Higher-dimensional Origin of $D = 3$ Coset Symmetries

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ABSTRACT

It is well known that the toroidal dimensional reduction of supergravities gives rise in three dimensions to theories whose bosonic sectors are described purely in terms of scalar degrees of freedom, which parameterise sigma-model coset spaces. For example, the reduction of eleven-dimensional supergravity gives rise to an $E_8/SO(16)$ coset Lagrangian. In this paper, we dispense with the restrictions of supersymmetry, and study all the three-dimensional scalar sigma models $G/H$ where $G$ is a maximally-non-compact simple group, with $H$ its maximal compact subgroup, and find the highest dimensions from which they can be obtained by Kaluza-Klein reduction. A magic triangle emerges with a duality between rank and dimension. Interesting also are the cases of Hermitian symmetric spaces and quaternionic spaces.

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1 Introduction

It is well known that the dimensional reduction of eleven-dimensional supergravity to $D = 3$ gives rise to a scalar coset theory with an $E_{8(+8)}$ global symmetry. Conversely, we may say that the “oxidation endpoint” of the three-dimensional $E_{8(+8)}$ scalar coset is the bosonic sector of $D = 11$ supergravity. In [2], a classification of three-dimensional scalar cosets associated with the various groups that arise from the dimensional reduction of $D = 4$ gravity coupled to matter systems was given. In many cases, these four-dimensional theories can themselves be obtained from the dimensional reduction of yet higher dimensional theories of matter coupled to gravity. In most of this paper, we shall study the “oxidation endpoints” (also called group disintegration endpoints) of the three-dimensional scalar cosets given in [2]. By the “oxidation endpoint” of a three-dimensional scalar coset, we mean the (bosonic) theory in the highest possible dimension $D_{\text{max}}$ whose toroidal dimensional reduction gives back precisely the three-dimensional scalar coset. We shall concentrate on the examples where the numerator group of the three-dimensional scalar coset is maximally non-compact. It turns out that these cases are simpler and exhibit more regularity than the cases of non-maximally non-compact symmetry groups first studied systematically for four dimensional supergravities in [4].

A particular example, namely the $SL(n, \mathbb{R})/O(n)$ scalar coset, was studied in [3]. It was shown that it can be obtained from the dimensional reduction of pure gravity in $D = 2 + n$ dimensions. This in fact provides a useful clue in more general cases as to what is the oxidation endpoint of a given three-dimensional scalar coset. Namely, if the global symmetry group of a three-dimensional scalar coset contains a subgroup $SL(n, \mathbb{R})$, then one might expect that the oxidation endpoint of the coset would be in $D_{\text{max}} = n + 2$ dimensions. We find that this rule works for all the maximally non-compact simple or semi-simple groups, with the exception of $C_k$. The reason for this can be clarified by recalling that the reduction to two dimensions leads to a set of equations invariant under the Kac-Moody affine symmetry. This symmetry extends the three-dimensional finite dimensional symmetry group in such a way that the affine Dynkin diagram reveals the subgroup $SL(n, \mathbb{R})$ by direct inspection (for instance the (Freudenthal-) subgroup $SL(9, \mathbb{R})$ of $E_8$). One can then check that $C_k$ cannot be oxidised beyond 4 dimensions. The point is that the affine vertex of the extended diagram should be the endpoint of this $SL(n, \mathbb{R})$ (see, for instance, [4]). Let us also point out that the toroidal dimensional reduction of any $D$-dimensional generally-covariant theory to $D = 3$ will automatically admit a $GL(D-3, \mathbb{R})$ rigid internal symmetry. Part of the magic here is that $SL(D - 2, \mathbb{R})$ invariance holds, as one can check in a case by
Having found the oxidation endpoints for the various three-dimensional scalar cosets, one can then look also at the global symmetry groups that result when these higher-dimensional theories are reduced to various intermediate dimensions $3 < D < D_{\text{max}}$. Each oxidation endpoint theory gives rise to a particular sequence of such intermediate theories, which we may label by the associated global symmetry group in $D = 3$. The most well-known example is thus the “$E_8$ sequence,” which comes from the dimensional reduction of $D_{\text{max}} = 11$ supergravity, giving $E_{11-D}$ in $D$ dimensions. Note that, unlike the situation discussed in [2, 3, 5], we shall principally be concerned with the case where the dimensional reduction is on a purely spacelike torus. The only difference, from the group-theoretic point of view, is that the denominator groups $H$ for the scalar cosets $G/H$ will be compact rather than non-compact.

2 Simply-laced scalar coset sequences

2.1 $A_n$ sequences

The maximally non-compact form of $A_n$ is $SL(n+1, \mathbb{R})$. As mentioned in the Introduction, the associated three-dimensional scalar cosets $SL(n+1, \mathbb{R})/O(n+1)$ can be obtained by the dimensional reduction of pure gravity in $D_{\text{max}} = n + 3$ dimensions [5].

If we perform the dimensional reduction to $D = 3$ of the pure Einstein-Hilbert Lagrangian in $D_{\text{max}} = n + 3$, we obtain

$$L_3 = R \ast 1 - \frac{1}{2} \ast d\vec{\phi} \wedge d\vec{\phi} - \frac{1}{2} \sum_i e^{\vec{b}_i \cdot \vec{\phi}} \ast F_{(2)}^i \wedge F_{(2)}^i - \frac{1}{2} \sum_{i<j} e^{\vec{b}_{ij} \cdot \vec{\phi}} \ast F_{(1)}^i \wedge F_{(1)}^j .$$

(2.1)

After dualising the potentials $A_{(1)}^i$ to give further axions $\chi_i$, one obtains the scalar Lagrangian

$$L_3 = R \ast 1 - \frac{1}{2} \ast d\vec{\phi} \wedge d\vec{\phi} - \frac{1}{2} \sum_i e^{\vec{b}_i \cdot \vec{\phi}} \ast G_{(1)i} \wedge G_{(1)i} - \frac{1}{2} \sum_{i<j} e^{\vec{b}_{ij} \cdot \vec{\phi}} \ast F_{(1)}^i \wedge F_{(1)}^j .$$

(2.2)

where $\ast F_{(2)}^i e^{\vec{b}_i \cdot \vec{\phi}} = G_{(1)i} \equiv \gamma_j^i d\chi_j$. It is now evident that we can extend the range of the $i$ index to $a = (0, i)$ (with $0 < i$ here), and define axions $\bar{A}_{(0)i}^a$ for all $a < b$:

$$\bar{A}_{(0)i}^0 = \chi_i , \quad \bar{A}_{(0)i}^i = A_{(0)i}^i .$$

(2.3)

(The bar over the potential indicates the extended set.) Defining also $\bar{\gamma}_{ab}^i$ as in Appendix A, but for the extended set of axionic potentials $\bar{A}_{(0)i}^a$, by $\gamma_{i}^0 = -\chi_i$ and $\gamma_{ij}^i = \gamma_{ij}^i$, we see...
that (2.2) assumes the form
\[ \mathcal{L} = R \ast 1 - \frac{1}{2} e \ast d\vec{\phi} \wedge d\vec{\phi} - \frac{1}{2} \sum_{a<b} e^{\bar{b}_{ab} \cdot \vec{\phi}} F_a^{(1)} \wedge F_b^{(1)}, \]
(2.4)

Thus the usual proof of the existence of the \( SL(n, \mathbb{R}) \) symmetry now establishes that we have an \( SL(n + 1, \mathbb{R}) \) symmetry in this three-dimensional case (see, for example, [3]).

The dilaton vectors \( \vec{b}_{ij} \) and \( -\vec{b}_i \) form the positive roots of \( SL(n + 1, \mathbb{R}) \), and it is easily seen that \( -\vec{b}_1 \) and \( \vec{b}_{i,i+1} \) (with \( 1 \leq i \leq n - 1 \)) are the simple roots. Thus we have the Dynkin diagram [3]

\[ -\vec{b}_1 \quad \vec{b}_{12} \quad \vec{b}_{23} \quad \ldots \quad \vec{b}_{n-2,n-1} \quad \vec{b}_{n-1,n} \]

Diagram 1: In \( D = 3 \), \( -\vec{b}_1 \) and \( \vec{b}_{i,i+1} \) generate the \( SL(n + 1, \mathbb{R}) \) Dynkin diagram. Vertices appear one by one from the right, starting in dimension \( 1 + n \), except for the left-most one, which is a disconnected \( \mathbb{R} \) from dimensions \( 2 + n \) down to 3, where it connects as an \( SL(2, \mathbb{R}) \) subgroup.

If the pure gravity theory in \( D_{\text{max}} = n + 3 \) dimensions is reduced to intermediate dimensions \( D = D_{\text{max}} - m > 3 \) on an \( m \)-torus, then the global symmetry group will be \( GL(m, \mathbb{R}) \).

2.2 \( D_n \) sequences

The maximally non-compact form of \( D_n \) is \( O(n, n) \). These groups are familiar as the T-duality symmetry groups of string theory, dimensionally reduced on a torus. For example, if one reduces the NS-NS sector of ten-dimensional type II string to \( D \geq 5 \) on \( T^{10-D} \), one obtains an \( O(10-D, 10-D) \times \mathbb{R} \) global symmetry. There are symmetry enhancements in \( D = 4 \) and \( D = 3 \), leading to \( O(6,6) \times SL(2, \mathbb{R}) \), and \( O(8,8) \) respectively [3, 4, 10]. More generally, we can consider the Lagrangian
\[ \mathcal{L}_D = R \ast 1 - \frac{1}{2} e^{a \varphi} \ast F^{(3)} \wedge F^{(3)} \] (2.5)
in an arbitrary dimension \( D_{\text{max}} \), where the constant \( a \) is given by \( a^2 = 8/(D_{\text{max}} - 2) \). Upon toroidal reduction to \( D = 3 \), we obtain
\[ \mathcal{L}_3 = R \ast 1 - \frac{1}{2} e^{\varphi} \ast d\varphi \wedge d\varphi - \frac{1}{2} \sum_i e^{\bar{b}_{ij} \cdot \vec{\phi}} F_i^{(2)} \wedge F_i^{(2)} - \frac{1}{2} \sum_{i<j} e^{\bar{b}_{ij} \cdot \vec{\phi}} F_{(1)i}^{(1)} \wedge F_{(1)j}^{(1)} \]
\[-\frac{1}{2} \sum_i e^{\vec{a}_i \cdot \vec{\phi}} * F_{(2)i} \wedge F_{(2)i} - \frac{1}{2} \sum_{i<j} e^{\vec{a}_{ij} \cdot \vec{\phi}} * F_{(1)ij} \wedge F_{(1)ij}. \tag{2.6}\]

Note that here \(\vec{\phi}\) denotes the set of dilatons \((\phi_1, \phi_2, \ldots, \phi_{D_{\text{max}}-3})\) introduced in Appendix A, augmented by \(\varphi\) as a zero'th component; \(\vec{\phi} = (\varphi, \phi_1, \phi_2, \ldots, \phi_{D_{\text{max}}-3})\). Similarly, the various dilaton vectors are those defined in Appendix A, augmented by a first component that is equal to the constant \(a\) in the case of \(\vec{a}_i\) and \(\vec{a}_{ij}\), and is equal to zero in the case of \(\vec{b}_i\) and \(\vec{b}_{ij}\).

After dualising the 1-form potentials \(A_{(1)i}\) and \(A_{(1)ij}\) to axions \(\chi_i\) and \(\psi^j\) respectively, the three-dimensional Lagrangian (2.6) can be written as the purely scalar Lagrangian

\[
\mathcal{L}_3 = R * 1 - \frac{1}{2} * d\vec{\phi} \wedge d\vec{\phi} - \frac{1}{2} \sum_i e^{-\vec{b}_i \cdot \vec{\phi}} * G_{(1)i} \wedge G_{(1)i} - \frac{1}{2} \sum_{i<j} e^{\vec{b}_{ij} \cdot \vec{\phi}} * F_{(1)ij} \wedge F_{(1)ij}
- \frac{1}{2} \sum_i e^{-\vec{a}_i \cdot \vec{\phi}} * G_{(1)i} \wedge G_{(1)i} - \frac{1}{2} \sum_{i<j} e^{\vec{a}_{ij} \cdot \vec{\phi}} * F_{(1)ij} \wedge F_{(1)ij}, \tag{2.7}\]

where the dualised field strengths are given by

\[G_{(1)i} = \gamma^{j\ i} (d\chi_j - A_{(0)kj} d\psi^k), \quad G_{(1)i} = \gamma^{j\ i} d\psi^j. \tag{2.8}\]

(See Appendix A for details of how the dualisations work.)

It is now straightforward to see that if we take \(D_{\text{max}} = n + 2\), then the three-dimensional Lagrangian (2.7) has a \(D_n\) global symmetry, where \((\vec{b}_{ij}, -\vec{b}_i, \vec{a}_{ij}, -\vec{a}_i)\) are its positive roots, with the simple roots being \(\vec{a}_{12}, \vec{b}_{i,j+1} (i \leq n - 1)\) and \(-\vec{a}_n\). Thus we have the Dynkin diagram

\[
\begin{array}{cccccccc}
\vec{b}_{12} & \vec{b}_{23} & \vec{b}_{34} & \vec{b}_{n-1,n} & -\vec{a}_n \\
0 & 0 & 0 & \cdots & 0 & 0
\end{array}
\]

|          |
| 0        |
| \vec{a}_{12} |

**Diagram 2:** The dilaton vectors \(\vec{a}_{12}, \vec{b}_{i,i+1}\) and \(-\vec{a}_n\) generate the \(D_n\) Dynkin diagram. The vertices appear one by one, starting from the extremities of the horizontal line which are both \(\mathbb{R}\) factors in dimension \(n + 1\). The \(SL(2, \mathbb{R})\) on the left appears in dimension \(n\), the short leg being still \(\mathbb{R}\), and the inner roots appear from the left. The diagram becomes irreducible in dimension 3.
In other words, we have shown that a three-dimensional scalar coset Lagrangian with
\( G/H = O(n, n)/(O(n) \times O(n)) \) can be dimensionally oxidised to the endpoint dimension
\( D_{\text{max}} = n + 2 \), where it is described by the Lagrangian (2.5).

The same Dynkin diagram can be used in order to determine the global symmetry
groups of the intermediate steps of oxidation of the three-dimensional Lagrangian (2.7),
by removing from the right those vertices that are associated with fields that exist only in
dimensions lower than the dimension under consideration. The vertex \(-\vec{a}_n\) itself must be
dhandled carefully, since it is associated with a dualised field. In four dimensions it should be
replaced by \(-\vec{a}\), whilst it corresponds in dimensions greater than four to the top dimension
(dilaton) scalar field. Consequently, the global symmetry group of the theory obtained by
oxidising the \( D_n \)-symmetric three-dimensional scalar Lagrangian to \( D \leq D_{\text{max}} = n + 2 \)
dimensions is \( D_m \times \mathbb{R} \), where \( m = D_{\text{max}} - D \), if \( D \geq 5 \), while it is \( D_{n-2} \times SL(2, \mathbb{R}) \)
if \( D = 4 \). (The additional \( \mathbb{R} \) factor in \( D \geq 5 \) is descended from the \( \mathbb{R} \) symmetry of the
\((n+2)\)-dimensional Lagrangian (2.5); similarly, the Kalb-Ramond field yields a full \( SL(2, \mathbb{R}) \)
subgroup connected to the left part of the diagram from dimension \( n - 1 \) downwards.) The
connection of the KR degrees of freedom to the second node on the left is due to the fact
that it is a second-rank antisymmetric tensor. We shall see shortly that a third-rank RR
form leads to the \( E_n \) series for a similar reason.

2.3 \( E_n \) sequences

2.3.1 \( E_8 \) sequence

This is the well-known sequence of \( D \)-dimensional scalar coset theories obtained by the
dimensional reduction of eleven-dimensional supergravity on the \((D - 11)\)-torus [11], whose
bosonic sector is given by

\[
\mathcal{L}_{11} = R^* \mathbf{1} - \frac{1}{2} F^{(4)} \wedge F^{(4)} - \frac{1}{6} dA^{(3)} \wedge dA^{(3)} \wedge A^{(3)}. \tag{2.9}
\]

After dualising all higher-degree fields where necessary, the scalar sector of the \( D \)-dimensional
theory has \( E_{11-D} \), in its maximally non-compact form, as its global symmetry group. A
discussion of these symmetries, in the language we are using in this paper, may be found in
[7]. The scalar fields in \( D \) dimensions are the dilatons \( \phi \), the axions \( A_{(0)}^{i} \)
coming from the metric and \( A_{(0)i j k} \) coming from the eleven-dimensional 3-form \( A^{(3)} \), together with further
axions in \( D \leq 5 \) coming from the dualisation of certain higher-degree gauge fields. The
simple-root vectors are \( \vec{b}_{i, i+1} \) and \( \vec{a}_{123} \) (associated with the dilaton vectors \( \vec{b}_{ij} \) and \( \vec{a}_{ijk} \)
for \( A_{(0)}^{i} \) and \( A_{(0)i j k} \) respectively). These give the Dynkin diagrams depicted in Diagram 3:
Diagram 3: The dilaton vectors $\vec{b}_{i,i+1}$ and $\vec{a}_{123}$ generate the $E_8$ Dynkin diagram. The short leg starts as $\mathbb{R}$ in $D = 10$, becomes $SL(2, \mathbb{R})$ in $D = 8$, and connects below that.

In a given dimension $D$, only those vertices with index values $i \leq 11 - D$ survive, giving rise to the Dynkin diagram for $E_{11-D}$. (We adopt the standard convention where $E_5 \sim D_5$, $E_4 \sim A_4$, $E_3 \sim A_2 \times A_1$, $E_2 \sim GL(2, \mathbb{R})$ and $E_1 \sim \mathbb{R}$.)

2.3.2 $E_7$ sequence of scalar cosets

To obtain this sequence, we consider a consistent (albeit non-supersymmetric) truncation of $D = 9$ maximal supergravity, to the theory whose bosonic sector comprises just the metric, a dilaton, a vector potential and a 3-form potential. The Lagrangian is

$$\mathcal{L}_9 = R * \mathbf{1} - \frac{1}{2} * d\varphi \wedge d\varphi - \frac{1}{2} e^{-\sqrt{7} \varphi} * F_{(4)} \wedge F_{(4)} - \frac{1}{2} e^{-\sqrt{7} \varphi} * F_{(2)} \wedge F_{(2)} - \frac{1}{2} dA_{(3)} \wedge dA_{(3)} \wedge A_{(1)} . \quad (2.10)$$

To be precise, the potential $A_{(1)}$, with field strength $F_{(2)} = dA_{(1)}$, is the potential $A_{(1)12}$ coming from $A_{(3)}$ after two steps of toroidal reduction, in the notation of [12, 7]. The dilaton $\varphi$ is the linear combination of the two dilatons of $D = 9$ maximal supergravity that is parallel to $\vec{a}$ (and hence also parallel to $\vec{a}_{12}$) in the notation of [12, 7]. (In other words, $\varphi = \frac{1}{7} \sqrt{7} \vec{a} \cdot \vec{\phi} = -\frac{1}{7} \sqrt{7} \vec{a}_{12} \cdot \vec{\phi}$.) However, having noted the $D = 11$ origin of the truncated theory, we shall now use the more convenient notation where we simply denote the vector potential by $A_{(1)}$. Further dimensional reductions will use a notation in the spirit of [12, 7], but where the first reduction step, with index $i = 1$, is from $D = 9$ to $D = 8$, and so on.

Note that in obtaining the Lagrangian (2.10) the largest possible number of fields has been consistently truncated out, while still retaining the 3-form potential $A_{(3)}$. In particular,\footnote{By a “consistent truncation,” we mean that all the equations of motion, including those of the fields that are set to zero, are satisfied.}
the FFA term in (2.10) makes it impossible to make a further truncation of the vector potential $A_{(1)}$ while keeping $A_{(3)}$. Another way of understanding the consistency of the truncation is by looking at maximal $D = 9$ supergravity from the point of view of type IIB supergravity. The truncation to (2.10) is then the one that results from keeping just the metric and the self-dual 5-form in $D = 10$. In fact, this ten-dimensional theory can also be viewed as a natural oxidation endpoint for the $E_7$ sequence of theories that we are considering in this subsection. The truncation can easily be seen to be consistent, because all the singlet fields of the $SL(2,\mathbb{R})$ global symmetry in $D = 9$ (or $D = 10$ type IIB) are retained. Note that $SL(8,\mathbb{R})$ is indeed a subgroup of $E_7$, and so the $SL(D - 2,\mathbb{R})$ rule mentioned in the Introduction is satisfied.

Performing the dimensional reduction, using a notation analogous to that in [12, 11], we obtain the $D$-dimensional Lagrangian

$$\mathcal{L}_D = R \mathbb{I} - \frac{1}{2} d\vec{\phi} \wedge d\vec{\phi} - e^{\vec{a}_{ij} \cdot \vec{\phi}} F_{(4)} \wedge F_{(4)} - \frac{1}{2} \sum_i e^{\vec{a}_{ij} \cdot \vec{\phi}} F_{(3)i} \wedge F_{(3)i}
- \frac{1}{2} \sum_{i<j} e^{\vec{a}_{ij} \cdot \vec{\phi}} F_{(2)ij} \wedge F_{(2)ij}
- \frac{1}{2} \sum_i e^{\vec{b}_{i} \cdot \vec{\phi}} F_{(1)i} \wedge F_{(1)i}
- \frac{1}{2} \sum_i e^{\vec{c}_{i} \cdot \vec{\phi}} F_{(2)i} \wedge F_{(2)i}
- \frac{1}{2} \sum_{i<j} e^{\vec{b}_{ij} \cdot \vec{\phi}} F_{(1)ij} \wedge F_{(1)ij}
+ \mathcal{L}_{FFA}. \quad (2.11)$$

Here, the notation for the dilatons is that $\vec{\phi} = (\varphi, \phi_1, \phi_2, \ldots, \phi_{9-D})$.

Let us now focus on the reduction to $D = 3$, and study the global symmetry group. There are 7 dilatons $\vec{\phi}$; $20 + 6 + 15 = 41$ axions $A_{(0)ijk}$, $A_{(0)i}$ and $A_{(0)ij}$ coming from $A_{(3)}$, $A_{(1)}$ and the metric; and $15 + 1 + 6 = 22$ vectors $A_{(1)ij}$, $A_{(1)}$ and $A_{(1)i}$ (coming from the same three sources). We can dualise the vectors to give further axions, whose dilaton vectors will be the negatives of those for the vector potentials, namely $-\vec{a}_{ij}$, $-\vec{c}$ and $-\vec{b}_i$. In total, we therefore now have 7 dilatons and 63 axions. This is precisely the dimension of the coset $E_7/SU(8)$. As usual, the global symmetry is best understood by noting that the dilaton vectors for the axions of the fully-dualised three-dimensional theory form the positive roots of $E_7$. This is easily seen by observing that the dilaton vectors $\vec{b}_{i,i+1}$ $(1 \leq i \leq 5)$, $\vec{a}_{123}$ and $\vec{c}_1$ can be taken as the simple roots, and all the other dilaton vectors can then be written as linear combinations of these with non-negative integer coefficients. From the defining properties of the dilaton vectors, we see that these simple roots give the $E_7$ Dynkin diagram:
Diagram 4: The dilaton vectors $\vec{b}_{i,i+1}, \vec{c}_1$ and $\vec{a}_{123}$ generate the $E_7$ Dynkin diagram.

The roots add from left to right as one descends through the dimensions. The short leg starts as $\mathbb{R}$ in dimension 9, and becomes $SL(2,\mathbb{R})$ in dimension 6.

The full system of $E_7$ positive roots is then generated by

$$
\vec{b}_{ij} = \vec{b}_{ik} + \vec{b}_{kj}, \quad \vec{c}_j = \vec{b}_{ij} + \vec{c}_i,$$
$$\vec{a}_{ijk} = \vec{b}_{i\ell} + \vec{a}_{ijk}, \text{ etc.} \quad -\vec{a}_{kj} = \vec{b}_{ik} + \vec{a}_{ij}, \text{ etc.}
$$
$$-\vec{a}_{12} = 2 \vec{b}_{12} + 3 \vec{b}_{23} + 3 \vec{b}_{34} + 2 \vec{b}_{45} + \vec{b}_{56} + \vec{a}_{123} + \vec{c}_1,$$
$$-\vec{c} = \vec{b}_{12} + 2 \vec{b}_{23} + 3 \vec{b}_{34} + 2 \vec{b}_{45} + \vec{b}_{56} + 2 \vec{a}_{123} . \quad (2.12)
$$

It is easy to verify that these summation rules imply that all the dilaton vectors for the axions can be expressed as linear combinations of the simple-root dilaton vectors, with non-negative integer coefficients. The first four lines are trivial, implying that the fields carrying $i, j, \ldots$ indices fall into representations under $SL(6,\mathbb{R})$, which originates from the general-coordinate transformations of the internal 6-torus. The last two lines give the non-trivial extension from $SL(6,\mathbb{R})$ to $E_7$.

In an intermediate dimension $3 < D < D_{\text{max}} = 10$, we can read off the global symmetry groups by removing the dots in the Dynkin diagram 3 whose indices lie outside the range 1 to $9 - D$. Thus the oxidation of the $E_7$-symmetric three-dimensional scalar Lagrangian to $D$ dimensions gives theories with the following global symmetry groups:
| Dim.   | $G$       | $H$     |
|--------|-----------|---------|
| $D_{\text{max}} = 10$ | —         | —       |
| $D = 9$ | $O(1,1)$  | —       |
| $D = 8$ | $GL(2, \mathbb{R})$ | $O(2)$  |
| $D = 7$ | $GL(3, \mathbb{R})$ | $O(3)$  |
| $D = 6$ | $SL(4, \mathbb{R}) \times SL(2, \mathbb{R})$ | $O(4) \times O(2)$ |
| $D = 5$ | $SL(6, \mathbb{R})$ | $O(6)$  |
| $D = 4$ | $O(6,6)$  | $O(6) \times O(6)$ |
| $D = 3$ | $E_7$     | $SU(8)$ |

Table 1: Scalar cosets for the $E_7$ sequence

Note that in $D \geq 7$, the global symmetry is simply the $GL(9-D, \mathbb{R})$ general coordinate symmetry of the internal $(9-D)$-torus. In $D \leq 6$, the global symmetry contains a part that lies outside the general coordinate transformations. It should be emphasised that in all the dimensions, the global symmetries given in Table 1 arise when all fields are dualised if this results in a reduction of their degrees. Again the short leg connects at the appropriate place for a (self-dual) 4-form potential.

2.3.3 $E_6$ sequence of scalar cosets

The oxidation endpoint of the $E_6$ sequence is $D = 8$, and again the associated theory is the smallest one obtainable as a consistent truncation of maximal supergravity in which the 3-form potential is retained. It comprises the metric, a dilaton and an axion, together with the 3-form potential. The eight-dimensional Lagrangian is given by

$$L_8 = R \ast \mathbb{1} - \frac{1}{2} d\bar{\varphi} \wedge d\varphi - \frac{1}{2} e^{2\varphi} \ast d\chi \wedge d\chi - \frac{1}{2} e^{-\varphi} \ast F_{(4)} \wedge F_{(4)} + \chi dA_{(3)} \wedge dA_{(3)},$$

(2.13)

where $F_{(4)} = dA_{(3)}$. The theory has a global $SL(2, \mathbb{R})$ symmetry at the level of its equations of motion, with $F_{(4)}$ and its dual $e^{-\varphi}F_{(4)}$ forming a doublet. The details of the Lagrangians in $D$ dimensions follow straightforwardly from the discussion in Appendix A.

As in the previous case, let us now focus on the reduction to $D = 3$, and study the global symmetry group. There are 6 dilatons $\bar{\varphi}$; $10 + 10 + 1 = 21$ axions $A_{(0)ijk}$, $A_{(0)ij}$ and $\chi$; and $10 + 5 = 15$ vectors $A_{(1)ij}$ and $A_{(1)i}$. We can dualise the vectors to give further axions, whose dilaton vectors will be the negatives of those for the vector potentials. In total, we therefore now have 6 dilatons and 36 axions, which is precisely the dimension of the coset $E_6/USp(8)$. 

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We now find that the dilaton vectors $\vec{b}_{i,i+1}$ \(1 \leq i \leq 4\), $\vec{a}_{123}$ and $\vec{c}$ can be taken as the simple roots, and all the other dilaton vectors can then be written as linear combinations of these with non-negative integer coefficients. Here, we are using $\vec{c}$ to denote the dilaton vector for the axion $\chi$. From the defining properties of the dilaton vectors, we see that these simple roots give the $E_6$ Dynkin diagram:

\[
\begin{array}{cccc}
\vec{b}_{12} & \vec{b}_{23} & \vec{b}_{34} & \vec{b}_{45} \\
\circ & \circ & \circ & \circ \\
\circ & \vec{a}_{123} & \circ & \circ \\
\circ & \circ & \vec{c} & \circ
\end{array}
\]

**Diagram 5:** The dilaton vectors $\vec{b}_{i,i+1}$, $\vec{c}$ and $\vec{a}_{123}$ generate the $E_6$ Dynkin diagram.

The roots add from left to right as one descends through the dimensions. In $D = 5$, a root combines with the $\mathbb{R}$ factor from 7 dimensions to form a second $SL(3, \mathbb{R})$ containing the top-dimension scalars; the group becomes simple below that.

We may again also read off the global symmetry groups in all intermediate dimensions $3 < D < D_{\text{max}} = 8$, by making the appropriate truncations of the Dynkin diagram. Thus we find for the $E_6$ sequence

| Dim. | $G$           | $H$           |
|------|---------------|---------------|
| $D_{\text{max}} = 8$ | $SL(2, \mathbb{R})$ | $O(2)$         |
| $D = 7$ | $GL(2, \mathbb{R})$ | $O(2)$         |
| $D = 6$ | $GL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ | $O(2) \times O(2)$ |
| $D = 5$ | $SL(3, \mathbb{R}) \times SL(3, \mathbb{R})$ | $O(3) \times O(3)$ |
| $D = 4$ | $SL(6, \mathbb{R})$ | $O(6)$         |
| $D = 3$ | $E_6$         | $USp(8)$      |

**Table 2:** Scalar cosets for the $E_7$ sequence

One notes again the Ehlers-type phenomenon of a build-up from a trivial $\mathbb{R}$ symmetry to give a simple duality group in lower dimensions. Here, it is $SL(3, \mathbb{R})$ in dimension 5.
2.3.4 $E_n$ ($n \leq 5$) sequences of scalar cosets

The above discussion can be extended to the set of cases where we begin in $D \leq 7$ with a maximal consistent truncation of the corresponding maximal supergravity, in which just the metric, a dilaton and a 3-form potential are retained. (In $D = 4$, the 3-form is non-dynamical, and will be set to zero.) In all these cases, the Lagrangian can be expressed as

$$\mathcal{L}_D = R \ast 1 - \frac{1}{2} \ast d \phi \wedge d \phi - \frac{1}{2} e^{a \phi} \ast F(4) \wedge F(4), \quad (2.14)$$

where $a^2 = \frac{2(11-D)}{D-2}$. After dimensional reduction to $D = 3$ and dualisation of the vectors, the scalar coset has the global symmetry $E_{D-2}$, where $D \leq 7$ is understood here to be the oxidation endpoint dimension $D_{\text{max}}$. To be more precise, we have

| $D = 3$ Scalar coset | Oxidation Endpoint |
|----------------------|---------------------|
| $O(5,5)/(O(5) \times O(5))$ | $D_{\text{max}} = 7$ |
| $SL(5, \mathbb{R})/O(5)$ | $D_{\text{max}} = 6$ |
| $SL(3, \mathbb{R}) \times SL(2, \mathbb{R})/(O(3) \times O(2))$ | $D_{\text{max}} = 5$ |
| $GL(2, \mathbb{R})/O(2)$ | $D_{\text{max}} = 4$ |
| $\mathbb{R}$ | $D_{\text{max}} = 3$ |

for the global symmetries of the three-dimensional theories coming from the reductions of (2.14) in the indicated endpoint dimensions.

These results can be understood very simply as follows. In the case $D_{\text{max}} = 7$, the 3-form potential can be dualised to a 2-form, and hence the Lagrangian is like the low-energy effective theory for a seven-dimensional bosonic string. This means that the global symmetry group will be at least the usual $O(n,n)$ T-duality group when the theory is compactified on an $n$-torus. When $n = 3$, the resulting four-dimensional theory will actually have an $O(3,3) \times SL(2, \mathbb{R})$ symmetry, where the extra $SL(2, \mathbb{R})$ factor is the electric/magnetic S-duality $[13]$. When $n = 4$, the three-dimensional theory has $O(5,5)$ rather than simply $O(4,4)$ as its global symmetry.

For the $E_4$ sequence, which starts in $D_{\text{max}} = 6$, the 3-form potential is dual to a vector, and in fact the dualised theory can be obtained by dimensional reduction from pure gravity in $D = 7$. Thus the global symmetry of the three-dimensional theory will be the same as that for pure gravity reduced from seven dimensions, namely $SL(5, \mathbb{R}) \sim E_4$. Again, this
is in accordance with the $SL(D - 2, \mathbb{R})$ rule mentioned in the Introduction.

For the $E_3$ sequence, starting in $D_{\text{max}} = 5$, the 3-form is dual to an axion, and hence the five-dimensional theory comprises the metric and a dilaton-axion system with an $SL(2, \mathbb{R})$ symmetry. Upon reduction to three dimensions, the gravity sector contributes an $SL(3, \mathbb{R})$ factor, giving $SL(3, \mathbb{R}) \times SL(2, \mathbb{R}) \sim E_3$ in total.

For the $E_2$ sequence, beginning in $D_{\text{max}} = 4$, we have just pure gravity plus a dilaton. The global symmetry of the reduced three-dimensional theory is therefore $GL(2, \mathbb{R}) \sim E_2$, where the $SL(2, \mathbb{R})$ factor is the usual one from the reduction of gravity to $D = 3$, and the extra $\mathbb{R}$ factor is from the shift symmetry of the four-dimensional dilaton.

Finally, for the $E_1 \sim \mathbb{R}$ sequence, we simply have gravity plus a scalar in $D = 3$, and so no oxidation is possible.

2.3.5 Summary

It is well known that the dimensional reduction of eleven-dimensional supergravity to $D = 3$ gives rise to a scalar coset theory with an $E_8$ global symmetry. In other words, the oxidation endpoint of the three-dimensional $E_8/SO(16)$ scalar coset is $D = 11$ supergravity. In this section we studied the oxidation endpoints of the three-dimensional scalar cosets whose global symmetry groups are the $E_n$ subgroups of $E_8$, with $2 \leq n \leq 8$. We showed that in the oxidation endpoint dimension for the subgroup $E_n$ is generically given by $D_{\text{max}} = n + 2$. In all cases, the endpoint theory includes the metric, a dilaton, and a 3-form potential, and in $D_{\text{max}} \leq 7$ there are no additional fields. In $D = 10$, corresponding to $E_8$, there is also a 2-form potential and a vector potential. In $D = 9$, corresponding to $E_7$, there is instead just an additional vector potential. In $D = 8$, there is instead an additional 0-form potential, or axion.

There are, as we saw, three special cases that arise. For $E_8$ the “endpoint” implied by the generic discussion, namely $D = 10$, can be further oxidised to the bosonic sector of $D = 11$ supergravity. For $E_7$, the generic discussion leads to an endpoint in $D = 9$, but again a further oxidation is possible, giving, in this case, the truncation of type IIB supergravity in $D = 10$ to the metric plus the self-dual 5-form. The third special case is $E_4$, for which the “endpoint” in $D = 6$ can be further oxidised to pure gravity in $D = 7$, after first dualising the 3-form potential to a vector in $D = 6$. These are the only three cases among the $E_n$ groups that contain $SL(n + 1, \mathbb{R})$ as subgroups, which explains why in each case a further oxidation by one step is possible. (A $D$-dimensional theory involving gravity, when compactified to three dimensions, is expected to exhibit an $SL(D - 2, \mathbb{R})$
Our results for the oxidation sequences for the various \( E_n \) cosets in \( D = 3 \) may be presented in the following table, which serves to make apparent a rather intriguing symmetry. The oxidation sequence for each of the three-dimensional \( E_n \) cosets is presented vertically, with \( n \) plotted horizontally (as usual, \( E_5 \sim D_5, E_4 \sim A_4, E_3 \sim A_1 \times A_2, E_2 \sim \mathbb{R} \times A_1, E_1 \sim \mathbb{R}, \) and \( E_0 \) is trivial):

| \( D = 11 \) | \( \mathbb{R} \times A_1 \) | \( \mathbb{R} \) |
| \( D = 10 \) | \( \mathbb{R} \) |
| \( D = 9 \) | \( \mathbb{R} \times A_2 \) | \( \mathbb{R} \times A_1 \) | \( A_1 \) |
| \( D = 8 \) | \( A_1 \times A_2 \) | \( \mathbb{R} \times A_2 \) |
| \( D = 7 \) | \( E_4 \) | \( \mathbb{R} \times A_2 \) | \( \mathbb{R} \times A_1 \) | \( \mathbb{R} \) | \(-\) |
| \( D = 6 \) | \( E_5 \) | \( A_1 \times A_3 \) | \( \mathbb{R} \times A_1^2 \) | \( \mathbb{R}^2 \) | \( \mathbb{R} \) |
| \( D = 5 \) | \( E_6 \) | \( A_5 \) | \( A_2^2 \) | \( \mathbb{R} \times A_1^2 \) | \( \mathbb{R} \times A_1 \) | \( A_1 \) |
| \( D = 4 \) | \( E_7 \) | \( D_6 \) | \( A_5 \) | \( A_1 \times A_3 \) | \( \mathbb{R} \times A_2 \) | \( \mathbb{R} \times A_1 \) | \( \mathbb{R} \) |
| \( D = 3 \) | \( E_8 \) | \( E_7 \) | \( E_6 \) | \( E_5 \) | \( E_4 \) | \( A_1 \times A_2 \) | \( \mathbb{R} \times A_1 \) | \( \mathbb{R} \) |

**Table 3:** Disintegration (i.e. Oxidation) for \( E_n \) Cosets

Note that in each sequence of theories represented by a vertical column in Table 3, the sequence of symmetry groups is obtained by successively deleting the vertices of the \( D = 3 \) Dynkin diagram as one ascends through the dimensions. In particular, in the step from \( D = 3 \) to \( D = 4 \), the vertex of \( E_n \) (with \( n \geq 4 \)) that is deleted is always the one that connects to the “extra” vertex of the extended Dynkin diagram for \( E_n \). Note also that an entry “−” indicates that the sequence has an oxidation endpoint with no global symmetry in that dimension. (We have not included the ten-dimensional ancestor of \( E_7 \), nor the seven-dimensional ancestor of \( E_4 \).)

Most striking is the “magic” reflection symmetry across the diagonal, which is much more complete than the approximate symmetry mentioned in [8] for the non-maximally non-compact duality groups of \( N = 4 \) pure supergravities. This might lead to a more systematic and deeper understanding of U-dualities, since we now know of three “magic tables.” As a comment we should point out that the fourth slot of the magic square of Tits-Rosenfeld-Freudenthal is absent; this does not seem to follow from a special choice of real forms but rather it suggests that a new geometry remains to be discovered here.
3 Non-simply-laced scalar coset sequences

In all the examples in the previous section, the global symmetry groups of the three-
dimensional scalar cosets were simply-laced. There are further examples in the classification of [2] for which the three-dimensional coset manifold has a non-simply-laced symmetry
group, namely $B_n$, $C_n$, $G_2$ and $F_4$. Here, we shall consider the cases where the group is
maximally non-compact.

3.1 $B_n$ sequences

The maximally non-compact form of $B_n$ is $O(n+1, n)$. The $D = 3$ scalar coset manifold for
this case can be obtained by starting in $D_{\text{max}} = n + 2$ dimensions with a theory comprising
the metric, a dilaton, a 2-form potential and a vector potential. The Lagrangian is given by
\[
\mathcal{L} = R \ast 1 - \frac{1}{2} e^{a\varphi} \ast F_{(3)} \wedge F_{(3)} - \frac{1}{2} e^{\frac{1}{2}a\varphi} \ast F_{(2)} \wedge F_{(2)} ,
\]
where $a^2 = 8/(D_{\text{max}} - 2)$, and
\[
F_{(3)} = dA_{(2)} + \frac{1}{2} A_{(1)} \wedge dA_{(1)} , \quad F_{(2)} = dA_{(1)} .
\]
(This is really a special case of the standard type of construction in string theory: If one
starts instead with $N$ vectors in $\tilde{D}$ dimensions, one gets an $O(\tilde{D} - D + N, \tilde{D} - 2)$ symmetry
in $D$ dimensions, enlarging to $O(\tilde{D} - 2 + N, \tilde{D} - 2)$ in $D = 3$ [10].)

Note that the special (stringy?) case of 1 vector added to a bosonic type I sector in
dimension 10 would possibly lead to an hyperbolic algebra of maximal rank 10, very much
like the conjectured $E_{10}$ symmetry or the overextended $D_8$ [8] of the type II and I theories.
For a classification of hyperbolic Coxeter groups see, for instance, [15].

An interesting special case arises when $n = 3$. The oxidation endpoint following the
above discussion would be in $D_{\text{max}} = 5$. In fact, this theory can be oxidised further, to
$D_{\text{max}} = 6$, to $N = 1$ supergravity, whose bosonic sector comprises the metric and a self-dual
3-form. The dimensional reduction of this theory to $D = 5$ and $D = 4$ was discussed in
detail in [14]. In particular, in $D = 4$, the global symmetry group is $O(2, 2)$. When it
is further reduced to $D = 3$, the global symmetry becomes $O(4, 3)$. We use the standard
notation, where we take $\vec{a}$ to denote the dilaton vectors for the fields coming from the
reduction of the self-dual 3-form, and $\vec{b}$ for those coming from the metric. In $D = 3$, there
are a total of 3 dilatons and 9 axions, after dualising the three Kaluza-Klein vectors. We
can verify that the 9 axion dilaton vectors form the full set of positive roots of $O(4, 3)$, with
$\vec{b}_{12}$, $\vec{b}_{23}$ and $\vec{a}_{12}$ being the simple roots:
Diagram 6: The dilaton vectors $\vec{b}_{12}, \vec{b}_{23}$ and $\vec{a}_{12}$ generate the $B_3$ Dynkin diagram, with the black dot indicating the shorter root. The 2-form potential leads to a branching off here of the shorter root.

The reason why the further oxidation to $D_{\text{max}} = 6$ is possible in this case is that the $O(4, 3)$ group contains $SL(4, \mathbb{R})$ as a subgroup. This $SL(4, \mathbb{R})$ is associated with the global symmetry already present in any theory involving gravity, indicating an oxidation endpoint in $D_{\text{max}} = 6$.

For higher $n$’s the growth of the linear subgroup of $B_n$ is away from the small root.

3.2 $C_n$ sequences

The maximally non-compact form of $C_n$ is $Sp(2n, \mathbb{R})$, and the corresponding three-dimensional scalar cosets are $Sp(2n, \mathbb{R})/U(n)$. These cosets are different from the others that we have discussed, in that their oxidation endpoints are all in four dimensions, and no further oxidations beyond $D = 4$ are possible. The endpoints in $D_{\text{max}} = 4$ have the global symmetry $Sp(2n - 2, \mathbb{R})$.

To see this, let $\vec{\sigma}_\alpha$ be the $(n - 1)$-component positive-root vectors of $Sp(2n - 2, \mathbb{R})$. These can be written in terms of a basis of $(n - 1)$ unit vectors $\vec{e}_i$ as

$$\vec{e}_i \pm \vec{e}_j, \quad i > j, \quad \text{and} \quad 2 \vec{e}_i,$$

where we are defining positivity in terms of the sign of the first non-vanishing component starting from the right. The simple roots are

$$\vec{e}_{i+1} - \vec{e}_i, \quad \text{and} \quad 2 \vec{e}_1.$$

The full set of positive roots for $Sp(2n, \mathbb{R})$ is then given by the $n$-component vectors

$$(\vec{\sigma}_\alpha, 0), \quad (\pm \vec{e}_i, 1), \quad (\vec{0}, 2).$$

These have a simple Kaluza-Klein interpretation as a reduction from $D = 4$ to $D = 3$. The root $(\vec{0}, 2)$ arises as the dilaton vector associated with the dual of the Kaluza-Klein
vector, while the roots \((\vec{\sigma}_\alpha, 0)\) originate from the dilaton vectors of the axions in an \(Sp(2n-2, \mathbb{R})/U(n-1)\) scalar coset in \(D = 4\), with \(\vec{\sigma}_\alpha\) as positive root vectors. The roots \((\vec{e}_i, 1)\) arise as the dilaton vectors for the direct dimensional reductions of a set of \((n-1)\) vector potentials \(A^i_{(1)}\), to give axions \(A^i_{(0)}\) in \(D = 3\). In addition, the vector potentials will also give rise to vector potentials in \(D = 3\), and after dualising these to axions, their associated dilaton vectors will be \((-\vec{e}_i, 1)\), accounting for the remaining root vectors in (3.5). Thus the \(Sp(2n, \mathbb{R})/U(n)\) sigma model in \(D = 3\) oxidises to give the Lagrangian

\[
\mathcal{L}_4 = -\frac{1}{2} \star \partial \vec{\phi} \wedge d \vec{\phi} - \frac{1}{2} \sum_\alpha e^{\vec{\phi} \cdot \vec{\sigma}_\alpha} G^\alpha_{(1)} \wedge G^\alpha_{(1)} - \frac{1}{2} \sum_{i=1}^{n-1} e^{\vec{e}_i \cdot \vec{\phi}} F^i_{(2)} \wedge F^i_{(2)} \tag{3.6}
\]

in \(D = 4\). Here, the \((n-1)\) dilatons \(\vec{\phi} = (\phi_1, \cdots, \phi_{n-1})\) are associated with the Cartan subalgebra of \(Sp(2n-2, \mathbb{R})\), and the \(\frac{1}{2} n(n-1)\) 1-form field strengths \(G^\alpha_{(1)}\) are formed from the axions associated with the positive roots of \(Sp(2n-2, \mathbb{R})\). The vector potentials \(A^i_{(1)}\) transform under the fundamental representation of \(Sp(2n-2, \mathbb{R})\).

It is easily seen that no further oxidation beyond \(D_{\text{max}} = 4\) is possible for this \(C_n\) sequence of three-dimensional scalar cosets. One way to see this is to note that the field strengths \(F^i_{(2)}\) in (3.6) have dilaton vectors \(\vec{e}_i\) that all have \((\text{length})^2 = 1\). If (3.6) were to come from the Kaluza-Klein reduction of a theory in \(D = 5\), then the Kaluza-Klein vector from this reduction step would have to be interpreted as one of the vector potentials in \(D = 4\). This is not possible, however, since the Kaluza-Klein vector would have a dilaton vector with \((\text{length})^2 = 3\), rather than \((\text{length})^2 = 1\). Thus \(D_{\text{max}} = 4\) is the oxidation endpoint for the \(C_n\) sequence of three-dimensional scalar cosets.

We may recall here that the affine extension of the group symmetry after reduction to 2 dimensions includes a manifest \(SL(D_{\text{max}} - 2, \mathbb{R})\) symmetry. A look at the Dynkin diagram of the affine \(C_n^1\) immediately shows that \(D_{\text{max}} = 4\), since the linear subgroup should start from one of the long root vertices.

### 3.3 \(G_2\) sequence

There is a \(G_2/(SU(2) \times SU(2))\) maximally non-compact scalar coset theory in \(D = 3\) [2]. We find that it can be oxidised back to an endpoint in \(D_{\text{max}} = 5\), where it becomes the bosonic sector of simple supergravity. This is the Einstein-Maxwell system in \(D = 5\), with an FFA term:

\[
\mathcal{L}_5 = R \star \mathbf{1} - \frac{1}{2} \ast F_{(2)} \wedge F_{(2)} + \frac{1}{3 \sqrt{3}} F_{(2)} \wedge F_{(2)} \wedge A_{(1)} \tag{3.7}
\]

Upon reduction to \(D = 3\), we obtain, in the standard notation, the Lagrangian

\[
\mathcal{L} = R \star \mathbf{1} - \frac{1}{2} d \vec{\phi} \wedge d \vec{\phi} - \frac{1}{2} e^{\vec{\phi}_2 - \sqrt{3} \phi_1} \star F^{(1)}_{(1)2} \wedge F^{(1)}_{(1)2} + \frac{1}{2} e^{\sqrt{3} \phi_1} \star F^{(1)}_{(1)1} \wedge F^{(1)}_{(1)1}
\]

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\[-\frac{1}{2}e^{\phi_2} - \frac{1}{\sqrt{3}} \phi_1 \ast F_{(1)2} \wedge F_{(1)2} - \frac{1}{2} e^{-\phi_2 - \sqrt{3}\phi_1} \ast F_{(2)1} \wedge F_{(2)1} \]
\[-\frac{1}{2}e^{-2\phi_2} \ast F_{(2)1} \wedge F_{(2)1} - \frac{1}{2} e^{-\phi_2 - \sqrt{3}\phi_1} \ast F_{(2)} \wedge F_{(2)} + \frac{2}{\sqrt{3}} dA_{(0)1} \wedge dA_{(0)2} \wedge A_{(1)} .\]  

(3.8)

After dualising the vector potentials to give axions, we see that there will be six axions, together with the two dilatons. We find that we may take the dilatons \( \vec{\alpha}_1 = (-\sqrt{3}, 1) \) and \( \vec{\alpha}_2 = (\frac{\sqrt{3}}{\sqrt{3}}, 0) \), corresponding to the axions \( A_{1(0)2} \) and \( A_{(0)1} \), as the simple roots of \( G_2 \), with the remaining dilaton vectors expressed in terms of these as
\[( -\frac{1}{\sqrt{3}}, 1) = \vec{\alpha}_1 + \vec{\alpha}_2 , \quad (\frac{1}{\sqrt{3}}, 1) = \vec{\alpha}_1 + 2\vec{\alpha}_2 , \quad (\sqrt{3}, 1) = \vec{\alpha}_1 + 3\vec{\alpha}_2 \),  
\[(0, 2) = 2\vec{\alpha}_1 + 3\vec{\alpha}_3 .\]  

(3.9)

It is easily verified that \( \vec{\alpha}_1 \) and \( \vec{\alpha}_2 \) generate the Dynkin diagram for \( G_2 \):
\[
\circ \equiv \equiv \equiv \bullet .
\]  

(3.10)

It is interesting to look also at the global symmetry of the oxidation of the \( G_2 \) coset to \( D = 4 \). From the \( G_2 \) Dynkin diagram, we expect that there should be an \( SL(2, \mathbb{R}) \) symmetry. The \( D = 4 \) Lagrangian is
\[
\mathcal{L}_4 \equiv R \ast \mathbf{1} + \frac{1}{2} \ast d\phi_1 \wedge d\phi_1 - \frac{1}{2} e^{\sqrt{3}\phi_1} \ast F_{(1)1} \wedge F_{(1)1} - \frac{1}{2} e^{-\sqrt{3}\phi_1} \ast F_{(2)1} \wedge F_{(2)1} - \frac{1}{2} e^{-\phi_2 - \sqrt{3}\phi_1} \ast F_{(2)} \wedge F_{(2)} + \frac{2}{\sqrt{3}} dA_{(0)1} \wedge dA_{(0)2} \wedge A_{(1)} .
\]  

(3.11)

It can be seen from this that the scalar manifold, comprising the dilaton \( \phi_1 \) and the axion \( A_{(0)1} \), indeed has an \( SL(2, \mathbb{R}) \) symmetry. It can also be seen that the 1-component dilaton vectors for the two vector potentials, together with their negatives, form the weights of the 4-dimensional representation of \( SL(2, \mathbb{R}) \).

A discussion of the dimensional reduction of simple supergravity in \( D = 5 \), and its global symmetry, was also given in [16].

3.4 \( F_4 \) sequence

We find that the oxidation endpoint of the \( F_4 \) sequence is a \( D_{\text{max}} = 6 \) dimensional theory, containing the metric, a dilaton, an axion, two vectors, a 2-form potential and a self-dual 3-form field strength. The Lagrangian is given by
\[
\mathcal{L}_6 \equiv R \ast \mathbf{1} - \frac{1}{2} \ast d\phi \wedge d\phi - \frac{1}{2} e^{\sqrt{2}\phi} \ast d\chi \wedge d\chi - \frac{1}{2} e^{-\sqrt{2}\phi} \ast F_{(3)1} \wedge F_{(3)1} - \frac{1}{2} e^{-\sqrt{2}\phi} \ast F_{(2)1} \wedge F_{(2)1} - \frac{1}{2} e^{-\sqrt{2}\phi} \ast F_{(2)} \wedge F_{(2)} - \frac{1}{2} e^{-\sqrt{2}\phi} \ast F_{(3)} \wedge F_{(3)} - \frac{1}{2} e^{-\sqrt{2}\phi} \ast F_{(2)1} \wedge F_{(3)} \wedge F_{(2)} - \frac{1}{2} e^{-\sqrt{2}\phi} \ast F_{(2)1} \wedge F_{(2)} .
\]  

(3.12)
The self-duality condition on the 3-form $G_{(3)}$ is to be imposed after having obtained the equations of motion. The field strengths are given in terms of potentials as follows:

$$F_{(3)} = dA_{(2)} + \frac{1}{2} A_{(1)}^- \wedge dA_{(1)}^-, \quad G_{(3)} = dB_{(2)} - \frac{1}{\sqrt{2}} \chi F_{(3)} - \frac{1}{2} A_{(1)}^+ \wedge dA_{(1)}^-, \quad F_{(2)}^+ = dA_{(1)}^+ + \frac{1}{\sqrt{2}} \chi dA_{(1)}^-,$$

$$F_{(2)}^- = dA_{(1)}^- + \frac{1}{\sqrt{2}} \chi dA_{(1)}^-, \quad F_{(2)} = dA_{(1)}^- . \quad (3.13)$$

It is easily verified that the self-duality constraint is consistent with the equations of motion and Bianchi identities. It can also be seen that the theory described by (3.12) has a global $SL(2, \mathbb{R})$ symmetry at the level of the equations of motion, with the two vectors $A_{(1)}^+$ and $A_{(1)}^-$ forming a doublet, and the 3-form fields $F_{(3)}$, $*F_{(3)}$ and $G_{(3)}$ forming the $(+, -, 2)$ components of a triplet. As we shall explain in section 3.5, the Lagrangian (3.12) can be obtained by starting with the eight-dimensional Lagrangian (2.13) which is the oxidation endpoint of the $E_6$ sequence, and reducing to $D = 6$. After performing the reduction, the six-dimensional Lagrangian has a $GL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ global symmetry. A consistent truncation can then be performed, under which a diagonal $SL(2, \mathbb{R})$ survives; this truncated theory is described by the Lagrangian (3.12) together with the self-duality constraint $G_{(3)} = *G_{(3)}$. This six-dimensional theory is the bosonic sector of a supersymmetric $N = 1$ chiral supergravity, with two anti-self-dual tensor multiplets and two vector multiplets. This is similar to the class of chiral six-dimensional supergravities constructed in [17]; see Appendix B for a detailed discussion.

In the standard notation, and associating dilaton vectors $\vec{a}$, $\vec{c}$, $\vec{c}^+$, $\vec{c}^-$, $\vec{b}$ and $\vec{h}$ with the fields $F_{(3)}$, $G_{(3)}$, $F_{(2)}^+$, $F_{(2)}^-$, $\mathcal{F}$ and $\chi$ respectively, we find, after reduction to $D = 3$ and dualisation of the vectors, that the dilaton vectors of all the 24 axions form the positive roots of the exceptional Lie algebra $F_4$. In particular, $\vec{b}_{23}$, $\vec{b}_{12}$, $\vec{c}_1^+$ and $\vec{h}$ are the positive simple roots, giving

$$\begin{align*}
\vec{b}_{23} & \quad \vec{b}_{12} & \quad \vec{c}_1^+ & \quad \vec{h} \\
\circ & \quad \circ & \quad \bullet & \quad \bullet
\end{align*}$$

**Diagram 7:** The dilaton vectors $\vec{b}_{23}$, $\vec{b}_{12}$, $\vec{c}_1^+$ and $\vec{h}$ generate the $F_4$ Dynkin diagram. The build-up of the symmetry is now towards the left end.

From this Dynkin diagram, it is straightforward to see that the oxidations of the three-dimensional $F_4$ theory to $D = 4$ and $D = 5$ will have global symmetries $Sp(6, \mathbb{R})$ and $SL(3, \mathbb{R})$ respectively.
3.5 Embedding in simply-laced cosets

The non-simply-laced algebras can all be interpreted as embeddings in larger simply-laced algebras. At the level of their Dynkin diagrams, these embeddings can be viewed as “folding over” the diagrams for the simply-laced diagrams, so that two or more vertices become identified. These are described in terms of Satake diagrams in [18]. These identifications in turn have an interpretation at the level of the coset Lagrangians, as consistent truncations of the Lagrangians for the simply-laced groups to give the Lagrangians for the non-simply-laced groups. We shall discuss the various cases below.

3.5.1 $B_{n-1}$ embedded in $D_n$

One can obtain the $B_{n-1}$ Dynkin diagram from the $D_n$ Dynkin diagram, given in Diagram 2, by folding over and identifying the vertices $\vec{b}_{12}$ and $\vec{a}_{12}$. This implies the existence of a consistent truncation of the three-dimensional Lagrangian (2.7) in which the axions $A^{(0)(1)}$ are equated to $A^{(0)(1)}_1$, and the axion $\chi_1$ is equated to $\psi_1$. In fact the easiest way to see this is to look at the Lagrangian in $D=n+1$, obtained by performing a single circle reduction of the oxidation endpoint Lagrangian (2.5) for the $D_n$ coset. In $D=n+1$, we obtain the Lagrangian

\[
\mathcal{L} = R \ast 1 - \frac{1}{2} \ast d\vec{\phi} \wedge d\vec{\phi} - \frac{1}{2} e^{\vec{a}_1 \cdot \vec{\phi}} \ast F^{(2)}(1) \wedge F^{(2)}(1) - \frac{1}{2} e^{\vec{b}_1 \cdot \vec{\phi}} \ast F^{(1)}(2) \wedge F^{(1)}(2) - \frac{1}{2} e^{\vec{a}_1 \cdot \vec{\phi}} \ast F^{(3)}(1) \wedge F^{(3)}(1).
\]

(3.14)

It is easily seen that the two 2-form field strengths $F^{(2)}(1)$ and $F^{(1)}(2)$ can be equated provided that the two dilatons in $\vec{\phi}$ are also truncated to a single dilaton combination, such that $(\vec{a}_1 - \vec{b}_1) \cdot \vec{\phi} = 0$. Defining

\[
a = \frac{1}{2} |\vec{a}_1 + \vec{b}_1|, \quad \frac{1}{2} (\vec{a}_1 + \vec{b}_1) \cdot \vec{\phi} = a \varphi, \quad F^{(2)} = \sqrt{2} F^{(2)}(1) = \sqrt{2} F^{(1)}(1),
\]

(3.15)

it follows that (3.14) can be consistently truncated to give (3.1), in $D = n + 1$. This is indeed the Lagrangian that describes the oxidation endpoint of the $B_{n-1}$ scalar coset in $D = 3$. It is easily seen that the identification of the two 1-form potentials in $D = n + 1$ implies precisely the set of identifications of axions given above equation (3.14).

3.5.2 $G_2$ embedded in $D_4$

The Dynkin diagram for $G_2$ can be obtained from that of $D_4$, by identifying the three vertices corresponding to the three mutually orthogonal simple roots. As in the $B_{n-1}$ example above, the associated consistent truncation can best be described in the endpoint
dimension of the non-simply-laced coset, namely \( D_{\text{max}} = 5 \) for this \( G_2 \) case. The endpoint dimension for the \( D_4 \) coset is \( D_{\text{max}} = 6 \), where there is the metric, a dilaton and a 2-form potential, as described by \((2.5)\). Upon reduction to \( D = 5 \), this theory can be consistently truncated to the bosonic sector of \( D = 5 \) simple supergravity, with Lagrangian given by \((3.7)\). This is achieved by equating the three 2-form field strengths \( F^{(2)}_1, F^{(2)}_{21} \) and \( \ast F^{(3)} \). It is easy to see that from the three-dimensional viewpoint, these identifications lead to identifications of the axions that imply precisely the identification of the three simple roots \( \vec{b}_{12}, \vec{b}_{34} \) and \( \vec{a}_{12} \) that form the “ears” of the \( D_4 \) Dynkin diagram in Diagram 2.

3.5.3 \( F_4 \) embedded in \( E_6 \)

This embedding corresponds to an identification of the pair of simple roots \( \vec{b}_{12} \) and \( \vec{c} \), and the pair of simple roots \( \vec{b}_{23} \) and \( \vec{a}_{123} \), of the \( E_6 \) algebra given in Diagram 5. The corresponding consistent truncation of the Lagrangian can be most conveniently performed in the oxidation endpoint dimension \( D_{\text{max}} = 6 \) of the three-dimensional \( F_4 \) scalar Lagrangian. On the other hand the endpoint of the \( E_6 \) sequence, as discussed in section 2.3.2, is in \( D_{\text{max}} = 8 \). Dimensionally reducing the Lagrangian \((2.13)\) to \( D = 6 \), we can make a consistent truncation to give the Lagrangian \((3.12)\), by equating the axions \( \chi \) and \( A^{(1)}_{(0)2} \), and the 2-form fields \( F^{(2)}_3 \) and \( F^{(2)}_{(0)12} \). In this truncation, the dual of the 4-form field strength \( F^{(4)} \) is set equal to \( F^{(1)}_{(2)} \), and the field \( F^{(3)}_{(3)2} \) is required to be self-dual. It is easy to see that the consequent identifications of fields in \( D = 3 \) imply precisely the identification of simple roots listed above, describing the embedding of \( F_4 \) in \( E_6 \).

3.5.4 \( C_n \) embedded in \( A_{2n-1} \)

This embedding is described by folding over the Dynkin diagram of \( A_{2n-1} \) at its midpoint, so that \((n-1)\) pairs of simple roots are identified. As usual, we make the corresponding consistent truncation in the Lagrangian in the oxidation endpoint dimension, namely \( D_{\text{max}} = 4 \) for this \( C_n \) sequence. The oxidation endpoint for the \( A_{2n-1} \) sequence is pure gravity in \( D_{\text{max}} = 2n + 2 \). Dimensionally reducing this to \( D = 4 \), we can make a consistent truncation given by

\[
F^{i}_{(2)} = \ast F^{2n-1-i}_{(2)}, \quad F^{i}_{(1)j} = F^{2n-1-j}_{(1)2n-1-i}.
\]

This truncation gives precisely the Lagrangian \((3.6)\). The dilaton vectors \( \vec{e}_i \) in \((3.6)\) are given by \( \vec{e}_i = \vec{b}_i - \vec{b}_{2n-1-i} \), and they are indeed orthonormal. Reducing one step further to \( D = 3 \), we can see that the consequent identifications of axions imply the necessary identification rules for obtaining the Dynkin diagram of \( C_n \) from that of \( A_{2n-1} \).
4 Non-maximally non-compact cosets

The discussion of three-dimensional $G/H$ scalar coset theories where $G$ is not maximally non-compact becomes somewhat involved, and we shall not attempt a complete classification of their oxidation endpoints here. The principal examples, from our point of view, are the sets of cosets $O(p,q)/(O(p) \times O(q))$, $SU(p,q)/S(U(p) \times U(q))$ and $SO^*(2n)/U(n)$. The first of these sets is easily understood from a higher-dimensional standpoint. It is well known that a ten-dimensional $N = 1$ string with $m$ vector potentials gives rise upon toroidal dimensional reduction to a theory with $O(10+m-D, 10-D)$ global symmetry in $5 \leq D \leq 10$ dimensions, $O(6+m,6) \times SL(2, \mathbb{R})$ in $D = 4$, and $O(8+m,8)$ in $D = 3$. This generalises straightforwardly to other starting dimensions. Thus if we suppose, without loss of generality, that $p > q$, we may oxidise the three-dimensional $O(q+m,q)/(O(q+m) \times O(q))$ scalar coset to $D_{\text{max}} = q+2$ dimensions, where it corresponds to a theory similar to that of an $N = 1$ string, comprising the metric, a dilaton, a 2-form potential and $m$ 1-form potentials $A^i_{(1)}$, with the Lagrangian

$$
\mathcal{L} = R \star 1 - \frac{1}{2} d\varphi \wedge d\varphi - \frac{1}{2} e^{a\varphi} \star F_{(3)} \wedge F_{(3)} - \frac{1}{2} e^{\frac{1}{2} a\varphi} \sum_i \star F^i_{(2)} \wedge F^i_{(2)},
$$

(4.1)

where $a^2 = 8/(D_{\text{max}} - 2)$, and

$$
F_{(3)} = dA_{(2)} + \frac{1}{2} A^i_{(1)} \wedge dA^i_{(1)}, \quad F^i_{(2)} = dA^i_{(1)}.
$$

(4.2)

Other non-maximally non-compact cosets, including some further isolated examples, are discussed in [2], where their oxidations to $D = 4$ are given. It may be that these cases, like the $C_n$ sequences discussed in section 3.2, cannot be further oxidised beyond four dimensions, but we have not investigated this in detail. One result is that all cosets that are Hermitean symmetric spaces have oxidation endpoints in $D = 4$. Another observation is that all quaternionic spaces technically homogeneous under the Borel subgroup (also called Alekseevskian spaces) have maximal oxidation dimension 6. For a discussion of the relation of these spaces to supergravities, see for instance, [19].

One example among those conjectured in [4, 8] was constructed recently as an anomalous but classically supersymmetric theory, in [20].

5 Summary and conclusions

We have examined $G/H$ scalar coset Lagrangians in three dimensions, for all the cases where $G$ is a simple maximally-non-compact group, and $H$ is its maximal compact subgroup. In all cases, the three-dimensional scalar Lagrangian can be re-interpreted as coming from
the toroidal dimensional reduction of some higher-dimensional theory comprising gravity, possibly with additional antisymmetric tensor fields and dilatons. Thus we have introduced the notion of the “oxidation endpoint” of a three-dimensional scalar coset Lagrangian, by which we mean the theory in the highest possible dimension whose toroidal dimensional reduction yields (after dualisation of all vector potentials to axions) precisely the originally-given three-dimensional theory. In many cases the fields of the three-dimensional theory can be produced rather “economically” from the higher dimension, in that many three-dimensional fields are generated from each of a few higher-dimensional ones. By contrast there are other cases, most notably the $Sp(2n, \mathbb{R})/U(n)$ three-dimensional scalar cosets, where less advantage is gained by oxidation. For example, the $Sp(2n, \mathbb{R})/U(n)$ theories can be oxidised only as far as $D = 4$.

We may summarise the oxidation endpoint dimensions for all the simple maximally-non-compact three-dimensional $G/H$ cosets in the following table. Complete details, together with any particular discussion for special cases, may be found in the earlier sections. (Note that for the $Sp(2n, \mathbb{R})/U(n)$ cosets the field contents of the $D_{\text{max}} = 4$ theory includes $(n-1)$ dilatons $\vec{\phi}$, $\frac{1}{2}n(n-1)$ axions $\chi^\alpha$, and $(n-1)$ 1-forms $A_{(1)}^i$, as described in section 3.2.)

| $G$         | $H$                     | $D_{\text{max}}$ | Fields                |
|-------------|-------------------------|-------------------|-----------------------|
| $SL(n+1, \mathbb{R})$ | $O(n+1)$                | $n + 3$           | $g_{\mu\nu}$          |
| $O(n, n)$   | $O(n) \times O(n)$      | $n + 2$           | $g_{\mu\nu}, \varphi, A_{(2)}$ |
| $E_8$       | $O(16)$                 | 11                | $g_{\mu\nu}, A_{(3)}$ |
| $E_7$       | $SU(8)$                 | 10                | $g_{\mu\nu}, B_{(4)}(\text{self-dual})$ |
| $E_6$       | $USp(8)$                | 8                 | $g_{\mu\nu}, \varphi, \chi, A_{(3)}$ |
| $O(n+1, n)$ | $O(n+1) \times O(n)$    | $n + 2$           | $g_{\mu\nu}, \varphi, A_{(1)}, A_{(2)}$ |
| $Sp(2n, \mathbb{R})$ | $U(n)$                  | 4                 | $g_{\mu\nu}, \vec{\phi}, \chi^\alpha, A_{(1)}^i$ |
| $G_2$       | $SU(2) \times SU(2)$    | 5                 | $g_{\mu\nu}, A_{(1)}$ |
| $F_4$       | $USp(6) \times SU(2)$   | 6                 | $g_{\mu\nu}, \phi, \chi, A_{(1)}^+, \vec{A}_{(1)}^-, A_{(2)}, B_{(2)}(\text{self-dual})$ |

Table 4: Oxidation endpoints for maximally-non-compact cosets

A compact way to summarise the build-up of hidden dimensions is that $E_8, E_7, G_2, B_3$ and $A_n$ can be lifted to a dimension equal to their rank plus three, $E_6, F_4, D_n$ and $B_n$ can be lifted to their rank plus 2 dimensions, and finally $C_n$ can be lifted to dimension 4. The oxidation endpoints of the $A_1, G_2, B_3, F_4, E_8, D_8$ and $B_8$ scalar cosets are theories
that admit supersymmetric extensions, while the endpoints for all other cosets are not the bosonic sectors of any supersymmetric theory.

**Appendices**

**A Dimensional Reduction**

Here, we give the general formulae for the toroidal reduction of the Einstein-Hilbert Lagrangian in $D$ dimensions, coupled to a single $p$-form field strength. We consider a reduction on an $n$-torus. The $D$-dimensional Lagrangian is

$$\mathcal{L}_D = R \ast 1 - \frac{1}{2} \ast F_{(p)} \wedge F_{(p)}. \quad (A.1)$$

The metric will be reduced using the standard Kaluza-Klein ansatz which, in the notation of [3, 7], is

$$ds_D^2 = e^{\tilde{s} \cdot \phi} ds_{D-n}^2 + \sum_{i=1}^{n} e^{2\hat{\gamma}_i \cdot \phi} (h^i)^2, \quad (A.2)$$

where

$$h^i = dz^i + A^i_{(0)j} dz^j + A^i_{(1)} (dz^j + \hat{A}^i_{(1)j}), \quad (A.3)$$

and

$$\hat{\gamma}^i_j = \delta^i_j + A^i_{(0)j}.$$  

We define also $\gamma^i_j = (\hat{\gamma}^{-1})^i_j$, as in [7].

The constant vectors $\vec{s}$ and $\vec{\gamma}_i$ are given by

$$\vec{s} = (s_1, s_2, \ldots, s_n), \quad \vec{\gamma}_i = \frac{1}{2} \vec{s} - \frac{1}{2} \vec{f}_i, \quad (A.4)$$

where

$$s_i = \sqrt{\frac{2}{(D-1-i)(D-2-i)}}, \quad \vec{f}_i = \left(0, 0, \ldots, 0, (\tilde{D} - 1 - i)s_i, s_{i+1}, s_{i+2}, \ldots, s_n \right), \quad (A.5)$$

The potential $A_{(p-1)}$ is reduced according to the standard procedure

$$A_{(p-1)} \rightarrow A_{(p-1)} + A_{(p-2)i} dz^i + \frac{1}{2} A_{(p-1)ij} dz^i \wedge dz^j \cdots. \quad (A.6)$$

After reduction on the $n$-torus, the Lagrangian in $(D - n)$ dimensions is given by

$$\mathcal{L} = R \ast 1 - \frac{1}{2} \ast d\tilde{\phi} \wedge d\phi - \frac{1}{2e^{\tilde{s} \cdot \tilde{\phi}}} \ast F_{(p)} \wedge F_{(p)} - \frac{1}{2} \sum_i e^{\tilde{a}_i \cdot \phi} \ast F_{(p-1)i} \wedge F_{(p-1)i}$$

$$- \frac{1}{2} \sum_{i < j} e^{\tilde{a}_{ij} \cdot \phi} \ast F_{(p-2)ij} \wedge F_{(p-2)ij} - \cdots$$

$$- \frac{1}{2} \sum_{i_1 < i_2 < \cdots < i_{p-1}} e^{\tilde{b}_{i_1 i_2 \cdots i_{p-1}} \cdot \phi} \ast F_{(1) i_1 i_2 \cdots i_{p-1}} \wedge F_{(1) i_1 i_2 \cdots i_{p-1}}$$

$$- \frac{1}{2} \sum_i e^{\tilde{b}_i \cdot \phi} \ast F_{(1)i} \wedge F_{(1)i} - \frac{1}{2} \sum_{i < j} e^{\tilde{b}_{ij} \cdot \phi} \ast F_{(1)ij} \wedge F_{(1)ij}. \quad (A.7)$$

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The dilaton vectors are given by
\[ \vec{a} = -(p-1) \vec{s}, \quad \vec{a}_i = \vec{f}_i - (p-1) \vec{s}, \quad \vec{a}_{ij} = \vec{f}_i + \vec{f}_j - (p-1) \vec{s}, \ldots \]
\[ \vec{a}_{i_1 \cdots i_{p-1}} = \vec{f}_{i_1} + \vec{f}_{i_2} + \cdots + \vec{f}_{i_{p-1}} - (p-1) \vec{s}, \]
\[ \vec{b}_i = -\vec{f}_i, \quad \vec{b}_{ij} = -\vec{f}_i + \vec{f}_j. \]  
(A.8)

The field strengths are given by
\[ F_{(q)i_1i_2 \cdots i_{p-q}} = \gamma^{i_1}_{i_1} \gamma^{i_2}_{i_2} \cdots \gamma^{i_{p-q}}_{i_{p-q}} \vec{f}_{(q)i_1i_2 \cdots i_{p-q}}, \quad \vec{F}^i = \gamma^i_j \vec{F}^j, \]  
(A.9)

with
\[ \vec{F}_{(p)} = dA^{(p-1)} - dA^{(p-2)i} \hat{A}^i_{(1)} + \frac{1}{2} dA^{(p-3)ij} \hat{A}^i_{(1)} \hat{A}^j_{(1)} - \frac{1}{6} dA^{(p-4)ijk} \hat{A}^i_{(1)} \hat{A}^j_{(1)} \hat{A}^k_{(1)} + \cdots, \]
\[ F_{(p-1)i} = dA^{(p-2)} + dA^{(p-3)ij} \hat{A}^j_{(1)} + \frac{1}{2} dA^{(p-4)ijk} \hat{A}^j_{(1)} \hat{A}^k_{(1)} + \frac{1}{6} dA^{(p-5)ijkt} \hat{A}^j_{(1)} \hat{A}^k_{(1)} \hat{A}^\ell_{(1)} + \cdots, \]
\[ F_{(1)i_1 \cdots i_{p-1}} = dA^{(0)i_1 \cdots i_{p-1}}, \quad \vec{F}^i_{(1)} = \gamma^i_j F^j_{(1)}, \]  
(A.10)

If we take the case where \( D = n+3 \), so that the reduction goes down to three dimensions, the Lagrangian will be simply
\[ \mathcal{L}_3 = R \ast 1 - \frac{1}{2} d\vec{\phi} \wedge d\vec{\phi} - \frac{1}{2} \sum_i e^{\vec{b}_i \cdot \vec{\phi}} * \vec{F}^i_{(2)} \wedge \vec{F}^i_{(2)} \]
\[ -\frac{1}{2} \sum_{i < j} e^{\vec{b}_{ij} \cdot \vec{\phi}} * F^{(1)ij} \wedge F^{(1)ij} - \frac{1}{2} \sum_{i_1 < \cdots < i_{p-2}} e^{\vec{a}_{i_1 \cdots i_{p-2}} \cdot \vec{\phi}} * F^{(2)i_1 \cdots i_{p-2}} \wedge F^{(2)i_1 \cdots i_{p-2}} \]
\[ -\frac{1}{2} \sum_{i_1 < \cdots < i_{p-1}} e^{\vec{a}_{i_1 \cdots i_{p-1}} \cdot \vec{\phi}} * F^{(1)i_1 \cdots i_{p-1}} \wedge F^{(2)i_1 \cdots i_{p-1}}. \]  
(A.11)

This is obtained from (A.7) by dropping all field strengths associated with forms of degree higher than 2. We may then follow the standard procedure for dualising the 1-form potentials \( A^i_{(1)} \) and \( A^{i_1 \cdots i_{p-2}}_{(1)} \) to axionic scalars \( \chi_i \) and \( \psi^{i_1 \cdots i_{p-2}} \), by introducing the axions as Lagrange multipliers for the Bianchi identities for the 2-form field strengths (see, for example, [2]). Upon doing so, we arrive at the purely scalar three-dimensional Lagrangian
\[ \mathcal{L}_3 = R \ast 1 - \frac{1}{2} d\vec{\phi} \wedge d\vec{\phi} - \frac{1}{2} \sum_i e^{-\vec{b}_i \cdot \vec{\phi}} * G^{(1)i} \wedge G^{(1)i} \]
\[ -\frac{1}{2} \sum_{i < j} e^{\vec{b}_{ij} \cdot \vec{\phi}} * F^{(1)ij} \wedge F^{(1)ij} - \frac{1}{2} \sum_{i_1 < \cdots < i_{p-2}} e^{-\vec{a}_{i_1 \cdots i_{p-2}} \cdot \vec{\phi}} * G^{(1)i_1 \cdots i_{p-2}} \wedge G^{(1)i_1 \cdots i_{p-2}} \]
\[ -\frac{1}{2} \sum_{i_1 < \cdots < i_{p-1}} e^{-\vec{a}_{i_1 \cdots i_{p-1}} \cdot \vec{\phi}} * F^{(1)i_1 \cdots i_{p-1}} \wedge F^{(2)i_1 \cdots i_{p-1}}. \]  
(A.12)
where

\[ F_{i_1 \cdots i_p}^{(1)} = \gamma_{i_1}^{j_1} \cdots \gamma_{i_{p-1}}^{j_{p-1}} dA_{(0)j_1 \cdots j_{p-1}}, \]
\[ G_{i_1 \cdots i_{p-2}}^{(1)} = \tilde{\gamma}^{j_1}_{i_1} \cdots \tilde{\gamma}^{j_{p-1}}_{i_{p-1}} d\psi_{j_1(0) \cdots j_{p-1}}, \]
\[ G_{(1)i} = \gamma^j_i (d\chi_j - \sum_{k_1 < \cdots < k_{p-2}} A_{(0)k_1 \cdots k_{p-2}j} d\psi_{k_1 \cdots k_{p-2}}), \]
\[ F_{(1)ij} = \gamma^k_{ij} dA_{(0)k}^{(i)}. \] (A.13)

**B The \( F_4 \) oxidation endpoint and chiral \( D = 6 \) supergravity**

In general, the theories corresponding to the oxidation endpoints of the various three-dimensional sigma-model Lagrangians that we have considered in this paper are rather straightforward to analyse. One case is slightly more involved, namely the six-dimensional oxidation endpoint of the \( F_4 \) scalar coset, discussed in section 3.4. Here, we present the details of the proof that it corresponds to the bosonic sector of an \( N = 1 \) chiral supergravity in \( D = 6 \), with two anti-self-dual tensor multiplets, and two abelian vector multiplets.

From (3.12), we see that \( G_{(3)} \) satisfies \( d^* G_{(3)} = -\frac{1}{\sqrt{2}} d(\chi F_{(3)}) - \frac{1}{2} dA_{(1)}^{+} \wedge dA_{(1)}^{-} \), while from (3.13), we see that the Bianchi identity for \( dG_{(3)} \) gives an identical expression. This shows that we can indeed consistently impose the self-duality constraint \( G_{(3)} = *G_{(3)} \) after having obtained the equations of motion.

The equation of motion for \( F_{(3)} \) is

\[ -d(e^{-\sqrt{2} \phi} *F_{(3)}) + \frac{7}{2} d(\chi^2 F_{(3)}) + \sqrt{2} d(\chi G_{(3)}) - \frac{1}{2} d(A_{(1)}^{+} F_{(3)}^{+}) + \frac{1}{2\sqrt{2}} d(\chi A_{(1)}^{+} F_{(2)}^{-}) = 0, \] (B.1)

which can be integrated to give

\[ *F_{(3)} = e^{\sqrt{2} \phi} (dC_{(2)} + \sqrt{2} \chi G_{(3)} + \frac{1}{2} \chi^2 F_{(3)} - \frac{1}{2} A_{(1)}^{+} F_{(2)}^{+} + \frac{1}{2\sqrt{2}} \chi A_{(1)}^{+} F_{(2)}^{-}), \] (B.2)

where we have introduced the third 2-form potential \( C_{(2)} \). Thus, we have

\[ F_{(3)} = F_{(3)}^{-}, \]
\[ G_{(3)} = F_{(3)}^{-} - \frac{1}{\sqrt{2}} \chi F_{(3)}^{-}, \]
\[ *F_{(3)} = e^{\sqrt{2} \phi} (F_{(3)}^{+} + \sqrt{2} \chi F_{(3)}^{2} - \frac{1}{2} \chi^2 F_{(3)}^{-}), \] (B.3)

where we have defined

\[ F_{(3)}^{-} \equiv dA_{(2)} + \frac{1}{2} A_{(1)}^{+} dA_{(1)}^{-}, \quad F_{(3)}^{2} \equiv dB_{(2)} - \frac{1}{2} A_{(1)}^{+} dA_{(1)}^{-}, \quad F_{(3)}^{+} \equiv dC_{(2)} - \frac{1}{2} A_{(1)}^{+} dA_{(1)}^{+}. \] (B.4)

We wish to make a comparison with the chiral \( N = 1 \) six-dimensional supergravity described in [17]. In [17], there is 1 self-dual 3-form \( H_{(3)} \), and \( n \) anti-self-dual 3-forms \( K_{(3)}^m \).
We have the opposite convention, so we have 1 anti-self-dual \(H_{(3)}\), and, in our case, \(n = 2\) self-dual 3-forms \(K_{(3)}^{m}\), with \(m = 1, 2\). We may relate these fields and our fields \(G_{(3)}\) and \(F_{(3)}\), as follows:

\[
\begin{align*}
H_{(3)} &= \frac{1}{2} e^{-\frac{\sqrt{2}}{4} \phi} (F_{(3)} - *F_{(3)}), \\
K_{(3)}^1 &= \frac{1}{2} e^{-\frac{\sqrt{2}}{4} \phi} (F_{(3)} + *F_{(3)}), \\
K_{(3)}^2 &= G_{(3)}.
\end{align*}
\]

We can now express these in terms of our three field strengths defined in (B.4), by making use of the results in (B.3). In fact it is advantageous at this point to define

\[
F_{(3)}^\pm = F_{(3)}^1 + F_{(3)}^0, \quad F_{(3)}^0 = F_{(3)}^1 - F_{(3)}^0, \quad F_{(3)}^2.
\]

The fields \(F_{(3)}^0\), \(F_{(3)}^1\) and \(F_{(3)}^2\) will now correspond to the \(F_{(3)}^r\) \((r = 0, 1, 2)\) fields of [17]. Thus we have

\[
\begin{align*}
H_{(3)} &= \cos \frac{\phi}{\sqrt{2}} F_{(3)}^0 - \sinh \frac{\phi}{\sqrt{2}} F_{(3)}^1 - \frac{1}{\sqrt{2}} \chi e^{\sqrt{2} \phi} F_{(3)}^2 + \frac{1}{4} e^{\sqrt{2} \phi} \chi^2 (F_{(3)}^0 + F_{(3)}^1), \\
K_{(3)}^1 &= \cos \frac{\phi}{\sqrt{2}} F_{(3)}^1 - \sinh \frac{\phi}{\sqrt{2}} F_{(3)}^0 + \frac{1}{\sqrt{2}} \chi e^{\sqrt{2} \phi} F_{(3)}^2 - \frac{1}{4} e^{\sqrt{2} \phi} \chi^2 (F_{(3)}^0 + F_{(3)}^1), \\
K_{(3)}^2 &= F_{(3)}^0 - \frac{1}{\sqrt{2}} \chi (F_{(3)}^1 + F_{(3)}^0).
\end{align*}
\]

This can be directly compared with equations (3.5) in [17], namely

\[
H_{(3)} = v^r F_{(3)}^r, \quad K_{(3)}^m = x_i^m F_{(3)}^i,
\]

allowing us to read off the quantities \(v^r\) and \(x_i^m\). We find that the matrix \(V\) defined in [17], \(i.e.\)

\[
V = \begin{pmatrix}
v_0 & v_1 & v_2 \\
x_0^1 & x_1^1 & x_2^1 \\
x_0^2 & x_1^2 & x_2^2
\end{pmatrix},
\]

is given by

\[
V = \begin{pmatrix}
\cos \frac{\phi}{\sqrt{2}} + \frac{1}{4} e^{\sqrt{2} \phi} \chi^2 & -\sinh \frac{\phi}{\sqrt{2}} + \frac{1}{4} e^{\sqrt{2} \phi} \chi^2 & -\frac{1}{\sqrt{2}} e^{\sqrt{2} \phi} \chi \\
-\sinh \frac{\phi}{\sqrt{2}} - \frac{1}{4} e^{\sqrt{2} \phi} \chi^2 & \cos \frac{\phi}{\sqrt{2}} - \frac{1}{4} e^{\sqrt{2} \phi} \chi^2 & \frac{1}{\sqrt{2}} e^{\sqrt{2} \phi} \chi \\
-\frac{1}{\sqrt{2}} \chi & -\frac{1}{\sqrt{2}} \chi & 1
\end{pmatrix}.
\]

It is easily verified that \(V\) is an \(SO(1, 2)\) matrix, satisfying \(V^T \eta V = \eta\), where \(\eta = \text{diag}(1, -1, -1)\). It can also be seen that

\[
dV V^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix}
0 & -d\phi & -\frac{1}{\sqrt{2}} e^{\sqrt{2} \phi} \chi \\
-d\phi & 0 & \frac{1}{\sqrt{2}} e^{\sqrt{2} \phi} \chi \\
-\frac{1}{\sqrt{2}} e^{\sqrt{2} \phi} \chi & -\frac{1}{\sqrt{2}} e^{\sqrt{2} \phi} \chi & 0
\end{pmatrix}.
\]
In [17], the field strengths \( F^r_{(3)} \) are defined to be

\[ F^r_{(3)} = dA^r_{(2)} + c^r_z \omega^z, \] (B.12)

summed over the various factors in the Yang-Mills gauge group, labelled by \( z \), where \( \omega^z \) are the Chern-Simons forms for each factor and \( c^r_z \) are constants. In our case, the situation will be slightly different: we have two \( U(1) \) factors, with 1-form potentials that we may call \( A^+_1 \) and \( A^-_1 \); they form a spinor of \( SO(1, 2) \). We shall need a third vector combining the previous two, namely

\[ A_1 = A^-_1 - A^+_1. \] (B.13)

Actually the \( SO(1, 2) \) covariance will be preserved by choosing for the coefficients \( c^r_z \) the Clebsch-Gordans of this group.\(^2\) The corresponding three Chern-Simons forms are

\[ \omega^+ = A^+_1 dA^+_1, \quad \omega^- = A^-_1 dA^-_1, \quad \omega^3 = A_1 dA_1. \] (B.14)

(In each case, one is free to add any total derivative to \( \omega \), since the only important point is that \( d\omega = F_{(2)} \wedge F_{(2)} \) in each factor.)

Note that \( A^+_1 \) and \( A^-_1 \) are precisely the two \( U(1) \) gauge potentials appearing in our Lagrangian (B.12). From (B.4) and (B.6), we have

\[
\begin{align*}
F^0_{(3)} &= \frac{1}{2} d(A_2) + C_{(2)} + \frac{1}{4} A^-_1 dA^-_1 + \frac{1}{4} A^+_1 dA^+_1, \\
F^1_{(3)} &= \frac{1}{2} d(A_2) + C_{(2)} + \frac{1}{4} A^-_1 dA^-_1 - \frac{1}{4} A^+_1 dA^+_1, \\
F^2_{(3)} &= d(B_2) + \frac{1}{2} A^+_1 dA^-_1 - \frac{1}{4} A^+_1 dA^-_1 - \frac{1}{4} A^-_1 dA^+_1.
\end{align*}
\] (B.15)

Comparing with (B.12) and (B.14), bearing in mind (B.13), we can read off the values of the constants \( c^r_z \) for each \( r = (0, 1, 2) \) and each \( z = (+, -, 3) \), leading to

\[
\begin{align*}
c^0_+ &= \frac{1}{4}, & c^0_- &= \frac{1}{4}, & c^0_3 &= 0, \\
c^1_+ &= -\frac{1}{4}, & c^1_- &= \frac{1}{4}, & c^1_3 &= 0, \\
c^2_+ &= -\frac{1}{4}, & c^2_- &= -\frac{1}{4}, & c^2_3 &= \frac{1}{4}.
\end{align*}
\] (B.16)

For the purposes of comparison with [17], we have used the diagonal constants \( c^r_z \) in eqn (B.16).

\(^2\)In fact it is natural to extend the class of theories discussed in [17], at least for abelian vector fields, by considering Chern-Simons corrections to the 3-forms with the more general structure \( F^r_{(3)} = dA^r_{(3)} + c^r_{\alpha\beta} A^\alpha_{(3)} dA^\beta_{(3)} \). The constants \( c^r_{\alpha\beta} \) in our case are then precisely the Clebsch-Gordan coefficients relating one spin-1 to the product of two spin-\( \frac{1}{2} \) \( SL(2, \mathbb{R}) \) representations. We believe that the proof of supersymmetry for the “diagonal” cases \( c^r_z \sim c^r_{\alpha\alpha} \) discussed in [17] extends straightforwardly to these more general structures.
This completes our comparison of the theory described by (3.12) with the family of chiral $D = 6$ supergravities described in [17]). It corresponds, as we have seen, to the situation where the supergravity multiplet, with its anti-self-dual tensor multiplet, is supplemented by two self-dual tensor matter multiplets, and two abelian vector matter multiplets. The structure of the Chern-Simons terms is a slight modification of that given in [17].

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