Information and complexity measures in the interface of a metal and a superconductor

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Abstract

Fisher information, Shannon information entropy and Statistical Complexity are calculated for the interface of a normal metal and a superconductor, as a function of the temperature for several materials. The order parameter $\Psi(r)$ derived from the Ginzburg-Landau theory is used as an input together with experimental values of critical transition temperature $T_c$ and the superconducting coherence length $\xi_0$. Analytical expressions are obtained for information and complexity measures. Thus $T_c$ is directly related in a simple way with disorder and complexity. An analytical relation is found of the Fisher Information with the energy profile of superconductivity i.e. the ratio of surface free energy and the bulk free energy. We verify that a simple relation holds between Shannon and Fisher information i.e. a decomposition of a global information quantity (Shannon) in terms of two local ones (Fisher information), previously derived and verified for atoms and molecules by Liu et al. Finally, we find analytical expressions for generalized information measures like the Tsallis entropy and Fisher information. We conclude that the proper value of the non-extensivity parameter $q \simeq 1$, in agreement with previous work using a different model, where $q \simeq 1.005$.

Keywords: Information Theory; Fisher Information; Shannon Entropy; Tsallis Entropy; Statistical Complexity; Superconductivity; Order Parameter.

1 Introduction

Fisher information [1] has two basic roles to play in theory [2]. First, it is a measure of the ability to estimate a parameter; this makes it a cornerstone of the statistical field of study called parameter estimation. Second, it is a measure of the state of disorder of a system or phenomenon, an important aspect of physical theory [2]. Fisher’s information measure (FIM) is defined for the simplest case of one-dimensional probability distribution $P(x)$ as a functional of $P(x)$ i.e.

$$
I_F \equiv \int P(x) \left( \frac{d\ln P(x)}{dx} \right)^2 dx = \int \frac{1}{P(x)} \left( \frac{dP(x)}{dx} \right)^2 dx.
$$

(1)

$I_F$ can also be written in terms of the so called Fisher information density $i(x) = \frac{1}{P(x)} \left( \frac{dP(x)}{dx} \right)^2$ i.e. $I_F = \int i(x) dx$. The FIM, according to its definition, is an account of the sharpness of the probability density. A sharp and strongly localized probability density gives rise to a larger value of Fisher information. Its appealing features differ appreciably from other information measures because of its local character, in contrast with the global nature of several other functionals, such as the Shannon [3], Tsallis [4, 5] and Renyi [6] entropies. The local character of Fisher information shows an enhanced sensitivity to strong changes, even over a very small-sized region in the domain of definition, because it is as a functional of the distribution gradient.

Very interesting applications of Fisher information have been made in quantum systems [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17] including atoms, molecules, nuclei, or in mathematical physics in general [18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39]. However, in spite of extensive applications of information theory in quantum many body systems, most of the them focus on the behavior of systems in a single phase. Actually, there are a limited number of information-theoretical applications to systems that undergo a phase transition (see for example [40, 41, 42, 43, 44, 45, 46]). The main motivation of the present work is to extend the application of FIM in order to include systems in a phase transition by employing the phenomenological theory of Ginzburg-Landau. More precisely, we consider the order parameter $\Psi(r)$ as the basic ingredient of the FIM functional. In particular, we replace the probability distribution $P(x)$ with the inhomogeneous distribution of the superconducting phase defined in a proper way. In this model, the Fisher information measure is introduced...
in a convenient way, directly related with the characteristics of a phase transition. We focus on the case of the interface between a normal metal and a superconductor. We find the dependence of the FIM on specific properties of the interface of superconductors including the superconducting coherence length as well as the critical transition temperature.

Next, we calculate the Shannon information entropy \[3\] and the LMC (statistical) complexity \[47, 48\]. The dependence of these measures on the temperature is examined and interesting comments are made. The question of generalized (non-extensive) information measures is also addressed. An analytical relationship between Shannon and Fisher information, previously proved and shown to hold for atoms and molecules, is demonstrated in the present case of the superconducting interface as well.

This letter is organized as follows. In section 2 we review briefly the Ginzburg-Landau theory for inhomogeneous systems. In Section 3 we present the Fisher information measure using a probability distribution related with the order parameter. In section 4 we calculate and discuss some additional information and complexity measures including generalized ones. In section 5 we demonstrate Liu’s identity, while in section 6 we discuss the possibility of alternative probability distributions. Finally, section 7 contains our concluding remarks.

## 2 Ginzburg-Landau theory for inhomogeneous systems

The Ginzburg-Landau theory is a theory of second-order phase transition, where one introduces an order parameter \(\Psi(\mathbf{r})\) which is zero above the transition temperature \(T_c\), but takes a finite value for \(T < T_c\) and uses the symmetry of the relevant Hamiltonian to restrict the form of the free energy as a functional of \(\Psi(\mathbf{r})\) \[49\]. Following the discussion by Leggett \[49\], we assume that \(F[\Psi(\mathbf{r}), T]\) is the space integral of a free energy density \(F\) which is a function only of \(\Psi(\mathbf{r})\) and its space derivatives and also their complex conjugates. The form of the free energy functional is \[49\]

\[
F_s(T) = \int F_s[\Psi(\mathbf{r}), T] d\mathbf{r},
\]

\[
F_s[\Psi(\mathbf{r}), T] \equiv F_n(T) + \alpha(T)|\Psi(\mathbf{r})|^2 + \frac{1}{2}\beta(T)|\Psi(\mathbf{r})|^4 + \gamma(T)|\nabla\Psi(\mathbf{r})|^2,
\]

where \(F_s\) is the superconducting free energy density and \(F_n\) is the normal-state free energy density. In eq. \[3\] the normalization of the order parameter \(\Psi(\mathbf{r})\) is arbitrary. One demands that \(\Psi(\mathbf{r})\) should be zero above \(T_c\) and take a uniform non-zero value for \(T < T_c\) leading to the following equalities for the coefficients \(\alpha(T), \beta(T)\) and \(\gamma(T)\) \[49\]

\[
\alpha(T) \equiv \alpha_0(T - T_c),
\]

\[
\beta(T) \equiv \beta(T_c) \equiv \beta,
\]

\[
\gamma(T) \equiv \gamma(T_c) \equiv \gamma.
\]

To obtain the total free energy we must integrate this over the system \[49, 51\]

\[
F_s(T) = F_n(T) + \int \left(\alpha(T)|\Psi(\mathbf{r})|^2 + \frac{1}{2}\beta|\Psi(\mathbf{r})|^4 + \gamma|\nabla\Psi(\mathbf{r})|^2\right) d\mathbf{r}.
\]

According to Eq. \[5\] the free energy is a functional of the scalar functions \(\Psi(\mathbf{r})\) and \(\Psi^*(\mathbf{r})\). In order to find the order parameter \(\Psi(\mathbf{r})\) we must minimize the total free energy of the system. The condition for the minimum free energy is found by performing a functional differentiation with respect to the above functions that is to solve the following two equations

\[
\frac{\delta F_s[\Psi(\mathbf{r}), T]}{\delta \Psi(\mathbf{r})} = 0, \quad \frac{\delta F_s[\Psi(\mathbf{r}), T]}{\delta \Psi^*(\mathbf{r})} = 0.
\]

The above conditions can be satisfied only when \(\Psi(\mathbf{r})\) obeys

\[
- \gamma \nabla^2 \Psi(\mathbf{r}) + \left(\alpha + \beta|\Psi(\mathbf{r})|^2\right) \Psi(\mathbf{r}) = 0.
\]

Equation \[7\] has several applications including the properties of the surfaces and interface of superconductors. Following Refs. \[50, 51\] we consider a simple model for the interface between a normal metal and a superconductor. The interface lies in the \(yz\) plane separating the normal metal in the \(x < 0\) region from the superconductor in the \(x > 0\) region. On the normal metal side of the interface the superconducting order parameter \(\Psi(\mathbf{r})\) must be zero. Now, assuming that \(\Psi(\mathbf{r})\) must be continuous, then the following one dimensional nonlinear type Schrödinger equation must be solved,

\[
- \gamma \frac{d^2\Psi(x)}{dx^2} + \alpha(T)\Psi(x) + \beta\Psi^3(x) = 0
\]
in the region $x > 0$ with the boundary condition $\Psi(0) = 0$, eq. (8) can be solved analytically with the result

$$
\Psi(x) = \Psi_0 \tanh \left( \frac{x}{\sqrt{2} \xi(T)} \right).
$$

(9)

Fig. 1 sketches the spatial variation of the order parameter $\Psi(x)$ at the interface between a normal and superconducting metal in the more general case of the presence of a magnetic field. The effective penetration depth $\lambda_{\text{eff}}$ of the magnetic field is also displayed (for more details see Ref. [50]). In Eq. (9) $\Psi_0$ is the value of the order parameter in the bulk far from the surface and the parameter $\xi(T)$ is defined as

$$
\xi(T) = \left( \frac{\gamma}{a(T)} \right)^{1/2} = \left( \frac{\gamma}{a_0} \right)^{1/2} \frac{1}{\sqrt{T_c - T}}.
$$

(10)

Considering that $\xi(0) \equiv \xi_0$ is the value of $\xi$ for $T = 0$ the above relation is rewritten as

$$
\xi(T) = \xi_0 \frac{1}{\sqrt{1 - \frac{T}{T_c}}}, \quad \xi_0 = \left( \frac{\gamma}{a_0 T_c} \right)^{1/2}.
$$

(11)

The quantity $\xi$ has dimensions of length and is known as the Ginzburg-Landau coherence length or healing length. The physical significance of this length, in the condensed phase, is that it is a measure of the minimum distance over which one can "bend" the order parameter either in magnitude or in phase, before the bending energy becomes comparable to the condensation energy [19]. The coherence length plays for the superconducting systems a role closely analogous to the healing length in a dilute Bose condensate [49] and arises in all problems related with inhomogeneous superconductors, including surfaces, interfaces, defects and vortices [51].

3 The Fisher information measure as a functional of the order parameter

The next step is to construct a bridge to connect the order parameter $\Psi(r)$ with the Fisher information measure. At this point it will be helpful to follow the related discussion of Leggett [49]. According to this analysis Ginzburg and Landau guess that the order parameter $\Psi(r)$ has the nature of a macroscopic wave function. Actually, according to BCS theory it is indeed (up to normalization) the center-of mass wave function of the Cooper pairs. It is characterized also macroscopic in the sense that it is that unique eigenfunction of the two-particle density matrix which is associated with a macroscopic eigenvalue [49].

According to the above analysis one can proceed one more step by considering that (since $\Psi(r)$ is in general a complex quantity) the function $|\Psi(r)|^2$ represents a probability distribution for the Cooper pairs in a superconductors [51]. Moreover, in the present specific problem we consider that a suitable choice of the probability distribution $P(x)$ involved in Eq. (1) is the following

$$
P(x) = \frac{1}{\sqrt{2}\xi} \left( 1 - \left( \frac{\Psi(x)}{\Psi_0} \right)^2 \right) = \frac{1}{\sqrt{2}\xi} \left( 1 - \tanh^2 \left( \frac{x}{\sqrt{2}\xi} \right) \right).
$$

(12)

The probability distribution $P(x)$ as a function of the distance $x$ is displayed in Fig. 2(a). Since $\int_0^\infty |\Psi(x)|^2 dx$ diverges, (the number of particles is not conserved in superconductivity) we employ an associated probability distribution $P(x)$ to describe the transition from metal to the superconducting phase (the interface region). The proposed distribution firstly ensures the convergence of the integral of $P(x)$ and secondly satisfies the normalization condition $\int_{x=0}^\infty P(x) dx = 1$. Moreover, $P(x)$ given by (12) incorporates all the characteristics of the order parameter $\Psi(x)$. Obviously, $P(x)$ has the dimension of inverse length and consequently the FIM has the dimension of inverse length squared. We comment on alternative, possible choices of $P(x)$ in section 6.

Now, the Fisher information measure, defined in Eq. (1), by employing the probability distribution (12), is given by

$$
I_F = \frac{4}{\sqrt{2}\xi} \int_0^\infty \left( \frac{\tanh(x)}{\cosh(x)} \right)^2 dx.
$$

(13)

The integral is easily calculated and gives the factor $1/3$. In total, the FIM exhibits the following analytical and very simple relation with the coherence length $\xi$

$$
I_F = \frac{2}{3} \frac{1}{\xi^2}.
$$

(14)
The above dependence is displayed in Fig. 2(b). This result confirms both qualitatively and quantitatively the statement that FIM is a measure of the state of the disorder of a system. In particular, the smaller $\xi$ (fast recovery of the order parameter to its bulk value), the higher is the value of the Fisher measure. Combining (11) and (14), we obtain $I_F$ as a function of $T/T_c$ (see Fig. 2(c) and also Fig. 3).

It is of interest to examine also the dependence of FIM on other physical quantities related with the phase transition. Thus from Eqs. (11) and (14) we get its temperature dependence:

$$I_F = \frac{2\alpha_0}{3\gamma}(T - T_c). \quad (15)$$

The FIM attains its maximum value $I_F^{\text{max}} = -2\alpha_0 T_C/3\gamma$ at temperature $T = 0$, that is for $\xi = \xi_0$, while it becomes zero when the superconducting phase disappears (for $T \geq T_c$). It is worth to point out that the FIM is connected with some experimental quantities specifically the critical temperature $T_c$ as well as the values of $\xi(0) = \xi_0$. In particular, the zero temperature value of $\xi$ in BCS theory is related with the size of a single Cooper pair [51]. In Figs. 2(c) we display also the linear dependence of the ratio $I_F/I_F^{\text{max}}$ on $T/T_c$. Hence, in a way, the Fisher information is an alternative measure of the extent of the interface between a normal metal and a superconductor.

Moreover, the Fisher information related with the free energy characterizes a superconductor. In particular, the surface free energy $F_{\text{surf}}(T)$ is the difference between the condensation free energy $F_s(T) - F_0(T)$ defined in Eq. (15) (a measurable quantity [49]) and the bulk free energy density $F_{\text{bulk}}(T)$ which corresponds to an uniformly superconducting medium in the region from $x = 0$ to $x = \infty$. It was found that [50, 51]

$$F_{\text{surf}}(T) = \frac{4\sqrt{2}}{3}F_{\text{bulk}}(T)\xi(T). \quad (16)$$

Now, by combining Eqs. (14) and (16) we obtain the relation

$$I_F = \left(\frac{4}{3}\right)^3 \left(\frac{F_{\text{surf}}}{F_{\text{bulk}}}\right)^{-2} \quad (17)$$

Eq. (17) exhibits a simple relation between the FIM and the energy profile of superconductivity. Obviously, the Fisher information is a measure of the surface energy (in units of the bulk energy). In Fig. 3 we plot the Fisher measure as a function of temperature for four materials. We observe a strong dependence of $I_F$ on $\xi_0$. In any case, it is remarkable to connect directly a theoretical quantity $I_F$ with two experimentally measured quantities $T_c$ and $\xi_0$.

4 Shannon entropy and statistical complexity. Generalized information measures

It is instructive to calculate the Shannon entropy [3], and LMC statistical complexity [47, 48]. In particular, the Shannon entropy defined as

$$S = \int s(x)dx = - \int P(x)\ln P(x)dx, \quad (18)$$

where $s(x) = -P(x)\ln P(x)$ is the Shannon information density. Using the probability distribution (12), $S$ exhibits the following dependence on $\xi$

$$S = 2 + \ln \left(\frac{\xi}{2^{4/3}}\right). \quad (19)$$

Relations (11) and (14) provide the temperature dependence of $S$, shown in Fig. 4. $S$ is a measure of the information content stored in a quantum system described by $P(x)$. The units of $S$ are nats for a natural logarithm or bits, if the base of the logarithm is 2.

The disequilibrium $D$ defined as $D = \int [P(x)]^2dx$ is given by

$$D = \frac{\sqrt{2}}{3\xi} \quad (20)$$

$D$ is the disequilibrium of the system i.e. the distance from its actual state to equilibrium. The quantity $D$ is an experimentally measurable quantity i.e. in quantum chemistry [52] known as quantum self-similarity [53, 54] or information energy [55] or linear entropy [57, 58]. The dependence of $D$ on $T$ is plotted in Fig. 5.

The statistical measure of complexity $C$ introduced by Lopez-Ruiz, Calbet and Mancini (LMC) [48] employs the information entropy $S$ or information stored in a system and its distance $D$ to the equilibrium probability
distribution as the two basic ingredients giving the correct asymptotic properties of a well-behaved measure of complexity. The LMC measure is easily computable (in contrast with other definitions e.g. algorithmic complexity [29, 60] defined as the length of the shortest possible program necessary to reproduce a given object), defined as a product \( C = S \cdot D \). As expected intuitively it vanishes for the two extreme cases of a perfect crystal (perfect order) and ideal gas (complete disorder). An investigation of \( C \) in a quantum many-body system was carried out in atoms, for continuous electron distributions [61] and discrete ones [62]. An alternative definition of complexity is the SDL measure \( \Gamma_{\alpha \beta} [47] \) (Shiner, Davison, Landsberg) defined and calculated in a similar way as \( C \). It has been applied in atoms starting from [63]. A welcome property of a definition of complexity might be the following: If one complicates the system by varying some of its parameters, and this leads to an increase of the adapted measure applied in atoms starting from [63]. A welcome property of a definition of complexity might be the following: If one complicates the system by varying some of its parameters, and this leads to an increase of the adapted measure, then one could argue that this measure describes the complexity of the system properly. A detailed discussion of the physical meaning of \( C \) can be found in [61], section 4.

It is noted that \( C \) cannot be measured experimentally, but it is possible to calculate it, starting from a reasonable probabilistic definition (e.g. LMC [48] or SDL [47]) by using an information-theoretical method, developed in previous work see e.g. [61, 62, 63].

The LMC complexity \( C = S \cdot D \) is a product of two global information quantities \( S \) and \( D \). It is appropriate to examine \( \hat{C} = S \cdot I_F \) as well which is defined as a product of one global quantity \( S \) by a local one, specifically the Fisher information \( I_F \). Finally, we find that the statistical complexities \( C \) and \( \hat{C} \) are given by

\[
C = \frac{\sqrt{2}}{3\xi} \left[ 2 + \ln \left( \frac{\xi}{2^{3/2}} \right) \right],
\]

\[
\hat{C} = \frac{2}{3\xi^2} \left[ 2 + \ln \left( \frac{\xi}{2^{3/2}} \right) \right].
\]

The dependence of \( C \) on \( T \) is seen in Fig. 6(a) while \( \hat{C} \) is shown in Fig. 6(b). The calculated values of \( I_F, C \) and \( D \), follow the same trend as a function of \( Z \) compared with experimental \( T_c \) (see Fig. 7). The only exception is the inverse behavior of \( S \). Although \( S \) and \( D \) are reciprocal, their product (LMC complexity) \( C = S \cdot D \) shows a trend similar to \( T_c \). This is expected according to physical intuition: Larger (smaller) values of \( T_c \) correspond to larger (smaller) complexity \( C, D, I_F \). In the present work we find that this holds for the interface metal-superconductor.

It is worth to extend also our study by including a generalization of the Shannon Entropy and Fisher measure in the framework of the nonextensive statistical mechanics. This is based on the generalization of the definition of the logarithm according to the expression [41]

\[
\ln_q x = \frac{x^{1-q} - 1}{1-q}
\]

In view of the above definition, the Fisher information \( I_q \) and the Tsallis entropy \( T_q \) (a generalization of the Shannon entropy) are defined as

\[
I_q = \int P(x) \left( \frac{d \ln_q P(x)}{dx} \right)^2 dx = \int |P(x)|^{1-2q} \left( \frac{dP(x)}{dx} \right)^2 dx
\]

and

\[
T_q = \int P(x) \ln_q \left( \frac{1}{P(x)} \right) dx = \frac{1}{q-1} \left( 1 - \int |P(x)|^q dx \right)
\]

Our results are

\[
I_q = \frac{2^{-3+q}\xi^2q^{-4}}{(2q-5)(2q-3)\Gamma (\frac{4}{2q} - 2q)} \times \left( -\frac{\sqrt{\pi}(q-1)\Gamma (6-2q)}{q-2} + 2 \Gamma \left( \frac{9}{2} - 2q \right) \left[ 5 - 2q + (3 - 2q)2F_1 (1, -2 + 2q, 6-2q, -1) \right] \right)
\]

and

\[
T_q = \frac{1}{q-1} \left( 1 - \frac{\Gamma(q)}{\Gamma(1+q)} 2^{\frac{q}{2}(3q-1)} \xi^{1-q} F_1 (q, 2q, 1 + q, -1) \right)
\]

where \( \Gamma(x) \) is the usual gamma function and and \( _2F_1 (a, b, c, x) \) is the corresponding hypergeometric function. The statistical complexity, in the framework of the nonextensive statistical mechanics, is given by

\[
C_q = D_q \cdot T_q = \frac{\sqrt{2}}{3\xi} \frac{1}{q-1} \left( 1 - \frac{\Gamma(q)}{\Gamma(1+q)} 2^{\frac{q}{2}(3q-1)} \xi^{1-q} F_1 (q, 2q, 1 + q, -1) \right).
\]
It is noted that Eqs. (26) and (27) for $q \to 1$ converge to the corresponding ones (14) and (19) respectively. In Fig. 9 we display the Tsallis entropy $T_q$ as a function of temperature for Sn for various values of $q$. Previously, $T_q$ has been employed in more sophisticated attempts to estimate an optimal value of $q(q \approx 1)$ in superconductivity [64, 65]. It was found that small deviations with respect to $q = 1$ i.e. $q \approx 1.005$ provide better agreement with experimental results. A value of $q \approx 1$ can also be inferred with the present model from Fig. 9 by inspection. We present only the plot of $T_q$, while the other quantities $I_F$ and $C_q$ lead to the same approximate conclusion about $q$.

5 Analytical relation between Shannon and Fisher information densities

Under the condition, for systems such as atoms and molecules whose asymptotic density decays strongly, Liu et al. [34, 35] obtained a specific relationship between their respective densities, $s(r)$, $\rho(r)$, $i_F(r)$, $i’_F(r)$ and consequently of the integral forms of $S$ and $I_F$

$$s(r) = -\rho(r) + \frac{1}{4\pi} \int \frac{i_F(r')}{|r - r'|} dr' - \frac{1}{4\pi} \int \frac{i'_F(r')}{|r - r'|} dr'. \quad (29)$$

Integration of both sides gives

$$S[\rho] = -N + \frac{1}{4\pi} \int \int \frac{i_F(r')}{|r - r'|} dr' dr - \frac{1}{4\pi} \int \int \frac{i'_F(r')}{|r - r'|} dr' dr. \quad (30)$$

The above equations are identities. Here

$$i_F(r) = \frac{\nabla \rho(r) \cdot \nabla \rho(r)}{\rho(r)} \quad (31)$$

and

$$i'_F(r) = -\nabla^2 \rho(r) \ln \rho(r) dr. \quad (32)$$

We also have

$$I_F = \int i_F(r) dr \quad (33)$$

$$I'_F = \int i'_F(r) dr. \quad (34)$$

The densities $i_F(r)$, $i'_F(r)$ are different functions of the vector $r$, but their integrals are equal, as expected. For a strongly decaying local quantity $q(r)$ the authors of Ref. [34, 35] state that $q(r)$ should decay faster than $1/|r|$ and its derivative faster than $1/r^2$ as a requirement for their identities.

Liu’s identity can be written in compact form:

$$S = -N + I_F + I'_F \quad (35)$$

quite a remarkable decomposition of a global information quantity $S$ in terms of (two) local ones, $I_F$ and $I'_F$.

In our present work we show that an analogous relation holds for the special case of the one-dimensional density $P(x)$ and verify it in superconductivity. Again, we checked that $P(x)$ satisfies the same criterion of strong decay for $x \to \infty$. In particular we start from the identity

$$q(x) \equiv q(0) + \int_0^x \frac{d^2 q(x')}{dx'^2} (x - x') dx' \quad (36)$$

which holds under the condition

$$\left(\frac{dq(x)}{dx}\right)_{x=0} = 0. \quad (36)$$

Now, we define the local Shannon information density

$$s(x) = -P(x) \ln P(x) \quad (37)$$

which obviously leads to the Shannon entropy

$$S = \int_0^\infty s(x) dx = - \int_0^\infty -P(x) \ln P(x) dx. \quad (38)$$
Now, according to the identity (36) we have also

\[ s(x) = s(0) + \int_0^x \frac{d^2 s(x')}{dx'^2} (x-x')dx' \]  

(39)

and using Eq. (37) we obtain

\[
\begin{align*}
    s(x) &= s(0) - P(x) + P(0) - \int_0^x \frac{d^2 P(x')}{dx'^2} \ln P(x')(x-x')dx' - \int_0^x \frac{1}{P(x')} \left( \frac{dP(x')}{dx'} \right)^2 (x-x')dx' \\
    &= -P(x) + P(0) - \int_0^x \frac{1}{P(x')} \left( \frac{dP(x')}{dx'} \right)^2 (x-x')dx' \\
    &+ s(0) - \int_0^x \frac{d^2 P(x')}{dx'^2} \ln P(x')(x-x')dx' = -P(x) + I_{1f}(x) + I_{2f}(x)
\end{align*}
\]

(40)

where

\[
I_{1f}(x) = P(0) - \int_0^x \frac{1}{P(x')} \left( \frac{dP(x')}{dx'} \right)^2 (x-x')dx' 
\]

(41)

\[
I_{2f}(x) = s(0) - \int_0^x \frac{d^2 P(x')}{dx'^2} \ln P(x')(x-x')dx'
\]

(42)

Finally, integrating Eq. (40) over \( x \) we get

\[ S = -1 + I_{F1} + I_{F2} \]  

(43)

where

\[
I_{1F} = \int_0^\infty I_{1f}(x)dx 
\]

(44)

\[
I_{2F} = \int_0^\infty I_{2f}(x)dx 
\]

(45)

Equation (43) is analogous with the Liu identity (35) for \( N = 1 \), rewritten for the one dimensional case.

Table 1: Experimental values of the coherence length \( \xi \) [66], the Shannon information entropy \( S \), the terms \( I_{1F} \) and \( I_{2F} \) and the sum \(-1 + I_{1F} + I_{2F}\) in order to confirm relation (43).

| Atom | \( Z \) | \( \xi_0 \) (nm) | \( S \) | \( I_{1F} \) | \( I_{2F} \) | \(-1 + I_{1F} + I_{2F}\) |
|------|-------|------------------|-------|----------|----------|------------------|
| Al   | 13    | 1600             | 8.338 | -13.337  | 22.675   | 8.338            |
| Nb   | 41    | 38               | 4.598 | -828.154 | 833.752  | 4.598            |
| In   | 49    | 360              | 6.846 | -207.527 | 215.374  | 6.846            |
| Sn   | 50    | 230              | 6.398 | -551.682 | 559.080  | 6.398            |
| Ga   | 64    | 760              | 7.594 | -26.265  | 34.859   | 7.594            |
| Ta   | 73    | 93               | 5.493 | -288.979 | 295.471  | 5.493            |
| Pb   | 82    | 83               | 5.379 | -370.474 | 376.853  | 5.379            |

6 Alternative probability distributions

The question naturally arises if instead of \( P(x) \), relation (12), we may consider different, alternative definitions, repeat all our calculations and examine the consequence on our results, both qualitatively and quantitatively. First we see that the integral

\[ \int_0^\infty \left| \frac{\Psi(x)}{\Psi_0} \right|^2 dx = \int_0^\infty \tanh^2 \left( \frac{x}{\sqrt{2\xi}} \right) dx \]

diverges. We may try a cutoff, i.e. integrate from 0 to \( n\xi \), where \( n = 1, 2, 3, \ldots \). We note that it is not possible to know in advance or a priori the correct upper limit if such a limit does exist. We see that for \( n = 5 \), \( x = 5\xi \):

\[ \tanh \left( \frac{x}{\sqrt{2\xi}} \right) \approx 0.9983 \]
for any material. Hence we define

\[ \tilde{P}(x) = \mathcal{N} \left( \frac{\Psi(x)}{\Psi_0} \right)^2 = \mathcal{N} \tanh^2 \left( \frac{x}{\sqrt{2\xi}} \right) \]  

(46)

The normalization condition \( \int_{-\infty}^{\infty} \tilde{P}(x) dx = 1 \) leads to

\[ \mathcal{N} = \left[ \sqrt{2\xi} \left( \frac{n}{\sqrt{2}} - \tanh \left( \frac{n}{\sqrt{2}} \right) \right) \right]^{-1} \]  

(47)

The Shannon entropy and the Fisher information take the analytical forms

\[
S = -\ln(\mathcal{N}) - \mathcal{N} \xi \left[ 4n \text{arccoth} \left( e^{\sqrt{2n}} \right) + n \ln \left( \tanh^2 \left( \frac{n}{\sqrt{2}} \right) \right) - \sqrt{2} \, \text{Li}_2 \left( -e^{-\sqrt{2n}} \right) \right. \\
+ \left. \sqrt{2} \, \text{Li}_2 \left( e^{-\sqrt{2n}} \right) - \sqrt{2} \left( -2 + \ln \left( \tanh^2 \left( \frac{n}{\sqrt{2}} \right) \right) \right) \tanh \left( \frac{n}{\sqrt{2}} \right) - \frac{\pi^2}{2\sqrt{2}} \right] \\
I_F = \mathcal{N} \frac{\sqrt{2}}{3\xi} \operatorname{sech}^3 \left( \frac{n}{\sqrt{2}} \right) \left[ 3 \sinh \left( \frac{n}{\sqrt{2}} \right) + \sinh \left( \frac{3n}{\sqrt{2}} \right) \right]
\]  

(48)

(49)

where \( \text{Li}_n(x) \) is the dilogarithm (or Spence’s function) which is defined for \(|x| \leq 1 \) either a) by a power series in \( x \) i.e. \( \text{Li}_2(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^2} \) or b) by the integral i.e. \( \text{Li}_2(x) = -\int_0^x \ln(1-u)/udu \). We choose \( n = 5 \) (integration from 0 to 5\( \xi \)), we calculate again \( S, I_F, D \) and \( C \) and present our results for various materials in Table 3 and plot in Fig. 8. We see that the absolute values of the relevant quantities depend on the choice of \( n \), as expected and are different than Table 2. However, the qualitative trend of our calculated quantities as function of \( Z \) is the same that is the zig zag pattern in Figs 7 and 8. In addition, we mention that the absolute values of information quantities are not important, since \( \ln \xi \) is known up to an additive constant \( c \) (see relation (19)). For example, if one wishes to change or eliminate units of \( \xi \) divides \( \xi \) inside the logarithm \( \xi \) by \( c \). Accordingly Fisher information is known up to a multiplicative factor (see relation (14)). Thus, we select to employ our definition \( P(x) \) of relation (12), in order to avoid the ambiguity of a cutoff. This choice may also be justified by Occam’s razor, though falsifiability is the ultimate criterion of the quality of a model in physical science. An additional merit of \( P(x) \) is that it enables us to use an infinite integration interval leading to a direct demonstration of Liu’s identity. The fact that for two different distributions \( P(x) \) and \( \tilde{P}(x) \) we get the same behavior corroborates the general property that the normalization of \( \Psi(x) \) is arbitrary. It is enough to carry out consistent calculations for information and complexity.

7 Concluding remarks

Concluding, we note that the Fisher information is a qualitative as well as a quantitative measure of the superconducting phase between a normal metal and a superconductor. It shows a simple analytical dependence on experimentally measurable quantities like the correlation length \( \xi \) and consequently the critical temperature \( T_c \).

Some useful insights have been obtained by calculating a few information and complexity measures. Our formalism allows us to verify an identity derived previously by Liu et al. [31, 33] shown to hold for atoms and molecules. Thus we are able to decompose a global information measure (Shannon) in terms of two local ones (Fisher) for the metal-superconductor interface. Liu’s identity is an example of a relationship between Fisher information (local) with Shannon entropy (global) for a specific class of continuous probability distributions (asymptotic behavior). In fact, Fisher information may be considered as the limiting form of many different measures of information or in the words of B. Roy Frieden (Ref. [2] p.39) it is a kind of ”mother” information. To justify this statement one can show that \( I_F \) is the cross-entropy between a \( P(x) \) and its infinitesimally shifted version \( P(x + \Delta x) \). In a sense, \( I_F \) more generally results as a ”cross-information” between \( P(x) \) and \( P(x + \Delta x) \) for a host of different types of information measures (Ref. [68]).

A planned future work will extend our study to include other phases of superconductors in connection with the Fisher information measure (including surfaces, defects, vortices e.t.c). A final comment seems appropriate, quoting [11]. It is difficult to quantify complexity, a context dependent and multi-faceted concept. Choosing a pragmatic approach, we employ as a starting point a definition of complexity based on a probabilistic description of a quantum system, in the present case a metal-superconductor interface.

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Table 2: Experimental values of the coherence length $\xi_0$ [66] and the critical temperature $T_c$ [67] for various materials in comparison with calculated information and complexity measures including the Shannon entropy $S$, the Fisher measure $I_F$, the Disequilibrium $D$ and the LMC Complexity $C$. The probability distribution $P(x)$ is used, relation (12).

| Atom | $Z$ | $\xi_0$ (nm) | $T_c$(0K) | $S$ | $I_F$ ($\times 10^{-5}$) | $D$ ($\times 10^{-4}$) | $C$ ($\times 10^{-3}$) |
|------|----|--------------|-----------|-----|----------------|----------------|----------------|
| Al   | 13 | 1600         | 1.175     | 8.338 | 0.0260       | 2.946         | 2.471          |
| Nb   | 41 | 38           | 9.25      | 4.598 | 46.168       | 124.054       | 5.704          |
| In   | 49 | 360          | 3.41      | 6.846 | 0.514        | 13.095        | 8.965          |
| Sn   | 50 | 230          | 3.72      | 6.398 | 1.260        | 20.496        | 13.113         |
| Ga   | 64 | 760          | 1.083     | 7.594 | 0.115        | 6.200         | 4.708          |
| Ta   | 73 | 93           | 4.47      | 5.493 | 7.700        | 50.680        | 27.840         |
| Pb   | 82 | 83           | 7.2       | 5.379 | 9.670        | 56.790        | 30.547         |

Table 3: The same as in Table 2, using the probability distribution $\tilde{P}(x)$, relation (46).

| Atom | $Z$ | $\xi_0$ (nm) | $T_c$(0K) | $S$ | $I_F$ ($\times 10^{-5}$) | $D$ ($\times 10^{-4}$) | $C$ ($\times 10^{-3}$) |
|------|----|--------------|-----------|-----|----------------|----------------|----------------|
| Al   | 13 | 1600         | 1.175     | 8.339 | 0.0205       | 1.514         | 1.338          |
| Nb   | 41 | 38           | 9.25      | 5.099 | 36.400       | 63.750        | 32.506         |
| In   | 49 | 360          | 3.41      | 7.347 | 0.405        | 6.730         | 4.945          |
| Sn   | 50 | 230          | 3.72      | 6.899 | 0.993        | 10.533        | 7.267          |
| Ga   | 64 | 760          | 1.083     | 7.594 | 0.091        | 3.188         | 2.581          |
| Ta   | 73 | 93           | 4.47      | 5.994 | 6.076        | 26.050        | 15.614         |
| Pb   | 82 | 83           | 7.2       | 5.880 | 7.628        | 29.188        | 17.163         |

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Figure 1: A schematic picture for the spatial variation of the order parameter $\Psi(x)$ at the interface between a normal and superconducting metal in the presence of a magnetic field. The effective penetration depth $\lambda_{\text{eff}}$ is also displayed. For more details see text and Ref. [50].

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Figure 2: (a) The probability distribution function \( P(x_R) = \sqrt{2\xi} P(x) \) as a function of the distance ratio \( x_R = x/\sqrt{2\xi} \); (b) the Fisher’s information measure \( I_F \) (in units of \( I_F^{max} \)) as a function of the coherence length ratio \( R = \xi/\xi_0 \) and (c) of the temperature ratio \( T/T_c \).

Figure 3: The Fisher measure as a function of temperature for four materials.

Figure 4: Temperature dependence of the Shannon information \( S \) for four materials.
Figure 5: Temperature dependence of the Disequilibrium $D$ for four materials.

Figure 6: a) The LMC complexity $C = S \cdot D$ as a function of temperature for four materials. (b) The complexity $\tilde{C} = S \cdot I_F$ as a function of temperature for four materials.
Figure 7: The critical temperature $T_c$ and the calculated values of LMC complexity $C$, the Fisher measure $I_F$, the Shannon entropy $S$ and the disequilibrium $D$ as functions of the atomic number $Z$ for various materials (see also Table 2). We use the probability distribution $P(x)$, relation (12).

Figure 8: The same as in Fig. 7, but here we use the probability distribution $\tilde{P}(x)$, relation (46) (see also Table 3).
Figure 9: The Tsallis entropy as a function of the temperature, for Sn and $0.95 < q < 1.03$. 