AN EQUIVALENT FORMULATION OF CHROMATIC QUASI-POLYNOMIALS

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ABSTRACT. Given a central integral arrangement, the reduction of the arrangement modulo positive integers \( q \) gives rise to a subgroup arrangement in \( (\mathbb{Z}/q\mathbb{Z})^\ell \). Kamiya-Takemura-Terao (2008) introduced the notion of characteristic quasi-polynomials, which uses to evaluate the cardinality of the complement of the subgroup arrangement. Chen-Wang (2012) found a similar but more general setting that replacing the integral arrangement by its restriction to a subspace of \( \mathbb{R}^\ell \), and evaluating the cardinality of the \( q \)-reduction complement will also lead to a quasi-polynomial in \( q \). On an independent study, Brändén-Moci (2014) defined the so-called chromatic quasi-polynomial, and initiated the study of \( q \)-colorings on a finite list of elements in a finitely generated abelian group. The main purpose of this paper is to verify that the Chen-Wang’s quasi-polynomial and the Brändén-Moci’s chromatic quasi-polynomial are equivalent in the sense that the quasi-polynomials enumerate the cardinalities of isomorphic sets.

1. INTRODUCTION

Background. In the simplest setting, when a finite list \( A \) of integer vectors in \( \mathbb{Z}^\ell \) is given, we may naturally associate to it a central hyperplane arrangement \( \mathcal{A}(\mathbb{R}) \) in \( \mathbb{R}^\ell \), which we call integral arrangement. The study of a hyperplane arrangement typically goes along with the study of its characteristic polynomial as the polynomial carries combinatorial and topological information of the arrangement (e.g., [OS80]). In this paper, we are mainly interested in an arithmetical method, generally known as “finite field method”, for studying the integral arrangements. The method probably was first initiated by [CR70] and developed into a systematic tool by Athanasiadis [Ath96], after closely related techniques have been used by Björner-Ekedahl [BE97], and Blass-Sagan [BS98] to solve problems related to subspace arrangements. Roughly speaking, suppose that the integral arrangement \( \mathcal{A}(\mathbb{R}) \) associated to the list \( A \) is given, we can take coefficients...
modulo a positive integer \( q \) and get an arrangement \( \mathcal{A}(\mathbb{Z}/q\mathbb{Z}) \) of subgroups in \( (\mathbb{Z}/q\mathbb{Z})^\ell \). One of the reasons why the method is regarded as the “finite field method” presumably comes from one of the most well-known and fundamental results in the theory. It states that when \( q \) is a sufficiently large prime, the arrangement \( \mathcal{A}(\mathbb{Z}/q\mathbb{Z}) \) now is defined over the finite field \( \mathbb{F}_q \), and the cardinality of its complement coincides with \( \chi_{\mathcal{A}(\mathbb{R})}(q) \), the evaluation of the characteristic polynomial \( \chi_{\mathcal{A}(\mathbb{R})}(t) \) of \( \mathcal{A}(\mathbb{R}) \) at \( q \) (e.g., [Ath96, Theorem 2.2]).

The fundamental theorem mentioned above is efficiently applicable to compute the characteristic polynomials of several arrangements arising from root systems (e.g., [Ath96]). Kamiya-Takemura-Terao [KTT08] showed that the cardinality of the complement is actually a quasi-polynomial in \( q \), and named this the \emph{characteristic quasi-polynomial} of \( \mathcal{A} \) as its 1-constituent is identical with \( \chi_{\mathcal{A}(\mathbb{R})}(t) \). Chen-Wang [CW12] considered the restriction of the integral arrangement to a subspace of \( \mathbb{R}^\ell \), and proved a stronger result that after taking reduction modulo \( q \) of the restricted arrangement, the cardinality of the complement of is also a quasi-polynomial in \( q \). Later on, Yoshinaga ([Yos16], [Yos18]) extended the analysis on the deformations of root system arrangements and enhanced the calculation of the characteristic quasi-polynomials via the connection with Ehrhart quasi-polynomials.

In yet another consideration, given a finite list \( \mathcal{A} \) in \( \mathbb{Z}^\ell \), we can also associate it a \emph{toric arrangement} \( \mathcal{A}(G) \) in the torus \( G^\ell \) with \( G \) is \( \mathbb{S}^1 \) or \( \mathbb{C}^\times \) (e.g., [DCP05], [Moc12]). Although we are concerned with the subtori (hypersurfaces) instead of the hyperplanes, the “finite field method” remains alive and is formulated in various ways (e.g., [Law11], [ERS09], [ACH15]). To compute and even to make a broader understanding the characteristic polynomial, an arithmetical generalization of the ordinary Tutte polynomial [Tut54], the arithmetic Tutte polynomial was introduced [Moc12]. These polynomials are currently receiving increasing attention (e.g., [DM13], [FM16]). Brändén-Moci [BM14] defined the \emph{Tutte quasi-polynomial} associated to a finite list of elements in a finitely generated abelian group. This quasi-polynomial not only produces the interpolation between the Tutte polynomial and the arithmetic Tutte polynomial but also gives rise to \emph{chromatic quasi-polynomial} and \emph{flow quasi-polynomial}. These are group-theoretical counterparts of the graphic chromatic and flow polynomials which proved to have an application to colorings and flows on CW complexes [DM].

\textbf{Our result.} A newly introduced notion of \emph{G-Tutte polynomials} [LYT17] establishes a common generalization of several “Tutte-like” polynomials including all of the (quasi-)polynomials mentioned previously. In particular, the notion of \emph{G-Tutte polynomials} is useful that enables us to unify the
objects. We take the advantage to get the result that the Chen-Wang’s quasi-polynomial and the Brändén-Moci’s chromatic quasi-polynomial are “the same” in the sense that any Chen-Wang’s quasi-polynomial is a chromatic quasi-polynomial and vice versa (equalities (3.3) and (3.4)).

**Organization of the paper.** The remainder of the paper is organized as follows. In Section 2, we recall definitions of the characteristic, Chen-Wang’s quasi-polynomials (§2.1), the chromatic quasi-polynomials (§2.2) and briefly recall the motivations of why they have been defined. In Section 3, after recalling the basic facts of $G$-Tutte polynomials, we show the equivalence of the chromatic quasi-polynomials and the Chen-Wang’s quasi-polynomials (equalities (3.3) and (3.4)). In §3.2, we give a Deletion-Contraction formula (Theorem 3.7) for the chromatic quasi-polynomials. By using the language of chromatic quasi-polynomials, we also give a discussion to a problem asked by Chen-Wang (Problem 3.11). In Section 4, we generalize the fundamental theorem in the primary “finite field method” applied to any $\mathbb{R}$-plexification. That is, the 1-constituent of the chromatic quasi-polynomial of a list $A$ either is 0 or agrees with the characteristic polynomial of the corresponding $\mathbb{R}$-plexification $A(\mathbb{R})$, and the characteristic polynomial of any $\mathbb{R}$-plexification $A(\mathbb{R})$ can be computed by the 1-constituent of the chromatic quasi-polynomial of the deletion list of $A$ by the list of its torsion elements (Theorem 4.4 and Proposition 4.5).

2. **Preliminaries**

Let us first fix some definitions and notations throughout the paper.

A function $f : \mathbb{Z} \to \mathbb{Z}$ is called a *quasi-polynomial* if there exist $\rho \in \mathbb{Z}_{>0}$ and polynomials $g^k(t) \in \mathbb{Z}[t]$ ($1 \leq k \leq \rho$) such that for any $q \in \mathbb{Z}_{>0}$,

$$f(q) = g^k(q),$$

when $q \equiv k \mod \rho$. The number $\rho$ is called a period and the polynomial $g^k(t)$ is called the $k$-constituent of $f(q)$.

Let $\Gamma$ be a finitely generated abelian group, and let $A \subseteq \Gamma$ be a finite list (multiset) of elements in $\Gamma$. We will use the term *pair* $(A, \Gamma)$ to refer to these objects.

For each sublist $S \subseteq \Gamma$, we denote by $r_S$ the *rank* (as an abelian group) of the subgroup $\langle S \rangle \leq \Gamma$ generated by $S$. By the Structure Theorem, we may write $\Gamma / \langle S \rangle \simeq \bigoplus_{i=1}^{n_S} \mathbb{Z} / d_{S,i} \mathbb{Z} \oplus \mathbb{Z}^{r_S - r}$ where $n_S \geq 0$ and $1 < d_{S,i} | d_{S,i+1}$. The LCM-period $\rho_A$ of $A$ is defined by

$$\rho_A := \text{lcm}(d_{S,n_S} \mid S \subseteq A).$$

Given a group $K$, denote by $K_{\text{tor}}$ the torsion subgroup of $K$. Denote

$$S_{\text{tor}} := S \cap \Gamma_{\text{tor}}.$$
We are going to investigate one of typical problems in enumerative combinatorics: counting the sizes of sets which depend on positive integers \( q \). Our motivated example is the graphic chromatic polynomials. Let \( G = (V, E) \) be a graph. Enumerating the set \( c_G(q) \) of all proper \( q \)-colorings, i.e., labelings \( x \in \{1, \ldots, q\}^V \) such that adjacent vertices get different labels: if \((ij) \in E \) then \( x_i \neq x_j \), gives rise to a polynomial in \( q \). The polynomial \( c_G(q) \) is broadly known as the chromatic polynomial of \( G \), going back to Birkhoff and Whitney. More generally, it happens quite often that enumerating the cardinalities of sets will lead to quasi-polynomials. One of the most famous examples is that given a rational polytope \( P \subseteq \mathbb{R}^d \), the function \( \#(qP \cap \mathbb{Z}^d) \) for \( q \in \mathbb{Z}_{>0} \) agrees with a quasi-polynomial, called the Ehrhart quasi-polynomial.

### 2.1. Characteristic and Chen-Wang’s quasi-polynomials

Our next and important example that a quasi-polynomial appears in the counting problem list is that of characteristic quasi-polynomials. We will define the characteristic quasi-polynomials in a slightly different language to what has been stated in the Introduction part. For instance, the arrangement \( A(\mathbb{Z}/q\mathbb{Z}) \) called \( q \)-reduction arrangement in [KTT08] and its complement will not appear here but later in Section 3.1 after invoking the notion of \( G \)-plexifications. We specify \( \Gamma = \mathbb{Z}^\ell \). Let \( q \in \mathbb{Z}_{>0} \), and set \((\mathbb{Z}/q\mathbb{Z})^\times := (\mathbb{Z}/q\mathbb{Z}) \setminus \{0\}\). For simplicity of notation, we use the same symbols \( A \) and \( z \) for the realizations of the list \( A \subseteq \mathbb{Z}^\ell \) and the element \( z \in (\mathbb{Z}/q\mathbb{Z})^\ell \) as matrices of size \( \ell \times \#A \) and \( 1 \times \ell \), respectively. Denote

\[
\text{KTT}(A, \mathbb{Z}^\ell; q) := \{z \in (\mathbb{Z}/q\mathbb{Z})^\ell \mid z \cdot A \in ((\mathbb{Z}/q\mathbb{Z})^\times)^\#A \}.
\]

We agree that \( \text{KTT}(0, \mathbb{Z}^\ell; q) = (\mathbb{Z}/q\mathbb{Z})^\ell \), thought of as no constraints on \( z \). Kamiya-Takemura-Terao [KTT08] produced two different methods with one of them relies on the theory of Ehrhart quasi-polynomials to show that \( \#\text{KTT}(A, \mathbb{Z}^\ell; q) \) is a monic quasi-polynomial in \( q \) with a period \( \rho_A \). The quasi-polynomial is called the characteristic quasi-polynomial of \( A \). The name really explains the main reason of why the quasi-polynomial was introduced as its generality influences the study of real hyperplane arrangements. That is, its 1-constituent coincides with the characteristic polynomial (e.g., [OT92, Definition 2.52]) of the hyperplane arrangement \( \{H_\alpha \mid \alpha \in A\} \) in \( \mathbb{R}^\ell \) with \( H_\alpha \) is the hyperplane orthogonal to \( \alpha \) (e.g., [Ath96], [KTT08]).

Let \( B \) be another finite list in \( \mathbb{Z}^\ell \). Chen-Wang [CW12] considered a more general setting

\[
\text{CW}(A, B, \mathbb{Z}^\ell; q) := \left\{z \in (\mathbb{Z}/q\mathbb{Z})^\ell \mid \begin{array}{l} z \cdot A \in ((\mathbb{Z}/q\mathbb{Z})^\times)^\#A \\ z \cdot B = (0)^\#B \end{array} \right\},
\]
and applied the elementary divisor method of [KTT08] to show that the cardinality \( \#\text{CW}(\mathcal{A}, \mathcal{B}, \mathbb{Z}^\ell; q) \) is also a quasi-polynomial in \( q \). The notion of Chen-Wang’s quasi-polynomials strictly generalizes that of characteristic quasi-polynomials because \( \text{KTT}(\mathcal{A}, \mathbb{Z}^\ell; q) = \text{CW}(\mathcal{A}, \mathbb{B}, \mathbb{Z}^\ell; q) \) when \( \mathbb{B} \) is the zero matrix, and \( \#\text{KTT}(\emptyset, \mathbb{Z}^\ell; q) = q^\ell \) while \( \#\text{CW}(\emptyset, \mathbb{B}, \mathbb{Z}^\ell; q) \) still depends on \( \mathbb{B} \).

2.2. Chromatic quasi-polynomials. Let \((\mathcal{A}, \Gamma)\) be any pair. Brändén-Moci [BM14] defined the following set

\[
\text{BM}(\mathcal{A}, \Gamma; q) := \{ \varphi \in \text{Hom}(\Gamma, \mathbb{Z}/q\mathbb{Z}) \mid \varphi(\alpha) \neq 0 \text{ for all } \alpha \in \mathcal{A} \},
\]

and proved that its cardinality \( \#\text{BM}(\mathcal{A}, \Gamma; q) \) is a quasi-polynomial in \( q \) for which \( \rho_{\mathcal{A}} \) is a period.\(^\text{1}\) Thus any characteristic quasi-polynomial is indeed a Brändén-Moci’s quasi-polynomial by the following way. Fix a standard basis (of unit vectors) \( \{\epsilon_1, \ldots, \epsilon_\ell\} \) for \( \mathbb{Z}^\ell \), and apply the isomorphism \( \text{Hom}(\mathbb{Z}^\ell, \mathbb{Z}/q\mathbb{Z}) \cong (\mathbb{Z}/q\mathbb{Z})^\ell \) to obtain \( \text{KTT}(\mathcal{A}, \mathbb{Z}^\ell; q) = \text{BM}(\mathcal{A}, \mathbb{Z}^\ell; q) \).

The authors named the quasi-polynomial the chromatic quasi-polynomial as it generalizes the concept of chromatic polynomials defined on graphs. We briefly recall how it can be seen. Let \( \mathcal{G} = (V, E) \) be a graph. Define a list of vectors \( \mathcal{L} = \{\alpha_e \mid e \in E\} \) in \( \mathbb{Z}^{#V} \) as follows. If \( e = (ij) \in E \), let \( \alpha_e \) be the vector with entry \( j \) is 1, entry \( i \) is \(-1\), and the other entries are 0. Then the chromatic polynomial \( c_\mathcal{G}(q) \) of \( \mathcal{G} \) can be expressed as \( c_\mathcal{G}(q) = \#\text{BM}(\mathcal{L}, \mathbb{Z}^{#V}; q) \) with the unique constituent coincides with the characteristic polynomial of the real graphical arrangement \( \{x_i = x_j \mid (ij) \in E\} \) in variable \( q \) (e.g., [OT92, Theorem 2.88]).

3. Unify the quasi-polynomials

3.1. \(G\)-Tutte polynomials. The Chen-Wang’s quasi-polynomial and the Brändén-Moci’s chromatic quasi-polynomial arise independently in different contexts and may seem unrelated at first glance. We will show that the notion of \(G\)-Tutte polynomials is useful to unify them.

Let \( \mathcal{G} \) be an arbitrary abelian group. We recall the notions of \(G\)-plexifications and \(G\)-Tutte polynomials of \( \mathcal{A} \) following [LTY17, §3]. We regard \( \text{Hom}(\Gamma, G) \) as our total group. For each \( \alpha \in \mathcal{A} \), we define the \(G\)-hyperplane associated to \( \alpha \) as follows:

\[
H_{\alpha, G} := \{ \varphi \in \text{Hom}(\Gamma, G) \mid \varphi(\alpha) = 0 \}.
\]

Then the \(G\)-plexification \( \mathcal{A}(G) \) of \( \mathcal{A} \) is the collection of the subgroups \( H_{\alpha, G} \)

\[
\mathcal{A}(G) := \{ H_{\alpha, G} \mid \alpha \in \mathcal{A} \}.
\]

\(^\text{1}\)Brändén-Moci actually defined a somewhat different period. However, the fact that \( \rho_{\mathcal{A}} \) is also a period becomes clear after proving Theorem 3.5.
The $G$-complement $\mathcal{M}(A; \Gamma, G)$ of $\mathcal{A}(G)$ is defined by

$$\mathcal{M}(A; \Gamma, G) := \text{Hom}(\Gamma, G) \smallsetminus \bigcup_{\alpha \in A} H_{\alpha, G}.$$  

For any sublist $S \subseteq A$, the deletion $A \setminus S$ is defined as a list of elements in the same group $\Gamma$. We also define the contraction $A/S$ as the list of cosets $\{[\alpha] \mid \alpha \in A \setminus S\}$ in the group $\Gamma/\langle S \rangle$. The method of identifying sets discussed in [LTY17, §3.2] enables us to write

$$M(A/S; \Gamma/\langle S \rangle, G) = \left\{ \varphi \in \text{Hom}(\Gamma, G) \mid \varphi(\alpha) = 0, \text{ for all } \alpha \in S \right\} \cup \left\{ \varphi \in \text{Hom}(\Gamma, G) \mid \varphi(\alpha) \neq 0, \text{ for all } \alpha \in A \setminus S \right\}.$$  

An abelian group $G$ is said to be torsion-wise finite if $G[d] := \{x \in G \mid d \cdot x = 0\}$ is finite for all $d \in \mathbb{Z}_{>0}$. In what follows, we assume that $G$ is torsion-wise finite.

**Definition 3.1.** The $G$-multiplicity $m(S; G)$ for each $S \subseteq A$ is defined by

$$m(S; G) := \# \text{Hom}((\Gamma/\langle S \rangle)_{\text{tor}}, G).$$  

**Definition 3.2.**

1. The $G$-Tutte polynomial $T_G^A(x, y)$ of $A$ is defined by

$$T_G^A(x, y) := \sum_{S \subseteq A} m(S; G)(x - 1)^{r_A - r_S}(y - 1)^{#S - r_S}.$$  

2. The $G$-characteristic polynomial $\chi_G^A(t)$ of $A$ is defined by

$$\chi_G^A(t) := (-1)^{r_A} \cdot t^{r_A - r_S} \cdot T_G^A(1 - t, 0).$$  

The notion of $q$-reduction arrangements we mentioned in §2.1 is specialization of that of $\mathbb{Z}/q\mathbb{Z}$-plexifications. For general $\Gamma$, it turns out that

$$\text{BM}(A, \Gamma; q) = \mathcal{M}(A; \Gamma, \mathbb{Z}/q\mathbb{Z}).$$

Using formula (3.1) and the equality (3.2) above, we can write

$$\text{CW}(A, B, \mathbb{Z}^l; q) = \text{BM}((A \sqcup B)/B, \mathbb{Z}^l/\langle B \rangle; q).$$

Thus any Chen-Wang’s quasi-polynomial is a chromatic quasi-polynomial defined on a certain contraction list. The converse is also true as we will see in the lemma below.

**Lemma 3.3.** Given a pair $(A, \Gamma)$ with $\Gamma \simeq \mathbb{Z}^r \oplus \mathbb{Z}/d_1 \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_s \mathbb{Z}$, we can find two lists $Q \subseteq \mathcal{L} \subseteq \mathbb{Z}^{r+s}$ with $r_Q = s$ such that $A = \mathcal{L}/Q$.

**Proof.** We can view $\Gamma \simeq \mathbb{Z}^{r+s}/\langle Q \rangle$, where $Q = \{q_1, \ldots, q_s\} \subseteq \mathbb{Z}^{r+s}$, $q_i$ has $d_i$ in the $(r + i)$-th coordinate and 0 elsewhere. Thus $\mathcal{A}$ can be identified with a list of cosets $\mathcal{A} = \{[\alpha_1], \ldots, [\alpha_k]\}$ with $\alpha_i \in \mathbb{Z}^{r+s}$. We choose a representative $\alpha_i \in \mathbb{Z}^{r+s}$ for each coset, which is determined up to a linear
combination of elements from $Q$. Define $\tilde{A} := \{a_1, \ldots, a_k\} \subseteq \mathbb{Z}^{r+s}$, and $L := \tilde{A} \cup Q \subseteq \mathbb{Z}^{r+s}$. Thus $\mathcal{A} = L/Q$. \hfill \Box

**Remark 3.4.** The construction of the lists $\tilde{A}$ and $L$ presented in Lemma 3.3 is probably well-known among experts, for instance [DM13, §3.4], wherein it plays a crucial role in proving the representability of the duals of arithmetic matroids.

With the notation as in Lemma 3.3, for any pair $(\mathcal{A}, \Gamma)$ we can write
\[
BM(\mathcal{A}, \Gamma; q) = CW(\tilde{A}, Q, \mathbb{Z}^{r+s}; q).
\]

We have verified that the Chen-Wang's quasi-polynomial and the Brändén-Moci's chromatic quasi-polynomial are "equivalent" in the sense that the quasi-polynomials enumerate the cardinalities of isomorphic sets.

3.2. **More on chromatic quasi-polynomials.** Given any pair $(\mathcal{A}, \Gamma)$, let us denote by $\chi_{\mathcal{A}}^{\text{quasi}}(q)$ the chromatic quasi-polynomial of $\mathcal{A}$ i.e., $\chi_{\mathcal{A}}^{\text{quasi}}(q) = \#\mathcal{M}(\mathcal{A}; \Gamma, \mathbb{Z}/q\mathbb{Z})$. We also write $f_k^{\mathcal{A}}(t)$ for the $k$-constituent of $\chi_{\mathcal{A}}^{\text{quasi}}(q)$ ($1 \leq k \leq \rho_{\mathcal{A}}$).

**Theorem 3.5** ([BM14], [LTY17]).
\[\chi_{\mathcal{A}}^{\text{quasi}}(q) = \chi_{\mathcal{A}'}^{\mathbb{Z}/q\mathbb{Z}}(q).\]

**Proposition 3.6** ([CW12]).

1. For any $k$ with $1 \leq k \leq \rho_{\mathcal{A}}$, $f_k^{\mathcal{A}}(t) = \chi_{\mathcal{A}}^{\mathbb{Z}/q\mathbb{Z}}(t)$.
2. $\chi_{\mathcal{A}}^{\text{quasi}}(q)$ satisfies the GCD-property i.e. $f_k^{\mathcal{A}}(t) = f_k^{\mathcal{A'}}(t)$ if $\gcd(a, \rho_{\mathcal{A}}) = \gcd(b, \rho_{\mathcal{A}})$.
3. For any $k$ with $1 \leq k \leq \rho_{\mathcal{A}}$, if $\gcd(q, \rho_{\mathcal{A}}) = k$, then $\chi_{\mathcal{A}}^{\text{quasi}}(q) = f_k^{\mathcal{A}}(q)$.

Fix $\alpha \in \mathcal{A}$. Denote $\mathcal{A}' := \mathcal{A} \setminus \{\alpha\}$, and $\mathcal{A}'' := \mathcal{A}/\{\alpha\}$.

**Theorem 3.7** (Deletion-Contraction formula).
\[\chi_{\mathcal{A}}^{\text{quasi}}(q) = \chi_{\mathcal{A}'}^{\mathbb{Z}/q\mathbb{Z}}(q) - \chi_{\mathcal{A}''}^{\mathbb{Z}/q\mathbb{Z}}(q).\]

**Proof.** This follows directly from [LTY17, Corollary 4.11] and Theorem 3.5 by letting $G = \mathbb{Z}/q\mathbb{Z}$. We can also obtain it from [LTY17, Proposition 3.4]. \hfill \Box

**Remark 3.8.** Using Theorem 3.7, the Deletion-Restriction formula in [CW12, Lemma 3.3] can be exhibited by setting $\mathcal{A}$ as the contraction list $(\mathcal{A} \cup \mathcal{B})/\mathcal{B}$, where $\mathcal{A} \neq \emptyset$ and $\mathcal{B}$ are finite lists in $\mathbb{Z}^\ell$.

**Corollary 3.9.** If $k \leq \min\{\rho_{\mathcal{A}'}, \rho_{\mathcal{A}''}\}$, then the $k$-constituents satisfy
\[f_k^{\mathcal{A}}(t) = f_k^{\mathcal{A'}}(t) - f_k^{\mathcal{A''}}(t).\]

**Proof.** Note that the LCM-period of any deletion/contract list is a divisor of the LCM-period of the parent list. \hfill \Box
Remark 3.10. For a pair $(\mathcal{A}, \Gamma)$, the LCM-period of $\chi_\mathcal{A}^\text{quasi}(q)$ is not necessarily the minimum period. We clarify it by an example. Let $\Gamma = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, $\mathcal{A} = \{\alpha, \beta\} \subseteq \Gamma$ with $\alpha = (0, 0)$ and $\beta = (1, 0)$. Then $\rho_\mathcal{A} = 2$, while the minimum period is actually 1 and $\chi_\mathcal{A}^\text{quasi}(q) = 0$ for every $q$. Note that this fact can also be clarified by another class of examples originated from [CW12, Example 4.2].

We close this section by giving a discussion on a problem asked in [CW12, Problem 2].

Problem 3.11. Let $\mathcal{A}_1, \mathcal{A}_2$ be finite lists in $\mathbb{Z}_\ell$ with $r_{\mathcal{A}_2} = \ell$. Assume that $\#\text{CW}(\mathcal{A}_1, \mathcal{A}_2, \mathbb{Z}_\ell; q) = 0$ for every $q \in \mathbb{Z}_{>0}$. Then there exists $\alpha \in \mathcal{A}_1$ such that $\alpha \not\in \langle \mathcal{A}_2 \rangle$.

Discussion. The statement is true if and only if $\ell = 1$. Assume that $\ell = 1$. By equality (3.3), we rewrite the assumption as $\#\text{BM}(\mathcal{A}, \Gamma; q) = 0$ with $\mathcal{A} = (\mathcal{A}_1 \sqcup \mathcal{A}_2)/\mathcal{A}_2$, and $\Gamma = \mathbb{Z}/\langle \mathcal{A}_2 \rangle \cong \mathbb{Z}/d\mathbb{Z}$ for some $d \in \mathbb{Z}_{>0}$. Suppose to the contrary that for every $\alpha \in \mathcal{A}_1$, $\alpha \not\in \langle \mathcal{A}_2 \rangle$. It is equivalent to saying that $\overline{\alpha} \neq 0$ for all $\overline{\alpha} \in \mathcal{A}$. Set $T := \{z \in \mathbb{C} \mid |z|^d = 1\}$. For each $\overline{\alpha} \in \mathcal{A}$ with $0 \leq a \leq d - 1$, set $T_a := \{z \in T \mid z^a = 1\}$. Thus

$$f^\mathcal{A}_\mathcal{A}(t) = \chi^\mathbb{Z}/\rho_{\mathcal{A}^\mathbb{Z}}(t) = \# \left( T \setminus \bigcup_{\overline{\alpha} \in \mathcal{A}} T_a \right) > 0,$$

which is a contradiction. For $\ell \geq 2$, we show that the statement is not true by providing a counterexample. Let us first prove the following fact: if $\Gamma = \mathbb{Z}/d_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_\ell\mathbb{Z}$ is a finite abelian group containing at least two distinct nonidentity elements of order 2, say $\beta_1, \beta_2$, and $\mathcal{A} = \{\alpha \in \Gamma \mid \alpha \neq 0_\Gamma\}$, then $\#\text{BM}(\mathcal{A}, \Gamma; q) = 0$ for every $q \in \mathbb{Z}_{>0}$. Indeed by definition,

$$\text{BM}(\mathcal{A}, \Gamma; q) = \{\varphi \in \text{Hom}(\Gamma, \mathbb{Z}/q\mathbb{Z}) \mid \varphi(\alpha) \neq 0, \text{ for all } \alpha \in \mathcal{A}\},$$

$$= \{\varphi \in \text{Hom}(\Gamma, \mathbb{Z}/q\mathbb{Z}) \mid \varphi \text{ is injective}\}.$$

If the set above is nonempty, then $\varphi(\alpha), \varphi(\beta)$ are distinct and both have order 2 in $\mathbb{Z}/q\mathbb{Z}$. This contradiction implies that $\#\text{BM}(\mathcal{A}, \Gamma; q) = 0$. With the notation as in equality (3.4), $\#\text{CW}(\mathcal{A}, \mathbb{Q}, \mathbb{Q}_\ell; q) = 0$. Now choose $\Gamma = (\mathbb{Z}/2\mathbb{Z})^{\ell}$ with $\ell \geq 2$, and let $\mathcal{A}_1 = \mathcal{A}, \mathcal{A}_2 = Q$.

4. Application to Hyperplane Arrangements

The aim of this section is to generalize the result in [KTT08] that the 1-constituent of $\chi_\mathcal{A}^\text{quasi}(q)$ agrees with the characteristic polynomial of some real arrangement. A natural choice is $\mathcal{A}(\mathbb{R})$. However, as long as $\Gamma$ is any finitely generated abelian group and the list $\mathcal{A}$ may contain torsion elements
of $\Gamma$, we need to know what $A(\mathbb{R})$ is all about. It turns out that we can realize $A(\mathbb{R})$ as a certain (restriction of) integral arrangement. Then we also would like to compute the characteristic polynomial of any $\mathbb{R}$-plexification $A(\mathbb{R})$. These facts will be made clear in Theorem 4.4 and Propositions 4.3, 4.5. More generally, we will give other interpretations for every chromatic quasi-polynomial and their constituents through subspace and toric viewpoints in our forthcoming paper [TY18].

In the following setting and until before Proposition 4.3, we restrict our attention to the case $\Gamma = \mathbb{Z}^\ell$, and view $A$ as a finite list of nonzero vectors in $\mathbb{Z}^\ell$. We regard $\{\epsilon_1, \ldots, \epsilon_\ell\}$ as the standard basis for $\mathbb{R}^\ell$, and equip it the standard inner product $(\cdot, \cdot)$. Then the $\mathbb{R}$-plexification $A(\mathbb{R})$ is an arrangement of (possibly repeated) hyperplanes in $\mathbb{R}^\ell$ with each hyperplane $H_\alpha, \mathbb{R}$ can be identified with $H_\alpha = \{x \in \mathbb{R}^\ell \mid (\alpha, x) = 0\}$. Such $\mathbb{R}$-plexifications are integral arrangements. Let $L_{A(\mathbb{R})}$ be the intersection poset (e.g., [OT92, §2.1]) of $A(\mathbb{R})$. Note that we require the intersection poset to be a set, not multiset. Also, the ambient space $\mathbb{R}^\ell$ can be added to the arrangement without affecting the arrangement’s intersection poset. For each $X \in L_{A(\mathbb{R})}$, the localization of $A(\mathbb{R})$ on $X$ is defined by
\[ A(\mathbb{R})_X := \{ H \in A(\mathbb{R}) \mid X \subseteq H \}, \]
and the restriction $A(\mathbb{R})^X$ of $A(\mathbb{R})$ to $X$ is defined by
\[ A(\mathbb{R})^X := \{ H \cap X \mid H \in A(\mathbb{R}) \setminus A(\mathbb{R})_X \}. \]

Denote by $X^\perp$ the orthogonal complement of $X$ in $\mathbb{R}^\ell$. Set
\[ A_X := A \cap X^\perp \subseteq A. \]

**Proposition 4.1.** The following formulas are valid at level of multisets:

(1) $A(\mathbb{R})_X = (A_X)(\mathbb{R})$.
(2) $A(\mathbb{R})^X = (A/A_X)(\mathbb{R})$.

**Proof.** The proof of (1) is straightforward. To prove (2), for every $X \in L_{A(\mathbb{R})}$ with $X \neq \mathbb{R}^\ell$, we use $X = \bigcap_{H \in A(\mathbb{R})_X} H$, the longest expression of $X$ in terms of intersection of the hyperplanes in $A(\mathbb{R})$. To see $A(\mathbb{R})^X = (A/A_X)(\mathbb{R})$ as multisets, note that the number of occurrences of each element $H_{\beta, \mathbb{R}} \cap X$ in these multisets is equal to $\#\{ \gamma \in A \setminus A_X \mid \gamma \in \text{span}_R \{\beta, A_X\}\}$. \qed

Denote by $\chi_H(t)$ the characteristic polynomial of the real arrangement $H$ (e.g., [OT92, Definition 2.52]). The following result is essentially appeared in [CW12, Corollary 2.4] (see also [Ath96, Corollary 6.1]). The idea of the proof is to use Whitney’s theorem (e.g., [Sta07, Theorem 2.4]) and Proposition 4.1.
Lemma 4.2. $\chi_{\mathcal{A}(\mathbb{R})}^X(t) = f_{\mathcal{A}/\mathcal{A}_X}^1(t)$.

Now we give an arrangement theoretic realization for $\mathcal{A}(\mathbb{R})$.

Proposition 4.3. Given a pair $(\mathcal{A}, \Gamma)$, if $\mathcal{A}^{\text{tor}} = \emptyset$ then $\mathcal{A}(\mathbb{R})$ is an integral arrangement, and also can be realized as a restriction of $\mathcal{L}(\mathbb{R})$ where $\mathcal{L}$ is a finite list in some $\mathbb{Z}^r$.

Proof. We use the notation as in Lemma 3.3. Set $X := \bigcap_{q \in Q} H_{q, \mathbb{R}} \subseteq \mathcal{L}(\mathbb{R})$. The condition $\mathcal{A}^{\text{tor}} = \emptyset$ is crucial, otherwise it may happen that $Q \subset \mathcal{L}_X$. By Proposition 4.1, $\mathcal{A}(\mathbb{R}) = (\mathcal{L}/\mathcal{L}_X)(\mathbb{R}) = \mathcal{L}(\mathbb{R})^X$. This means that $\mathcal{A}(\mathbb{R})$ is the restriction of $\mathcal{L}(\mathbb{R})$ to $X$, and also can be identified with an integral arrangement in $\mathbb{R}^{r\Gamma}$. □

Next, we prove an important property of $\chi_{\mathcal{A}}^{\text{quasi}}(q)$, which is the main theorem of this section.

Theorem 4.4. Let $(\mathcal{A}, \Gamma)$ be any pair. Then

$$\chi_{\mathcal{A}(\mathbb{R})}(t) = f_{\mathcal{A}\setminus\mathcal{A}^{\text{tor}}}(t).$$

Proof. If $\mathcal{A}^{\text{tor}} = \emptyset$, we apply Proposition 4.3 and Lemma 4.2. If $\mathcal{A}^{\text{tor}} \neq \emptyset$, note that $\mathcal{A}(\mathbb{R})$ and $(\mathcal{A}\setminus\mathcal{A}^{\text{tor}})(\mathbb{R})$ have the same intersection poset. □

The 1-constituent $f_{\mathcal{A}}^1(t)$ sometimes can be regarded as the chromatic polynomial defined on a graph, for example, via connection with graphical arrangements discussed in §2.2. It is well known (and easy to show) that the graphic chromatic polynomial is identical to 0 if the graph contains some (graph theoretic) loop. Recall from [DM13, §4.4] that an element $\alpha \in \mathcal{A}$ is called a loop (resp. coloop) if $\alpha \in \Gamma^{\text{tor}}$ (resp. $r_{\mathcal{A}} = r_{\mathcal{A}'} + 1$). An element $\alpha \in \mathcal{A}$ that is neither a loop nor a coloop is said to be proper. We will prove in the proposition below that a similar result holds for $f_{\mathcal{A}}^1(t)$.

Proposition 4.5. Let $(\mathcal{A}, \Gamma)$ be any pair with $\mathcal{A}^{\text{tor}} \neq \emptyset$. Then

$$f_{\mathcal{A}}^1(t) = 0.$$

Proof. Use Corollary 3.9 (viewing as $k = 1$) to reduce the problem to the case that $\mathcal{A} = \mathcal{F} \sqcup \mathcal{T}$ with $\mathcal{F}$ and $\mathcal{T} \neq \emptyset$ consist of only coloops and loops, respectively. Then apply Proposition 3.6(1). □

Remark 4.6. There is a neater proof: fix $\alpha \in \mathcal{A}^{\text{tor}}$ and break $f_{\mathcal{A}}^1(t)$ into two summations with one of them is taken over $\mathcal{B} \subseteq \mathcal{A}$, $\alpha \in \mathcal{B}$. 

Example 4.7. Let $\Gamma = \mathbb{Z}^2 \oplus \mathbb{Z}/4\mathbb{Z}$, $\mathcal{A} = \{\alpha, \beta, \gamma\} \subset \Gamma$ with $\alpha = (2, 2, 1)$, $\beta = (0, 2, 3)$ and $\gamma = (0, 0, 3)$. Then $\rho_{\mathcal{A}} = \rho_{\mathcal{A} \setminus \{\gamma\}} = 8$, and
\[
\chi_{\mathcal{A}}^{\text{quasi}}(q) = \begin{cases} 
0 & \text{if } \gcd(q, 8) = 1, \\
q^2 & \text{if } \gcd(q, 8) = 2, \\
3q^2 - 4q + 4 & \text{if } \gcd(q, 8) = 4, \\
3q^2 - 12q + 12 & \text{if } \gcd(q, 8) = 8.
\end{cases}
\]

\[
\chi_{\mathcal{A} \setminus \{\gamma\}}^{\text{quasi}}(q) = \begin{cases} 
q^2 - 2q + 1 & \text{if } \gcd(q, 8) = 1, \\
2q^2 - 4q + 4 & \text{if } \gcd(q, 8) = 2, \\
4q^2 - 8q + 8 & \text{if } \gcd(q, 8) = 4, \\
4q^2 - 16q + 16 & \text{if } \gcd(q, 8) = 8.
\end{cases}
\]

Note that $(\mathcal{A} \setminus \{\gamma\})(\mathbb{R}) = \mathcal{L}(\mathbb{R})^X$, where $\mathcal{L}(\mathbb{R}) = \{\{2x + 2y + z = 0\}, \{2y + 3z = 0\}, \{z = 0\}\} \subseteq \mathbb{R}^3$ and $X = \{z = 0\}$, which can also be identified with the integral arrangement $\{\{x + y = 0\}, \{y = 0\}\}$ in $\mathbb{R}^2$. In either way,
\[
\chi_{(\mathcal{A} \setminus \{\gamma\})(\mathbb{R})}(t) = f_{\mathcal{A} \setminus \{\gamma\}}^1(t) = t^2 - 2t + 1.
\]

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