ON POINTWISE GRADIENT ESTIMATES FOR THE COMPLEX MONGE-AMPERE EQUATION

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Abstract

In this note, a gradient estimate for the complex Monge-Ampère equation is established. It differs from previous estimates of Yau, Hanani, Blocki, P. Guan, B. Guan - Q. Li in that it is pointwise, and depends only on the infimum of the solution instead of its $C^0$ norm.

1 Introduction

A priori estimates for complex Monge-Ampère equations are of considerable interest in non-linear partial differential equations and complex geometry. Among them, gradient estimates are somewhat special. In Yau’s classic work on the Calabi conjecture [Y], they could be bypassed, since he was able to derive a priori estimates for the Laplacian $\Delta \varphi$ of the solution $\varphi$, assuming only a priori estimates for $\|\varphi\|_{C^0}$. General linear elliptic theory allows then to control $\|\nabla \varphi\|_{C^0}$ in terms of $\|\Delta \varphi\|_{C^0}$ and $\|\varphi\|_{C^0}$. Subsequently, several more direct estimates for $\|\nabla \varphi\|_{C^0}$ were obtained by Hanani [H], Blocki [B1], P. Guan [G], and B. Guan-Q. Li [GL] which did not require estimating $\|\Delta \varphi\|_{C^0}$ first. This results in an improved dependence on the ambient geometry, together with greater flexibility for various generalizations, as we shall describe in more detail in §2 below. In the case of the Dirichlet problem, a different argument was given by Chen [C1] using blow-ups.

The purpose of this note is to present a new gradient estimate, which has at least two distinct advantages. The first is that it depends only on $\inf \varphi$, and not on $\|\varphi\|_{C^0}$ as in all earlier gradient estimates. The second is that it is a more precise pointwise estimate for $\nabla \varphi(z)$ which remains valid even when $\|\nabla \varphi\|_{C^0}$ is unbounded. Such features were essential in the construction in [PS4] of geodesic rays starting from a test configuration, and in fact, a version of the present gradient estimates had been established there. That formulation was however somewhat obscured by the particular geometric set-up of [PS4], and it seems worthwhile to extract a simpler and more general gradient estimate, even though the proof is essentially the same, in the expectation that it will find other uses.

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2 The gradient estimate

Let \((M, \omega)\) be a compact Kähler manifold with smooth boundary \(\partial M\) (which may be empty) and complex dimension \(n\). We consider the Monge-Ampère equation on \(\bar{M}\)

\[
(\omega + \frac{i}{2} \partial \bar{\partial} \varphi)^n = F(z, \varphi) \omega^n.
\]

(2.1)

Here \(F(z, t)\) is a \(C^2\) function on \(\bar{M} \times \mathbb{R}\) which is assumed to be positive on the set \(\bar{M} \times [\inf \varphi, \infty)\). We impose also the Dirichlet condition

\[
\varphi = \varphi_b \text{ on } \partial M \text{ if } \partial M \neq \emptyset
\]

(2.2)

for a given \(C^2\) function \(\varphi_b\). Our main goal is to obtain gradient estimates with constants which depend only on the lower bound for \(\varphi\) (and not on \(\sup_M |\varphi|\)). There are two versions of such gradient estimates. In the first version, the function \(\varphi\) is assumed to be \(C^4\) on \(\bar{M}\). In the second, \(\varphi\) is assumed to be \(C^4\) on \(\bar{M} \setminus S\), where \(S\) is a set which does not intersect \(\partial M\), and \(\varphi(z)\) is assumed to tend to \(\infty\) as \(z \to S\). All covariant derivatives and curvatures listed below are with respect to the metric \(\omega \equiv \frac{1}{2} g_{i\bar{j}} dz^i \wedge d\bar{z}^j\).

**Theorem 1** Let \((M, \omega)\) be a compact Kähler manifold with boundary \(\partial M\) as described above, and let \(F \in C^2(\bar{M} \times \mathbb{R})\) satisfy \(F > 0\).

(a) Let \(\varphi \in C^4(\bar{M})\) be a solution of the equation (2.1) (with the Dirichlet condition (2.2) if \(\partial M\) is not empty). Then we have the a priori estimate

\[
|\nabla \varphi(z)|^2 \leq C_1 \exp(A_1 \varphi(z)), \quad z \in \bar{M}
\]

(2.3)

where \(C_1\) and \(A_1\) are constants that depend only on upper bounds on

\[
\Lambda = -\inf_{a, b \in T M} \frac{R_{j\bar{k}a} \omega^{a \bar{b} \bar{c}}}{{|a|^2 |b|^2}},
\]

(2.4)

and

\[
\inf_{\partial M} \varphi, \sup_{M \times [\inf \varphi, \infty)} \{\nabla F^\frac{1}{n} \}^2, \sup_{M \times [\inf \varphi, \infty)} \left(|\nabla F^\frac{1}{n}\right) + |\partial_t F^\frac{1}{n}|\),
\]

\[
\sup_{\partial M} |\nabla \varphi|, \sup_{\partial M} |\nabla \varphi|.
\]

(2.5)

(b) Let now \(\varphi \in C^4(\bar{M} \setminus S)\) be a solution of the equation (2.1) in \(\bar{M} \setminus S\) with the Dirichlet condition (2.2). Assume further that there exists a constant \(B > 0\) so that

\[
\varphi(z) \to +\infty \text{ as } z \to S,
\]

\[
\log |\nabla \varphi(z)|^2 - B \varphi(z) \to -\infty \text{ as } z \to S.
\]

(2.6)

Then we have

\[
|\nabla \varphi(z)|^2 \leq C_1 \exp(A_1 \varphi(z)), \quad z \in \bar{M} \setminus S,
\]

(2.7)

where the constant \(A_1 = \max(B, D)\), and the constants \(C_1, D\) again depend only on the quantities listed in (2.4) and (2.5).
Before giving the proof of Theorem 1 we make a few remarks:

(1) The gradient estimates of Theorem 1 are the natural analogues of the classic estimates of Yau and Aubin [Y, A] for the Laplacian $\Delta \varphi$ of $\varphi$. More precisely, under the same hypotheses as (a) of Theorem 1, the estimates of Yau and Aubin are

$$|\Delta \varphi(z)| \leq C_2 \exp( A_2 (\varphi(z) - \inf_M \varphi) ) , \quad z \in \bar{M}$$

(2.8)

where $A_2$ and $C_2$ are constants depending only on upper bounds for $-\Delta \log F$, the scalar curvature of $\omega$, $\Lambda$, $\|\varphi\|_{C^0(\partial M)}$, and $\sup_{\partial M} (n + \Delta \varphi)$. On the other hand, when there is no boundary, and the function $F(z, \varphi)$ is a function $F(z)$ of $z$ alone, the equation (2.1) is unchanged under shifts of $\varphi$ by an additive constant. Thus the infimum of $\varphi(z)$ can be normalized to be 0 by replacing $\varphi(z) \to \varphi(z) - \inf \varphi$, so we obtain the estimate

$$|\nabla \varphi(z)|^2 \leq C_1 \exp(A_1 (\varphi(z) - \inf_M \varphi)) \quad z \in \bar{M}$$

(2.9)

where the constant $C_1$ does not depend on $\inf \varphi$, but depends only on the other quantities in (2.5). Thus we see that the apriori estimate (2.8) for the Laplacian and the apriori estimate (2.9) for the gradient have the same structure.

(2) Not surprisingly, the constants $\sup_{M \times [\inf \varphi, \infty]} F$ and $\sup_{M \times [\inf \varphi, \infty]} |\nabla F^\pm| + |\partial_t F^\pm|$ in (2.5) can be replaced by $\sup_{M \times [\inf \varphi, \sup \varphi]} F$ and $\sup_{M \times [\inf \varphi, \sup \varphi]} |\nabla F^\pm| + |\partial_t F^\pm|$ respectively. Thus, when $\|\varphi\|_{C^0}$ is bounded, we obtain gradient bounds for $\varphi$ for completely general smooth and strictly positive functions $F(z, \varphi)$. We have however stated them in the original form since we are mainly interested in the cases when there is no upper bound for $\sup \varphi$.

(3) As we had stressed, the point of the above gradient estimates is that they depend only on $\inf_M \varphi$. If a dependence on $\|\varphi\|_{C^0}$ is allowed, then there are many earlier direct approaches. The first appears to be due to Hanani [H]. More recently, Blocki [B1] gave a different proof, and our approach builds directly on his. The method of P. Guan [G] can be extended to Hessian equations, while the method of B. Guan-Q. Li [GL] allows a general Hermitian metric $\omega$ as well as a more general right hand side $F(z)\chi^n$, where $\chi$ is a Kähler form.

**Proof of Theorem 1:** We adapt the proof from the earlier paper [PS4]. The key ingredient is an inequality due to Blocki [B1].

Let $g'_{kj} = g_{kj} + \partial_j \partial_k \varphi$. We denote covariant derivatives with respect to $g'_{kj}$ by $\nabla'$. It is also convenient to formulate inequalities in terms of the relative endomorphism

$$h^j_k = g^{j\beta} g'_{\beta k}.$$  

(2.10)

Let $\gamma(x)$ be a function with $\gamma'(x) > 0$ and $-\gamma''(x) > 0$ for $x \in [\inf_M \varphi, \infty)$. Set

$$\beta(z) = |\nabla \varphi(z)|^2, \quad \alpha(z) = \log |\nabla \varphi(z)|^2 - \gamma(\varphi).$$

(2.11)
Then Blocki shows that at an interior maximum for $\alpha$, the following inequality holds:
$$\Delta' \alpha \geq \frac{1}{\beta} |\nabla' \nabla_1 \varphi|^2 + (\gamma'(\varphi) - \Lambda - \frac{F_1}{\beta^2}) \text{Tr} h^{-1} + (-\gamma''(\varphi) + 2 \frac{\gamma'(\varphi)}{\beta}) |\nabla' \varphi|^2$$
$$- (n + 2) \gamma'(\varphi) - \frac{2}{\beta}. \quad (2.12)$$

Here $\Lambda$ is the lower bound for the bisectional curvature defined in (2.4),
$$F_1 = 2 \sup |\nabla (F(z, \varphi(z))^\frac{1}{n})|, \quad (2.13)$$
and we have followed [PS4] in reformulating the inequality in terms of covariant derivatives and the endomorphism $h$ and its inverse $h^{-1}$.

Since $\partial_z (F(z, \varphi(z))^\frac{1}{n}) = (\partial_z F^\frac{1}{n})(z, \varphi) + \partial_z \varphi(z)(\partial_t F^\frac{1}{n})(z, \varphi)$, (2.12) implies
$$\Delta' \alpha \geq \frac{1}{\beta} |\nabla' \nabla_1 \varphi|^2 + (\gamma'(\varphi) - \Lambda - F_1' - \frac{F_1''}{\beta^2}) \text{Tr} h^{-1} + (-\gamma''(\varphi) + 2 \frac{\gamma'(\varphi)}{\beta}) |\nabla' \varphi|^2$$
$$- (n + 2) \gamma'(\varphi) - \frac{2}{\beta}. \quad (2.14)$$

where we have set
$$F'_1 = \sup_{M \times [\inf \varphi, \infty]} |\partial_t F^\frac{1}{n}|, \quad F''_1 = \sup_{M \times [\inf \varphi, \infty]} |\nabla_z F^\frac{1}{n}| \quad (2.15)$$

We now prove Part (b) of Theorem 1 (Part (a) is an immediate consequence of Part (b), simply by taking $S$ to be empty). Let the function $\gamma(x)$ be chosen to be
$$\gamma(x) = A_2 x - \frac{1}{x + C_4} \quad (2.16)$$
where $C_4 = -\inf_M \varphi + 1$, and $A_2$ will be chosen later. Note that in the range $x + C_4 \geq 1$
$$A_2 x - 1 \leq \gamma(x) \leq A_2 x,$n$$A_2 \leq \gamma'(x) \leq A_2 + 1,$n$$\gamma''(x) = \frac{1}{(x + C_4)^3}. \quad (2.17)$$

Let $\alpha(z)$ be the corresponding function as in (2.11). If we choose $A_2 \geq B$ then by hypothesis, the function $\alpha(z)$ attains its maximum somewhere at a point $p$ in $\bar{M} \setminus S$. It suffices to show that there is a constant $C_4$ depending only on the quantities in (2.5) so that
$$\alpha(p) \leq C_5. \quad (2.18)$$

It follows that for any $z \in \bar{M} \setminus S$,
$$\log |\nabla \varphi(z)|^2 - A_2 \varphi(z) \leq C_5 + \frac{1}{\varphi(z) + C_4} \leq C_6, \quad (2.19)$$
which implies the desired estimate.

If $p$ is on $\partial M$, $\alpha(p)$ is immediately bounded by constants of the form (2.5), and we are done.

Assume then that $p$ is an interior maximum point. Then $\Delta'^\alpha(p) \leq 0$. We can assume that $\beta(p) \geq 1$, otherwise

$$\alpha(p) = \log \beta(p) - A_2\varphi(p) \leq A_2(-\inf_M \varphi) \leq C_6$$

(2.20)

and we are again done. We apply now Blocki’s identity in the form (2.14), and simplify the right hand side by dropping the terms $|\nabla'\nabla\varphi|^2$, $\gamma''(\varphi)$ on the right hand side,

$$0 \geq \Delta'^\alpha \geq (A_2 - \Lambda - F_1' - F''_1)\text{Tr} h^{-1} - \gamma''(\varphi)|\nabla'\varphi|^2 - C_7.$$  

(2.21)

Choose $A_2 = \max(B, \Lambda + F_1' + F''_1 + 1)$. This implies that $\text{Tr} h^{-1}$ is bounded above, and hence $\lambda_i^{-1}$ are all bounded above, where $\lambda_i$ are the eigenvalues of $h$. By the Monge-Ampère equation, the product of the $\lambda_i$ is bounded by $\sup_M F$. Thus the eigenvalues $\lambda_i$ are also bounded above. This implies that

$$|\nabla\varphi(p)|^2 \leq C_8|\nabla'\varphi(p)|^2.$$  

(2.22)

It follows from the previous inequality and the explicit expression for $\gamma''$ that

$$\frac{1}{(\varphi + C_4)^3}|\nabla\varphi|^2 \leq C_9,$$  

(2.23)

that is

$$|\nabla\varphi(p)|^2 \leq C_9(\varphi(p) + C_4)^3.$$  

(2.24)

Now we may also assume that

$$\alpha(p) \geq 0$$  

(2.25)

since otherwise there is nothing to prove. But then

$$A_2\varphi(p) - 1 \leq \gamma(\varphi(p)) \leq \log |\nabla\varphi(p)|^2$$  

(2.26)

and thus

$$\varphi(p) \leq C_{10}\log |\nabla\varphi(p)|^2 + C_{11}.$$  

(2.27)

Altogether we obtain

$$|\nabla\varphi(p)|^2 \leq C_9(C_{10}\log |\nabla\varphi(p)|^2 + C_{12})^3.$$  

(2.28)

This shows that $|\nabla\varphi(p)|^2 \leq C_{13}$, and since we still have

$$\alpha(p) = \log |\nabla\varphi(p)|^2 - \gamma(\varphi(p)) \leq \log C_{13} + C_{14} = C_{15}.$$  

(2.29)

The theorem is proved.
3 Application to an observation of Tsuji

We illustrate the application of Theorem 1 to the construction of geodesic rays from a test configuration, as in [PS4]. In the geodesic problem, we encounter a Dirichlet problem for a Monge-Ampère equation with a background $(1,1)$-form $\omega_0$ which is closed, but which may be degenerate (more precisely, strictly positive but with no positive lower bound),

$$ (\omega_0 + \frac{i}{2} \partial \bar{\partial} \varphi)^n = G(z) \omega_0^n, \quad \varphi = \varphi_b \text{ on } \partial M. \quad (3.1) $$

The degeneracy of $\omega_0$ prevents the application of the standard gradient estimates. However, in the situation of [PS4], there is a number $\kappa > 0$ and an effective divisor $E$ disjoint from $\partial M$, with $\omega_0 - \kappa [E] > 0$, that is, $O(E)$ admits a metric $H(z)$ so that

$$ \omega \equiv \omega_0 + \frac{i}{2} \partial \bar{\partial} \log H(z)^\kappa > 0. \quad (3.2) $$

Let $\sigma(z)$ be the canonical section of $O(E)$ which vanishes exactly on $E$. Then a useful observation going back to Tsuji [T] is that the original equation can now be re-written as a new equation with non-degenerate background,

$$ (\omega + \frac{i}{2} \partial \bar{\partial} \psi)^n = F \omega^n, \quad (3.3) $$

where $F \equiv G(\omega_0^n/\omega^n)$, and the new unknown $\psi$ is defined by

$$ \psi(z) = \varphi(z) - \log \|\sigma(z)\|^\kappa \quad (3.4) $$

with $\|\sigma(z)\|^2 \equiv |\sigma(z)|^2 H(z)$ the square of the norm of $\sigma(z)$ with respect to the metric $H(z)$. If $\varphi(z)$ is bounded from below, then $\psi(z)$ is bounded from below, and Theorem 1 applied to the shifted equation (3.3) gives upper bounds for $|\nabla \psi(z)|$, and thus for $|\nabla \varphi(z)|$ away from the divisor $E$. Precise statements are given in Theorems 1-3 of [PS4]. The main implication is that the geodesic rays associated to general test configurations by Bergman approximations [PS1, PS2, PS3] are $C^{1,\alpha}$ for any $0 < \alpha < 1$. Other related $C^{1,\alpha}$ regularity results for geodesic rays or solutions of the degenerate Monge-Ampère equation in various geometric situations can be found in [PS3, C2, CT, BD, SZ1, SZ2].

4 Related estimates and other versions

This section is devoted to a few simple remarks and extensions of the previous gradient estimates.
4.1 The $C^2$ estimate

We observed earlier that the gradient estimates of Theorem 1 a) are natural analogues of the classic estimates of Yau and Aubin [Y, A] for the Laplacian $\Delta \varphi$ of $\varphi$. This analogy extends to the case (b) as well: assume now that $\varphi(z) \to +\infty$ as $z \to S$, and that there exists a constant $C$ so that

$$\log (n + \Delta \varphi(z)) - C \varphi(z) \to -\infty \text{ as } z \to S. \quad (4.1)$$

Then the same a priori estimate as in (2.8) holds, for $z \in \bar{M} \setminus S$. This is proved by the same argument as in Yau and Aubin, and is familiar to experts in the field. See related estimates in e.g. [EGZ, DP, TZ, ST1, ST2, T], where the preliminary $C^0$ estimates are built on those of Kolodziej [K].

4.2 A gradient estimate for general Hermitian backgrounds

The same argument applies to the case of $\omega$ just a Hermitian metric which is not necessarily Kähler. This is because Blocki’s identity can also be adapted to the Hermitian case (see e.g. eqs. (2.7)-(2.8) of [PS4]) if we replace the lower bound for the bisectional curvature by the constant $\Lambda^H$ defined by

$$M^{k\bar{l}} R^{p}_{k\bar{l}} \bar{m} N_{m\bar{p}} \geq -\Lambda^H (\text{Tr} M)(\text{Tr} N) \quad (4.2)$$

for all Hermitian non-negative matrices $M$ and $N$. We can then write

$$(g')^{k\bar{l}} \partial_p u R^{p}_{k\bar{l}} \bar{m} \partial_{\bar{m}} u \geq -\Lambda^H \beta \text{Tr} h^{-1}. \quad (4.3)$$

and obtain the exact same inequality as (2.12), with $\Lambda$ replaced by $\Lambda^H$. The rest of the proof proceeds as before.

We note that there has been considerable progress recently in the study of the Monge-Ampère equation on general Hermitian manifolds [GL, GL1, TW, DK].

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