On chaotic minimal center of attraction of a Lagrange stable motion for topological semi flows

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Abstract

Let $f: \mathbb{R}_+ \times X \to X$ be a topological semi flow on a Polish space $X$. In 1977, Karl Sigmund conjectured that if there is a point $x$ in $X$ such that the motion $f(t, x)$ has just $X$ as its minimal center of attraction, then the set of all such $x$ is residual in $X$. In this paper, we present a positive solution to this conjecture and apply it to the study of chaotic dynamics of minimal center of attraction of a motion.

Keywords: Minimal center of attraction · Chaotic motion · Semi flow

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1. Introduction

By a $C^0$ semi flow over a metric space $X$, we here mean a transformation $f: \mathbb{R}_+ \times X \to X$ where $\mathbb{R}_+ = [0, \infty)$, which satisfies the following three conditions:

1. The initial condition: $f(0, x) = x$ for all $x \in X$.
2. The condition of continuity: if there be given two convergent sequences $t_n \to t_0$ in $\mathbb{R}_+$ and $x_n \to x_0$ in $X$, then $f(t_n, x_n) \to f(t_0, x_0)$ as $n \to \infty$.
3. The semigroup condition: $f(t_2, f(t_1, x)) = f(t_1 + t_2, x)$ for any $x$ in $X$ and any times $t_1, t_2$ in $\mathbb{R}_+$.

Sometimes we write $f(t, x) = f^t(x)$ for any $t \geq 0$ and $x \in X$; and for any given point $x \in X$ we call $f(t, x)$ a motion in $X$ and $O_f(x) = f(\mathbb{R}_+, x)$ the orbit starting from the point $x$. If $O_f(x)$ is precompact (i.e. $\overline{O_f(x)}$ is compact) in $X$, then we say that $f(t, x)$ is Lagrange stable.

We refer to any subset $\Lambda$ of $X$ as an invariant set if $f(t, x) \in \Lambda$ for each pint $x \in \Lambda$ and any time $t \geq 0$. In dynamical systems, statistical mechanics and ergodic theory, we shall have to do with “probability of sojourn of a motion $f(t, x)$ in a given region $E$ of $X$” as $t \to +\infty$:

$$P(f(t, x) \in E) = \lim_{T \to +\infty} \frac{1}{T} \int_0^T \mathbb{1}_E(f(t, x)) dt,$$

where $\mathbb{1}_E(x)$ is the characteristic function of the set $E$ on $X$.

This motivates H.F. Hilmy to introduce following important concept, which was discussed in [13, 15, 16, 12], for example.
Lemma 1.2. Let \( f : \mathbb{R} \times X \to X \) be a \( \mathcal{C}^0 \)-semi flow on a metric space \( X \). Then each Lagrange stable motion \( f(t, x) \) always possesses the minimal center of attraction.

First of all, by the classical Cantor-Baire theorem and Zorn’s lemma we can obtain the following basic existence lemma.

Definition 1.1 (Hilmy 1936 [10]). Given any \( x \in X \), a closed subset \( C_x \) of \( X \) is called the center of attraction of the motion \( f(t, x) \) as \( t \to +\infty \) if \( \mathcal{P}(f(t, x) \in B_{\varepsilon}(C_x)) = 1 \) for all \( \varepsilon > 0 \), where \( B_{\varepsilon}(C_x) \) denotes the \( \varepsilon \)-neighborhood around \( C_x \) in \( X \). If the set \( C_x \) does not admit a proper subset which is likewise a center of attraction of the motion \( f(t, x) \) as \( t \to +\infty \), then \( C_x \) is called the minimal center of attraction of the motion \( f(t, x) \) as \( t \to +\infty \).

Our argument of Theorem 1.4 below also works for discrete-time case. Thus we can obtain the following:

Corollary. For any continuous transformation \( f \) of a Polish space \( X \), the set of points \( x \in X \) with \( C_x = X \), if nonempty, is residual in \( X \).

We now turn to some applications of Theorem 1.4. For our convenience, we first introduce two notions for a \( \mathcal{C}^0 \)-semi flow \( f : \mathbb{R}_+ \times X \to X \) on a Polish space \( X \).

Definition 1.5. An \( f \)-invariant subset \( \Lambda \) of \( X \) is referred to as generic if there exists some point \( x \in \Lambda \) with \( \Lambda = C_x \).

According to Conjecture 1.3 (or precisely speaking Theorem 1.4) for any generic minimal center of attraction \( C_x \) of a motion \( f(t, x) \), the set of points \( y \in C_x \) with \( C_y = C_x \) is residual in \( C_x \).

Definition 1.6. We say that a motion \( f(t, x) \) is chaotic for \( f \) if there can be found some point \( y \in X \) such that

\[
\liminf_{t \to +\infty} d(f(t, x), y) = 0, \quad \limsup_{t \to +\infty} d(f(t, x), y) > 0
\]

and

\[
\liminf_{t \to +\infty} d(f(t, x), f(t, y)) = 0, \quad \limsup_{t \to +\infty} d(f(t, x), f(t, y)) > 0.
\]
By using Theorem 1.4, we will show that if $C_x$ is not generic, then the chaotic behavior occurs near $C_x$; see Theorems 3.1 and 3.2 stated and proved in Section 3. On the other hand whenever $C_x$ is generic and it is not “very simple”, then chaotic motions are generic in $C_x$; that is the following

**Theorem 1.7.** Let $f(t, p)$ be a Lagrange stable motion in a Polish space $X$. If $C_p$ is generic and itself is not a minimal subset of $(X, f)$, then there exists a residual subset $S$ of $C_p$ such that $f(t, x)$ is chaotic for each $x \in S$.

In Section 4 we will consider a relationship of the minimal center of attraction of a motion with the pointwise recurrence.

Finally we shall consider the multiply attracting of the minimal center of attraction of a motion in Section 5.

2. Proof of Sigmund’s conjecture

This section will be devoted to proving Karl Sigmund’s Conjecture 1.3 in the continuous-time case; that is, Theorem 1.4.

Recall that if a continuous surjective map $T: X \to X$ is topologically transitive, then for a countable basis $U_1, U_2, \ldots, U_n, \ldots$ of the underlying space $X$, the set

$$\{ x \in X \mid \overline{O_T(x)} = X \} = \bigcap_{n=1}^{+\infty} \bigcup_{m=0}^{+\infty} T^{-m}(U_n),$$

where $O_T(x) = \{ T^n x : n \in \mathbb{Z}_+ \}$, is a dense $G_\delta$ set in $X$ because $\bigcup_{m=0}^{+\infty} T^{-m}(U_n)$ is open and dense in $X$. It is easy to see that this standard argument for topological transitivity does not work here for Sigmund’s conjecture (or Theorem 1.4). So we need a new idea that will explore the times of sojourn of a motion in a domain.

Given any $x \in X$, let $\mathcal{U}_x$ be the neighborhood system of the point $x$ in $X$. To prove Conjecture 1.3, we will need a classical result, which shows that $C_x$ consists of the points $y \in X$ which are interesting for $x$.

**Lemma 2.1** ([10, 15] for $C^0$-flow). Let $f: \mathbb{R}_+ \times X \to X$ be a $C^0$-semi flow on a metric space $X$. Then for any $x \in X$, there holds

$$C_x = \left\{ y \in X \mid \limsup_{T \to +\infty} \frac{1}{T} \int_0^T \mathbb{1}_U(f(t, x))dt > 0 \ \forall U \in \mathcal{U}_x \right\}.$$

**Proof.** For self-closeness, we present an independent proof for this lemma which is shorter than that of [15]. Let $x \in X$ and write

$$I(x) = \left\{ y \in X \mid \limsup_{T \to +\infty} \frac{1}{T} \int_0^T \mathbb{1}_U(f(t, x))dt > 0 \ \forall U \in \mathcal{U}_x \right\}.$$

We first claim that $C_x \subseteq I(x)$. Indeed, for any $q \in C_x$, let $U \in \mathcal{U}_q$ be a neighborhood of $q$ in $X$; then

$$\limsup_{T \to +\infty} \frac{1}{T} \int_0^T \mathbb{1}_U(f(t, x))dt > 0.$$
Otherwise, one would find some $\varepsilon > 0$ so that

$$\lim_{T \to +\infty} \frac{1}{T} \int_0^T 1_{B_3(q)}(f(t,x))\,dt = 0.$$  

Further $C_x - B_2(q)$ is a center of attraction of the motion $f(t,x)$ and we thus arrive at a contradiction to the minimality of $C_x$.

Finally we assert that $C_x \supseteq I(x)$. By contradiction, let $q \in I(x) - C_x$ and then we can take some $\varepsilon > 0$ such that $d(q,C_x) \geq 3\varepsilon$. Set

$$N(B_\varepsilon(q)) = \{ t \geq 0 \mid f(t,x) \in B_\varepsilon(q) \text{ for } 1 \leq i \leq l \}$$

and

$$N(B_\varepsilon(C_x)) = \{ t \geq 0 \mid f(t,x) \in B_\varepsilon(C_x) \text{ for } 1 \leq i \leq l \}.$$  

Clearly $N(B_\varepsilon(q)) \cap N(B_\varepsilon(C_x)) = \emptyset$. However, by definitions, $N(B_\varepsilon(q))$ has positive upper density and $N(B_\varepsilon(C_x))$ has density 1 in $[0,\infty)$. This is a contradiction.

The proof of Lemma 2.1 is thus completed.

In Definition 1.1 we do not require the $f$-invariance of $C_x$. However, it is actually $f$-invariant by Lemma 2.1.

**Corollary 2.2.** Let $f : \mathbb{R}_+ \times X \to X$ be a $C^0$-semi flow on a metric space $X$. For any $x \in X$, $C_x$ is $f$-invariant.

Since for any real number $\theta > 0$ and any integer $N \geq 0$ there holds

$$0 \leq \int_0^\theta 1_U(f(N\theta + t,x))\,dt \leq \theta,$$

then Lemma 2.1 implies immediately the following.

**Corollary 2.3.** Let $f : \mathbb{R}_+ \times X \to X$ be a $C^0$-semi flow on a Polish space $X$. For any $x \in X$ and any $\theta > 0$, there holds

$$C_x = \left\{ y \in X \mid \limsup_{N \to +\infty} \frac{1}{N} \int_0^{N\theta} 1_U(f(t,x))\,dt > 0 \forall U \in \mathcal{U}_y \right\}.$$  

Here $\mathbb{N} = \{ 1, 2, \ldots \}$.

**Proof.** The statement follows from that $T = N_T \theta + r_T$ where $N_T \in \mathbb{N}$, $0 \leq r_T < \theta$ for any $T \geq 1$ and that

$$\limsup_{T \to +\infty} \frac{1}{T} \int_0^T 1_U(f(t,x))\,dt = \limsup_{N \to +\infty} \frac{1}{N\theta} \int_0^{N\theta} 1_U(f(t,x))\,dt$$

for any $U \in \mathcal{U}_y$ and any $y \in X$.  

We are now ready to prove one of our main statements—Theorem 1.4—by applying Lemma 2.1 and Corollaries 2.2 and 2.3.
**Proof of Theorem 1.4.** Write $\Theta = \{ x \in X : C_x = X \}$ and let $x \in X$ be such that $C_x = X$. Then from Lemma 2.1, it follows that the orbit $O_f(x) = f(\mathbb{R}_+, x)$ is dense in $X$ and that $O_f(x) \subseteq \Theta$ such that for any $y \in O_f(x)$,\[ \limsup_{N \to \infty} \frac{1}{N} \int_0^N 1_U(f(t,x))dt = \limsup_{N \to \infty} \frac{1}{N} \int_0^N 1_U(f(t,y))dt, \]
for every nonempty $U \in \mathcal{F}_X$, where $\mathcal{F}_X$ is the topology of the space $X$.

Let $\mathcal{W} = \{ U_i \}_{i=1}^\infty$ be an any given countable base of the topology $\mathcal{F}_X$ of the state space $X$. Then by Corollary 2.3, we can choose a sequence of positive integers $L_i, i = 1, 2, \ldots$, such that\[ \limsup_{N \to \infty} \frac{1}{N} \int_0^N 1_{U_i}(f(t,x))dt > \frac{1}{L_i}, \]
for all $i = 1, 2, \ldots$. For each $i$, write\[ \Theta_i = \left\{ y \in X \mid \forall n_0 \in \mathbb{N}, \exists n > n_0 \text{ with } \int_0^{nL_i} 1_{U_i}(f(t,y))dt > n \right\}. \]
From the continuity of $f(t,y)$ with respect to $(t,y)$ and\[ \Theta_i = \bigcap_{n=1}^{\infty} \bigcup_{n_0=n}^{\infty} \left\{ y \in X \mid \int_0^{nL_i} 1_{U_i}(f(t,y))dt > n \right\}, \]
it follows that $\Theta_i$ is a $G_\delta$ subset of $X$. Because $x$ belongs to $\Theta_i$ by noting $N = n_0L_i + r_N$ and $\lim_{N \to \infty} \frac{r_N}{N} = 0$ where $0 \leq r_N < L_i$ and then similarly $O_f(x) \subseteq \Theta_i$, there follows $\bigcap_{i=1}^{\infty} \Theta_i$ is a dense and $G_\delta$ set in $X$.

Since for any $y \in \bigcap_{i=1}^{\infty} \Theta_i$ we have\[ \limsup_{N \to \infty} \frac{1}{N} \int_0^N 1_{U_i}(f(t,y))dt \geq \frac{1}{L_i} > 0 \]
for any $U_i \in \mathcal{W}$ and $\mathcal{W}$ is a base of the topology $\mathcal{F}_X$ of the space $X$, we see that $y \in \Theta$ from Corollary 2.3. Therefore, $\Theta$ is a residual set in $X$.

This completes the proof of Theorem 1.4. \hfill \square

Finally we mention that the statement of Theorem 1.4 is also valid for a continuous transformation $f : X \to X$ of a Polish space $X$.

### 3. Li-Yorke chaotic pairs and sensitive dependence on initial data

In this section, we shall apply Theorem 1.4 to the study of chaos of a topological dynamical system on a Polish space $X$.

Let $f : \mathbb{R}_+ \times X \to X$ be a $C^0$-semi flow on the Polish space $(X,d)$. Recall that two points $x, y \in X$ is called a Li-Yorke chaotic pair for $f$ if\[ \limsup_{t \to +\infty} d(f(t,x), f(t,y)) > 0 \quad \text{and} \quad \liminf_{t \to +\infty} d(f(t,x), f(t,y)) = 0. \]
That is to say, $x$ is proximal to $y$ but not asymptotical. If there can be found an uncountable set $S \subset X$ such that every pair of points $x, y \in S, x \neq y$, is a Li-Yorke chaotic pair for $f$, then we say $f$ is Li-Yorke chaotic; see, e.g., Li and Yorke 1975 \cite{14}.
3.1. Nongeneric case

Let \( f : \mathbb{R}_+ \times X \to X \) be a \( C^0 \)-semi flow on a compact metric space. The following theorem shows that if \( C_x \) has no the generic dynamics in the sense of Definition 1.5, then \( f \) has the chaotic dynamics.

**Theorem 3.1.** Given any \( x \in X \), if \( C_x \) is not generic, then one can find some point \( q \in \Delta \), for any closed \( f \)-invariant subset \( \Delta \subseteq C_x \), such that \((x, q)\) is a Li-Yorke chaotic pair for \( f \).

**Proof.** Given any \( x \in X \), let \( C_x \) be not generic in the sense of Definition 1.5. Let \( \Delta \) be an \( f \)-invariant nonempty closed subset of \( C_x \). Then by Theorem 1.4 it follows that \( x \notin C_x \). Moreover from Definition 1.1, we can obtain that \( x \) is proximal to \( \Delta \); that is,

\[
\liminf_{t \to +\infty} d(f(t, x), f(t, \Delta)) = 0.
\]

Then from [1, 6] also [7, Proposition 8.6], it follows that there exists some point \( q \in \Delta \) such that

\[
\liminf_{t \to +\infty} d(f(t, x), f(t, q)) = 0.
\]

We claim that \( \limsup_{t \to +\infty} d(f(t, x), f(t, q)) \geq 0 \). Otherwise, \( \lim_{t \to +\infty} d(f(t, x), f(t, q)) = 0 \) and it follows that \( C_x = C_q \); and then \( C_x \) is generic by Theorem 1.4.

The proof of Theorem 3.1 is therefore complete.

**Corollary 1.** Given any \( x \in X \), if \( C_x \) is not generic, then one can find some point \( q \in C_x \) such that \((x, q)\) is a Li-Yorke chaotic pair for \( f \) and the set

\[
N_f(x, B_\varepsilon(q)) = \{ t \geq 0 : f(t, x) \in B_\varepsilon(q) \}
\]

is a central set in \( \mathbb{R}_+ \), which has positive upper density.

**Proof.** Let \( \Delta \subseteq C_x \) be a minimal set. Then there can be found by Theorem 3.1 a point \( q \in \Delta \) such that \((x, q)\) is a Li-Yorke chaotic pair for \( f \). By definition (cf. [7, Definition 8.3] and [5, Definition 7.2]) \( N_f(x, B_\varepsilon(q)) \) is a central set of \( \mathbb{R}_+ \) for each \( \varepsilon > 0 \). In addition, by Lemma 2.1 it follows that \( N_f(x, B_\varepsilon(q)) \) has positive upper density. This proves the corollary.

**Corollary 2.** Let there exist a fixed point or a periodic orbit in the minimal center \( C_x \) of attraction of a motion \( f(t, x) \). Then we can find a Li-Yorke chaotic pair near \( C_x \).

**Proof.** First if \( C_x \) is generic in the sense of Definition 1.5, then \( f \) restricted to it is topologically transitive and has a fixed point or a periodic orbit. Then by Huang and Ye 2002 [11, Theorem 4.1], it follows that \( f \) restricted to \( C_x \) is chaotic in the sense of Li and Yorke.

On the other hand, if \( C_x \) is not generic in the sense of Definition 1.5, then from Theorem 3.1 it follows that there always exists a Li-Yorke chaotic pair near \( C_x \).

The proof of the corollary is thus complete.

Recall that a motion \( f(t, x) \) is referred to as a Birkhoff recurrent motion of \( f \) if \( \overline{O_f(x)} \) is minimal (cf. [15, 3]). It is also called “uniformly recurrent” in [7] and “almost periodic” of von Neumann in [9] in the discrete-time case.

Motivated by [2, 8, 4] we can obtain the following theorem on sensitivity on initial data near the minimal center of attraction of a motion.

\[ \text{ } \]
Theorem 3.2. Let $C_x$, for a motion $f(t, x)$, be not generic. If the Birkhoff recurrent points of $f$ are dense in $C_x$, then $f$ has the sensitive dependence on initial data near $C_x$ in the sense that one can find a sensitive constant $\epsilon > 0$ such that for any $a \in X, \hat{x} \in C_x$ and any $U \in \mathcal{B}_x$, there exists some point $c \in U$ with $\lim \sup_{t \to +\infty} d(f(t, a), f(t, c)) \geq \epsilon$.

Proof. Since $C_x$ is not generic, by Theorem 1.4 it follows that it is not minimal and so it contains at least two different motions of $f$ far away each other. Thus one can find a number $\delta > 0$ such that for all $\hat{x} \in C_x$ there exists a corresponding motion $f(t, q)$ in $C_x$, not necessarily recurrent, such that

$$d(\hat{x}, \overline{O_f(q)}) \geq \delta,$$

where $d(\hat{x}, A) = \inf_{a \in A} d(\hat{x}, a)$ for any subset $A$ of $X$. We will show that $f$ has sensitive dependence on initial data with sensitivity constant $\epsilon = \delta/4$ following the idea of [4, Theorem 4].

For this, we let $\hat{x}$ be an arbitrary point in $C_x$ and let $U$ be an arbitrary neighborhood of $\hat{x}$ in $X$. Since the Birkhoff recurrent motions of $(X, f)$ are dense in $C_x$ from assumption of the theorem, there exists a Birkhoff recurrent point $p \in U \cap B_{r/2}(\hat{x}) \cap C_x$, where $B_r(\hat{x})$ is the open ball of radius $r$ centered at $\hat{x}$ in $X$. As we noted above, there must exist another point $q \in C_x$ whose orbit $O_f(q)$ is of distance at least $4\epsilon$ from the given point $\hat{x}$.

Let $\eta > 0$ be such that $\eta < \epsilon/2$. Then from the Birkhoff recurrence of the motion $f(t, p)$, it follows that one can find a constant $T = T(\eta, p) > 0$ such that for any $\gamma \geq 0$, there is some moment $t_j \in [\gamma, \gamma + T]$ verifying that

$$d(p, f^{\gamma}(p)) < \eta.$$

For the given $q$, we simply write

$$V = \bigcap_{n \in [0, 2T]} f^{-n}(B_\epsilon(f^n(q))), \text{ where } f^{-n}(\cdot) = f(t, \cdot)^{-1}.$$

Clearly from the continuity of topological flow, it follows that $V$ is a neighborhood of $q$ in $X$ but not necessarily open, and it is nonempty since $q \in V$.

Since $\hat{x}$ is the minimal center of attraction of the motion $f(t, x)$, from Lemma 2.1 it follows that there is at least one point $z \in U \cap B_r(\hat{x})$ such that $f^N(z) \in V$ for some sufficiently large number $N \gg T$. Let

$$N = jT + r \text{ where } 0 \leq r < T, \ j \in \mathbb{N},$$

and

$$t_jT \in [jT, (j + 1)T) \text{ such that } d(p, f^{t_jT}(p)) < \eta.$$

Then $0 \leq t_jT - N < 2T$.

By construction, one has

$$f^{t_jT}(z) = f^{t_jT - N}(f^N(z)) \in f^{t_jT - N}(V) \subseteq B_\epsilon(f^{t_jT - N}(q)).$$

From the triangle inequality of metric, it follows that

$$d(f^{t_jT}(p), f^{t_jT}(z)) \geq d(p, f^{t_jT}(z)) - \eta$$

$$\geq d(\hat{x}, f^{t_jT}(z)) - d(p, \hat{x}) - \eta$$

$$\geq d(\hat{x}, f^{t_jT - N}(q)) - d(f^{t_jT - N}(q), f^{t_jT}(z)) - d(p, \hat{x}) - \eta.$$
Consequently, since $\eta < \epsilon/2$, $p \in B_{\epsilon/2}(\xi)$ and $f^{j\sigma}(z) \in B_{\epsilon}(f^{jN}(q))$, it holds that
\[ d(f^{j\sigma}(p), f^{j\sigma}(z)) > 2\epsilon. \]

Therefore from the triangle inequality again, one can obtain either
\[ d(f^{j\sigma}(\xi), f^{j\sigma}(z)) > \epsilon \]
or
\[ d(f^{j\sigma}(\xi), f^{j\sigma}(p)) > \epsilon. \]

Repeating this argument for another likewise $N$ bigger than $(j + 2)T$, one can find a sequence $t_n = j_nT \uparrow +\infty$ as $n \to +\infty$ such that either
\[ d(f^{j_n}(\xi), f^{j_n}(z)) > \epsilon \]
or
\[ d(f^{j_n}(\xi), f^{j_n}(p)) > \epsilon, \]
for all $n \geq 1$. Thus in either case, we have found a point $\tilde{y} \in U$ such that
\[ \limsup_{t \to +\infty} d(f^{j_n}(\xi), f^{j_n}(\tilde{y})) \geq \epsilon. \]

Now for any $a \in X$, using the triangle inequality once more, we see either
\[ \limsup_{t \to +\infty} d(f^{j_n}(\xi), f^{j_n}(a)) \geq \frac{\epsilon}{3} \]
or
\[ \limsup_{t \to +\infty} d(f^{j_n}(\tilde{y}), f^{j_n}(a)) \geq \frac{\epsilon}{3}. \]

Since $\xi, U$ both are arbitrary and $\tilde{y} \in U$, hence the proof of Theorem 3.2 is complete.

We note that if $(C_x, f)$ is distal (cf. [7, Definition 8.2]) and not minimal and even not topologically transitive, then the conditions of Theorem 3.2 hold; i.e., the Birkhoff recurrent points are dense in $C_x$. In fact, $C_x$ consists of Birkhoff recurrent points ([7, Corollary of Theorem 8.7]) and it is not topologically transitive.

### 3.2. Generic case

Let $f : \mathbb{R} \times X \to X$ be a $C^0$-semi flow on a Polish space and $f(t, p)$ a Lagrange stable motion. Then the minimal center $C_p$ of attraction of the motion $f(t, p)$ is always existent. Theorem 1.7 is just a corollary of the following

**Theorem 3.3.** Let $C_p$ be generic and not a minimal subset of $(X, f)$. Then there exists a residual subset $S$ of $C_p$ such that for any $x \in S$ and any minimal subset $\Lambda \subset C_p$, there corresponds some point $y \in \Lambda$ with the properties: $x, y$ form a Li-Yorke chaotic pair for $f$ and
\[
\liminf_{t \to +\infty} d(f(t, x), y) = 0 \quad \text{and} \quad \limsup_{t \to +\infty} d(f(t, x), y) \geq \frac{1}{2} \text{diam}(C_p).
\]
Proof. Since $C_p$ is generic in the sense of Definition 1.5, there exists some point $q \in C_p$ with $C_q = C_p$. Therefore by Theorem 1.4, there is a residual subset $S$ of $C_p$ such that $C_x = C_p$ for all point $x$ in $S$. Because $C_p$ is not a minimal subset of $X$ by hypothesis of Theorem 1.7, $O_f(x)$ is not minimal for each $x \in S$.

Let $\Lambda$ be a minimal subset of $C_p$. Then each $x \in S$ is proximal to $\Lambda$. Moreover by [7, Theorem 8.7], it follows that for every $x \in S$, there corresponds some point $y \in \Lambda$ such that $x$ is proximal to $y$ and $f(t, y)$ is Birkhoff recurrent (or uniformly recurrent). Clearly, $x$ and $y$ is a Li-Yorke chaotic pair for $f$. In addition, from Lemma 2.1 follows that

\[ \limsup_{t \to +\infty} d(f(t, x), y) \geq \frac{1}{2} \text{diam}(C_p) \]

and

\[ \liminf_{t \to +\infty} d(f(t, x), y) = 0. \]

This completes the proof of Theorem 3.3.

Therefore by Theorems 3.1 and 3.3, it follows that every Lagrange stable motion $f(t, x)$ is chaotic in the sense of Definition 1.6 if its minimal center $C_x$ of attraction is not a minimal subset of $(X, f)$.

4. Quasi-weakly almost periodic motion

In this section, we let $f : \mathbb{R}_+ \times X \to X$ be a $C^0$-semi flow on the compact metric space $X$.

Definition 4.1 (Huang-Zhou 2012 [12]). A point $x$ in $X$ is called a quasi-weakly almost periodic point of $f$ if for any $\varepsilon > 0$ there exists an integer $N = N(\varepsilon) \geq 1$ and an increasing positive integer sequence $\{n_j\}$ with the property that for each $j$ there are $0 = t_0 < t_1 < \cdots < t_{n_j} < n_j N$ with $t_{n_j} - t_i \geq 1$ such that $f(t_j, x) \in B_\varepsilon(x)$ for all $i = 1, \ldots, n_j$.

As results of the statements of Lemma 2.1 and Theorem 1.4, we can obtain the following two results.

**Proposition 4.2.** The following statements are equivalent to each other.

1. $x \in X$ is a quasi-weakly almost periodic point of $f$.
2. $x \in C_x$.

**Proof.** (1)$\Rightarrow$(2) follows from Lemma 2.1. (2)$\Rightarrow$(1) follows from Lemma 2.1 and the local section theorem of Bebutov (cf. [15, Theorem V.2.14]).

Proposition 4.2 has been proved in [12] in the case where $x$ is a Poisson stable point of $f$, i.e., there is a sequence $t_n \to \infty$ such that $f(t_n, x) \to x$ as $n \to \infty$.

**Proposition 4.3.** If $x \in C_x$, then the set $\{y \in C_x | y \in C_y = C_x\}$ is dense and $G_\delta$ relative to the subspace $C_x$.

**Proof.** The statement follows from Theorem 1.4 with $C_x$ replacing of $X$. 

9
5. Minimal center of multi-attraction of a motion

Let \( f: \mathbb{R}_+ \times X \to X \) be a C^\alpha-semi flow on a Polish space \( X \). From now on, by \( \lambda(dt) \) we denote the standard Haar (Lebesque) measure on \( \mathbb{R} \). We will need the following simple but useful fact.

**Lemma 5.1.** Let \( S \) be a measurable subset of \( \mathbb{R}_+ \) and \( \tau > 0 \). If \( S \) has the density \( \alpha \), i.e.,
\[
D(S) := \lim_{T \to +\infty} \frac{\lambda(S \cap [0, T])}{T} = \alpha,
\]
then \( \tau S = \{ \tau t: t \in S \} \) also has the density \( \alpha \) in \( \mathbb{R}_+ \).

**Proof.** Let \( \tau > 0 \) be any given. Since \( \lambda(\tau A) = \tau \lambda(A) \) and \( \lambda(\tau S \cap [0, T]) = \tau \lambda(S \cap [0, T\tau^{-1}]) \), hence it follows that \( D(\tau S) = 1 \). This proves the lemma. \( \Box \)

It should be noted here that there is no an analogous result for the discrete-time \( \mathbb{Z}_+ \); for example, \( S = \{0, 2, 4, 6, \ldots\} \) has the density \( \frac{1}{2} \) but \( \frac{1}{\tau} S \) has the density 1 in \( \mathbb{Z}_+ \).

The following lemma shows that every minimal center of attraction of a motion \( f(t, x) \) is multiply attracting as \( t \to +\infty \).

**Lemma 5.2.** Let \( C_x \) be existent for a motion \( f(t, x) \) as \( t \to +\infty \). Then for any \( t_1 > 0, \ldots, t_l > 0 \) where \( l \in \mathbb{N} \) and any \( \epsilon > 0 \),
\[
\lim_{T \to +\infty} \frac{1}{T} \int_0^T \mathbb{1}_{B_{\epsilon}(C_x)}(f(t_1 t, x)) \cdots \mathbb{1}_{B_{\epsilon}(C_x)}(f(t_l t, x)) dt = 1.
\]

**Proof.** For any \( \tau > 0 \) and any open set \( U \), define an open set
\[
N_\epsilon(x, U) = \{ t: 0 \leq t < +\infty, f(\tau t, x) \in U \}.
\]
It is easy to check that \( \tau^{-1} N_\epsilon(x, U) = N_\epsilon(x, U) \). Then by Lemma 5.1 and Definition 1.1, it follows that \( N_\epsilon(x, B_{\epsilon}(C_x)), \ldots, N_\epsilon(x, B_{\epsilon}(C_x)) \) all have the density 1. Thus
\[
N_{\epsilon, \ldots, \epsilon}(x, B_{\epsilon}(C_x)) := N_\epsilon(x, B_{\epsilon}(C_x)) \cap \cdots \cap N_\epsilon(x, B_{\epsilon}(C_x))
\]
also has the density 1 in \( \mathbb{R}_+ \). This completes the proof of Lemma 5.2. \( \Box \)

Recall that a motion \( f(t, x) \) is said to be Lagrange stable as \( t \to +\infty \) if the orbit-closure \( \overline{O_f(x)} \) is compact in \( X \) (cf. [15]). As a direct result of Lemma 5.2, we can obtain the following.

**Corollary 5.3.** For any Lagrange stable motion \( f(t, x) \) as \( t \to +\infty \), \( C_x \) is the minimal closed subset of \( X \) such that for any \( t_1 > 0, \ldots, t_l > 0 \) and any \( \epsilon > 0 \),
\[
\lim_{T \to +\infty} \frac{1}{T} \int_0^T \mathbb{1}_{B_{\epsilon}(C_x)}(f(t_1 t, x)) \cdots \mathbb{1}_{B_{\epsilon}(C_x)}(f(t_l t, x)) dt = 1.
\]

This result shows that \( C_x \) is the “minimal center of multi-attraction” of a Lagrange stable motion \( f(t, x) \) as \( t \to +\infty \).

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References

[1] J. Auslander, *On the proximal relation in topological dynamics*, Proc. Amer. Math. Soc., 11 (1960), 890–895.
[2] J. Banks, J. Brooks, G. Cairns, G. Davis, and P. Stacey, *On Devaney’s definition of chaos*, Amer. Math. Monthly, 99 (1992), 332–334.
[3] B. Chen and X. Dai, *On uniformly recurrent motions of topological semigroup actions*, Discret. Contin. Dyn. Syst., in press, 2015.
[4] X. Dai, *Chaotic dynamics of continuous-time topological semiflow on Polish spaces*, J. Differential Equations, 258 (2015), 2794–2805.
[5] X. Dai, *On the Furstenberg-Zimmer structure theorems for Noetherian modules over syndetic rings*, Preprint, 2015.
[6] R. Ellis, *A semigroup associated with a transformation group*, Trans. Amer. Math. Soc., 94 (1960), 272–281.
[7] H. Furstenberg, *Recurrence in Ergodic Theory and Combinatorial Number Theory*, Princeton University Press, Princeton, New Jersey, 1981.
[8] E. Glasner and B. Weiss, *Sensitive dependence on initial conditions*, Nonlinearity, 6 (1993), 1067–1075.
[9] W.H. Gottschalk and G.A. Hedlund, *Topological Dynamics*, Amer. Math. Soc. Colloq. Publ., vol. 36, Amer. Math. Soc., Providence, RI, 1955.
[10] H.F. Hilmy, *Sur les centres d’attraction minimaux des systèmes dynamiques*, Compositio Math., 3 (1936), 227–238.
[11] W. Huang and Y.-D. Ye, *Devaney’s chaos or 2-scattering implies Li-Yorke’s chaos*, Topology and its Appl., 117 (2012), 259–272.
[12] Y. Huang and Z.-L. Zhou, *Two new recurrent levels for C^0-flows*, Acta Appl. Math., 118 (2012), 125–145.
[13] K. Jacobs, *Einige Grundbegriffe der topologischen Dynamik*, Math. Phys. Semesterberichte, 14 (1967), 129–150.
[14] T. Li and J. Yorke, *Period 3 implies chaos*, Amer. Math. Monthly, 82 (1975), 985–992.
[15] V.V. Nemytskii and V.V. Stepanov, *Qualitative Theory of Differential Equations*, Princeton University Press, Princeton, New Jersey 1960.
[16] K. Sigmund, *On minimal centers of attraction and generic points*, J. Reine Angew. Math., 295 (1977), 72–79.