A non-commutative topology on rep A

Lieven Le Bruyn
Departement Wiskunde en Informatica
Universiteit Antwerpen
B-2020 Antwerp (Belgium)
lieven.lebruyn@ua.ac.be

Abstract

We extend the Zariski topology on simp A, the set of all simple finite dimensional representations of A, to a non-commutative topology (in the sense of Fred Van Oystaeyen) on rep A, the set of all finite dimensional representations of A, using Jordan-Hölder filtrations. The non-commutativity of the topology is enforced by the order of the composition factors.

All algebras will be affine associative k-algebras with unit over an algebraically closed field k. The non-commutative affine 'scheme' associated to an algebra A is, as a set, the disjoint union

\[ \text{rep} A = \bigsqcup_n \text{rep}_n A \]

where rep_n A is the (commutative) affine scheme of n-dimensional representations of A. In this note we will equip rep A with a non-commutative topology in the sense of Fred Van Oystaeyen [5 §7.2] (or, more precisely, a slight generalization of it).

Here is the main idea. The twosided prime ideal spectrum spec A is an (ordinary) topological space via the Zariski topology, see for example [4] or [11] §II.6]. Hence, the subset simp A of all simple finite dimensional A-representations can be equipped with the induced topology. This topology can then be extended to a non-commutative topology on rep A using Jordan-Hölder filtrations. The non-commutative nature of the topology is enforced by the order of the composition factors.

We give a few examples, connect this notion with that of Reineke’s composition monoid and remark on the difference between quotient varieties and moduli spaces from the perspective of non-commutative topology. Finally, we note that this construction can be generalized verbatim to any Artinian Abelian category as soon as we have a topology on the set of simple objects.
1 The Zariski topology on $\text{simp } A$.

Recall that a prime ideal $P$ of $A$ is a twosided ideal satisfying the property that if $I, J \subseteq P$ then $I \subseteq P$ or $J \subseteq P$ for any pair of twosided ideals $I, J$ of $A$. The prime spectrum $\text{spec } A$ is the set of all twosided prime ideals of $A$. The Zariski topology on $\text{spec } A$ has as its closed subsets

$$V(S) = \{ P \in \text{spec } A \mid S \subseteq P \}$$

where $S$ varies over all subsets of $A$, see for example [1, Prop. II.6.2]. Note that an algebra morphism $\phi : A \rightarrow B$ does not necessarily induce a continuous map $\phi^* : \text{spec } B \rightarrow \text{spec } A$ but is does so in the case $\phi$ is a central extension in the sense of [1, §II.6].

If $M \in \text{rep}_n A$ is a simple $n$-dimensional representation, there is a defining epimorphism $\psi_M : A \twoheadrightarrow M_n(\mathbb{k})$ and the kernel of this morphism $\ker \psi_M$ is a twosided maximal (hence prime) ideal of $A$. We define the Zariski topology on the set of all simple finite dimensional representations $\text{simp } A$ by taking as its closed subsets

$$V(S) = \{ M \in \text{simp } A \mid S \subseteq \ker \psi_M \}$$

Again, one should be careful that whereas an algebra map $\phi : A \rightarrow B$ induces a map $\phi^* : \text{rep } B \rightarrow \text{rep } A$ it does not in general map $\text{simp } B$ to $\text{simp } A$ (unless $\phi$ is a central extension).

With $\mathcal{L}_A$ we will denote the set of all open subsets of $\text{simp } A$. $\mathcal{L}_A$ will be the set of letters on which to base our non-commutative topology.

2 Non-commutative topologies (and generalizations).

In [5, Chp. 7] Fred Van Oystaeyen defined non-commutative topologies which are generalizations of usual topologies in which it is no longer true that $A \cap A$ is equal to $A$ for an open set $A$. In order to keep dichotomies of possible definitions to a minimum he imposed left-right symmetric conditions on the definition. However, for applications to representation theory it seems that the most natural non-commutative topologies are truly one-sided. For this reason we take some time to generalize some definitions and results of [5, Chp. 7].

We fix a partially ordered set $(\Lambda, \leq)$ with a unique minimal element 0 and a unique maximal element 1, equipped with two operations $\wedge$ and $\vee$. With $i_\Lambda$ we will denote the set of all idempotent elements of $\Lambda$, that is, those $x \in \Lambda$ such that $x \wedge x = x$. A finite global cover is a finite subset $\{ \lambda_1, \ldots, \lambda_n \}$ such that $1 = \lambda_1 \vee \ldots \vee \lambda_n$. In the table below we have listed the conditions for a (one-sided) non-commutative topology. Note that some requirements are less essential than others. For example, the covering condition (A10) is only needed if we want to fit non-commutative topologies in the framework of non-commutative Grothendieck topologies [5] and the weak modularity condition (A9) is not required if every basic open is $\vee$-idempotent (as is the case in most examples).
\( (x \lor y) \land \cdots \land (x \lor y) = x \quad (\forall x \lor y) \)

\( v \lor (q \land v) \leq (v \lor q) \land v \quad q \lor (v \land v) \leq (q \lor v) \land v \quad (6V) \)

\( h \land z \leq x \land z \Leftrightarrow h \leq x \quad z \land h \leq z \land x \Leftrightarrow h \leq x \quad (8V) \)

\( z \land h \land x = (z \land h) \land x = z \land (h \land x) \quad (7V) \)

\( x = x \land 0 \quad x = 0 \land x \quad (9V) \)

\( 1 = x \land 1 \quad 1 = 1 \land x \)

\( h \land x \leq h \quad (gV) \quad h \land x \leq x \quad (9V) \)

\( z \lor h \leq z \lor x \Leftrightarrow h \leq x \quad h \lor z \leq z \lor x \Leftrightarrow h \leq x \quad (10V) \)

\( z \lor h \lor x = (z \lor h) \lor x = z \lor (h \lor x) \quad (8V) \)

\( 0 = x \lor 0 \quad 0 = 0 \lor x \quad (9V) \)

\( x = x \lor 1 \quad x = 1 \lor x \)

\( h \leq h \lor x \quad x \geq h \lor x \quad (1V) \)

---

**Definition 1**

Let \((\Lambda, \leq)\) be a partially ordered set with minimal and maximal elements 0 and 1 and operations \(\land\) and \(\lor\). Then, \(\Lambda\) is said to be a left non-commutative topology if and only if the left and middle column conditions of (A1)-(A10) are valid for all \(x, y, z \in \Lambda\), all \(a, b \in i\), with \(a \leq b\) and all finite global covers \(\{\Lambda, \ldots, \Lambda_n\}\).
\( \Lambda \) is said to be a right non-commutative topology if and only if the middle and right column conditions of (A1)-(A10) are valid for all \( x, y, z \in \Lambda \), all \( a, b \in i_\Lambda \) with \( a \leq b \) and all finite global covers \( \{ \lambda_1, \ldots, \lambda_n \} \).

\( \Lambda \) is said to be a non-commutative topology if and only if the conditions (A1)-(A10) are valid for all \( x, y, z \in \Lambda \), all \( a, b \in i_\Lambda \) with \( a \leq b \) and all finite global covers \( \{ \lambda_1, \ldots, \lambda_n \} \).

There are at least two ways of building a genuine non-commutative topology out of these sets of basic opens. We briefly sketch the procedures here and refer to the forthcoming monograph [6] for details in the symmetric case (the one-sided versions present no real problems).

Let \( T(\Lambda) \) be the set of all finite \((\land, \lor)\)-words in the contractible idempotent elements \( i_\Lambda \) (that is, \( \lambda \in i_\Lambda \) such that for all \( \lambda_1, \lambda_2 \) with \( \lambda \leq \lambda_1 \lor \lambda_2 \) we have that \( \lambda = (\lambda \land \lambda_1) \lor (\lambda \land \lambda_2) \)). If \( \Lambda \) is a (left,right) non-commutative topology, then so is \( T(\Lambda) \). The \( \lor \)-complete topology of virtual opens \( T'(\Lambda) \) is then the set of all \((\land, \lor)\)-words in the contractible idempotents of finite length in \( \land \) (but not necessarily of finite length in \( \lor \)). This non-commutative topology has properties very similar to that of an ordinary topology and, in fact, has associated to it a commutative shadow.

The second construction, leading to the pattern topology, starts with the equivalence classes of directed systems \( S \subset \Lambda \) (that is, if for all \( x, y \in S \) there is a \( z \in S \) such that \( z \leq x \) and \( z \leq y \)) and where the equivalence relation \( S \sim S' \) is defined by

\[
\forall a \in S, \exists a' \in S, a' \leq a \text{ and } b \leq a' \leq b' \text{ for some } b, b' \in S'
\]

\[
\forall b \in S', \exists b' \in S', b' \leq b \text{ and } a \leq b' \leq a' \text{ for some } a, a' \in S
\]

One can extend the \( \land, \lor \) operations on \( \Lambda \) to the equivalence classes \( C(\Lambda) = \{ [S] \mid S \text{ directed } \} \) in the obvious way such that also \( C(\Lambda) \) is a (left,right) non-commutative topology. A directed set \( S \subset \Lambda \) is said to be idempotent if for all \( a \in S \), there is an \( a' \in S \cap i_\Lambda \) such that \( a' \leq a \). If \( S \) is idempotent then \( [S] \in i_{C(\Lambda)} \) and those idempotents will be called strong idempotents. The pattern topology \( \Pi(\Lambda) \) is the (left,right) non-commutative topology of finite \((\land, \lor)\)-words in the strong idempotents of \( C(\Lambda) \). A directed system \([S]\) is called a point iff \([S] \leq \lor[S_\alpha]\) implies that \([S] \leq [S_\alpha]\) for some \( \alpha \).

### 3 The basic opens.

For an \( n \)-dimensional representation \( M \) of \( A \) we call a finite filtration of length \( u \)

\[
\mathcal{F}^u : 0 = M_0 \subset M_1 \subset \ldots \subset M_u = M
\]

of \( A \)-representations a Jordan-Hölder filtration if the successive quotients

\[
\mathcal{F}_i = \frac{M_i}{M_{i-1}}
\]
are simple $A$-representations. Recall that $\mathcal{L}_A$ is the set of all open subsets $V$ of $\text{simp } A$. With $\mathbb{W}_A$ we denote the non-commutative words in these letters

$$\mathbb{W}_A = \{V_1 \ldots V_k \mid V_i \in \mathcal{L}_A, k \in \mathbb{N}\}$$

For a given word $w = V_1 V_2 \ldots V_k \in \mathbb{W}_A$ we define the left basic open set

$$O^l_w = \{M \in \text{rep } A \mid \exists \mathcal{F}^u \text{ Jordan-H" older filtration on } M \text{ such that } \mathcal{F}_i \in V_i\}$$

and the right basic open set

$$O^r_w = \{M \in \text{rep } A \mid \exists \mathcal{F}^u \text{ Jordan-H" older filtration on } M \text{ such that } \mathcal{F}_{u-i} \in V_{k-i}\}$$

Finally, to make these definitions symmetric we define the basic open set

$$O_w = \{M \in \text{rep } A \mid \exists \mathcal{F}^u \text{ Jordan-H" older filtration on } M \text{ such that } \mathcal{F}_{i_j} \in V_j$$

for some $1 \leq i_1 < i_2 < \ldots < i_k \leq u \}$

Clearly, $O^l_w$ consists of those representations having prescribed bottom structure, whereas $O^r_w$ consists of those with prescribed top structure. In order to avoid three sets of definitions we will denote from now on $O^\bullet_w$ whenever we mean $\bullet \in \{l, r, \emptyset\}$.

If $w = L_1 \ldots L_k$ and $w' = M_1 \ldots M_l$, we will denote with $w \cup w'$ the multi-set $\{N_1, \ldots, N_m\}$ where each $N_i$ is one of $L_j, M_j$ and $N_i$ occurs in $w \cup w'$ as many times as its maximum number of factors in $w$ or $w'$. With $\text{rep}(w \cup w')$ we denote the subset of $\text{rep } A$ consisting of the representations of $M$ having a Jordan-H" older filtration having factor-multi-set containing $w \cup w'$. For any triple of words $w, w'$ and $w''$ we denote $O^\bullet_{w''}(w \cup w') = O^\bullet_{w''} \cap \text{rep}(w \cup w')$.

We define an equivalence relation on the basic open sets by

$$O^\bullet_w \approx O^\bullet_{w'} \iff O^\bullet_w (w \cup w') = O^\bullet_{w'} (w \cup w')$$

The reason for this definition is that the condition of $M \in O^\bullet_w$ is void if $M$ does not have enough Jordan-H" older components to get all factors of $w$ which makes it impossible to define equality of basic open sets defined by different words.

We can now define the partially ordered sets $\Lambda^\bullet_A$ as consisting of all basic open subsets $O^\bullet_w$ of $\text{rep } A$. The partial ordering $\leq$ is induced by set-theoretic inclusion modulo equivalence, that is,

$$O^\bullet_w \leq O^\bullet_{w'} \iff O^\bullet_w (w \cup w') \subseteq O^\bullet_{w'} (w \cup w')$$

As a consequence, equality $=$ in the set $\Lambda^\bullet_A$ coincides with equivalence $\approx$. Observe that these partially ordered sets have a unique minimal and a unique maximal element (upto equivalence)

$$0 = \emptyset = O^\bullet_\emptyset \quad \text{and} \quad 1 = \text{rep } A = O^\bullet_{\text{simp } A}$$

The operations $\lor$ and $\land$ are defined as follows: $\lor$ is induced by ordinary set-theoretic union and $\land$ is induced by concatenation of words, that is

$$O^\bullet_w \land O^\bullet_{w'} \approx O^\bullet_{ww'}$$
Theorem 1 With notations as before:

- \((\Lambda^l_A, \leq, \simeq, 0, 1, \lor, \land)\) is a left non-commutative topology on \(\text{rep} \ A\).

- \((\Lambda^r_A, \leq, \simeq, 0, 1, \lor, \land)\) is a right non-commutative topology on \(\text{rep} \ A\).

Proof. The tedious verification is left to the reader. Here, we only stress the importance of the equivalence relation for example in verifying \(x \land 1 = x\). So, let \(w = L_1 \ldots L_k\) then

\[\mathcal{O}_w^l \land 1 = \mathcal{O}_{L_1 \ldots L_k \# \text{simp} A}^l \subset \mathcal{O}_w^l\]

and this inclusion is proper (look at elements in \(\mathcal{O}_w^l\) having exactly \(k\) composition factors). However, as soon as the representation has \(k + 1\) composition factors, it is contained in the left hand side whence \(\mathcal{O}_w^l \land 1 \simeq \mathcal{O}_w^l\). A similar argument is needed in the covering condition. □

Note however that \((\Lambda_A, \leq, \simeq, 0, 1, \lor, \land)\) is not necessarily a non-commutative topology: the problematic conditions are \(\mathcal{O}_w \land 1 = \mathcal{O}_w = 1 \land \mathcal{O}_w\) and the covering condition. The reason is that for \(w = L_1 \ldots L_k\) as before and \(M \in \mathcal{O}_w\) having \(> k\) factors, it may happen that the last factor is the one in \(L_k\) leaving no room for a successive factor in \(\text{simp} A\) (whence \(\mathcal{O}_w \cap 1\) is not equivalent to \(\mathcal{O}_w\)).

Example 1 Let \(A\) be a finite dimensional algebra, then \(A\) has a finite number of simple representations \(\text{simp} A = \{S_1, \ldots, S_n\}\) and the Zariski topology is the discrete topology. If for some \(1 \leq i, j \leq n\) we have that

\[\text{Ext}_A^1(S_i, S_j) = 0 \quad \text{and} \quad \text{Ext}_A^1(S_j, S_i) \neq 0\]

then \(\Lambda^l_A\) is a genuinely non-commutative topology, for example

\[\mathcal{O}_{S_i}^l \land \mathcal{O}_{S_j}^l = \mathcal{O}_{S_i S_j}^l \neq \mathcal{O}_{S_j S_i}^l = \mathcal{O}_{S_j}^l \land \mathcal{O}_{S_i}^l\]

as a non-trivial extension \(0 \longrightarrow S_i \longrightarrow X \longrightarrow S_j \longrightarrow 0\) belongs to \(\mathcal{O}_{S_i S_j}^l(S_i S_j \cup S_j S_i)\) but not to \(\mathcal{O}_{S_j S_i}^l(S_i S_j \cup S_j S_i)\).

4 Reineke’s mon(str)oid.

When \(A\) is the path algebra of a quiver without oriented cycles we can generalize the foregoing example and connect the previous definitions to the composition monoid introduced and studied by Markus Reineke in [2].

Let \(Q\) be a quiver without oriented cycles, then its path algebra \(A = \mathbb{k}Q\) is finite dimensional hereditary with all simple representations one-dimensional and in one-to-one correspondence with the vertices of \(Q\). For every dimension \(n\) we have that

\[\text{rep}_n A = \bigcup_{|\alpha|=n} GL_n \times^{GL(\alpha)} \text{rep}_\alpha Q\]
where \( \alpha \) runs over all dimension vectors of total dimension \( n \) and where \( \text{rep}_{\alpha} Q \) is the affine space of all \( \alpha \)-dimensional representations of the quiver \( Q \) with base-change group action by \( GL(\alpha) \).

The \textit{Reineke monstroid} \( \mathcal{M}(Q) \) has as its elements the set of all irreducible closed \( GL(\alpha) \)-stable subvarieties of \( \text{rep}_{\alpha} Q \) for all dimension vectors \( \alpha \), equipped with a product

\[
\mathcal{A} \ast \mathcal{B} = \{ X \in \text{rep}_{\alpha+\beta} Q \mid \text{there is an exact sequence}
\]

\[
0 \longrightarrow M \longrightarrow X \longrightarrow N \longrightarrow 0 \quad M \in \mathcal{A}, N \in \mathcal{B}
\]

if \( \mathcal{A} \) (resp. \( \mathcal{B} \)) is an element of \( \mathcal{M}(Q) \) contained in \( \text{rep}_{\alpha} Q \) (resp. in \( \text{rep}_{\beta} Q \)). It is proved in \cite{2} lemma 2.2 that \( \mathcal{A} \ast \mathcal{B} \) is again an element of \( \mathcal{M}(Q) \). This defines a monoid structure on \( \mathcal{M}(Q) \) which is too unwieldy to study directly. Observe that we changed the order of the terms wrt. the definition given in \cite{2}. That is, we will work with the \textit{opposite} monoid of \cite{2}.

On the other hand, the \textit{Reineke composition monoid} is very tractable. It is the submonoid \( \mathcal{C}(Q) \) of \( \mathcal{M}(Q) \) generated by the vertex-representation spaces \( R_i = \text{rep}_{\delta_i} Q \). These generators satisfy specific commutation relations which can be read off from the quiver structure, see \cite{2} \S 5. For example, if there are no arrows between \( v_i \) and \( v_j \) then

\[
R_i \ast R_j = R_j \ast R_i
\]

and if there are no arrows from \( v_i \) to \( v_j \) but \( n \) arrows from \( v_j \) to \( v_i \), then

\[
\begin{align*}
\left(R_i^{r(n+1)} \ast R_j \ast R_i^* \right) &= R_i^{en} \ast R_j \ast R_i^* \\
R_i \ast R_j^{p(n+1)} &= R_j \ast R_i \ast R_j^*
\end{align*}
\]

For more details on the structure of \( \mathcal{C}(Q) \) we refer to \cite{2} \S 5.

There is a relation between \( \mathcal{C}(Q) \) and the left- and right- non-commutative topologies \( \Lambda^l_A \) and \( \Lambda^r_A \). Because the Zariski topology on \( \text{simp} A \) is the discrete topology on the set \( \{S_1, \ldots, S_k\} \) of vertex simples, it is important to understand \( \mathcal{O}_w^r \), where \( w \) is a word in the \( S_i \), say \( w = S_{i_1} S_{i_2} \ldots S_{i_u} \). In fact, we could have based our definition of a one-sided non-commutative topology on the set \( \mathcal{L}_A \) of irreducible open subsets of \( \text{simp} A \) and then these basic opens would be all. If \( C \) is a \( GL(\alpha) \)-stable subset of \( \text{rep}_{\alpha} Q \) with \( |\alpha| = n \), we will denote the subset \( GL_n \times^{GL(\alpha)} \mathcal{C} \) of \( \text{rep}_{\alpha} A \) by \( \tilde{C} \).

\textbf{Proposition 1}

\[
\mathcal{O}_w^l = \bigcup_{w'} \tilde{A}_{w'} \quad \text{resp.} \quad \mathcal{O}_w^r = \bigcup_{w'} \tilde{A}_{w'}
\]

where \( \tilde{A}_{w'} \) is a \(*\)-word in the generators \( R_i \) of the composition monoid such that \( w' \) can be rewritten (using the relations in \( \mathcal{C}(Q) \)) in the form

\[
w' = R_{i_1} \ast R_{i_2} \ast \ldots \ast R_{i_u} \ast w'' \quad \text{resp.} \quad w' = w'' \ast R_{i_1} \ast R_{i_2} \ast \ldots \ast R_{i_u}
\]

for another \(*\)-word \( w'' \).
Also, the equivalence relation introduced before can be expressed in terms of $C(Q)$. If $w = S_{i_1}S_{i_2}...S_{i_u}$ and $w' = S_{j_1}S_{j_2}...S_{j_v}$ such that $w \cup w' = \{S_{k_1},...,S_{k_w}\}$, then

**Proposition 2** \(O^l_w \approx O^l_{w'}\) if and only if every \(*\)-word $v = R_{a_1}*...*R_{a_z}$ containing in it distinct factors $R_{k_1},...,R_{k_w}$ which can be brought in $C(Q)$ in the form

$$v = R_{i_1} \ast \ldots \ast R_{i_u} \ast v'$$

can also be written in the form

$$v = R_{j_1} \ast \ldots \ast R_{j_v} \ast v''$$

(and conversely). A similar result describes $O^r_w \approx O^r_{w'}$.

In particular, in this setting there will be hardly any idempotent basic opens (that is, satisfying $O^l_w \land O^l_{w'} \approx O^l_w$). Clearly, if \(\{S_{e_{a_1}},...,S_{e_{a_b}}\}\) are simples such that the quiver restricted to \(\{v_{e_{a_1}},...,v_{e_{a_b}}\}\) has no arrows, then any word $w$ in the $S_{e_j}$ gives an idempotent $O^l_w$. In the following section we will give an example where every basic open is idempotent and hence we get a commutative topology.

## 5 The commutative case.

If $A$ is a commutative affine $k$-algebra, then any simple representation is one-dimensional, $\text{simp} A = X_A$ the affine (commutative) variety corresponding to $A$ and the Zariski topologies on both sets coincide. Still, one can define the non-commutative topologies on $\text{rep} A$. However,

**Proposition 3** If $A$ is a commutative affine $k$-algebra, then both $\Lambda^l_A$ and $\Lambda^r_A$ are commutative topologies. That is, for all words $w$ and $w'$ in $\mathcal{L}_A$ we have

$$O^l_w \land O^l_{w'} \approx O^l_w \land O^l_{w'} \quad \text{and} \quad O^r_w \land O^r_{w'} \approx O^r_w \land O^r_{w'}$$

**Proof.** We claim that every basic open $O^l_w$ is idempotent. Observe that all simple $A$-representations are one-dimensional and that there are only self-extensions of those, that is, if $S$ and $T$ are non-isomorphic simples, then $\text{Ext}^1_A(S,T) = 0 = \text{Ext}^1_A(T,S)$. However, there are self-extensions with the dimension of $\text{Ext}^1_A(S,S)$ being equal to the dimension of the tangent space at $X_A$ in the point corresponding to $S$. As a consequence we have for any Zariski open subsets $U$ and $V$ of $X_A$ that

$$O^l_{U \cap V} = O^l_{V \cap U}$$

as we can change the order of the filtration factors (a representation $M$ is the direct sum of submodules $M_1 \oplus \ldots \oplus M_s$ with each $M_i$ concentrated in a single simple $S_i$ and we can add the successive $S_i$ factors of $M$ at any wanted place in the filtration sequence). Hence, for every word $w$ we have that

$$O^l_w \approx O^l_w \land O^l_w$$
and also for any pair of words \( w \) and \( w' \) we have that

\[
\mathcal{O}_w \wedge \mathcal{O}_{w'} = \mathcal{O}_{w+w'} = \mathcal{O}_{w'} \wedge \mathcal{O}_w
\]

Observe that in \([3]\) it is proved that a non-commutative topology in which every basic open is idempotent is commutative. We cannot use this here as the proof of that result uses both the left- and right- conditions. However, we are dealing here with a very simple example.

6 Quotient varieties versus moduli spaces.

Having defined a one-sided non-commutative topology on \( \text{rep} \, A \) we can ask about the induced topology on the quotient variety \( \text{iss} \, A \) of all isomorphism classes of semi-simple \( A \)-representations or on the moduli space \( \text{moduli}_\theta \, A \) with respect to a certain stability structure \( \theta \). Experience tells us that it is a lot easier to work with quotient varieties than with moduli spaces and non-commutative topology may give a partial explanation for this.

Indeed, as the points of \( \text{iss} \, A \) are semi-simple representations, it is clear that the induced non-commutative topology on \( \text{iss} \, A \) is in fact commutative. However, as the points of \( \text{moduli}_\theta \, A \) correspond to isomorphism classes of direct sums of stable representations (not simples!), the induced non-commutative topology on \( \text{moduli}_\theta \, A \) will in general remain non-commutative. Still, in nice examples, such as representations of quivers, one can define another non-commutative topology on \( \text{moduli}_\theta \, A \) which does become commutative. Use universal localization to cover \( \text{moduli}_\theta \, A \) by opens isomorphic to \( \text{iss} \, A \_\Sigma \) for some families \( \Sigma \) of maps between projectives and equip \( \text{moduli}_\theta \, A \) with a non-commutative topology (which then will be commutative!) obtained by gluing the induced non-commutative topologies on the \( \text{rep} \, A \_\Sigma \).

7 Generalizations.

It should be evident that our construction can be carried out verbatim in the setting of any Artinian Abelian category (that is, an Abelian category having Jordan-Hölder sequences) as soon as we have a natural topology on the set of simple objects. In fact, the same procedure can be applied when we have a left (or right) non-commutative topology on the simples.

In fact, the construction may even be useful in Abelian categories in which every object is filtered by special objects on which we can define a (one-sided) (non-commutative) topology.

References

[1] Claudio Procesi, *Rings with polynomial identities*, Marcel Dekker (1973)
[2] Markus Reineke, *The monoid of families of quiver representations*, Proc. London Math. Soc. **84** (2002) 663-685

[3] Alexei Rudakov, *Stability for an Abelian category*, J. Alg. **197** (1997) 231-245

[4] Fred Van Oystaeyen, *Prime spectra in noncommutative algebra*, Lect. Notes Math. 444, Springer (1975)

[5] Fred Van Oystaeyen, *Algebraic geometry for associative algebras*, Marcel Dekker (2000)

[6] Fred Van Oystaeyen, *Virtual topology and functor geometry*, monograph, to appear.