TIME SERIES MODELS FOR REALIZED COVARIANCE MATRICES
BASED ON THE MATRIX-F DISTRIBUTION

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We propose a new Conditional BEKK matrix-F (CBF) model for the time-varying realized covariance (RCOV) matrices. This CBF model is capable of capturing heavy-tailed RCOV, which is an important stylized fact but could not be handled adequately by the Wishart-based models. To further mimic the long memory feature of the RCOV, a special CBF model with the conditional heterogeneous autoregressive (HAR) structure is introduced. Moreover, we give a systematical study on the probabilistic properties and statistical inferences of the CBF model, including exploring its stationarity, establishing the asymptotics of its maximum likelihood estimator, and giving some new inner-product-based tests for its model checking. In order to handle a large dimensional RCOV matrix, we construct two reduced CBF models — the variance-target CBF model (for moderate but fixed dimensional RCOV matrix) and the factor CBF model (for high dimensional RCOV matrix). For both reduced models, the asymptotic theory of the estimated parameters is derived. The importance of our entire methodology is illustrated by simulation results and two real examples.

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1. Introduction. Modeling the multivariate volatility of many asset returns is crucial for asset pricing, portfolio selection, and risk management. After the seminal work of Barndorff-Nielsen and Shephard (2002, 2004) and Andersen et al. (2003), the realized covariance (RCOV) matrix, estimated from the intra-day high frequency return data, has been recognized as a better estimator than the daily squared returns for daily volatility. Consequently, increasing attention has been focused on the modeling and forecasting of these RCOVs; see, e.g., McAleer and Medeiros (2008), Hansen et al. (2012), Noureldin et al. (2012), Bollerslev et al. (2016), and many others.

Existing models for the RCOV matrices can be roughly categorized into two types: transformation-based models and likelihood-based models. Models in the first category capture the dynamics of the RCOV matrices in an indirect way via transformation. Bauer and Vorkink (2011) used a factor model for the vectorization of the log transformation of RCOV matrix; Chiriac and Voev (2011) applied a vector autoregressive fractionally integrated moving average process to model the Cholesky decomposition of RCOV matrix; Callot et al. (2017) transformed the RCOV matrix into a large vector by the $\text{vech}$ operator, and then fitted this transformed vector by a vector autoregressive model. In the first two models, the dimension of RCOV matrix has to be moderate (e.g., less than 6) for a feasible manipulation. In the third model, the dimension of RCOV matrix is allowed to be 30 in applications with the help of the LASSO method.

Models in the second category deal with RCOV matrices directly by assuming that the innovation, which drives the RCOV time series, has a specific matrix distribution that generates random positive definite matrices automatically without imposing additional constraints. This important feature results in positive-definite estimated RCOV matrices. Unlike scalar or vector distributions, so far only a few matrix distributions have been found to have explicit forms. The primary choice for the innovation distribution is Wishart, leading to the Wishart autoregressive (WAR) model in Gouriéroux et al. (2009), the conditional autoregressive Wishart (CAW) model in Golosnoy et al. (2012), and the generalized CAW model in Yu et al. (2017) to name a few. The other choice for the innovation distribution is matrix-F, which was recently adopted by Opschoor et al. (2018). Generally speaking, matrix-F distribution is the generalization of the usual F distribution, while Wishart distribution is the generalization of the $\chi^2$ distribution (see, e.g., Konno (1991) and Opschoor et al. (2018) for more discussions). Therefore, matrix-F distribution could
be more appropriate than Wishart distribution in capturing the heavy-tailed innovation, which is an important stylized fact in many applications (see, e.g., Bollerslev (1987), Fan et al. (2014), Zhu and Li (2015), and Oh and Patton (2017)). These likelihood models have at least three edges over the transformation-based models. First, the likelihood-based models preserve the useful and important matrix structural information, which makes them more interpretable compared with transformation-based models. Second, the number of estimated parameters in the transformation-based models has order $O(n^4)$, while the one in the likelihood-based models has order $O(n^2)$, where $n$ is the dimension of the RCOV matrix. When $n$ is large, the likelihood-based models can bring more convenience and a less daunting task in computation. Third, the likelihood-based models make use of the likelihood function of the RCOV matrices, and hence their statistical inference methods could be easily provided.

This paper contributes to the literature from three aspects. First, we propose a new Conditional BEKK matrix-F (CBF) model to study the time-varying RCOV matrices. Our CBF model has matrix-F distributed innovations with two degrees of freedom parameters $\nu_1$ and $\nu_2$. When $\nu_2 \to \infty$, our CBF model reduces to the CAW model (Golosnoy et al. (2012)), which has Wishart distributed innovations. Hence, the degrees of freedom $\nu_2$ is designed to capture the heavy-tailedness of the RCOV. Since the RCOV is also well documented to have long memory phenomenon, we further introduce a special CBF model which has a similar conditional heterogeneous autoregressive (HAR) structure as in Corsi (2009). This special model is coined the CBF-HAR model. Although the CBF-HAR model is not formally a long memory model, it gives rise to persistence in the RCOV time series. Two real examples demonstrate that our CBF model (especially the CBF-HAR model) can have a significantly better forecasting performance than the corresponding CAW model, and hence a simple incorporation of $\nu_2$ to capture the heavy-tailed RCOV is necessary from a practical viewpoint.

Second, we provide a systematically statistical inference procedure for the CBF model. Specifically, we explore its stationarity conditions, establish the strong consistency and asymptotic normality of its maximum likelihood estimator (MLE), and investigate some new inner-product-based tests for model diagnostic checking. Moreover, the performance of our entire methodology is assessed by simulation studies. Compared to the existing BEKK-type multivariate time series models, our proofs of the entire inference procedure
are much involved, since the CBF model is tailored for matrix time series. Particularly, our inner-product-based tests seem to be the first diagnostic checking tool for matrix time series models, and the related idea can be easily extended to other models.

Third, we construct two reduced CBF models — the variance targeted (VT) CBF (VT-CBF) model and the factor CBF (F-CBF) model, to handle moderately large and high dimensional RCOV matrix respectively. For both reduced models, the asymptotic theory of the estimated parameters is derived. The dimension of the RCOV matrix is allowed to be a moderate but fixed number in the VT-CBF model, while it is allowed to grow with the sample size $T$ and the intra-day sample size in the F-CBF model. Therefore, this makes the prediction of large dimensional RCOV matrices feasible in many cases. The importance of both reduced models is illustrated by two real applications.

The remainder of the paper is organized as follows. Section 2 introduces the CBF model and studies its probabilistic properties. Section 3 investigates the asymptotics of the MLE. Section 4 presents inner-product-based tests to check the model adequacy. Two reduced CBF models and their related asymptotic theories are provided in Section 5. Some simulation studies are carried out in Section 6. Applications are given in Section 7. Section 8 concludes this paper. Proofs of all theorems are relegated to the Appendices. The remaining proofs are provided in the supplementary material.

Some notations are used throughout the paper. $I_n$ is the identity matrix of order $n$, and $\otimes$ represents the Kronecker product. For an $n \times n$ matrix $A$, $tr(A)$ is its trace, $A'$ is its transpose, $|A|$ is its determinant, $\rho(A)$ is its biggest eigenvalue, $\|A\| = \sqrt{tr(A'A)}$ is its Euclidean (or Frobenius) norm, $\|A\|_{spec} = \sqrt{\rho(A'A)}$ is its spectral norm, $vec(A)$ is a vector obtained by stacking all the columns of $A$, $vech(A)$ is a vector obtained by stacking all columns of the lower triangular part of $A$, and $A^{\otimes 2} = A \otimes A$.

2. Model and Properties.

2.1. Model Specification. Let $Y^*_t$ be the integrated volatility matrix of $n$ asset returns $X_t$ at time $t = 1,...,T$. After the seminal work of Barndorff-Nielsen and Shephard (2002, 2004) and Andersen et al. (2003), the $n \times n$ positive definite realized covariance (RCOV) matrix $Y_t$ calculated from the high-frequency return data of $X_t$ has been widely applied to estimate $Y^*_t$ in the literature; see, e.g., Barndorff-Nielsen et al. (2011), Lunde et al. (2016), Aït-Sahalia and Xiu (2017), Kim et al. (2018) and references therein. Moreover, $Y_t$...
is often viewed as a precise estimate for the conditional variances and covariances of these
n low-frequency asset returns $X_t$, and hence how to predict $Y_t$ by some dynamic models is
important in practice. Motivated by this, a new dynamic model for $Y_t$ is proposed in the
current paper.

Let $\mathcal{G}_t = \sigma(Y_s; s \leq t)$ be a filtration up to time $t$. We assume that

$$Y_t = \Sigma_t^{1/2} \Delta_t \Sigma_t^{1/2},$$

where $\{\Delta_t\}_{t=1}^T$ is a sequence of independent and identically distributed (i.i.d.) $n \times n$ positive
definite random innovation matrices with $E(\Delta_t|\mathcal{G}_{t-1}) = I_n$, each $\Delta_t$ follows the matrix-F
distribution $F(\nu, \frac{\nu_2-n-1}{\nu_1} I_n)$, and the density of $F(\nu, \Sigma)$ is

$$f(x; \nu, \Sigma) = \Lambda(\nu) \times \frac{|\Sigma|^{-\nu_1/2} |x|^{(\nu_1-n-1)/2}}{|I_n + \Sigma^{-1}x|^{(\nu_1+n_2)/2}}, \quad \text{for } x \in \mathbb{R}^{n \times n},$$

where $\nu = (\nu_1, \nu_2)'$ with degrees of freedom $\nu_1 > n + 1$ and $\nu_2 > n + 1$, $\Sigma$ is an $n \times n$
positive definite matrix, and

$$\Lambda(\nu) = \frac{\Gamma_n((\nu_1 + \nu_2)/2)}{\Gamma_n(\nu_1/2)\Gamma(\nu_2/2)} \quad \text{with } \Gamma_n(x) = \pi^{n(n-1)/2} \prod_{i=1}^n \Gamma(x + (1 - i)/2);$$

moreover, $\Sigma_t^{1/2} \in \mathcal{G}_{t-1}$ is the square root of the $n \times n$ positive definite matrix $\Sigma_t$, which
has a BEKK-type dynamic structure (see Engle and Kroner, 1995):

$$\Sigma_t = \Omega + \sum_{i=1}^P \sum_{k=1}^K A_{ki} Y_{t-i} A'_{ki} + \sum_{j=1}^Q \sum_{k=1}^K B_{kj} \Sigma_{t-j} B'_{kj},$$

where $\Omega, A_{ki}, B_{kj}$ are all $n \times n$ real matrices, the integers $P, Q, K$ are known as the orders
of the model, and $\Omega$ as well as the initial states $\Sigma_0, \Sigma_1, ..., \Sigma_{Q+1}$ are all positive definite.
Under model (2.1),

$$Y_t|\mathcal{G}_{t-1} \sim F \left( \nu, \frac{\nu_2-n-1}{\nu_1} \Sigma_t \right)$$

with $E(Y_t|\mathcal{G}_{t-1}) = \Sigma_t$, that is, the conditional distribution of $Y_t$ is matrix-F with a BEKK-
type mean structure. In this sense, we call model (2.1) the Conditional BEKK matrix-F
(CBF) model.

The CBF model is related to the CAW model in Golosnoy et al. (2012), in which $\Delta_t$
follows the Wishart distribution. To see it clearly, we follow Konno (1991) and Leung and
Lo (1996) to re-write $Y_t$ in model (2.1) as

$$Y_t = \left( \frac{\nu_2-n-1}{\nu_1} \right)^{1/2} L_t^{1/2} R_t^{-1} L_t^{1/2} \Sigma_t^{1/2},$$
where \( L_t \sim \text{Wishart}(\nu_1, I_n) \) and \( R_t \sim \text{Wishart}(\nu_2, I_n) \) are independent. As \( \lim_{\nu_2 \to \infty} \nu_2^{-1} R_t = I_n \) in probability, the identity (2.5) implies that when \( \nu_2 \to \infty \), \( Y_t | G_{t-1} \sim \text{Wishart}(\nu_1, \nu_1^{-1} \Sigma_t) \), which is exactly the CAW model. Therefore, compared to the CAW model, the degrees of freedom \( \nu_2 \) in the CBF model accommodates the heavy-tailed RCOV (see, e.g., Opschoor et al. (2018) for more discussions and examples). Clearly, the identity (2.5) also guarantees \( Y_t \) to be symmetric and positive definite, and it can be used to generate \( Y_t \) by using Wishart random variables.

Besides the heavy-tailedness, long memory is another well documented feature for the RCOV, and it has been taken into account by many RCOV models, including the heterogeneous autoregressive (HAR) model in Corsi (2009) as a benchmark. Although the HAR model does not formally belong to the class of long memory models, it is able to reproduce the persistence of RCOV observed in many empirical data. Inspired by the HAR model, we consider a special CBF model, which has the following specification for \( \Sigma_t \):

\[
\Sigma_t = \Omega + A(d)Y_{t-1,d}A'(d) + A(w)Y_{t-1,w}A'(w) + A(m)Y_{t-1,m}A'(m),
\]

where \( Y_{t-1,d} = Y_{t-1}, Y_{t-1,w} = (1/5) \sum_{i=1}^{5} Y_{t-i}, \) and \( Y_{t-1,m} = (1/22) \sum_{i=1}^{22} Y_{t-i} \) are the daily, weekly, and monthly averages of RCOV matrices, respectively. In this case, we label model (2.1) as the CBF-HAR model, since we put “HAR dynamics” on \( \Sigma_t \). Clearly, the CBF-HAR model is simply a constrained CBF model with \( P = 22, K = 3 \) and \( Q = 0 \).

Figure 1 plots the sample autocorrelation functions (ACFs) up to lag 200 of one simulated data from the CBF-HAR model with \( \nu = (20, 10) \) and

\[
\Omega = \begin{pmatrix} 0.5 & 0.2 & 0.3 \\ 0.2 & 0.5 & 0.25 \\ 0.3 & 0.25 & 0.5 \end{pmatrix}, \quad A(d) = \begin{pmatrix} 0.7 & 0 & 0 \\ 0 & 0.65 & 0 \\ 0 & 0 & 0.75 \end{pmatrix},
\]

\[
A(w) = \begin{pmatrix} 0.6 & 0 & 0 \\ 0 & 0.6 & 0 \\ 0 & 0 & 0.55 \end{pmatrix}, \quad A(m) = \begin{pmatrix} 0.4 & 0 & 0 \\ 0 & 0.45 & 0 \\ 0 & 0 & 0.4 \end{pmatrix}.
\]

From this figure, we can find that all entries of \( Y_t \) exhibit long memory phenomenon as expected.

Note that when \( K = 1 \), sufficient identifiability conditions of model (2.3) are that the main diagonal elements of \( \Omega \) and the first diagonal element of each \( A_{1i}, B_{1j} \) are positive;
when $K > 1$, some sufficient identifiability conditions of model (2.3) can be found in Engle and Kroner (1995). For simplicity, we assume subsequently that model (2.3) is identifiable.

Of course, the BEKK specification in model (2.3) is not the only way to describe the dynamics of $\Sigma_t$. The multivariate ARCH-type models such as the VEC model in Bollerslev et al. (1988), the component model in Engle and Lee (1999), the dynamic conditional correlation model in Engle (2002) and many others can also be adopted to model $\Sigma_t$. Using these models together with the matrix-F distribution to fit and predict the RCOV matrices could be a promising direction for future study.

2.2. Stationarity. Stationarity is an important issue for most RCOV models, but so far it has been rarely studied. Denote $M = max(P, Q)$. For $i = 1, 2, \cdots, M$, let

$$A_i^* = \sum_{k=1}^{K} A_{ik}^{\otimes 2} \quad \text{and} \quad B_i^* = \sum_{k=1}^{K} B_{ik}^{\otimes 2},$$

where $A_{ik} = 0$ for $i > P$ and $B_{ik} = 0$ for $i > Q$. A sufficient condition for the stationarity of the CBF model is given below, and it works for other general distributions of $\Delta_t$.

**Theorem 2.1.** Suppose that $\{\Delta_t\}$ in model (2.1) is a sequence of i.i.d. $n \times n$ positive definite random matrices with $E\|\Delta_t\| < \infty$, and
(H1) the distribution of \( \Delta_1 \), denoted by \( \Gamma \), is absolute continuous with respect to the Lebesgue measure;

(H2) the point \( I_n \) is in the interior of the support of \( \Gamma \);

(H3) \( \rho \left( \sum_{i=1}^{M} (A_i^* + B_i^*) \right) < 1. \)

Then, \( Y_t \) in model (2.1) is strict stationary with \( E\|Y_t\| < \infty \). Moreover, \( Y_t \) is positive Harris recurrent and geometrically ergodic.

**Remark 1.** The results of Theorem 2.1 are similar to those in Boussama et al. (2011), where the stationarity of the BEKK model is studied. Like Boussama et al. (2011), the proof of Theorem 2.1 is based on the semi-polynomial Markov chains technique, however, it is much involved due to the matrix nature of model (2.1).

Under conditions (H1) and (H2), condition (H3) is necessary and sufficient for the strict stationarity of \( Y_t \) with a finite first moment. As a special case, the results in Theorem 2.1 hold for the CAW model, in which \( \Delta_t \) follows the Wishart distribution.

### 3. Maximum Likelihood Estimation.

Let \( \theta = (\gamma', \nu')' \in \Theta \) be the unknown parameter of model (2.1) with the true value \( \theta_0 = (\gamma_0', \nu_0')' \), where \( \Theta = \Theta_\gamma \times \Theta_\nu \) is the parametric space with \( \Theta_\gamma \subset \mathbb{R}^\tau_1 \) and \( \Theta_\nu \subset \mathbb{R}^2 \), \( \gamma = (w', u')' \), \( w = vech(\Omega) \), \( u = (vec(A_{11})', ..., vec(A_{KP})', vec(B_{11})', ..., vec(B_{KQ})')' \), and \( \tau_1 = \frac{1}{2}n + [(P+Q)K + \frac{1}{2}]n^2 \). Below, we assume that \( \Theta_\gamma \) and \( \Theta_\nu \) are compact and \( \theta_0 \) is an interior point of \( \Theta \).

Given the observations \( \{Y_t\}_t=1 \) and the initial values \( \{Y_t\}_{t \leq 0} \), the negative log-likelihood function based on (2.4) is

\[
L(\theta) = \frac{1}{T} \sum_{t=1}^{T} l_t(\theta),
\]

where

\[
l_t(\theta) = \frac{\nu_1}{2} \log \left| \frac{\nu_2 - n - 1}{\nu_1} \Sigma_t(\gamma) \right| - \frac{\nu_1 - n - 1}{2} \log |Y_t| + \frac{\nu_1 + \nu_2}{2} \log |I_n + \frac{\nu_1}{\nu_2 - n - 1} \Sigma_t^{-1}(\gamma) Y_t| + C(\nu)
\]

with \( C(\nu) = -\log \Lambda(\nu) \) and \( \Sigma_t(\gamma) \) calculated recursively by

\[
\Sigma_t(\gamma) = \Omega + \sum_{i=1}^{P} \sum_{k=1}^{K} A_{ki} Y_{t-i} A'_{ki} + \sum_{j=1}^{Q} \sum_{k=1}^{K} B_{kj} \Sigma_{t-j}(\gamma) B'_{kj}.
\]

Clearly, \( \Sigma_t(\gamma_0) = \Sigma_t \).
As the initial values \( \{Y_t\}_{t \leq 0} \) are not observable, we shall modify \( L(\theta) \) as

\[
(3.3) \quad \hat{L}(\theta) = \frac{1}{T} \sum_{t=1}^{T} \hat{l}_t(\theta),
\]

where \( \hat{l}_t(\theta) \) is defined in the same way as \( l_t(\theta) \) with \( \Sigma_t(\gamma) \) being replaced by \( \hat{\Sigma}_t(\gamma) \), and \( \hat{\Sigma}_t(\gamma) \) is calculated in the same way as \( \Sigma_t(\gamma) \) based on a sequence of given constant matrices \( h := \{Y_0, \ldots, Y_{-M+1}, \Sigma_0, \ldots, \Sigma_{-M+1}\} \). The minimizer, \( \hat{\theta} = (\hat{\gamma}', \hat{\nu}')' \), of \( \hat{L}(\theta) \) on \( \Theta \) is called the maximum likelihood estimator (MLE) of \( \theta_0 \). That is,

\[
(3.4) \quad \hat{\theta} = (\hat{\gamma}', \hat{\nu}')' = \arg \min_{\theta \in \Theta} \hat{L}(\theta).
\]

To study the asymptotic properties of \( \hat{\theta} \), we need two assumptions below.

**Assumption 3.1.** \( Y_t \) is strictly stationary and ergodic.

**Assumption 3.2.** For \( \gamma \in \Theta_\gamma \), if \( \gamma \neq \gamma_0 \), \( \Sigma_t(\gamma) \neq \Sigma_t(\gamma_0) \) almost surely (a.s.) for all \( t \).

Assumption 3.1 is standard, and Assumption 3.2 which is in line with Comte and Lieberman (2003) and Hafner and Preminger (2009) is the identification condition. The following two theorems give the consistency and asymptotic normality of \( \hat{\theta} \), respectively.

**Theorem 3.1.** Suppose that Assumptions 3.1-3.2 hold and \( E\|Y_t\| < \infty \). Then, \( \hat{\theta} \overset{a.s.}{\to} \theta_0 \) as \( T \to \infty \).

**Theorem 3.2.** Suppose that Assumptions 3.1-3.2 hold, \( E\|Y_t\|^3 < \infty \), and

\[
(3.5) \quad \mathcal{O} = E \left( \frac{\partial^2 l_t(\theta_0)}{\partial \theta \partial \theta'} \right) \text{ is invertible.}
\]

Then, \( \sqrt{T}(\hat{\theta} - \theta_0) \overset{d}{\to} N(0, \mathcal{O}^{-1}) \) as \( T \to \infty \).

Based on the observations \( \{Y_t\}_{t=1}^{T} \) and a sequence of given constant matrices \( h \), we can use the analytic expression of \( \partial^2 l_t(\theta)/(\partial \theta \partial \theta') \) (see Appendix D in the supplementary material) to estimate \( \mathcal{O} \) by its sample counterpart. As for the univariate ARCH-type models, the coefficients on the main diagonal of \( \Omega \) are positive to ensure the positive definiteness of \( \Sigma_t \). Hence, the classical \( t \) or Wald test, which is constructed by the estimate of \( \mathcal{O} \), can not be used to detect whether their values are zeros or not; see Li et al. (2018) for more discussions in this context.
4. Model Diagnostic Checking. Diagnostic tests are crucial for model checking in multivariate time series analysis; see, e.g., Li and McLeod (1981), Ling and Li (1997), Tse (2002) and many others. However, no attempt has been made for the stationary matrix time series. In this section, we propose some new inner-product-based tests to check the adequacy of model (2.1).

Let \( Z_t(\gamma) = \text{vec}(\Sigma^{-1/2}(\gamma)Y_t\Sigma^{-1/2}(\gamma) - I_n) \) be the vectorized residual for a given \( \gamma \), and \( b_{t,j}(\gamma) = Z'_t(\gamma)Z_{t-j}(\gamma) \) be the inner product of two vectorized residuals at lag \( j \). Then, we stack \( b_{t,j}(\gamma) \) up to lag \( l \) to construct \( V_l(\gamma) \), where

\[
V_l(\gamma) = \frac{1}{T} \sum_{t=l+1}^{T} (b_{t,1}(\gamma), b_{t,2}(\gamma), \ldots, b_{t,l}(\gamma))',
\]

and \( l \geq 1 \) is a given integer. Our testing idea is motivated by the fact that if model (2.1) is adequate, \( Z_t(\gamma_0) \) is a sequence of i.i.d. random vectors with mean zero, and hence the value of \( V_l(\hat{\gamma}) \) is expected to be close to zero. To implement our test, we need study the asymptotic property of \( V_l(\hat{\gamma}) \) in the following theorem.

**Theorem 4.1.** Suppose that Assumptions 3.1-3.2 hold, \( E\|Y_t\|^4 < \infty \), and (3.5) holds. Then, if model (2.1) is correctly specified, \( \sqrt{T}V_l(\hat{\gamma}) \xrightarrow{d} N(0, V) \) as \( T \to \infty \), where \( V = (I_l, \mathcal{R}_1)\mathcal{R}_2(I_l, \mathcal{R}_1)' \) with

\[
\mathcal{R}_1 = E \left( \begin{array}{c}
3'_{l-1}(\gamma_0) (\partial \mathcal{R}_t(\gamma_0) / \partial \theta') \\
3'_{l-2}(\gamma_0) (\partial \mathcal{R}_t(\gamma_0) / \partial \theta') \\
\vdots \\
3'_{l-l}(\gamma_0) (\partial \mathcal{R}_t(\gamma_0) / \partial \theta')
\end{array} \right) \times O^{-1} \text{ and } \mathcal{R}_2 = \left( \begin{array}{cc}
\text{tr} \{ E^2[3'_t(\gamma_0)\mathcal{R}_t(\gamma_0)] \} I_l & 0 \\
0 & O
\end{array} \right).
\]

Based on Theorem 4.1, we construct the inner-product-based test statistic

\[
\Pi(l) = T[V'_l(\hat{\gamma})\tilde{V}^{-1}V_l(\hat{\gamma})]
\]

to detect the adequacy of model (2.1), where \( \tilde{V} \) is the sample counterpart of \( V \). If \( \Pi(l) \) is larger than the upper-tailed critical value of \( \chi^2(l) \), the fitted model (2.1) is not adequate at a given significance level. Otherwise, it could be deemed as adequate.

Note that if we consider a test based on \( \{3_t(\hat{\gamma})\} \) directly, the resulting limiting distribution shall still be chi-squared, but its degrees of freedom increases fast with the dimension \( n \). To avoid this dilemma, we use the inner product of the residuals to propose our test \( \Pi(l) \). This new idea is different from the portmanteau test in Ling and Li (1997) in which...
the test statistic is constructed based on the auto-correlations of the transformed scale residuals, while our test $\Pi(l)$ is based on the auto-covariances of the original vectorized residuals. Clearly, our idea can be easily extended to the framework in Ling and Li (1997). Meanwhile, our inner-product-based test $\Pi(l)$ takes the auto-covariances of all entries of $Z_t(\hat{\gamma})$ into account, while the idea of regression-based test in Tse (2002) only considers one entry of $Z_t(\hat{\gamma})$ at a time. In view of this, we prefer to use the proposed inner-product idea for testing purpose.

5. The Reduced CBF Models. As the number of parameters in the CBF model is $O(n^2)$, the estimation of the CBF model could be very computationally demanding when $n$ is large. This section introduces two reduced CBF models, which are feasible in fitting RCOV matrices with a large $n$.

5.1. The VT-CBF model. This subsection proposes a reduced CBF model by using the variance target (VT) technique in Engle and Mezrich (1996). The idea of VT is to re-parameterize the drift matrix $\Omega$ by using the theoretical mean of $Y_t$, so that the estimation of $\Omega$ is excluded in the implementation of the maximum likelihood estimation. Other related studies on the VT time series models can be found in Francq et al. (2011) and Pedersen and Rahbek (2014).

To define our reduced model, we assume that $Y_t$ is strictly stationary with a finite mean $S = E(Y_t)$. By taking expectation on both sides of (2.3), we have

\begin{equation}
\Omega = S - \sum_{i=1}^{P} \sum_{k=1}^{K} A_{ki} S A_{ki}' - \sum_{j=1}^{Q} \sum_{k=1}^{K} B_{kj} S B_{kj}',
\end{equation}

(5.1)

due to the fact that $S = E(Y_t) = E(\Sigma_t)$. With the help of (5.1), model (2.1) becomes

\begin{equation}
Y_t = \Sigma_t^{1/2} \Delta_t \Sigma_t^{1/2},
\end{equation}

(5.2)

where all notations are inherited from model (2.1), except that

\begin{equation}
\Sigma_t = S - \sum_{i=1}^{P} \sum_{k=1}^{K} A_{ki} S A_{ki}' - \sum_{j=1}^{Q} \sum_{k=1}^{K} B_{kj} S B_{kj}'
+ \sum_{i=1}^{P} \sum_{k=1}^{K} A_{ki} Y_{t-i} A_{ki}' + \sum_{j=1}^{Q} \sum_{k=1}^{K} B_{kj} \Sigma_{t-j} B_{kj}'.
\end{equation}

(5.3)

We call model (5.2) the VT-CBF model. Clearly, this reduced model shares the same probabilistic properties as the full CBF model. Although the VT-CBF model has the
same amount of parameters as the full CBF model, its two-step estimator given below is computationally easier than the MLE for the full CBF model.

To present this two-step estimator, we let \( \theta_v = (\delta', \nu')' \in \Theta_v \) be the unknown parameters of model (5.2) and its true value be \( \theta_{v0} = (\delta'_0, \nu'_0)' \), where \( \Theta_v = \Theta_\delta \times \Theta_\nu \) is the parametric space with \( \Theta_\delta = \Theta_s \times \Theta_u \subset \mathbb{R}^{2}, \tau_2 = [(P + Q)K + 1]n^2 \) and \( \Theta_\nu \subset \mathbb{R}^2 \). Let \( \delta = (s', u')' \) with \( s = vec(S), \Theta_s \in \mathbb{R}^{n^2} \) and \( \Theta_u \in \mathbb{R}^{(P+Q)Kn^2} \). As before, we assume that \( \Theta_\delta \) and \( \Theta_\nu \) are compact and \( \theta_{v0} \) is an interior point of \( \Theta_v \).

In the first step, we estimate \( s \) by \( \hat{s}_v \), where \( \hat{s}_v = vec(\hat{Y}_t) := vec\left( \frac{1}{T} \sum_{t=1}^{T} Y_t \right) \). In the second step, we estimate the remaining parameters \( \zeta = (u', \nu')' \) by the constrained MLE based on the following modified log-likelihood function:

\[
(5.4) \quad \hat{L}_v(\theta_v) = \frac{1}{T} \sum_{t=1}^{T} \hat{l}_vt(\theta_v),
\]

where

\[
\hat{l}_vt(\theta_v) = \frac{\nu_1}{2} \log \left| \frac{\nu_2 - n - 1}{\nu_1} \hat{\Sigma}_{vt}(\delta) \right| - \frac{\nu_1 - n - 1}{2} \log |Y_t| \nonumber \\
+ \frac{\nu_1 + \nu_2}{2} \log \left| I_n + \frac{\nu_1}{\nu_2 - n - 1} \hat{\Sigma}_{vt}^{-1}(\delta)Y_t \right| + C(\nu),
\]

and \( \hat{\Sigma}_{vt}(\delta) \) is calculated recursively by

\[
(5.5) \quad \hat{\Sigma}_{vt}(\delta) = S - \sum_{i=1}^{P} \sum_{k=1}^{K} A_{ki} S A'_{ki} - \sum_{j=1}^{Q} \sum_{k=1}^{K} B_{kj} S B'_{kj} \nonumber \\
+ \sum_{i=1}^{P} \sum_{k=1}^{K} A_{ki} Y_{t-i} A'_{ki} + \sum_{j=1}^{Q} \sum_{k=1}^{K} B_{kj} \hat{\Sigma}_{vt-j}(\delta)B'_{kj},
\]

based on a sequence of given constant matrices \( h \). Clearly, \( \hat{L}_v(\theta_v) \) is analogous to \( \hat{L}(\theta) \) in (3.3), and it is the modification of the following log-likelihood function:

\[
(5.6) \quad L_v(\theta_v) = \frac{1}{T} \sum_{t=1}^{T} l_v(t(\theta_v),
\]

where \( l_v(t(\theta_v) \) is defined in the same way as \( \hat{l}_vt(\theta_v) \) with \( \hat{\Sigma}_{vt}(\delta) \) being replaced by \( \Sigma_{vt}(\delta), \) and \( \Sigma_{vt}(\delta) \) is calculated recursively by

\[
\Sigma_{vt}(\delta) = S - \sum_{i=1}^{P} \sum_{k=1}^{K} A_{ki} S A'_{ki} - \sum_{j=1}^{Q} \sum_{k=1}^{K} B_{kj} S B'_{kj} \nonumber \\
+ \sum_{i=1}^{P} \sum_{k=1}^{K} A_{ki} Y_{t-i} A'_{ki} + \sum_{j=1}^{Q} \sum_{k=1}^{K} B_{kj} \Sigma_{vt-j}(\delta)B'_{kj},
\]
based on the observations \( \{Y_t\}_{t=1}^T \) and the initial values \( \{Y_t\}_{t \leq 0} \). The minimizer, \( \hat{\zeta}_v = (\hat{u}_v', \hat{\nu}_v')' \), of \( \hat{L}_v(\hat{s}_v, \zeta) \) on \( \Theta_u \times \Theta_v \) is the constrained MLE of \( (u_0', \nu_0')' \). That is,

\[
(\hat{u}_v', \hat{\nu}_v')' = \arg \min \hat{L}_v(\hat{s}_v, \zeta).
\]

Now, we call \( \hat{\theta}_v = (\hat{s}_v', \hat{\zeta}_v')' \) the two-step estimator of \( \theta_v \) in model (5.2). Let

\[
\Psi(u) = \left( I_{n^2} - \sum_{i=1}^M A_i^* - \sum_{i=1}^M B_i^* \right)^{-1} \left( I_{n^2} - \sum_{i=1}^M B_i^* \right) \text{vec}(Y_t - \Sigma_v \delta_t),
\]

\[
w_t(\theta_v) = \left( \frac{\Psi(u) \text{vec}(Y_t - \Sigma_v \delta_t)}{\partial \ell_v(\theta_v) / \partial \zeta} \right).
\]

The following two theorems give the consistency and asymptotic normality of \( \hat{\theta}_v \), respectively.

**Theorem 5.1.** Suppose that Assumptions 3.1-3.2 hold and \( E\|Y_t\| < \infty \). Then, \( \hat{\theta}_v \xrightarrow{a.s.} \theta_{v_0} \) as \( T \to \infty \).

**Theorem 5.2.** Suppose that Assumptions 3.1-3.2 hold, \( E\|Y_t\|^3 < \infty \), and

\[
J_1 = E\left[ \frac{\partial^2 \ell_v(\theta_{v_0})}{\partial \zeta \partial \zeta'} \right] \text{ is invertible.}
\]

Then, \( \sqrt{T}(\hat{\theta}_v - \theta_{v_0}) \xrightarrow{d} N(0, O_v) \) as \( T \to \infty \), where

\[
O_v = \left( \begin{array}{cc}
I_{n^2} & 0 \\
-J_1^{-1}J_2 & -J_1^{-1}
\end{array} \right) E(w_t w_t') \left( \begin{array}{cc}
I_{n^2} & 0 \\
-J_1^{-1}J_2 & -J_1^{-1}
\end{array} \right)'
\]

with \( J_2 = E\left[ \frac{\partial^2 \ell_v(\theta_{v_0})}{\partial \theta_v \partial \theta_v'} \right] \) and \( w_t = w_t(\theta_{v_0}) \).

As before, we can use the sample counterpart of the analytic expressions of \( \partial \ell_v(\theta_v)/\partial \theta_v \) and \( \partial^2 \ell_v(\theta_v)/\partial \theta_v \partial \theta_v' \) to estimate \( O_v \). Although the VT-CBF model can be estimated by the aforementioned two-step estimation procedure, it still has to handle a large number of estimated parameters with order \( O(n^2) \) caused by the parameter matrices \( A_{ki} \) and \( B_{kj} \).

To make a more parsimonious VT-CBF model, we can further impose some restrictions on \( A_{ki} \) and \( B_{kj} \). McCurdy and Stengos (1992) and Engle and Kroner (1995) have suggested to use diagonal volatility models, which not only avoid over-parameterization, but also reflect the fact that the variances and the covariances rely more on its own past than the history of other variances or covariances. Motivated by this, we can assume that all \( A_{ki} \) and \( B_{kj} \) have a diagonal structure, leading to a diagonal VT-CBF model. Clearly, the number of estimated parameters in the diagonal VT-CBF model has order \( O(n) \), which is feasible to be handled for a moderate large but fixed \( n \).
Next, similar to $\Pi(l)$ in (4.1), we can construct the inner-product-based test statistics to check the adequacy of model (2.1) based on the two-step estimator $\hat{\theta}_v$. Let $\delta_0 = (s'_0, u'_0)'$, $\tilde{\delta}_v = (\tilde{s}'_v, \tilde{u}'_v)'$, $\hat{y}_vt(\delta) = \text{vec}(\Sigma^{-1/2}_vt)Y_t\Sigma^{-1/2}_vt - I_n$ be the residual vector for a given $\delta$, $b_{vt,j}(\delta) = \hat{y}_vt(\delta)\hat{y}_vt-j(\delta)$ be the inner product of the residuals at lag $j$, and

$$\mathcal{V}_{vl}(\delta) = \frac{1}{T} \sum_{t=l+1}^{T} (b_{vt,1}(\delta), b_{vt,2}(\delta), \ldots, b_{vt,l}(\delta))'.$$

The asymptotic property of $\mathcal{V}_{vl}(\tilde{\delta}_v)$ is given in the following theorem.

**Theorem 5.3.** Suppose that Assumptions 3.1-3.2 hold, $E\|Y_t\|^4 < \infty$, and (5.9) holds. Then, if model (2.1) is correctly specified, $\sqrt{T} \mathcal{V}_{vl}(\tilde{\delta}_v) \rightarrow N(0, \mathbf{V}_v)$ as $T \rightarrow \infty$, where

$$\mathbf{V}_v = (I_t, \mathcal{R}_{1v})\mathcal{R}_{2v}(I_t, \mathcal{R}_{1v})'$$

with

$$\mathcal{R}_{1v} = E \begin{pmatrix} \hat{y}_vt(\delta_0) & \partial \hat{y}_vt(\delta_0)/\partial \theta' \\ \hat{y}_vt-1(\delta_0) & \partial \hat{y}_vt-1(\delta_0)/\partial \theta' \\ \vdots & \vdots \\ \hat{y}_vt-l(\delta_0) & \partial \hat{y}_vt-l(\delta_0)/\partial \theta' \end{pmatrix} \times \begin{pmatrix} I_n^2 & 0 \\ -J_1^{-1}J_2 & -J_1^{-1} \end{pmatrix},$$

and

$$\mathcal{R}_{2v} = \begin{pmatrix} \text{tr}\{E^2[\hat{y}_vt(\delta_0)'\hat{y}_vt(\delta_0)]\}I_t & 0 \\ 0 & E(w_t w_t') \end{pmatrix}.$$

By the preceding theorem, we can adopt the test statistic

$$\Pi_v(l) = T[\mathcal{V}_{vl}(\tilde{\delta}_v)\hat{\mathbf{V}}_v^{-1}\mathcal{V}_{vl}(\tilde{\delta}_v)]$$

(5.10)

to detect the adequacy of model (2.1), where $\hat{\mathbf{V}}_v$ is the sample counterpart of $\mathbf{V}_v$. If $\Pi_v(l)$ is larger than the upper-tailed critical value of $\chi^2(l)$ at a given significance level, the fitted model (2.1) is inadequate. Otherwise, it is adequate.

**5.2. The Factor CBF Model.** In modern data analysis, the dimension $n$ could be growing with the sample size $T$ in many cases, and this makes the CBF (or VT-CBF) models computationally infeasible. Also, the dimension $n$ may be proportional to $m$ (the average intra-day sample size across all assets and all days), and then the methods to calculate $Y_t$ used for the fixed $n$ deliver an inconsistent estimator of $Y_t^*$. See, e.g., Wang and Zou (2010) and Tao et al. (2011) for surveys. To overcome this difficulty, we use the thresholding average realized volatility matrix estimator (TARVM) in Tao et al. (2011) to calculate $Y_t$, and this estimator is consistent for very large $n$, which is allowed to grow with $T$ and
For more recent works in this direction, we refer to Aït-Sahalia and Xiu (2017), Kim et al. (2018), and the references therein.

Since the dimension of $Y_t$ could be very large, it seems hard to study the dynamics of $Y_t$ without imposing some specific structure. Here, we adopt the factor model proposed by Tao et al. (2011) by assuming that

$Y_t^* = FY_{ft}^* + Y_0^*$,  

(5.11)

where $Y_{ft}^*$ is an $r \times r$ positive definite factor covariance matrix with $r$ being a fixed integer (much smaller than $n$), $Y_0^*$ is an $n \times n$ positive definite constant matrix, and $F$ is an $n \times r$ factor loading matrix normalized by the constraint $F'F = I_r$. In model (5.11), the dynamic structure of $Y_t^*$ is driven by that of a lower-dimensional latent process $Y_{ft}^*$, while $Y_0^*$ represents the static part of $Y_t^*$.

Define

$\nabla^* = \frac{1}{T} \sum_{t=1}^{T} Y_t^*$, $\nabla^* = \frac{1}{T} \sum_{t=1}^{T} \{ Y_t^* - \nabla^* \}^2,$

and

$\overline{\nabla} = \frac{1}{T} \sum_{t=1}^{T} Y_t$, $\overline{\nabla} = \frac{1}{T} \sum_{t=1}^{T} \{ Y_t - \overline{\nabla} \}^2.$

Then, we estimate $Y_{ft}^*$, $Y_0^*$ and $F$ by

$\hat{Y}_{ft} = \hat{F}' \hat{Y}_t \hat{F}$, $\hat{Y}_0 = \overline{\nabla} - \hat{F}' \hat{F} \overline{\nabla}$ and $\hat{F} = (\hat{f}_1, \cdots, \hat{f}_r)$,

(5.12)

respectively, where $\hat{f}_1, \cdots, \hat{f}_r$ are the eigenvectors of $\nabla$ corresponding to its $r$ largest eigenvalues. As suggested by Lam and Yao (2012) and Ahn and Horenstein (2013), we may select $r$ such that the $r$ largest ratios of adjacent eigenvalues are significantly larger.

In order to study the asymptotics of the proposed estimators, we introduce the following technical assumptions.

**Assumption 5.1.** All row vectors of $F'$ and $Y_0^*$ satisfy the sparsity condition below. For an $n$-dimensional vector $(x_1, \cdots, x_n)$, we say it is sparse if it satisfies

$\sum_{i=1}^{n} |x_i|^{\delta_*} \leq U \pi(n),$

where $\delta_* \in [0, 1)$, $U$ is a positive constant, and $\pi(n)$ is a deterministic function of $n$ that grows slowly in $n$ with typical examples $\pi(n) = 1$ or $\log(n)$.  

Assumption 5.2. The factor model (5.11) has \( r \) fixed factors, and matrices \( Y_0^* \) and \( Y_{ft}^* \) satisfy \( \|Y_0^*\| < \infty \) and \( \max_{1 \leq t \leq T} \|Y_{ft,jj}^*\| = O_p(B(T)) \) for \( j = 1, 2, \ldots, r \), where \( Y_{ft,jj}^* \) is the \( j \)-th diagonal entry of \( Y_{ft}^* \), and \( 1 \leq B(T) = o(T) \).

Assumption 5.3. \( \max_{1 \leq t \leq T} \|Y_t^* - Y_t\| = O_p(A(n, m, T)) \) for some rate function \( A(n, m, T) \) such that \( A(n, m, T)B^5(T) = o(1) \).

Assumptions 5.1-5.3 are sufficient to prove the consistency of \( \hat{Y}_{ft} \). For TARVM, we can take \( A(n, m, T) = \pi(n)[e_m(n^2T)^{1/\beta}]^{1-\delta} \log T \) and \( B(T) = \log T \) with \( e_m = m^{-1/6} \) so that \( A(n, m, T)B^5(T) = o(1) \) for large \( \beta \); see Tao et al. (2011). For other estimators, the rate \( A(n, m, T) \) may be improved; see Tao et al. (2013) for more discussions.

Theorem 5.4. Suppose that Assumptions 5.1-5.3 and the conditions in Theorem 3.2 hold. Then, as \( n, m, T \) go to infinity,

(i) \( F\hat{F} - I_r = O_p(A(n, m, T)B(T)) \),

(ii) \( \hat{Y}_{ft} - Y_{ft} = O_p(A^{1/2}(n, m, T)B^{3/2}(T)) \),

where \( Y_{ft} = Y_{ft}^* + FY_0^*F \), and \( F = (f_1, \ldots, f_r) \) with \( f_1, \ldots, f_r \) being the eigenvectors of \( \hat{S}^* \) corresponding to its \( r \) largest eigenvalues.

The above theorem indicates that \( \hat{Y}_{ft} \) is a consistent estimator of \( Y_{ft} \) rather than \( Y_{ft}^* \).

Next, we assume that \( Y_{ft} \) satisfies the CBF model, that is,

\[
Y_{ft} | \mathcal{G}_{t-1} \sim F \left( \nu, \frac{\nu_2 - n - 1}{\nu_1} \Sigma_{ft} \right)
\]

with \( E(Y_{ft} | \mathcal{G}_{t-1}) = \Sigma_{ft} \), where \( \Sigma_{ft} \) is defined in the same way as \( \Sigma_{t} \) in (2.3) with \( Y_t \) replaced by \( Y_{ft} \), and the remaining notations and set-ups inherent from model (2.1). We call models (5.11) and (5.13) the factor CBF (F-CBF) model. Particularly, if \( \Sigma_{ft} \) has the HAR dynamical structure as in (2.6), the resulting model is called the factor CBF-HAR (F-CBF-HAR) model. Based on this model, we have \( Y_t^* = F(Y_{ft} - FY_0^*F)F' + Y_0^* \). Since \( Y_t \approx Y_t^* \), it implies that we can study the large dimensional matrix \( Y_t \) by using an \( r \times r \) low-dimensional matrix \( Y_{ft} \).

As \( Y_{ft} \) is not observable, we should estimate model (5.13) based on \( \hat{Y}_{ft} \), and hence we consider a feasible MLE of \( \theta_0 \) in model (5.13) given by

\[
\hat{\theta}_{1f} = (\hat{\gamma}_{1f}', \hat{\nu}_{1f}')' = \arg \min_{\theta \in \Theta} \hat{L}_f(\theta),
\]
where \( \hat{L}_f(\theta) \) is defined in the same way as \( \hat{L}(\theta) \) in (3.3) with \( Y_t \) and \( \hat{\Sigma}_t(\gamma) \) replaced by \( \hat{Y}_f \) and \( \hat{\Sigma}_{ft}(\gamma) \), respectively. The following theorem shows that \( \hat{\theta}_{1f} \) is consistent with the ideal MLE \( \hat{\theta}_{2f} \) based on \( Y_{ft} \), where

\[
\hat{\theta}_{2f} = (\hat{\gamma}_{2f}', \hat{\nu}_{2f}')' = \arg\min_{\theta \in \Theta} L_f(\theta),
\]

and \( L_f(\theta) \) is defined in the same way as \( L(\theta) \) in (3.1) with \( Y_t \) and \( \Sigma_t(\gamma) \) replaced by \( Y_{ft} \) and \( \Sigma_{ft}(\gamma) \), respectively.

**Theorem 5.5.** Suppose that the conditions in Theorem 5.4 hold. Then, as \( n, m, T \) go to infinity, \( \hat{\theta}_{1f} - \hat{\theta}_{2f} = O_p(B(T)/T) + O_p(A^{1/2}(n, m, T)B^{5/2}(T)) \).

Since the dimension of \( Y_{ft} \) is \( r \) (much smaller than \( n \)), the calculation of \( \hat{\theta}_{1f} \) is computationally feasible. In order to further reduce the number of parameters in model (5.13), we can also assume that \( Y_{ft} \) follows a VT-CBF model. This leads to the F-VT-CBF model, which includes the F-VT-CBF-HAR model as a special case. For this F-VT-CBF model, we consider its feasible two-step estimator \( \hat{\theta}_{1fv} = (\hat{s}_{1fv}', \hat{\zeta}_{1fv}')' \), where

\[
\hat{s}_{1fv} = \frac{1}{T} \sum_{t=1}^{T} \hat{Y}_{ft}, \quad \hat{\zeta}_{1fv} = (\hat{u}_{1fv}', \hat{\nu}_{1fv}')' = \arg\min_{\zeta \in \Theta_{\nu} \times \Theta_{\nu}} \hat{L}_{fv}(\hat{s}_{1fv}, \zeta),
\]

and \( \hat{L}_{fv}(\theta_v) \) is defined in the same way as \( \hat{L}_v(\theta_v) \) in (5.4) with \( Y_t \) and \( \hat{\Sigma}_{vt}(\delta) \) replaced by \( \hat{Y}_f \) and \( \hat{\Sigma}_{ftv}(\delta) \), respectively. Similar to Theorem 5.5, \( \hat{\theta}_{1fv} \) is consistent with the ideal two-step estimator \( \hat{\theta}_{2fv} = (\hat{s}_{2fv}', \hat{\zeta}_{2fv}')' \) based on \( Y_{ft} \), where

\[
\hat{s}_{2fv} = \frac{1}{T} \sum_{t=1}^{T} Y_{ft}, \quad \hat{\zeta}_{2fv} = (\hat{u}_{2fv}', \hat{\nu}_{2fv}')' = \arg\min_{\zeta \in \Theta_{\nu} \times \Theta_{\nu}} L_{fv}(\hat{s}_{2fv}, \zeta),
\]

and \( L_{fv}(\theta_v) \) is defined in the same way as \( L(\theta_v) \) in (5.6) with \( Y_t \) and \( \Sigma_v(\delta) \) replaced by \( Y_{ft} \) and \( \Sigma_{ftv}(\delta) \), respectively.

**Theorem 5.6.** Suppose that the conditions in Theorem 5.4 hold. Then, as \( n, m, T \) go to infinity,

(i) \( \hat{s}_{1fv} - \hat{s}_{2fv} = O_p(A^{1/2}(n, m, T)B^{3/2}(T)) \),

(ii) \( \hat{\zeta}_{1fv} - \hat{\zeta}_{2fv} = O_p(B(T)/T) + O_p(A^{1/2}(n, m, T)B^{5/2}(T)) \).

Particularly, if \( Y_{ft} \) follows a diagonal VT-CBF model, the number of estimated parameters in model (5.13) is \( O(r) \), which is easy to calculate in practice. In view of model (5.11) and the fact that \( F'F = I_r \), we can predict \( Y_t \) by either \( \hat{F}\hat{\Sigma}_{ft}(\hat{\gamma}_{1f})\hat{F}' + \hat{Y}_o \) based on \( \hat{\theta}_{1f} \) or \( \hat{F}\hat{\Sigma}_{ftv}(\hat{\delta}_{1fv})\hat{F}' + \hat{Y}_o \) based on \( \hat{\theta}_{1fv} \), where \( \hat{\delta}_{1fv} = (\hat{s}_{1fv}', \hat{u}_{1fv}')' \).
6. Simulation. In this section, we first assess the performance of the MLE $\hat{\theta}$ and the two-step estimator $\hat{\theta}_v$ in the finite sample. We generate 1000 replications of sample size $T = 1000$ and 2000 from the following model:

$$Y_t = \Sigma_t^{1/2} \Delta_t \Sigma_t^{1/2} \text{ with } \Sigma_t = \Omega_0 + A_{10} Y_{t-1} A'_{10} + B_{10} \Sigma_{t-1} B'_{10}, \quad (6.1)$$

where

$$\Omega_0 = \begin{pmatrix} 0.5 & 0.2 & 0.3 \\ 0.2 & 0.5 & 0.25 \\ 0.3 & 0.25 & 0.5 \end{pmatrix}, \quad A_{10} = \begin{pmatrix} 0.4 & 0 & 0 \\ 0 & 0.55 & 0 \\ 0 & 0 & 0.5 \end{pmatrix}, \quad B_{10} = \begin{pmatrix} 0.4 & 0 & 0 \end{pmatrix},$$

$\{\Delta_t\}$ is a sequence of independent $F(\nu_0, \frac{\nu_0 - n - 1}{\nu_{10}} I_n)$ distributed random matrices with $n = 3$, and $\nu_0 = (10, 8), (15, 10)$ or $(20, 10)$. For each repetition, we calculate $\hat{\theta}$, $\hat{\theta}_v$, and their related asymptotic standard deviations. For $\hat{\theta}_v$, we report the results related to $\Omega$ instead of $S$, and hence the asymptotic standard deviation of the estimated parameters in $\Omega$ is absent in this case.

Table 1 reports the sample bias, the sample standard deviation (SD) and the average asymptotic standard deviation (AD) of $\hat{\theta}$ and $\hat{\theta}_v$. From this table, we can see that the biases of both estimators are small comparing to the magnitude of the parameters, and they become smaller as the sample size $T$ increases. This assures the accuracy of both estimators. Furthermore, we find that the SDs are generally close to the ADs for both estimators, and all of the SDs and ADs become smaller as $T$ increases from 1000 to 2000. In terms of ADs or SDs, $\hat{\theta}$ is generally more efficient than $\hat{\theta}_v$, although this efficiency advantage is weak for many parameters. However, the estimation time for $\hat{\theta}_v$ is almost 70% of the time for $\hat{\theta}$, and this computation advantage can be more significant when $n$ increases.

Next, we examine the performance of the inner-product-based tests $\Pi(l)$ and $\Pi_v(l)$ in the finite sample. We generate 1000 replications of sample size $T = 1000$ and 2000 from the following model:

$$Y_t = \Sigma_t^{1/2} \Delta_t \Sigma_t^{1/2} \text{ with } \Sigma_t = \Omega_0 + A_{10} Y_{t-1} A'_{10} + A_{20} Y_{t-2} A'_{20} + B_{10} \Sigma_{t-1} B'_{10}, \quad (6.2)$$

where the values of $\Omega_0$, $A_{10}$ and $B_{10}$ are chosen as in (6.1), $A_{20} = \text{diag}\{\lambda, \lambda, \lambda\}$ is a diagonal matrix with $\lambda = 0, 0.05, 0.1, 0.15, 0.2$, and $\{\Delta_t\}$ is a sequence of independent $F(\nu_0, \frac{\nu_0 - n - 1}{\nu_{10}} I_n)$ distributed random matrices with $n = 3$ and $\nu_0 = (10, 8)$. We fit each
| Case | T   | \( \hat{\theta} \) Bias | ESD | ASD | \( \hat{\theta}_v \) Bias | ESD | ASD | ESD | ASD | ESD | ASD | ESD | ASD |
|------|-----|-----------------------|-----|-----|-----------------------|-----|-----|-----|-----|-----|-----|-----|-----|
| Case 1 | 1000 | 0.0320 | 0.0160 | -0.0014 | -0.0029 | -0.0009 | -0.0151 | -0.0112 | -0.0102 | -0.0005 | 0.0028 | 0.0057 | -0.0009 | 0.0037 | 0.0053 |
|      |     | 0.3914 | 0.2452 | 0.0255 | 0.0249 | 0.0240 | 0.1170 | 0.0964 | 0.0728 | 0.0600 | 0.0188 | 0.0337 | 0.0419 | 0.0248 | 0.0601 |
|      |     | 0.4111 | 0.2563 | 0.0258 | 0.0259 | 0.0241 | 0.1103 | 0.0892 | 0.0652 | 0.0586 | 0.0179 | 0.0323 | 0.0402 | 0.0232 | 0.0562 |
| Case 2 | 1000 | 0.0188 | 0.0072 | -0.0003 | -0.0018 | -0.0002 | -0.0111 | -0.0036 | -0.0034 | 0.0019 | 0.0014 | 0.0030 | -0.0008 | 0.0012 | 0.0014 |
|      |     | 0.2767 | 0.1733 | 0.0174 | 0.0179 | 0.0168 | 0.0797 | 0.0633 | 0.0459 | 0.0431 | 0.0130 | 0.0231 | 0.0293 | 0.0169 | 0.0405 |
|      |     | 0.2880 | 0.1797 | 0.0181 | 0.0182 | 0.0169 | 0.0767 | 0.0615 | 0.0447 | 0.0417 | 0.0114 | 0.0226 | 0.0287 | 0.0162 | 0.0395 |
| Case 3 | 1000 | 2.0792 | 0.0000 | -0.0005 | -0.0011 | 0.0000 | -0.0074 | -0.0021 | -0.0041 | -0.0008 | 0.0009 | 0.0020 | -0.0017 | 0.0010 | 0.0020 |
|      |     | 0.5912 | 0.2515 | 0.0175 | 0.0171 | 0.0158 | 0.0801 | 0.0665 | 0.0467 | 0.0434 | 0.0123 | 0.0228 | 0.0299 | 0.0163 | 0.0412 |
|      |     | 0.5871 | 0.2517 | 0.0175 | 0.0171 | 0.0159 | 0.0805 | 0.0640 | 0.0461 | 0.0437 | 0.0121 | 0.0230 | 0.0293 | 0.0161 | 0.0406 |
|      |     | 0.5874 | 0.2532 | 0.0175 | 0.0182 | 0.0163 | 0.0800 | 0.0664 | 0.0467 | 0.0437 | 0.0138 | 0.0235 | 0.0305 | 0.0176 | 0.0413 |
|      |     | 0.5792 | 0.2537 | 0.0177 | 0.0178 | 0.0163 | 0.0827 | 0.0668 | 0.0462 | 0.0437 | 0.0138 | 0.0235 | 0.0305 | 0.0176 | 0.0413 |
|      |     | 1.0521 | 0.2024 | -0.0013 | -0.0021 | -0.0007 | -0.0165 | -0.0092 | -0.0082 | -0.0023 | 0.0026 | 0.0046 | -0.0026 | 0.0032 | 0.0031 |
|      |     | 1.4019 | 0.3340 | 0.0237 | 0.0237 | 0.0213 | 0.1253 | 0.0979 | 0.0712 | 0.0615 | 0.0173 | 0.0353 | 0.0418 | 0.0231 | 0.0399 |
|      |     | 1.4496 | 0.3442 | 0.0242 | 0.0235 | 0.0220 | 0.1127 | 0.0904 | 0.0654 | 0.0586 | 0.0169 | 0.0319 | 0.0399 | 0.0224 | 0.0463 |
| Case 4 | 1000 | 0.0737 | 0.0190 | -0.0006 | -0.0009 | 0.0001 | -0.0061 | -0.0047 | -0.0052 | -0.0016 | 0.0010 | 0.0018 | -0.0010 | 0.0016 | 0.0022 |
|      |     | 1.0087 | 0.2469 | 0.0169 | 0.0163 | 0.0149 | 0.0794 | 0.0671 | 0.0480 | 0.0418 | 0.0118 | 0.0227 | 0.0295 | 0.0161 | 0.0418 |
|      |     | 1.0057 | 0.2411 | 0.0170 | 0.0165 | 0.0155 | 0.0787 | 0.0630 | 0.0453 | 0.0429 | 0.0117 | 0.0225 | 0.0286 | 0.0157 | 0.0397 |
|      |     | 0.0192 | 0.0286 | -0.0004 | -0.0004 | 0.0000 | -0.0058 | -0.0045 | -0.0051 | -0.0020 | 0.0008 | 0.0013 | -0.0012 | 0.0011 | 0.0015 |
|      |     | 1.0022 | 0.2511 | 0.0170 | 0.0171 | 0.0153 | 0.0795 | 0.0673 | 0.0480 | 0.0419 | 0.0131 | 0.0231 | 0.0300 | 0.0176 | 0.0418 |
|      |     | 0.9923 | 0.2434 | 0.0172 | 0.0173 | 0.0158 | 0.0817 | 0.0652 | 0.0462 | 0.0437 | 0.0124 | 0.0213 | 0.0286 | 0.0142 | 0.0287 |

Note: Cases 1-3 correspond to \( \nu_0 = (10, 8), (15, 10) \) and \( (20, 10) \), respectively.
replication by the CBF model with \((K, P, Q) = (1, 1, 1)\), and use \(\Pi(l)\) and \(\Pi_v(l)\) to check whether the fitted model is adequate. Here, we set the significance level \(\alpha = 0.05\) and \(l = 2, 3, 4, 5, 6\). The empirical sizes and powers of both tests are reported in Table 2, and their sizes correspond to the results for the case of \(\lambda = 0\). From Table 2, we can find that both \(\Pi(l)\) and \(\Pi_v(l)\) always have accurate sizes, although they are slightly oversized for small \(T\). For the power of both tests, it is generally as expected. First, all the power becomes larger as \(T\) increases. Second, both tests become more powerful as \(\lambda\) becomes larger. Third, the power of \(\Pi(l)\) and \(\Pi_v(l)\) is comparable, but the former need more computational time. Note that when \(\nu_0 = (15, 10)\) and \((20, 10)\), the testing results are similar to these for \(\nu_0 = (10, 8)\), and hence they are not reported for saving space.

### Table 2

The results of \(\Pi(l)\) and \(\Pi_v(l)\) for model (6.2)

| \(\lambda\) | \(T\) | \(l = 2\) | \(l = 3\) | \(l = 4\) | \(l = 5\) | \(l = 6\) |
| --- | --- | --- | --- | --- | --- | --- |
| \(\Pi(l)\) | \(\Pi_v(l)\) | \(\Pi(l)\) | \(\Pi_v(l)\) | \(\Pi(l)\) | \(\Pi_v(l)\) | \(\Pi(l)\) | \(\Pi_v(l)\) |
| 0 | 1000 | 0.043 | 0.047 | 0.052 | 0.047 | 0.049 | 0.054 |
| 2000 | 0.048 | 0.056 | 0.058 | 0.053 | 0.052 | 0.051 | 0.054 |
| 0.05 | 1000 | 0.048 | 0.051 | 0.058 | 0.060 | 0.061 | 0.062 |
| 2000 | 0.060 | 0.063 | 0.064 | 0.063 | 0.067 | 0.058 | 0.074 |
| 0.1 | 1000 | 0.238 | 0.210 | 0.196 | 0.196 | 0.179 | 0.183 |
| 2000 | 0.414 | 0.371 | 0.350 | 0.309 | 0.316 | 0.320 |
| 0.15 | 1000 | 0.885 | 0.847 | 0.818 | 0.784 | 0.768 | 0.746 |
| 2000 | 0.974 | 0.966 | 0.956 | 0.946 | 0.941 | 0.919 |
| 0.2 | 1000 | 0.976 | 0.972 | 0.964 | 0.961 | 0.956 | 0.887 |
| 2000 | 0.992 | 0.989 | 0.987 | 0.987 | 0.985 | 0.910 |

Overall, both estimators \(\hat{\theta}\) and \(\hat{\theta}_v\) and both tests \(\Pi(l)\) and \(\Pi_v(l)\) have a good performance especially when the sample size \(T\) gets larger. When the dimension of \(Y_t\) is small, our simulation results show that \(\hat{\theta}_v\) is only slightly less efficient than \(\hat{\theta}\), and \(\Pi_v(l)\) is generally as powerful as \(\Pi(l)\). When the dimension of \(Y_t\) is large, \(\hat{\theta}_v\) and \(\Pi_v(l)\) can enjoy a faster computation speed than \(\hat{\theta}\) and \(\Pi(l)\), respectively. Based on these grounds, we would recommend using \(\hat{\theta}_v\) and \(\Pi_v(l)\) in practice.

### 7. Applications.

In this section, we consider two applications on the U.S. stock market. Application 1 studies the low dimensional RCOV matrix series calculated by composite realized kernels (CRK) in Lunde, Shephard and Sheppard (2016). Application 2 studies
the high dimensional RCOV series calculated by TARVM estimator in Tao et al. (2011).

7.1. Application 1. In this application, we revisit the RCOV matrix data of Hewlett-Packard Development Company, L.P. (HPQ), International Business Machines Corporation (IBM) and Microsoft Corporation (MSFT) in Lunde, Shephard and Sheppard (2016). This data set, denoted by \(\{Y_t\}_{t=1}^{1474}\), ranges from January 2006 to December 2011 with 1474 observations in total. Here, two flash crashes are flagged in 6 May, 2010 and 9 August, 2011 and replaced by an average of the nearest five preceding and following matrices.

Figure 2 plots the diagonal and off-diagonal components of \(\{Y_t\}_{t=1}^{1474}\), exhibiting that \(Y_t\) has a clear clustering feature. Meanwhile, Figure 3 plots their sample autocorrelation functions (ACFs), which show the significant temporal dependence of \(Y_t\). Based on these facts, we first fit \(\{Y_t\}_{t=1}^{1474}\) by a diagonal VT-CBF model with \((P, Q, K) = (3, 1, 1)\), where the order \(K\) is taken as one for ease of model identification, and the orders \(P\) and \(Q\) are selected by the Bayesian information criterion (BIC). Specifically, this diagonal VT-CBF model is estimated using the two-step estimation procedure, and the corresponding estimates are give in Table 3. Second, since the sample ACFs of each component in Figure 3 decay slowly, we also fit \(\{Y_t\}_{t=1}^{1474}\) by a diagonal VT-CBF-HAR model, and the related estimation results are also listed in Table 3. From this table, we find that the estimates of the degrees of freedom (especially for \(\nu_2\)) in both fitted models are close to each other, and
both estimates of $\nu_2$ are small indicating the heavy-tailedness of the examined data. For the estimates of the mean parameter matrix $S$, its standard errors based on the VT-CBF model are smaller than those based on the VT-CBF-HAR model. For other estimates of parameter matrices, the estimated diagonal components in each parameter matrix seem to have close values, meaning that the examined three stocks possibly have similar temporal structures. This similarity can also be seen from the values of persistence of each stock in Table 3, where the persistence of stock $s$ is defined by $\sum_{i=1}^{P} A_{1i,ss}^2 + \sum_{j=1}^{Q} B_{1j,ss}^2$ for the VT-CBF model and $A_{(d),ss}^2 + A_{(w),ss}^2 + A_{(m),ss}^2$ for the VT-CBF-HAR model. After estimation, we then apply our test statistics $\Pi_v(l)$ to both fitted models, and the results summarized in Table 4 imply that both fitted models are adequate at the 5% level.

Next, we consider the forecasting performance of our proposed diagonal VT-CBF and VT-CBF-HAR models. Specifically, we compute the 1-step, 5-step and 10-step predictions of the RCOV matrices, based on a rolling window procedure with window size equal to 800. That is, we start from $t = 800$ to compute the predictions of RCOV matrices for $t+1$, $t+5$, and $t+10$, where the model is always estimated by using the latest 800 observations. To examine the importance of $\nu_2$ in the CBF models, we also apply the diagonal VT-CAW and VT-CAW-HAR models to do prediction for the purpose of comparison. The diagonal VT-CAW and VT-CAW-HAR models are defined in the same way as the diagonal VT-CBF and VT-CBF-HAR models, except that the matrix-F distribution for $\Delta_t$ in the latter two
Table 3
The results of the estimated diagonal VT-CBF and VT-CBF-HAR models

| Diagonal VT-CBF model | Diagonal VT-CBF-HAR model |
|-----------------------|---------------------------|
| $\hat{\nu}$ | $\hat{\nu}$ | $\hat{\nu}$ | $\hat{\nu}$ | $\hat{\nu}$ | $\hat{\nu}$ | $\hat{\nu}$ | $\hat{\nu}$ | $\hat{\nu}$ | $\hat{\nu}$ | $\hat{\nu}$ | $\hat{\nu}$ |
| 74.0110 | 3.1523 | 1.1099 | 1.1635 | 0.7207 | 0.5358 | 0.0117 | 0.4129 | 0.9771 |
| (10.7545) | (1.8844) | (0.9031) | (0.7705) | (0.0223) | (0.0365) | (0.0176) | (0.0354) |
| 40.5849 | 1.1099 | 2.3683 | 1.0965 | 0.7200 | 0.5620 | 0.0119 | 0.3800 | 0.9788 |
| (3.9787) | (0.9031) | (2.1165) | (0.9209) | (0.0246) | (0.0289) | (0.0177) | (0.0382) |
| 1.1635 | 1.0965 | 2.7883 | 0.7118 | 0.5579 | 0.0127 | 0.3977 | 0.9762 |
| (0.7705) | (0.9209) | (1.3276) | (0.0211) | (0.0292) | (0.0190) | (0.0354) |

Note: The asymptotic standard errors are given in the parenthesis.

Table 4
The results of $\Pi_v(l)$ for the diagonal VT-CBF and VT-CBF-HAR models

| $l$ | Diagonal VT-CBF model | Diagonal VT-CBF-HAR model |
|-----|-----------------------|---------------------------|
| $\Pi_v(l)$ | $\Pi_v(l)$ | $\Pi_v(l)$ | $\Pi_v(l)$ | $\Pi_v(l)$ | $\Pi_v(l)$ | $\Pi_v(l)$ |
| 2 | 3 | 4 | 5 | 6 | 2 | 3 | 4 | 5 | 6 |
| 1.494 | 4.170 | 8.004 | 9.428 | 11.513 | 4.385 | 6.127 | 7.004 | 10.310 | 11.583 |
| 0.474 | 0.244 | 0.091 | 0.093 | 0.074 | 0.112 | 0.106 | 0.136 | 0.067 | 0.072 |

models is replaced by the Wishart distribution. Besides the CAW-type models, we further include a diagonal VAR-HAR model for comparison, where this VAR model uses an HAR structure with the diagonal autoregressive parameter matrices to fit $y_t = vech(Y_t)$.

Table 5 gives the average of forecasting errors in Frobenius and spectral norms for all models. From this table, we can find that regardless of the prediction horizon, the diagonal VT-CBF-HAR model always has the smallest forecasting error in both norms. Moreover, we apply the DM test (Diebold and Mariano, 1995) to examine whether the diagonal VT-CBF-HAR model has a significant forecasting accuracy over other four competing models. The corresponding testing results are given in Table 5, and they show that the VT-CBF-HAR model is significantly better than its four competing models in terms of 5-step and 10-step forecasts. For 1-step forecasts, the VT-CBF-HAR and VT-CBF model models have comparable forecasting accuracy, and the VT-CBF-HAR model is significantly better than the remaining three models at level 10%. Note that the VAR-HAR model always performs worst in all examined cases, and this is probably because the VAR-HAR model brutally disentangles the matrix-structure of the RCOV matrices, which may have some intrinsic and useful value for forecasts.
Table 5  
**Forecasting errors based on different models and the related DM testing results**

| Model                  | 1-step |           | 5-step |           | 10-step |           |
|------------------------|--------|-----------|--------|-----------|---------|-----------|
|                        | Frobenius | Spectral | Frobenius | Spectral | Frobenius | Spectral |
| Diagonal VT-CBF-HAR    | 1.5284 | 1.4607    | 1.9725 | 1.8850    | 2.2108  | 2.1091    |
| Diagonal VT-CBF        | 1.5349 | 1.4664    | 1.9955 | 1.9069    | 2.2802° | 2.1755°   |
| Diagonal VT-CAW-HAR    | 1.5383° | 1.4703°   | 2.0029° | 1.9147°   | 2.2864° | 2.1813°   |
| Diagonal VT-CAW        | 1.5390† | 1.4699†   | 2.0253° | 1.9351°   | 2.3364° | 2.2286°   |
| Diagonal VAR-HAR       | 1.6472° | 1.5661°   | 2.1700° | 2.0626°   | 2.6088° | 2.4711°   |

Note: The DM test is used to compare the prediction accuracy between the diagonal VT-CBF-HAR and the other four competing models. The result of each competing model is marked with "†", "∗" or "⋄", if the DM test implies the Diagonal VT-CBF-HAR model gives significantly more accurate predictions than this competing model at level 10%, 5% or 1%, respectively.

7.2. **Application 2.** In this section, we consider intra-day data of 112 stocks from four major sectors constituting S&P 500 index: 31 stocks from financial sector, 31 stocks from industrial sector, 25 stocks from health care sector, and 25 stocks from consumer discretionary sector (see Appendix D in the supplementary material for the details of all chosen stocks). All intra-day price data are downloaded from Wharton Research Data Services (WRDS) database, and they are taken from 1 July, 2009 to 30 December, 2016, including 1890 non-missing dates of trading data in total. Based on 100 times log of the price data, the daily RCOV matrices \( \{ Y_t \}_{t=1}^{1890} \) are calculated by the TARVM method in Tao et al. (2011) for each sector.

For each sector, since the dimension of the RCOV matrix is large, we fit the RCOV matrix data by the diagonal F-VT-CBF and F-VT-CBF-HAR models. To do this, we first look for the value of \( r \) in model (5.11) by plotting the ratios \( \frac{\lambda_i}{\lambda_{i+1}} \) for each sector in Fig 4, where \( \{ \lambda_i \} \) are the eigenvalues of \( \bar{S} \) in descending order. From Fig 4, we can choose \( r = 3 \) for financial sector, \( r = 2 \) for industrial sector, \( r = 2 \) for health care sector, and \( r = 1 \) for consumer discretionary sector. To get more information, we also plot the ratios \( \frac{\lambda_i}{\lambda_{i+1}} \) for all four pooled sectors in Fig 5, from which \( r = 3 \) is suggested. This implies that all 112 stocks considered may be driven by 3 latent factors, but among which only two may affect the industrial and health care sectors, and only one may affect the consumer discretionary sector. Hence, it is more reasonable to study the RCOV matrix data across sectors rather than together.

Next, we estimate the diagonal F-VT-CBF and F-VT-CBF-HAR models and choose the orders by a similar procedure as in Application 1, and the related results are reported in Table 6. From this table, we can find that except for the mean parameter matrix, the diagonal components of other parameter matrices seem to have different values, meaning
that each component of $Y_{ft}$ has a different dynamical structure. Moreover, the values of persistence for $Y_{ft,ss}$ show clear differences across four sectors, with the largest persistence in financial sector and the smallest persistence in health care sector. This finding indicates that the effect of past stock returns to its current volatility decays very slowly in the
financial sector, while it behaves oppositely in the health care sector.

### Table 6

The results of the estimated diagonal F-VT-CBF and F-VT-CBF-HAR models

| Sector     | Diagonal F-VT-CBF model | Diagonal F-VT-CBF-HAR model |
|------------|-------------------------|----------------------------|
|            | \( \hat{S}_{tv} \)     | \( \hat{Y}_{tv} \)         |
|            | \( \hat{S}_{tv} \)     | \( \hat{Y}_{tv} \)         |
| Financial  | (2.9679) (11.0314) (2.3577) (0.6519) (0.0348) (0.0518) (10.141) (0.0696) | (0.1389) (0.0211) (1.6309) (0.7292) (0.3010) (0.4190) (0.3817) |
| Industrial | (9.0931) (2.3577) (0.0211) (0.6844) (0.5382) (0.3172) (0.6068) (0.1831) | (0.1389) (0.0211) (1.6309) (0.7292) (0.3010) (0.4190) (0.3817) |
| Health Care | (1.730) (0.0608) (0.1181) (0.1831) (0.0699) | (0.1389) (0.0211) (1.6309) (0.7292) (0.3010) (0.4190) (0.3817) |
| Consumer   | (4.0371) (2.9442) (1.5757) (0.7505) | (4.0371) (2.9442) (1.5757) (0.7505) |
| Discretionary | (4.9315) (3.4402) (5.1577) (0.4998) | (4.9315) (3.4402) (5.1577) (0.4998) |
|            | (0.7277) (0.6488)    | (0.7277) (0.6488) |
|            | (0.7277) (0.6488)    | (0.7277) (0.6488) |

Note: The asymptotic standard errors given in the parenthesis are based on process \( Y_{tv} \) rather than \( Y_{tv} \).

In the end, we examine the forecasting performance of our F-CBF models. As in Application 1, five different diagonal factor models (see Table 7) are considered to forecast \( Y_{tv} \), based on a rolling window procedure with window size equal to 1000. Their forecasting performance is evaluated by the average of forecasting errors in Frobenius and spectral norms as well as the results of the related DM test in Table 7. From this table, we can see that except for the health care sector, the diagonal F-VT-CBF-HAR model always has the smallest forecasting error and the diagonal F-VAR-HAR model has the largest forecasting error. For 1-step forecasts in the health care sector, the diagonal F-VT-CAW-HAR has slightly smaller forecasting error compared with the diagonal F-VT-CBF-HAR.
model. In view of the results of DM test, the diagonal F-VT-CBF-HAR model has a significantly better performance than the other four competing models in terms of 5-step and 10-step forecasts, but this advantage is slightly weak in terms of 1-step forecasts, for which the diagonal F-VT-CBF and F-VT-CAW-HAR models have similar performance in the industrial sector, and the diagonal F-VT-CAW-HAR and F-VAR-HAR models have comparative performance in the health care sector.

8. Concluding remarks. This paper proposes a new CBF model to study the dynamics of the RCOV matrix. For this CBF model, we explore its stationarity property, establish the asymptotics of its maximum likelihood estimator, and investigate the inner-product-based tests for its model checking. Hence, a systematic inferential tool of this CBF model is available for empirical researchers. In order to deal with large dimensional RCOV matrices, we also construct two reduced CBF models: the VT-CBF model and the F-CBF model. For both reduced models, the asymptotic theory of the estimated parameters is derived. Compared with the CAW model with Wishart innovations, the CBF model with matrix-F innovations is more able in capturing the heavy-tailed RCOV. This advantage is demonstrated by two real examples on U.S. stock markets. As motivated by Chiriac and Voev (2011), one obvious future work is to introduce the fractional integration structure
into our CBF models. Another interesting potential future work could extend the idea of using the matrix-F innovation in a number of ways resulting in a large family of models, which shall be important to study the positive definite dynamics.

APPENDIX A: PROOF OF THEOREM 2.1.

This appendix contains the proof of Theorem 2.1. To facilitate the proof, we recall some results in Boussama et al. (2011).

THEOREM A.1. Let there be a multivariate semi-polynomial Markov Chain, which is of the form $X_{t+1} = \mathcal{E}(X_t, \delta_t)$, where $X_t$ is of dimension $m_1$, $\delta_t$ is i.i.d. sequence of dimension $m_2$, and $\mathcal{E}$ is a $C^1$ continuous map. Let $V \subseteq \mathbb{R}^{m_1}$ be an algebraic variety and $U$ be an open subset of $\mathbb{R}^{m_1}$.

Suppose there exist $C^1$ continuous maps $L$ and $\nu$ to satisfy the decomposition $\mathcal{E}(z, y) = L(z, \nu(z, y))$ and the regularity conditions in Section 3 of Boussama et al. (2011) hold.

Then if the following assumptions (S1)-(S4) hold, there exists a unique strict stationary solution to $X_t$ which is Harris-recurrent and geometrically $\beta$-mixing.

(S1) $\delta_t$ is i.i.d. with distribution $\Gamma$ which is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^{m_2}$.

(S2) Define for all $k \in \mathbb{N}^* \setminus \{1\}$, the function $\mathcal{E}^k(z, \delta_1, \ldots, \delta_k) := \mathcal{E}(\mathcal{E}^{k-1}(z, \delta_1, \ldots, \delta_{k-1}), \delta_k)$ for $z \in U$, $\delta_1, \ldots, \delta_k \in \mathbb{R}^{m_1}$. Then for any $z \in V \cap U$ we can define an orbit:

$$S_z := \bigcup_{k \in \mathbb{N}^*} \left\{ \mathcal{E}^k(z, y_1, \ldots, y_k) : y_1, \ldots, y_k \in E \right\} = \bigcup_{k \in \mathbb{N}^*} \mathcal{E}^k(z, E^k),$$

where $E$ denotes the support of $\Gamma$. There exist a point $a_0 \in \text{int}(E)$ and a point $\Lambda \in W \cap U$, where $W := \overline{ZS_A}$ as the Zariski closure of the orbit $S_A$, such that for all $z \in W \cap U$ the sequence $\{X_i^z : X_i^z = F(X_{i-1}^z, a_0), X_0^z = z\}$ converges to the point $\Lambda$.

(S3) The strict stationary solution of the Markov chain $X_t = \mathcal{E}(X_{t-1}, \delta_t)$ takes its values in the algebraic variety $W \cap U$.

(S4) The Forster-Lyapunov (FL) condition hold, i.e., there exist a function $V : U \rightarrow [1, \infty]$ and positive constants $\alpha < 1$, $b < \infty$ as well as a Borel set $K$ in $W \cap U$ such that the (FL) condition hold, i.e.

$$PV(x) \leq \alpha V(x) + b \cdot 1_K(x), \quad \forall x \in W \cap U.$$
PROOF OF THEOREM 2.1. Applying \( vec(\cdot) \) operation to both sides of model (2.3), we have \( \sigma_t = t + \sum_{i=1}^{M} (A_i^*y_{t-i} + B_i^*\sigma_{t-i}) \), where \( \sigma_t = vec(\Sigma_t) \), \( y_t = vec(Y_t) \), and \( t = vec(\Omega) \). Define process \( X_t \) as

\[
(A.1) \quad X_t = \begin{pmatrix}
\sigma_t \\
\vdots \\
\sigma_{t-M+1} \\
y_t \\
\vdots \\
y_{t-M+1}
\end{pmatrix} = \begin{pmatrix}
\sigma_t \\
\vdots \\
\sigma_{t-M+1} \\
y_t \\
\vdots \\
y_{t-M+1}
\end{pmatrix}.
\]

Then, by (S1)-(S4), there exist some maps \( \mathcal{E}, \mathcal{L} \) and \( \nu \) such that

\[
X_t = \mathcal{E}(X_{t-1}, \delta_t) = \mathcal{L}(X_{t-1}, y_t) = \mathcal{L}(X_{t-1}, \nu(X_{t-1}, \delta_t)),
\]

where \( y_t = \nu(X_{t-1}, \delta_t) \) and \( \delta_t = vec(\Delta_t) \). Since \( \mathcal{E}, \mathcal{L}, \nu \) are \( C^1 \) continuous by lemma 4.1 of Boussama et al. (2011), it is obvious that the CBF model has stationary solution if and only if (A.1) has stationary solution, which is the case by (S1)-(S4) according to Theorem A.1. Hence, the proof is completed if (S1)-(S4) hold. Notice (S1) automatically holds by (H1). Then, it suffices to check (S2)-(S4) by Lemmas A.1-A.3 below, respectively. \( \square \)

LEMMA A.1. Suppose that (H1)-(H3) hold. For the constructed markov chain \( Z_t \), (S2) holds by choosing \( a_0 = vec(I_n) \) and \( A \) defined via the following equation: \( A = (t', 0, \ldots, 0)' + \Psi A \), where \( \Psi = \begin{pmatrix}
B + A & 0 \\
0 & B + A
\end{pmatrix} \in \mathbb{R}^{2Mn^2 \times 2Mn^2} \) with

\[
A = \begin{pmatrix}
A_1^* & A_2^* & \cdots & A_{M-1}^* & A_M^* \\
0 & 0 & \cdots & 0 & 0 \\
0 & \ddots & \ddots & \vdots & \vdots \\
\vdots & \ddots & 0 & 0 & 0 \\
0 & \cdots & 0 & 0 & 0
\end{pmatrix} \in \mathbb{R}^{Mn^2 \times Mn^2},
\]

\[
B = \begin{pmatrix}
B_1^* & B_2^* & \cdots & B_{M-1}^* & B_M^* \\
I_{n^2} & 0 & \cdots & 0 & 0 \\
0 & \ddots & \ddots & \vdots & \vdots \\
\vdots & \ddots & I_{n^2} & 0 & 0 \\
0 & \cdots & 0 & I_{n^2} & 0
\end{pmatrix} \in \mathbb{R}^{Mn^2 \times Mn^2}.
\]
Lemma A.2. Suppose that (H1)-(H3) hold. Then, (S3) holds, i.e., the strict stationary solution of $X_t$ takes value in $W \cap U$.

Lemma A.3. Suppose that (H1)-(H3) hold. Then, the (FL) condition in (S4) holds.

The proofs of Lemmas A.1-A.3 can be found in the supplementary material.

APPENDIX B: PROOFS OF THEOREMS 3.1-5.6.

In this appendix, we only give the proofs of Theorems 5.1-5.6. The proofs of Theorems 3.1-3.2 and 4.1 are essentially similar and less complicated, and hence they are omitted. To facilitate the proofs, we define

\[ \mathcal{Y}_t = \left( \text{vec}(Y_t)', \ldots, \text{vec}(Y_{t-M})' \right)' \in \mathcal{R}^{Mn^2 \times 1}, \]

\[ \mathcal{H}_t(\delta) = \left( \text{vec}(\Sigma_{vt}(\delta))', \ldots, \text{vec}(\Sigma_{vt-M}(\delta))' \right)' \in \mathcal{R}^{Mn^2 \times 1}, \]

\[ \hat{\mathcal{H}}_t(\delta) = \left( \text{vec}(\hat{\Sigma}_{vt}(\delta))', \ldots, \text{vec}(\hat{\Sigma}_{vt-M}(\delta))' \right)' \in \mathcal{R}^{Mn^2 \times 1}, \]

\[ r(\delta) = \begin{pmatrix} s' \left[ I_{n^2} - \sum_{i=1}^{M} (A_i^* + B_i^*) \right]' \end{pmatrix}' \in \mathcal{R}^{Mn^2 \times 1}. \]

Then, the recursion (5.5) can be rewritten as

\[ \hat{\mathcal{H}}_t(\delta) = r(\delta) + A(u)\mathcal{Y}_{t-1} + B(u)\hat{\mathcal{H}}_{t-1}(\delta), \quad (B.1) \]

where $A$ and $B$ defined as in Lemma A.1 are functions of $u$, $\mathcal{Y}_0 = \mathcal{Y}_0^*$ and $\hat{\mathcal{H}}_0(\delta) = \hat{\mathcal{H}}_0^*$ are calculated based on the sequence of given initial constant matrices $h$. Similarly, the recursion (5.7) can be rewritten as

\[ \mathcal{H}_t(\delta) = r(\delta) + A(u)\mathcal{Y}_{t-1} + B(u)\mathcal{H}_{t-1}(\delta). \quad (B.2) \]

It is worth noting that when $E\|Y_t\| < \infty$, by Theorem 2.1 and a similar argument as for (B.15) in Pedersen and Rahbek (2014), there exists $0 < \phi < 1$, such that

\[ \sup_{u \in \Theta_u} \|B^i(u)\| \leq U \phi^i \] for any integer $i \geq 0, \quad (B.3) \]

where $U > 0$ is a generic constant in the sequel.

Moreover, we give five technical lemmas. Lemma B.1 provides a list of useful results in matrix algebra. Lemma B.2 presents some moment conditions related to $\Sigma_t(\delta)$. Lemma B.3 ensures that the effect of the first-step estimation and the initial values is negligible.
for the second-step estimation. Lemma B.4 is standard to prove the strong consistency of \( \hat{\theta}_v \). Lemma B.5 is needed for the identifiability of \( \hat{\theta}_v \). The proofs of Lemmas B.1-B.5 can be found in the supplementary material.

**Lemma B.1.** Suppose that \( A, B, C \) and \( D \) are \( n \times n \) square matrices. Then,

(i) \( \text{tr}(ABCD) = \text{vec}(D')'(C \otimes A)\text{vec}(B) = (\text{vec}(D))'(A \otimes C')\text{vec}(B') \);

(ii) \( \text{tr}(A \otimes B) = \text{tr}(A)\text{tr}(B) \);

(iii) \( \|\text{tr}(AB)\| \leq \|A\|\|B\| \);

(iv) \( \|A\|_{\text{spec}} \leq \|A\| \leq \sqrt{n}\|A\|_{\text{spec}} \);

(v) \( \|AB\| \leq \|A\|_{\text{spec}}\|B\| \) and \( \|A + B\|_{\text{spec}} \leq \|A\|_{\text{spec}} + \|B\|_{\text{spec}} \);

(vi) For \( A > 0 \), \( \|A\| \leq \text{tr}(A) \) and \( \|(I + A)^{-1}\| \leq \sqrt{n} \);

(vii) For \( A > 0 \), \( \log |A| \leq \text{tr}(A) \), \( \log |A| \leq n\log \|A\|_{\text{spec}} \), and \( \|\log |A|\| \leq \text{tr}(A) + \text{tr}(A^{-1}) \);

(viii) For \( A > 0 \), \( |A + B| \geq |B| \);

(ix) For \( A \geq 0 \) and \( B > 0 \), \( 0 < \text{tr} \left( (A + B)^{-1} \right) \leq \text{tr}(B^{-1}) \);

(x) For \( A > 0 \) and \( B > 0 \), \( \|\log |AB^{-1}|\| \leq n\|A - B\| \left( \|B^{-1}\| + \|A^{-1}\| \right) \).

**Lemma B.2.** Let \( \delta_i \) be the \( i \)-th entry of \( \delta \). Suppose that Assumption 3.1 holds. Then,

(i) \( \sup_{\delta \in \Theta} \|\Sigma_{\nu \ell}^{-1}(\delta)\| \leq U \);

(ii) \( \sup_{\delta \in \Theta} \|\widehat{\Sigma}_{\nu \ell}^{-1}(\delta)\| \leq U \);

(iii) If \( E\|Y_{\ell}\|^k < \infty \), \( E \left( \sup_{\delta \in \Theta} \|\Sigma_{\nu \ell}(\delta)\| \right)^k \) for some \( k \geq 1 \);

(iv) If \( E\|Y_{\ell}\|^k < \infty \), \( E \left( \sup_{\delta \in \Theta} \left\| \frac{\partial \Sigma_{\nu \ell}(\delta)}{\partial \delta_i} \right\| \right)^k \) for some \( k \geq 1 \) and each \( i = 1, 2, \ldots, \tau_2 \);

(v) If \( E\|Y_{\ell}\|^k < \infty \), \( E \left( \sup_{\delta \in \Theta} \left\| \frac{\partial^2 \Sigma_{\nu \ell}(\delta)}{\partial \delta_i \partial \delta_j} \right\| \right)^k \) for some \( k \geq 1 \) and each \( i, j = 1, 2, \ldots, \tau_2 \).

**Lemma B.3.** Suppose that Assumptions 3.1 and 3.2 hold and \( E\|Y_{\ell}\| < \infty \). Then,

\[ \sup_{(u, \nu) \in \Theta_a \times \Theta_v} \|L_v(s_0, u, \nu) - \widehat{L}_v(\widehat{s}_v, u, \nu)\| \xrightarrow{a.s.} 0 \text{ as } T \to \infty. \]

**Lemma B.4.** Suppose that Assumptions 3.1 and 3.2 hold and \( E\|Y_{\ell}\| < \infty \). Then,

(i) \( E \left( \sup_{\theta_v \in \Theta_v} \|l_{\nu \ell}(\theta_v)\| \right) \) for some \( k \geq 1 \);

(ii) \( \sup_{\theta_v \in \Theta_v} \|L_v(\theta_v) - E[l_{\nu \ell}(\theta_v)]\| \xrightarrow{a.s.} 0 \text{ as } T \to \infty. \)
LEMMA B.5. For any \((u_0, v_0) \neq (u, v)\), \(E[l_{vt}(s_0, u_0, v_0)] < E[l_{vt}(s_0, u, v)]\).

PROOF OF THEOREM 5.1. First, by the ergodic theorem, we have
\[
\hat{s}_v \xrightarrow{a.s.} s_0 \text{ as } T \to \infty.
\]
Second, we can show that when \(T\) is large, for any \(\varepsilon > 0\),
\[
E[l_{vt}(s_0, \hat{u}_v, \hat{v}_v)] < \hat{L}_v(s_0, \hat{u}_v, \hat{v}_v) + \frac{\varepsilon}{5} \text{ by Lemma B.3(ii)};
\]
\[
\hat{L}_v(s_0, \hat{u}_v, \hat{v}_v) < \hat{L}_v(\hat{s}_v, \hat{u}_v, \hat{v}_v) + \frac{\varepsilon}{5} \text{ by Lemma B.3};
\]
\[
\hat{L}_v(\hat{s}_v, \hat{u}_v, \hat{v}_v) < \hat{L}_v(\hat{s}_v, \hat{u}_v, \hat{v}_v) + \frac{\varepsilon}{5} \text{ by definition of } \hat{u}_v, \hat{v}_v;
\]
\[
\hat{L}_v(\hat{s}_v, u_0, \nu_0) < \hat{L}_v(s_0, u_0, \nu_0) + \frac{\varepsilon}{5} \text{ by Lemma B.2};
\]
\[
\hat{L}_v(s_0, u_0, \nu_0) < E[l_{vt}(s_0, u_0, \nu_0)] + \frac{\varepsilon}{5} \text{ by Lemma B.4(ii)}.
\]
Thus, when \(T\) is large, for any \(\varepsilon > 0\), \(E[l_{vt}(s_0, \hat{u}_v, \hat{v}_v)] < E[l_{vt}(s_0, u_0, \nu_0)] + \varepsilon\). By Lemma B.5 and the continuity of the log-likelihood function, it follows that \((\hat{u}_v, \hat{v}_v) \xrightarrow{a.s.} (u_0, \nu_0)\) by Theorem 2.1 in Newey and McFadden (1994). This completes the proof. \(\Box\)

In order to prove Theorem 5.2, we need four more lemmas. Lemmas B.6-B.8 present some standard technical conditions, and Lemma B.9 ensures the negligibility of the initial values. The proofs of Lemmas B.6-B.9 can be found in the supplementary material.

LEMMA B.6. Let \(\theta_{vi}\) be the \(i\)-th entry of \(\theta_v\). Suppose that Assumptions 3.1 and 3.2 hold and \(E\|Y_t\|^2 < \infty\). Then,
(i) \(E\left[\sup_{G_t \in \Theta} \left| \frac{\partial^2 l_{vt}(\theta_v)}{\partial \theta_{vi} \partial \theta_{vj}} \right| \right] < \infty\);
(ii) \(\sup_{\theta_v \in \Theta} \left\| \frac{\partial^2 L_v(\theta_v)}{\partial \theta_{vi} \partial \theta_{vj}} - E \left[ \frac{\partial^2 l_{vt}(\theta_v)}{\partial \theta_{vi} \partial \theta_{vj}} \right] \right\| \xrightarrow{a.s.} 0 \text{ as } T \to \infty\),
for each \(i, j = 1, 2, \cdots, \tau_2\).

LEMMA B.7. Suppose that Assumptions 3.1 and 3.2 hold and \(E\|Y_t\|^2 < \infty\). Then,
\[
\sqrt{T} \left( \frac{s_v - s_0}{\partial L_v(\theta_{v0})/\partial \zeta} \right) = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} w_t + o_p(1),
\]
where \(w_t\) is defined as in Theorem 5.2 and \(E(w_t | G_{t-1}) = 0\).

LEMMA B.8. Suppose that Assumptions 3.1 and 3.2 hold and \(E\|Y_t\|^2 < \infty\). Then,
\[
\frac{1}{\sqrt{T}} \sum_{t=1}^{T} w_t \xrightarrow{d} N(0, E\left[ w_t w'_t \right]) \text{ as } T \to \infty.
\]
LEMMA B.9. Suppose that Assumptions 3.1 and 3.2 hold and $E\|Y_t\|^3 < \infty$. Then,

\[
\begin{align*}
(i) \sup_{\theta_0 \in \Theta_0} & \left\| \sqrt{T} \left( \frac{\partial L_v(\theta_0)}{\partial \theta_{vi}} - \frac{\partial \hat{L}_v(\theta_0)}{\partial \theta_{vi}} \right) \right\|_p \to 0 \text{ as } T \to \infty; \\
(ii) \sup_{\theta_0 \in \Theta_0} & \left\| \frac{\partial^2 L_v(\theta_0)}{\partial \theta_{vi} \partial \theta_{v}} - \frac{\partial^2 \hat{L}_v(\theta_0)}{\partial \theta_{vi} \partial \theta_{v}} \right\|_p \to 0 \text{ as } T \to \infty,
\end{align*}
\]

for each $i, j = 1, 2, \ldots, \tau_2$, where $\theta_{vi}$ is the $i$-th entry of $\theta_0$.

PROOF OF THEOREM 5.2. By the mean value theorem, there exists $\theta_*$ between $\theta_{o0}$ and $\hat{\theta}_v$ such that $0 = \frac{\partial L_v(\theta_{o0})}{\partial \zeta} + \frac{\partial^2 L_v(\theta_{o0})}{\partial \zeta^2}(\hat{s}_v - s_0) + \frac{\partial^2 \hat{L}_v(\theta_0)}{\partial \zeta^2}(\hat{\zeta}_v - \zeta_0)$. Then, by Lemma B.9, we have

\[
0 = \sqrt{T} \frac{\partial L_v(\theta_{o0})}{\partial \zeta} + [J_{1T}^* + o_p(1)] \left[ \sqrt{T} (\hat{s}_v - s_0) \right]
\]

\[
+ [J_{1T}^* + o_p(1)] \left[ \sqrt{T} (\hat{\zeta}_v - \zeta_0) \right] + o_p(1),
\]

where $J_{1T}^* = \frac{\partial^2 L_v(\theta_{o0})}{\partial \zeta^2}$ and $J_{2T}^* = \frac{\partial^2 \hat{L}_v(\theta_0)}{\partial \zeta^2}$. By Lemma B.6 and Theorem 3.1 in Ling and McAleer (2003), we have $J_{1T}^* = J_1 + o_p(1)$ and $J_{2T}^* = J_2 + o_p(1)$. Hence, by (B.4) and Lemma B.7, it follows that

\[
\sqrt{T}(\hat{\theta}_v - \theta_{o0}) = \begin{pmatrix} I_{n^2} & 0 \\ -J_2^{-1} & -J_1^{-1} \end{pmatrix} \sqrt{T} \left( \frac{\partial L_v(\theta_{o0})}{\partial \zeta} \right) + o_p(1).
\]

Finally, the proof is completed by Slutzky’s theorem and Lemma B.8.

PROOF OF THEOREM 5.3. By Taylor’s expansion and Theorem 5.2, we can show that

\[
\sqrt{T} Y_{vt}(\hat{\delta}_v) = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \begin{pmatrix} b_{vt,1}(\hat{\delta}_v) \\ b_{vt,2}(\hat{\delta}_v) \\ \vdots \\ b_{vt,1}(\hat{\delta}_v) \end{pmatrix} + \frac{1}{T} \sum_{t=1}^{T} \begin{pmatrix} 3'_{vt}(\hat{\delta}_v) (\partial 3_{vt}(\delta_0)/\partial \theta') \\ 3'_{vt-1}(\hat{\delta}_v) (\partial 3_{vt}(\delta_0)/\partial \theta') \\ \vdots \\ 3'_{vt-l}(\hat{\delta}_v) (\partial 3_{vt}(\delta_0)/\partial \theta') \end{pmatrix} \times \frac{1}{\sqrt{T}} \begin{pmatrix} I_{n^2} & 0 \\ -J_2^{-1} & -J_1^{-1} \end{pmatrix} \sum_{t=1}^{T} w_t(\delta_0) + o_p(1)
\]

\[
= (I_t, \mathcal{R}_v) \frac{1}{\sqrt{T}} \sum_{t=1}^{T} e_{vt} + o_p(1).
\]

Since $e_{vt}$ is a martingale difference sequence, the proof follows by standard arguments.

Next, we consider the proofs of Theorems 5.4 and 5.6. Since the proof of Theorem 5.5 is essentially similar as the one for Theorem 5.6, it is omitted for simplicity.
Proof of Theorem 5.4. Based on Assumptions 5.1-5.3, the proof is the same as the one for Theorem 1 in Shen et al. (2018), hence it is omitted here.

Proof of Theorem 5.6. First, it is straightforward to show that (i) holds by Theorem 5.4(ii). Next, we can claim that

\[ \sup_{\zeta \in \Theta_u \times \Theta_v} \left\| \frac{\partial L_{f_t}(\hat{s}_{1f_t}, \zeta)}{\partial \zeta} - \frac{\partial L_{f_t}(\hat{s}_{2f_t}, \zeta)}{\partial \zeta} \right\| = O_p(B(T)/T) + O_p(A^{1/2}(n, m, T)B^{5/2}(T)). \]

In order to prove (B.6), we define

\[ Y_{f_t} = (\text{vec}(Y_{f_t})', \ldots, \text{vec}(Y_{f_t-M}))' \in \mathcal{R}^{M^2 n}, \]
\[ \hat{Y}_{f_t} = (\text{vec}(\hat{Y}_{f_t})', \ldots, \text{vec}(\hat{Y}_{f_t-M}))' \in \mathcal{R}^{M^2 n}, \]
\[ H_{f_t}(\delta) = (\text{vec}(\Sigma_{f_t}(\delta))', \ldots, \text{vec}(\Sigma_{f_t-M}(\delta))')' \in \mathcal{R}^{M^2 n}, \]
\[ \hat{H}_{f_t}(\delta) = (\text{vec}(\hat{\Sigma}_{f_t}(\delta))', \ldots, \text{vec}(\hat{\Sigma}_{f_t-M}(\delta))')' \in \mathcal{R}^{M^2 n}. \]

Then, as for (B.1)-(B.2), we have

\[ \hat{H}_{f_t}(\hat{s}_{1f_t}, \zeta) - H_{f_t}(\hat{s}_{2f_t}, \zeta) = [r(\hat{s}_{1f_t}, \zeta) - r(\hat{s}_{2f_t}, \zeta)] + A(u)[\hat{Y}_{f_t} - Y_{f_t}] + B(u)[\hat{H}_{f_t-1}(\hat{s}_{1f_t}, \zeta) - H_{f_t-1}(\hat{s}_{2f_t}, \zeta)], \]

and since \( \rho(\sum_{i=1}^{M} B^*_i) < 1 \), it implies that

\[ \hat{H}_{f_t}(\hat{s}_{1f_t}, \zeta) - H_{f_t}(\hat{s}_{2f_t}, \zeta) = B^t(u)(\hat{H}_{f_0} - H_{f_0}(\hat{s}_{2f_t}, \zeta)) + \sum_{i=0}^{t-1} B^i(u) \left\{ [r(\hat{s}_{1f_t}, \zeta) - r(\hat{s}_{2f_t}, \zeta)] + A(u)[\hat{Y}_{f_t} - Y_{f_t}] \right\}, \]

where \( \hat{H}_{f_0} \) is a given initial value. By (B.7), we can show that

\[ \sup_{\zeta \in \Theta_u \times \Theta_v} \left\| \hat{\Sigma}_{fvt}(\hat{s}_{1f_t}, \zeta) - \Sigma_{fvt}(\hat{s}_{2f_t}, \zeta) \right\| = O_p(\phi^t) + O_p(\phi^t/\sqrt{T}) + O_p(A^{1/2}(n, m, T)B^{3/2}(T)). \]
and
\[
\sup_{\zeta \in \Theta_u \times \Theta_v} \left\| \hat{\Sigma}_{f t t}^{-1}(\hat{s}_{1 f v}, \zeta) \hat{y}_{f t} - \Sigma_{f t t}^{-1}(\hat{s}_{2 f v}, \zeta) Y_{f t} \right\|
\]
\[
= \sup_{\zeta \in \Theta_u \times \Theta_v} \left\| \hat{\Sigma}_{f t t}^{-1}(\hat{s}_{1 f v}, \zeta)(\hat{y}_{f t} - Y_{f t}) \right. \\
- \Sigma_{f t t}^{-1}(\hat{s}_{2 f v}, \zeta) \left[ \Sigma_{f t t}(\hat{s}_{2 f v}, \zeta) - \hat{\Sigma}_{f t t}(\hat{s}_{2 f v}, \zeta) \right] \hat{\Sigma}_{f t t}^{-1}(\hat{s}_{1 f v}, \zeta) Y_{f t} \right\|
\]
(B.9)
\[
= \left[ O_p(\phi^t) + O_p(\phi^{t \sqrt{T}}) + O_p(A^{1/2}(n, m, T)B^{3/2}(T)) \right] \left[ 1 + O_p(B(T)) \right],
\]
where (B.8) holds by (B.3), the compactness of \( \Theta_u \) and \( \Theta_v \), Lemma B.2(iii)-(iv) and Theorems 5.1 and 5.4, and (B.9) holds by the triangular inequality, Lemma B.2(i)-(ii), (B.8), and Assumption 5.2.

Now, by (B.8)-(B.9) and Lemma B.1(x), we can show that \( \sup_{\zeta \in \Theta_u \times \Theta_v} \left\| \frac{\partial L_{f v}(\hat{s}_{1 f v}, \zeta)}{\partial \zeta} \right\| = \sum_{t=1}^{T} [O_p(\phi^t) + O_p(\phi^{t \sqrt{T}}) + O_p(A^{1/2}(n, m, T)B^{3/2}(T))] [1 + O_p(B(T))], \)
i.e., (B.6) holds. By (B.6) and Taylor’s expansion, we have
\[
0 = \frac{\partial L_{f v}(\hat{s}_{1 f v}, \hat{\zeta}_{1 f v})}{\partial \zeta} = \frac{\partial L_{f v}(\hat{s}_{2 f v}, \hat{\zeta}_{2 f v})}{\partial \zeta} + O_p(B(T)/T) + O_p(A^{1/2}(n, m, T)B^{5/2}(T)) \\
+ \frac{\partial L_{f v}(\hat{s}_{2 f v}, \hat{\zeta}_{2 f v})}{\partial \zeta} \frac{\partial L_{f v}(\hat{s}_{2 f v}, \hat{\zeta}_{2 f v})}{\partial \zeta'} (\hat{\zeta}_{1 f v} - \hat{\zeta}_{2 f v}) \\
+ O_p(B(T)/T) + O_p(A^{1/2}(n, m, T)B^{5/2}(T)) \\
+ O_p(B(T)/T) + O_p(A^{1/2}(n, m, T)B^{5/2}(T)),
\]
where \( \hat{\zeta}_{f v} \) lies between \( \hat{\zeta}_{1 f v} \) and \( \hat{\zeta}_{2 f v} \), and the fourth equality holds by Lemma B.6 and the law of large numbers theorem for stationary sequence. Hence, by Lemma B.6 again, it follows that (ii) holds. This competes all of the proofs. \( \square \)

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