Quantum correction to the entropy of noncommutative BTZ black hole

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In this paper we consider the generalized uncertainty principle (GUP) in the tunneling formalism via Hamilton-Jacobi method to determine the quantum-corrected Hawking temperature and entropy for noncommutative BTZ black hole. In our results we obtain several types of corrections including the expected logarithmic correction to the area entropy associated with the noncommutative BTZ black holes. We also show that the area entropy product of the noncommutative BTZ black holes is dependent on mass and by analyzing the nature of the specific heat capacity we have observed that the noncommutative BTZ black hole is stable at some range of parameters.

I. INTRODUCTION

In recent years a number of studies in three-dimensional gravity has increased due to the discovery of various aspects of black-hole solutions. Black holes constitute an important class of exact solutions of Einstein’s equations which are characterized by mass (M), electric charge (Q) and angular momentum (J) [1] and play a central role in both classical and quantum gravitational physics. In the literature there are various ways to implement the noncommutative in the physics of black holes. Thus, interest in the study of noncommutative black holes have been investigated by many authors in recent years (for a review see [2]). In particular, the noncommutative Banados-Teitelboim-Zanelli (BTZ) black holes were first analyzed in [3] and in [4] the noncommutative BTZ metric was found based on the three dimensional equivalence between gravity and the Chern-Simons theory which is a 3-dimensional topological quantum field theory and using the Seiberg-Witten map with the commutative BTZ solution [5]. The BTZ black hole is a solution of (2+1) dimensional gravity with negative cosmological constant and has become an important field of investigations. It is now well accepted that three-dimensional gravity is an excellent laboratory in order to explore and test some of the ideas behind the AdS/CFT correspondence [6]. In [7] the authors analyzed the gravitational Aharonov-Bohm effect due to BTZ black hole in a noncommutative background. Moreover, in [8] has been analyzed the behavior of a particle test in the noncommutative BTZ space-time. The thermodynamic properties of the charged BTZ black hole were investigated in [9].

A semiclassical approach considering the Hawking radiation as a tunneling phenomenon across the horizon has been proposed in recent years [10, 11]. In this approach the positive energy particle created just inside the horizon can tunnel through the geometric barrier quantum mechanically, and it is observed as the Hawking flux at infinity. There are several approaches to obtain the Hawking radiation and the entropy of black holes. One of them is the Hamilton-Jacobi method which is based on the work of Padmanabhan and collaborators [12] and also the effects of the self-gravitation of the particle are discarded. In this way, the method uses the WKB approximation in the tunneling formalism for the computation of the imaginary part of the action. The authors Parikh and Wilczek [10] using the method of radial null geodesic determined the Hawking temperature and in [13] this method was used by the authors for calculating the Hawking temperature for different spacetimes. In Ref. [14] has been analyzed Hawking radiation considering self-gravitation and back reaction effects in tunneling formalism. It has also been investigated in [15] the back reaction effects for self-dual black hole using the tunneling formalism by Hamilton-Jacobi method. In [16] has been studied the effects of the generalized uncertainty principle (GUP) in the tunneling formalism for
Hawking radiation to evaluate the quantum-corrected Hawking temperature and entropy of a Schwarzschild black hole. Moreover, the authors in [17] have discussed the Hawking radiation for acoustic black hole using tunneling formalism and in [18] the thermodynamical properties of self-dual black holes, using the Hamilton-Jacobi version of the tunneling formalism were investigated. It was analyzed in [19] the corrections for the thermodynamics of black holes assuming that the GUP corrected entropy-area relation is universal for all black objects.

In the literature there are several works on the statistical origin of black hole entropy — see for instance [20–23]. In Ref. [24], Kaul and Majumdar computed the lowest order corrections to the Bekenstein-Hawking entropy. They find that the leading correction is logarithmic. On the other hand, in Ref. [25] was shown that there is an additional logarithmic corrections that depend on conserved charges. In addition, for an understanding of the origin of black hole entropy, the brick-wall method proposed by G. ’t Hooft has been used for calculations on black holes. Thus, According to G. ’t Hooft, black hole entropy is just the entropy of quantum fields outside the black hole horizon. However, when one calculates the black hole statistical entropy by this method, to avoid the divergence of states density near black hole horizon, an ultraviolet cut-off must be introduced. In Ref. [26] was investigated (1 + 1)-dimensional acoustic black hole entropy by the brick-wall method.

The other related idea in order to cure the divergences is to consider models in which the Heisenberg uncertainty relation is modified. Thus, using the modified Heisenberg uncertainty relation the divergence in the brick-wall model are eliminated as discussed in [27]. The statistical entropy of various black holes has also been calculated via corrected state density of the GUP [28]. Thus, the results show that near the horizon quantum state density and its statistical entropy are finite. In [29] a relation for the corrected states density by GUP has been proposed. The authors in [30] using a new equation of state density due to GUP [31], the statistical entropy of a 2+1-dimensional rotating acoustic black hole has been analyzed. It was shown that considering the effect due to GUP on the equation of state density, no cut-off is needed [32] and the divergence in the brick-wall model disappears.

In this paper, inspired by all of these previous work we shall focus on the Hamilton-Jacobi method to determine the entropy of a noncommutative BTZ black hole using the GUP and considering the WKB approximation in the tunneling formalism to calculate the imaginary part of the action in order to determine the Hawking temperature and entropy for BTZ black holes. We anticipate that we have obtained the Bekenstein-Hawking entropy of BTZ black holes and its quantum corrections that are logarithm and also of other types.

The paper is organized as follows. In Sec. II we briefly review the noncommutative BTZ black holes geometry and address the entropy product. We notice that differently of the commutative case, for noncommutative BTZ black holes such a product depends on the mass of the black holes. We also use the Hamilton-Jacob method to determine the Hawking temperature and that the noncommutative correction for Hawking temperature occurs only at second order in the noncommutativity parameter. In Sec. III we consider the GUP in the tunneling formalism via the Hamilton-Jacobi method to find the quantum corrections to the Hawking temperature, entropy and specific heat capacity of a noncommutative BTZ black hole. Finally in Sec. IV we present our final comments.

II. NONCOMMUTATIVE BTZ BLACK HOLES

In this section we consider the metric of the BTZ black hole in a noncommutative background given by [4, 7]:

$$ds^2 = -F dt^2 + N^{-1} dr^2 + 2r^2 N^\phi dtd\phi + \left( r^2 - \frac{\theta B}{2} \right) d\phi^2,$$

where the metric components are

$$F = \frac{r^2 - r_+^2 - r_-^2}{l^2} - \frac{\theta B}{2l^2},$$

$$N = \frac{1}{r^2 l^2} \left[ (r^2 - r_+^2)(r^2 - r_-^2) - \frac{\theta B}{2}(2r^2 - r_+^2 - r_-^2) \right],$$

$$N^\phi = \frac{-r_+ r_-}{l^2}.$$

Here $B$ is the magnitude of a $U(1)$ flux in a noncommutative $U(1,1) \times U(1,1)$ Chern-Simons theory and $\theta$ is the noncommutative parameter of dimension $\text{length}^2$. The Seiberg-Witten map is carried out up to first order in $\theta$ — see Refs. [4] for further details. For the noncommutative BTZ black hole, the event horizons are given by

$$r_\pm^2 = r_+^2 + \frac{\theta B}{2} + O(\theta^2),$$
and
\[
\hat{r}_+\hat{r}_- = \sqrt{\hat{r}_+^2\hat{r}_-^2 + \frac{\theta B}{2}(\hat{r}_+^2 + \hat{r}_-^2) + O(\theta^2)} = \sqrt{\frac{I^2J^2}{4} + \frac{\theta B I^2 M}{2} + O(\theta^2)} = \left[1 + \frac{\theta B M}{J^2} + O(\theta^2)\right],
\]
where \(\hat{r}_+\) is the outer event horizon and \(\hat{r}_-\) is the inner event horizon of the commutative BTZ black hole. Note that for \(\theta = 0\) the event horizons of the commutative case are recovered.

In order to analyze the product of entropy, we will first consider the product and sum of the horizon radii that are given by
\[
\hat{r}_+ + \hat{r}_- = \sqrt{\hat{r}_+^2 + \frac{\theta B}{2}(\hat{r}_+^2 + \hat{r}_-^2) + O(\theta^2)} = \sqrt{\frac{I^2J^2}{4} + \frac{\theta B I^2 M}{2} + O(\theta^2)} = \frac{I J}{2} \left[1 + \frac{\theta B M}{J^2} + O(\theta^2)\right],
\]
and
\[
\hat{r}_+^2 + \hat{r}_-^2 = r_+^2 + r_-^2 + \theta B = I^2 M + \theta B + O(\theta^2).
\]
Observe that the product and the sum depend on mass parameter. On the other hand, for \(\theta = 0\), the product \(\hat{r}_+\hat{r}_- = I J/2\) is independent on mass.

Let us consider that \(\hat{S}_+ = 4\pi\hat{r}_+\) is the entropy of the noncommutative BTZ black holes. Thus, the product \(\hat{S}_+\hat{S}_-\) is given by
\[
\hat{S}_+\hat{S}_- = 16\pi^2\hat{r}_+\hat{r}_- = 16\pi^2\sqrt{\frac{I^2J^2}{4} + \frac{\theta B I^2 M}{2} + O(\theta^2)} = 8\pi^2 I J \left[1 + \frac{\theta B M}{J^2}\right] + O(\theta^2).
\]
Notice that, at least up to first order in \(\theta\), the entropy product of the noncommutative BTZ black holes is dependent on mass. On the other hand, for \(\theta = 0\), the result is independent on mass [33]. It is conjectured that the product of the areas for multi-horizon stationary black holes are in some cases independent on the mass of the black hole [34]. However, there are studies in the literature where the areas product is dependent on the mass [35]. For example, it has been shown that for Schwarzschild-de Sitter black hole in (3+1) dimensions the product of event horizon area and cosmological horizon area is not mass independent. Recently, it was also shown in Ref. [36] for acoustic black hole that the universal aspects of the areas product depends only on quantized quantities such as analogues of conserved electric charge and angular momentum.

The metric of noncommutative BTZ black hole can be rewritten as
\[
ds^2 = -f dt^2 + Q^{-1} dr^2 + \frac{J}{r} d\varphi dt + \left(1 - \frac{\theta B}{2 r^2}\right) r^2 d\phi^2,
\]
where
\[
f = -M + \frac{r^2}{l^2} - \frac{\theta B}{2 l^2},
\]
\[
Q = -M + \frac{r^2}{l^2} + \frac{J^2}{4 r^2} - \frac{\theta B}{2} \left(\frac{2}{l^2} - \frac{M}{r^2}\right).
\]
At this point we will consider the case where \(J = 0\). Thus, near the event horizon of a noncommutative BTZ black hole, we can rewrite the metric (10) as follows
\[
ds^2 = -\tilde{f} dt^2 + \tilde{Q}^{-1} dr^2 + \left(1 - \frac{\theta B}{2 r^2}\right) r^2 d\phi^2,
\]
where \(\tilde{f} = f'(\hat{r}_+)(r - \hat{r}_+)\) and \(\tilde{Q} = Q'(\hat{r}_+)(r - \hat{r}_+)\).

Now we use the Hamilton-Jacob method to determine the Hawking temperature. Using the Klein-Gordon equation for a scalar field \(\phi\) given by
\[
\left[\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu) - \frac{m^2}{\hbar^2}\right] \Phi = 0,
\]
and applying the WKB approximation
\[ \Phi = \exp \left[ \frac{i}{\hbar} I(t, r, x^i) \right], \] (15)
we obtain
\[ g^{\mu\nu} \partial_\mu I \partial_\nu I + m^2 = 0, \] (16)
that in terms of the metric (13), becomes
\[- \frac{1}{f}(\partial_I)^2 + \tilde{Q}(\partial_r I)^2 + \frac{1}{r^2}(\partial_\phi I)^2 + m^2 = 0. \] (17)
Now we can assume a solution to the Klein-Gordon equation via separation of variables as follows
\[ I = -Et + W(r) + J_\phi \phi, \] (18)
where \( J_\phi \) is a constant. By substituting (18) into equation (17) and solving for \( W(r) \) the spatial part of the classical action reads
\[ W(r) = \int_C \frac{\sqrt{E^2 - f'(\hat{r}_+)(r - \hat{r}_+)^2 \left( \frac{2j_+^2}{2r_+^2} + m^2 \right)}}{\sqrt{f'(\hat{r}_+)Q'(\hat{r}_+)}}} = \frac{2\pi i}{\kappa} E, \] (19)
where
\[ \kappa = \sqrt{f'(\hat{r}_+)Q'(\hat{r}_+) = \sqrt{\frac{4j_+^2}{l^4} - \frac{2\theta BM}{l^2 r_+^2}}}. \] (20)
On the other hand, the probability of a particle overcoming the potential barrier is given by
\[ \Gamma = \exp[-2\text{Im}(I)] \Rightarrow \Gamma = \exp \left( -\frac{4\pi E}{\kappa} \right). \] (21)
Now comparing (21) with the Boltzmann factor \( \exp(-E/T_H) \) we obtain the Hawking temperature of the BTZ black hole in a noncommutative background
\[ \tilde{T}_H = \frac{\kappa}{4\pi} = \frac{1}{4\pi} \sqrt{\frac{4j_+^2}{l^4} - \frac{2\theta BM}{l^2 r_+^2}} \hat{r}_+ = \frac{\hat{r}_+}{2\pi l^2} \left( 1 - \frac{\theta B r_+^2}{4\hat{r}_+^4} \right) + \cdots, \] (22)
that in terms of \( r_+ = \sqrt{l^2 M} \), we have
\[ \tilde{T}_H = \frac{r_+}{2\pi l^2} \left( 1 - \frac{\theta^2 B^2}{16r_+^2} \right) + \cdots = T_h - \frac{\theta^2 B^2}{256\pi^4 l^6 T_h^3} + \cdots, \] (23)
where \( T_h = r_+/2(2\pi l^2) \) is the Hawking temperature of the BTZ black hole. The above result shows that the non-commutative correction for Hawking temperature occurs only at second order in the parameter \( \theta \).

III. QUANTUM-CORRECTED HAWKING TEMPERATURE AND ENTROPY

In this section we consider the GUP in the tunneling formalism via the Hamilton-Jacobi method to find the quantum corrections to the Hawking temperature, entropy and specific heat capacity of a noncommutative BTZ black hole. Thus our starting point is the GUP [39, 40], which is an extension of [41] given by
\[ \Delta x \Delta p \geq \hbar \left( 1 - \frac{\alpha l_p}{\hbar} \Delta p + \frac{\alpha^2 l_p^2}{\hbar^2} (\Delta p)^2 \right), \] (24)
where $\alpha$ is a dimensionless positive parameter, $l_p = \sqrt{\hbar G/c^3} = M_p c/\sqrt{c} \approx 10^{-35}$m is the Planck length, $M_p = \sqrt{\hbar c/G}$ is the Planck mass and $c$ is the velocity of light. Since $G$ is the Newtonian coupling constant, the correction terms in the uncertainty relation (24) are due to the effects of gravity. Although this formula is written in four spacetime dimensions it also works in 2+1 dimensions under certain assumptions — see below. Let us compute the total position uncertainty [43] by considering BTZ black holes, i.e.,

$$\Delta x = \Delta x_1 + \Delta x_2 \simeq \frac{\lambda}{\sin \phi} + \frac{l}{2} \frac{8G_3 \Delta M}{\sqrt{8G_3 M}} \geq \lambda + \frac{4lG_3}{\sqrt{8G_3 M}} \frac{1}{\lambda}. \quad (25)$$

Here $\Delta x_1$ is the usual Heisenberg’s position uncertainty and $\Delta x_2 = r_+ (M + \Delta M) - r_+ (M)$ is the additional uncertainty due to the BTZ black hole for $J = 0$ and $\Delta M \ll M$. This implies the quadratic GUP

$$\Delta x \Delta p \geq 1 + \alpha^2 l_p (\Delta p)^2, \quad \alpha^2 = \frac{4}{\sqrt{8G_3 M}} \Delta p \sim \frac{1}{\lambda} \quad (26)$$

where we have reinstated the Newtonian constant at 2+1 dimensions $G_3 \propto l_p$. However, for latter convenience making $l_p^2 = 1$ at (24) or $l l_p = 1$ at (26) makes the quadratic parts of these GUPs formally the same. Furthermore, the quadratic part of the GUP is naturally consistent to a noncommutative geometric generalization of position space [41]. In addition, the linear part of (24) is also consistent with the noncommutativity of the spacetime [44] and Doubly Special Relativity (DSR) theories [40].

Now the equation (24) can be written as follows.

$$\Delta p \geq \frac{h(\Delta x + \alpha l_p)}{2a^2 l_p^2} \left(1 - \sqrt{1 - \frac{4\alpha^2 l_p^2}{(\Delta x + \alpha l_p)^2}}\right). \quad (27)$$

where we have chosen the negative sign. Since $l_p/\Delta x$ is relatively small compared to unity we can expand the equation above in Taylor series

$$\Delta p \geq \frac{1}{2\Delta x} \left[1 - \frac{\alpha}{2\Delta x} + \frac{\alpha^2}{2(\Delta x)^2} + \cdots\right]. \quad (28)$$

As we have chosen $G = c = k_B = 1$, so we also choose $h = 1$, and we have $l_p = 1$. In these units the uncertainty principle becomes

$$\Delta x \Delta p \geq 1. \quad (29)$$

Now using the saturated form of the uncertainty principle we have

$$E \Delta x \geq 1, \quad (30)$$

which follows from the saturated form of the Heisenberg uncertainty principle, $\Delta x \Delta p \geq 1$, where $E$ is the energy of a quantum particle. Therefore, we can rewrite equation (28) in the form

$$E_G \geq E \left[1 - \frac{\alpha}{2(\Delta x)} + \frac{\alpha^2}{2(\Delta x)^2} + \cdots\right], \quad (31)$$

So by using Hamilton-Jacobi method the tunneling probability of a particle with corrected energy $E_G$ becomes

$$\Gamma \approx \exp[-2\text{Im}(\hat{T})] = \exp \left[\frac{-2\pi E_G}{a}\right]. \quad (32)$$

Again, comparing with the Boltzmann factor ($e^{-E/T}$), we obtain the noncommutative BTZ black hole temperature

$$T = \hat{T}_H \left[1 - \frac{\alpha}{2(\Delta x)} + \frac{\alpha^2}{2(\Delta x)^2} + \cdots\right]^{-1}. \quad (33)$$

Here we will choose $\Delta x = 2\hat{r}_+$. Thus, we have the corrected temperature due to the GUP

$$T_{\text{GUP}} = \frac{\hat{r}_+}{2\pi l^2} \left(1 - \frac{\theta B r_+^2}{4 \hat{r}_+^2} + \cdots\right) \left(1 - \frac{\alpha}{4 \hat{r}_+} + \frac{\alpha^2}{8 \hat{r}_+^2} + \cdots\right)^{-1}
= \frac{\hat{r}_+}{2\pi l^2} \left(1 - \frac{\theta B r_+^2}{4 \hat{r}_+^2} + \cdots\right) \left(1 + \frac{\alpha}{4 \hat{r}_+} - \frac{\alpha^2}{8 \hat{r}_+^2} + \cdots\right), \quad (34)$$
that in terms of the \( r_+ = l\sqrt{M} \), we have
\[
T_{\text{GUP}} = \frac{r_+}{2\pi l^2} \left( 1 - \frac{\theta^2 B^2}{16r_+^4} + \cdots \right) \left[ 1 + \frac{\alpha}{4r_+} \left( 1 - \frac{\theta B}{4r_+^2} + \cdots \right) - \frac{\alpha^2}{8r_+^2} \left( 1 - \frac{\theta B}{2r_+^2} + \cdots \right) + \cdots \right].
\] (35)

or in terms of the Hawking temperature \( T_h = r_+/2(2\pi l^2) \) of the BTZ black hole, we obtain
\[
T_{\text{GUP}} = T_h - \frac{\theta^2 B^2}{256\pi^4 T_h^3} \left[ 1 + \frac{\alpha}{8\pi l^2} - \frac{\alpha \theta B}{128\pi^3 l^4 T_h^2} - \frac{\alpha^2}{32\pi^2 l^4 T_h} + \frac{\alpha^2 \theta B}{256\pi^4 l^8 T_h^3} + \cdots \right],
\] (36)

It is interesting to note that the third term in the above equation is independent on the horizon radius.

In the following we will analyze the quantum corrections due to GUP for energy density, specific heat capacity at constant volume and entropy. The corrections to the black hole of energy density can be calculated as follows
\[
\rho_{\text{GUP}} = -\frac{3}{l^2} \int S'(A)A^{-2}dA,
\] (37)
where, \( S'(A) = \frac{dS}{dA} \). Thus
\[
\rho_{\text{GUP}} = \frac{3}{l^2 A^2} - \frac{3}{2l^2} \frac{\alpha \pi}{A^3} + \frac{2}{l^2} \frac{\pi}{A^3} + \frac{3}{l^2} \frac{\alpha \theta B}{A^3} + \frac{48}{5l^2} \frac{\theta^2 B^2}{A^4} - \frac{48}{5l^2} \frac{\alpha^2 B}{A^4},
\] (38)

and considering that \( \rho = \frac{3}{r_+^2} \), we have
\[
\rho_{\text{GUP}} = \rho - \frac{1}{6l^2 \pi} \alpha l^2 \rho^2 + \frac{2}{2l^2} \pi^2 \alpha^2 l^4 \rho^3 + \frac{1}{27l^2} \pi^3 \alpha \theta B l^6 \rho^4 + \frac{16}{405l^4} \pi^4 \theta^2 B^2 l^8 \rho^5 - \frac{16}{405l^4} \pi^4 \alpha \theta B l^8 \rho^5.
\] (39)

At this point, we use the laws of thermodynamics of black holes to determine the entropy of the BTZ black hole in a noncommutative background as follows
\[
S_{\text{GUP}} = \frac{1}{T_{\text{GUP}}} dM = 4\pi l \sqrt{M} - \frac{\pi \theta^2 B^2}{12l^3 \sqrt{M}^3/2} - \pi \alpha \ln(l \sqrt{M}) - \frac{1}{8} \frac{\pi \alpha B}{l^2 l^2} - 1 \frac{\pi \alpha^2 B}{2 l^2 \sqrt{M}^2} + \cdots
\] (40)
and that in the terms of the entropy \( S = 4\pi r_+ = 4\pi l \sqrt{M} \), we have
\[
S_{\text{GUP}} = S - \frac{16 \pi^4 \theta^2 B^2}{3S^3} - \alpha \pi \ln(S) - \frac{2}{S^2} \frac{\alpha \pi \theta B}{S} - \frac{2 \pi \alpha^2}{S} - \frac{16}{3} \frac{\pi \alpha \theta B}{S} + \cdots
\] (41)
We have obtained corrections to the entropy through tunneling formalism using the Hamilton-Jacobi method due to the effects of GUP. Notice that from the above equation for \( \alpha = 0 \), we have that the noncommutative correction to the entropy occurs only at second order in the parameter \( \theta \). Besides, we have obtained logarithmic corrections to the entropy of the BTZ black hole.

The correction of the specific heat capacity at constant volume \( C_v = T_h \left( \frac{dS}{dT_h} \right)_\rho = 8\pi l^2 T_h = 4\pi r_+ \), reads
\[
C_{v,\text{GUP}} = 8\pi^2 l^2 dM = 8\pi^2 l^2 \left[ \frac{r_+}{2\pi l^2} - \frac{\theta^2 B^2}{32\pi^2 l^4 r_+^4} + \frac{\alpha \theta B}{8\pi l^2} - \frac{\alpha^2}{32\pi l^2 r_+^4} - \frac{\alpha^2 \theta B}{16\pi l^4 r_+^4} + \cdots \right],
\] (42)

or in terms of the \( T_h \) and \( C_v \) becomes
\[
C_{v,\text{GUP}} = \frac{8\pi^2 l^2}{C_v} \left[ T_h - \frac{\theta^2 B^2}{256\pi^4 l^8 T_h^3} + \frac{\alpha}{8\pi l^2} - \frac{\alpha \theta B}{128\pi^3 l^6 T_h^2} - \frac{\alpha^2}{32\pi^2 l^4 T_h} + \frac{\alpha^2 \theta B}{256\pi^4 l^8 T_h^3} + \cdots \right],
\] (43)

Observe that Figs. 1 and 2 show the behavior of the specific heat capacity at constant volume. In Fig. 1, the graph shows that \( C_{v,\text{GUP}} \) is positive, indicating that the BTZ black hole in a noncommutative background is stable. In Fig. 2 was analyzed for the case \( \theta B = \alpha \) (or up to first order in \( \alpha \) with \( \theta B \neq \alpha \)) and the graph shows that \( C_{v,\text{GUP}} \) vanishes in the critical event horizon \( r_\theta = \sqrt{\theta B}/2 \), and is negative for \( r_+ < r_\theta \) (an unphysical region). On the other hand, for \( r_+ > r_\theta \), \( C_{v,\text{GUP}} \) is positive, so for this region the BTZ black hole in a noncommutative background is stable. Therefore, our result is similar to that found by the authors in Ref. [42].
FIG. 1: Specific heat capacity. Plot $C_{v_{GUP}}$ vs. $r_+$. For $\alpha = 0.5$ and $\theta B = 0.1$. $C_v$ is specific heat capacity for $\theta B = \alpha = 0$.

FIG. 2: Specific heat capacity. Plot $C_{v_{GUP}}$ vs. $r_+$. For $\alpha = \theta B = 0.5$.

IV. CONCLUSIONS

In summary, using the Hamilton-Jacobi version of the tunneling formalism we have considered the metric of a noncommutative BTZ black hole and we have shown that the noncommutative corrections for Hawking temperature and entropy occur only in second order in the parameter $\theta$. In addition, we have also shown that the product of entropy is dependent on the mass parameter $M$ and becomes independent of the mass when $\theta = 0$. Besides, by considering the GUP via the Hamilton-Jacobi method to calculate the imaginary part of the action, we have obtained quantum corrections for Hawking temperature, entropy and specific heat capacity of a noncommutative BTZ black hole. Moreover, in our calculations the GUP was introduced by the correction to the energy of a particle due to gravity near the horizon. Thus, in our model the GUP allows us to find logarithmic corrections to the area law. Also, we have analyzed the nature of the specific heat capacity and we have observed from Fig. 1 that the noncommutative BTZ black hole is stable. On the other hand analyzing Fig. 2 we have found that the noncommutative BTZ black hole is stable only for $r_+ > r_{\theta} = \sqrt{\theta B}/2$. 

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