Damping of supernova neutrino transitions in stochastic shock-wave density profiles

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Abstract. Supernova neutrino flavour transitions during shock-wave propagation are known to encode relevant information not only about the matter density profile but also about unknown neutrino properties, such as the mass hierarchy (normal or inverted) and the mixing angle $\theta_{13}$. While previous studies have focused on ‘deterministic’ density profiles, we investigate the effect of possible stochastic matter density fluctuations in the wake of supernova shock-waves. In particular, we study the impact of small-scale fluctuations on the electron (anti)neutrino survival probability, and on the observable spectra of inverse beta-decay events in future water–Cherenkov detectors. We find that such fluctuations, even with relatively small amplitudes, can have significant damping effects on the flavour transition pattern, and can partly erase the shock-wave imprint on the observable time spectra, especially for $\sin^2 2\theta_{13} \gtrsim \mathcal{O}(10^{-3})$.

Keywords: neutrino experiments, supernova neutrinos

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1. Introduction

Future observations of supernova (SN) neutrinos in underground detectors represent a subject of intensive investigation in astroparticle physics. In this context, matter effects associated to neutrino flavour transitions in the SN envelope have been widely studied as a unique tool to probe, at the same time, neutrino properties and supernova astrophysics (see, e.g. [1]–[4] for recent reviews). In particular, the effects of supernova shock-waves on neutrino flavour transitions in matter [5] are gaining increasing attention in the recent literature [6]–[11]. Indeed, for a few seconds after the SN core bounce, the strong time dependence of the shock-wave profile can leave peculiar signatures on the time structure of the neutrino events, which could be monitored in large, real-time detectors [12]–[14].

So far, studies of supernova neutrino flavour transitions during the shock-wave propagation have been based on ‘deterministic’ matter density profiles, assumed to be known (or at least knowable, in principle) both in time and in radial dependence. However, stochastic density fluctuations and inhomogeneities, of various magnitudes and correlation lengths, may reasonably arise in the wake of a shock front. Possible causes of these inhomogeneities include microscopic fluctuations in the nascent neutron star [15] and large-scale fluctuations between the proto-neutron star and the supernova envelope due to hydrodynamical instabilities [16]–[18]. At early times (\(\lesssim 1\) s after bounce), post-shock convection overturns can also produce large density anisotropies. A supernova neutrino ‘beam’ travelling to the Earth might thus experience stochastic matter effects while traversing the stellar envelope.

Concerning neutrino oscillations, the phenomenology of possible stochastic matter density fluctuations has been investigated in several contexts, with emphasis on general properties [19]–[22] and on the solution to the solar neutrino problem [23]–[28]. Supernova
neutrinos have specifically been considered in relatively few cases [29], and only for a static profile. In general, it is found that the typical effect of random fluctuations on neutrino oscillations is to wash out the phase information (if any) and to damp the flavour transition pattern in the energy domain. It seems thus worthwhile to revisit this topic in the context of dynamic SN density profiles, which offer a complementary handle in the time domain\(^3\). Moreover, stochastic density fluctuations are expected to arise more easily in the wake of a SN shock-wave than in a relatively static stellar environment.

In this work we try to explore quantitatively the entangled effects of shock-waves and of possible stochastic fluctuations (behind the shock front) on supernova neutrino flavour transition probabilities and observables. Unfortunately, current core-collapse SN simulations do not yet offer a clear input for such phenomena, both because the details of the explosion mechanism are not well understood yet, and because current computer resources do not allow one to resolve density variations at scales smaller than, say, \(O(10)\) km. Therefore, some assumptions and simplifications are unavoidable. In particular, we shall limit ourselves to small-scale and small-amplitude fluctuations, which are definitely not excluded by current simulations, and which allow a simple perturbative approach.

Our work is organized as follows. In section 2 we introduce the notation for the neutrino mass and mixing parameters and for the electron (anti)neutrino survival probability \(P_{ee}\) characterizing supernova neutrino transitions. In section 3 we parameterize the ‘fluctuating’ shock-wave profile and discuss an approximate analytic expression for the survival probability \(P_{ee}\), which neatly includes the damping effect induced by density fluctuations. In particular, the analytical derivation of \(P_{ee}\) from the neutrino master equation for the density matrix, together with a comparison with representative numerical solutions, are given in the appendix. We find that, for \(\sin^2 \theta_{13} \gtrsim \mathcal{O}(10^{-3})\), small-scale stochastic fluctuations can suppress the imprint of the shock-wave on the flavour transition pattern in the time domain. In section 4 we discuss a specific experimental application for the case of antineutrinos in inverted hierarchy, by calculating the effects of fluctuations on positron event spectra observable in water–Cherenkov detectors through inverse beta-decay. Conclusions and prospects for further developments are given in section 5. The bottom line of our work is that some ‘smearing’ of shock-wave signatures in neutrino observables is to be expected, if stochastic matter fluctuations are present in the wake of the shock front. Improvements in supernova hydrodynamical simulations will greatly help to clarify this issue.

2. Three-neutrino mixing framework

In this section we set the notation for three-neutrino mixing and for the survival probability of electron (anti)neutrinos \(P_{ee}\). In the current ‘standard’ 3\(\nu\) scenario, the dominant parameters relevant to flavour transitions in supernovae are the largest squared mass difference \(\Delta m^2\) and the mixing angles \(\theta_{12}\) and \(\theta_{13}\) (see, e.g., [30,31]). In numerical examples, we take as reference values (close to their best fits [32])

\[
\Delta m^2 \simeq 2.4 \times 10^{-3} \text{ eV}^2
\]

\[
\sin^2 \theta_{12} \simeq 0.3.
\]

\(^3\) This possibility was mentioned in passing in the seminal [5].
The sign of $\Delta m^2$ distinguishes the cases of normal hierarchy (NH: $+\Delta m^2$) and inverted hierarchy (IH: $-\Delta m^2$). Concerning the mixing parameter $\sin^2 \theta_{13}$, we shall use representative values below the current upper limits ($\sin^2 \theta_{13} < \text{few} \times 10^{-2}$ [33]).

The smallest squared mass difference ($\delta m^2 \approx 8 \times 10^{-5}$ eV$^2$ [32]) is such that $\delta m^2 / \Delta m^2 \ll 1$; together with the smallness of $\sin^2 \theta_{13}$, this fact guarantees, to a very good approximation, the factorization of the 3$\nu$ dynamics into a 'low' ($L$) and a 'high' ($H$) 2$\nu$ subsystem. In other words, the electron (anti)neutrino survival probability $P_{ee}$, up to small terms of $O(\sin^2 \theta_{13}, \delta m^2 / \Delta m^2)$, can be expressed as

$$P_{ee} \approx P_{ee}^L \cdot P_{ee}^H,$$  \hspace{1cm} (3)

where $P_{ee}^L$ and $P_{ee}^H$ represent effective electron (anti)neutrino survival probabilities in the 2$\nu$ subsystems (see [30,31], [34]–[36] and references therein).

Neglecting Earth matter effects (not included in this work), one has [8,30,31]

$$P_{ee}^L \approx \begin{cases} 
\sin^2 \theta_{12} & \text{(for } \nu, \text{ any hierarchy)}, \\
\cos^2 \theta_{12} & \text{(for } \overline{\nu}, \text{ any hierarchy)},
\end{cases}$$  \hspace{1cm} (4)

and

$$P_{ee}^H \approx \begin{cases} 
P^H_+ & \text{(for } \nu \text{ in NH or } \overline{\nu} \text{ in IH)}, \\
P^H_- & \text{(for } \overline{\nu} \text{ in NH or } \nu \text{ in IH)},
\end{cases}$$  \hspace{1cm} (5)

where $P^H_{\pm}$ are defined below.

In general, $P_{ee}^H$ depends on $\theta_{13}$, on the hierarchy, and on the wavenumber $k_H$ in the $H$ subsystem,

$$\pm k_H = \pm \Delta m^2 / 2E \quad (+ \text{for NH, } - \text{ for IH}),$$  \hspace{1cm} (6)

and on the (anti)neutrino potential in matter [37],

$$\pm V(x) = \pm \sqrt{2} G_F N_e(x) \quad (+ \text{ for } \nu, - \text{ for } \overline{\nu})$$  \hspace{1cm} (7)

where $N_e$ is the electron density at the supernova radius $x$.

Strong matter effects are generally expected when the 'level crossing condition' is satisfied at (more than) one point $x_c$ (see [31]). The crossing condition requires equal signs for $V$ and $k_H$, so it is not realized for $\overline{\nu}$ in NH or $\nu$ in IH. The physics of matter effects can be encoded in terms of the so-called crossing probability $P_c$ [30,38]

$$P_c = \begin{cases} 
P_c(k_H, \sin^2 \theta_{13}, V) & \text{(for } \nu \text{ in NH or } \overline{\nu} \text{ in IH)}, \\
0 & \text{(for } \overline{\nu} \text{ in NH or } \nu \text{ in IH)},
\end{cases}$$  \hspace{1cm} (9)

where $P_c \neq 0 \ (= 0)$ defines the case of non-adiabatic (adiabatic) matter transitions$^4$.

$^4$ Concerning the early literature on two-level crossings, it is amusing to note that the classic Landau–Zener–Stückelberg results for $P_c$ [39]–[41] were similarly obtained by Majorana [42] in the context of spin-flip transitions in variable magnetic fields; see the discussion in [43]. Curiously, this contribution by Majorana (well known in atomic physics) is largely ignored in the neutrino literature.
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In the standard case (with no stochastic fluctuations) the survival probability $P^H_{\pm}$ at the Earth (averaged over many oscillation cycles) is related to the crossing probability $P_c$ through the well-known Parke’s formula [44],

$$P^H_{\pm} \simeq \frac{1}{2} + \left(\frac{1}{2} - P_c\right) \cos 2\theta_{13} \cos 2\tilde{\theta}_{13}(x_0)$$

(no fluctuations),

where $x_0$ is the neutrino production point, and

$$\cos 2\tilde{\theta}_{13}(x) = \frac{\cos 2\theta_{13} + V(x)/k_{H}}{\sqrt{(\cos 2\theta_{13} + V(x)/k_{H})^2 + (\sin 2\theta_{13})^2}},$$

defines the mixing angle $\tilde{\theta}_{13}$ in matter. In equation (11), the upper and lower signs correspond to the upper and lower signs of $P^H_{\pm}$ in equation (5). Due to the high matter density at the origin ($|V(x_0)/k_{H}| \gg 1$), in equation (10) one can simply put

$$\cos 2\tilde{\theta}_{13}(x_0) \simeq \mp 1.$$  

A final remark is in order. In the presence of density fluctuations with small amplitude, one does not expect that the crossing probability $P_c$ is significantly perturbed, being related to a ‘local’ non-adiabatic effect. The relation between $P^H_{\pm}$ and $P_c$ in equation (10) is instead expected to change, being related to the ‘global’ propagation history within the supernova matter. This intuitive picture, which enters in the analytical generalization of equation (10) discussed in the next section, is confirmed by numerical calculations (see the appendix). Similarly, one does not expect that small-amplitude fluctuations can ruin the effective $L \otimes H$ factorization, which relies only on the smallness of $\delta m^2/\Delta m^2$. This expectation is also supported by the smallness of fluctuation effects within the $L$ subsystem in our framework (see the last paragraph in section 3.3). A full numerical confirmation of the $L \otimes H$ factorization in the presence of stochastic effects, however, would involve rather heavy calculations of exact $3\nu$ solutions, which are beyond the scope of this work.

3. Effects of stochastic matter fluctuations on SN neutrino transitions

In this section we parameterize the SN stochastic density fluctuations through some simplifying assumptions (section 3.1), characterize their effects through an analytical prescription for the calculation of the probability $P^H_{\pm}$ (section 3.2), and discuss the time dependence of $P^H_{\pm}$ in the presence of fluctuations with increasing amplitude (section 3.3).

3.1. Shock-wave density profile and stochastic fluctuations

We make the reasonable assumption that stochastic fluctuations arise only after the passage of the shock-wave. Behind the shock front, fluctuations can be described as fractional (random) variations $\xi(x)$ of an ‘average’ neutrino potential $V_0$,

$$V(x) = V_0(x) + \delta V(x) = V_0(x)[1 + \xi(x)],$$

with vanishing mean $\langle \xi(x) \rangle = 0$, and with variance $\xi^2 = \langle \xi(x)^2 \rangle$. In the absence of further information, we simply assume that $\xi$ is both constant and small [O(few%)].

In a different context (Sun density profile), the combination of all available solar and KamLAND data sets an upper limit $\xi < 5\%$ at 70\% CL for the solar density fluctuations, assuming a fluctuation length scale $L_0 = 10$ km [28].
We also assume that the fluctuations arise at relatively small length scales $L_0$, as compared with the neutrino oscillation length in matter $\lambda_m$. In the region where matter effects are relevant ($V \simeq k_H$), this condition reads

$$L_0 \ll \lambda_m \simeq \frac{2\pi}{k_H \sin 2\theta_{13}} \simeq 60 \text{ km} \left(\frac{E}{10 \text{ MeV}}\right) \left(\frac{2 \times 10^{-3} \text{ eV}^2}{\Delta m^2}\right) \left(\frac{0.2}{\sin 2\theta_{13}}\right).$$

(14)

In all numerical examples we shall assume a representative value

$$L_0 \simeq 10 \text{ km}.$$  \hspace{1cm} (15)

Notice that this scale is of the same order as the radial hydrodynamical fluctuations described in [15], as well as of the neutrino-sphere radius [1], which sets the transverse size of the SN neutrino beam in the Earth direction.

The small-scale assumption in equations (14) and (15) implies that the autocorrelation function of the random field $\xi(x)$ can be written as an effective delta-function [20]–[23]

$$\langle \xi(x_1)\xi(x_2) \rangle = 2L_0\xi^2\delta(x_1 - x_2).$$

(16)

In a nutshell, fluctuations between points much closer (much farther) than one neutrino oscillation length in matter are assumed to be fully correlated (totally uncorrelated). Given equation (16), the master equation for the neutrino flavour evolution is determined (see the appendix).

Figure 1 shows the (fluctuating) shock-wave profiles used in this work. We take the unperturbed shock-wave potentials $V_0(x)$ from our previous simplified parameterizations in [8,12]. In the region behind the shock front, the thicker curves represent a band of fluctuations with fractional deviation $\xi \leq 4\%$. The upper panel refers to the neutrino potential $V$ in the presence of the forward shock only. In the lower panel we have taken into account also a possible reverse shock, characterized by a smaller density jump at the front [9]. In both panels, we also show the band spanned by the neutrino wavenumber $k_H = \Delta m^2/2E$ for $E \in [2, 60]$ MeV. Notice that the (crossing) condition for large matter effects ($V \simeq k_H$) is basically unperturbed by fluctuations as small as those in figure 1.

3.2. Analytical results

As anticipated in section 2, the survival probability $P^H_\pm$ in the $H$ subsystem is modified by matter density fluctuations. In general, the calculation of $P^H_\pm$ requires that the stochastic master equation (see the appendix) is solved numerically. Nevertheless, as shown in [24] and described in detail in the appendix, small-scale and small-amplitude fluctuations allow one to use perturbative techniques which, already at first order, provide rather accurate analytical approximations. In the context of SN neutrinos, the analytical prescription reads

$$P^H_\pm \simeq \frac{1}{2} \pm \left(\frac{1}{2} - P_c\right) e^{-\Gamma_\pm} \cos 2\theta_{13},$$

(17)

which generalizes [24] Parke’s formula (10) (in the conditions of equation (12)) through a simple exponential damping factor $e^{-\Gamma_\pm}$, whose exponent is given by

$$\Gamma_\pm = \int_{x_0}^{x_s} D(x) \sin^2 2\theta_{13}^\pm(x) \, dx,$$

(18)
Figure 1. Simplified radial profiles of the neutrino potential $V(x)$ at different post-bounce times $t$ (1, 2, 4, 8, and 16 s). The thicker curves behind the shock front represent fluctuations of amplitude $\xi \leq 4\%$ of the local matter potential $V$. Upper panel: forward shock only. Lower panel: forward plus reverse shock. In both cases, the static (pre-bounce) profile ($t \leq 0$ s) is also shown. The band within dashed lines marks the region where SN matter effects are potentially important ($V \simeq k_H$, with $E = 2$–60 MeV).

where

$$D(x) = \xi^2 V_0^2(x) L_0,$$

and the mixing angle in matter ($\theta^{\pm}_{13}$) is defined as in equation (11). The domain of the damping integral $\Gamma_{\pm}$ is the region behind the forward shock front position $x_s$, where stochastic fluctuations are assumed to arise. For strong damping ($\Gamma_{\pm} \gg 1$) one gets the limit $P^{H}_{c} \to 1/2$, corresponding to a sort of complete ‘flavour depolarization’, where the two effective $\nu$ states in the $H$ subsystem are democratically mixed.

All stochastic matter effects are embedded in the damping factor $e^{-\Gamma_{\pm}}$, while the crossing probability $P_c = P_c(k_H, \sin^2 \theta_{13}, V(x))$ in equation (17) is basically
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unperturbed\(^6\). In particular, \(P_c\) can be evaluated analytically by considering the ordered product of crossing matrices along the average (non-monotonic) supernova shock-wave density profile \(V_0(x)\), as previously discussed in [8, 12]. Therefore, apart from the numerical evaluation of the integral in equation (18), \(P_{\pm}^H\) can be computed through convenient analytical approximations.

3.3. Analysis of the survival probability \(P_{\pm}^H\) in the \(H\) subsystem, and comments on \(P_{\pm}^L\) in the \(L\) subsystem

In this section we describe the behaviour of \(P_{\pm}^H\) as a function of time, in the presence of stochastic fluctuations with increasing amplitude \(\xi\). The case of no fluctuations is recovered for \(\xi = 0\).

Figure 2 shows the variations of the survival probability \(P_{\pm}^H\) (relevant for \(\nu\) in NH or \(\overline{\nu}\) in IH),

\[
P_{\pm}^H \simeq \frac{1}{2} - \left(\frac{1}{2} - P_c\right) e^{-\Gamma_+} \cos 2\theta_{13},
\]

in the time interval \(t \in [1, 13]\ s\), for \(\xi = 0\) (solid curves), \(\xi = 2\%\) (dashed curves) and \(\xi = 4\%\) (dotted curves). The neutrino energy is fixed at \(E = 30\ \text{MeV}\), while \(\sin^2 \theta_{13}\) ranges from \(10^{-2}\) (upper panels) to \(10^{-5}\) (lower panels). We include both the case of forward shock only (left panels) and of forward plus reverse shock (right panels).

In the absence of fluctuations, the function \(P_{\pm}^H(t)\) in figure 2 shows strong, non-monotonic variations: a clear signature of the strongly non-adiabatic flavour transitions along the shock front, operative in the time window \(t \in \sim [3, 9]\ s\) (as discussed at length in [8, 12]). In the presence of stochastic fluctuations behind the shock front, however, the variations are partly suppressed, as a result of the ‘flavour depolarization’ effect. As the amplitude \(\xi\) increases, the survival probability \(P_{\pm}^H\) gets closer to the saturation value \(1/2\).

The effect of fluctuations is relevant in the same time window where shock effects are operative, since the occurrence of the condition \(V \simeq k_H\) (in multiple points \(x_c\)) lead to \(\sin^2 2\theta_{13} \simeq 1\) and thus to a ‘large’ damping factor (see equation (18)) around \(x_c\). At earlier or later times, the enhancement of \(\theta_{13}\) in matter is more modest and so are fluctuation effects.

In figure 2, the smearing effect of fluctuations appears to be particularly dangerous for the identification of shock-wave signatures: for instance, for \(\xi = 4\%\), the double-peak structure associated to the forward plus reverse shock (right panels) may be washed out, becoming more similar to the case of forward shock only (left panels). A ‘confusion scenario’ might arise, limiting the potential to monitor the shock-wave propagation through neutrino flavour transitions.

Figure 2 also shows that, in general, damping effects are suppressed at small values of \(\theta_{13}\) (say, \(\sin^2 \theta_{13} \lesssim 10^{-3}\)). This fact can be understood by expanding the damping integral (equation (18)) around the crossing points \(x_c\), where the integrand is locally maximal. At
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Figure 2. Probability function $P^H_+(t)$ (relevant for $\nu$ in NH or $\bar{\nu}$ in IH) at $E = 30$ MeV, for four representative values of $\sin^2 \theta_{13}$, in the presence of density fluctuations with fractional amplitude $\xi$ equal to 0% (solid curves), 2% (dashed curves) and 4% (dotted curves). Left panels: forward shock only. Right panels: forward plus reverse shock.

First order in $x - x_c$, and using the fact that $V \simeq k_H$ around $x_c$, one gets then

$$\Gamma_+ \simeq \pi \sin 2\theta_{13} \sum_{x_c} D(x_c) \left| \frac{d \ln V}{dx} \right|_{x_c}^{-1}, \quad (21)$$

which shows that the damping factor is basically proportional to $\theta_{13}$. In conclusion, for values of $\sin^2 \theta_{13} \gtrsim O(10^{-3})$, small-scale stochastic fluctuations with a fractional amplitude of a few per cent might significantly suppress shock-wave effects on the electron neutrino survival probability $P^H_+$. For smaller values of $\sin^2 \theta_{13}$, fluctuation effects appear to be less dangerous, but unfortunately the overall effect of shock-waves on $P^H_+$ is also small.
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Figure 3. As in figure 2, but for the probability function $P_{\nu}^H(t)$ (relevant for $\nu$ in IH or $\bar{\nu}$ in NH).

Figure 3 shows the time evolution of the survival probability $P_{\nu}^H$ (relevant for $\nu$ in IH or $\bar{\nu}$ in NH, where $P_c = 0$),

$$P_{\nu}^H \simeq \frac{1}{2} \left( 1 + e^{-\Gamma_-} \cos 2\theta_{13} \right),$$

(22)

in the same format as that of figure 2. In the absence of matter fluctuations ($\xi = 0$), it is $\Gamma_- = 0$ and $P_{\nu}^H \simeq 1$ trivially (being $\cos 2\theta_{13} \simeq 1$ in all cases shown in figure 3). In the presence of fluctuations, the damping term $e^{-\Gamma_-}$ tends to lower $P_{\nu}^H$ (down to the plateau value $1/2$ for large $\Gamma_-.$). However, the relative variations of the function $P_{\nu}^H$ in figure 3 are not as strong as for $P_{\nu}^H$ in figure 2, since the integrand in $\Gamma_-$ is never ‘resonant’, as happens instead for $\Gamma_+.$ For small $\theta_{13}$, it is simply

$$\Gamma_- \simeq \sin^2 2\theta_{13} \int_{x_0}^{x_s} \frac{D(x)}{(1 + V(x)/k_H)^2},$$

(23)
namely, the damping factor $\Gamma_-$ decreases with $\sim \theta_{13}^2$, and its effect vanishes more rapidly than for $\Gamma_+$. This fact explains why, in figure 3, fluctuation effects are visible only at larger values of $\theta_{13}$ (upper panels, $\sin^2 \theta_{13} = 10^{-2}$) as compared with figure 2. The absence of level crossing effects in $P^H$ strongly reduces the sensitivity to the details of the shock-wave profile, so that for $\xi > 0$ there is hardly any difference between the left and right upper panels in figure 3.

From the previous discussion, stochastic density fluctuations appear to be most relevant in the cases where $P^H$ (rather than $P^H$) is involved, i.e., neutrinos in normal hierarchy or antineutrinos in inverted hierarchy. For this reason, in the next section we shall examine the impact of these results in a phenomenologically relevant case ($\nu$ in IH), by calculating observable spectra of positrons induced by $\nu_e$ through inverse beta-decay.

Finally, we comment on the effects of fluctuations on the survival probability $P_{\ell e}$ in the 'low' ($L$) subsystem. In the presence of density fluctuations, this probability can be calculated as

$$P_{\ell e} \simeq \frac{1}{2} \mp \frac{1}{2} e^{-\Gamma_{\pm}(L)} \cos 2\theta_{12},$$

which is analogous to equation (17) for the adiabatic case $P_\ell = 0$ (the adiabaticity of the $L$ subsystem is discussed in [8]). The damping integral in the above equations reads

$$\Gamma_{\pm}(L) = \int_{x_0}^{x_s} D(x) \sin^2 2\tilde{\theta}_{12}(x) \, dx,$$

where the definition of $\tilde{\theta}_{12}$ is analogous to that in equation (11), but in terms of $\theta_{12}$ and of $k_L = \delta m^2 / 2E$. Since $k_L$ is a factor $\sim 30$ smaller than $k_H$, one has typically $V/k_L \gg 1$ for $x < x_s$, and thus $\sin^2 2\tilde{\theta}_{12}(x) \ll 1$ in equation (25), except possibly at late times ($t \gtrsim 10 \, s$; see figure 1), where the crossing condition $V \simeq k_L$ can be realized, leading to $\sin^2 \tilde{\theta}_{12} \simeq 1$ for neutrinos. However, at late times the neutrino luminosity is also small. Therefore, the damping effects of $\Gamma_{\pm}(L)$ are either small or suppressed by a low luminosity, and can be safely neglected for the purposes of our work. One can simply take $e^{-\Gamma_{\pm}(L)} \simeq 1$ in equation (24), thus recovering equation (4).

4. Observable positron spectra from SN $\nu$ in inverted hierarchy

As an application of the previous results, we study the effects of stochastic matter fluctuations on the observable positron time spectra detectable through the inverse beta-decay reaction

$$\nu_e + p \rightarrow n + e^+,$$

which represents the main SN neutrino detection channel in present [45] and planned Cherenkov detectors [46]–[49], as well as in scintillation detectors [50]–[52]. Therefore, in this section we focus on electron antineutrinos, in the relevant case of inverted hierarchy\(^7\), where shock-wave (plus fluctuation) effects become potentially observable through the $H$ subsystem dynamics,

$$P_{\ell e} \simeq \cos^2 \theta_{12} P^H_+.$$

\(^7\) An analogous study (not presented here) could be performed for the neutrino channel in normal hierarchy, in the context of $\nu_e$-sensitive detectors, such as liquid argon time proportional chambers [53].
Positron event rates are calculated for a reference 0.4 Mton water–Cherenkov detector, with neutrino spectra at the supernova source, interaction cross sections, and detection parameters fixed as in our previous work [12], to which the reader is referred for further details. Here we just mention that in [12] two representative ‘low’ and ‘high’ positron energy bins (at $E_{\text{pos}} = 20 \pm 5$ and $45 \pm 5$ MeV, respectively) were identified as good relative tracers of shock-wave effects. Indeed, as discussed in [12] (for no fluctuations), the time spectra at relatively high energy ($E_{\text{pos}} = 45 \pm 5$ MeV) show strong signatures of the shock-wave propagation, especially for increasing $\sin^2 \theta_{13}$, through non-monotonic time variations of the event rates. Conversely, the shock-induced time variations in the bin $E_{\text{pos}} = 20 \pm 5$ MeV are significantly smaller; at such energy, in fact, the initial electron and non-electron antineutrino fluxes (as used in our calculation input) happen to be approximately equal, and the effects of flavour transitions largely cancel.

Figure 4 shows absolute time spectra of events within the two reference positron energy bins ($E_{\text{pos}} = 20 \pm 5$ MeV and $E_{\text{pos}} = 45 \pm 5$ MeV) for four representative values of $\sin^2 \theta_{13}$, and for both $\xi = 0$ (no fluctuations, solid histograms) and $\xi = 4\%$ (dashed histograms). The left and right panels refer to the case of forward shock only and of forward plus reverse shock, respectively. In the presence of stochastic density fluctuations ($\xi = 4\%$ in figure 4), the spectra for the low-energy positron bin are basically unaffected, since all flavour transition effects (modified or not by fluctuations) are small. The time spectra for the high-energy positron bin ($E_{\text{pos}} = 45 \pm 5$ MeV) are instead significantly smoothed out by fluctuation effects for $\sin^2 \theta_{13} \gtrsim \mathcal{O}(10^{-3})$, as expected from the discussion of figure 2. Therefore, even at small amplitude ($\xi = 4\%$), fluctuations can suppress the shock imprint on the time spectra, and make them qualitatively similar to those in normal hierarchy (where they are expected a priori to be smooth and monotonic).

As suggested in [12] (for the case of no fluctuations), the relative flavour transition effects at low and high energy are best displayed by showing the ratio of the spectra at 20 ± 5 MeV and 45 ± 5 MeV, where the first spectrum acts as a ‘normalization’ factor. Figure 5 shows such a ratio (in inverted hierarchy) for the same values of $\xi$ and $\sin^2 \theta_{13}$ of figure 4. In the absence of stochastic fluctuations ($\xi = 0$), we recover the fact [12] that the spectral ratio can track the time variations of the electron antineutrino survival probability $P^H_\text{e}$, as is evident from a visual comparison of the solid curves ($\xi = 0$) in figures 2 and 5.8 Unfortunately, stochastic density fluctuations (dashed curves with $\xi = 4\%$ in figure 5) destroy this nice correspondence for $\sin^2 \theta_{13} \gtrsim \mathcal{O}(10^{-3})$. The spectral ratio becomes smoother, and the ‘peaks and valleys’ induced by the shock-wave on $P^H_\text{e}$ disappear. The damping effect implies a net loss of information about the shock-wave in the observable neutrino spectra.

5. Summary and conclusions

Stochastic density fluctuations in supernova matter may produce observable effects on the supernova neutrino signal. In this context, previous works have been focused on static and

8 The analysis in [12] actually refers to the crossing probability $P_\text{c}$. However, in the absence of fluctuations, the smallness of $\theta_{13}$ implies that $P^H_\text{e} \approx P_\text{c}$; see equation (10).
Figure 4. Absolute time spectra of positron events induced by $\nu_e$ in IH in a 0.4 Mton water–Cherenkov detector, in the presence of forward shock (left panels) and forward plus reverse shock (right panels). Four representative values of $\sin^2\theta_{13}$ are considered. The solid histograms refer to the case of no fluctuation ($\xi = 0$), while the dashed ones refer to the case of fluctuations with $\xi = 4\%$. In each panel, the upper (lower) couple of histograms refer to the positron energy bin $E_{\text{pos}} = 20 \pm 5$ MeV ($E_{\text{pos}} = 45 \pm 5$ MeV).

monotonic profiles. In this paper, we have investigated the case of time-dependent and non-monotonic profiles, embedding forward (plus reverse) shock propagation, as suggested by recent SN numerical simulations. In the hypothesis of small-scale ($L_0 \sim \mathcal{O}(10 \text{ km})$) and small-amplitude ($\xi \lesssim \text{few}\%$) fluctuations, we have discussed an analytical recipe to evaluate the SN electron (anti)neutrino survival probability $P_{ee}$, which accounts for both
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Figure 5. Time dependence of the ratio of events between the energy bins $E_{\text{pos}} = 20 \pm 5$ MeV and $E_{\text{pos}} = 45 \pm 5$ MeV in the case of forward shock only (left panels) and forward plus reverse shock (right panels). The ratio refers to $\nu_e$ in inverted hierarchy for no fluctuation ($\xi = 0$, solid curves) and a fluctuation amplitude $\xi = 4\%$ of the local matter potential (dashed curves).

We find that stochastic fluctuations—possibly arising after the shock front passage—may significantly suppress the imprint of SN shock-waves on $P_{ee}$ in the time domain, the more the larger $\sin^2 \theta_{13}$. For decreasing $\sin^2 \theta_{13}$, fluctuation effects decrease, but shock-wave signatures also become less pronounced. An application to observable time spectra of positron events in large water–Cherenkov detectors shows that, for the phenomenologically relevant case of inverted hierarchy, the time spectra can easily lose any imprint of the

standard matter transitions and additional ‘flavour-depolarization’ effects induced by the fluctuations, and which accurately reproduces the numerical results (see the appendix).
shock-wave for $\sin^2 \theta_{13} \gtrsim \mathcal{O}(10^{-3})$. Therefore, in the presence of stochastic fluctuations, it might be difficult to ‘monitor’ the shock-wave in real time in a future galactic SN explosion, or to find unmistakable signatures of inverted mass hierarchy effects.

In this context, future work might include a ‘fluctuation analysis’ of the SN 1987A neutrino events [54] and of the expected supernova relic neutrino spectrum [55]–[57]. For instance, some studies of the SN 1987A $\nu_e$ in the case of inverted hierarchy [58] found a tension between data and theory in the case of adiabatic transitions (i.e., $\sin^2 \theta_{13} \gtrsim 10^{-4}$) and inverted hierarchy, using a static SN profile. The effect of both non-adiabatic crossings and of damping behind the shock front might put these results in a different perspective.

In any case, further studies will greatly benefit from improvements of numerical SN simulations, so as to follow the shock-wave evolution with much higher space–time resolution that currently possible. A better theoretical understanding of the possibility and property of stochastic density fluctuations in the SN envelope might also help to remove the simplifying assumptions adopted in this work.

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Appendix. Stochastic neutrino master equation: analytical and numerical solutions

For the sake of completeness, in this appendix we explicitly derive equation (17) (based on previous literature on the subject) and compare it with our numerical results for representative SN shock-wave cases. To avoid an excessive ‘±’ notation, we consider here only the case of neutrinos in normal hierarchy. The cases of antineutrinos and of inverted hierarchy are obtained by the replacements $V \rightarrow -V$ and $k_H \rightarrow -k_H$, respectively.

In the presence of stochastic matter fluctuations, one must consider the neutrino evolution within a statistical ensemble of density profiles. Averaging over the ensemble generally produces a loss of the coherence in the system. In this context, the appropriate formalism involves the neutrino density matrix $\rho$.

For a 2$\nu$ mixing problem (as in the $H$ subsystem), the density matrix in the flavour basis $(\nu_e, \nu_\alpha)^T$, where $\nu_\alpha$ is any linear combination of $\nu_\mu$ and $\nu_\tau$, can be expressed in terms of a ‘polarization vector’ $\mathbf{P}$ [59]–[62]:

$$\rho = \frac{1}{2}(1 + \mathbf{P} \cdot \sigma), \quad (A.1)$$

where

$$\mathbf{P} = \begin{pmatrix} 2 \text{Re}(\nu_e^* \nu_\alpha) \\ 2 \text{Im}(\nu_e^* \nu_\alpha) \\ 2|\nu_e|^2 - 1 \end{pmatrix}, \quad (A.2)$$

and $\sigma$ is the vector of Pauli matrices. The length $|\mathbf{P}|$ of the polarization vector ($|\mathbf{P}|^2 = \text{Tr} \rho^2$) measures the degree of coherence of the system: $|\mathbf{P}| = 1$ corresponds to a pure state, $0 < |\mathbf{P}| < 1$ to a partially mixed state, and $|\mathbf{P}| = 0$ to a completely mixed state.
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With or without fluctuations, the Hamiltonian for neutrinos propagating in matter in the ‘high’ subsystem is the usual one [37]:

\[ H \equiv \frac{1}{2} \mathbf{h} \cdot \sigma = \frac{1}{2} [k_H \sin 2\theta_{13}] \sigma_1 + \frac{1}{2} [V(x) - k_H \cos 2\theta_{13}] \sigma_3, \]  
(A.3)

with

\[ \mathbf{h} = (k_H \sin 2\theta_{13}, 0, V(x) - k_H \cos 2\theta_{13}). \]  
(A.4)

In the presence of stochastic matter density fluctuations, the potential has been expressed as \[ V(x) = V_0(x)[1 + \xi(x)], \] with \( \delta \)-correlated Gaussian fluctuations (see section 3.1). In this case, the Liouville evolution of the fluctuation-averaged density matrix reads [20]–[22]

\[ \frac{d}{dx} \rho = -i[H_0, \rho] - \frac{1}{4} D[\sigma_3, [\sigma_3, \rho]], \]  
(A.5)

where \( H_0 \) is the unperturbed Hamiltonian (for \( \xi = 0 \)) and \( D \) is the same damping parameter as in equation (19).

Equation (A.5) has the well-known Lindblad form for dissipative systems [63], where the first term in the right-hand side reproduces the standard Liouville equation for the \( \nu \) density matrix, while the second term violates the conservation of \( \text{Tr} \rho^2 \) and allows transitions from pure to mixed states. In terms of the polarization vector, equation (A.5) assumes the neat form of a Bloch equation [60,65]:

\[ \frac{d}{dx} \mathbf{P} = h_0 \times \mathbf{P} - DP_T, \]  
(A.6)

where \( h_0 \) corresponds to equation (A.4) with \( \xi = 0 \), and \( P_T = \mathbf{P} - P_3 \) is the ‘transverse’ part of the polarization vector. In this picture, the first term in the right-hand side represents the usual coherent ‘precession’ of the polarization vector \( \mathbf{P} \) around the vector \( h_0 \) [59]–[62], [64], while the second (damping) term produces a loss of coherence, i.e., a shortening of the vector length \( |\mathbf{P}| \) [60,65].

Using equations (A.2)–(A.4), one can write equation (A.6) in components [20]–[24]

\[ \frac{d}{dx} \begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix} = \begin{pmatrix} -D(x) & -V_0(x) + k_H c_{2\theta_{13}} & 0 \\ V_0(x) - k_H c_{2\theta_{13}} & -D(x) & -k_H s_{2\theta_{13}} \\ 0 & k_H s_{2\theta_{13}} & 0 \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix} \equiv \mathcal{K} \begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix}, \]  
(A.7)

where \( c_{2\theta_{13}} \equiv \cos 2\theta_{13} \) and \( s_{2\theta_{13}} \equiv \sin 2\theta_{13} \). The (fluctuation-averaged) survival probability for electron neutrinos \( \nu = (1,0)^T \) is then given by

\[ P_+^H = \text{Tr}[^{\nu} \rho \nu] = \frac{1 + P_3}{2}. \]  
(A.8)

In general, the system of equations (equation (A.7)) has to be solved numerically, as done for example in [20,21,23]. However, in the case of small-amplitude fluctuations, the damping term \( D \) in equation (A.7) can be treated as a perturbation. In this way, it is possible to integrate equation (A.7) analytically. Here we closely follow the derivation presented in [24].

\footnote{For neutrinos propagating in a stochastic matter, the density matrix (equation (A.1)) and the polarization vector (equation (A.2)) must be understood as averaged over the ensemble of density profiles.}
We start by diagonalizing the $K$ matrix in equation (A.7) at zeroth order in $D$, establishing the ‘instantaneous diagonal basis’

$$
\begin{pmatrix}
\tilde{P}_1 \\
\tilde{P}_2 \\
\tilde{P}_3 
\end{pmatrix} = U
\begin{pmatrix}
P_1 \\
P_2 \\
P_3 
\end{pmatrix}.
$$

(A.9)

The matrix $U = [\hat{u}_1, \hat{u}_2, \hat{u}_0]$ diagonalizes $K$ as

$$
K_d = \text{diag}(\lambda_1, \lambda_2, \lambda_0) = U K U^\dagger,
$$

where the three eigenvalues are

$$
\lambda_0 = 0,
$$

(A.11)

$$
\lambda_{1,2} = \pm i |h_0| = \pm i \sqrt{(k_H \cos 2\theta_{13} - V)^2 + (k_H \sin 2\theta_{13})^2},
$$

(A.12)

and the corresponding eigenvectors are

$$
\hat{u}_0 = \hat{h}_0,
$$

(A.13)

$$
\hat{u}_{1,2} = \frac{1}{\sqrt{2}} \left( \hat{h}_0 \times \hat{s} \pm i \hat{s} \right),
$$

(A.14)

with $\hat{h}_0 = h_0/|h_0|$, while $\hat{s}$ is an arbitrary unitary vector perpendicular to $h_0$. By choosing $\hat{s} = (0, 1, 0)$, one obtains

$$
\hat{u}_0 = \left( \begin{array}{c}
\sin 2\tilde{\theta}_{13}^+ \\
0 \\
-\cos 2\tilde{\theta}_{13}^+
\end{array} \right), \quad \hat{u}_1 = \frac{1}{\sqrt{2}} \left( \begin{array}{c}
\cos 2\tilde{\theta}_{13}^+ \\
\sin 2\tilde{\theta}_{13}^+
\end{array} \right), \quad \hat{u}_2 = \frac{1}{\sqrt{2}} \left( \begin{array}{c}
\cos 2\tilde{\theta}_{13}^+ \\
-\sin 2\tilde{\theta}_{13}^+
\end{array} \right)
$$

(A.15)

while the matrix $U$ takes the form

$$
U = \left( \begin{array}{ccc}
\frac{1}{\sqrt{2}} \cos 2\tilde{\theta}_{13}^+ & \frac{1}{\sqrt{2}} \cos 2\tilde{\theta}_{13}^+ & \sin 2\tilde{\theta}_{13}^+ \\
\frac{1}{\sqrt{2}} \sin 2\tilde{\theta}_{13}^+ & -\frac{1}{\sqrt{2}} & -\cos 2\tilde{\theta}_{13}^+
\end{array} \right),
$$

(A.16)

where $\tilde{\theta}_{13}^+$ is the mixing angle in matter defined by

$$
\sin 2\tilde{\theta}_{13}^+ = \frac{\sin 2\theta_{13}}{\sqrt{(\cos 2\theta_{13} - V/k_H)^2 + (\sin 2\theta_{13})^2}},
$$

(A.17)

$$
\cos 2\tilde{\theta}_{13}^+ = \frac{\cos 2\theta_{13} - V/k_H}{\sqrt{(\cos 2\theta_{13} - V/k_H)^2 + (\sin 2\theta_{13})^2}}.
$$

(A.18)

In the diagonal basis, equation (A.7) becomes

$$
\frac{d}{dx} \tilde{P} = K_d \tilde{P} - U \frac{dU^\dagger}{dx} \tilde{P}.
$$

(A.19)
We assume that the propagation is adiabatic (\(dU/dx = 0\)), except near one ‘crossing point’ \(x_c\) where \(V(x_c) \simeq k_H\) (the generalization to multiple crossings is straightforward). In this approximation, equation (A.19) has thus a formal solution:

\[
P(x) = U(x)S(x_c, x)Q(x_c)S(x_0, x_c)U(x_0)P(x_0),
\]

where \(x_0\) is the neutrino production point, and

\[
S(x_1, x_2) = \text{diag} \left[ e^{i\phi(x_1, x_2)}, e^{-i\phi(x_1, x_2)}, 1 \right],
\]

is the adiabatic evolution operator, while \(\phi(x_1, x_2) = \int_{x_1}^{x_2} dy |h_0(y)|\), and the non-diagonal operator \(Q(x_c)\) takes into account the non-adiabatic transition (level crossing) around the point \(x_c\).

By inserting the above expressions in equation (A.8), and averaging over the oscillatory terms, one gets

\[
P^H_+ = \frac{1 + [Q(x_c)]_{33} \cos 2\tilde{\theta}_{13}^+(x) \cos 2\tilde{\theta}_{13}^+(x_0)}{2}.
\]

The comparison of equation (A.22) with equation (10) leads to the identification \([Q(x_c)]_{33} = 1 - 2P_c\), where \(P_c\) is the crossing probability between the two mass eigenstates in matter. At zeroth order in \(D\) one thus recovers the usual (no-fluctuation) Parke’s formula:

\[
P^H_+ = \frac{1}{2} + \left( \frac{1}{2} - P_c \right) \cos 2\tilde{\theta}_{13}^+(x) \cos 2\tilde{\theta}_{13}^+(x_0).
\]

The first-order corrections to the eigenvalues (equations (A.11), (A.12)) and eigenvectors (equations (A.13), (A.14)) are calculated through the standard perturbation theory as

\[
\lambda_1 = \lambda_a + \delta \lambda_a = \lambda_a - D\hat{u}_a \hat{G} \hat{u}_a,
\]

where \(a, b = 0, 1, 2\) and \(\hat{G} = \text{diag}(1, 1, 1)\). Since \(\hat{u}_a \hat{G} \hat{u}_a = \delta_{ab} - (\hat{u}_a)_3 (\hat{u}_b)_3\), one obtains

\[
\begin{align*}
\lambda'_1 &= -D \sin^2 2\tilde{\theta}_{13}^+, \\
\lambda'_{1,2} &= \pm i |h_0| - D \left( 1 - \frac{1}{2} \sin^2 2\tilde{\theta}_{13}^+ \right),
\end{align*}
\]

and

\[
\begin{align*}
\hat{u}'_0 &= \hat{u}_0 + D \frac{\sin 2\tilde{\theta}_{13}^+ \cos 2\tilde{\theta}_{13}^+}{|h_0|} \hat{s}, \\
\hat{u}'_{1,2} &= \hat{u}_{1,2} \mp i D \frac{\sin 2\tilde{\theta}_{13}^+}{\sqrt{2} |h_0|} \left[ - \cos 2\tilde{\theta}_{13}^+ \hat{u}_0 + \frac{\sin 2\tilde{\theta}_{13}^+}{2\sqrt{2}} \hat{u}_{2,1} \right].
\end{align*}
\]

By substituting the first-order eigenvalues in equation (A.20), the matrix \(S(x_1, x_2)\) can be rewritten as

\[
S(x_1, x_2) = \text{diag} \left[ e^{i\phi(x_1, x_2) + \int_{x_1}^{x_2} dx \delta \lambda_1}, e^{-i\phi(x_1, x_2) + \int_{x_1}^{x_2} dx \delta \lambda_2}, e^{i \int_{x_1}^{x_2} dx \delta \lambda_0} \right].
\]

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Figure A.1. Comparison of analytical and numerical calculations of the electron neutrino survival probability $P^H_+$ in the $H$ subsystem (solid and dashed curves, respectively) for $E = 30$ MeV, $\sin^2 \theta_{13} = 10^{-2}$, and fluctuations with amplitude $\xi = 2\%$ (upper panels) and $\xi = 4\%$ (lower panels). The left panels refer to the case of forward shock only, while the right panels refer to the case of forward plus reverse shock.

The expressions of $\delta \lambda_{1,2}$ are irrelevant for our purposes, since they disappear after averaging over the oscillating terms in $P^H_+$. Notice that, in equation (A.20), only the operator $S$ is corrected at first order. The matrix $U(x_0)$ remains unchanged, since the high-density condition $V(x_0)/k_H \gg 1$ implies that $\sin 2\tilde{\theta}_{13}^+(x_0) \simeq 0$ in equation (A.27). The matrix $U(x)$ at the detection point is also unchanged, since $\tilde{\theta}_{13}^+(x) = \theta_{13}$ and $D = 0$ at the Earth.

Finally, for small fluctuations, we assume that the operator $Q(x_c)$ (which only depends on the crossing condition $V(x_c) \simeq k_H$) is unchanged. After averaging on the oscillating
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terms one then obtains

\[ P^H_+ \simeq \frac{1}{2} + \left( \frac{1}{2} - P_e \right) e^{-\Gamma_+} \cos 2\tilde{\theta}_{13}(x) \cos 2\tilde{\theta}_{13}(x_0), \quad (A.29) \]

where

\[ \Gamma_+ = \int_{x_0}^x dy \lambda_0'(y) = \int_{x_0}^x dy \mathcal{D}(y) \sin^2 2\tilde{\theta}_{13}(y). \quad (A.30) \]

In the case of detected supernova neutrinos, \( \tilde{\theta}_{13}(x) \) corresponds to the vacuum value \( \theta_{13} \), while the initial high-density condition implies \( \cos 2\tilde{\theta}_{13}(x_0) \simeq -1 \). With these positions, equation (A.29) gives the desired expression

\[ P^H_+ \simeq \frac{1}{2} - \left( \frac{1}{2} - P_e \right) e^{-\Gamma_+} \cos 2\theta_{13}. \quad (A.31) \]

In order to check the reliability of the analytical approximation of \( P^H_+ \) in equation (A.31), we have compared it with the results of a numerical (Runge–Kutta) evolution of neutrino master equation (equation (A.7)) along representative fluctuating shock-wave profiles. Figure A.1 shows our calculation of \( P^H_+(t) \) at fixed neutrino energy \( (E = 30 \text{ MeV}) \) for \( \sin^2\theta_{13} = 10^{-2} \), and for two representative fluctuation amplitudes, \( \xi = 2\% \) (upper panels) and \( \xi = 4\% \) (lower panels). We consider both the case of forward shock only (left panels) and forward plus reverse shock (right panels). It can be seen that the analytical calculations (solid curves) and the numerical Runge–Kutta calculations (dashed curves) are in very good agreement in the whole time interval \( t \in [1, 13] \text{ s} \). The results of figure A.1 (and of other checks that we have performed for different values of \( \sin^2\theta_{13} \) and of the neutrino energy) show that the analytical calculation of \( P^H_+ \) reproduces the numerical results with high accuracy. The same reassuring check has been performed also in the case of \( P^H_- \) (not shown).

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