Wave Breaking limit in Arbitrary Mass Ratio Warm Plasmas

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The maximum sustainable amplitude, so-called wave breaking limit, of a nonlinear plasma wave in arbitrary mass ratio warm plasmas is obtained in the non-relativistic regime. Using the method of Sagdeev potential a general wave breaking formula is derived by taking into account the dynamics of both the species having finite temperature. It is found, that the maximum amplitude of the plasma wave decreases monotonically with the increase in temperature and mildly increases with increase in mass ratio.

Studies on wave breaking is a topic of fundamental interest in plasmas due to its various applications such as plasma heating \cite{1,2}, plasma based particle acceleration schemes \cite{3,4}, etc. The wave breaking \cite{5,6} limit of a nonlinear plasma oscillation and/or wave decides its maximum sustainable amplitude beyond which the coherent nature of the wave is destroyed. At the wave breaking point, the velocity of the plasma fluid element at the crest of the wave exceeds the phase velocity of the wave. In certain situations, a wave breaks at a lower amplitude than its usual breaking limit because of a novel phenomena called ‘phase mixing’ \cite{11–14}. Phase mixing is an important physical process through which an oscillation or/and wave undergoes a gradual loss of phase coherence, thus leading to breaking of the wave at a finite time, even if the amplitude of the wave is well below the breaking amplitude \cite{15}. Phase mixing occurs when the frequency of the wave becomes space-dependent and this may occur through various nonlinear processes like inhomogeneity \cite{16}, relativistic mass variation \cite{17}, etc.

The maximum amplitude of the wave breaking amplitude of an electron plasma wave was first introduced by Akhiezer and Polovin (AP) \cite{18} in a cold relativistic plasma where the massive ions were assumed to provide a fixed charge neutralizing background. Although, they never use the term “wave breaking” in their article, the derived wave breaking limit was $E_{wb} = (mc\omega_{pe}/e)\sqrt{2(\gamma - 1)}$, where $\gamma = [1 - (v_{ph}/c)^2]^{-1/2}$, $v_{ph}$ is the phase velocity of the wave, $\omega_{pe}$ is the electron plasma frequency and $m$ is the mass of an electron. Later, Dawson\cite{7} derived the wave breaking amplitude $E_{wb}$ in the nonrelativistic limit using a Lagrangian description which gives $E_{wb} = mv_{ph}\omega_{pe}/e$. Again in the nonrelativistic regime, thermal effects were included in the study of wave breaking by Coffey \cite{8} using a 1D waterbag model for electrons. The maximum sustainable amplitude derived by Coffey was $E_{wb} = (mv_{ph}\omega_{pe}/e)\left(1 - \frac{3}{2}\beta - \frac{8}{3}\beta^{1/4} + 2\beta^{1/2}\right)^{1/2}$, where $\beta = 3T/mv_{ph}^2$, $T$ is the temperature in the energy unit. It was found that the inclusion of the temperature through the plasma pressure reduces the wave breaking amplitude. Katsouleas and Mori \cite{20} investigated the wave breaking limit in the relativistic regime and obtained the wave breaking limit as $E_{wb} = (mc\omega_{pe}/e)\beta^{-1/4}(\ln 2\gamma^{1/2}\beta^{1/4})^{1/2}$. Thus the subject of wave breaking of large amplitude plasma waves with immobile ions has been thoroughly investigated with both cold and warm electrons, spanning the entire domain, starting from non-relativistic to relativistic regime. Khachatryan \cite{21} extended these studies on wave breaking by including ion motion in a cold relativistic electron-ion plasma. It was reported that, with the increase of electron to ion mass ratio, the wave breaking amplitude also increases. As a corollary, it was shown that the wave breaking limit in non-relativistic cold pair-ion plasmas (e.g., electron-positron plasmas/pair-ion plasmas) is higher than electron-ion plasmas ($E_{wb} = 1.08(mv_{ph}\omega_{pe}/e)$). However, to the best of our knowledge wave breaking studies with both warm electrons and warm ions have never been attempted. In this letter, we study finite temperature effects on wave breaking in warm unmagnetized arbitrary mass ratio plasmas, in the non-relativistic regime.

To obtain the general wave breaking amplitude, we have considered an unmagnetized, homogeneous warm nonrelativistic plasma with two species having equal and opposite charges. In one space-dimension, the basic equations that govern the propagation of nonlinear electrostatic waves in warm plasmas are the continuity equations, momentum equations for positively (ions or positrons) and negatively (electrons or negative ions) charged particles, and the Poisson’s equation, which are respectively written as follows:

\begin{equation}
\frac{\partial n_\pm}{\partial t} + \frac{\partial}{\partial x}(n_\pm u_\pm) = 0, \tag{1}
\end{equation}

\begin{equation}
\left(\frac{\partial}{\partial t} + u_\pm \frac{\partial}{\partial x}\right)u_\pm = \pm eE_{\pm} \quad \frac{1}{m_\pm n_\pm} \frac{\partial P_\pm}{\partial x}, \tag{2}
\end{equation}

and

\begin{equation}
\frac{\partial E}{\partial x} = 4\pi e(n_+ - n_-), \tag{3}
\end{equation}

where $n_\pm$, $u_\pm$, $P_\pm$ and $m_\pm$ are densities, velocities, partial pressures and masses of positively and negatively charged

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particles respectively; and \( E \) is the electric field. The plasma is overall quasi-neutral, and for singly charged particles we assume quasineutrality condition as \( n_{+0} = n_{-0} = n_0 \) (say), where \( n_{+0} \) and \( n_{-0} \) are respectively the equilibrium number density of positively and negatively charged particles. Using the adiabatic equation of state for both positively and negatively charged particles with \( \gamma = 3 \) (the ratio of specific heats, for a 1-D system), the equations of motion may be written as

\[
\left( \frac{\partial}{\partial t} + u_\pm \frac{\partial}{\partial x} \right) u_\pm = \pm \frac{eE}{m_\pm} - \frac{v_{th\pm}^2}{2} \frac{\partial n_\pm}{\partial x},
\]

where \( v_{th\pm} = \sqrt{3T_\pm/m_\pm} \) is the thermal speed of each species with \( T_+ \) and \( T_- \) as the equilibrium temperatures of the positively and negatively charged particles respectively. The governing equations \( \ref{eq:1} \), \( \ref{eq:4} \) and \( \ref{eq:3} \) are exact within the framework of waterbag model, i.e. these equations may be directly derived by taking moments of the Vlasov equation assuming waterbag distribution for the warm species. For such a distribution the heat flux turns out to be identically zero so that there is closure to the hierarchy of moments of the Vlasov equation \( \ref{eq:3} \).

For traveling wave solution of equations \( \ref{eq:1} \), \( \ref{eq:4} \) and \( \ref{eq:3} \), it is convenient to move into a wave frame with the variable transformation \( \psi = k_p (x - v_{ph} t) \), \( k_p = \omega_p/v_{ph} \), where plasma frequency \( \omega_p = \sqrt{4\pi n_0 e^2/m_-} \) and \( v_{ph} \) is the phase velocity of the wave. With this transformation the equations \( \ref{eq:1} \), \( \ref{eq:4} \) and \( \ref{eq:3} \) transforms, respectively into the following ordinary differential equations:

\[
\frac{d}{d\psi} \left[ \hat{n}_\pm (1 - \hat{u}_\pm) \right] = 0, \quad \frac{d}{d\psi} \left[ \hat{n}_\pm^2 - 2\hat{u}_\pm \right] = \pm 2\mu_\pm \hat{E} - \beta_\pm \frac{d}{d\psi} (\hat{n}_\pm^2),
\]

and

\[
\frac{d\hat{E}}{d\psi} = \hat{n}_+ - \hat{n}_- \quad \frac{d\hat{E}}{d\psi} = \hat{n}_+ - \hat{n}_- \quad \frac{d\hat{E}}{d\psi} = \hat{n}_+ - \hat{n}_- \quad \frac{d\hat{E}}{d\psi} = \hat{n}_+ - \hat{n}_-
\]

where \( \hat{n}_\pm, \hat{u}_\pm \) and \( \hat{E} \) are the normalized densities, velocities and electric field, normalized by equilibrium density \( n_0 \), phase velocity \( v_{ph} \) and \( m_- \omega_p v_{ph}/e \), respectively. The other plasma parameters are mass ratio \( \mu_\pm = m_-/m_\pm \) (here \( \mu_- = 1 \) and \( \mu_+ = m_-/m_+ = \mu \leq 1 \) (say)), and \( \beta_\pm = v_{th\pm}^2/v_{ph}^2 \). The normalized electric field can be expressed as 

\[
\hat{E} = -\frac{\partial \hat{\phi}}{\partial \psi}, \quad \hat{\phi} = e\phi/m_- v_{ph}^2
\]

is the normalized electric potential. Using this expression for normalized electric field, Eqs. \( \ref{eq:1} \) and \( \ref{eq:3} \) respectively reduces to

\[
\hat{n}_\pm = \frac{1}{(1 - \hat{u}_\pm)}
\]

\[
(1 - \hat{u}_\pm)^2 = \frac{\phi_\pm + \sqrt{\phi_\pm^2 - 4\beta_\pm}}{2}
\]

where we have chosen \( \hat{n}_\pm = 1 \), and \( \hat{\phi} = 0 \) for \( \hat{u}_\pm = 0 \).

In (Eq. \( \ref{eq:9} \)), we have set \( \hat{n}_- = 1 + \beta_- + 2\hat{\phi} \) and \( \hat{\phi} = (1 + \beta_+) + \mu(1 + \beta_-) - \mu \hat{\phi}_- \). Using Eqs. \( \ref{eq:8} \) and \( \ref{eq:9} \), from Eq. \( \ref{eq:7} \) we obtain the following second order ordinary differential equation:

\[
\frac{1}{2} \frac{d^2\phi_-}{d\psi^2} + \frac{\sqrt{2}}{(\phi_+ + \sqrt{\phi_-^2 - 4\beta_-})^{1/2}} \left( \frac{\sqrt{2} \phi_-}{\phi_+ + \sqrt{\phi_-^2 - 4\beta_-}} \right)^{1/2} = 0
\]

Here we have used \( \hat{E} = -(1/2)(d\phi_-/d\psi) \). The above equation may be further rewritten as

\[
\frac{1}{2} \frac{d^2\phi_-}{d\psi^2} + \frac{dU}{d\phi_-} = 0
\]

where \( U(\phi_-) \) is the Sagdeev potential, which is given by

\[
U(\phi_-) = \sqrt{2} \left[ (\sqrt{2} - \xi_2) + \frac{\sqrt{2}\beta_-}{3} \left( 1 - 2\sqrt{2} \xi_2^3 \right) \right] + \frac{\sqrt{2}}{\mu} \left[ (\sqrt{2} - \xi_1) + \frac{\sqrt{2}\beta_+}{3} \left( 1 - 2\sqrt{2} \xi_3^3 \right) \right]
\]
the first value of $U$ up to later one is for warm pair ion plasmas. From Fig. 1, it is the magnitude of wave breaking amplitude depends on mass ratio warm plasmas. It is clear from Eq. (12) that is the expression for wave breaking amplitude in arbitrary mass ratio warm plasmas for some typical values of chosen to be equal to zero at $\hat{\beta}$, i.e., the wave breaking amplitude is $\hat{U}$ equation (11) and is the maximum permissible value of total energy of a fictitious particle obeying the differential equation (11) gives

\[
\xi = (\phi_{\pm} + \frac{\sqrt{\phi_{\pm}^2 - 4\beta_{\pm}}}{2})^{1/2}
\]

and $U(\phi_{-})$ is chosen to be equal to zero at $\hat{\phi} = 0$, i.e., at $\phi_{-} = 1 + \beta_{-}$. It is clear from expression (12) that for real values of $U(\phi_{-})$ the range of $\phi_{-}$ is $\phi_{1} \leq \phi_{-} \leq \phi_{2}$ where $\phi_{1} = 2\beta_{-}^{1/2}$ and $\phi_{2} = 1 + \beta_{-} + (1/\mu)(1 - \beta_{-}^{1/2})^{2}$. Within this range, periodic solutions to (11) may exist. The first integral of equation (11) gives

\[
\hat{E}_{w}^{2} + U = U_{\text{max}},
\]

where $U_{\text{max}}$ is the integration constant indicating the total energy of a fictitious particle obeying the differential equation (11) and is the maximum permissible value of $U(\phi_{-})$. Therefore, the maximum achievable electric field, i.e., the wave breaking amplitude is $\hat{E}_{w} = \sqrt{U_{\text{max}}}$. This is the expression for wave breaking amplitude in arbitrary mass ratio warm plasmas. It is clear from Eq. (12) that the magnitude of wave breaking amplitude depends on the parameters $\mu$ and $\beta_{\pm}$.

Below we explore the wave breaking amplitude in arbitrary mass ratio warm plasmas for some typical values of $\mu$; we have chosen the value of $\mu = 1/1836$ and $\mu = 1.0$, the first value of $\mu$ is for warm electron-ion plasmas and later one is for warm pair ion plasmas. From Fig. 1 it is clear that for $\mu = 1/1836$, periodic solutions are possible upto $U_{\text{max}}$ calculated at $\phi_{-} = \phi_{1}$ for $\beta_{+} \geq \beta_{-}$, which implies that for $\mu = 1/1836$ and $\beta_{+} \geq \beta_{-}$

\[
\hat{E}_{w} = \sqrt{U(\phi_{1})},
\]

where

\[
U(\phi_{1}) = 2 \left[ \frac{1}{\mu} + 1 + \frac{\beta_{+}}{3\mu} + \frac{\beta_{-}}{3} - \frac{4}{3} \beta_{1/4} \right] - \frac{3\sqrt{2}}{\mu} \left( \phi_{2} + \sqrt{\phi_{2}^2 - 4\beta_{-}} \right)^{2} + 4\sqrt{2}\beta_{-}
\]

Here we define

\[
\phi_{*} = \phi_{+}|_{\phi_{-} = \phi_{1}} = (1 + \beta_{+}) + \mu(1 - \beta_{-}^{1/2})^{2}.
\]

Therefore, for $\mu = 1/1836$, wave breaking amplitude does not depend on the relative values of $\beta_{-}$ and $\beta_{+}$. However, for $\mu = 1$, wave breaking amplitude does depend on the relative values of $\beta_{-}$ and $\beta_{+}$ (see Fig. 2). There are two different wave breaking limit for $\mu = 1.0$ depending on whether $\beta_{-} > \beta_{+}$, or $\beta_{-} < \beta_{+}$, i.e., equivalently, $T_{-} > T_{+}$ or $T_{-} < T_{+}$. Fig. 2 shows that, the wave breaking amplitude $\hat{E}_{w} = \sqrt{U(\phi_{2})}$ for $\beta_{-} > \beta_{+}$ i.e., equation (13) with $\mu = 1.0$. In contrast, for $\beta_{-} < \beta_{+}$, Fig. 2 shows that, the wave breaking amplitude $\hat{E}_{w} = \sqrt{U(\phi_{2})}$ where, $\phi_{2} = 1 + \beta_{-} + (1 - \beta_{-}^{1/2})^{2}$. Therefore, the wave breaking amplitude of the plasma wave for $\mu = 1.0$ and $\beta_{-} < \beta_{+}$ is

\[
\hat{E}_{w} = \sqrt{U(\phi_{2})},
\]

where

\[
U(\phi_{2}) = 2 \left[ \frac{1}{\mu} + \frac{1}{\mu} + \frac{\beta_{+}}{3\mu} + \frac{\beta_{-}}{3} - \frac{4}{3} \beta_{1/4} \right] - \frac{3\sqrt{2}}{\mu} \left( \phi_{2} + \sqrt{\phi_{2}^2 - 4\beta_{-}} \right)^{2} + 4\sqrt{2}\beta_{-}
\]

Here we note that wave breaking amplitudes Eq. (14) and Eq. (15) for $\mu = 1.0$ and $\beta_{-} = \beta_{+}$ are same.

For large amplitude electron plasma waves in a warm plasma with immobile ions, from Eq. (13), with $\mu \rightarrow 0$, $\beta_{+} = 0$ and $\beta_{-} = \beta$, we recover Coffey’s limit as

\[
\hat{E}_{w} = \left( 1 - \frac{1}{3} \beta - \frac{8}{3} \beta^{1/4} + 2\beta^{3/2} \right)^{1/2},
\]

When ion dynamics is included, for large amplitude electron-ion waves in cold plasmas, by setting $\beta_{+} = 0$ and $\beta_{-} = 0$ in Eq. (14) we obtain

\[
\hat{E}_{w} = \left[ \frac{2\sqrt{1 + \mu}}{\mu} \left( \sqrt{1 + \mu} - 1 \right) \right]^{1/2}
\]

This is the non-relativistic version of the wave breaking limit derived by Khachatryan [21]. By setting $\mu = 1$
in the above equation, we obtain $\hat{E}_{wb} = 1.08$ for large amplitude waves in pair-ion plasmas.

In Fig. 3 and 4 we have plotted the normalized wave breaking electric field ($\hat{E}_{wb}$) vs. normalized thermal speed $\beta_-(\beta_+)$ respectively given by Eq. (14) and Eq. (15) for fixed $\beta_+(\beta_-)$. In Fig. 3 Coffey’s wave breaking limit ($\mu \to 0$) is compared with the wave breaking limit for ($\mu = 1/1836$) and the wave breaking limit for pair ion plasmas ($\mu = 1.0$). In all the cases, $\hat{E}_{wb}$ decreases with the increase of $\beta_-$ for a fixed $\beta_+$ (for Coffey’s case $\beta_+ = 0$). It is also observed that for a fixed temperature (i.e., $\beta_-$), the wave breaking amplitude mildly increases with increasing $\mu$. This feature has also been reported by Khachatryan [21] for the cold relativistic case. In Fig. 4 we have shown the variation of $\hat{E}_{wb}$ given by Eq. (15) vs. $\beta_+$ for a fixed $\beta_- = 0.0001$.

![FIG. 3: Maximum sustainable amplitude $\hat{E}_{wb}$ vs. $\beta_-$ for $\mu = 0$ (Coffey’s $\hat{E}_{wb}$); fixed $\beta_+ = 0.0001$ for $\mu = 1/1836$ ($\hat{E}_{wb}$ for electron-ion plasmas); fixed $\beta_+ = 0.0001$ for $\mu = 1.0$ ($\hat{E}_{wb}$ for pair-ion plasmas)](image)

![FIG. 4: Maximum sustainable amplitude $\hat{E}_{wb}$ vs. $\beta_+$ for a fixed $\beta_- = 0.0001$ and $\mu = 1.0$.](image)

In conclusion, we have derived the general wave breaking amplitude in nonrelativistic unmagnetized plasmas with finite temperature for both species and different mass ratios. From the general wave breaking amplitude, we have recovered the earlier results derived by Coffey [19] and Khachatryan [21]. It is found that the wave breaking amplitude ($\hat{E}_{wb}$) of a plasma wave in arbitrary mass ratio plasmas depends on the thermal to phase velocity ratio ($\beta_\pm$) and it decreases monotonically with the increase of $\beta_\pm$.

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