Multidimensional gravity in non-relativistic limit

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It is found the exact solution of the Poisson equation for the multidimensional space with topology $M_{3+d} = \mathbb{R}^3 \times T^d$. This solution describes smooth transition from the newtonian behavior $1/r_3$ for distances bigger than periods of tori (the extra dimension sizes) to multidimensional behavior $1/r_3^{1+d}$ in opposite limit. In the case of one extra dimension $d = 1$, the gravitational potential is expressed via compact and elegant formula. It is shown that the corrections to the gravitational constant in the Cavendish-type experiment can be within the measurement accuracy of Newton’s gravitational constant $G_N$. It is proposed models where the test masses are smeared over some (or all) extra dimensions. In 10-dimensional spacetime with 3 smeared extra dimensions, it is shown that the size of 3 rest extra dimensions can be enlarged up to submillimeter for the case of 1TeV fundamental Planck scale $M_{Pl(10)}$. In the models where all extra dimensions are smeared, the gravitational potential exactly coincides with the newtonian one. Nevertheless, the hierarchy problem can be solved in these models.

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Introduction There are two well-known problems which are related to each other. They are the discrepancies in gravitational constant experimental data and the hierarchy problem. Discrepancies (see e.g. Figure 2 in the "CODATA Recommended Values of the Fundamental Constants: 2006") are usually explained by extreme weakness of gravity. It is very difficult to measure the Newton’s gravitational constant $G_N$. Certainly, for this reason geometry of an experimental setup can effect on data. However, it may well be that, the discrepancies can also be explained (at least partly) by underlying fundamental theory. Formulas for an effective gravitational constant following from such theory can be sensitive to the geometry of experiments. For example, if correction to the Newton’s gravitational potential has the form of Yukawa potential, then the force due to this potential at a given minimum separation per unit-body mass is least for two spheres and greatest for two planes (see e.g. [1]). The hierarchy problem - the huge gap between the fundamental potential (see e.g. [2]): $\varphi \sim 1/r_3$ for $r_3 >> b$ and $\varphi \sim 1/r_3^{1+d}$ for $r_3+d << b$ where $r_3+d$ is magnitude of a radius vector in $(3+d)$-dimensional space.

To get the exact expression for $D$-dimensional gravitational potential, we start with the Poisson equation:

$$\Delta_D \varphi_D = S_D G_D \rho_D(r_D), \quad (1)$$

where $S_D = 2\pi^D/\Gamma(D/2)$ is a total solid angle (square of $(D-1)$-dimensional sphere of a unit radius), $G_D$ is a gravitational constant in $(D = D+1)$-dimensional spacetime and $\rho_D(r_D) = m \delta(x_1) \delta(x_2) ... \delta(x_D)$. In the case of topology $\mathbb{R}^D$, Eq. (1) has the following solution:

$$\varphi_D(r_D) = -\frac{G_D m}{(D-2)r_D^{D-2}}, \quad D \geq 3. \quad (2)$$

This is the unique solution of Eq. (1) which satisfies the boundary condition: $ lim_{r_D \to \infty} \varphi_D(r_D) = 0$. Gravitational constant $G_D$ in (1) is normalized in such a way that the strength of gravitational field (acceleration of a test body) takes the form: $-d\varphi_D/dr_D = -G_D m/r_D^{D-1}$.

If topology of space is $\mathbb{R}^3 \times T^d$, then it is natural to impose periodic boundary conditions in the directions of the extra dimensions: $
abla_D(r_3, \xi_1, \xi_2, ..., \xi_d) = \varphi_D(r_3, \xi_1, \xi_2, ..., \xi_i + a_i, ..., \xi_d), \quad i = 1, ..., d$, where $a_i$ denotes a period in the direction of the extra dimension.
Then, Poisson equation has solution (cf. also with (3) [14]):
\[
\varphi_D(r_3, \xi_1, \ldots, \xi_d) = -\frac{G_N m}{r_3} \\
\times \sum_{k_1=-\infty}^{+\infty} \cdots \sum_{k_d=-\infty}^{+\infty} \exp \left[ -2\pi \left( \sum_{i=1}^{d} \left( \frac{k_i}{a_i} \right)^2 \right)^{1/2} \right] \\
\times \cos \left( \frac{2\pi k_1}{a_1} \xi_1 \right) \cdots \cos \left( \frac{2\pi k_d}{a_d} \xi_d \right). \quad (3)
\]
To get this result we, first, use the formula \( \delta(\xi_i) = \frac{1}{a_i} \sum_{k_i=-\infty}^{+\infty} \cos \left( \frac{2\pi k_i}{a_i} \xi_i \right) \) and, second, put the following relation between gravitational constants in four- and \( D \)-dimensional spacetimes:
\[
\frac{S_D}{S_3} \cdot \frac{G_D}{\prod_{i=1}^{d} a_i} = G_N. \quad (4)
\]
The letter relation provides correct limit when all \( a_i \to 0 \). In this limit zero modes \( k_i = 0 \) give the main contribution and we obtain \( \varphi_D(r_3, \xi_1, \ldots, \xi_d) \to -G_N m/r_3 \). Eq. (1) was widely used in the concept of large extra dimensions which gives possibility to solve the hierarchy problem [2, 3]. It is also convenient to rewrite (4) via fundamental Planck scales:
\[
\frac{S_D}{S_3} \cdot M_{Pl(4)}^d = M_{Pl(D)}^{2+d} \prod_{i=1}^{d} a_i, \quad (5)
\]
where \( M_{Pl(4)} = G_N^{-1/2} = 1.2 \times 10^{19} \text{GeV} \) and \( M_{Pl(D)} \equiv G_D^{-1/2+d} \) are fundamental Planck scales in four and \( D \) spacetime dimensions, respectively.

In opposite limit when all \( a_i \to +\infty \) the sums in Eq. (3) can be replaced by integrals. Using the standard integrals (e.g. from (3) and relation (4)), we can easily show that, for example, in particular cases \( d = 1, 2 \) we get desire result: \( \varphi_D(r_3, \xi_1, \ldots, \xi_d) \to -G_N m/(D-2) r_3^{1+d} \).

One extra dimension In the case of one extra dimension \( d = 1 \) we can perform summation of series in Eq. (3). To do it, we can apply the Abel-Plana formula or simply use the tables of series [5]. As a result, we arrive at compact and nice expression:
\[
\varphi_4(r_3, \xi) = -\frac{G_N m}{r_3} \frac{\sinh \left( \frac{2\pi r_3}{a} \right)}{\cosh \left( \frac{2\pi r_3}{a} \right) - \cos \left( \frac{2\pi \xi}{a} \right)}, \quad (6)
\]
where \( r_3 \in [0, +\infty) \) and \( \xi \in [0, a] \). It is not difficult to verify that this formula has correct asymptotes when \( r_3 \gg a \) and \( r_4 \ll a \). Fig. 1 demonstrates the shape of this potential. Dimensionless variables \( \eta_i = r_3/a \in [0, +\infty) \) and \( \eta_2 \equiv \xi/a \in [0, 1] \). With respect to variable \( \eta_2 \), this potential has two minima at \( \eta_2 = 0, 1 \) and one maximum at \( \eta_2 = 1/2 \). We continue the graph to negative values of \( \eta_2 \in [-1, 1] \) to show in more detail the form of minimum at \( \eta_2 = 0 \). The potential [6] is finite for any value of \( r_3 \) if \( \xi \neq 0, a \) and goes to \(-\infty\) as \(-1/r_3^2\) if simultaneously \( r_3 \to 0 \) and \( \xi \to 0, a \) (see Fig. 2). We would like to mention that in particular case \( \xi = 0 \) formula (4) was also found in [6].

\[ \text{FIG. 1: Graph of function } \tilde{\varphi}(\eta_1, \eta_2) \equiv \varphi_4(r_3, \xi)/(G_N m/a) = -\exp\left(\frac{2\pi \eta_1}{\eta_2} - 1\right) \text{ for } \eta_1 \to +\infty \text{ and to } -1/(\eta_2^5) \text{ (dashed line) for } \eta_1 \to 0. \]

\[ \text{FIG. 2: Section } \xi = 0 \text{ of potential } \tilde{\varphi}. \text{ Solid line is } \tilde{\varphi}(\eta_1, 0) = -\exp\left(\frac{2\pi \eta_1}{\eta_2} - 1\right) \text{ which goes to } -1/\eta_2 \text{ (dotted line) for } \eta_1 \to +\infty \text{ and to } -1/(\eta_2^5) \text{ (dashed line) for } \eta_1 \to 0. \]

Having at hand formulas (3) and (6), we can apply it for calculation of some elementary physical problems and compare obtained results with known Newtonian expressions. Some of these calculations can be found in our preprint [8]. For a working approximation, it is usually sufficient to summarize in (3) up to the first Kaluza-Klein term. Some of these calculations can be found in our preprint [8]. For a working approximation, it is usually sufficient to summarize in (3) up to the first Kaluza-Klein term.
Yukawa contribution. The overall diagram of the experimental constraints can be found in [1] (see Figure 5) and we shall use these data for limitation \(a\) for given \(\alpha\).

In this approximation, the gravitational force between two spheres with masses \(m_1, m_2\), radii \(R_1, R_2\) and distance \(r_3\) between the centers of the spheres reads:

\[
F = -\frac{G_N(m_1m_2)}{r_3^2},
\]

where

\[
G_N(m_1r_3) \approx G_N \left( 1 + \frac{9}{2} \frac{s}{2\pi R_1} \left( \frac{a}{2\pi R_2} \right)^2 \right)
\]

\[
\times \frac{2\pi r_3}{a} \exp \left[-\frac{2\pi}{a}(r_3 - R_1 - R_2)\right] \equiv G_N(1 + \delta_G). \tag{9}
\]

Here, we used conditions: \(r_3 \geq R_1 + R_2\) and \(R_1, R_2 \gg a/2\pi\).

**Smeared extra dimensions** In what follows, we consider asymmetrical extra dimension model (cf. [8]) with topology

\[ M_D = \mathbb{R}^3 \times T^d \times T^p, \quad p \leq d, \tag{10} \]

where we suppose that \((d - p)\) tori have the same "large" period \(a\) and \(p\) tori have "small" equal periods \(b\). In this case, the fundamental Planck scale relation (5) reads

\[
\frac{S_D}{S_3} M_{Pl(4)}^2 = \frac{M_{Pl(D)}^{2+d} a^{d-p} p^p}{r_3}. \tag{11}
\]

Additionally, we assume that test bodies are uniformly smeared/spreaded over small extra dimensions. Thus, test bodies have a finite thickness in small extra dimensions (thick brane approximation). For short, we shall call such small extra dimensions as "smeared" extra dimensions. If \(p = d\) then all extra dimensions are smeared.

It is not difficult to show that the gravitational potential does not feel smeared extra dimensions. We can prove this statement by three different methods. First, we can directly solve D-dimensional Poisson equation (11) with the periodic boundary conditions for the extra dimensions \(\xi_{p+1}, \ldots, \xi_d\) and the mass density \(\rho = (m/\prod_{i=1}^p a_i) \delta(r_3) \delta(\xi_{p+1}) \ldots \delta(\xi_d)\). Second, we can average solutions (8) and (10) over dimensions \(\xi_1, \ldots, \xi_d\) and take into account that \(\int_0^a \cos(2\pi \xi/a) d\xi = 0\). In particular case of one extra dimensional, we can also show that

\[
-\frac{G_N m}{ar_3} \sinh \left( \frac{2\pi r_3}{a} \right) \int_0^a \left[ \cosh \left( \frac{2\pi r_3}{a} \right) \right]^{-1} d\xi = -\frac{G_N m}{r_3}. \tag{12}
\]

Finally, it is clear that in the case of test masses smeared over extra dimensions, the gravitational field vector \(E_D = -\nabla_D \phi_D\) does not have components with respect to extra dimensions: \(E_D = E_D n_{r_3}\). Thus, applying the Gauss’s flux theorem to the Poisson equation, we obtain: \(E_D(r_3) = -G_N m/r_3^2 \rightarrow \phi_D(r_3) = -G_N m/r_3\). Therefore, all these 3 approaches show that in the case of \(p\) smeared extra dimensions the wave numbers \(k_1, \ldots, k_p\) disappear from Eq. (3) and we should perform summation only with respect to \(k_{p+1}, \ldots, k_d\).

**Effective gravitational constant** Coming back to the effective gravitational constant \(G_N\) in the case of topology (11) with \(\rho = 0 \rightarrow \alpha = 2\) and \((D = 10)\)-dimensional model with \(d = 1, p = 0 \rightarrow \alpha = 2\) and \((D = 10)\)-dimensional model with \(d = 6, p = 3 \rightarrow \alpha = 6\). For these values of \(\alpha\), Figure 5 in [11] gives the upper limits for \(\lambda = a/(2\pi)\) correspondingly \(\lambda \approx 2 \times 10^{-4}\text{cm}^3\) and \(\lambda \approx 1.3 \times 10^{-2}\text{cm}\). To calculate \(\delta_G\), we take parameters of the Moscow experiment [11]: \(R_1 \approx 0.087\text{cm}\) for a platinum ball with the mass \(m_1 = 59.25 \times 10^{-3}\text{g}\), \(R_2 \approx 0.206\text{cm}\) for a tungsten ball with the mass \(m_2 = 706 \times 10^{-3}\text{g}\) and \(r_3 = 0.377\text{cm}\). For both of these models we obtain \(\delta_G \approx 0.0006247\) and \(\delta_G \approx 0.0000532\), respectively. Both of these values are very close to the measurement accuracy of \(G_N\) in [11]. So, if the same accuracy can be achieved in the Moscow-type experiments, then, changing values of \(R_1, 2\) and \(r_3\), we can reveal extra dimensions or obtain experimental limitations on considered models.

**Model:** \(D = 10\) with \(d = 6, p = 3\). Let us consider in more detail \((D = 10)\)-dimensional model with 3 smeared dimensions. Here, we have very symmetrical with respect to a number of spacial dimensions structure: 3 our external dimensions, 3 large extra dimensions with periods \(a\) and three small smeared extra dimensions with periods \(b\). For \(b\) we put a limitation: \(b \leq b_{max} = 10^{-17}\text{cm}\) which is usually taken for thick brane approximation. As we mentioned above, in the case of \(\alpha = 6\), for \(a\) we should take a limitation \(a \leq a_{max} = 8.2 \times 10^{-2}\text{cm}\). To solve the hierarchy problem, the multidimensional Planck scale is usually considered from 1TeV up to approximately 130 TeV (see e.g. [12]). To make some estimates, we take \(M_{Pl(10)} = 1\text{TeV} \lesssim M_{Pl(10)} \lesssim M_{max} = 50\text{TeV}\). Thus, as it follows from Eq. (11), the allowed values of \(a\) and \(b\) should satisfy inequalities:

\[
\frac{S_D}{S_3} M_{Pl(4)}^2 \leq a^3 b^3 \leq \frac{S_D}{S_3} M_{Pl(4)}^2 \tag{13}
\]

Counting all limitations, we find allowed region for \(a\) and \(b\) (shadow area in Fig. 11). In this trapezium, the upper horizontal and right vertical lines are the decimal logarithms of \(a_{max}\) and \(b_{max}\), respectively. The right and left inclined lines correspond to \(M_{Pl(10)} = 1\text{TeV}\).
and $M_{P(10)} = 50$ TeV, respectively. To illustrate this picture, we consider two points A and B on the line $M_{P(10)} = 1$ TeV. Here, we have $a = 0.82 \times 10^{-4}$ cm, $b = 10^{-18.6}$ cm for A and $a = 10^{-4}$ cm, $b = 10^{-21.5}$ cm for B. These values of large extra dimensions are much bigger than in the standard approach $a \sim 10^{32/6} \times 10^{-17}$ cm $\approx 10^{-11.7}$ cm.

![FIG. 3: Allowed region (shadow area) for periods of large (a) and smeared (b) dimensions in the model $D = 10$ with $d = 6, p = 3$.](image)

Model: $D$-arbitrary and $d = p$ In this model, the test masses are smeared over all extra dimensions. Therefore, in non-relativistic limit, there is no deviation from the Newton’s law at all. It worth of noting, that this result does not depend on the size of smeared extra dimensions. The ISL experiments will not show any deviation from the Newton’s law without regard to the size $b$ (see also Eq. (12) where $s = d - p = 0$). Similar reasoning are also applicable to Coulomb’s law. It is necessary to suggest other experiments which can reveal the multidimensionality of our spacetime. Nevertheless, we can solve the hierarchy problem in this model because Eq. (11) (where $d = p$) still works. For example, in the case of bosonic string dimension $D = 26$ we find $M_{P(26)} \approx 31$ TeV for $b = 10^{-17}$ cm. In the case $D = 10$ we get $M_{P(10)} \approx 30$ TeV for $b = 5.59 \times 10^{-14}$ cm.

Conclusions We have considered generalization of the Newton’s potential to the case of extra dimensions where multidimensional space has topology $M_D = \mathbb{R}^3 \times T^d$. We obtained the exact solution which describes smooth transition from the newtonian behavior $1/r^3$ for distances bigger than periods of tori (the extra dimension sizes) to multidimensional behavior $1/r^{D-2}$ in opposite limit. In the case of one extra dimension, the gravitational potential is expressed via compact and elegant formula (6).

As an Yukawa potential approximation, it was shown that the corrections to the gravitational constant in the Cavendish-type experiment can be within the measurement accuracy of $G_N$. It may reveal the extra dimensions or provide experimental limitations on parameters of multidimensional models.

Then, we proposed models where test masses can be smeared over extra dimensions. In this case, the gravitational potential does not feel smeared dimensions. The number of smeared dimensions can be equal or less than the total number of the extra dimensions. Such approach opens new remarkable possibilities.

For example in the case $D = 10$ with 3 large and 3 smeared extra dimensions and $M_{P(10)} = 1$ TeV, the large extra dimensions can be as big as the upper limit established by the ISL experiments for $\alpha = 6$, i.e. $a \approx 0.82 \times 10^{-2}$ cm. This value of $a$ is in many orders of magnitude bigger than the rough estimate $a \approx 10^{-11.7}$ cm obtained from the fundamental Planck scale relation of the form of Eq. (5).

The limiting case where all extra dimensions are smeared is another interesting example. Here, there is no deviation from the Newton’s law at all. Nevertheless, the hierarchy problem can be solved in this model.

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