Abstract

This paper defines new intersection homology groups. The basic idea is this. Ordinary homology is locally trivial. Intersection homology is not. It may have significant local cycles. A local-global cycle is defined to be a family of such local cycles that is, at the same time, a global cycle. The motivating problem is the numerical characterisation of the flag vectors of convex polytopes. Central is a study of the cycles on a cone and a cylinder, in terms of those on the base. This leads to the topological definition of local-global intersection homology, and a formula for the expected Betti numbers of toric varieties. Various related questions are also discussed.

1 Introduction

This paper defines new intersection homology groups. They record, in a global way, local information about the singularities. They give rise to new information, both globally and locally, and vanish on nonsingular varieties. Such groups are required, to obtain a satisfactory understanding of general convex polytopes. They also have other applications.

The basic idea is this. Ordinary homology is locally trivial. Intersection homology is not. It may have significant local cycles. A local-global cycle is a family of such local cycles that is, at the same time, a global cycle. The chains, that produce the homology relations between the cycles, are to have a similar local-global nature.

The theory of toric algebraic varieties, which associates an algebraic variety \( P_\Delta \) to each convex polytope \( \Delta \) (provided \( \Delta \) has rational vertices) establishes a dictionary between convex polytopes and algebraic varieties. Convex polytopes (or, if one prefers, the associated varieties) provide the simplest examples for these new concepts.

The basic problem is to understand general polytopes in the same way as simple polytopes are already understood. Suppose \( \Delta \) is a simple polytope. Loosely speaking, this means that the associated variety \( P_\Delta \) is nonsingular. The associated homology ring \( H_\bullet \Delta \) has the following properties. It is generated by the facets of \( \Delta \). It satisfies the Poincaré duality and strong Lefschetz theorems. The associated Betti numbers \( h_\Delta \) are a linear function of the face vector \( f_\Delta \), and vice versa.

These facts are central to Stanley’s proof \([24]\) of the necessity of McMullen’s numerical conditions \([20]\) on the face vectors of simple polytopes. (An ingenious construction of Billera and Lee \([4]\) proves sufficiency.) One would like to understand general polytopes in a similar way.

The first results in this direction are due to Bayer and Billera \([2]\). They consider the flag vector, not the face vector. For simple polytopes Poincaré duality represents what are known as the Dehn-Somerville equations on the face vector. Bayer and Billera describe the generalised Dehn-Somerville
equations on the flag vector. They also show that the flag vectors of \( n \)-dimensional polytopes span a space whose dimension is the \((n+1)\)st Fibonacci number \( F_{n+1} \).

The problem of characterizing the flag vectors of general polytopes has guided the development of local-global intersection homology. The usual middle perversity intersection homology theory produces \( \lceil n/2 \rceil + 1 \) independent independent linear functions of the flag vector. (This is the Bernstein-Khovanskii-MacPherson formula for the mpih Betti numbers \([6, 7, 25]\).) Clearly, more Betti numbers are needed, to record the whole of the flag vector. In addition, some analogue or extension to the usual ring structure on the homology of a nonsingular variety is required.

The general polytope problem makes it clear that some extension of intersection homology, and of the ring structure, is required. Topology by itself has failed to indicate clearly either the need for such an extension, or its form. (There are intersection homology groups for non-middle perversities, and ‘change of perversity’ groups, but these have the same problems as ordinary homology.) Finally, there are polytopes whose combinatorial structure is such that it cannot be realised with a polytope that has rational vertices \([13, \text{p94}]\). Thus, for general polytopes a theory that does not rely on algebraic geometry is required.

The root problem of this paper is as follows. Suppose \( Z \) is a possibly singular projective algebraic variety. In terms of the cycles on \( Z \), what are the cycles on \( CZ \) and \( IZ \)? Here, \( CZ \) is the projective cone on \( Z \), while \( IZ \) is the product of \( Z \) with \( P_1 \). The answer depends on what one understands a cycle to be, or in other words on some perhaps implicit choice of a homology theory. Suppose this question has been answered. One will then have a wide range of examples. These will determine the corresponding definition of a cycle, in the same way that a number of points will determine a plane.

These examples will also determine a linear function \( h\Delta \) of the flag vector \( f\Delta \) of convex polytopes. This is because of the following. There are operators \( I \) and \( C \) on polytopes, analogous to the \( I \) and \( C \) operators on varieties. An \( IC \) polytope is any polytope that can be obtained by repeatedly applying \( I \) and \( C \) to the point polytope. Any polytope flag vector can be written as a linear combination of the flag vectors of \( IC \) polytopes. The examples determine \( h\Delta \) on the \( IC \) polytopes.

This paper is organised as follows. First (§2) notation and definitions are established, and some basic results stated. Next (§3) the root problem is discussed and a solution presented. This serves to motivate the definition of the (extended) \( h \)-vector \( h\Delta \) of convex polytopes (§4), and the topological definition of local-global intersection homology (§5). To finish (§6), there is a summary, and a discussion of related questions.

This paper considers the topological and combinatorial aspects of local-global homology. There are others, to be presented elsewhere. The linear algebra \([9]\) allows \( h\Delta \) to be interpreted as the outcome of a ‘vector weighted inclusion-exclusion’ construction. The intersection theory \([10]\) provides a structure that reduces, in the simple or nonsingular case, to the ring structure on ordinary homology.

To date, the theory of local-global intersection homology consists of a series of definitions appropriate for each of the four aspects, together with examples and special cases, and various linking results. Much remains to be done, to fill in the ‘convex hull’ of the four aspects.

This paper has been written to be accessible to those who are unfamiliar with perhaps one or both of intersection homology and the combinatorics of convex polytopes. The reader who is in a hurry can find a summary in the final section. Formulae (1)–(7) define the extended \( h \)-vector. The topological definition is in §5. Text in parentheses (except for short comments) can be omitted on a first reading. The reader who is having difficulties should first understand the mpih part of the theory (i.e. ignore terms involving any of \( A \), \( \bar{A} \) or \( \{k\} \)).
2 Preliminaries

This section introduces notation and conventions. It also states results to be used later. This material is organised into six topics, namely language and conventions, cones and cylinders, local homology, the strong Lefschetz theorem, polytope flag vectors, and the index set for $h\Delta$.

First, language and conventions. Ordinary homology fails to have suitable properties, and so the word ‘homology’ when used without qualification will refer either to middle perversity intersection homology (mpih), or some local-global variant thereof. The abbreviation mpih will always refer to the usual intersection homology (as in [11]), with of course middle perversity. Each local-global homology group has an order, usually denoted by $r$. The usual mpih groups are order zero local-global groups. The higher order groups will be called strictly local-global. Unless otherwise stated, homology will always be with rational (or real) coefficients.

The analogy between convex polytopes and algebraic varieties is very important, particularly in §3 and §4. Throughout $\Delta$ will be a convex polytope of dimension $n$, and $Z$ a (projective) algebraic variety, also of dimension $n$. When $\Delta$ has rational vertices a projective toric variety $P_\Delta$ (of dimension $n$) can be constructed (as in, say, [3]). If $\Delta$ is the $n$-simplex $\sigma_n$ then $P_\Delta$ is projective $n$-space $\mathbb{P}_n$.

To strengthen the analogy, for toric varieties homology will be indexed by the complex dimension (half the normally used real dimension). The mpih Betti numbers of $P_\Delta$ are zero in the odd (real) dimensions, and the same is expected to hold for the local-global extension. Thus, this indexing convention amounts to ignoring the homology groups that are expected in any case to be zero.

The concept of a cone is one of the most important in this paper. In fact, the same word will be used for three closely related constructions, that apply respectively to topological spaces, projective algebraic varieties, and convex polytopes.

Suppose $B$ is a topological space. The cylinder $IB$ on $B$ is the product $[0,1] \times B$ of $B$ with the interval $I = [0,1]$, equipped with the product topology. If $p = (\mu, l)$ is a point of $IB$ and $\lambda \in I$ is a scalar then $\lambda p = (\lambda \mu, l)$ is also a point on $IB$. The cone $CB$ is the cylinder $IB$, with $\{0\} \times B$ identified (collapsed) to a single point, the apex of the cone. In §5, local-global cycles will be described as global cycles that can be collapsed in some specified way. The locus $\{1\} \times B$ is called the base of $CB$. There is a $\lambda$-action on $CB$ also.

Now suppose $Z \subset P_N$ is a projective algebraic variety. The (projective) cone $CZ \subset P_{N+1}$ with base $Z$ is constructed as follows. Each point $p \in P_N$ represents a line $l_p$ through the origin in $\mathbb{A}^{N+1}$. A point $v$ lies in the affine cone $\tilde{Z} \subset \mathbb{A}^{N+1}$ just in case it lies on some $l_z$, with $z \in Z$. A ‘hyperplane at infinity’ can be added to $\mathbb{A}^{N+1}$, to produce $P_{N+1}$. This hyperplane is a copy of $P_N$. The cone $CZ$ is the closure of $\tilde{Z}$. The base of $CZ$ is the copy $\{\infty\} \times Z$ of $Z$ that lies on the $P_N$ at infinity. The origin of $\mathbb{A}^{N+1}$ is the apex of the cone.

The interaction between the cone structure and relations among cycles is central to §3. The complex numbers act by multiplication on the ‘finite part’ $\tilde{Z}$ of $CZ$. Thus, if $\eta$ is a cycle on $CZ$, lying entirely on $\tilde{Z}$, it can as be ‘coned away’ by the $\lambda$-cone $C\lambda \eta$, where $\lambda$ ranges over $[0,1]$. This is a chain whose boundary is $\eta$ (unless $\eta$ has dimension zero). Now suppose $\eta$ on $CZ$ avoids the apex. In this case each point of $\eta$ lies on a unique line through the apex, and so there is a boundary that ‘moves’ $\eta$ to an equivalent cycle $\eta_\infty$ that lies entirely on the base $Z$ of $CZ$. Finally, suppose that $\eta$ is a cycle lying on the base $Z$ of $CZ$. Each point of $\eta$ determines a line in $CZ$, and so $\eta$ determines a cycle $C\eta$ on $CZ$. However, it may not be possible to find an $\eta'$ lying entirely on $\tilde{Z}$, that is equivalent to $\eta$.

(The reason for this is subtle. If it were always possible, then it would be possible to ‘cone away’ the cycle due to the hyperplane at infinity in $P_{n+1}$ (the cone on $P_n$). But this cycle is not homologous to zero. Although $\eta$ can locally be moved away from the base of $CZ$, in a manner that
is unique up to ‘phase’, it may not be possible to get all the phases to match up.)

The cylinder \(IZ\) is the product of \(Z\) with \(P_1\), which via the Segre embedding is to be thought of as a subvariety of \(P_{2N+1}\). The variety \(Z\) is the base of the cylinder. If \(Z\) is nonsingular, then so is \(IZ\), whereas \(CZ\) will in general have a singularity at its apex.

Analogous operators \(I\) and \(C\) can be defined for convex polytopes. If \(\Delta\) is a convex polytope then the cone (or pyramid) with base \(\Delta\) is the convex hull of \(\Delta\) with a point (the apex) that does not lie in the affine span of \(\Delta\). Similarly, the cylinder (or prism) \(I\Delta\) with base \(\Delta\) is the Cartesian product \([0,1] \times \Delta\) of \(\Delta\) with an interval \(I = [0,1]\). These operators respect the dictionary between convex polytopes and toric algebraic varieties.

The symbol ‘\(\bullet\)’ will be used to denote both the projective variety \(P_0\) (a single point), and the single point convex polytope. Thus, \(ICC\bullet\) can denote either \(P_1 \times P_2\) or a triangular prism. An \(IC\) polytope is one obtained by successively applying \(I\) and \(C\) to the point polytope, and similarly for an \(IC\) variety. For every word in \(I\) and \(C\), the latter is the toric variety associated to the former. Sometimes the two concepts will be identified.

In both cases one can also define the join of two objects. Suppose that \(Z_1\) and \(Z_2\) are subvarieties of some \(P_N\), and that their affine linear spans are disjoint. In that case, their join consists of all points that lie on some line \(l(z_1, z_2)\) that joins a point \(z_1 \in Z_1\) to another \(z_2 \in Z_2\). Similarly, if \(\Delta_1\) and \(\Delta_2\) have disjoint affine linear spans, then their join is the convex hull of \(\Delta_1 \cup \Delta_2\). For both polytopes and varieties, a cone is the join of the base to the apex. One could also join an object not to a point but to a projective line (respectively, an interval). This is the same as forming the cone on the cone. It will have an apex line (resp. apex edge) rather than an apex.

Intersection homology differs from ordinary homology in that for it nontrivial local cycles can exist. If \(s\) is a point on a complex algebraic variety \(Z\) any sufficiently small ball centered at \(s\) is homeomorphic to the (topological) cone \(CL_s\), whose boundary is \(s\). That something, which does not depend on the sufficiently small ball, is the link \(L_s\) at \(s\). Now suppose \(\eta\) is a cycle on \(CL_s\). The \(\lambda\)-action on a cone can then be applied to \(\eta\), to produce a chain \(C_\lambda \eta\), whose boundary is \(\eta\). (Strictly speaking, this is true only if \(\dim \eta > 0\).) Local cycles of dimension zero are trivial, and will not be counted by \(h\Delta\). Ordinary homology allows this coning away of local cycles. The perversity conditions of intersection homology however can be used to prohibit the use of \(C_\lambda \eta\) to generate a boundary.

The local (intersection) homology groups can be defined as follows. A local cycle \(\eta\) at \(s\) consists of a cycle \(\eta_U\) for any sufficiently small open set \(U\) containing \(s\), such that if \(U' \subset U\), one has on \(U\) that \(\eta_U\) and \(\eta_{U'}\) are homologous. Similarly, a local boundary \(\xi\) at \(s\) consists of a chain \(\xi_U\) on each sufficiently small open set \(U\), whose boundary \(\eta\) (the system \(\eta_U = d\xi_U\)) is a local cycle. This definition avoids use of the cone structure. In §4, another definition will be given.

If \(Z\) is a complex algebraic variety then it can be decomposed into a disjoint union of strata \(S_i\), where each \(S_i\) is either empty or has complex dimension \(i\), and along \(S_i\) the local topology of \(Z\) is locally constant. From this it follows that the local homology groups are also locally constant along \(S_i\), and so form what is known as a local system. This concept is used only in §5. However, it is closely related to an example of local-global homology.

(This paragraph and the next can be omitted on a first reading.) Suppose that \(i < j\) and that the stratum \(S_i\) is in the closure of \(S_j\). More particularly, suppose that \(\gamma : [0,1] \rightarrow Z\) is a path, with \(\gamma(0) \in S_i\) and \(\gamma(\lambda) \in S_j\) otherwise. Now let \(\eta\) be a local cycle at \(\gamma(1)\). By local constancy, it can be moved along \(\gamma\) until it is very close to \(\gamma(0)\). At this point the translate \(\eta_\lambda\) of \(\eta\) can be thought of as a local cycle at \(\gamma(0)\) on \(S_i\). In other words, each path from \(S_j\) to \(S_i\) (with \(j > i\)) transfers local cycles from \(S_j\) to \(S_i\). Note that the reverse process will not in general be possible. For example, if \(S_0\) is an isolated singularity, then a local cycle \(\eta\) at \(S_0\) cannot be moved away from \(S_0\).

Now consider \(H_0(S_i, L_i)\), where \(L_i\) is the local system formed from the local homology groups
along $S_i$. A cycle $\eta \in H_0(S_i, L_i)$ is a formal sum of local homology cycles (about points on $S_i$) subject to the equivalence due to motion along paths. As already described, these groups can be ‘glued together’ (certain elements identified) for different values of $i$. Provided one uses all $S_i$ whose dimension is at least some value value $j$, the result is independent of the stratification. (This is left to the reader. Then main point is that new strata have real codimension at least two, and so existing paths can be altered to avoid new strata.) These groups are examples of local-global homology.

The strong Lefschetz theorem is one of the central results in the homology of nonsingular algebraic varieties. It was stated by Lefschetz in 1924, but his proof was not satisfactory. The first proof is due to Hodge (1933–6, see [15, p117]). It also follows from Deligne’s proof of the Weil conjectures [17]. Strong Lefschetz also holds for middle perversity intersection homology. Here, Deligne’s proof is the only method known. For more background see [18]. The infinitesimal form of Minkowski’s facet area theorem for polytopes [22, 13, p.332] is a special case of both strong Lefschetz and the Riemann-Hodge inequalities. This seems not to have been noticed before.

Suppose $Z \subset \mathbf{P}_N$ is a projective algebraic variety. For convenience, complex dimension will be used to index its homology groups $H_i(Z)$. If $i + j = n$ (the dimension of $Z$), then by Poincaré duality $H_i(Z)$ and $H_j(Z)$ have the same dimension, for they are dual vector spaces. The embedding $Z \subset \mathbf{P}_N$ determines a hyperplane class $\omega = \omega_Z$ in $H_{n-1}(Z)$ with the following properties. First, the cap product $\omega \cap \eta$ is defined for any homology class $\eta$ on $Z$. This operation lowers degree by one. Now assume $i < j$ and also $i + j = n$. The strong Lefschetz theorem asserts that the map

$$\omega^{j-i} : H_j(Z) \to H_i(Z)$$

is an isomorphism.

This result provides a decomposition of $H_0(Z)$. Suppose that the above isomorphism takes $\eta$ to $\eta' = \omega^{j-i} \eta$. Say that $\eta$ is primitive and $\eta'$ is coprimitive, if $\omega \eta'$ is zero. It is a standard result, that $H_\bullet(Z)$ is the direct sum of $\omega^i P_j(Z)$, where $P_j(Z) \subseteq H_j(Z)$ are the primitive classes, and $i + j \leq n$. The Lefschetz isomorphism allows an ‘inverse’ $\omega^{-1}$ to $\omega$ to be defined. Define $\omega^{-1}$ to be the result of first applying the inverse of the Lefschetz isomorphism, then $\omega$, and then the Lefschetz isomorphism. It has degree $-1$. (The Riemann-Hodge inequality is that on $P_j(Z)$ the quadratic form $\omega \cap \omega^{j-i} \cap \eta$ is negative definite.)

The primitive and coprimitive cycles have a special rôle in the study of the root problem, namely the cycles on $CZ$ in terms of those on $Z$. They can occur only in certain dimensions, for which it is useful to have special adjectives. Say that a cycle $\eta$ on $Z$ is upper (respectively strictly upper) if its dimension is at least (resp. more than) half that of $Z$. Similarly, at most (resp. less than) define lower (resp. strictly lower). Primitives and coprimitives occur in the upper and lower dimensions respectively. A cycle dimension that is not lower is strictly upper, and vice versa. The middle dimension is both upper and lower.

The hyperplane class $\omega_Z$ on $Z$ can be represented as a Weil divisor (formal sum of codimension one subvarieties) on $Z$, namely the hyperplane section. For use in §3, note that the hyperplane class $\omega_{CZ}$ on a cone can be represented either as the cone $C\omega_Z$ on the class of the base, or as the base $Z$ of the cone (by intersecting $CZ \subset \mathbf{P}_{N+1}$ with the $\mathbf{P}_N$ at infinity).

If $\Delta$ is a simple convex polytope (this means that at each vertex there are $n = \dim \Delta$ edges) then $\mathbf{P}_\Delta$ behaves like a nonsingular algebraic variety, so far as its homology (with rational coefficients) is concerned. Its Betti numbers $h_i(\Delta) = h_i(\mathbf{P}_\Delta)$ are then a linear function of the face vector $f = f(\Delta) = (f_0, f_1, \ldots, f_n)$, where $f_i$ is the number of $i$-dimensional faces on $\Delta$. In fact, if one writes $f(x,y) = \sum f_i x^i y^{n-i}$, and $h(x,y)$ similarly, then the equation $h(x, x + y) = f(x, y)$ expresses the relation between $f$ and $h$. 


If \( \Delta \) is a general convex polytope, then flags should be counted. A \textit{flag} is a sequence

\[ \delta = (\delta_1 \subset \delta_2 \subset \ldots \subset \delta_r \subset \Delta) \]

of faces, each strictly contained in the next. Its \textit{dimension vector} (or \textit{dimension} for short) is the sequence

\[ d = (d_1 < d_2 < \ldots < d_r < n) \]

of the dimensions \( d_i \) of its terms \( \delta_i \). Altogether, there are \( 2^n \) possible flag dimensions. The component \( f_d \Delta \) of the \textit{flag vector} \( f = f\Delta \) of \( \Delta \) counts how many flags there on \( \Delta \), whose dimension is \( d \). (If \( \Delta \) is simple, the flag vector is a linear function of the face vector, and so contains no new information.)

For simple polytopes the \textit{Dehn-Somerville} equations state that \( h(x, y) \) is equal to \( h(y, x) \), or that the \( h \)-vector is \textit{palindromic}. (It is analogous to Poincaré duality.) For general polytopes the \textit{generalised Dehn-Somerville (gDS) equations} \[2\] imply that \( f\Delta \) has the Fibonacci number \( F_{n+1} \) linearly independent components. A similarly elegant interpretation of these equations is lacking.

For flag vectors the \textit{IC equation} \[8\]

\[ (I - C)C I = I (I - C)C \]

holds, in the following sense. Apply both sides to a convex polytope \( \Delta \), to obtain convex polytopes \( IC\Delta \) etc. The corresponding equation then holds among the flag vectors of these polytopes. The flag vectors of the \( IC \) polytopes span all polytope flag vectors, and those than contain neither ‘\( II \)’ nor ‘\( I\bullet \)’ form a basis.

It follows that if linear operators \( \bar{I} \) and \( \bar{C} \) are given that satisfy the \( IC \) equation, together with an initial value \( \bar{h}(\bullet) \) for which \( \bar{I} \bar{h}(\bullet) = \bar{C}h(\bullet) \), then there is a unique linear function \( \bar{h} \) on polytope flag vectors, for which the equations

\[ \bar{h}(IC\Delta) = \bar{I}h(\Delta); \quad \bar{h}(C\Delta) = \bar{C}h(\Delta); \]

are satisfied. This is used in §4, to define the extended \( h \)-vector.

Finally, note that the polytope flag vectors span a proper subspace of the span of all flag vectors. To provide a linear function on this subspace is not the same as to provide such on the larger space. Conversely, different linear functions on the larger space can agree on the subspace. Related to this is the idea that equivalent homology theories (on projective varieties) can be given different definitions, and that different triangulations can be found for a given space.

The last topic is the \textit{index set}. The extended \( h \)-vector \( h\Delta \), to be defined in §4, will be a formal sum of terms, of a particular type. Although the terms to be used will arise in a natural way, it is convenient to gather in one place a description of them. In some sense, their structure is a result of a geometric requirement (that the Betti numbers be organised into sequences, whose length grows with the dimension \( n \) of the polytope) and a combinatorial requirement (the Fibonacci numbers).

First, expressions such as \( (a, b, c) \) will stand for the homogeneous polynomial \( ax^2 + bxy + cy^2 \) in commuting variables \( x \) and \( y \). To save space, commas will where possible be omitted. Thus, \( (10) = x, (01) = y, (11) = x + y \) and \( (1) = 1 \). Similarly, the expression \( [abc] \) (short for \( [a, b, c] \)) will stand for \( aX^2 + bXY + cY^2 \), where \( X \) and \( Y \) are a different pair of commuting variables. Clearly, \( [1] = (1) = 1 \). Each of \( x, y, X \) and \( Y \) will have degree one.

Roughly speaking, to each coprimitive cycle on the base of a cone (or in the link along a face) the symbol \( \{k\} \) will correspond, where \( k \) is the dimension of the cycle. This symbol also corresponds to a local \( k \)-cycle. Because in the present context such can occur only in the strictly lower dimensions, \( \{k\} \) will be given degree \( 2k + 1 \). The symbol \( \{0\} \) will not be used. This corresponds to treating the
class of a point as a trivial local cycle. The difference $b - a$, which 'counts' dimension 1 coprimitives, will be denoted by $b'$. Similarly, $c' = c - b$, and so on up to halfway. In addition, a 'padding' symbol $A$ (or $\overline{A}$) is required. It has degree one.

The extended $h$-vector $h\Delta$ will be a sum of terms of the form $x^iy^jW$, where $W$ is a word in $A$ and $\{k\}$. Each such term will have degree equal to the dimension of $\Delta$. The word $W$ is allowed to be empty. This corresponds to the $\text{mpih}$ part of $h\Delta$. The last symbol in $W$ is not to be an $A$. (This can be achieved by supposing that there is a terminating symbol 'one prefers) of $\Delta$. It follows from the flag vector concept that $f$ can be interpreted as follows. Define $f(= A$ and setting $(This can be achieved by supposing that there is a terminating symbol ' at the end of each word, and setting $A$ equal to zero.) An auxiliary vector $h\Delta$ is used in §4. It is a sum of $X^i Y^j W$ terms, where $W$ is a word in $A$ and $\{k\}$. Its terms are otherwise the same as those of $h\Delta$.

The numerology of $x^i y^j W$ is interesting. Recall that $h\Delta$ has $F_{n+1}$ independent components. There are $F_{n+2}$ terms satisfying the above conditions, whose degree is $n$. Of these $F_{n+1}$ satisfy $i \leq j$, (and they correspond to a maximal set of independent components in $h\Delta$). Thus, $F_n (= F_{n+2} - F_{n+1})$ terms satisfy $i > j$. Similarly, $F_n$ terms satisfy $j > i$. Thus, $F_{n-1}$ terms satisfy $i = j$. The number of words $W$ of degree at most $n$ is $F_n$, for $n \geq 1$. (These results are not used, and so are stated without proof.)

Also, the equation

$$1 + F_1 + F_2 + \ldots + F_n = F_{n+2}$$

can be interpreted as follows. Define $f(\Delta)$ to be the sum of the flag vectors of the $i$-faces (or $i$-links if one prefers) of $\Delta$. It follows from the flag vector concept that $f(\Delta)$ and $f(\bullet) = (1, f(1), f(2), \ldots, f(n-1))$ are linear functions of each other. (The '1' corresponds to the 'empty' face, or to $\Delta$ itself.) Each $f(i)$ has $F_{i+1}$ independent components, and so $f(\bullet)\Delta$ (and hence $f(\Delta)$) has $F_{n+2}$ components whose dependence does not follow from the gDS equations on the faces. This helps justify $F_{n+2}$ as the number of components in $h\Delta$, for one wishes $h\Delta$ to permit an elegant expression of the generalised Dehn-Somerville equations.

### 3 Cycles on cones and cylinders

This section describes the $\text{mpih}$ and local-global cycles on a cone and a cylinder in terms of those on the base. First, the $\text{mpih}$ cycles are constructed and described. The local-global cycles are then a variant of the $\text{mpih}$ cycles. They make use of information, that $\text{mpih}$ ignores.

Suppose that $\eta$ is a cycle on $Z$. Later, this statement will acquire a richer meaning, but for now suppose that $\eta$ is a formal sum of embedded simplices, whose boundary is zero. The cycle $\eta$ on $Z$ determines three cycles on $IZ$, which can be denoted by $\{0\} \times \eta$, $\{\infty\} \times \eta$, and $I\eta$. The first two, which are equivalent, arise from the two 'poles' 0 and $\infty$ on $P_1$, each of which determines an embedding of $Z$ in $IZ$. The third, $I\eta$, is the product of $\eta$ with $P_1$. Similarly, on $CZ$ one will have $\{\infty\} \times \eta$ and $C\eta$. (There is also the apex, which will not be needed.)

Relations, as well as cycles, must be considered. On $P_1$ let $I_\lambda$ denote the chain that is a path from 0 to $\infty$. Similarly, let $I_\lambda \eta$ denote the chain on $IZ$, whose boundary is $\{\infty\} \times \eta - \{0\} \times \eta$.

The relations on a cone are more complicated. As noted in §2, in general it is not possible to cone a cycle $\eta$ on the base to form a relation $C_\lambda \eta$, whose boundary is $\eta$. However, if a cycle $\eta$ on $CZ$ is equivalent to an $\eta'$ that does not meet the base $Z$, then $\eta$ (via $\eta'$) be coned away to produce a relation $C_\lambda \eta$. Note that if such an $\eta'$ can be found, then the cap product of $\eta$ with the base (if defined) will be equivalent to zero, for $\eta'$ does not meet the base.

When no restrictions are placed on the cycles and relations, ordinary homology is the result. Based on the preceding discussion, one might expect the ordinary homology of $IZ$ to be the tensor product of that of $Z$ with that of $P_1$ (the Künneth formula), while for $CZ$ one might expect the 'cone' on the ordinary homology of the base. By this is meant the base homology raised by one
in degree, with the class of a point appended in degree zero. There are similar expected formulae for the ordinary homology Betti numbers. However, ordinary homology does not in general satisfy Poincaré duality and strong Lefschetz. Also, its Betti numbers are not a linear function of the flag vector. This is discussed further in §5.

Now consider middle perversity intersection homology, or more precisely, a theory that satisfies the Poincaré and Lefschetz theorems. These properties, particularly strong Lefschetz, will leave one with little choice as to what the cycles on CZ and IZ are, and hence lead to the usual middle perversity conditions on the cycles.

The task is to control the cycles and relations, so that the I and C operators preserve the Poincaré and Lefschetz properties. For I the usual Künneth formula will do this, a result that is left to the reader. For C, more care is needed.

Suppose \( \{\infty\} \times \eta \) is a cycle on CZ. Now use the \( C\omega_Z \) form of the hyperplane class. Clearly, one will have \( C\omega_Z \sim (\{\infty\} \times \eta) \) as the hyperplane action. Similarly, if \( C\eta \) is a cycle on CZ, use the ‘base’ form of the hyperplane class to obtain

\[
\omega_{CZ} \sim C\eta \sim \omega = \{\infty\} \times \eta
\]

as the hyperplane action. Now suppose that \( \eta \) is a primitive \( j \)-cycle on \( C \), by virtue of \( \omega^{i+1}_{CZ}C\eta \sim 0 \).

The relation

\[
\omega^{i+1}_{CZ}C\eta \sim 0
\]

follows from the above. Thus, \( C\eta \) is a primitive on CS, whenever \( \eta \) is a primitive on \( Z \). Moreover, \( \{\infty\} \times \eta \) is equal to \( \omega_{CZ} \sim C\eta \), and so cannot be primitive. All this assumes that \( C\eta \) is allowed as a cycle, when \( \eta \) on \( Z \) primitive.

These properties (Künneth and the coning of primitives) suffice to determine the homology of IZ and CZ respectively, in terms of that of \( Z \). The task now is to express these groups in terms of topological cycles and relations. First consider CZ. By assumption, if \( \eta \) is primitive on \( Z \), then \( C\eta \) is permitted on \( CZ \) (and is there primitive). Primitive is not a topological concept; it depends on the projective embedding \( Z \subset P_N \). However, the primitive cycles have upper dimension, and that is a topological notion. Thus, permit \( \xi = C\eta \) as a cycle on \( CZ \) whenever \( \eta \) on \( Z \) is upper, or in other words when \( \xi \) on \( CZ \) is strictly upper.

Now suppose that \( \eta \) on \( Z \) is coprimitive. It follows at once that

\[
\{\infty\} \times \eta \sim \{\infty\} \times Z \sim 0
\]

and so there is nothing in the homology of \( CZ \), that prevents \( \{\infty\} \times \eta \) being moved away from the base \( \{\infty\} \times Z \). Suppose that this can be done, to produce \( \eta' \sim \{\infty\} \times \eta \). As noted in §2, the \( \lambda \)-coning operation can be applied to \( \eta \), via \( \eta' \), to produce \( C\lambda\eta \). If \( \eta \) on \( Z \) is nonzero, then on \( CZ \) it is also by assumption nonzero. Thus, the ‘coning away to a point’ \( C\lambda\eta \) cannot be permitted (except perhaps if \( \dim \eta = 0 \)). As before, even though coprimitive is not a topological notion, the lower range of dimensions is. This leads to \( \xi = C\lambda\eta \) on \( CZ \) being prohibited as a chain, whenever \( \xi \) is lower.

These two examples (primitive and coprimitive) establish the middle dimension as the cut-off point for cycles and chains being permitted or prohibited respectively. This applies to how they meet the 0-strata. To obtain the remainder of the middle perversity conditions, study the cycles on IZ due to \( \eta \) on \( Z \), where now \( \eta \) is a cycle that satisfies the conditions that are already known. In this way, the rest can be built up, to produce the already known middle perversity intersection homology conditions on cycles and chains. This is left to the reader.

The assumption, that if \( \eta \) on \( Z \) is coprimitive, then \( \{\infty\} \times \eta \) is equivalent to an \( \eta' \) that does not meet the base \( Z \) of the cone, is quite strong. Previously, it was assumed that this might happen
from time to time, and so the associated coning aways were prohibited. That there are such prohibited coning aways is a topological property, which is local to the apex of \( CZ \). Local-global homology will count such ‘coning aways’, but calls them local-global cycles. It will as a heuristic principle be assumed that any coprimitive can be moved to avoid the base, and so be coned away. Such assumptions support the calculation in §4 of the expected values \( h\Delta \) of the local-global Betti numbers.

It is time to take stock. Recall that the purpose of this section is to describe the cycles on \( IZ \) and \( CZ \) in terms of those on \( Z \). When no restrictions are imposed, ordinary homology is the result. Requiring Poincaré duality and strong Lefschetz produces middle perversity intersection homology. For mpih the cycles on \( CZ \) are all of the form \( C\eta \) or \( \{\infty\} \times \eta \), for \( \eta \) a cycle on \( Z \). For the local-global extension, one also has the ‘coning away’ or local-global cycle \( C\lambda \eta \), for \( \eta \) any coprimitive cycle on the base \( Z \). (The apex of the cone \( Z \) is also called the apex of the local-global cycle \( C\lambda \eta \).) Now assume that on \( Z \) itself there is such a local-global cycle. Further local-global cycles may arise on \( CZ \) and \( IZ \), as a result of this local-global cycle on \( Z \).

As in the mpih case, on \( IZ \) it will be assumed that the Künneth formula continues to hold. In other words, any cycle on \( IZ \) can be expressed using \( \{0\} \times \eta \) (or \( \{\infty\} \times \eta \)) and \( I\eta \), where \( \eta \) ranges over the cycles on \( Z \). If \( \eta \) is a local-global cycle, say \( C\xi \), then \( I\eta \) is a new kind of local-global cycle. (Its apex locus is \( I \) applied to that of \( \eta \).) If \( \eta \) is thought of as a local cycle, then \( I\eta \) is a family of local cycles. In addition, note that by Künneth \( \{0\} \times \eta \) and \( \{\infty\} \times \eta \) are equivalent cycles on \( IZ \), and so local-global cycles must on occasion be allowed to move along the singular locus. (In fact, the rule will be that if they can move, then they are allowed to move.)

At this point it is possible to give some examples. In dimensions 0, 1 and 2 one has

\[
h(\bullet) = (1) \ ; \quad h(C\bullet) = h(\mathbf{1}\bullet) = (11) \ ; \quad h(CC\bullet) = (111) \ ; \quad h(ICC\bullet) = (121) ;
\]

of course. In dimension 3 one has

\[
h(CCCC\bullet) = (1111) \ ; \quad h(I\mathbf{1}C\bullet) = (1221) \ ; \quad h(I\mathbf{1}C\bullet) = (1331) ;
\]

as the nonsingular (or simple) examples, while

\[
h(CIC\bullet) = (1221) + (1)\{1\} ;
\]

is the only singular example. Here, \( \{1\} \) counts the local-global cycles on \( CZ \) due to the only nontrivial coprimitive on \( Z \), where \( Z \) is \( P_1 \times P_1 \) (or a square).

In dimension 4 something new happens. The simple cases

\[
h(CC\mathbf{1}C\bullet) = (11111) \ ; \quad h(I\mathbf{1}CC\bullet) = (12221) ;
\]

\[
h(I\mathbf{1}CG\bullet) = (13431) ; \quad h(I\mathbf{1}IC\bullet) = (14641) ;
\]

are just as before. The cones on the simple dimension 3 examples come next. They are

\[
h(CIC\mathbf{1}C\bullet) = (12221) + (1)\{1\}A ; \quad h(CI\mathbf{1}C\bullet) = (13331) + (2)\{1\}A ;
\]

where as before \( \{1\}A \) counts the nontrivial coprimitives on the base. The remaining examples are \( I \) and \( C \) applied to \( C\mathbf{1}C\bullet \), the only non-simple dimension 3 example.

The polytopes (or varieties) \( I\mathbf{1}C\bullet \) and \( CC\mathbf{1}C\bullet \) are more similar than they might at first sight appear. The polytope \( C\mathbf{1}C\bullet \) has an apex, and so \( I\mathbf{1}C\bullet \) has an apex edge. Now consider \( CC\mathbf{1}C\bullet \). From one point of view, this has two apexes, namely the apex of its base \( C\mathbf{1}C\bullet \), and the apex of \( CC\mathbf{1}C\bullet \) itself. However, \( CC\mathbf{1}C\bullet \) is also the join of \( I\mathbf{1}C\bullet \) with an interval, and so there is no
geometric way of distinguishing its two apexes. In other words, like $ICIC \bullet$, it too has an apex edge. Combinatorially, the two polytopes are the same along their respective apex edges. (This fact is at the heart of the IC equation.)

The mpih parts of $h(ICIC \bullet)$ and $h(CCIC \bullet)$ are $(13431)$ and $(12221)$ respectively. The remaining contribution comes from the strictly local-global cycles along the apex edge. Clearly, if $\eta$ is the cycle that contributes $(1)|1|$ to $CIC \bullet$, then $\eta$ and $\{0\} \times \eta$ will contribute $\{0\}|1$ and $(10)|1$ respectively to $ICIC \bullet$. However, on $CIC \bullet$ there is a single coprimitive cycle (it has dimension one), and so on $CCIC \bullet$ there will be a local-global of type $\{1\}A$. Now note that if $h \Delta$ is to be a linear function function of the flag vector, the non-mpih parts of the $h$-vectors of the two polytopes should be the same. To achieve this, the values

$$h(ICIC \bullet) = (13431) + (11)|1| + \{1\}A$$
$$h(CCIC \bullet) = (12221) + (11)|1| + \{1\}A$$

will be postulated.

This is to take any strictly local-global contribution that can occur for either $ICIC \bullet$ or $CICC \bullet$, and to insist that it can occur in the other. This forces $\{0\} \times \eta$ on $ICIC \bullet$ to contribute not only $(10)|1$ as already noted, but also $(1)|1A$. The cycle $\{\infty\} \times \eta$ on $CZ$ will make a similar contribution to $hCCIC \bullet$. Also, some sort of coning $C\eta$ of the cycle $\eta$ must be allowed, to obtain on $CCIC \bullet$ a contribution of $(10)|1$. This discussion is an example of how topology and combinatorics work together to determine the structure of the theory of local-global homology.

Now suppose that $\eta$ is a cycle (possibly of local-global type) on $Z$. Already, the associated cycles on $IZ$ have been described. The task now is to determine and describe the associated cycles on $CZ$. There are three basic possibilities. First, one can form $\{\infty\} \times \eta$, which is a cycle lying on the base $Z$ of $CZ$. Second, one can cone $\eta$ to form $C\eta$. Sometimes, as in the lower dimensions of mpih, this cycle is not needed. Finally, if $\{\infty\} \times \eta$ can be moved to an equivalent cycle $\eta'$, that does not meet the base $Z$ of $CZ$, one can form the ‘coning away’ $C_\lambda \eta$. These possibilities will be considered, one at a time.

First, the cycle $\{\infty\} \times \eta$ will always be admitted. There are no conditions imposed on $\eta$. One reason for this is that about their respective bases, the cylinder and the cone are combinatorially the same, and so that which is permitted for the one should be permitted for the other. But for the cylinder, the Künneth principle causes $\{\infty\} \times \eta$ to be admitted. As in the cylinder, this cycle may contribute to several distinct local-global homology groups.

Next consider $C\eta$. The example of $CCIC \bullet$ shows that this case requires more thought. As already noted, the cycle $\eta$ on $CIC \bullet$ contributes $(11)|1| + (1)|1A$ to $CCIC \bullet$. Clearly, $\{\infty\} \times \eta$ contributes $(10)|1$ and $(1)|1A$. The remainder, $(01)|1$ will have to come from $C\eta$. Thus, at least in this case, $C\eta$ must be allowed. But this seems to contradict the mpih case, where no use of $C\eta$ was made in the lower dimensions, and where any $C_\lambda \eta$ ‘coning away’ was explicitly prohibited. However, it is possible to harmonise the two cases. This involves looking again at the mpih situation.

Suppose that $\eta$ is an mpih cycle on $Z$. First consider $C\eta$ on $CZ$, as a purely formal object. Its main property is that $\omega_{\infty} \sim C\eta$ is equivalent to $\{\infty\} \times \eta$. In addition, if $\eta$ and $\xi$ on $Z$ have complementary dimensions (and so intersect to give a number), then $C\eta \sim \{\infty\} \times \xi = \eta \sim \xi$. These properties are not enough, in general, to determine $C\eta$ as a homology class on $CZ$. When $\xi$ is upper, $\omega$ is injective, and so $\omega \sim \xi$ determines $\xi$. This does not help in the other dimensions. Here, the Lefschetz isomorphism will be used. Write $\eta$ as $\omega_Z^{-1} \eta'$, where $\eta$ and $\eta'$ have complementary dimensions. This representation is always possible and unique, provided $\eta$ is lower. Provided $r > 0$, one can take $\{\infty\} \times (\omega_Z^{-1} \eta')$ as the cycle on $CZ$ that represents the formal object $C\eta$. Between
them, these two cover all the cases. Thus, there is no formal obstacle to thinking of $C\eta$ as a homology cycle on $CZ$.

The following construction, at least in certain cases, leads to a geometric form for $C\eta$, in the lower dimensions. As motivation, think of $CP_n$ (the cone on $P_n$, not complex projective $n$-space). Here, each cycle $\eta$ on $P_n$ can be coned to give a cycle $C\eta$ on $CP_n$, provided the apex is not part of the stratification. Adding the apex as a stratum will not however change the homology. It will thus be possible to move $\eta$ a little bit, so that it avoids the apex (at least in the lower dimensions). This change can be confined to a small ball centered at the apex. Think now of $C\eta$ on $CZ$ as follows. Cone $\eta$ to form $C\eta$, and form a small ball $B$ about the apex. Outside of $B$, there is no fault with $C\eta$. The task now is to change $C\eta$ within $B$, so as to avoid the apex. Consider now the intersection $R = S \cap C\eta$ of $C\eta$ with the boundary $S$ of $B$. For certain values of $R \subset S$, it will be possible to ‘fill-in’ $R$ within $B$, to obtain part of an intersection homology cycle, and for others it will not. (The difference between any two solutions is, clearly, a local intersection homology cycle.) For certain $\eta$ it will be possible to solve the associated $R \subset S$ problem. For heuristic purposes, it will be assumed that this is always possible. Intersection properties can be used to resolve the indeterminacy due to local cycles at the apex. In this way it is possible (modulo some assumptions) to treat $C\eta$ as a cycle on $CZ$, when $\eta$ is any mpih cycle on $Z$. The key is to if necessary modify the geometric form of $C\eta$ within a small ball centered at the apex.

Now consider $C\eta$ on $CZ$, where $\eta$ is a local-global cycle. The example of $CCIC\bullet$ forces one to allow this cycle, in some form of the other. The previous paragraph shows how this might be done. The geometric form of $C\eta$ must be modified in a small ball centered about the apex of $CZ$, or perhaps more exactly, replaced by something else.

In fact, this $C\eta$ problem need only be solved for mpih cycles. Each local-global cycle can be thought of as a $\lambda$-family of cycles. One can then solve this problem in the simpler case of $\lambda = 1$, and then define $C\eta$ to be the result of applying scalar multiplication to the this solution.

The third type of cycle on $CZ$ are those obtained by moving a cycle $\{\infty\} \times \eta$ away from the base $Z$, and then ‘coning it away’. This was the point of departure, for the local-global theory. (The mpih theory prohibits the use of such objects, to generate homology relations. The local-global theory treats such objects as a cycle, but of a new type.) This construction can be iterated. Here is an example. First, let $\eta$ be the local-global cycle on $CIC\bullet$. Now let $Z$ be $IIICIC\bullet$, and on $II\bullet$ let $\xi$ be a coprimitive cycle. On $Z$ there is a local-global cycle that can be written as $\xi \otimes \eta$. Now consider $CZ$. Provided $\{\infty\} \times (\xi \otimes \eta)$ can be moved within its class, so as to avoid the base $Z$, it can be ‘coned away’. This is an example of a second-order local-global intersection homology cycle.

Earlier in this section, $h\Delta$ was presented for all the $IC$ polytopes of dimension at most 4. To conclude, much the same will be done for dimension 5. However, to save space this will be done only for certain polytopes, whose flag vectors provide a basis for all polytope flag vectors. They are the ones in which neither $II$ nor $I\bullet$ occur. There are $F_6 = 8$ such polytopes.

For these basis polytopes the $h$-vectors are as follows.

\[
\begin{align*}
  h(CCCC\bullet) &= (111111) \\
  h(CCCI\bullet) &= (122221) + (111)\{1\} + (11)A\{1\} + (1)AA\{1\} \\
  h(CICI\bullet) &= (122221) + (11)A\{1\} + (1)AA\{1\} \\
  h(CICC\bullet) &= (122221) + (1)AA\{1\} \\
  h(CCCI\bullet) &= (134431) + (11)\{1\} + (11)A\{1\} + (2)AA\{1\} + (1)\{2\} \\
  h(IICC\bullet) &= (122221) \\
  h(IICC\bullet) &= (134431) + (121)\{1\} + (12)A\{1\} + (1)AA\{1\} \\
  h(IICI\bullet) &= (134431) + (11)A\{1\} + (1)AA\{1\}
\end{align*}
\]

Here is a summary of the discussion of the cycles on $IZ$ and $CZ$. On $IZ$ the cycles are as given
by the Künneth principle. Each cycle on $IZ$ is a sum of products of a cycle on $I$ (or $P_1$) with a cycle on $Z$. For $CZ$ the situation is more complicated. If $\eta$ is a cycle on $Z$, then one always has $\{\infty\} \times \eta$ on $CZ$. Provided the details are satisfied, as to what happens near to the apex, one will also have $C\eta$. (When $\eta$ is upper, these details are vacuous.) Finally, if $\{\infty\} \times \eta$ can be moved so as to avoid the base $Z$ of $CZ$, one also has its ‘coning away’, the $\lambda$-cone $C\lambda\eta$. (A necessary, and perhaps sufficient, condition for doing this is that $\omega_{CZ} \sim \{\infty\} \times \eta$, which is equal to $\{\infty\} \times (\omega_Z \sim \eta)$ be homologous to zero.)

This description of cycles motivates both the definition of the extended $h$-vector ($\S$4), and the topological definition of local-global homology ($\S$5). In both cases, there are two aspects to the discussion. The first is the cycles themselves, the focus of this section. The second is how they are to be counted.

Consider once again $ICIC\bullet$ and $CCIC\bullet$. There, it was seen that the same cycle may contribute to several different parts of $h\Delta$. This is something that is quite new. The basic idea is this. A local-global homology group is spanned by all cycles that satisfy certain conditions. If these conditions are relaxed, another local-global group is obtained. (The same happens in intersection homology, when the perversity is relaxed.) This is why the same local-global cycle may contribute to several components of $h\Delta$. The conditions are related to where the cycle may be found. For example (dim = 4), one can count all local 1-cycles (subject to equivalence), or one can allow only those that have some degree of freedom, as to their location. The former are counted by $\{1\} A$, the later by $x\{1\}$. For example, $CICC\bullet$ has the former but not the later, while $ICIC\bullet$ has both. These conditions control both the cycles and the relations. Sometimes (the 4-cross polytope for example), relaxing the conditions may allow new relations to appear amongst existing cycles. The varieties produced by $I$ and $C$ are special, in that this never happens. This makes the computation of their $h$-vectors much easier. This fact is exploited by the next section.

4 The extended $h$-vector

This section defines, for every convex polytope $\Delta$, an extended $h$-vector $h\Delta$. It does this by using rules $\tilde{I}$ and $\tilde{C}$ that satisfy the $IC$ equation. These rules are motivated by the previous section. In the next section, local-global homology groups will be defined for algebraic varieties. Provided various assumptions are satisfied, for $\Delta$ an $IC$ polytope the extended $h$-vector $h\Delta$ will give the local-global Betti numbers of the associated toric variety $P_\Delta$. (The same may not be true for other rational polytopes, and even it true will most likely be much harder to prove. Such would be both a formula for the local-global Betti numbers, and a system of linear inequalities on the flag vectors of rational polytopes.)

There are two stages to the definition of $h\Delta$. The previous section described the local-global cycles on $IZ$ and $CZ$ in terms of those on $Z$. It also noted that the same cycle might contribute in several ways to the local-global homology. The first stage is to define operators $\tilde{I}$ and $\tilde{C}$ that count the local-global cycles, but without regard to the multiple contributions. The second stage is to make a change of variable, to accomodate the multiple contributions. This corresponds to knowing the implications among the various conditions satisfied by local-global cycles. The second stage is vacuous for the mpih part of the theory. (For the $IC$ polytopes, each local-global cycle is determined by the corresponding global cycle, together with a statement, as to the $\lambda$-coning conditions it satisfies. The first stage counts each local-global cycle only once, at the most stringent conditions it satisfies. This process is meaningful only for $IC$ and similar polytopes. In general, relaxation of conditions will admit new relations, as well as new cycles.)

The first stage is to introduce an auxiliary vector $\tilde{h}\Delta$, defined via rules $\tilde{I}$ and $\tilde{C}$. The quantity
\( \tilde{h}\Delta \) will be a sum of terms such as \([abcd]W\). As noted in §2, \([abcd]\) stands for the homogeneous polynomial \(aX^3 + bX^2Y + cXY^2 + dY^3\), while \(W\) will be a word in \(\{k\}\) and \(A\). The rules \(\tilde{I}\) and \(\tilde{C}\) will be defined by their action on such terms.

The rule for \(\tilde{I}\) is to multiply by \([11] = X + Y\). It corresponds to the Künneth formula for cycles. The equation

\[
\tilde{I}[abcd]W = [11][abcd]W = [a, a + b, b + c, c + d, d]W
\]

is an example of this rule. Note that this rule preserves the property of being palindromic. If swapping \(X\) and \(Y\) leaves \(\tilde{h}\Delta\) unchanged, then the same is true of \(\tilde{I}\tilde{h}\Delta\).

The rule for \(\tilde{C}\) is more complicated. It has three parts. The first part \(\tilde{C}_1\) leaves \(W\) unchanged. It corresponds to the idea, that if \(\eta\) on \(Z\) is primitive, then so is \(C\eta\) on \(CZ\). Here are some examples of the rule

\[
\begin{align*}
\tilde{C}_1[a]W &= [aa]W; \\
\tilde{C}_1[ab]W &= [aab]W; \\
\tilde{C}_1[abc]W &= [abbc]W; \\
\tilde{C}_1[abcd]W &= [abccde]W; \\
\tilde{C}_1[abcdef]W &= [abcdeef]W;
\end{align*}
\]

for this part. It is to repeat the exactly middle, or failing that the just before middle, term in the \([\ldots]\) sequence.

The reader is asked to verify that the equation

\[
(\tilde{I} - \tilde{C}_1)\tilde{C}_1 = [010]
\]

holds, in the sense the applying the left hand side to, say, \([abde]\) will produce \([0abde0]\). As \([11]\) and \([010]\) commute, \(\tilde{I}\) and \(\tilde{C}_1\) satisfy the \(IC\) equation. Using just this part of the rule for \(\tilde{C}\) (together with the rule for \(\tilde{I}\), and \(h(\bullet) = [1]\) as an initial condition) will generate the \(\text{mph}\) part of \(h\Delta\).

The second part \(\tilde{C}_2\) corresponds to the \(\lambda\)-coning of a cycle \(\eta\) on the base. Such a cycle \(\eta\) must be coprimitive. The numbers \(b' = b - a\), \(c' = c - b\) and so on count the coprimitives. New words will be obtained by prepending to \(W\) a record, in the form \(\tilde{A}^j\{k\}\), of the coprimitive that has been \(\lambda\)-coned. Here are some examples of the rule

\[
\begin{align*}
\tilde{C}_2[a]W &= 0; \\
\tilde{C}_2[ab]W &= 0; \\
\tilde{C}_2[abc]W &= [b']\{1\}W; \\
\tilde{C}_2[abcd]W &= [b']\tilde{A}\{1\}W; \\
\tilde{C}_2[abcde]W &= [b']\tilde{A}^2\{1\}W + [c']\{2\}W; \\
\tilde{C}_2[abcdef]W &= [b']\tilde{A}^3\{1\}W + [c']\tilde{A}\{2\}W;
\end{align*}
\]

for this part. In \(\tilde{A}^j\{k\}\) the \(k\) records the degree of the coprimitive, and the \(j\) ‘takes up the slack’, to ensure homogeneity. As noted in the previous section, the trivial coprimitives (which correspond to \(a' = a\)) are not counted.

The sum \(\tilde{C}_1 + \tilde{C}_2\) of these two parts is not enough (or more exactly, is too much). One reason is that when used with \(\tilde{I}\), the result does not satisfy the \(IC\) equation. The third part is a correction, that balances the books. It is to subtract \([a]\tilde{A}^j\), for the appropriate power of \(j\). The geometric meaning of this correction will be presented later.

Here now is the rule for \(\tilde{C}\). The examples

\[
\begin{align*}
\tilde{C}[a] &= [aa] - [a]\tilde{A} \\
\tilde{C}[ab] &= [aab] - [a]\tilde{A}^2 \\
\tilde{C}[abc] &= [abc] - [a]\tilde{A}^3 + [b']\{1\} \\
\tilde{C}[abcd] &= [abcd] - [a]\tilde{A}^4 + [b']\tilde{A}\{1\} \\
\tilde{C}[abcde] &= [abcde] - [a]\tilde{A}^5 + [b']\tilde{A}^2\{1\} + [c']\{2\} \\
\tilde{C}[abcdef] &= [abcde] - [a]\tilde{A}^6 + [b']\tilde{A}^3\{1\} + [c']\tilde{A}\{2\}
\end{align*}
\]
suffice to show the general rule. In the above, it is to be understood that both sides have been multiplied on the right by a word \( W \) in the symbols \( A \) and \( \{ k \} \).

This rule, and the rule for \( \widetilde{I} \), together satisfy the IC equation. Here is an example. The calculation

\[
\widetilde{IC}[abcde] = [11][abcde] - [aa]A^5 + [b'b']A^2\{1\} + [c'c']\{2\}
\]

follows immediately from the above. The calculation for \( \tilde{C}C[abcde] \) is more involved. One has

\[
\tilde{C}[abcde] = [abcde] - [a]A^6 + [b']A^2\{1\} + [c']A\{2\}
\]

and also

\[
\begin{align*}
-\tilde{C}[a]A^5 &= -[aa]A^5 + [a]A\tilde{A}^5 \\
\tilde{C}[b']A^2\{1\} &= [b'b']A^2\{1\} - [b']A\tilde{A}^2\{1\} \\
\tilde{C}[c']A^2\{1\} &= [c'c']\{1\} - [c']A\{1\}
\end{align*}
\]

as the various contributions. Now compute the difference. All but two of the terms cancel. One has

\[
(\widetilde{IC} - \tilde{C}C)[abcde] = [11][abcde] - [abcde]
\]

which is, as for \( \tilde{C}_2 \), is equal to \([010][abcde] \). As this example is completely typical, the result follows.

To complete this definition of \( h\Delta \), one must supply an initial value \( \tilde{h}(\bullet) \), such that \( \tilde{C}\tilde{h}(\bullet) \) and \( \tilde{I}\tilde{h}(\bullet) \) are equal. Here a problem arises. The value \( \tilde{h}(\bullet) = [1] \) does not quite work. The quantities \( \tilde{C}[1] = [11] - [1]\tilde{A} \) and \( \tilde{I}[1] = [11] \) are not equal. Here is the solution. Recall that \( h\Delta \) is to be a sum of terms of the form \( hW\Delta\cdot W \), for some family of symbols \( W \). Thus, one should really write \( \tilde{h}(\bullet) = [1]W_0 \), for some symbol \( W_0 \). Consistency is then equivalent to the equation \( AW_0 = 0 \). However, it is more convenient to use \( \bullet \) as the initial value. The initial values

\[
\tilde{h}(\bullet) = [1]\bullet; \quad \tilde{A}\bullet = 0 \quad (3)
\]

conclude the definition. The multiple use of the symbol \( \bullet \) in practice causes no confusion.

(The geometric meaning of the correction \(-[a]A^4\) is as follows. Suppose that \( \Delta \) is an IC polytope, and that say \([abc]W \) appears in \( h\Delta \). This term \([abc]W \) is due to the \( I \) and \( C \) operations that constructed \( \Delta \). In particular, going back in this case two steps, one obtains the polytope \( \Delta_1 \), from which \( \Delta \) is derived by applying \( I \) and \( C \), and in \( h\Delta_1 \) the term \([a]W \) appears. Going back another step, one has \( \Delta_1 = C\Delta_0 \), and on \( \Delta_0 \) there are \( a \) independent coprimitive cycles \( \eta \) that contribute \([a]W \) to \( \tilde{h}\Delta_1 \). Now consider \( C\Delta \). These coprimitive cycles \( \eta \) (multiplied by \( \{ \infty \} \) as appropriate) continue to exist on \( \Delta \), and the \( \tilde{C}_2 \) part of the rule for \( \tilde{C} \) will thus cause them to contribute \([a]\tilde{A}W \). However, their contribution to \( \tilde{h}C\Delta \) has already been counted, as the ‘\( a \)’ part of \( \tilde{C}_1[abc]W = [abc]W \). This is because \( \tilde{h}\Delta \) counts cycles on IC polytopes, at the most stringent condition they satisfy. Hence the correction \(-[a]\tilde{A}^4\).

Finally, note that if \( h\Delta \) is palindromic then so is \( \tilde{C}\tilde{h}\Delta \). As the same is true for \( \tilde{I} \), it follows that \( \tilde{h}\Delta \) is indeed palindromic, first for all IC polytopes and then (by linearity) for all polytopes. This completes the definition of the auxiliary vector \( \tilde{h}\Delta \).

The second stage in the definition of the extended h-vector is to apply a linear change of variable to the above quantity \( \tilde{h}\Delta \). The meaning of this transformation is as follows. Recall that for \( \Delta \) an IC polytope, \( \tilde{h}\Delta \) counts each local-global cycle once, according to the most stringent conditions it satisfies. The transformation is the process of relaxation of conditions, as it applies to cycles on IC polytopes.
Previously, little attention has been given to the conditions satisfied by the local-global cycles. These cycles have been generated from coprimitives via the $\lambda$-coning operation, but the topological properties of the cycles so obtained have not been explicitly formulated. Instead, the $X, Y, \{k\}$ and $\hat{A}$ symbols have been used in a somewhat formal way, to record the relevant facts relating to the construction of the associated cycles.

Recall that in the discussion of mpih, one first obtained the cycles, and then the conditions that they satisfied. Something similar will be done with local-global homology. A local-global cycle on an IC variety $\Delta$ consists of a cycle, as counted by $\tilde{h}\Delta$, together with a condition that it satisfies. Such conditions, which control how the apex locus meets the strata, will be formulated in the next section. However, by assumption each word in $x, y, A$ and $\{k\}$ determines a condition, or in other words a type of local-global cycle. The present task is to describe how to pass from $\tilde{h}\Delta$ to $h\Delta$. If done properly, it will implicitly determine the set of conditions, that will in the next section be explicitly stated.

To each term such as $\hat{A}\{1\}$ or $X\{1\}$ in $\tilde{h}\Delta$, there is to correspond a condition that applies to local-global cycles. The coefficient in $\tilde{h}\Delta$ of such a term counts how many cycles there are on $\Delta$ that satisfy that condition, but not any condition that is more stringent. (This makes sense only for IC polytopes. Other polytopes may produce negative coefficients.) The transformation from $\tilde{h}$ to $h$ is thus determined once one knows the partial order on conditions, that is associated to stringency or implication. In other words, the partial order is that of implication among the associated conditions.

Here is an example. In $\tilde{h}(CCIC\bullet)$ the term $X\{1\}$ occurs, with coefficient 1. (As noted in §3, it also so occurs in $h(ICCIC\bullet)$.) This term corresponds to a particular type of local-global cycle, namely one that can be found anywhere along the apex edge. Now consider $\tilde{h}(CICC\bullet)$. Here the term $A\{1\}$ occurs, with coefficient 1. This too corresponds to a local-global cycle, but of a different type. It corresponds to a local-global cycle that can be found only about the apex of $CICC\bullet$. (In the previous case, there was an apex edge, anywhere along which the local-global cycle could be found. In this case there is no apex edge.) Thus the condition $X\{1\}$ is more stringent than $A\{1\}$.

Each term such as $A\{1\}$ or $X\{1\}$ in $\tilde{h}\Delta$ will contribute $A\{1\}$ and $x\{1\}$ respectively to $h\Delta$. These ‘diagonal’ terms arise because ‘the most stringent condition satisfied by a cycle’ is also ‘a condition satisfied by a cycle’. In addition, as just noted, if a cycle satisfies $X\{1\}$ then it will also satisfy $\hat{A}\{1\}$ (but not as the most stringent condition), and so $X\{1\}$ will contribute $A\{1\}$, in addition to $x\{1\}$. The transformation of

$$\tilde{h}(CCIC\bullet) = [12221]+[11]\{1\}$$

is therefore

$$h(CCIC\bullet) = (12221) + (1)A\{1\} + (11)\{1\}$$

which is as postulated in the previous section.

Here is another example. In involves second order local-global cycles. The symbol $\{1\}\{1\}$ represents a local 1-dimensional family of local 1-cycles. In dimension 7 each of the terms $A\{1\}$, $X\{1\}\{1\}$ and $\{1\}A\{1\}$ is to represent a different condition on a local 1-family of local 1-cycles. To begin to understand these conditions, for each of these terms an IC polytope will be produced, in whose $\tilde{h}$-vector the term occurs. Here is a list

- in $\tilde{h}(CICCCIC\bullet)$ the term $\hat{A}\{1\}\{1\}$ occurs
- in $\tilde{h}(CCICCIC\bullet)$ the term $X\{1\}\{1\}$ occurs
- in $\tilde{h}(CICCCIC\bullet)$ the term $\{1\}\hat{A}\{1\}$ occurs

of such polytopes. This the reader is invited to verify.
From this list the conditions, or more exactly the partial order, will be obtained. As IC varieties, each of the above will have a minimal stratification. The conditions control how the apex loci of the cycle meets the strata. (The apex locus is due to the $\lambda$-coning or ‘locality’ of the cycle.) Here is a list, which again the reader is invited to verify,

the strata for $\bar{A}\{1\}\{1\}$ have dimension 0, 4 and 7
the strata for $X\{1\}\{1\}$ have dimension 1, 4 and 7
the strata for $\{1\}\bar{A}\{1\}$ have dimension 0, 3 and 7

of the strata dimensions for the associated IC varieties.

The present discussion is of a local 1-dimensional family of local 1-cycles, and so there are two apex loci to consider. One of them lies on the strata of dimension 0 or 1, the other on strata of dimension 3 or 4. It is more stringent to require that an apex locus be found on a 1-strata than on a 0-strata, and similarly for 4-strata and 3-strata. Thus

$$X\{1\}\{1\} \Rightarrow \bar{A}\{1\}\{1\} \Rightarrow \{1\}\bar{A}\{1\}$$

is the partial order on conditions. As in the example of $\bar{A}\{1\}$ and $X\{1\}$, this partial order allows $h\Delta$ to be transformed into $h\Delta$.

It is now possible to define the partial order on terms. As in the previous example, associate to each term an IC polytope, whose $\bar{h}$-vector realises the term. Each such polytope, thought of as a variety, has a minimal stratification. The partial order on terms is then the partial order on the dimension vectors of the stratification associated to each term.

Here are the details. The partial order does not compare, say, $\{1\}\{2\}$ and $\{2\}\{1\}$, or $X\{1\}$ and $Y\{1\}$. The associated cycles differ by more than a change in conditions. Say that two expressions in $X$, $Y$, $\bar{A}$ and $\{k\}$ are broadly similar if, when $X$ and $\bar{A}$ have been deleted, they are identical. They should also have the same degree. The partial order applies only to broadly similar terms. Each term will appear in $\bar{h}\Delta$, for one or more IC polytopes $\Delta$. Choose one of these polytopes (it does not matter which) for which the associated stratification has as few terms as possible. It will in fact have $r + 1$ terms, where $r$ is the order of the term. This process associates a stratification dimension vector $d = (d_1 < d_2 \ldots < d_{r+1} = n)$ to each term. The partial order is that $d$ will imply $d'$ just in case $r = r'$, and $d_i \geq d'_i$ for $i = 1, 2, \ldots, r + 1$.

The above is a description, based on geometry, of the partial order on conditions. The stratification dimension vector $d$ associated to a term can also be computed directly. Suppose, for example, the term is $X^2Y^3\bar{A}^4\{5\}\bar{A}^2\{6\}$. The degree of this term is

$$2 + 3 + 4 + (2 \times 5 + 1) + 2 + (2 \times 6 + 1) = 35$$

and so 35 is the dimension of the top stratum. For the broadly similar term $X^i\{5\}\{6\}$, where $i$ makes the degree up to 35, the dimension vector written backwards is

$$35 > 35 - \deg\{6\} > 35 - \deg\{6\} - \deg\{5\}$$

(or $11 < 22 < 35$ the normal way around). For the given term $X^2Y^3\bar{A}^4\{5\}\bar{A}^2\{6\}$ the dimension vector is

$$35 > 35 - \deg\{6\} - 2 > 35 - \deg\{6\} - 2 - \deg\{5\} - 4$$

and from this the partial order can be given a combinatorial description. (The proofs have been left to the reader.)

The partial order on terms can be expressed in the following way. The terms implied by some given term, say $X^2Y^3\bar{A}^4\{5\}\bar{A}^2\{6\}$, can all be obtained in the following way. First of all, any number
of occurrences of $X$ can be replaced by $\bar{A}$. Such $\bar{A}$ should of course be placed after the $X^iY^j$ term. Next, any occurrence of $\bar{A}$ can be ‘slid’ rightwards over any $\{k\}$ symbol. Finally, note that the terminating symbol ‘$\bullet$’ is assumed to be at the end of each word, and that $\bar{A}\bullet$ is zero.

It is now possible to give an algebraic description of the rules that transform $\tilde{h}\Delta$ to the extended $h$-vector $h\Delta$. First of all the change of variable

$$X = x + \bar{A}; \quad Y = y;$$

is made. To ensure that equations such as

$$X^2 = x^2 + x\bar{A} + \bar{A}^2;$$
$$X^3 = x^3 + x^2\bar{A} + x\bar{A}^2 + \bar{A}^3;$$

hold, the equation

$$\bar{A}x = 0$$

is postulated. These rules allow $X$ and $Y$ to be replaced by $x$ and $y$. Of course, $X$ and $Y$ commute, as do $x$ and $y$. Thus, $y$ and $\bar{A}$ are also to commute.

The equation

$$\bar{A}\{k\} = A\{k\} + \{k\}\bar{A}$$

allows the ‘rightward slide’ of $\bar{A}$ over $\{k\}$. The $\bar{A}$ on the left hand side can remain where it is, to produce $A\{k\}$; or it can slide over the $\{k\}$. If so slid, it could be slid again. Thus, it is $\{k\}A$ rather than $\{k\}\bar{A}$ on the right hand side. Finally, in $\bar{A}A$ the second $\bar{A}$ is ‘non-sliding’. This stops the $\bar{A}$ from sliding. Thus the equation

$$\bar{A}A = AA$$

is also postulated.

The rules in the previous two paragraphs define the extended $h$-vector $h\Delta$ of any convex polytope $\Delta$, via the auxiliary vector $\tilde{h}\Delta$. Implicit in this are topological conditions on local-global cycles. In the next section these conditions will be made explicit.

5 Topology and local-global homology

In this section $Z$ will be a complex algebraic variety, considered as a stratified topological space. Local-global intersection homology groups will be defined for $Z$. The starting point is a particular means of expressing the concept of a local cycle, and of course the basic concepts of intersection homology. By considering families of such cycles, the full concept of a local-global cycle is developed. This is done by extending the notion of a simplex.

Recall that in the previous section there were two stages to the definition of the $h$-vector $h\Delta$. Loosely speaking the first stage, the definition of $\tilde{h}\Delta$, corresponded to the definition of a local-global cycle. The second stage was concerned with the conditions (on how it meets the strata) satisfied by the cycle. The topological definition of local-global homology similarly has two stages, the definition of the cycles, and the definition of the conditions. Most of the justification for the cycle definition comes from §3, while the conditions are those implicit in §4.

To begin with, consider the concept of a local cycle. In §2 such was thought of as a cycle $\eta$ lying on any sufficiently small neighbourhood $U$ of the point $s$, about which the cycle is to be local. That the local topology of $Z$ about $s$ is the cone on the link can clarify this concept. In this section, all this will be incorporated into the definition of a local cycle. Some preliminaries are required.

Homology can be defined using embedded simplices. Here is a review. An embedded $k$-simplex is simply a continuous map $f : \sigma_k \to Z$ from the $k$-simplex $\sigma_k$. A $k$-chain is a formal sum of

$$\sum a_i \sigma_i,$$
(embedded) $k$-simplices. Each simplex $\sigma_k$ has a boundary $d\sigma_k$, which is a formal sum of $(k-1)$-simplices. In the same way, each $k$-chain $\eta$ has a boundary $d\eta$, which is a $(k-1)$-chain. A $k$-cycle is a $k$-chain $\eta$ whose boundary $d\eta$ is zero (as a formal sum of embedded simplices). Because $d \circ d = 0$, if $\xi$ is a chain then $\eta = d\xi$ is a cycle. Such a cycle $\eta$ will be called a boundary. The $k$-th ordinary homology group of $Z$ consists of the $k$-cycles modulo the $k$-boundaries.

Intersection homology [11] places restrictions on how the embedded simplices can meet the strata. The conditions are expressed using a perversity, which is a sequence of numbers. In this paper only the middle perversity is used. It imposes the following condition on an embedded $k$-simplex $f : \sigma_k \to Z$. Let $S_i$ be the complex $i$-dimensional stratum of $Z$. Consider $f^{-1}(S_i)$, or more exactly its dimension. Let the empty set have dimension $-\infty$. If the inequality

$$\dim f^{-1}(S_i) \leq k - (n - i)$$

holds for every stratum $S_i$, then $f : \sigma_k \to Z$ is allowed (for the middle perversity). An allowed cycle $\eta$ is a formal sum $\eta$ of embedded simplices, whose boundary is zero. An allowed boundary $\eta = d\xi$ is a formal sum $\xi$ of allowed simplices, whose boundary $d\xi$ (which necessarily is a cycle) is also allowed. The $k$-th (middle perversity) intersection homology group $\zeta$ consists of the allowed cycles modulo the allowed boundaries. An important technical result [12] [10] is that this group is independent of the stratification $S_i$ chosen for $Z$.

Consider the concept of a local cycle. Each point $s$ on $Z$ has a neighbourhood that is homeomorphic to the cone $CL_s$ on something, namely the link $L_s$ at $s$. One can use $CL_s$ as the neighbourhood $U$ in which the cycles $\eta$ local to $s$ can be found. Because $CL_s$ is a cone (and $\eta$ avoids the apex of the cone), the cycle $\eta$ is equivalent to a cycle $\eta'$, that is supported on the base $L_s$ of the cone. (Use the cone structure to move the cycle $\eta$ to the base of the cone.) In fact, for each $0 < \lambda \leq 1$ one obtains a cycle $\eta_\lambda$, while the 'limit' $\eta_0$ of the family is the apex $CL_s$, which is the point $s$ of $Z$. In other words, a local cycle is a cycle that can be 'coned away' to a point, except that the perversity conditions may disallow this.

The concept of a local cycle can be formulated without using either $U$ or $CL_s$. Define a coned $k$-simplex to be the cone $C\sigma_k$ on a $k$-simplex. (Although isomorphic to $\sigma_{k+1}$, it will not be treated as such. Later, more complicated objects will be coned.) Local homology will be constructed using such simplices. An embedded coned $k$-simplex is simply a continuous map $f : C\sigma_k \to Z$. The apex of $f$ is the image under $f$ of the apex of $C\sigma_k$. A coned $k$-chain is a formal sum of (embedded) coned $k$-simplices. Each coned simplex $C\sigma_k$ has a boundary, which is a formal sum of coned $(k-1)$-simplices. (The boundary is taken only the the $\sigma_k$ direction, and not in the $C$ direction.) In the same way, each coned $k$-chain $\xi$ has a boundary $d\xi$, which is a coned $(k-1)$-chain. A coned $k$-cycle is a coned $k$-chain whose boundary $d\eta$ is zero (as a formal sum of embedded coned $(k-1)$-simplices.) Because $d \circ d = 0$, if $\xi$ is a coned chain, then $d\xi$ is a coned cycle. Such a cycle $\eta = d\xi$ will be called a coned boundary.

Finally, only certain cycles and boundaries will be allowed. Use $\lambda$ and $p$ to denote the cone and simplex variable respectively. For each $0 < \lambda \leq 1$, use $f_\lambda$ to denote the embedded simplex defined by the rule $f_\lambda(p) = f(\lambda, p)$, where $f$ is an embedded coned simplex. Say that a coned cycle $\eta$ is allowed if $\eta_\lambda$ is similarly allowed. Say that a coned boundary $\eta = d\xi$ is allowed if $\eta_\lambda = d\xi_\lambda$ is allowed, for $0 < \lambda \leq 1$. Now fix a point $z \in Z$. Say that an embedded coned simplex $f$ is local to $s$ if $s$ is the apex of $f$. In that case, $s$ will also be the apex of the boundary of $f$. The local $k$-homology at $s$ consists of the coned $k$-cycles modulo the coned $k$-boundaries, where the cycles and boundaries are allowed (by the perversity), and are constructed using only the embedded coned simplices local to $s$.

Local-global cycles differ from local cycles in that the base point or apex is allowed to move. To do this a new sort of simplex is required. Consider $\sigma_1 \times C\sigma_k$. This can be thought of as a
1-dimensional family of coned \( k \)-simplices. It has not a single point as its apex, but an *apex edge*. Its boundary in the \( \sigma_1 \) direction is a pair of coned \( k \)-simplices, one at each end of the apex edge. It also has a boundary in the \( \sigma_k \) direction, which is a formal sum of \( \sigma_1 \times C\sigma_{k-1} \) ‘simplices’. (As before, no boundary is taken in the cone direction.)

The task now is to define the *local-global \( k \)-homology* of \( Z \). The cycles are as in local \( k \)-homology, except that it is not required that the embedded coned \( k \)-simplices be local to some fixed point \( s \) of \( Z \). Clearly, such a cycle can be written as a formal sum of local \( k \)-cycles, based at different points \( s_1, \ldots, s_N \) of \( Z \).

The boundaries require more thought. Previously the simplices (possibly coned) were indexed by a single number \( k \), and the boundary operator \( d \) reduced the index by one. The present situation is that \( C\sigma_k \) arises from both \( C\sigma_{k+1} \) and \( \sigma_1 \times C\sigma_k \), and the latter also produces \( \sigma_1 \times C\sigma_{k-1} \). When \( C\sigma_k \) is written as \( \sigma_0 \times C\sigma_k \), it becomes clear that the ‘simplices’ are now indexed by a pair of numbers, and that the boundary operator \( d \) can reduce either one or the other by one.

Because of this, the concept of \( \eta = d\xi \) being a *local-global \( k \)-boundary* can be formulated in several, possibly inequivalent, ways. In each case \( \xi \) will be a formal sum of embedded \( \sigma_i \times C\sigma_j \) simplices, with \( i + j = k + 1 \). One also requires that \( \eta \) is the boundary of \( \xi \), and that for each \( 0 < \lambda \leq 1 \), one has the \( \eta_\lambda = d\xi_\lambda \) is allowed (in the usual way). The different concepts arise, according to the values of \( i \) and \( j \) allowed. Say that \( \xi \) is very pure if all its ‘simplices’ have the same type. Clearly, very pure boundaries should be allowed. In the present case, \( \sigma_0 \times C\sigma_{k+1} \) is required for local equivalence, while \( \sigma_1 \times C\sigma_k \) allows the apex to move. The sum of two very pure boundaries need not be very pure. Say that \( \xi \) is pure if all the ‘simplices’ are capable, by virtue of their ‘dimension’, of participating in a very pure boundary. In the present case, it means that \( \xi \) is built out of a mixture of \( \sigma_0 \times C\sigma_{k+1} \) and \( \sigma_1 \times C\sigma_k \). Finally, say that \( \xi \) is mixed if it is built out of any mixture whatsoever of \( \sigma_i \times C\sigma_j \) simplices’.

In principle, the three concepts are different. The author expects very pure and pure to give the same boundaries. (This means that if \( \eta = d\xi \) is a pure boundary, then there are very pure \( \eta_i = d\xi_i \), with \( \eta = \sum \eta_i \). It is not required that \( \xi = \sum \xi_i \).) The mixed concept permits \( \sigma_{k+1} \times C\sigma_0 \) to play a rôle. This seems to be wrong. In this paper, local-global homology will be defined using the pure concept. Experience will show if this is correct. (Already, the desired Betti numbers are known.)

The general definition of local-global cycles and boundaries proceeds in the same way. (Recall that the conditions that control how the apex loci meets the strata have not yet been considered.) To begin with, let \( k = (k_0, k_1, \ldots, k_r) \) be a sequence of nonnegative integers. Define the \( k \)-simplex \( \sigma_k \) to be the convex polytope

\[
\sigma_k = \sigma_r \times C(\sigma_{r-1} \times C(\ldots (\sigma_1 \times C\sigma_0)\ldots))
\]

where each \( \sigma_i \) is an ordinary simplex of dimension \( k_i \). In other words, \( \sigma_k \) is \( \sigma_r \times C\sigma_{k'} \), where \( k' = (k_0, k_1, \ldots, k_{r-1}) \). The number \( r \) is the order of \( \sigma_k \) (and of \( k \)). It is also the number of coning operators. If \( r = 0 \) then \( k = (k_0) \), and \( \sigma_k \) is an ordinary simplex, of dimension \( k_0 \).

For each choice \( \lambda = (\lambda_1, \ldots, \lambda_r) \) of a nonzero value for each of the coning variables in \( \sigma_k \), one obtains a ‘simplex’ \( \sigma_{k,\lambda} \cong \sigma_r \times \ldots \times \sigma_0 \). A formal sum \( \eta \) of continuous maps \( f : \sigma_k \to Z \) is a \( k \)-cycle if the boundary \( d\eta \) is zero, and for each such \( \lambda \) the maps \( f_\lambda : \sigma_{k,\lambda} \to Z \) associated to \( \eta \) are allowed (by the perversity conditions). The \( k \)-cycle \( \eta \) is a \( k \)-boundary if there is a formal sum \( \xi \) of \( k' \)-simplices such that \( \eta = d\xi \), and again the \( f_\lambda \) due to \( \xi \) are also allowed (by the perversity conditions). Here, because pure boundaries are being used, the index \( k' \) ranges over the \( r+1 \) indices obtained via choosing one of the \( k_i \) in \( k \), and raising it by 1. Note that the boundary components of \( \xi \), whose index is not \( k \), must all cancel to zero.
This defines the local-global cycles and boundaries. Each local-global homology group is determined by the choice of an index $k$ (which gives the ‘dimension’ of the cycles), and a choice of the conditions that control how the cycles and boundaries meet the strata. The rest of this section is devoted to the study of these conditions.

Recall that $Z$ is a complex algebraic variety considered, as a stratified topological space. Each stratum $S_i$ has real dimension $2i$. The basic building block for local-global homology, the $k$-simplex $\sigma_k$, can also be thought of as a stratified object. However, instead of strata it has apex loci. Altogether $\sigma_k$ will have $r$ (its order) apex loci. The $k$-simplex $\sigma_k$ is produced using $r$ coning operators. Each coning introduces an apex. Multiplication by the $\sigma_i$, and the subsequent coning operations, will convert this apex into an apex locus; except that any new apex does not belong to the already existing apex locus. The closure of the $i$-th apex locus is a $\sigma_k$-simplex, where $k' = (k_i, k_{i+1}, \ldots, k_r)$.

The conditions, that control how the apex loci meet the strata, are to give rise to groups that are independent of the stratification. From this, the nature of the conditions can be deduced. Here is an example. Suppose $A$ and $B$ are nonsingular projective varieties. Let $Z = A \times CB$ be the product of $A$ with the (projective) cone on $B$, with $Z$ and the apex locus $A$ as the closures of the strata. Each coprimitive $\eta$ on $B$ will (if it can be moved away from the base) determine a local-global cycle $C_\lambda \eta$ on $CB$ and then, by Künneth, the choice of a cycle $\xi$ on $A$ determines a local-global cycle $\xi \otimes C_\lambda \eta$ on $Z$. Suppose now that $\xi$ is the fundamental class $[A]$ on $A$, and that some condition permits $[A] \otimes C_\lambda \eta$. The apex locus of this cycle is the apex locus $A$ of $Z$. Now add strata to $Z$, that lie inside the apex locus $A$. There is no possibility of moving $[A] \otimes C_\lambda \eta$ within its equivalence class, in such a way that the apex locus of the cycle is changed. Thus, this addition of new strata to $A$ will not affect the admissibility of $[A] \otimes C_\lambda \eta$.

Return now to the general case. The previous example can be expressed in the following way. Use the stratification $S_i$ to $Z$ to define the filtration

$$U_i = Z - S_0 - S_1 - \ldots - S_{i-1} = S_i \cup S_{i+1} \cup \ldots \cup S_n$$

of $Z$ by open sets $Z = U_0 \supseteq U_1 \ldots \supseteq U_n$. For each apex locus $A$ of a local-global cycle, ask the following question: what is the largest $i$ such that $A \cap U_i$ is dense in $A$? Call this the $w$-codimension $w(A)$ of $A$. (It tells one where $A$ is generically to be found.) Clearly, adding strata as in the previous example will not change $w(A)$.

Consider once again $Z = A \times CB$. Suppose one wants a local-global cycle on $Z$, whose $w$-codimension has some fixed value $i \leq \dim A$. Such can be achieved, provided $Z$ is suitably stratified. To begin with, let $A_i \subseteq A$ be a subvariety of dimension $i$. Now form the local-global cycle $[A_i] \otimes C_\lambda \eta$, where $[A_i]$ is the class of $A_i$ in $H_i(A)$. If $Z$ and $A$ are the only closures of strata, then $w(A_i)$ will be $\dim A$. However, one can always add $A_i$ to the stratification, and then $w(A_i)$ will be equal to $\dim A_i$, which by construction is equal to $i$.

The meaning of this example is as follows. Strata impose conditions on cycles. The more strata, the more conditions. Suppose a condition allows the $w$-codimension of the apex locus of a cycle to be some number $i$, smaller than $\dim A$. Cycles meeting this condition can be found, when a suitable stratum (the subvariety $A_i$) is added. Thus, the condition must allow the cycle $[A_i] \otimes C_\lambda \eta$, even when $Z$ is given its minimal stratification, if the result is to be stratification independent. Reducing the stratification can only reduce the $w$-codimension. Thus, if a condition allows $i$ as a $w$-codimension, it should also allow all values smaller than $i$. In other words, $i$ should be maximum allowed value for the $w$-codimension.

(Another argument in favour of this conclusion is as follows. The apex locus of a local-global cycle need not be connected. Suppose the apex loci $A$ and $A'$ respectively of $\eta$ and $\eta'$ have $w$-codimensions $i$ and $i'$, with $i < i'$. Suppose also that $i$ is a permitted $w$-codimension. Thus, $\eta$ is
permitted. Now consider $\eta + \eta'$. Although its apex locus need not in every case be $A \cup A'$, so far as $w$-codimension is concerned, it might as well be. Thus, $\eta + \eta'$ will also have $i$ as the $w$-codimension of its apex locus, and so is permitted. Because both $\eta$ and $\eta + \eta'$ are permitted, the difference $(\eta + \eta') - \eta = \eta'$ must also be allowed. In other words, $i$ is a minimum allowed value.)

The local-global intersection homology groups $H_{k,w}(Z)$ can now be defined. The subscript

$k = (k_0, k_1, \ldots, k_r)$

is a sequence of positive integers. The cycles and boundaries of $H_{k,w}(Z)$ are constructed out of embedded $k$-simplices, as defined earlier in this section. The $w$-condition $w$ is a sequence

$w_1 \geq w_2 \geq \ldots \geq w_r \geq 0$

of nonnegative integers, which is used as follows. Each cycle (or boundary) will have $r$ apex loci, to be denoted by $A_1, A_2, \ldots, A_r$. Each apex locus $A_i$ will have a $w$-codimension $w(A_i)$. Only those cycles and boundaries for which

$w_i(A) \leq w_i \quad i = 1, \ldots, r$

holds are to be used in the construction of $H_{k,w}(Z)$. (In this construction, it is assumed that the middle perversity is used. If the $\eta_\lambda$ and $d\xi_\lambda = \eta_\lambda$ are to satisfy some other condition $p$, it can be added to the definition, and also to the notation $H^p_{k,w}(Z)$, like so. Only for certain values of $k$, $w$ and $p$ will nonzero groups be possible. The notation of the previous section takes this into account.)

6 Summary and Conclusions

To close this paper, its main points will be summarized, and then various questions arising are discussed. These are firstly, are the local-global intersection homology groups $H_{k,w}Z$ independent of the stratification of $Z$? Second, does $h\Delta$ compute the (local-global) Betti numbers of $P\Delta$? Third, can the (local-global) homology $H_\bullet\Delta$ of $\Delta$ be constructed without recourse to $P\Delta$? Fourth, what happens when integer coefficients are used? (This question is related to the resolution of singularities.) Fifth, does local-global homology have ring- or functor-like properties. Finally, there is a brief history of the genesis of this paper, and acknowledgements.

The relative importance of the various parts of this paper depend on one’s point of view. The construction ($\S 3$) of local-global cycles on the $IC$ varieties was chosen as the starting point, from which both the formula for $h\Delta$ ($\S 4$) and the topological definition ($\S 5$) followed. For one interested in more general varieties the topological definition is perhaps most important. The examples in $\S 3$ then become merely the application of a more general concept. Finally, not all general polytopes can be studied via topology, and so this gives $h\Delta$ ($\S 4$) and the questions arising from it a special importance. In some sense, $\lambda$-coning and apex loci are the key new concepts, over and above perversity conditions on cycles and boundaries. At the risk of pleasing none, this paper has tried to please all.

The concept of local-global homology has two aspects, namely the cycles and boundaries on the one side, and the strata conditions on the other. In the notation $H_{k,w}Z$, it is $k$ that controls the type of cycles and boundaries, while $w$ supplies the strata conditions. A local-global cycle $\eta$ is a global cycle $\eta(1)$, which can be coned in any of $r$ (the order of $k$ and of $\eta$) $\lambda$-directions. The subscript in $\eta(1)$ indicates that $(1,1,\ldots,1) = (1)$ are the values $\lambda_i$ of the coning variables. The
cycle $\eta$ is to be a suitable family $\eta_\lambda$ of global cycles, which collapses in various ways as each coning variable $\lambda_i$ goes to zero.

These $\lambda_i$ are not independent. If $\lambda_j = 0$ then $\eta_\lambda$ does not depend on $\lambda_i$, for $i < j$. In other words, there is a sequence of collapsings

$$\eta_0 = \eta \gg \eta_1 \gg \eta_2 \gg \ldots \gg \eta_r$$

of the family $\eta = \eta_\lambda$ of cycles. Each $\eta_i$ is the result of placing $\lambda_i = 0$, and is called an apex locus. The sequence $k = (k_0, k_1, \ldots, k_r)$ encodes the dimensions (more exactly the relative dimensions) of these families. These collapsings provide local information about the cycle $\eta$.

Of interest is not only how $\eta$ collapses to $\eta_i$, but also where. By this is meant how $\eta_i$ meets the strata $S_j$ of $Z$. More exactly, $w(\eta_i)$ is defined to be the largest $j$ such that

$$\eta_i \cap (S_j \cup S_{j+1} \cup \ldots \cup S_n)$$

is dense in $\eta_i$. This is a measure of how special a locus is required, to generically effect the collapsing of $\eta$ to $\eta_i$. The index $w = (w_1, w_2, \ldots, w_r)$ in $H_{k,w}Z$ controls the cycles and boundaries used as follows. A cycle $\eta$ (or boundary $\xi$) is to be used only if $w(\eta_i) \geqslant w(\xi_i)$ is at least $w_i$. Thus, reducing $w$ will potentially allow more cycles (and more relations) to participate in $H_{k,w}Z$. Whether this increases or decreases the size of $H_{k,w}Z$ will just depend on the circumstances. Sections 3 and 4 apply these concepts to $IC$ and toric varieties. This ends the summary.

The next topic is stratification independence. It is important that the $H_{k,w}Z$ not depend on the choice of the stratification. This is already known for mphp. First suppose $w$ is zero, so the ‘where’ conditions are vacuous. Suppose also that $\eta$ is a k-cycle, namely a $\lambda$-coning of the global cycle $\eta_{(1)}$. Already known is that if the stratification is changed, an $\eta'$ equivalent to $\eta_{(1)}$ can be found. This is a local result, and so the $\eta'$ can be found close to $\eta_{(1)}$. Thus, in this special case stratification independence will follow if the $\lambda$-coning structure that converts $\eta_{(1)}$ to $\eta$ can be extended to a small neighbourhood of $\eta$. This is more a statement about the local conical structure of a stratified topological space, than about the particular cycle $\eta$.

Now suppose $w$ is nonzero. As before, $\eta'$ can be found close to $\eta_{(1)}$. The new difficulty is the $w(\eta_i)$. By definition, this will cause no new problem, unless $w(\eta_i)$ is reduced. In this case, more must be done. Previously, $\eta_{(1)}$ was moved to $\eta'$, and it was assumed that $\eta'$ could be collapsed to the $\eta_i$ used for $\eta$. In this case, it is necessary to move the $\eta_i$. To do this, note that the collapsing $\eta \gg \eta_i$ will be a permissible coning away, for some perversity other that the middle. As stratification independence is known for all perversities, this may provide a means of producing the $\eta'_i$. As before, one hopes that uniformity of the $\lambda$-coning will complete the proof.

These arguments indicate that stratification independence for local-global homology will follow from uniformity of the $\lambda$-coning and some modification of the existing methods. There is however another approach. King [13] was able to prove stratification independence directly, without use of sheaves in the derived category. Habegger and Saper [14] found that this leads to an intersection homology theory for generalised perversities and local systems. Usually, a local system on $Z$ is determined by its value on the generic stratum $S_n \subset Z$. When generalised, information about behaviour at the boundary (the smaller $S_i$) is recorded by the local system. It may be that local-global homology can be expressed as the homology of such a local system. From this, stratification independence would follow.

(Certainly, the concepts are related. If $\eta$ is a local-global cycle, collapsing under $\lambda$ to $\eta_1$, and if $\eta_1 \cap S_j$ is dense in $\eta_1$, then the following holds. There is a local system $L_j$ (in the usual sense) on $S_j$ such that $\eta$ both determines and is determined by a cycle $\eta'$ on $S_j$ with coefficients in $L_j$.

(Because $S_j$ may not be closed, $\eta'$ might not be a compact cycle. This is a technical matter.)
Consider this $\eta'$. The Goresky-MacPherson theory imposes conditions on the dimension in which the closure $\bar{\eta'}$ of $\eta'$ meets the strata. For local-global homology, $\eta'$ determines a local-global cycle $\eta_0$ in a neighbourhood of $S_j$. Whether or not such an $\eta_0$ can be closed up to produce an $\eta$ is a delicate matter, which depends not so much on how $\eta'$ meets the smaller strata, but on how the local topology of $Z$ about these strata interacts with $\eta_0$. The generalised concept of a local system may provide a place where this information can be stored, and used to control $\eta'$.

This discussion has assumed that existing techniques, perhaps adapted, will be applied to local-global homology. However, stratification independence is primarily a technical result on the local topology of stratified spaces. It may be that local-global homology will provide a suitable language for describing this local topology. One must show that it causes no obstruction to the motion of cycles and boundaries, that is required when the stratification is refined. If this holds, then the concept of local-global homology is already implicit in the proof of stratification independence, and the consequences of King’s paper become less surprising.

Each Betti number of $P_{\Delta}$ is the dimension of a vector space, and so is nonnegative. Thus, a linear function that computes such a Betti number from $f\Delta$ is also a linear inequality on $f\Delta$, at least when $\Delta$ has rational vertices. Of special interest therefore are those parts of homology theory, for which the Betti numbers are indeed linear functions of $f\Delta$. Ordinary homology does not have this property [19]. Middle perversity intersection homology does [3]. This is a consequence of deep results in algebraic geometry, namely Deligne’s proof [17] of the Weil conjectures and the purity of mpih [3]. It is possible that at least some part of the local-global homology theory will also have this purity property. An example of Bayer (personal communication) shows that certain components of the extended $h$-vector are sometimes negative, and so are not always Betti numbers.

(Bayer’s example is $\Delta = BICCC\bullet$, where $B$ is the bipyramid operator, the combinatorial dual to the $I$ operator. In $h\Delta$ the term $xA\{1\}$ occurs with coefficient $-2$. The interpretation of this result requires some care. It does not show that the whole of $h\Delta$ is unsuitable, or that local-global homology is a flawed concept. The formula for $h\Delta$ is the extrapolation to all polytopes of the heuristically calculated formula for the various local-global homology Betti numbers of the $IC$ varieties. The component $xA\{1\}$ corresponds to the gluing together along paths of local cycles, a construction that is already known to yield a topological invariant. More exactly, $xA\{1\}$ counts local 1-cycles under this equivalence, on a 4-dimensional variety that has had its 0-strata removed. The corresponding Betti number is as computed by $h\Delta$, for the $IC$ varieties, but clearly not in general.)

(Ordinary homology can be approached in the same way. Use the simple case formula $h(x, x + y) = f(x, y)$ to define a ‘pseudo $h$-vector’ $h^2\Delta$ for general polytopes. The rules

$$h^2(I\Delta) = (11)h^2(\Delta); \quad h^2(C\Delta) = (100\ldots0) + yh^2(\Delta)$$

give its transformation under $I$ and $C$. The author suspects, but does not know, that $h^2\Delta$ gives the ordinary homology Betti numbers, when $\Delta$ is an $IC$ variety. However, when extrapolated to the octahedron, $(1, -1, 5, 1)$ is the result. Despite this negative value, ordinary homology is still a topological invariant. However, it is not suitable for the study of general polytope flag vectors.)

For $\Delta$ rational the mpih part $(h_0, h_1, \ldots, h_n)$ of $h\Delta$ is not only nonnegative but also unimodal. This means that $h_0 \leq h_1 \leq \ldots$ and so on up to halfway. It is a consequence of strong Lefschetz. The author suspects that the analogous result for general polytopes will be as follows. First compute $hC\Delta$. This is of course a linear function of $h\Delta$. Now look at the coefficients of $x^0y^0W$, where $W$ is a word in $A$ and $\{k\}$. These numbers are expected to be nonnegative. This is an extension of the mpih result. Of course, algebraic geometry can prove such results only when $\Delta$ is rational, for only then does $P_{\Delta}$ exist.
The only method that will produce such results for general polytopes, that the author can envision, is *exact calculation of homology*. This means constructing a complex of vector spaces (of length at most $n$), whose Euler characteristic (alternating sum of dimensions) is the desired Betti number. Exact calculation then consists of proving that this complex is exact, except possibly at one location. The homology of this complex at that location is then a vector space which as a consequence of exactness has the desired dimension. In [9] the author introduces such complexes. The proof of exactness is expected to be difficult. McMullen’s proof [21] of strong Lefschetz for simple polytopes, without the use of algebraic varieties, probably contains a prototype of the arguments that will be needed. There, the Riemann-Hodge inequalities were vital in supporting the induction on the dimension of the polytope. Thus, one would like local-global homology to support similar inequalities.

The process of constructing the complexes for exact calculation requires that the linear function $h_\Delta$ on polytope flag vectors be defined for all flag vectors, whether of a polytope or not. This allows one to talk about the contribution made by an individual flag to $h_\Delta$. (One then interprets this contribution as the dimension of a vector space associated to the flag, and then assembles all these vector spaces into a complex. The boundary map is induced by the deletion of a single term from the flag. In [9], the numerical contribution due to a flag is derived from a study of the associated vector spaces, which is the starting point. This approach better respects the inner logic of the exact calculation concept.)

The extension of $h_\Delta$ to all flag vectors (not just those of polytopes) can be done in the following way. Suppose $\delta$ is an $i$-face on $\Delta$. The local combinatorial structure of $\Delta$ along $\delta$ can be represented by an $(n - i - 1)$ dimensional polytope, the *link* $L_\delta$ of $\Delta$ along $\delta$. If $g_i$ is a linear function on flag vectors then the expression

$$h_\Delta = \sum_{\delta \subseteq \Delta} g_i B$$

is a linear function on the flag vector of $\Delta$. (Here, $\delta$ runs over all faces of $\Delta$, $i$ is the dimension of $\delta$, and $B$ is the link along $\delta$.) In this paragraph, $h_\Delta$ is a linear function that might or might not be the previously defined extended $h$-vector. Now use the rules

$$g_0 B = ChB - yhB$$

$$g_{i+1} B = yg_i B - g_i CB$$

and the initial value $g_0(\emptyset) = (1)$ to produce a recursive definition of $g$ and $h$. The $ChB$ in [9] stands for the rule $\tilde{C}$ of §4, translated into a $x, y, A$ and $\{k\}$ rule, and then applied to $hB$. (Here, $B$ has dimension less than that of $\Delta$, and so $hB$ is by induction already defined.) This defines a linear function $h_\Delta$, which however can now be applied to non-polytope flag vectors. For polytope flag vectors, it agrees with the previously defined value. (Central to the proof of this is the expression of the links on $I\Delta$ and $C\Delta$ in terms of those on $\Delta$. This approach does not rely on the $IC$ equation.)

The mpih part of (8) is the usual formula [6, 7, 25] for these Betti numbers. The whole of (8) can be ‘unwound’ to express $h_\Delta$ as a sum of numerical contributions due to individual flags. The mpih part has been presented in [1]. Note that just as the topological space $P_\Delta$ can be decomposed into cells in many ways, so its homology can be computed in various ways. To each suitable such method an extension of $h_\Delta$ to all flag vectors will follow, and vice versa. An extension which on simple polytopes reduces to the $h(x, x + y) = f(x, y)$ formula could be very useful. Finally, note that when $h_\Delta$ is a sum of flag contributions one can use ‘Morse theory’ or shelling to compute $h_\Delta$. Choose a linear ‘height’ function, so that each vertex on $\Delta$ has a distinct height. Define the index of a vertex $v$ to be the sum of the contributions due to the flag whose first term has $v$ as its highest point. It immediately follows that $h_\Delta$ is the sum, over all vertices, of their index.
Central to the problem of resolution of singularities (which is still open in finite characteristic) is the discovery of suitable invariants of singular varieties. These should be well behaved under monoidal transformation, and always permit the choice of a centre of transformation, which will reduce the invariant. When the invariant becomes zero, the variety should be nonsingular. It may be that some form of local-global homology will have such properties. One aspect of the problem is this. Suppose the singular locus consists of two lines meeting at a point. Along each line there is a locally constant singularity, whose resolution is essentially a lower dimensional problem, which can by induction be assumed solved. The difficulty is at the meeting point of the two lines. Along each line there is a resolution process. One needs to know whether and when a transformation should be centered at the common point.

In the study of this process one should use not only the local cycles due to mpih, but also those due to the higher order local-global homology. (In addition, one should use integer rather than rational coefficients, but more on this later.) The local cycles generic along each of the two lines will influence if not control the resolution process along that line. The concepts of local-global homology allow the interaction to be studied at their common point, of these ‘controls’ along the two lines of the resolution process. This argument indicates that local-global homology contains the right sort of information, for it to act as a suitable control on the resolution process. It does not show that it contains enough such information.

Torsion is when a cycle is not a boundary, but some multiple of it is. Using rational or real coefficients sets all such torsion cycles equal to zero. This simplifies the theory, and for the study of Betti numbers and the flag vectors of convex polytopes, such complications are not needed. For resolution of singularities, and more subtle geometric problems, the situation is otherwise. Here is an example. Consider the affine surface \(X = \{xy = z^k\}\), for \(k \geq 2\). This has a singularity at the origin. However, there are no non-trivial local-global cycles on \(X\), according to the definitions of §5. The divisor class group \(\text{Cl}(X)\) however is nontrivial, and all its elements are torsion. For example, the line \(L = \{x = z = 0\}\) is not defined by a principal ideal, whereas \(kL\) is defined by \(\{x = 0\}\).

Such information can be recorded by local-global homology, provided integer coefficients and a different concept of a local cycle are used. It is known that duality has a local expression, which pairs compact cycles avoiding say the apex of a cone, and cycles of complementary dimension, that meet the apex. Duality, of course, ignores torsion cycles. However, non-trivial but torsion local cycles exist for \(X = \{xy = z^k\}\), when the complementary concept of cycle is used. It is interesting that the construction of local-global cycles associated to the formula (8) produces such cycles, rather than compact local-global cycles.

It is natural to ask: does the vanishing of all such local-global cycles (except mpih of course) imply that the variety is either nonsingular, or a topological manifold? Because the Brieskorn hypersurface singularity \(x_1^2 + x_2^2 + x_3^2 + y^2 + z^5 = 0\) is locally homeomorphic to the cone on an exotic 7-sphere \(\mathbb{R}\), which is knotted inside a 9-sphere, purely topological invariants are not enough to ensure the nonsingularity of a variety \(Z\). The same may not be true for an embedded variety \(Z \subset \mathbb{P}_n\).

Ordinary homology and cohomology are functors. This distinguishes them from intersection homology. For mpih and, one hopes, a good part of the local-global homology, it is purity and formulae for Betti numbers that are the characteristic properties. Both concepts agree, of course, on nonsingular varieties. Thus, one of the geometric requirements on local-global homology is as follows. Suppose \(f : Z_1 \to Z_2\) is a map between, say, two projective varieties. Suppose also that the strictly local-global homology of both \(Z_1\) and \(Z_2\) vanishes. It may be that this condition ensures that the \(Z_i\) are, say, rational homology manifolds. If this is so, then \(f\) induces a map \(f_* : H_\bullet Z_1 \to H_\bullet Z_2\), which can be thought of as a point in \((H_\bullet Z_1)^* \otimes H_\bullet Z_2\). In the general case one
would want $f$ to induce some similar map or object in a space, which reduces to the previous $f_*$ when the strictly local-global homology vanishes.

Similarly, when $Z$ is nonsingular its homology carries a ring structure. (It is this structure that supplies the pseudopower inequalities \[20\] on the flag vector of a simple polytope.) Elsewhere \[10\], the author provides for a general compact $Z$ a similar structure, that in the nonsingular case reduces to the homology ring.

To close, some personal historical remarks, and acknowledgments. In 1985, upon reading the fundamental paper \[2\] of Bayer and Billera, it became clear to the author that a full understanding of general convex polytopes would require a far-reaching extension to the theory of intersection homology. The background to this insight came from Stanley’s proof \[24\] of the necessity of McMullen’s conjectured conditions \[20\] on the face vectors of simple polytopes, and the proof by Billera and Lee of their sufficiency. Danilov’s exposition \[3\] of toric varieties, and McConnell’s result \[19\], also contributed. It was also clear that once the extended $h$-vector was known, the rest would soon follow.

Already in the simple case there is an interplay between the topological definition of homology, $h$-vectors, and ‘combinatorial linear algebra’ or exact calculation. The same holds for middle perversity intersection homology. In 1985 the Bernstein-Khovanski-MacPherson formula was known, although not published until 1987 \[25\]. Because of this circle of ideas, knowledge of say the $h$-vector is sufficient in practice to determine the other parts of the theory. This led the author to find the $IC$ equation \[8\] (again, known in 1985 but not published until much later). This moved the focus on to the rules for the transformation of $h\Delta$ under $I$ and $C$.

In the early years of the search for these rule, two related and erroneous ideas were influential. The first is that the rule for $I$ (as via Künneth) should be multiplication by $(x + y)$. The second is that the generalised Dehn-Somerville equations should be expressed by $h\Delta$ being palindromic. (Also unhelpful was an undue concentration on the formula for $h\Delta$.) The crucial step that lead to these assumptions being dropped took place in 1993. Loosely speaking, it was the discovery of special cases of the ‘gluing local cycles together along paths’ construction. At that time how to form families of local cycles, or in other words the ‘local-global’ concept, was still mysterious.

In late 1995 the present formula for $h\Delta$ was discovered, as a solution to the various geometric, topological and combinatorial constraints that were known. It was not at that time properly understood by the author. This definition was in 1996 pushed around the circle of ideas, to produce first a combinatorial linear algebra construction for $H_k\Delta$, and then the topological definition. That all this can be done indicates that the definition of $h\Delta$ is correct. Once these advances had been understood, it was then possible to return to the derivation of the formula for $h\Delta$, and put it on a proper footing. It was only at this point that the fundamental concepts became clear. In 1997, the analogue to the ring structure was found. Its relation to the concepts presented here is, at the time of writing, still under investigation.

Many of the results in this paper were first made available as preprints and the like, in 1996 and 1997. The difficulties encountered by the readers have lead to many revisions in the exposition, and clarification of the basic concepts, both on paper and in the author’s understanding. Part of the difficulty is the extent of the circle of ideas, which passes through several areas of mathematics. Any one of topology, combinatorics, linear algebra and intersection theory can be chosen as the starting point. Another difficulty is that much of the intuition and guidance comes from perhaps uncomplicated examples and points of view that have, by and large, not yet been put into print.

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