On Renormalization Group Flows
and Exactly Marginal Operators in Three Dimensions

Matthew J. Strassler

School of Natural Sciences, Institute for Advanced Study, Olden Lane, Princeton, NJ 08540, USA
strasslr@ias.edu

Abstract

As in two and four dimensions, supersymmetric conformal field theories in three dimensions can have exactly marginal operators. These are illustrated in a number of examples with $\mathcal{N} = 4$ and $\mathcal{N} = 2$ supersymmetry. The $\mathcal{N} = 2$ theory of three chiral multiplets $X, Y, Z$ and superpotential $W = XYZ$ has an exactly marginal operator; $\mathcal{N} = 2 U(1)$ with one electron, which is mirror to this theory, has one also. Many $\mathcal{N} = 4$ fixed points with superpotentials $W \sim \Phi Q_i \tilde{Q}^i$ have exactly marginal deformations consisting of a combination of $\Phi^2$ and $(Q_i \tilde{Q}^i)^2$. However, $\mathcal{N} = 4 U(1)$ with one electron does not; in fact the operator $\Phi^2$ is marginally irrelevant. The situation in non-abelian theories is similar. The relation of the marginal operators to brane rotations is briefly discussed; this is particularly simple for self-dual examples where the precise form of the marginal operator may be guessed using mirror symmetry.
In recent years there has been progress on many fronts in our understanding of supersymmetric field theories in three and higher dimensions. There is a vast literature on conformal field theories in two dimensions, and superconformal field theories (SCFTs) are well studied there. However, SCFTs were not thought common in higher dimensions until quite recently. Isolated SCFTs were known in the 1970s for $SU(N)$ gauge theories with $N_f = 3N(1 - \epsilon)$ flavors of matter fields in the fundamental representation. A long list of finite theories (continuously infinite sets of fixed points indexed by a gauge coupling) in four dimensions with $\mathcal{N} = 4$, $\mathcal{N} = 2$ and $\mathcal{N} = 1$ supersymmetry were discovered in the 1980s (see [1,2] for a list of references) in which a combination of the gauge kinetic term and an interaction among matter fields serves as an exactly marginal deformation of the free field theory. There are also well-known isolated SCFTs in theories in three dimensions, stemming both from supersymmetric generalizations of $\phi^4$ theory and from gauge theories at large $N_f$. Although it is easy to find exactly marginal operators in theories based on supersymmetric $\phi^4$ theory, I am unaware of any research on marginal operators in three-dimensional theories prior to recent developments.

Seiberg [3] showed that SCFTs in four dimensions are much more common than previously realized. In [1], using techniques similar to those used in two dimensions but employing also special properties of four dimensions, it was shown that many of these non-trivial SCFTs have exactly marginal operators, and that the finite theories previously studied are just special cases. After the work of [4–11], our understanding of three dimensional SCFTs greatly increased. Exactly marginal operators were noted at that time, using some of the same arguments as in two dimensions [12–16]. However, nothing was published on this subject. This article is intended to fill the gap in the literature.

While elementary application of the superconformal algebra is often sufficient to show that a given SCFT contains an operator which is marginal, it is not sufficient to show that the operator is \textit{exactly} marginal — that it remains marginal when it is added to the Lagrangian as a perturbation. Instead, an argument must be given that the SCFT in question lies inside a continuous space of SCFTs; motion within this space corresponds to perturbation by an exactly marginal operator. In finite theories in four dimensions, all-orders arguments based on perturbation theory were given (see [1,2] for a list of references.) However, these approaches cannot be used if the free theory is not inside the space of SCFTs under study. A more powerful but very simple argument for the existence of continuous sets of SCFTs is the following. Each beta function is a function of all of the gauge and superpotential couplings. Suppose the total number of couplings is $n$. Since there is one beta function for each coupling, the requirement that all beta functions vanish puts $n$ constraints on $n$ couplings; any solution to these constraints is generally isolated and has no exactly marginal deformations. However, if only $p$ of the beta functions are linearly independent as functions of the $n$ couplings, then the general solution to the vanishing of the beta functions will be an $n - p$ dimensional subspace of the space of couplings. Any given SCFT on that subspace will have $n - p$ linearly independent exactly marginal deformations. Of course, it is possible that there are no solutions, or multiple solutions, to the conditions of vanishing beta functions.

\*An “exactly marginal operator” is one which, when added to the Lagrangian of a CFT, preserves conformal invariance.
In [1], the arguments proving the existence of exactly marginal operators are based on exact formulas for the beta functions. These formulas follow from special properties of $\mathcal{N} = 1$ ($\mathcal{N} = 2$) supersymmetric gauge theories in four (two and three) dimensions. Such theories have a holomorphic superpotential. Holomorphy implies severe restrictions; in particular, couplings of chiral fields in the superpotential are not perturbatively renormalized [17–19]. Non-perturbative renormalizations of the superpotential are restricted by holomorphy [19,20]. Still, any physical coupling is renormalized, and its running can be expressed in terms of its canonical dimension and the anomalous dimensions of the fields that it couples. That is, corresponding to the superpotential $W = h\phi_1 \ldots \phi_n$ there is a $\beta$-function

$$\beta_h \equiv \frac{\partial h(\mu)}{\partial \ln \mu} = h(\mu)\left( -d_W + \sum_k d(\phi_k) \right) = \frac{1}{2}h \left[ n(d-2) - 2(d-1) + \sum_{k=1}^n \gamma(\phi_k) \right] \equiv \frac{1}{2}h(\mu)A_h$$  

(1)

where $d_W = d-1$ is the canonical dimension of the superpotential and $d(\phi_k) = \frac{1}{2}[d-2+\gamma(\phi_k)]$ is the dimension of the superfield field $\phi_k$, with $\gamma(\phi_k)$ is its anomalous mass dimension. I will refer to $A_h$ as a scaling coefficient; it is twice the physical dimension of the operator $\phi_1 \ldots \phi_n$.

In four dimensions, exact formulas are also known for the running of gauge couplings [21–24]; these formulas follow from anomalies, and relate the gauge beta functions to anomalous dimensions of the charged fields in a similar way to (1). This makes all of the beta functions linear functionals of the anomalous dimensions, and it is quite easy to find theories in which the beta functions are linearly dependent.

In three dimensions similar formulas for the gauge couplings have not yet been found, and it is not clear that they exist. (It is also clear that they cannot depend merely on the anomalous dimensions of the charged fields, as they do in four dimensions; this will be explained later.) Despite the absence of such formulas, it is still possible to demonstrate the existence of exactly marginal operators, since application of the argument only requires that linear dependence of two or more of the beta functions be established. For this reason, our lack of knowledge of the beta function for gauge couplings in three dimensions is not a hindrance, as long as we consider linear dependence of beta functions for couplings in the superpotential, expressed through Eq. (1). In other words, the superpotentials with exactly marginal operators in three dimensions will have much the same form as in two dimensions, and will not have the more general form possible in four dimensions where linear dependence of gauge beta functions may be included. On the other hand, gauge interactions still play an essential role in three dimensions by expanding the number of possible exactly marginal operators, as will be explained below. In this sense, the results described in this letter are intermediate between those of two and four dimensions.

The simplest $\mathcal{N} = 2$ superconformal field theory (SCFT) is the supersymmetric generalization of $\phi^4$ theory. The $\lambda\phi^4$ perturbation of a free scalar field $\phi$ is relevant in three dimensions, and flows to a well-studied fixed point. The perturbation $W = \lambda\Phi^3$ of a free chiral $\mathcal{N} = 2$ superfield $\Phi$ is similarly relevant, since $\lambda$ has mass dimension $+\frac{1}{2}$ at the free field theory. The beta function for $\lambda$ is exactly

$$\beta_\lambda = \lambda(-2 + 3d_\phi) = \frac{1}{2}\lambda(-1 + 3\gamma_\phi)$$  

(2)
(λ has mass dimension $+\frac{1}{2}$ at the free field theory) and drives the dimension of $\Phi$ upward.

$$d_\Phi(\lambda) = \frac{1}{2} + \mathcal{O}(\lambda^2) > d_\Phi(0)$$

(3)

This growth continues (presumably monotonically, though no techniques as yet can prove it) until $d_\Phi = 2/3$. At this point λ is dimensionless and its beta function vanishes. It is to be expected that this fixed point is stable; the beta function is negative (positive) if λ is smaller (larger) than its fixed point value as long as $d_\Phi$ passes monotonically through 2/3 in the vicinity of the fixed point. Since it would require fine tuning for $d_\Phi(\lambda)$ to reach a maximum of 2/3 precisely at the fixed point value of λ, it is almost certain that the fixed point is stable. Stability can also be checked in an epsilon expansion, although this does not preserve supersymmetry.

This SCFT can be used to create a theory with an exactly marginal operator. Consider three chiral superfields $X, Y, Z$ with superpotential $W = \lambda_X X^3 + \lambda_Y Y^3 + \lambda_Z Z^3$. At the fixed point, $\lambda_X = \lambda_Y = \lambda_Z$ and $d_X = d_Y = d_Z = 2/3$. The perturbation $\Delta W = hXYZ$ leads to an exactly marginal operator:

$$A_{\lambda_X} = -1 + 3\gamma_X ; \quad A_{\lambda_Y} = -1 + 3\gamma_Y ; \quad A_{\lambda_Z} = -1 + 3\gamma_Z ; \quad A_{\lambda_H} = -1 + \gamma_X + \gamma_Y + \gamma_Z .$$

(4)

Only three of these scaling coefficients are linearly independent, so there is one exactly marginal operator, lying by symmetry in the subspace $\lambda_X = \lambda_Y = \lambda_Z \equiv \lambda_0$ (where also $\gamma_X = \gamma_Y = \gamma_Z$) defined by the condition that all scaling coefficients vanish, namely the single constraint on two couplings $\gamma_X(\lambda_0, h) = 1/3$. (Note that we could have assumed the symmetry among $X, Y, Z$ from the beginning and arrived at the same number of marginal operators; in future I will often shorten the analysis by making analogous assumptions.) Within the two-dimensional complex space of couplings $\lambda_0$ versus $h$, there will be a one-complex-dimensional subspace, separating the regions of $\gamma_X < 1/3$ and $\gamma_X > 1/3$, on which the theory is conformal, as shown in Fig. 1. The SCFTs $W = \lambda(X^3 + Y^3 + Z^3)$ and $W = hXYZ$ are thus connected by a line of SCFTs, as shown in Fig. 1a. It is also possible there are multiple subspaces, as shown in Fig. 1b and Fig. 1c, in which case the last statement may or may not be true.
FIG. 1. The theory $W = \lambda(X^3 + Y^3 + Z^3) + hXYZ$ has one (a) or more (b,c) one-complex-dimensional spaces of SCFTs, which may (a,c) or may not (b) connect the SCFTs with $\lambda = 0$ and $h = 0$. The SCFTs separate regions where $\gamma > 1/3$ from those with $\gamma < 1/3$. Renormalization group flow toward the infrared is indicated by arrows. The dotted line in (b) indicates a line of infrared unstable SCFTs.

Note that the perturbations $X^2Y, Y^2Z,$ and so forth are not marginal deformations of the theory with $W = \lambda(X^3 + Y^3 + Z^3)$. These perturbations are redundant, as they can be removed by field redefinitions. The only non-redundant perturbation is the operator $XYZ$.

Similar statements apply to generalizations of this model, such as $W = \sum_{i=1}^{6} \lambda X_i^3$ which has exactly marginal operators $X_1X_2X_3 + X_4X_5X_6, X_2X_3X_4 + X_5X_6X_1$, etc.

No other interesting SCFTs can be built using chiral superfields without gauge interactions. The perturbation $\Delta W = hX_1X_2X_3X_4$ of any SCFT (including a free theory) cannot be exactly marginal, on general grounds following from the conformal algebra. The dimension of at least one of the fields $X_i$ must be 1/2 or less for $X_1X_2X_3X_4$ to be marginal. However, at a fixed point no gauge invariant field can have dimension less than 1/2 (and therefore all four fields must have dimension 1/2 for $X_1X_2X_3X_4$ to be marginal) and fields of dimension 1/2 are free (and therefore $h$ cannot be non-zero at a fixed point.) Thus these perturbations must be marginally irrelevant. Note that the same reasoning shows that $\phi^6$ and $\phi^2\bar{\psi}\psi$ perturbations of non-supersymmetric theories of free scalars and fermions are irrelevant.

Since the above limitation stems from having all fields gauge invariant, it is natural as a next step to introduce gauge interactions in hope of evading it. The simplest theories to study are $\mathcal{N} = 2$ and $\mathcal{N} = 4$ supersymmetric theories with a $U(1)$ gauge group and some charged matter.

It is useful to review the properties of $\mathcal{N} = 4$ gauge theories in the language of $\mathcal{N} = 2$ supersymmetry. In $\mathcal{N} = 2$ language, the $\mathcal{N} = 4$ theory of $U(1)$ with $N_f$ hypermultiplets of charge 1 has a $U(1)$ vector multiplet, a neutral chiral multiplet $\Phi$, chiral multiplets $Q_i, \bar{Q}^i$ of charge 1 and $-1$, and a superpotential $W = \Phi \sum_i Q_i \bar{Q}^i$. At the origin of moduli space this theory flows in the infrared to an SCFT. While symmetry tells us $d_Q = d_{\bar{Q}}$, and $\mathcal{N} = 2$ supersymmetry tells us through the dimension of the superpotential that $d_\Phi + 2d_Q = 2$, all of $\mathcal{N} = 4$ supersymmetry is required to conclude that $d_\Phi = 1$ and $d_Q = \frac{1}{2}$, the latter being its canonical dimension. In short, only $\Phi$ picks up an anomalous dimension. The shift in dimension of $\Phi$ from $\frac{1}{2}$ to 1 is directly linked to the shift of the gauge coupling $g$ from
dimension $\frac{1}{2}$ to 0; the dimension of $g \Phi$ is constant. Without $\mathcal{N} = 4$ supersymmetry, the field $\Phi$ would not be tied to the gauge boson by any symmetry and there would be no direct connection between the dimension of $g$ and the dimension of $\Phi$. These statements — that $\Phi$ has dimension 1 and $Q$ dimension $\frac{1}{2}$ — are true in all $\mathcal{N} = 4$ SCFTs which stem from local Lagrangians, including theories with multiple and/or non-abelian gauge groups.

As an aside, notice that this implies that the beta function for the gauge coupling cannot simply be proportional to a linear combination of the beta functions of charged fields, as it is in four dimensions. In the $\mathcal{N} = 4$ $U(1)$ theories, the anomalous dimensions of $Q, \tilde{Q}$ are zero, both in the free theory and at the SCFT, and the fields $\Phi$ are neutral. Since the beta function is non-zero in the ultraviolet and vanishes in the infrared, it must get a non-trivial contribution from sources other than the charged fields.

Mirror symmetry [5], a relation between two SCFTs, will also be useful to us in the following. Under mirror symmetry, the SCFT of $\mathcal{N} = 4$ $U(1)$ theories, the anomalous dimensions of $Q, \tilde{Q}$ are zero, both in the free theory and at the SCFT, and the fields $\Phi$ are neutral. Since the beta function is non-zero in the ultraviolet and vanishes in the infrared, it must get a non-trivial contribution from sources other than the charged fields.

Mirror symmetry [5], a relation between two SCFTs, will also be useful to us in the following. Under mirror symmetry, the SCFT of $\mathcal{N} = 4$ $U(1)$ theories, the anomalous dimensions of $Q, \tilde{Q}$ are zero, both in the free theory and at the SCFT, and the fields $\Phi$ are neutral. Since the beta function is non-zero in the ultraviolet and vanishes in the infrared, it must get a non-trivial contribution from sources other than the charged fields.

Next, let us consider $\mathcal{N} = 2$ $U(1)$ with one flavor, the same as the previous theory but with the field $\Phi$ removed and a superpotential $W = 0$. As shown in [10], this theory is mirror to a theory of three singlets $S, q, \tilde{q}$ with superpotential $W = Sq \tilde{q}$. This is precisely of the form $W = XYZ$, a theory considered earlier and shown to have an exactly marginal operator when $X^3 + Y^3 + Z^3$ is added as a perturbation. The mapping of operators is $Q\tilde{Q}, V_+, V_- \rightarrow S, q, \tilde{q}$. From the discussion of the $W = XYZ$ model we learn that $Q\tilde{Q}$ has dimension $2/3$, and thus $d_Q = d_{\tilde{Q}} = 1/3$. This means that the operator $(Q\tilde{Q})^2$ is relevant, as are $V_+^2$ and $V_-^2$, while $(Q\tilde{Q})^3 + V_+^3 + V_-^3$ is an exactly marginal operator corresponding to the mirror of $S^3 + q^3 + \tilde{q}^3$.

It is interesting to consider the relevant operator $(Q\tilde{Q})^2$ in this theory. The importance of this operator was first emphasized in [1], where it was shown that in four dimensions it connects the electric-magnetic duality of finite $\mathcal{N} = 2$ theories to that of self-dual $\mathcal{N} = 1$ theories. It has also been considered in [23,10,20]. Here, it plays a similar interesting role. As in [10] one may rewrite the superpotential $W = \frac{1}{2} hQ\tilde{Q}QQ \tilde{Q}$ by introducing a gauge singlet auxiliary field $M$ with superpotential $W = MQ\tilde{Q} - \frac{1}{2h} M^2$. Although $M$ is introduced as an auxiliary field, it develops a propagator through loop diagrams and is indistinguishable from a canonical field as far as infrared physics is concerned. Thus, although the theory
with the new field is not the same as the original one, it has the same infrared behavior. I claim that the physical value of \( h \) flows to infinity in the infrared; consequently the mass of \( M \) goes to zero, and we are left in the infrared with the \( \mathcal{N} = 4 \) SCFT theory of \( U(1) \) with one flavor considered just above. To check this claim, consider the mirror description: the field \( S \) becomes massive, leading to a quartic superpotential

\[
W = Sq\bar{q} + \frac{1}{2}hs^2 \rightarrow -\frac{1}{2h}(q\bar{q})^2
\]

which is marginally irrelevant. The fields \( q, \bar{q} \) are thus free in the infrared, and constitute a free \( \mathcal{N} = 4 \) supersymmetric hypermultiplet.

From this we learn more about the global behavior of the renormalization group flow connecting these two theories. If we begin with the free \( \mathcal{N} = 4 \) theory in the ultraviolet, the perturbation \( \frac{1}{2}m\Phi^2 \) is relevant. The theory flows toward the free \( \mathcal{N} = 2 \) theory. However, it does so with an operator \( (Q\bar{Q})^2 \) in the superpotential. If \( m \gg g^2 \) then a classical analysis applies, and the operator \( (Q\bar{Q})^2 \) is just barely relevant; the theory flows very close to the \( \mathcal{N} = 2 \) free theory, then very close to the \( \mathcal{N} = 2 \) SCFT, then away from the \( \mathcal{N} = 2 \) fixed point toward the \( \mathcal{N} = 4 \) SCFT. On the other hand, if \( m \ll g^2 \), then the gauge coupling almost reaches its \( \mathcal{N} = 4 \) fixed point before the effect of \( m \neq 0 \) drives the theory away from the \( \mathcal{N} = 4 \) supersymmetric theory. However, it cannot get too far, as the \( (Q\bar{Q})^2 \) operator rapidly becomes relevant, driving the theory back to the \( \mathcal{N} = 4 \) SCFT. The flow is schematically shown in Fig. 2.

![Fig. 2. Renormalization group flow connecting the \( \mathcal{N} = 4 \) and \( \mathcal{N} = 2 \) theories of \( U(1) \) with \( N_f = 1 \). The horizontal direction represents the gauge coupling; the vertical represents the coupling of \( \Phi^2 \) (which grows as one moves up the diagram) or equivalently the coupling of \( (Q\bar{Q})^2 \) (which grows as one moves down the diagram.)](image)

Now let us consider \( N_f > 1 \), beginning with the \( \mathcal{N} = 2 \) case. These theories have \( W = 0 \) and flow to fixed points whose mirror descriptions have gauge groups \( U(1)^{N_f-1} \) and \( N_f \) triplets of mirror matter fields \( S_i, q_i, \bar{q}_i \) (the last two having charges under the \( i^{th} \) and \( (i-1)^{th} \) gauge groups) with cubic superpotentials \( W = S_i q_i \bar{q}_i \). However, no symmetry relates \( S_i \) to \( q_i \), in contrast to the case for \( N_f = 1 \). Therefore, the anomalous dimensions of these fields cannot be determined. In the absence of any superpotential in the original
variables, which might permit the use of Eq. (1), and in the absence of knowledge of the low-energy anomalous dimensions, it is impossible at this time to determine whether any of these \( \mathcal{N} = 2 \) fixed points have exactly marginal operators.

However, the fields \( S_i \), which under mirror symmetry are mapped to linear combinations of \( Q_i \tilde{Q}^i \), are unlikely to have dimension greater than 1. We have seen that for \( N_f = 1 \) the dimension of \( Q\tilde{Q} \) is 2/3, while for large \( N_f \) one can show the dimension of \( Q_i \tilde{Q}^i \) is less than 1 by an effect of order \( 1/N_f \). This strongly suggests that bilinear terms in \( S_i \) are always relevant, and thus quartic terms \( Q_i \tilde{Q}^i Q_m \tilde{Q}^m \) are relevant perturbations of the low-energy fixed point which then cause the theory to flow, possibly to a new \( \mathcal{N} = 2 \) SCFT. Let us therefore consider the fixed points of \( U(1) \) with \( N_f > 1 \) and the superpotential

\[
W = \sum_{k=0}^{[N_f/2]-1} \frac{1}{2} y_k \left( \sum_{n=1}^{N_f} e^{2\pi i k n / N_f} Q_n \tilde{Q}^n \right) \left( \sum_{n=1}^{N_f} e^{-2\pi i k n / N_f} Q_n \tilde{Q}^n \right)
\]  

(6)

(Here \( [N_f/2] \) means the integer part of \( N_f/2 \).) This is by no means the most general quartic superpotential, but it will serve to illustrate some important points.

First, the superpotential preserves a \( Z_N \) symmetry relating the \( Q_n \) and \( \tilde{Q}_n \) to one another, and so they all have the same anomalous dimension \( \gamma_Q(y_k, g) \). This is essential to ensure that each coupling \( y_k \) in (6) does not break up into multiple couplings under the renormalization group flow; if \( Q_1 \) and \( Q_2 \) have different anomalous dimensions, then the couplings multiplying \( (Q_1 \tilde{Q}^1)^2 \) and \( (Q_2 \tilde{Q}^2)^2 \) will run differently. With this symmetry, all couplings run with \( A_{y_k} \propto 4\gamma_Q(y_k, g) \) except the gauge coupling, which has a zero at some point \( y_k = 0, g = g_0^* \). If there is some point \( g \neq 0, y_k \neq 0 \), where \( \gamma_Q(g, y_k) = 0 \) and \( \beta_g(g, y_k) = 0 \) (two constraints on \( [N_f/2] + 1 \) couplings) then there will be a space of SCFTs of complex dimension \( [N_f/2] - 1 \) passing through that point.

The limit \( y_k \rightarrow 0 \) for \( k > 0 \) and \( y_0 \rightarrow \infty \) is a special one. As discussed earlier we may make the replacement

\[
\frac{1}{2} y_0 \left( \sum_{n=1}^{N_f} Q_n \tilde{Q}^n \right)^2 \rightarrow W = -\frac{1}{2y_0} \Phi^2 + \Phi \sum_{n=1}^{N_f} Q_n \tilde{Q}^n
\]  

(7)

so that in the above limit we might obtain the \( \mathcal{N} = 4 \) theory of \( U(1) \) with \( N_f \) hypermultiplets. Since we know the \( \mathcal{N} = 4 \) theory has a fixed point with \( \gamma_Q = 0 \), we learn that the theory (6) does have an \( [N_f/2] - 1 \) dimensional space of fixed points, and that it contains the \( \mathcal{N} = 4 \) SCFT. The physical picture as a function of the gauge coupling \( g \), the coupling \( y_0 \), and the other couplings \( y_k \) (treated as a single axis) is shown in Fig. 3.
FIG. 3. For $N_f > 1$, the $U(1)$ $\mathcal{N} = 4$ fixed point, at $y_0 = \infty$, $y_k = 0$, has exactly marginal deformations which preserve $\mathcal{N} = 2$ supersymmetry. In the same space, the $\mathcal{N} = 2$ theory with $W = 0$ is an isolated SCFT. There could be more SCFTs than shown.

The same result may be obtained from the mirror of the theory in Eq. (6). The quartic terms in the $Q_i$ correspond to quadratic terms in the fields $S_i$, which in turn lead to quartic terms in the mirror fields $q_i \tilde{q}^i$. An analysis of these terms leads to the same conclusion concerning the number of marginal operators, and also demonstrates that the $\mathcal{N} = 4$ SCFT is present in the limit $y_0 \to \infty$ with the other $y_k = 0$.

This analysis may be easily generalized to theories with more abelian gauge groups, and also for theories with non-abelian gauge groups. The conclusion is the same. If a $\mathcal{N} = 4$ SCFT has only a Coulomb branch, and thus is mirror to a theory of free hypermultiplets with no gauge fields, then it has no exactly marginal operators. Otherwise, the marginal masses for the fields $\Phi_n$ and the marginal quartic terms in the hypermultiplets $Q_i$ can generally be balanced off against the original superpotential $\Phi_n Q_i \tilde{Q}^i$ to make exactly marginal operators.

A special case with more interesting structure involves those theories which are self-dual under mirror symmetry. I use the simplest case, $U(1)$ with $N_f = 2$, for illustration. The mirror superpotential is

$$W = \phi(q_1 \tilde{q}^1 + q_2 \tilde{q}^2)$$

Mirror symmetry maps operators in the following way:

$$\Phi \leftrightarrow (q_1 \tilde{q}^1 - q_2 \tilde{q}^2) ; \quad Q_1 \tilde{Q}^1 - Q_2 \tilde{Q}^2 \leftrightarrow \phi$$

This means that the superpotential

$^1$Special properties of quartic superpotentials in self-dual theories in four dimensions were studied in [2]; those discussed here are similar but not identical.
is mirror to a theory with
\[ W = \phi(q_1q^1 + q_2q^2) + \frac{1}{2}h(q_1q^1 - q_2q^2)^2 + \frac{1}{2}k\phi^2 = -\frac{1}{2h}(q_1\tilde{Q}^1 \pm Q_2\tilde{Q}^2) + \frac{k}{2}(q_1\tilde{Q}^1 \pm Q_2\tilde{Q}^2)^2 \]  

Self-duality is maintained for \( h = k \), and the line of SCFTs will lie along this line by symmetry. This is shown if Fig. 4. In the limit \( h,k \to \infty \), by introducing the auxiliary scalar as in Eq. (5), we obtain the same SCFT as for \( h = k = 0 \) (the sign in the superpotential can be removed by a field redefinition.)

As shown in [6,7] in the context of Type IIB string theory, the field theory of \( U(1) \) with \( N_f \) hypermultiplets can be constructed by suspending a D3 brane between two NS5 branes, which gives a \( U(1) \) gauge theory whose photon is a 3-3 string (a string with both ends attached to the D3 brane,) and placing across it \( N_f \) D5 branes, which gives 5-3 strings which are hypermultiplets charged under the \( U(1) \) gauge theory (Fig. 5). All of the branes fill three dimensions; the NS5 branes also fill dimensions \( x^3, x^4, x^5 \), the D5 branes fills dimensions \( x^7, x^8, x^9 \), and the D3 branes stretch across \( x^6 \). We may consider rotating NS5 branes or D5 branes in the \( (x^4, x^5) - (x^8, x^9) \) plane. This breaks \( \mathcal{N} = 4 \) supersymmetry to \( \mathcal{N} = 2 \). As shown in [27,11], an angle between NS5 branes leads to a mass term for the field \( \Phi \), while rotation of a D5 brane changes the coupling of its hypermultiplet to \( \Phi \). It is easy to see that if both D5 and NS5 rotations are considered, and the field \( \Phi \) is integrated out, the rotations correspond to varying the couplings of quartic terms in the fields \( Q, \tilde{Q} \), as in Eq. (6). While a complete and detailed analysis will not be performed here, it is easy to see that in the case \( N_f = 2 \), the angle between the two NS5 branes corresponds to \( h \) and that between the two D5 branes corresponds to \( k \) in Eq. (11). Clearly self-duality is maintained only if the angles are identical — in short, if \( h = k \) — so that the brane construction remains invariant under exchange of NS5 and D5 branes. Note that the classical brane construction gives
incomplete insight, however, into the issue of whether the rotation is exactly marginal. This is especially obvious in the case of $N_f = 1$ where the relative rotation of the NS5 branes is relevant away from the fixed point but marginally irrelevant in the SCFT. It would be nice if this could be understood using a quantum mechanical treatment of the branes.

![FIG. 5. Brane construction of $U(1)$ with $N_f = 2$ out of a D3 brane stretched between two NS branes with two D5 branes placed along it. Strings with both ends on the D3 are in a vector multiplet; strings with ends on D3 and D5 are in a hypermultiplet. Rotations of the D5 and NS5 branes break $\mathcal{N} = 4$ supersymmetry to $\mathcal{N} = 2$ and generate the $\Phi^2$ and $(Q\bar{Q})^2$ terms required for the exactly marginal operators. In this self-dual case, the exactly marginal operator preserves self-duality and lies at $\phi_{NS} = \phi_D$, as shown.]

It is also interesting to consider the elliptic models of [7], the simplest being the self-mirror theory shown in Fig. 6. Again, if the two NS5 branes and two D5 branes have equal relative angles, the theory will remain self-dual and the angle will correspond to an exactly marginal operator. Under T-duality this theory corresponds to D2 branes moving on an $R^4/Z_2$ orbifold with two D6 branes, or in M theory to M2 branes on an $R^4/Z_2 \times R^4/Z_2$ orbifold. Rotations of both types of branes corresponds to deforming both $Z_2$ orbifolds toward conifold singularities; if the deformation preserves the $Z_2$ which exchanges them, then self-duality is preserved and the deformation is marginal.

![FIG. 6. A self-dual elliptic model; here the number of D3 branes is arbitrary.]

These statements can also be generalized to the non-abelian case. The precise field theory mapping between operators under mirror symmetry has not yet been carried out, although it is clearly very similar to the abelian case. It would be interesting to understand the
exactly marginal deformations of the $\mathcal{N} = 4$ theories in the limit of many D3 branes, where the AdS/CFT correspondence can be used. The example in Fig. 6 and its deformation by brane rotations is similar to the example of the $\mathbb{Z}_2$ orbifold deformed to the conifold, which corresponds to the same model with the D5 branes removed. In four dimensions, the rotation of the NS5 branes is relevant, and the theory flows to a new fixed point in which rotation of the NS5 branes is exactly marginal. In the three dimensional case, the classical deformations are similar but the associated dynamics are quite different. It would be useful to understand how this is manifested in the supergravity description of these theories.

I thank J. Bagger, K. Intriligator, A. Kapustin, N. Seiberg, R. Tatar and A. Uranga for discussions. I especially thank R. Tatar for encouraging me to publish this work and for sharing preliminary versions of his own results with K. Oh. This research was supported in part by National Science Foundation grant NSF PHY-9513835 and by the W.M. Keck Foundation.
REFERENCES

[1] R. G. Leigh and M. J. Strassler, “Exactly marginal operators and duality in four-dimensional N=1 supersymmetric gauge theory,” *Nucl. Phys.* **B447** (1995) 95–136, [hep-th/9503121](https://arxiv.org/abs/hep-th/9503121).

[2] M. J. Strassler, “Manifolds of fixed points and duality in supersymmetric gauge theories,” *Prog. Theor. Phys. Suppl.* **123** (1996) 373–380, [hep-th/9602021](https://arxiv.org/abs/hep-th/9602021).

[3] N. Seiberg, “Electric - magnetic duality in supersymmetric nonabelian gauge theories,” *Nucl. Phys.* **B435** (1995) 129–146, [hep-th/9411149](https://arxiv.org/abs/hep-th/9411149).

[4] N. Seiberg and E. Witten, “Gauge dynamics and compactification to three dimensions,” [hep-th/9607163](https://arxiv.org/abs/hep-th/9607163).

[5] K. Intriligator and N. Seiberg, “Mirror symmetry in three-dimensional gauge theories,” *Phys. Lett.* **B387** (1996) 513–519, [hep-th/9607207](https://arxiv.org/abs/hep-th/9607207).

[6] J. de Boer, K. Hori, H. Ooguri, Y. Oz, and Z. Yin, “Mirror symmetry in three-dimensional theories, SL(2,Z) and D-brane moduli spaces,” *Nucl. Phys.* **B493** (1997) 148–176, [hep-th/9612131](https://arxiv.org/abs/hep-th/9612131).

[7] A. Hanany and E. Witten, “Type IIB superstrings, BPS monopoles, and three-dimensional gauge dynamics,” *Nucl. Phys.* **B492** (1997) 152–190, [hep-th/9611230](https://arxiv.org/abs/hep-th/9611230).

[8] J. de Boer, K. Hori, H. Ooguri, and Y. Oz, “Mirror symmetry in three-dimensional gauge theories, quivers and D-branes,” *Nucl. Phys.* **B493** (1997) 101–147, [hep-th/9611063](https://arxiv.org/abs/hep-th/9611063).

[9] J. de Boer, K. Hori, and Y. Oz, “Dynamics of N=2 supersymmetric gauge theories in three- dimensions,” *Nucl. Phys.* **B500** (1997) 163, [hep-th/9703100](https://arxiv.org/abs/hep-th/9703100).

[10] O. Aharony, A. Hanany, K. Intriligator, N. Seiberg, and M. J. Strassler, “Aspects of N=2 supersymmetric gauge theories in three- dimensions,” *Nucl. Phys.* **B499** (1997) 67, [hep-th/9703110](https://arxiv.org/abs/hep-th/9703110).

[11] O. Aharony and A. Hanany, “Branes, superpotentials and superconformal fixed points,” *Nucl. Phys.* **B504** (1997) 239, [hep-th/9704170](https://arxiv.org/abs/hep-th/9704170).

[12] L. J. Dixon, “Some world sheet properties of superstring compactifications, on orbifolds and otherwise,”. Lectures given at the 1987 ICTP Summer Workshop in High Energy Physics and Cosmology, Trieste, Italy.

[13] E. J. Martinec, “Algebraic geometry and effective lagrangians,” *Phys. Lett.* **B217** (1989) 431.

[14] B. R. Greene, C. Vafa, and N. P. Warner, “Calabi-Yau manifolds and renormalization group flows,” *Nucl. Phys.* **B324** (1989) 371.

[15] C. Vafa and N. Warner, “Catastrophes and the classification of conformal theories,” *Phys. Lett.* **B218** (1989) 51.

[16] W. Lerche, C. Vafa, and N. P. Warner, “Chiral rings in N=2 superconformal theories,” *Nucl. Phys.* **B324** (1989) 427.

[17] M. T. Grisaru, W. Siegel, and M. Rocek, “Improved methods for supergraphs,” *Nucl. Phys.* **B159** (1979) 429.

[18] N. Seiberg, “Naturalness versus supersymmetric nonrenormalization theorems,” *Phys. Lett.* **B318** (1993) 469–475, [hep-ph/9309335](https://arxiv.org/abs/hep-ph/9309335).

[19] N. Seiberg, “The power of holomorphy: Exact results in 4-d SUSY field theories,” [hep-th/9408013](https://arxiv.org/abs/hep-th/9408013) talk given at PASCOS 1994, Syracuse, NY, USA.
[20] N. Seiberg, “Exact results on the space of vacua of four-dimensional SUSY gauge theories,” *Phys. Rev.* D49 (1994) 6857–6863, [hep-th/9402044](http://arxiv.org/abs/hep-th/9402044).

[21] V. A. Novikov, M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, “Exact Gell-Mann-Low function of supersymmetric Yang-Mills theories from instanton calculus,” *Nucl. Phys.* B229 (1983) 381.

[22] V. A. Novikov, M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, “Beta function in supersymmetric gauge theories: Instantons versus traditional approach,” *Phys. Lett.* 166B (1986) 329–333.

[23] M. A. Shifman and A. I. Vainshtein, “Solution of the anomaly puzzle in susy gauge theories and the wilson operator expansion,” *Nucl. Phys.* B277 (1986) 456.

[24] M. A. Shifman and A. I. Vainshtein, “On holomorphic dependence and infrared effects in supersymmetric gauge theories,” *Nucl. Phys.* B359 (1991) 571–580.

[25] K. Intriligator and N. Seiberg, “Duality, monopoles, dyons, confinement and oblique confinement in supersymmetric SO(N(c)) gauge theories,” *Nucl. Phys.* B444 (1995) 125–160, [hep-th/9503179](http://arxiv.org/abs/hep-th/9503179).

[26] I. R. Klebanov and E. Witten, “Superconformal field theory on three-branes at a Calabi-Yau singularity,” [hep-th/9807080](http://arxiv.org/abs/hep-th/9807080).

[27] J. L. F. Barbon, “Rotated branes and N=1 duality,” *Phys. Lett.* B402 (1997) 59–63, [hep-th/9703051](http://arxiv.org/abs/hep-th/9703051).

[28] K. Oh and R. Tatar, “Three dimensional SCFT from M2 branes at conifold singularities,” in preparation.