Kinematic generation of Darboux cyclides

Niels Lubbes, Josef Schicho

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Abstract

We state a relation between two families of lines that cover a quadric surface in the Study quadric and two families of circles that cover a Darboux cyclide.

Keywords: Study quadric, Darboux cyclides

1 Introduction

The Study quadric is a projective compactification of the group of Euclidean displacements. If we fix a point in 3-space, then projective varieties inside the Study quadric — considered as sets of displacements — give rise to orbit varieties in 3-space.

Let us consider the relation between a class of varieties in the Study quadric and their orbits in 3-space. A classical example is presented by the class of lines in the Study quadric; the orbit of a line is either a circle, line or point in 3-space (see Lemma 1). The orbit of a conic in the Study quadric is a rational quartic curve with full cyclicity [10]. The case of rational curves of arbitrary degree in the Study quadric is studied in [4]: a general rational curve of degree $d$ in the Study quadric has an orbit of degree $2d$. We also have a uniqueness result: for any rational curve of degree $d$ and cyclicity $2c$, there is a unique rational curve in the Study quadric of degree $d - c$ in the Study quadric defining that orbit [7, Theorem 2].
In this paper we show that the orbit of a doubly ruled quadric surface in the Study quadric is a Darboux cyclide. Darboux cyclides are surfaces that contain at least two and at most six circles through each point. These surfaces have been recently studied in [5, 6, 8, 9]. Theorem 1 is again a uniqueness result: for two families of circles that cover a Darboux cyclide there exists a unique doubly ruled quadric surface in the Study quadric.

2 The orbit map

The dual quaternions are defined as the noncommutative associative algebra

$$\mathbb{DH} := \mathbb{R}[i, j, k, \epsilon]/(i^2 + 1, j^2 + 1, ij + ji, k - ij, \epsilon^2, ci - ic, ej - je).$$

We consider the following coordinates for $h \in \mathbb{DH}$ and $\overline{h} \in \mathbb{DH}$:

$$h = p + q\epsilon = (p_0 + p_1i + p_2j + p_3k) + (q_4 + q_5i + q_6j + q_7k)\epsilon,$$

$$\overline{h} = p + \overline{q}\epsilon = (p_0 - p_1i - p_2j - p_3k) + (q_4 - q_5i - q_6j - q_7k)\epsilon.$$

We denote by $N : \mathbb{DH} \to \mathbb{D}$, $h \mapsto \overline{h}$, the dual quaternion norm. By projectivizing $\mathbb{DH}$ as a real 8-dimensional vector space, we obtain $\mathbb{P}^7$. The Study quadric is defined as

$$S := \{ h \in \mathbb{P}^7 \mid h\overline{h} \in \mathbb{R} \} = \{ p + q\epsilon \in \mathbb{P}^7 \mid p_0q_0 + p_1q_1 + p_2q_2 + p_3q_3 = 0 \}.$$

The Study boundary $B \subset S$ is defined as $B := \{ h \in S \mid h\overline{h} = 0 \}$. If we identify $\mathbb{R}^3$ with $\{ v \in \mathbb{DH} \mid v = v_1i + v_2j + v_3k \}$, then the Study kinematic mapping is a group action

$$\varphi : (S \setminus B) \times \mathbb{R}^3 \to \mathbb{R}^3, \quad (p + q\epsilon, v) \mapsto \frac{\bar{p}vp + \bar{q}vq - q\bar{p}}{\bar{p}p},$$

and $S \setminus B \cong SE(3)$ via this action [3, Section 2.1]. We choose the following coordinates for the 3-dimensional Möbius sphere:

$$S^3 := \{ x \in \mathbb{P}^4 \mid 4x_0x_4 - x_1^2 - x_2^2 - x_3^2 = 0 \}.$$
With this somewhat unusual choice of coordinates the stereographic projection with center \((0 : 0 : 0 : 0 : 1) \in S^3\) is defined as

\[
\tau : S^3 \to \mathbb{P}^3, \quad (x_0 : \ldots : x_4) \mapsto (x_0 : x_1 : x_2 : x_3).
\]

For any point \(u = (u_0 : \ldots : u_4) \in S^3\) such that \(u_0 \neq 0\), the orbit map is defined as

\[
\text{orb}_u : S \setminus F_u \to S^3, \quad p + q \epsilon \mapsto (p\bar{p} : w_1 : w_2 : w_3 : q\bar{q})
\]

where \(p\bar{p} + p\bar{q} - q\bar{p} = w_1i + w_2j + w_3k\) with \(v = \frac{u_1}{u_0}i + \frac{u_2}{u_0}j + \frac{u_3}{u_0}k\) the dehomogenization of \(\tau(u)\) and

\[
F_u := \{ h = p + q \epsilon \in \mathbb{D} \mathbb{H} \mid h\bar{h} = p\bar{p} + p\bar{q} - q\bar{p} = p\bar{p} = q\bar{q} = 0 \} \subset B.
\]

Notice that the orbit map is the composition of the projective closure of \(\varphi(\cdot, v) : S \to \mathbb{R}^3\) with the inverse stereographic projection. We fix notation for the identity \(e := (1 : 0 : \ldots : 0) \in S\) and the origin \(o := (1 : 0 : \ldots : 0) \in S^3\).

**Proposition 1.** The Zariski closure of the image of \(\text{orb}_u\) is \(S^3\) and \(F_u \subset S\) is a quartic 4-fold.

**Proof.** An Euclidean isometry of \(\mathbb{R}^3\) is realized by an automorphism of \(S\) that preserves \(B\). Thus there exists \(e \in S \setminus B\) such that \(\text{orb}_\epsilon(e) = u\). Since the automorphisms of \(S \setminus B\) and \(\mathbb{R}^3\) are transitive, we may assume without loss of generality that \(e = \epsilon\) and \(u = o\).

We notice that \(N(p\bar{q} - q\bar{p}) = w_1^2 + w_2^2 + w_3^2\). We claim that the image of \(\text{orb}_o\) is contained in \(S^3\) and thus \(4p\bar{p}q\bar{q} = N(p\bar{q} - q\bar{p})\) for \(h = p + \epsilon q \in S \setminus F_o\). Since \(h\bar{h} \in \mathbb{R}\) it follows that \(p\bar{q} + q\bar{p} = 0\) and thus \(p\bar{q} - q\bar{p} = 2p\bar{q}\). The norm is a homomorphism and thus \(N(p\bar{q} - q\bar{p}) = N(2p\bar{q}) = N(2)N(p)N(q) = 4p\bar{p}q\bar{q}\).

The locus where \(\text{orb}_o\) is not defined equals

\[
F_o := \{ p + q \epsilon \in \mathbb{D} \mathbb{H} \mid p\bar{q} = q\bar{p} = p\bar{p} = q\bar{q} = 0 \} \subset S.
\]
We computed with a computer algebra system the Hilbert function of the ideal of $F_o$ and find that $\dim F_o = \deg F_o = 4$. The image of $\text{orb}_o$ is Zariski dense in $S^3$, since $S \setminus B$ is isomorphic to $SE(3)$ via the Study kinematic mapping. This concludes the proof of this lemma.

A circle is an irreducible conic in $S^3$. Let

$$\mathcal{L}_e := \{ \ell \subset S \mid \ell \text{ is a line such that } e \in \ell \}$$

and

$$\mathcal{C}_o := \{ C \subset S^3 \mid C \text{ is either a circle or a point such that } o \in C \}.$$ 

**Lemma 1.** If $\ell \subset S$ is a line, then $\text{orb}_a(\ell)$ is a circle or a point. Moreover, the following map is almost everywhere one-to-one

$$\psi: \mathcal{L}_e \rightarrow \mathcal{C}_o, \quad \ell \mapsto \text{orb}_o(\ell).$$

**Proof.** Lines in the Study quadric correspond to either rotations or translations [3, Section 2.5] and orbits under these 1-parameter subgroups are circles, lines or points in $\mathbb{R}^3$ und thus via the stereographic projection $\tau$ points or circles in $S^3$.

We can associate to a circle in the 4-dimensional set $\mathcal{C}_o$, a unique 1-parameter subgroup of rotations corresponding to a line in $\mathcal{L}_e$. There is a 2-dimensional set of lines in $\mathcal{L}_e$ such that the rotational axis of these lines passes through $o$ and the corresponding Lie circle is the point $o$. Thus $\psi$ is one-to-one except for a lower dimensional subset as was claimed.

### 3 Quadric surfaces in the Study quadric

A *Darboux cyclide* is defined as a quartic weak del Pezzo surface in $S^3$ [2, Section 8.6.2]. Such surfaces are the intersection of $S^3$ with a quadric hypersurface [2, Theorem 8.6.2]. Let $\mathcal{U}_o$ denote the set of quadric surfaces $Q \subset S$ such that either there exists $V \cong \mathbb{P}^3$ such that $Q \subset V \subset S$, or $Q \subset F_o \subset S$. 


Lemma 2. If $Q \subset S$ is a doubly ruled quadric surface such that $Q \notin \mathcal{U}_o$, then $\text{orb}_o(Q) \subset \mathbb{S}^3$ is a Darboux cyclide.

Proof. There exists bilinear homogeneous $a + eb \in \mathbb{DH}[s_0, s_1, t_0, t_1]$ such that $Q$ is parametrized by

$$\mu: \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow Q \subset S, \quad (s_0 : s_1 : t_0 : t_1) \mapsto a + eb,$$

and $a \bar{b} - b \bar{a} = w_1i + w_2j + w_3k$ with $w_1, w_2, w_3 \in \mathbb{R}[s_0, s_1, t_0, t_1]$ so that

$$\text{orb}_o \circ \mu: \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow D \subset \mathbb{S}^3,$$

$$(s, t) \mapsto (a \bar{a} : w_1 : w_2 : w_3 : b \bar{b}).$$

Thus the map $\text{orb}_o \circ \mu$ is of bidegree $(2, 2)$ into $\mathbb{P}^4$. By Proposition 1, the map $\text{orb}_o$ is not defined at a quartic 4-fold $F_o \subset S$. We observe that $Q$ is the intersection of $S$ with a 3-space, since $Q \notin \mathcal{U}_o$. It follows that $Q \cap F_o$ consists of 4 points (counted with multiplicity). Thus $\text{orb}_o \circ \mu$ has 4 base points. A basis of all bidegree $(2, 2)$ functions on $\mathbb{P}^1 \times \mathbb{P}^1$ defines a map whose image is a degree 8 weak del Pezzo surface $X \subset \mathbb{P}^8$. A basis of bidegree $(2, 2)$ functions that pass through a basepoint, defines a map whose image is a projection of $X$ from a point so that the degree and embedding dimension drops by one. Such a projection realizes the blowup of $X$ in a point and is again a weak del Pezzo surface [2, Proposition 8.1.23]. Thus $M$ can be obtained as 4 subsequent projections of $X$, which results in a quartic weak del Pezzo surface in $\mathbb{P}^4$. This concludes the proof of this lemma, since the image of $\text{orb}_o$ is $\mathbb{S}^3$ by Proposition 1.

\[\square\]

Figure 1. A smooth Darboux cyclide contains six circles through each point and admits $\binom{6}{2} - 3$ pairs of families of circles $F$ and $F'$ so that $F \cdot F' = 1$. 

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A family of curves of a surface $X$ is defined as an irreducible hypersurface $F \subset X \times \mathbb{P}^1$ such that the closure of the first projection of $F$ equals $X$. A curve $F_t \subset X$ in the family $F$ for some $t \in \mathbb{P}^1$ is defined as $\pi_1(F \cap X \times \{t\})$. If $F$ and $F'$ are families of $X$, then we denote by $F \cdot F'$ the number of intersections of a general curve in $F$ and a general curve in $F'$.

**Lemma 3.** If two different Darboux cyclides in $S^3$ intersect in three circles, then two of these circles are co-spherical.

**Proof.** Suppose that $D, D' \subset S^3$ are Darboux cyclides. We can associate to the weak del Pezzo surface $D$ its Picard group, which is a quadratic lattice $\langle \alpha_0, \alpha_1, \ldots, \alpha_5 \rangle \mathbb{Z}$ with intersection pairing $\alpha_i^2 = 1$, $\alpha_i \cdot \alpha_j = 0$ for $i \neq j$ [2, Section 8.2.1]. We associate to a curve $C \subset D$ its divisor class $[C]$ in the Picard group of $D$. The class of any hyperplane section of $D$ is equal to the anticanonical class $-\kappa = 3 \alpha_0 - \alpha_1 - \ldots - \alpha_5$. Both $D$ and $D'$ are intersections of $S^3$ with a quadric hypersurface so that $[D \cap D'] = -2\kappa$ and $\deg D \cap D' = 8$. Since $D \cap D'$ contains 3 circles by assumption, it follows that $D \cap D'$ consists of 4 conics. The class of a conic in $M$ is either $\alpha_0 - \alpha_i$ or $2\alpha_0 + \alpha_i - \alpha_1 - \ldots - \alpha_5$ for some $1 \leq i \leq 5$. The classes of the conics have to add up to $-2\kappa$, and must be of the following form

$$(\alpha_0 - \alpha_i) + (2\alpha_0 + \alpha_i - \alpha_1 - \ldots - \alpha_5) + (\alpha_0 - \alpha_j) + (2\alpha_0 + \alpha_j - \alpha_1 - \ldots - \alpha_5),$$

for some $1 \leq i, j \leq 5$. Since $(\alpha_0 - \alpha_i) \cdot (2\alpha_0 + \alpha_i - \alpha_1 - \ldots - \alpha_5) = 2$ there are two co-spherical circles. \hfill \Box

**Lemma 4.** If $F, F' \subset D \times \mathbb{P}^1$ are families of circles on a Darboux cyclide $D \subset S^3$ such that $F \cdot F' = 1$ and $o \in D$, then there exists a unique doubly ruled quadric surface $Q \subset S$ such that $e \in Q$, $Q \notin U_o$, $\text{orb}_o(Q) = D$ and the two rulings of $Q$ correspond via $\text{orb}_o$ to $F$ and $F'$.

**Proof.** Let $C, C' \subset D$ be two circles in $F$ and $F'$ respectively, such that $C \cap C' = o$. By Lemma 1 there exist unique lines $\ell, \ell' \subset S$ containing $e$ such that $\text{orb}_o(\ell) = C$ and $\text{orb}_o(\ell') = C'$. We choose some point on $h \in \ell'$ and let $C''$ be the unique circle in the family $F$ that passes through $p :=
orbₜ(h). We apply Lemma 1 with ℓ replaced by p and obtain a unique line $L \subset S$ containing e. It follows from the construction that $C'' = \text{orb}_ℓ(ℓ'')$, where $ℓ'' := hL$ means the image of each point in $L$ multiplied with the dual quaternion $h$.

Thus we obtain three intersecting lines $ℓ, ℓ', ℓ'' \subset S$ spanning a 3-space $V$ and three circles $C, C', C'' \subset D$ that pairwise intersect in at most one point. We have that $V \not\subset S$, otherwise $\text{orb}_ℓ(ℓ''')$ would be either a point or a 2-sphere by Lemma 1. Thus $V \cap S$ defines a unique quadric surface $Q \subset S$ such that $ℓ, ℓ', ℓ'' \subset Q$ and $Q \not\in U_0$. It follows from Lemma 2 that $D' := \text{orb}_p(Q)$ is a Darboux cyclide such that $C, C', C'' \subset D \cap D'$. It follows from Lemma 3 that $D$ and $D'$ must be equal. By Lemma 1, lines in $S$ correspond to circles in $\mathbb{S}^3$ and a quadric $Q \subset S$ is covered by two families of lines. This concludes the proof of this lemma.

**Remark 1.** For the existence statement in Lemma 4, it is also possible to give an algebraic proof without using Lemma 3. By [11, Theorem 11], there exists a parametrization for Darboux cyclides of bidegree $2(2,2)$ so that the parameter curves are the circles in $F_1$ and $F_2$, respectively. Lifting the parametrization to $\mathbb{S}^3$, we obtain 5 biquadratic polynomials $X_0, \ldots, X_5 \in \mathbb{R}[s,t]$ such that $4X_0X_4 = X_1^2 + X_2^2 + X_3^2$ and $(X_0 : X_1 : X_2 : X_3 : X_4)$ is a parametrization of the Darboux cyclide $D$. By [6, Theorem 3], there exist bilinear polynomials $A, B \in \mathbb{H}[s,t]$ with quaternion coefficients such that

$$N(A) = X_0, \quad N(B) = X_4, \quad AB = X_1i + X_2j + X_3k.$$

The bilinear polynomial $H := A + \epsilon B \in \mathbb{DH}[s,t]$ then defines a parametrization of a nonsingular ruled quadric in the Study quadric $S$, and the image of this quadric via $\text{orb}_p$ is exactly $D$. 

\[\square\]
**Theorem 1.** The map $\text{orb}_o : S \setminus F_o \to \mathbb{S}^3$ defines a one-to-one correspondence between two families of lines that cover a quadric surface $Q \subset S$ such that $e \in Q$ with $Q \notin \mathcal{U}_o$ — and — two non-cospherial families of circles that cover a Darboux cyclide $D \subset \mathbb{S}^3$ such that $o \in D$.

**Proof.** The left to right direction is a consequence of Lemma 2 and Lemma 1. The converse direction follows from Lemma 4.

A smooth Darboux cyclide $D \subset \mathbb{S}^3$ that contains $o$, admits 6 families of circles and thus 15 pairs of such families [1, 9]. There are exactly three pairs $(F, F')$ of families of circles such that $F \cdot F' = 2$ (see Figure 1). For the remaining 12 pairs $(F, F')$ of families one has $F \cdot F' = 1$ and thus by Theorem 1 there are 12 quadric surfaces $Q \subset S$ such that $e \in Q$ and $\text{orb}_o(Q) = D$.

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N. Lubbes, Johann Radon Institute for Computational and Applied Mathematics (RICAM), Austrian Academy of Sciences

email: niels.lubbes@gmail.com

J. Schicho, Research Institute for Symbolic Computation (RISC), Johannes Kepler University

email: josef.schicho@risc.jku.at