Abstract. Given two graphs $G$ and $H$, there is a bi-resolving (or bi-covering) graph homomorphism from $G$ to $H$ if and only if their adjacency matrices satisfy certain matrix relations. We investigate the bi-covering extensions of bi-resolving homomorphisms and give several sufficient conditions for a bi-resolving homomorphism to have a bi-covering extension with an irreducible domain. Using these results, we prove that a bi-closing code between subshifts can be extended to an $n$-to-1 code between irreducible shifts of finite type for all large $n$.

1. Introduction

Resolving homomorphisms arose independently under different names in different fields of mathematics. These homomorphisms were introduced in the field of symbolic dynamics to solve the finite equivalence problem [1, 2] and they form a fundamental class of finite-to-one codes between subshifts. In particular, all the known general constructions of finite-to-one factor codes between irreducible shifts of finite type with equal entropy use resolving codes [3, 7, 17]. Covering homomorphisms, resolving ones with the lifting property, are closely related to the graph divisors and equitable partitions in the theory of spectra of graphs [10] and its references). They also appear in the categorical approach of graph fibration with the name of fibrations and opfibrations [5], and play a significant role in the theory of graph embeddings as “voltage graphs” [11, 12].

This paper is an attempt to investigate the existence and the extension of bi-resolving homomorphisms, i.e., both left and right resolving ones. They have more rigid structure than left or right resolving ones [18]. Even if two graphs $G$ and $H$ admit a left covering homomorphism and a right covering homomorphism between them, they need not admit a bi-covering one. We show that there is a bi-resolving (resp. bi-covering) homomorphism from a graph $G$ to another graph $H$ if and only if there is a subamalgamation matrix $S$ such that $A_G S \leq S A_H$ and $S^T A_G \leq A_H S^T$ (resp. $A_G S = S A_H$ and $S^T A_G = A_H S^T$), where $A_G$ and $A_H$ are the adjacency matrices of $G$ and $H$, respectively (see Theorems 3.1 and 3.2). These results can be considered as an analogue of the well-known description that there is a right resolving homomorphism from $G$ to $H$ if and only if there is a subamalgamation matrix $S$ such

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that $A_G S \leq S A_H$ (e.g., in [17]). We also investigate bi-covering extensions of bi-resolving homomorphisms in §3. Every bi-resolving homomorphism $\Phi : G \to H$ can be extended to a bi-covering homomorphism $\tilde{\Phi} : \tilde{G} \to H$ by enlarging the domain. We present sufficient conditions for $\tilde{G}$ to be irreducible when $H$ is irreducible (see Theorem 3.3).

There has been considerable attention on extending sliding block codes in symbolic dynamics. One can consider the following extension problem: Given a code $\phi : X \to Y$ and $\tilde{X} \supset X$ with certain properties, extend $\phi$ to a factor code from $\tilde{X}$ onto $Y$, respecting the properties. There are several results to this extension problem for infinite-to-one codes [6, 9, 13], for finite-to-one closing codes [3], and for inert automorphisms [14]. In particular, Ashley proved that if $X$ and $Y$ are mixing shifts of finite type, then any right closing code from a shift of finite type $Z \subset X$ can be extended to a right closing code from $X$ to $Y$ [3]. An analogous statement for bi-closing codes is false (see Example 4.4), thus we are led to consider the weaker version of the extension problem: Given a code $\phi : X \to Y$ with certain properties, construct an enlarged domain $\tilde{X}$ and extend $\phi$ to a factor code on $\tilde{X}$, respecting the properties.

The paper [5] provides many results to this weaker version of the extension problem. One of them concerns bi-closing codes: If $\phi : X \to Y$ is a bi-closing code between irreducible shifts of finite type, then there are an irreducible shift of finite type $\tilde{X}$ and a bi-closing extension $\tilde{\phi} : \tilde{X} \to Y$ of $\phi$. In §4 we extend this result as follows: Given a bi-closing code from a subshift $X$ to an irreducible shift of finite type $Y$ with $h(X) < h(Y)$, for all large $n$ there exist an irreducible shift of finite type $\tilde{X}$ and an $n$-to-1 (hence bi-closing) extension from $\tilde{X}$ onto $Y$. If $X$ is of finite type, then this result holds for every $n$ greater than the maximum number of $\phi$-preimages (see Theorem 4.3). This is related to the result in [15], which says that for a mixing shift of finite type $X$, there is a family of mixing shifts of finite type each of which is a constant-to-one extension of $X$. (In [15], each extension is a skew-product of $X$ with a group of the form $Z/pZ$).

2. Background

In this section, we recall some terminology and elementary results. For further details, see [17]. A (directed) graph $G$ is defined to be a pair $(V, E)$, where $V = V(G)$ is a finite set of vertices and $E = E(G)$ is a finite set of edges. We call $G$ irreducible if for each pair $(I, J)$ of vertices there exists a path from $I$ to $J$. A graph is weakly connected if its underlying graph is connected (i.e., it is possible to reach any vertex starting from any other vertex by traversing edges in some direction). An irreducible component of a graph is a maximal irreducible subgraph.

Let $G$ and $H$ be graphs. A (graph) homomorphism from $G$ into $H$ is a pair $\Phi = (\Phi_V, \Phi_E)$ of mappings $\Phi_V : V(G) \to V(H)$ and $\Phi_E : E(G) \to E(H)$ which respect adjacency. The edge map $\Phi_E$ naturally extends to paths. We say $\Phi : G \to H$ is right resolving (resp. right covering) if $\Phi_E|_{E_I(G)} : E_I(G) \to E_{\Phi_V(I)}(H)$ is injective (resp. bijective) for each $I \in V(G)$, where $E_I(G)$ is the set of all edges starting from $I$. Similarly, left resolving and left covering homomorphisms can be defined. If $\Phi$ is both left and right resolving (resp. covering) then it is called bi-resolving (resp.
bi-covering). If $\Phi : G \to H$ is bi-covering, then for each path $\pi$ in $H$ the paths in $\Phi^{-1}(\pi)$ are mutually separated, i.e., they do not share a vertex at the same time.

A 0-1 matrix is called a subamalgamation matrix if it has exactly one 1 in each row. An amalgamation matrix is a subamalgamation matrix which has at least one 1 in each column. For two graphs $G$ and $H$, any subamalgamation matrix $S$ indexed by $\mathcal{V}(G) \times \mathcal{V}(H)$ uniquely determines a vertex mapping $\Phi_V$ (and vice versa) by letting $\Phi_V(i) = I$ if and only if $S_{i,I} = 1$.

In §4, we apply the results on graph homomorphisms to obtain certain results in symbolic dynamics. We assume some familiarity with symbolic dynamics. See [16][17] for more on symbolic dynamics.

3. Existence and Extension of Bi-Resolving Homomorphisms

In this section, we investigate the existence and the extension of bi-resolving homomorphisms. We first show that a known necessary condition for the existence of a bi-resolving (resp. bi-covering) homomorphism is also sufficient.

**Theorem 3.1.** Let $G$ and $H$ be graphs. Then there exists a bi-covering homomorphism from $G$ to $H$ if and only if there exists a subamalgamation matrix $S$ with $A_G S = S A_H$ and $A_G S^T = A_H S^T$.

**Proof.** The ‘only if’ part is well known [17 §8.2]. We will show the converse.

Let $A_G = (a_{i,j}), A_H = (b_{i,j})$ and $S = (S_{i,I})$. Let $\Phi_V : \mathcal{V}(G) \to \mathcal{V}(H)$ be the vertex mapping induced by $S$, i.e., for each $i \in \mathcal{V}(G)$, $\Phi_V(i)$ is a unique vertex $I \in \mathcal{V}(H)$ with $S_{i,I} = 1$. Let $\mathcal{V}_I = \Phi_V^{-1}(I)$ for $I \in \mathcal{V}(H)$. For $I, J \in \mathcal{V}(H)$ with $\mathcal{V}_I, \mathcal{V}_J$ nonempty, let $A_{I,J} = (a_{i,j})_{i \in \mathcal{V}_I, j \in \mathcal{V}_J}$ be the (rectangular) submatrix of $A_G$, and let $G_{I,J}$ be the subgraph of $G$ determined by $A_{I,J}$, i.e., $G_{I,J}$ has the vertex set $\mathcal{V}_I \cup \mathcal{V}_J$ and its edge set, say $\mathcal{E}_{I,J}$, which is the set of edges going from a vertex in $\mathcal{V}_I$ to a vertex in $\mathcal{V}_J$. Since $A_G S = S A_H$, it follows that $\sum_{j \in \mathcal{V}_J} a_{i,j} = b_{I,J}$ for $I, J \in \mathcal{V}(H)$ and $i \in \mathcal{V}_I$, which implies that if $\mathcal{V}_I \neq \emptyset$ and $\mathcal{V}_J \neq \emptyset$, then every row sum of the matrix $A_{I,J}$ is equal to $b_{I,J}$, and that if $\mathcal{V}_I \neq \emptyset$ and $\mathcal{V}_J = \emptyset$, then $b_{I,J} = 0$. Similarly, since $A_G S^T = A_H S^T$, we see that if $\mathcal{V}_I \neq \emptyset$ and $\mathcal{V}_J \neq \emptyset$, then every column sum of the matrix $A_{I,J}$ is equal to $b_{I,J}$, and that if $\mathcal{V}_I = \emptyset$ and $\mathcal{V}_J \neq \emptyset$, then $b_{I,J} = 0$. Therefore, if $\mathcal{V}_I \neq \emptyset, \mathcal{V}_J \neq \emptyset$ and $b_{I,J} \neq 0$, then $|\mathcal{V}_I| = |\mathcal{V}_J|$, so that $A_{I,J}$ is a nonnegative integral square matrix with every row and column sum equal to $b_{I,J}$. It is well known that a nonnegative integral square matrix with every row and column sum equal to $R$ is the sum of $R$ permutation matrices (e.g. [20] §5). Therefore $A_{I,J}$ is the sum of $b_{I,J}$ permutation matrices, so that $\mathcal{E}_{I,J}$ is partitioned into disjoint $b_{I,J}$ subsets each of which consists of vertex-separated $|\mathcal{V}_I|$ edges (i.e. every distinct two of the $|\mathcal{V}_I|$ edges go neither from the same vertex nor to the same vertex). Hence we can define a graph homomorphism $\Phi_{I,J} : G_{I,J} \to H$ which sends all edges in every one of the $b_{I,J}$ subsets to some one of the $b_{I,J}$ edges going from $I$ to $J$ in $H$ so that the edges in distinct subsets may be sent to distinct edges.

There exists a graph homomorphism $\Phi : \tilde{G} \to H$ which is an extension of $\Phi_{I,J}$ for all $I, J \in \mathcal{V}(H)$ with $b_{I,J} \neq 0$. It follows that $\Phi$ is bi-covering. \(\square\)
Theorem 3.2. Let $G$ and $H$ be graphs. Then there exists a bi-resolving homomorphism from $G$ to $H$ if and only if there exists a subamalgamation matrix $S$ with $A_G S \leq SA_H$ and $S^T A_G \leq A_H S^T$.

Proof. As in Theorem 3.1, the ‘only if’ part is well known [17, §8.2]. We will show the converse. Notation being the same as in the proof above, let $d = \max\{|V_I| : I \in \mathcal{V}(H)\}$. Let $\hat{G}$ and $\hat{\Phi}_V : \mathcal{V}(\hat{G}) \to \mathcal{V}(H)$ be the extensions of $G$ and $\Phi_V$, respectively, such that $\hat{G}$ is obtained from $G$ by adding new $d - |V_I|$ (isolated) vertices, which are mapped to $I$ by $\hat{\Phi}_V$, for every $I \in \mathcal{V}(H)$. Then $|\mathcal{V}(\hat{G})| = d|\mathcal{V}(H)|$. If we define $\hat{S} = (\hat{S}_{i,J})$ to be the $|\mathcal{V}(\hat{G})| \times |\mathcal{V}(H)|$ 0-1 matrix such that $\hat{\Phi}_V(i) = I$ if and only if $\hat{S}_{i,J} = 1$, then $\hat{S}$ is an amalgamation matrix with $A_G \hat{S} \leq SA_H$ and $\hat{S}^T A_G \leq A_H \hat{S}^T$.

Let $\hat{V}_I = \hat{\Phi}_V^{-1}(I)$ for $I \in \mathcal{V}(H)$. Let $\hat{A}_{I,J} = (\hat{a}_{i,j})_{i \in \hat{V}_I, j \in \hat{V}_J}$ be the $d \times d$ submatrix of $A_{\hat{G}} = (\hat{a}_{i,j})$ for $I, J \in \mathcal{V}(H)$. Then since $A_{\hat{G}} \hat{S} \leq SA_H$, it follows that $\sum_{j \in \hat{V}_J} \hat{a}_{i,j} \leq b_{I,J}$ for $I, J \in \mathcal{V}(H)$ and $i \in \hat{V}_I$, so that each row sum of $\hat{A}_{I,J}$ is not greater than $b_{I,J}$ for each $I, J \in \mathcal{V}(H)$. Similarly, since $\hat{S}^T A_G \leq A_H \hat{S}^T$, each column sum of $\hat{A}_{I,J}$ is not greater than $b_{I,J}$ for each $I, J \in \mathcal{V}(H)$. We can obtain from $\hat{A}_{I,J}$ a nonnegative integral $d \times d$ matrix $\hat{A}_{I,J}$ with every row and column sum equal to $b_{I,J}$ by adding a necessary number of “1”s to the components of $\hat{A}_{I,J}$ (i.e. by adding a necessary number of new edges going from $\hat{V}_I$ to $\hat{V}_J$). For generally, if $A$ is a nonnegative integral $d \times d$ matrix with every row and column sum not greater than $b$ and with the total sum of components equal to $t$, then $bd - t$ times additions of “1” to an appropriate component each time, give a matrix $\hat{A}$ with every row and column sum equal to $b$. (This is straightforwardly proved by induction on $bd - t$.) Let $\hat{G}_{I,J}$ be the graph determined by $\hat{A}_{I,J}$. There exists a minimal extension $\hat{G}$ of $G$ such that $\hat{G}_{I,J}$ is a subgraph of $\hat{G}$ for all $I, J \in \mathcal{V}(H)$. Since $\hat{A}_{I,J}$ has every row and column sum equal to $b_{I,J}$ for all $I, J \in \mathcal{V}(H)$, it follows from the proof of Theorem 3.1 that there exists a bi-covering homomorphism $\hat{\Phi} : \hat{G} \to H$. The restriction of $\hat{\Phi}$ on $G$ is a desired bi-resolving homomorphism. □

An extension of a homomorphism $\Phi : G \to H$ is a homomorphism $\hat{\Phi} : \hat{G} \to H$ such that $\hat{G}$ is a graph containing $G$ and $\hat{\Phi}|_G = \Phi$. In the remainder of this section, we investigate bi-covering extensions of bi-resolving homomorphisms. In what follows, the degree of a homomorphism $\Phi : G \to H$, denoted by $\deg \Phi$, is the maximum number of preimages of vertices in $H$ under $\Phi_V$. For a graph $G$, denote by $\lambda_G$ the spectral radius of its adjacency matrix $A_G$.

Theorem 3.3. Let $\Phi : G \to H$ be a bi-resolving homomorphism with $H$ irreducible. Let $d = \deg \Phi$.

(1) If $G$ is weakly connected, then there exists a bi-covering extension $\tilde{\Phi} : \tilde{G} \to H$ of $\Phi$ with $\tilde{G}$ irreducible and $\deg \tilde{\Phi} = d$.

(2) If $\lambda_H > \lambda_G$ and $n > d$, then there exists a bi-covering extension $\tilde{\Phi} : \tilde{G} \to H$ of $\Phi$ with $\tilde{G}$ irreducible and $\deg \tilde{\Phi} = n$.

Proof. Let $\mathcal{V}_I = \Phi_V^{-1}(I)$ for $I \in \mathcal{V}(H)$. We may assume that $|\mathcal{V}_I| = d$ by adding new $d - |\mathcal{V}_I|$ (isolated) vertices for every $I \in \mathcal{V}(H)$. Further, we may assume that $n = d + 1$ in (2) by adding new $n - d - 1$ vertices for every $I \in \mathcal{V}(H)$. For $I, J \in \mathcal{V}(H)$, let $B_{I,J}$ the set of edges going from $I$ to $J$. 


Let $I, J \in \mathcal{V}(H)$ with $B_{I,J} \neq \emptyset$. Let $G_{I,J}$ be the subgraph of $G$ whose vertex set is $V_I \cup V_J$ and whose edge set, say $E_{I,J}$, is the set of all edges going from a vertex in $V_I$ to a vertex in $V_J$. Note that $E_{I,J}$ may be empty. Let $\Phi_{I,J} : G_{I,J} \to H$ be the restriction of $\Phi$ on $G_{I,J}$. Since $\Phi$ is bi-resolving, for each $b \in B_{I,J}$, $\Phi_{I,J}^{-1}(b)$ consists of vertex-separated edges. For each $b \in B_{I,J}$, if $|\Phi_{I,J}^{-1}(b)| < d$, we can add new $d - |\Phi_{I,J}^{-1}(b)|$ edges, which will be called the new $b$-edges, to the graph $G_{I,J}$ so that the new $b$-edges together with the edges in $\Phi_{I,J}^{-1}(b)$, which we will call the old $b$-edges, may be all vertex-separated. Let $G^*_{I,J}$ be the graph extension of $G_{I,J}$ with the new $b$-edges added for all $b \in B_{I,J}$. Let $\Phi_{I,J} : G^*_{I,J} \to H$ be the extension of $\Phi_{I,J}$ which sends all old and new $b$-edges to $b$ for all $b \in B_{I,J}$.

There exists a minimal extension $\tilde{G}$ of $G$ such that $\tilde{G}_{I,J}$ is a subgraph of $\tilde{G}$ for all $I, J \in \mathcal{V}(H)$ with $B_{I,J} \neq \emptyset$. There exists an extension $\tilde{\Phi} : \tilde{G} \to H$ of $\Phi$ whose restriction of $\tilde{\Phi}$ on $\tilde{G}_{I,J}$ is $\Phi_{I,J}$ for all $I, J \in \mathcal{V}(H)$ with $B_{I,J} \neq \emptyset$. Since $\tilde{\Phi} : \tilde{G} \to H$ is bi-covering and $H$ is irreducible, $\tilde{G}$ is the disjoint union of finitely many irreducible graphs. Therefore if $G$ is weakly connected, then there exists an irreducible component $\tilde{G}$ such that $G$ is a subgraph of $\tilde{G}$. Hence the restriction $\tilde{\Phi}$ of $\tilde{\Phi}$ on $\tilde{G}$ is a bi-covering homomorphism desired in (1). (Indeed, one can check that $\tilde{G} = \tilde{G}$ in this case.)

Suppose now $\lambda_G < \lambda_H$. Then each irreducible component of $\tilde{G}$ has a new $b$-edge for some edge $b$ in $H$. Let $\tilde{G}_1, \ldots, \tilde{G}_m$ be the irreducible components of $\tilde{G}$ and let $e_k$ be a new $b_k$-edge in $\tilde{G}_k$ with $b_k = \tilde{\Phi}(e_k)$ for $k = 1, \ldots, m$. Let $\tilde{G}_0$ be a copy of $H$. We assume that for each edge $b$ in $H$, the copy $e_b$ (in $\tilde{G}_0$) of $b$ is a new $b$-edge. For $k = 0, \ldots, m$, we define an irreducible graph $\tilde{G}_k$ inductively. Let $\tilde{G}_0 = \tilde{G}_0$. Assuming that $\tilde{G}_{k-1}$ is an irreducible graph which has a new $b$-edge for all edge $b$ in $H$, we define $\tilde{G}_k$ as follows: let the new $b_k$-edge $e_k$ and one of the new $b_k$-edges of $\tilde{G}_{k-1}$ exchange their terminal vertices; then we can merge $\tilde{G}_k$ and $\tilde{G}_{k-1}$ into one irreducible graph $\tilde{G}_k$, which has a new $b$-edge for all edge $b$ in $H$.

Let $\tilde{G} = \tilde{G}_m$. We have a graph homomorphism $\tilde{\Phi} : \tilde{G} \to H$ which sends all old and new $b$-edges to $b$ for all edges $b$ in $H$. Then $\tilde{\Phi}$ is a bi-covering extension of $\Phi$ with $\tilde{G}$ irreducible and $\deg \tilde{\Phi} = d + 1 = n$. Hence (2) is proved. \hfill \Box

The proof of Theorem 3.3(1) shows that every bi-resolving homomorphism can be extended to a bi-covering one with the same degree by enlarging the domain. We remark that Theorem 3.3 also holds if we replace irreducible with weakly connected. Note that the assumption $\lambda_H > \lambda_G$ in the theorem is crucial. Indeed, an application of Perron-Frobenius theorem shows that if $H$ is irreducible, $G$ is not irreducible and $\lambda_G = \lambda_H$, then $\tilde{G}$ cannot be irreducible for any bi-covering extension $\tilde{\Phi} : \tilde{G} \to H$ of $\Phi$.

**Example 3.4.** Let $G$ and $H$ be graphs as below and $\Phi : G \to H$ a subscript dropping homomorphism. It is easy to check that there is no bi-covering extension of $\Phi$ with degree 2 and with a weakly connected domain. This example shows that the assumption $n > \deg \Phi$ in Theorem 3.3(2) is crucial.
4. Extension of bi-closing codes

In this section, we investigate the extension property of bi-closing codes between general shift spaces. We prove that a bi-closing code between subshifts can be extended to an \( n \)-to-1 code between irreducible shifts of finite type for all large \( n \). When the domain is of finite type, we give a lower bound of degrees of extensions in the sense that there is \( N \in \mathbb{N} \) such that the above result holds for every \( n \geq N \).

We recall some definitions. A shift space (or subshift) is a closed shift-invariant subset of a full shift. A subshift is indecomposable if it is not the union of two disjoint nonempty subshifts \([8]\), and irreducible if it has a dense forward orbit. For a subshift \( X \), denote by \( B_n(X) \) the set of all words of length \( n \) appearing in the points of \( X \), and by \( h(X) \) the topological entropy of \( X \). A code is a continuous shift-commuting map between shift spaces. A code is right closing (resp. left closing) if it never collapses two distinct left (resp. right) asymptotic points, and bi-closing if it is both left and right closing. The edge shift \( X_G \) is the set of all bi-infinite trips on a graph \( G \). Every homomorphism \( \Phi : G \to H \) induces the code \( \phi : X_G \to X_H \) by letting \( \phi(x) = \Phi(x) \). A subshift is called a shift of finite type if it is conjugate to an edge shift.

We adopt the ideas from \([8, \text{Lemma 2.4}]\) to prove the following lemma on recoding.

**Lemma 4.1.** Let \( X \) and \( \bar{X} \) be shift spaces with \( X \subset \bar{X} \) and \( \phi : X \to Y \) a conjugacy. Then there exist a shift space \( \bar{Y} \supset Y \) and a conjugacy \( \bar{\phi} : \bar{X} \to \bar{Y} \) such that \( \bar{\phi}|_{X} = \phi \).

**Proof.** We may assume that \( \phi \) is 1-block and \( \phi^{-1} \) has \( N \in \mathbb{N} \) as its memory and anticipation, and that no \( X \)-word occurs as a symbol for \( Y \). Define an alphabet \( \mathcal{A} = B_1(Y) \cup (B_{2N+1}(\bar{X}) \setminus B_{2N+1}(X)) \). We will regard \( u \in B_{2N+1}(\bar{X}) \setminus B_{2N+1}(X) \) as a symbol in \( \mathcal{A} \). Define \( \bar{\phi} : \bar{X} \to \mathcal{A}^2 \) by

\[
\bar{\phi}(x)_i = \begin{cases} 
\phi(x_i) & \text{if } x_{[-N+i,N+i]} \in B_{2N+1}(X) \\
x_{[-N+i,N+i]} & \text{otherwise.}
\end{cases}
\]

and let \( \bar{Y} = \bar{\phi}(\bar{X}) \). Note that if \( \phi(x) = y \), then each \( y_{[-N+i,N+i]} \) determines \( x_i \) uniquely. Thus \( \bar{\phi} \) is a conjugacy onto its image \( \bar{Y} \). \( \square \)

For a graph \( G \) and \( N \in \mathbb{N} \), denote by \( G^{[N]} \) the \( N \)-th higher graph of \( G \) \([17, \text{§2.3}]\). A graph homomorphism \( \Phi : G \to H \) naturally induces the graph homomorphism \( \Phi^{[N]} : G^{[N]} \to H^{[N]} \) for each \( N \). If \( \Phi \) is bi-resolving, then clearly so is \( \Phi^{[N]} \). For a shift space \( X \) and \( N \in \mathbb{N} \), denote by \( X^{[N]} \) the \( N \)-th higher block shift of \( X \). Then \( X \) is conjugate to \( X^{[N]} \) by the conjugacy \( \beta_{N,X} : X \to X^{[N]} \) where \( \beta_{N,X}(x)_i = x_{[i,i+N-1]} \).

It is known that a code between irreducible shifts of finite type is constant-to-one if and only if it is conjugate to a code induced by a bi-covering homomorphism \([18]\). In this case the number of preimages of each point under the code is equal to the degree
of the homomorphism. In what follows, a graph is called essential if each vertex has an incoming edge and an outgoing edge.

**Lemma 4.2.** Let $\Phi : G \to H$ be a bi-resolving homomorphism where $G$ and $H$ are essential. Let $d = \max\{|\phi^{-1}(y)| : y \in X_H\}$, where $\phi : X_G \to X_H$ is the code induced by $\Phi$. Then there exists $N \in \mathbb{N}$ such that $\deg \Phi[N] = d$.

**Proof.** Since $\Phi$ is bi-resolving, the preimages of a given path must be mutually separated. Since $G$ and $H$ are essential, it follows from compactness that there is $n \in \mathbb{N}$ such that $|\Phi^{-1}(\pi)| \leq d$ for all paths $\pi$ of length greater than $n$. Take $N = n + 2$. Then $|(\Phi[N])^{-1}(J)| \leq d$ for all $J \in \mathcal{V}(G[N])$. Thus $\deg \Phi[N] \leq d$.

Suppose $\deg \Phi[N] < d$ and let $\phi[N]$ be the code induced by $\Phi[N]$. Since $\Phi[N]$ is bi-resolving, the number of preimages of a point under $\phi[N]$ must be less than $d$. This contradicts that $\phi$ is conjugate to $\phi[N]$. Thus $\deg \Phi[N] = d$. \qed

Now we prove the main theorem of this section which says that, in a sense, every bi-closing code sits in a constant-to-one code between irreducible shifts of finite type.

**Theorem 4.3.** Let $X$ be a shift space, $Y$ an irreducible shift of finite type with $h(X) < h(Y)$, and $\phi : X \to Y$ a bi-closing code. Let $d = \max\{|\phi^{-1}(y)| : y \in Y\}$.

1. If $X$ is an indecomposable shift of finite type, then for all $n \geq d$, there exist an irreducible shift of finite type $\tilde{X} \supset X$ and an $n$-to-1, onto extension $\tilde{\phi} : \tilde{X} \to Y$ of $\phi$.
2. If $X$ is of finite type, then the conclusion of (1) holds if “for all $n \geq d$” is replaced by “for all $n \geq d + 1$”.
3. If $X$ is a shift space, then the conclusion of (1) holds if “for all $n \geq d$” is replaced by “for all $n \geq m$ with some $m \geq d$”.

**Proof.** (1)/(2) Since $X$ and $Y$ are shifts of finite type and $Y$ is irreducible, using the higher block presentations and the recoding construction of [16, §4.3], we know that there exist an essential graph $G$, an irreducible graph $H$, a bi-resolving homomorphism $\Psi : G \to H$, a conjugacy $\alpha : X \to X_G$, and a higher block code $\beta : Y \to X_H$ such that $\phi = \beta^{-1}\psi\alpha$, where $\psi$ is the 1-block code induced by $\Psi$. By Lemma 4.2, there exists $N \geq 1$ such that $\deg \Psi[N] = d$. Letting $G_1 = G[N], H_1 = H[N]$ and $\Psi_1 = \Psi[N]$, we have $\phi = \beta^{-1}\beta_{N,X_H}\psi_1\beta_{N,X_G}\alpha$, where $\psi_1$ is the 1-block code induced by $\Psi_1$.

By Theorem 3.3 there exists a bi-covering extension $\tilde{\Psi}_1 : \tilde{G}_1 \to H_1$ of $\Psi_1$ with $\tilde{G}_1$ irreducible and $\deg \tilde{\Psi}_1 = n$. By Lemma 4.1, there exist an irreducible shift of finite type $\tilde{X}$ with $\tilde{X} \supset X$ and a conjugacy $\theta : \tilde{X} \to X_{\tilde{G}_1}$ such that $\theta|_X = \beta_{N,X_G}\alpha$. If we let $\tilde{\phi} = \beta^{-1}\beta_{N,X_H}\tilde{\psi}_1\theta$, where $\tilde{\psi}_1$ is the 1-block code induced by $\tilde{\Psi}_1$, then $\tilde{\phi}$ is an extension desired in (1) and (2).

3. Let $A = B_1(X)$. Then $X$ is a subshift over the alphabet $A$. Define $\phi$ by an $M$-block map $\Phi : B_M(X) \to B_1(Y)$ with memory $m$ and anticipation $a$ with $m + a + 1 = M$ and let $N$ be a number such that $Y[N]$ is an edge shift. For $k \geq M + N$, let $X_k$ be the shift of finite type defined by the set $F_k = A^k \setminus B_k(X)$ of forbidden blocks (i.e., $X_k$ is the $k$-step Markov approximation of $X$). Then we can define the code $\phi_k : X_k \to Y$ with the $M$-block map $\Phi$ with memory $m$ and anticipation $a$ (note that $\phi_k(X_k) \subset Y$). Clearly $X_{k+1} \subset X_k$ for all $k$ and $X = \bigcap_k X_k$. Since $h(X) < h(Y)$,
for all large $k$ we have $h(X_k) < h(Y)$. Since $\phi$ is bi-closing, it follows by a standard compactness argument that $\phi_k$ is bi-closing for all large $k$. Therefore we can apply (2) for $\phi_k$ with sufficiently large $k$ to prove (3). $\square$

This theorem may be viewed as another aspect of the extension result in [4, §4]. Indeed it gives more information except for the closing delay which is defined only for 1-block codes on 1-step shifts of finite type. Note that in Theorem 4.3 if $X$ and $Y$ are mixing then $\tilde{X}$.

Our last example shows that we cannot improve the extension theorem of Ashley [3] by replacing right closing with bi-closing.

**Example 4.4.** Let $X = Y = X_\mathcal{A}$ where $A = \left(\begin{smallmatrix} 1 & 2 \\ 1 & 0 \end{smallmatrix}\right)$ and $Z$ be a periodic orbit of $X$ of length greater than 1. Let $\phi : Z \rightarrow Y$ be the code that maps every point of $Z$ to the unique fixed point of $Y$. Clearly $\phi$ is bi-closing. If there is a bi-closing extension $\phi : X \rightarrow Y$ of $\phi$, then it must be constantly d-to-1 with $d > 1$. However, since $-1$ is an eigenvalue of $A$, it follows that $X$ only admits endomorphisms of degree one [19], which is a contradiction. Thus $\phi$ cannot be extended to a bi-closing code from $X$ onto $Y$.

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