Extracting work from correlations

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We consider the problem of extracting work from isolated quantum systems composed of \( n \) subsystems. In this scenario the work can be naturally divided into two contributions: a local contribution from each subsystem, and a global contribution originating from correlations between subsystems. Here we focus on the latter and consider quantum systems which are locally thermal, thus from which work can only be extracted from correlations. We derive bounds on the extractable work for general quantum states, separable states, and states with fixed entropy. Our results show that while entanglement gives an advantage for small quantum systems, this gain vanishes for a large number of subsystems.

Introduction

Thermodynamics and information theory are deeply related [1]. Recently much attention has been dedicated to the problem of understanding thermodynamics when considering quantum systems. This has led notably to the development of a resource theory for quantum thermodynamics [2, 3], and the study of quantum thermal machines [4, 5, 7–9]. The role and significance of quantum resources offer a significant advantage only for small quantum systems, this gain vanishes for a large number of quantum resources.

A problem of particular importance in quantum thermodynamics is to understand which quantum states allow for the storage and extraction of work from quantum systems [15, 16]. Such states are called non-passive, while states from which no work can be extracted are referred to as passive. Among passive states, one also distinguishes states which are completely passive, that is, from which no work can be extracted even when an arbitrary number of copies of the state are jointly processed. Completely passive states are in fact simply thermal states [15, 16].

In general, there exist two different ways in which work can be stored in a non-passive quantum system. First, and most commonly discussed, one stores work locally in the system, by basically rearranging the populations of the energy levels. However, there exists another possibility for storing work in a non-passive state, which makes use of correlations between subsystems. The main goal of the present work is to investigate this second scenario and understand how to optimally make use of correlations among quantum systems for work storage.

Specifically, we consider a quantum system composed of \( n \) subsystems (particles or modes). Each subsystem is assumed to be in a thermal state, at the same temperature \( T \). Hence each subsystem is passive and no work can be extracted from it. However, when the system can be jointly processed, here via a cyclic Hamiltonian process (or alternatively a global unitary operation), work can be extracted. This is because the subsystems are correlated, hence the global state is not passive in general. The main point of this approach is that it captures exactly the amount of work stored in the correlations, since locally (at the level of each individual subsystem) no work is available. First, we will see that if no restriction on the global state is made, then it is possible to store in the system the maximal amount of work compatible with the requirement that the reduced states are thermal. In other words, at the end of the protocol, the system is left in the ground state. Notably this is possible thanks to quantum entanglement. It is then natural to ask if the same amount of work can be stored using a separable, or even a purely classical state diagonal in the product energy eigenbasis, that is, with no coherences among different energy levels. We will see that, although the amount of work that can be stored in unentangled states is strictly smaller than to the entangled case for any finite \( n \), in the thermodynamic limit \( (n \to \infty) \) purely classical states already become optimal. In fact, quantum resources offer a significant advantage only for small \( n \), while neither entanglement nor energy coherences are needed for optimal work storage in the thermodynamic limit.

We also consider other natural limitations on the global state, such as fixing a bound on the entropy of the state, and investigate the role of quantum coherence and entanglement in this case. Finally, we show that our results are also applicable to a different framework where the system does not remain isolated but one has access to a bath or a reservoir.
Framework

We consider an isolated quantum system which consists of $n$ $d$-level subsystems. The local Hamiltonian $h = \sum_j E_j |j\rangle \langle j|$ is taken to be the same for each subsystem, and without loss of generality, it is assumed that the ground state energy is zero. We consider the situation where there is no interaction Hamiltonian between the subsystems, such that the total Hamiltonian $H$ is simply the sum of the individual local Hamiltonians $H = \sum_i h_i$.

The class of operations that we consider is the class of cyclic Hamiltonian processes, i.e. we can apply any time dependent interaction $V(t)$ between the $n$ subsystems for a time $\tau$, such that $V(t)$ is non-vanishing only when $0 \leq t \leq \tau$. The corresponding evolution can be described by a unitary operator $U(\tau) = \exp(-i \int_0^\tau dt (H + V(t)))$, where $\exp$ denotes the time-ordered exponential. By varying over all $V(t)$ we can generate any unitary operator $U = U(\tau)$ and therefore this class of operations can alternatively been as seen the ability to apply any global unitary on the system.

The task we are interested in is work extraction via a cyclic Hamiltonian process. Since the system is taken to be isolated, we quantify the extracted work by the change in average energy of the system under such a process. More precisely, we define the extracted work $W$ as

$$W = \text{Tr}(\rho H) - \text{Tr}(U \rho U^\dagger H).$$

Within this framework, it is well known that work can be extracted from a system if and only if the system is non-passive, where a passive system, with Hamiltonian $H = \sum_i \mathcal{E}_i |i\rangle \langle i|$, is one which satisfies

$$\rho_{\text{passive}} = \sum_i p_i |i\rangle \langle i|$$

such that $p_{i+1} \leq p_i$, $\mathcal{E}_{i+1} \geq \mathcal{E}_i$ (2)

i.e. a system is passive if and only if it is diagonal in its energy eigenbasis and its eigenvalues are non-increasing with respect to energy. Given non-passive state $\rho$, the extracted work $W$ is maximized by $\rho_{\text{passive}}$:

$$W_{\text{max}} = \text{Tr}(\rho H) - \text{Tr}(\rho_{\text{passive}} H)$$

where $\rho$ and $\rho_{\text{passive}}$ have the same spectrum.

Importantly, we see that passivity is a global property of a system, and thus this raises interesting possibilities when considering a system comprised of a number of subsystems, as we do here. Indeed, global operations are capable of extracting more work than local ones, as a state can be locally passive but globally not. Such enhancing may have two origins: activation or correlations between subsystems. Activation occurs when $\rho_{\text{passive}} \otimes k$ becomes a non-passive state for some $k$. Interestingly, thermal states are the only passive states that do not allow for activation, as any number of copies of thermal states is also thermal. On the other hand, states that are locally passive but have a non-product structure (i.e., they are correlated) also offer the possibility for work extraction. An extreme case, which is the focus of this article, is a set of correlated locally thermal states, as in such a case the global contribution uniquely comes from correlations. Our goal, in fact, is to understand how correlations allow for work extraction in systems that locally look completely passive.

We will then focus on a subset of all possible states of the system, specifically those which are locally thermal, that is states $\rho$ such that the reduced local state of subsystem $i$ satisfies

$$\rho_i = \text{Tr}_i \rho = \tau_{\beta}$$

for all $i$, where $\text{Tr}_i$ denotes the partial trace over all subsystems except subsystem $i$. Here $\tau_{\beta}$ is the thermal state of the subsystem at (a fixed but arbitrary) inverse temperature $\beta = 1/T$:

$$\tau_{\beta} = \frac{e^{-\beta h}}{Z},$$

where $Z = \text{Tr} e^{-\beta h}$ is the partition function. Since $\rho$ is locally thermal, and since $H$ is a sum of local Hamiltonians, the first term of the right hand side of (1) is fixed and given by $\text{Tr}(\rho H) = n E_{\beta}$ where $E_{\beta} = \text{Tr}(\tau_{\beta} H)$ is the average energy of the local thermal state.

Apart from understanding how to exploit correlations to store work in the system, we will also study the role of entanglement and energy coherences in these processes. We consider three natural sets of correlated states: (i) arbitrarily correlated, and thus entangled, states, (ii) separable states, and (iii) states diagonal in the product eigenbasis, which we also name classical. Clearly, these do not have any quantum coherence among the different energy levels, which are just classically correlated. We will study work extraction for these three different sets of correlated quantum states.

Optimal work extraction from correlations

We first show that within the above framework there is no restriction on the amount of work that can be stored in the correlations of a quantum system. That is, from equation (1), given that the initial average energy is fixed, maximal work extraction amounts to minimising the final average energy of the system, a non-negative quantity (given the convention that the ground state energy vanishes). Thus by ending in the ground state one has clearly extracted the optimal amount of work, equal to $W = n E_{\beta}$. To that end, consider the state

$$|\phi\rangle = \frac{1}{\sqrt{2}} \sum_{i=0}^{d-1} e^{-\beta E_i} |i\rangle \otimes n.$$
It is straightforward to verify that $\text{Tr}_i |\phi \rangle \langle \phi | = \tau_\beta$ for all $i$ and as such that $|\phi \rangle$ is locally thermal. Moreover, since the state $|6\rangle$ is pure, there exists a unitary matrix $U$ such that $U|\phi \rangle = |0\rangle^\otimes n$. Thus all work can be extracted from state $|\phi \rangle$.

However, it is clear that the state $|6\rangle$ is entangled. Hence it is natural to ask whether the amount of extractable work would change if furthermore we restrict ourselves to separable, or even classical (i.e. diagonal), states? If this is the case, then entanglement is clearly enhancing work extraction in the scenario we consider.

Maximal work extraction from separable and classical states

We start by considering the case where the system is initially in a separable state $\rho$. Hence the conditional entropy of $\rho$ is necessarily positive, that is $S(\rho) \geq S(\tau_\beta(h))$. This condition places an upper bound on the maximal extractable work from separable states. Indeed, the global state with the least energy compatible with a given entropy is the thermal state $|7\rangle$.

$$\rho_{\text{th}} = \tau_\beta^{\otimes n} \quad (7)$$

where $\beta'$ is defined implicitly through the relation $S(\tau_{\beta'}) = S(\tau_\beta(h))/n$, requiring that the thermal state $\rho_{\text{th}}$ has the same fixed entropy as the initial state. The state $\rho_{\text{th}}$ is the (possibly unachievable) optimal final state, as the minimal amount of energy is left in the state. The bound on the extractable work for separable states is thus

$$W_{\text{sep}} \leq nE_\beta \left( 1 - \frac{1}{E_\beta} \text{Tr} (\tau_{\beta'} h) \right). \quad (8)$$

Hence optimal work extraction is impossible with separable states, as the above bound is always strictly smaller than $nE_\beta$. Moreover, notice that it is not clear whether the above bound can be achieved, since we have assumed only conservation of entropy and did not take into consideration either the restriction to unitary processes and the constraint of being locally thermal. The tightness of this bound will be discussed in more detail later.

Let us now move to the case of purely classical states, i.e. states which are diagonal in the (global) energy eigenbasis. Consider state

$$\rho_{\text{cl}} = \frac{1}{Z} \sum_{i=0}^{d-1} e^{-\beta E_i/2} |i\rangle \langle i|^{\otimes n}, \quad (9)$$

which is simply the state $|6\rangle$ after being dephased in the (global) energy eigenbasis. Notice that $|6\rangle$ saturates the bound $S(\rho) \geq S(\tau_\beta(h))$, and in Appendix A we show that it is the only separable state that satisfies it. The maximal extractable work from $|5\rangle$ is found, as before, by finding its associated passive state, and then computing the average energy difference, see $[3]$. Since $\rho_{\text{cl}}$ is already diagonal (with $d$ non-zero eigenvalues), it is only necessary to rearrange these non-zero eigenvalues to the lowest possible energy levels. Let us assume that $n > d-1$, (i.e. that we are in the regime of sufficiently many subsystems). The $d-1$ largest eigenvalues can then simply be moved into the first excited subspace (with energy $E_1$). This construction provides a lower bound on the maximal amount of work that can be extracted from purely classical states:

$$W_{\text{cl}} \geq nE_\beta \left( 1 - \frac{(Z-1)E_1}{nZE_\beta} \right). \quad (10)$$

As the number of subsystems $n$ increases, we see that $W_{\text{cl}}$ rapidly converges to $W_{\text{max}} = nE_\beta$. This shows that, in the thermodynamic limit ($n \to \infty$), the difference in capacity between storing work in an entangled quantum state and a diagonal state vanishes, hence quantum coherences and entanglement play essentially no role here. However, for finite $n$ there will still be a difference, and in particular in the regime of $n$ relatively small the ability to store work in entanglement or coherences offers a significant advantage (see Fig.1).

Protocol for maximal work extraction given an entropy constraint

The previous results can be intuitively understood from entropy considerations. When the correlations in the state are not restricted, it is possible to satisfy the requirement of local thermality with a pure entangled states, therefore attaining optimal work extraction. When the state is separable, the global entropy of the state cannot be zero as it is lower bounded by the local entropy. This explains the gap between entangled and separable states. However, it is still possible to find the
classical state that, apart from being locally thermal, has an entropy that does not scale with \( n \). In the limit of a large number of subsystems, this global entropy becomes negligible and the classical state turns out to be effectively optimal. In view of these considerations, it is a natural question to study how the previous bounds are affected when the global entropy of the state is bounded. In fact, it is a natural scenario to consider systems whose entropy scales with the number of subsystems, for example \( S \propto n \), that is, systems with a non-vanishing entropy per subsystem.

Following the same line of reasoning as in the previous section, we obtain the following bound on the extractable work given a constraint on the entropy of the initial state \( S(\rho) = S \),

\[
W_S \leq nE_\beta \left( 1 - \frac{1}{E_\beta} \text{Tr}(\mathcal{H}_{\tau_{\beta'}}) \right). \tag{11}
\]

where \( \beta' \) is defined implicitly through the relation \( S(\tau_{\beta'}) = S/n \). In what follows we provide protocols obtaining this bound for all \( n \), i.e. there is an initial state \( \rho \) which is locally thermal and can be brought to a product of \( n \) thermal states at temperature \( \beta' \) by application of a suitable unitary. We provide a detailed analysis for the case of qubits; for the general case of qudits see [3], where the same protocol is applied for creating correlations at minimal energy cost.

For clarity, we work backwards, and exhibit a unitary which takes the initial state \( \sigma_{\alpha}^{\otimes n} \) to an initial state \( \rho \) which is locally thermal, at any temperature \( \beta \leq \beta' \). We first consider the simplest case of two qubits. We define the unitary transformation \( U_\alpha \):

\[
U_\alpha |00 \rangle = \cos \alpha |00 \rangle + \sin \alpha |11 \rangle, \quad U_\alpha |01 \rangle = |01 \rangle \\
U_\alpha |11 \rangle = -\sin \alpha |00 \rangle + \cos \alpha |11 \rangle, \quad U_\alpha |10 \rangle = |10 \rangle \tag{12}
\]

and consider the initial state \( \rho = U_\alpha \tau_{\beta'}^{\otimes 2} U_\alpha^\dagger \). Since \( U_\alpha \) only generates coherences in the subspace where both qubits are flipped, it is clear that the reduced state of each qubit is diagonal. To calculate the local temperature \( \beta \) it is convenient to introduce the parameter \( z = \langle 0|\rho_1|0 \rangle - \langle 1|\rho_1|1 \rangle \), i.e. the “bias” of the local (qubit) subsystem in state \( \rho_1 \). Indeed, the bias and temperature are related through \( z = \tanh(\beta E/2) \). A straightforward calculation shows that under the action of \( U_\alpha \), the state \( \tau_{\beta'} \) (with bias \( z' \)) transforms to an initial state \( \rho \) with bias \( z = \cos(2\alpha) z' \). That is, we can achieve any bias \( z \) such that \( |z| \leq z' \). As such the local temperature of the initial state, which is simply given by \( \beta = \frac{2}{\beta'} \tanh^{-1}(\cos(2\alpha) z') \), can take any temperature \( \beta \leq \beta' \) by an appropriate choice of \( \alpha \).

The above protocol can be readily generalised to the case of \( n \) qubits. Let us denote by \( \mathbb{I} = i_1 \cdots i_n \) an \( n \)-bit string, \( |i \rangle = \sum_k i_k \) the weight (number of 1s) of the string, \( |i \rangle = |i_1 \rangle \cdots |i_n \rangle \) the \( 2^n \) energy eigenstates of \( H \). Also, denoting \( \alpha_j = \alpha_1 \cdots \alpha_j \) a vector of \( j = \lfloor n/2 \rfloor \) angles, we define \( U_\alpha \) as

\[
U_\alpha |i \rangle = \cos \alpha_k |i \rangle + \sin \alpha_k |\bar{i} \rangle, \quad |i \rangle = k < \frac{n}{2} \\
U_\alpha |i \rangle = -\sin \alpha_k |i \rangle + \cos \alpha_k |\bar{i} \rangle, \quad |i \rangle = k > \frac{n}{2}, \tag{13}
\]

where \( \bar{i} \) denotes the bit-wise negated string (with \( |i \rangle = n - |\bar{i} \rangle \)), i.e. such that \( |\bar{i} \rangle = \sigma_{\alpha}^{\otimes n} |i \rangle \). \( U_\alpha \) is seen to perform rotations in all 2-dimensional subspaces comprised of states \( |i \rangle \), with energy \( \langle i |ih\rangle = E|i \rangle \), and the flipped state \( |\bar{i} \rangle \), of energy \( nE - \langle i |ih\rangle = E(n - |i \rangle) \), by an angle \( \alpha_k \) with \( k = |\bar{i} \rangle \). As we show in Appendix B, choosing \( \alpha_j = \alpha \) for all \( j \) the state \( \rho = U_\alpha \sigma_{\alpha}^{\otimes n} U_\alpha^\dagger \) is locally thermal, and, exactly as in the case of two qubits, the local bias \( z \) and temperature \( \beta \) are given by

\[
z = \cos(2\alpha) z', \quad \beta = \frac{2}{\beta'} \tanh^{-1}(\cos(2\alpha) z').
\]

Again, any bias \( |z| \leq z' \) can be reached, hence the initial state can take any temperature \( \beta \leq \beta' \). To summarise, we have constructed states \( \rho \) which are optimally correlated in the sense that the maximal extractable work is equal to the thermodynamic bound, which arises only from entropy considerations. As required they have local thermal marginals (which can take any possible temperature), in such a way that the only origin of the extracted work is the correlation among the subsystems.

Notice that the above protocol exploits coherence in all two-dimensional subspaces spanned by \( |i \rangle \) and \( |\bar{i} \rangle \). It turns out that, in the limit of large \( n \), such coherences imply the presence of entanglement (see Appendix C for details). Nevertheless, by adapting the above protocol, it is possible to extract the maximal amount of work without generating any coherences. Specifically, by taking \( n \) sufficiently large, one can optimally correlate the state in a fully classical manner by considering \( U_\alpha \) which invert the population (and therefore temperature) between states which have \( |i \rangle = k \) and \( |\bar{i} \rangle = n - k \), corresponding to energy \( kE \) and \( (n - k)E \) respectively, for a suitable choice of \( k \). Choosing \( k \approx \frac{n}{2} \), we obtain a bias \( z = z'(1 - O(1/\sqrt{n})) \). Thus for \( n \) sufficiently large, we can achieve approximately any \( |z| \leq z' \), and hence any temperature \( \beta \leq \beta' \) (see Appendix D for details).

**Work from energy coherences**

In this section we consider states whose diagonal (in the energy eigenbasis) is set to be equal to that of a global

\[\text{[1] Except for the case, which only occurs for } n \text{ even, when } \langle i |ih\rangle = \langle i |ih\rangle = n/2, \text{ in which case it is unnecessary to perform any rotation.}\]
thermal state, together with the initial condition of local thermality. More formally, this approach is equivalent to imposing that all moments of the energy distribution are those of the thermal state, i.e.

$$\text{Tr}(H^k \rho) = \text{Tr} \left( H^k \rho_{\beta}^{\text{deg}} \right)$$  \hspace{1cm} (14)

for all $k$. This contrasts with the previous sections where only the first moment (i.e. the average energy) was constrained. Moreover, notice that the entropy of the initial state is here unconstrained.

Focusing again first on the case of $n$ qubits, we consider states which are maximally entangled in every degenerate subspace:

$$\rho_{\text{deg}} = \sum_{k=0}^{n} \binom{n}{k} p^k (1 - p)^{(n-k)} |D_{n,k}\rangle \langle D_{n,k}|$$  \hspace{1cm} (15)

where $p = e^{-\beta E} / Z$, and $|D_{n,k}\rangle = 1 / \sqrt{\binom{n}{k}} \sum_{|i|=k} |i\rangle$ is the Dicke state of $n$ qubits with $k$ excitations. It is straightforward to verify that the above state satisfies equations (4) and (14).

The passive state associated to (15) can be found as follows. Notice that the state (15) is a mixture of $n+1$ orthogonal states. Therefore the optimal unitary amounts to rotating each of these states to the $n$ lowest energy levels. The work extracted via this transformation is thus given by

$$W_{\text{deg}} = n E_{\beta} - \left( 1 - \frac{n}{p^n} \right) p^n (1 - p)^{(1-p)n} E.$$  \hspace{1cm} (16)

Therefore it is possible to extract all the work contained in the initial state up to a correction of $O(1)$. Moreover, a similar result holds for the general case of $n$ qudits (see Appendix E).

An interesting question is whether the state $\rho_{\text{deg}}$ features entanglement. Intuition suggests that this may be the case, as large coherences are crucial in this scenario. However, using the techniques developed in [28], we have not been able to witness entanglement for $n \leq 50$.

### Access to a bath

Finally, we consider a different scenario in which the system is no longer isolated but one has access to a bath at the same (local) temperature. Then it is well-known that the extractable work is bounded by the free energy difference, $W \leq \Delta F$. In our set-up it reads,

$$W_{\text{S,bath}} \leq T \left( nS(\tau_{\beta}) - S \right),$$  \hspace{1cm} (17)

which reduces to the quantum mutual information in the bipartite case (thus enforcing our argument that in this setting work is only extracted from correlations). The bound (17) is strictly bigger than (11), which is natural as we consider a bigger set of operations. Equality in (17) can be obtained by a quasistatic process [19, 20], which essentially takes an infinite time whereas (11) can be reached with an appropriate controlled unitary operation, which is a fast process (thus enforcing the idea of a tradeoff between time and extracted work). On the other hand, the states (6) and (9) maximize the right hand side of (17), i.e. the free energy content is maximal, for entangled and separable states respectively, and thus our previous considerations also hold in this framework.

For the case of extracting work from energy coherences, one can readily use (17) by computing the entropy of (15). As $\rho_{\text{deg}}$ follows a binomial distribution, its entropy can be straightforwardly calculated,

$$S(\rho_{\text{deg}}) = \frac{1}{2} \ln n + \frac{1}{2} \ln (2\pi e p(1-p)) + O\left(\frac{1}{n}\right).$$  \hspace{1cm} (18)

Therefore, $\rho_{\text{deg}}$ allows for storing all work in coherences except for a $\ln n$ correcting term.

Our results thus complement previous studies in this setting, such as a detailed analysis on the extractable work from local/non-local operations [21–23], from correlated states [24, 25], from entanglement with feedback control [26], and also for deterministic work extraction [27]. It is also worth mentioning that when the correlations are not present between subsystems but rather between the system and the bath, they become a source of irreversibility [9].

### Conclusion

We have investigated the problem of storing work in correlations between subsystems. To ensure that no work can be extracted from subsystems locally, we focused on quantum states which are locally thermal. Hence all extractable work must come from correlations between subsystems. This gives a new perspective on the problem of passivity, in particular for the case of composed systems.

In the absence of any further constraint, all work can be extracted from the system. Importantly, entanglement was shown to be necessary in this case. Imposing additional constraints on the initial state, such as being separable, having a fixed entropy, or fixing all moments of the energy distribution, led to a reduction of the amount of extractable work. Nevertheless, in the thermodynamic limit, we found that essentially all work can be extracted in the above three cases. This was demonstrated by giving explicit examples of states.

\[\text{Footnote:}2\text{ That is, the system needs to undergo an arbitrarily large number of infinitesimal collisions with the bath, so that during each interaction the back action is negligible.}\]
An interesting open question is to investigate the scenario in which not only local marginals are thermal, but also $k$-body reduced states (nearest neighbours) are locally thermal. This may give insight into the role of different types of multipartite entanglement in the context of work extraction. A further interesting question is to derive bounds in the other direction, i.e., correlated states with minimal work content. This question will be addressed in further work [29].

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Appendix A: Maximally separable correlated states

What is found here is the separable state $\Omega$ of $N$ systems (with $d$ levels $\{E_{k}\}_{k=1}^{d}$) each such that it has the minimal entropy compatible with marginals all being thermal states at inverse temperature $\beta$:

$$\tau = \frac{1}{Z} e^{-\beta H} = \sum_{k=1}^{d} e^{-\beta E_{k}} |k\rangle \langle k| \equiv \sum_{k=1}^{d} p_{k} |k\rangle \langle k|. \quad (19)$$

Considering, e.g., the first system $(S^{i})$ versus the rest $(R = S^{2} \otimes \cdots \otimes S^{N})$ and keeping in mind that the partial states of $S$s are all $\tau$, we have

$$S(\Omega) - S(\tau) = -S(\Omega) \left( \tau \otimes \frac{I_{R}}{d_{R}} \right) + \ln d_{R}. \quad (20)$$

Since $\Omega$ is separable, it can be written in the following form:

$$\Omega = \sum_{x} \lambda_{x} \rho_{x}^{S_{1}} \otimes \rho_{x}^{R} = \sum_{x} \lambda_{x} \rho_{x}^{S_{1}} \otimes \rho_{x}^{S_{2}} \otimes \cdots \otimes \rho_{x}^{S_{N}} \quad (21)$$

for some discrete index $x$, nonnegative $\lambda_{x}$s summing up to 1, and some normalised states $\rho_{x}^{S_{i}}$ over $S$. Given the condition that the state of $S^{i}$, $\sum_{x} \lambda_{x} \rho_{x}^{S_{i}}$, is equal to $\tau$ and the joint convexity of the relative entropy [1], we have

$$S(\Omega) - S(\tau) = \ln d_{R} - S \left( \sum_{x} \lambda_{x} \rho_{x}^{S_{1}} \otimes \rho_{x}^{R} \right) \left( \sum_{x} \lambda_{x} \rho_{x}^{S_{1}} \otimes \frac{I_{R}}{d_{R}} \right) \geq \ln d_{R} - \sum_{x} \lambda_{x} S \left( \rho_{x}^{S_{1}} \otimes \rho_{x}^{R} \right) = \sum_{x} \lambda_{x} S(\rho_{x}^{S_{1}}) \geq 0. \quad (22)$$

So, the minimal possible value for $S(\Omega)$ is $S(\tau)$; and to find the purest $\Omega$ we have to saturate both inequalities in

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the chain \([22]\). The second inequality is resolved trivially, giving that \(\rho_{x}^{d} = \rho_{x}^{S} \otimes \cdots \otimes \rho_{x}^{S} \) for all values of \(x\) are pure. We denote these states as \(|R_{a}\rangle = |S_{x}^{2} \rangle \otimes \cdots \otimes |S_{x}^{N} \rangle\) (and \([S_{x}] = |S_{x}\rangle \langle S_{x}|\)). Doing the same with respect to, e.g., \(S_{2}^{d}\), we will get that \(\rho_{x}^{S_{2}^{d}}\) are also pure (and, as above, are denoted as \([S_{2}^{d}]\)).

The equality conditions for the first inequality of \([22]\) are less trivial \([1]\). If we only consider the nonzero \(\lambda_{x}\) and denote their number by \(L\), Theorem 8 of \([1]\) [2] will give us

\[
\left(\lambda_{x} \rho_{x}^{S_{1}^{d}} \otimes \rho_{x}^{R_{a}}\right)^{\lambda_{x}} \left(\lambda_{x} \rho_{x}^{S_{1}^{d}} \otimes \frac{I_{R_{a}}}{d_{R_{a}}}\right)^{-\lambda_{x}} = \Omega^{\lambda_{x}} \left(\tau \otimes \frac{I_{R_{a}}}{d_{R_{a}}}\right)^{-\lambda_{x}}
\]

for \(\forall t > 0\) and \(x = 1, \ldots, L\); \(\lambda_{x} > d\).

\[
\lambda_{x} = \langle S_{x} | \tau | S_{x} \rangle. \tag{25}
\]

Now we take \(|\{i\}|_{i=1}^{d}\), the eigenbasis of \(\tau\) in the Hilbert space of the subsystem \([19]\), and construct the matrix \(a_{xk} = \langle \langle S_{x} | k \rangle \rangle^{2} \geq 0\). With this we rewrite \([25]\) as

\[
\sum_{k=1}^{d} a_{xk} p_{k} = \lambda_{x}. \tag{26}
\]

Also, from the normalization we have

\[
\sum_{k} a_{xk} = 1 \quad \text{for} \quad \forall x. \tag{27}
\]

Finally, the condition that all partial states are \(\tau\): \(\sum_{x} \lambda_{x} [S_{x}] = \tau\), leads us to

\[
\sum_{x=1}^{L} \lambda_{x} a_{xk} = p_{k}. \tag{28}
\]

First, let us show that \(L > d\) cannot be true. Indeed, substitute \([26]\) into \([28]\), \(\sum_{x} a_{xk} a_{x'} p_{l} = p_{k}\), multiply the LHS by \(a_{xk}\) and sum over \(k\) and use \(\sum_{x} \lambda_{x} = 1 = \sum_{xk} a_{xk} p_{k}\):

\[
x \left(\sum_{k} a_{xk}^{2}\right) \left(\sum_{l} a_{xl} p_{l}\right) = 1. \tag{29}
\]

Given that it must hold that \(\sum_{xk} a_{xk} p_{k} = 1\) we see that \([29]\) can be true only if

\[
\sum_{k} a_{xk}^{2} = 1 \quad \text{for} \quad \forall x. \tag{30}
\]

But we have \([27]\) and that \(0 \leq a_{xk} \leq 1\) (30) can be true only if each row of \(\alpha = \{a_{xk}\}\) consists of zeroes and only one 1. Since none of \(p_{k}\) is zero, \([28]\) implies that there must be at least one 1 on each column of \(\alpha\). Let arrange the \(x\) so that the first \(d\) rows of \(\alpha\) look like an identity matrix. Then we get

\[
\lambda_{x} = p_{x} \quad \text{for} \quad x = 1, \ldots, d. \tag{31}
\]

Since \(\sum_{x} \lambda_{x} = 1\) we have that \(\lambda_{x} = 0\) for all \(x \geq d + 1\). Which is impossible because of \([26]\) and the fact that there must be at least one 1 on each row.

With the same argument, also \(d > L\) is not possible.

So, \(d = L\) and \([41]\) holds. Also, since now \(a = I\), \(|S_{x}\rangle = |x\rangle\), therefore

\[
\Omega = \sum_{k=1}^{d} p_{k} |k \cdots k\rangle \langle k \cdots k| \tag{32}
\]

**Appendix B: Protocol for maximal work extraction given an entropy constraint**

In this appendix we will show that the unitary \(U_{\alpha}\), with \(\alpha = \alpha \cdots \alpha\), given by

\[
U_{\alpha} |i\rangle = \cos |\alpha |i\rangle + \sin |\alpha |i\rangle, \quad (i \rangle H_{0} |i\rangle < \frac{n}{2}
\]

\[
U_{\alpha} |i\rangle = -\sin |\alpha |i\rangle + \cos |\alpha |i\rangle, \quad (i \rangle H_{0} |i\rangle < \frac{n}{2} \tag{33}
\]

\[
U_{\alpha} |i\rangle = |i\rangle, \quad (i \rangle H_{0} |i\rangle = \frac{n}{2}
\]

produces a state \(\rho = U_{\alpha} \tau \rho' (H_{S}) \otimes \rho' U_{\alpha} \dagger\) that is locally thermal with local bias \(z \) and temperature \(\beta \) given by

\[
z = \cos (2\alpha) z' \tag{34}
\]

\[
\beta = \frac{2}{\rho} \tan^{-1} (\cos (2\alpha) z') \tag{35}
\]

where \(z' = \langle 0 | \tau \rho' | 0\rangle - \langle 1 | \tau \rho' | 1\rangle = Tr (\tau \rho \tau')\) is the bias of \(\tau \rho'\) (where, for the sake of brevity, we now write \(\tau \rho'\) in place of \(\tau \rho' (H_{S})\) since no confusion should arise). To see that this is the case, we note first that \(\rho\) is symmetric under permutations, since both the initial state \(\tau \rho' (H_{S}) \otimes \rho'\) and \(U_{\alpha}\) are symmetric. Therefore it suffices to calculate \(z_{1} = \langle 0 | \rho_{1} | 0\rangle - \langle 1 | \rho_{1} | 1\rangle\). We note first that this can be re-written as follows

\[
z_{1} = Tr (\sigma_{z} \rho_{1}) = Tr (\sigma_{z} \otimes I_{n-1} \rho)
\]

\[
= \sum_{1}^{n} \langle i | (-1)^{i} \rho | i\rangle
\]

Now, it is straightforward to see that

\[
\langle i | \rho | i\rangle = \langle i | U_{\alpha} \tau \rho' \otimes U_{\alpha} \dagger | i\rangle
\]

\[
= \cos^{2} \alpha \langle i | \tau \rho' | i\rangle + \sin^{2} \alpha \langle i | \tau \rho' | i\rangle \tag{35}
\]
holds for all \(|i\rangle\), and furthermore that \(\langle i|\tau_{\beta'}|i\rangle = \frac{1}{2}(1 + (-1)^i z')\), which follows from the definition of \(z'\) as the bias. Put together, this allows one to re-express \(z_1\) as

\[
z_1 = \sum_{i_1 \cdots i_n} (-1)^{i_1} \left( \frac{\cos^2 \alpha}{2^n} \prod_{k} (1 + (-1)^i z') \right)
+ \left( \frac{\sin^2 \alpha}{2^n} \prod_{k} (1 + (-1)^i (-z')) \right)
\]

which, upon interchanging the order of the product and sum becomes

\[
z_1 = \frac{\cos^2 \alpha}{2^n} \prod_{i_1 \cdots i_n} (-1)^{i_1} (1 + (-1)^i z')
+ \frac{\sin^2 \alpha}{2^n} \prod_{i_1 \cdots i_n} (-1)^{i_1} (1 + (-1)^i (-z'))
\]

(36)

For \(k \neq 1\), \(\sum_{i_1 \cdots i_n} (-1)^{i_1} (1 + (-1)^i z') = 2\), whilst for \(k = 1\), \(\sum_{i_1 \cdots i_n} (-1)^{i_1} (1 + (-1)^i z') = 2z'\), from which we finally obtain

\[
z_1 = \cos^2(\alpha)z' + \sin^2(\alpha)(-z')
= \cos(2\alpha)z'
\]

(38)

**Appendix C: Presence of entanglement in the state**

Consider the state \(\rho = \frac{1}{2^n} U_\alpha \tau_{\beta'} U_\alpha^\dagger\), where each \(\tau_{\beta'}\) is a qubit with population at the excited state \(p = e^{-\beta'\epsilon}/\mathbb{Z}\). As it has an X-like shape, applying the PPT criterion with respect to a bipartition \(A|A\) to \(\rho\) will yield an independent positivity condition for each pair of coherences \(|i\rangle\rho|i\rangle\), \(|\bar{i}\rangle\rho|\bar{i}\rangle\), given by

\[
||\langle i|\rho|\bar{i}\rangle| - \sqrt{\langle i|\Pi_{A|A}\langle i|\Pi_{A|A}|\bar{i}\rangle|\bar{i}\rangle} \geq 0
\]

(39)

where \(\Pi_{A|A}\) is the permutation operator acting on the two-copy Hilbert space exchanging partition \(A\) between the two copies. Focusing on \(|i\rangle = |00\cdots0\rangle\), \(|\bar{i}\rangle = |11\cdots1\rangle\) and on the bipartition \((n/2)|n/2\), the condition for non-separability reads:

\[
\sin \alpha \cos (1 - e^{-\beta'\epsilon n}) - e^{-\beta'\epsilon n/2} \geq 0.
\]

(40)

Clearly, for sufficiently large \(n\), entanglement will be present in the state for any \(\alpha\).

**Appendix D: Maximal work extraction in asymptotic limit**

In this appendix we will show that in the asymptotic limit it is possible to approximately achieve maximal work extraction given an entropy constraint from a state which is classical. To do so we shall apply the unitary \(U_\alpha\) with \(\alpha\) chosen appropriately. Consider that \(\alpha_k\) is non zero (and equal to \(\pi/2\)) only for \(k = np' - \mu\), i.e. between the subspaces with \(|i\rangle = np' - \mu\) and \(|\bar{i}\rangle = n(1 - p') + \mu\), where \(p' = (1|\tau_{\beta'}|1) = \frac{1}{2}(1 - z')\) is the excited state probability in \(\tau_{\beta'}\). That is, we consider the unitary \(V\)

\[
V|i\rangle = |\bar{i}\rangle, \quad \langle |i| = np' - \mu
V|\bar{i}\rangle = -|i\rangle, \quad \langle \bar{i}| = np' - \mu
\]

(41)

\[
V|i\rangle = |\bar{i}\rangle, \quad \langle \bar{i}| = np' - \mu
V|\bar{i}\rangle = -|i\rangle, \quad \langle i| = np' - \mu
\]

(42)

In which case, denoting by \(\ell = np' - \mu\), we see that \(z_1\) is now given by

\[
z_1 = z' - \sum_{|i| = \ell} ((-1)^{i_1} |i||\tau_{\beta'}|\bar{i}\rangle + (-1)^{|\bar{i}|} \langle \bar{i}|\tau_{\beta'}|i\rangle)
+ \sum_{|i| = \ell} ((-1)^{i_1} |i|V\tau_{\beta'}V^\dagger|i\rangle + (-1)^{|\bar{i}|} \langle \bar{i}|V\tau_{\beta'}V^\dagger|\bar{i}\rangle)
\]

(43)

Using the definition of \(V\), and the fact that \((-1)^{\bar{i}} = 1\), this can be re-expressed as

\[
z_1 = z' - 2 \sum_{|i| = \ell} ((-1)^{i_1} |i||\tau_{\beta'}|\bar{i}\rangle + (-1)^{|\bar{i}|} \langle \bar{i}|\tau_{\beta'}|i\rangle)
\]

\[
= z' - 2 \left( (p')^\ell (1 - p')^n - \frac{n - 1}{\ell} \right)
+ (p')^n - \frac{n}{\ell - 1} \left( (n - 1) - \frac{n}{\ell - 1} \right)
\]

\[
= z' - 2 \left( (p')^\ell (1 - p')^n - (p')^{n - \ell} (1 - p')^\ell \right)
\]

(44)

where to obtain the second line we used the fact that the probability of a state in the subspace with \(|i\rangle = 0\) is \((p')^\ell (1 - p')^n - \ell\) and then divided the \(\binom{n}{\ell}\) states in the subspace into the \(\binom{n - 1}{\ell - 1}\) for which \(i_1 = 0\) and \(\binom{n - 1}{\ell - 1}\) for which \(i_1 = 1\) (and analogously for the subspace with \(|\bar{i}\rangle = n - \ell\).

Using now the definition of \(\ell\), and the following asymptotic expansion

\[
(p')^n p' - \mu (1 - p')^n + \frac{n}{np' - \mu} \left( 1 - e^{-\beta'E(nz'+2\mu)} \right)
\]

\[
= \sqrt{\frac{1}{2\pi np'(1 - p')}} e^{-\frac{z'^2}{2np'(1 - p')}} + O\left( \frac{1}{n} \right)
\]

(45)

after some straightforward manipulations we finally arrive at

\[
z = z' \left( 1 - \sqrt{\frac{1}{2\pi np'(1 - p')}} \left( 1 - e^{-\beta'\epsilon(nz'+2\mu)} \right) \right) + O\left( \frac{1}{n} \right)
\]
which demonstrates that we achieve \( z = z' (1 - O(1/\sqrt{n})) \) by swapping only the population between two subspaces within the typical subspace. By applying a sequence of unitaries of the form \( V \) with for different values of \( \mu \) (i.e. corresponding to different subspaces) we therefore see that we can change the local bias (and hence local temperature) of the state by increments of order \( 1/\sqrt{n} \), which can be made arbitrarily small by choosing \( n \) sufficiently large. We note however that the above analysis does not hold if \( p' \) becomes too small, approximately of the order \( 1/\sqrt{n} \). This thus constrains the entropy \( S \) of the initial state, which must grow approximately as \( \sqrt{n} \).

Appendix E: Correlations in degenerate subspaces

Consider the total Hamiltonian

\[
H = \sum_{i=1}^{n} h_i = \sum_{i=1}^{n} E_i \Pi_i, \quad g_i = \text{Tr} \Pi_i \quad (46)
\]

where each \( h_i = h := \sum_{k=0}^{d} \epsilon_k |k\rangle \langle k| \) (with \( \epsilon_0 = 0 \) ) has local dimension \( d \), which we assume to be finite. The number different global energies, \( n_l \) is found to be

\[
n_l = C_{n+d-1}^{d-1} = \frac{(n + d - 1)!}{n!(d - 1)!} \quad (47)
\]

which corresponds to the number of non-zero eigenvalues of \( |15\rangle \). Note that we switched the notation for the binomial coefficients from \( \binom{n}{k} \) to \( C_n^k \). In order to find the passive state associated to \( |15\rangle \), one has to move such eigenvalues to the lowest energy levels. This operation requires knowledge of the spectrum of \( h_i \). Nevertheless, it will suffice for our purposes to move them to a sufficiently degenerated energy. The degeneracy of a global energy \( E_i = \sum_j k^{(i)}_j \epsilon_j \) is equal to \( C_{n+d-1}^{d-1} \). The point is then to find the lowest energy, \( E_{\text{min}} \), satisfying \( C_{n+d-1}^{d-1} \geq C_{n+d-1}^{d-1} \), so that the work extracted after such a transformation is simply given by

\[
W_{\text{deg}} = E_{\rho_{\text{ens}}} - E_{\text{min}}. \quad (48)
\]

Now, for large \( n \) notice that

\[
\lim_{n \to \infty} \frac{C_{n+d-1}^{d-1}}{C_{n+d-1}^{d-1} k_j \ldots k_d} = 0, \quad \sum_{j=2}^{d} k_j = d \quad (49)
\]

with \( E' = \sum_{j=2}^{d} k_j \epsilon_j \). Notice that \( E' \) is of the order of the energy of one subsystem (for instance, choosing \( k_2 = d \) and \( k_j = 0 \) for \( j > 2 \), we obtain \( E' = d \epsilon_2 \)), and therefore we can take \( E_{\text{min}} = E' \) obtaining the desired result.

[1] A. Jenčová and M. B. Ruskai, “A unified treatment of convexity of relative entropy and related trace functions, with conditions for equality”, Rev. Math. Phys. 22, 1099 (2010).

[2] We use the part (d) of the theorem in the mentioned paper. Their and our notation is parallelized as follows: their \( K \) is unity in our case, there \( A \) = \( \sum_j A_j \) while here \( \Omega = \sum_x \lambda_x \rho_x^{A} \otimes \rho_x^{B} \), and \( B = \sum_j B_j \) is \( \tau \otimes \frac{I_n}{n} = \sum_x \lambda_x \rho_x^{A} \otimes \frac{I_n}{n} \).

[3] In that notation \( \Omega = \sum_x \lambda_x P_x \).