Special Bohr - Sommerfeld geometry.

Nikolay A. Tyurin
BLTPh (Dubna)*

and

NRU HSE (Moscow)†

August 28, 2015

Abstract

We introduce a new notion — special lagrangian submanifolds, which satisfy the Bohr - Sommerfeld condition — for algebraic varieties. We show that this leads to the construction of finite dimensional moduli space of special Bohr - Sommerfeld lagrangian submanifolds with respect to any ample linear bundle. The construction can be used in the studies of Mirror Symmetry.

The essence of Mirror Symmetry in the broadest context was expressed by Yu. I. Manin as “duality between symplectic geometry and complex Kahler geometry” (see [1]). Two algebraic Kahler manifolds $M, W$ are understood as “mirror partners”, if certain derived objects, constructed in the frameworks of algebraic and symplectic geometries of $M, W$ are cross equivalent: for example in Homological Mirror Symmetry due to M. Kontsevich (see [2]) the derived category of coherent sheaves on $M$ must be equivalent to the Fukaya - Floer category of $W$ and vice versa.

A.N. Tyurin, who spent many years studying stable vector bundles, suggested more geometrical correspondence: certain duality between vector bundles and lagrangian submanifolds (see, e.g. [3]). Namely for a pair of threefolds $M, W$ even cohomologies represent the Chern classes of vector bundles, combined into finite dimensional moduli spaces of stable vector bundles, and the middle odd cohomology can be realized by lagrangian submanifolds, should be combined into finite dimensional moduli spaces; and then on the comparing of these moduli spaces one could define the duality, which should present the essence of Mirror Symmetry. The main problem arises in this way is in “infinitness” of lagrangian geometry in contrast to algebraic geometry.

The problem can be solved if one introduces certain speciality condition on lagrangian submanifolds: realizing the ideology of calibrated lagrangian cycles J. MacLean and N. Hitchin (see [4]) proposed a special condition on lagrangian submanifolds in Calabi - Yau varieties which led to finite dimensional moduli spaces. Briefly, any Calabi - Yau variety by the very definition is endowed by top holomorphic non vanishing form $\theta \in \Omega^{3,0}$, and its restriction to any lagrangian

\*Joint Institute for Nuclear research, Joliot - Curie, 6, Moscow region, Dubna 141980, Russia

†AG Laboratory, HSE, 7 Vavilova str., Moscow, Russia, 117312
S is non vanishing as well; therefore the condition \( \theta|_S = \psi \mu(g) \) gives a correctly defined complex function \( \psi_S : S \rightarrow \mathbb{C}^* \). One says that \( S \) is special (or SpLAG) iff \( \psi \) has constant argument. Local deformations of the SpLAG submanifolds are finite dimensional and unobstructed, so the moduli space of special lagrangian submanifolds in a Calabi - Yau threefold has dimension \( b_1(S) \) (details can be found in [4]).

The introduction of SpLAG geometry led to the realization of Mirror Symmetry as T - duality: the famous SYZ - conjecture for Calabi - Yau threefolds (see [5]) explains Mirror Symmetry in terms of fibrations on SpLAG tori. According to this conjecture, kahlerian Calabi - Yau threefolds can be fibered on special lagrangian tori, parameterized by certain three dimensional base \( B \) (note that \( b_1(T^3) = 3 \)), and the mirror partner is given by the fibration on dual tori over the same base \( B \) (see [5]).

Unfortunately the existence problem for such a special lagrangian fibrations on Calabi - Yau threefolds is still open despite of crucial attempts to proof, and after years of high level popularity of SpLAG geometry the community of "mirror symmetrists" has changed the focus to Kontsevich’s homological approach. D. Auroux in [6] revisited the subject, extending the notion of SpLAG submanifolds to the case of open Calabi - Yau varieties, given by cut of a divisor from the anticanonical linear system on Fano varieties. In this approach the modified speciality condition turns to be relative — for a given Fano variety (or, more rigourisly, for a variety, whose anticanonical bundle admits holomorphic sections) this condition depends on the choice of the divisor and a lagrangian submanifold must lie in the complement of this divisor to be special, and since this complement can be understood as an open Calabi - Yau manifold, the Auroux’s approach can be seen as a modification of SpLAG geometry. The simplest example of special lagrangian fibration of an open Calabi - Yau is given by toric geometry: removing three lines \( l_i = \{ z_i = 0 \} \) from the projective plane \( \mathbb{CP}^2 \) one gets a special lagrangian fibration by Clifford tori. In [6] it was constructed a special lagrangian fibration for the complement of a reducible cubic equals to the union of nondegenerated quadric and projective line, and then it was conjectured that such a fibration exists for the complement of a smooth cubic curve in \( \mathbb{CP}^2 \), but this conjecture is still open. Except a single example from [7] there is no activity in this way.

In [8] one developed ideas of lagrangian approach to geometric quantization, namely one studied the moduli space of lagrangian submanifolds which satisfy the Bohr - Sommerfeld condition. In a previous paper A.N. Tyurin observed that the Bohr - Sommerfeld condition is in a sense "transversal" to the SpLAG condition in the Calabi - Yau case, which leads to possible definitions of certain finite invariants for Calabi - Yau threefolds, mirror to the Cassons invariants (see [3]). We develop this idea and introduce a new speciality condition for Bohr - Sommerfeld lagrangian submanifolds only, and this new notion of special Bohr - Sommerfeld submanifolds opens a chain of interesting observations, collected by Special Bohr - Sommerfeld geometry (SBS geometry for short).

Let \( (M, \omega) \) be a compact simply connected symplectic manifold, satisfied the Bohr - Sommerfeld condition condition. In a previous paper A.N. Tyurin observed that Bohr - Sommerfeld condition is in a sense "transversal" to the SpLAG condition in the Calabi - Yau case, which leads to possible definitions of certain finite invariants for Calabi - Yau threefolds, mirror to the Cassons invariants (see [3]). We develop this idea and introduce a new speciality condition for Bohr - Sommerfeld lagrangian submanifolds only, and this new notion of special Bohr - Sommerfeld submanifolds opens a chain of interesting observations, collected by Special Bohr - Sommerfeld geometry (SBS geometry for short).

Let \( (M, \omega) \) be a compact simply connected symplectic manifold, satisfied the Bohr - Sommerfeld condition of symplectic manifolds: the cohomology class \( [\omega] \in H^2(M, \mathbb{R}) \) is integer. Then fix the so called prequantization data: a linear bundle \( L \rightarrow M \) endowed by a hermitian structure and a hermitian connection \( a \) on it such that the curvature form \( F_a = 2\pi i \omega \). This condition uniquely defines the connection up to gauge transformations. A lagrangian submanifold
$S \subset M$ satisfies the Bohr - Sommerfeld condition (BS for short) iff the restriction $(L, a)|_S$ admits a covariantly constant section $\sigma_S$.

Let $s \in \Gamma(M, L)$ be any smooth section of $L$.

**Definition.** We call a BS- lagrangian submanifold $S$ special with respect to section $s$, iff $s|_S$ nowhere vanishes and the proportionality coefficient $\alpha(s, S)$, defined by the equality $s|_S = \alpha(S, s)\sigma_S$, has constant argument.

Since the definition doesn’t depend on the choice of $\sigma_S$ and of rescaling of $s$, it induces an “incidence cycle” in the direct product

$$U_{SBS} \subset \mathbb{P}(\Gamma(M, L)) \times B_S,$$

where the last symbol denotes the moduli space of Bohr - Sommerfeld lagrangian cycles of fixed topological type (see [8]). Namely, a pair $([s], S)$ belongs to $U_{SBS}$ iff $S$ is SBS w.r.t section $s$, if $s$ represents the class $[s]$ in the projectivized space. Naturally there are two projections $p_1, p_2$ to the first and to the second direct summands.

“Finiteness” of the set of SBS submanifolds is reflected by the following fact:

**Theorem A.** The projection $p_1 : U_{SBS} \to \mathbb{P}(\Gamma(M, L))$ has discrete fibers over the image $\text{Imp}_1 \subset \mathbb{P}(\Gamma(M, L))$.

On the other hand, one has

**Theorem B.** The image $\text{Imp}_1 \subset \mathbb{P}(\Gamma(M, L))$ is an open subset in the projective space.

Together they give

**Corollary A.** The space $U_{SBS}$ admits a Kahler structure.

Note that in general this fact leads to possible applications in Geometric Quantization. However we are strongly interested in the specific case: suppose that our symplectic manifold $(M, \omega)$ admits an integrable complex structure $I$, compatible with $\omega$. This means that $M$ is algebraic variety with principal polarization, defined by a holomorphic line bundle $L$. Then we get a finite dimensional subspace $\mathbb{P}(H^0(M_1, L)) \subset \mathbb{P}(\Gamma(M, L))$ formed by classes of holomorphic sections and therefore a reduced “incidence cycle”

$$\mathcal{M}_{SBS} \subset \mathbb{P}(H^0(M_1, L)) \times B_S$$

together with two natural projections to the direct summands which we again denote as $p_1, p_2$. Then we have

**Corollary B.** The moduli space $\mathcal{M}_{SBS}$ is finite dimensional possible singular Kahler variety.

Thus for any compact simply connected algebraic variety $X$ one can construct the following family of moduli spaces: for each very ample $L \to X$ the corresponding complete linear system defines the embedding of $X$ to the projective space, dual to $\mathbb{P}H^0(X, L)$; we can lift to $X$ the standard Kahler form from the projective space and consider it as a symplectic form on $X$ — and this is exactly the case where SBS geometry can be switched on. For fixed topological type of $S$ and the class $[S] \in H_0(X, \mathbb{Z})$ in the middle cohomology of $X$ we get the corresponding moduli space $\mathcal{M}_{SBS}$ marked by the data

$$\mathcal{M}_{SBS} = \mathcal{M}_{SBS}(S, [S], c_1(L)), \quad c_1(L) \in H^2(X, \mathbb{Z}).$$

Note however that even for close bundles these moduli spaces can be drastically different, and for the same bundle $L$ and the same class $[S]$ but for different
The topological types of the moduli space can be different as well: below we give the examples.

**Example 1.** Consider as $M$ the simplest compact simply connected symplectic manifold — complex projective line $\mathbb{CP}^1$ endowed by the standard Kahler structure. If we take as $L$ the bundle $\mathcal{O}(1)$, then the moduli space $\mathcal{M}_{SBS}(S^1, 0, h)$ is empty since no smooth loops on $\mathbb{CP}^1$ satisfy the Bohr - Sommerfeld condition (see [8]). But if we take as $L$ the bundle $\mathcal{O}(2)$, then the moduli space $\mathcal{M}_{SBS}(S^1, 0, 2h)$ is naturally isomorphic to $\mathbb{CP}^2\setminus Q$, where conic $Q$ is the image of $\mathbb{CP}^1$ under the Veronese embedding (see Section 3).

**Example 2.** For $M = \mathbb{CP}^2$ with the standard Kahler structure if we take $\mathcal{O}(2)$ as $L$ then the moduli space $\mathcal{calM}_{SBS}(T^2, 0, 2h)$ is empty while the moduli space $\mathcal{M}_{SBS}(\mathbb{RP}^2, 0, 2h)$ is nonempty. The last fact can be seen from the following arguments: fix coordinates $[z_0 : z_1 : z_2]$ compatible with the fixed Kahler structure and consider $S = \mathbb{RP}^2 = \{z_i \in \mathbb{R}\}$. Then it is not hard to see that $S$ is SBS w.r.t. the holomorphic section with zeros on the Fermat conic $Q = \{z_0^2 + z_1^2 + z_2^2 = 0\}$.

These results are derived in view of the following observation: the special Bohr - Sommerfeld condition with respect to a holomorphic section is naturally related to the Morse theory of plurisubharmonic functions on the complements to divisors. As we show below, if $s$ is a holomorphic section with zeroset $D_s \subset M$ then one takes $\phi_s = -ln|s|^2$ which is plurisubharmonic on $M\setminus D_s$ and then SBS condition for lagrangian submanifold $S \subset M$ reads as $\text{grad}\phi_s \subset TS$ at each point of $S$. Equivalently $S$ is preserved by the gradient flow of $\phi_s$. In particular this means that if $\phi_s$ is Morse outside of $D_s$ then the number and the type of critical points of $\phi_s$ dictate the possible types of lagrangian submanifolds. These critical points must be critical points of a Morse function on $S$, therefore to have a SBS torus of dimension 2 one must have at least 4 critical points of $\phi_s$ outside of $D_s$, which is non realistic for holomorphic sections of $\mathcal{O}(2)$; in contrast the set of critical points for $\phi_s$ where $s$ corresponds to the Fermat conic is very big — they form exactly $\mathbb{RP}^2$, and it is SBS due to the reformulated in Kahler terms SBS condition.

These arguments lead to

**Theorem C.** For a generic holomorphic section $s \in \mathcal{H}^0(M_I, L)$ the set of special Bohr - Sommerfeld lagrangian submanifolds is finite.

We expect that it is true for any holomorphic section, and that one has in general certain ramified covering

$$\mathcal{M}_{SBS} \rightarrow \mathbb{P}(\mathcal{H}^0(M_I, L)),$$

so should get a reach geometrical picture combined lagrangian and Kahler geometries of $M$.

The present paper is the first step and draft, so we focus on ideas and leave aside some technical details and computations in proofs.

**Acknowledgments.** The defintion of special Bohr - Sommerfeld submanifold arose in the discussion with Andery Losev, and the author would like to thank him first of all. I'm grateful to Andrey Shafarevich for the dicussion of general constructions and Vsevold Schechvishin who pointed out the relations to the theory of Kahler potential. I would like to thank Anastassia Tyurina for the computation in the toy example $M = \mathbb{CP}^1$.

The article was prepared within the framework of a subsidy granted to the
HSE by the Government of the Russian Federation for the implementation of the Global Competitiveness Program.

1 Special Bohr - Sommerfeld submanifolds

Consider a compact simply connected symplectic manifold $(M, \omega)$ of real dimension $2n$ such that the symplectic form $\omega$ has integer class in the de Rham cohomology of $M$: $[\omega] \in H^2(M, \mathbb{Z})$. In this case one says that the symplectic manifold $(M, \omega)$ satisfies the Bohr - Sommerfeld condition for manifolds. Fix the prequantization data $(L, a)$: linear hermitian bundle $L$ whose first Chern class is $c_1(L) = [\omega] \in H^2(M, \mathbb{Z})$ and a hermitian connection $a \in A_h(L)$, with the curvature form is $F_a = 2\pi i \omega$. In the simply connected case this condition defines a uniquely up to gauge transformations. Recall that the quadruple $(M, \omega, L, a)$ is the input data for the ALAG programme, see [8].

A submanifold $S \subset M$ of real dimension $n$ is lagrangian iff the restriction $\omega|_S \equiv 0$ is trivial. Therefore for any lagrangian submanifold the restriction $(L, a)|_S$ is topologically trivial bundle with a flat connection. In this paper we mostly consider the case of compact orientable lagrangian submanifolds although certain results can be extended to more general cases.

**Definition 1.** Lagrangian submanifold $S$ satisfies the Bohr - Sommerfeld condition iff the restriction $(L, a)|_S$ admits a covariantly constant section $\sigma_S \in \Gamma(S, L|_S)$.

In what follows we will abbreviate it as BS - condition.

It is not hard to see that BS - condition doesn’t depend on the choice of $a$ in the equivalence class up to gauge transformations. Indeed, **Definition 1** is equivalent to the following condition: for any loop $\gamma \subset S$ and any disc $D \subset M$ such that $\partial D = \gamma$ the symplectic area of disc $D$ is integer: $\int_D \omega \in \mathbb{Z}$. At the same time for a fixed $a$ the covariantly constant section $\sigma_S$ is defined up to $\mathbb{C}^*$.

Fix a smooth section $s \in \Gamma(M, L)$ of the prequantization bundle $L \to M$. Then for a BS - submanifold $S \subset M$ we give the following

**Definition 2.** We say that BS- submanifold $S$ is special w.r.t section $s$, iff $s|_S$ doesn’t vanish on $S$ and the proportionality coefficient $\alpha(s, S)$, defined by the equality $s|_S = \alpha(S, s) \sigma_S$, has constant argument. In other words $s|_S = fs e^{i\theta} \sigma_S$, where $c$ is a real constant and $f$ — a real positive function.

For short we will call such an $S$ as $s$ - SBS - submanifold or just SBS - submanifold if $s$ is coming from the context.

**Remark.** Our SBS - condition presented above is essentially different from the speciality conditions given by N. Hitchin in [4] and by D. Auroux in [6], despite of the fact that the second one depends on the section as in our case. Our SBS - condition is based on the Bohr - Sommerfeld condition and forother lagrangian submanifold it can’t be either generalized nor reformulated. At the same time certain weak relation takes place. Namely consider the case when $K_M = k[\omega]$, where $K_M = \text{det}(T^*M)^{1,0}$ is the canonical class of a Kahler manifold $M$, and connection $a$ is defined by the condition $F_a = 2\pi ik\omega$. Then both in the Calaby - Yau case in [4] ($k = 0$) and in the Fano case in [6] ($k < 0$) one can take the determinant Levi - Civita connection as $a$ (since the Kahler metric is the Kahler - Einstein), and if the mean curvature of our lagrangian submanifold $S$ identically vanishes then the covariantly constant section $\sigma_S$ is up to multiple the volume form of the restricted Kahler metric to $S$. Therefore in the both
cases minimal and special in the sense of Hitchin or Auroux. Lagrangian submanifolds are SBS - submanifolds with respect to the corresponding sections of the anticanonical bundle, given by top holomorphic form $\theta$ either on whole $M$ as in [4], or on the complement to the corresponding divisor form the anticanonical system as in [6]. Note however that lagrangian submanifold $S$ is minimal only if it is Bohr - Sommerfeld with respect to the determinant Levi - Civita connection, but BS - condition is not sufficient in the case so there are SpLag - submanifolds and BS - lagrangian submanifolds which are not SBS. However we think that it is reasonable to use term "special" in our Definition 1 above.

Recall that ALAG - programme from [8] supplies one in the situation described above with certain moduli space $\mathcal{B}_S$ of Bohr - Sommerfeld lagrangian submanifolds of fixed topological type (see [8]). Every such a moduli space is defined and numerated by the following discrete data: by the corresponding class $[S] \in H_n(M, \mathbb{Z})$, realized by the lagrangian submanifolds, and by the topological type of $S$ (i.e. in the case $n = 2$ the last type is fixed by the genus $g(S)$); we indicate this dependence as $\mathcal{B}_S(S, [S])$. This moduli space is an infinite dimensional smooth by Frechet real manifolds. At each point its tangent space is modelled by the real space $C^\infty(S, \mathbb{R})/\text{const}$ (the details can be found in [8]).

It’s clear that SBS - condition is stable with respect to rescaling of smooth section $s$, hence this condition induces certain “incidence cycle” in the direct product $\mathbb{P}(\Gamma(M, L) \times \mathcal{B}_S$. Namely define a subset $\mathbb{P}(\Gamma(M, L) \times \mathcal{B}_S \supset M_{SB} = \{(p, S)\}$ by the condition that BS - submanifold $S \subset M$ is $s$- SBS - submanifolds w.r.t the section $s \in \Gamma(M, L)$, representing the point $p \in \mathbb{P}(\Gamma(M, L)$.

As usual, for our “incidence cycle” $\mathcal{U}_{SB}$ one has two canonical projections

$$p_1 : \mathcal{U}_{SB} \to \mathbb{P}(\Gamma(M, L)), \quad p_2 : \mathcal{U}_{SB} \to \mathcal{B}_S$$

to the direct summands of the ambient direct product. The main part of the present work is to study the properties of these projections since the geometrical properties of the summands $\mathbb{P}(\Gamma(M, L))$ are $\mathcal{B}_S$ already known. Moreover both of them play essential roles in the Geometric Quantization constructions.

**Digression: Geometric Quantization.** Infinite dimensional projective space $\mathbb{P}(\Gamma(M, L)$ — one of the principal objects in Geometric Quantization. Usually in this projective space one cuts subspaces which represent quantum phase spaces of the quanatized system. However the second summand — the moduli space $\mathcal{B}_S$ — has been exploited in other approaches to the quantization problem for classical mechanical systems named as Lagrangian Geometric Quantization. In this set up one exploited certain building over $\mathcal{B}_S$, given in [8], when after ”halfweighting” one constructs $\mathcal{B}_S^{hw,r}$ — the moduli space of halfweighted Bohr - Sommerfeld lagrangian cycles of fixed topological type and volume. The aim of this building was a ”complexification” of real moduli space $\mathcal{B}_S$. The moduli space $\mathcal{B}_S^{hw,r}$ was used in the construction named as ALG(a) - quantization, see [9]. But the original aim wasn’t reached in this way since this building admits an almost Kahler structure with the constant Kahler angle which is not integrable. But in any case our ”incidence cycle” could play an important and interesting role: being universal in the direct product it could helps to carry geometrical data from $\mathbb{P}(\Gamma(M, L))$ to $\mathcal{B}_S$ and back. Thus it could give a connection between different approaches to Geometric Quantization problem. Moreover, since (as we will see below) the projection $p_2$ is epimorphic and since $\mathcal{U}_{SB}$ admits a Kahler structure lifted from the first summand, then our ”incidence cycle” can be regarded as a complexification of $\mathcal{B}_S$. 

6
Definition 2 implies the following simple topological observation. In the direct product $\mathbb{P}(\Gamma(M,L)) \times B_S$ take the determinantal subset $\Delta \subset \mathbb{P}(\Gamma(M,L)) \times B_S$ by the condition: section $s$ after the restriction to $S$ has zeros. Then the complement $\mathbb{P}(\Gamma(M,L)) \times B_S \setminus \Delta$ can be divided into connected components $K_i$, possible of infinite number. For each connected component $K_i$ one can define the cohomology class $m(K_i) \in H^1(S,\mathbb{Z})$ by the condition: for pair $(p,S) \subset K_i$ the restriction of $s$ to $S$ gives the function $\alpha(s,S) \in C^\infty(S,\mathbb{C}^*)$, so

$$m(K_i) = \alpha(s,S)^* \mu$$

, where $\mu$ is the generator in $H^1(\mathbb{C}^*,\mathbb{Z})$. Inside of the connected component the class $m(K_i)$ can’t change being integer valued therefore the definition of $m(K_i)$ doesn’t depend on the choice of $(p,S) \in K_i$.

Then one has

**Proposition 1.** The moduli space $\mathcal{U}_{SBS}$ has non trivial intersection with the component $K_i$ iff $m(K_i) = 0$.

The proof is obvious.

The next proposition is more interesting: it reflects the most important for us geometrical property of the first projection $p_1$.

**Proposition 2.** For any smooth section $s \in \Gamma(X,L)$ the set of $s$ - BS submanifolds of fixed topological type is discrete.

**Proof.** Suppose in contrary that there is one dimensional family $S_t$, $t \in [0,1]$, of BS - lagrangian submanifolds for a fixed smooth section $s_0$. Consider a Darboux - Weinstein neighborhood $O_0$ of the origin lagrangian submanifold $S_0$; then there is a sufficiently small segment $[0;\epsilon]$ such that every $S_t \subset O_0$ if $t \in [0;\epsilon]$ and consequently each $S_t$ is presented by the corresponding exact 1 - form $d\psi_t \in \Gamma(S_0,T^*S_0)$. Hence each $S_t$ intersects $S_0$ at least at two points $p_t^+, p_t^-$ if $t \in [0;\epsilon]$ since every function on a compact set must have at least two critical points. Join these two points by pathes $\gamma_0, \gamma_t$ which lie on the submanifolds $S_0$ and $S_t$ respectively. Choice covariantly constant sections $\sigma_0 \in \Gamma(S_0,L|_{S_0})$ and $\sigma_t \in \Gamma(S_t,L|_{S_t})$ such that $\sigma_0(p^-t) = \sigma_t(p_t^-)$. Then it is clear that the phase difference at the second common point $p_t^+$ can be expressed via the symplectic area of the disc $D_t$, bounded by $\gamma_0$ and $\gamma_t$:

$$\frac{\sigma_t(p_t^+)}{\sigma_0(p_t^-)} = \exp[2\pi i \int_{D_t} \omega] = 2\pi i (\psi_t(p_t^+) - \psi_t(p_t^-))$$

But both the BS - submanifolds $S_0$ and $S_t$ are special w.r.t the same global section $s_0$, therefore the phase of $\sigma_0$ and $\sigma_t$ must coincide at both the points $p_t^+$, which is impossible for sufficiently small $t$ if $\psi_t(p_t^+) - \psi_t(p_t^-)$ is not zero. But if it is then the maximum and the minimum of our function coincide so the function is constant and hence $S_t = S_0$.

**Remark.** If one removes from the Definition 2 the non vanishing condition for the restriction of $s$ to $S$ (we would call this "stability condition") then it is easy to construct a continuous deformation of BS - submanifolds for the same section $s_0$. Indeed, as the simplest example we take $\mathbb{C}$ with the standard symplectic form $\frac{1}{2}(dz \wedge d\bar{z})$, then the prequantization bundle it trivial $L = C^\infty(\mathbb{C},\mathbb{C})$. The prequantization connection with respect to the natural trivialization is given by 1-form $\frac{1}{2}(zdz - \bar{z}d\bar{z})$, and any real line of the form $z = at$ is BS - submanifold such that the correspondings covariantly constant
sections are given by constant functions. Holomorphic section \( f(z) = z \) vanishes at \( z = 0 \), and the pencil of real line \( z = ct \) is a continuous family of SBS - submanifolds, intersecting exactly at the origin. The situation can be modified to the compact case if we complete \( \mathbb{C} \) to the projective line \( \mathbb{CP}^1 \) and consider the family of meridians passing through the Poles — all of them are SBS in this weaker sense (non stable) w.r.t to the section given by the homogenous polynomial \( z_0z_1 \), for the prequantization bundle \( L = O(2) \).

Also note that \( p_1 \) is never epimorphic: one can take a smooth section \( s \in \Gamma(M,L) \) with very big set of zeros \( (s) = \{ x \in M | s(x) = 0 \} \subset M \) such that the complement \( M \setminus (s) \) would not contain a disc \( D \) with integer symplectic area (the argument doesn’t work when \( M \) admits family of shrinking lagrangian spheres).

Local computation ensures that

**Proposition 3.** Over a generic point \( p \in \text{Imp} \) the differential of \( dp_1 \) is an isomorphism.

Therefore we can state that

**Theorem 1.** The projection \( p_1 : \mathcal{U}_{SBS} \to \mathbb{P}(\Gamma(M,L)) \) has the structure of covering over the image \( \text{Imp}_1 \subset \mathbb{P}(\Gamma(M,L)) \).

Below we continue the studies of the first projection \( p_1 \).

## 2 Speciality and calibration

In this section we present an alternative description of special Bohr - Sommerfeld in calibration terms. Recall that calibration in the sense of Harvey and Lawson [11] is given by the vanishing condition for a set of distinguished form after the restriction to submanifolds. Below we show that SBS - condition is equivalent to vanishing of certain 1 - form, constructed in terms of section \( s \).

Consider \( s \in \Gamma(M,L) \) and denote its zeroset as \( D_s \subset M \). Then on the complement \( M \setminus D_s \) it is correctly defined the following complex 1 - form \( \rho_s \in \Omega^1_M \otimes \mathbb{C} \)

\[
\rho_s = \frac{\nabla_a s}{s} = \frac{\nabla_a s, s}{s}.
\]

where \( \nabla_a \) is the covariant derivative of the prequantization connection \( a \in \mathcal{A}_h(L) \).

Indeed, \( \nabla_a s \in \Gamma(L) \otimes \Omega^1_M \), and since \( s \) on \( M \setminus D_s \) doesn’t vanish we can express \( \nabla_a s \) as a section of \( L \otimes T^* M \) in the form \( s \otimes \rho_s \) outside of zeroset of \( s \).

**Proposition 4.** Complex 1 - form \( \rho_s \) doesn’t change under rescaling of \( s \) by a constant. The real part of \( \rho_s \) is exact. The differential of the imaginary part of \( \rho_s \) equals \( 2\pi \omega \).

First, rescale \( s \) by a constant \( c \in \mathbb{C} \) doesn’t change the zeroset \( D_s \), at the same time

\[
\frac{\nabla_a cs}{cs} = \frac{c\nabla_a s}{cs} = \frac{\nabla_a s}{s}
\]

on \( M \setminus D_s = M \setminus D_{cs} \). Therefore pair \( (D_s \subset M; 1 - \text{form } \rho_s) \) corresponds to a point from \( \mathbb{P}(\Gamma(M,L)) \).

Second, the real part of \( \rho_s \) can be found from the equality:

\[
d < s, s >= < \nabla_a s, s > + < s, \nabla_a s > = 2\text{Re} \rho_s < s, s >.
\]
since $\alpha$ is hermitian. Therefore

$$\text{Re} \rho_s = \frac{1}{2} \text{d}(\ln |s|^2),$$

so the part is exact on $M \setminus D_s$.

Third, the calculations of the imaginary part of $\rho_s$ can be done as follows: take on $M \setminus D_s$ (non hermitian) connection $\alpha_s$ such that $\nabla_{\alpha_s} s = 0$. Then the connection $\alpha_s$ is trivial and defined by the trivialization given by the section $s$ of the restricted bundle $L|_{M \setminus D_s}$. But the difference $\nabla_s - \nabla_{\alpha_s}$ is exactly $\rho_s$, and the differential of $\rho_s$ must be equal to the differential of the curvature forms $F_s - F_{\alpha_s} = d \rho_s$. The first one by the very definition equals to $2\pi i \omega$, and the second one is trivial. It follows $d \rho_s = 2\pi i \omega$, so

$$d(\text{Re} \rho_s) = 0, \quad d(\text{Im} \rho_s) = 2\pi i \omega,$$

which ends the proof.

**Remark.** Proposition 4 implies that any section $s \in \Gamma(M, L)$ defines the following structure on the open part $M \setminus D_s$ where $D_s$ is the zero set of $s$: a smooth function $\phi_s = -\ln |s|$ bounded below and a vector field $X_s$ given by $\frac{1}{2\pi} \omega^{-1}(\text{Im} \rho_s)$, which is a Liouville vector field since the Lie derivative $\text{Lie}_{X_s} \omega = \omega$. Both $\phi_s$ and $X_s$ are encoded by our 1-form $\rho_s$. The structure given by $(\omega, X_s)$ is very well known as the Liouville structure, and moreover if the vector field $X_s$ is gradient-like for the function $\phi_s$ then we get a Weinstein structure on $M \setminus D_s$. It seems that if happens iff there exists a compatible almost complex structure $J$ such that the section $s$ is pseudo holomorphic w.r.t. $J$. We will exploit the way “back from Weinstein to Stein” built up by Y. Eliashberg and K. Cieliebak\footnote{see the lecture course at www.mathematik.uni-muenchen.de/~kai/research/stein.pdf} in the next section devoted to the algebraic case.

The point is that our 1-form $\text{Im} \rho_s$ defines the required calibration:

**Theorem 2.** Smooth orientable submanifold $S \subset M$ of dimension $n$ is $s$ - special Bohr - Sommerfeld lagrangian iff the restriction of 1-form $\text{Im} \rho_s$ to $S$ identically vanishes.

Note that the condition $(\text{Im} \rho_s)|_S \equiv 0$ implies that $S \cap D_s = \emptyset$ since the form $\rho_s$ has pole along $D_s$, therefore we don’t mention the last condition in the Theorem formulation.

**Proof.** Let $S$ be special w.r.t. $s$ lagrangian BS - submanifold. Then the restriction of $\rho_s$ on $S$ can be calculated as follows:

$$\rho_s|_S = \nabla_s|_S (s|_S) = \frac{\nabla_s fe^{ic} \sigma_s}{f e^{ic} \sigma_s} = \frac{df}{f} = d(\ln f),$$

where $f$ is real positive function (see Definition 2). Thus $(\text{Im} \rho_s)|_S \equiv 0$.

Note that during the calculation above we establish the meaning of positive function $f$ — it is exactly $|s|$, restricted to $S$.

Now let certain smooth orientable half dimensional submanifold $S \subset M \setminus D_s$ satisfies the condition $(\text{Im} \rho_s)|_S \equiv 0$. According to Proposition 4 the differential of this 1-form equals to $2\pi i \omega$ which implies that $S$ must be lagrangian. The restrictions of $a$ and $\alpha_s$ on $S$ are both flat and their difference is real exact form $\rho_s|_S = d(\ln |s||_S)$. Since $s$ is covariantly constant w.r.t. $a_s$ it follows that $\rho_s|_S$ must be covariantly constant w.r.t. $a|_S$, therefore $S$ must be BS - submanifold.
The speciality condition obviously follows from the same argument, and it ends the proof.

Topology comes to the discussion at this step: according to Proposition 4 our calibrating 1-form from $\text{Im}\rho_s$ gives a cohomology class being restricted to any lagrangian submanifolds. Indeed, the restriction is a closed 1-form, and the corresponding cohomology class has been discussed above:

**Proposition 5.** Let pair $(p(s), S) \subset \mathbb{P}(\Gamma(M, L)) \times B_S$ belongs to a connected component $K_1$ from the Proposition 1 above. Then the cohomology class $[\text{Im}(\rho_s)] = m(K_i) \in H^1(S, \mathbb{Z}).$

Indeed, let $s|S = \alpha(s, S)\sigma_S$ be as in Proposition 1. The proportionality coefficient $\alpha(s, S)$ is a complex valued non vanishing function on $S$. The logarithmic derivative of this function is exactly our 1-form $\rho_s$ after restriction to $S$; the real part is exact so all cohomological data is concentrated in $\text{Im}\rho_S$, which gives the statement of Proposition 5.

SBS - submanifolds can be understood as zeros of certain vector field on a subset of the moduli space $B_S$. Let us fix a smooth section $s \in \Gamma(M, L)$ and consider the connected components $K_i(s) \subset B_S$, defined as $p_2(K_i \cap p_1^{-1}(p(s)))$ where $K_i$ were defined in Proposition 1 above, and then take the components with the trivial classes $m_i(K)$. Then for any point $[S] \in K_i(s)$ the restriction of the form $\text{Im}\rho_s|_S$ is exact. But the exact forms are tangent vectors to $\mathcal{B}$ at point $[S]$ (see [8], [9]), therefore our section $s$ generates a smooth vector field $\tau_s$ on each appropriate component $K_i(s)$. The point is that zeros of this vector field $\tau_s$ present SBS-submanifolds. Moreover, this vector field $\tau_s$ is transversal: zeros of $\tau_s$ is isolated. And even more can be said for the field: recall that the moduli space $B_S$ is covered by Darboux-Weinstein neighborhoods which play the role of charts in the natural atlas (see [8]). Then we have the following

**Theorem 3.** Any Darboux-Weinstein neighborhood contains at most only one $s$-special Bohr-Sommerfeld lagrangian submanifold w.r.t. a fixed smooth section $s$.

Let the “center” of a Darboux-Weinstein neighborhood — a BS-submanifold $S_0 \subset M\setminus D_s$ — is itself SBS w.r.t. to our fixed section $s$. Then according to Theorem 2 the restriction $\text{Im}\rho_s|_S$ identically vanishes. Transport our 1-form $\text{Im}\rho_s$ to a small neighborhood of the zero section $\mathcal{O}_x(S_0) \subset T^*S_0$ and denote this form as $\beta_s$. Then near zero section in $T^*S_0$ we have two 1-forms — the canonical 1-form $\alpha$ and new 1-form $\frac{1}{2}\beta_s$ with the same differentials. The difference of these forms is closed and identically vanishes on the zero section.

According to the Darboux-Weinstein theorem any close BS-submanifold of the same type is presented by the graph of an exact 1-form $df$ for certain real function $f \in C^\infty(S_0, \mathbb{R})$. Every function on a compact set admits at least two critical points which we denote as $x_+, x_-$ for function $f$. These points belongs to the intersection of the graph $\Gamma(df)$ and the zero section $S_0$ in $T^*S_0$. Consider two pathes $\gamma_0, \gamma_f = \Gamma(df(\gamma_0))$, connecting the points along $S_0$ and $\Gamma(df)$ respectively. Then the integrals along closed loop $\gamma_0 \cup \gamma_f$ for the forms $\alpha$ and $\frac{1}{2}\beta_s$ must be the same since both the forms have the same differential. But if we suppose that the graph $\Gamma(df)$ is again special w.r.t. to the same section $s$, then the integral for the second form $\frac{1}{2}\beta_s$ must be trivial. Indeed, by the Theorem 2 our 1-form $\beta_s$ must vanishes being restricted to $\Gamma(df)$ as well as to the zero section. But the integral for $\alpha$ along the same loop equals to the difference $f(x_+) - f(x_-)$, so it could happen iff the function $f$ is constant. But then $\Gamma(df) = S_0$. 

10
Then we extend the family of transformations $A_t$ identical since for any fixed section $s$, the proof of injectivity for all $A_t$ is defined for small exact forms only since the graph $\Gamma(\eta)$ is correctly defined for small exact forms only. Near the boundary of the Darboux - Weinstein neighborhood, this function can be smoothly deformed to zero, and the denote the result again as $\psi.$ Then if the fixed section $s$ is deformed by the following family $s_t = se^{-it\psi}, t \in [0,1], s_0 = s,$ and we take the corresponding calibrating 1-forms $\beta_s = \beta_t$ for each $t \in [0,1]$ one can define a non-linear transformation of the space of exact 1-forms $B^1(S_0)$. For each $\beta_t$ take an exact 1-form $\eta \in B^1(S_0)$ and consider

$$A_t(\eta) = \pi_*\beta_t|_{\Gamma(\eta)} \in B^1(S_0)$$

where $\Gamma(\eta) \subset T^*S_0$ is the graph of $\eta$. Note however that this transformation is correctly defined for small exact 1-forms only since the graph $\Gamma(\eta)$ must lie in the Darboux - Weinstein neighborhood. This means that $A_t$ is defined for certain small ball in $B^1(S_0).$ Our claim is that if $S_0$ is $s = s_0$ special BS then $A_t$ is locally surjective for any $t$. Locality here means that we are interested in a small ball around zero in $B^1(S_0).$ The claim is based on two facts: first, for any $t \in [0,1]$ the map $A_t$ injective; second, for $t = 1$ the map is surjective being identical since for $t = 1$ our form $\beta_1$ coincides with the canonical form $\alpha$. The proof of injectivity for all $t$ follows the same arguments as for Theorem 3. Now let our fixed section $s$ be deformed to $s_0$, and the corresponding 1-form is $\beta_0$. Then we extend the family of transformations $A_t, t \in [0,1]$ to $A_t, t \in [0,1 + \delta]$ where $\delta$ is small enough. Again all $A_t$ are injective, so one can expect that for sufficiently small $\delta$ they are locally surjective, and then for small $\delta$ zero 1-form belongs to the image $A_t$ therefore the corresponding preimage gives Bohr - Sommerfeld deformation of $S_0$ which is $s_{BS}$-special. This ends the sketch of the proof of

**Theorem 4.** The image of the first projection $\text{Imp}_1 \subset \mathbb{P}(\Gamma(M,L))$ is open subset.

Combining Theorem 1 and Theorem 4 we get

**Corollary.** The space $U_{BS}$ admits a Kähler structure.
Indeed, we can just lift the standard Kahler structure from \( \mathbb{P}(\Gamma(M, L)) \) using \( p_1 \).

Thus the space \( \mathcal{U}_{SBS} \) can be regarded as certain "complexification" of the moduli space \( \mathcal{B}_S \), however it is not a complexification in usual sense: the "dimension" of \( \mathcal{B}_S \) is much less than the "half dimension" of \( \mathcal{U}_{SBS} \).

3 Algebraic case

Now suppose that our symplectic manifold \((M, \omega)\) admits a compatible complex structure \( I \) which is integrable. This means that \( M \) is endowed with a Kahler metric of the Hodge type (since \( \omega \) defines an integer cohomology class) therefore \((M, \omega, I)\) can be regarded as an algebraic variety, see [10]. On the other hand the prequantization data \((L, a)\) in the case correspond to a holomorphic line bundle since the curvature \( F_a \) has type \((1,1)\) w.r.t. \( I \) therefore it induces a holomorphic structure on our hermitian line bundle \( L \). This means that fixing \( I \) we cut a finite dimensional subspace \( H^0(M_I, L) \subset \Gamma(M, L) \) formed by holomorphic sections of \( L \). This subspace is finite dimensional, and it is natural to construct the corresponding finite dimensional version of the space \( calU_{SBS} \) defined above. Take in the direct product \( \mathbb{P}H^0(M_I, L) \times \mathcal{B}_S \) the subset defined by the specialty condition as it was above and get the moduli space of SBS lagrangian submanifolds over \((M, \omega, I)\) defined as the preimage

\[
\mathcal{M}_{SBS} = p_1^{-1}(\mathbb{P}H^0(M_I, L)) \subset \mathcal{U}_{SBS}
\]

of the first projection of the projectivization of the holomorphic section subspace.

Since the properties of projection \( p_1 \) were studied above we know that

\[
p_1 : \mathcal{M}_{SBS} \rightarrow \mathbb{P}H^0(M_I, L)
\]

has discrete fibers, so the moduli space \( \mathcal{M}_{SBS} \) is a finite dimensional set fibered over an open subset in the projective space \( \mathbb{P}H^0(M_I, L) \). In the rest of the present text we show that for a generic holomorphic section the number of preimages at the corresponding fiber is finite and propose a constructive way how to find SBS lagrangian submanifolds. On the other hand in the known examples the open part of \( \mathbb{P}H^0(M_I, L) \) has very natural form: we just remove an algebraic subvariety from the projective space. This hints that the moduli space \( \mathcal{M}_{SBS} \) admits certain natural compactification.

The key observation for the algebraic case is the following:

**Proposition 6.** For a holomorphic section \( s \in H^0(M_I, L) \) the corresponding calibrating 1- form \( \text{Im}(\rho_s) \) equals to \(-I(d|\ln|s|)|\).

First, the form \( \rho_s \) has type \((1,0)\) w.r.t. the complex structure \( I \). Indeed, since

\[
\rho_s = \frac{<\nabla_a s, s>}{<s, s>}
\]

but \( s \) is holomorphic w.r.t. \( \partial_a \) therefore \( \nabla_a s \in L \otimes \Omega^{1,0} \) and the resting operations do not change the type.

Second, for the real and imaginary parts of a \((1, 0)\) - form we have the corresponding relation, and since we know the real part of \( \rho_s \) which equals to \( d|\ln|s|)| \) we get the statement of Proposition 6.
Therefore for a holomorphic section \( s \in H^0(M_1, L) \) we get the corresponding real smooth function
\[
\phi_s = -\frac{1}{2\pi} \ln |s|
\]
which is correctly defined on the complement \( M \setminus D_s \) and which is plurisubharmonic or strongly convex w.r.t. \( I \) on the complement since \( d(I(d\phi_s)) = d^c d\phi_s = \omega \) implied from Proposition 4. The convexity property is very useful for our investigations since we have the following remark.

**Proposition 7.** A lagrangian submanifold \( S \subset M \) is SBS w.r.t. a holomorphic section \( s \in H^0(M_1, L) \) iff \( \text{grad} \phi_s \) is parallel to \( TS \) at each point of \( S \).

Indeed, we know from Theorem 2 above that SBS condition is equivalent to \( \text{Im} \rho_s |_S \equiv 0 \), but if \( S \) is lagrangian and \( \text{Im} \rho_s = -I(\text{dln}|s|) \) then the SBS condition is equivalent to the fact that \( \omega(v, \text{grad} \phi_s) \) identically vanishes if \( v \) is tangent to \( S \) at each point. Thus \( \text{grad} \phi_s \) must be parallel to \( TS \).

Now suppose that for a holomorphic section \( s \in H^0(M_1, L) \) the corresponding function \( \phi_s \) is a Morse function on the complement \( M \setminus D_s \). It means that the real function \( |s|^2 \) is relatively Morse so it admits degenerated absolute minimum at \( D_s \) but all other critical points are non degenerated. For this case consider the critical points \( x_1, \ldots, x_k \in M \setminus D_s \) of the function \( \phi_s \). This function being restricted to any compact lagrangian submanifold \( S \) must have critical points (not less the Morse inequality dictates) and if \( S \) is SBS w.r.t. \( s \) it implies that every critical point of the restriction \( \phi_s |_S \) must be critical for global \( \phi_s \) at the same point of \( S \) considering as a point of \( M \setminus D_s \). Indeed, at these points both \( d\phi_s \) and \( I(d\phi_s) \) vanish being restricted to \( S \) (note that we suppose that \( S \) is SBS so the last form vanishes identically on \( S \)). Consequently any SBS lagrangian submanifold must contain several critical points from the set \( x_1, \ldots, x_k \).

For example if \( S \) is a lagrangian torus then the number of critical points it must contain is not less than \( n \). And if for a generic holomorphic section \( k < n \) (as it is for the case of \( CP^2 \) and \( L = O(2) \)) then SBS lagrangian tori do not exist.

Moreover, Proposition 7 shows that the gradient flow generated by \( \phi_s \) must preserve SBS lagrangian submanifold \( S \). It means that SBS submanifold \( S \) must contain not just critical points but trajectories of the gradient flow. In simple cases it hints how to completely solve the problem.

**Example.** Consider the projective line \( CP^1 \) with the standard Kahler structure and rescale the Kahler form by 2. The corresponding line bundle is \( O(2) \) so any holomorphic section \( s \) is completely determined by its zeros up to \( CP^1 \), and since our SBS condition depends on the class up to \( CP^1 \) it means that every pair of points \( (x_1, x_2) \) should define SBS lagrangian loops. Suppose first that the section is multiple so the points coincide \( x_1 = x_2 \). Then the function \( \phi_s \) has only one critical point \( x_{\min} \) on \( CP^1 \) and for this section SBS lagrangian loop doesn’t exist. Indeed, if it exists then \( \phi_s |_S \) must have at least two different critical points which must be global critical points of \( \phi_s \) on the punctured sphere but there one has only one critical point \( x_{\min} \). In contrast for a section \( s \) with two different zeros \( x_1, x_2 \) a SBS loop must exist due to pure topological reasons. Indeed, the vector field \( \text{grad} \phi_s \) in this case has at least three singular points \( x_1, x_2, x_{\min} \) with positive indices. But the Euler characteristics of sphere is 2 which implies that \( \phi_s \) admits a saddle point \( x_s \). Therefore the gradient flow must have one separatrix \( \gamma \) passing \( x_{\min} \) and \( x_s \) which separates trajectories which go to \( x_1 \) and \( x_2 \). This \( \gamma \) is preserved by the gradient flow which implies
that \( \gamma \) is SBS with respect to \( s \). There are exceptional cases when the zeros of \( s \) are antipodal, and in this case \( \phi_s \) is not Morse on the complement but admits a critical equatorial loop which is again SBS.

This presents another principle detecting SBS submanifolds: if for a holomorphic \( s \) the function \( \phi_s \) is not Morse but has sufficiently big critical subset (of maximal possible dimension \( n \)) it must be SBS lagrangian w.r.t. this section. For example for \( \mathbb{CP}^2 \) and \( L = \mathcal{O}(2) \) the holomorphic section \( s \) corresponding to conic \( \sum z_i^2 = 0 \) defines \( \phi_s \) which is not Morse on the complement but whose critical set is exactly the real part \( \mathbb{RP}^2 \subset \mathbb{CP}^2 \) of the projective plane which gives us an example of SBS lagrangian \( \mathbb{RP}^2 \).

All these show that the set \( \text{Crit} \phi_s \subset M \setminus D_s \) of critical for \( \phi_s \) points together with the lines of the gradient flow of \( \phi_s \) connecting the critical points of “finite type” (so we exclude infinite maximums of \( \phi_s \) at zeros \( D_s \)) form in a sense the “base set” \( B \) for SBS lagrangian submanifolds. More precisely, consider on the complement \( M \setminus D_s \) the set of critical points of \( \phi_s \).

**Definition 3.** For a holomorphic section \( s \in H^0(M, L) \) we define the base set \( B_s \subset M \setminus D_s \) as a subset of the complement \( M \setminus D_s \) which contains all critical points of \( \phi_s \) together with all finite trajectories of the gradient flow of \( \phi_s \) connecting pairs of critical points.

So if a pair \( x_k, x_l \) of the critical points is connected by a finite trajectory \( \gamma \) then the points of \( \gamma \) together with \( x_k \) and \( x_l \) must belong to \( B \).

Proposition 7 implies the following corollary: every SBS lagrangian submanifold \( S \) must lie in \( B \).

According to an old result of Milnor, [12], if \( \phi_s \) is strongly convex on the complex manifold \( M \setminus D_s \) then every Morse critical point \( x_i \) of \( \phi_s \) has the Morse index less or equal to \( n \) where \( n \) is the complex dimension of \( M \). Moreover, the negative subspace \( T_{-x} M \subset T_{x} M \) which correspond to incoming trajectories for the gradient flow must be isotropical w.r.t. our Kahler form \( \omega \). Suppose that \( S \subset M \setminus D_s \) is special w.r.t. \( s \) and suppose that \( S \) is compact. The restriction of \( \phi_s \) to \( S \) admits at least one maximum \( x_m \subset S \). Due to the arguments around Proposition 7 we know that \( x_m \) is a critical point of \( \phi_s \) on \( M \setminus D_s \). But it implies that \( x_m \) must have the Morse index equals exactly \( n \). The corresponding \( n \) -dimensional family of incoming trajectories of the gradient flow must be contained by \( S \). This implies the fact that the number of SBS lagrangian submanifolds is less or equal to the number of critical points of \( \phi_s \) on the Morse index \( n \). Thus we have sketched the proof of the following

**Theorem 5.** For a generic holomorphic section \( s \in H^0(M, L) \) the number of SBS lagrangian submanifolds is finite.

At the same time the restriction on the number of SBS submanifolds doesn’t automatically imply the existence of a single one, but the Morse theory for strongly convex functions \( \phi_s \) says that the existence theorem can be formulated after certain extension of the definition of SBS - submanifolds to SBS -cycles. This extension can be illustrated by the following

**Example.** Consider \( M = \mathbb{CP}^1 \) and \( L = \mathcal{O}(d) \). In this case holomorphic sections of \( L \) are presented by homogenous polynomials of degree \( d \) in variables \( [z_0 : z_1] \). In [13] one finds the separatrix trajectories for the gradient flow generated by generic holomorphic sections of \( \mathcal{O}(d) \) using the natural reformulation of the problem in terms of the polynomials. Taking the section \( s \in H^0(\mathbb{CP}^1, \mathcal{O}(d)) \) which corresponds to \( P_d = z_0^d + z_1^d, d > 2 \), we get for the function \( \phi_s \) three types of singular points on \( \mathbb{CP}^1 \): first, two minimal points at \([1 : 0]\) and \([0 : 1]\); second,
$d$ saddle points at $[1 : e^{2\pi ki/d}]$ and, third, $d$ infinite maximal points coming with zeros of $P_d$. Thus as the separatrix trajectories we have $d$ open segments $\gamma_k, i = 1, \ldots, d$, joining the poles and passing through the corresponding saddle point $[1 : e^{2\pi ki/d}]$. In the even case we have the coincidence of tangents to $\gamma_k$ and $\gamma_{k+d/2}$ at the poles $[1 : 0], [0 : 1]$ therefore we can combine closed smooth separatrix trajectories but it doesn’t happen if $d$ is odd. Moreover, it is not hard to see that for a generic polynomial of degree $d$ we should get essentially the same picture, but losing the symmetry of $P_d$ even in the case of even $d$ we should lost smooth combined trajectories: the coincidences of the tangents at the poles should be lost. This hints the way how the theory can be regularized: we must allow submanifolds with certain types of singularities and extend the considerations from SBS submanifolds to SBS cycles. In the case $\mathbb{C}P^1, \mathcal{O}(d)$ we allow loops with finite numbers of “corners” and it leads to the following description: there are $d(d-1)/2$ SBS cycles for a generic holomorphic section of $\mathcal{O}(d)$. In [13] one studies in details other fibers of the projection $p_1 : M_{SBS} \to \mathbb{P}H^0(\mathbb{C}P^1, \mathcal{O}(d))$ and describes the ramification structure of it.

Thus we shall consider not only lagrangian embedding but as well lagrangian immersions $S \subset M$, and, follow [8], we call these $S$ lagrangian cycles. As it was pointed out in [8] the Bohr - Sommerfeld condition can be imposed on lagrangian immersions as well as on smooth embedding: it is clear from the remarks after Definition 1 above since for immersed $S \subset M$ one can consider loops on $S$, discs with boundaries on the loops and the symplectic area of the discs which is an integral so it is correctly defined even in the case of non smooth loops.

The speciality condition can be extended to the immersions as well: we say that a BS lagrangian cycle is special w.r.t. a holomorphic section $s \in H^0(M_I, L)$ iff the calibration form $\text{Im} \rho_s = -I(d\text{ln}|s|)$ identically vanishes on $S$. For a singular point of $S$ it means that any tangent vector from the tangent cone annihilates the calibration form. As we have seen it could happen if the singular point of $S$ is a critical point of the Kahler potential $\phi_s$.

The main conjectures which can be formulated at the present time are the following

**Conjecture.** The number of special Bohr - Sommerfeld lagrangian cycles is invariant for generic holomorphic sections from $H^0(M_I, L)$.

If this is true then one gets certain system of lagrangian invariants for algebraic varieties.

On the other hand as it was mentioned above we are interested in the moduli spaces $M_{SBS}$ which now is defined as the moduli space of SBS lagrangian cycles. These moduli spaces admitsthe same as above projections to the projectivized spaces of holomorphic sections thus we can expect that these moduli spaces are finite dimensional Kahler manifolds which can be naturally compactified. This is our second conjecture.

### 4 Example: complex 2-dimensional quadric

In this section we discuss the first “non toy” example and present some technical arguments which should be exploited in general case of algebraic varieties.

Our $M$ is complex 2-dimensional quadric $Q$ realized either as the direct product $\mathbb{C}P^1 \times \mathbb{C}P^1$ or as a subvariety in $\mathbb{C}P^3$ defined by a quadratic polynomial. Our line bundle $L$ is taken to be $\mathcal{O}(1, 1)$ so the tensor product of two copies
of $\mathcal{O}(1)$ on both $\mathbb{CP}^1$’s lifted by the projections to the direct summands. The symplectic form is taken to be the direct sum of the lifted from the direct summands Fubini-Study forms. All details on Kahler geometry can be found in [10].

Recall first that $Q$ admits lagrangian 2-spheres: to see this let us fix coordinates $[x_0 : x_1]$ and $[y_0 : y_1]$ on the projective lines and consider the subset $S_0 = \{y_i = x_i\} \subset \mathbb{CP}^1 \times \mathbb{CP}^1$

which is obviously 2-dimensional sphere, and short calculation shows that it is lagrangian. Note that every lagrangian sphere automatically satisfies the Bohr-Sommerfeld condition.

Then holomorphic sections of $\mathcal{O}(1,1)$ are presented by linear combinations $\sum \alpha_{ij} x_i y_j$ and it is not hard to see that $S_0$ is special Bohr-Sommerfeld w.r.t. the section given by $s_0 = \{\alpha_{00} = \alpha_{11} = 1, \alpha_{01} = \alpha_{10} = 0\}$. Are there other $s_0$-SBS lagrangian spheres? The answer is negative: we take the function $\phi_{s_0}$ and find that the base set $B_s \subset Q \setminus D_{s_0}$ from Definition 3 above in this case coincides with $S_0$ so no other SBS spheres.

If we realize $Q$ as a quadratic surface in $\mathbb{CP}^3$ then the holomorphic sections of $\mathcal{O}(1,1)$ up to scale are presented by projective planes $H_s \subset \mathbb{CP}^3$ so the zero sets $D_s$ are presented by the intersections $H_s \cap Q$, see [10]. We have distinguished sections given by tangent planes so the intersections and consequently $D_s \subset Q$ for these cases are not smooth. For generic $s$ the subvariety $D_s$ is a smooth conic topologically equivalent to $S^2$, and for the tangent planes the corresponding sections $D_{s_0}$ are given by pair of intersecting projective lines so topologically these are equivalent to pair of $S^2$ tranversally intersecting at a point. Such a section is given by a reducible expression $\sum \alpha_{ij} x_i y_j = (a_0 x_0 + a_1 x_1)(b_0 y_0 + b_1 y_1)$, and two lines are $[-a_1 : a_0] \times \mathbb{CP}^1 \cup \mathbb{CP}^1 \times [-b_1 : b_0]$. Since the hermitian norm of a holomorphic section of (1) on $\mathbb{CP}^3$ has exactly two critical points it is not hard to see that the tensor product of two sections lifted to $Q$ from the direct summands must have exactly one critical point on $Q \setminus D_s$ therefore the base set $B_s$ for a reducible section consists of exactly one point. This implies that for reducible holomorphic sections there are no SBS lagrangian cycles.

Now we claim that for a generic holomorphic section $s \in H^0(Q, \mathcal{O}(1,1))$ there exists unique SBS lagrangian sphere. Indeed, if $s$ is a generic holomorphic section then the function $\phi_s$ on $Q \setminus D_s$ has exactly two critical points: minimal point $p_{\min}$ and “saddle” point $p_s$ with Morse index 2. Then the base set $B_s$ is again a 2-sphere: we take two dimensional family of incoming to $p_s$ trajectories whose tangents at $p_s$ span the negative definite subspace for the Hessian. Due to the Milnor remark ([12]), this subspace must be lagrangian, but the Liouville vector field $\text{grad}\phi_s$ preserves the lagrangian condition, so reversing the gradient flow of $\phi_s$ we get that the tangent space to $B_s$ at each point is lagrangian. Therefore we get a lagrangian sphere which is by the construction SBS w.r.t. the choosen section.

Thus the main strategy in construction of SBS cycles is the Morse theory of $\phi_s = -\ln|s|$ for holomorphic sections! It is related to the Morse theory of $|s|^2$ which is never Morse (we have $D_s$ as the subset of non isolated minima) but the case when $|s|^2$ has isolated non degenerated critical points on the complement $M \setminus D_s$ can be studied. Here we present the arguments for $Q$ when $D_s$ is smooth. Suppose that $s$ is relatively Morse so that $|s|^2$ is Morse on the complement $Q \setminus D_s$. 

16
The gradient of $|s|^2$ is outgoing near $D_s$: for any small tubular neighborhood of $D_s$ at the boundary points the gradient vectors see outside of the neighborhood.

It is well known that the subset of Morse functions is dense in the function space therefore $|s|$ can be deformed to a Morse function $\psi$ which has the same critical points on $Q \setminus D_s$. But let us choose $\psi$ in such a way that $\psi|_{D_s}$ is strictly Morse so it has only two critical points — maximal and minimal. Due to the fact that the gradient is outgoing near $D_s$ it follows that $\psi$ has exactly two additional critical points of indecies $0$ (the minimum on $D_s$) and $2$ (the maximum on $D_s$) to the set of critical points of $|s|$. If $\psi$ is strictly Morse we are done: the topology of $Q$ dictates that it must be $4$ critical points of indecies $0$, $2$, $2$, $4$, and the last pair is the desired $p_{\text{min}}$ and $p_s$ for our base set $B_s$. Another pair of related by the gradient flow critical points with the index difference $2$ can’t exist - otherwise we would get another cell in $H^2(Q, \mathbb{Z})$ (it is interesting that in this picture two cells from $H^2(Q, \mathbb{Z})$ are realized by holomorphic $(D_s)$ and lagrangian $2$ - spheres.

What happens if the section $s$ is reducible? Then $D_s$ takes not $2$ but $3$ critical points of $\psi$ and for $\phi_s$ on $Q \setminus D_s$ it remains unique critical point.

All these facts can be checked directly via computations for critical points of $|s|^2$: in coordinates $[x_0 : x_1], [y_0 : y_1]$: we take the expression

$$F_{\lambda, \mu} = | \sum \alpha_{ij} x_i y_j |^2 - \lambda(|x_0|^2 + |x_1|^2 - 1) - \mu(|y_0|^2 + |y_1|^2 - 1)$$

and solve the system

$$\frac{\partial F_{\lambda, \mu}}{\partial z} = 0$$

where $z = x_i, \bar{x}_i, y_j, \bar{y}_j$ (as usual for the computations of conditional extremums).

The system can be solved which gives us the set of critical points for the section $s$.

Summing up we get the following answer: for the case $M = Q$ with the standard Kahler structure and $L = \mathcal{O}(1, 1)$ with the standard hermitian structure, for the topological data $S \cong S^2$, $[S] = (1, -1) \in H^2(Q, \mathbb{Z})$, the moduli space $\mathcal{M}_{\text{SBS}}$ is naturally isomorphic to $\mathbb{CP}^3 \setminus Q'$ where $Q'$ is a quadric (projevively dual to $Q$). Another fact: this moduli space has a natural compactification isomorphic to $\mathbb{CP}^3$.

5 Final remark

At the end we would like to mention the following natural construction.

In [14] one constructs a natural $\mathbb{C}$ - bundle $\mathcal{L}$ with a fixed hermitian structure over the moduli space $\mathcal{B}_S$. On the other hand the projective space $\mathbb{P}H^0(M_1, L)$ is naturally endowed with the line bundle $\mathcal{O}(1)$ together with a hermitian connection $a$, whose curvature form is proportional to the Kahler form. In the presence of a complex structure $I$ on $M$ the bundle $\mathcal{L}$ is naturally endowed with a connection $A_I$ which depends on the choice of $I$. Therefore on the moduli space $\mathcal{M}_{\text{SBS}} \subset \mathbb{P}H^0(M_1, L) \times \mathcal{B}_S$ of SBS lagrangian cycles one naturally gets a bundle $\mathcal{E}$, which is the tensor product $p_1^* \mathcal{O}(1) \otimes p_2^* \mathcal{L}$ restricted to the moduli space $\mathcal{M}_{\text{SBS}}$. This bundle is endowed with a hermitian connection $\mathcal{A}$, given by the connections $a$ and $A_I$ on the tensor product components. Since the curvature of such a connection equals to the sum of the curvatures of $a$ and $A_I$ if the last one is trivial then it were possible to consider $\mathcal{A}$ as a connection whose curvature is
again proportional to the Kahler form (recall that we lift the Kahler structure from the projective space). Then we would come to the situation when all is ready for further “quantization”...

Recall the construction from [14]: in the direct product \( M \times \mathcal{B}_S \) one has the “incidence cycle”

\[ \mathcal{N} = \{(x,S) \mid x \in S\} \subset M \times \mathcal{B}_S \]

with natural projections \( q_i \) to the direct summands. Then the lift \( q_1^*(L,a) \) admits one-dimensional 0-cohomology space along the fibers of \( q_2 \) (this follows from the Bohr-Sommerfeld condition), therefore

\[ \mathcal{L} = R^0(q_2)_* q_1^*(L,a) \to \mathcal{B}_S \]

is a line bundle. Over point \( S \subset \mathcal{B}_S \) the fiber is spanned by the section \( \sigma_S \), consequently there is the natural \( U(1) \)-action, generated by the \( U(1) \)-action on the prequantization bundle \( L \): element \( e^{ic} \in U(1) \) just twists all \( \sigma_S \mapsto e^{ic}\sigma_S \).

The connection \( A_I \) on this bundle \( \mathcal{L} \to \mathcal{B}_S \) can be constructed as follows. For any point \( S \subset \mathcal{B}_S \) consider the Darboux-Weinstein neighborhood \( \mathcal{O}_{DW}(S) \subset M \), so all close to \( S \) points of the moduli space \( \mathcal{B}_S \) are given by graphs \( S_f = \Gamma(df) \) of the differential for functions \( f \in C^\infty(S, \mathbb{R}) \) and the corresponding covariantly constant sections of the restrictions \( (L,a)|_{S_f} \) are given by twisting of the form \( e^{i(f+c)}\sigma_S \). If there is a natural way how to choose for any function \( f \) a representative from the class \( f + c, c \in \mathbb{R} \) universally then it gives a local section of the bundle \( \mathcal{L} \) over the neighborhood \( S \subset \mathcal{B}_S \). In general this universal choice doesn’t exist, however in our case when a compatible complex structure on \( M \) is fixed therefore we have a fixed riemannian metric \( g \), then we can take the restriction \( g|_S \) and define the corresponding volume form \( d\mu(g|_S) \). Then the natural condition \( \int_S f d\mu(g|_S) = 0 \) specifies an \( f \) from the corresponding class \( C^\infty(S, \mathbb{R})/\text{const} \) (and this choice doesn’t depend on the orientation of \( S \)). Since for \( S \) and \( S_f \) the restrictions of \( g \) a priori give different volume forms it doesn’t directly lead to a local section of \( \mathcal{L} \) but for each point it gives a horizontal subspace in the corresponding point of the tangent bundle \( T(\text{tot}\mathcal{L}) \), and the universal \( U(1) \)-action lifts this horizontal subspace to a horizontal distribution which is by the construction \( U(1) \)-invariant. This is our hermitian connection \( A_I \).

Now the problem is to calculate the curvature for this connection \( A_I \). If it is flat then we would get on the moduli space \( \mathcal{M}_{SBS} \) a hermitian line bundle together with a hermitian connection whose curvature form is proportional to the Kahler form.

The work on the problems listed above is in progress, so one hopes that new results will be found soon in special Bohr-Sommerfeld geometry.

References:

[1] Yu. I. Manin, “Foreword for the 3d volume”, A.N. Tyurin’s selected papers, (in Russian), Moscow - Izhevsk, 2004;
[2] M. Kontsevich, “Homological algebra of mirror symmetry”, Proceedings of ICM (Zurich, 1994), Birkhouser, Basel 1995, pp. 120 - 139;
[3] A.N. Tyurin, “Geometric quantization and mirror symmetry”, arXiv: math/9902027v1;
[4] N. Hitchin, “Lectures on special lagrangian submanifolds”, Winter school on Mirror Symmetry (Cambridge MA, 1999), AMS/IP Stud. Adv. Math, 23, AMS 2001, Providence, pp. 151 - 182;
[5] A. Strominger, S.-T. Yau, E. Zaslow, “Mirror symmetry is T-duality”, Winter school on Mirror Symmetry (Cambridge MA, 1999), AMS/IP Stud. Adv. Math, 23, AMS 2001, Providence, pp. 333 - 347;

[6] D. Auroux, “Mirror symmetry and T-duality in the complement of an anticanonical divisor”, J. Gokova Geom. Topol. (2007), pp. 51 - 91;

[7] N.A. Tyurin, “Special lagrangian fibrations on the flag variety”, Theor. and Math. Phys. 167: 2 (2011), pp. 193 - 205;

[8] A.L. Gorodentsev, A.N. Tyurin, “Abelian lagrangian algebraic geometry”, Izvestiya math. 65: 3 (2001), pp. 15 - 50;

[9] N.A. Tyurin, “Geometric quantization and algebraic Lagrangian geometry”, LMS, Lecture Note series 338, Cambridge 2007, pp. 279 - 318;

[10] P. Griffiths, J. Harris, “Principles of Algebraic geometry”, NY, Wiley, 1978;

[11] R. Harvey, H. Lawson, “Calibrated geometries, Acta Math., 148 (1982), pp. 47 - 157;

[12] Y. Eliashberg, "Topological characterization of Stein manifolds of dim > 2", Internat. J. Math., 1, no. 1, pp. 29 -46 (1990);

[13] A.N. Tyurina, "Special Bohr-Sommerfeld lagrangian cycles on $\mathbb{CP}^1$", Batchelor diploma, NRU HSE (Moscow, 2016), in preparation;

[14] N.A. Tyurin, “Algebraic lagrangian geometry: three geometric observations”, Izvestiya math. 69 : 1 (2005), pp. 179 - 194.

Uspeniev den', Dubna