ASYMMETRIC TRUNCATED TOEPLITZ OPERATORS EQUAL TO THE ZERO OPERATOR

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ABSTRACT. Asymmetric truncated Toeplitz operators are compressions of multiplication operators acting between two model spaces. These operators are natural generalizations of truncated Toeplitz operators. In this paper we describe symbols of asymmetric truncated Toeplitz operators equal to the zero operator.

1. Introduction

Let $H^2$ denote the Hardy space of the unit disk $\mathbb{D} = \{ z : |z| < 1 \}$, that is, the space of functions analytic in $\mathbb{D}$ with square summable Maclaurin coefficients.

Using the boundary values, one can identify $H^2$ with a closed subspace of $L^2(\partial \mathbb{D})$, the subspace of functions whose Fourier coefficients with negative indices vanish. The orthogonal projection $P$ from $L^2(\partial \mathbb{D})$ onto $H^2$, called the Szegö projection, is given by

$$Pf(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(e^{it})dt}{1 - e^{-it}z}, \quad f \in L^2(\partial \mathbb{D}).$$

Note that if $f \in L^1(\partial \mathbb{D})$, then the above integral still defines a function $Pf$ analytic in $\mathbb{D}$.

The classical Toeplitz operator $T_\varphi$ with symbol $\varphi \in L^2(\partial \mathbb{D})$ is defined on $H^2$ by

$$T_\varphi f = P(\varphi f).$$

It is known that $T_\varphi$ is bounded if and only if $\varphi \in L^\infty(\partial \mathbb{D})$. The operator $S = T_z$ is called the unilateral shift and its adjoint $S^* = T_{\bar{z}}$ is called the backward shift. We have $Sf(z) = zf(z)$ and

$$S^* f(z) = \frac{f(z) - f(0)}{z}.$$ 

Let $H^\infty$ be the algebra of bounded analytic functions on $\mathbb{D}$ and let $\alpha \in H^\infty$ be an arbitrary inner function, that is, $|\alpha| = 1$ a.e. on $\partial \mathbb{D}$.

By the theorem of A. Beurling (see, for example, [7] Thm. 8.1.1), every nontrivial, closed $S$-invariant subspace of $H^2$ can be expressed as $\alpha H^2$ for some inner function $\alpha$. Consequently, every nontrivial, closed $S^*$-invariant subspace of $H^2$ is of the form

$$K_\alpha = H^2 \ominus \alpha H^2$$

with $\alpha$ inner. The space $K_\alpha$ is called the model space corresponding to $\alpha$.

The kernel function

$$k_\alpha^\alpha(w,z) = \frac{1 - \overline{\alpha(w)}\alpha(z)}{1 - \overline{w}z}, \quad w,z \in \mathbb{D},$$

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is a reproducing kernel for the model space $K_\alpha$, i.e., for each $f \in K_\alpha$ and $w \in \mathbb{D}$,

$$f(w) = \langle f, k^\alpha_w \rangle$$

($\langle \cdot, \cdot \rangle$ being the usual integral inner product). Observe that $k^\alpha_w$ is a bounded function for every $w \in \mathbb{D}$. It follows that the subspace $K_\alpha^\infty = K_\alpha \cap H^\infty$ is dense in $K_\alpha$. If $\alpha(w) = 0$, then $k^\alpha_w = k_w$, where $k_w$ is the Szegö kernel given by $k_w(z) = (1 - \overline{w}z)^{-1}$.

The function $\alpha$ is said to have a nontangential limit at $\eta \in \partial \mathbb{D}$ if there exists $\alpha(\eta)$ such that $\alpha(z)$ tends to $\alpha(\eta)$ as $z \in \mathbb{D}$ tends to $\eta$ nontangentially (with $|z - \eta| \leq C(1 - |z|)$ for some fixed $C > 0$). We say that $\alpha$ has an angular derivative in the sense of Carathéodory (an ADC) at $\eta \in \partial \mathbb{D}$ if both $\alpha$ and $\alpha'$ have nontangential limits at $\eta$ and $|\alpha(\eta)| = 1$ (for more details see [9 pp. 33–37]). P. R. Ahern and D. N. Clark proved in [1, 2], that the operator $A$ is a reproducing kernel for the model space $K_\alpha$. We denote $A(\eta)P$ the operator from $L^2(\partial \mathbb{D})$ onto $K_\alpha$. Then

$$P_\alpha f(z) = \langle f, k^\alpha_z \rangle, \quad f \in L^2(\partial \mathbb{D}), \quad z \in \mathbb{D}.$$

Just like with the Szegö projection, $P_\alpha f$ is a function analytic in $\mathbb{D}$ for all $f \in L^1(\partial \mathbb{D})$.

A truncated Toeplitz operator with a symbol $\varphi \in L^2(\partial \mathbb{D})$ is the operator $A^\alpha_\varphi$ defined on the model space $K_\alpha$ by

$$A^\alpha_\varphi f = P_\alpha(\varphi f).$$

Densely defined on bounded functions, the operator $A^\alpha_\varphi$ can be seen as a compression to $K_\alpha$ of the classical Toeplitz operator $T_\varphi$ on $H^2$.

The study of truncated Toeplitz operators as a class began in 2007 with D. Sarason’s paper [13]. In spite of similar definitions, there are many differences between truncated Toeplitz operators and the classical ones. One of the first results from [13] states that, unlike in the classical case, a truncated Toeplitz operator is not uniquely determined by its symbol. More precisely, $A^\alpha_\varphi = 0$ if and only if $\varphi \in \alpha H^2 + \alpha H^2$ ([13 Thm. 3.1]). As a consequence, unbounded symbols can produce bounded truncated Toeplitz operators. Moreover, there exist bounded truncated Toeplitz operators for which no bounded symbols exist (see [3]). For more interesting results see [6, 9, 10, 11, 12].

Recently, the authors in [4] and [5] introduced a generalization of truncated Toeplitz operators, the so-called asymmetric truncated Toeplitz operators. Let $\alpha$, $\beta$ be two inner functions and let $\varphi \in L^2(\partial \mathbb{D})$. An asymmetric truncated Toeplitz operator $A^\alpha_\varphi$ is the operator from $K_\alpha$ into $K_\beta$ given by

$$A^\alpha_\varphi f = P_\beta(\varphi f), \quad f \in K_\alpha.$$

The operator $A^\alpha_\varphi$ is densely defined. Clearly, $A^\alpha_\varphi = A^\alpha_\varphi^\alpha$.

We denote

$$T(\alpha, \beta) = \{ A^\alpha_\varphi : \varphi \in L^2(\partial \mathbb{D}) \text{ and } A^\alpha_\varphi \text{ is bounded} \}$$

and $T(\alpha) = T(\alpha, \alpha)$.

The purpose of this paper is to describe when an operator from $T(\alpha, \beta)$ is equal to the zero operator. The description is given in terms of the symbol of the operator. This was done in [4] and [5] for the case when $\beta$ divides $\alpha$, that is, when $\alpha/\beta$ is an inner function. It was proved in [4] and [5] that $A^\alpha_\varphi = 0$ if and only if $\varphi \in \alpha H^2 + \beta H^2$. Here we show that this is true for all inner functions $\alpha$ and $\beta$. We also give some examples of rank-one asymmetric truncated Toeplitz operators.
2. Main result

In this section we prove the following.

**Theorem 2.1.** Let \( \alpha, \beta \) be two nonconstant inner functions and let \( A_{\varphi}^{\alpha,\beta} : K_\alpha \to K_\beta \) be a bounded asymmetric truncated Toeplitz operator with \( \varphi \in L^2(\partial \mathbb{D}) \). Then \( A_{\varphi}^{\alpha,\beta} = 0 \) if and only if \( \varphi \in \alpha H^2 + \beta H^2 \).

We start with a simple technical lemma.

**Lemma 2.2.** Let \( \alpha, \beta \) be two arbitrary inner functions. If
\[
K_\alpha \subset \beta H^2,
\]
then both \( \alpha \) and \( \beta \) have no zeros in \( \mathbb{D} \), or at least one of the functions \( \alpha \) or \( \beta \) is a constant function.

**Proof.** Assume that \( K_\alpha \subset \beta H^2 \) holds. If \( \beta(z_0) = 0 \) for some \( z_0 \in \mathbb{D} \), then \( f(z_0) = 0 \) for every \( f \in K_\alpha \). For \( f = k_\alpha \) we get
\[
k_\alpha(z_0) = \|k_\alpha\|^2 = \frac{1 - |\alpha(z_0)|^2}{1 - |z_0|^2} = 0,
\]
which implies that \( |\alpha(z_0)| = 1 \). By the maximum modulus principle, \( \alpha \) is a constant function. Hence, the inclusion \( K_\alpha \subset \beta H^2 \) implies that \( \beta \) has no zeros in \( \mathbb{D} \), or \( \alpha \) is a constant function. But \( K_\alpha \subset \alpha H^2 \),

and, by the same reasoning, \( K_\alpha \subset \beta H^2 \) also implies that \( \alpha \) has no zeros in \( \mathbb{D} \), or \( \beta \) is a constant function. This completes the proof. \( \square \)

Lemma 2.2 can be rephrased as follows. If \( \alpha, \beta \) are two nonconstant inner functions and at least one of them has a zero in \( \mathbb{D} \), then the inclusion \( K_\alpha \subset \beta H^2 \) does not hold. This allows us to prove the following version of Theorem 2.1.

**Proposition 2.3.** Let \( \alpha, \beta \) be two nonconstant inner functions such that each of them has a zero in \( \mathbb{D} \) and let \( A_{\varphi}^{\alpha,\beta} : K_\alpha \to K_\beta \) be a bounded asymmetric truncated Toeplitz operator with \( \varphi \in L^2(\partial \mathbb{D}) \). Then \( A_{\varphi}^{\alpha,\beta} = 0 \) if and only if \( \varphi \in \alpha H^2 + \beta H^2 \).

**Proof.** The fact that \( \varphi \in \alpha H^2 + \beta H^2 \) implies \( A_{\varphi}^{\alpha,\beta} = 0 \) was proved in [1, Thm. 4.3]. For the convenience of the reader we repeat the reasoning from [1].

Assume that \( \varphi = \overline{\alpha h_1} + \beta h_2 \) with \( h_1, h_2 \in H^2 \). Then, for every \( f \in K_\alpha^\infty \),
\[
A_{\varphi}^{\alpha,\beta}f = P_\beta(\overline{\alpha h_1} f + \beta h_2 f) = P_\beta(\overline{\alpha h_1} f).
\]
Since \( f \perp \alpha H^2 \), we see that \( \overline{\alpha h_1} f \perp H^2 \) and \( P_\beta(\overline{\alpha h_1} f) = 0 \). The density of \( K_\alpha^\infty \) implies that \( A_{\varphi}^{\alpha,\beta} = 0 \). Note that this part of the proof does not depend on the existence of zeros of \( \alpha \) and \( \beta \).

Let us now assume that \( A_{\varphi}^{\alpha,\beta} = 0 \). By the first part of the proof, we can also assume that \( \varphi = \chi + \psi \) for some \( \chi \in K_\alpha, \psi \in K_\beta \). Let \( z_0 \in \mathbb{D} \) be a zero of \( \alpha \). Then \( k_\alpha = k_{z_0} \) and
\[
A_{\chi,\varphi}^{\alpha,\beta} k_\alpha = P_\beta(k_\alpha(z_0))
= P_\beta \left( \frac{\chi(z) - \chi(z_0)}{z - z_0} + \frac{\chi(z_0)}{z_0} k_\alpha(z_0) \right)
= \chi(z_0) k_\alpha(z_0),
\]
because the quotient \( (\chi(z) - \chi(z_0))/(z - z_0) \) belongs to \( K_\alpha \) (see [13] Subsection 2.6)).

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Hence,

\[0 = A^{\alpha,\beta}_{k_{z_0}} = A^{\alpha,\beta}_{\chi+\psi} k_{z_0} = \chi(z_0) k_{z_0} + A^{\alpha,\beta}_{\psi} k_{z_0} = P_\beta \left[ (\chi(z_0) + \psi) k_{z_0} \right],\]

which means that

\[(\chi(z_0) + \psi) k_{z_0} \in \beta H^2\]

and, consequently,

\[(2.2) \quad \chi(z_0) + \psi \in \beta H^2.\]

On the other hand ([4, Lem. 3.2]),

\[A^{\beta,\alpha}_{\chi+\psi} = \left( A^{\alpha,\beta}_{\chi+\psi} \right)^* = 0,\]

and a similar reasoning can be used to show that if \(\beta(w_0) = 0, w_0 \in \mathbb{D},\) then

\[(2.3) \quad \chi + \psi(w_0) \in \alpha H^2.\]

By (2.2), (2.3) and the first part of the proof we get

\[A^{\alpha,\beta}_{\chi+\psi(w_0)+\chi(z_0)+\psi} = 0,\]

and

\[A^{\alpha,\beta}_{\psi(w_0)+\chi(z_0)+\psi} = -A^{\alpha,\beta}_{\chi+\psi} = 0.\]

From this,

\[P_\beta \left[ (\psi(w_0) + \chi(z_0)) f \right] = 0\]

for all \(f \in K_\alpha.\)

If \(\psi(w_0) + \chi(z_0) \neq 0,\) then the above means that \(P_\beta(f) = 0\) for all \(f \in K_\alpha,\) that is, \(K_\alpha \subset \beta H^2.\)

However, by Lemma 2.2 this cannot be the case here. So

\[\psi(w_0) + \chi(z_0) = 0\]

and

\[\varphi = \chi + \psi = \chi + \psi(w_0) + \chi(z_0) + \psi \in \alpha H^2 + \beta H^2.\]

To give a proof of Theorem 2.1, we use the so-called Crofoot transform. For any inner function \(\alpha\) and \(w \in \mathbb{D},\) the Crofoot transform \(J^\alpha_w\) is the multiplication operator given by

\[(2.4) \quad J^\alpha_w f(z) = \frac{\sqrt{1 - |w|^2}}{1 - \overline{w}\alpha(z)} f(z).\]

The Crofoot transform \(J^\alpha_w\) is a unitary operator from \(K_\alpha\) onto \(K_{\alpha w},\) where

\[(2.5) \quad \alpha_w(z) = \frac{w - \alpha(z)}{1 - \overline{w}\alpha(z)},\]

(see, for example, [8, Thm. 10] and [13, pp. 521–523]). Moreover,

\[(J^\alpha_w)^* f = (J^\alpha_w)^{-1} f = J^{\alpha_w} f = \frac{\sqrt{1 - |w|^2}}{1 - \overline{w}\alpha_w} f = \frac{1 - \overline{w}\alpha}{\sqrt{1 - |w|^2}} f.\]
Lemma 2.4. Let $\alpha$ be an inner function and $w \in \mathbb{D}$. For every $z \in \mathbb{D}$ we have

\begin{equation}
(2.6) \quad k^{\alpha}_z = \frac{1 - |w|^2}{(1 - w\alpha(z))(1 - \overline{w}\alpha)} k^{\alpha}_z.
\end{equation}

Proof. Fix $w, z \in \mathbb{D}$. The reproducing kernel $k^{\alpha}_z$ is given by

\[ k^{\alpha}_z(\lambda) = \frac{1 - \alpha_w(z)\alpha_w(\lambda)}{1 - \overline{z}\lambda}, \quad \lambda \in \mathbb{D}. \]

Since

\[ 1 - \alpha_w(z)\alpha_w(\lambda) = 1 - \frac{\overline{w} - \overline{\alpha(z)}}{1 - w\alpha(z)} \frac{w - \alpha(\lambda)}{1 - \overline{w}\alpha(\lambda)} \]

we have

\[ k^{\alpha}_z(\lambda) = \frac{1 - |w|^2}{(1 - w\alpha(z))(1 - \overline{w}\alpha(\lambda))} \frac{1 - \alpha(z)\alpha(\lambda)}{1 - \overline{z}\lambda} \frac{1 - \overline{\alpha(z)}\overline{\alpha}(\lambda)}{1 - \overline{\alpha}(\lambda)} k^{\alpha}_z(\lambda). \]

It is known that the map

\[ A \mapsto J^{\alpha}_w A (J^{\alpha}_w)^{-1}, \quad A \in \mathcal{F}(\alpha), \]

carries $\mathcal{F}(\alpha)$ onto $\mathcal{F}(\alpha_w)$ (see [6]). A similar result is true for the asymmetric truncated Toeplitz operators.

Proposition 2.5. Let $\alpha, \beta$ be two inner functions. Let $a, b \in \mathbb{D}$ and let the functions $\alpha_a, \beta_b$ and the operators $J^{\alpha}_a : K_a \to K_{\alpha_a}$, $J^{\beta}_b : K_\beta \to K_{\beta_b}$ be defined as in (2.3) and (2.4), respectively. If $A$ is a bounded linear operator from $K_a$ into $K_\beta$, then $A$ belongs to $\mathcal{F}(\alpha, \beta)$ if and only if $J^{\beta}_b A (J^{\alpha}_a)^{-1}$ belongs to $\mathcal{F}(\alpha_a, \beta_b)$. Moreover, if $A = A^{\alpha, \beta}_\varphi$, then $J^{\beta}_b A (J^{\alpha}_a)^{-1} = A^{\alpha_a, \beta_b}_\phi$ with

\begin{equation}
(2.7) \quad \phi = \frac{(1 - \overline{a}\alpha)(1 - b\overline{\beta})}{\sqrt{1 - |a|^2} \sqrt{1 - |b|^2}} \varphi.
\end{equation}

Proof. Let $A$ be a bounded linear operator from $K_a$ into $K_\beta$. Assume first that $A$ belongs to $\mathcal{F}(\alpha, \beta)$, $A = A^{\alpha, \beta}_\varphi$ for $\varphi \in L^2(\partial \mathbb{D})$. We show that $J^{\beta}_b A (J^{\alpha}_a)^{-1} = A^{\alpha_a, \beta_b}_\phi$ with $\phi$ as in (2.7).

For every $f \in K^{\alpha_a}_\alpha$ and $z \in \mathbb{D}$ we have

\[ J^{\beta}_b A^{\alpha, \beta}_\varphi (J^{\alpha}_a)^{-1} f(z) = \sqrt{\frac{1 - |b|^2}{1 - b\beta(z)}} P_b \left( \frac{1 - \overline{a}\alpha}{\sqrt{1 - |a|^2}} \varphi f \right)(z) \]

where

\[ k^{\beta}_b(z) = \left\langle \frac{1 - \overline{\alpha_a}}{\sqrt{1 - |a|^2}} \varphi; k^{\alpha}_z \right\rangle. \]
By (2.6),

\[ J_b^\beta A_\phi^{\alpha,\beta} \left( J_a^\alpha \right)^{-1} f(z) = \frac{\sqrt{1 - |b|^2}}{1 - b\beta(z)} \left\langle \frac{1 - \alpha\alpha}{1 - |a|^2} \varphi f; \frac{1 - b\beta(z)}{1 - |b|^2} k^\beta_z \right\rangle \]

\[ = \left\langle \frac{1 - b\beta}{\sqrt{1 - |b|^2}} \frac{1 - \alpha\alpha}{\sqrt{1 - |a|^2}} \varphi f; k^\beta_z \right\rangle \]

\[ = P_{\beta} \left( \frac{1 - b\beta}{\sqrt{1 - |b|^2}} \frac{1 - \alpha\alpha}{\sqrt{1 - |a|^2}} \varphi f \right)(z) \]

\[ = A_\phi^{\alpha,\beta} f(z). \]

Thus \( A \in \mathcal{F}(\alpha, \beta) \) implies that \( J_b^\beta A \left( J_a^\alpha \right)^{-1} \in \mathcal{F}(\alpha, \beta) \).

To prove the other implication assume that \( J_b^\beta A \left( J_a^\alpha \right)^{-1} = A_\phi^{\alpha,\beta} \in \mathcal{F}(\alpha, \beta) \) for some \( \phi \in L^2(\partial \mathbb{D}) \). Then

\[ A = (J^\beta_b)^{-1} A_\phi^{\alpha,\beta} J_a^\alpha = J_b^{\beta} A_\phi^{\alpha,\beta} \left( J_a^\alpha \right)^{-1}. \]

But \((\alpha_a)_a = \alpha\) and \((\beta_b)_b = \beta\), and, by the first part of the proof,

\[ A = J_b^{\beta} A_\phi^{\alpha,\beta} \left( J_a^\alpha \right)^{-1} = A_\phi^{\alpha,\beta} \]

with

\[ \varphi = \frac{(1 - \alpha\alpha)(1 - b\beta)}{\sqrt{1 - |a|^2} \sqrt{1 - |b|^2}}. \]

Hence, \( A \in \mathcal{F}(\alpha, \beta) \). An easy computation shows that \( \phi \) satisfies (2.7). \( \square \)

**Proof of Theorem 2.7.** The fact that \( \varphi \in \alpha H^2 + \beta H^2 \) implies \( A_\phi^{\alpha,\beta} = 0 \) was established in the proof of Proposition 2.3. Assume now that \( \varphi \in L^2(\partial \mathbb{D}) \) and \( A_\phi^{\alpha,\beta} = 0 \).

If \( \alpha(0) = \beta(0) = 0 \), then \( \varphi \in \frac{\alpha H^2}{\beta} + \beta H^2 \) by Proposition 2.3. If \( \alpha(0) \neq 0 \) or \( \beta(0) \neq 0 \), put \( a = \alpha(0), b = \beta(0) \). By Proposition 2.3

\[ 0 = J_b^\beta A_\phi^{\alpha,\beta} \left( J_a^\alpha \right)^{-1} = A_\phi^{\alpha,\beta}, \]

where

\[ \phi = \frac{(1 - \alpha\alpha)(1 - b\beta)}{\sqrt{1 - |a|^2} \sqrt{1 - |b|^2}}. \]

Since \( \alpha(0) = \beta(0) = 0 \), by Proposition 2.3

\[ \phi \in \frac{\alpha H^2}{\beta} + \beta H^2. \]

Therefore, there exist \( h_1, h_2 \in H^2 \) such that

\[ \frac{(1 - \alpha\alpha)(1 - b\beta)}{\sqrt{1 - |a|^2} \sqrt{1 - |b|^2}} = \frac{\pi - \pi}{1 - \alpha\alpha} h_1 + \frac{b - \beta}{1 - b\beta} h_2, \]

and

\[ \phi = \frac{\pi - \pi}{1 - \alpha\alpha} \sqrt{1 - |a|^2} \sqrt{1 - |b|^2} h_1 + \frac{b - \beta}{1 - b\beta} \sqrt{1 - |a|^2} \sqrt{1 - |b|^2} h_2. \]

Since \( |\alpha| = 1 \) and \( |\beta| = 1 \) on the unit circle \( \partial \mathbb{D} \), we see that

\[ \frac{\pi - \pi}{1 - \alpha\alpha} = -\pi \quad \text{and} \quad \frac{b - \beta}{1 - b\beta} = -\beta \quad \text{on} \quad \partial \mathbb{D}. \]
and

\[ \varphi = \alpha g_1 + \beta g_2 \]

with

\[ g_1 = -\frac{\sqrt{1 - |a|^2}\sqrt{1 - |b|^2}}{(1 - \overline{a}\alpha)(1 - \overline{b}\beta)}h_1, \quad g_2 = \frac{\sqrt{1 - |a|^2}\sqrt{1 - |b|^2}}{(1 - \overline{a}\alpha)(1 - \overline{b}\beta)}h_2. \]

\( g_1, g_2 \in H^2 \). This completes the proof. \( \square \)

**Corollary 2.6.** If \( \varphi \) is in \( L^2(\partial \mathbb{D}) \), then there is a pair of functions \( \chi \in K_\alpha, \psi \in K_\beta \), such that

\[ A_{\varphi}^{\alpha,\beta} = A_{\chi + \psi}^{\alpha,\beta}. \]

If \( \chi, \psi \) is one such pair, then the most general such pair is of the form \( \chi - \overline{c}k_0^\alpha, \psi + ck_0^\beta \), with \( c \) a scalar.

**Proof.** The proof is analogous to the proofs given in [13] and [14].

The function \( \varphi \in L^2(\partial \mathbb{D}) \) can be written as \( \varphi = \varphi_+ + \varphi_- \) with \( \varphi_+, \varphi_- \in H^2 \). If \( \chi = P_\alpha(\varphi_-) \) and \( \psi = P_\beta(\varphi_+) \), then \( \varphi - \chi - \psi \in \alpha H^2 + \beta H^2 \). By Theorem 2.1, \( A_{\varphi}^{\alpha,\beta} = A_{\chi + \psi}^{\alpha,\beta} \).

Note that for \( f \in K_\alpha \),

\[ A_{\chi + \psi}^{\alpha,\beta} f = P_\beta f \left( f - \overline{\beta}(0)f \right) = \overline{\beta} f = A_1^{\alpha,\beta} f. \]

Since \( \alpha f \perp H^2 \) for \( f \in K_\alpha \), we get

\[ A_{\chi + \psi}^{\alpha,\beta} f = P_\beta f \left( f - \alpha(0)f \right) = \overline{\beta} f = A_1^{\alpha,\beta} f. \]

Therefore, if \( A_{\varphi}^{\alpha,\beta} = A_{\chi + \psi}^{\alpha,\beta} \) with \( \chi \in K_\alpha, \psi \in K_\beta \) as above and \( \chi_1 = \chi - \overline{c}k_0^\alpha, \psi_1 = \psi + ck_0^\beta \) for some constant \( c \in \mathbb{C} \), then

\[ A_{\chi_1 + \psi_1}^{\alpha,\beta} = A_{\chi + \psi}^{\alpha,\beta} - cA_1^{\alpha,\beta} + A_{\psi}^{\alpha,\beta} + cA_1^{\alpha,\beta} = A_{\varphi}^{\alpha,\beta}. \]

Moreover, if \( A_{\varphi}^{\alpha,\beta} = A_{\chi + \psi}^{\alpha,\beta} \) for any other \( \chi_1 \in K_\alpha, \psi_1 \in K_\beta \), then, by Theorem 2.1, there exist \( h_1, h_2 \in H^2 \) such that

\[ \chi + \psi - \chi_1 - \psi_1 = \overline{\alpha} h_1 + \beta h_2. \]

Hence

\[ \psi - \psi_1 = \beta h_2 + \overline{\alpha} h_1 - \chi_1 - \chi \]

and

\[ \psi - \psi_1 = P_\beta(\psi - \psi_1) = P_\beta(\overline{\alpha} h_1 + \chi_1 - \chi) = c_1 P_\beta 1 = c_1 k_0^\beta \]

for some constant \( c_1 \). Similarly,

\[ \chi - \chi_1 = \alpha h_1 + \overline{\beta} h_2 + \psi_1 - \psi \]

and

\[ \chi - \chi_1 = P_\alpha(\chi - \chi_1) = P_\alpha(\overline{\beta} h_2 + \psi_1 - \psi) = c_2 k_0^\alpha \]

for some constant \( c_2 \).

From this,

\[ 0 = A_{\chi + \psi - \chi_1 - \psi_1}^{\alpha,\beta} = \tau_2 A_{\chi + \psi}^{\alpha,\beta} + c_1 A_1^{\alpha,\beta} = (\tau_2 + c_1) A_1^{\alpha,\beta} = (\tau_2 + c_1) P_\beta|K_\alpha. \]

By Lemma 2.2, \( \tau_2 + c_1 = 0 \). Putting \( c = -c_1 = \tau_0 \) we have \( \psi_1 = \psi + ck_0^\beta \) and \( \chi_1 = \chi - \overline{c}k_0^\alpha \). \( \square \)
3. Rank-one operators in $\mathcal{T}(\alpha, \beta)$

Recall, that the model space $K_\alpha$ is equipped with a natural conjugation (antilinear, isometric involution) $C_\alpha: K_\alpha \to K_\alpha$, defined in terms of the boundary values by

$$C_\alpha f(z) = \alpha(z) \overline{f(z)}, \quad |z| = 1$$

(see [13] Subsection 2.3], for more details). A short calculation shows that the conjugate kernel $k_\alpha^\alpha = C_\alpha k_\alpha$ is given by

$$k_\alpha^\alpha(z) = \frac{\alpha(z) - \alpha(w)}{z - w}.$$ 

If $\eta \in \partial \mathbb{D}$ and $k_\eta^\alpha \in K_\alpha$, then

$$k_\eta^\alpha(z) = \frac{\alpha(z) - \alpha(\eta)}{z - \eta} = \alpha(\eta) k_\eta^\alpha(z).$$

We can now give some examples of rank-one asymmetric truncated Toeplitz operators (compare with [13] Thm. 5.1]).

**Proposition 3.1.** Let $\alpha, \beta$ be two nonconstant inner functions.

(a) For $w \in \mathbb{D}$, the operators $k_\alpha^\alpha \otimes k_\alpha^\alpha$ and $k_\beta^\beta \otimes k_\beta^\beta$ belong to $\mathcal{T}(\alpha, \beta)$,

$$\tilde{k}_\alpha^\alpha \otimes k_\alpha^\alpha = A_{\alpha,\beta}^{\alpha,\beta}, \quad k_\beta^\beta \otimes k_\beta^\beta = A_{\alpha,\beta}^{\alpha,\beta}.$$

(b) If both $\alpha$ and $\beta$ have an ADC at the point $\eta$ of $\partial \mathbb{D}$, then the operator $k_\eta^\beta \otimes k_\eta^\alpha$ belongs to $\mathcal{T}(\alpha, \beta)$,

$$k_\eta^\beta \otimes k_\eta^\alpha = A_{\alpha,\beta}^{\beta,\alpha}.$$ 

**Proof.** (a) Let $w \in \mathbb{D}$ and $f \in K_\alpha$. Since $\frac{f(z) - f(w)}{z - w} \in K_\alpha$ ([13] Subsection 2.6]), we have

$$A_{\alpha,\beta}^{\alpha,\beta} f = P_\beta \left( \frac{\beta(z)}{z - w} f(z) \right) = P_\beta \left( \frac{f(z) - f(w)}{z - w} + f(w) \frac{\beta(z) - \beta(w)}{z - w} + f(w) \frac{\beta(w)}{z - w} \right) = f(w) P_\beta \left( \frac{\beta(z)}{z - w} \right) + f(w) \beta(w) P_\beta \left( \frac{z}{1 - wz} \right) = f(w) k_\beta^\beta = (f, k_\alpha^\alpha) k_\beta^\beta = k_\beta^\beta \otimes k_\alpha^\alpha(f).$$

Similarly,

$$A_{\alpha,\beta}^{\alpha,\beta} f = P_\beta \left( \frac{\alpha(z)}{z - w} f(z) \right) = P_\beta \left( \frac{\alpha(z) f(z)}{z - w} \right) = P_\beta \left( \frac{C_\alpha f(z)}{z - w} \right) = \left( C_\alpha f(k_\alpha^\alpha) \right) = \left( C_\alpha f(k_\beta^\beta) \right) = C_\alpha f(k_\beta^\beta) = (C_\alpha f, k_\alpha^\alpha) k_\beta^\beta = (f, k_\alpha^\alpha) k_\beta^\beta = k_\beta^\beta \otimes k_\alpha^\alpha(f).$$

(b) Let $w \in \mathbb{D}$. Then

$$A_{k_\alpha^\alpha}^{\alpha,\beta} = A_{k_\alpha^\alpha}^{\alpha,\beta}$$

and

$$A_{k_\beta^\beta}^{\alpha,\beta} = A_{k_\beta^\beta}^{\alpha,\beta}.$$ 

Indeed,

$$A_{k_\alpha^\alpha}^{\alpha,\beta} f = P_\beta \left( (1 - \beta(w)) k_\beta^\beta f \right) = P_\beta \left( k_\beta^\beta f \right) = A_{k_\beta^\beta}^{\alpha,\beta} f,$$
for every $f \in K_\alpha$. From this,
\[ A_{k_w}^{\alpha,\beta} = \left( A_{k_w}^{\beta,\alpha} \right)^* = \left( A_{k_w}^{\beta,\alpha} \right)^* = A_{k_w}^{\alpha,\beta}. \]

Since for $w \neq 0$ and $|z| = 1$,
\[ \frac{\beta(z)}{z - w} = \frac{\beta(z) - \beta(w)}{z - w} + \frac{\beta(w)}{z - w} = k^\beta_w(z) + \frac{\beta(w)}{w} \frac{w \overline{\eta}}{1 - w^2} = \overline{k^\beta_w(z)} + \frac{\beta(w)}{w} (\overline{k^\beta_w(z)} - 1), \]
we have, by part (a),
\[ \overline{k^\beta_w} \otimes k^\alpha_w = A_{k_w}^{\alpha,\beta} = A_{k_w}^{\beta,\alpha}(\overline{k^\beta_w} - 1) = A_{k_w}^{\alpha,\beta}(\overline{k^\beta_w} - 1). \]

If $\alpha$ and $\beta$ have an ADC at $\eta \in \partial \mathbb{D}$, then $k^\alpha_w$ and $k^\beta_w$ converge in norm to $k^\alpha_\eta$ and $k^\beta_\eta$, respectively, as $w$ tends to $\eta$ non-tangentially. Hence $k^\beta_w \otimes k^\alpha_w$ tends to $k^\beta_\eta \otimes k^\alpha_\eta$ in the operator norm. On the other hand,
\[ \overline{k^\beta_w} + \frac{\beta(w)}{w} (\overline{k^\alpha_w - k^\beta_0}) \to \overline{k^\beta_\eta} + \frac{\beta(\eta)}{\eta} (\overline{k^\alpha_\eta - k^\beta_0}) \text{ in } L^2(\partial \mathbb{D}), \]
which implies that
\[ A_{k_w}^{\alpha,\beta} \to A_{k_\eta}^{\alpha,\beta} \text{ in } \mathcal{H}^2, \]
for every $f \in K^\alpha_\infty$. Therefore,
\[ \overline{k^\beta_\eta} \otimes k^\alpha_\eta = A_{k_\eta}^{\alpha,\beta}(\overline{k^\beta_\eta} - 1). \]

But
\[ \overline{k^\beta_\eta}(z) = \frac{\beta(z) - \beta(\eta)}{z - \eta} = \frac{\beta(\eta)}{\eta} k^\beta_\eta(z), \]
and
\[ k^\beta_\eta \otimes k^\alpha_\eta = \frac{\eta}{\beta(\eta)} \overline{k^\beta_\eta} \otimes k^\alpha_\eta = \frac{\eta}{\beta(\eta)} A_{k_\eta}^{\alpha,\beta}(k^\beta_\eta - k^\beta_0) \]
\[ = A_{k_\eta}^{\alpha,\beta}(k^\beta_\eta - k^\beta_0 - 1). \]

\[ \square \]

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