Supersymmetric extension of Moyal algebra
and its application to the matrix model

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We construct operator representation of Moyal algebra in the presence of fermionic fields. The result is used to describe the matrix model in Moyal formalism, that treat gauge degrees of freedom and outer degrees of freedom equally.

1. Introduction

Outer degrees of freedom can be converted into gauge degrees of freedom through compactification. The correspondence between a commutator of matrices and a Poisson bracket of functions used both in M-theory and IIB matrix model implies the fact. When we use the Moyal formalism we can interpolate these two degrees of freedom. There is, however, a problem which we have to overcome before we apply the Moyal formalism to the matrix model. Namely there exists no Moyal formulation of fermionic fields, which is appropriate to describe a supersymmetric theory. Fairlie was the first who wrote the matrix model in Moyal formalism. The supersymmetry, however, has not been fully explored. The main purpose of this article is to construct representation of Moyal algebra to describe the matrix model in Moyal formalism, which treat outer degrees of freedom and the gauge ones equally.

Ishibashi et al. have proposed a matrix model which looks like the Green-Schwarz action of type IIB string in the Schild gauge as a constructive definition of string theory. This matrix model has the manifest Lorentz invariance and $N = 2$ space-time supersymmetry.

They claim that IIB superstring theory can be regarded as a sort of classical limit of a part of the matrix model. This correspondence is based on the relationship between $su(N)$ and Poisson algebra. It is, therefore, very interesting if we could express the matrix model lagrangian and the Green-Schwarz action in a unified form in terms of Moyal algebra, since it is the unique one-parameter associative deformation of the Poisson algebra. We had a problem, however, that the corre-
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The correspondence between \( su(N) \) and Poisson algebra has been shown in the case that the theory involves only bosonic fields. On the other hand, we suggested an operator formalism of Moyal algebra in \(^{[1]}\), which is a generalization of Hamiltonian vector field. We think that the operator formalism is suitable for a description of matrix model. Therefore it is desirable to have an operator formalism of Moyal algebra including fermion fields. We will construct this in this article.

In the following two sections we review briefly the IIB Matrix model and \( su(N) \leftrightarrow \text{Poisson} \) correspondence. We introduce our fermionic Moyal algebra in \( \S 4 \). Based on these arguments, we extend the matrix model in \( \S 5 \). This extended model has coordinates which parameterize the world-sheet without the large \( N \) limit. Our procedure is not restricted to the matrix model but can be applied to any system that has \( U(N) \) gauge invariance and supersymmetry. We will present, as an example, the Moyal extension of the \( \mathcal{N} = 1 \) SYM in \( \S 6 \).

2. IIB matrix model

A large \( N \) reduced model has been proposed as a nonperturbative formulation of type IIB superstring theory\(^{[4]}\). It is defined by the following action:

\[
S_{\text{IKKT}} = -\frac{1}{g^2} \text{Tr} \left( \frac{1}{4} [A_\mu, A_\nu] [A_\mu, A_\nu] + \frac{1}{2} \bar{\psi} \Gamma^\mu [A_\mu, \psi] \right),
\]

(1)

here \( \psi \) is a ten dimensional Majorana-Weyl spinor field, and \( A_\mu \) and \( \psi \) are \( N \times N \) Hermitian matrices. It is formulated in a manifestly covariant way which they believe is a definite advantage over the light-cone formulation\(^{[7]}\) to study the non-perturbative issues of superstring theory.

This action can be related to the Green-Schwarz action of superstring\(^{[2]}\) by using the semiclassical correspondence in the large \( N \) limit:

\[
-\frac{i}{\sqrt{g}} \leftrightarrow \frac{1}{N} \{ \cdot, \cdot \}_P,
\]

\[
\text{Tr} \leftrightarrow N \int d^2 \sigma \sqrt{\hat{g}},
\]

(2)

where \( \{ \cdot, \cdot \}_P \) is a Poisson bracket. In fact eq.(1) reduces to the Green-Schwarz action in the Schild gauge\(^{[3]}\):

\[
S_{\text{Schild}} = \int d^2 \sigma \left[ \sqrt{\hat{g}} \alpha \left( \frac{1}{4} (X^\mu, X^\nu)^2_P - \frac{i}{2} \bar{\psi} \Gamma^\mu \{ X^\mu, \psi \}_P \right) + \beta \sqrt{\hat{g}} \right].
\]

(3)

Through this correspondence, the eigenvalues of \( A_\mu \) matrices are identified with the space-time coordinates \( X^\mu(\sigma) \). The \( \mathcal{N} = 2 \) supersymmetry manifests itself in \( S_{\text{Schild}} \) as\(^{[4]}\):

\[
\delta^{(1)} \psi = -\frac{1}{2} \sigma_{\mu\nu} \Gamma^{\mu\nu} \epsilon_1, \quad (\sigma_{\mu\nu} := \partial_0 X_\mu \partial_1 X_\nu - \partial_1 X_\mu \partial_0 X_\nu)
\]

\[
\delta^{(1)} X_\mu = i \epsilon_1 \Gamma_\mu \psi,
\]

(4)
and

\[
\delta^{(2)} \psi = \epsilon_2, \\
\delta^{(2)} X_\mu = 0.
\]  

(5)

The \( N = 2 \) supersymmetry (4) and (5) are directly translated into the symmetry of \( S_{IKKT} \) as

\[
\delta^{(1)} \psi = \frac{i}{2} [A_\mu, A_\nu] \Gamma^{\mu\nu}\epsilon_1, \quad ([A_\mu, A_\nu] := A_\mu A_\nu - A_\nu A_\mu)
\]

(6)

\[
\delta^{(1)} A_\mu = i\epsilon_1 \Gamma_\mu \psi;
\]

(7)

and

\[
\delta^{(2)} \psi = \epsilon_2, \\
\delta^{(2)} A_\mu = 0.
\]

(8)

3. Algebraic background

We consider the algebraic background of the correspondence (2).

The bases of \( \text{su}(N) \) algebra can be written as

\[
J_{(m_1,m_2)} = \omega^{m_1m_2/2} g^{m_1} h^{m_2},
\]

where \( g \) and \( h \) are matrices

\[
g = \begin{pmatrix}
1 \\
\omega \\
\omega^2 \\
\vdots \\
\omega^{N-1}
\end{pmatrix}, \\
h = \begin{pmatrix}
0 & 1 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 \\
-1 & 0 & \cdots & 0
\end{pmatrix},
\]

which satisfy

\[
g^N = h^N = -1, \quad hg = \omega gh, \quad \omega = \exp(2\pi i/N).
\]

With these bases, \( \text{su}(N) \) is expressed as, using the notation \( m = (m_1, m_2) \),

\[
[J_m, J_n] = -2i\sin\left[\frac{\pi}{N}(m \times n)\right] J_{m+n},
\]

(9)

\((0 \leq m_i, n_i \leq N - 1, \quad m, n \neq 0)\)

On the other hand the Poisson operator,

\[
X_f = \frac{\partial f}{\partial q} \frac{\partial}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial}{\partial q} = \nabla f \times \nabla
\]

satisfies the commutation relation

\[
[X_f, X_g] = X_{(f,g)}.
\]
which can be expressed as
\[ [X_m, X_n] = -(m \times n)X_{m+n}, \quad X_m := -ie^{-im\cdot q}m \times \nabla \] (10)
in Fourier components through the transformation \( f(q) = \sum_m f_m e^{-im\cdot q} \). Therefore (9) coincides, up to a constant factor, with (10) in the \( N \to \infty \) limit. It is this accordance that underlies the correspondence (2).

It is well-known that Moyal bracket
\[ \{f, g\}_M = \lim_{q' \to q \quad p' \to p} \lambda \sin \left( \lambda (\partial_{q'} \partial_p - \partial_{p'} \partial_Q) \right) f(q', p')g(q, p) \]
is the unique one-parameter associative deformation of the Poisson bracket, and the algebra (10) is modified into
\[ [K_m, K_n] = \frac{1}{\lambda} \sin \left( \lambda (m \times n) \right) K_{m+n}, \] (11)
Thus we can see that Moyal algebra corresponds to \( \text{su}(N) \) when the parameter \( \lambda \) is set to \( \pi/N \), and to Poisson algebra in the \( \lambda \to 0 \) limit.

4. Moyal operator for a fermionic field
A supersymmetric extension of the algebra (11) was discussed in
\[ \{F'_m, F'_n\} = \cos(\lambda(m \times n))K_{m+n} \]
\[ [K_m, F'_n] = \frac{1}{\lambda} \sin(\lambda(m \times n))F'_{m+n} \] (12)
They are realized by
\[ K_m = \frac{1}{i\lambda} F'_m = \frac{1}{2i\lambda} e^{i(2\lambda m_1 \hat{\rho} + m_2 \hat{\phi})}, \quad [\hat{x}, \hat{p}] = i\lambda \] (13)
as well as by
\[ K_m = \frac{1}{\lambda} F'_m = \frac{i}{2\lambda} e^{im\cdot q} \exp[-\lambda(m \times \nabla)]. \] (14)

We want to generalize this superalgebra to include fields. It should be done by using the basis-independent differential operator realization \( K_f \), which was introduced in
\[ K_f := \frac{1}{2i\lambda} f \left( x + i\lambda \frac{\partial}{\partial p}, p - i\lambda \frac{\partial}{\partial x} \right). \] (15)
However, a problem arises if we incorporate a fermionic field in a similar way and use the algebra (12). In (12) the effect of statistics has been taken into account without reference to fields.
There is an alternative realization of the Moyal bracket for bosonic fields. We have proposed in [6] a deformation of Hamilton vector fields which provides Moyal bracket in the place of Poisson bracket. It is a little modification of (15), which we denote as $B_f$:

$$B_f := \lim_{q' \to q} \frac{i}{\lambda} \sin \left[ \lambda \left( \frac{\partial}{\partial q'} \frac{\partial}{\partial p'} - \frac{\partial}{\partial p'} \frac{\partial}{\partial q} \right) \right] f(q', p') \quad (16)$$

$$= \lim_{q' \to q} \frac{i}{\lambda} \sin \left[ \lambda \left( \nabla' \times \nabla \right) \right] f(q') \quad (17)$$

We can check that the commutation relation among the operators

$$[B_f, B_g] = B_i \{f, g\}_M \quad (18)$$

holds. There must be a fermionic counterpart of this operator in order to have a supersymmetric algebra. To this end we introduce the following ‘operator’

$$F_\psi := - \lim_{q' \to q} \cos \left[ \lambda \left( \frac{\partial}{\partial q'} \frac{\partial}{\partial p'} - \frac{\partial}{\partial p'} \frac{\partial}{\partial q} \right) \right] \psi(q', p') \quad (19)$$

$$= - \lim_{q' \to q} \cos \left[ \lambda \left( \nabla' \times \nabla \right) \right] \psi(q') \quad (20)$$

associated with a fermionic field $\psi$. Then we are ready to convince ourselves that the commutation relations

$$\{F_\psi, F_\chi\} = \lambda^2 B_i \{\psi, \chi\}_M$$

$$[B_f, F_\psi] = F_i \{f, \psi\}_M \quad (21)$$

are correct, when the statistics of the fields are considered, where the Moyal bracket of fermions is defined as

$$\{\psi, \chi\}_M = \lim_{q' \to q} \frac{1}{\lambda} \sin \left[ \lambda \left( \frac{\partial}{\partial q} \frac{\partial}{\partial p} - \frac{\partial}{\partial p} \frac{\partial}{\partial q'} \right) \right] \psi(q, p) \chi(q', p')$$

$$= \lim_{q' \to q} \frac{1}{\lambda} \sin \left[ \lambda \left( \nabla \times \nabla' \right) \right] \psi(q) \chi(q').$$

Moreover we can see that behind this superalgebra ([8, 21]) there exists an algebra of generators which admit the $\text{su}(N)$ reduction. To show this we write the operators in Fourier components:

$$B_f = \sum_m f_m B_m, \quad \left( f(q) = \sum_m f_m e^{imq} \right),$$

$$F_\psi = \sum_m \psi_m F_m, \quad \left( \psi(q) = \sum_m \psi_m e^{imq} \right), \quad (22)$$
where
\[ B_m := \frac{1}{\lambda} e^{-imq} \sinh[\lambda(m \times \nabla)] \]
\[ F_m := -e^{-iqm} \cosh[\lambda(m \times \nabla)]. \] (23)

Then the generators satisfy a closed algebra
\[
[B_m, B_n] = -i\lambda \sin \{\lambda(m \times n)\} B_{m+n}
\]
\[
[F_m, F_n] = -i\lambda \sin \{\lambda(m \times n)\} B_{m+n}
\]
\[
[B_m, F_n] = -i\lambda \sin \{\lambda(m \times n)\} F_{m+n} \] (24)

The structure constants of these commutators are not only all common, but also agree with one of (11). Therefore the reduction to the su(N) algebra still remains supersymmetric. We like to emphasize the apparent difference of our commutators from those of (12). The anticommutation relation between fermionic operators in (21) arises due to the statistics of their fields.

5. Moyal operator formalism in the matrix model

Based on the arguments presented in the previous sections, we can express the matrix model lagrangian and the Green-Schwarz action in a unified form in Moyal formalism as
\[
S = -\frac{1}{g^2} \text{tr} \left( \frac{1}{4}[B_{\mu}, B_{\nu}][B_{\mu}, B_{\nu}] + \frac{1}{2} F_{\psi} \Gamma^\mu [B_{\mu}, F_{\psi}] \right) \] (25)

where \([ , , ]\) is a commutator of operators and not of matrices, and “tr” denotes the integration over the world sheet parameters and the sum over the complete set of functions on the world sheet. This action is invariant under the \(N = 2\) transformations
\[
\delta^{(1)} F_{\psi} = -\frac{1}{2} \Sigma_{\mu\nu} \Gamma^{\mu\nu} \epsilon_1, \quad (\Sigma_{\mu\nu} = \partial_0 B_{\mu} \partial_1 B_{\nu} - \partial_1 B_{\mu} \partial_0 B_{\nu})
\]
\[
\delta^{(1)} B_{\mu} = i\epsilon_1 \Gamma_{\mu} F_{\psi}, \] (26)

and
\[
\delta^{(2)} F_{\psi} = \epsilon_2,
\]
\[
\delta^{(2)} B_{\mu} = 0. \] (27)

It already has coordinates to parameterize the world-sheet without the \(\lambda \to 0\) limit.

We would like to mention the difference of our approach from one by Fairlie. The action proposed in this theory is
\[
S_{\text{Fairlie}} = \frac{1}{2\pi \alpha'} \int \text{d}^4x \text{d}^4\sigma \text{d}t \left( (D_{\mu} X)^2 + \cos \{\theta^T, D\theta\} + g_s^2 \text{tr} F_{\mu\nu}^2 \right. \\
- \left. \left( \frac{1}{\lambda g_s} \sin \{X^\mu, X^\nu\} \right)^2 + \frac{1}{g_s} \cos \left\{ \psi^T \Gamma_{\mu}, \frac{1}{\lambda} \sin \{X^\mu, \psi\} \right\} \right) \]
here the phase space variables \( \alpha, \beta \) are introduced to parametrise the fields instead of matrix indices. In this theory the commutators of matrices are deformed, while we deform the matrices themselves and leave the operation of product as the same as the operation of matrices. Moreover the action \((23)\) has manifest \( \mathcal{N} = 2 \) supersymmetry.

6. Conclusion

We considered the algebraic relationship behind the correspondence between IIB matrix model and the Green-Schwarz action and noticed that the both can be described in a unified form in Moyal formalism if the correspondence can be shown even in the presence of fermionic fields. We have shown this correspondence by constructing a closed algebra. Although the extension is originally motivated by the matrix model, our procedure is not restricted to that case but can be applied to any system that has both U(\(N\)) gauge invariance and supersymmetry. As an example, we present a Moyal extension of the \( \mathcal{N} = 1 \) SYM.

The \( \mathcal{N} = 1 \) SYM lagrangian density

\[
\mathcal{L}_{\text{SYM}} = -\frac{1}{4} F_{\mu \nu}^a F^{\mu \nu a} - i \lambda^a \bar{\sigma}^\mu D_\mu \lambda^a + \frac{1}{2} D^a D^a
\]

can be extended to the lagrangian density \( \mathcal{L}_M \), where

\[
\mathcal{L}_M = -\text{tr} \left( \frac{1}{4} Y_{\mu \nu} Y^{\mu \nu} + i F_{\lambda i} \bar{\sigma}^\mu D_\mu F_\lambda - \frac{1}{2} B_D B_D - \frac{i}{8} \epsilon^{\mu \nu \rho \sigma} Y_{\mu \nu} Y_{\rho \sigma} \right)
\]

(28)

\[
Y_{\mu \nu} = \partial_\mu B_{\lambda \nu} - \partial_\nu B_{\lambda \mu} + i g [B_{\lambda \mu}, B_{\lambda \nu}]
\]

(29)

and

\[
D_\mu F_\lambda = \partial_\mu F_\lambda + i g [B_{\lambda \mu}, F_\lambda].
\]

This lagrangian is invariant under supertransformations

\[
\delta B_{\lambda \mu} = -\frac{1}{\sqrt{2}} \left[ \epsilon^i \bar{\sigma}_i F_\lambda + F_{\lambda i} \bar{\sigma}_i \epsilon \right],
\]

(30)

\[
\delta F_{\lambda i} = -\frac{i}{2\sqrt{2}} (\sigma^\rho \bar{\sigma}^\sigma)_{\alpha \rho \sigma} Y_{\rho \sigma} + \frac{1}{\sqrt{2}} \epsilon_\alpha B_D,
\]

(31)

\[
\delta B_D = \frac{i}{\sqrt{2}} \left[ \epsilon^i \bar{\sigma}_i D_\mu F_\lambda - D_\mu F_{\lambda i} \bar{\sigma}^\mu \epsilon \right]
\]

(32)

as well as “gauge” transformations

\[
\delta_M B_{\lambda \mu} = -\partial_\mu F_\lambda - i g [B_{\lambda \mu}, B_\lambda]
\]

\[
\delta_M F_\lambda = -i g [F_\lambda, B_\lambda],
\]

where \( \Lambda \) is a transformation parameter.

When we describe \( S_{\text{IKKT}} \) in Moyal formalism new degrees of freedom appear. How should we interpret these? It gives a clue to remember \( S_{\text{IKKT}} \) is originally a low energy effective action of \( N \) coincident \((-1)\)-branes. The theory with a \( p \)-brane
compactified in a direction perpendicular to the brane is T-dual to the theory with a \((p+1)\)-brane compactified in a direction parallel to the brane. So the theory with \((-1)\)-branes compactified twice in directions perpendicular to the branes is T-dual to the theory with 1-branes compactified in directions parallel to the branes. Thus, in dual picture, there are two parameters to parameterize the 1-branes. The two parameters which appear when we describe the matrix model in Moyal formalism indicate that gauge degrees of freedom are obtained from outer degrees of freedom through compactification and that Moyal formalism treats these two degrees of freedom equally.

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