THE MANIN-MUMFORD CONJECTURE IN GENUS 2 AND RATIONAL CURVES ON K3 SURFACES

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Abstract. Let $A$ be a simple abelian surface over an algebraically closed field $k$. Let $S \subset A(k)$ be the set of torsion points $x$ of $A$ such that there exists a genus 2 curve $C$ and a map $f : C \to A$ such that $x$ is in the image of $f$, and $f$ sends a Weierstrass point of $C$ to the origin of $A$. The purpose of this note is to show that if $k$ has characteristic zero, then $S$ is finite — this is in contrast to the situation where $k$ is the algebraic closure of a finite field, where $S = A(k)$, as shown by Bogomolov and Tschinkel. We deduce that if $k = \overline{\mathbb{Q}}$, the Kummer surface associated to $A$ has infinitely many $k$-points not contained in a rational curve arising from a genus 2 curve in $A$, again in contrast to the situation over the algebraic closure of a finite field.

1. Introduction

Bogomolov and Tschinkel have observed that for a Kummer K3 surface $X$ over an algebraic closure of a finite field $\mathbb{F}_q$, there exists a rational curve passing through every $\mathbb{F}_q$-point of $X$ [BT05, Theorem 1.1]. Equivalently, for an abelian surface $A/\mathbb{F}_q$, any point $x \in A(\mathbb{F}_q)$ lies in the image of a morphism $f : C \to A$ from a hyperelliptic curve $C$ that sends a Weierstrass point $O_C$ to the origin $O_A$. They point out that as yet we have not ruled out a similar phenomenon over $\mathbb{Q}$ [BT00, BT05].

The goal of this note is to rule out a slightly weaker phenomenon. Bogomolov and Tschinkel in fact show that every point $x \in A(\mathbb{F}_q)$ in an abelian surface lies in the image of a map $f : (C, O_C) \to (A, O_A)$ from a genus 2 curve $C$, with $O_C$ a marked Weierstrass point. In contrast, we show:

Theorem 1. Let $A$ be a simple abelian surface over an algebraically closed field of characteristic zero. The set of torsion points $x \in A$, such that there exists a map $f : (C, O_C) \to (A, O_A)$ from a genus 2 hyperelliptic curve with a marked Weierstrass point, containing $x$ in its image, is finite.

In principle, the bound obtained is explicit in terms of the Galois representation on the Tate module of $A$ over a finitely generated field.

The condition that $f$ send a Weierstrass point to the origin is crucial, as the translates of one image $f(C)$ will cover $A$. Moreover the result is false if one does not assume simplicity; for example, if $A = E \times E$ is the square of a CM elliptic curve, every torsion point is in the image of a map from a genus 2 curve sending a Weierstrass point to the origin. Finally, note that as the Torelli map in genus 2 is dominant, there are in general infinitely many isomorphism classes of genus 2 curves mapping to a given abelian surface. Our theorem thus shows that a countably infinite collection of curves in $A$ contains only finitely many torsion points.

Now let $A$ be an abelian surface, and $X := \text{Bl}_{A[2]} A/(\pm 1)$ the associated Kummer K3 surface. Given a rational curve $C$ in $X$, we let $C'$ be the preimage in $A$. This curve is necessarily irreducible because $A$ contains no rational curves. Let $h(C)$ be the genus of its normalization. As a corollary to Theorem 1, we have

Corollary 2. Let $X/\overline{\mathbb{Q}}$ be a Kummer K3 surface associated to a simple abelian surface over $\overline{\mathbb{Q}}$. Then there exist infinitely many $\overline{\mathbb{Q}}$-points of $X$ not contained in any rational curve $C$ with $h(C) = 2$.

Indeed, the image in $X$ of any point of $A$ of finite order $N \gg 0$ will not be contained in such a curve $C$, by Theorem 1. Again, this is in contrast to Bogomolov and Tschinkel’s result over the algebraic closure of a finite field.

The primary antecedent to this result is the main result of [BM10], which shows that for certain elliptic K3 surfaces over $\overline{\mathbb{Q}}$, there exist $\overline{\mathbb{Q}}$-points not contained in any smooth rational curves. In contrast, not all of the rational curves in Corollary 2 (that is, those $C$ with $h(C) = 2$) are smooth.

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2. Proof of Theorem 1

Theorem 1 follows from an analysis of a proof of the Manin-Mumford conjecture, essentially due to Lang [Lan65] and Serre [Ser13, Letter to Ribet, 1/1/1981] (see e.g. [BR03, Section 2] for a discussion of the proof). In particular, the main input is a variant of a result of Serre on homotheties in the image of the Galois action on the Tate module of an abelian surface over a number field, due to Wintenberger.

Let \( k \) be a field of characteristic zero and \( \overline{k} \) an algebraic closure. For an abelian variety \( A/k \), let \( T_\ell(A) = \lim_n A[\ell^n](\overline{k}) \) be the \( \ell \)-adic Tate module of \( A \), \( T(A) = \prod_\ell T_\ell(A) \) the total Tate module of \( A \), and

\[
\rho_A,\ell : \text{Gal}(\overline{k}/k) \to GL(T_\ell(A)), \quad \rho_A : \text{Gal}(\overline{k}/k) \to GL(T(A))
\]

the associated \( \ell \)-adic (resp. adelic) Galois representations. For each \( \ell \), let \( H_\ell \subset GL(T_\ell(A)) \) be the subgroup consisting of \( \ell \)-adic scalar matrices (i.e. homotheties), and let \( H = \prod_\ell H_\ell \subset GL(T(A)) \) be the adelic scalar matrices. Wintenberger shows (verifying a conjecture of Lang in dimension \( \leq 4 \)):

**Theorem 3** (Wintenberger, [Win02, Corollaire 1]). Let \( A \) be an abelian variety over a finitely generated field \( k \) of characteristic zero, with \( \dim(A) \leq 4 \). Then the intersection of \( \rho_A(\text{Gal}(\overline{k}/k)) \) with \( H \) is open.

**Remark 4.** Wintenberger only states this result for \( k \) a number field, but the result for finitely generated fields of characteristic zero follows as in [Ser13, Letter to Ribet 1/1/1981, §1].

For \( A \) as in the theorem, let \( C(A) \) denote the index of

\[
[\rho_A(\text{Gal}(\overline{k}/k)) \cap H : H].
\]

By the theorem, \( C(A) \) is finite. In particular, Wintenberger’s theorem implies that for \( \dim(A) \leq 4 \), the set of \( \ell \)-adic homotheties in the image of Galois is all of \( \mathbb{Z}_\ell \cdot \text{id} \), for almost all \( \ell \).

Similarly, Serre shows [Ser13, Letter to Ribet, 1/1/1981] that for any abelian variety \( A \) (with no restriction on the dimension) over a finitely generated field of characteristic zero, the constants

\[
C_\ell(A) := [\rho_A,\ell(\text{Gal}(\overline{k}/k)) \cap H_\ell : H_\ell]
\]

are bounded independent of \( \ell \). Let \( S(A) := \max_\ell C_\ell(A) \) be the maximum of all of these indices. Note that if \( C(A) \) is finite, we have \( S(A) \leq C(A) \).

**Theorem 5** (Manin-Mumford Conjecture, Ribet, [BR03, Section 2]). Let \( X \) be a smooth projective curve of genus \( g \geq 2 \) over a finitely generated field \( k \) of characteristic zero, and let \( x \in X(k) \) be a point. Then there exists an integer \( N = N(S(\text{Jac}(X))) \) such that if \( X \) is embedded in \( \text{Jac}(X) \) via the Abel-Jacobi map associated to \( x \), then the image of \( X(\overline{k}) \) in \( \text{Jac}(X)(\overline{k}) \) does not contain any \( m \)-torsion points with \( m > N \).

Here \( S(\text{Jac}(X)) \) is the constant defined above. Aside from this “explicit” form of the Manin-Mumford conjecture, the main observation going into the proof of Theorem 1 is the following lemma on the fields of definition of genus 2 curves mapping to an abelian surface.

**Lemma 6.** Let \( A \) be a geometrically simple abelian surface over a finitely generated field \( k \) of characteristic zero. Then there exist constants \( C' = C'(A) \) such that if \( X \) is a curve of genus 2 over \( \overline{k} \) and \( f : X \to A_{\overline{k}} \) is a non-constant map sending a Weierstrass point of \( X \) to the origin, then:

- there exists a finite extension \( L \subset \overline{k} \) of \( k \), a curve \( X'/L \), a Weierstrass point \( x \in X'(L) \), a map \( f' : X' \to A_L \) sending \( x \) to the origin, and an identification of \( f'_L \) with \( f \), such that
- The constant \( C(\text{Jac}(X')) \) satisfies \( C(\text{Jac}(X')) < C' \).

**Proof.** Since \( A \) is simple, the map \( f_* : \text{Jac}(X_{\overline{k}}) \to A_{\overline{k}} \) is an isogeny, and the induced map \( T(\text{Jac}(X_{\overline{k}})) \to T(A) \) has a finite index image \( T' \subset T(A) \). Let \( G' \subset \text{Gal}(\overline{k}/k) \) be the stabilizer of \( T' \) under \( \rho_A \) and let \( k' \) be the fixed field of \( G' \). The Jacobian \( \text{Jac}(X_{\overline{k}}) \) and the map \( f_* \) descend to an abelian variety \( J' \) and a map \( g : J' \to A_{k'} \) over \( k' \) (as, for example, the kernel of the isogeny dual to \( f_* \) is stable under the action of the absolute Galois group of \( k' \)).

We now descend the principal polarization on \( \text{Jac}(X) \) to \( J' \), after a field extension of absolutely bounded degree. Let \( NS(J') \) be the Néron-Severi group of \( J' \); this is a free abelian group of rank at most 4. The action of \( \text{Gal}(\overline{k}/k') \) on \( NS(J') \) has finite image. Hence the order of this image is bounded above, for example, by the order of the maximal finite subgroup of \( GL_4(\mathbb{Z}) \), hence is independent of \( X \). Let \( k'' \) be the fixed field of the kernel of this action, so that each component of \( \text{Pic}(\text{Jac}(X)) \) descends to \( k'' \). After possibly replacing \( k'' \) by a finite extension of absolutely bounded degree, we may assume in addition that the absolute Galois group of \( k'' \) acts trivially on the 2-torsion group \( \text{Pic}^0(J')_{k''} [2](\overline{k}) \).
Choose an embedding of \( X \) in \( \text{Jac}(X) \) sending a Weierstrass point to the origin, and let \( \mathcal{L} := \mathcal{O}_{\text{Jac}(X)}(X) \in \text{Pic}(\text{Jac}(X)) \) be the corresponding line bundle, a.k.a. the theta bundle. The line bundle \( \mathcal{L} \) yields a geometric point \([\mathcal{L}]\) of \( \text{Pic}(J)^{tors} \), and for any \( \sigma \in \text{Gal}(\overline{k}/k) \), \([\mathcal{L}] − [\mathcal{L}]^\sigma\) is a 2-torsion point of \( \text{Pic}^0(J)^{tors} \) (as the property of sending a Weierstrass point to the origin may be checked after base change to \( \overline{k} \), and any two Weierstrass points differ by a 2-torsion point).

Thus the orbit of \([\mathcal{L}]\) under \( \text{Pic}^0(J)^{tors}[2] \simeq (\mathbb{Z}/2\mathbb{Z})^4 \) is defined over \( k'' \); there exists an extension \( L \) of \( k'' \) of absolutely bounded degree splitting this \( \text{Pic}^0(J)^{tors}[2] \)-torsor. (Explicitly, the torsor itself is the spectrum of an étale \( k'' \)-algebra of degree 16, and it will split over the residue field of any of its closed points).

Now the line bundle \( \mathcal{L} \) descends to \( L \); let \( s \in H^0(\mathcal{L}) \) be any non-zero global section. As \( \mathcal{L} \) is the theta bundle, \( h^0(\mathcal{L}) = 1 \), and so \( X' := V(s) \) is a descent of \( X \) to \( L \). Moreover the embedding of \( X' \) in its Jacobian given by \( \mathcal{L} \) sends a Weierstrass point to the origin by construction; hence \( X' \) is equipped with a rational Weierstrass point \( x \). We now need only verify that the condition of the second bullet point is satisfied, i.e. we must bound \( C(\text{Jac}(X')) \) purely in terms of \( C(A) \).

First, note that \( C(J') \) is equal to \( C(A) \), as any homothety in \( GL(T(A)) \) stabilizes any subgroup of \( T(A) \). That is, the index of the image of the absolute Galois group of \( k' \) in the homotheties of \( T(A) \) is bounded above by \( C(A) \). Now observe that the degrees of \( k'' \) and \( L \) over \( k' \) are all absolutely bounded, so we’re done.

We may now prove the main result:

**Proof of Theorem 1.** Let \( A \) be a simple abelian surface over an algebraically closed field as in the statement of the theorem. \( A \) descends to a geometrically simple abelian surface over some finitely generated field \( k \) of characteristic zero. Lemma 6 gives us a constant \( C' = C'(A) \) such that for any genus \( 2 \) curve \( X \) with a nonconstant map \( f : X \to A \), sending a Weierstrass point to the origin, \((X, f)\) is defined over a finitely generated field \( k' \) such that

\[
C_f(\text{Jac}(X)) := [\rho_{\text{Jac}(X)}(\text{Gal}(\overline{k}/k')) \cap H_f : H_f]
\]

is bounded above by \( C' \). Thus by Theorem 5, there exists a constant \( N(C') \) such that, under an embedding \( X \to \text{Jac}(X) \) arising from a Weierstrass point of \( X \), any torsion point of \( \text{Jac}(X) \) lying in the image of \( X \) has order bounded above by \( N(C') \). We claim that any torsion point of \( A \) in the image of \( f \) also has order bounded above by \( N(C') \).

But by the Albanese property of \( \text{Jac}(X) \), the map \( f \) factors through a map \( \text{Jac}(X) \to A \); this map is necessarily an isogeny as \( A \) is geometrically simple. Let \( x \) be a torsion point in the image of \( f \); its preimages in \( \text{Jac}(X) \) are torsion, and at least one lies in the image of the Abel-Jacobi map. Hence its order is bounded by \( N(C') \), by the previous paragraph. This completes the proof.

It is natural to ask if the analogous statement where one allows the curves \( C \) to vary over hyperelliptic curves of all genera holds:

**Question 7.** Let \( A \) be a simple abelian surface over \( \overline{Q} \). Let \( S \subset A(\overline{Q}) \) be the set of torsion points in \( A \) in the image of some map \( f : (C, O_C) \to (A, O_A) \) with \( C \) hyperelliptic, \( O_C \) Weierstrass. Is \( S \) finite?

A positive answer would resolve Bogomolov-Tschinkel’s question about rational points contained in a rational curve on a K3 surface over \( \overline{Q} \), i.e. it would show that there exist Kummer K3 surfaces over \( \overline{Q} \) with a \( \overline{Q} \)-point not contained in any rational curve.

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