Convergence of Spherical Harmonic Series Expansion of the Earth’s Gravitational Potential

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Abstract  Given a continuous boundary value on the boundary of a “simply closed surface” \( \partial S \) that encloses the whole Earth, a regular harmonic fictitious field \( V^*(P) \) in the domain outside an inner sphere \( iK \) that lies inside the Earth could be determined, and it is proved that \( V^*(P) \) coincides with the Earth’s real field \( V(P) \) in the whole domain outside the Earth. Since in the domain outside the inner sphere \( iK \) and the fictitious regular harmonic function \( V^*(P) \) could be expressed as a uniformly convergent spherical harmonic series, it is concluded that the Earth’s potential field could be expressed as a uniformly convergent spherical harmonic expansion series in the whole domain outside the Earth.

Keywords  potential field; fictitious field; uniqueness of the solution; spherical harmonic series; uniform convergence

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Introduction

One of the main tasks of physical geodesy is to determine the Earth’s external gravitational potential field \( V(P) \). If the mass density distribution \( \rho \) is known, the field \( V(P) \) can be determined by using the well-known Newtonian potential formula:

\[
V(r, \theta, \lambda) = G \int_{\Omega} \frac{\rho}{l_{QP}} \, d\tau, \quad (r, \theta, \lambda) \in \bar{\Omega}
\]

(1)

where \( G \) is the gravitational constant, \( \Omega \) the open domain occupied by the Earth (i.e., \( \Omega \) does not include the Earth’s surface), \( \bar{\Omega} \) the domain outside the Earth, including the boundary \( \partial \Omega \), \( r \) the distance between the coordinate origin and the field point \( P(r, \theta, \lambda) \), \( \theta \) the co-latitude, \( \lambda \) the longitude, \( l_{QP} \) the distance between the field point \( P(r, \theta, \lambda) \) and the integration moving point \( Q(r', \theta', \lambda') \), \( d\tau \) the integration volume element at the moving point \( Q \). Here, it is assumed that a spherical coordinate \( o = r\theta\lambda \) has been previously introduced, and the coordinate origin \( o \) is not necessarily coinciding with the mass center of the Earth.

Since it is almost impossible to obtain a complete knowledge about the Earth’s mass density distribution \( \rho \), the potential field \( V(P) \) cannot be determined with adequate precision by Eq. (1). To determine the real field \( V(P)(P \in \bar{\Omega}) \) with adequate precision, the boundary value problem is generally considered: given the boundary value \( V_{\partial\Omega} \) (or other kinds of boundary conditions) on the boundary \( \partial \Omega \) of the Earth, determine the external field \( V(P)(P \in \bar{\Omega}) \).
In practice, however, it is almost impossible to measure the potential (or gravity) on the whole boundary $\partial \Omega$ of the Earth (especially in the mountain areas and ocean areas). What we could obtain is the gravity data distributed on a closed spatial surface $S$ that encloses the whole Earth, e.g., through CHAMP mission\cite{1}, GRACE mission\cite{2} or GOCE mission\cite{3}. Now, suppose the boundary value $V_{\partial S}$ on the boundary $\partial S$ is given, e.g., using the energy integral approach\cite{4}. For simplicity and without loss of generality, we first suppose the surface $\partial S$ is a spherical surface $\partial K_s$ (for the case of a non-spherical surface $\partial S$, see Theorem 4 in Sec.3). In this case, one can use Poisson integral to determine the field $V(P)$ in the domain $K_s$, the domain outside the boundary $\partial K_s$. How to determine the Earth’s real field $V(P)$ in the domain between the Earth’s surface $\Omega$ and the spherical surface $\partial K_s$ is a question of “downward continuation”, which is a problem that has not been completely solved because it is an “ill-posed problem”\cite{5-7}.

Or, one can use the spherical harmonic series, which is much more useful in practical applications. With this method, the key problem is whether the spherical harmonic series can be continued down till the Earth’s surface, or in other words, whether there exists a spherical harmonic series that uniformly converges and coincides with the Earth’s real field $V(P)$ in the whole domain outside the Earth. This is a problem on the (uniform) convergence of the spherical harmonic series of the Earth’s potential in the domain $\Omega$ (the domain outside the Earth), which has not been completely solved\cite{5,6,8-14}. There is no doubt about the conclusion that there exists a spherical harmonic series that uniformly converges in the domain outside Brillouin sphere $B_K$ \cite{15} and coincides with the Earth’s real field $V(P)$ in that domain\cite{5,6}, noting that in this case the origin $o$ coincides with the center of the Brillouin sphere $K_b$ (Cf. Fig.1). In the domain outside Brillouin sphere, the considered series converges. The divergence might occur in the domain between the two surfaces $\partial \Omega$ and $\partial K_s$. The left problem is whether the spherical harmonic series converges in the domain between the Earth’s surface $\partial \Omega$ and the surface $\partial K_b$ of $K_b$. Since there still exist a lot of controversies\cite{5,6,8-19}, it is generally agreed that the convergence problem is still open. For instance, Wang (1997)\cite{12} and Ågren(2004)\cite{13} stated that it is wise to leave the uniform convergence problem open, “an arbitrary small change of the external potential may change convergence into divergence, and vice versa”\cite{5}.

In this paper, it is demonstrated that the Earth’s gravitational potential field $V(P)$ can be expressed as a uniformly convergent spherical harmonic series in the whole domain outside the Earth, and consequently, the conventionally applied approach that the spherical harmonic expansion series is truncated has a firm foundation.

For convenience, it will be defined that the “simply closed surface” $\partial S$ always means that it is a smooth closed surface that encloses the whole Earth and satisfies the following conditions: for every straight ray $oR_\infty$ from origin $o$ to infinity, if $oR_\infty$ contains one and only one point $Q$ that belongs to the surface $\partial S$, then we say that $\partial S$ is a “simply closed surface” with respect to (w.r.t.) the origin $o$.

1 The fictitious field

Suppose the following statement is true: given the gravitational potential boundary value $V(P)|_{\partial \Omega} \equiv V_{\partial \Omega}$ generated by the Earth on the Earth’s physical surface $\partial \Omega$, the Earth’s external potential field $V(P)$ is (theoretically) uniquely determined. It is well known that this statement is universally accepted by geodesists\cite{20}.

Choose an inner sphere $K_i$, which always lies inside the Earth, and an external sphere $K_f$, which always encloses the whole Earth, with their spherical centers coinciding with the coordinate origin $o$ that lies inside the Earth, and the origin $o$ is chosen in
such a way that any future-appeared surface \( \partial S \) that encloses the whole Earth is a “simply closed surface” w.r.t. the origin.

### 1.1 Poisson integral

Suppose the boundary value \( V(P) |_{\partial K} \equiv V_{\partial K} \) generated by the Earth on the external sphere’s boundary \( \partial K \) is given, the potential field \( V^S(P) \) is uniquely determined in the domain \( K \) (including the boundary \( \partial K \)), the domain outside the external sphere \( K \), by Poisson integral\(^{[20,21]}\):

\[
V^S(P) = \frac{r^2 - R_i^2}{4\pi R_i} \int_{\partial K} \frac{V_{\partial K}}{l_0^2} d\sigma, P \in \tilde{K}
\]

where \( R_i \) is the radius of the external sphere \( K \), \( d\sigma \) the integration surface element, the moving point \( Q \) lies on the surface \( \partial K \) of \( K \), \( l_{0,p} \) is the distance between the moving point \( Q \) and the field point \( P \). Obviously, the following equation holds:

\[
V^S(P) = V(P), P \in \tilde{K}
\]

where \( V(P) \) is the Earth’s real field defined in \( \tilde{K} \), the domain outside the Earth. Note that, as stated at the beginning of Sec.1, theoretically, \( V(P) (P \in \tilde{K}) \) is uniquely determined by using the boundary value \( V(P) |_{\partial K} \) on the boundary \( \partial \Omega \) of the Earth.

### 1.2 The fictitious field \( V^r(P) \)

Suppose there exists a continuous fictitious distribution \( V^r(P) |_{\partial K} \) on the boundary \( \partial K \) of the inner sphere \( K_i \), then, a regular harmonic fictitious field \( V^r(P) \) that is defined in the domain \( K \) (including the boundary \( \partial K \)), the domain outside the inner sphere \( K_i \), is uniquely determined by Poisson integral\(^{[20]}\):

\[
V^r(P) = \frac{r^2 - R_i^2}{4\pi R_i} \int_{\partial K} \frac{V_{\partial K}}{l_0^2} d\sigma, P \in K
\]

where \( R_i \) is the radius of the inner sphere \( K_i \). The moving point \( Q_i \) lies on the surface \( \partial K_i \) of \( K_i \), so the following relation holds:

\[
V^r(P) = V^S(P), P \in K
\]

which is equivalent to the following relation:

\[
V^r(P) = V^S(P), P \in \partial K
\]

because, if \( V^r(P) \) coincides with \( V^S(P) \) on the boundary \( \partial K \), then, they must coincide with each other in the domain \( K \). Note that in the domain \( K \), \( V^S(P) \) coincides with the real field \( V(P) \), because the boundary value \( V^S(P) |_{\partial K} \) is generated by the Earth.

Now, inverting the problem, supposing the boundary value \( V^S(P) |_{\partial K} \) is given, one should determine a fictitious distribution \( V^r_{\partial K} \) on \( \partial K \), just as Bjerhammar (1964)\(^{[22]}\) did. It should be noted that in Bjerhammar (1964)\(^{[22]}\), \( \partial S \) and \( \partial K \) are the Earth’s boundary \( \partial \Omega \) and the surface of Bjerhammar sphere, respectively, and the given boundary value is the gravity anomaly on \( \partial \Omega \). Combining Eqs. (4) and (6) one gets

\[
\frac{r^2 - R_i^2}{4\pi R_i} \int_{\partial K} \frac{V^r_{\partial K}}{l_0^2} d\sigma = V^S(P), P \in \partial K
\]

Eq.(7) is the well-known Abel-Poisson integral equation, or Fredholm integral equation of the first kind, the solution (i.e., \( V^r_{\partial K} \)) of which is referred to as the inverse of Abel-Poisson integral in literature\(^{[13,23]}\). If the solution of Eq.(7) exists, it must be unique, the proof of which is simple, as shown later in Sec.1.3. Then, the key problem is whether the Eq.(7) has a solution, which has been studied by many scientists/geodesists and the answer is positive\(^{[5,13,22-27]}\).

Hence, we have achieved the following conclusion (details will be provided later): given boundary value \( V^S(P) |_{\partial K} \) on the boundary \( \partial K \) of an external sphere \( K_i \), the fictitious boundary value \( V^r(P) |_{\partial K} \) on the boundary \( \partial K_i \) of an inner sphere \( K_i \) is uniquely determined, and consequently, a regular harmonic fictitious field \( V^r(P) \) in \( K_i \) (the domain outside the inner sphere \( K_i \)) is uniquely determined by Eq.(4), and \( V^r(P) \) coincides with the real field \( V(P) \) in \( K \). Note that \( V^r(P) \) does not represent the Earth’s real field in the Earth’s interior, and consequently, this is the reason why \( V^r(P) (P \in K) \) is called a fictitious field. As to whether the fictitious field \( V^r(P) \) coincides with the real field in the domain \( \tilde{K} - K \), it is a problem to be solved later (Cf. Sec.2)

### 1.3 The uniqueness theorems

We first establish the following uniqueness theorem.
Theorem 1  Fixing an inner sphere $K_i$, and given boundary value $V^S(P)|_{\partial K_i}$ generated by the Earth on the boundary $\partial K_s$ of an external sphere $K_s$, if the solution $V^*(P)|_{\partial K_i}$ of Eq.(7) exists, it must be unique.

Proof  Suppose there exist two solutions $V^*(P)|_{\partial K_i}$ and $V^{**}(P)|_{\partial K_i}$. Based on Poisson integral we have two regular harmonic functions in the domain $\Omega_i$:

$$V^*(P) = \frac{r_i^2 - R_0^2}{4\pi R_i} \int_{\partial K_i} \frac{V^S(P)}{l_{Q,P}} d\sigma$$

$$V^{**}(P) = \frac{r_i^2 - R_0^2}{4\pi R_i} \int_{\partial K_i} \frac{V^{**}(P)}{l_{Q,P}} d\sigma$$  

(8)

Since $V^*(P)$ and $V^{**}(P)$ are regular harmonic functions in the domain $\Omega_i$, they can be expressed as uniformly convergent harmonic series$^{[21]}$.

$$V^*(P) = \frac{GM}{r} \sum_{n=0}^{\infty} \sum_{m=1}^{n} \left( \frac{R_n}{r} \right)^n \left( a_{nm}^{*} \cos \lambda + b_{nm}^{*} \sin \lambda \right) P_{nm}(\cos \theta)$$

$$V^{**}(P) = \frac{GM}{r} \sum_{n=0}^{\infty} \sum_{m=1}^{n} \left( \frac{R_n}{r} \right)^n \left( a_{nm}^{**} \cos \lambda + b_{nm}^{**} \sin \lambda \right) P_{nm}(\cos \theta)$$

(9)

where $P_{nm}(\cos \theta)$ are the associated Legendre functions$^{[5]}$, and $a_{nm}^{*}, b_{nm}^{*}, a_{nm}^{**}, b_{nm}^{**}$ are spherical harmonic coefficients to be determined. Since in the domain $\Omega_i$ $V^*(P) = V^{**}(P) = V^S(P) = V(P)$ holds, the following relations must hold:

$$a_{nm}^{*} = a_{nm}^{**}, b_{nm}^{*} = b_{nm}^{**}$$  

(10)

Consequently, we have

$$V^*(P) = V^{**}(P), P \in \Omega_i$$  

(11)

The proof is completed.

It should be noted that, as stated in the subsection 1.2, the existence of the solution $V^*(P)|_{\partial K_i}$ of Eq.(7) has been generally accepted by scientists/geodesists$^{[5,13,22-25,27]}$. It is worth mentioning that the uniquely existing solution can be obtained by using the fictitious compress recovery method$^{[26]}$.

Theorem 2  Given two boundary values $V^S(P)|_{\partial K_i}$ and $V^S(P)|_{\partial K_s}$ generated by the Earth on the boundaries $\partial K_s$ and $\partial K_s$ of two external spheres $K_s$ and $K_{s'}$, respectively, if the inner sphere $K_i$ is fixed, the solutions $V^*(P)|_{\partial K_i}$ and $V^{**}(P)|_{\partial K_i}$, which are determined by the boundary values $V^S(P)|_{\partial K_i}$ and $V^S(P)|_{\partial K_s}$, respectively, have no difference: $V^*(P)|_{\partial K_i} = V^{**}(P)|_{\partial K_i}, P \in \Omega_i$.

The proof of Theorem 2 is omitted, because it is very similar to the proof of Theorem 1. Here it is noted that there exists a larger (but finite) external sphere $K_{s''}$ that encloses both $K_s$ and $K_{s'}$, and meet the condition $V^*(P) = V^{**}(P) = V^S(P) = V(P)$, $P \in \Omega_{s''}$, where $\Omega_{s''}$ is the domain outside $K_{s''}$, and $V^S(P)(P \in \Omega_{s''})$ is the field determined based on the boundary value $V^S(P)|_{\partial K_{s''}}$.

2 The convergence of the spherical harmonic series

As has been pointed out in Sec.1, since the boundary value $V^S(P)|_{\partial K_i}$ is generated by the Earth, the following equation must hold:

$$V^*(P) = V^{**}(P) = V^S(P) = V(P), P \in K_i$$  

(12)

It is noted that $V(P)(P \in \Omega)$ is the real external field generated by the Earth.

Theorem 3  Fixing an inner sphere, and given boundary value $V^S(P)|_{\partial K_i}$ generated by the Earth on the boundary $\partial K_s$ of an external sphere $K_s$, the Earth’s real external field $V(P)(P \in \Omega)$ can be not only uniquely determined (consequently solving the “downward continuation” problem), but also expressed as a uniformly convergent harmonic series (hence solving the convergence problem).

Proof  Based on Theorem 2, the fictitious field $V^*(P)(P \in \Omega_i)$ is uniquely determined by the boundary value $V^*(P)|_{\partial K_i} = V^S(P)|_{\partial K_i}$. It can be proved that the following relation holds:

$$V^*(P) = V(P), P \in \Omega$$  

(13)

The technical details are beyond the scope of this paper.

Based on Theorem 3, the fictitious solution $V^*(P)$ is regular and harmonic in $\Omega_i$, and consequently it can be expressed as a uniformly convergent harmonic series$^{[21]}$, given by the first expression of Eq.(9), re-written as

$$V^*(P) = \frac{GM}{r} \sum_{n=0}^{\infty} \sum_{m=1}^{n} \left( \frac{R_n}{r} \right)^n \left( a_{nm}^{*} \cos \lambda + b_{nm}^{*} \sin \lambda \right) P_{nm}(\cos \theta), P \in \Omega_i$$

(14)

where the coefficients $a_{nm}^{*}$ and $b_{nm}^{*}$ are determined based on the boundary value $V^*(R, \theta, \lambda)|_{\partial K_i}$ on the
boundary $\partial K_i^{[20]}$. 

Since $V^*(P)$ coincides with $V(P)$ in the domain $\Omega$, $V(P)$ can be expressed as a uniformly convergent harmonic series in that domain:

$$V(P) = \frac{GM}{r} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left( \frac{R}{r} \right)^n (a_{nm}^* \cos \lambda + b_{nm}^* \sin \lambda) P_{nm}(\cos \theta), P \in \Omega$$  \hspace{1cm} (15)

To determine $V(P)$ by using the boundary value, the following procedure can be executed. In the domain $K_s$, the real field $V(P)$ can be expressed as the following uniformly convergent harmonic series:

$$V^S(P) = \frac{GM}{r} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left( \frac{R}{r} \right)^n (a_{nm}^* \cos \lambda + b_{nm}^* \sin \lambda) P_{nm}(\cos \theta), P \in K_s$$  \hspace{1cm} (16)

where the coefficients $a_{nm}^*$ and $b_{nm}^*$ can be determined based on the boundary value $V^S(R_s, \theta, \lambda)|_{\partial K_s}$ on the boundary $\partial K_s^{[20]}$. Since Eq.(14) holds also in $K_s$, substituting Eqs.(14) and (16) into Eq.(12), we have:

$$a_{nm}^* = \frac{R_S}{R_i} a_{nm}^*, \quad b_{nm}^* = \frac{R_S}{R_i} b_{nm}^*$$  \hspace{1cm} (17)

Substituting $a_{nm}^*$ and $b_{nm}^*$ into Eq.(15), the field $V(P)(P \in \Omega)$ can be determined.

**Theorem 4** Fixing an inner sphere $K_i$, and given a continuous boundary value $V^S(P)|_{\partial S}$ generated by the Earth on a “simply closed surface” $\partial S$ that encloses the whole Earth, then, theoretically, a fictitious field $V(P)(P \in \Omega)$ is uniquely determined, which coincides with the Earth’s real field $V(P)$ in the whole domain outside the Earth, solving both the “downward continuation” problem and the convergence problem.

**Proof** Since the boundary value $V^S(P)|_{\partial S}$ generated by the Earth is given, theoretically, the Earth’s real field $V(P)$ in the domain $\bar{S}$ (the domain outside the surface $\partial S$) is uniquely determined. Then, one can choose a large enough external sphere $K'$ that encloses the surface $\partial S$, and the boundary value $V'(P)|_{\partial S}$, generated by $V^S(P)|_{\partial S}$ and consequently generated by the Earth on the surface of the external sphere $K'$, is known (Cf. Fig.2, $\partial S$ is a “simply closed surface” that encloses the whole Earth, $\partial K'$, $\partial S$ and $\partial K$ are the boundaries of an external sphere $K'$ that encloses the surface $\partial S$, the Earth and the inner sphere $K$, respectively.). On the basis of Theorem 3, the proof is completed.

### 3 Conclusion

Given the boundary value $V^S(P)|_{\partial K_s}$ generated by the Earth on the boundary $\partial K_s$ of an external sphere $K_s$, the Earth’s real external field $V(P)(P \in \Omega)$ can be uniquely determined by Eqs.(15), (16) and (17). Therefore, the convergence problem has been solved.

If the boundary $\partial S$ is a very general “simply closed surface” (e.g., a closed surface corresponding to the satellite altitude), theoretically, we know that there exists a unique solution, which can be expressed as a uniformly convergent harmonic series in the domain outside the Earth (Cf. theorem 4). Now, the key problem is whether this solution can be found. As is well-known, to find the solution, one cannot apply Green’s method or Poisson integral. The truncated spherical harmonic series might be applied in practice. In another aspect, one can also apply the fictitious compress recovery method$^{[28,29]}$.

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