The Mean Square of Divisor Function

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Abstract. Let $d(n)$ be the divisor function. In 1916, S. Ramanujan stated but without proof that
\[ \sum_{n \leq x} d^2(n) = xP(\log x) + E(x), \]
where $P(y)$ is a cubic polynomial in $y$ and
\[ E(x) = O(x^{\frac{3}{2} + \varepsilon}), \]
where $\varepsilon$ is a sufficiently small positive constant. He also stated that, assuming the Riemann Hypothesis (RH),
\[ E(x) = O(x^{\frac{1}{2} + \varepsilon}). \]

In 1922, B. M. Wilson proved the above result unconditionally. The direct application of the RH would produce
\[ E(x) = O(x^{\frac{1}{2}} (\log x)^5 \log \log x). \]
In 2003, K. Ramachandra and A. Sankaranarayanan proved the above result without any assumption.

In this paper, we shall prove
\[ E(x) = O(x^{\frac{1}{2}} (\log x)^5). \]

1. Introduction

Let $d(n)$ be the divisor function. In 1916, S. Ramanujan [9] stated but without proof that
\[ d^2(1) + d^2(2) + d^2(3) + \cdots + d^2(n) = An(\log n)^3 + Bn(\log n)^2 + Cn \log n + Dn + O(n^{\frac{3}{2} + \varepsilon}), \]
here

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\[ A = \frac{1}{\pi^2}, \quad B = \frac{12\gamma - 3}{\pi^2} - \frac{36}{\pi^4} \zeta'(2), \]

where \( \gamma \) is Euler’s constant, \( C, D \) are more complicated constants, \( \varepsilon \) is a sufficiently small positive constant. S. Ramanujan [9] also stated that, assuming the Riemann Hypothesis (RH), the error term in (1.1) can be improved to \( O(n^{\frac{1}{2} + \varepsilon}) \).

Write
\[ E(x) = \sum_{n \leq x} d^2(n) - xP(\log x), \quad (1.2) \]
where
\[ P(x) = Ax^3 + Bx^2 + Cx + D. \]

Then the statement of Ramanujan is that
\[ E(x) = O(x^{\frac{3}{2} + \varepsilon}), \quad (1.3) \]
and assuming the RH,
\[ E(x) = O(x^{\frac{1}{2} + \varepsilon}). \quad (1.4) \]

In 1922, B. M. Wilson [13] proved (1.4) unconditionally. By a general theorem of M. Kühleitner and W. G. Nowak (see (5.4) in [5]), we know
\[ E(x) = \Omega(x^{\frac{3}{2}}). \quad (1.5) \]

Let \( d_4(n) \) be the general divisor function which is the number of representations of \( n = d_1d_2d_3d_4 \). In 1973, assuming
\[ \sum_{n \leq x} d_4(n) = \frac{1}{6}x \log^2 x + (2\gamma - \frac{1}{2})x \log^2 x + ax \log x + bx + O(x^\alpha), \]
where \( \gamma \) is Euler’s constant, \( a, b \) are constants, \( \alpha \) is a constant strictly less than \( \frac{1}{2} \), D. Suryanarayana and R. Sitaramachandra Rao [10] proved
\[ E(x) = O(x^{\frac{3}{2}} \exp(-c(\log x)^{\frac{3}{4}}(\log \log x)^{-\frac{1}{2}})), \quad (1.6) \]
where \( c \) is a positive constant.

By Vinogradov’s estimate, if \( \frac{T}{2} \leq t \leq T \), then
\[ \frac{1}{\zeta(1 + 2it)} \ll (\log T)^{\frac{3}{4}}(\log \log T)^{\frac{1}{2}}. \]
So it is not difficult to prove

\[ E(x) = O(x^{\frac{1}{2}}(\log x)^{\frac{17}{3}}(\log \log x)^{\frac{1}{3}}). \]  

(1.7)

The direct application of the RH (or even the quasi-RH) would produce

\[ E(x) = O(x^{\frac{1}{2}}(\log x)^{5} \log \log x). \]  

(1.8)

In 2003, K. Ramachandra and A. Sankaranarayanan[8] proved (1.8) without any assumption and put forward the following conjecture.

**Conjecture** (Ramachandra-Sankaranarayanan). Assuming the RH, we have

\[ E(x) = O(x^{\frac{1}{2}}). \]  

(1.9)

For the average situation, in 2005, H. Maier and A. Sankaranarayanan[7] proved,

\[ \frac{1}{X} \int_{X}^{2X} E^2(x) dx \ll X \exp(-c(\log X)^{\frac{3}{5}}(\log \log X)^{-\frac{1}{5}}), \]  

(1.10)

where \( c \) is a positive constant.

In this paper, we shall prove the following theorem.

**Theorem.** If \( E(x) \) is defined in (1.2), then unconditionally we have

\[ E(x) = O(x^{\frac{1}{2}}(\log x)^{5}). \]  

(1.11)

Throughout this paper, we assume that \( \varepsilon \) is a sufficiently small positive constant and that \( T \) is sufficiently large.

2. Some lemmas

**Lemma 1** (Borel-Carathéodory). Suppose that \( f(z) \) is holomorphic in the disk \( |z - z_0| \leq R \) and that in the circle \( z = z_0 + Re^{i\theta} (0 \leq \theta \leq 2\pi) \),

\[ \text{Re} f(z) \leq M. \]

Then in the disk \( |z - z_0| \leq r(< R) \), we have

\[ |f(z)| \leq \frac{2r}{R - r} M + \frac{R + r}{R - r} |f(z_0)|. \]

See Section 5.5 of [11].
Lemma 2 (Hadamard). Suppose that \( f(z) \) is holomorphic in the disk \(|z - z_0| \leq R_3, R_1 < R_2 < R_3\). Write

\[
M_j = \max_{|z - z_0| = R_j} |f(z)|, \quad j = 1, 2, 3.
\]

Then we have

\[
\log M_2 \leq \frac{\log \left(\frac{R_3}{R_2}\right)}{\log \left(\frac{R_3}{R_1}\right)} \cdot \log M_1 + \frac{\log \left(\frac{R_2}{R_1}\right)}{\log \left(\frac{R_3}{R_1}\right)} \cdot \log M_3.
\]

See Section 5.3 of [11].

Lemma 3. For \( \alpha > 0 \) and \( x > 0 \), we have

\[
\frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} \Gamma(s) x^{-s} ds = e^{-x}.
\]

See (2.15.2) in page 33 of [12].

Lemma 4. For \(-1 \leq \sigma \leq 2\) and \(|t| \geq 1\), we have

\[
\Gamma(\sigma + it) \ll |t|^\sigma - \frac{x}{2} e^{-\frac{\pi}{2}|t|}.
\]

See (4.12.2) in page 78 of [12].

Lemma 5. For \( \text{Re}(s) > 1 \), let

\[
f(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s},
\]

where \( a(n) = O(\psi(n)) \), \( \psi(n) \) is non-decreasing, and as \( \sigma \to 1^+ \),

\[
\sum_{n=1}^{\infty} \frac{|a(n)|}{n^\sigma} = O\left(\frac{1}{(\sigma - 1)^\alpha}\right).
\]

Then if \( c > 1 \), \( x \) is not an integer, and \( N \) is the integer nearest to \( x \),

\[
\sum_{n<x} a(n) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} f(s) \frac{x^s}{s} ds + O\left(\frac{x^c}{T(c - 1)^\alpha}\right) + O\left(\frac{\psi(2x)x \log x}{T}\right) + O\left(\frac{\psi(N)x}{T|x - N|}\right).
\]

See Lemma 3.12 in page 60 of [12].

Lemma 6. For \( \text{Re}(s) > 1 \), we have

\[
\sum_{n=1}^{\infty} \frac{d^2(n)}{n^s} = \frac{\zeta^4(s)}{\zeta(2s)}.
\]
See (1.2.10) in page 5 of [12].

**Lemma 7.** For $\text{Re}(s) \geq \frac{1}{2}$ and $|s - 1| > 1$, we have

$$\zeta(s) = O(|s|).$$

See (2.12.2) in page 29 of [12].

**Lemma 8.** For $\sigma \geq 1$ and $t \geq 1$, we have

$$\frac{1}{\zeta(\sigma + it)} = O(\log t).$$

See (3.11.8) in page 60 of [12].

**Lemma 9.** For $t \geq 1$, we have

$$\zeta(\frac{1}{2} + it) = O(t^{\frac{1}{6} + \varepsilon}).$$

See Theorem 5.5 in page 99 of [12].

**Remark.** The bounds stated in Lemmas 8 and 9 suffice for our purpose though better upper bounds are known.

**Lemma 10.** For $\frac{1}{2} \leq \sigma \leq 1 + \varepsilon$ and $t \geq 1$, we have

$$\zeta(\sigma + it) = O(t^{\frac{1}{6}(1-\sigma)+\varepsilon}).$$

It follows from Lemma 9 and the explanation in Chapter 5 of [12].

**Lemma 11.** We have

$$\int_1^T |\zeta(\frac{1}{2} + it)|^4 dt = O(T \log^4 T).$$

See (7.6.1) in page 147 of [12].

**Lemma 12**(Huxley). For $\sigma \geq \frac{1}{2}$, let $N(\sigma, T, 2T)$ denote the number of zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ which satisfy $\beta \geq \sigma$ and $T \leq \gamma \leq 2T$. Then

$$N(\sigma, T, 2T) \ll T^{\frac{12}{\sigma}(1-\sigma)+\varepsilon}.$$
This is Lemma 1 in [4].

Define

\[ D(s; \frac{h}{k}) = \sum_{l=1}^{\infty} \frac{d(l)}{l^s} e(l \frac{h}{k}). \]  

(2.1)

\textbf{Lemma 14 (Estermann).} Suppose that \((h, k) = 1\). The function \(D(s; \frac{h}{k})\) is meromorphic in the whole plane with only one pole of order 2 at \(s = 1\). In the neighborhood of \(s = 1\),

\[ D(s; \frac{h}{k}) = \frac{1}{k} \cdot \frac{1}{(s-1)^2} + \frac{2}{k} (\gamma - \log k) \cdot \frac{1}{(s-1)} + \cdots, \]

where \(\gamma\) is Euler’s constant. At \(s = 0\), we have

\[ D(0; \frac{h}{k}) = \frac{1}{4} - \frac{1}{\pi^2} \sum_{a=1}^{k} \beta(a, k) \sum_{0 < b < \frac{k}{2}} \eta(b, k) e(ab \frac{h}{k}), \]

where \(h h \equiv 1 \pmod{k}\),

\[ \beta(a, k) = \begin{cases} \frac{1}{1-e(-\frac{a}{k})}, & \text{if } 1 \leq a < k, \\ \frac{1}{2}, & \text{if } a = k, \end{cases} \]

and when \(0 < b < \frac{k}{2}\),

\[ 0 < \eta(b, k) < \frac{1}{b}. \]

Moreover, \(D(s; \frac{h}{k})\) satisfies the functional equation

\[ D(s; \frac{h}{k}) = 2G^2(s) k^{1-2s} \left( D(1-s; \frac{\overline{h}}{k}) - \cos(\pi s) D(1-s; -\frac{\overline{h}}{k}) \right), \]

where

\[ G(s) = (2\pi)^{s-1} \Gamma(1-s). \]

See (21), (34), (32), (29) and (19) in [1].

\textbf{Lemma 15.} If \((m_1, m_2) = (n_1, n_2) = 1\), then

\[ (m_1 n_1^2, m_2 n_2^2) = (m_1, n_1^2) (m_2, n_2^2). \]

Proof. We have

\[ (m_1 n_1^2, m_2 n_2^2) = (m_1, n_2^2) \left( \frac{m_1}{n_1^2}, m_2 \frac{n_2^2}{n_1^2} \right). \]
Since
\[
\left( \frac{m_1}{(m_1, n_2^2)}, m_2 \right) = 1, \quad \left( \frac{m_1}{(m_1, n_2^2)}, \frac{n_2^2}{(m_1, n_2^2)} \right) = 1,
\]
we have
\[
\left( \frac{m_1}{(m_1, n_2^2)}, \frac{n_2^2}{(m_1, n_2^2)} \right) = \left( \frac{n_2^2}{(m_1, n_2^2)} \right) = \left( \frac{n_1^2}{m_2} \right).
\]
Thus, the conclusion of Lemma 15 follows.

**Lemma 16.** If \(a\) is a positive integer, then
\[
\sum_{M < m \leq 2^eM} (m, a) \ll Md(a).
\]

Proof. We have
\[
\sum_{M < m \leq 2^eM} (m, a) = \sum_{d \mid a} \sum_{m, \leq \frac{M}{d}} 1
\]
\[
= \sum_{d \mid a} \sum_{m, \leq \frac{\frac{M}{d^2}}{d}} 1
\]
\[
\leq \sum_{d \mid a} \sum_{M, \leq \frac{M}{d}} 1
\]
\[
\ll \sum_{d \mid a} \frac{M}{d} = Md(a).
\]

**Lemma 17.** Suppose that \(0 < A < B < 2q\) and that \(b\) is a positive integer. Then
\[
\sum_{A < a < B \atop (a, q) = 1} e\left( \frac{\overline{A}}{q} \right) \ll (l, q)^{1/2}q^{1/2}b^\varepsilon.
\]
Here \(\overline{a}\) is the integer such that \(a\overline{a} \equiv 1 \pmod{q}\).

Proof. By Lemma 3 of [4], for \(0 < A < B < 2q\), we have
\[
\sum_{A < a < B \atop (a, q) = 1} e\left( \frac{\overline{A}}{q} \right) \ll (l, q)^{1/2}q^{1/2}b^\varepsilon.
\]
Hence,
\[
\sum_{A < a \leq B \atop (a, q) = 1} e(l \frac{\tau}{q}) = \sum_{A < a \leq B \atop (a, q) = 1} \left( \sum_{d | (a, b)} \mu(d) \right) e(l \frac{\tau}{q})
\]
\[
= \sum_{d | b} \mu(d) \sum_{A < a \leq B \atop (a, q) = 1 \atop d | a} e(l \frac{\tau}{q})
\]
\[
= \sum_{d | b} \mu(d) \sum_{A < t \leq B \atop (dt, q) = 1} e(l \frac{\tau}{d})
\]
\[
\ll \sum_{d | b} |\mu(d)| \cdot (l d \cdot q)^{\frac{1}{2}} q^{\frac{1}{2} + \varepsilon}
\]
\[
\ll (l, q)^{\frac{1}{2}} q^{\frac{1}{2} + \varepsilon} \sum_{d | b} 1
\]
\[
\ll (l, q)^{\frac{1}{2}} q^{\frac{1}{2} + \varepsilon} b^\varepsilon.
\]

Thus, Lemma 17 is proved.

3. An asymptotic expression of \(\zeta(1 + it)\)

Let
\[
\rho_1 = \beta_1 + i \gamma_1, \quad \rho_2 = \beta_2 + i \gamma_2, \quad \ldots, \quad \rho_J = \beta_J + i \gamma_J
\]
be all zeros of \(\zeta(s)\) which satisfy \(\beta \geq 1 - 4\varepsilon, \ T \leq \gamma \leq 2T\). By Lemma 12,
\[
J = N(1 - 4\varepsilon, \ T, \ 2T) \ll T^{11\varepsilon}. \quad (3.1)
\]

We write domain \(D\) as
\[
D = \{s = \sigma + it : \ 1 - 4\varepsilon \leq \sigma, \ T \leq t \leq 2T\}.
\]
Write
\[ U_1 = \bigcup_{j=1}^{J} (\gamma_j - (\log T)^{10}, \gamma_j + (\log T)^{10}), \]
\[ U_2 = \bigcup_{j=1}^{J} (\gamma_j - 2(\log T)^{10}, \gamma_j + 2(\log T)^{10}), \]
\[ U_3 = \bigcup_{j=1}^{J} (\gamma_j - 3(\log T)^{10}, \gamma_j + 3(\log T)^{10}), \]
\[ U_4 = \bigcup_{j=1}^{J} (\gamma_j - 4(\log T)^{10}, \gamma_j + 4(\log T)^{10}). \] (3.2)

After removing all domains of the form \( \{ s = \sigma + it : 1 - 4\varepsilon \leq \sigma < 1, t \in U_1 \} \) in \( \mathbb{D} \), we denote the remained domain as \( D_1 \). \( D_1 \) is a connected domain in which \( \zeta(s) \neq 0 \) so that we can define a holomorphic function \( \log \zeta(s) \) in \( D_1 \). For \( \text{Re}(s) > 1 \), Euler's product formula produces
\[ \log \zeta(s) = -\sum_{p} \log \left(1 - \frac{1}{p^s}\right) = \sum_{p} \sum_{m=1}^{\infty} \frac{1}{mp^{ms}} = \sum_{n=2}^{\infty} \frac{\Lambda_1(n)}{n^s}, \] (3.3)
where
\[ \Lambda_1(n) = \frac{\Lambda(n)}{\log n}. \]

After removing all domains of the form \( \{ s = \sigma + it : 1 - 4\varepsilon \leq \sigma, t \in U_2 \} \) in \( \mathbb{D} \), we denote the remained domain as \( D_2 \). Now Lemma 1 can be applied. Take \( f(z) = \log \zeta(z) \). For \( s = \sigma + it \in D_2, 1 - 2\varepsilon \leq \sigma \leq 2 \), let the center of circle be \( z_0 = 2 + it \), the radius of bigger circle be \( R = 2 - (1 - 4\varepsilon) = 1 + 4\varepsilon \), the radius of smaller circle be \( r = 2 - (1 - 2\varepsilon) = 1 + 2\varepsilon \). On the bigger circle, by Lemma 7,
\[ \text{Re} \log \zeta(z) = \log |\zeta(z)| \leq C \log T, \]
where \( C \) is a positive constant. Thus, for \( s \) in the smaller circle, Lemma 1 yields
\[ |\log \zeta(s)| \leq \frac{2r}{R-r} \cdot C \log T + \frac{R+r}{R-r} \cdot |\log \zeta(2 + it)| \ll \log T. \]
For \( \text{Re}(s) \geq 2 \), it is easy to see
\[ \log \zeta(s) = O(1). \]
Hence, for \( s = \sigma + it \in D_2, \sigma \geq 1 - 2\varepsilon \), we have

\[
|\log \zeta(s)| \ll \log T. \tag{3.4}
\]

After removing all domains of the form \( \{ s = \sigma + it : 1 - 4\varepsilon \leq \sigma, t \in U_3 \} \) in \( D \), then limiting \( \sigma \geq 1 - 2\varepsilon \), we denote the obtained domain as \( D_3 \). Now Lemma 2 can be applied. Take \( f(z) = \log \zeta(z) \). For \( s = \sigma + it \in D_3, 1 - \varepsilon \leq \sigma \leq 1 + \varepsilon \), let the center of circle be \( z_0 = 2 + it, R_3 = 2 - (1 - 2\varepsilon) = 1 + 2\varepsilon, R_2 = 2 - (1 - \varepsilon) = 1 + \varepsilon, R_1 = 2 - (1 + \varepsilon) = 1 - \varepsilon \). By (3.4), \( M_3 \ll \log T \). It is obvious that \( M_1 = O(1) \). Lemma 2 yields

\[
\log M_2 \leq \frac{\log(\frac{1+2\varepsilon}{1+\varepsilon})}{\log(\frac{1+2\varepsilon}{1-\varepsilon})} \cdot \log M_1 + \frac{\log(\frac{1+\varepsilon}{1-\varepsilon})}{\log(\frac{1+2\varepsilon}{1-\varepsilon})} \cdot \log M_3
\]

\[
\leq O(1) + \frac{2\varepsilon + O(\varepsilon^2)}{3\varepsilon + O(\varepsilon^2)} \cdot \log \log T
\]

\[
= O(1) + \frac{2}{3} O(\varepsilon) \log \log T
\]

\[
\leq \frac{3}{4} \log \log T.
\]

Hence, for \( s = \sigma + it \in D_3, 1 - \varepsilon \leq \sigma \leq 1 + \varepsilon \), we have

\[
|\log \zeta(s)| \leq (\log T)^{\frac{3}{4}}.
\]

For \( \text{Re}(s) \geq 1 + \varepsilon \), it is obvious that

\[
\frac{1}{\zeta(s)} = O_\varepsilon(1).
\]

Thus, for \( s = \sigma + it \in D_3, \sigma \geq 1 - \varepsilon \), we have

\[
\frac{1}{\zeta(s)} \ll \exp((\log T)^{\frac{3}{4}}). \tag{3.5}
\]

After removing all domains of the form \( \{ s = \sigma + it : 1 - 4\varepsilon \leq \sigma, t \in U_4 \} \) in \( D \), then limiting \( \sigma \geq 1 - \varepsilon \), we denote the obtained domain as \( D_4 \). For \( s \in D_4, u \geq 0, |v| \leq (\log T)^3 \), we have

\[
\frac{1}{\zeta(s + u + iv)} \ll \exp((\log T)^{\frac{3}{4}}). \tag{3.6}
\]
For $s = 1 + it \in D_4$, $w = u + iv$, $X > 1$, we have

$$
\frac{1}{2\pi i} \int_{u=\epsilon, |v| \leq \log T^3} \frac{1}{\zeta(s + w)} \cdot \Gamma(w)X^w \, dw \\
= \frac{1}{2\pi i} \int_{u=\epsilon, |v| \leq \log T^3} \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{s+w}} \cdot \Gamma(w)X^w \, dw \\
= \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \cdot \frac{1}{2\pi i} \int_{u=\epsilon, |v| \leq \log T^3} \Gamma(w) \left( \frac{X}{n} \right)^w \, dw.
$$

By Lemma 4, if $|v| \geq 1$, then on the vertical line $u = \epsilon$, we have

$$
\Gamma(w) \ll |v|^{\frac{1}{2} + \frac{1}{2}e^{-\frac{\pi}{2}|v|}}.
$$

Hence,

$$
\frac{1}{2\pi i} \int_{u=\epsilon, |v| > \log T^3} \Gamma(w) \left( \frac{X}{n} \right)^w \, dw \\
\ll \left( \frac{X}{n} \right)^{\epsilon} \int_{u=\epsilon, |v| > \log T^3} |\Gamma(w)||dw| \\
\ll \left( \frac{X}{n} \right)^{\epsilon} \int_{|v| > \log T^3} |v|^{\frac{1}{2} + \frac{1}{2}e^{-\frac{\pi}{2}|v|}} \, dv \\
\ll \left( \frac{X}{n} \right)^{\epsilon} \int_{\log T^3}^{\infty} e^{-\frac{\pi}{2}v} \, dv \\
\ll \left( \frac{X}{n} \right)^{\epsilon} \exp\left( -\frac{\pi}{2} (\log T)^3 \right).
$$

By Lemma 3,

$$
\frac{1}{2\pi i} \int_{\epsilon - i\infty}^{\epsilon + i\infty} \Gamma(w) \left( \frac{X}{n} \right)^w \, dw = e^{-\frac{n}{X}}.
$$

Therefore it follows that

$$
\frac{1}{2\pi i} \int_{u=\epsilon, |v| \leq \log T^3} \frac{1}{\zeta(s + w)} \cdot \Gamma(w)X^w \, dw \\
= \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \left( e^{-\frac{n}{X}} + O\left( \left( \frac{X}{n} \right)^{\epsilon} \exp\left( -\frac{\pi}{2} (\log T)^3 \right) \right) \right) \\
= \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} e^{-\frac{n}{X}} + O\left( X^{\epsilon} \exp\left( -\frac{\pi}{2} (\log T)^3 \right) \right).
$$

We move the line of integration to Re($w$) = $-\epsilon$. At $w = 0$, $\Gamma(w)$ has a pole of order 1 with residue 1. Hence, the residue of $\frac{1}{\zeta(s + w)} \cdot \Gamma(w)X^w$ at
$w = 0$ is $\frac{1}{\zeta(s)}$. In two horizontal lines, by (3.6),

$$\frac{1}{2\pi i} \int_{-\varepsilon \leq u \leq \varepsilon, |v| = (\log T)^3} \frac{1}{\zeta(s + w)} \cdot \Gamma(w) X^w dw \ll X^\varepsilon \exp((\log T)^{\frac{3}{4}}) \int_{-\varepsilon}^{\varepsilon} e^{-\frac{\pi}{2}(\log T)^{\frac{3}{4}}} du \ll X^\varepsilon \exp(-(\log T)^3).$$

The integration on Re($w$) = $-\varepsilon$ is

$$\frac{1}{2\pi i} \int_{u = -\varepsilon, |v| \leq (\log T)^3} \frac{1}{\zeta(s + w)} \cdot \Gamma(w) X^w dw \ll X^{-\varepsilon} \exp((\log T)^{\frac{3}{4}}) \left( \int_{u = -\varepsilon, |v| \leq (\log T)^3} |\Gamma(w)||dw| \right) \ll X^{-\varepsilon} \exp((\log T)^{\frac{3}{4}}) \left( \int_{u = -\varepsilon, |v| \leq 1} |\Gamma(w)||dw| \right) \ll X^{-\varepsilon} \exp((\log T)^{\frac{3}{4}}) \left( \int_{|u| \leq (\log T)^3} \frac{|dw|}{|u|} \right) \ll X^{-\varepsilon} \exp((\log T)^{\frac{3}{4}}) \left( \int_{1 \leq |v| \leq (\log T)^3} |v|^{-\varepsilon - \frac{1}{2}} e^{-\frac{\pi}{2}|v|} dv \right) \ll \varepsilon^{-\varepsilon} X^{-\varepsilon} \exp((\log T)^{\frac{3}{4}}).$$

Combining all of the above, we get (with $s = 1 + it$)

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} e^{-\frac{n}{X}} + O(X^\varepsilon \exp(-(\log T)^{\frac{3}{4}})) + O(X^{-\varepsilon} \exp((\log T)^{\frac{3}{4}})) \quad (3.7)$$

Therefore we obtain an asymptotic expression of $\zeta(1 + it)$ as follows.

**Proposition 1.** Suppose that $T \leq t \leq 2T$, $t \notin U_4$ and

$$X = \exp(\frac{2}{\varepsilon}(\log T)^{\frac{3}{4}}). \quad (3.8)$$

Then we have

$$\frac{1}{\zeta(1 + it)} = \sum_{n \leq X} \frac{\mu(n)}{n^{1+it}} e^{-\frac{n}{X}} + O(1). \quad (3.9)$$

4. A mean value estimate on $\zeta(s)$
In this section, we shall prove the following mean value estimate on $\zeta(s)$.

**Proposition 2.** If $k$ is any given positive number, then we have

$$\int_{T}^{T} \frac{|\zeta(\frac{1}{2} + it)|^4}{|\zeta(1 + 2it)|^k} dt \ll T \log^4 T.$$ 

Firstly we shall prove the following Proposition 3. We use the method of Iwaniec\[4\] essentially but with some modification and refinement.

**Proposition 3.** Suppose that $N \ll T^{\frac{1}{16} - \varepsilon}$ and that for $N < n \leq 2^\varepsilon N$, $a(n) = O(N^{-1+\varepsilon})$. Then

$$\int_{T}^{T} \sum_{N < n \leq 2^\varepsilon N} \frac{|a(n)|^2}{n^{2it}} dt \ll T \log^4 T \frac{T}{N^{1-8\varepsilon}}.$$ 

Proof. By the discussion in Section 2 of [4], we shall estimate

$$\log T \sum_{r \leq 2^{\varepsilon \log_2 T} + O(1)} \int_{0}^{\infty} e^{-t} |\zeta(\frac{1}{2} + it)|^2.$$ 

$$\cdot \sum_{2^{\varepsilon r} < m \leq 2^{\varepsilon} \cdot 2^{\varepsilon}} \left\{ \frac{1}{m^{1+it}} \right\}^2 \sum_{N < n \leq 2^{\varepsilon} N} \frac{|a(n)|^2}{n^{2it}} dt.$$ 

Write

$$\left| \left( \sum_{M < m \leq 2^{\varepsilon} M} \frac{1}{m^{1+it}} \right) \left( \sum_{N < n \leq 2^{\varepsilon} N} \frac{a(n)}{n^{2it}} \right) \right|^2$$

$$= \left| \sum_{K < k \leq 8^\varepsilon K} b(k) \right|^2 = \sum_{K < k, h \leq 8^\varepsilon K} b(k) b(h) \frac{h}{k}^{1+it},$$

where $M = 2^{\varepsilon r}$, $M \ll T^{\frac{1}{16}}$, $K = M N^2$,

$$b(k) = \sum_{\frac{mn^2 = k}{M < m \leq 2^\varepsilon M} \frac{N < n \leq 2^\varepsilon N}} \frac{a(n)}{m^2}.$$ 

In the following we shall estimate

$$\int_{0}^{\infty} e^{-t} |\zeta(\frac{1}{2} + it)|^2 \sum_{K < k \leq 8^\varepsilon K} \frac{b(k)}{k^{1+it}} dt$$

$$= \sum_{K < k, h \leq 8^\varepsilon K} b(k) b(h) \int_{0}^{\infty} e^{-(\frac{1}{2} - i \log(h))t} |\zeta(\frac{1}{2} + it)|^2 dt.$$ 

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Let
\[ z = \frac{1}{T} - i \log \left( \frac{h}{k} \right) \] (4.2)
and note that
\[ |z| \leq \frac{1}{T} + \left| \log \left( \frac{h}{k} \right) \right| < 4\varepsilon \]
for \( K < k, h \leq 8\varepsilon K \).

By Lemma 13,
\[
\int_0^\infty e^{-zt} |\zeta(\frac{1}{2} + it)|^2 dt = 2\pi e^{i\frac{T}{2}} \left( \frac{h}{k} \right)^{1/2} \sum_{l=1}^\infty d(l) e(l \cdot \frac{h}{k}) \exp(2\pi ilx) + O(1) \] (4.3)

where
\[ x = \frac{h}{k} (e^{\frac{T}{2}} - 1). \] (4.4)

The contribution of the term \( O(1) \) to (4.1) is
\[
O \left( \sum_{K < k, h \leq 8\varepsilon K} |b(k)b(h)| \right) \ll \sum_{M < m_1 \leq 2^x M} \frac{1}{m_1} \sum_{M < m_2 \leq 2^x M} \frac{1}{m_2} \sum_{N < n_1 \leq 2^x N} |a(n_1)| \sum_{N < n_2 \leq 2^x N} |a(n_2)| \ll M \frac{1}{N^{1-\varepsilon}} \ll MN^{2\varepsilon} \ll \frac{T}{N^{1-8\varepsilon}}.
\]

Let
\[ S(x; \frac{h}{k}) = \sum_{l=1}^\infty d(l) e(l \cdot \frac{h}{k}) \exp(2\pi ix). \] (4.5)

Write
\[ \zeta = -2\pi i x = 4\pi \left( \frac{h}{k} \right) \sin \left( \frac{1}{2T} \right) e^{\frac{T}{h}}. \] (4.6)

By the discussion in Section 3 of [4], we know
\[ S(x; \frac{h}{k}) = \frac{1}{2\pi i} \int_{1+\varepsilon}^{1+\varepsilon+i\infty} D(s; \frac{h}{k}) \Gamma(s) \zeta^{-s} ds, \] (4.7)
where
\[
D(s; \frac{h}{k}) = \sum_{l=1}^{\infty} \frac{d(l)}{l^s} e(l\frac{h}{k}).
\]

In the following we write
\[
k^* = \frac{k}{(k, h)}, \quad h^* = \frac{h}{(k, h)}.
\] (4.8)

We move the line of integration from \(\text{Re}(s) = 1 + \epsilon\) to \(\text{Re}(s) = -\epsilon\), and get
\[
S(x; \frac{h}{k}) = \frac{1}{2\pi i} \int_{1+\epsilon-i\infty}^{1+\epsilon+i\infty} D(s; \frac{h^*}{k^*}) \Gamma(s) s^{-s} ds
\]
\[
= \frac{1}{2\pi i} \int_{-\epsilon-i\infty}^{-\epsilon+i\infty} D(s; \frac{h^*}{k^*}) \Gamma(s) s^{-s} ds + R_1(T; h, k) + R_0(T; h, k) \quad (4.9)
\]
\[
= R(T; h, k) + R_1(T; h, k) + R_0(T; h, k),
\]
where
\[
R(T; h, k) = \frac{1}{2\pi i} \int_{-\epsilon-i\infty}^{\epsilon+i\infty} D(s; \frac{h^*}{k^*}) \Gamma(s) s^{-s} ds, \quad (4.10)
\]
\[R_1(T; h, k)\) and \(R_0(T; h, k)\) are residues of \(D(s; \frac{h^*}{k^*}) \Gamma(s) s^{-s}\) coming from the poles at \(s = 1\) and \(s = 0\) respectively.

By the discussion in Section 3 of [4] and Lemma 14, we know that
\[
R_1(T; h, k) = \frac{1}{3k^*}(\gamma - \log 3 - 2\log k^*) \ll \frac{T \log T}{k^*}, \quad (4.11)
\]
\[
R_0(T; h, k) = D(0; \frac{h^*}{k^*}) \quad (4.12)
\]
\[
= \frac{1}{4} - \frac{1}{\pi i} \sum_{a=1}^{k^*} \beta(a, k^*) \sum_{0 < b < k^*} \eta(b, k^*) e(ab \frac{h^*}{k^*}).
\]

Now we see the contribution of \(R_1(T; h, k), R(T; h, k)\) and \(R_0(T; h, k)\) to (4.1).

1. The contribution of \(R_1(T; h, k)\)

We note that \(\frac{h}{k} \ll 1\) for \(K < h, k \leq 8^K\). Therefore the contribution of \(R_1(T; h, k)\) is
\[
\ll \sum_{K<k, h \leq 8^K} |b(k)b(h)||R_1(T; h, k)|
\]
\[
\ll \sum_{K<k, h \leq 8^K} |b(k)b(h)| \cdot \frac{T \log T}{k} (k, h)
\]
By Lemmas 15 and 16,

\[
\ll T \log T \sum_{M < m_1 \leq 2^M} \sum_{M < m_2 \leq 2^M} \sum_{N < n_1 \leq 2^N} \sum_{N < n_2 \leq 2^N} \frac{|a(n_1)|}{m_1^{1/2}} \cdot \frac{|a(n_2)|}{m_2^{1/2}} (m_1 n_1^2, m_2 n_2^2)
\]

(4.13)

\[
\ll T \log T \cdot \frac{1}{MN^{2-2\varepsilon}} \sum_{M < m_1 \leq 2^M} \sum_{M < m_2 \leq 2^M} \sum_{N < n_1 \leq 2^N} \sum_{N < n_2 \leq 2^N} (m_1 n_1^2, m_2 n_2^2) 
\]

\[= T \log T \cdot \frac{1}{M^2 N^{4-2\varepsilon}} \sum_{M < m_1 \leq 2^M} \sum_{M < m_2 \leq 2^M} \sum_{N < n_1 \leq 2^N} \sum_{N < n_2 \leq 2^N} (m_1 n_1^2, m_2 n_2^2). \]

By Lemmas 15 and 16,

\[
\sum_{M < m_1 \leq 2^M} \sum_{M < m_2 \leq 2^M} \sum_{N < n_1 \leq 2^N} \sum_{N < n_2 \leq 2^N} (m_1 n_1^2, m_2 n_2^2)
\]

\[= \sum_{d \leq 2^M} \sum_{M/d < m_1' \leq 2^M} \sum_{M/d < m_2' \leq 2^M} \sum_{r \leq 2^N} \sum_{N/r < n_1' \leq 2^N} r^2 \sum_{N/r < n_2' \leq 2^N} \sum_{(m_1', m_2') = 1} \sum_{(n_1', n_2') = 1} (m_1' n_1'^2, m_2' n_2'^2)
\]

\[= \sum_{d \leq 2^M} \sum_{M/d < m_1' \leq 2^M} \sum_{M/d < m_2' \leq 2^M} \sum_{r \leq 2^N} \sum_{N/r < n_1' \leq 2^N} \sum_{N/r < n_2' \leq 2^N} (m_1', n_1'^2)(m_2', n_1'^2)
\]

\[\leq \sum_{d \leq 2^M} \sum_{M/d < m_1' \leq 2^M} \sum_{M/d < m_2' \leq 2^M} \sum_{r \leq 2^N} \sum_{N/r < n_1' \leq 2^N} \sum_{N/r < n_2' \leq 2^N} (m_1', n_1'^2)(m_2', n_1'^2),
\]
Hence, the contribution of $R_1(T; h, k)$ is
\[
\ll \frac{T \log T}{M^2 N^{4-2\varepsilon}} \cdot M^2 N^{3+2\varepsilon} \log(2M) \ll \frac{T \log^2 T}{N^{1-8\varepsilon}}.
\]

2. The contribution of $R(T; h, k)$

By the functional equation in Lemma 14, we get
\[
R(T; h, k) = \frac{1}{2\pi i} \int_{-\varepsilon-i\infty}^{-\varepsilon+i\infty} D(s; \frac{h^*}{k^*}) \Gamma(s) \zeta^{-s} ds
\]
\[
= \frac{1}{2\pi i} \int_{-\varepsilon-i\infty}^{-\varepsilon+i\infty} 2G^2(s)k^*(1-2s) \left( D(1-s; \frac{h^*}{k^*}) - \cos(\pi s)D(1-s; -\frac{h^*}{k^*}) \right) \Gamma(s) \zeta^{-s} ds
\]
\[
= k^* \sum_{l=1}^{\infty} \frac{d(l)}{l} \cdot \frac{1}{2\pi i} \int_{-\varepsilon-i\infty}^{-\varepsilon+i\infty} 2G^2(s) \cdot \frac{l^s}{(h^* k^*)^s} \cdot \left( e(l \frac{h^*}{k^*}) - \cos(\pi s) e(-l \frac{h^*}{k^*}) \right) \Gamma(s) \left( 4\pi \sin\left(\frac{1}{2T}e^{\frac{\pi}{2T}}\right) \right)^{-s} ds
\]
\[
= k^* \sum_{l=1}^{\infty} \frac{d(l)}{l} \cdot \frac{1}{2\pi i} \int_{-\varepsilon-i\infty}^{-\varepsilon+i\infty} U(s, T)(\frac{l}{h^* k^*})^s \cdot \left( e(l \frac{h^*}{k^*}) - \cos(\pi s) e(-l \frac{h^*}{k^*}) \right) ds,
\]

where
\[
U(s, T) = 2G^2(s)\Gamma(s)\left( 4\pi \sin\left(\frac{1}{2T}e^{\frac{\pi}{2T}}\right) \right)^{-s}.
\]

The contribution of $R(T; h, k)$ is
\[
\ll \left| \sum_{K<k, h \leq 8^K} b(k)b(h)(\frac{h}{k})^{\frac{1}{2}} R(T; h, k) \right|.
\]
where

\[
\sum_{K<k,h \leq 8^sK} b(k)b(h) \left( \frac{h}{k} \right)^{\frac{1}{2}} R(T; h, k)
\]

= \sum_{K<k \leq 8^sK} \sum_{K<h \leq 8^sK} \frac{b(k)}{k^{\frac{1}{2}}} \cdot \frac{b(h)}{h^{\frac{1}{2}}} \cdot k^* \sum_{l=1}^{\infty} \frac{d(l)}{l} \cdot \frac{1}{2\pi i} \int_{-\varepsilon-i\infty}^{-\varepsilon+i\infty} \frac{U(s, T)(l)}{h^*k^*} e\left(\frac{lh}{k} - \cos(\pi s) e\left(-\frac{lh}{k} \right)\right) ds
\]

= \sum_{l=1}^{d(l)} \frac{d(l)}{l} \cdot \frac{1}{2\pi i} \int_{-\varepsilon-i\infty}^{-\varepsilon+i\infty} U(s, T) l^s \left( \sum_{K<k \leq 8^sK} \sum_{K<h \leq 8^sK} \frac{b(k)}{k^{\frac{1}{2}}} \cdot \frac{b(h)}{h^{\frac{1}{2}}} \cdot \frac{k^*}{(h^*k^*)^s} e\left(-\frac{lh}{k} \right) \right) ds
\]

= \sum_{l=1}^{\infty} \frac{d(l)}{l} \cdot \frac{1}{2\pi i} \int_{-\varepsilon-i\infty}^{-\varepsilon+i\infty} U(s, T) l^s (Q(l, s) - \cos(\pi s)Q(-l, s)) ds,
\]

where

\[
Q(l, s) = \sum_{K<k \leq 8^sK} \sum_{K<h \leq 8^sK} \frac{b(k)}{k^{\frac{1}{2}}} \cdot \frac{b(h)}{h^{\frac{1}{2}}} \cdot \frac{k^*}{(h^*k^*)^s} e\left(-\frac{lh}{k} \right).
\]

Equation (4.16)

For \( s = -\varepsilon + it \), by the discussion in Section 5 of [4],

\[
U(s, T) l^s \ll (\frac{T}{t})^c (|t| + 1)^{\frac{1}{2} + \varepsilon} \exp\left(\frac{1}{2T} - \frac{3}{2} \pi \right)|t|,
\]

\[
U(s, T) l^s \cos(\pi s) \ll (\frac{T}{t})^c (|t| + 1)^{\frac{1}{2} + \varepsilon} \exp\left(\frac{1}{2T} - \frac{\pi}{2} \right)|t|.
\]

In the following we shall estimate \( Q(l, s) \) for \( s = -\varepsilon + it \).

\[
Q(l, s) = \sum_{K<k \leq 8^sK} \sum_{K<h \leq 8^sK} b(k)b(h) \cdot \frac{1}{(k^*h^*)^{s-\frac{1}{2}}} e\left(l\frac{h}{k} \right)
\]

= \sum_{K<k \leq 8^sK} \sum_{K<h \leq 8^sK} b(k)b(h) \cdot \frac{(k^*h^*)^{2s-1}}{(kh)^{s-\frac{1}{2}}} e\left(l\frac{h}{k} \right)
\]

= \sum_{d \leq 8^sK} d^{2s-1} \sum_{K<k \leq 8^sK} \sum_{(k,h)=d} b(k)b(h) \cdot \frac{(kh)^{s-\frac{1}{2}} e\left(l\frac{h}{k} \right)}{d}
where

\[ M < M \]

for \( 1 \). Hence,

\[ d \]

We shall estimate

\[ B(l, s, n_1, n_2, d) = \sum_{d \leq 8^s K} \sum_{N < n_1 \leq 2^s N} \sum_{N < n_2 \leq 2^s N} \sum_{M < m_1 \leq 2^s M} \frac{a(n_2) a(n_1)}{m_2^{\frac{s}{2}} m_1^{\frac{s}{2}} (m_1 m_2 n_1 n_2^2)^{s-\frac{1}{2}}} e\left( l \left( \frac{m_1 n_1^2}{d} \right) \right). (4.21) \]

for \( M < M_1 \leq 2^s M \). Let \((m_1, d) = d_1\). Write \( d = d_1 d_2 \). We see \((d_2, \frac{m_1}{d_1}) = 1\). Hence, \( d \mid m_1 n_1^2 = d_2 \mid n_1^2 \Rightarrow d_2 \leq 4^s N_2 \). By Lemma 17,

\[ \sum_{M < m_1 \leq M_1} e\left( l \left( \frac{m_1 n_1^2}{d} \right) \right) \]

\[ = \sum_{d_1 \mid d} \left( \sum_{M < m_1 \leq M_1} \sum_{(m_1, d_1) = d_1} e\left( l \left( \frac{m_1 n_1^2}{d} \right) \right) \right) \]

\[ = \sum_{d_1 \mid d} \left( \sum_{M < m_1 \leq M_1} \sum_{(m_1, m_2) = 1} e\left( l \left( \frac{m_1 n_1^2}{d} \right) \right) \right) \]

\[ = \sum_{d_1 \mid d} \left( \sum_{M < m_1 \leq M_1} \sum_{(m_1, m_2) = 1} e\left( l \left( \frac{m_1 n_1^2}{d} \right) \right) \right) \]

\[ = \sum_{d_1 \mid d} \left( \sum_{M < m_1 \leq M_1} \sum_{(m_1, m_2) = 1} e\left( l \left( \frac{m_1 n_1^2}{d} \right) \right) \right) \]

\[ = \sum_{d_1 \mid d} \left( \sum_{M < m_1 \leq M_1} \sum_{(m_1, m_2) = 1} e\left( l \left( \frac{m_1 n_1^2}{d} \right) \right) \right) \]

\[ = \sum_{d_1 \mid d} \left( \sum_{M < m_1 \leq M_1} \sum_{(m_1, m_2) = 1} e\left( l \left( \frac{m_1 n_1^2}{d} \right) \right) \right) \]

\[ = \sum_{d_1 \mid d} \left( \sum_{M < m_1 \leq M_1} \sum_{(m_1, m_2) = 1} e\left( l \left( \frac{m_1 n_1^2}{d} \right) \right) \right) \]

\[ = \sum_{d_1 \mid d} \left( \sum_{M < m_1 \leq M_1} \sum_{(m_1, m_2) = 1} e\left( l \left( \frac{m_1 n_1^2}{d} \right) \right) \right) \]

\[ = \sum_{d_1 \mid d} \left( \sum_{M < m_1 \leq M_1} \sum_{(m_1, m_2) = 1} e\left( l \left( \frac{m_1 n_1^2}{d} \right) \right) \right) \]

\[ = \sum_{d_1 \mid d} \left( \sum_{M < m_1 \leq M_1} \sum_{(m_1, m_2) = 1} e\left( l \left( \frac{m_1 n_1^2}{d} \right) \right) \right) \]
By Lemma 16, we get

\[
\sum_{\frac{n_1^2}{d_1}} \left( \sum_{\frac{n_1^2}{d_2}} e\left( \frac{n_1^2}{d_2} \cdot \frac{m'}{d_2} \right) \right)
\]

By Lemma 16, here we note

\[
\sum_{\frac{n_1^2}{d_1}} \left( l \frac{n_1^2}{d_2} , \frac{m_2 n_2^2}{d} \right) \frac{1}{2} \left( \frac{m_2 n_2^2}{d} \right)^{\frac{1}{2} + \varepsilon} d_2^\varepsilon
\]

\[
\leq \sum_{\frac{n_1^2}{d_1}} \left( l \frac{n_1^2}{d_2} , \frac{m_2 n_2^2}{d} \right) \frac{1}{2} \left( \frac{m_2 n_2^2}{d} \right)^{\frac{1}{2} + \varepsilon} d_2^\varepsilon
\]

\[
\leq \sum_{\frac{n_1^2}{d_1}} \left( l \frac{n_1^2}{d_2} , \frac{m_2 n_2^2}{d} \right) \frac{1}{2} \left( MN^2 \right)^{\frac{1}{2} + \varepsilon} d_2^\varepsilon,
\]

here we note \( d_2 \leq 4^\varepsilon N^2 \implies \frac{M}{d_1} < \frac{2m_2 n_2^2}{d}. \)

By the above estimate and partial summation, for \( s = -\varepsilon + it \), we have

\[
\sum_{\frac{M}{d_1} \leq m_1 \leq \frac{2M}{d_1}} \sum_{\frac{m_2 n_2^2}{d}} \frac{1}{m_1} e\left( \frac{m_1 n_1^2}{d_2} \right) \leq (|t| + 1) M^\varepsilon \left( \frac{m_2 n_2^2}{d} \right)^{\frac{1}{2}} \left( \frac{MN^2}{d} \right)^{\frac{1}{2} + \varepsilon} d_2^\varepsilon.
\]

By Lemma 16,

\[
B(l, s, n_1, n_2, d) \leq (|t| + 1) \left( \frac{MN^2}{d} \right)^{\frac{1}{2} + \varepsilon} M^{2\varepsilon} d_2^\varepsilon \sum_{\frac{M}{d_1} \leq m_2 \leq \frac{2M}{d_1}} \left( l, \frac{m_2 n_2^2}{d} \right)^{\frac{1}{2}}
\]

\[
\leq (|t| + 1) \left( \frac{MN^2}{d} \right)^{\frac{1}{2} + \varepsilon} M^{2\varepsilon} d_2^\varepsilon \sum_{\frac{M}{d_1} \leq m_2 \leq \frac{2M}{d_1}} \left( l, \frac{m_2 n_2^2}{d} \right)^{\frac{1}{2}}
\]

\[
\leq (|t| + 1) \left( \frac{MN^2}{d} \right)^{\frac{1}{2} + \varepsilon} M^{1+2\varepsilon} d_2^\varepsilon \left( l, \frac{n_2^2}{d} \right)^{\frac{1}{2}} l^\frac{1}{4}.
\]

By Lemma 16 again, we get

\[
Q(l, s) \leq (|t| + 1) \left( MN^2 \right)^{\frac{1}{2} + \varepsilon} M^{1+2\varepsilon} \sum_{d \leq 8^\varepsilon K} \frac{1}{d^{\frac{3}{2} + \varepsilon}} \sum_{N \leq n_1 \leq 2^\varepsilon N} \sum_{N \leq n_2 \leq 2^\varepsilon N} \left| \alpha(n_1) \alpha(n_2) \right| \left( N^{2(1+2\varepsilon)} \left( l, \frac{n_2^2}{d} \right)^{\frac{1}{2}} l^\frac{1}{4} \right)
\]

\[
\leq (|t| + 1) M^{\frac{1}{2} + 3\varepsilon} N^{2+8\varepsilon} \sum_{N \leq n_2 \leq 2^\varepsilon N} \left( n_2, l \right)^\frac{1}{2}
\]

\[
\leq (|t| + 1) M^{\frac{1}{2} + 3\varepsilon} N^{3+8\varepsilon} l^\frac{1}{4}.
\]

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The contribution of $R(T; h, k)$ is
\[
\ll \sum_{l=1}^{\infty} \frac{d(l)}{l} \left( \frac{T^\varepsilon}{l} \right) \int_{-\infty}^{\infty} (|t| + 1)^{\frac{3}{2} + \varepsilon} \exp\left( \frac{1}{2T} - \frac{\pi}{2} |t| \right) dt \cdot M^{\frac{3}{2} + 3\varepsilon} N^{3 + 8\varepsilon} \left( \frac{T}{l} \right)^{\frac{1}{2} + \varepsilon}
\]
\[
\ll T^\varepsilon M^{\frac{3}{2} + 3\varepsilon} N^{3 + 8\varepsilon} \sum_{l=1}^{\infty} \frac{d(l)}{l^{1 + \frac{3}{2} + \varepsilon}}
\]
\[
\ll T^\varepsilon M^{\frac{3}{2} + 3\varepsilon} N^{3 + 8\varepsilon}
\]
\[
\ll \frac{T}{N^{1 - 8\varepsilon}}.
\]

3. The contribution of $R_0(T; h, k)$

Using Lemma 14, (4.12) and the estimates in 2., we get that the contribution of $R_0(T; h, k)$ is
\[
\ll \left| \sum_{K < k, h \leq 8^s K} b(k) \overline{b(h)} \left( \frac{h}{k} \right)^{\frac{1}{2}} R_0(T; h, k) \right|,
\]
while
\[
\sum_{K < k, h \leq 8^s K} b(k) \overline{b(h)} \left( \frac{h}{k} \right)^{\frac{1}{2}} R_0(T; h, k)
\]
\[
= \sum_{K < k \leq 8^s K} \sum_{K < h \leq 8^s K} b(k) \overline{b(h)} \left( \frac{h}{k} \right)^{\frac{1}{2}} \left( \frac{1}{4} - \frac{1}{\pi i} \sum_{a=1}^{k^*} \beta(a, k^*) \right) \cdot \sum_{0 < b < \frac{k^*}{2}} \eta(b, k^*) e\left( \frac{ab}{k^*} \right)
\]
\[
= \sum_{d \leq 8^s K} \sum_{K < k \leq 8^s K} \sum_{h \leq 8^s K} b(k) \overline{b(h)} \left( \frac{h}{k} \right)^{\frac{1}{2}} \left( \frac{1}{4} - \frac{1}{\pi i} \sum_{a=1}^{k^*} \beta(a, \frac{k}{d}) \right) \cdot \sum_{0 < b < \frac{k^*}{d}} \eta(b, \frac{k}{d}) e\left( \frac{ab}{k^*} \right)
\]
\[
= \sum_{d \leq 8^s K} \sum_{N < n_1 \leq 2^s N} \sum_{N < n_2 \leq 2^s N} \sum_{M < n_1 \leq 2^s M} \sum_{M < n_2 \leq 2^s M} \frac{a(n_2)}{m_2^{\frac{1}{2}}} \frac{a(n_1)}{m_1^{\frac{1}{2}}}
\]
\[
\cdot \left( \frac{m_1 n_1^2}{m_2 n_2^2} \right)^{\frac{1}{2}} \left( \frac{1}{4} - \frac{1}{\pi i} \sum_{a=1}^{m_2 n_2^2} \beta(a, \frac{m_2 n_2^2}{d}) \right) \sum_{0 < b < \frac{m_2 n_2^2}{d}} \eta(b, \frac{m_2 n_2^2}{d}) e\left( \frac{ab}{m_2 n_2^2} \right)
\]
\[
= \sum_{d \leq 8^s K} \sum_{N < n_1 \leq 2^s N} \sum_{N < n_2 \leq 2^s N} \sum_{M < n_2 \leq 2^s M} \frac{a(n_2) a(n_1)}{m_1^{\frac{1}{2}} m_2^{\frac{1}{2}}} \sum_{\frac{m_1 n_1}{n_2}} \frac{1}{m_2}.
\]
\[
\sum_{M < m_1 \leq 2^e M} \left( \frac{1}{4} - \frac{1}{\pi i} \sum_{a=1}^{d} \beta(a, \frac{m_2 n_2^2}{d}) \sum_{0 < b < \frac{m_2 n_2^2}{2d}} \eta(b, \frac{m_2 n_2^2}{d}) e\left(ab \frac{m_1 n_1^2}{d}ight) \right)
\]

\[
\ll \sum_{d \leq 8^e K} \sum_{N < n_1 \leq 2^e N} \sum_{N < n_2 \leq 2^e N} |a(n_1) a(n_2)| \sum_{M < m_2 \leq 2^e M} \frac{1}{m_2} \sum_{M < m_1 \leq 2^e M} \frac{1}{m_1}
\]

\[
+ \sum_{d \leq 8^e K} \sum_{M < m_1 \leq 2^e M} \sum_{N < n_1 \leq 2^e N} \sum_{N < n_2 \leq 2^e N} |a(n_1) a(n_2)| \sum_{M < m_2 \leq 2^e M} \frac{1}{m_2}.
\]

\[
\cdot \sum_{a=1}^{m_2 n_2^2 \frac{d}{M}} |\beta(a, \frac{m_2 n_2^2}{d})| \sum_{0 < b < \frac{m_2 n_2^2}{2d}} \eta(b, \frac{m_2 n_2^2}{d}) \sum_{M < m_1 \leq 2^e M} e\left(ab \frac{m_1 n_1^2}{d}ight)
\]

\[
\ll N^{-2+2\varepsilon} M^{-1} (M N)^2 + N^{-2+2\varepsilon} M^{-1} \sum_{d \leq 8^e K} \sum_{N < n_1 \leq 2^e N} \sum_{N < n_2 \leq 2^e N} \sum_{M < m_2 \leq 2^e M} \sum_{d \mid m_2 n_2^2} \left( \frac{M N^2}{d} \right)^{\frac{1}{2}+\varepsilon} d^{2\varepsilon}
\]

\[
\cdot \sum_{M < m_2 \leq 2^e M} \sum_{d \mid m_2 n_2^2} \left( \frac{M N^2}{d} \right)^{\frac{1}{2}+\varepsilon} d^{2\varepsilon} \sum_{a=1}^{m_2 n_2^2 \frac{d}{M}} |\beta(a, \frac{m_2 n_2^2}{d})| \left( a, \frac{m_2 n_2^2}{d} \right)^{\frac{1}{2}}.
\]

\[
\cdot \sum_{0 < b < \frac{m_2 n_2^2}{2d}} \frac{1}{b} \left( a, \frac{m_2 n_2^2}{d} \right)^{\frac{1}{2}}.
\]

We have

\[
\sum_{0 < b < \frac{m_2 n_2^2}{2d}} \frac{1}{b} \left( a, \frac{m_2 n_2^2}{d} \right)^{\frac{1}{2}} = \sum_{r \mid \frac{m_2 n_2^2}{d}} r^{\frac{1}{2}} \sum_{0 < b < \frac{m_2 n_2^2}{2d}} \frac{1}{b}.
\]
\[
\leq \sum_{r \mid m^2 n^2} r^\frac{1}{2} \sum_{0 < b \leq \frac{m^2 n^2}{2d}} \frac{1}{b} \log \left( \frac{2m^2 n^2}{d} \right)
\]

\[
\leq \sum_{r \mid m^2 n^2} \frac{1}{r^\frac{1}{2}} \log \left( \frac{2m^2 n^2}{d} \right)
\]

\[
\leq \left( \frac{m^2 n^2}{d} \right)^{\frac{1}{2}}
\]

\[
\leq \left( \frac{MN^2}{d} \right)^{\frac{1}{2}}
\]

and

\[
\sum_{a = 1}^{\frac{m^2 n^2}{d}} \left| \beta(a, \frac{m^2 n^2}{d}) \right| \left( a, \frac{m^2 n^2}{d} \right)^{\frac{1}{2}}
\]

\[
\leq \frac{m^2 n^2}{d} + \sum_{1 \leq a \leq \frac{m^2 n^2}{2d}} \frac{m^2 n^2}{d} \left( a, \frac{m^2 n^2}{d} \right)^{\frac{1}{2}}
\]

\[
+ \sum_{\frac{m^2 n^2}{2d} < a \leq \frac{m^2 n^2}{d} - 1} \frac{m^2 n^2}{d} \left( \frac{m^2 n^2}{d} - a, \frac{m^2 n^2}{d} \right)^{\frac{1}{2}}
\]

\[
\leq \frac{m^2 n^2}{d} + \frac{m^2 n^2}{d} + \sum_{1 \leq a \leq \frac{m^2 n^2}{2d}} \frac{1}{a} \left( a, \frac{m^2 n^2}{d} \right)^{\frac{1}{2}}
\]

\[
\leq \left( \frac{MN^2}{d} \right)^{1 + \frac{2}{d}}
\]

Therefore the contribution of \( R_0(T; h, k) \) is

\[
\ll MN^{2\varepsilon} + N^{-2 + 2\varepsilon} M^{-1} \sum_{d \leq 8^\varepsilon K} \sum_{N < n_1 \leq 2^\varepsilon N} \sum_{N < n_2 \leq 2^\varepsilon N} \sum_{M < m_2 \leq 2^\varepsilon M} \left( \frac{MN^2}{d} \right)^{\frac{1}{2} + \varepsilon} d^{2\varepsilon} \left( \frac{MN^2}{d} \right)^{1 + \frac{2}{d}}
\]

\[
\ll MN^{2\varepsilon} + N^{2\varepsilon} \left( MN^2 \right)^{\frac{3}{2} + \frac{3}{d}} \sum_{d \leq 8^\varepsilon K} d^{\frac{3}{d} - \frac{\varepsilon}{2}}
\]

\[
\ll MN^{2\varepsilon} + M^{3 + \frac{3}{d}} N^{3 + 5\varepsilon}
\]

\[
\ll \frac{T}{N^{1-8\varepsilon}}
\]
Combining all of the above, we get

\[
\int_0^\infty e^{-\frac{t}{T}} |\zeta\left(\frac{1}{2} + it\right)|^2 \left| \sum_{K<K\leq 8^K} b(k) e^{-\frac{k}{2it}} \right|^2 \ll \frac{T \log^2 T}{N^{1-8\varepsilon}}.
\]

Hence,

\[
\log T \sum_{r \leq \frac{1}{2 \log 2} \log T + O(1)} \int_0^\infty e^{-\frac{t}{T}} |\zeta\left(\frac{1}{2} + it\right)|^2.
\]

\[
\cdot \left| \sum_{2^r < m \leq 2^{2r}} \frac{1}{m^{1+it}} \right|^2 \left| \sum_{N<n \leq 2^r N} a(n) n^{2it} \right|^2 dt \ll \frac{T \log^4 T}{N^{1-8\varepsilon}}.
\]

So far the proof of Proposition 3 is finished.

Proof of Proposition 2. We observe that the measure of the set of all \( t \) such that \( \frac{T}{2} \leq t \leq T \) and \( 2t \in U_4 \) is \( \ll T^{11\varepsilon} (\log T)^{10} \). We suppose firstly that \( k = 2m \) with positive integer \( m \). By Proposition 1, Lemmas 8, 9 and 11,

\[
\int_{\frac{T}{4}}^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^4 \left| \zeta\left(1 + 2it\right) \right|^{-2m} dt
\]

\[
= \int_{\frac{T}{2} \leq t \leq T, \ 2t \notin U_4} \left| \zeta\left(\frac{1}{2} + it\right) \right|^4 \left| \zeta\left(1 + 2it\right) \right|^{-2m} dt
\]

\[+ \int_{\frac{T}{2} \leq t \leq T, \ 2t \in U_4} \left| \zeta\left(\frac{1}{2} + it\right) \right|^4 \left| \zeta\left(1 + 2it\right) \right|^{-2m} dt
\]

\[\ll \int_{\frac{T}{2} \leq t \leq T, \ 2t \notin U_4} \left| \zeta\left(\frac{1}{2} + it\right) \right|^4 \left( \sum_{l \leq X} \frac{\mu(l)}{l^{1+it}} \cdot e^{-\frac{l}{2it}} \right)^{2m} + O(1) dt + O(T \log^4 T)
\]

\[\ll \int_{\frac{T}{4}}^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^4 \left| \sum_{l \leq X} \frac{\mu(l)}{l^{1+it}} \cdot e^{-\frac{X}{2it}} \right|^{2m} dt + O(T \log^4 T)
\]

\[= \int_{\frac{T}{4}}^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^4 \left| \sum_{n \leq X^m} a(n) \right|^2 dt + O(T \log^4 T),
\]

where

\[a(n) = \frac{1}{n} \sum_{l_1, \ldots, l_m=n} \mu(l_1) \cdots \mu(l_m) \exp\left(-\frac{(l_1 + \cdots + l_m)}{X}\right),\]

\(U_4\) is defined as in (3.2), \(X\) is defined as in (3.8). We can see

\[X^m = \exp\left(\frac{2m}{\varepsilon} (\log T)^4\right) \ll T^{\frac{1}{16}} - 2\varepsilon\]
and
\[ a(n) = O(n^{-1+\varepsilon}). \]

By Cauchy’s inequality,
\[
\left| \sum_{n \leq X} \frac{a(n)}{n^{2it}} \right|^2 = \left| \sum_{s \leq \frac{m \log X}{\varepsilon \log 2} - 1} \frac{1}{2^{it}} \cdot 2^{\frac{es}{2}} \sum_{2^s < n \leq 2^s \varepsilon} \frac{a(n)}{n^{2it}} \right|^2 
\leq \sum_{s \leq \frac{m \log X}{\varepsilon \log 2} - 1} \frac{1}{2^{it}} \sum_{s \leq \frac{m \log X}{\varepsilon \log 2} - 1} 2^{\frac{es}{2}} \left| \sum_{2^s < n \leq 2^s \varepsilon} \frac{a(n)}{n^{2it}} \right|^2 
\ll \sum_{s \leq \frac{m \log X}{\varepsilon \log 2} - 1} 2^{\frac{es}{2}} \left| \sum_{2^s < n \leq 2^s \varepsilon} \frac{a(n)}{n^{2it}} \right|^2.
\]

Hence, Proposition 3 yields
\[
\int_{T}^{T+1} \left| \zeta\left(\frac{1}{2} + it\right) \right|^4 \left| \sum_{n \leq X} \frac{a(n)}{n^{2it}} \right|^2 dt 
\ll \sum_{s \leq \frac{m \log X}{\varepsilon \log 2} - 1} 2^{\frac{es}{2}} \cdot \frac{T \log^4 T}{2^s (1 - 8\varepsilon)} 
\ll T \log^4 T.
\]

Thus,
\[
\int_{T}^{T+1} \left| \zeta\left(\frac{1}{2} + it\right) \right|^4 \left| \zeta(1 + 2it) \right|^{-2m} dt \ll T \log^4 T.
\]

Therefore
\[
\int_{1}^{T} \left| \zeta\left(\frac{1}{2} + it\right) \right|^4 \left| \zeta(1 + 2it) \right|^{-2m} dt \ll T \log^4 T.
\]

For the general \( k > 0 \), we have an even integer \( 2m \) such that \( k < 2m \).

By Hölder’s inequality,
\[
\int_{1}^{T} \left| \zeta\left(\frac{1}{2} + it\right) \right|^4 \left| \zeta(1 + 2it) \right|^{-k} dt 
= \int_{1}^{T} \left| \zeta\left(\frac{1}{2} + it\right) \right|^4 \frac{2m-k}{2m} \cdot \left| \zeta\left(\frac{1}{2} + it\right) \right|^4 \frac{2m}{2m-k} \left| \zeta(1 + 2it) \right|^{-k} dt 
\leq \left( \int_{1}^{T} \left| \zeta\left(\frac{1}{2} + it\right) \right|^4 \frac{2m-k}{2m} \cdot \frac{2m}{2m-k} dt \right)^{\frac{2m-k}{2m}} 
\cdot \left( \int_{1}^{T} \left| \zeta\left(\frac{1}{2} + it\right) \right|^4 \frac{k}{2m} \cdot \frac{2m}{k} \left| \zeta(1 + 2it) \right|^{-k} \frac{2m}{k} dt \right)^{\frac{2m}{k}} 
\ll T \log^4 T.
\]
\[
\left( \int_1^T |\zeta(\frac{1}{2} + it)|^4 dt \right)^{\frac{2n-k}{2m}} \left( \int_1^T |\zeta(\frac{1}{2} + it)|^4 |\zeta(1 + 2it)|^{-2n} dt \right)^{\frac{k}{2m}} \ll T \log^4 T.
\]

So far the proof of Proposition 2 is finished.

5. The proof of Theorem

We shall apply Lemma 5. For Re(s) > 1, let

\[
f(s) = \sum_{n=1}^{\infty} \frac{d^2(n)}{n^s}.
\]

By Lemma 6,

\[
f(s) = \frac{\zeta^4(s)}{\zeta(2s)}.
\]

It is easy to see that \(\psi(n) = n^\varepsilon\) which is non-decreasing. As \(\sigma \to 1^+\),

\[
\sum_{n=1}^{\infty} \frac{d^2(n)}{n^\sigma} = \frac{\zeta^4(\sigma)}{\zeta(2\sigma)} = O\left(\frac{1}{(\sigma - 1)^4}\right).
\]

Let \(c = 1 + \varepsilon\), \(Y = \lfloor x \rfloor + \frac{1}{2}\), \(T = x^\frac{3}{4}\). Then

\[
\sum_{n \leq x} d^2(n) = \sum_{n < Y} d^2(n) + O_\varepsilon(x^\varepsilon)
\]

\[
= \frac{1}{2\pi i} \int_{1+\varepsilon-iT}^{1+\varepsilon+iT} \frac{\zeta^4(s)}{\zeta(2s)} \cdot \frac{Y^s}{s} ds + O_\varepsilon(x^{\frac{1}{4}+2\varepsilon}).
\]

We move the line of integration to Re(s) = \(\frac{1}{2}\). The residue of \(\frac{\zeta^4(s)}{\zeta(2s)} \cdot \frac{Y^s}{s}\) at \(s = 1\) is

\[
YP(\log Y) = xP(\log x) + O(x^\varepsilon).
\]

By Lemmas 8 and 10,

\[
\frac{1}{2\pi i} \int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \frac{\zeta^4(s)}{\zeta(2s)} \cdot \frac{Y^s}{s} ds \ll \max_{\frac{1}{2} \leq \sigma \leq 1+\varepsilon} T^{\frac{1}{2}(1-\sigma)+4\varepsilon} \log T \cdot \frac{x^\sigma}{T} \ll x^{\frac{1}{4}+4\varepsilon}.
\]

In the same way,

\[
\frac{1}{2\pi i} \int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \frac{\zeta^4(s)}{\zeta(2s)} \cdot \frac{Y^s}{s} ds \ll x^{\frac{1}{4}+4\varepsilon}.
\]

Hence,

\[
E(x) = \frac{1}{2\pi i} \int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \frac{\zeta^4(s)}{\zeta(2s)} \cdot \frac{Y^s}{s} ds + O(x^{\frac{1}{4}+4\varepsilon}).
\]
It follows from Proposition 2 that,

\[
E(x) \ll x^{\frac{1}{2}} \sum_{k \leq \log \frac{T}{\log x}} \int_{2^{k-1}}^{2^k} \frac{\left| \zeta\left( \frac{1}{2} + it \right) \right|^4}{\left| \zeta(1 + 2it) \right|} \, dt + O(x^{\frac{1}{2}})
\]

\[
\ll x^{\frac{1}{2}} \sum_{k \leq \log \frac{T}{\log x}} \frac{1}{2^k} \int_{1}^{2^k} \frac{\left| \zeta\left( \frac{1}{2} + it \right) \right|^4}{\left| \zeta(1 + 2it) \right|} \, dt + O(x^{\frac{1}{2}})
\]

\[
\ll x^{\frac{1}{2}} \sum_{k \leq \log \frac{T}{\log x}} \frac{1}{2^k} \cdot 2^k k^4
\]

\[
\ll x^{\frac{1}{2}} \log^5 x.
\]

Thus, the proof of the Theorem is complete.

6. Some remarks

By the method of this paper, we can prove that if \( k \) is any given positive number, \( a \) is a given positive integer, then

\[
\int_{1}^{T} \frac{\left| \zeta\left( \frac{1}{2} + it \right) \right|^4}{\left| \zeta(1 + at) \right|} \, dt \ll k, a \log^4 T.
\]

We note that if \( \mathrm{Re}(s) > 1 \),

\[
\sum_{n=1}^{\infty} \frac{d(n^3)}{n^s} = \frac{\zeta^4(s)}{\zeta^3(2s)} \cdot G_1(s),
\]

where

\[
G_1(s) = \prod_p \frac{(1 + \frac{2}{p})}{(1 - \frac{1}{p})(1 + \frac{1}{p^2})^3},
\]

\( G_1(s) \) is absolutely convergent for \( \mathrm{Re}(s) > \frac{1}{3} \). One can see page 95 in [2]. Using the method similar to that in this paper, we can prove the following proposition.

**Proposition 4.** We have

\[
\sum_{n \leq x} d(n^3) = x P_1(\log x) + O(x^{\frac{3}{2}}(\log x)^5),
\]

where \( P_1(y) \) is a suitable cubic polynomial in \( y \).

In 2006, M. Z. Garaev, F. Luca and W. G. Nowak[2] proved that as \( x \to \infty \), if \( y = y(x) \) satisfies

\[
\frac{y}{x^{\frac{3}{2}} \log x \log \log x} \to \infty,
\]

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then
\[ \sum_{x < n \leq x+y} d^2(n) \sim \frac{6}{\pi^2} y (\log x)^3, \]
and that as \( x \to \infty \), if \( y = y(x) \) satisfies
\[ \frac{y}{x^{\frac{3}{2}} \log x (\log \log x)^3} \to \infty, \]
then
\[ \sum_{x < n \leq x+y} d(n^3) \sim B_0 y (\log x)^3, \]
where \( B_0 \) is a positive constant.

Combining the method of this paper with that of [2], we can prove the following proposition.

**Proposition 5.** As \( x \to \infty \), if \( y = y(x) \) satisfies
\[ \frac{y}{x^{\frac{3}{2}} \log x} \to \infty, \]
then
\[ \sum_{x < n \leq x+y} d^2(n) \sim \frac{6}{\pi^2} y (\log x)^3 \]
and
\[ \sum_{x < n \leq x+y} d(n^3) \sim B_0 y (\log x)^3. \]

Let \( r(n) \) be the number of representations of \( n \) as the sum of two squares. In 2004, M. Kühleitner and W. G. Nowak[6] proved that
\[ \sum_{n \leq x} r^2(n) = 4x \log x + B_1 x + O(x^{\frac{3}{2}} (\log x)^3 \log \log x), \quad (6.2) \]
where \( B_1 \) is a positive constant, and that
\[ \sum_{n \leq x} r(n^3) = A_2 x \log x + B_2 x + O(x^{\frac{3}{2}} (\log x)^3 (\log \log x)^2), \quad (6.3) \]
where \( A_2, B_2 \) are positive constants.

Let \( \mathbf{K} = \mathbb{Q}(i) \), \( \zeta_{\mathbf{K}}(s) \) be the Dedekind \( \zeta \) function in the field \( \mathbf{K} \). If \( \Re(s) > 1 \),
\[ \sum_{n=1}^{\infty} \frac{r^2(n)}{n^s} = \frac{16 \zeta_{\mathbf{K}}^2(s)}{(1 + 2^{-s}) \zeta(2s)}, \]
\[ \sum_{n=1}^{\infty} \frac{r(n^3)}{n^s} = \frac{\zeta_{\mathbf{K}}^2(s)}{\zeta(2s) \zeta(2s)} \cdot G_2(s), \]
where $G_2(s)$ is holomorphic and bounded for $\text{Re}(s) > \frac{1}{3} + \varepsilon$. One can see (4.1) and (4.4) in [6].

If the result of Iwaniec[4] could be generalized to $\zeta_K(s)$, then the error terms in (6.2) and (6.3) could be improved to $O(x^{\frac{3}{7}}(\log x)^3)$. Furthermore, the sums studied in [2]

$$\sum_{x<n\leq x+y} r^2(n), \quad \sum_{x<n\leq x+y} r(n^3), \quad \sum_{x<n\leq x+y} d(n)r(n)$$

could also be improved correspondingly.

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