Higher Poisson Brackets and Differential Forms

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Abstract. We show how the relation between Poisson brackets and symplectic forms can be extended to the case of inhomogeneous multivector fields and inhomogeneous differential forms (or pseudodifferential forms). In particular we arrive at a notion which is a generalization of a symplectic structure and gives rise to higher Poisson brackets. We also obtain a construction of Koszul type brackets in this setting.

Keywords: Higher Poisson bracket, strongly homotopy Lie algebra, supermanifold, symplectic form, Legendre transformation, higher Koszul bracket

PACS: 02., 02.20.Tw, 02.40.-k, 45.20.Jj

1. INTRODUCTION

Consider a Poisson manifold $M$ with a Poisson tensor $P = (P^{ab})$. It is a known fact that raising indices with the help of $P^{ab}$ gives the following commutative diagram:

$$
\begin{array}{ccc}
\mathfrak{A}^k(M) & \xrightarrow{d_p} & \mathfrak{A}^{k+1}(M) \\
\uparrow & & \uparrow \\
\Omega^k(M) & \xrightarrow{d} & \Omega^{k+1}(M) \\
\end{array}
$$

(1)

Here we denote by $\mathfrak{A}^k(M)$ the space of multivector fields on $M$ of degree $k$. In the sequel we also use the notations such as $\mathfrak{A}(M)$ and $\Omega(M)$ for the algebras of multivector fields and differential forms, respectively. (On supermanifolds one should speak of ‘pseudodifferential forms’ and ‘pseudomultivector fields’, but we shall stick to a simplified usage unless it may lead a confusion.) The vertical arrows are the operations of raising indices with the help of $P$ and the top horizontal arrow is the Lichnerowicz differential $d_p = [P, ]$. The bracket is the canonical Schouten bracket of multivector fields. This diagram leads to a natural map $H^k(M, \mathbb{R}) \rightarrow H^k(\mathfrak{A}(M), d_p)$ from the de Rham to Poisson cohomology, which is an isomorphism when the bracket is symplectic.

The transformation $\Omega(M) \rightarrow \mathfrak{A}(M)$ also preserves the brackets, so it is a morphism of differential Lie superalgebras. Here the space of multivector fields $\mathfrak{A}(M)$ is considered with the canonical Schouten bracket and the space of forms $\Omega(M)$, with an odd bracket known as the Koszul bracket, induced by the Poisson structure on $M$. It is noteworthy that the differential on $\Omega(M)$ is canonical while the bracket depends on the Poisson tensor $P$, and the case of $\mathfrak{A}(M)$ is the opposite: the bracket is canonical but the differential depends on $P$; the map $\Omega(M) \rightarrow \mathfrak{A}(M)$ exchanges a canonical structure with one defined by $P$.

In this paper we show three things. Firstly, we shall show how the map $\Omega(M) \rightarrow \mathfrak{A}(M)$ and the diagram (1) can be generalized to the case when a bivector field $P$ is
replaced by an arbitrary even multivector field, on a supermanifold. Secondly, we shall explain what plays the role of a symplectic structure in such case (when a bivector $P$ is replaced by an inhomogeneous object). We shall show how an inhomogeneous even form with an appropriate non-degeneracy condition generates a sequence of ‘higher’ Poisson brackets making the space, $C^\infty(M)$, a homotopy Poisson algebra. This might be called a generalized or homotopy symplectic structure on $M$. It is remarkable that the role of the matrix inverse (for bivectors and 2-forms) is taken by the Legendre transform.

Thirdly, we shall also explain what is the replacement of the Koszul bracket for higher Poisson structure.

Here and in the main text, by a homotopy Poisson algebra we mean an $L_\infty$-algebra of Lada and Stasheff [6] — the superized version — endowed with a commutative associative multiplication w.r.t. which each bracket is a multiderivation. This is more restrictive than other notions discussed in the literature [2], but seems quite fitting for differential-geometric purposes. Similarly one defines a homotopy Schouten algebra.

The constructions that we discuss have direct analogs for an odd Poisson structure on $M$, as well as for Lie algebroids. (In fact, the roots of this work are in our studies of odd Laplacians in [3].) There is also a remarkable analogy with well-known constructions of classical mechanics. A more detailed text containing proofs will appear elsewhere.

A note about usage: to simplify the language, we usually speak about ‘manifolds’, ‘Lie algebras’, etc., meaning ‘supermanifolds’ and ‘superalgebras’ respectively, unless this may cause a confusion or we need to emphasize that we are dealing with a ‘super’ object. By a $Q$-manifold we mean a differential manifold, i.e., a supermanifold with a homological vector field. The reader should be warned that parity of objects (i.e., $\mathbb{Z}_2$-grading) and a $\mathbb{Z}$-grading such as degree of forms, where it make sense, are in general independent; when we speak about an object which is ‘even’, that means ‘even in the parity sense’. We use different notations for different types of brackets; the canonical Schouten brackets and the Koszul-type brackets of forms are denoted by the square brackets, while the canonical Poisson brackets, by the parentheses (round brackets). All other Poisson-type brackets are denoted by the braces (curly brackets). A subscript may be used to indicate a Poisson-type structure defining the bracket.

2. MAIN CONSTRUCTIONS

Let $M$ be a manifold (or supermanifold, which we shall still call a manifold according to our convention). There are two supermanifolds naturally associated with it: the tangent bundle with the reversed parity $\Pi TM$ and the cotangent bundle with the reversed parity $\Pi T^*M$. We denote by $\Pi$ the parity reversion functor.

Recall that for an ordinary manifold $M$, the differential forms on $M$ can be identified with the functions on the supermanifold $\Pi TM$, and the multivector fields on $M$, with the functions on $\Pi T^*M$,

$$\Omega(M) = C^\infty(\Pi TM),$$

$$\mathfrak{X}(M) = C^\infty(\Pi T^*M).$$

For a supermanifold $M$ we shall take these as the definitions. (This simple approach will be sufficient for our needs. We are not going into a deeper investigation of analogs of
differential forms on supermanifolds here.

If \( x^a \) are local coordinates on \( M \), then \( dx^a \) and \( x^*_a \) are the induced coordinates in the fibers of the vector bundles \( \Pi TM \) and \( \Pi T^*M \) respectively. They have parities opposite to that of \( x^a \): \( \tilde{dx}^a = \tilde{x}^*_a = \tilde{a} + 1 \). (We use the tilde to denote parity and \( \tilde{a} \) stands for the parity of the coordinate \( x^a \).) The transformation laws for them are

\[
dx^a = dx^a' \frac{\partial x^a}{\partial x^a'} \quad \text{and} \quad x^*_a = \frac{\partial x^d}{\partial x^a} x^*_d
\]

(mind the order).

Let us fix an arbitrary even multivector field \( P \in \mathfrak{A}(M) \). We shall define a bundle map \( \varphi_P : \Pi T^*M \to \Pi TM \) by the formula

\[
\varphi^*_P(dx^a) = (-1)^{\tilde{a}+1} \frac{\partial P}{\partial x^*_a}.
\]

One can show that the map is well-defined.

**Example 1.** If \( P = \frac{1}{2} P^{ab} (x) x^*_b x^*_a \), then the pull-back \( \varphi^*_P : \Omega(M) \to \mathfrak{A}(M) \) coincides with raising indices with the help of the tensor \( P^{ab} \).

Now we shall study an analog of the diagram (1). Consider the diagram

\[
\begin{array}{ccc}
\mathfrak{A}(M) & \xrightarrow{d_P} & \mathfrak{A}(M) \\
\varphi^*_P \uparrow & & \uparrow \varphi^*_P \\
\Omega(M) & \xrightarrow{d} & \Omega(M),
\end{array}
\]

where \( d_P := \text{ad} P = [P, ] \) (the Schouten bracket). The linear operator \( d_P \) is odd. In general \( d_P^2 \neq 0 \) and

\[
d^2 = \frac{1}{2} \text{ad}[P,P].
\]

The diagram (5) is, in general, not commutative. To describe its discrepancy we need one technical tool.

Any multivector field \( Q \in \mathfrak{A}(M) = C^\infty(\Pi T^*M) \) defines a derivation from the algebra \( \Omega(M) \) to the tensor product \( \Omega(M) \otimes_{C^\infty(M)} \mathfrak{A}(M) \) over the natural homomorphism \( \Omega(M) \to \Omega(M) \otimes_{C^\infty(M)} \mathfrak{A}(M) \), \( \omega \mapsto \omega \otimes 1 \). We denote it \( \varkappa_Q \). In coordinates,

\[
\varkappa_Q = (-1)^{\tilde{Q}+1(\tilde{a}+1)} \frac{\partial Q}{\partial x^*_a} \frac{\partial}{\partial dx^a}.
\]

**Theorem 1.** The discrepancy of the diagram (5) is given by the formula:

\[
\varphi^*_P \circ d_P \circ \varphi_P^* = -\frac{1}{2} \varphi^*_P \varkappa_{[P,P]}.
\]
Corollary. If $[P,P] = 0$, then the diagram \( \mathcal{D} \) is commutative and thus $\varphi_P$ is a map of $Q$-manifolds

\[
(\Pi T^* M, d_P) \to (\Pi TM, d).
\]

An even multivector field $P$ satisfying $[P,P] = 0$ is a generalization of a Poisson tensor. It defines a sequence of ‘higher Poisson brackets’ on $M$, i.e., a sequence of $n$-ary operations, $n = 0, 1, 2, 3, \ldots$, on the space $\mathcal{C}^\infty(M)$,

\[
\{f_1, \ldots, f_n\}_P := \ldots [P, f_1], \ldots, f_n]_M. \quad (8)
\]

(Although it is not manifest in the formula, the bracket is antisymmetric in $f_1, \ldots, f_n$.) When no confusion is possible we suppress the subscript $P$ for the Poisson brackets. Each operation is a multiderivation w.r.t. the associative multiplication, i.e., a derivation in each argument. The condition $[P,P] = 0$ ensures that $\mathcal{C}^\infty(M)$ becomes an $L_\infty$-algebra w.r.t. the brackets. (See [3], [11].)

For an ordinary Poisson structure on a manifold $M$, it is non-degenerate if the components of the Poisson bivector $P_{ab}$ make a non-degenerate matrix. Then the entries of the inverse matrix $(P_{ab})^{-1}$ are the components of a non-degenerate closed 2-form. Conversely, any non-degenerate closed 2-form $\omega$ defines a Poisson bracket for which the components of the Poisson bivector are the entries of the inverse matrix $(\omega_{ab})^{-1}$. This is the relation between Poisson brackets and symplectic structures. How one should extend it to the case of inhomogeneous (and, possibly, even not fiberwise-polynomial, in the super case) multivector fields and forms?

**Proposition 1.** The map $\varphi_P: \Pi T^* M \to \Pi TM$ is a diffeomorphism (at least, near the zero section) if the matrix of second partial derivatives

\[
\frac{\partial^2 P}{\partial x_a^* \partial x_b^*}
\]

is invertible at $x_a^* = 0$.

This is what replaces the non-degeneracy condition for a bivector field. What should stand for the inverse matrix?

Suppose the map $\varphi_P: \Pi T^* M \to \Pi TM$ is invertible. Consider the **fiberwise Legendre transformation** of the multivector field $P$:

\[
\omega := \tilde{\varphi} := (\varphi_P)^{-1} \left( (-1)^{d+1} \frac{\partial P}{\partial x_a^*} x_a^* - P(x, x^*) \right) \quad (9)
\]

(the first term in brackets is just $dx^a x_a^*$ if we apply the corresponding isomorphism). It is an even differential form on $M$. By changing the order in the first term,

\[
\omega = \tilde{\varphi} = (\varphi_P)^{-1} \left( x_a^* \frac{\partial P}{\partial x_a^*} - P(x, x^*) \right). \quad (10)
\]

**Proposition 2.** The inverse map $\varphi_P^{-1}: \Pi TM \to \Pi T^* M$ is defined by the form $\omega$ in the similar way as the original map is defined by $P$:

\[
\varphi_P^{-1} = \psi_\omega \quad \text{where} \quad \psi_\omega(x_a^*) = \frac{\partial \omega}{\partial x_a^*}.
\]
The multivector field $P$ can be recovered as the fiberwise Legendre transformation of the form $\omega$.

This statement follows from the well known properties of the Legendre transformation. (Geometrically, we have a Lagrangian submanifold in each fiber of the sum $\Pi TM \oplus \Pi T^*M$, which can be described as the graph of the ‘gradient’ of a function of either $x_a^*$ or $dx^a$; then the corresponding functions are related by the mutually-inverse Legendre transforms.)

**Example 2.** For the classical case $P$ is quadratic, $P = \frac{1}{2} P^{ab} x_b^* x_a^*$, therefore $x_a^* \frac{\partial P}{\partial x^a} = P = P^{ab} x_b^*$, and the Legendre transformation of $P$ is simply $(\varphi_P^*)^{-1}(P)$. We have

$$\varphi_P^*(dx^a) = (-1)^{\hat{a}+1} \frac{\partial P}{\partial x^a} = P_a^b x_b^*,$$

where $P_{ab}$ stand for the matrix entries of the inverse matrix for $P^{ab}$. Hence we arrive at the 2-form

$$\omega = \frac{1}{2} dx^a P_{ab} dx^b = \frac{1}{2} dx^a dx^b P_{ab} (-1)^{\hat{a}(\hat{b}+1)} = \frac{1}{2} dx^a dx^b P_{ba} (-1)^{\hat{a}+1}$$

or

$$\omega = \frac{1}{2} dx^a dx^b \omega_{ba} \quad \text{where} \quad \omega_{ab} = P_{ab} (-1)^{\hat{b}+1}.$$

(Note the symmetry properties: $P^{ab} = (-1)^{\hat{a}(\hat{b}+1)} P^{ba}$, $\omega_{ab} = (-1)^{\hat{a}(\hat{b}+1)} \omega_{ba}$, and $P_{ab} = -(-1)^{\hat{a}\hat{b}} P_{ba}$.)

The following theorem is an extension of the classical relation existing for 2-forms and bivector fields.

**Theorem 2.** The exterior differential of the Legendre transform of the multivector field $P$, the form $\omega$, is given by the formula:

$$d\omega = -\frac{1}{2} (\varphi_P^*)^{-1}([P,P]).$$

(11)

**Corollary.** The multivector $P$ satisfies $[P,P] = 0$ if and only if $d\omega = 0$.

**Remark 1.** The statement in one direction can be also deduced from Theorem [1]. Suppose a form $\omega'$ maps to the multivector field $P$ under the map $\varphi_P^*$. Under the assumption that $[P,P] = 0$, the diagram (5) commutes and $\varphi_P^*(d\omega') = d_P(\varphi_P^* \omega') = d_P(P) = 0$. Assuming that $\varphi_P^*$ is invertible, we arrive at $d\omega' = 0$. The question arises, what is the relation between the form $\omega' = (\varphi_P^*)^{-1}P$ and the form $\omega = \hat{P}$, the Legendre transform of $P$. The formula (10) for the Legendre transform can be re-written as follows:

$$\omega = (\varphi_P^*)^{-1}(E(P) - P).$$

(12)

where $E$ is the fiberwise Euler vector field on $\Pi T^*M$. In the classical case $\omega$ and $\omega'$ coincide, as we have seen, since for a bivector $E(P) = 2P$. We may note the following useful relation for arbitrary multivector fields:

$$E([P,Q]) = [E(P),Q] + [P,E(Q)] - [P,Q].$$

(13)
(which is an expression of the fact that the Schouten bracket has weight $-1$ in the natural $\mathbb{Z}$-grading). Therefore $[P, E(P)] = \frac{1}{2} ([E(P), P] + [P, E(P)]) = \frac{1}{2} (E[P, P] + [P, P])$ and $[P, E(P) - P] = \frac{1}{2} (E[P, P] - [P, P])$. We see that if $[P, P] = 0$, then $[P, E(P)] = 0$ too and the forms $\omega$ and $\omega'$ are both closed. The choice of $\omega$, not $\omega'$, as the correct analog of a symplectic form corresponding to $P$ is determined by the fact that the inverse map $(\phi_P)^{-1}$ is given by $\psi_\omega$ defined from $\omega$ in the same way as $\phi_P$ is defined from $P$, with the same type of non-degeneracy conditions.

The above constructions can be summarized in the following definitions.

**Definition 1.** A higher Poisson structure on $M$ given by a multivector field $P$ is non-degenerate if the map $\phi_P : \Pi^* TM \to \Pi T^* M$ is invertible (at least on a neighborhood of $M$ in $\Pi^* T^* M$).

In terms of the higher Poisson brackets $\{f_1, \ldots, f_k\}_P$ generated by $P$, this is equivalent to the non-degeneracy of just the binary bracket $\{f, g\}_P$. Since however this bracket satisfies the Jacobi identity only up to homotopy, it is not the same as an ordinary symplectic structure.

**Definition 2.** A generalized or homotopy, symplectic structure on $M$ is a closed even (pseudo)differential form $\omega \in \Omega(M)$ such that the map $\psi_\omega : \Pi T^* M \to \Pi^* T^* M$ defined by the formula

$$\psi_\omega^* \omega_{x^a} = \frac{\partial \omega}{\partial x^a}$$

is a diffeomorphism. (As above, we relax this condition by requiring $\psi_\omega$ to be a diffeomorphism only on a neighborhood of the zero section $M \subset \Pi T^* M$.) Such a form $\omega$ is called a generalized symplectic form.

The generalized symplectic forms $\omega \in \Omega(M)$ are in one-to-one correspondence with the non-degenerate higher Poisson structures given by multivector fields $P \in \mathfrak{M}(M)$ and the correspondence is given by the mutually-inverse Legendre transforms.

**Proposition 3.** The non-degeneracy condition for a generalized symplectic form is equivalent to requiring that the matrix of second partial derivatives

$$\frac{\partial^2 \omega}{\partial dx^a \partial dx^b}$$

is invertible at $dx^a = 0$.

**Example 3.** If an even form $\omega \in \Omega(M)$ can be written as

$$\omega = \omega_0 + \omega_1 + \omega_2 + \omega_3 + \ldots$$

where $\omega_k \in \Omega^k(M)$, it is a generalized symplectic form if and only if $\omega_2$ is an ordinary symplectic form and the other terms are arbitrary closed forms. (Note that on a supermanifold an even in the sense of parity form may have components both in even and odd degrees.) As we shall see in the next section, the Poisson brackets defined by (15) will not reduce to the ordinary Poisson bracket defined by the symplectic 2-form $\omega_2$, but include ‘higher corrections’.
3. EXAMPLES

Let $\omega \in \Omega(M)$ be a generalized symplectic form on $M$ with the corresponding non-degenerate Poisson multivector field $P \in \mathfrak{a}(M)$. What are the (higher) Poisson brackets generated by this form?

For an arbitrary higher Poisson structure defined by $P \in \mathfrak{a}(M)$, not necessarily non-degenerate, the higher Poisson brackets of functions on $M$ are given by (8). The Hamiltonian vector fields of ordinary Poisson geometry are replaced by multivector fields. For a function $f \in C^\infty(M)$, the multivector field $Q_f \in \mathfrak{a}(M)$ defined by the formula

$$Q_f := \phi^*_P(df)$$

may be called the Hamiltonian multivector field corresponding to $f$. By Theorem 1,

$$Q_f = [P, f].$$

Indeed, from the commutative diagram, $Q_f = \phi^*_P(df) = df = [P, f]$, because for functions $\phi^*_P(f) = f$. The formula (8) for higher Poisson brackets may be reformulated as

$$\{f_1, \ldots, f_k\} = \ldots [[Q_{f_1}, f_2], \ldots, f_k]|_M.$$  

Let $\omega$ be a generalized symplectic form. To calculate the Poisson brackets corresponding to $\omega$, it is sufficient to find the multivector field $Q_f$ for an arbitrary function $f \in C^\infty(M)$. We find it from the relation

$$Q_f = (\psi^*_\omega)^{-1}(df) \quad \text{or} \quad \psi^*_\omega Q_f = df.$$

Let us consider particular examples.

Remark 2. Although the case of ordinary manifolds is not at all trivial, more interesting examples should be related with supermanifolds. Indeed, for an ordinary manifold, $\omega$ is just an inhomogeneous differential form of the appearance (15). Since $\omega$ is supposed to be even, then only the $\omega_{2k}$ may be non-zero. As we shall see, this implies the vanishing of the differential, i.e., the unary bracket, on functions. However the higher brackets may still be non-zero and satisfy non-trivial identities.

Example 4. We start from an ordinary symplectic structure for further comparison. Suppose

$$\omega = \omega_2 = \frac{1}{2} dx^a dx^b \omega_{ba}(x).$$

Then we have the equation

$$x^*_a = dx^b \omega_{ba}$$

for determining the variables $dx^a$. Here and in the sequel we shall suppress the notations for the pull-backs $\psi^*_\omega$ and $\varphi^*_P$. From here

$$dx^a = x^*_b \omega^{ba} = (-1)^{\hat{a}+1} \omega^{ab} x^*_b$$
where $\omega_{ab} \omega^{cb} = \delta_a^b$ (note that $\omega^{ab} = -(-1)^{\alpha_b} \omega^{ba}$), and

$$Q_f = x_b^a \omega^{ba} \partial_a f = (-1)^{\tilde{f}(\tilde{a}+1)} \partial_a f x_b^a \omega^{ba} = -(-1)^{(\tilde{a}+1)} \partial_a f \omega^{ab} x_b^a.$$  

Therefore the only non-trivial bracket is, of course, binary, and it is given by

$$\{f, g\} = -(-1)^{\tilde{f}(\tilde{a}+1)} \partial_a f \omega^{ab} \partial_b g = (-1)^{\tilde{a}+1} \omega^{ab} \partial_b f \partial_a g.$$  

The Poisson tensor is given by

$$P = \frac{1}{2} (-1)^{\tilde{a}+1} \omega^{ab} x_b^a x_a^*.$$  

Example 5. Suppose now there is an extra linear term in $\omega$:

$$\omega = \omega_1 + \omega_2 = dx^a \omega_a + \frac{1}{2} dx^a dx^b \omega_{ba}.$$  

Note that in particular $d\omega_1 = 0$, hence locally $\omega_1 = d\chi$ for some odd function $\chi$. We have the equation

$$x_a^* = \omega_a + dx^b \omega_{ba}$$  

for determining $dx^a$ (where locally $\omega_a = \partial_a \chi$). As before we obtain

$$dx^a = (x_b^a - \omega_b) \omega^{ba} = (x_b^a - \partial_b \chi) \omega^{ba}.$$  

It is instructive to find the Poisson multivector field $P$. When we calculate the Legendre transform, the term $\omega_1$ makes no input into $E(\omega) - \omega$ and results only in the shift of the argument:

$$P = \frac{1}{2} (-1)^{\tilde{a}+1} \omega^{ab} (x_b^a - \partial_b \chi)(x_a^* - \partial_a \chi).$$  

Hence we have

$$P = P_0 + P_1 + P_2 = \frac{1}{2} (-1)^{\tilde{a}+1} \omega^{ab} \partial_a \chi \partial_a \chi - (-1)^{\tilde{a}+1} \omega^{ab} \partial_a \chi x_a^* + \frac{1}{2} (-1)^{\tilde{a}+1} \omega^{ab} x_b^a x_a^*.$$  

This leads to the following 0-, 1-, and 2-brackets:

$$\{\emptyset\} = \frac{1}{2} (-1)^{\tilde{a}+1} \omega^{ab} \partial_b \chi \partial_a \chi = \frac{1}{2} \{\chi, \chi\},$$  

$$\{f\} = (-1)^{\tilde{a}+1} \omega^{ab} \partial_b \chi \partial_a f = \{\chi, f\},$$  

$$\{f, g\} = (-1)^{\tilde{f}+1} \omega^{ab} \partial_b f \partial_a g,$$

and there are no higher brackets. Hence we have the binary Poisson bracket that satisfies the ordinary Jacobi identity. Besides it we are given an odd vector field $X = X_\chi$ locally-Hamiltonian w.r.t. this bracket (and thus automatically a derivation) and an even function $P_0 = \frac{1}{2} \{\chi, \chi\}$ such that $X^2 = X P_0$. The field $X$ is homological if $\{\chi, \chi\}$ is a local constant. (One may consider, unrelatedly to generalized symplectic structures, a structure similar to the above consisting of a Poisson bracket together with an odd function $\chi$ defining 0- and 1-brackets by the above formulas, where the corresponding vector field is homological if $\{\chi, \chi\}$ is a Casimir function.)
**Example 6.** Consider now a generalized symplectic form

\[ \omega = \omega_1 + \omega_2 + \omega_3 = dx^a \omega_a + \frac{1}{2} dx^a dx^b \omega_{ba}(x) + \frac{\lambda}{3!} dx^a dx^b dx^c \omega_{cba} \]

involving a cubic term. Notice that we have included a parameter \( \lambda \). We may start as before and obtain the relation

\[ x^*_a = \omega_a + dx^b \omega_{ba} + \frac{\lambda}{2} dx^b dx^c \omega_{cba} \]

for determining \( dx^a \). In order to obtain the solution introduce \( \xi_a = dx^b \omega_{ba} \). Hence \( dx^a = (-1)^{\bar{a}+1} \omega^{ab} \xi_b \), and we have the equation

\[ \xi_a + \frac{\lambda}{2} \omega^{kl} \xi_k \xi_k = \theta_a \]

for determining \( \xi_a \) (if we denote \( \theta_a = x^*_a - \omega_a \)). Here we raise indices with the help of \( \omega^{ab} \) with the following sign convention:

\[ \omega^{kl} = \omega^{kb} \omega^{lc} \omega_{cba} (-1)^{(l+1)(\bar{b}+1)+\bar{a}(\bar{k}+\bar{l})} \].

The equation for \( \xi_a \) can be solved by iterations, expressing the answer as an infinite power series in \( \lambda \). In the first order in \( \lambda \),

\[ dx^a = (-1)^{\bar{a}+1} \omega^{ab} (x^*_b - \omega_b) - \frac{\lambda}{2} (-1)^{\bar{a}+\bar{c}} \omega^{abc} (x^*_c - \omega_c) (x^*_b - \omega_b) + O(\lambda^2) \].

Here

\[ \omega^{abc} = \omega^{ap} \omega^{bq} \omega^{cr} \omega_{qrp} (-1)^{\bar{p}(\bar{b}+\bar{c})+\bar{q}(\bar{c}+1)} \].

To obtain the Legendre transform, this should be substituted into \( E(\omega) - \omega = \omega_2 + 2 \omega_3 \). We arrive at

\[ P = P_0 + P_1 + P_2 + P_3 + \ldots \]

where

\[ P_0 = \frac{1}{2} (-1)^{\bar{a}+1} \omega^{ab} \omega_b \omega_a + \frac{\lambda}{6} (-1)^{\bar{a}+\bar{c}} \omega^{abc} \omega_c \omega_b \omega_a + O(\lambda^2), \]

\[ P_1 = \left( (-1)^{\bar{a}} \omega^{ab} \omega_b - \frac{\lambda}{2} (-1)^{\bar{a}+\bar{c}} \omega^{abc} \omega_c \omega_b + O(\lambda^2) \right) x^*_a, \]

\[ P_2 = \frac{1}{2} \left( (-1)^{\bar{a}+1} \omega^{ab} + \lambda (-1)^{\bar{a}+\bar{c}} \omega^{abc} \omega_c + O(\lambda^2) \right) x^*_b x^*_a, \]

\[ P_3 = -\frac{\lambda}{6} (-1)^{\bar{a}+\bar{c}} \omega^{abc} x^*_c x^*_b x^*_a + O(\lambda^2), \]

\[ P_4 = O(\lambda^2), \text{ etc.} \]

Therefore there will be an infinite series of brackets and each bracket is given by an infinite series in the parameter \( \lambda \); all brackets higher than ternary are of order \( \geq 2 \) in \( \lambda \). They satisfy non-trivial Jacobi identities with \( n \) arguments for all \( n = 0, 1, 2, 3, \ldots \). Note the presence of higher corrections in the binary bracket.
4. HIGHER KOSZUL BRACKETS

We shall discuss now what replaces the Koszul bracket in the case of a higher Poisson structure, i.e., an even multivector field $P \in \mathfrak{a}(M)$ such that $[P, P] = 0$.

Let us recall the ordinary case. In the classical situation the Koszul bracket corresponding to a Poisson structure on a manifold $M$ given by a bivector field $P$ may be defined axiomatically as a unique odd Poisson (Schouten, Gerstenhaber, ...) bracket on the algebra of forms $\Omega(M)$ obeying the following 'initial conditions':

$$[f, g]_P = 0, \quad [f, dg]_P = (-1)^{\hat{f}} \{f, g\}_P, \quad \text{and} \quad [df, dg]_P = -(-1)^{\hat{f}} d\{f, g\}_P, \quad (19)$$

where the curly bracket $\{, \}_P$ stands for the Poisson bracket of functions and $[\ , \ ]_P$ stands for the Koszul bracket of forms. In particular, for coordinates and their differentials we have

$$[x^a, x^b]_P = 0, \quad [x^a, dx^b]_P = -P^{ab}, \quad \text{and} \quad [dx^a, dx^b]_P = dP^{ab}. \quad (20)$$

(The Lie bracket of 1-forms on a Poisson manifold was probably first introduced by B. Fuchssteiner [1], but it had a rich pre-history, see [4]. The bracket on the algebra of all forms was introduced by Koszul [5], as the bracket generated by a second-order operator on forms playing the role of the boundary operator for the Poisson homology $\mathfrak{h}^1$. See also [2] and references therein. Particular signs in formulas such as (19), (20) depend on conventions.)

In this case the Koszul bracket can be also defined using the diagrams (1) or (5). Namely, if we assume the invertibility of the matrix $(P^{ab})$, then one can consider

$$[\omega, \sigma]_P := (\varphi_P^*)^{-1}(\{\varphi_P^*\omega, \varphi_P^*\sigma\}) \quad (21)$$

By the construction it is an odd Poisson bracket on the algebra $\Omega(M)$. One can see that an explicit formula obtained from (21) does not include the inverse matrix for $(P^{ab})$ (this substantially relies on the identity $[P, P] = 0$) and for coordinates gives exactly (20). Therefore formula (21) ‘survives the limit’ when one passes to an arbitrary Poisson bivector.

All the above holds true for the classical (i.e., binary) case only. It is not a priori obvious how one can extend the axiomatic definition to the general case since there are now many ‘higher’ Poisson brackets of functions. If one would try to use naively the formula (21) defining an odd binary bracket on forms as an operation isomorphic to the canonical Schouten bracket of multivector fields, then it would not survive the limit when the condition of the non-degeneracy is dropped unlike for the classical case.

Therefore we need a different approach.

One should expect that to a higher Poisson structure on functions there corresponds a higher structure on forms as well rather than a single bracket, i.e., a sequence of ‘higher Koszul brackets’. We shall indeed define them — directly in terms of the multivector field $P \in \mathfrak{a}(M)$.

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1 It mimics the well-known expression of the canonical Schouten bracket on multivector fields in terms of a divergence operator, which is also a model for the Batalin–Vilkovisky formalism.
Recall that odd brackets on functions on a given manifold \( N \) are generated by an odd ‘master Hamiltonian’ \( S \), i.e., a function on the cotangent bundle \( T^*N \) satisfying \( \{ S, S \} = 0 \) for the canonical Poisson bracket. See, e.g., [10]. In our case we need odd brackets on the algebra \( \Omega(M) = C^\infty(\Pi T^*M) \). Therefore we should look for an odd function on the cotangent bundle \( T^*(\Pi T^*M) \). How can one get it from a given even function \( P \in C^\infty(\Pi T^*M) \)?

**Theorem 3.** There is a natural odd linear map

\[
\alpha: C^\infty(\Pi T^*M) \to C^\infty(T^*(\Pi T^*M)) \tag{22}
\]

that takes the canonical Schouten bracket on \( \Pi T^*M \) to the canonical Poisson bracket on \( T^*(\Pi T^*M) \), up to a sign:

\[
\alpha([P, Q]) = (-1)^{\beta+1}(\alpha(P), \alpha(Q)), \tag{23}
\]

for arbitrary \( P, Q \in C^\infty(\Pi T^*M) \).

(Here by the square brackets we denote the canonical Schouten bracket and by the parentheses, the canonical Poisson bracket. The sign in (23) depends on conventions.)

Sketch of a proof. By the theorem of Mackenzie and Xu [8] (see also [10]) there is a symplectomorphism between \( T^*(\Pi T^*M) \) and \( T^*(\Pi T^*M) \). Now, given a function on \( \Pi T^*M \), one can associate with it the corresponding Hamiltonian vector field of the opposite parity w.r.t. the canonical bracket on \( \Pi T^*M \). The Schouten bracket of functions on \( \Pi T^*M \) (which are multivector fields on \( M \)) maps to the commutator of vector fields. In turn, to each vector field on any manifold we can assign a fiberwise-linear function on the cotangent bundle so that the commutator of vector fields maps to the Poisson bracket of the corresponding Hamiltonians. Hence we have a sequence of linear maps preserving the brackets:

\[ C^\infty(\Pi T^*M) \to \mathfrak{F}(\Pi T^*M) \to C^\infty(T^*(\Pi T^*M)) \to C^\infty(T^*(\Pi T^*M)) \],

where the last arrow is induced by the identification \( T^*(\Pi T^*M) \cong T^*(\Pi T^*M) \). We define \( \alpha \) as the through map. It is odd and takes brackets to brackets.

**Corollary.** To each even \( P \in \mathfrak{F}(M) \) such that \( [P, P] = 0 \) there corresponds an odd \( K = K_P \in C^\infty(T^*(\Pi T^*M)) \) such that \( (K, K) = 0 \).

This odd Hamiltonian \( K = K_P \) defines the higher Koszul brackets on the algebra of forms \( \Omega(M) \) corresponding to a higher Poisson structure on \( M \) defined by the multivector field \( P \).

We may calculate the Hamiltonian \( K \) explicitly. If \( P = P(x, x^\pi) \), then

\[
K = (-1)^{\alpha} \frac{\partial P}{\partial x^\pi}(x, \pi) p_a + dx^a \frac{\partial P}{\partial x^a}(x, \pi), \tag{24}
\]

where \( \pi = (\pi_a) \) and we denote by \( p_a, \pi_a \) the momenta conjugate to the coordinates \( x^a, dx^a \) on \( \Pi T^*M \), respectively. Note the linear dependence on the coordinate \( dx^a \).

**Example 7.** For the quadratic \( P = \frac{1}{2} P^{ab} x_a^b x_a^b \) we get

\[
K = -P^{ab} \pi_b p_a + \frac{1}{2} dP^{ab} \pi_b \pi_a, \tag{25}
\]
which leads to the binary brackets
\[ [x^a, x^b]_P = ((K, x^a), x^b) = 0, \quad [x^a, d x^b]_P = ((K, x^a), d x^b) = -P^{ab}, \quad \text{and} \]
\[ [dx^a, d x^b]_P = ((K, dx^a), dx^b) = d P^{ab} \]
coinciding with (20). Therefore in this case our construction reproduces the classical Koszul bracket.

In general, the odd Hamiltonian \( K = K_P \) defines a sequence of odd \( n \)-ary brackets on \( \Omega(M) \),
\[ [\omega_1, \ldots, \omega_n]_P = (\ldots (K, \omega_1), \ldots, \omega_n)|_{\Pi T M}, \quad n = 0, 1, 2, \ldots, \tag{26} \]
which makes it a particular case of a ‘homotopy Schouten algebra’. (It is an \( L_\infty \)-algebra such that each bracket is a multiderivation of the associative multiplication.) It is instructive to have a look at the \((k + l)\)-bracket of \( k \) functions \( f_1, \ldots, f_k \) and \( l \) differentials \( df_{k+1}, \ldots, df_{k+l} \). Either from (24) or directly from the construction in Theorem 3 we obtain the following formulas:
\[ [f]_P = \{f\}_P \quad \text{and} \quad [f_1, \ldots, f_k] = 0 \quad \text{for} \quad k \geq 2, \tag{27} \]
\[ [f_1, df_{i+1}, \ldots, df_n]_P = (-1)^{i} \{f_1, f_{i+1}, \ldots, f_n\}, \tag{28} \]
\[ [df_1, \ldots, df_n]_P = (-1)^{i+1} d\{f_1, \ldots, f_n\}, \tag{29} \]
where \( i = n - 1 \). From here we see that our constructions yield precisely an \( L_\infty \)-algebroid structure on the cotangent bundle \( T^*M \). Such a structure on a vector bundle \( E \) consists of a sequence of higher Lie brackets of sections making their space an \( L_\infty \)-algebra and a sequence of ‘higher anchors’ (multilinear maps into the tangent bundle) so that, for each \( n \), the \( n \)-anchor appears in the Leibniz formula for the \( n \)-bracket. Since for the cotangent bundle, the differentials of functions span the space of sections over functions, it is sufficient to know the brackets as well as the action of the anchors just for differentials, as the rest can be recovered by the Leibniz rule. Therefore, taken together with (27), formulas (28) define the anchors and formulas (29) and (28), the brackets of sections for \( T^*M \). This extends the classical construction for ordinary Poisson manifolds, see [7]. Note finally that an \( L_\infty \)-algebroid structure on an arbitrary \( E \) is defined by a homological vector field on the total space \( \Omega E \) (for ordinary Lie algebroids this field has to be homogeneous of degree +1). What is this field in our case? One can immediately see that it is just the odd Hamiltonian vector field \( X_P \in \chi(\Pi T^*M) \) corresponding to the function \( P \in C^\infty(\Pi T^*M) \). This gives an alternative proof of Theorem 3. The higher Koszul brackets on \( \Omega(M) \) appear simply as the extension of the Lie brackets in this \( L_\infty \)-algebroid to the algebra \( \Omega(M) \) as multiderivations, in a complete analogy with the classical case.

A question remains about the arrow \( \phi_P^* : \Omega(M) \to \mathfrak{A}(M) \). Since there is only one non-zero bracket on \( \mathfrak{A}(M) \) and a whole sequence of brackets on \( \Omega(M) \), it cannot just map brackets to brackets as in the classical case. A hope is that it extends to an \( L_\infty \)-morphism. (This will be studied elsewhere.)
5. DISCUSSION

Instead of a Poisson manifold (with an even Poisson structure), one may consider an odd Poisson manifold. There is an analog of diagram (I) and of the Koszul brackets (3). Considerations of this paper can be extended to this case as well yielding ‘homotopy odd symplectic structures’ and higher Koszul brackets for higher Schouten structures. This corresponds to a map $T^*M \to \Pi TM$. Note that a map $T^*M \to TM$ is what is used in classical mechanics when passing from the Lagrangian to the Hamiltonian picture and back; it makes sense to study a map $\Pi T^*M \to TM$. Finally, one may wish to replace Poisson manifolds by Lie bialgebroids and their analogs. No doubt that the constructions of this paper can be carried over to them as well.

ACKNOWLEDGMENTS

It is a pleasure to thank the organizers of the annual Workshops on Geometric Methods in Physics in Białowieża where this work was first reported and particularly Prof. Anatol Odzijewicz for the hospitality and the exceptionally inspiring atmosphere at the meetings. Most cordial thanks are due to Prof. James Stasheff for numerous remarks on the first version of the text and his help in improving the exposition.

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