FIBRATIONS BY NON-SMOOTH PROJECTIVE CURVES OF ARITHMETIC GENUS TWO IN CHARACTERISTIC TWO

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Abstract. Looking in positive characteristic for failures of the Bertini–Sard theorem, we determine, up to birational equivalence, the separable proper morphisms of smooth algebraic varieties in characteristic two, whose fibres are non-smooth curves of arithmetic genus two.

Introduction

Bertini’s theorem on moving singularities, published in the last but one decade of the ninetens century, has become a fundamental tool in Algebraic Geometry. Nowadays, due to its similarities to Sard’s theorem on differentiable maps, it is also called the Bertini–Sard theorem. It assures that almost all fibres of a dominant morphism between smooth algebraic varieties are smooth.

However, in the 1940s Zariski [Z1] observed that the theorem may fail in positive characteristic. He had constructed a fibration \( \phi: T \rightarrow B \) by algebraic curves that admits moving singularities, though the total space \( T \) is smooth. A moving singularity of \( \phi \) can be viewed as a horizontal prime divisor on \( T \) with the property that each of its points is a singular point of the fibre to which it belongs.

Translated in modern language, Zariski argued that, though the generic fibre is a regular scheme over the base field \( k(B) \), it may not be smooth, that is, the geometric generic fibre, defined by extending the base field to its algebraic closure \( \overline{k(B)} \), may have singularities. This means that the function field \( k(T) \mid k(B) \) may be non-conservative, that is, its genus \( g \) may decrease by tensoring with \( k(B) \). For more explications we refer to Section 1.

To rescue Bertini’s theorem in positive characteristic \( p \), we are conducted to classify its exceptions. As follows from a theorem of Tate [T1], Bertini’s theorem can only fail if \( p \leq 2g + 1 \). Non-conservative function fields of genus 1 were classified by Queen [Q], and of genus 2 in odd characteristic by Borges Neto [Bo]. General results on non-conservative function fields and their singular primes were developed by Stichtenoth, Bedoya and the second author in the papers St1, BS and St1.

In their program to extend Enriques’ classification of algebraic surfaces to arbitrary characteristic, Bombieri and Mumford BM encountered quasi-elliptic fibrations, i.e., fibrations by cuspidal curves of arithmetic genus 1. There is a large number of recent papers on classification theory of algebraic varieties and singularities in positive characteristic, too large to put in our references, which can be

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found by starting the search with [BM] and looking successively for citations and references. Singularities of generic fibres in positive characteristic were analyzed by Schröer [Sc]. Fibrations by non-smooth curves of arithmetic genus 3 in characteristic 3, 5 and 7 were studied by Salomão [Sa1], [Sa2] and the second author [St3], [St4].

In the present paper we study in characteristic two the fibrations by non-smooth curves of arithmetic genus two. We realize the fibres by tri-canonical embeddings as curves on a cone in $\mathbb{P}^4$. The discussion naturally divides into two cases. If the function field of the generic fibre is separable over its canonical quadratic rational subfield, then we prove that almost every fibre is geometrically elliptic, i.e., its non-singular model is an elliptic curve (see Theorem 3.1). In the second case the fibres are rational, as discussed in Theorem 4.1.

We discover a 6-dimensional smooth algebraic variety $Z \subset \mathbb{P}^4 \times \mathbb{A}^5$ such that almost all fibres of the projection morphism $\pi : Z \to \mathbb{A}^5$ are cuspidal geometrically elliptic curves of arithmetic genus two (see Theorem 5.1). We describe how the elliptic modular invariant of the fibres varies, determine the singular points of all fibres, and discuss how the singularities move.

Theorem 5.2 is the main result of this paper. It states that each proper separable morphism between smooth algebraic varieties, whose fibres are geometrically elliptic curves of arithmetic genus two, is birational equivalent to a base extension of the fibration $\pi : Z \to \mathbb{A}^5$. A similar result for fibrations by rational curves of arithmetic genus two is also obtained (see Theorem 5.4).

1. Moving singularities of fibrations by algebraic curves

In this introductory section we present prerequisites on moving singularities of fibrations by algebraic curves, needed to understand our paper.

Let $\phi : T \to B$ be a dominant morphism of irreducible algebraic varieties defined over an algebraically closed field $k$. We assume that $\dim T = \dim B + 1$ or, equivalently, almost all fibres are algebraic curves (see [Sh, p. 74]). Thus by restricting if necessary the base variety $B$ to a dense open subvariety we get a fibration by algebraic curves.

By identifying the rational functions on the base $B$ with rational functions on the total space $T$ that are constant along each fibre, we can view the field $k(B)$ of the base as a subfield of $k(T)$. We assume that almost all fibres are integral. By a theorem of Matsusaka this means that $k(B)$ is algebraically closed in $k(T)$ and that $k(T)$ is separable over $k(B)$ (see [Mi, Sh pp. 256–257]). Thus the field $k(T)$ of the total space, which is a higher dimensional function field over the constant field $k$, becomes a one-dimensional separable function field over the base field $k(B)$.

In this sense, the fibrations by integral algebraic curves over the variety $B$, up to birational equivalence, correspond bijectively to the isomorphism classes of the one-dimensional separable function fields over $k(B)$.

In the setting of schemes, the function field $k(T)/k(B)$ is the field of the generic fibre $T \times_B \text{Spec} k(B)$, where the calligraphic letters $T$ and $B$ stand for the integral schemes whose points correspond bijectively to the closed irreducible subsets of $T$ and $B$, respectively. The generic fibre is a geometrically integral curve over $k(B)$.

Its closed points, which are exactly its non-generic points, correspond bijectively to the horizontal prime divisors of the fibre $\phi : T \to B$, that is, to the prime divisors of the total space $T$ whose images are dense in the base variety $B$.
local ring of the scheme $T$ (and also the local ring of the generic fibre) at a closed point of the generic fibre is equal to the local ring of the total space $T$ along the corresponding horizontal prime divisor, and its residue field is isomorphic to the field of rational functions on the horizontal prime divisor.

As the non-smooth locus of the morphism $\phi : T \to B$ (i.e., the union of the non-smooth loci of the fibres of $\phi$) is closed in $T$, and as even the non-smooth locus of the corresponding morphism $\Phi : T \to B$ of schemes is closed in $T$ (cf. [L, p. 224, Cor. 2.12]), we deduce that a closed irreducible subset $H$ of $T$ is contained in the non-smooth locus of $\phi$ if and only if the corresponding integral scheme $\mathcal{H}$ or, equivalently, its generic point is contained in the non-smooth locus of $\Phi$. Applying this remark to the horizontal prime divisors, we obtain:

**Proposition 1.1.** The horizontal prime divisors contained in the non-smooth locus of the fibration $\phi : T \to B$ correspond bijectively to the non-smooth closed points of the generic fibre $T \times_B \text{Spec} \ k(B)$.

These horizontal prime divisors, whose points are singular points of the fibres to which they belong, are called the moving singularities of the fibration. Here we do not consider singularities that move in subvarieties of codimension larger than 1.

We always assume that the dominant morphism $\phi : T \to B$ is proper. This implies that its fibres are complete, that $\phi$ is surjective and that even the restrictions of $\phi$ to the horizontal prime divisors are surjective. We further assume that the total space $T$ is smooth. In particular, it is regular in codimension one. Thus the generic fibre is a regular complete geometrically integral algebraic curve over $k(B)$, or more precisely,

$$T \times_B \text{Spec} \ k(B) = \mathcal{R}_{k(T)|k(B)}$$

where $\mathcal{R}_{k(T)|k(B)}$ denotes the regular complete model of the one-dimensional function field $k(T)|k(B)$. The closed points of $\mathcal{R}_{k(T)|k(B)}$ are exactly the primes $p$ of $k(T)|k(B)$, and their local rings are the corresponding discrete valuation rings $\mathcal{O}_p$. By Proposition 1.1 a horizontal prime divisor is a moving singularity if and only if the corresponding prime $p$ is a singular prime in the sense that $p$ is a non-smooth point of $\mathcal{R}_{k(T)|k(B)}$, i.e., the one-dimensional semi-local ring $\mathcal{O}_p \otimes_{k(B)} k(B)$ is non-regular, i.e., over the point $p$ of the generic fibre there lies a singular point of the geometric generic fibre

$$T \times_B \text{Spec} \ k(B) = (T \times_B \text{Spec} \ k(B)) \times_{\text{Spec} \ k(B)} \text{Spec} \ k(B) = \mathcal{R}_{k(T)|k(B)} \times_{\text{Spec} \ k(B)} \text{Spec} \ k(B).$$

By Rosenlicht’s genus drop formula (see Section 2) the number of the singular primes of $k(T)|k(B)$, counted according to their singularity degrees, is equal to $g - \overline{g}$, where $g$ and $\overline{g}$ denote the genera of the function fields $k(T)|k(B)$ and $k(T) \otimes_{k(B)} k(B)|k(B)$, respectively. Thus the fibration $\phi : T \to B$ admits moving singularities if and only if $\overline{g} < g$. As the genus remains invariant under separable base field extensions, we obtain Bertini’s theorem on moving singular points.

**Bertini–Sard Theorem.** In characteristic zero the fibration $\phi : T \to B$ does not admit moving singularities, i.e., almost all fibres are smooth.

Moreover, if the characteristic is a prime $p$, then by a theorem of Tate [T1] the genus drop $g - \overline{g}$ is a multiple of $\frac{p-1}{2}$, and so Bertini’s theorem can only fail if $p \leq 2g + 1$. 
By restricting the base $B$ to a dense open subvariety, we may assume that all fibres are of dimension one, and that not only the total space $T$ but also the base $B$ is smooth. Then the morphism $\phi : T \to B$ is flat (see [E, Theorem 18.16]), and so the arithmetic genus of each fibre is equal to the arithmetic genus of the generic fibre $T \times_B \text{Spec} \ k(B)$ (see [Ha, Ch. III, Theorem 9.9]). As the generic fibre is equal to $R^0_k(T) \mid k(B)$, its arithmetic genus is equal to the genus $g$ of the function field $k(T)[k(B)]$. Moreover, as the arithmetic genus is invariant under base field extensions, the genus $g$ is also equal to the arithmetic genus of the geometric generic fibre $R^0_k(T) \times_B \text{Spec} \ k(B)$.

The geometric generic fibre is a complete integral curve over $k(B)$ of geometric genus $g$. It reflects the properties of the closed fibres in a better way than the generic fibre. By semi-continuity, the geometric genus of each closed fibre is smaller than or equal to $g$, and equality holds for almost all fibres.

2. Curves of arithmetic genus 2 on a cone in $\mathbb{P}^4$

As becomes clear from the preceding section, we can apply the theory of function fields in order to study the generic fibres of morphisms.

Let $F \mid K$ be a one-dimensional function field of genus $g = 2$, and let $\epsilon$ be a canonical divisor of $F \mid K$. By the Riemann–Roch theorem its degree and the dimension of its space of global sections are equal to

$$\deg(\epsilon) = 2g - 2 = 2 \quad \text{and} \quad \dim H^0(\epsilon) = g = 2.$$ 

As $H^0(\epsilon) \neq 0$, the canonical divisor $\epsilon$ is linearly equivalent to a positive divisor, and so we can assume that $\epsilon$ is positive. Since the dimension of $H^0(\epsilon)$ is larger than 1, there is a function $x \in H^0(\epsilon) \setminus K$. As $\text{div}_\infty(x) \leq \epsilon$, the fundamental equality $[F : K(x)] = \deg \text{div}_\infty(x)$ implies $[F : K(x)] \leq \deg(\epsilon) = 2$. Since $g \neq 0$, and therefore $F \neq K(x)$, we conclude that

$$[F : K(x)] = 2 \quad \text{and} \quad \text{div}_\infty(x) = \epsilon.$$ 

As $H^0(\epsilon) = K \oplus Kx$, the canonical positive divisors different from $\epsilon$ are just of the form

$$\text{div}_0(x - a) \quad \text{where} \quad a \in K,$$

and the canonical subfield of $F \mid K$, i.e., the field generated by the global sections of any canonical positive canonical divisor, is equal to the rational quadratic subfield $K(x)$ of $F \mid K$. Clearly

$$H^0(\epsilon^n) \cap K(x) = Kx^0 \oplus Kx^1 \oplus Kx^n \quad \text{for each} \quad n \geq 2.$$ 

Moreover, by Riemann’s theorem

$$\dim H^0(\epsilon^n) = 2n - 1 \quad \text{for each} \quad n \geq 2.$$ 

Thus there is a function $y \in H^0(\epsilon^3) \setminus K(x)$, and we obtain

$$H^0(\epsilon^n) = \bigoplus_{i=0}^n Kx^i \oplus \bigoplus_{i=0}^{n-3} Kx^iy \quad \text{for each} \quad n \geq 3.$$ 

As $y^2 \in H^0(\epsilon^6)$ there is an equation

$$y^2 + a(x)y + b(x) = 0.$$
where \( a(x) = \sum_{i=0}^{3} a_i x^i \) and \( b(x) = \sum_{i=0}^{6} b_i x^i \) are polynomials with coefficients in \( K \) of formal degree 3 and 6, respectively. Since \( [F : K(x)] = 2 \) and \( y \notin K(x) \) we have

\[
F = K(x, y)
\]

and the above equation is the minimal equation of \( y \) over \( K(x) \).

Let \( \tilde{F} = K(\tilde{x}, \tilde{y}) \), where \( \tilde{y}^2 + a(\tilde{x})\tilde{y} + b(\tilde{x}) = 0 \), be another genus-2 function field in the above normal form. As \( K(\tilde{x}) \) and \( K(x) \) are the corresponding discrete valuation rings \( \mathcal{O}_p \) of \( \tilde{F} \) and \( F[K] \), respectively, it is easy to check that the \( K \)-isomorphisms \( \tilde{F} \rightarrow F \) are just given by the transformations

\[
\tilde{x} \mapsto \frac{\alpha_{11} x + \alpha_{12}}{\alpha_{21} x + \alpha_{22}} \quad \text{and} \quad \tilde{y} \mapsto \frac{\beta y + \sum_{i=0}^{3} \gamma_i x^i}{(\alpha_{21} x + \alpha_{22})^3}
\]

where \( (\alpha_{ij}) \in \text{GL}_2(K) \), \( \beta \in K^* \) and \( \gamma_0, \gamma_1, \gamma_2, \gamma_3 \in K \) such that

\[
\beta a(x) = (\alpha_{21} x + \alpha_{22})^3 \cdot \tilde{a} \left( \frac{\alpha_{11} x + \alpha_{12}}{\alpha_{21} x + \alpha_{22}} \right) + 2 \sum_{i=0}^{3} \gamma_i x^i
\]

and

\[
\beta^2 b(x) = (\alpha_{21} x + \alpha_{22})^6 \cdot \tilde{b} \left( \frac{\alpha_{11} x + \alpha_{12}}{\alpha_{21} x + \alpha_{22}} \right) + \left( \sum_{i=0}^{3} \gamma_i x^i \right)^2 + (\alpha_{21} x + \alpha_{22})^3 \cdot \tilde{a} \left( \frac{\alpha_{11} x + \alpha_{12}}{\alpha_{21} x + \alpha_{22}} \right) \sum_{i=0}^{3} \gamma_i x^i.
\]

A \( K \)-isomorphism of \( \tilde{F} \) onto \( F \) determines the transformation coefficients uniquely up to the \( \mathbb{G}_m \)-action defined for each \( c \in \mathbb{G}_m(K) = K^* \) by the assignment

\[
(a_{11}, a_{12}, a_{21}, a_{22}, \beta, \gamma_0, \gamma_1, \gamma_2, \gamma_3) \mapsto (c a_{11}, c a_{12}, c a_{21}, c a_{22}, c^3 \beta, c^3 \gamma_0, c^3 \gamma_1, c^3 \gamma_2, c^3 \gamma_3).
\]

If \( p \neq 2 \) then by completing the square we can normalize \( a(x) = 0 \), and the freedom to transform is restricted by the conditions \( \gamma_0 = \cdots = \gamma_3 = 0 \).

We will always assume that the function field \( F[K] \) is separable, that is, it admits a separating variable. This means that \( x \) or \( y \) is a separating variable, that is, \( p \neq 2 \) or \( a(x) \neq 0 \) or \( b'(x) \neq 0 \). Then the condition that the base field \( K \) is algebraically closed in \( F \) means that the minimal polynomial

\[
f(X, Y) := Y^2 + a(X)Y + b(X) \in K[X, Y]
\]

is absolutely irreducible, that is, it remains irreducible over the algebraic closure \( \overline{K} \) of \( K \). As \( f \) is monic of degree 2 in \( Y \), this means that there does not exist a polynomial \( c(X) \in \overline{K}[X] \) such that \( f(X, c(X)) = 0 \). In particular, if \( a(X) = 0 \) then \( b(X) \) is not a square in \( \overline{K}[X] \). As \( f \) is absolutely irreducible, we can consider the base field extension

\[
F \cdot \overline{K} := F \otimes_K \overline{K} = \overline{K}(x)[Y]/\overline{K}(x)[Y],
\]

that is, \( F \cdot \overline{K} = \overline{K}(x, y) \) where \( x \) is transcendental over \( \overline{K} \) and \( f(x, y) = 0 \).

Let \( \mathcal{R} = \mathcal{R}_{F[K]} \) be the regular complete model of the function field \( F[K] \). It is a regular complete curve over \( K \), or more precisely, a geometrically integral regular complete one-dimensional scheme of finite type over \( \text{Spec}(K) \). The algebraic set \( R = R_{F[K]} \) of its closed points consists of the primes \( p \) of \( F[K] \), whose local rings are the corresponding discrete valuation rings \( \mathcal{O}_p \) of \( F[K] \). The generic point is the only non-closed point of the scheme \( \mathcal{R} \), and its local ring is the function field \( F \).

The extended scheme \( \mathcal{R} \otimes_{\text{Spec}(K)} \text{Spec}(\overline{K}) \) is an integral complete curve over \( \overline{K} \), whose function field is equal to \( F \cdot \overline{K} := F \otimes_K \overline{K} \). The points of
\( R \otimes_K \bar{K} \) lying over a prime \( p \) of \( F|K \) correspond bijectively to the maximal ideals of the semi-local ring \( \mathcal{O}_p \bar{K} = \mathcal{O}_p \otimes_K \bar{K} \), and their local rings are the corresponding localizations. Though the curve \( \mathcal{R} \) is regular, it may not be smooth, i.e., the extended curve \( R \otimes_k \bar{K} \) may have singular points (see [Z2]). Recall that a prime \( p \) of \( F|K \) is called singular, if the domain \( \mathcal{O}_p \bar{K} \) is not normal, i.e., there is a singular point of the extended curve \( R \otimes_K \bar{K} \) lying over \( p \). Since the arithmetic genus \( p_a \) is preserved under base field extensions, we have

\[
p_a(R \otimes_K \bar{K}) = g = 2.
\]

As \( R \otimes_K \bar{K} \) is an integral complete hyperelliptic curve of arithmetic genus two, the global sections of the tri-canonical divisor \( c_3 \) define an embedding

\[
(1 : x : x^2 : x^3 : y) : R \otimes_K \bar{K} \hookrightarrow \mathbb{P}^4(\bar{K})
\]

(see [St2] Theorem 2.1]), and so the extended curve \( R \otimes_K \bar{K} \) can be realized as a curve

\[
S := \left\{ (u_0 : u_1 : u_2 : u_3 : v) \mid \text{rank} \begin{pmatrix} u_0 & u_1 & u_2 & u_3 \\ u_1 & u_2 & u_3 & 0 \end{pmatrix} < 2 \right\}
\]

in the 4-dimensional projective space \( \mathbb{P}^4(\bar{K}) \).

In the remainder of this section, we invert the preceding considerations. Given an absolutely irreducible polynomial \( f = Y^2 + a(X)Y + b(X) \in K[X,Y] \), where \( a(X) \) and \( b(X) \) are polynomials of formal degree 3 and 6 , respectively. We consider the function field

\[
F|K = K(x,y)|K \quad \text{where} \quad y^2 + a(x)y + b(x) = 0,
\]

and we assume that it is separable, that is, \( p \neq 2 \) or \( a(x) \neq 0 \) or \( b'(x) \neq 0 \).

If \( p \neq 2 \), then it is well known that the genus \( g \) of \( F|K \) is not larger than two, and equality holds if and only if the discriminant \( a(x)^2 - 4b(x) \) is square-free in \( K[x] \) and has degree 5 or 6 . If \( p = 2 \), then it is more difficult to determine the genus \( g \).

We will consider a possibly singular projective model of \( F\bar{K}|\bar{K} \) lying on the cone \( S \subset \mathbb{P}^4(\bar{K}) \). We note that the cone is the union of the projective lines

\[
L_u := \{(1 : u : u^2 : u^3 : v) \mid v \in \bar{K}\} \cup \{Q\} \quad (u \in \bar{K})
\]

and

\[
L_\infty := \{(0 : 0 : 0 : 1 : v) \mid v \in \bar{K}\} \cup \{Q\},
\]

which have the vertex \( Q := (0 : 0 : 0 : 0 : 1) \) as their only common point. The smooth locus \( S \setminus \{Q\} \) is described by the atlas consisting of the two charts

\[
W := S \setminus L_\infty = \{(1 : u : u^2 : u^3 : v) \mid (u,v) \in \bar{K}^{\oplus 2}\} \hookrightarrow \bar{K}^{\oplus 2}
\]

and

\[
\hat{W} := S \setminus L_0 = \{(\hat{u}^3 : \hat{u}^2 : \hat{u} : 1 : \hat{v}) \mid (\hat{u},\hat{v}) \in \bar{K}^{\oplus 2}\} \hookrightarrow \bar{K}^{\oplus 2}.
\]

Let \( C \subset S \) be the projective integral curve over \( \bar{K} \) described in the first chart \( W \) by the minimal equation

\[
y^2 + a(x)y + b(x) = 0,
\]

where the elements \( x \) and \( y \) of the function field \( F \) have been realized as the rational functions on \( C \) that map each point \((1 : u : u^2 : u^3 : v) \) of \( C \cap W \) onto \( u \) and \( v \),
respectively. With respect to the second chart \( \tilde{W} \) we have the local coordinate functions
\[
\tilde{x} := x^{-1} \quad \text{and} \quad \tilde{y} := x^{-3} y
\]
which satisfy the minimal equation
\[
\tilde{y}^2 + (a_0 \tilde{x}^3 + a_1 \tilde{x}^2 + a_2 \tilde{x} + a_3)\tilde{y} + b_0 \tilde{x}^6 + b_1 \tilde{x}^5 + \cdots + b_6 = 0.
\]
Without using charts, the curve \( C \) can be defined as the intersection of the cone \( S \) and the quadratic hypersurface cut out by the equation
\[
v^2 + \sum_{i=0}^{3} a_i u_i v + \sum_{i=0}^{3} b_{2i} u_i^2 + \sum_{i=0}^{2} b_{2i+1} u_i u_{i+1} = 0.
\]
In particular, the vertex \( Q \) does not lie on the curve. By calculating the Hilbert polynomial of the curve \( C \subset \mathbb{P}^4 \) (see [RS, p. 196]) we obtain the arithmetic genus:
\[
p_a(C) = 2.
\]

**Theorem 2.1.** The curve \( C \) on the cone \( S \) is isomorphic to the extended curve \( R_{F/K} \otimes_K K \) if and only if the genus \( g \) of the function field \( F|K \) is equal to two.

**Proof.** If \( R \otimes_K K \cong C \) then a fortiori \( p_a(R \otimes_K K) = p_a(C) \), that is, \( g = p_a(C) \) and therefore \( g = 2 \) by the preceding equation. The opposite direction follows from the first part of this section. \( \square \)

By Hironaka’s genus formula [Hi] the geometric genus \( p_g(C) \) of the curve \( C \), that is, the genus \( \overline{g} \) of its function field \( F\overline{K}/K \), is equal to
\[
\overline{g} = p_a(C) - \sum \dim(\overline{O}_{C,P}/\overline{O}_{C,P})
\]
where the sum is taken over the singular points \( P \) of \( C \), and where \( \overline{O}_{C,P} \) denote the normalizations of the local rings \( O_{C,P} \). Applying Hironaka’s genus formula to the extended curve \( R \otimes_K K \) and localizing, we obtain Rosenlicht’s genus drop formula
\[
g - \overline{g} = \sum \dim(\overline{O}_p \overline{K}/\overline{O}_p \overline{K})
\]
where \( p \) varies over the singular primes of \( F|K \) (cf. [Ro, Theorem 11]). To determine the genera \( \overline{g} \) and \( g \), we have to compute the dimensions of \( \overline{O}_{C,P}/\overline{O}_{C,P} \) and \( \overline{O}_p \overline{K}/\overline{O}_p \overline{K} \), which are called the singularity degrees of the points \( P \) and the primes \( p \), respectively.

As the curve \( C \) lies on the punctured cone \( S \setminus \{Q\} = W \cup \tilde{W} \), which via the two charts is locally isomorphic to the affine plane, the singular points of the curve can be computed by the Jacobian criterion, and their singularity degrees can be determined by a finite number of blowups.

As the genus \( g \) is preserved under separable base field extensions, in order to determine \( g \), we may assume that the base field \( K \) is separably closed. In this case the primes of \( F|K \) correspond bijectively to the primes of \( F\overline{K}/K \) and hence to the branches of the curve \( C \). A branch of \( C \) that corresponds to a singular prime of \( F|K \) is necessarily a singular branch and therefore centered at a singular point of \( C \). As the minimal equations in the two charts are monic of degree 2 in \( y \) and \( \tilde{y} \), such a singular point is necessarily unibranch of multiplicity 2. The singularity degrees of the primes of \( F|K \) can be determined by an algorithm developed in [BS].
Let $F|K$ be a one-dimensional separable function field of positive characteristic $p$. As $F|K$ is separable, its Frobenius pullback 

$$F_1|K := F^p K|K$$

is the only subfield of $F|K$ such that the extension $F|F_1$ is inseparable of degree $p$. On the other hand, the Frobenius pullback can be realized as a base field extension of $F|K$, or more precisely,

$$F_1|K \cong FK^p|K^p.$$ 

In particular, as the genus does not increase under base field extensions, we obtain 

$$g \geq g_1 \geq \mathfrak{g}$$

where $g_1$ denotes the genus of the Frobenius pullback $F_1|K$.

The function field $F|K$ is called \textit{conservative} if its genus $g$ is equal to the genus $\mathfrak{g}$ of $FK|K$. By Rosenlicht’s genus drop formula it is non-conservative (that is, $\mathfrak{g} < g$) if and only if it admits a singular prime. The function field $F|K$ is called \textit{geometrically elliptic} (resp., \textit{geometrically rational}) if $FK|K$ is elliptic (resp., rational), that is, $\mathfrak{g} = 1$ (resp., $\mathfrak{g} = 0$).

The genus-2 function field, written in the normal form $y^2 + a(x)y + b(x) = 0$ of Section 2, is called of \textit{separable type} if it is separable over its canonical quadratic rational subfield $K(x)$, that is, $p \neq 2$ or $a(x) \neq 0$.

\textbf{Theorem 3.1.} A one-dimensional separable function field $F|K$ of genus $g = 2$ in characteristic $p = 2$ is geometrically elliptic if and only if it is non-conservative and of separable type. Such a function field can be put into the normal form

$$y^2 + (a_2 x^2 + a_0)y + b_6 x^6 + b_4 x^4 + b_0 = 0$$

where $a_2, a_0, b_6, b_4, b_0 \in K$ and 

$$\Delta := b_6^2 (a_2^6 b_0 + a_0^2 a_2^4 b_4 + a_0^3 a_2^3 b_6 + a_0^4 b_6^2) \neq 0.$$ 

The modular invariant $\mathfrak{g}$ of the elliptic function field $FK|K$ is equal to 

$$\mathfrak{g} = (j_1)^{\frac{1}{2}}$$

where 

$$j_1 = \frac{a_0^{12}}{\Delta}$$

is the modular invariant of the Frobenius pullback $F_1|K = K(x^2, y)$.

Conversely, if $a_2, a_0, b_6, b_4$ and $b_0$ are elements of the base field $K$ satisfying $\Delta \neq 0$, then the above polynomial equation defines a geometrically elliptic function field. Its genus is equal to two, with the only exceptions that either $j_1 \in (K^*)^2$ and $a_0 a_2 \in K^2$, or $j_1 = 0$ and $a_0 b_6 \in K^2$.

In proving the theorem we will also decide when two of these function fields are isomorphic. We start the proof by considering a separable genus-2 function field $F|K = K(x, y)|K$ given in the normal form

$$y^2 + a(x)y + b(x) = 0$$

where $a(x) = \sum_{i=0}^{3} a_i x^i$ and $b(x) = \sum_{i=0}^{6} b_i x^i$.

As $p = 2$ the separability of $F|K$ means that $a(x) \neq 0$ or $b(x) \neq 0$. If $F|K$ is of inseparable type, i.e., $a(x) = 0$, then $F_1 = K(x^2, b(x)) = K(x)$, $FK = K(x, b(x)^{1/2}) = K(x^{1/2})$ and therefore $g_1 = \mathfrak{g} = 0$. 


Now we assume that \( F|K \) is of separable type, i.e., \( a(x) \neq 0 \). We further assume that \( F|K \) is non-conservative, that is, \( \overline{\gamma} < 2 \). Let \( K' \) be the separable closure of \( K \) in \( \overline{K} \). As the genus is preserved under separable base field extensions, the function field \( FK'|K' \) is also non-conservative, i.e., it admits a singular prime \( p' \). Let \( \overline{\gamma} \) be the unique prime of \( F\overline{K}|\overline{K} \) lying over \( p' \) and let \( P \) be the corresponding singular point of the curve \( C \) on the cone \( S \subset \mathbb{P}^1(\overline{K}) \).

We first assume that \( P \) does not lie on the line \( L_\infty \), that is, \( P \in W \). Then by the Jacobian criterion, the coordinates \( \overline{x} = x(\overline{\gamma}) \) and \( \overline{\gamma} = y(\overline{\gamma}) \) of \( P \) in the first chart satisfy
\[
a'(\overline{\gamma}) \overline{y} + b'(\overline{\gamma}) = 0, \quad a(\overline{\gamma}) = 0 \quad \text{and} \quad \overline{y}^2 = b(\overline{\gamma}).
\]
Moreover, as by Section \( 2 \) the point \( P \) is unibranch, we deduce \( a'(\overline{\gamma}) = 0 \) and therefore \( b'(\overline{\gamma}) = 0 \). Thus, \( \overline{x} \) is a zero of \( a(x) \) of order larger than one.

If \( P \in L_\infty \) then a similar reasoning in the second chart shows that the polynomial \( a(x) \) of formal degree 3 has order larger than one at \( \overline{\gamma} = x(\overline{\gamma}) = \infty \), that is, \( \deg a(x) \leq 1 \). Moreover, \( b_5 = 0 \) by analogy with the equation \( b'(\overline{\gamma}) = 0 \) of the previous case.

In both cases the point \( P \) is the only point of the curve \( C \) lying on the line \( L_\infty \). As the polynomial \( a(x) \) of formal degree 3 can only admit one multiple zero, we deduce that \( p' \) is the only singular prime of \( FK'|K' \). In particular, denoting by \( p \) the prime of \( F|K \) lying below \( p' \), we conclude that \( p \) is the only singular prime of \( F|K \).

We will first assume that the multiple zero \( \overline{x} \) of \( a(x) \) has order two. Replacing, if necessary, \( x \) and \( y \) by \( x^{-1} \) and \( x^{-3} y \), respectively, we can assume that \( \overline{\gamma} \neq \infty \), that is, \( \overline{x} \in \overline{K} \). If \( \deg a_3(x) = 3 \), then as \( p = 2 \), by Vieta's formula \( a_1/a_0 \) is a simple root of \( a(x) \) belonging to the base field \( K \). Hence, replacing \( x \) and \( y \) by \( (x - a_1/a_0)^{-1} \) and \( (x - a_1/a_0)^{-3} y \), respectively, we can arrange that \( \deg a_2(x) = 2 \), that is, \( a(x) = a_2 x^2 + a_0 \) and \( a_2 \neq 0 \).

If \( \overline{x} \) is a triple zero of \( a(x) \) and \( \overline{\gamma} \neq \infty \), then \( \overline{x} = a_1/a_0 \), and so by transforming as above we can arrange that \( \infty \) is a triple zero of \( a(x) \), that is, \( \deg a(x) = 0 \). Thus in each of the two cases we can normalize
\[
a(x) = a_2 x^2 + a_0 \neq 0.
\]

The transformations that preserve this normalization preserve the line \( L_\infty \), and so by Section \( 2 \) they are just of the form
\[
(x, y) \mapsto (\alpha x + \delta, \beta y + \sum_{i=0}^{3} \gamma_i x^i)
\]
where \( \alpha, \beta \in K^* \) and \( \gamma_0, \gamma_1, \gamma_2, \gamma_3, \delta \in K \). To make further normalizations, we are just allowed to substitute
\[
a(x) \mapsto \beta^{-1} a(\alpha x + \delta)
\]
and
\[
b(x) \mapsto \beta^{-2} \left( b(\alpha x + \delta) + \sum_{i=0}^{3} \gamma_i^2 x^{2i} + a(\alpha x + \delta) \sum_{i=0}^{3} \gamma_i x^i \right).
\]
If \( a_2 \neq 0 \) then we can normalize \( b_5 = b_3 = b_2 = 0 \), and furthermore we get \( b_1 = b'(\overline{\gamma}) = 0 \). If \( a_2 = 0 \) then \( b_5 = 0 \) and we can normalize \( b_5 = b_3 = b_2 = 0 \). Thus in both cases we have
\[
b(x) = b_6 x^6 + b_4 x^4 + b_0,
\]
and the freedom to transform is restricted by the conditions
\[ \gamma_1 = \gamma_3 = 0 \quad \text{and} \quad \alpha \gamma_0 + (a_2 \delta^2 + a_0) \alpha^{-2} \gamma_2 = \delta^4 b_6. \]

The coefficients of the minimal equation transform as follows:
\[
\begin{align*}
\alpha &\mapsto \alpha^2 \beta^{-1} a_2 \\
\beta &\mapsto \beta^{-1} (a_0 + a_2 \delta^2) \\
b_0 &\mapsto a^6 \beta^{-2} b_6 \\
b_4 &\mapsto a^4 \beta^{-2} (b_4 + \alpha^{-4} \gamma_2^2 + \alpha^{-2} \gamma_2 a_2 + \delta^2 b_6) \\
b_0 &\mapsto \beta^{-2} (b_0 + \gamma_0^2 + \gamma_0 a_2 \delta^2 + \gamma_0 a_0 + b_4 \delta^4 + b_0 \delta^6). \end{align*}
\]

In particular, the class \( b_6 \mod (K^*)^2 \) is an invariant of the function field \( F|K \).
Moreover, if \( a_2 \neq 0 \) (resp., \( a_2 = 0 \)), then we can normalize \( a_2 = 1 \) (resp., \( a_0 = 1 \)), and the freedom to transform is furthermore restricted by the condition \( \beta = \alpha^2 \) (resp., \( \beta = 1 \)).
Allowing a quadratic base field extension if necessary, we could also normalize \( b_4 = 0 \) (resp., \( b_0 = 0 \)).

As a hyperelliptic function fields admits exactly one quadratic subfield of genus zero (see [Ch. Chapter IV, Theorem 9]) and as \( F|K \) is of separable type, we conclude that the genus \( g_1 \) of the Frobenius pullback \( F_1|K = F^2 K|K \) is different from zero. As \( F_1 = K(x^2, y^2) \) and \( a(x) \neq 0 \), we have \( y \in F_1 \) and therefore \( F_1 = K(z, y) \) where \( z := x^2 \) and
\[ y^2 + (a_0 + a_2 z) y + b_6 z^3 + b_4 z^2 + b_0 = 0. \]

We notice that \( b_6 \neq 0 \), because otherwise by the Jacobian criterion \( F_1|K \) would be the function field of a plane projective smooth conic curve (respectively, of a projective line) if \( a_0^6 b_4 \neq a_0^2 b_6 \) (respectively, \( a_0^6 b_4 = a_0^2 b_6 \)), in contradiction with \( g_1 \neq 0 \). Moreover, we have \( \Delta \neq 0 \), because otherwise \( F_1|K \) would be rational as the function field of a plane projective geometrically integral cubic curve with a rational non-smooth point. Thus \( F_1|K \) is an elliptic function field and therefore \( g_1 = 7 = 1 \).

Let \( j_1 \) be the modular invariant of \( F_1|K \) as introduced in characteristic two by Tate [T2]. To compute \( j_1 \) we replace \( x \) and \( y \) by \( b_0 x \) and \( b_0 y \), respectively, in order to get a minimal equation that is monic in the two coordinate functions, and then we obtain \( j_1 = a_1^2/\Delta \) from Tate’s formulae. As the Frobenius map provides an isomorphism between the function fields \( FK^{1/2}|K^{1/2} \) and \( F_1|K \), we conclude that the elliptic function fields \( FK^{1/2}|K^{1/2} \) and \( F_1|K \) have the invariant \( j_1^{1/2} \).

To prove the last part of the theorem, let be given a polynomial
\[ f(X, Y) = Y^2 + (a_2 X^2 + a_0) Y + b_6 X^6 + b_4 X^4 + b_0 \in K[X, Y] \]
whose coefficients satisfy \( \Delta \neq 0 \). Then \( b_6 \neq 0 \) and \((a_0, a_2) \neq (0, 0)\), and this implies that \( f(X, Y) \) is absolutely irreducible. Indeed, if there would exist a polynomial \( c(X) \in \overline{K}[X] \) such that \( f(x, c(X)) = 0 \), then as \( b_6 \neq 0 \) its degree would be equal to 3 and by comparing the terms of degree 3 and 5 we would get the contradiction \( a_0 = a_2 = 0 \).

Let \( F|K = K(x, y)|K \) be the separable function field given by the absolutely irreducible equation \( f(x, y) = 0 \). As \( \Delta \neq 0 \), the Frobenius pullback \( F_1|K = K(x^2, y^2)|K = K(x^2, y)|K \) is an elliptic function field, and so \( g_1 = 7 = 1 \).

To express the genus \( g \) of \( F|K \) in terms of the coefficients, we will apply Rosenlicht’s genus drop formula. To determine the singularity degree of a prime \( p \) of
$F|K$, we look for a natural number $n$ such that the restriction $p_n$ of $p$ to the $n$-th Frobenius pullback $F_n := F^{2^n}K|K$ is rational. Such an integer exists if $K$ is separably closed (see [BS, Lemma 2.1]). We write the $2^n$-power of the separating variable $x$ as a Laurent series in a local parameter at $p_n$. Then the singularity degree of $p$, as well as other properties of the local ring $O_p$ can be determined from this Laurent series expansion (see [BS]). To finish the proof of the theorem, we divide the discussion into three cases, according to the degree of $p$.

The first case happens if and only if $j_1 = 0$.

Proof. (i) We assume that $a_2 = 0$, and normalize $a_0 = 1$. Let $p$ be a singular prime of $F|K$. As the only singular point of the curve $C$ lies on the line $L_\infty$, we conclude that $p$ is a pole of $x$. Thus, the restriction $p_1$ of $p$ to the Frobenius pullback $F_1|K = K(x^2,y)|K$ is the only pole of $z = x^2$. Hence $\deg(p_1) = 1$, $\text{ord}_{p_1}(z) = -2$, $\text{ord}_{p_1}(y) = -3$ and $t := z/y$ is a local parameter at $p_1$. To write $z = x^2$ as a Laurent series in $t$, we notice that

$$t^{-2}z^2 + t^{-1}z = b_0z^3 + b_4z^2 + b_0$$

and by comparing successively coefficients we obtain

$$z = b_0^{-1}(t^{-2} + t^{-1}b_4 + b_0t + \cdots)$$

where the dots stand for terms of order larger than 1. Now we can apply [BS, Proposition 4.1]: If $b_0 \notin K^2$ (resp., $b_0 \in K^2$) then the singularity degree of $p$ is equal to one (resp., zero) and so by Rosenlicht’s genus drop formula $g = 2$ (resp., $g = 1$).

(ii) We assume that $a_2 \neq 0$ and $a_0/a_2 \in K^2$, and normalize $a_2 = 1$ and $a_0 = 0$. Let $p$ be a singular prime of $F|K$. As the only singular point of the curve $C \subset \mathbb{P}^4$ lies on the line $L_0$, we conclude that $p$ is a zero of $x$. Thus the restriction $p_1$ of $p$ to $F_1|K = K(x^2,z)|K$ is the only zero of $z = x^2$.

We first assume that $b_0$ is a square, say $b_0 = c^2$ where $c \in K^*$. Then $F_1 = K(t,z)$ where $t := y + c$ and

$$t^2 + cz + t + b_4z^2 + b_0z^3 = 0.$$
(iii) We assume that \( a_2 \neq 0 \) and \( a_0/a_2 \notin K^2 \), normalize \( a_2 = 1 \), and so we have \( a_0 \notin K^2 \). Let \( p \) be a singular prime of \( F|K \). As the only singular point of \( C \) lies on the line \( L_{a_0^{1/2}} \), we conclude that \( p \) lies over the \((x^2 + a_0)\)-adic prime of the quadratic rational subfield \( K(x) \) of \( F|K \). Denoting by \( \overline{y} \) the residue class of \( y \) mod \( p \), we have
\[
\overline{y} = (b_0a_0^3 + b_4a_0^2 + b_0)^{1/2} = b_0^{-1}j_1^{-1/2} + a_0^2b_6.
\]
We will first assume that \( \overline{y} \in K \), that is, \( j_1 \in (K^*)^2 \). Then \( p_1 \) is a rational prime of \( F_1|K = K(x^2, y)|K \), and \( t := y + \overline{y} \) is a local parameter at \( p_1 \). To write \( z = x^2 \) as a Laurent series in \( t \), we enter into the polynomial equation
\[
(t + \overline{y})^2 + (z + a_0)t + \overline{y} + b_0z^3 + b_4z^2 + b_0 = 0
\]
and obtain
\[
x^2 = a_0 + b_0j_1^{1/2}t^2 + b_0^2j_1t^3 + \cdots.
\]
It now follows from [BS, Proposition 4.1] that the singularity degree of \( p \) is equal to one, and therefore \( g = 2 \).

Now we assume that \( \overline{y} \notin K \), that is, \( j_1 \notin K^2 \). As \( F_1 = K(x^2, y) \) we obtain
\[
F_2 = K(x^4, y^2) = K(x^4, t)
\]
where \( t := y^2 + \overline{y}^2 \) is a local parameter at the rational prime \( p_2 \). From the polynomial equation
\[
(t + \overline{y}^2)^2 + (x^4 + a_0^2)(t + \overline{y}^2) + b_0^2x^{12} + b_4^2x^8 + b_6^2 = 0
\]
we get the power series expansion
\[
x^4 = a_0^2 + b_0^2j_1t^2 + b_0^2j_1^2t^3 + \cdots.
\]
By [BS] Proposition 4.1 the residue field of \( p_1 \) is equal to \( K(j_1^{1/2}) \).

If \( a_0 \notin K^2(j_1) \) then by [BS] Proposition 4.3 the residue field of \( p \) is equal to \( K(j_1^{1/2}, a_0^{1/2}) \), the singularity degree of \( p \) is equal to 1, and therefore \( g = 2 \).

If \( a_0 \in K^2(j_1) \), that is, \( a_0 \in K^2(\overline{y}^2) \) say \( a_0 = \alpha^2 + \beta^2\overline{y}^2 \) where \( \alpha, \beta \in K \), then defining \( w := x + \alpha + \beta y \), we get the expansion
\[
w^4 = (b_0^2j_1 + \beta^4)t^2 + b_0^2j_1^2t^3 + \cdots,
\]
and so by [BS] Proposition 4.3 \( p|p_1 \) is ramified, the singularity degree of \( p \) is equal to 1, and therefore \( g = 2 \). \( \square \)

4. Genus-2 function fields of inseparable type

Let \( F|K \) be a one-dimensional separable function field of genus 2, written in the normal form \( y^2 + a(x)y + b(x) = 0 \) of Section 2. We assume that \( F|K \) is of inseparable type or, equivalently, it is an inseparable extension of its canonical quadratic rational subfield \( K(x) \), that is, \( p = 2 \) and \( a(x) = 0 \). Therefore
\[
y^2 = b(x) = \sum_{i=0}^{6} b_ix^i \in K[x] \quad \text{and} \quad b'(x) = b_5x^4 + b_3x^2 + b_1 \neq 0.
\]

By Section 2 the polynomial \( b(x) \) is uniquely determined by the isomorphism class of \( F|K \) up to the substitutions
\[
b(x) \mapsto \beta^{-2}(\alpha_{21}x + \alpha_{22})^6 b \left( \frac{\alpha_{11}x + \alpha_{12}}{\alpha_{21}x + \alpha_{22}} \right) + \sum_{i=0}^{3} \gamma_i^2x^{2i}
\]
where \((\alpha_{ij}) \in \text{GL}_2(K)\) and \(\beta \in K^*\) and \(\gamma_0, \gamma_1, \gamma_2, \gamma_3 \in K\). In particular, 
\[
b'(x) \mapsto \beta^{-2} \det(\alpha_{ij}) (\alpha_{21} x + \alpha_{22})^4 b' \left( \frac{\alpha_{11} x + \alpha_{12}}{\alpha_{21} x + \alpha_{22}} \right).
\]
Replacing if necessary \(x\) by \(\frac{1}{x}\) or \(1 + \frac{1}{x}\), we can arrange that \(b_5 \neq 0\), and so we can normalize \(b_5 = 1\).

**Theorem 4.1.** A one-dimensional separable function field of genus \(g = 2\) in characteristic \(p = 2\) is geometrically rational if and only if it is of inseparable type, that is, it can be put into the normal form
\[
y^2 = b(x) = \sum_{i=0}^{6} b_i x^i \quad \text{where } b_5 = 1 \text{ and } b_0, b_1, b_2, b_3, b_4, b_6 \in K.
\]
If \(b_3 \neq 0\) and (after an eventual quadratic separable base field extension) the two roots of the polynomial \(b(T^{1/2}) = T^2 + b_3 T + b_1\) belongs to the base field \(K\), then the function field of inseparable type has genus two if and only if each such root \(c\) satisfies \(c \notin K^2\) or \(b_0 + b_2 c + b_4 c^2 + b_6 c^3 \notin K^2\).

If \(b_3 = 0\), then the genus is equal to two if and only if one of the following three cases occurs:

(i) \(b_1 \in K \setminus K^2\)

(ii) \(b_1 \in K^2 \setminus K^4\), and \(b_0 + b_1 b_4 \notin K^2\) or \(b_2 + b_1 b_6 \notin K^2\)

(iii) \(b_1 \in K^4\) and \(\sum_{i=0}^{3} b_2 i^2 f_i^2 \notin K^2\).

**Proof.** The first part of the theorem follows from the first part of Theorem [5.1].

Let \(F | K\) be the function field given by the equation \(y^2 = b(x)\). As \(b'(x) \neq 0\), the \(n\)-th Frobenius pullback is equal to
\[
F_n = F^{2^n} K = K(x^{2^n}, y^{2^n}) = K(x^{2^n}, b(x)^{2^{n-1}}) = K(x^{2^n-1})
\]
for each natural number \(n\). If \(p\) is a singular prime of \(F | K\), then by the Jacobian criterion it is necessarily a zero of \(b'(x) = x^4 + b_3 x^2 + b_1\).

We will first assume that \(b_3 \neq 0\), and that we can factorize
\[
b'(x) = (x^2 + c)(x^2 + d) \quad \text{where } c, d \in K \text{ and } c \neq d.
\]
Let \(p \in \mathcal{R}_{F | K}\) be the zero of \(x^2 + c\). If \(c \in K^2\) then \(p_1\) is a rational prime, \(t := x + c^{1/2}\) is a local parameter at \(p_1\),
\[
y^2 = (b_0 + b_2 c + b_4 c^2 + b_6 c^3) t^0 + (b_2 + c^{1/2} (c + d)) t^2 + (c + d) t^3 + \cdots,
\]
and it follows from [BS] Proposition 4.1 that the singularity degree of \(p\) is equal to 1 (respectively, 0) if \(b_0 + b_2 c + b_4 c^2 + b_6 c^3\) does not belong (respectively, belongs) to \(K^2\).

Now we assume that \(c \notin K^2\). Then \(p_1\) is the \((x^2 + c)\)-adic prime of \(F_1 = K(x)\), \(p_2\) is rational, \(t := x + c\) is a local parameter at \(p_2\), and
\[
y^4 = (b_0 + b_2 c + b_4 c^2 + b_6 c^3) t^0 + (b_2 + c (c + d)^2) t^2 + (c + d)^2 t^3 + \cdots.
\]
The residue class of \(y\) mod \(p\) is equal to \(\overline{y} = (b_0 + b_2 c + b_4 c^2 + b_6 c^3)^{1/2}\).

If \(\overline{y} \notin K(c^{1/2})\) then \(p | p_1\) is inertial and by [BS] Theorem 3.2] the singularity degree of \(p\) is equal to 1.

If \(\overline{y} \in K(c^{1/2})\) say \(\overline{y} = \alpha + \beta c^{1/2}\) where \(\alpha, \beta \in K\), and if \(z := y + \alpha + \beta x\) then
\[
z^4 = (c (c + d)^2 + b_0^2 + \beta^4) t^2 + (c + d)^2 t^3 + \cdots,
\]
and, as $c \notin K^2$, $c \neq d$ and hence the coefficient of $t^2$ is non-zero, we deduce that $p|p_1$ is ramified, $\text{ord}_p(z) = 1$ and by [BS] Theorem 3.2 we again conclude that the singularity degree of $p$ is equal to 1.

Thus the singularity degree of the two zeros of $b'(x)$ are not larger than one, and so by Rosenlicht’s genus drop formula the genus $g$ is equal to two if and only if the two singularity degrees are equal to one.

Now we assume that $b_1 = 0$. Let $p \in R_{F/K}$ be the zero of $b'(x) = x^4 + b_1$.

(i) We assume that $b_1 \in K \setminus K^2$. Then $p_1$ is the $(x^4 + b_1)$-adic prime of $F_1 = K(x)$, $p_3$ is rational, $t := x^4 + b_1$ is a local parameter at $p_3$, and

$$y^8 = (b_0^2 + b_2^2 b_1 + b_4^2 b_1^2 + b_6^2 b_1^3) t^0 + (b_2 + b_1 b_0)^4 t^2 + (b_1 + b_4 + b_6 b_1) t^4 + t^5 + b_0 t^6.$$ 

The residue class of $y$ mod $p$ is equal to $y = (b_0^2 + b_2^2 b_1 + b_4^2 b_1^2 + b_6^2 b_1^3)^{1/4}$.

If $y \notin K(b_1^{1/4})$ then $p|p_1$ is inertial and by [BS] Theorem 3.2 the singularity degree of $p$ is equal to two.

Now we assume that $\gamma \in K(b_1^{1/4})$ say $y^4 = \alpha^4 + \beta^4 b_1 + \gamma^4 b_1^2 + \delta^4 b_1^3$ where $\alpha, \beta, \gamma, \delta \in K$. Then $(b_0 + b_1 b_2 + \alpha^2 + \gamma^2 b_1)^2 = b_1 (b_2 + b_6 b_1 + \beta^2 + \delta^2 b_1)^2$. As $b_1 \notin K^2$, this means

$$b_0 = b_1 b_2 + \alpha^2 + \gamma^2 b_1$$ 

and

$$b_2 = b_6 b_1 + \beta^2 + \delta^2 b_1.$$ 

Defining $z := y + \alpha + \beta x + \gamma x^2 + \delta x^3$ we obtain

$$z^8 = (b_1 + b_4^2 + b_6 b_1^2 + \gamma b_1^3) t^4 + t^5 + (b_0 + \delta) t^6.$$ 

As $b_1$ is not a square, the coefficient of $t^4$ is non-zero, hence $p|p_1$ is ramified, $z$ is a local parameter at $p$, and by [BS] Theorem 3.2 the singularity degree of $p$ is again equal to two.

(ii) We assume that $b_1 \in K^2 \setminus K^4$ say $b_1 = c^2$ where $c \in K \setminus K^2$. Then $p_1$ is the $(x^2 + c)$-adic prime of $F_1 = K(x)$, $p_2$ is rational, $t := x^2 + c$ is a local parameter at $p_2$, and

$$y^4 = (b_0 + b_2 c + b_4 c^2 + b_6 c^3)^4 t^0 + (b_2 + b_6 c^2)^2 t^2 + (c + b_4^2 + b_6 c^2) t^4 + t^5 + b_0 t^6.$$ 

The residue class of $y$ mod $p$ is equal to $y = (b_0 + b_2 c + b_4 c^2 + b_6 c^3)^{1/2}$.

If $y \notin K(c^{1/2})$ then $p|p_1$ is inertial and by [BS] Theorem 3.2 the singularity degree of $p$ is equal to two.

Now we assume that $\gamma \in K(c^{1/2})$ say $y^4 = \alpha^2 + \beta^2 c$, i.e., $b_0 + b_2 c + \alpha^2 = (b_2 + b_6 c^2 + \beta^2) c$ where $\alpha, \beta \in K$. Defining $z := y + \alpha + \beta x$ we obtain

$$z^4 = (b_2 + b_6 c^2 + \beta^2) t^2 + (c + b_4^2 + b_6 c^2) t^4 + t^5 + b_0 t^6.$$ 

If $b_2 \neq b_6 c^2 + \beta^2$, then $p|p_1$ is ramified, $z$ is a local parameter at $p$, and by [BS] Theorem 3.2 the singularity degree of $p$ is again equal to two. If $b_2 = b_6 c^2 + \beta^2$, i.e., $b_0 = b_4 c^2 + \alpha^2$, then the singularity degree of $p$ is equal to zero.

(iii) We assume that $b_1 \in K^4$ say $b_1 = c^4$ where $c \in K$. Then $p_1$ is rational, $t := x + c$ is a local parameter at $p_1$, and

$$y^2 = (b_0 + b_2 c^2 + b_4 c^4 + b_6 c^6) t^0 + (b_2 + b_6 c^4)^2 t^2 + (c + b_4 + b_6 c^2) t^4 + t^5 + b_0 t^6.$$ 

Now we can apply [BS] Proposition 4.1: If $\sum_{i=0}^3 b_{2i} c^{2i} \notin K^2$ then the singularity degree of $p$ is equal to two. If $\sum_{i=0}^3 b_{2i} c^{2i} \in K^2$ then the singularity degree of $p$ is equal to one (respectively, zero) if $b_2 + b_6 c^4 \notin K^2$ (respectively, $b_2 + b_6 c^4 \in K^2$). \qed
5. Fibrations by non-smooth curves of arithmetic genus two in characteristic 2.

Let $k$ be an algebraically closed field of characteristic 2, and

$$S = S(k) := \left\{ (u_0 : u_1 : u_2 : u_3 : v) \in \mathbb{P}^4(k) \mid \operatorname{rank} \begin{pmatrix} u_0 & u_1 & u_2 \\ u_1 & u_2 & u_3 \end{pmatrix} < 2 \right\}$$

be the cone that in Section 2 has been considered over the field $\mathbb{K}$. Influenced by Section 3 we announce:

**Theorem 5.1.** The algebraic variety

$$Z := \left\{ ((u_0 : u_1 : u_2 : u_3 : v), (a_0, a_2, b_0, b_4, b_6)) \in S \times \mathbb{A}^5 \mid v^2 + (a_0 u_0 + a_2 u_2) v + b_0 u_0^2 + b_4 u_2^2 + b_6 u_3^2 = 0 \right\}$$

is an irreducible smooth sixfold. The projection morphism

$$\pi : Z \longrightarrow \mathbb{A}^5$$

is proper and flat, and its fibres are non-smooth projective curves of arithmetic genus 2, which do not pass through the vertices.

The fibre over the point $(a_0, a_2, b_0, b_4, b_6) \in \mathbb{A}^5(k)$ is a geometrically elliptic curve (i.e., an integral curve of geometric genus 1) if and only if

$$\Delta := b_6^2 (a_0^2 b_0 + a_0^2 a_2^3 b_4 + a_0^2 a_3^2 b_6 + a_0^4 b_0^2) \neq 0.$$ 

In this case, the fibre has a cusp as its only singularity, and the elliptic modular invariant of its non-singular projective model is equal to $a_0^2/\Delta^{1/2}$.

During the proof of the theorem we will describe the singularities of all fibres, and discuss how they move. We start the proof by noting that the variety $Z$ is the closed subset of $S \times \mathbb{A}^5$ contained in the smooth subvariety $(S \setminus \{Q\}) \times \mathbb{A}^5$ that in the charts $W \times \mathbb{A}^5 \longrightarrow \mathbb{A}^7$ and $\tilde{W} \times \mathbb{A}^5 \longrightarrow \mathbb{A}^7$ is given by the equations

$$v^2 + (a_0 + a_2 u^2) v + b_0 + b_4 u^4 + b_6 u^6 = 0$$

and

$$\tilde{v}^2 + (a_0 \tilde{u}^3 + a_2 \tilde{u}) \tilde{v} + b_0 \tilde{u}^6 + b_4 \tilde{u}^2 + b_6 = 0,$$

respectively. Hence $Z$ is irreducible, smooth and of dimension 6. As $S$ is projective, the projection morphism $\pi$ is proper. The fibration $\pi : Z \rightarrow \mathbb{A}^5$ provides a 5-dimensional family of projective curves on the punctured cone $S \setminus \{Q\}$. For each point $(a_0, a_2, b_0, b_4, b_6)$ of the base $\mathbb{A}^5$ the corresponding curve is given in the two charts $W$ and $\tilde{W}$ by the above equations. As the base $\mathbb{A}^5$ is smooth, as the total space $Z$ is Cohen-Macaulay as a smooth variety, and as the dimension of each fibre is equal to $\dim(Z) - \dim(\mathbb{A}^5) = 1$, we conclude that the morphism $\pi : Z \rightarrow \mathbb{A}^5$ is flat (see [18] Theorem 18.16), and so the fibres have the same arithmetic genus.

The singular points of the fibres are obtained by applying the Jacobian criterion to the curves in the two charts of the punctured cone. The types of the singularities can be read off from the blowup sequences. Each curve of the family has at the point

$$\left( a_2^{3/2} : a_0^{1/2} a_2 : a_0 a_2^{1/2} : a_0^{3/2} : (b_0 a_2^6 + b_4 a_0 a_2^2 + b_6 a_0^6)^{1/2} \right)$$

a singularity with double tangent line. It is a cusp if and only if the second factor of $\Delta$ is non-zero. If the second factor vanishes and $a_2 \neq 0$, then it is a tacnode, i.e., a two-branched point of singularity degree two. If $a_2 = 0$, $a_0 \neq 0$, $b_6 = 0$ and $b_4 \neq 0$,
0, then it is a ramphoid cusp, i.e., a unibranch point of singularity degree two. If $a_2 = 0$, $a_0 \neq 0$, $b_0 = 0$ and $b_4 = 0$, then it is a two-branched point of multiplicity three.

If $a_2 \neq 0$ and $b_0 = 0$, then the curve has a second singularity, namely a node at the point $(0 : 0 : 0 : 1 : 0)$ on the line $L_\infty$, which collides with the first singularity if $a_2$ tends to zero. If $(a_0, a_2) \neq (0, 0)$, then there are no other singularities on the fibre. If $(a_0, a_2) = (0, 0)$, then the fibre is a double smooth rational curve, and so it is non-reduced and its points are singular.

If $(a_2, a_0) \neq (0, 0)$, then the fibre is non-integral if and only if $b_0 = 0$ and $a_2 b_4 = a_2^2 b_0$. In this case it is the union of two smooth rational curves. If $a_2 \neq 0$, then the two components intersect at the two singular points with the multiplicities 2 and 1. If $a_2 = 0$, then the two components intersect at the only singular point with multiplicity 3.

**Remark.** If by homogenizing we enlarge the base of $\pi$ from $A^5$ to $P^5$, then the total space acquires singularities at $\{Q\} \times (P^5 \setminus A^5)$, and the fibres over $P^5 \setminus A^5$ pass through the vertices.

**Theorem 5.2.** If $\phi : T \to B$ is a proper morphism of irreducible smooth algebraic varieties such that almost all fibres are geometrically elliptic curves of arithmetic genus 2, then the fibration $\phi : T \to B$ is, up to birational equivalence, a base extension of the fibration $\pi : Z \to A^5$.

**Proof.** By Section [1] the one-dimensional function field $k(T)/k(B)$ is geometrically elliptic of genus 2, and so by Theorem [3.1] it can be put into the normal form

$$k(T) = k(B)(x, y) \quad \text{where} \quad y^2 + (a_0 + a_2 x^2)y + b_0 + b_4 x^4 + b_6 x^6 = 0,$$

$a_0, a_2, b_0, b_4, b_6 \in k(B)$ and $\Delta(a_0, a_2, b_0, b_4, b_6) \neq 0$. Let $B'$ be the closed irreducible affine subvariety of $A^5$ with the coordinate algebra $k[a_0, a_2, b_0, b_4, b_6]$, let

$$T' := \pi^{-1}(B') \subseteq Z \subset S \times A^5$$

and let $\phi' : T' \to B'$ be the corresponding closed subfibration of $\pi : Z \to A^5$. By Theorem [5.1] the morphism $\phi'$ is proper and its fibres are non-smooth projective curves of arithmetic genus 2. As $\Delta \neq 0$, almost every fibre is a geometrically elliptic curve with a cusp as its only singularity. By construction

$$k(B') = k(a_0, a_2, b_0, b_4, b_6), \quad k(T') = k(B')(x, y)$$

and therefore

$$k(T) \cong k(T') \otimes_{k(B')} k(B).$$

By restricting the base $B$ of the fibration $\phi : T \to B$ to a dense open subset, we can arrange that the rational functions $a_2, a_0, b_0, b_4$ and $b_6$ become regular on $B$, and hence define a dominant morphism $B \to B'$. Thus we have the fibre product $T' \times_{B'} B$, whose function field is isomorphic to $k(T') \otimes_{k(B')} k(B)$ and hence isomorphic to $k(T)$. More precisely, the inclusion $k(T') \subseteq k(T)$ induces a rational map $T \dashrightarrow T'$ and hence a rational map of $B$-schemes $T \dashrightarrow T' \times_{B'} B$, which is birational, because the induced homomorphism between the function fields is the above isomorphism.

To diminish the dimension of the base of the fibration $\pi$, according to Corollary [3.2] we divide the discussion into the cases $7 \neq 0$ and $7 = 0$, normalize $a_2 = 1$ and $a_0 = 1$, respectively, and obtain as base varieties the affine spaces $A^4$ and $A^3$, respectively. By admitting separable quadratic base extensions if necessary, we can
further normalize $b_4 = 0$ and $b_0 = 0$, respectively, and diminish the dimensions of the bases by 1.

In the first case we get the irreducible smooth fourfold

$$X := \{(u_0 : u_1 : u_2 : u_3 : v), (a_0, b_0, b_6) \in S \times \mathbb{A}^3 | v^2 + (a_0 u_0 + u_2)v + b_0 u_0^2 + b_6 u_3^2 = 0\}$$

equipped with the proper and flat projection morphism $\chi : X \rightarrow \mathbb{A}^3$.

If $\Delta := b_0^2 (b_0 + a_0^2 b_6 + a_0^4 b_6^2) \neq 0$, then the fibre over the point $(a_0, b_0, b_6)$ is geometrically elliptic with the modular invariant $\Delta^{-1} \neq 0$, and has a cusp as its only singularity. This describes the generic behavior of the fibres of $\chi$.

By the discussion following Theorem 5.1, we also know the structure of the bad fibres. If $b_6 \neq 0$ and the second factor of $\Delta$ vanishes, then the fibre is a rational curve with a tacnode as its only singularity. If $b_0 = 0$ and $b_0 \neq 0$, then the fibre is a rational curve with a cusp and a node as its only singularities. In the remaining case where $b_0 = b_6 = 0$, the fibre is a union of two smooth rational curves, which meet in the two singular points with intersection multiplicities two and one.

In the second case where $\gamma = 0$, we get the irreducible smooth threefold

$$Y := \{(u_0 : u_1 : u_2 : u_3 : v), (b_4, b_6) \in S \times \mathbb{A}^2 | v^2 + u_0 v + b_4 u_2^2 + b_6 u_3^2 = 0\}$$

and the proper and flat projection morphism $\eta : Z \rightarrow \mathbb{A}^2$. If $b_6 \neq 0$, then the fibre over the point $(b_4, b_6)$ is geometrically elliptic with the modular invariant 0, and has a cusp as its only singularity.

The bad fibres are described as follows: If $b_6 = 0$ and $b_4 \neq 0$, then the fibre is a rational curve with a ramphoid cusp as its only singularity. If $b_4 = b_6 = 0$, then the fibre is a union of two smooth rational curves, which meet in only one point with intersection multiplicity three.

**Corollary 5.3.** Let $\phi : T \rightarrow B$ be a proper morphism of irreducible smooth varieties such that almost all fibres are geometrically elliptic curves of arithmetic genus two. If the modular invariants of the fibres are not identically zero (respectively, equal to zero), then the fibration $\phi : T \rightarrow B$, after an eventual separable quadratic base extension, is birational equivalent to a base extension of the fibration $\chi : X \rightarrow \mathbb{A}^3$ (respectively, $\eta : Y \rightarrow \mathbb{A}^2$).

¿From Section 4 we obtain by similar arguments the following result.

**Theorem 5.4.** The algebraic variety

$$V := \{(u_0 : u_1 : u_2 : u_3 : v), (b_0, \ldots, b_4, b_6) \in S \times \mathbb{A}^6 | v^2 + b_0 u_0^2 + b_1 u_0 u_1 + b_2 u_1^2 + b_3 u_1 u_2 + b_4 u_2^2 + u_2 u_3 + b_6 u_3^2 = 0\}$$

is an irreducible smooth sevenfold. The projection morphism $\mu : V \rightarrow \mathbb{A}^6$ is proper and flat, and its fibres are rational curves of arithmetic genus two. If $b_3 \neq 0$ (respectively, $b_3 = 0$), then the fibre over the point $(b_0, \ldots, b_4, b_6) \in \mathbb{A}^6(k)$ has two cusps (respectively, a ramphoid cusp).

If $\phi : T \rightarrow B$ is a proper morphism of irreducible non-smooth varieties such that almost all fibres are rational curves of arithmetic genus two, then the fibration $\phi : T \rightarrow B$ is, up to birational equivalence, a base extension of the fibration $\mu : V \rightarrow \mathbb{A}^6$. 
References

[BS] H. Bedoya, K.-O. Stöhr, An algorithm to calculate discrete invariants of singular primes in function fields, J. Number Theory 27 (1987) 310–323.

[BM] E. Bombieri, D. Mumford, Enriques’ classification of surfaces in characteristic p, III, Invent. Math. 35 (1976) 197–232.

[Bo] H. Borges Neto, Mudança de género e classificação de corpos de género 2, Tese de Doutorado, IMPA, Rio de Janeiro (1979).

[Ch] C. Chevalley, Introduction to the theory algebraic functions of one variable, Math. Surveys, No. IV, Amer. Math. Soc., New York (1951).

[E] D. Eisenbud, Commutative algebra with a view toward algebraic geometry, Graduate Texts in Mathematics 150 (1995), Springer-Verlag.

[Ha] R. Hartshorne, Algebraic Geometry, Springer-Verlag, New York (1977).

[Hi] H. Hironaka, On the arithmetic genera and the effective genera of algebraic curves, Mem. Coll. Sci. Univ. Ser. A Math. 35 (1976) 197–232.

[Bo] H. Borges Neto, Mudança de gênero e classificação de corpos de gênero 2, Tese de Doutorado, IMPA, Rio de Janeiro (1979).

[Ch] C. Chevalley, Introduction to the theory algebraic functions of one variable, Math. Surveys, No. IV, Amer. Math. Soc., New York (1951).

[E] D. Eisenbud, Commutative algebra with a view toward algebraic geometry, Graduate Texts in Mathematics 150 (1995), Springer-Verlag.

[Ha] R. Hartshorne, Algebraic Geometry, Springer-Verlag, New York (1977).

[Hi] H. Hironaka, On the arithmetic genera and the effective genera of algebraic curves, Mem. Coll. Sci. Univ. Ser. A Math. 35 (1976) 197–232.

[Bo] H. Borges Neto, Mudança de gênero e classificação de corpos de gênero 2, Tese de Doutorado, IMPA, Rio de Janeiro (1979).

[Ch] C. Chevalley, Introduction to the theory algebraic functions of one variable, Math. Surveys, No. IV, Amer. Math. Soc., New York (1951).

[E] D. Eisenbud, Commutative algebra with a view toward algebraic geometry, Graduate Texts in Mathematics 150 (1995), Springer-Verlag.

[Ha] R. Hartshorne, Algebraic Geometry, Springer-Verlag, New York (1977).

[Hi] H. Hironaka, On the arithmetic genera and the effective genera of algebraic curves, Mem. Coll. Sci. Univ. Ser. A Math. 35 (1976) 197–232.

[Bo] H. Borges Neto, Mudança de gênero e classificação de corpos de gênero 2, Tese de Doutorado, IMPA, Rio de Janeiro (1979).

[Ch] C. Chevalley, Introduction to the theory algebraic functions of one variable, Math. Surveys, No. IV, Amer. Math. Soc., New York (1951).

[E] D. Eisenbud, Commutative algebra with a view toward algebraic geometry, Graduate Texts in Mathematics 150 (1995), Springer-Verlag.

[Ha] R. Hartshorne, Algebraic Geometry, Springer-Verlag, New York (1977).

[Hi] H. Hironaka, On the arithmetic genera and the effective genera of algebraic curves, Mem. Coll. Sci. Univ. Ser. A Math. 35 (1976) 197–232.

[Bo] H. Borges Neto, Mudança de gênero e classificação de corpos de gênero 2, Tese de Doutorado, IMPA, Rio de Janeiro (1979).

[Ch] C. Chevalley, Introduction to the theory algebraic functions of one variable, Math. Surveys, No. IV, Amer. Math. Soc., New York (1951).

[E] D. Eisenbud, Commutative algebra with a view toward algebraic geometry, Graduate Texts in Mathematics 150 (1995), Springer-Verlag.

[Ha] R. Hartshorne, Algebraic Geometry, Springer-Verlag, New York (1977).

[Hi] H. Hironaka, On the arithmetic genera and the effective genera of algebraic curves, Mem. Coll. Sci. Univ. Ser. A Math. 35 (1976) 197–232.

[Bo] H. Borges Neto, Mudança de gênero e classificação de corpos de gênero 2, Tese de Doutorado, IMPA, Rio de Janeiro (1979).

[Ch] C. Chevalley, Introduction to the theory algebraic functions of one variable, Math. Surveys, No. IV, Amer. Math. Soc., New York (1951).

[E] D. Eisenbud, Commutative algebra with a view toward algebraic geometry, Graduate Texts in Mathematics 150 (1995), Springer-Verlag.

[Ha] R. Hartshorne, Algebraic Geometry, Springer-Verlag, New York (1977).

[Hi] H. Hironaka, On the arithmetic genera and the effective genera of algebraic curves, Mem. Coll. Sci. Univ. Ser. A Math. 35 (1976) 197–232.