Form Factor $g$ In Longitudinal Space Charge Impedance

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Abstract

In carrying out calculations of the effect of longitudinal space charge on longitudinal motion, the transverse beam size appears in a form factor which is usually written as $g = 1 + 2 \ln(b/a)$. In fact, this expression applies to particles with vanishing betatron amplitude in a beam with uniform transverse distribution. It is argued that an average over the transverse distribution should be used instead of the value on axis. It is shown that for the realistic ‘binomial’ family of distributions the 1 in the above expression for $g$ should be replaced by a value near 0.5 if $a$ is interpreted as twice the rms width of the beam.

1 Introduction

In an ideal machine with no space charge, the three degrees of freedom of motion can be investigated separately. With space charge, this is no longer possible: the longitudinal force on a particle depends upon its transverse position in the bunch and vice versa. It is still profitable to some extent to study longitudinal and transverse motions separately, but one must keep in mind that an approximation is being made. It has become standard practice to calculate longitudinal space charge effects using the on-axis force because it was felt that this corresponds to a ‘worst case’. However, there is no basis for this approach unless one is interested in the absolute longitudinal space charge limit (even in that case, the ‘on-axis’ approach is debatable).
Since synchrotron motion is slow compared with betatron motion, it is clear that in longitudinal simulations the longitudinal force should be averaged over transverse positions. (Conversely, in transverse simulations, the transverse force should be taken as that corresponding to an instantaneous longitudinal position.) This note is devoted to deriving a formula for the longitudinal force as averaged over transverse distributions for the binomial family of distributions. The beam is assumed to be round (radius=\(a\)) in a concentric round beam pipe (radius=\(b\)). It is found that the usual transverse geometric form factor \((g)\), which is \(1 + 2\ln(b/a)\) for the on-axis case of the uniform distribution, can be written as \(K + 2\ln(b/a)\), where \(K\) depends upon the distribution and the definition of \(a\). To be fair it should be pointed out that the error incurred by these assumptions (neither the beam nor the beam pipe is round nor of constant dimension) may very well be larger than the error incurred by using the on-axis formula.

### 2 The Distribution

The assumed distribution is

\[
\rho = (\mu + 1) \frac{\lambda}{\pi a^2} \left(1 - \frac{r^2}{a^2}\right)^\mu
\]  

(1)

for \(r < a\) and \(\rho = 0\) otherwise. The normalization is \(\iint \rho dx dy = 2\pi \int \rho r dr = \lambda\), the line density. For \(\mu = 0\) this is the Kapchinsky-Vladimirsky distribution (round case). For \(\mu \to \infty\) it is gaussian provided \(a \to \infty\) as well. This can be seen by finding the standard deviation.

\[
\overline{r^2} = \frac{\int \rho r^2 r dr d\theta}{\int \rho r dr d\theta} = \frac{2\pi}{\lambda} \int_0^a \rho r^3 dr = \frac{a^2}{\mu + 2}.
\]  

(2)

Since \(\overline{r^2} = x^2 + y^2 = x^2 + y^2 = 2x^2 \equiv 2\sigma_x^2\), this can be written as

\[
a = \sqrt{2(\mu + 2)\sigma_x}.
\]  

(3)

In the Sacherer-Lapostolle convention, the beam size is characterized not by \(\sigma_x\) but by \(\bar{x} \equiv 2\sigma_x\). This is handy because for the K-V case \(\bar{x} = a\). For the general case,

\[
\bar{x} = \sqrt{\frac{2}{\mu + 2}} a.
\]  

(4)
3 The On-Axis Potential

From Gauss’ law, the electric field is radial with magnitude

\[ E(r) = \frac{1}{\varepsilon_0} \int_0^r \rho(r') r' dr' = \frac{\lambda}{2\pi\varepsilon_0 r} \left\{ \begin{array}{ll} 1 - \left( \frac{r^2}{a^2} \right)^{\mu+1} & \text{if } r < a \\ 1 & \text{otherwise} \end{array} \right. . \] (5)

With the beam pipe (radius=\( b \)) at ground, the potential as a function of radius is

\[ V = -\int_r^b E(r') dr'. \] (6)

Assuming all the beam is in the pipe (\( a < b \)), the potential in the beam is

\[ V = \frac{-\lambda}{2\pi\varepsilon_0} \left\{ \int_r^a \left[ 1 - \left( \frac{r^2}{a^2} \right)^{\mu+1} \right] \frac{dr}{r} + \ln \left( \frac{b}{a} \right) \right\} , \] \tag{7}

which for \( \mu=\text{integer} \) can be expanded as

\[ V = \frac{-\lambda}{4\pi\varepsilon_0} \left[ \sum_{k=1}^{\mu+1} \frac{(1 - r^2/a^2)^{k}}{k} + 2 \ln \left( \frac{b}{a} \right) \right] . \] (8)

The restriction of \( \mu \) to an integer is of course not necessary, but gives simpler analytic expressions.

The expression in square brackets can be identified as \( g \). For the uniform distribution (\( \mu=0 \)) on axis (\( r=0 \)), we recover

\[ g = 1 + 2 \ln(b/a). \] \tag{9}

Generalizing to \( g = K_0 + 2 \ln(b/a) \), we find \( K_0 = \sum_{k=1}^{\mu+1} \frac{1}{k} \). See Table 1.

For the gaussian, the sum diverges logarithmically, but for constant \( \bar{x} \), \( \ln(b/a) \) diverges as well and the two infinities cancel.\(^2\) Defining \( \bar{K}_0 \) by \( g = \bar{K}_0 + 2 \ln(b/\bar{x}) \), we get

\[ \bar{K}_0 = K_0 - \ln(\mu/2 + 1) . \] \tag{10}

\( \bar{K}_0(\mu) \) is also given in Table 1. For large \( \mu \), \( K_0 \) approaches \( \gamma + \ln(\mu + 3/2) \), where \( \gamma \) is Euler’s constant 0.57721566\( \cdots \). Hence, for the gaussian, \( \bar{K}_0 = \gamma + \ln 2 \).

\(^1\)Strictly speaking this is only correct if \( \lambda \) is constant. However it is a good approximation if, as is usually the case with proton machines, the length scale of the longitudinal variation is large compared with the beam pipe size, i.e. bunch length \( \gg b \).

\(^2\)Actually, this violates our assumption \( a < b \). However, in realistic cases \( b > 4\sigma_x \) so there is so little beam outside \( r = b \) that the approximation is not significant.
Table 1: On-axis values for $K$

| $\mu$ | $K_0$   | $\tilde{K}_0$ |
|-------|---------|---------------|
| 0     | 1.0000  | 1.0000        |
| 1     | 1.5000  | 1.0945        |
| 2     | 1.8333  | 1.1402        |
| 3     | 2.0833  | 1.1670        |
| 5     | 2.4500  | 1.1972        |
| 10    | 3.0199  | 1.2281        |
| $\infty$ | $\infty$ | 1.2704 |

4 The Average Potential

One can find an average by first averaging the betatron motion for a given particle and then averaging over all the betatron amplitudes in the beam. However, for a stationary transverse distribution, one can find the result more easily by simply averaging the potential over all particles in the beam at a given instant in time without regard to betatron amplitudes. In this way the average potential is

$$\langle V \rangle = \frac{\int V \rho drd\theta}{\int \rho drd\theta} = \frac{2\pi}{\lambda} \int_0^a V \rho dr$$

(11)

Integrating term by term we find

$$\langle V \rangle = -\frac{\lambda}{4\pi\epsilon_0} \left[ \sum_{k=1}^{\mu+1} \left( \frac{1}{k} - \frac{1}{k+\mu+1} \right) + 2 \ln \left( \frac{b}{a} \right) \right].$$

(12)

The summation can be identified as $\langle K \rangle$. Again, it diverges as $\mu \to \infty$, but remains finite if the beam size is characterized by $\tilde{x}$ instead of $a$. See Table 2. For those interested in special functions, the solution for general $\mu$ is

$$\langle \tilde{K} \rangle = \gamma + 2\psi(\beta) - \psi(2\beta - 1) - \ln(\beta/2),$$

(13)

where $\beta = \mu + 2$ and $\psi$ is the psi (or digamma) function. An approximation that is within 0.2% for $\mu \geq 1$ is $\langle \tilde{K} \rangle \approx \gamma - 1/(4\mu + 11)$. 

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Table 2: Average values for $\bar{K}$

| $\mu$ | $\langle K \rangle$ | $\langle \bar{K} \rangle$ |
|-------|---------------------|------------------|
| 0     | 0.5000              | 0.5000           |
| 1     | 0.9167              | 0.5112           |
| 2     | 1.2167              | 0.5235           |
| 3     | 1.4488              | 0.5325           |
| 5     | 1.7968              | 0.5440           |
| 10    | 2.3489              | 0.5572           |
| $\infty$ | $\infty$ | 0.5772 |

5 Conclusion

The value of $\bar{K}$ averaged over the beam is remarkably stable; varying by only 15% over all the binomial distributions from K-V to gaussian. It is therefore safe to conclude that for reasonably ‘good’ transverse distributions (i.e. smooth, peaked at the centre, no halo), the form factor $g$ which appears in the longitudinal space charge force and in the space charge impedance is given to a good approximation by

$$g = \frac{1}{2} + 2 \ln \left( \frac{b}{\bar{x}} \right)$$

(14)

where $b$ is the beam pipe radius and $\bar{x}$ is twice the rms beam size. This should be used in place of $g = 1 + 2 \ln(b/a)$ for the calculation of space charge effects when longitudinal motion is considered independently of transverse motion.