ON THE REAL POLYNOMIAL BOHNENBLUST–HILLE INEQUALITY

J. R. CAMPOS, P. JIMÉNEZ-RODRÍGUEZ, G.A. MUÑOZ-FERNÁNDEZ, D. PELLEGRINO, AND J.B. SEOANE-SEPÚLVEDA

Abstract. It was recently proved by Bayart et al. that the complex polynomial Bohnenblust–Hille inequality is subexponential. We show that, for real scalars, this does no longer hold. Moreover, we show that, if $D_{\mathbb{R}, m}$ stands for the real Bohnenblust–Hille constant for $m$-homogeneous polynomials, then

$$\limsup_m D_{\mathbb{R}, m}^{1/m} = 2.$$ 

1. Introduction

If $E$ is a Banach space, real or complex, we say that $P$ is a homogeneous polynomial on $E$ of degree $m \in \mathbb{N}$ if there exists an $m$-linear form $L$ on $E^m$ such that $P(x) = L(x, \ldots, x)$ for all $x \in E$. It is customary to denote by $\hat{L}$ the restriction of $L$ to the diagonal of $E^m$. An old and widely known algebraic result establishes that for every homogeneous polynomial $P$ of degree $m$ on $E$ there exists a unique symmetric $m$-linear form $L$ on $E^m$, called the polar of $P$, such that $P = \hat{L}$. We denote by $\mathcal{P}^{(m)}(E)$, $\mathcal{L}^{(m)}(E)$ and $\mathcal{L}^{(m)}_c(E)$ the spaces of continuous $m$-homogeneous polynomials, continuous $m$-linear forms and continuous symmetric $m$-linear forms on $E$ respectively. It is well known that homogeneous polynomials or $m$-multilinear forms are continuous on $E$ if and only if they are bounded, respectively, over the unit ball $B_E$ of $E$ or $B^m_E$. In that case

$$\|P\| := \sup\{|P(x)| : x \in B_E\},$$

$$\|L\| := \sup\{|L(x_1, \ldots, x_m)| : x_1, \ldots, x_m \in B_E\},$$

define a norm in $\mathcal{P}^{(m)}(E)$ and $\mathcal{L}^{(m)}(E)$ respectively. If $P \in \mathcal{P}^{(m)}(E)$, we shall refer to $\|P\|$ as the polynomial norm of $P$ in $E$. This norm is very difficult to compute in most cases, for which reason it would be interesting to obtain reasonably good estimates on it. The $\ell_p$ norm of the coefficients of a given polynomial on $\mathbb{K}^n$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) has also been widely used in mathematics and is much easier to handle. Observe that an $m$-homogeneous polynomial in $\mathbb{K}^n$ can be written as

$$P(x) = \sum_{|\alpha| = m} a_\alpha x^\alpha,$$

where $x = (x_1, \ldots, x_n) \in \mathbb{K}^n$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. Thus we define the $\ell_p$ norm of $P$, with $p \geq 1$, as If $E$ has finite dimension $n$, then the polynomial norm $\|\cdot\|$ and the $\ell_p$ norm $\|\cdot\|_p$ ($p \geq 1$) are equivalent, and therefore there exist constants $k(m, n), K(m, n) > 0$ such that

$$k(m, n)|P|_p \leq \|P\| \leq K(m, n)|P|_p,$$

for all $P \in \mathcal{P}^{(m)}(E)$. The latter inequalities may provide a good estimate on $\|P\|$ as long as we know the exact value of the best possible constants $k(m, n)$ and $K(m, n)$ appearing in (1.1). The problem presented above is an extension of the well known polynomial Bohnenblust-Hille inequality (polynomial BH inequality for short). It was proved in [5] that there exists a constant $D_m \geq 1$ such that for every $P \in \mathcal{P}^{(m)}(E)$ we have

$$|P|_{2m+1} \leq D_m \|P\|.$$

Observe that (1.2) coincides with the first inequality in (1.1) for $p = \frac{2m}{m+1}$ except for the fact that $D_m$ in (1.2) can be chosen in such a way that it is independent from the dimension $n$. Actually Bohnenblust...
and Hille showed that \( \frac{2m}{m+1} \) is optimal in (1.2) in the sense that for \( p < \frac{2m}{m+1} \), any constant \( D \) fitting in the inequality 
\[ |P|_p \leq D \|P\|, \]
for all \( P \in \mathcal{P}(m \ell_\infty^n) \) depends necessarily on \( n \).

The polynomial and multilinear Bohnenblust–Hille inequalities were overlooked for a long period (see [3]) and were only rediscovered in the last few years; now these inequalities can be seen as an extension of the successful theory of absolutely summing operators (see [2]) and have fundamental importance in different fields of Mathematics and Physics, such as Operator Theory, Fourier and Harmonic Analysis, Complex Analysis, Analytic Number Theory and Quantum Information Theory (see [2, 6, 8, 9, 11–14] and references therein).

The best constants in (1.2) may depend on whether we consider the real or the complex version of \( \ell_\infty^n \), which motivates the following definition
\[ D_{K,m} := \inf \left\{ D > 0 : |P|_\frac{2m}{m+1} \leq D \|P\|, \text{ for all } n \in \mathbb{N} \text{ and } P \in \mathcal{P}(m \ell_\infty^n) \right\}. \]

If we restrict attention to \( \mathcal{P}(m \ell_\infty^n) \) for some \( n \in \mathbb{N} \), then we define
\[ D_{K,m}(n) := \inf \left\{ D > 0 : |P|_\frac{2m}{m+1} \leq D \|P\|, \text{ for all } P \in \mathcal{P}(m \ell_\infty^n) \right\}. \]

Note that \( D_{K,m}(n) \leq D_{K,m} \) for all \( n \in \mathbb{N} \).

It was recently shown in [2] that the complex polynomial Bohnenblust–Hille inequality is, at most, subexponential, i.e., for any \( \varepsilon > 0 \), there is a constant \( C_\varepsilon > 0 \) such that \( D_{C,m} \leq C_\varepsilon (1 + \varepsilon)^m \) for all positive integers \( m \).

The upper estimate

\[ \limsup_m D_{K,m}^{1/m} = 2. \]

2. THE UPPER ESTIMATE

The proof of the subexponentiality of the complex BH inequality given in [2] lies heavily in arguments restricted to complex scalars (it uses, for instance estimates from [1] for complex scalars); so a simple adaptation for the real case does not work. The calculation of the upper estimate of the BH inequality is quite simplified by the use of complexifications of polynomials. In particular we are interested in the following deep result due to Visser [15], which generalizes and old result of Chebyshev:

**Theorem 2.1** (Visser, [15], 1946). Let
\[ P(y_1, \ldots, y_n) = \sum_{|\alpha| \leq m} a_\alpha y_1^{\alpha_1} \cdots y_n^{\alpha_n}, \]
with \( \alpha = (\alpha_1, \ldots, \alpha_n) \), \( |\alpha| = \alpha_1 + \cdots + \alpha_n \), be a polynomial of total degree at most \( m \in \mathbb{N} \) in the variables \( y_1, \ldots, y_n \) and with real coefficients \( a_\alpha \). Suppose \( 0 \leq k \leq m \) and \( P_k \) is the homogeneous polynomial of degree \( k \) defined by
\[ P_k(y_1, \ldots, y_n) = \sum_{|\alpha| = k} a_\alpha y_1^{\alpha_1} \cdots y_n^{\alpha_n}. \]

Then we have
\[ \max_{z_1, \ldots, z_n \in \mathbb{D}} |P_m(z_1, \ldots, z_n)| \leq 2^{m-1}, \quad \max_{x_1, \ldots, x_n \in [-1,1]} |P(x_1, \ldots, x_n)|, \]
where \( \mathbb{D} \) stands for the closed unit disk in \( \mathbb{C} \). In particular, if \( P \) is homogeneous, then
\[ \max_{z_1, \ldots, z_n \in \mathbb{D}} |P(z_1, \ldots, z_n)| \leq 2^{m-1}, \quad \max_{x_1, \ldots, x_n \in [-1,1]} |P(x_1, \ldots, x_n)|. \]

Moreover, the constant \( 2^{m-1} \) cannot be replaced by any smaller one.

Let \( P : \ell_\infty^n(\mathbb{R}) \to \mathbb{R} \) be an \( m \)-homogeneous polynomial
\[ P(x) = \sum_{|\alpha| = m} a_\alpha x^\alpha. \]
and consider the complexification $P_C : \ell_\infty^\ell(C) \to C$ of $P$ given by

$$P_C(z) = \sum_{|\alpha| = m} a_\alpha z^\alpha.$$ 

From Theorem 2.1 above we know that

$$\|P_C\| \leq 2^{m-1} \|P\|.$$ 

Thus, since the complex polynomial Bohnenblust–Hille inequality is subexponential, for all $\varepsilon > 0$ there exists $C_\varepsilon > 1$ such that

$$|P|_{\bar{z}^m} = |P_C|_{\bar{z}^m} \leq C_\varepsilon (1 + \varepsilon)^m \|P_C\|$$

and combining (2.1) and (2.2) we conclude that

$$\limsup_m D_R^{1/m}_{\bar{z},m} \leq 2.$$ 

As we mentioned earlier, Bayart et al. proved, recently, that the complex polynomial Bohnenblust–Hille inequality is subexponential (see [2]). The following result shows that the exponential growth of the real polynomial BH inequality is sharp in a very strong way: the exponential bound can not be reduced in any sense, i.e., there is an exponential lower bound for $D_{R,m}$ which holds for every $m \in \mathbb{N}$.

**Theorem 2.2.**

$$D_{R,m} > \left(\frac{2 \sqrt[3]{3}}{\sqrt[5]{5}}\right)^m > (1.17)^m$$

for all positive integers $m > 1$.

**Proof.** Let $m$ be an even integer. Consider the $m$-homogeneous polynomial

$$R_m(x_1, \ldots, x_m) = (x_1^2 - x_2^2 + x_1 x_2)(x_3^2 - x_4^2 + x_3 x_4) \cdots (x_{2m-1}^2 - x_{2m}^2 + x_{2m-1} x_{2m}).$$

Since $\|R_2\| = 5/4$, it is simple to see that

$$\|R_m\| = (5/4)^{m/2}.$$ 

From the BH inequality for $R_m$ we have

$$\left(\sum_{|\alpha| = m} |a_\alpha| \frac{2^m}{m+1}\right)^{\frac{m+1}{2m}} \leq D_{R,m} \|R_m\|,$$

that is,

$$D_{R,m} \geq \left(\frac{3}{4}\right)^{\frac{m+1}{2m}} \geq \left(\frac{\sqrt{3}}{\sqrt{5}}\right)^{\frac{m+1}{2m}} > \left(\frac{2 \sqrt[3]{3}}{\sqrt[5]{5}}\right)^m.$$ 

Now let us suppose that $m$ is odd. Keeping the previous notation, consider the $m$ homogeneous polynomial

$$R_m(x_1, \ldots, x_{2m}) = (x_{2m} + x_{2m-1}) R_{m-1}(x_1, \ldots, x_{m-1}) + (x_{2m} - x_{2m-1}) R_{m-1}(x_m, \ldots, x_{2m-2}).$$

So we have

$$D_{R,m} \geq \left(\frac{4 \cdot 3^{m-1}}{2 \cdot \left(\frac{3}{4}\right)^{m-1}}\right)^{\frac{m+1}{2m}} = 2^{m-1} \left(\frac{\sqrt[3]{3}}{\sqrt[5]{5}}\right)^{m-1} > \left(\frac{2 \sqrt[3]{3}}{\sqrt[5]{5}}\right)^{m-1}.$$ 

□
3. The lower estimate

Using our previous results, in order to show that \( \lim \sup_m D_{\mathbb{R},m}^{1/m} = 2 \), we just need the following theorem:

**Theorem 3.1.** If \( k \in \mathbb{N} \) is fixed, then

\[
\lim \sup_m D_{\mathbb{R},m}^{1/m}(2^k) \geq 2^{1-2^{-k}}.
\]

Therefore,

\[
\lim \sup_m D_{\mathbb{R},m}^{1/m} \geq 2.
\]

**Proof.** Consider the sequence of polynomials (with norm 1) defined recursively by

\[
Q_2(x_1, x_2) = x_1^2 - x_2^2,
\]

\[
Q_{2m}(x_1, \ldots, x_{2k}) = Q_{2m-1}(x_1, \ldots, x_{2^{m-1}})^2 - Q_{2m-1}(x_{2^{m-1}+1}, \ldots, x_{2^m})^2.
\]

Let us show (by induction on \( m \)) that

\[
|Q_n^m|_\infty \geq \left( \frac{2^n}{n+1} \right)^{2^{m-1}}
\]

for every natural number \( n \). The case \( m = 1 \) comes from the fact that, since

\[
2^n = \sum_{k=0}^{n} \binom{n}{k} \leq (n+1) \max_{0 \leq k \leq n} \binom{n}{k},
\]

the \( 2n \)-homogeneous polynomial \( Q_2^n \) admits the following estimate:

\[
|Q_2^n|_{\infty} = \max_{0 \leq k \leq n} \binom{n}{k} \geq \frac{2^n}{n+1}.
\]

Let us now suppose that equation (3.1) holds for some \( m \), and notice that

\[
|Q_{2^{m+1}}^n|_{\infty} = |Q_{2m}^n|_{\infty} \geq \max_{0 \leq k \leq n} \binom{n}{k} \geq \frac{2^n}{n+1},
\]

by equation (3.2). Therefore, the formula given in (3.1) holds for every positive integer \( m \).

Next, every \( n \)-homogeneous polynomial \( P \) admits the clear estimate given by

\[
|P|_{\infty} \geq |P|_{\infty},
\]

from which equation (3.1) yields that

\[
D_{\mathbb{R},2^m}(2^m) \geq \left( \frac{2^n}{n+1} \right)^{2^{m-1}},
\]

and the proof follows straightforwardly. \( \square \)
4. Contractivity in finite dimensions: complex versus real scalars

We remark that the complex polynomial Bohnenblust-Hille constants for polynomials on \( \mathbb{C}^n \), with \( n \in \mathbb{N} \) fixed, are contractive.

**Proposition 4.1.** For all \( n \geq 2 \) the complex polynomial BH inequality is contractive in \( P(m) \). More precisely, for all fixed \( n \in \mathbb{N} \), there are constants \( D_m \), with \( \lim_{m \to \infty} D_m = 1 \), so that

\[
|P|_{m+1} \leq D_m ||P||
\]

for all \( P \in P(m) \).

**Proof.** Let \( P(z) = \sum_{|\alpha|=m} c_\alpha z^\alpha \) and \( f(t) = P(e^{it_1}, \ldots, e^{it_n}) = \sum_{|\alpha|=m} c_\alpha e^{i\alpha t} \), where \( t = (t_1, \ldots, t_n) \in \mathbb{R}^n \). Observe that if \( \|f\| \) denotes the sup norm of \( f \) on \([-\pi, \pi] \), by the Maximum Modulus Principle \( \|f\| = ||P|| \). Also, due to the orthogonality of the system \( \{e^{iks} : k \in \mathbb{Z} \} \) in \( L^2([-\pi, \pi]) \) we have

\[
\|P\|^2 = \|f\|^2 \geq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t)|^2 dt = \sum_{|\alpha|=m} |c_\alpha|^2 = |P|_2^2,
\]

from which \( |P|_2 \leq ||P|| \). On the other hand it is well known that in \( \mathbb{K}^d \) we have

\[
| \cdot |_q \leq | \cdot |_p \leq d^{1/2} | \cdot |_q,
\]

for all \( 1 \leq p \leq q \). Since the dimension of \( P(m) \) is \( \binom{m+n-1}{n-1} \), the result follows from \( |P|_2 \leq ||P|| \) by setting in (4.1) \( p = \frac{2m}{m+1} \), \( q = 2 \) and \( d = \binom{m+n-1}{n-1} \). So \( D_m = (\frac{m+n-1}{n-1})^{\frac{1}{m+1}} \) and since

\[
\lim_{m \to \infty} (\frac{m+n-1}{n-1})^{\frac{1}{m+1}} = 1,
\]

the proof is done.

The next result shows that the **real version** of Proposition 4.1 is not valid; we stress that Theorem 2.2 cannot be used here since it uses polynomials in a growing number of variables.

**Theorem 4.2.** For all fixed positive integer \( N \geq 2 \), the exponentiality of the real polynomial Bohnenblust-Hille inequality in \( P(m) \) cannot be improved. More precisely,

\[
\limsup_m D_{\mathbb{R},m}^{1/m} (N) \geq \sqrt[27]{\sqrt{27}} \approx 1.5098
\]

for all \( N \geq 2 \).

**Proof.** If suffices to set \( N = 2 \) and prove that, for \( m = 4n \),

\[
D_{\mathbb{R},4n} (2) \geq \sqrt{\frac{4}{m\pi}} (\sqrt[27]{\sqrt{27}})^n.
\]

Consider the 4-homogeneous polynomial given by

\[
P_4(x,y) = x^3 y - xy^3 = xy(x^2 - y^2).
\]

A straightforward calculation shows that \( P_4 \) attains its norm at \((\pm \frac{1}{\sqrt{3}}, 1)\) and \((1, \pm \frac{1}{\sqrt{3}})\) and that \( \|P_4\| = 2\sqrt{27} \). On the other hand \( \|P_4^n\| = \left( \frac{2\sqrt{27}}{n} \right)^n \) and

\[
P_4(x,y)^n = x^n y^n \sum_{k=0}^{n} \binom{n}{k} (-1)^k x^{2k} y^{2(n-k)}.
\]
Hence, if \( \mathbf{a} \) is the vector of the coefficients of \( P_4 \), using the fact that \( | \cdot |_{\frac{n+1}{n+1}} \geq | \cdot |_2 \) (notice that here \( | \cdot |_2 \) is the Euclidian norm), we have

\[
D_{R,4n}(2) \geq \frac{|\mathbf{a}|_{\frac{n+1}{n+1}}}{\|P_4\|_n^n} = \left( \sum_{k=0}^{n} \binom{n}{k} \right)^{\frac{n+1}{n+1}} \left( \frac{2\sqrt{3}}{9} \right)^n
\]

\[\geq \left( \sum_{k=0}^{n} \binom{n}{k} \right)^{\frac{3}{2}} \left( \frac{2\sqrt{3}}{9} \right)^n = \left( \frac{\sqrt{2n}}{n!} \right) \sqrt{\frac{n!}{n^n}} \sqrt{\frac{2\sqrt{3}}{9}} \left( \frac{2\sqrt{3}}{9} \right)^n n! \]

Above we have used the well known formula

\[
\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.
\]

Using Stirling’s approximation formula

\[
n! \sim \sqrt{2\pi n} \left( \frac{n}{e} \right)^n
\]

in (4.2) we have, for \( m = 4n \),

\[
D_{R,m}(2) = D_{R,4n}(2) \geq \frac{\sqrt{2\pi n}}{\sqrt{2\pi n}} \left( \frac{2\pi n}{e} \right)^{2n} \sqrt{2\sqrt{3}} \left( \frac{2\sqrt{3}}{9} \right)^n \left( \frac{n}{e} \right)^n = \left( \frac{4}{m\pi} \right) \left( \sqrt{\frac{27}{2}} \right)^m.
\]

\[\square\]

References

[1] F. Bayart, Hardy spaces of Dirichlet series and their composition operators, Monatsh. Math. 136 (2002), 203–236.
[2] F. Bayart, D. Pellegrino, and J. B. Seoane-Sepúlveda, The Bohr radius of the n-dimensional polydisk is equal to \( \sqrt{(\log n)/\pi} \), arXiv:1310.2834 [math.FA].
[3] H. P. Boas, The football player and the infinite series, Notices Amer. Math. Soc. 44 (1997), no. 11, 1430–1435.
[4] H. P. Boas and D. Khavinson, Bohr’s power series theorem in several variables, Proc. Amer. Math. Soc. 125 (1997), no. 10, 2975–2979.
[5] H. F. Bohnenblust and E. Hille, On the absolute convergence of Dirichlet series, Ann. of Math. (2) 32 (1931), no. 3, 600–622.
[6] A. Defant, L. Frerick, J. Ortega-Cerdà, M. Ounaïes, and K. Seip, The Bohnenblust-Hille inequality for homogeneous polynomials is hypercontractive, Ann. of Math. (2) 174 (2011), no. 1, 485–497.
[7] J. Diestel, H. Jarchow, and A. Tonge, Absolutely summing operators, Cambridge Studies in Advanced Mathematics, vol. 43, Cambridge University Press, Cambridge, 1995.
[8] D. Diniz, G. A. Muñoz-Fernández, D. Pellegrino, and J. B. Seoane-Sepúlveda, The asymptotic growth of the constants in the Bohnenblust-Hille inequality is optimal, J. Funct. Anal. 263 (2012), 415–428.
[9] D. Diniz, G. A. Muñoz-Fernández, D. Pellegrino, and J. B. Seoane-Sepúlveda, Lower bounds for the constants in the Bohnenblust-Hille inequality: the case of real scalars, Proc. Amer. Math. Soc. 142 (2014), 575–580.
[10] P. Erdős, Some remarks on polynomials, Bull. Amer. Math. Soc. 53 (1947), 1169–1176.
[11] A. Montanaro, Some applications of hypercontractive inequalities in quantum information theory, J. Math. Physics 53 (2012).
[12] D. Núñez-Alarcón, A note on the polynomial Bohnenblust-Hille inequality, J. Math. Anal. Appl. 407 (2013), no. 1, 179–181.
[13] D. Pellegrino and J. B. Seoane-Sepúlveda, New upper bounds for the constants in the Bohnenblust-Hille inequality, J. Math. Anal. Appl. 386 (2012), no. 1, 300–307.
[14] H. Queffélec, H. Bohr’s vision of ordinary Dirichlet series: old and new results, J. Anal. 3 (1995), 43–60.
[15] C. Visser, A generalization of Tchebycheff’s inequality to polynomials in more than one variable, Nederl. Akad. Wetensch., Proc. 49 (1946), 455–456 = Indagationes Math. 8, 310–311 (1946).
Departamento de Análisis Matemático, 
Facultad de Ciencias Matemáticas, 
Plaza de Ciencias 3, 
Universidad Complutense de Madrid, 
Madrid, 28040, Spain. 
E-mail address: pablo.jimenez@mat.ucm.es

Departamento de Análisis Matemático, 
Facultad de Ciencias Matemáticas, 
Plaza de Ciencias 3, 
Universidad Complutense de Madrid, 
Madrid, 28040, Spain. 
E-mail address: gustavo.fernandez@mat.ucm.es

Departamento de Matemática, 
Universidade Federal da Paraíba, 
58.051-900 - João Pessoa, Brazil. 
E-mail address: pellegrino@pq.cnpq.br and dmpellegrino@gmail.com

Departamento de Análisis Matemático, 
Facultad de Ciencias Matemáticas, 
Plaza de Ciencias 3, 
Universidad Complutense de Madrid, 
Madrid, 28040, Spain. 
E-mail address: jseoane@mat.ucm.es