Existence results for non–coercive variational problems

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Abstract. We consider the problem (P):

\[
\min \left\{ \int_0^T \left[ g(t, u) + f(t, u') \right] dt \mid u \in W_{1,1}([0, T], \mathbb{R}^m), u(0) = a, u(T) = b \right\}
\]

with neither coercivity nor convexity assumptions. More precisely, assuming that \( f(t, \xi) \) is a lower semicontinuous function, bounded from below, Lipschitz continuous with respect to \( t \), satisfying

\[
\lim_{n \to +\infty} \left[ f^{**}(t_n, \xi_n) - \langle \nabla f^{**}(t_n, \xi_n), \xi_n \rangle \right] = -\infty
\]

for every sequence \( \{t_n\} \in [0, T] \) and for every choice of points \( \xi_n \) of differentiability of \( f^{**}(t_n, \cdot) \) such that \( \lim_n |\xi_n| = +\infty \), and assuming that \( g(t, x) \) is a continuous function, Lipschitz continuous with respect to \( t \) and concave with respect to \( x \), such that \( g(t, x) \geq -\alpha - \beta|x| \) for every \( (t, x) \in [0, T] \times \mathbb{R}^m \), and for suitable constants \( \alpha \) and \( \beta \geq 0 \), we show that the problem (P) has a solution in the space \( W^{1,\infty}([0, T], \mathbb{R}^m) \). The main tools for the proof are an existence theorem for the problem (P), with \( f(t, \xi) \) convex with respect to \( \xi \) and \( g \) continuous, and the closure of the convex hull of the epigraph of the function \( f(t, \cdot) \).

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1. Introduction

It is well known that, if $L$ is a continuous function, such that $\xi \mapsto L(t, x, \xi)$ is convex and superlinear, then the variational problem

\[
\min \left\{ \int_0^T L(t, u, u') dt \mid u \in W^{1,1}([0, T], \mathbb{R}^m), u(0) = a, u(T) = b \right\},
\]

has a solution (see for instance [7]).

In recent years, the possibility of avoiding the convexity or the superlinearity assumption was investigated by many authors.

Some existence results for non–convex coercive problems were obtained in the case $L(t, x, \xi) = g(t, x) + f(t, \xi)$ (see for instance [5], [14], [16] and the references therein). In particular, in [5] it was proved that the convexity assumption on $f(t, \cdot)$ can be replaced by the condition of concavity of $g(t, \cdot)$.

More recently, some techniques were developed in order to treat convex but non–coercive problems. In this case, even if the functionals considered are lower semicontinuous in the weak topology of $W^{1,1}([0, T], \mathbb{R}^m)$, the direct method of the Calculus of Variations can not be applied, due to the lack of compactness of the minimizing sequences.

In [10], it was studied the problem (1.1) with $L$ continuous, bounded from below and convex with respect to $\xi$, the superlinearity being replaced by a weaker condition which permits to construct a relatively compact minimizing sequence, obtained by considering the minima of suitable coercive approximating problems. The main step in the proof of the existence result in [10] was to show that every minimum point of the approximating problems solves a generalized DuBois–Reymond condition, which implies that the minimizing sequence is bounded in the space $W^{1,\infty}([0, T], \mathbb{R}^m)$.

A similar approach was used in [6] for the autonomous problem $L(t, x, \xi) = g(x) + f(\xi)$, where $g$ is a nonnegative continuous function, and $f \in C^1(\mathbb{R}^m, \mathbb{R})$ is a strictly convex function bounded from below, such that

\[
\lim_{|\xi| \to +\infty} [f(\xi) - \langle \nabla f(\xi), \xi \rangle] = -\infty.
\]

In that paper, it was proved that, for every rectifiable curve $C$ in $\mathbb{R}^m$ joining $a$ to $b$ there exists a unique solution to the problem (1.1) restricted to the class of all absolutely continuous parameterizations $u: I \to \mathbb{R}^m$ of $C$. Thus, every element $u_n$ of a minimizing sequence can be replaced by the minimum corresponding to the curve parameterized by $u_n$. It can be shown, still using a DuBois–Reymond condition satisfied by those minima,
and by (1.2), that this new sequence is bounded in $W^{1,\infty}([0,T], \mathbb{R}^m)$, so that there exists a minimum point for (1.1) in this space.

In [12] both the superlinearity and the convexity assumptions were dropped for lagrangians of the form $L(t,x,\xi) = \langle a(t), x \rangle + f(\xi)$ where $f$ is a lower semicontinuous function whose convexification $f^{**}$ satisfies (1.2) for every diverging sequence of points of differentiability of the Lipschitz continuous function $f^{**}$. The existence of a minimum is proved by a technique relying only on a Lyapunov type theorem due to Olech (see [15]).

For other results concerning non–coercive problems we mention [1], [2] and [3].

In this paper we consider non–autonomous problems of the form

(1.3) $\min \left\{ \int_0^T \left[ g(t,u) + f(t,u') \right] \,dt \mid u \in W^{1,1}([0,T], \mathbb{R}^m), \, u(0) = a, \, u(T) = b \right\}$

with neither coercivity nor convexity assumptions. More precisely, we introduce the class $\mathcal{E}$ of all functions $\psi : [0,T] \times \mathbb{R}^m \to \mathbb{R}$, bounded from below, such that $\psi(\cdot, \xi)$ is Lipschitz continuous for every fixed $\xi \in \mathbb{R}^m$, $\psi(t, \cdot)$ is lower semicontinuous and satisfies

$$\lim_{n \to +\infty} \left[ \psi^{**}(t^n, \xi^n) - \langle \nabla \psi^{**}(t^n, \xi^n), \xi^n \rangle \right] = -\infty$$

for every sequence $\{t^n\} \in [0,T]$ and for every choice of points $\xi^n$ of differentiability of $\psi^{**}(t^n, \cdot)$ such that $\lim_n |\xi^n| = +\infty$. We show that, if $f \in \mathcal{E}$ and there exists two constants $A$ and $B$, $B > 0$ such that $f(t,\xi) \geq -A + B|\xi|$ for every $(t,\xi) \in [0,T] \times \mathbb{R}^m$, and $g(t,x)$ is a continuous function, Lipschitz continuous with respect to $t$, concave with respect to $x$, satisfying $g(t,x) \geq -\alpha - \beta |x|$ for every $(t,x) \in [0,T] \times \mathbb{R}^m$, and for suitable constants $\alpha$ and $0 \leq \beta \leq B/T$, then the problem (1.3) has a solution in the space $W^{1,\infty}([0,T], \mathbb{R}^m)$. This result is the analogue for a class of non–coercive functionals of the one in [5], but it is not a generalization of that result, due to the additional requirement of the Lipschitz continuity of the lagrangian with respect to the variable $t$. However this extra regularity allows us to obtain the necessary conditions that, used at an intermediate step, give also a regularity result, interesting by itself.

As a first step we prove an existence result for (1.3) requiring that $f$ be convex with respect to $\xi$ and dropping the concavity assumption on $g$. This can be done following [10] and making suitable changes, due to the the fact that the lagrangian is not bounded from below. The second step, linking the convex to the non–convex case, is based on a result concerning the closure of the convex hull of the epigraph of functions whose convexification is strictly convex at infinity (that is, the graph of the convexification contains no rays).
This result is an extension of the classical theorem that holds for superlinear functions (see [13]). We want to remark that the notion of strict convexity at infinity was still used in [11] in order to study non–coercive problems of the type (1.1) with the additional state constraint $\|u\|_{L^\infty} < R$. We shall prove that every function in the class $E$ is strictly convex at infinity for every fixed $t$, so that, by using the previous results and the Lyapunov theorem on the range of non–atomic measures, the existence result for the non–convex problems follows. The regularity of the solution of (1.3) is a consequence of the regularity of the solution to the relaxed problem.

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### 2. Preliminaries

We shall denote by $\langle x, y \rangle$ the standard scalar product of two vectors $x, y \in \mathbb{R}^m$. For every $1 \leq p \leq +\infty$, we shall denote by $L^p(I, \mathbb{R}^m)$ and $W^{1,p}(I, \mathbb{R}^m)$, respectively, the usual Lebesgue and Sobolev spaces of functions from the interval $I = [0, T]$ into $\mathbb{R}^m$. We shall use the symbol $\| \cdot \|_{L^p}$ to denote the norm in $L^p(I, \mathbb{R}^m)$.

If $A \subset \mathbb{R}^m$, we shall denote by $\text{int} A$ the interior of $A$, and by $\text{co} A$ the convex hull of $A$, that is, the smallest convex set which contains $A$. It is well known that, by Carathéodory’s theorem, the convex hull of $A$ can be characterized by

$$
\text{co} A = \left\{ x \in \mathbb{R}^m \mid x = \sum_{i=1}^{m+1} \lambda_i x_i, \; \tilde{\lambda} \in E_{m+1}, \; x_i \in A, \; i = 1, \ldots, m+1 \right\},
$$

where $\tilde{\lambda} = (\lambda_1, \ldots, \lambda_{m+1})$, and $E_{m+1}$ denotes the standard simplex:

$$
E_{m+1} = \left\{ (\lambda_1, \ldots, \lambda_{m+1}) \in \mathbb{R}^{m+1} \mid \lambda_i \geq 0 \; \forall i = 1, \ldots, m+1, \; \sum_{i=1}^{m+1} \lambda_i = 1 \right\}.
$$

Given a function $\psi: \mathbb{R}^m \to \mathbb{R}$, we shall denote by $\text{dom}(\psi)$ its effective domain, defined as the subset of $\mathbb{R}^m$ $\{ \xi \mid \psi(\xi) < +\infty \}$, and by $\text{epi} \psi$ its epigraph, that is the set:

$$
\text{epi} \psi = \{ (x, a) \in \mathbb{R}^m \times \mathbb{R} \mid \psi(x) \leq a \}.
$$

If $\psi: \mathbb{R}^m \to \mathbb{R}$ is Lipschitz continuous in a neighborhood of a point $\xi$, we shall denote by $\partial \psi(\xi)$ the generalized gradient of $\psi$ at $\xi$, defined by

$$
\partial \psi(\xi) \doteq \text{co} \left\{ \lim_{i \to +\infty} \nabla \psi(\xi_i) \mid \xi_i \to \xi, \; \xi_i \in \mathcal{D}(\psi) \right\},
$$

where $\mathcal{D}(\psi)$ is the domain of $\psi$. The regularity of the solution of (1.3) is a consequence of the regularity of the solution to the relaxed problem.
where $\mathcal{D}(\psi)$ denotes the set of points of differentiability of $\psi$. If $\psi$ is differentiable at $\xi$, then 

$$
\partial \psi(\xi) = \{\nabla \psi(\xi)\}.
$$

We recall that a Lipschitz continuous function $\psi$ is almost everywhere differentiable in $\text{int}(\text{dom}(\psi))$.

A function $\psi: \mathbb{R}^m \to (-\infty, +\infty]$ is convex if, for every $\xi, \eta \in \mathbb{R}^m$ and for every $\lambda \in [0, 1]$, we have $\psi(\lambda \xi + (1 - \lambda)\eta) \leq \lambda \psi(\xi) + (1 - \lambda)\psi(\eta)$. We say that $\psi$ is concave if $-\psi$ is convex.

Given a function $\psi: \mathbb{R}^m \to (-\infty, +\infty]$, we shall denote by $\psi^*$ its dual function, defined for every $p \in \mathbb{R}^m$ by

$$
\psi^*(p) = \sup_{\xi \in \mathbb{R}^m} \{\langle p, \xi \rangle - \psi(\xi)\}.
$$

It is well known that the bidual function $\psi^{**}$ coincides with the convexification of $\psi$, which is the largest convex function $\varphi$ satisfying $\varphi \leq \psi$.

If $\psi: \mathbb{R}^m \to (-\infty, +\infty]$ is convex, then the generalized gradient of $\psi$ coincides in $\text{int}(\text{dom}(\psi))$ with the subgradient of $\psi$ in the sense of convex analysis, defined at every point $\xi \in \text{dom}(\psi)$ by

$$
\partial \psi(\xi) = \{p \in \mathbb{R}^m \mid \psi(\eta) \geq \psi(\xi) + \langle p, \eta - \xi \rangle, \text{ for every } \eta \in \mathbb{R}^m\}.
$$

(see [8], Proposition 2.2.7). By definition, we set $\partial \psi(\xi) = \emptyset$ for every $\xi \notin \text{dom}(\psi)$.

In the following proposition we collect some well known properties of the subgradient (see [8] and [13]).

**Proposition 2.1**  
Let $\psi: \mathbb{R}^m \to (-\infty, +\infty]$ be a convex function. Then the following properties hold:

(i) if $\psi$ is bounded from above in a non–empty open set $A$, then $\psi$ is locally Lipschitz continuous in $A$;

(ii) for every $\xi \in \mathbb{R}^m$, the set $\partial \psi(\xi)$ (possibly empty) is convex and closed in $\mathbb{R}^m$;

(iii) if $\xi \in \text{int}(\text{dom}(\psi))$, then $\partial \psi(\xi)$ is a non–empty compact set.

### 3. The closure result

In this section we shall prove a result concerning the closure of the convex hull of the epigraph of functions possibly without superlinear growth.

We recall the notion of strict convexity at infinity, introduced by Clarke and Loewen in [11].

**Definition 3.1**  
A convex function $\psi: \mathbb{R}^m \to \mathbb{R}$ is said to be strictly convex at infinity if its graph contains no rays, that is for every $\nu \in \mathbb{R}^m$, $\nu \neq 0$, and for every $\xi \in \mathbb{R}^m$,
the function $\psi_{\nu,\xi}(s) = \psi(s\nu + \xi)$ has the following property: for every $s_0 \in \mathcal{D}(\psi_{\nu,\xi})$ there exists $s_1 \in \mathcal{D}(\psi_{\nu,\xi})$, $s_1 > s_0$, such that $\psi'_{\nu,\xi}(s_1) > \psi'_{\nu,\xi}(s_0)$.

**Remark 3.2** It is easy to see that, if $\psi: \mathbb{R}^m \rightarrow \mathbb{R}$ is convex, then $\psi$ is strictly convex at infinity if and only if $\partial \psi^*(p)$ is either empty or bounded for every $p \in \mathbb{R}^m$.

**Definition 3.3** We shall denote by $\mathcal{G}$ the family of all lower semicontinuous functions $\psi: \mathbb{R}^m \rightarrow \mathbb{R}$ such that $\psi^{**} \neq -\infty$ and $\psi^{**}$ is strictly convex at infinity.

**Remark 3.4** Clearly, every strictly convex function is strictly convex at infinity. Moreover, every lower semicontinuous superlinear function $\psi: \mathbb{R}^m \rightarrow \mathbb{R}$ belongs to $\mathcal{G}$. Indeed, denoting by $\varphi$ the convexification $\psi^{**}$, for every fixed $\nu, \xi \in \mathbb{R}^m$, $\nu \neq 0$, by (2.3) it follows that the inequality $\langle \nabla \varphi(s\nu + \xi), s\nu \rangle \geq \varphi(s\nu + \xi) - \varphi(\xi)$ holds for every $s \in \mathcal{D}(\varphi_{\nu,\xi})$. This implies that

$$\varphi'_{\nu,\xi}(s) = \langle \nabla \varphi(s\nu + \xi), \nu \rangle \geq \frac{\varphi(s\nu + \xi) - \varphi(\xi)}{s}, \text{ for every } s \in \mathcal{D}(\varphi_{\nu,\xi}), \ s > 0.$$ 

Since $\psi$ is superlinear, the last term tends to $+\infty$ as $s$ goes to $+\infty$.

**Lemma 3.5** For every function $\psi \in \mathcal{G}$, satisfying $\psi \geq 0$ and $\psi(0) = 0$, there exist two positive constants $C, \rho$ such that $\psi(\xi) \geq C|\xi|$ for every $|\xi| > \rho$.

**Proof.** We can certainly assume that $\psi$ is convex, for if not, we replace $\psi$ by $\psi^{**}$. We start by proving that $\psi$ is coercive, that is $\psi(\xi) \rightarrow +\infty$ as $|\xi| \rightarrow +\infty$. Since $\psi$ is convex, the sets $\psi^a = \{\xi \in \mathbb{R}^m \mid \psi(\xi) < a\}$ are convex subsets of $\mathbb{R}^m$ for every $a \geq 0$. By contradiction, suppose that there exists $a > 0$ such that $\psi^a$ is unbounded. Since $\psi^a$ is convex, it contains at least one half line $\{s\nu \mid s \geq 0\}$ for some $\nu \in \mathbb{R}^m$, $\nu \neq 0$. This means that $\psi_{\nu,0}(s) < a$ for every $s \geq 0$. Since $\psi_{\nu,0}$ is an absolutely continuous function, then for every $\tau > 0$ we have

$$0 \leq \psi_{\nu,0}(\tau) - \psi_{\nu,0}(0) = \int_0^\tau \psi'_{\nu,0}(\sigma) \, d\sigma.$$ 

Hence, there exists $s_0 \in \mathcal{D}(\psi_{\nu,0}) \cap [0, \tau]$ such that $\psi'_{\nu,0}(s_0) \geq 0$. Since $\psi$ is strictly convex at infinity, there exists $s_1 \in \mathcal{D}(\psi_{\nu,0})$, $s_1 > s_0$, such that $\psi'_{\nu,0}(s_1) > 0$. By the convexity of $\psi_{\nu,0}$ it follows that

$$\psi_{\nu,0}(s) \geq \psi_{\nu,0}(s_1) + (s - s_1)\psi'_{\nu,0}(s_1), \text{ for every } s \geq 0,$$

and this implies that $\lim_{s \rightarrow +\infty} \psi_{\nu,0}(s) = +\infty$, in contradiction with $\psi_{\nu,0} < a$. 

Since \( \psi \) is coercive, there exist two positive constants \( \rho, \delta \) such that:

\[
\psi(\eta) \geq \delta, \quad \text{for all } |\eta| = \rho.
\]

If \( |\xi| > \rho \), let us define \( \lambda = \rho/|\xi| \) and \( \eta = \lambda \xi \). By the convexity of \( \psi \), and recalling that \( \psi(0) = 0 \), we get

\[
\psi(\xi) \geq \frac{1}{\lambda} \psi(\eta) = \frac{\psi(\eta)}{\rho} |\xi| \geq \frac{\delta}{\rho} |\xi|,
\]

so that we have done by choosing \( C = \delta/\rho \).

We are now in a position to prove the closure result. The proof is based on the fact that, if \( f \) belongs to the class \( \mathcal{G} \), then for every support hyperplane \( r \) of \( f \), the function \( f - r \) belongs to \( \mathcal{G} \). Applying the estimate of Lemma 3.5 to this function, we can follow the lines of the proof of Lemma IX.3.3 in [13].

**Theorem 3.6**  For every \( f \in \mathcal{G} \) the set \( \text{co epi } f \) is closed.

**Proof.** Let \( (\xi, a) \in \partial(\text{co epi } f) \), where \( \partial S \) denotes the boundary of the set \( S \), and let \( r(\eta) \equiv \langle c, \eta \rangle + d \) be an affine function such that the hyperplane \( H \equiv \{ (\eta, r(\eta)) \} \) weakly separates \( \text{co epi } f \) and the point \( (\xi, a) \). Let us define the function

\[
\phi(\eta) \equiv f(\eta + \xi) - r(\eta + \xi).
\]

We have \( \phi^{**}(\eta) = f^{**}(\eta + \xi) - r(\eta + \xi) \), \( \phi^{**} \geq 0 \), \( \phi^{**}(0) = 0 \). Moreover, for every \( \nu \in \mathbb{R}^m \), \( \nu \neq 0 \), for every \( \eta \in \mathbb{R}^m \) and for every \( s \in \mathcal{D}(f^{**}_{\nu,\xi+\eta}) \) we have \( (\phi^{**}_{\nu,\eta})'(s) = (f^{**}_{\nu,\xi+\eta})'(s) - \langle c, \nu \rangle \). Since \( f^{**} \) is strictly convex at infinity, then so is \( \phi^{**} \). By Lemma 3.5, there exist two positive constants \( C, \rho \) such that

\[
(3.1) \quad \phi^{**}(\eta) \geq C|\eta|, \quad \text{for every } |\eta| \geq \rho.
\]

Notice that \( (\xi, a) \in \text{co epi } f \) if and only if \( (0, 0) \in \text{co epi } \phi \). Moreover, \( (\xi, a) \in \partial(\text{co epi } f) \) if and only if \( (0, 0) \in \partial(\text{co epi } \phi) \). Hence, to prove the proposition, it suffices to show that \( (0, 0) \in \text{co epi } \phi \).

Let \( (\xi^n, a^n) \in \text{co epi } \phi \) be such that \( \lim_n (\xi^n, a^n) = (0, 0) \). By the characterization (2.1) of the convex hull, for every \( n \) there exist \( \lambda^n = E_{m+2} \) and \( (\xi^n_j, a^n_j) \in \text{co epi } \phi, j = 1, \ldots, m+2 \), such that

\[
\sum_{j=1}^{m+2} \lambda^n_j (\xi^n_j, a^n_j) = (\xi^n, a^n).
\]
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By the very definition of epigraph it follows that

$$a^n = \sum_{j=1}^{m+2} \lambda_j^n a_j^n \geq \sum_{j=1}^{m+2} \lambda_j^n \phi(\xi^n_j).$$

Moreover, (3.2) and the fact that $\phi \geq \phi^{**}$ imply that $a^n \geq \sum_{j=1}^{m+2} \lambda_j^n \phi^{**}(\xi^n_j)$. Since $\phi^{**} \geq 0$, the inequality

$$a^n \geq \sum_{j=1}^{m+2} \lambda_j^n \phi^{**}(\xi^n_j)$$

holds for every $j = 1, \ldots, m + 2$. Let $J \subset \{1, \ldots, m + 2\}$ be the set of all $j$ such that $\{|\xi^n_j|\}_{n}$ is unbounded, and let $I = \{1, \ldots, m + 2\} \setminus J$. By passing to a subsequence, we can assume that there exist $\xi_j, j \in I$, and $\lambda \in E_{m+2}$, such that

$$\lim_{n \to +\infty} |\xi^n_j| = +\infty, \quad j \in J,$$

$$\lim_{n \to +\infty} \xi^n_j = \xi_j, \quad j \in I,$$

$$\lim_{n \to +\infty} \lambda^n_j = \lambda_j, \quad j \in \{1, \ldots, m + 2\}.$$

For every $j \in J$, we have $|\xi^n_j| > \rho$ for $n$ large enough, and then from (3.1) and (3.3) it follows that $a^n \geq C\lambda_j^n|\xi^n_j|$. Since $\lim_n a^n = 0$, we get

$$\lim_{n \to +\infty} \lambda^n_j |\xi^n_j| = 0, \quad j \in J.$$

From (3.4), and recalling that $\lim_n \xi^n = 0$, we deduce that

$$\sum_{j \in I} \lambda_j^n \xi_j = \lim_{n \to +\infty} \sum_{j \in I} \lambda_j^n \xi^n_j =$$

$$= \lim_{n \to +\infty} \left( \sum_{j=1}^{m+2} \lambda_j^n \xi^n_j - \sum_{j \in J} \lambda_j^n \xi^n_j \right) = \lim_{n \to +\infty} \left( \xi^n - \sum_{j \in J} \lambda_j^n \xi^n_j \right) = 0.$$

Moreover, since $\lim_n \lambda_j^n = 0$ for every $j \in J$, we obtain

$$\sum_{j \in I} \lambda_j = \lim_{n \to +\infty} \sum_{j \in I} \lambda_j^n = 1.$$

Since $\phi$ is a non-negative lower semicontinuous function, we get

$$0 \leq \sum_{j \in I} \lambda_j \phi(\xi_j) \leq \liminf_{n \to +\infty} \sum_{j \in I} \lambda_j^n \phi(\xi^n_j) \leq \liminf_{n \to +\infty} a^n = 0.$$

There is no loss of generality in assuming that $\lambda_j > 0$ for every $j \in I$, hence (3.7) implies that $\phi(\xi_j) = 0$ for every $j \in I$, that is $(\xi_j, 0) \in \text{epi} \phi$ for every $j \in I$. Thus, by (3.5) and (3.6), we can conclude that $(0, 0)$ belongs to co epi $\phi$. \qed
Now we state two direct consequences of Theorem 3.6.

**Corollary 3.7** If \( f \in \mathcal{G} \), then

\[
f^{**}(\xi) = \min \left\{ \sum_{j=1}^{m+1} \lambda_j f(\xi_j) \mid \sum_{j=1}^{m+1} \lambda_j \xi_j = \xi, \ \bar{\lambda} \in E_{m+1} \right\},
\]

for every \( \xi \in \mathbb{R}^m \).

**Proof.** See [13], Lemma IX.3.3.

We recall that a function \( f: I \times \mathbb{R}^m \to \mathbb{R} \) is said to be a normal integrand (see [13]) if \( f(t, \cdot) \) is lower semicontinuous for a.e. \( t \in I \), and there exists a Borel function \( \tilde{f}: I \times \mathbb{R}^m \to \mathbb{R} \) such that \( \tilde{f}(t, \cdot) = f(t, \cdot) \) for a.e. \( t \in I \).

**Corollary 3.8** Let \( f: I \times \mathbb{R}^m \to \mathbb{R} \) be a normal integrand, and suppose that \( f(t, \cdot) \in \mathcal{G} \) for every \( t \in I \). Then for any measurable mapping \( p: [0, T] \to \mathbb{R}^m \), there exist a measurable mapping \( \bar{\lambda}: [0, T] \to E_{m+1} \) and \( m+1 \) measurable mappings \( q_j: [0, T] \to \mathbb{R}^m \), such that

\[
\sum_{j=1}^{m+1} \lambda_j(t)q_j(t) = p(t), \quad \sum_{j=1}^{m+1} \lambda_j(t)f(t, q_j(t)) = f^{**}(t, p(t)),
\]

for almost all \( t \in [0, T] \).

**Proof.** See [13], Proposition IX.3.1.

4. Existence results for variational problems

In this section we shall show that the existence result proved by Cellina and Colombo in [5] holds even for functions of the class \( \mathcal{E} \) defined below. In the following, the convexification and the gradient of a function \( \psi(t, \xi) \) are understood with respect to \( \xi \).

**Definition 4.1** We shall denote by \( \mathcal{E} \) the family of all functions \( \psi: I \times \mathbb{R}^m \to \mathbb{R} \), bounded from below, such that \( \psi(\cdot, \xi) \) is Lipschitz continuous for every fixed \( \xi \in \mathbb{R}^m \), \( \psi(t, \cdot) \) is lower semicontinuous for every fixed \( t \in I \), and

\[
\lim_{R \to +\infty} \sup_{t \in I} \sup_{|\xi| > R} \left\{ \psi^{**}(t, \xi) - \langle p, \xi \rangle \mid p \in \partial \xi \psi^{**}(t, \xi) \right\} = -\infty.
\]

The following proposition gives a characterization of the family \( \mathcal{E} \). The proof is similar to the one of Proposition 3.2 in [12].
Proposition 4.2 The condition (4.1) in Definition 4.1 is equivalent to:

\[
\lim_{n \to +\infty} \left[ \psi^*(t^n, \xi^n) - \langle \nabla \psi^*(t^n, \xi^n), \xi^n \rangle \right] = -\infty
\]

for every sequence \((t^n, \xi^n) \in I \times \mathbb{R}^m\) such that \(\xi^n \in \mathcal{D}(\psi^*(t^n, \cdot))\), \(\lim_n |\xi^n| = +\infty\).

**Proof.** We have to prove that (4.2) implies (4.1), the other implication being trivial. Let us denote by \(\chi(R)\) the argument of the limit in (4.1), and let \(\{R_n\}\) be a diverging sequence. For every fixed \(n \in \mathbb{N}\), by definition of supremum, there exists \((t^n, \xi^n, p^n) \in I \times \mathbb{R}^m \times \mathbb{R}^m\), with \(p^n \in \partial \xi \psi^*(t^n, \xi^n)\) and \(|\xi^n| > R_n\), such that

\[
\chi(R_n) \leq \psi^*(t^n, \xi^n) - \langle p^n, \xi^n \rangle + 1.
\]

From (2.2) and (2.1), there exist \(p^n_j \in \partial \xi \psi^*(t^n, \xi^n), \xi^n_j \in \mathcal{D}(\psi^*(t^n, \cdot))\), with \(|\xi^n_j - \xi^n| < 1\), \(j \in J = \{1, \ldots, m+1\}\), and \(\tilde{\lambda}^n \in E_{m+1}\), such that

\[
p^n = \sum_{j=1}^{m+1} \lambda^n_j p^n_j, \quad |\nabla \psi^*(t^n, \xi^n) - p^n_j| < \frac{1}{|\xi^n| + 1}, \quad \text{for every } j \in J.
\]

For every \(j \in J\), the last inequality and the fact that \(|\xi^n_j - \xi^n| < 1\) imply that

\[
|\nabla \psi^*(t^n, \xi^n) - \psi^*(t^n, \xi^n)| < \frac{|\xi^n_j|}{|\xi^n| + 1} < 1.
\]

By the convexity of \(\psi^*(t^n, \cdot)\) we have

\[
\psi^*(t^n, \xi^n) - \psi^*(t^n, \xi^n_j) \leq \langle p^n_j, \xi^n - \xi^n_j \rangle, \quad \text{for every } j \in J.
\]

Using (4.4) and (4.5) we obtain

\[
\psi^*(t^n, \xi^n) - \langle p^n_j, \xi^n \rangle \leq \psi^*(t^n, \xi^n) - \langle \nabla \psi^*(t^n, \xi^n), \xi^n \rangle + 1.
\]

Multiplying (4.6) by \(\lambda^n_j\) and summing over \(j\) it follows that \(\psi^*(t^n, \xi^n) - \langle p^n, \xi^n \rangle \leq \mu^n\), where \(\mu^n = 1 + \max_j \{\psi^*(t^n, \xi^n_j) - \langle \nabla \psi^*(t^n, \xi^n_j), \xi^n_j \rangle\}\).

Since \(\lim_n |\xi^n_j| = +\infty\) for every \(j \in J\), (4.2) implies that \(\lim_n \mu^n = -\infty\). Hence, by (4.3), it follows that

\[
\lim_{n \to +\infty} \chi(R_n) \leq \lim_{n \to +\infty} (\mu^n + 1) = -\infty.
\]

Since \(\chi\) is a monotone non–increasing function, (4.1) holds.
Remark 4.3  The Definition 4.1 agrees with the one given in [6] and [12], respectively in the case of convex time–independent smooth functions and non–convex time–independent functions.

Lemma 4.4  If \( \psi \in \mathcal{E} \), then \( \psi(t, \cdot) \in \mathcal{G} \) for every \( t \in I \).

Proof.  Let us fix \( t \in I \), and denote by \( \varphi \) the convexification with respect to \( \xi \) of \( \psi(t, \xi) \). By Lemma 3.3 in [12], the effective domain \( \text{dom}(\varphi^*) \) of \( \varphi^* \) is an open subset of \( \mathbb{R}^m \). Hence, by Proposition 2.1(iii), \( \partial \varphi^*(p) \) is either bounded, if \( p \in \text{dom}(\varphi^*) \), or empty, if \( p \notin \text{dom}(\varphi^*) \). By Remark 3.2, the result is thus proved. \( \square \)

Lemma 4.5  Let \( \varphi : I \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R} \) be a lower semicontinuous function, Lipschitz continuous with respect to the first variable. Assume that \( \varphi(t, x, \cdot) \) is convex for a.e. \( t \in I \) and for every \( x \in \mathbb{R}^m \), and that there exist three constants \( C_i, i = 0, 1, 2 \), such that 

\[
(4.7) \quad |v| \leq C_0|\varphi(t, x, \xi)| + C_1|x| + C_2,
\]

for every \( (t, x, \xi) \in I \times \mathbb{R}^m \times \mathbb{R}^m \) and for every \( v \in \partial t \varphi(t, x, \xi) \), where \( \partial_t \varphi \) denotes the generalized gradient of \( \varphi \) with respect to \( t \).

Let \( u \in W^{1,1}(I, \mathbb{R}^m) \), and assume that the function \( t \mapsto \varphi(t, u(t), u'(t)) \) belongs to \( L^1(I) \). Then there exists \( k_0 \in L^1(I) \) such that 

\[
|\varphi(s_2, u(t), u'(t)) - \varphi(s_1, u(t), u'(t))| \leq k_0(t)|s_2 - s_1|,
\]

for every \( t, s_1, s_2 \in I \).

Proof.  For every fixed \( t_1, t_2 \in I \), let us define the function 

\[
g(s) = |\varphi(t_1 + sd, x, \xi) - \varphi(t_1, x, \xi)|, \quad s \in [0, 1],
\]

where \( d = t_2 - t_1 \). By (4.7), it follows that for a.e. \( s \in [0, 1] \)

\[
g'(s) \leq |d||\partial_t \varphi(t_1 + sd, x, \xi)| \leq |d|(C_0g(s) + C_0|\varphi(t_1, x, \xi)| + C_1|x| + C_2).
\]

We can apply Gronwall’s inequality to the non–negative absolutely continuous function \( g \), obtaining 

\[
(4.8) \quad |\varphi(t_2, x, \xi) - \varphi(t_1, x, \xi)| = g(1) \leq |t_2 - t_1|e^{C_0T}(C_0|\varphi(t_1, x, \xi)| + C_1|x| + C_2).
\]
This inequality, with $t_1 = t$ and $t_2 = s_1$, implies that

(4.9) \[ |\varphi(s_1, x, \xi)| \leq |\varphi(t, x, \xi)| + T e^{C_0 T} (C_0 |\varphi(t, x, \xi)| + C_1 |x| + C_2). \]

Again by (4.8), with $t_1 = s_1$, $t_2 = s_2$, and by (4.9), it follows that

\[ |\varphi(s_2, x, \xi) - \varphi(s_1, x, \xi)| \leq |s_2 - s_1| (C_0 |\varphi(t, x, \xi)| + \tilde{C}_1 |x| + \tilde{C}_2), \]

where $\tilde{C}_i = C_i e^{C_0 T} (1 + T C_0 e^{C_0 T})$, $i = 0, 1, 2$. Finally, by hypothesis, the function

\[ k_0(t) = \tilde{C}_0 |\varphi(t, u(t), u'(t))| + \tilde{C}_1 |u(t)| + \tilde{C}_2 \]

belongs to $L^1(I)$, completing the proof.

**Definition 4.6** We shall say that $\theta \in C^1((0, +\infty), \mathbb{R})$ is a Nagumo function if $\theta$ is convex, increasing and it satisfies $\lim_{r \to +\infty} \theta(r)/r = +\infty$.

We begin the study of minimization problems, starting with an existence result for convex functionals. We collect here the basic hypotheses on the integrand.

\begin{itemize}
  \item [(H_0)] $f \in \mathcal{E}$, and $f(t, \cdot)$ is a convex function for every $t \in I$.
  \item [(H_1)] There exist two constants $A$ and $B$, with $B > 0$, such that $f(t, \xi) \geq -A + B|\xi|$ for every $(t, \xi) \in I \times \mathbb{R}^m$.
  \item [(H_2)] $g : I \times \mathbb{R}^m \to \mathbb{R}$ is Lipschitz continuous with respect to the first variable, continuous with respect to the second, and there exist two constants $\alpha$, $\beta$, with $0 \leq \beta < B/T$, such that $g(t, x) \geq -\alpha - \beta |x|$ for every $(t, x) \in I \times \mathbb{R}^m$.
  \item [(H_3)] There exist three constants $C_i$, $i = 0, 1, 2$, such that the condition (4.7) holds with $\varphi(t, x, \xi) \geq g(t, x) + f(t, \xi)$.
\end{itemize}

**Remark 4.7** If $f \in \mathcal{E}$ is independent of $t$, then it is easily seen that Lemma 3.5 and Lemma 4.4 imply that condition (H_1) is always satisfied for suitable constants $A$, $B$, with $B > 0$.

**Theorem 4.8** Let $f$ and $g$ satisfy the hypotheses $(H_0)$, $(H_1)$, $(H_2)$, $(H_3)$. Then there exists a solution $\tilde{u}$ to the problem

(4.10) \[ \min \left\{ F(u) \mid u \in W^{1,1}(I, \mathbb{R}^m), u(0) = a, u(T) = b \right\} \]

where

\[ F(u) \doteq \int_I [f(t, u'(t)) + g(t, u(t))] \, dt. \]
Moreover $\tilde{u}$ belongs to $W^{1,\infty}(I, \mathbb{R}^m)$ and satisfies for a.e. $t \in I$

$$f(t, \tilde{u}(t)) - \langle p(t), \tilde{u}'(t) \rangle + g(t, \tilde{u}(t)) = c + \int_0^t v(\tau) \, d\tau,$$

where $c$ is a constant, and $(v(t), p(t)) \in (\partial_t f(t, \tilde{u}'(t)) + \partial_t g(t, \tilde{u}(t)), \partial_x f(t, \tilde{u}'(t)))$ for almost every $t \in I$.

**Proof.** The proof follows the lines of the one of Theorem 3 in [10], with some changes due to the fact that in this case the lagrangian is not bounded from below. As in [10] one can prove, using the De Giorgi’s semicontinuity result (see [4]) and the Dunford–Pettis criterion of weak compactness in $L^1(I, \mathbb{R}^m)$, that for every Nagumo function $\theta$ and for every $l > 0$ there exists a solution $u_l$ to the problem

$$\min \left\{ F(u) \mid u \in AC^l_\theta(I, \mathbb{R}^m), u(0) = a, u(T) = b \right\},$$

where $AC^l_\theta(I, \mathbb{R}^m)$ denotes the class of all function $u \in W^{1,1}(I, \mathbb{R}^m)$ such that $\Theta(u) \leq l$, with $\Theta(u) = \int_I \theta(|u'(t)|) \, dt$. Let us set $V_\theta(l) = F(u_l)$.

One can easily check that, if $V_\theta(l) = V_\theta(l_0)$ for every $l \geq l_0$, then $u_{l_0}$ is a solution to the problem

$$\min \left\{ F(u) \mid u \in W^{1,1}(I, \mathbb{R}^m), \Theta(u) < +\infty, u(0) = a, u(T) = b \right\}.$$  

Finally, as in [10], if we are able to prove that $u_{l_0}$ belongs to $W^{1,\infty}(I, \mathbb{R}^m)$, then we can conclude that such a function is a solution to (4.10).

Thus it remains to prove that $V_\theta$ is eventually constant and that, for $l$ large enough, $u_l$ belongs to $W^{1,\infty}(I, \mathbb{R}^m)$ and satisfies (4.11). Since $V_\theta$ is lower semicontinuous, for every $l > 0$ there exists a proximal subgradient (see [9]) of $V_\theta$ at $l$ and, since $V_\theta$ is nonincreasing, it is nonpositive. If $V_\theta$ is not eventually constant, by Proposition 6.1 in [10], there exists a diverging sequence $\{l_k\}$ such that the proximal subgradient of $V_\theta$ at $l_k$ takes the form $-r_k$, with $r_k > 0$. Moreover, it is easy to check that, if we set $u_k \equiv u_{l_k}$, then $\Theta(u_k) = l_k$, so that

$$\lim_{k \to +\infty} \|u_k'\|_{L^\infty} \geq \lim_{k \to +\infty} \theta^{-1}(l_k/T) = +\infty.$$  

By definition of $r_k$ and the fact that $\Theta(u_k) = l_k$, it follows that for every $k \in \mathbb{N}$ there exists a positive constant $\sigma_k$ such that, if we define

$$G(u) = F(u) + r_k \Theta(u) + \sigma_k |\Theta(u) - \Theta(u_k)|^2,$$
then we get that \( G(u_k) \leq G(u) \) for every \( u \) admissible for (4.12) and such that \( \Theta(u) \) is sufficiently near to \( \Theta(u_k) \) (see [10]). By \((H_3)\) and Lemma 4.5, it follows that there exists \( k_0 \in L^1(I) \) such that for every \( s_1, s_2, t \in I \)

\[
|f(s_1, u_k'(t)) + g(s_1, u_k(t)) - f(s_2, u_k'(t)) - g(s_2, u_k(t))| \leq k_0(t)|s_1 - s_2|,
\]

so that we can apply Theorem 5 of [10]. Thus we obtain that \( u_k \) satisfies

\[
E_f(t, u_k'(t)) + g(t, u_k(t)) + r_k E_{\theta}(|u_k'(t)|) = c_k + \int_0^t v_k(\tau) \, d\tau,
\]

where \( c_k \) is a constant, \( E_f(t, u_k'(t)) = f(t, u_k'(t)) - \langle p(t), u_k'(t) \rangle \), with \( (v_k(t), p_k(t)) \in (\partial_t f(t, u_k(t)) + \partial_t g(t, u_k(t)), \partial_z f(t, u_k(t))) \) for a.e. \( t \in I \), and \( E_{\theta}(s) \doteq \theta(s) - s\theta'(s) \).

Moreover there exists \( M_1 > 0 \) such that \( \|u_k\|_{L^\infty} \leq M_1 \) for every \( k \in \mathbb{N} \). Actually, if there exists \( t_k \in I \) such that \( \limsup_{k \to +\infty} |u_k(t_k)| = +\infty \), then

\[
\limsup_{k \to +\infty} \int_I |u_k'(t)| \, dt \geq \limsup_{k \to +\infty} \left| \int_0^{t_k} u_k'(t) \, dt \right| = \limsup_{k \to +\infty} |u_k(t_k) - a| = +\infty,
\]

while, if we define \( u_0(t) \doteq a + \xi t \), with \( \xi = (b - a)/T \), then \( u_0 \) is admissible for (4.12), \( F(u_0) < +\infty \), and

\[
F(u_0) \geq F(u_k) \geq (-A - \alpha)T + B\|u_k'\|_{L^1} - \beta\|u_k\|_{L^1} \geq \tilde{A} + (B - \beta T)\|u_k'\|_{L^1},
\]

so that, by \((H_2)\), \( \{u_k'\} \) must be bounded in \( L^1(I, \mathbb{R}^m) \).

The boundedness of \( \{u_k\} \) in \( L^\infty(I, \mathbb{R}^m) \) and the continuity of \( g \) guarantee that there exists \( M_2 \) such that

\[
|g(t, u_k(t))| \leq M_2,
\]

for a.e. \( t \in I \) and for every \( k \). Moreover, by \((H_3)\) we obtain

\[
\left| \int_0^t v_k(s) \, ds \right| \leq \int_I [C_0 |f(s, u_k'(s)) + g(s, u_k(s))| + C_1 |u_k'(s)| + C_2] \, ds \leq
\]

\[
\leq \int_I \left[ C_0 \|\alpha + \beta|u_k(s)| + f(s, u_k'(s)) + g(s, u_k(s))| + \tilde{C}_1 |u_k(s)| + \tilde{C}_2 \right] \, ds,
\]

where \( \tilde{C}_1 = C_0\beta + C_1 \) and \( \tilde{C}_2 = C_0|\alpha| + C_2 \). Without loss of generality we can assume that \( f \) is positive, so that, thanks to \((H_2)\), it follows that for every \( k \in \mathbb{N} \)

\[
f(s, u_k'(s)) + g(s, u_k(s)) + \alpha + \beta|u_k(s)| \geq 0, \quad \text{a.e. } s \in I.
\]
By (4.15), (4.17) and (4.18) there exist $M_3 > 0$ and two constants $\hat{C}_1$, $\hat{C}_2$ such that

\[(4.19) \quad \left| \int_0^t v_k(s) \, ds \right| \leq C_0 F(u_k) + \hat{C}_1 \|u_k\|_{L^1} + \hat{C}_2 \leq M_3, \quad \text{for every } t \in I.\]

By (4.14), (4.16), and (4.19) we obtain

\[E_f(t, u'_k(t)) + r_k E_\theta(|u'_k(t)|) \leq c_k + M_2 + M_3,\]

for every $t \in I$ and for every $k \in \mathbb{N}$.

We claim that it is not possible that there exists a subsequence of \{c_k\}, still denoted by \{c_k\}, such that $\lim k \, c_k = -\infty$. Indeed, if this is the case, then for every $t \in I$ we should have

\[(4.20) \quad \lim_{k \to +\infty} E_f(t, u'_k(t)) + r_k E_\theta(|u'_k(t)|) = -\infty.\]

Since $f \in \mathcal{E}$ and $\theta$ is superlinear, (4.20) implies that $\lim_k |u'_k(t)| = +\infty$ for every $t \in I$, which, by Fatou’s Lemma, contradicts the boundedness of $u'_k$ in $L^1(I, \mathbb{R}^m)$.

Thus there exists $c^*$ such that $c_k \geq c^*$ for every $k$. From (4.14) we obtain, for every $t \in I$,

\[(4.21) \quad E_f(t, u'_k(t)) + r_k E_\theta(|u'_k(t)|) \geq c^* - M_2 - M_3.\]

Now let us suppose that for every $k$ there exists $t_k \in I$ such that $\limsup_k |\xi_k| = +\infty$, where $\xi_k = u'_k(t_k)$. Since $f$ and $\theta$ belong to $\mathcal{E}$, we have

\[\liminf_{k \to +\infty} \left[ E_f(t_k, \xi_k) + r_k E_\theta(|\xi_k|) \right] \leq \liminf_{k \to +\infty} \sup_{t \in I} \left\{ E_f(t, \xi_k) + r_k E_\theta(|\xi_k|) \right\} = -\infty,\]

in contradiction with (4.21). This implies that $\|u'_k\|_{L^\infty}$ is bounded, which contradicts (4.13).

So we can conclude that $V_\theta$ is eventually constant. Hence for $k$ sufficiently large $u_k \in W^{1,\infty}(I, \mathbb{R}^m)$ is a solution of (4.12). Moreover $r_k = 0$, so that $u_k$ satisfies (4.11). Then the proof is complete.

The last part of this section is devoted to the study of the non–convex case. The hypotheses $(H_0)$ and $(H_3)$ will be replaced respectively by:

\[(H_0') \quad f \in \mathcal{E}.\]

\[(H_3') \quad \text{There exist three constants } C_i, \ i = 0, 1, 2, \text{ such that the condition (4.7) holds with } \varphi(t, x, \xi) = g(t, x) + f^{**}(t, \xi).\]
Notice that $(H'_3)$ requires the Lipschitz continuity of $f^\ast\ast$ with respect to $t$. The following two lemmas show that this conclusion follows from $(H'_0)$ and $(H_4)$ For every $R > 0$ there exists a constant $L$ such that

$$|f(t, \xi) - f(s, \xi)| \leq L|t - s|, \quad \text{for every } t, s \in I, \text{ and } \xi \in \overline{B}_R,$$

where $\overline{B}_R$ denotes the closed ball centered at the origin and with radius $R$.

**Lemma 4.9** Let $\psi \in \mathcal{E}$, and let us define, for every $(t, p) \in I \times \mathbb{R}^m$, the set

$$W(t, p) = \{ \xi \in \mathbb{R}^m \mid p \in \partial_\xi \psi^\ast\ast(t, \xi) \}.$$  

Then for every $r > 0$ there exists $R > 0$ such that for every $(t, p) \in I \times \mathbb{R}^m$ the condition $W(t, p) \cap \overline{B}_r \neq \emptyset$ implies $W(t, p) \subset \overline{B}_R$.

**Proof.** Suppose, by contradiction, that there exist sequences $(t_n, p_n) \subset I \times \mathbb{R}^m$, $(\eta_n) \subset \overline{B}_r$, $(\xi_n) \subset \mathbb{R}^m$, with $\lim_n |\xi_n| = +\infty$, such that, for every $n \in \mathbb{N}$,

$$(4.22) \quad p_n \in \partial_\xi \psi^\ast\ast(t_n, \eta_n), \quad p_n \in \partial_\xi \psi^\ast\ast(t_n, \xi_n).$$

From (4.22) it follows that, for every $n \in \mathbb{N}$,

$$(4.23) \quad \psi^\ast\ast(t_n, \eta_n) - \langle p_n, \eta_n \rangle = \psi^\ast\ast(t_n, \xi_n) - \langle p_n, \xi_n \rangle.$$  

Since $(\eta_n)$ is a bounded sequence, there exists a constant $C$ such that the left hand side of (4.23) is bounded from below by $C$. Thus

$$(4.24) \quad C \leq \psi^\ast\ast(t_n, \xi_n) - \langle p_n, \xi_n \rangle \leq \chi(|\xi_n|), \quad \text{for every } n \in \mathbb{N},$$

where $\chi(R)$ is the argument of the limit in (4.1). Since $\lim_n |\xi_n| = +\infty$, from (4.1) we have that $\lim_n \chi(|\xi_n|) = -\infty$, which contradicts (4.24). $\square$

**Remark 4.10** Let us fix $\xi \in \mathbb{R}^m$. Let $t \in I$, $\tilde{\lambda} \in E_{m+1}$, $\xi_j \in \mathbb{R}^m$, $j = 1, \ldots, m+1$ satisfy

$$f^\ast\ast(t, \xi) = \sum_{j=1}^{m+1} \lambda_j f(t, \xi_j), \quad \xi = \sum_{j=1}^{m+1} \lambda_j \xi_j.$$  

Since for every $j$ there exists $p_j \in \partial_\xi f^\ast\ast(t, \xi)$ such that $\xi_j \in W(t, p_j)$, by Lemma 4.9 we obtain that there exists $R > 0$, depending only on $|\xi|$, such that $\xi_j \in \overline{B}_R$ for every $j = 1, \ldots, m+1$. 


Lemma 4.11  If \( f \in E \) satisfies \((H_4)\), then \( f^{**}(\cdot, \xi) \) is Lipschitz continuous for every \( \xi \in \mathbb{R}^m \).

Proof. Let us fix \( \xi \in \mathbb{R}^m \), and consider \( t, s \in I \). By Corollary 3.7, there exist \( \lambda, \mu \in E_{m+1}, \xi_j, \eta_j \in \mathbb{R}^m \), \( j = 1, \ldots, m + 1 \), such that

\[
f^{**}(t, \xi) = \sum_{j=1}^{m+1} \lambda_j f(t, \xi_j), \quad f^{**}(s, \xi) = \sum_{j=1}^{m+1} \mu_j f(s, \eta_j),
\]

and \( \xi = \sum_j \lambda_j \xi_j = \sum_j \mu_j \eta_j \). Moreover, one has

\[
|f^{**}(s, \xi) - f^{**}(t, \xi)| \leq \sum_{j=1}^{m+1} \lambda_j |f(s, \xi_j) - f(t, \xi_j)| \leq \sum_{j=1}^{m+1} \lambda_j |t - s| = L|t - s|.
\]

In the same way one obtains

\[
|f^{**}(t, \xi) - f^{**}(s, \xi)| \leq \sum_{j=1}^{m+1} \mu_j |f(t, \eta_j) - f(s, \eta_j)| \leq L|t - s|,
\]

completing the proof. \(\square\)

We are now in a position to prove the existence result for the non–convex case.

Theorem 4.12  Let \( g \) and \( f \) satisfy the basic hypotheses \((H'_0), (H_1), (H_2), (H'_3), (H_4)\), and assume that \( g(t, \cdot) \) is concave for every \( t \in I \). Then the problem (4.10) has a solution \( u \in W^{1,\infty}([0, T], \mathbb{R}^m) \).

Proof. The proof follows the same lines of the one of Theorem 1 in [5]. It is enough to use Theorem 4.8 to obtain a solution \( \tilde{u} \in W^{1,\infty}([0, T], \mathbb{R}^m) \) of the relaxed problem, and to replace Lemma IX.3.3 and Proposition IX.3.1 of [13] with Corollaries 3.7 and 3.8. Since \( \tilde{u}' \in L^\infty([0, T], \mathbb{R}^m) \), it is easily seen that we obtain a solution \( u \in W^{1,\infty}([0, T], \mathbb{R}^m) \). \(\square\)
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