plus other higher order derivative terms.

### 3.2 Brane vector and its couplings

Next consider a curved 5D space-time with the general Kaluza-Klein metric of eq. (1). The Gauss-Coddazi equations and the induced Einstein equation now become more complicated producing a set of differential equations for the spin-2 (4D graviton \( g_{\mu\nu} \)), spin-1 (4D vector \( A_\mu \)) and spin-0 (4D dilaton \( \rho \)). As discussed in refs. [1, 4, 5], the Higgs mechanism is operational. Naively, one simply replaces \( \partial_\mu \phi \rightarrow \partial_\mu \phi + A_\mu \rightarrow X_\mu \) in eq. (22). Some care is required, however, since \( \partial_\mu \partial_\nu \phi \) has the ambiguity of being replaced by either \( \partial_\mu X_\nu \) or \( \partial_\nu X_\mu \). Moreover, there are also terms dependent on the field strength \( F_{\mu\nu} \). To resolve any ambiguity, one must figure out the relation between \( n_\mu \) and \( X_\mu \). To do so, the 5D Kaluza-Klein metric (1) is used to calculate the Christoffel connections and extrinsic curvature which are shown to depend only on \( X_\mu \) and \( g_{\mu\nu} \).

Consider the transformation laws of the various fields. A bulk vector field \( V^M \) transforms under a general coordinate transformation as

\[
\delta_\epsilon V^M = \epsilon^K \partial_K V^M - V^K \partial_K \epsilon^M
\]

(26)

where \( \epsilon^M = (\epsilon^\mu(x, y), \epsilon^5(x, y)) \). As in the usual 5D Kaluza-Klein theories, we take \( \epsilon^\mu = 0 \), and \( \epsilon^5 = \epsilon(x) \). This corresponds to a particular 5D general coordinate transformation (or a gauge transformation for \( A_\mu \))

\[
Y'^\mu = Y^\mu(x), \quad Y'^5 = Y^5 - \epsilon(x)
\]

(27)

so that \( \phi(x) = Y^5(x) \). In addition, the zero mode fields \( g_{\mu\nu}(x), A_\mu(x), \rho(x), V^\mu(x), V^5(x) \) transform as

\[
\begin{align*}
\delta_\epsilon \phi &= -\epsilon(x), \quad \delta_\epsilon g_{\mu\nu} = 0, \\
\delta_\epsilon A_\mu &= \partial_\mu \epsilon(x), \quad \delta_\epsilon \rho = 0, \\
\delta_\epsilon V^5 &= -V^\mu \partial_\mu \epsilon(x), \quad \delta_\epsilon V^\mu = 0.
\end{align*}
\]

(28)
It is easy to see that $X_\mu \equiv A_\mu + \partial_\mu \phi$, $\hat{V}^5 \equiv A_\mu V^\mu + V^5$, $\hat{V}_\mu \equiv V^\mu$, $\rho$ and $g_{\mu\nu}$ are all invariant under the transformation (27). From eq. (4), it follows that the induced metric $h_{\mu\nu} = \rho^{-\frac{1}{2}}(g_{\mu\nu} + \rho X_\mu X_\nu)$ and all intrinsic geometric quantities on the brane, such as the Christoffel connection, Riemannian tensor, Ricci tensor and Ricci scalar are invariant as well. The normal vector and the extrinsic curvature can then be computed.

In Section 2, we discussed the role that $\rho$ plays in modifying the mass of brane vectors. Here, to simplify the calculation, we set $\rho = \kappa = 1$ and decompose the 5D metric as

$$G_{MN} = \begin{pmatrix} g_{\mu\nu} + A_\mu A_\nu & A_\mu \\ A_\nu & 1 \end{pmatrix} = S^T \hat{G} S$$

with $S = \begin{pmatrix} \delta^\tau_\nu & 0 \\ A_\nu & 1 \end{pmatrix}$ and $\hat{G} = \begin{pmatrix} g_{\rho\tau} & 0 \\ 0 & 1 \end{pmatrix}$. The matrix $S$ when acting on a bulk vector shifts only the fifth component so that, for example,

$$\hat{n}^M \equiv S^M_K n^K = (n^\mu, n^5 + A_\mu n^\nu).$$

This provides an invariant form under the transformation (27) provided one takes $V^M = n^M$. Acting on $\tilde{e}_M^{\mu} = Y_M^{\mu}$ in the static gauge gives

$$\hat{Y}_M^{\mu} \equiv S^M_K Y^K_{\mu} = (\delta^\nu_\mu, A_\mu + \partial_\mu \phi) = (\delta^\nu_\mu, X_\mu).$$

The embedding conditions can be written in this “shifted” frame as

$$h_{\mu\nu} = g_{\mu\nu} + X_\mu X_\nu, \quad \hat{n}^\mu = -X^\mu \hat{n}^5, \quad g_{\mu\nu} \hat{n}^\mu \hat{n}^\nu + (\hat{n}^5)^2 = 1$$

which can be readily solved yielding

$$\hat{n}^\mu = \frac{-X^\mu}{\sqrt{1 + X^\mu X_\mu}}, \quad \hat{n}^5 = \frac{1}{\sqrt{1 + X^\mu X_\mu}}$$

where $X^\mu = g^{\mu\nu} X_\nu$. To compute $\Gamma^\lambda_{\mu\nu}$ and $K_{\mu\nu}$, only the first equation (Gauss equation) of (17) needs to be solved. Multiplying by the matrix $S^L_M$ on both sides gives

$$S^L_M \nabla_\mu Y^M_{\nu,\lambda} = \Gamma^\lambda_{\mu\nu} S^L_M Y^M_{\lambda,\lambda} - K_{\mu\nu} S^L_M n^M = \Gamma^\lambda_{\mu\nu} \hat{Y}^\nu_{\lambda,\lambda} - K_{\mu\nu} \hat{n}^L.$$  

To compute the left hand side, we use the 5D metric $G_{MN}$ to compute the connections

$$\tilde{\Gamma}^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \frac{1}{2} g^\lambda_\rho (A_\mu F_{\rho\nu} + A_\nu F_{\rho\mu}),$$

$$\tilde{\Gamma}^5_{\mu\nu} = \frac{1}{2} A^\rho (A_\mu F_{\rho\nu} + A_\nu F_{\rho\mu}) + \frac{1}{2} (\tilde{\nabla}_\nu A_\mu + \tilde{\nabla}_\mu A_\nu),$$

$$\Gamma^\lambda_{5\mu} = -\frac{1}{2} g^\lambda_\rho F_{\rho\mu}, \quad \Gamma^5_{5\mu} = \frac{1}{2} A^\rho F_{\rho\mu}$$
where \( F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} = X_{\nu,\mu} - X_{\mu,\nu} \), and \( \tilde{\Gamma}^{\lambda}_{\mu\nu} \) is built from \( g_{\mu\nu} \). It follows that

\[
S^L_M \nabla_\mu Y^M_{\mu,\nu} = \begin{cases} 
\tilde{\Gamma}^{\lambda}_{\mu\nu} - \frac{1}{2}g^{\nu\rho}(F_{\rho\mu}X_\nu + F_{\rho\nu}X_\mu) \quad ; \quad L=\lambda \\
\frac{1}{2}(X_{\mu,\nu} + X_{\nu,\mu}) \quad ; \quad L=5
\end{cases}
\]  

(36)

Now \( \Gamma_{\tau\mu\nu} \) and \( K_{\mu\nu} \) are computed as

\[
\Gamma_{\tau\mu\nu} = \tilde{\Gamma}_{\tau\mu\nu} - \frac{1}{2}(F_{\tau\mu}X_\nu + F_{\tau\nu}X_\mu) + \frac{1}{2}X_\tau(X_{\mu,\nu} + X_{\nu,\mu}),
\]

\[
K_{\mu\nu} = -\frac{1}{2\sqrt{1 + X_\mu X_\mu}}[X^\rho(F_{\rho\mu}X_\nu + F_{\rho\nu}X_\mu) + (\tilde{\nabla}_\mu X_\nu + \tilde{\nabla}_\nu X_\mu)]
\]

(37)

where \( \tilde{\nabla}_\mu X_\nu \equiv X_{\mu,\nu} - \tilde{\Gamma}^{\lambda}_{\mu\nu}X_\lambda \). The Christoffel connection \( \Gamma_{\tau\mu\nu} \) coincides with the result computed directly from the induced metric. When taking the flat 5D space-time limit, the extrinsic curvature \( K_{\mu\nu} \) reduces to the previously obtained result [24]. This expression is a generalization of the so-called ADM formulation of gravity [18]. Since we did not include the higher Kaluza-Klein modes, a term like \( \partial_\theta g_{\mu\nu} \) vanishes in \( K_{\mu\nu} \). Note that besides \( g_{\mu\nu} \), the only other field dependence in the induced metric, intrinsic curvature, extrinsic curvature etc. occurs through the combination \( X_\mu = A_\mu + \partial_\theta \phi \). Moreover, as noted previously, the Maxwell term \( F_{\mu\nu}F^{\mu\nu} \) for \( A_\mu \), which was obtained from the decomposition of 5D Einstein-Hilbert term \( \hat{R}^{(5)} \), does not change when replacing \( A_\mu \) by \( X_\mu \) so

\[
\hat{R}^{(5)} = R^{(4)}(g) + \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \quad F_{\mu\nu} = X_{\nu,\mu} - X_{\mu,\nu}.
\]

(38)

On the other hand, one can derive a different decomposition of \( \hat{R}^{(5)} \) using eq. (21) as

\[
\hat{R}^{(5)} = R^{(4)}(h) + \text{Tr}K^2 - (\text{Tr}K)^2 + 2\nabla_M(n^M\nabla_Nn^N - n^N\nabla_Nn^M)
\]

(39)

where \( \text{Tr}K^2 = K_{\mu\nu}K^{\mu\nu} \), \( \text{Tr}K = K_{\mu\nu}h^{\mu\nu} \) and \( h_{\mu\nu} = g_{\mu\nu} + X_\mu X_\nu \) (Note that \( R^{(4)}(h) \) is calculated from the induced metric \( h_{\mu\nu} \)). Integrating and noting that we only include the zero modes, one obtains

\[
\int d^4x\sqrt{g} \left[ R^{(4)}(h) - R^{(4)}(g) + \text{Tr}K^2 - (\text{Tr}K)^2 \right] = \int d^4x\sqrt{g} \frac{1}{4}F^2
\]

(40)

which is a simple relation among the scalar curvature \( R^{(4)} \), the extrinsic curvature terms \( \text{Tr}K^2, (\text{Tr}K)^2 \) and the Maxwell term \( F^2 \). It is easy to check this identity at the order of \( \mathcal{O}(X^2) \) and this provides an alternative way to build the Maxwell term, which will be discussed from the point of view of a 4D non-linear realization in the next section.

The expression for the normal vector can be used to extract the couplings of \( X_\mu \) to gravity and the Standard Model. Notice that the right hand side of the first equation in (22) contains the term \( \hat{T}_{MN}n^Mn^Nh_{\mu\nu} \). Since both \( \hat{T}_{MN} \) and \( n^M \) do not contain \( h_{\mu\nu} \) explicitly, this term in the Einstein equation must correspond to an action term

\[
\sqrt{\text{det}h_{\mu\nu}} \hat{T}_{MN}n^Mn^N = \sqrt{h} \left( T^{SM}_{\mu\nu}n^\mu n^\nu + \ldots \right)
\]

(41)
where we assume that the stress-energy tensor of the Standard model $T^{SM}_{\mu\nu}$ is included in $\hat{T}_{MN}$ as in ref. [10] and the ellipsis represents other components of the $\hat{T}_{MN}$ term. Using the form for $\hat{n}^\mu$ (c.f. eq. (33))

$$n^\mu = \hat{n}^\mu = \frac{-X^\mu}{\sqrt{1 + X^\mu X_\mu}} = -X^\mu + O(X^2) .$$

(42)

Plugging into (41) and expanding in the power series of $X_\mu$, one readily extracts the lowest order couplings of $X_\mu$ to the Standard Model as i.e.

$$\sqrt{\hbar} T^{SM}_{\mu\nu} n^\mu n^\nu \sim \sqrt{\eta} X^\mu X^\nu T^{SM}_{\mu\nu} + O(X^4) .$$

(43)

This is the non-derivative coupling of brane vector $X_\mu$ to the Standard Model fields.

Next consider the derivative couplings to the Standard Model. The $X_\mu$ field strength

$$F_{\mu\nu} = X_{\nu,\mu} - X_{\mu,\nu} = \tilde{\nabla}_\mu X_\nu - \tilde{\nabla}_\nu X_\mu$$

and (c.f. eq. (37))

$$K_{\mu\nu} = -\frac{1}{2}(\tilde{\nabla}_\mu X_\nu + \tilde{\nabla}_\nu X_\mu) + O(X^2)$$

(45)

contain the anti-symmetric and symmetric pieces of $\partial_\mu X_\nu$ respectively. Since both $F_{\mu\nu}$ and $K_{\mu\nu}$ are invariant under the transformation (27), so is their product

$$F_{\mu\rho}K_{\rho\nu} = F_{\mu\rho}g^{\rho\tau}K_{\tau\nu} = \tilde{\nabla}_{[\rho}X_{\sigma]}\tilde{\nabla}_{\tau\rho}X_{[\nu]} + \frac{1}{2}(\tilde{\nabla}_\rho X_\mu \tilde{\nabla}_\rho X_\nu - \tilde{\nabla}_\mu X_\rho \tilde{\nabla}_\nu X_\rho) + O(X^3) .$$

(46)

Notice that the first term of last line of (46) is anti-symmetric in $\mu, \nu$ while the second term is symmetric. Since $X_\mu$ is a singlet under the Standard Model gauge group, the above combination couples invariantly to the $SU(3) \times SU(2) \times U(1)$ singlet antisymmetric hypercharge field strength $B_{\mu\nu}$ and its dual $\tilde{B}_{\mu\nu}$ as

$$(\kappa_1 B^{\mu\nu} + \kappa_2 \tilde{B}^{\mu\nu})F_{\mu\rho}K^\rho_{\nu} = (\kappa_1 B^{\mu\nu} + \kappa_2 \tilde{B}^{\mu\nu})\tilde{\nabla}_{[\rho}X_{\sigma]}\tilde{\nabla}_{\tau\rho}X_{[\nu]} + O(X^3) .$$

(47)

Here $\kappa_1, \kappa_2$ are dimensionless parameters. This interaction has the same dimension as the $X^\mu X^\nu T_{\mu\nu}$ terms. In addition, $X_\mu$ has invariant couplings to the Standard Model scalar doublet bilinear, $\varphi^\dagger \varphi$, which can contribute to the decay rate of the Standard Model Higgs boson [3]. Combining terms (and taking the $g_{\mu\nu} = \eta_{\mu\nu}$ limit ) yields the effective action

$$S_{4D \text{ eff}} = \int d^4x [ L_{SM} - \frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{1}{2} m_X^2 X^\mu X_\mu ]$$

$$+ \frac{m_X^2}{2F_X^4} \left( T^{SM}_{\mu\nu} X_\mu X_\nu + (\kappa_1 B^{\mu\nu} + \kappa_2 \tilde{B}^{\mu\nu})F_{\mu\rho}K^\rho_{\nu} ight)$$

$$+ (\lambda_1 K_{\mu\nu}K^{\mu\nu} + \lambda_2 F_{\mu\nu}F^{\mu\nu} + \lambda_3 F_{\mu\nu}F^{\mu\nu})(\varphi^\dagger \varphi - \frac{v^2}{2}) ] .$$

(48)
The phenomenology based on the effective action (48) has been studied in ref. [3]. For \( N \geq 2 \) isotropic codimensions, there is an \( SO(N) \) symmetry associated with the isometry of the co-dimensional space. Under this symmetry, the \( X_i^\mu \) vectors transform non-trivially while all Standard Model fields are \( SO(N) \) singlets. As such, invariant couplings of the \( X_i^\mu \) vectors must occur in pairs and the vectors are stable. For the \( N = 1 \) case, the stability is insured provided there is a discrete reflection symmetry in the extra dimension under which the vector is odd, \( X_\mu \rightarrow -X_\mu \).

4 Connection to Coset Method

In the last two sections, we deduced the couplings of the brane vectors from the bulk-brane world point of view using the embedding geometry. So doing, we constructed a four dimensional effective action detailing their interactions with gravity and matter fields. The non-linear realization, or coset, method, provides another approach to four dimensional effective theories. Previously, the relation between the embedding geometry and the coset method has been considered [1, 4, 12, 13, 15] for the case of embedding a hypersurface into a flat higher dimensional space-time. Here we consider the properties of a \((p+1)\)-dimensional hypersurface embedded into a curved \(D\)-dimensional space-time with dynamical gravity. In fact, since the brane vectors arise from the off-diagonal components of the higher dimensional metric, their presence requires that the higher dimensional space-time be curved. Previously [2] we showed how to generalize the coset formulation in order to include the gravitational fields. In this section, we examine the connection between the two formalisms.

4.1 Embedding Geometry in Moving Frames

Thus far, we have employed the coordinate bases \((\tilde{e}_\mu = Y^M_\mu \partial_M, \ n_i = n_i^M \partial_M)\) to describe the embedding geometry. However, in order to make connection with the coset approach and construct the Maurer-Cartan forms, we need Lie algebra valued matrices. To achieve this correspondence, the embedding conditions can be recast [1, 4, 12, 13, 15] as

\[
E^A_M \partial_\mu Y^M (U^{-1})^a_A = e^a_\mu, \\
E^A_M \partial_\mu Y^M (U^{-1})^i_A = e^i_\mu = 0 \tag{49}
\]

where \(E^A_M\) is the higher dimensional vielbein and \(U^B_A = (U^B_a, U^B_i)\) are \(SO(1, D-1)\) matrices whose inverse is \((U^{-1})^A_B \equiv \eta^{AC} \eta_{BD} U^D_C \equiv ((U^{-1})^a_B, (U^{-1})^i_B)\). Conditions (49) show that \(E^A_M \partial_\mu Y^M\) is not the induced vielbein on the brane and one has to perform a Lorentz rotation \(U^B_A\) to ensure that the induced vielbein and the induced metric satisfy \(h_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab} \) [1, 4]. First we briefly review the properties of the \(U^B_A\) matrices, which have been discussed in detail in [1, 4, 12, 13]. By definition, they satisfy

\[
U^A_C U^B_D \eta^{CD} = \eta^{AB}, \quad U^A_C U^B_D \eta_{AB} = \eta_{CD} \tag{50}
\]
which are invariant under the independent left and right $SO(1, D - 1)_L \times SO(1, D - 1)_R$ transformations

$$U^A_B \longrightarrow \tilde{U}^A_B = (A_L)^A_CU^C_D(A_R^{-1})^D_B.$$  \hfill (51)

Equations (49) are invariant under the transformations $SO(1,p) \times SO(N)$ and thus the group $SO(1, D - 1)_L$ is broken down to $SO(1,p) \times SO(N)$, while the $SO(1, D - 1)_R$ group is unbroken. Therefore these $U^A_B$ matrices parametrize a coset manifold $SO(1,D-1)/(SO(p) \times SO(N))$ and the embedding is Lorentz covariant due to the unbroken $SO(1, D - 1)_R$ symmetry.

Next we build connections between these two frames (actually the connections among the coordinate basis, non-coordinate basis induced on the brane and the bulk geometry). The first equation of (49) shows how $e^a_\mu$ is related to $E^A_M \partial_\mu Y^M$. If we define the normal vectors as

$$n^i_M \equiv E^A_M(U^{-1})^A_i,$$  \hfill (52)

then the second equation of (49) is just the orthogonality condition $n^i_M \partial_\mu Y^M = 0$. With definition (52), it is easy to see that $G_{MNP}n^i_Mn^j_P = \delta_{ij}$ which is the normalization condition of vectors $n_i$. This shows that our definition for $n_i$ satisfies both orthogonality and normalization conditions for normal vectors. Note, however, that $n_i$ is fixed only up to an $SO(N)$ rotation. The relations can be summarized as

$$e_a = U^A_aE_A = U^A_aE^M_A \partial_M = e^a_\mu \partial_\mu Y^M \partial_M,$$

$$n_i = U^i_AE_A = U^i_AE^M_A \partial_M = n^i_M \partial_M.$$  \hfill (53)

Also recall the coordinate basis used in the previous section

$$\tilde{e}_\mu = \partial_\mu Y^M \partial_M,$$

$$n_i = n^i_M \partial_M.$$  \hfill (54)

Now we write the Gauss-Weingarten equations in the non-coordinate basis, with new coefficients $\omega^e_a$, $K^{ij}_{ab}$ and $B^j_{bij}$ to be determined, as

$$\nabla_{e_a}e_b = \omega^e_ae_c - K^i_{ab}n_i,$$

$$\nabla_{e_a}n^i = K^i_{ac}e_c + B^j_{bij}n_j.$$  \hfill (55)

After some straightforward, albeit lengthy, calculation, we find the Gauss-Coddazi-Ricci equations in the non-coordinate basis are given by

$$\hat{R}_{ABCD}U^A_dU^B_dU^C_dU^D_d = R_{abcd} + K^i_{ac}K_{bdi} - K^i_{bc}K_{adi},$$

$$\hat{R}_{ABCD}U^A_bU^B_dU^C_dU^D_d = \hat{\nabla}_{c}K_{bdi} - \nabla_{d}K_{bci},$$

$$\hat{R}^C_{AB}U^A_dU^B_dU^C_dU^D_d = F^j_{ab} + K^i_{ac}K^j_{b} - K^i_{bc}K^j_{a}.$$  \hfill (56)

where $U^j_C = (U^{-1})^j_C$. In obtaining this result, we have employed the torsion free condition $\omega^e_a - \omega^e_b = C^e_{ab}$. Note that the curvatures built from the spin connection and twist potential
contain the anholonomy coefficient $C_{\alpha \beta}$ terms as
\[
R_{\alpha \beta}^c = e_b e^c \omega_{\alpha d} - e_a \omega_{\alpha d} + \omega_{d \alpha} \omega_{\beta d} - \omega_{d \alpha} \omega_{\beta d} + C_{\alpha \beta}^c \omega_{\beta d} ,
\]
\[
F_{\alpha \beta}^i = e_a B_{\alpha i}^d - e_b B_{\beta i}^d + B_{\alpha i}^k B_{\beta k}^d - B_{\alpha i}^k B_{\beta k}^d - C_{\alpha \beta}^i B_{\alpha \beta}^d ,
\]
\[
\tilde{\nabla}_a K_{bc}^i = e_a K_{bc}^i - \omega_{ac b} K_{bd}^i - \omega_{ab} K_{dc}^i - B_{ab} K_{dc}^i .
\] (57)

Using equations (53), the components of eq. (55) can be written in terms of the $U$ matrices as
\[
\tilde{\nabla}_a U^A_b = \omega_{ab} U^A_c - K_{ab}^i U^A_i ,
\]
\[
\tilde{\nabla}_a U^A_i = K_{ab} U^A_b + B_{ai} U^A_j .
\] (58)

where $\tilde{\nabla}_a \equiv U^A_a (E_A + \Omega_A)$ and $\tilde{\nabla}_a U^A_m = U^D (E_D^M \partial_M U^A_m + \Omega_A^M U^D_m)$, for $m = (b, i)$. Here $\Omega_{BC}^A$ is the higher dimensional spin connection. From equation (50), we obtain the expressions for the extrinsic curvature, spin connection and twist potential respectively as
\[
K_{ab}^i = -(U^{-1})^i_a \tilde{\nabla}_a U^A_b ,
\]
\[
\omega_{ab} = +(U^{-1})^c_a \tilde{\nabla}_a U^A_b ,
\]
\[
B_{ai} = +(U^{-1})^j_A \tilde{\nabla}_a U^A_i .
\] (59)

One may recognize the $U^{-1} \tilde{\nabla} U$ pattern in the above equations and consider them as the components of a covariant Cartan form (This will be shown manifestly in the next section through the coset approach)
\[
(U^{-1} \tilde{\nabla} U)^A_B = \begin{pmatrix}
\omega_{ab}^a & K_{ab}^j \\
K_{ab}^i & B_{ai}^j
\end{pmatrix}
\] (60)

with the one-forms $K_{ab}^i \equiv e^a K_{ab}^i$, $B_{ai}^j \equiv e^a B_{ai}^j$, $\omega_{ab}^c \equiv e^a \omega_{ab}^c$. It follows that the $K_{ab}^i$ and $K_{i \mu}^i$, $B_{ai}^j$ and $B_{ij}^i$, $\omega_{ab}^c$ and $\Gamma_{\mu \nu}^\lambda$ are simply related as
\[
K_{ab}^i = e_d^\mu e_b^\nu K_{\mu \nu}^i ,
\]
\[
B_{ai}^j = e_d^\mu B_{\mu i}^j ,
\]
\[
\omega_{ab}^c = e_d^\mu e_b^\nu (\partial_\nu e_d^\mu + \Gamma_{\nu \lambda}^\mu e_b^\lambda) = e_d^\mu e_b^\nu \nabla_\nu e_d^\mu .
\] (61)

Note the last equation in (61) is just the usual relation between spin connection and Christoffel symbol. As mentioned earlier, the coset manifold of $SO(1,D) / SO(1,1) \times SO(D)$ is parametrized by the matrices $U^A_B = (e^{i a M^a})^A_B$ where $M^a$ are the broken Lorentz generators. Following [4] one obtains
\[
(U^{-1})^A_B = \begin{pmatrix}
\cos \sqrt{uv} & \sin \sqrt{uv} \\
-i \sin \sqrt{uv} & \cos \sqrt{uv}
\end{pmatrix}
\] (62)
where \( v = v^a_i, \dot{v} = v^i_j = \delta^j_i \eta_{ab} v^a_i \). The embedding condition \( E^A_M \partial_\mu Y^M (U^{-1})^i_A = 0 \) imposes \( 4N \) constraints which are the same as the number of the Nambu-Goldstone fields \( v^a_i \). Therefore these constraints completely fix \( v^a_i \) in terms of \( E^A_M \partial_\mu Y^M \), though in general it is difficult to solve these constraints. Here we only need the explicit expression for the induced vielbein \( e^a_\mu \) (for calculation details see [4])

\[
e^a_\mu = e^a_\mu (1 + M)^{\frac{1}{2}} \lambda_\mu, \quad M = (e^T \eta e)_{\parallel}^{-1} e_{\perp}^T \delta e_{\perp}
\]

where \( e^a_\mu = E^a_M \partial_\mu Y^M, \ e^i_\perp \mu = E^i_M \partial_\mu Y^M \) and \( \eta = \eta_{ab}, \delta = \delta_{ij} \). Taking the Kaluza-Klein vielbein as (indices with bars are the co-volume world ones)

\[
E^A_M = \begin{pmatrix}
\mathcal{E}^a_{\mu} & 0 \\
\mathcal{E}^i_{\bar{k} A} \xi^k_{\mu} & \mathcal{E}^i_{\bar{j}}
\end{pmatrix}
\]

yields the induced vielbein on the brane

\[
e^a_\mu = \mathcal{E}^a_\lambda (\delta^\lambda_\mu + X^\lambda_i X_{\mu i})^{\frac{1}{2}}
\]

which depends only on \( \mathcal{E}^a_\lambda \) and \( X^i_\mu \) (note that \( \mathcal{E}^a_\mu \mathcal{E}^b_\nu \eta_{ab} = g_{\mu \nu} \)). Therefore one can start with the embedding frame (coordinate basis) and compute \( K^i_\mu, B^i_\mu \) in that frame and then use the induced vielbein to convert them to the ones in the non-coordinate frame, and finally obtain the covariant Cartan forms which may be used to build an invariant effective action in 4D space-time.

### 4.2 Connecting with the Coset Approach

Previously, we presented [2] a detailed construction of the \( X \) vector coupling to gravity and the Standard Model using coset methods. In that case, a \( p \)-brane is embedded in \( D \) dimensional space-time resulting in the spontaneous breakdown of the symmetry group from \( \text{ISO}(1, D-1) \) to \( \text{ISO}(1, p) \times \text{SO}(N), N = D - p - 1 \). The \( \text{ISO}(1, D-1) \) generators \( (M_{AB}, \ P_C) \) are decomposed into those of the stability group \( \text{SO}(1, p) \times \text{SO}(N) \) generators \( (M_{ab}, M_{ij}) \), the broken Lorentz generators \( M_{ai} \), the broken translation generator \( P_i \) and the unbroken translation generators \( P_a \). The coset element is taken to be

\[
\Omega(x) = e^{ix^a P_a} e^{i\phi(x) P_i} e^{ix^a(x) M_{ai}}.
\]

A connection term which includes gravitational fields is then added to the Maurer-Cartan form so that

\[
\omega = \Omega^{-1} \nabla \Omega \equiv \Omega^{-1} (d + i\dot{E}) \Omega
\]

transforms analogously to the way it did in the global case

\[
\omega'(x') = h(x) \omega(x) h^{-1}(x) + h(x) dh^{-1}(x),
\]

\[
\omega'(x') = h(x) \omega(x) h^{-1}(x) + h(x) dh^{-1}(x),
\]

\[
\omega'(x') = h(x) \omega(x) h^{-1}(x) + h(x) dh^{-1}(x),
\]

\[
\omega'(x') = h(x) \omega(x) h^{-1}(x) + h(x) dh^{-1}(x),
\]
with the stability group element \( h(x) \in SO(1,p) \times SO(N) \).

To ascertain the meaning of the embedding condition in the coset method, consider the general one form
\[
\omega = \mathcal{G}^{-1}(d + i \tilde{E})\mathcal{G}
\]
with \( \mathcal{G} = \mathcal{P} \mathcal{U}, \mathcal{P} \equiv e^{Y^A P_A}, \mathcal{U} \equiv e^{\frac{1}{2} \Omega^B_{\cdot \cdot \cdot} M_{BC}} \) and \( \tilde{E} \equiv dY^M (\tilde{E}^A_{\cdot \cdot \cdot} P_A - \frac{1}{2} \Omega^B_{\cdot \cdot \cdot} M_{BC}) \). Thus
\[
\omega = \mathcal{U}^{-1} \mathcal{P}^{-1}(d + i \tilde{E})\mathcal{P} \mathcal{U}
\]
\[
= dY^M [i E^A_M \mathcal{U}^{-1} P_A \mathcal{U} + \mathcal{U}^{-1} \partial_M \mathcal{U} - \frac{i}{2} \mathcal{U}^{-1} \Omega^B_{\cdot \cdot \cdot} M_{BC} \mathcal{U}] \tag{70}
\]
where \( E^A_M \equiv \tilde{E}^A_{\cdot \cdot \cdot} + \delta^A_M - \Omega^A_{\cdot \cdot \cdot} Y_B \) is the shifted vielbein [2]. The above one form has been decomposed according to the generators of \( SO(1,D-1) \), i.e. the first term of the last line in (70) takes values on \( P_A \), the second and third terms take values on \( M_{BC} \). Now consider the embedding of the brane whose position is described by the embedding function \( Y^M = Y^M(x^\mu) \) as before. Notice that the first term can be written as
\[
i dY^M E^A_M \mathcal{U}^{-1} P_A \mathcal{U} = i dx^\mu \partial_\mu Y^M E^A_M \mathcal{U}^{-1} P_A \mathcal{U} = i dx^\mu \partial_\mu Y^M E^A_M L_A^B \mathcal{P} \mathcal{U} \tag{71}
\]
where \( L_A^B \) forms a vector representation of the \( SO(1,D-1) \) Lorentz group. Imposing the embedding condition as in (72)
\[
E^A_M \partial_\mu Y^M L_A^a = e^a_\mu, \quad E^A_M \partial_\mu Y^M L_A^i = e^i_\mu = 0
\]
restricts the \( SO(1,D-1) \) Lorentz matrices \( L_A^B \) to be coset elements of \( SO(1,D-1) \times SO(N) \) as we have shown below equation (50). In other words, we may parametrize \( \mathcal{U} = e^{iv^a M_a} \), instead of a general \( SO(1,D-1) \) Lorentz group element \( e^{iv^a M_a} \), from the outset. Consequently \( \mathcal{G} = e^{Y^A P_A e^{iv^a M_a}} \) is exactly the coset element in (53), which after taking the static gauge, i.e. \( Y^a = x^a, Y^i = \phi^i \), takes the form
\[
e^{Y^A P_A e^{iv^a M_a}} = e^{ix^a P_a} e^{i\phi^i(x)} P_i e^{iv^a(x) M_a} \tag{73}
\]
while \( \mathcal{G}^{-1}(d + i \tilde{E})\mathcal{G} \) becomes the covariant Maurer-Cartan one form.

As in the flat background case of higher dimensional space-time [12,15], it follows using the Poincare algebra \( ISO(1,D-1) \) commutators that
\[
\mathcal{U}^{-1} P_a \mathcal{U} = (\cos \sqrt{v^a v_a}) b b P_a - (\frac{\sin \sqrt{v^a v_a}}{\sqrt{v^a v_a}}) c v^c j P_j ,
\]
\[
\mathcal{U}^{-1} P_a \mathcal{U} = i b (\frac{\sin \sqrt{v^a v_a}}{\sqrt{v^a v_a}}) c v^c \phi^i P_i + (\cos \sqrt{v^a v_a}) i P_j . \tag{74}
\]
Comparing with (62), it is easy to see that by taking the vector representations for the broken Lorentz generators \( M_a \) that \( (L^T)^B_A = (U^{-1})^B_A \). The angular momentum generator piece in the decomposition of (70) is then computed as
\[
\omega = dY^M [\mathcal{U}^{-1} \partial_M \mathcal{U} - \frac{i}{2} \mathcal{U}^{-1} \Omega^B_{\cdot \cdot \cdot} M_{BC} \mathcal{U}] \tag{75}
\]
with $U = e^{i\omega_A M A}$. Once again, taking the matrix representation for all $SO(1, D-1)$ Lorentz generators as $(M^{AB})^C_D = i\eta^{CD} (\delta_E^A \delta^B_D - \delta^A_D \delta_E^B)$, then $(U)^B_A = U^B_A$, and eq. (75) becomes

$$
\omega = dY^M [U^{-1} \partial_M U + U^{-1} \Omega_M U] = dx^\mu \partial_\mu Y^M E_M^B [(U^{-1})_A^B (U^{-1})^E_B U + (U^{-1})_B^E (U^{-1})^A_B U] = dx^\mu e^a_\mu (U^{-1} \hat{\nabla}_a U)
$$

with $\hat{\nabla}_a = U^A_a (E_A + \Omega_A)$ (c.f. below equation (58)). Here the embedding condition has been used in obtaining the last identity. Writing $\omega = \frac{1}{2} \omega_{AB} M^{AB}$ and using the vector representation for $M^{AB}$, the identification

$$
\omega^A_B = \begin{pmatrix} \omega^a_b \omega^a_j \\ \omega^i_b \omega^j_b \end{pmatrix} = \begin{pmatrix} \omega^a_b & K^a_j \\ -K^i_b & B^i_j \end{pmatrix}
$$

is secured. Thus, using the embedding condition (49), it is established that the covariant Maurer-Cartan 1-form components, the induced vielbein, the induced spin connection, the extrinsic curvature and the twist potential, all have geometrical meanings. The coset approach and the embedding geometry construction yield identical results. As an example, consider the 5D space-time case where the twist potential vanishes, $B^ij = 0$, while $K^i\mu$ and $\Gamma^\lambda_{\mu\nu}$ are given in eq. (37), and the induced vielbein is simply $e^a_\mu = \mathcal{E}^a_\lambda (\delta^\lambda_\mu + X^\lambda X^\mu)^{\frac{1}{2}}$ (c.f. eq. (65)). Hence all the components of the covariant Maurer-Cartan 1-form can be explicitly expressed in terms of gravitational vielbein $\mathcal{E}^a_\mu, X_\mu$ and their derivatives.

We end this section by considering other embedding conditions. So far we have analyzed eq. (16) using the metric $G_{MN}$ and eq. (49) using the vielbein $E^A_M$. Note that the metric tensor can be expressed in two different forms, called the K-K form and the ADM form, as

$$
G_{MN} = \begin{pmatrix} g_{\mu\nu} + g_{m\bar{m}} A^m_\mu A_{\bar{m}}^\bar{\nu} \\ g_{i\bar{m}} A^m_\mu \\ g_{ij} \end{pmatrix} \quad \text{K-K}
$$

$$
= \begin{pmatrix} h_{\mu\nu} \\ N_{i\bar{m}} N_{\bar{m}}^{\mu} \\ N_{ij} + N_{i\bar{m}} N_{\bar{m}}^{\mu} h_{\mu\nu} N_{\bar{m}}^{\bar{\nu}} \end{pmatrix} \quad \text{ADM}
$$

These two forms come from the different decompositions of the metric tensor, i.e. $G_{MN} = \eta_{AB} \mathcal{E}^A_M \mathcal{E}^B_N = \eta_{AB} e^A_M e^B_N$ where the K-K vielbein $\mathcal{E}^A_M$ and the ADM vielbein $e^A_M$ will be given below. The fields $(g_{\mu\nu}, A^m_\mu, g_{ij})$ and $(h_{\mu\nu}, N_{i\bar{m}}, N_{ij})$ are related as

$$
h_{\mu\nu} = g_{\mu\nu} + g_{m\bar{m}} A^m_\mu A_{\bar{m}}^\bar{\nu}, \quad (N^{-1})^{\bar{m}\bar{\nu}} = g^m_{\bar{\nu}} + A^\lambda m A_{\bar{\lambda}}^\bar{\nu}, \quad N_{i\bar{m}} = (N^{-1})^{\bar{m}\bar{\nu}} A_{\bar{\nu}i}\ .
$$

Now consider embedding the brane into the bulk spacetime. The original embedding condition (49) corresponds to the ADM form

$$
E^A_M \partial_\mu Y^M (U^{-1})^B_A = \begin{pmatrix} e^a_\mu \\ 0 \end{pmatrix} \iff e^A_M = \begin{pmatrix} e^a_\mu \\ 0 \end{pmatrix} \eta^{ab} e^\lambda b N_{\lambda k}^k N_{ij}^\lambda \ .
$$
That is, an arbitrary higher dimensional vielbein $E^i_A$ is projected by $\partial_\mu Y^M$ and rotated by $U^{-1}$ into the first column of the ADM vielbein. Alternatively, a rotation by $U_{2}^{-1}$ produces the K-K vielbein as

$$E^i_A \partial_\mu Y^M (U_{2}^{-1})^B_A = \{\bar{E}^a_{\mu} \bar{E}^i_{k} X^k_{\mu} \leftrightarrow \mathcal{E}^A_M = \left( \begin{array}{cc} \mathcal{E}^a_{\mu} & 0 \\ \mathcal{E}^i_{k} \mathcal{A}^k_{\mu} & \mathcal{E}^i_{j} \end{array} \right) \right\}.$$ (81)

This corresponds to the condition used in the coset construction [2]. Note that $X^i_{\mu}$ and $A^i_{\mu}$ coincide in the unitary gauge for the Nambu-Goldstone fields, i.e. $\phi^i = 0$. Since these two vielbeins are related by an $SO(1, D-1)$ matrix $T$ as $\mathcal{E}^A_M = T^A_B e^B_M$, one can multiply (80) by the matrix $T^A_B$, giving

$$U_{2}^B = T U_1,$$ (82)

The embedding condition (81) also requires that the expression of normal vectors be modified. Formally, $n^i_M = E^i_A (U_{2}^{-1})^j_B T^A_B$ or more precisely

$$n^i_M = (N^{-\frac{1}{2}})^j_{\bar{j}} [E^A_M (U_{2}^{-1})^j_A - \bar{e}^\nu_{\mu} \mathcal{E}^i_{k} X^k_{\nu}].$$ (83)

where $\bar{e}^\nu_{\mu} = G_{MN} h^{\mu\nu} \partial_\mu Y^N$ is given in section 2. Note that the normal vectors are determined up to $SO(N)$ rotations. Both conditions (80) and (81), with corresponding expressions (82) and (83) for the normal vectors, lead to the same embedding condition (16). The condition (80) is related to the embedding geometry more closely, while (81) splits the vielbein directly into the graviton and the brane vectors and is more convenient for phenomenological applications.

5 Conclusions

It has been shown that Kaluza-Klein gravity in higher dimensional space-time, combined with the brane world scenario, leads to extra vectors which couple to 4D gravity and the Standard Model. The off diagonal components of the higher dimensional metric become massive vector fields $X^i_{\mu}$ as a consequence of the gravitational Higgs mechanism. As an example, a 4 dimensional brane embedded in a 5D space-time was considered and intrinsic and extrinsic geometrical objects, such as the induced metric, connections, extrinsic curvature and so on were calculated. All these quantities depend on the 4D graviton and the vector $X^i_{\mu}$. It follows that $X^i_{\mu}$ is a salient dynamical degree of freedom for describing the fluctuation of the brane. Both non-derivative and derivative couplings between $X^i_{\mu}$ and the Standard Model fields were studied and a four dimensional effective action was constructed from the higher dimensional theories and embedding geometry. Finally the relation between the embedding and the coset approach was clarified by comparing the covariant Maurer-Cartan forms.
Acknowledgments

The work of TEC, STL and CX was supported in part by the U.S. Department of Energy under grant DE-FG02-91ER40681 (Task B). The work of M.N. is supported in part by Grant-in-Aid for Scientific Research (No. 20740141) from the Ministry of Education, Culture, Sports, Science and Technology-Japan. The work of TtV was supported in part by a Cottrell Award from the Research Corporation and by the NSF under grant PHY-0758073. CX would like to thank Martin Kruczenski for the discussions on string theory.

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