Black hole entropy in Loop Quantum Gravity

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Abstract

We calculate the black hole entropy in Loop Quantum Gravity as a function of the horizon area and provide the exact formula for the leading and sub-leading terms. By comparison with the Bekenstein–Hawking formula we uniquely fix the value of the 'quantum of area' in the theory.

The Bekenstein–Hawking formula gives the leading term in the entropy of the black hole in the form

\[ S = \frac{1}{4} \frac{A}{l_P^2} \]  

where \( l_P \) is the Planck length and \( A \) is the black hole horizon area. According to the common belief, the entropy is always connected with the logarithm of the number of microscopic states realizing a given macroscopic state. The fact that the entropy in (1) is proportional to the area (and not as is usual to the volume) has led to the formulation of the so called holographic principle. It was however difficult to find the microscopic states that could account for such an entropy.

The problem was attacked in two different approaches – in string theory [1]-[5] and in Loop Quantum Gravity [6]-[10] (see also the review [11]). The latter approach is based on a quantum theory of geometry. The basic geometric operators were introduced in [12, 13] and a detailed description of the quantum horizon geometry was introduced using the \( U(1) \) Chern-Simons theory in [10] where one can find the detailed discussion of the states on the horizon of the black hole (see also [14]). However, the procedure introduced in [10] for state counting contained a spurious constraint on admissible sequences and the number of the relevant horizon states is underestimated. The correct method of counting was proposed in [14] but that analysis provides only lower and upper bounds on the number of states. The purpose of this paper is to rectify this situation. We will provide the correct value
of the Barbero-Immirzi parameter that is needed to obtain agreement with
the Hawking-Bekenstein formula for large black holes. Furthermore, since we
are able to calculate the number of states to sufficient accuracy, we also rig-
orously obtain the precise sub-leading quantum correction to the Hawking-
Bekenstein formula. The physical aspects of this result will be discussed
elsewhere [15].

As it was shown in [14] the proper counting of states is given by all
sequences (of arbitrary length) of \( m_i \in \mathbb{Z}/2 \), \( m_i \neq 0 \) such that

\[
\sum_i \sqrt{|m_i|(|m_i| + 1)} < a \quad (2)
\]

where

\[
a = \frac{A}{8\pi \gamma l_p} \quad (3)
\]

and \( \gamma \) is a parameter introduced in [16]. Additionally we have to impose
the condition

\[
\sum_i m_i = 0 \quad (4)
\]

It is our task to calculate the number of sequences satisfying (2) and (4).

We start with the sequences without (4) imposed. If we denote by \( N(a) \)
the number of sequences satisfying (2) then we can split the counting into
the first number \( m_1 \) and the rest and write the recurrence relation

\[
N(a) = \theta(a - \sqrt{3}/2) \left( 2N(a - \sqrt{3}/2) + 2N(a - \sqrt{2}) + \ldots \\
+ 2N(a - \sqrt{|m_i|(|m_i| + 1)} + \ldots + 2 \left[ \sqrt{4a^2 + 1} - 1 \right] \right) \quad (5)
\]

where the symbol \([\ldots]\) denotes the integer part. The first term on the RHS
corresponds to sequences with at least two elements and \( m_1 = \pm 1/2 \), the
second to sequences with at least two elements and \( m_1 = \pm 1 \) and so on and
the last term to sequences consisting of just \( m_1 \) and nothing else.

We now assume (we will later prove that it is indeed the case) that the
leading part of the solution of the recurrence relation (5) is for large \( a \) given by

\[
N(a) = Ce^{2\pi \gamma_M a} \quad (6)
\]

where \( \gamma_M \) is a constant to be determined.
Plugging the form (6) into (5) we get

\[ 1 = \sum_{k=1}^{\infty} 2 e^{-2\pi \gamma_M \sqrt{k(k+2)/4}} \]  

(7)

where we neglected terms vanishing in the limit \( a \to \infty \). The solution to this equation can be found numerically and reads

\[ \gamma_M = 0.23753295796592\ldots \]  

(8)

We see that the coefficient of growth \( \gamma_M \) is between the lower and upper bounds given in [14] i.e. \( \frac{\ln 2}{\pi} < \gamma_M < \frac{\ln 3}{\pi} \). It seems that \( \gamma_M \) defined by (7) cannot be calculated analytically or given as a solution of any “simpler” equation. In the present paper we will also use \( \tilde{\gamma} \) where

\[ \tilde{\gamma}_M := 2\pi \gamma = 1.492463591462379\ldots \]  

(9)

Since the condition (4) (as we will later prove) gives only power-like corrections to \( N(a) \) therefore for large \( a \) we have from (6)

\[ S = \ln N(a) = \frac{\gamma_M}{4\gamma} \frac{A}{l_P^2} + O(\ln A). \]  

(10)

The comparison with the Bekenstein–Hawking formula (1) gives therefore the unambiguous result

\[ \gamma = \gamma_M \]  

(11)

so this physical requirement uniquely fixes the value of \( \gamma \) and hence the ‘quantum of area’ in the theory.

One may note that the exponential ansatz (6) works in many other cases for example if the number of states for a given spin \( j \) was \((2j+1)\) instead of 2 one would have in (7) \((k+1)\) instead of 2 in front of the exponential and the appropriate \( \gamma_M \) would be equal to 0.273985635\ldots.

We now prove that the assumption (6) is actually correct and at the end we add the condition (4) – the final result will not modify (11) and will enable us to find the sub–leading (logarithmic) correction to the formula for entropy (10).

We introduce the Laplace transform of \( N(a) \):

\[ P(s) := \int_{0}^{\infty} da \, N(a) \, e^{-sa}. \]  

(12)
The transform is well defined since we know that $N(a)$ is piecewise continuous and of exponential order i.e. it is bounded by $Ke^{-\beta a}$ for some $K$ and $\beta$ ([14]). Integrating the relation (5) over $a$ from 0 to $\infty$ we get an exact result

$$P(s) = \frac{2 \sum_{k=1}^{\infty} e^{-s\sqrt{k(k+2)/4}}}{s \left(1 - 2 \sum_{k=1}^{\infty} e^{-s\sqrt{k(k+2)/4}}\right)}.$$  \hspace{1cm} (13)

If all the poles of the Laplace transform are distinct and of finite order then we can make analytical continuation to the complex half-plane $\text{Re}(s) > 0$, excluding positions of the poles, i.e. beyond the region allowed by the definition (12). We now use the well known fact that the growth of $N(a)$ is determined by the poles of $P(s)$ i.e.

$$N(a) = \sum_{s_i, \text{Re}(s_i)>0} \text{res}_{s_i} e^{s_i a} + O(a^n)$$ \hspace{1cm} (14)

(we assumed here that all the poles are simple – otherwise we would have to add some powers of $a$ in front of the exponentials). Therefore our problem boils down to determination of such complex $s_i$ that satisfy

$$G(s_i) - \frac{1}{2} = 0$$ \hspace{1cm} (15)

where we defined

$$G(s) := \sum_{k=1}^{\infty} e^{-s\sqrt{k(k+1)/4}}.$$ \hspace{1cm} (16)

It is important to analyze the distribution of zeroes of $(G(s) - \frac{1}{2})$ in more detail. Since $G(s)$ is complex analytic in the open domain $\text{Re}(s) > 0$ all the zeroes are distinct, of finite order and there is no point of accumulation of zeroes inside this domain. Therefore we have to analyze the domain’s border i.e. the imaginary line $\text{Re}(s) = 0$. To analyze the limit when $\text{Re}(s) \searrow 0$ define a new function $G_r(s)$ by subtracting two analytic (for $\text{Re}(s) > 0$) functions

$$G_r(s) := G(s) - \sum_{k=1}^{\infty} e^{-s(k+1)/2} - \frac{s}{4} \sum_{k=1}^{\infty} e^{-s(k+1)/2} \frac{1}{k}.$$ \hspace{1cm} (17)

The function $G_r(s)$ is well defined in the whole domain $\text{Re}(s) \geq 0$, continuous and doesn’t have any singularities in this domain. On the other hand two
subtracted functions have a well defined limit when \( \text{Re}(s) \searrow 0 \) so we can write
\[
G(s) = G_r(s) + \frac{e^{-s/2}}{e^{s/2} - 1} - \frac{s e^{-s/2}}{4} \ln \left( 1 - e^{-s/2} \right).
\]  (18)

We see that apart from singularities at \( s = 4\pi n \), the function \( G(s) \) has a well defined limit when \( \text{Re}(s) \searrow 0 \) and in the domain \( \text{Re}(s) > 0 \) it is complex analytic so in the whole domain \( \text{Re}(s) \geq 0 \) there can be no points of accumulation of zeroes – therefore all zeroes of the function \( G(s) - \frac{1}{2} \) in the whole domain \( \text{Re}(s) \geq 0 \) must be distinct and of finite order.

The only real zero is given by (9):
\[
s = \tilde{\gamma}_M. \tag{19}
\]

Close to this zero the function \( P(s) \) behaves as
\[
P(s) \sim \frac{C_M}{s - \tilde{\gamma}_M} \tag{20}
\]
where
\[
C_M = -\frac{1}{2\tilde{\gamma}_M G''(\tilde{\gamma}_M)} = 0.509202564 \ldots \tag{21}
\]
Therefore the leading behaviour for large \( a \) is given by
\[
N(a) = C_M e^{\tilde{\gamma}_Ma} \tag{22}
\]
as was to be shown.

The complex zeroes of the function \( G(s) - \frac{1}{2} \) for \( \text{Re}(s) > 1 \) and \( |\text{Im}(s)| < 100 \) are listed below:

| \( \text{Re}(s) \) | \( \text{Im}(s) \) |
|-----------------|------------------|
| 1.49246359\ldots | 0                |
| 1.22393017\ldots | ± 22.1530069\ldots |
| 1.41016352\ldots | ± 35.9362749\ldots |
| 1.30363023\ldots | ± 58.0472739\ldots |
| 1.18587535\ldots | ± 71.8281531\ldots |
| 1.17654302\ldots | ± 79.9673974\ldots |
| 1.07191106\ldots | ± 87.4120423\ldots |
| 1.21660746\ldots | ± 93.9713059\ldots |
It is easy to prove that all the complex zeroes (i.e. poles of $P(s)$) with nonvanishing imaginary part must have smaller real part than $\tilde{\gamma}_M$ so in comparison to the leading behaviour (22) they give exponentially small (and rapidly oscillating) contribution to $N(a)$.

To illustrate the method in the cases where the number of sequences is explicitly known let us give two examples. For sequences of half-integers with the condition $\sum |m_i| < a$ the number of sequences is $3^{[2a]} - 1$ and by the method described above we get

$$P(s) = \frac{2}{s \left( e^{s/2} - 3 \right)}$$

and indeed the real pole is equal to $2 \ln 3$. For sequences of half-integers with the condition $\sum (|m_i| + 1/2) < a$ the number of sequences is $\frac{1}{3} (2^{[2a]} + 1) - 1$ (where + is for even and - for odd $[2a]$) and

$$P(s) = \frac{2}{s \left( e^s - e^{s/2} - 2 \right)}$$

and indeed the positive real pole is equal to $2 \ln 2$. The actual numbers of sequences in both cases indeed show that the leading behaviour is given precisely by the largest positive real pole of $P(s)$.

It can be useful to discuss how different ways of defining the sequences are reflected in the properties of $P(s)$. We have essentially three different “statistics”.

- Boltzmann statistics (used in this paper): all sequences satisfying the condition are allowed and are treated as different. Therefore $P(s)$ is given by eq. (13):

$$P(s) = \frac{2}{s \left( 1 - 2 \sum_{k=1}^{\infty} e^{-s\sqrt{k(k+2)/4}} \right)}$$

which has the largest positive real pole equal to $\gamma_M$.

- the intermediate statistics (between Boltzmann and Bose): it requires that $|m_{i+1}| \geq |m_i|$ but treats for example $(+\frac{1}{2}, -\frac{1}{2}, +\frac{1}{2}, \ldots)$ as different from $(-\frac{1}{2}, +\frac{1}{2}, +\frac{1}{2}, \ldots)$ – then we have

$$P(s) = \prod_{k=1}^{\infty} \frac{1}{1 - 2e^{-s\sqrt{k(k+2)/4}}}$$
which has the largest positive real pole given by $k = 1$: 

$$s = \frac{2 \ln 2}{\sqrt{3}}.$$ 

- Bose statistics: it requires not only that $|m_{i+1}| \geq |m_i|$ but treats for example $(+ \frac{1}{2}, - \frac{1}{2}, + \frac{1}{2}, \ldots)$ as the same as $(- \frac{1}{2}, + \frac{1}{2}, + \frac{1}{2}, \ldots)$ i.e. the allowed sequences have the form $(\frac{1}{2}, \ldots, - \frac{1}{2}, \ldots, +1, \ldots, -1, \ldots)$. Then we have

$$P(s) = \prod_{k=1}^{\infty} \frac{1}{\left(1 - e^{-s} \sqrt{k(k+2)/4}\right)^2} \tag{27}$$

and this $P(s)$ does not have a pole with Re($s$) > 0 at all so $N(a)$ must have milder growth than $e^{c'a}$. However when $s \to 0$ the function $P(s)$ behaves as $P(s) \sim e^{\frac{2\zeta(2)}{s}}$ where $\zeta(s)$ is the Riemann’s zeta function. Hence we have accumulation of zeroes at $s = 0$ and the inverse Laplace transform gives the exponential growth with $\sqrt{a}$:

$$N(a) \sim a^c e^{\sqrt{8\zeta(2)a}} \tag{28}$$

(this type of statistics and therefore similar formula gives for example the degeneracy of states in string theory).

At the end we impose the condition (4) i.e. $\sum m_i = 0$. We introduce $N(a, p)$ as the number of sequences satisfying

$$\sum_i \sqrt{|m_i|(|m_i + 1|)} < a \tag{29}$$

and

$$\sum_i m_i = p \tag{30}$$

where $p \in \mathbb{Z}/2$ and $a \geq \sqrt{|p||p| + 1})$. We can write the recurrence relation as

$$N(a, p) = \theta(a - \sqrt{3}/2)\theta(a - \sqrt{|p||p| + 1}) \times$$

$$\left( N(a - \sqrt{3}/2, p - 1/2) + (N(a - \sqrt{3}/2, p + 1/2) +$$

$$+ N(a - \sqrt{2}, p - 1) + N(a - \sqrt{2}, p + 1) + \ldots + 1 \right) \tag{31}$$

where the first term inside the brackets on the RHS corresponds to sequences with at least two elements and $m_1 = +\frac{1}{2}$, the second to sequences with at
least two elements and $m_1 = -\frac{1}{2}$ and so on and the last term to sequences
consisting of just $m_1$ (which must be equal to $p$).

Next we define $P(s, \omega)$ as the Fourier transform with respect to $p$ and
Laplace transform with respect to $a$ of $N(a, p)$:

$$P(s, \omega) := \sum_{p \in \mathbb{Z}/2} \int_0^\infty da \, N(a, p) \, e^{i\omega p} \, e^{-sa}. \quad (32)$$

Then summing and integrating eq. (31) we get

$$P(s, \omega) = 2s \sum_{k=1}^\infty e^{-s\sqrt{k(k+2)/4}} \left( 1 - 2 \sum_{k=1}^\infty e^{-s\sqrt{k(k+2)/4}} \cos(k\omega/2) \right)^{-1} \quad (33)$$

where we neglected some contributions from the endpoints ($p \sim a$) what will
be justified \textit{a posteriori}. Therefore we have to find the zeroes of the function
inside the bracket on the RHS of (33). With $\omega = 0$ we know that the only
real zero (i.e. the one that gives the leading behaviour) is equal to $s = \tilde{\gamma}_M$.
Expanding around $s = \tilde{\gamma}_M$ we get

$$1 - 2 \sum_{k=1}^\infty e^{-\tilde{\gamma}_M \sqrt{k(k+2)/4}} \left( 1 - (s - \tilde{\gamma}_M) \sqrt{k(k+2)/4} - k^2 \omega^2/8 + O(\omega^4) \right) = 0. \quad (34)$$

Hence the pole of $P(s, \omega)$ is given by

$$s = \tilde{\gamma}_M - \beta_M \omega^2 + O(\omega^4) \quad (35)$$

where

$$\beta_M = -\frac{\sum_{k=1}^\infty k^2 e^{-\tilde{\gamma}_M \sqrt{k(k+2)/4}}}{8G'(\tilde{\gamma}_M)} = 0.475841255\ldots \quad (36)$$

Therefore the leading behaviour of $N(a, p)$ is given by

$$N(a, p) = \frac{C_M}{2\pi} \int d\omega \, e^{-i\omega p} \, e^{\tilde{\gamma}_M a - \beta_M \omega^2 a} = \frac{C_M}{\sqrt{4\pi \beta_M a}} \, e^{2\pi\gamma_M a} \, e^{-p^2/(4\beta_M a)}. \quad (37)$$

As we see the endpoints $p \sim a$ are exponentially small justifying (33). Also we
see \textit{a posteriori} that the terms $O(\omega^4)$ from (35) would give in (37) corrections
of relative size $1/a$ so for large $a$ we can neglect them.
Note that the gaussian distribution with respect to \( p \) in (37) with the mean square deviation of the order of \( \sqrt{a} \) can be expected by the analogy with the random walk where the average distance is zero and the mean square deviation is of the order of square root of the number of steps.

Imposing the condition \( p = 0 \) we get the final result

\[
N(a, 0) = \frac{C_M}{\sqrt{4\pi \beta M a}} e^{2\pi \gamma M a}.
\]  

Therefore the entropy is given by

\[
S = \ln N(a, 0) = \frac{\gamma M \, A}{4\gamma \, l_P^2} - \frac{\ln(1/l_P^2)}{2} + O(1)
\]  

so for large \( A/l_P^2 \) we confirm (10) as the leading behaviour and we can unambiguously both get the physical value of \( \gamma \) equal to \( \gamma_M \) (given in (8)) and resolve the controversy (summarized in [17]) about the coefficient of the logarithmic term fixing it to \(-\frac{1}{2}\).

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