On a functional equation appearing in characterization of distributions by the optimality of an estimate

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Let $X$ be a second countable locally compact Abelian group containing no subgroup topologically isomorphic to the circle group $\mathbb{T}$. Let $\mu$ be a probability distribution on $X$ such that its characteristic function $\hat{\mu}(y)$ does not vanish and $\hat{\mu}(y)$ for some $n \geq 3$ satisfies the equation

$$
\prod_{j=1}^{n} \hat{\mu}(y_j + y) = \prod_{j=1}^{n} \hat{\mu}(y_j - y), \quad \sum_{j=1}^{n} y_j = 0, \quad y_1, \ldots, y_n, y \in Y.
$$

Then $\mu$ is a convolution of a Gaussian distribution and a distribution supported in the subgroup of $X$ generated by elements of order 2.

The present note is devoted to study a functional equation on a locally compact Abelian group which appears in characterization of probability distributions by the optimality of an estimate.

Let $X$ be a second countable locally compact Abelian group, $Y = X^*$ be its character group, $(x, y)$ be the value of a character $y \in Y$ at an element $x \in X$. Denote by $M^1(X)$ the convolution semigroup of probability distribution on the group $X$, and denote by

$$
\hat{\mu}(y) = \int_X (x, y) d\mu(x)
$$

the characteristic function of a distribution $\mu \in M^1(X)$. For $\mu \in M^1(X)$ define the distribution $\bar{\mu} \in M^1(X)$ by the formula $\bar{\mu}(B) = \mu(-B)$ for all Borel sets of $X$. Then $\hat{\bar{\mu}}(y) = \hat{\mu}(y)$.

A distribution $\gamma \in M^1(X)$ is called Gaussian if its characteristic function is represented in the form

$$
\hat{\gamma}(y) = (x, y) \exp\{-\varphi(y)\}, \quad (1)
$$

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where \( x \in X \), and \( \varphi(y) \) is a continuous nonnegative function on the group \( Y \) satisfying the equation

\[
\varphi(y_1 + y_2) + \varphi(y_1 - y_2) = 2[\varphi(y_1) + \varphi(y_2)], \quad y_1, y_2 \in Y.
\]  

A Gaussian distribution is called symmetric if in (1) \( x = 0 \). Denote by \( \Gamma(X) \) the set of Gaussian distributions on the group \( X \).

Consider a probability space \((X, \mathcal{B}, \mu)\), where \( \mathcal{B} \) is a \( \sigma \)-algebra of Borel subsets of \( X \), and \( \mu \in M^1(X) \). Form a family of distributions \( \mu_\theta(A) = \mu(A - \theta) \), \( A \in \mathcal{B}, \theta \in X \). Denote by \( \Pi \) a class of estimates \( f : X^n \rightarrow X \) satisfying the condition \( f(x_1 + c, \ldots, x_n + c) = f(x_1, \ldots, x_n) + nc \) for all \( x_1, \ldots, x_n, c \in X \). According to \( [1] \) (see also \( [2], [3, Ch. 7, \S 7.10] \)), an estimate \( f_0 \in \Pi \) of a parameter \( n\theta \) is called an optimal estimate in the class \( \Pi \) for a sample volume \( n \) if for any estimate \( f \in \Pi \) and for all \( y \in Y \) the inequality

\[
E_\theta[(f_0(x), y) - (n\theta, y)]^2 \leq E_\theta[(f(x), y) - (n\theta, y)]^2
\]

holds. It turns out that the existence of an optimal estimate of the parameter \( n\theta \) gives the possibility in some cases to describe completely the possible distributions \( \mu \).

As has been proved in \( [1] \), if an estimate \( f_0 \) is represented in the form

\[
(f_0(x), y) = (f(x), y) \frac{E_\theta[(f(x), -y)|z]}{E_\theta[(f(x), -y)]}, \quad y \in Y,
\]

where \( f \in \Pi \), and \( z = (x_2 - x_1, \ldots, x_n - x_1) \), then \( f_0 \in \Pi \), \( f_0 \) does not depend on the choice of \( f \) and \( f_0 \) is an optimal estimate. It follows from \( [3] \) that \( f_0 \) is an optimal estimate if and only if \( \arg E_\theta[(f_0(x), y)|z] = 0 \). When \( f_0(x) = \sum_{j=1}^n x_j \) it follows from this that the characteristic function \( \hat{\mu}(y) \) satisfies the equation

\[
\prod_{j=1}^n \hat{\mu}(y_j + y) = \prod_{j=1}^n \hat{\mu}(y_j - y), \quad \sum_{j=1}^n y_j = 0, \quad y_1, \ldots, y_n, y \in Y,
\]

and \( \hat{\mu}^n(y) > 0 \). When \( n \geq 3 \) this implies that if a group \( X \) contains no elements of order 2, then \( \mu \in \Gamma(X) \) (see \( [1] \)).

This note is devoted to solving of equation \( (1) \) in a general case when \( X \) is a locally compact Abelian group. Let us fix the notation. Denote by \( f_2 : X \rightarrow X \) the endomorphism of \( X \) defined by the formula \( f_2(x) = 2x \). Put \( X^{(2)} = \text{Ker} f_2, X^{(2)} = \text{Im} f_2 \). Denote by \( \mathbb{T} \) the circle group, and by \( \mathbb{Z} \) the group of integers.

Let \( \psi(y) \) be an arbitrary function on the group \( Y \) and \( h \in Y \). Denote by \( \Delta_h \) the finite difference operator

\[
\Delta_h \psi(y) = \psi(y + h) - \psi(y), \quad y \in Y.
\]

A continuous function \( \psi(y) \) on the group \( Y \) is called a polynomial if

\[
\Delta_h^{m+1} \psi(y) = 0
\]

for some \( m \) and for all \( y, h \in Y \). The minimal \( m \) for which \( (5) \) holds is called the degree of the polynomial \( \psi(y) \).
From analytical point of view the result proved in [1] can be reformulated in the following way. Let \( \mu \in M^1(X) \), the characteristic function \( \hat{\mu}(y) \) satisfy equation (4) for some \( n \geq 3 \) and \( \hat{\mu}^n(y) > 0 \). Then if the group \( X \) contains no elements of order 2, then \( \mu \in \Gamma(X) \).

It is easy to see that if \( \gamma \) is a symmetric Gaussian distribution on the group \( X \) and \( \pi \in M^1(X_{(2)}) \), then the characteristic functions \( \hat{\gamma}(y) \) and \( \hat{\pi}(y) \) satisfy equation (4), and hence the characteristic function of the distribution \( \mu = \gamma \ast \pi \) also satisfies equation (4). Describe first the groups \( X \) for which the converse statement is true.

**Theorem 1.** Let \( X \) be a second countable locally compact Abelian group, \( \mu \in M^1(X) \). Let the characteristic function \( \hat{\mu}(y) \) satisfy equation (4) for some \( n \geq 3 \) and \( \hat{\mu}(y) \neq 0 \). Assume that the following condition holds: (i) the group \( X \) contains no subgroup topologically isomorphic to the circle group \( \mathbb{T} \). Then \( \mu = \gamma \ast \pi \), where \( \gamma \in \Gamma(X) \) and \( \pi \in M^1(X_{(2)}) \).

**Proof.** Set \( \nu = \mu \ast \bar{\mu} \). Then \( \hat{\nu}(y) = |\hat{\mu}(y)|^2 > 0 \). Put \( \psi(y) = -\ln \hat{\nu}(y) \). Equation (4) is equivalent to the equation

\[
\sum_{j=1}^{n} \psi(y_j + y) = \sum_{j=1}^{n} \psi(y_j - y), \quad \sum_{j=1}^{n} y_j = 0, \quad y_1, \ldots, y_n, y \in Y. \tag{6}
\]

We also note that

\[
\psi(-y) = \psi(y), \quad y \in Y. \tag{7}
\]

Substituting in (6) \( y_3 = -y_1 - y_2, \ y_4 = \cdots = y_n = 0 \) and taking into account (7), we get

\[
\psi(y_1 + y_2 + y) - \psi(y_1 + y_2 - y) = \psi(y_1 + y) - \psi(y_1 - y) + \psi(y_2 + y) - \psi(y_2 - y), \quad y_1, y_2, \ y \in Y. \tag{8}
\]

Setting successively \( y = y_1 + y_2, \ y = y_1, \ y = y_2 \), we find from (7) that

\[
\psi(2y_1 + 2y_2) = \psi(2y_1) + 2\psi(y_1 + y_2) - 2\psi(y_1 - y_2) + \psi(2y_2), \quad y_1, y_2 \in Y.
\]

This implies that

\[
\psi(2y_1 + 2y_2) + \psi(2y_1 - 2y_2) = 2[\psi(2y_1) + \psi(2y_2)], \quad y_1, y_2 \in Y,
\]

i.e. the function \( \psi(y) \) satisfies equation (2) on the subgroup \( Y^{(2)} \), and hence, the function \( \psi(y) \) satisfies equation (2) on the subgroup \( Y^{(2)} \). Denote by \( \varphi_0(y) \) the restriction of the function \( \psi(y) \) to the subgroup \( Y^{(2)} \).

It is well known that we can associate to each function \( \varphi(y) \) satisfying equation (2) a symmetric 2-additive function

\[
\Phi(u, v) = \frac{1}{2}[\varphi(u + v) - \varphi(u) - \varphi(v)], \quad u, v \in Y.
\]

Then \( \varphi(y) = \Phi(y, y) \). Using this representation it is not difficult to verify that the function \( \psi(y) \) on the subgroup \( Y^{(2)} \) satisfies the equation

\[
\Delta^2_k \Delta_{2h} \psi(y) = 0, \quad k, \ y \in Y^{(2)}, \quad h \in Y. \tag{9}
\]
Return to equation (9) and apply the finite difference method to solve it. Let \( h_1 \) be an arbitrary element of the group \( Y \). Put \( k_1 = h_1 \). Substitute \( y_2 + h_1 \) for \( y_2 \) and \( y + k_1 \) for \( y \) in equation (9). Subtracting equation (9) from the resulting equation we obtain

\[
\Delta_{2h_1} \psi(y_1 + y_2 + y) = \Delta_{h_1} \psi(y_1 + y) - \Delta_{-h_1} \psi(y_1 - y) + \Delta_{2h_1} \psi(y_2 + y). \tag{10}
\]

Next, let \( h_2 \) be an arbitrary element of the group \( Y \). Put \( k_2 = -h_2 \). Substitute \( y_2 + h_2 \) for \( y_2 \) and \( y + k_2 \) for \( y \) in equation (10). Subtracting equation (10) from the resulting equation we get

\[
\Delta_{-h_2} \Delta_{h_1} \psi(y_1 + y) - \Delta_{h_2} \Delta_{-h_1} \psi(y_1 - y) = 0.
\]

Reasoning similarly we find from this

\[
\Delta_{2h_3} \Delta_{-h_2} \Delta_{h_1} \psi(y_1 + y) = 0,
\]

and finally

\[
\Delta_{2h_3} \Delta_{-h_2} \Delta_{h_1} \Delta_{y_1} \psi(y) = 0. \tag{11}
\]

Note that \( h_j, y_1, y \) are arbitrary elements of the group \( Y \). Setting in (11) \( h_3 = h, -h_2 = h_1 = y_1 = k \), we find

\[
\Delta_k \Delta_{2h} \psi(y) = 0, \quad k, \ h, \ y \in Y. \tag{12}
\]

Fix \( h \in Y \). On the one hand, it follows from (12) that the function \( \Delta_{2h} \psi(y) \) is a polynomial of degree \( \leq 2 \) on the group \( Y \). On the other hand, as follows from (9) the function \( \Delta_{2h} \psi(y) \) is a polynomial of degree \( \leq 1 \) on the subgroup \( Y^{(2)} \). Then as not difficult to verify, the function \( \Delta_{2h} \psi(y) \) must be a polynomial of degree \( \leq 1 \) on the group \( Y \), i.e.

\[
\Delta_k \Delta_{2h} \psi(y) = 0, \quad k, \ h, \ y \in Y. \tag{13}
\]

Theorem 1 follows now from the following lemma.

**Lemma 1** ([5, Prop. 1]). Let \( X \) be a second countable locally compact Abelian group containing no subgroup topologically isomorphic to the circle group \( \mathbb{T} \). Let \( \mu \in \mathcal{M}^1(X) \), \( \nu = \mu * \bar{\mu} \) and \( \tilde{\nu}(y) = \exp\{-\psi(y)\} \), where the function \( \psi(y) \) satisfies equation (13). Then \( \mu = \gamma * \pi \), where \( \gamma \in \Gamma(X) \) and \( \pi \in \mathcal{M}^1(X^{(2)}) \).

**Remark 1.** Obviously, the above mentioned Rukhin’s theorem follows directly from Theorem 1.

**Remark 2.** Let \( X \) be a second countable locally compact Abelian group containing a subgroup topologically isomorphic to the circle group \( \mathbb{T} \). Then we can consider any distribution \( \mu \) on the circle group \( \mathbb{T} \) as a distribution on \( X \). Note that \( \mathbb{Z} \) is the character group of \( \mathbb{T} \). Following to [1] consider on the group \( \mathbb{Z} \) the function

\[
f(m) = \begin{cases} 
\exp\{-m^2\}, & \text{if } m \in \mathbb{Z}^{(2)}, \\
\exp\{-m^2 + \varepsilon\}, & \text{if } m \notin \mathbb{Z}^{(2)},
\end{cases}
\]

where \( \varepsilon > 0 \).
where \( \varepsilon > 0 \) is small enough. Then
\[
\rho(t) = \sum_{m=-\infty}^{\infty} f(m)e^{-imt} > 0.
\]

Let \( \mu \) be a distribution on \( \mathbb{T} \) with density \( \rho(t) \) with respect to the Lebesgue measure. Then \( f(m) \) is the characteristic function of a distribution \( \mu \) on the circle group \( \mathbb{T} \). Considering \( \mu \) as a distribution on the group \( X \), we see that \( \hat{\mu}(y) > 0 \) and the characteristic function \( \hat{\mu}(y) \) satisfies equation (13), but as easily seen, \( \mu \notin \Gamma(X) \ast M^1(X_{(2)}) \). This example shows that condition (i) in Theorem 1 is sharp.

**Remark 3.** Let \( X \) be a second countable locally compact Abelian group. In the articles [4] and [5] (see also [6, §16]) were studied group analogs of the well-known Heyde theorem, where a Gaussian distribution is characterized by the symmetry of the conditional distribution of a linear form \( L_2 = \beta_1 \xi_1 + \cdots + \beta_n \xi_n \) of independent random variables \( \xi_j \) given \( L_1 = \alpha_1 \xi_1 + \cdots + \alpha_n \xi_n \) (coefficients of the forms are topological automorphisms of the group \( X \)). Let \( \hat{\mu}_j(y) \) be the characteristic function of the random variable \( \xi_j \). It is interesting to remark that if the number of independent random variables \( n = 2 \), then the functions \( \psi_j(y) = -\ln |\hat{\mu}_j(y)|^2 \) also satisfy equation (13). For the groups \( X \) containing no subgroup topologically isomorphic to the circle group \( \mathbb{T} \), and also for the two-dimensional torus \( X = \mathbb{T}^2 \) this implies that all \( \mu_j \in \Gamma(X) \ast M^1(X_{(2)}) \).

We use Theorem 1 to prove the following statement, a significant part of which refers to the case when the group \( X \) contains a subgroup topologically isomorphic to the circle group \( \mathbb{T} \).

**Theorem 2.** Let \( X \) be a second countable locally compact Abelian group. Let \( \mu \in M^1(X) \), let the characteristic function \( \hat{\mu}(y) \) satisfy equation (13) for some odd \( n \), and \( \hat{\mu}^n(y) > 0 \). Assume that the group \( X \) satisfies the condition: (i) the subgroup \( X_{(2)} \) is finite. Then \( \mu = \gamma_0 \ast \pi \), where \( \gamma_0 \in \Gamma(X) \), and \( \pi \) is a signed measure on \( X_{(2)} \).

**Proof.** Put \( \psi(y) = -\ln |\hat{\mu}(y)| \). Then the function \( \psi(y) \) satisfies equation (13). As has been proved in [5] in this case the function \( \psi(y) \) is represented in the form
\[
\psi(y) = \varphi(y) + r_\alpha, \quad y \in y_\alpha + \overline{Y_{(2)}},
\]
where \( \varphi(y) \) is a continuous function satisfying equation (2), and \( Y = \bigcup_\alpha (y_\alpha + \overline{Y_{(2)}}) \) is a decomposition of the group \( Y \) with respect to the subgroup \( \overline{Y_{(2)}} \). Since \( X_{(2)} \) is a finite subgroup, it is easy to see that the function \( g(y) = \exp\{-r_\alpha\}, \quad y \in y_\alpha + \overline{Y_{(2)}} \), is the characteristic function of a signed measure \( \pi \) on the subgroup \( X_{(2)} \). It follows from this that
\[
|\hat{\mu}(y)| = \hat{\gamma}(y)\hat{\pi}(y),
\]
where \( \gamma \in \Gamma(X) \) and \( \hat{\gamma}(y) = \exp\{-\varphi(y)\} \).

Set \( l(y) = \hat{\mu}(y)/|\hat{\mu}(y)| \) and check that the function \( l(y) \) is a character of the group \( Y \). Hence, Theorem 2 will be proved.
Note that the function \( l(y) \) satisfies equation \((14)\) and
\[
l(-y) = \overline{l(y)}, \quad l^n(y) = 1, \quad y \in Y.
\] (14)

Put in \((14)\) \( y_2 = -y_1, y_3 = \cdots = y_n = 0 \). We get
\[
l^{n-2}(y)l(y_1 + y)l(-y_1 + y) = l^{n-2}(-y)l(y_1 - y)l(-y_1 - y), \quad y, y_1, y_2 \in Y.
\]

Taking into account \((14)\), it follows from this that
\[
l^2(y + y_1)l^2(y - y_1) = l^4(y), \quad y, y_1 \in Y.
\]

Set \( m(y) = l^2(y) \). Then the function \( m(y) \) satisfies the equation
\[
m(u + v)m(u - v) = m^2(u), \quad u, v \in Y. \tag{15}
\]

We find by induction from \((15)\) that
\[
m(py) = m^p(y), \quad p \in \mathbb{Z}, \quad y \in Y. \tag{16}
\]

Now we formulate as a lemma the following statement.

**Lemma 2.** Let \( Y = Y_1 + Y_2 \), let a continuous function \( m(y) \) on \( Y \) satisfy equation \((15)\) and \( m^n(y) = 1 \) for some odd \( n \). Then, if the restriction of the function \( m(y) \) to \( Y_j \) is a character of the group \( Y_j \), \( j = 1, 2 \), then \( m(y) \) is a character of the group \( Y \).

**Proof.** Denote by \( y = (y_1, y_2), y_1 \in Y_1, y_2 \in Y_2 \) elements of the group \( Y \). Put \( a(y_1, y_2) = m(y_1, 0)m(0, y_2), b(y_1, y_2) = m(y_1, y_2)/a(y_1, y_2) \). Then \( b(y_1, 0) = b(0, y_2) = 1, \quad (y_1, y_2) \in Y_1, \quad (y_1, y_2) \in Y_2 \).

It is obvious that the function \( b(y_1, y_2) \) also satisfies equation \((15)\). Substitute in \((15)\) \( u = (y_1, 0), v = (y_1, y_2) \). We have
\[
b(2y_1, y_2)b(0, -y_2) = b^2(y_1, 0), \quad y_1 \in Y_1, \quad y_2 \in Y_2.
\]

This implies that \( b(2y_1, y_2) = 1 \) for \( y_1 \in Y_1, \quad y_2 \in Y_2 \). In particular, \( b(2y_1, 2y_2) = 1 \). But it follows from \((16)\) that \( b(2y_1, 2y_2) = b^2(y_1, y_2) \). Hence, \( b(y_1, y_2) = \pm 1 \). Since \( b^n(y_1, y_2) = 1 \) and \( n \) is odd, we have \( b(y_1, y_2) = 1 \) for \( y_1 \in Y_1, \quad y_2 \in Y_2 \), i.e. \( m(y_1, y_2) = a(y_1, y_2) \) is a character of the group \( Y \).

Continue the proof of Theorem 2. Since, by the assumption, \( X_{(2)} \) is a finite subgroup, there exist \( q \geq 0 \) such that the group \( X \) contains a subgroup topologically isomorphic to the group \( \mathbb{T}^q \), but \( X \) does not contain a subgroup topologically isomorphic to the group \( \mathbb{T}^{q+1} \). It is well known that a subgroup of \( X \) topologically isomorphic to a group of the form \( \mathbb{T}^n \) is a topologically direct summand in \( X \). For this reason the group \( X \) is represented in the form \( X = \mathbb{T}^q + G \), where the group \( G \) contains no subgroup topologically isomorphic to the circle group \( \mathbb{T} \). We have \( Y \cong \mathbb{Z}^q + H, \quad H = G^* \). It follows from Lemma 2 and \((16)\) by induction that the function \( m(y) \) on the group \( \mathbb{Z}^q \), satisfying equation \((15)\) and the condition \( m^n(y) = 1 \) is a character of the group \( \mathbb{Z}^q \). By Theorem 1 the restriction of the function \( \hat{\mu}(y) \) to \( H \) is a product of the characteristic function of a Gaussian distribution on the group \( G \) and the characteristic function of a Gaussian distribution on the group \( \mathbb{Z}^q \).
function of a distribution on the subgroup $G(2)$. Taking into account that the characteristic function of any distribution on $G(2)$ takes only real values, it follows from the equality

$$\hat{\mu}^2(y) = |\hat{\mu}(y)|^2 m(y), \quad y \in Y,$$

that the restriction of the function $m(y)$ to $H$ is a character of the subgroup $H$. Applying again Lemma 2 to the group $Y$, we obtain that $m(y)$ is a character of the group $Y$.

Since $n$ is odd, we have $2r + ns = 1$ for some integers $r$ and $s$. Taking into account (14) this implies that $l(y) = (l(y))^{2r+ns} = (m(y))^r$ is a character of the group $Y$. Theorem 2 is completely proved.

We note that the example given in Remark 2 shows that a signed measure $\pi$ needs not be a measure.

**Remark 4.** Consider the infinite-dimensional torus $X = \mathbb{T}^{\aleph_0}$. Then $Y \cong \mathbb{Z}^{\aleph_0*}$, where $\mathbb{Z}^{\aleph_0*}$ is the group of all sequences of integers such that in each sequence only finite number of members are not equal to zero.

Consider on the group $\mathbb{Z}$ the sequence of the functions

$$f_k(m) = \begin{cases} \exp\{-a_km^2\}, & \text{if } m \in \mathbb{Z}^{(2)}, \\ \exp\{-a_km^2 + k\}, & \text{if } m \not\in \mathbb{Z}^{(2)}, \end{cases}$$

where $k = 1, 2, \ldots$. Put

$$f(m_1, \ldots, m_l, 0, \ldots) = \prod_{k=1}^{l} f_k(m_k), \quad (m_1, \ldots, m_l, 0, \ldots) \in \mathbb{Z}^{\aleph_0*}.$$

Take $a_k > 0$ such that

$$\sum_{(m_1, \ldots, m_l, 0, \ldots) \in \mathbb{Z}^{\aleph_0*}} f(m_1, \ldots, m_l, 0, \ldots) < 2.$$

Then

$$\rho(t_1, \ldots, t_l, \ldots) = \sum_{(m_1, \ldots, m_l, 0, \ldots) \in \mathbb{Z}^{\aleph_0*}} f(m_1, \ldots, m_l, 0, \ldots) e^{-i(m_1t_1 + \cdots + m_l t_l + \cdots)} > 0, \quad t_j \in \mathbb{R}.$$

It follows from this that $f(m_1, \ldots, m_l, 0, \ldots)$ is the characteristic function of a distribution $\mu \in M^1(\mathbb{T}^{\aleph_0})$ such that $\hat{\mu}(y) > 0$ and $\hat{\mu}(y)$ satisfies equation (14), but $\mu$ can not be represented as a convolution $\mu = \gamma * \pi$, where $\gamma \in \Gamma(\mathbb{T}^{\aleph_0})$, and $\pi$ a signed measure on the group $\mathbb{T}^{\aleph_0(2)}$. The subgroup $\mathbb{T}^{\aleph_0(2)}$ is infinite in this case. This example shows that condition (i) in Theorem 2 is sharp.

**Remark 5.** We assumed in Theorem 2 that $n$ is odd. This condition can not be omitted even for the circle group $X = \mathbb{T}$. Indeed, let $n = 4$. Take $a$ in such a way that the function

$$f(m) = \exp\{-am^2 + \frac{i}{2}m^3\}, \quad m \in \mathbb{Z}$$

is the characteristic function of a distribution $\mu \in M^1(\mathbb{T}^{\aleph_0})$ such that $\hat{\mu}(y) > 0$ and $\hat{\mu}(y)$ satisfies equation (14), but $\mu$ can not be represented as a convolution $\mu = \gamma * \pi$, where $\gamma \in \Gamma(\mathbb{T}^{\aleph_0})$, and $\pi$ a signed measure on the group $\mathbb{T}_{(2)}^{\aleph_0}$. The subgroup $\mathbb{T}_{(2)}^{\aleph_0}$ is infinite in this case. This example shows that condition (i) in Theorem 2 is sharp.
be the characteristic function of a distribution $\mu \in M^1(\mathbb{T})$. On the one hand, it is obvious that the function $f(m)$ satisfies equation (4) and $f^4(m) > 0$, $m \in \mathbb{Z}$. On the other hand, the distribution $\mu$ can not be represented in the form $\mu = \gamma \ast \pi$, where $\gamma \in \Gamma(\mathbb{T})$, and $\pi$ is a signed measure on $\mathbb{T}$. This example also shows that a function $f(y)$ satisfying equation (4), generally speaking, needs not be real.
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