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Application of fractional optimal control theory for the mitigating of novel coronavirus in Algeria

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A B S T R A C T

In this paper, we investigate the dynamics of novel coronavirus infection (COVID-19) using a fractional mathematical model in Caputo sense. Based on the spread of COVID-19 virus observed in Algeria, we formulate the model by dividing the infected population into two sub-classes namely the reported and unreported infective individuals. The existence and uniqueness of the model solution are given by using the well-known Picard–Lindelöf approach. The basic reproduction number \(R_0\) is obtained and its value is estimated from the actual cases reported in Algeria. The model equilibriums and their stability analysis are analyzed. The impact of various constant control parameters is depicted for integer and fractional values of \(a\). Further, we perform the sensitivity analysis showing the most sensitive parameters of the model versus \(R_0\) to predict the incidence of the infection in the population. Further, based on the sensitivity analysis, the Caputo model with constant controls is extended to time-dependent variable controls in order obtain a fractional optimal control problem. The associated four time-dependent control variables are considered for the prevention, treatment, testing and vaccination. The fractional optimality condition for the control COVID-19 transmission model is presented. The existence of the Caputo optimal control model is studied and necessary condition for optimality in the Caputo case is derived from Pontryagin’s Maximum Principle. Finally, the effectiveness of the proposed control strategies are demonstrated through numerical simulations. The graphical results revealed that the implantation of time-dependent controls significantly reduces the number of infective cases and are useful in mitigating the infection.

Introduction

COVID-19 caused by a novel virus from the family of coronaviruses is considered one of the dangerous respiratory diseases. This infection emerged in Wuhan, China in December 2019. The transmissibility of COVID-19 is very high and it has been spread to the whole world within a short time period. It was declared as a pandemic by the world health organization (WHO) on March 2020. The particularity of the COVID-19 virus is that once a person exposed to the virus, the disease symptoms show up within 2 to 14 days. A COVID-19 infected person becomes contagious in a short period of time usually up to two days before the appearance of symptoms and they remain infectious to others for 10–20 days. COVID-19 is mainly transmitted from person to person due to direct touching with contaminated surfaces. It can also be transmitted through respiratory droplets inhalation from infected individuals. Most common COVID-19 symptoms include loss of taste or smell, fever, cough and in severe cases difficulty of breathing. Another particularity of the COVID-19 pandemic is that the symptoms of this infection vary from mild to severe. In some situation, people infected with COVID-19 have mild illness and even have no symptoms at all. While on the other hand, some cases with COVID-19 lead to severe respiratory problems, lasting lung, nervous system problems, heart muscle damage and even kidney failure. COVID-19 is diagnosed through laboratory tests. Parallel to laboratory testing, chest CT scans are also helpful to diagnose the presence of infection in individuals with a high clinical suspicion of infection. There are two basic types of viral or diagnostic tests which can confirm that a person is infected with the COVID-19 virus and an
antibody test that can show if a person was previously exposed to or infected with the COVID-19 virus. Still the scientists are investigating an effective treatment for this novel infection. Various high effective vaccines have been introduced which are helpful to slow down the pandemic incidence [1,2].

Mathematical modeling is a useful tool in understanding various dynamical aspects of an infectious disease. Usually, to formulate a compartmental epidemic model, the cumulative population is divided into various epidemiological sub-groups which are mutually-exclusive and mostly named as susceptible, exposed, infected and recovered etc. The compartmental model allows to estimate how the number of individuals in each compartment varies over time and estimate the duration of the infection. In the available literature, many models for different diseases including the novel COVID-19 have been presented. For instance, in [3] a deterministic model for assessing the TB infection dynamics based on the reported cases in Khyber Pakhtunkhwa was studied. The temporal dynamics of COVID-19 outbreak are analyzed in China, Italy and France in [4]. The dynamics of COVID-19 using a novel modeling approach is analyzed in [5]. The transmission dynamics of pandemic is studied using stochastic modeling approach in [6]. A similar study via nonlinear differential equations has been carried out in [7]. In addition to mathematical modeling, the theory of optimal control in epidemiology plays an important role that provides appropriate preventive control strategies. Application of optimal control theory is can be found in mathematical epidemic models. In [8] a mathematical modeling analysis is presented to explore the dynamical aspects of the COVID-19 in absence of treatment and vaccination. The authors in [8] introduced two time dependent control. In [9], the authors investigated an optimal control problem couple with cost-effective analysis for the transmission dynamics of COVID-19. Recently, the role of some selected control measures including time dependent vaccine strategy on the incidence of COVID-19 is studied in [10]. Application of logistic modeling approach with a detail numerical analysis to explore the dynamical aspects of pandemic has been conducted in [11].

Most of the existing mathematical models in epidemiology are constructed by differential systems of classical integer-order. In the recent past, the application of fractional calculus has been widely found in modeling and analyzing transmission dynamics of epidemiology models and other field of science [12–14]. The superiority of fractional over the classical integer order derivatives is due to they possess memory and has shown its effectiveness in modeling various science phenomena [15–18]. Moreover, in the recent literature, the classical optimal control is generalized into fractional order optimal control which the differential equations are as fractional differential equations. The implementation of fractional optimal control analysis for the mitigation of pandemic can be found in [19].

The present study deals with a fractional optimal control problem by incorporating four time-dependent control variables is developed and analyzed. In the problem formulation, the fractional order derivative in Caputo case is taken into account. Further, the population of infected class is divided into undetected and detected (or reported) sub-classes. The control strategies adopted as prevention, testing, ameliorating treatment and vaccination. The actual COVID-19 infected cases for a specific time period in Algeria are utilized to estimate the parameters values. Various scenarios have been considered and the impact of each case with and without vaccination on the infection eradication is depicted graphically. In addition, we investigate the effect of the fractional order (memory index) α on the transmission dynamics of COVID-19. The organization of the manuscript is described as follows. Section “Preliminaries On fractional order calculus” gives some basic concepts of fractional derivative. In Section “Caputo fractional order COVID-19 epidemic model”, we describe the proposed COVID-19 model and parameters estimation. Section “Analysis of the model” presents the qualitative studies of the model. In Section “Optimal control analysis”, the fractional optimal control model is analyzed. Further, in this section, detailed numerical simulations are given for the fractional model with and without optimal control measures. Finally, we summarized with a conclusion in Section “Conclusion”.

Preliminaries on fractional order calculus

Fractional differential operators are known to be an efficient tools in mathematical modeling and have been effectively used in areas like life science and epidemiology. Here we give some basic definitions and properties that we will need in rest of the paper [20,21].

Definition 1. Let \( f(t) \in L^\infty(\mathbb{R}) \cap \mathbb{C}(\mathbb{R}) \). For a \( \alpha > 0 \) the Riemann–Liouville fractional integral of order \( \alpha \) is defined by

\[
\mathcal{I}_R^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) \, ds, \quad t > 0.
\]

where \( \alpha > 0 \) and \( \Gamma(\cdot) \) is the gamma function.

Definition 2. Let \( f(t) \in L^\infty(\mathbb{R}) \cap \mathbb{C}(\mathbb{R}) \), the Riemann–Liouville type derivative of \( f \) having order \( \alpha > 0 \), is described by

\[
\mathcal{D}_R^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+n-1}} \, ds, \quad n-1 < \alpha < n, \quad n = [\alpha], \quad a = n \in \mathbb{N}.
\]

Definition 3. Let \( f(t) \in L^\infty(\mathbb{R}) \cap \mathbb{C}(\mathbb{R}) \), the Caputo fractional derivative of order \( \alpha > 0 \) of \( f \) is defined by

\[
\mathcal{D}_C^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+n-1}} \, ds, \quad n-1 < \alpha < n, \quad n = [\alpha], \quad a = n \in \mathbb{N}.
\]

The following proposition summaries some useful formula on Riemann–Liouville integral (1), the Riemann–Liouville and Caputo fractional operator.

Proposition 4. Let \( f(t) \in L^\infty(\mathbb{R}) \cap \mathbb{C}(\mathbb{R}) \), and \( a \in \mathbb{R}, n-1 < a < n, n \in \mathbb{N} \), then the following hold

1. \( \mathcal{D}_C^a f(t) = f(t) \).
2. \( (\mathcal{I}_R^a \mathcal{D}_C^a f(t)) = f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \).
3. In particular, if \( 0 < a < 1 \), then \( (\mathcal{I}_R^a \mathcal{D}_C^a f(t)) = f(t) - f(0) \).
4. The constant function. \( \mathcal{D}_C^0 f(t) = 0 \), and \( \mathcal{D}_R^0 f(t) = \frac{f(t)}{\Gamma(1-a)} t^{-a} \).

Definition 5. The two parameters Mittag-Leffler function is defined by

\[
E_{a,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(ak + \beta)}.
\]

Theorem 6. The Laplace transform of the Caputo fractional differential operator of order \( \alpha \), \( n-1 < \alpha < n \) is given by

\[
\mathcal{L}^{\mathcal{D}_C^\alpha} f(s) = s^\alpha F(s) - \sum_{k=0}^{n-1} \frac{s^{\alpha-k-1}}{\Gamma(k+1)} f^{(k)}(0).
\]

In the above equation \( F(s) \) represents the respective Laplace transform of \( f(t) \).

Theorem 7. The Laplace transformation of two parameters function of Mittag Leffler case is given by

\[
\mathcal{L}^{E_{a,\beta}} f(t) = \frac{z^{\alpha(1-x)}}{\Gamma(\alpha(1-x))}.
\]

Caputo fractional order COVID-19 epidemic model

In this section we consider a fractional compartmental COVID-19 epidemic model. The cumulative population \( N(t) \), is divided in five sub-class namely susceptible population \( S(t) \), exposed people \( E(t) \),
undetected infected people $I_u(t)$, the detected infected people (or reported) $I_r(t)$, and the recovered people $R(t)$. Thus, $N(t) = S(t) + E(t) + I_u(t) + I_r(t) + R(t)$.

The susceptible population are recruited at rate $\Delta$, and get infected by the coronavirus through contacts with both undetected and detected infected people where the force of infection is given by $\frac{\beta(v_1 I_u + v_2 I_r)}{N}$. In the force of infection term, $\beta$ denotes the disease transmission rate while $v_1$ and $v_2$ are relative infectiousness of unreported and reported infected individuals respectively. A fraction $\rho$ of the exposed people moves to the detected infected compartment at rate $\sigma$, while the other fraction $1 - \rho$ joins the undetected infected compartment $I_u(t)$ at the same rate. Some population from undetected infected compartment $I_u(t)$ and the detected infected compartment $I_r(t)$ progress to the recovery compartment $R(t)$ at rate $\gamma_{I_u}$ and $\gamma_{I_r}$ respectively. Notice that $\delta$ is the effort rate of undetected infected people to become detected infected people, by testing with self procurement test. The parameter $\mu$ represents the natural death rate in each compartment considered in the model while the parameters $d_1$ and $d_2$ represent the coronavirus related death rates of the undetected infected and the detected infected people respectively. The consideration of the parameter $d_1$ in the respective class is because the dead person in the compartment $I_u$ is reported infected of the virus when it is transported to the emergency. The transmission among various compartments is described in Fig. 1.

We translate the dynamics of COVID-19 in each compartment by the following system Caputo type differential equation

\[
\begin{align*}
C^\alpha_t S &= \Delta - \frac{\beta(v_1 I_u + v_2 I_r)}{N} S - \mu S, \\
C^\alpha_t E &= \frac{\beta(v_1 I_u + v_2 I_r)}{N} S - (\sigma + \mu) E, \\
C^\alpha_t I_u &= \sigma(1 - \rho) E - (\mu + d_1 + \delta + \gamma_{I_u}) I_u, \\
C^\alpha_t I_r &= \sigma \rho E + \delta I_u - (\mu + d_2 + \gamma_{I_r}) I_r, \\
C^\alpha_t R &= \gamma_{I_u} I_u + \gamma_{I_r} I_r - \mu R.
\end{align*}
\]

(7)

Associated to the following initial values

\[
\begin{align*}
S(0) &= S_0 > 0; & E(0) &= E_0 \geq 0; & I_u(0) &= I_{u0} \geq 0; \\
I_r(0) &= I_{r0} \geq 0; & R(0) &= R_0 \geq 0.
\end{align*}
\]

(8)

**Estimation of parameters**

In order to make the study biologically more feasible, the actual infected cases of Algeria for a selected period of pandemic are utilized to estimate the parameters values using the least square approach. The total population of Algeria at the start of pandemic is considered as susceptible to COVID-19. Therefore, we consider as $\mathbb{S}(0) = 43.409.568$ [22]. The initial reported infected cases detected as positive are $I_r(0) = 1$ [1]. The initial values of rest of population classes are assumed as $E(0) = 6000$, $I_u(0) = 0$, $R(0) = 0$. The model predicted simulation to the actual cases is shown in Fig. 2 and Parameters estimated values with biological description and relevant source are listed in Table 1.

**Fig. 2** shows the confirmed cases of COVID-19 infection in a period of 180 days from March 22, 2020. The increasing levels of confirmed cases suggest an exponential evolution of the infection. This confirms what is observed in several countries, and the model to predict cumulative infected cases remain in a same pattern.

**Analysis of the model**

**Positivity and boundedness**

We investigate the positivity and boundedness of the solution for the system (7). To do so we state the lemma to prove the result about the positivity and boundedness.

**Lemma 8 (Generalized Mean Value Theorem [23]).** Let $g(t) \in \mathbb{C}([0,T])$ and $C^\alpha_t g(t) \in \mathbb{C}([0,T])$ for $\alpha \in (0,1]$, then this theorem states

\[
g(t) = g(0) + \frac{1}{\Gamma(\alpha)} \int_0^t C^\alpha_t g(\xi) t^\alpha d\xi,
\]

where $\xi \in [0,t], \forall t \in [0,T]$.

**Remark 9.** The following interpretation is obtained from (8)

1. The function $g$ is non-decreasing $\forall \xi \in (0,T)$, if $C^\alpha_t g(t) \geq 0$.

2. The function shown by $g$ is non-increasing $\forall \xi \in (0,T)$, if $C^\alpha_t g(t) \leq 0$.

**Theorem 10.** The set $\Omega = \{(S, E, I_u, I_r, R) \in \mathbb{R}_+^5; 0 \leq S + E + I_u + I_r + R \leq \frac{\Delta}{\mu}\}$ is positively invariant and bounded set for all $t \in [0,T_0]; T_0 > 0$.

**Proof.** From the system (7), we get

\[
C^\alpha_t S|_{t=0} = \Delta \geq 0.
\]
As \( N \to \infty \), we obtain
\[
\lim_{t \to \infty} \frac{N(t)}{S(t)} = 1 - \frac{1}{\mu T(1)}.
\]

Further, \( N(t) \leq \frac{4}{\mu T(1)} \).

Applying Laplace transform, Theorem 6 we get as \( 0 < a < 1 \),
\[
N(s) \leq \frac{4}{s(s + \mu)} + \frac{4(s - 1)(s + \mu)}{(s + \mu)^2} N(0).
\]

By the inverse Laplace transform, and in view of [20] and by considering Theorem 7 we get
\[
N(t) \leq \frac{4}{\mu T(a)} + (N(0) - \frac{4}{\mu}) \tilde{E}_{a,1}(\mu t^a).
\]

As \( t \to \infty \) then (see [20])
\[
\lim_{t \to \infty} N(t) \leq \frac{4}{\mu T(a)}.
\]

Using property 3 in Proposition 4 we obtain the following equations in integral form
\[
S(t) - S(0) = I_0^a(\Delta - \frac{\beta(v_1 I_u + v_2 I_s) S}{N} - \mu S),
\]
\[
E(t) - E(0) = I_0^a(\frac{\beta(v_1 I_u + v_2 I_s) S}{N} - (\sigma + \mu) E),
\]
\[
I_0^a(\sigma E + \delta I_u - (\mu + d_2 + \gamma_r) I_r).
\]
We have

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\[ \begin{align*}
\mathcal{L}_a(t) - L_0(0) &= \mathcal{T}_a^\dagger((1-\rho)E - (\mu + d_1 + \delta + \gamma_{\text{in}})I_a), \\
\mathcal{L}_b(t) - L_0(0) &= \mathcal{T}_b^\dagger(\sigma \rho E + \delta L_b - (\mu + d_2 + \gamma_{\text{out}})I_b), \\
R(t) - R(0) &= \mathcal{T}_a^\dagger((\gamma_{\text{in}}) I_a + \gamma_{\text{in}} I_b - \mu R).
\end{align*} \]

Then we get

\[ \begin{align*}
S(t) &= S(0) + \left( \frac{1}{T(a)} \right) \int_0^t (t-s)^{a-1} F_1(s, S(s)) ds, \\
E(t) &= E(0) + \left( \frac{1}{T(a)} \right) \int_0^t (t-s)^{a-1} F_2(s, E(s)) ds, \\
I_a(t) &= I_a(0) + \left( \frac{1}{T(a)} \right) \int_0^t (t-s)^{a-1} F_3(s, I_a(s)) ds, \\
I_b(t) &= I_b(0) + \left( \frac{1}{T(a)} \right) \int_0^t (t-s)^{a-1} F_4(s, I_b(s)) ds, \\
R(t) &= R(0) + \left( \frac{1}{T(a)} \right) \int_0^t (t-s)^{a-1} F_5(s, R(s)) ds.
\end{align*} \]

Further, utilizing the Picard iteration leads as follow

\[ \begin{align*}
S_a(t) &= S_a(0) + \left( \frac{1}{T(a)} \right) \int_0^t (t-s)^{a-1} F_1(s, S_a(s)) ds, \\
E_a(t) &= E_a(0) + \left( \frac{1}{T(a)} \right) \int_0^t (t-s)^{a-1} F_2(s, E_a(s)) ds, \\
I_{a,u}(t) &= I_{a,u}(0) + \left( \frac{1}{T(a)} \right) \int_0^t (t-s)^{a-1} F_3(s, I_{a,u}(s)) ds, \\
I_{b,u}(t) &= I_{b,u}(0) + \left( \frac{1}{T(a)} \right) \int_0^t (t-s)^{a-1} F_4(s, I_{b,u}(s)) ds, \\
R_a(t) &= R_a(0) + \left( \frac{1}{T(a)} \right) \int_0^t (t-s)^{a-1} F_5(s, R_a(s)) ds.
\end{align*} \]

Finally, the initial value problem (10) transforms as follows

\[ X(t) = X(0) + 1/T(a) \int_0^t F(s, X(s))(t-s)^{a-1} ds. \] (12)

The following two lemmas are presented.

**Lemma 11.** The vector \( F(t, X) \) described in (11) fulfills the well-known Lipschitz condition in the variable \( X \) on a set \([0, T) \times \mathbb{R}^5_+\), with the Lipschitz constant

\[ L = \max(3(\nu_1 + \nu_2), (\sigma + \mu), (\mu + d_1 + \delta + \gamma_{\text{in}}), (\mu + d_2 + \gamma_{\text{in}}), \mu). \]

**Proof.** We have

\[ \|F(t, S_1(t)) - F(t, S_2(t))\| = \frac{\beta(u_1 + \nu_1)}{N} \| (\beta(v_1 + \nu_1) I_1 - \mu(S_1 - S_2) \| \leq (\beta(v_1 + \nu_1) + \mu)\|S_1 - S_2\|, \]

and also we have

\[ \|F(t, E_1(t)) - F(t, E_2(t))\| \leq (\sigma + \mu)\|E_1 - E_2\|, \]

\[ \|F(t, I_{a,u}(t)) - F(t, I_{a,u}(t))\| \leq (\mu + d_1 + \delta + \gamma_{\text{in}})\|I_{a,u} - I_{a,u}\|, \]

\[ \|F(t, I_{b,u}(t)) - F(t, I_{b,u}(t))\| \leq (\mu + d_2 + \gamma_{\text{in}})\|I_{b,u} - I_{b,u}\|. \]

We conclude that

\[ \|F(t, X_1) - F(t, X_2)\| \leq L\|X_1 - X_2\|, \] (13)

where \( L = \max(3(\nu_1 + \nu_2), (\sigma + \mu), (\mu + d_1 + \delta + \gamma_{\text{in}}), (\mu + d_2 + \gamma_{\text{in}}), \mu). \)

**Lemma 12.** Suppose we have (13), then the initial value problem described in (7) and (8) has a unique solution \( X(t) \in \mathbb{R}^5_+ \).

**Proof.** The fixed point theory coupled with Picard–Lindelöf method is utilized in the proof desired below. The system (7)–(8) solution can be written as

\[ X(t) = T(X(t)). \]

In the above equation, \( T \) describes the Picard operator given as follow

\[ T : \mathbb{R}^5_+ \rightarrow \mathbb{R}^5_+ \]

\[ T[X(t)] = X(0) + 1/T(a) \int_0^t F(s, X(s))(t-s)^{a-1} ds. \]

Further, we lead to

\[ \|T[X(t)] - T[X_0(t)\| = \left\| \frac{1}{T(a)} \int_0^t (t-s)^{a-1} \times [F(s, X_0(s)) - F(s, X_0(s))] ds \right\| \leq \left\| \frac{1}{T(a)} \int_0^t (t-s)^{a-1} \times \|F(s, X_0(s))\| ds \right\| \leq \left( \frac{L}{aT(a)} \right) \int_0^t (t-s)^{a-1} ds \leq \frac{L}{aT(a)} T. \]

If we have \( \frac{L}{aT(a)} T < 1 \), then \( T \) shows a contraction, therefore, the initial value problem (7)–(8) has a unique solution. \( \square \)

**The basic reproduction number**

The basic reproductive number \( R_0 \) has a biological significance and plays a key role in epidemiology. The evaluation of \( R_0 \) is usually carried out using the next generation matrix approach [24].

**Proposition 13.** The basic reproduction numbers denoted \( R_0 \) of the system (7), is

\[ R_0 = \frac{1}{(\sigma + \mu)} (R_a + R_a + R_c), \] (14)

where

\[ R_a = \frac{\sigma \rho (1-\rho)}{(\delta + \gamma_{\text{in}} + \mu + d_1)}, \]

\[ R_b = \frac{\sigma \rho (1-\rho)}{(\delta + \gamma_{\text{in}} + \mu + d_2)}, \]

\[ R_c = \frac{\sigma \rho \gamma_{\text{in}} (1-\rho)}{(\delta + \gamma_{\text{in}} + \mu + d_1)}. \]

**Remark 14.** Based on the parameters values listed in Table 1, the basic reproduction number for the dynamic of coronavirus given in system (7) is calculated as \( R_0 \approx 1.2696 \). Thus using the proposed model we obtained \( R_0 \approx 1.2696 > 1 \).

**Existence of equilibrium points**

The disease free equilibrium point for the system (7) is \( X_0 = (\frac{2}{\rho_0}, 0, 0, 0, 0) \), and the endemic equilibrium point is \( X^* = (S^*, E^*, I_a^*, I_b^*, R^*), \)

where

\[ E^* = \frac{\Delta R_0 - 1}{\rho_0} \]

\[ S^* = \frac{(1 + \rho_0) R_0}{R_0 - 1} E^*, \]

\[ I_a^* = \frac{R_a}{\rho_0} E^*, \]

\[ I_b^* = \frac{R_b}{\rho_0} E^*, \]

\[ R^* = \frac{R_c}{\rho_0} (R_a + R_c) E^*. \]

where \( \rho_0 = \sigma + \mu \), and \( R_a, R_b, R_c \) are defined in Proposition 11.

**Local and global stability analysis**

This part of manuscript describes the stability of the equilibrium points for both DFE and EE cases \( X_0 \) and \( X^* \) respectively. Before doing
the aforementioned analysis, notice in the system (7) the last equation involving $R$ does not occur in the remaining equations. Therefore, for simplicity the following system is considered for the stability analysis

\begin{align}
C D_t^\nu S &= - \beta (v_1 L_x + v_2 I_x) S, \\
C D_t^\nu E &= \beta (v_1 L_x + v_2 I_x) S - (\sigma + \mu) E, \\
C D_t^\nu I &= \sigma E - \delta I - (\mu + d_1 + \gamma_i) I, \\
C D_t^\nu R &= \delta I - (\mu + d_2 + \gamma_r) R.
\end{align}

with initial conditions

\[ S(0) = S_0 \geq 0; \quad E(0) = E_0 \geq 0; \quad I_x(0) = I_{x0} \geq 0; \quad I_r(0) = I_{r0} \geq 0. \]

\text{Local stability}

We state the results in the following theorems.

\textbf{Theorem 15.} The disease free equilibrium $X_0$ of the system (7) is locally asymptotically stable if $R_0 < 1$, and unstable if $R_0 > 1$.

\textbf{Proof.} The associated Jacobian matrix $J_{X_0}$ of the system (15) evaluated at $X_0$ is given by

\[
J_{X_0} = \begin{pmatrix}
-\mu & 0 & -\beta v_1 \\
0 & -\rho & 0 \\
\sigma (1 - \rho) & -\rho_1 & 0 \\
\sigma \rho & \delta & -\rho_2
\end{pmatrix}
\]

Solving the characteristic polynomial, the eigenvalue can obtained as

\[ \det(J_{X_0} - \lambda I) = 0. \]

Calculation allows as to obtain the following algebraic equation

\[ (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)(\lambda - \lambda_4) = 0, \]

where

\[
\begin{align*}
\lambda_1 &= -\mu, \\
\lambda_2 &= (\sigma + \mu), \\
\lambda_3 &= (\mu + d_1 + \delta + \gamma_i) R_x, \\
\lambda_4 &= (\mu + d_2 + \gamma_r) R_x.
\end{align*}
\]

We have the following situation

1. All the eigenvalue have negative real part if $R_0 < 1$ and the equilibrium point $X_0$ is asymptotically stable.
2. If $R_0 = 1$ then $\lambda_4 = 0$, so the equilibrium point $X_0$ is stable.
3. If $R_0 > 1$ and $R_x < 1$ then $\lambda_4 > 0$, and the equilibrium point $X_0$ is unstable.

This complete the proof. $\square$

\textbf{Global stability}

In order to establish results on global stability, the following two lemmas are recalled.

\textbf{Lemma 16 ([25]).} Let $x(t)$ a real continuous and derivable function then for any $t \geq t_0$, we have

\[ \frac{1}{2} \int_{t_0}^{t} C D_t^\nu (x^2(t)) \leq x(t) \int_{t_0}^{t} C D_t^\nu x, \forall a \in [0, 1]. \]

\[ \text{(17)} \]

\textbf{Lemma 17 ([25]).} Let $x(t)$ a real positive continuous and derivable function then for any $t \geq t_0$, we have

\[ \int_{t_0}^{t} \left[ x^2(t) - x^2 \ln x(t) \right] \leq \left( 1 - \frac{x(t)}{x(t)} \right) \int_{t_0}^{t} C D_t^\nu x, \forall a \in [0, 1]. \]

\[ \text{(18)} \]

\textbf{Theorem 18.} The disease free equilibrium $X_0 = (\frac{S}{S_0}, 0, 0, 0, 0)$ of the system (7) is globally asymptotically stable in $\Omega$, if $R_0 \leq 1$ and unstable if $R_0 > 1$.

\textbf{Proof.} We define a Lyapunov function $L_0(t)$ given by

\[ L_0(t) = \frac{\beta v_1 (\Gamma(a))}{2 S_0} E(t)^2 + \frac{\beta v_2 (\Gamma(a))}{2 S_0 p_1} I_x(t)^2 + \frac{\beta v_2 (\Gamma(a))}{2 S_0 p_2} I_r(t)^2. \]

\[ \text{(19)} \]

where $S_0 = \frac{a}{\mu}$. It is clear that the function $L_0(t)$ is continuous and positive definite for all $t \geq 0$. Now using result of Lemma 16 we have

\[ C D_t^\nu (L_0(t)) \leq \frac{\beta v_1 (\Gamma(a))}{2 S_0} C D_t^\nu E(t)^2 + \frac{\beta v_2 (\Gamma(a))}{2 S_0 p_1} C D_t^\nu I_x(t)^2 + \frac{\beta v_2 (\Gamma(a))}{2 S_0 p_2} C D_t^\nu I_r(t)^2 \]

\[ \text{(20)} \]

As $(E(t), I_x(t), I_r(t)) \in \Omega$ we have

\[ C D_t^\nu (L_0(t)) \leq \frac{\lambda}{u F(a)} \left( \frac{\Gamma(a)}{2 S_0} C D_t^\nu E(t) + \frac{\beta v_1 (\Gamma(a))}{2 S_0 p_1} C D_t^\nu I_x(t) + \frac{\beta v_2 (\Gamma(a))}{2 S_0 p_2} C D_t^\nu I_r(t) \right) \]

\[ \text{(21)} \]

\textbf{Theorem 19.} The endemic equilibrium $X^* = (S^*, E^*, I_x^*, I_r^*, R^*)$ of the system (7) is globally asymptotically stable in $\Omega$, when $R_0 \geq 1$.

\textbf{Proof.} We begin by setting $N = 1$ is the system (15), this is achieved by dividing $S, E, I_x, I_r$ and $R$ by $N$. Let $L_1(t)$ a Lyapunov function defined by

\[ L_1(t) = (S - S^* - S^* \ln \frac{S}{S^*}) + (E - E^* - E^* \ln \frac{E}{E^*}) \]

\[ + \frac{\beta v_1 S^*}{p_1} (I_x - I_x^* - I_x^* \ln \frac{I_x}{I_x^*}) + \frac{\beta v_2 S^*}{p_2} (I_r - I_r^* - I_r^* \ln \frac{I_r}{I_r^*}). \]

\[ \text{(20)} \]

Using result of Lemma 17 we have

\begin{align*}
&\int_{t_0}^{t} C D_t^\nu (L_1(t)) \leq \left( 1 - \frac{S}{S_0} \right) C D_t^\nu S(t) + \left( 1 - \frac{E}{E^*} \right) C D_t^\nu E(t) \\
&+ \frac{\beta v_1 S^*}{p_1} (1 - \frac{I_x}{I_x^*}) C D_t^\nu I_x(t) \\
&+ \frac{\beta v_2 S^*}{p_2} (1 - \frac{I_r}{I_r^*}) C D_t^\nu I_r(t).
\end{align*}

\[ \text{(21)} \]
At the endemic equilibrium point we have
\[ \Delta = \beta(v_i I'_u + v_x I'_x)S^* + \mu S^*, \]
\[ \rho_0 = \frac{p_1 I'_u}{E^*}, \]
\[ \sigma(1 - \rho) = \frac{p_x I'_x}{E^*}, \]
\[ \sigma = \frac{\delta I'_u + p_x I'_x}{E^*}. \]

Introducing Eqs. (22)–(25) into (21), and direct calculation gives
\[ C D_t^{\alpha} (L(t)) \leq 2\mu S^*(2 - \frac{S^*}{S}) + \beta(v_i I'_u S^* + \frac{E I'_u}{E^*} - \frac{S E I'_u}{S^* E^*}) \]
\[ + \beta v_x S^* I'_x (3 - \frac{S^*}{S}) - \frac{S E I'_x}{S^* E^*}), \]
\[ \text{and if in addition we have} \]
\[ \left(1 - \frac{1}{T_1} \right) \frac{1}{T_2} \frac{E}{E^*} \leq 0, \]
\[ \text{then, } C D_t^{\alpha} (X) \leq 0. \text{ Furthermore, } C D_t^{\alpha} (X^*) = 0 \text{ iff } (S(t), E(t), I(t), I_x(t), R(t)) = X^*. \text{ In a result, the maximal invariant set for } (S(t), E(t), I(t), I_x(t), R(t)) \in \mathbb{R}^5, \text{ is } X^* \text{ and thus utilizing, the LaSalle’s invariance principle the endemic equilibrium } X^* \text{ is globally asymptotically stable when, } R_0 > 1. \]

**Numerical solution of the COVID-19 model**

The dynamics of the model (7) is analyzed graphically in this part of the paper. This section mainly aims to demonstrate the influence of fractional order \( \alpha \) as will the key parameters especially the transmission rates \( \beta, v_i \) and \( v_x \). The well-known second order Runge Kutta Method [26] is utilized to construct a sequences of approximations. The estimated parameters with value in Table 1 are utilized in the graphical results. The model dynamics are examined by taking time level as 30 days. To proceed, consider the nonlinear fractional-order system as follows:

\[ \begin{cases} 
C D_t^{\alpha} X(t) = F(t, X(t)), \\
X(0) = X_0,
\end{cases} \]

\[ X(t) = X_0 + \frac{h^\alpha}{(\alpha + 1)} K_1 + \frac{h^\alpha}{(\alpha + 1)} K_2, \]

where, \( K_1 = F(t_{x}, X_{x}), \) and \( K_2 = F(t_{x} + \frac{h^\alpha}{(\alpha + 1)}, X_{x} + \frac{h^\alpha}{(\alpha + 1)} K_1). \)

Fig. 3(a–f) shows the impact of different value of the fractional order \( \alpha \) on the solution of system (7) with initial condition given in (8). In the sub-plot 3(a), we have depicted the dynamics of individuals in susceptible class. As can be seen the susceptible population class deceases for all considered values of index memory \( \alpha \) to a specific positive value after day 25. The same behaviors are observed in Fig. 3(b) for individuals in the exposed class except that they approach a specific positive value close to zero. The dynamical aspects of individuals in unreported and reported infected compartments are shown in sub-plots 3(c) and (d) respectively. It is observed that the peak of the infected individuals decreases slightly and diminishes according to the smallest value of \( a \). Fig. 3(f) illustrates the profile of recovered individuals. Finally, in the Fig. 3(e), we analyzed the dynamics of the cumulative infective individuals \( I_x + I_x \) considering baseline values of parameters given in the Table 1 and for different values of \( a \) (see Fig. 3).

Furthermore, in Figs. 4–6 the dynamics of cumulative infective people profile \( (I_x + I_x) \) are analyzed for variation in the infection transmission rates \( \beta, v_i \) and \( v_x \). The simulation results are carried out for \( \alpha = 0.85, \alpha = 0.90, \alpha = 0.95, \) and \( \alpha = 1.00. \) These graphical results revealed that the peaks of cumulative infective individuals decrease with the reduction in the aforementioned parameters from its baseline values with different levels. It is further noticed that the influence of transmission coefficient \( \beta \) is more significant than the influence of the parameters \( v_i \) and \( v_x \).

**Optimal control analysis**

This section develops a fractional optimal control problem to combat the infection. The optimality control concepts are utilized to formulate the control system. Before constructing an appropriate control problem, we perform the sensitivity analysis of the model parameters to point out the most sensitive parameter to the disease incidence. We proceed as follows:

**Sensitivity analysis of \( R_0 \)**

As the disease incidence is related to \( R_0 \), we studied the role of each parameter introduced in the system (7) on \( R_0 \). This leads us to determine the most potential parameters in the spread of the infection. The natural approach is to calculate the partial derivatives of the value of \( R_0 \), defined in (14) with respect to the parameters values of system (7). In fact, we use a forward normalized sensitivity index of \( R_0 \).

**Definition 20 ([27])**. The variable normalized forward sensitivity index to measure the relative change in \( R_0 \), to the change in the model parameters is defined by

\[ P_x = \frac{\partial R_0}{\partial x} \times \frac{x}{R_0}. \]

Using **Definition 20**, the normalized sensitivity index measuring the relative change of \( R_0 \) with respect the different parameters associated to the mathematical model (7) are

\[ P_\rho = 1, \]
\[ P_{v_i} = 1 - \frac{R_3 + R_4}{R_3 + R_4 + R_5}, \]
\[ P_{v_x} = 1 - \frac{R_3 + R_4 + R_5}{R_3}, \]
\[ P_\mu = 1 - \frac{R_3 + R_4}{\mu}, \]
\[ P_\gamma = 1 - \frac{1}{(1 - \rho)} \frac{R_3 + R_4 + R_5}{R_3 + R_4 + R_5}, \]
\[ P_\delta = \frac{\delta}{(\delta + \gamma)} \frac{R_3 + R_4 + R_5}{R_3 + R_4 + R_5}, \]
\[ P_{\gamma_1} = \frac{\gamma_1}{(\delta + \gamma_1 + \mu + d_1)} \frac{R_3 + R_4 + R_5}{R_3 + R_4 + R_5}, \]
\[ P_{\gamma_2} = \frac{\gamma_2}{(\delta + \gamma_2 + \mu + d_2)} \frac{R_3 + R_4 + R_5}{R_3 + R_4 + R_5}, \]
\[ P_{\mu} = \frac{\mu - \frac{\mu}{\mu + \delta}}{\sigma + \mu} \frac{R_3 + R_4 + R_5}{R_3 + R_4 + R_5 + R_5}, \]
\[ P_{\mu_2} = \frac{\mu - \frac{\mu}{\mu + \delta}}{(\delta + \gamma_1 + \mu + d_1)} \frac{R_3 + R_4 + R_5}{R_3 + R_4 + R_5 + R_5}, \]
\[ P_{\mu_3} = \frac{\mu - \frac{\mu}{\mu + \delta}}{(\delta + \gamma_2 + \mu + d_2)} \frac{R_3 + R_4 + R_5}{R_3 + R_4 + R_5 + R_5}. \]
Utilizing the estimated values of parameters listed in Table 1, we have the following table which gives values of the sensitivity indices of $R_0$ compared to the parameters system (7).

Based on sensitivity indices given in Table 2, the most influential parameters to disease incidence are the transmission rate $\beta$, the transmission rates of unreported and reported infected people $\nu_1$ and $\nu_2$, the incubation rate $\sigma$, the fraction of individuals move to the reported compartment $\rho$ and the recovery rate of reported people $\gamma_{I_r}$. Thus in a result, for the better eradication of the virus transmission, it is recommended to decrease the virus transmission rates $\beta$, $\nu_1$ and $\nu_2$. Additionally, increase the rate of recovery of both unreported people and reported people $\gamma_{I_u}$ and $\gamma_{I_r}$, and the fraction portion parameter $\rho$ will lead to a decrease in $R_0$.

**Optimal control of the model**

We extend the COVID-19 Caputo epidemic model described by system (7) to include the effect of time-dependent four control measures. For this purpose, we introduce four variable controls namely $\varphi_1(t), \varphi_2(t), \varphi_3(t)$ and $\varphi_4(t)$. In fact, the controls are represented as Lebesgue measurable function of time $t$. The notation and the role of each control are defined as follows:

1. **Prevention measure of COVID-19.** The control variable $\varphi_1(t)$ denotes the level of prevention efforts in order to reduce the effective contacts. Prevention efforts include wearing medical mask, often cleaning hands by using soap and water, or alcohol-based hand rap. These preventive measures decrease the transmission rate of the unreported infected people $I_u$ and the reported infected people $I_r$.

2. **COVID-19 testing.** The time-dependent control function $\varphi_2(t)$ is used for COVID-19 testing efforts of unreported people. The effective testing policy will be helpful to bring the unreported infected people to the reported infected class and will reduce the disease incidence.

3. **Ameliorating treatment.** A patient with severe infection requires high-flow oxygen therapy. When this is not enough other therapy can be adopted such as antibiotic therapy, anti-inflammation therapy, etc. This increases the recovery rate. The control variable $\varphi_3$ is utilized for ameliorating the treatment of the infected people.

4. **Vaccination.** Finally, the fourth control variable $\varphi_4(t)$ denotes the immunization rate of susceptible class due to vaccination.

---

**Table 2**

| Parameters | $\beta$ | $\nu_1$ | $\nu_2$ | $\sigma$ | $\rho$ | $\delta$ | $\gamma_{I_u}$ | $\gamma_{I_r}$ | $d_1$ | $d_2$ | $\mu$ |
|------------|--------|--------|--------|--------|-----|--------|-------------|-------------|-----|-----|-----|
| Value      | 0.6979 | 0.3958 | 0.4941 | 0.3688 | 0.3368 | 0.5142 | 0.2245      | 0.3014      | 5.3007 $\times 10^{-4}$ | 0.4321 $\times 10^{-4}$ | 0.3535 $\times 10^{-4}$ |
| Sens. Index| 1      | 0.2136 | 0.7864 | 0.9999 | $-0.0074$ | $-0.2010$ | $-0.2029$ | $-0.7860$ | $-0.0005$ | $-0.0004$ | $-0.0002$ |
The system of state Eqs. (7) with control functions $\varphi_i$, $i = 1, \ldots, 4$, is formulated as follows:

$$
D^\alpha_S \Delta = \frac{(1 - \varphi_1)\theta(v_1 I_u + v_2 I_3)S}{N} - \mu S - \varphi_4 S,
$$

$$
D^\alpha_S E = \frac{(1 - \varphi_1)\theta(v_1 I_u + v_2 I_3)S}{N} - (\sigma + \mu)E,
$$

$$
D^\alpha_I I_u = \sigma (1 - \rho) E - (\mu + d_1 + (1 - \kappa)\delta + \gamma_1) I_u - \kappa \delta \varphi_2 I_u,
$$

$$
D^\alpha_I I_w = \sigma \rho E + (1 - \kappa)\delta I_u - (\mu + d_2 + \gamma_1) I_w - \varphi_3 I_w + \kappa \delta \varphi_2 I_u,
$$

$$
D^\alpha_R R = \gamma_1 I_u + \gamma_1 I_w - \mu R + \varphi_5 I_u + \varphi_7 S,
$$

with initial condition described in (8).

$$
S(0) = S_0 > 0, \quad E(0) = E_0 \geq 0, \quad I_u(0) = I_{u0} \geq 0, \quad I_w(0) = I_{w0} \geq 0, \quad R(0) = R_0 \geq 0
$$

(28)

The objective goal is to reduce the number of exposed, unreported infected, reported infected people and increase the number of recovered people under the cost of introducing control strategies. Then, our optimal control problem is to minimize the number of exposed people in the class $E$, number of unreported infected people in the class $I_u$, and number of reported infected people in the class $I_w$, and maximize the number of recovered people in the class $R$. To do so we introduce the following objective functional

$$
J(\varphi(\cdot)) = \int_{t_f}^{t_0} \left[ C_1 E(t) + C_2 I_u(t) + C_3 I_w(t) - C_4 R(t) + \frac{q_1}{2} \varphi_1^2(t) + \frac{q_2}{2} \varphi_2^2(t) \right] dt + \frac{q_3}{2} \varphi_3^2(t) + \frac{q_4}{2} \varphi_4^2(t) dt,
$$

(30)

where $C_i$, $i = 1, 2, 3, 4$ are positive balancing coefficients which represent the cost of implementing the four controls and the positives constants $q_i$, $i = 1, 2, 3, 4$ correspond to the effort used to limit the spread of the virus, and $t_f$ represent the final time.

The fractional order optimal control problem under consideration is to find the optimal control $\varphi(t)$ that minimize the objective given functional stated as

$$
\min J(\varphi), \quad \varphi(0) \in \overline{U}
$$

subject to the state system (28) and initial condition (29). Here $\overline{U} = \{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$ is the control vector, and

$$
\overline{U} = \{\varphi \in (L^\alpha([0, t_f]))^4, 0 \leq \varphi_i \leq 1, i = 1, 2, 3, 4\},
$$

is the set of admissible controls, which is closed and bounded by construction.

**Existence and optimality**

The system (7)–(8) can be written in following classical form

$$
C D^\alpha X(t) = F(t, X(t)) + G(t, X(t))\overline{\varphi}, \quad 0 \leq t \leq t_f,
$$

$X(0) = X_0$.

In above system, the vector $X = (S(t), E(t), I_u(t), I_w(t), R(t))^T$ denotes the state variables, $\varphi(t) = (\varphi_1(t), \varphi_2(t), \varphi_3(t), \varphi_4(t))$ is the control function.
We prove the existence of an optimal four control under consideration by proving the following condition \cite{19}:

- The set of all feasible solution to the control problem is non-empty.
- The admissible controls set is convex, bounded and closed.
- The function \( F(t, X(t)) + G(t, X(t))\) is bounded by a linear function in the state and control variables.
- The convexity of

\[
C_1 E(t) + C_2 I_u(t) + C_3 I_s(t) - C_4 R(t) + \frac{\theta_1}{2} \tilde{q}_1(t) + \frac{\theta_2}{2} \tilde{q}_2(t) + \frac{\theta_3}{2} \tilde{q}_3(t) + \frac{\theta_4}{2} \tilde{q}_4(t)
\]

on the set \( \overline{U} \).

**Remark 21.**

1. It is obvious to see that \( \varphi_1 = 1, \varphi_2 = \varphi_3 = \varphi_4 = 0 \) are control in \( \overline{U} \) and the solution \( \overline{X} = (S, E, I_u, I_s, R) \) of system (7)--(8) is a solution corresponding to controls \( \varphi_1 = 1, \varphi_2 = \varphi_3 = \varphi_4 = 0 \), then the set of all feasible solution to the control problem is non-empty.
2. By definition of \( \overline{U} \), the admissible controls set is convex, bounded and closed.
3. The boundedness and the existence of the solution of system (28), and initial condition (29) is a consequence of theorem (feasible of solution), and theorem (existence and uniqueness), since 0 \( \leq \varphi_i \leq 1, \ i = 1, 2, 3, 4 \).

**Lemma 22.** The function \( F(t, X(t)) + G(t, X(t))\) satisfy for a solution \( \overline{X} = (S, E, I_u, I_s, R) \)

\[
\| F(t, \overline{X}) + G(t, \overline{X}) \| \leq \max(C_1, C_2)(\| \overline{X} \| + \| \overline{\varphi} \|),
\]

where

\[
C_1 = \max(1 + \beta(v_1 + v_2) + \mu, \sigma + \mu, \mu + d_1 + (1 - \kappa)\delta + \gamma_u).
\]
Fig. 6. Impact of \( \nu_2 \) on the cumulative infected people for various values of \( \alpha \) where, (a) \( \alpha = 1.00 \), (b) \( \alpha = 0.95 \), (c) \( \alpha = 0.90 \), (d) \( \alpha = 0.85 \).

\[
F(t, X(t)) = \begin{pmatrix}
\frac{4}{3} - \beta(v_1 I_u + v_2 I_r) & -\mu & 0 & 0 & 0 & 0 \\
\frac{N}{\beta(v_1 I_u + v_2 I_r)} & -(\sigma + \mu) & 0 & 0 & 0 & 0 \\
0 & \sigma(1 - \rho) & -(\mu + d_1 + (1 - \kappa)\delta + \gamma_{I_u}) & 0 & 0 & 0 \\
0 & \sigma \rho & (1 - \kappa)\delta & -\mu & \gamma_{I_v} & 0 \\
0 & 0 & \gamma_{I_u} & -\mu & \gamma_{I_v} & 0 \\
\end{pmatrix}
\begin{pmatrix}
S \\
E \\
I_u \\
I_r \\
R \\
\end{pmatrix}
\]

Box 1.

\[ C_2 = \max(\beta(v_1 + v_2), \kappa \delta, 1). \]

**Proof.** We write \( F(t, X(t)) \) that is given in Box 1

Using the fact that \( \Delta \leq S \), and the solution is bounded, we obtain

\[
\| F(t, X) \| \leq \max(1 + \beta(v_1 + v_2) + \mu, \sigma + \mu, \mu + d_1 + (1 - \kappa)\delta + \gamma_{I_u}, \mu + d_2 + \gamma_{I_v}, \mu)\| X \|
\]

Same procedure gives for \( G(t, \overline{X}) \),

\[
\| F(t, \overline{X}) \| \leq \max(\beta(v_1 + v_2), \kappa \delta, 1)\| \overline{\varphi} \|. \]

**Lemma 23.** The functional

\[
L(t, \overline{X}, \overline{\varphi}) = C_1 E(t) + C_2 I_u(t) + C_3 I_v(t) - C_4 R(t) + \frac{q_1}{2} \varphi_1(t) + \frac{q_2}{2} \varphi_2(t) + \frac{q_3}{2} \varphi_3(t) + \frac{q_4}{2} \varphi_4(t).
\]
Is convex in $\overline{U}$, and there exist non negative constant $q$ such that

$$L(t, \overline{X}, \overline{\psi}) \geq q\|\overline{\psi}\|.$$

**Proof.** The Hessian matrix of the functional $L(t, \overline{X}, \overline{\psi})$ is given by

$$
\begin{bmatrix}
2\varphi_1 & 0 & 0 & 0 \\
0 & 2\varphi_2 & 0 & 0 \\
0 & 0 & 2\varphi_3 & 0 \\
0 & 0 & 0 & 2\varphi_4
\end{bmatrix}
$$

We see that the Hessian of the functional is positive definite in $\overline{U}$, then the functional $L(t, \overline{X}, \overline{\psi})$ is strictly convex in $\overline{U}$.

Further more, for $q = \min\left(\frac{q_1}{\varphi_1^2}, \frac{q_2}{\varphi_2^2}, \frac{q_3}{\varphi_3^2}, \frac{q_4}{\varphi_4^2}\right)$ we have

$$L(t, \overline{X}, \overline{\psi}) = C_1 E(t) + C_2 I_E(t) + C_3 I_s(t) + C_4 R(t)$$

$$+ \frac{q_1}{2} \varphi_1^2(t) + \frac{q_2}{2} \varphi_2^2(t) + \frac{q_3}{2} \varphi_3^2(t) + \frac{q_4}{2} \varphi_4^2(t)$$

$$\geq \varphi(t)^T \begin{bmatrix}
\frac{1}{2} q_1 & 0 & 0 & 0 \\
0 & \frac{1}{2} q_2 & 0 & 0 \\
0 & 0 & \frac{1}{2} q_3 & 0 \\
0 & 0 & 0 & \frac{1}{2} q_4
\end{bmatrix} \varphi(t)$$

under the condition $C_1 E(t) + C_2 I_E(t) + C_3 I_s(t) \geq C_4 R(t)$. □

From Remark 21, and Lemmas 22 and 23 we have the result of the existence of an optimal control and we have the following theorem.

**Theorem 24.** There exist an optimal control $\overline{\varphi} = (\varphi_1^*, \varphi_2^*, \varphi_3^*, \varphi_4^*)$ and a corresponding solution $\overline{X} = (S^*, E^*, I_s^*, R^*)$ that minimize the objective functional $J(\overline{\varphi})$ on the set $\overline{U}$, that is

$$\min_{\overline{\varphi} \in \overline{U}} J(\overline{\varphi}) = J(\overline{\varphi}^*).$$

**Necessary condition for optimality**

The necessary condition needed to be satisfied by optimal control come from the Pontryagin’s Maximum principle [28], this is achieved by transforming system Eqs. (28) and (30) in a problem of point

$$\overline{C}(X, \overline{X}, \Lambda) = C_1 E(t) + C_2 I_E(t) + C_3 I_s(t) + C_4 R(t)$$

$$+ \frac{q_1}{2} \varphi_1^2(t) + \frac{q_2}{2} \varphi_2^2(t) + \frac{q_3}{2} \varphi_3^2(t) + \frac{q_4}{2} \varphi_4^2(t)$$

$$+ \lambda_S \left(1 - (1 - \varphi_1)\beta(v_1 I_s + v_2 I_s) S - \beta S - \varphi_4 S \right)$$

$$+ \lambda_E \left(1 - (1 - \varphi_2)\beta(v_1 I_s + v_2 I_s) S - \beta S - \varphi_4 S \right)$$

$$+ \lambda_s \left(1 - (1 - \varphi_3)\beta(v_1 I_s + v_2 I_s) S - \beta S - \varphi_4 S \right)$$

$$+ \lambda_R \left(1 - (1 - \varphi_4)\beta(v_1 I_s + v_2 I_s) S - \beta S - \varphi_4 S \right)$$

$$+ \lambda_S (\sigma(1 - \rho)E - (\mu + d_1 + (1 - \kappa)S + \gamma_1 I_s - \kappa \delta I_s) E)$$

$$+ \lambda_E (\sigma(1 - \rho)E - (\mu + d_1 + (1 - \kappa)S + \gamma_1 I_s - \kappa \delta I_s) E)$$

$$\times I_s - \varphi_3 I_s + \kappa \delta I_s) E$$

$$+ \lambda_R (\gamma_1 I_s + \gamma_1 I_s - \mu R + \varphi_3 I_s + \varphi_3 S),$$

where, $\lambda = (\lambda_S, \lambda_E, \lambda_s, \lambda_R)$, denote the adjoint variables associated to their respective state variables.

**Theorem 25.** Given an optimal control $\varphi_i^*$, $i = 1, 2, 3, 4$ and solution of state variable $S^*, E^*, I_s^*, R^*$ of the associated system (28) which minimize the objective functional (30) over the set $\overline{U}$, then there exists a function $\lambda_S, \lambda_E, \lambda_s, \lambda_R$ and such that

$$c \lambda_S^* \lambda_S = (\lambda_S - \lambda_E)(1 - \varphi_1)\beta(v_1 I_s + v_2 I_s) \left(1 - \frac{1}{N} \right)$$

$$+ (\lambda_S - \lambda_E) \sigma \varphi_4 + \mu \lambda_S,$$

$$c \lambda_E^* \lambda_E = (\lambda_E - \lambda_S)(1 - \varphi_2)\beta(v_1 I_s + v_2 I_s) \left(1 - \frac{1}{N} \right)$$

$$+ (\lambda_E - \lambda_S) \sigma \varphi_4 + \mu \lambda_E,$$

$$c \lambda_S^* \lambda_s = (\lambda_S - \lambda_E)(1 - \varphi_3)\beta(v_1 I_s + v_2 I_s) \left(1 - \frac{1}{N} \right)$$

$$+ (\lambda_S - \lambda_E) \sigma \varphi_4 + \mu \lambda_S,$$

$$c \lambda_R^* \lambda_R = (\lambda_R - \lambda_S)(1 - \varphi_4)\beta(v_1 I_s + v_2 I_s) \left(1 - \frac{1}{N} \right)$$

$$+ (\lambda_R - \lambda_S) \sigma \varphi_4 + \mu \lambda_R + C_4,$$

with the transversality conditions

$$\lambda_S(t_f) = \lambda_E(t_f) = \lambda_s(t_f) = \lambda_R(t_f) = \lambda_R(t_f) = 0.$$ (35)

where the Hamiltonian function $H$ is defined by Eq. (34). Furthermore, the optimal control are characterized as follow:

$$\varphi_1^* = \min \left(1, \frac{(\lambda_S - \lambda_E) \varphi_1}{q_1}, \frac{(\lambda_S - \lambda_E) \beta(v_1 I_s + v_2 I_s) S}{N}\right),$$

$$\varphi_2^* = \min \left(1, \frac{(\lambda_S - \lambda_E) \varphi_2}{q_2}, \frac{(\lambda_S - \lambda_E) \beta(v_1 I_s + v_2 I_s) S}{N}\right),$$

$$\varphi_3^* = \min \left(1, \frac{(\lambda_S - \lambda_E) \varphi_3}{q_3}, \frac{(\lambda_S - \lambda_E) \beta(v_1 I_s + v_2 I_s) S}{N}\right),$$

$$\varphi_4^* = \min \left(1, \frac{(\lambda_S - \lambda_E) \varphi_4}{q_4}, \frac{(\lambda_S - \lambda_E) \beta(v_1 I_s + v_2 I_s) S}{N}\right).$$

**Proof.** The existence of the functions $\lambda_S, \lambda_E, \lambda_s, \lambda_R$ and $\lambda_R$ are obtained by the Pontryagin’s Maximum principle [28]. The characterizing of each fractional control defined in (36)–(39) is computed by solving the equations

$$\frac{dH}{d\varphi_1} = \frac{dH}{d\varphi_2} = \frac{dH}{d\varphi_3} = \frac{dH}{d\varphi_4} = \frac{dH}{d\varphi_4} = 0.$$ (40)

On the interior of the admissible controls set $\overline{U}$ defined in (31). □

**Numerical solution of the optimal control problem**

In this part of the study, we analyze the numerical simulation of the considered COVID-19 model with controls and without controls in order to show the significance of the time-dependent interventions on the disease dynamics. The numerical approach used to find out the solution model is given in Section “Numerical solution of the COVID-19 model”. Furthermore, we consider the following equal weight effort $q_i = 1$, $i = 1 : 4$, and the equal balancing coefficients $C_i = 1$ $i = 1 : 4$ for the numerical simulation of the optimal control problem. The simulation results are depicted for two values of the fractional order $\alpha$, $\alpha = 0.80$ and $\alpha = 1.00$. The value of the parameters in the simulation remain the same and are listed in the Table 1. Additionally, we assumed that 20% of undetected people can be tested therefore, the estimated value of $\kappa = 0.2$. Further, two different approach for the eradication of the COVID-19 are studied. In the first approach, we perform the simulation by considering all controls without the vaccination strategy while in the second approach the control model is simulated with activating the vaccination strategy only. Moreover, in the first approach, we discuss three possible scenarios.

**First part: without vaccination**

In this part we simulate the control model by activating the prevention, testing and treatment controls in the absence of vaccination intervention.

1. **Scenario with prevention measure $\varphi_1 \neq 0$.**

In the first scenario, we consider the prevention measure only by activating $\varphi_1$ and making $\varphi_2 = \varphi_3 = \varphi_4 = 0$. The graphical results with $a = 1.00$ and $a = 0.80$ are demonstrated in Figs. 7 and 8.
Fig. 7. Impact of the control \( \psi_1 \) on the dynamics of various population classes with \( \alpha = 1.00 \).

respectively. The graphical interpretation shows the effect of the prevention measures on the COVID-19 dynamics. It seems that by considering this set of controls \( \psi_1 \), the exposed, undetected, detected and recovered are decreasing significantly, whereas the susceptible population increases compared to the case of no control solutions. It is observed that for \( \alpha = 0.80 \) the graphical solution of the COVID-19 model decreases slightly faster as compared to \( \alpha = 1.00 \).
2. **Scenario with testing efforts** $\varphi_2 \neq 0$.

In the second scenario, we simulate the control model (28) in order to analyze the impact of testing measure on the incidence of COVID-19. The graphical results by considering the control $\varphi_2$ are depicted in Figs. 9 and 10. It can be observed that the testing control strategy has no significant impact on the COVID-19 dynamics as compared with no control solution. Therefore, the implementation of only testing efforts is not effective to minimize the infection.

3. **Scenario with treatment** $\varphi_3 \neq 0$. 

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Fig. 8. Impact of the control $\varphi_1$ on the dynamics of various population classes with $\alpha = 0.08$. 

- (a) 
- (b) 
- (c) 
- (d) 
- (e)
In this case, we analyze the impact of ameliorating treatment by considering the time-dependent control variable $\varphi_3$. As expected, individuals in the detected class $I_r(t)$ decrease significantly by introducing the control $\varphi_3$. The graphical interpretation for $\alpha = 1$ and $\alpha = 0.80$ is depicted in Figs. 11 and 12 respectively. Compared to scenario 1 and scenario 2, the amelioration in treatment intervention is more effective to mitigate the spread of COVID-19.
Part 2 with vaccination only

In this strategy, we analyze the effect of the implementation of the vaccination strategies denoted by $\varphi_4$ only on the dynamics of various population classes. In this case, we take $\varphi_1 = \varphi_2 = \varphi_3 = 0$ and only activate the $\varphi_4 \neq 0$. Figs. 13 and 14 show the impact of this case graphically on the COVID-19 dynamics for $\alpha = 1$ and $\alpha = 0.80$ respectively. One can observe that the individuals in susceptible class...
Fig. 11. Impact of the treatment control $\varphi_3$ on the dynamics of various population classes with $\alpha = 1.00$. 
Fig. 12. Impact of the treatment control $\phi_3$ on the dynamics of various population classes with $\sigma = 0.80$. 
are decreased significantly. Further, individuals in exposed, undetected, and detected classes are reduced slightly as compared to the first part. The individuals in the recovered class are increased significantly. This concludes that the vaccination strategy remains a suitable way to deal with the spread of the COVID-19.

**Conclusion**

In this paper, we presented the transmission dynamics of COVID-19 using the Caputo fractional order mathematical model. Some of the basic proprieties of the model solution such as positivity, boundedness,
existence and uniqueness are studied. The basic reproduction number is obtained theoretically and its value is estimated based on the actual COVID-19 cases in Algeria. The stability results of the fractional model with respect to $R_0$ are carried out in detail. The diseases free equilibrium point is locally and globally asymptotically stable when $R_0 < 1$.

Further, we established the local and global stability of the endemic equilibrium point of the Caputo case epidemic model. The model without variable control functions is solved numerically and detailed simulation results are provided showing the impact of fractional order (memory index $\alpha$) and model key parameters on the disease dynamics.
In the second phase of the study, we perform the sensitivity analysis demonstrating that $R_0$ is most sensitive to the disease transmission rate $β$, the transmission rates of both undetected and detected people $v_1$ and $v_2$ respectively, the recovered rate of undetected people $γ_u$ and finally the recovered rate of detected people $γ_d$. Therefore, based on the sensitivity analysis, we constructed and analyzed an optimal control problem by implementing four controls, time-dependent control functions i.e., prevention measures, testing, ameliorating treatment and vaccination denoted by $ϕ_1$, $ϕ_2$, $ϕ_3$, and $ϕ_4$ respectively. The Pontryagin’s Maximum principle is taken into account to provide the existence solution of the optimal control problem. Also, we use the well known second order fractional Runge–Kutta Method to establish the numerical solution for the system with constant and the optimal control problem. Finally, to demonstrate the efficacy and the impact of controls on the spread of the COVID-19, we performed detailed simulation results by considering different combination of control measures. From numerical simulation we demonstrated the efficiency of the vaccination and the treatment amelioration strategies compared to the testing and prevention control measures. The outcomes of this study will be beneficial for the government and health departments in order to combat the COVID-19 pandemic in the community.

CRediT authorship contribution statement

Yacine El hadj Moussa: Conceptualization, Writing – original draft.
Ahmed Boudaoui: Conceptualization, Writing – original draft.
Saif Ullah: Methodology, Formal analysis, Investigation, Writing – original draft.
Khursheed Muzammil: Validation, Writing – original draft.
Muhammad Bilal Riaz: Formal analysis, Investigation, Writing – original draft.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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