CYCLIC SIEVING OF FINITE GRASSMANNIANS AND FLAG VARIETIES

ANDREW BERGET AND JIA HUANG

Abstract. In this paper we prove instances of the cyclic sieving phenomenon for finite Grassmannians and partial flag varieties, which carry the action of various tori in the finite general linear group $GL_n(F_q)$. The polynomials involved are sums of certain weights of the minimal length parabolic coset representatives of the symmetric group $S_n$, where the weight of a coset representative can be written as a product over its inversions.

1. Introduction

The cyclic sieving phenomenon (CSP) was introduced by Reiner, Stanton and White [9], generalizing Stembridge’s $q = -1$ phenomenon [12]. It has been the subject of a flurry of recent papers in combinatorics, including a survey by Sagan [10]. As defined in [1], the CSP pertains to a finite set $X$, carrying a permutation action of a finite abelian group written explicitly as a product $C := C_1 \times \cdots \times C_m$ of cyclic groups $C_i$, and a polynomial $X(t) := X(t_1, \ldots, t_m)$ in $\mathbb{Z}[t]$. The polynomial is thought of as a generating function for the elements of $X$ according to some natural statistic. One says that the triple $(X, X(t), C)$ exhibits the CSP if after choosing embeddings of groups $\omega_i : C_i \hookrightarrow \mathbb{C}^\times$, one has for every $c = (c_1, \ldots, c_m)$ in $C$ that the cardinality of the fixed point set of $c$ is given by

$$[X(t)]_{t_i = \omega_i(c_i)} = |\{ x \in X : c(x) = x \}|.$$ 

The goal of this paper is to prove several instances of the CSP that pertain to the action of a maximal torus on a finite Grassmannian, and more generally on a finite partial flag variety. We will always take our flag varieties to be defined over $F_q$, a finite field with $q$ elements. Our main theorem is technical to state explicitly, so here we content ourselves with the following rendition:

**Theorem.** Let $n$ and $k$ be positive integers, $k \leq n$ and $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ a composition of $n$. There is an action of the torus $T_\alpha := \prod_{i=1}^\ell F_q^{\times \alpha_i}$ on the Grassmannian $G_k(n)$ of $k$-dimensional $F_q$-subspaces of $F_q^n$. For each partition $\lambda$ in a $k$-by-$(n-k)$ box there is an associated weight $\text{wt}(\lambda; \alpha, k)$, which is a product over the cells of $\lambda$ and is a polynomial in $t_1, \ldots, t_\ell$, such that the triple

$$\left( G_k(n), \sum_{\lambda \subseteq (n-k)^k} \text{wt}(\lambda; \alpha, k), T_\alpha \right)$$

exhibits the cyclic sieving phenomenon.

Berget was partially supported by NSF grant DMS-0636297. Huang was partially supported by NSF grant DMS-1001933.
More generally, if \( \beta = (\beta_1, \ldots, \beta_k) \) is a composition of \( n \), and \( \mathcal{F}(\beta) \) is the variety of partial flags of subspaces of \( \mathbb{F}^n_q \) whose dimensions are given by the partial sums \( \beta_1 + \cdots + \beta_i \), then there is an action of \( T_\alpha \) on \( \mathcal{F}(\beta) \) and a polynomial \( \chi_{\alpha,\beta}(t) \) such that \((\mathcal{F}(\beta), \chi_{\alpha,\beta}(t), T_\alpha)\) exhibits the cyclic sieving phenomenon.

Some remarks are in order. The first is that the determination of the weights \( \text{wt}(\lambda; \alpha, k) \) is entirely elementary. It consists of breaking up \( \lambda \) into smaller partitions, determined by \( \alpha \) and \( k \), and then evaluating the relative positions of the cells of the smaller partitions within certain boxes. For partial flag varieties the polynomials \( \chi_{\alpha,\beta}(t) \) are determined in a similar fashion.

The second remark is that in the extreme case \( \alpha = (n) \), our polynomials \( \chi_{\alpha,\beta}(t) \) coincide with the \((q, t)\)-multinomial coefficients \( \binom{n}{\beta} \) of Reiner, Stanton and White [9, §9]. In this special case, our main theorem was one of the foundational examples of the CSP in [9].

Thirdly, the weights will visibly have the property

\[
\text{wt}(\lambda; \alpha, k)|_{t_1 = 1} = q^{\lambda}
\]

where \( |\lambda| \) is the sum of its parts. Thus, our formula is a carefully crafted refinement of the well-known expression

\[
|G_k(n)| = \sum_{\lambda \subset (n-k)^k} q^{\lambda}.
\]

It is equally well-known that this expression factors as

\[
|G_k(n)| = \sum_{\lambda \subset (n-k)^k} q^{\lambda} = \frac{(q^n - 1)(q^n - q) \cdots (q^n - q^{n-k+1})}{(q^k - 1)(q^k - q) \cdots (q^k - q^{k-1})}.
\]

For general \( \alpha \) the generating function \( \sum_{\lambda} \text{wt}(\alpha; \lambda, k) \) does not appear to have a similar factorization. However, we will show that a natural refinement of it does, and specializing at \( t_1, \ldots, t_\ell = 1 \) gives rise to the well-known \( q \)-Vandermonde identity. In fact, our proof for the CSP on partial flag varieties gives a generalized \( q \)-Vandermonde identity

\[
\left[ \alpha_1 + \cdots + \alpha_\ell \atop \beta_1, \ldots, \beta_m \right]_q = \sum_{\beta^{(1)}, \ldots, \beta^{(\ell)}} \prod_{r=1}^\ell \left[ \alpha_r \atop \beta^{(r)} \right]_q \prod_{1 \leq s < r \leq \ell} q^{i_j \beta^{(r)}},
\]

summed over all weak compositions \( \beta^{(r)} = (\beta^{(r)}_1, \ldots, \beta^{(r)}_m) \) of \( \alpha_r \), \( 1 \leq r \leq \ell \), with component-wise sum \( \beta^{(1)} + \cdots + \beta^{(\ell)} = \beta \).

Lastly, there is also a “\( q = 1 \)” version of our main result on flags that pertains to flags of sets. For \( r = 1, \ldots, \ell \), let \( C_r \) be a cyclic group generated by a regular element \( c_r \) in the symmetric group \( \mathfrak{S}_n \), i.e., \( C_r \) acts nearly freely on \( [\alpha_r] \). The triple

\[
\left( \left[ \begin{array}{c} \alpha_r \\ \beta^{(r)} \end{array} \right], \left[ \begin{array}{c} \alpha_r \\ \beta^{(r)} \end{array} \right]_{c_r}, C_r \right)
\]

exhibits the CSP, where \( \beta^{(r)} \) is a weak composition of \( \alpha_r \), and \( \left[ \begin{array}{c} \alpha_r \\ \beta^{(r)} \end{array} \right] \) is the set of all partial flags of subsets of cardinality \( \beta^{(r)}_1 + \beta^{(r)}_2 + \cdots \) in the set \( [\alpha_r] = \{1, \ldots, \alpha_r\} \); see [9, §1]. The embedding \( \mathfrak{S}_{\alpha_1} \times \cdots \times \mathfrak{S}_{\alpha_\ell} \to \mathfrak{S}_n \) sends \( C_1 \times \cdots \times C_\ell \) to a subgroup \( T_\alpha \).
of \( S_n \). A moment’s thought reveals that the triple

\[
\left( \begin{bmatrix} [n] \end{bmatrix}, \ Y_{\alpha,\beta}(t), \ T_{\alpha} \right)
\]

exhibits the cyclic sieving phenomenon for any composition \( \beta \) of \( n \), with

\[
Y_{\alpha,\beta}(t) = \sum_{\beta^{(1)}, \ldots, \beta^{(\ell)}} \left[ \begin{array}{c} \alpha_1 \\ \beta^{(1)} \end{array} \right]_{t_1} \cdots \left[ \begin{array}{c} \alpha_{\ell} \\ \beta^{(\ell)} \end{array} \right]_{t_{\ell}},
\]

summer over all weak compositions \( \beta^{(r)} \) of \( \alpha_r \), \( r = 1, \ldots, \ell \), whose component-wise sum is \( \beta \). One will see in Remark 4.5 that \( Y_{\alpha,\beta}(t) \) can be viewed as a “\( q = 1 \)” version of \( X_{\alpha,\beta}(t) \).

Our paper is organized as follows. In Section 2 we set up the complete statement of our main theorem for cyclic sieving of finite Grassmannians. In Section 3 we then prove the main theorem, which will follow from a series of lemmas. In Section 4 we extend the main result to the finite flag varieties.

2. Precise statement of the main result

In this section we carefully define the polynomials involved in the statement of our main theorem, as well as the torus actions. Throughout this discussion we have fixed two integers \( n \) and \( k \), \( k \leq n \), which define the Grassmannian that we are working in.

2.1. Partition statistics. A (weak) composition \( \alpha = (\alpha_1, \ldots, \alpha_\ell) \) of \( n \) is a sequence of positive (nonnegative) integers \( \alpha_i \) such that \( n = \alpha_1 + \cdots + \alpha_\ell \). The integers \( \alpha_i \) are called parts of \( \alpha \). Fix a composition \( \alpha \) with exactly \( \ell \) parts.

Associated to each partition \( \lambda \subset (n-k)k \) is a (reduced) row echelon form, which is a matrix of 1’s (pivots), 0’s, and *’s. The matrix is in row echelon form, in the sense that every pivot 1 sees zeros in directions east, north and south. The columns containing the 1’s are called pivot columns, and their positions are \( \{ \lambda_i + k - i + 1 : 1 \leq i \leq k \} \). The remaining spots are filled with stars so that the \( i \)-th row contains exactly \( \lambda_i \) stars.

The composition \( \alpha \) breaks up the columns of this matrix into blocks of sizes \( \alpha_1, \ldots, \alpha_\ell \), labeled by 1, \ldots, \ell, from right to left. We set \( \beta(\lambda) = (\beta_1, \ldots, \beta_\ell) \) so that \( \beta_j \) is the number of 1’s in the \( j \)-th block of columns. To be formal,

\[
\beta_j := \# \{ i : \alpha_1 + \cdots + \alpha_j < \lambda_i + k - i + 1 \leq \alpha_1 + \cdots + \alpha_j \}.
\]

Note that \( \beta(\lambda) \) is a weak composition of \( k \) into \( \ell \) parts, and this association is not injective.

**Example 2.1.** Take \( n = 9 \), \( k = 4 \) and \( \alpha = (4, 2, 3) \). The composition associated to \( \lambda = (5, 4, 1, 1) \) is \( \beta(\lambda) = (2, 0, 2) \), as is evidenced by the matrix below or using the formula above.

\[
\begin{bmatrix}
* & 0 & 0 & * & * & * & 0 & * & 1 \\
* & 0 & 0 & * & * & 1 & 0 & 0 \\
* & 0 & 1 & 0 & 0 \\
* & 1 & 0 & 0
\end{bmatrix}
\]

We have indicated the composition \( \alpha \) as blocks of columns, and the composition \( \beta \) as blocks of rows (the double line has an empty block in between its lines).
As above, we can partition the row echelon form of every $\lambda$ into a block triangular matrix. The $(r,s)$-th block is of size $\beta_r$-by-$\alpha_s$, and the $s$'s in it give rise to a partition that fits in a $\beta_r$-by-$(\alpha_s - \beta_s)$ box. Denote this partition by $\lambda^{r,s}$, which only depends on $\alpha$, $\lambda$ and $\ell$. Note that $\lambda^{r,r}$ can be arbitrary, while $\lambda^{r,s}$ is always a $\beta_r$-by-$(\alpha_s - \beta_s)$ rectangle if $r < s$, or empty if $r > s$. By sending $\lambda$ to $[\lambda^{r,s}]$ one obtains bijection

\[
(2) \quad \{ \lambda : \beta(\lambda) = \beta \} \xrightarrow{\sim} \{ [\lambda^{r,s}] : \lambda^{r,r} \subseteq (\alpha_r - \beta_r)^{\beta_r}, \lambda^{r,s} = (\alpha_s - \beta_s)^{\beta_r} (r < s), \lambda^{r,s} = \emptyset (r > s) \}.
\]

Denote by $[a,b]$ the \textit{q-number}

\[
[a,b] := \frac{a^q - b^q}{a-b} = \sum_{i+j=q-1} a^i b^j.
\]

If $a$ and $b$ are $(q-1)$-th roots of unity then $[a,b] = 1$ when $a \neq b$ or $[a,b] = q$ when $a = b$.

We define the weight of a cell $x \in \lambda^{r,s}$ to be

\[
\text{wt}(x; t_r, t_s) := [t_q^{(x)(r)(s)}], [t_q^{(x)(r)}], [t_q^{(r)}], [t_q^{(x)}], [t_q^{(x)(r)(s)}] + [t_q^{(r)}].
\]

Here $i(x)$ and $j(x)$ are the horizontal and vertical distances of $x$ from the bottom-left cell of the $\beta_r$-by-$(\alpha_s - \beta_s)$ box in which $\lambda^{r,s}$ fits. Define the weight of $\lambda^{r,s}$ to be

\[
\text{wt}(\lambda^{r,s}; t_r, t_s) := \prod_{x \in \lambda^{r,s}} \text{wt}(x; t_r, t_s).
\]

Define the weight of $\lambda$ to be

\[
\text{wt}(\lambda; \alpha, k) := \prod_{1 \leq r \leq s \leq \ell} \text{wt}(\lambda^{r,s}; t_r, t_s).
\]

We see at once that $\text{wt}(\lambda; \alpha, k)$ is a polynomial in $t_1, \ldots, t_{\ell}$. This choice of a weight was directly influenced by the results of Reiner and Stanton in \[8\]. In the case $\alpha = (n)$ our weight is exactly the weight they associate to $\lambda$.

**Example 2.2.** As before, take $n = 9$, $k = 4$, $\alpha = (4,2,3)$, and $\lambda = (5,4,1,1)$. The labeling of all the cells of $\lambda$ with their weights gives the following diagram:

\[
\begin{bmatrix}
[t_1^4, t_1^2] & [t_1^4, t_1^2] & [t_1^4, t_2^2] & [t_1^4, t_1^2] & [t_1^4, t_2^2] \\
[t_1^2, t_1^4] & [t_1^2, t_2^2] & [t_1^2, t_2^2] & [t_1^2, t_1^4] & [t_1^2, t_1^4] \\
[t_2^2, t_1^4] & [t_2^2, t_1^4] & [t_2^2, t_1^4] & [t_2^2, t_1^4] & [t_2^2, t_1^4] \\
[t_3^2, t_1^2] & [t_3^2, t_1^2] & [t_3^2, t_1^2] & [t_3^2, t_1^2] & [t_3^2, t_1^2]
\end{bmatrix}
\]

**Example 2.3.** If $\alpha = (1,1,\ldots,1)$ then $\beta(\lambda)$ is what is sometimes referred to as the \textit{“code”} associated to $\lambda$. That is, $\beta(\lambda)$ has a $1$ in positions $n-k-\lambda_i+i$, $1 \leq i \leq k$, and zeros elsewhere. The weight of $\lambda$ is

\[
\text{wt}(\lambda; 1^n, k) = \prod_{1 \leq i \leq k} \{ t_{k-\lambda'_i+j}, t_{n-k-\lambda_i+i} \}.
\]

Here, $\lambda'$ denotes the conjugate partition, and $\lambda'_j$ is its $j$-th part, i.e. the number of parts of $\lambda$ that are at least $j$. 

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2.2. Tori in $\text{GL}_n(\mathbb{F}_q)$. Let $\text{GL}_n(K)$ denote the group of $n$-by-$n$ matrices with non-zero determinant in the field $K$. If $K$ is an algebraically closed field then a maximal torus in $\text{GL}_n(k)$ is a subgroup isomorphic to the group of diagonal matrices $(K^\times)^n \subset \text{GL}_n(K)$. Since $\mathbb{F}_q$ is not algebraically closed, the definition of a torus in this group is more subtle.

Let $F$ be denote Frobenius automorphism $F(x) = x^q$ of the algebraic closure $K$ of $\mathbb{F}_q$. A maximal torus $T \subset \text{GL}_n(K)$ is $F$-stable if $F(T) = T$. A maximal torus in $\text{GL}_n(\mathbb{F}_q)$ consists of the $F$-fixed points of an $F$-stable torus $T$. See Carter [3, Chapter 3] for more details related to these objects.

We now define a class of maximal tori in $\text{GL}_n(\mathbb{F}_q)$ labeled by compositions of $n$. Given a composition $\alpha = (\alpha_1, \ldots, \alpha_t)$ of $n$, consider the $n$-dimensional $\mathbb{F}_q$-vector space

$$V_\alpha := \mathbb{F}_{q^{\alpha_1}} \oplus \cdots \oplus \mathbb{F}_{q^{\alpha_t}}.$$ 

The multiplications of the factors give rise to $\mathbb{F}_q$-linear automorphisms of $V_\alpha$. Choosing an isomorphism $V_\alpha \cong \mathbb{F}_q^n$, one has an injection of groups

$$T_\alpha := \mathbb{F}_q^{x_1} \times \cdots \times \mathbb{F}_q^{x_t} \hookrightarrow \text{GL}_n(\mathbb{F}_q)$$

whose image is a maximal torus in $\text{GL}_n(\mathbb{F}_q)$ (this will be seen explicitly in Corollary 2.2). The Grassmannian $G_k(V_\alpha) \cong G_k(n)$ of the $k$-dimensional subspace of $\mathbb{F}_q^n$ can be thought of as the collection of full rank $k$-by-$n$ matrices with entries in $\mathbb{F}_q$, modulo the natural left action by $\text{GL}_k(\mathbb{F}_q)$. The group $\text{GL}_n(\mathbb{F}_q)$ acts on the right of $G_k(n)$ by multiplication. It follows that after embedding $T_\alpha$ in $\text{GL}_n(\mathbb{F}_q)$ there is a right action of $T_\alpha$ on $G_k(n)$.

**Proposition 2.4.** The number of fixed points of an element $T_\alpha$ as it acts on the Grassmannian $G_k(n)$ does not depend on the embedding of $T_\alpha$ in $\text{GL}_n(\mathbb{F}_q)$.

*Proof.* Suppose we have two different injections $\phi, \psi : T_\alpha \hookrightarrow \text{GL}_n(\mathbb{F}_q)$, both coming from isomorphisms $V_\alpha \cong \mathbb{F}_q^n$, as above. We see that $\phi(t)$ and $\psi(t)$ are conjugate in $\text{GL}_n(\mathbb{F}_q)$ for any $t \in T_\alpha$. The fixed sets of $\phi(t)$ are $\psi(t)$ are translates of each other, hence equicardinal. 

In the case that $\alpha = (1^n)$, the torus $T_\alpha$ is said to be *maximally split*. When we are in the case of a maximally split torus acting on a Grassmannian or a partial flag variety our analysis will be considerably simplified.

2.3. Precise statement of the theorem. We are now in a position to give the full statement of our main theorem, where the notation is as in the previous subsections.

**Theorem 2.5** (c.f. Reiner–Stanton [3]). The triple

$$(G_k(V_\alpha), \sum_{\lambda \subset (n-k)^k} \text{wt}(\lambda; \alpha, k), T_\alpha)$$

exhibits the cyclic sieving phenomenon.

**Corollary 2.6.** If $(t_1, \ldots, t_n)$ is an element of a maximally split torus of $\text{GL}_n(\mathbb{F}_q)$ then it fixes exactly

$$\sum_{\lambda \subset (n-k)^k} \prod_{1 \leq i < j \leq k} \omega(t_{k-\lambda_j+i})^{\gamma} - \omega(t_{n-k-\lambda_i+i})^{\gamma} \prod_{1 \leq j \leq \lambda_1} \omega(t_{k-\lambda_j+i}) - \omega(t_{n-k-\lambda_1+i})$$

elements in $G_k(n)$, where $\omega$ is an injection of groups $\mathbb{F}_q^\times \hookrightarrow \mathbb{C}^\times$. 

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Proof. This is the special case of the theorem when $\alpha = (1^n)$.

The theorem gives an effective method for computing certain complex characters of $GL_n(\mathbb{F}_q)$ at elements that are semi-simple over an algebraic closure of $\mathbb{F}_q$. Let $\chi$ be the character of the induction $\text{Ind}_{P_k}^{GL_n(\mathbb{F}_q)} \mathbb{C}$ of the trivial representation $\mathbb{C}$ of $P_k$, the parabolic subgroup of block upper triangular matrices with two invertible diagonal blocks of size $k$ and $n-k$. An element of $GL_n(\mathbb{F}_q)$ is semi-simple over the algebraic closure of $\mathbb{F}_q$ if and only if it occurs in the image of some $T_\alpha$. A moment’s thought reveals that the computation in Theorem 2.5 is equivalent to the computation of $\chi$ at semi-simple elements.

This character can be computed using the classical formula for induction. However, this formula is of little use to us since it simply states the tautology: The induced character evaluated at $u$ is the number of fixed points $u$ on $GL_n(\mathbb{F}_q)/P_k = G_k(n)$. Our theorem reduces the computation of this character to fewer than

$$\binom{n}{k} \cdot k(n-k)$$

evaluations of rational functions of the form $[a,b] = (a^q - b^q)/(a - b)$ at roots of unity. This can be done using floating point arithmetic, since all answers can be rounded to the nearest integer.

3. Proof of Theorem 2.5

We now begin to prove our main theorem. To do this we give a number of preliminary lemmas. Once we have these in hand we will explicitly count the number of subspaces fixed by a given torus element, using the Cecioni–Frobenius theorem and a result of Reiner and Stanton.

As before, $\alpha$ is a composition of $n$ and

$$V_\alpha = \mathbb{F}_{q^{\alpha_1}} \oplus \cdots \oplus \mathbb{F}_{q^{\alpha_\ell}}$$
$$T_\alpha = \mathbb{F}_{q^{\alpha_1}}^\times \times \cdots \times \mathbb{F}_{q^{\alpha_\ell}}^\times.$$

There is a right action of $T_\alpha$ on $G_k(n)$ that comes from choosing an isomorphism $V_\alpha \cong \mathbb{F}_q^n$, and hence an embedding $T_\alpha \hookrightarrow GL_n(\mathbb{F}_q)$.

Lemma 3.1. Let $u$ be an element in $\mathbb{F}_q^\times$ with minimal polynomial

$$x^d - a_{d-1}x^{d-1} - \cdots - a_0.$$

Then there exists a basis for $\mathbb{F}_q^n$ such that the right action of $u$ on $G_k(n)$ is represented by the right multiplication by a block diagonal matrix $\text{diag}(U, \ldots, U)$ in $GL_n(\mathbb{F}_q)$, where

$$U = \begin{bmatrix}
0 & 1 & & \\
0 & 1 & & \\
& & \ddots & \\
 a_0 & a_1 & \cdots & a_{d-1}
\end{bmatrix}$$

is the companion matrix of the minimal polynomial of $u$. 

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Proof. Since there is a tower of fields $\mathbb{F}_q \subseteq \mathbb{F}_q[u] \subseteq \mathbb{F}_q^\ell$, we see that $\mathbb{F}_q$ is of degree $n/d$ over $\mathbb{F}_q[u]$. Let $v_1, \ldots, v_{n/d}$ denote an $\mathbb{F}_q[u]$-basis for $\mathbb{F}_q^\ell$. It follows from Lang [6, Proposition V.1.2] that
\[
\{u^iv_j : 0 \leq i \leq d-1, 1 \leq j \leq n/d\}
\]
is a $\mathbb{F}_q$-basis for $\mathbb{F}_q^\ell$. There is now an isomorphism of $\mathbb{F}_q[u]$-modules,
\[
\mathbb{F}_q^\ell \cong (\mathbb{F}_q[u])^{\oplus n/d},
\]
given by grouping the basis elements above by the index of $v$. The matrix of $u$ acting on each copy of $\mathbb{F}_q[u]$ is $U$, so we are done. \hfill \Box

**Corollary 3.2.** Let $u = (u_1, \ldots, u_\ell)$ be an element of $T_\alpha$. Then there exists a basis for $V_\alpha$ such that the action of $u$ on $V_\alpha$ is represented by the right multiplication by the block diagonal matrix
\[
\text{diag}(U_1, \ldots, U_1, U_2, \ldots, U_2, \ldots, U_\ell, \ldots, U_\ell).
\]
Here, as before, $U_i$ is the companion matrix of the minimal polynomial of $u_i$ and each matrix $U_i$ is repeated $[\mathbb{F}_{q^\nu_i} : \mathbb{F}_q[u_i]]$ times.

If $Z \subseteq V_\alpha$ is a $k$-dimensional subspace, we obtain a sequence of nonnegative integers $\beta(Z) = (\beta_1, \ldots, \beta_\ell)$ by letting
\[
\beta_r = \dim \pi_r(\ker(\pi_{r-1}) \cap Z)
\]
where $\pi_r$ is the projection $V_\alpha = \bigoplus_{i=1}^\ell \mathbb{F}_{q^\nu_i} \rightarrow \bigoplus_{i=1}^r \mathbb{F}_{q^\nu_i}$.

The numbers $\beta(Z)$ can be determined in coordinates quite easily. Choose a basis for $V_\alpha$ which is a concatenation of bases for $\mathbb{F}_{q^\nu_1}, \ldots, \mathbb{F}_{q^\nu_\ell}$. Write the subspace $Z$ as the row space of a $k$-by-$n$ matrix in row echelon form with respect this basis, and let $\lambda$ be the associated partition. Then $\alpha$ has partitioned the columns of this matrix into blocks, labeled by $1, \ldots, \ell$ from right to left, and $\beta_r$ is the number of $1$'s in the $r$th block. In other words, $\beta(Z) = \beta(\lambda)$ as in Section 21. It is a weak composition of $k$ into $\ell$ parts, breaking the rows of a $k$-by-$\nu$ matrix into blocks, labeled by $1, \ldots, \ell$ from top to bottom (some may be empty). Therefore one can write $Z = [Z_{rs}]_{r,s=1}^\ell$ as a block matrix.

**Lemma 3.3.** Let $u = (u_1, \ldots, u_\ell) \in T_\alpha$ with $[\mathbb{F}_q[u_\ell] : \mathbb{F}_q] = d_r$. Let $Z$ be a $k$-dimensional subspace of $V_\alpha$ with $\beta(Z) = (\beta_1, \ldots, \beta_\ell)$. If $Z$ is fixed by $u$, then under the basis given in Corollary 3.2 $Z$ has row echelon form equal to a block matrix $[Z_{rs}]_{1 \leq r \leq s \leq \ell}$ where each block $Z_{rs}$ has dimension $\beta_r$-by-$\nu_s$ and $Z_{rs} = 0$ whenever $r > s$. Furthermore, each anti-diagonal block $Z_{rr}$ is in a block row echelon form
\[
Z_{rr} = \left[ \begin{array}{cccccc}
* & \ldots & * & 0 & \ldots & 0 \\
\vdots & & \vdots & \ddots & \ddots & \vdots \\
\ldots & & \ldots & \ldots & \ldots & \ldots \\
0 & \ldots & 0 & \ldots & * & * \\
0 & \ldots & \ldots & \ldots & * & 0 \\
\end{array} \right].
\]

Here the blocks $0$, $I_{d_r}$, and $*$ are all of size $d_r$-by-$d_r$.

**Proof.** Let $W_r \subseteq \mathbb{F}_{q^{\nu_r}}$ be defined as
\[
W_r = \pi_r(\ker(\pi_{r-1}) \cap Z) \subseteq \mathbb{F}_{q^{\nu_r}},
\]
where \( \pi_r \) is the natural projection \( V_\infty \rightarrow \bigoplus_{i=1}^r F_p^{\alpha_i} \). By definition, \( W_r \) is a \( \beta_r \)-dimensional subspace of \( F_p^{\alpha_r} \).

We have \( W_r u_r = W_r \), since \( Z \) being fixed by \( u \) implies

\[
\pi_r ((\ker(\pi_{r-1}) \cap Z)u_r = \pi_r ((\ker(\pi_{r-1}) \cap Z)u) = \pi_r ((\ker(\pi_{r-1}) \cap Z)).
\]

Thus \( W_r \) is an \( F[u_r] \)-submodule of \( F_p^{\alpha_r} \). Using a basis of \( F_p^{\alpha_r} \) furnished by the isomorphism (see the proof of Lemma 3.1)

\[
F_p^{\alpha_r} \cong (F_q[u_r])^{\oplus \alpha_r/d_r},
\]
we write \( W_r \) as a block matrix in row echelon form. Each pivot block must be an identity matrix, since \( F_q[u_r] \) is irreducible as a module over itself.

Doing the above procedure for each \( r \) proves that \( Z \) can be written as a block triangular matrix \( [Z_{rs}] \) where the anti-diagonal blocks have the desired form. Then one can do block row reduction using anti-diagonal blocks to put the entire matrix \( [Z_{rs}] \) in row echelon form.

Remark 3.4. The *’s in \( Z_{rr} \) are \( d_r \)-by-\( d_r \) matrices with entries in \( F_q \) that satisfy the equation \( U_r X = X U_r \), \( U_r \) being the companion matrix of the minimal polynomial of \( u_r \). The solutions to this equation bijectively correspond to elements in \( F_q^{d_r} = F_p[u_r] \).

Lemma 3.5 (Cecioni–Frobenius). Let \( A \) and \( B \) be matrices of size \( a \times a \) and \( b \times b \) with entries in a field \( F \). The \( F \)-dimension of the space of solutions of the equation \( AX = XB \), for \( X \) a matrix of size \( a \times b \), is given by the sum

\[
\sum_{i,j} \deg \gcd(d_i(A), d_j(B)),
\]
where \( d_i(A) \) is the \( i \)th invariant factor of \( \lambda I_n - A \) over \( F[\lambda] \), and similarly for \( d_j(B) \).

Proof. This is a classical, if somewhat unknown result. See [7, Theorem 46.3].

Lemma 3.6. For \( i = 1, 2 \), let \( u_i \) be an element in \( F_q^{\alpha_i} \), with \( F_q[u_i] = F_q^{d_i} \), let \( U_i \) be the companion matrix of the minimal polynomial of \( u_i \), and let \( m_i \) be a positive integer divisible by \( d_i \). Then the number of \( m_1 \)-by-\( m_2 \) matrices \( X \) with entries in \( F_q \) that satisfy

\[
diag(U_1, \ldots, U_1)X = X diag(U_2, \ldots, U_2)
\]

is equal to

\[
\prod_{j=1}^{m_1} \prod_{i=1}^{m_2} \left[ \omega_1(u_1)^{q^{i+j}} \omega_2(u_2)^{q^{i+m_1}} \right]
\]

where \( \omega_i : F_q^{\times} \rightarrow \mathbb{C}^{\times} \) is a fixed injection of groups, \( i = 1, 2 \).

We extend our previous notation and let \([a, b]_q = (a^q - b^q)/(a - b)\), since we occasionally use the notation \([a, b]_{q^2} \), etc..

Proof. Write \( X \) as a block matrix \( [X_{rs}] \) where each block \( X_{rs} \) has dimension \( d_1 \)-by-\( d_2 \). Then Equation (3) is equivalent to \( U_1 X_{rs} = X_{rs} U_2 \) for all \( X_{rs} \).
Take an arbitrary block $X_{rs}$ and let $z$ its the bottom-left entry. It follows from
\[ \omega_i(u_i)^{d_i} = \omega_i(u_i), \quad i = 1, 2, \]
that
\[
\prod_{x \in X_{rs}} \left[ \omega_1(u_1)^{q^i + j(x)} z, \omega_2(u_2)^{q^i + m_1} \right] = \prod_{j=0}^{d_1-1} \prod_{i=0}^{d_2-1} \left[ \omega_1(u_1)^{q^{i+j(x)+i+j}}, \omega_2(u_2)^{q^{i+j(x)+i+j}} \right]
\]
\[
= \prod_{j=0}^{d_1-1} \prod_{i=0}^{d_2-1} \left[ \omega_1(u_1)^{q^{i+j(x)}}, \omega_2(u_2)^{q^i} \right]
\]
where $i(x), j(x), i(z), j(z)$ are all taken in the $m_1$-by-$m_2$ rectangle. Thus it suffices to show that the last product above is equal to the number of solutions to a single equation $U_1 X_{rs} = X_{rs} U_2$.

Let $f_i$ be the minimal polynomial over $F_q$ of $u_i$ for $i = 1, 2$. Since $f_i$ is irreducible over $F_q$, the invariant factors of $\lambda U_i - U_i$ are $1, \ldots, 1, f_i$ for $i = 1, 2$. The roots of $f_i$ are $u_i, u_1^q, \ldots, u_1^{q^{d_i-1}}$, which are distinct and form the orbit of $u_i$ under the Frobenius automorphism $u \mapsto u^q$.

Suppose that $f_1 \neq f_2$. Then $u_1^{q^i} \neq u_2^{q^j}$ for all $i, j$. Hence
\[
\prod_{j=0}^{d_1-1} \prod_{i=0}^{d_2-1} \left[ \omega_1(u_1)^{q^{i+j}}, \omega_2(u_2)^{q^i} \right] = \prod_{j=0}^{d_1-1} \prod_{i=0}^{d_2-1} \frac{\omega_1(u_1)^{q^{i+j+1}} - \omega_2(u_2)^{q^{i+1}}}{\omega_1(u_1)^{q^{i+j}} - \omega_2(u_2)^{q^i}}
\]
\[
= \prod_{j=0}^{d_1-1} \prod_{i=0}^{d_2-1} \left( \omega_1(u_1)^{q^{i+j}} - \omega_2(u_2)^{q^i} \right)
\]
\[
= 1.
\]
Here again the last equality follows from $\omega_i(u_i)^{d_i} = \omega_i(u_i)$. Thus, our product formula predicts that the number of solutions $N$ is 1, which agrees with Cecioni–Frobenius.

Now assume $f_1 = f_2 = f$, which implies $d_1 = d_2 = d$. Since $u_1$ and $u_2$ are both roots of $f$, there exists a unique $k$ such that $0 \leq k \leq d - 1$ and $u_1^{q^k} = u_2$. Hence, if $0 \leq j \leq d - 1$ then
\[
\left[ \omega_1(u_1)^{q^j}, \omega_2(u_2)^{q^j} \right] = \begin{cases} 1, & j \neq k, \\ q^d, & j = k. \end{cases}
\]
One also sees from the definition that
\[
\prod_{i=0}^{d-1} \left[ t_1^{q^i}, t_2^{q^i} \right] = \left[ t_1^{q^i}, t_2^{q^i} \right].
\]
We predict that,
\[
N = \prod_{i,j=0}^{d-1} \left[ \omega_1(u_1)^{q^{i+j}}, \omega_2(u_2)^{q^j} \right] = \prod_{j=0}^{d-1} \left[ \omega_1(u_1)^{q^j}, \omega_2(u_2)^{q^j} \right] = q^d.
\]
This agrees with Cecioni–Frobenius, which gives the dimension of the solution space of $U_1 X_{rs} = X_{rs} U_2$ to be $d$.

We are finally in a good position to prove our theorem. Recall that we have fixed injections $\omega : F_q^\times \rightarrow C^\times$ of groups throughout this discussion.
Proof of Theorem 2.5. Let \( \beta = (\beta_1, \ldots, \beta_\ell) \) be a weak composition of \( k \) with \( \beta_r \leq \alpha_r \) for \( r = 1, \ldots, \ell \). It suffices to show that the number of subspaces \( Z \) that are fixed by an element \( u = (u_1, \ldots, u_\ell) \) in \( T_\alpha \) and have \( \beta(Z) = \beta \) is equal to

\[
\sum_{\lambda; \beta(\lambda) = \beta} \text{wt}(\lambda; \alpha, k)|_{t_r = \omega_r(u_r)}.
\]

Let \( Z \) be such a subspace, and let \( [Z_{rs}]_{1 \leq r \leq s \leq \ell} \) be the row echelon form of \( Z \) given by Lemma 3.3. Using Corollary 3.2 one has

\[
Zu = [Z_{rs}] \cdot \text{diag}(U_1, \ldots, U_1, U_2, \ldots, U_2, \ldots, U_\ell, \ldots, U_\ell)
\]

where each \( U_r \) appears \( \alpha_r / d_r \) times. Since the pivots in \( [Z_{rs}] \) form identity blocks \( I_{d_r} \), one can bring \( Zu \) back to a row echelon form by left-multiplying by

\[
\text{diag}(U_1, \ldots, U_1, U_2, \ldots, U_2, \ldots, U_\ell, \ldots, U_\ell)^{-1},
\]

where each \( U_r \) appears \( \beta_r / d_r \) times, and the result must agree with the row-echelon form of \( Z \). Using Lemma 3.6 one sees that the number of solutions for \( Z_{rs} \), \( r < s \), is given by

\[
(4) \quad \text{wt}(\lambda^{r,s}; \omega_r(u_r), \omega_s(u_s)) = \prod_{j=0}^{\beta_r-1} \prod_{i=0}^{\alpha_s-1} [\omega_r(u_r)^{q^{i+j}}, \omega_s(u_s)^{q^{i+j}s}] = \beta_r^{-1} \alpha_s^{-1} \beta_s^{-1}.
\]

where \( \lambda^{r,s} \) is a \( \beta_r \)-by-\( (\alpha_s - \beta_s) \) rectangle.

The number of solutions for the anti-diagonal block \( Z_{rs} \) to be fixed by \( u_r \) is the number of \( \beta_r \)-dimensional \( \mathbb{F}_q \)-subspaces of \( \mathbb{F}_{q^r} \) fixed by \( u_r \), which was computed by Reiner and Stanton to be an evaluation of their \((q, t)\)-binomial coefficient. They proved in their [5] Theorem 5.2] that this number is

\[
(5) \quad \sum_{\lambda; \beta(\lambda) = \beta} \text{wt}(\lambda^{r,s}; \omega_r(u_r), \omega_s(u_r)).
\]

Combining equations (4) and (5) together with the bijection (2) shows that the total number of subspaces \( Z \) with \( \beta(Z) = \beta \) that are left fixed by \( u \) is

\[
\sum_{\lambda; \beta(\lambda) = \beta} \text{wt}(\lambda; \alpha, k)|_{t_r = \omega_r(u_r)}.
\]

This is what we needed to show. \( \square \)

Corollary 3.7. Let \( C_\lambda \) be the Schubert cell of \( Gr_k(V_\alpha) \) indexed by \( \lambda \vdash n \). For any weak composition \( \beta = (\beta_1, \ldots, \beta_\ell) \) of \( k \) with \( \beta_r \leq \alpha_r, 1 \leq r \leq \ell \), the triple

\[
\left( \bigcup_{\lambda; \beta(\lambda) = \beta} C_\lambda, \sum_{\lambda; \beta(\lambda) = \beta} \text{wt}(\lambda; \alpha, k), T_\alpha \right)
\]

exhibits the cyclic sieving phenomenon. In addition, the polynomial \( \sum_{\lambda; \beta(\lambda) = \beta} \text{wt}(\lambda; \alpha, k) \) factors as

\[
\left( \prod_{r=1}^\ell \prod_{i=0}^{\beta_r-1} \frac{q^{q^{ir}r}}{q^{q^{ir}r}-t_r^{q^{ir}r}} \right) \left( \prod_{1 \leq r < s \leq \ell} \prod_{j=0}^{\beta_r-1} \prod_{i=0}^{\alpha_s-1} \prod_{s}^{\beta_s-1} \left[ q^{q^{i+j}s}, t_r^{q^{i+j}s} \right] \right).
\]
Proof. One can refine the action of $T_{\alpha}$ to $\bigcup_{\beta(\lambda)=\beta} C_{\lambda}$ since
\[ \pi_r(\ker(\pi_{r-1}) \cap Zu) = \pi_r(\ker(\pi_{r-1}) \cap Zu). \]

The proof of Theorem 2.5 shows precisely the cyclic sieving phenomenon for this refined action. The second assertion follows from the product formulation of the $(q, t)$-binomial coefficient (Reiner–Stanton \cite{8} p.1) and Equation (4). \qed

Remark 3.8. It follows the above corollary that
\[
\begin{bmatrix} n \\ k \end{bmatrix}_q = \lim_{t_1, \ldots, t_{\ell} \to 1} \sum_{\beta_1 + \cdots + \beta_i = k} \sum_{\lambda, \beta(\lambda) = \beta} \text{wt}(\lambda; \alpha, k)
= \sum_{\beta_1 + \cdots + \beta_i = k} \prod_{r=1}^{\ell} \left[ \frac{\alpha_r}{\beta_r} \right] q^{\beta_r(\alpha_s - \beta_s)}.
\]

When $\ell = 2$ this gives the well-known $q$-Vandermonde identity.

4. Partial Flag Varieties

In this section we generalize the previous results to the partial flag varieties. We do this at the cost of some repetition, as all of our previous results are subsumed in the forthcoming pages. We find this to be pedagogically sound since the proofs presented by themselves would be opaque without the presentation of the Grassmannians as “warm-up” cases.

We start by giving the relevant definitions, and then consider the two extreme cases of the $(1^n)$ and $(n)$ tori, $(\mathbb{F}_q \times)^n$ and $\mathbb{F}_q^n$. Following this we define the polynomials which gives the CSP for the partial flag varieties and prove our main theorem.

4.1. Partial flag varieties and Schubert decomposition. Our discussion begins with some geometry and combinatorics of partial flag varieties.

Let $\beta = (\beta_1, \ldots, \beta_m)$ be a (weak) composition of $n$. The partial flag variety of type $\beta$ is
\[ F(\ell(\beta)) = \{ 0 \subset V_1 \subset V_2 \subset \cdots \subset V_m = V : \dim(V_i) = \beta_1 + \cdots + \beta_i \}, \]
where $V$ is a $n$-dimensional vector space over $\mathbb{F}_q$. We will usually take $V = V_\alpha$, as in Section 3.

The parabolic subgroup (or Young subgroup) $W_\beta = S_\beta$ of the symmetric group $W = S_n$ is the direct product $S_{\beta_1} \times \cdots \times S_{\beta_m}$ where $S_{\beta_i}$ is the permutation group on the integers $\beta_1 + \cdots + \beta_{i-1} + 1, \ldots, \beta_1 + \cdots + \beta_i$.

Written in one-line notation, the permutations $w = w(1)w(2)\ldots w(n)$ in $S_n$ are naturally partitioned by $\beta$ into blocks. Any coset $wS_\beta$ can be represented by the element with the minimal length, i.e., the element obtained from $w$ by sorting every block of $w$ into increasing order. For example, if $\beta = (4, 2, 2)$ and $w = 5268\, \vert \, 73\, \vert \, 14$ then the minimal coset representative of $wS_\beta$ is $2568\, \vert \, 37\, \vert \, 14$. Let $W^\beta = W/W_\beta$ be the set of all these minimal coset representatives.

Let $0 \subset V_1 \subset V_2 \subset \cdots \subset V_m$ be a flag in $F(\ell(\beta))$ represented by an $n$-by-$n$ matrix $F$ whose first $\beta_1 + \cdots + \beta_i$ rows span $V_i$ for all $i = 1, \ldots, m$. Then there exists a unique permutation $w \in W^\beta$ such that $F$ can be reduced by row operations fixing the partial flag to a row echelon form $[a_{ij}]_{i,j=1}^n$ with
• $a_{ij} = 1$, called a pivot, if $j = w(i)$,
• $a_{ij}$ is arbitrary if $i < w^{-1}(j)$, $w(i) > j$,
• $a_{ij} = 0$ otherwise.

In other words, $F$ is obtained from the permutation matrix of $w$ by replacing those zeros in the positions corresponding to the inversions of $w$ with arbitrary numbers in $\mathbb{F}_q$. The Schubert cell $C_w$ indexed by $w \in W^0$ consists of those flags whose associated permutation is $w$.

For instance, if $\beta = (1, 2, 2)$ and $w = 23514$, then $C_w$ consists of all partial flags that can be represented in the form

\[
\begin{bmatrix}
* & 1 & 0 & 0 & 0 \\
* & 0 & 1 & 0 & 0 \\
* & 0 & 0 & * & 1 \\
0 & 0 & 0 & 1 & 0
\end{bmatrix}.
\]

Here the dashed lines indicate which rows span subspaces of the associated partial flag. We shall omit them if they are clear from context. We refer to Fulton [4, §10.2] for the geometry of the complete flag variety.

4.2. The $1^n$-torus action on $\mathcal{F}\ell(\beta)$. The torus $T_1^n$ acts on $\mathcal{F}\ell(\beta)$ by rescaling the columns of a matrix representing a flag. For example, if $F$ is a flag as in the above example, then

\[
F = \begin{bmatrix}
a & 1 & 0 & 0 & 0 \\
b & 0 & 1 & 0 & 0 \\
c & 0 & 0 & d & 1 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{bmatrix} \xrightarrow{u} \begin{bmatrix}
au_1 & u_2 & 0 & 0 & 0 \\
bu_1 & 0 & u_3 & 0 & 0 \\
cu_1 & 0 & 0 & du_4 & u_5 \\
u_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & u_4 & 0
\end{bmatrix}.
\]

Applying row operations that do not effect $F \cdot u$ we obtain

\[
\begin{bmatrix}
u_2^{-1}au_1 & 1 & 0 & 0 & 0 \\
u_3^{-1}bu_4 & 0 & 1 & 0 & 0 \\
u_5^{-1}cu_1 & 0 & 0 & u_5^{-1}du_4 & 1 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{bmatrix}.
\]

By the uniqueness of the row echelon form, $F$ is fixed by $u$ if and only if

\[
u_2a = au_1, \quad u_3b = bu_1, \quad u_5c = cu_1, \quad u_5d = du_4.
\]

The number of solutions to these equations is given by

\[
[t_2, t_1][t_3, t_1][t_5, t_1][t_5, t_4]|_{t_i = \omega(u_i)}
\]

where $\omega : \mathbb{F}_q^\times \rightarrow \mathbb{C}^\times$ is a fixed injection of groups.

One can easily extend this example to the action of $1^n$-torus on the partial flag variety $\mathcal{F}\ell(\beta)$ for all compositions $\beta$ of $n$ and show that

\[
\left(\mathcal{F}\ell(\beta), X_{1^n, \beta}(t), (\mathbb{F}_q^\times)^{\times n}\right)
\]

\[\text{12}\]
exhibits the cyclic sieving phenomenon, where
\[ X_{1^n, \beta}(t) = \sum_{w \in W^\beta} \prod_{(i,j) \in \text{Inv}(w)} [t_i, t_j]. \]
Here \( \text{Inv}(w) = \{(i,j) : i < j, w(i) > w(j)\} \) is the usual set of inversions of \( w \).

4.3. The \( n \)-torus action on \( \mathcal{F} \ell(\beta) \). Reiner, Stanton, and White \cite{ReinerStantonWhite} considered the cyclic action of \( \mathbb{F}_q^\ell \) on the partial flag variety \( \mathcal{F} \ell(\beta) \). They observed that a flag \( 0 \subset V_1 \subset \cdots \subset V_m = V \) in \( F(\beta, \mathbb{F}_q) \) is fixed by an element \( u \in \mathbb{F}_q^\ell \) if and only if all \( V_i \) are \( \mathbb{F}_q^{d_i} \)-spaces, where \( d = [\mathbb{F}_q[u] : \mathbb{F}_q] \). On the other hand, they defined the \((q,t)\)-multinomial coefficient
\[ \left[ n \right]_{\beta, q, t} : = \frac{n_{1,q,t}}{\beta_1!_{q,t} \cdot \beta_2!_{q,t} \cdot \beta_3!_{q,t} \cdots}, \]
where \( n_{1,q,t} = (1 - t^{q^n-1})(1 - t^{q^n-q}) \cdots (1 - t^{q^n-q^{n-1}}) \), and showed that
\[ \left[ n/d \right]_{\beta, q, \omega(u)} : = \left[ \beta/d \right]_{q, t}^d, \]
where \( \beta/d = (\beta_1/d, \beta_2/d, \ldots) \) and \( \omega : \mathbb{F}_q^\ell \hookrightarrow \mathbb{C}^\times \) is an injection of groups. Hence
\[(\mathcal{F} \ell(\beta), X_{n, \beta}(t), \mathbb{F}_q^\ell)\]
exhibits the cyclic sieving phenomenon, with \( X_{n, \beta}(t) = \left[ n \right]_{\beta, q, t}. \)

Reiner and Stanton \cite{ReinerStanton} Section 8 defined a weight function that allows one to write the \((q,t)\)-multinomial coefficient as
\[ \left[ n \right]_{\beta, q, t} = \sum_{w \in W^\beta} \text{wt}(w; t). \]
They used a recurrence relation to define the weight, which was later shown by Hivert and Reiner \cite{HivertReiner} to take the form
\[ \text{wt}(w; t) = \prod_{(i,j) \in \text{Inv}(w)} \text{wt}((i,j); t), \]
for some weights associated to the inversions of \( w \), which we now define.

Given a word \( w = w_1 \ldots w_\ell \), recursively define a labeled tree by taking the smallest letter \( w_s \) of \( w \) as the root and attaching to it the trees obtained from the subwords \( w_1 \ldots w_{s-1} \) and \( w_{s+1} \ldots w_\ell \) as left and right subtrees. For instance, the tree associated to \( w = 385216479 \) is
\[
\begin{array}{c}
1 \\
\begin{array}{cc}
2 & 4 \\
3 & \quad \begin{array}{c}
6 \\
5 \\
\quad \begin{array}{c}
7 \\
9 \\
8
\end{array}
\end{array}
\end{array}
\end{array}
\]
For any inversion \((i,j)\) of \( w \), find the smallest \( w(k) \) with \( i \leq k \leq j \). In the tree of \( w \), \( w(k) \) is the join (lowest common parent) of \( w(i) \) and \( w(j) \). Let \( \ell \) (resp. \( r \)) be the set of
all vertices in the left (resp. right) subtree of $w(k)$ whose label is at least $w(i)$ (resp. at most $w(j)$). Then

$$\text{wt}((i,j);t) = [k-1+r, k-1+r-\ell].$$

Here the notation is $[a,b] = [t^n, t^b]_q$.

The weights of the inversions of $w = 385216479$ are given in the following table.

| $(w(i), w(j))$ | $k$ | $\ell$ | $r$ | $\text{wt}$ |
|----------------|-----|--------|-----|------------|
| $(2, 1)$       | 5   | 4      | 0   | $[4,0]$    |
| $(3, 1)$       | 5   | 3      | 0   | $[4,1]$    |
| $(3, 2)$       | 4   | 3      | 0   | $[3,0]$    |
| $(5, 1)$       | 5   | 2      | 0   | $[4,2]$    |
| $(5, 2)$       | 4   | 2      | 0   | $[3,1]$    |
| $(8, 1)$       | 5   | 1      | 0   | $[4,3]$    |
| $(8, 2)$       | 4   | 1      | 0   | $[3,2]$    |
| $(5, 4)$       | 5   | 2      | 1   | $[5,3]$    |
| $(6, 4)$       | 7   | 1      | 0   | $[6,5]$    |
| $(8, 4)$       | 5   | 1      | 1   | $[5,4]$    |
| $(8, 5)$       | 3   | 1      | 0   | $[2,1]$    |
| $(8, 6)$       | 5   | 1      | 2   | $[6,5]$    |
| $(8, 7)$       | 5   | 1      | 3   | $[7,6]$    |

Reiner and Stanton [8] also observed that

$$\lim_{t\to 1} \frac{n}{\beta} = \left[ \begin{array}{c} n \\ \beta \end{array} \right]_q,$$

$$\lim_{q\to 1} \frac{n}{\beta} = \left( \begin{array}{c} n \\ \beta \end{array} \right)_t$$

4.4. **Statement of the main result.** Let $\alpha$ and $\beta$ be compositions of $n$. The action of the torus $T_\alpha$ on $V = V_{\alpha}$ induces an action on the partial flag variety $\mathcal{F} \ell(\beta)$. Our goal in this subsection is to define a multivariate polynomial $X_{\alpha, \beta}(t)$ so that the triple

$$(\mathcal{F} \ell(\beta), X_{\alpha, \beta}(t), T_\alpha)$$

exhibits the cyclic sieving phenomenon.

The compositions $\alpha$ and $\beta$ give set partitions of $[n]$ as in

$$A_r = \{\alpha_1 + \cdots + \alpha_{r-1} + 1, \ldots, \alpha_1 + \cdots + \alpha_r\}$$

for $r = 1, \ldots, \ell$, and likewise define $B_1, \ldots, B_m$ from $\beta$. These will allow us to break up a permutation $w \in W^\beta$ into “sub-permutations”, as we did with partitions in the Grassmannian case.

Let $C_w$ be a Schubert cell of $\mathcal{F} \ell(\beta)$ represented by its associated row-echelon form

$$F = [a_{ij}]_{i,j=1}^n$$

of 0’s, 1’s and s’s. Define $F_{rs}$ to be the submatrix of $F$ with column indices in $A_s$ and row indices in $w^{-1}(A_r)$, for $1 \leq r, s \leq \ell$.

If $F$ represents the partial flag $0 \subset V_1 \subset \cdots \subset V_m = V_{\alpha}$, and

$$\pi_s : V_\alpha = \mathbb{F}_{q^1} \oplus \cdots \oplus \mathbb{F}_{q^\ell} \to \mathbb{F}_{q^{m_s}} \oplus \cdots \oplus \mathbb{F}_{q^{m_\ell}}.$$

is the projection map, $1 \leq s \leq \ell$, then define weak compositions $\beta^{(s)}(F)$ by

$$\beta^{(s)}(F) = \dim \pi_s(\ker(\pi_{s+1}) \cap V_k), 1 \leq k \leq m.$$  

This is equivalent to

$$\beta^{(s)}(F) = |\{i : i \in B_k, w(i) \in A_s\}|.$$  

One sees that $\beta^{(1)}(F), \ldots, \beta^{(\ell)}(F)$ are weak compositions of $\alpha_1, \ldots, \alpha_\ell$, respectively, and their component-wise sum is $\beta$, i.e.

$$\beta^{(1)}(F) + \cdots + \beta^{(\ell)}(F) = \beta,$$  

$k = 1, \ldots, m$.  

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Since the definition depends only on the Schubert cell $C_w$ that contains $F$, we can write $\beta^{(s)}(w) = \beta^{(s)}(F)$. The matrix $[\beta^{(s)}(w)_{k,a}]$ indexes the double coset $S_{\alpha}wS_{\beta}$.

Each “diagonal” submatrix $F_{rs}$ determines a permutation $w_s$ in $S_{\alpha}/S_{\beta^{(r)}(w)}$. Conversely, given permutations $w_s$ in $S_{\alpha}/S_{\beta^{(r)}(w)}$, $1 \leq s \leq \ell$, one can recover the permutation $w$ in $S_n$ in a unique way.

**Example 4.1.** Let $\alpha = (4,4)$, $\beta = (1,3,4)$, and $w = 53461278$. Then $C_w$ is represented by

$$F = \begin{bmatrix}
* & * & * & 1 \\
* & * & 1 & \\
* & \\
1 & 1 & \\
1 & 1 & \\
1 & 1 & \\
\end{bmatrix}.$$

It is divided by $\alpha$ into submatrices

$$F_{11} = \begin{bmatrix} *
* & * & 1 \\
* & * & \\
1 & \\
\end{bmatrix}, \quad F_{12} = 0, \quad F_{21} = \begin{bmatrix} * & * & * \\
* & * & \\
1 & \\
\end{bmatrix}, \quad F_{22} = \begin{bmatrix} 1 & \\
1 & \\
1 & \\
\end{bmatrix}.$$

One sees that $\beta^{(1)}(w) = (0,2,2)$, $\beta^{(2)}(w) = (1,1,2)$, $w_1 = 3412$, $w_2 = 5678$.

**Lemma 4.2.** (a) If $r < s$ then $F_{rs} = 0$.
(b) If $r = s$ then $F_{ss}$ is the row echelon form for the Schubert cell $C_{w_s}$ of $F(\beta^{(s)}(w))$.
(c) If $r > s$ then $F_{rs}$ contains only stars and zeros.
(d) Let $a_{ij}$ be a star in $F$ that falls into some $F_{rs}$ with $r > s$. If $i \in B_k$, then $a_{ij}$ is also a star for all $i' \in B_k$ with $w(i') \in A_r$; if $w^{-1}(j) \in B_k$ then $a_{ij'}$ is a star for all $j' \in A_s$ with $w^{-1}(j') \in B_k$.

**Proof.** By the definition, $a_{i,w(i)}$ is a 1 for $i = 1, \ldots, n$, $a_{ij}$ is a star, whenever $i < w^{-1}(j)$ and $w(i) > j$, and $a_{ij}$ is a 0 otherwise. This at once yields (a) and (c) and a moment’s thought gives (b).

Finally, to prove (d), let $a_{ij}$ be a star that falls in some $F_{rs}$ with $r > s$, i.e. $i < w^{-1}(j)$, $w(i) > j$, $w(i) \in A_r$, $j \in A_s$.

Suppose that $i$ and $j'$ are both in $B_k$ for some $k$, and $w(i')$ is in $A_r$. If $w^{-1}(j) \in B_k$, then $i \in B_k$, $i < w^{-1}(j)$, and $w(i) > j$ give a contradiction to $w \in W^d$. Hence $w^{-1}(j) \notin B_k$, and then $i < w^{-1}(j)$ implies $i' < w^{-1}(j)$. Since $w(i), w(i') \in A_r$, $j \in A_s$, and $w(i) > j$, one also has $w(i') > j$. Therefore $a_{ij'}$ is a star.

Similarly, if $w^{-1}(j)$ and $w^{-1}(j')$ are both in $B_k$ for some $k$, and $j'$ is in $A_s$, then $i \notin B_k$ and $i < w^{-1}(j)$ imply $i < w^{-1}(j')$, and $w(i) > j$ implies $w(i) > j'$. Thus $a_{ij'}$ is a star.

It follows from (d) that each $F_{rs}$, $r > s$, consists of one $\beta^{(s)}_{rb}$ by $\beta^{(r)}_{rn}$ rectangle of stars for all pairs $(a,b)$ with $1 \leq a < b \leq m$, and zeros in the remaining spots. Define its
weight to be

$$\text{wt}(F_{rs}; t_r, t_s) := \prod_{1 \leq a < b \leq m} \prod_{i=1}^{\beta_a} \prod_{j=1}^{\beta_b} \begin{bmatrix} t_{r+j}^i & t_{s+j}^i \end{bmatrix}.$$  

In the previous example, $F_{21}$ contains three 1-by-2 rectangles of stars, so its weight is

$$\text{wt}(F_{21}; t_2, t_1) = \left( [t_2^1, t_1^1] [t_2^2, t_1^2] \right)^3.$$  

Let $\text{wt}(F_{ss}; t_s, t_a) := \text{wt}(w_s; t_s)$ as in (6), $1 \leq s \leq \ell$. Define the weight of a Schubert cell $C_w$ to be

$$\text{wt}(w; \alpha) := \prod_{1 \leq s \leq r \leq \ell} \text{wt}(F_{rs}; t_r, t_s).$$

This weight does not depend on $\beta$. Finally define

$$X_{\alpha, \beta}(t) := \sum_{w \in W^\beta} \text{wt}(w; \alpha).$$

Now we can state the main result on flag varieties.

**Theorem 4.3.** The triple $(C, X_{\alpha, \beta}(t), T_\alpha)$ exhibits the cyclic sieving phenomenon.

**4.5. Proof of the main result.** The reader can check that the following is a straightforward generalization of Lemma 3.3 to flag varieties.

**Lemma 4.4.** Let $u = (u_1, \ldots, u_\ell)$ be an element in $T_\alpha$ with $[F_q[u_s] : F_q] = d_s$, $s = 1, \ldots, \ell$, and let $F$ be a flag in $F(\ell, F_q)$ fixed by $u$. Then under the basis for $V_\alpha$ given in Corollary 3.2, $F$ has row echelon form $[F_{rs}]_{r,s=1}^\ell$ in which the pivots form block matrices equal to $I_{d_s}$.

We are now in a good position to prove the main result on the flag varieties.

**Proof of Theorem 4.3.** Take weak compositions $\beta^{(1)}, \ldots, \beta^{(\ell)}$ of $\alpha_1, \ldots, \alpha_\ell$, respectively, such that the component-wise sum of $\beta^{(1)}, \ldots, \beta^{(\ell)}$ is equal to $\beta$. Let $u = (u_1, \ldots, u_\ell)$ be an element in $T_\alpha$ with $[F_q[u_s] : F_q] = d_s$, $s = 1, \ldots, \ell$. Fix the embeddings $\omega_s : F_q \rightarrow \mathbb{C}^\times$. Consider all flags $F$ in $F(\ell, F_q)$ which are fixed by $u$ and have $\beta^{(s)}(F) = \beta^{(s)}$, $s = 1, \ldots, \ell$.

Let $[F_{ss}]_{r,s=1}^\ell$ be the row echelon form of $F$ given by Lemma 4.4. Similarly to the proof of Theorem 3.3, it follows from Corollary 3.2, Lemma 4.2(a), and Lemma 4.4 that $F_{ss}$ is fixed by $u_s$ for $s = 1, \ldots, \ell$, and

$$\text{diag}(U_r, \ldots, U_r) F_{rs} = F_{rs} \text{diag}(U_s, \ldots, U_s)$$

for all $1 \leq s < r \leq \ell$. By equation (8), the number of choices for $F_{ss}$ is

$$\begin{bmatrix} \alpha_s \\
\beta^{(s)} \end{bmatrix}_{q, \omega_s(u_s)} = \sum_{w_s \in G_{\alpha_s} / G_{\beta^{(s)}}} \text{wt}(w_s; \omega_s(u_s)).$$

By Lemma 4.2(d) and Lemma 3.6 the number of solutions for $F_{rs}$ is

$$\text{wt}(F_{rs}; \omega_r(u_r), \omega_s(u_s)).$$

Multiplying all choices for the various submatrices $F_{rs}$ one obtains the number of solutions for $F$ from

$$\sum_{w : \langle \beta^{(s)} \rangle(w) = \beta^{(s)}} \text{wt}(w; \alpha).$$
by setting $t_s = \omega_s(u_s)$, $1 \leq s \leq \ell$.

**Remark 4.5.** Similarly to Corollary 3.7 and as the above proof shows, for fixed weak compositions $\beta^{(1)}, \ldots, \beta^{(\ell)}$ of $\alpha_1, \ldots, \alpha_\ell$ whose component-wise sum equals $\beta$, one can refine the cyclic sieving phenomenon to the triple

$$\left( \bigcup_{w: (\forall r) \beta^{(r)}(w) = \beta^{(r)}} C_w, \sum_{w: (\forall r) \beta^{(r)}(w) = \beta^{(r)}} \text{wt}(\alpha; w), T_\alpha \right)$$

and the polynomial $\sum_{w: (\forall r) \beta^{(r)}(w) = \beta^{(r)}} \text{wt}(w, \alpha)$ factors nicely. Taking $t_1, \ldots, t_\ell \to 1$ leads to

$$[\alpha_1 + \cdots + \alpha_\ell]_{\beta_1, \ldots, \beta_m}_q = \sum_{\beta^{(1)}, \ldots, \beta^{(\ell)}} \prod_{r=1}^\ell \prod_{1 \leq \alpha \leq \ell} \prod_{1 \leq i < j \leq m} q^{\beta_i^{(r)} \beta_j^{(r)}},$$

summed over all weak compositions $\beta^{(r)}$ of $\alpha_r$, $1 \leq r \leq \ell$, with component-wise sum $\beta^{(1)} + \cdots + \beta^{(\ell)} = \beta$. This generalizes the $q$-Vandermonde identity.

On the other hand, by Equation (7) and the following limit

$$\lim_{q \to 1} [a, b]_q = 1, \quad a \neq b,$$

one can view $Y_{\alpha, \beta}(t)$, defined in Equation (11), as a “$q = 1$” version of $X_{\alpha, \beta}(t)$.

5. Further Questions

The partial flag variety $\mathcal{F}_\ell(\beta)$ can be identified with the parabolic cosets $G/P_\beta$, where $G = GL(n, F_q)$ and $P_\beta$ is the parabolic subgroup of all block upper triangular matrices with invertible diagonal blocks of size $\beta_1, \ldots, \beta_m$. If $C$ denotes $F_q^n$, then Springer’s theorem [11] asserts that, as $F_q[G \times C]$-bimodules, the coinvariant algebra $F_q[x]/(F_q[x]^C)$ and the group algebra $F_q[G]$ have the same composition factors. The cyclic sieving phenomenon for the triple $(G/P_\beta, X_{\alpha, \beta}(t), C)$ is a consequence of Springer’s theorem, with

$$X_{\alpha, \beta}(t) = \left[ \begin{array}{c} n \\ \beta \end{array} \right]_{q, t} = \text{Hilb} \left( F_q[x]/(F_q[x]^C), t \right).$$

See Broer, Reiner, Smith, and Webb [2] for details and generalizations.

Is there a Springer-type result for the $F_q[G \times T_\alpha]$-module $F_q[G]$ which would imply the cyclic sieving phenomenon for the $T_\alpha$-action on $G/P_\beta$? There might be some clue suggested by the factorization of the sieving polynomial in Corollary 3.7 or Remark 4.5 (even for the “$q = 1$” version).

Now let $G$ be a linear algebraic group defined over the algebraic closure of $F_q$ and fix a maximal split torus. Let $J$ be a subset of the positive roots of $G$. This subset $J$ defines a parabolic subgroup $P_J \subset G$ and the quotient $G/P_J$ is a generalized flag variety. The set of $F_q$-rational points of $G/P_J$ is invariant under the natural action of a maximal torus $T \subset G(F_q)$. Finally, let $W$ denote the Weyl group of $G$, $W_J$ a parabolic subgroup and $W^J$ the shortest length coset representatives of $W/W_J$.

When $G = GL_k$, we have succeeded in expressing all of our cyclic sieving polynomials in the form

$$\sum_{w \in W^J} \text{wt}(w; T),$$
i.e., to each coset representative \( w \) we have associated a polynomial weight depending on \( T \), which is a product over those positive roots made negative by \( w \). Can this be extended to groups \( G \) other than \( \text{GL}_n \)?

6. Acknowledgements

The authors would like to thank Julian Gold for preliminary computations done during a summer 2010 REU at UC Davis with the first author. Thanks also to Victor Reiner and Dennis Stanton for helpful comments and suggestions.

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Department of Mathematics, University of California, Davis, CA 95616
E-mail address: berget@math.ucdavis.edu

Department of Mathematics, University of Minnesota, Minneapolis, MN 55455
E-mail address: huang338@math.umn.edu