Abstract. Take $k$ a field and $A, B, C$ $k$-algebras with $C \subset A, B$ as subalgebra, then there is the pushout $D = A \cup_C B$, which is again a $k$-algebra. We would like to know when is $D$ an Artin-Schelter regular algebra if $A, B, C$ are. We identify a class of 3 dimensional regular algebras $A, B, C$ where $D$ is regular of dimension 4.

1. Introduction

There is an ongoing effort to classify quantum $\mathbb{P}^3$s, noncommutative projective 3-spaces, or their algebraic correspondence, Artin-Schelter regular algebras of global dimension four. Many families of regular algebras have been discovered recently in [5, 6, 7, 8, 9, 11, 12, 13, 14, 15, 16, 17]. In this paper we construct a new class of Artin-Schelter regular algebras of dimension four as the pushout of two regular algebras of dimension 3.

Take $k$ a field and $A, B, C$ $k$-algebras with $C \subset A, B$ as subalgebra, then there is the $k$-algebra pushout $D = A \cup_C B$. In general $D$ is not regular if $A, B, C$ are. For a counter example we have the free algebra $k\langle x, y \rangle = k\langle x \rangle \cup_k k\langle y \rangle$. For simplicity we only work with regular algebras generated in degree 1. As such algebras are well understood up to dimension three [1, 2, 3], the first interesting case is when $D$ has global dimension 4. By the work of [6], such an algebra $D$ is generated by 2, 3, or 4 elements and the projective resolution of the trivial module $k_D$ is given in [6, Proposition 1.4]. When $D$ is generated by 4 elements, then $D$ has 6 quadratic relations, and the projective resolution of $k_D$ is of the form

$$0 \rightarrow D(-4) \rightarrow D(-3)^{\oplus 4} \rightarrow D(-2)^{\oplus 6} \rightarrow D(-1)^{\oplus 4} \rightarrow D \rightarrow k_D \rightarrow 0.$$

Suggested by the form of the above resolution, we say such an algebra is of type $(14641)$. In this paper we mainly deal with algebras of type $(14641)$. An algebra of type $(14641)$ is quadratic and Koszul. It is easy to see then both $A$ and $B$ are quadratic regular algebras generated by 3 elements. The algebras $A$ and $B$ are symmetrical, thus we only list the projective resolution of $k_A$

$$0 \rightarrow A(-3) \rightarrow A(-2)^{\oplus 3} \rightarrow A(-1)^{\oplus 3} \rightarrow A \rightarrow k_A \rightarrow 0.$$

Following the notation for $D$, we say $A$(and $B$) is of type $(1331)$. As $A$ and $B$ both have 3 quadratic relations, and $D$ has 6 quadratic relations, the quadratic relations of $D$ must be the same as the quadratic relations of $A$ and $B$, hence the subalgebra $C$ can not have quadratic relations. A regular algebra with global dimension 2 has a quadratic relation, and a regular algebra with global dimension 3 has either 3 quadratic relations, or 2 cubic relations. Thus $C$ must be a regular algebra of global dimension 3, with 2 cubic relations.
There are too many dimension 3 algebras to cover at once. In this paper we restrict our algebras $A, B, C$ to the following situation, here all relations are listed in descending order, with $x_4 > x_3 > x_2 > x_1$.

$$A = k\langle x_3, x_1, x_2 \rangle/(r_1, r_2, r_3)$$
$$B = k\langle x_1, x_2, x_4 \rangle/(r_4, r_5, r_6)$$
$$C = k\langle x_1, x_2 \rangle/(r_7, r_8)$$

with the relations $r_i$ defined as follows

\[
\begin{align*}
    r_1 &= x_3^2 - p_1(x_1, x_2, x_3) \\
    r_2 &= x_3 x_1 - p_2(x_1, x_2, x_3) \\
    r_3 &= x_3 x_2 - p_3(x_1, x_2, x_3) \\
    r_4 &= x_4 x_1 - p_4(x_1, x_2, x_4) \\
    r_5 &= x_4 x_2 - p_5(x_1, x_2, x_4) \\
    r_6 &= x_4^2 - p_6(x_1, x_2, x_4) \\
    r_7 &= x_2 x_1^2 - p_7(x_1, x_2) \\
    r_8 &= x_2 x_1^2 - p_8(x_1, x_2)
\end{align*}
\]

Our main theorem is the following

**Theorem 1.1.** (Theorem 4.10) Under the above restriction on the 3 dimensional regular algebras $C \subset A, B$, any pushout algebra $D = A \cup B$ is AS-regular of global dimension 4.

There are many interesting questions we would like to answer

(a) We would like to continue the computation for all dimension 3 algebras $A, B, C$. We hypothesize that for any quadratic regular algebra $A, B$ and cubic subalgebra $C$, their pushout is “generically” a regular algebra of global dimension 4.

(b) We would like to know what kind of algebraic properties is inherited by the pushout algebra. For example, all 3 dimensional regular algebras are Noetherian and we would like to show the dimension 4 pushout algebras are also Noetherian.

(c) We ask the similar question about geometric objects. For example, how the point modules of pushout algebra relate to its subalgebras.

(d) We are interested in higher dimensional extensions.

Here is an outline of the paper: in section 2 we review some basic definitions; in section 3 we give an example that demonstrates many aspect of our computation; in section 4 we define pushout regular algebras and prove their regularity; in section 5 we compute all possible 3 dimensional cubic subalgebras $C$; in section 6 we compute all possible 3 dimensional quadratic algebra $A$; in section 7 we give a list of all dimension 4 pushout regular algebras.

2. Definitions

Throughout $k$ is an algebraically closed base field. Everything is over $k$; in particular, an algebra or a ring is a $k$-algebra. An algebra $D$ is called connected graded if

$$D = k \oplus D_1 \oplus D_2 \oplus \cdots$$
with $1 \in k = D_0$ and $D_i, D_j \subset D_{i+j}$ for all $i, j$. If $D$ is connected graded, then $k$ also denotes the trivial graded module $D/D_{\geq 1}$. In this paper we are working on connected graded algebras. One basic concept we will use is the Artin-Schelter regularity, which we now review. A connected graded algebra $D$ is called Artin-Schelter regular or regular for short if the following three conditions hold.

(AS1) $D$ has finite global dimension $d$, and
(AS2) $D$ is Gorenstein, namely, there is an integer $l$ such that,
\[
\text{Ext}^i_D(Dk, D) = \begin{cases} 
  k(l) & \text{if } i = d \\
  0 & \text{if } i \neq d 
\end{cases}
\]
where $Dk$ is the left trivial $D$-module; and the same condition holds for the right trivial $D$-module $k_D$.
(AS3) $D$ has finite Gelfand-Kirillov dimension, i.e., there is a positive number $c$ such that $\dim D_n < c n^c$ for all $n \in \mathbb{N}$.

If $D$ is regular, then the global dimension of $D$ is called the dimension of $D$. The notation $(l)$ in (AS2) is the $l$-th degree shift of graded modules.

In this paper we further assume all graded algebras are generated in degree 1. If $D$ is regular, then by [10, Proposition 3.1.1], the trivial right $D$-module $k_D$ has a minimal free resolution of the form
\[(E2.0.1) \quad 0 \rightarrow P_d \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow k_D \rightarrow 0\]
where $P_w = \bigoplus_{s=1}^{\infty} D(-i_{w,s})$ for some finite integers $n_w$ and $i_{w,s}$. The Gorenstein condition (AS2) implies that the above free resolution is symmetric in the sense that the dual complex of (E2.0.1) is a free resolution of the trivial left $D$-module after a degree shift. As a consequence, we have $P_0 = D$, $P_d = D(-l)$, $n_w = n_{d-w}$, and $i_{w,s} + i_{d-w,n_{w-s+1}} = l$ for all $w, s$.

Regular algebras of dimension three have been classified by Artin, Schelter, Tate and Van den Bergh [123]. A regular algebra of dimension three is generated by either two or three elements. If $A$ is generated by three elements, then $A$ is Koszul and the trivial right $A$-module $k_A$ has a minimal free resolution of form
\[ 0 \rightarrow A(-3) \rightarrow A(-2)^{\oplus 3} \rightarrow A(-1)^{\oplus 3} \rightarrow A \rightarrow k_A \rightarrow 0. \]
If $C$ is generated by two elements, then $C$ is not Koszul and the trivial right $C$-module $k_C$ has a minimal free resolution of the form
\[ 0 \rightarrow C(-4) \rightarrow C(-3)^{\oplus 2} \rightarrow C(-1)^{\oplus 2} \rightarrow C \rightarrow k_C \rightarrow 0. \]

If $D$ is a Noetherian regular algebra of dimension four, then $D$ is generated by 2, 3, or 4 elements [6, Proposition 1.4]. Minimal free resolutions of the trivial module $k$ is listed in [6, Proposition 1.4]. The following lemma is well-known. The transpose of a matrix $M$ is denoted by $M^t$.

**Lemma 2.1.** Let $D$ be a regular graded domain of dimension four. Suppose $D$ is generated by elements $x_1, x_2, x_3, x_4$ (of degree 1).

(a) $D$ is of type $(14641)$, namely, the trivial right $D$-module $k$ has a free resolution
\[(E2.1.1) \quad 0 \rightarrow D(-4) \xrightarrow{\partial_4} D^{\oplus 4}(-3) \xrightarrow{\partial_3} D^{\oplus 6}(-2) \xrightarrow{\partial_2} D^{\oplus 4}(-1) \xrightarrow{\partial_1} D \xrightarrow{\partial_0} k_D \rightarrow 0\]
where $D^{\oplus n}$ is the free right $D$-module written as an $n \times 1$ matrix.
(b) $\partial_0$ is the augmentation map with $\text{ker} \partial_0 = D_{\geq 1}$. 

(c) $\partial_1$ is given by the left multiplication by $(x_1, x_2, x_3, x_4)$.
(d) $\partial_2$ is the left multiplication by a $4 \times 6$-matrix $R = (r_{ij})_{4 \times 6}$ such that $r_i = \sum_{i=1}^{4} x_i r_{ij}$, for $i = 1, \ldots, 6$, are the 6 relations of $D$.
(e) $\partial_3$ is the left multiplication by a $6 \times 4$-matrix $T = (t_{ij})_{6 \times 4}$.
(f) $\partial_4$ is the left multiplication by $(x'_1, x'_2, x'_3, x'_4)^t$ where $\{x'_1, x'_2, x'_3, x'_4\}$ is a set of generators of $D$. (So each $x'_i$ is a $k$-linear combination of $\{x_i\}_{i=1}^4$)

(2.5) $0 \rightarrow D \rightarrow \cdots$

The dual complex of $(2.1.1)$ is obtained by applying the functor $(-)^\vee = \text{Hom}_D(-, D)$ to $(2.1.1)$. Condition (AS2) implies that the dual complex of $(2.1.1)$ is a free resolution of the left $D$-module $k(4)$:

$0 \leftarrow k(4) \leftarrow D(4) \leftarrow D(3) \leftarrow D(2) \leftarrow D(1) \leftarrow D \leftarrow 0$.

Lemma 2.1(f) follows from this observation. Other parts of Lemma 2.1 are clear.

We introduce the following technical condition, referred to as TC1, as we use it frequently.

**Definition 2.2.** Technical Condition TC1:

We assume $D$ is a quadratic domain of the form $D = k < x_1, x_2, x_3, x_4 > \langle r_1, r_2, r_3, r_4, r_5, r_6 \rangle$

where $\{x_1, \ldots, x_4\}$ is a set of degree one generators and $\{r_1, \ldots, r_6\}$ is a set of quadratic relations. Further assume $D$ has Hilbert Series $H_D(t) = (1 - t)^{-4}$ (AS3), and a complex (not necessarily a resolution) of the form

$(2.3) \quad 0 \rightarrow D(-4) \xrightarrow{X^t} D(-3)^\oplus 4 \xrightarrow{ST} D(-2)^\oplus 6 \xrightarrow{T} D(-1)^\oplus 6 \xrightarrow{X} D \rightarrow k_D \rightarrow 0$

Where $X = [x_1, x_2, x_3, x_4]$. $R$ is a $4 \times 6$ matrix and $T$ is a $6 \times 4$ matrix, both with entries in $A_1$, defined by

$XR = [r_1, r_2, r_3, r_4, r_5, r_6]$

$TX^t = [r_1, r_2, r_3, r_4, r_5, r_6]^t$

$S$ is a $6 \times 6$ invertible matrix with entries in $k$.

If the algebra $D$ satisfies all of the above conditions, then we say $D$ satisfies TC1.

This condition mimics the $k$ resolution from lemma 2.1 with $\partial_3 = ST$ and $\partial_4 = X^t$.

**Lemma 2.4.** Assume $D$ satisfies TC1. If in addition the partial complex

$(2.5) \quad 0 \rightarrow D(-4) \xrightarrow{X^t} D(-3)^\oplus 4 \xrightarrow{T} D(-2)^\oplus 6 \rightarrow \cdots$

is exact, then the complex $(2.3)$ is exact and the algebra $D$ has global dimension four (AS1).

**Proof.** Since $S$ is invertible, the exactness of the complex $(2.5)$ implies the complex $(2.3)$ is exact at the $D(-3)^\oplus 4$ and $D(-4)$ positions. That the complex $(2.3)$ is exact at the $k_D$ and $D$ positions is clear. A result of Govorov [1] gives $(2.3)$ exact at the $D(-1)^\oplus 4$ position. Lastly $H_D(t) = (1 - t)^{-4}$ and exactness at all other positions of $(2.3)$ give us exactness at $D(-2)^\oplus 6$ position.

Thus the complex $(2.3)$ is exact and gldim$(D) = \text{pdim}(k_D) = 4$. \(\square\)

We can apply lemma 2.4 to the dual complex of $(2.3)$ to get
Lemma 2.6. Assume $D$ satisfies the conditions of lemma 2.4 hence (AS1) and (AS3), and the following partial complex is exact.

$$0 \to D^{op}(-4) \xrightarrow{X^t} D^{op}(-3)^{\oplus 4} \xrightarrow{R^t} D^{op}(-2)^{\oplus 6} \to \cdots$$

Then the dual complex of (2.3) is also exact and $D$ satisfies (AS2), hence is regular. In particular, if $D^{op} \cong D$ then $D$ is AS-regular.

3. An Example

Before introduce the formal definition of pushout regular algebra, we give a short example to help establish notation, and to help demonstrate the computation techniques we use.

Define algebra $A$ as follows,

$$A = k\langle x_3, x_1, x_2 \rangle / (r_1 = x_3^2 - x_1 x_2 - x_2 x_1, r_2 = x_3 x_1 + x_1 x_3, r_3 = x_3 x_2 + x_2 x_3)$$

It is easy to see that $A$ is a quadratic AS-regular algebra of global dimension 3. In the algebra $A$ there are two different ways of expressing $x_3^2 x_1$. One way is to first use $r_1$ to write $(x_3^2) x_1 = (x_1 x_2 + x_2 x_1) x_1$. The other way is to use $r_2$ to write $x_3(x_3 x_1) = x_3(-x_1 x_1)$, then simplify using $r_2$ and $r_1$. The process of writing $x_3^2 x_1$ as $(x_3^2) x_1$ and $x_3(x_3 x_1)$ is called resolving. In this case resolving $x_3^2 x_1$ shows that

$$x_3^2 x_1 = (x_3^2) x_1 = x_2 x_1^2 + x_1 x_2 x_1 = x_3(x_3 x_1) = -x_3 x_1 x_3 = x_1 x_3^2 = x_1 x_2 x_1 + x_1^2 x_2$$

Thus we have the cubic relation $x_2 x_1^2 - x_1 x_3^2$. We resolve $x_3^2 x_2$ and $x_3$ similarly. This shows that in $A$ we also have the following two cubic relations:

$$r_7 = x_3 x_1 - x_1 x_3^2$$
$$r_8 = x_3 x_2 - x_1 x_2 x_1$$

It is an easy check, using Bergman’s Diamond lemma, that the algebra $A$ has no other ambiguities.

We define algebras $B$ and $C$ as

$$B = k\langle x_1, x_2, x_4 \rangle / (r_4 = x_4 x_1 + x_1 x_4, r_5 = x_4 x_2 + x_2 x_4, r_6 = x_4 - x_1 x_2 - x_2 x_1)$$
$$C = k\langle x_1, x_2 \rangle / (r_7 = x_2 x_1 - x_1 x_2, r_8 = x_2 x_1^2 - x_1 x_2^2)$$

In this case $B \cong A$ and $C$ is a dimension 3 regular subalgebra of both $A$ and $B$.

Define $D = A \cup_C B$, $D$ is the pushout of $C \hookrightarrow A$, and $C \hookrightarrow B$. In terms of generators and relations, we have

$$D = k\langle x_3, x_1, x_2, x_4 \rangle / (r_1, r_2, r_3, r_4, r_5, r_6)$$

We give explicit $k$ resolutions for $A$ and $D$. A minimal free resolution of $k_A$ is

$$0 \to A(-3) \xrightarrow{X_A} A(-2)^{\oplus 3} \xrightarrow{R_A} A(-1)^{\oplus 3} X_A A \to k_A \to 0$$

Where $X_A = [x_3, x_1, x_2]$, and

$$R_A = \begin{bmatrix} x_3 & x_1 & x_2 \\ -x_2 & x_3 & 0 \\ -x_1 & 0 & x_3 \end{bmatrix}, \quad S_A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix},$$

$$T_A = \begin{bmatrix} x_3 & -x_2 & -x_1 \\ x_1 & x_3 & 0 \\ x_2 & 0 & x_3 \end{bmatrix}, \quad Q_A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix},$$

$$A$$
\[ R_A S_A = Q_A T_A = \begin{bmatrix} x_3 & -x_2 & -x_1 \\ -x_2 & 0 & -x_3 \\ -x_1 & -x_3 & 0 \end{bmatrix} \]

A minimal free resolution of \( k_D \) is

\[ 0 \to D(-4) \xrightarrow{X'} D(-3) \oplus 4 \xrightarrow{ST} D(-2) \oplus 6 \xrightarrow{R} D(-1) \oplus 4 \xrightarrow{X} D \to k_D \to 0 \]

Where \( X = \{x_3, x_1, x_2, x_4\} \), and

\[
R = \begin{bmatrix} x_3 & x_1 & x_2 & 0 & 0 & 0 \\ -x_2 & x_3 & 0 & x_4 & 0 & -x_2 \\ -x_1 & 0 & x_3 & 0 & x_4 & -x_1 \\ 0 & 0 & 0 & x_1 & x_2 & x_4 \end{bmatrix}
\]

\[
T = \begin{bmatrix} x_3 & -x_2 & -x_1 & 0 \\ x_1 & x_3 & 0 & 0 \\ x_2 & 0 & x_3 & 0 \\ 0 & x_4 & 0 & x_1 \\ 0 & 0 & x_4 & x_2 \\ 0 & -x_2 & -x_1 & x_4 \end{bmatrix}
\]

\[
S = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}
\]

With the matrix \( RST = 0 \) given as follows,

\[
\begin{bmatrix} x_3^2 - x_1 x_2 - x_2 x_1 & -x_3 x_2 - x_2 x_3 & -x_1 x_3 - x_3 x_1 & 0 \\ -x_2 x_3 - x_3 x_2 & x_2^2 - x_3^2 & x_2 x_1 - x_3^2 + x_2 x_1 & x_2 x_4 + x_4 x_2 \\ -x_1 x_3 - x_3 x_1 & x_1 x_2 - x_2^2 + x_2 x_1 & x_1^2 - x_2^2 & x_1 x_4 + x_4 x_1 \\ 0 & x_1 x_2 + x_2 x_1 - x_2^2 & x_4 x_1 + x_1 x_4 & x_1 x_2 + x_2 x_1 - x_2^2 \end{bmatrix}
\]

The complex \( 3.2 \) is exact by theorem 4.9 of next section. It is easy to see that \( H_D(t) = (1 - t)^{-4} \) and \( D^{op} \cong D \), thus by lemma 2.6 the algebra \( D \) is regular.

**Remark 3.3.**

(a) In the remainder of the paper we will follow the notations in the above example.

(b) In this example we choose \( B \cong A \). In general \( B \) do not have to be isomorphic to \( A \).

(c) Although we have the relations of algebra \( D \), the resolution of \( k_D \) is one chain longer than the resolution of \( k_A \). It is a non-trivial task to find a \( 6 \times 6 \) invertible matrix \( S \) as required by TC1.

4. **Definition and Regularity of Pushout Algebra**

We introduce an ordering on the monomials. We order first by degree, from highest to lowest. If two monomials have the same degree, we compare them from left to right according to the ordering \( x_4 > x_3 > x_2 > x_1 \). By Bergman’s diamond lemma, we can reduce any element to a sum of basis monomials with minimal order. The basis we thus obtain is similar to Gröbner basis for commutative algebras. Our ordering extend naturally to a finite set of relations. From now on by a set of relations we mean a set of relations with minimal order.
Definition 4.1. Pushout Algebra:
Take $A, B, C$, each a regular algebra of global dimension 3, given as follows,

$$A = k\{x_3, x_1, x_2\}/(r_1, r_2, r_3)$$

$$B = k\{x_1, x_2, x_4\}/(r_4, r_5, r_6)$$

$$C = k\{x_1, x_2\}/(r_7, r_8)$$

where the set of relations $\{r_i\}$, in minimal ordering, is given as follows,

$$r_1 = x_3^2 - p_1(x_1, x_2, x_3)$$
$$r_2 = x_3 x_1 - p_2(x_1, x_2, x_3)$$
$$r_3 = x_3 x_2 - p_3(x_1, x_2, x_3)$$
$$r_4 = x_4 x_1 - p_4(x_1, x_2, x_4)$$
$$r_5 = x_4 x_2 - p_5(x_1, x_2, x_4)$$
$$r_6 = x_4^2 - p_6(x_1, x_2, x_4)$$
$$r_7 = x_2^2 x_1 - pr(x_1, x_2)$$
$$r_8 = x_2 x_1^2 - p(x_1, x_2)$$

We further require that $C = A \cap B$ as subalgebra of $A$ and $B$. All ambiguities in $C$ resolve, and after reduction by $r_7$ and $r_8$, all ambiguities in $A$ and $B$ resolve.

Under the above assumptions, we can form the $k$-algebra pushout $D = A \cup_C B$. From now on, we use the term Pushout Algebra for the above situation only.

Remark 4.2. As we are interested in $D$ quadratic algebra only. It is clear that $A, B$ both have to be quadratic and $C$ must be cubic. The additional requirement on the relations is to break the computation into more manageable pieces. For instance we have the following quick computation of Hilbert Series.

Lemma 4.3. The algebra $D$ has Hilbert Series $H_D(t) = (1-t)^{-4}$ and basis elements

$$x_1^4(x_2 x_1)^l x_2^m x_3^{0.1} (x_4 x_3)^n x_4^{0.1}$$

Proof. Algebra $D$ has the same Hilbert Series and basis as the following algebra

$$k\{x_3, x_1, x_2, x_4\}/(x_1 x_2 x_1 x_2 x_4 x_4 x_1, x_2^3 x_3 x_2 x_3 x_1, x_2^2 x_1 x_2 x_1)$$

Define $X_A = [x_3, x_1, x_2], X_B = [x_1, x_2, x_4], and T_A, T_B 3 \times 3$ matrices with $T_A X_A^t = [r_1, r_2, r_3]^t, T_B X_B^t = [r_4, r_5, r_6]^t$. Since $A$ and $B$ are regular, we have $k$ resolutions

$$0 \rightarrow A(-3) \xrightarrow{X_A} A(-2) \oplus 3 Q_A A(-1) \oplus 3 X_A \rightarrow A \rightarrow k_A \rightarrow 0$$

$$0 \rightarrow B(-3) \xrightarrow{X_B} B(-2) \oplus 3 Q_B B(-1) \oplus 3 X_B \rightarrow B \rightarrow k_B \rightarrow 0$$

Where $Q_A, Q_B$ are $3 \times 3$ non-singular scalar matrices.

For the following results we assume our pushout algebra $D$ satisfies $TC1$, namely there is a complex of right $D$ modules

$$0 \rightarrow D(-4) \xrightarrow{X} D(-3) \oplus 4 \xrightarrow{ST} D(-2) \oplus 6 \xrightarrow{R} D(-1) \oplus 4 \xrightarrow{X} D \rightarrow k_D \rightarrow 0$$

where $X = [x_3, x_1, x_2, x_4], R$ is a $4 \times 6$ matrix with $XR = [r_1, \ldots, r_6], T$ is a $6 \times 4$ matrix with $TX^t = [r_1, \ldots, r_6]^t$ and $S$ a $6 \times 6$ scalar matrix.
We will show that the complex $[4.6]$ is exact at the $D(3)$ position. Notice the upper left and lower right $3 \times 3$ blocks of the matrix $T$ is respectively the matrices $T_A$ and $T_B$, with the remaining entries 0. From the exact sequences $[4.4]$ and $[4.5]$ we have $\text{im}(X_A^t) = \text{ker}(T_A)$ and $\text{im}(X_B^t) = \text{ker}(T_B)$. View the algebra $A$ as subalgebra of $D$, we have the following,

**Lemma 4.7.** Take $[d_1, d_2, d_3]^t \in D^{\oplus 3}$ with $T_A[d_1, d_2, d_3]^t = 0$, then

$$[d_1, d_2, d_3]^t \in [x_3, x_1, x_2]^t D$$

**Proof.** In lemma $[4.3]$ we have a basis element of $D$ to have the form

$$x_3^p(x_2x_1)^t x_3^m x_3 x_4^{0,1}$$

Define $g_{2n} = (x_4x_3)^n$ and $g_{2n+1} = (x_4x_3)^n x_4$. Then an element $d \in D$ can be uniquely written as

$$d = \sum a_n g_n$$

with each $a_n \in A$ a sum of basis elements in $A$. Take another element $a' \in A$, then

$$a'd = a'(\sum a_n g_n) = \sum (a'a_n)g_n = \sum a'_n g_n$$

Here $a'_n = a'a_n \in A$ is also a sum of basis elements in $A$, and $a'_n g_n$ is a sum of basis elements in $D$. Follow this notation, we write

$$[d_1, d_2, d_3]^t = [\sum a_{1n}g_n, \sum a_{2n}g_n, \sum a_{3n}g_n]^t$$

Let $[t_1, t_2, t_3] \in A^{\oplus 3}$ be a row in $T_A$, we have

$$0 = [t_1, t_2, t_3][d_1, d_2, d_3]^t$$

$$= [t_1, t_2, t_3][\sum a_{1n}g_n, \sum a_{2n}g_n, \sum a_{3n}g_n]^t$$

$$= \sum t_1 a_{1n}g_n + \sum t_2 a_{2n}g_n + \sum t_3 a_{3n}g_n$$

$$= \sum (t_1 a_{1n} + t_2 a_{2n} + t_3 a_{3n})g_n$$

From the above we conclude that for each $n$, $0 = t_1 a_{1n} + t_2 a_{2n} + t_3 a_{3n}$. Since we have $\text{im}(X_A^t) = \text{ker}(T_A)$, we have for each $n$,

$$[a_{1n}, a_{2n}, a_{3n}]^t \in [x_3, x_1, x_2]^t A$$

This shows that

$$[d_1, d_2, d_3]^t = \sum [a_{1n}, a_{2n}, a_{3n}]^t g_n \in [x_3, x_1, x_2]^t D$$

Applying the same argument to $B \subset D$, we have

**Lemma 4.8.** Take $[d_1, d_2, d_3]^t \in D^{\oplus 3}$ with $T_B[d_1, d_2, d_3]^t = 0$, then

$$[d_1, d_2, d_3]^t \in [x_1, x_2, x_4]^t D$$

**Theorem 4.9.** If the pushout algebra $D$ satisfies TC1, then $\text{gldim}(D) = 4$.

**Proof.** By lemma $[4.3]$ $D$ has Hilbert Series $H_D(t) = (1-t)^{-4}$. From our form of basis we can easily see that left multiplication by $x_1$ is injective, hence left multiplication by $X^t$ is also injective. By lemma $[2.4]$ to show $\text{gldim}(D) =$
4, we only need to show ker($T_D$) $\subset$ im($X'$).

Define matrices $T_1, T_2$ as follows,

\[
T_1 = \begin{bmatrix}
T_A & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix}, \quad T_2 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & T_B
\end{bmatrix}
\]

By lemma 4.7, 4.8 we have

\[
\ker(T_1) = \begin{bmatrix} x_3d_1 \\ x_1d_1 \\ x_2d_1 \\ d_2 \end{bmatrix}, \quad \ker(T_2) = \begin{bmatrix} d_4 \\ x_1d_3 \\ x_2d_3 \\ x_4d_3 \end{bmatrix}
\]

From the above we have

\[
\ker(T_D) \subset \ker(T_1) \cap \ker(T_2) = \begin{bmatrix} x_3d_1 \\ x_1d_1 \\ x_2d_1 \\ d_2 \end{bmatrix} \cap \begin{bmatrix} d_4 \\ x_1d_3 \\ x_2d_3 \\ x_4d_3 \end{bmatrix} = \text{im}(\begin{bmatrix} x_3 \\ x_1 \\ x_2 \\ x_4 \end{bmatrix})
\]

By lemma 7.3 of section 7 we have that all possible pushout algebra $D$’s satisfy $TCI$. By lemma 6.2 of section 6 we have for all pushout algebra $D$, $D^{op} \cong D$. Thus combining theorem 4.9 and lemma 2.6 we have our main result

**Theorem 4.10.** All pushout algebras, defined as in 4.1, are regular of global dimension 4.

5. **Classification of Algebra $C$**

In this section we identify all possible dimension 3 regular cubic algebra $C$ with relations

\[
C = k(x_1, x_2)/(r_7, r_8)
\]

where the relations $r_7$ and $r_8$ are

\[
r_7 = x_2^2x_1 - c_1x_2x_1x_2 - c_2x_1x_2^2 - c_3x_1x_2x_1 - c_4x_1^2x_2 - c_5x_1^3 \\
r_8 = x_2x_1^2 - c_6x_1x_2^2 - c_7x_1x_2x_1 - c_8x_1^2x_2 - c_9x_1^3
\]

with coefficients $c_i \in k$. 

---

**Note:** The image contains a page from a document discussing Artin-Schelter regular algebras, specifically focusing on the identification of algebra $C$. The content involves mathematical proofs and definitions, including matrix definitions and theorems related to the classification of certain algebraic structures.
Resolving the ambiguity \(x_2^2x_1^2\) in \(C\) gives us the equation
\[
0 = (x_2^2x_1)x_1 - x_2(x_2x_1^2)
\]
\[
= -c_6x_2x_1x_2^2 + (c_1 - c_7)x_2x_1x_2x_1 - c_6c_8x_1x_2^3
+ (c_1c_2 - c_1c_6c_9 - c_7c_8)x_1x_2x_1x_2 + (c_2^2 + c_3c_6 - c_8^2 - c_8c_9 - c_2c_6c_9)x_1^2x_2^2
+ (c_2c_3 + c_4 + c_3c_7 - c_3c_6c_9 - c_8c_9 - c_2^2c_9)x_1^2x_2
+ (c_2c_4 + c_3c_8 - c_4c_6c_9 - c_8c_9 - c_7c_8c_9)x_1^4x_2

+ (c_5 + c_2c_5 + c_3c_9 - c_3c_6c_9 - c_2^2 - c_7c_9)x_1^4
\]
Solving for the coefficients \(c_i\)'s in the above equation gives the following set of equations,
\[
\begin{align*}
c_6 &= 0 \\
c_7 &= c_1 \\
0 &= c_1(c_2 - c_8) \\
0 &= (c_2 - c_8)(c_2 + c_8) \\
0 &= (c_1 + c_2)c_3 + c_4 - (c_1^2 + c_8)c_9 \\
0 &= c_2c_4 + c_3c_8 - (1 + c_1)c_8c_9 \\
0 &= (c_2 + 1)c_5 + c_3c_9 - (1 + c_1)c_9^2
\end{align*}
\]
Some of the solutions to the above system give algebras that are not domain, hence not regular. We have the following lemma for the other solutions

**Lemma 5.1.** Any domain satisfying the above system of equations are regular.

**Proof.** To verify a solution is a domain is computationally intensive. Fortunately we only have to exclude the algebras that are obviously not domain. It is easy to see the remaining solutions have Hilbert Series equal to \((1 - t)^{-2}(1 - t^2)^{-1}\), hence satisfy (AS3). We verify (AS1) by constructing a resolution \(P \to kC\) for each algebra. We omit the resolutions here as they play no future role. Finally for each solution \(C\), we have \(C^{op} \cong C\), and the dual resolution of \(P \to kC\) gives us (AS2). \(\square\)

We list the relations of regular algebra \(C\)'s below. It is worth noting that some of the algebras are only determined up to (linear) isomorphisms. We try to choose the basis that give us shorter relations but our choices are by no means optimal.

**C0** This is a special case of the next algebra, C1.
\[
\begin{align*}
r_7 &= x_2^2x_1 - x_1x_2^2 \\
r_8 &= x_2x_1^2 - x_1^2x_2
\end{align*}
\]

**C1**
\[
\begin{align*}
c_2 &\neq 0 \\
r_7 &= x_2^2x_1 - c_1x_2x_1x_2 - c_2x_1x_2^2 \\
r_8 &= x_2x_1^2 - c_1x_2x_1x_2 - c_2x_1^2x_2
\end{align*}
\]

**C2**
\[
\begin{align*}
c_2 &\neq 0 \\
r_7 &= x_2^2x_1 - c_2x_1x_2^2 \\
r_8 &= x_2x_1^2 + c_2x_1^2x_2
\end{align*}
\]
C3

\begin{align*}
    r_7 &= x_2^2x_1 + x_1x_2^2 - x_1^3 \\
    r_8 &= x_2x_1^2 - x_1^2x_2
\end{align*}

C4

\begin{align*}
    c_1 &\neq 2 \\
    r_7 &= x_2^2x_1 - c_1x_1x_2x_2 + x_1x_2^2 - c_5x_1^3 \\
    r_8 &= x_2x_1^2 - c_1x_1x_2x_1 + x_1^2x_2
\end{align*}

C5

\begin{align*}
    c_9 &\neq 0 \\
    r_7 &= x_2^2x_1 - 2x_2x_1x_2 + x_1x_2^2 - 3c_9x_1x_2x_1 \\
    r_8 &= x_2x_1^2 - 2x_1x_2x_1 + x_1^2x_2 - c_9x_1^3
\end{align*}

C6

\begin{align*}
    c_3 &\neq 0 \\
    r_7 &= x_2^2x_1 - 2x_2x_1x_2 + x_1x_2^2 - c_3x_1x_2x_1 + c_3x_1^2x_2 - c_5x_1^3 \\
    r_8 &= x_2x_1^2 - 2x_1x_2x_1 + x_1^2x_2
\end{align*}

C7 One of \( c_4, c_9 \) is non-zero.

\begin{align*}
    r_7 &= x_2^2x_1 - x_1x_2^2 - (c_9 - c_4)x_1x_2x_1 - c_4x_1^2x_2 - \frac{1}{2}c_4c_9x_1^3 \\
    r_8 &= x_2x_1^2 - x_1^2x_2 - c_9x_1^3
\end{align*}

6. Classification of Algebra \( A \)

In this section we identify all possible regular quadratic algebras \( A \) over each choice of algebra \( C \). By symmetry the algebra \( B \)'s have the same classification. We list the relations \( r_i \) of \( A \) below

\[ A = k \langle x_3, x_1, x_2 \rangle / (r_1, r_2, r_3) \]

with relations \( r_1, r_2, r_3 \) given as

\begin{align*}
    r_1 &= x_2^2 - a_6x_2x_3 - a_5x_2^2 - a_4x_2x_1 - a_3x_1x_3 - a_2x_1x_2 - a_1x_1^2 \\
    r_2 &= x_3x_1 - a_26x_2x_3 - a_25x_2^2 - a_24x_2x_1 - a_23x_1x_3 - a_22x_1x_2 - a_21x_1^2 \\
    r_3 &= x_3x_2 - a_16x_2x_3 - a_15x_2^2 - a_14x_2x_1 - a_13x_1x_3 - a_12x_1x_2 - a_11x_1^2
\end{align*}

We reduce the relations by linear change of variable. Any change of variables must preserve the relations of \( C \) inside \( A \), thus we can scale \( x_2 \), or change \( x_3 \) to \( x_3 + \alpha x_1 + \beta x_2 \). If \( a_{26} = 0 \) then we can choose \( a_{24} = a_{12} = 0 \) after the change of variable \( x_3 \rightarrow x_3 + (a_{13}a_{24} + a_{12})x_1 + a_{24}x_2 \). Otherwise \( a_{26} \neq 0 \) and first we can choose \( a_{26} = 1 \) by scaling \( x_2 \), then we can choose \( a_{24} = a_{25} = 0 \) after the change of variable \( x_3 \rightarrow x_3 - (a_{24} + a_{25})x_1 - a_{25}x_2 \). To sum it up, we have, in general

\begin{align*}
    r_1 &= x_2^2 - a_6x_2x_3 - a_5x_2^2 - a_4x_2x_1 - a_3x_1x_3 - a_2x_1x_2 - a_1x_1^2 \\
    r_2 &= x_3x_1 - a_26x_2x_3 - a_25x_2^2 - a_23x_1x_3 - a_22x_1x_2 - a_21x_1^2 \\
    r_3 &= x_3x_2 - a_16x_2x_3 - a_15x_2^2 - a_14x_2x_1 - a_13x_1x_3 - a_12x_1x_2 - a_11x_1^2
\end{align*}

with either \( a_{26} = a_{12} = 0 \), or \( a_{25} = 0, a_{26} = 1 \).
In the special case of algebra C0, in addition to the previous change of variable we can also change $x_2$ to $x_2 + \gamma x_1$. In this case we can reduce $r_1, r_2, r_3$ further to have

\begin{align*}
    r_1 &= x_3^2 - a_6 x_2 x_3 - a_5 x_2^2 - a_4 x_2 x_1 - a_3 x_1 x_3 - a_2 x_1 x_2 - a_1 x_1^2 \\
    r_2 &= x_3 x_1 - a_2 x_2 x_3 - a_25 x_2^2 - a_23 x_1 x_3 - a_22 x_1 x_2 - a_21 x_1^2 \\
    r_3 &= x_3 x_2 - a_16 x_2 x_3 - a_{15} x_2^2 - a_{14} x_2 x_1 - a_{13} x_1 x_3 - a_{12} x_1 x_2 - a_{11} x_1^2 \\
    r_7 &= x_2^2 x_1 - x_1 x_2^2 \\
    r_8 &= x_2 x_1^2 - x_1^2 x_2
\end{align*}

with either $a_{26} = 1, a_{25} = a_{23} = 0$ or $a_{13} = a_{26} = a_{12} = 0$.

**Remark 6.1.** We use $A_i j$ to denote the algebra of type $j$, lying over $C_i$.

Only algebras C0, C1, and C6 have possible regular algebra $A$ containing them. We have the following lemma

**Lemma 6.2.** The regular algebras A6.1, A1.1, A1.2, A1.3, A0.1, A0.2, A0.3, A0.4 are the only possible algebras containing any algebra $C$. In addition, for each algebra $A$, we have $A^{\text{op}} \cong A$ by the change of variable $x_1$ to $-x_1$. Thus for any pushout algebra $D = A \cup_C B$ we also have $D^{\text{op}} \cong D$, by the same change of variable.

**Proof.** Here we give an outline, while omitting the details. In principle the computations can be carried out explicitly. We use the package “Affine” of the program “Maxima” for most of the Gröbner basis computation. This leads to several systems of equations which we then use “Maple” to solve. First we solve the system of equations come from resolving the ambiguities $x_3^2, x_2^2 x_1, x_2 x_2^2, x_3 x_2 x_1, x_3 x_2 x_1^2, x_2^2 x_1^2$. Then we check if the solutions generates the cubic relations $r_7$ and $r_8$. Once this is done we then find a linear resolution for $k_A$ for each algebra. This is equivalent to find a non-singular $3 \times 3$ scalar matrix $S_A$ with $R_A S_A X_A^t = 0$. We then notice from the list below that for each solution we have $A^{\text{op}} \cong A$. \qed

**A6.1**

\[
p^2 = 1
\]

\begin{align*}
    r_1 &= x_3^2 - x_2 x_1 + x_1 x_2 + a_1 x_1^2 \\
    r_2 &= x_3 x_1 - p x_1 x_3 \\
    r_3 &= x_3 x_2 - p x_2 x_3 + p(1 - a_1) x_1 x_3 \\
    r_7 &= x_2 x_1 - 2 x_2 x_1 x_2 + x_1 x_2^2 - 2 x_1 x_2 x_1 + 2 x_1^2 x_2 + 2 a_1(1 - a_1) x_1^3 \\
    r_8 &= x_2 x_1^2 - 2 x_1 x_2 x_1 + x_1^2 x_2
\end{align*}

**A1.1**

\[a_{23} \neq 0, c_2 \neq 0\]

\begin{align*}
    r_1 &= x_3^2 - x_2 x_1 - c_2 x_2 x_3 \\
    r_2 &= x_3 x_1 - a_{23} x_1 x_3 \\
    r_3 &= x_3 x_2 - a_{23}^{-1} x_2 x_3 \\
    r_7 &= x_2 x_1 - (a_{23}^{-1} - c_2 x_2 x_3) x_2 x_1 x_2 - c_2 x_1 x_2^2 \\
    r_8 &= x_2 x_1^2 - (a_{23}^{-1} - c_2 x_2 x_3) x_1 x_2 x_1 - c_2 x_1^2 x_2
\end{align*}
A1.2

\[ p^2 = -1 \]

\[ r_1 = x_3^2 - x_2^2 \]
\[ r_2 = x_3x_1 - px_1x_3 \]
\[ r_3 = x_3x_2 - x_2x_3 - x_1^2 \]
\[ r_7 = x_2^2x_1 + x_1x_2^2 \]
\[ r_8 = x_2x_1^2 + x_1^2x_2 \]

A1.3

\[ p^2 - p + 1 = 0, \ a_{11}a_{25} \neq 1 - p \]

\[ r_1 = x_3^2 - x_2x_1 + px_1x_2 \]
\[ r_2 = x_3x_1 - a_{25}x_2^2 - (1 - p)x_1x_3 \]
\[ r_3 = x_3x_2 - px_2x_3 - a_{11}x_1^2 \]
\[ r_7 = x_2^2x_1 - (p - 1)x_1x_2^2 \]
\[ r_8 = x_2x_1^2 - (p - 1)x_1^2x_2 \]

A0.1 This is the special case of A1.1 with \( a_{23} = p \) and \( c_2 = 1 \).

\[ p^2 = -1 \]

\[ r_1 = x_3^2 - x_2x_1 + x_1x_2 \]
\[ r_2 = x_3x_1 - px_1x_3 \]
\[ r_3 = x_3x_2 + px_2x_3 \]
\[ r_7 = x_2^2x_1 - x_1x_2^2 \]
\[ r_8 = x_2x_1^2 - x_1^2x_2 \]

A0.2

\[ r_1 = x_3^2 - x_2^2 \]
\[ r_2 = x_3x_1 - x_1x_3 \]
\[ r_3 = x_3x_2 + x_2x_3 - a_{15}x_2^2 - x_1^2 \]
\[ r_7 = x_2^2x_1 - x_1x_2^2 \]
\[ r_8 = x_2x_1^2 - x_1^2x_2 \]

A0.3

\[ a_{11}(a_{15} - a_{11}) \neq 1 \]

\[ r_1 = x_3^2 - x_2^2 - x_1^2 \]
\[ r_2 = x_3x_1 - x_1x_3 \]
\[ r_3 = x_3x_2 + x_2x_3 - a_{15}x_2^2 - a_{11}x_1^2 \]
\[ r_7 = x_2^2x_1 - x_1x_2^2 \]
\[ r_8 = x_2x_1^2 - x_1^2x_2 \]
A0.4 Generically regular, with the coefficients $a_i$ satisfying lemma 6.3

\begin{align*}
    r_1 &= x_3^2 - a_5 x_2^2 - a_4 x_2 x_1 - a_4 x_1 x_2 - a_1 x_1^2 \\
    r_2 &= x_3 x_1 - a_5 x_2^2 + x_1 x_3 - a_21 x_1^2 \\
    r_3 &= x_3 x_2 + x_2 x_3 - a_15 x_2^2 - a_11 x_1^2 \\
    r_7 &= x_2^2 x_1 - x_1 x_2^2 \\
    r_8 &= x_2 x_1^2 - x_1 x_2 \\
\end{align*}

The algebra $A$ from section 3 is of this type.

For the algebra A0.4, if we define

\begin{align*}
    k_1 &= -a_4 a_{15} - a_1 a_{25} \\
    k_2 &= a_4 a_{21} + a_5 a_{11} \\
    k_3 &= a_5 - a_21 a_{25} \\
    k_4 &= a_4 + a_{11} a_{25} \\
    k_5 &= a_{11} a_{15} - a_1 \\
\end{align*}

then resolving the ambiguities $x_3^3, x_2^2 x_1, x_2^2 x_2$ give us the following cubic relations,

\begin{align*}
    k_1 r_7 + k_2 r_8 &= 0, \\
    k_3 r_7 + k_4 r_8 &= 0, \\
    -k_4 r_7 + k_5 r_8 &= 0 \\
\end{align*}

Thus for $r_1, r_2, r_3$ to generate $r_7, r_8$, we need the matrix

\[
    \begin{bmatrix}
        k_1 & k_2 \\
        k_3 & k_4 \\
        -k_4 & k_5 \\
    \end{bmatrix}
\]

to have rank 2. This is same as requiring one of the $2 \times 2$ minors to be non-singular. Thus one of $k_1 k_4 - k_2 k_3$ and $k_3 k_5 + k_4^2$ must be non-zero.

In addition to generating $r_7$ and $r_8$, we need to solve for a $3 \times 3$ non-singular scalar matrix $S_A$ with $R_A S_A X_A^t = 0$. Our solution has the form

\[
    S_A = \begin{bmatrix}
        s_1 & s_2 & s_3 \\
        s_2 & -(a_{11} s_3 + a_{21} s_2 + a_1 s_1) & -a_4 s_1 \\
        s_3 & -a_4 s_1 & -(a_{15} s_3 + a_{25} s_2 + a_5 s_1) \\
    \end{bmatrix}
\]

subject to the constraints

\begin{align*}
    0 &= -k_4 s_3 + k_3 s_2 + (-a_{15} k_4 + a_{25} k_5) s_1 \\
    0 &= k_5 s_3 + k_4 s_2 + (a_{11} k_3 + a_{21} k_4) s_1 \\
\end{align*}

A solution for a non-singular $S_A$ is the same as a non-trivial solutions for $s_1, s_2, s_3$, subject to the above two equations, and $\det(S_A) \neq 0$, and one of $k_1 k_4 - k_2 k_3$ or $k_3 k_5 + k_4^2$ non-zero. This is true for generic choices of the coefficients $a_i$. We sum up the above argument in the following lemma,
Lemma 6.3. The algebra $A0.4$ is regular if there is a non-trivial set of solutions $s_1, s_2, s_3$ satisfying the following set of conditions,

\[
\begin{align*}
0 &= -k_4 s_3 + k_3 s_2 + (-a_{15} k_4 + a_{25} k_5) s_1 \\
0 &= k_5 s_3 + k_4 s_2 + (a_{11} k_3 + a_{21} k_4) s_1 \\
0 &\neq \det(S_A) \\
0 &\neq k_1 k_4 - k_2 k_3 \quad \text{or} \quad 0 \neq k_3 k_5 + k_4^2
\end{align*}
\]

Where $k_1 = -a_{4} a_{15} - a_{1} a_{25}$, $k_2 = a_{4} a_{21} + a_{5} a_{11}$, $k_3 = a_{5} - a_{21} a_{25}$, $k_4 = a_{4} + a_{11} a_{25}$, $k_5 = a_{11} a_{15} - a_{1}$, and

\[
S_A = \begin{bmatrix}
s_1 & s_2 & s_3 \\
s_2 & -(a_{11} s_3 + a_{21} s_2 + a_3 s_1) & -a_{4} s_1 \\
s_3 & -a_{4} s_1 & -(a_{15} s_3 + a_{25} s_2 + a_5 s_1)
\end{bmatrix}
\]

Thus the algebra $A0.4$ is regular for generic choices of coefficients $a_i$.

Proof. There are three variables $s_1, s_2, s_3$ with two equations

\[
\begin{align*}
0 &= -k_4 s_3 + k_3 s_2 + (-a_{15} k_4 + a_{25} k_5) s_1 \\
0 &= k_5 s_3 + k_4 s_2 + (a_{11} k_3 + a_{21} k_4) s_1
\end{align*}
\]

and two open conditions

\[
\begin{align*}
0 &\neq \det(S_A) \\
0 &\neq k_1 k_4 - k_2 k_3 \quad \text{or} \quad 0 \neq k_3 k_5 + k_4^2
\end{align*}
\]

Hence there exist non-trivial solutions to $s_1, s_2, s_3$ for an open dense set of coefficients $a_i$’s. Thus the algebra $A0.4$ is regular for generic $a_i$’s. \qed

7. Classification of Pushout Algebras

From the list of algebra $A$’s, we can write down the relations for each possible pushout algebra $D$. It remains to check the condition $TC1$, that is, for each algebra $D$, solve for a non-singular $6 \times 6$ matrix $S = [s_{ij}]$, $i, j \in \{1 \ldots 6\}$ such that $RST = 0$. Here the matrices $R, S, T$ is as defined in [22]. This is done through direct computation, together with the help of the following lemma.

Lemma 7.1. If $D$ is a possible pushout algebra, then its $6 \times 6$ matrix $S$ from $TC1$ has the block form

\[
S = \begin{bmatrix}
S_1 & 0 \\
0 & S_2
\end{bmatrix}
\]

where $S_1$ and $S_2$ are $3 \times 3$ non-singular matrices.

Proof. For this lemma we only need to use the leading term of each relations. Write $D = k(x_3, x_1, x_2, x_4)/(x_3^2 - f_1, x_3 x_1 - f_2, x_3 x_2 - f_3, x_4 x_1 - f_4, x_4 x_2 - f_5, x_4^2 - f_6)$

We expand the entry $RST_{12} = \sum R_{1i} S_{ij} T_{j2}$ in lowest Gröbner basis. The only $x_3 x_4$ term in it is $s_{14} x_3 x_4$. Since $RST = 0$ and there are no relation with $x_3 x_4$, we must have $s_{14} = 0$. Again in $RST_{12}$ we obtain an $x_1 x_4$ term only if $i = 2$, $j = 4$. The monomials whose reduction give possible $x_1 x_4$ terms are $x_1 x_1, x_1 x_2, x_1^2$. These do not appear in $RST_{12}$. Thus the $x_1 x_4$ term in $RST_{12}$ is $s_{24} x_1 x_4$, so we conclude $s_{24} = 0$. Similarly, the $x_2 x_4$ term in $RST_{12}$ is $s_{34} x_2 x_4$, so we conclude $s_{34} = 0$.

If we carry out the same analysis for $x_3 x_4, x_1 x_4, x_2 x_4$ terms in $RST_{13}, RST_{14}$ we get $s_{15} = s_{25} = s_{35} = s_{16} = s_{26} = s_{36} = 0$. Thus the upper right $3 \times 3$ block of
S are 0’s. Similarly, by computing \(x_1x_3, x_2x_3, x_4x_3\) terms in \(RST_{41}, RST_{42}, RST_{43}\) we have the lower left 3 \(\times\) 3 block of S are 0’s.

\[\square\]

**Remark 7.2.** The matrices \(S_1, S_2\) do not have to be the same as the 3 \(\times\) 3 matrices \(S_A, S_B\) from the algebras \(A\) and \(B\). We notice from our computation that if we choose \(\det(S_1) = 1\), then we always have \(\det(S_2) = -1\). We do not yet know the reason behind it.

The following lemma is proved by solving for a matrix \(S\) that satisfies \(RST = 0\) for each algebra. We state it without showing the details of computation.

**Lemma 7.3.** All possible pushout algebras satisfy condition TC1.

**Remark 7.4.** We use \(D_{i,j,k}\) to denote the algebra \(A_{i,j} \cup C \cup B_{i,k}\). Here \(B_{i,k}\) is isomorphic to \(A_{i,k}\), with \(x_3\) replaced by \(x_4\) and any coefficient \(a_n\) replaced by \(b_n\).

As there is a large number of pushout algebras, we do not list the relations for most of them and list only any additional constraints arising from solving for condition TC1.

**Algebras without \(A_{0,4}\).**

For the algebra \(D_{6,11}\) we have \(a_1 = b_1\). There is no additional constraint for the algebras \(D_{1,11}, D_{1,22}, D_{1,33}\). For the algebra \(D_{1,12}\) we have \(c_2 = -1, a_{23}^2 = 1\). For the algebra \(D_{1,13}\) we have \(c_2 = p - 1, a_{23}^4 = p - 1\). It is not possible to form \(A_{1,2} \cup B_{1,3}\) as they require incompatible coefficients in the algebra \(C_1\). There is no additional constraints for the algebras \(D_{0,11}, D_{0,12}, D_{0,13}, D_{0,22}, D_{0,23}, D_{0,33}\). All of the above constraints are exactly what needed to match coefficients in the algebra \(C\)’s.

**Algebras \(D_{0,41}, D_{0,42}, D_{0,43}\).**

For the algebras \(D_{0,41}, D_{0,42}, D_{0,43}, D_{0,44}\) the solutions of the 6 \(\times\) 6 matrix \(S\) has the form

\[
S = \begin{bmatrix}
S_A & 0 \\
0 & S_2
\end{bmatrix}
\]

That is, the upper left 3 \(\times\) 3 block of \(S\) is the same as the 3 \(\times\) 3 matrix \(S_A\) from the algebra \(A_{0,4}\). Thus we can keep using the notations introduced in lemma 6.3. Let \(k_i\)’s and \(s_i\)’s as defined in lemma 6.3. Let also

\[
s_4 = -a_{11}k_3s_3 + (a_{25}k_5 - a_{15}k_4 - a_{21}k_3)s_2 \\
+ (k_3k_5 + k_4^2 - a_{11}a_{25}k_4 - a_{11}a_{15}k_3)s_1
\]

Then the algebras \(D_{0,41}, D_{0,42}, D_{0,43}\) are regular if the coefficients \(a_i\) satisfy lemma 6.3 and the extra open condition \(s_4 \neq 0\). It is easy to see that this is again true for generic choices of \(a_i\)’s.

**Algebra \(D_{0,44}\).**
Finally we list the relations of algebra $D0.44 = A0.4 \cup_{C_0} B0.4$ as follows,

$$
\begin{align*}
    r_1 &= x_1^2 - a_5 x_2^2 - a_4 x_2 x_1 - a_4 x_1 x_2 - a_1 x_1^2 \\
    r_2 &= x_3 x_1 - a_2 x_2^2 + x_1 x_3 - a_21 x_1^2 \\
    r_3 &= x_3 x_2 + x_2 x_3 - a_15 x_2^2 - a_11 x_1^2 \\
    r_4 &= x_4 x_1 - b_25 x_2^2 + x_1 x_4 - b_21 x_1^2 \\
    r_5 &= x_4 x_2 + x_2 x_4 - b_15 x_2^2 - b_11 x_1^2 \\
    r_6 &= x_4^2 - b_3 x_2^2 - b_4 x_2 x_1 - b_4 x_1 x_2 - b_1 x_1^2
\end{align*}
$$

Since $B0.4 \cong A0.4$, after identifying $a_i$'s with $b_i$'s, the coefficients $b_i$'s also satisfy lemma [6.3]. For the algebra $B0.4$ we define $l_i$, $S_B$, $t_1, t_2, t_3, t_4$ from the coefficients $b_i$'s similarly as we defined $k_i$, $S_A$, $s_1, s_2, s_3, s_4$ in lemma [6.3] from the coefficients $a_i$'s. The algebra $D0.44$ is regular if there is a non-trivial set of solutions $\{s_1, s_2, s_3, t_1, t_2, t_3\}$ satisfying

$$
\begin{align*}
    0 &= -k_4 s_3 + k_3 s_2 + (-a_{15} k_4 + a_{25} k_5) s_1 \\
    0 &= k_5 s_3 + k_4 s_2 + (a_{11} k_3 + a_{21} k_4) s_1 \\
    0 &\neq \det(S_A) \\
    0 &\neq k_1 k_4 - k_2 k_3 \quad \text{or} \quad 0 \neq k_3 k_5 + k_4^2 \\
    0 &= -l_4 t_3 + l_3 t_2 + (-b_{15} l_4 + b_{25} l_5) t_1 \\
    0 &= l_5 t_3 + l_4 t_2 + (b_{11} l_3 + b_{21} l_4) t_1 \\
    0 &\neq \det(S_B) \\
    0 &\neq l_1 l_4 - l_2 l_3 \quad \text{or} \quad 0 \neq l_3 l_5 + l_4^2 \\
    0 &= s_4 + t_4
\end{align*}
$$

There are six variables, five equations, with extra open conditions, hence we have non-trivial solutions for an open set of $a_i$'s and $b_j$'s. We conclude the algebra $D0.44$ is generically regular. The algebra $D$ from section [6.3] is of type $D0.44$.

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