ON THE CAUCHY PROBLEM FOR PARABOLIC INTEGRO-DIFFERENTIAL EQUATIONS IN GENERALIZED HÖLDER SPACES

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Abstract. An integro-differential Kolmogorov equation is considered in Hölder-type spaces defined by a scalable Lévy measure. Some properties of those spaces and estimates of the solution are derived by using probabilistic representations.

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1. Introduction

Let \((Ω, ℱ, P)\) be a complete probability space. Given a Lévy measure \(ν\) on \(R^d_0 = R^d \setminus \{0\}\), we suppose there exists an adapted Poisson random measure \(J(ds, dy)\) on \((Ω, ℱ, P)\) such that

\[
E[J(ds, dy)] = ν(dy) ds,
\]

\[
\tilde{J}(ds, dy) = J(ds, dy) - ν(dy) ds.
\]

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Then there is a Lévy process $Z^\nu_t$ associated to $\nu$ in the way that
\[
Z^\nu_t = \int_0^t \int_{\mathbb{R}^d} \chi_\alpha (y) y \tilde{J} (ds, dy) + \int_0^t \int_{\mathbb{R}^d} (1 - \chi_\alpha (y)) y J (ds, dy),
\]
where, as a convention,
\[
\alpha := \inf \{ \sigma \in (0, 2) : \int |y|^{\sigma} \nu (dy) < \infty \}
\]
is the order of $\nu$, and $\chi_\alpha (y) := 1_{\alpha \in (1, 2)} + 1_{\alpha = 1} 1_{|y| \leq 1}$.

The aim of this paper is twofold. One is to introduce function spaces of generalized smoothness and reveal the embedding relations among them. The other is to study the Cauchy problem of the following parabolic-type Kolmogorov equation within the framework of such generalized smoothness:
\[
\partial_t u (t, x) = Lu (t, x) - \lambda u (t, x) + f (t, x), \lambda \geq 0,
\]
\[
0 = u (0, x), \quad (t, x) \in [0, T] \times \mathbb{R}^d,
\]
where $L$ is the infinitesimal generator of $Z^\nu_t$.

Study on function spaces of generalized smoothness dates back to the seventies, signified by the work of H. Triebel [13], G.A. Kalyabin [4], P.I. Lizorkin [5] and so on. It is a natural development after the theory of differentiable functions of multi-variables and has been thriving for decades due to its close relation to interpolation theory, potential theory and the theory of differential operators. What is of most interest to us is the possibility to use the language of generalized smoothness to describe and investigate some special Lévy processes, (1.1) in particular. We know by the Lévy-Khinchine formula that each Lévy process $(Z^\nu_t)_{t \geq 0}$ is determined by a continuous negative definite function which is called the symbol. Generally speaking, by assuming the symbol $\tilde{\psi}$ of $(Z^\nu_t)_{t \geq 0}$ behaves up to a perturbation like $\psi$, one could expect the scales of spaces associated with $\psi$ plays the same role for $\tilde{\psi}$ as the classical Besov spaces do for elliptic operators. This was illustrated in [2] and [3] and was a motivation for defining such spaces. In this paper, we utilize a continuous function $w$ to capture the discrepancy generated by scaling and we support this viewpoint by investigating (1.2) in $w$-scaled Besov spaces.

**Definition 1.** A continuous function $w : (0, \infty) \to (0, \infty)$ is called a scaling function if
\[
\lim_{r \to 0} w (r) = 0, \quad \lim_{R \to \infty} w (R) = \infty
\]
and if there is a nondecreasing continuous function $l (\varepsilon), \varepsilon > 0$ such that
\[
\lim_{\varepsilon \to 0} l (\varepsilon) = 0
\]
and
\[
w (\varepsilon r) \leq l (\varepsilon) w (r), \quad \forall r, \varepsilon > 0.
\]
We call $l$ the scaling factor of $w$. 
Assumptions throughout this paper will be summarized in Section 2.3, denoted by $A(w,l)$. Intuitively, a measure satisfying $A(w,l)$ is non-degenerate and has a scaling effect on integrability that can be compensated by $w$, which voices for a large family of Lévy measures, including $\alpha$-stable measures, $\alpha$-stable-like measures and certain radical-and-angular expressed measures. (See Section 2.3.) We will fix a Lévy measure $\mu$ that meets $A(w,l)$ as our reference measure and use $w$ to define generalized Besov (resp. Hölder) norms $|\cdot|_{\beta,\infty}$ (resp. $|\cdot|_{\beta}$) and generalized Besov (resp. Hölder) spaces $\tilde{C}^{\beta}_{\infty,\infty}, \beta > 0$ (resp. $\tilde{C}^{\beta}$). (See Section 2.2.) Write $H_T = [0,T] \times \mathbb{R}^d$. One of the main results of this paper is:

**Theorem 1.1.** Let $\beta \in (0,\infty), \lambda \geq 0$ and $\nu$ be a Lévy measure satisfying $A(w,l)$. If $f(t,x) \in \tilde{C}^{\beta}_{\infty,\infty}(H_T)$. Then there is a unique solution $u(t,x) \in \tilde{C}^{1+\beta}_{\infty,\infty}(H_T)$ to

$$\partial_t u(t,x) = L^{\nu} u(t,x) - \lambda u(t,x) + f(t,x), \lambda \geq 0,$$

$$u(0,x) = 0, \quad (t,x) \in H_T,$$

where for any function $\varphi \in C^2_b(\mathbb{R}^d)$,

$$L^{\nu} \varphi(x) := \int [\varphi(x + y) - \varphi(x) - \chi_\alpha(y) y \cdot \nabla \varphi(x)] \nu(dy).$$

Moreover, there exists a constant $C$ depending on $\kappa, \beta, d, T, \mu, \nu$ such that

$$|u|_{\beta,\infty} \leq C \lambda^{-1} \wedge T \cdot |f|_{\beta,\infty},$$

$$|u|_{1+\beta,\infty} \leq C |f|_{\beta,\infty}$$

And there is a constant $C$ depending on $\kappa, \beta, d, T, \mu, \nu$ such that for all $0 \leq s < t \leq T, \kappa \in [0,1],$

$$|u(t,\cdot) - u(s,\cdot)|_{\kappa+\beta,\infty} \leq C |t-s|^{-\kappa} |f|_{\beta,\infty}.$$

As it will be seen later, when $\nu$ behaves like an $\alpha$-stable measure, (1.6)-(1.8) are ordinary Besov (equiv. Hölder-Zygmund) regularity estimates.

By norm equivalence stated in Section 3, Theorem 1.1 implies immediately

**Theorem 1.2.** Let $\beta \in (0,\infty), \lambda \geq 0$ and $\nu$ be a Lévy measure satisfying $A(w,l)$. If $f(t,x) \in \tilde{C}^{\beta}(H_T)$ and

$$\int_0^1 l(t)^{\beta} \frac{dt}{t} + \int_1^\infty l(t)^{\beta} \frac{dt}{t^2} < \infty.$$

Then there is a unique solution $u(t,x) \in \tilde{C}^{1+\beta}(H_T)$ to (1.4). Moreover, there exists a constant $C$ depending on $\kappa, \beta, d, T, \mu, \nu$ such that

$$|u|_{\beta} \leq C \lambda^{-1} \wedge T \cdot |f|_{\beta},$$

$$|u|_{1+\beta} \leq C |f|_{\beta}$$

And there is a constant $C$ depending on $\kappa, \beta, d, T, \mu, \nu$ such that for all $0 \leq s < t \leq T, \kappa \in [0,1],$

$$|u(t,\cdot) - u(s,\cdot)|_{\kappa+\beta} \leq C |t-s|^{-\kappa} |f|_{\beta}.$$
In [11], a parabolic-type Kolmogorov equation with an operator \( L = A + B \) was considered in the standard Hölder-Zygmund space, where \( B \) is the lower order part and the principal part \( A \) assumes a form of
\[
Au(t, x) := \int [u(t, x + y) - u(t, x) - \chi_\alpha(y) y \cdot \nabla u(t, x)] \rho(t, x, y) \frac{dy}{|y|^{d+\alpha}}.
\]
While in [10], a parabolic integro-differential equation perturbed by Gaussian noise was studied in the stochastic Hölder spaces. Operators were introduced as
\[
Lu(t, x) := \int [u(t, x + y) - u(t, x) - 1_{\alpha \geq 1} 1_{|y| \leq 1} y \cdot \nabla u(t, x)] \nu(t, x, dy) + 1_{\alpha = 2} a^{ij}(t, x) \partial^2_{ij} u(t, x) + 1_{\alpha \geq 1} \tilde{b}^i(t, x) \partial_i u(t, x) + l(t, x) u(t, x),
\]
and results were expressed in terms of moments. A similar operator was adopted in [9] and the corresponding deterministic model was studied in the little Hölder-Zygmund spaces. Besides, the Cauchy problem for a linear parabolic SPDE of the second order was considered in [8] and [12] in standard Hölder classes.

Our note is organized as follows.

In section 2, notation is introduced and spaces are defined. Meanwhile, we collect all assumptions that are needed in this paper and provide with examples that satisfy all the assumptions. A few more defining properties of our model are discussed as well.

In section 3, we elaborate embedding relations among function spaces. Probability representations are used to extend operations to all functions in \( C^\infty_b(\mathbb{R}^d) \). After the extension, those operations become bijections on \( C^\infty_b(\mathbb{R}^d) \). Norm equivalence then follows from continuity of operators.

Regularity estimates in the case of smooth inputs are collected in section 4, while those for Besov (equiv. Hölder) inputs are put in section 5. Section 6 accommodates existing results that are used in our proofs.

2. Notation, Spaces and Models

2.1. Basic Notation. \( \mathbb{N} = \{0, 1, 2, 3, \ldots\} \), \( \mathbb{N}_+ = \mathbb{N} \setminus \{0\} \). \( H_T = [0, T] \times \mathbb{R}^d \). \( S^{d-1} \) is the unit sphere in \( \mathbb{R}^d \). \( \Re \) is a notation for the real part of a complex-valued quantity.

For a function \( u = u(t, x) \) on \( H_T \), we denote its partial derivatives by \( \partial_t u = \partial u / \partial t \), \( \partial_i u = \partial u / \partial x_i \), \( \partial^2_{ij} u = \partial^2 u / \partial x_i x_j \), and denote its gradient with respect to \( x \) by \( \nabla u = (\partial_1 u, \ldots, \partial_d u) \) and \( D^\gamma u = \partial^\gamma u / \partial x_1^{\gamma_1} \ldots \partial x_d^{\gamma_d} \), where \( \gamma = (\gamma_1, \ldots, \gamma_d) \in \mathbb{N}^d \) is a multi-index.

We use \( C^\infty_b(\mathbb{R}^d) \) to denote the set of infinitely differentiable functions on \( \mathbb{R}^d \) whose derivative of arbitrary order is finite, and \( C^k(\mathbb{R}^d) \), \( k \in \mathbb{N} \) the class of \( k \)-times continuously differentiable functions.

\( S(\mathbb{R}^d) \) denotes the Schwartz space on \( \mathbb{R}^d \) and \( S'(\mathbb{R}^d) \) denotes the space of continuous functionals on \( S(\mathbb{R}^d) \), i.e. the space of tempered distributions.
We adopt the normalized definition for Fourier and its inverse transforms for functions in $S(\mathbb{R}^d)$, i.e.,

\[
\mathcal{F}\varphi(\xi) = \hat{\varphi}(\xi) := \int e^{-i2\pi x \cdot \xi} \varphi(x) \, dx,
\]
\[
\mathcal{F}^{-1}\varphi(x) = \check{\varphi}(x) := \int e^{i2\pi x \cdot \xi} \varphi(\xi) \, d\xi, \quad \varphi \in S(\mathbb{R}^d).
\]

Recall that Fourier transform can be extended to a bijection on $\mathcal{S}'(\mathbb{R}^d)$.

$\mu$ always refers to our reference measure, and $\alpha$ denotes its order unless otherwise specified.

Throughout the sequel, $Z_t^{\nu}$ represents the Lévy process associated to the Lévy measure $\nu$ in the way as (1.1).

For any Lévy measure $\nu$, any $R > 0$ and $\forall B \in \mathcal{B}(\mathbb{R}^d)$,

\[
\nu_R(B) = \int 1_B(y/R) \nu(dy),
\]
\[
\bar{\nu}_R(dy) := w(R) \nu_R(dy).
\]

Without loss of generality, we normalize $w$ by a constant so that $w(1) = 1$ and $\bar{\nu}_1(dy) = \nu(dy)$. Meanwhile, we introduce for any Lévy measure $\nu$,

\[
\bar{\nu}(dy) := \frac{1}{2}(\nu(dy) + \nu(-dy)).
\]

We have specific values assigned for $\alpha_1, \alpha_2, c_0, c_1, c_2, N_0, N_1$, but we allow $C$ to vary from line to line. In particular, $C(\cdots)$ represents a constant depending only on quantities in the parentheses.

### 2.2. Function Spaces of Generalized Smoothness

By definition of the scaling factor, there is a constant $N > 3$ such that $l(N^{-1}) < 1 < l(N)$. For such a $N$, by Lemma 6.1.7 in [1] and appropriate scaling, there exists $\phi \in C_0^\infty(\mathbb{R}^d)$ such that $\text{supp}(\phi) = \{\xi : \frac{1}{N} \leq |\xi| \leq N\}$, $\phi(\xi) > 0$ in the interior of its support, and

\[
\sum_{j=-\infty}^{\infty} \phi(N^{-j}\xi) = 1 \quad \text{if } \xi \neq 0.
\]

We denote throughout this paper

\[
\varphi_j = \mathcal{F}^{-1} \left[ \phi(N^{-j}\xi) \right], \quad j = 1, 2, \ldots, \xi \in \mathbb{R}^d,
\]
\[
\varphi_0 = \mathcal{F}^{-1} \left[ 1 - \sum_{j=1}^{\infty} \phi(N^{-j}\xi) \right].
\]

Apparently, $\varphi_j \in S(\mathbb{R}^d)$, $j \in \mathbb{N}$. If we write

\[
\tilde{\varphi}_j = \varphi_{j-1} + \varphi_j + \varphi_{j+1}, \quad j \geq 2,
\]
\[
\tilde{\varphi}_1 = \tilde{\varphi}_0 = \varphi_0 + \varphi_1,
\]
\[
\tilde{\varphi}_0 = \varphi_0 + \varphi_1,
\]
then,
\[ \mathcal{F} \tilde{\varphi}_j (\xi) = \hat{\varphi}_j (\xi) = \mathcal{F} \tilde{\varphi} (N^{-j} \xi) , \quad \xi \in \mathbb{R}^d, j \geq 1, \]
where
\[ \mathcal{F} \tilde{\varphi} (\xi) = \phi (N \xi) + \phi (\xi) + \phi (N^{-1} \xi). \]
Note that \( \phi \) is necessarily 0 on the boundary of its support. Then,
\[ \mathcal{F} \varphi_j = \mathcal{F} \varphi_j \mathcal{F} \tilde{\varphi}_j, j \geq 0, \]
and then
\[ (2.4) \quad \varphi_j = \varphi_j * \tilde{\varphi}_j, j \geq 0, \]
where in particular
\[ \tilde{\varphi}_j (x) = N^{jd} \tilde{\varphi} (N^j x), j \geq 1. \]
\( \varphi_j, j \geq 0 \) are convolution functions we use to define **generalized Besov spaces**. Namely, we write \( \tilde{C}^\beta_{\infty, \infty} (\mathbb{R}^d) \) as the set of functions in \( S' (\mathbb{R}^d) \) for which the norm
\[ |u|_{\beta, \infty} := \sup_j w (N^{-j})^{-\beta} |u * \varphi_j|_0 < \infty, \quad \beta \in (0, \infty). \]
For \( \kappa \in [0, 1], \beta \in (0, \infty), C^{\mu, \kappa, \beta} (\mathbb{R}^d) \) denotes the collection of functions in \( S' (\mathbb{R}^d) \) whose norm
\[ |u|_{\mu, \kappa, \beta} := |u|_0 + |\mathcal{F}^{-1} [\psi^{\mu, \kappa} \mathcal{F} u]|_{\beta, \infty} = |u|_0 + |L^{\mu, \kappa} u|_{\beta, \infty} < \infty, \]
where
\[ \psi^{\mu} (\xi) := \int \left[ e^{i2\pi \xi \cdot y} - 1 - i2\pi \chi_\alpha (y) \xi \cdot y \right] \mu (dy), \xi \in \mathbb{R}^d, \]
is the Lévy symbol associated to \( L^\mu \),
\[ \psi^{\mu, \kappa} := \begin{cases} \psi^{\mu} & \text{if } \kappa = 1, \\ -(\Re \psi^{\mu})^\kappa & \text{if } \kappa \in (0, 1), \\ 1 & \text{if } \kappa = 0, \end{cases} \]
and
\[ (2.5) \quad L^{\mu, \kappa} u := \mathcal{F}^{-1} [\psi^{\mu, \kappa} \mathcal{F} u], u \in S' (\mathbb{R}^d). \]
When \( \kappa = 0 \), \( L^{\mu, \kappa} u := u \), then \( C^{\mu, \kappa, \beta} (\mathbb{R}^d) \) is \( \tilde{C}^\beta_{\infty, \infty} (\mathbb{R}^d) \). When \( \kappa = 1 \), we simply write \( L^{\mu, \kappa} = L^\mu \) and write \( |u|_{\mu, \kappa, \beta} \) as \( |u|_{\mu, \beta} \). In this case, if \( u \in C^1_b (\mathbb{R}^d) \), then definition (2.5) coincides with
\[ (2.6) \quad L^\mu \varphi (x) := \int [\varphi (x + y) - \varphi (x) - \chi_\alpha (y) y \cdot \nabla \varphi (x)] \mu (dy). \]
For \( \kappa \in [0, 1], \beta \in (0, \infty), \tilde{C}^{\mu, \kappa, \beta} (\mathbb{R}^d) \) is the class of functions in \( S' (\mathbb{R}^d) \) whose norm
\[ ||u||_{\mu, \kappa, \beta} := |(I - L^\mu)^\kappa u|_{\beta, \infty} < \infty, \]
where

\[(I - L^\mu)^\kappa u := \begin{cases} (I - L^\mu) u & \text{if } \kappa = 1, \\ \mathcal{F}^{-1}[(1 - \Re \psi^\mu)^\kappa \mathcal{F} u] & \text{if } \kappa \in [0, 1). \end{cases}\]

When \(\kappa = 0\), \((I - L^\mu)^\kappa u := u\), then \(\tilde{C}^{\mu, \kappa, \beta}(\mathbb{R}^d)\) is again \(\tilde{C}^\beta_{\infty, \infty}(\mathbb{R}^d)\). When \(\kappa = 1\), we simply write \(\|u\|_{\mu, \kappa, \beta}\) as \(\|u\|_{\mu, \beta}\).

We will see in Section 3 that \(L^{\mu, \kappa}\) and \((I - L^\mu)^\kappa\) could be defined for functions in \(C^\infty_b(\mathbb{R}^d)\) even if \(\kappa \in (1, 2)\). That is

\[L^{\mu, \kappa} := L^{\mu, \kappa/2} \circ L^{\mu, \kappa/2}, \quad (I - L^\mu)^\kappa := (I - L^\mu)^{\kappa/2} \circ (I - L^\mu)^{\kappa/2},\]

where \(\circ\) means composition.

There will also be \textbf{generalized H"older spaces}. Using the scaling function, we write for \(\beta \in (0, 1/\alpha)\)

\[|u|_0 = \sup_{t, x} |u(t, x)|, \quad [u]_\beta = \sup_{t, x, h \neq 0} \frac{|u(t, x + h) - u(t, x)|}{w(|h|)^\beta}.\]

\(\tilde{C}^{\beta}(\mathbb{R}^d)\) denotes the set of functions such that the norm

\[|u|_\beta := |u|_0 + [u]_\beta < \infty, \quad \beta \in (0, 1/\alpha).\]

And \(\tilde{C}^{1+\beta}(\mathbb{R}^d)\) denotes the set of functions such that the norm

\[|u|_{1+\beta} := |u|_0 + |L^\mu u|_0 + [L^\mu u]_\beta < \infty, \quad \beta \in (0, 1/\alpha).\]

\section*{2.3. Assumptions and Examples}

All the assumptions needed in this paper are collected in this section. Because of their dependence on \(w, l\), we denote them by \(A(w, l)\). Let \(\nu\) be a Lévy measure, i.e.

\[\int_{\mathbb{R}^d} \left(1 + |y|^2\right) \nu(dy) < \infty.\]

Recall definitions (2.1) and (2.2), \(A(w, l)\).

(i) For all \(R > 0\), \(\check{\nu}_R(dy) \geq \mu^0(dy)\), where \(\mu^0\) is a Lévy measure supported on the unit ball \(B(0)\) and

\[\int |y|^2 \mu^0(dy) + \int |\xi|^4 [1 + v(\xi)]^{d+3} \exp\{-\zeta^0(\xi)\} d\xi < \infty,\]

in which

\[v(\xi) = \int \chi_\alpha(y) |y| \left(\left(|\xi| |y|\right) \wedge 1\right) \mu^0(dy), \quad \zeta^0(\xi) = \int \left[1 - \cos(2\pi \xi \cdot y)\right] \mu^0(dy).\]

In addition, for all \(\xi \in S_{d-1} = \{\xi \in \mathbb{R}^d : |\xi| = 1\}\), there is a constant \(c_1 > 0\), such that

\[\int_{|y| \leq 1} |\xi \cdot y|^2 \mu^0(dy) \geq c_0.\]
(ii) If $\alpha = 1$, then
\begin{equation}
\int_{r<|y|<R} y \nu \, (dy) = 0 \quad \text{for all } 0 < r < R < \infty.
\end{equation}

(iii) There exist constants $\alpha_1 \geq \alpha_2$ such that $\alpha_1, \alpha_2 \in (0, 1)$ if $\alpha \in (0, 1)$, $\alpha_1, \alpha_2 \in (1, 2)$ if $\alpha \in (1, 2)$, $\alpha_1 \in (1, 2]$ and $\alpha_2 \in [0, 1)$ if $\alpha = 1$, and
\begin{equation}
\int_{|y|\leq1} |y|^{\alpha_1} \tilde{\nu}_R \, (dy) + \int_{|y|>1} |y|^{\alpha_2} \tilde{\nu}_R \, (dy) \leq N_0
\end{equation}
for some positive constant $N_0$ that is independent of $R$.

(iv) $\varsigma (r) := \nu (|y| > r), r > 0$ is continuous in $r$ and
\begin{equation}
\int_0^1 s \varsigma (rs) \varsigma (r)^{-1} \, ds \leq C_0,
\end{equation}
for some $C_0 > 0$ independent of $r$.

We assume both the reference measure $\mu$ and the operator measure $\nu$ satisfy $\mathbf{A}(w, l)$.

Though looking heavy, $\mathbf{A}(w, l)$ embraces various models that have been receiving wide attention. For instance, in [14], $\nu$ is confined by two $\alpha$-stable Lévy measures, namely,
\begin{equation}
\int_{S_{d-1}} \int_0^\infty 1_B (rw) \frac{dr}{r^{1+\alpha}} \Sigma_1 \, (dw)
\end{equation}
for any Borel measurable set $B$, where $\Sigma_1$ and $\Sigma_2$ are two finite measures defined on the unit sphere and $\Sigma_1$ is nondegenerate. As a result, $\nu$ satisfies $\mathbf{A}(w, l)$ for $w (r) = l (r) = r^\alpha, r > 0$.

To see some other examples, let us adopt for now the radial and angular coordinate system and write $\nu$ as
\begin{equation}
\nu (B) = \int_0^\infty \int_{|w|=1} 1_B (rw) \, a (r, w) \, j (r) \, r^{d-1} S \, (dw) \, dr, \quad \forall B \in \mathcal{B} \left( \mathbb{R}^d_0 \right),
\end{equation}
where $S \, (dw)$ is a finite measure on the unit sphere.

Suppose $\Lambda \, (dt)$ is a measure on $(0, \infty)$ such that $\int_0^\infty (1 \wedge t) \, \Lambda \, (dt) < \infty$, and $\phi (r) = \int_0^\infty (1 - e^{-rt}) \, \Lambda \, (dt), r \geq 0$ is the associated Bernstein function. Set in (2.10) $S \, (dw)$ to be the usual Lebesgue measure, $a (r, w) = 1$, and
\begin{equation}
j (r) = \int_0^\infty (4\pi t)^{-d/2} \exp \left( -\frac{r^2}{4t} \right) \, \Lambda \, (dt), r > 0.
\end{equation}

Furthermore, assume $H$. (i) There is $C > 1$ such that
\begin{equation}
C^{-1} \phi (r^{-2}) \, r^{-d} \leq j (r) \leq C \phi (r^{-2}) \, r^{-d}.
\end{equation}
(ii) There are \( 0 < \delta_1 \leq \delta_2 < 1 \) and \( C > 0 \) such that for all \( 0 < r \leq R \)
\[
C^{-1} \left( \frac{R}{r} \right)^{\delta_1} \leq \frac{\phi(R)}{\phi(r)} \leq C \left( \frac{R}{r} \right)^{\delta_2}.
\]

G. There is a function \( \rho_0 (w) \) defined on the unit sphere such that \( \rho_0 (w) \leq a (r, w) \leq 1 \), \( \forall r > 0 \), and for all \( |\xi| = 1 \),
\[
\int_{S^{d-1}} |\xi \cdot w|^2 \rho_0 (w) \geq c > 0.
\]

Options for such \( \Lambda \) and thus \( \phi \) could be

1. \( \phi (r) = \sum_{i=1}^{n} r^\alpha_i, \alpha_i \in (0, 1), i = 1, \ldots, n; \)
2. \( \phi (r) = (r + r^\alpha)^\beta, \alpha, \beta \in (0, 1); \)
3. \( \phi (r) = r^\alpha (\ln (1 + r))^\beta, \alpha \in (0, 1), \beta \in (0, 1 - \alpha); \)
4. \( \phi (r) = [\ln (\cosh \sqrt{r})]^\alpha, \alpha \in (0, 1). \)

It can be shown that Assumptions H and G offer us quite a few candidates of the \( A(w,l) \), with the scaling function \( w (r) = j (r) - 1 r^{-1} r^{-d}, r > 0 \) and the scaling factor
\[
l (r) = \begin{cases} 
C r^{2 \delta_1} & \text{if } r \leq 1, \\
C r^{2 \delta_2} & \text{if } r > 1
\end{cases}
\]
for some \( C > 0 \). (See [6] for details.) In [6] and [7], Cauchy problems have been considered in the \( L^p \)-space and \( H^{\mu,s}_p \)-space respectively which are defined by Lévy measures from the \( A(w,l) \) class.

2.4. More Discussion about the Model. Some estimates on magnitude of the scaling function \( w \) and the scaling factor \( l \) can be extracted merely from their definitions.

**Lemma 1.** Let \( w : (0, \infty) \to (0, \infty) \) be a scaling function and \( l \) be an associated scaling factor which satisfies \( l \left( N^{-1} \right) < 1 < l \left( N \right) \). \( r_1 = \inf \{ r > 0 : N^r \geq l (N) \} \), \( r_2 = \sup \{ r > 0 : N^{-r} \geq l \left( N^{-1} \right) \} \). Then

(i) there exist \( c_0, C_0 > 0 \) such that
\[
c_0 \leq w (x) \leq C_0, \quad \forall x \in \left[ N^{-1}, N \right],
\]

(ii) \( r_1 \geq r_2 \), and for \( c_0, C_0 \) above,
\[
c_0 \left( x^{r_1} \wedge x^{r_2} \right) \leq w (x) \leq C_0 \left( x^{r_1} \vee x^{r_2} \right), \quad \forall x \in \mathbb{R}_+,
\]

(iii) for the same \( c_0 \) and \( C_0 \),
\[
l (x) \geq \frac{c_0}{C_0} \left( x^{r_1} \wedge x^{r_2} \right), \quad \forall x \in \mathbb{R}_+.
\]

(iv) \( \gamma (x) := \inf \{ s : l (s) \geq x \} \). For the same \( c_0 \) and \( C_0 \),
\[
\gamma (x) \leq \frac{C_0}{c_0} \left( \frac{1}{x^{r_1}} \vee \frac{1}{x^{r_2}} \right), \quad \forall x \in \mathbb{R}_+.
\]
Proof. (i) Utilize (1.3) and monotonicity of \( \alpha \), that (ii) holds. In Lemma 3, we shall show that the order \( r \).

Remark: Suggested by the bounds in (ii), we redefine \( r \).

As a summary, \( r \).

(iii) By (ii) and (1.3), \( r \).

(iv) is a direct conclusion from (iii). \( \square \)

Lemma 2. Let \( \nu \) be a Lévy measure and \( w \) be the scaling function which \( \nu \) satisfies \( A \) for. Then,

a) there are constants \( C_1, C_2 > 0 \) such that

\[
C_1 \leq w(r)^{-1} \leq C_2, \quad \forall r > 0.
\]

b) \( \int_{|y|\leq 1} w(|y|) \nu(dy) = +\infty. \)

c) For any \( \varepsilon > 0 \), \( \int_{|y|\leq 1} |y|^{1+\varepsilon} \nu(dy) < \infty. \)

d) For any \( \varepsilon > 0 \), \( \int_{|y|\leq 1} |y|^\varepsilon w(|y|) \nu(dy) < \infty. \)

Proof. a) First,

\[
w(r)^{-1} \int_{|y|>1} \nu_r(dy) = \int_{|y|>1} \nu(dy) = \zeta(r), \quad \forall r > 0,
\]
then by (iii) in $A(w,l)$,
\begin{equation}
\zeta (r) \leq C w (r)^{-1}, \quad \forall r > 0.
\end{equation}

On the other hand, for all $r > 0$,
\begin{equation}
 w (r)^{-1} \int_{|y| \leq 1} |y|^2 \tilde{u}_r (dy) = r^{-2} \int_{|y| \leq r} |y|^2 \nu (dy) = -r^{-2} \int_0^r s^2 \zeta (s).
\end{equation}

By the generalized formula of integration by parts from stochastic calculus,
\begin{equation}
\int_0^r s^2 \zeta (s) ds = \lim_{\varepsilon \to 0} \int_\varepsilon^r s^2 \zeta (s) ds
= \lim_{\varepsilon \to 0} \left( s^2 \zeta (s) \bigg|_{\varepsilon}^r - 2 \int_\varepsilon^r s \zeta (s) ds \right)
= r^2 \zeta (r) - 2 \int_0^r s \zeta (s) ds.
\end{equation}

Note that in above derivation, we used the fact that $\lim_{\varepsilon \to 0} \varepsilon^2 \zeta (\varepsilon) = 0$. This is due to $A(w,l)(i)$, \eqref{2.13} and \eqref{2.14}, which implies
\begin{equation}
\lim_{\varepsilon \to 0} \varepsilon^2 \zeta (\varepsilon) \leq C \lim_{\varepsilon \to 0} \varepsilon^2 w (\varepsilon)^{-1} \leq C \lim_{\varepsilon \to 0} \int_{|y| \leq \varepsilon} |y|^2 \nu (dy) = 0.
\end{equation}

Combine \eqref{2.12}, \eqref{2.14} and \eqref{2.15}.
\begin{equation}
 w (r)^{-1} \int \left( |y|^2 \wedge 1 \right) \tilde{u}_r (dy) = 2 r^{-2} \int_0^r s \zeta (s) ds = 2 \int_0^1 s \zeta (rs) ds.
\end{equation}

Again by $A(w,l)(i)$,
\begin{equation}
 w (r)^{-1} \leq C \zeta (r) \int_0^1 s \zeta (rs) \zeta (r)^{-1} ds \leq C \zeta (r).
\end{equation}

b) Use a) and the Itô formula.
\begin{equation}
\int_{|y| \leq 1} w (|y|) \nu (dy) = -\int_0^1 w (r) d\zeta (r)
\leq -C \int_0^1 \zeta (r)^{-1} d\zeta (r) = -C \lim_{\varepsilon \to 0} \int_\varepsilon^1 \zeta (r)^{-1} d\zeta (r)
= C \left( \lim_{\varepsilon \to 0} \ln \zeta (\varepsilon) - \ln \zeta (1) \right) = \infty.
\end{equation}

c) For any $\varepsilon > 0$,
\begin{equation}
\int_{|y| \leq 1} w (|y|)^{1+\varepsilon} \nu (dy) \leq C \int_0^1 \zeta (r)^{1-\varepsilon} d\zeta (r) = C \zeta (r)^{-\varepsilon} |1^0 < \infty.
\end{equation}
d) By c), for any $\epsilon > 0$, $\sigma' > \inf \left\{ \sigma \in (0, 2) : \limsup_{r \to 0} \frac{r^\sigma}{w(r)} = 0 \right\}$,
\[ \int_{|y| \leq 1} |y|^\epsilon w(|y|) \nu(dy) = \int_{|y| \leq 1} \left( \frac{|y|^{\sigma'}}{w(|y|)} \right)^{\frac{\epsilon}{\sigma'}} w(|y|)^{1+\frac{\epsilon}{\sigma'}} \nu(dy) \leq C \int_{|y| \leq 1} w(|y|)^{1+\frac{\epsilon}{\sigma'}} \nu(dy) < \infty. \]

\[ \Box \]

**Lemma 3.** Let $\nu$ be a Lévy measure and $w$ be the scaling function which $\nu$ satisfies $A(w, l)$ for. $\alpha$ is the order of $\nu$. Then
\[ \alpha = \inf \left\{ \sigma : \limsup_{r \to 0} \frac{r^\sigma}{w(r)} = 0 \right\}. \]

**Proof.** Denote $\alpha' = \inf \left\{ \sigma \in (0, \infty) : \limsup_{r \to 0} \frac{r^\sigma}{w(r)} = 0 \right\}$. We first show that $\alpha' \leq \alpha$. Note if $\sigma \in (0, \alpha')$, $\limsup_{r \to 0} \frac{r^\sigma}{w(r)} = \infty$. Otherwise
\[ \limsup_{r \to 0} \frac{r^{\sigma+\alpha'}}{w(r)} \leq \limsup_{r \to 0} \frac{r^\sigma}{w(r)} \limsup_{r \to 0} \frac{r^{\alpha'}}{2} = 0, \]
which contradicts with the definition of $\alpha'$. Now take $0 < r \leq 1$. For any $\sigma \in (\alpha, \infty)$,
\[ \int_{|y| \leq 1} |y|^\sigma \nu(dy) \geq \int_{|y| \leq r} |y|^\sigma \nu(dy) = \frac{r^\sigma}{w(r)} \int_{|y| \leq 1} |y|^\sigma \tilde{\nu}_r \geq \frac{r^\sigma}{w(r)} \int_{|y| \leq 1} |y|^2 \tilde{\nu}_r. \]
By (i) in $A(w, l)$, $\{\tilde{\nu}_r : r > 0\}$ are non-degenerate. Hence
\[ c \frac{r^\sigma}{w(r)} \leq \frac{r^\sigma}{w(r)} \int_{|y| \leq 1} |y|^2 \tilde{\nu}_r \leq \int_{|y| \leq 1} |y|^\sigma \nu(dy) < C \]
for some $c, C > 0$ independent of $r$. Thus $\alpha' \leq \sigma$, and thus $\alpha' \leq \alpha$.

For the other direction, assume to the contrary $\alpha' < \alpha$. Then by Lemma 2 for $\alpha' < \sigma' < \sigma < \alpha$,
\[ \int_{|y| \leq 1} |y|^\sigma \nu(dy) = \int_{|y| \leq 1} \frac{|y|^{\sigma'}}{w(|y|)} |y|^{\sigma-\sigma'} w(|y|) \nu(dy) \leq C \int_{|y| \leq 1} |y|^{\sigma-\sigma'} w(|y|) \nu(dy) < \infty. \]
But this contradicts with the definition of $\alpha$. Therefore, $\alpha \leq \alpha'$. Combining the argument above, we obtain $\alpha' = \alpha$. \[ \Box \]

According to Lemma 3, we can claim immediately that two Lévy measures which satisfy $A(w, l)$ for the same $w, l$ have the same order.

The last two lemmas of this section explain why we restricted the Hölder order $\beta \in (0, 1/\alpha)$ when defining generalized Hölder spaces in section 2.2. If $\beta$ exceeds or equals to $1/\alpha$, the function may reduce to a trivial case that is no longer of interest.
Lemma 4. Set \( \alpha' = \inf \{ \sigma \in (0, 2) : \lim \sup_{r \to 0} \frac{w^\sigma}{w(r)} = 0 \} \).

a) If \( 0 < \frac{1}{\beta} < \alpha' \) and \( \sup_{x,y} \frac{|f(x) - f(x+y)|}{w(|y|)^\beta} < \infty \), then \( f \) is a constant.

b) If \( \frac{1}{\beta} > \alpha' \) and \( f \) is a bounded Lipschitz function, then for each \( \varepsilon \in (0, 1) \), there is a positive constant \( C_\varepsilon \) depending on \( \varepsilon \) but independent of \( f \) so that

\[
\sup_{x,y} \frac{|f(x) - f(x+y)|}{w(|y|)^\beta} \leq \varepsilon \sup_{x,y} \frac{|f(x+y) - f(x)|}{|y|} + C_\varepsilon |f|_0.
\]

Namely, the space \( C^\beta \) contains all bounded Lipschitz functions.

Proof. a) Let \( \varepsilon \in (0, \beta \alpha' - 1) \), \( \beta'(1 + \varepsilon) = \beta \). Then \( \frac{1}{\beta} \leq \frac{1}{\beta'} < \alpha' \) and we can find a sequence \( y_n \to 0 \) so that \( \frac{|y_n|^{1/\beta'}}{w(|y_n|)} \geq C > 0 \). Let \( f_\varepsilon = f * w_\varepsilon \) where \( w_\varepsilon \) is the standard mollifier. We then have

\[
\sup_{x,y} \frac{|f(x) - f(x+y)|}{w(|y|)^\beta} \geq C |f_\varepsilon(x) - f_\varepsilon(x+y)\| w(|y_n|)^{\beta'} \beta'(1+\varepsilon)
\]

Hence \( \nabla f_\varepsilon(x) = 0, x \in \mathbb{R}^d, \forall \varepsilon \in (0, \beta \alpha' - 1) \), which implies \( f_\varepsilon(x) = C_\varepsilon, \forall \varepsilon \in (0, \beta \alpha' - 1) \). Obviously, \( f \) is continuous, then \( f_\varepsilon \to f \) uniformly on any compact subsets, and thus \( f \) is a constant.

b) Since \( \lim \sup_{r \to 0} \frac{1}{w(r)} = 0 \), then for each \( \varepsilon \in (0, 1) \) there is \( \delta > 0 \) so that \( \frac{|y|^{1/\beta}}{w(|y|)} \leq \varepsilon^{\beta} \) if \( |y| \leq \delta \). Hence,

\[
\frac{|f(x) - f(x+y)|}{w(|y|)^\beta} = \frac{|f(x) - f(x+y)|}{|y|} \left( \frac{|y|^{1/\beta}}{w(|y|)} \right)^\beta
\]

if \( |y| \leq \delta \), and

\[
\frac{|f(x) - f(x+y)|}{w(|y|)^\beta} \leq 2w(\delta)^{-\beta} l(1)^{\beta} |f|_0 \leq C_\varepsilon |f|_0
\]

if \( |y| > \delta \). \(\square\)

Lemma 5. Let \( \alpha' = \inf \{ \sigma \in (0, 2) : \lim \sup_{r \to 0} \frac{\sigma}{w(r)} = 0 \} \).

a) If \( \lim \sup_{r \to 0} \frac{\alpha'}{w(r)} = 0 \) and \( f \) is a bounded Lipschitz function, then for each
\( \varepsilon \in (0, 1) \) there is a positive constant \( C_\varepsilon \) depending on \( \varepsilon \) but independent of \( f \) so that
\[
\sup_{x,y} \frac{|f(x) - f(x + y)|}{w(|y|)^{1/\alpha'}} \leq \varepsilon \sup_{x,y} \frac{|f(x + y) - f(x)|}{|y|} + C_\varepsilon |f|_0.
\]
Namely, the space \( \tilde{C}^{1/\alpha'} \) contains all bounded Lipschitz functions.

b) If \( \limsup_{r \to 0} \frac{w^{\alpha'}}{w(r)} \in (0, \infty) \), then \( \tilde{C}^{1/\alpha'} \) is the space of bounded Lipschitz functions.

c) If \( \limsup_{r \to 0} \frac{w^{\alpha'}}{w(r)} = \infty \), then \( \tilde{C}^{1/\alpha'} \) consists of constants only.

**Proof.**

a) The proof is identical to part b) of Lemma 4.

b) Let \( w_\varepsilon \) be a standard mollifier and \( f_\varepsilon = f \ast w_\varepsilon \). For any \( f \in \tilde{C}^{1/\alpha'} \), there is a sequence \( y_n \to 0 \) so that \( \frac{|y_n|^{\alpha'}}{w(|y_n|)} \geq c > 0 \). Then
\[
\frac{|f_\varepsilon(x) - f_\varepsilon(x + y_n)|}{w(|y_n|)^{1/\alpha'}} = \frac{|f_\varepsilon(x) - f_\varepsilon(x + y_n)|}{|y_n|} \left( \frac{|y_n|^{\alpha'}}{w(|y_n|)} \right)^{1/\alpha'} \\
\geq c \frac{|f_\varepsilon(x) - f_\varepsilon(x + y_n)|}{|y_n|}.
\]

Thus, \( |\nabla f_\varepsilon|_0 \leq C |f|_{1/\alpha'} \), and thus \( |\nabla f|_0 \leq C |f|_{1/\alpha'} \).

On the other hand, \( \frac{|y|^{\alpha'}}{w(|y|)} \leq C |y| \leq 1 \), then
\[
\frac{|f(x) - f(x + y)|}{w(|y|)^{1/\alpha'}} = \frac{|f(x) - f(x + y)|}{|y|} \left( \frac{|y|^{\alpha'}}{w(|y|)} \right)^{1/\alpha'} \\
\leq C \frac{|f(x) - f(x + y)|}{|y|},
\]
if \( |y| \leq 1 \), and for \( |y| > 1 \),
\[
\frac{|f(x) - f(x + y)|}{w(|y|)^{1/\alpha'}} \leq 2w(1)^{-1/\alpha'} l(1)^{1/\alpha'} |f|_0 \leq C |f|_0.
\]

Hence \( f \in \tilde{C}^{1/\alpha'} \) if \( f \) is a bounded Lipschitz function.

c) There is a sequence \( \{y_n : n \in \mathbb{N}\} \) so that \( y_n \to 0 \) and for any \( n \in \mathbb{N} \),
\( \frac{|y_k|^{\alpha'}}{w(|y_k|)} \geq n \) if \( k \geq n \). Then for \( k \geq n \),
\[
n^\frac{1}{\alpha'} \frac{|f_\varepsilon(x) - f_\varepsilon(x + y_k)|}{|y_k|} \leq \frac{|f_\varepsilon(x) - f_\varepsilon(x + y_k)|}{w(|y_k|)^{1/\alpha'}} \leq |f_\varepsilon|_{1/\alpha'} \leq C |f|_{1/\alpha'}.
\]

Thus \( \nabla f_\varepsilon = 0 \), \( \forall x \in \mathbb{R}^d \), \( \forall \varepsilon \in (0, 1) \), and thus \( f \) is a constant. \( \square \)
3. Characterization of Spaces and Norm Equivalence

Our target spaces of general smoothness are \( \tilde{C}^\beta, \tilde{C}^\beta_{\infty, \infty}, C^{\mu, \kappa, \beta}, \tilde{C}^{\mu, \kappa, \beta} \) endowed with norms \(|\cdot|_{\beta}, |\cdot|_{\beta, \infty}, |\cdot|_{\mu, \kappa, \beta}, \|\cdot\|_{\mu, \kappa, \beta}\) respectively, and our goal in this section is to establish norm equivalence among them.

**Lemma 6.** Let \( \beta \in (0, \infty) \). If \( u \in \tilde{C}^\beta_{\infty, \infty}(\mathbb{R}^d) \), then \( u \in C(\mathbb{R}^d) \) and

\[
|u|_0 \leq \sum_{j=0}^{\infty} |u * \varphi_j|_0 \leq C(\beta) |u|_{\beta, \infty}.
\]

**Proof.** Note that \( u * \varphi_j \in C(\mathbb{R}^d), \forall j \in \mathbb{N} \), so is \( \sum_{j=0}^{n} u * \varphi_j, \forall n \in \mathbb{N}_+ \). Since

\[
\sum_{j=0}^{\infty} |u * \varphi_j|_0 = \sum_{j=0}^{\infty} w(N^{-j})^\beta w(N^{-j})^{-\beta} |u * \varphi_j|_0
\]

\[
\leq \sup_{j \geq 0} w(N^{-j})^{-\beta} |u * \varphi_j|_0 \sum_{j=0}^{\infty} w(N^{-j})^\beta
\]

\[
\leq C |u|_{\beta, \infty} \sum_{j=0}^{\infty} l(N^{-1})^j < \infty,
\]

we have \( \sum_{j=0}^{n} u * \varphi_j \to \sum_{j=0}^{\infty} u * \varphi_j \) uniformly in \( \mathbb{R}^d \) as \( n \to \infty \). Therefore, \( \sum_{j=0}^{\infty} u * \varphi_j \in C(\mathbb{R}^d) \), and \( \sum_{j=0}^{n} u * \varphi_j \xrightarrow{n \to \infty} \sum_{j=0}^{\infty} u * \varphi_j \) in the topology of \( S'(\mathbb{R}^d) \). By continuity of the Fourier transform,

\[
\mathcal{F}\left(\sum_{j=0}^{\infty} u * \varphi_j\right) = \lim_{n \to \infty} \mathcal{F}\left(\sum_{j=0}^{n} u * \varphi_j\right) = \lim_{n \to \infty} \sum_{j=0}^{n} \hat{u} \hat{\varphi}_j = \sum_{j=0}^{\infty} \hat{u} \hat{\varphi}_j = \hat{u}.
\]

And therefore, \( u = \sum_{j=0}^{\infty} u * \varphi_j \in C(\mathbb{R}^d) \). \( \square \)

**Proposition 1.** Let \( \beta \in (0, 1) \) and

\[
(3.2) \quad \int_{0}^{1} l(t)^\beta \frac{dt}{t} + \int_{1}^{\infty} l(t)^\beta \frac{dt}{t^2} < \infty.
\]

Then norm \( |u|_\beta \) and norm \( |u|_{\beta, \infty} \) are equivalent. Namely, there is a constant positive \( C \) depending only on \( d, \beta, N \) such that

\[
C^{-1} |u|_\beta \leq |u|_{\beta, \infty} \leq C |u|_\beta, \forall u \in C(\mathbb{R}^d).
\]

**Proof.** Suppose \( |u|_\beta < \infty \). If \( j = 0 \), then

\[
w(1)^{-\beta} |u * \varphi_0|_0 \leq w(1)^{-\beta} |u|_0 \int |\varphi_0(y)| dy \leq C |u|_\beta.
\]
If $j \neq 0$, then by the construction of $\varphi_j$, $\int \varphi_j(y) \, dy = \hat{\varphi}_j(0) = 0$. Therefore,

$$w (N^{-j})^{-\beta} |u * \varphi_j|_0$$

$$= w (N^{-j})^{-\beta} \left| \int [u(y) - u(x)] \varphi_j(x - y) \, dy \right|_0$$

$$\leq w (N^{-j})^{-\beta} |u|_\beta \int w(|y - x|)^\beta N^j |\tilde{\varphi} (N^j (x - y))| \, dy$$

$$= w (N^{-j})^{-\beta} |u|_\beta \int w (N^{-j} |y|)^\beta |\tilde{\varphi} (y)| \, dy$$

$$\leq |u|_\beta \int l(|y|)^\beta |\tilde{\varphi} (y)| \, dy \leq C |u|_\beta .$$

That is to say $|u|_{\beta, \infty} \leq C |u|_\beta$ for some constant $C (\beta, d) > 0$.

For the other direction, by Lemma 6 $|u|_0 \leq C |u|_{\beta, \infty}$. Meanwhile, we can write

$$[u]_\beta = \sup_{x,y} \frac{|u(x + y) - u(x)|}{w(|y|)^\beta} \leq \sup_{t > 0} \frac{\sup_{|y| \leq t} |u(x + y) - u(x)|_0}{w(t)^\beta} := \sup_{t > 0} \varpi(t,u) w(t)^\beta,$$

where $\varpi(t,u) := \sup_{|y| \leq t} |u(x + y) - u(x)|_0$ is increasing in $t$. Then, for $k \geq 0$,

$$\varpi(N^{-k-1},u) \leq \varpi(t,u) \leq \varpi(N^{-k},u) \quad \text{if } N^{-k-1} \leq t < N^{-k},$$

and then by monotonicity of $l$, for $N^{-k-1} \leq t < N^{-k},$

$$l(N)^{-\beta} \frac{\varpi(N^{-k-1},u)}{w(N^{-k-1})^\beta} \leq \frac{\varpi(t,u)}{w(t)^\beta} \leq l(N)^{-\beta} \frac{\varpi(N^{-k},u)}{w(N^{-k})^\beta}.$$
Therefore by Lemma 6, for each $k$:

$$C_k \leq C(N^j y \wedge 1) |u * \varphi_j|_0, \quad j \geq 1.$$ 

Hence, for some positive constants $C$ and therefore,

$$1 \leq C \sup_{|y| \leq N^{-k}} \sum_{j=0}^{\infty} (N^j |y| \wedge 1) |u * \varphi_j|_0,$$

and therefore,

$$\varpi(N^{-k}, u) \leq C |u|_{\beta, \infty} \sup_{|y| \leq N^{-k}} \sum_{j=0}^{\infty} (N^j |y| \wedge 1) w(N^{-j})^\beta \leq C |u|_{\beta, \infty} \left[ \sum_{j=0}^{k} N^{j-k} w(N^{-j})^\beta + \sum_{j=k+1}^{\infty} w(N^{-j})^\beta \right].$$

Clearly, for all $j \in \mathbb{N}$, $j \leq x \leq j + 1$,

$$l(1)^{-\beta} w(N^{-x})^\beta \leq w(N^{-j})^\beta \leq l(N)^\beta w(N^{-x})^\beta,$$

$$\frac{l(1)^{-\beta}}{N} w(N^x w(N^{-x})^\beta \leq N^j w(N^{-j})^\beta \leq l(N)^\beta N^x w(N^{-x})^\beta.$$ 

Then for all $k$:

$$C_1 \int_0^{k+1} N^x w(N^{-x}) dx \leq \sum_{j=0}^{k} N^j w(N^{-j})^\beta \leq C_2 \int_0^{k+1} N^x w(N^{-x})^\beta dx$$

for some positive constants $C_1, C_2$ that do not depend on $k, j$. Hence,

$$\sum_{j=0}^{k} N^{j-k} w(N^{-j})^\beta = N^{-k} \sum_{j=0}^{k} N^j w(N^{-j})^\beta \leq CN^{-k} \int_0^{k+1} N^x w(N^{-x})^\beta dt = CN^{-k} \int_1^{N^{k+1}} w(t^{-1})^\beta dt \leq Cw(N^{-k})^\beta \int_{N^{-1}}^{N^k} l(r)^\beta r^{-2} dr.$$
Meanwhile,

\[
\sum_{j=k+1}^{\infty} w(N^{-j})^\beta \leq C \int_{k+1}^{\infty} w(N^{-x})^\beta \, dx \leq C w(N^{-k})^\beta \int_{k+1}^{\infty} l(N^k N^{-x})^\beta \, dx
\]

\[
= C w(N^{-k})^\beta \int_0^{N-1} l(r)^\beta \, \frac{dr}{r}.
\]

Therefore, under the assumption (3.2),

\[
w(N^{-k})^{-\beta} w(N^{-k}, u) \leq C w(N^{-k})^{-\beta} |u|_{\beta,\infty} \left[ \sum_{j=0}^{k} N^{j-k} w(N^{-j})^\beta + \sum_{j=k+1}^{\infty} w(N^{-j})^\beta \right]
\]

\[
\leq C |u|_{\beta,\infty} \left[ \int_{N-1}^{\infty} l(r)^\beta r^{-2} \, dr + \int_0^{N-1} l(r)^\beta \, \frac{dr}{r} \right] \leq C |u|_{\beta,\infty}.
\]

That ends the proof. \qed

**Remark:** When \( \mu(dy) = \frac{dy}{|y|^{d+\alpha}} \), one of Lévy measures that are of the most research interest, or when in case (14), \( w(t) = l(t) = t^\alpha \), (3.2) reduces to \( \beta < 1/\alpha \), which corresponds to the classical equivalence of the Hölder-Zygmund norm and the Besov norm.

The next lemma is fundamental to this paper.

**Lemma 7.** Let \( \nu \) be a Lévy measure satisfying (iii) in \( A(w,l) \). For any function \( \varphi \in C_b^\infty(R^d) \),

\[ L^{\tilde{\nu}}\varphi(x) := \int [\varphi(x+y) - \varphi(x) - \chi_\alpha(y) y \cdot \nabla \varphi(x)] \tilde{\nu}(dy), \quad R > 0. \]

Then,

\[ \int |\varphi(x+y) - \varphi(x) - \chi_\alpha(y) y \cdot \nabla \varphi(x)| \tilde{\nu}(dy) < C(\alpha, d, \varphi, \alpha_1, \alpha_2). \]

Moreover, \( L^{\tilde{\nu}}(\varphi) \in C_b^\infty(R^d) \) and \( D^\gamma L^{\tilde{\nu}}\varphi = L^{\tilde{\nu}}(D^\gamma \varphi) \), where \( \gamma \in \mathbb{N}^d \) is a multi-index. If \( \varphi(x) \in \mathcal{S}(R^d) \), then \( L^{\tilde{\nu}}\varphi(x) \in L^1(R^d) \) and \( \left| L^{\tilde{\nu}}\varphi \right|_{L^1(R^d)} \leq C \) for some positive \( C \) that is uniform with respect to \( R \).
**Proof.** Obviously,

\[(3.4) \quad \int |\varphi(x + y) - \varphi(x) - \chi_\alpha(y) y \cdot \nabla \varphi(x)| \tilde{\nu}_R(dy) \leq 1_{\alpha \in (0,1)} \int_{|y| \leq 1} \int_0^1 |\nabla \varphi(x + \theta y)| |y| d\theta \tilde{\nu}_R(dy) + 1_{\alpha \in [1,2]} \int_{|y| \leq 1} \int_0^1 |\nabla^2 \varphi(x + \theta_1 \theta_2 y)| |y|^2 d\theta_1 d\theta_2 \tilde{\nu}_R(dy) + \int_{|y| > 1} (|\varphi(x + y)| + |\varphi(x)| + \chi_\alpha(y) |y| |\nabla \varphi(x)|) \tilde{\nu}_R(dy). \]

If \(\varphi(x) \in C_b^\infty(\mathbb{R}^d)\), by (iii) in \(A(w,l)\),

\[
\int |\varphi(x + y) - \varphi(x) - \chi_\alpha(y) y \cdot \nabla \varphi(x)| \tilde{\nu}_R(dy) \leq 1_{\alpha \in (0,1)} \int_{|y| \leq 1} |\nabla \varphi|_0 |y| \tilde{\nu}_R(dy) + 1_{\alpha \in [1,2]} \int_{|y| \leq 1} |\nabla^2 \varphi|_0 |y|^2 \tilde{\nu}_R(dy) + \int_{|y| > 1} (2 |\varphi|_0 + \chi_\alpha(y) |y| |\nabla \varphi|_0) \tilde{\nu}_R(dy) \]

\[(3.5) \leq C \left( \int_{|y| \leq 1} |y|^\alpha_1 \tilde{\nu}_R(dy) + \int_{|y| > 1} |y|^\alpha_2 \tilde{\nu}_R(dy) \right) < C. \]

Since \(\partial_i \varphi \in C_b^\infty(\mathbb{R}^d), i = 1, 2, \ldots, d\), the same steps can be applied to \(\partial_i \varphi\). Then (3.5) indicates that \(L^{\tilde{\nu}_R} \partial_i \varphi \in C_b(\mathbb{R}^d)\) and \(\partial_i L^{\tilde{\nu}_R} \varphi = L^{\tilde{\nu}_R} \partial_i \varphi\) by the dominated convergence theorem. Then, \(L^{\tilde{\nu}_R} \varphi \in C_b^\infty(\mathbb{R}^d)\) and \(D^\gamma L^{\tilde{\nu}_R} \varphi = L^{\tilde{\nu}_R} D^\gamma \varphi, \gamma \in \mathbb{N}^d\) is a consequence of induction.

If \(\varphi(x) \in \mathcal{S}(\mathbb{R}^d)\), then by (3.4),

\[
|L^{\tilde{\nu}_R} \varphi|_{L^1(\mathbb{R}^d)} \leq \int \int |\varphi(x + y) - \varphi(x) - \chi_\alpha(y) y \cdot \nabla \varphi(x)| \tilde{\nu}_R(dy) dx \leq 1_{\alpha \in (0,1)} \int_{|y| \leq 1} \int_0^1 |\nabla \varphi(x)| dx |y| d\theta \tilde{\nu}_R(dy) + 1_{\alpha \in [1,2]} \int_{|y| \leq 1} \int_0^1 |\nabla^2 \varphi(x)| dx |y|^2 d\theta_1 d\theta_2 \tilde{\nu}_R(dy) + \int_{|y| > 1} \int (2 |\varphi(x)| + \chi_\alpha(y) |y| |\nabla \varphi(x)|) dx \tilde{\nu}_R(dy),
\]

again by (iii) in \(A(w,l)\),

\[
|L^{\tilde{\nu}_R} \varphi|_{L^1(\mathbb{R}^d)} \leq C \left( \int_{|y| \leq 1} |y|^\alpha_1 \tilde{\nu}_R(dy) + \int_{|y| > 1} |y|^\alpha_2 \tilde{\nu}_R(dy) \right) \leq C(\alpha, d, \varphi, \alpha_1, \alpha_2). \]

\(\Box\)
where $C$. Thus,

\[ \int_{\infty}^{t} \phi \] (3.6)

(0 $\phi$ if $L$

Besides, $\phi$ $C$ $0$

The lemma below is about integrability of $L^{\nu, \kappa} \phi, \kappa \in (0, 1)$ and its probabilistic representation which we shall use repeatedly.

**Lemma 8.** Let $\nu$ be a Lévy measure satisfying (iii) in $A(w, l)$ and $\kappa \in (0, 1)$. $L^{\nu, \kappa}, R > 0$ is the associated operator defined as (2.5). Then for any $\phi(x) \in C_{0}^{\infty}(\mathbb{R}^{d})$,

\[ L^{\nu, \kappa} \phi(x) = C \int_{0}^{\infty} t^{-\kappa-1} E \left[ \phi \left( x + Z_{t}^{\nu R} \right) - \phi(x) \right] dt, R > 0, \]

where $C^{-1} = \int_{0}^{\infty} t^{-\kappa-1} \left( 1 - e^{-t} \right) dt$ and

\[ \overline{\nu}_{R}(dy) = \frac{1}{2} (\nu_{R}(dy) + \nu_{R}(-dy)), R > 0. \]

Besides, $L^{\nu, \kappa} \phi \in C_{0}^{\infty}(\mathbb{R}^{d})$. And $|L^{\nu, \kappa} \phi|_{L^{1}(\mathbb{R}^{d})} < C'$ for some $C' > 0$ independent of $R$ if $\phi(x) \in S(\mathbb{R}^{d})$.

**Proof.** Clearly, for all $R > 0, \xi \in \mathbb{R}^{d}$, $\Re \psi^{\nu R}(\xi) \leq 0$. Then for any $\kappa \in (0, 1)$,

\[ \int_{0}^{\infty} t^{-\kappa-1} \left( 1 - \exp\{\Re \psi^{\nu R}(\xi) t\} \right) dt = (-\Re \psi^{\nu R}(\xi))^{\kappa} \int_{0}^{\infty} t^{-\kappa-1} \left( 1 - e^{-t} \right) dt. \]

Thus,

\[ L^{\nu, \kappa} \phi(x) = C F^{-1} \left[ \int_{0}^{\infty} t^{-\kappa-1} \left( \exp\{\Re \psi^{\nu R}(\xi) t\} - 1 \right) F \phi dt \right](x), \]

where $C^{-1} = \int_{0}^{\infty} t^{-\kappa-1} \left( 1 - e^{-t} \right) dt$. Since $\Re \psi^{\nu R}(\xi) = \psi^{\nu R}(\xi)$, by the Lévy-Khintchine formula,

\[ L^{\nu, \kappa} \phi(x) = C F^{-1} \left[ \int_{0}^{\infty} t^{-\kappa-1} \left( \exp\{\psi^{\nu R}(\xi) t\} - 1 \right) F \phi dt \right](x) \]

\[ = C F^{-1} \left[ \int_{0}^{\infty} t^{-\kappa-1} \mathbb{E} \left( \phi \left( x + Z_{t}^{\nu R} \right) - \phi(x) \right) dt \right](x) \]

\[ = C F^{-1} \left[ \int_{0}^{\infty} t^{-\kappa-1} \mathbb{E} \left( \phi \left( x + Z_{t}^{\nu R} \right) - \phi(x) \right) dt \right](x) \]

if $\phi(x) \in S(\mathbb{R}^{d})$. Note

\[ \int_{0}^{t} t^{-\kappa-1} \left| \mathbb{E} \left[ \phi \left( x + Z_{t}^{\nu R} \right) - \phi(x) \right] \right| dt \]

\[ \leq \int_{0}^{1} t^{-\kappa-1} \int_{0}^{t} \left| L^{\nu R} \phi \left( x + Z_{r}^{\nu R} \right) \right| dr dt + \int_{1}^{\infty} t^{-\kappa-1} \mathbb{E} \left| \phi \left( x + Z_{t}^{\nu R} \right) - \phi(x) \right| dt, \]
and note \( \tilde{R} = v_R \). If \( \nu \) satisfies (iii) in \( A(w, l) \), so does \( v \). Then by Lemma 7

\[
\int_0^\infty t^{-\kappa-1} \int E \left[ \varphi \left( x + Z_t^R \right) - \varphi(x) \right] dt dx \\
\leq \int_0^1 t^{-\kappa-1} \int t \int L_{\tilde{R}} \varphi(x) dx + 2 \int_1^\infty t^{-\kappa-1} \int |\varphi(x)| dx dt \\
(3.7) \leq C'
\]

for some \( C' > 0 \) independent of \( R \). Thus Fubini's theorem applies, and thus

\[
L_{\tilde{R}}^{\varphi, \kappa} \varphi(x) = C \int_0^\infty t^{-\kappa-1} E \left[ \varphi \left( x + Z_t^R \right) - \varphi(x) \right] dt.
\]

\( L^1 \) integrability of \( L_{\tilde{R}}^{\varphi, \kappa} \varphi \) has been shown in (3.7).

For \( \varphi \in C_b^\infty (\mathbb{R}^d) \), we introduce \( \{ \zeta_n : n \in \mathbb{N} \} \subseteq C_0^\infty (\mathbb{R}^d) \) such that \( 0 \leq \zeta_n(x) \leq 1, \forall n \in \mathbb{N}, \forall x \in \mathbb{R}^d \) and \( \zeta_n(x) = 1, \forall x \in \{ x \in \mathbb{R}^d : |x| \leq n \} \). Then \( \varphi \zeta_n \xrightarrow{n \to \infty} \varphi \) pointwise, which by the dominated convergence theorem implies that \( \varphi \zeta_n \xrightarrow{n \to \infty} \varphi \) in the weak topology on \( S' (\mathbb{R}^d) \). Clearly, (3.6) holds for \( \varphi \zeta_n \). Hence,

\[
< L_{\tilde{R}}^{\varphi, \kappa} \varphi \zeta_n, \eta > \\
= C \int_0^\infty t^{-\kappa-1} E \left[ \varphi \zeta_n \left( x + Z_t^R \right) - \varphi \zeta_n(x) \right] dt \eta(x) dx \\
= C \int_0^\infty t^{-\kappa-1} E \left[ \eta \left( x - Z_t^R \right) - \eta(x) \right] dt \varphi \zeta_n(x) dx, \forall \eta \in S (\mathbb{R}^d).
\]

Let \( n \to \infty \).

\[
\lim_{n \to \infty} < L_{\tilde{R}}^{\varphi, \kappa} \varphi \zeta_n, \eta > \\
= C \int_0^\infty t^{-\kappa-1} E \left[ \eta \left( x - Z_t^R \right) - \eta(x) \right] dt \varphi(x) dx \\
= C \int_0^\infty t^{-\kappa-1} E \left[ \varphi \left( x + Z_t^R \right) - \varphi(x) \right] dt \eta(x) dx, \forall \eta \in S (\mathbb{R}^d).
\]

This is to say \( L_{\tilde{R}}^{\varphi, \kappa} \varphi \zeta_n \xrightarrow{n \to \infty} \int_0^\infty t^{-\kappa-1} E \left[ \varphi \left( x + Z_t^R \right) - \varphi(x) \right] dt \) in the topology of \( S' (\mathbb{R}^d) \). By continuity of the Fourier transform,

\[
C \mathcal{F} \int_0^\infty t^{-\kappa-1} E \left[ \varphi \left( x + Z_t^R \right) - \varphi(x) \right] dt \\
= \lim_{n \to \infty} \mathcal{F} \left[ L_{\tilde{R}}^{\varphi, \kappa} \varphi \zeta_n \right] = -(-\Re \psi_{\tilde{R}})^\kappa \lim_{n \to \infty} \mathcal{F} \left[ \varphi \zeta_n \right] \\
= -(-\Re \psi_{\tilde{R}})^\kappa \mathcal{F} \varphi.
\]

Therefore, (2.5) is well-defined for all functions in \( C_b^\infty (\mathbb{R}^d) \) and (3.6) applies. That \( L_{\tilde{R}}^{\varphi, \kappa} \varphi \in C_b^\infty (\mathbb{R}^d) \) is a result of dominated convergence theorem and induction. \( \square \)
Remark: Lemma 8 claims $L^{\mu_{R,\kappa}}, \forall R > 0$ is a closed operation in $C^\infty_b(R^d)$. Because of that, we can set $L^{\mu,\kappa} = L^{\mu_{\kappa/2}} \circ L^{\mu_{\kappa/2}}$ if $\kappa \in (1, 2)$, where $\circ$ means composition of two operations. Clearly, (2.5) is well-defined for all $\kappa \in (1, 2)$.

**Corollary 1.** Let $\nu$ be a Lévy measure satisfying $A(w, l)$ and $\kappa \in (0, 1]$. Denote $g_j = F^{-1} [F g (N^{-j} \cdot)], \forall g (x) \in S(R^d)$. Then there exists a constant $C > 0$ independent of $j$ such that

$$|L^{\mu,\kappa} g_j |_{L^1(R^d)} < C w (N^{-j})^{-\kappa}.$$  

**Proof.** If $j = 0$, this is a straightforward consequence of Lemmas 7 and 8. Now consider $j \neq 0$. By (2.8) in $A(w, l)$,

$$\psi^\nu (\xi) = w (N^{-j})^{-1} \psi_{N^{-j}} (N^{-j} \xi), \xi \in R^d, \forall j \in N_+,$$

therefore, by Lemma 7

$$|L^{\mu,\kappa} g_j |_{L^1(R^d)} = \int |F^{-1} [\psi^\nu (\xi) F g (N^{-j} \xi)] (x)| \, dx = \int |F^{-1} [w (N^{-j})^{-1} \psi_{N^{-j}} (N^{-j} \xi) F g (N^{-j} \xi)] (x)| \, dx = w (N^{-j})^{-1} \int |F^{-1} [\psi_{N^{-j}} (\xi) F g (\xi)] (x)| \, dx = w (N^{-j})^{-1} |L^{\psi_{N^{-j}}} g |_{L^1(R^d)} < C (\alpha, d, \alpha_1, \alpha_2) w (N^{-j})^{-1},$$

and by Lemma 8

$$|L^{\mu,\kappa} g_j |_{L^1(R^d)} = \int |F^{-1} [-w (N^{-j})^{-1} \Re \psi_{N^{-j}} (N^{-j} \xi)] (x)| \, dx = \int |F^{-1} [-w (N^{-j})^{-1} \Re \psi_{N^{-j}} (N^{-j} \xi)] (x)| \, dx = w (N^{-j})^{-\kappa} \int |F^{-1} [-\Re \psi_{N^{-j}} (\xi)] (x)| \, dx = w (N^{-j})^{-\kappa} |L^{\psi_{N^{-j}}} g |_{L^1(R^d)} < C (\alpha, d, \alpha_1, \alpha_2) w (N^{-j})^{-\kappa}.$$

\qed

The following two lemmas are crucial for proofs of norm equivalence.
Lemma 9. Let $a \in (0, \infty)$ and $\nu$ be a Lévy measure satisfying (iii) in $A(\mathbf{w}, \mathbf{l})$. Then the operator $aI - L^\nu$ defines a bijection on $C^\infty_b \left( \mathbb{R}^d \right)$. Moreover, for any function $\varphi \in C^\infty_b \left( \mathbb{R}^d \right)$,

\begin{align}
\varphi(x) &= \int_0^\infty e^{-at} E(aI - L^\nu) \varphi(x + Z_t^\nu) \, dt, \\
(aI - L^\nu)^{-1} \varphi(x) &= \int_0^\infty e^{-at} E\varphi(x + Z_t^\nu) \, dt, \quad x \in \mathbb{R}^d,
\end{align}

where $Z_t^\nu$ is the Lévy process associated to $\nu$.

Proof. By Lemma 7, $aI - L^\nu$ maps from $C^\infty_b \left( \mathbb{R}^d \right)$ to $C^\infty_b \left( \mathbb{R}^d \right)$. Apply the Itô formula to $e^{-at}\varphi(x + Z_t^\nu)$ on $[0, S]$ with respect to $t$, and take expectation afterwards, then

\begin{align*}
e^{-aS} E\varphi(x + Z_t^\nu) - \varphi(x) &= \int_0^S -ae^{-at} E\varphi(x + Z_t^\nu) \, dt + \int_0^S e^{-at} E\varphi(x + Z_t^\nu) \, dt.
\end{align*}

Note both $\varphi$ and $L^\nu \varphi$ are bounded. Let $S \to \infty$ and we obtain (3.9), which by Fubini theorem can also be written as

\begin{equation}
\varphi(x) = (aI - L^\nu) \int_0^\infty e^{-at} E\varphi(x + Z_t^\nu) \, dt,
\end{equation}

namely, $aI - L^\nu$ is a surjection. Meanwhile, if $\varphi$ is a function in $C^\infty_b \left( \mathbb{R}^d \right)$ such that $(aI - L^\nu) \varphi = 0$, then applying the same procedure, we arrive at (3.9), which claims $\varphi = 0$ and thus $aI - L^\nu$ is bijective. It follows immediately that

\begin{equation*}
(aI - L^\nu)^{-1} \varphi(x) = \int_0^\infty e^{-at} E\varphi(x + Z_t^\nu) \, dt, \quad x \in \mathbb{R}^d.
\end{equation*}

\[\Box\]

Similar results for $(I - L^\nu)^\kappa$, $\kappa \in (0, 1)$ are stated in next lemma. Denote

\[\mathcal{A}(\mathbb{R}^d) = \{ \varphi \in \mathcal{S}'(\mathbb{R}^d) : (a - \Re \psi^\nu)^\kappa \mathcal{F} \varphi, (a - \Re \psi^\nu)^{-\kappa} \mathcal{F} \varphi \in \mathcal{S}'(\mathbb{R}^d) \}.\]

Define for all $\varphi \in \mathcal{A}(\mathbb{R}^d)$,

\begin{align}
(aI - L^\nu)^\kappa \varphi &= \mathcal{F}^{-1} \left[ (a - \Re \psi^\nu)^\kappa \mathcal{F} \varphi \right], \\
(aI - L^\nu)^{-\kappa} \varphi &= \mathcal{F}^{-1} \left[ (a - \Re \psi^\nu)^{-\kappa} \mathcal{F} \varphi \right].
\end{align}

Obviously, (3.12) and (3.13) offer a bijection on $\mathcal{A}(\mathbb{R}^d)$.

Lemma 10. Let $\kappa \in (0, 1)$, $a \in (0, \infty)$ and $\nu$ be a Lévy measure satisfying (iii) in $A(\mathbf{w}, \mathbf{l})$. Then, $C^\infty_b \left( \mathbb{R}^d \right) \subset \mathcal{A}(\mathbb{R}^d)$ and thus $(aI - L^\nu)^\kappa$ is a
bijection on it. Moreover, for any function $\varphi \in C^\infty_b(\mathbb{R}^d)$,

$$\begin{align*}
(3.14) \quad (aI - L')^\kappa \varphi(x) &= C \int_0^\infty t^{-\kappa-1} \left[ \varphi(x) - e^{-at} \mathbb{E} \varphi(x + Z_t^0) \right] dt, \\
(3.15) \quad (aI - L^\nu)^{-\kappa} \varphi(x) &= C' \int_0^\infty t^{\kappa-1} e^{-at} \mathbb{E}\varphi(x + Z_t^0) dt,
\end{align*}$$

where $C^{-1} = \int_0^\infty t^{-\kappa-1} (1 - e^{-t}) dt$, $C'^{-1} = \int_0^\infty t^{\kappa-1} e^{-t} dt$, $Z_t^0$ is the Lévy process associated to $\bar{\nu}$ and $\bar{\nu}(dy) := \frac{1}{2} (\nu(dy) + \nu(-dy))$.

**Proof.** Since $a - \Re \psi^\nu(\xi) > 0, \forall \xi \in \mathbb{R}^d$ and

$$\int_0^\infty t^{-\kappa-1} (1 - \exp\{\Re \psi^\nu(\xi) t - at\}) dt = (a - \Re \psi^\nu(\xi))^\kappa \int_0^\infty t^{-\kappa-1} (1 - e^{-t}) dt,$$

then for all $\varphi \in S(\mathbb{R}^d)$,

$$(a - \Re \psi^\nu(\xi))^\kappa \mathcal{F} \varphi = C \int_0^\infty t^{-\kappa-1} (1 - \exp\{\Re \psi^\nu(\xi) t - at\}) \mathcal{F} \varphi dt,$$

(3.16)

$$= C \int_0^\infty t^{-\kappa-1} \mathbb{E} \left[ \varphi(x) - e^{-at} \varphi(x + Z_t^0) \right] dt,$$

where $C^{-1} = \int_0^\infty t^{-\kappa-1} (1 - e^{-t}) dt$. Note for $t \in (0, 1)$, we have

$$|\mathbb{E} \left[ \varphi(x) - e^{-at} \varphi(x + Z_t^0) \right]| \leq |1 - e^{-at}| \left| \varphi(x) \right| + \mathbb{E} \int_0^t \left| L^\varphi (x + Z_r^0) \right| dr.$$

By Lemma 7, Fubini’s theorem applies to (3.16), which implies

$$\int_0^\infty t^{-\kappa-1} \mathbb{E} \left[ \varphi(x) - e^{-at} \varphi(x + Z_t^0) \right] dt \in C^\infty_b(\mathbb{R}^d),$$

and

$$(a - \Re \psi^\nu(\xi))^\kappa \mathcal{F} \varphi = C \mathcal{F} \int_0^\infty t^{-\kappa-1} \mathbb{E} \left[ \varphi(x) - e^{-at} \varphi(x + Z_t^0) \right] dt \in S'(\mathbb{R}^d).$$

Thus (3.12) is well-defined. As a result,

$$\begin{align*}
(3.18) \quad (aI - L')^\kappa \varphi(x) &= C \int_0^\infty t^{-\kappa-1} \mathbb{E} \left[ \varphi(x) - e^{-at} \varphi(x + Z_t^0) \right] dt, \\
\int_0^\infty t^{\kappa-1} \exp\{\Re \psi^\nu(\xi) t - at\} dt &= (a - \Re \psi^\nu(\xi))^{-\kappa} \int_0^\infty t^{\kappa-1} e^{-t} dt,
\end{align*}$$

then for all $\varphi \in S(\mathbb{R}^d)$,

$$(a - \Re \psi^\nu(\xi))^{-\kappa} \mathcal{F} \varphi = C' \mathcal{F} \int_0^\infty t^{\kappa-1} e^{-at} \mathbb{E}\varphi(x + Z_t^0) dt \in S'(\mathbb{R}^d),$$

where $C'^{-1} = \int_0^\infty t^{\kappa-1} e^{-t} dt$. And

$$\begin{align*}
(3.19) \quad (aI - L^\nu)^{-\kappa} \varphi(x) &= C' \int_0^\infty t^{\kappa-1} e^{-at} \mathbb{E}\varphi(x + Z_t^0) dt.
\end{align*}$$
To extend (3.18), (3.19) to all \( \varphi \in C^\infty_b(\mathbb{R}^d) \), we repeat what we did in Lemma 9 and introduce \( \{ \zeta_n : n \in \mathbb{N} \} \subseteq C^\infty_0(\mathbb{R}^d) \) such that \( 0 \leq \zeta_n(x) \leq 1, \forall n \in \mathbb{N}, \forall x \in \mathbb{R}^d \) and \( \zeta_n(x) = 1, \forall x \in \{ x \in \mathbb{R}^d : |x| \leq n \} \). Then, \( \varphi \zeta_n \xrightarrow{n \to \infty} \varphi \) and

\[
(aI - L^\nu)^{-\kappa} \varphi \zeta_n \xrightarrow{n \to \infty} C' \int_0^\infty t^{\kappa-1} e^{-at} E \varphi(x + Z_t^\nu) dt,
\]

\[
(aI - L^\nu)^{\kappa} \varphi \zeta_n \xrightarrow{n \to \infty} C \int_0^\infty t^{-\kappa-1} [\varphi(x) - e^{-at} E \varphi(x + Z_t^\nu)] dt
\]

all in the topology of \( S'(\mathbb{R}^d) \). Applying continuity of the Fourier transform, we know that \( C^\infty_b(\mathbb{R}^d) \subseteq A(\mathbb{R}^d) \) and (3.14), (3.15) hold on it. \( \square \)

**Remark:** Lemmas 9 and 10 have shown that \( (aI - L^\nu)^{\kappa}, \kappa \in (0, 1] \) is a closed operation in \( C^\infty_b(\mathbb{R}^d) \). Naturally, we may define \( (aI - L^\nu)^{\kappa} \) for all \( \kappa \in (1, 2) \) and all \( \varphi \in C^\infty_b(\mathbb{R}^d) \) as follows:

\[
(aI - L^\nu)^{\kappa} \varphi = (aI - L^\nu)^{\kappa/2} \circ (aI - L^\nu)^{\kappa/2} \varphi,
\]

\[
(aI - L^\nu)^{-\kappa} \varphi = (aI - L^\nu)^{-\kappa/2} \circ (aI - L^\nu)^{-\kappa/2} \varphi.
\]

Here \( \circ \) represents composition of two operations. This definition is compatible with (3.12), (3.13) when \( \kappa \in (1, 2) \). The corollary below says that the probabilistic representation of \( (aI - L^\nu)^{-\kappa} \) for \( \kappa \in (0, 1) \) also applies to \( \kappa \in (1, 2) \).

**Corollary 2.** Let \( \kappa \in (0, 2), a \in (0, \infty) \) and \( \nu \) be a Lévy measure satisfying (iii) in \( A(w,l) \). Then \( (aI - L^\nu)^{\kappa} \) is a bijection on \( C^\infty_b(\mathbb{R}^d) \). For any function \( \varphi \in C^\infty_b(\mathbb{R}^d) \),

\[
(3.20) \quad (aI - L^\nu)^{-\kappa} \varphi(x) = C \int_0^\infty t^{\kappa-1} e^{-at} E \varphi(x + Z_t) dt,
\]

where \( C \) is a constant only depending on \( \kappa \), and \( Z_t = Z_t^\nu \) if \( \kappa = 1 \), \( Z_t = Z_t^0 \) otherwise.

**Proof.** That \( (aI - L^\nu)^{\kappa}, \kappa \in (0, 2) \) is a bijection follows from the definition. Suppose \( \kappa \in (1, 2) \) and \( \varphi \in C^\infty_b(\mathbb{R}^d) \). Use (3.15).

\[
(aI - L^\nu)^{-\kappa} \varphi(x) = C \int_0^\infty t^{\kappa/2-1} e^{-at} E (aI - L^\nu)^{-\kappa/2} \varphi(x + Z_t^\nu) dt
\]

\[
= C \int_0^\infty t^{\kappa/2-1} e^{-at} E \int_0^\infty s^{\kappa/2-1} e^{-as} E \varphi(x + Z_t^\nu + Z_s^0) ds dt.
\]
\( Z_t^\nu, Z_s^\nu \) denote two independent and identically distributed Lévy processes associated to \( \tilde{\nu} \). Therefore,

\[
(aI - L^\nu)^{-\kappa} \varphi(x) = C \int_0^\infty t^{\kappa/2-1} e^{-at} \int_0^\infty s^{\kappa/2-1} e^{-as} \mathbb{E} \varphi(x + Z_{t+s}^\nu) \, ds \, dt
\]

where \( p(t, z) \) is the probability density of \( Z_t^\nu \). Then by changing variables and applying Fubini theorem, we obtain

\[
(aI - L^\nu)^{-\kappa} \varphi(x) = C \int_0^\infty t^{\kappa-1} \int_0^\infty r^{\kappa/2-1} e^{-atr} \int \varphi(x + z) p(t + tr, z) \, dz \, dr \, dt.
\]

**Proposition 2.** Let \( \nu \) be a Lévy measure satisfying \( A(w, l), \beta \in (0, \infty) \), \( \kappa \in (0, 1] \). Then norm \( |u|_{\nu, \kappa, \beta} \) and norm \( \|u\|_{\nu, \kappa, \beta} \) are equivalent in \( C_0^\infty(\mathbb{R}^d) \).

**Proof.** For the purpose of clarity, we state our proof in parts.

**Part 1:** Show \( |u|_{\nu, \kappa, \beta} \leq C \|u\|_{\nu, \kappa, \beta} \) for all \( \kappa \in (0, 1] \).

By (3.10), (3.15) and (3.1), for all \( \kappa \in (0, 1] \),

\[
|\langle I - L^\nu \rangle^{-\kappa} u \rangle_0| \leq C |u|_0 \leq C |u|_{\beta, \infty}, \forall u \in C_0^\infty(\mathbb{R}^d).
\]

Since \( (I - L^\nu)^\kappa \) is a bijection, it implies

\[
|u|_0 \leq C |(I - L^\nu)^\kappa u|_{\beta, \infty}, \forall u \in C_0^\infty(\mathbb{R}^d).
\]

In the mean time, by (3.10), for all \( \kappa \in (0, 1] \), \( j \in \mathbb{N} \), \( u \in C_0^\infty(\mathbb{R}^d) \),

\[
|\langle (I - L^\nu)^{-\kappa} u \rangle \star \varphi_j \rangle_0| = \left| \int_0^\infty t^{\kappa-1} e^{-t} \mathbb{E} [u \star \varphi_j(x + Z_t^\nu)] \, dt \right| \leq C |u \star \varphi_j|_0,
\]

where \( Z_t = Z_t^\nu \) if \( \kappa = 1 \) and \( Z_t = Z_t^\nu \) otherwise. Given that \( (I - L^\nu)^\kappa \) is bijective, it leads to

\[
|u \star \varphi_j|_0 \leq C |\langle (I - L^\nu)^{-\kappa} u \rangle \star \varphi_j \rangle_0|, \forall j \in \mathbb{N},
\]

namely, for all \( \kappa \in (0, 1] \),

\[
|u|_{\beta, \infty} \leq C |(I - L^\nu)^\kappa u|_{\beta, \infty}, \forall u \in C_0^\infty(\mathbb{R}^d).
\]
Therefore,

\[(3.23) \quad |L^\nu u|_{\beta,\infty} \leq |(I - L^\nu) u|_{\beta,\infty} + |u|_{\beta,\infty} \leq C |(I - L^\nu) u|_{\beta,\infty}.\]

Similarly, for \( \kappa \in (0, 1) \), by \((3.6)\) and \((3.14)\),

\[
|L^{\nu,\kappa} u \ast \varphi_j|_0 \leq |(I - L^\nu)\kappa u \ast \varphi_j|_0 + C \left| \int_0^\infty t^{-\kappa-1} (1 - e^{-t}) E u \ast \varphi_j (x + Z^\nu_t) \, dt \right|_0 \\
\leq |(I - L^\nu)\kappa u \ast \varphi_j|_0 + C |u \ast \varphi_j|_0, \forall j \in \mathbb{N},
\]

which together with \((3.22)\) indicate

\[(3.24) \quad |L^{\nu,\kappa} u|_{\beta,\infty} \leq |(I - L^\nu)\kappa u|_{\beta,\infty} + C |u|_{\beta,\infty} \leq C |(I - L^\nu)\kappa u|_{\beta,\infty}.\]

Combine \((3.21), (3.23), (3.24)\). \(|u|_{\nu,\kappa,\beta} \leq C \|u\|_{\nu,\kappa,\beta}, \forall \kappa \in (0, 1)\).

**Part 2:** Show \(|u|_{\nu,\kappa,\beta} \leq C |u|_{\nu,\kappa,\beta}\) for all \( \kappa \in (0, 1) \).

By \((3.6)\) and \((3.14)\) again,

\[
|(I - L^\nu)\kappa u \ast \varphi_j|_0 \leq |L^{\nu,\kappa} u \ast \varphi_j|_0 + C \left| \int_0^\infty t^{-\kappa-1} (1 - e^{-t}) E u \ast \varphi_j (x + Z^\nu_t) \, dt \right|_0 \\
\leq |L^{\nu,\kappa} u \ast \varphi_j|_0 + C |u \ast \varphi_j|_0, \forall j \in \mathbb{N}.
\]

This is to say

\[(3.25) \quad |(I - L^\nu) u|_{\beta,\infty} \leq |L^{\nu,\kappa} u|_{\beta,\infty} + C |u|_{\beta,\infty}.\]

It then suffices to prove \(|u|_{\beta,\infty} \leq C \left( |u|_0 + |L^{\nu,\kappa} u|_{\beta,\infty} \right) \) for \( \kappa \in (0, 1) \).

Note,

\[(3.26) \quad |u \ast \varphi_0|_0 \leq |\varphi_0|_{L^1(\mathbb{R}^d)} |u|_0 \leq C |u|_0.\]

For \( j \neq 0 \),

\[
|u \ast \varphi_j|_0 = \left| \mathcal{F}^{-1} \left[ \left( -\Re \psi^\nu (\xi) \right)^{-\kappa} \hat{\varphi}_j (\xi) \hat{\varphi}_j (\xi) \left( -\Re \psi^\nu (\xi) \right)^{\kappa} \mathcal{F} u \right] \right|_0 \\
= \left| \left( \mathcal{F}^{-1} g_j \right) \ast (L^{\nu,\kappa} u \ast \varphi_j) \right|_0,
\]

where

\[
g_j (\xi) = - \left( -\Re \psi^\nu (\xi) \right)^{-\kappa} \mathcal{F} \varphi \left( N^{-j} \xi \right).
\]

We would like to show that \( |\mathcal{F}^{-1} g_j|_{L^1(\mathbb{R}^d)} < C \) for some \( C \) independent of \( j \). Because in that case,

\[
|u \ast \varphi_j|_0 \leq |\mathcal{F}^{-1} g_j|_{L^1(\mathbb{R}^d)} |L^{\nu,\kappa} u \ast \varphi_j|_0 \leq C |L^{\nu,\kappa} u \ast \varphi_j|_0, j \in \mathbb{N}_+,
\]
which together with (3.26) lead to $|u|_{\beta,\infty} \leq C \left(|u|_0 + |L^{\nu,\kappa} u|_{\beta,\infty}\right)$. Indeed,

$$\int |F^{-1}g_j(x)| \, dx \leq C \left| F^{-1} \int_0^\infty t^{\kappa-1} \exp\{\Re(\nu) \cdot t\} \mathcal{F} \tilde{\phi}(N^{-j} \xi) \, dt \right|_{L^1(\mathbb{R}^d)}$$

$$= C \left| F^{-1} \int_0^\infty t^{\kappa-1} \exp\{w(N^{-j})^{-1} \Re(\nu_{N^{-j}})(N^{-j} \xi)\} \mathcal{F} \tilde{\phi}(N^{-j} \xi) \, dt \right|_{L^1(\mathbb{R}^d)}$$

$$= C \left| F^{-1} \int_0^\infty t^{\kappa-1} \exp\{w(N^{-j})^{-1} \Re(\nu_{N^{-j}})(\xi)\} \mathcal{F} \tilde{\phi}(\xi) \, dt \right|_{L^1(\mathbb{R}^d)}. $$

Note $\Re(\nu_{N^{-j}})(\xi) = \nu_{N^{-j}}(\xi)$, where $\nu_{N^{-j}}(dy) = \tilde{\nu}_{N^{-j}}$. Then,

$$\int |F^{-1}g_j(x)| \, dx \leq C \left| F^{-1} \int_0^\infty t^{\kappa-1} \mathcal{F} \mathcal{E}(x + Z_{\tilde{\nu}_{N^{-j}}}^{-1}t) \, dt \right|_{L^1(\mathbb{R}^d)}. $$

Recall that $\text{supp} \mathcal{F} \tilde{\phi} = \{\xi : N^{-2} \leq |\xi| \leq N^2\}$. By Lemma 18 in Appendix, there are positive constants $C_1, C_2$ independent of $j$ ($j \neq 0$), such that

$$\int_{\mathbb{R}^d} \left| \mathcal{E}(x + Z_{\tilde{\nu}_{N^{-j}}}^{-1}t) \right| \, dx \leq C_1 e^{-C_2 t}. $$

Therefore,

$$\int |F^{-1}g_j(x)| \, dx \leq C \left| \int_0^\infty t^{\kappa-1} \mathcal{E}(x + Z_{\tilde{\nu}_{N^{-j}}}^{-1}t) \, dt \right|_{L^1(\mathbb{R}^d)}$$

$$\leq C \int_0^\infty t^{\kappa-1} \exp\{-C_2 t\} \, dt$$

$$\leq C w(N^{-j})^\kappa \leq C$$

for some $C$ independent of $j$.

**Part 3:** Show $\|u\|_{\nu,\kappa,\beta} \leq C \|u\|_{\nu,\kappa,\beta}$ for all $\kappa = 1$.

Since $|(I - L^\nu) u |_{\beta,\infty} \leq |L^\nu u|_{\beta,\infty} + |u|_{\beta,\infty}$. Similarly as Part 2, we just need to show $|u|_{\beta,\infty} \leq C \left(|u|_0 + |L^\nu u|_{\beta,\infty}\right)$. Again,

$$|u \ast \varphi_0|_0 \leq |\varphi_0|_{L^1(\mathbb{R}^d)} |u|_0 \leq C |u|_0. $$

And for $j \neq 0$,

$$|u \ast \varphi_j|_0 = \left| F^{-1} \left[ \left(\psi^\nu(\xi)\right)^{-1} \tilde{\phi}_j(\xi) \hat{\varphi}_j(\xi) \psi^\nu(\xi) \mathcal{F} u \right] \right|_0$$

$$= \left| \left(F^{-1}g_j \ast (L^\nu u \ast \varphi_j)\right|_0$$

$$\leq \left| F^{-1}g_j \right|_{L^1(\mathbb{R}^d)} |L^\nu u \ast \varphi_j|_0, $$

where

$$g_j(\xi) = \left(\psi^\nu(\xi)\right)^{-1} \mathcal{F} \tilde{\phi}(N^{-j} \xi).$$
By Lemma [16] in Appendix, for each \( j \in \mathbb{N}_+ \), \( \Re \psi^{\nu} < 0 \), thus \( g_j(\xi) \) is well-defined. The rest of this proof is devoted to looking for an upper bound of \( \| F^{-1}g_j \|_{L^1(\mathbb{R}^d)} \) that is uniform with respect to \( j \). Analogously as before, applying Lemma [18] in Appendix,

\[
\int |F^{-1}g_j(x)| \, dx = \left| F^{-1} \left[ \int_0^\infty \exp\{\psi^{\nu}(\xi) t\} F\tilde{\varphi}(N^{-j} \xi) \, dt \right] \right|_{L^1(\mathbb{R}^d)} \\
\leq \int_{\mathbb{R}^d} \int_0^\infty \left| E\tilde{\varphi} \left( x + Z^\beta u(N^{-j})^{-1} t \right) \right| \, dt \, dx \\
\leq C w(N^{-j}) \leq C.
\]

This concludes the proof.

\[\square\]

**Proposition 3.** Let \( \nu \) be a Lévy measure satisfying \( A(w, u), \beta \in (0, \infty), \kappa \in (0, 1) \). Then norm \( |u|_{\nu, \kappa, \beta} \) and norm \( |u|_{\nu, \kappa, \beta} \) are equivalent in \( C_0^\infty(\mathbb{R}^d) \).

**Proof.** We first assume the finiteness of \( |u|_{\nu, \kappa, \beta} \). It was showed in Lemma [3] that \( |u|_0 \leq C |u|_{\nu, \kappa, \beta} \). To prove \( |L^{\nu, \kappa} u|_{\beta, \infty} \leq C |u|_{\nu, \kappa, \beta} \) for some \( C > 0 \), it suffices to show for each \( j \in \mathbb{N} \),

\[
|(L^{\nu, \kappa} u) * \varphi_j|_0 \leq C w(N^{-j})^{-\kappa} |u * \varphi_j|_0, \quad \kappa \in (0, 1].
\]

In fact, by Corollary [1]

\[
|(L^{\nu, \kappa} u) * \varphi_j|_0 = |L^{\nu, \kappa} (u * \varphi_j + \tilde{\varphi}_j)|_0 = |(L^{\nu, \kappa} \tilde{\varphi}_j) * (u * \varphi_j)|_0 \\
\leq |L^{\nu, \kappa} \tilde{\varphi}_j|_{L^1(\mathbb{R}^d)} |u * \varphi_j|_0 \leq C w(N^{-j})^{-\kappa} |u * \varphi_j|_0.
\]

This is to say for all \( \kappa \in (0, 1] \),

\[
(3.27) \quad |u|_{\nu, \kappa, \beta} = |u|_0 + |L^{\nu, \kappa} u|_{\beta, \infty} < C |u|_{\nu, \kappa, \beta}.
\]

To show \( |u|_{\nu, \kappa, \beta} < C |u|_{\nu, \kappa, \beta} \), according to Proposition [2], we just need to show \( |u|_{\nu, \kappa, \beta} < C |u|_{\nu, \kappa, \beta} \). By (3.10),

\[
|(I - L^{\nu})^{-1} (u * \varphi_j)|_0 = \int_0^\infty e^{-t} F^{-1} \left[ \exp\{\psi^{\nu}(\xi) t\} \tilde{\varphi}_j(\xi) \varphi_j(\xi) \hat{u}(\xi) \right] \, dt_0, \forall j \in \mathbb{N}.
\]

By (3.15), for all \( \kappa \in (0, 1) \),

\[
|(I - L^{\nu})^{-\kappa} (u * \varphi_j)|_0 = C \int_0^\infty t^{\kappa - 1} e^{-t} F^{-1} \left[ \exp\{\psi^{\nu}(\xi) t\} \tilde{\varphi}_j(\xi) \varphi_j(\xi) \hat{u}(\xi) \right] \, dt_0, \forall j \in \mathbb{N}.
\]

First we consider \( j = 0 \). Set \( Z_t = Z^{\nu}_t \) if \( \kappa = 1 \) and \( Z_t = Z^{\nu}_0 \) otherwise. For all \( \kappa \in (0, 1] \),

\[
|(I - L^{\nu})^{-\kappa} (u * \varphi_0)|_0 \leq |u * \varphi_0|_0 \int_0^\infty t^{\kappa - 1} e^{-t} \| E\varphi_0 \cdot (\cdot + Z_t) \|_{L^1(\mathbb{R}^d)} \, dt \leq C |u * \varphi_0|_0.
\]
For $j \neq 0$, use (3.8).

\[
\left| (I - L_f)^{-1} (u * \varphi_j) \right|_0 \\
= \left| \int_0^\infty e^{-t} \mathcal{F}^{-1} \left[ \exp \{ w \left( N^{-j} \psi_{N^{-j}} (N^{-j} \xi) \right) \} \mathcal{F} \varphi \left( (N^{-j} \xi) \right) \} \right] (u * \varphi_j) dt \right|_0 \\
\leq |u * \varphi_j|_0 \left| \int_0^\infty e^{-t} \left[ \exp \{ \psi_{N^{-j}} (\xi) w \left( (N^{-j})^{-1} t \right) \mathcal{F} \varphi \left( \xi \right) \} \right] \right|_{L^1(\mathbb{R}^d)} dt \\
\leq |u * \varphi_j|_0 \left| \int_0^\infty e^{-t} \mathcal{E} \varphi \left( \cdot + Z_{w(N^{-j})^{-1}t} \right) \right|_{L^1(\mathbb{R}^d)} dt,
\]

which, by Lemma 18 in Appendix, leads to

\[
(3.29) \quad \left| (I - L_f)^{-1} (u * \varphi_j) \right|_0 \\
\leq C |u * \varphi_j|_0 \int_0^\infty e^{-C_2 w(N^{-j})^{-1}t} dt \leq C w \left( N^{-j} \right) |u * \varphi_j|_0.
\]

Similarly, for $\kappa \in (0, 1)$,

\[
\left| (I - L_f)^{-\kappa} (u * \varphi_j) \right|_0 \\
\leq C |u * \varphi_j|_0 \int_0^\infty t^{-\kappa - 1} e^{-t} \left| \mathcal{E} \varphi \left( \cdot + Z_{w(N^{-j})^{-1}t} \right) \right|_{L^1(\mathbb{R}^d)} dt \\
(3.30) \quad \leq C w \left( N^{-j} \right) ^\kappa |u * \varphi_j|_0.
\]

Combine (3.28) - (3.30).

\[
\left| (I - L_f)^{-\kappa} u \right|_{\kappa + \beta, \infty} \leq C |u|_{\beta, \infty}, \forall \kappa \in (0, 1), \forall u \in C_b^\infty \left( \mathbb{R}^d \right).
\]

By Lemmas 9 and 10 that means

\[
|u|_{\kappa + \beta, \infty} \leq C \left| (I - L_f)^{\kappa} u \right|_{\beta, \infty}, \forall u \in C_b^\infty \left( \mathbb{R}^d \right).
\]

Therefore, $|u|_{\kappa + \beta, \infty} \leq C \left\| u \right\|_{\nu, \kappa, \beta} < C |u|_{\nu, \kappa, \beta}$.

\[\square\]

**Corollary 3.** Let $\nu$ be a Lévy measure satisfying $A(w, l)$, $\beta \in (0, \infty)$, $\kappa \in (0, 1)$. Then norm $\left\| u \right\|_{\nu, \kappa, \beta}$ and norm $\left| u \right|_{\kappa + \beta, \infty}$ are equivalent in $C_b^\infty \left( \mathbb{R}^d \right)$.

**Proof.** This is a consequence of Propositions 2 and 3. \[\square\]

**Corollary 4.** Let $\nu$ be a Lévy measure satisfying $A(w, l)$, $\kappa \in (0, 2)$ and $\beta \in (0, \infty)$, $u \in C_b^\infty \left( \mathbb{R}^d \right) \cap C_b^{\kappa + \beta} \left( \mathbb{R}^d \right)$. Then there exists a constant $C > 0$ independent of $j$ such that

\[
\left| L_f^{\nu, \kappa} u \right|_0 \leq \left| L_f^{\nu, \kappa} u \right|_{\beta, \infty} \leq C \left| u \right|_{\beta + \kappa, \infty}.
\]

**Proof.** By Proposition 3 if $\kappa \in (0, 1)$,

\[
\left| L_f^{\nu, \kappa} u \right|_0 \leq \left| L_f^{\nu, \kappa} u \right|_{\beta, \infty} \leq C \left| u \right|_{\kappa + \beta, \infty}.
\]
Now suppose $\kappa \in (1,2)$. $L^{\nu,\kappa}u := L^{\nu,\kappa/2} \circ L^{\nu,\kappa/2}u$. Then by Corollary 1

$$|L^{\nu,\kappa}u \ast \varphi_j|_0 = |u \ast \varphi_j \ast L^{\nu,\kappa/2}\tilde{\varphi}_j \ast L^{\nu,\kappa/2}\tilde{\varphi}_j|_0$$

$$\leq |u \ast \varphi_j|_0 \left( L^{\nu,\kappa/2}\tilde{\varphi}_j \right)_{L^1(R^d)} \left( L^{\nu,\kappa/2}\tilde{\varphi}_j \right)_{L^1(R^d)}$$

$$\leq Cw(N^{-j})^{-\kappa} |u \ast \varphi_j|_0, \forall j \in \mathbb{N}.$$  

Therefore, $L^{\nu,\kappa}u \in \tilde{C}^{\beta}_{\infty,\infty}(R^d)$ and $|L^{\nu,\kappa}u|_0 \leq |L^{\nu,\kappa}u|_{\beta,\infty} \leq C|u|_{\kappa+\beta,\infty}$. □

**Proposition 4.** Let $0 < \beta' < \beta$. Then for any $\varepsilon \in (0,1)$ and any bounded function $u$ in $R^d$,

$$|u|_{\beta',\infty} \leq \varepsilon |u|_{\beta,\infty} + C_{\varepsilon} |u|_0,$$

where $C_{\varepsilon}$ is independent of $u$.

**Proof.** It is sufficient to show that for $\forall j \in \mathbb{N},$

$$w(N^{-j})^{-\beta'} |u \ast \varphi_j|_0 \leq \varepsilon |u|_{\beta,\infty} + C_{\varepsilon} |u|_0.$$  

Apply Young’s inequality. For any $\varepsilon \in (0,1)$ and any pair of $p,q$ such that $\frac{1}{p} + \frac{1}{q} = 1$,

$$|u \ast \varphi_j|_0 = \left( \epsilon w(N^{-j})^{\frac{\beta' - \beta}{p}} |u \ast \varphi_j|_0^{rac{1}{p}} \right) \left( \epsilon^{-1} w(N^{-j})^{\frac{\beta - \beta'}{q}} |u \ast \varphi_j|_0^{rac{1}{q}} \right)$$

$$\leq \frac{e^p w(N^{-j})^{\beta' - \beta}}{p} |u \ast \varphi_j|_0 + \frac{w(N^{-j})^{\frac{(\beta - \beta')q}{p}}}{qe^q} |u \ast \varphi_j|_0 \forall j \in \mathbb{N},$$

thus,

$$w(N^{-j})^{-\beta'} |u \ast \varphi_j|_0$$

$$\leq \frac{e^p w(N^{-j})^{-\beta}}{p} |u \ast \varphi_j|_0 + \frac{w(N^{-j})^{\frac{(\beta - \beta')q}{p}}}{qe^q} |u \ast \varphi_j|_0$$

$$\leq \frac{e^p}{p} |u|_{\beta,\infty} + \frac{1}{qe^q} w(N^{-j})^{\frac{(\beta - \beta')q}{p}} |u \ast \varphi_j|_0 \forall j \in \mathbb{N}.$$  

Choose $p,q$ such that $\frac{(\beta - \beta')q}{p} - \beta' \geq 0$, then for some $C > 0$,

$$\frac{1}{qe^q} w(N^{-j})^{\frac{(\beta - \beta')q}{p}} |u \ast \varphi_j|_0 \leq \frac{C}{qe^q} |u \ast \varphi_j|_0 \leq \frac{C}{qe^q} |u|_0 \forall j \in \mathbb{N}.$$  

Take $\varepsilon$ such that $\frac{e^p}{p} = \varepsilon$ and this is the end the proof. □

**Proposition 5.** Let $\beta \in (0,\infty)$, $u \in \tilde{C}^{\beta}_{\infty,\infty}(R^d)$. Then there exists a sequence $u_n \in C_0^\infty(R^d)$ such that

$$|u|_{\beta,\infty} \leq \liminf_n |u_n|_{\beta,\infty}, \quad |u_n|_{\beta,\infty} \leq C |u|_{\beta,\infty}$$
for some $C > 0$ that only depends on $d, N$, and for any $0 < \beta' < \beta$,

$$|u_n - u|_{\beta', \infty} \to 0 \text{ as } n \to \infty.$$  

Proof. Set $u_n(x) = \sum_{j=0}^{n+2} (u * \varphi_j)(x), n \in \mathbb{N}$. Then

$$|u_n|_{\beta, \infty} = \sup_j \left| \sum_{k=0}^{n+2} u \ast \varphi_k \ast \varphi_j \right|_0 w(N^{-j})^{-\beta}.$$  

By construction of $\varphi_j, j \in \mathbb{N}$ in this note, if $j \geq 1, n \geq j - 1$,

$$\left| \sum_{k=0}^{n+2} u \ast \varphi_k \ast \varphi_j \right|_0 \leq \left| u \ast \varphi_j \ast \varphi_{j-1} \right|_0 + \left| u \ast \varphi_j \ast \varphi_j \right|_0 = 2 |F^{-1}\phi|_{L^1(\mathbb{R}^d)} |u \ast \varphi_j|_0.$$  

Besides,

$$\left| \sum_{k=0}^{n+2} u \ast \varphi_k \ast \varphi_0 \right|_0 \leq \left| u \ast \varphi_0 \ast \varphi_0 \right|_0 + \left| u \ast \varphi_0 \ast \varphi_1 \right|_0 = \left( |\varphi_0|_{L^1(\mathbb{R}^d)} + |\varphi_1|_{L^1(\mathbb{R}^d)} \right) |u \ast \varphi_j|_0.$$  

Therefore, for all $n \in \mathbb{N}$,

$$|u_n|_{\beta, \infty} \leq C \sup_j |u \ast \varphi_j|_0 w(N^{-j})^{-\beta} \leq C |u|_{\beta, \infty}.$$  

On the other hand, by Lemma 6, $u(x) = \sum_{k=0}^{\infty} (u \ast \varphi_k)(x)$. Then in the same vein as above,

$$|u \ast \varphi_j|_0 = \left| \sum_{k=0}^{n+2} u \ast \varphi_k \ast \varphi_j + \sum_{k=n+3}^{\infty} u \ast \varphi_k \ast \varphi_j \right|_0 \leq |u_n \ast \varphi_j|_0, \quad \forall n \geq j - 1, \forall j \in \mathbb{N},$$  

thus,

$$|u \ast \varphi_j|_0 w(N^{-j})^{-\beta} \leq |u_n|_{\beta, \infty}, \quad \forall n \geq j - 1, \forall j \in \mathbb{N},$$  

and thus $|u|_{\beta, \infty} \leq \lim \inf_n |u_n|_{\beta, \infty}$. 

At last,

\[ |u - u_n|_{\beta', \infty} = \sup_j \left| \sum_{k=n+3}^{\infty} u * \varphi_k * \varphi_j \right| w(N^{-j})^{-\beta'} \]

\[ = \sup_{j \geq n+2} \left| \sum_{k=n+3}^{\infty} u * \varphi_k * \varphi_j \right| w(N^{-j})^{-\beta} w(N^{-j})^{-\beta'} \]

\[ \leq C \sup_{j \geq n+2} |u * \varphi_j|_0 w(N^{-j})^{-\beta} w(N^{-j})^{-\beta'} \]

\[ \leq C |u|_{\beta, \infty} l(N^{-n})^{-\beta'} \rightarrow 0 \text{ as } n \rightarrow \infty. \]

Using the approximating sequence introduced in the lemma above, we can extend \( L^{\nu, \kappa}u, \kappa \in (0, 2) \) to all \( u \in \tilde{C}^{\kappa+\beta}_0(R^d) \), \( \beta > 0 \) as follows:

\[ L^{\nu, \kappa}u(x) = \lim_{n \rightarrow \infty} L^{\nu, \kappa}u_n(x), x \in R^d. \]

The next proposition justifies this definition and addresses continuity of the operator defined in this sense.

**Proposition 6.** Let \( \nu \) be a Lévy measure satisfying \( A(w,l) \), \( \beta \in (0, \infty) \) and \( \kappa \in (0, 2) \). Then \( L^{\nu, \kappa}u(x) \) is well-defined for all \( \kappa \) and all \( u \in \tilde{C}^{\kappa+\beta}_0(R^d) \),

\[ L^{\nu, \kappa}u(x) = \lim_{n \rightarrow \infty} L^{\nu, \kappa}u_n(x), x \in R^d, \]

and this convergence is uniform with respect to \( x \). Moreover,

\[ |L^{\nu, \kappa}u|_0 \leq |L^{\nu, \kappa}u|_{\beta, \infty} \leq C |u|_{\kappa+\beta, \infty} \]

for some \( C > 0 \) independent of \( u \).

**Proof.** Since \( u \in \tilde{C}^{\kappa+\beta}_0(R^d) \), by Proposition 3 there is a a sequence \( u_n \in C^\infty_b(R^d) \) such that

\[ |u|_{\kappa+\beta, \infty} \leq \liminf_n |u_n|_{\kappa+\beta, \infty}, \quad |u_n|_{\kappa+\beta, \infty} \leq C |u|_{\kappa+\beta, \infty} \]

for some \( C > 0 \) independent of \( u \), and for any \( 0 < \beta' < \beta \),

\[ |u_n - u|_{\kappa+\beta', \infty} \rightarrow 0 \text{ as } n \rightarrow \infty, \]

which, according to Lemma 6 and (3.1), indicates \( u \in C(R^d) \). Meanwhile, \( |u_n - u|_0 \rightarrow 0 \text{ as } n \rightarrow \infty \) and thus \( u_n \xrightarrow{n \rightarrow \infty} u \) in the weak topology of \( S'(R^d) \). For such a sequence, by Corollary 4

\[ |L^{\nu, \kappa}u_n|_0 \leq |L^{\nu, \kappa}u_n|_{\beta, \infty} \leq C |u_n|_{\beta+\kappa, \infty}, \]

\[ |L^{\nu, \kappa}u_n - L^{\nu, \kappa}u_m|_0 \leq C |u_n - u_m|_{\beta'+\kappa, \infty} \xrightarrow{n,m \rightarrow \infty} 0. \]
Therefore, both $L^{\nu,\kappa}u_n, \forall n \in \mathbb{N}$ and $\lim_{n \to \infty} L^{\nu,\kappa}u_n$ are continuous functions, and therefore,

$$F \left[ \lim_{n \to \infty} L^{\nu}u_n \right] = \lim_{n \to \infty} \psi^{\nu}Fv_n = \psi^{\nu}Fv \in S'(\mathbb{R}^d),$$

$$F \left[ \lim_{n \to \infty} L^{\nu,\kappa}u_n \right] = \lim_{n \to \infty} (-R\psi^{\nu})^k Fv_n = -(-R\psi^{\nu})^k Fv \in S'(\mathbb{R}^d), \kappa \neq 1.$$  

Namely,

$$L^{\nu,\kappa}u(x) = \lim_{n \to \infty} L^{\nu,\kappa}u_n(x), x \in \mathbb{R}^d.$$  

Clearly, this convergence is uniform over $x$. Now given any $\beta \in (0, \infty)$,

$$w(N^{-j})^{-\beta} |L^{\nu,\kappa}u_n \ast \varphi_j|_0 = \lim_{n \to \infty} w(N^{-j})^{-\beta} |L^{\nu,\kappa}u_n \ast \varphi_j|_0$$

$$\leq \limsup_{n \to \infty} |u_n|_{\beta+\kappa,\infty} \leq C |u|_{\beta+\kappa,\infty}, \forall \beta \in \mathbb{N}.$$  

Namely, $|L^{\nu,\kappa}u|_{\beta,\infty} \leq C |u|_{\beta+\kappa,\infty}$. \hfill \qed

**Theorem 3.1.** Let $\nu$ be a Lévy measure satisfying $A(w,l)$, $\beta \in (0, \infty)$, $\kappa \in (0, 1)$. Then norm $|u|_{\nu,\kappa,\beta}$ and norm $|u|_{\nu,\kappa,\beta}^\circ$ are equivalent.

**Proof.** As a consequence of (3.31) and Proposition 3 there exists a positive constant $C$ independent of $u$ such that

$$|u|_0 + |L^{\nu,\kappa}u|_{\beta,\infty} \leq C |u|_{\kappa+\beta,\infty}.$$  

Now suppose $|u|_0 + |L^{\nu,\kappa}u|_{\beta,\infty} < \infty$. First, $u$ is a bounded function. Then $u_n = \sum_{i=0}^{n+2} u \ast \varphi_i \in C^\infty_b \left( \mathbb{R}^d \right), \forall n \in \mathbb{N}$. Meanwhile, recall that

$$(L^{\nu,\kappa}u)_n := \sum_{i=0}^{n+2} (L^{\nu,\kappa}u) \ast \varphi_i = L^{\nu,\kappa}u_n \in \mathcal{C}^{\kappa+\beta}_{\infty,\infty} \left( \mathbb{R}^d \right)$$

approximates $L^{\nu,\kappa}u$ and $|L^{\nu,\kappa}u_n|_{\beta,\infty} \leq C |L^{\nu,\kappa}u|_{\beta,\infty}$. Therefore, by Proposition 3

$$|u_n|_{\kappa+\beta,\infty} \leq C \left( |u_n|_0 + |L^{\nu,\kappa}u_n|_{\beta,\infty} \right) \leq C |L^{\nu,\kappa}u_n|_{\beta,\infty} \leq C |L^{\nu,\kappa}u|_{\beta,\infty}.$$  

That is to say, for any $j \in \mathbb{N}$,

$$w(N^{-j})^{-\beta} |u_n \ast \varphi_j|_0 \leq C |L^{\nu,\kappa}u|_{\beta,\infty}.$$  

It suffices to observe that for $j \geq 2, n \geq j - 1$,

$$|u_n \ast \varphi_j|_0 = |u \ast \tilde{\varphi}_j|_0 \leq C w(N^{-j})^{\kappa+\beta} |L^{\nu,\kappa}u|_{\beta,\infty},$$  

and $|u_n \ast \varphi_j|_0 = |u \ast \varphi_j|_0 \leq C |u|_0, j = 0$ or $1, n \geq j - 1$. Therefore, $|u|_{\kappa+\beta,\infty} \leq C \left( |u|_0 + |L^{\nu,\kappa}u|_{\beta,\infty} \right)$. \hfill \qed

**Theorem 3.2.** Let $\nu$ be a Lévy measure satisfying $A(w,l)$, $\beta \in (0, \infty)$, $\kappa \in (0, 1)$. Then norm $|u|_{\nu,\kappa,\beta}$ and norm $|u|_{\kappa+\beta,\infty}$ are equivalent.
Proof. $\kappa = 1$ has been covered by Theorem 3.1 and Proposition 4. Let us consider $\kappa \in (0,1)$.

First assume the finiteness of $|u|_{\kappa+\beta,\infty}$. Then by Lemma 8, $u$ is a bounded and continuous function. Set $u_n = \sum_{i=0}^{n+2} u \ast \varphi_i \in C_b^\infty(\mathbb{R}^d)$, $n \in \mathbb{N}$. We have known that $|u_n - u|_0 \leq C |u_n - u|_{\kappa+\beta',\infty}$ \(\rightarrow\infty\) 0, $\forall \beta' \in (0,\beta)$. Hence,

\[
\mathcal{F}\left[\lim_{n \to \infty} (I - L^\nu)^\kappa u_n\right] = \lim_{n \to \infty} (1 - \Re\psi^\mu)^\kappa \mathcal{F}u_n = (1 - \Re\psi^\mu)^\kappa \mathcal{F}u \in \mathcal{S}'(\mathbb{R}^d),
\]

namely, $(I - L^\nu)^\kappa u$ is well-defined and $(I - L^\nu)^\kappa u = \lim_{n \to \infty} (I - L^\nu)^\kappa u_n$. By Lemma 8 and Corollary 3,

\[
\lim_{n,m \to \infty} |(I - L^\nu)^\kappa u_n - (I - L^\nu)^\kappa u_m|_0 \leq C \lim_{n,m \to \infty} |(I - L^\nu)^\kappa u_n - (I - L^\nu)^\kappa u_m|_{\beta'} \leq C \lim_{n \to \infty} |u_n - u_m|_{\kappa+\beta'} = 0.
\]

Then the convergence is uniform on $\mathbb{R}^d$. Hence, for any $j \in \mathbb{N}$,

\[
w(N^{-j})^{-\beta} |(I - L^\nu)^\kappa u \ast \varphi_j|_0 = w(N^{-j})^{-\beta} \lim_{n \to \infty} |(I - L^\nu)^\kappa u_n \ast \varphi_j|_0 \leq C \lim_{n \to \infty} |u_n|_{\kappa+\beta} \leq C |u|_{\kappa+\beta},
\]

i.e. $\|u\|_{\nu,\kappa,\beta} \leq C |u|_{\kappa+\beta}$.

If $\|u\|_{\nu,\kappa,\beta}$ is finite, then the approximating functions of $(I - L^\nu)^\kappa u$

\[
((I - L^\nu)^\kappa u)_n = (I - L^\nu)^\kappa u_n \in C_b^\infty(\mathbb{R}^d) \cap \bar{C}_{\kappa,\infty}^\beta(\mathbb{R}^d).
\]

Because $(I - L^\nu)^\kappa$ is a bijection on $C_b^\infty(\mathbb{R}^d)$, $u_n \in C_b^\infty(\mathbb{R}^d)$. Then Corollary 3 implies immediately

\[
|u_n|_0 \leq C |u_n|_{\kappa+\beta,\infty} \leq C |(I - L^\nu)^\kappa u_n|_{\beta,\infty} \leq C |(I - L^\nu)^\kappa u|_{\beta,\infty}.
\]

For $j \geq 2, n \geq j - 1$,

\[
|u_n \ast \varphi_j|_0 = |u \ast \varphi_j \ast \varphi_j|_0 = |u \ast \varphi_j|_0 \leq C w(N^{-j})^{\kappa+\beta} |(I - L^\nu)^\kappa u|_{\beta,\infty},
\]

and for $j = 0$ or 1, $n \geq j - 1$,

\[
|u_n \ast \varphi_j|_0 = |u \ast \varphi_j|_0 \leq C \sup_n |u_n|_0 \leq C |(I - L^\nu)^\kappa u|_{\beta,\infty}.
\]

Therefore, $|u|_{\kappa+\beta,\infty} \leq C |(I - L^\nu)^\kappa u|_{\beta,\infty}$. \(\square\)

**Theorem 3.3.** Let $\nu$ be a Lévy measure satisfying $A(\nu,\mu)$, $\beta \in (0,\infty)$, $\kappa \in (0,1)$. Then norm $\|u\|_{\nu,\kappa,\beta}$ and norm $|u|_{\nu,\kappa,\beta}$ are equivalent.

**Proof.** This is an immediate consequence of Theorems 3.1 and 3.2. \(\square\)
4. Solution Estimates for Smooth Inputs

4.1. Existence and Uniqueness.

**Theorem 4.1.** Let \( \nu \) be a Lévy measure, \( \alpha \in (0, 2), \beta \in (0, 1), \lambda \geq 0 \). Assume that \( f (t, x) \in C^\infty_b (H_T) \cap \tilde{C}^{\beta}_{\infty, \infty} (H_T) \). Then there is a unique solution \( u \in (t, x) \in C^\infty_b (H_T) \) to

\[
\partial_t u (t, x) = L^\nu u (t, x) - \lambda u (t, x) + f (t, x),
\]

\[
u = 0, \quad (t, x) \in [0, T] \times \mathbb{R}^d.
\]

**Proof.** EXISTENCE. Denote \( F (r, Z^\nu_r) = e^{-\lambda (r-s)} f (s, x + Z^\nu_r - Z^\nu_s), s \leq r \leq t \), and apply the Itô formula to \( F (r, Z^\nu_r) \) on \([s, t] \).

\[
e^{-\lambda (t-s)} f (s, x + Z^\nu_t - Z^\nu_s) - f (s, x)
\]

\[
= -\lambda \int_s^t F (r, Z^\nu_r) \, dr + \int_s^t \int \chi_\alpha (y) \, y \cdot \nabla F (r, Z^\nu_r) \, J (dr, dy)
\]

\[
+ \int_s^t \left[ F (r, Z^\nu_r + y) - F (r, Z^\nu_r) - \chi_\alpha (y) \, y \cdot \nabla F (r, Z^\nu_r) \right] J (dr, dy).
\]

Take expectation for both sides and use the stochastic Fubini theorem,

\[
e^{-\lambda (t-s)} \mathbb{E} f (s, x + Z^\nu_t - Z^\nu_s) - f (s, x)
\]

\[
= -\lambda \int_s^t e^{-\lambda (r-s)} \mathbb{E} f (s, x + Z^\nu_r - Z^\nu_s) \, dr + \int_s^t L^\nu e^{-\lambda (r-s)} \mathbb{E} f (s, x + Z^\nu_r - Z^\nu_s) \, dr.
\]

Integrate both sides over \([0, t] \) with respect to \( s \) and obtain

\[
\int_0^t e^{-\lambda (t-s)} \mathbb{E} f (s, x + Z^\nu_t - Z^\nu_s) \, ds - \int_0^t f (s, x) \, ds
\]

\[
= -\lambda \int_0^t \int_0^r e^{-\lambda (r-s)} \mathbb{E} f (s, x + Z^\nu_r - Z^\nu_s) \, ds \, dr
\]

\[
+ \int_0^t L^\nu \int_0^r e^{-\lambda (r-s)} \mathbb{E} f (s, x + Z^\nu_r - Z^\nu_s) \, ds \, dr,
\]

which shows \( u (t, x) = \int_0^t e^{-\lambda (t-s)} \mathbb{E} f (s, x + Z^\nu_{t-s}) \, ds \) solves \((4.1)\) in the integral sense. Obviously, as a result of the dominated convergence theorem and Fubini’s theorem, \( u \in C^\infty_b (H_T) \). And by the equation, \( u \) is continuously differentiable in \( t \).

UNIQUENESS. Suppose there are two solutions \( u_1, u_2 \) solving the equation, then \( u := u_1 - u_2 \) solves

\[
\partial_t u (t, x) = L^\nu u (t, x) - \lambda u (t, x),
\]

\[
u (0, x) = 0.
\]
Fix any \( t \in [0, T] \). Apply the Itô formula to \( v(t - s, Z_{t-s}^\nu) := e^{-\lambda s} u(t - s, x + Z_{t-s}^\nu) \), \( 0 \leq s \leq t \), over \([0, t]\) and take expectation for both sides of the resulting identity, then
\[
u(t, x) = -\mathbb{E} \int_0^t e^{-\lambda s} \left[ (-\partial_t u - \lambda u + L^\nu u) \circ (t - s, x + Z_{t-s}^\nu) \right] ds = 0.
\]


\[\tag{4.3}\]

\[\tag{4.4}\]

\[\tag{4.5}\]

4.2. H"older-Zygmund Estimates of the Solution. Since \( f(t, x) \in C^\infty_t (H_T) \cap C^\beta(\infty_\infty (H_T)) \), by Lemma 4
\[
f(t, x) = (f(t, \cdot) * \varphi_0 (\cdot)) (x) + \sum_{j=1}^\infty (f(t, \cdot) * \varphi_j (\cdot)) (x) := f_0 (t, x) + \sum_{j=1}^\infty f_j (t, x).
\]

Accordingly, \( u_j (t, x) = u(t, x) * \varphi_j (x) = \int_0^t e^{-\lambda(t-s)} \mathbb{E} f_j (s, x + Z_{t-s}^\nu) ds, j = 0, 1, \ldots \) is the solution to (4.1) with input \( f_j = f * \varphi_j \). Then by Lemmas 8 and 7 for \( \kappa \in (0, 1] \),
\[
L^{\mu, \kappa} u_j (t, x) = u_j * L^{\mu, \kappa} \hat{\varphi}_j
= \int_0^t e^{-\lambda(t-s)} \mathbb{E} \int f_j (s, x - z + Z_{t-s}^\nu) L^{\mu, \kappa} \hat{\varphi}_j (z) dz ds
= \int_0^t e^{-\lambda(t-s)} \int f_j (s, z) \mathbb{E} L^{\mu, \kappa} \hat{\varphi}_j (x - z + Z_{t-s}^\nu) dz ds,
\]
and then
\[
L^{\mu, \kappa} u_j (t, x) = \int_0^t \int \mathcal{F}^{-1} \left[ e^{(\psi^\nu (\xi) - \lambda)(t-s)} \mathbb{E} L^{\mu, \kappa} \hat{\varphi}_j \right] (x - z) f_j (s, z) dz ds
= C \int_0^t \int \mathcal{F}^{-1} \left[ \tilde{F}_t^{j, \kappa} (\xi) \right] (z) f_j (s, x - z) dz ds,
\]
where for \( j \in \mathbb{N} \),
\[
\tilde{F}_t^{j, \kappa} (\xi) := \left\{ \begin{array} {ll}
- e^{(\psi^\nu (\xi) - \lambda)t} (-\Re \psi^{\mu^j (\xi)})^\kappa \hat{\varphi}_j (\xi), & \xi \in \mathbb{R}^d, \kappa \in (0, 1), \\
e^{(\psi^\nu (\xi) - \lambda)t} \psi^{\mu^j (\xi)} \hat{\varphi}_j (\xi), & \xi \in \mathbb{R}^d, \kappa = 1, \\
e^{(\psi^\nu (\xi) - \lambda)t} \hat{\varphi}_j (\xi), & \xi \in \mathbb{R}^d, \kappa = 0.
\end{array} \right.
\]

In particular, when \( j \in \mathbb{N}_+ \),
\[
\mathcal{F}^{-1} \left[ \tilde{F}_t^{j, \kappa} (\xi) \right] (x) = e^{-\lambda t} w \left( N^{-j} \right)^{-\kappa} N^{jd} H_{w(N^{-j} - 1, t)} (N^j x),
\]
where for \( j \in \mathbb{N}_+ \),
\[
H_{t}^{j, \kappa} := \left\{ \begin{array} {ll}
\mathcal{F}^{-1} \left[ - \exp \{ \psi^p N^{-j} (\xi) t \} (-\Re \psi^{\mu^j (\xi)})^\kappa \hat{\varphi} (\xi) \right], & \kappa \in (0, 1), \\
\mathcal{F}^{-1} \left[ \exp \{ \psi^p N^{-j} (\xi) t \} \psi^{\mu^j (\xi)} \hat{\varphi} (\xi) \right], & \kappa = 1, \\
\mathcal{F}^{-1} \left[ \exp \{ \psi^p N^{-j} (\xi) t \} \hat{\varphi} (\xi) \right], & \kappa = 0.
\end{array} \right.
\]
Lemma 11. Let $\kappa \in [0, 1]$. For all $t \in [0, T]$ and $j \in \mathbb{N}_+$, there is $C_1, C_2 > 0$ depending only on $\alpha, d, N, \alpha_1, \alpha_2, \kappa$ such that
\[
\int |H_t^{\kappa}(x)| \, dx \leq C_1 e^{-C_2 t}.
\]

Proof. Recall that $\hat{\phi}(\xi) = \phi(N\xi) + \phi(\xi) + \phi(N^{-1}\xi)$. If we introduce $\tilde{\phi}$ such that $\hat{\phi}(\xi) = \phi(N^2\xi) + \phi(N\xi) + \phi(\xi) + \phi(N^{-1}\xi) + \phi(N^{-2}\xi)$, then $\tilde{\phi} = \hat{\phi}\phi$ and $\tilde{\phi}, \tilde{\phi} \in C_0^\infty(\mathbb{R}^d)$. We write
\[
H_t^{0,0} = \mathcal{F}^{-1}[\hat{\phi}(\xi) \exp \{ \psi^{N^{-j}}(\xi) t \} \hat{\phi}(\xi)],
\]
\[
H_t^{1,1} = \mathcal{F}^{-1}[\psi^{N^{-j}}(\xi) \hat{\phi}(\xi) \exp \{ \psi^{N^{-j}}(\xi) t \} \hat{\phi}(\xi)],
\]
\[
H_t^{j,\kappa} = \mathcal{F}^{-1}[(-\Re \psi^{N^{-j}}(\xi))^{\kappa} \hat{\phi}(\xi) \exp \{ \psi^{N^{-j}}(\xi) t \} \hat{\phi}(\xi)], \quad \kappa \in (0, 1).
\]

Thus, for all $\kappa \in [0, 1],
\[
H_t^{j,\kappa}(x) = \int [L^{\tilde{\mu},N^{-j},\kappa}_{t} \tilde{\phi}(x - z) \cdot \mathbf{E}\tilde{\phi} \left(z + Z_t^{N^{-j}}\right)] \, dz, x \in \mathbb{R}^d,
\]
and thus
\[
\left|H_t^{j,\kappa}\right|_{L^1(\mathbb{R}^d)} \leq \left|L^{\tilde{\mu},N^{-j},\kappa}_{t} \tilde{\phi}\right|_{L^1(\mathbb{R}^d)} \left|\mathbf{E}\tilde{\phi} \left(z + Z_t^{N^{-j}}\right)\right|_{L^1(\mathbb{R}^d)}.
\]

Since $\nu$ verifies $A(\mathbf{w}, \mathbf{l})$, by Lemma 15 there exist positive constants $C_1, C_2$ depending only on $\alpha, d, N, \alpha_1, \alpha_2$, such that
\[
\left|\mathbf{E}\tilde{\phi} \left(z + Z_t^{N^{-j}}\right)\right|_{L^1(\mathbb{R}^d)} < C_1 e^{-C_2 t}.
\]

Combining Lemmas 7 and 8 we then arrive at the conclusion. \hfill \Box

Lemma 12. Let $\kappa \in [0, 1]$. For all $t \in [0, T]$ and $j \in \mathbb{N}$, there is $C > 0$ depending only on $\alpha, d, N, \alpha_1, \alpha_2$ such that
\[
\int |\mathcal{F}^{-1}[\tilde{\mathcal{F}}^0_{t}(\xi)](x)| \, dx < C,
\]
\[
\int_0^t \int |\mathcal{F}^{-1}[\tilde{\mathcal{F}}^{j,\kappa}_{t}(\xi)](x)| \, dx \, dr < C, \quad j \in \mathbb{N}_+.
\]

Proof. First by Lemmas 7 and 8
\[
\int |\mathcal{F}^{-1}[\tilde{\mathcal{F}}^0_{t}(\xi)](x)| \, dx \leq \int |\mathbf{EL}^{\mu,\kappa} \tilde{\varphi}_0(x + Z_t^\nu)| \, dx \leq \int |L^{\mu,\kappa} \tilde{\varphi}_0(x)| \, dx < C.
\]

For $j \in \mathbb{N}_+$, use Lemma 11
\[
\int_0^t \int |\mathcal{F}^{-1}[\tilde{\mathcal{F}}^{j,\kappa}_{t}(\xi)](x)| \, dx \, dr \leq \int_0^t \int w(N^{-j})^{-\kappa} H^{j,\kappa}_{w(N^{-j})^{-1}r}(x) \, dx \, dr \leq w(N^{-j})^{1-\kappa} \int_0^\infty \int |H^{j,\kappa}_{r}(x)| \, dx dr < C.
\]
Corollary 5. Let $\kappa \in [0,1]$ and $u$ be the solution to (1.1) and $\mu$ be the reference measure. Then there exists $C > 0$ depending only on $\alpha, d, N, \kappa, \alpha_1, \alpha_2, T$ such that

$$|L^{\mu,\kappa} u_j|_0 \leq C |f_j|_0, j \in N.$$ 

Proof. Recall that

$$L^{\mu,\kappa} u_j (t, x) = C \int_0^t \int F^{-1} \left[ \tilde{F}^{j,\kappa}_{t-s} (\xi) \right] (z) f_j (s, x - z) \, dz \, ds, j \in N.$$ 

Therefore, by Lemma [12] for all $t \in [0, T]$,

$$|L^{\mu,\kappa} u_j|_0 \leq C |f_j|_0 \int_0^T \int |F^{-1} \left[ \tilde{F}^{j,\kappa}_{t-s} (\xi) \right] (z)| \, dz \, ds \leq C |f_j|_0, j \in N.$$ 



Lemma 13. Let $\kappa \in [0,1]$ and $\mu$ be the reference measure. Both $\mu$ and $\nu$ satisfy $A (u, l)$. Then there is $C > 0$ depending only on $\alpha, d, N, \alpha_1, \alpha_2, \kappa$, such that for all $0 \leq s < t$,

$$|E \left[ L^{\mu,\kappa} \tilde{\varphi}_0 (\cdot + Z_s^\nu) - L^{\mu,\kappa} \tilde{\varphi}_0 (\cdot + Z_t^\nu) \right] |_{L^1 (R^d)} \leq C (t - s).$$

Proof. Denote $\tilde{\varphi}_0 = F^{-1} \left[ F_0 (\xi) + \phi (N^{-2} \xi) \right]$, then $\tilde{\varphi}_0 \in S \left( R^d \right)$ and $F \tilde{\varphi}_0 F \tilde{\varphi}_0 = F \tilde{\varphi}_0$. And then,

$$|E \left[ L^{\mu} \tilde{\varphi}_0 (\cdot + Z_s^\nu) - L^{\mu} \tilde{\varphi}_0 (\cdot + Z_t^\nu) \right] |_{L^1 (R^d)} = \left| E \left[ F^{-1} \left[ \tilde{\varphi}_0 (\xi) \left( e^{\psi (\xi) t} - e^{\psi (\xi) s} \right) \tilde{\varphi}_0 (\xi) \right] \right] \right|_{L^1 (R^d)} \leq \left| L^{\mu} \tilde{\varphi}_0 |_{L^1 (R^d)} \right| E \left[ \tilde{\varphi}_0 (\cdot + Z_s^\nu) - \tilde{\varphi}_0 (\cdot + Z_t^\nu) \right] |_{L^1 (R^d)} \leq \left| L^{\mu} \tilde{\varphi}_0 |_{L^1 (R^d)} \right| E \int_s^t \nu \tilde{\varphi}_0 (\cdot + Z_r^\nu) \, dr |_{L^1 (R^d)},$$

thus, by Lemmas [7, 8]

$$|E \left[ L^{\mu} \tilde{\varphi}_0 (\cdot + Z_s^\nu) - L^{\mu} \tilde{\varphi}_0 (\cdot + Z_t^\nu) \right] |_{L^1 (R^d)} \leq C (t - s).$$

Similarly, for $\kappa \in (0,1),$

$$|E \left[ L^{\mu,\kappa} \tilde{\varphi}_0 (\cdot + Z_s^\nu) - L^{\mu,\kappa} \tilde{\varphi}_0 (\cdot + Z_t^\nu) \right] |_{L^1 (R^d)}$$

$$= \left| E \left[ F^{-1} \left[ - (R_{ij})^{\mu,\kappa}_{N-j} (\xi) \tilde{\varphi}_0 (\xi) \left( e^{\psi (\xi) t} - e^{\psi (\xi) s} \right) \tilde{\varphi}_0 (\xi) \right] \right] \right|_{L^1 (R^d)} \leq \left| L^{\mu,\kappa} \tilde{\varphi}_0 |_{L^1 (R^d)} \right| E \int_s^t \nu \tilde{\varphi}_0 (\cdot + Z_r^\nu) \, dr |_{L^1 (R^d)} \leq C (t - s),$$
and for $\kappa = 0,$
\[
|E[\tilde{\varphi}_0 (\cdot + Z^\nu_t) - \varphi_0 (\cdot + Z^\nu_s)]|_{L^1(\mathbb{R}^d)} \leq |\tilde{\varphi}_0|_{L^1(\mathbb{R}^d)} \left| E \int_s^t L^\nu \tilde{\varphi}_0 (\cdot + Z^\nu_r) \, dr \right|_{L^1(\mathbb{R}^d)} \leq C (t - s).
\]

The next Lemma is a stronger version of Lemma 13 because the Fourier transform of the underlying Schwartz function has a compact support that is away from 0.

**Lemma 14.** Let $\kappa \in [0, 1]$ and $j \in \mathbb{N}_+$. Then there are $C_1, C_2 > 0$ depending only on $\alpha, d, N, \alpha_1, \alpha_2, \kappa$, such that for all $0 \leq s < t$,
\[
\int \left| H^j_1 (x) - H^j_1 (x) \right| \, dx \leq C_1 e^{-C_2 s} (t - s).
\]

**Proof.** Similarly as what we did in Lemma 11 we introduce $\tilde{\phi}$ such that $\tilde{\phi} (\xi) = \phi (N^2 \xi) + \tilde{\phi} (\xi) + \phi (N^{-2} \xi).$ As a consequence, $\hat{\phi} = \hat{\tilde{\phi}} \hat{\phi}$, and $\hat{\tilde{\varphi}}, \tilde{\phi}, \hat{\phi} \in C_0^\infty (\mathbb{R}^d)$. Then
\[
H^{j, 0}_t - H^{j, 0}_s = \mathcal{F}^{-1} \left[ \tilde{\phi} (\xi) e^{\psi^j (\xi)} \hat{\phi} (\xi) \left( e^{\psi^j (\xi)(t-s)} - 1 \right) \hat{\phi} (\xi) \right],
\]
and
\[
H^{j, 1}_t - H^{j, 1}_s = \mathcal{F}^{-1} \left[ - (\Re \psi^j (\xi)) \hat{\phi} (\xi) e^{\psi^j (\xi)} \hat{\phi} (\xi) \left( e^{\psi^j (\xi)(t-s)} - 1 \right) \hat{\phi} (\xi) \right],
\]
and for $\kappa \in (0, 1)$,
\[
H^{j, \kappa}_t - H^{j, \kappa}_s = \mathcal{F}^{-1} \left[ - (\Re \psi^j (\xi)) \hat{\phi} (\xi) e^{\psi^j (\xi)} \hat{\phi} (\xi) \left( e^{\psi^j (\xi)(t-s)} - 1 \right) \hat{\phi} (\xi) \right].
\]
Thus, for all $\kappa \in [0, 1]$,
\[
H^{j, \kappa}_t - H^{j, \kappa}_s = L^{\tilde{\mu}^j_{N-j, \kappa}} \tilde{\phi} (\cdot) \ast E \tilde{\phi} (\cdot + Z^{\tilde{\mu}^{N-j}}_{t-s}) \ast E \left[ \hat{\phi} (\cdot + Z^{\tilde{\mu}^{N-j}}_{t-s}) - \hat{\phi} (\cdot) \right] = L^{\tilde{\mu}^j_{N-j, \kappa}} \tilde{\phi} (\cdot) \ast E \tilde{\phi} (\cdot + Z^{\tilde{\mu}^{N-j}}_{t-s}) \ast E \int_0^{t-s} L^{\tilde{\mu}^{N-j}} \tilde{\phi} (\cdot + Z^{\tilde{\mu}^{N-j}}_{r}) \, dr,
\]
and thus by Lemmas [7] [8] and [18]
\[
\int \left| H^{j, \kappa}_t (x) - H^{j, \kappa}_s (x) \right| \, dx \leq \int_0^{t-s} \left| L^{\tilde{\mu}^j_{N-j, \kappa}} \tilde{\phi} (\cdot + Z^{\tilde{\mu}^{N-j}}_{r}) \right|_{L^1(\mathbb{R}^d)} \left| L^{\tilde{\mu}^{N-j}} \tilde{\phi} \right|_{L^1(\mathbb{R}^d)} \leq C_1 e^{-C_2 s} (t - s).
\]
Lemma 15. Let \( u \) be the solution to (4.1), \( \mu \) be the reference measure and \( \kappa \in [0, 1] \). Then there exists \( C > 0 \) depending only on \( \alpha, d, N, \alpha_1, \alpha_2, \kappa, T \) such that for all \( 0 \leq s < t \leq T \),

\[
|L^{\mu, \kappa} u_0 (t, x) - L^{\mu, \kappa} u_0 (s, x)| \leq C (t - s) |f_0|_0, \forall x \in \mathbb{R}^d,
\]

\[
|L^{\mu, \kappa} u_j (t, x) - L^{\mu, \kappa} u_j (s, x)| \leq C (t - s)^{1-\kappa} |f_j|_0, \forall x \in \mathbb{R}^d, j \in \mathbb{N}._{+}.
\]

Proof. According to (4.3),

\[
|L^{\mu, \kappa} u_j (t, x) - L^{\mu, \kappa} u_j (s, x)|
\leq C |f_j|_0 \int_s^t \left| \mathcal{F}^{-1} \left[ \tilde{F}^{j, \kappa}_{t-r} (\xi) \right] (z) \right| dz dr
\]

\[
+ C |f_j|_0 \int_0^s \left| \mathcal{F}^{-1} \left[ \tilde{F}^{j, \kappa}_{t-r} (\xi) - \tilde{F}^{j, \kappa}_{s-r} (\xi) \right] (z) \right| dz dr
\]

\[
:= C |f_j|_0 (I_1 + I_2), \quad j \in \mathbb{N}.
\]

When \( j = 0 \), Lemma [12] implies

\[
I_1 = \int_s^t \int \mathcal{F}^{-1} \left[ \tilde{F}^{j, \kappa}_{t-r} (\xi) \right] (z) \left| dz dr \leq C (t - s), \forall \kappa \in [0, 1].
\]

When \( j \neq 0 \), recall (4.5).

\[
I_1 \leq \int_s^t \int \left| w \left( \frac{t-s}{1-\kappa} \right) H^{j, \kappa}_{w(\frac{t-s}{1-\kappa})^{-1}} (z) \right| dz dr
\]

\[
= w \left( \frac{t-s}{1-\kappa} \right) \int_0^{w(\frac{t-s}{1-\kappa})^{-1}(t-s)} \int \left| H^{j, \kappa}_t (z) \right| dz dr.
\]

If \( w \left( \frac{t-s}{1-\kappa} \right) \leq 1 \), since \( \int \left| H^{j, \kappa}_t (z) \right| dz \leq C \) by Lemma [11]

\[
I_1 \leq C w \left( \frac{t-s}{1-\kappa} \right) \int_0^{w(\frac{t-s}{1-\kappa})^{-1}(t-s)} \int \left| H^{j, \kappa}_t (z) \right| dz dr.
\]

If \( w \left( \frac{t-s}{1-\kappa} \right) > 1 \), again use Lemma [11]

\[
I_1 \leq w \left( \frac{t-s}{1-\kappa} \right) \int_0^{\infty} \int \left| H^{j, \kappa}_t (z) \right| dz dr \leq C w \left( \frac{t-s}{1-\kappa} \right) \int_0^{w(\frac{t-s}{1-\kappa})^{-1}(t-s)} \int \left| H^{j, \kappa}_t (z) \right| dz dr.
\]

Next we investigate \( I_2 \). Recall definitions (4.4) – (4.6). When \( j = 0 \) and \( \kappa \in [0, 1] \),

\[
I_2 = \int_0^s \int \left| \mathcal{F}^{-1} \left[ \tilde{F}^{0, \kappa}_{t-r} (\xi) - \tilde{F}^{0, \kappa}_{s-r} (\xi) \right] (z) \right| dz dr
\]

\[
\leq e^{-\lambda(t-s)} - 1 \int_0^s e^{-\lambda(s-r)} \int \mathcal{F}^{-1} \left[ -e^{\psi(\xi)(t-r)} \mathcal{F} L^{\mu, \kappa} \varphi_0 (\xi) \right] (z) \left| dz dr
\]

\[
+ \int_0^s \int \mathcal{F}^{-1} \left[ -\left( e^{\psi(\xi)(t-r)} - e^{\psi(\xi)(s-r)} \right) \mathcal{F} L^{\mu, \kappa} \varphi_0 (\xi) \right] (z) \left| dz dr
\]

:= I_{21} + I_{22}.
Therefore, by Lemmas 7 and 8,

\[ I_{21} \leq \frac{2T}{\lambda} |e^{-\lambda(t-s)} - 1| E L^{j,\kappa} \varphi_0 (\cdot + Z_{t-r}^\nu) \big|_{L^1(\mathbb{R}^d)} \]

\[ \leq \frac{2T}{\lambda} |e^{-\lambda(t-s)} - 1| L^{j,\kappa} \varphi_0 \big|_{L^1(\mathbb{R}^d)} \leq C (t-s), \quad \kappa \in [0,1]. \]

Meanwhile, by Lemma 13,

\[ I_{22} \leq T \left| E \left[ L^{j,\kappa} \varphi_0 (\cdot + Z_{t-r}^\nu) - L^{j,\kappa} \varphi_0 (\cdot + Z_{s-r}^\nu) \right] \right|_{L^1(\mathbb{R}^d)} \]

\[ \leq C (t-s), \quad \kappa \in [0,1]. \]

When \( j \neq 0, \)

\[ I_2 \leq |e^{-\lambda(t-s)} - 1| w \left( (N-j)^{-\kappa} \right) \int_0^s |e^{-\lambda(s-r)} \int |H^{j,\kappa}_{w(N-j)^{-1}} (z) | dz dr \]

\[ + w \left( (N-j)^{-\kappa} \right) \int_0^s \left| H^{j,\kappa}_{w(N-j)^{-1}} (z) - H^{j,\kappa}_{w(N-j)^{-1}} (s-r) \right| dz dr \]

\[ := I'_{21} + I'_{22}. \]

By Lemma 11,

\[ I'_{21} \leq \left| e^{-\lambda(t-s)} - 1 \right| w \left( (N-j)^{-\kappa} \right) \int_0^s \left| e^{-\lambda r} \int H^{j,\kappa}_{w(N-j)^{-1}} (z) \right| dz dr \]

\[ \leq C \left| e^{-\lambda(t-s)} - 1 \right| w \left( (N-j)^{-\kappa} \right) \int_0^s \left| e^{-\lambda r} e^{-C'w(N-j)^{-1}} \right| dr. \]

If \( w \left( (N-j)^{-1} (t-s) \right) \leq 1, \)

\[ I'_{21} \leq C \left| e^{-\lambda(t-s)} - 1 \right| w \left( (N-j)^{-\kappa} \right) \int_0^s e^{-\lambda r} dr \]

\[ \leq C (t-s) w \left( (N-j)^{-\kappa} \right) \leq C (t-s)^{1-\kappa}. \]

If \( w \left( (N-j)^{-1} (t-s) \right) > 1, \) use Lemma 11

\[ I'_{21} \leq C w \left( (N-j)^{-\kappa} \right) \int_0^s e^{-C'w(N-j)^{-1}} r dr \leq C w \left( (N-j)^{-1-\kappa} \right) \leq C (t-s)^{1-\kappa}. \]

On the other hand,

\[ I'_{22} = w \left( (N-j)^{1-\kappa} \right) \int_0^s w \left( (N-j)^{-1} \right) \int H^{j,\kappa}_{w(N-j)^{-1}} (t-s+r) (z) - H^{j,\kappa}_{w(N-j)^{-1}} (z) \big| dz dr \]

\[ \leq w \left( (N-j)^{1-\kappa} \right) \int_0^\infty H^{j,\kappa}_{w(N-j)^{-1}} (t-s+r) (z) - H^{j,\kappa}_{w(N-j)^{-1}} (z) \big| dz dr. \]

If \( w \left( (N-j)^{-1} (t-s) \right) \leq 1, \) use Lemma 14

\[ I'_{22} \leq C w \left( (N-j)^{-\kappa} \right) w \left( (N-j)^{-1} (t-s) \right) \leq C (t-s)^{1-\kappa}. \]

If \( w \left( (N-j)^{-1} (t-s) \right) > 1, \) use Lemma 11

\[ I'_{22} \leq 2 w \left( (N-j)^{1-\kappa} \right) \int_0^\infty H^{j,\kappa}_{w(N-j)^{-1}} (z) \big| dz dr \leq C w \left( (N-j)^{-1-\kappa} \right) \leq C (t-s)^{1-\kappa}. \]
This is the end of the proof. □

**Theorem 4.2.** Let \( \nu \) be a Lévy measure satisfying \( A(w,l) \) and \( \beta \in (0, \infty) \). Then the unique solution \( u \in (t, x) \) to (4.1) satisfies

\[
(u)_{\beta, \infty} \leq C (\lambda^{-1} \wedge T) |f|_{\beta, \infty},
\]

\[
|u|_{1+\beta, \infty} \leq C |f|_{\beta, \infty}
\]

for some \( C \) depending on \( \alpha, \alpha_1, \alpha_2, \beta, N, d, T \). Meanwhile, for all \( \kappa \in [0, 1] \), there exists a constant \( C \) depending on \( \alpha, \kappa, \beta, \alpha_1, \alpha_2, N, d, T, \nu \) such that for all \( 0 \leq s < t \leq T \),

\[
|u(t, \cdot) - u(s, \cdot)|_{\kappa+\beta, \infty} \leq C |t - s|^{1-\kappa} |f|_{\beta, \infty}.
\]

**Proof.** Denote as before \( u_j = u * \varphi_j, j \in \mathbb{N} \). We have known that

\[
u_j(t, x) = \int_0^t e^{-\lambda(t-s)} E \nu_j(s, x + Z_{t-s}) ds, \forall j \in \mathbb{N}.
\]

Obviously,

\[
|u_j| \leq |f_j| \int_0^t e^{-\lambda s} ds \leq C (\lambda^{-1} \wedge T) |f_j|, \forall j \in \mathbb{N},
\]

which implies \( |u|_{\beta, \infty} \leq C (\lambda^{-1} \wedge T) |f|_{\beta, \infty}, \forall \beta \in (0, \infty) \). Recall (5.1).

\[
|u|_{1+\beta, \infty} \leq C (|u|_0 + |L^\mu u|_{\beta, \infty}) \leq C |f|_{\beta, \infty}.
\]

Similarly, by Lemma 15 we know that for all \( j \in \mathbb{N} \),

\[
|L^\mu u_j(t, x) - L^\mu u_j(s, x)| \leq C (t - s)^{1-\kappa} |f_j|, \forall x \in \mathbb{R}^d, \kappa \in [0, 1],
\]

namely, for all \( \beta \in (0, \infty) \),

\[
|L^\mu u(t, \cdot) - L^\mu u(s, \cdot)|_{\beta, \infty} \leq C (t - s)^{1-\kappa} |f|_{\beta, \infty}, \kappa \in [0, 1].
\]

Therefore, for all \( \kappa \in [0, 1] \) and all \( 0 \leq s < t \leq T \),

\[
|u(t, \cdot) - u(s, \cdot)|_{\mu, \kappa, \beta} \leq |u(t, \cdot) - u(s, \cdot)|_{\beta, \infty} + |L^\mu u(t, \cdot) - L^\mu u(s, \cdot)|_{\beta, \infty} \leq C |t - s|^{1-\kappa} |f|_{\beta, \infty}.
\]

By Proposition 3, this is equivalent to

\[
|u(t, \cdot) - u(s, \cdot)|_{\kappa+\beta, \infty} \leq C |t - s|^{1-\kappa} |f|_{\beta, \infty}.
\]
5. **Proof of Theorem 1.1: Generalized Hölder-Zygmund inputs**

Existence and Estimates. Given \( f \in \tilde{C}^{\beta}_{\infty, \infty}(H_T) \), by Proposition 3, we can find a sequence of functions \( f_n \) in \( C^\infty_b(H_T) \) such that

\[
|f_n|_{\beta, \infty} \leq C |f|_{\beta, \infty}, \quad |f|_{\beta, \infty} \leq \lim_{n \to \infty} |f_n|_{\beta, \infty},
\]

and for any \( 0 < \beta' < \beta \),

\[
|f_n - f|_{\beta', \infty} \to 0 \quad \text{as} \quad n \to \infty.
\]

According to Theorems 4.1 and 4.2 for each pair of functions \( f_m, f_n \), there are corresponding solutions \( u_m, u_n \in C^\infty_b(H_T) \) verifying

\[
|u_m - u_n|_{1+\beta', \infty} \leq C |f_m - f_n|_{\beta', \infty} \to 0, \quad \text{as} \quad m, n \to \infty
\]

for all \( \beta' \in (0, \beta) \), which by Proposition 3 implies

\[
|u_n - u_m|_0 \to 0 \quad \text{as} \quad m, n \to \infty.
\]

Clearly, \( \{u_n : n \geq 0\} \) has a limit in the space of continuous functions. We denote it by \( u \). \( \lim_{n \to \infty} |u_n - u|_0 = 0 \). Therefore, for any given \( j \in \mathbb{N} \),

\[
w(N^{-j})^{-1-\beta} |u \ast \varphi_j|_0 = \lim_{n \to \infty} w(N^{-j})^{-1-\beta} |u_n \ast \varphi_j|_0 \leq \limsup_{n \to \infty} |u_n|_{1+\beta, \infty} \leq C \limsup_{n \to \infty} |f_n|_{\beta, \infty} \leq C |f|_{\beta, \infty},
\]

which indicates \( u \in \tilde{C}^{1+\beta}_{\infty, \infty}(H_T) \) and \( |u|_{1+\beta, \infty} \leq C |f|_{\beta, \infty} \). Meanwhile, for any given \( j \in \mathbb{N} \) and any \( \beta' \in (0, \beta) \),

\[
\lim_{n \to \infty} w(N^{-j})^{-1-\beta'} |(u_n - u) \ast \varphi_j|_0 = \lim_{n \to \infty} \lim_{m \to \infty} w(N^{-j})^{-1-\beta'} |(u_n - u_m) \ast \varphi_j|_0 \\
\leq \lim_{n \to \infty} \lim_{m \to \infty} |(u_n - u_m)|_{1+\beta', \infty} \\
\leq C \lim_{n \to \infty} \lim_{m \to \infty} |f_n - f_m|_{\beta', \infty} \to 0,
\]

Namely, for all \( \beta' \in (0, \beta) \).

\[
\text{(5.1)} \quad \lim_{n \to \infty} |u_n - u|_{1+\beta', \infty} \to 0, \quad \text{as} \quad n \to \infty.
\]

Analogously, for any given \( j \in \mathbb{N} \),

\[
w(N^{-j})^{-\beta} |u \ast \varphi_j|_0 = \lim_{n \to \infty} w(N^{-j})^{-\beta} |u_n \ast \varphi_j|_0 \leq \limsup_{n \to \infty} |u_n|_{\beta, \infty} \leq C (\lambda^{-1} \wedge T) \limsup_{n \to \infty} |f_n|_{\beta, \infty} \leq C (\lambda^{-1} \wedge T) |f|_{\beta, \infty}.
\]

This implies \( |u|_{\beta, \infty} \leq C (\lambda^{-1} \wedge T) |f|_{\beta, \infty} \).
Using Theorems 4.1 and 4.2 we can show in the same vein that for all
\[0 \leq s \leq t \leq T, \ \kappa \in [0, 1],\]
\[u(t, \cdot) - u(s, \cdot) = \lim_{n \to \infty} (u_n(t, \cdot) - u_n(s, \cdot)) \in \tilde{C}_{\infty, \infty}^{\kappa+\beta}(\mathbb{R}^d),\]
\[|u(t, \cdot) - u(s, \cdot)|_{\kappa+\beta, \infty} \leq C \limsup_{n \to \infty} |u_n(t, \cdot) - u_n(s, \cdot)|_{\kappa+\beta, \infty}\]
\[\leq C |t-s|^{1-\kappa} |f|_{\beta, \infty}.\]

Now we claim that such a function \(u\) solves (4.1), i.e.,
\[u(t, x) = \int_0^t \left[L \nu u(r, x) - \lambda u(r, x) + f(r, x)\right] dr, \quad (t, x) \in [0, T] \times \mathbb{R}^d.\]

Indeed, according to (5.1) and Proposition 6 \(L \nu u = \lim_{n \to \infty} L \nu u_n\) and \(\lim_{n \to \infty} |L \nu u_n - L \nu u|_0 = 0\). Passing the limit on both sides of
\[u_n(t, x) = \int_0^t \left[L \nu u_n(r, x) - \lambda u_n(r, x) + f_n(r, x)\right] dr,\]
we obtain (5.2).

**Uniqueness.** Suppose there are two solutions \(u_1, u_2 \in \tilde{C}_{\infty, \infty}^{1+\beta}(H_T)\) to (5.2), then \(u := u_1 - u_2\) solves
\[u(t, x) = \int_0^t \left[L \nu u(r, x) - \lambda u(r, x)\right] dr, \quad (t, x) \in [0, T] \times \mathbb{R}^d.\]

By Proposition 5 there is a sequence of functions \(u_n \in C_b^\infty(H_T)\) such that for any \(0 < \beta' < \beta\),
\[|u_n - u|_{1+\beta', \infty} \to 0 \text{ as } n \to \infty.\]

Clearly, \(\tilde{u}_n(t, x) := \int_0^t u_n(s, x) ds\) solves
\[\int_0^t u_n(s, x) ds = \int_0^t \left[L \nu \int_0^s u_n(r, x) dr - \lambda \int_0^s u_n(r, x) dr + \left(u_n(s, x) - L \nu \int_0^s u_n(r, x) dr + \lambda \int_0^s u_n(r, x) dr\right)\right] ds.\]

By Lemma 7 \(u_n(t, x) - L \nu \int_0^t u_n(s, x) ds + \lambda \int_0^t u_n(s, x) ds \in C_b^\infty(H_T)\). Then according to Theorem 4.2
\[|\tilde{u}_n|_{1+\beta', \infty} \leq \left|u_n(t, x) - L \nu \int_0^t u_n(s, x) ds + \lambda \int_0^t u_n(s, x) ds\right|_{\beta', \infty}.\]
Use (5.3), (3.1) and Proposition 6
\[ \left| \int_0^t u(s, x) \, ds \right|_{1+\beta', \infty} = \lim_{n \to \infty} \left| \int_0^t u_n(s, x) \, ds \right|_{1+\beta', \infty} \]
\[ \leq \liminf_{n \to \infty} \left| u_n(t, x) - L^\nu \int_0^t u_n(s, x) \, ds + \lambda \int_0^t u(s, x) \, ds \right|_{\beta', \infty} \]
\[ \leq \left| u(t, x) - L^\nu \int_0^t u(s, x) \, ds + \lambda \int_0^t u(s, x) \, ds \right|_{\beta', \infty} = 0. \]

By (3.1) again, \( \int_0^t u(s, x) \, ds = 0 \) for all \( t \in [0, T] \) and \( x \in \mathbb{R}^d \), and thus
\( u = 0 \) \( (t, x) \text{-a.e.} \).

6. Appendix

We simply state a few results that were used in this paper. Please look up in the references for proofs if you are interested.

Recall all parameters introduced in assumptions \( A(w, l) \).

Lemma 16. [6] Lemma 7] Let \( \nu \) be a Lévy measure of order \( \alpha \) and \( w \) be a scaling function.
(i) Suppose there exists \( N_2 > 0 \) such that for all \( R > 0 \),
\[ \int (|y| \wedge 1) \tilde{v}_R(dy) \leq N_2 \text{ if } \alpha \in (0, 1), \]
\[ \int \left( |y|^2 \wedge 1 \right) \tilde{v}_R(dy) \leq N_2 \text{ if } \alpha = 1, \]
\[ \int \left( |y|^2 \wedge |y| \right) \tilde{v}_R(dy) \leq N_2 \text{ if } \alpha \in (1, 2). \]

Then there is a constant \( C > 0 \) depending only on \( c_1, N_0, N_1, N_2 \) such that for all \( \xi \in \mathbb{R}^d \),
\[ \int [1 - \cos(2\pi \xi \cdot y)] \nu(dy) \leq C \cdot w\left(|\xi|^{-1}\right)^{-1}, \]
\[ \int [\sin(2\pi \xi \cdot y) - 2\pi \chi_\alpha(y) \xi \cdot y] \nu(dy) \leq C \cdot w\left(|\xi|^{-1}\right)^{-1}, \]
where \( w\left(|\xi|^{-1}\right)^{-1} := 0 \text{ if } \xi = 0. \)

(ii) Suppose there is \( n_1 > 0 \) such that for all \( R > 0 \) and all \( \xi \in \{ \xi \in \mathbb{R}^d : |\xi| = 1 \} \),
\[ \int_{|y| \leq 1} |\xi \cdot y|^2 \tilde{v}_R(dy) \geq n_1. \]

Then there is a constant \( c > 0 \) depending only on \( c_1, N_0, N_1, N_2, n_1 \) such that for all \( \xi \in \mathbb{R}^d \),
\[ \int [1 - \cos(2\pi \xi \cdot y)] \nu(dy) \geq c \cdot w\left(|\xi|^{-1}\right)^{-1}, \]
where \( w \left( |\xi|^{-1} \right)^{-1} := 0 \) if \( \xi = 0 \).

**Lemma 17.** [6] Lemma 5] Let \( \nu \) be a Lévy measure satisfying \( A(w,l) \). \( Z_t^\nu R \) is the Lévy process associated to \( \tilde{\nu}_R, R > 0 \). For each \( t, R \), \( Z_t^\nu R \) has a bounded and continuous density function \( p_R(t,x) \), \( t \in (0, \infty), x \in \mathbb{R}^d \). And \( p_R(t,x) \) has bounded and continuous derivatives up to order 4. Meanwhile, for any multi-index \( |\vartheta| \leq 4 \),

\[
\int |\partial^{\vartheta} p_R(t,x)| \, dx \leq C \gamma(t)^{-|\vartheta|},
\]

\[
\sup_{x \in \mathbb{R}^d} |\partial^{\vartheta} p_R(t,x)| \leq C \gamma(t)^{-d-|\vartheta|},
\]

where \( C > 0 \) is independent of \( t, R \). For any \( \beta \in (0,1) \) such that \( |\vartheta| + \beta < 4 \),

\[
\int |\partial^{\vartheta} \partial^{\vartheta} p_R(t,x)| \, dx \leq C \gamma(t)^{-|\vartheta|} - \beta.
\]

For any \( a > 0 \), there is a constant \( C > 0 \) independent of \( t, R \), so that

\[
\int_{|x| > a} |\partial^{\vartheta} p_R(t,x)| \, dx \leq C \left( \gamma(t)^{2-|\vartheta|} + t \gamma(t)^{-|\vartheta|} \right).
\]

**Lemma 18.** [7] Lemma 2] Let \( \nu \) be a Lévy measure satisfying \( A(w,l) \) and \( Z_t^\nu R \) be the Lévy process associated to \( \tilde{\nu}_R \). Then for any \( \varphi, \varphi_0 \in \mathcal{S} \left( \mathbb{R}^d \right) \) such that \( \mathcal{F} \varphi_0 \in C_0^\infty \left( \mathbb{R}^d \right), \supp (\mathcal{F} \varphi) \subseteq \{ \xi : 0 < R_1 \leq |\xi| \leq R_2 \} \), and

\[
\max_{|\gamma| \leq [d/2]+3} |D^\gamma \varphi(\xi)| \leq N_2, R_1 \leq |\xi| \leq R_2.
\]

Then there are constants \( C_1, C_2 > 0 \) depending only on \( c_1, N_0, N_1, N_2, R_1, R_2, d \) such that

\[
\int (1 + |x|^{a_2}) |E \varphi \left( x + Z_t^\nu R \right)| \, dx \leq C_1 e^{-C_2 t}, \quad t \geq 0,
\]

\[
\int |x|^{a_2} |E \varphi_0 \left( x + Z_t^\nu R \right)| \, dx \leq C_1 (1 + t), \quad t \geq 0,
\]

\[
\int |E \varphi_0 \left( x + Z_t^\nu R \right)| \, dx \leq C_1, \quad t \geq 0.
\]

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