Parameter estimation based on cumulative Kullback-Leibler divergence

Yaser Mehrali\textsuperscript{1}, and Majid Asadi\textsuperscript{2}
Department of Statistics, University of Isfahan, Isfahan, 81744, Iran
E-mail: yasermehrali@gmail.com\textsuperscript{1}, m.asadi@sci.ui.ac.ir\textsuperscript{2}

April 9, 2018

Abstract

In this paper, we propose some estimators for the parameters of a statistical model based on Kullback-Leibler divergence of the survival function in continuous setting. We prove that the proposed estimators are subclass of “generalized estimating equations” estimators. The asymptotic properties of the estimators such as consistency, asymptotic normality, asymptotic confidence interval and asymptotic hypothesis testing are investigated.

Key words and Phrases: Entropy, Estimation, Generalized Estimating Equations, Information Measures.

2010 Mathematics Subject Classification: 62B10, 94A15, 94A17, 62G30, 62E20, 62F03, 62F05, 62F10, 62F12, 62F25.

1 Introduction

The Kullback-Leibler (KL) divergence or relative entropy is a measure of discrimination between two probability distributions. If \( X \) and \( Y \) have probability density functions \( f \) and \( g \), respectively, the KL divergence of \( f \) relative to \( g \) is defined as

\[
D(f||g) = \int_{\mathbb{R}} f(x) \log \frac{f(x)}{g(x)} \, dx,
\]

for \( x \) such that \( g(x) \neq 0 \). The function \( D(f||g) \) is always nonnegative and it is zero if and only if \( f = g \) a.s..

Let \( f(x; \theta) \) belong to a parametric family with \( k \)-dimensional parameter vector \( \theta \in \Theta \subset \mathbb{R}^k \) and \( f_n \) be kernel density estimation of \( f \) based on \( n \) random variables \( \{X_1, \ldots, X_n\} \) of distribution \( X \). Basu and Lindsay (1994) used KL divergence of \( f_n \) relative to \( f \) as

\[
D(f_n||f) = \int_{\mathbb{R}} f_n(x) \log \frac{f_n(x)}{f(x; \theta)} \, dx,
\]

and defined the minimum KL divergence estimator of \( \theta \) as

\[
\hat{\theta} = \arg \inf_{\theta \in \Theta} D(f_n(x)||f(x; \theta)).
\]
Lindsay (1994) proposed a version of (1) in discrete setting. In recent years, many authors such as Morales et al. (1995), Jiménez and Shao (2001), Broniatowski and Keziou (2009), Broniatowski (2014), Cherfi (2011, 2012, 2014) studied the properties of minimum divergence estimators under different conditions. Basu et al. (2011) discussed in their book about the statistical inference with the minimum distance approach.

Although the method of estimation based on $D(f_n||f)$ has very interesting features, the definition is based on $f$ which, in general, may not exist and also depends on $f_n$ which even if the number of samples tends to infinity, there is no guarantee that converges to its true measure.

Let $X$ be a random variable with cumulative distribution function (c.d.f) $F(x)$ and survival function (s.f) $\bar{F}(x)$. Based on $n$ observations $\{X_1, \ldots, X_n\}$ of distribution $F$, define the empirical cumulative distribution and survival functions, respectively, by

$$F_n(x) = \sum_{i=1}^{n} \frac{i}{n} I_{[x_{(i)}, x_{(i+1)}]}(x),$$

and

$$\bar{F}_n(x) = \sum_{i=0}^{n-1} \left(1 - \frac{i}{n}\right) I_{[x_{(i)}, x_{(i+1)}]}(x),$$

where $I$ is the indicator function and $(0 = X_{(0)} \leq X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ are the ordered sample. $F_n(\bar{F}_n)$ is known in the literature as "empirical estimator" of $F(\bar{F})$.

In the case when $X$ and $Y$ are continuous nonnegative random variables with s.f’s $\bar{F}$ and $\bar{G}$, respectively, a version of $KL$ divergence estimator ($CKL$) of $\bar{F}_n$ and survival function $\bar{F}$ (we call it $CKL\left(\bar{F}_n||\bar{F}\right)$ as

$$CKL\left(\bar{F}_n||\bar{F}\right) = \int_{0}^{\infty} \bar{F}_n(x) \log \frac{\bar{F}_n(x)}{\bar{F}(x; \theta)} - \left[\bar{F}_n(x) - \bar{F}(x; \theta)\right] dx$$

$$= \int_{0}^{\infty} \bar{F}_n(x) \log \bar{F}_n(x) dx - \int_{0}^{\infty} \bar{F}_n(x) \log \bar{F}(x; \theta) dx - [\bar{x} - E_{\theta}(X)].$$

The properties of this divergence measure are studied by some authors such as Liu (2007) and Baratpour and Habibi Rad (2012).

In order to estimate the parameters of the model, Liu (2007) proposed cumulative $KL$ divergence between the empirical survival function $\bar{F}_n$ and survival function $\bar{F}$ (we call it $CKL\left(\bar{F}_n||\bar{F}\right)$) as

$$\widehat{\theta} = \arg \inf_{\theta \in \Theta} CKL\left(\bar{F}_n(x)||\bar{F}(x; \theta)\right).$$

If consider the parts of $CKL\left(\bar{F}_n||\bar{F}\right)$ that depends on $\theta$ and define

$$g(\theta) = E_{\theta}(X) - \int_{0}^{\infty} \bar{F}_n(x) \log \bar{F}(x; \theta) dx,$$

then the $MCKL$ of $\theta$ can equivalently be defined by

$$\widehat{\theta} = \arg \inf_{\theta \in \Theta} g(\theta).$$
Two important advantages of this estimator are that one does not need to have the density function and for large values of $n$ the empirical estimator $F_n$ tends to the distribution function $F$.

Liu (2007) applied this estimator in uniform and exponential models and Yari and Saghaﬁ (2012) and Yari et al. (2013) used it for estimating parameters of Weibull distribution; see also Park et al. (2012) and Hwang and Park (2013). Yari et al. (2013) found a simple form of (4) as

$$g(\theta) = E_{\theta}(X) - \frac{1}{n} \sum_{i=1}^{n} h(x_i) = E_{\theta}(X) - \overline{h(x)},$$

where $\overline{h(x)} = \frac{1}{n} \sum_{i=1}^{n} h(x_i)$ for any function $h$ on $x$, and

$$h(x) = \int_{0}^{x} \log \overline{F}(y; \theta) dy.$$  \quad (6)

They also proved that

$$E(h(X)) = \int_{0}^{\infty} \overline{F}(x; \theta) \log \overline{F}(x; \theta) dx,$$

which shows that if $n$ tends to infinity, then $CKL(\overline{F}_n || \overline{F})$ converges to zero.

The aim of the present paper is to investigate properties of $MCKLE$. The rest of the paper is organized as follows: In section 2, we propose an extension of the $MCKLE$ in the case when the support of the distribution is real line and provide some examples. In Section 3, we show that the proposed estimator is in the class of generalized estimating equations ($GEE$). Asymptotic properties of $MCKLE$ such as consistency, normality are investigated in this section. In Section 3, we also provide some asymptotic confidence intervals and asymptotic tests statistics based on $MCKLE$ to make some inference on the parameters of the distribution.

## 2 An Extension of $MCKLE$

In this section, we propose an extension of the $MCKLE$ for the case when $X$ is assumed to be a continuous random variable with support $\mathbb{R}$. It is known that

$$E_{\theta} |X| = \int_{-\infty}^{0} F(x) dx + \int_{0}^{\infty} F(x) dx,$$

(see, Rohatgi and Saleh, 2015).

We define the $CKL$ divergence and $CKL$ estimator in Liu approach as follows.

**Definition 1.** Let $X$ and $Y$ be random variables on $\mathbb{R}$ with c.d.f’s $F(x)$ and $G(x)$, s.f’s $\overline{F}(x)$ and $\overline{G}(x)$, finite means $E(X)$ and $E(Y)$, respectively. The $CKL$ divergence of $\overline{F}$ relative to $\overline{G}$ is defined as

$$CKL(\overline{F} || \overline{G}) = \int_{-\infty}^{0} \left\{ F(x) \log \frac{F(x)}{G(x)} - [F(x) - G(x)] \right\} dx$$

$$+ \int_{0}^{\infty} \left\{ \overline{F}(x) \log \frac{\overline{F}(x)}{\overline{G}(x)} - [\overline{F}(x) - \overline{G}(x)] \right\} dx$$

$$= \int_{-\infty}^{0} F(x) \log \frac{F(x)}{G(x)} dx + \int_{0}^{\infty} \overline{F}(x) \log \frac{\overline{F}(x)}{\overline{G}(x)} dx - [E|X| - E|Y|].$$
An application of the log-sum inequality and the fact that \( x \log \frac{y}{x} \geq x - y, \forall x, y > 0 \) (equality holds if and only if \( x = y \)) show that the CKL is non-negative. Using the fact that in log-sum inequality, equality holds if and only if \( F = G, \ a.s. \), one gets that \( CKL(\bar{F}||\bar{G}) = 0 \) if and only if \( F = G, \ a.s. \).

Let \( F(x; \theta) \) be the population c.d.f. with unknown parameters \( \theta \) and \( F_n(x) \) be the empirical c.d.f. based on a random sample \( X_1, X_2, \ldots, X_n \) from \( F(x; \theta) \). Based on above definition, the CKL divergence of \( \bar{F}_n \) relative to \( \bar{F} \) is defined as

\[
CKL(\bar{F}_n||\bar{F}) = \int_{-\infty}^{0} F_n(x) \log \frac{F_n(x)}{F(x; \theta)} \, dx + \int_{0}^{\infty} \bar{F}_n(x) \log \frac{\bar{F}_n(x)}{F(x; \theta)} \, dx - \lfloor \bar{x} \rfloor - E_{\theta} |X|,
\]

where \( |\bar{x}| \) is the mean of absolute values of the observations. Let us also define

\[
g(\theta) = E_{\theta} |X| - \int_{-\infty}^{0} F_n(x) \log F(x; \theta) \, dx - \int_{0}^{\infty} \bar{F}_n(x) \log \bar{F}(x; \theta) \, dx.
\]

(7)

If \( E_{\theta} |X| < \infty \) and \( g''(\theta) \) is positive definite, then we define MCKLE of \( \theta \) to be a value in the parameter space \( \Theta \) which minimizes \( g(\theta) \). If \( k = 0 \) (i.e., \( X \) is nonnegative), then \( g(\theta) \) in (7) reduces to (4). So the results of Liu (2007), Yari and Saghafi (2012), Yari et al. (2013), Park et al. (2012) and Hwang and Park (2013) yield as special case.

It should be noted that by the law of large numbers \( F_n(x) \) converges to \( F(x) \) and \( \bar{F}_n(x) \) converges to \( \bar{F}(x) \) as \( n \) tends to infinity. Consequently \( CKL(\bar{F}_n||\bar{F}) \) converges to zero. As a consequence, if we take \( \hat{\theta}_n = T(F_n) \), then it is Fisher consistent, i.e., \( T(F) = \theta \) (see, Fisher (1922) and Lindsay (1994)).

In order to study the properties of the estimator, we first find a simple form of (7). Let us introduce the following notations.

\[
u(x) = \int_{x}^{0} \log F(y; \theta) \, dy,
\]

and

\[s(x) = I_{(-\infty,0)}(x) \, u(x) + I_{[0,\infty)}(x) \, h(x),\]

(8)

where \( h \) is defined in (6). Assuming that \( x(1), x(2), \ldots, x(n) \) denote the ordered observed values of the sample and that \( x(k) < 0 \leq x(k+1) \), for some value of \( k, k = 0, \ldots, n \). Then by (2), (3)
and (7), we have

\[
\int_{-\infty}^{0} F_n(x) \log F(x; \theta) \, dx = \int_{-\infty}^{0} \sum_{i=1}^{n} \frac{i}{n} I_{[x(i), x(i+1))}(x) \log F(x; \theta) \, dx
\]

\[
= \sum_{i=1}^{n} \frac{i}{n} \int_{-\infty}^{\infty} I_{(-\infty, 0)}(x) I_{[x(i), x(i+1))}(x) \log F(x; \theta) \, dx
\]

\[
= \sum_{i=1}^{k-1} \frac{i}{n} \int_{x(i)}^{x(i+1)} \log F(x; \theta) \, dx + \frac{k}{n} \int_{x(k)}^{0} \log F(x; \theta) \, dx
\]

\[
= \frac{1}{n} \sum_{i=1}^{k-1} i [u(x(i)) - u(x(i+1))] + \frac{k}{n} u(x(k))
\]

\[
= \frac{1}{n} \sum_{i=1}^{k-1} [i u(x(i)) - (i + 1) u(x(i+1))] + \frac{1}{n} \sum_{i=1}^{k-1} u(x(i+1)) + \frac{k}{n} u(x(k))
\]

\[
= \frac{1}{n} \sum_{i=1}^{k} u(x(i)).
\]

Using the same steps, we have

\[
\int_{0}^{\infty} \tilde{F}_n(x) \log \tilde{F}(x; \theta) \, dx = \frac{1}{n} \sum_{i=k+1}^{n} h(x(i)).
\]

So \( g(\theta) \) in (7) gets the simple form

\[
g(\theta) = E_{\theta} |X| - \frac{1}{n} \sum_{i=1}^{k} u(x(i)) - \frac{1}{n} \sum_{i=k+1}^{n} h(x(i))
\]

\[
= E_{\theta} |X| - \frac{1}{n} \sum_{i=1}^{n} [I_{(-\infty, 0)}(x_i) u(x_i) + I_{[0, \infty)}(x_i) h(x_i)]
\]

\[
= E_{\theta} |X| - \frac{1}{n} \sum_{i=1}^{n} s(x_i) = E_{\theta} |X| - s(x).
\]  

(9)

If \( k = 0 \) (i.e., \( X \) is nonnegative), then \( g(\theta) \) in (9) reduces to (5). It can be easily seen that

\[
E(s(X)) = \int_{-\infty}^{0} F(x; \theta) \log F(x; \theta) \, dx + \int_{0}^{\infty} \tilde{F}(x; \theta) \log \tilde{F}(x; \theta) \, dx,
\]

which proves that if \( n \) tends to infinity, then \( CKL(\tilde{F}_n||\tilde{F}) \) converges to zero.

In The following, we give some examples.

**Example 2.** Let \( \{X_1, \ldots, X_n\} \) be sequence of i.i.d. Normal random variables with probability density function

\[
\phi(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left( -\frac{1}{2} \left( \frac{x - \mu}{\sigma} \right)^2 \right), \quad x \in \mathbb{R}.
\]
In this case $E(|X|) = \mu \left[2\Phi \left(\frac{\mu}{\sigma}\right) - 1\right] + 2\sigma \phi \left(\frac{\mu}{\sigma}\right)$, where $\Phi$ denotes the distribution function of standard normal. For this distribution, $h(x)$, $u(x)$ and $g(\mu, \sigma)$ do not have close forms. The derivative of $g(\mu, \sigma)$ with respect to $\mu$ and $\sigma$ and setting the results to zero gives respectively

$$
2n\Phi \left(\frac{\mu}{\sigma}\right) - n - \sum_{i=1}^{k} \log \Phi \left(\frac{x_i - \mu}{\sigma}\right) + k \log \Phi \left(-\frac{\mu}{\sigma}\right) + \sum_{i=k+1}^{n} \log \Phi \left(\frac{\mu - x_i}{\sigma}\right) - (n - k) \log \Phi \left(\frac{\mu}{\sigma}\right) = 0,
$$

and

$$
2n\phi \left(\frac{\mu}{\sigma}\right) + \sum_{i=1}^{k} \int_{x_i < 0} \frac{z\phi(z)}{\Phi(z)} dz - \sum_{i=k+1}^{n} \int_{x_i \geq 0} \frac{z\phi(z)}{1 - \Phi(z)} dz = 0. \quad (10)
$$

To obtain our estimators, we need to solve these equations which should be solved numerically. For computational purposes, the following equivalent equation can be solved instead of (10).

$$
2\phi \left(\frac{\mu}{\sigma}\right) + \int_{x(1)-\mu}^{\frac{-\mu}{\sigma}} F_{n}(\mu + \sigma z) \frac{z\phi(z)}{\Phi(z)} dz - \int_{-\frac{\mu}{\sigma}}^{x(0)-\mu} \bar{F}_{n}(\mu + \sigma z) \frac{z\phi(z)}{1 - \Phi(z)} dz = 0.
$$

Figure 1 represents $g(\mu, \sigma)$ for a simulated sample of size 100 from Normal distribution with parameters $(\mu = 2, \sigma = 3)$. This figure shows that in this case $g(\mu, \sigma)$ has minimum and hence the estimators of $\mu$ and $\sigma$ are the values that minimize $g(\mu, \sigma)$. 

Figure 1: $g(\mu, \sigma)$ for a simulated sample of size 100 from Normal distribution with parameters $(\mu = 2, \sigma = 3)$.
Figure 2 compares these estimators with the corresponding MLE’s. In order to compare our estimators and the MLE’s we made a simulation study in which we used samples of sizes 10 to 55 by 5 with 10000 repeats, where we assume that the true values of the model parameters are \( \mu_{\text{true}} = 2 \) and \( \sigma_{\text{true}} = 3 \). It is evident from the plots that the MCKLE approximately coincides with the MLE in both cases.

**Example 3.** Let \( \{X_1, \ldots, X_n\} \) be sequence of i.i.d. Laplace random variables with probability density function

\[
f(x; \theta) = \frac{1}{\theta} \exp \left( \frac{x}{\theta} \right), \quad x \in \mathbb{R}.
\]

We simply have MCKLE of \( \theta \) as

\[
\hat{\theta} = \sqrt{\frac{X^2}{2}}.
\]

For asymptotic properties of this estimator see Section 3.

### 3 Asymptotic properties of estimators

In this section we study asymptotic properties of MCKLE’s. For this purpose, first we give a brief review on GEE. Some related references on GEE are Huber (1964), Serfling (1980, chapter 7), Qin and Lawless (1994), van der Vaart (2000, chapter 5), Pawitan (2001, chapter 14), Shao (2003, chapter 5), Huber and Ronchetti (2009, chapter 3) and Hampel et al. (2011).

Throughout this section, we use the terminology used by Shao (2003). We assume that \( X_1, \ldots, X_n \) represents independent (not necessarily identically distributed) random vectors, in
which the dimension of $X_i$ is $d_i$, $i = 1, \ldots, n$ ($\sup d_i < \infty$). We also assume that in the population model the vector $\theta$ is a $k$-vector of unknown parameters. The GEE method is a general method in statistical inference for deriving point estimators. Let $\Theta \subset \mathbb{R}^k$ be the range of $\theta$, $\psi_i$ be a Borel function from $\mathbb{R}^{d_i} \times \Theta$ to $\mathbb{R}^k$, $i = 1, \ldots, n$, and

$$s_n(\gamma) = \sum_{i=1}^n \psi_i (X_i, \gamma), \quad \gamma \in \Theta.$$  

If $\hat{\theta} \in \Theta$ is an estimator of $\theta$ which satisfies $s_n(\hat{\theta}) = 0$, then $\hat{\theta}$ is called a GEE estimator. The equation $s_n(\gamma) = 0$ is called a GEE. Most of the estimation methods such as likelihood estimators, moment estimators and M-estimators are special cases of GEE estimators. Usually GEE’s are chosen such that

$$E [s_n (\theta)] = \sum_{i=1}^n E [\psi_i (X_i, \theta)] = 0. \quad (11)$$

If the exact expectation does not exist, then the expectation $E$ may be replaced by an asymptotic expectation. The consistency and asymptotic normality of the GEE are studied by the authors under different conditions (see, for example Shao, 2003).  

### 3.1 Consistency and asymptotic normality of the MCKLE

Let $\hat{\theta}_n$ be MCKLE by minimizing $g(\theta)$ in (9) with $s(x)$ as defined in (8). Here, we show that the MCKLE’s are special cases of GEE. Using this, we show consistency and asymptotic normality of MCKLE’s.

#### Theorem 4. MCKLE’s, by minimizing $g(\theta)$ in (9), are special cases of GEE estimators.

**Proof:** In order to minimize $g(\theta)$ in (9), we get the derivative of $g(\theta)$, under the assumption that it exists,

$$\frac{\partial}{\partial \theta} g(\theta) = \frac{\partial}{\partial \theta} E_\theta |X| - \frac{1}{n} \sum_{i=1}^n \frac{\partial}{\partial \theta} s(x_i) = 0,$$

which is equivalent to GEE $s_n(\theta) = 0$ where

$$s_n(\theta) = \sum_{i=1}^n \left[ \frac{\partial}{\partial \theta} E_\theta |X| - \frac{\partial}{\partial \theta} s(x_i) \right] = \sum_{i=1}^n \psi(x_i, \theta), \quad (12)$$

with

$$\psi(x, \theta) = \frac{\partial}{\partial \theta} E_\theta |X| - \frac{\partial}{\partial \theta} s(x). \quad (13)$$

We must prove that $E [s_n (\theta)] = 0$ or equivalently $E [\psi (X, \theta)] = 0$. We have

$$E [\psi (X, \theta)] = \frac{\partial}{\partial \theta} E_\theta |X| - E \left[ \frac{\partial}{\partial \theta} s (X) \right].$$

So, it is enough to show that

$$E \left[ \frac{\partial}{\partial \theta} s (X) \right] = \frac{\partial}{\partial \theta} E_\theta |X|. \quad (14)$$
By simple algebra we have

\[
E \left[ \frac{\partial}{\partial \theta} s(X) \right] = \int_{-\infty}^{0} \frac{\partial}{\partial \theta} u(x; \theta) f(x; \theta) \, dx + \int_{0}^{\infty} \frac{\partial}{\partial \theta} h(x) f(x; \theta) \, dx
\]

\[
= \frac{\partial}{\partial \theta} \left\{ \int_{-\infty}^{0} F(y; \theta) \, dy + \int_{0}^{\infty} F(y; \theta) \, dy \right\}
\]

\[
= \frac{\partial}{\partial \theta} E \theta |X| ,
\]

which proves the result.  

**Corollary 5.** In special case that support of \(X\) is \(\mathbb{R}^+\), MCKLE is an special case of GEE estimators, where

\[
s_n(\theta) = \sum_{i=1}^{n} \left[ \frac{\partial}{\partial \theta} E \theta (X) - \frac{\partial}{\partial \theta} h(x_i) \right] = \sum_{i=1}^{n} \psi(x_i, \theta) ,
\]

with

\[
\psi(x, \theta) = \frac{\partial}{\partial \theta} E \theta (X) - \frac{\partial}{\partial \theta} h(x) .
\]

We now study other conditions under which MCKLE’s are consistent. For each \(n\), let \(\hat{\theta}_n\) be an MCKLE or equivalently a GEE estimator, i.e., \(s_n(\hat{\theta}_n) = 0\), where \(s_n(\theta)\) is defined as (12) or (15). In the next Theorem, we study the regular consistency of \(\hat{\theta}_n\).

**Theorem 6.** For each \(n\), let \(\hat{\theta}_n\) be an MCKLE or equivalently a GEE estimator. Suppose that \(\psi\) which is defined in (13) or (16) is a bounded and continuous function of \(\theta\). Let

\[
\Psi(\theta) = E [\psi(X, \theta)] ,
\]

where we assume that \(\Psi'(\theta)\) exists and is full rank. Then \(\hat{\theta}_n \overset{p}{\to} \theta\).

**Proof:** The result follows from Proposition 5.2 of Shao (2003) using the fact that (11) holds.

Asymptotic normality of a consistent sequence of MCKLE’s can be established under some conditions. We first consider the special case where \(\theta\) is scalar and \(X_1, ..., X_n\) are i.i.d.

**Theorem 7.** For each \(n\), let \(\hat{\theta}_n\) be an MCKLE or equivalently a GEE estimator. Then

\[
\sqrt{n} \left( \hat{\theta}_n - \theta \right) \overset{d}{\to} N (0, \sigma_F^2) ,
\]

where \(\sigma_F^2 = A/B^2\), with

\[
A = E \left[ \frac{\partial}{\partial \theta} s(X) \right]^2 - \left[ \frac{\partial}{\partial \theta} E \theta |X| \right]^2 ,
\]

and

\[
B = \int_{-\infty}^{0} \left[ \frac{\partial}{\partial \theta} F(x; \theta) \right]^2 \, dx + \int_{0}^{\infty} \left[ \frac{\partial}{\partial \theta} \bar{F}(x; \theta) \right]^2 \, dx .
\]
Proof: Using Theorem 4 we have $E[\psi(X, \theta)] = 0$. So if consider $\psi$ defined in (13)

$$E[\psi(X, \theta)]^2 = \text{Var}[\psi(X, \theta)]$$

$$= \text{Var}\left[ \frac{\partial}{\partial \theta} F_{\theta} | X \right] - \frac{\partial}{\partial \theta} s(X)$$

$$= \text{Var}\left[ \frac{\partial}{\partial \theta} s(X) \right]$$

$$= E\left[ \frac{\partial}{\partial \theta} s(X) \right]^2 - E^2 \left[ \frac{\partial}{\partial \theta} s(X) \right]$$

$$= E\left[ \frac{\partial}{\partial \theta} s(X) \right]^2 - \left[ \frac{\partial}{\partial \theta} E_{\theta} | X \right]^2,$$

where the last equality follows from (14). On the other hand

$$\Psi'(\theta) = \frac{\partial^2}{\partial \theta^2} E_{\theta} | X \right] - E\left[ \frac{\partial^2}{\partial \theta^2} s(X) \right],$$

and

$$E\left[ \frac{\partial^2}{\partial \theta^2} s(X) \right] = \int_{-\infty}^{0} \int_{x}^{0} \frac{\partial^2}{\partial \theta^2} \log F(y; \theta) dyf(x; \theta) dx + \int_{0}^{\infty} \int_{x}^{\infty} \frac{\partial^2}{\partial \theta^2} \log \bar{F}(y; \theta) dyf(x; \theta) dx$$

$$= \int_{-\infty}^{0} \left\{ \frac{\partial^2}{\partial \theta^2} F(y; \theta) \frac{\partial \bar{F}(y; \theta)}{\partial \theta} - \left[ \frac{\partial}{\partial \theta} \bar{F}(y; \theta) \right]^2 \right\} F(y; \theta) dy$$

$$+ \int_{0}^{\infty} \left\{ \frac{\partial^2}{\partial \theta^2} \bar{F}(y; \theta) \frac{\partial \bar{F}(y; \theta)}{\partial \theta} - \left[ \frac{\partial}{\partial \theta} \bar{F}(y; \theta) \right]^2 \right\} \bar{F}(y; \theta) dy$$

$$= \frac{\partial^2}{\partial \theta^2} \int_{-\infty}^{0} F(x; \theta) dx - \int_{-\infty}^{0} \left[ \frac{\partial}{\partial \theta} F(x; \theta) \right]^2 dx$$

$$+ \frac{\partial^2}{\partial \theta^2} \int_{0}^{\infty} \bar{F}(x; \theta) dx - \int_{0}^{\infty} \left[ \frac{\partial}{\partial \theta} \bar{F}(x; \theta) \right]^2 dx$$

$$= \frac{\partial^2}{\partial \theta^2} E_{\theta} | X \right] - \int_{-\infty}^{0} \left[ \frac{\partial}{\partial \theta} F(x; \theta) \right]^2 dx - \int_{0}^{\infty} \left[ \frac{\partial}{\partial \theta} \bar{F}(x; \theta) \right]^2 dx.$$

So

$$\Psi'(\theta) = \int_{-\infty}^{0} \left[ \frac{\partial}{\partial \theta} F(x; \theta) \right]^2 dx + \int_{0}^{\infty} \left[ \frac{\partial}{\partial \theta} \bar{F}(x; \theta) \right]^2 dx.$$

Now, using Theorem 5.13 of Shao (2003), $\sigma_F^2$ will be found. \hfill \blacksquare

The next Theorem shows asymptotic normality of MCKLE’s, when $\theta$ is vector and $X_1, \ldots, X_n$ are i.i.d.

**Theorem 8.** Under the conditions of Theorem 5.14 of Shao (2003),

$$V_n^{-1/2} \left( \hat{\theta}_n - \theta \right) \xrightarrow{d} N_k(0, I_k),$$

where $\hat{\theta}_n$ is the MCKLE estimator of $\theta$. \hfill \blacksquare
where \( V_n = \frac{1}{n} B^{-1} A B^{-1} \) with

\[
A = \left[ \frac{\partial}{\partial \theta} s(X) \right] \left[ \frac{\partial}{\partial \theta} s(X) \right]^T - \left[ \frac{\partial}{\partial \theta} E_{\theta} |X| \right] \left[ \frac{\partial}{\partial \theta} E_{\theta} |X| \right]^T,
\]

and

\[
B = \int_{-\infty}^{0} \frac{\partial}{\partial \theta} F(x; \theta) \left[ \frac{\partial}{\partial \theta} F(x; \theta) \right]^T dx + \int_{0}^{\infty} \frac{\partial}{\partial \theta} \bar{F}(x; \theta) \left[ \frac{\partial}{\partial \theta} \bar{F}(x; \theta) \right]^T dx,
\]

provided that \( B \) is invertible matrix.

\textbf{Proof:} The proof is similar to that of Theorem 7. \( \blacksquare \)

\textbf{Remark 9.} In Theorems 7 and 8, for special case that support of \( X \) is \( \mathbb{R}^+ \), \( A \) and \( B \) are given, respectively, by

\[
A = E \left[ \frac{\partial}{\partial \theta} h(X) \right]^2 - \left[ \frac{\partial}{\partial \theta} E_{\theta} (X) \right]^2,
\]

\[
B = \int_{0}^{\infty} \frac{\partial}{\partial \theta} \bar{F}(x; \theta) \left[ \frac{\partial}{\partial \theta} \bar{F}(x; \theta) \right]^T dx,
\]

and

\[
A = E \left[ \frac{\partial}{\partial \theta} h(X) \right] \left[ \frac{\partial}{\partial \theta} h(X) \right]^T - \left[ \frac{\partial}{\partial \theta} E_{\theta} (X) \right] \left[ \frac{\partial}{\partial \theta} E_{\theta} (X) \right]^T,
\]

\[
B = \int_{0}^{\infty} \frac{\partial}{\partial \theta} \bar{F}(x; \theta) \left[ \frac{\partial}{\partial \theta} \bar{F}(x; \theta) \right]^T dx.
\]

Now, following Pawitan (2001), we can find sample version of the variance formula for the \textit{MCKLE} as follows. Given \( x_1, ..., x_n \) let

\[
J = \hat{E} \left[ \psi(X, \theta) \right]^2 = \frac{1}{n} \sum_{i=1}^{n} \psi^2 \left( x_i, \hat{\theta} \right), \quad (17)
\]

where in the vector case we would simply use \( \psi \left( x_i, \hat{\theta} \right) \psi^T \left( x_i, \hat{\theta} \right) \) in the summation, and

\[
I = -\hat{E} \frac{\partial}{\partial \theta} \psi(X, \theta) = -\frac{1}{n} \sum_{i=1}^{n} \frac{\partial}{\partial \theta} \psi \left( x_i, \hat{\theta} \right). \quad (18)
\]

Then, we have the following result.

\textbf{Theorem 10.} Using notations defined in (17) and (18), and an application of Slutsky’s Theorem yields

\[
\hat{V}_n^{-1/2} \left( \hat{\theta}_n - \theta \right) \overset{d}{\rightarrow} N_k (0, I_k),
\]

where

\[
\hat{V}_n = \frac{1}{n} I^{-1} J I^{-1}, \quad (19)
\]

provided that \( I \) is invertible matrix, or equivalently \( g(\theta) \) has infimum value on parameter space \( \Theta \).
In Theorems 7 and 8, the estimator $\hat{V}_n$ is a sample version of $V_n$, see also Basu and Lindsay (1994). It is also known that the sample variance (19) is a robust estimation which is known as the 'sandwich' estimator, with $I^{-1}$ as the bread and $J$ the filling (see, Huber, 1967). In likelihood approach, the quantity $I$ is the usual observed Fisher information.

**Example 11.** Let $\{X_1, \ldots, X_n\}$ be sequence of i.i.d. exponential random variables with probability density function

$$f(x; \lambda) = \lambda e^{-\lambda x}, \quad x > 0, \quad \lambda > 0.$$  

We simply have MCKLE of $\lambda$ as

$$\hat{\lambda} = \sqrt{\frac{2}{X^2}}.$$  

This estimator is function of linear combinations of $x_i^2$’s, and so by strong law of large numbers (SLLN), $\hat{\lambda}$ is strongly consistent for $\lambda$, as well as the MME of $\lambda$.

Now, by CLT and delta method or using Theorem 7, one can show that

$$\sqrt{n} \left( \hat{\lambda} - \lambda \right) \xrightarrow{d} N \left( 0, \frac{5\lambda^2}{4} \right),$$

and $n^{-1}$ order asymptotic bias of $\hat{\lambda}$ is $15\lambda/8n$. It is well known that the MLE of $\lambda$ is $\hat{\lambda}_m = 1/\bar{X}$ with asymptotic distribution

$$\sqrt{n} \left( \hat{\lambda}_m - \lambda \right) \xrightarrow{d} N \left( 0, \lambda^2 \right),$$

and $n^{-1}$ order asymptotic bias of $\hat{\lambda}_m$ is $\lambda/n$.

Notice that using asymptotic bias of $\hat{\lambda}$, we can find some unbiasing factors to improve our estimator. Since the MLE has inverse Gamma distribution, the unbiased estimator of $\lambda$ is $\hat{\lambda}_{um} = (n - 1)/n\bar{X}$ (see, Forbes et al., 2011). In Liu approach an approximately unbiased estimator of $\lambda$ is

$$\hat{\lambda}_u = \frac{8n}{8n + 15} \sqrt{\frac{2}{\bar{X}^2}}. \tag{20}$$

Figure 3 compares these estimators. In order to compare our estimator and the MLE, we made a simulation study in which we used samples of sizes 10 to 55 by 5 with 10000 repeats, where we assumed that the true value of the model parameter is $\lambda_{true} = 5$. The plots in Figure 3 show that the MCKLE has more biased than the MLE, but MCKLE in (20) which is approximately unbiased coincides with the unbiased MLE.

**Remark 12.** In Example 3, note that $|X|$ has exponential distribution. So, using Example 11, one can easily find asymptotic properties of $\theta$ in Laplace distribution.

**Example 13.** Let $\{X_1, \ldots, X_n\}$ be sequence of i.i.d. two parameter exponential random variables with probability density function

$$f(x; \mu, \sigma) = \frac{1}{\sigma} e^{-(x-\mu)/\sigma}, \quad x \geq \mu, \quad \mu \in \mathbb{R}, \quad \sigma > 0.$$  

It is not difficult to show that MCKLE of $\mu$ and $\sigma$ are, respectively,

$$\hat{\mu} = \bar{X} - \sqrt{\bar{X}^2 - \bar{X}^2}, \quad \hat{\sigma} = \sqrt{\bar{X}^2 - \bar{X}^2}.$$
These estimators are functions of linear combinations of $x_i$’s and $x_i^2$’s, and hence by SLLN, $(\hat{\mu}, \hat{\sigma})$ are strongly consistent for $(\mu, \sigma)$, as well as the MME of $(\mu, \sigma)$.

Now, by CLT and delta method or using Theorem 7, one can show that

$$V_n^{-1/2} \left( \frac{\hat{\mu} - \mu}{\hat{\sigma} - \sigma} \right) \xrightarrow{d} N_2 (0, I_2),$$

where

$$V_n = \frac{\sigma^2}{n} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}.$$

Figure 4 represents $g(\mu, \sigma)$ for a simulated sample of size 100 from two parameter exponential distribution with parameters $(\mu = 3, \sigma = 2)$. The figure shows that the estimators of $\mu$ and $\sigma$ are the values that minimize $g(\mu, \sigma)$.

**Example 14.** Let $\{X_1, \ldots, X_n\}$ be sequence of i.i.d. Pareto random variables with probability density function

$$f(x; \alpha, \beta) = \frac{\alpha \beta^\alpha}{x^{\alpha+1}}, \quad x \geq \beta, \quad \alpha > 0, \quad \beta > 0.$$ 

So we simply have

$$g(\alpha, \beta) = \frac{\alpha \beta}{\alpha - 1} + \alpha x \log x - \alpha \bar{x} (\log \beta + 1) + \alpha \beta, \quad \alpha > 1.$$ 

Differentiating $g(\alpha, \beta)$ with respect to $\beta$ and setting zero gives

$$\widehat{\beta}_\alpha = \frac{\bar{x} (\alpha - 1)}{\alpha}.$$
Figure 4: $g(\mu, \sigma)$ for a simulated sample of size 100 from two parameter exponential distribution with parameters $(\mu = 3, \sigma = 2)$
Figure 5: $g(\alpha, \beta)$ for a simulated sample of size 100 from Pareto distribution with parameters $(\alpha = 2, \beta = 5)$

So, if we define the function $g$ of $\alpha$ as follows

$$g(\alpha) = g(\alpha, \hat{\beta}_\alpha) = \alpha \bar{x} \log \frac{\alpha}{\alpha - 1} + \alpha x \log x - \alpha \bar{x} \log \bar{x}, \quad \alpha > 1,$$

then, derivative of $g(\alpha)$ with respect to $\alpha$ and setting zero gives

$$\log \frac{\alpha}{\alpha - 1} - \frac{1}{\alpha - 1} + \frac{x \log x}{\bar{x}} - \log \bar{x} = 0.$$

This equation can be solved numerically to find MCKLE of parameters. Now, using Theorem 8, one can show that

$$V_n^{-1/2} \begin{pmatrix} \hat{\alpha} - \alpha \\ \hat{\beta} - \beta \end{pmatrix} \overset{d}{\to} N_2(0, I_2),$$

where

$$V_n = \frac{1}{n(\alpha - 2)^2} \begin{bmatrix} 2\alpha (\alpha - 1)^4 & \alpha \beta (\alpha - 1)^2 \\ \alpha \beta (\alpha - 1)^2 & \beta^2 \alpha (\alpha^2 - 2\alpha + 2) \end{bmatrix}, \quad \alpha > 2.$$

Figure 5 represents $g(\alpha, \beta)$ for a simulated sample of size 100 from Pareto distribution with parameters $(\alpha = 2, \beta = 5)$. This figure shows that the estimators of $\alpha$ and $\beta$ are the values that minimize $g(\alpha, \beta)$.

### 3.2 Asymptotic confidence interval

In the following we assume that $\mathbf{\theta}$ is a scalar. Using Theorem 7, we can find an asymptotic confidence interval for $\mathbf{\theta}$. Under the conditions of Theorem 7, an asymptotic $100(1 - \alpha)\%$
confidence interval for $\theta$ is defined as
\[
P \left( -z_{\alpha/2} < \frac{\sqrt{n} (\hat{\theta} - \theta)}{\sigma_F} < z_{\alpha/2} \right) = 1 - \alpha, \tag{21}\]

where $z_{\alpha}$ is the $(1 - \alpha)$-quantile of the $N(0, 1)$ and $\sigma_F$ is defined in Theorem 7. If inequalities in (21) are not invertible, then we can use $\hat{\sigma}_F$ instead of $\sigma_F$ to obtain an approximate confidence interval, where $\hat{\sigma}_F$ is $\sigma_F$ that evaluated at $\theta = \hat{\theta}$.

Pawitan (2001) presented an approach which is called likelihood interval for parameters. Using his approach, one can find a divergence interval for the parameter. Similar to the likelihood interval that is defined by Pawitan (2001), we define a divergence interval as a set of parameter values with low enough divergence:
\[
\left\{ \theta \text{ s.t. } \exp \left[ g(\hat{\theta}) - g(\theta) \right] > k \right\}, \tag{22}\]

for some cutoff point $k$, where $\exp \left[ g(\hat{\theta}) - g(\theta) \right]$ is the normalized divergence with $g(\theta)$ as (5) or (9); see Basu et al. (2011, chapter 5). Let us define the quantity $Q$ as
\[
Q(\hat{\theta}, \theta) = \frac{2n \left[ g(\theta) - g(\hat{\theta}) \right]}{\sigma_F^2 \cdot g''(\hat{\theta})},
\]
where
\[
g''(\theta) = \frac{\partial^2}{\partial \theta^2} g(\theta).
\]

Using Theorem 7, we show that this quantity is asymptotically a pivotal quantity. In other words, under the conditions of Theorem 7,
\[
Q(\hat{\theta}, \theta) \xrightarrow{d} \chi^2_1. \tag{23}\]

This is so, because using Taylor expansion of $g(\theta)$ around $\hat{\theta}$, we have
\[
Q(\hat{\theta}, \theta) \approx \frac{n (\theta - \hat{\theta})^2}{\sigma_F^2}. \tag{24}\]

Now using this fact, we can find the divergence interval for $\theta$.

**Theorem 15.** Under the conditions of Theorem 7, the asymptotic $100(1 - \alpha)$% divergence interval for $\theta$ is defined as (22), with
\[
k = \exp \left\{ -\frac{1}{2n} c(\hat{\theta}) \chi_{\alpha, 1}^2 \right\},
\]
where $\chi_{\alpha, 1}^2$ is the $(1 - \alpha)$-quantile of the $\chi_1^2$ and
\[
c(\theta) = \sigma_F^2 \cdot g''(\theta).
\]
**Proof:** Using (23), the probability that divergence interval (22) covers $\theta$ is

$$P\left(\exp\left[g\left(\hat{\theta}\right) - g\left(\theta\right)\right] > k\right) = P \left( Q\left(\hat{\theta}, \theta\right) < -\frac{2n \log k}{\sigma_F^2 \cdot g''(\hat{\theta})} \right)$$

$$= P \left( \chi^2_1 < -\frac{2n \log k}{\sigma_F^2 \cdot g''(\hat{\theta})} \right).$$

So, for some $0 < \alpha < 1$ we choose a cutoff

$$k = \exp\left\{-\frac{1}{2n}\sigma_F^2 \cdot g''(\hat{\theta}) \chi^2_{\alpha,1}\right\}.$$

Since $\sigma_F^2$ is unknown, we estimate it with $\hat{\sigma}_F^2$. This completes the proof. ■

**Remark 16.** Form (24), The asymptotic confidence interval in (21) with $\hat{\sigma}_F$ instead of $\sigma_F$, is approximately equivalent with that in (22). Also, in (21) and (22), we can practically use sample version of $\sigma_F^2$ that is defined in Theorem 10.

**Example 17.** In Example 11, the asymptotic 100 (1 - $\alpha$)% divergence interval for $\lambda$ is in form (22) with

$$k = \exp\left\{-\frac{5}{4n} \chi^2 \frac{X^2}{2} \chi^2_{\alpha,1}\right\}.$$

In other words, the confidence interval is in form

$$(L, U) = \frac{b \pm \sqrt{b^2 - 2X^2}}{X},$$

with $b = - \log k + \sqrt{2X^2}$. For a simulated sample of size $n = 30$ from exponential distribution with parameter $\lambda = 3$, Figure 6 shows normalized divergence and asymptotic 95% confidence interval for $\lambda$. In this typical sample $x^2 = 0.2063127$, $\hat{\lambda} = 3.113522$, $\lambda_u = 2.930374$, $k = 0.9498908$ and $(L, U) = (2.092375, 4.633022)$.

**Remark 18.** When $\dim(\theta) > 1$, we can’t easily find a pivotal quantity. In these cases, using quantiles of $g^*$ from repeated samples, we can find cutoffs of divergence-based confidence regions.

**Example 19.** In Example 13, Using 10000 replicated simulated samples of size 100 from two-parameter exponential distribution with parameters ($\mu = 3$, $\sigma = 2$), we can find asymptotic cutoffs of divergence-based confidence regions for ($\mu, \sigma$). Figure 7 shows asymptotic 90%, 70%, ..., 10% confidence regions for ($\mu, \sigma$).

### 3.3 Asymptotic hypothesis testing

Let $\dim(\theta) = 1$ and $\Theta_0$ and $\Theta_1$ be two subsets of $\Theta$ such that

$$\Theta_0 \cup \Theta_1 = \Theta, \ \Theta_0 \cap \Theta_1 = \phi.$$

We are interested in testing hypotheses

$$H_0 : \theta \in \Theta_0 \text{ vs } H_1 : \theta \in \Theta_1.$$  (25)
Figure 6: Normalized divergence and asymptotic 95% confidence interval for $\lambda$, in a simulated sample of size $n = 30$ from exponential distribution with parameter $\lambda = 3$.

Figure 7: Asymptotic 90%, 70%, ..., 10% confidence regions for $(\mu, \sigma)$, using 10000 replicated simulated samples of size 100 from two parameter exponential distribution with parameters $(\mu = 3, \sigma = 2)$
It is clear that by inverting asymptotic confidence interval in (21), we can find a critical region for statistical tests (asymptotically of level $\alpha$)

$$H_0 : \theta = \theta_0 \text{ vs } H_1 : \theta \neq \theta_0,$$

for a given $\theta_0$. Similar to approach of generalized likelihood ratio test, Basu (1993) and Lindsay (1994) presented a divergence difference test (DDT) statistic based on (1) for testing hypotheses in (26) in continuous and discrete cases; also see Basu et al. (2011, chapter 5). Here, we perform an alternative statistical test based on CKL divergence. For testing hypotheses in (25), we define the generalized divergence difference test (GDDT) statistic as

$$GDDT = 2n \left[ \inf_{\theta \in \Theta_0} g(\theta) - \inf_{\theta \in \Theta} g(\theta) \right].$$

We consider behavior of $GDDT$ as a test statistic for a null hypothesis of the form $H_0 : \theta \in \Theta_0$.

**Theorem 20.** Under the conditions of Theorem 7 and the null hypothesis $H_0 : \theta \in \Theta_0$,

$$GDDT \overset{d}{\to} c(\hat{\theta}_0) \chi^2_1.$$

**Proof:** Using Taylor expansion of $g(\hat{\theta}_0)$ around $\hat{\theta}$ we get

$$2n \left[ g(\hat{\theta}_0) - g(\bar{\theta}) \right] \approx 2n \left[ g(\bar{\theta}) + (\hat{\theta}_0 - \bar{\theta}) g'(\bar{\theta}) + \frac{(\hat{\theta}_0 - \bar{\theta})^2}{2} g''(\bar{\theta}) - g(\bar{\theta}) \right]$$

$$= \frac{n}{2} (\hat{\theta}_0 - \bar{\theta})^2 g''(\bar{\theta}).$$

Under $H_0$, the quantity $g''(\bar{\theta})$ convergence to $g''(\hat{\theta}_0)$. Thus

$$GDDT \approx c(\hat{\theta}_0) \frac{n(\hat{\theta}_0 - \bar{\theta})^2}{\sigma^2 F_0},$$

where $\sigma^2 F_0$ is $\sigma^2 F$ that evaluated at $\theta = \hat{\theta}_0$. Now using Theorem 7 the proof is complete.  

**Remark 21.** Under the conditions of Theorem 20, we can obtain the following approximation for the power function in a given $\theta_1 \in \Theta_1$ as

$$\beta(\theta_1) \approx P \left( \chi^2_1 > \frac{2n \left[ g(\theta_1) - g(\hat{\theta}_0) \right] + c(\hat{\theta}_0) \chi^2_{\alpha,1}}{c(\theta_1)} \right).$$

As an important application of the above approximation, one can find the approximate sample size that guarantees a specific power $\beta$ for a given $\theta_1 \in \Theta_1$. Let $n_0$ be the positive root of the equation

$$\beta = P \left( \chi^2_1 > \frac{2n \left[ g(\theta_1) - g(\hat{\theta}_0) \right] + c(\hat{\theta}_0) \chi^2_{\alpha,1}}{c(\theta_1)} \right).$$
i.e.,
\[ n_0 = \frac{c(\theta_1) \chi^2_{\alpha,1} - c(\bar{\theta}_0) \chi^2_{\alpha,1}}{2 \left[ g(\theta_1) - g(\bar{\theta}_0) \right]} \]

The required sample size is then
\[ n^* = \left[ n_0 \right] + 1, \tag{27} \]

where \( \lceil \cdot \rceil \) is used here to denote “integer part of”.

Remark 22. In special case that \( \Theta_0 = \{ \theta_0 \} \), we can find a critical region for statistical tests (asymptotically of level \( \alpha \)) in (26). One can do this by replacing \( \bar{\theta}_0 \) with \( \theta_0 \).

Example 23. In Example 11, the statistical test (asymptotically of level \( \alpha \)) of null hypothesis \( H_0 : \lambda = \lambda_0 \) against the alternative \( H_1 : \lambda \neq \lambda_0 \) is defined with the critical region
\[ \bar{X}^2 > \left( \frac{b + \sqrt{b^2 - 4ac}}{2a} \right)^2 \quad \text{or} \quad \bar{X}^2 < \left( \frac{b - \sqrt{b^2 - 4ac}}{2a} \right)^2, \]

where \( a = n \lambda_0^2 \), \( b = 2\sqrt{2n} \lambda_0 \) and \( c = 2n - \frac{5}{2} \chi^2_{\alpha,1} \).

References
Baratpour, S., & Habibi Rad, A. (2012). Testing goodness-of-fit for exponential distribution based on cumulative residual entropy. *Communications in Statistics-Theory and Methods*, 41(8), 1387-1396.

Basu, A. (1993). *Minimum disparity estimation: applications to robust tests of hypotheses*. Technical Report, Center for Statistical Sciences, University of Texas at Austin.

Basu, A., & Lindsay, B. G. (1994). Minimum disparity estimation for continuous models: efficiency, distributions and robustness. *Annals of the Institute of Statistical Mathematics*, 46(4), 683-705.

Basu, A., Shioya, H., & Park, C. (2011). *Statistical inference: the minimum distance approach*. CRC Press.

Broniatowski, M. (2014). Minimum divergence estimators, maximum likelihood and exponential families. *Statistics & Probability Letters*, 93, 27-33.

Broniatowski, M., & Keziou, A. (2009). Parametric estimation and tests through divergences and the duality technique. *Journal of Multivariate Analysis*, 100(1), 16-36.

Cherfi, M. (2011). Dual \( \phi \)-divergences estimation in normal models. *arXiv preprint arXiv:1108.2999*.

Cherfi, M. (2012). Dual divergences estimation for censored survival data. *Journal of Statistical Planning and Inference*, 142(7), 1746-1756.

Cherfi, M. (2014). On Bayesian estimation via divergences. *Comptes Rendus Mathematique*, 352(9), 749-754.
Fisher, R. A. (1922). On the mathematical foundations of theoretical statistics. *Philosophical Transactions of the Royal Society of London. Series A, Containing Papers of a Mathematical or Physical Character, 222*, 309-368.

Forbes, C., Evans, M., Hastings, N., & Peacock, B. (2011). *Statistical distributions*. John Wiley & Sons.

Hampel, F. R., Ronchetti, E. M., Rousseeuw, P. J., & Stahel, W. A. (2011). *Robust statistics: the approach based on influence functions* (Vol. 114). John Wiley & Sons.

Huber, P. J. (1964). Robust estimation of a location parameter. *The Annals of Mathematical Statistics, 35*(1), 73-101.

Huber, P. J. (1967). The behavior of maximum likelihood estimates under nonstandard conditions. *Proceedings of the fifth Berkeley symposium on mathematical statistics and probability*, 221-233.

Huber, P., & Ronchetti, E. (2009). *Robust Statistics*, Wiley: New York.

Hwang, I., & Park, S. (2013). On scaled cumulative residual Kullback-Leibler information. *Journal of the Korean Data and Information Science Society, 24*(6), 1497-1501.

Jiménez, R., & Shao, Y. (2001). On robustness and efficiency of minimum divergence estimators. *Test, 10*(2), 241-248.

Lindsay, B. G. (1994). Efficiency versus robustness: the case for minimum Hellinger distance and related methods. *The Annals of Statistics, 22*(2), 1081-1114.

Liu, J. (2007). *Information theoretic content and probability*. Ph.D. Thesis, University of Florida.

Morales, D., Pardo, L., & Vajda, I. (1995). Asymptotic divergence of estimates of discrete distributions. *Journal of Statistical Planning and Inference, 48*(3), 347-369.

Park, S., Rao, M., & Shin, D. W. (2012). On cumulative residual Kullback–Leibler information. *Statistics & Probability Letters, 82*(11), 2025-2032.

Pawitan, Y. (2001). *In all likelihood: statistical modelling and inference using likelihood*: Oxford University Press.

Qin, J., & Lawless, J. (1994). Empirical likelihood and general estimating equations. *The Annals of Statistics, 22*(1), 300-325.

Rohatgi, V. K., & Saleh, A. M. E. (2015). *An introduction to probability and statistics (2 ed.)*. John Wiley. New York.

Serfling, R. (1980). *Approximation Theorems of Mathematical Statistics* John Wiley. New York.

Shao, J. (2003). *Mathematical Statistics (2 ed.)*. Springer, New York, USA.

van der Vaart, A. W. (2000). *Asymptotic statistics*. Cambridge university press.

Yari, G., Mirhabibi, A., & Saghafi, A. (2013). Estimation of the Weibull parameters by Kullback-Leibler divergence of Survival functions. *Appl. Math, 7*(1), 187-192.
Yari, G., & Saghafi, A. (2012). Unbiased Weibull Modulus Estimation Using Differential Cumulative Entropy. *Communications in Statistics-Simulation and Computation, 41*(8), 1372-1378.

3, 4