New formulas counting one-face maps and Chapuy’s recursion

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Abstract

In this paper, we begin with the Lehman-Walsh formula counting one-face maps and construct two involutions on pairs of permutations to obtain a new formula for the number \( A(n, g) \) of one-face maps of genus \( g \). Our new formula is in the form of a convolution of the Stirling numbers of the first kind which immediately implies a formula for the generating function

\[
A_n(x) = \sum_{g \geq 0} A(n, g)x^{n+1-2g}
\]

other than the well-known Harer-Zagier formula. By reformulating our expression for \( A_n(x) \) in terms of the backward shift operator \( E : f(x) \rightarrow f(x-1) \) and proving a property satisfied by polynomials of the form \( p(E)f(x) \), we easily establish the recursion obtained by Chapuy for \( A(n, g) \). Moreover, we give a simple combinatorial interpretation for the Harer-Zagier recurrence.

Keywords: Lehman-Walsh formula; Chapuy’s recurrence; Harer-Zagier recurrence; one-face map; log-concavity
Mathematics Subject Classifications: 05A19; 05A05

1 Introduction

A one-face map is a graph embedded in a closed orientable surface such that the complement is homeomorphic to an open disk. It is well known that the combinatorial counterpart of one-face maps are fatgraphs with one boundary component [4]. A fatgraph is a graph with a specified cyclic order of the ends of edges incident to each vertex of the graph. In this paper we will not differentiate between maps and fatgraphs.

A fatgraph having \( n \) edges and one boundary component can be encoded as a triple of permutations \((\alpha, \beta, \gamma)\) on \([2n] = \{1, 2, \ldots, 2n\}\) where \( \gamma \) is the long cycle \((1, 2, 3, \ldots, 2n)\). This can be seen as follows: given a fatgraph \( F \) with one boundary component, we call the (two) ends of an edge half edges. Pick a half edge, label it 1, and start to travel the fatgraph counterclockwise. Label all visited half edges entering a vertex sequentially by labels \( 2, 3, \ldots, 2n \). This induces two permutations \( \alpha \) and \( \beta \), where \( \alpha \) is an involution
without fixed points such that each $\alpha$-cycle consists of the labels of the two half edges of the same edge and each $\beta$-cycle represents the counterclockwise cyclic arrangement of all half edges incident to the same vertex. By construction, $\gamma = (1, 2, \ldots, 2n) = \alpha\beta$, represents the unique boundary component of the fatgraph $F$. An example of a fatgraph is illustrated in Figure 1, its corresponding triple of permutations are:

$$\alpha = (1, 4)(2, 5)(3, 6), \quad \beta = (1, 5, 3)(4, 2, 6), \quad \gamma = (1, 2, 3, 4, 5, 6).$$

![Figure 1: A fatgraph with 6 half edges, the dashed curve represents its boundary component.](image)

A rooted one-face map is a one-face map where one half-edge is particularly marked and called the root. We shall always label the root of a rooted one-face map with the label 1, that is, we start from the root when traveling the boundary of the map. Now given two rooted one-face maps which are respectively encoded into the triples $(\alpha, \beta, \gamma)$ and $(\alpha', \beta', \gamma')$, they will be viewed as equivalent if there exists a permutation $\pi$ such that

$$\alpha' = \pi \alpha \pi^{-1}, \quad \gamma' = \pi \gamma \pi^{-1}, \quad \pi(1) = 1.$$

Following Euler’s characteristic formula, the number of edges $n$, the number of vertices $v$ and the genus $g$ of a one-face map satisfy

$$v - n + 1 = 2 - 2g.$$

The enumeration of rooted one-face maps has been extensively studied, see for instance [1, 2, 3, 4, 5, 6, 7, 8, 12] and the references therein. Let $A(n, g)$ denote the number of rooted one-face maps (up to equivalence) of genus $g$ having $n$ edges and let $A_n(x) = \sum_{g \geq 0} A(n, g)x^{n+1-2g}$ be the corresponding generating function. Four decades ago, Walsh and Lehman [12, eq. (13)], using a direct recursive method and formal power series, obtained an explicit formula for $A(n, g)$ which can be reformulated as follows:

$$A(n, g) = \sum_{\lambda \vdash g} \frac{(n + 1)n \cdots (n + 2 - 2g - \ell(\lambda))}{2^{2g} \prod_i c_i!(2i + 1)^{c_i}} \frac{(2n)!}{(n + 1)!n!}, \quad (1)$$

where the summation is taken over partitions $\lambda$ of $g$, $c_i$ is the number of parts $i$ in $\lambda$, and $\ell(\lambda)$ is the total number of parts.
More than a decade later, Harer and Zagier \cite{H83} obtained in the context of computing the virtual Euler characteristics of a curve:

\[ A(n, g) = \frac{(2n)!}{(n+1)!(n-2g)!} [x^{2g}] \left( \frac{x/2}{\tanh x/2} \right)^{n+1}, \]  

(2)

where \([x^k]f(x)\) denotes the coefficient of \(x^k\) in the expansion of the function \(f(x)\). Considering the relation between the RHS of eq. (2) and its derivatives, they obtained the following three-term recurrence, known as the Harer-Zagier recurrence:

\[(n+1)A(n, g) = 2(2n-1)A(n-1, g) + (2n-1)(n-1)(2n-3)A(n-2, g-1). \]  

(3)

They furthermore obtained the so-called Harer-Zagier formula:

\[ A_n(x) = \frac{(2n)!}{2^n n!} \sum_{k \geq 1} 2^{k-1} \binom{n}{k-1} \binom{n}{k} x^k. \]  

(4)

There is a body of work on how to derive these results \cite{C06, G04, L02, R02, S02}. A direct bijection for the Harer-Zagier formula was given in \cite{L02}. Combinatorial arguments to obtain the Lehman-Walsh formula and the Harer-Zagier recurrence were recently given in \cite{G04}. One of the most recent advances is a new recurrence for \(A(n, g)\) obtained by Chapuy \cite{C06} via a bijective approach:

\[ 2gA(n, g) = \sum_{k=1}^g \binom{n+1-2(g-k)}{2k+1} A(n, g-k). \]  

(5)

See also \cite{R02} for a refinement of the recurrence and certain generalizations via plane permutations.

In this paper, we will prove a new explicit formula for \(A(n, g)\) and obtain a new formula for \(A_n(x)\) which is more regular than the Harer-Zagier formula. We will also derive Chapuy’s recursion.

A brief outline of the paper is as follows. In section 2, we shall employ an alternative interpretation of the Lehman-Walsh formula and construct two involutions on pairs of permutations to obtain the following new formula for \(A(n, g)\):

\[ A(n, g) = \frac{(2n)!}{2^n n!(n+1)!} \sum_{k=0}^n \binom{n}{k} \sum_{i+j=n+2-2g} C(n-k+1, i)(-1)^{k+1-j}C(k+1, j), \]  

(6)

where \(C(n, k)\) denotes the number of permutations on \(n\) elements with \(k\) cycles, i.e., the unsigned Stirling numbers of the first kind. This immediately gives us a new formula for the generating functions \(A_n(x), n \geq 1:\)

\[ A_n(x) = \frac{(2n)!}{2^n n!} \sum_{k \geq 0} \binom{n}{k} \binom{x+n-k}{n+1}. \]  

(7)
Utilizing the alternative interpretation of the Lehman-Walsh formula, another combinatorial explanation of the Harer-Zagier recurrence will be presented as well.

In section 3, by reformulating our expression for $A_n(x)$ in terms of the backward shift operator $E : f(x) \rightarrow f(x - 1)$ and proving a property satisfied by polynomials of the form $p(E)f(x)$, we easily establish Chapuy’s recursion. Furthermore, by applying another property of polynomials of the form $p(E)f(x)$ proved in Stanley [11], we obtain the log-concavity of the numbers $A(n, g)$.

2 New formulas for $A(n, g)$ and $A_n(x)$

In the following, we will first prove a new formula for $A(n, g)$ by constructing two involutions on pairs of permutations.

We call a cycle of a permutation odd and even if it contains an odd and even number of elements, respectively. Let $O(n + 1, g)$ denote the number of permutations on $[n + 1]$ which consist of $n + 1 - 2g$ odd (disjoint) cycles. For readers familiar with the formula for the number of permutations of a specific cycle type, see Stanley [10, Prop. 1.3.2], it may be immediately realized that the Lehman-Walsh expression can be rewritten as

$$
\sum_{\lambda \vdash g} \binom{n}{\lambda_1} \cdots \binom{n + 2 - 2g - \ell(\lambda)}{\cdots} \frac{(2n)!}{(n + 1)!n!2^{2g}O(n + 1, g)} = \frac{(2n)!}{(n + 1)!n!2^{2g}}O(n + 1, g),
$$

thus we have more fundamental objects (permutations instead of maps) to work with.

Let $S = \{a, 1, 2, \ldots, n, b\}$. Let $T_{A,l}$ denote the set of pairs $(\alpha, \beta)$ where $\alpha$ is a permutation on $A \subset S$ while $\beta$ is a permutation on $S \setminus A$, such that the sum of the number of $\alpha$- and $\beta$-cycles equals $l$. Let

$$
T_l = \bigcup_{A \subset S} T_{A,l},
$$

where the union is taken over all $A \subset S$ such that $a \in A$ and $b \notin A$. For each pair $(\alpha, \beta) \in T_l$, we denote the difference between $|S \setminus A|$ and the number of $\beta$-cycles as $d(\beta)$ and set $W[(\alpha, \beta)] = (-1)^{d(\beta)}$. Then, it is clear that

Lemma 2.1.

$$
\sum_{(\alpha, \beta) \in T_l} W[(\alpha, \beta)] = \sum_{k=0}^{n} \binom{n}{k} \sum_{i+j=l} C(n - k + 1, i)(-1)^{k+1-j}C(k + 1, j), \quad (8)
$$

where $C(n, k)$ denotes the number of permutations on $n$ elements with $k$ cycles.

Let $T' \subset T_l$ consist of pairs $(\alpha, \beta)$ where $\alpha(a) = a$ and $b$ is contained in an odd cycle. We will construct our first involution $\phi$, which leads to

Lemma 2.2.

$$
\sum_{(\alpha, \beta) \in T_l} W[(\alpha, \beta)] = \sum_{(\alpha, \beta) \in T'} W[(\alpha, \beta)].
$$
Proof. Given \((\alpha, \beta) \in \mathcal{T}_t\), write both \(\alpha\) and \(\beta\) in their cycle decompositions and denote the length of the cycle containing \(b\) as \(B\). Define a map \(\phi : \mathcal{T}_t \rightarrow \mathcal{T}_t\) as follows:

- **Case 1:** if \((\alpha, \beta) \in \mathcal{T}'\), then \(\phi : (\alpha, \beta) \mapsto (\alpha, \beta)\);
- **Case 2:** if \((\alpha, \beta) \notin \mathcal{T}'\), we distinguish two scenarios:
  - If \(B\) is odd and \(\alpha(a) \neq a\), then \(\phi : (\alpha, \beta) \mapsto (\alpha', \beta')\), where \(\alpha'\) is obtained by deleting \(\alpha(a)\) from the cycle decomposition of \(\alpha\) while \(\beta'\) is obtained by inserting \(\alpha(a)\) between \(b\) and \(\beta(b)\) in the cycle containing \(b\) and if \(b = \beta(b)\), we map the cycle \((b)\) to \((b, \alpha(a))\).
  - If \(B\) is even then \(b \neq \beta(b)\). Define \(\phi : (\alpha, \beta) \mapsto (\alpha', \beta')\), where \(\alpha'\) is obtained by inserting \(\beta(b)\) between \(a\) and \(\alpha(a)\) and for \(a = \alpha(a)\), we map \((a)\) to \((a, \beta(b))\).

We inspect that \(\phi\) is an involution, whose fixed points are exactly all Case 1-pairs. Furthermore, \(\phi\) preserves the total number of cycles within the pairs. Accordingly, for Case 1-pairs, \(\phi\) will preserve weights. For Case 2-pairs, \(\beta\) and \(\beta'\) differ in the number of elements by 1 but have the same number of cycles, so that \(\phi\) will change the sign of the weight, i.e., \(W[(\alpha, \beta)] = -W[(\alpha', \beta')]\). Hence, the total weight of the \(\phi\)-orbit of any Case 2-pair is 0. Thus, \(\sum_{(\alpha, \beta) \in \mathcal{T}_t} W[(\alpha, \beta)]\) is equal to the total weight of all Case 1-pairs, completing the proof.

Let \(\mathcal{T}''\) be the set of pairs \((\alpha, \beta) \in \mathcal{T}'\) such that all cycles in \(\alpha\) and \(\beta\) are odd.

**Lemma 2.3.**

\[
\sum_{(\alpha, \beta) \in \mathcal{T}'} W[(\alpha, \beta)] = \sum_{(\alpha, \beta) \in \mathcal{T}''} W[(\alpha, \beta)].
\]

**Proof.** Define a map \(\varphi : \mathcal{T}' \rightarrow \mathcal{T}'\) as follows:

- **Case 1:** if \((\alpha, \beta) \in \mathcal{T}''\), then \(\varphi : (\alpha, \beta) \mapsto (\alpha, \beta)\);
- **Case 2:** if \((\alpha, \beta) \notin \mathcal{T}''\), there is at least one even cycle in the collection of cycles from both \(\alpha\) and \(\beta\). Obviously, there is a unique even cycle, denoted by \(C\), containing the minimal element among the union of elements from all even cycles. Let \(\varphi : (\alpha, \beta) \mapsto (\alpha', \beta')\), where
  - If \(C\) is a cycle in \(\alpha\), then \(\alpha' = \alpha \setminus C\) and \(\beta' = \beta \cup C\);
  - Otherwise \(\alpha' = \alpha \cup C\) and \(\beta' = \beta \setminus C\).

It is easy to see that \(\varphi\) is an involution having Case 1 pairs as the only fixed points. Furthermore, for Case 1 pairs, \(\varphi\) preserves weights. For Case 2 pairs, \(\beta\) and \(\beta'\) differ in the number of elements by an even number while they differ in the number of cycles by 1. Hence \(W[(\alpha, \beta)] = -W[\varphi((\alpha, \beta))]\). As a result, the total weight over \((\alpha, \beta)\) in Case 2 is 0, completing the proof.

Based on these lemmas above, we obtain
Theorem 2.1. For \( n, l \geq 0 \), we have
\[
\sum_{k=0}^{n} \binom{n}{k} \sum_{i+j=l} \binom{n-k+1, i}{1}(-1)^{k+1-j}C(k+1, j) = 2^{l-2}O(n+1, \frac{n+2-l}{2}). \tag{9}
\]

Proof. Lemma 2.1, Lemma 2.2 and Lemma 2.3 imply, that the LHS of eq. (9) equals \( \sum_{(\alpha, \beta) \in \mathcal{T}'} W[(\alpha, \beta)], \) where the total number of cycles in \( \alpha \) and \( \beta \) is \( l \). Since in \( \mathcal{T}' \) all cycles are odd, the number of elements and the number of cycles in \( \beta \) have the same parity. Thus, for any \( (\alpha, \beta) \in \mathcal{T}' \), \( W[(\alpha, \beta)] = 1 \), i.e the total weight over \( \mathcal{T}' \) equals the total number of elements in \( \mathcal{T}' \).

Since \( a \) is a fixed point in \( \alpha \) for any \( (\alpha, \beta) \in \mathcal{T}' \), each pair \( (\alpha, \beta) \in \mathcal{T}' \) can be viewed as a partition of all \( l-1 \) odd cycles of a permutation on \([n] \cup \{b\}\), except the cycle containing \( b \), into two ordered parts. Conversely, given a permutation on \([n] \cup \{b\}\) with \( l-1 \) odd cycles, there are \( 2^{l-2} \) different ways to partition all cycles except the one containing \( b \) into two ordered parts. Therefore, we have
\[
\sum_{(\alpha, \beta) \in \mathcal{T}'} W[(\alpha, \beta)] = |\mathcal{T}'| = 2^{l-2}O(n+1, \frac{n+2-l}{2}),
\]
completing the proof.

As a corollary, we obtain a new recurrence for the unsigned Stirling numbers of the first kind:

Corollary 2.1. For \( n \geq 0 \) and \( l \neq n \mod 2 \), we have
\[
\sum_{k=0}^{n} \binom{n}{k} \sum_{i+j=l} (-1)^{k+1-j}C(n-k+1, i)C(k+1, j) = 0. \tag{10}
\]

Proof. Since every cycle is odd for any \( (\alpha, \beta) \in \mathcal{T}' \), the number of total elements \( n+2 \) has the same parity as the total number of cycles \( l \). If \( l \neq n \mod 2 \), \( \mathcal{T}' = \emptyset \), i.e., \( O(n+1, \frac{n+2-l}{2}) = 0 \), whence
\[
\sum_{k=0}^{n} \binom{n}{k} \sum_{i+j=l} (-1)^{k+1-j}C(n-k+1, i)C(k+1, j) = 0.
\]

\( \square \)

Setting \( l = n + 2 - 2g \) in eq. (9), we have

Corollary 2.2. For \( n, g \geq 0 \), we have
\[
\sum_{k=0}^{n} \binom{n}{k} \sum_{i+j=n+2-2g} C(n-k+1, i)(-1)^{k+1-j}C(k+1, j) = 2^{n-2g}O(n+1, g). \tag{11}
\]

Accordingly, multiplying \( \frac{(2n)!}{2^n(n+1)!} \) on both sides of eq. (11) we obtain a new explicit formula for \( A(n, g) \):
Theorem 2.2. \(A(n, g) = \frac{\binom{2n}{n}^n}{2^n n!} \tilde{A}(n, g)\) where

\[
\tilde{A}(n, g) = \sum_{k=0}^{n} \binom{n}{k} \sum_{i+j=n+2-2g} C(n-k+1, i)(-1)^{k+1-i}C(k+1, j).
\] (12)

Then, we immediately obtain

Theorem 2.3. The generating functions \(A_n(x)\) for \(n \geq 0\) satisfy

\[
\sum_{g \geq 0} A(n, g) x^{n+1-2g} = \frac{(2n)!}{2^n n!} \sum_{k \geq 0} \binom{n}{k} \binom{x + n - k}{n + 1}.
\] (13)

Proof. According to [10] we have

\[
x(x+1)(x+2)\cdots(x+n-1) = \sum_{k \geq 1} C(n, k)x^k,
\]

\[
x(x-1)(x-2)\cdots(x-n+1) = \sum_{k \geq 1} (-1)^{n-k}C(n, k)x^k.
\]

From these facts and eq. (12), we immediately compute

\[
\tilde{A}(n, g)
\]

\[
= \sum_{k=0}^{n} \binom{n}{k} \binom{x^{n+2-2g}[(x+n-k)(x+n-k-1)\cdots x[x(x-1)\cdots (x-k)]]}{(n+1)!}
\]

\[
= \sum_{k=0}^{n} \binom{n}{k} \binom{x^{n+1-2g}[(x+n-k)(x+n-k-1)\cdots x(x-1)\cdots (x-k)]}{(n+1)!}
\]

from which the theorem follows.

Remark 2.4. Clearly, depending on \(n\), \(A_n(x)\) represents, either an odd or even function, which is not obvious from Harer-Zagier’s formula. Our new formula on the RHS of eq. (13) makes this feature immediately evident: let \((a)_n\) denote the falling factorial \(a(a-1)\cdots(a-n+1)\). Then,

\[
\binom{n}{k} \frac{(x+n-k)_{n+1}}{(n+1)!} = \binom{n}{k} \frac{(x+n-k)(x+n-k-1)\cdots(x-k)}{(n+1)!}
\]

\[
= (-1)^{n+1} \binom{n}{k} \frac{[-(x+n-k)][-(x+n-k-1)]\cdots[-(x-k)]}{(n+1)!}
\]

\[
= (-1)^{n+1} \binom{n}{n-k} \frac{(-x+n-(n-k))_{n+1}}{(n+1)!},
\]

which implies \(A_n(x) = (-1)^{n+1}A_n(-x)\).

For the three-term Harer-Zagier recurrence eq. (3), the proof in [2] is based on a combinatorial isomorphism. Here, we give another indirect combinatorial proof for it, which relies on the three-term recurrence for \(O(n, g)\) evident in the following lemma.
Lemma 2.4. The numbers $O(n, g)$ satisfy

$$O(n + 1, g) = O(n, g) + n(n - 1)O(n - 1, g - 1). \quad (14)$$

Proof. Firstly, we claim that

Claim. The numbers $O(n, g)$ satisfy

$$O(n + 1, g) = O(n, g) + \sum_{k=1}^{g} (n)_{2k} O(n - 2k, g - k). \quad (15)$$

This can be seen by classifying all permutations on $[n+1]$ with $n+1-2g$ odd cycles based on the length $\ell_1$ of the cycle containing the element 1. Obviously, the class with $\ell_1 = 1$ has $O(n, g)$ permutations. For the class with $\ell_1 = 2k+1$, $k \geq 1$, there are $(n)_{2k}$ ways to choose $2k$ elements from $[n+1] \setminus \{1\}$ which together with the element 1 form a cycle of length $2k+1$. The remaining $n-2k$ elements can arbitrarily form $n-2g = (n-2k) - 2(g-k)$ odd cycles, and there are $O(n-2k, g-k)$ different ways to do that. Thus, eq. (15) follows.

Next, we note eq. (15) implies

$$O(n - 1, g - 1) = O(n - 2, g - 1) + \sum_{k=1}^{g-1} (n-2)_{2k} O(n - 2k - 2, g - 1 - k).$$

Thus we obtain

$$O(n + 1, g) = O(n, g) + \sum_{k=1}^{g} (n)_{2k} O(n - 2k, g - k)$$

$$= O(n, g) + n(n - 1)[O(n - 2, g - 1) + \sum_{k=2}^{g} (n-2)_{2k-2} O(n - 2k, g - k)]$$

$$= O(n, g) + n(n - 1)O(n - 1, g - 1),$$

completing the proof of the lemma.

Proposition 2.1 (Harer-Zagier recurrence).

$$(n+1)A(n, g) = 2(2n-1)A(n-1, g) + (2n-1)(n-1)(2n-3)A(n-2, g-1).$$

Proof. Using eq. (14), we have

$$A(n, g) = \frac{(2n)!}{(n+1)!n!2^g} O(n + 1, g)$$

$$= \frac{(2n)!}{(n+1)!n!2^g} O(n, g) + \frac{(2n)!}{(n+1)!n!2^g} n(n-1)O(n - 1, g - 1)$$

$$= \frac{2n(2n-1)}{(n+1)n} \cdot \frac{(2n-2)!}{n!(n-1)!2^{2g}} O(n, g)$$

$$+ \frac{2n(2n-1)(2n-2)(2n-3)}{(n+1)n(n-1)!2^{2g}} \cdot \frac{(2n-4)!n(n-1)}{(n-1)!(n-2)!2^{2g-2}} O(n - 1, g - 1)$$

$$= \frac{2(2n-1)}{n+1} A(n - 1, g) + \frac{(2n-1)(n-1)(2n-3)}{n+1} A(n - 2, g - 1),$$

whence the proposition.

\[8\]
3 Chapuy’s recursion and log-concavity

In this section, by reformulating our expression for $A_n(x)$ in terms of the backward shift operator $E : f(x) \rightarrow f(x - 1)$ and proving a property satisfied by polynomials of the form $p(E)f(x)$, we easily establish Chapuy’s recursion. Furthermore, by applying another property of polynomials of the form $p(E)f(x)$ proved in Stanley [11], we obtain the log-concavity of the numbers $A(n, g)$.

First, our new formula of $A_n(x)$ implies

Proposition 3.1.

$$\sum_{g \geq 0} A(n, g)x^{n+1-2g} = \frac{(2n)!}{2^n n!(n+1)!}(1 + E)^n (x + n)_{n+1}. \quad (16)$$

Proof. This is evident from the following computation:

$$\sum_{g \geq 0} A(n, g)x^{n+1-2g} = \frac{(2n)!}{2^n n!(n+1)!} \sum_{k \geq 0} \binom{n}{k} (x + n - k)^n + 1$$

$$= \frac{(2n)!}{2^n n!(n+1)!} \sum_{k \geq 0} \binom{n}{k} (x - k + n)_{n+1}$$

$$= \frac{(2n)!}{2^n n!(n+1)!} \sum_{k \geq 0} \binom{n}{k} E^k(x + n)_{n+1}$$

$$= \frac{(2n)!}{2^n n!(n+1)!} (1 + E)^n (x + n)_{n+1}. \quad \square$$

We proceed by showing that any polynomial of the form $p(E)(x + n)_{n+1}$ satisfies:

Theorem 3.1. Let $p(t) = \sum_{k=0}^{n} a_k t^k \text{ and } F(x) = p(E)(x + n)_{n+1}$. If $\frac{a_k}{a_0} = \frac{a_{n-k}}{a_n}$ and $ka_k + (n - k + 2)a_{k-2} = \frac{a_k}{a_0} a_{k-1}$, for $2 \leq k \leq n$, then

$$(n + 2 + \frac{a_1}{a_0})F(x) = x(F(x + 1) - F(x - 1)). \quad (17)$$

Moreover, let $b_k = [x^k]p(E)(x + n)_{n+1}$, then we have

$$\left(\frac{n + 2 + \frac{a_1}{a_0}}{2} - k\right) b_k = \sum_{j \geq 1} \binom{k + 2j}{2j + 1} b_{k+2j}. \quad (18)$$

Proof. Note that by assumption, the RHS of eq. (17) is equal to

$$\sum_{k=0}^{n} \{a_k x(x + 1 + n - k)_{n+1} - a_k x(x - 1 + n - k)_{n+1}\}.$$
which can be reformulated as

\[
\frac{n + 2 + \frac{a_1}{a_0}}{2} F(x) - x F'(x) = \sum_{k \geq 0} \left( \frac{n + 2 + \frac{a_1}{a_0}}{2} - k \right) b_k x^k = \sum_{j \geq 1} \frac{x F(2j+1)(x)}{(2j+1)!}.
\]
Comparing the coefficients of the last equation based on the fact that
\[
x F^{(2j+1)}(x) \quad (2j + 1)! = \sum_{i \geq 0} \binom{i}{2j+1} b_i x^{i-2j},
\]
we obtain eq. (18) and the proof of the theorem is complete.

As a consequence of Theorem 3.1, we immediately obtain Chapuy’s recurrence.

**Corollary 3.1** (Chapuy’s recursion).

\[
2gA(n, g) = \sum_{k=1}^{g} \binom{n + 1 - 2(g-k)}{2k+1} A(n, g-k).
\]

Proof. For \(A_n(x)\), Proposition 3.1 gives us \(p(t) = \sum_{k=0}^{n} a_k t^k\) where \(a_k = \frac{(2n)!}{2^n n!(n+1)!} \binom{n}{k}\). It is obvious that \(\frac{a_{k+1}}{a_k} = \frac{a_{k-1}}{a_k} = n\). Furthermore,

\[
ka_k + (n - k + 2)a_{k-2} = \frac{(2n)!}{2^n n!(n+1)!} \left[ k \binom{n}{k} + [n - (k-2)] \binom{n}{k-2} \right]
\]

\[
= \frac{(2n)!}{2^n n!(n+1)!} \left[ n \binom{n-1}{k-1} + n \binom{n-1}{k-2} \right]
\]

\[
= \frac{(2n)!}{2^n n!(n+1)!} n \binom{n}{k-1} = n a_{k-1}.
\]

Hence, we can apply Theorem 3.1 to \(A_n(x)\) and obtain

\[
2gA(n, g) = \binom{n+2+n}{2} - (n + 1 - 2g) [x^{n+1-2g}] A_n(x)
\]

\[
= \sum_{j \geq 1} \binom{n+1-2g+2j}{2j+1} [x^{n+1-2g+2j}] A_n(x)
\]

\[
= \sum_{k=1}^{g} \binom{n+1-2(g-k)}{2k+1} A(n, g-k),
\]

which is Chapuy’s recurrence. □

### 3.1 Log-concavity of \(A(n, g)\)

Proposition 3.1 has a close relative in [11, eq. (6)] in terms of a formula using the backward shift operator which can be reformulated as

\[
\sum_{g \geq 0} A(n, g) x^{n+1-2g} = \frac{1}{2^n n!(2n+1)} (1 - E^2)^n (x+2n)_{2n+1}. \tag{20}
\]

This formula was obtained using the character theory of the symmetric group. In addition, it was proved in [11] that the RHS of the last equation is a degree \(n+1\) polynomial despite it appears to be a degree \(2n+1\) polynomial.
In [11], it is further proved that every zero of the polynomial on the RHS of eq. (20) is purely imaginary, based on the following result:

**Proposition 3.2** (Stanley [11]). Let \( p(t) \) be a complex polynomial of degree exactly \( d \), such that every zero of \( p(t) \) lies on the circle \( |z| = 1 \). Suppose that the multiplicity of 1 as a root of \( p(t) \) is \( m \geq 0 \). Let \( P(x) = p(E)(x + n - 1)_n \). If \( d \leq n - 1 \), then

\[
P(x) = (x + n - d - 1)_{n-d}Q(x),
\]

where \( Q(x) \) is a polynomial of degree \( d - m \) for which every zero has real part \( \frac{d-n+1}{2} \).

Applying Proposition 3.2 to the RHS of eq. (16), we have \( p(t) = (1 + t)^n, d = n, m = 0 \) so that \( \sum_{g \geq 0} A(n, g)x^{n+1-2g} = xQ(x) \) with every zero of \( Q(x) \) being purely imaginary. Let \( H(x) = \sum_{g \geq 0} A(n, g)x^{\left\lfloor \frac{n+1}{2} \right\rfloor - g} \). Then, we have

**Corollary 3.2.** \( H(x) \) has only non-positive real zeros, and for fixed \( n \geq 1 \), the sequence \( A(n, g) \) is log-concave, i.e., \( A(n, g)^2 \geq A(n, g-1)A(n, g+1) \).

**Proof.** By construction, we have either \( H(x^2) = xQ(x) \) or \( H(x^2) = Q(x) \), depending on the parity of \( n \). In either case, it implies \( H(x) \) has only non-positive real zeros. It is well known that having only real zeros implies log-concavity of the coefficients (see [9] for instance). This completes the proof. \( \Box \)

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