Quantum Diffusive Magneto-transport in Massive Dirac Materials with Chiral Symmetry Breaking

Bo Fu, Huan-Wen Wang, and Shun-Qing Shen

Department of Physics, The University of Hong Kong, Pokfulam Road, Hong Kong, China

Massive Dirac fermions break the chiral symmetry explicitly and also make the Berry curvature of the band structure non-Abelian. By utilizing the Green’s function technique, we develop a microscopic theory to establish a set of quantum diffusive equations for massive Dirac materials in the presence of electric and magnetic fields. It is found that the longitudinal magnetoresistance is always negative and quadratic in the magnetic field, and decays quickly with the mass. The theory is applicable to the systems with non-Abelian Berry curvature and resolves the puzzles of anomalous magnetotransport properties measured in topological materials.

Introduction. Symmetries and their corresponding conservation laws play an important role in understanding the fundamental nature of matter. However, a classical conservation law might turn out to be violated in its quantized version, i.e., the so-called quantum anomaly [1, 2]. A well-known example is that the massless relativistic Dirac fermions or Weyl fermions in three spatial dimensions possess the chiral anomaly [3, 4]. In 1983, Nielsen and Ninomiya [5] proposed that the chiral anomaly of the Weyl fermions could be realized in the Weyl semimetals based on the picture of the Landau levels of the Weyl fermions in a finite magnetic field as shown in Fig. 1(left), and a negative magnetoresistance is regarded as a substantial signature of the effect. Since then, a lot of theoretical approaches have been developed for the anomaly-induced magnetoresistivity for massless Weyl fermions [6–11]. Recent advances in topological materials demonstrate a series of topological materials may host the chiral quasi-particles [12–19], which provide a practical route to detect the signatures of the purely quantum mechanical effect. The longitudinal negative magnetoresistance has been reported experimentally in a large class of topological materials [20–28]. However, a puzzle arises as some topological materials with negative magnetoresistance are actually not Weyl semimetals: for example, ZrTe$_3$ [28–31] and Cd$_2$As$_3$ [32] actually have a tiny direct band gap, and Bi$_2$Se$_3$ is a typical topological insulator [33–35]. It is known that massive Dirac fermions break the chiral symmetry explicitly. Another direct consequence of the chiral symmetry breaking is that the Berry curvature of the band structure is non-Abelian [36–39]. Thus it becomes an open issue whether the measured negative magnetoresistance could be still attributed to the chiral anomaly in the case of massive Dirac fermions. Some mechanisms have been proposed for topological and trivial states without invoking chiral anomaly [40–45]. However, it is desirable to develop a unified quantum magnetotransport theory for the topological materials with either Abelian or non-Abelian Berry curvature to clarify the puzzle.

In this Letter, we develop a quantum diffusive theory for massive Dirac materials in a finite uniform magnetic field by using the diagrammatic perturbation theory. A set of the coupled diffusive equations is derived for all the 16 relevant physical observables in terms of the Dirac matrices. The quantum fluctuation for the axial charge and current density still survives for massive Dirac fermions, and consequently, the longitudinal magnetococonductivity is found to be positive and quadratic in a magnetic field and decays quickly with the mass. A renormalized continuity equation for the axial density and currents is obtained in the presence of the electric and magnetic field. Our calculation also demonstrates that the anomaly correction is rooted in the current vertex renormalization from the axial charge density.

Figure 1. Schematics for chiral anomaly related magnetoresistance mechanism for massless (left panel) and massive (right panel) Dirac materials in the parallel electromagnetic field. The occupied and unoccupied states are shown as solid and open dots for the lowest Landau level, respectively, and the color of the dots indicates the averaged chirality ($\langle \gamma^5 \rangle$) as a function of the momentum. For massive case of $m \neq 0$, the states are mixed near $k_z = 0$. The populations for two chiralities are different due to the presence of parallel electric and magnetic field. The black solid line arrows demonstrate the charge transfer driven by the electric field. The exceeding right hand electrons (red solid dots) are scattered back to left (blue open dots) as indicated by the dashed green line arrow, and the relaxation time is characterized by $\tau_a$.
Model Hamiltonian. We start with the Hamiltonian for massive Dirac fermions,
\[ \mathcal{H}_0 = \int d^3x \bar{\Psi}(x) (\not{\partial} - \gamma \hat{\mathbf{v}} + \varepsilon_F \gamma^0 + m^2) \Psi(x), \] (1)
where \( m \) is the Dirac mass, \( \hat{\mathbf{v}} \) is the effective velocity, \( \varepsilon_F \) is the chemical potential, and \( \not{\partial} = -i\hbar{\nabla} \) is the momentum operator. \( \Psi(x) \) is the four-component Dirac spinor with the time-space position four vector \( x^\mu = (t, \mathbf{x}) \), that the Greek indices (\( \mu, \nu \), etc.) run over all the spacetime indices (0, 1, 2, 3). \( \gamma^\mu \) are the Dirac gamma matrices in Weyl representation \( \gamma^0 = \tau^3 \otimes \sigma_0 \) and \( \gamma^i = i\tau^2 \otimes \sigma^i \) (\( i = 1, 2, 3 \)) with \( \tau^i \) and \( \sigma^i \) are the Pauli matrices, acting on the orbital and spin degrees of freedom correspondingly. The chirality operator is \( \gamma^5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3 = \tau^1 \otimes \sigma^0 \). The mass term breaks the chiral symmetry, and also modifies the Nielsen-Ninomiya’s picture for chiral anomaly at a finite field. The two 0th Landau levels are mixed together near the crossing point as shown in Fig. 1(right). The charge tunneling process can be realized through the smoothly connected region, which is analogous to the massless case that the chiral charge pumping is through the infinite Dirac sea [8]. The conservation law for the axial charge is modified to be [10]
\[ \partial_\mu J^{\mu} = 2m^2 \hat{n}_P + \frac{e^4}{2\pi^2\hbar^2} \mathbf{E} \cdot \mathbf{B} \] (2)
where \( \hat{n}_P = \bar{\Psi} \gamma^5 \Psi \) is the pseudo-scalar density, indicating that the axial charges are even not conserved at the classical level in the presence of the Dirac mass. Furthermore, the anomaly term arises as a consequence of the ultraviolet divergence of the “VVA” triangle diagrams which cannot be cured by a finite mass [10]. In a uniform magnetic field, say along the z-direction, the kinetic momentum operator \( \hat{\mathbf{p}} \) is replaced by the canonical momentum operator in Eq. (1), \( \hat{\mathbf{p}} = -i\hbar(\nabla - i\mathbf{A}) \) with the gauge field chosen as \( \mathbf{A} = (-Bx_2, 0, 0) \). In the case the model is solvable, and the energy dispersion becomes discrete to form the Landau levels [13]. The Green’s functions for the free Dirac fermions at a magnetic field can be obtained analytically [see Sec. S1 in [17]].

Using the five Dirac gamma matrices \( \gamma^\mu (\mu = 0, 1, 2, 3 \text{ and } 5) \) and their descendants, we can define 16 physical quantities as shown in Table I. The enlarged (pseudospin \( \otimes \) spin) gamma matrices will allow us to obtain a microscopic theory of diffusive transport for all the possible coupled physical observables in the presence of an external field. The Dirac structure of the Hamitltonian also allows various types of disorder. For simplicity, we only concentrate on impurities with time reversal and parity invariance
\[ \mathcal{H}_{\text{dis}} = \int d^3x \left[ V(x) \bar{\Psi}(x) \gamma^0 \Psi(x) + V_m(x) \bar{\Psi}(x) \Psi(x) \right], \] (3)
which corresponds to the random chemical potential and mass respectively. All the disorders are quenched, random variables behaving as white noises, \( \langle \langle V(x)V(x') \rangle \rangle = \Delta \delta^3(\mathbf{x} - \mathbf{x}') \) and \( \langle \langle V_m(x)V_m(x') \rangle \rangle = \Delta m \delta^3(\mathbf{x} - \mathbf{x}') \), and further assume the random chemical potential dominates the elastic scattering processes \( \Delta \gg \Delta m \). The self-energy is calculated in the Born approximation and we neglect the imaginary part of self-energy \( \Im \Sigma_R = -\frac{\pi}{2k_F} (\Delta + \Delta m)(\gamma^0 + \eta \mathbf{1}) \), where \( \rho = k_F^2/(2\pi^2\hbar v_F) \) is the density of states with Fermi wavevector \( k_F = \sqrt{\mathbf{p}^2 + m^2v_F^2}/\hbar \) and Fermi velocity \( v_F = (\hbar^2k_F^2)/(\rho \pi^2) \). The orbital polarization is defined by \( \eta \equiv \langle \gamma^0 \rangle = m \epsilon^2/\rho v_F \). Thus the total relaxation time (or quasiparticle lifetime) is given by \( \tau = h/|\pi\rho(\Delta + \Delta m)(1 + \eta^2)| \) and the random mass induced relaxation time is \( \tau_m = h/|\pi\rho m(1 + \eta^2)| \).

Quantum diffusive equations in the real space. With the help of Table I, we can introduce the 16-dimensional vectors \( \mathbf{S}_A(x) = d_A \bar{\Psi}(x) \gamma^A \Psi(x) \) for all the possible physical observables in terms of the Dirac matrices, where \( d_A = e \) for the charge density operators (the first four quantities in Table I) and \( d_A = ev \) for the current operators. To investigate the response to the external potential \( \mathcal{A}(x) \), we consider the generic external perturbation \( \mathcal{H}(t) = \sum_A d^A \oint \mathbf{S}_A(x) \mathcal{A}_A(x) \) with \( \mathcal{A}_A \) are also 16-dimensional vectors. The observables can be evaluated within the framework of the linear response theory [52], \( \mathbf{S}_A \approx \mathbf{S}_A^{(0)} + \mathbf{S}_A^{(1)} + \mathcal{O}(A^2) \). \( \mathbf{S}_A^{(0)} = -d_A \text{Tr}[\gamma^A G(x, x)] \) is the zeroth order term in \( \mathcal{H}_1 \) and \( G(x, x') \) is the fermion propagator for \( \mathcal{H}_0 \), \( \mathbf{S}_A^{(1)} = \oint d^4x' \gamma^A G(x, x') \mathcal{A}_A(x') \) is the first order response to \( \mathcal{H}_1 \), with \( \gamma^A_{AB}(x, x') \) is the retarded response function which can be evaluated by analytical continuation of the imaginary time expression. In order to establish a set of the diffusion equations to describe the coupled dynamics of all the physical quantities, we

| table I. Various types of physical quantities and disorder represented by fermionic bilinears (\( i = 1, 2, 3 \)), their symmetries under time-reversal (\( T \)), parity (\( \mathcal{P} \)), and continuous chiral rotation (\( C \)). The time-reversal symmetry \( T \) is generated by an anti-unitary operator \( \gamma^1 \gamma^5 \mathcal{K} \), where \( \mathcal{K} \) is complex conjugation, such that \( T^2 = -\mathbf{1} \). The parity operator is generated by \( \gamma^5 \). Here \( \gamma^0 \) and \( \gamma^i \) signifies even and odd under a symmetry operation, respectively. And, we use the Latin capital letters \( A, B, ... \) for indices when the index runs through the entire hypercomplex system from 1 to 16. |
need to consider the vertex renormalization due to the multi-scattering which appears perturbatively as a series of impurity line ladder diagrams, in which only combinations of retarded and advanced Green’s functions, having poles on opposite sides of the real axis, will contribute. As a consequence, the bare vertex $\gamma^B$ in the linear’s response theory should be replaced by the dressed vertex $\Gamma^B(x, x'; \omega)$, which satisfies the Bethe-Salpter equation. By expanding the renormalized vertex $\Gamma^B$ in terms of the Dirac matrix $\gamma^C$, we yield a $16 \times 16$ matrix $\Gamma^B$ with its elements defined as $\overline{\Gamma^{BC}} = \frac{1}{2} \text{Tr} [\Gamma^B \gamma^C]$. In the diffusive regime or hydrodynamic regime, the spatial variations of $\Gamma(x, x'; \omega)$ are small on the scale of the mean free path $\ell_c = \frac{v_F}{\nu}$. Therefore, we expand $\Gamma(x_1, x'; \omega)$ about $x_1 = x$: $\Gamma(x_1, x'; \omega) \approx \Gamma(x, x'; \omega) + (x_1 - x) \cdot \nabla_x \Gamma(x, x'; \omega) + \frac{1}{6} (x_1 - x)^2 \nabla^2_x \Gamma(x, x'; \omega)$ and substitute it into the Bethe-Salpter equation. Finally, we obtain the differential equations for $\tilde{\Gamma}(x, x'; \omega)$, $\partial_x^{-1} \tilde{\Gamma}^T(x, x'; \omega) = \mathcal{W}^{-1} \delta(x - x')$ with $\mathcal{W}^{-1}$ is the diffusion operator and $\mathcal{W}$ is an impurity related diagonal matrix $\gamma^C$. The linear response $S^{(1)}(x, \omega)$ can be expressed in terms of $(\tilde{\Gamma}(x, x'; \omega), D^{-1})$:

$$\mathcal{W}^{-1} \delta^{(1)}(x, \omega) = \left( \begin{array}{cc} i\omega + \frac{\Delta}{\tau} - D \partial_x^2 & \frac{\varphi_1}{\tilde{\Gamma}_v \varphi_2} (1 - \eta^2)(i\omega + \frac{\Delta_0}{\tau} - D \partial_x^2) \\ \frac{\varphi_0}{\tilde{\Gamma}_v \varphi_2} \left( \eta(i\omega - \frac{\Delta}{\tau} - D \partial_x^2) \right) & \frac{\varphi_0}{\eta(1 - \eta^2)} \\ \frac{\varphi_0}{\eta(1 - \eta^2)} \lambda \varphi_1 & \frac{\varphi_0}{\eta(1 - \eta^2)} \lambda \varphi_1 \end{array} \right).$$

The coefficient $\varphi = \frac{1}{2} (\frac{\nu}{\pi \rho \ell_B})^2$ with the magnetic length $\ell_B = \sqrt{\hbar / eB}$ and the dimensionless diffusion channel relaxation rates are: $\Lambda_0 = \eta^2, \Lambda_\beta = \frac{1}{\eta}, A_3 = 2(\Delta + 2\Delta_m + \eta^2(2\Delta + \Delta_m))$ and the axial relaxation rate is $\Lambda_{ax} = 2(\Delta + \Delta_m)(1 - \eta^2)$ from the chiral symmetry breaking. By substituting $\varphi$ into Eq. (1) and using the explicit form of $\mathcal{W} = \Delta \mathbf{1}_4 + \Delta_\beta \mathbf{1}_4 \delta (1, -1, 1, -1)$ for this $4 \times 4$ case, we arrive at the coupled charge-current dynamics equations in the presence of electromagnetic field. By transforming into frequency-momentum space, the coupled charge-current equations can be solved in the diffusive regime ($\omega, D \tau q^2 \ll 1$):

$$\delta J^0 = \frac{\varepsilon_F}{v_B k_F} \frac{\omega}{i\omega + D^* q^2} \frac{e^3}{2 \pi^2} EBr_a;$$

$$\delta J^3 = \frac{i\omega \sigma_D E}{i\omega + D^* q^2} \left( 1 + \frac{3 \tau_a}{4 \tau^*} k_F^4 \eta q^2 D^* \right);$$

$$\delta J^0 = \frac{i\omega \sigma_D E}{i\omega + D^* q^2} \left( 1 + \frac{3 \tau_a}{4 \tau^*} k_F^4 \eta q^2 D^* \right) \frac{i\omega}{i\omega + D^* q^2};$$

$$\delta n_\beta = \eta \delta J^\beta,$$

where $D^* = 3 \frac{1}{\Delta + 2\Delta_m} \frac{(1 + \eta^2)\Delta + \Delta_m}{\eta(1 - \eta^2)} D$ is the renormalized diffusion coefficient with the classical diffusion constant $D = v_B^2 \tau / 3$, and $\sigma_D = 2e^2 \rho D^*$ is the Drude conductivity. We have introduced the ratio $\frac{\tau^*}{\tau_a} = \frac{1}{\frac{\tau}{\tau_a}}$, where the axial relaxation time $\tau_a$ describes the attenuation time of the axial charge when it propagates in the disordered medium and $\tau^* = \tau(1 + \eta^2)$.

The solution in Eq. (7) gives the dynamic longitudinal conductivity $\sigma_{zz}(\omega, q, B) = \delta J^3(\omega, q, B)/E(\omega, q)$ in a finite magnetic field. In the "slow limit", $\lim_{q \to 0} \lim_{B \to 0} (\omega / q^2 D^*) \to 0$, such that the perturbing potential is nearly constant on the timescale $1/q^2 D^*$. Consequently, in the thermodynamic equilibrium no current will be generated: $\lim_{q \to 0} \lim_{B \to 0} \sigma_{zz}(\omega, q, B) = 0$, which is also a requirement of the gauge invariance that a purely longitudinal and static vector potential cannot induce any physical current. In the "rapid limit", $\lim_{q \to 0} \lim_{B \to 0} (\omega / q^2 D^*/\omega) \to 0$. In this case, we obtain the remarkable result, $\lim_{q \to 0} \lim_{B \to 0} \sigma_{zz}(\omega, q, B) = \sigma_D + \sigma_{CA}(\eta, B)$, with the anomaly-induced magnetoconductivity as

$$\sigma_{CA}(\eta, B) = \frac{3 \tau_a}{4 \tau^*} \frac{1}{k_F^4 \eta q^2 D^*} \left( \frac{B}{B_F} \right)^2 \sigma_D. \quad (10)$$

The magnetoconductivity is positive and quadratic in $B$. The longitudinal magnetoconductance is $A$. A general solution of the quantum diffusive equations is quite complicated. As an application to explore the longitudinal magnetoconductance, we focus on the linear response for the electric field also along the z-direction. The perturbation part of the Hamiltonian is $\mathcal{H}(t) = d^2 \mathbf{X}_0 \psi(x) \gamma^0 \psi(x)$. In this case, the diffusion operator $\mathcal{D}^{-1}$ can be reduced into a block diagonal form: among all the 16 physical quantities, only 4 observables we are interested in are coupled together in the quantum diffusive regime. Thus we extract the following coupled $4 \times 4$ sub-matrix of $\mathcal{D}^{-1}$ which is spanned by $(J^0, J^a_0, n_\beta, J^3)$:

$$\mathcal{W}^{-1} \delta^{(1)}(x, \omega) = \frac{2}{\pi} \left[ \mathcal{W}^{-1} - \mathcal{D}^{-1} \right] \mathcal{W}^{-1} i\omega A(x, \omega). \quad (4)$$

The longitudinal magnetoconductance is $A$. A general solution of the quantum diffusive equations is quite complicated. As an application to explore the longitudinal magnetoconductance, we focus on the linear response for the electric field also along the z-direction. The perturbation part of the Hamiltonian is $\mathcal{H}(t) = d^2 \mathbf{X}_0 \psi(x) \gamma^0 \psi(x)$. In this case, the diffusion operator $\mathcal{D}^{-1}$ can be reduced into a block diagonal form: among all the 16 physical quantities, only 4 observables we are interested in are coupled together in the quantum diffusive regime. Thus we extract the following coupled $4 \times 4$ sub-matrix of $\mathcal{D}^{-1}$ which is spanned by $(J^0, J^a_0, n_\beta, J^3)$:

$$\mathcal{W}^{-1} \delta^{(1)}(x, \omega) = \frac{2}{\pi} \left[ \mathcal{W}^{-1} - \mathcal{D}^{-1} \right] \mathcal{W}^{-1} i\omega A(x, \omega). \quad (4)$$

The longitudinal magnetoconductance is $A$. A general solution of the quantum diffusive equations is quite complicated. As an application to explore the longitudinal magnetoconductance, we focus on the linear response for the electric field also along the z-direction. The perturbation part of the Hamiltonian is $\mathcal{H}(t) = d^2 \mathbf{X}_0 \psi(x) \gamma^0 \psi(x)$. In this case, the diffusion operator $\mathcal{D}^{-1}$ can be reduced into a block diagonal form: among all the 16 physical quantities, only 4 observables we are interested in are coupled together in the quantum diffusive regime. Thus we extract the following coupled $4 \times 4$ sub-matrix of $\mathcal{D}^{-1}$ which is spanned by $(J^0, J^a_0, n_\beta, J^3)$:

$$\mathcal{W}^{-1} \delta^{(1)}(x, \omega) = \frac{2}{\pi} \left[ \mathcal{W}^{-1} - \mathcal{D}^{-1} \right] \mathcal{W}^{-1} i\omega A(x, \omega). \quad (4)$$

The longitudinal magnetoconductance is $A$. A general solution of the quantum diffusive equations is quite complicated. As an application to explore the longitudinal magnetoconductance, we focus on the linear response for the electric field also along the z-direction. The perturbation part of the Hamiltonian is $\mathcal{H}(t) = d^2 \mathbf{X}_0 \psi(x) \gamma^0 \psi(x)$. In this case, the diffusion operator $\mathcal{D}^{-1}$ can be reduced into a block diagonal form: among all the 16 physical quantities, only 4 observables we are interested in are coupled together in the quantum diffusive regime. Thus we extract the following coupled $4 \times 4$ sub-matrix of $\mathcal{D}^{-1}$ which is spanned by $(J^0, J^a_0, n_\beta, J^3)$:

$$\mathcal{W}^{-1} \delta^{(1)}(x, \omega) = \frac{2}{\pi} \left[ \mathcal{W}^{-1} - \mathcal{D}^{-1} \right] \mathcal{W}^{-1} i\omega A(x, \omega). \quad (4)$$
and $B_F = \frac{e^2}{4\pi^2}\Delta F$. In the massless case of $\eta = 0$, only the impurities which break the chiral symmetry (the impurity matrix anticommutes with the chiral symmetry operator) can cause the scattering between different chiralities (nodes) control the axial relaxation time, and we may reproduce the previous results from the semiclassical theory $\lim_{\eta \to 0} \sigma_{CA}(\eta, B) = \frac{3}{4\pi^2 \eta} \left( \frac{\text{Re}B}{\eta} \right)^2 \frac{\pi}{\varepsilon_F} \tau_D$. [3]

In a massive case of $\eta \neq 0$, the chiral symmetry is broken explicitly due to the Dirac mass $m$, and the eigenstates near the Fermi level mix the chiralities. The disorder with the chiral symmetry (e.g. the chemical potential randomness) can cause the backscattering between opposite helicity, giving an axial relaxation time proportional to the inverse of $\eta^2$. As shown in Eq. (10), when the carrier density of the system is fixed, the relative magnetoconductivity $\sigma_{CA}(\eta, B) / \sigma_D$ is only determined by two parameters, the relative impurity strength $\tau / \tau_m = \Delta_m / (\Delta + \Delta_m)$ and the orbital polarization $\eta$. As shown in Fig. 2(a), when the relative impurity strength $\tau / \tau_m$ is fixed, the relative anomaly-induced magnetoconductivity $\sigma_{CA}(\eta, B) / \sigma_{CA}(\eta = 0)$ is suppressed as the parameter $\eta$ grows. When $\tau / \tau_m$ goes to zero, the mass becomes dominant in chiral symmetry breaking, and the anomaly-induced magnetoconductivity quenches to zero more quickly as $\eta$ grows. We also plot the absolute value of $\sigma_{CA}$ by using the realistic parameters according to Ref. 28. As shown in Fig. 2 (b), $\sigma_{CA}$ is strongly suppressed as $\tau / \tau_m$ and $\eta$ grows. Experimental observation of the anomaly-induced magnetoconductivity requires a long axial current relaxation time $\tau_a$, which stems from the near conservation of chiral charge, and a lower carrier density. A finite mass cannot forbid such an effect, but only suppress its contribution. From Eq. (9), in the "rapid limit", the chirality imbalance $\delta J^{\text{rapid}} = \frac{\eta}{\varepsilon_F} \frac{E B a}{\Delta F}$ is also self-consistently obtained, further confirming our calculations. The chiral anomaly-induced magnetoconductivity is rooted in the current vertex renormalization from the axial charge density in the presence of parallel electromagnetic field [see Sec. S5 in 17]. Therefore, we could not obtain such an anomaly correction only in the Drude approximation by considering the bubble diagram.

**Anomaly induced magneto field correction to the diffusive motion of the electrons.** The conservation of total charge also makes an anomaly-induced correction to the dynamical polarization function or the density-density response function $\chi_{00}(\omega, \mathbf{q}, \mathbf{B})$. The gauge invariance poses some constraints on the elements of the response function: $\chi_{00} = -\frac{2}{3} \chi_{03}$ with $\chi_{03} = \frac{2}{3} \chi_{00}$, we yield the following compact form for the polarization function from Eq. (8),

$$\chi_{00}(\omega, \mathbf{q}, \mathbf{B}) = 2 \frac{\eta^2 \mathcal{D}(\mathbf{B})}{\eta^2 \mathcal{D}(\mathbf{B}) + i \omega}, \tag{11}$$

where $\mathcal{D}(\mathbf{B}) = \mathcal{D}^* (1 + \frac{3}{4} \frac{\eta^2}{\Delta^2} \frac{1}{\varepsilon_F})$ is the field-dependent diffusion constant. This factor $(1/[\eta^2 \mathcal{D}(\mathbf{B}) + i \omega])$ is known as the "diffusion pole", which emerges from the repeated elastic scattering (the ladder diagram), and also reflects the conservation of total charge. Many-body effects are directly associated with this diffusion pole. For example, when the electron-electron interaction cannot be neglected, each electron will be influenced by the electronic density fluctuation from other electrons described by $\chi_{00}$. As a consequence, the spectral and transport properties are modified by the interaction effect. One way to detect the effect is to measure the tunnel conductance, which directly reflects the variation of the density of states $\delta \rho$ due to the Coulomb interaction. The reduction of the tunnel conductance is given by $\delta G_t(V)/G_i = \delta \rho(V)/\rho \propto (\sqrt{\varepsilon_F} \delta(B) - C)/\mathcal{D}(\mathbf{B})$ where $V$ is the voltage difference between two leads and C is a constant independent of the bias. Since the change is maximal around the Fermi energy $\varepsilon_F$, the tunneling spectrum will display a downward cusp at the Fermi level, i.e., the so-called zero-bias anomaly. Due to the magnetic field dependence of the diffusion constant, we can expect the zero bias downward cusp should be weakened under the magnetic field. Furthermore, this interaction correction in conductivity shows a strong dependence on the configuration of the electric and magnetic field in sharp contrast with the contribution from weak localization, providing a fruitful way to distinguish the two effects.

**Discussion and conclusion** The pseudo-scalar density $\tilde{n}_P$ modifies directly the continuity equation for the axial charge and current density in Eq. (9). In the presence of the electric and magnetic field, it is found that the expectation value of the pseudo-scalar density has the form $\langle \tilde{n}_P \rangle = \frac{B^2}{\sqrt{\varepsilon_F} \varepsilon_F} (\frac{m_F^2}{\alpha} - 1) \frac{e^3}{16\pi^2} E \cdot B$, which vanishes when $m = 0$. Thus the anomaly equation is reduced to

$$\partial_{\mu} \tilde{T}^a_{\mu} = \frac{\varepsilon_F}{\hbar v_F} \frac{e^3}{2\pi^2 \hbar^2} E \cdot B. \tag{12}$$

![Figure 2. (a) The universal behavior of the relative anomaly magnetoconductivity correction for different $\tau/\tau_m$ as a function of the parameter $\eta = mv^2/\varepsilon_F$. (b) The anomaly-related positive magneto conductivities at $B = 1T$ are plotted as a function of Dirac mass $mv^2$ for several different $\tau/\tau_m$ with the chemical potential $\varepsilon_F = 90 meV$ and Ohm resistance at zero field $R = \sigma^{-1} = 1.2 m\Omega \cdot cm$. The other parameter is chosen as $\hbar = 6 \times 10^{-5} meV \cdot cm$.](image-url)
As the chemical potential $\varepsilon_F = \sqrt{(v \hbar k_F)^2 + m^2 v^4}$, the prefactor $\frac{\varepsilon_F}{\hbar v k_F} = \sqrt{1 + \left( \frac{m v}{\hbar k_F} \right)^2}$ is always larger than 1 for a finite mass. Assume the electric field is caused by a spatially varying chemical potential $\varepsilon_F$. Integrating the anomaly equation leads to the dissipationless axial current $\mathbf{J}^a = -e \mathbf{B} / \hbar k_F \varepsilon_F^2$ for massive Dirac fermions, i.e., the so-called the chiral separation effect. The axial current will induce a chiral charge separation, i.e., a nonzero chiral chemical potential $\mu_5$. In a homogeneous case, the chiral charge transfer rate is $\partial_t J^a = \frac{\varepsilon_F^2}{\hbar v k_F^2} \mathbf{E} \cdot \mathbf{B}$ and the chiral separation becomes nonzero $\mu_5$. The energy cost for the energy transferring $\partial_t J^a \mu_5$ should be supplied by the Joule heating, $\mathbf{J} \cdot \mathbf{E}$ [56]. In this way it follows that $\mathbf{J} = \frac{\varepsilon_F^2}{\hbar v k_F^2} \mathbf{E} \mu_5 \mathbf{B}$, i.e., the chiral magnetic effect for massive Dirac fermions. In the relaxation time approximation, we can obtain the anomaly-induced magneto-conductivity in Eq. (10).

In short, we have established a set of the quantum diffusive equations for massive Dirac fermions as an application, we calculated the longitudinal magnetoresistance and the dynamical polarization functions. The longitudinal magnetoresistance is always negative for both massive Dirac fermions and Weyl fermion and decays quickly with the mass or energy gap. It resolves the puzzle of the longitudinal negative magnetoresistance observed in several topological Dirac semimetals and even in topological insulators. Finally, it is worth stressing that this quantum diffusive theory is applicable to the systems with either the Abelian or non-Abelian Berry curvature.

ACKNOWLEDGMENTS

This work was supported by the Research Grants Council, University Grants Committee, Hong Kong under Grant No. 17301116.

[1] W. A. Bardeen, Phys. Rev. 184, 1848 (1969).
[2] K. Fujikawa, H. Suzuki, Path Integrals and Quantum Anomalies, (Oxford University Press on Demand, 2004), Vol.122.
[3] S. L. Adler, Phys. Rev. 177, 2426 (1969).
[4] J. S. Bell and R. Jackiw, Nuovo Cimento A 60, 47 (1969).
[5] H. Nielsen and M. Ninomiya, Phys. Lett. B 130, 389 (1983).
[6] M. Stephanov and Y. Yin, Phys. Rev. Lett. 109, 162001 (2012).
[7] D. T. Son and B. Z. Spivak, Phys. Rev. B 88, 104412 (2013).
[8] G. Basar, D. E. Kharzeev, and H. U. Yee, Phys. Rev. B 89, 035142 (2014).
[9] A. A. Burkov, Phys. Rev. Lett. 113, 247203 (2014).
[10] A. A. Burkov, Phys. Rev. B 91, 245157 (2015).
[11] H. Z. Lu and S. Q. Shen, Front. Phys. 12, 127201 (2017).
[12] S. Murakami, New J. Phys. 9, 356 (2007).
[13] X. Wan, A. M. Turner, A. Vishwanath, and S. Y. Savrasov, Phys. Rev. B 83, 205101 (2011).
[14] Z. Wang, H. Weng, Q. Wu, X. Dai, and Z. Fang, Phys. Rev. B 88, 125427 (2013).
[15] S. Y. Xu, I. Belopolski, N. Alidoust, M. Neupane, G. Bian, C. Zhang, R. Sankar, G. Chang, Z. Yuan, C. C. Lee et al., Science 349, 613 (2015).
[16] S. M. Huang, S. Y. Xu, I. Belopolski, C. C. Lee, G. Chang, B. Wang, N. Alidoust, G. Bian, M. Neupane, C. Zhang et al., Nat. Commu. 6, 7373 (2015).
[17] B. Q. Lv, H. Weng, B. Fu, X. Wang, H. Miao, J. Ma, S. Richard, X. Huang, L. Zhao, G. Chen, et al., Phys. Rev. X 5, 031013 (2015).
[18] L. Lu, Z. Wang, D. Ye, L. Ran, L. Fu, J. D. Joannopoulos, and M. Soljacic, Science 349, 622 (2015).
[19] N. P. Armitage, E. J. Mele, and A. Vishwanath, Rev. Mod. Phys. 90, 015001 (2018).
[20] H. J. Kim, K. S. Kim, J. P. Wang, M. Sasaki, N. Satoh, A. Ohnishi, M. Kitaura, M. Yang, and L. Li, Phys. Rev. Lett. 111, 246603 (2013).
[21] X. Huang, L. Zhao, Y. Long, P. Wang, D. Chen, Z. Yang, H. Liang, M. Xue, H. Weng, Z. Fang et al., Phys. Rev. X 5, 031023 (2015).
[22] J. Xiong, S. K. Kushwaha, T. Liang, J. W. Krizan, M. Hirschberger, W. Wang, R. J. Cava, and N. P. Ong, Science 350, 413 (2015).
[23] C. Z. Li, L. X. Wang, H. Liu, J. Wang, Z. M. Liao, and D. P. Yu, Nat. Commu. 6, 10137 (2015).
[24] C. L. Zhang, S. Y. Xu, I. Belopolski, Z. Yuan, Z. Lin, B. Tang, G. Bian, N. Alidoust, C. C. Lee, S. M. Huang et al., Nat. Commu. 7, 10735 (2016).
[25] F. Arnold, C. Shekhar, S. C. Wu, Y. Sun, R. D. Dos Reis, N. Kumar, M. Naumann, M. O. Ajeesh, M. Schmidt, A. G. Grushin et al., Nat. Commu. 7, 11615 (2016).
[26] H. Li, H. He, H. Z. Lu, H. Zhang, H. Liu, R. Ma, Z. Fan, S. Q. Shen, and J. Wang, Nat. Commu. 7, 10301 (2016).
[27] S. Liang, J. Lin, S. Kushwaha, J. Xing, N. Ni, R. J. Cava, and N. P. Ong, Phys. Rev. X 8, 031002 (2018).
[28] Q. Li, D. E. Kharzeev, C. Zhang, Y. Huang, I. Pletikosic, A. V. Fedorov, R. D. Zhong, J. A. Schneeloch, G. D. Gu, and T. Valla, Nat. Phys. 12, 550 (2016).
[29] Y. Zhang, C. Wang, L. Yu, G. Liu, A. Liang, J. Huang, S. Nie, X. Sun, Y. Zhang, B. Shen et al., Nat. Commu. 8, 15512 (2017).
[30] B. Xu, L. Zhao, P. Marsik, E. Sheveleva, F. Lyzwa, Y. Dai, G. Chen, X. Qiu, and C. Bernhard, Phys. Rev. Lett. 121, 187401 (2018).
[31] J. Mutch, W. C. Chen, P. Went, T. Qian, I. Z. Wilson, A. Andreev, C. C. Chen, and J. H. Chu, Sci. Adv. 5, eaav9771 (2019).
[32] S. Jeon, B. B. Zhou, A. Gyenis, B. E. Feldman, I. Kimchi, A. C. Potter, Q. D. Gibson, R. J. Cava, A. Vishwanath,
and A. Yazdani, Nat. Mater. 13, 851 (2014).

[33] J. Wang, H. Li, C. Chang, K. He, J. S. Lee, H. Lu, Y. Sun, X. Ma, N. Samarth, S. Shen et al., Nano Res. 5, 739 (2012).

[34] S. Wiedmann, A. Jost, B. Fauque J. Van Dijk, M. Mei- jer, T. Khouri, S. Pezzini, S. Grauer, S. Schreyeck et al., Phys. Rev. B 94, 081302(R) (2016).

[35] O. Breunig, Z. Wang, A. Taskin, J. Lux, A. Rosch, and Y. Ando, Nat. Commu. 8, 15545 (2017).

[36] R. Shindou and K. I. Imura, Nucl. Phys. B 720, 399 (2005).

[37] M. C. Chang and Q. Niu, J. Phys.: Condens. Matter 20, 193202 (2008).

[38] C. P. Chuu, M. C. Chang, and Q. Niu, Solid State Commun. 150, 533 (2010).

[39] J. W. Chen, J. Y. Pang, S. Pu, and Q. Wang, Phys. Rev. D 89, 094003 (2014).

[40] P. Goswami, J. Pixley, and S. D. Sarma, Phys. Rev. B 92, 075205 (2015).

[41] Y. Gao, S. A. Yang, and Q. Niu, Phys. Rev. B 95, 165135 (2017).

[42] X. Dai, Z. Z. Du, and H. Z. Lu, Phys. Rev. Lett. 119, 166601 (2017).

[43] A. V. Andreev and B. Z. Spivak, Phys. Rev. Lett. 120, 026601 (2018).

[44] H. W. Wang, B. Fu, and S. Q. Shen, Phys. Rev. B 98, 081202 (R) (2018).

[45] B. Fu, H. W. Wang, and S. Q. Shen, Phys. Rev. Lett. 122, 246601 (2019).

[46] A. Zee, Quantum Field Theory in a Nutshell, (Princeton university press, 2010), Vol. 7.

[47] See Supplemental Material at [URL to be added by publisher] for details of (S1) the Green’s function in Landau level representation, (S2) the diffusion operator in the real space, (S3), the evaluation of the diffusion operator in the Landau level representation, (S4) the calculation of the pseudoscalar density, and (S5) the anomalous coupling between the axial charge and vector current, which includes Refs. [48–50].

[48] V. A. Miransky and I. A. Shovkovy, Phys. Rep. 576, 1 (2015).

[49] R. H. Fang, J. Y. Pang, Q. Wang, and X. N. Wang, Phys. Rev. D 95, 014032 (2017).

[50] S. Lin and L. Yang, Phys. Rev. D 98, 114022 (2018).

[51] G. D. Mahan, Many-particle Physics (Springer Science and Business Media, 2013).

[52] R. Kubo, J. Phys. Soc. Jpn. 12, 570 (1957).

[53] V. Zlatic and R. Monnier, Modern Theory of Thermoelectricity (Oxford University Press, Oxford, 2014).

[54] B. L. Altshuler and A. G. Aronov, Modern Problems in Condensed Matter Sciences, (Elsevier, 1985), Vol. 10.

[55] E. V. Gorbar, V. A. Miransky, I. A. Shovkovy, and X. Wang, Phys. Rev. D 88, 025025 (2013).

[56] S. Q. Shen, Topological Insulators: Dirac Equation in Condensed Matter, 2nd ed. (Springer, Singapore, 2017).