LOGARITHMIC INTERTWINING OPERATORS AND VERTEX OPERATORS

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ABSTRACT. This is the first in a series of papers where we study logarithmic intertwining operators for various vertex subalgebras of Heisenberg and lattice vertex algebras. In this paper we examine logarithmic intertwining operators associated with rank one Heisenberg vertex operator algebra $M(1)_a$, of central charge $1 - 12a^2$. We classify these operators in terms of depth and provide explicit constructions in all cases. Our intertwining operators resemble puncture operators appearing in quantum Liouville field theory. Furthermore, for $a = 0$ we focus on the vertex operator subalgebra $L(1, 0)$ of $M(1)_0$ and obtain logarithmic intertwining operators among indecomposable Virasoro algebra modules. In particular, we construct explicitly a family of hidden logarithmic intertwining operators, i.e., those that operate among two ordinary and one genuine logarithmic $L(1, 0)$-module.

0. INTRODUCTION

The theory of vertex algebras continues to be very effective in proving rigorous results in two-dimensional Conformal Field Theory (CFT) (for some recent breakthrough see [H1], [H2], [Le]).

In 1993, Gurarie [Gu] studied a CFT-like structure with two features absent in the ordinary CFT: logarithmic behavior of matrix coefficients and appearance of indecomposable representations of the Virasoro algebra underlying the theory. There are several additional examples of “logarithmic” models that have been discovered since then (see for instance [GK1], [F1], [G1], and especially [FFHST], [F2], [G2] and references therein). By now, a structure that involves a family of modules for the chiral algebra, closed under the “fusion”, with logarithmic terms in the operator product expansion is usually called a logarithmic conformal field theory (LCFT). The most important examples of LCFTs are the so-called rational LCFTs. These involve only finitely many inequivalent irreducible representations, but also some indecomposable logarithmic representations so that the modular invariance is preserved (e.g., the triplet model [GK1], [GK2], [GK3], [BF], etc.). We stress here that neither LCFT nor rational LCFT are mathematically precise notions.

In [M1] we proposed a purely algebraic approach to LCFT based on the notion of logarithmic modules and logarithmic intertwining operators. The key idea is to introduce a deformation parameter $\log(x)$ and to define logarithmic intertwining operators as expressions involving intertwining-like operators multiplied with appropriate powers of $\log(x)$, such that the translation invariance is preserved. These operators can be used to explain appearance of logarithms in correlation functions. In our setup we do not require an extension of the space of “states” for the underlying vertex algebra. There are other proposals in the literature [FFHST] where $\log(x)$ is also viewed as a deformation parameter, but with an important difference that $\log(x)$ is also part of an extended chiral algebra (or an OPE algebra). Even though the construction in [FFHST] has been shown successful in explaining various logarithmic behaviors of CFTs, it is unclear to us if the approach in [FFHST] can be used to address the problem of fusion. On the other hand, logarithmic intertwining operators have been used by Huang Lepowsky and Zhang in [HLZ] as a convenient tool to develop a generalization of Huang-Lepowsky’s tensor product theory [HL] to non-semisimple
tensor categories. Another important contribution is Miyamoto’s generalization of Zhu’s modular invariance theorem for vertex operator algebras satisfying the $C_2$-condition \cite{My1}, which possibly involve logarithmic modules. In view of \cite{My1} we tend to believe that rational LCFT give rise from vertex algebras satisfying the $C_2$-cofiniteness condition.

From everything being said it appears that several important aspects of LCFTs can be studied in the framework of vertex (operator) algebras. This paper continues naturally on \cite{M1} and \cite{M2}. In the present work we focus on a simple, yet interesting class of vertex operator algebras—those associated with Heisenberg Lie algebras and the Virasoro algebra. The aim here is to construct a family of logarithmic intertwining operators associated with certain weak $M(1)_a$-modules, which can be used for building logarithmic intertwining operators among indecomposable representations of the Virasoro algebra and various $W$-algebras \cite{AM,M3}. The most interesting part of our work is an explicit construction of the so-called hidden logarithmic intertwining operators, i.e., those which intertwine a pair of ordinary and one logarithmic module. We should stress here that our logarithmic intertwining operators are closely related to puncture operators appearing in quantum Liouville field theory \cite{Se}, \cite{ZZ}.

Let us elaborate the construction on an example. Consider the Feigin-Fuchs module $M(1,\lambda)_a$ of central charge $c = 1 - 12a^2$ and lowest conformal weight $\frac{a^2}{2} - a\lambda$. Tensor the module $M(1,\lambda)_a$ with a two-dimensional space $\Omega$, where $h(0)$ acts on $\Omega$ (in some basis) as

$$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

so we obtain a weak $M(1)_a$-module $M(1,\lambda)_a \otimes \Omega$. Now it is easy to see that $M(1,\lambda)_a \otimes \Omega$ is an ordinary module if and only if $\lambda = a$, so that $M(1, a)_a \otimes \Omega$ is of lowest conformal weight $-\frac{a^2}{2}$.

Our main results (cf. Theorem 5.5 and Theorem 7.5) then provides us with a genuine logarithmic intertwining operator of type

$$W \left( M(1, a)_a \otimes \Omega \bigg| M(1, a)_a \otimes \Omega \right),$$

where $W$ is a logarithmic module. Notice that for special $a$ this result is in “agreement” with some results in the physics literature. For instance, for $c = -2$ and $a = \frac{1}{2}$, the lowest conformal weight of $M(1, \frac{1}{2})_a \otimes \Omega$ is $\frac{1}{8}$. This model is known to be logarithmic after \cite{Gu}. More generally, one takes

$$c = 1 - \frac{6(p-1)^2}{p}, \quad p \geq 2 \text{ (cf. } [AM], [M3]).$$

We should say here that Feigin-Fuchs modules $M(1, a)_a$ with the lowest conformal weight $-\frac{a^2}{2}$ and central charge $1 - 12a^2$ are indeed special from at least two points of view. These modules are self-dual (cf. (3.9)) and in addition we have

$$h - \frac{c}{24} = \frac{-1}{24},$$

so that

$$\text{tr}_{M(1, a)_a} q^{L(0)-c/24} = \frac{1}{\eta(\tau)},$$

where $\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$ is the Dedekind $\eta$-function. In fact these are the only Feigin-Fuchs modules whose modified graded dimensions are modular functions.

Let us briefly outline the content of the paper. In Section 2 we recall the notions of logarithmic module and of logarithmic intertwining operator. In sections 3 and 4 we prove some standard
results about extended Heisenberg Lie algebras, the corresponding vertex operator algebras and associated logarithmic intertwining operators. In Section 5 we derive a logarithmic version of the Li-Tsuchiya-Kanie’s “Nuclear Democracy Theorem” for logarithmic intertwining operators needed for our construction. In sections 5 and 7 we provide a classification of logarithmic intertwining operators among a triple of logarithmic $M(1)_a$-modules. Finally, in sections 9 and 10 we use certain restriction theorems for construction of logarithmic intertwining operators among triples of logarithmic $L(1,0)$-modules. In particular, we obtain a family of hidden intertwining operators from a pair of ordinary $L(1,0)$-modules. Section 11 is an introduction to [AM] and [M3].

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1. LOGARITHMIC MODULES AND LOGARITHMIC INTERTWINING OPERATORS

In this section notation and definitions are mostly from [[DLM], [HLZ], [LL] and [M1]]. For an accessible introduction to vertex (operator) algebras and their representations we refer the reader to [LL].

Definition 1.1. Let $(V,Y,1,\omega)$, $V = \bigoplus_{i \in \mathbb{Z}} V_i$, be a vertex operator algebra. A logarithmic $V$-module $M$ is a weak $V$-module [DLM] which admits a decomposition

$$M = \bigoplus_{h \in \{h_i + \mathbb{Z}, i=1,...,k\}} M_h, \ h_i \in \mathbb{C}, \text{ for some } k \in \mathbb{N}$$

$$M_h = \{v \in M : (L(0) - hI)^m v = 0, \text{ for some } m\}, \ \dim(M_h) < +\infty.$$  

Here $M_h$ denotes the generalized homogeneous subspace of $M$ of generalized weight $h$.

It is important to mention that we use the definition of weak module as in [DLM], so that $M$ carries an action of the full Virasoro algebra and not only the $L(-1)$-operator. Hence, for every $v \in V_i$ and $m \in \mathbb{Z}$ we have

$$v_m M_h \subseteq M_{h+i-m-1}.$$  

The category of logarithmic $V$-modules $\text{LOG}$ is essentially just a different name for the category $\mathcal{O}$ introduced in [DLM]. In particular every logarithmic module is admissible [DLM]. We say that a weak $V$-module is a generalized logarithmic module if it decomposes into (not necessarily finite-dimensional) generalized $L(0)$-eigenspaces. Furthermore, we will say that a logarithmic $V$-module $M$ is genuine if under the action of $L(0)$ it admits at least one Jordan block of size 2 or more.

We recall the definition of logarithmic intertwining operators from [M1]. More precisely, here we are using a slightly more general definition following [HLZ]. In what follows $\log(x)$ is just a formal variable satisfying $\frac{d}{dx} \log(x) = \frac{1}{x}$. 

Definition 1.2. A logarithmic intertwining operator among a triple of logarithmic $V$-modules $W_1$, $W_2$ and $W_3$ is a linear map

\[ \mathcal{Y}(\cdot, x) : W_1 \otimes W_2 \rightarrow W_3 \{ x \} \{ \log(x) \} \]

satisfying:

(i) (Truncation condition) For every $w_1 \in W_1$, $w_2 \in W_2$ and $\alpha \in \mathbb{C}$,

\[ (w_1)^{(k)}_{\alpha+m} w_2 = 0, \text{ for } m \in \mathbb{N}, \quad m > 0. \]

(ii) (Translation invariance) For every $w_1 \in W_1$,

\[ [L(-1), \mathcal{Y}(w_1, x)] = \frac{d}{dx} \mathcal{Y}(w_1, x). \]

(iii) (Jacobi identity) For every $w_i \in W_i$, $i = 1, 2$ and $v \in V$, we have

\[ x_0^{-1} \frac{\delta \left( x_1 - x_2 \right)}{x_0} \mathcal{Y}(v, x_1) \mathcal{Y}(w_1, x_2) w_2 - x_0^{-1} \frac{\delta \left( -x_2 + x_1 \right)}{x_0} \mathcal{Y}(w_1, x_2) \mathcal{Y}(v, x_1) w_2 = x_2^{-1} \frac{\delta \left( x_1 - x_0 \right)}{x_2} \mathcal{Y}(v, x_0) w_1, x_2) w_2. \]

We will denote the vector space of logarithmic intertwining operators among $W_1$, $W_2$ and $W_3$ by

\[ I \left( \begin{array}{c} W_3 \\ W_1 \\ W_2 \end{array} \right). \]

We say that a logarithmic intertwining operator is genuine if $(w_1)^{(k)}_{\alpha}$ is nonzero for some $k \geq 1$ and some $w_1 \in W_1$. A genuine logarithmic intertwining operator of type $\left( \begin{array}{c} W_3 \\ W_1 \\ W_2 \end{array} \right)$ is called hidden if two modules $W_i$ and $W_j$ are ordinary $V$-modules, where $\{ i, j \} \subset \{ 1, 2, 3 \}$.

Definition 1.3. A logarithmic intertwining operator $\mathcal{Y}$ of type $\left( \begin{array}{c} W_3 \\ W_1 \\ W_2 \end{array} \right)$ is said to be strong if

\[ \text{depth}(\mathcal{Y}) := \sup \{ k : (w_1)^{(k)}_{\alpha} w_2 \neq 0, \alpha \in \mathbb{C}, w_1 \in W_1, w_2 \in W_2, k \in \mathbb{N} \} \]

is finite.

In other words, for strong logarithmic intertwining operators the powers of $\log(x)$ are globally bounded.

Definition 1.4. A logarithmic intertwining operator $\mathcal{Y}$ is said to be locally strong if

\[ \mathcal{Y}(w_1, x) \in \text{Hom}(W_2, W_3) \{ x \} \{ \log(x) \}, \text{ for every } w_1 \in W_1. \]

The previous definition was used in [Milas 1] as the definition of logarithmic intertwining operators. In this paper we shall only study strong intertwining operators. From the previous definitions we clearly have a chain of embeddings

\[ I_{\text{ord}} \left( \begin{array}{c} W_3 \\ W_1 \\ W_2 \end{array} \right) \subseteq I_{\text{st}} \left( \begin{array}{c} W_3 \\ W_1 \\ W_2 \end{array} \right) \subseteq I_{\text{lst}} \left( \begin{array}{c} W_3 \\ W_1 \\ W_2 \end{array} \right) \subseteq I \left( \begin{array}{c} W_3 \\ W_1 \\ W_2 \end{array} \right). \]
where $I_{st}$, $I_{lst}$ and $I_{ord}$ stand for the vector space of strong, locally strong and ordinary intertwining operators, respectively.

**Proposition 1.5.**

(i) Every strong logarithmic intertwining operator $\mathcal{Y}$ of type $(W_3 W_1 W_2)$ defines an ordinary intertwining operator of the same type.

(ii) Every locally strong logarithmic intertwining operator $\mathcal{Y}$ among a triple of finitely generated modules is strong.

**Proof.** Let $k = \text{depth}(\mathcal{Y})$. From

$$\mathcal{Y}(w, x) = \sum_{i=0}^{k} \mathcal{Y}^{(i)}(w, x) \log^{i}(x)$$

it is clear that the truncation condition and Jacobi identity hold for $\mathcal{Y}^{(k)}$. From $\frac{d}{dx} \log^{k}(x) = \frac{k}{x} \log^{k-1}(x)$ it follows that

$$[L(-1), \mathcal{Y}^{(k)}(w, x)] = \frac{d}{dx} \mathcal{Y}^{(k)}(w, x).$$

To prove (ii) it suffices to assume that $W_1$ is finitely generated. Let $\{w_{1,1}, \ldots, w_{1,m}\}$ be a generating set of $W_1$ so that for every $w_1 \in W_1$, there exist $v_i \in V$ and $n_i \in \mathbb{Z}$ such that $w_1 = \sum_{i=1}^{k} (v_i)_{n_i} w_{1,i}$. Let

$$\mathcal{Y}(w_{1,i}, x) = \sum_{i=1}^{r_i} \mathcal{Y}^{(i)}(w_{1,i}, x) \log^{i}(x), \ r_i \in \mathbb{N}.$$

The Jacobi identity now gives

$$\mathcal{Y}(Y(v_i, x_0) w_{1,i}, x) = \sum_{i=1}^{r_i} \mathcal{Y}^{(i)}(w_{1,i}, x) \log^{i}(x), \ r_i \in \mathbb{N}.$$

After we take $\text{Coeff}_{x^{-n_i-1}}$ in (1.5) we see that the powers of $\log(x)$ in $\mathcal{Y}(v_i, x_0) w_{1,i, x}$ are bounded by the highest power of $\log(x)$ in $\mathcal{Y}(w_{1,i}, x)$. Because of the finiteness of the generating set $W_1$, the powers of $\log(x)$ in $\mathcal{Y}$ are globally bounded.

**Remark 1.** In the previous proposition we considered a canonical map

$$I_{st} \left( \begin{array}{c} W_3 \\ W_1 W_2 \end{array} \right) \rightarrow I_{ord} \left( \begin{array}{c} W_3 \\ W_1 W_2 \end{array} \right),$$

sending a strong logarithmic intertwining operator to its "top" (or the depth) component. This map is clearly surjective, but it is far from being injective (see Lemma 5.4). Therefore, in general, the logarithmic fusion rules (i.e., $\dim I^{(W_3 W_1 W_2)}$) are not the same as the nonlogarithmic fusion rules (i.e., $\dim I^{ord (W_3 W_1 W_2)}$).

**Remark 2.** It is tempting to relax the condition in (1.1) and assume instead that

$$\mathcal{Y}(\cdot, x) : W_1 \otimes W_2 \rightarrow W_3 \{x\}\{\log(x)\}.$$ 

We will show (cf. Section 8) that there is a downside for doing that. One could even put further restriction on (1.1) and force a freshman calculus "formula"

$$e^{\log(x)} = x.$$
This identity, in our formal variable setup, is unnatural and it should be avoided. The main problem is that $e^{\log(x)} - x$ is a unit in the formal ring $\mathbb{C}[[x, \log(x)]]$.

2. An extended Heisenberg Lie algebra

Let $\mathfrak{h}$ be a finite-dimensional complex abelian Lie algebra with an inner product $(\cdot, \cdot)$. Consider the central extension of the affinization of $\mathfrak{h}$, denoted by $\hat{\mathfrak{h}}$, generated by $\hat{\mathfrak{h}}(n) := \mathfrak{h} \otimes t^n$, $n \in \mathbb{Z}$, $h \in \mathfrak{h}$, with the bracket relations:

$$[a(m), b(n)] = m(a, b)\delta_{m+n,0}C,$$

$$[C, h(m)] = 0,$$

where $a, b \in \mathfrak{h}$, $m, n \in \mathbb{Z}$ and $C$ is the central element.

For purposes of this paper we will assume that $\dim(\mathfrak{h}) = 1$ with a fixed unit vector $h$, so that

$$[h(m), h(n)] = m\delta_{m+n,0}C.$$ 

The Lie algebra $\hat{\mathfrak{h}}$ is an example of an extended Heisenberg Lie algebra (of course, the Heisenberg Lie algebra associated with $\mathfrak{h}$ does not involve $\mathfrak{h}(0)$). Let

$$\hat{\mathfrak{h}} = \hat{\mathfrak{h}}_{<0} \oplus \hat{\mathfrak{h}}_{>0} \oplus \mathbb{C}h(0) \oplus \mathbb{C}C,$$

where $\hat{\mathfrak{h}}_{<0}$ and $\hat{\mathfrak{h}}_{>0}$ are defined as usual. Denote by $\mathcal{C}_k$ the category of restricted $\mathbb{Z}$-graded, $\hat{\mathfrak{h}}$-modules of level $k$, i.e., the category of $\mathbb{Z}$-graded modules $W = \coprod_{n \in \mathbb{Z}} W_n$,

$$h(n)W_m \subseteq W_{m-n}, \quad \text{for every } m, n \in \mathbb{Z},$$

such that there exists $N \in \mathbb{N}$, so that $W_n = 0$ for $n < N$, and the central element $C$ acts as the multiplication with $k$.

Let us denote by $M(k) \equiv \mathcal{U}(\hat{\mathfrak{h}}_{<0}) \cdot 1$ essentially unique irreducible lowest weight module of level $k$, where $h(n) \cdot 1 = 0$, for $n \geq 0$ and the grading is the obvious one (see [FLM]).

Then we have a version of the Stone-Von Neumann theorem for the extended Heisenberg algebra $\hat{\mathfrak{h}}$:

**Lemma 2.1.** Let $W$ be a restricted $\hat{\mathfrak{h}}$-module of level $k$. Then

$$W \cong M(k) \otimes \Omega(W),$$

where $\Omega(W) = \{w \in W : h(n)w = 0, n > 0\}$ (the vacuum space of $W$) is $\mathfrak{h}$-stable.

**Proof:** It is known (see Theorem 1.7.3 in [FLM], for instance) that every restricted module for the Heisenberg algebra $\hat{\mathfrak{h}} = \hat{\mathfrak{h}} \setminus \mathbb{C}h(0)$ admits a decomposition (2.8) where $\Omega(W)$ is the vacuum space of $W$. Now, $h(0)$ commutes with $h(n)$ for every $n$, hence it preserves the vacuum space $\Omega(W)$.

3. Logarithmic $M(1)_a$-modules

The Heisenberg vertex operator algebra is omnipresent in conformal field theory and representation theory. Despite of its simplicity, it is the main tool for building (more interesting) rational vertex operator algebras (e.g., lattice VOAs [D], [FLM]). Moreover, Heisenberg vertex operator algebras also contain many interesting subalgebras (e.g., $W$-algebras [FKRW]).
It is well-known (cf. [FrB], [FLM], [K2]) that \((M(1), Y, 1, \omega_a)\) has a vertex operator algebra structure of central charge \(c = 1 - 12a^2\), where
\[
\omega_a = \frac{h(-1)^2 1}{2} + ah(-2)1.
\]
This VOA will be denoted by \(M(1)_a\). It is known that \(M(1)_a\) has infinitely many inequivalent irreducible modules, which can be easily classified [LL]. For every highest weight irreducible \(M(1)_a\)-module \(W\) there exists \(\lambda \in \mathbb{C}\) such that \(W \cong M(1)_a \otimes \Omega_\lambda\), where \(\Omega_\lambda\) is one-dimensional and \(h(0)\) acts as the multiplication with \(\lambda\). Such a module will be denoted by \(M(1, \lambda)_a\). The restricted \(\mathfrak{h}\)-modules are essentially logarithmic \(M(1)_a\)-modules. Since \(M(1)_a\) is at the same time \(\hat{\mathfrak{h}}\), \(M(1)_a\) and Virasoro algebra module, we stress some differences under taking the contragradient module. If we denote by \(M(1, \lambda)^*\) the dual \(\mathfrak{h}\)-module of \(M(1, \lambda)\), under the standard anti-involution \(h(n) \mapsto -h(-n)\), then we have
\[
M(1, \lambda)^* \cong M(1, -\lambda),
\]
viewed as \(\hat{\mathfrak{h}}\). On the other hand, if we denote by \(M(1, \lambda)'_a\) the contragradient \(M(1)_a\)-module (or the dual Virasoro algebra module) of \(M(1, \lambda)_a\) (cf. [FHL]), then we have
\[
M(1, \lambda)'_a \cong M(1, 2a - \lambda)_a.
\]

(3.9)

Let us also mention the automorphism \(\tau\) of \(M(1)_a\), uniquely determined by \(\tau(h(-n_1) \cdots h(-n_k)1) = (-1)^k h(-n_1) \cdots h(-n_k)1\). This map does not preserve the Virasoro element \(\omega_a\), unless \(a = 0\).

As in Lemma 2.1 we now describe all logarithmic modules for the vertex operator algebra \(M(1)_a\).

**Proposition 3.1.** Suppose that \(W\) is a logarithmic \(M(1)_a\)-module. Then, viewed as an \(\mathfrak{h}\)-module, \(W \cong M(1)_a \otimes \Omega(W)\).

**Proof:** Every weak \(M(1)_a\)-module carries a level one representation of \(\mathfrak{h}\) via the expansion
\[
Y(h(-1)1, x) = \sum_{n \in \mathbb{Z}} h(n)x^{-n-1}.
\]
Because \(W\) is logarithmic, it is also a restricted \(\mathbb{Z}\)-graded \(\mathfrak{h}\)-module. An application of Lemma 2.1 yields \(W = M(1)_a \otimes \Omega(W)\), for some vacuum space \(\Omega(W)\).

**Corollary 3.2.** Suppose that \(\left(\frac{h^2(0)}{2} - ah(0)\right)\vert_\Omega\) admits a Jordan block of size at least 2. Then \(M(1)_a \otimes \Omega\) is a genuine logarithmic \(M(1)_a\)-module.

In the case of irreducible \(M(1)\)-modules it is possible to classify all intertwining operators and the corresponding fusion rules. The following result seems to be known (see [FrB] for instance).

**Proposition 3.3.** Let \(M(1, \lambda)_a\), \(M(1, \tau)_a\) and \(M(1, \nu)_a\) be three (ordinary) \(M(1)_a\)-modules. Then the vector space of ordinary intertwining operators \(I_{\text{ord}}(M(1, \nu)_a M(1, \tau)_a)\) is nontrivial if and only if \(\nu = \lambda + \tau\). If so, \(I_{\text{ord}}(M(1, \lambda + \tau)_a M(1, \tau)_a)\) is one-dimensional.

In Section 5 we will substantially generalize this result.
4. GENERALIZED LOGARITHMIC INTERTWINING OPERATORS

Here we show that certain spaces of "logarithmic operators" give rise to (generalized) logarithmic modules. In particular, this construction will be useful for classification of logarithmic intertwining operators among certain triples of logarithmic $M(1)_c$-modules. We will closely follow Li’s work [Li1] (see also [Li2]). The following definition is a logarithmic version of Definition 6.1.1 in [Li2] (see also [Li1]).

**Definition 4.1.** Let $V$ be a vertex operator algebra and $W_1$ and $W_2$ a pair of logarithmic $V$-modules. An operator valued formal series

$$\phi(x) = \sum_{\alpha \in \mathbb{C}} \sum_{n \in \mathbb{N}} \phi^{(n)}(x) \log^n(x)x^{-\alpha-1} \in \text{Hom}(W_1, W_2) \{x\}[[\log(x)]]$$

is called a generalized logarithmic intertwining operator of generalized weight $h$ if it satisfies the following conditions:

(i) For every $w \in W_1$ we have

$$\phi(x)w \in W_2 \{x\} [\log(x)],$$

and for every $i$

$$\phi^{(i)}_{\alpha+m}w = 0, \text{ for } m \text{ large enough.}$$

(ii) $[L(-1), \phi(x)] = \frac{d}{dx} \phi(x)$,

(iii) For every $v \in V$, there is a positive integer $n_v$ such that

$$(x_1 - x_2)^{n_v} Y(v, x_1) \phi(x_2) = (x_1 - x_2)^{n_v} \phi(x_2) Y(v, x_1).$$

(iv) There exists $k \in \mathbb{N}$ such that

$$\left(\text{ad}_{L(0)} - x \frac{d}{dx} - h \right)^k \phi(x) = 0,$$

where $\text{ad}_{L(0)} \phi(x) = [L(0), \phi(x)]$.

We will denote by $G_h(W_1, W_2)$ the vector space of generalized logarithmic intertwining operators of generalized weight $h$ and by $G_{\log}(W_1, W_2) = \bigoplus_{h \in \mathbb{C}} G_h(W_1, W_2)$ the vector space of generalized logarithmic intertwining operators.

**Proposition 4.2.** The vector space $G_{\log}(W_1, W_2)$ has a generalized logarithmic $V$-module structure.

**Proof.** We will closely follow Section 6 in [Li2] (or Theorem 7.1.6 in [Li1]), with appropriate modifications due to logarithms. As in Definition 6.1.2 in [Li2] we let

$$Y(u, x_0) \circ \phi(x_2) = \text{Res}_{x_1} \left( x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) Y(u, x_1) \phi(x_2) - x_0^{-1} \delta \left( -\frac{x_2 + x_1}{x_0} \right) \phi(x_2) Y(u, x_1) \right).$$

The $L(-1)$-property is then proven as in [Li2]. Let $a \in V$ be a homogeneous vector of weight $\text{wt}(a)$ and $\phi(x)$ a generalized logarithmic intertwining operator of generalized weight $h$, so that

$$(\text{ad}_{L(0)} - x \frac{d}{dx} - h)^k \phi(x) = 0,$$
for some \( k \in \mathbb{N} \). We claim that
\[
\left( \text{ad}_{L(0)} - h - \text{wt}(a) - x_0 \frac{\partial}{\partial x_0} - x_2 \frac{\partial}{\partial x_2} \right)^k Y(a, x_0) \circ \phi(x_2) = 0.
\]
By using Lemma 7.1.4 in \([\text{Li}1]\) we get the identity
\[
[L(0), Y(a, x_0) \circ \phi(x_2)]
= \left( \text{wt}(a) + x_0 \frac{\partial}{\partial x_0} + x_2 \frac{\partial}{\partial x_2} \right) Y(a, x_0) \circ \phi(x_2) + Y(a, x_0) \circ \left( \text{ad}_{L(0)} \phi(x_2) - x_2 \frac{\partial}{\partial x_2} \phi(x_2) \right).
\]
Thus
\[
\left( \text{ad}_{L(0)} - \text{wt}(a) - h - x_0 \frac{\partial}{\partial x_0} - x_2 \frac{\partial}{\partial x_2} \right) Y(a, x_0) \circ \phi(x_2) = Y(a, x_0) \circ \left( \text{ad}_{L(0)} - x_2 \frac{\partial}{\partial x_2} - h \right) \phi(x_2).
\]
From the previous formula and property (iv) in Definition 4.1 the claim now follows. Finally, the Jacobi identity is the verbatim repetition of the proof of Theorem 6.1.7 in \([\text{Li}2]\).

In parallel with the ordinary case, generalized logarithmic intertwining operators are closely related to logarithmic intertwining operators.

**Lemma 4.3.** Let \( W \) be a logarithmic \( V \)-module and \( \phi \in \text{Hom}_V(W, G_{\log}(W_1, W_2)) \). Define
\[
I_\phi(\cdot, x) : W \rightarrow \text{Hom}(W_1, W_2) \{ x \} [[\log(x)]],
I_\phi(w, x) = \phi(w)(x).
\]
Then \( I_\phi(\cdot, x) \) is a logarithmic intertwining operator of type \( \frac{W_2}{W_1} \).

**Proof.** Firstly,
\[
I_\phi(w, x)w_1 \in W_2 \{ x \} [[\log(x)]].
\]
Now we have to check properties (i)-(iii) in Definition \([\text{Li}2]\). Let
\[
I_\phi(w, x) = \sum_{\alpha \in \mathbb{C}} \sum_{n \geq 0} w_\alpha^{(n)} \log(x)^n x^{-\alpha - 1}.
\]
Then the truncation property
\[
w_\alpha^{(i)} w_{\alpha+m} = 0 \text{ for } m >> 0
\]
is just a consequence of Definition \([\text{Li}1]\)(i). Similarly, the \( L(-1) \)-property holds. The Jacobi identity is then proven as in formula (6.2.3) in \([\text{Li}2]\).

The map between \( \text{Hom}_V(W, G_{\log}(W_1, W_2)) \) and \( I \left( \frac{W_2}{W_1} \right) \) defined in the previous lemma is clearly injective. Conversely, to every logarithmic intertwining operator \( \mathcal{Y} \) of type \( \frac{W_2}{W_1} \) we associate a map \( \psi \in \text{Hom}_V(W, G_{\log}(W_1, W_2)) \), via
\[
\psi(w) = \mathcal{Y}(w, x), \ w \in W.
\]
Combined together we obtain:

**Theorem 4.4.** Let \( W, W_1 \) and \( W_2 \) be logarithmic \( V \)-modules. Then the vector space \( I \left( \frac{W_2}{W_1} \right) \) is naturally isomorphic to \( \text{Hom}_V(W, G_{\log}(W_1, W_2)) \).

The next result is a logarithmic analogue of Li’s Theorem 7.3.1 \([\text{Li}1]\) (after Tsuchiya and Kanie who proved an important spacial case \( V = \text{sl}_2(k, 0), k \in \mathbb{N} \)). We do not need this theorem in full generality so we just focus on a special case \( V = M(1)_a \).
**Theorem 4.5.** Let $W_1$ and $W_2$ be two logarithmic $M(1)_a$-modules. Let $\Omega$ be a finite-dimensional $\mathfrak{h}$-module and $\mathcal{Y}(\cdot, x)$ a linear map from $\Omega$ to $\text{Hom}(W_1, W_2)[[[\log(x)]]\{x\}$ satisfying the truncation condition, the $L(-1)$-property and

\[(x_1 - x_2)^{(a)-1}Y(a, x_1)\mathcal{Y}(w, x_2) - (-x_2 + x_1)^{(a)-1}\mathcal{Y}(w, x_2)Y(a, x_1)\]

(4.10)

for every homogeneous $a \in V$ and $w \in \Omega$. Then $\mathcal{Y}$ extends uniquely to an intertwining operator of type $\left(W_2, (M(1)_a \otimes \Omega) W_1 \right)$.

**Proof.** From (4.10) it follows that $\mathcal{Y}(u, x)$ is a generalized intertwining operator for every $u \in \Omega$. Moreover, $\Omega$ is also an $A(M(1)_a)$-module (where the Zhu’s algebra $A(M(1)_a)$ is just the polynomial ring in one variable). Now we may proceed as in [Li1]. The linear map $\mathcal{Y}$ extends to an intertwining map from $M(1)_a \otimes \Omega$ (The universal Verma $M(1)_a$-module $\bar{M}(\Omega)$ appearing in Li’s theorem is $M(1)_a \otimes \Omega$.) If there is another intertwining operator $\mathcal{Y}'$ extending $\mathcal{Y}|_{\Omega}$, then $\mathcal{Y} - \mathcal{Y}'$ would be trivial on $\Omega$. But $M(1)_a \otimes \Omega$ is generated by $\Omega$, so $\mathcal{Y} = \mathcal{Y}'$.

5. LOGARITHMIC INTERTWining OPERATORS AMong $M(1)_a$-mODULES

In this section we will give a sharp upper bound on the dimension of the vector space of strong logarithmic intertwining operators among certain logarithmic $M(1)_a$-modules. From now on we shall assume that every logarithmic $M(1)_a$-module is of the form $M(1)_a \otimes \Omega$, where $\Omega$ is a finite-dimensional $\mathfrak{h}$-module such that $(h(0) - \lambda)^n|_{\Omega} = 0$ for some $\lambda$ and $n$ large enough. Then Proposition 1.10 in [M1] implies that every intertwining operator among a triple of such modules is strong. If we remove the finite-dimensionality condition on $\Omega$ it is not hard to construct logarithmic intertwining operators that are neither strong nor locally strong.

We prove a few lemmas first.

**Lemma 5.1.** Suppose that $(h(0) - \lambda)^{m_1}|_{\Omega_1} = 0$ and $(h(0) - \nu)^{m_2}|_{\Omega_2} = 0$ for some $m_i \in \mathbb{N}$, $i = 1, 2$. Let $\mathcal{Y} \in I \left(W, (M(1)_a \otimes \Omega_1) M(1)_a \otimes \Omega_2 \right)$. Then for every $w_1 \in M(1)_a \otimes \Omega_1$ and $w_2 \in M(1)_a \otimes \Omega_2$ we have

$$(h(0) - \lambda - \nu)^{m_1 + m_2 - 1}\mathcal{Y}(w_1, x)w_2 = 0.$$ 

Moreover, if $\Omega_1$ and $\Omega_2$ are one-dimensional $\mathfrak{h}$-modules, then there are no genuine logarithmic intertwining operators of type $\left(W, (M(1)_a \otimes \Omega_1) M(1)_a \otimes \Omega_2 \right)$.

**Proof.** Let $w_1 \in M(1)_a \otimes \Omega_1$ and $w_2 \in M(1)_a \otimes \Omega_2$ so that $(h(0) - \lambda)^{m_1} \cdot w_1 = (h(0) - \nu)^{m_2} \cdot w_2 = 0$. From the Jacobi identity it follows that

\[\sum_{i_1 \geq 0, i_2 \geq 0 \atop i_1 + i_2 = m_1 + m_2 - 1} (m_1 + m_2 - 1 \choose i_1) \mathcal{Y}((h(0) - \lambda)^{i_1}w_1, x)(h(0) - \nu)^{i_2}w_2.

Now, it is easy to see that every term on the right hand side is zero.
If $m_1 = m_2 = 1$ then $m_1 + m_2 - 1 = 1$, so $h(0)$ is diagonalizable on the image of $\mathcal{Y}$, but so is $L(0) = \frac{1}{2}h(0)^2 - ah(0) + \sum_{n > 0} h(-n)h(n)$. Now, apply Proposition 1.10 in [M1].

Let us recall (cf. [M1], [HLZ]) that for every $w_i \in \mathcal{W}_i$ of generalized weight $h_i$, $i = 1, 2$ and every $\mathcal{Y} \in I_{\mathcal{W}_1 \mathcal{W}_2}$, the vector

$$(w_1)^{(k)} w_2$$

is of generalized weight $h_1 + h_2 - \alpha - 1$ (independently of $k$).

**Lemma 5.2.** Suppose that $(h(0) - \lambda)^{m_1} |_{\Omega_1} = (h(0) - \nu)^{m_2} |_{\Omega_2} = 0$ and let $\mathcal{Y} \in I_{(M(1)_a \otimes \Omega_1)(M(1)_a \otimes \Omega_2)}$. Then

$$\mathcal{Y}(w_1, x)w_2 \in x^{\mu \lambda} W((x)) \oplus x^{\mu \lambda} \log(x) W((x)) \oplus \cdots \oplus x^{\mu \lambda} \log(x)^{m_1 + m_2 - 2} W((x)),$$

for every $w_i \in M(1) \otimes \Omega_i$, $i = 1, 2$. Equivalently,

$$\text{depth}(\mathcal{Y}) \leq m_1 + m_2 - 2.$$

**Proof.** The proof goes by induction on $m_1 + m_2 \geq 2$. For $m_1 + m_2 = 2$ the statement holds by the previous lemma. From the same lemma and

$$(5.11) \quad L(0)|_{\Omega_1} = \frac{1}{2}h(0)^2 - ah(0)$$

it follows that $L(0)|_{M(1)_a \otimes \Omega_1}$ (resp. $L(0)|_{M(1)_a \otimes \Omega_2}$) does not admit a Jordan block of size larger than $m_1$ (resp. $m_2$). For the induction step we apply the bracket relation between $L(0)$ and $\mathcal{Y}(w_1, x)$ and an elementary ODE argument as in Proposition 1.10, [M1].

**Lemma 5.3.** Let $\mathcal{Y}$ and $\Omega_i$ be as in Lemma 5.2. Suppose further that $\lambda = 0$. Then for every $w_i \in M(1)_a \otimes \Omega_i$, $i = 1, 2$ we have

$$(5.12) \quad \mathcal{Y}(w_1, x)w_2 \in W((x)) \oplus \log(x) W((x)) \oplus \cdots \oplus \log^{m_1-1}(x) W((x)).$$

If $\nu = 0$, then (5.12) holds with $m_1$ replaced by $m_2$.

**Proof.** By using the isomorphism $I_{(W_1 \mathcal{W}_2)} \cong I_{(W_2 \mathcal{W}_1)}$, it is sufficient to consider the $\lambda = 0$ case, so that $h(0)|_{\Omega_1}$ is a nilpotent operator. We prove the formula (5.12) by induction on $m_1$. Firstly, let $m_1 = 1$, so $M(1)_a \otimes \Omega_1 \cong M(1)_a$. In this case we have to show that there are no genuine logarithmic intertwining operators (i.e., $\mathcal{Y}(w_1, x)w_2 \in W((x))$). From the $L(-1)$-property and $L(-1) 1 = 0$, it follows that

$$\langle w'_3, \mathcal{Y}(L(-1) 1, x)w_2 \rangle = \frac{d}{dx} \langle w'_3, \mathcal{Y}(1, x)w_2 \rangle = 0, \quad w'_3 \in W_3'.$$

Thus $\mathcal{Y}(1, x)$ is a constant term (operator) and it does not involve powers of $\log(x)$. Similarly, from the Jacobi identity, it follows that $\mathcal{Y}(w, x)w_1$ does not involve nonzero powers of $\log(x)$ for every $w \in M(1)_a$ and $w_1 \in M(1)_a \otimes \Omega_2$.

Now, suppose that (5.12) holds for every $m < m_1$. For $w_1 \in \Omega_1$, and $w_2 \in M(1)_a \otimes \Omega_2$ and $w'_3 \in \Omega(W') = \Omega(W)'$, we clearly have

$$\langle w'_3, \mathcal{Y}(L(-1)w_1, x)w_2 \rangle = \frac{d}{dx} \langle w'_3, \mathcal{Y}(w_1, x)w_2 \rangle.$$

On the other hand, the Jacobi identity gives

$$\mathcal{Y}(L(-1)w_1, x) = \mathcal{Y}(h(-1)h(0)w_1, x) = \frac{d}{dx} \mathcal{Y}(h(0)w_1, x).$$
Proof. The truncation condition and the Jacobi identity clearly hold for prove the $L$-property. If we distribute the $L$-property for $\mathcal{V}(\cdot, x)$ among $\mathcal{V}^{(i)}(\cdot, x)$ we get

$$[L(-1), \mathcal{V}^{(k)}(w, x)] = \frac{d}{dx} \mathcal{V}^{(k)}(w, x),$$

and

$$[L(-1), \mathcal{V}^{(i)}(w, x)] = \frac{d}{dx} \mathcal{V}^{(i)}(w, x) + \frac{i + 1}{x} \mathcal{V}^{(i+1)}(w, x),$$

for $i \leq k - 1$. Thus

$$[L(-1), \mathcal{V}_{-1}(w, x)] = \sum_{i=0}^{k-1} (i + 1) [L(-1), \mathcal{V}^{(i+1)}(w, x)] \log^i(x)$$

$$= k \frac{d}{dx} \left( \mathcal{V}^{(k)}(w, x) \right) \log^{k-1}(x) + \sum_{i=0}^{k-2} (i + 1) \left( \frac{d}{dx} \mathcal{V}^{(i+1)}(w, x) + \frac{i + 2}{x} \mathcal{V}^{(i+2)}(w, x) \right) \log^i(x)$$

$$= \sum_{i=0}^{k-1} (i + 1) \frac{d}{dx} \left( \mathcal{V}^{(i+1)}(w, x) \log^i(x) \right).$$

The following result gives a sharp upper bound on the depth of (strong) logarithmic intertwining operators among a triple of logarithmic $M(1)_{\alpha}$-modules.

Theorem 5.5. Let $h(0)|_{\Omega_1}$ and $h(0)|_{\Omega_2}$ such that

$$(h(0) - \lambda)^{m_1}|_{\Omega_1} = (h(0) - \nu)^{m_2}|_{\Omega_2} = 0,$$

and

$$(h(0) - \lambda)^{m_1-1}|_{\Omega_1} \neq 0, (h(0) - \nu)^{m_2-1}|_{\Omega_2} \neq 0,$$
for some $\lambda, \nu \in \mathbb{C}$. Then we have

(i) For every $\mathcal{Y} \in I \left( M(1)_a \otimes \Omega, M(1)_a \otimes \Omega \right)$,

$$0 \leq \text{depth}(\mathcal{Y}) \leq k = \begin{cases} 
    m_1 + m_2 - 2 & \text{for } \lambda \nu \neq 0, \\
    m_1 - 1 & \text{for } \lambda = 0 \text{ and } \nu \neq 0, \\
    m_2 - 1 & \text{for } \lambda \neq 0 \text{ and } \nu = 0, \\
    \min(m_1 - 1, m_2 - 1) & \text{for } \lambda = \nu = 0
  \end{cases}$$

(ii) There exists a canonical embedding

$$I \left( M(1)_a \otimes \Omega, M(1)_a \otimes \Omega \right) \hookrightarrow \text{Hom}_h(\Omega \otimes \Omega, \Omega(W))^{\otimes k},$$

where $k$ is as in (5.14).

(iii) The range of depth$(\mathcal{Y})$ in (5.14) is the best possible. More precisely, for every nonnegative integer $m \leq k$ there exists a logarithmic intertwining operator of depth exactly $m$.

Proof. Here we prove (i) and (ii) only. We will complete the proof of (iii) in Section 7.

The assertion (i) follows from Lemma 5.2 and 5.3. Let $\mathcal{Y} \in I \left( M(1)_a \otimes \Omega, M(1)_a \otimes \Omega \right)$. Then $\mathcal{Y}$ admits a canonical expansion

$$\mathcal{Y}(w_1, x) = \sum_{i=0}^{m_1+m_2-2} \mathcal{Y}^{(i)}(w_1, x) \log^i(x),$$

for every $w_1 \in M(1)_a \otimes \Omega$. By Lemma 5.2 for every $i$,

$$\mathcal{Y}^{(i)}(w_1, w_2) w_2 \in x^{\lambda \nu} W((x)).$$

Also, for $w_1$ and $w_2$ satisfying $(L(0) - h_1)^{m_1} w_1 = (L(0) - h_2)^{m_2} w_2 = 0$, the vector $(w_1)^{(i)} w_2$ is homogeneous of generalized weight $h_1 + h_2 - \alpha - 1$ for every $i$. Suppose that $w_1 \in \Omega_1$ and $w_2 \in \Omega_2$ are of generalized weight $\lambda^2 - a \lambda$ and $\nu^2 - a \nu$, respectively. Then for every $i$,

$$w_1 \otimes w_2 \mapsto (w_1)^{(i)} \otimes (w_2).$$

defines an $h$-module map

$$F^{(i)}_{\mathcal{Y}} : \Omega_1 \otimes \Omega_2 \rightarrow \Omega(W).$$

We claim that

$$F : I \left( M(1)_a \otimes \Omega, M(1)_a \otimes \Omega \right) \rightarrow \text{Hom}_h(\Omega \otimes \Omega, \Omega(W))^{\otimes k},$$

$$F(\mathcal{Y}) = (F^{(0)}_{\mathcal{Y}}, \ldots, F^{(m_1+m_2-2)}_{\mathcal{Y}}),$$

defines an embedding. Suppose that $F^{(i)}_{\mathcal{Y}} = 0$ for every $i$. We have to show that $\mathcal{Y} \equiv 0$.

Firstly, we prove that $\mathcal{Y}(w_1, x) w_2 = 0$ for every $w_1 \in \Omega_1$ and $w_2 \in \Omega_2$. In order to prove that we observe first that the contragradient (or dual) module of $W$, denoted by $W'$, is again a logarithmic module generated by $\Omega(W') = \Omega(W)^t$. By the assumption we have

$$\langle w_3', \mathcal{Y}(w_1, x) w_2 \rangle = 0, \quad w_3 \in \Omega(W').$$
Furthermore, for every $w'_3 \in \Omega(W')$ and $n \geq 1$ we have
\[ (h(-n) \cdot w'_3, \mathcal{Y}(w_1, x)w_2) = -(w'_3, x^{-n} \mathcal{Y}(h(0)w_1, x)w_2) = 0. \]
Since $W'$ is generated by $h(n)$, $n \leq -1$ from $\Omega(W')$, the previous formula gives
\[ (w'_3, \mathcal{Y}(w_1, x)w_2) = 0, \]
for every $w'_3 \in W'$. Hence, $\mathcal{Y}(w_1, x)w_2 = 0$ for every $w_1 \in \Omega_1$ and $w_2 \in \Omega_2$. Finally, $M(1)_a \otimes \Omega_i$ is generated by $\Omega_i$, so $\mathcal{Y}(w_1, x)w_2 = 0$ for every $w_1 \in M(1)_a \otimes \Omega_i$, $i = 1, 2$. This proves the injectivity.

In Section 7 we will complete the proof of (iii). Now, let us assume that there exists $\mathcal{Y}$ and $W$ such that $\text{depth}(\mathcal{Y}) = k$ where $k$ is as in (5.14) (i.e., $\text{depth}(\mathcal{Y})$ reaches its upper bound). Then to construct $\mathcal{Y}_i$ with $\text{depth}(\mathcal{Y}_i) = k - i$, for $i = 1, \ldots, k$ we simply apply Lemma 5.4.

6. THE OPERATOR "$x^{h(0)}$"

Let us recall that every endomorphism of a finite-dimensional complex vector space admits a unique decomposition
\[ h(0) = h_s(0) + h_n(0), \]
where $h_s(0)$ and $h_n(0)$ are the semisimple and nilpotent part of $h(0)$, respectively such that $h_s(0)$ and $h_n(0)$ commute.

The following elementary fact will be of use in the next section: Let $\Omega$ be a finite dimensional vector space and $h(0)$ an endomorphism of $\Omega$. Then
\[ e^{\log(x)h_n(0)}x^{h_s(0)}, \]
is a solution of the ODE
\[ x \frac{d}{dx} A(x) = h(0)A(x), \]
where
\[ A(x) : V \longrightarrow V \{x\}[\log(x)], \]
is a linear map. The operator valued expression (6.16) is a replacement for $x^{h(0)}$ that (in the case when $h(0)$ is semisimple) appears on many places in the literature. Clearly, the "operator" $x^{h(0)}$ for $h(0)$ nonsemisimple is not well-defined.

7. LOGARITHMIC INTERTWINING OPERATORS FOR $M(1)_a$: THE PROOF OF THE EXISTENCE

In this section to every $T^{\Omega_1, \Omega_2}_{\Omega_1, \Omega_2} \in \text{Hom}_b(\Omega_1 \otimes \Omega_2, \Omega_3)$ we associate a linear map
\[ \mathcal{Y}(\cdot, x) : \Omega_1 \otimes W_2 \longrightarrow W_3 \{x\}[\log(x)], \]

\[ W_i = M(1)_a \otimes \Omega_i, \quad i = 2, 3, \]
with the following properties:
\[ [\mathcal{Y}(w, x), \mathcal{Y}(w, x)] = x^n \mathcal{Y}(h(0) \cdot w, x), \quad n \in \mathbb{Z}, \]
\[ [L(-1), \mathcal{Y}(w, x)] = x \frac{d}{dx} \mathcal{Y}(w, x), \]
for every $w \in \Omega_1$. We will construct this map explicitly. Our construction mimics the well-known formulas when $\Omega_i$ are all one-dimensional, which will be a special case of our construction.
Firstly, we fix a basis $B = \{w_1, \ldots, w_n\}$ for $\Omega_1$ in which $h(0)|_{\Omega_1}$ admits a Jordan form consisting of a single Jordan block so that
\begin{align}
    h(0) \cdot w_1 &= \lambda w_1, \\
    h(0) \cdot w_i &= \lambda w_i + w_{i-1}, \text{ for } 2 \leq i \leq n.
\end{align}
(7.19)

Let
\begin{align*}
\int^+ h(x) &= h(0) \log(x) + \sum_{m>0} \frac{h(m)x^{-m}}{-m}, \\
\int^- h(x) &= \sum_{m<0} \frac{h(m)x^{-m}}{-m}.
\end{align*}

Formal differentiation yields
\[
\frac{d}{dx} \left( \int^+ h(x) + \int^- h(x) \right) = \sum_{n \in \mathbb{Z}} h(n)x^{-n-1}.
\]

Let
\begin{align*}
E^+(\lambda, x) &= \exp \left( \sum_{m>0} \frac{\lambda h(m)x^{-m}}{-m} \right), \\
E^-(\lambda, x) &= \exp \left( \sum_{m<0} \frac{\lambda h(m)x^{-m}}{-m} \right).
\end{align*}

Here we use a slightly different notation compared with [LL]; $E^\pm(\lambda, x)$ are usually denoted by $E^\pm(-\lambda, x)$. The following lemma is easy to prove (see for instance [FLM], [K1], [LL], etc.).

**Lemma 7.1.**
\[
[L(-1), E^-(\lambda, x)] = \sum_{n \leq -2} \lambda h(n)x^{-n-1} E^-(\lambda, x),
\]
(7.20)
\[
[L(-1), E^+(\lambda, x)] = E^+(\lambda, x) \sum_{n \geq 0} \lambda h(n)x^{-n-1}.
\]

Similarly,
\[
[h(n), E^-(\lambda, x)E^+(\lambda, x)] = x^n \lambda E^-(\lambda, x)E^+(\lambda, x), \text{ for } n \neq 0,
\]
\[
[h(0), E^-(\lambda, x)E^+(\lambda, x)] = 0.
\]

Also,
\[
[h(0), T^{\Omega_3}_{\Omega_2}(w)] = T^{\Omega_3}_{\Omega_2}(h(0) \cdot w), \text{ for } w \in \Omega_1,
\]
so that
\[
[h(0), E^-(\lambda, x)E^+(\lambda, x)T^{\Omega_3}_{\Omega_2}(w)e^{\log(x)\lambda h_n(0)x^{\lambda h_n(0)}}\]
\[
= E^-(\lambda, x)E^+(\lambda, x)T^{\Omega_3}_{\Omega_2} (h(0) \cdot w)e^{\log(x)\lambda h_n(0)x^{\lambda h_n(0)}}.
\]
(7.21)
Lemma 7.2. For every $w \in \Omega_1$ we have

$$[L(-1), E^-(\lambda, x)E^+(\lambda, x)T_{\Omega_2}^{\Omega_3}(w)e^{\log(x)\lambda h_n(0)}x^{\lambda h_s(0)}]$$

$$= \left( \sum_{n \leq -2} \lambda h(n)x^{-n-1} \right) E^-(\lambda, x)E^+(\lambda, x)T_{\Omega_2}^{\Omega_3}(w)e^{\log(x)\lambda h_n(0)}x^{\lambda h_s(0)}$$

$$+ E^-(\lambda, x)E^+(\lambda, x)T_{\Omega_2}^{\Omega_3}(w)e^{\log(x)\lambda h_n(0)}x^{\lambda h_s(0)} \left( \sum_{n \geq 0} \lambda h(n)x^{-n-1} \right)$$

$$+ h(-1)E^-(\lambda, x)E^+(\lambda, x)T_{\Omega_2}^{\Omega_3}(h(0) \cdot w)e^{\log(x)\lambda h_n(0)}x^{\lambda h_s(0)}.$$  \hfill (7.22)

In particular if $h(0) \cdot w = \lambda w$, then

$$[L(-1), E^-(\lambda, x)E^+(\lambda, x)T_{\Omega_2}^{\Omega_3}(w)e^{\log(x)\lambda h_n(0)}x^{\lambda h_s(0)}] =$$

$$\frac{d}{dx} \left( E^-(\lambda, x)E^+(\lambda, x)T_{\Omega_2}^{\Omega_3}(w)e^{\log(x)\lambda h_n(0)}x^{\lambda h_s(0)} \right).$$  \hfill (7.23)

Proof. Let us recall that $L(-1) = \frac{1}{2} \sum_{n \in \mathbb{Z}} : h(-n - 1) h(n) :$, which does not depend on $a$. For simplicity let

$$A(\lambda, w, x) = E^-(\lambda, x)E^+(\lambda, x)T_{\Omega_2}^{\Omega_3}(w)e^{\log(x)\lambda h_n(0)}x^{\lambda h_s(0)}.$$ 

By using

$$[L(-1), T_{\Omega_2}^{\Omega_3}(w)] = h(-1)T_{\Omega_2}^{\Omega_3}(h(0) \cdot w).$$
we have

\[
\begin{align*}
&[L(-1), A(\lambda, w, x)] = \left( \sum_{n \leq -2} \lambda h(n)x^{-n-1} \right) E^-(\lambda, x)E^+(\lambda, x)T^{\Omega_3}_{\Omega_2}(w)e^{\log(x)\lambda h_n(0)x\lambda h_s(0)} \\
&+ E^-(\lambda, x)E^+(\lambda, x) \left( \sum_{n \geq 0} \lambda h(n)x^{-n-1} \right) T^{\Omega_3}_{\Omega_2}(w)e^{\log(x)\lambda h_n(0)x\lambda h_s(0)} \\
&+ E^-(\lambda, x)E^+(\lambda, x)h(-1)T^{\Omega_3}_{\Omega_2}(h(0) \cdot w)e^{\log(x)\lambda h_n(0)x\lambda h_s(0)} \\
&= \left( \sum_{n \leq -2} \lambda h(n)x^{-n-1} \right) E^-(\lambda, x)E^+(\lambda, x)T^{\Omega_3}_{\Omega_2}(w)e^{\log(x)\lambda h_n(0)x\lambda h_s(0)} \\
&+ \lambda x^{-1} E^-(\lambda, x)E^+(\lambda, x)T^{\Omega_3}_{\Omega_2}(h(0) \cdot w)e^{\log(x)\lambda h_n(0)x\lambda h_s(0)} \\
&- x^{-1} E^-(\lambda, x)E^+(\lambda, x)T^{\Omega_3}_{\Omega_2}(h(0) \cdot w)e^{\log(x)\lambda h_n(0)x\lambda h_s(0)} + \\
&h(-1)E^-(\lambda, x)E^+(\lambda, x)T^{\Omega_3}_{\Omega_2}(h(0) \cdot w)e^{\log(x)\lambda h_n(0)x\lambda h_s(0)} \\
&= \left( \sum_{n \leq -2} \lambda h(n)x^{-n-1} \right) E^-(\lambda, x)E^+(\lambda, x)T^{\Omega_3}_{\Omega_2}(w)e^{\log(x)\lambda h_n(0)x\lambda h_s(0)} \\
&+ E^-(\lambda, x)E^+(\lambda, x)T^{\Omega_3}_{\Omega_2}(w)e^{\log(x)\lambda h_n(0)x\lambda h_s(0)} \left( \sum_{n \geq 0} \lambda h(n)x^{-n-1} \right) \\
&+ h(-1)E^-(\lambda, x)E^+(\lambda, x)T^{\Omega_3}_{\Omega_2}(h(0) \cdot w)e^{\log(x)\lambda h_n(0)x\lambda h_s(0)}.
\end{align*}
\]

Lemma 7.3.

\[
\begin{align*}
[h(n), E^-(\lambda, x)E^+(\lambda, x)T^{\Omega_3}_{\Omega_2}(w)e^{\log(x)\lambda h_n(0)x\lambda h_s(0)}] \\
&= \lambda x^n E^-(\lambda, x)E^+(\lambda, x)T^{\Omega_3}_{\Omega_2}(w)e^{\log(x)\lambda h_n(0)x\lambda h_s(0)}, \text{ for } n \neq 0, \\
[h(0), E^-(\lambda, x)E^+(\lambda, x)T^{\Omega_3}_{\Omega_2}(w)e^{\log(x)\lambda h_n(0)x\lambda h_s(0)}] \\
&= E^-(\lambda, x)E^+(\lambda, x)T^{\Omega_3}_{\Omega_2}(h(0) \cdot w)e^{\log(x)\lambda h_n(0)x\lambda h_s(0)}.
\end{align*}
\]

The following theorem gives a solution to equations (7.17) and (7.18) for a Jordan block of length \( n \).

Theorem 7.4. Assume that \( w_i \) satisfy (7.19). Let

\[(7.24) \quad \mathcal{Y}(w_1, x) = E^-(\lambda, x)E^+(\lambda, x)T^{\Omega_3}_{\Omega_2}(w_1)e^{\log(x)\lambda h_n(0)x\lambda h_s(0)}, \]


and for $2 \leq i \leq n$, let

\begin{equation}
\mathcal{Y}(w_i, x) = \sum_{i=1}^{n} \sum_{j=0}^{i-1} \left( \frac{(j-1)!}{j!} \right) E^{-}(\lambda, x) E^{+}(\lambda, x) T_{\Omega_{2}}^{\Omega_{2}}(w_1) e^{\log(x)\lambda_{h_n}(0)} x^{\lambda_{h_s}(0)} \left( \frac{f^+ h(x)^{i-j}}{(i-l-j)!} \right).
\end{equation}

Then

\begin{equation}
[L(-1), \mathcal{Y}(w_i, x)] = \frac{d}{dx} \mathcal{Y}(w_i, x),
\end{equation}

\begin{equation}
[h(n), \mathcal{Y}(w_i, x)] = x^n \mathcal{Y}(h(0) \cdot w_i, x).
\end{equation}

\textbf{Proof.} We already proved the formula in the $i = 1$ case. So we may assume $i \geq 2$. For every $j \geq 1$ we have

\begin{align*}
[L(-1), \frac{(j-1)!}{j!} E^{-}(\lambda, x) E^{+}(\lambda, x) T_{\Omega_{2}}^{\Omega_{2}}(w_1) e^{\log(x)\lambda_{h_n}(0)} x^{\lambda_{h_s}(0)} & \left( \frac{f^+ h(x)^{i-j}}{(i-l-j)!} \right)] = \\
\left( \sum_{n \leq -2} h(n)x^{-n-1} \right) \frac{(j-1)!}{j!} E^{-}(\lambda, x) E^{+}(\lambda, x) T_{\Omega_{2}}^{\Omega_{2}}(w_1) e^{\log(x)\lambda_{h_n}(0)} x^{\lambda_{h_s}(0)} \left( \frac{f^+ h(x)^{i-j}}{(i-l-j)!} \right) \\
+ \frac{(j-1)!}{j!} \frac{d}{dx} \left( E^{-}(\lambda, x) E^{+}(\lambda, x) T_{\Omega_{2}}^{\Omega_{2}}(w_1) e^{\log(x)\lambda_{h_n}(0)} x^{\lambda_{h_s}(0)} \right) \left( \frac{f^+ h(x)^{i-j}}{(i-l-j)!} \right) \\
+ \frac{(j-1)!}{j!} \frac{d}{dx} \left( E^{-}(\lambda, x) E^{+}(\lambda, x) T_{\Omega_{2}}^{\Omega_{2}}(w_1) e^{\log(x)\lambda_{h_n}(0)} x^{\lambda_{h_s}(0)} \right) \left( \frac{f^+ h(x)^{i-j-1}}{(i-l-j-1)!} \right) \left( \sum_{n \geq 0} h(n)x^{-n-1} \right)
\end{align*}

\begin{align*}
&= \frac{d}{dx} \left( \frac{(j-1)!}{j!} E^{-}(\lambda, x) E^{+}(\lambda, x) T_{\Omega_{2}}^{\Omega_{2}}(w_1) e^{\log(x)\lambda_{h_n}(0)} x^{\lambda_{h_s}(0)} \left( \frac{f^+ h(x)^{i-j}}{(i-l-j)!} \right) \right) \\
&- h(-1) \left( \frac{(j-1)!}{(j-1)!} E^{-}(\lambda, x) E^{+}(\lambda, x) T_{\Omega_{2}}^{\Omega_{2}}(w_1) e^{\log(x)\lambda_{h_n}(0)} x^{\lambda_{h_s}(0)} \left( \frac{f^+ h(x)^{i-j}}{(i-l-j)!} \right) \right).
\end{align*}
For \(2 \leq l \leq i - 1\), by using Lemma 7.2 we have

\[
\begin{align*}
[L(-1), \left(\frac{\int - h(x)}{j!} E^{-(\lambda, x)} E^{+ (\lambda, x)} T^{\Omega_2}_{\Omega_2} (u_l) e^{\log(x) \lambda_{\nu}(0) x \lambda_{\nu}(0)} \left(\frac{f^+ h(x)}{(i - l - j)!}\right)\right) &= \\
&= \left(\sum_{n \leq -2} h(n) x^{n-1}\right) \left(\frac{f^+ h(x)}{(j - 1)!}\right) E^{-(\lambda, x)} E^{+ (\lambda, x)} T^{\Omega_2}_{\Omega_2} (u_l) e^{\log(x) \lambda_{\nu}(0) x \lambda_{\nu}(0)} \left(\frac{f^+ h(x)}{(i - l - j)!}\right)\right) \\
&+ \left(\sum_{n \leq -2} \lambda h(n) x^{n-1}\right) E^{-(\lambda, x)} E^{+ (\lambda, x)} T^{\Omega_2}_{\Omega_2} (u_l) e^{\log(x) \lambda_{\nu}(0) x \lambda_{\nu}(0)} \left(\sum_{n \geq 0} \lambda h(n) x^{n-1}\right) \\
&+ h(-1) E^{-(\lambda, x)} E^{+ (\lambda, x)} T^{\Omega_2}_{\Omega_2} (u_l) e^{\log(x) \lambda_{\nu}(0) x \lambda_{\nu}(0)} \left(\frac{f^+ h(x)}{(i - l - j)!}\right) \\
&+ \left(\frac{\int - h(x)}{j!}\right) E^{-(\lambda, x)} E^{+ (\lambda, x)} T^{\Omega_2}_{\Omega_2} (u_l) e^{\log(x) \lambda_{\nu}(0) x \lambda_{\nu}(0)} \left(\frac{f^+ h(x)}{(i - l - j)!}\right) \\
&+ \left(\frac{\int - h(x)}{j!}\right) E^{-(\lambda, x)} E^{+ (\lambda, x)} T^{\Omega_2}_{\Omega_2} (u_l) e^{\log(x) \lambda_{\nu}(0) x \lambda_{\nu}(0)} \left(\sum_{n \geq 0} h(n) x^{n-1}\right) \\
&+ h(-1) E^{-(\lambda, x)} E^{+ (\lambda, x)} T^{\Omega_2}_{\Omega_2} (u_{l-1}) e^{\log(x) \lambda_{\nu}(0) x \lambda_{\nu}(0)} \left(\frac{f^+ h(x)}{(i - l - j)!}\right) \\
&+ \left(\frac{\int - h(x)}{j!}\right) E^{-(\lambda, x)} E^{+ (\lambda, x)} T^{\Omega_2}_{\Omega_2} (u_l) e^{\log(x) \lambda_{\nu}(0) x \lambda_{\nu}(0)} \left(\sum_{n \geq 0} h(n) x^{n-1}\right) \\
&= \frac{d}{dx} \left(\frac{\int - h(x)}{j!}\right) E^{-(\lambda, x)} E^{+ (\lambda, x)} T^{\Omega_2}_{\Omega_2} (u_l) e^{\log(x) \lambda_{\nu}(0) x \lambda_{\nu}(0)} \left(\frac{f^+ h(x)}{(i - l - j)!}\right) \\
&- h(-1) \left(\frac{\int - h(x)}{j!}\right) E^{-(\lambda, x)} E^{+ (\lambda, x)} T^{\Omega_2}_{\Omega_2} (u_l) e^{\log(x) \lambda_{\nu}(0) x \lambda_{\nu}(0)} \left(\frac{f^+ h(x)}{(i - l - j)!}\right) \\
&+ h(-1) \left(\frac{\int - h(x)}{j!}\right) E^{-(\lambda, x)} E^{+ (\lambda, x)} T^{\Omega_2}_{\Omega_2} (u_{l-1}) e^{\log(x) \lambda_{\nu}(0) x \lambda_{\nu}(0)} \left(\frac{f^+ h(x)}{(i - l - j)!}\right). \\
\end{align*}
\]
Also, by Lemma 7.2 we have

\[ [L(-1), E^{-}(\lambda, x)E^{+}(\lambda, x)T_{\Omega_{2}}^{\Omega_{3}}(w_{1})e^{\log(x)\lambda h_{n}(0)x\lambda h_{s}(0)}] \]

\[ = \frac{d}{dx} \left( E^{-}(\lambda, x)E^{+}(\lambda, x)T_{\Omega_{2}}^{\Omega_{3}}(w_{1})e^{\log(x)\lambda h_{n}(0)x\lambda h_{s}(0)} \right) \]

\[ + h(-1)E^{-}(\lambda, x)E^{+}(\lambda, x)T_{\Omega_{2}}^{\Omega_{3}}(w_{1-1})e^{\log(x)\lambda h_{n}(0)} . \]

By combining the previous three formulas we obtain

\[ \sum_{l=1}^{i} \sum_{j=0}^{i-l} [L(-1), \frac{(-h(x))^{j}}{j!}E^{-}(\lambda, x)E^{+}(\lambda, x)T_{\Omega_{2}}^{\Omega_{3}}(w_{l})e^{\log(x)\lambda h_{n}(0)x\lambda h_{s}(0)} \left( \frac{f^{+}h(x)^{i-l-j}}{(i-l-j)!} \right)] \]

\[ = \sum_{l=1}^{i} \sum_{j=0}^{i-l} \frac{d}{dx} \left( \frac{(-h(x))^{j}}{j!}E^{-}(\lambda, x)E^{+}(\lambda, x)T_{\Omega_{2}}^{\Omega_{3}}(w_{l})e^{\log(x)\lambda h_{n}(0)x\lambda h_{s}(0)} \left( \frac{f^{+}h(x)^{i-l-j}}{(i-l-j)!} \right) \right) + \]

\[ -h(-1) \sum_{l=1}^{i-1} \sum_{j=1}^{i-l} \left( \frac{(-h(x))^{j-1}}{(j-1)!}E^{-}(\lambda, x)E^{+}(\lambda, x)T_{\Omega_{2}}^{\Omega_{3}}(w_{l})e^{\log(x)\lambda h_{n}(0)x\lambda h_{s}(0)} \left( \frac{f^{+}h(x)^{i-l-j}}{(i-l-j)!} \right) \right) \]

\[ + h(-1) \sum_{l=2}^{i} \sum_{j=0}^{i-l} \left( \frac{(-h(x))^{j}}{j!}E^{-}(\lambda, x)E^{+}(\lambda, x)T_{\Omega_{2}}^{\Omega_{3}}(w_{l-1})e^{\log(x)\lambda h_{n}(0)x\lambda h_{s}(0)} \left( \frac{f^{+}h(x)^{i-l-j}}{(i-l-j)!} \right) \right) \]

\[ = \frac{d}{dx} \mathcal{Y}(w_{1}, x) \]

\[ -h(-1) \sum_{l=1}^{i-1} \sum_{j=1}^{i-l} \left( \frac{(-h(x))^{j-1}}{(j-1)!}E^{-}(\lambda, x)E^{+}(\lambda, x)T_{\Omega_{2}}^{\Omega_{3}}(w_{l})e^{\log(x)\lambda h_{n}(0)x\lambda h_{s}(0)} \left( \frac{f^{+}h(x)^{i-l-j}}{(i-l-j)!} \right) \right) \]

\[ + h(-1) \sum_{l=2}^{i} \sum_{j=0}^{i-l} \left( \frac{(-h(x))^{j}}{j!}E^{-}(\lambda, x)E^{+}(\lambda, x)T_{\Omega_{2}}^{\Omega_{3}}(w_{l-1})e^{\log(x)\lambda h_{n}(0)x\lambda h_{s}(0)} \left( \frac{f^{+}h(x)^{i-l-j}}{(i-l-j)!} \right) \right) \]

\[ = \frac{d}{dx} \mathcal{Y}(w_{1}, x) . \]
Assume now that \( n > 0 \). The formula (7.26) certainly holds for \( i = 1 \). Then for \( i \geq 2 \), by Lemma 7.3 we get

\[
[h(n), \mathcal{Y}(w_i, x)] = x^n \mathcal{Y}(h(0) \cdot w_i, x).
\]

It is easy to see that the previous formula holds for \( n < 0 \) as well. Finally, by Lemma 7.3

\[
[h(0), \mathcal{Y}(w_i, x)] = \mathcal{Y}(h(0) \cdot w_i, x).
\]

**Remark 3.** Even though the intertwining operator \( \mathcal{Y}(w, x) \) associated with \( T_{\Omega_1, \Omega_2}^{\Omega_3} \) was defined for a single Jordan block only, it is now straightforward to define \( \mathcal{Y}(w, x) \) for an arbitrary finite-dimensional \( \mathfrak{h} \)-module \( \Omega_1 \).

Finally, we have a description of logarithmic intertwining operators among a triple of logarithmic modules with finite-dimensional vacuum spaces.

**Theorem 7.5.** Let \( W_1 = M(1)_a \otimes \Omega_1 \), \( W_2 = M(1)_a \otimes \Omega_2 \) and \( W_3 = M(1)_a \otimes \Omega_3 \) as above and

\[
T_{\Omega_1, \Omega_2}^{\Omega_3} \in \text{Hom}_\mathfrak{h}(\Omega_1, \text{Hom}(\Omega_2, \Omega_3)),
\]

then \( \mathcal{Y}(w, x) \) associated with \( T_{\Omega_1, \Omega_2}^{\Omega_3} \) as in Theorem 7.4 and defined for \( w \in \Omega_1 \) only, extends uniquely to a logarithmic intertwining operator of type \((W_1, W_3)_{W_2}\).

**Proof.** Because of Theorem 7.4 and Theorem 4.5 it only remains to show (4.10). The formula (4.10) clearly holds in the case when \( v = h(-1)1 \) by Theorem 7.3. But the vertex operator algebra is generated by the vector \( h(-1)1 \) so (4.10) holds for every homogeneous \( v \in M(1)_a \) (cf. Proposition 7.1.5 and 7.3.3 in [Li1]).
Proof of Theorem 5.5 (iii). We choose \( \Omega(W) = \Omega_1 \otimes \Omega_2 \), such that \( T^{\Omega_1 \otimes \Omega_2}_{\Omega_1, \Omega_2} = Id \in \text{Hom}_h(\Omega_1 \otimes \Omega_2, \Omega_1 \otimes \Omega_2) \). Let \( \mathcal{Y} \in I \left( \begin{array}{c} M(1)_a \otimes \Omega_1 \otimes \Omega_2 \\ M(1)_a \otimes \Omega_1 \end{array} \right) \) as constructed in Theorem 7.5 and Theorem 7.4. It is not hard to see that for such \( \mathcal{Y} \) and \( \lambda \nu \neq 0 \), we have \( \text{depth}(\mathcal{Y}) = m_1 + m_2 - 2 \), because of (7.24) and (7.25). Similarly for \( \lambda = 0 \) or \( \nu = 0 \).

8. MOCK LOGARITHMIC INTERTWINERS AMONG ORDINARY MODULES

In this section (which is completely independent from the rest of the paper) we construct certain operators related to logarithmic intertwining operators studied in earlier sections. These operators involve logarithms but operate among ordinary \( M(1)_a \)-modules. The next result has already been proven.

Proposition 8.1. The only logarithmic intertwining operators among \( M(1, \lambda)_a, M(1, \nu)_a \) and \( M(1, \lambda + \nu)_a \) are the ordinary intertwining operators.

Proposition 8.2. Suppose that in Definition 1.2 we allow \( \mathcal{Y}(\cdot, x) \cdot \) to satisfy (1.6). Under this new definition, there exists a nontrivial (mock) logarithmic intertwining operator among every triple of ordinary \( M(1)_a \)-modules \( M(1, \lambda)_a, M(1, \nu)_a \) and \( M(1, \lambda + \nu)_a \), provided that \( \lambda \nu \neq 0 \).

Proof. Let \( \mathcal{Y} \in I \left( \begin{array}{c} M(1, \lambda + \nu)_a \\ M(1, \lambda)_a \end{array} \right) \). It is easy to see that the operator

\[ \bar{\mathcal{Y}}(\cdot, x) = \mathcal{Y}(\cdot, x) x^{-\lambda h(0)} \]

satisfies the Jacobi identity, but it doesn’t satisfy the \( L(-1) \)-property. For the same reason

\[ \mathcal{Y}(\cdot, x) x^{-\lambda h(0)} h(0)^k \log(x)^k \]

satisfies the Jacobi identity (notice that \( h(0)^k \) acting on \( M(1, \nu)_a \) is merely \( \nu^k \)), but again it does not satisfy the \( L(-1) \)-property. Finally, we consider

\[ \mathcal{Y}_{\log}(\cdot, x) = \mathcal{Y}(\cdot, x) x^{-\lambda h(0)} e^{\lambda h(0) \log(x)} \]

where

\[ e^{\lambda h(0) \log(x)} = \sum_{n=0}^{\infty} \frac{\lambda^n h(0)^n \log(x)^n}{n!} \]

Now, as before \( \mathcal{Y}_{\log}(\cdot, x) \) satisfies the Jacobi identity but also the \( L(-1) \)-property

\[ [L(-1), \mathcal{Y}_{\log}(w, x)] = \frac{d}{dx} \mathcal{Y}_{\log}(w, x), \]

which follows from \( L(-1) \)-property for \( \mathcal{Y}(\cdot, x) \) and the formula

\[ \frac{d}{dx} (x^{-\lambda h(0)} e^{\lambda h(0) \log(x)}) = 0. \]
9. INDECOMPOSABLE AND LOGARITHMIC REPRESENTATIONS OF THE VIRA SORO ALGEBRA OF CENTRAL CHARGE $c = 1$

In Section 7 we described logarithmic intertwining operators associated with logarithmic $M(1)_a$-modules. Here we restrict our construction to an important vertex operator subalgebra $L(1,0) = \mathcal{U}(Vir) \cdot \mathbf{1} \subset M(1)_0$, where $L(c,h), (c,h) \in \mathbb{C}^2$ denote the irreducible lowest weight irreducible module for the Virasoro algebra of central charge $c$ and lowest conformal weight $h$ [KR]. For simplicity, we shall also use $M(1)$ instead of $M(1)_0$. Results from this section form logarithmic extension of several results from [M2].

Let us introduce some notation. Let $W$ be a $Vir$-module. By $\bullet$ we will denote a lowest weight vector inside $W$ such that $\mathcal{U}(Vir) \cdot \bullet$ is an irreducible lowest weight module (every $\bullet$ is of course a singular vector in $W$). By $\circ$ we will denote a vector that becomes a lowest weight vector in the quotient of $W$ by moding out the submodule generated by all lowest weight vectors. These vectors will be called subsingular vectors. Similarly $\triangleright$ will denote a vector that becomes a lowest vector after quotienting with the submodule of $W$ generated by all lowest weight vectors and all subsingular vectors. Such a vector is called a sub-subsingular vector. One can continue in this manner and introduce vectors that become lowest weight vectors after quotienting with the submodule generated by the lowest weight, subsingular and sub-subsingular vectors, but we shall not need those in the paper. An arrow $\circ \to \bullet$ indicates that $\bullet$ is contained in the submodule generated by $\circ$, etc. We also recall here that cosingular vectors are those vectors that are being mapped to singular vectors in the contragradient module. There is also a pairing between singular vectors and equivalence classes of cosingular vectors in a module.

Let us illustrate these definition with an example: Let $W$ be a $Vir$-module such that

$$0 \to L(1,m^2) \to W \to L(1,(m+1)^2) \to 0,$$

is a nonsplit extension, so that $W$ is generated by a subsingular vector of weight $(m+1)^2$. This extension may be visualized as follows

\[
\begin{array}{c}
\bullet \\
\circ \\
\end{array}
\]

where the arrow pointing up indicates that the conformal weight of $\bullet$ is smaller than of $\circ$. More complicated diagrams will appear later.

In the previous example we implicitly assumed the following result (see [M2]).

**Proposition 9.1.** For every $k, m \in \mathbb{Z}$, we have

$$\text{Ext}^1_{Vir,L(0)}(L(1,k^2), L(1,m^2)) = \mathbb{C},$$

if and only if $|k - m| = 1$. In all other cases $\text{Ext}^1_{Vir,L(0)}$ is trivial.

Our aim is to determine Virasoro submodule structure of $M(1) \otimes \Omega$ (viewed as a $Vir$-module) for some special $h(0)|\Omega$.

The first part in the following result is well-known (see for instance [KR]). For the second part see [DG] or [M2].
Theorem 9.2. Viewed as a Virasoro module $M(1)$ decomposes as a direct sum of irreducible Virasoro modules

\[ M(1) = \bigoplus_{m=0}^{\infty} L(1,m^2). \]

If $u^m$ denotes the lowest weight vector (unique up to a nonzero scalar) of $L(1,m^2)$, then the Virasoro module generated by $Y(u^n,x)u^m$ decomposes as

\[ L(1,(m + n)^2) \oplus L(1,(m - n)^2). \]

The previous theorem can be used for construction of some intertwining operators among irreducible $L(1,0)$-modules. This construction relies on non-vanishing of certain $3j$-symbols [DG], [M2]. For instance, \( \dim \left( L(1,1) \right) \cong M(1) \otimes L(1,0) \) = 1, but this “fusion rule” is not covered by Theorem 9.2.

Here is a useful consequence of Theorem 9.2

Corollary 9.3. The Virasoro module generated by

\[ \{ h(-n)u^m : n \in \mathbb{Z} \} \]

is isomorphic to

\[ L(1,(m - 1)^2) \oplus L(1,(m + 1)^2), \]

for $m \geq 1$. If $m = 0$, the first summand in (9.29) is trivial.

**Proof.** We just have to observe that $h(-1)1 = u^1$, so that $Y(u^1,x) = \sum_{n \in \mathbb{Z}} h(n)x^{-n-1}$. Now, apply the previous theorem.

Lemma 9.4. Let $h(0)|\Omega$ be a nilpotent operator. Then $M(1) \otimes \Omega$ is a self-dual $M(1)$-module, i.e., \((M(1) \otimes \Omega)' \cong M(1) \otimes \Omega\). Clearly, the same is true if $M(1) \otimes \Omega$ is viewed as a Virasoro algebra module.

**Proof.** Every logarithmic $M(1)$-module $W$ is uniquely determined by the $h(0)$-action on $\Omega(W)$. Thus, two logarithmic $M(1)$-modules $M(1) \otimes \Omega_1$ and $M(1) \otimes \Omega_2$ are equivalent if and only if there exists $\Psi : \Omega_1 \longrightarrow \Omega_2$ such that $h(0)|\Omega_1 = \Psi^{-1}h(0)|\Omega_2\Psi$. The module $(M(1) \otimes \Omega)'$ is isomorphic to $M(1) \otimes \Omega'$, where the action of $h(0)$ on the dual space $\Omega'$ is given by

\[ \langle h(0) \cdot w', w \rangle = -\langle w', h(0) \cdot w \rangle, \quad w \in \Omega, w' \in \Omega', \]

so that $h(0)|\Omega' = -h^*(0)$, where $h^*(0)$ is the dual map. The operator $h(0)$ is nilpotent, thus $-h^*(0)$ and consequently $h(0)|\Omega'$ are nilpotent as well. But $-h(0)^*$ and $h(0)$ admit the same Jordan form, so there exists $\Psi$ with wanted properties.
Theorem 9.5. Let $\Omega$ be a two-dimensional space and $h(0)|_{\Omega} \neq 0$, $h(0)^2|_{\Omega} = 0$. Then viewed as a Virasoro module, $M(1) \otimes \Omega$ is generated by a sequence of subsingular vectors as on the following diagram

(9.30)

where the $s$-th $\bullet$ and $\diamond$, counting from the top, have conformal weight $(s-1)^2$, $s \geq 1$. Here dotted arrows indicate the action of the transpose of $h(0)$, which uniquely determines every $\diamond$.

Proof. Firstly, we may and will choose a basis $\{w_1, w_2\}$ of $\Omega$ such that $h(0) \cdot w_1 = 0$, $h(0) \cdot w_2 = w_1$, so that $h(0)^T \cdot w_1 = w_2$ and $h(0)^T \cdot w_2 = 0$. Since $h(0)^2|_{\Omega} = 0$, the module $M(1) \otimes \Omega$ is $L(0)$-diagonalizable. From the Virasoro algebra embedding $M(1) \hookrightarrow M(1) \otimes \Omega$, where we use the identification $M(1) = \mathcal{U}(h) \cdot w_1$, and Theorem 9.2, it is clear that $M(1) \otimes \Omega$ contains a sequence of singular vectors of weight $m^2$ for every $m \geq 0$. These singular vectors, displayed in the left column of (9.30) by $\bullet$ are determined up to a constant. Let $u^m = P_m(h)w_1$ denote such a vector of weight $m^2$, where $P_m(h)$ is a polynomial in $h(-i)$, $i \geq 1$, of degree $m^2$. As we already mentioned there are also vectors in $M(1) \otimes \Omega$ that become singular after quotienting with $M(1)$. These vectors are uniquely determined if we assume that every $\diamond$ is obtained from $\bullet$ by applying $h(0)^T$ to $u^m$. These vectors will be denoted by $u^{2,m}$, $m \geq 0$, so that $u^{2,m} = h(0)^Tu^m = P_m(h) \cdot w_2$. It is clear that $M(1) \otimes \Omega$ is generated by $S = \{u^m : m \geq 0\} \cup \{u^{2,m} : m \geq 0\}$. It remains to prove that we can reduce the generating set $S$ down to $\{u^{2,m} : m \geq 0\}$. The short exact sequence of $\text{Vir}$-modules

$$0 \longrightarrow M(1) \longrightarrow M(1) \otimes \Omega \overset{\pi}{\longrightarrow} M(1) \longrightarrow 0,$$

together with Theorem 9.2 gives

$$0 \longrightarrow \bigoplus_{m \geq 0} L(1, m^2) \longrightarrow M(1) \otimes \Omega \overset{\pi}{\longrightarrow} \bigoplus_{m \geq 0} L(1, m^2) \longrightarrow 0.$$ 

Thus $M(1) \otimes \Omega$, which is $L(0)$-diagonalizable, gives a nonzero element in

(9.31) \hspace{1cm} \text{Ext}^1_{\text{Vir},L(0)}(\bigoplus_{m \geq 0} L(1, m^2), \bigoplus_{n \geq 0} L(1, n^2)) \cong \prod_m \prod_n \text{Ext}^1_{\text{Vir},L(0)}(L(1, m^2), L(1, n^2)).

Now, Proposition 9.1 implies that

(9.32) \hspace{1cm} \text{Ext}^1_{\text{Vir},L(0)}(\bigoplus_{m \geq 0} L(1, m^2), \bigoplus_{n \geq 0} L(1, n^2)) \cong \prod_{|m-n|=1} \text{Ext}^1_{\text{Vir},L(0)}(L(1, m^2), L(1, n^2)).

\footnote{These subsingular vectors are also cosingular.}
Already from the previous formula it is clear that there could be at most two arrows exiting from \( u^{2,m} \). Now, we determine these arrows for every \( m \). For \( m = 0 \), there is precisely one outgoing arrow from \( u^{2,0} \) pointing to \( u^1 \). This follows from \( L(-1)w_2 = h(-1)h(0)w_2 = h(-1)w_1 = u^1 \), where \( u^1 \) is the lowest weight vector of \( L(1,1) \subset \mathcal{M}(1) \).

**Claim:** For \( m \geq 1 \), from each \( u^{2,m} \) there are precisely two outgoing arrows; one pointing to \( u^{m-1} \) and the other pointing to \( u^{m+1} \).

To see that we write the generator \( L(n), n \in \mathbb{Z} \) as

\[
h(0)h(n) + \frac{1}{2} \sum_{k+l=n, kl \neq 0} :h(k)h(l):.
\]

Now,

\[
L(n)u^{2,m} = L(n)P_m(h)w_2 = \left( h(0)h(n) + \frac{1}{2} \sum_{k+l=n, kl \neq 0} :h(k)h(l): \right) P_m(h)w_2 = h(n)P_m(h)w_1 + \bar{L}(n)P_m(h)w_2,
\]

where

\[
\bar{L}(n) = \frac{1}{2} \left( \sum_{k+l=n, kl \neq 0} :h(k)h(l): \right).
\]

By using Corollary 9.3 we have

\[
h(n)P_m(h)w_1 \in L(1, (m - 1)^2) \oplus L(1, (m + 1)^2), \quad n \in \mathbb{Z}.
\]

Combined with \( \bar{L}(n)u^{2,m} = 0, n \geq 1 \), it follows that there exists \( a^+ \in \mathcal{U}(Vir_{>0}) \), such that

\[
a^+ \cdot u^{2,m} = u^{m-1}.
\]

This proves that there is an arrow pointing to \( u^{m-1} \). Let us recall that under taking the dual the singular vectors in \( M(1) \otimes \Omega \) are mapped to cosingular vectors and vice-versa. In addition, orientations of arrows are reversed. Now, Lemma 9.4 yields an isomorphism between \( M(1) \otimes \Omega \) and its dual, which maps singular vectors \( u^m \) to cosingular vectors \( u^{2,m} \) and cosingular vectors \( u^{2,m} \) to singular vectors \( w^m \) such that \( w^m \) and \( u^{2,m} \) form a Jordan block with respect to \( h(0) \) (i.e., \( h(0) \cdot w^m = 0, h(0) \cdot u^{2,m} = w^m \)). Clearly, \( w^m \) generates the lowest weight module \( L(1, m^2) \). Thus, there will be an arrow pointing down from \( u^{2,m-1} \) to \( w^m \). Now, by using the same argument as before we argue that there is an arrow pointing up from every \( u^{2,m} \) to \( w^{m-1} \). The proof follows.

Here is a useful consequence of the previous theorem. For simplicity we only consider \( \Omega \) with \( \dim(\Omega) = 3 \).

**Corollary 9.6.** Let \( \Omega \) be a three dimensional \( h \)-module such that \( h(0)^3|_\Omega = 0 \) and \( h(0)^2|_\Omega \neq 0 \). Then, viewed as a \( Vir \)-module, \( M(1) \otimes \Omega \) is generated by a sequence of sub-subsingular vectors (denoted by \( \bowtie \))
as on the following diagram

(9.33)

where every \( \circ \) (resp. \( \triangleright \)) is obtained from \( \bullet \) (resp. \( \circ \)) of the same generalized weight by applying the transpose of \( h(0) \). For simplicity we do not display dotted arrows and arrows obtained by the "addition of arrows" rule.

Proof. From an embedding \( M(1) \otimes \Omega_2 \hookrightarrow M(1) \otimes \Omega_3 \) it is clear that there will be arrows connecting \( u^{2,m} \) and \( u^{m \pm 1} \) as in Theorem 9.5. Let \( u^{3,n} = h^T(0)u^{2,n} \), so \( u^{3,n} \) is represented by a \( \triangleright \). By arguing as before, from each \( u^{3,m} \) there will be arrows pointing to \( u^{2,m+1} \) and \( u^{2,m-1} \). So we only have to show that there are no additional arrows from \( u^{3,m} \) except those displayed on (9.33). From the formula \( L(0)u^{3,n} = n^2u^{3,n} + u^n \) it follows that \( u^n \) and \( u^{3,n} \) form a Jordan block with respect to \( L(0) \). But \( u^n \) can be also reached from \( u^{3,n} \) via an oriented path \( u^{3,n} \rightarrow u^{2,n+1} \rightarrow u^n \) so there is no need to display an arrow from \( u^{3,n} \) to \( u^n \), because the submodule generated by \( u^{3,n} \) contains \( u^n \). Similarly with \( u^{3,n} \) and \( u^{n+2} \).

10. Hidden logarithmic intertwining operators

Suppose that \( h(0)|_{\Omega_2} \) in some basis \( \{w_1, w_2\} \) for \( \Omega_2 \) is represented by

\[
\begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
\]

and \( h(0)|_{\Omega_3} \), in some basis \( \{\tilde{w}_1, \tilde{w}_2, \tilde{w}_3\} \) for \( \Omega_3 \), is represented by

\[
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
\]

Then a surjective map

\[
T_{\Omega_3, \Omega_2} : \text{Hom}(\Omega_2 \otimes \Omega_2, \Omega_3),
\]

defined by

(10.34)

\[
\\begin{align*}
w_2 \otimes w_2 & \mapsto \tilde{w}_3, & w_2 \otimes w_1 & \mapsto \frac{\tilde{w}_2}{2}, & w_1 \otimes w_2 & \mapsto \frac{\tilde{w}_2}{2}, & w_1 \otimes w_1 & \mapsto \frac{\tilde{w}_1}{2} \\
\\end{align*}
\]

commutes with \( h(0) \).
Let us denote by $W_2(1, m^2) \subset M(1) \otimes \Omega_2$ a cyclic $Vir$-module generated by $u^{2,m}$ of weight $m^2$. For $m > 0$, $W_2(1, m^2)$ can be visualized as a “wedge” in $\Omega_2$

or a single arrow

in the $m = 0$ case. Similarly, we denote by $W_3(1, m^2) \subset M(1) \otimes \Omega_3$ (cf. Corollary 9.6) the module generated by $\triangleright$, of generalized weight $m^2$. For every $m > 1$ this module can be visualized as

Similarly, $W_3(1, 0)$ may be visualized as

Again, we shall assume that $\triangleright$ (resp. $\triangleright$) is obtained from $\circ$ (resp. $\bullet$) of the same generalized weight by applying $h(0)^T$. The following Lemma is just a consequence of $L(0)u^{3,m} = m^2u^{3,m} + \frac{1}{2}u^{m}$, so we omit the proof.

**Lemma 10.1.** For every $m \geq 0$ the module $W_3(1, m^2)$ is a genuine logarithmic module.

The module $W_3(1, m^2)$ is a nonsplit extension of $L(1, m^2)$ by $W_2(1, (m - 1)^2) + W_2(1, (m + 1)^2)$, where if $m = 0$ the first summand is trivial.

Now we have a logarithmic version of Theorem 9.2.
Theorem 10.2. Let $\Omega_2$ and $\Omega_3$ be as above. Then there exists a nontrivial $\mathcal{Y} \in I \left( \begin{pmatrix} M(1) \otimes \Omega_3 \\ M(1) \otimes \Omega_2 M(1) \otimes \Omega_2 \end{pmatrix} \right)$, such that $\mathcal{Y}$ projects down to a hidden logarithmic intertwining operator
\[
\tilde{\mathcal{Y}} \in I \left( \begin{pmatrix} W \\ W_2(1, m^2) W_2(1, n^2) \end{pmatrix} \right)
\]
of depth one, where
\[
W = \sum_{|m-n| \leq k \leq m+n} W_3(1, k^2).
\]
This sum is not direct, whenever $mn \neq 0$.

Proof. Let $\mathcal{Y}$ be as in Theorem 7.5 with $T_{\Omega_2, \Omega_2}^{\Omega_3}$ as in (10.34). Let
\[
\tilde{\mathcal{Y}}(\cdot, x) = \mathcal{Y}(\cdot, x) \cdot |w_2(1, m^2) \otimes w_2(1, n^2)|.
\]
We recall (cf. Section 7) that in this case
\[
\mathcal{Y}(u^0, x) = \mathcal{Y}(w_1, x) = T_{\Omega_2, \Omega_2}^{\Omega_3}(w_1),
\]
\[
\mathcal{Y}(u^2, x) = \mathcal{Y}(w_2, x) = \int h(x) T_{\Omega_2, \Omega_2}^{\Omega_3}(w_1) + T_{\Omega_2, \Omega_2}^{\Omega_3}(w_1) \int h(x) + T_{\Omega_2, \Omega_2}^{\Omega_3}(w_2).
\]
Also,
\[
\mathcal{Y}(u^{2, m}, x) = \mathcal{Y}(P_m(h) u^{2, 0}, x).
\]
We need a more precise information about the image of $\mathcal{Y}$. Since $W_2(1, m^2)$ is cyclic, and generated by $u^{2, m}$, the image of $\tilde{\mathcal{Y}}$, denoted by $\tilde{W}$, is actually the Virasoro submodule generated by the Fourier coefficients of $\tilde{\mathcal{Y}}(u^{2, m}, x) u^{2n}$, $\tilde{\mathcal{Y}}(u^{m+1}, x) u^{2n}$, $\tilde{\mathcal{Y}}(u^{2, m}, x) u^{n+1}$ and $\tilde{\mathcal{Y}}(u^{m+1}, x) u^{n+1}$. As before, let $u^{3, k}$ denote a generator of $W_3(1, k^2)$ of generalized weight $k^2$. Since $L(1, (m-1)^2) \oplus L(1, (m+1)^2) \subset W_2(1, m^2)$, then Theorem 9.2 (10.35) and (10.37) imply that the submodule of $\tilde{W}$ generated by $\tilde{\mathcal{Y}}(u^{m+1}, x) u^{n+1}$ is precisely
\[
L(1, (m-n-2)^2) \oplus L(1, (m-n)^2) \oplus \cdots \oplus L(1, (m+n+2)^2).
\]
Now we move "one step higher" or "deeper" in the filtration and determine the Virasoro submodule generated by $\tilde{\mathcal{Y}}(u^{2, m}, x) u^n$. From the formula
\[
h(0) \mathcal{Y}(u^{2, m}, x) u^n = \mathcal{Y}(u^m, x) u^n,
\]
and the previous discussion it follows that the submodule generated by $\mathcal{Y}(u^{2, m}, x) u^{n+1}$ is contained inside $W_2(1, (m-n-1)^2) + \cdots + W_2(1, (m+n+1)^2)$ and possibly some $L(1, k^2)$, $k^2 \notin \{(m-n-2)^2, ..., (m+n+2)^2\}$. But having in the image such $L(1, k^2)$ would contradict to Theorem 9.2. Similarly, the submodule generated by $\tilde{\mathcal{Y}}(u^{n+1}, x) u^{2n}$ lies again inside the sum $W_2(1, (m-n-1)^2) + \cdots + W_2(1, (m+n-1)^2)$. Furthermore, from
\[
h(0) \mathcal{Y}(u^{2, m}, x) u^{2n} = \mathcal{Y}(u^m, x) u^{2n} + \mathcal{Y}(u^{2, m}, x) u^n
\]
it follows that the Virasoro module generated by the coefficients of $\tilde{\mathcal{Y}}(u^{2, m}, x) u^{2n}$ is contained inside $W_3(1, (m-n)^2) + \cdots + W_3(1, (m+n)^2)$ and possibly some irreducible module $L(1, k^2)$ not included in (10.38). But this would again contradict to Theorem 9.2. Thus, we have shown that
the image $\hat{W}$ is contained inside $W$ (cf. 10.35). In fact, it is not hard to show that $u^{3,k} \in \hat{W}$ for $k^2 \in \{(m-n)^2, \ldots, (m+n)^2\}$, which would imply $\hat{W} = W$.

Finally, from (10.36) and (10.37) it is clear that $\bar{Y}$ is a genuine logarithmic intertwining operator of depth one.

The module $W$ in the previous theorem has the following diagram representation

\[
\begin{array}{c}
\bigtriangleup \\
\\
\bigtriangleup \\
\vdots \\
\vdots \\
\bigtriangleup \\
\end{array}
\]

\[
\begin{array}{c}
\biglozenge \\
\\
\biglozenge \\
\vdots \\
\vdots \\
\biglozenge \\
\end{array}
\]

\[
\begin{array}{c}
\biglozenge \\
\\
\biglozenge \\
\vdots \\
\vdots \\
\biglozenge \\
\end{array}
\]

\[
\begin{array}{c}
\biglozenge \\
\\
\biglozenge \\
\vdots \\
\vdots \\
\biglozenge \\
\end{array}
\]

Remark 4. It is not hard to generalize the results from this section to logarithmic modules with Jordan blocks of arbitrary size.

11. Hidden logarithmic intertwining operators among Feigin-Fuchs modules at $c = 1 - 12a^2$

Let us recall that $M(1, \lambda)_a$ is a $M(1)_a$-module of lowest conformal weight $\frac{\lambda^2}{2} - a\lambda$. As we have already mentioned in the introduction, the $\lambda = a$ case (e.g., $\lambda = 0$ for $a = 0$) is indeed very special. Here is a consequence of Corollary 3.2.

**Lemma 11.1.** Let $M(1)_a \otimes \Omega$, where $\Omega$ is two-dimensional and $h(0)|_\Omega$ is represented by $\left[\begin{array}{cc} \lambda & 1 \\ 0 & \lambda \end{array}\right]$, in some basis. Then $M(1)_a \otimes \Omega$ is an ordinary $M(1)_a$-module if and only if $\lambda = a$, in which case the Feigin-Fuchs module $M(1, a)_a$ is of lowest conformal weight $\frac{-a^2}{2}$.

Let $\Omega$ be as in the lemma. It is easy to see that $M(1)_a \otimes (\Omega \otimes \Omega)$ is a genuine logarithmic $M(1)_a$-modules. Now, we have a consequence of Theorem 7.5.

**Corollary 11.2.** Let $M(1)_a \otimes \Omega$ be as in Lemma 11.1. Then there exists a genuine logarithmic intertwining operator of type $\left(M(1)_a \otimes (\Omega \otimes \Omega) \right) \rightarrow \left(M(1)_a \otimes \Omega \right) \rightarrow \left(M(1)_a \otimes \Omega \right)$.

**Example.** For $a = \frac{1}{2}$ the vertex operator algebra $M(1)_{\frac{1}{2}}$ has central charge $c = -2$. Then $M(1)_{\frac{1}{2}}$-module $M(1)_{\frac{1}{2}} \otimes \Omega$, where $h(0)|_\Omega$ is represented by

\[
\left[\begin{array}{cc} \frac{1}{2} & 1 \\ 0 & \frac{3}{2} \end{array}\right]
\]

is an ordinary $M(1)_{\frac{1}{2}}$-module with the lowest conformal weight $-\frac{1}{8}$. The intertwining operator constructed in Corollary 11.2 is closely related to logarithmic operators studied in [Gu].

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