Semiclassical Transition in $\phi^4$ Theory

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Abstract

We have shown an example of semiclassical transition in $\phi^4$ theory with positive coupling constant. This process can be described by the classical $O(4)$-invariant solution, considered on a contour in the complex time plane. The transition is technically analogous to the one-instanton transition in the electroweak model. It is suppressed by the factor $\exp(-2S_0)$, where $S_0$ is Lipatov instanton action. This process describes a semiclassical transition between two coherent states with much smaller number of particles in the initial state than in the final state. Therefore, it could be relevant to the problem of calculation of amplitudes for multiparticle production in $\phi^4$-type models.

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1 Introduction

Recently, considerable efforts have been made to calculate amplitudes for multiparticle production in weakly coupled field theories. The study of this problem was initiated by the observation of the fact [1] that baryon-number violating processes in the electroweak theory, associated with multiparticle production, could become relevant at energy scale $E \sim 10 \text{ TeV}$. This problem gave impulse to study multiparticle amplitudes in the simpler case of $\phi^4$ model [2, 3], considered before in the context of large orders of perturbation theory [4].

The semiclassical methods for computing such amplitudes in the electroweak theory use Euclidean classical solutions of the equations of motion — instantons [5]. The similar calculations in $\phi^4$ theory [3] are based on the existence of the instanton-like solutions in $\phi^4$ model with negative coupling constant [4]. These instanton-like calculations show an exponential growth of the total cross sections with energy in the leading order of perturbation theory around the instanton [6, 7]. A naive extrapolation of these results to the high energy scale violates the unitarity bound for the cross section.

The problem is that instanton solutions describe transitions between vacuum states. The real process, however, describes the production of many final-state particles in high energy, two-particle collisions. Obviously, we cannot ignore the effects of the external particles. Indeed, at high energies (for example energies in the order of the sphaleron energy in the electroweak model [8]) instanton calculations become inappropriate, which can be seen in the fact that corrections to the leading-order transition probability become large [6, 7]. Therefore, in order to estimate accurately the external particle effects we have to modify the instanton-based approach.

Formally, we cannot calculate the transition probability for the process $\text{two} \rightarrow \text{many}$ particles in the semiclassical manner at all, because of the non-semiclassical nature of the initial two-particle state. Instead, as proposed in Ref. [10], we can calculate the probability of transition between a semiclassical initial state with a “small” number of particles and a final semiclassical state with a “large” number of particles. The probability of such a transition can be considered as some approximation to the two particle cross section in one-instanton sector and gives us an upper bound for this cross section.
In this approach the problem can be reduced to the solution of the classical field equations with some specific boundary conditions, determined by the initial and final states. Such a formalism, based on the coherent state representation of the $S$-matrix elements [12], has been used to find the transition probability for processes mediated by so-called “periodic instantons” [11].

Finding the exact form of the relevant high-energy instanton-like configuration is, however, a very difficult problem, even in the massless limit of the theory. To avoid this problem, the formalism of Refs. [10, 11] has been recently modified [13] to use exact Minkowskian classical solutions, which can be easily found for a number of models. This modification allows one to calculate, in principle, the semiclassical scattering above the sphaleron energy in the electroweak model. It has been shown in [13], that a Minkowskian solution of the $O(3)$-invariant two-dimensional $\sigma$-model, analytically continued to the complex time plane, can be used to describe instanton-like processes in this model with a “strong” violation of the number of particles ($n_{\text{initial}} \ll n_{\text{final}}$).

In this paper we consider four-dimensional massless $\phi^4$ theory. While $\phi^4$ theory with positive coupling constant does not allow direct instanton-like calculations (we have to use Lipatov’s trick [4] and consider first the theory with negative coupling constant), the formalism of Ref. [13] can be used for direct semiclassical calculations in this theory.

We show that $\phi^4$ theory allows a semiclassical transition even for the case of positive coupling constant. This transition is described by a classical $O(4)$-invariant solution, considered on a contour in the complex time plane. The “type” of the transition is determined by the position of this contour with respect to the positions of the singularities of the classical solution.

We consider the transition technically analogous to the one-instanton transitions in the electroweak model. It is suppressed by the factor $\exp (-2S_0)$, where $S_0$ is equal to Lipatov instanton action – the action of the classical solution in the Euclidean theory with negative coupling constant [4]. To interpret the process, we analyze a similar “transition” in one-dimensional quantum mechanics.

This process describes a classically-forbidden transition between two coherent states with a much smaller number of particles in the initial state than in the final state – $n_{\text{final}} \sim n_{\text{initial}}^{5/7}/\lambda^{2/7}$ (where $\lambda$ is a small coupling constant). Therefore, it could be relevant to the calculation of amplitudes for multiparticle production in $\phi^4$-type models. We suppose that the contribu-
tion of such a process must be included into the corresponding multiparticle amplitude and, probably, can slow down the factorial growth of the perturbative amplitude [2].

The paper is organized as follows. In the second section we describe the basic formalism [13]. The third section is devoted to the description of the classical $O(4)$-invariant solutions of the $\phi^4$ theory and calculation of the transition probability. As an example, we consider also the theory with negative coupling constant, where the results can be compared with previous calculations [14]. In section (4) we find the initial and final coherent states as asymptotics of the classical solution and the corresponding numbers of the initial and final particles. The last section contains concluding remarks.

2 Formalism

In this section we describe the slightly modified formalism of the Refs. [11, 13], based on the coherent state representation of the $S$-matrix elements [12]. The $S$-matrix element in this representation is a generating functional for transition amplitudes between states with definite numbers of particles. Moreover, this formalism allows us to estimate easily the influence of the external particles on the semiclassical transition.

We calculate here the probability of transition between two coherent states. For the purpose of calculating multiparticle amplitudes, the coherent state is a good approximation to the final multiparticle state. Unfortunately, it cannot describe an initial two-particle state. The hope is, however, that the two-particle cross section can be approximated by the probability of transition between coherent states with a “small” number of particles in the initial state [14]. Then, the problem of calculating the transition probability can be converted to the problem solving the field equations with some specific boundary conditions [11, 13], determined by the initial and final states. However, we cannot solve these equations for arbitrary states. So we find first any real solution of the equations of motion and then determine which boundary conditions (initial and final states) correspond to the solution. We will see below that these states and “types” of transition are closely related.
to the structure of the singularities of the classical solution in the complex
time plane.

First, let us consider a matrix element

\[ A_E(b^*, a) = \langle \{b_k\} \mid SP_E \mid \{a_k\}\rangle \]

which describes the amplitude for a transition at fixed energy \( E \) from the
initial coherent state \( | \{a_k\} \rangle \) (projected onto this energy ) to the final coherent
state \( | \{b_k\} \rangle \). The operator \( P_E \) is a projector onto subspace of definite energy
\( E \); \( S \) is the \( S \)-matrix. The system is considered in the center of mass frame
so we do not need to project onto the space of definite spatial momentum.

Using the completeness condition we write this element as

\[ \langle \{b_k\} \mid SP_E \mid \{a_k\}\rangle = \int d\phi_i d\phi_f \langle \{b_k e^{i\omega_k T_f}\} \mid \phi_f \rangle \langle \phi_f \mid U(T_f, T_i) \mid \phi_i\rangle \langle \phi_i \mid P_E \mid \{a_k e^{-i\omega_k T_i}\}\rangle \]

In this expression \( \langle \{b_k e^{i\omega_k T_f}\} \mid \phi_f \rangle \) and \( \langle \phi_i \mid P_E \mid \{a_k e^{-i\omega_k T_i}\}\rangle \) represent the
wave functions of the coherent states in \( \phi \) representation (the initial state is
projected onto a state of definite energy), \( \phi_{i,f}(x) = \phi(x, T_{i,f}) \) and \( T_i \to -\infty, \)
\( T_f \to +\infty \) are initial and final moments of time. The matrix element of the
evolution operator \( U \) can be expressed by the functional integral

\[ \langle \phi_f \mid U(T_f, T_i) \mid \phi_i\rangle = \int_{\phi(T_i) = \phi_i}^{\phi(T_f) = \phi_f} D\phi e^{iS(\phi)}. \]

The projection operator \( P_E \) can be written in the form

\[ \langle \{b_k\} \mid P_E \mid \{a_k\}\rangle = \int d\xi e^{-iE\xi} \langle \{b_k\} \mid e^{iH_0\xi} \mid \{a_k\}\rangle = \int d\xi e^{-iE\xi} \langle \{b_k\} \mid \{a_k e^{i\omega_k \xi}\}\rangle = \int d\xi \exp\{-iE\xi + \int d\kappa b^*_k a_k e^{i\omega_k \xi}\}, \]

where \( H_0 \) is the free Hamiltonian.

Therefore, the amplitude \( A_E(b^*, a) \), divided by the norm of the initial
state \( N_a \)

\[ N_a = \exp\left\{ \frac{1}{2} \int d\kappa a^*_k a_k \right\} \]

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and the norm of the final state $N_b$

$$N_b = \exp\left\{ \frac{1}{2} \int d k b_k ^* b_k \right\},$$

has the following integral representation \[^1\]

$$A_E (b ^*, a) = \int d \xi \ d \phi _i \ d \phi _f \ D \phi \ \exp \{ -iE \xi + B_i (a_k e ^{i \omega_k \xi}, \phi _i) +$$

$$B_f (b_k ^*, \phi _f) + i \int _{T_i} ^{T_f} d t L (\phi) - \frac{1}{2} \int d k a_k ^* a_k - \frac{1}{2} \int d k b_k ^* b_k \right\}. \quad (1)$$

Here $\phi$ stands for all bosonic fields of the theory and $B_i (a_k, \phi _f)$ and $B_f (b_k ^*, \phi _f)$ are the boundary terms ($\exp B_i$ and $\exp B_f$ are the wave functions of the coherent states in the $\phi$ representation)

$$B_i (a_k, \phi _f) = - \frac{1}{2} \int d k a_k a_{-k} e ^{-2i \omega_k T_i} - \frac{1}{2} \int d k \omega_k \phi _i (k) \phi _i (-k) +$$

$$+ \int d k \sqrt{2 \omega_k} e ^{-i \omega_k T_i} a_k \phi _i (k),$$

$$B_f (b_k ^*, \phi _f) = - \frac{1}{2} \int d k b_k ^* b_{-k}^* e ^{2i \omega_k T_f} - \frac{1}{2} \int d k \omega_k \phi _f (k) \phi _f (-k) +$$

$$+ \int d k \sqrt{2 \omega_k} e ^{i \omega_k T_f} b_k ^* \phi _f (k), \quad (2)$$

where $\phi _{i,f} (k)$ are the spatial Fourier components of the field $\phi$ at the initial and final moments of time.

When $E \sim 1/\lambda$ and $a_k, b_k \sim 1/\sqrt{\lambda}$ for small $\lambda$, we can evaluate the transition amplitude (1) in the saddle-point approximation.

The saddle-point field configuration is determined by the field equation $\delta S/\delta \phi = 0$. The variation with respect to $\phi _i$ and $\phi _f$ gives

$$- i \dot{\phi _i} (k) - \omega_k \phi _i (k) + \sqrt{2 \omega_k} a_k e ^{-i \omega_k T_i + i \omega_k \xi} = 0 \quad (3)$$

$$i \dot{\phi _f} (k) - \omega_k \phi _f (k) + \sqrt{2 \omega_k} b_{-k}^* e ^{i \omega_k T_f} = 0 \quad (4)$$

We assume below that the field $\phi$ becomes free at large initial and final time, which means that its spatial Fourier transform can be written as a
superposition of plane waves. Therefore, we have for a large positive time 
\( t \to +\infty \)
\[
\phi(k, t) = \frac{1}{\sqrt{2\omega_k}} (g_k e^{-i\omega_k t} + f_{-k} e^{i\omega_k t})
\]
Using the boundary conditions (4) we obtain immediately
\[
g_k^* = b_k^*,
\]
i.e. the positive frequency part of the field is determined by the final state.

Consider now the integration with respect to \( \xi \). The real part of \( \xi \) corre-
sponds to time translation so we choose \( \xi \) to be imaginary
\[
\xi \to i \xi
\]
and also allow time to be complex.

The complex time formalism has been used in quantum mechanical calcula-
tions of tunneling events for a long time [15], and has been introduced re-
cently into instanton calculations in Ref. [11]. The “naive” argument in favour
of this step is that we cannot describe simultaneously the classically-allowed
events (such as free evolution of the initial and final states) and classically-
forbidden events (for example tunneling) in framework of the pure Minkowski
or Euclidean time semiclassical calculations. For the classically allowed prop-
agation, there exists a “classical trajectory”, whereas a classically-forbidden
event does not have a “trajectory” in the real time).

Consider now the contour in the complex time plane shown on Fig.(1)
[11, 13]. The part A of this contour is shifted upward and runs parallel to the
real axis \( t = t' + iT \). Evolution of the system with respect to \( t' \) corre-
sponds to initial state propagation, while the real part of the contour describes final
state propagation. We can interpret evolution along the imaginary part of the
contour as some classically-forbidden event (for example, it can correspond
under some conditions to a tunneling event [11]).

On part A of the contour, the field at early time \( t' \to -\infty \) has the form
\[
\phi(k, t') = \frac{1}{\sqrt{2\omega_k}} (f_k e^{-i\omega_k t'} + \overline{f_{-k}} e^{i\omega_k t'})
\] (5)
(\( \overline{f_k} \) is not complex conjugate to \( f_k \)). Substitution of this field into the first
boundary condition (3) gives
\[
f_k = a_k e^{\omega_k T - \omega_k \xi},
\]
which determines the negative frequency part of the field.

![Diagram](image.png)

Fig.1

Thus, to find the transition probability, we have to solve the field equations with fixed negative frequency part of the field at early time and positive frequency part of the field at late time. This is an extremely difficult problem for arbitrary initial and final states, even in the case of the $\phi^4$ theory. So we are forced to restrict ourselves to a less general problem [13]: we find first some real Minkowski-time solution and then find the corresponding initial and final states as asymptotics of this solution. The “inverted” equations

\[ b_k^* = g_k^* \quad (6) \]
\[ a_k = f_k e^{\omega_k \xi - \omega_k T} \quad (7) \]

determine the initial and final states.

We have to make some remarks about the choice of the “appropriate” solution.

First, we consider only real solutions because, as it has been shown in Ref.[13]), the probability of the transition from the given initial state to all possible final states is saturated by a single final state which is real at real time. Therefore, the real saddle-point configuration corresponds to the transition from the given initial state to the most probable final state. Then, we immediately obtain for the final state $\overline{f}_k = g_k^*$.

The second condition is that this solution should have an appropriate singularity structure in the complex time plane - we have to be able to choose the contour of Fig.(1) and avoid any singularities of the solution.
We will show in the next section that $\phi^4$ theory possesses such solutions. To find the amplitude of the transition we have to substitute the saddle-point field configuration and boundary values (6) and (7) into the integral (1). Then, the amplitude $A$ (where $A = A_E(g^*, f)$) is expressed by

$$A = i \int d\xi \exp\{ E\xi + iS + \frac{1}{2} \int d\mathbf{k} \mathcal{T}_k f_k - \frac{1}{2} \int d\mathbf{k} f_k^* f_k e^{-2\omega_k T + 2\omega_k \xi} \} \quad (8)$$

where we have neglected the contributions of rapidly oscillating terms. In this expression $S$ represents the classical action of the saddle-point configuration (the time integration is done along the contour of Fig.(1)).

The integral with respect to $\xi$ can be done in the saddle-point approximation. The saddle-point value $\xi_0$ determines the energy as a function of the other variables

$$E = \int d\mathbf{k} \omega_k f_k^* f_k e^{-\omega_k T_0} \quad (9)$$

Here

$$T_0 = 2T - 2\xi_0$$

is determined by the deviation of the shift of the contour $T$ from the saddle-point value of $\xi$. Equation (9) determines the parameter $T_0$ in terms of energy.

After substitution of the saddle point value of $\xi$ into (8), the probability of the transition in the saddle-point approximation is determined by

$$\sigma = |A|^2 = \exp\{ 2E\xi_0 - 2\text{Im } S + \int d\mathbf{k} \text{Re}(\mathcal{T}_k f_k) - \int d\mathbf{k} f_k^* f_k e^{-\omega_k T_0} \} =$$

$$= \exp\{ -2\text{Im } S + 2E(T - \frac{T_0}{2}) + \int d\mathbf{k} \text{Re}(\mathcal{T}_k f_k) - \int d\mathbf{k} f_k^* f_k e^{-\omega_k T_0} \}, \quad (10)$$

where $\text{Im } S$ is imaginary part of the classical action, calculated along the time contour of Fig.(1).

Finally, we make some remarks about the choice of the contour of Fig.(1). Changing $T$ corresponds to the shift of part A of the contour upward or downward. This shift, however, does not change the initial state if no singularities of the solution have been crossed. This can be shown in the following way. If we move contour between lines $\text{Im } t = T$ and $\text{Im } t = T'$ (both lying between same two singularities) then the negative frequency components of the field are related by ($\eta = T' - T$)

$$f_k' = f_k e^{i\omega_k \eta}.$$
Here we used Eq. (5) and analytically continued time $t'$ to $t' + i(T' - T)$. Then it is easy to see from Eq. (7) that
\[
a'_k = f'_k e^{i \omega_k \xi_0 - \omega_k T'} = f_k e^{i \omega_k (T' - T) - \omega_k T'} = f_k e^{i \omega_k \xi_0 - \omega_k T} = a_k.
\]

Contours with $T$ and $T'$, separated by singularities, correspond to the completely different states (it will be shown in next sections for $\phi^4$ model, see also Refs. [13, 16] for other models).

At nonzero $T_0$, the energy is expressed in a nonstandard way with respect to the negative frequency part of the field (but it is expressed usually in terms of initial state – $E = \int d\v{k} a'^*_k a_k$). In this case the average of arbitrary bilinear operator $O = \int d\v{k} F(\v{k}) A^*_k A_k$ can be written in terms of Fourier components of the field as [13]
\[
\langle O \rangle = \int d\v{k} F(\v{k}) f'^*_k f_k e^{-\omega_k T_0},
\]
where $T_0$ should be defined by Eq. (9). Hence, the number of the initial particles is expressed by
\[
n_{\text{initial}} = \int d\v{k} f'^*_k f_k e^{-\omega_k T_0} \tag{11}
\]

The probability of the transition (10) does not depend on the choice of $T$ and we can move the contour upward or downward until we reach a singularity of the classical solution. If it is possible to choose $T$ to satisfy the condition $T_0 = 0$, which is equivalent to $T = \xi_0$, then the energy and the number of particles are expressed in a standard way. In this case the probability (10) corresponds to the expression derived in [13].

The term $-2 \text{Im} S$ in the probability exponent (10) is the suppression instanton-like factor, while the other terms account for the presence of the initial particles.

In the next section we apply this formalism to the $\phi^4$ model.
3 The semiclassical process in $\phi^4$ theory

This section is devoted to consideration of massless $\phi^4$ theory (we assume that the energy scale is much larger than the mass scale when the massless limit is a reasonable approximation). We describe here a real $O(4)$-invariant solution of the theory and investigate the structure of the singularities of this solution in the complex time plane. It is shown below that the contour of Fig.(1) can correspond to a classically-forbidden transition between two coherent states and the corresponding suppression factor in the transition probability is calculated. To interpret this transition, we analyze a similar “process” in one-dimensional quantum mechanics. Finally, as an illustration, we consider $\phi^4$ theory with negative coupling constant where we can compare our results with previous calculations of the instanton-like processes [14].

1. The action of the model (we consider a real scalar field), written in conformally invariant form [17], is

$$S = \int d^4x \left( -\frac{1}{2} \phi \partial_{\mu} \partial^{\mu} \phi - \frac{\lambda}{4} \phi^4 \right),$$

where $\lambda > 0$ is the small coupling constant. The corresponding classical field equation is

$$\partial^2 \phi + \lambda \phi^3 = 0 \quad (12)$$

$O(4)$-invariant solutions of this equation [18] can be easily found using the invariance of the massless theory under the Minkowski conformal group. This invariance can be made explicit by projecting the theory onto the surface of the hypertorus [19]. Then, $O(4)$-invariant solutions can be found by solving a one-dimensional equation and they correspond to the oscillations with amplitude $a$ in the one-dimensional potential $V(x) = 1/2x^2 + 1/4\lambda x^4$.

The $O(4)$-invariant solution can be expressed in terms of elliptic functions

$$\phi(\vec{x}, t) = \frac{1}{\sqrt{\lambda}} \frac{2a}{\sqrt{(r^2 - (t - i)^2)(r^2 - (t + i)^2)}} \text{cn}(\sqrt{1 + a^2} \zeta - \zeta_0, k^2), \quad (13)$$

where $r = |\vec{x}|$, $k^2 = a^2/(2(1 + a^2))$ and

$$\zeta = \frac{1}{2t} \ln \left( \frac{r^2 - (t - i)^2}{r^2 - (t + i)^2} \right).$$
Here \( \text{cn} \) stands for the Jacobi elliptic cosine (see, for example, [20]) and \( k \) is the modulus of this function. The arbitrary integration constants are \( a \) and \( \zeta_0 \). We choose \( \zeta_0 = K \) (where \( K = \int_0^{\pi/2} dx/\sqrt{1 - k^2 \sin^2 x} \) is the complete elliptic integral), in which case \( \phi = 0 \) at \( t = 0 \). The constant \( a \), as we will see below, is related to the energy.

According to the approach, described in section (2), we are going to calculate a transition corresponding to the saddle point configuration (13) considered on the contour of Fig.(1) for some value of parameter \( T = \text{Im} t \).

First, we investigate the analytic structure of the solution in the complex time plane.

This solution is real on the real time axis, so, as has been shown in Ref. [13], it corresponds to a transition from the given initial state to the most probable final state.

The solution has essential singularities at \( t = \pm x \pm i \). Hence, we have to choose \( T < 1 \) for the contour of Fig.(1) not to cross the ”light-cone” singularity.

In addition, there are singularities (poles) at the “points” where

\[
\sqrt{1 + a^2} \zeta - K = 2mK + (2n + 1)i K'.
\]

Here \( K'(k^2) = K(1 - k^2) \), \( m, n = 0, \pm 1, \pm 2, \ldots \). These “points” are poles of the elliptic cosine [20]. Because \( \zeta \) is a function of radial coordinate and complex time, the solutions of this equation determine the singularity curves in the coordinate axes \( r, \text{Re} t, \text{Im} t \).

We will consider below only the case \( a \ll 1 \), which, as will be shown in the next section, corresponds to the case of a “small” number of final-state particles \( n_{\text{final}} \ll 1/\lambda \). In this limit \( K \approx \pi/2 \) and only \( m = -1 \) and \( n \geq 0 \) case corresponds to the singularities in the region \( \text{Im} t \geq 0, \text{Re} t \leq 0 \). The singularities curves (numerated by integer number \( n \)) \( t = t_n(r) \) run asymptotically “parallel” to the “light-cone” and have \( \text{Im} t \) coordinate close to 1

\[
t_n = i \left( 1 - \left( \frac{a^2}{16} \right)^{2n+1} \right) - r
\]

at \( r \to +\infty \) and \( n = 0, 1, 2, \ldots \). We have shown in Fig. (2) two curves in the region \( \text{Im} t \geq 0, \text{Re} t \leq 0 \).

We can see that the structure of the singularities of this solution is “appropriate” – we are able to choose the contour of Fig.(1) and not to cross.
any singularities. We choose the contour with exactly one singularity curve under it (i.e. with \(1 - a^2/16 < T < 1 - (a^2/16)^3\)). It will be shown below that this choice corresponds to a classically-forbidden (exponentially suppressed) transition.

\[\sigma \sim \exp(-2 \text{Im } S).\]

Here \(\text{Im } S\) should be calculated along the contour of Fig.(1). To calculate the imaginary part of the action we use the method of Ref.[13].

The action of the model is

\[S = \frac{\lambda}{4} \int d^3x \int_C dt \phi^4(\vec{x}, t),\]

where we have used the equation of motion. For every \(x\) the time integral along the contour of Fig. (1) is equal to the sum of the integral along the
real time axis (which is real) and contribution of the pole $t_0$, corresponding to the singularity (14) at $n = 0$. The pole contribution can be calculated using the expression for the $\text{cn}$ near the singularity $-2K + iK' [20]$

$$\text{cn} (-2K + iK' + u) = -\frac{1}{iku} - \frac{1}{6iku} (1 - 2k^2) u + O(u^2)$$

and expanding $\zeta$ in Taylor series up to the fourth order. Poles of the first, second and third orders give contribution to the imaginary part of the action and after lengthy calculations the pole contribution is equal to the integral

$$\text{Im} S = \text{Im} S_{\text{pole}} = \frac{2\pi^2}{\lambda} \int_0^\infty \frac{-80 t^3 (r^2 - (t - i)^2)(r^2 - (t + i)^2)^2}{(1 + t^2 + r^2)^2} r^2 dr \quad (15)$$

evaluated at $t = t_0(r)$. After substitution of the exact equation of the singularity line in the form

$$t_0 = p - \sqrt{p^2 + r^2 + 1}$$

(where $p$ is $i (1-a^2/16)$ for small $a$) the integral (15) is reduced to the integral

$$\text{Im} S = \frac{20\pi^2}{\lambda} \int_0^\infty \frac{(1 + p^2)^2 r^2}{(1 + r^2 + p^2)^{7/2}} dr \quad (16)$$

By substitution $y^2 = r^2/(1 + p^2)$ factor $p$ can be scaled out of integral and the final result is

$$\text{Im} S = \frac{20\pi^2}{\lambda} \int_0^\infty \frac{y^2 dy}{(1 + y^2)^{7/2}} = \frac{8\pi^2}{3\lambda} \quad (17)$$

It is exactly equal to Lipatov instanton action: the Euclidean action of the classical solution in $\phi^4$ theory with negative coupling constant [4] (our normalization of $\lambda$ differs from the normalization of $\lambda$ in [4] by factor 6). Thus, the choice of the contour between the first and the second singularity line corresponds to the classically forbidden transition suppressed by the factor

$$\sigma \sim \exp(-2S_0),$$

where $S_0$ is equal to Lipatov instanton action. So this process is analogous to the one-instanton transition in the electroweak model or to the “instanton-like” transition in $\phi^4$ theory with negative coupling constant.
The existence of such “instanton-like” processes in the $\phi^4$ theory with positive coupling constant seems surprising. However, we can find analogy in one-dimensional quantum mechanics.

2. Let us consider scattering above the potential barrier $V(x)$ in the semiclassical approximation, following the approach described in the textbook of Landau and Lifshitz [21]. The transmission above the barrier is classically-allowed, so, in the first approximation, the transmission probability is equal to one $T = 1$ and the reflection probability is zero (i.e. exponentially small) $R = 0$. We know, however, that there should exist some classically forbidden reflection from the barrier.

According to the general approach of [21], to calculate the classically-forbidden probability of transition from some initial state to some final state we have to find classical “trajectory” connecting initial and final “points” and calculate action $S(q_1, q_0) + S(q_0, q_2)$ (the classical action is $S = \int p \, dx$, where $p(x) = \sqrt{2m(E - V(x))}$ is a classical momentum) for the evolution of the system from the initial “point” $q_1$ to the “turning point” $q_0$ (singular point of the classical momentum) and then from the $q_0$ to the final “point” $q_2$. Then, the probability of the process is proportional to

$$\omega \sim \exp\left\{-\frac{2}{\hbar}\Im(S_1(q_1, q_0) + S_2(q_0, q_2))\right\}.$$

In our case the singular “turning point” is some complex coordinate $x_0$ (and complex conjugate coordinate $x_0^\ast$) determined by the requirement $V(x_0) = E$. The classically-allowed contribution to the transmission probability corresponds to the action on the trajectory along the real $x$-axis (or a trajectory which can be deformed to the real axis) and connects points $x_1 \to -\infty$ and $x_2 \to +\infty$ (Fig.(3), trajectory A).

This trajectory gives a contribution equal to one to the transmission probability.

The classically forbidden reflection is determined by the trajectory in the complex coordinate plane which connects points $x_1, x_2 \to -\infty$ and “winds” around the “turning point” $x_0$ (Fig.(3), trajectory B). The reflection probability is determined by the imaginary part of the classical action on this trajectory

$$R = |A_R|^2 \sim \exp\left\{-\frac{2}{\hbar}\Im\int_C p \, dx\right\},$$
where $A_R$ is the amplitude of reflection.

Because for the $T = 1$ and $R \neq 0$ the unitarity condition $R + T = 1$ is violated, in order to “unitarize” the amplitudes we have to take into account the classically forbidden contribution to the transmission amplitude corresponding to the “transmission after reflections”. This contribution is determined by the trajectory connecting points $x_1 \to -\infty$ and $x_2 \to +\infty$ and “winding” around the “turning points” $x_0$ and $x_0^*$. This contribution to the amplitude of the transmission is proportional to

$$A_T \sim \exp\left(-\frac{1}{\hbar} \text{Im} \int_C p\,dx\right),$$

where $C$ is a contour $C$ on the Fig.(3).

Thus, the transmission amplitude is dominated in the semiclassical approximation by the saddle-point contributions, corresponding to the trajectories in the complex coordinate plane. The trajectory along the real axis
(or a trajectory which can be deformed to the real axis) corresponds to the classically allowed contribution, and trajectories, which cannot be deformed to the real axis (without crossing the singular “turning points”), describe classically-forbidden contributions corresponding to “transmissions after reflections”.

Of course, we cannot relate directly quantum mechanical and $\phi^4$ field theory examples. But both models have a common feature, namely that the amplitude of transition in the semiclassical approximation is dominated by complex saddle-point configurations, representing a classical solution analytically continued to the complex “coordinate” plane. These contributions can be “classified” by the position of the corresponding “trajectories” with respect to the singularities of this solution.

Part A of the contour of Fig.(1) describes the free propagation of an incoming spherically-symmetric shell (13) at early time. Evolution along the imaginary part of the contour can be interpreted as a classically-forbidden reflection in the $\phi^4$ potential. The Minkowski part of the contour corresponds to an outgoing wave at late time. Therefore, we can call the semiclassical process in the $\phi^4$ model, described by the nontrivial “trajectory” (lying between the singular lines) in the complex time plane, as a “transmission after reflections”. Like the quantum mechanical example, including the contribution of such processes into the corresponding amplitude can, probably, unitarize the perturbative amplitude.

3. Now, as an illustration, we want to investigate the $\phi^4$ model with negative coupling constant. This theory allows instanton-like processes, which can be considered as models for the “shadow processes“ [22] (processes describing transitions from initial particles in the false vacuum to final-state particles in the false vacuum through an intermediate state containing a bubble of the true vacuum). The probability of such processes has been derived in the framework of the instanton formalism in Ref.[14]. It is proportional to the $\sigma \sim \exp(-2S_0)$, where $S_0$ is the Lipatov instanton action. We will show below that such processes can be described by the classical solution considered in the complex time plane (at least up to the suppression factor in the transition probability).

We analyze here only the massless case. Of course, a mass term has to be added to make the $\phi = 0$ state at least metastable, but we expect that the mass corrections to the transition probability shall not affect the leading suppression term $\sigma \sim \exp(-2 \text{Im } S)$. 17
The $O(4)$-invariant classical solutions of Eq. (2) with negative coupling constant $\lambda = -|\lambda|$ can be found using the approach of Ref. [19]. After “reduction” of the theory to a one-dimensional model, they correspond to oscillations with an amplitude $a$ in the one-dimensional potential $V = 1/2\phi^2 - \lambda/4\phi^4$ (so even massless theory is metastable in the “subspace” of the $O(4)$-invariant solutions). The solution can be written in the form

$$\phi(\vec{x}, t) = \frac{1}{\sqrt{|\lambda|}} \frac{2a}{\sqrt{(r^2 - (t - i)^2)(r^2 - (t + i)^2)}} \text{sn} \left( \sqrt{1 - \frac{a^2}{2}} \zeta - \zeta_0, k^2 \right),$$

where again $r = |\vec{x}|$ and $\zeta = 1/(2i) \ln ((r^2 - (t - i)^2)/(r^2 - (t + i)^2))$, $k^2 = a^2/(2 - a^2)$. We choose $\zeta_0$ to be $K$, which corresponds to the case $\partial \phi/\partial t = 0$ at $t = 0$. One can see that the structure of the singularities in the complex time plane does not change relative to the case with positive coupling constant. Choosing the contour between the first and second singularity lines, and using the expansion of the elliptic sin function near the singularity

$$\text{sn} (-2K + i K' + u) = -\frac{1}{ku} - \frac{1}{6k} (1 + k^2) u + O(u^2),$$

we obtain the result that the imaginary part of the action on this contour is exactly equal to the Lipatov instanton action $S_0 = 8\pi^2/(3|\lambda|)$. Thus, the transition probability of this process is suppressed by a factor $\sigma \sim \exp(-2S_0)$ and this process indeed describes (at least up to the suppression factor in the probability) a transition in the “one-instanton” sector, previously considered in Ref. [14].

4 The initial and final states

In this section we calculate the Fourier components of the initial and final states and find the corresponding energy and average number of particles. As has been mentioned before, the final state is determined via Eq. (6) by the asymptotics of the classical solution on the Minkowski part of the contour of Fig. (1) in the limit $t \to +\infty$, while the initial state corresponds to the
asymptotics of the solution on part A of the contour in the limit $t' \to -\infty$, where $t = iT + t'$ (Eq.(7)). We consider only case $a << 1$ which, we will see below, corresponds to the case $n_{\text{final}} << 1/\lambda$.

First, we determine the final state. In the limit $a << 1$, the Fourier components can be easily found by using the first-order approximation for the elliptic cosine ($k$ here is a modulus of the elliptic function)

$$\text{cn}(u, k)|_{k \to 0} = \cos u$$

(18)

The Fourier components are

$$g_k = a \sqrt{\frac{2\pi}{2k\lambda}} i e^{-k}$$

where $k = |k|$, for the negative frequency part of the field and

$$g^*_k = g_k = -a \sqrt{\frac{2\pi}{2k\lambda}} i e^{-k}$$

for the positive frequency part of the field.

The energy, the number of final particles $n_{\text{final}}$ and the average momentum $k_{\text{average}} \approx E/n_{\text{final}}$ are determined as

$$E = \int dk k g^*_k g_k = \frac{\pi^2 a^2}{\lambda}$$

$$n_{\text{final}} = \int dk g^*_k g_k = \frac{\pi^2 a^2}{\lambda}$$

and

$$k_{\text{initial}} \approx 1.$$

To find the initial state is not as easy. The initial state should be determined by the asymptotics of the solution on part A of the contour of Fig.(1). In this case we cannot use a first-order approximation (18) since we have to integrate through the region in the vicinity of the singularity of the elliptic cosine. Instead, we consider the integral determining the Fourier components of the solution

$$\phi(k, t) = \frac{\sqrt{2k}}{(2\pi)^{3/2}} \int d^3x \phi(x, t) e^{ikx} = \sqrt{\frac{2\pi}{2k}} \int_{-\infty}^{+\infty} \phi(x, t) e^{ikr} \frac{r}{ik} dr$$

(19)
in the complex plane of the variable \( r \). The solution has complicated singularity structure at complex \( r \), including poles and branch points. However, in the case \( a \ll 1 \), we can calculate leading contributions to the Fourier components using the following trick.

Consider again the final state. If \( \phi(x,t) \) is an exact solution, the integral (19), calculated along the real \( r \)-axis, corresponds to the exact Fourier components of the solution. The solution \( \phi(x,t) \) has poles at points \( r = \pm i (1 - (a^2/16)^{2n+1}) \pm t \) and branch points at \( r = \pm i \pm t \). If we want to calculate the Fourier components of the initial state we have to make an analytical continuation of the integral (19) to the complex time \( t \to t + iT \). As the result of this continuation the pole \( r = -t + i(1 - a^2/16) \) crosses the real \( r \)-axis from above and the pole \( r = t - i(1 - a^2/16) \) crosses the real \( r \)-axis from below (other singularities do not cross the real \( r \)-axis). For \( T = 1 - \epsilon \) the expression for the “new” position of the poles is

\[
r = -t + i(1 - \frac{a^2}{16}) \quad \rightarrow \quad r(I) = -t - i\left(\frac{a^2}{16} - \epsilon\right)
\]

\[
r = t - i\left(\frac{a^2}{16}\right) \quad \rightarrow \quad r(II) = t + i\left(\frac{a^2}{16} - \epsilon\right).
\]

(20)

![Diagram](Fig. 4)

We can see that the difference between the Fourier components of the final state, analytically continued to complex time, and the Fourier components of the initial state is given by the contributions of the poles (18) to the integral (19) (see Fig. (4)).
We can write it symbolically as

\[ \phi_{\text{initial}}(k, t) = \phi_{\text{final}}(k, t \to t + iT) - \text{pole(I)} + \text{pole(II)}, \tag{21} \]

where \( \text{pole(I, II)} \) presents the contributions of poles to the integral (19) and \( \phi_{\text{final}} \) is the contribution of the final state, analytically continued to complex time. In the leading approximation at small \( a \) we can use again the approximation (18) to calculate the contribution of the final state in Eq.(21).

To calculate contributions of the poles, we find the residues of the classical solutions at the singular points

\[ \text{res(I), res(II)} = -\frac{\sqrt{2}i}{a} \frac{\sqrt{(r - t - i)(r - t + i)(r + t + i)(r + t - i)}}{(-4tr)} \]

and substitute \( r = r(I), r(II); t \to t + iT \). Here we use the expression for the residue of the elliptic cosine [20], which equals \(-i/k\), and corresponds to the singularities determined by numbers \( m = 0, n = 0 \) in Eq.(14).

The contribution of \( \text{pole(I)} \) cancels the leading term, proportional \( a \), in the first term of Eq.(21). We obtain the leading contribution to the negative frequency part of the field in the limit \( \text{Re} t \to -\infty \),

\[ f_k \approx \sqrt{\frac{2\pi}{2k\lambda}} \frac{ia}{i} (a e^{-k\epsilon} - a e^{-k\epsilon + ka^2/16}) \approx \]

\[ \approx -\sqrt{\frac{2\pi}{2k\lambda}} \frac{a^3}{16} e^{-k\epsilon}. \]

In this expression we neglect terms of the order \( a^3 \) in the final state contribution, coming from the \( a^2 \) term in the expansion of the elliptic cosine, because these terms are proportional to \( 1/\sqrt{k}e^{-k\epsilon} \) and give suppressed contributions to the energy and number of particles.

The positive frequency part of the initial state is proportional to \( a \)

\[ \bar{f}_k \approx \sqrt{\frac{2\pi}{2k\lambda}} i a (-e^{-(2-\epsilon)k} + e^{-ka^2/16+i\epsilon k}). \]

To find number of the initial particles we use Eq.(11) because we cannot choose \( T \) (or \( \epsilon \)) to satisfy condition \( E = \int d\boldsymbol{k} f_k^* f_k \). The parameter \( T_0 \) in Eq.(11) is determined by the requirement

\[ E = \int d\boldsymbol{k} f_k^* f_k e^{-kT_0} \tag{22} \]
After substitution of the negative frequency part of the Fourier components into (22), we obtain

\[ E = \frac{\pi^2 a^2}{\lambda} = \frac{3 \pi^2 a^6}{4^4 (\epsilon + T_0/2)^{5/2}} \lambda. \]

This gives us the expression for \( T_0 \),

\[ \epsilon + \frac{1}{2} T_0 = \left( \frac{3}{4^4} \right)^{1/5} a^{4/5} \]

The number of the initial particles is given by (11) and equals

\[ n_{\text{initial}} = \int dk f_k^* f_k e^{-k T_0} \]

\[ = \frac{1}{2} E (\epsilon + \frac{1}{2} T_0) = \left( \frac{3}{4^4} \right)^{1/5} \frac{\pi^2 a^2}{2 \lambda} a^{4/5} = \frac{1}{2} \left( \frac{3}{4^4} \right)^{1/5} a^{4/5} n_{\text{final}}. \]

This result implies

\[ n_{\text{final}} \sim \frac{n_{\text{initial}}^{5/7}}{\lambda^{2/7}} \]

and

\[ k_{\text{average}} \sim \frac{k_{\text{final average}}}{a^{4/5}}. \]

We can see that for small coupling constant the number of the final “soft” particles is much larger than the number of the initial “hard” particles.

Thus, the classical solution, considered on the contour of Fig.(1) in the complex time plane above the singularity line, corresponds to the transition between two coherent states with a “strong” violation of particle number, \( n_{\text{final}} \gg n_{\text{initial}} \).

5 Concluding remarks

In the previous sections we have studied the semiclassical process in \( \phi^4 \) theory with positive coupling constant, which describes transition between
two coherent states. This transition is suppressed by the factor \( \exp(-2S_0) \), where \( S_0 \) is equal to the Lipatov instanton action – the Euclidean action of the classical solution in the theory with negative coupling constant.

The initial and final states, corresponding to this transition, have different numbers of particles \( (n_{\text{final}} >> n_{\text{initial}}) \) and different average momenta \( (k_{\text{final}} << k_{\text{initial}}) \), so this transition approximates some multiparticle scattering process with a large number of “soft” final particles.

The process is technically analogous to the one-instanton transition in electroweak model and could serve as a good model for studying the instanton effects. It seems that we can also describe some “multi-instanton” processes using the solution (13) and choosing the contour of Fig.(1) above several singularity lines.

We believe that we have to include the contributions of these instanton-like processes into the corresponding “total” amplitude for multiparticle production. Such contributions might slow down the factorial growth of the perturbative amplitude and unitarize the high energy cross section.

The energy dependence of the transition probability of this process is a very interesting problem. It requires a detail investigation of Eq.(10). The growth of the transition probability is related to the presence of the external particles. The first term in Eq.(10) describes the suppression factor while other terms describe the contributions of the external particles. These terms are trying to overcome the suppression factor and can be, in principle, large. An accurate estimation of the contribution of the external particles requires, however, including mass term effects into consideration.

We have to add the mass term for the following reason. Calculation of the transition probability requires summing the contributions from different “sizes” of the classical field. In this paper we consider only the contribution of the solution with a “unit” size (field configuration (13)). This integration is divergent at the large “sizes” and should be regularized by introducing a mass term into the action in the manner of the “constrained instanton” approach [23]. We do not consider the effects of the mass term in this paper, so this problem requires a more detailed investigation.

The important point is that the framework of the formalism allows one to analyze, in principle, the case \( n_{\text{final}} \geq 1/\lambda \). This case is analogous to multi-particle scattering at the sphaleron energy in the standard model, where the behavior of multi-particle cross section is still far from being understood.
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