The four-dimensional Martínez Alonso–Shabat equation: reductions and nonlocal symmetries

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We consider the four-dimensional integrable Martínez Alonso–Shabat equation, and present three integrable three-dimensional reductions thereof. One of these reductions, the basic Veronese web equation, provides a new example of an integrable three-dimensional PDE.

We also construct an infinite hierarchy of commuting nonlocal symmetries (and not just the shadows, as it is usually the case in the literature) for the Martínez Alonso–Shabat equation.

\textbf{Keywords:} integrable systems, nonlocal symmetries, differential coverings

Introduction

Consider the four-dimensional Martínez Alonso–Shabat equation

\[ u_{ty} = u_{z} u_{xy} - u_{y} u_{xz} \]  

introduced in \cite{16}. It has \cite{18} a Lax pair

\[ q_{y} = \lambda u_{y} q_{x}, \quad q_{z} = \lambda (u_{z} q_{x} - q_{t}) \]  

with a non-removable parameter $\lambda \neq 0$, and a recursion operator, and is therefore integrable. Another important property of (1) was established in \cite{2}: its cotangent covering has \textit{local} higher symmetries, which appears to be quite uncommon for nonlinear PDEs in more than two independent variables.

Below we present three reductions of (1) to integrable three-dimensional equations: the so-called rdDym equation \cite{4, 22, 19, 21}, the universal hierarchy equation \cite{16}, and the equation (4) which, to the best of our knowledge, is a new integrable (2+1)-dimensional PDE. It is covered by the Veronese web equation, see \cite{25, 7} for the latter, and for this reason we refer to (4) as to the \textit{basic Veronese web equation}.

We also construct an infinite hierarchy of commuting nonlocal symmetries in the sense of \cite{14, 15} for equation (1) using a technique from \cite{23}. It should be stressed that finding full-fledged nonlocal symmetries rather than mere shadows is quite uncommon, cf. e.g. \cite{5, 23} and the discussion at the end of Section 2.

1 Reductions of the Martínez Alonso–Shabat equation

We have found three reductions of (1) to integrable three-dimensional PDEs:

1.1 The rdDym equation

The reduction $z = x$ yields the rdDym equation \cite{4, 22, 19, 21} that arises as the $r \to \infty$ limit of the so-called $r$th dispersionless Harry Dym equation \cite{4}:

\[ u_{ty} = u_{x} u_{xy} - u_{y} u_{xx}. \]  

The Lax pair (2) after the reduction boils down to the known Lax pair for (3),

\[ q_{t} = (u_{x} - \lambda^{-1}) q_{x}, \quad q_{y} = \lambda u_{y} q_{x}. \]
1.2 The universal hierarchy equation

Putting \( t = y \) in (1) and (2) yields the universal hierarchy equation

\[
\frac{\partial^2 u}{\partial y \partial y} = \frac{\partial^2 u}{\partial z \partial x}\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x \partial z}
\]

and its Lax pair

\[
q_y = \lambda u_y q_x, \quad q_z = \lambda (u_z - \lambda u_y) q_x.
\]

1.3 The basic Veronese web equation

Another interesting reduction admitted by (1) arises when we put \( z = t \), which produces the equation

\[
\frac{\partial u}{\partial y} = \frac{\partial u}{\partial t}\frac{\partial u}{\partial x} - u_x\frac{\partial u}{\partial t}.
\]

Performing this reduction also in (2) yields a Lax representation for (4),

\[
q_t = \lambda u_t q_x / (\lambda + 1), \quad q_y = \lambda u_y q_x.
\]

Upon eliminating \( u \) from (5) we arrive at the equation

\[
q_y q_{tx} = (\lambda + 1) q_t q_{xy} - \lambda q_x q_{ty},
\]

which is, up to the removal of non-essential parameters, nothing but the so-called ABC equation

\[
A q_x q_t y + B q_y q_t x + C q_t q_{xy} = 0, \quad A + B + C = 0,
\]

which describes three-dimensional Veronese webs. This equation is of importance in geometry: as shown in [9], to any smooth solution of (7) one can associate a three-dimensional Einstein–Weyl structure. Note that in [6] it was shown that (7) with \( A + B + C \neq 0 \) is also integrable but with a nonisospectral Lax pair.

In geometric terms, equation (5) defines a covering over (4). In this covering, (6) is the covering equation while (4) is the base equation. For this reason we refer to (4) as to the basic Veronese web equation.

2 Nonlocal symmetries for the Martínez Alonso–Shabat equation

Consider the Lax operators associated with the Lax pair (2),

\[
L_1 = D_z + \lambda (D_t - u_z D_x), \quad L_2 = D_y - \lambda u_y D_x,
\]

where \( D_t, D_x, D_y, D_z \) are total derivatives in the covering (2) over (1). We can now construct another covering over (1) as follows. Let \( M = s D_x \). Then it is readily checked that the equations \( [L_i, M] = 0, \ i = 1, 2 \), boil down to the following equations for \( s \):

\[
s_y = \lambda (u_y s_x - u_{xy} s), \quad s_z = \lambda (u_z s_x - s_t - u_{xz} s).
\]

These equations are compatible by virtue of (1), and thus define a covering over (1) with the pseudopotential \( s \).

The covering (8) is of interest inter alia for the following reason: it is easily checked that \( U = s \) satisfies the linearized version of (1).

\[
U_{ty} = u_z U_{xy} - u_y U_{xz} + u_{xy} U_z - u_{xz} U_y,
\]

i.e., \( s \) is a shadow (in the sense of [14, 15, 5]) of nonlocal symmetry for (1).

To any shadow one can associate a covering in which this shadow is lifted to a nonlocal symmetry using the construction of [12, 13]. However, this requires an introduction of an additional infinite series of nonlocal variables, and these series are different for different shadows.
It is therefore a remarkable fact that for equation (11) we were able to construct a fairly simple covering (13) for which there exists an infinite commuting series of nonlocal symmetries of (11) expressible solely in terms of pseudopotentials $w_i$ of this covering. The technique employed by us to this end below mimics the one from [23].

Namely, following [23], consider a copy of the covering (8) with $\lambda$ replaced by another parameter $\mu$ and the pseudopotential denoted by $w$ instead of $s$:

$$w_y = \mu (u_y w_x - u_{xy} w), \quad w_z = \mu (u_z w_x - w_t - u_{xz} w). \tag{10}$$

According to the above, $w$ is a shadow of nonlocal symmetry for (11). Informally this can be restated as follows: suppose that $u, w, s$ also depend on an independent variable $\tau$, then equation $u_\tau = w$ is compatible with (11), (10).

Moreover, it turns out that the system

$$u_\tau = w, \quad s_\tau = \frac{\lambda\mu}{\mu - \lambda} (w s_x - s w_x) \tag{11}$$

is compatible with (1), (8) by virtue of (1), (8), (10) and (11). It is important to stress that the extension to $s$ of the first part of the flow, $u_\tau = w$, is not uniquely defined, see e.g. [14], i.e., the choice made for the right-hand side of the second equation of (11) is, generally speaking, not a canonical one.

Now slightly alter the notation to stress the dependence of all relevant objects on $\mu$: write $\tau(\mu)$ instead of $\tau$ and $w(\mu)$ instead of $w$. In this notation $s = w(\lambda)$, so if we assume, following (11), that there holds

$$(w(\mu))_{\tau(\nu)} = \frac{\mu \nu}{\nu - \mu} (w(\nu) (w(\mu))_x - w(\mu) (w(\nu))_x),$$

and its counterpart with $\mu$ and $\nu$ interchanged, then the flows with the times $\tau(\mu)$ and $\tau(\nu)$ are readily checked to commute for all $\mu \neq \nu$:

$$\frac{\partial^2 u}{\partial \tau(\mu) \partial \tau(\nu)} = \frac{\partial^2 u}{\partial \tau(\nu) \partial \tau(\mu)}, \quad \frac{\partial^2 s}{\partial \tau(\mu) \partial \tau(\nu)} = \frac{\partial^2 s}{\partial \tau(\nu) \partial \tau(\mu)}. \tag{12}$$

Expanding $w(\mu)$ into formal power series in $\mu$, $w(\mu) = \sum_{i=0}^{\infty} w_i \mu^i$, now yields a new covering over (11) with the pseudopotentials $w_i$ defined by the system

$$(w_i)_y = u_y (w_{i-1})_x - u_{xy} w_{i-1}, \quad (w_i)_z = u_z (w_{i-1})_x - (w_{i-1})_t - u_{xz} w_{i-1} \tag{13}$$

for $i \in \mathbb{N}$ with an arbitrary smooth function $w_0 = w_0(t, x)$.

The pseudopotentials $w_i, i \in \mathbb{N}$, are readily checked to be shadows of nonlocal symmetries of equation (11) in the covering (13). It turns out that we reproduce the action of the inverse recursion operator $R^{-1}$ for (11) from [18]: we have $w_i = R^{-1} w_{i-1}, i \in \mathbb{N}$.

We now intend to show that the shadows $w_i$ can be lifted to full-fledged nonlocal symmetries for (11) in the sense of [14, 15]. To this end we write, following the spirit of theory of generating functions for commuting flows, cf. e.g. equation (4) in [11], and references therein, a formal expansion

$$\frac{\partial}{\partial \tau(\mu)} = \sum_{i=0}^{\infty} \mu^i \frac{\partial}{\partial \tau_i},$$

and substitute this, along with the formal expansion for $w$ in $\mu$, into (11).

This results in the following equations:

$$u_{\tau_i} = w_i, \quad s_{\tau_i} = \sum_{k=0}^{i-1} \lambda^{k+i+1} ((w_k)_x s - w_k s_x), \quad i \in \mathbb{N}. \tag{14}$$

It is readily checked that for any given $i \in \mathbb{N}$ the system (14) is compatible with (11), (8) by virtue of (1), (8), (13) and (14).
As \( s \equiv w(\lambda) \), we also have a formal expansion \( s = \sum_{j=0}^{\infty} w_j \lambda^j \). Substituting this into the second equation of (14) yields the system

\[
\begin{align*}
    u_{\tau_i} &= w_i, \\
    (w_j)_{\tau_i} &= \sum_{k=0}^{i-1} (w_{i+j-k-1}(w_k)_x - w_k (w_{i+j-k-1})_x),
\end{align*}
\]

which is compatible with (11) and (13); here \( i, j \in \mathbb{N} \).

It is easily seen that the flows (15) commute, i.e., for all \( i, j, k \in \mathbb{N} \)

\[
\begin{align*}
    \frac{\partial^2 u}{\partial \tau_i \partial \tau_j} &= \frac{\partial^2 u}{\partial \tau_j \partial \tau_i}, & \frac{\partial^2 w_k}{\partial \tau_i \partial \tau_j} &= \frac{\partial^2 w_k}{\partial \tau_j \partial \tau_i},
\end{align*}
\]

In fact, this result can be extracted directly from (12).

Thus, we arrive at the following theorem, which can also be proved by direct computation instead of the above reasoning:

**Theorem 1** The infinite prolongations of the vector fields

\[
Q_i = w_i \frac{\partial}{\partial u} + \sum_{j=1}^{\infty} \sum_{k=0}^{i-1} (w_{i+j-k-1}(w_k)_x - w_k (w_{i+j-k-1})_x) \frac{\partial}{\partial w_j},
\]

where \( i \in \mathbb{N} \), upon restriction to (1) and (13) form an infinite series of commuting nonlocal symmetries for equation (1) in the covering (13).

The commutativity in this context means that the Lie brackets of the infinite prolongations of \( Q_i \) and \( Q_j \) (or, equivalently, the Jacobi brackets of their characteristics, cf. [5]) vanish for all \( i \) and \( j \). This is equivalent to (16). Note that the existence of an infinite hierarchy of commuting flows is one of the most important hallmarks of integrability, see e.g. [8, 1, 20, 3, 17] and references therein.

As we have \( w_i = R^{-1}w_{i-1}, i \in \mathbb{N} \), where \( R \) is the recursion operator for (1) found in [18], cf. above, the commutativity (16) of the flows (14) suggests that \( R^{-1} \) and \( R \) should be hereditary (cf. e.g. [8, 3, 20] for more details on this property) in some appropriate sense, at least on the linear span of \( w_i \). It is not quite clear, however, whether this claim can be made precise, let alone proved, because of the complicated structure of nonlocal terms in \( R^{-1} \) and \( R \).

To conclude, let us stress again that finding an explicit form of an infinite-dimensional algebra of nonlocal symmetries (rather than just the shadows, cf. e.g. [14, 15] for the details on differences among the two) for a non-overdetermined multidimensional PDE is quite rare. We were able to find just two similar results in the literature: the first is the commutative self-dual Yang–Mills hierarchy [1], and the second [10] is an infinite-dimensional noncommutative algebra of nonlocal symmetries for the (2+1)-dimensional integrable Boyer–Finley equation. It would therefore be very interesting to obtain the results similar to our Theorem 1 for other multidimensional integrable systems, for instance, those recently found in [24].

**Acknowledgments**

This research of AS was supported in part by the Ministry of Education, Youth and Sports of the Czech Republic (MSMT CR) under RVO funding for IC47813059, and by the Grant Agency of the Czech Republic (GA CR) under grant P201/12/G028.

This research was initiated in the course of visits of OIM to the Silesian University in Opava and of AS to the University of Tromsø. Both authors thank the universities they visited for the warm hospitality extended to them.
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