Simulation of the hydroabrasive erosion of a bucket:
A multiscale model with projective integration to circumvent the spatio-temporal scale separation

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Abstract. The hydroabrasive erosion of hydraulic machine components, especially Pelton buckets and needles, is a widespread problem that entails significant costs. The multiscale nature of the process renders its simulation very computationally demanding unless specific strategies are used to approach it. A previously validated multiscale model of erosion that tackles the problem of spatial scale separation is presented. It involves two coupled submodels that describe the microscopic sediment impacts and the macroscopic turbulent sediment transport, respectively, without introducing the uncertainty inherent to empirical erosion correlations. As a further step and in order to circumvent the temporal scale separation, a novel projective integration scheme is introduced. It allows simulating the erosion process for long time periods, including the eroded surface evolution and its effect on the flow field. The proposed model is tested on a 2D case involving a bucket being eroded by a slurry jet. The results are compared qualitatively with experimental data on Pelton buckets. The main features of the erosion distribution, the surface transformation and its effect on the flow are captured correctly.

1. Introduction

Hydroabrasive erosion is the gradual removal of material from a surface in contact with a sediment-laden flow. The hydroabrasive erosion of hydraulic turbines, especially Pelton buckets and needles, is a widespread problem that entails efficiency degradation, cavitation enhancement, and outage for expensive repairs [1, 2]. There is a wide range of erosion damage mitigation strategies related to the design and operation of these machines, for instance the use of surface coatings, an ad hoc turbine geometry, strategic shutdown in periods of high sediment concentration, periodic welding repairs, etc. [3]. The ability to estimate the erosion damage a turbine will experience under given conditions would be useful to find the best compromise among the aforementioned mitigation strategies. One example could be calculating the sediment concentration above which it becomes less expensive to shut down production instead of paying for the repair costs that would be required otherwise.

Experimental studies of the erosion phenomenon date back half a century [4–6], and have provided important insight into the mechanisms involved and a series of analytical models that are used in a variety of empirical erosion correlations used to date. Apart from general
descriptions of the erosion of hydraulic machines [1, 7], the specificities of the process have also been studied, such as the erosion resistance of several martensitic steels and coatings [8], or the mechanisms involved in the erosion of a model Pelton turbine [9]. However, it proves difficult to use these results to predict the erosion of a real machine, partly because of the lack of similarity between the laboratory and prototype conditions. Although empirical models can be calibrated using field data from a specific power plant [3], their transferability remains a problem.

Numerical simulation has also been used to study the erosion problem through two different approaches. On the first approach, detailed simulations of the sediment impact problem have been performed with both finite element methods [10, 11] and smoothed particle hydrodynamics [12, 13]; Although limited to a microscopic scale, the advantage of this approach is the use of comprehensive physical models to predict the erosion process. On the second approach, computational fluid dynamics has been used to track the sediment trajectories through a macroscopic domain, together with empirical correlations to estimate the erosion caused by each impact [14, 15]. The advantage of this approach is that it allows for real world conditions: sediment size, impact angle and impact velocity distributions over large surfaces. However, the use of erosion correlations instead of material modeling greatly decreases the predictive power of this approach [16].

The present investigation extends a previously validated multiscale model of erosion [17] by using projective integration, allowing for the simulation of the erosion process of industrial-scale components without introducing the uncertainty inherent to empirical erosion correlations.

2. Modeling Methodology

Many problems of interest to engineers have a multiscale character, such that the observable macroscopic behavior emerges from countless microscopic interactions. Although it is sometimes possible to use a simplified macroscopic description by itself, including information from the microscale is oftentimes required for accuracy. Multiscale models have been proposed for a variety of problems [18–20]. In this type of approach the results of a detailed microscale model are used to improve the macroscale model which describes the problem of interest.

The hydroabrasive erosion phenomenon is inherently multiscale: It is a gradual process over large-scale surfaces, yet it is caused by countless microscopic sediment impacts. This concept is illustrated for the case of a Pelton turbine in figure 1: Whereas the sediments involved have a typical size of about 100 \( \mu m \) [1] and their impacts last less than a microsecond [21], the erosion damage they cause appears after hundreds of hours of operation and affects the whole machine. To further complicate things, a Pelton turbine may encounter about 1\( \times \)10\(^9\) sediments per second under everyday conditions. The following subsections introduce the approach used to tackle the problems of spatial and temporal scale separation illustrated in figure 1.

2.1. Multiscale Model of Erosion

With the aim of formulating a model with improved accuracy and generality compared to the state-of-the-art, the use of empirical erosion correlations has been avoided altogether. Instead, the sediment impacts and the material erosion are simulated using comprehensive physical models. However, since extending the level of detail required to simulate the sediment impacts throughout the domain of interest is not computationally feasible, we have formulated a multiscale model. Only a short description will be provided here since all the details can be found elsewhere [17], including the governing equations for all submodels, as well as the convergence analysis and validation.

The multiscale model is composed of two parts: the microscale model, which describes the sediment impacts and the erosion of the material, and the macroscale model, which describes the turbulent sediment transport through the domain of interest. In the microscale model, the mass and momentum conservation laws are solved for the solid, which is modeled as homogeneous,
isotropic and elasto-plastic; The temperature-corrected Mie-Grünen equation of state is used to close the system. The yield stress and failure plastic strain of the solid, which depend on the strain hardening, strain-rate and temperature, are captured by the Johnson-Cook strength and damage models used [22]. The material frictional and thermoplastic heating are also considered. The sediments are modeled as spherical and rigid.

In the macroscale model, the mass and momentum conservation laws are solved for the fluid, which is modeled as Newtonian and weakly compressible; The Tait equation of state and the standard $k$-$\epsilon$ turbulence model are used for closure. The sediments, considered as point masses, are tracked through the domain using a one-way coupling scheme; the hydrodynamic force considers the effect of drag, added mass, pressure gradient and shear lift. The important effect of turbulence on the sediments is considered via a continuous random walk model [23].

A sequential multiscale coupling strategy is used: First, the space of possible impact conditions is explored by independent microscale simulations, each of which involves 100 to 300 impacts at constant angle and velocity on a microscopic solid block. These simulations render the restitution coefficients and the steady-state erosion ratio (mass eroded/sediment mass) for each impact condition studied. As a second step, the macroscale sediment transport simulation on the domain of interest is carried out; every time an impact is detected, the microscale simulation results are interpolated in order to determine the sediment rebound velocity (using the restitution coefficients) and the amount of mass removed (using the steady-state erosion ratio and the sediment mass). The impact conditions and the amount of eroded mass are stored at each impact location, such that at the end of the macroscale simulation one obtains the distributions of eroded mass, erodent mass, impact angle and velocity on the surface of interest.

2.2. The Finite Volume Particle Method
The aforementioned multiscale model is simulated by discretizing its governing equations using the finite volume particle method (FVPM), which can be understood as a generalization of the classical finite volume method that consists of overlapping spherical volumes. FVPM is based on an arbitrary Lagrangian-Eulerian formulation, allowing the computational nodes to move with an arbitrary velocity. This flexibility results in a suitting description of moving interfaces, such as a fluid free surface or an erodable solid. A detailed derivation of the 3D-FVPM formulation, several validation cases and applications are made available in references [22, 24–26].

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**Figure 1.** Scales involved in the erosion simulation of a Pelton turbine. For each process, the bounds represent the simulation duration, time step, domain size and discretization size.

**Figure 2.** Projective integration of a variable $M$, whose slow dynamics depends on other parameters with fast time scales. The outer (projective) step is calculated using the variable derivative in the inner time interval, whose integration takes into account the fast dynamics.
2.3. Projective Integration Scheme

Stiff systems are characterized by a wide gap in their eigenvalue spectrum, implying an important separation between their fast and slow time scales. This kind of problem is very expensive to solve using common explicit time integration due to the very small time step required for stability of the fast dynamics; However, it can be solved efficiently by using projective integration methods [27, 28]. Consider a macroscopic variable $M$, and two time intervals, $\Delta T^i$ and $\Delta T^o$. Using a Taylor expansion, we can write

$$M(t + \Delta T^i + \Delta T^o) \approx M(t + \Delta T^i) + \Delta T^o \frac{\partial M}{\partial t} \bigg|_{(t+\Delta T^i)}.$$  \hspace{1cm} (1)

As a first step, an inner integrator with a sufficiently small time step to ensure stability of the fast dynamics is used to calculate $M$ in the inner time interval $\Delta T^i$. As a second step, $M$ is calculated after an outer time interval $\Delta T^o$ using the solution time derivative calculated using the available values in the inner interval. This idea is illustrated in figure 2. Provided $\Delta T^o \gg \Delta T^i$, projective integration provides a significant computational advantage since the expensive inner integration is only used in the inner interval $\Delta T^i$. Although the aforementioned scheme is a simple first-order forward Euler, higher-order projective integrators have been proposed [29].

As can be seen in figure 1, the erosion process is a very stiff problem: the time scale over which the surface evolves is many orders of magnitude greater than the time scale of the phenomenon which determines that evolution, namely the hydrodynamic sediment transport process. We have therefore implemented a projective integration scheme which allows using the aforementioned multiscale model to simulate the erosion of an industrial component over long time periods, including the feedback the surface modification has on the sediment transport process.

In the proposed scheme, the inner integrator corresponds to the multiscale model itself, which is run for a time period $\Delta T^i$ after which the eroded mass is stored in each of the FVPM spherical volumes (referred to as particles) which discretize the macroscopic surface being eroded. The inner integrator uses a time step small enough to ensure stability of the fluid and sediment transport models, i.e. respecting the CFL criterion; its overall integration period $\Delta T^i$ must be long enough to ensure that the erosion and impact condition distributions on the domain of interest have converged.

The outer integration step takes charge of the slow evolution of the eroded surface. The displacement of each FVPM particle on the surface due to erosion is expressed as

$$r_j = \Delta T^o \frac{m_{e,j}}{\rho A_j \Delta T^i} n_j,$$ \hspace{1cm} (2)

where $m_{e,j}$ is the amount of eroded mass accumulated on particle $j$ during the inner time period $\Delta T^i$, $\rho$ is the eroded material density, $A_j$ is the surface area of particle $j$, and $n_j$ is its inward-pointing surface normal. Note that this expression corresponds to the last term of the general example provided in equation (1). Once the surface particle positions have been updated to account for erosion, the surface normals are recomputed for consistency.

After each outer integration step, the inner multiscale integrator is resumed: The fluid field and sediment positions at the end of the previous inner integration are used as initial conditions for the following one. The outer time period $\Delta T^o$ is therefore constrained: it must be small enough to avoid drastic changes of the surface position. As highlighted in figure 2, $\Delta T^o$ is also limited in order to keep the integration error, caused by the assumption of constant erosion rate throughout the interval, within reasonable levels. Also note that during the inner step the surface erosion has no effect on the hydrodynamics; it is simply accumulated as a scalar value on each of the FVPM particles which describe the surface. In other words, the surface does not move during the inner period $\Delta T^i$. This is justified by the fact that the surface movement time scale $\Delta T^o \gg \Delta T^i$. 

3. Test Case and Results

As already mentioned, the multiscale model has been previously validated and its convergence analyzed. In this investigation, the model has been extended by means of projective integration; in this section the implementation is tested and its capabilities are illustrated in a simple case.

3.1. Test Case Description

A static 2D Pelton-like copper bucket impacted by a slurry jet, illustrated in figure 3, is selected as test case. The water jet has a diameter $D_o = 0.10$ [m], velocity $C_o = 40$ [m s$^{-1}$], turbulence intensity of 3.25 % and turbulence length scale equal to 0.03$D_o$. The bucket is defined by Bézier curves such that the half-splitter angle $\beta = 18^\circ$, the outlet angle $\gamma = 12^\circ$, the bucket width $B = 0.323$ [m] and its depth is equal to 0.25$B$. The quartz sediment diameters are randomly drawn from a Weibull distribution with scale parameter $\lambda = 175$ [$\mu$m] and shape parameter $k = 2.5$; The minimum and maximum diameter values allowed are 50 [$\mu$m] and 250 [$\mu$m].

The inner integration period $\Delta T_i = 80$ [ms], which is equal to 10 times the characteristic time $B/C_o$. During this period 64,000 sediments are injected; it has been verified that the erosion distribution on the bucket reaches a steady-state. The outer period is chosen such that the maximum wall particle displacement due to erosion is equal to half its diameter, which results in $\Delta T_o = 240 - 900$ [s]. A total of 12 outer integration steps are taken, summing up to 80 [min].

3.2. Main Simulation Results

Some of the results presented in this section involve spatial distributions over the bucket surface, which have been averaged about the symmetry plane defined by the splitter. Therefore they are presented as distributions spanning from the splitter to the bucket outlet edge.

The average sediment impact angle and velocity along the bucket surface are presented in figure 4 for three inner integration periods representative of the beginning, middle and end of the simulation. The highest average impact angle, 50$^\circ$, occurs at the splitter, where the jet impinges perpendicularly. Further downstream the impact angle stabilizes to a fairly small value of about 5$^\circ$. The minimum average impact velocity occurs at the splitter, due to the stagnation region present, and is equal to about 0.2$C_o$; this highlights the fact that the sediments are fairly responsive to the fluid acceleration. Further downstream they tend to impact at higher velocity, reaching 0.8$C_o$ near the outlet edge.

The averages presented in figure 4 are taken over specific inner intervals; they are not the average up to a given time. Therefore they represent the instantaneous average impact conditions at several moments of the bucket erosion process. It is evidenced that the surface evolution only has a minor effect on the impact conditions, the most significant of which is a small increase.
of the impact angle near the outlet edge. As discussed further down, this has to do with the well-known ripple pattern created by the erosion.

The sediment flux against the bucket surface, normalized by the inlet sediment flux, is presented in figure 5 for three inner integration periods representative of the beginning, middle and end of the simulation. The results for the first inner integration \((t = 80 \text{ [ms]})\) show a fairly smooth distribution which has an isolated peak at the splitter, and a maximum about 60 % along the bucket surface. This distribution is tightly linked to the surface curvature and turbulence intensity, which determine the centrifugal force and fluctuating velocity component, respectively, that take the sediments towards the bucket. Indeed, the sediment flux is maximum at the zone of highest curvature near the deepest part of the bucket, which coincides with the highest turbulence intensity as seen in figure 3.

The surface erosion is not perfectly uniform, so small dents appear as illustrated in the inset of figure 6. The downstream half of these dents is more likely to be impacted by the passing sediments, whereas the upstream side is relatively sheltered. This explains why the sediment flux distribution becomes spiky as the simulation advances, especially towards the bucket outlet.

The erosion rate over the bucket surface is presented in figure 6 for three inner integration periods. These distributions depend on where and how the sediments impact, i.e. on the aforementioned sediment flux, impact angle and impact velocity distributions. The results for the first inner integration \((t = 80 \text{ [ms]})\) show a smooth erosion rate distribution which peaks towards the bucket outlet, where both the sediment flux and impact velocity are high. As the surface evolves, the erosion rate distribution changes dramatically: Strong peaks appear towards the bucket exit and increase in magnitude as the surface gets eroded. This is partly explained by the appearance of dents that imply a local increase of the sediment flux, as already explained. Furthermore, as illustrated in figure 4, these dents also imply a local increase of the impact angle, which causes a greater erosion ratio according to the microscale model results.

The inset of figure 6 illustrates the progressive erosion of the bucket surface near the left outlet edge at five outer time steps. The surfaces have been shifted horizontally to allow for a clear illustration; the arrows are fixed in space so they serve as reference to assess the surface movement. It is clear that the surface does not evolve evenly, but develops dents and bulges. The downstream halves of these dents become hotspots for erosion due to the increase sediment flux and impact angle they entail.
So far we have presented spatial distributions of the quantities of interest over the bucket surface at three instants during its erosion process. Now let us analyze how certain global parameters change in time as a consequence of the bucket surface evolution.

The vertical force felt by the bucket as well as the average outflow angle as functions of time are presented in figure 7, where the error bars represent one standard deviation. The inset contains the vertical force time history for one inner integration interval $\Delta T_i$. The bucket erosion, which is most significant near the outlet, causes an increase in the outlet angle $\gamma$, which in turn implies an increase in the outflow angle. As a consequence, the vertical force felt by the bucket decreases in virtue of momentum conservation.

It is well known that the fluid does not exit the bucket at exactly the outlet angle $\gamma$, but rather at a greater angle; the difference between the outflow angle and the bucket outlet angle $\gamma$ has typical values of $5 - 8^\circ$ [30]. The first instants of the simulation, when the water is exiting the bucket for the first time, show a thin water sheet that exits at an outflow angle similar to $\gamma$. However, by the time the simulation reaches the steady state illustrated in Fig 3, the outlet water sheet is quite thick due to the bidimensional nature of the simulation; this implies an average outflow angle which is about $17^\circ$ greater than $\gamma$, as presented in figure 7 for $t = 0$.

There is an unexpected behavior during the inner integration periods corresponding to $t = 56$ and 60 [min], which lead to an unusual jump of the average outflow angle, and consequently a sudden drop in the force felt by the bucket, at $t = 64$ [min]. This behavior is in fact a consequence of the non-linear erosion process: It was found that a surface dent, similar to the ones visible in the inset of figure 6, had formed near the right bucket outlet and induced a local decrease of $\gamma$ that affected the outflow. As the erosion process continued, the dent eventually grew all the way to the outlet edge, causing the sudden increase in outflow angle presented in figure 7 at $t = 64$ [min]. After this event, the average outflow angle follows the increasing trend evidenced since the start of the erosion process.

The integral of the erosion rate over the domain is presented in figure 8 as a function of the cumulative simulation time. In line with the aforementioned erosion distribution results, an increase of the integral erosion rate is evidenced as the erosion process advances. It appears to stabilize towards the end of the simulation, reaching a value which is about 50 % higher than the initial integral erosion rate. These results highlight the important feedback between the evolution of the bucket surface caused by the erosion and the erosion process itself, which could not have been possible without the projective integration scheme.
3.3. Qualitative Comparison with Experimental Data

A comparison between the simulation results and available experimental data is not straightforward for a number of reasons. First and foremost, even if the simulated bucket resembles a Pelton bucket, its bidimensionality implies an inherent dissimilarity which makes the results incompatible. Second, the simulated bucket is made of copper due to the availability of the required material model parameters in the literature. It is clear that the erosion behavior of such a soft metal will be significantly different from the behavior of the martensitic stainless steel typically used in Pelton turbines. Third, the test case parameters used (jet velocity, sediment size distribution, bucket curvature, etc.) are realistic but do not exactly match any given Pelton turbine installation.

In order to minimize these discrepancies to allow for a qualitative comparison with experimental data, the following actions have been taken. First, the simulated erosion depth profile has been multiplied by a factor $f = 1 - 0.75 \frac{l}{l^*}$, where $l$ is the distance along the bucket surface starting at the splitter, and $l^*$ is this distance at the bucket outlet edge. This factor scales down the erosion depth proportionally to the distance from the splitter, such that $f = 1$ at the splitter and $f = 0.25$ at the bucket outlet. The motivation for such a correction is the fact that the simulation does not take into account the three-dimensional spreading of the water jet as it travels through the Pelton bucket; as such, the sediment flux and erosion are overrepresented proportionally to the distance from the splitter. The factor slope, 0.75, is chosen based on the observation that the water sheet extension at a Pelton bucket outlet is about 4 times the jet diameter [25]. Second, the erosion depth distribution is uniformly scaled to have a magnitude comparable to the available experimental data; this normalization tackles the difference caused by comparing soft copper and stainless steel buckets, as well as temporal differences (constant jet impingement in the simulation compared with the periodic impingement of the buckets of a Pelton turbine). Third, the erosion depth and spatial coordinates are normalized by the respective bucket width $B$ to compare buckets that are not exactly the same size.

Having followed the aforementioned normalization, the simulation erosion depth at $t = 38$ [min] is qualitatively compared with field data on uncoated Pelton buckets, available in literature [7], in figure 9. The experimental data is an average over four profiles: the two sides of the two bucket that were measured; the standard deviation bands therefore illustrate the erosion variation arising from otherwise identical conditions. Taking into account the above-stated
differences between the simulation and experiment, the erosion profiles agree to a considerable extent. Although the simulation underpredicts the erosion of the splitter, the overall shape of the erosion depth distribution is well captured: an isolated peak at the splitter followed by an increasing amount of erosion towards the bucket outlet. As illustrated in figure 3, the initial splitter width is 1\% B, which is more than twice the typical splitter width [30]. Since a given amount of eroded mass will imply a greater erosion depth on a sharper splitter, this difference might partly explain the apparent underestimation of the erosion depth at the splitter.

4. Discussion and Conclusions
One important advantage in terms of accuracy is that the erosion is simulated using comprehensive physical models instead of relying on less general empirical erosion correlations. The downside is the need to perform the material characterization to obtain the model parameters. We have used copper as the bucket material because of the availability of these parameters, although we are currently undertaking the characterization of martensitic stainless steel to allow for realistic simulations of the erosion of Pelton turbines.

The multiscale model has been extended by means of projective integration, which turns out to be instrumental in capturing the long-term effects of erosion. Indeed, without the use of projective integration, it would be impossible to simulate the erosion process for long enough to encounter the aforementioned effects of the surface transformation on the flow, even for this relatively inexpensive 2D simulation. Given $\Delta T^i = 80$ [ms] and the average $\Delta T^o = 400$ [s], reproducing the bucket erosion test case by using only the inner multiscale integrator would be 5,000 times more expensive than using projective integration. This figure translates to about 15 million CPU core-hours for a problem that does not efficiently scale to more than 8 cores, in which case 214 years of computation would be needed.

Note that the time frame which constitutes the long-term behavior is relative. For the case of a copper bucket, 80 [min] is enough to cause significant erosion. For a steel bucket, one would expect a significantly lower erosion rate which would allow for longer outer intervals $\Delta T^o$. The simulation would encompass a significantly longer physical time but with a similar number of outer steps and therefore a comparable computational cost.

Although the proposed model predicts the formation of dents and bulges in the eroded surface which tend to grow and move in the direction of the flow, it is not clear if these ripples exactly correspond to the physical ripples known to be caused by erosion. Further investigation on three-dimensional geometries is required to assess whether the erosion simulation actually generates ripples whose characteristics and dynamics correspond with experimental data.

The main achievement of this work is having proposed a model that is capable of providing physically sound insight into the long-term erosion process of an industrial component. It has been shown that the model captures the effect of the surface curvature and turbulence intensity on the sediment flux against the wall, the details of the impact angle and impact velocity distributions on the surface, the effect of erosion on the outflow angle and the vertical force felt by the bucket, and the important feedback effect whereby the erosion of the bucket generates a subsequent increase of the rate of erosion. Once the model is thoroughly validated with suitable experimental data, it will serve as a predictive tool whose insight might be included in the shape optimization loop of modern Pelton buckets and in the management of Pelton turbine installations by estimating the sediment concentration threshold for economically reasonable preventive shut-off or in the scheduling of repair campaigns.

Apart from the characterization of martensitic stainless steel to obtain the necessary material model parameters, ongoing work focuses on the optimization of the GPU version of our code to run on high-performance computing clusters. These developments will allow performing multiscale simulations with projective integration of rotating Pelton buckets in the near future.
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