GROTHENDIECK CATEGORIES AND THEIR TENSOR PRODUCT AS FILTERED COLIMITS

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ABSTRACT. We present two ways of recovering a Grothendieck category as a filtered colimit of small categories by means of the construction of the (2-)filtered (bi)colimit of categories from [9]. The first one, making use of the fact that Grothendieck categories are locally presentable, allows to recover a Grothendieck category as a filtered colimit of its subcategories of \( \alpha \)-presentable objects, for \( \alpha \) varying in the family of small regular cardinals. The second one, making use of the fact that Grothendieck categories are precisely the linear topoi, permits to recover a Grothendieck category as a filtered colimit of its linear site presentations. We then show that the tensor product of Grothendieck categories from [18] can be recovered as a filtered colimit of Kelly’s \( \alpha \)-cocomplete tensor product of the categories of \( \alpha \)-presentable objects with \( \alpha \) varying in the family of small regular cardinals. We use this construction to translate the functoriality, associativity and symmetry of Kelly’s tensor product to the tensor product of Grothendieck categories.

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1. INTRODUCTION

We fix a commutative ring \( k \) for the rest of the article. The Gabriel-Popescu theorem shows that Grothendieck \( k \)-linear categories are precisely the \( k \)-linear topoi [15]. Making use of this perspective, a tensor product of Grothendieck categories is defined in [18] based upon their representations as categories of linear sheaves. In particular, this tensor product is shown to be an instance of the tensor product of locally presentable \( k \)-linear categories \( \otimes_{LP} \) [18, Thm 5.4]. The 2-category of locally presentable \( k \)-linear categories endowed with \( \otimes_{LP} \) is a closed symmetric monoidal bicategory in the sense

The author is a Postdoctoral Fellow of the Research Foundation - Flanders (FWO). She acknowledges as well the support of the Research Foundation Flanders (FWO) under Grant No. G.0112.13N during the time in which most of the results of this paper were obtained.
of [10] (see [8, Lem 2.7], [12, §6.5], [3, Exerc 1.1]), with the inner hom given by the co-
continuous \( k \)-linear morphisms. More precisely, given \( \mathcal{A}, \mathcal{B}, \mathcal{C} \) locally presentable \( k \)-linear categories, we have the universal property

\[
\text{Cocont}(\mathcal{A} \boxtimes_{LP} \mathcal{B}, \mathcal{C}) \cong \text{Cocont}(\mathcal{A}, \text{Cocont}(\mathcal{B}, \mathcal{C}))
\]

(1)

in the 2-category of locally presentable \( k \)-linear categories, where \( \text{Cocont} \) denotes the \( k \)-linear cocontinuous functors. Observe that from the universal property one can easily deduce the associativity, symmetry and functoriality with respect to cocontinuous functors of \( \boxtimes_{LP} \) (up to equivalence of categories). This immediately endows the tensor product of Grothendieck categories with the same nice properties.

On the other hand, the tensor product of locally presentable \( k \)-linear categories and the tensor product of Grothendieck \( k \)-linear categories are closely related to Kelly’s tensor product of \( \alpha \)-cocomplete \( k \)-linear categories, for any regular cardinal \( \alpha \) [12], [11]. This relation is provided by Gabriel-Ulmer duality. More precisely, given \( \mathcal{A}, \mathcal{B} \) two \( \alpha \)-locally presentable \( k \)-linear categories, the subcategory of \( \alpha \)-presentable objects \( (\mathcal{A} \boxtimes_{LP} \mathcal{B})^\alpha \) of their tensor product is given by Kelly’s tensor product \( \mathcal{A}^\alpha \boxtimes_{P} \mathcal{B}^\alpha \) of the corresponding \( k \)-linear subcategories of \( \alpha \)-presentable objects. The category \( \text{Cat}^\alpha(k) \) of small \( \alpha \)-cocomplete \( k \)-linear categories endowed with Kelly’s tensor product \( \boxtimes_{P} \) is proved in [12, §6.5] to be a closed symmetric monoidal bicategory in the sense of [10], with the inner hom given by the \( \alpha \)-cocontinuous \( k \)-linear functors. More precisely, given \( a, b, c \) in \( \text{Cat}^\alpha(k) \), we have the universal property

\[
\text{Cocont}_{P}(\alpha \boxtimes_{P} b, c) \cong \text{Cocont}_{P}(a, \text{Cocont}_{P}(b, c))
\]

(2)

in \( \text{Cat}^\alpha(k) \), where \( \text{Cocont}_{P} \) denotes the \( \alpha \)-cocontinuous \( k \)-linear functors.

The main aim of this work is to show that the functoriality, associativity and symmetry of the tensor product of Grothendieck categories can be also derived from the same properties of Kelly’s tensor product, without reference to the universal property (1) on the level of the large categories. It is important to keep in mind that this result will allow us to make use of the functoriality of Kelly’s tensor product in order to compute tensor products of functors between Grothendieck categories, which provides an advantage when dealing with concrete examples.

Our key tool along the paper will be the construction of the \((2-)\)filtered (bi)colimit of categories from [9], which will allow to work with Grothendieck categories as filtered colimits of small categories. In first place, given the enriched nature of our framework, we show that the particular instance of 2-filtered bicolimits in which we will be interested, namely the filtered colimits (i.e. 2-filtered bicolimits where the indexing category is just a filtered 1-category), is well behaved with respect to the linear enrichment. More precisely, we prove the following.

**Proposition 1.1** (Proposition 2.5). Given \( J \) a filtered category and \( F : J \to \text{Cat} \) a pseudofunctor that factors through the 2-category \( \text{Cat}(k) \) of \( k \)-linear categories with \( k \)-linear functors and \( k \)-linear natural transformations, we have that the filtered colimit of \( F \) is in particular a \( k \)-linear category.

We will then provide two different ways of representing a Grothendieck category as a filtered colimit of small linear categories. The first one, generalizable to any locally presentable category, states the following.

**Proposition 1.2** (Corollary 4.3). Let \( \mathcal{C} \) be a locally presentable \( k \)-linear category. Then \( \mathcal{C} \) is a filtered colimit of its family of subcategories of locally \( \alpha \)-presentable objects \( (\mathcal{C}^\alpha)_{\alpha} \), where \( \alpha \) varies in the totally ordered set of small regular cardinals.

The second one makes use of the topos theoretical nature of Grothendieck categories. Given a \( k \)-linear Grothendieck category \( \mathcal{C} \), we consider category \( \beta_{\mathcal{C}} \) of all site
presentations of \( \mathcal{C} \), this is all the LC morphisms \((a, \mathcal{T}) \rightarrow \mathcal{C}\) from a \( k \)-linear site \((a, \mathcal{T})\) to \( \mathcal{C} \) (see Definition 3.11), and we show it is filtered (see Proposition 4.5). We then consider the functor \( G_\mathcal{C} : \mathcal{D} \rightarrow \text{Cat} \) assigning to each LC morphism its domain. We prove the following.

**Theorem 1.3** (Theorem 4.6). Given a Grothendieck category \( \mathcal{C} \), we have that \( \mathcal{C} \) is the \( k \)-linear filtered colimit of \( G_\mathcal{C} \).

Consider now \( \mathcal{C}, \mathcal{D} \) two locally \( k \)-presentable categories. Relying on Proposition 1.2 and the relation between Kelly’s tensor product and the tensor product of Grothendieck categories, one can show that the tensor product \( \mathcal{C} \otimes_{k} \mathcal{D} \) of locally presentable categories can be recovered as the filtered colimit of the family \((\mathcal{C} \otimes_{k} \mathcal{D})^\alpha\) of subcategories of \( \alpha \)-presentable objects, with the transition functors given by the natural embeddings \( \mathcal{C} \otimes_{k} \mathcal{D}^\alpha \subseteq \mathcal{C} \otimes_{k} \mathcal{D}^\beta \). However, observe that these transition functors cannot be immediately seen to be functorial with respect to Kelly’s tensor product. We show in §5, based upon the properties of LC morphisms, that if we restrict to Grothendieck categories, the tensor product can be obtained as a filtered colimit of Kelly’s tensor product of the subcategories of \( \alpha \)-presentable objects in a functorial way. Namely, we show the following.

**Theorem 1.4** (Theorem 5.4). Let \( \mathcal{C}, \mathcal{D} \) be two Grothendieck categories. The tensor product \( \mathcal{C} \otimes_{k} \mathcal{D} \) of Grothendieck categories can be expressed as the filtered colimit of the tensor products \((\mathcal{C} \otimes_{k} \mathcal{D})^\alpha\) of categories of \( \alpha \)-presentable objects, where \( \alpha \) takes values in the totally ordered set of small regular cardinals, and transition maps \( \mathcal{C} \otimes_{k} \mathcal{D}^\alpha \rightarrow \mathcal{C} \otimes_{k} \mathcal{D}^\beta \) given by those induced by the universal property of Kelly’s tensor product (2) by the canonical embeddings \( \alpha^a \subseteq \beta^a \) and \( \alpha^a \subseteq \beta^a \) for all \( \alpha \leq \beta \).

Finally, this result allows us in §6 to describe the functoriality of the tensor product of Grothendieck categories via Kelly’s tensor product of \( \alpha \)-cocomplete categories (see Definition 6.4). Moreover, we translate the associativity and symmetry of Kelly’s tensor product to the same properties of the tensor product of Grothendieck categories (see Proposition 6.6, Proposition 6.7), as desired.

**Acknowledgements.** This article presents and extends part of the work carried out by the author in her PhD thesis under the supervision of Wendy Lowen and Boris Shoikhet. I am very grateful to both of them for the interesting discussions and their helpful comments.

### 2. Generalities on the 2-filtered bicolimit of categories

Given a (pseudo)functor \( F : \mathcal{A} \rightarrow \text{Cat} \) where \( \mathcal{A} \) is a category and \( \text{Cat} \) denotes the 2-category of small categories, we can consider Grothendieck’s construction of the colimit category \( \lim_{\mathcal{A}} F \) [2, Exposé VI]. In particular, this construction can be performed when \( \mathcal{A} \) is a filtered category. In [9] a suitable generalization of the Grothendieck construction to the 2-categorical realm is provided for the filtered case, and referred to as 2-filtered bicolimit. In this section we provide a short overview of the 2-filtered bicolimit of loc.cit. More concretely, we focus on its construction for the particular 1-categorical case of a pseudofunctor \( F : \mathcal{A} \rightarrow \text{Cat} \) where \( \mathcal{A} \) is a filtered 1-category. In addition, we show that in such case, if the functor \( F \) factors through the 2-category of \( k \)-linear categories \( \text{Cat}(k) \), the filtered colimit is also \( k \)-linear.

We first fix some notations for the rest of the paper. Given a bicategory, we denote by \( \bullet \) the vertical composition of 2-morphisms and by \( \circ \) the horizontal composition of
2-morphisms, following the convention in [19]. In particular, given a diagram

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow \text{Id}_f & & \downarrow \alpha \\
B & \xleftarrow{g} & C
\end{array}
\]

in a bicategory C, we denote by \( \alpha \circ f \) to the horizontal composition \( \alpha \circ \text{Id}_f \).

We recall some important definitions.

**Definition 2.1.** Let \( F, G : A \to \mathcal{B} \) be two pseudofunctors between 2-categories \( A, \mathcal{B} \).

1. A **pseudonatural transformation** \( \Phi : F \Rightarrow G \) is given by a family of 1-morphisms
   \[
   (\Phi_A : F(A) \to G(A))_{A \in \text{Obj}(A)}
   \]
   and a family of invertible 2-morphisms
   \[
   (\Phi_f : \Phi(A') \circ F(f) \Rightarrow G(f) \circ \Phi(A))_{f : A \to A'}
   \]
   with the corresponding coherence laws.

2. Given two pseudonatural transformations \( \Phi, \Psi : F \Rightarrow G \), a morphism of pseudonatural transformations \( r \) between \( \Phi \) and \( \Psi \) is a modification, that is a family of 2-morphisms
   \[
   (r_A : \Phi_A \Rightarrow \Psi_A)_{A \in A}
   \]
   such that
   \[
   (G(f) \circ r_A) \circ \Phi_f = \Psi_f \circ (r_{A'} \circ F(f))
   \]
   for all \( f : A \to A' \) in \( A \).

We denote by \( \text{Psnat}(F, G) \) the category of pseudonatural transformations between \( F \) and \( G \), with morphisms given by the modifications (see [9] or [13]).

The notion of **2-filtered 2-category** is introduced in [9, §2] as a suitable generalisation in the 2-categorical realm of the classical notion of filtered category. In particular, as it is already mentioned in the introduction of [9], any (1-)category considered as a trivial 2-category is 2-filtered if and only if it is filtered as an ordinary category. Throughout this paper we will always use an indexing category which is of this latter type, hence we can safely avoid going through the technicalities of the construction of 2-filtered 2-category for a general indexing 2-category.

**Definition 2.2 ([9, Thm. 1.19]).** Given a pseudofunctor \( F : \mathcal{I} \to \text{Cat} \), where \( \mathcal{I} \) is a 2-filtered 2-category\(^1\), the **2-filtered bicolimit of** \( F \) is a category \( \mathcal{B} \) together with a pseudonatural transformation \( F \Rightarrow \mathcal{B} \) from \( F \) to the constant 2-functor \( \mathcal{I} \to \text{Cat} \) taking the value \( \mathcal{B} \) such that, for every category \( \mathcal{C} \), it induces via composition an isomorphism of categories

\[
\text{Cat}(\mathcal{B}, \mathcal{C}) = \text{Psnat}(F, \mathcal{C})
\]

between the category of functors \( \mathcal{B} \to \mathcal{C} \) and the category of pseudonatural transformations between the pseudofunctor \( F \) and the constant 2-functor taking the value \( \mathcal{C} \).

**Remark 2.3.** Note that such a category is uniquely determined up to a unique equivalence. We will denote it by \( \mathcal{L}(F) \), following the notations from [9].

**Remark 2.4.** Observe that the original definition (see [9, Thm. 1.19]) only considers 2-filtered bicolimits of \( F \) with \( F \) a strict 2-functor. For our purposes we need to consider the more general situation in which \( F \) is a pseudofunctor.

\(^1\)The bicolimit and its construction actually work when the indexing category is a *pre2-filtered 2-category*, which is a weaker notion than that of 2-filtered 2-category, as pointed out in the introduction of [9].
The main result of [9] is, given a 2-functor $F : J \to \text{Cat}$, the construction of the bicolimit $\mathcal{L}(F)$ in an intrinsic way in terms of the 2-functor $F$. One can observe that, when the indexing category is just an ordinary filtered (1-)category the construction is greatly simplified. For this choice of indexing category, one can easily extend the construction from [9] to the case in which $F$ is a pseudofunctor by means of a slight generalization of the results explained in loc.cit.\footnote{The general construction for $F$ a pseudofunctor should be possible in full generality, without restrictions of the indexing category, just by readjusting the notion of homotopy in [9, 1.5(iii)], as we have done in our particular case.}

We flesh out below the construction of the (bi)coterm for our particular situation, i.e. when

- $F : J \to \text{Cat}$ is not necessarily a strict 2-functor but a pseudofunctor,
- the indexing category $J$ is a filtered (1-)category.

**Description of the objects:**

Objects of $\mathcal{L}(F)$ are pairs $(x, A)$ where $A \in J$ and $x \in F(A)$.

**Description of the morphisms:**

First we describe the class of premorphisms:

- Consider two objects $(x, A)$ and $(y, B)$. A premorphism $(x, A) \to (y, B)$ consists of a triple $(u, f, v)$, where $u : A \to C$, $v : B \to C$ and $f : F(u)(x) \to F(v)(y)$ in $F(C)$. In order to make the object $C$ explicit in the notation, we will write $(u, f, v) : (x, A) \to (y, B)$.
- Two pre-morphisms $(u_1, f, v_1)_{C_1}, (u_2, g, v_2)_{C_2} : (x, A) \to (y, B)$ are said to be homotopical if there exists an object $C \in C$ and morphisms $w_i : C_i \to C$, for $i = 1, 2$ such that $w_1 \circ v_1 = w_2 \circ v_2, \quad w_1 \circ u_1 = w_2 \circ u_2$ and the following diagram commutes

$$
\begin{array}{ccc}
F(w_1) & \cong & F(w_2) \\
\downarrow & & \downarrow \\
F(u_1) & \cong & F(u_2)
\end{array}
$$

This relation is an equivalence relation and we denote the equivalence class of a premorphism $(u, f, v)_{C} : (x, A) \to (y, B)$ by $[(u, f, v)]_C$.

By means of the homotopy relation, we define the morphisms:

- **Morphisms** in $\mathcal{L}(F)$ between two objects are given by premorphisms between those two objects modulo homotopy.
- The identity morphism of an object $(x, A)$ in $\mathcal{L}(F)$ is given by $[(\text{Id}_A, \text{Id}_x, \text{Id}_A)]_A$.
- Given two morphisms

$$
[(u_1, f_1, v_1)]_C : (x, A) \to (y, B),
$$
$$
[(u_2, f_2, v_2)]_C : (y, B) \to (z, C),
$$

the composition $[(u_1, f_1, v_1)]_C \circ [(u_2, f_2, v_2)]_C$ in $\mathcal{L}(F)$ is given by the morphism $[(s_1 \circ u_1, f, s_2 \circ v_2)]$ for $s_i : C_i \to C$ for $i = 1, 2$ such that $s_1 \circ v_1 = s_2 \circ u_2$ and where the morphism $f : F(s_1 \circ u_1)(x) \to F(s_2 \circ v_2)(z)$ is defined as the following composition:

$$
f : F(s_1 \circ u_1)(x) \cong F(s_1) \circ F(u_1)(x) \xrightarrow{F(s_1) \circ f_1} F(s_1) \circ F(v_1)(y) \cong F(s_1 \circ v_1)(y) = F(s_2 \circ v_2)(y) \cong F(s_2) \circ F(u_2)(y) \xrightarrow{F(s_2) \circ f_2} F(s_2) \circ F(v_2)(z) \cong F(s_2 \circ v_2)(z)
$$
One can check that this is well-defined. We do not write the details, but essentially, the argument goes as follows. By choosing two different possible representatives of the composition, one can find natural candidates for a homotopy between them by using the fact that $A$ is filtered. In order to check that any of these natural choices is indeed a homotopy, one just needs to use the fact that the isomorphisms $F(g) \circ F(f) \Rightarrow F(g \circ f)$ are natural in both $f$ and $g$ for $f$ and $g$ composable morphisms in $\mathcal{J}$.

The category $\mathcal{L}(F)$ fulfills the universal property (3) above.

We are interested in $k$-linear categories and their 2-filtered bicorestrictions. More explicitly, we are interested in 2-filtered bicorestrictions of pseudofunctors $F : \mathcal{J} \rightarrow \text{Cat}$ that take values in the 2-category $\text{Cat}$ of $k$-linear categories with $k$-linear functors and $k$-linear natural transformations, that is, in functors $F : \mathcal{J} \rightarrow \text{Cat}$ that factor through the forgetful functor $\text{Cat} \rightarrow \text{Cat}$ defined and does not depend on the choice of $f$.

In addition, one can easily show that the composition is $k$-linear. To show this, consider two morphisms $[(u_1, f_1, v_1)_{C_1}, (u_2, f_2, v_2)_{C_2}] : (x, A) \rightarrow (y, B)$ and an element $\lambda \in k$, we define $[(u_1, f_1, v_1)_{C_1} + \lambda(u_2, f_2, v_2)_{C_2}] = [(u_1 \circ u_1, F(u_1)(f_1) + \lambda F(u_2)(f_2), v_1 \circ v_2)_{C_2}]$ where $u_1 : C_1 \rightarrow C$, $u_2 : C_2 \rightarrow C$ and $u_1 \circ u_1 = u_2 \circ u_2$ and $v_1 \circ v_2 = v_2 \circ v_2$. Observe that such $u_1, u_2$ exist because $J$ is a filtered category. An easy check shows that this is well-defined and does not depend on the choice of $u_1$ and $u_2$. The fact that it provides a $k$-module structure is directly deduced from the fact that, by hypothesis, for each object $C \in J$, $F(C)$ is a $k$-linear category and for each morphism $D \rightarrow E$ in $J$, the functor $F(D \rightarrow E) : F(D) \rightarrow F(E)$ is $k$-linear.

In addition, one can easily show that the composition is $k$-linear. To show this, consider morphisms $[(u_1, f_1, v_1)_{C_1}, (u_2, f_2, v_2)_{C_2}] : (x, A) \rightarrow (y, B), [(s_1, g_1, t_1)_{D_1}, (s_2, g_2, t_2)_{D_2}] : (y, B) \rightarrow (z, C)$ and elements $\lambda, \lambda' \in k$. Without loss of generality we can assume that $D = C_1 = C_2 = D_2, u = u_1 = u_2 : A \rightarrow D, v = v_1 = v_2 = s_1 = s_2 : B \rightarrow D$ and $t = t_1 = t_2 : C \rightarrow D$. We have that

\[
[(v, g_1, t)_{D} \circ [(u, f_1, v)_{D} + \lambda(u, f_2, v)_{D}] = [(v, g_1, t) \circ ((u, f_1 + \lambda f_2, v)_{D}) = [(u, g_1 \circ f_1 + \lambda g_1 \circ f_2, t)_{D}] = [(v, g_1, t)_{D} \circ [(u, f_1, v)_{D} + \lambda'(v, g_2, t)_{D}] \circ [(u, f_2, v)_{D}].
\]

Similarly, one proves that

\[
[(v, g_1, t)_{D} + \lambda'(v, g_2, t)_{D}] \circ [(u, f_1, v)_{D}] = [(v, g_1, t)_{D}].
\]

Hence, the composition is $k$-linear as desired. \qed

Remark 2.6. Observe that one could define a $k$-linear 2-filtered bicorestriction by replacing in Definition 2.2 above $\text{Cat}$ by $\text{Cat}(k)$ and the category $\text{Psnat}(F, C)$ by its $k$-linear analogue. Notice then that, given a pseudofunctor $F$ as in Proposition 2.5 above, we have that the (2-)filtered (bi)corestriction $\mathcal{L}(F)$ coincides with the $k$-linear (2-)filtered (bi)corestriction.
of $F$. In other words, the forgetful functor $U : \text{Cat}(k) \to \text{Cat}$ preserves and reflects filtered colimits, as it happens with the forgetful functor $\text{Ab} \to \text{Set}$ [4, Prop 2.13.5].

3. GENERALITIES ON LINEAR SITES

In this section we revise the basic notions and results concerning linear sites, as they will be an essential tool in the rest of the paper. For a more complete account we point the reader to [17, §2] and [21, §2].

Linear sites and Grothendieck categories can be seen as the linear counterpart of the classical Grothendieck sites and Grothendieck topoi from [1]. We point the reader to [7] for more general enhancements of sites and topoi.

Let $a$ be a small $k$-linear category and consider the category of (right) $a$-modules $\text{Mod}(a) := \text{Fun}_a(a^{\text{op}}, \text{Mod}(k))$.

**Definition 3.1.** Given an object $A \in a$, a (linear) sieve on $A$ is a subobject $R$ of the representable module $a(-, A)$ in the category $\text{Mod}(a)$. Given $S = (f_i : A_i \to A)_{i \in I}$ a family of morphisms in $a$, the sieve generated by $S$ is the smallest sieve $R$ on $A$ such that $f_i \in R(A_i)$ for all $i \in I$.

**Definition 3.2.** A cover system $\mathcal{R}$ on $a$ consists of providing for each $A \in a$ a family of sieves $\mathcal{R}(A)$ on $A$. The sieves in a cover system $\mathcal{R}$ are called covering sieves or covers (for $\mathcal{R}$). We will say that a family $(f_i : A_i \to A)_{i \in I}$ is a cover, or a covering family, if the sieve it generates is a cover.

**Definition 3.3.** Given $R$ a sieve on $A \in a$ and $g : A' \to A$ a morphism in $a$, the pullback of $R$ along $g$, denoted $g^{-1}R$, is the sieve on $A'$ obtained as the pullback

$$
g^{-1}R \leftarrow a(-, A')
\downarrow
R \leftarrow a(-, A)
g \circ f \in R(A')
$$

in $\text{Mod}(a)$. In particular, we have that $g^{-1}R(A'') = \{f : A'' \to A' | g \circ f \in R(A'')\}$ for all $A'' \in a$.

**Definition 3.4.** A cover system $\mathcal{T}$ on $a$ is localizing if it satisfies the following:

1. **(Id)** Identity axiom: Given any object $A \in a$, the sieve generated by $\text{Id}_A$ is a cover for $\mathcal{T}$, i.e., $a(-, A) \in \mathcal{T}(A)$ for all $A \in a$;

2. **(Pb)** Pullback axiom: Given a covering sieve $R \subseteq \mathcal{T}(A)$ and $g : A' \to A$ a morphism in $a$, the pullback sieve $g^{-1}R$ is also a covering sieve for $\mathcal{T}$.

If moreover $\mathcal{T}$ also satisfies the following:

3. **(Glue)** Glueing axiom: Let $R$ be a sieve on $A$. If there exists a sieve $S$ on $A$ such that for all morphism $g : A' \to A$ in $S$ the pullback sieve $g^{-1}R \in \mathcal{T}(A')$, then $R \in \mathcal{T}(A)$;

we say $\mathcal{T}$ is a Grothendieck topology.

**Definition 3.5.** A linear site is a pair $(a, \mathcal{T})$ where $a$ is a $k$-linear category and $\mathcal{T}$ is a Grothendieck topology on $a$.

Given a linear site $(a, \mathcal{T})$ one can define linearised versions of presheaves and sheaves, in analogy with the classical notions.

**Definition 3.6.**

- A presheaf $F$ on $(a, \mathcal{T})$ is an $a$-module, this is $F \in \text{Mod}(a)$. 

We denote the class of LC morphisms by LC.

Theorem 3.9. The categories of sheaves over linear sites are precisely the Grothendieck categories.

Remark 3.8. We will frequently consider Grothendieck categories themselves as (large) sites, endowed with their canonical topology. In this particular case, the covering families are the jointly epimorphic families and Sh(\mathcal{C}, \mathcal{T}_{\text{can}}) \cong \mathcal{C}.

The following is a consequence of Gabriel-Popescu theorem [20] in combination with enriched topos theory [7].

Theorem 3.9. The categories of sheaves over linear sites are precisely the Grothendieck categories.

Definition 3.10. We say that \( f \) is a continuous morphism of sites if the functor \( f^*: \text{Mod}(b) \to \text{Mod}(a) \) preserves sheaves. We denote by \( f_*: \text{Sh}(b, \mathcal{T}_b) \to \text{Sh}(a, \mathcal{T}_a) \) the corresponding restriction functor, and by \( f^*: \text{Sh}(a, \mathcal{T}_a) \to \text{Sh}(b, \mathcal{T}_b) \) its left adjoint.

The class of LC morphisms between sites, where LC stands for “Lemme de comparaison” [15, §4], will be extensively used in the following sections:

Definition 3.11 ([18, Def 3.4]). Consider a \( k \)-linear functor \( f: a \to c \).

1. Suppose \( c \) is endowed with a cover system \( \mathcal{T}_c \). We say that \( f: a \to (c, \mathcal{T}_c) \) satisfies
   - (G) if for every \( C \in c \) there is a covering family \( (f(A_i) \to C_i)_i \) for \( \mathcal{T}_c \).
2. Suppose \( a \) is endowed with a cover system \( \mathcal{T}_a \). We say that \( f: (a, \mathcal{T}_a) \to c \) satisfies
   - (F) if for every \( c : f(A) \to f(A') \) in \( c \) there exists a covering family \( a_i: A_i \to A \) for \( \mathcal{T}_a \) and \( f_i: A_i \to A' \) with \( c f(a_i) = f(f_i) \);
   - (FF) if for every \( a : A \to A' \) in \( a \) with \( f(a) = 0 \) there exists a covering family \( a_i: A_i \to A \) for \( \mathcal{T}_a \) with \( a a_i = 0 \).
3. Suppose \( a \) and \( c \) are endowed with cover systems \( \mathcal{T}_a \) and \( \mathcal{T}_c \) respectively. We say that \( f: (a, \mathcal{T}_a) \to (c, \mathcal{T}_c) \) satisfies
   - (LC) if \( f \) satisfies (G) with respect to \( \mathcal{T}_c \), (F) and (FF) with respect to \( \mathcal{T}_a \), and we further have \( \mathcal{T}_a = f^{-1} \mathcal{T}_c \).

We denote the class of LC morphisms by LC.

Remark 3.12. Observe that the identity morphism of a linear site \( (a, \mathcal{T}) \) belongs to LC. In addition, LC is closed under composition [15, Prop 4.4].

The importance of LC morphisms between linear sites relies in the fact that they are continuous morphisms inducing equivalences between the corresponding sheaf categories [15, Cor 4.5] together with the following key result:
**Theorem 3.13** ([21, Thm 5]). Let $(a, \mathcal{T}_a)$ and $(b, \mathcal{T}_b)$ be linear sites and consider a colimit preserving functor $F : \mathcal{Sh}(a, \mathcal{T}_a) \to \mathcal{Sh}(b, \mathcal{T}_b)$. There exist a subcanonical site $(c, \mathcal{T}_c)$ and a diagram

\begin{equation}
\begin{tikzcd}
a \ar{r}{c} \ar{dr}{w} & b \ar{dr}{f'} \ar{d}{f} & \ar{d}{w'} \\
\end{tikzcd}
\end{equation}

where $f$ is a continuous morphism of sites and $w$ is an LC morphism, such that

\begin{equation}
\begin{tikzcd}
\mathcal{Sh}(a, \mathcal{T}_a) \ar{r}{F} & \mathcal{Sh}(b, \mathcal{T}_b) \ar{d}{w'} \ar{d}{w} \ar{d}{F} \\
\end{tikzcd}
\end{equation}

is a commutative diagram up to isomorphism.

In [18] a tensor product $\boxtimes$ of Grothendieck categories was introduced based upon the definition of a tensor product of linear sites. Given two linear sites $(a, \mathcal{T}_a), (b, \mathcal{T}_b)$, their tensor product $(a, \mathcal{T}_a) \boxtimes (b, \mathcal{T}_b)$ is provided by endowing $a \boxtimes b$ with a tensor product topology $\mathcal{T}_a \boxtimes \mathcal{T}_b$ such that $\mathcal{Sh}(a \boxtimes b, \mathcal{T}_a \boxtimes \mathcal{T}_b) \subseteq \text{Mod}(a \boxtimes b)$ is given by the full subcategory of bimodules $F : a^{\text{op}} \otimes b^{\text{op}} \to \text{Mod}(k)$ for which $F(A, -) \in \mathcal{Sh}(b, \mathcal{T}_b)$ for all $A \in a$ and $F(-, B) \in \mathcal{Sh}(a, \mathcal{T}_a)$ for all $B \in b$. The tensor product of the Grothendieck categories $\mathcal{A} = \mathcal{Sh}(a, \mathcal{T}_a)$ and $\mathcal{B} = \mathcal{Sh}(b, \mathcal{T}_b)$ is defined in loc.cit. as

$$\mathcal{A} \boxtimes \mathcal{B} = \mathcal{Sh}(a \otimes b, \mathcal{T}_a \boxtimes \mathcal{T}_b)$$

and seen to be independent of the site presentations of $\mathcal{A}$ and $\mathcal{B}$ chosen. The proof of this relies on the following result regarding LC morphisms.

**Proposition 3.14** ([18, Prop 3.14]). Consider two functors $f : (a, \mathcal{T}_a) \to (b, \mathcal{T}_b)$ and $g : (c, \mathcal{T}_c) \to (b, \mathcal{T}_b)$. If $f$ and $g$ are LC morphisms, so is the functor $f \otimes g : (a \otimes c, \mathcal{T}_a \boxtimes \mathcal{T}_c) \to (b \otimes b, \mathcal{T}_b \boxtimes \mathcal{T}_b)$.

Recall that Grothendieck categories are in particular locally presentable categories [6, Prop 3.4.16] and that there exists a tensor product $\boxtimes_{\text{LP}}$ of locally presentable categories with the following universal property

$$\text{Cocont}(\mathcal{A} \boxtimes_{\text{LP}} \mathcal{B}, \mathcal{C}) \cong \text{Cocont}(\mathcal{A}, \text{Cocont}(\mathcal{B}, \mathcal{C})),$$

for $\mathcal{A}, \mathcal{B}, \mathcal{C}$ locally presentable categories (see for example [8]). The following result was proven in [18, Thm 5.4].

**Theorem 3.15.** Given two Grothendieck categories $\mathcal{A}, \mathcal{B}$, we have that

$$\mathcal{A} \boxtimes \mathcal{B} \cong \mathcal{A} \boxtimes_{\text{LP}} \mathcal{B}.$$
of \( \mathcal{U} \)-small (right) \( \mathcal{U} \)-modules \( \mathcal{U} \text{-Mod}(a) \) := \text{Fun}_K(\alpha^{op}, \mathcal{U} \text{-Mod}(k)) \), and the category of \( \alpha \)-left exact \( \alpha \)-modules \( \text{Lex}_\alpha(a) \subseteq \mathcal{U} \text{-Mod}(a) \), this is the \( k \)-linear functors \( \alpha^{op} \rightarrow \mathcal{U} \text{-Mod}(k) \) preserving \( \alpha \)-small limits.

By definition, a \( \mathcal{U} \)-locally presentable category is a \( k \)-linear category with \( \mathcal{U} \)-small colimits and a \( \mathcal{U} \)-small set of \( \alpha \)-presentable strong generators for some \( \mathcal{U} \)-small regular cardinal \( \alpha \). Let \( \mathcal{V} \) be a larger universe such that for all \( \mathcal{U} \)-small cardinal \( \alpha \), all the categories \( \text{Lex}_\alpha(a) \) with \( a \in \text{Cat}^\alpha(k) \) are \( \mathcal{V} \)-small, and so is the category \( \mathcal{X} \) given by the totally ordered class of \( \mathcal{U} \)-small regular cardinals. Observe that given a \( \mathcal{U} \)-locally-\( \alpha \)-presentable category \( \mathcal{C} \), its subcategory of \( \alpha \)-presentable objects \( \mathcal{C}^\alpha \) is essentially \( \mathcal{U} \)-small and hence we can still consider it as an element in \( \text{Cat}^\alpha(k) \). In the rest of the chapter, we will omit the universes \( \mathcal{U} \) and \( \mathcal{V} \) from our notations and terminology.

**Theorem 4.1.** Let \( \mathcal{C} \) be a category which is a union of full small subcategories indexed by a directed poset. Then, \( \mathcal{C} \) is the filtered bicolimit of that family of subcategories. More precisely, if \( \mathcal{C} \) is a category such that \( \mathcal{C} = \bigcup_{i \in I} \mathcal{C}_i \), where \( \mathcal{C}_i \subseteq \mathcal{C} \) are full small subcategories, \( I \) is a directed poset and \( \mathcal{C}_i \subseteq \mathcal{C}_j \) if and only if \( i \leq j \), then \( \mathcal{C} \) is a filtered bicolimit of the family \( \{ \mathcal{C}_i \}_{i \in I} \).

**Proof.** Denote by \( \mathcal{J} \) the filtered category given by the directed poset \( I \), and denote by \( t_{i,j} : \mathcal{C}_i \rightarrow \mathcal{C}_j \) the natural embeddings for \( i \leq j \). We define the 2-functor

\[
F_C : \mathcal{J} \rightarrow \text{Cat}
\]

to be given by \( F_C(i) = \mathcal{C}_i \) for every \( i \in \mathcal{J} \) and \( F_C(i \leq j) = t_{i,j} : \mathcal{C}_i \rightarrow \mathcal{C}_j \) for every morphism \( i \leq j \) in \( \mathcal{J} \).

We build a functor

\[
\phi : \mathcal{L}(F_C) \rightarrow \mathcal{C}
\]

defined as follows:

- \( \phi(x, i) = x \in \mathcal{C}_i \subseteq \mathcal{C} \) for every \( (x, i) \in \mathcal{L}(F_C) \);
- \( \phi \left( \{(i \leq k, f : t_{i,k}(x) \rightarrow t_{i,k}(y), j \leq k)\} \right) = f \in \mathcal{C}_k(t_{i,k}(x), t_{i,k}(y)) = \mathcal{C}(x, y) \) for every morphism \( \{(i \leq k, f : t_{i,k}(x) \rightarrow t_{i,k}(y), j \leq k)\} : (x, i) \rightarrow (y, j) \) in \( \mathcal{L}(F_C) \).

One can readily check that this is well defined and it defines a functor. As \( \mathcal{C} \cong \bigcup_{i \in \mathcal{J}} \mathcal{C}_i \), one trivially has that this functor is essentially surjective, and because \( t_{i,j} \) are fully faithful for all \( i, j \in \mathcal{J} \), one easily deduces that the functor is fully faithful.

**Remark 4.2.** Assume \( \mathcal{C} \) is \( k \)-linear, and hence \( \mathcal{C}_i \) is \( k \)-linear for all \( i \in \mathcal{J} \) and so are the fully faithful functors \( t_{i,j} : \mathcal{C}_i \rightarrow \mathcal{C}_j \) for \( i \leq j \). Then we have that both \( \mathcal{J} \) and \( F_C \) are in the hypothesis of Proposition 2.5 above, hence \( \mathcal{L}(F_C) \) is \( k \)-linear. Observe that the functor \( \phi \) defined in the proof above is as well \( k \)-linear, and thus \( \mathcal{C} \) and \( \mathcal{L}(F_C) \) are equivalent as \( k \)-linear categories.

As indicated above, we denote by \( \mathcal{X} \) the category obtained from the totally ordered set of regular cardinals. Recall that given \( \mathcal{C} \) a locally presentable \( k \)-linear category and \( \alpha \in \mathcal{X} \), we denote by \( \mathcal{C}^\alpha \) the full \( k \)-linear subcategory of \( \mathcal{C} \) consisting of the \( \alpha \)-presentable objects. Recall that \( \mathcal{C}^\alpha \) is an \( \alpha \)-cocomplete category and that if \( \alpha \leq \beta \), we have a fully faithful embedding \( \mathcal{C}^\alpha \subseteq \mathcal{C}^\beta \). In addition, one has that \( \mathcal{C} \cong \bigcup_{\alpha \in \mathcal{X}} \mathcal{C}^\alpha \). For these and other basic facts concerning locally presentable categories we point the reader to [3, Ch 1].

**Corollary 4.3.** Let \( \mathcal{C} \) be a locally presentable \( k \)-linear category. Then \( \mathcal{C} \) is the \( k \)-linear filtered bicolimit of its family of subcategories of locally presentable objects \( (\mathcal{C}^\alpha)_\alpha \), where \( \alpha \) varies in \( \mathcal{X} \).

**Proof.** It follows from Theorem 4.1 and Remark 4.2. \( \square \)
Remark 4.4. Observe that the statement is true for Grothendieck $k$-linear categories, as they are an instance of locally presentable $k$-linear categories.

We now introduce another presentation of a Grothendieck category as a filtered colimit, where the indexing filtered category will be a certain category of linear sites.

Consider the $k$-linear category $\mathcal{C}$ defined as follows:

- Objects of $\mathcal{C}$ are given by $\{ u_a : (a, \mathcal{T}_a) \to \mathcal{C} \mid (a, \mathcal{T}_a) \text{ is a } k \text{-linear site } \}$ where $\mathcal{C}$ is endowed with the canonical topology (see Remark 3.8). For readability, we will frequently omit the topology from our notations and write $u_a : a \to \mathcal{C}$.
- Morphisms between two objects $u_a : a \to \mathcal{C}$ and $u_b : b \to \mathcal{C}$ are given by the $k$-linear functors $f : a \to b$ which belong to $\text{LC}$ and such that $u_b \circ f = u_a$. We write $f : u_a \to u_b$.

One can readily check this is a well defined category as a direct consequence of Remark 3.12.

**Proposition 4.5.** Given a Grothendieck category $\mathcal{C}$, the category $\mathcal{C}$ constructed above is filtered.

**Proof.** Observe that the category $\mathcal{C}$ is not empty. Given two objects $u_a : a \to \mathcal{C}$ and $u_b : b \to \mathcal{C}$, we want to find a third object $u_c : c \to \mathcal{C}$ and morphisms $f : u_a \to u_c$ and $g : u_b \to u_c$. Let $c$ be the full subcategory of $\mathcal{C}$ spanned by $\{ u_a(A) \}_{A \in a} \cup \{ u_b(B) \}_{B \in b}$ and endow it with the topology induced by the canonical topology in $\mathcal{C}$. We hence have that the embedding $u_c : c \to \mathcal{C}$ is an LC morphism. Now consider the corestrictions of $u_a : a \to c$ and $u_b : b \to c$. We trivially have that these define morphisms $u_a \to u_c$ and $u_b \to u_c$.

Consider now two morphisms $f, g : u_a \to u_b$. We want to find an object $u_c : c \to \mathcal{C}$ and a morphism $h : u_b \to u_c$ equalizing $f$ and $g$. Take $c$ to be the full subcategory of $\mathcal{C}$ spanned by $u_b(b)$ and endow it with the (restriction of the) canonical topology. Take $u_c : c \to \mathcal{C}$ to be the embedding (which is LC) and $h : b \to c$ the corestriction of $u_b : b \to \mathcal{C}$ to $c$. Then, by definition, one has that $h u_c = u_b$. Furthermore, we have that $h f = h g$ as a direct consequence of the fact that $u_b f = u_a = u_b g$.

We can thus conclude that $\mathcal{C}$ is a filtered category. \hfill $\square$

We now consider the $k$-linear functor $G_C : \mathcal{C} \to \text{Cat}$ given by forgetting the “slice structure”, i.e. defined by sending each object $u_a : a \to \mathcal{C}$ to the small $k$-linear category $a$ and each morphism $f : u_a \to u_b$ to itself seen as a $k$-linear morphism $f : a \to b$.

We proceed to describe $\mathcal{L}(G_C)$ using the construction from §2 above. Observe that the description in this case will be simplified because $G_C$ is a strict functor:

- Objects are $\{(x, u_a : a \to \mathcal{C}) \mid (u_a : a \to \mathcal{C}) \in \mathcal{C}, x \in a\}$
- Morphisms $(x, u_a : a \to \mathcal{C}) \to (y, u_b : b \to \mathcal{C})$ are given by homotopy classes of triples $(u, f, v)$ where $u : u_a \to u_b$, $v : u_b \to u_c$ are morphisms in $\mathcal{C}$ and $f : u(x) \to v(y)$ is a morphism in $c$. As before, we use the notation $(u, f, v)_{u_a}$ to make explicit the codomain of $u$ and $v$. Two morphisms $(u_1, f, v)_{u_{a_1}}$ and $(u_2, g, v)_{u_{a_2}}$ are homotopic if there exist morphisms $w_1 : u_{a_1} \to u_c$ and $w_2 : u_{a_2} \to u_c$ such that $w_1 u_1 = w_2 u_2$, $w_1 v_1 = w_2 v_2$ and $w_1(f) = w_2(g)$. As in §2, we denote the homotopy class of $(u_1, f, v)_{u_{a_1}}$ by $[(u_1, f, v)_{u_{a_1}}]$.

Observe that $G_C$ factors through $\text{Cat}(k)$ and hence, by Proposition 2.5, we have that $\mathcal{L}(G_C)$ is a $k$-linear category.
Theorem 4.6. Given a Grothendieck category $\mathcal{C}$, we have that $\mathcal{C}$ is the $k$-linear filtered colimit of $G_\mathcal{C}$.

Proof. By Proposition 2.5, we have that the filtered colimit $\mathcal{L}(G_\mathcal{C})$ is a $k$-linear category. To conclude, it suffices to construct a $k$-linear equivalence $\psi_\mathcal{C} : \mathcal{L}(G_\mathcal{C}) \to \mathcal{C}$. With the notations introduced in §2 for the objects and morphisms of $\mathcal{L}(G_\mathcal{C})$, we consider the following assignations:

- To every object $(x, u_a : a \to \mathcal{C}) \in \mathcal{J}_\mathcal{C}$ we assign the object
  $$\psi_\mathcal{C}(x, u_a : a \to \mathcal{C}) := u_a(x) \in \mathcal{C}.$$  

- To every morphism $[(u, f, v)_{u_a}] : (x, u_a : a \to \mathcal{C}) \to (y, u_b : b \to \mathcal{C})$ we assign the morphism
  $$\psi_\mathcal{C}([(u, f, v)_{u_a}]) = (u_c(f) : u_c(x) \to u_c(y) = u_b(y)).$$

We have the following:

1. The assignment on morphisms is well-defined. Consider two homotopical morphisms $(u_1, f_1, v_1)_{u_a_1}, (u_2, f_2, v_2)_{u_a_2} : (x, a \to \mathcal{C}) \to (y, b \to \mathcal{C})$, this is, there exist morphisms $w_1 : u_{c_1} \to u_c$ and $w_2 : u_{c_2} \to u_c$ such that $w_1 u_1 = w_2 u_2$, $w_1 v_1 = w_2 v_2$ and $w_1 f_1 = w_2 f_2$. We want to show that $u_c(f_1) = u_c(f_2)$. Observe that $u_{c_1}(f_1) = u_{c_2}(f_2) = u_c(f_2)$, as desired.

2. The assignations define a functor $\psi_\mathcal{C} : \mathcal{L}(G_\mathcal{C}) \to \mathcal{C}$. First observe that it preserves identities. Indeed, the identity morphism $[(Id_a, Id_a, Id_a)_{u_a}]$ of the object $(x, u_a : a \to \mathcal{C})$ gets sent to $u_a(Id_a) = Id_{u_a(x)}$. We now check that it preserves compositions. Consider two composable morphisms $[(u_1, f_1, v_1)_{u_a_1}] : (a, u_a) \to (b, u_b)$ and $[(u_2, f_2, v_2)_{u_a_2}] : (b, u_b) \to (c, u_c)$ in $\mathcal{L}(G_\mathcal{C})$. Because $\mathcal{J}_\mathcal{C}$ is filtered, we can assume that $u_b = u_{c_1} = u_{c_2}$, i.e., that the morphisms $u_1, v_1, u_2, v_2$ in $\mathcal{J}_\mathcal{C}$ have the same codomain $u_c : \emptyset \to \mathcal{C}$, and that $v_1 = v_2$. Their composition in $\mathcal{L}(G_\mathcal{C})$ is given by $[(u_1, f_2 \circ f_1, v_2)_{u_a_2}]$ and gets sent to $u_b(f_2 f_1)$. On the other hand, we have that $[(u_1, f_2, v_2)_{u_a_1}]$ and $[(u_2, f_2, v_2)_{u_a_2}]$ get sent to $u_a(f_2)$ and $u_b(f_2)$ respectively, whose composition is $u_b(f_2)u_a(f_1)$. As $u_a : \emptyset \to \mathcal{C}$ is a functor, we have that $u_b(f_2 f_1) = u_b(f_2)u_a(f_1)$ as desired.

3. The functor $\psi_\mathcal{C}$ is $k$-linear. Let $[(u_1, f_1, v_1)_{u_a_1}] : (a, u_a) \to (b, u_b)$ be two morphisms in $\mathcal{L}(G_\mathcal{C})$. Because $\mathcal{J}_\mathcal{C}$ is filtered, we may assume that $u_c = u_{c_1} = u_{c_2}$ and that $u = u_1 = u_2$, $v = v_1 = v_2$. Then, given $\lambda \in k$, we have that $\psi_\mathcal{C}([(u, f_1, v)_{u_a_1} + \lambda[(u, f_2, v)_{u_a_2}]] = \psi_\mathcal{C}([(u_1, f_1 + \lambda f_2, v)_{u_a_1}]) = u_b(f_1 + \lambda f_2)$. On the other hand, we have that $\psi_\mathcal{C}([(u_2, f_2, v)_{u_a_2}]) = u_b(f_2)$ and $\lambda u_b(f_2)$. As $u_a$ is a $k$-linear functor, we have that $u_b(f_1 + \lambda f_2) = u_a(f_1) + \lambda u_a(f_2)$ as desired.

4. The functor $\psi_\mathcal{C}$ is essentially surjective. Let $y$ be an object in $\mathcal{C}$. Consider the small full subcategory $a$ of $\mathcal{C}$ spanned by the objects $\{y\} \cup \{g\}_{g \in G}$, where $G$ is a small set of generators of $\mathcal{C}$. We endow $a$ with the topology induced by the canonical topology in $\mathcal{C}$. Then the embedding $i : a \to \mathcal{C}$ is trivially an LC morphism and hence we have that $\psi_\mathcal{C}(i, y : a \to \mathcal{C}) = i(y) = y$, as desired.

5. The functor $\psi_\mathcal{C}$ is faithful. Let $[(u_1, f_1, v_1)_{u_a_1}] : (x, u_a) \to (y, u_b)$ be two morphisms, such that $u_{c_1}(f_1) = u_{c_2}(f_2)$. Consider the full subcategory $\mathcal{E}$ of $\mathcal{C}$ spanned by the objects $\{u_{c_1}(e_1)\} \cup \{u_{c_2}(e_2)\}$ endowed with the topology induced by the canonical topology in $\mathcal{C}$ and the associated embedding $\iota : \emptyset \to \mathcal{C}$.
We hence have proven that \( \psi \) is a \( k \)-linear equivalence of categories as desired.

\[ \square \]

**Remark 4.7.** Observe that, in order to recover any locally presentable category \( \mathcal{C} \) as a filtered colimit using the construction from Corollary 4.3, we can always use the same filtered category, namely the category \( \mathcal{K} \) associated to the total ordered class of small regular cardinals. Notice that this is not the case for this last presentation of Grothendieck categories provided by Theorem 4.6, as the filtered category \( \mathcal{F} \) is dependent on the Grothendieck category \( \mathcal{C} \) we want to recover.

5. **THE TENSOR PRODUCT OF GROTHENDIECK CATEGORIES AS A FILTERED COLIMIT**

In this section we analyse the tensor product of Grothendieck categories from [18] (see §3) in terms of the realization of Grothendieck categories as filtered colimits provided in §4.

Recall there is a well-defined notion of tensor product of \( \alpha \)-cocomplete \( k \)-linear categories. In particular, the bicategory \( \text{Cat}^\alpha(k) \) of \( \alpha \)-cocomplete \( k \)-linear categories as defined above is, together with this tensor product, a closed monoidal bicategory [12, §6.5]. More precisely, we have the following:

**Definition 5.1.** Given \( a, b \) two \( \alpha \)-cocomplete \( k \)-linear categories, there exists another \( \alpha \)-cocomplete \( k \)-linear category \( a \otimes_k b \) and a \( k \)-linear functor \( u_{a,b} : a \otimes_k b \to a \otimes a \).
is \( \alpha \)-cocontinuous in each variable, such that, for every \( \alpha \)-cocomplete \( k \)-linear category \( c \), composition with \( \iota_{a,b} \) induces an equivalence
\[
(8) \quad \text{Cocont}_c(a \otimes_a b, c) \cong \text{Cocont}_c(a, \text{Cocont}_c(b, c))
\]
in \( \text{Cat}^\alpha(k) \).

Consider \( a, b \in \text{Cat}^\alpha(k) \). The category \( a \otimes_a b \) can be constructed as the closure under \( \alpha \)-small colimits of the image of the composition
\[
a \otimes b \longmapsto Y_{\text{Mod}(a \otimes b)} R_{\text{Lex}_a(a, b)}
\]
where \( \text{Lex}_a(a, b) \subseteq \text{Mod}(a \otimes b) \) is defined as the full subcategory with objects the bimodules \( F : a^{op} \otimes b^{op} \rightarrow \text{Mod}(k) \) that preserve \( \alpha \)-small limits in each variable, the functor \( Y : a \otimes b \leftarrow \text{Mod}(a \otimes b) \) is the Yoneda embedding and the functor \( R : \text{Mod}(a \otimes b) \rightarrow \text{Lex}_a(a, b) \) is the left adjoint to the embedding \( \text{Lex}_a(a, b) \leftarrow \text{Mod}(a \otimes b) \). In addition, we know that given locally \( \alpha \)-presentable categories \( A, B \), we have that \( A \otimes_{\text{Lp}} B = \text{Lex}_a(A^\alpha, B^\beta) \) is \( \alpha \)-locally presentable and its subcategory of \( \alpha \)-presentable objects is given by \( A^\alpha \otimes_a B^\alpha \). For these results, we point the reader to [11] and [12], or to [14] for the case \( \alpha = \kappa \).

Let \( A, B \) be two locally presentable \( k \)-linear categories and choose the smallest regular cardinal \( \kappa \) such that both are locally \( \kappa \)-presentable. Consider regular cardinals \( \alpha \leq \beta \) and denote by \( \iota_{A} : A \rightarrow A^{\beta}, \iota_{B} : B \rightarrow B^{\beta} \) the natural embeddings, which in particular are \( \alpha \)-cocontinuous. Observe that we have a canonical morphism
\[
(9) \quad f_{\alpha, \beta} : A^\alpha \otimes_a B^\alpha \rightarrow A^\beta \otimes_{\beta} B^\beta
\]
that makes the diagram
\[
\begin{array}{ccc}
A^\alpha \otimes B^\alpha & \xrightarrow{\iota_{A} \otimes B^\alpha} & A^{\beta} \otimes B^\beta \\
\downarrow{u_{\alpha, \beta, \beta}} & & \downarrow{u_{\alpha, \beta, \beta}} \\
A^\alpha \otimes_a B^\alpha & \xrightarrow{f_{\alpha, \beta}} & A^{\beta} \otimes_{\beta} B^\beta
\end{array}
\]
commutative. Indeed, \( f_{\alpha, \beta} \) is defined as the image via the universal property (8) in \( \text{Cocont}_c(A^\alpha \otimes_a B^\alpha, A^\beta \otimes_{\beta} B^\beta) \) of the composition
\[
A^\alpha \otimes B^\alpha \xrightarrow{\iota_{A} \otimes_{\beta} B^\alpha} A^\beta \otimes_{\beta} B^\beta \xrightarrow{u_{\alpha, \beta, \beta}} A^{\beta} \otimes_{\beta} B^\beta,
\]
which is \( \alpha \)-cocontinuous in each variable.

First observe that, as a direct consequence of Theorem 4.1, we have that \( A \otimes_{\text{Lp}} B \) is the filtered bicolicmit of the family \( \{(A \otimes_{\text{Lp}} B)^\gamma\}_{\gamma \in \mathcal{K}} \) and that from \( \alpha \geq \kappa \) we have that \( (A \otimes_{\text{Lp}} B)^\gamma \cong A^\alpha \otimes_a B^\gamma \). However, it is not directly obvious whether the fully faithful functors \( A^\alpha \otimes_a B^\alpha \rightarrow A^\beta \otimes_{\beta} B^\beta \) with \( \beta \geq \alpha \geq \kappa \) provided by the inclusion \( (A \otimes_{\text{Lp}} B)^\gamma \subseteq (A \otimes_{\text{Lp}} B)^\gamma \) coincide with the canonical functors \( f_{\alpha, \beta} \), and hence whether the filtered bicolicmit is compatible with the \( \alpha \)-cocomplete tensor products for \( \alpha \) varying in \( \mathcal{K} \).

We will now provide a positive answer in the context of Grothendieck categories.

**Theorem 5.2.** Let \( A, B \) be two Grothendieck \( k \)-linear categories and choose the smallest regular cardinal \( \kappa \) such that both \( A \) and \( B \) are locally \( \kappa \)-presentable. Then, for all \( \alpha, \beta \in \mathcal{K} \) such that \( \beta \geq \alpha \geq \kappa \) the canonical functor
\[
(10) \quad f_{\alpha, \beta} : A^\alpha \otimes_a B^\alpha \rightarrow A^\beta \otimes_{\beta} B^\beta
\]
defined in (9) is fully faithful. In particular, the functor \( f_{\alpha, \beta} \) coincides, up to the equivalences \( (A \otimes B)^\gamma \cong A^\alpha \otimes_a B^\gamma \) and \( (A \otimes B)^\gamma \cong A^\beta \otimes_{\beta} B^\beta \), with the canonical inclusion \( (A \otimes B)^\gamma \subseteq (A \otimes B)^\beta \).
Proof. Consider a locally $\kappa$-presentable Grothendieck $k$-linear category $\mathcal{C}$. Consider a regular cardinal $\alpha$ such that $\alpha \geq \kappa$. We endow $\mathcal{C}^{\alpha}$ with the topology induced by the canonical topology in $\mathcal{C}$ via the natural embedding $i_{\alpha}^{\mathcal{C}} : \mathcal{C}^{\alpha} \hookrightarrow \mathcal{C}$. With this choice of topology $i_{\alpha}^{\mathcal{C}}$ is an LC morphism and as a direct consequence of Gabriel-Popescu theorem together with the representation theorem of locally presentable categories (see [18, Thm 5.3]), we have that the functor

$$Y_{\alpha} : \mathcal{C} \rightarrow \text{Mod}(\mathcal{C}^{\alpha}) : C \mapsto \mathcal{C}(i_{\alpha}^{\mathcal{C}}(-), C)$$

factors through an equivalence

$$\xymatrix{ \mathcal{C} \ar[r]^{Y_{\alpha}} \ar[rd]_{\text{Lex}_{\alpha}(\mathcal{C}^{\alpha})}^{\simeq} & \text{Mod}(\mathcal{C}^{\alpha}). \ar[ld]_{\text{Sh}(\mathcal{C}^{\alpha})}^{\simeq} }$$

Take now $\alpha, \beta \geq \kappa$ regular cardinals with $\alpha \leq \beta$. If we endow $\mathcal{C}^{\alpha}$ and $\mathcal{C}^{\beta}$ with the topology induced by the canonical topology of $\mathcal{C}$, we have that not only the embeddings $i_{\alpha}^{\mathcal{C}}$ and $i_{\beta}^{\mathcal{C}}$ are LC morphisms, but also the embedding $i_{\alpha,\beta}^{\mathcal{C}} : \mathcal{C}^{\alpha} \hookrightarrow \mathcal{C}^{\beta}$. This implies that the induced functor

$$(i_{\alpha,\beta}^{\mathcal{C}})_* : \text{Lex}_{\alpha}(\mathcal{C}^{\alpha}) \rightarrow \text{Lex}_{\beta}(\mathcal{C}^{\beta})$$

between the sheaf categories is an equivalence, with quasi-inverse given by $(i_{\alpha,\beta}^{\mathcal{C}})^*$ (see Definition 3.10). Observe that $(i_{\alpha,\beta}^{\mathcal{C}})_*E_{\beta}^{\mathcal{C}}(C) = \mathcal{C}(i_{\alpha,\beta}^{\mathcal{C}}(\alpha)(-), C) = E_{\alpha}^{\mathcal{C}}(C)$, for every $C \in \mathcal{C}$. Consequently, we have that $E_{\beta} = (i_{\alpha,\beta}^{\mathcal{C}})^*E_{\alpha}^{\mathcal{C}}$ and hence the diagram

$$\xymatrix{ \mathcal{C}^{\alpha} \ar@{^{(}->}[d] & \mathcal{C}^{\beta} \ar@{^{(}->}[d] \ar[ld]_{\simeq}^{E_{\alpha}^{\mathcal{C}}} \ar[rd]^{E_{\beta}^{\mathcal{C}}} \ar@{^{(}->}[r]_{(i_{\alpha,\beta}^{\mathcal{C}})^*} & \text{Lex}_{\beta}(\mathcal{C}^{\beta}) \ar@{^{(}->}[d] \ar[ld]_{\simeq}^{E_{\alpha}^{\mathcal{C}}} \ar[rd]^{E_{\beta}^{\mathcal{C}}} & \text{Lex}_{\alpha}(\mathcal{C}^{\alpha}) \ar@{^{(}->}[d] \ar[ld]_{\simeq}^{E_{\alpha}^{\mathcal{C}}} \ar[rd]^{E_{\beta}^{\mathcal{C}}} \ar@{^{(}->}[r]_{i_{\alpha,\beta}^{\mathcal{C}}} & \mathcal{C}^{\alpha} \ar@{^{(}->}[d] }$$

is commutative.

Consider $\mathcal{A}$ and $\mathcal{B}$ as in the statement. As the tensor product of LC morphisms remains an LC morphism (see Proposition 3.14), for all $\beta \geq \alpha \geq \kappa$ we have that $i_{\alpha,\beta}^{\mathcal{A}} \otimes i_{\alpha,\beta}^{\mathcal{B}} : \mathcal{A}^{\alpha} \otimes \mathcal{B}^{\alpha} \rightarrow \mathcal{A}^{\beta} \otimes \mathcal{B}^{\beta}$ is an LC morphism when $\mathcal{A}^{\alpha} \otimes \mathcal{B}^{\alpha}$ and $\mathcal{A}^{\beta} \otimes \mathcal{B}^{\beta}$ are endowed with the tensor product of the induced topologies. Consequently, the functor

$$(i_{\alpha,\beta}^{\mathcal{A}} \otimes i_{\alpha,\beta}^{\mathcal{B}})_* : \text{Lex}_{\alpha}(\mathcal{A}^{\alpha}, \mathcal{B}^{\alpha}) \rightarrow \text{Lex}_{\beta}(\mathcal{A}^{\beta}, \mathcal{B}^{\beta})$$

is an equivalence with quasi-inverse given by $(i_{\alpha,\beta}^{\mathcal{A}} \otimes i_{\alpha,\beta}^{\mathcal{B}})^*$. We thus have a diagram

$$\xymatrix{ \mathcal{A}^{\alpha} \otimes \mathcal{B}^{\alpha} \ar[r]^{i_{\alpha,\beta}^{\mathcal{A}} \otimes i_{\alpha,\beta}^{\mathcal{B}}} \ar[d]_{\text{Mod}(\mathcal{A}^{\alpha} \otimes \mathcal{B}^{\alpha})} \ar@/_4ex/[drr]_{\text{Lex}_{\alpha}(\mathcal{A}^{\alpha}, \mathcal{B}^{\alpha})} & \mathcal{A}^{\beta} \otimes \mathcal{B}^{\beta} \ar[d]_{\text{Mod}(\mathcal{A}^{\beta} \otimes \mathcal{B}^{\beta})} \ar@/_4ex/[drr]_{\text{Lex}_{\beta}(\mathcal{A}^{\beta}, \mathcal{B}^{\beta})} \ar[l]_{\text{Mod}(\mathcal{A}^{\alpha} \otimes \mathcal{B}^{\alpha})} \ar[r]_{i_{\alpha,\beta}^{\mathcal{A}} \otimes i_{\alpha,\beta}^{\mathcal{B}}} & \mathcal{A}^{\beta} \otimes \mathcal{B}^{\beta} \ar[d]_{\text{Mod}(\mathcal{A}^{\beta} \otimes \mathcal{B}^{\beta})} \ar@/_4ex/[drr]_{\text{Lex}_{\beta}(\mathcal{A}^{\beta}, \mathcal{B}^{\beta})} \ar[l]_{\text{Mod}(\mathcal{A}^{\alpha} \otimes \mathcal{B}^{\alpha})} }$$

where the upper square, the two squares on the sides and the bigger square are commutative. From this one readily deduces using the universal property of $\otimes_\kappa$ that the lower square is also commutative. Observe that the vertical arrows in that lower square are
fully faithful and the lower horizontal arrow is an equivalence. Consequently, \( f_{\alpha, \beta} \) is fully-faithful as desired.

In addition, we have that the following diagram

\[
\begin{array}{ccc}
(A \otimes B)^{\alpha} & \xrightarrow{f_{\alpha, \beta}} & (A \otimes B)^{\beta} \\
\cong & & \cong \\
\mathbf{Lex}_\alpha(A_\alpha \otimes B_\alpha) & \xrightarrow{\kappa \otimes \eta_\alpha} & \mathbf{Lex}_\beta(A_\beta \otimes B_\beta) \\
\cong & & \cong \\
(A \otimes B) & \xrightarrow{\kappa \otimes \eta_{\alpha, \beta}} & (A \otimes B)^{\alpha \otimes B^{\beta}} \\
\end{array}
\]

is commutative, which shows that \( f_{\alpha, \beta} \) coincides, up to the equivalences \( E_{\alpha, \beta}^{A \otimes B} \mid (A \otimes B)^{\alpha} \) and \( E_{\beta}^{A \otimes B} \mid (A \otimes B)^{\beta} \), with the embedding \( \iota_{\alpha, \beta} : (A \otimes B)^{\alpha} \hookrightarrow (A \otimes B)^{\beta} \) as desired. \( \square \)

We can now define

\[(12) \quad F_{A, B} : \mathcal{K} \to \text{Cat} \]

the pseudofunctor given by

- \( F_{A, B}(\alpha) = F_A(\alpha) \otimes_a F_B(\alpha) = A_\alpha \otimes_a B_\alpha \) for every \( \alpha \in \mathcal{K} \);
- \( F_{A, B}(\alpha \leq \beta) = f_{\alpha, \beta} \) for every morphism \( \alpha \leq \beta \) in \( \mathcal{K} \);

with the notations from §4 above.

**Remark 5.3.** Observe that \( F_{A, B} \) is a pseudofunctor and not a strict 2-functor.

**Theorem 5.4.** Given \( A \) and \( B \) two Grothendieck \( k \)-linear categories, one has that \( A \otimes B \) is the \( k \)-linear filtered colimit of \( F_{A, B} \).

**Proof.** By Proposition 2.5, \( \mathcal{L}(F_{A, B}) \) is a \( k \)-linear category. In addition, we know that there exists a \( \kappa \in \mathcal{K} \) such that \( A \otimes B \) is locally \( \alpha \)-presentable and \( (A \otimes B)^{\alpha} \cong A_\alpha \otimes_a B_\alpha \) for every \( \alpha \geq \kappa \). Consider \( \kappa \) the smallest regular cardinal with such property. We build a \( k \)-linear functor

\[ \phi : \mathcal{L}(F_{A, B}) \to A \otimes B \]

as follows.

For an object \((x, \alpha) \in \mathcal{L}(F_{A, B})\), we put

\[ \phi(x, \alpha) = \begin{cases} x \in A_\alpha \otimes_a B_\alpha \subseteq A \otimes B & \text{if } \alpha \geq \kappa \\ f_{\alpha, \kappa}(x) \in A_\kappa \otimes_a B_\kappa \subseteq A \otimes B & \text{if } \alpha < \kappa \end{cases} \]

For a morphism \( [[(\alpha \leq \gamma, g, \beta \leq \gamma)]] : (x, \alpha) \to (y, \beta) \), making use of Theorem 5.2, we define \( \phi([[\alpha \leq \gamma, g, \beta \leq \gamma]]) \) as follows:

- If \( \alpha, \beta \geq \kappa \), we put

  \[ \phi([[\alpha \leq \gamma, g, \beta \leq \gamma]]) = g \in A_\gamma \otimes_a B_\gamma (f_{\alpha, \gamma}(x), f_{\beta, \gamma}(y)) \cong A \otimes B(x, y). \]

Observe that in this case \( \gamma \geq \kappa \) holds.
• If $a < \kappa$ and $\beta \geq \kappa$, we put
  \[
  \phi([\alpha \leq \gamma, \varnothing, \beta \leq \gamma]) = g \in \mathcal{A}_\gamma \otimes \mathcal{B}_\gamma(f_{\alpha,\gamma}(x), f_{\beta,\gamma}(y)) \\
  \cong \mathcal{A} \otimes \mathcal{B}(f_{\alpha,\kappa}(x), y).
  \]

  Observe that in this case $\gamma \geq \kappa$ holds.

• If $\alpha \geq \kappa$ and $\beta < \kappa$, we put
  \[
  \phi([\alpha \leq \gamma, \varnothing, \beta \leq \gamma]) = g \in \mathcal{A}_\gamma \otimes \mathcal{B}_\gamma(f_{\alpha,\gamma}(x), f_{\beta,\gamma}(y)) \\
  \cong \mathcal{A} \otimes \mathcal{B}(x, f_{\beta,\kappa}(y)).
  \]

  Observe that in this case $\gamma \geq \kappa$ holds.

• If $\alpha, \beta < \kappa$ and $\gamma \geq \kappa$, we put
  \[
  \phi([\alpha \leq \gamma, \varnothing, \beta \leq \gamma]) = f_{\alpha,\kappa}(g) \in \mathcal{A}_\kappa \otimes \mathcal{B}_\kappa(f_{\alpha,\kappa}(x), f_{\beta,\kappa}(y)) \\
  \cong \mathcal{A} \otimes \mathcal{B}(f_{\alpha,\kappa}(x), f_{\beta,\kappa}(y)).
  \]

  Observe this functor is well-defined and $k$-linear. In addition, we have that
  \[
  \mathcal{A} \otimes \mathcal{B} \cong \bigcup_{a \in \kappa} (\mathcal{A} \otimes \mathcal{B})^a \cong \bigcup_{a \in \kappa} (\mathcal{A} \otimes \mathcal{B})^a \cong \bigcup_{a \in \kappa} \mathcal{A}^a \otimes \mathcal{B}^a.
  \]

  Consequently, the functor is essentially surjective. We also have that all the transition functors $f_{a,\beta}$ are fully-faithful for $\beta \geq \alpha \geq \kappa$ by Theorem 5.2 above, hence one can conclude that the functor is fully-faithful as desired. □

**Remark 5.5.** One may wonder if an analogous approach would allow to obtain a realization of the tensor product of Grothendieck categories as a filtered colimit by using, instead of the tensor product of $\alpha$-cocomplete categories, the tensor product of linear sites and LC morphisms from §3 and, instead of the realization of Grothendieck categories as filtered colimits of $\alpha$-presentable objects from Corollary 4.3, the realization of Grothendieck categories as filtered colimits of linear sites from Theorem 4.6. We will explain why this is not the case. Roughly, the argument goes as follows:

Let $\mathcal{A}, \mathcal{B}$ be Grothendieck categories. We use the notations introduced in §4 for the rest of the remark. Consider the filtered categories $\mathcal{J}_\mathcal{A}$ (resp. $\mathcal{J}_\mathcal{B}$) with objects the LC morphisms $u:(c, \mathcal{T}_c) \to (\mathcal{A}, \mathcal{F}_{\mathcal{A}_{can}})$ (resp. the LC morphisms $v:(0, \mathcal{T}_0) \to (\mathcal{B}, \mathcal{F}_{\mathcal{B}_{can}})$). Denote by $\mathcal{J}_{\mathcal{A},\mathcal{B}}$ the category with objects given by tensor products $u \otimes v:(c \otimes 0, \mathcal{T}_c \otimes \mathcal{T}_0) \to (\mathcal{A} \otimes \mathcal{B}, \mathcal{F}_{\mathcal{A}_{can}} \otimes \mathcal{F}_{\mathcal{B}_{can}})$ of objects in $\mathcal{J}_\mathcal{A}$ with objects of $\mathcal{J}_\mathcal{B}$, and with morphisms given by the tensor product of LC morphisms. In particular, we have that $\mathcal{J}_{\mathcal{A},\mathcal{B}}$ is filtered. Moreover, as LC is closed under composition (see Remark 3.12) and tensor products (see Proposition 3.14), composition with the natural LC morphism $(\mathcal{A} \otimes \mathcal{B}, \mathcal{F}_{\mathcal{A}_{can}} \otimes \mathcal{F}_{\mathcal{B}_{can}}) \to (\mathcal{A} \otimes \mathcal{B}, \mathcal{F}_{\mathcal{A}_{can}} \otimes \mathcal{F}_{\mathcal{B}_{can}})$ defines a faithful functor $\mathcal{S} : \mathcal{J}_{\mathcal{A},\mathcal{B}} \to \mathcal{J}_{\mathcal{A} \otimes \mathcal{B}}$, where $\mathcal{J}_{\mathcal{A} \otimes \mathcal{B}}$ is the category with objects the LC morphisms $(c, \mathcal{T}_c) \to (\mathcal{A} \otimes \mathcal{B}, \mathcal{F}_{\mathcal{A}_{can}} \otimes \mathcal{F}_{\mathcal{B}_{can}})$. Consider now the functor

\[
\hat{G}_{\mathcal{A},\mathcal{B}} : \mathcal{J}_{\mathcal{A},\mathcal{B}} \to \mathcal{J}_{\mathcal{A} \otimes \mathcal{B}} \to \mathcal{Cat}.
\]

One can then construct a $k$-linear functor

\[
\psi_{\mathcal{A},\mathcal{B}} : \mathcal{L}(\hat{G}_{\mathcal{A},\mathcal{B}}) \to \mathcal{A} \otimes \mathcal{B},
\]

by restricting the functor $\psi_{\mathcal{A} \otimes \mathcal{B}} : \mathcal{L}(G_{\mathcal{A} \otimes \mathcal{B}}) \to \mathcal{A} \otimes \mathcal{B}$ constructed in the proof of theorem 4.6 above. However, $\psi_{\mathcal{A},\mathcal{B}}$ is not an equivalence. This follows from the fact that the natural functor $\mathcal{A} \otimes \mathcal{B} \to \mathcal{A} \otimes \mathcal{B}$ is not essentially surjective (consider, for example, the natural functor $\text{Mod}(a) \otimes \text{Mod}(b) \to \text{Mod}(a \otimes b)$), which implies that $\psi_{\mathcal{A},\mathcal{B}}$ is also not essentially surjective.
6. The Tensor Product of Grothendieck Categories: Functoriality, Associativity and Symmetry

In this section, based upon Theorem 5.4 above and the properties of the tensor product of $\alpha$-cocomplete categories, we prove that the tensor product of Grothendieck categories is functorial with respect to cocontinuous functors, associative and symmetric up to equivalence of categories.

Consider $\mathcal{A}, \mathcal{B}$ two locally presentable categories and a regular cardinal $\alpha$. A functor $F : \mathcal{A} \rightarrow \mathcal{B}$ is said to have rank $\alpha$ if it preserves $\alpha$-filtered colimits [5, §5.5]. It is trivial to see that if a functor $F$ has rank $\alpha$, then it has rank $\beta$ for every $\beta \geq \alpha$. We say a functor has rank if there exists a regular cardinal $\alpha$ such that it has rank $\alpha$. We have the following useful proposition.

**Proposition 6.1 ([5, Prop. 5.5.6]).** Let $G : \mathcal{A} \rightarrow \mathcal{B}$ a functor between locally presentable categories. If $G$ has a left adjoint, then $G$ has a rank.

The following is easy to show, but we provide a proof for the convenience of the reader.

**Proposition 6.2.** Let $F : \mathcal{A} \rightarrow \mathcal{B}$ a cocontinuous functor between locally presentable categories. Then, there exists a regular cardinal $\alpha$ such that $F(\mathcal{A}^\alpha) \subseteq \mathcal{B}^\beta$ for every $\beta \geq \alpha$.

**Proof.** By the dual of the Special Adjoint Functor Theorem [4, Thm. 3.3.4], we have that $F$ has a right adjoint $G$. In particular, by Proposition 6.1, $G$ has rank. Fix the smallest $\alpha$ such that $G$ has rank $\alpha$. Then, given an element $C \in \mathcal{A}^\alpha$, we have that

$$\mathcal{B}(F(C), \text{colim}_i D_i) = \mathcal{A}(C, G(\text{colim}_i D_i)) = \mathcal{A}(C, \text{colim}_i G(D_i)) = \text{colim}_i \mathcal{A}(C, G(D_i)) = \text{colim}_i (F(C), D_i)$$

where colim$_i D_i$ is any $\alpha$-filtered colimit in $\mathcal{B}$. Hence $F(C) \in \mathcal{B}^\alpha$ as desired. \hfill $\Box$

**Remark 6.3.** Given $F : \mathcal{A} \rightarrow \mathcal{B}$ as in the proposition, note that the restriction-corestriction $F^\alpha : \mathcal{A}^\alpha \rightarrow \mathcal{B}^\beta$ of $F$ is $\beta$-cocontinuous for all $\beta \geq \alpha$.

Consider Grothendieck $k$-linear categories $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and $\mathcal{D}$ and cocontinuous functors $F : \mathcal{A} \rightarrow \mathcal{C}$ and $F' : \mathcal{B} \rightarrow \mathcal{D}$. Take $\kappa$ the smallest regular cardinal for which both $F$ and $F'$ preserve $\alpha$-presentable objects for every $\alpha \geq \kappa$. We define a $k$-linear pseudonatural transformation

$$\Phi : F_{A,B} \Rightarrow C \otimes D,$$

where $F_{A,B}$ is defined as in (12), as follows. For each $\alpha \in \mathcal{K}$, we put

- If $\alpha < \kappa$, we define $\Phi_{\alpha}$ as the natural composition

$$\mathcal{A}^\alpha \otimes \mathcal{B}^\alpha \xrightarrow{F^\alpha} \mathcal{A}^\alpha \otimes \mathcal{C}^\alpha \mathcal{C}^\alpha \otimes \mathcal{D}^\alpha \leftarrow \mathcal{C} \otimes \mathcal{D};$$

- If $\alpha \geq \kappa$, we define $\Phi_{\alpha}$ as the natural composition

$$\mathcal{A}^\alpha \otimes \mathcal{B}^\alpha \xrightarrow{F_{\alpha} \otimes F'_{\alpha}} \mathcal{C}^\alpha \otimes \mathcal{D}^\alpha \leftarrow \mathcal{C} \otimes \mathcal{D};$$

where, for any regular cardinal $\gamma$, $F_\gamma : \mathcal{A}^\gamma \otimes \mathcal{B}^\gamma \rightarrow \mathcal{C}^\gamma \otimes \mathcal{D}^\gamma$ denotes the natural functor obtained from $F_\gamma \otimes F'_\gamma : \mathcal{A}^\gamma \otimes \mathcal{B}^\gamma \rightarrow \mathcal{C}^\gamma \otimes \mathcal{D}^\gamma$ via the universal property of $\otimes$. For each morphism $\alpha \leq \beta$ in $\mathcal{K}$, we set the invertible natural transformations

$$\Phi_{\alpha \leq \beta} : \Phi_{\beta} \circ F_{A,B}(\alpha \leq \beta) \Rightarrow \Phi_{\alpha}$$

to be the natural ones induced by the universal properties involved.
We are already in position to provide the desired associativity for the tensor product of Grothendieck categories with respect to cocontinuous functors. Indeed, we have the following:

**Definition 6.4.** Given cocontinuous functors \( F : \mathcal{A} \rightarrow \mathcal{C} \) and \( F' : \mathcal{B} \rightarrow \mathcal{D} \) as above, we define \( F \boxtimes F' : \mathcal{A} \boxtimes \mathcal{B} \rightarrow \mathcal{C} \boxtimes \mathcal{D} \) to be the functor associated to the pseudonatural transformation \( F_{\mathcal{A}, \mathcal{B}} \Rightarrow \mathcal{C} \boxtimes \mathcal{D} \) above via the universal property of \( \mathcal{L}(F_{\mathcal{A}, \mathcal{B}}) \).

**Remark 6.5.** Note that \( F \boxtimes F' \) is also cocontinuous. The filtered nature of the bicolimit plays an important role in the proof. Roughly it can be shown as follows. Consider \( \text{colim} \mathcal{X}_i \) the colimit of a small family of objects in \( \mathcal{A} \boxtimes \mathcal{B} \). Then, we can choose a regular cardinal \( \alpha \) such that \( \text{colim} \mathcal{X}_i \) is an \( \alpha \)-small colimit, all the \( \mathcal{X}_i \) are \( \alpha \)-presentable and \( F \) and \( F' \) preserve \( \alpha \)-presentable objects. Then we can see \( \text{colim} \mathcal{X}_i \) as an element in \( \mathcal{A}^\alpha \boxtimes_a \mathcal{B}^\alpha \) and we have that

\[
F \boxtimes F'(\text{colim} \mathcal{X}_i) = F_a \otimes_a F'_a(\text{colim} \mathcal{X}_i) = \text{colim}(F_a \otimes_a F'_a)(\mathcal{X}_i) = \text{colim}(F \boxtimes F')(\mathcal{X}_i),
\]

where we have used that \( F_a \otimes_a F'_a : \mathcal{A}^\alpha \boxtimes_a \mathcal{B}^\alpha \rightarrow \mathcal{C}^\alpha \boxtimes_a \mathcal{D}^\alpha \) preserves \( \alpha \)-small colimits by the universal property of \( \mathcal{L}(F_a) \).

Now, we proceed to prove the associativity and the symmetry of the tensor product of Grothendieck categories by using the filtered bicolimit construction we have provided in §5.

Consider Grothendieck \( k \)-linear categories \( \mathcal{A}, \mathcal{B} \) and \( \mathcal{C} \). Define now

\[
F_{\mathcal{A}, \mathcal{B}, \mathcal{C}} : \mathcal{K} \rightarrow \text{Cat}
\]

by \( F_{\mathcal{A}, \mathcal{B}, \mathcal{C}}(\alpha) = (\mathcal{A}^\alpha \boxtimes \mathcal{B}^\alpha) \boxtimes \mathcal{C}^\alpha \) with the natural transition functors

\[
F_{\mathcal{A}, \mathcal{B}, \mathcal{C}}(\alpha) : (\mathcal{A}^\alpha \boxtimes \mathcal{B}^\alpha) \boxtimes \mathcal{C}^\alpha \rightarrow (\mathcal{A}^\beta \boxtimes \mathcal{B}^\beta) \boxtimes \mathcal{C}^\beta
\]

induced by the universal property of \( \mathcal{L}(\mathcal{A}) \). Analogously, we put

\[
F_{\mathcal{A}, \mathcal{B}, \mathcal{C}}(\alpha) : \mathcal{K} \rightarrow \text{Cat}
\]

with \( F_{\mathcal{A}, \mathcal{B}, \mathcal{C}}(\alpha) = \mathcal{A}^\alpha \boxtimes (\mathcal{B}^\alpha \boxtimes \mathcal{C}^\alpha) \) with the natural transition functors

\[
F_{\mathcal{A}, \mathcal{B}, \mathcal{C}}(\alpha) : \mathcal{A}^\alpha \boxtimes (\mathcal{B}^\alpha \boxtimes \mathcal{C}^\alpha) \rightarrow \mathcal{A}^\beta \boxtimes (\mathcal{B}^\beta \boxtimes \mathcal{C}^\beta).
\]

In a similar fashion to Theorem 5.4, one can show that

\[
\mathcal{L}(F_{\mathcal{A}, \mathcal{B}, \mathcal{C}}) \cong (\mathcal{A} \boxtimes \mathcal{B}) \boxtimes \mathcal{C},
\]

and analogously

\[
\mathcal{L}(F_{\mathcal{A}, \mathcal{B}, \mathcal{C}}) \cong \mathcal{A} \boxtimes (\mathcal{B} \boxtimes \mathcal{C}).
\]

We know that, for any regular cardinal \( \alpha \), the category \( \text{Cat}^\alpha(k) \) of \( \alpha \)-cocomplete small categories endowed with \( \otimes_a \) is a closed monoidal symmetric bicategory. In particular, we have that

\[
(\alpha \otimes_b \beta) \otimes_c \gamma \cong \alpha \otimes_a (\beta \otimes_c \gamma)
\]

for all \( \alpha \)-cocomplete categories \( \alpha, \beta, \gamma \).

Consequently, there is a canonical isomorphism \( F_{\mathcal{A}, \mathcal{B}, \mathcal{C}}(\alpha) \cong F_{\mathcal{A}, \mathcal{B}, \mathcal{C}}(\alpha) \) for each \( \alpha \), and it behaves functorially. We thus have

\[
(13) \quad F_{\mathcal{A}, \mathcal{B}, \mathcal{C}} \cong F_{\mathcal{A}, \mathcal{B}, \mathcal{C}}.
\]

We are already in position to provide the desired associativity for the tensor product of Grothendieck categories:

**Proposition 6.6.** Let \( \mathcal{A}, \mathcal{B} \) and \( \mathcal{C} \) be Grothendieck categories, then, there exists an equivalence

\[
(\mathcal{A} \boxtimes \mathcal{B}) \boxtimes \mathcal{C} \cong \mathcal{A} \boxtimes (\mathcal{B} \boxtimes \mathcal{C}).
\]
Proof. It follows from applying filtered colimits to \((13)\).

The argument to prove the symmetry of the tensor product of Grothendieck categories is analogous. Consider two Grothendieck categories \(\mathcal{A}\) and \(\mathcal{B}\). As the monoidal bicategory \((\text{Cat}^\text{c}(k), \otimes_a)\) is symmetric, we have
\[
a \otimes_a b \cong b \otimes_a a
\]
for all \(\alpha\)-cocomplete categories \(a, b\).

Thus, reasoning as above, we have a canonical isomorphism
\[
F_{\mathcal{A}, \mathcal{B}} \cong F_{\mathcal{B}, \mathcal{A}},
\]
where \(F_{\mathcal{A}, \mathcal{B}}\) and \(F_{\mathcal{B}, \mathcal{A}}\) are defined as in \((12)\).

**Proposition 6.7.** Let \(\mathcal{A}, \mathcal{B}\) be Grothendieck categories. Then, there exists an equivalence
\[
\mathcal{A} \boxtimes \mathcal{B} \cong \mathcal{B} \boxtimes \mathcal{A}.
\]

**Proof.** It follows from applying filtered colimits to \((14)\). \(\square\)

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