Spinor-vector supersymmetry algebra in three dimensions

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Abstract

We focus on a spin-3/2 supersymmetry (SUSY) algebra of Baaklini in \( D = 3 \) and explicitly show a nonlinear realization of the SUSY algebra. The unitary representation of the spin-3/2 SUSY algebra is discussed and compared with the ordinary (spin-1/2) SUSY algebra.

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Supersymmetry (SUSY) algebra based on a spinor-vector generator, which is called the spin-3/2 SUSY, was first introduced by Baaklini [1], in which a nonlinear (NL) representation of the algebra has been realized by introducing a spin-3/2 Nambu-Goldstone (NG) fermion. This work was done by extending the NL realization of the ordinary (spin-1/2) SUSY algebra in terms of a spin-1/2 NG fermion known as the Volkov-Akulov (VA) model [2]. For the spin-1/2 SUSY algebra, the relation between the NL and the linear (L) SUSY [3], i.e., the algebraic equivalence of the NL SUSY VA action with various (renormalizable) spontaneously broken linear (L) supermultiplets has been investigated in [4]-[7]. According to this fact, corresponding L supermultiplets to a spin-3/2 NL SUSY action [1] are also expected for the case of the spin-3/2 SUSY through a linearization.

As a first step to investigate the L realization of the spin-3/2 algebra and the linearization of the spin-3/2 NL SUSY, we have recently studied the unitary representation of the spin-3/2 SUSY algebra of Baaklini [8] and a (N = 1) spin-3/2 L SUSY invariance of a free action in terms of spin-(0, 1/2, 1, 3/2) fields [9]. Those works are important preliminary not only to find out the (spontaneously broken) L SUSY supermultiplets which are equivalent to the NL realization of the spin-3/2 SUSY algebra, but also to obtain some informations for linearizing the interacting global NL SUSY theory with spin-3/2 (NG) fields in curved spacetime [12]. It may give new insight into an analogous mechanism with the super-Higgs one [13] for high spin fields which appear in a composite unified theory based on SO(10) super-Poincaré (SP) group (the superon-graviton model) [14, 15].

In order to study further the spin-3/2 L supermultiplet structure, e.g., the closure on commutator algebras of L SUSY transformations, the structure of auxiliary fields, the mechanism of spontaneous SUSY breaking and the linearization of NL SUSY etc., it is useful to consider lower dimensional cases for simplicity of calculations. In this letter we focus on the spin-3/2 SUSY algebra of Baaklini in $D = 3$ and explicitly show the NL realization of the SUSY algebra. The unitary representation of the spin-3/2 SUSY algebra is further discussed and results are compared with those of the spin-1/2 SUSY algebra.

Let us introduce the spin-3/2 SUSY (SP) algebra of Baaklini in $D = 3$ based on...
a spinor-vector generator $Q^a_\alpha$ as
\[ [P^a, P^b] = 0, \]
\[ [P^a, J^{bc}] = i(\eta^{ab} P^c - \eta^{ac} P^b), \]
\[ [J^{ab}, J^{cd}] = -i(\eta^{ac} J^{bd} - \eta^{ad} J^{bc} + \eta^{bc} J^{ad} + \eta^{bd} J^{ac}), \]
\[ [Q^a_\alpha, P^b] = 0, \]
\[ [Q^a_\alpha, J^{bc}] = \frac{1}{2}(\sigma^{bc})_{\alpha\beta} Q^a_\beta + i(\eta^{ab} Q^c_\alpha - \eta^{ac} Q^b_\alpha), \]
\[ = -\frac{1}{2} \epsilon^{bcd}(\gamma_d)_\alpha^\beta Q^a_\beta + i(\eta^{ab} Q^c_\alpha - \eta^{ac} Q^b_\alpha), \]
\[ \{Q^a_\alpha, Q^b_\beta\} = i \epsilon^{abc}(C)_{\alpha\beta} P_c, \]
where $P^a$ and $J^{ab}$ are translational and Lorentz generators of the Poincaré group. It can be shown that Eqs.(1)-(6) satisfy all the Jacobi identities, in particular, the identities,
\[ [Q^a_\alpha, [J^{bc}, J^{de}]] + [J^{de}, [Q^a_\alpha, J^{bc}]] + [J^{bc}, [J^{de}, Q^a_\alpha]] = 0, \]
\[ \{Q^a_\alpha, [Q^b_\beta, J^{cd}]] + [J^{cd}, \{Q^a_\alpha, Q^b_\beta\}] + \{Q^b_\beta, [Q^a_\alpha, J^{cd}]] = 0, \]
by means of the relation $\eta^{ab} \epsilon^{cde} = \eta^{ae} \epsilon^{bde} + \eta^{ad} \epsilon^{bec} + \eta^{ae} \epsilon^{cab}$.

As parallel discussions with [1, 2], a NL representation of the spin-3/2 SUSY algebra which reflects Eq.(6) can be easily realized in terms of a Majorana spin-3/2 NG field $\psi^a$. Indeed, supertranslations of the $\psi^a$ and the Minkowski coordinate $x^a$ parametrized by a global Majorana spinor-vector parameter $\zeta^a$ are given by
\[ \delta \psi^a = \zeta^a, \]
\[ \delta x^a = \kappa \epsilon^{abc} \psi^b \zeta_c, \]
where $\kappa$ is a constant whose dimension is (mass)$^{-3}$. Eq.(8) means NL SUSY transformation of $\psi^a$ at a fixed spacetime point as
\[ \delta Q \psi^a = \zeta^a - \kappa \epsilon^{bcd} \psi^b \zeta_c \partial_d \psi^a, \]
which gives the closed off-shell commutator algebra,
\[ [\delta Q_1, \delta Q_2] = \delta P(\Xi^a), \]
\[ ^3\text{Minkowski spacetime indices are denoted by } a, b, \cdots = 0, 1, 2 \text{ and two-component spinor indices}\]
\[ \text{are } \alpha, \beta = (1), (2). \text{ The Minkowski spacetime metric is } \frac{1}{2}\{\gamma^a, \gamma^b\} = \eta^{ab} = \text{diag}(+,-,-) \text{ and }\]
\[ \sigma^{ab} = \frac{i}{2}[\gamma^a, \gamma^b] = -\epsilon^{abc}\gamma_c (\epsilon^{012} = -1 = \epsilon^{012}), \text{ where we use the } \gamma \text{ matrices defined as } \gamma^0 = -\sigma^2,\]
\[ \gamma^1 = i\sigma^3, \gamma^2 = -i\sigma^3 \text{ with } \sigma^I(I = 1, 2, 3) \text{ being Pauli matrices. We also use the charge conjugation matrix, } C = \sigma^2 = -\gamma^0.\]
where $\delta P(\Xi^a)$ means a translation with a generator $\Xi^a = -2\kappa\epsilon^{abc}\bar{\psi}_b\psi_c$. Based upon a spin-3/2 NL SUSY invariant differential one-form defined as

$$\omega^a = dx^a + \kappa\epsilon^{abc}\bar{\psi}_b\psi_c$$

$$= (\delta^a_b + \kappa\epsilon^{acd}\bar{\psi}_c\partial_b\psi_d)\,dx^b$$

$$= (\delta^a_b + t^a_b)\,dx^b$$

$$= w^a_b\,dx^b,$$

(11)

an action, which is invariant under the spin-3/2 NL SUSY transformation (9), is constructed as the volume form in $D = 3$:

$$S = -\frac{1}{2\kappa}\int \omega^0 \wedge \omega^1 \wedge \omega^2$$

$$= -\frac{1}{2\kappa}\int d^3x \,|w|$$

$$= -\frac{1}{2\kappa}\int d^3x \left[ 1 + t^a_a + \frac{1}{2!}(t^a_b t^b_a - t^a_b t^b_a) + \frac{1}{3!}\epsilon_{abc}\epsilon^{def}t^a_d t^b_e t^c_f \right],$$

(12)

where the second term, $-(1/2\kappa)\,t^a_a = -(1/2)\epsilon^{abc}\bar{\psi}_a\partial_b\psi_c$, means the kinetic term for $\psi^a$.

To know the $L$ supermultiplet structure as corresponding one to the above spin-3/2 NL SUSY model, we study the unitary representation of the spin-3/2 SUSY algebra (1)-(6). By the use of similar methods to the case of spin-1/2 SUSY in $D = 3$ (for example, see [16]), operators which raise or lower the helicity of states for massive representations are constructed from the commutation relation (5) as explains below: In fact, Eq.(5) for the single helicity operator $J = J^{12}$ is

$$[Q^0_{(1)}, J] = \frac{i}{2}Q^0_{(2)}, \quad [Q^0_{(2)}, J] = -\frac{i}{2}Q^0_{(1)},$$

$$[Q^1_{(1)}, J] = \frac{i}{2}Q^1_{(2)} - iQ^2_{(1)}, \quad [Q^1_{(2)}, J] = -\frac{i}{2}Q^1_{(1)} - iQ^2_{(2)},$$

$$[Q^2_{(1)}, J] = \frac{i}{2}Q^2_{(2)} + iQ^1_{(1)}, \quad [Q^2_{(2)}, J] = -\frac{i}{2}Q^2_{(1)} + iQ^1_{(2)}.$$ 

(13)

If we define the following operators as linear combinations of the real charges $Q^a_{(1)}$ and $Q^a_{(2)}$,

$$R^{a\pm} = \frac{1}{2}(Q^a_{(1)} \pm iQ^a_{(2)}), \quad (R^{a\pm})^\dagger = \frac{1}{2}(Q^a_{(1)} \mp iQ^a_{(2)}) = R^{a\mp},$$

(14)

$$S^\pm = R^{1\pm} + iR^{2\pm} = \frac{1}{2}\{(Q^1_{(1)} \pm iQ^1_{(2)}) + i(Q^2_{(1)} \pm iQ^2_{(2)})\},$$

(15)

$$S^\pm = R^{1\mp} - iR^{2\mp} = \frac{1}{2}\{(Q^1_{(1)} \mp iQ^1_{(2)}) - i(Q^2_{(1)} \mp iQ^2_{(2)})\},$$

(16)
then commutators between the operators (14)-(16) and the $J$ become

$$[R^0-, J] = -\frac{1}{2} R^0-, \quad [R^0+, J] = \frac{1}{2} R^0+,$$  \hspace{1cm} (17)

$$[S^+, J] = -\frac{1}{2} S^+, \quad [(S^+)^\dagger, J] = \frac{1}{2}(S^+)^\dagger,$$  \hspace{1cm} (18)

$$[S^-, J] = -\frac{3}{2} S^-, \quad [(S^-)^\dagger, J] = \frac{3}{2}(S^-)^\dagger.$$  \hspace{1cm} (19)

due to Eq.(13). Eqs.(17) and (18) mean that $R^0\pm$ and $(S^+, (S^+)^\dagger)$, raise or lower the helicity of states by $1/2$, while Eq.(19) shows that $(S^-, (S^-)^\dagger)$ raise or lower the helicity of states by $3/2$.

Furthermore, according to the anticommutation relation (6), which is explicitly written in the component form as

$$\{Q^1_{(1)}, Q^2_{(2)}\} = -P_0, \quad \{Q^1_{(2)}, Q^2_{(1)}\} = P_0,$$

$$\{Q^2_{(1)}, Q^0_{(2)}\} = -P_1, \quad \{Q^2_{(2)}, Q^0_{(1)}\} = P_1,$$

$$\{Q^0_{(1)}, Q^1_{(2)}\} = -P_2, \quad \{Q^0_{(2)}, Q^1_{(1)}\} = P_2$$  \hspace{1cm} (20)

with all other anticommutators being zero, nonvanishing ones among the operators (14)-(16) are

$$\{S^+, (S^+)^\dagger\} = P_0,$$  \hspace{1cm} (21)

$$\{S^-, (S^-)^\dagger\} = -P_0,$$  \hspace{1cm} (22)

$$\{R^0-, S^+\} = -\frac{1}{2}(P_1 + iP_2),$$  \hspace{1cm} (23)

$$\{R^0+, S^-\} = \frac{1}{2}(P_1 + iP_2),$$  \hspace{1cm} (24)

$$\{R^0-, (S^-)^\dagger\} = \frac{1}{2}(P_1 - iP_2),$$  \hspace{1cm} (25)

$$\{R^0+, (S^+)^\dagger\} = -\frac{1}{2}(P_1 - iP_2).$$  \hspace{1cm} (26)

Note that $\{R^0-, R^0+\} = 0$ for the $R^{0\pm}$ of Eq.(17). Obviously, by taking $P_a = (m, 0, 0)$ for the massive case $P^2 = m^2$ and by defining creation and annihilation operators,

$$a_1 = \frac{1}{\sqrt{m}} S^+, \quad a_1^\dagger = \frac{1}{\sqrt{m}}(S^+)^\dagger,$$

$$a_2 = \frac{1}{\sqrt{m}} S^-, \quad a_2^\dagger = \frac{1}{\sqrt{m}}(S^-)^\dagger.$$  \hspace{1cm} (27)
Eqs. (21)-(26) become
\[
\{a_1, a_1^\dagger\} = 1, \quad \{a_2, a_2^\dagger\} = -1, \\
\{a_i, a_j\} = 0, \quad \{a_i^\dagger, a_j^\dagger\} = 0,
\]
where \(i, j = 1, 2\). Eq. (28) shows that only the \((a_1, a_1^\dagger)\) are the operators in the Fock space which shift the helicity of states by 1/2, while the \((a_2, a_2^\dagger)\) which shift by 3/2 gives the negative norm (the non-unitary representation). Therefore, a (physical) massive irreducible representation for the spin-3/2 SUSY algebra in \(D = 3\) induced only from \((a_1, a_1^\dagger)\) contains the stuffs with helicity \((\lambda, \lambda + 1/2)\), e.g., \((0, 1/2)\) as the simplest case. Namely, as for the physical modes of the massive multiplets in \(D = 3\) has the same \(L\) supermultiplet structure as that of the \(N = 1\) spin-1/2 SUSY in \(D = 4\) [17], although the multiplicity is another. Also they are compatible with the \(N = 1\) spin-1/2 SUSY in \(D = 3\) case [16] obtained in the different context.

On the other hand, we are not able to construct consistent unitary representations for the massless case \(P^2 = 0\), e.g., with a reference frame \(P_a = (\epsilon, 0, \epsilon)\), since Eqs. (23)-(26) does not vanish. This is the similar situation to the spin-1/2 SUSY in \(D = 3\), where the helicity in the massless case is no more good concept. In this case it may be necessary to use the method of constructing massless spin-1/2 \(D = 3\) supermultiplets by extending (the representation space of) SUSY algebra [18]. We finally note that in \(D = 2\) case of the spin-3/2 SUSY algebra the anticommutator \(\{Q^a_\alpha, Q^b_\beta\}\) of Baaklini’s type vanishes.

To summarize, we focus in this letter on the spin-3/2 SUSY algebra of Baaklini in \(D = 3\) and show the NL realization of the SUSY algebra (1)-(6); namely, we explicitly construct the action (12) in terms of the spin-3/2 NG fermion \(\psi^a\) which is invariant under the NL SUSY transformation (9). We further discuss on the unitary representation of the spin-3/2 SUSY algebra in \(D = 3\) towards the linearization of the spin-3/2 NL SUSY. For the massive case, the physical irreducible representation is induced only from the operators \((a_1, a_1^\dagger)\) which shift the helicity of states by 1/2 in Eq. (28). On the other hand, no consistent unitary representation is constructed for the massless case by means of the nonvanishing anticommutation relations (23)-(26). This fact reflects the similar situation to the spin-1/2 SUSY in \(D = 3\) with no good concept of helicity in the massless case, and we need further investigations to construct massless spin-3/2 \(D = 3\) supermultiplets.
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