Solvable Quotients of Kähler Groups

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Abstract

We prove several results on the structure of solvable quotients of fundamental groups of compact Kähler manifolds (Kähler groups).

1. Introduction.

We first recall a definition from [AN].

Definition 1.1 A solvable group $\Gamma$ has finite rank, if there is a decreasing sequence $\Gamma = \Gamma_0 \supset \Gamma_1 \supset \ldots \supset \Gamma_{m+1} = \{1\}$ of subgroups, each normal in its predecessor, such that $\Gamma_i/\Gamma_{i+1}$ is abelian and $\mathbb{Q} \otimes (\Gamma_i/\Gamma_{i+1})$ is finite dimensional for all $i$.

In what follows $F_r$ denotes a free group with the number of generators $r \in \mathbb{Z}_+ \cup \{\infty\}$. Our main result is

Theorem 1.2 Let $M$ be a compact Kähler manifold. Assume that the fundamental group $\pi_1(M)$ is defined by the sequence

$$\{1\} \rightarrow F \rightarrow \pi_1(M) \xrightarrow{p} H \rightarrow \{1\}$$

where $H$ is a solvable group of finite rank of the form

$$\{0\} \rightarrow A \rightarrow H \rightarrow B \rightarrow \{0\}$$

with non-trivial abelian groups $A, B$ so that $\mathbb{Q} \otimes A \cong \mathbb{Q}^m$ and $m \geq 1$. Assume also that $p^{-1}(A) \subset \pi_1(M)$ does not admit a surjective homomorphism onto $F_{\infty}$. Then all eigen-characters of the conjugate action of $B$ on the vector space $\mathbb{Q} \otimes A$ are torsion.

In Lemma 2.3 we will show that the condition for $p^{-1}(A)$ holds if $F$ does not admit a surjective homomorphism onto $F_{\infty}$.

Using Theorem 1.2 we prove a result on solvable quotients of Kähler groups.

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Theorem 1.3 Assume that a Kähler group $G$ is defined by the sequence

$$
\{1\} \xrightarrow{} F \xrightarrow{} G \xrightarrow{q} H \xrightarrow{} \{1\}
$$

where $F$ does not admit a surjective homomorphism onto $F_\infty$ and $H$ is a solvable group of finite rank. Then there exist normal subgroups $H_1 \supset H_2$ of $H$ so that

(a) $H_1$ has finite index in $H$;
(b) $H_1/H_2$ is nilpotent, and
(c) $H_2$ is torsion.

Remark 1.4 (1) Clearly the conclusion of Theorem 1.2 is valid for $H$ being an extension of $\mathbb{Z}^n$ by $\mathbb{Z}^m$, $n, m \geq 1$, and $F$ being a finitely generated group. Assume that not all eigen-characters of the action of $\mathbb{Z}^n$ on $\mathbb{Q} \otimes \mathbb{Z}^m$ are torsion. Then a semidirect product of such $F$ and $H$ (where $F$ is a normal subgroup of this product) is not a Kähler group.

(2) Let $G$ be a Kähler group. By $DG = D^1G$ we denote the derived subgroup of $G$, and set $D^iG = DD^{i-1}G$. Assume that $H := G/D^nG$, $n \geq 1$, is a solvable group of finite rank. Then it was proved in [AN, Th. 4.9] and [Ca, Th. 2.2] that $H$ satisfies conditions (a)-(c) of Theorem 1.3. It is a consequence of the fact that $G$ does not admit a surjective homomorphism onto $F_r$ with $2 \leq r < \infty$.

2. Proof of Theorem 1.2.

In what follows $T_n(\mathbb{C}) \subset GL_n(\mathbb{C})$ denotes the Lie group of upper triangular matrices. Let $T_2 \subset T_2(\mathbb{C})$ be the Lie group of matrices of the form

$$
\begin{pmatrix}
  a & b \\
  0 & 1
\end{pmatrix} \quad (a \in \mathbb{C}^*, \ b \in \mathbb{C}),
$$

$D_2 \subset T_2$ and $N_2 \subset T_2$ be the groups of diagonal and unipotent matrices. Let $M$ be a compact Kähler manifold. For a homomorphism $\rho \in Hom(\pi_1(M), T_2)$ we let $\rho_\alpha \in Hom(\pi_1(M), \mathbb{C}^*)$ denote the upper diagonal character of $\rho$. The main result used in our proofs is the following

**Proposition 2.1** Assume that $\pi_1(M)$ is defined by the sequence

$$
\{1\} \xrightarrow{} F \xrightarrow{} \pi_1(M) \xrightarrow{} H \xrightarrow{} \{1\}
$$

where the normal subgroup $F$ does not admit a surjective homomorphism onto $F_\infty$. Assume that $\rho \in Hom(\pi_1(M), T_2)$ satisfies $F \subset Ker(\rho_\alpha)$ but $F \notin Ker(\rho)$. Then $\rho_\alpha$ is a torsion character.
Proof. Given a character $\xi \in \text{Hom}(\pi_1(M), \mathbb{C}^*)$, let $\mathbb{C}_\xi$ denote the associated $\pi_1(M)$-module. We define $\Sigma^1(M)$ to be the set of characters $\xi$ such that $H^1(\pi_1(M), \mathbb{C}_\xi)$ is nonzero. The structure of $\Sigma^1(M)$ was described in the consequent papers of Beauville [Be], Simpson [S], Campana [Ca].

**BSC Theorem.** There is a finite number of surjective holomorphic maps with connected fibres $f_i : M \to C_i$ onto smooth compact complex curves of genus $\geq 1$ and torsion characters $\rho_i, \xi_j \in \text{Hom}(\pi_1(M), \mathbb{C}^*)$ such that

$$
\Sigma^1(M) = \bigcup_i \rho_i f_i^* \text{Hom}(\pi_1(C_i), \mathbb{C}^*) \cup \bigcup_j \{\xi_j\}
$$

Further, the group $N_2$ acts on $T_2$ by conjugation. Any two homomorphisms from $\text{Hom}(\pi_1(M), T_2)$ belonging to the orbit of this action will be called equivalent.

Let $\rho \in \text{Hom}(\pi_1(M), T_2)$ satisfy the conditions of Proposition 2.1. Then it is well known that the class of equivalence of $\rho$ is uniquely defined by an element $c_\rho \in H^1(\pi_1(M), \mathbb{C}_{\rho_a})$ (see e.g. [A, Prop. 2]). In particular, if $c_\rho = 0$, $\rho$ is equivalent to a representation into $D_2$. In our case, $c_\rho \neq 0$ because $F \not\subseteq \text{Ker}(\rho)$. Thus $\rho_a$ satisfies the conditions of BSC Theorem. If $\rho_a$ coincides with one of $\xi_j$ then it is torsion by the above theorem. So assume that $\rho_a = \rho_i f_i^* \phi$ for some torsion character $\rho_i$ and $\phi \in \text{Hom}(\pi_1(C_i), \mathbb{C}^*)$. Let $K := \text{Ker}(\rho_i) \subset \pi_1(M)$ and $p : M_1 \to M$ be the Galois covering of $M$ corresponding to the finite abelian Galois group $\pi_1(M)/K$.

Let $M_1 \xrightarrow{\varrho} C \xrightarrow{h} C_1$ be the Stein factorization of $f_i \circ p$. Here $g$ is a morphism with connected fibres onto a smooth curve $C$ and $h$ is a finite morphism. Assume, to the contrary, that $\rho_a$ is not torsion. Then we prove

**Lemma 2.2** $F \subset \text{Ker}(g_*)$.

**Proof.** Set $G := (f_i)_* F \subset \pi_1(C_1)$. According to the assumptions of Proposition 2.1 we have either (a) $G$ is a subgroup of finite index in $\pi_1(C_1)$, or (b) $G$ is isomorphic to $F_\varrho$ with $r < \infty$. Let us consider (a). Since by definition $F \subset \text{Ker}(\rho_a)$ and $\rho_i$ is torsion, $\phi(G)$ is a finite abelian group. Then $G_1 := \text{Ker} \phi \cap G$ is a subgroup of finite index in $G$. In particular, $G_1$ is a subgroup of finite index in $\pi_1(C_1)$ and so it is not free. But by our assumption, $\phi$ is not torsion and so $\text{Ker}(\phi) \cong F_\infty$. Thus $G_1$ is also free as a subgroup of $F_\infty$. This shows that (a) is never happen.

Consider now (b). Using the fact that $(f_i)_*$ is a surjection, we conclude that $G$ is a normal subgroup of $\pi_1(C_1)$. Let $S \to C_1$ be a regular covering corresponding to $Q := \pi_1(C_1)/G$. If $r \geq 2$ then the group $\text{Iso}(S)$ of isometries of $S$ (with respect to the hyperbolic metric) is finite and since $Q$ is infinite we have $r \leq 1$. If $r = 1$, any discrete subgroup of $\text{Iso}(S)$ is virtually cyclic and in particular does not act cocompactly on $S$. Thus $r = 0$ which means that $G = \{e\}$ and $F \subset \text{Ker}(f_i)_*$. Then the assumption of the proposition implies that $F \subset K = \pi_1(M_1)$.

Now consider $\tilde{\rho} := \rho_{\pi_1(M_1)}$ with the upper diagonal character $\tilde{\rho}_a := \rho_{a|\pi_1(M_1)}$. Since $\pi_1(M_1) \subset \pi_1(M)$ is a subgroup of finite index, $\tilde{\rho}_a$ is also not torsion. Set $\tilde{\phi} := h^* \phi$. Then $\tilde{\rho}_a = g^* \tilde{\phi}$. Let $G_1 := g_* F \subset \pi_1(C)$. Then the same argument for $G_1$ as above for $G$ (with $\tilde{\rho}_a$ and $\tilde{\phi}$ instead of $\rho_a$ and $\phi$) yields $F \subset \text{Ker}(g_*)$. \hfill $\square$

According to Lemma 2.2 and the assumptions of Proposition 2.1 we have that $\tilde{\rho}|_{\text{Ker}(g_*)}$ is non-trivial and $\tilde{\rho}_a$ is the pullback of a character from $\text{Hom}(\pi_1(C), \mathbb{C}^*)$. 

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Then from [Br, Proposition 3.6] it follows that \( \tilde{\rho}_a \) is torsion. Therefore \( \rho_a \) is torsion, as well. This contradiction proves the proposition. \( \square \)

We are ready to prove Theorem [1.2]. According to the assumptions of the theorem there is a homomorphism \( i \) of \( H \) into the Lie group \( R \) of the form

\[
\{0\} \rightarrow \mathbb{C}^m \rightarrow R \rightarrow B \rightarrow \{0\}
\]

whose kernel is \( Tor(A) \). Here we identify \( \mathbb{C}^m \) with \( \mathbb{C} \otimes A \). Consider the action \( s : B \rightarrow GL_m(\mathbb{C}) \) by conjugation. Since, by the definition of \( \pi_1(M) \), \( B \) is a finitely generated abelian group, \( s = \oplus_{j=1}^d s_j \) where \( s_j \) is equivalent to a nilpotent representation \( B \rightarrow T_{m_j}(\mathbb{C}) \) with a diagonal character \( \rho_j \). Here \( \sum_{j=1}^d m_j = m \). From this decomposition it follows that there is an invariant \( B \)-submodule \( V_j \subset \mathbb{C}^m \) of \( dim_{\mathbb{C}}V_j = m - 1 \) such that \( W_j = \mathbb{C}^m/V_j \) is a one-dimensional \( B \)-module and the action of \( B \) on \( W_j \) is defined as multiplication by the character \( \rho_j \). By definition, \( V_j \) is a normal subgroup of \( R \) and the quotient group \( R_j = R/V_j \) is defined by the sequence

\[
\{0\} \rightarrow \mathbb{C} \rightarrow R_j \rightarrow B \rightarrow \{0\}.
\]

Here the action of \( B \) on \( \mathbb{C} \) is multiplication by the character \( \rho_j \). (As before the associated \( B \)-module is denoted by \( \mathbb{C}_{\rho_j} \).) Let us denote by \( t_j \) the composite homomorphism \( \pi_1(M) \rightarrow H \rightarrow R \rightarrow R_j \). Further, the equivalence class of extensions of \( B \) by \( \mathbb{C} \) isomorphic to \( R_j \) is defined by an element \( c_j \in H^2(B, \mathbb{C}_{\rho_j}) \). We assume that the character \( \rho_j \) is non-trivial (for otherwise, \( \rho_j \) is clearly torsion). Then \( H^2(B, \mathbb{C}_{\rho_j}) = 0 \) (for the proof see e.g. [AN, Lemma 4.2]). This shows that \( R_j \) is isomorphic to the semidirect product of \( \mathbb{C} \) and \( B \), i.e., \( R_j = \mathbb{C} \times B \) with multiplication

\[
(v_1, g_1) \cdot (v_2, g_2) = (v_1 + \rho_j(g_1) \cdot v_2, g_1 \cdot g_2), \quad v_1, v_2 \in \mathbb{C}, \ g_1, g_2 \in B.
\]

Let us determine a map \( \phi_j \) of \( R_j \) to \( T_2 \) by the formula

\[
\phi_j(v, g) = \begin{pmatrix}
\rho_j(g) & v \\
0 & 1
\end{pmatrix}
\]

Obviously, \( \phi_j \) is a correctly defined homomorphism with upper diagonal character \( \rho_j \). Hence \( \phi_j \circ t_j : \pi_1(M) \rightarrow T_2(\mathbb{C}) \) is a homomorphism non-trivial on \( p^{-1}(A) \subset \pi_1(M) \) by its definition. Also \( p^{-1}(A) \subset Ker(\rho_j \circ t_j) \). Since by our assumptions \( p^{-1}(A) \) does not admit a surjective homomorphism onto \( F_\infty \), Proposition [2.1] applied to \( \phi_j \circ t_j \) implies that \( \rho_j \) is torsion. This completes the proof of the theorem. \( \square \)

We prove now the following result.

**Lemma 2.3** Assume that a group \( G \) is defined by the sequence

\[
\{1\} \rightarrow G_1 \rightarrow G \rightarrow G_2 \rightarrow \{1\}
\]

where \( G_1, G_2 \) do not admit surjective homomorphisms onto \( F_\infty \). Then \( G \) satisfies the similar property.
Proof of Lemma 2.3. Assume, to the contrary, that there is a surjective homomorphism \( \phi : G \to F_\infty \). Then \( \tilde{G}_1 := \phi(G_1) \) is a normal subgroup of \( F_\infty \) and \( \tilde{G}_2 := F_\infty / \tilde{G}_1 \) is a quotient of \( G_2 \). Now the assumption of the lemma implies that \( \tilde{G}_1 \cong F_r \) with \( r < \infty \). Let \( X \) be a complex hyperbolic surface with \( \pi_1(X) = F_\infty \) and \( S \to X \) be the regular covering corresponding to \( \tilde{G}_2 \). Assume first that \( r \geq 1 \). Since \( \pi_1(S) = \tilde{G}_1 \), any subgroup of the group \( \text{Iso}(S) \) of isometries of \( S \) (with respect to the hyperbolic metric) is finitely generated. In particular \( \tilde{G}_2 \) is finitely generated (as well as \( \tilde{G}_1 \)). This implies that \( F_\infty \) should be finitely generated which is wrong. Thus \( r = 0 \) and \( \tilde{G}_2 = F_\infty \). This contradicts to our assumption and shows that there is no such \( \phi \). \( \Box \)

3. Proof of Theorem 1.4.

For a group \( L \) set \( L^{ab} := L/\text{DL} \). We say that an \( L \)-module \( V \) is quasi-unipotent if there is a subgroup \( L' \subseteq L \) of finite index whose elements act unipotently on \( V \).

To prove the theorem we will check the following condition from [AN, Lemma 4.8].

Lemma 3.1 Let \( H' \subseteq H \) be a subgroup of finite index. Then \( H' \) acts quasi-unipotently on the finite-dimensional vector space \( \mathbb{Q} \otimes (H' \cap DH)^{ab} \).

Proof. We set \( K := H'/(H' \cap DH) \), \( G' := q^{-1}(H') \), \( S := D(H' \cap DH) \), and \( S' := q^{-1}(S) \). Here \( K \) is a finitely generated abelian group. Indeed, \( H' \) is finitely generated as a subgroup of finite index of the finitely generated group \( H (= \text{the image of the finitely generated group } G) \). Thus \( K \) is finitely generated as the image of \( H' \). The fact that \( K \) is abelian follows directly from the definition. Further, since \( H' \) is a subgroup of finite index in \( H \), \( G' \) is a subgroup of finite index in \( G \). In particular, it is Kähler. Moreover, we have

\[
\{1\} \to S' \to G' \to L \to \{1\}
\]

where \( L \) is a solvable group of finite rank defined by the sequence

\[
\{0\} \to (H' \cap DH)^{ab} \to L \to K \to \{0\}.
\]

Now the statement of the lemma is equivalent to the fact that all eigen-characters of the conjugate action of \( K \) on \( \mathbb{Q} \otimes (H' \cap DH)^{ab} \) are torsion. To prove that it suffices to show that \( S' \) does not admit a surjective homomorphism onto \( F_\infty \), and then to apply Lemma 2.3 and Theorem 1.2.

Note that \( S' \) is defined by the sequence

\[
\{1\} \to F \to S' \xrightarrow{q} S \to \{1\}
\]

where \( S \) is a solvable group of finite rank. By our assumption \( F \) does not admit a surjective homomorphism onto \( F_\infty \). Thus by Lemma 2.3, \( S' \) satisfies the same property. \( \Box \)

Now Theorem 1.3 is the consequence of Lemma 3.1 and [AN, Lemma 4.8]. \( \Box \)
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