Static Quantization of Two-dimensional Dilaton Gravity and Black Holes

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Abstract
Two-dimensional matterless dilaton gravity is a topological theory and can be classically reduced to a (0+1)-dimensional theory with a finite number of degrees of freedom. If quantization is performed, a simple gauge invariant quantum mechanics is obtained. The properties of the gauge invariant operators and of the Hilbert space of physical states can be determined. In particular, for $N$-dimensional pure gravity with $(N-2)$-dimensional spherical symmetry, the square of the ADM mass operator is self-adjoint, not the mass itself.

1 Introduction
Recently, the investigation of lower-dimensional gravitational models [1] has received a large amount of attention because of its relation to higher-dimensional gravity, integrable systems and black hole physics.

The aim of this paper is to discuss some interesting classical and quantum properties of the (1+1)-dimensional model of dilaton gravity [2]-[6]

$$S_{DG} = \int d^2x \sqrt{-h} \left( \phi R^{(2)}(h) + W(\phi) \right), \quad (1)$$

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where $\phi$ is the dilaton field, $W(\phi)$ is the dilaton potential, and $R^{(2)}(h)$ is the two-dimensional Ricci scalar. (Here and throughout the paper we use Landau-Lifshits conventions \[7\] for the Ricci scalar and natural units.)

Theories of the form (1) may arise from dimensional reduction of higher dimensional gravity in presence of symmetries. A remarkable example is dimensional reduction of spherically symmetric vacuum Einstein-Hilbert gravity in $N > 2$ dimensions

$$S^{(N)} = \frac{1}{16\pi l_{pl}^{N-2}} \int d^N y \sqrt{-g} R^{(N)}(g).$$

Using for the $N$-dimensional metric the $(N - 2)$-spherically symmetric ansatz ($\mu, \nu = 0, 1$)

$$ds^2_N = [\phi(x)]^{-(N-3)/(N-2)} h_{\mu\nu}(x) dx^\mu dx^\nu + [\gamma\phi(x)]^{2/(N-2)} d\Omega^2_{N-2}, \quad \phi > 0,$$

Eq. (2) can be cast in the form (1) where

$$W(\phi) = (N - 2)(N - 3)(\gamma^2 \phi)^{-1/(N-2)},$$

$$\gamma = 16\pi l_{pl}^{N-2}/V_{N-2} \quad \text{and} \quad V_{N-2} = 2\pi^{(N-1)/2}/\Gamma((N - 1)/2) \text{ is the volume of the } (N - 2)-\text{dimensional unit sphere } \Omega^2_{N-2}. \quad \text{Note that we have neglected the surface term}$$

$$\partial S = -\frac{N-1}{N-2} \int d^2 x \sqrt{-h} \nabla^2 \phi.$$ 

Dilaton gravity theories defined by Eq. (1) are classically integrable \[3\]. A number of statements are equivalent:

i) Dilaton gravity theories reduce to (0+1)-dimensional theories. Any solution can be represented (in suitable coordinates) as function of only one coordinate, a property sometimes referred to as “stacticity property” though the Killing vector is not timelike and orthogonal to a spacelike hypersurface on the entire manifold.

ii) Dilaton gravity theories are topological theories.

iii) A locally conserved (gauge invariant) quantity exists (for spherically symmetric gravity this coincides with the ADM \[8\] mass of the system) and defines the horizon(s).

iv) The only gauge invariant quantities are the locally conserved quantity and its conjugate momentum.

These properties can be easily proved using the dilaton and the locally conserved quantity (and their conjugates) as coordinates in the phase space (“geometrodynamical variables” \[3\]).

A further, conjectured property that should be mentioned is the equivalence of dilaton gravity theories to a couple of D’Alembert (free) fields (plus a single degree of freedom).
This is proved through the explicit identification of a canonical transformation in the case of a potential of the form

\[ W(\phi) = ae^{c\phi} + be^{-d\phi}. \]

where \(a, b, c, d\) are constant parameters. However, in the general case the proof of existence of a free field representation [3] is not sufficient for the explicit construction of the canonical transformation.

These classical properties are an essential guideline in choosing the quantization scheme so as to preserve them. This is evident in the scheme that has been worked out explicitly in the case of the CGHS model [3, 5, 10] using free fields, and in the general case using geometrodynamical variables [4].

Thanks to the staticity property, dilaton gravity can be quantized according to two alternative approaches [11]. The first approach is implemented by the explicit reduction of the (1+1)-dimensional system to the couple of gauge invariant variables (the locally conserved quantity and its conjugate momentum) (see e.g. [4, 12, 13]). In the second approach the system is classically reduced to a gauge (0+1)-dimensional problem and then quantized, leading to a quantum mechanical theory. In both cases the ensuing quantum theory is described by a Hilbert space spanned by the eigenstates of the quantum operator corresponding to the gauge invariant quantity (alternatively, its conjugate) and the two approaches formally lead to the same Hilbert space (“Quantum Birkhoff Theorem” [4, 10, 11]).

Although the two approaches are formally equivalent, the second method has the advantage that canonical quantities are explicitly represented as differential operators, the Hilbert space is explicitly defined, and the Hermiticity properties of operators are controlled. This harvest of results is typical of the quantum mechanical approach and cannot be obtained by the (1+1)-dimensional method, neither by the direct reduction to the couple of gauge invariant quantities nor by reduction to free fields. The quantum mechanical approach has been worked in detail for the Schwarzschild black hole system [14, 15]. In this case, a remarkable result of the (0+1)-dimensional method is that the square of the ADM mass, not the ADM mass itself, is self-adjoint.

In this paper we are interested in extending the quantum mechanical treatment originally developed for the Schwarzschild black hole to the entire class of dilaton gravity models described by Eq. (1). Our purpose is to show that the self-adjointness properties of the quantum operators corresponding to the gauge invariant quantities depend on the particular model under consideration. In particular, we will show that the non self-adjointness of the ADM mass is not a general property of dilaton gravity theories: it holds for pure dilaton theories that correspond to hyperspherically symmetric metrics and does not depend on the dimension of space time.

These results suggest that the role of the Schwarzschild black hole mass in gravity can be clarified by the simple quantum procedure model considered in this paper and that the root of positivity of the ADM mass in the Schwarzschild black hole geometry can be found in its quantum realization.

The outline of the paper is as follows. In the next section we present the classical
canonical theory. In Sect. 3 we focus attention on the models derived from dimensional reduction of spherically symmetric gravity. Finally, in Sect. 4 we deal with the quantum theory and discuss the self-adjointness of the relevant operators.

# 2 General Canonical Theory

We parametrize the two-dimensional metric $h_{\mu\nu}$ as

$$ g_{\mu\nu} = \rho \begin{pmatrix} \alpha^2 - \beta^2 & \beta \\ \beta & -1 \end{pmatrix}. \quad (7) $$

Here $\alpha > 0$ and $\beta > 0$ play the role of the lapse function and of the shift vector respectively; $\rho(x_0, x_1)$ represents the dynamical gravitational degree of freedom. The coordinates $x_0, x_1$ are both defined on $\mathbb{R}$.

The Hamiltonian form of Eq. (1) is

$$ S = \int d^2x \left[ \dot{\rho} \pi_\rho + \dot{\phi} \pi_\phi - \alpha \mathcal{H} - \beta \mathcal{P} \right], \quad (8) $$

where the super-Hamiltonian and super-momentum are

$$ \mathcal{H} = \rho \pi_\rho \pi_\phi + \frac{\rho'}{\rho} \phi' - 2\phi'' - \rho W(\phi), \quad (9) $$

$$ \mathcal{P} = -\phi' \pi_\phi + \rho' \pi_\rho + 2\rho \pi'_\rho, \quad (10) $$

respectively. (We neglect boundary terms as they are irrelevant to the following discussion. See e.g. [4, 9, 12] and references therein.)

According to the statement i) of Sect. 1 any classical solution, in suitable coordinates, can be written as a function of a single coordinate $\mathbb{R}$. Thus the problem is reduced to a problem of finite degrees of freedom. In the canonical framework we can impose the staticity condition by requiring that both the metric and the dilaton and their momenta depend on a single variable. Setting

$$ \alpha \equiv \alpha(x_0), \quad \rho \equiv \rho(x_0), \quad \phi \equiv \phi(x_0), \quad \pi_\rho \equiv \pi_\rho(x_0), \quad \pi_\phi \equiv \pi_\phi(x_0), \quad (11) $$

the action (8) is cast in the form

$$ S = \int dx_0 \left[ \dot{\rho} \pi_\rho + \dot{\phi} \pi_\phi - l(x_0) \mathcal{H} \right], \quad (12) $$

where

$$ l(x_0) \equiv \alpha \rho(x_0) \quad (13) $$

is a Lagrange multiplier enforcing the constraint $H = 0$. The super-momentum constraint defined in Eq. (10) vanishes identically and $H$ corresponds to the (0+1)-dimensional slice of the super-Hamiltonian in Eq. (11). This is given by

$$ H = \pi_\rho \pi_\phi - W(\phi). \quad (14) $$
Two remarks are in order. Firstly, the action \((12)\) should be interpreted as a density action in the coordinate \(x_1\), i.e. \([S] = [\text{length}^{-1}]\). Alternatively, the coordinate \(x_1\) can be made compact. In this case we set for simplicity \(\text{Vol}(x_1) = 1\). The second remark concerns the definition of the Lagrange multiplier \(l(x_0)\). Equation \((13)\) is meaningful only if \(\rho\) has definite sign. Indeed, the gauge evolution parameter is a monotonic increasing function in \(x_0\) on all trajectories provided that the Lagrange multiplier has definite sign – at least on the constraint surface. (Possibly, some simple zeroes may be harmless but one cannot make any general statement about this.) Therefore, in the following we will restrict attention on strictly positive values of \(\rho\). (The discussion for \(\rho < 0\) is analogous and leads to the same canonical equations, the only difference being the overall sign of the gauge parameter.) This condition can be lifted if one requires the continuity of the canonical variables at any space time point. Indeed, on the constraint surface the equation \(\rho = 0\) defines the horizon(s) of the two-dimensional metric \((7)\) – see Eq. \((15)\) below. So, by requiring the continuity of the canonical variables across the horizon(s) the dynamics generated by Eqs. \((12)\) and \((14)\) holds for any value of \(\rho\).

The gauge equations or, alternatively, the (unconstrained) equations of motion can be easily integrated. The result is

\[
\frac{\tau}{2I}, \quad \frac{\pi_\phi}{2I} = 2I[H + W(\phi)],
\]

\[
\rho = 4I^2[N(\phi) + H\phi - J/2], \quad \frac{\pi_\rho}{2I} = \frac{1}{2I},
\]

where \(I\) and \(J\) are two gauge invariant quantities, \(N(\phi) = \int d\phi W(\phi)\), and

\[
\tau(x_0) = \int_0^{x_0} d\bar{x}_0 \, l(\bar{x}_0)
\]  

(16)

is the gauge parameter. The gauge invariant quantities \(I\) and \(J\) can be written as functions in the phase space:

\[
I = \frac{1}{2\pi_\rho}, \quad J = 2[N(\phi) + H\phi - \rho\pi_\rho^2].
\]

(17)

(18)

Clearly, since \(I\) and \(J\) are gauge invariant, their Poisson brackets with \(H\) must vanish (at least weakly). By direct calculation one can prove that actually the Poisson brackets vanish strongly. Moreover one also finds \([J, I]_P = 1\). By completing the triplet \(I, J, H\) by

\[
Y = \frac{\phi}{\pi_\rho}, \quad [Y, H]_P = 1,
\]

(19)

one obtains a maximal set of gauge invariant canonical variables \([16]\), often referred as “Shanmugadhasan” variables \([17]\). Using the Shanmugadhasan variables the action \((12)\) assumes the simpler form

\[
S = \int dx_0 \left[\dot{J}I + \dot{Y}H - l(x_0)H\right].
\]

(20)
For sake of completeness let us note that the gauge invariant quantity \( J \) is related to the (0+1)-dimensional slice of the conserved local quantity \( M \) as defined by [3] (see also [2, 18])

\[
M = \int_0^\phi W(\phi) d\phi - \frac{\rho^2 \pi^2 - \phi^2}{\rho}.
\]  

By a simple algebra one can prove that \( M|_{0+1} = J/2 - H\phi \). We will see later that in the spherically symmetric reduced models \( J \) coincides (apart from some numerical factors) with the ADM mass of the system.

Let us conclude this section by an interesting remark concerning the support of the gauge invariant quantity \( I \). From the first Eq. (15) we see that \( I \) has the sign of \( \phi \). Indeed, since \( l(x_0) > 0 \), we can take \( \tau \) positive. (In quantum mechanics one never gets the Feynman propagator without this positivity restriction, see e.g. [19, 20].) This property will be essential in the following.

3 Spherically Symmetric Gravity

We have mentioned in the introduction that \((N - 2)\)-spherically symmetric gravity in \( N \) dimensions can be described by Eq. (11) where the dilaton potential is given by Eq. (10). The connection with \( N \)-dimensional spherically symmetric gravity in the standard Schwarzschild form can be better exploited using the “Schwarzschild-like” canonical variables \((a, \pi_a; b, \pi_b)\) [14, 15] defined by the canonical transformation

\[
\rho = 2ab, \quad \pi_\rho = \frac{\pi_a}{2b},
\]

\[
\phi = \frac{1}{\gamma} b^{N-2}, \quad \pi_\phi = \frac{1}{\gamma} \frac{b\pi_b - a\pi_a}{b^{N-2}}.
\]

By defining the new Lagrange multiplier

\[
\tilde{l}(x_0) = \frac{\gamma}{N - 2} \frac{1}{b^{N-3}} l(x_0),
\]

the (0+1)-dimensional action (12) becomes

\[
S = \int dx_0 \left[ a\pi_a + b\pi_b - \tilde{l}(x_0)\mathcal{H} \right],
\]

where

\[
\mathcal{H} = \frac{\pi_a}{2b^2} (b\pi_b - a\pi_a) - k b^{N-4}, \quad k = (N - 2)^2 (N - 3) \gamma^{-\frac{N-1}{2}}.
\]

Using the new canonical chart the \( N \)-dimensional metric (3) reads

\[
ds_N^2 = 2 \gamma^{\frac{N-4}{2}} b^4 r^{-N} \left[-adt^2 + ndr^2\right] + b^2 d\Omega^2_{N-2}.
\]
where we have set $\alpha = \sqrt{n/a}$ and $x_0 = r$, $x_1 = t$. Finally, in terms of the Schwarzschild variables the Lagrange multiplier is

$$\mathcal{I}(r) = \frac{2\gamma \sqrt{an}}{N - 2 b^{N-4}}. \quad (27)$$

For sake of completeness we give the (on-shell) expression of the gauge invariant variables $I$ and $J$ in terms of the new canonical variables:

$$I_{\mathcal{P}=0} = \frac{b}{\pi a}, \quad (28)$$

$$J_{\mathcal{P}=0} = \frac{2(N - 2)^3}{\gamma (N-1)/(N-2)} b^{N-3} - \pi_a \pi_b. \quad (29)$$

When $N = 4$ Eqs. (25)-(29) coincide with the corresponding quantities of [14, 15].

Inserting the solution (15) in Eq. (26) we have

$$ds^2 = -\tilde{I}^2 \left(1 - \frac{\tilde{J}}{b^{N-3}} - \frac{\tilde{H}}{b^{N-4}}\right) dt^2 + \left(1 - \frac{\tilde{J}}{b^{N-3}} - \frac{\tilde{H}}{b^{N-4}}\right)^{-1} db^2 + b^2 d\Omega_{N-2}^2, \quad (30)$$

where

$$\tilde{I} = \frac{2(N - 2)}{\gamma^{1/(N-2)}} I_{|\mathcal{H}=0}, \quad (31)$$

$$\tilde{J} = \frac{\gamma^{(N-1)/(N-2)}}{2(N - 2)^2} J_{|\mathcal{P}=0}, \quad (32)$$

$$\tilde{H} = \frac{\gamma^{(N-1)/(N-2)}}{(N - 2)^2} \mathcal{P}. \quad (33)$$

The horizons of the $N$-dimensional black hole are defined on the constraint shell $\tilde{H} = 0$ by

$$b^{N-3} = \tilde{J}. \quad (34)$$

The ADM mass on the constraint shell is

$$M_{\text{ADM}} = \frac{N - 2}{\gamma} \tilde{J} = \frac{\gamma^{1/(N-2)}}{2(N - 2)} J_{|\mathcal{H}=0}. \quad (35)$$

Let us discuss the support of the canonical variables and of the gauge invariant quantities. From Eqs. (3) and (4) we have $\phi > 0$, $\phi = 0$ being a singularity of the metric. (Clearly, the same conclusion is obtained using the Schwarzschild canonical variables. Indeed, starting from the metric (20) we have $b > 0$ and the canonical transformation (22) implies $\phi > 0$.) From the discussion at the end of the previous section it follows $I > 0$. This property will play an essential role in the quantization of the system. The gauge
invariant variable $J$ (and thus the ADM mass) does not have a definite sign in general. Both positive and negative masses are allowed. In order to exclude negative ADM masses in spherically symmetric geometries we have to invoke a further, physically required, ad hoc principle as it is usually the case, or we have to look for some mechanism responsible for this property. Curiously, this result is a bonus of the quantum theory of spherically symmetric geometries, see next section.

4 Quantization

The quantization of the model described in the previous sections leads to a quantum mechanical system with gauge invariance. Here our treatment follows closely [15]. We implement the quantization by the Dirac method, quantizing first and fixing the gauge after having solved the Wheeler-de Witt equation.

The quantization of the system is straightforward in the Shanmugadhasan representation. Formally, the quantization is achieved by imposing first the commutation relations

\[
[\hat{J}, \hat{I}] = \im\hbar, \quad [\hat{Y}, \hat{H}] = \im\hbar, \tag{36}
\]

and then by imposing the constraint as a null operator on the states in the Hilbert space

\[
\hat{H} \Psi = 0. \tag{37}
\]

In order to represent the canonical coordinates as differential operators we must first choose a pair of commuting variables as coordinates in the Hilbert space and establish the form of the (non-gauge fixed) Hilbert measure $d\mu$. The measure $d\mu$ is determined by the requirement that it is invariant under the symmetry transformations of the system, namely under the rigid transformations generated by a couple of suitable gauge invariant quantities $F(I, J)$ and $G(I, J)$ and under the gauge transformations generated by $H$. In this process the support of the canonical variables is essential.

Let us suppose that $I$ and $J$ are defined on the whole real axis. This happens for instance when the dilaton potential $W(\phi) - \text{see Eq. (4)}$ is a well defined functional of the dilaton for any value of $\phi$. In this case we can choose $Y$ and $I$ as coordinates in the Hilbert space. (Alternatively, we might choose $Y$ and $J$, the two representations being related by a Fourier transform.) Denoting by $y$, $x$, $j$ the (continuous) eigenvalues of $\hat{Y}$, $\hat{I}$, $\hat{J}$, respectively, the gauge and rigid invariant measure in the Hilbert space is

\[
d\mu = dxdy. \tag{38}
\]

The differential representation of the operators is

\[
\hat{I} = x, \quad \hat{J} = \im\hbar \frac{\partial}{\partial x}, \quad \hat{Y} = y, \quad \hat{H} = -\im\hbar \frac{\partial}{\partial y}. \tag{39}
\]
By imposing the quantum constraint (37) we find that the physical states do not depend on \( y \). A basis in the gauge fixed \((y = \text{const})\) Hilbert space – see [15] for details – is given by the set of eigenstates of \( \hat{J} \) with eigenvalue \( j \)

\[
\Psi_j(x) = \frac{1}{\sqrt{2\pi m_{\text{pl}}}} e^{-ijx/m_{\text{pl}}}. \tag{40}
\]

Let us now suppose that the support of \( \phi \) does not coincide with the entire real axis and consider for simplicity \( \phi \in \mathbb{R}^+ \). We have seen in the previous section that models describing spherically symmetric Einstein gravity in \( N \)-dimensions belong to this class. In this case \( I > 0 \) from (15) and we cannot use the rigid symmetry generated by \( J \) to fix the Hilbert measure since it changes the sign of \( I \). Following [14, 15] we define the gauge invariant “dilatation” generator \( N = IJ \). (The dilatation operator was also introduced in [21], to avoid negative masses; here and in [14, 15] this result is obtained as a consequence of the support properties of the conjugate variable \( I \) and of quantization.) It is easy to check that \( N \) generates a symmetry that preserves the sign of \( I \) and can be used to determine the Hilbert measure. Imposing the invariance under the rigid symmetries generated by \( N \) and \( I \) the Hilbert measure is

\[
d\mu = \frac{dx}{x} dy, \quad x > 0. \tag{41}
\]

The measure (41) implies that the operator \( \hat{J} \) is not self-adjoint being conjugate to a positive definite operator. Indeed, according to (11) the differential representation of \( \hat{J} \) is

\[
\hat{J} = im_{\text{pl}} \sqrt{x} \frac{\partial}{\partial x} \frac{1}{\sqrt{x}}. \tag{42}
\]

Consequently, the eigenstates of \( \hat{J} \) are

\[
\Psi_j(x) = c(j) \sqrt{x} e^{-ijx/m_{\text{pl}}}. \tag{43}
\]

It is straightforward to verify that \( \hat{J} \) is not self-adjoint in the space defined by (11). A self-adjoint operator in the space (11) is rather \( \hat{J}^2 \). Thus in spherically symmetric gravity the square of the ADM mass operator, not the ADM mass operator, is self-adjoint.

For sake of completeness, let us give the eigenfunctions of \( \hat{J}^2 \) (with eigenvalue \( j^2 \)). We have two separate sets \((j > 0)\)

\[
\Psi_{j^2}^{(1)}(x) = \frac{1}{\sqrt{\pi jm_{\text{pl}}}} \sqrt{x} \sin(jx/m_{\text{pl}}), \tag{44}
\]

\[
\Psi_{j^2}^{(2)}(x) = \frac{1}{\sqrt{\pi jm_{\text{pl}}}} \sqrt{x} \cos(jx/m_{\text{pl}}). \tag{45}
\]

The effect of the non self-adjoint operator \( \hat{J} \) is to transform the set (44) into the set (13) and viceversa.
5 Conclusions

The (0+1)-dimensional (“static”) canonical quantization of two-dimensional matterless dilaton gravity shows that the self-adjointness of gauge invariant operators depends on the global properties of the model. In particular, the gauge invariant operator $\hat{J}$ that identifies the horizon(s) of the metric may not have a self-adjoint extension. This happens for models describing spherically symmetric gravity in $N$ dimensions. In this case $\hat{J}$ is – apart from a numerical factor – the gauge invariant operator corresponding to the ADM mass of the geometry. Consequently, the ADM mass operator is not self-adjoint. Instead, its square is self-adjoint and its eigenfunctions can be defined in the Hilbert space (with positive eigenvalues of course). This result (obtained in [14] and [15] for the Schwarzschild black hole) may be the key to dispose of an ad hoc principle to eliminate negative masses in spherical geometries, since the only admissible operator is the square of the mass. See also the discussion contained in [21] where the use of the operator $\hat{N}$ is advocated as a principle to avoid negative values of the mass.

Regardless of the dilaton potential chosen in Eq. (1), both $\hat{J}^2$ and $\hat{J}$ (when the latter can be defined) have continuous spectra. This result is in agreement with the group theoretical quantization of SO(3)-symmetric four-dimensional gravity via reduction to a SL(2,R)/SO(2) non-linear sigma model coupled to three-dimensional gravity [22, 23].

Quantization of the mass can be achieved by changing the boundary conditions. For examples of this procedure, we refer to [13] and especially to [21] where further references can be found. Let us remark that there are indications for a discrete mass spectrum to be obtained by inclusion of matter in the system. Evidence supporting this conjecture can be found in [24] where the quantization of spherically symmetric gravity coupled to a thin dust shell is derived.

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