Renormalization in Minkowski space-time

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The functional renormalization group method is applied for a scalar theory in Minkowski space-time. It is argued that the appropriate choice of the subtraction point is more important in Minkowski than in Euclidean space-time. The parameters of the cutoff theory, defined by a subtraction point in the quasi-particle domain, are complex due to the mass-shell contributions to the blocking and the renormalization group flow becomes more involved. The Landau poles are avoided when the parameters are complexified. The absence of the UV pole owing to the marginal parameters makes the scalar theory asymptotically free in four dimensions. However the continuation of the trajectory beyond the regularized IR pole at the UV-IR crossover in the phase with spontaneously broken symmetry is possible only if the non-trivial saddle points to the blocking are taken into account.

I. INTRODUCTION

Quantum Field Theories are defined in the Minkowski space-time [1–4], and their renormalization, namely the removal of their UV divergences, has been developed accordingly [5–7]. The availability of simpler regulators in Euclidean space-time [8] and the similarity of the introduction of the renormalized parameters in Quantum Field Theory with critical phenomena [9, 10] led to the recasting of the renormalization group method in imaginary time and developing it further in that context. For instance, the functional formalism of the renormalization group method to find non-perturbative solutions of Quantum Field Theory models, have exclusively been presented in Euclidean space-time [11–17]. The goal of this work is to assess the changes in the renormalization group method within the framework of
the $\phi^4$ scalar model, induced by returning to Minkowski space-time.

The renormalization group strategy is usually introduced by splitting the degrees of freedom into the sets of IR and UV variables, separated by the cutoff, and after that the effective dynamics is sought for the IR variables by eliminating the UV one. The power of this method stems from solving an overdetermined problem [18]: The IR cutoff theory is supposed to reproduce all expectation values of the IR degrees of freedom. Hence each expectation value represents a constraint on the cutoff theory. Rather than dealing with such an involved constrained dynamical system an intuitively appealing procedure of statistical mechanics, the blocking [19], is followed in constructing the effective IR dynamics. This idea is feasible if the effective action is not too complicated. The overdetermined nature of this problem comes from this step, the description of all expectation value by a restricted set of parameters of the effective action.

The way to get around this difficulty is the exploitation of a peculiar feature of the physical laws, their dependence on the scales. It seems that the complexity of the physical phenomena can drastically be reduced by viewing them in layered fashion and by dealing with the layers, organized by their scale, one-by-one, in an iterative manner. Each layer has its own dynamics which can be characterized by a restricted set of parameters. The result is a scale dependent set of effective parameters, the renormalized trajectory. The cutoff theory, a model equipped with an UV cutoff, describes the dynamics in terms of the parameters, belonging to the layer at the scale of the cutoff. The systematic reduction of the degrees of freedom by eliminating the of the particle modes around the cutoff scale can easily be implemented by the help of these parameters.

Such a strategy produces a reasonable approximation if the retained parameters characterize a large number of expectation values at the cutoff scale. These parameters are given by certain vertex functions, one particle irreducible Green functions of the IR theory, evaluated at appropriately chosen energy-momentum of the external legs, called subtraction point. The name originates from the multiplicative renormalization group where a renormalization conditions is imposed for each renormalizable parameter with the goal to define the counterterms which subtract the UV divergences. The procedure of the functional renormalization group scheme is different, one follows a number of parameters, beyond the class of renormalizable one and the subtraction point is defined by the variable of the blocked action, an IR field configuration.
The blocking in Minkowski space-time is addressed in this work within the framework of the functional renormalization group method to be introduced in section II. The details of building up an approximative evolution equations are presented in section III in several steps: We start with the renormalization group flow in Euclidean space-time, defined by an $O(d)$ invariant regulator, the scheme $E_d$. After that the symmetry of the regulator is reduced to $O(d-1)$ in the scheme $E_{d-1}$ to make the return to Minkowski space-time smoother. The first Minkowski space-time renormalization group scheme $M_{d-1}$ is based on the same Wick rotation invariant subtraction point, imposed at vanishing energy, as $E_{d-1}$ to minimize the changes generated by the inverse Wick rotation. But the applicability of this scheme is questionable due to a qualitative difference between theories in Minkowski and Euclidean in space-time, namely the oscillatory functions, appearing in the former in the place of the monotonic exponential functions of the latter render the choice of the subtraction scale more important in weakly coupled Minkowski theories than in their Euclidean counterparts. Hence further two improved subtraction schemes are introduced where the subtraction point is placed within the quasi-particle domain. The quasi-particles are defined by the partially resummed propagator containing a complex self energy term and the quasi-particles domain covers the region of the energy-momentum space around the peak of the absolute magnitude of the propagator. The schemes $M_{pw}$ and $M_{L}$ are defined by the help of a monochromatic plane wave and a packet of Lorentzian modes, respectively.

The main difference between the blocking in imaginary and real time arises from the presence of the mass mass-shell singularities in the latter. The mass-shell denotes the null-space of the linear operator of the equation of motion of free, stable particles. The propagator is rendered finite on the mass-shell by Feynman's $i\epsilon$ prescription, a regularization of the mass-shell singularities by the introduction of the imaginary part $-i\epsilon$ to the bare mass square which amounts to the spreading of the contribution of a discrete point over the continuous part of the spectrum. One may expect that such a regularization can be avoided in theories with gapless excitation spectrum since the decay of the quasi-particles induces finite imaginary part to the self energy. However the regularization of the mass-shell singularities is needed even in that case because it leaves behind a finite trace, irreversibility [20]. The dissipative forces are canceled in the usual formalism of Quantum Field Theory, based on the transition amplitude between pure states, however we keep $\epsilon$ finite during the integration of the evolution equations and perform the limit $\epsilon \to 0^+$ at the end of the calculation to
pave the way for the later application of the renormalization group method within a more appropriate formalism which tackles the issue of dissipation.

It is well known that the unitarity of the time evolution makes certain values of the Green functions complex. Therefore the running coupling constants, given in terms of the vertex functions, may be complex, too. The internal logic of the renormalization group method is that the theory is defined by the same set of parameters at any scale and the scale-dependence is reflected only in the numerical values of these parameters. Thus the complexification of the IR parameters, guided by the vertex functions, leads to a reduplication of the real running parameters in theories, defined in Minkowski space-time.

The comparison of different blocking schemes sheds a new light onto universality. This concept has been introduced to explain the independence of physical observations, carried out at finite scale, from the vast majority of microscopic parameters and can be summarized by the equivalence of the relevant (negative critical exponent) and the renormalizable (allow the removal of the cutoff) operators. The regulator consists of irrelevant operators with prescribed tree-level cutoff-dependence hence the change of the blocking scheme should preserve the universality class. However the argument holds for analytic changes of the renormalization scheme and the mass-shell singularity, emerging during the inverse Wick rotation, may violate this condition. Section IV contains some details about this issue by looking into the asymptotic scaling around the Gaussian fixed point. The scheme $M_{pw}$ produces singular scaling laws but the other schemes preserve the Gaussian fixed point with a linearizable evolution around it. While the critical exponents of the linearized scaling laws around four dimensions are identical in these schemes the scaling operators display a scheme dependence.

Despite the recovery of a formal universality for the leading order critical exponents the higher loop corrections are non-universal and induce an interesting effect in four dimensions. There are actually no marginal operators in either space-time because the higher loop contributions always generate non-vanishing contributions to the critical exponents. The classically marginal operator, the quartic vertex, receives negative contribution to its critical exponent when the parameters are real and becomes weakly irrelevant, reflecting the triviality of scalar theories. The particular way the degeneracy of the leading order critical exponents of the real and the imaginary components of the quartic coupling strength is broken leads to a weakly relevant scaling law and makes the scalar theory asymptotically
free. The triviality, the impossibility of extending the quartic interaction beyond the scale of UV Landau pole, is avoided by the complexification of interaction strength.

Some global issues of the renormalization group flow, such as the phase structure and crossovers, are addressed in section V. The two separatrices are located, one of them belongs to the spontaneous symmetry breaking with \( \phi \) as order parameter and the UV-IR crossover is found rather sharp in the symmetry broken phase. An additional crossover is found in the scheme \( M_{pw} \) where the singular, non-linear scaling laws go over a linearizable UV scaling as the cutoff is lowered. However the non-linear scaling installs such a linearizable scaling that the interaction is suppressed at and beyond the UV-IR crossover as \( \epsilon \to 0 \).

This work opens more questions than answers. Some of them are listed in section VI.

II. BLOCKING

Our blocking step is infinitesimal and consists of the decrease of the cutoff, \( k \to k - \Delta k \). The field variable,

\[
\phi_p = \int d^d x e^{i p \cdot x} \phi_x, \tag{1}
\]

in the Fourier space is split into the sum \( \phi \to \phi + \varphi \), where \( \phi (\varphi) \) belongs to the IR, under the cutoff (UV, beyond the cutoff) component. The change of the action \( S_k[\phi] \), corresponding to the cutoff \( k \), is given by the blocking relation,

\[
e^{-S_{E,k-\Delta k}(\phi)} = \int D[\varphi] e^{-S_{E,k}[\phi+\varphi]},
\]

\[
e^{i S_{M,k-\Delta k}(\phi)} = \int D[\varphi] e^{i S_{M,k}[\phi+\varphi]}, \tag{2}
\]

in Euclidean and Minkowski space-time, respectively (\( \hbar = 1 \)). The resulting one-loop equations are

\[
\dot{S}_E[\phi] = -\frac{k}{2} \text{Tr} \ln \left[ \frac{\delta^2 S_E}{\delta \phi \delta \phi} \right],
\]

\[
\dot{S}_M[\phi] = -i \frac{k}{2} \text{Tr} \ln \left[ \frac{\delta^2 S_M}{\delta \phi \delta \phi} \right], \tag{3}
\]

where \( \dot{f} = \partial_\tau f, \tau = \ln(k/k_{in}) \), \( k_{in} \) being the initial value of the cutoff. The initial conditions, imposed on the initial, bare theory, are \( g_n(k_{in}) = g_{in,n} \). These equation are exact since the limit \( \Delta k \to 0 \) suppresses the higher loop contributions. An IR field-independent constant, arising from the complex Fresnel integral, is ignored in the second line. Note that the
complex conjugate of the action follows an inverted evolution in real time, more precisely $S_M[\phi]$ and $-S_M^*[\phi]$ obey the same evolution equation, reminiscent of the time inversion of the equation of motion.

The action is truncated onto the local potential approximation,

$$S_{E,k}[\phi] = \int d^d x \left[ \frac{1}{2}(\partial_0 \phi_x)^2 + \frac{1}{2}(\partial_x \phi_x)^2 + U_k(\phi_x) \right],$$

$$S_k[\phi] = \int d^d x \left[ \frac{1}{2}(\partial_0 \phi_x)^2 - \frac{1}{2}(\partial_x \phi_x)^2 - U_k(\phi_x) \right],$$

in this work and the potential

$$U_k(\phi) = \sum_{n=0}^{N} \frac{g^{2n}(k)}{2n!} \phi^{2n}$$

is restricted to quartic terms, $N = 2$, in the numerical calculation and its evolution is found by evaluating the blocking relations at the subtraction point, defined by the IR field $\phi^{(IR)}_{\omega,p} = \Phi \delta_{p,0} (\delta^{(\gamma)}_{\omega,\omega_r} + \delta^{(\gamma)}_{\omega,-\omega_r})/2$, a mode packet with a natural Lorentzian spread,

$$\delta^{(\gamma)}_{\omega} = \frac{\gamma}{\pi} \frac{1}{\omega^2 + \gamma^2}.$$  

The expression of the IR field in the space-time is $\phi^{(IR)}_{\omega,x} = \Phi \chi_t$, where

$$\chi_t = i \frac{\gamma}{2\pi} e^{-\gamma|t|} \left( \frac{e^{i\omega_r t}}{\omega_r + i\gamma} - \frac{e^{-i\omega_r t}}{\omega_r - i\gamma} \right).$$

The limit $\gamma \to 0$ corresponds to a monochromatic oscillatory field, $\phi^{(IR)}_{\omega,p} = \Phi \delta_{p,0} (\delta_{\omega,\omega_r} + \delta_{\omega,-\omega_r})/2$ and $\chi_t = (e^{i\omega_r t} + e^{-i\omega_r t})/2$.

Few remarks are in order about the evolution equations: (i) One should retain the possible non-trivial saddle points of the blocking (2). Being space-time dependent they induce non-local contributions to the blocked action. Hence the use of the local potential approximation, the projection of the evolution equation onto the functional space onto (4), implies the omission of the non-trivial saddle points. (ii) The representation of the local potential by a polynomial of finite order can only be justified for weakly coupled theories. (iii) Note that the mass square, $g_2$, must have non-vanishing imaginary part in Minkowski space-time to regulate the mass-shell singularities of the loop integrals. (iv) The IR field is the same on both sides of the evolution equation, (2), hence its cut-off-dependence does not contribute to the left hand side of eqs. (3). (v) We need $\text{Im}(g_N) < 0$ to assure the convergence of the path integral. The boundedness of the energy from below requires $\text{Re}(g_N) \geq 0$ in a theory with real
parameters. The issue of stability becomes more involved with complex parameters because the quasi-particles of finite life-time can not destabilize the system with unbounded energy and one has to take into account the radiation energy loss to the environment to construct the asymptotic states. This problem is postponed to a later time and the convergence of the path integral, \( \text{Im}(g_N) < 0 \), is used in this calculation as the only restriction on the parameters. (vi) The return to real time in the renormalization group equation poses an unexpected problem, the difficulty of maintaining the boost invariance in non-perturbative schemes [21]. This issue is circumvented here by relying non-relativistic cutoff and leaving the issue of a possible restoration of the boost symmetry in the renormalized theory for a later time. To minimize the symmetry breaking effects of the cutoff the running action is projected back to the symmetric form (4) after the blocking.

III. RENORMALIZATION SCHEMES

The power of the renormalization group method arises from trying to solve an over-determined problem [18]. The IR cutoff theory should reproduce the expectation value of all observables, constructed by the help of the IR degrees of freedom. Hence each expectation value represents a constraint on the cutoff theory. Rather than dealing with such a complicated system of equations the evolution equation (3) is used to introduce effective, cutoff-dependent parameters for the cutoff action whose number is significantly smaller than the number of IR expectation values. Such a strategy produces a reasonable approximation only if the retained effective parameters, limited in number, characterize a large number of IR expectation values at the cutoff scale. The cutoff dependent, running parameters of the blocked action (4) characterize the physics at the scale of the cutoff and are given by certain vertex functions, one particle irreducible Green functions, of the IR theory, evaluated at an appropriately chosen energy-momentum values of the external legs, usually called the subtraction points.

To find the optimal subtraction point we consider a weakly coupled theory where the IR quasi-particles have long life-time. The dressed propagator develops poles along the complex quasi-particle dispersion relation, \( \omega_{qp}(p) \), sketched in Fig. 1 on the complex energy plane. The propagator is a multiplicative factor in the integrand of the finite loop integral and the vertex function is dominated by the energy range \( \omega \sim \text{Re} \omega_{qp}(p) \) (\( \omega \sim \text{Im} \omega_{qp}(p) \)) in a
FIG. 1: A quasi-particle pole, denoted by the heavy dot, on the physical sheet of the complex energy plane of the dressed propagator. The typical dependence of the propagator on the Minkowski and the Euclidean energy is shown by the dashed lines along the real and imaginary axes, respectively.

Minkowski and an Euclidean theory. Hence the subtraction point should be chosen to find the best approximation for the self energy in this regions. The imaginary part of the quasi particle energy is generated by the interactions therefore $|\Im \omega_{qp}(p)| < |\Re \omega_{qp}(p)|$. Thus the subtraction point should be $\omega = \Re \omega_{qp}(p)$ ($\omega = \Im \omega_{qp}(p)$) in Minkowski (Euclidean) models where the quasi-particle peak is narrow (wide), rendering the choice of the subtraction point important (less important).

The qualitatively novel feature of the blocking in real time is that the parameters of the blocked action develop imaginary part. In fact, a model with real parameters except the mass, $\Im g_2 = -\epsilon$, describes unitary dynamics and the optical theorem assures the emergence of a complex self energy in the kinematical region allowing the mass-shell particle modes to contribute to the internal lines of the corresponding Feynman graph. The emerging complex mass in turn generates imaginary part to the other parameters during the further blocking. Thus we have twice as many real parameters, $g_n = g_{nr} + ig_{ni}$, in the action and even the qualitative, topological features of the flow diagram together with the fixed point and the phase structure may be changed.

We compare below the renormalization group flow obtained by five different schemes in $d$-dimensional space-time, each of them defined by the cutoff and the subtraction point.

A. $O(d)$ invariant Euclidean theory with $\omega_r = 0$

The scheme $E_d$ has the sharp $O(d)$ symmetrical cutoff, suppressing the field variable beyond a sphere of radius $k$ in the momentum space, $\phi_p \neq 0$ for $p^2 < k^2$ and the IR field (1) at the subtraction point is given by $\omega_r = 0$. The renormalization group flow, generated by
the first equation of (3) is given by the Wegner-Houghton equation [11],

$$\dot{U} = -\frac{1}{2} k^d \alpha_d \ln(k^2 + U'),$$  \hspace{1cm} (8)

where $\alpha_d = 2\pi^{d/2}/(2\pi)^d \Gamma(d/2)$ and $U = U(\Phi)$. The identification of the same powers of $\Phi$ defines the beta functions,

$$\dot{g}_n = \beta_n = -\frac{1}{2} k^d \alpha_d \partial_\Phi \ln(k^2 + U') \bigg|_{\Phi=0}. \hspace{1cm} (9)$$

The distinguishing feature of this scheme is the vanishing of the higher orders of the gradient expansion and the non-local terms in the blocked action. These terms receive contributions during the blocking from the loop integral, containing the product of propagator with variable shifted by an external momentum,

$$\int \frac{d^d p}{(2\pi)^d} \cdots D^{UV}_p D^{UV}_{p+q} \cdots, \hspace{1cm} (10)$$

where $D^{UV}_p = \chi^{UV}_p / (p^2 + g_2)$ represents the particle modes to be eliminated, the characteristic function $\chi^{UV}_p = 1$ within the shell $k - dk < |p| < k$ and is vanishing otherwise. The shell of width $\Delta k$ is curved and the common region of two such shells which are shifted with respect to each other by the external momentum $q$ is $O(\Delta k^2)$ and is negligible in the limit $\Delta k \rightarrow 0$. This argument is reminiscent of the absence of nesting in case of curved Fermi surface in solids, the role of the Fermi surface and the temperature being played by the "Wegner-Houghton surface", $p^2 = k^2$ and $\Delta k$, respectively.

B. $O(d-1)$ invariant Euclidean theory with $\omega_r = 0$

We use in the scheme $E_{d-1}$ an $O(d-1)$ symmetrical cutoff in an Euclidean theory, restricting the IR field within a cylinder in the momentum space $p = (\omega, \mathbf{p})$, i.e. $\phi_{\omega, \mathbf{p}} \neq 0$ if $|\mathbf{p}| < k$ and $|\omega| < \Omega_k$, $\Omega_k$ being a suitably chosen function of $k$. The evolution equation, based on the zero momentum subtraction point, realized by the evaluation of the first equation of (3) at $\phi_x = \Phi$, is

$$\dot{U} = -\frac{\alpha_{d-1}}{2\pi} \left[ k^{d-1} \int_0^{\Omega_k} d\omega (\omega^2 + k^2 + U'') + \dot{\Omega}_k \int_0^k dp \int_0^{\Omega_k} d\omega (\omega^2 + p^2 + U'') \right]. \hspace{1cm} (11)$$
The integrals can easily be carried out for integer dimensions, we present the result for \( d = 4 \),

\[
\dot{U} = -\frac{1}{4\pi^3} \left[ k^3 \left( 2\omega_k \arctan \frac{\Omega_k}{\sqrt{k^2 + U^\prime}} + \Omega_k [-2 + \ln(\Omega^2_k + k^2 + U'')] \right) \\
+ \frac{k}{3} \dot{\Omega}_k \left( 2(\Omega^2_k + U'') - \frac{2k^2}{3} - 2\frac{(\Omega^2_k + U'')^2}{k} \arctan \frac{k}{\sqrt{\Omega^2_k + U''}} \\
+ k^2 \ln(\Omega^2_k + k^2 + U'') \right) \right].
\]

(12)

In the case of a momentum cutoff, \( \Omega_k = \infty \), the particle modes of a given momentum are completely eliminated by the blocking and one finds

\[
\dot{U} = -\frac{k^3 \omega_k}{2\pi^2}
\]

(13)

with \( \omega_k = \sqrt{k^2 + U''} \), \( \Re(\omega_k) \geq 0 \), the product of a kinematical factor and the zero point fluctuation energy of the eliminated modes.

We might have simplified the blocking by using only momentum cutoff because the energy integrals of the loop-expansion are finite. However as soon as one allows the \( \phi \)-dependence in the wave function renormalization constant or in the possible higher order terms of the gradient expansion in the action the energy integrals diverge and require \( \Omega_k < \infty \). These divergences are quantum mechanical, i.e.
are proportional to a positive power of the Planck constant and are independent of the number of degrees of freedom. They are reminiscent of the divergences of quantum mechanics with operator mixing, in the presence of anharmonic terms in the Hamiltonian, containing both \( x \) and \( p \) [22].

C. Minkowski space-time with \( \omega_r = 0 \)

The scheme \( M_{d-1} \) is identical to the scheme \( E_{d-1} \) except that it is for Minkowski space-time. One finds the evolution equation

\[
\dot{U} = i^{d-1} \alpha \left[ k^{d-1} \int_0^{\Omega_k} d\omega \ln(\omega^2 - \omega_k^2) + \dot{\Omega}_k \int_0^k dp \int_0^{p^2 - U''} d\omega \ln(\Omega_k^2 + \omega^2 - U'') \right],
\]

(14)
in particular

\[
\dot{U} = \frac{i}{4\pi^3} \left[ k^3 \left( 2\omega_k \text{arctanh} \frac{\Omega_k}{\omega_k} + \Omega_k [-2 + \ln(\Omega^2_k + \omega^2_k)] \right) \\
+ \frac{k}{3} \dot{\Omega}_k \left( -2(\Omega^2_k - U'') \right) \\
+ \frac{2}{3} k^2 + 2\frac{(\Omega^2_k - U'')^2}{k} \arctanh \frac{k}{\sqrt{\Omega^2_k - U''}} + k^2 \ln(\Omega^2_k - \omega_k^2) \right]
\]

(15)
for the four dimensional theory. This reduces to

\[ \dot{U} = -\tau_{rad} \frac{k^3 \omega_k}{2\pi^2} \frac{\omega_k}{2}, \]

in the case of a momentum cutoff, \( \Omega_k = \infty \), differing from (16) by the presence of a sign factor, \( \tau_{rad} = -\text{sign}(\text{Im}(U^\prime)) \), representing a formal direction of time.

The particle modes are fully eliminated when the choice \( \Omega_K = \infty \) is made and such a blocking preserves the unitarity. Since there are no real particle modes at the subtraction point at vanishing energy the parameters of the blocked action remain real, more precisely the imaginary part of \( g_2 \) remains infinitesimal. The complex nature of the evolution equation (15) is therefore only a cutoff effect, generated by a finite \( \Omega_k \).

The comparison of eqs. (13) and (16) reveals a characteristic difference between the imaginary and real time beta functions, obtained by expanding the evolution equation in the power of the field and writing it as \( \dot{g}_n = \beta_n(g) \). These functions govern the dressing of the theory by modes within a finite scale interval and as such, they are continuous functions of the running parameters in imaginary time [23, 24]. This is not the case anymore in real time dynamics where the change of the sign of \( g_2 \) can be interpreted either as a time inversion or an exchange of the UV and the IR directions and induces a finite discontinuity in the beta functions.

## D. Subtraction point in the on-shell region

The scheme \( M_L \) has the same cutoff as the previous case however the subtraction point is chosen to be at at the maximum of the quasi-particle propagator, \( \omega_r = \sqrt{|k^2 + \text{Re}(U^\prime)|} \geq 0 \), taking place at the scale of the cutoff. The monochromatic limit, \( \gamma \to 0 \), defines the plane-wave scheme \( M_{pw} \).

The contribution of the potential energy of the action (4), evaluated at the IR field, \(-L^{d-1}u(\Phi)\) yields the energy density

\[ u(\Phi) = \sum_n \frac{g_n u_n}{n!} \Phi^n, \]

where \( L \) denotes the spatial size of the quantization box and

\[ u_n = \int dt \chi_t^n. \]
The expansion of the right hand side of the evolution equation (3) in $\Phi$ yields

$$\dot{u}(\phi_0) = i \frac{k^{d-1}}{2} \alpha_d \left[ \int \frac{d\omega}{2\pi} \ln(\omega^2 - \omega_k^2) ight. \right.$$

$$\left. - \sum_{n=1}^{\infty} \frac{1}{n} \int dt_1 \cdots dt_2 \cdots \int dt_{2n} D_{t_{2n}-1}^{(k)} \cdot \prod_{i=1}^{n} \left[ \Sigma_{t_{2i}-t_{2i-1}} - \Sigma_{t_{2n-1}-t_{2n}} \right] \right],$$

(19)

involving the free propagator for the modes to be eliminated in the absence of the IR field,

$$D_{t}^{(k)} = \int d\omega \frac{e^{-i\omega t}}{2\pi \omega^2 - \omega_k^2} = i \text{sign}(g_2 i) \frac{e^{i\text{sign}(g_2 i)\omega_k |t|}}{2\omega_k}$$

(20)

and the self energy $\Sigma_{t,t'} = \delta_{t,t'} g_4 \phi_t^{(IR)} / 2$. The Feynman graphs, contributing to the first two orders are depicted in Fig. 2.

The identification of the $O(\Phi^2)$ terms of (19) yields the evolution equations for $g_2$,

$$\dot{g}_2 = \text{sign}(g_2 i) \frac{k^{d-1}}{4\omega_k} g_4.$$  

(21)

The graph of Fig. 2 (a) is independent of the external energy-momentum, the choice of the subtraction point, and remains real for real $g_4$ and infinitesimal $g_2 i$. The $O(\Phi^4)$ terms yield the evolution equation

$$\dot{g}_4 = \alpha_d \frac{3k^{d-1}}{8\omega_k^2} - \frac{g_4^2}{4\omega_k} B_4$$

(22)

with

$$B_4 = i \frac{1}{u_4} \int dt dt' \chi_t^2 e^{2i\text{sign}(g_2 i)\omega_k |t-t'|} \chi_{t'}^2.$$  

(23)

The quasi-particle poles of the two propagators of Fig. 2 (b) are close for $\omega \sim -\omega_r$ and a finite, non-negligible imaginary part is generated for $\dot{g}_4$ even for infinitesimal $g_2 i$. The loop-integrals are easy to carry out, leading to

$$B_4 = 32 \text{sign}(g_2 i) \begin{cases} i \frac{\omega_k}{2g_2 r} - \frac{1}{\omega_k} & g_2 r > -\omega_r^2, \\ \frac{3}{2\omega_k} & g_2 r < -\omega_r^2. \end{cases}$$  

(24)

in the plane wave limit $\gamma \rightarrow 0$ of the scheme $M_{pw}$. The singular $1/g_2 i$ term emerges in a manner similar to the nesting of the Fermi surfaces, the common origin being a perturbative loop-integral, the right hand side of eq. (22) in our case, where the mass-shell poles of
FIG. 2: Feynman graphs to $\dot{g}_2$ (a) and $\dot{g}_4$ (b). External line frequency is $\omega_e = \sigma_e \omega_r$ with $\sigma_e = \pm 1$ and the loop frequencies are are shown at the internal lines.

Internal lines approach each others. The coefficient $B_4$ turns out to be

$$B_4 = \frac{2\gamma(\omega_r^2 + \gamma^2)(\omega_r^2 + 4\gamma^2)}{3\omega_r^8 - 6\omega_r^6 \gamma^2 + 7\omega_r^4 \gamma^4 + 32\gamma^8} \times \left[ i \left( \omega_r^4 + \omega_r^2 \gamma (2 \text{sign}(g_{2i})\omega_k - \gamma) - 2(\text{sign}(g_{2i})\omega_k + i\gamma)^2 \gamma^2 \right)^2 \right. $$

$$+ \frac{B_n}{2(\text{sign}(g_{2i})\omega_k - i\gamma)(\text{sign}(g_{2i})\omega_k + i\gamma)^2 - \omega_r^2](\omega_r^2 + 4\gamma^2)\gamma(\omega_r^2 + \gamma^2)^5} \right] \quad (25)$$

with

$$B_n = \omega_k^2(3\omega_r^8 - 6\omega_r^6 \gamma^2 + 7\omega_r^4 \gamma^4 + 32\gamma^8) + i\text{sign}(g_{2i})\omega_k(9\omega_r^8 \gamma - 18\omega_r^6 \gamma^3 + 13\omega_r^4 \gamma^5 - 40\omega_r^2 \gamma^7 + 64\gamma^9)$$

$$- 2(\omega_r^2 + 4\gamma^2)(\omega_r^2 - \omega_r^2 \gamma^2 + 2\gamma^4)^2 \quad (26)$$

in the scheme $M_L$.

The flow of the simplest scheme $E_d$ is well known, it serves as a starting point. The scheme $E_{d-1}$ is to demonstrate the impact of the non-relativistic cutoff in the traditional Euclidean setting. The schemes $E_{d-1}$ and $M_{d-1}$ are defined by the same subtraction point, their difference reflects the effects of the Wick rotation. The subtraction point of these schemes is set in the virtual domain, at vanishing energy-momentum as opposed to the schemes $M_{pw}$ and $M_L$ where it is placed in a kinematic region which is dominated by the quasi-particle modes. The IR field is a monochromatic plane wave, placed at the spectral peak for the former and a Lorentzian spread version for the latter.
E. Singular beta functions

Before closing this section we comment a peculiarity of the last two schemes, with subtraction point in the quasi particle regime. The beta functions represent the change of the parameters of the theory during an infinitesimal change of the cutoff, $\Delta g_n = \Delta k \beta_n$ and the continuity of the renormalized trajectory, $\Delta g_n \to 0$ as $\Delta k \to 0$, is a central point of the renormalization group method and can be proven rigorously in Euclidean space-time [23, 24]. It explains for instance that despite the regularity of the physical laws at any given scale the singularities of a critical system do arise from the diverging scale window between the microscopic cutoff. The $1/|g_{2i}|$ singularity in the non-Gaussian beta function of the scheme $M_{pw}$ is generated by the mass-shell singularities as $g_{2i} \to 0$.

The smearing of the IR field in the scheme $M_L$ removes this divergence but the beta functions remain non-differentiable at $g_{2i} = 0$. Such a non-differentiability poses further serious problems. The change of the sign indicates the flipping of the direction of the time, a rather surprising phenomenon of the blocking. An even more disturbing feature is that the trajectory where $g_{2i}$ changes sign changes is not unique. In fact, the initial conditions determine uniquely a local solution only if the beta functions are continuous in the parameters. This is an untenable conclusion in a closed system and can be avoided either by (i) finding corrections the evolution equation around $g_{2i} = 0$ or (ii) demonstrating that the trajectories avoid the singularity and sign($g_{2i}$) is conserved during the evolution.

The two alternatives are actually related, they are about the presence or absence of a censor mechanism to exclude singularities by piling up higher loop contributions during the evolution: The blocking consists of a small but finite decrease of the cutoff, $k \to k - \Delta k$, and the resulting evolution is a finite difference equation where the higher loop contributions are suppressed by the small parameter $\Delta k \hbar$ rather than by the usual $\hbar$ factor. Thereby the leading order one-loop finite difference equation, an approximation, becomes an exact differential equation in the limit $\Delta k \to 0$. The possibility (i) is realized if the large magnitude of the propagator, close to the quasi-particle peak, upsets this simplification and the true evolution equation contains multi-loop contributions. But this complication might be avoided by a more careful integration of the evolution equation which is supposed to resum the higher loop contributions. In fact, a one-loop contribution to a running coupling constant during a blocking appears in that coupling constant, multiplying a the vertex of a one-loop integral
at the next blocking step. Each blocking performs a partial resumming of the next order of the loop expansion and the diverging number of blocking steps, arising as $\Delta k \to 0$, leads to a complete resummation. Thus one can maintain the full resummation by assuring that the step size $\Delta k$ of the finite difference evolution equation is small enough.

A similar problem occurs in the presence of a condensate where the truncated Euclidean evolution equation reaches a singularity at finite $k$ where a saddle point appears and generates tree-level contributions to the blocking [25, 26]. However the improvement of the truncation and the retaining the evolution of more vertices may lead to the accumulation of higher order contributions of the blocking which censors the evolution and keeps the trajectory away from the singularity [27]. The possibility (ii) corresponds to the presence of such a censor in real time dynamics.

\section*{IV. ASYMPTOTIC SCALING}

We start the exploration of the difference of the renormalization group flow of the schemes, introduced above, by considering the asymptotic scaling regime around the Gaussian fixed point around four dimensions. The evolution equations, written in terms of the dimensionless parameters, $\tilde{g}_n = k^{n\frac{d-2}{2}}g_n$, simplify in this regime to

$$
\dot{\tilde{g}}_2 = -2\tilde{g}_2 + b_2\tilde{g}_4, \\
\dot{\tilde{g}}_4 = -\eta\tilde{g}_4 + b_4\tilde{g}_4^2, 
$$

(27)

by retaining the leading order terms in $\eta = 4 - d$, $\tilde{g}_2$ and $\tilde{g}_4$. The coefficients $b_2$ and $b_4$ are given in table I. The linearization in the parameters produces the double degenerated critical exponents, $\nu = -2$ and $-\eta$ since the linearized beta functions can be written without using the complex conjugate parameters. The scaling combinations of the parameters are the real and the imaginary parts of $g_2$ and $g_4 + g_2b_2/2$, respectively.

\subsection*{A. Virtual domain}

The result of the blocking is projected onto an $O(d)$ symmetrical action and the $O(d)$ space-time symmetry is reinforced in this calculation and the reduction $O(d) \to O(d-1)$ of the space-time symmetry affects only the running of the almost marginal scaling parameters.
TABLE I: Beta function coefficients

| Scheme          | $\pi^2 b_2$ | $\pi^2 b_4$ |
|-----------------|-------------|-------------|
| $E_d$           | $-\frac{1}{16}$ | $\frac{3}{16}$ |
| $E_{d-1}, \Omega = \infty$ | $-\frac{1}{8}$ | $\frac{3}{16}$ |
| $M_{d-1}, \Omega = \infty$ | $\text{sign}(g_{2i})\frac{1}{8}$ | $-\text{sign}(g_{2i})\frac{3}{16}$ |
| $M_L$           | $\text{sign}(g_{2i})\frac{1}{8}$ | $\frac{3}{16}B_4^{(0)}$ |

FIG. 3: The beta function of $g_4$ in the scheme $M_{d-1}$, displayed on the complex $g_4$ plane for $g_2 = 0.5 - 0.01i$ in $d = 4$.

The inverse Wick rotation to real time makes generates discontinuous beta functions and splits the parameter space into two domains of analyticy. The Wilson-Fisher fixed point, $\tilde{g}_{WF_2} = \eta b_2/2b_4$, $\tilde{g}_{WF_4} = \eta/b_4$, is slightly shifted by the reduction of the space-time symmetry and corresponds to an ill-defined theory with $g_{2i} = g_{4i} = 0$ in the scheme $M_{d-1}$.

The beta function of $\tilde{g}_4$, the right hand side of the second equation (27), generates a more involved flow pattern for complex parameters, shown in Fig. 3 in the scheme $M_{d-1}$, where $g_{2i} \neq 0$ introduces an asymmetry with respect to complex conjugation, $g_4 \rightarrow g_4^*$. The evolution of the Euclidean theory with real parameters from the Gaussian to the Wilson-Fisher fixed point follows the separatrix, a straight line of length $O(\epsilon)$ and the weak coupling domain with $0 < g_4 < g_{WF4}$ disappears as $d \rightarrow 4$. When the flow is extended over complex parameters these trajectories survive in $d = 4$ as a manifold of closed curves on the complex $g_4$ plane.

It is instructive to consider the theory in four dimensions where the weakly irrelevant parameters becomes weakly relevant for $\text{sign}(g_{2i})g_{4r} > 0$ and small $|g_{4i}|$, c.f. the lower left
quadrant of Fig (3). In fact, the solution of the second equation of (27),

\[ g_4(\tau) = \frac{g_4(\tau_{in})}{1 - b_4 g_4(\tau_{in})\tau}, \tag{28} \]

shows that the four dimensional model is asymptotically free. The usual UV Landau pole at \( k = k_{in} \exp(1/\beta_4 g_4(\tau_{in})) \), indicating problems in extending the UV physics with the given analytical form of the action, can be avoided by complexifying the parameters.

### B. Quasi-particle domain

The mass-shell singularity of the free propagator generates diverging \( \beta_4 \) as \( g_2 \to 0 \) in the scheme \( M_{pw} \). It is shown below that despite this divergence the free massless theory of the scheme \( M_{pw} \) represents a non-linear fixed point. The \( 1/g_2 \) divergence is regulated by spreading the IR field in the scheme \( M_L \), rendering the limit \( g_2(\tau_{in}) \to 0 \) uniform. Furthermore, \( \beta_4 \) becomes continuous at the Gaussian fixed point since the dependence of (25) on \( \text{sign}(g_2) \) is suppressed as \( g_2 \to 0 \) yielding

\[ B_4^{(0)} = -\frac{68 - 93\gamma^2 + 18\gamma^4 - 109\gamma^6 + 240\gamma^8 + 96\gamma^{10} + i\frac{4}{\gamma}(1 + 5\gamma^2 + 4\gamma^4)(2 - 3\gamma^2 - 2\gamma^4)^2}{(4 + \gamma^2)^2(3 - 6\gamma^2 + 7\gamma^4 + 32\gamma^8)}, \tag{29} \]

assuming \( \gamma = k\tilde{\gamma} \). The regulator \( \gamma \) can safely be removed with the decrease of the cutoff owing to the finite life-time of the quasi particles, generated during the evolution.

### V. GLOBAL ISSUES OF THE RENORMALIZATION GROUP FLOW

The parameter space of the theory is equipped by a particular structure by the renormalization group flow where the different physical theories, making up a space of one co-dimension are evolving as the function of their cutoff. Such a sliced structure can display two different global issues. One usually finds several scaling regimes, separated by crossover(s), when a theory, corresponding to a fixed physical content, is followed as the function of the cutoff. When the theories, corresponding to nearby physical content are compared as the functions of the cutoff then one may find separatrices. The hypersurface of the parameters space, spanned by these trajectories corresponds to a phase transition, defined by a singular relation between the UV and the IR parameters, developing during infinitely long evolution in \( \tau \). We restrict our attention to four dimensional theories below.
A. Phase transitions

The phase transitions, the singular dependence of the IR parameters on the UV one, are along the separatrices of the renormalization group flow. Let us define the generalized scaling combinations of the parameters, \( e^{(\alpha)} \), by the normalized eigenvectors of the real matrix \( \partial \tilde{\beta}_j / \partial \tilde{g}_k, j, k = 1, \ldots, 4 \). The directions of eigenvectors with negative eigenvalue are generalized relevant scaling combinations and the vanishing of the beta function along their direction, \( \sum_k \partial \tilde{\beta}_j / \partial \tilde{g}_ke_k^{(\alpha)} = 0 \), defines the separatrices. There are two relevant parameters around the Gaussian fixed point thus one expect two phase transitions. Each separatrix is a hypersurface of one co-dimension and their intersection with a two dimensional plane is a line. The tangent of the separatrices on the planes \((g_{2r}, g_{4r})\) and \((g_{2i}, g_{4i})\) at the origin is \((b_2/2, 1)\) according to the asymptotic beta function of \(g_2\) in (27).

The symmetry, related to the separatrix on the plane \((g_{2r}, g_{4r})\) is expected to be the internal space inversion, \( \phi_x \to -\phi_x \), namely the regions right or left of the separatrix belong to the symmetric and spontaneously symmetry broken phases, respectively, the status of the symmetry being monitored by \( \text{Re}(\langle \phi_x \rangle) \). To access expectation values in the present formalism we restrict our attention to closed bare theories, defined by \( g_{in,2} \to 0^- \) and \( g_{in,4i} = 0 \) where the initial values of the parameters are denoted by \( g_{in,n} = g_n(\tau_{in}) \).

One would expect a similar phase structure around the separatrix on the plane \((g_{2i}, g_{4i})\), as well, based on the complex conjugation, \( g_n \to g_n^* \), where \( \text{Im}(\langle \phi_x \rangle) \) distinguishes the phases. But this is not the case in Minkowski space-time where the complex conjugation of the parameters makes the path integral diverging and to recover finite transition amplitudes one has upgrade it to a (real) time inversion, \( iS_M \to -iS_M^* \). Since the transition amplitude is computed between identical states, the perturbative vacuum, the amplitude and the expectation value of time independent observables are time inversion invariant for closed bare theories. We do not attempt to identify a symmetry breaking, related to the phase transition at the separatrix on the plane \((g_{2i}, g_{4i})\) in this work.

B. Crossovers

The crossover of the scheme \( E_d \) can easily be seen in the beta function (9), the product of a kinematic phase space factor, \( k^d \), and a dynamical expression, a polynomial of \( 1/(k^2 + U''(s)) \),
FIG. 4: The limit $g_{in,2i} \to 0$ in the symmetric phase of the scheme $M_{pw}$. The quantities (a): $g_{4r}$, (b): $g_{4i}$, (c), (d): $\tilde{g}_{2i}$, (e): $|w_{fl}|$ and (f): $\text{Im}(\ln(w_{fl})/\pi)$ are shown as the functions of $t = \log(k/k_{in})$ for the initial values $\tilde{g}_{in,2r} = 0.1$, $g_{in,4r} = 0.1$, $g_{in,4i} = 0$ and $\tilde{g}_{in,2i} = -10^{-3}$ (continuous line), $10^{-4}$ (dashed line), $10^{-5}$ dotted line) and $10^{-6}$ (dotted-dashed line).

the propagator of the particle modes to be eliminated. The UV and IR scaling laws results from the approximation $1/(k^2 + U'') \approx 1/k^2$ and $1/(k^2 + U'') \approx 1/U''$, respectively. The asymptotic UV scaling is dominated by the kinetic energy and weakly coupled, leading to $\tilde{g}_2(\tau) = \tilde{g}_{in,2i}e^{-2\tau}$ and the crossover at $\tau_{UV-IR} = \mathcal{O}(g_4^0)$ indicates a change of the dynamical factor which is more involved for real time owing to the quasi particle peak of the propagator.

The kinematical and the dynamical factors are given here by $k^{d-1}$ and the loop integrals in (19). The UV-IR crossover can conveniently be located by following the square of the dimensionless weight of the quantum fluctuations at the cutoff, $w_{fl} = (-2i\tilde{D}_0^{(k)})^2 = 1/\omega_k^2 = 1/(1 + g_2)$, c.f. Eq. (20) which is approximately constant and drops exponentially in $t$ in the UV and the IR scaling regime, respectively.

The limit $g_{in,2i} \to 0$ generates another crossover at $\tau_{nl-l} = \mathcal{O}(g_{in,2i})$ in the scheme $M_{pw}$, separating the non-linear and the linearizable, Gaussian UV scaling regimes. The evolution of the Gaussian parameter $g_2$ remains linearizable in the UV regime and the resulting exponential scaling can be approximated by a constant, $g_2(k) = g_{in,2i}$, in the short non-linear scaling regime. The evolution equation for $g_4$, (22) with (24), can easily be
integrated with the result
\[ g_4(\tau) = \frac{g_{in,4}}{1 - i b_{4\text{int}} g_{in,4} \tau}, \] (30)
with \( b_{4\text{int}} = 3/\pi^2 \tilde{g}_{in,2i} \) yielding the crossover at the smeared Landau pole, \( \tau_{nl-l} = \pi^2 \tilde{g}_{in,2i}/3 g_{in,4} \).

\[
g_4 = \begin{cases} 
  g_{in,4r} - 2\tau b_4 g_{in,4r} g_{in,4i} & \tau \ll \tau_{cr} \\
  \frac{2 g_{in,4r} g_{in,4i}}{b_4 |g_{in,4}|^2 \tau} & \tau \gg \tau_{cr} 
\end{cases}
\]

\[
g_4i = \begin{cases} 
  g_{in,4i} + \tau b_4 (g_{in,4r}^2 - g_{in,4i}^2) & \tau \ll \tau_{cr} \\
  \frac{g_{in,4r}^2 - g_{in,4i}^2}{b_4 |g_{in,4}|^2 \tau} & \tau \gg \tau_{cr} 
\end{cases}
\] (31)

The numerical results, shown in Fig 4 correspond to a weakly coupled theory. The evolution of \( g_4 \), given by (30), is recovered in Figs. 4 (a) and (b) and is followed by the freeze-out in the IR scaling regime. The evolution of the mass parameter is dominated by the rescaling term, in particular \( \tilde{g}_{2i}(\tau) = \tilde{g}_{in,2i} e^{-2\tau} \) is supported by Fig. 4 (c). Fig. 4 (d) shows that the mass can be considered as constant in the non-linear scaling regime. The magnitude and the phase of the free propagator are reproduced in Figs. 4 (e) and (f) and can be used to locate the UV-IR crossover.

The smaller \( -g_{in,2i} \) increases \( -g_{4i} \) and makes the non-linear scaling shorter. Since the UV-IR crossover is \( g_{in,2i} \)-independent the increased length of the linearizable UV scaling where both \( g_4r \) and \( g_4i \) are strongly irrelevant suppresses stronger these coupling strengths. The end result is a non-uniform convergence as \( g_{in,2i} \to 0 \) to a renormalizable theory with some interaction close to the cutoff but with free dynamics, \( g_4r = \mathcal{O}(\tilde{g}_{in,2i}^2) \) and \( g_4i = \mathcal{O}(\tilde{g}_{in,2i}) \), at and below the UV-IR crossover.

The renormalization group flow of the scheme \( M_L \) approaches those of \( M_{pw} \) in the limit \( \gamma \to 0 \) however the length of the non-linear scaling regime shrinks in \( M_L \) as \( g_{in,2i}/\gamma \to 0 \) and the limit \( g_{in,2i} \to 0 \) is safe with finite \( \gamma \) and a Gaussian fixed point is recovered, c.f. Table I and Fig. 5. After a very short remnant of the non-linear scaling regime \( g_{2i} \) assumes the weakly coupled scaling which converges as \( g_{in,2i} \to 0 \) according to Fig. 5 (a). The evolution of the coupling strength \( g_4 \), shown in Figs. 5 (b) and (c), is \( g_{in,2i} \)-independent and displays the UV-IR crossover beyond which the magnitude of the free propagator drops and the phase indicates a finite life-time of the quasi-particles, converging when \( g_{in,2i} \to 0 \) in Figs. 5 (d) and (e).
FIG. 5: The limit $g_{in,2i} \to 0$ in the symmetric phase of the scheme $M_L$. The quantities (a): $\tilde{g}_{2i}$, (b): $g_{4r}$, (c): $g_{4i}$, (d): $|w_{fl}|$ and (e): $\text{Im}(\ln(w_{fl})/\pi)$ is shown for $\tilde{\gamma} = 1$ and the initial values $g_{in,2r} = 0.1$, $g_{in,4r} = 0.1$, $g_{in,4i} = 0$ and $\tilde{g}_{in,2i} = -10^{-3}$ (continuous line), $10^{-5}$ (dashed line), $10^{-7}$ dotted line) and $10^{-9}$ (dotted-dashed line).

A novelty of the scheme $M_L$ emerges as the symmetry broken phase is approached. The evolution in the vicinity of the separatrix is shown in Figs. 6 with two trajectories in both phases according to Fig. 6 (a). The coupling strength $g_4$, shown in Figs. 6 (d) and (e) develops a jump sharply at the UV-IR crossover in the symmetry broken phase, driven by the quantum fluctuations whose weight is shown in Figs. 6 (f) and (g). The sharp peaks in $g_{4r}$ and $g_{4i}$ correspond to the scale where $\text{Re}(w_{fl})$ changes sign. This is the scale where we enter into the spinodal instable region, indicated by the flipping of $\text{sign}(1 + \tilde{g}_{2r})$.

The possibility of flipping $\text{sign}(g_{2i})$ in Figs. 6 (c) and (g) raises the question whether the renormalized trajectories can really develop a non-continuous tangent vector, a rather unusual behaviour, or there is some kind of accumulation of higher order loop contributions, a quantum censor, to prevent such singularity to take place. A typical trajectory within the phase with spontaneously broken symmetry is reproduced in Fig. 7. The change of $\text{sign}(g_{2i})$ takes place earlier as $g_{in,2i}$ approaches 0 according to Fig. 7 (a), without affecting the sharp variation of $g_4$ and $|w_{fl}|$ in Figs. 7 (b), (c) and (d). The phase of the propagator, displayed in Fig. 7 (e) indicates that the sharp changes belong to the jump of $\text{sign}(\text{Re}(w_{fl}))$. The
FIG. 6: The dependence of (a): $\tilde{g}_{2r}$, (b): $\tilde{g}_{2i}$, (c): $-\tilde{g}_{2i}$ for $\tilde{g}_{2i} < 0$, (d): $g_{4r}$, (e): $g_{4i}$, (f): $|w_{fl}|$ and (g): $\text{Im}(\ln(w_{fl}))/\pi$ in the scheme $M_L$ in the vicinity of the phase transition for $\tilde{\gamma} = 1$ and the initial values $\tilde{g}_{in,2i} = -10^{13}$, $g_{in,4r} = 0.1$, $g_{in,4i} = 0$ and $\tilde{g}_{in,2r} = -0.4 \cdot 10^{-3}$ (continuous line), $-0.56 \cdot 10^{-3}$ (dashed line), $-0.73 \cdot 10^{-3}$ dotted line) and $-9 \cdot 10^{-4}$ (dotted-dashed line).

stability of $\text{sign}(g_{2i})$ around the narrow peak of the magnitude of the propagator suggests that $\Delta k$ is sufficiently small in the numerical integration. The scale where $\text{sign}(g_{2i})$ flips seems to approach the location of the peak of the magnitude of the propagator, suggesting their coincidence in closed bare theories where $g_{in,2i} \rightarrow 0$. The fluctuations around the trivial saddle point in the spinodal instable domain contains positive (negative) energy fluctuations in the distant future (past), contrary to a stable theory.

However our renormalized trajectory is not be reliable below the UV-IR crossover, defined by $k_{UV-IR}^2 + g_{2r}(k_{UV-IR}) = 0$, in the symmetry broken phase owing to the saddle point contributions to the blocking. The saddle points are delocalized field configurations with spatial momentum $|p| = k$ and arbitrary time dependence. The lowest lying saddle point
of the closed bare theory is the trivial one. It is easy to see that the bounds $g_{2i}, g_{2i} < 0$ are sufficient in weakly coupled theories to keep the absolute magnitude of the classical prefactor, multiplying the saddle point contributions to (2), dominant among all possible saddle points. However the flipping of sign($g_{2i}$) at the UV-IR crossover makes the non-trivial saddle point competitive. More dedicated work is needed to establish the renormalized trajectories in the IR regime.

VI. SUMMARY

Different aspects of the use of the functional renormalization group method in Minkowski space-time are touched upon in this work in the context of the $\phi^4$ scalar model. The effective parameters, defined by the appropriate vertex functions evaluated at the subtraction point, are complex and the scaling laws to the complexified parameters are considered. The Euclidean theories possess the discrete symmetry under the complex conjugation, $g_n \rightarrow g_n^*$. The symmetry is broken explicitly by the factor $i$ in front of the Minkowski action in the exponent of the path integral but can be recovered by upgrading the complex conjugation.
to time inversion.

The pole of the resummed propagator is closer to the real than the imaginary energy axis in weakly coupled theories. This forces us to abandon the Wick rotation invariant subtraction schemes and to define the effective parameters in the quasi-particle domain, using inhomogeneous IR field in deriving the evolution equations. The move of the subtraction point from the virtual to the quasi-particle domain changes the scaling laws in a fundamental manner. The blocking relations around the massless free theory become singular unless the energy of the subtraction point is smeared. One recovers linearizable scaling laws around the Gaussian fixed point in such a smeared scheme.

The complex initial value for the mass regularizes not only the mass-shell divergences of a closed dynamics, it removes the singularity of the Landau poles appearing for real parameters, too. The removal of the UV Landau pole renders the scalar theory asymptotically free in four dimensions, and avoids the IR Landau poles at the UV-IR crossover in the vacuum with condensate. But the construction of the renormalized trajectory within the IR scaling regime remains problematic due to the non-trivial saddle points to the blocking. The limit $g_{in,2i} \to 0$ makes the UV-IR crossover rather sharp in weakly coupled closed bare theories.

These results are preliminary, suggesting the need of a more thorough construction of the functional renormalization group method in Minkowski space-time. Few questions, calling for further inquires are the following:

The usual strategy of the renormalization group to obtain Green’s functions is the successive elimination of the modes in the path integral. This is certainly a mathematically correct way to deal with multi-dimensional integrals however there is no way to interpret physically the blocked theory with lowered cutoff in the quantum case. The reason is that the cutoff theory always describes an open dynamics hence its handling requires the Closed Time Path formalism. The quantum fluctuations of the bra and the ket components are independent in a closed dynamics and lead to a formal redoubling of the dynamical variables. These copies of the IR field become coupled by the IR-UV entanglement, leading to involved scaling laws.

The real time dynamics confronts us with an unexpected complication in Quantum Field Theories, namely the difficulties of finding boost invariant non-perturbative regu-
tor [28]. This problem is circumvented here by employing the sharp momentum cutoff, a
non-relativistic regulator and the action is projected onto a relativistically invariant form
after each blocking. This problem should be clarified rather than swept under the rug and
a careful analysis of the status of the boost invariance is required.

Finally, the asymptotically free nature of the four dimensional scalar theory offers a new
point of view on the renormalization of massive Abelian gauge theories. In fact, the gauge
invariant sector of these theories has a marginal coupling constant and the UV Landau pole
might be avoided by the complex parameters.

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