Bessel–like birth–death process

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Abstract

We consider models of the population or opinion dynamics which result in non-linear stochastic differential equations (SDEs) exhibiting spurious long-range memory. In this context, the correspondence between the description of birth-death processes as continuous-time Markov chains and continuous SDEs is of high importance for the alternatives of modeling. We propose and generalize Bessel-like birth-death process having clear representation by SDEs. The new process helps to integrate alternatives of description and to derive equations for the probability density function (PDF) of burst and inter-burst duration of the proposed continuous time birth-death processes. This PDF might be used to discriminate between spurious memory and true long-range memory in complex systems exhibiting continuing fluctuations.

1 Introduction

Birth-death processes or continuous time Markov chains are of great interest in the modeling of biological and social systems [1]. First of all, birth-death processes are useful in demography, population dynamics, genetics, epidemic dynamics, ecology, queuing theory, [2–4]. In the case of global agent interactions or the randomly generated networks such systems exhibit continuing stochastic fluctuations in collective behavior [5,6] with power-law in first and second order statistics [7,9] and thus are of great importance in finance and other social systems [10,11].

We aim to find a reliable method of how to discriminate macroscopic behavior of such Markov systems from alternative one exhibiting true long-range memory properties such as in fBm. The main idea is to analyze PDF of burst and inter-burst duration, which is invariant for the various non-linear transformations of observed stochastic time series [12,14]. This opportunity is given by the well defined general power-law form of PDF $P(\tau) \sim \tau^{2-H}$ for the burst and inter-burst duration $\tau$ of stochastic time series with Hurst exponent $H$ [15]. For the Markov time series $H = 1/2$ thus the exponent of PDF has the value $3/2$ general for all one-dimensional stochastic processes with uncorrelated increments.

The scientific uncertainty of these power-laws is related to the divergence of PDF and its moments. Fortunately, this divergence is rather formal mathematical, and one can overcome
this problem dealing with a finite number of agents or a finite number of states. Indeed, an infinite number of agents or the continuous limit of stochastic processes is a mathematical abstraction. Systems we aim to model usually have a limited number of agents and thus behave as discrete stochastic processes. Thus we consider the continuous time birth-death process with discrete states \( \{0, 1, 2, 3, \ldots, m, \ldots, N\} \) or equivalently a system of \( N \) agents with possible two states of individual agents \( \{0, 1\} \).

The problem of burst and inter-burst duration in such system is just a part of general first passage time theory \([13,14]\), where the moments of first passage time for birth-death processes have the closed-form expressions \([16]\). Nevertheless, the explicit form of first passage time PDF is known for a few cases and usually has the infinite sums of special functions \([17]\). The Bessel process written as one-dimensional SDE has an explicit solution for the first passage time PDF. We will integrate this knowledge introducing a Bessel-like birth-death process and will show that the solution for the continuous process can be used to approximate the burst and inter-burst duration PDF of proposed Bessel-like birth-death process.

In the second section, we shortly discuss the relation between birth-death processes and one-dimensional SDEs. Then we introduce the Bessel-like birth-death process and derive PDF of burst and inter-burst duration. Finally, we discuss the results and make some conclusions.

2 Modeling by birth-death processes and continuous SDE’s

We will use the notations of transition rate \( \lambda(m, N) \) from state \( m \) to \( m + 1 \) and transition rate \( \mu(m, N) \) from state \( m \) to \( m - 1 \), \( \lambda_N = 0, \mu_0 = 0 \) and both rates are \( > 0 \) otherwise. Let us start from the primary our interest to model social systems \([14,18]\) using well-defined herding model with rates written as

\[
\lambda(m, N) = (N - m)(\varepsilon_1 + m), \quad \mu(m, N) = m(\varepsilon_2 + (N - m)).
\]  

The master equation for PDF of macroscopic state evolution \( P(m, t) \) can be written using transition rates \( \lambda_m \) and \( \mu_m \). In the limit of a high number of agents \( N \) one can define Fokker-Planck equation for continuous variable \( x = m/N \) PDF, see \([13]\) for details, or corresponding stochastic differential equation (SDE) in Ito sens:

\[
dx = \Delta x(\lambda - \mu)dt + \sqrt{\Delta x^2(\lambda + \mu)}dW,
\]

where \( \Delta x = 1/N \) is the change of \( x \) during one step of the birth-death process, and \( W \) is the standard Wiener noise. Transition rates in general case depend on \( x \) and \( N \). For the herding model, SDE is

\[
dx = [\varepsilon_1(1 - x) - \varepsilon_2 x]dt + \sqrt{2x(1 - x)}dW,
\]
For the certain forms of transition rates, when \((\lambda(x, N) + \mu(x, N)) \sim N^2\), the stochastic term in Eq. \(\frac{\lambda(x, N) + \mu(x, N)}{N^2}\) does not disappears and non-equilibrium fluctuations in the agent system are sustained, the case of non-extensive statics \[19\], herding transition rates can be considered as an example \[8,9\].

We consider here the burst and inter-burst duration introduced in previous work \[13\]. In other words, the burst duration means here the first passage time to the threshold starting from the first state above a threshold and inter-burst duration starting from the first state below a threshold. Statistical properties of burst and inter-burst duration are invariant regarding non-linear transforms of the time series when one transforms the thresholds as well. SDE for population ratio \(y = \frac{x}{1-x}\) can be written as, Eq. (17) from \[9\]:

\[
dy = \left[\frac{\varepsilon_1}{y} + (2 - \varepsilon_2)\right]y (1+y) dt + \sqrt{2}y (1+y) dW. \tag{4}
\]

Note that this SDE is invariant for the transformation of variable \(y \to \frac{1}{y}\) and exhibits spurious long-range memory \[13\]. By this transformation bursts and inter-bursts interchanges.

In the case of continuous time discrete state birth-death process, the inter-burst duration is equivalent to the first passage time from state \(m-1\) to state \(m\) and the burst duration starting from state \(m+1\) to state \(m\). The correspondence of bursting time (burst or inter-burst duration) between continuous and discrete descriptions is the main idea of this contribution. It will help to solve the problem of burst and inter-burst duration statistics for the non-extensive Bessel-like birth-death process.

After Lamperti transformation \(z(y) = \sqrt{2} \arctan \sqrt{y}\) and assuming \(\varepsilon_1 = \varepsilon_2 = \varepsilon\) the herding model can be written as

\[
dz = \frac{2\varepsilon - 1}{\sqrt{2}} \cot(z\sqrt{2}) dt + dW, \tag{5}
\]

As we deal with number of agents \(N\), continuous and discrete variables are defined in the following intervals: \(0 \leq x \leq 1\); \(0 \leq y \leq \infty\); \(0 \leq z \leq \pi/\sqrt{2}\); \(0 \leq m \leq N\). In the limit \(z \to 0\) Eq. \(\frac{5}{5}\) becomes equivalent to the Bessel process

\[
dz = \frac{2\varepsilon - 1}{2z} dt + dW, \tag{6}
\]

3 Bessel-like birth-death process

We consider the Bessel process as an asymptotic limit of herding model, see \[20\] for details. Let us to partition the interval \(0 \leq z \leq \pi/\sqrt{2}\) into equally spaced intervals \(\Delta z = \pi/\sqrt{2}/N\) seeking to define an alternative non-extensive Bessel-like birth-death process. Transition rates of such process are as follows

\[
\lambda_b(m, N) = \frac{N^2}{\pi^2} \left(1 + \frac{\varepsilon - 1/2}{m}\right), \quad \mu_b(m, N) = \frac{N^2}{\pi^2} \left(1 - \frac{\varepsilon - 1/2}{m}\right). \tag{7}
\]
Substitution of rates (7) into Eq. (2) for \(z\) gives Eq. (6). Although the Bessel-like birth death process is not bounded, first passage times from any lower values of \(m\) to higher values of \(m\) are well defined and are related with passage time solution for the SDE (6). Let us demonstrate the correspondence between discrete first passage time \(\tau_m\) from the state \(m - 1\) to state \(m\) and well defined first passage time in continuous time SDE (6). The PDF of passage time \(\tau_m\) is given in [21] as

\[
P^{(\nu)}_b(\tau_m) = \frac{z^{\nu-2}_m}{z^{\nu-2}_{m-1}} \sum_{k=1}^{k_m} j_{\nu,k} J_{\nu} \left( \frac{z_{m-1}}{z_m} j_{\nu,k} \right) \exp \left( -\frac{j_{\nu,k}^2}{2z_m^2} \tau_m \right),
\]

where \(P^{(\nu)}_b(\tau_m)\) is a probability density function of the first passage times at level \(z_m\) of Bessel process with index \(\nu = \varepsilon - 1\) starting from \(z_{m-1}\), \(J_{\nu}\) is a Bessel function of the first kind of order \(\nu\) and \(j_{\nu,k}\) is a \(k\)-th zero of \(J_{\nu}\). The number of terms in the sum \(k_m = \infty\) for the continuous Bessel process. The main discrepancy of PDF (8) for continuous Bessel process and its discrete version is in the region \(\tau_m \rightarrow 0\). In Fig. 1 (a) we compare numerical calculations (Gillespie algorithm) of the first passage time PDF from state \(m = 59\) to state \(m = 60\) with corresponding continuous time PDF (8). It is obvious that for discrete space of variable values there are natural limits of diffusion as system can not passage the states lower than the state with \(\mu_b(m_0, N) = 0\) and for the smallest \(\tau_m\) probability density approaches exponential form for direct jumps from state \(m - 1\) to \(m\). For the Bessel-like process the real space of diffusion depends on the parameter \(\varepsilon\) as well. Probably there is some limit values \(k_m\) of index \(k\) in sum of exponential terms of Eq. (8), where the last term \(\exp \left( -\frac{j_{\nu,k}^2}{2z_m^2} \tau_m \right)\) describes direct jumps from state \(m - 1\) to state \(m\). We noticed that the last exponential rate \(\frac{j_{\nu,k}^2}{2z_m^2}\) should be equal to the biggest eigenvalue \(\xi_m\) from Keilson theorem [22, 23]. With such assumption \(k_m\) can be defined from the following equation

\[
\frac{j_{\nu,k}^2}{2z_m^2} = \xi_m = 2(\lambda(m - 1) + \mu(m - 1)),
\]

where \(\xi_m\) is related to the conditional passage from state \(m - 1\) to \(m\), not hitting the state \(m - 2\), see [16]. A simple equations \(k_m = \frac{2\pi}{\nu}\) follows from Eq. (9), when we substitute Bessel like rates (7) and use periodic property of Bessel zeros \(j_{\nu,k}\).

Using periodic property of Bessel zeros \(j_{\nu,k} = \pi k\) and ratio \(s_m = \frac{z_{m-1}}{z_m} = \frac{m-1}{m}\) we can simplify PDF (8) as follows

\[
P^{(\nu)}_b(\tau_m) = \frac{\nu + 1/2}{\nu} \sum_{k=1}^{k_m} \pi k \sin \left( \frac{k \pi}{m} \right) \exp \left( -\frac{\pi^2 k^2}{2z_m^2} \tau_m \right).
\]

For the parameter value \(\nu = \varepsilon - 1 = 0.5\) PDFs (8) and (10) coincide. It is worth to note that the integral of normalization \(S(k_m) = \int_0^{k_m} P^{(\nu)}_b(\tau_m) d\tau_m\) is an around 1 oscillating function.
Figure 1: The comparison of numerical $\tau_m$ PDF (red points) with Eqs. (8) and (10) for the Bessel-like process (7), $N = 100$. a) Eq. (8), $m = 60$, $k_m = 37$ (green line), $k_m = 60$ (cyan line), $k_m = 85$ (black line), $k_m = 97$ (blue line), $k_m = 120$ (orange line), $\varepsilon = 1.5$. b) Eq. (10), $m = 90$ (green line), $m = 30$ (blue line), $m = 10$ (black line), $\varepsilon = 5.5$; c) Eq. (8), $\varepsilon = 5.5$, $m = 90$ (green line), $m = 30$ (blue line), $m = 10$ (black line).

of $k_m$

$$S(k_m) = \frac{2}{\pi s_m} \sum_{k=1}^{k_m} \frac{1}{k} \sin \left( \frac{k \pi}{m} \right) =$$

$$= \frac{2}{\pi s_m} \text{SinIntegral} \left( \frac{k_m \pi}{m} \right) \simeq$$

$$\simeq \frac{2}{\pi s_m} \left[ \frac{(k_m - 1)\pi}{m} - \frac{1}{18} \left( \frac{k_m \pi}{m} \right)^3 + \frac{1}{600} \left( \frac{k_m \pi}{m} \right)^5 \right].$$

Here $S(k_m) \to 1$, when $k_m \to \infty$, but the value of $k_m$ calculated from Eq. (9) approximately is equal to the value needed for $S(k_m)$ to reach 1 for the first time. Nevertheless, the power expansion of $S(k_m)$ in Eq. (11) is applicable up to $k_m = m$ and gives a slightly lower value of $k_m$ needed for the normalization. Our numerical evaluation confirms, that Eq. (11) is better suited to define $k_m$ for the best fit of theoretical PDF to the numerical $\tau_m$ histogram.

Note that for other values of parameter $\nu \neq 0.5$ one has to integrate PDF (8) instead of (10) in order to define normalization $S(k_m)$ and $k_m$ needed for the discrete modeling.

To confirm this form of PDF in Fig. 1 we compare numerical calculations using Gillespie algorithm [24] of first passage time $\tau_m$ (inter-burst duration) PDF with Eqs. (8) and (10) having only $k_m$ exponential terms in the sum.

Finite number $k_m$ of exponential terms in PDF (8) explains the PDF behavior when $\tau_m \to 0$. Differences between continuous Bessel process and Bessel-like birth-death process with finite number of agents $N$ are important in this region. In both cases we have well-defined first passage time $\tau_m$ PDF, coinciding for the all $\tau_m$ values with the exception in the limit for very small values. This discrepancy appears because only direct jump $m - 1 \to m$ is possible for the birth-death process when in the continuous diffusion case there is an infinite number of states in between.

We do seek to extend the applications of Eq. (8) for the wider class of birth-death processes and first of all for the cases with bounded diffusion. Let’s add additional terms $-\frac{\varepsilon^2 - 1/2}{N-m}$ to the birth rate $\lambda_b$ and $\frac{\varepsilon^2 - 1/2}{N-m}$ to the death rate $\mu_b$ Eq. (7). This defines the bounded Bessel-like
The comparison of numerical inter-burst time $\tau_m$ PDF for the Bessel-like birth-death process (blue points) with the bounded Bessel-like process (red points). Parameters are as follows: $N = 100$, $m = 70$, a) $\varepsilon = 3.5$, b) $\varepsilon = 1.5$.

The continuous SDE corresponding to this new non-extensive birth-death process, for simplicity in the case $\varepsilon = \varepsilon_1 = \varepsilon_2$,

$$dz = \frac{(\pi/\sqrt{2} - 2z)(\varepsilon - 1/2)}{z(\pi/\sqrt{2} - z)}dt + dW. \quad (13)$$

Compare Eq. (13) with Eq. (5), both describe the Brownian particle in symmetric potential well.

Introducing new birth-death rates Eq. (12) we generalize Bessel-like birth-death process Eq. (7). This bounded version has a clear relation with classical Bessel process and represents the case of diffusion in the same interval $0 \leq z \leq \pi/\sqrt{2}$ as the corresponding transformation of Kirman’s model Eq. (5). We do expect to adjust results of the first passage and bursting time PDF, described above, for this generalized version of birth-death process.

First, let us make the numerical comparison of inter-burst duration $\tau_m$ PDF calculated with bounded and not bounded versions of Bessel-like birth-death process. The numerical comparison of bounded and unbounded versions, see examples in Fig. 2, reveals that differences of PDF appear only for the tail part representing exponential cut off of the distribution. In order to account this peculiar behavior for the bounded version of Bessel-like birth-death process we propose to add one more exponential term to the PDF of $\tau_m$ given by Eq. (8).
Figure 3: The comparison of numerical inter-burst duration $\tau_m$ PDF (red points) with Eq. (14) for the bounded Bessel-like process (12), $N = 1000$, a) $\varepsilon = 1.5$, $m = 800$ (green line); b) $\varepsilon = 3.5$, $m = 700$ (blue line).

with the number of exponential terms $k_m$ estimated from the normalization.

$$P_{bb}(\tau_m) = (1 - \rho)P_b(\tau_m) + \frac{\rho}{\tau_{m0}} \exp \left( -\frac{\tau_m}{\tau_{m0}} \right). \quad (14)$$

Here are two parameters in this new form of PDF. $\rho$ defines the weight of the new exponential term and $\tau_{m0}$ is a scale of exponential cut off. We define these parameters from the first $\tau_{m,1}$ and second $\tau_{m,2}$ moments of the first passage time, calculated from the rates of birth-death process, see [16]. Two equations for the needed parameters are as follows

$$\begin{align*}
\tau_{m,1} &= (1 - \rho)Q_1 + \rho \tau_{m0}, \\
\tau_{m,2} &= (1 - \rho)Q_2 + 2\rho \tau_{m0}^2.
\end{align*} \quad (15)$$

Here $Q_1$ and $Q_2$ are coefficients defined as sums of Bessel functions.

$$\begin{align*}
Q_1 &= 4z_m^2 \left( \frac{m}{m - 1} \right)^\nu \frac{\nu}{\nu - 1} \sum_{k=1}^{k_m} \frac{J_{\nu - 1}(m)J_{\nu - 1}(m)}{J_{\nu,k}(m)J_{\nu,k}(m)}, \\
Q_2 &= 16z_m^4 \left( \frac{m}{m - 1} \right)^\nu \frac{\nu}{\nu - 1} \sum_{k=1}^{k_m} \frac{J_{\nu - 1}(m)J_{\nu - 1}(m)}{J_{\nu,k}(m)J_{\nu,k}(m)}.
\end{align*} \quad (16)$$

We confirm this approximation by the comparison in Fig. 3 of numerical inter-burst duration PDF (Gillespie algorithm) for bounded Bessel-like birth-death process Eq. (12) and proposed PDF (14) with $\rho$ and $\tau_{m0}$ evaluated from Eqs. (15).

4 Discussion and conclusions

The relation between the stochastic and agent-based description of social systems is of great interest. The most direct connection is possible through the correspondence of birth-death
processes to the one-dimensional SDEs. Being a well defined mathematical tool in population and opinion dynamics birth-death processes can be easily treated as an outcome of heterogeneous agent system with two opinions. The macroscopic description of such social systems results in nonlinear SDEs exhibiting first and second orders power-law statistics [10,20,25]. The herding model serves as the most popular birth-death process used in the contemporary models of social systems [10]. Here we proposed the Bessel-like birth-death process as an alternative to the herding model and demonstrated that both are asymptotically related. Bessel-like birth-death process is a convenient, discrete version of modeling having the well established corresponding continuous version of Bessel process Eq. (6). We explored this correspondence, first of all, for a better understanding of the statistical properties of burst and inter-burst duration. If in the continuous description PDFs of burst and inter-burst duration are divergent for the small time intervals, PDFs are normalized and well defined for the birth-death processes. We do expect that proposed simple form of PDF (16) can be very useful for the empirical analyses of the time series exhibiting power-law statistical properties and helpful defining whether a process has spurious or true long-range memory.

Indeed, continuous SDE (13) after variable transform $y = z\sqrt{2}\pi^{1/2}$ and time scaling $t_s = \frac{2t}{\pi^2}$, see for comparison variable transformations used with herding model Eq. (4), can be written as

$$dy = \left[\frac{\varepsilon - 1}{y} - \left(\varepsilon - \frac{3}{2}\right)\right](1 + y)^3 dt_s + (1 + y)^2 dW_s.$$  \hspace{1cm} (17)

This proves that Bessel-like birth-death process with a sufficiently high number of agents $N$ may generate time series having the same power-law statistics as defined by the class of non-linear SDEs, considered in the series of papers [19,25,27]

$$dx = \left(\eta - \frac{\lambda}{2}\right)x^{\eta-1}dt_s + x^\eta dW_s.$$  \hspace{1cm} (18)

These SDEs generate stationary power-law PDF of the variable $P(x) \sim x^{-\lambda}$ with exponent $\lambda$ and power spectral density $S(f) \sim \frac{1}{f^\beta}$ with exponent $\beta = 1 + \frac{\lambda - 3}{2\eta - 2}$. For the Bessel-like process Eq. (17), $\eta = 2$ and $\lambda = 2\varepsilon + 1$. Observed power-law statistical properties can be easily confused with true long-range memory. Here presented Bessel-like birth-death process, previously considered Kirman’s model and other non-extensive birth-death processes [28] probably can be used for the modeling of social systems exhibiting continuing fluctuations [29,30]. All these Markov chains may generate very similar statistical properties of time series and should have very similar PDFs of burst and inter-burst duration. Presented correspondence between birth-death processes and a class of non-linear SDEs can be very helpful in building the theoretical background of various social systems exhibiting power-law behavior.
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