Topology of quantum discord

Nga T T Nguyen and Robert Joynt

Department of Physics, University of Wisconsin-Madison, Madison, WI 53706, United States of America

E-mail: rjjoynt@wisc.edu

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Abstract
Quantum discord is an important measure of quantum correlations that can serve as a resource for certain types of quantum information processing. Like entanglement, discord is subject to destruction by external noise. The routes by which this destruction can take place depends on the shape of the hypersurface of zero discord $C$ in the space of generalized Bloch vectors. For 2 qubits, we show that with a few points subtracted, this hypersurface is a simply-connected 9-dimensional manifold embedded in a 15-dimensional background space. We do this by constructing an explicit homeomorphism from a known manifold to the subtracted version of $C$. We also construct a coordinate map on $C$ that can be used for integration or other purposes. This topological characterization of $C$ has important implications for the classification of the possible time evolutions of discord in physical models. The classification for discord contrasts sharply with the possible evolutions of entanglement. We classify the possible joint evolutions of entanglement and discord. There are 9 allowed categories: 6 categories for a Markovian process and 3 categories for a non-Markovian process, respectively. We illustrate these conclusions with an anisotropic $XY$ spin model. All 9 categories can be obtained by adjusting parameters in this model.

Keywords: quantum information theory, quantum correlations, quantum discord, quantum entanglement, (non)Markovian process, topology, time evolution

(Some figures may appear in colour only in the online journal)

1. Introduction

Some of the most characteristic features of quantum mechanics show up in the correlations of two subsystems that are independently measurable. The most famous is entanglement [1], but this notion does not exhaust everything that is quantum about correlations. Even two systems
that are separable and thus have zero entanglement may not satisfy the usual relationships between mutual information and conditional probabilities. One quantity that measures the additional quantumness of correlations is quantum discord $D$ [2–4]. Roughly speaking, $D$ is the difference between the total quantum correlation and its classical counterpart. Discord can serve as a resource for the accomplishment of certain tasks in a way somewhat similar to the way that entanglement can. For computation, the quantum algorithm DQC1 does seem to use discord rather than entanglement [5–7] as a resource, and discord is the appropriate quantity for computing changes in costs of state merging [8] and dense coding capacity after measurements on a subsystem [9–11]. More general statements about the uses of discord are difficult to make at this stage. Quantum discord and other quantum correlation measures have recently received an extensive review [12].

One question that is of experimental importance is how quantum correlations are erased by external noise. In the case of entanglement, there is a rather rich range of possible behaviors of the time evolution of the concurrence (the function $C(t); t \in [0, \infty)$) as a composite system loses its quantum correlations [13]. A somewhat similar, though distinctly more limited, range of behaviors has been found in numerical studies of the discord evolution (the function $D(t); t \in [0, \infty)$) [14–22]. For entanglement, a general classification of time evolutions was seen to depend on understanding the topology of entanglement: essentially the structure of the set of separable states $\mathcal{S}$ [23, 24]. The purpose of the current work is to achieve the same goal for discord. We will first determine the relevant topological properties of the set $\mathcal{C}$ of concordant states, i.e. the set of states for which $D$ vanishes. Once the topology of $\mathcal{C}$ is clear we will be able to deduce the topology of other level sets of $D$. This leads to a general classification of the types of evolution of the discord. Furthermore, we shall give examples of physical models that realize the various types of evolution. The paper will focus on the case of 2 qubits.

The most basic result about $\mathcal{C}$, established by Ferraro et al [16], is that it is of zero 15-volume. To understand the significance of this, we first note that the set of 2-qubit density matrices, which we shall call $\mathcal{M}$, is a convex subset of a real 15-dimensional vector space. $\mathcal{M}$ itself is a 15-dimensional manifold with boundary: any interior point of $\mathcal{M}$ has a neighborhood that is homeomorphic to a neighborhood in $\mathbb{R}^{15}$. $\mathcal{C}$ is a subset of $\mathcal{M}$. The fact that it has zero 15-volume means that the dimension of any neighborhood of any point in $\mathcal{C}$ is less than 15, but gives no further information. We shall show that (except for one point) the precise number for the local dimensionality of $\mathcal{C}$ is 9. It has been shown previously that $\mathcal{C}$ is path-connected; we shall prove the stronger result that $\mathcal{C}$ (with one point removed) is simply connected. The zero-volume statement already implies a very important point about discord evolution: sudden death of discord is not possible. This was conjectured early on from results of numerical studies and the connection with the geometry of $\mathcal{S}$ was understood. Other phenomena, such as frozen discord [15], have also been shown to benefit from a geometric analysis [25]. These analyses have been carried out in the 3-dimensional set of Bell-diagonal states.

Our aim here is to extend this framework to the full 15-dimensional space. This will allow us to characterize in a topological fashion all joint evolutions of entanglement and discord that lead to the disappearance of both. Some evolutions have been computed by previous authors [26, 27]. The characterization of joint evolutions depends not only on the shape of $\mathcal{C}$, but also on the shape of $\mathcal{S}$, the set of separable states, so we also describe the relation of these two sets.

The paper is organized as follows. Section 2 establishes concepts and notation. Section 3 establishes the basic facts about the geometrical and topological nature of $\mathcal{C}$ and $\mathcal{S}$. Section 4 applies the results of section 3 to the dynamical evolution of the discord, first establishing a categorization of the possible evolutions, then illustrating this categorization. In section 5 we give a discussion and the outlook for future work.
The definition of quantum discord that best expresses its foundation in information theory is:

\[ D(B|A) = I(A : B) - J(B|A), \]

where \( I(A : B) \) is the quantum mutual information:

\[ I(A : B) = S(A) + S(B) - S(A, B), \]

\( S(A) \) is the usual von Neumann entropy, while \( J(B|A) \) is a measure of the total classical correlation present \([2–4]\). \( J(B|A) \) is defined in stages. First note that if system \( A \) is measured by an operator \( E_a \) and is found to be in the state \( a \), then the density matrix of \( B \) after the measurement is \( \rho_{AB} = \sum_a p_a \rho_a \otimes \rho_{B|a} \), where \( p_a \) is the probability of measuring the result \( a \) in the state \( \rho_{AB} \), i.e. \( p_a = \text{Tr}(E_a \rho_{AB}) \). We may then define a conditional entropy under the measurement of \( E_a : S(B|E_a) = \sum_a p_a S(\rho_{B|a}) \), and then we have a corresponding mutual-information-like quantity \( J(B|E_a) = S(B) - S(B|E_a) \). Quantum mechanics is distinguished from classical mechanics by the fact that this quantity depends on the choice of measurements. To remove this ambiguity, we maximize over the choice of \( \{E_a\} \) and arrive at a measure of the total classical correlation:

\[ J(B|A) = \max_{\{E_a\}} J(B|E_a). \]

\( D(B|A) \) is clearly not symmetric between systems \( A \) and \( B \), but it has the essential property of being invariant under local unitary operations.

For our purposes, the most important property of discord is that \( D(B|A) = 0 \) when \( \rho_{AB} \) is classical-quantum: \( \rho_{AB} = \sum_a p_a \Pi_a \otimes \rho(B|a) \). Here \( \{\Pi_a\} \) is any set of rank-one projectors and \( \rho(B|a) \) is the resulting partial density matrix for \( B \) if the result has been obtained from a measurement of \( A \). This gives an explicit definition of the set \( C \) of concordant states mentioned above.

We intend to investigate the topology of \( C \). To define a topology on any set requires a specification of its open subsets. A metric is the most convenient way to do this, and we will employ the metric on the set \( M \) of density matrices that follows from the Hilbert–Schmidt inner product:

\[ \rho \rho' = \text{Tr}(\rho \rho')^2. \]

To give a consistent treatment of discord, we also need a metric-based definition. Fortunately, there is the geometric discord, defined by \([28, 29]\):

\[ D_G(B|A) = \min_{\chi \in C} |\rho_{AB} - \chi| = \min_{\chi \in C} \text{Tr}\left[(\rho_{AB} - \chi)^2\right], \]

i.e. \( D_G(\rho_{AB}) \) is the Hilbert–Schmidt distance from \( \rho_{AB} \) to the nearest point of \( C \). This differs slightly from the information-theory based definition above. We will comment on the differences below. Since we intend to compare entanglement and discord, we also need a metric-based definition of entanglement, the geometric entanglement:

\[ E_G(B|A) = \min_{\chi \in S} |\rho_{AB} - \chi| = \min_{\chi \in S} \text{Tr}\left[(\rho_{AB} - \chi)^2\right], \]

where \( S \) is the set of separable states, i.e. \( \rho_{AB} \in S \) if and only if

\[ \rho_{AB} = \sum_a p_a \rho_A^a \otimes \rho_B^a, \]

where the \( p_a \) are probabilities and \( \rho_A^a, \rho_B^a \) refer to systems \( A \) and \( B \), respectively. We shall also have occasion to refer to classical states, which we take to be states of the form

\[ \rho_{AB} = \sum_a p_a \Pi_A^a \otimes \Pi_B^a, \]

where \( \Pi_A^a, \Pi_B^a \) are projections. The set of pure states, for which there is a basis in which \( \rho_{AB} \) is itself a projection operator, will be denoted by \( \mathcal{P} \).
For 2 qubits, a general state can be written using the basis of SU(4) generators:

$$\rho = \frac{1}{4} (\sigma_0 \otimes \sigma_0 + \sum_{i=1}^{3} N_{0i} \sigma_i \otimes \sigma_i + \sum_{i=1}^{3} \sum_{j=1}^{3} N_{ij} \sigma_i \otimes \sigma_j).$$

(2)

$\sigma_0$ is the $2 \times 2$ identity and $\sigma_{1,2,3}$ are the Pauli matrices that generate $SU(2)$. The 15 $SU(4)$ generators are $\sigma_i \otimes \sigma_j$ (where either $i > 0$ or $j > 0$). $N_{0i}$ and $N_{ij}$ are sometimes called local Bloch vectors of qubit $A$ and $B$, respectively. $N_{ij}$ with both $i > 0$ and $j > 0$ is sometimes termed the correlation tensor. This representation of the density matrix is variously called the Pauli basis, the polarization vector, the coherence vector, and the generalized Bloch vector. We will usually use the latter term. Since we will mainly use the geometric discord in this paper, it is important to clarify the distinction between the usual quantum discord and the geometric discord. The distinction is most important when we consider possible physical trajectories in the state space. For the moment, we will not worry about specific models but will just consider curves that are smoothly parametrized by $t$, which we may think of as a time variable. Unlike a quantum entanglement measure such as the concurrence and its geometric counterpart (distance to the nearest separable state), discord and geometric discord are not always monotonic functions of one another, i.e. it is possible that $dD/dt$ has the opposite sign from $dD_G/dt$ at points along some trajectory $\rho_{AB}(t)$ in the state vector space. An example is shown in figure 1 where quantum discord and geometric discord show different behavior for a trajectory restricted to the Bell-diagonal subclass of states defined by the fact that only the three components $N_{11}, N_{22}, N_{33}$ are non-zero. In figure 1, the trajectory moves along the straight line $N_{11} = -0.7, N_{22} = -0.3, N_{33} = -1 + 2t$ as $t$ varies from 0 to 1. It can be seen that there are values of $t$ such that $dD/dt < 0$ but $dD_G(t) > 0$.

The reason for this non-intuitive behavior can be seen from figure 2, where curves of constant $D$ and $D_G$ in the plane defined by $N_{11} = -0.7$ are depicted. All allowed states then lie inside the tilted rectangle in this plane. The only concordant point in this plane is $(N_{11}, N_{22}, N_{33}) = (-0.7, 0, 0)$—the center of the tilted rectangle. The curves of constant geometric discord are the circles. The other more complicated curves are the curves of constant quantum discord. The trajectory of figure 1 is the thick vertical line segment $N_{22} = -0.3$, staying inside the rectangle of the vertical line plotted in figure 2. This trajectory hits some of the geometric discord curves only once while it hits some of the discord curves two times, which is the reason for the two different time behaviors. It is easily seen that the trajectory must be carefully chosen for this to occur, which is the reason for the arbitrary-seeming values of the trajectory parameters. One can see from this discussion that while the two quantities $D$ and $D_G$ measure essentially the same thing, subtle differences in the actual functional dependences mean that the relation between the two is not monotone.

The use of the geometric discord based on the Hilbert–Schmidt metric in this paper rather than the quantum discord is for ease of computation and analysis, even though this correlation measure leaves much to be desired [30]. The corresponding work for other better measures of the quantum discord would appear to be more difficult and is left for future work.

2.1. Frozen discord

‘Frozen’ quantum discord occurs when $D(t)$ or $D_G(t)$ is constant positive number for a finite interval of time. During this time period, the quantum mutual information and the classical correlations decrease, but the difference $D = I - I_{\text{class}}$ remains fixed [15, 31]. Since surfaces
of zero discord can have simple shapes in $N$-space [25], surfaces of constant geometric discord can also have relatively simple shapes and simple plausible models can produce the phenomenon of frozen discord. This is much less likely to occur for the quantum discord, for which the shapes of the surfaces are typically complicated. Examples of the latter are shown in figure 3.

3. State space

3.1. Topology of $\mathcal{C}$

Optimizing the classical correlations over the measurements requires considerable effort [28, 32, 33]. The geometric discord $D_G$, defined in equation (1), is usually easier to compute. The minimization present in definition (1) can now be performed explicitly and the geometric discord is obtained in a fully analytical form [29, 34]

$$D_G = \frac{1}{4} \left( \sum_{i=1}^{3} \sum_{\alpha=0}^{1} N_{i\alpha}^2 - k_{\text{max}} \right).$$

(3)

where $k_{\text{max}}$ is the maximum eigenvalue of the matrix

$$L_{ij} = N_{ij}N_{00} + \sum_{k=1}^{3} N_{ik}N_{jk}.$$

We also note that the geometric discord satisfies [34, 35] $1 \geq 2D_G \geq D^2$ with equality corresponding to pure states that are maximally entangled.

The density matrix of the zero-discord state for a pair of qubits A and B has the form (details in appendix A):

$$\rho_{AB} = p|\Psi_0\rangle\langle\Psi_0| \otimes \rho_0 + (1 - p)|\Psi_1\rangle\langle\Psi_1| \otimes \rho_1.$$ 

(4)
Here, $0 < p < \frac{1}{2}$. $\rho_k$, $k = 0, 1$, is a marginal density matrix for qubit $B$. $|\psi_0\rangle$ and $|\psi_1\rangle$ are any two orthogonal states of qubit $A$. $D(\rho_{AB}) = D_G(\rho_{AB}) = 0$ if and only if $\rho_{AB}$ has this classical-quantum form. If so, then $\rho_{AB} \in \mathcal{C}$, the set of concordant states.

Figure 2. The plane of Bell-diagonal states having $N_{11} = -0.7$. Circles centered on the origin represent surfaces of constant geometric discord. Other more complex curves represent surfaces of constant discord. Only the states lying inside the tilted rectangle are physical states that satisfy positivity. The Bell-diagonal subclass of states lies in the tilted rectangle. The square is the corresponding separable subset of this subclass of states. The larger the (geometric) discord value, the further the constant (geometric) discord curve from the concordant (zero-discord) point $(N_{11}, N_{22}, N_{33}) = (-0.7, 0, 0)$. The vertical line with the segment inside the rectangle describes one possible trajectory that results in discord and its geometric measure of the system not mutually monotonic increasing with one another. This is the trajectory shown in figure 1.

Figure 3. Examples of curves of constant discord for two different sections of $N$-space. Only the two coordinates listed are nonzero. Coordinate axes are always straight surfaces of zero discord, and discord increases as the distance from the axes increases, but the precise functional dependence varies depending on which axis pair is considered. All states inside the square and the disk are separable.
Our goal in this section is to determine the topological structure of $C$. We shall show that if certain points are subtracted from $C$ we get a set $\tilde{C}$ that is a boundaryless 9-manifold. Thus nearly every point of $C$ has a neighborhood that is homeomorphic to an open set of $H^9$, the 9-dimensional half-space. This serves as a basis for understanding the dynamics of discord.

The strategy of the argument is first to establish a one-to-one continuous and invertible mapping $f$ from a known boundaryless 9-manifold $\mathcal{A}$ to a set $\tilde{C}$. We then consider extensions of $f$ in order to understand the relation of $\tilde{C}$ to $C$ itself. We can also show that the 9 tangent vectors of this mapping are linearly independent on $\tilde{C}$ so that we have a valid coordinate chart on $\tilde{C}$. Since the difference between $C$ and $\tilde{C}$ is a set of measure zero, the coordinate chart is sufficient for purposes of, for example, integration on $\tilde{C}$.

We consider the set $\mathcal{A} = J \times S_2 \times B_3 \times B_5 \times$ denotes the Cartesian product. $S$ denotes a boundaryless 1-manifold, is the open interval $(0, 1/2)$. Points belonging to $J$ will be labeled by $p$: $0 < p < 1/2$. Points belonging to $S_2$ will be denoted by $\vec{m}$ or $(m_1, m_2, m_3)$ with $|\vec{m}|^2 = m_1^2 + m_2^2 + m_3^2 = 1$. (Spherical polar coordinates will also be used later). $B_3$ is the open 3-ball which is a boundaryless 3-manifold. Points belonging to the first copy of $B_3$ will be denoted by $\vec{n}_0$ or $(n_{01}, n_{02}, n_{03})$ with $|\vec{n}_0|^2 = n_{01}^2 + n_{02}^2 + n_{03}^2 < 1$ and similarly for the second copy of $B_3$ and $\vec{n}_1$. Since the Cartesian product of simply-connected boundaryless manifolds is a boundaryless manifold, and the dimensions add, $\mathcal{A}$ is a simply-connected boundaryless 9-manifold. We now define a map $f(p, \vec{m}, \vec{n}_0, \vec{n}_1)$ from $\mathcal{A}$ to $\mathbb{R}^{15}$ (Euclidean 15-space) $f: \mathcal{A} \rightarrow \mathbb{R}^{15}$ by

- $N_{0i} = p n_{0i} + (1 - p) n_{1i} \quad (5)$
- $N_{10} = (2p - 1) m_1 \quad (6)$
- $N_{0j} = m_1 [ p n_{0j} - (1 - p) n_{1j} ] \quad (7)$

The various $N$‘s give the 15 components (appendix A) of $f$ and $i, j = 1, 2, 3$. These can be thought of as a generalized Bloch vector for states in $C$. It contains 3 components for the marginal density matrices of the two individual qubits and 9 for the correlations. Geometrically, the points of $N_{0i}$, considered as a set in $\mathbb{R}^3$, lie on the line joining the 3-vectors $\vec{n}_0$ and $\vec{n}_1$. Since $\vec{n}_0 \in B_3$ and $\vec{n}_1 \in B_3$, the set of points $N_{0i}$ (i.e. the image of $f$ restricted to the first three dimensions of $\mathbb{R}^{15}$) fills out an open 3-ball $B_3$, and this set is independent of the value of $p$. Similarly the set of possible values of $N_{ij}$ for any fixed $i$ is an open 3-ball of radius $m_i$ that is independent of $p$.

The physical meaning of the various parameters is clarified by computing the magnitude of $\vec{N} = (N_{01}, \ldots, N_{0i}, \ldots, N_{3j})^T$:

$$|\vec{N}|^2 = (2p - 1)^2 + 2p^2 |\vec{n}_0|^2 + 2(1 - p)^2 |\vec{n}_1|^2. \quad (8)$$

Pure states have $|\vec{N}|^2 = 3$ in our normalization, which implies that the pure states of $C$ have $p = 0$ and $|\vec{n}_0| = 1$. Since entanglement and discord are the same for pure states, these are product states, as is evident if we insert the conditions for $p$ and $\vec{n}_0$ in equation (4).

$f$ consists only of polynomial functions so it is obviously smooth. $C$ is the image of $f$ and it is defined by equations (5)–(7), and the restrictions on the input variables. $\text{im } f \subset \mathbb{R}^{15}$ and $f$ is surjective on $\tilde{C}$ by definition. $C$ is clearly compact.

It remains to show that $f$ is injective and therefore invertible. We note first from equation (6) that $N_{0i}$, considered as a 3-vector $\vec{N}_{0i}$, lies inside a ball of radius 1: $N_{0i}^2 + N_{20}^2 + N_{30}^2 < 1$. This follows from the fact that $0 < 1 - 2p < 1$. It is also the case that any point in $\mathbb{R}^{15}$ that
has \( \{N_{10}, N_{20}, N_{30}\} = \{0, 0, 0\} \) is not included in \( C_{-} \) since \( |\vec{m}| = 1 \) and \( p < 1/2 \). We will comment on this later. The restricted function \( N_{ij}(p, m_{ij}) \) is one-to-one for all \( \{N_{10}, N_{20}, N_{30}\} \) such that \( 0 < N_{10}^2 + N_{20}^2 + N_{30}^2 < 1 \), and the inverse function is \( (m_{1}, m_{2}, m_{3}) = (N_{10}, N_{20}, N_{30})/|N_{0}| \) and \( p = 1/2 - |\vec{N}|/2 \). Hence the specification of \( \vec{N}_{0} \) uniquely determines \( p \) and \( \vec{m} \). Once these quantities are known and \( N_{ij} \) and \( N_{ji} \) are given, we can form the combinations

\[
\frac{1}{2p} (N_{ij} + N_{ji}/m_{ij}) = n_{0i},
\]

\[
\frac{1}{2(1 - p)} (N_{ij} - N_{ji}/m_{ij}) = n_{1i},
\]

obtained by adding and subtracting equations (5) and (7). Because of the product form of \( N_{ij} \), any choice of \( j \) for which \( m_{j} \neq 0 \) (and at least one such must exist since \( |\vec{m}| = 1 \)) will do in these equations, which determine \( n_{0i} \) and \( n_{1i} \) uniquely. This completes the specification of \( f^{-1} \). \( f^{-1} \) maps every point in \( C_{-} \) to a unique point of \( A \). \( f \) is injective and \( f \) and \( f^{-1} \) are continuous, so \( f \) is a homeomorphism. Every compact subset of \( A \) is mapped to a compact subset of \( \text{im} f \), so \( f \) is an embedding and \( C_{-} \) is a boundaryless 9-manifold. Every point in \( C_{-} \) has a neighborhood that is homeomorphic to a neighborhood in \( \mathbb{R}^6 \).

The topology of \( \text{im} f \) is found by a parallel argument. \( C_{-} \) is homeomorphic to \( A \), which is simply-connected since it is a Cartesian product of simply-connected manifolds. Hence \( C_{-} \) is simply connected. Its algebraic topology is not entirely trivial, however, since the second homology group \( H_2(A) = \mathbb{Z} \) (because of the factor of \( S_2 \) in \( A \)), which implies that \( H_2(C_{-}) = \mathbb{Z} \) as well.

It remains to relate \( C_{-} \) to \( C \), the set of concordant states. To do so, we examine points in the closure of \( A \) and the associated extensions of \( f \). There are 3 classes of such points, which we now consider in turn.

1. \( |\vec{n}_{0i}| = 1 \) and \( |\vec{n}_{1i}| = 1 \). Addition of these points to \( A \) adds the boundary of a 3-dimensional half-space to the allowed \( N_{ij} \) and similarly for the set of allowed \( N_{ij} \) whenever \( m_{j} = 1 \). The points added to \( C_{-} \) have neighborhoods homeomorphic to a neighborhood of a boundary point of \( H^6 \), the 9-dimensional half-space, so they are typical boundary points. Physically, \( |\vec{n}_{0i}| = 1 \) or \( |\vec{n}_{1i}| = 1 \) indicates a pure state of qubit \( A \) in one term of superposition.

2. \( p = 0 \). For any continuous extension of \( f \) to the points with \( p = 0 \) we find that the new points for the generalized Bloch vector are given by

\[
N_{0i} = n_{1i},
\]

\[
N_{10} = -m_{i},
\]

\[
N_{ij} = -m_{i}n_{ij}.
\]

Again, since the set of allowed \( N_{0i} \) and \( N_{ij} \) is independent of \( p \), the only effect of varying \( p \) is to vary the magnitude of \( N_{0i} \). \( p = 0 \) corresponds to unit radius. Adding \( p = 0 \) to the domain of \( f \) thus adds the boundary of a 3-dimensional half-space to the allowed \( N_{ij} \) and again these are typical boundary points of \( C_{-} \). Physically, this value of \( p \) corresponds to a product state: qubit \( B \), in a mixed state for all \( p > 0 \), is now in a pure state defined by \( \vec{m} \) and qubit \( A \) is in an arbitrary mixed state specified by \( \vec{n}_{i} \). There is no correlation whatever between \( A \) and \( B \).

3. \( p = 1/2 \). These points also lie in the closure of \( A \). Now we obtain an extension of \( f \) whose image includes the new points
\[ N_{0i} = \frac{1}{2}(n_{0i} + n_{i}) \]  
\[ N_{00} = 0 \]  
\[ N_{ij} = \frac{1}{2}m_i(n_{0j} - n_{ij}). \]

We need only consider the change in the set of allowed \( N_{0i} \), since the set of allowed \( N_{00} \) and \( N_{ij} \) is not affected by \( p \), as already noted. The only points of \( \mathbb{R}^1 \) that are added to \( \text{im} f \) are those with \( N_{00} = 0 \)—otherwise there is no change. For any fixed \( p < 1/2 \), \( \mathcal{C} \) restricted to the 3-dimensional subspace \( N_{0i} \) is an open 3-ball with the origin subtracted out. For \( p = 1/2 \), the image of the three-dimensional subspace in which only the \( N_{0i} \) are nonvanishing is the origin. The origin is a 0-dimensional object, so any extension of \( f \) that includes \( p = 1/2 \) in its domain will not be invertible. The origin does lie in \( \mathcal{C} \), of course. However, it is easy to show that it is not a simple boundary point. All of the 15 coordinate axes belong to \( \mathcal{C} \) and they intersect at the origin. This implies that the origin does not have a neighborhood in \( \mathcal{C} \) that is homeomorphic to an open set of \( \mathbb{R}^3 \). Hence \( \mathcal{C} \) itself is not a manifold. Physically, at \( p = 1/2 \) the qubit \( B \) is in the completely mixed state and any partial density matrix is possible for qubit \( A \).

The addition of points in classes 1 and 2 do not affect the algebraic topology of \( \partial \mathcal{C} \). They are essentially boundary points and any path passing through these points can be deformed into a path that lies entirely in \( \mathcal{C} \). This is probably also the case for points in class 3, which leads to the conjecture that \( \mathcal{C} \) itself is simply-connected. We do not have a proof of this, however.

To summarize, we find that \( \partial \mathcal{C} \subset \mathcal{C} \) is a simply-connected 9-manifold without boundary.

The homeomorphism \( f \) provides an embedding of \( \partial \mathcal{C} \) into the 15-dimensional space \( \mathcal{M} \) of all density matrices, defining a 9-dimensional hypersurface that differs from \( \mathcal{C} \) itself by a set of measure zero. Some additional properties of the hypersurfaces are:

1. \( \mathcal{C} \) includes points infinitesimally close to the origin, (the point having \( N_{0i} = N_{00} = N_{ij} = 0 \). 
2. \( \mathcal{C} \) includes the origin itself.
3. \( \mathcal{C} \) includes intervals lying on all 15 of the coordinate axes (points for which only one of the \( N_{0i} \), \( N_{00} \), \( N_{ij} \) is nonzero). See appendix B for the proof. For example the \( N_{0x} \) axis corresponds to \( p = 1/2, n_{0x} = n_{1x} = 0, n_{0y} = n_{0z} = n_{1y} = n_{1z} = 0 \); the \( N_{00} \) axis corresponds to \( m_i = 1, n_{0y} = n_{0z}, n_{1x} = 0, n_{0y} = n_{1z} = 0 \). The \( N_{0y} \) axis corresponds to \( p = 1/2, m_i = 1, m_v = m_e = 0, n_{0y} = n_{1y} = 0 \), all others zero.
4. The four eigenvalues of \( \rho_{AB} \in \mathcal{C} \) are:
\[ \lambda_{4,2} = \frac{1}{2} p(1 \pm |\bar{n}_0|^2) \]
\[ \lambda_{3,4} = \frac{1}{2} (1 - p)(1 \pm |\bar{n}_1|^2). \]

Pure states have one eigenvalue equal to one and the others zero, which means \( p = 0 \), \( |\bar{n}_i| = 1 \) and \( n_i \) arbitrary. These points lie on \( \partial \mathcal{C} \), the boundary of \( \mathcal{C} \). The pure concordant states are just the usual pure product state and belong to a 4-manifold \( \mathcal{P}_C \). Expressed in terms of density matrices, any state of this type is classical–classical with a single product of projection operators, i.e. its density matrix is of the form \( \rho = |\Psi_0\rangle\langle\Psi_0| \otimes |\Psi_0\rangle\langle\Psi_0|. \)
In what follows, we will often refer to the set $C$ rather than $\bar{C}$, since many of our considerations do not depend on the fact that $C$ is not itself a manifold; $C$ and $\bar{C}$ differ by only a set of measure zero. The distinction between $C$ and $\bar{C}$ is rather technical. In fact, if we consider the topology of a different level set of the function $D$, say $C_\varepsilon$ defined as $C_\varepsilon = \{ \mathbf{n} | D(\mathbf{n}) = \varepsilon \}$, where $\varepsilon$ is positive but very small, then we expect that $C_\varepsilon$ is a manifold. The topology of general level sets for arbitrary $\varepsilon$ raises interesting questions that are beyond the scope of this paper.

3.2. Parameterization of $C$ (calculus on $C$)

We may calculate the 9 tangent vectors, namely $\mathbf{t}_i = \partial \mathbf{f} / \partial x_i$; $i = 1$ to 9, where $x_1 = \theta, x_2 = \phi$ are the spherical polar coordinates for $\mathbf{m}, x_3 = p, x_4 = n_0, \ldots$ etc. The explicit forms of the $\mathbf{t}_i \in \mathbb{R}^{15}$ are in appendix D. We show there that these 9 tangent vectors form a linearly independent set almost everywhere in $C$, i.e. that if there exists a set of real numbers $c_1, c_2, c_3, \ldots, c_9$ such that $c_1 \mathbf{t}_1 + c_2 \mathbf{t}_2 + c_3 \mathbf{t}_3 + \ldots + c_9 \mathbf{t}_9 = 0$ then $c_1 = c_2 = \ldots = c_9 = 0$. This procedure fails when any of the $\mathbf{t}_i$ vanish. This occurs at the purely coordinate singularities $\theta = 0$ and $\theta = \pi$, which are not truly singular points. It also happens at points with $p = 1/2, n_0 = \bar{n}_4$ and at the points $p = 0$.

As we have seen above, these correspond to real singular points. However, since they occupy a set of measure zero, the parametrization with $\theta, \phi, p, n_0, \bar{n}_4$ can be used for integration with the 9-surface element

$$dS = \sqrt{|g|} \ d\theta \ d\phi \ dp \ ... \ dn_{11} \ dn_{12} \ dn_{13}$$

where $g$ is the appropriate matrix tensor $g = \left( g_{ij} \right)$.

$g$ consists of elements $g_{ij} = \mathbf{t}_i \cdot \mathbf{t}_j$. Most of the off-diagonal elements of $g$’s are zero (see the full matrix form in appendix D). We obtain

$$\sqrt{|g|} = 16p^3(1-p)^3 \sin \theta \left\{ \sum_{i=1}^{3} [p m_{ii} - (1-p)n_{ii}]^2 + (1 - 2p)^2 \right\}.$$ 

3.3. 2-dimensional and 3-dimensional cross sections of $C$

It is difficult to visualize a 9-dimensional structure such as $C$. Accordingly, we consider sections of $C$: intersections of $C$ with coordinate planes obtained by setting some coordinates of $\mathbb{R}^{15}$ equal to zero. In particular, we will consider 2-sections for which 13 coordinates are zero, and 3-sections for which 12 coordinates are zero. This will help to make clear the differences between entanglement and discord. Because of the fact that $\mathcal{M}$ and $\mathcal{S}$ are convex 15-dimensional sets that include the origin, the 2-sections of $\mathcal{M}$ and $\mathcal{S}$ are all 2-dimensional convex sets. In fact all 2-sections of $\mathcal{M}$ are either squares or disks centered at the origin [23]. (Note that using a different basis, such as the Gell-Mann matrices [36], can result in the presence of other types of geometry for the 2-sections such as triangles and parabolas.) Zhou et al [23] were able to show that the occurrence of squares and disks is determined by the commutativity properties of the operators corresponding to the two axes: squares for commuting operators and disks for non-commuting operators. Since the shape of $\mathcal{M}$ is determined by positivity conditions on the eigenvalues; this is not so surprising: the contribution of the coefficients $N_{ij}$ add in quadrature to the eigenvalues of the non-commuting case.

Making a complete survey of the 2-sections of $C$ reveals interesting similarities and differences to those of $\mathcal{M}$, as shown in table 1. There are three geometries observed for the 2-sections of $C$: the square, the disk, and the cross. The first two are the same as for $\mathcal{M}$, and
presumably reflect similar physics, but the cross is new and it occupies about one third of the table. It is the union of the two line intervals $[-1, 1]$ lying in the two Cartesian axes. This is a locally 1-dimensional object (except at the origin, where the intersection of the intervals occurs), which reflects the lower dimensionality of $\mathcal{C}$, as compared to $S$ or $\mathcal{M}$. Furthermore, unlike entangled states, there are discordant states arbitrarily close to the origin. Using the explicit form for the 15 components of $N$'s for a concordant state as expressed in equations (5) and (7), the disk and square of $\mathcal{C}$ are always specified with 2 independent variables while this is not possible for the cross; the only two nonzero components of the intersecting plane of the state cannot be nonzero at the same time if we are to have zero discord. An explicit example for the square is the state $\rho = I_{10} \otimes I_{10} + I_{10} \otimes I_{21} + I_{21} \otimes I_{10}$, $I$ is the identity matrix. This is a concordant subset of $\mathcal{M}$ obtained when $n_{02} = n_{03} = n_{12} = n_{13} = m_2 = m_3 = 0, m_1 = \pm 1$, and $n_{01} (=n_{11})$ and $p$ are freely chosen from $A$ such that $N_{10} = \pm (2p - 1)$ and $N_{13} = \pm 2pm_{01}$.

For the cross geometry, consider the example $\rho = I_{10} + I_{10} \otimes I_0 + I_{10} \otimes I_{21} + I_{21} \otimes I_0$. The states of this set are discordant everywhere except on the coordinate axes. The concordant states have only a single nonzero component, either $N_{10} = (2p - 1)m_1$ or $N_{21} = 2pm_{01}$. Specifically, in order for both $N_{10}$ and $N_{21}$ to be nonzero, the product $2(2p - 1)mm_{01} = 0$. But this implies that $N_{11} = 2mp_{01} = 0, N_{20} = (2p - 1)m_2 = 0$.

Let us consider the positions of the cross geometry in more detail, since this geometry is unique to discord. States in a 2-section have the form

$$\rho = I_{ij} + \frac{1}{4}N_{ij} \otimes \sigma_j + \frac{1}{4}N_{kl} \otimes \sigma_l$$

so that we can refer to the $ij, kl$ section with $0 \leq i, j, k, l \leq 3$. We first note that crosses occur only when at least one correlation function is involved, i.e. at most one of the $j, l$ can be zero. (This observation is related to the fact that we have considered the 'left' discord measure, which is, in this case, on qubit A. Similar statements hold for $i, k$ for the 'right' discord measure.)

|   | 0X | 0Y | 0Z | 00 | 01 | 02 | 03 | 04 | 05 | 06 | 07 | 08 | 09 |
|---|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 1 | 1  | 2  | 3  | D  | D  |   |   |   |   |   |   |   |   |
| 2 | 4  | S  | S  |    |    |   |   |   |   |   |   |   |   |
| 3 | 5  | S  | S  | S  | D  |   |   |   |   |   |   |   |   |
| 4 | 6  | S  | S  | S  | D  |   |   |   |   |   |   |   |   |
| 5 | X  | 7  | 8  | 9  |   |   |   |   |   |   |   |   |   |
| 6 | Y  |   |   |   |   |   |   |   |   |   |   |   |   |
| 7 |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 8 |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 9 |   |   |   |   |   |   |   |   |   |   |   |   |   |
|10 |   |   |   |   |   |   |   |   |   |   |   |   |   |
|11 |   |   |   |   |   |   |   |   |   |   |   |   |   |
|12 |   |   |   |   |   |   |   |   |   |   |   |   |   |
|13 |   |   |   |   |   |   |   |   |   |   |   |   |   |
|14 |   |   |   |   |   |   |   |   |   |   |   |   |   |
|15 |   |   |   |   |   |   |   |   |   |   |   |   |   |
measure.) This is expected, since discord requires correlation. This emphasizes the fact that discord, as compared to entanglement, is much more resistant to dephasing, since states of this kind can be arbitrarily close to the origin where the system is completely dephased. This state contains quantum correlation because it combines the non-commuting operators $\sigma_1\sigma_2$ and $\sigma_3\sigma_4$.

The choice of how to measure qubit 1 (along the $x$-axis or along the $y$-axis) can have some effect on how much information we gain about qubit 2. Finally, we look at the case when all of the $i, j, k, l$ are nonzero. Crosses occur if and only if $i \neq k$ and $j \neq l$, e.g.

$$\rho = \frac{I}{4} + \frac{1}{4}N_1\sigma_i \otimes \sigma_k + \frac{1}{4}N_2\sigma_j \otimes \sigma_l$$

with $0 < |N_1| < 1$ and $0 < |N_2| < 1$ is separable but discordant, but

$$\rho = \frac{I}{4} + \frac{1}{4}N_1\sigma_i \otimes \sigma_k + \frac{1}{4}N_2\sigma_j \otimes \sigma_l$$

with $0 < |N_1| < 1$ and $0 < |N_2| < 1$ (a disk state) is separable and concordant. It seems that since measuring qubit 1 along the $y$ or $z$ axes gives no non-trivial information, the choice involved does not generate discord.

An examination of the 3-sections of $\mathcal{C}$ is also revealing. Such a state is of the form

$$\rho = \frac{I}{4} + \frac{1}{2}N_1\sigma_i \otimes \sigma_k + \frac{1}{2}N_2\sigma_j \otimes \sigma_l + \frac{1}{4}N_{mn}\sigma_m \otimes \sigma_n.$$ Since three nonzero coefficients are necessary to form a maximally entangled (pure) state, the 3-sections bring in qualitatively new physics. If the 3-section does include maximally entangled states, then these states occupy the vertices of a tetrahedron geometry, as shown previously for the Bell states [25]. We show that the 3-sections can have zero or nonzero 3-volume. Using this fact and the table of 2-section geometries (table 1) we can characterize all allowed 3-section geometries.

When a 3-section has nonzero volume, it is specified by a set of three independent parameters drawn from $A$. As a relatively simple example, consider the 3-section obtained by varying $p, m_0, n_0$, and $n_1$, setting $m_1 = m_2 = n_0 = n_2 = n_3 = 0$ and $m_3 = 1$. The 3 remaining parameters $p, n_0$, and $n_1$ are arbitrary. Consider $m_1 = 1$. Then we find that the allowed values of $N_{01} = pn_{01} + (1 - p)n_{11}, N_{02} = (2p - 1), N_{11} = pn_{01} - (1 - p)n_{11}$ form a tetrahedron. Note that $N_{01}, N_{02}, N_{11}$ defined in this way are independent of one another: the subclass of states having the density matrix $\rho = \frac{1}{4}(\sigma_0 \otimes \sigma_0 + N_{01}\sigma_0 \otimes \sigma_1 + N_{02}\sigma_0 \otimes \sigma_0 + N_{11}\sigma_1 \otimes \sigma_1)$ contains all physical states in a tetrahedron that is identical to the concordant tetrahedron specified before the set of parameters $\{p, n_{01}, n_{11}\}$.

We now make the observation that the 2-sections of this tetrahedral 3-section obtained by intersection with the $\{N_{01}, N_{02}\}, \{N_{01}, N_{11}\}$ and $\{N_{02}, N_{11}\}$ planes are all squares (see table (1)). This is in fact true of any combination of $\{N_{01}, N_{02}, N_{11}\}$ (9 combinations in total) that forms a tetrahedron of concordant states and leads to the classification of all nonzero-volume 3-sections into 4 types.

1. Tetrahedron in which all 3 2-sections are squares.
2. Unit ball in which all 3 2-sections are disks. For example, the combination $\{N_{01}, N_{02}, N_{03}\}$ by setting $p = 1/2, \vec{n}_0 = \vec{n}_1$ is in this class. Now, $N_{01} = n_{01}$ independent of $\vec{m}$. Similarly to the tetrahedron case discussed above, all three components are independent of one another, which indicates that any physical state made of three component $N_{01}$ is a concordant state. This property holds for other combinations such as $\{N_{01}, N_{22}, N_{33}\}$ by setting $\vec{n}_0 = \vec{n}_1 = 0$ or $\{N_{11}, N_{12, N_{13}}\}$ by setting $p = 1/2, \vec{n}_0 = -\vec{n}_1$, and $m_3 = m_1 = 0$, etc.
3. Union of 2 cones in which 1 2-section is a square and 2 are disks. The object can be thought of as 2 cones glued together at their bases or as the surface of revolution formed when a square is rotated about an axis that passes through its center. See figure 4(left).
4. A less easily described 3-dimensional object shown in figure 4(right), in which 1 2-section is a disk and 2 2-sections are squares. For example, the combination \( \{ N_{01}, N_{10}, N_{12} \} \) obtained by setting \( m_2 = m_3 = n_{03} = n_{13} = 0 \) and \( p n_{01} = (1 - p) n_{11} \) and \( p n_{02} = -(1 - p) n_{12} \). This 3-section has 2-sections that are a set of 2 squares \( \{ N_{01}, N_{10} \} \) and \( \{ N_{01}, N_{12} \} \) and 1 disk \( \{ N_{10}, N_{12} \} \), etc.

Last, we consider 3-sections with zero 3-volume. Its 2-sections include at least one cross. Analytically, such a 3-section is specified by a union of sets of equations and each set has at most 2 independent variables. An example is the 3-section with nonzero \( \{ N_{01}, N_{10}, N_{12} \} \) obtained by setting either 1) \( m_2 = m_3 = n_{02} = n_{03} = n_{12} = n_{13} = 0 \) and \( p n_{01} = (1 - p) n_{11} \) i.e. by at most 2 independent parameters \( (p,n_{01}) \) or \( (p,n_{11}) \) or 2) \( m_1 = m_3 = n_{02} = n_{03} = n_{12} = n_{13} = 0 \) and \( p = 1/2 \) i.e. by 2 independent parameters \( (n_{01}, n_{11}) \). If there are 3 crosses among the 2-sections, then the 3-section is locally 1-dimensional. The extreme example is the Bell-diagonal state with nonzero \( \{ N_{11}, N_{22}, N_{33} \} \) that has a 3-section that is the union of 3 1-section objects—the coordinate axes.

3.4. Relative topology of \( C \) and \( S \)

In order to understand the joint evolution of entanglement and discord, we need to understand the relationship of \( C \), the set on which discord vanishes, to \( S \), the set on which entanglement vanishes. The topology of \( S \) is relatively simple and well-understood. It is convex, has finite 15-volume (in contrast to \( S \)), includes both the origin and points on the surface of \( M \) (the set of all physical states). It can be thought of as a spheroid with a golf-ball-like shape, since parts of its surface are faceted and parts are curved. Of course \( C \subset S \subset M \). (In this section we neglect the difference between \( C \) and \( \mathcal{C} \).

The main question is the nature of the set \( S \setminus C \), since it is on this set that discord and entanglement differ in character. Since \( \dim S - \dim C = 6 > 1 \), the subtraction of \( C \) from \( S \) cannot disconnect \( S \). Indeed, the subtraction cannot affect the homology groups \( C_n(S) \) for \( n < \dim S - \dim C - 1 = 5 \). The evolution takes place on a 1-dimensional curve and is not affected by any possible high-order changes. We conclude that the possible joint evolutions are constrained only by the relatively trivial constraint of inclusion of \( C \) in \( S \), and by the individual constraints imposed by the separate topologies of \( C \) and \( S \).
4. Time evolution of discord

We now turn to the consequences of the topological analysis for the time evolution of the quantum discord in 2-qubit systems. We are mainly interested in decoherence and thus in the various ways in which discord eventually disappears. Hence we will focus on trajectories for which the initial state of the system has finite discord that decreases initially and that vanishes as $t \to \infty$. Other behaviors are possible and interesting. For example, we could consider the opposite behavior where discord vanishes in the initial state but is created by contact with a cold bath or some entangling 2-qubit gate. The analysis of such trajectories, which involves the understanding of the level sets of $D$ at higher values of discord, is left for future work. In this paper the trajectories mainly live in the vicinity of $C$.

Hence we consider functions $D_G(\vec{N}(t))$, where $\vec{N}$ is the 15-dimensional real generalized Bloch vector and $D_G$ is the geometric discord. We will further assume that the system tends to a limit as the time approaches infinity: 

$$\lim_{t \to \infty} \vec{N}(t) = \vec{N}_\infty$$ and 

$$D_G(\vec{N}_\infty) = D_G(\vec{N}_\infty).$$ If there are no self-intersections, the trajectory $\{\vec{N}(t) | 0 \leq t < \infty\}$ itself is a 1-dimensional manifold.

We briefly review the analysis of evolution entanglement. For any evolution, we define the set of times when the entanglement vanishes: 

$$T_0 = \{ t | C(\vec{N}(t)) = 0 \},$$ where $C$ is the concurrence [37, 38]. In previous work [24], transversality theory was applied to the intersections of trajectories with the set $S$ to analyze the possible forms of $T_0$. It was found that when the trajectory and $S$ are transversal, (the generic case), then there are 4 possible behaviors of entanglement disappearance as measured by, for example, the concurrence.

1. ‘Entering’ behavior abbreviated by $E$. This is also known as entanglement sudden death and corresponds to $T_0 = (t_e, \infty)$.

2. ‘Oscillating’ behavior abbreviated by $O$. This is sudden death followed after a finite interval by re-emergence of entanglement. The pattern may repeat a finite or countably infinite number of times. This corresponds to $T_0$ being a union of finite intervals.

3. ‘Approaching’ behavior abbreviated by $A$. In this scenario, entanglement measures approach zero asymptotically in time, (most typically exponentially). This corresponds to $T_0 = \emptyset$.

4. ‘Bouncing’ behavior abbreviated by $B$. Entanglement measures go to zero at finite times but the entanglement reappears immediately. $T_0$ is a collection of isolated points.

It follows from transversality theorems that $E$ and $O$ are both stable under small perturbations. $A$ and $B$ behaviors occur when transversality is violated, which requires a symmetry in the dynamics or other special conditions.

The condition for transversality theorems to hold is that the sum of the dimensions of the intersecting manifolds be greater than or equal to the dimension of the underlying space, which holds for trajectories (which have dimension 1) and $S$ since $\dim S + 1 = 16 > 15 = \dim M$. Since $\dim C = 9 < 14 = \dim M - 1$, we cannot use the same reasoning for discord evolution.

Let us define $T^0_0 = \{ t | D_G(\vec{N}(t)) = 0 \}$. If we assume that $\vec{N}(t)$ has a continuous first derivative and that $D_G(\vec{N}_\infty) = 0$, there are two possibilities: ($T^0_0 = \emptyset$, the null set) (‘half-life’) or $T^0_0$ is a collection of isolated points (bouncing behavior), and neither of these categories is stable with respect to small perturbations. They happen only as a result of particular choices, when the discord has a non-trivial relationship to the dynamics. This can happen naturally—for example, the origin $\vec{N} = 0$ belongs to $C$ and $\vec{N}_\infty = 0$ for a system in contact with a bath at high temperature.

Our aim in this section is to illustrate, going from the simplest to more complex models, the various possibilities for the disappearance of discord that are allowed by the topology of
We shall be particularly interested in evolutions in which the discord and the entanglement behave in qualitatively different ways. This will enable to draw conclusions later about the joint evolution of the two quantities.

4.1. Unitary evolution

4.1.1. Introduction. Having classified the various possibilities for the evolution of entanglement and discord, we now turn to the question of the realization of these evolutions in explicit models. In this regard, it is useful to distinguish between unitary evolution of the density matrix and non-unitary evolution. This distinction is of course crucial for the experimental investigation of all types of coherence: unitary time evolution is by definition coherent overall since all correlation measures should be unchanged by local unitary evolution, but the behavior of different correlation measures under nonlocal unitary time evolution can help to understand the distinctions between different measures.

4.1.2. Ising model. In this section we exhibit an example of a realistic Hamiltonian and a set of initial conditions for which discord is a constant of the motion.

The Ising Hamiltonian:

\[ H^I = J_{\sigma 3} \otimes \sigma_3 \] (12)
generates a two-qubit unitary operator of the system of the form:

\[ U = \exp(-iJt \sigma_3 \otimes \sigma_3) \]
\[ = \begin{pmatrix} e^{-iJt} & 0 & 0 & 0 \\ 0 & e^{iJt} & 0 & 0 \\ 0 & 0 & e^{iJt} & 0 \\ 0 & 0 & 0 & e^{-iJt} \end{pmatrix} \]
(13)

where we use the abbreviations \( S = \sin(Jt) \) and \( C = \cos(Jt) \). \( U \) is in the basis \( \{ (1, 0), (1, 0); (1, 0), (0, 1); (0, 1), (1, 0); (0, 1), (0, 1) \} \). This would be an appropriate Hamiltonian for well-separated superconducting flux qubits with the rings lying in the same plane when the applied field is zero, or for capacitively coupled double quantum dot qubits. It is known that this Hamiltonian, together with local qubit operations, can yield a CNOT gate, so in that sense it is capable of entangling operations. However, for a wide class of states the discord is a constant of the motion, as we now show.

Under the unitary transformation (13), any initial state having the general form given in equation (2) evolves as:

\[ \rho(t) = \frac{1}{4}(\sigma_0 \otimes \sigma_0 + N_{01}(t)\sigma_0 \otimes \sigma_1 + N_{02}(t)\sigma_0 \otimes \sigma_2 + N_{03}(t)\sigma_0 \otimes \sigma_3) \]
(14)

There are 7 constants of the motion: \( N_{01}, N_{02}, N_{12}, N_{21}, N_{11}, N_{22} \). If the initial condition is an \( X \)-state\(^1\), the discord depends only on these seven components of \( N \) and it is therefore independent of time. This model appears to be the most non-trivial model that has trivial dynamics for the discord for a reasonable wide class of states: \( D \) is completely independent of time for the 8-dimensional space of \( X \)-states.

\(^1\) A state is called \( X \)-state if its density matrix contains non-zero elements lying on only the diagonal and off-diagonal lines and this formation makes the density matrix shape like the letter \( X \).
In terms of the various sets previously mentioned, what this example shows is that it is simple enough to construct a trajectory that lies entirely in $\mathcal{S} \setminus \mathcal{C}$ except at discrete times.

4.1.3. **Heisenberg model.** In this section we exhibit a slightly more complicated Hamiltonian: the set of initial conditions for which discord is a constant of the motion is smaller, but still non-empty. The model also illustrates the fact that the entanglement and the discord can evolve in qualitatively different fashions.

A Heisenberg model with the presence of all XYZ terms i.e. $H^H = J \sum_{i=1}^{n} \sigma_i \otimes \sigma_i$ is appropriate for electron spin qubits with overlapping wave functions, which will then feel the exchange interaction. The unitary transformation generated by $H^H$ is of course much richer than for the Ising Hamiltonian. Nevertheless the class of states with constant discord is the 3-dimensional space of Bell-diagonal states:

$$\rho(0) = \rho_{\text{Bel}}(0) = \frac{1}{4} [\sigma_0 \otimes \sigma_0 + \sum_i (N_i \sigma_i \otimes \sigma_i)]$$

with arbitrary values for $N_{11}, N_{22},$ and $N_{33},$ each within the range $[-1, 1].$

The remaining 8 components evolve under the unitary transformation as:

$$
\begin{align*}
N_{00}(t) &= N_{00}C_3 + N_{32}S_3 \\
N_{12}(t) &= N_{12}C_3 - N_{01}S_3 \\
N_{02}(t) &= N_{02}C_3 - N_{31}S_3 \\
N_{31}(t) &= N_{31}C_3 + N_{02}S_3 \\
N_{10}(t) &= N_{10}C_3 + N_{23}S_3 \\
N_{23}(t) &= N_{23}C_3 - N_{10}S_3 \\
N_{20}(t) &= N_{20}C_3 - N_{13}S_3 \\
N_{13}(t) &= N_{13}C_3 + N_{20}S_3
\end{align*}
$$

(15)

where $S_3 = \sin(2Jt)$ and $C_3 = \cos(2Jt)$.

Note that under a unitary transformation the purity of the state is conserved, i.e. $|\tilde{N}(t)|^2 = |\tilde{N}|^2$. The two separate groups with time-dependent components of $\rho(t)$ in equations (15) and (16) are two groups of DQC1 separable states [5].

If the initial condition is a non-Bell state, then we can get nontrivial dynamics of $D$. We choose a state such that only $N_{20}$ is nonzero and $N_{20} = 1$, which is a concordant state. The system evolves as:

$$
\begin{align*}
N_{20}(t) &= C_3 \\
N_{13}(t) &= S_3
\end{align*}
$$

(17)

and is a separable state i.e. $C(t) = 0$ with concordant subset as the union of $N_{20}(t)$ and $N_{13}(t)$ axes (see table 1). Quantum trajectory of the system is the unit circle $N_{20}(t)^2 + N_{13}(t)^2 = 1$.

The quantum discord of (17) is (see detailed calculations in appendix E):

$$
D(t) = -\frac{1}{2} [(1 + C_3) \log(1 + C_3) + (1 - C_3) \log(1 - C_3)]
$$

$$
+ 1 - \frac{1}{2} [(1 - S_3) \log(1 - S_3) + (1 + S_3) \log(1 + S_3)]
$$

and the geometric quantum discord is:

$$
D_g(t) = \frac{1}{4} (1 - \max\{C_3^2, S_3^2\})
$$
which are shown in figure 5. The entanglement is identically zero, while the discord oscillates with maximum (≈0.2) at $t = \frac{n\pi}{16} + \frac{2\pi}{3}$ and vanishes at $t = n\frac{\pi}{8}$. In the semiclassical picture, the two spins precess about one another. We have chosen a starting state that is separable, and the mutual precession does not generate entanglement. This is true for nearly all separable initial conditions, so our choice of initial state is fairly generic. For the discord, however, the situation is quite different. To have zero discord, the classical states of one subsystem need to pair up with the mixed states of the other. This requires additional phase relations. Because these phase relations are oscillating, we get a periodic behavior of the discord. Note that the geometric and quantum discord behave very similarly, as is nearly always the case. The only significant distinction is the linear (quadratic) zeros for the quantum (geometric) discord corresponding to the linear and quadratic distance measures in the definitions.

In terms of our topological analysis, the trajectories pierce $C$ at certain points, while remaining always in the set of separable states. This raises some interesting questions concerning the relative topologies of $C$ and the set of separable states, but we cannot enter into that here.

### 4.1.4. Anisotropic XY-model.

In this section we exhibit a Hamiltonian that respects few symmetries. The discord has nontrivial evolution for nearly all states.

Consider an anisotropic exchange Hamiltonian with cross-product terms:

$$H^{XY} = J_{xy} \sigma_x \otimes \sigma_y + J_{yx} \sigma_y \otimes \sigma_x.$$  \hspace{1cm} (18)

The corresponding unitary operator is:

$$U(t) = e^{-i(J_{xy} \sigma_x \otimes \sigma_y + J_{yx} \sigma_y \otimes \sigma_x)} = C_1 C_2 \left( \begin{array}{ccc} \cos(J_{xy} + J_{yx})t & 0 & 0 & \sin(J_{xy} + J_{yx})t \\ 0 & \cos(J_{xy} - J_{yx})t & \sin(J_{xy} - J_{yx})t & 0 \\ 0 & -\sin(J_{xy} - J_{yx})t & \cos(J_{xy} - J_{yx})t & 0 \\ \sin(J_{xy} + J_{yx})t & 0 & 0 & \cos(J_{xy} + J_{yx})t \end{array} \right).$$ \hspace{1cm} (19)

![Figure 5. Time dependence of the quantum discord (red solid line), geometric discord (black dotted line), and concurrence (blue dashed line) of system described by the initial conditions in equation (17) and evolving according to the Heisenberg Hamiltonian. Notice that the discord measures show oscillatory behavior but the concurrence is identically zero.](image)
where \( S_1 = \sin(J_{xy}t), C_1 = \cos(J_{xy}t) \) and \( S_2 = \sin(J_{yx}t), C_2 = \cos(J_{yx}t) \). These cross-product terms reflect the fact that the number of constants of motion decreases as compared to that of the Ising model and we expect to see different evolution behaviors for the quantum correlations in the Bell-diagonal class. Let us consider the situation where \( J_{xy} = -J_{yx} \). The unitary operator simplifies

\[
U(t) = C_2^2 \sigma_0 \otimes \sigma_0 + S_2^2 \sigma_1 \otimes \sigma_3 + i S_2 C_2 (\sigma_1 \otimes \sigma_2 - \sigma_2 \otimes \sigma_1). \tag{20}
\]

This model can arise from the Dzyaloshinskii-Moriya interaction between two electron spins whose separation vector is along the \( z \)-axis. Consider the initial state of the general form in equation (2). The state at time \( t \) is given by:

\[
N_{03}(t) = \frac{1}{2}[N_{03} + N_{30} + (N_{03} - N_{30}) \cos(4J_{xy}t) + (N_{11} + N_{22}) \sin(4J_{xy}t)]
\]

\[
N_{30}(t) = \frac{1}{2}[N_{03} + N_{30} + (-N_{03} + N_{30}) \cos(4J_{xy}t) - (N_{11} + N_{22}) \sin(4J_{xy}t)]
\]

\[
N_{11}(t) = \frac{1}{2}[-N_{11} - N_{22} + (-N_{03} + N_{30}) \sin(4J_{xy}t) + (N_{11} + N_{22}) \cos(4J_{xy}t)]
\]

\[
N_{22}(t) = \frac{1}{2}[-N_{11} + N_{22} + (-N_{03} + N_{30}) \sin(4J_{xy}t) + (N_{11} + N_{22}) \cos(4J_{xy}t)]
\]

and

\[
N_{33}(t) = N_{33}
\]

\[
N_{12}(t) = N_{12}
\]

\[
N_{21}(t) = N_{21}
\]

and

\[
N_{01}(t) = N_{01} \cos(2J_{xy}t) - N_{13} \sin(2J_{xy}t)
\]

\[
N_{13}(t) = N_{01} \sin(2J_{xy}t) + N_{13} \cos(2J_{xy}t)
\]

\[
N_{02}(t) = N_{02} \cos(2J_{xy}t) - N_{23} \sin(2J_{xy}t)
\]

\[
N_{23}(t) = N_{02} \sin(2J_{xy}t) + N_{23} \cos(2J_{xy}t)
\]

\[
N_{10}(t) = N_{10} \cos(2J_{xy}t) + N_{31} \sin(2J_{xy}t)
\]

\[
N_{31}(t) = -N_{10} \sin(2J_{xy}t) + N_{31} \cos(2J_{xy}t)
\]

\[
N_{20}(t) = N_{20} \cos(2J_{xy}t) + N_{32} \sin(2J_{xy}t)
\]

\[
N_{32}(t) = -N_{20} \sin(2J_{xy}t) + N_{32} \cos(2J_{xy}t).
\]

In this \( XY \)-model, the discord of the Bell-diagonal class of states is no longer independent

time. In the \( X \)-type of class of states, only the states with only three nonzero components \( \{N_{25}(t), N_{31}(t), N_{33}(t)\} \) have time-independent discord. All physical states of this type lie in the tetrahedron similar to the geometry of the Bell-diagonal states with the concordant subset as the union of the three intervals in the Cartesian axes.
A Werner state \([39]\):
\[
\rho(0) = \rho_W(0) \equiv \frac{1}{4}[\sigma_0 \otimes \sigma_0 - \alpha \sum_i \sigma_i \otimes \sigma_i] \\
= \frac{1 - \alpha}{4} I + \alpha |\Psi^-\rangle\langle\Psi^-|
\]
where \(|\Psi^-\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}\) and \(0 \leq \alpha \leq 1\) can exhibit sudden death/birth and oscillating behavior unitarily in this model. It evolves as:
\[
\begin{align*}
N_{03}(t) &= -N_{30}(t) = -\alpha \sin(4J_{xt}t) \\
N_{14}(t) &= N_{22}(t) = -\alpha \cos(4J_{xt}t) \\
N_{30}(t) &= -\alpha.
\end{align*}
\]
With the introduction of the mixing parameter \(\alpha\) we can also find transitions between different evolution categories. \(\alpha = 1\) is a pure maximally entangled state, while \(\alpha = 0\) is the completely mixed state. This Werner state is separable when \(\alpha = 1\) (equation (27)). As \(\alpha\) is varied, we find in the Bloch vector representation that \(\alpha \rightarrow -\alpha \rightarrow \alpha \) so the vector lies on a line segment whose end points are the origin \(\alpha = 0\) and a point on the boundary of \(\mathcal{M}\).

If the initial condition is \(\alpha = 1\), the system evolves away from a maximally entangled situation. The concurrence and quantum discord are:
\[
C(t) = |\cos(4J_{xt}t)|
\]
\[
D(t) = 1 - \frac{1}{2} \left[(\cos(2J_{xt}t) + \sin(2J_{xt}t))^2 \log[(\cos(2J_{xt}t) + \sin(2J_{xt}t))^2]\right] \\
+ (\cos(2J_{xt}t) - \sin(2J_{xt}t))^2 \log[(\cos(2J_{xt}t) - \sin(2J_{xt}t))^2].
\]
The peaks of \(D(t)\) correspond to the two pure-state points of the 2 maximally entangled Bell states:\(N_1[J_{xt}t = (2n + 1)\frac{\pi}{4}] = N_{33}(t) = -N_{22}(t) = N_{03}(t) = N_{30}(t) = -1\) while the vanishing discord points are two of the 4 pure-state points of the concordant tetrahedron \([N_{03}, N_{03}, N_{33}]\) (see figure 6). Since the entanglement and the discord vanish at discrete points, this is \(BB\) joint evolution of entanglement and discord. In terms of the sets \(S\) and \(C\): the trajectory intersects \(S\) in a non-transversal fashion, while it pierces \(C\) repeatedly, these events happening simultaneously.

When the initial condition is \(\alpha = 1/2\), then the state starts as partially entangled. Evolution under the XY-Hamiltonian leads to entanglement death \(C = 0\) in the region \(\frac{\pi}{12} + n\frac{\pi}{4} \leq J_{xt}t \leq \frac{\pi}{6} + n\frac{\pi}{4}\) and rebirth as illustrated in figure 6. \(C\) can be computed explicitly as \(C(t) = \max \{0, \Delta_t\} \) with \(\Delta_t = \frac{1}{8}(9 + 4 \cos(8J_{xt}t) + 2\Delta - \sqrt{9 + 4 \cos(8J_{xt}t) - 2\Delta - 2})\) and \(\Delta = \sqrt{2} \cos(16J_{xt}t) + 18 \cos(8J_{xt}t) + 16\). This is \(O\)-type behavior. The quantum discord in this case is:
\[
D(t) = \frac{5}{8} \log 5 - \frac{3 - 2 \sin(4J_{xt}t)}{8} \log(3 - 2 \sin(4J_{xt}t)) \\
- \frac{3 + 2 \sin(4J_{xt}t)}{8} \log(3 + 2 \sin(4J_{xt}t)).
\]
\(D(t)\) vanishes at the discrete points \(J_{xt}t = \frac{\pi}{8} + n\frac{\pi}{4}\) as shown in figure 6(b). This is \(B\)-type behavior. These concordant points belong to the interior of the above tetrahedron. Thus the joint entanglement-discord evolution is of type \(OB\). The trajectory enters \(S\) transversally and pierces \(C\) repeatedly, but not simultaneously. Notice that the topological analysis does not
address the issue of the magnitude of the ratio of the entanglement measure and the discord measure but only whether the ratio is zero or infinite.

This Werner class of states belongs to a more general class of the Bell-diagonal type. In this larger class we find other types of joint evolution. For example, consider the initial state as the Bell-diagonal subclass with constraint $N_{11} = N_{22} = -\frac{N_{12}}{2} = \beta$ with $0 \leq \beta \leq 1/2$. The time dependent Bloch vector becomes:

\[
N_{03}(t) = -N_{00}(t) = \beta \sin(4J_{xy}t) \\
N_{11}(t) = N_{22}(t) = \beta \cos(4J_{xy}t) \\
N_{33}(t) = N_{33}.
\] (31)

The initial concurrence of this Bell-diagonal state is $C(0) = \max\{0, (4\beta - 1)/2\}$ which implies that it is partially entangled for $1/4 < \beta \leq 1/2$. The concurrence evolves as: $C(t) = \max\{0, \Delta_B\}$ where

\[
\Delta_B = \frac{1}{4} \left[ \sqrt{1 + 4\beta + 8\beta^2 \cos(4J_{xy}t)^2 + 4\beta \Gamma_B} \\
- \sqrt{1 + 4\beta + 8\beta^2 \cos(4J_{xy}t)^2 - 4\beta \Gamma_B} \\
- 2(1 - 2\beta) \right]
\]
with \( \Gamma_\beta = \sqrt{\cos(4 J_{\beta} t)^2 (1 + 4 \beta + 4 \beta^2 \cos(4 J_{\beta} t))^2} \); while the quantum discord in this case is given by:

\[
D(t) = \frac{5}{8} \log 5 - \frac{3 - 2 \sin(4 J_{\beta} t)}{8} \log(3 - 2 \sin(4 J_{\beta} t)) \\
- \frac{3 + 2 \sin(4 J_{\beta} t)}{8} \log(3 + 2 \sin(4 J_{\beta} t)).
\] (32)

\( D(t) \) vanishes at the discrete points \( J_{\beta} t = \frac{\pi}{8} + n \frac{\pi}{4} \) as shown in figure 6(b). For \( \beta = 1/2 \) the concurrence reduces to a simpler form \( C(t) = | \cos(4 J_{\beta} t) |^2 \) and the quantum discord and quantum entanglement evolve in a relatively similar manner (bouncing) so that the joint evolution is again \( BB \) as seen in figure 6(c). The Bloch vector has the time dependent form given in equation (31). The topological analysis is the same as in figure 6(a). When \( \beta = 1/4 \) we again have zero entanglement coexisting with oscillating discord as seen in figure 6(d). Here again the trajectory lies in \( S \setminus C \) except at discrete times.

4.2. Non-unitary evolution

4.2.1. Ising model with random telegraph noise. In this section we generalize the Ising Hamiltonian by adding external noise, producing non-unitary evolution and decay of correlations, as measured both by discord and by entanglement.

The system of two qubits decoheres because of random telegraph noise due to an unbiased single fluctuator. As we shall see, this can be a Markovian or non-Markovian process. The system is described by the following Hamiltonian:

\[ H = H_l + H_{RTN} + H_Z \] (33)

where \( H_l = J \sigma_x \otimes \sigma_x \), \( H_{RTN} = s(t) g \sigma_x \otimes \sigma_0 \), and the Zeeman energy \( H_Z = B \sigma_z \otimes \sigma_0 \), \( s(t) \) is the telegraph function that switches randomly between \(-1\) and \(+1\) at an average transition rate \( \gamma \). The quasi-Hamiltonian method \([40, 41]\) can be used to do the averaging over the realizations of \( s(t) \) exactly. Note that all the three terms in equation (33) are mutually commuting so that the solution for the entire system can be obtained by solving each single-term Hamiltonian separately (see appendix I).

Above we obtained full closed forms for all Bloch vector components of the Ising model as the system evolves unitarily where the \( X \)-type class has all 7 components as constants of motion. This means that all these components are affected only by the noise and applied field parts of \( H \). As a consequence, all states of \( X \)-type exhibit only categories \( A \) (for discord) and \( A \) and \( E \) (for entanglement) in Markovian regime. If the initial state is outside the \( X \)-type (union of the two subclasses (15) and (16)), its evolution type depends on \( J \). For example, equation (17) now becomes

\[
N_{20}(t) = e^{-\gamma t} \cos(2 J t) F(R_0) \\
N_{31}(t) = e^{-\gamma t} \sin(2 J t) F(R_0)
\] (34)

where \( R_0 = \sqrt{g^2 - \gamma^2/4} \) and

\[
F(R_0) = \frac{2 R_0 \cosh(2i R_0 t) - i \gamma \sinh(2i R_0 t)}{(2 R_0)}.
\] (35)

Note that these components are independent of the applied field \( B \). The dynamical process changes character sharply at \( \gamma/2 = g \) when \( R_0 \) changes from real to pure imaginary. When \( 2g, l, \gamma \ll 1 \) the characteristic times of the bath are small compared to those of the system,
and a Markov approximation is valid. When $2g/l_\gamma \gg 1$ the dynamics of the system are fully non-Markovian. The quantum discord (analytic form obtained in appendix F) of this system exhibits only category B and entanglement is 0 for all $t$.

The time evolution of the Bloch vector of the Bell-diagonal state in this model is:

\[
\begin{align*}
N_{11}(t) &= N_{11} e^{-\gamma t} F(R_0) \cos(2B_\gamma t) \\
N_{12}(t) &= -N_{11} e^{-\gamma t} F(R_0) \sin(2B_\gamma t) \\
N_{22}(t) &= N_{22} e^{-\gamma t} F(R_0) \sin(2B_\gamma t) \\
N_{23}(t) &= N_{23}.
\end{align*}
\]

This model yields a wide range of possible joint evolutions depending on initial conditions. For the Werner state with $\alpha = 1/4$, we get zero entanglement at all times, and the discord shows A behavior, as shown in figure 7(a). This is a trajectory that lies in $S \setminus C$ but approaches C only asymptotically. For $\alpha = 1/2$, we find E-type (sudden death) behavior for the entanglement, while the discord shows A behavior, an EA joint evolution. The trajectory enters S transversally, but approaches C asymptotically. For $\alpha = 1$, we find AA-type joint evolution, a joint asymptotic approach to a point that lies on the surface of $S$ which happens to belong $S \cap C$. These 3 evolutions are all in the Markovian regime. The discord at time $t$ of these states is obtained in appendix G. In case $\alpha = 1$:

\[
D(t) = 1 + \lambda_+ \log \lambda_+ + \lambda_- \log \lambda_- \\
\lambda_\pm = \frac{1 \pm \sqrt{\lambda(t)^2 + \lambda(t)^2}}{2}
\]

and the corresponding concurrence is:

\[
2C(t) = |N_{11}(t)| + \sqrt{1 - N_{12}(t)^2} - \left|N_{13}(t)\right| - \sqrt{1 - N_{12}(t)^2}.
\]

4.2.2. XY-model with random telegraph noise.

\[
H = H^{XY} + H^{RTN} + H^Z
\]

where $H^{XY} = J_{xy}(\sigma_x \otimes \sigma_y - \sigma_y \otimes \sigma_x)$. This is more complicated than the Ising case (see appendix H) as the XY-term does not commute with the noise and the B-field terms. Consider the initial state as Werner state (27). The Bloch vector of the system evolves as:

\[
\begin{align*}
N_{00}(t) &= -N_{00}(t) \\
N_{12}(t) &= -N_{12}(t) \\
N_{11}(t) &= N_{22}(t) \\
N_{33}(t) &= N_{33} (\neq -N_{21}(t)).
\end{align*}
\]

$N_{12}(t)$ and $N_{21}(t)$ are addition elements when noise is added as compared to the case without noise (see the corresponding unitary transformation).

In the evolutions generated by the XY-Hamiltonian, oscillations occur in the two correlation measures, and we find zero entanglement and B behavior for the discord for $\alpha = 1/4$, while $\alpha = 1/2$ leads to BB behavior for the joint evolution, and $\alpha = 1$ gives OB joint evolution, with the behavior of the entanglement given first, with the topological analyses as given above. The actual evolutions are shown in figures 7(d)–(f).

Note that the quantum discord never quite vanishes for this case of the applied field on the second qubit $B_z$ which guarantee two components $N_{12}(t) = -N_{21}(t) = 0$ (which are zero in the
case of the unitary transformation). Recall that for the unitary evolution, the quantum discord vanishes at \( \pi \) when \( t = \frac{\pi}{\gamma} \) is zero. At that point, the Bloch vector \( \vec{N}(t) = \{N_{03}(t), N_{02}(t), N_{33}(t)\} \) and this state lies in one of the concordant subsets.

In both Markovian and non-Markovian (see figure 8) regimes the interaction between the qubits is kept the leading contribution to the total energy of the entire system. As the noise strength is increased compared to the interaction term the discord decays much rapidly. We note some quantitative differences in the Markovian and non-Markovian evolutions, but the evolution categories do not change, since the origin of the categories is topological.

We note that the noise affects the quantum discord evolution in case of the Ising model \( H_I \) more strongly than in case of the anisotropic exchange Hamiltonian \( H_{XY} \). As \( \xi \) is increased the discord vanishes faster for \( H_I \) than for \( H_{XY} \).

4.2.3. Noise correlation effect. We obtain the closed forms of the time dependence of all Bloch vector components when the qubits interact with each other through the Ising spin exchange and with two separate uncorrelated RTN sources of different transition rates \( \gamma_1 = \gamma \), \( \gamma_2 = \xi \gamma \) (\( \xi > 0 \)). The full analytical solution for a general state is obtained in appendix I. As shown there, the RTN noise on qubit 1 does not affect the subclass of states specified in equation \( (15) \), the RTN noise on qubit 2 does not affect the subclass of equation \( (16) \), and the mutual qubit interaction does not affect the X-type class (see equation \( (I.3) \)). This is an example of decoherence-free subspaces [42]. The point is that the states specified in equations \( (16) \) and \( (I.3) \).

As a consequence, for \( J = 0 \), all components in each of the three separate subclasses have similar time-dependence and only differ by their initial conditions \( N_{ij}(t = 0) \). Enhancement of the noise effect is seen in the X-type of class: e.g. for \( \xi = 1 \) a Bell-diagonal state at time \( t \) has:

\[
N_{ii}(t) = N_{ii} e^{-2\gamma t} G(R_0); \quad i = 1, 2 \\
N_{33}(t) = N_{33}
\]

\[ (40) \]
\[\gamma = \pm \sqrt{1 - \alpha^2} = \pm J, \quad \alpha = 1/2, \quad \frac{\alpha}{2} = \frac{1}{4}\] where the Hamiltonian is given by (33) and \(\alpha = 0.8\). The bottom sketch describes another non-Markovian process for the correlated noise case where the trajectory never visits the origin and only approaches this point as \(t \to \infty\). Both quantum entanglement and discord never quite vanish but approximate a BB-type joint evolution.

Figure 8. Three categories (seen in both Ising and XY-models) for the joint evolution of quantum discord (solid lines) and the concurrence (dashed lines). The initial state is the Werner state defined in equation (27) with \(\alpha = 1/4\) for (a), \(\alpha = 1\) for (b), and \(\alpha = 1/2\) for (c). The evolution is non-Markovian. It is characterized by the parameters \(\gamma = 0.27, J = J_x = 3, B = 3B, t = 1\). (a) and (b) are obtained using the Ising model (equation (33)) and (c) the XY-model (equation (38)).

Figure 9. Joint evolution of discord (solid lines) and the concurrence (dashed lines) of a system subject to uncorrelated (a) and correlated (b) telegraph noise. The switching rate \(\gamma\) is the same for the (a) and (b). The dynamics are non-Markovian with an Ising interaction, with the Hamiltonian as given in equation (33) and the initial state is a Werner state with \(N_x = N_y = N_z = -\alpha = 0.8\). The sketches on the right-hand-side represent the oscillatory trajectory along a line in the direction \((-1, -1, -1)\) in the Bloch-vector space. In (a) the trajectory hits the origin in a finite time while in (b) it approaches the origin asymptotically.

where

\[
G(R_0) = \frac{1}{8R_0^2}\left[4g^2 + (4R_0^2 - \gamma^2) \cosh(4i\gamma) - 4i\gamma R_0 \sinh(4i)\right].
\] (41)

Figure 9(a) shows the decoherence for this model for a system initially prepared in a partially entangled Werner state (27) for \(\alpha = 0.8\). The joint evolution approximates the OB type.

The bottom sketch describes another non-Markovian process for the correlated noise case where the trajectory never visits the origin and only approaches this point as \(t \to \infty\). Both quantum entanglement and discord never quite vanish but approximate a BB-type joint evolution.
Conclusions

The time evolution of quantum entanglement and quantum discord in 2-qubit systems behave in fundamentally different ways. For the most part, this difference comes from the different topologies of the zero sets: the intersections of the system trajectory with $S$, the set of separable states, and with $C$, the set of concordant states. We do not include pathological trajectories with discontinuous derivatives or highly symmetrical models whose trajectories are confined to low-dimensional submanifolds.

The generic time evolutions for the disappearance of entanglement are of the $E$ (‘sudden death’) and $O$ (‘oscillating’) types, with $A$ (‘half-life’) and $B$ (‘bouncing’) possible for symmetric situations. The generic evolutions for discord disappearance are of $A$ and $B$ types, but discord disappearance depends on having the asymptotic limit point lie on a set of low dimension: it is more rare than entanglement disappearance, but it happens naturally in physical models, since, for example, the completely mixed state is a common limit point. $E$ and $O$ types of behavior are not allowed for discord. Furthermore, there are coexistence rules for joint evolution. All these facts are summarized in Table 2.

| Entanglement | A | E | B | O |
|--------------|---|---|---|---|
| Discord      | A |
| Joint evolutions | AA | EA, EB | BB | OB |

5. Conclusions

The time evolution of quantum entanglement and quantum discord in 2-qubit systems behave in fundamentally different ways. For the most part, this difference comes from the different topologies of the zero sets: the set of separable states and the set of concordant states, respectively. The set of separable states is a convex 15-manifold. The set of concordant states is a non-convex simply-connected when a certain set of zero measure has been subtracted out. The generic time evolutions for the disappearance of entanglement are of the $E$ (‘sudden death’) and $O$ (‘oscillating’) types, with $A$ (‘half-life’) and $B$ (‘bouncing’) possible for symmetric situations. The generic evolutions for discord disappearance are of $A$ and $B$ types, but discord disappearance depends on having the asymptotic limit point lie on a set of low dimension: it is more rare than entanglement disappearance, but it happens naturally in physical models, since, for example, the completely mixed state is a common limit point. $E$ and $O$ types of behavior are not allowed for discord. Furthermore, there are coexistence rules for joint evolution. All these facts are summarized in Table 2.

Roszak et al have computed the joint evolution of entanglement and geometric discord in a model of two excitonic quantum dot qubits dephased by noise from phonons [26]. They find the expected phenomenon of incomplete disappearance of discord at long times when the temperature is finite (and therefore the final state is not fully mixed). This case is not included in our analysis, though the generalization is straightforward. In cases where the disappearance is complete, they observe $EA$ and $EB$ behaviors for this model, as one would expect. Benedetti et al have done similar calculations for two qubits subjected to classical noise [27]. They observe $OB$ and $BB$ behavior, except when considering models that produce trajectories confined to a low-dimensional manifold—in their case a mixture of Bell states. Again this fits into our classification scheme.

Once the topology of the zero set is understood, the construction of explicit models that display the various behaviors is relatively straightforward. In particular one can show that qualitatively different behaviors of entanglement and discord can be observed in the same system. This is true even if the evolution is unitary: with an Ising interaction one can find oscillatory behavior of the discord even though the entanglement is strictly zero at all times, for a judicious choice of the initial state. With a slightly more complicated Hamiltonian still with unitary time development of the state, the coexistence of all reversible types of oscillatory evolution for entanglement and discord can be obtained. For example, $B$-type behavior of the discord is compatible with the $O$-type behavior of the concurrence in which the state is separable for an infinite number of finite time intervals. For non-unitary evolution, it is also found that all different kinds of evolutions of discord and entanglement can coexist. We are able to produce coexistence of $A$-type behavior of the discord with $E$-type as well as $A$ behavior for both; coexistence of the various kinds of decaying oscillatory behavior is also possible.
The main point of this paper is that these time evolutions correlate with various topologies of the intersections of the state trajectory and the zero sets.

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Appendix A. The Bloch vector of a concordant state

In this section, we obtain the 15 components of the Bloch vector for a concordant state. We define two projection operators: \( \Pi_k \equiv |\Psi_k\rangle\langle \Psi_k| = \frac{1}{2} (\sigma_0 \pm \vec{m} \cdot \vec{\sigma}) \) where \( \vec{m} \equiv (m_1, m_2, m_3) = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta) \) is a Bloch unit vector. Qubit B is \( \rho_k = \frac{1}{2}(\sigma_0 \pm \vec{n}_k \cdot \vec{\sigma}) \). We can rewrite equation (4) using these notations and the generalized Bloch vector becomes:

\[
N_{ij} = \frac{p}{2} \text{Tr}(\Pi_0 \otimes \sigma_0)(\sigma_i \otimes \sigma_j) + \frac{1-p}{2} \text{Tr}(\Pi_i \otimes \sigma_0)(\sigma_i \otimes \sigma_j) + \frac{p}{2} \sum_{k=1}^{3} n_{ik} \text{Tr}(\Pi_0 \otimes \sigma_k)(\sigma_i \otimes \sigma_j) + \frac{1-p}{2} \sum_{k=1}^{3} n_{ik} \text{Tr}(\Pi_i \otimes \sigma_k)(\sigma_i \otimes \sigma_j)
\]

\[\equiv N_{ij} = p \delta_{ij} 0 + (1-p) \delta_{ij} \text{Tr}(\Pi_0 \otimes \sigma_i) + p (1- \delta_{ij}) n_{ij} \text{Tr}(\Pi_0 \otimes \sigma_i) + (1-p) (1- \delta_{ij}) n_{ij} \text{Tr}(\Pi_i \otimes \sigma_i).
\]

These results can then be used to obtain the explicit forms for \( N_{ij} \) given in (5)–(7) of the main text.

Appendix B. Concordant states on the coordinate axes

We wish to show that \( C \) contains the coordinate axes.

\[
\rho = \frac{1}{4} (\sigma_0 \otimes \sigma_0 + i \sum_i N_{i0} \sigma_0 \otimes \sigma_i)
\]

\[= \frac{1}{2} \sigma_0 \otimes \sigma_0 + \frac{1}{2} \sum_i N_{i0} \sigma_i
\]

\[= \frac{1}{2} \Pi_0 \otimes \rho_0 + \frac{1}{2} \Pi_i \otimes \rho_0
\]

where \( \Pi_i ; i = 0, 1 \) form an orthonormal basis in qubit A while \( \rho_0 = \frac{1}{2}(\sigma_0 + \sum_i N_{i0} \sigma_i) \) is a general state of qubit B given \( N_0 \in [-1, 1] \). This is always satisfied using the condition of positivity for system qubit \( \rho \). The last line in equation (B.1) is the necessary and sufficient condition for state \( \rho \) to be concordant.
Appendix C. Symmetries in the concordant subset

Generally, concordant states are asymmetric under exchange of the two qubits. In this appendix, we show that a concordant state is symmetric under some certain conditions.

The left 0-discord state has the Bloch vector described as in equations (5)–(7) and ‘left’ geometric discord as in (3) where

\[
k_{\text{max}} = \sum_{i=1}^{3} N_{0i}^2 + \sum_{i,j=1}^{3} N_{ij}^2
\]

\[
= |N|^{2} - (N_{01}^{2} + N_{02}^{2} + N_{03}^{2}).
\]

A ‘right’ 0-discord state, instead, has:

\[
N'_{0i} = (2p' - 1)m'_i
\]

\[
n'_{0i} = p'n'_{0i} + (1 - p')n'_{i1}
\]

\[
n'_{ij} = m'[p'n'_{ij} - (1 - p')n'_{ij}]
\]

and its ‘right’ geometric discord is

\[
D_{G}^R = \frac{1}{4} [\text{Tr}(\hat{y}\hat{y}^{T}) + \text{Tr}(T^{*}T^{T}) - q_{\text{max}}]
\]

where \(\hat{y} = (N'_{0i}, N'_{20}, N'_{30})\) and \(q_{\text{max}}\) is the largest eigenvalues of \(\hat{y}\hat{y}^{T} + T^{*}T^{T}\). Generally, a left discordant state has \(D_{G}^L = 0 = D_{G}^R\) i.e. symmetric discordant states. If a (left) discordant state has \(N_{0i} = \pm N_{0i}\), which is equivalent to the condition \(p'n_{0i} + (1 - p')n_{i1} = \pm(2p - 1)\hat{m}\), then it is symmetric. Typical symmetric discordant point is the origin which has \(p_{1/2}\) and \(n_{0} = \hat{0}\). All pure discordant states satisfying two conditions \(p(1-p) = 0\) & \(|\hat{m}| = 1\) are also symmetric.

Appendix D. Linear independence of the tangent vectors

We obtain the explicit form for the 9 tangent vectors to the manifold \(C\) as follows:

\[
\tilde{\hat{n}} = \frac{\partial \hat{N}}{\partial \theta} = \hat{0} + (2p - 1) \sum_{i=1}^{3} \frac{\partial m_{i}}{\partial \theta} \hat{e}_{0i} + \sum_{i,j=1}^{3} [pm_{0j} - (1 - p)m_{ij}] \frac{\partial m_{ij}}{\partial \theta} \hat{e}_{ij} \quad \text{(D.1a)}
\]

\[
\tilde{\hat{e}}_{1} = \frac{\partial \hat{N}}{\partial \varphi} = \hat{0} + (2p - 1) \sum_{i=1}^{3} \frac{\partial m_{i}}{\partial \varphi} \hat{e}_{0i} + \sum_{i,j=1}^{3} [pm_{0j} - (1 - p)m_{ij}] \frac{\partial m_{ij}}{\partial \varphi} \hat{e}_{ij} \quad \text{(D.1b)}
\]

\[
\tilde{\hat{n}} = \frac{\partial \hat{N}}{\partial p} = \sum_{i=1}^{3} (n_{0i} - m_{i1}) \hat{e}_{0i} + 2 \sum_{i=1}^{3} m_{i} \hat{e}_{0i} + \sum_{i,j=1}^{3} m_{i}(n_{0j} - m_{ij}) \hat{e}_{ij} \quad \text{(D.1c)}
\]

\[
\tilde{\hat{u}} = \frac{\partial \hat{N}}{\partial m_{01}} = p \sum_{i=1}^{3} \delta_{i1} \hat{e}_{0i} + \hat{0} + p \sum_{i=1}^{3} m_{i} \hat{e}_{ij} \quad \text{(D.1d)}
\]

\[
\tilde{\hat{u}} = \frac{\partial \hat{N}}{\partial m_{02}} = p \sum_{i=1}^{3} \delta_{i2} \hat{e}_{0i} + \hat{0} + p \sum_{i=1}^{3} m_{i} \hat{e}_{ij} \quad \text{(D.1e)}
\]
\[
\tilde{t}_5 = \frac{\partial \tilde{N}}{\partial n_{03}} = p \sum_{i=1}^{3} \delta_{i,3} \tilde{e}_{0i} + \tilde{N} \sum_{i=1}^{3} m_i \tilde{e}_{ij} 
\]  
(D.1f)

\[
\tilde{t}_6 = \frac{\partial \tilde{N}}{\partial n_{11}} = (1 - p) \sum_{i=1}^{3} \delta_{i,1} \tilde{e}_{0i} + \tilde{N} (1 - p) \sum_{i=1}^{3} m_i \tilde{e}_{ij} 
\]  
(D.1g)

\[
\tilde{t}_7 = \frac{\partial \tilde{N}}{\partial n_{12}} = (1 - p) \sum_{i=1}^{3} \delta_{i,2} \tilde{e}_{0i} + \tilde{N} (1 - p) \sum_{i=1}^{3} m_i \tilde{e}_{ij} 
\]  
(D.1h)

\[
\tilde{t}_8 = \frac{\partial \tilde{N}}{\partial n_{13}} = (1 - p) \sum_{i=1}^{3} \delta_{i,3} \tilde{e}_{0i} + \tilde{N} (1 - p) \sum_{i=1}^{3} m_i \tilde{e}_{ij}. 
\]  
(D.1i)

The tensor matrix has the explicit form:

\[
g = \begin{pmatrix}
g_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & g_{22} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & g_{33} & g_{34} & g_{35} & g_{36} & 0 & 0 \\
0 & 0 & g_{43} & g_{44} & 0 & 0 & 0 & 0 \\
0 & 0 & g_{53} & g_{54} & 0 & 0 & 0 & 0 \\
0 & 0 & g_{63} & g_{65} & g_{66} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & g_{77} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & g_{88} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & g_{99} 
\end{pmatrix}
\]  
(D.2)

with \(g_{11} = (1 - 2p)^2 + \sum_{i=1}^{3} (pn_0 - (1-p)n_i)^2; \ g_{22} = ((1-2p)^2 + \sum_{i=1}^{3} (pn_0 - (1-p)n_i)^2) \sin^2 \theta_i; \ g_{33} = 2(1 + \sum_{i=1}^{3} (n_0 - n_i)^2); \ g_{44} = g_{55} = g_{66} = 2p^2; \ g_{77} = g_{88} = g_{99} = 2(1-p)^2; \ g_{34} = g_{43} = 2p(n_0 - n_1); g_{35} = g_{53} = 2p(n_0 - n_2); g_{36} = g_{63} = 2pn_3 - n_1).

### Appendix E. Quantum discord of the state in equation (17)

The corresponding density matrix \(\rho(t) = \frac{1}{2}(c_0 \otimes c_0 + S_3 c_1 \otimes c_3 + c_2 \otimes c_0)\) has 4 eigenvalues of \(\{\frac{1}{2}, \frac{1}{2}, 0, 0\}\) and \(S[\rho(t)] = 1\). The two subsystems are: \(\rho_1(t) = \frac{1}{2} \left( \begin{array}{cc} 1 & -iC_3 \\ iC_3 & 1 \end{array} \right) \) with two eigenvalues \(\lambda_1 = \frac{1}{2}(1 \pm C_3)\); and \(\rho_2(t) = c_0/2\).

The quantum mutual information is:

\[
\mathcal{I} = S[\rho_1(t)] + S[\rho_2(t)] - S[\rho(t)] = S[\rho_1(t)] \\
= 1 - \frac{1}{2}[(1 - C_3) \log(1 - C_3) + (1 + C_3) \log(1 + C_3)]. 
\]  
(E.1)

The classical mutual information is
\[ J^{\text{class}} = S[\rho_g(t)] - \min_{\{A_k\}} S[\rho(t)|A_k] \]
\[ = 1 - \min_{\{A_k\}} S[\rho(t)|A_k] \]  
(\text{E.2})

where \( \{A_k = V \Pi_k V^\dagger; \ k = 1, 2 \} \) defines a set of measurement on subsystem \( A; V = v_0 + \vec{v} \cdot \vec{\sigma} \) where \( t^2 + v_1^2 + v_2^2 + v_3^2 = 1 \) and \( \{ \Pi_k \} \) is some local orthogonal basis. Without loss of generality, we choose \( \Pi_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \) and \( \Pi_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \). Some useful expressions are:

\[ \sigma_1 \sigma_2 \sigma_3 = - \Pi \sigma_1 \Pi \sigma_2 \sigma_3 = - \Pi \]  
(\text{E.3})

\[ \sigma_1 \sigma_2 \sigma_3 = - \Pi \sigma_1 \sigma_2 \]  
(\text{E.4})

\[ \sigma_1 \sigma_2 \sigma_3 = - \Pi \sigma_1 \sigma_2 \]  
(\text{E.5})

After measurement \( \{A_k \} \) the system is sent to \( \rho(t) = \rho_A(t) \) set.

\[ \rho_A(t) = \frac{1}{4} (V \otimes \sigma_0) (\Pi_1 \otimes \sigma_0) (V^\dagger \otimes \sigma_0) (\Pi_1 \otimes \sigma_0) - \frac{1}{4} (V \otimes \sigma_0) (\Pi_2 \otimes \sigma_0) (V^\dagger \otimes \sigma_0) (\Pi_2 \otimes \sigma_0) \]  
(\text{E.6})

with \( X \) defined above and \( Y = z_3 S_3 \).

Similarly, \( \rho_2(t) \) is obtained as:

\[ \rho_2(t) = \frac{1}{4} (V \Pi_2 V^\dagger) \otimes [(1 + X) \sigma_0 + Y \sigma_3]. \]  
(\text{E.7})

Now, eigenvalues of \( \rho_1(t) \) are \( \frac{1}{4} (1 + X + Y); \frac{1}{4} (1 - X + Y) \) and of \( \rho_2(t) \) are \( \frac{1}{4} (1 + X + Y); \frac{1}{4} (1 - X - Y) \). One can obtain the conditional entropy as:

\[ S[\rho(t)|A_k] = p_1 S[\rho_1(t)] + p_2 S[\rho_2(t)] \]
\[ = 1 - \frac{1}{4} [(1 + X + Y) \log(1 + X + Y) + (1 + X - Y) \log(1 + X - Y) + (1 - X + Y) \log(1 - X + Y) + (1 - X - Y) \log(1 - X - Y)] \]
\[ - 2(1 + X) \log(1 + X) - 2(1 - X) \log(1 - X)]. \]  
(\text{E.8})

Note that \( X = 2 C_3 (v_1 + v_2 v_3) \leq 2 C_3; Y = 2 S_3 (-v_2 + v_1 v_3) \leq 2 S_3 \). (\text{E.8}) has identical minima at \( X = 0 \) & \( Y = \pm S_3 \). As a result,

\[ \min_{\{A_k\}} S[\rho(t)|A_k] = 1 - \frac{1}{2} [(1 - |S_3|) \log(1 + |S_3|) + (1 + |S_3|) \log(1 + |S_3|)] \]  
(\text{E.9})

Now substitute (\text{E.9}) into (\text{E.2}) and subtract (\text{E.2}) from (\text{E.1}) the quantum discord is obtained as in the main text.
Appendix F. Quantum discord of the state in equation (34)

Note that
\[ |\tilde{N}(t)| = \sqrt{N_{20}^2(t) + N_{13}^2(t)} \]
\[ = e^{-\gamma t}\cosh(2iR_0^t) - i\frac{\gamma}{2R_0} \sinh(2iR_0^t)). \]  

(F.1)

Entropy of the system (34) is
\begin{align*}
S_p &= -\left(\lambda_1 \log \lambda_1 + \lambda_2 \log \lambda_2 + \lambda_3 \log \lambda_3 \\
&\quad + \lambda_4 \log \lambda_4\right)
\end{align*}

(F.2)

where
\[ \lambda_{1,2} = \frac{1}{4}[1 + \sqrt{N_{20}^2(t) + N_{13}^2(t)}] \]
\[ \lambda_{3,4} = \frac{1}{4}[1 - \sqrt{N_{20}^2(t) + N_{13}^2(t)}]. \]  

(F.3)

Entropy of system A is
\begin{align*}
S_{\rho_A} &= 1 - \frac{1}{2}\left[\{1 + N_{20}(t)\log[1 + N_{20}(t)]
\right. \\
&\quad + \{1 - N_{20}(t)\log[1 - N_{20}(t)]\}
\end{align*}

(F.4)

and the classical mutual information after optimization is:
\begin{align*}
S_{\text{class.}} &= 1 - \frac{1}{2}\left[\{1 + N_{13}(t)\log[1 + N_{13}(t)]
\right. \\
&\quad + \{1 - N_{13}(t)\log[1 - N_{13}(t)]\}. 
\end{align*}

(F.5)

The time evolution of the quantum discord of the above system: \( D(t) = S_{\rho_A} - S_p + S_{\text{class.}} \).

Appendix G. Quantum discord of the state in equation (36)

\begin{align*}
D(t) &= 2 - \frac{1 + \alpha}{2} \log(1 + \alpha) - \frac{1 - \alpha}{2} \log(1 - \alpha) \\
&\quad - S_p 
\end{align*}

(G.1)

where \( S_p = -\lambda_0 \log \lambda_0 + \lambda_1 \log \lambda_1 - \lambda_2 \log \lambda_2 - \lambda_3 \log \lambda_3 \) with \( \lambda_0 = \lambda_1 = \frac{1 - \alpha}{4}, \lambda_2, 3 = \frac{1 + \alpha + 2\sqrt{N_{20}^2(t) + N_{13}^2(t)}}{4} \).

Appendix H. Quasi-Hamiltonian \( H_q \) of the XY-model

This section is for further discussions of solving the XY-model in a non-unitary evolution.

In a general XY-model in interaction with a single RTN fluctuator \( (B_z = 0) \), \( H_q \) has 15 different eigenvalues: \(-2i\gamma, -i\gamma, -2R_0, -i\gamma + 2R_0, -i\gamma \pm W_1, -i\gamma \pm W_2, \omega_1, \omega_2, \omega_3, \Omega_1, \Omega_2, \Omega_3\) where \( W_{1,2} = \sqrt{4J_x^2 + 2g_z^2 - \gamma^2 + \sqrt{g_z^2 + 4J_x^2g_z^2 - 4J_y^2\omega^2}}, \omega \) and \( \Omega \), respectively, are roots of polynomial \(-32J_y^2 - 4(4J_y^2 + g_z^2)\omega + 2i\gamma\omega^2 + \omega^3\) and \(-8ig_z^2\gamma - 4(4J_y^2 + g_z^2 + \gamma^2)\omega + 4i\gamma\Omega^2 + \Omega^3\).
Appendix I. Analytical solution of two uncorrelated RTN fluctuators with different transition rates $\gamma, \xi\gamma$

We obtain exact solutions for the Ising models in interaction with two different uncorrelated noise sources.

For the components constructing a general off-X-type of class (see the left-hand-side (LHS) of equations (15) and (16)), the solution is obtained by replacing $\gamma$ as in the single fluctuator case (see equation (34)) by $\gamma, \xi\gamma$ (respectively, equals $\gamma, \xi\gamma$) for the corresponding subclass. It is due to the fact that the LHS components in equation (15) commute with the term $\sigma_0 \otimes \sigma_3$ and the LHS components in equation (16) commute with the term $\sigma_3 \otimes \sigma_0$. As a result, each subclass is not affected by the noise from the other qubit. More generally, if a Hamiltonian consists of $k$ commuting terms then the combined solution is $N(t) = \sum_{1 \leq i < j \leq k} N_{i,j} \Pi_{m=0}^{3} F_{ij}^m(t)$ where $\sum_{m=1}^{k} = \sum_{N(t)}$ is the solution for the $m$-th term. Back to the above case, one has:

\[
N_{00}(t) = (N_{00}C_3 + N_{02}S_3)e^{-\xi\gamma F(R_2)}
\]

\[
N_{02}(t) = (N_{02}C_3 - N_{00}S_3)e^{-\xi\gamma F(R_2)}
\]

\[
N_{03}(t) = (N_{03}C_3 + N_{02}S_3)e^{-\xi\gamma F(R_2)};
\]

\[
N_{0i}(t) = (N_{0i}C_3 + N_{12}S_3)e^{-\gamma F(R_i)}
\]

\[
N_{02}(t) = (N_{32}C_3 - N_{12}S_3)e^{-\gamma F(R_i)}
\]

\[
N_{03}(t) = (N_{33}C_3 + N_{12}S_3)e^{-\gamma F(R_i)}
\]

where $R_{i,j} = \sqrt{4R_{i,j}^2 - \gamma_{i,j}^2}/2$.

The enhanced noise effect will be seen in the other class—the X-type:

\[
N_{i,j}(t) = N_{i,j}e^{-\gamma(1+\xi)\gamma H(R_0, \xi)}, \quad i,j = 1 \div 3
\]

\[
N_{33}(t) = N_{33}
\]

(1.3)

where

\[
H(R_0, \xi) = \frac{1}{4R_0 X_0} [2R_0 \cosh(2iR_0 t) - i\gamma \sinh(2iR_0 t)]
\]

\[
X_0 = \sqrt{4R_0^2 - \gamma^2(\gamma^2 - 1)/2}. \quad \text{If } \xi = 1 \text{ (i.e. } \gamma_1 = \gamma_2) \text{ then } X_0 = R_0 \text{ and } H(R_0, \xi) \equiv G(R_0) \text{ as defined in (41).}
\]

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