The extremal limits of the C-metric: Nariai, Bertotti-Robinson and anti-Nariai C-metrics

Óscar J. C. Dias*
Centro Multidisciplinar de Astrofísica - CENTRA,
Departamento de Física, Instituto Superior Técnico,
Av. Rovisco Pais 1, 1049-001 Lisbon

José P. S. Lemos†
Department of Physics,
Columbia University, New York, NY 10027

Centro Multidisciplinar de Astrofísica - CENTRA,
Departamento de Física, Instituto Superior Técnico,
Av. Rovisco Pais 1, 1049-001 Lisbon,
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In two previous papers we have analyzed the C-metric in a background with a cosmological constant \( \Lambda \), namely the de Sitter (dS) C-metric (\( \Lambda > 0 \)), and the anti-de Sitter (AdS) C-metric (\( \Lambda < 0 \)), extending thus the original work of Kinnersley and Walker for the C-metric in flat spacetime (\( \Lambda = 0 \)). These exact solutions describe a pair of accelerated black holes in the flat or cosmological constant background, with the acceleration \( A \) being provided by a strut in-between that pushes away the two black holes or, alternatively, by strings hanging from infinity that pull them in. In this paper we analyze the extremal limits of the C-metric in a background with generic cosmological constant \( \Lambda > 0 \), \( \Lambda = 0 \), and \( \Lambda < 0 \). We follow a procedure first introduced by Ginsparg and Perry in which the Nariai solution, a spacetime which is the direct topological product of the 2-dimensional dS and a 2-sphere, is generated from the four-dimensional dS-Schwarzschild solution by taking an appropriate limit, where the black hole event horizon approaches the cosmological horizon. Similarly, one can generate the Bertotti-Robinson metric from the Reissner-Nordström metric by taking the limit of the Cauchy horizon going into the event horizon of the black hole, as well as the anti-Nariai by taking an appropriate solution and limit. Using these methods we generate the C-metric counterparts of the Nariai, Bertotti-Robinson and anti-Nariai solutions, among others. These C-metric counterparts are conformal to the product of two 2-dimensional manifolds of constant curvature, the conformal factor depending on the angular coordinate. In addition, the C-metric extremal solutions have a conical singularity at least at one of the poles of their angular surfaces. We give a physical interpretation to these solutions, e.g., in the Nariai C-metric (with topology \( dS_2 \times \tilde{S}^2 \)) to each point in the deformed 2-sphere \( \tilde{S}^2 \) corresponds a \( dS_2 \) spacetime, except for one point which corresponds a \( dS_2 \) spacetime with an infinite straight strut or string. There are other important new features that appear. One expects that the solutions found in this paper are unstable and decay into a slightly non-extreme black hole pair accelerated by a strut or by strings. Moreover, the Euclidean version of these solutions mediate the quantum process of black hole pair creation, that accompanies the decay of the dS and AdS spaces.

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I. INTRODUCTION

Three important exact solutions of general relativity are de Sitter (dS) spacetime which is a spacetime with positive cosmological constant (\( \Lambda > 0 \)), Minkowski (or flat) spacetime with \( \Lambda = 0 \), and anti-de Sitter (AdS) spacetime with negative cosmological constant (\( \Lambda < 0 \)). These stainless spacetimes, with trivial topology \( \mathbb{R}^4 \), have nonetheless a rich internal structure best displayed through the Carter-Penrose diagrams [1]. They also serve as the background to spacetimes containing black holes which are then asymptotically dS, flat, or AdS. These black holes in background spacetimes with a cosmological constant - Schwarzschild, Reissner-Nordström, Kerr, and Kerr-Newman - have a complex causal and topological structure well described in [2].

There are other very interesting solutions of general relativity with generic cosmological constant, that are neither pure nor contain a black hole, and somehow can be considered intermediate type solutions. These are the Nariai, Bertotti-Robinson, and anti-Nariai solutions [3]. The Nariai solution [4, 5] solves exactly the Einstein equations with \( \Lambda > 0 \), without or with a Maxwell field, and has been discovered by Nariai in 1951 [4]. It is the direct topological product of \( dS_2 \times S^2 \), i.e., of a \((1+1)\)-dimensional dS spacetime with a round 2-sphere of fixed radius. The Bertotti-Robinson solution [6] is an exact

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*Electronic address: oscar@fisica.ist.utl.pt
†Electronic address: lemos@physics.columbia.edu
solution of the Einstein-Maxwell equations with any \( \Lambda \), and was found independently by Bertotti and Robinson in 1959. It is the direct topological product of AdS_2 \times S^2, i.e., of a (1+1)-dimensional AdS spacetime with a round 2-sphere of fixed radius. The anti-Nariai solution, i.e., the AdS counterpart of the Nariai solution, also exists [7] and is an exact solution of the Einstein equations with \( \Lambda < 0 \), without or with a Maxwell field. It is the direct topological product of AdS_2 \times H_2, with H_2 being a 2-hyperboloid.

Three decades after Nariai’s paper, Ginsparg and Perry [8] connected the Nariai solution with the Schwarzschild-dS solution. They showed that the Nariai solution can be generated from a near-extreme dS black hole, through an appropriate limiting procedure in which the black hole horizon approaches the cosmological horizon. A similar extremal limit generates the Bertotti-Robinson solution and the anti-Nariai solution from an appropriate near-extreme black hole (see, e.g. [7]). One of the aims of Ginsparg and Perry was to study the quantum stability of the Nariai and the Schwarzschild-dS solutions [8]. It was shown that the Nariai solution is in general unstable and, once created, decays through a quantum tunnelling process into a slightly non-extreme black hole pair (for a complete review and references on this subject see, e.g., Bousso [9], and later discussions on our paper). The same kind of process happens for the Bertotti-Robinson and anti-Nariai solutions.

There is yet another class of related metrics, the C-metric class, which represent not one, but two black holes, being accelerated apart from each other. These black holes can also inhabit a dS, flat or AdS background. Following the approach of Kinnersley and Walker [10] for the \( \Lambda = 0 \) C-metric, we have analyzed in detail, in two previous papers [11, 12], the physical interpretation and properties of the dS C-metric, i.e., the C-metric with \( \Lambda > 0 \) [11], and of the AdS C-metric, i.e., the C-metric with \( \Lambda < 0 \) [12]. As occurs with the flat C-metric, the cosmological constant C-metric describes a pair of accelerated black holes with the acceleration that drives them away being provided by a strut or by strings (although in the AdS C-metric case, this is true only when the acceleration parameter \( A \) and the cosmological constant are related by \( A > |\Lambda|/3 \) [12]). When the acceleration is zero the C-metric reduces to a single non-accelerated black hole with the usual properties.

It is therefore of great interest to apply the Ginsparg-Perry procedure to these metrics in order to find a new set of exact solutions with a clear physical and geometrical interpretation. In this paper we address this issue of the extremal limits of the C-metric with a generic \( \Lambda \) following [8], in order to generate the C-metric counterparts (\( A \neq 0 \)) of the Nariai, Bertotti-Robinson and anti-Nariai solutions (\( A = 0 \)), among others.

The plan of this paper is as follows. In section II, we describe the main features of the Nariai, Bertotti-Robinson and anti-Nariai solutions. In section III, we analyze the extremal limits of the dS C-metric. We generate the Nariai C-metric, the Bertotti-Robinson dS C-metric, and the Nariai Bertotti-Robinson dS C-metric. We study the topology and the causal structure of these solutions, and we give a physical interpretation. In section IV we present the extremal limits of the flat C-metric and Ernst solution, which are obtained from the solutions of section III by taking the direct \( \Lambda = 0 \) limit. The Euclidean version of one of these \( \Lambda = 0 \) solutions has already been used previously, but we discuss two new solutions that have not been discussed previously. In section V, we discuss the extremal limits of the AdS C-metric and, in particular, we generate the anti-Nariai C-metric. Finally, in section VI concluding remarks are presented.

II. EXTREMAL LIMITS OF BLACK HOLE SOLUTIONS IN dS, FLAT, AND AdS SPACETIMES: THE NARIAI, BERTOTTI-ROBINSON AND ANTI-NARIAI SOLUTIONS

In this section we will describe the main features of the Nariai, Bertotti-Robinson and anti-Nariai solutions. The extremal limits of the C-metric that will be generated in later sections reduce to these solutions when the acceleration parameter \( A \) is set to zero.

A. The Nariai solution

The neutral Nariai solution has been first introduced by Nariai [4, 5]. It satisfies the Einstein field equations in a positive cosmological constant \( \Lambda \) background, \( G_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \), and is given by

\[
ds^2 = \Lambda^{-1}(-\sin^2 \chi \, d\tau^2 + d\chi^2 + d\theta^2 + \sin^2 \theta \, d\phi^2),
\]

where \( \chi \) and \( \theta \) both run from 0 to \( \pi \), and \( \phi \) has period \( 2\pi \).

The electromagnetic extension of the Nariai solution has been introduced by Bertotti and Robinson [6]. Its gravitational field is given by

\[
ds^2 = \frac{R_0^2}{K_0} (-\sin^2 \chi \, d\tau^2 + d\chi^2) + R_0^2 (d\theta^2 + \sin^2 \theta \, d\phi^2)
\]

where \( R_0 \) is a positive constant and constant \( K_0 \) satisfies \( 0 < K_0 \leq 1 \), while the electromagnetic field of the Nariai solution is

\[
F = q \sin \theta \, d\theta \wedge d\phi
\]

in the purely magnetic case, and

\[
F = \frac{q}{K_0} \sin \chi \, d\tau \wedge d\chi
\]

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The Nariai 4-submanifold is determined by the two conformal
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It is not an asymptotically 4-dimensional dS spacetime
causal structure as (1+1)-dimensional dS spacetime, but through its embedding in the flat manifold.
The neutral Nariai solution (1) is obtained from the charged solution (2)-(3) when one sets \( K_0 = 1 \).
The \( \Lambda = 0 \) limit of the Nariai solution, which induced the metric (6) on the Nariai hypersurface (9).

Another way [5, 13] to see clearly the topological structure, have shown that the Nariai solutions are quantum mechanically unstable.
Indeed, due to quantum fluctuations, the radius \( R_0 \) of the 2-spheres oscillates along the non-compact spatial coordinate \( \chi \) and the degenerate horizon splits back into a black hole and a cosmological horizon. Those 2-spheres whose radius fluctuates into a radius smaller than \( R_0 \) will collapse into the dS black hole interior, while the 2-spheres that have a radius greater than \( R_0 \) will suffer an exponential expansion that generates an asymptotic dS region. Therefore, the Nariai solutions are unstable and, once created, they decay through the quantum tunnelling process into a slightly non-extreme dS–Reissner-Nordström black hole pair. This issue of the Nariai instability against perturbations and associated evaporation process has been further analyzed by Bousso and Hawking [14], by Nojiri and Odintsov [19], and by Kofman, Sahni, and Starobinski [20]. Second, the Nariai Euclidean solution plays a further role as an instanton, in the quantum decay of the dS space. This decay of the dS space is accompanied by the creation of a dS black hole pair, in a process that is the gravitational analogue of the Schwinger pair production of charged particles in an external electromagnetic field. Here, the energy necessary to materialize the black hole pair and to accelerate the black holes apart comes from the cosmological constant background. It is important to note that not all of the dS black holes can be pair produced through this quantum process of black hole pair creation. Only those black holes that have regular Eu-

and the charge of the Nariai solution are related to \( R_0 \) and \( K_0 \) by

\[
\Lambda = \frac{1 + K_0}{2R_0^2},
\]

\[
q^2 = \frac{1 - K_0}{2R_0^2}.
\]

Note that \( 0 < K_0 \leq 1 \), otherwise the charge is a complex number. The charged Nariai solution satisfies the field equations of the Einstein-Maxwell action in a positive cosmological constant background, \( G_{\mu\nu} + A_{\mu\nu} = 8\pi T_{\mu\nu} \), with \( T_{\mu\nu} \) being the energy-momentum tensor of the Maxwell field. Through a redefinition of coordinates, \( \sin^2 \chi = 1 - \frac{K_0}{R_0^2} R^2 \) and \( \tau = \sqrt{\frac{R_0}{R}} T \), the spacetime (2) can be rewritten in new static coordinates as

\[
ds^2 = -N(R) \, dT^2 + \frac{dR^2}{N(R)} + R_0^2 \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right),
\]

with

\[
N(R) = 1 - \frac{K_0}{R_0^2} R^2,
\]

and the electromagnetic field changes also accordingly to the coordinate transformation. Written in these coordinates, we clearly see that the Nariai solution is the direct topological product of \( dS_2 \times S^2 \), i.e., of a (1+1)-dimensional dS spacetime with a round 2-sphere of fixed radius \( R_0 \). This spacetime is homogeneous with the same causal structure as (1+1)-dimensional dS spacetime, but it is not an asymptotically 4-dimensional dS spacetime since the radius of the 2-spheres is constant (\( R_0 \)), contrarily to what happens in the dS solution where this radius increases as one approaches infinity.

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Quite remarkably, Ginsparg and Perry [8] (see also [14]) have shown that the neutral Nariai solution (1) can be obtained from the near-extreme Schwarzschild-dS black hole through an appropriate limiting procedure. By extreme we mean that the black hole horizon and the cosmological horizon coincide. Hawking and Ross [15], and Mann and Ross [16] have concluded that a similar limiting approach takes the near-extreme dS–Reissner-Nordström black hole into the charged Nariai solution (2). In this case, by extreme we mean that the cosmological and outer charged black hole horizons coincide.

We will make heavy use of this Ginsparg-Perry procedure later, so we will not sketch it here. These relations between the near-extreme dS black holes and the Nariai solutions are a priori quite unexpected since (i) the dS black holes have a curvature singularity while the Nariai solutions do not, (ii) the Nariai spacetime is homogeneous unlike the dS black holes spacetimes, (iii) the dS black holes approach asymptotically the 4-dimensional dS spacetime while the Nariai solutions do not. The Carter-Penrose diagram of the Nariai solution is equivalent to the diagram of the (1+1)-dimensional dS solution, as will be discussed in subsection III B.

An important role played by the Nariai solution in physics is at the quantum level (see Bousso [9] for a detailed review of what follows). First, there is the issue of the stability of the solution when perturbed quantically. Ginsparg and Perry [8], in the neutral case, and Bousso and Hawking [14] and Bousso [17], in the charged case, have shown that the Nariai solutions are quantum mechanically unstable. Indeed, due to quantum fluctuations, the radius \( R_0 \) of the 2-spheres oscillates along the non-compact spatial coordinate \( \chi \) and the degenerate horizon splits back into a black hole and a cosmological horizon. Those 2-spheres whose radius fluctuates into a radius smaller than \( R_0 \) will collapse into the dS black hole interior, while the 2-spheres that have a radius greater than \( R_0 \) will suffer an exponential expansion that generates an asymptotic dS region. Therefore, the Nariai solutions are unstable and, once created, they decay through the quantum tunnelling process into a slightly non-extreme dS–Reissner-Nordström black hole pair. This issue of the Nariai instability against perturbations and associated evaporation process has been further analyzed by Bousso and Hawking [18], by Nojiri and Odintsov [19], and by Kofman, Sahni, and Starobinski [20]. Second, the Nariai Euclidean solution plays a further role as an instanton, in the quantum decay of the dS space. This decay of the dS space is accompanied by the creation of a dS black hole pair, in a process that is the gravitational analogue of the Schwinger pair production of charged particles in an external electromagnetic field. Here, the energy necessary to materialize the black hole pair and to accelerate the black holes apart comes from the cosmological constant background. It is important to note that not all of the dS black holes can be pair produced through this quantum process of black hole pair creation. Only those black holes that have regular Eu-
B. The Bertotti-Robinson solution

The simplest Bertotti-Robinson solution [6] (see also [7, 25]) is a solution of the $\Lambda = 0$ Einstein-Maxwell equations. Its gravitational field is given by

$$ds^2 = \frac{1}{q^2} (-\sinh^2 \chi \, d\tau^2 + d\chi^2) + q^2 (d\theta^2 + \sin^2 \theta \, d\phi^2),$$

(10)

where $q$ is the charge of the solution, and $\chi$ is unbounded, $\theta$ runs from 0 to $\pi$ and $\phi$ has period $2\pi$. The electromagnetic field of the Bertotti-Robinson solution is

$$F = q \sin \theta \, d\theta \wedge d\phi,$$

(11)

and

$$F = -q \sin \chi \, d\tau \wedge d\chi,$$

(12)

in the magnetic and electric cases, respectively. Through a redefinition of coordinates, $\sinh^2 \chi = R^2/q^2 - 1$ and $\tau = T/q$, the spacetime (10) can be rewritten in new static coordinates as

$$ds^2 = -N(R) \, dT^2 + \frac{dR^2}{N(R)} + q^2 (d\theta^2 + \sin^2 \theta \, d\phi^2),$$

(13)

with

$$N(R) = R^2/q^2 - 1,$$

(14)

and the electromagnetic field changes also accordingly to the coordinate transformation. Written in these coordinates, we clearly see that the Bertotti-Robinson solution is the direct topological product of $AdS_2 \times S^2$, i.e., of a (1+1)-dimensional AdS spacetime with a round 2-sphere of fixed radius $q$. Another way to see clearly the topological structure of the Bertotti-Robinson solution is achieved by defining it through its embedding in the flat manifold $M^{2,4}$, with metric

$$ds^2 = -dz_0^2 + dz_1^2 - dz_2^2 + dz_3^2 + dz_4^2 + dz_5^2.$$  

(15)

The Bertotti-Robinson 4-submanifold is determined by the two constraints

$$-z_0^2 + z_1^2 - z_2^2 = -q^2,$$

$$z_3^2 + z_4^2 + z_5^2 = q^2.$$  

(16)

The first of these constraints defines the $AdS_2$ hyperboloid and the second defines the 2-sphere of radius $q$. The parametrization of $M^{2,4}$, $z_0 = \sqrt{R^2 - q^2} \sinh (T/q)$, $z_1 = \sqrt{R^2 - q^2} \cosh (T/q)$, $z_2 = R$, $z_3 = q \sin \theta \cos \phi$, $z_4 = q \sin \theta \sin \phi$ and $z_5 = q \cos \theta$, induces the metric (13) on the Bertotti-Robinson hypersurface (16). Note that since the parametrization $z_0 = \sqrt{q^2 + R^2} \sinh (T/q)$, $z_1 = R$ and $z_2 = \sqrt{q^2 + R^2} \cos (T/q)$, also obeys (15) and the first constraint of (16), the Bertotti-Robinson solution (13) can also be written as $ds^2 = -N(R) \, dT^2 + dR^2/N(R) + q^2 (d\theta^2 + \sin^2 \theta \, d\phi^2)$, with $N(R) = R^2/q^2 + 1$.

Following a similar procedure to the one applied in the Nariai solution, the Bertotti-Robinson solution can be obtained from the near-extreme Reissner-Nordström black hole through an appropriate limiting procedure (here, by extreme we mean that the inner black hole horizon and the outer black hole horizon coincide). Later on, we will make heavy use of the Ginzburg-Perry procedure, so we will not sketch it here. Generalizations of the Bertotti-Robinson solution to include a cosmological constant background also exist [6], and are an extremal limit of the near-extreme Reissner-Nordström black holes with a cosmological constant. The Carter-Penrose diagram of the Bertotti-Robinson solution (with or without $\Lambda$) is equivalent to the diagram of the (1+1)-dimensional AdS solution, as will be discussed in subsection III C.

The Hawking effect in the Bertotti-Robinson universe has been studied by Lapedes [25], and its thermodynamic properties have been analyzed by Zaslavsky [26], and by Mann and Solodukhin [27]. In [28] the authors have shown that quantum back-reaction effects prevent a near extreme charged black hole from ever reach a Bertotti-Robinson state of precise extremality. Recently, Ortaggio and Podolský [29] have found exact solutions that describe non-expanding impulsive waves propagating in the Bertotti-Robinson universe.

C. The anti-Nariai solution

The anti-Nariai solution has a gravitational field given by [7]

$$ds^2 = \frac{R_0^2}{k_0} ( -\sinh^2 \chi \, d\tau^2 + d\chi^2) + R_0^2 (d\theta^2 + \sin^2 \theta \, d\phi^2),$$

(17)
where $\chi$ and $\theta$ are unbounded, $\phi$ has period $2\pi$, $R_0$ is a positive constant, and the constant $K_0$ satisfies $1 \leq K_0 < 2$. The electromagnetic field of the anti-Nariai solution is

$$F = q \sinh \theta \, d\theta \wedge d\phi$$  \hspace{1cm} (18)

in the purely magnetic case, and

$$F = -\frac{q}{K_0} \sinh \chi \, d\tau \wedge d\chi$$  \hspace{1cm} (19)

in the purely electric case, with $q$ being the magnetic or electric charge, respectively. The cosmological constant, $\Lambda < 0$, and the charge of the anti-Nariai solution are related to $R_0$ and $K_0$ by

$$\Lambda = -\frac{1 + K_0}{2R_0^2} < 0,$$

$$q^2 = \frac{K_0 - 1}{2} R_0^2.$$  \hspace{1cm} (20)

The neutral anti-Nariai solution ($q = 0$) is obtained from the charged solution (17) when one sets $K_0 = 1$ which implies $R_0^2 = |\Lambda|^{-1}$. The $\Lambda = 0$ limit of the anti-Nariai solution, which is Minkowski spacetime, is taken in the Appendix A. The charged anti-Nariai solution satisfies the field equations of the Einstein-Maxwell action in a negative cosmological constant background. Through a redefinition of coordinates, $\sinh^2 \chi = 1 - \frac{K_0}{2} R^2$ and $\tau = \sqrt{\frac{K_0}{2}} T$, the spacetime (17) can be rewritten in static coordinates as

$$ds^2 = -N(R) \, dT^2 + \frac{dR^2}{N(R)} + R_0^2 \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right),$$  \hspace{1cm} (21)

with

$$N(R) = -1 + \frac{K_0}{R_0^2} R^2,$$  \hspace{1cm} (22)

and the electromagnetic field changes also accordingly to the coordinate transformation. Written in these coordinates, we clearly see that the anti-Nariai solution is the direct topological product of $AdS_2 \times H_2$, i.e., of a $(1+1)$-dimensional AdS spacetime with a $2$-hyperboloid of radius $R_0$. It is a homogeneous spacetime with the same causal structure as $(1+1)$-dimensional AdS spacetime, but it is not an asymptotically $4$-dimensional AdS spacetime since the size of the $2$-hyperboloid is constant ($R_0$), contrarily to what happens in the AdS solution where this radius increases as one approaches infinity. Another way to see clearly the topological structure of the anti-Nariai solution is achieved by defining it through its embedding in the flat manifold $M^{3,3}$ with metric

$$ds^2 = -dz_0^2 + dz_1^2 - dz_2^2 + dz_3^2 + dz_4^2 - dz_5^2.$$  \hspace{1cm} (23)

The anti-Nariai $4$-submanifold is determined by the two constraints

$$-z_0^2 + z_1^2 - z_2^2 = -\ell^2,$$

$$z_3^2 + z_4^2 - z_5^2 = -R_0^2.$$  \hspace{1cm} (24)

where $\ell^2 = R_0^2/K_0$. The first of these constraints defines the $AdS_2$ hyperboloid and the second defines the $2$-hyperboloid of radius $R_0$. The parametrization of $M^{3,3}$, $z_0 = \sqrt{R^2 - \ell^2} \sin \theta (T/\ell)$, $z_1 = \sqrt{R^2 - \ell^2} \cos \theta (T/\ell)$, $z_2 = R$, $z_3 = R_0 \sin \theta \cos \theta$, $z_4 = R_0 \sin \theta \sin \theta$ and $z_5 = R_0 \cos \theta$ induces the metric (21) on the anti-Nariai hypersurface (24).

Having in mind the example of the Nariai solution, we may ask if the anti-Nariai solution can be obtained, through a similar limiting Ginsparg-Perry procedure, from a near-extreme AdS black hole. There are black holes whose horizons have topologies different from spherical, such as toroidal horizons [30], and hyperboloidal horizons [31], also called topological black holes. The AdS black hole that generates the anti-Nariai solution is the hyperbolic one [as is clear from the angular part of Eq. (21)], and has a cosmological horizon. We will make a further reference to their properties in section V. The anti-Nariai solution can be obtained from the extremal limit of these black holes when the black hole horizon approaches the cosmological horizon.

A further study of the anti-Nariai solution was done in [29] where non-expanding impulsive waves propagating in the anti-Nariai universe are described.

III. EXTREMAL LIMITS OF THE dS C-METRIC

In the last section we saw that there is an appropriate extremal limiting procedure, introduced by Ginsparg and Perry [8], that generates from the near-extreme black hole solutions the Nariai, Bertotti-Robinson and anti-Nariai solutions. Analogously, we shall apply the procedure of [8] to generate new exact solutions from the near-extreme cases of the dS C-metric. Specifically the C-metric counterparts of the Nariai and Bertotti-Robinson solutions are found using this method. When the acceleration parameter $A$ is set to zero in these new solutions, we will recover the Nariai and Bertotti-Robinson solutions. First we will briefly review the dS C-metric [11].

A. The dS C-metric

The massive charged dS C-metric has been found by Plebański and Demiański [32], and its gravitational field can be written as (see, e.g., [11])

$$ds^2 = |A(x + y)|^{-2} (-F dt^2 + F^{-1} dy^2 + G^{-1} dx^2 + G dx^2),$$  \hspace{1cm} (25)
where
\[ \mathcal{F}(y) = -\frac{3 \Lambda + 3 A^2}{3 A^2} + y^2 - 2 m A y^2 + q^2 A^2 y^4, \]
\[ \mathcal{G}(x) = 1 - x^2 - 2 m A x^3 - q^2 A^2 x^4, \]  
and the Maxwell field in the magnetic case is given by
\[ F = -q \, dx \wedge d\phi, \]
while in the electric case it is given by
\[ F = -q \, dt \wedge dy. \]

This solution depends on four parameters namely, the cosmological constant \( \Lambda \), \( A > 0 \) which is the acceleration of the black holes, and \( m \) and \( q \) which are interpreted as the ADM mass and electromagnetic charge of the non-accelerated black hole. The general shape of \( \mathcal{G}(x) \) and \( \mathcal{F}(y) \) is represented in Fig. 1. The physical properties and interpretation of this solution have been analyzed by Dias and Lemos [11], and by Podolský and Griffiths [33].

The solution has a curvature singularity at \( y = +\infty \), and \( y \) must belong to the range \( -x \leq y < +\infty \). The point \( y = -x \) corresponds to a point that is infinitely far away from the curvature singularity, thus as \( y \) increases we approach the curvature singularity and \( y + x \) is the inverse of a radial coordinate. At most, \( \mathcal{F}(y) \) can have four real zeros which we label in ascending order by \( y_{\text{neg}} < 0 < y_A \leq y_+ \leq y_- \). The roots \( y_- \) and \( y_+ \) are respectively the inner and outer charged black hole horizons, and \( y_A \) is an acceleration horizon which coincides with the cosmological horizon and has a non-spherical shape, although the topology is spherical. The negative root \( y_{\text{neg}} \) satisfies \( y_{\text{neg}} < - x \) and has no physical significance, i.e., it does not belong to the range accessible to \( y \).

We will demand that \( x \) belongs to the interval \( x_a \leq x \leq x_n \), sketched in Fig. 1 where \( \mathcal{G}(x) \geq 0 \). By doing this we guarantee that the metric has the correct signature \((-+++)\) [see Eq. (25)] and that the angular surfaces \( t = \text{const} \) and \( y = \text{const} \) are compact. For a reason that we shall explain soon we will label these surfaces by \( S^2 \). In these angular surfaces we now define two new coordinates,
\[ \theta = \int_{x_a}^{x} \mathcal{G}^{-1/2} \, dx \quad \text{and} \quad \phi = z/\kappa, \]  
where \( \phi \) ranges between \([0, 2\pi]\) and \( \kappa \) is an arbitrary positive constant which will be discussed later. The coordinate \( \theta \) ranges between the north pole, \( \theta = \theta_n = 0 \), and the south pole, \( \theta = \theta_s \) (not necessarily at \( \pi \)). Rewritten as a function of these new coordinates, the angular part of the metric becomes \( ds^2 = d\theta^2 + \kappa^2 \mathcal{G} \, d\phi^2 \).

Note that when we set \( A = 0 \) we have \( x_a \leq x \leq x_n \), with \( x_n = -1 \) and \( x_n = 1 \), \( x = \cos \theta \), \( \mathcal{G} = 1 - x^2 = \sin^2 \theta \), and \( \kappa = 1 \). Therefore, in this case the compact angular surface \( S^2 \) is a round \( S^2 \) sphere which justifies the label given to the new angular coordinates. When we set \( A \neq 0 \) the compact angular surface turns into a deformed 2-sphere that we represent onwards by \( S^2 \).

Recently, the issue of the gravitational radiation emitted by uniformly accelerated sources without horizons has been analyzed by Bičák and Krtoň [34] and the radiative properties of accelerated black holes in dS background have been analyzed by Krtoň and Podolský [35].

**B. The Nariai C-metric**

We will generate the Nariai C-metric from a special extremal limit of the dS C-metric. First we will describe this particular near-extreme solution and then we will generate the Nariai C-metric.

We are interested in a particular extreme dS C-metric, for which the size of the acceleration horizon \( y_A \) is equal to the size of the outer charged horizon \( y_+ \). Let us label this degenerated horizon by \( \rho \), i.e., \( y_A = y_+ = \rho \) with \( \rho < y_- \). In this case, the function \( \mathcal{F}(y) \) can be written as
\[ \mathcal{F}(y) = \frac{\rho^2 - 3\gamma}{\rho^4} (y - y_{\text{neg}})(y - y_-)(y - \rho)^2, \]
where
\[ \gamma = \frac{\Lambda + 3 A^2}{3 A^2}, \]  
and the roots \( \rho, y_{\text{neg}} \) and \( y_- \) are given by
\[ \rho = \frac{3m}{4q^2 A} \left( 1 - \sqrt{1 - \frac{8q^2}{9m^2}} \right), \]
\[ y_{\text{neg}} = \frac{\gamma \rho}{\rho^2 - 3\gamma} \left( 1 - \sqrt{\frac{\rho^2 - 2\gamma}{\gamma}} \right), \]
\[ y_- = \frac{\gamma \rho}{\rho^2 - 3\gamma} \left( 1 + \sqrt{\frac{\rho^2 - 2\gamma}{\gamma}} \right). \]
The mass and the charge of the solution are written as functions of $\rho$ as

\[
m = \frac{1}{A\rho} \left(1 - \frac{2\gamma}{\rho^2}\right),
\]

\[
q^2 = \frac{1}{A^2\rho^2} \left(1 - \frac{3\gamma}{\rho^2}\right).
\]  

(35)

The conditions $\rho < y_-$ and $q^2 \geq 0$ require that the allowed range of $\rho$ is

\[
\sqrt{3\gamma} \leq \rho < \sqrt{6\gamma}.
\]  

(36)

The value of $y_-$ decreases monotonically with $\rho$ and we have $\sqrt{6\gamma} < y_- < +\infty$. The mass and the charge of the Nariai solution, denoted now as $m_N$ and $q_N$ respectively, are monotonically increasing functions of $\rho$, and as we go from $\rho = \sqrt{3\gamma}$ into $\rho = \sqrt{6\gamma}$ we have

\[
\frac{1}{3\sqrt{\Lambda + 3A^2}} \leq m_N < \frac{\sqrt{7}}{3\sqrt{\Lambda + 3A^2}},
\]

\[
0 \leq q_N < \frac{1}{2\sqrt{\Lambda + 3A^2}},
\]  

(37)

so the $A \neq 0$ extreme ($y_A = y_+$) solution has a lower maximum mass and charge, and has a lower minimum mass than the corresponding $A = 0$ solution [16, 21, 22], and, for a fixed $\Lambda$, as the acceleration parameter $A$ grows these extreme values decrease monotonically. For a fixed $\Lambda$ and for a fixed mass between $\sqrt{1/(9\Lambda)} < m < \sqrt{2/(9\Lambda)}$, the allowed acceleration varies as $\sqrt{1/(27m^2)} - \Lambda/3 \leq A < \sqrt{2/(27m^2)} - \Lambda/3$.

We are now ready to obtain the Nariai C-metric. In order to generate it from the above near-extreme dS C-metric we first set

\[
y_A = \rho - \varepsilon, \quad y_+ = \rho + \varepsilon, \quad \text{with } \varepsilon << 1,
\]  

(38)

in order that $\varepsilon$ measures the deviation from degeneracy, and the limit $y_A \to y_+$ is obtained when $\varepsilon \to 0$. Now, we introduce a new time coordinate $\tau$ and a new radial coordinate $\chi$,

\[
t = \frac{1}{\varepsilon \mathcal{K}} \tau, \quad y = \rho + \varepsilon \cos \chi,
\]  

(39)

where

\[
\mathcal{K} = \frac{\rho^2 - 3\gamma}{A^2\rho^2} (\rho - y_{\text{neg}})(\rho - y_-) = \frac{2(\Lambda + 3A^2)}{A^2\rho^2} - 1,
\]  

(40)

and condition (36) implies $0 < \mathcal{K} \leq 1$ with $q = 0 \Rightarrow \mathcal{K} = 1$. In the limit $\varepsilon \to 0$, from (25) and (30), we get the gravitational field of the Nariai C-metric

\[
ds^2 = \frac{\mathcal{R}^2(x)}{\mathcal{K}} \left(\sin^2 \chi \, dx^2 + d\chi^2\right) + \mathcal{R}^2(x) \left[\mathcal{G}^{-1}(x) dx^2 + \mathcal{G}(x) dz^2\right],
\]  

(41)

where $\chi$ runs from 0 to $\pi$ and

\[
\mathcal{R}^2(x) = \left(Ax + \sqrt{\frac{2(\Lambda + 3A^2)}{1 + \Lambda}}\right)^{-2},
\]

\[
\mathcal{G}(x) = 1 - x^2 - \frac{2A}{3} \sqrt{\frac{(1 + \chi)(2 - \chi)}{2(\Lambda + 3A^2)}} x^3 - \frac{4A^2}{3} \frac{1 - \Delta^2}{1 + \Lambda + 3A^2} x^4.
\]  

(42)

$\mathcal{G}(x)$ has only two real roots, the south pole $x_s$ and the north pole $x_n$. The angular coordinate $x$ can range between these two poles, whose values are calculated in Appendix B. Under the coordinate transformation (39), the Maxwell field for the magnetic case is still given by Eq. (27), while in the electric case, (28) becomes

\[
F = \frac{q}{\mathcal{K}} \sin \chi \, d\tau \wedge d\chi.
\]  

(43)

So, if we give the parameters $\Lambda, A$, and $q$ we can construct the Nariai C-metric. The Nariai C-metric is conformal to the topological product of two 2-dimensional manifolds, $dS_2 \times S^2$, with the conformal factor $\mathcal{R}^2(x)$ depending on the angular coordinate $x$, and $S^2$ being a deformed 2-sphere.

In order to obtain the $A = 0$ limit, we first set $\hat{\rho} = A\rho$ [see Eq. (32)], a parameter that has a finite and well-defined value when $A \to 0$. Then when $A \to 0$ we have $K \to K_0 = 2\Lambda/\rho^2 - 1$ and $\mathcal{R}^2(x) \to \mathcal{R}_0^2 = \hat{\rho}^{-2}$, with $\mathcal{R}_0^2$ and $K_0$ satisfying relations (5). This, together with transformations (29), show that the Nariai C-metric transforms into the Nariai solution (2) when $A = 0$.

The limiting procedure that has been applied in this subsection has generated a new exact solution that satisfies the Maxwell-Einstein equations in a positive cosmological constant background.

In order to give a physical interpretation to this extremal limit of the dS C-metric, we first recall the physical interpretation of the dS C-metric. This solution describes a pair of uniformly accelerated black holes in a dS background, with the acceleration being provided by the cosmological constant and by a strut between the black holes that pushes them away or, alternatively, by a string that connects and pulls the black holes in. The presence of the strut or of the string is associated to the conical singularities that exist in the C-metric (see, e. g., [11]). Indeed, in general, if we draw a small circle around the north or south pole, as the radius goes to zero, the limit circumference/radius is not $2\pi$. There is a deficit angle at the north pole given by (see [11, 12]), $\delta_n = 2\pi \left(1 - \frac{2G(x_n)}{s}\right)$ (where the prime means derivative with respect to $x$) and, analogously, a similar conical singularity ($\delta_s$) is present at the south pole. The so far arbitrary parameter $\kappa$ introduced in Eq. (29) plays its important role here. Indeed, if we choose $\kappa^{-1} = \frac{1}{2} G(x_s)$ we remove the conical singularity at the south pole ($\delta_s = 0$). However, since we only have a single constant $\kappa$ at our disposal and this has been fixed to avoid the conical singularity at the south pole ($\delta_s = 0$), we conclude that a conical singularity will be
present at the north pole with \( \delta_a < 0 \). This is associated to a strut (since \( \delta_a < 0 \)) that joins the two black holes along their north poles and provides their acceleration. This strut satisfies the relation \( p = -\mu > 0 \), where \( p \) and \( \mu = \delta_a/(8\pi) \) are respectively its pressure and its mass density (see [12]). There is another alternative. We can choose instead \( \kappa^{-1} = -\frac{1}{2}[G'(x_n)] \) and by doing so we avoid the deficit angle at the north pole (\( \delta_a = 0 \)), and leave a conical singularity at the south pole (\( \delta_a > 0 \)). This option leads to the presence of a string (with \( p = -\mu < 0 \)) connecting the black holes along their south poles that furnishes the acceleration. Summarizing, when the conical singularity is at the north pole, the pressure of the strut is positive, so it points outwards and pushes the black holes apart, furnishing their acceleration. When the conical singularity is at the south pole, it is associated to a string between the two black holes with negative pressure that pulls the black holes away from each other.

The causal structure of the dS C-metric has been analyzed in detail in [11]. The Carter-Penrose diagram of the non-extreme charged dS C-metric is sketched in Fig. 2. (a) (whole figure) and has a structure that, loosely speaking, can be divided into left, middle and right regions. The middle region contains the null infinity, the past infinity, \( I^- \), and the future infinity, \( I^+ \), and an accelerated Rindler-like horizon, \( h_A \) (that coincides with the cosmological horizon). The left and right regions both contain a timelike curvature singularity (the zig-zag line), and an inner (\( h_- \)) and an outer (\( h_+ \)) horizons associated to the charged character of the solution. This diagram represents two dS–Reissner-Nordström black holes that approach asymptotically the Rindler-like acceleration horizon (for a more detailed discussion see [11]). This is also schematically represented in Fig. 3 (whole figure), where we explicitly show the strut that connects the two black holes and provides their acceleration.

Now, as we have just seen, the Nariai C-metric can be appropriately obtained from the vicinity of the black hole and acceleration horizons in the limit in which the size of these two horizons approach each other. We now will see that the conical singularity of the dS C-metric survives the near-extremal limiting procedure that generates the Nariai C-metric. Following an elucidative illustration shown in [36] for the Bertotti-Robinson solution (with \( \Lambda < 0 \) and \( A = 0 \)), this near-horizon region is sketched in Fig. 2. (a) as a shaded area, and from it we can identify some of the features of the Nariai C-metric, e.g., the curvature singularity of the original dS black hole is lost in the near-extremal limiting procedure. But, more important, this shaded near-horizon region also allows us to construct straightforwardly the Carter-Penrose diagram of the Nariai C-metric drawn in Fig. 2. (b). The construction steps are as follows. First, as indicated by (38) and the second relation of (39), we let the cross lines that represent the black hole horizon [\( h_+ \) in Fig. 2. (a)] join together with the cross lines that represent the acceleration horizon [\( h_A \) in Fig. 2. (a)], and so on, i.e., we do this joining ad infinitum with all the cross lines \( h_+ \) and \( h_A \). After this step all that is left from the original diagram is a single cross line, i.e., all the spacetime that has originally contained in the shaded area of Fig. 2. (a) has collapsed into two mutually perpendicular lines at 45° at \( y = \rho \). Now, as indicated by the first relation of (39), when \( \varepsilon \to 0 \) the time suffers an infinite blow up. To this blow up in the time corresponds an infinite expansion in the Carter-Penrose diagram in the vicinity of \( y = \rho \). We then get again the shaded area of Fig. 2. (a), but now the cross lines of this shaded area are all identified into a single horizon, and they no longer have the original signature associated to \( h_+ \) and \( h_A \) that differentiated them. This is, in the shaded area of Fig. 2. (a) we must now erase the original labels \( h_+ \) and \( h_A \). The Carter-Penrose diagram of the Nariai C-metric is then given by Fig. 2. (b), which is equivalent to the diagram of the (1+1)-dimensional dS solution. Note that the diagram of the \( A = 0 \) dS–Reissner-Nordström solution is identical to the one of Fig. 2. (a), as long as we replace \( h_A \) (acceleration horizon) by \( h_c \) (cosmological horizon). Applying the same construction process described just above we find that the Carter-Penrose diagram of the Nariai solution (\( A = 0 \)), described in subsection IIA, is also given by Fig. 2. (b).

The Nariai near-horizon region is also sketched in Fig. 3 as a shaded area. This schematic figure is clarifying in the sense that it indicates that the strut that connects the two original black holes along their north pole directions survives to the near-extremal limiting process and will be present in the final result of the process, i.e., in the Nariai C-metric. Indeed, note that the endpoints of the strut are at the north pole of the event horizons of the two black holes and crosses the acceleration horizon. As we

\[ R = 0 \]

\[ \Lambda = 0 \]
saw just above, this region suffers first a collapse followed by an infinite blow up and during the process the strut is not lost. Thus the Nariai C-metric (41)-(42) describes a spacetime that is conformal to the product $dS_2 \times S^2$. To each point in the deformed 2-sphere corresponds a $dS_2$ spacetime, except for one point which corresponds a $dS_2$ spacetime with an infinite straight strut. This strut has negative mass density given by

$$\rho = \frac{1}{\gamma^2} \left( 1 - \frac{|G'(x_n)|}{G'(x_s)} \right),$$

where $G'(x)$ is the derivative of Eq. (42), and with a positive pressure $p = -\mu > 0$. Alternatively, if we remove the conical singularity at the north pole, the Nariai C-metric describes a string with positive mass density $\rho = (1/4) (1 - |G'(x_n)/G'(x_s)|)$ and negative pressure $p = -\mu < 0$.

As we have said in subsection II A, the Nariai solution ($A = 0$) is unstable and, once created, it decays through the quantum tunnelling process into a slightly non-extreme black hole pair. We then expect that the Nariai C-metric is also unstable and that it will decay into a slightly non-extreme pair of black holes accelerated by a strut or by a string. The Nariai C-metric instanton also plays an important role in the decay of the $dS$ space, since it can mediate the Schwinger-like quantum process of pair creation of black holes in a $dS$ background [37]. Indeed, as we said in subsection II A, the Nariai instanton ($A = 0$) has been used [16, 21–23] to study the pair creation of $dS$ black holes materialized and accelerated by the cosmological background field. Moreover, the Euclidean “Nariai” flat C-metric [38] and Ernst solution [39] (also discussed in this paper in section III A) have been used to analyze the process of pair production of $A = 0$ black holes, accelerated by a string or by an electromagnetic external field, respectively. Therefore, it is natural to expect that the Euclidean Nariai limit of the $dS$ C-metric mediates the process of pair creation of black holes in a cosmological background, that are then accelerated by a string, in addition to the cosmological background field. The picture would be that of the nucleation, in a $dS$ background, of a Nariai universe, whose string then breaks down and a pair of $dS$ black holes is created at the endpoints of the string. This expectation is confirmed in [37].

C. The Bertotti-Robinson $dS$ C-metric

Now, we are interested in another particular extreme $dS$ C-metric (usually called cold solution when $A = 0$ [16, 21, 22]) for which the size of the outer charged black hole horizon $y_+$ is equal to the size of the inner charged horizon $y_-$. Let us label this degenerated horizon by $\rho$, such that, $y_+ = y_- = \rho$ and $\rho > y_A$. This solution requires the presence of the electromagnetic charge. In this case, the function $F(y)$ can be written as

$$F(y) = \frac{\rho^2 - 3\gamma(y - y_{\text{neg}})(y - y_A)(y - \rho)^2}{\rho^4},$$

with $\gamma$ given by Eq. (31), the roots $\rho$ and $y_{\text{neg}}$ are defined by Eqs. (32) and (33), respectively, and $y_A$ is given by

$$y_A = \frac{\gamma\rho}{\rho^2 - 3\gamma} \left( 1 + \sqrt{\frac{\rho^2 - 2\gamma}{\gamma}} \right).$$

Eq. (35) defines the mass and the charge of the solution as a function of $\rho$, and, for a fixed $A$ and $\Lambda$, the ratio $q/m$ is higher than 1. The conditions $\rho > y_A$ and $q^2 > 0$ require that the allowed range of $\rho$ is

$$\rho > \sqrt{6\gamma}.$$

The value of $y_A$ decreases monotonically with $\rho$ and we have $\sqrt{\gamma} < y_A < \sqrt{6\gamma}$. Contrary to the Nariai case, the mass and the charge of the Bertotti-Robinson solution, $m_{\text{BR}}$ and $q_{\text{BR}}$, respectively, are monotonically decreasing functions of $\rho$, and as we come from $\rho = +\infty$ into $\rho = \sqrt{6\gamma}$ we have

$$0 < m_{\text{BR}} < \frac{\sqrt{2}}{3 \sqrt{\Lambda + 3A^2}},$$

$$0 < q_{\text{BR}} < \frac{1}{2 \sqrt{\Lambda + 3A^2}},$$

so the $A \neq 0$ extremal ($y_+ = y_-)$ solution has a lower maximum mass and charge than the corresponding $A = 0$ solution [16, 21, 22] and, for a fixed $A$, as the acceleration parameter $A$ grows this maximum value decreases.
monotonically. For a fixed $\Lambda$ and for a fixed mass below $\sqrt{2/(9\Lambda)}$, the maximum value of the acceleration is $\sqrt{2/(27m^2)} - \Lambda/3$.

We are now ready to generate the Bertotti-Robinson dS C-metric from the above cold dS C-metric. We first set

$$y_+ = \rho - \epsilon, \quad y_- = \rho + \epsilon, \quad \text{with } \epsilon << 1,$$

in order that $\epsilon$ measures the deviation from degeneracy, and the limit $y_+ \to y_-$ is obtained when $\epsilon \to 0$. Now, we introduce a new time coordinate $\tau$ and a new radial coordinate $\chi$,

$$t = \frac{1}{\epsilon K} \tau, \quad y = \rho + \epsilon \cosh \chi,$$

where

$$K = \frac{\rho^2 - 3\gamma}{\rho^4} (\rho - \rho_{\text{neg}})(\rho - \rho_A) = 1 - \frac{2(\Lambda + 3A^2)}{A^2 \rho^2},$$

and condition (47) implies $0 < K < 1$. In the limit $\epsilon \to 0$, from (25) and (45), the metric becomes

$$ds^2 = \frac{R^2(x)}{K} \left(-\sinh^2 \chi \, d\tau^2 + d\chi^2\right) + R^2(x) \left[G^{-1}(x) dx^2 + G(x) dz^2\right],$$

where $\chi$ is unbounded and

$$R^2(x) = \left(Ax + \sqrt{\frac{2(\Lambda + 3A^2)}{A^3 - \rho^2}}\right)^{-2},$$

$$G(x) = 1 - x^2 - \frac{2A}{3} \sqrt{\frac{1-K)(2+K)}{2(\Lambda + 3A^2)}} x^3 - \frac{A^2}{4} \frac{1-K^2}{\Lambda + 3A^2} x^4.$$  

$G(x)$ has only two real roots, the south pole $x_n$ and the north pole $x_n$. The angular coordinate $x$ can range between these two poles whose value is calculated in appendix B. Under the coordinate transformation (50), the Maxwell field for the magnetic case is still given by Eq. (27), while in the electric case, (28) becomes

$$F = -\frac{q}{K} \sinh \chi \, d\tau \wedge d\chi.$$  

So, if we give the parameters $\Lambda$, $A$, and $q$ we can construct the Bertotti-Robinson dS C-metric. This solution is conformal to the topological product of two 2-dimensional manifolds, $AdS_2 \times S^2$, with the conformal factor $R^2(x)$ depending on the angular coordinate $x$.

In order to obtain the $A = 0$ limit, we first set $\tilde{\rho} = A \rho$ [see Eq. (32)], a parameter that has a finite and well-defined value when $A \to 0$. Then when $A \to 0$ we have $K \to K_0 = 1 - 2\Lambda/\tilde{\rho}^2$ and $R^2(x) \to R_0^2 = \tilde{\rho}^{-2}$. This, together with transformations (29), show that the Bertotti-Robinson dS C-metric transforms into the dS counterpart of the Bertotti-Robinson solution discussed in subsection II B, when $A = 0$. The limiting procedure that has been applied in this subsection has generated a new exact solution that satisfies the Maxwell-Einstein equations in a positive cosmological constant background.

We have just seen that the Bertotti-Robinson dS C-metric can be appropriately obtained from the vicinity of the inner and outer black hole horizons in the limit in which the size of these two horizons approach each other. This near-horizon region is sketched in Fig. 4.(a) as a shaded area, and from it we can construct straightforwardly the Carter-Penrose diagram of the Bertotti-Robinson dS C-metric drawn in Fig. 4.(b). The construction steps are as follows. First, as indicated by (49) and the second relation of (50), we let the cross lines that represent the black hole Cauchy horizon $[h_- \text{ in Fig. 4.}(a)]$ join together with the cross lines that represent the black hole event horizon $[h_+ \text{ in Fig. 4.}(a)]$, and so on, i.e., we do this junction ad infinitum with all the cross lines $h_-$ and $h_+$. After this step all that is left is the original diagram is a single cross line, i.e, all the spacetime that has originally contained in the shaded area of Fig. 4.(a) has collapsed into two mutually perpendicular lines at $45^\circ$ at $y = \rho$. Now, as indicated by the first relation of (50), when $\epsilon \to 0$ the time suffers an infinite blow up. To this blow up in the time corresponds an infinite expansion in the Carter-Penrose diagram in the vicinity of $y = \rho$ that generates an AdS$_2$ region. We then get again the shaded area of Fig. 4.(a), but now the cross lines of this shaded area are all identified into a single line, and they no longer have the original labels associated to $h_-$ and $h_+$ that differentiated them. This is, in the shaded area of Fig. 4.(a) we must now erase the original labels $h_-$ and $h_+$. The Carter-Penrose diagram of the Bertotti-Robinson dS C-metric is then given by Fig. 4.(b), which is equivalent to the diagram of the (1+1)-dimensional AdS solution. Note that the diagram of the
We can choose the parameter \( \delta \) has a conical singularity at least at one of its poles. The strut does not survive to the limiting process.

The Bertotti-Robinson near-horizon region is also sketched in Fig. 5 as a shaded area. This schematic figure is clarifying in the sense that it indicates that the strut that connects the two original black holes along their north pole directions does not survive to the near-extremal limiting process and will not be present in the final result of the process, i.e., in the Bertotti-Robinson dS C-metric. However, a reminiscence of this strut remains in the final solution. Indeed, the angular factor of the Bertotti-Robinson dS C-metric [which, remind, describes a deformed 2-sphere \( \tilde{S}^2 \) with a fixed size \( \mathcal{R}^2(x) \) given by (53)] has a conical singularity at least at one of its poles. We can choose the parameter \( \kappa \), introduced in (29), in order to have a conical singularity only at the north pole (\( \delta_n = 0 \)), or only at the south pole (\( \delta_s = 0 \)), however we cannot eliminate both. When the parameter \( A \) is set to zero, the conical singularities disappear, the angular factor describes a round 2-sphere with fixed radius, and the Bertotti-Robinson dS C-metric reduces into the Bertotti-Robinson dS solution with topology \( AdS_2 \times S^2 \).

D. The Nariai Bertotti-Robinson dS C-metric

As the previous sections and the label suggest, the Nariai Bertotti-Robinson dS C-metric will be generated from the extremal limit of a very particular dS C-metric (usually called ultracold solution when \( A = 0 \) [16, 21, 22]) for which the size of the three horizons (\( y_A, y_+ \) and \( y_- \)) are equal, and let us label this degenerated horizon by \( A = 0 \) Reissner-Nordström–dS solution is identical to the one of Fig. 4.(a). Therefore, applying the same construction process described just above we find that the Carter–Penrose diagram of the Bertotti-Robinson dS solution (\( A = 0 \)) is similar to the one of Fig. 4.(b). The diagram of the Bertotti-Robinson solution with \( \Lambda = 0 \) described by (10) is also given by Fig. 4.(b).

The mass and the charge of the Nariai Bertotti-Robinson dS C-metric solution, \( m_{\text{NBR}} \) and \( q_{\text{NBR}} \), respectively, are given by

\[
\begin{align*}
m_{\text{NBR}} &= \sqrt{2} \sqrt{\frac{1}{3} \frac{1}{\Lambda + 3A^2}}, \\
q_{\text{NBR}} &= \frac{1}{2} \sqrt{\frac{1}{\Lambda + 3A^2}},
\end{align*}
\]

and these values are the maximum values of the mass and charge of both the Nariai C and Bertotti-Robinson C solutions, Eqs. (37) and (48), respectively. To clarify the nature of these solutions, a diagram \( m\sqrt{\Lambda + 3A^2} \times q\sqrt{\Lambda + 3A^2} \) is plotted in Fig. 6. For a fixed value of \( A \) and \( \Lambda \), the allowed range of the mass and charge of the Nariai C-metric, of the Bertotti-Robinson dS C-metric, and of the Nariai Bertotti-Robinson dS-C metric is shown.

We are now ready to generate the Nariai Bertotti-Robinson dS C-metric from the above ultracold dS C-metric. We first set

\[
\rho = \sqrt{6\gamma - \varepsilon}, \quad y_- = \sqrt{6\gamma + \varepsilon}, \quad \text{with } \varepsilon << 1.
\]

Now, we introduce a new time coordinate \( \tau \) and a new radial coordinate \( \chi \),

\[
t = \frac{1}{2\varepsilon^2\mathcal{K}} \tau, \quad y = \sqrt{6\gamma + \varepsilon \cosh(\sqrt{2\varepsilon^2\mathcal{K}} \chi)},
\]

where

\[
\mathcal{K} = \frac{\rho^2 - 3\gamma}{\rho^4} (\rho - y_{\text{neg}}) = \frac{1}{3} \frac{2A^2}{\Lambda + 3A^2}.
\]
In the limit $\epsilon \to 0$ the metric (25) becomes
\[ ds^2 = R^2(x) \left[ -\chi^2 dt^2 + dx^2 + G^{-1}(x) d\theta^2 + G(x) dz^2 \right], \] (60)
with
\[ R^2(x) = \left( Ax + \sqrt{2(\Lambda + 3A^2)} \right)^{-2}, \]
\[ G(x) = 1 - x^2 - \frac{2A}{\sqrt{2(\Lambda + 3A^2)}} x^3 - \frac{A^2}{(\Lambda + 3A^2)} x^4. \] (61)

$G(x)$ has only two real roots, the south pole $x_s$ and the north pole $x_n$. The angular coordinate $x$ can range between these two poles whose value is calculated in appendix B. Notice that the spacetime factor $-\chi^2 dt^2 + dx^2$ is just $M^{1,1}$ in Rindler coordinates. Therefore, under the usual coordinate transformation $\chi = \sqrt{x^2 - t^2}$ and $\tau = \text{arctanh}(t/x)$, this factor transforms into $-d\tilde{t}^2 + dx^2$.

Under the coordinate transformation (58), the Maxwell field for the magnetic case is still given by Eq. (27), while in the electric case, (28) becomes
\[ F = -q \chi d\tau \wedge d\chi. \] (62)

The Nariai Bertotti-Robinson dS C-metric is conformal to the topological product of two 2-dimensional manifolds, $M^{1,1} \times S^2$, with the conformal factor $R^2(x)$ depending on the angular coordinate $x$.

In the $A = 0$ limit, $R^2(x) \to (2\Lambda)^{-1}$, and one obtains the Nariai Bertotti-Robinson solution $ds^2 = (2\Lambda)^{-1}(-d\tilde{t}^2 + dx^2 + d\theta^2 + \sin^2 \theta d\phi^2)$, which has the topology $M^{1,1} \times S^2$ [16, 21, 22]).

We have just seen that the Nariai Bertotti-Robinson dS C-metric can be appropriately obtained from the vicinity of the accelerated and black hole horizons in the limit in which the size of these three horizons approach each other. This near-horizon region is sketched in Fig. 7.(a) as a shaded area, and can be viewed as the intersection of the shaded areas of Figs. 2.(a) and 4.(a). From it we can construct straightforwardly, following the construction steps already sketched in subsections III B and III C, the Kruskal diagram of the Nariai Bertotti-Robinson dS C-metric drawn in Fig. 7.(b). This diagram is equivalent to the Kruskal diagram of the Rindler solution. The strut that connects the two original black holes along their north pole directions survives to the near-extremal limiting process. Thus the Nariai Bertotti-Robinson dS C-metric describes a spacetime that is conformal to the product $M^{1,1} \times S^2$. To each point in the deformed 2-sphere corresponds a $M^{1,1}$ spacetime, except for one point which corresponds to a $M^{1,1}$ spacetime with an infinite straight strut or string, with a mass density and pressure satisfying $p = -\mu$. In an analogous way to the one that occurs with the Nariai C universe (see section IV B), we expect that the Nariai Bertotti-Robinson dS C universe is unstable and, once created, it decays through the quantum tunneling process into a slightly non-extreme black hole pair. The picture would be that of the nucleation, in a dS background, of a Nariai Bertotti-Robinson dS C universe, whose string then breaks down and a pair of dS black holes is created at the endpoints of the string. This expectation is confirmed in [37]. When the parameter $A$ is set to zero, the conical singularities disappear (and so the strut or string are no longer present) and the angular factor describes a round 2-sphere with fixed radius, and the topology of the Nariai Bertotti-Robinson dS C-metric reduces into $M^{1,1} \times S^2$.

IV. EXTREMAL LIMITS OF THE FLAT C-METRIC AND OF THE ERNST SOLUTION

The Euclidean version of the “Nariai” flat C-metric that will be discussed in subsection IV B has already been used previously [38–40] in the study of the quantum process of pair creation of black holes. However, as far as we know, the Bertotti-Robinson flat C-metric (discussed in subsection IV C) and the Nariai Bertotti-Robinson flat C-metric (discussed in subsection IV D) have not been written explicitly.

A. The flat C-metric and the Ernst solution

1. The flat C-metric

The gravitational field and the electromagnetic field of the massive charged flat C-metric is given by Eqs. (25)-(28) [see [10]], as long as we set $\Lambda = 0$ in Eq. (26), and therefore we have now $F(y) = -G(-y)$ [see Fig. 8 for the general shape of $F(y)$ and $G(x)$]. The discussion of subsection III A applies then directly also to the $\Lambda = 0$ case. In particular, in general the flat C-metric has a conical singularity at one of its poles that is interpreted [10] as a strut joining the north poles of the two black
The Bertotti-Robinson case, and the inner and outer charged horizons by the presence of an accelerated horizon is indicated by IV C), and in the Nariai Bertotti-Robinson case, in the electric solution one has where

\[ q \equiv \varepsilon \quad \text{and} \quad E_0 \equiv E_0, \]

i.e., \( q \) and \( E_0 \) are respectively the electric charge and the external electric field. In the magnetic solution one has \( q \equiv g \) and \( E_0 \equiv B_0 \), i.e., \( q \) and \( E_0 \) are respectively the magnetic charge and the external magnetic field. The electromagnetic potential of the magnetic Ernst solution is [42]

\[
A_z = \frac{2}{\Sigma B_0} \left( 1 + \frac{1}{2} g B_0 x \right),
\]

while for the electric Ernst solution it is given by [43]

\[
A_t = qy - \frac{E_0}{2A^2(x+y)^2} \left( 1 + e E_0 x - \frac{1}{2} e E_0 y \right)
- \frac{E_0}{2A^2} (1 + r - Ay)(1 + r + Ay)(1 - e E_0 y/2),
\]

with \( r_+ r_- = c^2 \) and \( r_+ + r_- = 2m \). This exact solution describes two oppositely charged Reissner-Nordström black holes accelerating away from each other in a magnetic Melvin or in an electric Melvin-like background, respectively.

Technically, the Harrison-type transformation employed to generate the Ernst solution introduces, in addition to the parameter \( \kappa \), a new parameter, the external field \( E_0 \) that when appropriately chosen allow us to eliminate the conical singularities at both poles. Indeed, applying a procedure analogous to the one employed in section III A, but now focused in spacetime (63), one introduces the new angular coordinates

\[
\tilde{\theta} = \int_{x_n}^x \tilde{\mathcal{G}}^{-1/2} dx, \quad \tilde{\phi} = z/\tilde{\kappa},
\]

where \( \tilde{\mathcal{G}}(x, y) = \Sigma^{-2}(x, y) \mathcal{G}(x) \). As before, the conical singularity at the south pole is avoided by choosing

\[
\tilde{\kappa}^{-1} = \frac{1}{2} \left| \mathcal{G}'(x_n) \right|,
\]

while the conical singularity at the north pole can now be also eliminated by choosing the value of the external field \( E_0 \) to satisfy

\[
\left| \mathcal{G}'(x_n) \right| = \left| \tilde{\mathcal{G}}'(x_n) \right|.
\]

An interesting support to the physical interpretation given to the Ernst solution is the fact that in the particle limit, i.e., for small values of \( mA \), the condition (69) implies the classical Newton’s law [42]

\[
q E_0 \approx mA.
\]

So, in this regime, the acceleration is indeed provided by the Lorentz force. For a more detailed discussion on the properties of the C-metric see [39, 40]. Note that in a cosmological constant background the Harrison transformation does not work, since it does not leave invariant the cosmological term in the action. Thus Ernst’s trick cannot be used.
B. The “Nariai” flat C-metric

The Nariai solution [4] is originally a solution in the \( \Lambda > 0 \) background which can be obtained from a near-extremal limit of the dS black hole when the outer black horizon approaches the cosmological horizon. Therefore, it would seem not appropriate to use the name Nariai to label a \( \Lambda = 0 \) solution. However, in the flat C-metric the acceleration horizon plays the role of the cosmological horizon. Moreover, the limit \( \Lambda = 0 \) of the solution discussed in this subsection is equal to the limit \( \Lambda = 0 \) of the Nariai solution (see Appendix A). In this context, we find appropriate to label this solution by “Nariai” in between commas, in order to maintain the nomenclature of the paper.

This solution can be obtained from a near-extremal limit of the flat C-metric when the outer black horizon approaches the acceleration horizon. This is the way it has been first constructed in [39, 40]. However, given the Nariai C-metric \( (\Lambda > 0) \) generated in subsection III B, we can construct the “Nariai” flat C-metric \( (\Lambda = 0) \) by taking directly the \( \Lambda = 0 \) limit of Eqs. (41)-(42). The gravitational field of the “Nariai” flat C-metric is then given by Eq. (41) with

\[
R^2(x) = A^{-2} \left( x + \sqrt{\frac{6}{1 + \kappa}} \right)^{-2},
\]

\[
G(x) = 1 - x^2 - \sqrt{\frac{2(1 + \kappa)(2 - \kappa)^2}{27}} x^3 - \frac{1 - \kappa^2}{12} x^4,
\]

where \( 0 < \kappa \leq 1 \) and \( \kappa = 1 \) when \( q = 0 \). \( G(x) \) has only two real roots, the south pole \( x_s \) and the north pole \( x_n \). The angular coordinate \( x \) can range between these two poles whose value is calculated in Appendix B.

There is a great difference between the Nariai C solution of the \( \Lambda > 0 \) case and the “Nariai” C solution of the \( \Lambda = 0 \) case. In subsection III B we saw that the Nariai C-metric \( (\Lambda > 0) \) has a conical singularity at one of the poles of its deformed 2-sphere. This feature is no longer present in the “Nariai” flat C-metric, i.e., it is free of conical singularities. Indeed, in the flat C-metric we have \( F(y) = -G(-y) \). Therefore, if the outer black hole horizon coincides with the acceleration horizon then the roots \( x_0 \) and \( x_0 \) of \( G(x) \) also coincide (see Fig. 8). This implies that the range of the angular coordinate \( x \) becomes \( x_s < x < x_n \) since the proper distance between \( x_s \) and \( x_n \) goes to infinity [39, 40]. The point \( x_s \) disappears from the \( x, z \) angular section which is no longer compact but becomes topologically \( \mathbb{R} \times S^1 \) or \( \mathbb{R}^2 \). So, we have a conical singularity only at \( x = x_n \) which can be avoided by choosing \( 2\kappa^{-1} = |G'(x_n)| \). Therefore, while the Nariai C-metric \( (\Lambda > 0) \) is topologically conformal to \( dS_2 \times S^2 \), the “Nariai” flat C-metric is topologically conformal to \( dS_2 \times \mathbb{R}^2 \). Its Carter-Penrose diagram is given by Fig. 2(b). We could construct the “Nariai” Ernst solution, however since its main motivation is related to the removal of the conical singularities present in the C-metric solution and in this case our “Nariai” flat C-metric is free of conical singularities, we will not do it.

At this point, it is appropriate to find the \( \Lambda = 0 \) limit of the “Nariai” flat C-metric. This limit is not direct [see (71)]. To achieve the suitable limit we first make the coordinate rescales: \( \tilde{x} = \chi/A, \tilde{y} = x/A, \) and \( \tilde{z} = z/A \). Then, setting \( \Lambda = 0 \), the solution becomes

\[
ds^2 = (1 + \kappa)/6 \left[ K^{-1}(\tilde{x}^2 d\tilde{y}^2 + d\tilde{x}^2) + d\tilde{y}^2 + d\tilde{z}^2 \right].
\]

This limit agrees with the one taken from the \( \Lambda = 0 \) Nariai solution (written in subsection II A) in the limit \( \Lambda = 0 \) (see Appendix A). Therefore, while the limit \( \Lambda = 0 \) of the Nariai C-metric \( (\Lambda > 0) \) is topologically \( dS_2 \times S^2 \), the \( \Lambda = 0 \) limit of the “Nariai” flat C-metric is topologically \( M^{1,1} \times \mathbb{R}^2 \). This is a reminiscence of the fact that with \( \Lambda = 0 \) when \( \Lambda \) goes to zero there is no acceleration horizon to play the role of cosmological horizon that supports the extremal limit taken in this subsection. As a final remark in this subsection, we note that Horowitz and Sheinblatt [44] have taken a different extremal limit, a final remark in this subsection, we note that Horowitz and Sheinblatt [44] have taken a different extremal limit,

C. The Bertotti-Robinson flat C-metric

Given the Bertotti-Robinson dS C-metric generated in subsection III C, we can construct the Bertotti-Robinson flat C-metric by taking the direct \( \Lambda = 0 \) limit of Eqs. (52)-(53). The gravitational field of the Bertotti-Robinson flat C-metric is then given by (52) with

\[
R^2(x) = \left(Ax + \sqrt{\frac{0.42}{1 + \kappa}} \right)^{-2},
\]

\[
G(x) = 1 - x^2 - \sqrt{\frac{2(1 - \kappa)(2 + \kappa)^2}{27}} x^3 - \frac{1 - \kappa^2}{12} x^4.
\]

and \( 0 < \kappa < 1 \). \( G(x) \) has only two real roots, the south pole \( x_s \) and the north pole \( x_n \). The angular coordinate \( x \) can range between these two poles whose value is calculated in Appendix B. As in the Bertotti-Robinson dS C-metric, this solution has topology conformal to \( AdS_2 \times S^2 \), and a Carter-Penrose diagram drawn in Fig. 4(b). The solution has a conical singularity at one of the poles of its deformed 2-sphere \( S^2 \). In the \( \Lambda = 0 \) limit, we have \( \kappa \to 1 \) and \( R^2(x) \to q^2 \) and we obtain the Bertotti-Robinson solution (10) discussed in subsection II B.

From the above solution we can generate the Bertotti-Robinson Ernst solution whose gravitational field is given by

\[
ds^2 = \Sigma^2(x) \frac{R^2(x)}{\kappa} \left( -\sinh^2 \chi d\tau^2 + d\chi^2 \right) + R^2(x) \left[ \frac{\Sigma^2(x)}{G(x)} dx^2 + \frac{G(x)}{\Sigma^2(x)} dz^2 \right].
\]

with \( R^2(x) \) and \( G(x) \) given by Eq. (72), and with

\[
\Sigma(x) = \left(1 + \frac{1}{2} q E_0 x \right)^2 + \frac{1}{4} E_0^2 G(x) R^2(x).
\]
Its electromagnetic field is given by Eq. (65), in the pure magnetic case, and by Eq. (66), in the pure electric case. Choosing $k$ satisfying Eq. (68), with $\mathcal{G}(x) = \Sigma^{-2}(x)\mathcal{F}(x)$, and $\mathcal{E}$ such that condition (69) is satisfied, the Bertotti-Robinson Ernst solution is free of conical singularities. As a final remark in this subsection, we note that Dowker, Gauntlett, Kastor and Traschen [40] have taken a different extremal limit, which differs from the one discussed in this subsection mainly because it preserves the asymptotic behavior and topology of the original flat C-metric solution.

D. The “Nariai” Bertotti-Robinson flat C-metric

From the Nariai Bertotti-Robinson dS C-metric generated in subsection IIID, we can construct the “Nariai” Bertotti-Robinson flat C-metric by taking the direct $\Lambda = 0$ limit of Eqs. (60)-(61). We can also construct the “Nariai” Bertotti-Robinson Ernst solution. We do not do these limits here because they are now straightforward.

V. EXTREMAL LIMITS OF THE AdS C-METRIC

In the AdS background there are three different families of C-metrics. When we set $A = 0$, each one of these families reduces to a different single AdS black hole. Before we deal with the AdS C-metric we will then briefly describe the main features of each one of these black holes.

The Einstein equations with $\Lambda < 0$ admit a three-family of black hole solutions whose gravitational field is described by

$$ds^2 = -V(r) dt^2 + \frac{dr^2}{V(r)} + r^2 d\Omega_5^2,$$

where $b$ can take the values 1, 0, −1 and

$$\left\{ \begin{array}{l}
d\Omega_5^2 = d\theta^2 + \sin^2 \theta d\phi^2 & \text{for } b = 1, \\
d\Omega_5^2 = d\theta^2 + d\phi^2 & \text{for } b = 0, \\
d\Omega_5^2 = d\theta^2 + \sin^2 \theta d\phi^2 & \text{for } b = -1. \end{array} \right.$$  

These three solutions describe three different kind of black holes. The black holes with $b = 1$ are the usual AdS black holes with spherical topology. The black holes with $b = 0$ have planar, cylindrical or toroidal (with genus $g \geq 1$) topology and were introduced and analyzed in [30]. The topology of the $b = -1$ black holes is hyperbolic or, upon compactification, toroidal with genus $g \geq 2$, and they have been analyzed in [31]. These hyperbolic black holes have negative mass. They have a cosmological horizon (contrarily to what happens with the $b = 1$ and $b = 0$ cases) and their charged version has only one black hole horizon (as opposed to the charged $b = 1$ and $b = 0$ cases which have an inner and an outer black hole horizon). The hyperbolic topology together with the presence of the cosmological horizon turn these $b = -1$ black holes into the appropriate solutions that allow the generation of the anti-Nariai solution (17) with the limiting Ginsparg-Perry procedure, when the black hole horizon approaches the cosmological horizon.

Now, to each one of these families corresponds a different AdS C-metric which has been found by Plebański and Demiański [32]. In what follows we will then describe each one of these solutions and, in particular, we will generate a new solution - the anti-Nariai C-metric.

A. Extremal limits of the AdS C-metric with spherical horizons

The gravitational field of the massive charged AdS C-metric [32] is given by Eq. (25) with

$$\mathcal{F}(y) = \frac{|A| - 3A^2}{3A^2} + y^2 - 2mA^3 + q^2A^2y^4,$$

and the electromagnetic field is given by (27) and (28). This solution describes a pair of accelerated black holes in the AdS background [12] when the acceleration $A$ and the cosmological constant are related by $A > |\Lambda|/3$. When we set $A = 0$ [12, 45] we obtain a single non-accelerated AdS black hole with spherical topology described by Eqs. (75)-(77) with $b = 1$.

In a way analogous to the one described in last sections we can generate new solutions from the extremal limits of the AdS C-metric. Since this follows straightforwardly, we do not do it here. The new relevant feature of this case is the fact that the Nariai-like extremal solution only exists when $A > |\Lambda|/3$. In this case, and only in this one, the acceleration horizon is present in the AdS C-metric and we can have the outer black hole horizon approaching it.

B. Extremal limits of the AdS C-metric with toroidal horizons

The gravitational field of the massive charged toroidal AdS C-metric (see [32, 46]) is given by Eq. (25) with

$$\mathcal{F}(y) = \frac{|A| - 3A^2}{3A^2} - 2mA^3 + q^2A^2y^4,$$

and the electromagnetic field is given by (27) and (28). When we set $A = 0$ we obtain the AdS black hole with planar, cylindrical or toroidal topology described by Eqs. (75)-(77) with $b = 0$ [30].
In a way analogous to the one described in section III we can generate new solutions from the extremal limits of the toroidal AdS C-metric, that are the toroidal AdS counterparts of the Nariai and Bertotti-Robinson C-metrics. Since this follows straightforwardly, we do not do it here.

C. Extremal limits of the AdS C-metric with hyperbolic horizons

The gravitational field of the massive charged AdS C-metric with hyperbolic horizons, the hyperbolic AdS C-metric, is given by Eq. (25) with [32, 46]

\[
F(y) = \frac{|A| + 3A^2}{3A^2} - y^2 - 2mAy^3 + q^2A^2y^4,
\]

\[
G(x) = -1 + x^2 - 2mA^2x^2 - q^2A^2x^4
\]

(80)

(represented in Fig. 9), and the electromagnetic field is given by (27) and (28).

This solution depends on four parameters namely, the cosmological constant \(\Lambda < 0\), the acceleration parameter \(A > 0\), and \(m\) and \(q\) which are mass and electromagnetic charge parameters, respectively. When \(A = 0\) this solution reduces to the hyperbolic black holes [case \(b = -1\) discussed in Eqs. (75)-(77)].

![Diagram](image)

FIG. 9: Shape of \(G(x)\) and \(F(y)\) for a general non-extremal charged massive hyperbolic AdS C-metric studied in section V C. The allowed range of \(x\) is between \(x_s\) and \(x_n\) where \(G(x)\) is positive and compact. The permitted range of \(y\) is \(-x \leq y < +\infty\). The presence of an accelerated horizon is indicated by \(y_A\) and the black hole horizon by \(y_+\). In the anti-Nariai case considered in subsection V C.1, \(y_A\) and \(y_+\) coincide. [For completeness we comment on a case not represented in the figure but discussed on the text: when \(q = 0\), the zero \(x_n\) of \(G(x)\) disappears, and \(G(x)\) grows monotonically from \(x = x_s\) into \(x = +\infty\).]

1. The anti-Nariai C-metric

We are interested in a particular extreme hyperbolic AdS C-metric, for which \(y_A = y_+\) (see Fig. 9), and let us label this degenerated horizon by \(y\): \(y_A = y_+ = \rho\). In this case, the function \(F(y)\) can be written as

\[
F(y) = -\frac{\rho^2 - 3\gamma}{\rho^4} (y - y_{\text{neg}}) (y - y'_{\text{neg}}) (y - \rho)^2,
\]

where

\[
\gamma = \frac{|A| + 3A^2}{3A^2},
\]

and the degenerate root \(\rho\), and the negative roots \(y_{\text{neg}}\) and \(y'_{\text{neg}}\) are given by

\[
\rho = \frac{3m}{4q^2A} \left(1 + \sqrt{1 + \frac{8q^2}{9m^2}}\right),
\]

\[
y_{\text{neg}} = -\frac{\gamma\rho}{3\gamma - \rho^2} \left(1 + \sqrt{\frac{\rho^2 - 2\gamma}{\gamma}}\right),
\]

\[
y'_{\text{neg}} = -\frac{\gamma\rho}{3\gamma - \rho^2} \left(1 - \sqrt{\frac{\rho^2 - 2\gamma}{\gamma}}\right).
\]

The mass parameter and the charge parameter of the solution are written as a function of \(\rho\) as

\[
m = -\frac{1}{A\rho} \left(1 - \frac{2\gamma}{\rho^2}\right),
\]

\[
q^2 = \frac{1}{A^2\rho^2} \left(\frac{3\gamma}{\rho^2} - 1\right).
\]

The requirement that \(y_{\text{neg}}\) and \(y'_{\text{neg}}\) are real roots and the condition \(q^2 \geq 0\) require that the allowed range of \(\rho\) is

\[
2\gamma < \rho^2 \leq 3\gamma.
\]

The mass and the charge of the anti-Nariai type solution, \(m_{\text{aN}}\) and \(q_{\text{aN}}\), respectively, are both monotonically decreasing functions of \(\rho\), and as one comes from \(\rho = \sqrt{3\gamma}\) into \(\rho = \sqrt{\frac{3\gamma}{2}}\) one has

\[
-\frac{1}{3\sqrt{|A| + 3A^2}} \leq m_{\text{aN}} < 0,
\]

\[
0 \leq q_{\text{aN}} < \frac{\sqrt{3}}{2\sqrt{|A| + 3A^2}}.
\]

In order to generate the anti-Nariai C-metric from the near-extreme topological AdS C-metric we first set

\[
y_A = \rho - \epsilon, \quad y_+ = \rho + \epsilon, \quad \text{with } \epsilon < 0,\]

in order that \(\epsilon\) measures the deviation from degeneracy, and the limit \(y_A \rightarrow y_+\) is obtained when \(\epsilon \rightarrow 0\). Now, we introduce a new time coordinate \(\tau\) and a new radial coordinate \(\chi\),

\[
t = \frac{1}{\epsilon\chi} \tau, \quad y = \rho + \epsilon \cosh \chi,
\]
when
\[ K = \frac{3\gamma - \rho^2}{\rho^4} (\rho - y_{\text{neg}})(\rho - y'_{\text{neg}}) = \frac{2(|A| + 3A^2)}{A^2 \rho^2} - 1, \]
(89)
and \(2\gamma < \rho^2 \leq 3\gamma\) implies \(1 \leq K < 2\) with \(q = 0 \Rightarrow K = 1\). In the limit \(\varepsilon \to 0\), from Eq. (25) and Eq. (81), the metric becomes
\[ ds^2 = \frac{R^2(x)}{K} \left( - \sinh^2 \chi \, dr^2 + d\chi^2 \right) + R^2(x) \left[ G^{-1}(x)dx^2 + G(x)dz^2 \right]. \]
(90)
where
\[ R^2(x) = \left( Ax + \sqrt{2(\Lambda + 3A^2)} \right)^{-2}, \]
(91)
\[ G(x) = -1 + x^2 + \frac{4}{3} \sqrt{2(\Lambda + K)(2 - K)} x^3 + \frac{A^2}{4} \frac{K^2 - 1}{|A| + 3A^2} x^4. \]
(92)
Under the coordinate transformation (88), the Maxwell field for the magnetic case is still given by Eq. (27), while in the electric case, (28) becomes
\[ F = -\frac{q}{K} \sinh \chi \, d\tau \wedge d\chi. \]
(93)
The Carter-Penrose diagram of the anti-Nariai C-metric is also given by Fig. 4(b).

At this point, let us focus our attention on the angular surfaces with \(\tau = \text{constant}\) and \(\chi = \text{constant}\). When \(q \neq 0\), we choose \(x\) such that it belongs to the range \([x_n, x_h]\) (sketched in Fig. 9) where \(G(x) \geq 0\). In this way, the metric has the correct signature, the angular surfaces are compact, and the allowed range of \(y\) includes the acceleration \((y_A)\) and black hole \((y_+)\) horizons (if we had chosen the other possible interval of \(x\) where \(G(x) \geq 0\), sketched in Fig. 9, this last condition would not be satisfied). We unavoidably have a conical singularity at least at one of the poles of this compact surface (that we label by \(\hat{S}^2\), say). Thus, the charged anti-Nariai C-metric has a compact angular surface of fixed size with a conical singularity at one of its poles. It is topologically conformal to \(AdS_2 \times \hat{S}^2\). When \(q = 0\), the zero \(x_n\) of \(G(x)\) disappears, and \(G(x)\) grows monotonically from \(x = x_h\) into \(x = +\infty\). Then, the angular surfaces are not compact, and we have a single pole \((x = x_h)\) with a conical singularity, which can be eliminated. Thus, the neutral anti-Nariai C-metric has a non-compact angular surface of fixed size (a kind of a deformed 2-hyperboloid that we label by \(\tilde{H}_2\), say) which is free of conical singularities. It is topologically conformal to \(AdS_2 \times \tilde{H}_2\).

In order to obtain the \(A = 0\) limit, we first set \(\bar{\rho} = A \rho\) [see first relation of (83)], a parameter that has a finite and well-defined value when \(A \to 0\). Then when \(A \to 0\) we have \(K \to K_0 = 2|\Lambda|/\rho^2 - 1\) and \(R^2 \to R^2_0 = \bar{\rho}^{-2}\), with \(R^2_0\) and \(K_0\) satisfying relations (20). Moreover, when we set \(A = 0\), \(m \neq 0\) and \(q \neq 0\), the coordinate transformations \(\theta = \int_{x_n} x G^{-1/2} dx\) and \(\phi = \int y G^{-1/2} dy\) imply that \(x \in [x_n, +\infty[\), \(x = \cosh \theta\), and \(G = -1 + x^2 = \sinh^2 \theta\). The angular surface then reduces to a 2-hyperboloid, \(H_2\), of fixed size with line element \(d\theta^2 + \sinh^2 \theta \, d\phi^2\). Therefore, when \(A = 0\) the anti-Nariai C-metric reduces to the anti-Nariai solution (17) described in subsection II C, with topology \(AdS_2 \times H_2\). The limiting procedure that has been applied in this subsection has generated a new exact solution that satisfies the Maxwell-Einstein equations in a negative cosmological constant background.

2. Other extremal limits

We could also discuss other extremal limits of the charged topological AdS C-metric (see Fig. 9), but these do not seem to be so interesting.

VI. CONCLUSIONS

Following the limiting approach first introduced by Ginsparg and Perry [8], we have analyzed the extremal limits of the dS, flat and AdS C-metrics. Among other new exact solutions, we have generated the Nariai C-metric, the Bertotti-Robinson C-metric, the Nariai Bertotti-Robinson C-metric and the anti-Nariai C-metric. These solutions are the C-metric counterparts of the well know solutions found in the 1950’s. They are specified by an extra parameter: the acceleration parameter of the C-metric from which they are generated. When we set \(A = 0\) the solutions found in this paper reduce to the Nariai, the Bertotti-Robinson, the Nariai Bertotti-Robinson and the anti-Nariai solutions.

One of the features of these \(A = 0\) solutions is the fact that they are topologically the direct product of two 2-dimensional manifolds of constant curvature. Their C-metric counterparts are conformal to this topology, with the conformal factor depending on the angular coordinate. Moreover, the angular surfaces of these new C-solutions have a fixed size, but they lose the symmetry of the \(A = 0\) counterparts. For example, while the angular surfaces of the Nariai and Bertotti-Robinson solutions are round 2-spheres, the angular surfaces of their C-metric counterparts are deformed 2-spheres - they are compact but not round. Another important difference between the \(A = 0\) and \(A \neq 0\) solutions is the fact that the \(A \neq 0\) solutions have, in general, a conical singularity at least at one of the poles of their angular surfaces. This conical singularity is a reminiscence of the conical singularity that is present in the C-metric from which they were generated. In the C-metric these conical singularities are associated to the presence of a strut or string that furnishes the acceleration of the near-extremal black holes. In this context, we find that the Nariai C-metric generated from an extremal limit of the dS C-metric describes a spacetime that is conformal to the product \(dS_2 \times \hat{S}^2\). To each point
in the deformed 2-sphere corresponds a $dS_2$ spacetime, except for one point which corresponds a $dS_2$ spacetime with an infinite straight strut or string, with a mass density and pressure satisfying $p = -\mu$. Analogously, the Nariai Bertotti-Robinson $dS$ C-metric describes a spacetime that is conformal to the product $\mathbb{M}^{1,1} \times S^2$. To each point in the deformed 2-sphere corresponds a $\mathbb{M}^{1,1}$ spacetime, except for one point which corresponds a $\mathbb{M}^{1,1}$ spacetime with an infinite straight strut or string. In the case of the Bertotti-Robinson $dS$ C-metric (topologically conformal to $AdS_2 \times S^2$), the strut or string does not survive to the Ginsparg-Perry limiting procedure, and thus in the end of the process we only have a conical singularity.

In what concerns the causal structure, the Carter-Penrose diagrams of the $A \neq 0$ solutions are equal to those of the $A = 0$ solutions. For example, the diagram of the Nariai C-metric is equal to the one that describes the $(1+1)$-dimensional $dS$ solution, the diagram of the Bertotti-Robinson C-metric and of the anti-Nariai C-metric is equal to the one that describes the $(1+1)$-dimensional $AdS$ solution, and the diagram of the Nariai Bertotti-Robinson C-metric is given by the Rindler diagram.

Some of these solutions, perhaps all, are certainly of physical interest. Indeed, it is known that the Nariai solution ($A = 0$) is unstable and, once created, it decays through the quantum tunneling process into a slightly non-extreme black hole pair [9]. We then expect that the Nariai C-metric is also unstable and that it will decay into a slightly non-extreme pair of black holes accelerated by a strut or by a string. The solutions found in this paper also play an important role in the decay of the $dS$ or $AdS$ spaces, and therefore can mediate the Schwinger-like quantum process of pair creation of black holes. Indeed, the Nariai, and the $dS$ Nariai Bertotti-Robinson instants ($A = 0$) are one of the few Euclidean solutions that are regular, and have thus been used [16, 21–23] to study the pair creation of $dS$ black holes materialized and accelerated by the cosmological constant background field (an instanton is a solution of the Euclidean field equations that smoothly connects the spacelike sections of the initial state, the pure $dS$ space in this case, and the final state, the $dS$ black hole pair in this case). Moreover, the Euclidean “Nariai” flat C-metric [38] and Ernst solution [39, 40] (also discussed in this paper) have been used to analyze the process of pair production of $\Lambda = 0$ black holes, accelerated by a string or by an electromagnetic external field, respectively. Therefore, its natural to expect that the Euclidean extremal limits of the $dS$ C-metric and $AdS$ C-metric found in this paper mediate the process of pair creation of black holes in a cosmological background, that are then accelerated by a string, in addition to the cosmological field acceleration. For the Nariai case, e.g., the picture would be that of the nucleation, in a $dS$ background, of a Nariai C universe, whose string then breaks down and a pair of $dS$ black holes is created at the endpoints of the string. This expectation is confirmed in [37].

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**APPENDIX A: $\Lambda = 0$ LIMIT OF THE NARIAI AND ANTI-NARIAI SOLUTIONS**

In this appendix we find the $\Lambda = 0$ limit of the Nariai solution, Eq. (2), and of the anti-Nariai solution, Eq. (17). In this limit the line element of both solutions goes apparently to infinity since $R_0^2 \to \infty$. To achieve the suitable limit of the Nariai solution, we first make the coordinate rescales: $\tilde{x} = (R_0/\sqrt{\lambda_0}) x$, and $\tilde{\varrho} = R_0 \theta$. Then, taking the $\Lambda = 0$ limit, the solution becomes

$$ds^2 = (-\tilde{x}^2 d\tau^2 + d\tilde{x}^2) + (d\tilde{\varrho}^2 + \tilde{\varrho}^2 d\tilde{\varphi}^2). \quad (A1)$$

The spacetime sector is a Rindler factor, and the angular sector is a cylinder. Therefore, under the usual coordinate transformation $\tilde{x} = \sqrt{\lambda^2 - \tau^2}$ and $\tau = \text{arctanh}(\tilde{t}/\tilde{x})$, and unwinding the cylinder ($\tilde{y} = \tilde{\varrho} \cos \tilde{\varphi}$ and $\tilde{z} = \tilde{\varrho} \sin \tilde{\varphi}$), we have

$$ds^2 = (-d\tilde{t}^2 + d\tilde{x}^2) + (d\tilde{y}^2 + d\tilde{z}^2). \quad (A2)$$

Therefore, while the Nariai C-metric is topologically $dS_2 \times S^2$, its $\Lambda = 0$ limit is topologically $\mathbb{M}^{1,1} \times \mathbb{R}^2$, i.e., $\mathbb{R}^4$.

A similar procedure shows that taking the $\Lambda = 0$ limit of the anti-Nariai solution (17) ($AdS_2 \times H_2$) leads to Eq. (A2).

**APPENDIX B: DETERMINATION OF THE NORTH AND SOUTH POLES**

In this appendix we discuss the zeros of the function $G(x)$ that appears in the extremal limits of the $dS$ C-metric, and in the extremal limits of the flat C-metric. This function $G(x)$ has only two real zeros in the cases discussed in this paper, namely the Nariai, the Bertotti-Robinson, and the Nariai Bertotti-Robinson (both for $\Lambda > 0$ and $\Lambda = 0$). These two roots are the south pole and the north pole, and are respectively given by

$$x_n = -p + \frac{h}{2} - \frac{a}{4b} < 0,$$

$$x_s = +p + \frac{h}{2} - \frac{a}{4b} > 0. \quad (B1)$$
with

\[
p = \frac{1}{2} \left( \frac{s}{3} + \frac{a^2}{2b^2} - \frac{1 - 12b}{3s_b^2} - \frac{4}{3b} + n \right)^{1/2},
\]

\[
n = -\frac{a^3}{4b^3},
\]

\[
h = \sqrt{\frac{s}{3} + \frac{a^2}{4b^2} + \frac{1 - 12b}{3s_b^2} - \frac{2}{s}},
\]

\[
s = \frac{1}{21^{1/3}b} \left( \lambda - \sqrt{x^2 - 4(1 - 12b)^2} \right)^{1/3},
\]

where \(a\) and \(b\) are, respectively, the absolute values of the coefficients of \(x^3\) and \(x^4\) in Eqs. (42), (53), (61), (71), and (72). For the function \(G(x)\) written in a different polynomial form that facilitates the determination of its zeros see Hong and Teo [47].

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