Analytical estimation of the Earth’s magnetic field scale

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Abstract

In this paper we analytically estimate the magnetic field scale of planets with physical core conditions similar to that of Earth from a statistical point of view. We evaluate the magnetic field on the basis of the physical parameters of the center of the planet, such as density, temperature, and core size. We look at the contribution of the Peltier-Seebeck effect on the magnetic field, showing that an electrical thermal current can exist in a rotating fluid sphere. Finally, we apply our calculations to Earth and Jupiter. In each case we show that the thermal generation of currents leads to a magnetic field scale comparable to the observed fields of the two planets.

1 Introduction

The Earth’s magnetic field is a fascinating problem that has been faced by many authors and the reader can find rich literature on the topic. For many years the intuitive idea that the magnetic field is generated by heavy fluid in the center of Earth subjected to the rotational motion of our planet, has been conjectured. Many numerical works have started to shed light on the possible mechanism of the generation of Earth’s magnetic field. The basic model for the generation of Earth’s
magnetic field or of other planets, is based upon the dynamo effect of a turbulent convection in rotating fluids. This idea has received much attention in the past few years and many numerical studies based on the dynamo model attempted to reproduce some of the main properties of the magnetism of celestial bodies \[1, 2, 3, 4\], among them the phenomenon of magnetic field reversal. Magnetic field reversal, the phenomenon for which the positions of magnetic north and magnetic south are interchanged, is another important feature of the terrestrial magnetic field that has been studied intensively, and recently a similar phenomenon has been reproduced in the laboratory \[5\].

In a previous work \[6\] the authors showed that a magnetic field can be generated in the laminar region of a fluid velocity under the condition \(\rho = \eta \sigma \mu\), but such a condition is far from the usual condition of the celestial body and, more important, its magnitude would be of the order of \(B \sim \Omega R \sqrt{\rho \mu_0}\) where \(\Omega\) is the rotational velocity, \(R\) is the radius of the outer core, and \(\rho\) is the density of the fluid in the outer core of Earth \[7\]. Inserting Earth’s parameters would imply an intensity field of \(B \sim 10^5\) gauss that is very far from Earth’s actual magnetic field value. Independent from the path that the above value has been obtained, it is a fact that the magnetic field scale \(B \sim \Omega R \sqrt{\rho \mu_0}\) can be deduced by magnetohydrodynamic equations. This is an indication that the Earth’s magnetic field could have a different origin that is not strictly dynamic, since the value appears to be too high when compared with the observed field. The above discussion leads us to the following question: Does there exist for Earth a magnetic field scale as a function of the physical system parameters? We shall show that there exists such a characteristic magnetic field with an intensity that is very close to the actual value of Earth’s field.

2 Magnetohydrodynamic equations

Let us consider the set of equations for a plasma with finite conductivity and constant density \[8, 9\]

\[
\rho \left[ \frac{\partial}{\partial t} \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right] = -\nabla P + [\nabla \times \mathbf{H}] \times \mathbf{B} + \mathbf{f} + \sigma \quad (1)
\]

\[
\nabla \cdot \mathbf{v} = 0 \quad (2)
\]
\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times [\mathbf{v} \times \mathbf{B}] + \frac{1}{\mu \sigma} \nabla^2 \mathbf{B} \tag{3}
\]
\[
\nabla \cdot \mathbf{B} = 0, \tag{4}
\]
where \( \mathbf{v} \) is the flow velocity, \( \mathbf{H} \) is the magnetic field related to the magnetic induction \( \mathbf{B} \) via the relation \( \mathbf{B} = \mu \mathbf{H} \), \( P \) is the pressure of the gas, and \( \rho \) is the mass density. The gravity force density, \( \mathbf{f} \), takes the form \( \mathbf{f} = \rho \nabla \psi \). The vector \( \mathbf{\sigma} \) is defined through its components as follows \[10\]
\[
\sigma_i' = \frac{\partial \sigma'_{ik}}{\partial x_k}, \quad \sigma'_{ik} = \eta \left( \frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} \right), \tag{5}
\]
where \( \sigma'_{ik} \) is the viscous stress tensor, and \( \eta \) is the coefficient of viscosity which is assumed constant. We also used the convention of dropping the symbol of sum for the repeated indexes. The current density \( \mathbf{J} \) is given by the constitutive relation \( \mathbf{J} = \sigma (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \) where \( \sigma \) is the electrical conductivity of the fluid.

The dynamic system has to be implemented using the equation of heat transfer in magnetohydrodynamics \[8\]
\[
\rho c_p \left( \frac{\partial}{\partial t} T + \mathbf{v} \cdot \nabla T \right) = \sigma'_{ik} \frac{\partial v_i}{\partial x_k} + \kappa \nabla^2 T + \frac{J^2}{\sigma} + Q \tag{6}
\]
where \( c_p \) is the specific heat at constant pressure, \( \kappa \) is the thermal conductivity, \( T \) is the temperature of the fluid, and \( Q \) is the quantity of heat generated by external sources of heat contained in a unit volume of the fluid per unit time.

Let us define a dimensionless velocity \( \mathbf{u} = \frac{\mathbf{v}}{(\bar{\Omega} R)} \), a dimensionless operator \( \nabla' = \frac{R}{\bar{\Omega}} \nabla \), and a dimensionless time \( \tau = \frac{\bar{\Omega}}{R} t \). Combining (1) and (5) we can rewrite the hydromagnetics equations as

\[
\frac{\partial}{\partial \tau} \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{R_P} \nabla (p + \bar{\psi}) + \frac{1}{R_B} [\nabla \times \mathbf{b}] \times \mathbf{b} + \frac{1}{\text{Re}} \nabla^2 (\mathbf{u}) \tag{8}
\]
\[
\nabla \cdot \mathbf{u} = 0
\]
\[
\frac{\partial \mathbf{b}}{\partial \tau} = \nabla \times [\mathbf{u} \times \mathbf{b}] + \frac{1}{R_M} \nabla^2 \mathbf{b} \tag{9}
\]
\[
\nabla \cdot \mathbf{b} = 0 \tag{10}
\]
where we defined the dimensionless quantities \( p = \frac{P}{P_0}, \bar{\psi} = \rho \psi / P_0, \mathbf{b} = \frac{\mathbf{B}}{B_0} \) with \( P_0 \) as a characteristic pressure, and \( B_0 \) as a characteristic...
induction field. For brevity we redefined $\nabla'$ as $\nabla$. Finally we defined the numbers

$$Re = \frac{\rho \tilde{\Omega} R^2}{\eta}, \quad R_P = \frac{\rho \tilde{\Omega}^2 R^2}{P_0}, \quad R_B = \mu \frac{\rho \tilde{\Omega}^2 R^2}{B_0^2}, \quad R_M = \sigma \mu \tilde{\Omega} R^2$$

where $Re$ is the Reynolds number, $R_B$ the magnetic force number, and $R_M = \sigma \mu \tilde{\Omega} R^2$ is the Reynolds magnetic number.

As many authors have pointed out the analytical solution of this equation is a very hard task and only a few exact or approximate cases are known (see for example [11] for a review). Our main aim is to find an estimation for $B_0$ as a function of the physical parameters of the center of Earth (such as density, temperature, etc.) without necessarily solving the system (7)-(10).

3 Remarks on field velocity components for a rotating sphere

In this section we will show that for a rotating sphere, in general the $\phi$ component of the velocity is not enough to describe a time dependent motion, with it being understood that $v_\phi = \Omega r \sin \theta$, where $\Omega$ is the angular velocity, is an exact solution. Note that in Ref. [12] is shown evidence that the inner core of the Earth may spin faster than the rest of the planet so that the above exact solution does not hold for the fluid motion of the terrestrial core. This conclusion implies that the temperature distribution can not be a radial distribution due to the fact that Eq. (6) is, in general, coupled with the velocity field. To enforce this statement we shall show that in general a rotating fluid sphere, as previously stated, can not be described by the $\phi$ component of velocity. Let us then assume that a sphere starts to rotate with $v_\phi(t)$ such that $v_\phi(0) = 0$ and $v_\theta = v_r = 0$. We also assume that for symmetry all physical variables do not depend on $\phi$. Writing only the hydrodynamic part of the set of equations (1) - (4), i.e. setting $\mathbf{B} = 0$, we obtain

$$\cot \theta \frac{v_\phi^2}{r} = \frac{1}{\rho r} \frac{\partial P}{\partial \theta}$$

(11)
\[
\frac{v^2_\phi}{r} = \frac{1}{\rho} \frac{\partial P}{\partial r}, \quad (12)
\]
\[
\frac{\partial v_\phi}{\partial t} = \frac{\eta}{\rho} \nabla^2 v_\phi. \quad (13)
\]

Combining Eqs. (11) and (12) we obtain
\[
\cot \theta \frac{\partial}{\partial r} v^2_\phi = \frac{1}{r} \frac{\partial}{\partial \theta} v^2_\phi. \quad (14)
\]

The above equation is satisfied by a function \( v_\phi(r, \theta) \) of the form \( v_\phi(r, \theta) = v_\phi(r \sin \theta) \). Using this result we may rewrite Eq. (13) as
\[
\frac{\partial}{\partial t} v_\phi(t, x) = \frac{\eta}{\rho} \left( \frac{\partial^2}{\partial x^2} + \frac{1}{x} \frac{\partial}{\partial x} - \frac{1}{x^2} \right) v_\phi(t, x), \quad x \equiv r \sin \theta. \quad (15)
\]

The general solution of the above equation, via Laplace transform and with the condition \( v_\phi(0, x) = 0 \), is
\[
\hat{v}_\phi(s, x) = aI_1 \left( \frac{x \sqrt{s}}{\sqrt{D}} \right) + bK_1 \left( \frac{x \sqrt{s}}{\sqrt{D}} \right) \quad (16)
\]
where, by definition, \( \hat{v}_\phi(s, x) = \int_0^\infty \exp[-st]v_\phi(t, x)dt \) and \( D = \eta/\rho \).

Assuming a finite solution for \( r \to 0 \) and consequently \( x \to 0 \), then \( b = 0 \). It is evident by the form of Eq. (16) that the boundary condition for the velocity, i.e. \( \hat{v}_\phi(s, R \sin \theta) = aI_1 \left( R \sin \theta \sqrt{s/D} \right) \) where \( R \) is the sphere radius, can be satisfied only by a restricted class of functions.

One could consider adding another component, for example the radial component \( v_r \), to describe matter falling in the center. But the independency of the dynamic by the angular variable \( \phi \) implies that the \( \theta \) component of the velocity \( v_\theta \) also has to be considered. This fact comes directly from the continuity equation. Let us assume that there is another component of velocity, the radial component \( v_r \). The continuity equation is
\[
\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) = 0. \quad (17)
\]

The solution \( v_r = f(\theta)/r^2 \) diverges at the origin and can not vanish on the surface of the sphere \( r = R \). To avoid this inconsistency we are forced to add \( v_\theta \) to the flow. From this we can infer that from the early stages of Earth’s formation to the present, the velocity of the
fluid could not be described by only one component of the velocity vector. This analytical conclusion is in agreement with the numerical works presented in the references.

4 Thermal generation of magnetic field

Looking at Eq. (3), or Eq. (8), we notice that is a diffusive-like equation. For the time evolution of $B$ it is important to give an estimation of its initial value. It is accepted that the Earth’s core is made mainly of iron with a solid inner core the size $10^3$ km and an outer core of liquid about $2 \times 10^3$ km thick [7]. The temperature distribution of the core is not uniform and it ranges from approximatively $10^4$ °K at the very center to $10^3$ °K at the surface of the outer core. The non uniform temperature can generate a contribution to the electrical current that is proportional to the gradient of the temperature, known as the Peltier—Seebeck effect, so that we can write the total current as [13]

$$J = \sigma [E + v \times B - \alpha(T)\nabla T].$$

Note that Eq. (3) does not change if $\alpha(T)\nabla T$ can be written as a gradient of a function. To evaluate the coefficient $\alpha$, we have to consider the fact that the density of either the solid inner core, or the fluid outer core, is such that the electrons can be considered a degenerate Fermi’s gas. Indeed, according to the authors of Ref. [14] the core density is of the order of $10^4$ Kg m$^{-3}$. This implies that the Fermi energy of the electrons of the Earth’s core is

$$\varepsilon_F = \frac{\hbar^2}{2m} \left( \frac{3\pi^2 N}{V} \right)^{2/3} \approx 2 \times 10^{-18} \text{ Joule}$$

(19)

corresponding to a Fermi temperature $T_F \approx 1.4 \times 10^5$°K. This is at least one order of magnitude higher than the Earth’s core temperature therefore justifying the degenerate Fermi’s gas approximation. Quantum calculations show that [13]

$$|\alpha(T)| \sim k_B \frac{k_B T}{e\varepsilon_F}$$

(20)

where $k_B$ is the Boltzmann constant and $e$ is the electron charge. Let us consider a very simplified model of the early stage’s of Earth’s formation. Models suggest that Earth’s core was completely molten [15].
Since the temperature change of the core has a time scale on the order of Earth’s age [16] we infer that whatever is the contribution from the thermal term in Eq. (18) this contribution still holds today with the same order of magnitude. A widely accepted estimation of the core temperature is approximately $8 \times 10^3$ °K for the inner core and $4 \times 10^3$ °K for the outer core (see for example [14, 17, 18]). Let us consider the contribution to the magnetic field due to thermal term. From Maxwell’s equation we obtain

$$\nabla \times \mathbf{B} = \mu_0 \sigma \mathbf{J} = -\mu_0 \sigma \alpha(T) \nabla T.$$ \hspace{1cm} (18)

In general, the temperature distribution in time and space is coupled with the velocity field, Eq. (6), and as shown in the previous section, all components of the velocity are present so that the temperature distribution can not be only radial. Note also that we use $\mu_0$ as the value of the magnetic permeability since at such temperature we assume that there is no magnetization. We can deduce the field scale via the relation

$$\frac{B}{R} \approx \mu_0 \sigma \alpha(T) \frac{\Delta T}{R}$$ \hspace{1cm} (19)

where $\Delta T$ is the difference in temperature, and $R$ is the length scale of the system. We obtain the scale strength of the thermal magnetic field

$$B_T = \mu_0 \sigma \frac{k_B T_c}{\epsilon \varphi_F} k_B \Delta T.$$ \hspace{1cm} (20)

Using the value of temperature $T_c \approx 8 \times 10^3$ °K, $\Delta T \approx 4 \times 10^3$ °K, and $\sigma \approx 10^5$ S m$^{-1}$ [19] we obtain the numerical value for the strength of the core of Earth’s magnetic field

$$B_T \approx 0.0024 \text{ tesla} = 24 \text{ gauss}.$$ \hspace{1cm} (21)

The estimated strength of the core of Earth’s magnetic field is approximately $B_{est} \sim 25$ gauss [20] which is very close to the analytical value given by Eq. (20). We note that selecting different values for the temperature, according to the different models present in the literature, the value of the field would of course change consequently, but the scale magnitude remains of the order of tens of gauss. The thermal current gives a strong (if not total) contribution to the magnetic
field. It is worthy to stress that in principle the only phenomenological parameter, i.e. the conductivity $\sigma$, could be evaluated using quantum mechanics [21] so that we can conclude that the field given by the expression (22) may be written in terms of fundamental constant and physical parameters of the system such as density $N/V$ and temperature $T$. Note that dependence on the radius $R$ of the fluid region is implicit in the dependence of the temperature on $R$. This is the main reason why we kept explicit the temperature difference $\Delta T$ in Eq. (22). Once we clarify this, we can rewrite more concisely Eq. (22) as

$$B_T = \mu_0 \sigma \frac{(k_B \bar{T})^2}{e \varepsilon_F}$$

(24)

where $\bar{T}$ can be taken, for example, as the average temperature of the fluid region. We conclude that Eq. (24) represents the scale of the strength of the magnetic field of celestial bodies with an Earth-like physical condition for the core, from a statistical point of view.

5 Jupiter’s magnetic field estimation

In principle we can apply the ideas of the previous section to other celestial bodies, particularly in our solar system. The main difficulty with this is the scarcity of information about the physical internal condition of other planets, although we can make some general considerations. For example, Mercury and Mars are quite smaller than Earth. This fact surely contributes to a faster cooling of their interiors so we can expect that the cores of these planets are no longer in the fluid state. In fact the two planets have a very weak magnetic field.

Venus does not have a magnetic field [24] and there are several possible explanations for this. Venus is a planet very similar to Earth in dimension but it does not exhibit volcanic activity, and this could imply a cold core. Venus has a very slow rotational motion compared to Earth, and a rotational motion is considered to play a crucial role for the terrestrial magnetic field. Also we should consider the possibility that Venus could be in a reversal phase.

Jupiter is a good candidate to test our model. Even though little is known about the planet, its internal structure has been modeled by several authors (see for example [25, 27, 26]) and the physical informa-
tion is enough to allow a rough estimation of the scale of its magnetic field using Eq. (22). The electrical conductivity is \( \sigma \approx 10^5 \text{ S m}^{-1} \) [26], its temperature ranges from \( T \approx 2 \times 10^{45} \text{K} \) for the core boundary to \( T \approx 10^{46} \text{K} \) for the metallic hydrogen boundary, and the estimated density of the metallic hydrogen is \( \rho \approx 4 \times 10^3 \text{Kg m}^{-3} \) [25, 27]. Consequently the Fermi’s energy takes the value \( \varepsilon_F \approx 10^{-17} \text{Joule} \) corresponding to a Fermi temperature \( T_F \approx 7 \times 10^5 \text{K} \) so that we can apply the Fermi statistic for the electrons in the metallic region. Plugging these values into Eq. (22) we obtain for the magnetic field of Jupiter an estimation of its strength in the metallic hydrogen region \( B_J \approx 30 \text{ gauss} \). Taking into account that this region extends for a fraction, ranging from 0.7 to 0.78 times the Jupiter radius, we obtain for the surface value of the magnetic field a range from \( B_{JS} \approx B_J(0.7)^3 \approx 10 \text{ gauss} \) to \( B_{JS} \approx B_J(0.78)^3 \approx 14 \text{ gauss} \). This is in agreement with the scale of the observed values [28]. As said for Earth, according to several models, we can change the values of the parameters, but the strength of the field would still be of the order of the observed field.

6 Conclusion

We provided an analytical estimation of the magnetic field scale of planets with physical core conditions similar to that of Earth from a statistical point of view. The magnetic field strength was evaluated directly from the physical parameters of the center of the planet, considering density, temperature and core’s size. We showed that an electrical current generated by a thermal gradient can exist in a rotating fluid sphere and can give an important contribution to the magnetic field. Our conjecture was supported by estimating the magnetic field strengths of Earth and Jupiter that were in agreement with the observed magnetic field intensity of the two planets.

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