Dispersion surfaces and light propagation in homogeneous dielectric-magnetic uniaxial medium

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Abstract. We investigated light propagation in a homogeneous medium having double anisotropy (i.e. with anisotropy of both dielectric and magnetic permittivities) and an arbitrary orientation of its optical axis in the plane of incidence. We investigated wave surfaces for the considered medium. We showed that only some groups of surfaces can arise. We investigated the conditions for total independent of polarization reflection and the conditions for total transmission, as well as possibilities of the subject system of serving as: an omnidirectional reflector, a beam splitter and a phase retarder.

1. Introduction
Metamaterials are artificial composites containing sub-long-wave structures and manifesting such linear and non-linear optical properties as: negative refraction, reverse Doppler effect, electromagnetic wave energy propagation in the direction opposite to the wave vector, etc. [1, 2]. They also have other extraordinary applications, such as: perfect lenses [3], invisible cloaks [4, 5], perfect absorbers [6], etc. Though the easiest way of having negative refraction is the application of the isotropic metamaterials, this refraction can also be observed in anisotropic metamaterials. Moreover, in the last general case it is not necessary to require that all the elements of the dielectric and magnetic permittivities be negative [7]. Recently the investigation of such anisotropic metamaterials has been of great interest [8-24]. However, the literature mainly considers the cases when the dielectric and magnetic tensor principal elements are either parallel or perpendicular to the boundaries of the system. In [10], the peculiarities of super-light propagation in an anisotropic metamaterials were investigated, for an arbitrary orientation of the optical axis in the incidence plane. In [13], the case of the omnidirectional total transmission and possibility of existence of a negative Brewster's angle – at the boundary, isotropic medium-anisotropic medium – for an arbitrary orientation of the optical axis when \( \hat{\mu} = \hat{I} \) (\( \hat{\mu} \) is the magnetic permittivity tensor, \( \hat{I} \) is the unit matrix) are investigated. In [19], the possibilities of total reflection at the boundary, isotropic medium-anisotropic medium, are investigated and a condition for total reflection is obtained. In [20], the possibilities of total negative reflection at the boundary, isotropic medium-anisotropic medium, are investigated for an arbitrary orientation of the optical axis, again, for \( \hat{\mu} = \hat{I} \). In [16], the dispersion equations for anisotropic metamaterials are classified. In the present paper, the dispersion surface peculiarities and dependences of the dispersion curves on the optical axis orientation are investigated. We also investigated the conditions for independent of polarization total reflection and the conditions of the total transmission, as well as the possibilities of use of the subject system as: an omnidirectional reflector, a beam splitter and a phase retarder.
2. Dispersion Surfaces

Let us consider peculiarities of the dispersion surfaces of an anisotropic medium having arbitrary orientated optical axis in the incidence plane. We assume that the dielectric and magnetic permittivity local tensors of the medium can both be diagonalized, and the tensors \( \hat{\varepsilon}_0 \) and \( \hat{\mu}_0 \) have the following form in the corresponding frame:

\[
\hat{\varepsilon}_0 = \begin{pmatrix}
\varepsilon_1 & 0 & 0 \\
0 & \varepsilon_2 & 0 \\
0 & 0 & \varepsilon_3
\end{pmatrix}, \quad \hat{\mu}_0 = \begin{pmatrix}
\mu_1 & 0 & 0 \\
0 & \mu_2 & 0 \\
0 & 0 & \mu_3
\end{pmatrix},
\]

i.e. it is assumed that the principal axes of the dielectric and magnetic permittivity tensors coincide. Below we also assume that the medium is uniaxis, i.e. \( \varepsilon_2 = \varepsilon_3 \neq \varepsilon_1 \) and \( \mu_2 = \mu_3 \neq \mu_1 \). In the laboratory frame \( \hat{\varepsilon} \) and \( \hat{\mu} \) have the following form:

\[
\hat{\varepsilon} = \hat{T}[y, \phi] \hat{\varepsilon}_0 \hat{T}[y, \phi]^{-1}, \quad \hat{\mu} = \hat{T}[y, \phi] \hat{\mu}_0 \hat{T}[y, \phi]^{-1},
\]

where \( \hat{T}[y, \phi] \) is the rotation matrix for the y axis, at \( \phi \) angle.

A plane electromagnetic wave of \( \omega \) frequency and \( \hat{k} \) wave vector propagates in the mentioned medium. From Maxwell’s equations, we obtain the following dispersion equations for refractive index:

\[
\left(n^2(1-\delta_z) + \mu_n \varepsilon_n (\delta \pi - 1)(1-\delta_m) + \delta \pi \delta \zeta \right) \left(n^2(1-\delta_m) + \mu_m \varepsilon_m (\delta \pi - 1)(1-\delta_n) + \delta \pi \delta \zeta \right) = 0,
\]

where \( n^2 = n_x^2 + n_y^2 + n_z^2 \), \( \delta \zeta = 2(n_x \cos \phi + n_z \sin \phi)^2 \), \( \varepsilon_m = (\varepsilon_1 + \varepsilon_3) / 2 \), \( \mu_m = (\mu_1 + \mu_3) / 2 \), \( \varepsilon \) is the wavelength in the medium. The dispersion equation for plane waves in such a material can be factorized into two terms: One – dispersion equation for electric modes and the other dispersion equation for magnetic modes.

Let us reduce the dispersion equation of electric modes to the canonic form. To do it we represent \( n_x, n_y \) and \( n_z \) in the following forms:

\[
n_x = a n_x + n_z + n_3, \quad n_y = n_1 + b n_z, \quad n_z = n_1 + n_2 + c n_z,
\]

where

\[
a = 1, \quad b = 2 \frac{1 + \delta \zeta \sin 2\phi}{\delta \zeta - 1}, \quad c = \frac{1 + \delta \zeta \cos 2\phi + \delta \zeta \sin 2\phi}{1 + \delta \zeta \cos 2\phi - \delta \zeta \sin 2\phi}
\]

Then the dispersion equation for electric modes takes the form:

\[
\frac{n_x^2}{\lambda_1} + \frac{n_y^2}{\lambda_2} + \frac{n_z^2}{\lambda_3} = 1,
\]

\[
\lambda_1 = \varepsilon_m \mu_m \left(3 - 2\delta \zeta + 2\delta \zeta \sin 2\phi \right) \left(\delta \zeta - 1 \right) \left(1 - \delta_m \right), \quad \lambda_2 = \varepsilon_m \mu_m \left(1 + \delta \zeta \sin 2\phi \right) \left(1 + \delta_m \right),
\]

where

\[
l_1 = 2 \varepsilon_m \mu_m \left(1 + \delta \zeta \sin 2\phi \right) \left(3 - 2\delta \zeta + 2\delta \zeta \sin 2\phi \right) \left(1 + \delta_m \right), \quad \lambda_3 = 2 \varepsilon_m \mu_m \left(1 + \delta \zeta \sin 2\phi \right) \left(3 - 2\delta \zeta + 2\delta \zeta \sin 2\phi \right) \left(1 + \delta_m \right).
\]

Dispersion equation for magnetic modes also has the same form, but in this case \( \lambda_1, \lambda_2, \lambda_3, n_1, n_2, n_3 \) are obtained by the interchanges: \( \delta \zeta \rightarrow \delta \mu \) and \( \mu_m \rightarrow \varepsilon_m \), in (4) and (6).

Dispersion surfaces characterize the dependence of the electromagnetic wave refraction in the medium on the light propagation direction. Electromagnetic plane waves propagating inside the material, depending on the values of \( \lambda_1, \lambda_2, \lambda_3 \), can exhibit dispersion surfaces in the form of ellipsoids of revolution, hyperboloids of one sheet, or hyperboloids of two sheets. Furthermore, depending on the optical axis orientation, the intersections of these surfaces with the propagation plane can be circles, ellipses, hyperbolas, or straight lines. Now let us go into the details of the problem.

I. if in (5) \( \lambda_1, \lambda_2, \lambda_3 \) are positive, that is, when:

\[
f = \varepsilon_m \mu_m \left(\delta \zeta - 1 \right) \left(\delta \mu - 1 \right) > 0, \quad g = \left(\delta \mu + 1 \right) \left(3 - 2\delta \mu + 2\delta \mu \sin 2\phi \right) > 0 \quad \text{and} \quad h = \left(\delta \mu - 1 \right) \left(1 + \delta \mu \sin 2\phi \right) < 0,
\]
the dispersion surface of electric modes is an ellipsoids of revolution, with semiaxes along the directions: \( \vec{n}_1, \vec{n}_2 \) and \( \vec{n}_3 \), i.e. along the directions:

\[
\vec{n}_1 = (n_x \hat{x} + n_z \hat{z}) (1 + \delta_x \sin 2\phi) + (n_x \hat{x} - n_z \hat{z}) \delta_x \cos 2\phi + n_y \hat{y} (1 - \delta_x)
\]

\[
\vec{n}_2 = (n_x \hat{x} + n_z \hat{z}) (1 + \delta_x \sin 2\phi) + (n_x \hat{x} - n_z \hat{z}) \delta_x \cos 2\phi + n_y \hat{y} (1 - \delta_x)
\]

where \( \hat{x}, \hat{y}, \hat{z} \) are the unit vectors of the \( x, y \) and \( z \) axes.

II. If \( \lambda_1 < 0, \lambda_2 < 0 \) i.e. for \( f < 0, g > 0 \) and \( h < 0 \), the mode is evanescent.

III. If one of \( \lambda_1, \lambda_2, \lambda_3 \) is negative, and the others are positive, i.e. for \( f < 0 \) and \( g < 0 \), or \( f < 0 \) and \( h > 0 \), then the dispersion surface is a hyperboloids of one sheet, with semiaxes along the directions \( \vec{n}_1, \vec{n}_2 \) and \( \vec{n}_3 \).

IV. If one of \( \lambda_1, \lambda_2, \lambda_3 \) is positive, and the others are negative, i.e. for \( f > 0 \) and \( g < 0 \), or for \( f > 0 \) and \( h > 0 \), then the dispersion surface is a hyperboloids of two sheet, with semiaxes along the directions \( \vec{n}_1, \vec{n}_2 \) and \( \vec{n}_3 \).

V. If \( \delta_x = 1 \), the dispersion surface is a plane, and for \( \delta_x = 1 \), we have for electric modes:

\( n_x \cos \phi + n_z \sin \phi = 0 \), i.e. the dispersion surface becomes a plane. It should be noted, that for \( \delta_x = -1 \), the dispersion surface has the form: \( n_x^2 + (n_x \cos \phi - n_z \sin \phi) = 0 \). From this follows, that for \( n_x = 0 \), the dispersion surface is the straight line, \( n_y = n_x \cos \phi \). In the opposite case the mode is evanescent. Let us note, that the plane arises only for \( \delta_x = 1 \), and the straight line for \( \delta_x = -1 \).

On figure 1, we present (for various parameters of the medium) the possible (in the general case) pairs of dispersion surfaces (one – for the electric modes, the other for the magnetic modes). They can be defined from dispersion equation (3).

**Figure 1.** The dispersion surfaces for various parameters of the system.

- a: \( \epsilon_1 = 2.5, \epsilon_2 = 1.5, \phi = \pi / 3, \mu_1 = 1.7, \mu_2 = 2.9 \).
- b: \( \epsilon_1 = 3.1, \epsilon_2 = 2.5, \phi = \pi / 4, \mu_1 = -1.3, \mu_2 = 2.2 \).
- c: \( \epsilon_1 = 1.2, \epsilon_2 = -1.5, \phi = \pi / 5, \mu_1 = 1.3, \mu_2 = -1.1 \).
- d: \( \epsilon_1 = -2.2, \epsilon_2 = 3, \phi = \pi / 4, \mu_1 = 1.3, \mu_2 = -2.2 \).
- e: \( \epsilon_1 = 2.5, \epsilon_2 = 0, \phi = \pi / 4, \mu_1 = -0.9, \mu_2 = 3.7 \).

Let us note that in the general case the following pairs are impossible: an ellipsoid of revolution with a hyperboloid of one sheet, a hyperboloid of one sheet with a hyperboloid of two sheets, one evanescent mode with a hyperboloid of two sheets. It is natural, for one can show that \( \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \) are all positive, i.e. only even numbers of negative \( \lambda \) can exist. Here \( \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \) are the corresponding coefficients for magnetic modes. They are obtained by the interchanges \( \epsilon \leftrightarrow \mu \) and \( \mu \leftrightarrow \epsilon \) in the \( \lambda_1, \lambda_2, \lambda_3 \). Let us also note that if the dispersion surface of one of the modes is a plane, then the other is either a conical surface (turning either into a plane or a straight line in particular cases) presented in figure 1e, or an evanescent. And if the
dispersion surface of one of the modes is a straight line, then the other can be: a revolution ellipsoid; a hyperboloid of one sheet; a hyperboloid of two sheets; a straight line (figure 2).

![Figure 2](image)

**Figure 2.** The dispersion surfaces for the case when one of them is a straight line.

- a: \( \varepsilon_1 = 0, \varepsilon_2 = 0.7, \phi = \pi/3, \mu_1 = 1.2, \mu_2 = 1.1 \).
- b: \( \varepsilon_1 = 0, \varepsilon_2 = 0.7, \phi = \pi/3, \mu_1 = 1.2, \mu_2 = -1.5 \).
- c: \( \varepsilon_1 = 0, \varepsilon_2 = 0.7, \phi = \pi/3, \mu_1 = -1.2, \mu_2 = 1.5 \).
- d: \( \varepsilon_1 = 0, \varepsilon_2 = 0.7, \phi = \pi/3, \mu_1 = 1.5, \mu_2 = 0 \).

Now we pass on to detailed analysis of the dispersion equation (3) for \( n_z = 0 \), that is for fixed incident plane. Let’s investigate the dependences of dispersion curves on the optical axis orientation.

Figure 3a presents the dependences of dispersion curves of electric modes on the parameter, \( \phi \), for the same parameters of the problem for which the dispersion curve is an ellipse. If the rotation angle, \( \phi \), of the optical axis is equal to \( \pi k/2 \), then the ellipse semi-axes are directed along the directions: \( \hat{n}_x \) and \( \hat{n}_z \). For the other values of this angle, the ellipse semi-axes are shifted from the directions, \( \hat{n}_x \) and \( \hat{n}_z \).

Figure 3b presents the dependences of the dispersion curves of electric modes on the parameter, \( \phi \), for the same parameters of the problem for which the dispersion curve is a hyperbola.

![Figure 3](image)

**Figure 3.** The dependences of dispersion curves on the optical axis orientation.

- a: \( \varepsilon_1 = 1.2, \varepsilon_2 = 3.8, \mu_1 = 1.5, \mu_2 = 1.1 \).
- b: \( \varepsilon_1 = 1.2, \varepsilon_2 = -0.7, \mu_1 = 1.5, \mu_2 = 1.2 \).

As it is seen from that picture, if the angle, \( \phi \), changes, the dispersion curves, which are hyperbolas, rotate in the plane, \( n_x n_z \), and (for some specific value of that angle) these directions become asymptotes.

At the end of this section, let us note that the above considerations remain true also for the magnetic modes.

### 3. Omnidirectional reflection and total transmission

Let us consider light reflection and refraction on the border of an anisotropic medium — uniaxial anisotropic metamaterial. The medium is uniaxial and it occupies the half-space, \( z \geq 0 \), i.e. the medium border is parallel to the plane, \( xy \), and the incident plane coincides with the plane \( xz \) (\( xyz \) is the laboratory system). An electromagnetic wave of \( \omega \) frequency is incident, at the incidence angle \( \alpha \),
from isotropic and homogenous medium, having the parameters, \( \varepsilon \) and \( \mu \) (the dielectric and magnetic permittivities of the medium), on the subject half-space.

Now we consider the possibility of obtaining total reflection on the base of metamaterials such that it does not depend on the incidence angle and polarization and total transmission independent of polarization – for certain incidence angles.

Total reflection condition for \( p \)-polarization \( \Re_p = 1 \) (reflection coefficient for \( p \)-polarization) has the following form:

\[
(1 - \delta_z^2)
\left( n^2_{\varepsilon} - \varepsilon_{\varepsilon} \mu_{\mu} (1 - \delta_\mu) (1 - \delta_z \cos 2\phi) \right) > 0. \tag{8}
\]

The condition of total reflection, for \( p \)-polarization, for an arbitrary incident angle has the following form:

\[
\begin{align*}
1 - \delta_z^2 & > 0, \\
0 > \varepsilon_{\varepsilon} \mu_{\mu} (1 - \delta_\mu) \quad \text{or} \quad \varepsilon_{\varepsilon} \mu_{\mu} (1 - \delta_\mu) (1 - \delta_z \cos 2\phi) > \varepsilon_\varepsilon \mu_\mu,
\end{align*}
\tag{9}
\]

Doing the interchanges \( \delta_z \rightarrow \delta_\mu \) and \( \mu_\mu \rightarrow \varepsilon_\varepsilon \) in the (8) and (9) we will obtain analogous conditions for \( s \)-polarization, consequently, requiring the conditions for \( s \)- and \( p \)-polarizations simultaneously, we can obtain omnidirectional reflection. Our calculations show that, in particular, for \( \varepsilon_1 = 1.7, \varepsilon_2 = 2.8, \mu_1 = -1.0, \mu_2 = -1.4, \phi = \pi/4 \) omnidirectional reflection takes place.

For the condition:

\[
\begin{align*}
1 - \delta_z^2 = 0, \\
0 > \varepsilon_{\varepsilon} \mu_{\mu} (1 - \delta_\mu) \quad \text{or} \quad \varepsilon_{\varepsilon} \mu_{\mu} (1 - \delta_\mu) (1 - \delta_z \cos 2\phi) > \varepsilon_\varepsilon \mu_\mu,
\end{align*}
\tag{10}
\]

we have \( \Im_p = 1 \) (transmission coefficient for \( p \)-polarization).

From (10) follows that total transmission for \( p \)-polarization is possible for certain incidence angles, which are defined from (10) (the Brewster angle for \( p \)-polarization):

\[
\alpha_{B}^p = \sin^{-1}\left(\sqrt{\frac{\varepsilon_\varepsilon \mu_\mu (1 - \delta_\mu) (1 - \delta_z \cos 2\phi)}{\varepsilon_\varepsilon \mu_\mu (1 - \delta_\mu) (1 - \delta_z \cos 2\phi)}}\right). \tag{11}
\]

It is to be noted that, in contrast to the reflection on the border of two isotropic media when there is only one Brewster angle, here we have two of them (for \( p \)- and \( s \)-polarizations). Having same conditions for \( s \)-polarization, we can calculate the condition of \( \alpha_{B}^s = \alpha_{B}^p = \alpha_{B} \):

\[
\sin^2\phi = \frac{(\varepsilon_\varepsilon \mu_\mu - \varepsilon_\varepsilon \mu_\mu) (\varepsilon_\varepsilon \mu_\mu - \varepsilon_\varepsilon \mu_\mu)}{\varepsilon_\varepsilon \mu_\mu (\varepsilon_\varepsilon - \varepsilon_\varepsilon^2) - \mu_\mu (\varepsilon_\varepsilon^2 - \mu_\mu^2) - \mu_\mu (\varepsilon_\varepsilon - \varepsilon_\varepsilon^2) (\mu_\mu - \mu_\mu^2) + \varepsilon_\varepsilon \mu_\mu (\varepsilon_\varepsilon^2 - \mu_\mu^2).} \tag{12}
\]

Consequently, there is total transmission at the incident angle \( \alpha_{B} \), regardless of the polarization.

Now let us consider possibilities of anisotropic metamaterials as beam splitters. The refraction angles of the two forward eigen waves (\( P_{i1} > 0 \) and \( P_{i2} > 0 \), \( \vec{P}_i \) is the Poynting vector of the i-th eigen wave) and split angle are defined as follows:

\[
\begin{align*}
\theta_1 = \tan^{-1}\left(\frac{P_{i1}}{P_{i2}}\right), \quad \theta_2 = \tan^{-1}\left(\frac{P_{i2}}{P_{i1}}\right) \quad \text{and} \quad \Delta \theta = |\theta_2 - \theta_1|, \tag{13}
\end{align*}
\]

Figure 4 presents the dependence of \( \Delta \theta \) on the \( \varepsilon_\varepsilon \). It is to be noted that for each \( \varepsilon_\varepsilon \alpha = \alpha_{B} \) and the \( \phi \) angle is chosen in such a way that satisfies (12), i.e. here \( \Delta \theta \) is the beam splitting angle for total transmission. As it is seen from the figure 4, \( \Delta \theta \rightarrow \pi \), for the certain parameters. Consequently, it is possible to design miniature beam splitters without any intensity loss on the base of metamaterials. At \( \Delta \theta = 0 \), both refraction angles are the same. At this condition, the system can work as a phase retarder. From figure 4 we can see, that for certain parameters \( \Delta \theta = 0 \), therefore, on this parameters of the problem, the system can work as an ideal phase retarder, again, without any intensity loss.
Figure 4. The dependence of split angle, $\Delta \theta$, on the dielectric permittivity, $\varepsilon_1$.

The problem parameters are:

$\varepsilon_2 = 1.1$, $\mu_1 = 1.5$, $\mu_2 = -3.4$, $\varepsilon_1 = 1$, $\mu_\parallel = 1$, $\phi = \phi_\parallel$, $\alpha = \alpha_\parallel$

4. Conclusion

Concluding, let us to note that we investigated dispersion surfaces for the anisotropic metamaterials with dielectric and magnetic anisotropies. We showed that for an arbitrary orientation of the optical axis in the incidence plane, some groups of surfaces can arise. The conditions for independent of polarization total reflection and the conditions for total transmission are obtained, as well as obtained the conditions of the subject system of serving as: an omnidirectional reflector, a beam splitter and a phase retarder.

5. References

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