Research article

Superconvergence for optimal control problems governed by semilinear parabolic equations

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Abstract: In this paper, we first investigate optimal control problem for semilinear parabolic and introduce the standard $L^2(\Omega)$-orthogonal projection and the elliptic projection. Then we present some necessary intermediate variables and their error estimates. At last, we derive the error estimates between the finite element solutions and $L^2$-orthogonal projection or the elliptic projection of the exact solutions.

Keywords: finite element approximation; semilinear parabolic equation; optimal control problem; the elliptic projection; $L^2$-orthogonal projection; superconvergence

Mathematics Subject Classification: 49J20, 65N30

1. Introduction

As we know, optimal control theory is widely used in many subjects. In the past few decades, it has attracted the attention of more and more scholars, and is also related to some specific applications, from finance to aerospace industry, from biology to medicine and so on. For example, how a spacecraft to land on the moon surface at rest with minimal fuel consumption [1]? Under what circumstances the tumor can be eliminated [2]?

In fact, optimal control problem (OCP) for partial differential equations (PDEs) is a challenging research hotspot, and much has been done both on the mathematical analysis and on its numerical approximation. Among numerous numerical methods, finite element discretization of the state equation is widely applied. Finite element approximation of optimal control problems plays a great role in modern science, technology, engineering, etc. We can find systematic introduction of finite
element methods and optimal control problems governed by PDEs, for example [6,7,28]. There have been extensive studies in error estimates, convergence of finite element approximation for OCP. Casas, Mateos and Raymond [3,4] have studied a priori error estimation of semilinear elliptic boundary control problems. Chen and Huang [5,9] have gained a priori error estimates of stochastic elliptic PDEs, both a prior and a posterior error estimates of stokes equations with $H^1$-norm state constraint. For a posteriori error estimates of quadratic OCP governed by linear parabolic equation, see Liu and Yan [24], for optimal rates of convergence with Ritz-Galerkin approximations and numerical approximation of a parabolic time OCP, see Lasiecka and Knowles [10,11]. In particular, Liu and Yan studied the posteriori error estimates for control problems governed by elliptic equations [12], and extended it to the OCP dominated by parabolic equation [13] and Stokes equation [14].

Furthermore, the superconvergence properties of OCP is a research focus in the field of optimal control problem, because superconvergence has always been an important tool to obtain high performance finite element discretization, which can provide high-precision approximate solutions. The research on superconvergence began in the late 1970s, and obtained fruitful results, see, e.g. [8,15,21–23,27].

When the objective function in the OCP contains the gradient of scalar function, the mixed finite element method is an effective numerical method. In recent ten years, for the OCP of PDEs by the mixed finite element method, professor Chen’s team has studied this aspect deeply, and has made a series of research achievements, such as a priori error estimation, a posteriori error estimation, $L^\infty$-error estimates and superconvergence etc [16–20,25,26].

Among the numerous research, Chen and Dai in [27] showed the superconvergence for optimal control problems governed by semilinear elliptic equations. The purpose of this paper is to extend the superconvergence property of [27] to the semilinear parabolic control problems.

In this paper, given the state $y$ and the co-state $p$ variables together with their approximations $y_h$ and $p_h$, we say that the approx super converges if the state and co-state variables are approximated by the piecewise linear functions, the control variable is approximated by the piecewise constant functions, we can get the superconvergence properties for both the control variable and the state variables. We are interested in the following optimal control problem

$$
\min_{u \in K} \left\{ \frac{1}{2} \int_0^T \left( \|y(x,t) - y_a(x,t)\|^2_{\Omega} + \|u(x,t)\|^2_{\Omega} \right) \, dt \right\}
$$

(1.1)

$$
y_i(x,t) - \text{div}(A(x)\nabla y(x,t)) + \phi(y(x,t)) = f(x,t) + u(x,t), \quad x \in \Omega, \ t \in J,
$$

(1.2)

$$
y(x,t) = 0, \quad x \in \partial \Omega, \ t \in J,
$$

(1.3)

$$
y(x,0) = y_0(x), \quad x \in \Omega,
$$

(1.4)

where $\Omega$ is a bounded domain in $\mathbb{R}^n$ with a Lipschitz boundary $\partial \Omega$, $0 < T < +\infty$, $J = [0,T]$, $\eta_i(x,t)$ denotes the partial derivative of $y$ in time, $A(x) = (a_{ij}(x))_{n \times n} \in (W^{1,\infty}(\Omega))^{n \times n}$, such that $(A(x)\xi) \cdot \xi \geq c |\xi|^2, \ \forall \xi \in \mathbb{R}^n, \ c > 0$. We assume that the function $\phi(\cdot) \in W^{2,\infty}(-R,R)$ for any $R > 0$, $\phi'(y) \in L^2(\Omega)$ for any $y \in L^2(J;H^1(\Omega))$, and $\phi'(y) \geq 0$. Moreover, we assume that $y_a(x,t) \in C(J;L^2(\Omega))$, $y_0(x) \in H^1_0(\Omega)$ and $K$ is a nonempty closed convex set in $L^2(J;L^2(\Omega))$, defined by

$$
K = \left\{ v(x,t) \in L^2(J;L^2(\Omega)) : \int_\Omega \int_0^T v(x,t) dx dt \geq 0, \ \ a.e. \ x \in \Omega, \ t \in J \right\}.
$$
In this paper, we adopt the standard notation $W^{m,p}(\Omega)$ for Sobolev spaces on $\Omega$ with a norm $\| \cdot \|_{m,p}$ given by $\|v\|_{m,p}^p = \sum |v|_{\alpha}^p$, $\alpha \leq m$, a semi-norm $| \cdot |_{m,p}$ given by $|v|_{m,p}^p = \sum |D^\alpha v|_{L^p(\Omega)}^p$. We set $W^m_0(\Omega) = \{ v \in W^{m,p}(\Omega) : v|_{\partial \Omega} = 0 \}$. For $p = 2$, we denote $H^m(\Omega) = W^{m,2}(\Omega)$, $H^m_0(\Omega) = W^{m,2}_0(\Omega)$, and $\| \cdot \|_m = \| \cdot \|_{m,2}$, $\| \cdot \| = \| \cdot \|_{0,2}$. We denote by $L^s(0,T; W^{m,q}(\Omega))$ the Banach space of all $L^s$ integrable functions from $J$ into $W^{m,q}(\Omega)$ with norm $\|v\|_{L^s(J; W^{m,q}(\Omega))} = \left( \int_0^T \|v\|_{W^{m,q}(\Omega)}^s dt \right)^{1/s}$ for $s \in [1,\infty)$, and the standard modification for $s = \infty$. Similarly, one can define the spaces $H^k(0,T; W^{m,q}(\Omega))$ and $C^k(0,T; W^{m,q}(\Omega))$.

The paper is organized as follows: in section 2, we briefly review the finite element method, and then the approximation schemes for the model optimal control problem will be constructed. In section 3, some intermediate error estimates which is the base of the result will be gained. In section 4, superconvergence properties for both control and state variables are derived.

2. Finite element method for optimal control problems

In this section, we will discuss the finite element approximation of the quadratic optimal control problem governed by semilinear parabolic equations (1.1)–(1.4). We set $W = L^2(J; V)$ with $V = H^1_0(\Omega)$, $X = L^2(0,T; U)$ with $U = L^2(\Omega)$, $\| w \| = \| w \|_{H^1_0(\Omega)}$ and $\| \cdot \|_{L^2(\Omega)}$. Let

\[
A(v, w) = \int_{\Omega} (A(x) \nabla v, \nabla w) dx, \quad \forall v, w \in V,
\]

\[
(f_1, f_2) = \int_{\Omega} f_1 w \cdot f_2 dx, \quad \forall f_1, f_2 \in L^2(\Omega).
\]

It follows from Friedriechs’ inequality that

\[
A(v, v) \geq c \| v \|^2_V, \quad \forall v \in V,
\]

\[
\| A(v, w) \| \leq C \| v \|_V \| w \|_V, \quad \forall v, w \in V.
\]

We denote by $H^{-1}(\Omega)$ the dual space to $H^1_0(\Omega)$. If $f \in H^{-1}(\Omega)$, we note

\[
\| f \|_{H^{-1}(\Omega)} = \| f \|_{-1}, \quad \| f \|_{-1} = \sup_{u \in H^1_0(\Omega), \| u \|_{H^1_0(\Omega)} \leq 1} (f, u). \tag{2.1}
\]

Then the standard weak formula for the state equation reads: find $y(u)$ such that

\[
(y, w) + A(y(u), w) + (\phi(y(u)), w) = (f + u, w), \quad \forall w \in V.
\]

Thus the above equation has a solution.

We recast (1.1)–(1.4) in the following weak form: find $(y, u)$ such that

\[
\min_{u \in U} \left\{ \frac{1}{2} \int_0^T (\| y - y_d \|^2 + \| u \|^2) \, dt \right\} \tag{2.2}
\]

\[
(y, w) + A(y, w) + (\phi(y), w) = (f + u, w), \quad \forall w \in V = H^1_0(\Omega). \tag{2.3}
\]
It is well known (see, e.g., [28]) that the control problem (2.2)–(2.3) has a solution \( (y, u) \), and that if a pair \( (y, u) \) is the solution of (2.2)–(2.3), then there is a co-state \( p \in H^1(J; L^2(\Omega)) \cap W \) such that the triplet \( (y, p, u) \) satisfies optimality conditions as follows:

\[
(y, w) + a(y, w) + (\phi(y), w) = (f + u, w), \quad \forall w \in V, \tag{2.4}
\]

\[
-(p_i, q) + a(q, p) + (\phi'(y)p, q) = (y - y_d, q), \quad \forall q \in V, \tag{2.5}
\]

\[
\int_0^T (u + p, v - u) dt \geq 0, \quad \forall v \in K, \tag{2.6}
\]

\[
y(u)(x, 0) = y_0(x), \quad p(u)(x, T) = 0 \quad \forall x \in \Omega. \tag{2.7}
\]

Now we introduce the following significant result (see [29]).

**Lemma 2.1.** [29] A necessary and sufficient condition for the optimality of a control \( u \in K \) with corresponding state \( y(u) \) and co-state \( p(u) \), respectively, is the following relation:

\[
u = \max(0, \bar{p}) - p,
\]

where \( \bar{p} = \frac{\int_{\Omega} \int_0^T p dx dt}{\int_{\Omega} \int_0^T dx dt} \) denotes the integral average on \( \Omega \times J \) of the function \( p \).

In the following, we will consider the semi-discrete finite element for the problem.

Let \( \mathcal{T}^h \) be regular triangulations of \( \Omega \), such that \( \bar{\Omega} = \bigcup \tau \). Let \( h = \max_{\tau \in \mathcal{T}^h} h_\tau \), where \( h_\tau \) denotes the diameter of the element \( \tau \). Note two spaces as follows:

\[
U^h = \{ u_h \in U : u_{h_\tau} = \text{constant}, \tau \in \mathcal{T}^h \}, \tag{2.9}
\]

\[
V^h = \{ v_h \in C(\bar{\Omega}) : v_{h_\tau} \in P_1, \tau \in \mathcal{T}^h, y_{h_\tau}\mid_{\partial \Omega} = 0 \} \tag{2.10}
\]

\[
K^h := L^2(J; U^h) \cap K \tag{2.11}
\]

where \( P_1 \) is the space of polynomials of degree less than or equal to 1. In addition, \( c \) or \( C \) denotes a general positive constant independent of \( h \).

Now, the finite element approximation of the optimal control problem (2.2)–(2.3) is as follows:

\[
\begin{align*}
&\min_{u_h \in K^h} \left\{ \frac{1}{2} \int_0^T \left( \|y_h - y_d\|^2 + \|u_h\|^2 \right) dt \right\} \tag{2.12} \\
&(y_{h,T}, w_h) + a(y_h, w_h) + (\phi(y_h), w_h) = (f + u_h, w_h), \quad \forall w_h \in V^h. \tag{2.13}
\end{align*}
\]

The optimal control problem (2.12)–(2.13) has a solution \( (y_h, u_h) \), and that if a pair \( (y_h, u_h) \) is the solution of (2.12)–(2.13), then there is a co-state \( p_h \) such that the triplet \( (y_h, p_h, u_h) \) satisfying the following optimal conditions:

\[
(y_{h,T}, w_h) + a(y_h, w_h) + (\phi(y_h), w_h) = (f + u_h, w_h), \quad \forall w_h \in V^h, \tag{2.14}
\]

\[
-(p_{h,T}, q_h) + a(q_h, p_h) + (\phi'(y_h)p_h, q_h) = (y_h - y_d, q_h), \quad \forall q_h \in V^h, \tag{2.15}
\]

\[
\int_0^T (u_h + p_h, v_h - u_h) dt \geq 0, \quad \forall v_h \in K^h, \tag{2.16}
\]

\[
y_h(u_h)(x, 0) = y_0^h(x), \quad p_h(u_h)(x, T) = 0, \quad \forall x \in \Omega. \tag{2.17}
\]
Lemma 3.1. Let $u \in L^2(\Omega)$, then we have the following approximation properties (see e.g., [27] and [30]):

$$w = \max(0, \bar{p}_h) - p_h,$$

where $\bar{p}_h = \frac{\int^T_0 \int_\Omega p_h \, dx \, dt}{\int^T_0 \int_\Omega 1 \, dx \, dt}$ denotes the integral average on $\Omega \times J$ of the function $p_h$.

3. Intermediate error estimates

First of all, we will introduce some intermediate variables. For any $\tilde{u} \in K$, let $(y(\tilde{u}), p(\tilde{u}))$ be the solution of the following equations:

$$(y(\tilde{u}), w) + a(y(\tilde{u}), w) + (\phi(y(\tilde{u})), w) = (f + \tilde{u}, w), \quad \forall w \in V,$$  \hspace{1cm} (3.1)

$$(p(\tilde{u}), q) + a(q, p(\tilde{u})) + (\phi'(y(\tilde{u}))p(\tilde{u}), q) = (y(\tilde{u}) - y_d, q), \quad \forall q \in V.$$  \hspace{1cm} (3.2)

Then, for any $\tilde{u} \in K$, let $(y_h(\tilde{u}), p_h(\tilde{u}))$ be the solution of the following equations:

$$(y_h(\tilde{u}), w_h) + a(y_h(\tilde{u}), w_h) + (\phi(y_h(\tilde{u})), w_h) = (f + \tilde{u}, w_h), \quad \forall w_h \in V^h,$$  \hspace{1cm} (3.3)

$$(p_h(\tilde{u}), q_h) + a(q_h, p_h(\tilde{u})) + (\phi'(y_h(\tilde{u}))p_h(\tilde{u}), q_h) = (y_h(\tilde{u}) - y_d, q_h), \quad \forall q_h \in V^h.$$  \hspace{1cm} (3.4)

Note that $(y, p) = (y(u), p(u)), (y_h, p_h) = (y_h(u_h), p_h(u_h))$.

Now we give the standard $L^2(\Omega)$–orthogonal projection $Q_h : U \rightarrow U^h$, for $U = L^2(\Omega)$, which satisfies: for any $\psi \in U$

$$(\psi - Q_h\psi, u_h) = 0, \quad \forall u_h \in U^h, \hspace{1cm} (3.5)$$

and the elliptic projection $R_h : V \rightarrow V^h$, which satisfies: for all $v \in V$

$$a(v - R_hv, v_h) = 0, \quad \forall v_h \in V^h, \hspace{1cm} (3.6)$$

We have the following approximation properties (see e.g., [27] and [30]):

$$||\psi - Q_h\psi||_{L^s} \leq Ch^{1+s}||\psi||_{L^1}, \quad s = 0, 1,$$  \hspace{1cm} (3.7)

$$||w - R_hw|| \leq Ch^2||w||_{L^2}, \quad \text{for} \ w \in H^2(\Omega).$$  \hspace{1cm} (3.8)

Lemma 3.1. Let $u \in L^2(J; H^1(\Omega))$, for $h$ sufficiently small, there exists a positive constant $C$ such that

$$||y(Q_hu) - y(u)||_{L^2(J; L^2(\Omega))} \leq Ch^2, \hspace{1cm} (3.9)$$

$$||p(Q_hu) - p(u)||_{L^2(J; L^2(\Omega))} \leq Ch^2.$$  \hspace{1cm} (3.10)

Proof. Choose $\tilde{u} = Q_hu$ and $\bar{u} = u$ in (3.1)–(3.2), respectively, then we have the following error equations

$$(y(\tilde{u}), w) + a(y(\tilde{u}), w) + (\phi(y(Q_hu)) - \phi(y(u)), w) = ((Q_hu - u), w). \hspace{1cm} (3.11)$$
namely,

\[-(p_t(Qh u) - p_t(u), p(Qh u) - p(u)) + a(p(Qh u) - p(u), p(Qh u) - p(u))
\]

\[+(\phi'(y(Qh u))p(Qh u) - \phi'(y(u))p(u), p(Qh u) - p(u)) = (y(Qh u) - y(u), p(Qh u) - p(u)),\]  

(3.16)

for any \(w \in V\) and \(q \in V\).

First, choose \(w = y(Qh u) - y(u)\) in (3.11), we have

\[
(\gamma_t(Qh u) - \gamma_t(u), y(Qh u) - y(u)) + a(y(Qh u) - y(u), y(Qh u) - y(u))
\]

\[+(\phi(y(Qh u)) - \phi(y(u)), y(Qh u) - y(u)) = (Qh u - u, y(Qh u) - y(u)).\]  

(3.13)

Now, we estimate the right hand side of (3.13). Using (3.7), we have

\[
(Qh u - u, y(Qh u) - y(u)) = \leq C\|y(Qh u) - y(u)\|_1 \cdot \|Qh u - u\|_1
\]

\[\leq Ch^2\|u\|_1 \cdot \|y(Qh u) - y(u)\|_1.\]  

(3.14)

From (3.13) and (3.14), using \(\epsilon\)-Cauchy inequality and the assumption of \(A\) and \(\phi(\cdot)\), we have

\[
\frac{1}{2} \frac{d}{dt} \|y(Qh u) - y(u)\|_1^2 + c\|y(Qh u) - y(u)\|_1^2
\]

\[\leq (\gamma_t(Qh u) - \gamma_t(u), y(Qh u) - y(u)) + a(y(Qh u) - y(u), y(Qh u) - y(u))
\]

\[+(\phi(y(Qh u)) - \phi(y(u)), y(Qh u) - y(u)) = (Qh u - u, y(Qh u) - y(u))
\]

\[\leq Ch^2\|y(Qh u) - y(u)\|_1
\]

\[\leq Ch^4 + c\|y(Qh u) - y(u)\|_1^2.\]  

(3.15)

Note that

\[y(Qh u)(x, 0) - y(u)(x, 0) = 0,\]

next, integrating the both sides of (3.15) in time from 0 to \(t\), we get

\[\|y(Qh u) - y(u)\|_{L^2(J; L^2(\Omega))}^2 + c\|y(Qh u) - y(u)\|_{L^2(J; H^1(\Omega))}^2 \leq Ch^4,
\]

which implies (3.9).

Choose \(q = p(Qh u) - p(u)\) in (3.11), we have

\[-(p_t(Qh u) - p_t(u), p(Qh u) - p(u)) + a(p(Qh u) - p(u), p(Qh u) - p(u))
\]

\[+(\phi'(y(Qh u))p(Qh u) - \phi'(y(u))p(u), p(Qh u) - p(u)) = (y(Qh u) - y(u), p(Qh u) - p(u)),\]  

(3.16)

namely,

\[-(p_t(Qh u) - p_t(u), p(Qh u) - p(u)) + a(p(Qh u) - p(u), p(Qh u) - p(u))
\]

\[+(\phi'(y(Qh u))p(Qh u) - \phi'(y(u))p(u), p(Qh u) - p(u)) = (y(Qh u) - y(u), p(Qh u) - p(u))
\]

\[= (y(Qh u) - y(u), p(Qh u) - p(u))\]  

\[= (y(Qh u) - y(u), p(Qh u) - p(u)).\]  

(3.12)
for

∥

Notice that

Next, we consider the given condition

\[
(p(u)(\phi'(y(u)) - \phi'(y(Q_hu))), p(Q_hu) - p(u)).
\] (3.17)

Using the assumption for \(\phi(\cdot)\) and (3.9), we have

\[
(p(u)(\phi'(y(u)) - \phi'(y(Q_hu))), p(Q_hu) - p(u))
\leq C\|p(u)\|_{0,4}\|\phi'(y(u)) - \phi'(y(Q_hu))\| \cdot \|p(Q_hu) - p(u)\|_{0,4}
\leq C\|p(u)\|_1\|\phi\|_{W^{2,\infty}}\|y(u) - y(Q_hu)\| \cdot \|p(Q_hu) - p(u)\|_1
\leq Ch^2\|p(Q_hu) - p(u)\|_1
\leq Ch^4 + \|p(Q_hu) - p(u)\|^2_1.
\] (3.19)

where we used the embedding \(\|v\|_{0,4} \leq C\|v\|_1\). Then, using (3.17), (3.18), (3.19) and the assumption for \(\phi(\cdot)\), we have

\[
- \frac{1}{2} \frac{d}{dt} \|p(Q_hu) - p(u)\|^2 + c\|p(Q_hu) - p(u)\|^2_1
\leq - (p_t(Q_hu) - p_t(u), p(Q_hu) - p(u)) + a(p(Q_hu) - p_h(u), p(Q_hu) - p(u))
+ (\phi'(y(Q_hu)(p(Q_hu) - p(u)), p(Q_hu) - p(u))
\leq Ch^2\|p(Q_hu) - p(u)\|_1
\leq Ch^4 + \frac{c}{2}\|p(Q_hu) - p(u)\|^2_1.
\] (3.21)

Next, we consider the given condition

\[
p(Q_hu)(x, T) - p(u)(x, T) = 0,
\] (3.22)

then, we integrate in time from \(t\) to \(T\) in (3.11) and use Gronwall’s inequality, we have

\[
\|p(Q_hu) - p(u)\|^2_{L^2(J; L^2(\Omega))} + \|p(Q_hu) - p(u)\|^2_{L^2(J; H^1(\Omega))} \leq Ch^4,
\] (3.23)

which implies (3.10).

\[\square\]

**Lemma 3.2.** For any \(\tilde{u} \in K\), if the intermediate solution satisfies

\[
y(\tilde{u}), y_t(\tilde{u}), p(\tilde{u}), p_t(\tilde{u}) \in L^2(J; H^1(\Omega)) \cap L^2(J; H^2(\Omega)),
\]

and \(\Omega\) is convex, then we have

\[
\|y_t(\tilde{u}) - R_hy(\tilde{u})\|_{L^2(J; H^1(\Omega))} \leq Ch^2,
\]
(3.24)

\[
\|p_h(\tilde{u}) - R_hp(\tilde{u})\|_{L^2(J; H^1(\Omega))} \leq Ch^2.
\]
(3.25)
Proof. From (3.1)–(3.2) and (3.3)–(3.4), we have the following error equations:

\[
\begin{aligned}
(y_{h,t}(\tilde{u}) - y_t(\tilde{u}), w_h) + a(y_{h}(\tilde{u}) - y(\tilde{u}), w_h) + (\phi(y_{h}(\tilde{u})) - \phi(y(\tilde{u})), w_h) = 0, \\
- (p_{h,t}(\tilde{u}) - p_t(\tilde{u}), q_h) + a(q_h, p_{h}(\tilde{u}) - p(\tilde{u})) + (\phi'(y_{h}(\tilde{u}))p_{h}(\tilde{u}) - \phi'(y(\tilde{u})))p(\tilde{u}), q_h)
\end{aligned}
\]

(3.26)

for any \( w_h \in V_h \) and \( q_h \in V_h \). Using the definition of \( R_h \), the above equation can be restated as

\[
\begin{aligned}
(y_{h,t}(\tilde{u}) - R_h y_{h}(\tilde{u}), w_h) + a(y_{h}(\tilde{u}) - R_h y(\tilde{u}), w_h) + (\phi(y_{h}(\tilde{u})) - \phi(R_h y(\tilde{u})), w_h) \\
- (p_{h,t}(\tilde{u}) - R_h p_{h}(\tilde{u}), q_h) + a(q_h, p_{h}(\tilde{u}) - R_h p(\tilde{u})) + (\phi'(y_{h}(\tilde{u}))p_{h}(\tilde{u}) - \phi'(y(\tilde{u})))p(\tilde{u}), q_h)
\end{aligned}
\]

(3.28)

First, let \( w_h = y_{h}(\tilde{u}) - R_h y(\tilde{u}) \) in (3.28), using the \( \epsilon \)-Cauchy inequality and the assumptions for \( A \) and \( \phi(\cdot) \), we have

\[
\frac{1}{2} \frac{d}{dt} ||y_{h} - R_h y(\tilde{u})||^2 + c||y_{h} - R_h y(\tilde{u})||^2
\]

\[
\leq (y_{h}(\tilde{u}) - R_h y(\tilde{u}), y_{h}(\tilde{u}) - R_h y(\tilde{u}))
\]

\[
+ a(y_{h}(\tilde{u}) - R_h y(\tilde{u}), y_{h}(\tilde{u}) - R_h y(\tilde{u})) + (\phi(y_{h}(\tilde{u})) - \phi(R_h y(\tilde{u})), y_{h}(\tilde{u}) - R_h y(\tilde{u}))
\]

\[
= (y_{h}(\tilde{u}) - R_h y(\tilde{u}), y_{h}(\tilde{u}) - R_h y(\tilde{u})) + (\phi(y_{h}(\tilde{u})) - \phi(R_h y(\tilde{u})), y_{h}(\tilde{u}) - R_h y(\tilde{u}))
\]

\[
\leq C h^2 ||y_{h}(\tilde{u})||_{2} ||y_{h}(\tilde{u}) - R_h y(\tilde{u})|| + C ||\phi||_{W^{1,\infty}} ||y(\tilde{u})||_{2} \cdot ||y_{h}(\tilde{u}) - R_h y(\tilde{u})||
\]

\[
\leq C h^2 ||y_{h}(\tilde{u}) - R_h y(\tilde{u})||_{1}^2
\]

\[
\leq C h^4 + \frac{C}{2} ||y_{h}(\tilde{u}) - R_h y(\tilde{u})||^2.
\]

(3.30)

It is known that

\[
y_{h}(\tilde{u})(x, 0) - R_h y(\tilde{u})(x, 0) = y_{0}^h - R_h y_0 = 0,
\]

then integrating in time for (3.30) and using Gronwall’s inequality, we have

\[
||y_{h}(\tilde{u}) - R_h y(\tilde{u})||_{L^\infty(J; L^2(\Omega))} + ||y_{h}(\tilde{u}) - R_h y(\tilde{u})||_{L^2(I; H^1(\Omega))} \leq C h^2,
\]

(3.31)

which implies (3.24).

Then, let \( q_h = p_{h}(\tilde{u}) - R_h p(\tilde{u}) \) in (2.9). Note that

\[
(y_{h}(\tilde{u}) - y(\tilde{u}), p_{h}(\tilde{u}) - R_h p(\tilde{u})) \leq ||y_{h}(\tilde{u}) - y(\tilde{u})|| \cdot ||p_{h}(\tilde{u}) - R_h p(\tilde{u})||
\]

\[
\leq C h^2 ||y(\tilde{u})||_{2} \cdot ||p_{h}(\tilde{u}) - R_h p(\tilde{u})||
\]

\[
\leq C h^2 ||p_{h}(\tilde{u}) - R_h p(\tilde{u})||_{1},
\]

(3.32)

and

\[
(R_h p_{h}(\tilde{u}) - p_{h}(\tilde{u}), p_{h}(\tilde{u}) - R_h p(\tilde{u})) \leq C ||R_h p_{h}(\tilde{u}) - p_{h}(\tilde{u})|| \cdot ||p_{h}(\tilde{u}) - R_h p(\tilde{u})||
\]
Using the assumption for \( \phi(\cdot) \), we get

\[
(p(\bar{u})(\phi'(y(\bar{u})) - \phi'(y_h(\bar{u}))), p_h(\bar{u}) - R_h p(\bar{u}))
\]

\[
\leq C||p(\bar{u})||_{0,4}||\phi'(y(\bar{u})) - \phi'(y_h(\bar{u}))|| \cdot ||p_h(\bar{u}) - R_h p(\bar{u})||
\]

\[
\leq C h^2 ||p(\bar{u})||_1 \cdot ||\phi||_{W^{2,\infty}} ||y(\bar{u})||_2 \cdot ||p_h(\bar{u}) - R_h p(\bar{u})||
\]

\[
\leq C h^2 ||p_h(\bar{u}) - R_h p(\bar{u})||_1,
\]

where we used the embedding \( ||v||_{0,4} \leq C||v||_1 \). Then, using the definition of \( R_h \) and the assumption for \( \phi(\cdot) \), we get

\[
(\phi'(y_h(\bar{u}))(p(\bar{u}) - R_h p(\bar{u})), p_h(\bar{u}) - R_h p(\bar{u}))
\]

\[
\leq C||\phi||_{W^{1,\infty}} ||p(\bar{u}) - R_h p(\bar{u})|| \cdot ||p_h(\bar{u}) - R_h p(\bar{u})||
\]

\[
\leq C h^2 ||\phi||_{W^{1,\infty}} ||p(\bar{u})||_2 \cdot ||p_h(\bar{u}) - R_h p(\bar{u})||
\]

\[
\leq C h^2 ||p_h(\bar{u}) - R_h p(\bar{u})||_1.
\]

From (3.29) and (3.32)–(3.35), we have

\[
c||p_h(\bar{u}) - R_h p(\bar{u})||_1^2
\]

\[
\leq \alpha(p_h(\bar{u}) - R_h p(\bar{u}), p_h(\bar{u}) - R_h p(\bar{u})) + (\phi'(y_h(\bar{u}))(p_h(\bar{u}) - R_h p(\bar{u})), p_h(\bar{u}) - R_h p(\bar{u}))
\]

\[
= (y_h(\bar{u}) - y(\bar{u}), p_h(\bar{u}) - R_h p(\bar{u})) + (p(\bar{u})(\phi'(y(\bar{u})) - \phi'(y_h(\bar{u}))), p_h(\bar{u}) - R_h p(\bar{u}))
\]

\[
+ (\phi'(y_h(\bar{u}))(p(\bar{u}) - R_h p(\bar{u})), p_h(\bar{u}) - R_h p(\bar{u}))
\]

\[
\leq C h^2 ||p_h(\bar{u}) - R_h p(\bar{u})||_1.
\]

Note that

\[
p_h(\bar{u})(x, T) - R_h p(\bar{u})(x, T) = 0,
\]

then combining (3.32)–(3.36), and using the \( \epsilon \)-Cauchy inequality and the assumptions for \( A \) and \( \phi(\cdot) \), (3.29) can be rewritten as

\[
-\frac{1}{2} \frac{d}{dt} ||p_h(\bar{u}) - R_h p(\bar{u})||^2 + c ||p_h(\bar{u}) - R_h p(\bar{u})||^2 \leq C h^4 + \frac{1}{2} ||p_h(\bar{u}) - R_h p(\bar{u})||^2.
\]

Integrating the above inequality in time and using Gronwall’s inequality, we have

\[
||p_h(\bar{u}) - R_h p(\bar{u})||_{L^\infty(J; W)} + ||p_h(\bar{u}) - R_h p(\bar{u})||_{L^2(J; H^1(\Omega))} \leq C h^2,
\]

which implies (3.25).

\[\square\]

**Lemma 3.3.** For \( \bar{u} \in L^2(J; H^1(\Omega)) \), assume \( p(\bar{u}), p_h(\bar{u}), y(\bar{u}), y_h(\bar{u}) \in L^2(J; H^1(\Omega)) \cap L^2(J; H^2(\Omega)) \), then we have the estimate

\[
||p(\bar{u}) - p_h(\bar{u})||_{L^2(J; H^1(\Omega))} \leq C h^2.
\]
Proof. After rewriting

\[ p(\tilde{u}) - p_h(\tilde{u}) = p(\tilde{u}) - R_h p(\tilde{u}) + R_h p(Q_h \tilde{u}) + R_h p(Q_h \tilde{u}) - p_h(Q_h \tilde{u}) - p_h(\tilde{u}), \]

from Lemma 3.1 and assumption of \( p \), it is known that

\[ \| R_h p(\tilde{u}) - R_h p(Q_h \tilde{u}) \| \leq Ch^2, \]

and from Lemma 3.2, we get

\[ \| p_h(Q_h \tilde{u}) - R_h p(Q_h \tilde{u}) \| \leq Ch^2, \]

so we have

\[
\| p(\tilde{u}) - p_h(\tilde{u}) \| \leq \| p(\tilde{u}) - R_h p(\tilde{u}) \| + \| R_h p(\tilde{u}) - R_h p(Q_h \tilde{u}) \| + \| R_h p(Q_h \tilde{u}) - p_h(Q_h \tilde{u}) \| + \| p_h(Q_h \tilde{u}) - p_h(\tilde{u}) \| \leq Ch^2 + \| p_h(Q_h \tilde{u}) - p_h(\tilde{u}) \| .
\]

Choose \( \tilde{u} = Q_h \tilde{u}, w_h = y_h(Q_h \tilde{u}) - y_h(\tilde{u}) \) in (3.3), and let \( q_h = p_h(Q_h \tilde{u}) - p_h(\tilde{u}) \) in (3.4), then we obtain the following error equations

\[
\begin{align*}
& (y_h(Q_h \tilde{u}) - y_h(\tilde{u}), y_h(Q_h \tilde{u}) - y_h(\tilde{u})) + (\phi(y_h(Q_h \tilde{u})) - \phi(y_h(\tilde{u})), y_h(Q_h \tilde{u}) - y_h(\tilde{u})) \\
& + a(y_h(Q_h \tilde{u}) - y_h(\tilde{u}), y_h(Q_h \tilde{u}) - y_h(\tilde{u})) = (B(Q_h \tilde{u} - \tilde{u}), y_h(Q_h \tilde{u}) - y_h(\tilde{u})), \quad (3.41) \\
& - (p_h(Q_h \tilde{u}) - p_h(\tilde{u}), p_h(Q_h \tilde{u}) - p_h(\tilde{u})) \\
& + (\phi'(y_h(Q_h \tilde{u})) p_h(Q_h \tilde{u}) - \phi'(y_h(\tilde{u})) p_h(\tilde{u}), p_h(Q_h \tilde{u}) - p_h(\tilde{u})) \\
& + a(p_h(Q_h \tilde{u}) - p_h(\tilde{u}), p_h(Q_h \tilde{u}) - p_h(\tilde{u})) = (y_h(Q_h \tilde{u}) - y_h(\tilde{u}), p_h(Q_h \tilde{u}) - p_h(\tilde{u})). \quad (3.42)
\end{align*}
\]

Then from equality (3.41), using \( \epsilon \)-Cauchy inequality and (3.7), we derive

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \| y_h(Q_h \tilde{u}) - y_h(\tilde{u}) \|_1^2 + c \| y_h(Q_h \tilde{u}) - y_h(\tilde{u}) \|_1^2 \\
& \leq (B(Q_h \tilde{u} - \tilde{u}), y_h(Q_h \tilde{u}) - y_h(\tilde{u})) \\
& = (Q_h \tilde{u} - \tilde{u}, B'(y_h(Q_h \tilde{u}) - y_h(\tilde{u}))) \\
& \leq C \| \tilde{u} - Q_h \tilde{u} \|_1 \| B'(y(\tilde{u}) - y(Q_h \tilde{u})) \|_1 \\
& \leq Ch^2 \| y_h(Q_h \tilde{u}) - y_h(\tilde{u}) \|_1 \\
& \leq Ch^4 + c \| y_h(Q_h \tilde{u}) - y_h(\tilde{u}) \|_1^2. \quad (3.43)
\end{align*}
\]

Note that

\[ y_h(Q_h \tilde{u})(x, 0) - Q_h(\tilde{u})(x, 0) = 0, \]

next, integrating both sides of (3.43) in time, we obtain

\[
\| y_h(Q_h \tilde{u}) - y_h(\tilde{u}) \|_{L^2(J, U)} + c \| y_h(Q_h \tilde{u}) - y_h(\tilde{u}) \|_{L^2(J, H^1(\Omega))}^2 \leq Ch^4,
\]
and obviously
\[ \|y_h(Q_h\tilde{u}) - y_h(u)\|_{L^2(J; H^1(\Omega))} \leq C h^2. \] (3.44)

Next, we consider the equality (3.42) similar to the above idea.

\[-(p_{h,1}(Q_h\tilde{u}) - p_{h,1}(\tilde{u}), p_h(Q_h\tilde{u}) - p_h(\tilde{u})) + (\phi'(y_h(Q_h\tilde{u}))(p_h(Q_h\tilde{u}) - p_{h,1}(\tilde{u}), p_h(Q_h\tilde{u}) - p_h(\tilde{u}))
\]
\[+ a(p_h(Q_h\tilde{u}) - p_h(\tilde{u}), p_h(Q_h\tilde{u}) - p_h(\tilde{u}))
\]
\[= (y_h(Q_h\tilde{u}) - y_h(\tilde{u}), p_h(Q_h\tilde{u}) - p_h(\tilde{u}))
\]
\[+ (p_h(\tilde{u})(\phi'(y_h(\tilde{u})) - \phi'(y_h(Q_h\tilde{u}))), p_h(Q_h\tilde{u}) - p_h(\tilde{u})). \] (3.45)

Note that
\[ (y_h(Q_h\tilde{u}) - y(\tilde{u}), p_h(Q_h\tilde{u}) - p_h(\tilde{u})) \leq C\|y_h(Q_h\tilde{u}) - y_h(\tilde{u})\| \cdot |p_h(Q_h\tilde{u}) - p(\tilde{u})| \]
\[\leq C\|y_h(Q_h\tilde{u}) - y_h(\tilde{u})\|^2 + \|p_h(Q_h\tilde{u}) - p_h(\tilde{u})\|^2. \] (3.46)

Using the assumption for \(\phi(\cdot)\) and (3.9), we get
\[(p_h(\tilde{u})(\phi'(y_h(\tilde{u})) - \phi'(y_h(Q_h\tilde{u}))), p_h(Q_h\tilde{u}) - p_h(\tilde{u}))
\]
\[\leq C\|p_h(\tilde{u})\|_{L^0.4}\|\phi'(y_h(\tilde{u})) - \phi'(y_h(Q_h\tilde{u}))\| \cdot \|p_h(Q_h\tilde{u}) - p_h(\tilde{u})\|_{L^0.4}
\]
\[\leq C\|p_h(\tilde{u})\|_1 \||\phi||_{W^2,\infty}\|y_h(\tilde{u}) - y_h(Q_h\tilde{u})\| \cdot \|p_h(Q_h\tilde{u}) - p_h(\tilde{u})\|_1
\]
\[\leq C\|y_h(Q_h\tilde{u}) - y_h(\tilde{u})\|^2 + \|p_h(Q_h\tilde{u}) - p(\tilde{u})\|^2, \] (3.47)

where we used the embedding \(\|v\|_{L^0.4} \leq C\|v\|_1\). Then, using (3.45), (3.46), (3.47) and the assumption for \(\phi(\cdot)\), we have
\[-\frac{1}{2} \frac{d}{dt} \|p_h(Q_h\tilde{u}) - p_h(\tilde{u})\|^2 + C\|p_h(Q_h\tilde{u}) - p_h(\tilde{u})\|^2
\]
\[\leq -(p_h(Q_h\tilde{u}) - p_h(\tilde{u}), p_h(Q_h\tilde{u}) - p_h(\tilde{u})) + a(p_h(Q_h\tilde{u}) - p_h(\tilde{u}), p_h(Q_h\tilde{u}) - p_h(\tilde{u}))
\]
\[+ (\phi'(y_h(Q_h\tilde{u}))(p_h(Q_h\tilde{u}) - p_h(\tilde{u}), p_h(Q_h\tilde{u}) - p_h(\tilde{u}))
\]
\[= (y_h(Q_h\tilde{u}) - y_h(\tilde{u}), p_h(Q_h\tilde{u}) - p_h(\tilde{u}))
\]
\[+ (p_h(\tilde{u})(\phi'(y_h(\tilde{u})) - \phi'(y_h(Q_h\tilde{u}))), p_h(Q_h\tilde{u}) - p_h(\tilde{u})). \] (3.48)

Next, we consider the given condition
\[ p_h(Q_h\tilde{u})(x, T) - p_h(\tilde{u})(x, T) = 0, \] (3.49)

then, we integrate in time for (3.48) and use Gronwall’s inequality and (3.44), we have
\[ \|p_h(Q_h\tilde{u}) - p_h(\tilde{u})\|^2_{L^\infty(J; L^2(\Omega))} + \|p_h(Q_h\tilde{u}) - p_h(\tilde{u})\|^2_{L^2(J; H^1(\Omega))} \leq C h^4, \] (3.50)

which implies (3.39).
Let \( y(u) \) and \( y_h(u_h) \) are the solutions of (2.3) and (2.13), respectively. Let
\[
J(u) = \left\{ \frac{1}{2} \int_0^T \left( \| p - p_d \|^2 + \| y - y_d \|^2 + \| u \|^2 \right) dt \right\},
\]
\[
J_h(u_h) = \left\{ \frac{1}{2} \int_0^T \left( \| p_h(u_h) - p_d \|^2 + \| y_h(u_h) - y_d \|^2 + \| u_h \|^2 \right) dt \right\}.
\]

Then, the simplified problems of (2.2) and (2.12) read as
\[
\min_{u \in K} \{ J(u) \}, \quad (3.51)
\]
and
\[
\min_{u_h \in K_h} \{ J_h(u_h) \}, \quad (3.52)
\]
respectively. It can be shown that
\[
(J'(u), v) = \int_0^T (u + p, v) dt,
\]
\[
(J'(u_h), v) = \int_0^T (u_h + p(u_h), v) dt,
\]
\[
(J'(Q_h u), v) = \int_0^T (Q_h u + p(Q_h u), v) dt,
\]
\[
(J'_h(u_h), v) = \int_0^T (u_h + p_h, v) dt,
\]
where \( p(u_h) \) and \( p(Q_h u) \) are solutions of (3.1)–(3.2) for \( \tilde{u} = u_h \) and \( \tilde{u} = Q_h u \), respectively.

In many application, \( J(\cdot) \) is uniform convex near the solution \( u \). The convexity of \( J(\cdot) \) is bound up with the second order suffcient conditions of the control problem, which are supposed in many studies on numerical methods of the problem. Next, there is a constant \( c > 0 \), independent of \( h \), such that
\[
(J'(Q_h u) - J'(u_h), Q_h u - u_h) \geq c \| Q_h u - u_h \|_{L^2(I; U)}^2,
\]
(3.53)
where \( u \) and \( u_h \) are solutions of (3.51) and (3.52) respectively, \( Q_h u \) is the orthogonal projection of \( u \) which is introduced in (3.5). From beginning to end, we will use the above inequality in this paper. More discussion of this can be found in [3, 4].

4. Superconvergence

In this section, superconvergence for both the control variable and the state variables will be discussed. Let \( \pi^\circ \) defined in [31] is the average operator such that \( \pi^\circ u = Q_h u \). Let
\[
\Omega^+ = \{ \tau : \tau \subset \Omega, u|_\tau > 0 \},
\]
\[
\Omega^0 = \{ \tau : \tau \subset \Omega, u|_\tau = 0 \},
\]
\[
\Omega^- = \Omega \setminus (\Omega^+ \cup \Omega^0).
\]
In this paper, we assume that \( u \) and \( T_h \) are regular such that \( \text{meas}(\Omega^-) = \text{meas}(\Omega^0) \leq Ch \).
Theorem 4.1. Let $u$ be the solution of (2.4)–(2.6) and $u_h$ be the solution of (2.14)–(2.16). We assume that the exact control and state solution satisfy

$$u, \ u + p \in L^2(J; W^{1,\infty}(\Omega)),$$

and

$$y(u), \ p(u) \in (L^2(J; H^2(\Omega))).$$

Then, we have

$$\|Q_h u - u_h\|_{L^2(J;U)} \leq Ch^\frac{3}{2}. \quad (4.1)$$

Proof. Set $v = u_h$ in (2.6) and $v_h = Q_h u$ in (2.16), and add the two inequalities, then we get

$$\int_0^T \{ (u + p) - (u_h - p, Q_h u - u_h) + (u + p, Q_h u - u) \} dt \geq 0. \quad (4.2)$$

By using the definition of $Q_h$ and (4.2), we get

$$\int_0^T (Q_h u - u_h, Q_h u - u_h) dt \leq \int_0^T (u - u_h, Q_h u - u_h) dt \leq \int_0^T \{ (p - p, Q_h u - u_h) + (u + p, Q_h u - u) \} dt. \quad (4.3)$$

For the first term of (4.3), we separate it into three parts,

$$\int_0^T (p - p, Q_h u - u_h) dt = \int_0^T (p - p(u_h), Q_h u - u_h) dt + \int_0^T (p(u_h) - p(Q_h u), Q_h u - u_h) dt$$

$$+ \int_0^T (p(Q_h u) - p(u), Q_h u - u_h) dt, \quad (4.4)$$

from (4.3)–(4.4), we get that

$$\int_0^T \{ (Q_h u - u_h, Q_h u - u_h) - (p(u_h) - p(Q_h u), Q_h u - u_h) \} dt$$

$$\leq \int_0^T (p - p(u_h), Q_h u - u_h) dt + \int_0^T (p(Q_h u) - p(u), Q_h u - u_h) dt$$

$$+ \int_0^T (u + p, Q_h u - u) dt. \quad (4.5)$$

We can estimate the following by $\epsilon$-Cauchy inequality

$$\int_0^T (p(u_h) - p, Q_h u - u_h) dt \leq C \int_0^T \|p(u_h) - p\| \cdot \|Q_h u - u_h\| dt$$
which completes the proof of Theorem 4.1. □

For the second term of (4.3)

\[
\begin{align*}
\int_0^T (p(Q_h u) - p(u), Q_h u - u_h) dt &\leq C \int_0^T \|p(Q_h u) - p(u)\| \cdot \|Q_h u - u_h\| dt \\
&\leq \int_0^T \|p(Q_h u) - p(u)\|^2 dt + \epsilon \int_0^T \|Q_h u - u_h\|^2 dt \\
&= \|p(Q_h u) - p(u)\|^2_{L^2(J; H)} + \|Q_h u - u_h\|^2_{L^2(J; U)}.
\end{align*}
\] (4.7)

For the second term of (4.3)

\[
\int_0^T (u + p, Q_h u - u) dt = \int_0^T \left\{ \int_{\Omega^c} + \int_{\Omega^0} + \int_{\Omega^0} (u + p, Q_h u - u) dx \right\} dt.
\]

Obviously, \((Q_h u - u)|_{\Omega^0} = 0\). From (2.6), we have pointwise a.e. \((u + p) \geq 0\), we set \(\tilde{u}|_{\Omega^c} = 0\) and \(\tilde{u}|_{\Omega^0} = u\), so that \((u + p, u)|_{\Omega^c} \leq 0\). So, \((u + p)|_{\Omega^c} = 0\). Then

\[
\begin{align*}
\int_0^T (u + p, Q_h u - u) dt &= \int_0^T (u + p, Q_h u - u)_{\Omega^0} dt \\
&\leq \int_0^T (u + p - \pi\epsilon (u + p), Q_h u - u)_{\Omega^0} dt \\
&\leq Ch^2 \int_0^T \|u + p\|_{L^2(\Omega^c)} \|u\|_{L^2(\Omega^0)} dt \\
&\leq Ch^2 \int_0^T \|u + p\|_{L^2(\Omega)} \|u_{\text{meas}}(\Omega^0)\} dt \\
&\leq Ch^2.
\end{align*}
\] (4.8)

According to (3.53), the left hand of (4.5) can be restated as:

\[
\begin{align*}
\int_0^T \{ (Q_h u - u_h, Q_h u - u_h) - (p(u_h) - p(Q_h u), Q_h u - u_h) \} dt \\
= \int_0^T \{ (Q_h u + p(Q_h u), Q_h u - u_h) - (u_h + p(u_h), Q_h u - u_h) \} dt \\
= \int_0^T (J'(Q_h u) - J'(u_h), Q_h u - u_h) dt \\
\geq c \|Q_h u - u_h\|^2_{L^2(J; U)}.
\end{align*}
\] (4.9)

Then, combining (3.10), (3.39) and (4.5)–(4.9), we have

\[
\|Q_h u - u_h\|^2_{L^2(J; U)} \leq Ch^2,
\]

which completes the proof of Theorem 4.1.
Theorem 4.2. Let $u$ be the solution of (2.4)--(2.6), $u_h$ be the solution of (2.14)--(2.16) and $\Omega$ is convex. We assume that the exact control and state solution satisfy

$$u, \, u + p \in L^2(J; W^{1,\infty}(\Omega)),$$

and

$$y(u), \, p(u) \in L^2(J; H^1(\Omega)) \cap L^2(J; H^2(\Omega)).$$

Then, we have

$$\|y - R_h y\|_{L^2(J; H^1(\Omega))} \leq Ch^\frac{3}{2}, \quad (4.10)$$

and

$$\|p - R_h p\|_{L^2(J; H^1(\Omega))} \leq Ch^\frac{1}{2}. \quad (4.11)$$

Proof. From (2.4)--(2.5) and (2.14)--(2.15), We have the following error equations

$$(y_{h,t} - y_t, w_h) + a(y_h - y, w_h) + (\phi(y_h) - \phi(y), w_h) = (u_h - u, w_h), \quad \forall w_h \in V_h, \quad (4.12)$$

and

$$-(p_{h,t} - p_t, q_h) + a(q_h, p_h - p) + (\phi'(y_h)p_h - \phi'(y)p, q_h) = (y_h - y, q_h), \quad \forall q_h \in V_h. \quad (4.13)$$

Using the definition of $R_h$, we have

$$(y_{h,t} - R_h y, w_h) + a(y_h - R_h y, w_h) + (\phi(y_h) - \phi(R_h y), w_h)$$

$$= (y_t - R_h y, w_h) + (u_h - u, w_h) + (\phi(y) - \phi(R_h y), w_h), \quad (4.14)$$

and

$$-(p_{h,t} - R_h p, q_h) + a(q_h, p_h - R_h p) + (\phi'(y_h)(p_h - R_h p), q_h)$$

$$= (R_h p_t - p_t, q_h) + (y_h - y, q_h) + (\phi'(y_h)(p - R_h p), q_h) + (p(\phi'(y) - \phi'(y_h)), q_h), \quad (4.15)$$

for any $w_h$ and $q_h \in V_h$.

First, taking $w_h = y_h - R_h y$ in (4.14) and using the assumption of $\phi(\cdot)$, we have

$$\frac{1}{2} \frac{d}{dt} \|y - R_h y\|^2 + c\|y - R_h y\|^2$$

$$\leq (y_{h,t} - R_h y, y_h - R_h y) + a(y_h - R_h y, y_h - R_h y) + (\phi(y_h) - \phi(R_h y), y_h - R_h y)$$

$$= (y_t - R_h y, y_h - R_h y) + (u_h - Q_h u, y_h - R_h y)$$

$$+ (Q_h u - u, y_h - R_h y) + (\phi(y) - \phi(R_h y), y_h - R_h y)$$

$$\leq C\|y_t - R_h y\|_1 \|y_h - y_h\| + \|u_h - Q_h u\| \|y_h - R_h y\|$$

$$+ \|Q_h u - u\| \|y_h - R_h y\| + \|\phi\|_{1, \infty} \|y - R_h y\| \|y_h - R_h y\|$$

$$\leq C(h^2\|y_h - R_h y\| + \|u_h - Q_h u\| \|y_h - R_h y\| + h^2\|u\|_1 \|y_h - R_h y\|)$$

$$\leq C\left(h^4 + h^3 + \|y_h - R_h y\|^2 + \|y_h - R_h y\|^2\right)$$

$$\leq Ch^3 + \epsilon\|y_h - R_h y\|^2. \quad (4.16)$$

Note that

$$y_h(x, 0) - R_h y(x, 0) = 0,$$
integrating in time and using Gronwall’s inequality, we estimate
\[
\| y_h - R_h y \|_{L^\infty(J;U)}^2 + \| y_h - R_h y \|_{L^2(J;H^1(\Omega))}^2 \leq Ch^3, \tag{4.17}
\]
which implies (4.10).

Then, we take \( q_h = p_h - R_h p \) in (4.15). Notice that
\[
(R_h p_t - p_t, p_h - R_h p) \leq Ch^2 \| p_h - R_h p \|_1,
\]
and
\[
(y_h - y, p_h - R_h p) = (y_h - R_h y, p_h - R_h p) + (R_h y - y, p_h - R_h p) 
\leq C \left( h^4 + \| y_h - R_h y \|_1^2 + \| p_h - R_h p \|_1^2 \right). \tag{4.18}
\]
Using the definition of \( R_h \) and the assumption for \( \phi(\cdot) \), we have
\[
\| \phi'(y)(p - R_h p), p_h - R_h p \| \leq Ch^2 \| \phi \|_{1,\infty} \| p \|_2 \| p_h - R_h p \| 
\leq Ch^2 \| p_h - R_h p \|_1 
\leq C \left( h^4 + \| p_h - R_h p \|_1^2 \right). \tag{4.19}
\]
and
\[
(p(\phi'(y) - \phi'(y_h)), p_h - R_h p) 
\leq C \| \phi \|_{2,\infty} \| p(y - y_h) \|_2 \| p_h - R_h p \|_0 \| p_h - R_h p \|_4 
\leq C \| \phi \|_{2,\infty} \| y - y_h \|_2 \| p \|_4 \| p_h - R_h p \|_1 \| p_h - R_h p \|_1 
\leq C \left( h^4 + \| R_h y - y_h \|_1^2 + \| p_h - R_h p \|_1^2 \right). \tag{4.20}
\]
From (4.15) and (4.18)–(4.20), we have
\[
\frac{1}{2} \frac{d}{dt} \| p_h - R_h p \|_1^2 + c \| p_h - R_h p \|_1^2 
\leq -(p_{h,t} - R_h p_t, q_h) + a(p_h - R_h p, p_h - R_h p) + \phi'(y_h)(p_h - R_h p), p_h - R_h p 
= (R_h p_t - p_t, p_h - R_h p) + (y_h - y, p_h - R_h p) + \phi'(y_h)(p_h - R_h p), p_h - R_h p 
+ (p(\phi'(y) - \phi'(y_h)), p_h - R_h p) 
\leq C \left( h^4 + \| y_h - R_h y \|_1^2 + \| p_h - R_h p \|_1^2 \right). \tag{4.21}
\]
Note that
\[
p_h(x, T) - R_h p(x, T) = 0,
\]
integrating in time and using Gronwall’s inequality and (4.10), we estimate
\[
\| p_h - R_h p \|_{L^\infty(J;U)}^2 + \| p_h - R_h p \|_{L^2(J;H^1(\Omega))}^2 \leq Ch^3, \tag{4.22}
\]
which implies (4.11).
5. Conclusions

In this paper, we present finite element approximation method for solving semilinear parabolic OCP. When the state and co-state variables are approximated by the piecewise linear functions, the control variable is approximated by the piecewise constant functions, superconvergence properties for both the control variable and the state variables are discussed. In our future work, we shall use this method to deal with hyperbolic optimal control problems, including linear and nonlinear styles.

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Conflict of interest

The authors declare no conflict of interest.

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