SPIN CHAIN FROM MEMBRANE AND THE
NEUMANN-ROSOCHATIUS INTEGRABLE SYSTEM

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We find membrane configurations in $AdS_4 \times S^7$, which correspond to the continuous limit of the $SU(2)$ integrable spin chain, considered as a limit of the $SU(3)$ spin chain, arising in $\mathcal{N} = 4$ SYM in four dimensions, dual to strings in $AdS_5 \times S^5$. We also discuss the relationship with the Neumann-Rosochatius integrable system at the level of Lagrangians, comparing the string and membrane cases.

Keywords: M-theory, integrable systems, AdS-CFT duality.

1 Introduction

One of the predictions of AdS/CFT duality is that the string theory on $AdS_5 \times S^5$ should be dual to $\mathcal{N} = 4$ SYM theory in four dimensions [1], [2], [3]. The spectrum of the string states and of the operators in SYM should be the same. The first checks of this conjecture beyond the supergravity approximation revealed that there exist string configurations, which in the semiclassical limit are related to the anomalous dimensions of certain gauge invariant operators in the planar SYM [4], [5]. On the field theory side, it was found that the corresponding dilatation operator is connected to the Hamiltonian of integrable Heisenberg spin chain [6]. On the other hand, it was established in [7] that there is agreement at the level of actions between the continuous limit of the $SU(2)$ spin chain arising in $\mathcal{N} = 4$ SYM theory and a certain limit of the string action in $AdS_5 \times S^5$ background. Shortly after, it was shown that such equivalence also holds for the $SU(3)$ and $SL(2)$ cases [8], [9]. Information about the latest developments on the subject can be found for example in [20] and references therein. Let us also point out the work [21], where the connection between the worldvolume theory of membranes living on flat space and integrable quantum spin chains has been explored.

Here, we are interested in answering the question: is it possible to reproduce this type of string/spin chain correspondence from membranes on eleven dimensional curved backgrounds? It turns out that the answer is positive at least for the case of M2-branes

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2See also [10] [11] [12] [13] and [14] [15] [16] [17] [18] [19].
on $AdS_4 \times S^7$, as we will show later on. More precisely, we will find that the action for the continuous limit of $SU(2)$ integrable spin chain, considered as a certain limit of the $SU(3)$ chain, can be obtained from particular membrane configurations.

This investigation is motivated by the following reasons. First of all, it will shed light on some properties of the conjectured duality between membranes on $AdS_4 \times S^7$ and three dimensional CFT [1]. In particular, our results will give a support for the existence of an integrable sector on the field theory side. On the other hand, strings on $AdS_5 \times S^5$ and membranes on $AdS_4 \times S^7$ are dual to different gauge theories. Hence, there should exist common integrable sectors in the corresponding field theories.

The plan of the paper is as follows. In section 2, we reconsider the $SU(3) \to SU(2)$ string case in diagonal worldsheet gauge. Section 3 is devoted to membranes on $AdS_4 \times S^7$. In section 4, we discuss the relationship with the Neumann-Rosochatius integrable system at the level of Lagrangians, comparing the string and membrane cases.

## 2 Strings on $AdS_5 \times S^5$

It is known that the ferromagnetic integrable $SU(3)$ spin chain provides the one-loop anomalous dimension of single trace operators involving the three complex scalars of $\mathcal{N} = 4$ SYM. The nonlinear sigma-model, describing the continuum limit of the $SU(3)$ spin chain, corresponds to strings moving with large angular momentum on the five-sphere in $AdS_5 \times S^5$ [8].

In order to have more close analogy with the membrane case considered in the next section, we will reproduce the relevant string action in the framework of diagonal worldsheet gauge. In this gauge, the Polyakov action and constraints are given by

\[
S_S = \int d^2\xi L_S = \int d^2\xi \frac{1}{4\lambda^0} \left[ G_{00} - (2\lambda^0 T)^2 G_{11} \right],
\]

\[
G_{00} + (2\lambda^0 T)^2 G_{11} = 0,
\]

\[
G_{01} = 0,
\]

where

\[
G_{mn} = g_{MN} \partial_m X^M \partial_n X^N,
\]

\[
\left[ \partial_m = \partial/\partial \xi^m, \quad m = (0, 1), \quad (\xi^0, \xi^1) = (\tau, \sigma), \quad M = (0, 1, \ldots, 9) \right],
\]

is the induced metric and $\lambda^0$ is Lagrange multiplier. The commonly used conformal gauge corresponds to $2\lambda^0 T = 1$.

We choose to embed the string in $AdS_5 \times S^5$ as follows

\[
Z_s = R r_s (\xi^m) e^{i\phi_s (\xi^m)}, \quad s = (0, 1, 2), \quad \eta^{rs} r_r r_s + 1 = 0, \quad \eta^{rs} = (-1, 1, 1),
\]

\[
W_i = R r_i (\xi^m) e^{i\phi_i (\xi^m)}, \quad i = (1, 2, 3), \quad \delta_{ij} r_i r_j - 1 = 0,
\]

where $\phi_s$ and $\phi_i$ are the isometric coordinates on which the metric of $AdS_5$ and $S^5$ respectively does not depend. The embedding coordinates $Z_s, W_i$ are related to the ones
in (2.4) by the equalities (φ₀ = t is the time coordinate on AdS₅)

\[
Z_0 = R \cosh ρ e^{iφ₀},
Z_1 = R \sinh ρ \sin θ e^{iφ₁},
Z_2 = R \sinh ρ \cos θ e^{iφ₂},
W_1 = R \sin γ \cos ψ e^{iφ₁},
W_2 = R \sin γ \sin ψ e^{iφ₂},
W_3 = R \cos γ e^{iφ₃}.
\]

For this ansatz, \(G_{mm}\) reduces to

\[
G_{mm} = \eta^{rs} \partial_{(m} Z_r \partial_{n)} \bar{Z}_s + \delta_{ij} \partial_{(m} W_i \partial_{n)} \bar{W}_j = \begin{align*}
R^2 \left[ \sum_{r,s=0}^{2} \eta^{rs} \left( \partial_m r_i \partial_n r_s + r_i^2 \partial_m φ_i \partial_n φ_s \right) + \sum_{i=1}^{3} \left( \partial_m r_i \partial_n r_i + r_i^2 \partial_m φ_i \partial_n φ_i \right) \right],
\end{align*}
\]

where (…) on the first line means symmetrization. The expression (2.6) for \(G_{mn}\) must be used in (2.1), (2.2) and (2.3). Correspondingly, the string Lagrangian will be

\[
\mathcal{L} = \mathcal{L}_S + Λ_A (\eta^{rs} r_r r_s + 1) + Λ_S (δ_{ij} r_i r_j - 1),
\]

where Λₐ and Λₛ are Lagrange multipliers.

Here, we are interested in the following particular case of the string embedding (2.5)

\[
Z_0 = Re^{iκτ}, \quad Z_1 = Z_2 = 0,
\]

which implies

\[
r_0 = 1, \quad r_1 = r_2 = 0; \quad φ₀ = t = κτ, \quad κ = \text{const.}
\]

For this ansatz, \(G_{mm}\) reduces to

\[
G_{mm} = R^2 \left[ \sum_{i=1}^{3} \left( \partial_m r_i \partial_n r_i + r_i^2 \partial_m φ_i \partial_n φ_i \right) - δ_{n}^{0} δ_{n}^{0} κ^2 \right].
\]

We now introduce new coordinates according to the rule

\[
(φ₁, φ₂, φ₃) = (κτ + α + φ, κτ + α - φ, κτ + α + φ)
\]

and take the limit \(κ → ∞, \partial_0 → 0, κ\partial_0\)-finite. The result is

\[
\begin{align*}
G_{00} &= 2R^2 κ \left[ \partial_0 α + (r_1^2 - r_2^2) \partial_0 φ + r_3^2 \partial_0 φ \right],
G_{11} &= R^2 \left\{ \sum_{i=1}^{3} (\partial_i r_i)^2 + (r_1^2 + r_2^2) (\partial_i φ)^2 + r_3^2 (\partial_i φ)^2 \right. \\
&\left. + (\partial_1 α)^2 + 2∂_1 α \left[ (r_1^2 - r_2^2) \partial_1 φ + r_3^2 \partial_1 φ \right] \right\},
G_{01} &= R^2 κ \left[ \partial_1 α + (r_1^2 - r_2^2) \partial_1 φ + r_3^2 \partial_1 φ \right].
\end{align*}
\]

³Of course, other parameterizations of AdS₅ × S⁵, which ensure that the embedding constraints \(η^{rs} r_r r_s + 1 = 0\) and \(δ_{ij} r_i r_j - 1 = 0\) are satisfied identically, are also possible.
In order to eliminate $\partial_1 \alpha$, we use the constraint $G_{01} = 0$ and obtain the following two alternative expressions for the string action after introducing the variable $t = \kappa \tau$

$$S = \int d\tau d\sigma \mathcal{L}_{SC} = \frac{R^2 \kappa}{2 \lambda^0} \int d\tau d\sigma \left[ \partial_t \alpha + (r_1^2 - r_2^2) \partial_t \varphi + r_3^2 \partial_t \phi \right] - \frac{\lambda^0 (TR)^2}{\kappa} \int d\tau d\sigma \left\{ \sum_{i=1}^3 (\partial_1 r_i)^2 + \left[ (r_1^2 + r_2^2) - (r_1^2 - r_2^2)^2 \right] (\partial_1 \varphi)^2 + (r_1^2 + r_2^2) r_3^2 (\partial_1 \phi)^2 \right\}$$

$$- 2(r_1^2 - r_2^2) r_3^2 \partial_1 \varphi \partial_1 \phi + \frac{1}{\kappa} \int d\tau d\sigma \Lambda_s \left( \sum_{i=1}^3 r_i^2 - 1 \right)$$

$$= \frac{R^2 \kappa}{2 \lambda^0} \int d\tau d\sigma \left[ \partial_0 \alpha + (r_1^2 - r_2^2) \partial_0 \varphi + r_3^2 \partial_0 \phi \right] - \frac{\lambda^0 (TR)^2}{\kappa} \int d\tau d\sigma \left\{ \sum_{i=1}^3 (\partial_1 r_i)^2 \right\}$$

$$+ \frac{4 r_1^2 T^2}{r_1^2 + r_2^2} (\partial_1 \varphi)^2 + (r_1^2 + r_2^2) r_3^2 \left[ \frac{(r_1^2 - r_2^2)}{r_1^2 + r_2^2} \partial_1 \varphi - \partial_1 \phi \right]^2 \right\}$$

$$+ \frac{1}{\kappa} \int d\tau d\sigma \Lambda_s \left( \sum_{i=1}^3 r_i^2 - 1 \right).$$

The momentum $P_\alpha$ conjugated to $\alpha$ should be identified with the total angular momentum of the string $J$

$$P_\alpha = \frac{\pi R^2 \kappa}{\lambda^0} \equiv J.$$

Then the coefficients in the action become

$$\frac{R^2 \kappa}{2 \lambda^0} = \frac{J}{2 \pi}, \quad \frac{\lambda^0 (TR)^2}{\kappa} = \frac{\lambda}{4 \pi J},$$

where we have used the relation $TR^2 = \sqrt{\lambda}/2\pi$ between the string tension $T$ and the 't Hooft coupling $\lambda$.

If we parameterize the two-sphere in the following way

$$r_1 = \cos \psi \cos \theta, \quad r_2 = \sin \psi \cos \theta, \quad r_3 = \sin \theta,$$

(2.8) reduces to

$$S = \frac{J}{2 \pi} \int d\tau d\sigma \left[ \partial_t \alpha + \cos^2 \theta \cos(2\psi) \partial_t \varphi + \sin^2 \theta \partial_t \phi \right]$$

$$- \frac{\lambda}{4 \pi J} \int d\tau d\sigma \left\{ (\partial_\sigma \theta)^2 + \cos^2 \theta \left[ (\partial_\sigma \psi)^2 + \sin^2(2\psi) (\partial_\sigma \varphi)^2 \right] \right\}$$

$$+ \frac{1}{4} \sin^2(2\theta) \left[ \cos(2\psi) \partial_\sigma \varphi - \partial_\sigma \phi \right]^2 \right\}.$$

This is the string action corresponding to the thermodynamic limit of $SU(3)$ spin chain after the identification $J \equiv L$ is made, where $L$ is the length of the chain [8].
The particular case of $SU(2)$ spin chain corresponds to $r_3 = 0$ in (2.8) or $\theta = 0$ in (2.9). In order to make connection with the membrane case, let us fix $r_2^2 = \varepsilon^2$ and take the limit $\varepsilon^2 \to 0$ in (2.8). Neglecting the higher order terms, one obtains

$$S = \frac{R^2 \kappa}{2 \lambda_0} \int d\tau d\sigma \left[ \partial_0 \alpha + (r_1^2 - r_2^2) \partial_0 \varphi \right]$$

$$- \lambda^0 (TR)^2 \int d\tau d\sigma \left\{ \sum_{a=1}^2 (\partial_1 r_a)^2 + (r_1^2 + r_2^2) - (r_1^2 - r_2^2)^2 \right\} (\partial_1 \varphi)^2 \right\}$$

$$+ \int d\tau d\sigma \Lambda_S \left[ \sum_{a=1}^2 r_a^2 - (1 - \varepsilon^2) \right].$$

According to the constraint in the above action, the coordinates $r_1, r_2$ must lie on a circle with radius $(1 - \varepsilon^2)^{1/2}$. To satisfy this constraint identically, we choose

$$r_1 = (1 - \varepsilon^2)^{1/2} \cos \psi, \quad r_2 = (1 - \varepsilon^2)^{1/2} \sin \psi,$$

and receive

$$S/(1 - \varepsilon^2) = \frac{R^2 \kappa}{2 \lambda_0} \int d\tau d\sigma \left[ \partial_1 \tilde{\alpha} + \cos(2\psi) \partial_1 \varphi \right]$$

$$- \frac{\lambda^0 (TR)^2}{\kappa} \int d\tau d\sigma \left[ (\partial_\sigma \psi)^2 + \sin^2(2\psi)(\partial_\sigma \varphi)^2 \right],$$

where the new variable $\tilde{\alpha}$ has been introduced through the equality

$$\alpha = (1 - \varepsilon^2)\tilde{\alpha}.$$

Obviously, the right hand side of (2.11) coincides with the string action corresponding to the thermodynamic limit of the $SU(2)$ spin chain action.

### 3 Membranes on $AdS_4 \times S^7$

Turning to the membrane case, let us first write down the gauge fixed membrane action and constraints in diagonal worldvolume gauge, we are going to work with:

$$S_M = \int d^3 \xi \mathcal{L}_M = \int d^3 \xi \left\{ \frac{1}{4 \lambda_0^2} \left[ G_{00} - (2\lambda^0 T_2)^2 \det G_{ij} \right] + T_2 C_{012} \right\},$$

$$G_{00} + (2\lambda^0 T_2)^2 \det G_{ij} = 0,$$

$$G_{0i} = 0.$$

They coincide with the frequently used gauge fixed Polyakov type action and constraints after the identification $2\lambda^0 T_2 = L = \text{const}$, where $\lambda^0$ is Lagrange multiplier and $T_2$ is the membrane tension. In (3.1)-(3.3), the fields induced on the membrane worldvolume $G_{mn}$ and $C_{012}$ are given by

$$G_{mn} = g_{MN} \partial_m X^M \partial_n X^N, \quad C_{012} = c_{MNP} \partial_0 X^M \partial_1 X^N \partial_2 X^P,$$

$$\partial_m = \partial/\partial M^m, \quad m = (0, i) = (0, 1, 2),$$

$$\left( \xi^0, \xi^1, \xi^2 \right) = (\tau, \sigma_1, \sigma_2), \quad M = (0, 1, \ldots, 10).$$
where $g_{MN}$ and $c_{MNP}$ are the components of the target space metric and 3-form gauge field respectively.

Searching for membrane configurations in $AdS_4 \times S^7$ dual to integrable spin chains, we should first eliminate the membrane interaction with the background 3-form field on $AdS_4$, to ensure more close analogy with the strings on $AdS_5 \times S^5$. To make our choice, let us write down the background. It can be parameterized as follows

$$ds^2 = (2l_p R)^2 \left[ -\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho (d\alpha^2 + \sin^2 \alpha d\beta^2) + 4d\Omega_7^2 \right],$$

$$c^{(3)} = (2l_p R)^3 \sinh^3 \rho \sin \alpha dt \wedge d\alpha \wedge d\beta.$$

Since we want the membrane to have nonzero conserved energy and spin on $AdS$, the choice for which the interaction with the $c^{(3)}$ field disappears is

$$\alpha = \alpha_0 = \text{const}.$$

The metric of the corresponding subspace of $AdS_4$ is

$$ds^2_{\text{sub}} = (2l_p R)^2 \left[ -\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho \sin^2 \alpha_0 d\beta^2 \right] = (3.5)$$

Hence, the appropriate membrane embedding into $AdS_5 \times S^5$, analogous to the string embedding in $AdS_5 \times S^5$, is

$$Z_\mu = 2l_p R r_\mu (\xi^m) e^{i\phi_\mu (\xi^m)}, \quad \mu = (0, 1), \quad \phi_\mu = (\phi_0, \phi_1) = (t, \beta \sin \alpha_0),$$

$$\eta^{\mu\nu} r_\mu r_\nu + 1 = 0, \quad \eta^{\mu\nu} = (-1, 1),$$

$$W_a = 4l_p R r_a (\xi^m) e^{i\phi_a (\xi^m)}, \quad a = (1, 2, 3, 4), \quad \delta_{ab} r_a r_b - 1 = 0.$$(3.6)

For this embedding, the induced metric is given by

$$G_{mn} = \eta^{\mu\nu} \partial_m Z_\mu \partial_n Z_\nu + \delta_{ab} \partial_m W_a \partial_n W_b = (3.7)$$

We will use the expression (3.7) for $G_{mn}$ in (3.1), (3.2) and (3.3). Correspondingly, the membrane Lagrangian becomes

$$\mathcal{L} = \mathcal{L}_M + \Lambda_A (\eta^{\mu\nu} r_\mu r_\nu + 1) + \Lambda_S (\delta_{ab} r_a r_b - 1).$$

In this paper, we are interested in the following particular case of the membrane embedding (3.6)

$$Z_0 = 2l_p R e^{i\kappa t}, \quad Z_1 = 0,$$(3.9)

\footnote{Of course, we can fix the angle $\beta$ instead of $\alpha$. We choose to fix $\alpha$ because $\beta$ is one of the isometry coordinates in the initial $AdS_4$ space.}
which implies
\[ r_0 = 1, \quad r_1 = 0, \quad \phi_0 = t = \kappa \tau. \]

For this ansatz, the metric induced on the membrane worldvolume simplifies to
\[ G_{mn} = (4L_p R)^2 \left[ \sum_{a=1}^{4} \left( \partial_m r_a \partial_n r_a + r_a^2 \partial_m \varphi_a \partial_n \varphi_a \right) - \delta_0^a \delta_0^a (\kappa/2)^2 \right], \quad (3.10) \]
and the membrane Lagrangian is given by
\[ \mathcal{L} = \mathcal{L}_M + \Lambda_S \left( \sum_{a=1}^{4} r_a^2 - 1 \right). \quad (3.11) \]

Let us now introduce new coordinates by setting
\[ (\varphi_1, \varphi_2, \varphi_3, \varphi_4) = \left( \frac{\kappa}{2} \tau + \alpha + \varphi, \frac{\kappa}{2} \tau + \alpha - \varphi, \frac{\kappa}{2} \tau + \alpha + \phi, \frac{\kappa}{2} \tau + \alpha - \phi \right). \]

Then, the induced metric \((3.10)\) takes the form
\[
G_{mn} = (4L_p R)^2 \left\{ \frac{\kappa}{2} \left( \delta_m^\alpha \partial_n \alpha + \delta_n^\alpha \partial_m \alpha \right) + \frac{\kappa}{2} \delta_m^0 \left[ (r_1^2 - r_2^2) \partial_n \varphi + r_3^2 \partial_n \phi - r_4^2 \partial_n \tilde{\varphi} \right] \right. \\
+ \frac{\kappa}{2} \delta_n^0 \left[ (r_1^2 - r_2^2) \partial_m \varphi + r_3^2 \partial_m \phi - r_4^2 \partial_m \tilde{\varphi} \right] + \sum_{a=1}^{4} \partial_m r_a \partial_n r_a + \partial_m \alpha \partial_n \alpha \\
+ (r_1^2 + r_2^2) \partial_m \varphi \partial_n \alpha + r_3^2 \partial_m \phi \partial_n \alpha + r_4^2 \partial_m \tilde{\varphi} \partial_n \alpha \left. \right\}. \\

Our next step is to take the limit \(\kappa \to \infty, \partial_0 \to 0, \kappa \partial_0 \) finite. In this limit, \(G_{00}\) and \(G_{0i}\) simplify to
\[
G_{00} = (4L_p R)^2 \kappa \left[ \partial_0 \alpha + (r_1^2 - r_2^2) \partial_0 \varphi + r_3^2 \partial_0 \phi - r_4^2 \partial_0 \tilde{\varphi} \right], \\
G_{0i} = (4L_p R)^2 \frac{\kappa}{2} \left[ \partial_i \alpha + (r_1^2 - r_2^2) \partial_i \varphi + r_3^2 \partial_i \phi - r_4^2 \partial_i \tilde{\varphi} \right],
\]
while \(G_{ij}\) do not change. With the aim to take into account the constraints \(G_{0i} = 0\) and to eliminate simultaneously \(\partial_i \alpha\) from the membrane Lagrangian, we replace
\[- \partial_i \alpha = (r_1^2 - r_2^2) \partial_i \varphi + r_3^2 \partial_i \phi - r_4^2 \partial_i \tilde{\varphi} \]
into \(\det G_{ij}\) and obtain for \((3.11)\) the following expression
\[
\mathcal{L} = \frac{(2L_p R)^2}{\lambda_0^3} \kappa \left( \partial_0 \alpha + \sum_{k=1}^{3} \nu_k \partial_0 \rho_k \right) \quad (3.12)
- \lambda_0 T_2^2 (4L_p R)^4 \left\{ \sum_{a < b=1}^{4} \left( \partial_1 r_a \partial_2 r_b - \partial_2 r_a \partial_1 r_b \right)^2 + \sum_{a=1}^{4} \sum_{k=1}^{3} \mu_k (\partial_1 r_a \partial_2 \rho_k - \partial_2 r_a \partial_1 \rho_k)^2 \right. \\
- \sum_{a=1}^{4} \left( \partial_1 r_a \sum_{k=1}^{3} \nu_k \partial_2 \rho_k - \partial_2 r_a \sum_{k=1}^{3} \nu_k \partial_1 \rho_k \right)^2 + \sum_{k=1}^{3} \mu_k \nu_n (\partial_1 \rho_k \partial_2 \rho_n - \partial_2 \rho_k \partial_1 \rho_n)^2 \\
- \sum_{k=1}^{4} \mu_k \left( \partial_1 \rho_k \sum_{n=1}^{3} \nu_n \partial_2 \rho_n - \partial_2 \rho_k \sum_{n=1}^{3} \nu_n \partial_1 \rho_n \right)^2 \left. \right\} + \Lambda_S \left( \sum_{a=1}^{4} r_a^2 - 1 \right).
For simplifying reason, we introduced in (3.12) the notations
\[ (\mu_1, \mu_2, \mu_3) = (r_1^2 + r_2^2, r_3^2, r_4^2), \]
\[ (\nu_1, \nu_2, \nu_3) = (r_2^2 - r_3^2, r_3^2, -r_4^2), \]
\[ (\rho_1, \rho_2, \rho_3) = (\varphi, \phi, \dot{\phi}). \]

Now, we are ready to face our main problem: how to reduce the membrane Lagrangian (3.12) to the one corresponding to the thermodynamic limit of spin chain, without shrinking the membrane to string? We propose the following solution of this task:
\[ \alpha = \alpha(\tau, \sigma_1), \quad r_1 = r_1(\tau, \sigma_1), \quad r_2 = r_2(\tau, \sigma_1), \]
\[ r_3 = r_3(\tau, \sigma_2) = a \sin[b \sigma_2 + c(\tau)], \quad r_4 = r_4(\tau, \sigma_2) = a \cos[b \sigma_2 + c(\tau)], \quad \text{(3.13)} \]
\[ \varphi = \varphi(\tau, \sigma_1), \quad a, b, \phi, \dot{\phi} = \text{constants}, \quad a^2 < 1. \]

The restrictions (3.13) lead to
\[ \mathcal{L} = \frac{(2l_p R)^2}{\lambda^0} \kappa (\partial_0 \alpha + \nu_1 \partial_0 \rho_1) \]
\[ - \lambda^0 (ab T_2)^2(4l_p R)^4 \left[ \sum_{a=1}^2 (\partial_1 r_a)^2 + (\mu_1 - \nu_1^2)(\partial_1 \rho_1)^2 \right] + \Lambda_S \left[ \sum_{a=1}^2 r_a^2 - (1 - a^2) \right] \]
\[ = \frac{(2l_p R)^2}{\lambda^0} \kappa \left[ \partial_0 \alpha + (r_1^2 - r_2^2) \partial_0 \varphi \right] \]
\[ - \lambda^0 (ab T_2)^2(4l_p R)^4 \left\{ \sum_{a=1}^2 (\partial_1 r_a)^2 + \left[ (r_1^2 + r_2^2) - (r_1^2 - r_2^2)^2 \right] (\partial_1 \varphi)^2 \right\} \]
\[ + \Lambda_S \left[ \sum_{a=1}^2 r_a^2 - (1 - a^2) \right]. \]

The above membrane Lagrangian is fully analogous to the string Lagrangian in (2.10), obtained after fixing \( r_3^2 \) to \( \varepsilon^2 \to 0 \). Proceeding as in the string case, we introduce the parametrization
\[ r_1 = (1 - a^2)^{1/2} \cos \psi, \quad r_2 = (1 - a^2)^{1/2} \sin \psi, \]
the new variable \( \tilde{\alpha} \)
\[ \alpha = (1 - a^2) \tilde{\alpha}, \]
and take the limit \( a^2 \to 0 \). Thus, we receive
\[ \mathcal{L}/(1 - a^2) = \frac{(2l_p R)^2}{\lambda^0} \kappa \left[ \partial_0 \tilde{\alpha} + \cos(2\psi) \partial_0 \varphi \right] \]
\[ - \lambda^0 (ab T_2)^2(4l_p R)^4 \left[ (\partial_1 \psi)^2 + \sin^2(2\psi)(\partial_1 \varphi)^2 \right], \]
which should be compared with (2.11). As for the membrane action corresponding to the above Lagrangian, it can be represented in the form
\[ S_M = \frac{\mathcal{J}}{2\pi} \int dt d\sigma \left[ \partial_1 \tilde{\alpha} + \cos(2\psi) \partial_1 \varphi \right] \]
\[ - \frac{\lambda}{4\pi \mathcal{J}} \int dt d\sigma \left[ (\partial_\theta \psi)^2 + \sin^2(2\psi)(\partial_\theta \varphi)^2 \right]. \]
where $J$ is the angular momentum conjugated to $\tilde{\alpha}$, $t = \kappa \tau$ and

$$\tilde{\lambda} = 2^{15}[\pi^2(1 - a^2)abT_2]^2(l_pR)^6.$$ 

Obviously, the action (3.16) corresponds to the thermodynamic limit of $SU(2)$ integrable spin chain.

4 Relationship with the Neumann-Rosochatius integrable system

Here, we are going to discuss the connection between the thermodynamical limits of the integrable spin chains, viewed as arising from strings on $AdS_5 \times S^5$ and membranes on $AdS_4 \times S^7$, and the Neumann-Rosochatius integrable system [22, 23, 24], related to specific configurations of strings [25, 26, 27, 28] and membranes [29].

4.1 The string case

It is known that large class of classical string solutions in the type IIB $AdS_5 \times S^5$ background is related to the Neumann and Neumann-Rosochatius integrable systems. It was found in [28] that, working in conformal gauge, the spiky strings [30, 31, 32, 33, 34] and giant magnons [35] - [55], [34] can be also accommodated by a version of the Neumann-Rosochatius system. The appropriate string embedding of the type (2.5) is

$$Z_0 = Re^{i\kappa \tau}, \quad Z_1 = Z_2 = 0, \quad W_i = Rr_i(\xi)e^{[\omega_i \tau + f_i(\xi)]}, \quad \xi = \alpha \sigma + \beta \tau,$$

i.e.

$$r_0 = 1, \quad r_1 = r_2 = 0; \quad \phi_0 = t = \kappa \tau, \quad r_i = r_i(\xi), \quad \varphi_i = \omega_i \tau + f_i(\xi).$$

Correspondingly, the induced metric takes the form

$$G_{00} = R^2 \left\{ \sum_{i=1}^{3} \left[ \beta^2 (\partial_\xi r_i)^2 + r_i^2 (\beta \partial_\xi f_i + \omega_i)^2 \right] - \kappa^2 \right\},$$

$$G_{11} = R^2 \alpha^2 \sum_{i=1}^{3} \left[ (\partial_\xi r_i)^2 + r_i^2 (\partial_\xi f_i)^2 \right],$$

$$G_{01} = R^2 \alpha \sum_{i=1}^{3} \left\{ \beta \left[ (\partial_\xi r_i)^2 + r_i^2 (\partial_\xi f_i)^2 \right] + \omega_i r_i^2 \partial_\xi f_i \right\}.$$

On the Neumann-Rosochatius ansatz (4.1), the string Lagrangian is given by

$$\mathcal{L}_s = -\frac{R^2}{4 \lambda^0} \left\{ \sum_{i=1}^{3} \left[ (A^2 - \beta^2)(\partial_\xi r_i)^2 + (A^2 - \beta^2)r_i^2 \left( \partial_\xi f_i - \frac{\beta \omega_i}{A^2 - \beta^2} \right)^2 \right] - \frac{A^2}{A^2 - \beta^2 \omega_i^2 r_i^2} \right\} + \Lambda_s \left( \sum_{i=1}^{3} r_i^2 - 1 \right),$$

\footnote{As in section 2, we will use diagonal worldsheet gauge.}
where

\[ A^2 \equiv \left(2\lambda^0 T\alpha\right)^2. \]

After integrating the equations of motion for \( f_i \) once and replacing back the solution into (4.2), one arrives at

\[
\mathcal{L}_* = -\frac{R^2}{4\lambda^0} \left\{ \sum_{i=1}^{3} \left[ (A^2 - \beta^2)(\partial_t r_i)^2 + \frac{C_i^2}{(A^2 - \beta^2)r_i^2} \right] - \frac{A^2}{A^2 - \beta^2\omega_i^2 r_i^2} \right\} + \Lambda_S \left( \sum_{i=1}^{3} r_i^2 - 1 \right),
\]

where \( C_i \) are arbitrary integration constants. Following [28], we change the overall sign, discard the constant term \( \sim \kappa^2 \), and obtain

\[
\mathcal{L}_{NR} = \frac{R^2}{4\lambda^0} \sum_{i=1}^{3} \left[ (A^2 - \beta^2)(\partial_t r_i)^2 - \frac{C_i^2}{(A^2 - \beta^2)r_i^2} - \frac{A^2}{A^2 - \beta^2\omega_i^2 r_i^2} \right] + \Lambda_S \left( \sum_{i=1}^{3} r_i^2 - 1 \right),
\]

which is Lagrangian for the Neumann-Rosochatius integrable system.

Now, with the aim of comparison with the thermodynamic limit of the SU(3) spin chain, let us set

\[
\omega_1 = \omega_2 = \omega_3 = \kappa, \quad (f_1, f_2, f_3) = (\alpha + \varphi, \alpha - \varphi, \alpha + \phi),
\]

in (4.2) and in the constraint \( G_{01} = 0 \). The result is

\[
\mathcal{L}_* = -\frac{R^2}{4\lambda^0} \left\{ 2\kappa \beta \left[ \partial_t \alpha + (r_1^2 - r_2^2)\partial_t \varphi + r_3^2 \partial_t \phi \right] - (A^2 - \beta^2) \left\{ \sum_{i=1}^{3} (\partial_t r_i)^2 + (\partial_t \alpha)^2 + 2\partial_t \alpha \left[ (r_1^2 - r_2^2)\partial_t \varphi + r_3^2 \partial_t \phi \right] + (r_1^2 + r_2^2)(\partial_t \varphi)^2 + r_3^2(\partial_t \phi)^2 \right\} + \Lambda_S \left( \sum_{i=1}^{3} r_i^2 - 1 \right),
\]

\[
\kappa \left[ \partial_t \alpha + (r_1^2 - r_2^2)\partial_t \varphi + r_3^2 \partial_t \phi \right] + \beta \left\{ \sum_{i=1}^{3} (\partial_t r_i)^2 + (\partial_t \alpha)^2 + 2\partial_t \alpha \left[ (r_1^2 - r_2^2)\partial_t \varphi + r_3^2 \partial_t \phi \right] + (r_1^2 + r_2^2)(\partial_t \varphi)^2 + r_3^2(\partial_t \phi)^2 \right\} = 0.
\]

The next step is to take the limit

\[ \kappa \to \infty, \beta \to 0, \quad \kappa\beta - \text{finite}, \]

in which \( \mathcal{L}_* \) reduces to

\[
\mathcal{L}_{lim} = -\frac{R^2\kappa\beta}{2\lambda^0} \left[ \partial_t \alpha + (r_1^2 - r_2^2)\partial_t \varphi + r_3^2 \partial_t \phi \right] + \lambda^0 (TR\alpha)^2 \left\{ \sum_{i=1}^{3} (\partial_t r_i)^2 + (r_1^2 - r_2^2)^2 + (r_1^2 + r_2^2)^2 + (r_1^2 + r_2^2)r_3^2(\partial_t \phi)^2 \right\} + \Lambda_S \left( \sum_{i=1}^{3} r_i^2 - 1 \right).
\]
Now, we turn to the case of string configurations corresponding to the thermodynamic limit of the SU(3) integrable spin chain, and consider the particular case when all variables \((\alpha, \varphi, \phi, r_i)\) depend only on \(\xi = \alpha \sigma + \beta \tau\). Then, \(L_{SC}\) in (2.8) coincides with \(L_{\text{lim}}\). Let us point out that the constraint \(G_{01} = 0\) now reads

\[
\partial_\xi \alpha + (r_1^2 - r_2^2) \partial_\xi \varphi + r_3^2 \partial_\xi \phi = 0.
\]

### 4.2 The membrane case

The most general membrane embedding in \(AdS_4 \times S^7\) leading to the Neumann-Rosochatius integrable system is [29]

\[
Z_0 = 2l_p R e^{i \kappa \tau}, \quad Z_1 = 0, \quad W_a = 4l_p R r_a(\xi, \eta) e^{i[\omega a + g_a(\xi, \eta)]},
\]

\[
\xi = \alpha \sigma_1 + \beta \tau, \quad \eta = \gamma \sigma_2 + \delta \tau, \quad \alpha, \beta, \gamma, \delta = \text{constants},
\]

for

\[
r_1 = r_1(\xi), \quad r_2 = r_2(\xi), \quad r_3 = r_3(\eta) = a \sin(b \eta + c), \quad r_4 = r_4(\eta) = a \cos(b \eta + c), \quad a < 1,
\]

\[
g_1 = g_1(\xi), \quad g_2 = g_2(\xi), \quad a, b, c, g_3, g_4 = \text{constants}, \quad \delta = 0.
\]

The above ansatz reduces the membrane Lagrangian \(L\) in (3.8) to

\[
L_{M}^* = \lambda^0 \left\{ \sum_{a=1}^{2} \left[ (\tilde{A}^2 - \beta^2)(\partial_\xi r_a)^2 + (\tilde{A}^2 - \beta^2) r_a^2 \left( \partial_\xi g_a - \frac{\beta \omega_a}{A^2 - \beta^2} \right)^2 \right] - \tilde{A}^2 \frac{\omega_a^2 r_a^2}{A^2 - \beta^2} \right\} + \Lambda_S \left[ \sum_{a=1}^{2} r_a^2 - (1 - a^2) \right],
\]

where

\[
\tilde{A}^2 \equiv \left( 8 \lambda^0 T_2 l_p R a b c \right)^2.
\]

As shown in [29], \(L_{M}^*\) corresponds to the following Neumann-Rosochatius type Lagrangian

\[
L_{NR}^M = \lambda^0 \left\{ \sum_{a=1}^{2} \left[ (\tilde{A}^2 - \beta^2)(\partial_\xi r_a)^2 - \frac{C_a^2}{(A^2 - \beta^2) r_a^2} - \frac{\tilde{A}^2 \omega_a^2 r_a^2}{A^2 - \beta^2} \right] \right\} + \Lambda_S \left[ \sum_{a=1}^{2} r_a^2 - (1 - a^2) \right], \quad C_a = \text{constants}.
\]

Turning to comparison with the thermodynamic limit of the SU(2) spin chain, we set in (4.4) and in the constraint \(G_{01} = 0\)

\[
\omega_1 = \omega_2 = \omega_3 = \omega_4 = \kappa/2, \quad (g_1, g_2) = (\tilde{\alpha} + \varphi, \tilde{\alpha} - \varphi)
\]

\footnote{For the present case, the constraint \(G_{02} = 0\) is satisfied identically, due to \(\delta = 0\).}
which leads to

\[
\mathcal{L}_s^M = \frac{(2l_pR)^2}{\lambda^0} \left\{ (1-a^2)\partial_\xi \tilde{\alpha} + (r_1^2-r_2^2)\partial_\xi \varphi \right\}
\]

\[
- (\tilde{A}^2 - \beta^2) \left\{ \sum_{\alpha=1}^2 (\partial_\xi r_\alpha)^2 + (1-a^2)(\partial_\xi \tilde{\alpha})^2 + 2(r_1^2-r_2^2)\partial_\xi \tilde{\alpha} \partial_\xi \varphi \right\}
\]

\[
+ (r_1^2 + r_2^2)(\partial_\xi \varphi)^2 \right} + \Lambda_s \left[ \sum_{\alpha=1}^2 r_\alpha^2 - (1-a^2) \right],
\]

\[
\frac{\kappa}{2} [(1-a^2)\partial_\xi \tilde{\alpha} + (r_1^2-r_2^2)\partial_\xi \varphi]
\]

\[
+ \beta \left\{ \sum_{\alpha=1}^2 (\partial_\xi r_\alpha)^2 + (1-a^2)(\partial_\xi \tilde{\alpha})^2 + 2(r_1^2-r_2^2)\partial_\xi \tilde{\alpha} \partial_\xi \varphi + (r_1^2+r_2^2)(\partial_\xi \varphi)^2 \right\} = 0.
\]

As a next step, we take the limit

\[
\kappa \to \infty, \quad \beta \to 0, \quad \kappa \beta \text{ finite},
\]

in which \(\mathcal{L}_s^M\) and \(G_{01} = 0\) reduce to

\[
\mathcal{L}_{lim}^M = \frac{(2l_pR)^2}{\lambda^0} \kappa \beta \left\{ (1-a^2)\partial_\xi \tilde{\alpha} + (r_1^2-r_2^2)\partial_\xi \varphi \right\}
\]

\[
- \lambda^0 (T_2\omega \beta \gamma)^2 (4l_p R)^4 \left\{ \sum_{\alpha=1}^2 (\partial_\xi r_\alpha)^2 + (1-a^2) \left[ 1 - \left( \frac{r_1^2-r_2^2}{1-a^2} \right)^2 \right] (\partial_\xi \varphi)^2 \right\}
\]

\[
+ \Lambda_s \left[ \sum_{\alpha=1}^2 r_\alpha^2 - (1-a^2) \right],
\]

\[
(1-a^2)\partial_\xi \tilde{\alpha} + (r_1^2-r_2^2)\partial_\xi \varphi = 0.
\]

Parameterizing the circle

\[
\sum_{\alpha=1}^2 r_\alpha^2 - (1-a^2) = 0
\]

by

\[
r_1 = (1-a^2)^{1/2} \cos \psi, \quad r_2 = (1-a^2)^{1/2} \sin \psi,
\]

one obtains

\[
\mathcal{L}_{lim}^M/(1-a^2) = \frac{(2l_pR)^2}{\lambda^0} \kappa \beta \left\{ (\partial_\xi \tilde{\alpha} + \cos(2\psi)\partial_\xi \varphi \right\}
\]

\[
- \lambda^0 (T_2\omega \beta \gamma)^2 (4l_p R)^4 \left\{ (\partial_\xi \psi)^2 + \sin^2(2\psi)(\partial_\xi \varphi)^2 \right\}.
\]

Now, we turn to the case of membrane configurations corresponding to the thermodynamic limit of the \(SU(2)\) integrable spin chain. Let us suppose that all variables \((\tilde{\alpha}, \varphi, \psi)\) depend on \((\tau, \sigma_1)\) through the linear combination \(\xi = \alpha \sigma_1 + \beta \tau\) only. Then, \(\mathcal{L}\) in \((3.15)\) coincides with \(\mathcal{L}_{lim}^M\) in \((4.6)\) for \(\gamma = 1\).
5 Concluding remarks

In this paper we were able to find membrane configurations in $AdS_4 \times S^7$ of the type \(3.6\), given by \(3.9\) and \(3.13\), which in a certain limit reproduce the continuous limit of the $SU(2)$ integrable spin chain, arising in $\mathcal{N} = 4$ SYM, dual to strings on $AdS_5 \times S^5$ (see \(3.16\)).

Besides, we investigated the connection between the thermodynamical limits of the $SU(3)$ and $SU(2)$ integrable spin chains, viewed as arising from strings on $AdS_5 \times S^5$ and membranes on $AdS_4 \times S^7$, and the Neumann-Rosochatius integrable system, related to specific string and membrane configurations, at the level of Lagrangians. The conclusion is that we can act in the same way in the string and membrane cases. Namely, in order to obtain identical Lagrangians, we must impose restrictions on both sides of the correspondence. On the side of Neumann-Rosochatius model, we have to restrict ourselves to the case when all frequencies are equal and proportional to the parameter $\kappa$, related to the string/membrane energy, then take an appropriate limit. On the spin chain side, we must consider the particular case, when the dependence on the coordinates $(\tau, \sigma/\sigma_1)$ is only through their linear combination $\xi$.

As we pointed out in the introduction, in a recent work, the connection between membrane dynamics on flat space-time (without and with fluxes) and integrable quantum spin chains has been discussed [21]. In that paper, the relationship between the worldvolume theory of membranes and integrable quantum spin chains has been explored. More precisely, the author was able to show that the very general framework for translating Hamiltonian quantum mechanical matrix models to quantum spin chains in the large $N$ limit, which was first spelled out in [56], can be applied to explore the dynamics of a large class of membrane models. This framework can presumably be of interest to the membrane configurations, we are studying in this paper. Indeed, it would be very illuminating if the underlying connection between the integrable systems that were discussed in [21] and the ones arising here can be clarified.

On the other hand, according to AdS-CFT correspondence, strings on $AdS_5 \times S^5$ and membranes on $AdS_4 \times S^7$ are dual to different gauge theories. Therefore, one is tempting to conjecture that there should exist common integrable sectors on the field theory side.

Note added: While writing this paper, we learn about the work [57] on the same subject. There, a rotating probe membrane in $S^3$ inside $AdS_4 \times S^7$ background of M-theory is studied. With (partial) gauge fixing, it is shown that in the fast limit, the worldvolume of tensionless membrane reduces to either the XXX$_{1/2}$ spin chain or the two-dimensional $SU(2)$ Heisenberg spin model. After that, the author introduces anisotropy and coupling to an external magnetic field. The correspondence for higher dimensional (D)p-branes is also considered.

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