ON THE NUMBER OF REPRESENTATIONS OF INTEGERS AS DIFFERENCES BETWEEN PIATETSKI-SHAPIRO NUMBERS

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ABSTRACT. For $\alpha > 1$, set $\beta = 1/(\alpha - 1)$. We show that, for every $1 < \alpha < (\sqrt{21} + 4)/5 \approx 1.717$, the number of pairs $(m, n)$ of positive integers with $d = \lfloor n^\alpha \rfloor - \lfloor m^\alpha \rfloor$ is equal to $\beta \alpha^{-\beta} \zeta(\beta) d^{\beta - 1} + o(d^{\beta - 1})$ as $d \to \infty$, where $\zeta$ denotes the Riemann zeta function. We use this result to derive an asymptotic formula for the number of triplets $(l, m, n)$ of positive integers such that $l < n$ and $\lfloor l^\alpha \rfloor + \lfloor m^\alpha \rfloor = \lfloor n^\alpha \rfloor$. Furthermore, we prove that the additive energy of the sequence $(\lfloor n^\alpha \rfloor)_{n=1}^N$, i.e., the number of quadruples $(n_1, n_2, n_3, n_4)$ of positive integers with $\lfloor n_1^\alpha \rfloor + \lfloor n_2^\alpha \rfloor = \lfloor n_3^\alpha \rfloor + \lfloor n_4^\alpha \rfloor$ and $n_1, n_2, n_3, n_4 \leq N$, is equal to $O(\alpha^4 N^{4-\alpha})$ when $1 < \alpha \leq 4/3$.

1. Introduction

1.1. The number of solutions of the equation $x + y = z$ in $\text{PS}(\alpha)$. Let $\text{PS}(\alpha)$ be the set $\{\lfloor n^\alpha \rfloor : n \in \mathbb{N}\}$ for $\alpha \geq 1$, where $\lfloor x \rfloor$ (resp. $\lceil x \rceil$) denotes the greatest (resp. least) integer $\leq x$ (resp. $\geq x$) for a real number $x$, and $\mathbb{N}$ denotes the set of all positive integers. The set $\text{PS}(\alpha)$ has infinitely many solutions of the equation $x + y = z$ if $\alpha = 1, 2$, and has no solutions of the same equation if $\alpha \geq 3$ is an integer (Fermat’s last theorem) \cite{25}. When $\alpha = 2$, such a solution $(x, y, z)$ is called a Pythagorean triple, and several asymptotic formulas are known for the number of Pythagorean triples. For example, the number of Pythagorean triples with hypotenuse less than $x$ is estimated as \cite{22}

$$\#\{(l, m, n) \in \mathbb{N}^3 : n < x, \ l^2 + m^2 = n^2\} = \frac{1}{\pi} x \log x + Bx + O(x^{1/2} \exp(-C(\log x)^{3/5}(\log \log x)^{-1/5})) \quad (x \to \infty)$$

for an explicit constant $B$ and some $C > 0$. If the Riemann hypothesis is true, the above error term is improved \cite{16}. Also, the number of Pythagorean triples with both

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legs less than $x$ is estimated as \[3\]

\[
\{ (l, m, n) \in \mathbb{N}^3 : l, m < x, \ l^2 + m^2 = n^2 \} \\
= \frac{4 \log(1 + \sqrt{2})}{\pi^2} x \log x + O(x) \quad (x \to \infty).
\]

For other asymptotic formulas, see \[12, 14\] (primitive Pythagorean triples with perimeter less than $x$) and \[5, 12, 24, 27\] (primitive Pythagorean triples with area less than $x$).

Consider the case of non-integral $\alpha > 1$. As a special case of \[6, Proposition 5.1\], it is known that, for all $\alpha \in (1, 2)$ and sufficiently large $d \in \mathbb{N}$, the set $PS(\alpha)$ has a solution of the equation $d = z - y$. This fact immediately implies that the set $PS(\alpha)$ has infinitely many solutions of the equation $x + y = z$ if $\alpha \in (1, 2)$. However, we almost never know an asymptotic formula for the number of such solutions in the case $\alpha \in (1, 2)$. Indeed, we only know the following asymptotic formula \[26, Corollary 1.3\]:

\[
\lim_{x \to \infty} \frac{\# \{ (l, m, n) \in \mathbb{N}^3 : n < x, \ \lfloor n^{\alpha} \rfloor + \lfloor m^{\alpha} \rfloor = \lfloor l^{\alpha} \rfloor \} }{x^{\alpha - 1} (\beta - 1) + 1} = \frac{\beta \alpha^{-\beta} \zeta(\beta)}{\alpha (\beta - 1) + 1},
\]

where $\Gamma$ denotes the gamma function.

In this paper, we estimate the number of solutions of the equation $d = z - y$ in $PS(\alpha)$.

For a real number $\alpha > 1$ and an integer $d \geq 1$, define the number $N_\alpha(d)$ as

\[
N_\alpha(d) = \# \{ (m, n) \in \mathbb{N}^2 : \lfloor n^{\alpha} \rfloor - \lfloor m^{\alpha} \rfloor = d \}.
\]

Note that $N_\alpha(d)$ is finite for all $\alpha > 1$ and $d \in \mathbb{N}$.

**Theorem 1.1.** Let $1 < \alpha < (\sqrt{21} + 4)/5$ and $\beta = 1/ (\alpha - 1)$. Then

\[
\lim_{d \to \infty} \frac{N_\alpha(d)}{d^{\beta - 1}} = \beta \alpha^{-\beta} \zeta(\beta),
\]

where $\zeta$ denotes the Riemann zeta function.

**Theorem 1.1** yields the following asymptotic result immediately.

**Corollary 1.2.** Let $1 < \alpha < (\sqrt{21} + 4)/5$ and $\beta = 1/ (\alpha - 1)$. Then

\[
\lim_{x \to \infty} \frac{\# \{ (l, m, n) \in \mathbb{N}^3 : l < x, \ \lfloor l^{\alpha} \rfloor + \lfloor m^{\alpha} \rfloor = \lfloor n^{\alpha} \rfloor \} }{x^{\alpha(\beta - 1) + 1}} = \frac{\beta \alpha^{-\beta} \zeta(\beta)}{\alpha (\beta - 1) + 1}.
\]

**Proof of Corollary 1.2 assuming Theorem 1.1.** The above number of triplets $(l, m, n)$ is expressed as $\sum_{1 \leq l < x} N_\alpha(\lfloor l^{\alpha} \rfloor)$. Hence, **Corollary 1.2** follows form **Theorem 1.1**. \qed

The set $PS(\alpha)$ is called a *Piatetski-Shapiro sequence* when $\alpha > 1$ is non-integral. When $f$ is a positive-integer-valued function with suitable properties, we can apply **Theorem 1.1** to the equation $f(x) + y = z$ in the same way as the proof of **Corollary 1.2**.
Although Corollary 1.2 does not cover the case $\alpha \geq (\sqrt{21} + 4)/5$, we expect that the asymptotic formula in Corollary 1.2 should be true for every $1 < \alpha \leq 2$. If formally substituting $\alpha = 2$ in Corollary 1.2, then we obtain
\[
\lim_{x \to \infty} \frac{\# \{(l, m, n) \in \mathbb{N}^3 : l > x, l^2 + m^2 = n^2\}}{x} = \infty.
\]
Actually, one can verify the asymptotic formula
\[
\# \{(l, m, n) \in \mathbb{N}^3 : l > x, l^2 + m^2 = n^2\} = \frac{x(\log x)^2}{\pi^2} + O(x \log x) \quad (x \to \infty)
\]
in a similar way to [3].

1.2. Additive energies of Piatetski-Shapiro sequences. Recently, many researchers have paid attention to the additive energies of strictly increasing integer sequences $(a_n)_{n=1}^N$, i.e., the number of quadruples $(n_1, n_2, n_3, n_4) \in \mathbb{N}^4$ such that $a_{n_1} + a_{n_2} = a_{n_3} + a_{n_4}$ and $n_1, n_2, n_3, n_4 \leq N$, since it is closely related to the metric Poissonian property [1, 2, 4, 13]. From now on, we denote by $\mathcal{E}_\alpha(N)$ the additive energy of $(\lfloor \alpha n \rfloor)_{n=1}^N$.

Several bounds for $\mathcal{E}_\alpha(N)$ are known in existing studies. For example, it follows from [1, p. 505] (essentially [17, Theorem 2]) that $\mathcal{E}_\alpha(N) \ll \varepsilon N^{2+\varepsilon} + N^{4-\alpha+\varepsilon}$ for all $\alpha > 1, \varepsilon > 0$ and $N \in \mathbb{N}$. Also, it is known that $\mathcal{E}_\alpha(N) \ll \alpha N^{4-\alpha} \log(N + 1)$ for all $\alpha \in (1, 3/2]$ and $N \in \mathbb{N}$ [7, p. 1000]. Moreover, it is easy to check that $\mathcal{E}_\alpha(N) \gg N^2 + N^{4-\alpha}$ for all $\alpha \geq 1$ and $N \in \mathbb{N}$ (for instance, see [7, p. 1000]). However, this lower bound is not equal to the above upper bounds as the growth rate.

In this paper, we estimate $\mathcal{E}_\alpha(N)$ and obtain an upper bound for $\mathcal{E}_\alpha(N)$ when $1 < \alpha \leq 4/3$.

**Theorem 1.3.** For all $\alpha \in (1, 4/3]$ and $N \in \mathbb{N}$, the additive energy $\mathcal{E}_\alpha(N)$ of $(\lfloor \alpha n \rfloor)_{n=1}^N$ is equal to $O_\alpha(N^{4-\alpha})$.

The upper bound in Theorem 1.3 is best possible, since the known lower bound $N^2 + N^{4-\alpha}$ is greater than $N^{4-\alpha}$. We prove Theorem 1.3 by estimating a variant of $\mathcal{N}_\alpha(d)$ in Section 7.

However, one can also estimate $\mathcal{E}_\alpha(N)$ as follows. Let $\mathcal{R}_\alpha(N)$ be the number of solutions of the equation $x + y = N$ in $\text{PS}(\alpha)$. Then
\[
\mathcal{E}_\alpha(N) \leq \sum_{2 \leq n \leq 2N^\alpha} \mathcal{R}_\alpha(n)^2.
\]
Assume $\alpha \in (1, 4/3)$. Since $\mathcal{R}_\alpha(N) \asymp N^{2/\alpha - 1}$ as $N \to \infty$ [26, Corollary 1.6], it can easily be checked that $\mathcal{E}_\alpha(N) \ll_\alpha N^{4-\alpha}$. This way is different from the proof in Section 7 and is easier to come up with.
1.3. Related work on the number of solutions of a linear equation in $\text{PS}(\alpha)$.
There are existing studies that examined whether a linear equation in two or three variables has infinitely many solutions or not. An equation $f(x_1, \ldots, x_k) = 0$ is called solvable in a subset $\mathcal{S}$ of $\mathbb{N}$ if $\mathcal{S}^k$ contains infinitely many pairwise distinct tuples $(x_1, \ldots, x_k)$ with $f(x_1, \ldots, x_k) = 0$. Matsusaka and Saito [15] showed that, for all $t > s > 2$ and $a, b, c \in \mathbb{N}$, the set of all $\alpha \in [s, t]$ such that the equation $ax + by = cz$ is solvable in $\text{PS}(\alpha)$, has positive Hausdorff dimension. Glasscock [8] showed that if the equation $y = ax + b$ with real numbers $a \neq 0, 1$ and $b$ is solvable in $\mathbb{N}$, then, for Lebesgue-a.e. $\alpha \in (1, 2)$ (resp. $\alpha > 2$), the equation $y = ax + b$ is solvable (resp. not solvable) in $\text{PS}(\alpha)$.
Saito [18] improved Glasscock’s result in the case $a > b > 0$ as follows. Let $a \neq 1$ and $b$ be real numbers with $a > b \geq 0$. If the equation $y = ax + b$ is solvable in $\mathbb{N}$, then (i) for all $\alpha \in (1, 2)$, the equation $y = ax + b$ is solvable in $\text{PS}(\alpha)$; (ii) for all $t > s > 2$, the set of all $\alpha \in [s, t]$ such that the equation $y = ax + b$ is solvable in $\text{PS}(\alpha)$, has the Hausdorff dimension $2/s$. Moreover, Saito [19] investigated the Hausdorff dimension of the set of $\alpha \in [s, t]$ such that a linear equation $y = a_1x_1 + \cdots + a_nx_n$ with positive coefficients $a_1, \ldots, a_n$ has infinitely many solutions in $\text{PS}(\alpha)$. For example, he showed the following statement [19, Theorem 1.2]: for Lebesgue-a.e. $\alpha > 3$, the equation $x + y = z$ has at most finitely many solutions in $\text{PS}(\alpha)$.

2. More general statements

2.1. Arithmetic progressions with fixed common difference. We extend the definition of $\mathcal{N}_\alpha(d)$ a little. For a real number $\alpha > 1$ and integers $d \geq 1$ and $k \geq 2$, define the number $\mathcal{N}_{\alpha,k}(d)$ as

$$\mathcal{N}_{\alpha,k}(d) = \# \left\{ (n, r) \in \mathbb{N}^2 : \forall j = 0, 1, \ldots, k-1, \frac{\lfloor n \alpha \rfloor}{d} + dj \right\}.$$ 

In other words, $\mathcal{N}_{\alpha,k}(d)$ is the number of arithmetic progressions $P$ of length $k$ ($k$-APs) such that $(\lfloor n \alpha \rfloor)_{n \in P}$ is an AP with common difference $d$. The number $\mathcal{N}_{\alpha}(d)$ is equal to $\mathcal{N}_{\alpha,2}(d)$. Note that $\mathcal{N}_{\alpha,k}(d)$ is finite for all $\alpha > 1$, all integers $d \geq 1$ and $k \geq 2$. The following theorem is an extension of Theorem [11].

**Theorem 2.1.** Let $1 < \alpha < (\sqrt{21} + 4)/5$ and $\beta = 1/(\alpha - 1)$. Then, for every integer $k \geq 2$,

$$\lim_{d \to \infty} \frac{\mathcal{N}_{\alpha,k}(d)}{d^{\beta - 1}} = \frac{\beta \alpha^{-\beta} \zeta(\beta)}{k - 1}.$$ 

**Notation.** From now on, we denote by $\{x\} = x - \lfloor x \rfloor$ the fractional part of a real number $x$, and use the notations “$o(\cdot)$, $O(\cdot)$, $\sim, \asymp, \ll$” in the usual sense. If implicit constants depend on parameters $a_1, \ldots, a_n$, we often write “$O_{a_1, \ldots, a_n}(\cdot)$, $\asymp_{a_1, \ldots, a_n}$,”
<\alpha_1, \ldots, \alpha_n>" instead of "O(\cdot), \asymp, \ll". Also, the expression \( a/bc \) always means \( \frac{a}{b} \cdot c \) and does not mean \( \frac{a}{b}c \).

Theorem 2.1 is derived from the following propositions and lemmas immediately.

**Proposition 2.2.** Let \( 1 < \alpha < 2 \) and \( \beta = 1/(\alpha - 1) \). Then, for every integer \( k \geq 2 \),

\[
\lim_{d \to \infty} \frac{N_{\alpha,k}(d)}{d^{\beta - 1}} = \frac{\beta \alpha^{-\beta} \zeta(\beta)}{k - 1}.
\]

**Proposition 2.3.** Let \( 1 < \alpha < 2 \) and \( \beta = 1/(\alpha - 1) \). For an integer \( r \geq 1 \), define the strictly increasing function \( f_r : [0, \infty) \to [r^\alpha, \infty) \) as \( f_r(x) = (x + r)^\alpha - x^\alpha \). Then, for every integer \( k \geq 2 \),

\[
\lim_{d \to \infty} \frac{N_{\alpha,k}(d)}{d^{\beta - 1}} \leq \frac{\beta \alpha^{-\beta} \zeta(\beta)}{k - 1} + \lim_{d \to \infty} \frac{E_1(d) + E_2(d)}{d^{\beta - 1}},
\]

where \( E_1(d) \) and \( E_2(d) \) are defined as

\[
E_1(d) = \#\{ r \in \mathbb{N} : r \leq d^{1/\alpha}/4, \{ f_r^{-1}(d) \} + C_1 d^{\beta - 1}/r^\beta > 1 \}
\]

and

\[
E_2(d) = \#\{ n \in \mathbb{N} : n < C_2 d^{1/\alpha}, \{ (n^\alpha + d)^{1/\alpha} \} + 2d^{1/\alpha - 1} > 1 \}
\]

with suitable constants \( C_1, C_2 > 0 \) only depending on \( \alpha \).

**Lemma 2.4.** Let \( 1 < \alpha < 1 + 1/\sqrt{2}, \beta = 1/\alpha - 1 \) and \( C_1 > 0 \). Then \( E_1(d) \) defined in Proposition 2.3 is equal to \( o(d^{\beta - 1}) \) as \( d \to \infty \).

**Lemma 2.5.** Let \( 1 + 1/\sqrt{2} < \alpha < (\sqrt{21} + 4)/5, \beta = 1/(\alpha - 1) \) and \( C_1 > 0 \). Then \( E_1(d) \) defined in Proposition 2.3 is equal to \( o(d^{\beta - 1}) \) as \( d \to \infty \).

**Lemma 2.6.** Let \( 1 < \alpha < (\sqrt{10} + 2)/3, \beta = 1/(\alpha - 1) \) and \( C_2 > 0 \). Then \( E_2(d) \) defined in Proposition 2.3 is equal to \( o(d^{\beta - 1}) \) as \( d \to \infty \).

The above propositions and lemmas are proved in Sections 4 and 6.

2.2. **Statements in order to prove Theorem 1.3.** To prove Theorem 1.3, we extend the definition of \( \mathcal{N}_\alpha(d) \) a little in another way. For a real number \( \alpha > 1 \) and integers \( d, l \geq 1 \), define the number \( \mathcal{N}_{\alpha,l}(d) \) as

\[
\mathcal{N}_{\alpha,l}(d) = \#\{(n, r) \in \mathbb{N}^2 : r \geq l, \lfloor (n + r)^\alpha \rfloor - \lfloor n^\alpha \rfloor = d \}.
\]

If \( l = 1 \), then \( \mathcal{N}_{\alpha,l}(d) = \mathcal{N}_{\alpha,2}(d) = \mathcal{N}_\alpha(d) \). Theorem 1.3, which is proved in Section 7, is derived from the following proposition and lemmas.
Proposition 2.7. Let $1 < \alpha < 2$ and $\beta = 1/(\alpha - 1)$. Then, for all integers $d, l \geq 1$,
\[ N^{cl}_\alpha(d) \ll_{\alpha} d^{\beta-1}l^{1-\beta} + d^{2/\alpha-1} + E^{cl}_1(d-1) + E_2(d-1), \]
where $E_2(d)$ is defined in Proposition 2.3, and $E^{cl}_1(d)$ is defined as
\[ E^{cl}_1(d) = \# \{ r \in \mathbb{N} : l \leq r \leq d^{1/\alpha}/4, \{ f^{-1}_r(d) \} + C_1d^{3-1}/r^{\beta} > 1 \}
\]
with the same function $f_r$ and the same constant $C_1$ as in Proposition 2.3.

Lemma 2.8. Let $1 < \alpha \leq 3/2$ and $\beta = 1/(\alpha - 1)$. Then, for all integers $d, l \geq 1$,
\[ E^{cl}_1(d) \ll_{\alpha} d^{\beta-1}l^{1-\beta} + d^{3/3l(1-\beta)/3} + d^{2/\alpha-1}, \]
where $E^{cl}_1(d)$ is defined in Proposition 2.7.

Lemma 2.9. Let $1 < \alpha \leq 4/3$ and $\beta = 1/(\alpha - 1)$. Then, for all integers $d \geq 1$,
\[ E_2(d) \ll_{\alpha} d^{2/\alpha-1}, \]
where $E_2(d)$ is defined in Proposition 2.3.

The above proposition and lemmas are proved in Section 7.

3. Equidistribution modulo 1 and exponential sums

The following lemma is a key point in proving Propositions 2.2 and 2.3.

Lemma 3.1 (Equidistribution modulo 1). Let $1 < \alpha < 2$, $\beta = 1/(\alpha - 1)$, $r \in \mathbb{N}$, $c_1 < c_2$ and $c_2 - c_1 \in \mathbb{N}$. Then, for every convex set $C$ of $[0,1)^2$,
\[ \lim_{d \to \infty} \frac{1}{d^{\beta-1}} \# \left\{ n \in \mathbb{N} : \frac{(d + c_1)}{r\alpha} \leq n < \frac{(d + c_2)}{r\alpha}, \right\} \]
\[ \left( \{ n^\alpha \}, \{ r\alpha n^\alpha - 1 \} \right) \in C \]
\[ = \frac{\beta(c_2 - c_1)}{(r\alpha)^{\beta}} \mu(C), \]
where $\mu$ denotes the Lebesgue measure on $\mathbb{R}^2$.

Recently, Saito and Yoshida [20, Lemma 5.7] investigated the distribution of the sequence $((n^\alpha, r\alpha n^\alpha - 1))_{n=1}^\infty$ modulo 1 on short intervals. However, we cannot use their lemma here because the length
\[ \left( \frac{d + c_2}{r\alpha} \right)^{\beta} - \left( \frac{d + c_1}{r\alpha} \right)^{\beta} \sim \beta(c_2 - c_1) \frac{d^{\beta-1}}{(r\alpha)^{\beta}} \]
of the interval in Lemma 3.1 is too short. Nevertheless, the above equidistribution holds due to the assumption $c_2 - c_1 \in \mathbb{N}$. We prove Lemma 3.1 at the end of this section.
To prove Lemma 3.1, we need to estimate exponential sums. Denote by \( e(x) \) the function \( e^{2\pi ix} \), and by \( |I| \) the length of an interval \( I \) of \( \mathbb{R} \). The following lemmas are useful to estimate exponential sums.

**Lemma 3.2** (van der Corput). Let \( I \) be an interval of \( \mathbb{R} \) with \( |I| \geq 1 \) and \( f: I \to \mathbb{R} \) be a \( C^2 \) function, and let \( c \geq 1 \). If \( \lambda_2 > 0 \) satisfies that

\[
\lambda_2 \leq |f''(x)| \leq c\lambda_2
\]

for all \( x \in I \), then

\[
\sum_{n \in I \cap \mathbb{Z}} e(f(n)) \ll_e |I| \lambda_2^{1/2} + \lambda_2^{-1/2}.
\]

**Lemma 3.3** (Sargos–Gritsenko). Let \( I \) be an interval of \( \mathbb{R} \) with \( |I| \geq 1 \) and \( f: I \to \mathbb{R} \) be a \( C^3 \) function, and let \( c \geq 1 \). If \( \lambda_3 > 0 \) satisfies that

\[
\lambda_3 \leq |f'''(x)| \leq c\lambda_3
\]

for all \( x \in I \), then

\[
\sum_{n \in I \cap \mathbb{Z}} e(f(n)) \ll_e |I| \lambda_3^{1/6} + \lambda_3^{-1/3}.
\]

Lemma 3.2 is called the second derivative test; for its proof, see [9, Theorem 2.2] for instance. Lemma 3.3 was shown by Sargos [21] and Gritsenko [10] independently. Now, let us prove Lemma 3.1 by using Lemma 3.2.

**Proof of Lemma 3.1.** Define the function \( f \) as \( f(x) = x^\alpha \). If the following criterion holds, Lemma 3.1 follows in the same way as Weyl’s equidistribution theorem. **Weyl’s criterion:** for every non-zero \((h_1, h_2) \in \mathbb{Z}^2\),

\[
\lim_{d \to \infty} \frac{1}{N-M} \sum_{n=M}^{N-1} e(h_1 f(n) + h_2 f'(n)) = 0,
\]

where

\[
M = M(d) := \left\lfloor \left( \frac{d+c_1}{r\alpha} \right)^\beta \right\rfloor \quad \text{and} \quad N = N(d) := \left\lfloor \left( \frac{d+c_2}{r\alpha} \right)^\beta \right\rfloor.
\]
First, we show (3.1) when $h_1 = 0$ and $h_2 \neq 0$. Let $h_1 = 0$, and let $h_2 \neq 0$ be an integer.

Then

$$\left| \frac{1}{N - M} \sum_{n=M}^{N-1} e(h_2rf'(n)) \right|$$

$$\leq \frac{1}{N - M} \left| \sum_{n=M}^{N-1} e(h_2rf'(n)) - \sum_{n=M}^{N-1} e(h_2r(f'(M) + (n - M)f''(M))) \right|$$

$$+ \frac{1}{N - M} \left| \sum_{n=M}^{N-1} e(h_2r(f'(M) + (n - M)f''(M))) \right| \tag{3.2}$$

Taylor’s theorem implies that for every integer $M < n < N$, there exists $\theta_n \in (M, n)$ such that

$$f'(n) = f'(M) + (n - M)f''(M) + \frac{(n - M)^2}{2}f'''(\theta_n).$$

Due to this equality and the inequality $|e(y) - e(x)| \leq 2\pi |y - x|$, the first term of (3.2) bounded from above by

$$\frac{2\pi |h_2|}{N - M} \sum_{n=M}^{N-1} |f'(n) - (f'(M) + (n - M)f''(M))|$$

$$\leq \frac{2\pi |h_2|}{N - M} \sum_{n=M}^{N-1} \frac{(n - M)^2}{2} |f'''(\theta_n)| \tag{3.3}$$

To estimate (3.3), we note that

$$N - M \sim \left( \frac{d + c_2}{r\alpha} \right) - \left( \frac{d + c_1}{r\alpha} \right) \sim \frac{\beta(c_2 - c_1)}{r\alpha} \left( \frac{d}{r\alpha} \right)^{\beta - 1} (d \to \infty).$$

Eq. (3.3) is bounded from above by

$$\pi |h_2| r(N - M)^2 |f'''(M)| \ll_{\alpha, c_1, c_2, h_2} d^{2(\beta - 1)} d^{3(\alpha - 3)} = d^{-1},$$

whence (3.3) is equal to $O_{\alpha, c_1, c_2, h_2}(1/d)$, and so is the first term of (3.2). Also, the second term of (3.2) is equal to

$$\frac{1}{N - M} \left| \frac{1 - e(h_2r(N - M)f''(M))}{1 - e(h_2rf''(M))} \right|$$

$$= \frac{1}{N - M} \left| \frac{\sin(\pi h_2r(N - M)f''(M))}{\sin(\pi h_2rf''(M))} \right| \tag{3.4}.$$
Since the relations $N - M \gg_{\alpha, r, c_1} d^{\beta - 1}$ and $f''(M) \in (0, \pi/2)$ hold for sufficiently large $d \geq 1$, it follows that

$$(N - M) \left| \sin(\pi h_2 r f''(M)) \right| \geq (N - M) \cdot \frac{2}{\pi} \cdot \pi |h_2| r f''(M)$$

\[ \gg_{\alpha, r, c_1} d^{\beta - 1} d^{\beta(\alpha - 2)} = 1 \]

for sufficiently large $d \geq 1$. Thus, (3.4) is equal to

$$O_{\alpha, r, c_1} \left( \left| \sin(\pi h_2 r (N - M) f''(M)) \right| \right)$$

for sufficiently large $d \geq 1$, which vanishes as $d \to \infty$ because

$$r(N - M)f''(M) \sim r \cdot \frac{\beta(c_2 - c_1)}{r\alpha} \left( \frac{d}{r\alpha} \right)^{\beta - 1} \cdot \alpha(\alpha - 1) \left( \frac{d}{r\alpha} \right)^{\beta(\alpha - 2)}$$

$$= (c_2 - c_1) \left( \frac{d}{r\alpha} \right)^{\beta - 1} \left( \frac{d}{r\alpha} \right)^{1 - \beta}$$

$$= c_2 - c_1 \in \mathbb{N} \quad (d \to \infty).$$

Therefore, (3.4) vanishes as $d \to \infty$, and so does the second term of (3.2).

Next, we show (3.1) when $h_1 \neq 0$. Let $h_1 \neq 0$ and $h_2$ be integers. The second derivative of the function $g(x) := h_1 f(x) + h_2 r f'(x)$ satisfies that

$$|g''(x)| \leq |h_1| \alpha(\alpha - 1)x^{\alpha - 2}(1 + |h_2/h_1| r(2 - \alpha)x^{-1})$$

$$\leq 2|h_1| \alpha(\alpha - 1)x^{\alpha - 2}$$

and

$$|g''(x)| \geq |h_1| \alpha(\alpha - 1)x^{\alpha - 2}(1 - |h_2/h_1| r(2 - \alpha)x^{-1})$$

$$\geq (1/2)|h_1| \alpha(\alpha - 1)x^{\alpha - 2}$$

for every $x \geq M$ with sufficiently large $d \geq 1$. Thus, for every $M \leq x \leq N$ with sufficiently large $d \geq 1$,

$$|g''(x)| \asymp_{\alpha, r, h_1} d^{\beta(\alpha - 2)} = d^{1 - \beta}.$$  

By Lemma 3.2 and the inequality $N - M \gg_{\alpha, r, c_1} d^{\beta - 1}$, we obtain that

$$\left| \frac{1}{N - M} \sum_{n=M}^{N-1} c(h_1 f(n) + h_2 r f'(n)) \right|$$

$$\ll_{\alpha, r, h_1} \frac{(N - M)d^{(1-\beta)/2} + d^{(\beta-1)/2}}{N - M} \ll_{\alpha, r, c_1} d^{(1-\beta)/2}$$

for sufficiently large $d \geq 1$. Therefore, (3.1) follows. \qed
4. PROOFS OF PROPOSITIONS 2.2 AND 2.3

We prove Propositions 2.2 and 2.3 by using Lemma 3.1.

Proof of Proposition 2.2. Let $k \geq 2$ be an integer. Take arbitrary $\varepsilon \in (0, 1)$ and $R \in \mathbb{N}$. We also take $x_0 = x_0(k, \varepsilon, R) > 0$ such that

\begin{equation}
\frac{R^2(k - 1)^2}{2} \alpha(\alpha - 1)x_0^{\alpha-2} \leq \varepsilon.
\end{equation}

Let us show that

\[
\lim_{d \to \infty} \frac{N_{\alpha,k}(d)}{d^{\beta-1}} \geq \frac{(1 - \varepsilon)^2 \beta \alpha^{-\beta}}{k - 1} \sum_{r=1}^{R} \frac{1}{r^\beta}.
\]

Define the convex set $C_k^-(\varepsilon)$ of $\mathbb{R}^2$ as

\[ C_k^-(\varepsilon) = \{(y_0, y_1) \in \mathbb{R}^2 : 0 \leq y_0 < 1 - \varepsilon, 0 \leq y_0 + (k - 1)y_1 < 1 - \varepsilon\}. \]

Note that if $(y_0, y_1) \in C_k^-(\varepsilon)$, then $0 \leq y_0 + jy_1 < 1 - \varepsilon$ for all $j = 0, 1, \ldots, k - 1$. Taylor’s theorem implies that for all integers $n, r, j \geq 1$, $j \geq 0$ and $s$,

\begin{equation}
(n + rj)^\alpha = n^\alpha + rj\alpha n^{\alpha-1} + \frac{(rj)^2}{2} \alpha(\alpha - 1)(n + rj\theta)^{\alpha-2}
\end{equation}

\[ = \lfloor n^\alpha \rfloor + j(\lfloor r\alpha n^{\alpha-1} \rfloor + s) + \delta_s(n, r, j), \]

where $\theta = \theta(n, r, j) \in (0, 1)$ and

\[ \delta_s(n, r, j) := \lfloor n^\alpha \rfloor + j(\lfloor r\alpha n^{\alpha-1} \rfloor - s) + \frac{(rj)^2}{2} \alpha(\alpha - 1)(n + rj\theta)^{\alpha-2}. \]

Thus, if $s \in \mathbb{Z}$, $n \geq x_0$ and $(\lfloor n^\alpha \rfloor, \lfloor r\alpha n^{\alpha-1} \rfloor - s) \in C_k^-(\varepsilon)$, then $0 \leq \delta_s(n, r, j) < 1$ and $\lfloor (n + rj)^\alpha \rfloor = \lfloor n^\alpha \rfloor + j(\lfloor r\alpha n^{\alpha-1} \rfloor + s)$ for all $r = 1, 2, \ldots, R$ and $j = 0, 1, \ldots, k - 1$. (We have $s \in \{0, 1\}$ under the same assumptions, but this fact is not necessarily used here.) Also, the following equivalence holds:

\[ \lfloor r\alpha n^{\alpha-1} \rfloor + s = d \iff \left( \frac{d - s}{r\alpha} \right)^\beta \leq n < \left( \frac{d - s + 1}{r\alpha} \right)^\beta. \]

From the above facts, it follows that for sufficiently large $d \geq 1$,

\[ N_{\alpha,k}(d) \geq \sum_{r=1}^{R} \# \left\{ n \in \mathbb{N} : \forall j = 0, 1, \ldots, k - 1, \lfloor (n + rj)^\alpha \rfloor = \lfloor n^\alpha \rfloor + dj \right\} \]

\[ \geq \sum_{r=1}^{R} \sum_{s \in \mathbb{Z}} \# \left\{ n \geq x_0 : \left( \frac{d - s}{r\alpha} \right)^\beta \leq n < \left( \frac{d - s + 1}{r\alpha} \right)^\beta, (\lfloor n^\alpha \rfloor, \lfloor r\alpha n^{\alpha-1} \rfloor - s) \in C_k^-(\varepsilon) \right\}, \]
where the sum $\sum_{s \in \mathbb{Z}}$ is a finite sum due to the boundedness of $C_k^-(\varepsilon)$ (or due to $s \in \{0, 1\}$ as already stated). Using Lemma 3.1, we obtain

$$\lim_{d \to \infty} \frac{N_{\alpha,k}(d)}{d^{\beta-1}} \geq \sum_{r=1}^{R} \sum_{s \in \mathbb{Z}} \frac{\beta}{(r\alpha)^\beta} \mu\left(C_k^-(\varepsilon) \cap ([0,1) \times [-s, 1-s])\right)$$

$$= \sum_{r=1}^{R} \frac{\beta}{(r\alpha)^\beta} \mu(C_k^-(\varepsilon)) = \sum_{r=1}^{R} \frac{\beta}{(r\alpha)^\beta} \cdot \frac{(1-\varepsilon)^2}{k-1} = \frac{(1-\varepsilon)^2 \beta \alpha^{-\beta}}{k-1} \sum_{r=1}^{R} \frac{1}{r^\beta}.$$ 

Finally, letting $\varepsilon \to +0$ and $R \to \infty$, we complete the proof. \hfill \Box

**Proof of Proposition 2.3.** Let $k \geq 2$ be an integer. Take arbitrary $\varepsilon \in (0, 1)$ and $R \in \mathbb{N}$. We also take $x_0 = x_0(k, \varepsilon, R) > 0$ that satisfies (4.1). Let us show that

$$\lim_{d \to \infty} \frac{N_{\alpha,k}(d)}{d^{\beta-1}} \leq \frac{(1+\varepsilon)\beta \alpha^{-\beta}}{k-1} \sum_{r=1}^{R} \frac{1}{r^\beta}$$

$$+ \lim_{d \to \infty} \frac{E_1(d) + E_2(d)}{d^{\beta-1}} + O\left(\sum_{r>R} \frac{1}{r^\beta}\right),$$

where the implicit constant only depends on $\alpha \in (1, 2)$. Define the convex set $C_k^+(\varepsilon)$ of $\mathbb{R}^2$ as

$$C_k^+(\varepsilon) = \{(y_0, y_1) \in \mathbb{R}^2 : 0 \leq y_0 < 1, \ -\varepsilon \leq y_0 + (k-1)y_1 < 1\}.$$

**Step 1.** Take $x_0 = x_0(k, \varepsilon, R) > 0$ that satisfies (4.1). We show that if integers $d \geq 1$, $n \geq x_0$ and $1 \leq r \leq R$ satisfy $[(n+rj)^\alpha] = [n^\alpha] + dj$ for all $j = 0, 1, \ldots, k-1$, then there exists an integer $s < d$ such that

- the point $\{(n^\alpha), \{r\alpha n^{\alpha-1}\} - s\}$ lies in $C_k^+(\varepsilon)$, and
- $$\left(\frac{d-s}{r\alpha}\right)^\beta \leq n < \left(\frac{d-s+1}{r\alpha}\right)^\beta.$$ 

Set $s = d - \lfloor r\alpha n^{\alpha-1} \rfloor$. Then the second condition holds. Since Taylor’s theorem implies (4.2), it follows that

$$(n+rj)^\alpha = [n^\alpha] + j(\lfloor r\alpha n^{\alpha-1} \rfloor + s) + \delta_s(n, r, j)$$

$$= [n^\alpha] + dj + \delta_s(n, r, j) = [(n+rj)^\alpha] + \delta_s(n, r, j)$$

for all $j = 0, 1, \ldots, k-1$. This yields that $0 \leq \delta_s(n, r, j) < 1$ for all $j = 0, 1, \ldots, k-1$. Thus, the point $\{(n^\alpha), \{r\alpha n^{\alpha-1}\} - s\}$ lies in $C_k^+(\varepsilon)$.

**Step 2.** For an integer $d \geq 1$, define the number $E_0(d, R)$ as

$$E_0(d, R) = \#\{(n, r) \in \mathbb{N}^2 : r > R, \ [(n+r)^\alpha] - [n^\alpha] = d\}.$$
By Step 1, we have that for sufficiently large $d \geq 1$,

$$N_{\alpha,k}(d) = \sum_{r=1}^{R} \# \left\{ n \in \mathbb{N} : \forall j = 0, 1, \ldots, k - 1, \left\lfloor (n + rj)^\alpha \right\rfloor = \lfloor n^\alpha \rfloor + dj \right\} + E_0(d, R)$$

$$\leq \sum_{r=1}^{R} \sum_{s \in \mathbb{Z}} \# \left\{ n \geq x_0 : \left( \frac{d - s}{r^\alpha} \right)^\beta \leq n < \left( \frac{d - s + 1}{r^\alpha} \right)^\beta, \right\}$$

$$\leq R \sum_{r=1}^{R} \sum_{s \in \mathbb{Z}} \# \left\{ n \geq x_0 : \left( \frac{d - s}{r^\alpha} \right)^\beta \leq n < \left( \frac{d - s + 1}{r^\alpha} \right)^\beta, \right\}$$

$$+ R x_0 + E_0(d, R),$$

where the sum $\sum_{s \in \mathbb{Z}}$ is a finite sum due to the boundedness of $C_k^+(\varepsilon)$. Lemma 3.1 implies that

$$\lim_{d \to \infty} \frac{N_{\alpha,k}(d)}{d^{\beta-1}} \leq \sum_{r=1}^{R} \sum_{s \in \mathbb{Z}} \frac{\beta}{(r^\alpha)^\beta} \mu \left( C_k^+(\varepsilon) \cap \left( \left[ 0, 1 \right] \times [-s, 1 - s) \right) \right)$$

$$+ \lim_{d \to \infty} \frac{E_0(d, R)}{d^{\beta-1}}$$

$$= \sum_{r=1}^{R} \frac{\beta}{(r^\alpha)^\beta} \mu \left( C_k^+(\varepsilon) \right) + \lim_{d \to \infty} \frac{E_0(d, R)}{d^{\beta-1}}$$

$$= \frac{1 + \varepsilon}{k - 1} \sum_{r=1}^{R} \frac{1}{r^\beta} + \lim_{d \to \infty} \frac{E_0(d, R)}{d^{\beta-1}}.$$

Also, it follows that

$$E_0(d, R) \leq \# \{(n, r) \in \mathbb{N}^2 : r > R, \ d - 1 < f_r(n) < d + 1 \}$$

$$\leq \# \{(n, r) \in \mathbb{N}^2 : R < r \leq (d - 1)^{1/\alpha}/4, \ d - 1 < f_r(n) < d + 1 \}$$

$$+ \# \{(n, r) \in \mathbb{N}^2 : r > (d - 1)^{1/\alpha}/4, \ d - 1 < f_r(n) < d + 1 \}. \tag{4.5}$$

**Step 3.** Let us estimate the first term of (4.5). For every $1 \leq r \leq (d - 1)^{1/\alpha}$, the inverse function of $f_r : [0, \infty) \to [r^{\alpha}, \infty)$ is defined, and so are the values $f_r^{-1}(d - 1)$ and
\[ f_r^{-1}(d+1). \] Thus,
\[
\# \{(n,r) \in \mathbb{N}^2 : R < r \leq (d - 1)^{1/\alpha}/4, \; d - 1 < f_r(n) < d + 1\} 
= \sum_{R < r \leq (d - 1)^{1/\alpha}/4} \# \{n \in \mathbb{N} : f_r^{-1}(d - 1) < n < f_r^{-1}(d + 1)\} 
\leq \sum_{R < r \leq (d - 1)^{1/\alpha}/4} ([f_r^{-1}(d + 1)] - [f_r^{-1}(d - 1)]) 
= \sum_{R < r \leq (d - 1)^{1/\alpha}/4} F_r(d - 1) 
+ \sum_{R < r \leq (d - 1)^{1/\alpha}/4} ([f_r^{-1}(d - 1)] - [f_r^{-1}(d + 1)]),
\] (4.6)
where \( F_r(x) := f_r^{-1}(x + 2) - f_r^{-1}(x) \). The mean value theorem implies that
\[
F_r(d - 1) = \frac{2}{(f_r' \circ f_r^{-1})(d + \theta)} = \frac{2}{f_r'(y_\theta)} < \frac{2(y_\theta + r)^{2-\alpha}}{\alpha(\alpha - 1)r},
\]
where \( \theta = \theta(r,d) \in (-1,1) \) and \( y_\theta := f_r^{-1}(d + \theta) \). If \( d \geq 1 \) and \( 1 \leq r \leq (d - 1)^{1/\alpha}/4 < d^{1/\alpha} \), then the inequalities \( r < (d/r)^\beta \),
\[
y_\theta + r < \left(\frac{d + \theta}{r\alpha}\right)^\beta + (d/r)^\beta \ll_\alpha (d/r)^\beta,
F_r(d - 1) \ll_\alpha r^{-1}(d/r)^{\beta(2-\alpha)} = d^{\beta - 1}/r^\beta
\]
hold. Thus, the first sum of (4.6) is equal to \( O_\alpha(d^{3-1} \sum_{r > R} 1/r^\beta) \). The second sum of (4.6) is bounded from above by
\[
\# \{r \leq (d - 1)^{1/\alpha}/4 : [f_r^{-1}(d-1)] > [f_r^{-1}(d+1)]\} 
\leq \# \{r \leq (d - 1)^{1/\alpha}/4 : [f_r^{-1}(d-1)] + F_r(d-1) \geq 1\} 
\leq \# \{r \leq (d - 1)^{1/\alpha}/4 : [f_r^{-1}(d-1)] + C_1(d-1)^{\beta - 1}/r^\beta > 1\} 
= E_1(d - 1),
\]
where \( C_1 > 0 \) is a constant only depending on \( \alpha \). Therefore, the first term of (4.5) is less than or equals to \( O_\alpha(d^{3-1} \sum_{r > R} 1/r^\beta) + E_1(d - 1) \).

**Step 4.** Let us estimate the second term of (4.5). If \( d \geq 2 \), \( r > (d - 1)^{1/\alpha}/4 \) and \( f_r(n) < d + 1 \), then the mean value theorem implies that
\[
(d - 1)^{1/\alpha}n^{\alpha - 1} \ll (d - 1)^{1/\alpha}(\alpha/4)n^{\alpha - 1}
< ran^{\alpha - 1} < f_r(n) < d + 1 \ll d - 1,
\]
whence \( n \ll_{\alpha} (d - 1)^{1/\alpha} \). Thus, for some constant \( C_2 > 0 \) only depending on \( \alpha \),

\[
\# \{(n, r) \in \mathbb{N}^2 : r > (d - 1)^{1/\alpha}/4, \; d - 1 < f_r(n) < d + 1\}
\leq \# \left\{(n, r) \in \mathbb{N}^2 : n < C_2(d - 1)^{1/\alpha}, \; (n^\alpha + d - 1)^{1/\alpha} < n + r < (n^\alpha + d + 1)^{1/\alpha} \right\}
\leq \sum_{n < C_2(d - 1)^{1/\alpha}} \left( \lfloor (n^\alpha + d + 1)^{1/\alpha} \rfloor - \lfloor (n^\alpha + d - 1)^{1/\alpha} \rfloor \right)
= \sum_{n < C_2(d - 1)^{1/\alpha}} G_n(d - 1)
+ \sum_{n < C_2(d - 1)^{1/\alpha}} \left( \{n^\alpha + d + 1)^{1/\alpha}\} - \{n^\alpha + d - 1)^{1/\alpha}\} \right),
\]

where \( G_n(x) := (n^\alpha + x + 2)^{1/\alpha} - (n^\alpha + x)^{1/\alpha} \). Since the mean value theorem implies

\[
G_n(d - 1) < \frac{2}{\alpha}(n^\alpha + d - 1)^{1/\alpha-1} < 2(d - 1)^{1/\alpha-1},
\]

the first sum of (4.7) is bounded from above by

\[
C_2(d - 1)^{1/\alpha} : 2(d - 1)^{1/\alpha-1} \ll_{\alpha} d^{2/\alpha-1}.
\]

Moreover, the second sum of (4.7) is bounded from above by

\[
\# \{n < C_2(d - 1)^{1/\alpha} : \{n^\alpha + d - 1)^{1/\alpha}\} > \{n^\alpha + d + 1)^{1/\alpha}\} \leq \# \{n < C_2(d - 1)^{1/\alpha} : \{n^\alpha + d - 1)^{1/\alpha}\} + G_n(d - 1) \geq 1\}
\leq \# \{n < C_2(d - 1)^{1/\alpha} : \{n^\alpha + d - 1)^{1/\alpha}\} + 2(d - 1)^{1/\alpha-1} > 1\}
= E_2(d - 1).
\]

Therefore, the second term of (4.5) is less than or equal to \( O_{\alpha}(d^{2/\alpha-1}) + E_2(d - 1) \).

**Step 5.** By Steps 2–4, the inequality (4.3) holds. Letting \( \varepsilon \to +0 \) and \( R \to \infty \) in (4.3), we complete the proof. \( \square \)

5. **Discrepancy and Preliminary Lemmas**

This section is a preparation to prove Lemmas 2.4, 2.6. For a sequence \( (x_n)_{n=1}^N \) of real numbers, define the discrepancy \( D(x_1, \ldots, x_N) \) as

\[
D(x_1, \ldots, x_N)
= \sup_{0 \leq a < b \leq 1} \left| \frac{\# \{n \in \mathbb{N} : n \leq N, \; a \leq \{x_n\} < b\}}{N} - (b - a) \right|.
\]
Lemma 5.3. Let $0 < H < 1$. By Lemma 5.1, the estimation of discrepancies reduces to that of exponential sums. In order to apply Lemmas 3.2 and 3.3, we examine the second and third derivatives of the function $f_r^{-1}(d)$ of $r$, where $f_r$ is defined in Proposition 2.3.

Lemma 5.4. Let $1 < \alpha < 2$, $\beta = 1/(\alpha - 1)$, $d \in \mathbb{N}$ and $c > 0$. Define the function $y$ of $0 < r \leq d^{1/\alpha}$ as $y = f_r^{-1}(d)$. Then $y \leq y + r \leq (1 + c^{-1})y$ for all $0 < r \leq d^{1/\alpha}/f_1(c)^{1/\alpha}$.

Proof. Let $0 < r \leq d^{1/\alpha}/f_1(c)^{1/\alpha}$. Then $f_r(cr) = f_1(c)r^{\alpha} \leq d = f_r(y)$. Thus, $cr \leq y$ and $y \leq y + r \leq (1 + c^{-1})y$.

Lemma 5.5. Let $1 < \alpha < 2$, $\beta = 1/(\alpha - 1)$ and $d \in \mathbb{N}$. Define the function $y$ of $0 < r \leq d^{1/\alpha}$ as $y = f_r^{-1}(d)$. Then

\begin{equation}
(5.1) \quad y'' = \frac{d(\alpha - 1)}{((y + r)^{\alpha - 1} - y^{\alpha - 1})^2} y^{2-\alpha}(y + r)^{2-\alpha}.
\end{equation}

In particular, $y'' \approx d^3/r^{3\alpha + 2}$ for all $0 < r \leq d^{1/\alpha}/2$.

Proof. Differentiating both sides of $f_r(y) = d$ with respect to $r$, we obtain

\begin{equation}
(5.2) \quad -y' = \frac{(y + r)^{\alpha - 1}}{(y + r)^{\alpha - 1} - y^{\alpha - 1}} \quad \text{and} \quad -y' - 1 = \frac{y^{\alpha - 1}}{(y + r)^{\alpha - 1} - y^{\alpha - 1}}.
\end{equation}
Lemma 5.5. Let \( y \) suffocating \( \alpha \) such that \( f \). Since \( y \), it follows that \( y \), we have
\[
\frac{y''}{y'} = (\alpha - 1) \left( \frac{y'}{y + r} + \frac{(y + r)^{\alpha - 2} - y'y^{\alpha - 2}}{(y + r)^{\alpha - 1} - y^{\alpha - 1}} \right)
\]
\[
= (\alpha - 1) \frac{-(y' + 1)(y + r)^{\alpha - 1} + y'y^{\alpha - 2}}{(y + r)^{\alpha - 1} - y^{\alpha - 1}}
\]
\[
= (\alpha - 1)y^{\alpha - 2} \cdot \frac{-y + ry'}{(y + r)((y + r)^{\alpha - 1} - y^{\alpha - 1})}.
\]
Since
\[
y - ry' = \frac{y((y + r)^{\alpha - 1} - y^{\alpha - 1}) + r(y + r)^{\alpha - 1}}{(y + r)^{\alpha - 1} - y^{\alpha - 1}}
\]
\[
= \frac{(y + r)^{\alpha} - y^{\alpha}}{(y + r)^{\alpha - 1} - y^{\alpha - 1}} = \frac{d}{(y + r)^{\alpha - 1} - y^{\alpha - 1}},
\]
it follows that
\[
\frac{y''}{y'} = \frac{-d(\alpha - 1)y^{\alpha - 2}}{(y + r)((y + r)^{\alpha - 1} - y^{\alpha - 1})^2}
\]
and
\[
y'' = \frac{d(\alpha - 1)y^{\alpha - 2}(y + r)^{\alpha - 2}}{((y + r)^{\alpha - 1} - y^{\alpha - 1})^3}.
\]
Next, assume \( 0 < r \leq d^{1/\alpha}/2 \). Then \( f_1(1) = 2^{\alpha} - 1 < 2^{\alpha} \) and \( r \leq d^{1/\alpha}/2 < d^{1/\alpha}/f_1(1)^{1/\alpha} \). By Lemma 5.3, the inequality \( y \leq y + r \leq 2y \) holds. Also, the mean value theorem implies that
\[
 r^{\alpha}y^{\alpha - 1} < f_1(y) < r^{\alpha}(y + r)^{\alpha - 1}.
\]
Since \( f_r(y) = d \), it follows that \( (d/r^{\alpha})^3 < y + r \) and \( y < (d/r^{\alpha})^3 \). Therefore,
\[
y \asymp y + r \asymp (d/r^{\alpha})^3 \text{ and } (y + r)^{\alpha - 1} - y^{\alpha - 1} \asymp (r^{\alpha}y^{\alpha - 2}.
\]
whence \( y'' \asymp \frac{d^3}{r^{3/2}} \) for all \( 0 < r \leq d^{1/\alpha}/2 \).
Proof. Taking the logarithmic derivative of both sides in (5.1), we have

\[
\frac{y'''}{y''} = -3(\alpha - 1) \frac{(y' + 1)(y + r)^{\alpha-2} - y'y^{\alpha-2}}{(y + r)^{\alpha-1} - y^{\alpha-1}} - (2 - \alpha) \left( \frac{y'}{y} + \frac{y' + 1}{y + r} \right).
\]

By (5.2), it follows that

\[
\frac{y'''}{y''} = 3(\alpha - 1) \frac{y^{\alpha-1}(y + r)^{\alpha-2} - (y + r)^{\alpha-1}y^{\alpha-2}}{((y + r)^{\alpha-1} - y^{\alpha-1})^2}
+ \frac{2 - \alpha}{(y + r)^{\alpha-1} - y^{\alpha-1}} \left( \frac{y + r}{y} + \frac{y^{\alpha-1}}{y + r} \right).
\]

(5.5)

The first term of the right-hand side in (5.5) is equal to

\[
3(\alpha - 1) \frac{-ry^{\alpha-2}(y + r)^{\alpha-2}}{((y + r)^{\alpha-1} - y^{\alpha-1})^2}
= 3(\alpha - 1) \frac{-ry^{\alpha-1}(y + r)^{\alpha-1}}{((y + r)^{\alpha-1} - y^{\alpha-1})^2 y(y + r)},
\]

and the second term of the right-hand side in (5.5) is equal to

\[
(2 - \alpha) \frac{((y + r)^{\alpha-1} - y^{\alpha-1})(y + r)\alpha + y^\alpha)}{((y + r)^{\alpha-1} - y^{\alpha-1})^2 y(y + r)}
= (2 - \alpha) \frac{y + r)^{2\alpha-1} - y^{2\alpha-1} - ry^{\alpha-1}(y + r)^{\alpha-1}}{((y + r)^{\alpha-1} - y^{\alpha-1})^2 y(y + r)}.
\]

Thus,

\[
\frac{y'''}{y''}(y + r)^{\alpha-1} - y^{\alpha-1})^2 y(y + r)
= -3(\alpha - 1)ry^{\alpha-1}(y + r)^{\alpha-1}
+ (2 - \alpha)((y + r)^{2\alpha-1} - y^{2\alpha-1} - ry^{\alpha-1}(y + r)^{\alpha-1})
= -(2\alpha - 1)ry^{\alpha-1}(y + r)^{\alpha-1} + (2 - \alpha)((y + r)^{2\alpha-1} - y^{2\alpha-1}),
\]

which implies (5.4).
Next, assume $0 < r \leq d^{1/\alpha}/2$. Then (5.3) holds. By the mean value theorem, the numerator of the right-hand side in (5.4) is equal to

$$(2\alpha - 1)ry^{\alpha - 1}(y + r)^{\alpha - 1} - (2 - \alpha)((y + r)^{2\alpha - 1} - y^{2\alpha - 1})$$

$$= (2\alpha - 1)r(y^{\alpha - 1}(y + r)^{\alpha - 1} - (2 - \alpha)(y + r\theta)^{2\alpha - 2})$$

where $\theta = \theta(y, r) \in (0, 1)$. Since the inequality $y + r \leq 2y$ holds by Lemma 5.3 it turns out that

$$y^{\alpha - 1}(y + r)^{\alpha - 1} > y^{\alpha - 1}(y + r)^{\alpha - 1} - (2 - \alpha)(y + r\theta)^{2\alpha - 2}$$

$$> 2^{1-\alpha}(y + r)^{2\alpha - 2} - (2 - \alpha)(y + r)^{2\alpha - 2} = (2^{1-\alpha} + \alpha - 2)(y + r)^{2\alpha - 2}.$$  

Noting the inequality $2^{1-\alpha} + \alpha - 2 > 0$, we have

$$y^{\alpha - 1}(y + r)^{\alpha - 1} - (2 - \alpha)(y + r\theta)^{2\alpha - 2} \simeq y^{2\alpha - 2}.$$  

Thus, the numerator of the right-hand side in (5.4) is $\simeq \alpha ry^{2\alpha - 2}$. This, (5.4) and (5.3) yield that $-y'' \simeq \alpha d^{3}/r^{\beta + 3}$.

6. Proofs of Lemmas 2.4–2.6

We prove Lemmas 2.4–2.6 by using Lemmas 3.2, 3.3, 5.1, 5.4 and 5.5.

Proof of Lemma 2.4. Note that (i) the inequality $(3 - 2\beta)/(\beta - 1) < \beta - 1$ is equivalent to $\alpha < 1 + 1/\sqrt{2} \approx 1.707$ if $1 < \alpha < 2$; (ii) the inequality $(3 - 2\beta)/(\beta - 1) < 1/\alpha$ is equivalent to $\alpha < (\sqrt{10} + 2)/3 \approx 1.721$ if $1 < \alpha < 2$, since

$$(3 - 2\beta)/(\beta - 1) < 1/\alpha \iff 3\alpha - 5 - (3(\alpha - 1) - 2) < 1/\alpha$$

$$\iff 3\alpha^2 - 4\alpha - 2 < 0 \iff \alpha < (\sqrt{10} + 2)/3.$$  

Now, we can take a positive number $\gamma_1$ with $(3 - 2\beta)/(\beta - 1) < \gamma_1 < \min\{1/\alpha, \beta - 1\}$ due to the assumption $1 < \alpha < 1 + 1/\sqrt{2}$. Then, for sufficiently large $d \geq 1$,

$$E_1(d) = \#\{r \leq d^{1/\alpha}/4 : \{f_r^{-1}(d)\} + C_1d^{\beta - 1}/r^\beta > 1\}$$

$$\leq d^{\gamma_1} + \sum_{j=\lceil \gamma_1 \log_2 d \rceil}^{(1/\alpha)\log_2 d} \#\{2^j < r \leq 2^{j+1} : \{f_r^{-1}(d)\} + C_1d^{\beta - 1}/r^\beta > 1\}.$$  

(6.1)
Denote by $D_1(d, R)$ the discrepancy of the sequence $(f_r^{-1}(d))_{R < r \leq 2R}$. The second term of (6.1) is bounded from above by

$$\sum_{j = \lceil \gamma \log_2 d \rceil}^{\lfloor (1/\alpha) \log_2 d \rfloor - 2} 2^j \leq \sum_{j = \lceil \gamma \log_2 d \rceil}^{\lfloor (1/\alpha) \log_2 d \rfloor - 2} \left( C_1 d^{\beta - 1}/2^j \beta + D_1(d, 2^j) \right) \cdot 2^j \leq \alpha d^{(1 - \gamma)(\beta - 1)} + \sum_{j = \lceil \gamma \log_2 d \rceil}^{\lfloor (1/\alpha) \log_2 d \rfloor - 2} D_1(d, 2^j) \cdot 2^j.$$ (6.2)

Now, we estimate the discrepancy $D_1(d, R)$ when $R$ is an integer with $d^{\gamma_1}/2 < R \leq d^{1/\alpha}/2$. Set $H = d^{-\beta/3} R^{(\beta + 2)/3}$. By Lemma 5.1

$$D_1(d, R) R \ll H^{-1} R + \sum_{1 \leq h \leq H} \frac{1}{h} \left| \sum_{R < r \leq 2R} e(h f_r^{-1}(d)) \right|.$$ 

Since $2R \leq d^{1/\alpha}/2$, Lemmas 3.2 and 5.4 imply that

$$\sum_{R < r \leq 2R} e(h f_r^{-1}(d)) \ll \alpha R(h d^\beta / R^{\beta + 2})^{1/2} + h^{1/2} d^{\beta/2} R^{-\beta/2} + h^{-1/2} d^{-\beta/2} R^{(\beta + 2)/2}.$$

Thus,

$$D_1(d, R) R \ll H^{-1} R + \sum_{1 \leq h \leq H} \left( h^{1/2 - 1/2} d^{\beta/2} R^{-\beta/2} + h^{-1/2} d^{-\beta/2} R^{(\beta + 2)/2} \right) \ll H^{-1} R + H^{1/2} d^{\beta/2} R^{-\beta/2} + d^{-\beta/2} R^{(\beta + 2)/2} \ll d^{3/3} R^{(1 - \beta)/3} + d^{-\beta/2} R^{(\beta + 2)/2}.$$
The second term of (6.2) is bounded from above as follows:

\[
\sum_{j=[\gamma_1 \log_2 d]}^{\lfloor (1/\alpha) \log_2 d \rfloor - 2} D_1(d, 2^j) \cdot 2^j
\]

Since \( \gamma_1 > (3 - 2\beta)/(\beta - 1) \), it turns out that

\[
\beta/3 - \gamma_1(\beta - 1)/3 < \beta/3 - (3 - 2\beta)/3 = \beta - 1,
\]

whence the first term of (6.3) is equal to \( o(d^{3-1}) \). Also, noting the equality \( \alpha \beta = \beta + 1 \),

we have the following equivalence if \( \alpha > 1 \):

\[
1/2 \alpha < \beta - 1 \iff 1/2 < \beta + 1 - \alpha \iff -1/2 < \beta - \alpha.
\]

From the assumption \( 1 < \alpha < 1 + 1/\sqrt{2} \), it follows that

\[
\beta - \alpha > \sqrt{2} - (1 + 1/\sqrt{2}) = \sqrt{2}/2 - 1 > 1/2 - 1 = -1/2.
\]

Thus, the second term of (6.3) is equal to \( o(d^{3-1}) \). Moreover, (6.2) and (6.1) are also equal to \( o(d^{3-1}) \). Therefore, \( E_1(d) = o(d^{3-1}) \).

**Proof of Lemma 2.5.** Note that the inequality \( 1 < \beta \leq \sqrt{2} \) is equivalent to \( 1 + 1/\sqrt{2} \leq \alpha < 2 \). Also, if \( 5/4 < \alpha < 2 \) (i.e., \( 1 < \beta < 4 \)), then the following equivalences hold:

\[
\beta - 1 \leq \frac{3 - 2\beta}{\beta - 1} \iff \alpha \geq 1 + 1/\sqrt{2};
\]

\[
\frac{3 - 2\beta}{\beta - 1} < \frac{6\beta - 7}{4 - \beta} \iff \alpha < \frac{\sqrt{21} + 4}{5} \approx 1.717;
\]

\[
\frac{3 - 2\beta}{\beta - 1} < \frac{4\beta - 3}{\beta + 3} \iff \alpha < \frac{\sqrt{10} + 2}{3} \approx 1.721;
\]

\[
\frac{3\alpha - 5}{2 - \alpha} = \frac{3 - 2\beta}{\beta - 1} < 1/\alpha \iff \alpha < \frac{\sqrt{10} + 2}{3}.
\]

Now, we can take positive numbers \( \gamma_1 \) and \( \gamma_2 \) with

\[
0 < \gamma_2 < \beta - 1 \leq \frac{3 - 2\beta}{\beta - 1} < \gamma_1 < \min \left\{ \frac{6\beta - 7}{4 - \beta}, \frac{4\beta - 3}{\beta + 3}, 1/\alpha \right\}
\]
due to the assumption $1 + 1/\sqrt{2} \leq \alpha < (\sqrt{21} + 4)/5$. Then, for sufficiently large $d \geq 1$,

$$E_1(d) = \# \{ r \leq d^{1/\alpha}/4 : \{ f^{-1}_r(d) \} + C_1 d^\beta / r^\beta > 1 \}$$

$$\leq d^{\gamma_2} + \sum_{j = \lceil \gamma_1 \log_2 d \rceil}^{\lceil (1/\alpha \log_2 d) \rceil - 1} \# \{ 2^j < r \leq 2^j + 1 : \{ f^{-1}_r(d) \} + C_1 d^\beta / r^\beta > 1 \}$$

$$+ \sum_{j = \lceil \gamma_1 \log_2 d \rceil}^{\lceil (1/\alpha \log_2 d) \rceil - 2} \# \{ 2^j < r \leq 2^j + 1 : \{ f^{-1}_r(d) \} + C_1 d^\beta / r^\beta > 1 \}. $$

In the same way as the proof of Lemma 2.4, it follows that

$$E_1(d) \ll_{\alpha} d^{\gamma_2} + d^{(1-\gamma_2)(\beta-1)} + d^{(1-\gamma_1)(\beta-1)}$$

$$+ \sum_{j = \lceil \gamma_1 \log_2 d \rceil}^{\lceil (1/\alpha \log_2 d) \rceil - 1} D_1(d, 2^j) \cdot 2^j + \sum_{j = \lceil \gamma_1 \log_2 d \rceil}^{\lceil (1/\alpha \log_2 d) \rceil - 2} D_1(d, 2^j) \cdot 2^j. \quad (6.6)$$

Now, we estimate the discrepancy $D_1(d, R)$ when $R$ is an integer with $d^{\gamma_2}/2 < R \leq d^{\gamma_1}/2 < d^{1/\alpha}/4$. Set $H = d^{-\beta/7} R^{(\beta+3)/7}$. Since $2R \leq d^{1/\alpha}/2$, Lemmas 5.1, 3.3 and 5.5 imply that

$$D_1(d, R) \ll_{\alpha} H^{-1} R + H^{1/6} R \cdot (d^\beta / R^\beta + 3)^{1/6} + (d^\beta / R^\beta + 3)^{-1/3}$$

$$\ll d^{3/7} R^{(4-\beta)/7} + d^{-\beta/3} R^{(\beta+3)/3}. $$

Noting the inequality $1 < \beta \leq \sqrt{2} < 4$, we obtain

$$\ll_{\alpha} \sum_{j = \lceil \gamma_1 \log_2 d \rceil}^{\lceil \gamma_1 \log_2 d \rceil - 1} D_1(d, 2^j) \cdot 2^j \quad (6.7)$$

$$\ll_{\alpha} \sum_{j = \lceil \gamma_1 \log_2 d \rceil}^{\lceil \gamma_1 \log_2 d \rceil - 1} (d^{\beta/7} \cdot 2^{j(4-\beta)/7} + d^{-\beta/3} \cdot 2^{j(\beta+3)/3})$$

$$\ll_{\alpha} d^{3/7} \cdot d^{\gamma_1(4-\beta)/7} + d^{-\beta/3} \cdot d^{\gamma_1(\beta+3)/3}. $$

Also, the inequality (6.3) holds in the same way as the proof of Lemma 2.4. Thus, the inequalities (6.3), (6.6) and (6.7) yield that

$$E_1(d) \ll_{\alpha} d^{\gamma_2} + d^{(1-\gamma_2)(\beta-1)} + d^{(1-\gamma_1)(\beta-1)}$$

$$+ d^{\beta/7+\gamma_1(4-\beta)/7} + d^{-\beta/3+\gamma_1(\beta+3)/3}$$

$$+ d^{\beta/3-\gamma_1(\beta-1)/3} + d^{1/2\alpha}. \quad (6.8)$$
By (6.5), all terms of (6.8) except for the last term are equal to $o(d^{3-1})$. Also, noting the inequality $\alpha < (\sqrt{21} + 4)/5 < (\sqrt{10} + 2)/3$, we have

$$\beta - \alpha > 3/((\sqrt{10} - 1) - (\sqrt{10} + 2)/3) = (\sqrt{10} + 1)/3 - (\sqrt{10} + 2)/3 = -1/3 > -1/2.$$  

By (6.4), the last term of (6.8) is equal to $o(d^{3-1})$. Therefore, $E_1(d) = o(d^{3-1})$. □

**Proof of Lemma 2.6.** Denote by $D_2(d, N)$ the discrepancy of the sequence $((n^\alpha + d)^{1/\alpha})_{N \leq n < 2N}$. Then

$$E_2(d) = \# \{n < C_2d^{1/\alpha} : (n^\alpha + d)^{1/\alpha} + 2d^{1/\alpha - 1} > 1\}$$

$$\leq \sum_{j=0}^{[\log_2(C_2d^{1/\alpha})]} \# \{2^j \leq n < 2^{j+1} : (n^\alpha + d)^{1/\alpha} + 2d^{1/\alpha - 1} > 1\}$$

$$\leq \sum_{j=0}^{[\log_2(C_2d^{1/\alpha})]} (2d^{1/\alpha - 1} \cdot 2^j + D_2(d, 2^j) \cdot 2^j)$$

$$\ll_\alpha d^{1/\alpha - 1} \cdot d^{1/\alpha} + \sum_{j=0}^{[\log_2(C_2d^{1/\alpha})]} D_2(d, 2^j) \cdot 2^j.$$  

(6.10)

We estimate the discrepancy $D_2(d, N)$ when $d$ and $N$ are positive integers with $1 \leq N \leq C_2d^{1/\alpha}$. Set $H = d^{(\alpha - 1)/3\alpha} N^{(2 - \alpha)/3}$. By Lemma 5.1

$$D_2(d, N)N \ll H^{-1}N + \sum_{1 \leq h \leq H} \frac{1}{h} \left| \sum_{N \leq n < 2N} e(h(n^\alpha + d)^{1/\alpha}) \right|.$$  

The second derivative of the function $y = (x^\alpha + d)^{1/\alpha}$ is equal to

$$y'' = (\alpha - 1)dx^{\alpha - 2}(x^\alpha + d)^{1/\alpha - 2},$$

which satisfies that $y'' \approx_\alpha d^{1/\alpha - 1}N^{\alpha - 2}$ for all $N \leq x < 2N$ (because of $0 < x^d \ll d$). By Lemma 3.2

$$\sum_{N \leq n < 2N} e(h(n^\alpha + d)^{1/\alpha})$$

$$\ll_\alpha N(hd^{1/\alpha - 1}N^{\alpha - 2})^{1/2} + (hd^{1/\alpha - 1}N^{\alpha - 2})^{-1/2}$$

$$= h^{1/2}d^{(1-\alpha)/2\alpha}N^{\alpha/2} + h^{-1/2}d^{(\alpha-1)/2\alpha}N^{(2-\alpha)/2}. $$
Thus,
\[
D_2(d, N)N \ll_{\alpha} H^{-1}N + H^{1/2}d^{(1-\alpha)/2\alpha}N^{\alpha/2} + d^{(\alpha-1)/2\alpha}N^{(2-\alpha)/2}
\ll d^{(1-\alpha)/3\alpha}N^{(\alpha+1)/3} + d^{(\alpha-1)/2\alpha}N^{(2-\alpha)/2},
\]
and moreover, the second term of \(6.10\) is bounded from above as follows:
\[
\left\lfloor \log_2\left(\frac{C_2d_1}{\alpha}\right) \right\rfloor \sum_{j=0}^{[\log_2(C_2d^{1/\alpha})]} D_2(d, 2^j) \cdot 2^j
\ll_{\alpha} d^{(1-\alpha)/3\alpha} \cdot 2^{(\alpha+1)/3} + d^{(\alpha-1)/2\alpha} \cdot 2^{(2-\alpha)/2}
= d^{2/3\alpha} + d^{1/2\alpha} \ll d^{2/3\alpha}.
\]
Also, noting the equality \(\alpha \beta = \beta + 1\), we have the following equivalence if \(\alpha > 1\):
\[
2/3\alpha < \beta - 1 \iff 2/3 < \beta + 1 - \alpha \iff -1/3 < \beta - \alpha.
\]
Since \(6.9\) follows from the assumption \(1 < \alpha < (\sqrt{10}+2)/3\), the second term of \(6.10\) is equal to \(o(d^{\beta-1})\). Since the first term of \(6.10\) is also equal to \(o(d^{\beta-1})\), we complete the proof. \(\square\)

7. Proof of Theorem 1.3

We prove Proposition 2.7, Lemmas 2.8, 2.9 and Theorem 1.3.

Proof of Proposition 2.7. The number \(N_{\alpha}^{\geq l}(d)\) is equal to \(E_0(d, l - 1)\) defined in \((4.4)\). Thus, by \((4.5)\), we have
\[
N_{\alpha}^{\geq l}(d) \leq \#\{(n, r) \in \mathbb{N}^2 : l \leq r \leq (d - 1)^{1/\alpha}/4, \ d - 1 < f_r(n) < d + 1\}
+ \#\{(n, r) \in \mathbb{N}^2 : r > (d - 1)^{1/\alpha}/4, \ d - 1 < f_r(n) < d + 1\}.
\]
The first term of the right-hand side is equal to \(O_{\alpha}(d^{\beta-1} \sum_{r \geq l} 1/r^\beta) + E_{1}^{\geq l}(d - 1)\), and the second term of the right-hand side is equal to \(O_{\alpha}(d^{2/\alpha - 1}) + E_2(d - 1)\) in the same way as Steps 3 and 4 of the proof of Proposition 2.3. Since \(\sum_{r \geq l} 1/r^\beta \ll_{\alpha} l^{1-\beta}\), we obtain the desired inequality. \(\square\)
Proof of Lemma 2.8. By the definition of $E_1^{≥ l}(d)$,

$$E_1^{≥ l}(d) = \# \{ r \in \mathbb{N} : l \leq r \leq d^{1/\alpha}/4, \{ f_r^{-1}(d) \} + C_1d^{3-1}/r^\beta > 1 \}$$

$$\leq 1 + \sum_{j=\lceil \log_2 l \rceil} \# \{ 2^j < r \leq 2^{j+1} : \{ f_r^{-1}(d) \} + C_1d^{3-1}/r^\beta > 1 \}.$$ 

Thus,

$$E_1^{≥ l}(d) \ll_\alpha d^{\beta l - 1 - \beta} + d^{3/2}(1-\beta)/3 + d^{1/2\alpha}$$

in the same way as the proof of Lemmas 2.4 (replace $d^\gamma$ with $l^\alpha$). Since the inequality $1/2\alpha \leq 2/\alpha - 1$ follows from the assumption $1 < \alpha \leq 3/2$, we obtain the desired inequality. \hfill \Box

Proof of Lemma 2.9. By the proof of Lemmas 2.6,

$$E_2(d) \ll_\alpha d^{2/\alpha - 1} + d^{2/3\alpha}.$$ 

Since the inequality $2/3\alpha \leq 2/\alpha - 1$ follows from the assumption $1 < \alpha \leq 4/3$, we obtain the desired inequality. \hfill \Box

Proof of Theorem 1.3. Let $\alpha > 1$ and $N \in \mathbb{N}$. First, we show that

(7.1) $$\mathcal{E}_\alpha(N) \leq N^2 + 6 \sum_{l=1}^{N-1} \sum_{(l-1)\alpha N^{\alpha-1} \leq d \leq l\alpha N^{\alpha-1}} N_\alpha^{≥ l}(d)^2.$$ 

Noting that the equation $\lfloor n_1^\alpha \rfloor + \lfloor n_2^\alpha \rfloor = \lfloor n_3^\alpha \rfloor + \lfloor n_4^\alpha \rfloor$ is equivalent to

$$\exists d \in \mathbb{Z}, \lfloor n_1^\alpha \rfloor - \lfloor n_2^\alpha \rfloor = d \quad \text{and} \quad \lfloor n_3^\alpha \rfloor - \lfloor n_4^\alpha \rfloor = d,$$

we have

$$\mathcal{E}_\alpha(N) = N^2 + 2 \sum_{d=1}^{\infty} \left( \# \left\{ (n, r) \in \mathbb{N}^2 : \left\lfloor (n + r)^\alpha \right\rfloor - \lfloor n^\alpha \rfloor = d \right\} \right)^2.$$

(7.2) 

$$= N^2 + 2 \sum_{d=1}^{\infty} \sum_{r=1}^{N-1} \# \left\{ n \in \mathbb{N} : \left\lfloor (n + r)^\alpha \right\rfloor - \lfloor n^\alpha \rfloor = d \right\}^2.$$ 

Set the summands of the inner sum in (7.2) as

$$N_{r,d} = \# \left\{ n \in \mathbb{N} : \left\lfloor (n + r)^\alpha \right\rfloor - \lfloor n^\alpha \rfloor = d \right\} (d \geq 1, 1 \leq r < N).$$
Then

$$\mathcal{E}_\alpha(N) = N^2 + 2 \sum_{d=1}^\infty \left( \sum_{r=1}^{N-1} N_{r,d} \right)^2$$

(7.3)

$$= N^2 + 2 \sum_{d=1}^\infty \sum_{r=1}^{N-1} N_{r,d}^2 + 4 \sum_{d=1}^\infty \sum_{1 \leq r_1 < r_2 < N} N_{r_1,d} N_{r_2,d}.$$ 

Now, consider the equation \( \lfloor (n + r)^\alpha \rfloor - \lfloor n^\alpha \rfloor = d \) in the variables \( n, r \in \mathbb{N} \) when \( d \geq 1 \) is an integer. If \( n + r \leq N \), then the mean value theorem implies that

(7.4) 

\[ d - 1 < (n + r)^\alpha - n^\alpha < r\alpha(n + r)^{\alpha - 1} \leq r\alpha N^{\alpha - 1}. \]

By this,

(7.5) 

\[ \sum_{d=1}^\infty \sum_{r=1}^{N-1} N_{r,d}^2 = \sum_{r=1}^{N-1} \sum_{0 \leq d - 1 < r\alpha N^{\alpha - 1}} N_{r,d}^2 \leq \sum_{l=1}^{N-1} \sum_{(l-1)\alpha N^{\alpha - 1} \leq d - 1 < l\alpha N^{\alpha - 1}} N_{\alpha}^{\geq 1}(d)^2, \]

where we have used

(7.6) 

\[ \sum_{r=1}^{N-1} \sum_{0 \leq d - 1 < r\alpha N^{\alpha - 1}} = \sum_{r=1}^{N-1} \sum_{l=1}^{(l-1)\alpha N^{\alpha - 1} \leq d - 1 < l\alpha N^{\alpha - 1}} \sum_{l=1}^{N-1} \sum_{r=l}^{(l-1)\alpha N^{\alpha - 1} \leq d - 1 < l\alpha N^{\alpha - 1}} \]

to obtain the last inequality. Also,

(7.7) 

\[ \sum_{d=1}^\infty \sum_{1 \leq r_1 < r_2 < N} N_{r_1,d} N_{r_2,d} = \sum_{2 \leq r_2 < N} \sum_{d=1}^\infty \sum_{1 \leq r_1 < r_2} N_{r_1,d} N_{r_2,d}. \]
By (7.6) (when replacing \( r \) and \( N \) with \( r_1 \) and \( r_2 \), respectively) and the fact around (7.4),

\[
\sum_{d=1}^{\infty} \sum_{1 \leq r_1 < r_2} N_{r_1,d} N_{r_2,d} = \sum_{1 \leq r_1 < r_2} \sum_{0 \leq d-1 < r_1 \alpha N^{-1}} N_{r_1,d} N_{r_2,d}
\]

By this and (7.7),

\[
\sum_{d=1}^{\infty} \sum_{1 \leq r_1 < r_2 < N} N_{r_1,d} N_{r_2,d} \\
\leq \sum_{2 \leq r_2 < N} \sum_{1 \leq l \leq (r_2-1) \alpha N^{-1}} \sum_{d-1 < l \alpha N^{-1}} N_{r_1,d} N_{r_2,d}
\]

Next, assuming \( 1 < \alpha \leq 4/3 \), we show that \( \mathcal{E}_\alpha(N) \ll N^{4-\alpha} \). Proposition 2.7, Lemmas 2.8 and 2.9 imply that

\[
\mathcal{N}_{\alpha}^{\geq l}(d) \ll \alpha l^{1-\beta}d^{\beta-1} + l^{(1-\beta)/3}d^{3/3} + d^{2/\alpha-1}.
\]

This and (7.1) yield that

\[
\mathcal{E}_\alpha(N) \ll N^2 + \sum_{l=1}^{N-1} \sum_{(l-1) \alpha N^{-1} \leq d-1 < l \alpha N^{-1}} N_{\alpha}^{\geq l}(d)^2
\]

\[
\ll \alpha N^2 + \sum_{l=1}^{N-1} \sum_{(l-1) \alpha N^{-1} \leq d-1 < l \alpha N^{-1}} (l^{2-2\beta}d^{2\beta-2} + l^{2(1-\beta)/3}d^{23/3} + d^{4/\alpha-2}).
\]
The first sum with summands $l^2 - 2\beta d^2 - 2$ is
\[
\sum_{l=1}^{N-1} \left\{ \sum_{(l-1)\alpha N^{\alpha-1} \leq d-1 < l\alpha N^{\alpha-1}} l^2 - 2\beta d^2 - 2 \right\} \\
\leq \sum_{l=1}^{N-1} \int_{[(l-1)\alpha N^{\alpha-1}]+1}^{[l\alpha N^{\alpha-1}]+1} l^2 - 2\beta d^2 - 2 \, dx \\
\ll_{\alpha} \sum_{l=1}^{N-1} l^2 - 2\beta \cdot N^{\alpha-1} (lN^{\alpha-1})^2 - 2 \\
= \sum_{l=1}^{N-1} N^{3-\alpha} < N^{4-\alpha}.
\]

The second sum with summands $l^{2(1-\beta)/3} d^2 / 3$ is
\[
\sum_{l=1}^{N-1} \left\{ \sum_{(l-1)\alpha N^{\alpha-1} \leq d-1 < l\alpha N^{\alpha-1}} l^{2(1-\beta)/3} d^2 / 3 \right\} \\
\leq \sum_{l=1}^{N-1} \int_{[(l-1)\alpha N^{\alpha-1}]+1}^{[l\alpha N^{\alpha-1}]+1} l^{2(1-\beta)/3} d^2 / 3 \, dx \\
\ll_{\alpha} \sum_{l=1}^{N-1} l^{2(1-\beta)/3} \cdot N^{\alpha-1} (lN^{\alpha-1})^{2/3} \\
= \sum_{l=1}^{N-1} l^{2/3} N^{\alpha-1/3} < N^{\alpha+4/3}.
\]

The third sum with summands $d^{4/\alpha - 2}$ is
\[
\sum_{l=1}^{N-1} \left\{ \sum_{(l-1)\alpha N^{\alpha-1} \leq d-1 < l\alpha N^{\alpha-1}} d^{4/\alpha - 2} \right\} \\
\leq \sum_{l=1}^{N-1} \int_{[(l-1)\alpha N^{\alpha-1}]+1}^{[l\alpha N^{\alpha-1}]+1} x^{4/\alpha - 2} \, dx \ll_{\alpha} \sum_{l=1}^{N-1} N^{\alpha-1} (lN^{\alpha-1})^{4/\alpha - 2} \\
= \sum_{l=1}^{N-1} l^{4/\alpha - 2} N^{5-\alpha - 4/\alpha} < N^{4/\alpha - 1} N^{5-\alpha - 4/\alpha} = N^{4-\alpha}.
\]
Since the inequality \( \alpha + \frac{4}{3} \leq 4 - \alpha \) follows from the assumption \( 1 < \alpha \leq \frac{4}{3} \), we obtain \( E_\alpha(N) \ll \alpha N^{4-\alpha} \).

\[ \square \]

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