Probabilistic switching can solve the problem of costly prosocial exclusion

Wei-Wei Xie, Hong-Ye Liu*, Feng Gao, Jie-Ru Zhang
School of Management, Northwest Minzu University, GanSu, China, 730030
*Correspondent Author: Liuhy@xbmu.edu.cn

Abstract: Public Goods Game (PGG) originates from “the Tragedy of Commons”, which is argued by the American scholar Hardin. PGG is a type of evolution game that these individuals in this game can affect each other, and reached a stable state. This paper demonstrates a new version PGG model which is an exclusion PGG model. Besides, combining Parrondo’s Paradox with this PGG model, and it also shows a similar “winning” outcome. Parrondo’s Paradox shows that two individual-losing games, when alternate, yield a winning outcome. Moreover, the additional periodic switching mechanism also applies to the PGG model and obtains a similar result when compared with that of conventional Parrondo’s Paradox. The biggest contribution of this paper is that Parrondo’s Paradox can fully applicable to the exclusion PGG model, and that simulation result also presents an effective result.

1. Introduction
Cooperation is an enduring topic in our world, and it has a significant role in the evolution of human civilization. Cooperation problems generally involving multi-agents instead of two individuals in a realistic world. As such, Public goods game (PGG) was proposed to solve this problem, and PGG refers to a multi-individual version of prisoners' dilemma. However, collaboration is vulnerable because of the self-interest of individuals who try their best to maximize short-term benefits. In addition to that, the cooperation will incur costs, and defectors can take a free rider and share the cooperators' payoff. Therefore, how to overcome this problem becomes an emergency for the scholars and government. To date, scholars proposed several strategies to solve the problem, which involving reputation, costly punishment, group voting, and intergenerational compensation. For instance, Hauser et al. (2014) [1] and Barfuss et al. (2020) [2] argued that cooperating with the future generation can promote individuals to cooperate. Fehr and Gächter (2002) [3] experiment indicated that the loom of sanction could improve the positive effect on individuals' average contribution in public goods games. Still, the prerequisite is that the game repeated many times, and once the sanction is weak, the individual's contribution will decrease to the marginal level. Some scholars (Nowak and Sigmund, 1998 [4], Panchanathan and Boyd, 2004 [5]) found that reputation is an essential factor in the model of cooperation.

Besides, social exclusion is also researched to solve the cooperation problem. Social exclusion refers to deter individuals from participating the social activities or denied benefit obtained from joint ventures and groups. Recent papers indicated that social exclusion outperforms punishment strategy (Li et al., 2015 [6] and Liu et al., 2017 [7]). In addition to that, Chen (2014) [8] switched two defecting scenes, pure cooperators and defectors, independent punishers, and defectors, to produce an overall cooperation outcome. Inspired by this research, this paper combines two mechanisms: the first one is similar to Chen's first strategy (2014) [8], and another only involves excluding and defectors. Intriguingly, this compound strategy produces prevailing cooperation among group members. As demonstrated in
previous work (Li et al. 2015 [6]), probabilistic exclusion can outperform the defection. This paper takes a different idea that cooperators and excluders yield, when switch, overall cooperation. That similar to Parrondo's Paradox (Parrondo et al., 2000 [9], Harmer and Aboot,1999 [10]), which means two losing games yield, when alternate, a winning outcome.

The rest of the paper written with proceeding way: the second section constructs the PGG model and setting the various variable of this model. Besides, this section is also going to show the spatial pattern of cooperation and gradient of selection. The third section expands the variable's analysis from the mechanism of paradox, and the last part will conclude the paper.

2. Model construction

This section introduces the new version of the PGG model, composed of collaborators (excluders and collaborators) and defectors. This new model comprises two strategies and takes the Monte Carlo simulation method to demonstrate spatial population distribution change over time. Using the replicator equation shows the evolution of cooperation over the fraction of cooperators of the new models.

2.1. Variable setting

| New Model Variable | Variable description |
|--------------------|----------------------|
| Group size($n$)    | The number of members in a single group in the public goods game (PGG). |
| Cooperator($C$)    | Individuals willing to contribute to the common pool but do not take any measures for those who do not contribute. They are also called second-order free riders. |
| Excluder($E$)      | Since cooperators contribute to the common pool and exclude the individuals who do not contribute from the shared benefits, they would pay extra costs to take exclusion measures. |
| Defector($D$)      | Individuals are unwilling to contribute to the common, and they share the public benefits that cooperators and excludes contribute to the common pool. They are also called free riders. |
| Fraction of excluder($p$) | The fraction of the cooperators in each group size $n$ is selected and switching as excluder. |
| Contribution amount($c$) | The cooperators in the group will deliver their part of the benefit to the shared pool, and the collection in turns gives back to the whole group. |
| Enhancement factor($r$) | Since all cooperators contribute to the pool's shared pool and resources will grow gradually with a speed $r$, and benefit all group members. |
| Exclude costs($c_E$) | Excluders ($E$) contributed $c$ to the common pool and also paid another extra cost $c_E$ to exclude defectors from the shared benefits. |

In summary, the following content will demonstrate the relationship between these variables. In a public goods game (PGG), there is a group size of $n$, and each cooperator $C$ contributes an amount $c$ to the common pool, while defectors $D$ donate nothing. The aggregate amount of the common pool equals the amount $c$ times the number of cooperators, and the total amount will increase with an enhancement factor $r > 1$ in each unit time. After that, the common pool will, in turn, give back benefits to all of the group members with equal value. Subsequently, a fraction of cooperator $p$ is randomly selected from the cooperator group and switching as excluder ($E$), and each excluder not only contributes amount $c$ but also pay another extra cost $c_E$ to exclude the defectors from the shared benefits. If there is at least one excluder, and each defector in this group will be excluded from the common pool benefit. Also, all excluders ($E$) bear the exclude costs together, and the average excluding charges is expressed as $\frac{(n-n_E)c_E}{n_E}$. At the same time, $n_C$ denotes the number of cooperators and $n_E$ represents the number of excluders. In agreement with these rules, if contribution amount $c = 1$, and the average payoff of cooperators equals to $\Pi_C = \frac{rn_C}{n} - 1$. Moreover, the average payoff of excluders denoted as $\Pi_E = \frac{rn_C}{n} - 1 - \frac{(n-n_E)c_E}{n_E}$. However, if there are no excluders in the group and the average payoff of the defectors is given by $\Pi_D = \frac{rn_C}{n}$. Otherwise, the benefits of defectors will be omitted by the excluder, and defectors get nothing. To simplify the analysis, the model omits the maintain cost of authority organization because the cost is different in different places. It is worth mentioning that when there is only one excluder, and other group members are defectors, this excluder has to bear all costs, which means $(n-1)c_E$. This strategy is costly and tends to a general defection.
2.2. Gradient of selection in well-mixed populations

After completing the model construction, the studied PGG's evolutionary dynamics can be demonstrated with a simpler version, which means using the replicator equation of the fraction of cooperators in well-mixed populations regardless of whether they are excluded or not. The method of replicator equation is taken from the study of Chen (2014) [8], and the equation is given by

\[ \frac{df}{dt} = f(1-f)(\Pi_W - \Pi_b) \]  

(1)

Where \( \Pi_W = p\Pi_E + (1-p)\Pi_C \) is the average payoff of all cooperators, and other algebra has been introduced and not repeated here. Algebra \( f \) denotes the fraction of the cooperator within the group, and its range value is \( 0 \leq f \leq 1 \). Assuming the interaction group size \( n \) is randomly selected from the well-mixed population in each round, and exploring the evolution dynamics of \( f \). The average payoff of three roles are given by

\[ \Pi_E = \sum_{x=0}^{n} \binom{n-1}{x} f^x (1-f)^{n-1-x} \times \sum_{y=0}^{x} \binom{x}{y} p^y (1-p)^{x-y} \left[ \frac{(x+1)}{n} - 1 - \frac{a(n-1-x)}{y+1} \right] \]  

(2)

\[ \Pi_C = \sum_{x=0}^{n} \binom{n-1}{x} f^x (1-f)^{n-1-x} \times \sum_{y=0}^{x} \binom{x}{y} p^y (1-p)^{x-y} \left[ \frac{x(n+1)}{n} - 1 \right] \]  

(3)

and

\[ \Pi_b = \sum_{x=0}^{n} \binom{n-1}{x} f^x (1-f)^{n-1-x} (1-p)^{x+y} \]  

(4)

respectively. The sought payoff difference is given by

\[ \Pi_W - \Pi_b = p\Pi_E + (1-p)\Pi_C - \Pi_b \]  

(5)

and the replicator equation can be expressed as the following equation:

\[ \frac{df}{dt} = f(1-f)(p\Pi_E + (1-p)\Pi_C - \Pi_b) \]  

(6)

The algebra \( x \) denotes the number of cooperators in the group, and the alphabet \( y \) represents the number of excluders of this group. As such, the gradient of selection is demonstrated in figure 1.

Besides, the three replicator equation of the original model is given by

\[ \Pi_E = \sum_{x=0}^{n} \binom{n-1}{x} f^x (1-f)^{n-1-x} \times \sum_{y=0}^{x} \binom{x}{y} p^y (1-p)^{x-y} \left[ \frac{(x+1)}{n} - 1 - \frac{a(n-1-x)}{y+1} \right] \]  

(7)

\[ \Pi_C = \sum_{x=0}^{n} \binom{n-1}{x} f^x (1-f)^{n-1-x} \times \sum_{y=0}^{x} \binom{x}{y} p^y (1-p)^{x-y} \left[ \frac{x(n+1)}{n} - 1 \right] \]  

(8)

\[ \Pi_b = \sum_{x=0}^{n} \binom{n-1}{x} f^x (1-f)^{n-1-x} \sum_{y=0}^{x} \binom{x}{y} p^y (1-p)^{x-y} \left[ \frac{n-a}{n} \right] + \sum_{x=0}^{n} \binom{n-1}{x} f^x (1-f)^{n-1-x} (1-p)^{x+y} \]  

(9)

According to the equation's average, the gradient of each model’s selection can be obtained, and it shows below.

![Figure 1. Gradient of selection](image)

Figure 1. Probabilistic exclusion in well-mixed populations transform the public goods game (PGG) into a cooperation game, which game with full cooperation and defection as the two stable equilibrium. The depiction is the gradient of selection depends on the different fraction of cooperators. Steady-state is state \( f = 0 \) and \( f = 1 \).

It can be seen from Figure 1 above that the algebra \( f \) denotes the proportion of collaborators in the population. As mentioned earlier, prosocial exclusion strategies perform better than punishment.
strategies. To compare the exclusion strategy's advantages and the punishment strategy more clearly, the exclusionary needs to bear a higher exclusion cost than the punisher. Although the cost of exclusion ($c_E$) is higher than that of punishment, A1 shows that the basis for attracting cooperation is greater than defection. Also, A2 shows that the base of cooperative attraction is more significant than defection strategy. Despite its high cost, exclusion strategies are still useful. However, choosing to increase the fee of exclusion in this article is that under the premise of low exclusion cost, the basin of attraction is almost above the level line. This means the exclusion strategy is successful when the cost is low, and Li et al. (2015) [7] also found that the exclusion strategy could fail to promote cooperation under the high excluding charges. Figure A1 proved that the exclusion strategy might lead to failure when the cost is high enough.

2.3 Spatial pattern of cooperation

The well-mixed population is an ideal situation in PGG, while the individuals in the community will not randomly interact with each other but are limited to the other individuals. As such, this section takes a Monte Carlo simulation to demonstrate the dynamic evolution of cooperation. Using a simple square lattice map to illustrate this dynamic process beyond the well-mixed population assumption, the model can realize the interaction among these players within its inherent structure rather than random. Since Nowak and Sigmund (1998) [4] proposed the lattice method, many researchers studied the evolution game and continue to nowadays. The lattice method shows a significant difference in the evolution game outcome when the well-mixed population assumption is abandoned to use a structured population. Many research results have followed this approach and provided strong support for square lattice claims. Especially for games affected by group interactions, the use of square lattice can reveal all relevant evolutionary results, and these results are qualitatively independent of the interaction structure. Besides, the method's details can be seen in the Li et al. 2017[7] and Chen (2014) [8].

As such, the square lattice updating within these two strategies, and each Monte Carlo step gives each player a possibility to imitate others' approaches once on average beyond their aspirations. It’s worth mentioning that the fraction of cooperator $f$ (regardless of whether they exclude or cooperate) is determined in advance. Hence, $f$ is considered a time-independent variable.

![Figure 2](image-url)

Figure 2. PGG's spatial pattern is obtained through the 10000 times Monte Carlo step (MCS) and setting the excluding cost of the excluders $c_E = 3$. These figures above are results of 0, 5, 100, and 10000MCS,
and parameters are the same except for the Exclusion cost. The PGG's initial state consists of 50% cooperators (regardless of cooperators of excluders) and 50% defectors, that are comprised of well-mixed populations. The switching probability of cooperators \( p = \frac{1}{2} \) and square lattice size \( L = 100 \).

Figure 2 is a well-mixed population model of PGG, the detail of punishment strategy can be seen in Chen (2014) \[8\]. Therefore, the discussion of punishment strategy will not be repeated here, and the following content will demonstrate the scenario of Exclusion strategy. Panels 1-12 is the exclusion strategy scenario. As to pictures 9-12, they result from pure excluders and defectors, and it demonstrates that defectors prevail in this PGG when PGG is played at 100 MCS. Although the previous research concludes that the exclusion strategy possesses the most significant competition than other approaches (Liu and Chen, 2017) \[7\], excluders might fail to dominate the whole group of individuals under the condition of high excluding costs. However, panel 5-8 is a scenario where cooperators can be switched to excludes and eventually dominate the entire group. It means that probabilistic excluding can overcome the problem of high excluding cost, and lead to the complete cooperation of PGG players.

Figure 3. Simulation results of two switch mechanism in PGG

Figure 2 is a result of a well-mixed population, shows a similar Parrondian phenomenon. Moreover, the community is not randomly distributed among the space in the real world. Individuals who have similar characteristics tend to live together rather than randomly distributed, called a network. Hence, figure 3 is a simulation result of the structured population, and it also shows a similar effect to that of well-mixed communities. Figure 3 also shows a periodic switch between two individual losing strategies of two versions, which also fit the mechanism of Parrondo’s Paradox. The periodic switch not only makes the cooperator bear the cost (except for the contribution amount) but also can "conquer" the territory of defectors. In terms of the exclusion strategy, collaborators collect to defend the defectors' invasion in this round. These collaborators can be switched to repulsors with powerful strength to exclude defectors' payoff in the next PGG. Therefore, defectors will have a strong motivation to change as cooperators to lest excluded by these excluders. Figure c, g above shows the result of each strategy's 100th MCS. The territory of collaborators who adopt the periodical conversion strategy is larger than that of the probability conversion strategy under the same condition, which means that the periodic strategy is better than the probability strategy. This phenomenon is consistent with Parrondo’s Paradox, which means the periodic combination of two losing methods is better than the probabilistic combination.

The simulation result above proves that the rules of Parrondo’s Paradox can also apply in the PGG and can obtain better results when using it. The periodic and probabilistic strategy can also spread the costs among these cooperators, solve the costly punishment and exclusion problem, and boost the cooperation of the population to solve Commons' tragedy.

3. The analysis of the model result
According to the result of the simulation, cooperators alternate their roles can produce an overall
cooperation result. From the strategy of pure excluders and pure defectors, the result of competition among these individuals showing this strategy is an overall failure strategy under the condition of specified excluding cost. The analysis of the excluding cost range has been illustrated in many papers (Li et al. 2015 [6], Liu and Chen, 2019 [7]). Hence, no repeated here, and this section will analyze the mechanism of paradox.

The switching role between cooperators and excluders can outperform defectors under high excluding cost, and it has been proven in the simulation. When these cooperators are pure excluders, they need to bear the excluding costs for a long time. The imitation strategy is these individuals choose one neighbor as their reference objective. Once an excluder chooses the defector as their imitation objective, the probability of switching is more significant than 0.5. These excluders will turn to behave as defectors in the long run of PGG. However, if the excluder can switch as a defector randomly, they can overturn the losing result. It can be analyzed from two dimensions; the first one is that the stochastic switching role can make cooperators and excluders undertake the whole excluding cost. Those who did not switch to defectors will change their status per PGG, and hence they can share the excluding costs to decrease the burden of these excluders. In this way, inducing the cooperators and excluding help cooperation rather than defection. For the second reason, the changing role can change the invasion direction of excluders and continuously expand their territory. Because only if there are any excluders in the group, the payoff will decrease by zero, and they have a strong tendency to switching as a type of “cooperator” to obtain the positive yield next PGG.

As such, these collaborators (whether pure collaborators or excluders) can use probability conversion to invade the defector’s area. This strategy is similar to the transition between attack and defense. When these excluders present at the border, they can invade the defector’s territory and is like an attack strategy. However, when collaborators and excluders change roles, internal collaborators become excluders, and external excluders become collaborators. The inside excluders function as a defense strategy to prevent invasion from the defectors and assimilate the group’s defectors inside. In this way, eliminate the inside defectors and defense the invasion of outside defectors. In terms of pure cooperators and defectors, the cooperators can only assemble to defend the defectors' attack passively, and defectors can enjoy the cooperators' payoff and do nothing at all. It's obvious a weak strategy. For excluders and defectors, excluders prevail in the whole group because of the substantial exclusion and keep the defectors from enjoying the payoff. The recent papers also compared the competition power of these strategies and found that the exclusion strategy outperforms than other methods. However, if the exclusion cost reached a value, the excluders can also be defeated by defectors. The high prices lead to excluders tend to switch as defectors because of no contribution and no cost.

4. Conclusion
This paper researches the PGG problem with a Parrondian mechanism to realize PGG players' prevailing cooperation based on the PGG model of Chen (2014) [9]. This paper also uses attraction basins to demonstrate the PGG trend based on the different punishing costs, excluding costs, and the proportions of switching among these cooperators. Consistently, the evolution of PGG is also toward the direction of the attraction basin. The new model (exclusion strategy) shows that the result of the exclusion strategy is similar to that of the punishment strategy. However, the exclusion costs are higher than punishment costs in this PGG and proved that competition of exclusion strategy is more considerable than the punishment strategy. Besides, the excluders can be defeated by defectors under the condition of high exclusion costs, and sub-pictures 9-12 of figure 2 show the result of the PGG when there are only defectors and excluders. Similar to probabilistic punishment, probabilistic exclusion can also overcome the costly exclusion problem and achieve full cooperation. Besides, periodic switching of two types of cooperators can even realize the complete cooperation in this PGG regardless of high exclusion costs. In short, these conclusions have been proven in Parrondo’s Paradox can be applied to the PGG and achieve a winning result.

In summary, this paper demonstrates that probabilistic exclusion can solve the costly exclusion problem. Switching between the two types of collaborators (excluders and collaborators) periodically
can also solve this problem, even better than eliminating possibilities. In this way, the mechanism of exclusion can be a good measure to solve the social problem, and possess the bright future.

Reference
[1] Hauser, O. P., Rand, D. G., Peysakhovich, A., Nowak, M. A. (2014). Cooperating with the future. Nature 511, 220–223.
[2] Barfussa, W., Donges, J. F., Vasconcellos, V.V., Kurths, J., Levin, S. A. (2020). Caring for the future can turn tragedy into comedy for long-term collective action under risk of collapse. PNAS.
[3] Fehr, E., & Gächter, S. (2002). Erratum: Altruistic punishment in humans. Nature, 415(6868), 137–140.
[4] Nowak, M. A., & Sigmund, K. (1998). Evolution of indirect reciprocity by image scoring. Nature, 393(6685), 573–577.
[5] Panchanathan, K., & Boyd, R. (2004). Indirect reciprocity can stabilize cooperation without the second-order free rider problem. Nature, 432(7016), 499–502.
[6] Li, K., Cong, R., Wu, T. & Wang, L. (2015). Social exclusion in finite populations. Phys. Rev. E 91, 042810.
[7] Liu, L. Chen, X. J., Szolnoki, A. (2017). Competitions between prosocial exclusions and punishments in finite populations. Sci. Rep. 7, 46634.
[8] Chen, X. J., Szolnoki, A., Perc, M. (2014). Probabilistic sharing solves the problem of costly punishment. New J. Phys. 16, 083016.
[9] Parrondo, J. M. R. et al. (2000). New Paradoxical Games Based on Brownian Ratchets. Physical review Letters 85, 5226.
[10] Harmer, G. P., & Abbott, D. (1999). Game theory - Losing strategies can win by Parrondo’s Paradox. Nature 402, 864–864.