STABILITIES AND DYNAMIC TRANSITIONS OF THE FITZHUGH-NAGUMO SYSTEM

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Abstract. The article aims to examine the dynamic transition of the reaction-diffusion Fitzhugh-Nagumo system defined on a thin spherical shell and a 2D-rectangular domain. The mathematical tool employed is the theory of phase transition dynamics established for dissipative dynamical systems. The main results in this paper include two parts. First, for the system on a thin spherical shell, we only focus on the transition from a real simple eigenvalue. More precisely, if the first eigenspace is three-dimensional, the system undergoes either a continuous transition or a jump transition. Besides, a mix transition is also allowed if the first eigenspace is one-dimensional. Second, for the system on a rectangular domain, both the transitions from a simple real eigenvalue and a pair of simple complex eigenvalues are considered. Our results imply that two steady-state solutions bifurcate, which are either attractors or saddle points, and a Hopf bifurcation is also possible in the system on the rectangular domain.

1. Introduction. Fitzhugh-Nagumo (FN) system is originated in the work of Fitzhugh [6] and Nagumo et al. [19]. The system is obtained as a simplification for the Hodgkin-Huxley model describing nerve impulse propagation. Due to the essential feature to describe the initiation and propagation of action potentials in neurons [12], the FN system has been investigated from different angles including bifurcation [12, 8, 10, 17, 27], traveling wave solutions [1, 3, 4, 7, 25, 26] and other dynamic aspects [2, 23]. Schonbek [23] studies the local and global existence of solutions for the FN equations. Heijster and Sandstede [10] investigate the pitchfork bifurcations and Hopf bifurcation from stationary stable spots to traveling spots in a planar three-component FN system by employing the spectrum method and

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center manifold reductions. The condition for the occurrence of Turing bifurcation and amplitude equations are given by Zheng and Shen in [27]. Kunzutsov et al. [12] numerically study the dynamic behavior of a two-component reaction-diffusion system of FN type before the onset of subcritical Turing bifurcation. In addition, Gaduño and Krause [8] numerically investigate Turing and Turing–Hopf bifurcation of the reaction-diffusion FN system on growing tours and spherical domains.

Recently, Mao [17] explores the dynamic transition of the FN system on a bounded 1D domain from the perspective of phase transition dynamics [14]. The main philosophy of phase transition dynamics is to search for the full set of transition states, giving a complete characterization of stability and transition. The theory says that the phase transitions in nonlinear dissipative systems are classified into three categories. The first type is called continuous transition, which is essentially attractor bifurcation (bifurcation from a steady-state to an attractor). The other two types are respectively called jump transition and mixed transition. Due to the inherent physical advantages of the dynamic transition theory, it has been applied to study phase transitions in biological systems, e.g., the dynamic transition of Keller-Segel model [15] and dynamic transition of the Acetabularia whorl formation [18]. For more application of the theory, we refer readers to [5, 9, 11, 20, 21, 24].

Inspired by the recent work [17], in the present article, we aim to explore the dynamic transitions of the two-component reaction-diffusion FN system in a thin spherical shell and a rectangular domain. More precisely, we have two primary purposes. The first goal is to apply the dynamic transition theory [14] to derive a sufficient condition leading to a dynamic transition in the system. The sufficient condition can be derived by conducting the linear stability analysis, which is essentially determining the spectrum of the corresponding linear differential operator of the FN system and verifying the PES (principle of exchange of stability) condition. The second aim is to determine the possible types of dynamic transitions and give the full set of transition states. The types of dynamic transitions are to be described by the corresponding reduced equation derived from the reaction-diffusion FN system by employing the technique of center manifold reduction.

The main results in this paper are comprised of two parts. The first part gives the transitions of the reaction-diffusion FN system in a thin spherical shell, where the transition from a real eigenvalue is only considered. If the dimension of first eigenspace is three, it is shown that the FN system may undergo either a continuous transition or a jump transition. More precisely, a global attractor bifurcates in the FN system, which is a 2D homological sphere as the corresponding control parameter crosses its critical value. The second part describes the transitions of the FN system in a rectangular domain, where both the transitions from real an eigenvalue and a pair of conjugate complex simple eigenvalues are considered. Our results imply that the FN system bifurcates from zero solution to two new equilibrium solutions, which are either attractors or saddle points. Additionally, a periodic solution bifurcates, whose stability is determined by the sign of a non-dimensional coefficient. From the reduced equations, we conclude that the dynamic behavior in a thin spherical shell is more complex than that in a rectangular domain. Finally, we numerically give some examples to illustrate these results in a rectangular domain.

The rest of this paper is organized as follows. In section 2, we present the abstract operator form of the reaction-diffusion FN system. In section 3, the eigenvalues of linearized equations of the system are investigated, and the PES condition is verified. In section 4, based on the PES condition and the technique of center manifold
reduction, we present the main results involved the dynamic transitions, including
the transitions in a thin spherical shell and the transitions in a rectangular domain.
In section 5, some numerical experiments are illustrated to show our results.

2. Mathematical setting.

2.1. Mathematical model. In this paper, we aim to study the dynamics of the
FN system [12, 25] as follows

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= D\Delta u - u(u + \alpha)(u - 1) - v, \\
\frac{\partial v}{\partial t} &= \Delta v + \epsilon(u - v), \\
\end{aligned}
\]

subject to the following initial-boundary conditions

\[
\begin{aligned}
\frac{\partial u}{\partial \nu} |_{\partial \Omega} = 0, & \quad \frac{\partial v}{\partial \nu} |_{\partial \Omega} = 0, \\
u(0, x) = u_0(x), & \quad v(0, x) = v_0(x),
\end{aligned}
\]

where \( \Omega \) is a thin spherical shell or a rectangular domain on \( R^2 \), and \( \nu \) is an outward
normal vector on \( \partial \Omega \). Following [20, 13, 22], the thin spherical shell of interest in
present article is the Cartesian product \( S^2_r \times (0, 1) \) where \( S^2_r \) is the sphere with
radius \( r \), and the rectangular domain of interest is \( (0, l_1) \times (0, l_2) \). The Laplacian
operator \( \Delta \) in the thin spherical shell is given by

\[
\Delta u = \tilde{\Delta} u + \frac{\partial^2 u}{\partial z^2},
\]

in which \( \tilde{\Delta} \) is the Laplacian operator on the \( S^2_r \), defined by

\[
\tilde{\Delta} u = \frac{1}{r^2 \sin \theta} \left[ \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial u}{\partial \theta} \right) + \frac{1}{\sin \theta} \frac{\partial^2 u}{\partial \vartheta^2} \right], \quad \theta \in [0, 2\pi], \quad \vartheta \in [0, \pi].
\]

Besides, the Laplacian operator on the rectangular domain \( (0, l_1) \times (0, l_2) \) is standard, given by

\[
\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}, \quad 0 \leq x_1 \leq l_1, \quad 0 \leq x_2 \leq l_2.
\]

2.2. Abstract form of model. Let \( w_s = (u_s, v_s) \) be the non-negative steady-state
solution of the system (1), then \( u_s, v_s \) are determined by the following algebraic
equations

\[
\begin{aligned}
u_s(u_s + \alpha)(u_s - 1) + v_s &= 0, \\
u_s - v_s &= 0.
\end{aligned}
\]

From (4), we derive that

\[
u_s = v_s = 0, \quad \text{or} \quad u_s = v_s = \frac{1 - \alpha \pm \sqrt{\alpha^2 + 2\alpha - 3}}{2}.
\]

Apparently, the system (1) has the unique trivial equilibrium point \( (0, 0) \) provided
\(-3 < \alpha < 1\). In this paper, we shall consider the dynamics of the system (1) in the
vicinity of the trivial steady-state \( w_s = (0, 0) \). More precisely, we only focus on the
dynamic transitions from the trivial state and the corresponding transition types.
For the thin spherical shell domain \( \Omega = S^2_r \times (0, 1) \), we introduce the following spaces
\[
Y_1 = \left\{ w = (u, v) \in H^2(\Omega)^2 \left| \frac{\partial w}{\partial z} = 0 \text{ at } z = 0, 1 \right. \right\}, \quad Y = \{ w \in L^2(\Omega)^2 \}.
\]

Correspondingly, for the rectangular domain \( \Omega = (0, l_1) \times (0, l_2) \), we introduce the following function spaces
\[
X_1 = \left\{ w = (u, v) \in H^2(\Omega, \mathbb{R}^2) \left| \left| \frac{\partial w}{\partial \nu} \right|_{\partial \Omega} = 0 \right. \right\}, \quad X = \{ w \in L^2(\Omega, \mathbb{R}^2) \}.
\]

With the help of above function spaces, let us define the linear operator \( L_\alpha : Y_1 (X_1) \rightarrow Y (X) \) as follows
\[
L_\alpha w = -Aw + B_\alpha w,
\]
where
\[
-Aw = \begin{pmatrix} D\Delta & 0 \\ 0 & \Delta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \quad B_\alpha u = \begin{pmatrix} \alpha & -1 \\ \epsilon & -\epsilon \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.
\]

It is not tough to verify that \( L_\alpha \) is a family of completely continuous operator, which is continuously depending on the single control parameter \( \alpha \). In addition, we define the nonlinear operator \( G : Y_1 (X_1) \rightarrow Y (X) \) as follows
\[
G(w, \alpha) = \begin{pmatrix} -w^3 - (\alpha - 1)u^2 \\ 0 \end{pmatrix}.
\]

Combining (5) and (6), the system (1)-(3) is then put into the abstract form of
\[
\begin{cases}
dw \over dt &= L_\alpha w + G(w, \alpha), \\
w(0) &= w_0,
\end{cases}
\]
where \( w_0 = (u_0(x), v_0(x))^T \) is the initial value.

3. Eigenvalues and principle of exchange of stabilities. In this section, we shall investigate the eigenvalues of the linear part of (1)-(3) and verify the PES condition. The corresponding eigenvalue problem reads
\[
L_\alpha w = \beta w, \quad w \in Y_1 (X_1).
\]

To solve the preceding eigenvalue problem, we need to first solve the eigenvalue problem
\[
-\Delta \varphi = \rho \varphi, \quad \varphi \in Y_1 (X_1),
\]
whose all eigenvectors consist of a basis for the function space \( Y_1 (X_1) \).

If \( \Omega = S^2_r \times (0, 1) \), it is well known that there are \( 2l + 1 \) eigenvectors
\[
\varphi_{lmn} = Y_{lm}(\theta, \varphi) \cos(n\pi z), \quad m \in \{-l, -l + 1, \cdots, 0, \cdots, l - 1, l\}
\]
of the operator \( -\Delta \) on the space \( Y_1 \) with the corresponding eigenvalue
\[
\rho_{ln} = \frac{l(l + 1)}{r^2} + (n\pi)^2, \quad k = (l, n) \in \mathbb{N} \times \mathbb{Z},
\]
where \( Y_{lm} \) are spherical harmonics. If \( \Omega = (0, l_1) \times (0, l_2) \), each eigenvalue \( \rho_{ln} \) the operator \( -\Delta \) on the space \( X_1 \) explicitly given by
\[
\rho_{ln} = \left( \frac{l\pi}{l_1} \right)^2 + \left( \frac{n\pi}{l_2} \right)^2, \quad (l, n) \in \mathbb{Z} \times \mathbb{Z},
\]
and the corresponding eigenvector $\varphi_{ln}$ is

$$\varphi_{ln} = \frac{2}{\sqrt{l_1 l_2}} \cos \left( \frac{l_1 \pi x_1}{l_1} \right) \cos \left( \frac{n \pi x_2}{l_2} \right).$$

With the help of the exact expressions of eigenvalues of the Laplacian operator $-\Delta$ on the space $Y_1 (X_1)$, the eigenvalue problem (8) can be solved by analyzing a finite dimensional eigenvalue problem. More precisely, let us define

$$M_{ln}(\alpha) = \begin{pmatrix} -D\rho_{ln} + \alpha & -1 \\ -\rho_{ln} - \epsilon & -\rho_{ln} - \epsilon \end{pmatrix},$$

then each eigenvalue $\beta_{ln}$ of $L_\alpha$ and the corresponding eigenvectors $e_{lmn} = \eta_{ln}\varphi_{lmn}$ or $e_{ln} = \eta_{ln}\varphi_{ln}$

(14)

can be derived by solving

$$M_{ln}(\alpha)\eta_{ln} = \beta_{ln}\eta_{ln},$$

(15)

in which $\varphi_{lmn}$ ($\varphi_{ln}$) is the eigenvector of $-\Delta$ corresponding to its $\rho_{ln}$.

For the eigenvalue problem (15), the eigenvalue $\beta_{ln}$ can be solved from the quadratic equation

$$\beta_{ln}^2 - B_{ln}(\alpha)\beta_{ln} + C_{ln}(\alpha) = 0,$$

where

$$B_{ln}(\alpha) = \text{trace}(M_{ln}(\alpha)) = -D\rho_{ln} - \rho_{ln} + \alpha - \epsilon,$$

$$C_{ln}(\alpha) = \text{det}(M_{ln}(\alpha)) = D\rho_{ln}^2 + \epsilon D\rho_{ln} + \epsilon - \alpha(\rho_{ln} + \epsilon).$$

After some straightforward computations, one can see that each $\beta_{ln}$ has two values

$$\beta_{ln}^1 = \frac{B_{ln} + \sqrt{B_{ln}^2 - 4C_{ln}}}{2}, \quad \beta_{ln}^2 = \frac{B_{ln} - \sqrt{B_{ln}^2 - 4C_{ln}}}{2},$$

(16)

which are the eigenvalues of the eigenvalue problem (8).

The exact expression of $\beta_{ln}^1$ enables us to find the critical control parameter characterizing the lose of linear stability of the zero solution to the system (7) by solving the equation $\beta_{ln}^1 = 0$, which gives

$$\alpha = \frac{D\rho_{ln}^2 + \epsilon D\rho_{ln} + \epsilon}{\rho_{ln} + \epsilon}.$$

The right hand side of the preceding equation is determined by each $\rho_{ln}$ for each specified $D, \epsilon$ and the domain $\Omega$, which attains its minimum at some $(l, n) = (p, q)$. The minimum can be used to analyze the linear stability of the zero solution of the system (7). More precisely, let us define two parameters

$$\alpha_c = \epsilon, \quad \tilde{\alpha}_c = \frac{D\rho_{pq}^2 + \epsilon D\rho_{pq} + \epsilon}{\rho_{pq} + \epsilon} = \min_{\rho_{ln}} \frac{D\rho_{ln}^2 + \epsilon D\rho_{ln} + \epsilon}{\rho_{ln} + \epsilon},$$

(17)

then we have the following theorem.

**Theorem 3.1.** Let the parameter $\tilde{\alpha}_c$ be given as (17). Then, for the eigenvalue problem (8), the following assertions hold true
Thus, we obtain that 

\[ \beta_p^1(\alpha) = \begin{cases} 
0, & \alpha > \tilde{\alpha}_c, \\
< 0, & \alpha < \tilde{\alpha}_c, 
\end{cases} \]

by which one can have

\[ \text{Step 1, we show the assertion (1).} \]

The proof is divided into two steps.

**Remark 1.** If the control parameters \( D \) and \( \epsilon \) satisfy \( D\epsilon \geq 1 \), then the critical value \( \tilde{\alpha}_c \) attains at the unique point \( (p, q) = (0, 0) \).

**Proof.** This proof is divided into two steps.

**Step 1, we show the assertion (1).** By the definition of the parameters \( \alpha_c \) and \( \tilde{\alpha}_c \), we derive that

\[ C_{pq}(\tilde{\alpha}_c) = D\rho_{pq}^2 + \epsilon D\rho_{pq} + \epsilon - \tilde{\alpha}_c(\rho_{pq} + \epsilon) \]

\[ = D\rho_{pq}^2 + \epsilon D\rho_{pq} + \epsilon - (\rho_{pq} + \epsilon) \frac{D\rho_{pq}^2 + \epsilon D\rho_{pq} + \epsilon}{\rho_{pq} + \epsilon} = 0. \]

Moreover, we have

\[ B_{pq}(\tilde{\alpha}_c) = -D\rho_{pq} - \rho_{pq} + \tilde{\alpha}_c - \epsilon < -D\rho_{pq} - \rho_{pq} + \alpha_c - \epsilon \]

\[ = -D\rho_{pq} - \rho_{pq} \leq 0. \]

Thus, we obtain that \( \beta_p^1(\tilde{\alpha}_c) = 0 \) and \( \beta_p^2(\tilde{\alpha}_c) < 0 \).

If \( \alpha > \tilde{\alpha}_c \), we derive that

\[ C_{pq}(\alpha) = D\rho_{pq}^2 + \epsilon D\rho_{pq} + \epsilon - \alpha(\rho_{pq} + \epsilon) \]

\[ < D\rho_{pq}^2 + \epsilon D\rho_{pq} + \epsilon - \tilde{\alpha}_c(\rho_{pq} + \epsilon) = 0, \]

by which one can have

\[ \beta_p^1(\alpha) = \frac{B_{pq}(\alpha) + \sqrt{B_{pq}^2(\alpha) - 4C_{pq}(\alpha)}}{2} \]

\[ \geq \frac{B_{pq}(\alpha) + \sqrt{B_{pq}^2(\alpha)}}{2} \geq 0. \]

If \( \alpha < \tilde{\alpha}_c \). Similarly, we have

\[ B_{pq}(\alpha) = -D\rho_{pq} - \rho_{pq} + \alpha - \epsilon \]

\[ < -D\rho_{pq} - \rho_{pq} + \tilde{\alpha}_c - \epsilon \]

\[ < -D\rho_{pq} - \rho_{pq} + \alpha_c - \epsilon \leq 0, \]

\[ C_{pq}(\alpha) = D\rho_{pq}^2 + \epsilon D\rho_{pq} + \epsilon - \alpha(\rho_{pq} + \epsilon) \]

\[ > D\rho_{pq}^2 + \epsilon D\rho_{pq} + \epsilon - \tilde{\alpha}_c(\rho_{pq} + \epsilon) = 0, \]
which imply that $\beta_{pq}^1(\alpha) < 0$. Hence, the PES condition (18) hold true.

In addition, $(l, n) \neq (p, q)$, we have

$$B_{ln}(\tilde{\alpha}_c) = -D\rho_{ln} - \rho_{ln} + \tilde{\alpha}_c - \epsilon < -D\rho_{ln} - \rho_{ln} + \alpha_c - \epsilon \leq 0,$$

and

$$C_{ln}(\tilde{\alpha}_c) = D\rho_{ln}^2 + \epsilon D\rho_{ln} + \epsilon - \tilde{\alpha}_c(\rho_{ln} + \epsilon)$$

$$> D\rho_{ln}^2 + \epsilon D\rho_{ln} + \epsilon - (\rho_{ln} + \epsilon) \frac{D\rho_{ln}^2 + \epsilon D\rho_{ln} + \epsilon}{\rho_{ln} + \epsilon} = 0.$$

Consequently, the conclusion (1) stated in Theorem 3.1 is valid.

**Step 2. We shall prove the conclusion (2).** From the definition of the parameters $\alpha_c$ and $\tilde{\alpha}_c$, it is clear that

$$B_{00}(\alpha) \begin{cases} > 0, & \alpha > \alpha_c, \\ = 0, & \alpha = \alpha_c, \\ < 0, & \alpha < \alpha_c, \end{cases}$$

and $C_{00}(\alpha_c) = \epsilon(1 - \epsilon) \neq 0$. Hence, we obtain (19). Similarly, direct calculations give $B_{ln}(\alpha) < 0$ and $C_{ln}(\alpha_c) > 0$. Consequently, the conclusion (2) hold.

4. Nonlinear dynamic transitions.

4.1. Dynamic transition in a thin spherical shell. If $\tilde{\alpha}_c < \alpha_c$, the PES condition (18) says that the system (7) must undergo one of three types of transitions. In this section, we shall establish the nonlinear transition theorem for the system (7) defined on the thin spherical shell $S^2_r \times (0, 1)$.

From section 3, we have known that there are $2l + 1$ eigenvectors

$$\varphi_{lmn} = Y_{lm}(\theta, \vartheta) \cos(n\pi z), \quad m \in \{-l, -l + 1, \ldots, 0, \ldots, l - 1, l\}$$

corresponding to the eigenvalue

$$\rho_{ln} = \frac{l(l + 1)}{r^2} + (n\pi)^2, \quad k = (l, n) \in \mathbb{N} \times \mathbb{Z},$$

of the operator $-\Delta$. Then, (14) means that there are $2l + 1$ eigenvectors $e^i_{lnm}$ solving the eigenvalue problem

$$L_\alpha e^i_{lnm} = \beta^i_{ln} e^i_{lnm}, \quad e^i_{lnm} \in Y_1, \quad i = 1, 2,$$

where $\beta^1_{ln}$ and $\beta^2_{ln}$ are given by (16). If the eigenvalues $\beta^1_{ln}$ and $\beta^2_{ln}$ are real, then $e^i_{lnm}$ are explicitly given by

$$e^i_{lnm} = \left( \frac{\varphi_{lmn}}{\varphi_{lmn} h^i_{ln}} \right), \quad h^i_{ln} = \frac{\epsilon}{\beta^i_{ln} + \rho_{ln} + \epsilon}. \quad (20)$$

Denote the dual operator of $L_\alpha$ as $L^*_\alpha$, we find that the dual eigenvalue $\beta^*_i$ of $L^*_\alpha$ satisfy $\beta^*_i = \beta^i_{ln}(i = 1, 2)$, and dual eigenvectors $e^i_{lnm}$ corresponding to $\beta^*_i$ are

$$e^i_{lnm}^* = \left( \frac{\varphi_{lmn}}{\varphi_{lmn} h^*_i} \right), \quad h^*_i = -\frac{1}{\beta^*_i + \rho_{ln} + \epsilon}. \quad (21)$$

If $\tilde{\alpha}_c < \alpha_c = \epsilon$, then in the vicinity of the $\alpha_c$, the eigenvalue $\beta^1_{pq}$ given in (16) satisfies the PES condition (18) and the multiplicity of $\beta^1_{pq}$ is $2p + 1$. Hence, making use of these expressions given in (20)-(21), we establish the following theorem.
Theorem 4.1. If \( \tilde{\alpha}_c < \alpha_c \), then the stabilities and transitions of the system (7) in the vicinity of \( \tilde{\alpha}_c \) are determined by a system of 2p + 1 ODEs given by

\[
\frac{dx_m}{dt} = \beta_{pq}^1 x_m + \omega_{pq} x_m |x|^2 + \sum_{m_1+m_2=m \atop p_1+p_2=m_2, \atop i \in \{2,\ldots,2p\}} \xi_i^p_{|p|p,l} c_{p,p,l}^1 x_m x_{m_1} x_{m_2} x_{m_3} + o(3), \quad m = -p, \ldots, p,
\]

where \( |x|^2 = \sum_{m=-p}^p |x_m|^2 \) and the sum is over \( |m_1| \leq p \) as well as \( |p_1| \leq p \). In addition, \( c_{i,j,l}^1, \omega_{pq}, \xi_i^p_{|p|p,l} \) are defined by (26) and (30)-(32), and

\[
P_i^l = 1 - \frac{\epsilon}{|\beta_i^l + \rho_i + \epsilon|^2}
\]

for \( i = 1, 2 \). Particularly, for \( p = 1 \), the reduced equations are expressed by

\[
\begin{align*}
\frac{dx_{-1}}{dt} &= \beta_{11}^1 x_{-1} + \eta_1 x_{-1} |x|^2 + \frac{81}{1024 P_{14}^1} x_{-1} (x_{-1})^2 + o(3), \\
\frac{dx_{0}}{dt} &= \beta_{11}^1 x_{0} + \eta_1 x_{0} |x|^2 - \frac{162}{1024 P_{14}^1} x_{0} (x_{0})^2 + o(3), \\
\frac{dx_{1}}{dt} &= \beta_{11}^1 x_{1} + \eta_1 x_{1} |x|^2 + \frac{81}{1024 P_{14}^1} x_{1} (x_{1})^2 + o(3),
\end{align*}
\]

where \( \eta_1 = \omega_{1q} + \frac{1}{8 \pi} \sum_{i=1}^{2} \xi_i^{2,1,q} - \frac{162}{1024 P_{14}^1} \).

Proof. Let \( E_1 \) be the space

\[
E_1 = \left\{ \sum_{m=-p}^p x_m \epsilon_{pmq}^1 : x_{-m} = (-1)^m x_m, x_m \in R \right\}
\]

and define \( E_2 = E_1^+ \), then by the spectral theory of linearly complete continuous fields, we have the decomposition \( Y = E_1 \oplus E_2 \). Note that the solution to the system (7) can be decomposed into

\[
w = \sum_{m=-p}^p x_m \epsilon_{pmq}^1 + y,
\]

where \( y \in E_2 \). Thus, on the center manifold, the system (7) can be reduced to

\[
\frac{dx_m}{dt} = \beta_{pq}^1 x_m + \frac{2}{P_{pq}^1} \left\langle G \left( \sum_{m=-p}^p x_m \epsilon_{pmq}^1 + \Phi, \alpha \right), \epsilon_{pmq}^1 \right\rangle
\]

where \( m \in \{-p, -p+1, \ldots, 0, 1, \ldots, p\} \), and \( \Phi : E_1 \rightarrow E_2 \) is the corresponding center manifold function associated with the PES condition (18).

In what follows, we aim to calculate the leading order terms of the center manifold function \( \Phi \). Ma and Wang [14] have shown that \( \Phi \) solves the following equation

\[
-L_\alpha \Phi = P_2 G_2 \left( \sum_{m=-p}^p x_m \epsilon_{pmq}^1, \alpha \right) + o(|x|^2) + O \left( |\beta_{pq}^1| |x|^2 \right),
\]

where \( |x|^2 = \sum_{m=-p}^p |x_m|^2 \) and

\[
G_2(w, \alpha) = \begin{pmatrix} -\alpha - 1 & u^2 \\ 0 & 0 \end{pmatrix}.
\]
Direct calculations give
\[ G_2 \left( \sum_{m=-p}^{p} x_m c_{pmq}^i, \alpha \right) = \sum_{m_1=-p}^{p} X_{m_1 m_2} \cos^2(q\pi z) \begin{pmatrix} -(\alpha - 1) \\ 0 \end{pmatrix}, \]
in which
\[ X_{m_1 m_2} = Y_{pm_1} Y_{pm_2} x_{m_1} x_{m_2}. \]
Furthermore, utilizing the relation
\[ \cos^2(q\pi z) = \cos(2q\pi z) + 1 \]
and the orthogonality of the spherical harmonics (see appendix in [20]), the center manifold function \( \Phi \) can be expressed in form of
\[ \Phi = \sum_{i=1}^{2} y_{i000}^0 e_{i000}^0 + \sum_{|m|\leq l}^{l\in\{2 \ldots 2p\}} \sum_{i\in\{1,2\}}^{i\in\{1,2\}} y_{i0m0}^l e_{i0m0}^l + \sum_{|m|\leq l}^{l\in\{2 \ldots 2p\}} \sum_{i\in\{1,2\}}^{i\in\{1,2\}} y_{i1m2q}^l e_{i1m2q}^l + o(2), \quad (25) \]
where
\[ e_{i000}^0 = \left( \frac{1}{h^0_{i0}} \right) Y_{00}(\theta, \vartheta), \quad e_{i0m0}^l = \left( \frac{1}{h^l_{i0}} \right) Y_{lm}(\theta, \vartheta), \]
\[ e_{i1m2q}^l = \left( \frac{1}{h^l_{i2q}} \right) Y_{lm}(\theta, \vartheta) \cos(2q\pi z). \]
Upon performing direct some computations, we yield
\[ \beta_{00}^i y_{000}^i \left( e_{i000}^0, e_{i000}^0 \right) = \frac{r^2}{2} (\alpha - 1) Y_{00}^2, \]
\[ \beta_{10}^i y_{1m0}^l \left( e_{i1m0}^l, e_{i1m0}^l \right) = \frac{r^2}{2} (\alpha - 1) \sum_{m_1=-p}^{p} c_{p,m_1,m_2,m} x_{m_1} x_{m_2}, \]
\[ \beta_{12q}^i y_{1m2q}^l \left( e_{i1m2q}^l, e_{i1m2q}^l \right) = \frac{r^2}{4} (\alpha - 1) \sum_{m_1=-p}^{p} c_{p,m_1,m_2,m} x_{m_1} x_{m_2}, \]
where
\[ c_{p,m_1,m_2,m} = \frac{1}{r^{2l}} \int_{S^2} Y_{l_1,m_1} Y_{l_2,m_2} Y_{lm} dS^2. \quad (26) \]
Thus, we have
\[ y_{000}^i = \frac{(\alpha - 1)}{2\beta_{00}^i \beta_{00}^0} Y_{00}^2, \quad (27) \]
\[ y_{1m0}^l = \frac{(\alpha - 1)}{2\beta_{10}^l \beta_{10}^0} \sum_{m_1=-p}^{p} c_{p,m_1,m_2,m} x_{m_1} x_{m_2}, \quad (28) \]
\[ y_{1m2q}^l = \frac{(\alpha - 1)}{2\beta_{12q}^l \beta_{12q}^0} \sum_{m_1=-p}^{p} c_{p,m_1,m_2,m} x_{m_1} x_{m_2}. \quad (29) \]
Denote
\[ \Phi_1 = \sum_{i=1}^{2} y_{i00}^i Y_{00} + \sum_{l \in \{2, \ldots, 2p\}} y_{l0}^i Y_{lm} + \sum_{l \in \{2, \ldots, 2p\}} y_{lm2}^i Y_{lm} \cos(2q\pi z), \]
\[ \Phi_2 = \sum_{i=1}^{2} y_{i00}^i h_{00}^i + \sum_{l \in \{2, \ldots, 2p\}} y_{l0}^i Y_{lm} h_{l0}^i + \sum_{l \in \{2, \ldots, 2p\}} y_{lm2}^i Y_{lm} \cos(2q\pi z) h_{l2q}^i, \]
then the center manifold function can be rewritten as
\[ \Phi = (\Phi_1, \Phi_2)^T + o(2). \]

With the help of (25)-(29), we finally derive that
\[ \frac{2}{P_1 p_2^2} \left( G \left( \sum_{m_1 = -p}^{p} x_{m_1} e_{pm_1q} + \Phi, \alpha \right), e_{pm_1q}^T \right) \]
\[ = \omega_{pq} x_m |x|^2 - \sum_{m_1 + m_2 = m \atop p_1 + p_2 = p_2} \xi_{l,pq}^{m_1,m_2,m} x_{p_1,p_2,m_2} x_{p_1} x_{p_2} x_{m_1} \]
\[ - \frac{3}{4 P_1 p_2^2} \sum_{m_1 = -p \atop m_2 = -p \atop m_3 = -p} \tilde{b}_{p_1,p_2,p_2,p_3}^{m_1,m_2,m_3,m} x_{m_1} x_{m_2} x_{m_3} + o(3), \]
where
\[ \omega_{pq} = -\frac{(\alpha - 1)^2}{4\pi P_1 p_2^2} \sum_{i=1}^{2} \frac{1}{P_{i00} P_{00}^i}, \quad \xi_{l,pq}^{m_1,m_2,m} = -\frac{1}{P_1 p_2^2} \left( \frac{(\alpha - 1)^2}{2 P_{p_1} p_{2q}^2} + \frac{(\alpha - 1)^2}{2 P_{p_2} p_{1q}^2} \right), \]
\[ \tilde{b}_{p_1,p_2,p_2,p_3}^{m_1,m_2,m_3,m} = \frac{1}{\sqrt{2}} \int_{S^2} Y_{p_1 m_1} Y_{p_2 m_2} Y_{p_3 m_3} \tilde{Y}_{pm} dS^2. \]

Particularly, for \( p = 1 \), some direct computations give
\[ \sum_{m_1,m_2,m_3 = -1}^{1} b_{1,1,1}^{m_1,m_2,m_3,1} x_{m_1} x_{m_2} x_{m_3} = \frac{54}{256} x_{-1}^2 + \frac{27}{256} x_{-1} x_{-1}, \]
\[ \sum_{m_1,m_2,m_3 = -1}^{1} b_{1,1,1}^{m_1,m_2,m_3,0} x_{m_1} x_{m_2} x_{m_3} = \frac{54}{256} x_{0}^2 + \frac{108}{256} x_{0} x_{0}, \]
\[ \sum_{m_1,m_2,m_3 = -1}^{1} b_{1,1,1}^{m_1,m_2,m_3,1} x_{m_1} x_{m_2} x_{m_3} = \frac{54}{256} x_{1}^2 + \frac{27}{256} x_{1} x_{1}, \]
\[ \sum_{m_1+m_2=m \atop p_1+p_2=m_2 \atop i \in \{1,2\}}^{1} \xi_{2,1,q}^{m_1,m_2,m} c_{p_1,p_2,m_2} x_{p_1} x_{p_2} x_{m_1} = \frac{1}{\sqrt{2}} \sum_{i=1}^{2} \xi_{2,1,q}^{m_1,m_2,m} x_{m} |x|^2, \]
for $m = -1, 0, 1$. Finally, making use of (33)-(36), we derive equations (23).

Relying on the reduced equations given in the Theorem 4.1 and attractor bifurcation theorem in [14, 16], we aim to establish the corresponding transition theorem for the nonlinear system (7) by introducing the following parameter

$$\dot{q}(\alpha) = \omega_{1q} + \frac{1}{5\pi} \sum_{i=1}^{2} \xi_{2,iq} - \frac{81}{1024P_{1q}^{1}},$$

where $\omega_{1q}$ and $\xi_{2,iq}$ are given as (30)-(31), respectively.

**Theorem 4.2.** If $\hat{\alpha}_c < \alpha_c$ and $p = 1$, then the system (7) undergoes a dynamic transition. More precisely, the following assertions hold true:

1. If $\dot{q}(\hat{\alpha}_c) < 0$, then the system (7) undergoes a continuous transition. Namely, an attractor $A_{\alpha}$ with $2 \leq \dim A_{\alpha} \leq 3$ bifurcates on $\alpha > \hat{\alpha}_c$. The attractor $A_{\alpha}$ is an 2-dimensional homological sphere which attracts $Y \setminus \Gamma$, where $\Gamma$ is the stable manifold of $w = 0$ with codim 2.

2. If $\dot{q}(\hat{\alpha}_c) > 0$, then the system (7) undergoes a jump transition as $\alpha > \hat{\alpha}_c$. Namely, there exists an open and dense set $U$ of $w = 0$ in $Y$ such that for any $w_0 \in U$ and for every $\alpha \in (\hat{\alpha}_c, \alpha_c + \epsilon)$ with some $\epsilon > 0$, the solution of (7) satisfies

$$\limsup_{t \to \infty} ||w(t, w_0)|| = \delta > 0,$$

where $\delta$ is independent of $\alpha$.

**Proof.** From Theorem 4.1, the stability and dynamic transition of the system (7) are determined by the reduced equations (23). At the critical value $\alpha = \hat{\alpha}_c$, we have $\beta_{1q}^1 = 0$. Therefore, the reduced equations (23) at $\alpha = \hat{\alpha}_c$ become

$$\begin{cases}
\frac{dx_{-1}}{dt} = \eta_1 x_{-1} |x|^2 + \frac{81}{1024P_{1q}^{1}} x_{-1} (x_{-1})^2, \\
\frac{dx_0}{dt} = \eta_1 x_0 |x|^2 - \frac{162}{1024P_{1q}^{1}} x_0 (x_0)^2, \\
\frac{dx_1}{dt} = \eta_1 x_1 |x|^2 + \frac{81}{1024P_{1q}^{1}} x_1 (x_1)^2.
\end{cases}$$

For the preceding equations, their Lyapunov function is

$$V(x_{-1}, x_0, x_1) = \frac{1}{2} (x_{-1}^2 + x_0^2 + x_1^2),$$

satisfying $V(0, 0, 0) = 0$ and $V(x_{-1}, x_0, x_1) > 0$ for $(x_{-1}, x_0, x_1) \neq (0, 0, 0, 0)$. Moreover, some direct computations yield that

$$\frac{dV}{dt} = \left( \omega_{1q} + \frac{1}{5\pi} \sum_{i=1}^{2} \xi_{2,iq} - \frac{81}{1024P_{1q}^{1}} \right) |x|^4 - \frac{162}{1024P_{1q}^{1}} (x_{-1}^4 + x_0^4 + x_1^4).$$

Direct check finds that $\frac{dV}{dt} < 0$ if $\dot{q}(\hat{\alpha}_c) < 0$, i.e., $(x_{-1}, x_0, x_1) = (0, 0, 0)$ is asymptotically stable while $(x_{-1}, x_0, x_1) = (0, 0, 0)$ is unstable if $\dot{q}(\hat{\alpha}_c) > 0$. Then, the conclusions of Theorem 4.2 hold by the attractor bifurcation theorem in [14].

For the non-general case of $(p, q) = (0, 0)$, we have the mixed transition theorem.

**Theorem 4.3.** If $\hat{\alpha}_c < \alpha_c$ and $(p, q) = (0, 0)$, then we have the following assertions:
The system (7) undergoes mixed transition at \((0, \tilde{\alpha}_c)\). Namely, there is a neighborhood \(U \subset Y\) of \(w = 0\) such that \(U\) is decomposed into two disjoint open sets \(U_1^\alpha\) and \(U_2^\alpha\) by the stable manifold \(\Gamma_\alpha\) of \(w = 0\), satisfying

(i) \(U = U_1^\alpha + U_2^\alpha + \Gamma_\alpha\);

(ii) The transition in \(U_1^\alpha\) is continuous, and in \(U_2^\alpha\) is jump.

A singular point \(\tilde{w}(\alpha)\) in \(U_1^\alpha\) bifurcates on \(\alpha > \tilde{\alpha}_c\), satisfying

\[
\lim_{t \to \infty} ||w(t, w_0) - w(\alpha)|| = 0, \quad \text{for any } w_0 \in U_1^\alpha.
\]

A unique unstable point \(\tilde{\tilde{w}}(\alpha)\) in \(U_2^\alpha\) bifurcates on \(\alpha < \tilde{\alpha}_c\).

The bifurcating singular point \(w(\alpha)\) has the following approximation

\[
w(\alpha) = \frac{(4\pi)^2}{2} \frac{P_1^{1\alpha}}{(\alpha - 1)} e_0^{1\alpha} + o(3).
\]

**Proof.** Denote the space \(E_1 = \text{span}\{e_0^{1\alpha}\}\), and let \(E_2 = E_1^\perp\). With the method used in the proof of Theorem 4.2, the system (7) can be reduced to

\[
dx{t} = \beta_0^{1\alpha} x + \frac{1}{P_1^{1\alpha}} \langle G(xe_0^{1\alpha} + \Phi(x, \alpha), e_0^{1\alpha}) , e_0^{1\alpha} \rangle,
\]

where \(\Phi : E_1 \to E_2\) is the center manifold function.

From (24), we know that

\[
G_2(xe_0^{1\alpha}, \alpha) = x^2 Y_0 \begin{pmatrix} - (\alpha - 1) \\ 0 \end{pmatrix}.
\]

Therefore, owing to the following center manifold approximation formula

\[
-L_\alpha \Phi = P_2 G_2(xe_0^{1\alpha}, \alpha) + o(|x|^2) + O(|\beta_0^{1\alpha}| |x|^2),
\]

we obtain \(\Phi = o(|x|^2)\) by which we arrive at

\[
\left\langle \frac{1}{P_1^{1\alpha}} G(xe_0^{1\alpha} + \Phi(x, \alpha), e_0^{1\alpha}) , e_0^{1\alpha} \right\rangle = - \frac{(\alpha - 1)}{(4\pi)^2} \frac{P_1^{1\alpha}}{P_0^{1\alpha}} |x|^2 + o(|x|^3).
\]

Consequently, by the Theorem 2.3.2 in [14], we obtain all assertions.

**4.2. Dynamic transition in a rectangular domain.** In this subsection, we focus on the transition types of (7) in a rectangular domain \(\Omega = (0, l_1) \times (0, l_2)\).

From (15), if both of the eigenvalues \(\beta_{1n}^i\) \((i = 1, 2)\) are real, one can see that the corresponding eigenvectors \(e_{1n}^i\) are explicitly given by

\[
e_{1n}^i = \left( \frac{\varphi_{1n}}{\varphi_{1n} h_{1n}^i} \right), \quad h_{1n}^i = \frac{\epsilon}{\beta_{1n}^i + \rho_{1n} + \epsilon}.
\]

The eigenvalues \(\beta_{1n}^i\) of \(L_\alpha^*\) satisfy \(\beta_{1n}^i = \beta_{1n}^i\) \((i = 1, 2)\) and the dual eigenvectors \(e_{1n}^{*i}\) corresponding to \(\beta_{1n}^i\) are

\[
e_{1n}^{*i} = \left( \frac{\varphi_{1n}}{\varphi_{1n} h_{1n}^{*i}} \right), \quad h_{1n}^{*i} = -\frac{1}{\beta_{1n}^i + \rho_{1n} + \epsilon}.
\]
We introduce a parameter \( b(\alpha) \) defined by

\[
b(\alpha) = -2(\alpha - 1)^2 \frac{1}{l_1 l_2} \left[ \frac{1}{\beta_{00} P_{00}^4} + \frac{1}{\beta_{20} P_{20}^4} \right] + \left( \frac{1}{\beta_{20} P_{20}^4} + \frac{1}{\beta_{20} P_{20}^4} \right) + \left( \frac{1}{\beta_{02} P_{02}^4} + \frac{1}{\beta_{02} P_{02}^4} \right) + \frac{1}{4} \left( \frac{1}{\beta_{20} P_{20}^4} + \frac{1}{\beta_{20} P_{20}^4} \right) - \frac{9}{4} \frac{1}{l_1 l_2},
\]

where \( P_{ln}^i \) is given in (22). Then, relying on these explicit expressions (37)-(38), and with the method of center manifold reduction, we have the following theorem, which describes that two new steady-state solutions are bifurcated.

**Theorem 4.4.** If \( \hat{\alpha}_c < \alpha_c \) in a rectangular domain \((0, l_1) \times (0, l_2) \subset R^2 \) which satisfies \( l_1 \neq l_2 \) as well as \((p, q) \neq (0,0)\), then the following assertions hold:

1. If the parameter \( b(\hat{\alpha}_c) < 0 \), then the system (7) undergoes a continuous transition from \((0, \hat{\alpha}_c)\) and two stable singular points \( w^\pm \) bifurcate on \( \alpha > \hat{\alpha}_c \).
2. If the parameter \( b(\hat{\alpha}_c) > 0 \), then the system (7) undergoes a jump transition from \((0, \hat{\alpha}_c)\) and two saddle points \( w^\pm \) bifurcate on \( \alpha < \hat{\alpha}_c \).
3. The bifurcating points \( w^\pm \) are expressed by

\[
w^\pm = \pm \sqrt{-\frac{\beta_{pq}^1}{b(\hat{\alpha}_c)} \beta_{pq}^1 p_{pq}^1 e_{pq}^1 + \Phi(\frac{\beta_{pq}^1}{b(\hat{\alpha}_c)})} + o(3),
\]

where \( \Phi \) is given as (44).

**Proof.** Denote the linear space spanned by \( \{e_{pq}^1\} \) as \( E_1 \). Let \( E_2 \) be the space spanned by rest of eigenvectors. By the spectral theory of linearly complete continuous fields, the spaces \( X \) can be decomposed into \( X = E_1 \oplus E_2 \). Thus, in the vicinity of \( \hat{\alpha}_c \), the solution to the system (7) has the following decomposition

\[
w = x_{pq} e_{pq}^1 + y, \quad y = \sum_{(i,n)\neq(p,q)} x_{ln} e_{ln}^1 + \sum_{(i,n) = (0,0)} x_{ln} e_{ln}^2.
\]

where \( x_{pq} e_{pq}^1 \in E_1 \) and \( y \in E_2 \). Hence, on the center manifold, the system (7) can be reduced to

\[
\frac{dx_{pq}}{dt} = \beta_{pq}^1 x_{pq} + \frac{1}{(e_{pq}^1 e_{pq}^1)^{1+\alpha}} \langle G(x_{pq} e_{pq}^1 + \Phi, \alpha), e_{pq}^1 \rangle, \quad (40)
\]

where \( \Phi : E_1 \rightarrow E_2 \) is the corresponding center manifold function.

Next, we calculate the center manifold function \( \Phi \) which solves the equation

\[
-L \beta \Phi = P_2 G_2(x_{pq} e_{pq}^1, \alpha) + o(|x_{pq}|^{2}) + O(|\beta_{pq}^1||x_{pq}|^{2}),
\]

where \( P_2 : X \rightarrow E_2 \) is the canonical projection. From (24), we know that

\[
G_2(x_{pq} e_{pq}^1, \alpha) = x_{pq}^2 \beta_{pq}^1 \begin{pmatrix} (-\alpha - 1) \\ 0 \end{pmatrix},
\]

(42)
Note that
\[
\varphi_{pq}^2 = \frac{4}{l_1 l_2} \cos^2 \left( \frac{2p\pi}{l_1} x_1 \right) \cos^2 \left( \frac{2q\pi}{l_2} x_2 \right)
\]
\[
= \frac{1}{l_1 l_2} \left[ \cos \left( \frac{2p\pi}{l_1} x_1 \right) \cos \left( \frac{2q\pi}{l_2} x_2 \right)
\right] + \cos \left( \frac{2p\pi}{l_1} x_1 \right) + \cos \left( \frac{2q\pi}{l_2} x_2 \right) + 1,
\]
which implies that the center manifold function \( \Phi \) takes the form of
\[
\Phi = \Phi_{00}^1 e_{00}^1 + \Phi_{00}^2 e_{00}^2 + \Phi_{2p0}^1 e_{2p0}^1 + \Phi_{2p0}^2 e_{2p0}^2
\]
\[
+ \Phi_{02q}^1 e_{02q}^1 + \Phi_{02q}^2 e_{02q}^2 + \Phi_{2p2q}^1 e_{2p2q}^1
\]
\[
+ \Phi_{2p2q}^2 e_{2p2q}^2 \| x_{pq}^2 + o(2).
\]

With the help of (42), upon substituting the preceding expression into (41) and comparing the corresponding coefficients on both sides, one can derive that
\[
\Phi_{00}^1 = \frac{\alpha - 1}{\beta_{00}^1 P_{00}^1 \sqrt{l_1 l_2}}, \quad \Phi_{00}^2 = \frac{\alpha - 1}{\beta_{00}^2 P_{00}^2 \sqrt{l_1 l_2}}
\]
\[
\Phi_{2p0}^1 = \frac{\alpha - 1}{\beta_{2p0}^1 P_{2p0}^1 \sqrt{l_1 l_2}}, \quad \Phi_{2p0}^2 = \frac{\alpha - 1}{\beta_{2p0}^2 P_{2p0}^2 \sqrt{l_1 l_2}}
\]
\[
\Phi_{02q}^1 = \frac{\alpha - 1}{\beta_{02q}^1 P_{02q}^1 \sqrt{l_1 l_2}}, \quad \Phi_{02q}^2 = \frac{\alpha - 1}{\beta_{02q}^2 P_{02q}^2 \sqrt{l_1 l_2}}
\]
\[
\Phi_{2p2q}^1 = \frac{\alpha - 1}{2 \beta_{2p2q}^1 P_{2p2q}^1 \sqrt{l_1 l_2}}, \quad \Phi_{2p2q}^2 = \frac{\alpha - 1}{2 \beta_{2p2q}^2 P_{2p2q}^2 \sqrt{l_1 l_2}}.
\]

Denote
\[
\phi_1 = \Phi_{00}^1 e_{00}^1 + \Phi_{00}^2 e_{00}^2 + \Phi_{2p0}^1 e_{2p0}^1 + \Phi_{2p0}^2 e_{2p0}^2 + \Phi_{02q}^1 e_{02q}^1 \varphi_{02q}
\]
\[
+ \Phi_{02q}^2 e_{02q}^2 + \Phi_{2p2q}^1 e_{2p2q}^1 + \Phi_{2p2q}^2 e_{2p2q}^2
\]
\[
\phi_2 = \Phi_{00}^1 h_{00}^1 e_{00}^1 + \Phi_{00}^2 h_{00}^2 e_{00}^2 + \Phi_{02q}^1 h_{02q}^1 e_{02q}^1 \varphi_{02q}
\]
\[
+ \Phi_{02q}^2 h_{02q}^2 e_{02q}^2 + \Phi_{2p2q}^1 h_{2p2q}^1 e_{2p2q}^1 \varphi_{2p2q}
\]
\[
+ \Phi_{2p2q}^2 h_{2p2q}^2 e_{2p2q}^2 \| \| (43)
\]
then the center manifold function \( \Phi \) can be expressed by
\[
\Phi = (\phi_1, \phi_2)^T x_{pq}^2 + o(2), \quad (44)
\]

combining which and (43) we derive that
\[
\langle G(x_{pq} e_{pq}^1 + \Phi, \alpha), e_{pq}^1 \rangle = b(\alpha) x_{pq}^3 + o(|x_{pq}|^3).
\]
The preceding approximate expansion means (40) is reduced to
\[
\frac{dx_{pq}}{dt} = \beta_{pq}^1 x_{pq} + \frac{b(\alpha)}{P_{pq}^3} x_{pq}^3 + o(|x_{pq}|^3).
\]

Finally, the Theorem 2.4.1 in [14] implies that the assertions hold.

If \((p,q) = (0,0)\) in Theorem 3.1, with the same method, we can obtain that the system (7) undergoes a mixed transition. Hence, we omit the corresponding details. In what follows, we only focus on the transition from a pair of complex eigenvalues.
Let \( e^{1}_n, e^{2}_n \) and \( e^{1*}_n, e^{2*}_n \) be the eigenvectors of \( L_\alpha \) and dual operator \( L^*_\alpha \), respectively, with the corresponding complex eigenvalues \( \beta^{2}_n = \beta^{1*}_n = Re\beta^{1}_n + iIm\beta^{1}_n \).

Then we have

\[
\begin{align*}
L_\alpha e^{1}_n &= Re\beta^{1}_n e^{1}_n + Im\beta^{1}_n e^{2}_n, \\
L_\alpha e^{2}_n &= -Im\beta^{1}_n e^{1}_n + Re\beta^{1}_n e^{2}_n,
\end{align*}
\]

and

\[
\begin{align*}
L^*_\alpha e^{1*}_n &= Re\beta^{1*}_n e^{1*}_n - Im\beta^{1*}_n e^{2*}_n, \\
L^*_\alpha e^{2*}_n &= Im\beta^{1*}_n e^{1*}_n + Re\beta^{1*}_n e^{2*}_n.
\end{align*}
\]

Similarly, based on Theorem 3.1 and with the method of center manifold reduction, we can obtain the following results involved the nonlinear transitions. Before stating our results, let us introduce the following parameter

\[
\tilde{\eta}_1 \tilde{\eta}_2 = \left( \begin{array}{c} \tilde{\eta}_1 \\ \tilde{\eta}_2 \end{array} \right) = \left( \begin{array}{c} \eta_1 - \alpha \\ \eta_2 \end{array} \right), \quad \tilde{\eta}^*_1 \tilde{\eta}^*_2 = \left( \begin{array}{c} \tilde{\eta}^*_1 \\ \tilde{\eta}^*_2 \end{array} \right) = \left( \begin{array}{c} \eta_1^* - \alpha \\ \eta_2^* \end{array} \right).
\]

**Theorem 4.5.** If \( \alpha_0 < \tilde{\alpha}_0 = \tilde{\alpha}_0 \), then the system (7) undergoes a Hopf bifurcation at \( \alpha_0 \) and the following assertions hold:

1. If \( b(\alpha_0) < 0 \), then the system (7) undergoes a continuous transition and bifurcates from \( (0, \alpha_0) \) to a periodic solution on \( \alpha > \alpha_0 \), which is an attractor.
2. If \( b(\alpha_0) > 0 \), then the system (7) undergoes a jump transition and bifurcates from \( (0, \alpha_0) \) to a periodic solution on \( \alpha < \alpha_0 \), which is a repeller.
3. The approximate expression for the periodic solution is given by

\[
w = x e^{1*}_{00} + y e^{2*}_{00} + o(Re\beta^{1*}_{00}),
\]

where

\[
\begin{align*}
x(t) &= \sqrt{\frac{4Re\beta^{1*}_{00}(\alpha)}{-\pi b(\alpha)}} \cos(Im\beta^{1*}_{00}t) + o(Re\beta^{1*}_{00}), \\
y(t) &= \sqrt{\frac{4Re\beta^{1*}_{00}(\alpha)}{-\pi b(\alpha)}} \sin(Im\beta^{1*}_{00}t) + o(Re\beta^{1*}_{00}).
\end{align*}
\]

**Proof.** First, we calculate the eigenvectors \( e^{1*}_{00}(\alpha_0), e^{2*}_{00}(\alpha_0) \) and their conjugates \( e^{1*}_{00}(\alpha_0), e^{2*}_{00}(\alpha_0) \). To this end, let us denote

\[
\begin{align*}
\epsilon e^{1*}_{00} &= (\eta_1 \varphi_0, \eta_2 \varphi_0), & \epsilon e^{2*}_{00} &= (\bar{\eta}_1 \varphi_0, \bar{\eta}_2 \varphi_0), \\
\epsilon e^{1*}_{00} &= (\eta_1^* \varphi_0, \eta_2^* \varphi_0), & \epsilon e^{2*}_{00} &= (\bar{\eta}_1^* \varphi_0, \bar{\eta}_2^* \varphi_0).
\end{align*}
\]

At the critical value \( \alpha = \alpha_0 \), we know that \( Re\beta^{1*}_{00}(\alpha_0) = 0 \). Hence, substituting (48) into (45) and (46), respectively, we derive that

\[
\begin{align*}
(\alpha_0 - 1 \epsilon \eta_1) &= Re\beta^{1*}_{00} \left( \begin{array}{c} \tilde{\eta}_1 \\ \tilde{\eta}_2 \end{array} \right), & (\alpha_0 - 1 \epsilon \eta_2) &= -Im\beta^{1*}_{00} \left( \begin{array}{c} \tilde{\eta}_1 \\ \tilde{\eta}_2 \end{array} \right), \\
(\alpha_0 - 1 \epsilon \eta_1) &= -Im\beta^{1*}_{00} \left( \begin{array}{c} \tilde{\eta}_1^* \\ \tilde{\eta}_2^* \end{array} \right), & (\alpha_0 - 1 \epsilon \eta_2) &= Re\beta^{1*}_{00} \left( \begin{array}{c} \tilde{\eta}_1^* \\ \tilde{\eta}_2^* \end{array} \right).
\end{align*}
\]
Note that

$$\alpha_c = \epsilon, \quad (\text{Im} \beta_{00}^1)^2 = \epsilon (1-\epsilon).$$

Using (51), from (49)-(50), one can obtain

$$\eta_1 = 1 + \frac{1}{\epsilon} \text{Im} \beta_{00}^1, \quad \eta_2 = 1, \quad \tilde{\eta}_1 = 1 - \frac{1}{\epsilon} \text{Im} \beta_{00}^1, \quad \tilde{\eta}_2 = 1,$$

$$\eta_1^* = -\epsilon + \text{Im} \beta_{00}^1, \quad \eta_2^* = 1, \quad \tilde{\eta}_1^* = -\epsilon - \text{Im} \beta_{00}^1, \quad \tilde{\eta}_2^* = 1.$$

Then, one can obtain

$$e_{00}^i = (1 + \frac{1}{\epsilon} \text{Im} \beta_{00}^1,1) \varphi_{00}, \quad e_{00}^2 = (1 - \frac{1}{\epsilon} \text{Im} \beta_{00}^1,1) \varphi_{00},$$

$$e_{00}^{1*} = (-\epsilon + \text{Im} \beta_{00}^1,1) \varphi_{00}, \quad e_{00}^{2*} = (-\epsilon - \text{Im} \beta_{00}^1,1) \varphi_{00}.$$

Some direct calculations give

$$\langle e_{00}^1, e_{00}^{1*} \rangle = -\langle e_{00}^2, e_{00}^{1*} \rangle = -2 \text{Im} \beta_{00}^1,$$

$$\langle e_{00}^1, e_{00}^{2*} \rangle = \langle e_{00}^2, e_{00}^{2*} \rangle = 2 - 2\epsilon.$$ (52)

Let us set

$$\Psi_1^* = (\Psi_{11}^* \varphi_{00}, \Psi_{12}^* \varphi_{00}), \quad \Psi_2^* = (\Psi_{21}^* \varphi_{00}, \Psi_{22}^* \varphi_{00}),$$

where

$$\Psi_{11}^* = \frac{1}{\langle e_{00}^1, e_{00}^{1*} \rangle} [\langle e_{00}^1, e_{00}^{1*} \rangle \eta_1^* + \langle e_{00}^1, e_{00}^{2*} \rangle \tilde{\eta}_1^*],$$

$$\Psi_{12}^* = \frac{1}{\langle e_{00}^2, e_{00}^{1*} \rangle} [\langle e_{00}^2, e_{00}^{1*} \rangle \eta_2^* + \langle e_{00}^2, e_{00}^{2*} \rangle \tilde{\eta}_2^*],$$

$$\Psi_{21}^* = \frac{1}{\langle e_{00}^2, e_{00}^{2*} \rangle} [\langle e_{00}^2, e_{00}^{2*} \rangle \eta_1^* + \langle e_{00}^2, e_{00}^{1*} \rangle \tilde{\eta}_1^*],$$

$$\Psi_{22}^* = \frac{1}{\langle e_{00}^1, e_{00}^{2*} \rangle} [\langle e_{00}^1, e_{00}^{2*} \rangle \eta_2^* + \langle e_{00}^1, e_{00}^{1*} \rangle \tilde{\eta}_2^*].$$

Then, using (52)-(53), one can check that (55)-(58) satisfy (50).

Due to the decomposition $X = E_1 \oplus E_2$, where $E_1 = \text{span}\{e_{00}^1, e_{00}^2\}$ and $E_2 = E_1^\perp$, the solution of the system (7) is expressed by

$$w = xe_{00}^1 + ye_{00}^2 + z,$$

where $z \in E_2$. Making use of (52)-(58), some calculations yield to that

$$\langle e_{00}^1, \Psi_1^* \rangle = \langle e_{00}^2, \Psi_1^* \rangle = 0,$$

$$\langle e_{00}^1, \Psi_2^* \rangle = \langle e_{00}^2, \Psi_2^* \rangle = \frac{2}{1-\epsilon} \left[ (1-\epsilon)^2 + (\text{Im} \beta_{00}^1)^2 \right].$$

Therefore, the system (7) at the critical value $\alpha_c$ is reduced to

$$\frac{dx}{dt} = -\text{Im} \beta_{00}^1 y + \frac{1}{\langle e_{00}^1, \Psi_1^* \rangle} (G(xe_{00}^1 + ye_{00}^2 + \Phi, \alpha_c), \Psi_1^*),$$

$$\frac{dy}{dt} = \text{Im} \beta_{00}^1 x + \frac{1}{\langle e_{00}^2, \Psi_2^* \rangle} (G(xe_{00}^1 + ye_{00}^2 + \Phi, \alpha_c), \Psi_2^*),$$

where $\Phi : E_1 \to E_2$ is the center manifold function.

Next, we shall calculate the center manifold function $\Phi$ which is given by

$$\Phi = \Phi_1 + \Phi_2 + \Phi_3 + o(2),$$
where $\Phi_1, \Phi_2, \Phi_3$ solve, respectively,

$$-L_{\alpha_0} \Phi_1 = x^2 G_{11} + y^2 G_{22} + xy(G_{12} + G_{21}).$$

$$-([-L_{\alpha_0}]^2 + 4(Im\beta_{00}^1)^2) L_{\alpha_0} \Phi_2 = 2(Im\beta_{00}^1)^2[(x^2 - y^2)(G_{22} - G_{11})$$

$$- 2xy(G_{12} + G_{21})],$$

$$([-L_{\alpha_0}]^2 + 4(Im\beta_{00}^1)^2) \Phi_3 = Im\beta_{00}^1[(x^2 - y^2)(G_{12} + G_{21}) + 2xy(G_{11} - G_{22})],$$

in which $G_{ij} = P_2 G_2(e_{00}^i, e_{00}^j, \alpha_c), 1 \leq i, j \leq 2$. From (24), we obtain

$$G_2(e_{00}^i, e_{00}^j, \alpha_c) = \left(\begin{array}{c}
-(\alpha_c - 1)\eta_1\eta_1 \\
0
\end{array}\right) e_{00}^2, \ 1 \leq i, j \leq 2,$$

where

$$(\eta_{11}, \eta_{12}) = (1 + \frac{1}{\epsilon} Im\beta_{00}^1, 1), \ \ \ (\eta_{21}, \eta_{22}) = (1 - \frac{1}{\epsilon} Im\beta_{00}^1, 1).$$

Thus, we have

$$G_{ij} = P_2 G_2(e_{00}^i, e_{00}^j, \alpha_c) = \langle G_2(e_{00}^i, e_{00}^j, \alpha_c), e_{\ell n}^1 \rangle e_{\ell n}^1,$$

where $P_2 : X \to E_2 = E_2^1$ is the canonical projection which means

$$\langle G_2(e_{00}^i, e_{00}^j, \alpha_c), e_{\ell n}^1 \rangle = -(\alpha_c - 1)\eta_1\eta_1 \int_\Omega \varphi_{\ell n}^2 \varphi_{00}^2 dx = 0. \ \ \ (60)$$

By the preceding identity, from (59) and (60), it finds that

$$\Phi_1 = \Phi_2 = \Phi_3 = 0 \Rightarrow \Phi = o(2).$$

Furthermore, with the help of (54), we obtain

$$\langle G(x e_{00}^1 + ye_{00}^2 + \Phi, \alpha_c), \Psi_1^0 \rangle = \sum_{2 \leq p + q \leq 3} a_{pq}^1 x^p y^q + o(3),$$

$$\langle G(x e_{00}^1 + ye_{00}^2 + \Phi, \alpha_c), \Psi_2^0 \rangle = \sum_{2 \leq p + q \leq 3} a_{pq}^2 x^p y^q + o(3),$$

where

$$a_{20}^1 = - a_{20}^2 = \frac{(\alpha_c - 1) Im\beta_{00}^1}{(1 - \epsilon)^2 l_1 l_2},$$

$$a_{11}^1 = - a_{11}^2 = \frac{2(\alpha_c - 1) Im\beta_{00}^1}{(1 - \epsilon)^2 l_1 l_2},$$

$$a_{02}^1 = - a_{02}^2 = \frac{(\alpha_c - 1) Im\beta_{00}^1}{(1 - \epsilon)^2 l_1 l_2},$$

$$a_{30}^1 = - a_{30}^2 = \frac{Im\beta_{00}^1}{(1 - \epsilon)^2 l_1 l_2},$$

$$a_{21}^1 = - a_{21}^2 = \frac{3 Im\beta_{00}^1}{(1 - \epsilon)^2 l_1 l_2},$$

$$a_{12}^1 = - a_{12}^2 = \frac{3 Im\beta_{00}^1}{(1 - \epsilon)^2 l_1 l_2},$$

$$a_{03}^1 = - a_{03}^2 = \frac{Im\beta_{00}^1}{(1 - \epsilon)^2 l_1 l_2},$$

$$\langle G(x e_{00}^1 + ye_{00}^2 + \Phi, \alpha_c), \Psi_1^0 \rangle = \sum_{2 \leq p + q \leq 3} a_{pq}^1 x^p y^q + o(3). \ \ \ (61)$$
Finally, the system (7) is then reduced to
\[
\frac{dx}{dt} = -Im(\beta_1)00y + \frac{1 - \epsilon}{2[(1 - \epsilon)^2 + (Im(\beta_{100}))^2]^2} \sum_{2 \leq p+q \leq 3} a_{pq}^1 x^p y^q + o(3),
\]
\[
\frac{dy}{dt} = Im(\beta_1)00x + \frac{1 - \epsilon}{2[(1 - \epsilon)^2 + (Im(\beta_{100}))^2]^2} \sum_{2 \leq p+q \leq 3} a_{pq}^2 x^p y^q + o(3).
\]
by which we drive that
\[
\tilde{b}(\alpha_c) = \frac{3\pi}{4} (a_{10}^1 + a_{02}^2) + \frac{\pi}{4} (a_{12}^1 + a_{21}^2) + \frac{\pi}{2Im(\beta_{100})} (a_{02}^1 a_{02}^2 - a_{10}^1 a_{20}^2) + \frac{\pi}{4Im(\beta_{100})} (a_{11}^1 a_{20}^1 + a_{11}^2 a_{02}^1 - a_{11}^2 a_{10}^2 - a_{11}^1 a_{02}^2).
\]
Thus, based on the preceding reduced equations, and with the same method used to derive the Theorem 2.3.7 in [14], one can see that all assertions hold.

5. Numerical results. In the preceding section, we have established the nonlinear dynamic transition theorems for the FN system. For the purpose of illustration, for some specified control parameters, we aim to examine the specific transition types. To this end, taking \( \Omega = (0, 10) \times (0, 20) \), and choosing the control parameters \( \epsilon \in [0.055, 1] \) and \( D \in [0.001, 0.05] \), respectively, the numerical estimates of the critical parameter \( \tilde{\alpha}_c = \min_{\rho_{1n}} \frac{D\rho_{1n}}{\rho_{1n} + \epsilon} \) are shown in Figure 1.

![Figure 1](image_url)

**Figure 1.** Neutral stability surface \( \tilde{\alpha}_c(D, \epsilon) \).

From Figure 1, we can see that both the assertion (1) and assertion (2) of Theorem 3.1 can be satisfied. In fact, we know that the critical parameter values \( \tilde{\alpha}_c \) are less than the critical parameter value \( \alpha_c \) for \( \epsilon = 1 \) and each \( D \in [0.001, 0.05] \). Namely, the assertion (1) of Theorem 3.1 is satisfied. In addition, it is shown that the critical parameter value \( \tilde{\alpha}_c > \alpha_c \) if \( \epsilon \) and \( D \) are small enough i.e., the assertion (2) of Theorem 3.1 is satisfied. For instance, setting \( D = 0.02 \) and \( \epsilon = 0.01 \), we explicitly have \( \alpha_c = 0.01 < \tilde{\alpha}_c = 0.0281 \), where \( \tilde{\alpha}_c \) is obtained at \( (p, q) = (1, 5) \).

Making use of (39) and (47), we obtain the regions - \( (D, \epsilon) \) shown in Figure 2, from which one can apparently know the transition types. More precisely, the region \( A \) is the control parameter region \( (D, \epsilon) \) at each point of which we have \( \tilde{\alpha}_c < \alpha_c \) and \( b(\tilde{\alpha}_c) > 0 \), which implies that the system is capable of exhibiting a dynamic
transition of jump type. As a result, there exists a pitchfork bifurcation in the
system on $\alpha < \tilde{\alpha}_c$, and two saddle points are bifurcated from the equilibrium point 
$(0, 0)$. Similarly, we know that $\tilde{\alpha}_c > \alpha_c$ and $\tilde{b}(\alpha_c) > 0$ if $(D, \epsilon)$ is chosen in the
region B. That is, the system undergoes a dynamic transition of jump type and an
unstable periodic solution is bifurcated from equilibrium point $(0, 0)$ on $\alpha < \alpha_c$.

REFERENCES

[1] P. Carter and B. Sandstede, Fast pulses with oscillatory tails in the FitzHugh-Nagumo system,
SIAM Journal Mathematic Analysis, 47 (2015), 3393–3441.
[2] C.-N. Chen, S.-Y. Kung and Y. Morita, Planar standing wavefronts in the Fitzhugh-Nagumo
equations, SIAM Journal Mathematic Analysis, 46 (2014), 657–690.
[3] C.-N. Chen, C.-C. Chen and C.-C. Huang, Traveling waves for the Fitzhugh-Nagumo system
on an infinite channel, Journal of Differential Equations, 261 (2016), 3010–3041.
[4] P. Cornwell and C. K. R. T. Jones, On the existence and stability of fast traveling waves in
a doubly diffusive Fitzhugh-Nagumo system, SIAM Journal Applied Dynamical Systems, 17
(2018), 754–787.
[5] H. Dijkstra, T. Sengul, J. Shen and S. Wang, Dynamic transitions of quasi-geostrophic channel
flow, SIAM Journal on Applied Mathematics, 75 (2015), 2361–2378.
[6] R. Fitzzhugh, Impulses and physiological states in theoretical models of nerve membrane,
Biophysical Journal, 1 (1961), 445–466.
[7] M. O. Gani and T. Ogawa, Instability of periodic traveling wave solutions in a modified
Fitzhugh-Nagumo model for excitable media, Applied Mathematics and Computation, 256
(2015), 968–984.
[8] F. Sánchez-Garduño, A. L. Krause, J. A. Castillo and P. Padilla, Turing-Hopf patterns on
growing domains: The torus and the sphere, Journal of Theoretical Biology, 481 (2019),
136–150.
[9] D. Han, M. Hernandez and Q. Wang, Dynamical transitions of a low-dimensional model for
Rayleigh-Bénard convection under a vertical magnetic field, Chaos, Solitons and Fractals,
114 (2018), 370–380.
[10] P. van Heijster and B. Sandstede, Bifurcations to travelling planar spots in a three-component
Fitzhugh-Nagumo system, Physica D Nonlinear Phenomena, 275 (2014), 19–34.
[11] C. Kieu, T. Sengul, Q. Wang and D. Yan, On the Hopf (double Hopf) bifurcations and
transitions of two layer western boundary currents, Communications in Nonlinear Science
and Numerical Simulation, 65 (2018), 196–215.
[12] M. Kuznetsov, A. Kolobov and A. Polezhaev, Pattern formation in a reaction-diffusion system of Fitzhugh-Nagumo type before the onset of subcritical Turing bifurcation, *Physical Review E*, 95 (2017), 052208, 7 pp.

[13] J.-L. Lions, R. Teman and S. Wang, New formation of the primitive equations of atmosphere and applications, *Nonlinearity*, 5 (1992), 237–288.

[14] T. Ma and S. Wang, *Phase Transition Dynamics*, Springer, New York, 2014.

[15] T. Ma and S. Wang, Dynamic transition and pattern formation for chemotaxis systems, *Discrete and Continuous Dynamical System Series B*, 19 (2014), 2089–2835.

[16] T. Ma and S. Wang, *Bifurcation Theory and Application*, Series A: Monographs and Treatises, 53. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2005.

[17] Y. Mao, Dynamic transitions of the Fitzhugh-Nagumo equations on a finite domain, *Discrete and Continuous Dynamical System Series B*, 23 (2018), 3935–3947.

[18] Y. Mao, D. Yan and C. Lu, Dynamic transitions and stability for the acetabularia whorl formation, *Discrete and Continuous Dynamical System Series B*, 24 (2019), 5989–6004.

[19] J. Nagumo, S. Arimoto and S. Yoshizawa, An active pulse transmission line simulating nerve axon, *Proceedings of the IRE*, 50 (1962), 2061–2070.

[20] S. Özer, T. Şengül and Q. Wang, Multiple equilibria and transitions in spherical MHD equations, *Communications in Mathematical Sciences*, 17 (2019), 1531–1555.

[21] S. Özer and T. Şengül, Transitions of spherical thermohaline circulation to multiple equilibria, *Journal of Mathematical Fluid Mechanics*, 20 (2018), 499–515.

[22] J. Pedlosky, *Geophysical Fluid Dynamics*, Springer Science & Business Media, 2013.

[23] M. Schonbek, Boundary value problem for the Fitzhugh-Nagumo equations, *Journal of Differential Equations*, 30 (1978), 119–147.

[24] T. Sengul and S. Wang, Pattern formation and dynamic transition for magnetohydrodynamic convection, *Communications on Pure and Applied Analysis*, 13 (2014), 2609–2639.

[25] E. P. Zemskov and I. R. Epstein, Wave propagation in a Fitzhugh-Nagumo-type model with modified excitability, *Physical Review E*, 82 (2010), 026207, 6 pp.

[26] E. P. Zemskov, M. A. Tsiganov and W. Horsthemke, Multifront regime of a piecewise-linear Fitzhugh-Nagumo model with cross diffusion, *Physical Review E*, 99 (2019), 062214, 9 pp.

[27] Q. Zheng and J. Shen, Pattern formation in the Fitzhugh-Nagumo model, *Computers & Mathematics with Applications*, 70 (2015), 1082–1097.

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