Nucleon spin structure functions, considering target mass correction and higher twist effects at the NNLO accuracy and their transverse momentum dependence

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(Dated: March 29, 2022)

Using recent and updated world data on polarized structure functions \(g_1\) and \(g_2\) we perform QCD analysis at the next-next-to-leading-order (NNLO) accuracy. We include also target mass correction and higher twist effect to get more precise results in our fitting procedure. To confirm the validity of our fitting results several sum rules are examined and we do a comparison for them with results from other models. In our analysis we employ Jacobi polynomials approach to obtain analytical solutions of the DGLAP evolution equations for parton distribution functions (PDFs). Using the extracted PDFs from our data analysis as input we also compute the \(x\)- and \(p_T\)-dependence of some transverse momentum dependence (TMD) PDFs in polarized case, based on covariant parton model. These functions are naively even time-reversal (T-even) at twist-2 approximation. The results for TMDs are indicating proper and acceptable behaviour with respect to what are presented in other literatures.

CONTENTS

I. Introduction 1
II. Leading twist spin dependence of structure function 2
III. Jacobi polynomials expansion technique 3
IV. Target mass corrections in polarized case 3
V. Twist-3 contribution 4
VI. Fitting contents in QCD analysis 4
   A. Parametrization 5
   B. Overview of data sets 5
   C. \(\chi^2\) minimization 6
VII. The Sum Rules 7
   A. Bjorken sum rule 7
   B. Proton helicity sum rule 10
   C. The twist-3 reduced matrix element \(d_2\) 11
   D. Burkhardt-Cottingham (BC) sum rule 11
   E. Efremov-Leader-Teryaev (ELT) Sum Rule 11
VIII. Comparison for the spin structure functions 13
IX. Predictions for polarized TMDs 14
X. Conclusions 16

Acknowledgments 16
References 16

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I. INTRODUCTION

The determination of the nucleon’s spin into its quark and gluon components is still an important challenge in particle physics. The deep-inelastic scattering (DIS) experiments performed at DESY, SLAC, CERN, and JLAB have refined our understanding of the spin distributions and revealed the spin-dependent structure functions of the nucleon. The polarized structure functions \(g_1(x, Q^2)\) and \(g_2(x, Q^2)\) are measured in deep-inelastic scattering of a longitudinally polarized lepton on polarized nuclear targets. We do the required analysis on the polarized structure function to extract the desired parton densities at the initial \(Q_0\).

In exact consideration of inclusive processes it is required to take into account the distributions in which the role of transverse momentum is embedded. These distributions are known as transverse momentum dependent (TMD) distributions. TMDs are the generalization of PDFs which provide us an extensive knowledge to investigate the hadron structure function. In a native parton model in which the effect of transverse momentum of a quark is not outstanding, there is a proper computational frame which is called infinite momentum frame (IMF) [1, 2]. In this frame the target (nucleon) is moving fast, comparable to speed of light and because of Lorentz contraction the nucleon seems like a flat disc. In this case one can imagine a transverse space position of quark inside the disk with respect to the moving direction of target. This space coordinate is called impact parameter and denoted usually by \(b_T\). Corresponding to the impact parameter in coordinate space we can attribute to a quark inside the target a transverse momentum, \(k_T\), that is perpendicular to moving direction of nucleon. This momentum component is ignorable against the quark longitudinal momentum. This model then gives oversimplified relations between structure and distribution functions. In
an another model which is called covariant parton model (CPM) [3] a more exact but much more complex relations between structure and distribution functions are given. The original assumptions of this model is based on covariance of relations together with a spherically symmetric quark momenta distribution in the nucleon rest frame where one photon exchange is used in a charged lepton-quark interaction. The output of this model is such us the quark transverse momentum is as important as longitudinal one and the transverse momentum dependence of parton densities are obtained analytically [4].

The extended PDF is then describing the parton distribution with respect to both $x$ and $k_T$ variables. On the other words quarks can have transverse momentum with respect to the motion of parent hadron. The transverse momentum of parton at initial state and inside the parent hadron is called the intrinsic transverse momentum, denoted by $k_T$. In the final state the transverse momentum of parton with respect to the momentum of produced hadron is denoted by $p_T$. TMDs have outstanding effect on the momentum feature of produced hadron. They also have crucial role to describe the spin asymmetry in produced hadron [5] by analysing the semi inclusive DIS (SIDIS) processes [6, 7]. To achieve the three dimensional (3D) picture of nucleon, some processes like SIDIS are required in which one can measure the effect of transverse momentum of partons in created hadron. It is therefore required to consider the spin dependence of PDFs. Early applications to polarized structure functions were made by [8–10].

The PDFs in polarized case are two types. The first one is related to the longitudinal polarized quark inside longitudinal polarized nucleon, denoted by $g_1(x)$ that is called helicity function. The second one is related to transverse polarized quark inside the transverse polarized nucleon, denoted by $h_1(x)$ and is called the transversity function. The type of polarization is determined with respect to moving direction of nucleon. If the parton transverse momentum as an extra degree of freedom is also considered then total number of PDFs, involving polarized cases, are arising to eight ones [11]. In this article the polarized TMDs which are even time reversal functions, based on covariant parton model, are investigated.

The organization of this paper is as following. In Sec.II an overview on theoretical aspects of polarized structure function is done. In Sec.III the theoretical framework of Jacobi polynomials approach is reviewed. Sec.IV is devoted to discuss the target mass correction for $g_1$ and $g_2$ structure functions. Additionally in Sec.V higher twist effect is demonstrated for polarized structure functions. In Sec.VI which includes also some subsections we illustrate our QCD data analysis which we call it as MA22 analysis. To get more validation of our MA22 results, we examine in Sec.VII several sum rules. In Sec.VIII our prediction for polarized PDFs and structure functions are presented. Using the results of our MA22 analysis, some polarized TMDs can be calculated. We do it in Sec.IX. In the last part that is Sec.X our conclusions is given.

II. LEADING TWIST SPIN DEPENDENCE OF STRUCTURE FUNCTION

To achieve the main goal of this article to calculate the polarized TMDs we first need to analysis DIS structure function in polarized case. For this purpose linear combination of polarized parton densities and coefficient functions can be used to express the leading twist spin-dependent proton and neutron structure functions, $g_1^p(x, Q^2)$ and $g_1^n(x, Q^2)$ at the next-next-to-leading-order (NNLO) accuracy as it follows [12–14]:

$$g_1^p(x, Q^2) = \frac{1}{2} \sum_q e_q^2 \Delta q_s(x, Q^2) \otimes$$

$$\left( 1 + \frac{\alpha_s(Q^2)}{2\pi} \Delta C_q^{(1)} + \left( \frac{\alpha_s(Q^2)}{2\pi} \right)^2 \Delta C_q^{(2)} \right)$$

$$+ e_q^2 (\Delta q_s + \Delta g_s)(x, Q^2) \otimes$$

$$\left( 1 + \frac{\alpha_s(Q^2)}{2\pi} \Delta C_g^{(1)} + \left( \frac{\alpha_s(Q^2)}{2\pi} \right)^2 \Delta C_g^{(2)} \right) \otimes \Delta g(x, Q^2)$$

\begin{equation}
(1)
\end{equation}

Here $\Delta q_s$, $\Delta q_s$ and $\Delta g$ are the polarized valance, sea and gluon densities, respectively. The pQCD evolution kernel for PDFs is now available at the NNLO accuracy in Ref. [15–17]. The $\Delta C_q^{(1)}$ and $\Delta C_q^{(2)}$ in Eq.(1) are denoting to the NLO spin-dependent quark and gluon hard scattering coefficients, calculable in pQCD [18]. We now apply the hard scattering coefficients, extracted at NNLO approximation. At this order the Wilson coefficients are different for quarks and antiquarks. They are presented in Eq.(1) by $\Delta C_g^{(1)}$ and $\Delta C_g^{(2)}$ respectively and their analytical relations have been reported in [19]. The symbol $\otimes$ in Eq.(1) is representing typical convolution in $x$-space.

The neutron structure function, $g_1^n(x, Q^2)$, can be obtained from the proton one by considering isospin symmetry. Hence the deuteron structure function at leading twist would be available, utilizing the $g_1^p$ and $g_1^n$ structure functions such as:

$$g_1^{\tau(1)}(x, Q^2) = \frac{1}{2} \left[ g_1^p(x, Q^2) + g_1^n(x, Q^2) \right] \times (1 - 1.5 w_D),$$

\begin{equation}
(2)
\end{equation}

where $w_D = 0.05 \pm 0.01$ is the probability to find the deuteron in a $D$-state [20–22]. Using the Wandzura and Wilczek (WW) relation [23] the leading twist polarized
structure function of \( g_2^{\tau_2}(x, Q^2) \) can be fully determined via \( g_1^2(x, Q^2) \) structure function:

\[
g_2^{\tau_2}(x, Q^2) = g_2^{WW}(x, Q^2) = -g_1^{\tau_2}(x, Q^2) + \int_x^1 \frac{dy}{y} g_1^{\tau_2}(y, Q^2). \tag{3}
\]

This relation that is in the leading twist approximation can also be used when target mass correction (TMC) is included \([23]\).

The \( g_1^{\tau_2}(x, Q^2) \) and \( g_2^{\tau_2}(x, Q^2) \) structure functions at the leading twist order have valid definition in the Bjorken limit, i.e. \( Q^2 \rightarrow \infty \), \( x = \) fixed. But at the a moderate low \( Q^2 \) (\( \sim 1 - 5 \) GeV\(^2\)) and \( W^2(4 \text{ GeV}^2 < W^2 < 10 \text{ GeV}^2) \) where \( W^2 \) is the invariant mass of the hadronic system, both TMC along with higher twist corrections should be considered. We investigate them in Sec.IV and Sec.V.

Next section is devoted to illustrate the nucleon and deuteron structure functions, based on Jacobi polynomial approach which yield us these functions in momentum space.

### III. JACOBI POLYNOMIALS EXPANSION TECHNIQUE

To achieve the nucleon structure function in momentum \( n \)-space we resort to a method that is based on the Jacobi polynomials expansion. Practical aspects of this method including its major advantages are presented in our previous studies \([12, 13, 24-29]\). According to this method, one can easily expand the polarized structure functions \( xg_1^{QCD}(x, Q^2) \), in terms of the Jacobi polynomials \( \Theta_n^{\alpha, \beta}(x) \), as it follows \([30-42]\):

\[
xg_1^{\tau_2}(x, Q^2) = x^\beta (1 - x)^\alpha \sum_{n=0}^{N_{\text{max}}} a_n(Q^2) \Theta_n^{\alpha, \beta}(x), \tag{4}
\]

where \( N_{\text{max}} \) is the maximum order of expansion. The parameters \( \alpha \) and \( \beta \) are Jacobi polynomials free parameters which normally fixed on their best values. These parameters have to be chosen so as to achieve the fastest convergence of the series on the right-hand side of Eq. (4). In the polynomial fitting procedure, the evolution equation is combined with the truncated series to perform a direct fit to the structure functions.

The Jacobi moments, \( a_n(Q^2) \) are codifying the \( Q^2 \)-dependence of the polarized structure functions. The \( x \)-dependence will be provided by the weight function \( w_n^{\alpha, \beta}(x) \equiv x^\beta (1-x)^\alpha \) and the Jacobi polynomials \( \Theta_n^{\alpha, \beta}(x) \) which can be written as,

\[
\Theta_n^{\alpha, \beta}(x) = \sum_{j=0}^{n} c_j^{(n)}(\alpha, \beta) x^j, \tag{5}
\]

where the coefficients \( c_j^{(n)}(\alpha, \beta) \) are combinations of Gamma functions in terms of \( n, \alpha \) and \( \beta \). The above Jacobi polynomials are satisfying the following orthonormality condition:

\[
\int_0^1 dx x^\beta (1-x)^\alpha \Theta_n^{\alpha, \beta}(x) \Theta_l^{\alpha, \beta}(x) = \delta_{n,l}. \tag{6}
\]

Consequently the Jacobi moments, \( a_n(Q^2) \), can be obtained, using the above relation such as,

\[
a_n(Q^2) = \int_0^1 dx xg_1^{\tau_2}(x, Q^2) \Theta_n^{\alpha, \beta}(x)
= \sum_{j=0}^{n} c_j^{(n)}(\alpha, \beta) M[xg_1^{\tau_2}, j + 2](Q^2), \tag{7}
\]

where the Mellin transform \( M[xg_1^{\tau_2}, N](Q^2) \) is given by,

\[
M[xg_1^{\tau_2}, N](Q^2) = \int_0^1 dx x^{N-2} xg_1^{\tau_2}(x, Q^2). \tag{8}
\]

Using the QCD expressions for the Mellin moments, \( M[xg_1^{\tau_2}, N](Q^2) \), the polarized structure function \( xg_1^{\tau_2}(x, Q^2) \), can be constructed. Therefore, based on the method of Jacobi polynomial expansion, the \( xg_1^{\tau_2}(x, Q^2) \) is given by:

\[
xg_1^{\tau_2}(x, Q^2) = x^\beta (1-x)^\alpha \sum_{n=0}^{N_{\text{max}}} \Theta_n^{\alpha, \beta}(x)
\times \sum_{j=0}^{n} c_j^{(n)}(\alpha, \beta) M[xg_1^{\tau_2}, j + 2](Q^2). \tag{9}
\]

By setting \( N_{\text{max}} = 9, \alpha = 3 \) and \( \beta = 0.5 \), as we have shown in our previous analyses \([12, 13, 24-29]\), it is possible to obtain the optimal convergence of above expansion through the whole kinematic region that is constrained by the polarized DIS data.

In next section we improve our analysis of DIS polarized data, considering the TMC correction to the nucleon structure functions.

### IV. TARGET MASS CORRECTIONS IN POLARIZED CASE

Power suppressed corrections to the structure functions can have important contributions in some kinematic regions. Hence nucleon mass correction cannot be neglected in low \( Q^2 \) region. The TMCs can be calculated via an expression which is different from higher twist (HT) effects in dynamical case. In the case of polarized structure function we follow the suggested method by Blumlein and Tkabladze \([43]\) which is in fact the generalized one that was established by Georgi and Politzer \([44]\) for the unpolarized structure function.

Mellin inversion to \( x \)-space or the integer moments of structure function can be used to present these correc-
tions. Leading twist-2 expression for $g_1$, that is containing TMC, is given explicitly by [43]:
\[
g_1^{\tau_2 + \text{TMCs}}(x, Q^2) = x g_1^{\tau_2}(\xi, Q^2; M = 0) + \frac{4 M^2 x^2}{Q^2} \frac{1}{\xi} \int_\xi^{1} d\xi' \frac{1}{\xi'} g_1^{\tau_2}(\xi', Q^2; M = 0)
\]
\[
\times \frac{Q^2}{2(1 + 4 M^2 x^2/Q^2)^{3/2}} \int_\xi^{1} d\xi'' \frac{1}{\xi''} g_1^{\tau_2}(\xi'', Q^2; M = 0) .
\]

The twist-2 contribution for the $g_2$ structure function, including TMC is similarly presented by [43]:
\[
g_2^{\tau_2 + \text{TMCs}}(x, Q^2) = x g_2^{\tau_2}(\xi, Q^2; M = 0) + \frac{4 M^2 x^2}{Q^2} \frac{1}{\xi} \int_\xi^{1} d\xi' \frac{1}{\xi'} g_2^{\tau_2}(\xi', Q^2; M = 0)
\]
\[
\times \frac{Q^2}{2(1 + 4 M^2 x^2/Q^2)^{3/2}} \int_\xi^{1} d\xi'' \frac{1}{\xi''} g_2^{\tau_2}(\xi'', Q^2; M = 0) .
\]

Numerical illustrations for the target mass effects in $g_1$ and $g_2$ have been given in [45]. In both above equations $M$ is the nucleon mass and $\xi$ is called Nachtmann variable that is defined by [46]:
\[
\xi = \frac{2x}{1 + \sqrt{1 + 4 M^2 x^2/Q^2}} .
\]

It can be seen that by choosing the maximum value for the $x$-Bjorken variable, the maximum kinematic value of $\xi$ variable would be less than unity. This means that the target mass corrected structure functions at leading twist in both the polarized and unpolarized cases, as it is expected, do not vanish at maximum $x = 1$ value.

As we referred before, in addition to target mass correction, higher twist effects would also be dominant at low $Q^2$ values and make contribution to nucleon structure function in related kinematic region. Next section is devoted to this effects.

V. TWIST-3 CONTRIBUTION

The long-range nonperturbative multiparton correlations which have outstanding contributions at low values of $Q^2$ will lead to higher twist (HT) terms. A proper analysis of this effect can be found in [47]. For a developing phenomenological analysis an advantageous parametrization is made by the BLMP model [48] for HT terms. Following that HT distributions are constructed from convolution integrals that are containing light-cone wave functions. In this connection a simple model based on three valence quark and one gluon distributions with the total zero angular momentum are assumed.

Accordingly, we utilize the parameterized form, suggested by the BLMP model at the twist-3 order for $g_2$ structure function in an initial scale $Q_0$ as it follows [48, 49]:
\[
g_2^{\tau_3}(x) = A[\ln(x) + (1 - x) + \frac{1}{2}(1 - x)^2]
\]
\[
+ (1 - x)^3[B - C(1 - x) + D(1 - x)^2 - E(1 - x)^3] .
\]

The unknown coefficients in Eq.(13) are extracted by fitting the data. Since higher twist contributions are important in a region with large-$x$ values, a nonsinglet evolution equation is employed. The results of this approach can be compared with exact evolution equations where a gluon-quark-antiquark correlation is considered [48]. It is expecting that these two results are in good agreement with each other.

The twist-3 part of different spin-dependent structure functions, $g_1^{\tau_3}$ and $g_2^{\tau_3}$, are related by the following integral relation [43],
\[
g_1^{\tau_3}(x, Q^2) = \frac{4x^2 M^2}{Q^2} [g_2^{\tau_3}(x, Q^2)
\]
\[
- \frac{1}{2} \int_x^{1} dy \frac{1}{y} g_2^{\tau_3}(y, Q^2)] .
\]

The $Q^2$-dependence of the $g_1^{\tau_3}$ can be achieved within nonsinglet perturbative QCD evolution as
\[
g_2^{\tau_3}(n, Q^2) = g_2^{\tau_3}(n, Q^2) g_2^{\tau_3}(n) .
\]

Finally the spin-dependent structure functions, considering the TMCs and HT terms are given by,
\[
x g_1, 2^{\text{Full=pQCD+TMC+HT}}(x, Q^2) = x g_1, 2^{\tau_2 + \text{TMCs}}(x, Q^2) + x g_1, 2^{\tau_3}(x, Q^2) .
\]

One of the particular feature of $x g_1, 2^{\text{Full}}(x, Q^2)$ function is that the twist-3 term is not suppressed there by inverse powers of $Q^2$. Consequently to describe this function, this contribution is so important as the twist-2 contribution.

Since the required theoretical inputs are accessed by us, we can do now the concerned data analysis which is done in next section

VI. FITTING CONTENTS IN QCD ANALYSIS

The fitting procedure, including the recent and updated data for polarized structure functions which we do in our QCD analysis, are containing the following parts.
A. Parametrization

We start the QCD analysis considering the following parametrization at the initial scale of \( Q_0^2 = 1 \) GeV\(^2 \) where \( q = \{ u_v, d_v, g \} \):

\[
x \Delta q(x, Q_0^2) = \mathcal{N}_q \eta_q x^{a_q} (1 - x)^{b_q} (1 + c_q x) .
\]

The normalization constant \( \mathcal{N}_q \),

\[
\mathcal{N}_q^{-1} = \left( 1 + c_q \frac{a_q}{a_q + b_q + 1} \right) B(a_q, b_q + 1) ,
\]

is determined such that \( \eta_q \) in Eq. (18) is the first moment of the polarized parton distribution functions (PPDFs). Here \( B(a, b) \) is the Euler beta function. Considering SU(3) flavor symmetry, we assume \( \Delta \eta = \Delta \pi = \Delta \bar{\pi} = \Delta s = \Delta \bar{s} \).

The unknown free parameters can be extracted through a fit which involves a large degree of flexibility. Some of parameters can be determined via the following constrains, as described in below:

- The weak matrix elements \( F \) and \( D \) as measured in quark and hyperon \( \beta \) decays \([50]\) can be related to the first moments of the polarized valence quark densities. Considering these constrains, the numerical values \( \eta_u = 0.928 \pm 0.014 \) and \( \eta_d = -0.342 \pm 0.018 \) are obtained.

- Due to the present accuracy of the data, the \( c_q \) and \( c_g \) parameters are setting to zero. Considering nonzero values for them, there would not be observed any improvement in the fit.

- The large-\( x \) behavior of the polarized sea quarks and gluons are controlled by \( b_q \) and \( b_g \) parameters. In a region that is dominated by the valence distributions, these parameters have large uncertainties.

- Due to higher twist effect to the \( g_{2, \{ p, n, d \}} \) and consequently \( g_{1, \{ p, n, d \}} \), there are unknown parameters \( \{ A, B, C, D, E \} \), see Eq. (13). By a simultaneous fit to the all polarized structure function data of \( g_1 \) and \( g_2 \), these parameters can be determined.

- The values of some parameters are frozen in the first minimization procedure. They involve \( \{ \eta_u, \eta_d, c_q, c_g \} \) and finally the \( b \) parameter. As demonstrated in Tables I and II the \( \{ b_q, b_g, c_u, c_d \} \) and \( \{ A, B, C, D, E \} \) parameters are then fixed in the second minimization. Nine unknown parameters, including \( \alpha_s(M_Z^2) \), are left which are determined in the fit. They have enough flexibility to perform a reliable fit.

- The numerical value \( \alpha_s(M_Z^2) = 0.112804 \pm 0.001907 \) would be achieved in which we need to change the energy scale to the Z boson mass. It is while for the present world average, the value \( \alpha_s(M_Z^2) = 0.1179 \pm 8.5 \times 10^{-6} \) is reported \([51]\).

To extract the unknown parameters, it needs to access to all available concerned data sets which we describe them in below.

![Figure 1: Our MA22 results for the polarized PDFs at \( Q_0^2 = 1 \) GeV\(^2 \) as a function of \( x \) in the NNLO approximation. It is indicated by a solid curve along with their \( \Delta \chi^2 = 1 \) uncertainty bands which is computed, based on the Hessian approach \([52]\). The recent results of TKAA16 (dashed-dotted) \([12]\) is also shown in NNLO approximation without inclusion of HT terms and TMCs. Additionally the KTA17(dashed) \([13]\) in NNLO approximation is presented that including the HT terms and TMCs. The KTAO11(dashed-dotted) in NLO approximation \([24]\) is furthermore indicated. Finally the results of NAAMY21(dashed-dashed-dotted) \([29]\) in NLO approximation is also plotted.

B. Overview of data sets

In our recent analysis which we call it MA22 we focus on the polarized DIS data samples.. The required DIS data for all PPDFs are coming from the experiments at electron-proton collider and also in fixed-target including proton, neutron and heavier targets such as deuteron.

Although it is not possible to separate quarks from antiquarks, nonetheless it is the inclusive DIS data that are included in the fit. Additionally we take into our MA22
fitted scenario

Table I: Final parameter values and their statistical errors at the input scale $Q^2_0 = 1\text{ GeV}^2$ determined from two different global analyses. Those marked with (*) are fixed.

| Parameters | Full scenario | pQCD scenario |
|------------|---------------|---------------|
| $g_{u,v}$  | $0.928^*$     | $0.928^*$     |
| $a_{u,v}$  | $0.898 \pm 0.202$ | $0.277 \pm 0.0072$ |
| $b_{u,v}$  | $3.218 \pm 0.035$ | $2.725 \pm 0.029$ |
| $c_{u,v}$  | $3.88^*$       | $28.95^*$     |
| $d_{u,v}$  | $-0.342^*$     | $-0.342^*$     |
| $a_{d,v}$  | $0.217 \pm 0.027$ | $0.150 \pm 0.012$ |
| $b_{d,v}$  | $2.947 \pm 1.45$ | $2.591 \pm 0.087$ |
| $c_{d,v}$  | $9.335^*$       | $31.75^*$     |
| $\delta_q$ | $-0.0288 \pm 0.0002$ | $-0.0356 \pm 0.0033$ |
| $a_q$      | $1.227 \pm 0.068$ | $1.991 \pm 0.041$ |
| $b_q$      | $3.364^*$       | $11.163^*$     |
| $c_q$      | $0.0^*$         | $0.0^*$       |
| $\delta_g$ | $0.0921 \pm 0.022$ | $0.178 \pm 0.014$ |
| $a_g$      | $10.2 \pm 1.22$ | $26.33 \pm 0.49$ |
| $b_g$      | $46.32^*$       | $99.95^*$     |
| $c_g$      | $0.0^*$         | $0.0^*$       |
| $\alpha_s(Q^2)$ | $0.3362 \pm 0.002$ | $0.4688 \pm 0.0008$ |
| $\chi^2/ndf$ | $1111.789/957 = 1.161$ | $1580.761/957 = 1.651$ |

Table II: Parameter values for the coefficients of the twist-3 corrections at $Q^2 = 1\text{ GeV}^2$ obtained in the full scenario.

| Parameters | A | B | C | D | E |
|------------|---|---|---|---|---|
| $g_{z,p}^{u,v}$ | 0.0879 | 1.0196 | -0.8832 | -2.3765 | 2.4234 |
| $g_{z,n}^{u,v}$ | 1.0086 | 0.3009 | -0.6583 | 0.3466 | -2.7571 |
| $g_{z,d}^{u,v}$ | 0.8878 | 1.3430 | -2.1334 | 0.1878 | 2.2293 |

fitting procedure the $g_2$ structure function. Due to the technical difficulty in operating the required transversely polarized target, these data have been traditionally neglected before.

The data which we use in our recent analysis are to date and including more data than we employed in our previous analysis [13]. In fact we use all available $g_1$ data from E143, HERMES98, SMC, EMC, E155, HERMES06, COMPASS10, COMPASS16, JLAB06 and JLAB17 experiments [53-62], and $g_1$ data from HERMES98, E142, E154, HERMES06, Jlab03, Jlab04 and Jlab05 [54, 63-68] and finally the $g_2$ data from E143, SMC, HERMES06, E155, COMPASS05, COMPASS06 and COMPASS17 [53, 55, 58, 69-72]. The DIS data for $g_2^{p,n,d}$ from E143, E141, Jlab03, Jlab04, Jlab05, E155, Hermes12 and SMC [53, 63, 66-68, 73-75] also are included. These data sets are summarized in Table III. The kinematic coverage, the number of data points for each given target, and the fitted normalization shifts $N_i$ also presented in this Table. Our MA22 analysis algorithm computes the $Q^2$ evolution and extracts the structure function in $x$ space using Jacobi polynomials approach. It is corresponding to the fitting programs on the market which solve the DGLAP evolution equations in the Mellin space.

One of the important quantity which is used as a criteria to indicate the validation of fit, is the chi-square ($\chi^2$) test which is assessing the goodness of fit between observed values and those expected theoretically. We discuss about it in the following subsection.

C. $\chi^2$ minimization

The $\chi^2_{\text{global}}(p)$ quantifies the goodness of fit to the data for a set of $p$ independent parameters. To determine the best fit, it is needed to minimize the $\chi^2_{\text{global}}$ function with the free unknown parameters. We do it for PDFs at the NNLO approximation which additionally includes the QCD cut off parameter, $\Lambda_{QCD}$ which finally specifies the polarized PDFs at $Q^2 = 1\text{ GeV}^2$.

This function is presented as it follows:

$$\chi^2_{\text{global}}(p) = \sum_{n=1}^{N_{\text{exp}}} w_n \chi^2_n.$$  \hspace{1cm} (19)$$

In this equation, $w_n$ is a weight factor for the $n^{th}$ experiment and $\chi^2_n$ is defined by:

$$\chi^2_n(p) = \left(1 - \frac{N_i}{\Delta N_i}\right)^2 \sum_{i=1}^{N_i^{\text{data}}} \left( \frac{N_i^{\text{Exp}} g_{(1,2),i}^{(1,2),i}(p) - g_{(1,2),i}^{\text{Theory}}}{\Delta N_i^{\text{Exp}} g_{(1,2),i}^{(1,2),i}(p)} \right)^2.$$  \hspace{1cm} (20)$$
To do a proper fit an over normalization factor for the data of experiment \( n \) is needed which is denoted by \( N_n \). An uncertainty \( \Delta N_n \) is attributed to this factor which should be considered in the fit. These factors, considering the uncertainties, quoted by the experiments are used to relate different experimental data sets. In fact they are taken as a free parameters which are determined simultaneously with the other parameters in the fit. They are obtained in the pre-fitting procedure and then fixed at their best values in further steps. Numerical results for the unknown parameters, resulted from \( \chi^2 \) minimization, are listed in Table.I and II. Different data sets which are used in the fit, is presented in Table.III.

Now we are at stage to do some analytical computations for a more confirmation of the fitting validation, taken into account the several sum rules as we do it in the next section.

VII. THE SUM RULES

Sum rules like total momentum fraction carried by partons or the total contribution of parton spin to the spin of the nucleon are important tools to investigate some fundamental properties of the nucleon structure. Inclusion of TMCs and HT terms into the NNLO polarized structure function analysis leads to an improvement for the precision of PPDF determination as well as QCD sum rules and we are exploring herein their effects. In what are following by utilizing available experimental data, we describe some important polarized sum rules.

A. Bjorken sum rule

The polarized Bjorken sum rule expresses the integral over the spin distributions of quarks inside the nucleon in terms of its axial charge, \( g_A \) (as measured in neutron \( \beta \) decay), times a coefficient function, \( C_{Bj}(\alpha_s(Q^2)) \) \[78\], and considering higher twist (HT) corrections, it is given by

\[
\Gamma^{\text{NS}}_1(Q^2) = \Gamma^{\text{HT}}_1(Q^2) - \Gamma^{\ast}_1(Q^2) = \int_0^1 [g^T_i(x,Q^2) - g^H_i(x,Q^2)] dx \\
= \frac{1}{6} |g_A| C_{Bj}[\alpha_s(Q^2)] + \text{HT corrections}.
\]

Bjorken sum rule potentially provides a very precise handle on the \( \alpha_s \) as strong coupling constant. The value of coupling can be extracted via \( C_{Bj}[\alpha_s(Q^2)] \) expression from experimental data. This function has been calculated in 4-loop pQCD corrections in the massless \[79\] and very recently massive cases \[80\]. As previously reported in Ref. \[81\], determination of \( \alpha_s \) from the Bjorken sum rule suffers from small-\( x \) extrapolation ambiguities.
Table III: Summary of published polarized DIS experimental data points with measured $x$ and $Q^2$ ranges and the number of data points.

| Experiment     | Ref. | $[x_{\text{min}}, x_{\text{max}}]$ | $Q^2$ (GeV$^2$) | Num. of data poi. | $\chi^2$ | $N_i$ |
|----------------|------|------------------------------------|-----------------|------------------|----------|-------|
| SLAC/E143(p)   | [53] | [0.031–0.749]                      | 1.07–9.52       | 28               | 19.0218  | 0.99705 |
| HERMES(p)      | [54] | [0.028–0.66]                       | 1.01–7.36       | 39               | 55.2816  | 0.99982 |
| SMC(p)         | [58] | [0.005–0.480]                      | 1.30–58.0       | 12               | 8.9328   | 1.00009 |
| EMC(p)         | [56] | [0.015–0.466]                      | 3.50–29.5       | 10               | 3.8416   | 1.00592 |
| SLAC/E155      | [57] | [0.015–0.750]                      | 1.22–34.72      | 24               | 41.7453  | 0.99915 |
| HERMES06(p)    | [55] | [0.026–0.731]                      | 1.12–14.29      | 51               | 21.0559  | 0.99915 |
| COMPASS10(p)   | [59] | [0.005–0.568]                      | 1.10–62.10      | 15               | 23.1003  | 1.00073 |
| COMPASS16(p)   | [60] | [0.0035–0.575]                     | 1.03–96.1       | 54               | 52.6140  | 0.99926 |
| SLAC/E143(p)   | [53] | [0.031–0.749]                      | 2.3–5           | 84               | 41.7453  | 0.99915 |
| HERMES(p)      | [54] | [0.023–0.66]                       | 2.5             | 20               | 35.2073  | 0.99726 |
| SMC(p)         | [58] | [0.003–0.4]                        | 10              | 12               | 14.8318  | 1.00071 |
| Jlab06(p)      | [61] | [0.3771–0.9086]                    | 3.48–4.96       | 70               | 99.6438  | 1.00127 |
| Jlab17(p)      | [62] | [0.37696–0.94585]                  | 3.01503–5.75676 | 82               | 171.5716 | 1.00282 |
| SLAC/E143(d)   | [53] | [0.031–0.749]                      | 1.27–9.52       | 28               | 38.3735  | 1.00210 |
| SLAC/E155(d)   | [69] | [0.015–0.750]                      | 1.22–34.79      | 24               | 20.0319  | 1.00228 |
| SMC(d)         | [58] | [0.005–0.479]                      | 1.30–54.80      | 12               | 18.3574  | 1.00006 |
| HERMES06(d)    | [55] | [0.026–0.731]                      | 1.12–14.29      | 51               | 44.4642  | 1.00654 |
| COMPASS05(d)   | [70] | [0.0051–0.4740]                    | 1.03–71.4       | 43               | 36.2019  | 1.01090 |
| COMPASS06(d)   | [71] | [0.0046–0.566]                     | 1.10–55.3       | 15               | 8.4408   | 1.00052 |
| COMPASS17(d)   | [72] | [0.0045–0.569]                     | 1.03–74.1       | 84               | 127.5502 | 0.99981 |
| SLAC/E143(d)   | [53] | [0.031–0.749]                      | 2.3–5           | 84               | 127.5502 | 0.99981 |
| SLAC/E142(n)   | [63] | [0.035–0.466]                      | 1.10–5.50       | 8                | 8.0235   | 0.99881 |
| HERMES(n)      | [54] | [0.033–0.464]                      | 1.22–5.25       | 9                | 2.7585   | 0.99995 |
| E154(n)        | [65] | [0.017–0.564]                      | 1.20–15.00      | 17               | 14.6888  | 0.99908 |
| HERMES06(n)    | [64] | [0.026–0.731]                      | 1.12–14.29      | 51               | 18.1873  | 0.99913 |
| Jlab03(n)      | [66] | [0.14–0.22]                        | 1.09–1.46       | 4                | 1.803e-2 | 0.99950 |
| Jlab04(n)      | [67] | [0.33–0.60]                        | 2.71–4.8        | 3                | 2.2174   | 1.05642 |
| Jlab05(n)      | [68] | [0.19–0.20]                        | 1.13–1.34       | 2                | 3.2639   | 0.98666 |
| E143(p)        | [53] | [0.038–0.595]                      | 1.49–8.85       | 12               | 7.1338   | 1.00074 |
| E155(p)        | [73] | [0.038–0.780]                      | 1.1–8.4         | 8                | 11.9908  | 0.99886 |
| E142(n)        | [63] | [0.036–0.466]                      | 1.19–5.5        | 8                | 18.5955  | 0.99999 |
| Jlab03(n)      | [66] | [0.14–0.22]                        | 1.09–1.46       | 4                | 0.9362   | 0.99337 |
| Jlab04(n)      | [67] | [0.33–0.60]                        | 2.71–4.8        | 3                | 3.9915   | 1.10299 |
| Jlab05(n)      | [68] | [0.19–0.20]                        | 1.13–1.34       | 2                | 15.5600  | 0.98896 |

The $g_1^f$ is also available from accurate methods to compute the width decay of $\tau$-lepton and the $Z$-boson into hadrons [82, 83]. An important test of QCD consistency can be offered by comparing these values.

Our results for the Bjorken sum rule can be compared with experimental measurements such as E143 [53], SMC [79], HERMES06 [55] and COMPASS16 [60]. The comparisons indicate an adequate consistency as we list them in Table IV.
Our results, MA22, at the NNLO approximation (solid curve) are compared with KT A17 at the same approximation (dashed) [13] and with NAAMY21 at the NLO approximation (dashed-dotted) [20].

**Figure 4:** The spin-dependent proton, neutron and deuteron structure functions, $xg_1$, as a function of $x$ and $Q^2$. Our results, MA22, at the NNLO approximation (solid curve) are compared with KT A17 at the same approximation (dashed) [13].

**Figure 5:** The spin-dependent proton structure function, $xg_1$, as a function of $x$ and $Q^2$. Our result, MA22, at the NNLO approximation (solid curve) is compared with KT A17 at the same approximation (dashed) [13] and with NAAMY21 at the NLO approximation (dashed-dotted) [20].

**Figure 6:** The twist-3 contribution to $xg_1^p$ at $Q^2=4$ GeV$^2$ as a function of $x$ is compared with the results of LSS [101] and KTA17 [13] at NNLO approximation and with JAM [100] at the NLO approximation.

Table IV: Comparison of our computed MA22 result for the Bjorken sum rule, $\Gamma_{1NS}^N$, with world data from E143 [53], SMC [75], HERMES06 [55] and COMPASS16 [60]. Only HERMES06 [55] results are not extrapolated in full $x$ range (measured in region $0.021 \leq x \leq 0.9$).

|       | E143 53 | SMC 75 | HERMES06 55 | COMPASS16 60 | KTA17 13 | MA22 |
|-------|--------|--------|-------------|-------------|--------|------|
| $\Gamma_{1NS}^N$ | $0.164 \pm 0.021$ | $0.181 \pm 0.035$ | $0.148 \pm 0.017$ | $0.181 \pm 0.008$ | $0.167 \pm 0.005$ | $0.171 \pm 0.001$ |
Figure 7: The twist-3 contribution to $x g_2$ at $Q^2 = 4$ GeV$^2$ as a function of $x$. Our result, MA22 (solid curve), is compared with KTA17 at the NNLO approximation [13] (dashed-dotted), JAM [100] (dashed) and BLMP [48] (dashed dashed dotted) at the NLO approximation. E143 experimental data [53] have also been added.

Figure 8: The twist-3 contribution of $x g_2$ for the proton, neutron, and deuteron as a function of $x$ and for different values of $Q^2$ according to our result, MA22, at the NNLO analysis.

Figure 9: The twist-3 contribution of $x g_1$ for the proton, neutron, and deuteron as a function of $x$ and for different values of $Q^2$ according to our results, MA22, at the NNLO analysis.

B. Proton helicity sum rule

This sum rule is related to the extrapolation of proton spin among its constituents that is completing our knowledge in the field of nuclear physics [84]. An accurate picture of the quark and gluon helicity density are obtained, considering proton’s momentum sum rule that needs a precise extraction of PDFs.

The spin of the nucleon are carried by its constituents that is generally represented by

$$\frac{1}{2} = \frac{1}{2} \Delta \Sigma(Q^2) + \Delta G(Q^2) + L(Q^2).$$

Here $\Delta \Sigma(Q^2) = \sum_q \int_0^1 dx (\Delta q(x, Q^2) + \Delta \bar{q}(x, Q^2))$ denotes spin contribution of the singlet flavour, $\Delta G(Q^2) = \int_0^1 dx \Delta g(x, Q^2)$ is interpreted as the gluon spin contribution and finally $L(Q^2)$ represents the total contribution from quark and gluon orbital angular momentum. Each individual term in Eq.(22) is a function of $Q^2$ but the sum is not. Finding a way to measure them is a real challenge. Describing the measurement methods is the beyond the scope of this paper.

In Table V the amount of first moment for the singlet-quark and gluon are listed at $Q^2=10$ GeV$^2$. Our results are compared to those from the NNPDFpol1.0 [85], NNPDFpol1.1 [86] and DSSV08 [87] at both truncated
and full \( x \) region.

In Table \( \text{VI} \) our results, MA22, are presented and compared with the results of DSSV08 [87], BB10 [47], LSS10 [88], NNPDFpol1.0 [85] and KTA17 [13] at \( Q^2=4 \) GeV\(^2\).

As can be seen from the Table \( \text{V} \) and Table \( \text{VI} \) for the \( \Delta \Sigma \), our MA22 results are consistent within uncertainties with that of other groups. It is back to this reason that the first moment of polarized densities are mainly fixed by semileptonic decays. Very different values are reported by various groups when we turn to the gluon. Considering their large uncertainty are avoiding us to reach a firm conclusion about the full first moment of gluon.

Based on the extracted values presented in Table \( \text{VI} \) we can finally discuss the proton spin sum rule. Hence the amount of quark and gluon orbital angular momentum to the spin of the proton would be:

\[
L(Q^2 = 4 \text{ GeV}^2) = 0.3591 \pm 0.0779. \tag{23}
\]

A definite conclusion about the contribution of the total orbital angular momentum to the spin of the proton can not be done because of the large uncertainty that is mainly originating from the gluons. To obtain a precise determination of each individual contribution, it is required to improve the current level of experimental accuracy.

C. The twist-3 reduced matrix element \( d_2 \)

One of the quantity which is not considered as a sum rule but its numerical evaluation is remarkable to investigate the higher twist effect is the twist-3 reduced matrix element and is denoted by \( d_2 \). Detailed of higher twist analyses for \( g_1 \) polarized structure function have been performed in [47]. In operator product expansion (OPE) theorem [89] the effect of quark-gluon correlations can be studied through the moments of \( g_1 \) and \( g_2 \) structure functions. These moments lead to definition of reduced matrix element, \( d_2(Q^2) \), as it follows

\[
d_2(Q^2) = 3 \int_0^1 x^2 g_2(x, Q^2) \, dx
= \int_0^1 x^2 \left[ 3 g_2(x, Q^2) + 2 g_1(x, Q^2) \right] \, dx. \tag{24}
\]

In this equation \( g_2 = g_2 - g_2^{\text{WW}} \) where \( g_2^{\text{WW}} \) is given by Wandzura and Wilczek (WW) relation as in Eq.\( (3) \). The \( d_2(Q^2) \) that is in fact the twist-3 reduced matrix element of spin dependent operators in nucleon, can be used to measure the deviation of \( g_2 \) from \( g_2^{\text{WW}} \). Due to the \( x^2 \) weighting factor in Eq.\( (24) \), this matrix element is specially sensitive to the large-\( x \) behaviour of \( g_2 \). Some insights into the size of the multi-parton correlation terms can be obtained by extracting the \( d_2 \) which indicates its important.

The significance of higher twist terms in QCD analyses is revealed by having non-zero value for \( d_2 \). To achieve precise information on the higher twist operators and to improve model prediction, a much more accurate experimental measurement for \( d_2 \) is required. In Table \( \text{VII} \) we present our results for \( d_2 \) which are compared with the other theoretical predictions and also experimental values.

D. Burkhardt-Cottingham (BC) sum rule

Considering dispersion relations for virtual Compton scattering in all \( Q^2 \), Burkhardt and Cottingham predicted that the zeroth moment of \( g_2 \) goes to zero [90] such as:

\[
\Gamma_2 = \int_0^1 dx g_2(x, Q^2) = 0. \tag{25}
\]

This relation is called Burkhardt-Cottingham (BC) sum rule and is trivial consequence of the WW relation for \( g_2^0 \) (see Eq.\( (3) \)). It should be noted that zeroth moment of structure function does not exist in the light cone expansion and hence can not be described by local operator product expansion [91]. Even if the target mass corrected structure function is used, this sum rule is still established [43]. Consequently any violation of the BC sum rule is an evidence for the presence of HT contributions [74].

Our MA22 results for \( \Gamma_2 \) together with data from E143 [53], E155 [73], HERMES2012 [74], RSS [92], E01012 [93] groups for proton, deuteron and neutron are listed in table \( \text{VIII} \). The low-\( x \) behaviour of \( g_2 \) which is not yet precisely measured, has considerable effect on any conclusion which we might be get.

The BC sum rule can be obtained analytically from the covariant parton model as it is discussed in [94].

E. Efremov-Leader-Teryaev (ELT) Sum Rule

Considering the valence part of \( g_1 \) and \( g_2 \) structure functions and integrating them over \( x \) variable the Efremov-Leader-Teryaev (ELT) sum rule is obtained. The ELT sum rule is derived like the Bjorken sum rule and is trivial consequence of the WW relation for \( g_2^0 \) (see Eq.\( (3) \)). The \( \Gamma_1 \) is required. In Table \( \text{VII} \) we present our results for \( \Gamma_1 \) which is
defined [90, 91] numerically evaluated is remarkable to investigate the higher twist effect is the twist-3 reduced matrix element and is denoted by \( d_2 \). Detailed of higher twist analyses for \( g_1 \) polarized structure function have been performed in [47]. In operator product expansion (OPE) theorem [89] the effect of quark-gluon correlations can be studied through the moments of \( g_1 \) and \( g_2 \) structure functions. These moments lead to definition of reduced matrix element, \( d_2(Q^2) \), as it follows:

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Our MA22 results for \( \Gamma_2 \) together with data from E143 [53], E155 [73], HERMES2012 [74], RSS [92], E01012 [93] groups for proton, deuteron and neutron are listed in table \( \text{VIII} \). The low-\( x \) behaviour of \( g_2 \) which is not yet precisely measured, has considerable effect on any conclusion which we might be get.

The BC sum rule can be obtained analytically from the covariant parton model as it is discussed in [94].

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The BC sum rule can be obtained analytically from the covariant parton model as it is discussed in [94].
Table V: Results for the full and truncated first moments of the polarized singlet-quark 
\[ \Delta \Sigma(Q^2) = \sum_x \int_0^1 dx [\Delta q(x) + \Delta \bar{q}(x)] \] and gluon distributions at the scale \( Q^2 = 10 \text{ GeV}^2 \) in the MS-scheme. Also shown are 
the recent polarized global analysis of NNPDFpol1.0 [85], NNPDFpol1.1 [86] and DSSV08 [87].

| Full \( x \) region \([0,1]\) | DSSV08 [87] | NNPDFpol1.0 [85] | NNPDFpol1.1 [86] | KTA17 [13] | MA22 |
|-----------------------------|-------------|-----------------|-----------------|-------------|-------|
| \( \Delta \Sigma(Q^2) \)    | 0.242       | +0.16 ± 0.30    | +0.18 ± 0.21    | 0.2587 ± 0.044 | 0.2445 ± 0.0048 |
| \( \Delta G(Q^2) \)        | −0.084      | −0.95 ± 3.87    | 0.03 ± 3.24     | 0.2104 ± 0.034 | 0.1205 ± 0.03 |

Table VI: Same as Table V, but only for the full first moments of the polarized singlet-quark and gluon distributions at the scale \( Q_0^2 = 4 \text{ GeV}^2 \) in the MS-scheme. Those of DSSV08 [87], BB10 [47], LSS10 [88] and NNPDFpol1.0 [85] are presented for comparison.

| DSSV08 [87] | BB10 [47] | LSS10 [88] | NNPDFpol1.0 [85] | KTA17 [13] | MA22 |
|-------------|-----------|-----------|-----------------|-------------|-------|
| \( \Delta \Sigma(Q^2) \) | 0.245 | 0.193 ± 0.075 | 0.207 ± 0.034 | 0.18 ± 0.20 | 0.1774 ± 0.029 | 0.2607 ± 0.0065 |
| \( \Delta G(Q^2) \) | −0.096 | 0.462 ± 0.430 | 0.316 ± 0.190 | −0.9 ± 4.2 | 0.1882 ± 0.0294 | 0.1905 ± 0.027 |

By combining the data of E143 [53] and E155 [73] the numerical value for this sum rule at \( Q^2 = 5 \text{ GeV}^2 \) is −0.011 ± 0.008 and what we obtain at the same energy scale would be 0.01017 ± 0.000044.
VIII. COMPARISON FOR THE SPIN STRUCTURE FUNCTIONS

Since our QCD analysis has been validated by extracting the PDFs via the fitting processes and also obtaining their evolved outputs and in continuation by considering several sum rules, we are now at the position to investigate the polarized structure functions. In this regard, we first back to what we got before. Our results, MA22 PDFs, as a function of $x$ at $Q_0^2 = 1$ GeV$^2$ along with the corresponding uncertainty bounds, is presented in Fig. 1.

The evolution of MA22 polarized parton distributions for a selection of $Q^2$ values indicates in Fig. 2 while for comparison various parameterizations of KTA17 [13], KATO011 [24], TKAA16 [12], NAAMY21 [29] at the NLO approximation are illustrated there. It is seen that by increasing $Q^2$, except for the gluon density, the evolution of all distributions tends to flatten out the peak.

Now for the structure functions, we see that in different panels of Fig. 3, our MA22 predictions for the polarized structure functions of the proton $x g_1^p(x,Q^2)$, neutron $x g_1^n(x,Q^2)$ and deuteron $x g_1^d(x,Q^2)$ are compared with respect to the fixed-target DIS experimental data from E143. As we mentioned, MA22 refers to 'pQCD+TMC+HT' scenario. The results from KATO011 analysis in NLO approximation [24], TKAA16 analysis in NNLO approximation [12], KTA17 analysis in NNLO approximation [13], THK14 analysis in NLO approximation [76] and finally NAAMY21 analysis in NLO approximation [29] are also depicted there. We find our results are in good agreement with the experimental data and in accord with other determinations over the entire range of $x$ at $Q^2=5$ GeV$^2$.

Further illustrations of the fit quality are presented in different panels of Fig. 4, for the $x g_2^{p,n,d}(x,Q^2)$ polarized structure functions, obtained from Eq. (16). In comparison with the $g_1$ data, the $g_2$ data have generally larger uncertainties which indicates the lack of knowledge for the $g_2$ structure function. At the current level of accuracy, MA22 is in agreement with data within their uncertainties. We need to a large number of data with higher precision to get a precise quantitative extraction of the $x g_2(x,Q^2)$. In fact we concentrate on the general characteristic of the $x g_2(x,Q^2)$ structure function.

Fig. 5 is presenting our MA22 prediction for the polarized structure functions of the proton $x g_1^p(x,Q^2)$ while a comparison with the fixed-target DIS experimental data from JLAB17 [62] is done there.

Fig. 6 represents our $x g_2^{d}(x,Q^2)$ with the results from LSS [101] and JAM [100] groups. Analysis of the LSS group is based on splitting the measured $x$ region into seven bins to determine the HT correction to $g_1$. The HT contribution has been extracted by LSS group in a model-independent way while its scale dependence is ignored. On the other side an analytical form for the twist-3 part of $g_2$ is parameterized by the JAM group where using integral relation of Eq. (14) they calculated $q_1^3$ at the NLO accuracy in a global fit.

E143 collaboration at SLAC reported the twist-3 contribution to proton spin structure function $x g_2^3$ structure function with relatively large errors [53]. We employ them and present our MA22 results for twist-3 part of $g_2$ in Fig. 7 which are accompanied with those of JAM [100] and BLMP [48] groups.

However, within experimental precision the $g_2$ data are well described by the twist-2 contribution but the precision of the current data is not sufficient enough to distinguish model precision. Hence we compute twist-3 part of $g_2$ for different targets and depict them in Fig. 8 which has significant contribution even at large $Q^2$ values.

In continuation to have a comparison, we compute the $x g_2^3$ and indicate them in Fig. 9. We find that these functions vanish rapidly at $Q^2 > 5$ GeV$^2$ where in the limit of $Q^2 \to \infty$, the $x g_2^3$ remains nonzero.

Up here we focused on longitudinal polarized parton densities and structure functions. In next section we utilize our MA22 analysis which we have done before to illustrate the transversal case which are including the polarized TMDs.

![Figure 10: The TMD $q_1^f(x, p_T)$ for u- (upper panel) and d-quarks (lower panel). Left panel: $q_1^f(x, p_T)$ as function of $x$ for $p_T/M = 0.10$ (dashed), 0.13 (dotted), 0.20 (dash-dotted line). Right panel: $q_1^f(x, p_T)$ as function of $p_T/M$ for $x = 0.15$ (solid), 0.18 (dashed), 0.22 (dotted), 0.30 (dash-dotted line).](image-url)
IX. PREDICTIONS FOR POLARIZED TMDS

Since we achieved to sufficient information on longitudinal polarized parton distributions and structure function, we are now at a situation to utilize the covariant parton model [4, 102] and extract the transverse momentum dependent (TMD) distributions in polarized case. Indeed TMDs provide us new insight toward a more complete understanding of the quark-gluon structure in a nucleon [103–109]. Without a more accurate and realistic picture in three dimensions of the nucleon which includes naturally transverse motion, it would be hard to explain some experimental observations. In fact TMDs provide such pictures and their necessities feel more and more in nucleon investigations.

The first and simplest example of quark TMD is \( f_1^q(x, k_T) \). It arises when an unpolarized beam scatters off an unpolarized target hadron, and therefore does not carry quark/hadron spin information. The function \( f_1^q(x, k_T) \) provides the probability that a beam particle strikes a target quark of momentum fraction \( x \) and transverse momentum \( k_T \). It is related to the traditional DIS PDF \( f_1^q(x) \) by \( \int d^2k_T \ f_1^q(x, k_T) = f_1^q(x) \).

Similarly to \( f_1^q(x, k_T) \), we get the \( g_1^q(x, k_T) \) as longitudinal polarized TMD and \( h_1^q(x, k_T) \) as transverse polarized TMD, whose integrals are denoted respectively by \( g_1^q(x) \) (presented before by \( \Delta q_i(x) \)) and \( h_1^q(x) \) that we know them as quark longitudinal polarized (helicity) distribution and the quark transversity distribution.

In addition to the three above TMDs for quarks which are direct extension of the DIS PDFs, there are five other quark TMDs which depend not only on the magnitude of \( k_T \), but also on its direction. Therefore these TMDs vanish if simply integrated over \( k_T \), and do not directly connect to DIS PDFs. They are:

1. The Sivers distribution \( f_{1T}^{u,q}(x, k_T) \) which expresses, in a transversely polarized hadron, the asymmetric distribution of the quark transverse momentum, \( p_z \), around the center of the \( p_x \) and \( p_y \) plane [110]. The appearance of azimuthal asymmetric quark distribution in the transverse momentum space is often called the “Sivers effect”. This TMD has opposite signs in semi-inclusive DIS (SIDIS) with respect to Drell-Yan processes and it is therefore an odd time reversal function (T-odd function).

2. The Boer-Mulders function \( h_{1T}^{u,q}(x, k_T) \) characterizes the distribution of longitudinal polarized quarks in an unpolarized hadron [111]. It is also a T-odd function, like \( f_{1T}^{u,q} \). The rest tree TMDs are:

3-Function \( h_{1T}^{u,q}(x, k_T) \) which is describing a transverse polarized quark inside a transverse polarized nucleon while its direction is perpendicular to a polarized nucleon. It is called Pretzelosity function.

4-Function \( g_{1T}^{u,q}(x, k_T) \) that is describing the longitudinal polarized quark inside a transverse polarized nucleon and is named as Worm-gear-I function. And finally:
5- Worm-gear-II function, denoted by $h_{1L}^{-q}(x, k_T)$ and is describing the transverse polarized quark inside a longitudinal polarized nucleon.

Similarly to quark TMDs, gluon TMDs allow access to the gluonic orbital angular momentum, another possibly important contribution to the nucleon spin. Just as there are eight TMDs for quarks, there are eight gluon TMDs [112]. Gluon TMDs were first proposed in 2001 [113].

Here we only consider the Quark TMDs that are twist-2 naively and time-reversal even (T-even) functions. They have been extracted via covariant parton model (CPM) which is based on the Lorentz invariance and the assumption of a rotationally symmetric distribution of parton momenta in the nucleon rest frame [114].

As a result of CPM, T-even polarized TMDs can be obtained at the leading twist approximation, in terms of a single “generating function” $K^q(x, p_T)$. They are given by [115, 116]

$$
g_1^q(x, p_T) = \frac{1}{2x} \left( x + \frac{m}{M} \right)^2 - \frac{p_T^2}{M^2} \times K^q(x, p_T),
$$

$$
h_1^q(x, p_T) = \frac{1}{2x} \left( x + \frac{m}{M} \right)^2 \times K^q(x, p_T),
$$

$$
g_{1T}^{-q}(x, p_T) = \frac{1}{x} \left( x + \frac{m}{M} \right) \times K^q(x, p_T),
$$

$$
h_{1T}^{-q}(x, p_T) = -\frac{1}{x} \times K^q(x, p_T).$$

(27)

According to [114] $K^q(x, p_T)$ as generating function is defined in compact notation by

$$
K^q(x, p_T) = M^2 x \int d\{p^1\} 
$$

$$
d\{p^1\} \equiv \frac{dp^1}{p^0} \frac{H^q(p^0)}{p^0 + m} \delta \left( \frac{p^0 + p^1}{M} - x \right).$$

(29)

It can be shown that due to rotational symmetry the following relations hold [115]:

$$
K^q(x, p_T) = M^2 \frac{H^q(p^0)}{p^0 + m}, \quad p^0 = \frac{1}{2} x M \left( 1 + \frac{p_T^2 + m^2}{x^2 M^2} \right),
$$

$$
\pi x^2 M^2 H^q \left( \frac{M}{2x} \right) = 2 \int_x^1 \frac{dy}{y} g_1^q(y) + 3 g_1^q(x) - x \frac{dg_1^q(x)}{dx},
$$

(31)

In deriving Eq.(31) the limit $m \rightarrow 0$ has been taken. Consequently the following result in that limit would be obtained for the generating function [115]:

$$
K^q(x, p_T) = M^2 \frac{H^q(p^0)}{p^0 + m}, \quad p^0 = \frac{1}{2} x M \left( 1 + \frac{p_T^2 + m^2}{x^2 M^2} \right),
$$

(32)

Substituting the above relations in Eq.(27), the following result for the $g_1^q(x, p_T)$ would be obtained:

$$
g_1^q(x, p_T) = \frac{2 x - \xi}{\pi \xi^3 M^3} \left[ \int_{y=1}^1 \frac{dy}{y} g_1^q(y) + 3 g_1^q(\xi) - \xi \frac{dg_1^q(\xi)}{d\xi} \right].
$$

(33)

Based on above relation and using the MA22 analysis which we did in this paper for $g_1^q(x)$ at 4 GeV$^2$ in the NNLO approximation, we could obtain the result for $g_1^q(x, p_T)$ which has been shown in Fig.[10] for $u$ and $d$ quarks.

Using Eq.(27) and in the limit $m \rightarrow 0$ the other TMDs can be obtained. They are presented in below which which are different by simple $x$-dependent prefac-
\[ h_1^q(x, p_T) = \frac{x}{2} K_1^q(x, p_T), \]
\[ g_1^{L,q}(x, p_T) = K_1^q(x, p_T), \]
\[ h_1^{L,q}(x, p_T) = -\frac{1}{x} K_1^q(x, p_T). \] (34)

The result for \( h_1^q(x, p_T) \) is depicted in Fig. 11. In Fig. 12 the result for \( g_1^{L,q} \) with respect to \( x \) and \( p_T/M \) is shown. It does not need to plot \( h_1^{L,q} \) since in the used approach this TMD is equal to \(-g_1^{L,q} \) [114]. As can be seen from Fig. 10, \( g_1^q(x, p_T) \) is the only TMD which has positive and negative values. The other TMDs in other figures do not change sign which follows from Eqs. (27, 34).

We should note that among all TMDs, as we see from Fig. 13, \( h_1^{L,q}(x, p_T) \) as pretzelosity function has largest absolute value which is due to the prefactor \( 1/x \). This function has its own worth since in some quark models [117, 118], including the utilized approach in [119, 120], this function is related to quark orbital angular momentum.

\section{X. Conclusions}

Determining the nucleon spin structure functions \( g_1(x, Q^2) \) and \( q_2(x, Q^2) \) and their moments is the main goal of our present MA22 analysis. They are essential to test QCD sum rules and to evaluate the TMDs. We provided a unified and consistent PPDF through an achievement, containing an excellent description of the fitted data while we employed TMC and HT effects in our analysis. Within the known very large uncertainties arising from the lack of constraining data, our helicity distributions are in good consistency with other extractions. Here the TMCs and HT effects, which are relevant in the region of low \( Q^2 \), have also been studied for the several sum rules at the NNLO approximation. Our results for the reduced matrix element \( d_2 \) at the NNLO approximation have also been presented. We also studied Burkhardt-Cottinghan and Efremov-Leader-Teryaev sum rules. To scrutinize them more accurate data are needed.

Finally we studied the behavior of the TMD structure functions which are time-reversal even with respect to \( x \) and \( p_T/M \) variables at the NNLO approximation, based on the covariant parton model. Our MA22 results, containing analysis of up to date and last data on nucleon spin structure functions, with respect to what we did in [13], can be compared with the results from [115] which indicated adequate and acceptable behaviours.

This study can be extended to include other TMDs while higher twist effect is employed. We hope to report on this issue as our further research task.

\section*{Acknowledgments}

The Authors are indebted P. Zavada for reading the manuscript and providing useful comments. We are grateful to O. V. Teryaev for his useful comments and suggestion. We are appreciated P. Schweitzer to read our manuscript and give us his opinion about it. S. A. T. is thankful from the School of Particles and Accelerators, Institute for Research in Fundamental Sciences (IPM) to make the required facilities to do this project. A. M acknowledges the Yazd university for the provided facility to do this project.

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