Quark Mass Anomalous Dimension to $O(\alpha_s^4)$

K.G. Chetyrkin $^a, b$

$^a$Institute for Nuclear Research, Russian Academy of Sciences, 60th October Anniversary Prospect 7a, Moscow 117312, Russia

$^b$Max-Planck-Institut für Physik, Werner-Heisenberg-Institut, Föhringer Ring 6, 80805 Munich, Germany

Abstract

We present the results of analytic calculation of the quark mass anomalous dimension to $O(\alpha_s^4)$. 

Corresponding author: K.G. Chetyrkin, address: Max-Planck-Institut für Physik, Werner-Heisenberg-Institut, Föhringer Ring 6, 80805 Munich, Germany
e-mail: chet@mppmu.mpg.de
1 Introduction

The quark masses depend on the renormalization scheme and, within a given one, on the renormalization scale. The dependence on the latter is usually referred to as “running” and is governed by the quark mass anomalous dimension, $\gamma_m$.

The two-loop anomalous dimension is known for long [1]; the three-loop result is also available from Refs. [2,3].

The quark mass anomalous dimension $\gamma_m$ is defined as

$$\mu^2 \frac{d}{d\mu^2} m_{\mu, m_B} = m \gamma_m(a_s) \equiv -m \sum_{i \geq 0} \gamma_i a_s^{i+1},$$

(1)

where $a_s = \alpha_s / \pi = g^2 / (4\pi^2)$, $g$ is the strong coupling constant. To calculate $\gamma_m$ one needs to find the so-called quark mass renormalization constant, $Z_m$, which is defined as the ratio of the bare and renormalized quark masses, viz.

$$Z_m = \frac{m_B}{m} = 1 + \sum_{i,j}^{0 < j \leq i} (Z_m)_{ij} \left( \frac{\alpha_s}{\pi} \right)^i \frac{1}{\epsilon^j}.$$  

(2)

Within the MS scheme [4] the coefficients $(Z_m)_{ij}$ are just numbers, with $D = 4 - 2\epsilon$ standing for the space-time dimension.

In practice $Z_m$ is usually computed from the vector and scalar parts of the quark self-energy $\Sigma_V(p^2)$ and $\Sigma_S(p^2)$. In our convention, the bare quark propagator is proportional to $[\not{p} (1 + \Sigma^0_V(p^2)) - m_q (1 - \Sigma^0_S(p^2))]^{-1}$. Requiring the finiteness of the renormalized quark propagator and keeping only massless and linear in $m_q$ terms, one arrives at the following recursive equations to find $Z_m$

$$Z_m Z_2 = 1 + K_\epsilon \left\{ Z_m Z_2 \Sigma^0_V(p^2) \right\}, \quad Z_2 = 1 - K_\epsilon \left\{ Z_2 \Sigma^0_S(p^2) \right\},$$

(3)

where $K_\epsilon f(\epsilon)$ stands for the singular part of the Laurent expansion of $f(\epsilon)$ in $\epsilon$ near $\epsilon = 0$ and $Z_2$ is the quark wave function renormalization constant. Eqs. (3) express $Z_m$ through massless propagator-type (that is dependent on one external momentum only) Feynman integrals (FI), termed as $p$-integrals below.

In this letter we first present the results of the analytic calculation of the quark mass anomalous dimension to $\mathcal{O}(\alpha_s^4)$. Then we apply it to evaluate the running of a heavy quark mass in QCD with the number of active flavours $n_f$ varied from 3 to 6.
2 Results and discussion

Taken literally, Eqs. (3) require calculation of a host of one-, two-, three-, and four-loop p-integrals to find $Z_m$ to $\mathcal{O}(\alpha_s^4)$. At present a direct analytical calculation of a p-integral with the loop number not exceeding three is a rather easy business. First, there exists an elaborated algorithm — the method of integration by parts of Ref. [5,6] — which allows one to analytically evaluate divergent as well as finite parts of any three-loop p-integral. Second, the algorithm has been neatly and reliably implemented in the language FORM [7] as the package named MINCER in Ref. [8].

The situation with four-loop diagrams is quite different. At present there is simply no way to directly compute the divergent part of a four-loop p-integral. An indirect and rather involved approach to perform such calculations analytically is to use the method of Infrared Rearrangement (IRR) of Ref. [9] enforced by a special technique of dealing with infrared divergences — the $R^\ast$-operation of Refs. [10,11]. Recently, the technique has been dramatically simplified by explicitly resolving all the relevant combinatorics for a particular case of 4-loop diagrams contributing to the correlator of two scalar or vector currents in Refs. [12] and [13] respectively. We have extended these improvements for the problem at hand.

The four-loop diagrams contributing to Eqs. (3) to order $\alpha_s^4$ (about 6000) have been generated with the program QGRAF [14], then globally rearranged to a product of some three-loop p-integrals with a trivial (essentially one-loop) massive Feynman integral and, finally, computed with the program MINCER. All calculations have been done in MS-scheme. As is well-known, every anomalous dimension is the same for MS- and $\overline{\text{MS}}$ schemes [15], thus, the results are also valid for the latter one.

Our result for the quark mass anomalous dimension reads (in evaluating the colour factors we have assumed the case of the colour group $SU_c(N)$; $n_f$ stands for the number of quark flavours)

$$
\gamma_0 = \frac{N^2 - 1}{4N^2} \left[ \frac{3}{2} \right], \quad \gamma_1 = \frac{N^2 - 1}{16N^2} \left\{ -\frac{3}{8} + \frac{203}{24} N^2 + n_f \left[ -\frac{5}{6} N \right] \right\},
$$

$$
\gamma_2 = \frac{N^2 - 1}{64N^4} \left\{ \frac{129}{16} - \frac{129}{16} N^2 + \frac{11413}{216} N^4 
+ n_f \left[ \frac{23}{4} N - \frac{1177}{108} N^3 - 6N \zeta(3) - 6N^3 \zeta(3) \right] + n_f^2 \left[ -\frac{35}{54} N^2 \right] \right\},
$$

$$
\gamma_3 = \frac{N^2 - 1}{256N^4} \left\{ \frac{1261}{128} + \frac{50047}{384} N^2 - \frac{66577}{1152} N^4 + \frac{460151}{1152} N^6 + 21 \zeta(3) \right\}.
$$
\[-\frac{47}{2}N^2 \zeta(3) + 52N^4 \zeta(3) + \frac{1157}{18}N^6 \zeta(3) - 110N^4 \zeta(5) - 110N^6 \zeta(5) + n_f \left[ \frac{37}{6}N + \frac{10475}{216}N^3 - \frac{11908}{81}N^5 - \frac{111}{2}N\zeta(3) - 85N^3 \zeta(3) - \frac{889}{6}N^5 \zeta(3) \right. \\
\left. + 33N^3 \zeta(4) + 33N^5 \zeta(4) - 30N \zeta(5) + 50N^3 \zeta(5) + 80N^5 \zeta(5) \right] \] \\
\[ + n_f^2 \left[ -\frac{19}{27}N^2 + \frac{899}{324}N^4 + 10N^2 \zeta(3) + 10N^4 \zeta(3) - 6N^2 \zeta(4) - 6N^4 \zeta(4) \right] \]
\[ + n_f^3 \left[ -\frac{83}{162}N^3 + \frac{8}{9}N^3 \zeta(3) \right]. \]

Here $\zeta$ is the Riemann zeta-function ($\zeta(3) = 1.202056903 \ldots$, $\zeta(4) = \pi^4/90$ and $\zeta(5) = 1.036927755 \ldots$). With $N = 3$ one gets the following result for QCD

\[ \gamma_0 = 1, \quad \gamma_1 = \frac{1}{16} \left\{ \frac{202}{3} + n_f \left[ -\frac{20}{9} \right] \right\}, \]
\[ \gamma_2 = \frac{1}{64} \left\{ 1249 + n_f \left[ -\frac{2216}{27} - \frac{160}{3} \zeta(3) \right] + n_f^2 \left[ -\frac{140}{81} \right] \right\}, \]
\[ \gamma_3 = \frac{1}{256} \left\{ \frac{4603055}{162} + \frac{135680}{27} \zeta(3) - 8800 \zeta(5) - n_f \left[ -\frac{91723}{27} - \frac{34192}{9} \zeta(3) + 880 \zeta(4) + \frac{18400}{9} \zeta(5) \right. \right. \\
\left. \left. + n_f^2 \left[ \frac{5242}{243} + \frac{800}{9} \zeta(3) - \frac{160}{3} \zeta(4) \right] + n_f^3 \left[ -\frac{332}{243} + \frac{64}{27} \zeta(3) \right] \right\}. \]

Note that in three-loop order we exactly reproduce the known result of Ref. [2,3]. The four-loop term proportional to $n_f^3$ is in agreement to the one found in Ref. [16].

In numerical form $\gamma_m$ reads

\[ \gamma_m = -a_s - a_s^2 (4.20833 - 0.138889n_f) - a_s^3 (19.5156 - 2.28412n_f - 0.0270062n_f^2) - a_s^4 (98.9434 - 19.1075n_f + 0.276163n_f^2 + 0.00579322n_f^3). \]

It is amusing to compare the exact result $\gamma_3^{\text{exact}} = 44.2629 \ldots$ for $n_f = 3$ and a recent estimation of Ref. [17]

\[ \gamma_3^{\text{est}} (n_f = 3) = \frac{\gamma_2^2 (n_f = 3)}{\gamma_1 (n_f = 3)} = 40.6843. \]
For the case of QED with $n_f$ identical fermions our result reads

\[
\begin{align*}
\gamma_{0}^{\text{QED}} &= \frac{3}{4}, \\
\gamma_{1}^{\text{QED}} &= \frac{1}{16} \left\{ \frac{3}{2} + n_f \left[ -\frac{10}{3} \right] \right\}, \\
\gamma_{2}^{\text{QED}} &= \frac{1}{64} \left\{ \frac{129}{2} + n_f \left[ -46 + 48 \zeta(3) \right] + n_f^2 \left[ -\frac{140}{27} \right] \right\}, \\
\gamma_{3}^{\text{QED}} &= \frac{1}{256} \left\{ -\frac{1261}{8} - 336 \zeta(3) + n_f \left[ -\frac{88}{3} + 72 \zeta(3) - 480 \zeta(5) \right] \\
&\quad + n_f^2 \left[ \frac{304}{27} - 160 \zeta(3) + 96 \zeta(4) \right] + n_f^3 \left[ -\frac{664}{81} + \frac{128}{9} \zeta(3) \right] \right\}. 
\end{align*}
\]

The RG equation (1) is usually solved by

\[
\frac{m(\mu)}{m(\mu_0)} = \frac{c(a_s(\mu))}{c(a_s(\mu_0))}.
\]

where

\[
c(x) = (x)^\gamma_0 \left\{ 1 + (\bar{\gamma}_1 - \bar{\beta}_1 \bar{\gamma}_0) x \right. \\
+ \frac{1}{2} \left[ (\bar{\gamma}_1 - \bar{\beta}_1 \bar{\gamma}_0)^2 + \bar{\gamma}_2 + \bar{\beta}_1^2 \bar{\gamma}_0 - \bar{\beta}_1 \bar{\gamma}_1 - \bar{\beta}_2 \bar{\gamma}_0 \right] x^2 \\
+ \frac{1}{6} (\bar{\gamma}_1 - \bar{\beta}_1 \bar{\gamma}_0)^3 + \frac{1}{2} (\bar{\gamma}_1 - \bar{\beta}_1 \bar{\gamma}_0) (\bar{\gamma}_2 + \bar{\beta}_1^2 \bar{\gamma}_0 - \bar{\beta}_1 \bar{\gamma}_1 - \bar{\beta}_2 \bar{\gamma}_0) \\
+ \frac{1}{3} \left( \bar{\gamma}_3 - \bar{\beta}_1^3 \bar{\gamma}_0 + 2 \bar{\beta}_1 \bar{\beta}_2 \bar{\gamma}_0 - \bar{\beta}_3 \bar{\gamma}_0 + \bar{\beta}_1 \bar{\gamma}_1 - \bar{\beta}_2 \bar{\gamma}_1 - \bar{\beta}_1 \bar{\gamma}_2 \right) x^3 + \mathcal{O}(x^4) \right\}.
\]

Here $\bar{\gamma}_i = \gamma_i / \beta_0$, $\bar{\beta}_i = \beta_i / \beta_0$, (i=1,2,3) and $\beta_i$ are the coefficient of the QCD beta-function defined as:

\[
\mu^2 \frac{d}{d\mu^2} \left( \frac{\alpha_s(\mu)}{\pi} \right) \bigg|_{g_b,m_b} = \beta \equiv - \sum_{i \geq 0} \beta_i \left( \frac{\alpha_s}{\pi} \right)^{i+2}.
\]

Equation (15) explicitly demonstrates that the knowledge of the four-loop coefficients $\gamma_3$ and $\beta_3$ is absolutely necessary for the self-consistent running of the quark mass in the cases when the mass-dependent terms of order $\alpha_s^3$ are taken into account. Examples can be found in Refs. [12,17,18].

The four-loop beta-function has been recently analytically computed in Ref. [19] with the
result

\[
\beta_0 = \frac{1}{4} \left( 11 - \frac{2}{3} n_f \right), \quad \beta_1 = \frac{1}{16} \left( 102 - \frac{38}{3} n_f \right),
\]
\[
\beta_2 = \frac{1}{64} \left( \frac{2857}{2} - \frac{5033}{18} n_f + \frac{325}{54} n_f^2 \right),
\]
\[
\beta_3 = \frac{1}{256} \left( \frac{149753}{6} + 3564 \zeta(3) \right) - \left[ \frac{1078361}{162} + \frac{6508}{27} \zeta(3) \right] n_f + \left[ \frac{50065}{162} + \frac{6472}{81} \zeta(3) \right] n_f^2 + \frac{1093}{729} n_f^3.
\]

(17)

Now we are in position to evaluate the c-function numerically with four-loop accuracy. For strange, charm, bottom and top it reads:

\[
c_s(x) = x^{4/9} \left( 1 + 0.895062 x + 1.37143 x^2 + 1.95168 x^3 \right), \quad (n_f = 3),
\]
\[
c_c(x) = x^{12/25} \left( 1 + 1.01413 x + 1.38921 x^2 + 1.09054 x^3 \right), \quad (n_f = 4),
\]
\[
c_b(x) = x^{12/23} \left( 1 + 1.17549 x + 1.50071 x^2 + 0.172478 x^3 \right), \quad (n_f = 5),
\]
\[
c_t(x) = x^{4/7} \left( 1 + 1.39796 x + 1.79348 x^2 - 0.68343 x^3 \right), \quad (n_f = 6).
\]

(18)

**Acknowledgments**

I would like to thank Matthias Steinhauser for the support and careful reading the manuscript.
References

[1] R. Tarrach, Nucl. Phys. B 183 (1981) 384.
[2] O.V. Tarasov, preprint JINR P2-82-900 (1982).
[3] S.A. Larin, Preprint NIKHEF-H/92-18, hep-ph/9302240 (1992); In Proc. of the Int. Baksan School ”Particles and Cosmology” (April 22-27, 1993, Kabardino-Balkaria, Russia) eds. E.N. Alexeev, V.A. Matveev, Kh.S. Nirov, V.A. Rubakov (World Scientific, Singapore, 1994).
[4] G. ’t Hooft, Nucl. Phys. B 61 (1973) 455.
[5] F. V. Tkachov, Phys. Lett B 100 (1981) 65.
[6] K. G. Chetyrkin and F. V. Tkachov, Nucl. Phys. B 192 (1981) 159.
[7] J.A. M. Vermaseren, Symbolic Manipulation with FORM, Version 2, CAN, Amsterdam, 1991.
[8] S.A. Larin, F.V. Tkachov, J.A.M. Vermaseren, Preprint NIKHEF-H/91-18 (1991).
[9] A. A. Vladimirov, Teor. Mat. Fiz. 43 (1980) 210.
[10] K. G. Chetyrkin and V. A. Smirnov, Phys. Lett. B 144 (1984) 419.
[11] K. G. Chetyrkin, preprint MPI-PAE/PTH 13/91 (Munich, 1991).
[12] K.G. Chetyrkin, Phys. Lett. B 390 (1997) 309.
[13] K.G. Chetyrkin, Phys. Lett. B 391 (1997) 402.
[14] P. Nogueira, J. Comput. Phys. 105 (1993) 279.
[15] W.A. Bardeen, A.J. Buras, D.W. Duke and T. Muta, Phys. Rev. D18 (1978) 3998.
[16] D. Espiriu, A. Palanques-Mestre, P. Pascual and R. Tarrach, Z.Phys. C 13 (1982) 153.
[17] K.G. Chetyrkin, D. Pirjol and K. Schilcher, Preprint MZ-TH-96-27, hep-ph/9612394.
[18] K. G. Chetyrkin and J.H. Kuhn, Preprint MPI/PhT/96-84, hep-ph/9609202.
[19] T. van Ritbergen, J.A.M. Vermaseren, and S.A. Larin, Preprint UM-TH-97-01, hep-ph/9701307.