Dimensional collapse and fractal attractors of a system with fluctuating delay times

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A frequently encountered situation in the study of delay systems is that the length of the delay time changes with time, which is of relevance in many fields such as optics, mechanical machining, biology or physiology. A characteristic feature of such systems is that the dimension of the system dynamics collapses due to the fluctuations of delay times. In consequence, the support of the long-trajectory attractors of this kind of systems is found being fractal in contrast to the fuzzy attractors in most random systems.

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Introduction. In ubiquitous natural and laboratory situations the action of time delayed signals is an essential ingredient to understand the system dynamics. For instance, in optical and physiological systems 1-3, a finite transmission speed usually leads to delayed reactions to signals from remote origins and/or time shifts among signals taking different paths. In mechanical engineering the rotation of the workpiece is the origin of time delay effects 4, and in feedback control methods time delayed signals are utilized to stabilize the system behavior 5.

A frequently encountered situation which was rarely addressed previously 6, is that the length of the delay time often varies with time. A biological example is that the reproduction cycle of animals fluctuates with the environment change. In mechanical engineering, vibrations of the machine tool and/or the workpiece may also change the length of the delay time 6. As a modern control scheme, the time variation of delays has been found to enhance the control efficiency 7.

The relevance of fluctuating delay times for all these phenomena motivates us to gain a better understanding of its influence on the system dynamics. In this contribution, we will start from a very simple map system where the delay time takes only the value of one or zero discrete time steps. High dimensional systems with longer fluctuating delays show similar behavior 11,12.

A characteristic feature of such systems is that the system dynamics collapses to a low dimensional subspace due to the fluctuation of delay times. One would expect that such a dimensional collapse will have some interesting consequences on the system dynamics. As an example, we show in this paper its influence on the fractal properties of the system attractors.

Models. For simplicity, we focus on the map system

$$x_{t+1} = 1 - ax_t^2 + (b + k\epsilon_t)x_{t-1} \quad (1)$$

where $k$ and $b$ are constants and the time dependent random variable $\epsilon_t$ takes only value 0 or 1. The probability of $\epsilon_t$ taking the value 1 is denoted as $p$. For a parameter setting with $b = 0$ the length of the delay time of our system can vary randomly between one or zero time steps with the system dynamics simply switching between the Henon map ($\epsilon_t = 1$) and the logistic map ($\epsilon_t = 0$). To illustrate the importance of the delay time variation and the resulting dimensional collapse a comparison is made with the well-studied case of the random Henon map in 13 with $b \neq 0$ and $\epsilon_t$ being an uniform noise in the interval $[0,1]$, where the delay time is constantly 2. In the following the parameter $a$ is fixed as $a = 1.1$ if it is not stated otherwise.

Numerical results. As depicted in 13 to study the effect of randomness on attractors of a system like Eq.(1) one may consider two different situations, either the snapshot attractor formed by an ensemble of points at a single instant of time with all points starting from different initial conditions and evolving under a same given realization of the external noise, or the long-trajectory attractor formed by a long trajectory segment starting from a single initial condition and evolving under a given realization of the external noise. The main issue addressed here will be the difference between the long-trajectory attractors of the random delay case and the random Henon case.

It was shown 13 that for the random Henon case $b \neq 0$ the long-trajectory attractor is a fuzzy attractor with a smooth density of points while the corresponding snapshot attractor may be fractal. An example with $b = 0.3$, $k = 0.05$ is presented in Fig.1(a) and (b). In contrast, for the random delay case $b = 0$ we find that the long-trajectory attractor turns out to be fractal in consequence of the delay time variation and the dimensional collapse. In Fig.1(c) and (d) the long-trajectory and snapshot attractors are shown for a random delay case with $b = 0$, $k = 0.3$ and $p = 0.5$. Moreover, as demonstrated the long-trajectory attractors obtained are identical for two runs with different initial conditions and different realizations of the noise $\epsilon_t$. This is different from previously investigated random systems where in general different realizations of external noise lead to different attracting sets lying in different regions of the phase space. The snapshot attractor of the random delay case is time dependent and turns out to be at probability 1 a one-dimensional curve (see Fig.1(d)).

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To show quantitatively the difference in the fractal properties between the random delay case and the random Henon case we calculated the capacity and information dimension of the attractor via the box-counting algorithm. The variation of the number of non-empty boxes \( N(l) \) versus the box size \( l \) is plotted in Fig.2. By definition the slope of the log-log plot of \( N(l) \) versus \( 1/l \) gives the capacity dimension \( D_0 \) of the attractor. Four curves in Fig.2 correspond to the four attractors shown in Fig.1. To guide the eyes, three lines of slope 1, 2 and 1.26 (the capacity dimension of the Henon attractor) are plotted in the same figure. One can easily see from the plot that the capacity dimension of the long-trajectory attractor of the random Henon case and the random delay case is very close to 2 and 1.26 respectively. This confirms our above observation from Fig.1 that the long trajectory attractor of the random Henon case is a fuzzy attractor with a smooth density of points while the long trajectory attractor of the random delay case is a fractal. The capacity dimension of the typical snapshot attractor of the random Henon case and the random delay case is 1.14 and 1 respectively, as expected from the observation that the former attractor is fractal while the latter is a one-dimensional curve.

Moreover, we calculated also the information dimension \( D_1 \) of the four attractors shown in Fig.1. Results are given in Tab.1. Notice that the information dimension of the long trajectory attractor for the random delay case is close to 1.

To characterize the dynamical behavior of the system we calculate the Lyapunov spectrum of the random delay case via the standard method [10]. In general the system has one Lyapunov exponent of finite value and the other of value \(-\infty\) due to the dimension collapse. The Kaplan-Yorke dimension \( D_\lambda \) therefore has the value 1, consistent with the obtained value \( D_1 \approx 1 \) for the long trajectory attractor as depicted in Tab.1. Furthermore, as shown in Fig.3 the leading Lyapunov exponent of the system can be positive or negative depending on the probability \( p \). The capacity dimension \( D_0 \), however, stays constant in the regime \( p > 0.2 \) irrespective of the value of \( p \). The smaller value of \( D_0 \) obtained in the regime \( p < 0.2 \) is a numerical artifact due to an insufficient number of phase points used. Simulations show indeed that the observed value of \( D_0 \) for \( p < 0.2 \) increases with the number of phase points used and the regime of constant \( D_0 \) expands correspondingly. An additional interesting point in Fig.3 is that a randomly temporal switching between two nonchaotic dynamics with trivial attractors results in a chaotic dynamics holding a fractal attractor (e.g. for \( a = 0.9, k = 0.35 \) and \( p = 0.5 \)). Without considering the positive Lyapunov exponent corresponding to the external noise \( \epsilon_t \) the fractal attractor of the random delay case with a negative Lyapunov exponent may be viewed as a strange nonchaotic attractor [14].

**Dimension collapse.** A characteristic feature of the random delay system Eq.1 is that the length of the delay time fluctuates between 0 and 1 randomly. It is there-
fore convenient for our analysis to partition the evolution of the system dynamics into segments having either pure delay time 0 or 1, wherein the system dynamics simply corresponds to the logistic or the Henon map, respectively. For simplicity, we will denote them as 0- or 1-segment respectively. Whenever the system switches from a 1-segment to a 0-segment, the phase point representing the dynamic evolution of the system will fall immediately on the parabola $z_{n+1} = 1 - az_n$ and the evolution of the system dynamics sticks in such a one-dimensional subspace during the 0-segment. This is what we called dimension collapse. At the end of the 0-segment, the system enters a 1-segment and the dynamics is now just the Henon map without randomness until the appearance of the next 0-segment. It is obvious that the target of the collapses is not influenced by the history of the system dynamics or the delay variations. The dynamic evolution of the random delay system Eq. (1) can therefore be viewed as the re-injections to the parabola during the 0-segments and the iterations of the parabola under the Henon dynamics during the 1-segments. The long trajectory attractor of the system can thus be roughly viewed as the union of the parabola and its n-fold image under the Henon dynamics where n is the possible length of the 1-segments and $n = 1, 2, \cdots, +\infty$.

An expression of the natural measure $\rho^*$ of the attractor can be worked out accordingly. Based on the above discussions one may decompose $\rho^*$ in the following way

$$\rho^* = \sum_{n=0}^{\infty} \rho_{L^n} = \rho_{L^0} + \sum_{n=1}^{\infty} \rho_{L^n}$$

where $\rho_{L^0}$ is the part of the natural measure living on the parabola given by the logistic map and $\rho_{L^n}$ represents the part of the nature measure whose support is the n-fold iteration of the parabola under the Henon dynamics. On the other hand a self-consistent equation for $\rho^*$ can be written down as

$$\rho^* = p\hat{H}\rho^* + (1 - p)\hat{L}\rho^*$$

which is a Frobenius-Perron like equation for our random system Eq. (1). Here $\hat{H}$ and $\hat{L}$ denote the evolution operators for the density of the logistic and the Henon map respectively. Inserting (2) in (3) reads

$$\rho_{L^0} + \sum_{n=1}^{\infty} \rho_{L^n} = (1 - p)\hat{L}\rho^* + p \sum_{n=0}^{\infty} \hat{H}\rho_{L^n}. \quad (4)$$

Identifying terms on two sides of (4) which have the same phase space support leads to

$$\rho_{L^0} = (1 - p)\hat{L}\rho^*; \quad (5)$$

$$\rho_{L^n} = p^n \hat{H}^n \rho_{L^0} \text{ for } n > 0. \quad (6)$$

Inserting (5) in (2) generates a new expression of $\rho^*$

$$\rho^* = \sum_{n=0}^{\infty} p^n \hat{H}^n \rho_{L^0}. \quad (7)$$

As a consistence check one can insert (5) in (7), it reads

$$\rho^* = (1 - p) \sum_{n=0}^{\infty} p^n \hat{H}^n \hat{L}\rho^* \quad (8)$$

$$= (1 - p)(1 - p\hat{H})^{-1} \hat{L}\rho^* , \quad (9)$$

which is just a simple reformulation of the Frobenius-Perron equation given in (3).
Eq. (6) indicates that $\rho_{L,0}$ is the projection of the natural measure $\rho^*$ on the parabola of the logistic map. It therefore has a continuous density on this one dimensional subject for the used parameter. As discussed the support of $\rho_{L,n}$ is the $n$-fold iteration of the parabola under the action of the Henon map and it approaches the Henon attractor as $n$ goes to $+\infty$. Dimensional collapse means that the dimension of $\rho_{L,n}$ is smaller than the complete dimension of the phase space which in consequence implies that the iterations of $\rho_{L,n}$ under the action of the Henon map are overlapped only at measure zero points. One would thus expect from Eq. (6) that the capacity dimension $D_0$ of the (long trajectory) attractor is smaller than the dimension of the phase space while the information dimension $D_1$ should be determined by $\rho_{L,0}$ and has the value 1. This explains the appearance of a fractal attractor in our random delay system Eq. (1) and the above obtained values of $D_1$ of the corresponding attractors. It is necessary to point out that the capacity dimension $D_0$ of the (long trajectory) attractor is in general different from that of the corresponding Henon attractor (see the case $a = 0.9, b = 0.35$ in Fig. 3), although they are very close if the corresponding Henon dynamics is chaotic.

Moreover as can been from Eq. (6), while other Renyi dimensions may vary with the probability $p$ the capacity and information dimensions of the long-trajectory attractor are independent of $p$ (see Fig. 3 for numerical confirmation).

Note that the random occurrence of segments of logistic map dynamics in our random delay system interrupts the previous Henon map evolution and starts a new Henon evolution with a different point on the parabola of the logistic map as the initial condition. The randomness of the delay variation can only determine when the reinjection happens and prepare a new initial condition for the next Henon evolution. This is obviously irrelevant for the global geometrical properties of the long-trajectory attractor. Therefore different realizations of external noise $\epsilon_t$ would lead to the same long trajectory attractor as also indicated by Eq. (6) (see Fig. 1 (c)). The dynamical evolution of phase point surely depends on the realization of the external noise, see for example the variation of the shape of the snapshot attractor. In this sense the snapshot and long-trajectory attractors contain different information of the system dynamics. This is assumed due to the nonstationary of the evolution rule of random systems.

As can be seen from above analysis the collapse of the phase space dimension works as a new mechanism of generating fractal attractor in random delay systems like (1). The collapse ensures that all segments of Henon evolutions starts synchronously from the same low dimensional subspace, the parabola of the logistic map. Such a synchronization prevents the randomness to blur the fine structure of the attractors generated from the Henon evolutions.

Summary. As a first step towards a deep understanding of the behavior of systems with fluctuating delay time, we studied a simple map system with either one or two step memory of the past states. Numerical simulations and supplementary arguments showed that the dimension collapse of the random delay system has some very interesting consequences in the system dynamics for instance the fractal properties of the attractor.

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