Analytic aspects of exceptional Hermite polynomials and associated minimal surfaces

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Abstract

The main aim of this paper is to construct minimal surfaces associated with exceptional Hermite polynomials. That is, we derive an ordinary differential equation associated with a specific family of exceptional polynomials of codimension two. We show that these polynomials can be expressed in terms of classical Hermite polynomials. Based on this fact, we demonstrate that there exists a link between the norm of an exceptional Hermite polynomial and the gap sequence arising from the partition used to construct this polynomial. We find the analytical general solution of the exceptional Hermite differential equation which has no gap in its spectrum. We show that the spectrum is completed by non-polynomial solutions, and that the exceptional Hermite polynomials of odd and even degrees correspond respectively to the odd and even powers of a series that is part of the general solution. The paper contains a detailed exposition of the theory of exceptional Hermite polynomials as well as an exposition of the main tool used to construct their associated minimal surfaces obtained through the Enneper-Weierstrass immersion formula. We present 3D numerical displays of these surfaces. Several examples of minimal surfaces are included as illustrations of the presented theoretical results.

Keywords: Exceptional orthogonal polynomial, integrable system, minimal surface, Enneper-Weierstrass immersion formula.

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1 Introduction

Over the last decade, the problem of exceptional orthogonal polynomials (XOPs) has generated a great deal of interest and activity in several areas of mathematics and physics. Most of these activities focused on Jacobi and Laguerre XOPs (see e.g. [5, 6, 16, 17, 20, 25, 29, 33–35] and references therein), Hermite XOPs (see e.g. [7, 14, 15, 18, 19, 24]) and multi-indexed orthogonal polynomials (see e.g. [28, 30–32] and references therein). XOPs have been shown to play an essential role in several branches of physics, mostly related to the quantum harmonic oscillator. In particular, Jacobi XOPs were applied to the description of the Kepler-Coulomb quantum model in [22] and were seen as having an electrostatic interpretation in [10]. Moreover, Hermite XOPs were applied to the description of coherent states in [21].

The results obtained in [14] for the construction of Hermite XOPs were so promising that it seems to be worthwhile to study a particular case of Hermite XOPs corresponding to a codimension of two by an analytical method. We demonstrate the links between Hermite XOPs and minimal surfaces immersed in the su(2) Lie algebra. For this purpose, it is convenient to write the structural equations of these surfaces in terms of the moving frame using $2 \times 2$ matrices. In particular, in [9], such a description was applied to the investigation of classical orthogonal polynomials [36] and their link with special classes of minimal surfaces associated with the Hermite, Bessel, Chebyshev, Legendre, Laguerre, Gegenbauer and Jacobi polynomials.

In this paper, we examine certain aspects of Hermite XOPs for which we fix the partition. We set the partition defining a family of polynomials of codimension two, corresponding to the gap sequence in the spectrum of the exceptional Hermite differential operator. We show that setting the partition to a specific value allows us to determine the ordinary differential equation (ODE) whose general solution includes the Hermite XOPs and to determine the explicit form of the minimal surfaces associated with these polynomials. The methodological approach assumed in this paper is based on the general solution of such ODEs describing orthogonal polynomials as presented in [9, 11]. It allows us to identify a specific ODE describing XOPs corresponding to the reduced linear problem associated with minimal surfaces. The main idea is to investigate Hermite XOPs as the solutions of the linear problem for the moving frame. These polynomials are introduced in the linear problem, resulting in a moving frame directly determined by Hermite XOPs. The resolution of the linear problem then leads to the explicit computation of the associated Enneper-Weierstrass formula for the immersion of minimal surfaces in the 3-dimensional Euclidean space $\mathbb{E}^3$. This is, in short, the aim of this paper.

The paper is organized as follows. In section 2, we recall some basic notions and definitions on the theory of Hermite XOPs. In section 3, we use the theory of Hermite XOPs to study a family of exceptional polynomials of codimension two, for a fixed partition. We then study a formulation of Hermite XOPs in terms of classical Hermite polynomials. The orthogonality relation is presented. From the Hermite exceptional differential operator, we derive an ODE associated with a family of codimension two for this partition. We present a fundamental system for this ODE composed of analytic solutions obtained by the method of generalized series. We study the dependence link between Hermite XOPs of codimension two and these new solutions. In section 4, we construct minimal surfaces associated with Hermite XOPs of codimension two, benefiting from the link between the Enneper-Weierstrass representation of surfaces and the linear problem associated with the moving frame. We first present the immersion formula for minimal surfaces and associate it with a matrix representation in the su(2) algebra. We then apply a method for the reduction of the linear problem by a gauge transformation, which leads to a second-order linear ODE. This ODE is identified with the ODE associated with the Hermite XOPs of codimension two under consideration. We determine the arbitrary functions of the Enneper-Weierstrass representation, which lead to the explicit form of the minimal surfaces in the Euclidean space $\mathbb{E}^3$. These results are illustrated by numerical representations for different values of the parameter of the exceptional Hermite ODE. In Appendices A and B, we present some proofs by induction related to the general solution of the exceptional Hermite ODE.
2 Exceptional Hermite polynomials

2.1 Sturm-Liouville operator in terms of Hermite polynomials

To make the paper self-contained, we present in this section some known results concerning Hermite XOPs which are relevant for our purposes. A summary of recent developments on this subject can be found in the work of Gómez-Ullate et al [14]. As a starting point, consider the Sturm-Liouville problem

\[ L\psi = \lambda \psi \]  \hspace{1cm} (1)

for the Schrödinger operator possessing a potential \( U(z) \)

\[ L = -\frac{d^2}{dz^2} + U(z). \]  \hspace{1cm} (2)

If the operator (2) is without monodromy, then the potential is of the form [27]

\[ U(z) = -2\frac{d^2}{dz^2}\log Wr(H_{k_1}, H_{k_2}, ..., H_{k_n}) + z^2. \]  \hspace{1cm} (3)

In this context, \( \{k_i\}^n_{i=1} \) is a strictly increasing sequence of positive integers and \( H_n(z) \) is the \( n^{th} \) classical Hermite polynomial, which can be described by the Rodrigues formula [1]

\[ H_n(z) = (-1)^n e^{z^2} \frac{d^n}{dz^n} e^{-z^2}. \]  \hspace{1cm} (4)

The potential (3) is rational and has singularities corresponding to the zeros of the Wronskian

\[ Wr(H_{k_1}, H_{k_2}, ..., H_{k_n})(z). \]  \hspace{1cm} (5)

Zeros of Wronskians of Hermite polynomials have been studied in [13, 24]. The following theorem (formulated in a slightly different way in [14]) summarizes the results obtained by Krein [23] and Adler [2] concerning the zeros of the Wronskian of the eigenfunctions of the problem (1) in a more general context (general eigenfunctions). In our formulation, the sequence \( \{k_1, ..., k_n\} \) is expressed as a new sequence \( \{0, 1, ..., M_0', M_1, ..., M_1', ..., M_s, ..., M_s'\} \), in order to clarify its structure. Here, according to the notation used in [2], the symbol prime ’ denotes the biggest positive integer of the block \( \{M_l, ..., M_l'\} \), while \( M_l \) denotes the smallest positive integer of the same block.

**Theorem 1.** (Krein-Adler) Let \( \phi_j \) be the eigenfunctions of a pure-point Sturm-Liouville operator \( L = -\frac{d^2}{dx^2} + U \) defined on the real line

\[ L[\phi_j] = \lambda_j \phi_j, \quad j = 0, 1, 2, ..., \quad x \in (-\infty, \infty), \]  \hspace{1cm} (6)

with suitable boundary conditions. The Wronskian \( Wr(\phi_{k_1}, ..., \phi_{k_n}) \) has no zero on the real line if and only if the sequence of distinct positive integers \( \{k_1, ..., k_n\} \), when arranged in an ascending order, has the following structure

\[ \{0, 1, ..., M_0'\} \cup \{M_1, ..., M_1'\} \cup ... \cup \{M_s, ..., M_s'\} \]  \hspace{1cm} (7)

where \( M_l' + 1 < M_{l+1} \) for all \( l = 0, ..., s - 1 \). Here, the block \( \{0, 1, ..., M_0'\} \) may be absent and the blocks \( \{M_l, ..., M_l'\} \) consist of an even number of terms when \( l \geq 1 \).

Condition (7) means that the sequence is allowed to (but does not necessarily) begin with a sequence of arbitrary length, composed of consecutive positive integers starting with zero, followed by an arbitrary number of blocks of even length. The meaning of the inequality \( M_l' + 1 < M_{l+1} \) is that there is a gap greater or equal to 1 between the biggest positive integer \( M_l' \) of a block and the smallest positive integer \( M_{l+1} \) of the next block. These results are used to construct Hermite XOPs.
2.2 Construction of exceptional Hermite polynomials

The definition of Hermite XOPs begins with the choice of a partition \( \lambda = (\lambda_1, ..., \lambda_l) \) for a specific positive integer \( m \in \mathbb{N} \), which consists of a non-decreasing sequence

\[
0 \leq \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_l.
\]  

(8)

For a combinatorial interpretation of the partition associated with XOPs, see e.g. [5–7,13,15]. The sequence (8) is a partition of a unique positive integer \( m \) if

\[
m = \sum_{k=0}^{l} \lambda_k.
\]  

(9)

A sequence of type (8) determines a strictly increasing sequence called a gap sequence [18], of the form

\[
0 \leq k_1 < k_2 < ... < k_l,
\]  

(10)

where

\[
k_i = \lambda_i + i - 1.
\]  

(11)

**Definition 1.** A partition of length \( l \) is called a double partition if \( l \) is even and \( \lambda_{2i-1} = \lambda_{2i} \) for all \( i \).

From an arbitrary partition \( \lambda \), we define a double partition of length 2\( l \) by duplicating each term of the partition (8)

\[
\lambda^2 = (\lambda_1, \lambda_1, \lambda_2, \lambda_2, ..., \lambda_l, \lambda_l).
\]  

(12)

The sequence \( \{k_1, k_2, ..., k_n\} \) arising from the application of the relation (11) to any double partition of the form (12) respects the structure (7) of Theorem 1.

**Definition 2.** An Adler partition is either a double partition or a double partition preceded by a sequence of zeros of arbitrary length.

For each partition \( \lambda \), we consider the Wronskians

\[
H_{\lambda} := Wr(H_{k_1}, ..., H_{k_l}),
\]  

(13)

\[
H_{\lambda,n} := Wr(H_{k_1}, ..., H_{k_l}, H_n), \quad n \notin \{k_1, ..., k_l\}.
\]  

(14)

**Definition 3.** [14] For any double partition (12) of length 2\( l \), we define the \( X_{\lambda} \)-Hermite family of polynomials, denoted by \( \{H_{n}^{(\lambda)}\} \), as the following countable sequence

\[
H_{n}^{(\lambda)} = H_{\lambda^2,n}, \quad n \in \mathbb{N}\setminus\{k_1, k_1 + 1, ..., k_l, k_l + 1\}.
\]  

(15)

**Definition 4.** [17] Let \( \lambda = (\lambda_1, \lambda_2, ..., \lambda_l) \) be a partition of length \( l \). The positive integer

\[
2m := |\lambda^2| = \sum_{i=1}^{l} (\lambda_{2i-1} + \lambda_{2i}) = 2 \sum_{i=1}^{l} \lambda_{2i}
\]  

(16)

is called the codimension of the \( X_{\lambda} \)-Hermite family of polynomials.

It is known that Hermite XOPs exist only for even codimension 2\( m \) [14]. This is a consequence of Theorem 1 that leads to the choice of a double partition, as in equation (12). Hermite XOPs were applied to the description of coherent states in [21]. In what follows, the notation \( X_{2m}^{\lambda} \)-Hermite will be used to refer to the family of Hermite XOPs associated with a partition \( \lambda \) of some positive integer \( m \) with codimension 2\( m \).

From the Wronskian (14) and from Definition 3, we obtain that [14]

\[
\deg H_{n}^{(\lambda)}(x) = 2 \sum_{k=1}^{l} \lambda_k - 2l + n.
\]  

(17)

Equation (17) tells us that the degree of a polynomial of the \( X_{2m}^{\lambda} \)-Hermite family is \( n \) if and only if \( \lambda_1 > 0 \) and \( m = l \), i.e. the positive integer \( m \) is equal to the length of its partition, which leaves only one possibility for the \( m \)-components partition, namely \( \lambda = (1, 1, ..., 1) \).
2.3 Differential operator and orthogonality relation

We consider the classical Hermite differential operator

\[ T[y] := \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx}. \]  

(18)

The exceptional Hermite operator is obtained through the use of state-deleting Darboux-Crum transformations and intertwining relations, as in [14], and through the study of polynomial flags, as in [18]

\[ T_\lambda[y] := \frac{d^2 y}{dx^2} - 2 \left( x + \frac{H'_\lambda}{H_\lambda} \right) \frac{dy}{dx} + \left( \frac{H''_\lambda}{H_\lambda} + 2x \frac{H'_\lambda}{H_\lambda} \right) y, \]  

(19)

where the symbol prime ' denotes the derivative with respect to \( x \). Generally, the differential operator (19) will have singular rational coefficients for an arbitrary partition \( \lambda \). However, for an Adler partition \( \lambda^2 \), the operator \( T_\lambda^2 \) is non-singular on \( \mathbb{R} \) and is called the \( X_\lambda \)-Hermite operator or exceptional Hermite operator [14]. Oblomkov has studied regular singularities of the potential (3) in [27]. The following results were discussed in [14].

Proposition 1. [14] For every partition \( \lambda \), we have

\[ T_\lambda[H_\lambda,n] = 2(l - n)H_\lambda,n, \quad n \notin \{k_1, ..., k_l\} \]  

(20)

where \( l \) is the length of the partition.

Corollary 1. [14] The Hermite XOPs \( H_n^{(\lambda)} \) introduced in Definition 3 are eigenfunctions of the following second-order differential operator

\[ T_\lambda[H_n^{(\lambda)}] = 2(2l - n)H_n^{(\lambda)}, \quad n \in \mathbb{N}\{k_1, k_1 + 1, ..., k_l, k_l + 1\}, \]  

(21)

where \( T_\lambda \) is given by (19).

If we define the polynomial of degree \( l \)

\[ p_\lambda(x) := (x - k_1)(x - k_2)\cdots(x - k_l), \]  

(22)

we conclude that, for any double partition, \( p_\lambda(n) \geq 0 \forall n \in \mathbb{N} \). Proposition 2, which was formulated in [14,15], is true because Hermite XOPs are defined through an Adler partition \( \lambda^2 \), as in Definition 3.

Proposition 2. [14, 15] The Hermite XOPs \( H_n^{(\lambda)} \) satisfy the orthogonality relation

\[ \int_{-\infty}^{+\infty} H_m^{(\lambda)}(x)H_n^{(\lambda)}(x)W_{\lambda^2}(x)dx = \delta_{m,n}2^{n+2l}n!\sqrt{\pi}p_\lambda(n), \]  

(23)

where the orthogonality weight given by

\[ W_{\lambda^2}(x) = \frac{e^{-x^2}}{(H_{\lambda^2}(x))^2} \]  

(24)

is regular and positive definite.
3 General solution of the exceptional Hermite differential equation associated with the reduced double partition $\lambda = (1)$

In this section, we fix a partition $\lambda$ and use the theoretical results from section 2 to obtain the $X_{2m}^\lambda$-Hermite ODE associated with the chosen partition. We express the $X_{2m}^\lambda$-Hermite polynomials in terms of classical Hermite polynomials and we find the general solution of the $X_{2m}^\lambda$-Hermite ODE associated with the fixed partition $\lambda$.

3.1 Exceptional Hermite polynomials in terms of classical Hermite polynomials

The partition (8) can start with a sequence of zeros of arbitrary length. In what follows, we consider a reduced double partition, i.e. a partition for which $\lambda_1 > 0$ [14]. If we set $m = 1$, then the only possible reduced partition is $\lambda = (1)$. Therefore, $l = 1$ and $\lambda^2 = (1, 1)$ is a reduced double partition for which the associated strictly increasing sequence of length $2l$ is $\{k_1, k_2\}$ (the gap sequence). From relation (11), we obtain

\[
\begin{align*}
    k_1 &= \lambda_1 + 1 - 1 = 1, \\
    k_2 &= \lambda_2 + 2 - 1 = 2.
\end{align*}
\]

Due to Definition 4, the codimension of the family of polynomials which results from this choice of partition is $2m = 2$. We therefore consider the countable family of polynomials which constitutes the $X_2^{(1)}$-Hermite family

\[\{H_n^{(1)}(x) \mid n \in \mathbb{N}\setminus\{1, 2\}\}.
\]

From relation (17), we see that the degree of a polynomial of this family reduces to

\[\text{deg}H_n^{(1)}(x) = n \quad \forall n \in \mathbb{N}\setminus\{1, 2\}.
\]

The Wronskians defined in (13) and (14) become

\[
\begin{align*}
    H_{(1,1)}(x) &= Wr(H_1, H_2)(x) = 4(1 + 2x^2), \\
    H_n^{(1)}(x) &= H_{(1,1),n}(x) = Wr(H_1, H_2, H_n) = \begin{vmatrix} 2x & -2 + 4x^2 & H_n \\ 2 & 8x & H_n' \\ 0 & 8 & H_n'' \end{vmatrix}.
\end{align*}
\]

where $n \notin \{1, 2\}$. Under the above hypotheses, we obtain the following result.

**Theorem 2.** For the fixed partition $\lambda = (1)$, the polynomials (30) satisfy the relation

\[H_n^{(1)}(x) = 8(n-1)(n-2)\hat{H}_n(x), \quad \forall n \in \mathbb{N}\setminus\{1, 2\}
\]

where

\[\hat{H}_n(x) := H_n(x) + 4nH_{n-2}(x) + 4(n-3)H_{n-4}(x).
\]

**Proof.** Making use of the differential relation [1]

\[H_n'(x) = 2nH_{n-1}(x)
\]

and Definition 3, we find

\[H_n^{(1)} = H_{(1,1),n} = \begin{vmatrix} 2x & -2 + 4x^2 & H_n \\ 2 & 8x & H_n' \\ 0 & 8 & H_n'' \end{vmatrix} = 16 \left( H_n - 2nH_{n-1} + 2n(n-1)x^2H_{n-2} + n(n-1)H_{n-2} \right).
\]

Through successive applications of the recurrence relation [1]

\[2xH_{n+1}(x) = 2(n + 1)H_n(x) + H_{n+2}(x)
\]

to equation (34), we obtain

\[H_n^{(1)} = 8(n-1)(n-2)\left( H_n(x) + 4nH_{n-2}(x) + 4(n-3)H_{n-4}(x) \right).
\]
Remark 1. The function $\hat{H}_n(x)$ (32) is known from Cariñena et al [8], where we also find the analogue of the Rodrigues formula (4) adapted to relation (32)

$$\hat{H}_n(x) = (-1)^n e^{x^2} \left( \frac{d^n}{dx^n} + 4n \frac{d^{n-2}}{dx^{n-2}} + 4n(n-3) \frac{d^{n-4}}{dx^{n-4}} \right) e^{-x^2}. \quad (37)$$

3.2 Orthogonality relation of the $X_{2}^{(1)}$-Hermite polynomials

When $\lambda = (1)$, the polynomial (22) associated with the double partition $\lambda^2 = (1, 1)$ becomes

$$p_{(1,1)}(n) = (n-1)(n-2) \geq 0 \quad \forall \ n \in \mathbb{N}. \quad (38)$$

Replacing (29) in relation (24), the weight becomes

$$W_{(1,1)}(x) = e^{-x^2} \left( H_{(1,1)}^2(x) \right) \quad \forall x \in \mathbb{R}. \quad (39)$$

The orthogonality relation (23) becomes

$$\int_{-\infty}^{+\infty} H_{m}^{(1)}(x) H_{n}^{(1)}(x) \frac{e^{-x^2}}{(4(1+2x^2))^2} dx = \delta_{m,n} \sqrt{\pi} 2^{n+2} n!(n-1)(n-2), \quad (40)$$

where $m, n \in \mathbb{N}\{1, 2\}$. Equivalently, using Theorem 2, we find

$$\int_{-\infty}^{+\infty} \hat{H}_{m}(x) \hat{H}_{n}(x) \frac{e^{-x^2}}{(1+2x^2)^2} dx = \delta_{m,n} \frac{\sqrt{\pi} 2^n n!}{(n-1)(n-2)}, \quad (41)$$

which was shown independently in [8]. Because of an order relation between integrands, the integral (40) diverges if the integral (41) diverges. Indeed, for all $m = n \in \mathbb{N}\{1, 2\}$,

$$0 < 4^2 \hat{H}_{n}^2 < 4^3(n-1)^2(n-2)^2 \hat{H}_{n}^2. \quad (42)$$

Using Theorem 2, we find

$$0 < \hat{H}_{n}^2 < \left( \frac{H_{n}^{(1)}}{4} \right)^2, \quad (43)$$

and multiplying by $e^{-x^2}/(1+2x^2)^2$, we get

$$0 < \frac{\hat{H}_{n}^2 e^{-x^2}}{(1+2x^2)^2} < \left( \frac{H_{n}^{(1)}}{4(1+2x^2)^2} \right)^2 e^{-x^2}. \quad (44)$$

Integrating each side on the orthogonality interval $(-\infty, +\infty)$, we obtain

$$0 < \int_{-\infty}^{+\infty} \frac{\hat{H}_{n}^2 e^{-x^2}}{(1+2x^2)^2} dx < \int_{-\infty}^{+\infty} \left( \frac{H_{n}^{(1)}}{4(1+2x^2)^2} \right)^2 dx. \quad (45)$$

The norm of $X_{2}^{(1)}$-Hermite polynomials is therefore defined on $\mathbb{N}$ except for integer values which are zeros of the polynomial (38), namely $n = 1, 2$. For $\lambda = (1)$, these integer values correspond to the gap sequence (10).

Remark 2. It was proved in [14] that

$$\text{span} \left\{ H_{n}^{(1)}(x) \mid n \in \mathbb{N}\{1, 2\} \right\} \quad (46)$$

is dense in the Hilbert space $L^2(\mathbb{R}, W_{(1,1)}(x))$. 

\[\square\]
3.3 $X_2^{(1)}$-Hermite differential equation

Consider the first and second-order derivatives of the polynomial (29) obtained from the double partition $\lambda^2 = (1,1)$

\[
\begin{align*}
H_{(1,1)}(x) &= 4(1 + 2x^2), \\
H'_{(1,1)}(x) &= 16x, \\
H''_{(1,1)}(x) &= 16.
\end{align*}
\]

Making use of (47)-(49), equation (50) becomes

\[
H''_{\lambda^2,n} - 2 \left(x + \frac{H'_{\lambda^2}}{H_{\lambda^2}}\right) H'_{\lambda^2,n} + \left(\frac{H''_{\lambda^2}}{H_{\lambda^2}^2} + 2x H'_{\lambda^2}/H_{\lambda^2}\right) H_{\lambda^2,n} = (2l - 4n) H_{\lambda^2,n}.
\]

Making use of (47)-(49), equation (50) becomes

\[
H''_{(1,1),n} - 2 \left(x + \frac{16x}{4(1+2x^2)}\right) H'_{(1,1),n} + \left(\frac{16}{4(1+2x^2)} + 2x \frac{16x}{4(1+2x^2)}\right) H_{(1,1),n} = (2l - 4n) H_{(1,1),n},
\]

or equivalently,

\[
\left(H_n^{(1)}(x)\right)^{''} - 2 \left(x + \frac{4x}{1+2x^2}\right) \left(H_n^{(1)}(x)\right)' + 2n H_n^{(1)}(x) = 0.
\]

In other words, the polynomial $H_n^{(1)}(x)$ (30) is a solution of the second-order linear homogeneous ODE

\[
\omega''(x) - 2 \left(x + \frac{4x}{1+2x^2}\right) \omega'(x) + 2n\omega(x) = 0, \quad x \in \mathbb{R}, \quad n \in \mathbb{N}\setminus\{1,2\}.
\]

From this point on, we will refer to equation (53) as the $X_2^{(1)}$-Hermite ODE, which was presented in [26] as the exceptional Hermite differential equation. Consider the complex extension of the $X_2^{(1)}$-Hermite ODE (53)

\[
\omega''(z) - 2 \left(z + \frac{4z}{1+2z^2}\right) \omega'(z) + 2n\omega(z) = 0, \quad z \in \mathbb{C}.
\]

Equation (54) possesses three regular singular points at \{±i/√2, ∞\}. We use Fuchs Theorem, by setting

\[
p(z) = -2 \left(z + \frac{4z}{(z - \frac{i}{\sqrt{2}})(z + \frac{i}{\sqrt{2}})}\right), \quad q(z) = 2n.
\]

Let $B_δ(z_0)$ be the open ball of radius $δ = 2/√2$ around the point $z_0 \in \mathbb{C}$. Then the functions

\[
(z - i/\sqrt{2})p(z), \quad (z - i/\sqrt{2})^2q(z)
\]

are analytic on $B_δ(i/\sqrt{2})$ and the functions

\[
(z + i/\sqrt{2})p(z), \quad (z + i/\sqrt{2})^2q(z)
\]

are analytic on $B_δ(-i/\sqrt{2})$, which shows that \{±i/√2\} are regular singular points. We apply the Möbius transformation

\[
M : z \mapsto \zeta = \frac{1}{z - \frac{i}{\sqrt{2}}}
\]

to the ODE (54) and obtain

\[
\omega''(\zeta) - 2 \left(\frac{4i}{\sqrt{2}} \zeta^2 + 4\zeta + \frac{i}{\sqrt{2}} - \frac{1}{\zeta}\right) \omega'(\zeta) + 2n\omega(\zeta) = 0.
\]

We define the new coefficients

\[
\tilde{p}(\zeta) = -2 \left(\frac{4i}{\sqrt{2}} \zeta^2 + 4\zeta + \frac{i}{\sqrt{2}} + \frac{1}{\zeta}\right), \quad \tilde{q}(\zeta) = 2n.
\]

The functions $\zeta \tilde{p}(\zeta)$ and $\zeta^2 \tilde{q}(\zeta)$ are analytic on an arbitrary neighborhood of $z_0 = 0$, therefore \{∞\} is a regular singular point of the ODE (54).
3.4 Polynomial and non-polynomial solutions of the $X_{2}^{(1)}$-Hermite differential equation

In this section, we study the polynomial and non-polynomial solutions of the complex $X_{2}^{(1)}$-Hermite ODE (54). We find new solutions using the method of generalized series. We compare these solutions to the Hermite XOPs and we perform an extension of the classical Hermite polynomials to negative integers, leading to non-polynomial solutions.

**Corollary 2.** The function $\hat{H}_n(x)$ defined in (32) is a polynomial solution of the $X_{2}^{(1)}$-Hermite ODE (53) for all $n \in \mathbb{N}\backslash\{1,2\}$. Moreover, this solution and the polynomial $H_n^{(1)}(x)$ defined in (30) are linearly dependent for all $n \in \mathbb{N}\backslash\{1,2\}$.

**Proof.** Let $n \in \mathbb{N}\backslash\{1,2\}$. The proof is straightforward, considering that the polynomial $\hat{H}_n(x)$ is equal to the polynomial $H_n^{(1)}(x)$, up to a constant (which depends on $n$), by Theorem 2.

**Remark 3.** By Corollary 2, we know that the polynomials $H_n^{(1)}(z)$ and $\hat{H}_n(z)$ are linearly dependent solutions of the ODE (54) for $n \in \mathbb{N}\backslash\{1,2\}$. However, by the relation (30) and by Theorem 2, we see that $H_n^{(1)}(z)$ is a trivial solution of the ODE (54) for $n = 1, 2$. Performing the extension of the classical Hermite polynomials to negative integers, we notice that $\hat{H}_n(z)$ is a non-polynomial and nontrivial solution of the ODE (54) for $n = 1, 2$. Indeed, making use of the Rodrigues formula associated with classical Hermite polynomials (4), we define

\[
H_{-1}(z) := -e^{z^2} \int_{z}^{\infty} e^{-t^2} dt = \frac{\sqrt{\pi}}{2} e^{z^2} (1 - \text{erf}(z)),
\]

(61)

where erf$(z)$ is the Error function defined by [1]

\[
\text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_{0}^{z} e^{-t^2} dt.
\]

(62)

The non-polynomial extension to negative integers may then be established by the recurrence relation (35) and by substituting Definition (61) into relation (32). We find

\[
\hat{H}_1(z) = 4z + \sqrt{\pi} e^{z^2} (1 - 2e^{z^2}) (1 - \text{erf}(z)),
\]

\[
\hat{H}_2(z) = 2 + 4z^2 + 4\sqrt{\pi} ze^{z^2} (1 - \text{erf}(z)),
\]

(63)

which are non-polynomial and nontrivial solutions of the complex $X_{2}^{(1)}$-Hermite ODE (54) for $n = 1, 2$, such that the integrals

\[
\int_{-\infty}^{+\infty} \left(\hat{H}_1(x)\right)^2 \frac{e^{-x^2}}{(1+2x^2)^2} dx, \quad \int_{-\infty}^{+\infty} \left(\hat{H}_2(x)\right)^2 \frac{e^{-x^2}}{(1+2x^2)^2} dx,
\]

(64)

diverge. We recall that $\hat{H}_1(x)$ and $\hat{H}_2(x)$ are not part of the complete orthogonal polynomial system formed by the $X_{2}^{(1)}$-Hermite family of polynomials. However, when considering the non-polynomial extension of classical Hermite polynomials to negative integers, we can say that $\hat{H}_n(z)$ is a solution of the complex $X_{2}^{(1)}$-Hermite ODE (54) for all $n \in \mathbb{N}$.

**Remark 4.** We notice that the online application WolframAlpha does consider the extension of classical Hermite polynomials to negative integers.

**The method of generalized series** The variable coefficients $p(z)$ and $q(z)$ are analytic on $B_{\delta'}(0)$, where $\delta' = 1/\sqrt{2}$. Therefore, we can find at least one solution by the method of generalized series. Moreover, the principal part of the Laurent series around $z_0 = 0$ must vanish. Let

\[
\beta_n(z) = z^{\sigma_1} \sum_{k=0}^{\infty} c_k z^k, \quad \nu_n(z) = z^{\sigma_2} \sum_{k=0}^{\infty} \tilde{c}_k z^k,
\]

(65)
where \( \sigma_1 \) and \( \sigma_2 \) are the roots of the indicial equation associated with equation (54)

\[
\sigma(\sigma - 1) + a_0 \sigma + b_0 = 0.
\]  

(66)

By definition, we have

\[
a_0 = \lim_{z \to 0} z \cdot \left(-2z - \frac{8z}{1 + 2z^2}\right) = 0, \quad b_0 = \lim_{z \to 0} z^2 \cdot 2n = 0.
\]  

(67)

The indicial equation (66) reduces to

\[
\sigma(\sigma - 1) = 0,
\]

(68)

which possesses the roots \( \sigma_1 = 0 \) and \( \sigma_2 = 1 \). The two series (65) become

\[
\begin{align*}
\beta_n(z) &= \sum_{k=0}^{\infty} c_k z^k, \\
\nu_n(z) &= \sum_{k=0}^{\infty} \tilde{c}_k z^{k+1}.
\end{align*}
\]

(69)

(70)

**Case 1: root \( \sigma_1 = 0 \).** We substitute the series \( \beta_n(z) \) (69) and its derivatives up to order two into the ODE (54) and obtain

\[
(2c_2 + 2nc_0) + (6c_3 + 2(n - 5)c_1) z + \sum_{k=2}^{\infty} [(k + 2)(k + 1)c_{k+2} + 2(k(k - 6) + n)c_k + 4(n - k + 2)c_{k-2}] z^k = 0,
\]

(71)

where \( c_0 \) and \( c_1 \) are arbitrary constants. Let \( c_0 = c_1 = 1 \). Then the first coefficients take the form

\[
c_2 = -n, \quad c_3 = -\frac{1}{3} (n - 5), \quad c_4 = \frac{1}{6} n(n - 10),
\]

(72)

and we conclude that for all \( k \geq 4 \), the recurrence relation is as follows

\[
c_k = \frac{-2((k - 2)(k - 8) + n)c_{k-2} - 4(n - k + 4)c_{k-4}}{k(k - 1)}.
\]

(73)

The first even and odd coefficients are presented in Table 1.

| \( k \) | \( c_{2k} \) | \( c_{2k-1} \) |
|-------|------|------|
| 0     | 1    | -    |
| 1     | -n   | 1    |
| 2     | \( \frac{n^2 - 10n}{8} \) | \( \frac{-n - 5}{8} \) |
| 3     | \( \frac{n^3 - 30n^2 + 104n}{99} \) | \( \frac{n^2 - 20n + 51}{n^2 - 45n^2 + 311n - 555} \) |
| 4     | \( \frac{n^4 - 60n^3 + 524n^2 - 1200n}{2420} \) | \( \frac{n^3 - 80n^3 + 1046n^2 - 4720n + 6825}{n^2 - 80n^2 + 1046n^2 - 4720n + 6825} \) |
| 5     | \( \frac{n^5 - 100n^4 + 1580n^3 - 8720n^2 + 15744n}{113400} \) |

For \( k \geq 2 \), the denominators of the even coefficients \( c_{2k} \) from Table 1 take the form \( (2k)!/2^k \) while the denominators of the odd coefficients \( c_{2k-1} \) from Table 1 take the form \( (2k - 1)!/2^{k-1} \). This is due to the fact that the recurrence relation (73) may be written as

\[
c_k = -\frac{4((k - 2)(k - 8) + n)c_{k-2} - 2(n - k + 4)c_{k-4}}{2k(k - 1)}.
\]

(74)

The sign of the highest power of \( n \) appearing in the even and odd coefficients from Table 1 alternates. Therefore, the coefficients of the series \( \beta_n(z) \) (69) are of the form

\[
c_{2k} = (-1)^k \frac{p_k(n)}{(2k)!/2^k}, \quad c_{2k-1} = (-1)^{k+1} \frac{q_k(n)}{(2k - 1)!/2^{k-1}},
\]

(75)
where \( p_k(n) \) and \( q_k(n) \) are polynomials of the positive integer variable \( n \) of order \( k \) and \( k - 1 \), respectively. We denote by \( \lambda_p(k) \) and \( \lambda_q(k) \) the roots of the polynomials \( p_k(n) \) and \( q_k(n) \), respectively.

Table 2: Roots of the coefficients of the series \( \beta_n(z) \)

| \( k \) | \( \lambda_p(k) \) | \( \lambda_q(k) \) |
|-------|-----------------|-----------------|
| 0     | -               | -               |
| 1     | 0               | -               |
| 2     | 0, 10           | 5               |
| 3     | 0, 4, 26        | 3, 17           |
| 4     | 0, 4, 6, 50     | 3, 5, 37        |
| 5     | 0, 4, 6, 8, 82  | 3, 5, 7, 65     |

Table 2 shows the roots of the first terms associated with the series \( \beta_n(z) \) (69). For \( k \geq 3 \), the coefficients \( c_k \) (even and odd) have in particular as a root \( \lambda(k) = (k - 1)^2 + 1 \). These roots correspond to the positive integers in bold character. The coefficients \( c_{2k} \) then possess the factor

\[
(n - ((2k - 1)^2 + 1)), 
\]

while the coefficients \( c_{2k-1} \) possess the factor

\[
(n - (2(2k - 1)^2 + 1)). 
\]

If the coefficient is even, the remaining roots consist of a sequence of even positive integers \( 0, 4, 6, ..., k - 2 \), where the positive integer 2 is excluded. The even coefficients then possess the factors

\[
n \cdot \prod_{j=1}^{k-2} (n - 2(1 + j)). 
\]

If the coefficient is odd, the remaining roots consist of a sequence of odd positive integers \( 3, 5, 7, ..., k - 2 \), where the positive integer 1 is excluded. The odd coefficients then possess the factors

\[
\prod_{j=1}^{k-2} (n - 2(1 + j) + 1). 
\]

Fixing the positive integer \( n \) therefore truncates the series of even or odd coefficients, but not both, depending on the parity of \( n \). Since even and odd coefficients have no root in common, the series must be infinite. Moreover, the fact that the positive integers 1 and 2 are excluded indicates that the \( X_2^{(1)} \)-Hermite family of polynomials is defined on the spectrum \( \mathbb{N}\backslash\{1, 2\} \).

Making use of (76)-(79), we find

\[
p_2(n) = n(n - 10) 
\]

\[
p_k(n) = n(n - ((2k - 1)^2 + 1)) \prod_{j=1}^{k-2} (n - 2(1 + j)), \quad k \geq 3, 
\]

\[
q_2(n) = (n - 5) 
\]

\[
q_k(n) = (n - (2(2k - 1)^2 + 1)) \prod_{j=1}^{k-2} (n - 2(1 + j) + 1), \quad k \geq 3, 
\]

and taking (80) and (81) into account, the coefficients (75) become

\[
c_{2k} = (-1)^k \frac{n(n - ((2k - 1)^2 + 1)) \prod_{j=1}^{k-2} (n - 2(1 + j))}{(2k)!/2^k}, 
\]

\[
c_{2k-1} = (-1)^{k+1} \frac{(n - ((2(k - 1))^2 + 1)) \prod_{j=1}^{k-2} (n - 2(1 + j) + 1)}{(2k - 1)!/2^{k-1}}. 
\]
The series $\beta_n(z)$ (69) is therefore of the form
\[
\beta_n(z) = 1 + z - nz^2 - \frac{n-5}{3}z^3 + \frac{n(n-10)}{6}z^4
\]
\[+\sum_{k=3}^{\infty} \left[ (-1)^k \frac{n(n-((2k-1)^2+1)) \prod_{j=1}^{k-2}(n-2(1+j))}{(2k)!/2^k} z^{2k} \right.\]
\[\left. + (-1)^{k+1} \frac{n-((2(k-1)^2+1)) \prod_{j=1}^{k-2}(n-2(1+j)+1)}{(2k-1)!/2^{k-1}} z^{2k-1} \right].
\] (84)

The series $\beta_n(z)$ (84) converges. Indeed, if the series converges absolutely, then it may be written in terms of two separate series for the even and for the odd coefficients
\[
\beta_n(z) = 1 + z - nz^2 - \frac{n-5}{3}z^3 + \frac{n(n-10)}{6}z^4
\]
\[+\sum_{k=3}^{\infty} \left[ (-1)^k \frac{n(n-((2k-1)^2+1)) \prod_{j=1}^{k-2}(n-2(1+j))}{(2k)!/2^k} z^{2k} \right.\]
\[\left. + (-1)^{k+1} \frac{n-((2(k-1)^2+1)) \prod_{j=1}^{k-2}(n-2(1+j)+1)}{(2k-1)!/2^{k-1}} z^{2k-1} \right].
\] (85)

where the series (85) and (86) respectively have as a general term $c_{2k}$ (82) and $c_{2k-1}$ (83). We apply the D’Alembert ratio test and find
\[
\lim_{k \to \infty} \left| \frac{c_{2(k+1)}}{c_{2k}} \right| = \lim_{k \to \infty} \left| \frac{(-1)^{k+1} n(n-((2k+1)^2+1)) \prod_{j=1}^{(k+1)-2}(n-2(1+j))}{(2k)!/2^{k+1}} \right|
\]
\[
\left| \frac{(-1)^k n(n-((2k-1)^2+1)) \prod_{j=1}^{k-2}(n-2(1+j))}{(2k)!/2^k} \right|
\]
\[
= 2 \lim_{k \to \infty} \frac{n-((2k+1)^2+1)) \prod_{j=1}^{(k+1)-2}(n-2(1+j))}{(2k+1)(2k+2)(n-((2k-1)^2+1)) \prod_{j=1}^{k-2}(n-2(1+j))}
\]
\[
= 0 < 1,
\] (87)

\[
\lim_{k \to \infty} \left| \frac{c_{2k(k+1)-1}}{c_{2k-1}} \right| = \lim_{k \to \infty} \left| \frac{(-1)^{k+1} n-((2(k+1)-1)^2+1)) \prod_{j=1}^{(k+1)-1}(n-2(1+j)+1)}{(2k)!/2^{k+1}} \right|
\]
\[
\left| \frac{(-1)^k n-((2(k-1))^2+1)) \prod_{j=1}^{k-1}(n-2(1+j)+1)}{(2k-1)!/2^k} \right|
\]
\[
= \lim_{k \to \infty} \frac{n-((2(k+1)-1)^2+1)) \prod_{j=1}^{(k+1)-1}(n-2(1+j)+1)}{k(2k+1)(n-((2(k-1))^2+1)) \prod_{j=1}^{k-1}(n-2(1+j)+1)}
\]
\[
= 0 < 1.
\] (88)

We conclude that the series (85) and (86) converge. Moreover, these series converge absolutely, therefore the series $\beta_n(z)$ (84) converges.

**Case 2: root $\sigma_2 = 1$.** Consider the series $\nu_n(z)$ (70) associated with the root $\sigma_2$ of the indicial equation (66). Based on the above reasoning, we obtain
\[
\nu_n(z) = z - \frac{1}{3}(n-5)z^3 + \sum_{k=3}^{\infty} \left[ (-1)^{k+1} \frac{n-((2k-1)^2+1)) \prod_{j=1}^{k-2}(n-2(1+j)+1)}{(2k)!/2^{k-1}} z^{2k-1} \right].
\] (89)

which is a convergent series, by (88). The expansion of the series $\nu_n(z)$ (89) is finite for all values of $n$ which correspond to a root of the polynomial $q_k(n)$ (see the sequence $\lambda_q(k)$ in Table 2). The coefficients of the series $\nu_n(z)$ (89) correspond to the odd coefficients of the series $\beta_n(z)$ (84). For this reason, we will now define a notation that will be useful in what follows
\[
\mu_n(z) := 1 - nz^2 + \frac{n(n-10)}{6}z^4 + \sum_{k=3}^{\infty} \left[ (-1)^k \frac{n-((2k-1)^2+1)) \prod_{j=1}^{k-2}(n-2(1+j))}{(2k)!/2^k} z^{2k} \right],
\] (90)
so that the series $\beta_n(z)$ (84) may be rearranged as

$$\beta_n(z) = \mu_n(z) + \nu_n(z).$$  \hfill (91)

**Remark 5.** The series $\mu_n(z)$ (90) converges, by relation (87).

**Proposition 3.** The series $\beta_n(z)$ (84) is a non-polynomial solution of the complex $X_2^{(1)}$-Hermite ODE (54) for all $n \in \mathbb{N}$.

**Proof.** See Appendix A.

**Proposition 4.** The series $\mu_n(z)$ (90) is a polynomial solution of the complex $X_2^{(1)}$-Hermite ODE (54) for all $n \in 2\mathbb{N}\setminus\{2\}$, while the series $\nu_n(z)$ (84) is a polynomial solution of the complex $X_2^{(1)}$-Hermite ODE (54) for all $n \in (2\mathbb{N} - 1)\setminus\{1\}$.

**Proof.** See Appendix B.

We study the linear dependence relation between $\beta_n(z)$ and $\hat{H}_n(z)$, as well as the linear dependence relation between $\mu_n(z)$ and $H_n(z)$ and between $\nu_n(z)$ and $\hat{H}_n(z)$. We find

$$Wr\left(\hat{H}_n, \beta_n\right)(n, z) = \phi_1(n)e^{z^2}(1 + z^2)^2 = \phi_1(n)\frac{16}{W_{(1,1)}(z)},$$  \hfill (92)

for all $n \in \mathbb{N}\setminus\{1, 2\}$, where $|\phi_1| : \mathbb{N}\setminus\{1, 2\} \rightarrow \mathbb{N}\setminus\{0\}$ is a strictly increasing function. On the other hand, we find

$$Wr\left(\hat{H}_n, \mu_n\right)(z) = \begin{cases} \phi_2(n)e^{z^2}(1 + 2z^2)^2, & n \in (2\mathbb{N} - 1)\setminus\{1\}, \\ 0, & n \in (2\mathbb{N})\setminus\{2\}, \end{cases},$$  \hfill (93)

where $|\phi_2| : \mathbb{N}\setminus\{1, 2\} \rightarrow \mathbb{N}\setminus\{0\}$ is a strictly increasing function, and

$$Wr\left(\hat{H}_n, \nu_n\right)(z) = \begin{cases} 0, & n \in (2\mathbb{N} - 1)\setminus\{1\}, \\ \phi_3(n)e^{z^2}(1 + 2z^2)^2, & n \in 2\mathbb{N}\setminus\{2\}, \end{cases},$$  \hfill (94)

where $|\phi_3| : \mathbb{N}\setminus\{1, 2\} \rightarrow \mathbb{N}\setminus\{0\}$ is a strictly increasing function.

The numerators of the coefficients $c_{2k}$ (82) and $c_{2k-1}$ (83) have no factor in common, therefore the solution $\beta_n(z)$ (84) is non-polynomial for all values of $n$. Consider the solution $\nu_n(z)$ (89) and let

$$r_1(k) := (2(k - 1))^2 + 1, \quad k \geq 2,$$

$$r_2(j) := 2(1 + j) - 1, \quad 1 \leq j \leq k - 2.$$  \hfill (95, 96)

Then the solution $\nu_n(z)$ (89) may be written as

$$\nu_n(z) = z - \frac{1}{3}(n - r_1(2))z^3 + \sum_{k=3}^{\infty} \left(-1\right)^{k+1} \frac{(n - r_1(k)) \prod_{j=1}^{k-2} (n - r_2(j))}{(2k - 1)!/2^{k-1}} z^{2k-1}.$$  \hfill (97)

The roots of the polynomials $q_k(n)$ (81)

$$q_k(n) = (n - r_1(k)), \quad k = 2,$$

$$q_k(n)(n - r_1(k)) \prod_{j=1}^{k-2} (n - r_2(j)) \quad k \geq 3,$$  \hfill (98, 99)

are odd positive integers as shown in table 2

$$r_1(k) \in \{5, 17, 37, 65, 101, ...\},$$

$$r_2(j) \in \{3, 5, 7, ..., 2k - 3\}.$$  \hfill (100, 101)

Therefore, the solution $\nu_n(z)$ (89) is non-polynomial for all $n \in 2\mathbb{N} \cup \{1\}$, and polynomial for all odd values of $n$ except $n = 1$. The first polynomial cases of the solution $\nu_n(z)$ (89) are presented in Table 3.
the polynomial solution \( \nu \) to the roots \( \lambda \) is the polynomial associated with the fixed partition \( \lambda \) where

\[
(2k - 1) \text{ is the degree of the missing term.}
\]

Moreover, equation (94) means that when \( n = 2l - 1 \) for some positive integer \( l \geq 2 \), the series \( \nu_n(z) \) (89) is equal to the polynomial \( \hat{H}_n(z) \) (32), up to a constant, and we therefore see a part of the \( X_2^{(1)} \)-Hermite polynomials (corresponding to odd degrees \( 2l - 1 \geq 3 \)) arising from the construction of the solution of the ODE (54), using the method of generalized series.

**Theorem 3.** The solutions \( \hat{H}_n(z) \) (32) and \( \nu_n(z) \) (89) of equation (54) follow the proportionality relation

\[
\hat{H}_n(z) = M_1(n) \nu_n(z),
\]

for all \( n = 2l - 1, l \geq 2 \), where

\[
M_1(n) = \frac{(-1)^{(n+1)/2}n!2^{(n+1)/2}}{p_{(1,1)}(n) \prod_{j=1}^{(n-3)/2}(n - 2(1 + j) + 1)},
\]

where

\[
p_{(1,1)}(n) = (n - 1)(n - 2)
\]

is the polynomial associated with the fixed partition \( \lambda = (1) \) defined in (38), and where

\[
\prod_{j=1}^{0} \alpha(j) := 1.
\]
Proof. Let $n = 2l - 1$, $l \geq 2$, and let $\hat{H}_n(z)$ and $\nu_n(z)$ be the functions defined in (32) and (89), respectively. $\nu_n(z)$ is a polynomial of degree $n$. Indeed, by equations (28) and (31), we know that $\hat{H}_n(z)$ is a polynomial of degree $n$, and therefore equation (94) leads to the conclusion that $\nu_n(z)$ is also a polynomial of degree $n$. The missing term in the solution $\nu_n(z)$ illustrated in Table 3 corresponds to a power of $z$ that is always strictly smaller than $n$, because if there is a missing term, then $n = (2(k - 1))^2 + 1$ for some $k \geq 2$. Since $n = (2(k - 1))^2 + 1 > (2k - 1)$ for all $k \geq 2$, we conclude that the missing term is always associated with a power of $z$ smaller than $n$. The contrary would lead to a contradiction because $\nu_n(z)$ is a polynomial of degree $n$.

Let $\tilde{c}_n$ be the coefficient of the term $z^n$ in the polynomial $\nu_n(z)$. From equation (89), we have that for $k \geq 3$, the coefficients are defined by the rational expression

$$
\tilde{c}_k = (-1)^{k+1} \frac{(n - (2(k - 1))^2 + 1)) \prod_{j=1}^{k-2} (n - 2(1 + j) + 1)}{(2k - 1)!/2^{k-1}}. 
$$

The highest order term is of degree $n = 2k - 1$, which implies that

$$
k = \frac{n + 1}{2}. 
$$

Substituting the right-hand side of (110) into equation (109), we obtain

$$
\tilde{c}_n = (-1)^{(n+1)/2} p_{(1,1)}(n) 2^{(n-1)/2} \prod_{j=1}^{(n-3)/2} (n - 2(1 + j) + 1). 
$$

Let $\hat{c}_n$ be the coefficient of the term $z^n$ in the polynomial $\hat{H}_n(z)$. Then, by relation (32), we have that

$$
\hat{c}_n = 2^n. 
$$

Evaluating the ratio of the coefficients (111) and (112), we get

$$
M_1(n) = \frac{\hat{c}_n}{\tilde{c}_n} = \frac{(-1)^{(n+1)/2} p_{(1,1)}(n) 2^{(n-1)/2} \prod_{j=1}^{(n-3)/2} (n - 2(1 + j) + 1)}{n!}. 
$$

The only zeros of the denominator of $M_1(n)$ are the zeros of $p_{(1,1)}(n)$, namely the gap sequence $\{1, 2\}$, because

$$
\left\{ \prod_{j=1}^{(n-3)/2} (n - r_2(j)) : n = 3, 5, 7, \ldots \right\} = \{1, 2, 2 \cdot 4, 2 \cdot 4 \cdot 6, \ldots \}, 
$$

which means that $M_1(n)$ is not defined on the gap sequence. We complete the proof by verifying the case $k = 2$, i.e. $\hat{H}_3(z) = M_1(3) \nu_3(z)$ holds. Indeed,

$$
M_1(3) = 12, 
$$

which corresponds to the example given in (104).

□

From Theorem 3, we find that the function $\phi_2(n)$ in the Wronskian (93) corresponds to the additive inverse of the function $M_1(n)$ (106)

$$
Wr \left( \hat{H}_n, \mu_n \right)(z) = -M_1(n) e^{z^2 (1 + 2z^2)^2} \begin{cases} 1, & n \in (2N - 1) \setminus \{1\} \\ 0, & n \in (2N) \setminus \{2\} \end{cases}. 
$$

Moreover, Theorem 3 shed light on the fact that Hermite XOPs $\hat{H}_{2l-1}$ arise from the odd part of the series $\beta_n(z)$ (84). This motivates the search for Hermite XOPs $H_{2l}$ in the even part of the series $\beta_n(z)$ (84). Consider the series $\mu_n(z)$ (90) and let

$$
\begin{align*}
r_3(k) & : = (2k - 1)^2 + 1, & k \geq 2, \\
r_4(j) & : = 2(1 + j), & 1 \leq j \leq k - 2.
\end{align*}
$$
Then the series \( \mu_n(z) \) (90) may be written as

\[
\nu_n(z) = 1 - nz^2 + \frac{n(n - r_3(2))}{6} z^4 + \sum_{k=3}^{\infty} \left[ \frac{(-1)^k n(n - r_3(k)) \prod_{j=1}^{k-2} (n - r_4(j))}{(2k)!/2^k} z^{2k} \right].
\]  

(119)

The roots of the polynomials \( p_k(n) \) (80)

\[
p_k(n) = (n - r_3(k)), \quad k = 2,
\]

(120)

\[
p_k(n) = n(n - r_3(k)) \prod_{j=1}^{k-2} (n - r_4(j)), \quad k \geq 3,
\]

are even positive integers as shown in Table 2

\[
\begin{align*}
r_3(k) & \in \{10, 26, 50, 82, 122, \ldots\}, \\
r_4(j) & \in \{4, 6, 8, \ldots, 2k - 2\}.
\end{align*}
\]

(121)

(122)

Therefore, the series \( \mu_n(z) \) (90) is non-polynomial for all \( n \in (2N - 1) \cup \{2\} \), and polynomial for all even values of \( n \) except \( n = 2 \). The first polynomial cases of the series \( \mu_n(z) \) (90) are presented in Table 4.

Table 4: First polynomial cases of the series \( \mu_n(z) \)

| \( l \) | \( 2l \) | \( \mu_{2l}(z) \) |
|---|---|---|
| 0 | 0 | 1 |
| 2 | 4 | 1 - 4z^2 - 4z^4 |
| : | : | : |
| 4 | 8 | 1 - 8z^2 - \frac{8}{3} z^4 + \frac{32}{5} z^6 - \frac{16}{15} z^8 |
| 5 | 10 | 1 - 10z^2 + 0 \cdot z^4 + \frac{32}{3} z^6 - \frac{80}{21} z^8 + \frac{32}{105} z^{10} |
| 6 | 12 | 1 - 12z^2 + 4z^4 + \frac{224}{15} z^6 - \cdots - \frac{64}{945} z^{12} |
| : | : | : |
| 12 | 24 | 1 - 24z^2 + 56z^4 + \cdots - \frac{4096}{13749310575} z^{24} |
| 13 | 26 | 1 - 26z^2 + \frac{208}{3} z^4 + 0 \cdot z^6 - \cdots + \frac{8192}{516234433225} z^{26} |

Table 4 shows that for the values \( n = 2l \in \{10, 26, 50, \ldots\} \), one term is missing. These values correspond to the roots \( \lambda_p(k) \) in bold character from Table 2. This is due to the fact that in these cases, the degree of the polynomial \( \mu_n(z) \) (90) corresponds to a root of the polynomials \( p_k(n) \) (120), namely a value of the sequence (121), i.e.

\[
n \in \{r_3(k) \mid k \geq 2\} \subset \lambda_p(k),
\]

(123)

where \( (2k) \) is the degree of the missing term.

The constant term in the polynomials from Table 4 is normalized, because we made the arbitrary choice \( c_1 = 1 \) during the construction of the generalized series \( \beta_n(z) \) (69). We notice that for each \( n \in \{2l \mid l \geq 2\} \), there exists a proportionality constant that depends on \( l \), so that

\[
\hat{H}_{2l}(z) = M_2(l) \cdot \mu_{2l}(z), \quad l = 2, 3, \ldots,
\]

(124)

**Theorem 4.** The functions \( \hat{H}_n(z) \) (32) and \( \mu_n(z) \) (90) follow the proportionality relation

\[
\hat{H}_n(z) = M_2(n) \mu_n(z),
\]

(125)

for all \( n = 2l \), \( l \in \{0, 2, 3, 4, \ldots\} \), where

\[
M_2(n) = \frac{(-1)^{(n+2)/2} n! 2^{n/2}}{n \cdot p_{(1,1)}(n) \prod_{j=1}^{(n-4)/2} (n - 2(1 + j))},
\]

(126)

\[
M_2(0) := 1.
\]

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Proof. Let \( n = 2l \), \( l \geq 2 \), and let \( \hat{H}_n(z) \) and \( \mu_n(z) \) be the functions defined in \((32)\) and \((90)\), respectively. \( \mu_n(z) \) is a polynomial of degree \( n \). The missing term in the polynomial \( \mu_n(z) \) illustrated in Table 4 corresponds to a power of \( z \) that is always strictly smaller than \( n \), because if there is a missing term, then \( n = (2k-1)^2 + 1 \) for some \( k \geq 2 \). Since \( n = (2k-1)^2 + 1 > 2k \) for all \( k \geq 2 \), we conclude that the missing term is always associated with a power of \( z \) smaller than \( n \).

Let \( c_n \) be the coefficient of the term \( z^n \) in the polynomial \( \mu_n(z) \). From equation \((90)\), we have that for \( k \geq 3 \), the coefficients are defined by the rational expression

$$ c_k = (-1)^k \frac{n(n-((2k-1)^2+1)) \prod_{j=1}^{k-2}(n-2(1+j))}{(2k)!/2^k}. \quad (127) $$

The highest order term is of degree \( n = 2k \), which implies that

$$ k = n/2. \quad (128) $$

Substituting the right-hand side of \((128)\) into \((127)\), we obtain

$$ c_n = \frac{(-1)^{(n+1)/2}p_{(1,1)}(n)2^{(n-1)/2} \prod_{j=1}^{(n-3)/2}(n-2(1+j)+1)}{n!}. \quad (129) $$

Let \( \hat{c}_n \) be the coefficient of the term \( z^n \) in the polynomial \( \hat{H}_n(z) \). Then, by relation \((32)\), we have that

$$ \hat{c}_n = 2^{3n}. \quad (130) $$

Evaluating the ratio of the coefficients \((129)\) and \((130)\), we get

$$ M_2(n) = \frac{\hat{c}_n}{c_n} = \frac{(-1)^{(n+2)/2}n!2^{n/2}}{n \cdot p_{(1,1)}(n) \prod_{j=1}^{(n-4)/2}(n-2(1+j)+1)}. \quad (131) $$

The only zeros of the denominator of \( M_2(n) \) are the zeros of \( p_{(1,1)}(n) \) and zero itself, namely the gap sequence \( \{1,2\} \) and zero, because

$$ \left\{ \prod_{j=1}^{(n-4)/2} (n-r_4(j)) : n = 4, 6, 8, ... \right\} = \{1, 3, 3 \cdot 5, 3 \cdot 5 \cdot 7, ...\}, \quad (132) $$

which means that \( M_2(n) \) is not defined on the gap sequence, neither at \( n = 0 \), which is why we define this case separately in equations \((126)\). Remark 6 discusses another formulation of the constant \( M_2(n) \).

We complete the proof by verifying the case \( k = 2 \), i.e. \( \hat{H}_4(z) = M_2(4)\mu_4(z) \) holds. Indeed,

$$ \hat{H}_4(z) = M_2(4)\mu_4(z), \quad (133) $$

where \( M_2(4) = -4 \).

\( \square \)

Remark 6. Since \( n \) is even, \( M_2(n) \) can be equivalently expressed in term of the Euler gamma function \([1]\)

$$ M_2(n) = (-1)^{(n+2)/2}2^{n-1}n^{-1/2}\Gamma\left(\frac{n-1}{2}\right), \quad \forall n \in 2\mathbb{N}. \quad (134) $$

From Theorem 4, we find that the function \( \phi_3(n) \) in the Wronskian \((94)\) corresponds to the function \( M_2(n) \) \((126)\)

$$ Wr\left(\hat{H}_n, \nu_n\right)(z) = M_2(n)e^{-z^2}(1+2z^2)^2 \cdot \left\{ \begin{array}{ll} 0, & n \in (2\mathbb{N}-1)\setminus\{1\} \\ 1, & n \in 2\mathbb{N}\setminus\{2\} \end{array} \right. \quad (135) $$

From Theorems 3 and 4, we find that the function \( \phi_1(n) \) appearing in the Wronskian \((92)\) corresponds to the additive inverse of the function \( M_1(n) \) \((106)\) when \( n \) is odd, and to the function \( M_2(n) \) \((126)\) when \( n \) is even

$$ Wr\left(\hat{H}_n, \beta_n\right)(n, z) = e^{-z^2}(1 + z^2)^2 \cdot \left\{ \begin{array}{ll} -M_1(n), & n \in (2\mathbb{N}-1)\setminus\{1\} \\ M_2(n), & n \in 2\mathbb{N}\setminus\{2\} \end{array} \right. \quad (136) $$

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Corollary 3. The functions $H_n^{(1)}(z)$ (30), $\mu_n(z)$ (90) and $\nu_n(z)$ (89) follow the proportionality relations

$$H_n^{(1)}(z) = (-1)^{(n+1)/2}8n!2^{(n+1)/2}\nu_n(z) \prod_{j=1}^{(n-3)/2}(n-2(1+j)+1)$$ (137)

for all $n = 2l - 1$, $l \geq 2$,

$$H_n^{(1)}(z) = \frac{(-1)^{(n+2)/2}8n!2^{n/2}}{n!} \prod_{j=1}^{(n-4)/2}(n-2(1+j))\mu_n(z)$$ (138)

for all $n = 2l$, $l \geq 2$,

$$H_0^{(1)}(z) = 16\mu_0(z).$$ (139)

Proof. The proof is straightforward, making use of Theorems 2-4.

Corollary 4. The functions $\nu_n(z)$ (89) and $\mu_n(z)$ (90) follow the orthogonality relations

$$\int_{-\infty}^{\infty} \nu_m(x)\nu_n(x) \frac{e^{-x^2}}{(1 + 2x^2)^2} \, dx = \delta_{m,n} \frac{\sqrt{\pi}p(1,1)\prod_{j=1}^{(n-3)/2}(n-2(1+j)+1)}{2 \cdot n!}$$ (140)

for all $n = 2l - 1$, $l \geq 2$,

$$\int_{-\infty}^{\infty} \mu_m(x)\mu_n(x) \frac{e^{-x^2}}{(1 + 2x^2)^2} \, dx = \delta_{m,n} \frac{\sqrt{\pi}n \cdot p(1,1)\prod_{j=1}^{(n-4)/2}(n-2(1+j))}{(n-1)!}$$ (141)

for all $n = 2l$, $l \geq 2$,

$$\int_{-\infty}^{\infty} (\mu_0(x))^2 \frac{e^{-x^2}}{(1 + 2x^2)^2} \, dx = \frac{\sqrt{\pi}}{2}.$$ (142)

Proof. The orthogonality relation for $\hat{H}_n(x)$ is given by (41)

$$\int_{-\infty}^{\infty} \hat{H}_m(x)\hat{H}_n(x) \frac{e^{-x^2}}{(1 + 2x^2)^2} \, dx = \delta_{m,n} \frac{\sqrt{\pi}2^n n!}{(n-1)(n-2)}.$$ (143)

If $n = 2l - 1$, $l \geq 2$, making use of Theorem 3, then the orthogonality relation (143) becomes

$$M_1(m)M_1(n) \int_{-\infty}^{\infty} \nu_m(x)\nu_n(x) \frac{e^{-x^2}}{(1 + 2x^2)^2} \, dx = \delta_{m,n} \frac{\sqrt{\pi}2^n n!}{(n-1)(n-2)},$$ (144)

which implies that

$$\int_{-\infty}^{\infty} \nu_m(x)\nu_n(x) \frac{e^{-x^2}}{(1 + 2x^2)^2} \, dx = \delta_{m,n} \frac{\sqrt{\pi}p(1,1)\prod_{j=1}^{(n-3)/2}(n-2(1+j)+1)}{2 \cdot n!}.$$ (145)

The case $n = 0$ is easily verified. If $n = 2l$, $l \geq 2$, making use of Theorem 4, then the orthogonality relation (143) becomes

$$M_2(m)M_2(n) \int_{-\infty}^{\infty} \mu_m(x)\mu_n(x) \frac{e^{-x^2}}{(1 + 2x^2)^2} \, dx = \delta_{m,n} \frac{\sqrt{\pi}2^n n!}{(n-1)(n-2)},$$ (146)

which implies that

$$\int_{-\infty}^{\infty} \mu_m(x)\mu_n(x) \frac{e^{-x^2}}{(1 + 2x^2)^2} \, dx = \delta_{m,n} \frac{\sqrt{\pi}n \cdot p(1,1)\prod_{j=1}^{(n-4)/2}(n-2(1+j))}{(n-1)!}.$$ (147)
Remark 7. On the gap sequence, the functions $\hat{H}_1(z)$ and $\hat{H}_2(z)$ may be defined from the extension of classical Hermite polynomials to negative integers, as in equations (63), leading to non-polynomial solutions of the ODE (54). Theorems 3 and 4 state that there exists a proportionality constant between the polynomials $\hat{H}_{2l}(z)$ (32) and $\mu_{2l}(z)$ (90), where $l \in \{0, 2, 3, \ldots\}$ and between the polynomials $\hat{H}_{2l-1}(z)$ (32) and $\nu_{2l-1}(z)$ (89), where $l \in \{2, 3, 5, \ldots\}$. The same phenomenon holds between $H_2(z)$ and $\mu_2(z)$ and between $\hat{H}_1(z)$ and $\nu_1(z)$, namely on the gap sequence, that indicates that the extension of classical Hermite polynomials to negative integers arises naturally in the construction of the solution of the ODE (54).

We showed in the present section that the $X_2^{(1)}$-Hermite polynomials of even degree may be expressed as the even part of the series $\beta_n(z)$ (84), when $n$ is even, and that the $X_2^{(1)}$-Hermite polynomials of odd degree may be expressed as the odd part of the series $\beta_n(z)$ (84), when $n$ is odd. We would like to express the general solution of the complex $X_2^{(1)}$-Hermite ODE (54) as a linear combination of two separate functions, where the first function would include the $X_2^{(1)}$-Hermite polynomials together with the non-polynomial cases $n = 1, 2$ (the gap sequence), and where the second function would include non-polynomial solutions. Let $\hat{H}_1(z)$ and $\hat{H}_2(z)$ be defined as in (63). Then $\{\hat{H}_n(z)\}_{n=0}^\infty$ is a countable sequence of functions which includes the $X_2^{(1)}$-Hermite polynomials, up to a constant, and two non-polynomial solutions, namely $n = 1, 2$ (the gap sequence). Consider the proportionality constants $M_1(n)$ and $M_2(n)$ from Theorems 3 and 4, respectively, and let

$$M_3(n) := \begin{cases} 1, & n \in \{1, 2\} \\ M_1^{-1}(n), & n \in 2\mathbb{N} - 1 \{1\} \\ M_2^{-1}(n), & n \in 2\mathbb{N} \{2\} \end{cases}$$ (148)

We define

$$\alpha_n(z) := M_3(n)\hat{H}_n(z), \quad n \in \mathbb{N},$$ (149)

where the functions $\hat{H}_1(z)$ and $\hat{H}_2(z)$ are defined as in (63). The constant $M_3(n)$ in the definition (149) is not necessary. For $n \not\in \{1, 2\}$, the result is that the odd and even cases correspond exactly to the polynomial cases of $\nu_n(z)$ (89) and $\mu_n(z)$ (90), respectively, resulting in polynomials possessing a normalized coefficient associated with the smallest power of $z$, as shown in Tables 3 and 4.

Under the above hypotheses, we have the following theorem.

**Theorem 5. (Main result)** The general solution of the complex $X_2^{(1)}$-Hermite ODE (54) is

$$k_1\alpha_n(z) + k_2\beta_n(z), \quad k_1, k_2 \in \mathbb{C},$$ (150)

where the functions $\alpha(z)$ and $\beta(z)$ are defined as in (149) and (84), respectively. Moreover, the countable sequence

$$\{\alpha_n(z)\}_{n \in \mathbb{N} \\backslash \{1, 2\}}$$ (151)

corresponds to the exceptional Hermite orthogonal polynomials of codimension 2 associated with the partition $\lambda = (1)$ of the positive integer $m = 1$, up to a constant that depends on $n$, whereas the set

$$\{\alpha_n(z)\}_{n \in \{1, 2\}}$$ (152)

corresponds to non-polynomial solutions defined from the extension of classical Hermite polynomials to negative integers, which complete the gap $\{1, 2\}$ in the spectrum of the orthogonal polynomial system (151), but are not part of the orthogonal polynomial system (151) itself. The countable sequence

$$\{\beta_n(z)\}_{n \in \mathbb{N}}$$ (153)

is composed of non-polynomial functions.

**Proof.** Consider the definition of the function $\alpha_n(z)$ (149). By Corollary 2, we know that $\hat{H}_n(z)$ is a polynomial solution of the complex $X_2^{(1)}$-Hermite ODE (54) for all $n \in \mathbb{N} \\backslash \{1, 2\}$, which indicates that $\alpha_n(z)$ is a polynomial solution for these values of $n$. The non-polynomial cases $n = 1, 2$ are easily verified by substituting the expressions (63) and their derivatives up to order 2 into the ODE (54), when considering the extension of classical Hermite polynomials to negative integers. This information is confirmed making use of Theorems 3 and 4 and looking at the Wronskians

$$WR(\alpha_n, \mu_n)(z) = e^{z^2} (1 + 2z^2)^2 \cdot \begin{cases} -2, & n = 1 \\ 0, & n = 2 \\ -1, & n \in (2\mathbb{N} - 1) \{1\} \, , \\ 0, & n \in (2\mathbb{N}) \{2\} \end{cases}$$ (154)
\[ Wr(\alpha_n, \nu_n)(z) = e^{z^2}(1 + 2z^2)^2, \begin{cases} 0, & n = 1 \\ 2, & n = 2 \\ 0, & n \in (2N - 1) \setminus \{1\} \\ 1, & n \in 2N \setminus \{2\} \end{cases} \]  \tag{155}

showing that \( \{\hat{H}_l(z), \mu_2(z)\} \) and \( \{\hat{H}_{2l+1}(z), \nu_{2l+1}(z)\} \) are linearly dependent sets for all \( l \geq 0 \) (we recall that it was shown in Appendix B that \( \{\mu_2(z)\} \) and \( \{\nu_2(z)\} \) are solutions for all \( l \geq 0 \).

By Proposition 3, we know that \( \beta(z) \) is a non-polynomial solution of the ODE (54) for all \( n \in \mathbb{N} \). Moreover, the Wronskian

\[ Wr(\alpha_n, \beta_n)(n, z) = e^{z^2}(1 + z^2)^2, \begin{cases} (\sqrt{\pi} - 2), & n = 1 \\ 2(1 - 2\sqrt{\pi}), & n = 2 \\ -1, & n \in (2N - 1) \setminus \{1\} \\ 1, & n \in 2N \setminus \{2\} \end{cases} \]  \tag{156}

shows that \( \alpha_n(z) \) and \( \beta_n(z) \) are linearly independent functions.

Making use of the theoretical results from section 2, it was shown in detail in section 3 (by construction) that the countable sequence

\[ \{\alpha_n(z)\}_{n \in \mathbb{N} \setminus \{1, 2\}} \]  \tag{157}

corresponds to the exceptional Hermite orthogonal polynomials of codimension 2 associated with the partition \( \lambda = (1) \) of the positive integer \( m = 1 \), up to a constant that depends on \( n \). This was shown extensively in Theorems 2-4 and in Corollaries 2 and 3.

By the definition of \( \alpha_n(z) \) (149) and by equations (63), the set

\[ \{\alpha_n(z)\} \in \{1, 2\} \]  \tag{158}

corresponds to non-polynomial solutions defined from the extension of classical Hermite polynomials to negative integers.

The countable sequence

\[ \{\beta_n(z)\}_{n \in \mathbb{N}} \]  \tag{159}

is composed of non-polynomial functions. This is due to the form of the series \( \beta_n(z) \) (84), composed of coefficients \( c_{2k}(n) \) (82) associated with even powers of \( z \) which have no roots in common with the coefficients \( c_{2k-1}(n) \) (83) associated with odd powers of \( z \) (see the roots \( \lambda_p(k) \) and \( \lambda_q(k) \) in Table 2).

\[ \square \]

**Remark 8.** The linear combination (150) is the analytical general solution of the ODE (21) for \( n \in \mathbb{N} \) (no gap), on the complex plane, for the particular case \( \lambda = (1) \). Provided that \( \lambda^2 \) is an Adler partition, the differential operator \( T_{\lambda^2} \) is non-singular on \( \mathbb{R} \). We showed that in the case \( \lambda = (1) \), the operator possesses singularities at \( \pm i/\sqrt{2} \notin \mathbb{R} \). This is due to the fact that the operator \( T_\lambda \) (19) has singularities corresponding to the zeros of the Wronskian \( H_\lambda \) (13). Indeed, the general solution was built from an Adler partition and making use of the transformation (11), leading to a gap sequence which fulfills the hypotheses of the Krein-Adler Theorem 1, so the ODE arising from these choices has no singularity on \( \mathbb{R} \). Moreover, the general solution (150) was built using the method of generalized series, under the assumption that it could be expressed as a Taylor series around \( z = 0 \), resulting in an analytical function on \( \mathbb{C} \). The solution \( \alpha_n(z) \) (149) is non-polynomial on the gap sequence arising from the choice of the partition \( \lambda = (1) \). The mathematical expression of the coefficients of the series \( \beta_n(z) \) (84) constructed from the differential operator \( T_{(1, 1)} \) provides some clues explaining the existence of a gap in the eigenvalue spectrum of the differential operator. The coefficients \( c_{2k}(n) \) (82) and \( c_{2k-1}(n) \) (83) of the series \( \beta_n(z) \) (84), associated with even and odd powers of \( z \), respectively, are polynomials of the parameter \( n \), which possess no root on the gap sequence (see Table 2), leading to the non-polynomial solutions \( \nu_1(z) \) and \( \mu_2(z) \) for the values \( n = 1, 2 \).
4 Minimal surfaces associated with $X_2^{(1)}$-Hermite polynomials

In this section, we make use of the link between the classical Enneper-Weierstrass formula for the immersion of a minimal surface $F$ in the Euclidean space $E^3$ and the linear problem for the moving frame

$$\sigma = (\partial F, \partial F, N)^T$$

on the surface, where we used the notation for the holomorphic and antiholomorphic derivatives

$$\partial = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \bar{\partial} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right), \quad z = x + iy.$$

This link, expressed as a second-order linear ODE [11], allows us to investigate the behavior of Hermite XOPs represented by minimal surfaces. We calculate the explicit form of the immersion formula and we show a numerical display of these surfaces for different values of the parameter of the $X_2^{(1)}$-Hermite complex ODE (54).

4.1 Enneper-Weierstrass formula and $su(2)$ representation

Consider the Enneper-Weierstrass immersion formula [12, 37] describing a zero mean curvature surface (denoted by $H = 0$) in terms of two locally holomorphic arbitrary functions

$$F(\xi_0, \xi) = \frac{1}{2} \Re \left( \int_{\xi_0}^{\xi} (1 - \chi^2, i(1 + \chi^2), 2\chi)^T \eta^2 \, dz \right) \in E^3,$$

where

$$\bar{\partial} \eta = 0, \quad \bar{\partial} \chi = 0.$$

The integration in formula (162) is performed on an arbitrary path from the constant $\xi_0 \in \mathbb{C}$ to the complex variable $\xi \in \mathbb{C}\{\xi_0\}$.

Let $\tilde{F} \in su(2) \simeq E^3$ be the quaternionic description of the minimal surface described by formula (162). In order to determine the explicit form of this representation, we identify the Euclidean space $E^3$ with the imaginary quaternions by the formula [3]

$$\tilde{F} = -i \sum_{\alpha=1}^3 F_{\alpha} \sigma_{\alpha} \in Im\mathbb{H} \simeq su(2), \quad Tr(\tilde{F}) = 0, \quad \tilde{F}^\dagger = -\tilde{F},$$

where dagger $\dagger$ denotes the Hermitian conjugate of the considered expression. The matrices $\sigma_{\alpha}, \alpha = 1, 2, 3$ are the Pauli matrices, such that $\sigma_{\alpha}^\dagger = \sigma_{\alpha}$. The inner product is then

$$\langle X, Y \rangle = -\frac{1}{2} Tr(XY), \quad \forall X, Y \in su(2).$$

Substituting the components $F_{\alpha}, \alpha = 1, 2, 3$, of the Enneper-Weierstrass representation (162) into formula (164), we obtain a matrix formulation of the surface [9]

$$\tilde{F} = -\frac{i}{2} \begin{pmatrix}
\int_{\xi_0}^{\xi} \chi \eta^2 \, dz + \left( \int_{\xi_0}^{\xi} \chi \eta^2 \, dz \right)^* & \int_{\xi_0}^{\xi} \eta^2 \, dz - \left( \int_{\xi_0}^{\xi} \chi^2 \eta^2 \, dz \right)^* \\
- \int_{\xi_0}^{\xi} \chi \eta^2 \, dz + \left( \int_{\xi_0}^{\xi} \eta \eta^2 \, dz \right)^* & - \int_{\xi_0}^{\xi} \chi \eta^2 \, dz - \left( \int_{\xi_0}^{\xi} \chi \eta^2 \, dz \right)^*
\end{pmatrix},$$

where star $^*$ denotes the complex conjugate of the considered expression. The formula (166) for $\tilde{F}$ is a quaternionic representation of the surface immersed in the $su(2)$ Lie algebra, because $Tr(\tilde{F}) = 0, \tilde{F}^\dagger = -\tilde{F}$. 

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4.2 Holomorphic reduction of the linear problem for the moving frame

Making use of the Lie algebra isomorphism $\mathfrak{so}(3) \simeq \mathfrak{su}(2)$, the Gauss-Weingarten equations for the moving frame $\sigma$ (160) may be written in terms of $2 \times 2$ complex-valued matrices [3, 4], where the wavefunction $\Phi \in SU(2, \mathbb{C})$ satisfies the linear differential equations

$$\partial \Phi = U \Phi, \quad \overline{\partial} \Phi = V \Phi,$$

and where $U(z, \bar{z}), V(z, \bar{z}) \in \mathfrak{sl}(2, \mathbb{C})$. When the mean curvature vanishes ($H = 0$), the matrices $U$ and $V$ are of the form

$$U = \begin{pmatrix} \frac{1}{2} \partial u & -Qe^{-\frac{z}{2}} \\ 0 & -\frac{1}{2} \partial u \end{pmatrix}, \quad V = \begin{pmatrix} -\frac{1}{2} \partial u & 0 \\ Qe^{-\frac{z}{2}} & \frac{1}{2} \partial u \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{C}),$$

where $U^\dagger = -V$. We apply the gauge transformation $M$ to the wavefunction $\Phi \in SU(2, \mathbb{C})$ of the linear problem (167), as proposed in [11] and used afterwards in [9]

$$\Psi = M \Phi,$$

where

$$M = \begin{pmatrix} -\eta \chi & 0 \\ 0 & \eta \chi \end{pmatrix} \in SL(2, \mathbb{C}).$$

We obtain

$$\partial \Psi = \lambda \eta^2 \begin{pmatrix} \chi & -1 \\ \lambda^2 & -\chi \end{pmatrix} \Psi, \quad \overline{\partial} \Psi = 0,$$

where

$$\hat{U}(\lambda; z) = \lambda \eta^2 \begin{pmatrix} \chi & -1 \\ \lambda^2 & -\chi \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{C}).$$

The system (170) is a reduced linear problem for the holomorphic wavefunction $\Psi(z)$. The potential matrix $\hat{U}$ is parametrized by the spectral parameter $\lambda \in \mathbb{C} \setminus \{0\}$, where $\eta = re^{i\theta}$, $r \in \mathbb{R}^+$, $\theta \in [0, 2\pi]$ and $\lambda = \eta/\bar{\eta} = e^{2i\theta}$. The linear system (170) can be equivalently expressed by the system

$$\partial^2 \Psi_1 - 2 \frac{\partial \eta}{\eta} \partial \Psi_1 - \lambda \eta^2 \partial \chi \Psi_1 = 0,$$

$$\Psi_2 = \chi \Psi_1 - \frac{\partial \Psi_1}{\lambda \eta^2},$$

where $\Psi = (\Psi_1, \Psi_2)^T$. The coefficients of the linear second-order ODE (172) possess a degree of freedom involving two arbitrary locally holomorphic complex-valued functions $\eta(z)$ and $\chi(z)$. These functions correspond to the arbitrary functions from the Enneper-Weierstrass representation (162) describing minimal surfaces in $\mathbb{E}^3$.

4.3 Links between the linear problem and the $X_2^{(1)}$-Hermite differential equation

Making use of the approach described in [9], we carry out an association between the coefficients of the ODE (172) and the coefficients of the complex $X_2^{(1)}$-Hermite ODE (54). We obtain the system

$$-2 \frac{\partial \eta}{\eta} = -2 \left( z + \frac{4z}{1 + 2z^2} \right), \quad -\lambda \eta^2 \partial \chi = 2n.$$

The association made in (174) signifies that the component $\Psi_1(n; z)$ of the holomorphic wavefunction $\Psi$ corresponds to the general solution of equation (54). We obtain the explicit form of the arbitrary functions from the Enneper-Weierstrass representation (162)

$$\eta^2(z) = c_1^2 e^{z^2} (1 + 2z^2)^2 = \frac{16c_1^2}{W_{(1,1)}(z)},$$

$$\chi(n; \lambda; z) = -\frac{2n}{\lambda c_1^2} \left( c_2 + \frac{\sqrt{\pi}}{4} erf(z) + \frac{e^{-z^2}}{2(1 + 2z^2)} \right)$$

$$= -\frac{2n}{\lambda c_1^2} \left( c_2 + \frac{\sqrt{\pi}}{4} erf(z) + 8(1 + 2z^2) W_{(1,1)}(z) \right).$$
where $c_1 \in \mathbb{C}\setminus\{0\}$ and $c_2 \in \mathbb{C}$ are arbitrary constants. The functions $\eta$ (175) and $\chi$ (176) are written in terms of the complex extension of the weight (39). They can be compared to the arbitrary functions $\eta_0$ and $\chi_0$ arising from the identification with the classical complex Hermite ODE [36]

$$\omega''(z) - 2z\omega'(z) + 2n\omega(z) = 0, \quad z \in \mathbb{C}, \; n \in \mathbb{N}. \quad (177)$$

These were obtained in [9]

$$\eta^2_0(z) = c^2_1 e^{z^2}, \quad \chi_0(u; \lambda; z) = -\frac{2n}{\lambda c^2_1} \left(c_2 + \frac{\sqrt{\pi}}{2} \text{erf}(z)\right). \quad (178)$$

Substituting the functions $\eta$ (175) and $\chi$ (176), the components of the potential matrix $\hat{U}(u; \lambda; z) = (u_{ij})$ (171) become

$$u_{11} = -u_{22} = -\frac{32n}{W_{(1,1)}(z)} \left( c_2 + \frac{\sqrt{\pi}}{4} \text{erf}(z) + 8(1 + 2z^2)W_{(1,1)}(z) \right),$$

$$u_{12} = -\frac{16\lambda c^2_1}{W_{(1,1)}(z)}, \quad (179)$$

$$u_{21} = \frac{64n^2}{\lambda c^2_1 W_{(1,1)}(z)} \left( c_2 + \frac{\sqrt{\pi}}{4} \text{erf}(z) + 8(1 + 2z^2)W_{(1,1)}(z) \right)^2.$$

From equation (172) and from the association made in equations (174), we know that the component $\Psi_1$ of the wavefunction corresponds to the general solution of the complex $X_2^{(1)}$-Hermite ODE given by Theorem 5. The component $\Psi_2$ of the wavefunction is given by equation (173). We obtain

$$\Psi(n; \lambda; z) = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} k_1 \alpha_n(z) + k_2 \beta_n(z) \\ \chi (k_1 \alpha_n(z) + k_2 \beta_n(z)) - \frac{\partial(k_1 \alpha_n(z) + k_2 \beta_n(z))}{\lambda n^2} \end{pmatrix}. \quad (180)$$

Substituting the functions $\eta$ (175) and $\chi$ (176) into equation (180), we obtain

$$\Psi = \begin{pmatrix} k_1 \alpha_n(z) + k_2 \beta_n(z) \\ -\frac{2n}{\lambda c^2_1} \left( c_2 + \frac{\sqrt{\pi}}{4} \text{erf}(z) + 8(1 + 2z^2)W_{(1,1)}(z) \right) \left( k_1 \alpha_n(z) + k_2 \beta_n(z) \right) \end{pmatrix}, \quad (181)$$

where $c_1, \lambda \in \mathbb{C}\setminus\{0\}$ and $c_2, k_1, k_2 \in \mathbb{C}$.

### 4.4 Minimal surfaces describing the $X_2^{(1)}$-Hermite polynomials

The explicit form of the components of the Enneper-Weierstrass representation (162) is obtained by integration of the functions $\eta$ (175) and $\chi$ (176). Let

$$I_1 := \int_{\xi_0}^{\xi} \eta^2 \, dz, \quad I_2 := \int_{\xi_0}^{\xi} \chi^2 \eta^2 \, dz, \quad I_3 := \int_{\xi_0}^{\xi} \chi^2 \, dz. \quad (182)$$

Then the Enneper-Weierstrass immersion formula (162) describing a minimal surface immersed in $\mathbb{E}^3$ becomes

$$F(n; \lambda; \xi_0, \xi) = \begin{pmatrix} \frac{1}{2} \Re (I_1 - I_2), -\frac{1}{2} \Im (I_1 + I_2), \Re (I_3) \end{pmatrix}^T \in \mathbb{E}^3, \quad (183)$$

where

$$I_1 = c^2_1 \left[ \sqrt{\pi} \text{erf}(z) + e^z z(2z^2 - 1) \right]_{\xi_0}, \quad (184)$$

$$I_2 = \frac{4n^2}{\lambda^2 c^4_1} \left[ c^2_2 \sqrt{\pi} \text{erf}(z) + \frac{\sqrt{\pi}}{6} z^2 \text{erf}(z) + \frac{\sqrt{\pi}}{8} \text{erf}(z) + \frac{\sqrt{\pi}}{8} \text{erf}(z) + \frac{c_2^2}{2} z^2 F_2(1, 1; -1/2, 2; z^2) 
+ c_2^2 \frac{\sqrt{\pi}}{2} z^2 F_2(1, 1; 1/2, 2; z^2) + \frac{c_2^2 \sqrt{\pi}}{2} z^2 F_2(1, 1; 3/2, 2; z^2) + \frac{c_2^2 \sqrt{\pi}}{2} z^4 + \frac{2c_2^2 z}{3} - \frac{c_2^2}{2} z^2 \right],$$

$$I_3 = \int_{\xi_0}^{\xi} \chi^2 (2z^2 + 1) \text{erf}(z) \, dz, \quad (185)$$

$$+ c_2^2 \frac{2z^2 e^z - c_2^2 z^2 - \frac{1}{6} z^2 e^{-z} + \frac{5}{12} z e^{-z}}{4 \lambda^2 c^4_1} \int_{\xi_0}^{\xi} e^z (2z^2 + 1) \text{erf}(z) \, dz,$$

$$+ \frac{n^2 \pi}{4 \lambda^2 c^4_1} \int_{\xi_0}^{\xi} e^z (2z^2 + 1) \text{erf}(z) \, dz.$$
\[ I_3 = \frac{2n}{\lambda} \left[ c_2 \sqrt{\pi} \text{erfi}(z) + c_2 e^{z^2} (2z^2 - 1) - \frac{1}{4} z^2 \cdot 2 F_2(1, 1; -1/2, 2; z^2) \\
+ \frac{1}{2} z^2 \cdot 2 F_2(1, 1; 1/2, 2; z^2) + \frac{1}{4} z^2 \cdot 2 F_2(1, 1; 3/2, 2; z^2) - \frac{1}{4} z^4 + \frac{1}{3} z^3 - \frac{1}{4} z^2 + \frac{1}{2} z \right] \xi_o . \tag{186} \]

The function \( \text{erfi}(z) \) appearing in the components (184)-(186) is the imaginary Error function defined by [1]

\[ \text{erfi}(z) = -i \cdot \text{erf}(iz), \tag{187} \]

and the function \( \pFq{p}{q}{a_1, a_2, \ldots, a_p; b_1, b_2, \ldots, b_q; z} \) is the generalized hypergeometric function defined by [1]

\[ \pFq{p}{q}{a_1, a_2, \ldots, a_p; b_1, b_2, \ldots, b_q; z} = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k (b_2)_k \cdots (b_q)_k} \frac{z^k}{k!} , \tag{188} \]

where \((a)_k := a(a+1)\cdots(a+k-1)\) is the Pochhammer symbol. The components of the surface (183) take the form

\[
F_1 = \frac{1}{2} \Re \left[ \sqrt{\pi} \left( c_1^2 - \frac{4n^2c_2^2}{\lambda^2c_1^4} \right) \text{erfi}(z) \right]_{\xi_0} + \left( c_1^2 - \frac{4n^2}{\lambda^2c_1^4} \right) e^{z^2} z(2z^2 - 1) \right]_{\xi_0}
- \frac{4n^2}{\lambda^2c_1^4} \left[ \frac{\sqrt{\pi}}{2} \left( \frac{1}{3} z^2 \cdot 2 F_2(1, 1; 1/2, 2; z^2) + \frac{c_2 \sqrt{\pi}}{2} z^2 \cdot 2 F_2(1, 1; 3/2, 2; z^2) - \frac{c_2 \sqrt{\pi}}{2} z^4 + \frac{2c_2}{3} z^3 \right) \right. \\
\left. + c_2 \sqrt{\pi} z^2 \cdot 2 F_2(1, 1; 1/2, 2; z^2) + \frac{c_2 \sqrt{\pi}}{2} z^2 \cdot 2 F_2(1, 1; 3/2, 2; z^2) - \frac{c_2 \sqrt{\pi}}{2} z^4 + \frac{2c_2}{3} z^3 \right]_{\xi_0}
- \frac{c_2 \sqrt{\pi}}{2} z^2 + c_2 z + \frac{5}{12} e^{z^2} z - \frac{1}{2} e^{z^2} z \right]_{\xi_0} + \frac{\pi}{16} \int_{\xi_0}^{\xi} e^{z^2} (2z^2 + 1) \text{erf}^2(z)dz ,
\tag{189}
\]

\[
F_2 = \frac{1}{2} \Im \left[ \sqrt{\pi} \left( c_1^2 + \frac{4n^2c_2^2}{\lambda^2c_1^4} \right) \text{erfi}(z) \right]_{\xi_0} + \left( c_1^2 + \frac{4n^2}{\lambda^2c_1^4} \right) e^{z^2} z(2z^2 - 1) \right]_{\xi_0}
+ \frac{4n^2}{\lambda^2c_1^4} \left[ \frac{\sqrt{\pi}}{2} \left( \frac{1}{3} z^2 \cdot 2 F_2(1, 1; 1/2, 2; z^2) + \frac{c_2 \sqrt{\pi}}{2} z^2 \cdot 2 F_2(1, 1; 3/2, 2; z^2) - \frac{c_2 \sqrt{\pi}}{2} z^4 + \frac{2c_2}{3} z^3 \right) \right. \\
\left. + c_2 \sqrt{\pi} z^2 \cdot 2 F_2(1, 1; 1/2, 2; z^2) + \frac{c_2 \sqrt{\pi}}{2} z^2 \cdot 2 F_2(1, 1; 3/2, 2; z^2) - \frac{c_2 \sqrt{\pi}}{2} z^4 + \frac{2c_2}{3} z^3 \right]_{\xi_0}
- \frac{c_2 \sqrt{\pi}}{2} z^2 + c_2 z + \frac{5}{12} e^{z^2} z - \frac{1}{2} e^{z^2} z \right]_{\xi_0} - \frac{\pi}{16} \int_{\xi_0}^{\xi} e^{z^2} (2z^2 + 1) \text{erf}^2(z)dz ,
\tag{190}
\]

\[
F_3 = \Re \left[ \frac{2n}{\lambda} \left[ c_2 \sqrt{\pi} \text{erfi}(z) + c_2 e^{z^2} z(2z^2 - 1) - \frac{1}{4} z^2 \cdot 2 F_2(1, 1; -1/2, 2; z^2) \\
+ \frac{1}{2} z^2 \cdot 2 F_2(1, 1; 1/2, 2; z^2) + \frac{1}{4} z^2 \cdot 2 F_2(1, 1; 3/2, 2; z^2) - \frac{1}{4} z^4 + \frac{1}{3} z^3 - \frac{1}{4} z^2 + \frac{1}{2} z \right]_{\xi_0} \right] .
\tag{191}
\]

The integral in terms of the Error function (62)

\[ I_4 := \int_{\xi_0}^{\xi} e^{z^2} (2z^2 + 1) \text{erf}^2(z)dz \tag{192} \]

appearing in equations (189) and (190) may be numerically approximated for the plotting of the surface. It may also be reduced to the numerical approximation of the integral

\[ \int_{\xi_0}^{\xi} e^{z^2} \text{erf}^2(z)dz . \tag{193} \]

We integrate by parts by putting

\[ u = \text{erf}^2(z), \quad dv = e^{z^2} (2z^2 + 1)dz . \tag{194} \]
The integral $I_4$ (192) becomes

$$I_4 = \operatorname{erf}^2(z) \left( \sqrt{\pi} \operatorname{erfi}(z) + e^{z^2} z(2z^2 - 1) \right) \bigg|_{\xi_0}^{\xi} - \frac{1}{2\sqrt{\pi}} (4z^4 - 4z^2 - 1) \operatorname{erf}(z) \bigg|_{\xi_0}^{\xi}$$

$$- \frac{1}{\pi} e^{-z^2} (2z^3 + z) \bigg|_{\xi_0}^{\xi} - 4 \int_{\xi_0}^{\xi} e^{-z^2} \operatorname{erf}(z) \operatorname{erfi}(z) \, dz. \quad (195)$$

The integral

$$I_5 := \int_{\xi_0}^{\xi} e^{-z^2} \operatorname{erf}(z) \operatorname{erfi}(z) \, dz \quad (196)$$

appearing in equation (195) may also be integrated by parts by putting

$$s = \operatorname{erfi}(z), \quad dt = e^{-z^2} \operatorname{erf}(z) \, dz. \quad (197)$$

The integral $I_5$ (196) becomes

$$I_5 = \frac{\sqrt{\pi}}{4} \operatorname{erf}^2(z) \operatorname{erfi}(z) \bigg|_{\xi_0}^{\xi} - \frac{1}{2} \int_{\xi_0}^{\xi} e^{-z^2} \operatorname{erf}^2(z) \, dz. \quad (198)$$

and the integral $I_4$ (192) becomes

$$I_4 = \left[ \operatorname{erf}^2(z) \left( \sqrt{\pi} \operatorname{erfi}(z) + e^{z^2} z(2z^2 - 1) \right) - \sqrt{\pi} \operatorname{erf}^2(z) \operatorname{erfi}(z) \right]$$

$$- \frac{1}{2\sqrt{\pi}} (4z^4 - 4z^2 - 1) \operatorname{erf}(z) - \frac{1}{\pi} e^{-z^2} (2z^3 + z) \bigg|_{\xi_0}^{\xi} + 2 \int_{\xi_0}^{\xi} e^{-z^2} \operatorname{erf}^2(z) \, dz. \quad (199)$$

In terms of the integrals (182), the immersion formula (166) describing a minimal surface immersed in $\mathfrak{su}(2)$ becomes

$$\tilde{F}(n; \lambda; z) = -\frac{i}{2} \left( \begin{array}{cc} I_3 + I_3^* & I_1 - I_2^* \\ -I_2 + I_1^* & -(I_3 + I_3^*) \end{array} \right) \in \mathfrak{su}(2), \quad (200)$$

because $Tr(\tilde{F}) = 0$, $\tilde{F}^\dagger = -\tilde{F}$, and where $I_k^*$ denotes the complex conjugate of the integral $I_k$, $k = 1, 2, 3$. The expressions for the components of the surface $\tilde{F} \in \mathfrak{su}(2)$ are rather long so we omit them here. It suffices to substitute the integrals (184), (185) and (186) into formula (200).

### 4.5 Numerical representation of minimal surfaces describing $X_2^{(1)}$-Hermite polynomials

Even if the $X_2^{(1)}$-Hermite XOPs are not defined for $n = 1, 2$, we are able to construct the surfaces describing the behavior of the solutions of the $X_2^{(1)}$-Hermite complex ODE (54) for these values of $n$. This is due to the fact that the surface may be described by the holomorphic wavefunction $\Psi$ (the solution of the linear problem (170)), acting as the moving frame on the surface, which is determined by the general solution (150), defined for all $n \in \mathbb{N}$. For $n = 0$, the surface coincides with the plane $F_3 \equiv 0$. Figures 1 to 4 below show the evolution of the surface for $n = 1, 2, 3$ and 7. They were obtained using the Mathematica symbolic software and applying the Enneper-Weierstrass immersion formula (162) associated with $X_2^{(1)}$-Hermite polynomials. The components of the surface were calculated in (189), (190) and (191). The integration constants and the parameter were fixed as $c_1 = c_2 = 1$ and $\lambda = \sqrt{\pi}$, respectively. The integration was performed from $\xi_0 = 1 + 3i$ to $\xi = x + iy$, where $x \in [-1,1]$, $y \in [-1,1]$. The parameter $n$ is related to the $X_2^{(1)}$-Hermite complex ODE (54). As $n$ grows, the surface expands, but the evolution of the surface as the parameter $n$ grows suggests a global flattening phenomena for the third component $F_3$ (notice that there is a change of scale from one figure to another). A mirror symmetry about the plane $F_2 \equiv C$, for some $C < 0$, appears clearly in each image.
Figure 1: Representation of the $X_2^{(1)}$-Hermite polynomials for $n = 1$.

Figure 2: Representation of the $X_2^{(1)}$-Hermite polynomials for $n = 2$.

Figure 3: Representation of the $X_2^{(1)}$-Hermite polynomials for $n = 3$.

Figure 4: Representation of the $X_2^{(1)}$-Hermite polynomials for $n = 7$. 
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A Proof of Proposition 3

We show that the series $\beta_n(z)$ (84) is a non-polynomial solution of equation (54) for all $n \in \mathbb{N}$.

Proof. Consider the following proposition:

$$P_n(n): \quad \text{The function } \beta_n(z) \text{ (84) is a solution of equation (54) for all } n \in \mathbb{N}. \quad (201)$$

We proceed by induction. Multiplying by $(1 + 2z^2)$, equation (54) can be equivalently expressed as

$$\left(1 + 2z^2\right) \omega''(z) + (-4z^3 - 10z) \omega'(z) + \left(2n + 4n^2z^2\right) \omega(z) = 0.$$  

(202)

Case $n = 0$. Equation (202) becomes

$$\left(1 + 2z^2\right) \omega''(z) + (-4z^3 - 10z) \omega'(z) = 0. \quad (203)$$

Let

$$\Delta_1(k) := (-1)^k 2k^{-1} ((2(k - 1))^2 + 1) \prod_{j=1}^{k-2} (1 - 2(1 + j)). \quad (204)$$

Then the series (84) and its first and second-order derivatives take the form

$$\beta_0(z) = 1 + z + \frac{5}{3} z^3 + \sum_{k=3}^{\infty} \frac{\Delta_1(k)}{(2k - 1)!} z^{2k - 1}, \quad (205)$$

$$\frac{d\beta_0}{dz}(z) = 1 + 5z^2 + \sum_{k=3}^{\infty} \frac{\Delta_1(k)}{(2k - 2)!} z^{2k - 2}, \quad (206)$$

$$\frac{d^2\beta_0}{dz^2}(z) = 10z + \sum_{k=3}^{\infty} \frac{\Delta_1(k)}{(2k - 3)!} z^{2k - 3}. \quad (207)$$

Substituting (206) and (207) into the left-hand side (LHS) of equation (203), we obtain

$$G_1(0; z) = \left(1 + 2z^2\right) \cdot 10z + \left(-4z^3 - 10z\right) \cdot \left(1 + 5z^2\right) + \sum_{k=3}^{\infty} \frac{\Delta_1(k)}{(2k - 3)!} z^{2k - 3} \quad (208)$$

$$+ \sum_{k=3}^{\infty} \frac{2\Delta_1(k)}{(2k - 3)!} z^{2k - 1} - \sum_{k=4}^{\infty} \frac{4\Delta_1(k)}{(2k - 4)!} z^{2k + 1} - \sum_{k=3}^{\infty} \frac{10\Delta_1(k)}{(2k - 2)!} z^{2k - 1}.$$ 

In order to obtain powers corresponding to $(2k - 1)$ in all series, we perform a translation of the summation variable where necessary

$$G_1(0; z) = \left(1 + 2z^2\right) \cdot 10z + \left(-4z^3 - 10z\right) \cdot \left(1 + 5z^2\right) + \sum_{k=2}^{\infty} \frac{\Delta_1(k+1)}{(2k - 1)!} z^{2k - 1} \quad (209)$$

$$+ \sum_{k=3}^{\infty} \frac{2\Delta_1(k)}{(2k - 3)!} z^{2k - 1} - \sum_{k=4}^{\infty} \frac{4\Delta_1(k-1)}{(2k - 4)!} z^{2k - 1} - \sum_{k=3}^{\infty} \frac{10\Delta_1(k)}{(2k - 2)!} z^{2k - 1}.$$ 

Extracting the terms of degree $k \leq 3$, and considering the terms outside of a series in equation (209), we see that they cancel each other. Regrouping all series, we get

$$G_1(0; z) = \sum_{k=4}^{\infty} \left[ \frac{\Delta_1(k+1)}{(2k - 1)!} + \frac{2\Delta_1(k)}{(2k - 3)!} - \frac{4\Delta_1(k-1)}{(2k - 4)!} - \frac{10\Delta_1(k)}{(2k - 2)!} \right] z^{2k - 1}. \quad (210)$$
Evaluating $\Delta$ from relation (204), we obtain

$$G_1(0; z) = \sum_{k=4}^{\infty} \left[ \frac{(-1)^k 2^k \prod_{j=1}^{k-3} (1 + 2(1 + j))}{(2k - 4)!} \cdot \Delta_2(k) \right] z^{2k-1},$$ \hspace{1cm} (211)

where

$$\Delta_2(k) : = - \frac{(2k)^2 + 1 - 2(1 + (k - 2))(1 - 2(1 + (k - 1)))}{(2k - 3)(2k - 2)(2k - 1)} + \frac{(2(k - 1))^2 + 1 - 2(1 + (k - 2))}{(2k - 3)}$$

$$+ ((2k - 2)^2 + 1) - \frac{5((2(k - 1))^2 + 1)(1 - 2(1 + (k - 2)))}{(2k - 3)(2k - 2)} = 0,$$ \hspace{1cm} (212)

for all $k \geq 4$. We conclude that $P_1(0)$ is true.

**Induction hypothesis.** Suppose that $P_1(n - 1)$ is true for some $n \geq 0$, i.e.

$$(1 + 2z^2) (\beta_{n-1}(z)')' + (-4z^3 - 10z) (\beta_{n-1}(z)) + 2(n - 1) (1 + 2z^2) \beta_{n-1}(z) = 0.$$ \hspace{1cm} (213)

We want to show that $P_1(n)$ is true, i.e.

$$(1 + 2z^2) (\beta_n(z))'' + (-4z^3 - 10z) (\beta_n(z))' + 2n (1 + 2z^2) \beta_n(z) = 0.$$ \hspace{1cm} (214)

Subtracting the LHS of (213) and (214), we need to show the equality

$$(1 + 2z^2) [(\beta_{n-1})'' - (\beta_n)''] (z) + (-4z^3 - 10z) [(\beta_{n-1})' - (\beta_n)'] (z) + 2n (1 + 2z^2) [\beta_{n-1} - \beta_n] (z) = 0.$$ \hspace{1cm} (215)

Let

$$\Delta_3(k) : = (n - 1)(n - ((2k - 1)^2 + 2)) \prod_{j=1}^{k-2} (n - (2(1 + j) + 1)),$$ \hspace{1cm} (216)

$$\Delta_4(k) : = n(n - ((2k - 1)^2 + 1)) \prod_{j=1}^{k-2} (n - 2(1 + j)),$$ \hspace{1cm} (217)

$$\Delta_5(k) : = (n - ((2k - 1)^2 + 2)) \prod_{j=1}^{k-2} (n - 2(1 + j)),$$ \hspace{1cm} (218)

$$\Delta_6(k) : = (n - ((2k - 1)^2 + 1)) \prod_{j=1}^{k-2} (n - 2(1 + j) + 1).$$ \hspace{1cm} (219)

Then we get

$$[\beta_{n-1} - \beta_n] (z) = z^2 + \frac{1}{3} z^3 + \frac{11}{6} - 2n z^4$$

$$+ \sum_{k=3}^{\infty} \left[ \frac{(-1)^k 2^k}{(2k)!} (\Delta_3 - \Delta_4) z^{2k} + \frac{(-1)^k 2^k}{(2k-1)!} (\Delta_5 - \Delta_6) z^{2k-1} \right],$$ \hspace{1cm} (220)

$$[(\beta_{n-1})' - (\beta_n)'] (z) = 2z + z^2 + \frac{22}{3} - 4n z^3$$

$$+ \sum_{k=3}^{\infty} \left[ \frac{(-1)^k 2^k}{(2k-1)!} (\Delta_3 - \Delta_4) z^{2k-1} + \frac{(-1)^k 2^k}{(2k-2)!} (\Delta_5 - \Delta_6) z^{2k-2} \right],$$ \hspace{1cm} (221)

$$[(\beta_{n-1})'' - (\beta_n)'''] (z) = 2 + 2z + (22 - 4n) z^2$$

$$+ \sum_{k=3}^{\infty} \left[ \frac{(-1)^k 2^k}{(2k-2)!} (\Delta_3 - \Delta_4) z^{2k-2} + \frac{(-1)^k 2^k}{(2k-3)!} (\Delta_5 - \Delta_6) z^{2k-3} \right].$$ \hspace{1cm} (222)
Substituting (220)-(222) into the LHS of equation (215), we obtain

\[ G_1(n; z) = (1 + 2z^2) \left( 2z + (22 - 4n)z^2 \right) + (-4z^3 - 10z) \left( 2z + z^2 + \frac{22 - 4n}{3}z^3 \right) \]

+ 2n(1 + 2z^2) \left( z^2 + \frac{1}{3}z^3 + \frac{11}{6}z^4 \right) \]

- 2 \left( 1 + 2z^2 \right) \left( 1 + z - (n - 1)z^2 - \frac{n - 6}{3}z^3 + \frac{(n - 1)(n - 11)}{6}z^4 \right) \]

\[ + \sum_{k=3}^{\infty} \left[ \frac{(-1)^k 2^k}{(2k - 2)!} \left( \Delta_3 - \Delta_4 \right) z^{2k-2} + \frac{(-1)^k 2^k}{(2k - 3)!} \left( \Delta_5 - \Delta_6 \right) z^{2k-3} \right] \]

\[ + \sum_{k=3}^{\infty} \left[ \frac{(-1)^k 2^k + 1}{(2k - 2)!} \left( \Delta_3 - \Delta_4 \right) z^{2k} + \frac{(-1)^k 2^k + 1}{(2k - 3)!} \left( \Delta_5 - \Delta_6 \right) z^{2k-1} \right] \]

(223)

In order to obtain powers corresponding to \((2k)\) and \((2k - 1)\) in all series, we perform a translation of the summation variable where necessary. Extracting the terms of degree \(k \leq 3\), and considering the terms outside of a series in equation (223), we see that they cancel each other. Regrouping all series, we obtain

\[ G_1(n; z) = \sum_{k=4}^{\infty} \left[ \frac{(-1)^k 2^k + 1}{(2k - 3)!} \left( \prod_{j=1}^{k-3} (n - (2(1 + j) + 1)) \Delta_7 - \prod_{j=1}^{k-3} (n - (2(1 + j)) \Delta_8) \right) z^{2k} \]

\[ + \frac{(-1)^k 2^k + 1}{(2k - 4)!} \left( \prod_{j=1}^{k-3} (n - (2(1 + j)) \Delta_9 - \prod_{j=1}^{k-3} (n - (2(1 + j) + 1)) \Delta_{10} \right) z^{2k-1} \right], \]  

(224)
where

\[ \Delta_7(k) = \frac{-(n-1)(n-(2k+1)^2 + 2)((n - 2(k-1) + 1)(n - 2(k+1))}{(2k-2)(2k-1)(2k)} + (n-1)(n-(2k-1)^2 + 2)(n - 2(k-1) + 1) \]
\[ \cdot \left( \frac{1}{2k-2} - \frac{5}{(2k-2)(2k-1)} + \frac{n}{(2k-2)(2k-1)(2k)} \right) \]
\[ - \frac{1}{(2k-2)(2k-1)(2k)} (n-1)(n-(2k-1)^2 + 2)(n - 2(k-1) + 1) \]
\[ + \frac{1}{(2k-2)} (n-1)(n-(2k-3)^2 + 2) + (n-1)(n-(2k-3)^2 + 2) \left( 1 - \frac{n}{(2k-2)} \right) \]
\[ = 0, \]
\[ \Delta_8(k) = \frac{-n(n-(2k+1)^2 + 1))(n - 2(k-1))(n - 2k)}{(2k-2)(2k-1)(2k)} + n(n - (2k-1)^2 + 1)(n - 2(k-1)) \]
\[ \cdot \left( \frac{1}{2k-2} - \frac{5}{(2k-2)(2k-1)} + \frac{n}{(2k-2)(2k-1)(2k)} \right) \]
\[ + n(n-(2k-3)^2 + 1) \left( 1 - \frac{n}{(2k-2)} \right) \]
\[ = 0, \]
\[ \Delta_9(k) = \frac{-(n-(2k^2 + 2))(n - 2(k-1))(n - 2k)}{(2k-3)(2k-2)(2k-1)} + (n - (2(k-1))^2 + 2)(n - 2(k-1)) \]
\[ \cdot \left( \frac{1}{2k-3} - \frac{5}{(2k-3)(2k-2)} + \frac{n}{(2k-3)(2k-2)(2k-1)} \right) \]
\[ + (n - (2(k-1))^2 + 2) \left( 1 - \frac{n}{(2k-3)} \right) \]
\[ - (n - (2(k-1))^2 + 2)(n - 2(k-1)) + \frac{n - ((2(k-2))^2 + 2)}{(2k-3)} \]
\[ = 0, \]
\[ \Delta_{10}(k) = \frac{-(n-(2k^2 + 1))(n - 2(k-1) + 1)(n - 2k + 1)}{(2k-3)(2k-2)(2k-1)} + (n - (2(k-1))^2 + 1)(n - 2(k-1) + 1) \]
\[ \cdot \left( \frac{1}{2k-3} - \frac{5}{(2k-3)(2k-2)} + \frac{n}{(2k-3)(2k-2)(2k-1)} \right) \]
\[ + (n - (2(k-2))^2 + 1) \left( 1 - \frac{n}{(2k-3)} \right) \]
\[ = 0, \]

for all \( k \geq 4 \). We conclude that the induction hypothesis implies that \( P_1(n) \) is true. By construction, the solution \( \beta_n(z) \) (84) is non-polynomial, because the coefficients \( c_{2k}(n) \) (82) and \( c_{2k-1}(n) \) (83), associated with even and odd powers of \( z \), respectively, are polynomials of the parameter \( n \), possessing no root on the gap sequence (see Table 2), which completes the proof of Proposition 3.
B Proof of Proposition 4

We show that the series $\mu_n(z)$ (90) is a polynomial solution of equation (54) for all $n \in 2\mathbb{N}\setminus\{2\}$ and that the series $\nu_n(z)$ (89) is a polynomial solution of equation (54) for all $n \in (2\mathbb{N} - 1)\setminus\{1\}$.

**Proof.** Consider the following proposition:

$P_2(n)$ : The function $\mu_n(z)$ (90) is a solution of equation (54) for all $n \in \mathbb{N}$. 

(229)

We proceed by induction.

Case $n = 0$. The series $\mu_n(z)$ (90) and its first and second-order derivatives take the form

$$
\mu_0(z) = 1, \quad \frac{d\mu_0}{dz}(z) = 0, \quad \frac{d^2\mu_0}{dz^2}(z) = 0,
$$

(230)

so we see immediately that $\mu_0(z)$ is a solution of equation (203). We conclude that $P_2(0)$ is true.

**Induction hypothesis.** Suppose that $P_2(n - 1)$ is true for some $n \geq 0$, i.e.

$$(1 + 2z^2) (\mu_{n-1}(z))'' + (-4z^3 - 10z)(\mu_{n-1}(z))' + 2(n - 1) (1 + 2z^2) \mu_{n-1}(z) = 0. \tag{231}$$

We want to show that $P_2(n)$ is true, i.e.

$$(1 + 2z^2) (\mu_n(z))'' + (-4z^3 - 10z)(\mu_n(z))' + 2n(1 + 2z^2) \mu_n(z) = 0. \tag{232}$$

Subtracting the LHS of (231) and (232), we need to show the equality

$$
(1 + 2z^2) [((\mu_{n-1})'' - (\mu_n)') (z) + (-4z^3 - 10z) [(\mu_{n-1})' - (\mu_n)'] (z) + 2n(1 + 2z^2) [\mu_{n-1} - \mu_n] - 2(1 + 2z^2)\mu_{n-1}(z) = 0.
$$

(233)

We get

$$
[\mu_{n-1} - \mu_n] (z) = z^2 + \frac{11 - 2n}{6} z^4 + \sum_{k=3}^{\infty} \left[ \frac{(-1)^k 2^k}{(2k)!} (\Delta_3 - \Delta_4) z^{2k} \right].
$$

(234)

$$
[(\mu_{n-1})' - (\mu_n)'] (z) = 2z + \frac{22 - 4n}{3} z^3 + \sum_{k=3}^{\infty} \left[ \frac{(-1)^k 2^k}{(2k-1)!} (\Delta_3 - \Delta_4) z^{2k-1} \right],
$$

(235)

$$
[(\mu_{n-1})'' - (\mu_n)'''] (z) = 2 + (22 - 4n)z^2 + \sum_{k=3}^{\infty} \left[ \frac{(-1)^k 2^k}{(2k-2)!} (\Delta_3 - \Delta_4) z^{2k-2} \right].
$$

(236)

Substituting (234)-(236) into the LHS of equation (233), we obtain

$$
G_2(n; z) = (1 + 2z^2) \left( 2 + (22 - 4n)z^2 \right) + (-4z^3 - 10z) \left( 2z + \frac{22 - 4n}{3} z^3 \right)
$$

$$
+ 2n(1 + 2z^2) \left( z^2 + \frac{11 - 2n}{6} z^4 \right) - 2 \left( 1 + 2z^2 \right) \left( 1 - (n-1)z^2 + \frac{(n-1)(n-11)}{6} z^4 \right)
$$

$$
+ \sum_{k=3}^{\infty} \left[ \frac{(-1)^k 2^k}{(2k-2)!} (\Delta_3 - \Delta_4) z^{2k-2} \right] + \sum_{k=3}^{\infty} \left[ \frac{(-1)^k 2^k + 1}{(2k-1)!} (\Delta_3 - \Delta_4) z^{2k} \right]
$$

$$
+ \sum_{k=3}^{\infty} \left[ \frac{(-1)^k 2^k + 1}{(2k-1)!} (\Delta_3 - \Delta_4) z^{2k+2} \right] + \sum_{k=3}^{\infty} \left[ \frac{(-1)^k 2^k + 1}{(2k)!} (\Delta_3 - \Delta_4) z^{2k+1} \right]
$$

$$
+ \sum_{k=3}^{\infty} \left[ \frac{(-1)^k 2^k + 1}{(2k)!} (\Delta_3 - \Delta_4) z^{2k} \right] + \sum_{k=3}^{\infty} \left[ \frac{(-1)^k 2^k + 1}{(2k)!} (\Delta_3 - \Delta_4) z^{2k+2} \right] - \sum_{k=3}^{\infty} \left[ \frac{(-1)^k 2^k + 1}{(2k)!} (\Delta_3 - \Delta_4) z^{2k+2} \right].
$$

(237)
In order to obtain powers corresponding to \((2k)\) in all series, we perform a translation of the summation variable where necessary. Extracting the terms of degree \(k \leq 3\), and considering the terms outside of a series in equation (237), we see that they cancel each other. Regrouping all series, we obtain

\[
G_2(n; z) = \sum_{k=4}^{\infty} \frac{(-1)^k q^{k+1}}{(2k-3)!} \left( \prod_{j=1}^{k-3} (n - (2(1 + j) + 1)) \Delta_7 - \prod_{j=1}^{k-3} (n - 2(1 + j)) \Delta_8 \right) z^{2k},
\]

where we already showed by the relations (225) and (226) that \(\Delta_7(k) \equiv \Delta_8(k) \equiv 0\) for all \(k \geq 4\). We conclude that the induction hypothesis implies that \(P_2(n)\) is true. By construction, the coefficients \(c_{2k}(n)\) of the series \(\mu_n(z)\) (90) possess only even roots \(\lambda_p(k)\) from which \(n = 2\) is excluded, as illustrated in Table 2 and by the polynomials \(p_k(n)\) (80). Therefore the only polynomial cases are \(\mu_{2l}\), where \(l \in \{0, 2, 3, 4\ldots\}\).

Consider the following proposition:

\[
P_3(n) : \text{The function } \nu_n(z) \text{ (89) is a solution of equation (54) for all } n \in \mathbb{N}. \tag{238}
\]

**Case** \(n = 0\). The function \(\nu_n(z)\) (89) and its first and second-order derivatives take the form

\[
\nu_0(z) = z + \frac{5}{3} z^3 + \sum_{k=3}^{\infty} \frac{\Delta_1(k)}{(2k-1)!} z^{2k-1},
\]

\[
\frac{d\nu_0}{dz}(z) = 1 + 5z^2 + \sum_{k=3}^{\infty} \frac{\Delta_1(k)}{(2k-2)!} z^{2k-2},
\]

\[
\frac{d^2\nu_0}{dz^2}(z) = 10 + \sum_{k=3}^{\infty} \frac{\Delta_1(k)}{(2k-3)!} z^{2k-3}.
\]

Substituting (240) and (241) into equation (203), we obtain

\[
G_3(0; z) = (1 + 2z^2) \cdot 10z + (-4z^3 - 10z) \cdot (1 + 5z^2) + \sum_{k=3}^{\infty} \frac{\Delta_1(k)}{(2k-3)!} z^{2k-3} + \sum_{k=3}^{\infty} \frac{2\Delta_1(k)}{(2k-3)!} z^{2k-1} - \sum_{k=3}^{\infty} \frac{4\Delta_1(k)}{(2k-2)!} z^{2k+1} - \sum_{k=3}^{\infty} \frac{10\Delta_1(k)}{(2k-2)!} z^{2k-1}. \tag{242}
\]

In order to obtain powers corresponding to \((2k - 1)\) in all series, we perform a translation of the summation variable where necessary

\[
G_3(0; z) = (1 + 2z^2) \cdot 10z + (-4z^3 - 10z) \cdot (1 + 5z^2) + \sum_{k=2}^{\infty} \frac{\Delta_1(k + 1)}{(2k-1)!} z^{2k-1} + \sum_{k=3}^{\infty} \frac{2\Delta_1(k)}{(2k-3)!} z^{2k-1} - \sum_{k=4}^{\infty} \frac{4\Delta_1(k - 1)}{(2k-4)!} z^{2k-1} - \sum_{k=3}^{\infty} \frac{10\Delta_1(k)}{(2k-2)!} z^{2k-1}. \tag{243}
\]

Extracting the terms of degree \(k \leq 3\), and considering the terms outside of a series in equation (243), we see that they cancel each other. Regrouping all series, we get

\[
G_3(0; z) = \sum_{k=4}^{\infty} \left[ \frac{\Delta_1(k+1)}{(2k-1)!} + \frac{2\Delta_1(k)}{(2k-3)!} - \frac{4\Delta_1(k-1)}{(2k-4)!} - \frac{10\Delta_1(k)}{(2k-2)!} \right] z^{2k-1}. \tag{244}
\]

Evaluating \(\Delta_4\) from relation (204), we obtain

\[
G_3(0; z) = \sum_{k=4}^{\infty} \left[ \frac{(-1)^k q^k \prod_{j=1}^{k-3}(1 + 2(1 + j))}{(2k-4)!} \cdot \Delta_2(k) \right] z^{2k-1}. \tag{245}
\]

We already showed by the relation (212) that \(\Delta_2(k) \equiv 0\) for all \(k \geq 4\). We conclude that \(P_3(0)\) is true.
**Induction hypothesis.** Suppose that $P_3(n-1)$ is true for some $n \geq 0$, i.e.

$$(1 + 2z^2) (\nu_{n-1}(z))'' + (4z^3 - 10z) (\nu_{n-1}(z))' + 2(n - 1) (1 + 2z^2) \nu_{n-1}(z) = 0. \quad (246)$$

We want to show that $P_3(n)$ is true, i.e.

$$(1 + 2z^2) (\nu_n(z))'' + (4z^3 - 10z) (\nu_n(z))' + 2n (1 + 2z^2) \nu_n(z) = 0. \quad (247)$$

Subtracting the left-hand side (LHS) of (246) and (247), we need to show the equality

$$(1 + 2z^2) [(\nu_n(z))'' - (\nu_{n-1}(z))'' + (4z^3 - 10z) [(\nu_n(z))' - (\nu_{n-1}(z))' + 2n (1 + 2z^2) ] \nu_n(z) - 2(1 + 2z^2) \nu_{n-1}(z) = 0. \quad (248)$$

We get

$$[\nu_{n-1} - \nu_n] (z) = \frac{1}{3} z^3 + \sum_{k=3}^{\infty} \frac{(-1)^{k+1} 2^{k-1}}{(2k-1)!} (\Delta_5 - \Delta_6) z^{2k-1}, \quad (249)$$

$$[(\nu_n)' - (\nu_{n-1})'] (z) = z^2 + \sum_{k=3}^{\infty} \frac{(-1)^{k+1} 2^{k-1}}{(2k-2)!} (\Delta_5 - \Delta_6) z^{2k-2}, \quad (250)$$

$$[(\nu_n)'' - (\nu_{n-1})''] (z) = 2z + \sum_{k=3}^{\infty} \frac{(-1)^{k+1} 2^{k-1}}{(2k-3)!} (\Delta_5 - \Delta_6) z^{2k-3}. \quad (251)$$

Substituting (249)-(251) into the LHS of equation (248), we obtain

$$G_3(n; z) = (1 + 2z^2) \cdot 2z + (4z^3 - 10z) \cdot z^2 + 2n (1 + 2z^2) \cdot \frac{1}{3} z^3 - 2 (1 + 2z^2) \left( z - \frac{n - 6}{3} z^3 \right)$$

$$+ \sum_{k=3}^{\infty} \frac{(-1)^{k+1} 2^{k-1}}{(2k-3)!} (\Delta_5 - \Delta_6) z^{2k-3} + \sum_{k=3}^{\infty} \frac{(-1)^{k+1} 2^{k-1}}{(2k-2)!} (\Delta_5 - \Delta_6) z^{2k-2}$$

$$+ \sum_{k=3}^{\infty} \frac{(-1)^{k+1} 2^{k-1}}{(2k-1)!} (\Delta_5 - \Delta_6) z^{2k-1} + \sum_{k=3}^{\infty} \frac{5(-1)^{k+1} 2^{k-1}}{(2k-2)!} (\Delta_5 - \Delta_6) z^{2k-1}$$

$$+ \sum_{k=3}^{\infty} \frac{(-1)^{k+1} 2^{k-1}}{(2k-3)!} (\Delta_5 - \Delta_6) z^{2k-3} + \sum_{k=3}^{\infty} \frac{(-1)^{k+1} 2^{k+1}}{(2k-1)!} (\Delta_5 - \Delta_6) z^{2k+1}. \quad (252)$$

In order to obtain powers corresponding to $(2k-1)$ in all series, we perform a translation of the summation variable where necessary. Extracting the terms of degree $k \leq 3$, and considering the terms outside of a series in (252), we see that they cancel each other. Regrouping all series, we obtain

$$G_3(n; z) = \sum_{k=3}^{\infty} \frac{(-1)^{k+1} 2^k}{(2k-4)!} \left( \prod_{j=1}^{k-3} (n - 2(1+j)) \Delta_9 - \prod_{j=1}^{k-3} (n - (2(1+j) + 1)) \Delta_{10} \right) z^{2k-1}, \quad (253)$$

where we already showed by the relations (227) and (228) that $\Delta_9(k) \equiv \Delta_{10}(k) \equiv 0$ for all $k \geq 4$. We conclude that the induction hypothesis implies that $P_3(n)$ is true.

By construction, the coefficients $c_k(n)$ of the series $\nu_n(z)$ (89) possess only odd roots $\lambda_q(k)$ from which $n = 1$ is excluded, as illustrated by Table 2 and by the polynomials $q_k(n)$ (81). Therefore the only polynomial cases are $\nu_{2l-1}$, where $l \geq 2$, which completes the proof of Proposition 4. \hfill \Box
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