Nonlocal games, synchronous correlations, and Bell inequalities

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A nonlocal game with a synchronous correlation is a natural generalization of a function between two finite sets, and has recently appeared in the context of quantum graph homomorphisms. In this work we examine analogues of Bell’s inequalities for synchronous correlations. We show that, unlike general correlations and the CHSH inequality, there can be no quantum Bell violation among synchronous correlations with two measurement settings. However we exhibit explicit analogues of Bell’s inequalities for synchronous correlations with three measurement settings and two outputs, provide an analogue of Tsirl’son’s bound in this setting, and give explicit quantum correlations that saturate this bound.

I. NONLOCAL GAMES AND SYNCHRONOUS CORRELATIONS

It has long been recognized that quantum entanglement can provide two otherwise non-communicating parties stronger than classical correlation\textsuperscript{27,28}. An interesting example of this is so-called pseudo-telepathy, where two parties can consistently give correct answers to questions that would be classical impossible—without additional communication; a key example of this is a graph coloring that utilizes fewer colors than the chromatic number of the graph\textsuperscript{29,33}. Much as the chromatic number can be realized via the existence and nonexistence of graph homomorphisms into complete graphs of certain sizes, these quantum protocols can be viewed as a form of generalized graph homomorphisms. There has been a great deal of recent work exploring the structure of such nonlocal games in this and similar contexts\textsuperscript{31,41}. In this work we set aside the graph theoretic framework, and focus strictly on the structure of these nonlocal games that can be “won” with certainty as a generalization of functions.

Formally, given two finite sets $X, Y$, players Alice and Bob obtain from a referee elements $x_A, x_B \in X$ respectively and must produce elements $y_A, y_B \in Y$ respectively. We say they “win” the game if whenever they are presented $x_A = x_B$ then with certainty their outputs satisfy $y_A = y_B$. Such games generalize functions; each function $f : X \to Y$ defines a winning strategy: Alice and Bob agree to output $y_A = f(x_A)$ and $y_B = f(x_B)$ respectively. We will often need to discuss the set of all functions $X \to Y$, so will use the notation $Y^X$ for this set.

In general, a strategy is characterized by a conditional probability distribution, or simply correlation, $p(y_A, y_B | x_A, x_B)$. We call a correlation synchronous\textsuperscript{38,41} if Alice and Bob win with certainty:

$$p(y_A, y_B | x, x) = 0 \text{ if } x \in X \text{ and } y_A \neq y_B \text{ in } Y. \quad (1)$$

Note that the space of all synchronous correlations is convex. Moreover the correlation associated to the strategy of applying the function $f \in Y^X$, that is

$$p(y_A, y_B | x_A, x_B) = \mathbb{I}_{\{y_A = f(x_A)\}} \mathbb{I}_{\{y_B = f(x_B)\}}, \quad (2)$$

is synchronous.

A common technique to analyze such correlations is through polytopes of stochastic matrices\textsuperscript{42,43}. Namely, if $|X| = n$ and $|Y| = m$ then $p(y_A, y_B | x_A, x_B)$ forms the entries of an $m^2 \times n^2$ (column) stochastic matrix. The set of general correlations form a $(m^2 - 1)n^2$-dimensional polytope with $m^2n^2$ vertices, these vertices correspond to the deterministic strategies\textsuperscript{44}. The synchronous-ness condition\textsuperscript{11} slices this polytope in a simple way. The polytope of all synchronous correlations is $(m^2 - 1)n^2 - n(m^2 - m)$-dimensional with $m^2n^2 - m^2n + mn$ vertices, again these vertices corresponding to the deterministic synchronous correlations.

\textbf{Definition 1.} A correlation is symmetric if $p(y_A, y_B | x_A, x_B) = p(y_B, y_A | x_B, x_A)$.

Beyond synchronous correlations studied here, symmetric correlations have proven to be a fruitful study in that one can compute the probability of winning such games in many circumstances\textsuperscript{45}. Similarly, we will see symmetric synchronous correlations playing a special role below. Clearly the
synchronous correlations that arise from functions \( f \) are symmetric, and the convex sum of symmetric synchronous correlations is symmetric and synchronous.

In complete generality, correlations as given above allow arbitrary communication between Alice and Bob. In this paper we focus on “nonlocal” games, by which we mean Alice and Bob may utilize preshared information but cannot communicate once they receive their inputs \( x_A, x_B \). This includes strategies corresponding to a function: once Alice and Bob have agreed to apply \( f \), no further communication is required. To capture the notion of “no additional communication” we use the well-known nonsignaling conditions \( [17] \) defined as follows.

**Definition 2.** A correlation \( p \) is nonsignaling if it satisfies (i) for all \( y_A, x_A, x_B, x_B' \)

\[
\sum_{y_B} p(y_A, y_B | x_A, x_B) = \sum_{y_B} p(y_A, y_B | x_A, x_B'),
\]

and (ii) for all \( y_B, x_B, x_A, x_A' \)

\[
\sum_{y_A} p(y_A, y_B | x_A, x_B) = \sum_{y_A} p(y_A, y_B | x_A', x_B).
\]

**II. SYNCHRONOUS CLASSICAL AND QUANTUM CORRELATIONS**

**Definition 3.** A local hidden variables strategy, or simply classical correlation, is a correlation of the form

\[
p(y_A, y_B | x_A, x_B) = \sum_{\omega \in \Omega} \mu(\omega) p_A(y_A | x_A, \omega) p_B(y_B | x_B, \omega)
\]

for some finite set \( \Omega \) and probability distribution \( \mu \).

Here \( (\Omega, \mu) \) is shared randomness Alice and Bob may draw on, and \( p_A \) and \( p_B \) are local (conditional) probabilities they use to produce their respective outputs. Clearly every correlation of the form \( [3] \) will be nonsignaling. Without loss of generality, we may assume \( \mu(\omega) > 0 \) for all \( \omega \in \Omega \) as otherwise we simply restrict to the support of \( \mu \) and still have the same form.

In order that \( p \) be synchronous we must have for each \( x \in X \), whenever \( y_A \neq y_B \) that

\[
0 = \sum_{\omega \in \Omega} \mu(\omega) p_A(y_A | x, \omega) p_B(y_B | x, \omega).
\]

In particular, for each \( x \in X \) and \( \omega \in \Omega \) we must have whenever \( y_A \neq y_B \) that

\[
0 = p_A(y_A | x, \omega) p_B(y_B | x, \omega).
\]

Fix some \( x, \omega \). As \( \sum_y p_A(y | x, \omega) = 1 \) there exists a \( y_0 \) (depending on \( x, \omega \)) such that \( p_A(y_0 | x, \omega) > 0 \), and so from above \( p_B(y | x, \omega) = 0 \) whenever \( y \neq y_0 \). Thus

\[
1 = \sum_y p_B(y | x, \omega) = p_B(y_0 | x, \omega).
\]

Exchanging \( A \) and \( B \) shows \( p_A(y_0 | x, \omega) = 1 \) as well.

Therefore, for each \( \omega \in \Omega \) we obtain a function \( f_\omega : X \rightarrow Y \) given by \( f_\omega(x) = y_0 \) where \( y_0 \) is the value with \( p_A(y_0 | x, \omega) = p_B(y_0 | x, \omega) = 1 \). This allows us to map \( \Omega \) into \( Y^X \), proving the following results.

**Theorem 4.** The set of synchronous local hidden variables strategies on \( X \rightarrow Y \) is bijective to the set of probability distributions on \( Y^X \). Given such a probability distribution the associated strategy is: Alice and Bob sample a function \( f \in Y^X \) according the specified distribution, and upon receiving \( x_A, x_B \) they output \( y_A = f(x_A) \) and \( y_B = f(x_B) \).

**Corollary 5.** The extreme points of the synchronous local hidden variables strategies on \( X \rightarrow Y \) can be canonically identified with \( Y^X \).

**Corollary 6.** Every synchronous classical strategy is symmetric.

**Definition 7.** A quantum correlation is a correlation that takes the form

\[
p(y_A, y_B | x_A, x_B) = \text{tr}(\rho(E_{y_A}^{x_A} \otimes F_{y_B}^{x_B}))
\]

where \( \rho \) is a density operator on the Hilbert space \( \mathcal{H}_A \otimes \mathcal{H}_B \), and for each \( x \in X \) we have \( \{E_{y_A}^{x_A}\}_{y_A \in Y} \) and \( \{F_{y_B}^{x_B}\}_{y_B \in Y} \) are POVMs on \( \mathcal{H}_A \) and \( \mathcal{H}_B \) respectively.

Again, any correlation of the form \( [4] \) will be nonsignaling from the fact that \( \{E_{y_A}^{x_A}\}_{y_A \in Y} \) and \( \{F_{y_B}^{x_B}\}_{y_B \in Y} \) are POVMs. We will only treat the case when \( \mathcal{H}_A \) and \( \mathcal{H}_B \) are finite dimensional. Without loss of generality we can take \( \text{tr}_A(\rho) \) and \( \text{tr}_B(\rho) \) of maximal rank by restricting \( \mathcal{H}_A \) and \( \mathcal{H}_B \) if necessary. It is common to define quantum correlations using projections rather than general positive-operator valued measures as one can always enlarge \( \mathcal{H}_A \) and \( \mathcal{H}_B \) to achieve such. However for synchronous quantum correlations we have these POVMs must already be projection-valued. The proof of this is contained in \( [33, \text{Proposition 1}] \), which is for “quantum coloring games” but carries over to synchronous quantum correlations without modification; see also \( [33, 35, 38, 40] \).
Lemma 8. Let

\[ p(y_A, y_B | x_A, x_B) = \text{tr}(\rho(E^{x_A}_{y_A} \otimes F^{x_B}_{y_B})) \]

be a synchronous quantum correlation. Then the POVMs \( \{E^x_y\}_{y \in Y} \) and \( \{F^x_y\}_{y \in Y} \), for \( x \in X \), are projection-valued measures. Moreover each \( E^x_y \) commutes with \( \text{tr}_B(\rho) \) and each \( F^x_y \) commutes with \( \text{tr}_A(\rho) \).

The works cited above are primarily focused on the existence of a synchronous quantum correlation that satisfies some additional conditions, for example preserving graph adjacency. In this context, a common result is that if one such correlation exists then another exists whose state is maximally entangled: examples of such include [33, Proposition 1], [40, Lemma 4], [36, Theorem 2.1]. It is certainly not the case that every synchronous quantum correlation can be taken to have a maximally entangled state, as these include hidden variables strategies. Nonetheless we can prove that every synchronous quantum correlation is a convex sum of such.

Theorem 9. Every synchronous quantum correlation can be expressed as the convex combination of synchronous quantum correlations with maximally entangled pure states. In particular, if a synchronous quantum correlation \( \text{tr}(\rho(E^{x_A}_{y_A} \otimes F^{x_B}_{y_B})) \) is extremal then we may take \( \rho = |\psi\rangle \langle \psi| \) with \( |\psi\rangle \) maximally entangled.

Proof. Let \( p(y_A, y_B | x_A, x_B) = \text{tr}(\rho(E^{x_A}_{y_A} \otimes F^{x_B}_{y_B})) \) be a synchronous quantum correlation. As we may decompose \( \rho \) into a convex combination of pure states, we can assume \( \rho = |\psi\rangle \langle \psi| \). Suppose \( |\psi\rangle \) has \( r \) distinct Schmidt coefficients, and in particular let us write the Schmidt decomposition of \( |\psi\rangle \) as

\[ |\psi\rangle = \sum_{j=1}^r \sqrt{\sigma_j} \phi^A_j \otimes \phi^B_j. \]

Note \( \sum_j \sigma_j = 1 \). The spectral decomposition of the partial trace is then

\[ \text{tr}_B(|\psi\rangle \langle \psi|) = \sum_{j=1}^r \sigma_j \Pi^A_j, \]

where

\[ \Pi^A_j = \sum_{m=1}^{\ell_j} |\phi^A_{j,m}\rangle \langle \phi^A_{j,m}|. \]

These eigenprojections \( \{\Pi^A_j\} \) decompose Alice’s Hilbert space \( \mathcal{H}^A \) into an orthogonal sum of subspaces: \( \mathcal{H}^A = \mathcal{H}_0 \oplus \bigoplus \mathcal{H}^A_j \) where \( \mathcal{H}^A_j = \text{im}(\Pi^A_j) \) and \( \mathcal{H}^A_0 = \ker(\text{tr}_B(|\psi\rangle \langle \psi|)). \) Identically \( \text{tr}_A(|\psi\rangle \langle \psi|) = \sum_{j=1}^r \sigma_j \Pi^B_j \), inducing a decomposition of Bob’s space \( \mathcal{H}_B = \mathcal{H}_0 \oplus \bigoplus \mathcal{H}^B_j \).

From the lemma, each \( E^{x}_{y} \) commutes with \( \text{tr}_A(|\psi\rangle \langle \psi|) \) and so preserves the decomposition \( \mathcal{H}^A = \mathcal{H}_0 \oplus \bigoplus \mathcal{H}^A_j \), with a similar statement holding for the \( F^{x}_{y} \). Therefore

\[ p(y_A, y_B | x_A, x_B) = \langle \psi|E^{x_A}_{y_A} \otimes F^{x_B}_{y_B}|\psi\rangle \]

\[ = \sum_{j,k} \sqrt{\sigma_j} \sigma_k \sum_{m,n=1}^{\ell_j,\ell_k} \langle \phi^A_j | E^{x_A}_{y_A} | \phi^A_k \rangle \langle \phi^B_j | F^{x_B}_{y_B} | \phi^B_k \rangle \]

\[ = \sum_{j=1}^r \sum_{m,n=1}^{\ell_j,\ell_k} \langle \phi^A_j | E^{x_A}_{y_A} | \phi^A_j \rangle \langle \phi^B_j | F^{x_B}_{y_B} | \phi^B_j \rangle \]

\[ = \sum_{j=1}^r \sum_{m,n=1}^{\ell_j} \langle \phi^A_j | E^{x_A}_{y_A} | \phi^A_j \rangle \langle \phi^B_j | F^{x_B}_{y_B} | \phi^B_j \rangle \]

\[ = \sum_{j=1}^r \sum_{m,n=1}^{\ell_j} |\phi^A_{j,m}\rangle \otimes |\phi^B_{j,m}\rangle \text{ is maximally entangled on } \mathcal{H}^A_j \otimes \mathcal{H}^B_j. \]

This theorem shows that any extremal synchronous quantum correlation will be associated to some maximally entangled pure state, which can be taken canonically [33]. After restricting \( \mathcal{H}^A \) and \( \mathcal{H}^B \) to the support of the partial traces of \( |\psi\rangle \langle \psi| \) if necessary, we can take

\[ |\psi\rangle = (V \times \mathbb{1}) |\Omega\rangle \]

Here \( \{|\psi\rangle\}_{j=1}^d \) is a fixed orthonormal basis of \( \mathcal{H} = \mathcal{H}_B \) and \( V \) is an isometry of \( \mathcal{H}^A \) onto \( \mathcal{H}_B \), or unitary upon also identifying \( \mathcal{H}^A = \mathcal{H} \). Redefining \( V^\dagger E^x_y V \to E^x_y \) reduces our form for correlations with maximally entangled state to

\[ p(y_A, y_B | x_A, x_B) = \langle \Omega| E^{x_A}_{y_A} \otimes F^{x_B}_{y_B} |\Omega\rangle \]

\[ = \frac{1}{d} \sum_{j,k=1}^d \langle j| E^{x_A}_{y_A} | k \rangle \langle k| F^{x_B}_{y_B} | j \rangle \]

\[ = \frac{1}{d} \sum_{j,k=1}^d \langle j| E^{x_A}_{y_A} | k \rangle \langle k| E^{x_B}_{y_B} | j \rangle \]

\[ = \frac{1}{d} \text{tr}(E^{x_A}_{y_A} F^{x_B}_{y_B}). \]

Here \( F^{x}_{y} \) refers to the projection whose entries in the \( \{|\psi\rangle\}_{j=1}^d \) basis are the complex conjugates of those of \( E^{x}_{y} \); that is \( F^{x}_{y} \) is the transpose of \( E^{x}_{y} \) with respect to this basis.
As $p$ is synchronous we have

$$1 = \frac{1}{d} \sum_{y_A, y_B} \text{tr}(E^x_{y_A} F^x_{y_B}) = \frac{1}{d} \sum_{y} \text{tr}(E^x_{y} F^x_{y}).$$

But from Cauchy-Schwarz,

$$1 = \frac{1}{d} \sum_{y} \text{tr}(E^x_{y} F^x_{y}) \leq \left[ \frac{1}{d} \sum_{y} \text{tr}(E^x_{y} E^x_{y}) \right]^{\frac{1}{2}} \left[ \frac{1}{d} \sum_{y} \text{tr}(F^x_{y} F^x_{y}) \right]^{\frac{1}{2}} = \left[ \frac{1}{d} \text{tr}(\mathbb{I}) \right]^{\frac{1}{2}} \left[ \frac{1}{d} \text{tr}(\mathbb{I}) \right]^{\frac{1}{2}} = 1.$$

And so again by Cauchy-Schwarz $E^x_{y} = F^x_{y}$. This proves the following results, which can be found in greater generality as [38, Theorem 5.5].

**Theorem 10.** Let $X, Y$ be finite sets, $\mathcal{F}$ a $d$-dimensional Hilbert space, and for each $x \in X$ a projection-valued measure $\{E^x_{y}\}_{y \in Y}$ on $\mathcal{F}$. Then

$$p(y_A, y_B | x_A, x_B) = \frac{1}{d} \text{tr}(E^x_{y_A} E^x_{y_B})$$

defines a synchronous quantum correlation. Moreover every synchronous quantum correlation with maximally entangled pure state has this form.

**Corollary 11.** Every synchronous quantum correlation is symmetric.

### III. BELL INEQUALITIES FOR SYNCHRONOUS CORRELATIONS

Bell’s inequalities characterize hidden variables strategies among general nonsignaling ones. We are interested in synchronous correlations and so focus on “synchronous” Bell inequalities. In Theorem 4 we saw that the polytope of synchronous hidden variables strategies is the convex hull of the set of functions. Therefore, Bell’s inequalities are precisely the inequalities that describe the facets of this polytope.

However not all these inequalities are interesting. As every classical correlation satisfies the nonsignaling relations many of these facets, and hence their associated inequalities, will be inherited from the analogous polytope describing synchronous nonsignaling strategies. As our goal is to characterize when a quantum strategy is not classical, such inequalities are useless as any synchronous quantum correlations is nonsignaling as well. Hence the Bell inequalities we wish to examine are those that do not arise from the nonsignaling conditions.

Let us begin by examining the case of strategies with domain $\{0, 1\}$ into some finite set $Y$. In the case of general (not necessarily synchronous) nonsignaling correlations, some results along these lines have already been shown [44].

**Lemma 12.** Let $Y$ be a finite set and $u = u(y_A, y_B)$ and $v = v(y_A, y_B)$ be probability distributions on $Y^2$ such that for all $y \in Y$

1. $\sum_{y'} u(y, y') = \sum_{y'} v(y', y)$ and
2. $\sum_{y'} u(y', y) = \sum_{y'} v(y, y').$

Write $\theta(y)$ and $\phi(y)$ for these two sums respectively and define

$$p(y_A, y_B | 0, 0) = \mathbb{I}_{\{y_A = y_B\}} \theta(y_A)$$
$$p(y_A, y_B | 0, 1) = u(y_A, y_B)$$
$$p(y_A, y_B | 1, 0) = v(y_A, y_B)$$
$$p(y_A, y_B | 1, 1) = \mathbb{I}_{\{y_A = y_B\}} \phi(y_A).$$

Then $p$ is a synchronous nonsignaling correlation. Moreover every nonsignaling correlation with domain $\{0, 1\}$ arises this way.

**Proof.** A straightforward computation shows that $p$ as defined is synchronous and satisfies the nonsignaling conditions. Conversely, given a synchronous nonsignaling correlation $p$ from $\{0, 1\}$ to $Y$, define

$$u(y_A, y_B) = p(y_A, y_B | 0, 1)$$

and

$$v(y_A, y_B) = p(y_A, y_B | 1, 0).$$

Then

$$\sum_{y'} p(y, y' | 0, 0) = \sum_{y'} p(y, y' | 0, 1)$$
$$= \sum_{y'} u(y, y') = \theta(y)$$

where we take this as the definition of $\theta$. Then $p(y, y' | 0, 0) = 0$ when $y \neq y'$ and

$$p(y, y | 0, 0) = \sum_{y'} p(y, y' | 0, 0) = \theta(y)$$

and therefore $p(y, y' | 0, 0) = \mathbb{I}_{\{y = y'\}} \theta(y)$. An identical argument shows $p(y, y' | 1, 1) = \mathbb{I}_{\{y = y'\}} \phi(y)$
where \( \phi(y) = \sum_{y'} v(y, y') \). Finally, we compute
\[
\sum_{y'} u(y, y') = \sum_{y'} p(y, y' | 0, 1)
\]
\[
= \sum_{y'} p(y, y' | 0, 0)
\]
\[
= p(y, y | 0, 0)
\]
\[
= \sum_{y'} p(y', y | 0, 0)
\]
\[
= \sum_{y'} p(y', y | 1, 0)
\]
\[
= \sum_{y'} v(y', y).
\]
Again a similar argument shows \( \sum_{y'} u(y', y) = \sum_{y'} v(y, y') \), and therefore \( u \) and \( v \) satisfy the two constraints stated in the theorem. \( \square \)

**Lemma 13.** Let \( Y \) be a finite set and \( u = u(y_A, y_B) \) be probability distributions on \( Y^2 \). Write

1. \( \theta(y) = \sum_{y'} u(y, y') \) and
2. \( \phi(y) = \sum_{y'} u(y', y) \).

Define
\[
p(y_A, y_B | 0, 0) = \mathbb{I}_{\{y_A=y_B\}} \theta(y_A)
\]
\[
p(y_A, y_B | 0, 1) = u(y_A, y_B)
\]
\[
p(y_A, y_B | 1, 0) = u(y_B, y_A)
\]
\[
p(y_A, y_B | 1, 1) = \mathbb{I}_{\{y_A=y_B\}} \phi(y_A).
\]
Then \( p \) is a synchronous classical correlation. Moreover every nonsignaling correlation with domain \( \{0, 1\} \) arises this way.

**Proof.** Define a probability distribution on \( Y^{\{0, 1\}} \) by \( \mu(f) = u(f(0), f(1)) \). Then
\[
\sum_{f} u(f(0), f(1)) \mathbb{I}_{\{y_A=f(0)\}} \mathbb{I}_{\{y_B=f(1)\}}
\]
\[
= u(y_A, y_B)
\]
\[
= p(y_A, y_B | 0, 1)
\]
and identically
\[
p(y_A, y_B | 1, 0)
\]
\[
= \sum_{f} u(f(0), f(1)) \mathbb{I}_{\{y_A=f(1)\}} \mathbb{I}_{\{y_B=f(0)\}}.
\]
A similar computation shows
\[
\sum_{f} u(f(0), f(1)) \mathbb{I}_{\{y_A=f(0)\}} \mathbb{I}_{\{y_B=f(0)\}}
\]
\[
= \mathbb{I}_{\{y_A=y_B\}} \sum_{y'} u(y_A, y') = p(y_A, y_B | 0, 0)
\]
and
\[
p(y_A, y_B | 1, 1)
\]
\[
= \sum_{f} u(f(0), f(1)) \mathbb{I}_{\{y_A=f(1)\}} \mathbb{I}_{\{y_B=f(1)\}}.
\]
Therefore \( p \) is the classical strategy associated to the distribution \( \mu \).

Conversely, if \( p \) is classical then it is nonsignaling and so has the form as given in Lemma 13. But from Corollary 14 \( p \) is also symmetric and hence \( v(y_A, y_B) = u(y_B, y_A) \). \( \square \)

**Corollary 14.** A synchronous correlation from \( \{0, 1\} \) to a finite set is classical if and only if it is symmetric.

**Corollary 15.** Every synchronous quantum correlation from \( \{0, 1\} \) to a finite set is classical.

These results imply that for synchronous correlations with domain \( \{0, 1\} \) the Bell inequalities are actually equations: the equations for a correlation being symmetric. Consequently, one cannot achieve a Bell violation in this circumstance since all synchronous quantum correlations will also satisfy these equations.

To find nontrivial Bell inequalities we turn to the next simplest case to analyze: a two-point range. Again some progress along these lines have been made for general correlations by considering the extreme points of the associated polytopes [19]. Moreover, the polytope of hidden-variables strategies has been characterized and one nontrivial quantum strategy was found [50].

**Lemma 16.** Suppose \( |X| \geq 2 \) and let \( w = w(x_A, x_B) \) be a nonnegative function on \( X^2 \) such that for every \( x_A, x_B \in X \)

1. \( w(x_A, x_B) \leq w(x_A, x_A) \),
2. \( w(x_A, x_B) \leq w(x_B, x_B) \), and
3. \( w(x_A, x_A) + w(x_B, x_B) \leq 1 + w(x_A, x_B) \).
Define
\[
    p(0, 0 \mid x_A, x_B) = 1 + w(x_A, x_B) - w(x_A, x_B) - w(x_A, x_B)
\]
and
\[
    p(0, 1 \mid x_A, x_B) = w(x_B, x_B) - w(x_A, x_B)
\]
then
\[
    p(1, 0 \mid x_A, x_B) = w(x_A, x_B) - w(x_A, x_B)
\]
and
\[
    p(1, 1 \mid x_A, x_B) = w(x_A, x_B).
\]

Then \( p \) is a synchronous nonsignaling correlation from \( X \) to \( \{0, 1\} \). Moreover every synchronous nonsignaling correlation from \( X \) to \( \{0, 1\} \) arises in this way.

**Proof.** Suppose \( w \) is given as stated. Clearly the given conditions imply that \( p \) is a correlation. It is synchronous as
\[
    p(0, 1 \mid x, x) = p(1, 0 \mid x, x) = w(x, x) - w(x, x) = 0.
\]
Verifying the nonsignaling relations is simple and left to the reader.

Conversely, suppose \( p \) is a synchronous nonsignaling correlation from \( X \) to \( \{0, 1\} \), and define \( w(x_A, x_B) = p(1, 1 \mid x_A, x_B) \). From the nonsignaling conditions
\[
    p(1, 0 \mid x_A, x_B) + p(1, 1 \mid x_A, x_B) = F(x_A)
\]
for some function \( F = F(x_A) \) independent of \( x_B \). Taking \( x_B = x_A \) and appealing to synchronousness of \( p \) we have
\[
    F(x_A) = p(1, 1 \mid x_A, x_A) = w(x_A, x_A).
\]
Thus
\[
    p(1, 0 \mid x_A, x_B) = F(x_A) - p(1, 1 \mid x_A, x_B) = w(x_A, x_A) - w(x_A, x_B).
\]
Similarly,
\[
    p(0, 1 \mid x_A, x_B) = w(x_B, x_B) - w(x_A, x_B).
\]
Finally, since \( p \) is a correlation
\[
    p(0, 0 \mid x_A, x_B) = 1 - p(0, 1 \mid x_A, x_B) \quad \text{and} \quad p(1, 0 \mid x_A, x_B) - p(1, 1 \mid x_A, x_B)
\]
\[
    = 1 + w(x_A, x_B) - w(x_A, x_B) - w(x_A, x_B).
\]
Conditions 1, 2, and 3, and that \( w \) is nonnegative follow from \( p \) being nonnegative. \( \square \)

Specializing to the case \( X = \{0, 1, 2\} \), this lemma gives that the polytope of synchronous nonsignaling correlations can be parametrized by nine coordinates
\[
    w_{3x_A + x_B} = p(1, 1 \mid x_A, x_B).
\]
The conditions in the lemma form 24 linear inequalities:
\[
    \begin{align*}
    0 \leq w_1, & \quad w_0 + w_4 \leq 1 + w_1, \\
    0 \leq w_2, & \quad w_0 + w_4 \leq 1 + w_3, \\
    0 \leq w_3, & \quad w_0 + w_8 \leq 1 + w_2, \\
    0 \leq w_5, & \quad w_0 + w_8 \leq 1 + w_6, \\
    0 \leq w_6, & \quad w_4 + w_8 \leq 1 + w_5, \\
    0 \leq w_7, & \quad w_4 + w_8 \leq 1 + w_7.
    \end{align*}
\]
(5)

Note \( 0 \leq w_0, w_4, w_8 \) implicit in the lemma are implied by the above. This polytope in \( \mathbb{R}^9 \) has 80 vertices, which can be easily constructed using standard mathematical software \[51\].

There are 8 functions from \( \{0, 1, 2\} \) to \( \{0, 1\} \) and the synchronous hidden variable strategies are precisely the convex hull of these 8 points, forming a 6 dimensional polytope with these functions as vertices. The three equations defining the space in which this polytope lives are
\[
    w_1 = w_3, \quad w_2 = w_6, \quad w_5 = w_7,
\]
which express the symmetry of hidden variables strategies. Enforcing these conditions in the inequalities \[6\] that is restricting the nonsignaling polytope to this codimension three subspace, reduces to 12 linear inequalities:
\[
    \begin{align*}
    0 \leq w_3, & \quad w_0 + w_4 \leq 1 + w_3, \\
    0 \leq w_6, & \quad w_0 + w_8 \leq 1 + w_6, \\
    0 \leq w_7, & \quad w_4 + w_8 \leq 1 + w_7, \\
    w_3 \leq w_0, & \quad w_3 \leq w_4, \\
    w_6 \leq w_0, & \quad w_6 \leq w_8, \\
    w_7 \leq w_4, & \quad w_7 \leq w_8.
    \end{align*}
\]
(6)

However these alone do not define the hidden variables polytope. There are four additional inequalities that serve as the (interesting) Bell inequalities for synchronous correlations from \( \{0, 1, 2\} \) to \( \{0, 1\} \):
\[
    \begin{align*}
    J_0 = w_0 - w_3 + w_4 - w_6 - w_7 + w_8 \leq 1, \\
    J_1 = w_0 - w_3 - w_4 + w_7 \geq 0, \\
    J_2 = -w_3 + w_4 + w_6 - w_7 \geq 0, \\
    J_3 = w_3 - w_6 - w_7 + w_8 \geq 0.
    \end{align*}
\]
(7)

These four inequalities precisely define when a nonsignaling strategy is a hidden variables one.
IV. QUANTUM VIOLATIONS OF BELL’S INEQUALITIES

At the end of the previous section, we were able to characterize the polytopes of synchronous nonsignaling and classical strategies from \{0,1,2\} to \{0,1\}, and hence obtain Bell inequalities \[5\] for synchronous strategies. In this section we give concrete strategies that violate these inequalities. First we note that among nonsignaling correlations the Bell inequalities can be violated by a magnitude of at most \(\frac{1}{2}\), but only one inequality at a time.

**Proposition 17.** Every synchronous nonsignaling strategy satisfies \(J_0 \leq \frac{5}{8}\) and \(J_1, J_2, J_3 \geq -\frac{3}{8}\). However no individual correlation can violate more that one of the inequalities \(J_0 \leq 1\) and \(J_1, J_2, J_3 \geq 0\).

**Proof.** Enumerating all 80 vertices of the nonsignaling polytope is easily accomplished with standard mathematical software, again for example \[51\]. One discovers that there are 32 vertices that exhibit a violation; eight of these have \((1 - J_0, J_1, J_2, J_3) = (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})\) and similarly eight each with \((\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2})\), and \((\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})\). Consider a general synchronous nonsignaling strategy that exhibits a violation of one of the Bell inequalities \[10\]. The argument does not depend on the form of the \(J\) and so without loss of generality suppose \(1 - J_0 < 0\). Every synchronous nonsignaling strategy is a convex sum of the \(8\) extreme strategies; write \(c_j\) for the sum of the \(8\) coefficients in such a convex sum that involve the \(8\) vertices producing a \(J_j\)-violation. Discarding the contributions of the other \(48\) vertices, we have

\[-\frac{1}{2}c_0 + \frac{1}{2}(c_1 + c_2 + c_3) \leq 1 - J_0 < 0.\]

Consequently \(c_0 > c_1 + c_2 + c_3\). But then \(c_0 > c_1\) and so

\[0 < -\frac{1}{2}c_1 + \frac{1}{2}(c_0 + c_2 + c_3) \leq J_1.\]

Similarly \(J_2, J_3 > 0\) as well. \(\square\)

When restricted to synchronous quantum correlations, such inequalities are typically called Tsir'lis bounds after his seminal work on Bell’s inequality \[52\]. As quantum correlations are nonsignaling, the above proposition shows that only one inequality can be violated for any given correlation. Our analogue of Tsir’lis’ bound is that this violation can have magnitude at most \(\frac{1}{8}\).

**Theorem 18.** Every synchronous quantum correlation satisfies \(J_0 \leq \frac{9}{8}\) and \(J_1, J_2, J_3 \geq -\frac{1}{8}\). However no individual correlation can violate more than one of the inequalities \(J_0 \leq 1\) and \(J_1, J_2, J_3 \geq 0\).

**Proof.** As already indicated, the second claim follows from the fact that quantum correlations are nonsignaling and the proposition above. To prove the stated bounds, we note the space of quantum strategies is convex, and each \(J_j\) is linear, the extreme values will occur at extremal quantum strategies. By Theorem \[10\] these take the form

\[p(b_A, b_B \mid x_A, x_B) = \frac{1}{d} \text{tr}(E_{b_A}^x E_{b_B}^y),\]

for projection-valued measures \(\{E_x^0, E_x^1\}_{x=0,1,2}\) on \(\mathcal{H} = \mathbb{C}^d\). Define the \pm 1-valued observables \(M_x = E_0^x - E_1^x\), and traces

\[m_x = \frac{1}{d} \text{tr}(M_x)\] and \(m_{x\Lambda x\Lambda B} = \frac{1}{d} \text{tr}(M_{x\Lambda} M_{x\Lambda B}).\]

In particular,

\[m_x = p(0, 0 \mid x, x) - p(1, 1 \mid x, x),\] and

\[m_{x\Lambda x\Lambda B} = p(0, 0 \mid x_A, x_B) + p(1, 1 \mid x_A, x_B) - p(0, 1 \mid x_A, x_B) - p(1, 0 \mid x_A, x_B).\]

Then our correlation in these coordinates has

\[p(0, 0 \mid x_A, x_B) = \frac{1}{4}(1 + m_{x_A} + m_{x_B} + m_{x_A, x_B})\]

\[p(0, 1 \mid x_A, x_B) = \frac{1}{4}(1 + m_{x_A} - m_{x_B} - m_{x_A, x_B})\]

\[p(1, 0 \mid x_A, x_B) = \frac{1}{4}(1 - m_{x_A} + m_{x_B} - m_{x_A, x_B})\]

\[p(1, 1 \mid x_A, x_B) = \frac{1}{4}(1 - m_{x_A} - m_{x_B} + m_{x_A, x_B})\]

and we compute

\[1 - J_0 = \frac{1}{4} (1 + m_{01} + m_{02} + m_{12})\]

\[J_1 = \frac{1}{4} (1 - m_{01} - m_{02} + m_{12})\]

\[J_2 = \frac{1}{4} (1 - m_{01} + m_{02} - m_{12})\]

\[J_3 = \frac{1}{4} (1 + m_{01} - m_{02} - m_{12}).\]

Now,

\[\frac{1}{d} \text{tr}((M_0 + M_1 + M_2)^2)\]

\[= \frac{1}{d} \left[ \text{tr}(M_0^2) + \text{tr}(M_1^2) + \text{tr}(M_2^2) + 2\text{tr}(M_0 M_1) + 2\text{tr}(M_0 M_2) + 2\text{tr}(M_1 M_2) \right]\]

\[= 3 + 2(m_{01} + m_{02} + m_{12})\]

\[= 1 + 8(1 - J_0).\]
Therefore
\[
1 - J_0 = -\frac{1}{8} + \frac{1}{8d} \text{tr}((M_0 + M_1 + M_2)^2) \geq -\frac{1}{8}.
\]

Identical bound for \(J_1\), \(J_2\), and \(J_3\) can be derived from \(\text{tr}((-M_0 + M_1 + M_2)^2)\), \(\text{tr}((M_0 - M_1 + M_2)^2)\), and \(\text{tr}((M_0 + M_1 - M_2)^2)\) respectively.

**Theorem 19.** There exists synchronous quantum correlations from \(\{0, 1, 2\}\) to \(\{0, 1\}\) with \(H_A = H_B = \mathbb{C}^2\) that saturate the bounds of the previous theorem. Moreover each of these are achieved by a unique correlation.

**Proof.** Take a basis of \(\mathbb{C}^2\) so that \(E_1^0 = |1\rangle\langle 1|\). Write \(E_1^1 = |\phi_1\rangle\langle \phi_1|\) and \(E_2^2 = |\phi_2\rangle\langle \phi_2|\) where in Bloch sphere coordinates
\[
|\phi_1\rangle = \cos \alpha |0\rangle + e^{i\theta} \sin \alpha |1\rangle,
|\phi_2\rangle = \cos \gamma |0\rangle + e^{i\delta} \sin \gamma |1\rangle.
\]

As
\[
p(b_A, b_B | x_A, x_B) = \frac{1}{2} \text{tr}(E_{b_A}^x E_{b_B}^y) = \frac{1}{2} \text{tr}(U^\dagger E_{b_A}^x U U^\dagger E_{b_B}^y U),
\]
we may simultaneously conjugate these projection-valued by a global unitary. In particular \(U = e^{i\beta |\alpha, \gamma|/2}\) leaves \(E_1^0\) invariant and applies a phase shift by \(-\beta\) to \(E_1^1\) and \(E_2^2\). As \(\delta\) was arbitrary, we simply incorporate this shift into the definition of \(\delta\) and so may take
\[
|\phi_1\rangle = \cos \alpha |0\rangle + \sin \alpha |1\rangle,
|\phi_2\rangle = \cos \gamma |0\rangle + e^{i\delta} \sin \gamma |1\rangle.
\]

Now it is straightforward to compute each of the coordinates \(w_j\) in terms of the above. For example,
\[
w_1 = \frac{1}{2} \text{tr}(E_1^0 E_1^1) = \frac{1}{2} |\langle 1 | \phi_1 \rangle|^2 = \frac{1}{2} \sin^2 \alpha.
\]

From this we find
\[
1 - J_0 = \cos \alpha \cos \gamma \cos \delta \sin \alpha \sin \gamma + \sin^2 \alpha \sin^2 \gamma,
J_1 = \cos \alpha \cos \gamma \cos \delta \sin \alpha \sin \gamma + \cos^2 \alpha \cos^2 \gamma,
J_2 = -\cos \alpha \cos \gamma \cos \delta \sin \alpha \sin \gamma + \cos^2 \alpha \sin^2 \gamma,
J_3 = -\cos \alpha \cos \gamma \cos \delta \sin \alpha \sin \gamma + \sin^2 \alpha \cos^2 \gamma.
\]

Substituting \(\alpha = \frac{\pi - \theta}{2}\) and \(\gamma = \frac{\pi - \delta}{2}\) produces
\[
1 - J_0 = \frac{\cos \delta}{4} (\cos^2 \sigma - \cos^2 \rho)
+ \frac{\cos^2 \sigma + \cos^2 \rho}{4} - \frac{\cos \rho \cos \sigma}{2},
J_1 = \frac{\cos \delta}{4} (\cos^2 \sigma - \cos^2 \rho)
+ \frac{\cos^2 \sigma + \cos^2 \rho}{4} - \frac{\cos \rho \cos \sigma}{2},
J_2 = \frac{\cos \delta}{4} (\cos^2 \rho - \cos^2 \sigma)
+ \frac{\sin^2 \rho + \sin^2 \sigma}{4} - \frac{\sin \rho \sin \sigma}{2},
J_3 = \frac{\cos \delta}{4} (\cos^2 \rho - \cos^2 \sigma)
+ \frac{\sin^2 \rho + \sin^2 \sigma}{4} - \frac{\sin \rho \sin \sigma}{2}.
\]

Our goal is to verify \(-\frac{1}{8}\) is the lower bound for each of these expressions. For concreteness we focus on \(1 - J_0\). There two cases to consider:

1. \(\cos^2 \rho > \cos^2 \sigma\), for which the minimum occurs when \(\delta = 0\); and,
2. \(\cos^2 \rho < \cos^2 \sigma\), for which the minimum occurs when \(\delta = \pi\).

Note that when \(\cos^2 \rho = \cos^2 \sigma\) we have \(1 - J_0 \geq 0\) which produces no Bell violation. When \(\delta = 0\) we find
\[
(1 - J_0)_{|\delta = 0} = \frac{1}{2} (\cos^2 \sigma - \cos \rho \cos \sigma).
\]

As a quadratic in \(\cos \sigma\) its minimum occurs at \(\cos \sigma = \frac{1}{2} \cos \rho\), which is consistent with the constraint \(\cos^2 \rho > \cos^2 \sigma\). The minimum is then
\[
(1 - J_0)_{|\sigma = \cos^{-1}(\frac{1}{2} \cos \rho), \delta = 0} = -\frac{1}{8} \cos^2 \rho.
\]

This is minimized at \(\rho = 0, \pi\) with value \(-\frac{1}{8}\) as desired.

Similarly in the case \(\cos^2 \rho < \cos^2 \sigma\) with \(\delta = \pi\), we find the minima of \(1 - J_0\) are at \(\sigma = 0, \pi\) with \(\cos \rho = \frac{1}{2} \cos \sigma\), also taking value \(-\frac{1}{8}\). And so while we have produce a number of potential solutions,
\[
(\rho, \sigma, \delta) = (0, \pm \frac{\pi}{3}, 0), (\pi, \pm \frac{2\pi}{3}, 0),
(\pm \frac{\pi}{9}, 0, \pi), \text{ or } (\pm \frac{2\pi}{3}, \pi, \pi),
\]
and while these can lead to different expressions for \(|\phi_1\rangle\) and \(|\phi_2\rangle\), they all produce the same stochastic...
matrix. For example

$\begin{align*}
|\phi_0\rangle &= |1\rangle \\
|\phi_1\rangle &= \frac{\sqrt{2}}{2} |0\rangle + \frac{1}{2} |1\rangle \\
|\phi_2\rangle &= \frac{\sqrt{2}}{2} |0\rangle - \frac{1}{2} |1\rangle.
\end{align*}$

produces

$P_0 = \frac{1}{8} \begin{pmatrix} 4 & 1 & 1 & 1 & 4 & 1 & 1 & 1 & 4 \\
0 & 3 & 3 & 3 & 0 & 3 & 3 & 3 & 0 \\
0 & 3 & 3 & 3 & 0 & 3 & 3 & 3 & 0 \\
4 & 1 & 1 & 1 & 4 & 1 & 1 & 1 & 4 \end{pmatrix}.$

Strictly as an aside, $P_0$ has $J_1 = J_2 = J_3 = \frac{3}{8}$.

The analysis of the other three cases is similar, and so left to the reader. We provide the final matrices each saturating strategy, with an example of the measurements that realize them: for $J_1 = -\frac{1}{8}$ we find

$P_1 = \frac{1}{8} \begin{pmatrix} 4 & 3 & 3 & 3 & 4 & 1 & 3 & 1 & 4 \\
0 & 1 & 1 & 1 & 0 & 3 & 1 & 3 & 0 \\
0 & 1 & 1 & 1 & 0 & 3 & 1 & 3 & 0 \\
4 & 3 & 3 & 3 & 4 & 1 & 3 & 1 & 4 \end{pmatrix}$

is obtained from

$\begin{align*}
|\phi_0\rangle &= |1\rangle \\
|\phi_1\rangle &= \frac{1}{2} |0\rangle + \frac{\sqrt{2}}{2} |1\rangle \\
|\phi_2\rangle &= \frac{1}{2} |0\rangle - \frac{\sqrt{2}}{2} |1\rangle;
\end{align*}$

for $J_2 = -\frac{1}{8}$ we find

$P_2 = \frac{1}{8} \begin{pmatrix} 4 & 3 & 1 & 3 & 4 & 1 & 3 & 3 & 4 \\
0 & 1 & 3 & 1 & 0 & 1 & 3 & 1 & 0 \\
0 & 1 & 3 & 1 & 0 & 1 & 3 & 1 & 0 \\
4 & 3 & 1 & 3 & 4 & 1 & 3 & 1 & 4 \end{pmatrix}$

is obtained from

$\begin{align*}
|\phi_0\rangle &= |1\rangle \\
|\phi_1\rangle &= \frac{1}{2} |0\rangle + \frac{\sqrt{3}}{2} |1\rangle \\
|\phi_2\rangle &= \frac{1}{2} |0\rangle + \frac{\sqrt{3}}{2} |1\rangle;
\end{align*}$

for $J_3 = -\frac{1}{8}$ we find

$P_3 = \frac{1}{8} \begin{pmatrix} 4 & 1 & 3 & 1 & 4 & 3 & 3 & 3 & 4 \\
0 & 3 & 1 & 3 & 0 & 1 & 1 & 1 & 0 \\
0 & 3 & 1 & 3 & 0 & 1 & 1 & 1 & 0 \\
4 & 1 & 3 & 1 & 4 & 3 & 3 & 3 & 4 \end{pmatrix}$

is obtained from

$\begin{align*}
|\phi_0\rangle &= |1\rangle \\
|\phi_1\rangle &= \frac{\sqrt{3}}{2} |0\rangle + \frac{1}{2} |1\rangle \\
|\phi_2\rangle &= \frac{\sqrt{3}}{2} |0\rangle + \frac{1}{2} |1\rangle.
\end{align*}$

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