STABILITY AND DYNAMICS OF A WEAK VISCOELASTIC SYSTEM WITH MEMORY AND NONLINEAR TIME-VARYING DELAY

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Abstract. This paper is concerned with the stability and dynamics of a weak viscoelastic system with nonlinear time-varying delay. By imposing appropriate assumptions on the memory and sub-linear delay operator, we prove the global well-posedness and stability which generates a gradient system. The gradient system possesses finite fractal dimensional global and exponential attractors with unstable manifold structure. Moreover, the effect and balance between damping and time-varying delay are also presented.

1. Introduction. The propagation mechanism for the free vibration of a membrane can be described by the wave equation.

It is well-known that the memory term generates stabilizing mechanism in viscoelastic wave systems (see [1, 3, 9, 24])

\[ u_{tt} - \Delta u(t, x) + \int_0^t g(t - s)\Delta u(s, x)ds = 0 \quad (1) \]

There is also rich literature studying the stabilization of viscoelastic systems by some main factors: Damping and time delay (See [2], [14], [33], [34]). For instance, [12] investigates the viscoelastic system when the damping occurs

\[ u_{tt} - \Delta u(x, t) + \int_0^t g(t - \tau)\Delta u(\tau, x)d\tau + u_t(t, x) = 0, \quad (2) \]

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[3] studies the viscoelastic model with damping and delay as

\[ u_{tt} - \Delta u(t, x) + \int_0^{+\infty} \mu(s) \Delta u(t - s, x) ds + ku_t(t, x) = 0. \quad (3) \]

This paper concerns with the stability and dynamics for the following viscoelastic wave system with memory and time varying delay (sub-linear operator) in a bounded domain \( x \in \Omega \subseteq \mathbb{R}^n \) (\( n \geq 1 \)) and \( t \in \mathbb{R}^+ \):

\[
\begin{aligned}
u_{tt} - \Delta u(t, x) + \int_0^t g(t - s) \Delta u(s, x) ds + \mu_1 u_t(t, x) \\
+ \mu_2 u_t(t - \tau(t), x) + h(x),
\end{aligned}
\]

where \( \mu_1 \) and \( \mu_2 \) denote positive parameters, \( h \) be the source term, \( u_0, u_1 \) and \( f_0 \) represent the general initial and delay conditions respectively. The convolution term \( \int_0^t g(t - s) \Delta u(s, x) ds \) reflects the memory effects of materials due to viscoelasticity and the function \( \tau(t) : \mathbb{R}^+ \to \mathbb{R}^+ \) denotes the time-varying delay.

For the motivation of this work, we review some previous literature regarding the well-posedness and stability for the viscoelastic wave equations.

For the well-posedness and stability for viscoelastic systems with memory term: Messaoudi [23] investigates the general stability by using of the perturbed energy method for viscoelastic models without damping

\[ u_{tt} - \Delta u(t, x) + \alpha(t) \int_0^t g(t - \tau) \Delta u(\tau, x) d\tau = 0. \quad (5) \]

Appleby [4] proves the exponential decay of strong solution in a Hilbert space for a more generalized system with nonlinear operator \( A \)

\[ u_{tt} + Au(t, x) + \int_{-\infty}^t K(t - s) Au(s, x) ds = 0. \quad (6) \]

Gvesmia and Messaoudi [16] study more general model in the presence of past and infinite history memories

\[
u_{tt} - \Delta u(t, x) + \int_0^t g_1(t - s) \text{div}(a_1(x) \nabla u(s, x)) ds \\
+ \int_0^{+\infty} g_2(s) \text{div}(a_2(x) \nabla u(t - s, x)) ds = 0 \quad (7)
\]

and obtain general decay, exponential and polynomial decay rates for only special cases.

Moreover, for the viscoelastic systems with memory and damping, Cavalcanti et al [5] proves the solution for viscoelastic system with frictional damping

\[ u_{tt} - \Delta u(t, x) + \int_0^t g(t - s) \Delta u(s, x) ds + a(x) u_t + f(x, t, u) = 0 \quad (8) \]

satisfies exponential decay rate for certain assumptions on \( f(x, t, u) \) (such as \( f(x, t, u) = |u|^\gamma u \)), which improves the results of (2) in Fabrizio and Polidoro [12] with \( a(x) = 1 \) and \( f(x, t, u) = 0 \) in (8). Said-Houari [32] improves the model (8) with space-time dependent damping term as

\[ u_{tt} - \Delta u(t, x) + \int_0^t g(t - s) \Delta u(s, x) ds + b(t, x) u_t(t, x) + |u|^{p-1} u = 0 \quad (9) \]
and derives the $L^2$-decay rate of the solutions. Pata [29] studies the exponential stability for more generalized case

$$u_{tt} + \alpha Au(t, x) + \beta u_t(t, x) - \int_0^\infty \mu(s)Au(t-s, x)ds = 0. \quad (10)$$

Guo et al [17] studies the energy decay of a viscoelastic system with supcritical nonlinearities

$$u_{tt} - K(0)\Delta u(t, x) - \int_0^\infty K'(s)\Delta u(t-s, x)ds + |u_t|^{m-1}u_t = |u|^{p-1}u. \quad (11)$$

A very general viscoelastic model

$$u_{tt} - \Delta u(t, x) + \int_0^t g_1(t-s)\text{div}(a_1(x)\nabla u(s, x))ds + \int_0^{+\infty} g_2(s)\text{div}(a_2(x)\nabla u(t-s, x))ds = 0. \quad (12)$$

is also considered by [16].

Conti and Pata [8] proves the existence of regular global attractor for a viscoelastic system

$$u_{tt} - K(0)\Delta u(t, x) - \int_0^{+\infty} K'(s)\Delta u(t-s, x)ds + \alpha u_t + g(u_t) = f(t, x), \quad (13)$$

the dynamic system of its special case ($\alpha = 0$) is studied by [15].

On the other hand, the delay term is a source that may destabilize the asymptotic stability of solutions for an evolutionary system. This result is well justified in both mathematical analysis and physics examples, such as non-instant transmission phenomena as well as many biological models.

There is rich literature studying the effect of delay terms in the stabilization of viscoelastic systems, for instance [11, 25]. The reference [3] proves the exponential decay for the viscoelastic system (3) with continuous constant delay, its extended model

$$u_{tt} + Au(t, x) - \int_0^{+\infty} \mu(s)\Delta u(t-s, x)ds + b(t)u_t(t-\tau, x) = f(t, x), \quad (14)$$

is studied in [30]. For more generalized viscoelastic system with delay

$$u_{tt} - \phi_1(x)\Delta u(t, x) + \phi_2(t, x)\int_0^t g(t-s)\Delta u(s, x)ds + \mu_1 u_t(t, x) + \mu_2 u_t(t-\tau(t), x) = f(t, x), \quad (15)$$

for some special cases:

1. $\phi_1(x) = 1, \phi_2(t, x) = \beta(t)$, Liu [19], Liu and Diao [20] proves the energy decay of this weak viscoelastic equation with linear delay, that is $\tau(t)$ is constant.
2. $\phi_1(x) = \phi_2(t, x) = \phi(x)$, the delay term $\tau(t)$ is constant. Feng [13] obtained the general decay by perturbation method.

However, for this generalized model (15), the results with respect to the nonlinear delay are limits. This is the objective for our current and future work. Physically, nonlinear delays are more prevalent than linear delays in engineering systems.

Nevertheless, there are fruitful results for the well-posedness and asymptotic stability (such as general decay) of viscoelastic systems under both linear and nonlinear delay terms, see [18, 22, 25, 26, 34] and references therein. There are also some interesting topics not being well covered:
The viscoelastic model with nonlinear varying delay needs more delicate estimates for the study of asymptotic stability. A lot of literature used the semigroup method to obtain the well-posedness, while here we use Galerkin’s approximation method as we can achieve more priori estimates that are needed in both well-posedness and stability.

The existence and structure of attractors for the dynamic system. In general, the exponential attractor is bigger than global, hence the study of exponential attractor, the structure and finite fractal dimension of global attractors become more significant topics.

The balance between among the delay, memory and damping terms, which is a key condition for establishing exponential stability.

The main results and features of this present paper are summarized as follows:

(i) In other work [27], [28], [19], and [20], for the nonlinear time-varying delay \( \tau(t) \), the speed of the delay is assumed to satisfy \( \tau'(t) \leq d < 1 \), with the delay \( 0 < \tau(t) \leq \bar{\tau} \) being bounded from both above and below. The significant difference between our treatment and their work allows a more relaxed condition for the delay: \( \tau(t) > 0 \) without an upper bound, with the speed of delay satisfies \( 0 < \tau'(t) < 1 \). As a result, the parameters for the damping and the delay \( \mu_1 \) and \( \mu_2 \) enjoy a more relaxed condition \( \mu_2 < \mu_1 \) rather than \( \mu_2 \leq \sqrt{1-d} \cdot \mu_1 \) as in [27, 28, 19, 20].

(ii) From the new transformation and definition of phase space that deals with the delay term, we use Galerkin approximation method to establish the existence of global unique weak solution for viscoelastic model with the nonlinear delay (4). See Theorem 2.1. Using the perturbed energy method, see [21], we construct the exponential decay rate of energy functional. See Theorem 2.3. Our results have improved the works by Qin, Ren and Wei [31], Dai and Yang [10].

(iii) Under the basis of global well-posedness, the global weak solution generates a gradient system, which satisfies quasi-stability, and hence the finite fractal dimensional global and generalized exponential attractors have been derived. The structure of global attractor has been described by unstable manifold of equilibrium (stationary points). See Theorem 2.5. The dynamic behavior extends the results in Liu [19], Liu and Diao [20]. Moreover, our results have improved the works by Qin, Ren and Wei [31], Dai and Yang [10], especially the existence of exponential attractors, the finite fractal dimension and structure of global attractor for problem (4).

The rest part of this paper is organized as follows. In Section 2, the main results and its preparation has been stated. The proof of main results has been presented in Section 3. In Section 4, the further research and outlook are also shown.

2. Main results.

2.1. Assumptions. To proceed with the analysis, we make the following assumption on the delay term:

(H1) The varying delay \( \tau \) is a \( C^1 \) function satisfying:
\[
\tau(t) > 0, \quad 0 < \tau'(t) < 1, \quad \text{for} \quad t \geq 0
\]

We also make the following assumptions on the memory kernel \( g(t) \):

(H2) The memory kernel \( g(t) : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) is a \( C^1 \) function satisfying
\[
g(0) > 0, \quad 1 - \int_0^\infty g(s) \, ds = l > 0
\]
(H3) There exists a positive constant $\zeta_0$ such that
\[ g'(t) \leq -\zeta_0 g(t) \quad \text{for} \; t \geq 0. \]

(H4) Moreover, we assume that
\[ \mu_1 > \mu_2. \]

**Remark 1.** The parameters $\mu_1$ and $\mu_2$ are critical for the stability to system (19). It is shown in [25] that the dynamic system
\[
u_{tt} - \Delta u + \mu_1 u_t(x, t) + \mu_2 u_t(x, t - \tau) = 0 \tag{16}\]
is exponentially stable only when $\mu_1 > \mu_2$. When $\mu_1 \leq \mu_2$, instability occurs. In [18], the addition of memory term helps to save the case $\mu_1 = \mu_2$ for exponential stability while for $\mu_1 < \mu_2$ the system is still unstable. In our scenario, the unbounded time-varying delay term offsets the stabilizing effects of the memory term. Thus, to obtain suitable energy estimates for stability and dynamic systems, we need the assumption $\mu_1 > \mu_2$.

2.2. Equivalent equation. In order to study system (4), we introduce the following new variable to represent the delay term (See Datko et al. [11])
\[ z(x, \eta, t) = u_t(x, t - \eta \tau(t)), \tag{17} \]
then $z(x, \eta, t)$ satisfies
\[ \tau z_t(x, \eta, t) + (1 - \eta \tau') z_\eta(x, \eta, t) = 0 \quad \text{in} \; \Omega \times (0, 1) \times (0, +\infty). \tag{18} \]

Using the above transformation, system (4) can be written as equivalent form
\[
\begin{cases}
\nu_{tt}(x, t) - \Delta u(x, t) + \int_0^t g(t - s) \Delta u(x, s) \, ds \\
\quad + \mu_1 u_t(x, t) + \mu_2 z(x, 1, t) = h(x), \\
\tau z_t(x, \eta, t) + (1 - \eta \tau') z_\eta(x, \eta, t) = 0, \; x \in \Omega, \; 0 < \eta \leq 1, \; t \geq 0, \\
u(x, t) = 0, \; x \in \partial \Omega, \; t \geq 0, \\
z(x, 0) = u_t(x, t), \; x \in \Omega, \; t \geq 0, \\
u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \; x \in \Omega, \\
z(x, \eta, 0) = f_0(x, -\eta \tau(0)), \; x \in \Omega, \tag{19}
\end{cases}
\]

Our objective in this paper is to study the well-posedness, stability and dynamic systems for problem (19).

2.3. Some notations and settings. Throughout this paper, we use the simplified notation $\| \cdot \|_2$ to denote $\| \cdot \|^2_{L^2(\Omega)}$.

Now we assume that $\xi > 0$ satisfies
\[ \frac{\tau \mu_2}{1 - \tau} \leq \xi \leq \tau(2\mu_1 - \mu_2). \tag{20} \]

The phase space of global solution to system (19) can be defined as the following Hilbert space
\[ \mathcal{H} = H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega \times (0, 1)). \tag{21} \]
2.4. Global well-posedness: Existence and uniqueness.

**Theorem 2.1.** Assume that the hypotheses (H1)-(H-4) holds, \( h \in L^2(\Omega) \), then for the given initial data \( u_0 \in H^1_0(\Omega) \), \( u_1 \in L^2(\Omega) \) and \( f_0 \in L^2(\Omega \times (0,1)) \), the problem (19) possesses a unique global weak solution \((u,u_t,z)\) which satisfies

\[
\begin{align*}
  u &\in C([0,T],H^1_0(\Omega)) \cap C^1([0,T],L^2(\Omega)), \\
  u_t &\in C([0,T],H^1_0(\Omega)) \cap C([0,T],L^2(\Omega)), \\
  z &\in C([0,T],L^2(\Omega \times [0,1])),
\end{align*}
\]

which implies that \((u,u_t,z)\) \( \in C([0,T];H) \).

**Proof.** We can see Subsection 3.1 by the Galerkin approximation method to achieve.

2.5. Stability result: Exponential decay of energy functional. For evolu-

tionary systems, asymptotic stability always plays an important role in control

theory and application, especially the polynomial and exponential decay. In this

section, we are going to analyze the stability of system (19) through estimation

on the associated energy functional (23) by multiplier method, i.e., the perturbed

energy functional technique.

The energy functional associated to system (19) is defined as

\[
E(t) = \frac{1}{2} \left[ (1 - \int_0^t g(s) ds) \| \nabla u(t) \|_{L^2(\Omega)}^2 + \| u_t(t) \|_{L^2(\Omega)}^2 + (g \circ L^2(\Omega)u)(t) \right]
+ \frac{\xi}{2} \| z(t) \|_{L^2(\Omega \times [0,1])}^2 - \int \Omega hudx,
\]

where the operation “\( \circ \)” is described by

\[
(g \circ v)(t) := \int_0^t \int \Omega g(t-s)|v(t) - v(s)|^2 dx ds.
\]

For operation “\( \circ \)”, we give the following Lemma that will be frequently used throughout this work.

**Lemma 2.2.** let \( u \in H^1_0 \), \( g : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) be a bounded \( C^1 \)-function, then the following identity holds:

\[
\begin{align*}
\int_\Omega \nabla u(t) \int_0^t g(t-s) \nabla u(s) ds d\Omega &\quad = \quad \frac{1}{2} (g' \circ \nabla u)(t) - \frac{1}{2} (g \circ \nabla u)'(t) - \frac{1}{2} g(t) \int_\Omega |\nabla u|^2 d\Omega \\
&\quad + \frac{1}{2} \frac{d}{dt} \left\{ \int_0^t g(s) ds \int_\Omega |\nabla u|^2 d\Omega \right\}.
\end{align*}
\]

**Proof.** Taking the derivative of \((g \circ \nabla u)\) with respect to time variable, we can easily

obtain (25), here we skip the detail. We have following asymptotic stable result for the decay rate of the energy func-
tional (23).

**Theorem 2.3.** Assume that the hypotheses (H1)-(H-4) holds, \( h \in L^2(\Omega) \), then for the given initial data \( u_0 \in H^1_0(\Omega) \), \( u_1 \in L^2(\Omega) \) and \( f_0 \in L^2(\Omega \times (0,1)) \) for problem

(19), then the energy functional \( E(t) \) of global solution has exponent decay rate

\[
E(t) \leq C_0 E(0) e^{-\lambda t}
\]
for some positive constants \( C_0 \) and \( \lambda \).

Proof. By the perturbed energy functional technique, we can attain our stability, see Subsection 3.2.

2.6. **Dynamic systems:** Finite fractal dimensional global and exponential attractors. The existence of a global attractor requires three sufficient properties: continuity property of semigroup, dissipation and compactness, which we will briefly review the basic definitions and theory in Subsection 3.3. Especially, the gradient system has very good dissipation which implies the existence of finite fractal dimensional global and exponential attractor if the semigroup satisfies quasi-stability, see Lasiecka and Chueshov [6], [7].

**Definition 2.4.** \((X, S(t))\) is called a gradient system if it admits a strict Lyapunov function or functional, i.e., a functional \( \Phi : X \rightarrow \mathbb{R} \) is a strict Lyapunov function or functional for the system \((X, S(t))\) if

(i) the map \( t \rightarrow \Phi(S(t)z) \) is non-increasing for any \( z \in X \);
(ii) if \( \Phi(S(t)z) = \Phi(z) \) for all \( t \), then \( z \) is a stationary point of \( S(t) \).

**Proposition 1.** (Lasiecka and Chueshov [6], [7]) Let \((X, S(t))\) be a gradient system and suppose that the system is quasi-stable on every bounded positively invariant set \( B \subset X \). Then \((X, S(t))\) has a global attractor \( \mathcal{A} = M_+(N) \) with finite fractal dimension, here \( N \) is the set of equilibrium for \( S(t) \), \( M_+(N) \) is the unstable manifold for \( N \). Moreover, the generalized finite fractal dimensional exponential attractor also exists under suitable condition for \( S(t) \).

**Remark 2.** For hyperbolic systems, such as Timoshenko model and viscoelastic equation, the asymptotic stability usually obtained by perturbed energy functional method, which implies the existence of strict Lyapunov functional. It is easily to verify that the solution operator with its phase space \((X, S(t))\) is a gradient system.

Based on the global well-posedness for the system (19) in Theorem 2.1, the uniqueness of global weak solution generates a single parameter operator:

\[
S(t) : \mathcal{H} \rightarrow \mathcal{H},
S(t)(u_0, u_1, f_0) = (u(x, t), u_t(x, t), z(x, \eta, t)), \quad t > 0
\]

where \((u(x, t), u_t(x, t), z(x, \eta, t))\) is the weak solution corresponding to the initial data \((u_0, u_1, f_0)\). Obviously, the operator \( S(t) \) satisfies the property of continuous semigroup

\[
S(0) = I, \quad S(t+s) = S(t)S(s), \quad s, t \geq 0
\]

Thus, the problem (19) generates a gradient system \((\mathcal{H}, S(t))\).

Since the continuity is obtained by the property of continuous semigroup, the dissipation is easily derived by exponential stability and quasi-stability can proved in Subsection 3.4, we can state the following main theorem about dynamic systems for problem (19).

**Theorem 2.5.** Assume that the hypotheses (H1)-(H4) holds, \( h \in L^2(\Omega) \), then for the given initial data \( u_0 \in H^1_0(\Omega), u_1 \in L^2(\Omega) \) and \( f_0 \in L^2(\Omega \times (0,1)) \), we have the following results:

(1) The gradient system \((\mathcal{H}, S(t))\) generated by the problem (19) has a compact finite fractal dimensional global attractor \( \mathcal{A} \) in \( \mathcal{H} \).
The global attractor \( \mathcal{A} \) in \( \mathcal{H} \) has the structure:

\[
\mathcal{A} = M_+ (\mathcal{N})
\]

where \( \mathcal{N} = \{ y \in \mathcal{H}, \; S(t)y = y \} \) for all \( t > 0 \) is the set of stationary points and \( M_+ (\mathcal{N}) \) be the unstable manifold from the set emanating from the set \( \mathcal{N} \).

Moreover, the gradient system has a generalized exponential attractor \( \mathcal{A}^{exp} \subset \mathcal{H} \) with finite fractal dimension in \( \mathcal{H} \).

Proof. These results are established in Subsection 3.4 via deriving existence of the absorbing set and verifying quasi-stability for the semigroup.

Remark 3. (a) The finite fractal dimension implies the Hausdorff dimension is also finite.

(b) The quasi-stability of semigroup implies the contractive function method also true, this means the pullback asymptotic compactness of process holds for non-autonomous viscoelasitc systems and the family of pullback attractors also can be obtained.

3. Proof of main results.

3.1. Proof of Theorem 2.1: Well-posedness.

Proof. We use Gelerkin approximation method to achieve.

Step 1. Approximation sequence

Let \( \{ \varphi_i \}_{i=1}^\infty \) be the orthonormal basis for the Hilbert space \( H_0^1(\Omega) \) consisting the eigenvectors of the problem

\[
-\Delta w_i = \lambda_i w_i \quad \text{in} \; \Omega, \quad w_i = 0 \quad \text{on} \; \partial \Omega
\]

where \( 0 < \lambda_1 \leq \lambda_2 \leq \cdots \) are the corresponding eigenvalues.

The \( n \)-dimensional \( W_n = \text{span} \{ w_1, \cdots, w_n \} \) is the finite dimensional subspace of \( H_0^1(\Omega) \). Since \( H_0^1(\Omega) \) is dense in \( L^2(\Omega) \), we can choose two sequences \( \{ u_{0n} \}_{n=1}^\infty \) and \( \{ u_{1n} \}_{n=1}^\infty \) with \( u_{0n} \in W_n \) and \( u_{1n} \in W_n \) respectively, such that

\[
u_{0n} \to u_0 \quad \text{strongly in} \; H_0^1(\Omega), \quad u_{1n} \to u_1 \quad \text{strongly in} \; L^2(\Omega) \quad \text{as} \; n \to \infty.
\]

Let \( \{ \varphi_i(x, \eta) \}_{i=1}^\infty \) be the orthonormal basis for the Hilbert space \( L_2(\Omega \times [0,1]) \) satisfying \( \varphi_i(x, 1) = w_i(x) \). \( \varphi_i(x, \eta) \) can be generated by extending \( \varphi_i(x, 1) \) on \( L_2(\Omega \times [0,1]) \). Then the \( n \)-dimensional \( V_n = \text{span} \{ \varphi_1, \varphi_2, \cdots, \varphi_n \} \) is the finite dimensional subspace of \( L_2(\Omega \times [0,1]) \). We can choose a sequence \( \{ z_{0n} \}_{n=1}^\infty \) with \( z_{0n} \in V_n \) such that

\[
z_{0n} \to f_0(x, -\eta \tau(0)) \quad \text{strongly in} \; L_2(\Omega \times [0,1]), \; \text{as} \; n \to \infty.
\]

We define

\[
u_n(t, x) = \sum_{j=1}^n \alpha_j^n(t) w_j(x), \quad z_n(t, x, \eta) = \sum_{j=1}^n \beta_j^n(t) \varphi_j(x, \eta)
\]

as the approximated solutions which satisfy the Cauchy problem for \( 1 \leq j \leq n \)

\[
\begin{aligned}
&\left\{ \int_\Omega u_{nt} w_j dx + \int_\Omega \nabla u_{nt} \nabla w_j dx - \int_0^t \int_\Omega g(t-s) \nabla u_n(s) \nabla w_j dx ds \\
&+ \int_\Omega (\mu_1 u_{nt}(t, x) w_j + \mu_2 z_n(x, 1, t) w_j) dx = \int_\Omega h(x) w_j dx, \\
&z_n(x, 0, t) = u_{nt}(x, t), \\
&(u_n(0), u_{nt}(0)) = (u_{0n}, u_{1n})
\end{aligned}
\]

(32)
and
\[
\begin{align*}
\int_0^1 \int_{\Omega} (\tau z_{nt}(x, \eta, t) \varphi_j + (1 - \eta \tau')z_{n\eta}(x, \eta, t) \varphi_j) \, dx \, d\eta = 0,
\end{align*}
\]
(33)
and
\[
\begin{align*}
z_n(x, \eta, 0) &= z_{0n}.
\end{align*}
\]
Because \( \{w_j\} \) and \( \{\varphi_j\} \) are orthonormal and \( \varphi_j(x, 1) = w_j(x) \), we have that, for \( 1 \leq j \leq n \), (32) also writes
\[
\begin{align*}
\dot{\alpha}_j^n(t) &= \int_{\Omega} w_j \cdot w_j \, dx + \alpha_j^n(t) \int_{\Omega} \nabla w_j \cdot \nabla w_j \, dx \\
&\quad - \int_0^t \alpha_j^n(s) g(t - s) \int_{\Omega} \nabla w_j \cdot \nabla w_j \, dx \, ds \\
&\quad + \mu_1 \dot{\alpha}_j^n(t) \int_{\Omega} w_j \cdot w_j \, dx + \mu_2 \beta_j^n(t) \int_{\Omega} \varphi_j(x, 1) w_j \, dx = \int_{\Omega} h(x) w_j \, dx,
\end{align*}
\]
(34)
that is
\[
\begin{align*}
\dot{\alpha}_j^n(t) &= \int_{\Omega} w_j \cdot w_j \, dx + \alpha_j^n(t) \int_{\Omega} \nabla w_j \cdot \nabla w_j \, dx \\
&\quad - \int_0^t \alpha_j^n(s) g(t - s) \int_{\Omega} \nabla w_j \cdot \nabla w_j \, dx \, ds + \mu_2 \beta_j^n(t) \\
&\quad = \int_{\Omega} h(x) w_j \, dx,
\end{align*}
\]
(35)
Moreover, the problem (33) can be transformed as follows for \( 1 \leq j \leq n \) because \( \{\varphi_j\} \) is orthonormal
\[
\begin{align*}
\tau(t) \dot{\beta}_j^n(t) + \sum_{i=1}^n \beta_i^n(t) \int_0^1 \int_{\Omega} (1 - \eta \tau') (\varphi_i(x, \eta)) \varphi_j(x, \eta) \, dx \, d\eta = 0,
\end{align*}
\]
(36)
We can see (33) and (32) are a first and second order Cauchy systems respectively. By the local existence of unique solution for the Cauchy problem of ODE, thus (32) and (33) have unique local solutions \( \{\alpha_j^n(t)\}, \{\beta_j^n(t)\} (1 \leq j \leq n) \) on interval \([0, T_n]\) for some \( 0 < T_n < T \) respectively. Then this solution can be extended to global by using the following uniformly priori estimates.

**Step 2. Uniformly priori estimate**

Multiplying the equation (32) by \( \dot{\alpha}_j^n \) and then add those equations with respect to \( j \) from 1 to \( n \), we get
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \left( \int_{\Omega} u_{nt}^2(t) \, dx + \int_{\Omega} \nabla u_n^2(t) \, dx \right) \\
&\quad + \mu_1 \int_{\Omega} u_{nt}^2(t) \, dx + \mu_2 \int_{\Omega} z_n(x, 1, t) u_{nt}(t) \, dx \\
&\quad + \frac{1}{2} g(t) \int_{\Omega} |\nabla u_n(t)|^2 \, dx - \frac{1}{2} (g \circ \nabla u_n)(t) \\
&\quad + \frac{1}{2} \frac{d}{dt} \left\{ (g \circ \nabla u_n)(t) - \left( \int_0^t g(s) \, ds \right) \int_{\Omega} \nabla u_n^2(t) \, dx \right\} \\
&\quad = \int_{\Omega} h u_{nt}(t) \, dx.
\end{align*}
\]
(37)
Integrating in terms of $t$, we have
\[
\frac{1}{2} \left[ \left( 1 - \int_0^t g(s) \, ds \right) \| \nabla u_n(t) \|^2 + \| u_{nt}(t) \|^2 + (g \circ \nabla u_n)(t) \right] \\
+ \mu_1 \int_0^t \| u_{nt}(s) \|^2 \, ds + \mu_2 \int_0^t \int_\Omega z_n(x, 1, s)u_{nt}(s) \, dx \, ds \\
+ \frac{1}{2} \int_0^t g(s) \| \nabla u_n(s) \|^2 \, ds - \frac{1}{2} \int_0^t g' \circ \nabla u_n(s) \, ds \\
= \int_0^t \int_\Omega h_{nt} \, dx \, ds + \frac{1}{2} \left( \| \nabla u_0 \|^2 + \| u_1 \|^2 \right). \tag{38}
\]

Multiplying (33) by $\beta_n^2(t)/\tau(t)$ and then add those equations with respect to $j$ from 1 to $n$, we derive that
\[
\frac{1}{2} \frac{d}{dt} \int_0^1 \int_\Omega z_n^2(x, \eta, t) \, dx \, d\eta + \frac{1}{2\tau} \int_\Omega z_n^2(x, 1, t) - z_n^2(x, 0, t) \, dx \\
- \frac{\tau'}{2\tau} \int_\Omega z_n^2(x, 1, t) \, dx + \frac{\tau'}{2\tau} \int_0^1 \int_\Omega z_n^2(x, \eta, t) \, dx \, d\eta = 0. \tag{39}
\]

Integrating in terms of $t$, we obtain that
\[
\frac{1}{2} \int_0^1 \int_\Omega z_n^2(x, \eta, t) \, dx \, d\eta + \int_0^t \int_\Omega \frac{1}{2\tau} \left[ z_n^2(x, 1, s) - z_n^2(x, 0, s) \right] \, dx \, ds \\
- \int_0^t \int_\Omega \frac{\tau'}{2\tau} z_n^2(x, 1, s) \, dx \, ds + \int_0^t \int_\Omega \frac{\tau'}{2\tau} z_n^2(x, \eta, s) \, dx \, d\eta \, ds = \frac{1}{2} \| z_0 \|^2_{L^2(\Omega \times (0,1))}. \tag{40}
\]

Setting
\[
\mathcal{E}_n(t) = \frac{1}{2} \left[ \left( 1 - \int_0^t g(s) \, ds \right) \| \nabla u_n(t) \|^2 + \| u_{nt}(t) \|^2 + (g \circ \nabla u_n)(t) \right] \\
+ \frac{\xi}{2} \| z_n(t) \|^2_{L^2(\Omega \times [0,1])} \tag{41}
\]
and multiplying (40) by $\xi$, then summing up with (38), we have
\[
\mathcal{E}_n(t) + \left( \mu_1 - \frac{\xi}{2\tau} \right) \int_0^t \| u_{nt}(s) \|^2 \, ds + \frac{\xi(1 - \tau')}{2\tau} \int_0^t \| z_n(x, 1, s) \|^2 \, ds \\
+ \int_0^t \int_0^1 \int_\Omega \frac{\xi \tau'}{2\tau} z_n^2(x, \eta, s) \, dx \, d\eta \, ds + \mu_2 \int_0^t \int_\Omega z_n(x, 1, s)u_{nt}(s) \, dx \, ds \\
+ \frac{1}{2} \int_0^t g(s) \| \nabla u_n(s) \|^2 \, ds - \frac{1}{2} \int_0^t g' \circ \nabla u_n(s) \, ds \\
= \int_0^t \int_\Omega h_{nt} \, dx \, ds + \mathcal{E}_n(0). \tag{42}
\]

Next, we shall estimate (42) under the hypothesis $\mu_2 < \mu_1$. From the energy identity (42), we have
\[
\mathcal{E}_n(t) + \left( \mu_1 - \frac{\xi}{2\tau} - \frac{\mu_2}{2} \right) \int_0^t \| u_{nt}(s) \|^2 \, ds \\
+ \left( \frac{\xi(1 - \tau')}{2\tau} - \frac{\mu_2}{2} \right) \int_0^t \| z_n(x, 1, s) \|^2 \, ds
\]
we conclude that there exists a subsequence

Then since the embedding $H^1_0(\Omega) \hookrightarrow L^2(\Omega)$, we have

Since $h \in L^2(\Omega)$, we have

Since the sequence $\{u_n\}, \{u_1\}$ and \{z_n\} are the projections of $u_0$, $u_1$ and $f_0(x, -\eta(0))$ in space $W_n, W_1$ and $V_n$ respectively, we obtain that

Using hypotheses (H1) and (H2), there exists a constant $C$ independent of $n$ such that

Thus combine (45) and (47), it follows that

Then since the embedding $H^1_0(\Omega) \hookrightarrow L^2(\Omega)$ is compact, by Aubin-Lions lemma, we conclude that there exists a subsequence $\{u_{n_k}\}$ such that

$$ u_{n_k} \rightarrow u \text{ strongly in } L^2(0, T; H^1_0(\Omega)), $$

$$ u_{n_k} \rightarrow u(t) \text{ strongly in } L^2(0, T; L^2(\Omega)), $$
3.2. Proof of Theorem 2.3: Stability result.

Proof.Lemma 3.1. The energy defined in (23) for system (19) is decreasing and there exists a positive constant $C$ such that

$$E(t) \leq -C \left( \|u(t)\|_2^2 + \|z(x, t)\|_2^2 + \|z(x, \eta, t)\|_{L^2(\Omega \times (0, 1))}^2 + \int_0^t g(s) \, ds \right)$$

$$+ \frac{1}{2} \|g(t) \nabla u(t)\|_2^2 + \frac{1}{2} (g \circ \nabla u)(t).$$

Proof. Since

$$\frac{dE(t)}{dt} = -\frac{1}{2} g(t) \|\nabla u(t)\|_2^2 + \left( 1 - \int_0^t g(s) \, ds \right) \int_\Omega \nabla u(t) \cdot \nabla u(t) \, dx$$

$$+ \int_\Omega u_t(t) u_{tt}(t) \, dx - \int_\Omega h u_t(t) \, d\Omega$$

$$+ \frac{1}{2} \|g \circ \nabla u\|_2^2 + \frac{1}{2} \int_0^1 z(t) z_t(t) \, dx \, d\eta,$$
combining with (19), we have
\[
\int_{\Omega} u_t(t) u_{tt}(t) \, dx = \int_{\Omega} u_t(t) \Delta u(t) \, dx - \int_{\Omega} u_t(t) \int_{0}^{t} g(t-s) \Delta u(x,s) \, ds \, dx
- \mu_1 \int_{\Omega} u_t^2(t) \, dx - \mu_2 \int_{\Omega} u_t(t) z(x,1,t) \, dx + \int_{\Omega} h u_t(t) \, dx. \tag{61}
\]
By Lemma 2.2, we have
\[
\frac{1}{2} (g \circ \nabla u)'(t)
= - \int_{0}^{t} g(t-s) \nabla u_t(t) \nabla u(s) \, ds \, dx + \frac{1}{2} (g' \circ \nabla u)(t)
+ \int_{0}^{t} g(s) \, ds \int_{\Omega} \nabla u(t) \cdot \nabla u_t(t) \, ds.
\]
Moreover, we also have
\[
\int_{0}^{1} \int_{\Omega} z(t) z_t(t) \, dx \, dt 
= - \int_{0}^{1} \int_{\Omega} \frac{1-\eta \tau'}{\tau} z(t) z_\eta(t) \, dx \, dt
= - \frac{1}{2\tau} \int_{\Omega} z^2(x,1,t) - z^2(x,0,t) \, dx + \frac{\tau'}{2\tau} \int_{\Omega} z^2(x,1,t) \, dx
- \frac{\tau'}{2\tau} \int_{0}^{1} \int_{\Omega} z^2(x,\eta,t) \, dx \, d\eta
= - \frac{1-\tau'}{2\tau} \int_{\Omega} z^2(x,1,t) \, dx + \frac{1}{2\tau} \int_{\Omega} u_t^2(t) \, dx
- \frac{\tau'}{2\tau} \int_{0}^{1} \int_{\Omega} z^2(x,\eta,t) \, dx \, d\eta. \tag{63}
\]
Combining (61), (62) and (63) with (60), we conclude that
\[
\frac{dE(t)}{dt} = - \frac{1}{2} g(t) \| \nabla u(t) \|^2 + \frac{1}{2} (g' \circ \nabla u)(t) - \frac{\xi \tau'}{2\tau} \int_{0}^{t} \int_{\Omega} z^2(x,\eta,t) \, dx \, d\eta
- \left( \mu_1 - \frac{\xi}{2\tau} \right) \int_{\Omega} u_t^2(t) \, dx
- \frac{\xi (1-\tau')}{2\tau} \int_{\Omega} z^2(x,1,t) \, dx - \mu_2 \int_{\Omega} u_t(t) z(x,1,t) \, dx
\leq - \left( \mu_1 - \frac{\xi}{2\tau} - \frac{\mu_2}{2} \right) \int_{\Omega} u_t^2(t) \, dx - \left( \frac{\xi (1-\tau')}{2\tau} - \frac{\mu_2}{2} \right) \int_{\Omega} z^2(x,1,t) \, dx
- \frac{1}{2} g(t) \| \nabla u(t) \|^2 - \frac{\xi \tau'}{2\tau} \int_{0}^{1} \int_{\Omega} z^2(x,\eta,t) \, dx \, d\eta + \frac{1}{2} (g' \circ \nabla u)(t), \tag{64}
\]
which means the lemma holds. \qed
Next we want to construct a perturbed functional $F(t)$, which is equivalent to $E(t)$ and satisfies the exponential decay inequality

$$\frac{dF(t)}{dt} \leq -cF(t)$$

for some positive $c$. To do this, we need the following lemma.

**Lemma 3.2.** Defining

$$G_1(t) = \int_{\Omega} u_t(t)u(t) \, dx,$$  

Then we have

$$\frac{dG_1(t)}{dt} \leq C_1\|u_t(t)\|_2^2 + C_1\int_{\Omega} z^2(x,1,t) \, dx$$

$$+ \int_{\Omega} hu \, dx - C_2\|\nabla u(t)\|_2^2 + \frac{1-l}{2l} (g \circ \nabla u)(t)$$

for some positive constants $C_1$ and $C_2$.

**Proof.** Since

$$\frac{d}{dt} \int_{\Omega} u_t u \, dx$$

$$= \int_{\Omega} u_t u \, dx + \int_{\Omega} u_t^2 \, dx$$

$$= \int_{\Omega} \left[ \Delta u - \int_0^t g(t-s)\Delta u \, ds - \mu_1 u_t - \mu_2 z(x,1,t) + h \right] u \, dx + \|u_t(t)\|_2^2$$

$$= -\|\nabla u(t)\|_2^2 + \|u_t(t)\|_2^2 - \mu_1 \int_{\Omega} u_t u \, dx - \mu_2 \int_{\Omega} z(x,1,t) u \, dx$$

$$+ \int_{\Omega} hu \, dx + \int_{\Omega} \nabla u \int_{0}^{t} g(t-s)\nabla u(s) \, ds \, dx,$$  

then we shall estimate the last term in above formula as

$$\int_{\Omega} \nabla u(t) \int_{0}^{t} g(t-s)\nabla u(s) \, ds \, dx$$

$$\leq \frac{1}{2} \int_{\Omega} |\nabla u(t)|^2 \, dx + \frac{1}{2} \int_{\Omega} \left( \int_{0}^{t} g(t-s)(|\nabla u(s) - \nabla u(t)| + |\nabla u(t)|) \, ds \right)^2 \, dx$$

$$\leq \frac{1}{2} \int_{\Omega} |\nabla u(t)|^2 \, dx + \frac{1}{2} (1+k)(1-l)^2 \int_{\Omega} |\nabla u(t)|^2 \, dx$$

$$+ \frac{1}{2} (1 + \frac{1}{k}) (1-l) (g \circ \nabla u)(t).$$  

In addition, by Poincare’s inequality, we have

$$\left| -\mu_1 \int_{\Omega} u_t u \, dx - \mu_2 \int_{\Omega} z(x,1,t) u \, dx \right|$$

$$\leq \frac{1}{4\delta} \left( \mu_1 \int_{\Omega} u_t^2 \, dx + \mu_2 \int_{\Omega} z^2(x,1,t) \, dx \right) + \delta c_0^2 (\mu_1 + \mu_2) \int_{\Omega} |\nabla u(t)|^2 \, dx$$

for some positive constant $c_0 > 0$ and small $\delta > 0$. 


Combining the above formula and setting $k = \frac{1}{l-1}$, we deduce

\[
\frac{dG_1(t)}{dt} = \frac{d}{dt} \int_{\Omega} u_t(t)u(t) \, dx
\]

\[
\leq \left(1 + \frac{\mu_1}{4\delta}\right)\|u_t(t)\|^2 + \frac{\mu_2}{4\delta} \int_{\Omega} \varepsilon^2(x,1,t) \, dx + \int_{\Omega} hu \, dx
\]

\[
- \left(\frac{1}{2} - \delta c_0^2(\mu_1 + \mu_2)\right)\|\nabla u(t)\|^2_2 + \frac{1 - l}{2l}(g \circ \nabla u)(t),
\]

which implies Lemma 3.2.

**Proof of Theorem 2.3.** For $\varepsilon > 0$, we define the perturbed Lyapunov functional as

\[
F(t) \triangleq E(t) + \varepsilon G_1(t),
\]

it is easily to verify that there exist positive constants $\alpha_1$ and $\alpha_2$ such that

\[
\alpha_1 E(t) \leq F(t) \leq \alpha_2 E(t).
\]

By the estimates (59) and (67), we get

\[
\frac{dF(t)}{dt} \leq -\left(C - \varepsilon C_1\right)\|u_t\|^2_2 - \left[\frac{1}{2}g(t) + \varepsilon C_2\right]\|\nabla u\|^2_2 - C\|\varepsilon(z(\eta,t))\|^2_2_{L^2(\Omega \times (0,1))}
\]

\[
- \left(C - \varepsilon C_1\right)\|z(x,1,t)\|^2_2 + \frac{\varepsilon(1 - l)}{2l}(g \circ \nabla u)(t)
\]

\[
+ \frac{1}{2}(g' \circ \nabla u)(t) + \varepsilon \int_{\Omega} hu \, dx
\]

Choosing suitable value of $\varepsilon$, we can find some positive constants $\beta_1, \beta_2$ such that

\[
F'(t) \leq -\beta_1 E(t) + \beta_2 (g \circ \nabla u)(t).
\]

Then by the hypothesis (H3), it yields

\[
\zeta_0 F'(t) \leq -\beta_1 \zeta_0 E(t) + \beta_2 \zeta_0 (g \circ \nabla u)(t)
\]

\[
\leq -\beta_1 \zeta_0 E(t) - \beta_2 (g' \circ \nabla u)(t) \leq -\beta_1 \zeta_0 E(t) - 2\beta_2 E'(t).
\]

Therefore, we have

\[
\left(\zeta_0 F(t) + 2\beta_2 E(t)\right)' \leq -\beta_1 \zeta_0 E(t).
\]

Setting $F(t) = \zeta_0 F(t) + 2\beta_2 E(t)$, noting that $F(t) \sim E(t)$, we have $F(t) \sim E(t)$. Then the equation (77) can be represented as

\[
F'(t) \leq -\beta_1 \zeta_0 E(t) \leq -\beta_3 \zeta_0 F(t).
\]

Thus, $F(t)$ decays exponentially as $t \to \infty$, i.e.,

\[
F(t) \leq F(0)e^{-\beta_3 \zeta_0 t}.
\]

Since $F$ is equivalent to $E$, we also have

\[
E(t) \leq C E(0)e^{-\beta_3 \zeta_0 t}.
\]

Set $C_0 = C$ and $\lambda = \beta_3$, we finish the proof for Theorem 2.3. \qed
3.3. Theory of dynamic systems. In this section, we will review the quasi-stability theory as shown in Lasiecka and Chueshov [6], [7]. More details could be found in the original papers [6], [7].

• Some definitions:

Definition 3.3. (a) (Dissipation) A set $B_0 \subset X$ is called an absorbing set for the semigroup $S(t)$ ($t \geq 0$) if for any bounded set $B \subset X$ there exists a time $t_1 = t_1(B) > 0$ such that for all $t > t_1$, $S(t)B \subseteq B_0$.
(b) (Asymptotic smoothness) The semigroup $S(t)$ ($t \geq 0$) is said to be asymptotically smooth in $X$ if for any closed bounded subset $B \subset X$ satisfying $S(t)B \subset B$, there exists a nonempty compact set $K = K(B) \subset X$ such that $\text{dist}_X(S(t)B, K(B)) \to 0$ as $t \to \infty$.
(c) (Asymptotic compactness) A dynamical system $(X,S(t))$ is asymptotically compact if for any bounded set $B \subset X$, and sequence $\{x_k\} \subset B$, the sequence $\{S(t_k)x_k\}$ has convergent subsequence as $t_k \to \infty$.

Definition 3.4. A compact set $A \subset X$ is called a global attractor of the semigroup $S(t)$ if

(i) $A$ is strictly invariant with respect to $S(t)$, i.e., for all $t \geq 0$, $S(t)A = A$
(ii) $A$ attracts any bounded set $B \subset X$: for any $\varepsilon > 0$ there exists a time $t_1 = t_1(\varepsilon, B) > 0$ such that for all $t \geq t_1(\varepsilon, B)$, $S(t)B \subseteq O_\varepsilon(A)$, where $O_\varepsilon(A)$ is an $\varepsilon$-neighborhood of $A$ in $X$.

Definition 3.5. Given a compact set $M$ in a metric space $X$, the fractal dimension of $M$ is defined by

$$\dim^X M = \limsup_{\varepsilon \to 0} \frac{\ln \mathcal{N}(M, \varepsilon)}{\ln(1/\varepsilon)},$$

where $\mathcal{N}(M, \varepsilon)$ is the minimal number of closed balls with radius $\varepsilon > 0$ which covers $M$.

• Quasi-stability and global attractors:

Definition 3.6. The continuous semigroup generated by a dynamic system (or gradient system) possesses a global attractor $A$ if

(1) there exists an absorbing set for semigroup,
(2) the semigroup (or gradient system) has asymptotically smooth or compact property.

Definition 3.7. The unstable manifold $M_+(\mathcal{N})$ is defined as the family of $y \in X$ such that there exists a full trajectory $u(t)$ satisfying

$$u(0) = y, \quad \lim_{t \to -\infty} \text{dist}_X(u(t), \mathcal{N}) = 0,$$

here $\mathcal{N}$ is the set of equilibrium for $S(t)$.

Theorem 3.8. (See [6]) Assume that the gradient system $(S(t), X)$ with corresponding Lyapunov functional $\Phi$ is asymptotically compact. Moreover, assume that

(I) $\Phi(S(t)z) \to \infty$ if and only if $\|z\|_X \to \infty$,
(II) the set of equilibrium $\mathcal{N}$ is bounded.

Then, the gradient system $(S(t), X)$ possesses a compact global attractor $A \subset X$ which has the structure $A = M_+(\mathcal{N})$.

Definition 3.9. (See [6], [7]) The dynamic system $(S(t), X)$ is quasi-stable on a set $B \subset X$ if there exists a compact semi-norm $n_Y$ on $Y$, the subspace of $X$ and
nonnegative scalar functions \(a(t)\) and \(c(t)\), locally bounded on \([0, \infty)\) and \(b(t) \in L^1(\mathbb{R}^+)\) with \(\lim_{t \to \infty} b(t) = 0\), such that for \(U_1, U_2 \in B\)

\[
\|S(t)U_1 - S(t)U_2\|_X^2 \leq a(t)\|U_1 - U_2\|_X^2,
\]

(82)

\[
\|S(t)U_1 - S(t)U_2\|_X^2 \leq b(t)\|U_1 - U_2\|_X^2 + c(t) \sup_{0 < s < t} \left[ \nu_Y(y_1(s) - y_2(s)) \right]^2.
\]

(83)

Inequality (83) is usually called stabilizability inequality.

**Theorem 3.10.** (See [6], [7]) Based on the quasi-stability property of gradient system, we have

(a) Let \((X, S(t))\) be a dynamical system and suppose that the system is quasi-stable on every bounded positively invariant set \(B \subset X\). Then \((X, S(t))\) is asymptotically compact.

(b) Suppose that the dynamical system has a global attractor \(\mathcal{A}\) and it is quasi-stable. Then, the global attractor \(\mathcal{A}\) has finite fractal dimension.

- Fractal dimensional exponential attractors:

  Quasi-stability also implies the existence of finite fractal dimensional exponential attractors.

**Theorem 3.11.** (See [6], [7]) Assume \((X, S(t))\) is a dissipative dynamical system satisfying quasi-stable property on some bounded absorbing set \(B\), and there exists an external space \(\hat{X}\) with \(X \subset \hat{X}\), such that for every \(T > 0\)

\[
\|S(t_1)y - S(t_2)y\|_X \leq C_{GR}|t_1 - t_2|^{\eta}, \quad t_1, t_2 \in [0, T], \ y \in B,
\]

(84)

were \(C_{GR}\) and \(\eta \in (0, 1)\) are positive constants. Then this system has a generalized finite fractal dimensional exponential attractor \(\mathcal{A}^{exp}\) in \(\hat{X}\).

3.4. **Proof of Theorem 2.5: Dynamic systems.** The proof of Theorem 2.5 will be divided into the proofs of existence for absorbing set and the quasi-stability inequality for gradient system \((\mathcal{H}, S(t))\).

**Lemma 3.12.** Assume the hypotheses \((H1)-(H4)\) hold, the gradient system \((\mathcal{H}, S(t))\) corresponding to system (19) has a bounded absorbing set \(B\), i.e., dissipation property of semigroup.

**Proof.** We need the estimate of \(\|(u(t), u_t(t), z(t))\|_\mathcal{H}\). Considering

\[
\mathcal{E}(t) = \frac{1}{2} \left[ \left(1 - \int_0^t g(s) \, ds\right) \|\nabla u(t)\|^2 + \|u_t(t)\|^2 + (g \circ \nabla u)(t) \right]
+ \frac{\xi}{2} \|z(t)\|^2_{L^2(\Omega \times [0, 1])},
\]

(85)

Then we have

\[
\frac{d\mathcal{E}(t)}{dt} = -\frac{1}{2} g(t)\|\nabla u(t)\|^2 + \left(1 - \int_0^t g(s) \, ds\right) \int_\Omega \nabla u(t) \cdot \nabla u_t(t) \, dx
+ \int_\Omega u_t(t)u_{tt}(t) \, dx + \frac{1}{2} (g \circ \nabla u)'(t) + \xi \int_0^1 \int_\Omega z(t) z_t(t) \, dx \, d\eta.
\]

(86)
Similar to the proof of inequality (59), we derive the following estimate

\[
\frac{d\mathcal{E}(t)}{dt} = -\frac{1}{2} g(t) \|\nabla u(t)\|_2^2 + \frac{1}{2} (g' \circ \nabla u)(t) - \frac{\tau'}{2\tau} \int_0^t \int_{\Omega} z^2(x, \eta, t) \, dx \, d\eta \\
- \left( \mu_1 - \frac{\xi}{2\tau} \right) \int_{\Omega} u_t^2(t) \, dx - \frac{\xi(1 - \tau')}{2\tau} \int_{\Omega} z^2(x, 1, t) \, dx \\
- \mu_2 \int_{\Omega} u_t(z(x, 1, t) \, dx + \int_{\Omega} hu_t(t) \, dx \\
\leq -\left( \mu_1 - \frac{\xi}{2\tau} - \frac{\mu_2}{2} - \delta_1 \right) \int_{\Omega} u_t^2(t) \, dx - \left( \frac{\xi(1 - \tau')}{2\tau} - \frac{\mu_2}{2} \right) \int_{\Omega} z^2(x, 1, t) \, dx \\
- \frac{\xi\tau'}{2\tau} \int_0^1 \int_{\Omega} z^2(x, \eta, t) \, dx \, d\eta - \frac{1}{2} g(t) \|\nabla u(t)\|_2^2 \\
+ \frac{1}{2} (g' \circ \nabla u)(t) + \frac{1}{2\delta_1} \|h\|_2^2. \tag{87}
\]

Defining \(G_1(t)\) as the same functional in (66), using the same technique as in Lemma 3.2, we deduce

\[
\frac{dG_1(t)}{dt} = \frac{d}{dt} \int_{\Omega} u_t(t)u(t) \, dx \\
\leq \left( 1 + \frac{\mu_1}{4\delta_2} \right) \|u_t(t)\|_2^2 + \frac{\mu_2}{4\delta_2} \int_{\Omega} z^2(x, 1, t) \, dx + \frac{1}{4\delta_2} \|h\|_2^2 \\
- \left( \frac{l}{2} - \delta_2 C^2(\mu_1 + \mu_2) \right) \|\nabla u(t)\|_2^2 + \frac{1-l}{2l}(g \circ \nabla u)(t). \tag{88}
\]

Defining \(J(t)\) as

\[
J(t) = \mathcal{E}(t) + \varepsilon G_1(t), \tag{89}
\]

then by (87) and (88), we have

\[
\frac{dJ(t)}{dt} \leq - \left( \mu_1 - \frac{\xi}{2\tau} - \frac{\mu_2}{2} - \delta_1 \right) \|u_t\|_2^2 \\
- \left[ \frac{1}{2} g(t) + \varepsilon \left( \frac{l}{2} - \delta_2 C^2(\mu_1 + \mu_2) \right) \|\nabla u\|_2^2 \right] \\
- \frac{\xi\tau'}{2\tau} \|z(x, \eta, t)\|_2^2 \chi_{\Omega \times (0,1)} - \left( \frac{\xi(1 - \tau')}{2\tau} - \frac{\mu_2}{2} \right) \|z(x, 1, t)\|_2^2 \\
+ \frac{1-l}{2l}(g \circ \nabla u)(t) + \frac{1}{2} (g' \circ \nabla u)(t) + \left( \frac{1}{2\delta_1} + \frac{1}{4\delta_2} \right) \|h\|_2^2. \tag{90}
\]

Then choosing \(\varepsilon\) small enough, it yields that there exist some positive constants \(\gamma_1, \gamma_2\) and \(\gamma_3\) such that

\[
J'(t) \leq -\gamma_1 \mathcal{E}(t) + \gamma_2 (g \circ \nabla u)(t) + \gamma_3 \|h\|_2^2. \tag{91}
\]
Thus, by the similar technique in (76), we obtain
\[
\zeta_0 J'(t) \leq -\gamma_1 \zeta_0 E(t) + \gamma_2 \zeta_0 (g \circ \nabla u)(t) + \gamma_3 \zeta_0 \| h \|^2_2
\]
\[
\leq -\gamma_1 \zeta_0 E(t) - \gamma_2 (g' \circ \nabla u)(t) + \gamma_3 \zeta_0 \| h \|^2_2
\]
\[
\leq -\gamma_1 \zeta_0 E(t) - 2\gamma_2 E'(t) + (\gamma_3 \zeta_0 + \frac{\gamma_2}{\delta_1}) \| h \|^2_2, \quad (92)
\]
Therefore,
\[
(\zeta_0 J(t) + 2\gamma_2 E(t))' \leq -\gamma_1 \zeta_0 E(t) + \gamma_4 \| h \|^2_2. \quad (93)
\]
Setting \( J(t) = \zeta_0 J(t) + 2\gamma_2 E(t) \), then \( J(t) \) is equivalent to \( E(t) \) and satisfies the inequality
\[
J'(t) \leq -\gamma_1 \zeta_0 E(t) + \gamma_4 \| h \|^2_2 \leq -\gamma_0 J(t) + \gamma_4 \| h \|^2_2, \quad (94)
\]
which implies that
\[
J(t) \leq J(0)e^{-\gamma_0 t} + \gamma_4 \| h \|^2_2 \int_0^t e^{-\gamma_0 (t-s)} \, ds \leq J(0)e^{-\gamma_0 t} + \frac{\gamma_4}{\gamma_0} (1 - e^{-\gamma_0 t}) \| h \|^2_2. \quad (95)
\]
Therefore, there exist positive constants \( C_1 \) and \( C_2 \), such that
\[
E(t) \leq C_1 e^{-\gamma_0 t} + C_2 \| h \|^2_2. \quad (96)
\]
On the other hand, by the definition of phase space, we have
\[
\|(u(t), u_t(t), z(t))\|_H = \|\nabla u(t)\|^2_2 + \|u_t(t)\|^2_2 + \|z(t)\|^2_{L_2(\Omega \times [0,1])}
\]
\[
\leq \max(1, \xi) \left( \|\nabla u(t)\|^2_2 + \|u_t(t)\|^2_2 + \|z(t)\|^2_{L_2(\Omega \times [0,1])} + (g \circ \nabla u)(t) \right)
\]
\[
\leq \frac{2}{1-t} \max(1, \xi) \eta(t) \leq \frac{2C_1}{1-t} \max(1, \xi) e^{-\gamma_0 t} + \frac{2C_2}{1-t} \max(1, \xi) \| h \|^2_2. \quad (97)
\]
This implies any closed ball \( B(0, R) \) with radius
\[
R = \sqrt{\frac{2C_2}{1-t} \max(1, \xi) \cdot \| h \|^2_2 + 1} \quad (98)
\]
is a bounded absorbing set of gradient system \((H, S(t))\). Then the Lemma 3.12 is proved.

For the convenience of establishing the quasi-stability of system \((H, S(t))\), we set
\[
U^i_0 = (u^i_0(x,t), u^i_t(x,t), f^i_0(x,-\eta\tau(0))) \in H \quad (99)
\]
to be an initial condition of system (19) in space \( H \). Thus the solutions of (19) corresponding to \( U^i_0 \) is denoted by
\[
U^i = (u^i(x,t), u^i_t(x,t), z^i(x,\eta,t)). \quad (100)
\]

**Lemma 3.13.** Assume \((H1)-(H4)\) hold, the dynamical system \((H, S(t))\) corresponding to system (19) satisfies the following quasi-stability condition for any initial conditions \( U^i_0 \) \((i = 1, 2)\) in \( B(0, R) \) defined in (98)
\[
\| S(t) U^i_0 - S(t) U^2_0 \|^2_H \leq b(t) \| U^i_0 - U^2_0 \|^2_H + c(t) \sup_{0 \leq s \leq t} \| \nabla u^1(t) - \nabla u^2(t) \|^2_2. \quad (101)
\]
Proof. For any initial condition \((u^0_i, u^1_i, f^0_i) \in B(0, R)\), let \((u^1_i, u^2_i, z^i)\) be the corresponding solution with respect to the initial condition \((u^0_i, u^1_i, f^0_i)\) with \(i = 1, 2\). Let

\[
W(t) = (\Phi, \Phi_t, \mathcal{Z}) = (u^1 - u^2, (u^1 - u^2)_t, z^1 - z^2) = S(t)U^0_1 - S(t)U^2_0,
\]

then \(W(t)\) satisfies

\[
\begin{align*}
\Phi_{tt}(x, t) - \Delta \Phi(x, t) + \int_0^t g(t - s)\Phi(x, s)\, ds \\
+ \mu_1 \Phi_t(x, t) + \mu_2 \mathcal{Z}(x, 1, t) &= 0, \quad x \in \Omega, \quad t \geq 0, \\
\tau \mathcal{Z}_t(x, \eta, t) + (1 - \eta \tau') \mathcal{Z}_\eta &= 0, \quad x \in \Omega, \quad 0 < \eta \leq 1, \quad t \geq 0, \\
\Phi(x, t) &= 0, \quad x \in \partial \Omega, \quad t \geq 0, \\
\mathcal{Z}(x, 0, t) &= \Phi_0(x, t), \quad x \in \Omega, \quad t \geq 0, \\
\Phi_t(x, 0) &= u^1_0(x) - u^1_1(x), \quad \Phi_t(x, 0) = u^2_1(x) - u^2_0(x), \quad x \in \Omega, \\
\mathcal{Z}(x, \eta, 0) &= f^1_0(x, -\eta \tau(0)) - f^2_0(x, -\eta \tau(0)), \quad x \in \Omega.
\end{align*}
\]

The norm of \(W(t) = U^1(t) - U^2(t)\) in \(\mathcal{H}\) is equivalent to

\[
\mathcal{M}(t) = \frac{1}{2} \left[ ||\nabla \Phi(t)||_2^2 + ||\Phi_t(t)||_2^2 \right] + \frac{\xi}{2} \left[ ||\mathcal{Z}(t)||_{L_2(\Omega \times [0, 1])}^2 \right].
\]

Defining

\[
\mathcal{W}(t) = \frac{1}{2} \left[ \left(1 - \int_0^t g(s)\, ds\right) ||\nabla \Phi(t)||_2^2 + ||\Phi_t(t)||_2^2 + (g \circ \nabla \Phi)(t) \right] \\
+ \frac{\xi}{2} ||\mathcal{Z}(t)||_{L_2(\Omega \times [0, 1])}^2,
\]

then we have

\[
\mathcal{W}(t) = \mathcal{M}(t) - \frac{1}{2} \left( \int_0^t g(s)\, ds \right) ||\nabla \Phi(t)||_2^2 - (g \circ \nabla \Phi)(t),
\]

Next, we want to estimate \(\mathcal{M}(t)\) through \(\mathcal{W}(t)\). Since

\[
\frac{d\mathcal{W}(t)}{dt} = -\frac{1}{2} g(t) ||\nabla \Phi||_2^2 + \left(1 - \int_0^t g(s)\, ds\right) \int_{\Omega} \nabla \Phi(t) \cdot \nabla \Phi_t(t)\, dx \\
+ \int_{\Omega} \Phi_t(t) \Phi_{tt}(t)\, dx + \xi \int_0^1 \int_{\Omega} \mathcal{Z}(t) \mathcal{Z}_t(t)\, dx\, dt,
\]

using the similar technique as in the proof of inequality (59) and by Lemma 2.2, we have

\[
\frac{d\mathcal{W}(t)}{dt} \leq \frac{1}{2} \left(g' \circ \nabla \Phi\right)(t) - \frac{1}{2} g(t) \int_{\Omega} (\nabla \Phi)^2\, dx - \left(1 - \frac{\xi \tau'}{2\tau}\right) \int_{\Omega} \mathcal{Z}^2(x, \eta, t)\, dx\, dt \\
- \left(\mu_1 - \frac{\xi}{2\tau} - \frac{\mu_2}{2}\right) \int_{\Omega} \Phi_t^2(t)\, dx - \left(\frac{\xi (1 - \tau')}{2\tau} - \frac{\mu_2}{2}\right) \int_{\Omega} \mathcal{Z}^2(x, 1, t)\, dx.
\]

Defining

\[
\mathcal{V}(t) = \int_{\Omega} \Phi_t(t) \Phi(t)\, dx,
\]

(109)
Then it follows
\[
\frac{dV(t)}{dt} = \int_{\Omega} \Phi_t(t) \Phi(t) \, dx + \int_{\Omega} \Phi_t^2(t) \, dx
\]
\[
= \int_{\Omega} \Phi_t^2(t) \, dx + \int_{\Omega} \Phi_t(t) \Delta \Phi(t) \, dx - \int_{\Omega} \Phi_t(t) \int_0^t g(t - s) \Delta \Phi(x, s) \, ds \, dx
\]
\[- \mu_1 \int_{\Omega} \Phi_t^2(t) \, dx - \mu_2 \int_{\Omega} \Phi_t(t) \mathcal{Z}(x, 1, t) \, dx,
\]
by the similar method to deal with the proof of inequality (67), it yields
\[
\frac{dV(t)}{dt} \leq \left(1 + \mu_1 \frac{1}{4\delta} \right) \|\Phi_t(t)\|^2_2 - \left(\frac{1}{1 - \delta \mu_1 + \mu_2}\right) \|\nabla \Phi(t)\|^2_2
\]
\[+ \frac{\mu_2}{2} \int_{\Omega} \mathcal{Z}^2(x, 1, t) \, dx + \frac{1 - l}{2l} (g \circ \nabla \Phi)(t). \tag{111}\]
Defining
\[
\mathcal{G}(t) = \varpi(t) + \varepsilon \mathcal{V}(t), \tag{112}\]
it is easy to see that \(\mathcal{G}(t)\) is equivalent to \(\varpi(t)\). By estimates (108) and (110), we conclude that
\[
\frac{dG(t)}{dt} \leq - \left(\mu_1 - \frac{\varepsilon M}{2} - \frac{\mu_2}{2}\right) \|\Phi_t(t)\|^2_2
\]
\[- \left[\frac{1}{2} g(t) + \varepsilon \left(\frac{1}{2} - \delta \mu_2\right)\right] \|\nabla \Phi\|^2_2
\]
\[- \frac{\mu_2}{2} \|\mathcal{Z}(x, 1, t)\|^2_{L^2(\Omega \times (0, 1))} - \left[\frac{\varepsilon}{2} - \frac{\mu_2}{2}\right] \|\mathcal{Z}(x, 1, t)\|^2_2
\]
\[+ \frac{1 - l}{2l} (g \circ \nabla \Phi)(t) + \frac{1}{2} (g' \circ \nabla \Phi)(t). \tag{113}\]
Choosing \(\varepsilon > 0\) small enough, there exist \(\theta_1 > 0\) and \(\theta_2 > 0\) such that
\[
\mathcal{G}'(t) \leq - \theta_1 \varpi(t) + \theta_2 (g \circ \nabla \Phi)(t). \tag{114}\]
Then by (H3), we have
\[
\zeta_0 \mathcal{G}'(t) \leq \zeta_0 \theta_1 \varpi(t) + \zeta_0 \theta_2 (g \circ \nabla \Phi)(t)
\leq \zeta_0 \theta_1 \varpi(t) - \theta_2 (g' \circ \nabla \Phi)(t)
\leq - \zeta_0 \theta_1 \varpi(t) - 2 \theta_2 \varpi'(t). \tag{115}\]
Setting \(\mathcal{K}(t) = \zeta_0 \mathcal{G}(t) + 2 \theta_2 \varpi(t)\), then \(\mathcal{K}(t)\) is equivalent to \(\varpi(t)\). By equation (115), we have
\[
\mathcal{K}'(t) \leq - \zeta_0 \theta_1 \varpi(t) \leq - \zeta_0 \theta_3 \mathcal{K}(t). \tag{116}\]
Therefore \(\mathcal{K}(t)\) satisfies the following estimates
\[
\mathcal{K}(t) \leq \mathcal{K}(0) e^{-\zeta_0 \theta_3 t}. \tag{117}\]
Since \(\mathcal{K}(t) \sim \varpi(t)\), it follows
\[
\varpi(t) \leq C \varpi(0) e^{-\zeta_0 \theta_3 t}. \tag{118}\]
Using (106), we have
\[
\mathcal{M}(t) - \frac{1}{2} \left[\int_0^t g(s) \, ds \|\nabla \Phi(t)\|^2_2 - (g \circ \nabla \Phi)(t)\right] \leq C \mathcal{M}(0) e^{-\zeta_0 \theta_3 t}. \tag{119}\]
Therefore, we obtain that

\[ M(t) \leq C M(0) e^{-\zeta_0 t} + \frac{1}{2} \int_0^t g(s) ds \| \nabla \Phi(t) \|^2_2 - (g \circ \nabla \Phi)(t) \]

\[ \leq C M(0) e^{-\zeta_0 t} + \frac{1}{2} \int_0^t g(s) ds \| \nabla \Phi(t) \|^2_2 \]

\[ \leq C M(0) e^{-\zeta_0 t} + \frac{1}{2} \int_0^t g(s) ds \sup_{0 \leq s \leq t} \| \nabla \Phi(s) \|^2_2. \] (120)

Setting

\[ b(t) = C e^{-\zeta_0 t}, \quad c(t) = \frac{1}{2} \int_0^t g(s) ds, \] (121)

we can obtain the quasi stability inequality in (101). This means Lemma 3.13 is proved.

**Proof of Theorem 2.5.** Combining Lemmas 3.13 to verifying asymptotic compactness, 3.12 for dissipation, we can derive that the gradient system for (19) has a global attractor \( \mathcal{A} \). The Theorem 2.5 has been proved.

4. Further research and outlook. In this present paper, we have studied the stability and dynamic systems for a viscoelastic system with varying delay, the effect of damping and delay terms on the stability. However, the effect and balance among memory, delay and damping is still open.

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