Is the third coefficient of the Jones knot polynomial a quantum state of gravity?

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Abstract

Some time ago it was conjectured that the coefficients of an expansion of the Jones polynomial in terms of the cosmological constant could provide an infinite string of knot invariants that are solutions of the vacuum Hamiltonian constraint of quantum gravity in the loop representation. Here we discuss the status of this conjecture at third order in the cosmological constant. The calculation is performed in the extended loop representation, a generalization of the loop representation. It is shown that the Hamiltonian does not annihilate the third coefficient of the Jones polynomial \( J_3 \) for general extended loops. For ordinary loops the result acquires an interesting geometrical meaning and new possibilities appear for \( J_3 \) to represent a quantum state of gravity.

1 Introduction

One of the most promising achievements of the new variable canonical quantization program of gravity \cite{1} is the possibility to find in a generic case solutions to the Wheeler-DeWitt equation. This fact has allowed to remove one of the main difficulties that stopped the hamiltonian quantization of gravity for almost thirty years. If the space of states could be determined in any appropriate way (that is, in any well defined formalism associated with gravity), then one has the chance to advance one more step in the long avenue of the quantum theory of gravity. In the case of the Ashtekar formalism this avenue present special features (as the fact that the variables are complex and reality conditions need to be imposed at the quantum level) that could reveal unexpected difficulties to complete the whole program. But all future advance necessarily goes through the knowledge of the physical space of states of the theory, that is, of the solutions of the diffeomorphism and the Hamiltonian constraints.
There are mainly two ingredients of the new program that guide the search of solutions of the constraints of quantum gravity: the loop representation and its connection with knot theory \cite{2} and the relationship between Chern-Simons theory and the Kauffman bracket and Jones knot polynomials \cite{3}. The existence of a loop representation for quantum gravity is a direct consequence of the new variables introduced by Ashtekar (connections and triads instead of metrics and conjugated momenta). Whenever one has a theory given in terms of a Lie-algebra valued connection on a three manifold, one can introduce a loop representation for it \cite{4}. The fundamental goal of the loop representation is that it has allowed to find for the first time nonperturbative solutions of the Wheeler-De Witt equation.

The loop representation has an intrinsic geometrical content that simplifies notably the constraint equations. The invariance under diffeomorphism of the theory can be immediately coded in the requirement of knot invariance. This fact automatically solves the diffeomorphism constraint, so we have to deal here only with the Hamiltonian constraint.

One can take different points of view for the analysis of the Hamiltonian constraint in the loop representation. One possible approach is to consider the geometrical properties of loops rather than the explicit analytical expressions of the knot invariants. This point of view was the first adopted historically and it was based initially on the following observation: the action of the Hamiltonian is automatically zero over smooth loop wavefunctions (that is, over wavefunctions that are nonzero only for loops without kinks and intersections). Using this fact, it is possible to give a prescription that connects the space-time metric with some underlying structure constructed in terms of smoothened loops (the weaves) \cite{5}.

Another possible approach to the problem is by using the analytical expression of the knot invariants to evaluate the Hamiltonian. This apparently trivial observation is in fact amazing. We know only a few analytical expressions for the so many knot invariants that one can construct in knot theory. On the other hand, any explicit calculation of the Hamiltonian over a loop wavefunction implies to compute the loop derivative \cite{6} over some kind of intersection and this means to handle with hard computational and regularization problems. In fact, the only nondegenerate solution of the vacuum Hamiltonian constraint found up to now by this method corresponds to the second coefficient of the Alexander-Conway knot polynomial and the result is only formal \cite{7}. In spite of this, there exists a guideline that increase the expectative to find a systematic method to generate solutions of the Hamiltonian constraint. This is the content of the second ingredient mentioned above, the relationship between Chern-Simons theory and the Kauffman bracket and Jones knot polynomials. The possibility that these knot polynomials might be associated with quantum states of gravity is considered in the next section. The explicit exploration of this fact faces up with very nontrivial technical difficulties in the loop representation.

Recently a representation closely related to the loop representation was
introduced for quantum gravity [8, 9]. This representation is based on an extension of the group of loops to a local infinite dimensional Lie group [10] and it is called by this reason the extended loop representation. In spite that some problems appear at the basic level of the extended loop representation (the convergence problems associated with the matrix representations of the extended holonomies and its relationship with the gauge invariance properties of the representation [11]), the extended loop formalism was able to show an interesting ability to handle the regularization and renormalization problems and to increase the power of calculus of the theory. These and other related questions are discussed in reference [12].

The aim of this paper is to exhibit the computational power of this new representation and show that relevant information about the states of quantum gravity can be obtained. In particular we will show under which conditions the third coefficient of a certain expansion of the Jones polynomial can be viewed as a solution of the Wheeler-DeWitt equation. The main result is that, when restricted to ordinary loops the third coefficient could be in principle annihilated by the Hamiltonian for some particular topologies of the loop at the intersections. Moreover, it is shown that only for such privileged topologies the $J_3$ knot invariant would satisfy all the Mandelstam identities required for the quantum states of gravity in the loop representation.

The article is organized as follows: in Sect. 2 the loop transform of the exponential of the Chern-Simons form is considered. This state is the key that allows to relate solutions of the constraints of gravity in the connection and the loop representations. In Sect. 3 a result for $H_0 J_3$ is derived in terms of extended loops. This section includes a subsection where the tools of the extended loop framework are introduced. The reduction of the result from extended to ordinary loops is performed in Sect. 4. The implications of the procedure of reduction for quantum gravity are considered and, in particular, the Mandelstam identities of $J_3$ are discussed in Sect. 4.1. The conclusions are included in Sect. 5 and two appendixes with some useful derivations are added.

2 The exponential of the Chern-Simons form

Kodama [13] was the first to recognize that the exponential of the Chern-Simons form $\Psi_{CS}[A]$ constructed with the Ashtekar connection gives an exact quantum state of gravity with cosmological constant $\Lambda$:

$$H_{\Lambda}[A] \Psi_{CS}[A] = (H_0[A] + \frac{\Lambda}{6} \det q[A]) \Psi_{CS}[A] = 0$$

(1)

$H_0[A]$ and $\det q[A]$ are the Hamiltonian and the determinant of the three metric in the connection representation. The connection and the loop representations
are related through the (formal) loop transform
\[ \Psi[\gamma] = \int dA W_\gamma[A] \Psi[A] \tag{2} \]
where \( W_\gamma[A] := \text{Tr}(U_A(\gamma)) \) is the Wilson loop and \( U_A(\gamma) := P \exp(\int_\gamma A_a(x)dx^a) \) is the holonomy. At least formally one can consider the loop transform of \( \Psi_{CS}(A) \):
\[ \Psi_{CS}[\gamma] = \int dA W_\gamma[A] e^{-\frac{6}{\Lambda} S_{CS}(A)} . \tag{3} \]
This expression looks the same as the expectation value of the Wilson loop in a Chern-Simons theory and has been studied by many authors \[14, 15, 16\]. The result is that it is a knot invariant that is known as the Kauffman bracket knot polynomial. The loop transform of equation (1) promotes then, at least formally, the Kauffman bracket as a solution of the Hamiltonian constraint with cosmological constant in the loop representation. This fact was confirmed by Brügmann, Gambini and Pullin \[3\] up to the second order in the cosmological constant. The extended loop version of these calculations has showed that the result is also valid into a regularized and renormalized context \[9, 17\].

The Kauffman bracket and the Jones polynomial are related through the following expression
\[ K_{\Lambda}(\gamma) = e^{-\Lambda \varphi_G(\gamma)} J_{\Lambda}(\gamma) \tag{4} \]
where \( \varphi_G(\gamma) \) is the Gauss self-linking number of \( \gamma \) and we have rescaled the cosmological constant (\( \Lambda_{\text{new}} \equiv \Lambda_{\text{old}}/6 \)). Let \( K_{\Lambda}(\gamma) = \sum_{m=0}^\infty \Lambda^m K_m(\gamma) \) and \( J_{\Lambda}(\gamma) = \sum_{n=0}^\infty \Lambda^n J_n(\gamma) \) be the expansions of the Kauffman and Jones polynomial in terms of the cosmological constant, then
\[ H_{\Lambda} K_{\Lambda}(\gamma) = \sum_{m=0}^\infty \Lambda^m (H_0 + \Lambda \det q) K_m(\gamma) \]
\[ = \sum_{m=1}^\infty \Lambda^m [H_0 K_m(\gamma) + \det q K_{m-1}(\gamma)] \tag{5} \]
with
\[ K_m = \sum_{n=0}^m \frac{(-1)^m}{n!} \varphi_G^n J_{m-n} \tag{6} \]

The exponential of the Gauss self-linking number is by itself a solution of the Hamiltonian constraint with cosmological constant \[18, 17\]. This means that
\[ H_{\Lambda} e^{-\Lambda \varphi_G(\gamma)} = \sum_{n=0}^\infty \Lambda^n \frac{(-1)^n}{n!} [H_0 \varphi_G^n - n \det q \varphi_G^{n-1}] = 0 \tag{7} \]
Using this fact it is easy to show that (5) can be put in the form
\[ H_{\Lambda} K_{\Lambda}(\gamma) = \sum_{m=2}^\infty \Lambda^m \{H_0 J_m + \sum_{n=1}^{m-1} \frac{(-1)^n}{n!} [H_0 (\varphi_G^n J_{m-n}) - n \det q (\varphi_G^{n-1} J_{m-n})] \} \tag{8} \]
If the Kauffman bracket is annihilated by $H_\Lambda$, each term of the sum in $m$ has to vanish for separate. For $m = 2$ the above result reduces to (recall that $J_1 \equiv 0$

$$H_0 J_2(\gamma) = 0$$

(9)

$J_2(\gamma)$ coincides with the second coefficient of the Alexander-Conway polynomial, so this result is the same that the obtained by Brügmann, Gambini and Pullin through an explicit computation. This fact aimed these authors to make the conjecture that the same result could also hold for higher orders in $\Lambda$. This means that some cancellation mechanism must operate in order to make $H_0 J_m(\gamma) = 0$ for all $m$ (the cancellation of the sum of square brackets in (8)). If this is true, the expansion of the Jones polynomial in terms of the cosmological constant would provide an infinite string of knot invariants that are annihilated by the vacuum Hamiltonian constraint.

To third order in $\Lambda$ equation (8) reads

$$H_\Lambda K^{(3)}_\Lambda = \Lambda^3 \{ H_0 J_3 + \text{det} q J_2 - H_0 (\varphi_G J_2) \}$$

(10)

The analytical expressions of the knot invariants included in the above equation are the followings:

$$\varphi_G(\gamma) = \frac{3}{2} g_{\mu_1 \mu_2} X^{\mu_1 \mu_2}(\gamma)$$

(11)

$$J_2(\gamma) = -3 \{ h_{\mu_1 \mu_2 \mu_3} X^{\mu_1 \mu_2 \mu_3}(\gamma) + g_{\mu_1 \mu_3} g_{\mu_2 \mu_4} X^{\mu_1 \mu_2 \mu_3 \mu_4}(\gamma) \}$$

(12)

$$J_3(\gamma) = -6 \{ (2 g_{\mu_1 \mu_2} g_{\mu_2 \mu_3} g_{\mu_3 \mu_6} + \frac{1}{2} g_{(\mu_1 \mu_3} g_{\mu_2 \mu_5} g_{\mu_4 \mu_6)} c) X^{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5}(\gamma)$$

$$+ g(\mu_{13} h_{\mu_2 \mu_4 \mu_5}) c X^{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5}(\gamma)$$

$$+ (h_{\mu_1 \mu_2 \alpha} g^{\alpha \beta} h_{\mu_3 \mu_4 \beta} - h_{\mu_1 \mu_4 \alpha} g^{\alpha \beta} h_{\mu_2 \mu_3 \beta}) X^{\mu_1 \mu_2 \mu_3 \mu_4}(\gamma) \}$$

(13)

The loop dependence of the knot invariants are written in terms of the multitangents fields $X^{\mu_1 \ldots \mu_n}(\gamma)$, which are defined as distributions integrated along the loop $\gamma$:

$$X^{\mu_1 \ldots \mu_n}(\gamma) := \int_\gamma dy_1^{\mu_1} \ldots \int_\gamma dy_n^{\mu_n} \delta(x_n - y_n) \ldots \delta(x_1 - y_1) \Theta(\alpha, y_1, \ldots, y_n)$$

(14)

The greek indices of the multitangents represent a pair of vector index and space point ($\mu_i := (a_i, x_i)$). The $\Theta$ function orders the points of integration along $\gamma$ with origin $o$ and $g_{\mu_1 \mu_2}$ and $h_{\mu_1 \mu_2 \mu_3}$ are the two and three point propagators of the Chern-Simons theory, given by

$$g_{\mu_1 \mu_2} = -\frac{1}{4\pi} \epsilon_{a_1 a_2 k} (x_1 - x_2)^k | x_1 - x_2 |^3 = -\epsilon_{a_1 a_2 k} \frac{\partial^k}{\nabla^2} \delta(x_1 - x_2)$$

(15)

$$h_{\mu_1 \mu_2 \mu_3} = \epsilon^{a_1 \alpha_2 \alpha_3} g_{\mu_1 \alpha_1} g_{\mu_2 \alpha_2} g_{\mu_3 \alpha_3}$$

(16)

with

$$\epsilon^{a_1 \alpha_2 \alpha_3} = \epsilon^{c_1 c_2 c_3} \int d^3 t \delta(z_1 - t) \delta(z_2 - t) \delta(z_3 - t)$$

(17)
In equations (11)-(13) a generalized Einstein convention is assumed (the repeated vector indices are summed from 1 to 3 and the spatial variables are integrated in $R_3$).

The above expressions admit a direct translation to the extended loop space. In fact, extended loops were introduced as generalizations of the multitangent fields to include more general fields. These fields are multivector densities that satisfy two conditions: the differential and algebraic constraints, given by

\[ \partial_{\mu} X^{\mu_1 \ldots \mu_i \ldots \mu_n} = [\delta(x_i - x_{i-1}) - \delta(x_i - x_{i+1})] X^{\mu_1 \ldots \mu_i \ldots \mu_{i+1} \ldots \mu_n} \]  

\[ X^{\mu_1 \ldots \mu_k \mu_{k+1} \ldots \mu_n} = X^{\mu_1 \ldots \mu_{k} \mu_{k+1} \ldots \mu_n} \]  

In (19) the underline means a sum over all the permutations that preserve the order of the $k$ and $n-k$ indices among themselves. The differential constraint depends on a basepoint $o$ (we have $x_0 = x_{n+1} \equiv o$ in (13)), that coincides with the origin of the loops when $X^\mu = X^\mu(\gamma)$. The constraints are related to properties of the multitangent fields: the differential constraint (18) is associated with the behavior of the holonomies under gauge transformations and the algebraic constraint (19) is a consequence of the existence of an order of integration along the path (the $\Theta$ function that appears in (14)). In general we write the elements of the extended loop group as infinite strings of the form

\[ X = (X, X^{\mu_1}, \ldots, X^{\mu_1 \ldots \mu_n}, \ldots) = (X, X^\mu) \]  

where $X$ is a real number and $\mu = (\mu_1 \ldots \mu_n)$ represents a set of indices of rank $n(\mu) = n$ (the rank of a multivector is given by the number of paired indices). The extended group product is defined through the expression

\[ (X_1 \times X_2)^\mu = \delta_{\pi\theta}^{\alpha} X_1^\pi X_2^\theta \]  

where the matrix $\delta_{\alpha\beta}^{\mu}$ is given by the following product of discrete and continuous delta functions if the sets $\alpha$ and $\beta$ have equal rank and zero otherwise. In (21) the sets $\pi$ and $\theta$ are summed from rank zero to rank infinite. Notice that for $n(\pi) = 0$ and $n(\theta) = 0$ one gets the components of rank zero of the multivector strings.

The Hamiltonian and the determinant of the three metric can be implemented in the extended space. So one can verify if $H_\Lambda K^{(3)}_\Lambda = 0$ in the extended loop representation. The evaluation of $H_0 J_3(X)$ is possible but implies a long calculation. Here we adopt the following point of view: we accept that the Kauffman bracket is annihilated by $H_\Lambda$ and then we derive a result for $H_0 J_3(X)$ from the annulment of (10). The confirmation of the fact that $H_\Lambda K^{(3)}_\Lambda = 0$ will be given elsewhere [19].

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1 $\delta_{\beta_i}^{\alpha_i} := \delta_{d_i}^{c_i} \delta(z_i - y_i)$, with $\alpha_i := (c_i, z_i)$ and $\beta_i := (d_i, y_i)$.  

6
3 A result for $H_0 J_3(X)$

We start this section introducing the fundamental tools of the extended loop framework.

3.1 Extended loops and quantum gravity

The extended loop representation of gravity is constructed in the group $D_o$ whose elements satisfy only the differential constraint. The wavefunctions are linear in the multivector fields and the Mandelstam identities make them to depend on the following combination of $X$’s

$$\psi(X) \equiv \psi(R) = D_\mu R^\mu$$

(22)

with

$$R^\mu := \frac{1}{2}[X^\mu + X^{\bar{\mu}}]$$

(23)

The overline indicates two operations: the reversal of the sequence of indices and a sign that depends on the rank of the set; that is

$$X^{\bar{\mu}} := (-1)^{n(\mu)} X^{\mu^{-1}}$$

(24)

with $\mu^{-1} := (\mu_n \ldots \mu_1)$. The expression of the constraints in terms of extended loops are

$$C_{ax} \psi(R) = \psi(F_{ab}(x) \times R^{(bx)})$$

(25)

$$\frac{1}{2} H_0(x) \psi(R) = \psi(F_{ab}(x) \times R^{(ax,bx)})$$

(26)

The action of the diffeomorphism (25) and the Hamiltonian (26) operators reduce to a shift in the argument of the wavefunctions. The shifted arguments are given in both cases by the group product between an element of the algebra (the $F_{ab}(x)$) and a combination of $R$’s with one and two spatial points evaluated at $x$. The $F_{ab}(x)$ has only two nonvanishing components

$$F_{ab}^{a_1x_1}(x) = \delta^{a_1 a}_d \partial_d \delta(x_1 - x)$$

(27)

$$F_{ab}^{a_1x_1,a_2x_2}(x) = \delta^{a_1 a_2}_{a b} \delta(x_1 - x) \delta(x_2 - x) ,$$

(28)

and the “one-point-R” and the “two-point-R” are given by the following combinations of multivector fields:

$$[R^{(bx)}]^{\mu} \equiv R^{(bx)\mu} := R^{(bx)\mu}_c$$

(29)

$$[R^{(ax,bx)}]^{\mu} \equiv R^{(ax,bx)\mu} := \delta^{\mu}_{\pi\theta} R^{(ax, bx)\mu}_c$$

(30)

The subscript $c$ indicates cyclic permutation. For the determinant of the three metric one finds the result
\[ \frac{1}{2} \text{det} q(x) \psi(R) = \psi(\epsilon_{abc} R^{(ax, bx, cx)}) \]  

where the “three-point-R” is given by the following expression

\[ R^{(ax, bx, cx)} \mu := \delta^{\mu}_{\alpha\beta\gamma} [2 R^{(ax \beta bx \alpha cx \gamma \rho)}_{\mu} + R^{(ax \alpha bx \beta cx \gamma \rho)}_{\mu}] \]  

The diffeomorphism and the Hamiltonian have very similar expressions when they are written in terms of extended loops. The only difference is the object that one puts into the group product with \( F_{ab}(x) \). So, the difference of the results would depend on the properties of the one- and two-point-R. The difference lies basically in their symmetry and regularity properties. A multivector field with two indices evaluated at the same spatial point generates a divergence. This is due to the distributional character of the multitensors. A multitensor satisfying the differential constraint \( \mathcal{L} \) diverges when two successive indices are evaluated at the same spatial point. This divergence can be regularized introducing point-splitting smearing functions. In spite of this, we develop here only the formal calculation for the sake of simplicity. The regularization and renormalization of the formal result for \( H_0 J_3 \) involves several particular features that will be given elsewhere.

Notice the effect of \( F_{ab}(x) \) in the general expression of the constraints. This quantity has only two nonvanishing components of rank one and two, so

\[ \frac{1}{2} H_0(x) \psi(R) = \sum_{n=0}^{\infty} D_{\mu_1 \ldots \mu_n} [F_{ab}^{\mu_1}(x) R^{(ax, bx)}_{\mu_2 \ldots \mu_n} + F_{ab}^{\mu_1 \mu_2} R^{(ax, bx)}_{\mu_3 \ldots \mu_n}] \]  

Due to the Mandelstam identities the propagators are cyclic under permutation of the indices \( (D_{\mu_1 \ldots \mu_n} = D_{(\mu_1 \ldots \mu_n)}) \). This means that the indices of the propagator that are contracted with \( F_{ab}(x) \) really lie in any position of the \( D \)'s. In general the result of this contraction is to modify the structure of the propagator and to fix some indices of the propagator and of the two-point-R at \( x \).

### 3.2 The calculation

From the annulment of \( \mathcal{L} \) one has

\[ H_0 J_3 = H_0 (\varphi G) J_2 - \text{det} q J_2 \]  

In order to compute the first contribution of the r.h.s. we need to define the action of an operator onto the product of two invariants (remember that the operators only know to act on linear expressions of the multivector fields). The linear wavefunction that corresponds to the product of \( \varphi G \) and \( J_2 \) is taken to be the following:\footnote{The star product can be defined in a rigorous manner in the extended framework and it satisfies several interesting properties, see [20].}
\[ \varphi_G(X)J_2(X) \rightarrow (\varphi_G * J_2)(X) := -9\left\{ g_{\mu_1\mu_2} h_{\mu_3\mu_4\mu_5} X_{\mu_1\mu_2\mu_3\mu_4\mu_5}^{\mu_3\mu_4\mu_5} + g_{\mu_1\mu_2} g_{\mu_3\mu_5} g_{\mu_4\mu_6} X_{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6} \right\} \]

In \( D_0 \), \((\varphi_G * J_2)\) defines the wavefunction “product” of \( \varphi_G \) and \( J_2 \). For any multiterior that satisfy the algebraic constraint the above expression reduces to the usual product of the two diffeomorphism invariants. From (11), (12) and (33) we then have

\[ H_0(\varphi_G * J_2)(R) = -9\left\{ g_{\mu_1\mu_2} h_{\mu_3\mu_4} (F_{ab}(x) \times R^{(ax,bz)}_{(ax,bz)})_{\mu_1\mu_2\mu_3\mu_4} + g_{\mu_1\mu_2} g_{\mu_3\mu_5} g_{\mu_4\mu_6} (F_{ab}(x) \times R^{(ax,bz)}_{(ax,bz)})_{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6} \right\} \]

It is possible to prove that \([21]\)

\[ (F_{ab}(x) \times R^{(ax,bz)}_{(ax,bz)})_{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6} = \sum_{i=0}^{k} F_{ab}^{\mu_1...\mu_i}(x) R^{(ax,bz)}_{(ax,bz)}(\mu_{k+1}...\mu_{n}) + \sum_{i=k}^{n} F_{ab}^{\mu_1...\mu_i}(x) R^{(ax,bz)}_{(ax,bz)}(\mu_{k+1}...\mu_{n}) \]

(37)

This result follows from the fact that \( F_{ab}(x) \) satisfies the homogeneous algebraic constraint. Developing \([33]\) according to this rule one obtains

\[ H_0(\varphi_G * J_2)(R) = -9\left\{ g_{\mu_1\mu_2} h_{\mu_3\mu_4\mu_5} (F_{ab}(x) \mu_1 R^{(ax,bz)}_{(ax,bz)})_{\mu_2 \mu_3 \mu_4 \mu_5} + F_{ab}(x) h_{\mu_3 \mu_4 \mu_5} (R^{(ax,bz)}_{(ax,bz)})_{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5} + g_{\mu_1 \mu_2} g_{\mu_3 \mu_5} g_{\mu_4 \mu_6} (F_{ab}(x) \mu_1 R^{(ax,bz)}_{(ax,bz)})_{\mu_2 \mu_3 \mu_4 \mu_5 \mu_6} + F_{ab}(x) \mu_3 R^{(ax,bz)}_{(ax,bz)} \mu_1 \mu_2 \mu_3 \mu_5 \mu_6} \right\} \]

(38)

Next we have to evaluate the action of \( F_{ab}(x) \) onto the two and three point propagators. The following results are obtained

\[ F_{ab}^{\mu_1}(x) g_{\mu_1 \mu_2} = -\epsilon_{a b z} \delta(x - x_2) - \partial_{a z} g_{a x b x} \]

(39)

\[ F_{ab}^{\mu_1}(x) h_{\mu_1 \mu_2 \mu_3} = -g_{\mu_2} [a x b x]_{\mu_3} + (g_{a x b x} g_{a x b x})_{\mu_2 \mu_3} \]

\[ + \frac{1}{2} g_{a x b z} \epsilon_{c_1 c_2 c_3} [g_{c_1 c_2 c_3} + \partial_{a z} g_{c_1 c_2 c_3} - g_{c_1 c_2 c_3} \partial_{a z} g_{c_1 c_2 c_3}] \]

(40)

\[ F_{ab}^{\mu_1 \mu_2}(x) g_{\mu_1 \mu_3} g_{\mu_2 \mu_4} = g_{\mu_3} [a x b x]_{\mu_4} \]

(41)

\[ F_{ab}^{\mu_1 \mu_2}(x) h_{\mu_1 \mu_2 \mu_3} = 2 h_{a x b x} \]

(42)

In the last term of (38) an integration in \( z \) is assumed. Introducing (39), (42) into (38) and performing the integrations by parts indicated by the derivatives we get
\[ \frac{1}{9} H_0 (\varphi_G * J_2) (\mathbf{R}) = \epsilon_{abc} h_{\mu_1 \mu_2 \mu_3} R^{(ax, bx) cx \mu_1 \mu_2 \mu_3} + \epsilon_{abc} g_{\mu_1 \mu_3} g_{\mu_2 \mu_4} \left[ R^{(ax, bx) cx \mu_1 \mu_2 \mu_3 \mu_4} + R^{(ax, bx) \mu_1 \mu_2 \mu_3 cx \mu_4} \right] \]

\[
- g_{\mu_1 \mu_2} \left[ 2 h_{ax bx \mu_3} - \epsilon_{c3} c \right] g_{ax bz} g_{\mu_3 cz} g_{cz} g_{cz \mu_3} \right] R^{(ax, bx) \mu_1 \mu_2 \mu_3} \\
+ g_{\mu_1 \mu_2 \mu_3} \left[ g_{ax bx \partial cz} R^{(ax, bx) \mu_1 \mu_2 \mu_3 \mu_4} - (g_{ax bx} - g_{ax bx 4}) R^{(ax, bx) \mu_1 \mu_2 \mu_3 \mu_4} \right] \\
+ g_{ax bz} g_{\mu_1 \mu_3} g_{\mu_2 \mu_4} \partial cz R^{(ax, bx) cx \mu_1 \mu_2 \mu_3} + h_{\mu_1 \mu_2 \mu_3} \partial cz R^{(ax, bx) cx \mu_1 \mu_2 \mu_3} \]

\[
\text{It is easy to demonstrate that} \\
2 h_{ax bx \mu_3} = \epsilon_{c3} c g_{ax bz} \mu_3 cz g_{cz} g_{cz \mu_3} , \quad \partial cz R^{(ax, bx) \mu_1 \mu_2 \mu_3 cx \mu_4} = \left[ \delta (z - x_3) - \delta (z - x_4) \right] R^{(ax, bx) \mu_1 \mu_2 \mu_3 \mu_4} ,
\]

\[
\text{and} \quad \partial cz R^{(ax, bx) cx \mu_1 \ldots \mu_n} = 0 .
\]

Using these results one obtains

\[
\frac{1}{9} H_0 (\varphi_G * J_2) (\mathbf{R}) = \epsilon_{abc} h_{\mu_1 \mu_2 \mu_3} R^{(ax, bx) cx \mu_1 \mu_2 \mu_3} + \epsilon_{abc} g_{\mu_1 \mu_3} g_{\mu_2 \mu_4} \left[ R^{(ax, bx) cx \mu_1 \mu_2 \mu_3 \mu_4} + R^{(ax, bx) \mu_1 \mu_2 \mu_3 cx \mu_4} \right] \]

\[
\text{The determinant of the three metric on } J_2 \text{ gives} \\
det q J_2 (\mathbf{R}) = -6 \epsilon_{abc} [h_{\mu_1 \mu_2 \mu_3} R^{(ax, bx, cx) \mu_1 \mu_2 \mu_3} + g_{\mu_1 \mu_3} g_{\mu_2 \mu_4} R^{(ax, bx, cx) \mu_1 \mu_2 \mu_3 \mu_4} .
\]

In the appendix I it is demonstrated that the three-point-R that appears in \[ \det q \] can be related to the two-point-R of the Hamiltonian in the following way

\[
\epsilon_{abc} R^{(ax, bx, cx) \mu} = - \epsilon_{abc} \left[ R^{(ax, bx) cx \mu} + \frac{1}{2} R^{(ax, bx) cx \mu} \right] \]

if the set of indices \[ \mu \] is cyclic and

\[
R^{(ax, bx) cx \mu} := \delta^{\mu}_{\alpha \beta} R^{(ax, bx) \alpha \beta} \\
R^{(ax, bx) cx \mu} := \delta^{\mu}_{\alpha \beta} R^{(ax, bx) \alpha \beta}
\]

Notice that the above expressions correspond to the algebraic constraint combination \[ (49) \] underlying the index \[ cx \] (in \[ (51) \] one has besides to overline the set of indices that follows \[ cx \). Using \[ (49) \] we get from \[ (47) \] and \[ (48) \]

\[
H_0 (\varphi_G * J_2) - \det q (x) J_2 = 9 \epsilon_{abc} g_{\mu_1 \mu_2} g_{\mu_3 \mu_4} R^{(ax, bx) \mu_1 \mu_2 \mu_3 cx \mu_4} \\
+ 6 \epsilon_{abc} h_{\mu_1 \mu_2 \mu_3} \left[ R^{(ax, bx) cx \mu_1 \mu_2 \mu_3} - R^{(ax, bx) cx \mu_1 \mu_2 \mu_3} \right] \\
+ 6 \epsilon_{abc} g_{\mu_1 \mu_3} g_{\mu_2 \mu_4} \left[ R^{(ax, bx) cx \mu_1 \mu_2 \mu_3 \mu_4} - R^{(ax, bx) cx \mu_1 \mu_2 \mu_3 \mu_4} \right]
\]

It is easy to see that the first contribution of the r.h.s. can be put in the form
\[ \epsilon_{abc} g_{\mu_1 \mu_2} g_{\mu_3 \mu_4} R^{(ax, bx) \mu_1 \mu_2 \mu_3} R^{(cx \mu_4)} = 2 \epsilon_{abc} g_{\mu_1 \mu_2} g_{\mu_3 \mu_4} [R^{(ax, bx) \mu_1} (\mu_2 \mu_3 \mu_4) c + R^{(ax, bx) \mu_1 \mu_3} \mu_2 \mu_4] \]  

(53)

and that

\[ \epsilon_{abc} g_{\mu_1 \mu_2} g_{\mu_3 \mu_4} R^{(ax, bx) \mu_1} (\mu_2 \mu_3 \mu_4) c \equiv 0 \]

(54)

by symmetry considerations. These facts allow to write the following expression for the Hamiltonian on the third coefficient of the Jones polynomial

\[
H_0 J_3(R) = 2\epsilon_{abc} \{ J_2[R^{(ax, bx) \mu_1 \mu_2 \mu_3 \mu_4}] - J_2[R^{(ax, bx) \mu_1 \mu_2 \mu_3 \mu_4}] \} + 18\epsilon_{abc} g_{\mu_1 \mu_2} g_{\mu_3 \mu_4} [R^{(ax, bx) \mu_1 \mu_3 \mu_2 \mu_4} + R^{(ax, bx) \mu_1 \mu_3 \mu_4 \mu_2}] 
\]

(55)

How can this result be zero? The only general way to get the cancellation of (55) is by using symmetry considerations. The answer is negative: the Hamiltonian does not annihilates the third coefficient of the Jones polynomial for general extended loops. In fact, developing the two-point-R’s according to (30) and using the symmetry properties of the \(R\), the propagators and \(\epsilon_{abc}\) one gets

\[
\frac{1}{2} H_0 J_3(R) = \epsilon_{abc} \bar{h}_{\mu_1 \mu_2 \mu_3} R^{(ax \mu_1 bx \mu_2 cx \mu_3)} - 3 \epsilon_{abc} g_{\mu_1 \mu_2} g_{\mu_3 \mu_4} [R^{(ax \mu_1 \mu_3 bx \mu_2 \mu_4 cx)} + R^{(ax \mu_1 \mu_3 bx \mu_4 \mu_2 cx)} - R^{(ax \mu_1 bx \mu_3 \mu_2 \mu_4 cx)}] 
\]

(56)

No further reduction is possible. In the next section we shall see that new possibilities appear when (55) is specialized to ordinary loops.

### 4 \(H_0 J_3\) in terms of ordinary loops

The relationship between the extended loop and the loop representations can be formulated in general. It is demonstrated that the constraints (25) and (26) generate those of the conventional loop representation when extended loops are reduced to ordinary loops (9). We will use here the same procedure of reduction to express (55) in terms of ordinary loops (that is, when one imposes that \(R = R(\gamma)\) is a multitangent field).

Let us consider a loop \(\gamma\) that intersects itself \(p\) times at the spatial point \(x\) (we say that the loop has “multiplicity” \(p\) at \(x\)). Following (8) we write

\[ \gamma_o = \gamma^{(1)} x_o [\gamma_{xx}]^p \gamma^{(1)} x \]

(57)

where \(o\) is the origin of the loop and \(\gamma^z_y\) indicates an open path from \(y\) to \(z\). The loops \(\gamma_z^{(i)}\), \(i = 1, \ldots, p\) are the \(p\) “petals” (basepointed at \(x\)) of \(\gamma_0\) and

\[ [\gamma_{xx}]^k := \gamma^{(i)} x \gamma^{(i+1)} x \ldots \gamma^{(k)} x \]

(58)
Usually we denote this composition of loops simply by $\gamma_{ik}$. The loop $\gamma_o$ is completely described by the multitangent fields $X^\mu(\gamma_o)$ of all rank. As we know, these fields satisfy the algebraic and differential constraints. Besides, the multitangents have another property related to the possibility to write a loop as a composition of open paths. In general

$$X^{\mu_1 \cdots \mu_n}(\gamma_o) = \int_{\gamma_o} dz^a \, \delta(x_i - z) X^{\mu_1 \cdots \mu_{i-1}}(\gamma^x) X^{\mu_{i+1} \cdots \mu_n}(\gamma^o)$$  \hspace{1cm} (59)

If the index $\mu_i$ is fixed at the point $x$ one can write

$$X^{\mu_1 \cdots \mu_i \alpha x \mu_{i+1} \cdots \mu_n}(\gamma_o) = \sum_{m=1}^{p} X^{\mu_1 \cdots \mu_i \alpha x}(\gamma^{(1)x}(\gamma_{ax,m})^o_{1m} \gamma^{(1)x}(\gamma_{ax,m})^o_{1m}) \frac{T^{ax}_m}{X^{\mu_{i+1} \cdots \mu_n}(\gamma_{ax,m+1}^o x)}$$  \hspace{1cm} (60)

where $T^{ax}_m$ is the tangent at $x$ when the loop crosses the time $m$ to this point (in the above expression the following convention is assumed: $[\gamma_{ax,m}^o]_{m+1} \approx i_{ax}$, with $i_{ax}$ the null path). The property (60) can be easily generalized to the case of any number of spatial indices evaluated at $x$. In the appendix II we shall show that the following decomposition is valid for $R^{(\alpha x, bx)}\alpha \beta(\gamma_o)$:

$$\epsilon_{abc} R^{(\alpha x, bx)}\alpha \beta(\gamma_o) = -\frac{1}{2} \epsilon_{abc} T^{ax}_m T^{bx}_q T^{cx}_r \times
\begin{align*}
&[X^{(\alpha)}(\gamma_{ax,m}, \gamma_{ax,r}) X^{(\beta)}(\gamma_{ax}) + X^{(\alpha)}(\gamma_{ax,m}, \gamma_{ax,q}) X^{(\beta)}(\gamma_{ax}) + \\
&X^{(\alpha)}(\gamma_{ax,m}, \gamma_{ax,q}) X^{(\beta)}(\gamma_{ax}) + X^{(\alpha)}(\gamma_{ax,m}, \gamma_{ax,q}) X^{(\beta)}(\gamma_{ax}) +
\end{align*}$$

$$X^{(\alpha)}(\gamma_{ax,m}, \gamma_{ax,q}) X^{(\beta)}(\gamma_{ax}) + X^{(\alpha)}(\gamma_{ax,m}, \gamma_{ax,q}) X^{(\beta)}(\gamma_{ax}) +$$

$$\frac{X^{(\alpha)}(\gamma_{ax,m} \gamma_{ax,r}, \gamma_{ax,q}) X^{(\beta)}(\gamma_{ax}) + X^{(\alpha)}(\gamma_{ax,m} \gamma_{ax,q}, \gamma_{ax,r}) X^{(\beta)}(\gamma_{ax}) +}$$

$$\frac{X^{(\alpha)}(\gamma_{ax,m} \gamma_{ax,q}, \gamma_{ax,r}) X^{(\beta)}(\gamma_{ax}) + X^{(\alpha)}(\gamma_{ax,m} \gamma_{ax,q}, \gamma_{ax,r}) X^{(\beta)}(\gamma_{ax}) +}{(61)}$$

where $(\cdot || \cdot)_c$ indicates the cyclic permutation of the sets of indices and the sums run from 1 to $p - 2$ for $m$, from $m + 1$ to $p - 1$ for $q$ and from $q + 1$ to $p$ for $r$. In the above expression $\gamma_{ik} := \gamma_{ax[k]} \gamma_{ax[k-1]} \cdots \gamma_{ax[1]}$ defines the rerouting of the loop $\gamma_{ik}$ and

$$\gamma_{mq} := [\gamma_{ax}]^q_{m+1}$$  \hspace{1cm} (62)
$$\gamma_{qr} := [\gamma_{ax}]^r_{q+1}$$  \hspace{1cm} (63)
$$\gamma_{rm} := [\gamma_{ax}]^p_{r+1} \gamma^{(1)o}_{x o \gamma^{(1)x}(\gamma_{ax})}$$  \hspace{1cm} (64)

Notice now in which way the combinations defined by (62) and (63) are generated from (61): one has simply to contract (61) with $\delta^{\mu}_{\alpha \beta}$ and $\delta^{\mu}_{\alpha \beta}$ respectively.

This means that a group product is formed for all the terms of (61) and the only difference between the two cases will be the “rerouting” of the set of indices $\beta$. Using the fact that

$$\delta^{\mu}_{\alpha \beta} X^{\alpha}(\gamma) X^{\beta}(\gamma') = X^{\mu}(\gamma \gamma')$$  \hspace{1cm} (65)
and

\[ X^\gamma = X^\mu, \quad (66) \]

one gets from (61) the following results:

\[ \epsilon_{abc} R^{(ax, bx)} D^\mu (\gamma_o) = - \epsilon_{abc} \sum_{m, q, r} T^x_m T^{bx}_q T^{cx}_r \times \]

\[ \left[ R^\mu (\gamma_{mq} \gamma_{qr} \gamma_{rm}) + R^\mu (\gamma_{rm} \gamma_{mq} \gamma_{qr}) + R^\mu (\gamma_{rm} \gamma_{mq} \gamma_{qr}) \right. \]

\[ \left. + \frac{1}{2} \{ X^\mu (\gamma_{mq} \gamma_{qr} \gamma_{rm}) + X^\mu (\gamma_{rm} \gamma_{mq} \gamma_{qr}) \} + \frac{1}{2} \{ X^\mu (\gamma_{rm} \gamma_{mq} \gamma_{qr}) + X^\mu (\gamma_{mq} \gamma_{qr} \gamma_{rm}) \} \right] (67) \]

and

\[ \epsilon_{abc} R^{(ax, bx)} D^\mu (\gamma_o) = - \epsilon_{abc} \sum_{m, q, r} T^x_m T^{bx}_q T^{cx}_r \times \]

\[ \left[ R^\mu (\gamma_{mq} \gamma_{qr} \gamma_{rm}) + R^\mu (\gamma_{rm} \gamma_{mq} \gamma_{qr}) + R^\mu (\gamma_{rm} \gamma_{mq} \gamma_{qr}) \right. \]

\[ \left. + \frac{1}{2} \{ X^\mu (\gamma_{mq} \gamma_{qr} \gamma_{rm}) + X^\mu (\gamma_{rm} \gamma_{mq} \gamma_{qr}) \} + \frac{1}{2} \{ X^\mu (\gamma_{rm} \gamma_{mq} \gamma_{qr}) + X^\mu (\gamma_{mq} \gamma_{qr} \gamma_{rm}) \} \right] (68) \]

We see that the loop \( \gamma_o \) is decomposed into a “three petal structure” with a rerouted portion. Each “petal” consists of a combination of loops basepointed at \( x \). Suppose now that the greek indices of the above expressions are contracted with suitable propagators \( D^\mu \). Then using the cyclicity of \( D^\mu \) we get

\[ \epsilon_{abc} D^\mu R^{(ax, bx)} (\gamma_o) = - 2 \epsilon_{abc} \sum_{m, q, r} T^x_m T^{bx}_q T^{cx}_r \times \]

\[ \left[ \psi (\gamma_{mq} \gamma_{qr} \gamma_{rm}) + \psi (\gamma_{rm} \gamma_{mq} \gamma_{qr}) + \psi (\gamma_{rm} \gamma_{mq} \gamma_{qr}) \right] \]

(69)

and

\[ \epsilon_{abc} D^\mu R^{(ax, bx)} (\gamma_o) = - 2 \epsilon_{abc} \sum_{m, q, r} T^x_m T^{bx}_q T^{cx}_r \times \]

\[ \left[ \psi (\gamma_{mq} \gamma_{rm} \gamma_{qr}) + \psi (\gamma_{rm} \gamma_{mq} \gamma_{qr}) + \psi (\gamma_{rm} \gamma_{mq} \gamma_{qr}) \right] \]

(70)

with

\[ \psi (\gamma) := D^\mu R^\mu (\gamma) \]

If \( \psi (\gamma) \) is a knot invariant, the r.h.s of equations (69) and (70) reduces to a combination of invariants evaluated onto the three petal structure. This result shows that the algebraic combinations (60) and (51) are able to capture relevant geometrical information when they are specialized to ordinary loops.

What happens with the terms of the form \( \ldots R^{(ax, bx)} \ldots \) in (55)? These terms generate Gauss link invariants when ordinary loops are introduced. In effect, it is possible to show that
This expression can be simplified using the following Mandel stam identity valid due to the algebraic constraint,

\[ R^{\mu_1 \mu_3}(\gamma) = \frac{1}{2} X^{\mu_1}(\gamma) X^{\mu_3}(\gamma) \] (73)

and

\[ R^{\mu_2 \mu_4}(\gamma') = \frac{1}{2} [X^{\mu_2}(\gamma) - X^{\mu_2}(\gamma')][X^{\mu_4}(\gamma) - X^{\mu_4}(\gamma')] \] (74)

So one can write

\[ g_{\mu_1 \mu_2} R^{\nu_1 \nu_3}(\gamma) R^{\nu_2 \nu_4}(\gamma') \] (75)

with

\[ \varphi_G(\gamma, \gamma') := \frac{4}{3} g_{\alpha_1 \alpha_2} X^{\alpha_1}(\gamma) X^{\alpha_2}(\gamma') \] (76)

the Gauss linking number of the loops \( \gamma \) and \( \gamma' \). Introducing now equations (59), (71), (72) and (75) into (53) the following result is obtained

\[ H_0 J_3(\gamma_0) = 4 \epsilon_{abc} \sum_{m,q,r} T_m^{ax} T_q^{bx} T_r^{cx} \times \]

\[ \{ J_2(\gamma_{mq} \gamma_{qr} \gamma_{rm}) + J_2(\gamma_{mq} \gamma_{qr} \gamma_{rm}) + J_2(\gamma_{mr} \gamma_{qr} \gamma_{rm}) \]

\[ - J_2(\gamma_{mq} \gamma_{rm} \gamma_{mr}) - J_2(\gamma_{mr} \gamma_{qr} \gamma_{rm}) - J_2(\gamma_{mr} \gamma_{mq} \gamma_{rm}) \]

\[ - 6[\varphi_G(\gamma_{mq}, \gamma_{qr}) - \varphi_G(\gamma_{mq}, \gamma_{rm})]^2 - 6[\varphi_G(\gamma_{mr}, \gamma_{mq}) - \varphi_G(\gamma_{mr}, \gamma_{rm})]^2 \]

\[ - 6[\varphi_G(\gamma_{mr}, \gamma_{mq}) - \varphi_G(\gamma_{mr}, \gamma_{rm})]^2 \} \] (77)

This expression can be simplified using the following Mandelstam identity valid for \( J_2 \):

\[ J_2(\gamma_{mq} \gamma_{qr} \gamma_{rm}) - J_2(\gamma_{qr} \gamma_{mq} \gamma_{rm}) = J_2(\gamma_{mr} \gamma_{mq} \gamma_{rm}) - J_2(\gamma_{mq} \gamma_{qr} \gamma_{rm}) \] (78)

We then conclude

\[ H_0 J_3(\gamma_0) = -12 \epsilon_{abc} \sum_{m,q,r} T_m^{ax} T_q^{bx} T_r^{cx} \times \}

\[ \{ J_2(\gamma_{mq} \gamma_{qr} \gamma_{rm}) - J_2(\gamma_{mr} \gamma_{mq} \gamma_{rm}) \]

\[ + 2[\varphi_G(\gamma_{mq}, \gamma_{qr}) - \varphi_G(\gamma_{mq}, \gamma_{rm})]^2 + 2[\varphi_G(\gamma_{mr}, \gamma_{mq}) - \varphi_G(\gamma_{mr}, \gamma_{rm})]^2 \]

\[ + 2[\varphi_G(\gamma_{mr}, \gamma_{mq}) - \varphi_G(\gamma_{mr}, \gamma_{qr})]^2 \} \] (79)
This expression has a quite nontrivial geometrical content. The action of the Hamiltonian on the third coefficient of the Jones polynomial is given by a combination of knot \( J_2 \) and link \( \varphi_G \) invariants for a loop with an intersection of arbitrary multiplicity. Moreover, the knot and link invariants are evaluated onto a precise decomposition of the original loop into a three petal structure basepointed at \( x \).

The knot and link invariants appear combined into pairs. This fact suggests that \( J_3 \) could in principle be annihilated by \( H_0 \) by means of simple topological requirements. For example, for the unknot trefoil (79) reduces to

\[
H_0 J_3(\text{unknot trefoil}) = 12\epsilon_{abc}T_1^{ax}T_2^{bx}T_3^{cx}\{J_2(\gamma_2\gamma_1\gamma_3) - J_2(\gamma_1\gamma_2\gamma_3)\} \quad (80)
\]

that is nonzero in general. In spite that the cancellation does not take place for the simplest three petal structure, it seems plausible that it could happen for some topologies of that kind. A first approach to the problem has not revealed any immediate solution of this type. This topic is currently under progress.

### 4.1 The Mandelstam identities of \( J_3 \)

The Mandelstam identities \[22\] are requisites for the wavefunctions in the loop and extended loop representations (the identities follows from the properties of the Wilson loop and the Wilson functional is in the basis of the loop and extended loop transforms). One can see that \( J_3 \) does not satisfies all the Mandelstam identities in general (that is, for arbitrary loops or extended loops) \( ^3 \). From this point of view one can question the expectative that this knot invariant could represent a genuine quantum state of gravity \( ^4 \).

An intriguing fact that we are going to consider here is that the Mandelstam identities are recovered totally by the \( J_3 \) invariant if the topological conditions necessary for \( H_0 J_3 = 0 \) are fulfilled.

As it was mentioned in Sect. 2, the Kauffman bracket can be viewed as the expectation value of the Wilson loop. This means that the knot invariants \( K_m(\gamma) \) of the expansion (5) satisfy the Mandelstam identities by construction. According to \( ^5 \), each coefficient of the expansion is expressed by a sum of products of Gauss and Jones invariants. As the Mandelstam identities are nonlinear, the identities will not be inherited by \( J_n(\gamma) \) in general \( ^6 \) (the abelian property makes trivial the Mandelstam identities for the case of the Gauss invariants).

Let us limit the discussion to the case of interest. To third order one has

\[
K_3 = J_3 - J_2 \varphi_G - \frac{1}{3!} \varphi_G^3 \quad (81)
\]

\(^3\)I thank Rodolfo Gambini to point me out this fact.

\(^4\)This fact was not realized at the time the conjecture mentioned in Sect. 2 was proposed.

\(^5\)The second coefficient \( J_2 \) is an exception to this rule.
The product $J_2 \varphi_G$ is responsible that the property \[\text{(78)}\] will not be inherited by $J_3$. However, from the fact that $J_2$ and $\varphi_G$ satisfy the identity \[\text{(78)}\] it is straightforward to derive the following relationship for the product of the two invariants

\[
(J_2 \varphi_G)(\gamma_{mq} \gamma_{qr} \gamma_{rm}) =
(J_2 \varphi_G)(\gamma_{qr} \gamma_{mq} \gamma_{rm}) + (J_2 \varphi_G)(\gamma_{qr} \gamma_{mq} \overline{\gamma}_{rm}) - (J_2 \varphi_G)(\gamma_{rm} \gamma_{mq} \overline{\gamma}_{qr}) +
[J_2(\gamma_{qr} \gamma_{mq} \overline{\gamma}_{rm}) - J_2(\gamma_{mq} \gamma_{qr} \overline{\gamma}_{rm})][\varphi_G(\gamma_{qr} \gamma_{mq} \gamma_{rm}) - \varphi_G(\gamma_{mq} \gamma_{qr} \overline{\gamma}_{rm})] \tag{82}
\]

where we have used the fact that

\[
\varphi_G(\gamma_{mq} \gamma_{qr} \gamma_{rm}) = \varphi_G(\gamma_{qr} \gamma_{mq} \gamma_{rm}) \tag{83}
\]

The difference of the Gauss invariants is simply given by

\[
\varphi_G(\gamma_{qr} \gamma_{mq} \gamma_{rm}) - \varphi_G(\gamma_{mq} \gamma_{qr} \overline{\gamma}_{rm}) = 4[\varphi_G(\gamma_{mq}, \gamma_{rm}) + \varphi_G(\gamma_{qr}, \gamma_{rm})] \tag{84}
\]

and using \[\text{(78)}\] again we conclude from \[\text{(82)}\]

\[
(J_2 \varphi_G)(\gamma_{mq} \gamma_{qr} \gamma_{rm}) + (J_2 \varphi_G)(\gamma_{mq} \gamma_{qr} \overline{\gamma}_{rm}) =
(J_2 \varphi_G)(\gamma_{qr} \gamma_{mq} \gamma_{rm}) + (J_2 \varphi_G)(\gamma_{qr} \gamma_{mq} \overline{\gamma}_{rm}) +
4[J_2(\gamma_{mq} \gamma_{qr} \gamma_{rm}) - J_2(\gamma_{qr} \gamma_{mq} \gamma_{rm})][\varphi_G(\gamma_{mq}, \gamma_{rm}) + \varphi_G(\gamma_{qr}, \gamma_{rm})] \tag{85}
\]

Those three petal structures that are annihilated by the Hamiltonian constraint would verify that

\[
J_2(\gamma_{mq} \gamma_{qr} \gamma_{rm}) - J_2(\gamma_{qr} \gamma_{mq} \gamma_{rm}) = 0 \tag{86}
\]

For these cases, $J_2 \varphi_G$ recovers the property \[\text{(78)}\]. As $\varphi_G^2$ satisfies the Mandelstam identities in general, we conclude that $J_3$ will verify the Mandelstam identity \[\text{(78)}\] for those loops that makes $H_0 J_3(\gamma_o) = 0$ at the intersecting points.

## 5 Conclusions

The initial question about the third coefficient of the Jones polynomial in quantum gravity cannot be answered with a simple yes or no. In the analysis, we have passed from a negative answer in the extended loop manifold to a new expectation in the ordinary loop space. The new expectation is based in the nontrivial topological content of the result \[\text{(79)}\].

The possibility that this result could provide a new solution of the Wheeler-DeWitt equation is still unclear. Besides the proper difficulty associated with the topological conditions that have to be fulfilled, there exists some problems

\[\text{There are three basic Mandelstam identities: the cyclicity (}J_3(\gamma \gamma_2) = J_3(\gamma_2 \gamma_1))\], (the inversion (}J_3(\gamma) = J_3(\overline{\gamma})) and (78). The first two are valid in general for $J_3$.\]
on the general ground. The main problem follows from the restriction of the
domain of definition of the loop wavefunction. The limitation of the domain in
the loop space implies some kind of characteristic function that takes the value
one for a set of loops with a definite topology and zero otherwise. This fact
faces up two new difficulties: the action of the Hamiltonian on the Heaviside
part of the wavefunction could be nontrivial and the Mandelstam identities
of the restricted wavefunction break down. Up to present it is not clear how
to solve these questions in a general way. It is worth of emphasize that this
objections are shared by the smoothened loops of reference 5.

To finalize, a comment about the use of extended loops in quantum gravity
is in order. In a companion article 19 it is explicitely proved that the Kauffman
bracket is a solution of the Hamiltonian constraint with cosmological constant
to third order. This requires the explicit computation of the vacuum Hamil-
tonian on $J_3$, a very involved task from the point of view of the conventional
loop representation 7. Besides of this, the analysis developed in 19 shows
that a systematic of operation exists for the constraints in the extended loop
framework. This systematic allows to raise several interesting questions about
knot theory and quantum gravity, such as: Which are the analytical expressions
of the knot invariants in terms of the two and three point propagators of the
Chern-Simons theory that satisfy the Mandelstam identities?; or: There exist
loop wavefunctions of this type besides the Kauffman bracket, the exponential
of the Gauss number and the second coefficient of the Alexander-Conway coef-
ficient? These questions are of interest to the knowledge of the state of space of
quantum gravity. Extended loops are able to provide an answer to these (and
related) topics.

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Appendix I

The action of the determinant of the three metric is characterized in the ex-
tended loop representation by $R^{(ax, bx, cx)}$. We show here that this combination
of multitensor fields can be written in terms of the two-point-R that appears
in the Hamiltonian. The definition of the three-point-R is

\begin{equation}
R^{(ax, bx, cx)}_{\mu} = \delta^{\mu}_{\sigma_1, \sigma_2, \sigma_3} [2 R^{(ax, bx, cx)}_{\sigma_2, \sigma_1, \sigma_3} c + R^{(ax, bx, cx)}_{\sigma_1, \sigma_2, \sigma_3} c] \tag{87}
\end{equation}

\footnote{In general, the loop derivative obligates to limit the analysis to loops with some kind of simple
intersection (typically, intersections of multiplicity three). Notice that the results obtained via
extended loops are valid for arbitrary ordinary loops.}
From (87) we have

\[ R^{(ax, bx)\alpha cx \beta}_{\pi \theta}(\pi \sigma \beta) = \delta^{\alpha cx \beta}_{\pi \theta} R^{(ax \pi bx\beta)_{c}}(\pi \sigma \beta) \] (88)

and it is easy to see that

\[ \delta^{\alpha cx \beta}_{\pi \theta} = \delta^{\alpha cx \sigma \beta}_{\pi \theta} + \delta^{\alpha \sigma cx \beta}_{\pi \theta} \] (89)

Then

\[ R^{(ax, bx)\alpha cx \beta}_{\pi \sigma \beta} = \delta^{\beta}_{\sigma \theta} R^{(ax \alpha cx \sigma \sigma \beta)_{c}}(\pi \sigma \beta) - \delta^{\alpha \sigma cx \beta}_{\pi \theta} R^{(ax \pi bx \sigma \sigma \beta)_{c}}(\pi \sigma \beta) \] (90)

The \( R \)'s are invariant under the overline operation: \( \bar{R}^\mu = R^\mu \). Using this property to reverse the sequence of indices of the last term of the r.h.s we get

\[ R^{(ax, bx)\alpha cx \beta}_{\pi \sigma \beta} = \delta^{\beta}_{\sigma \theta} R^{(ax \alpha cx \sigma \sigma \beta)_{c}}(\pi \sigma \beta) + \delta^{\alpha \sigma cx \beta}_{\sigma \theta} R^{(ax \sigma \sigma \beta bx \sigma \beta)_{c}}(\pi \sigma \beta) \] (91)

Then,

\[ R^{(ax, bx)\alpha cx \beta}_{\pi \sigma \beta} = \delta^{\beta}_{\sigma \theta} R^{(ax \alpha cx \sigma \sigma \beta)_{c}}(\pi \sigma \beta) + \delta^{\alpha \sigma cx \beta}_{\sigma \theta} R^{(ax \sigma \sigma \beta bx \sigma \beta)_{c}}(\pi \sigma \beta) \] (92)

If the set of indices \( \mu \) is cyclic one has

\[ \delta^{(\mu)_{c}}_{\sigma \sigma \beta} = \delta^{(\mu)_{c}}_{\sigma \sigma \beta} \] (93)

and then we get from (92)

\[ \epsilon_{abc} R^{(ax, bx)\alpha cx \beta}_{\pi \sigma \beta} = -2\epsilon_{abc} \delta^{\mu}_{\sigma \sigma \beta} R^{(ax \alpha cx \sigma \sigma \beta)_{c}}(\pi \sigma \beta) \] (94)

A similar procedure can be applied to (51). One obtains in this case

\[ \epsilon_{abc} R^{(ax, bx)\alpha cx \beta}_{\pi \sigma \beta} = -2\epsilon_{abc} \delta^{\mu}_{\sigma \sigma \beta} R^{(ax \alpha cx \sigma \sigma \beta)_{c}}(\pi \sigma \beta) \] (95)

Introducing (94) and (95) into (87) we conclude that

\[ \epsilon_{abc} R^{(ax, bx, cx)\alpha cx \beta}_{\pi \sigma \beta} = -\epsilon_{abc} [R^{(ax, bx)\alpha cx \beta}_{\pi \sigma \beta} + \frac{1}{2} R^{(ax, bx)\alpha cx \beta}_{\pi \sigma \beta}] \] (96)

if the set of indices \( \mu \) is cyclic.

**Appendix II**

In this appendix we shall demonstrate that

\[ \epsilon_{abc} R^{(ax, bx)\alpha cx \beta}_{\pi \theta}(\gamma_{\theta}) = -\frac{1}{2} \epsilon_{abc} \sum_{m,q,r} T_{m}^{ax} T_{q}^{bx} T_{r}^{cx} \times \]
\[ [X^{\alpha}(\gamma_{mq} \gamma_{rm})X^{\beta}_{c}(\gamma_{qr}) + X^{\alpha}(\gamma_{rm} \gamma_{mq})X^{\beta}_{c}(\gamma_{qr}) + \]
\[ X^{\alpha}(\gamma_{qr} \gamma_{mq})X^{\beta}_{c}(\gamma_{rm}) + X^{\alpha}(\gamma_{rm} \gamma_{qr})X^{\beta}_{c}(\gamma_{mq}) + \]
\[ X^{\alpha}(\gamma_{rm} \gamma_{qr})X^{\beta}_{c}(\gamma_{mq}) + X^{\alpha}(\gamma_{qr} \gamma_{rm})X^{\beta}_{c}(\gamma_{mq})] \] (97)
for a loop with multiplicity \( p \) at \( x \). From (11) we know that

\[
\epsilon_{abc} R^{(ax, bx) \alpha \epsilon \beta} = -\epsilon_{abc} \left[ \delta^{\alpha}_{\sigma_1, \sigma_2} R^{(ax, bx) \beta \epsilon \sigma_1}_c + \delta^{\beta}_{\sigma_1, \sigma_2} R^{(ax, \alpha \sigma_1 cx \sigma_2)_c} \right]
\]

(98)

We start by considering in general the decomposition of \( R^{(ax, \sigma_1 bx \sigma_2 cx \sigma_3)_c} (\gamma_o) \). The expression of this quantity in terms of multivector fields is

\[
R^{(ax, \sigma_1 bx \sigma_2 cx \sigma_3)_c} = \frac{1}{2} \left[ X^{(ax, \sigma_1 bx \sigma_2 cx \sigma_3)_c} - X^{(ax, \sigma_3 cx \sigma_2 bx \sigma_1)_c} \right]
\]

(99)

Let us first develop \( X^{(ax, \sigma_1 bx \sigma_2 cx \sigma_3)_c} (\gamma_o) \). The cyclic combination of multivector fields with one spatial index evaluated at \( x \) can be written in this way

\[
X^{(ax, \sigma)_c} = \delta^{\sigma}_{\lambda_1, \lambda_2} X^{\lambda_2 ax \lambda_1}
\]

(100)

Then

\[
X^{(ax, \sigma_1 bx \sigma_2 cx \sigma_3)_c} = \delta^{\sigma_1}_{\lambda_1, \lambda_2} \delta^{\sigma_2}_{\lambda_1, \lambda_2} X^{\lambda_2 ax \lambda_1}
\]

(101)

The delta matrix with two spatial indices fixed at \( x \) admits the following decomposition

\[
\delta^{\sigma_1, bx \sigma_2 cx \sigma_3}_{\lambda_1, \lambda_2} = \delta^{\sigma_1, bx \sigma_2 cx \sigma_3}_{\lambda_1, \lambda_1} \delta^{\rho_1}{\lambda_2} + \delta^{\sigma_1, bx \sigma_2 cx \sigma_3}_{\lambda_1, \lambda_2} \delta^{\sigma_2}{\rho_1, \lambda_2} + \delta^{\sigma_1, bx \sigma_2 cx \sigma_3}_{\lambda_1, \lambda_2} \delta^{\sigma_2}{\rho_1, \lambda_2}
\]

(102)

Introducing (102) into (101) we get

\[
X^{(ax, \sigma_1 bx \sigma_2 cx \sigma_3)_c} = \delta^{\sigma_1, bx \sigma_2 cx \sigma_3}_{\lambda_1, \lambda_2} X^{\lambda_2 bx ax \lambda_1} + \delta^{\sigma_2, bx \sigma_2 cx \sigma_3}_{\lambda_1, \lambda_2} X^{\lambda_2 cx ax \sigma_1}_c
\]

(103)

A multitangent field with three indices evaluated at \( x \) decomposes in the following way for a loop with multiplicity \( p \) at \( x \)

\[
X^{ax, bx, \sigma' \epsilon', \lambda' \lambda} (\gamma_o) = \sum_{m, q, r} T^{ax}_m T^{bx}_q T^{cx}_r \times
X^{\lambda_2 \epsilon, \sigma_0 \epsilon \gamma_2 \gamma_{o}} m \cdot X^{\epsilon_0 \epsilon \sigma_0 \epsilon \gamma_{o} \gamma} m_2 \cdot X^{\epsilon_0 \epsilon \sigma_0 \epsilon \gamma_{o} \gamma} m_2 \cdot X^{\epsilon_0 \epsilon \sigma_0 \epsilon \gamma_{o} \gamma} m_2 \cdot
\]

\[
X^{(ax, bx, \sigma_1 bx, \sigma_2 ax \sigma_3)_c} (\gamma_o) = \sum_{m, q, r} T^{ax}_m T^{bx}_q T^{cx}_r \times
X^{\lambda_2 \epsilon, \sigma_0 \epsilon \gamma_2 \gamma_{o}} m \cdot X^{\epsilon_0 \epsilon \sigma_0 \epsilon \gamma_{o} \gamma} m_2 \cdot X^{\epsilon_0 \epsilon \sigma_0 \epsilon \gamma_{o} \gamma} m_2 \cdot X^{\epsilon_0 \epsilon \sigma_0 \epsilon \gamma_{o} \gamma} m_2 \cdot
\]

(104)

Notice that the sets \( \lambda_1 \) and \( \lambda_2 \) will be joined by a group product once this result is introduced into (103). Each of these terms generate then the following composition of loops

\[
\delta^{\sigma_1, bx \sigma_2 bx \sigma_3}_{\lambda_1, \lambda_2} X^{\lambda_2 bx \epsilon \gamma_{o} \gamma_2 m_2} \cdot X^{\lambda_2 \epsilon, \sigma_0 \epsilon \gamma_2 \gamma_{o}} m \cdot X^{\epsilon_0 \epsilon \sigma_0 \epsilon \gamma_{o} \gamma} m_2 \cdot X^{\epsilon_0 \epsilon \sigma_0 \epsilon \gamma_{o} \gamma} m_2 \cdot X^{\epsilon_0 \epsilon \sigma_0 \epsilon \gamma_{o} \gamma} m_2 \cdot
\]

(105)
We get from (103), (104) and (105)

$$
\epsilon_{abc} X^{(ax \sigma_1 \ bx \sigma_2 \ cx \sigma_3) c} (\gamma_o) = \epsilon_{abc} \sum_{m,q,r} T_m^{ax} T_q^{bx} T_r^{cx} X^{(\sigma_1)(\gamma_{rm})} X^{(\sigma_2)(\gamma_{mq})} X^{(\sigma_3)c} (\gamma_{qr})
$$

(106)

where \((\cdot || \cdot || \cdot) c\) means the cyclic permutation of the \(sets\) of indices. The other contribution of (99) can be developed in a similar way. In this case we have the reroutings generated by the overline operation:

$$
\epsilon_{abc} X^{(ax \overline{\sigma}_2 \ bx \sigma_2 \ cx \sigma_3) c} (\gamma_o) = \epsilon_{abc} \sum_{m,q,r} T_m^{ax} T_q^{bx} T_r^{cx} X^{(\overline{\sigma}_2)(\gamma_{rm})} X^{(\sigma_2)(\gamma_{mq})} X^{(\sigma_3)c} (\gamma_{qr})
$$

(107)

Using the above results one can write

$$
\epsilon_{abc} R^{(ax \sigma_1 \ bx \sigma_2 \ cx \sigma_3) c} = \frac{1}{2} \epsilon_{abc} \sum_{m,q,r} T_m^{ax} T_q^{bx} T_r^{cx} \times [X^{(\sigma_1)(\gamma_{rm})} X^{(\sigma_2)(\gamma_{mq})} X^{(\sigma_3)c} (\gamma_{qr}) + X^{(\sigma_3)(\gamma_{rm})} X^{(\sigma_2)(\gamma_{mq})} X^{(\sigma_1)c} (\gamma_{qr})]
$$

(108)

Now it is straightforward to evaluate the r.h.s of (98). For the first contribution of (98) we have

$$
\epsilon_{abc} \bar{\delta}^{\alpha} \bar{\sigma}_1 R^{(ax \sigma_2 \ bx \sigma_2 \ cx \sigma_3) c} (\gamma_o) = \frac{1}{2} \epsilon_{abc} \sum_{m,q,r} T_m^{ax} T_q^{bx} T_r^{cx} \times [X^{(\sigma_2)(\gamma_{rm})} X^{(\beta)(\gamma_{mq})} X^{(\sigma_3)c} (\gamma_{qr}) + X^{(\sigma_3)(\gamma_{rm})} X^{(\beta)(\gamma_{mq})} X^{(\sigma_2)c} (\gamma_{qr})]
$$

(109)

A similar result is obtained for the other contribution:

$$
\epsilon_{abc} \bar{\delta}^{\beta} \bar{\sigma}_3 R^{(ax \alpha \ bx \sigma_1 \ cx \sigma_2) c} (\gamma_o) = \frac{1}{2} \epsilon_{abc} \sum_{m,q,r} T_m^{ax} T_q^{bx} T_r^{cx} \times [X^{(\alpha)(\gamma_{mq})} X^{(\beta)(\gamma_{qr})} X^{(\sigma_1)(\gamma_{rm})} + X^{(\alpha)(\gamma_{qr})} X^{(\beta)(\gamma_{rm})} X^{(\sigma_1)(\gamma_{mq})} + X^{(\alpha)(\gamma_{rm})} X^{(\beta)(\sigma_1)(\gamma_{mq})} X^{(\gamma_{qr})} + X^{(\alpha)(\gamma_{rm})} X^{(\beta)(\sigma_1)(\gamma_{mq})} X^{(\gamma_{qr})} + X^{(\alpha)(\gamma_{qr})} X^{(\beta)(\sigma_1)(\gamma_{mq})} X^{(\gamma_{qr})} + X^{(\alpha)(\gamma_{rm})} X^{(\beta)(\sigma_1)(\gamma_{mq})} X^{(\gamma_{qr})}]
$$

(110)

Introducing now (109) and (110) into (98) we get the result (97).
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