Thermodynamics of a black hole in a cavity

Renaud Parentani\textsuperscript{1\S}, Joseph Katz\textsuperscript{2†} and Isao Okamoto\textsuperscript{2}

\textsuperscript{1} The Racah Institute of Physics, The Hebrew University of Jerusalem, Jerusalem 91904, Israel
\textsuperscript{2} Division of Theoretical Astrophysics, National Astronomical Observatory, Mizusawa, Iwate 023, Japan (E-mail: okamoto@gprx.miz.nao.ac.jp)

\textbf{Abstract.} We present a unified thermodynamical description of the configurations consisting on self-gravitating radiation with or without a black hole. We compute the thermal fluctuations and evaluate where will they induce a transition from metastable configurations towards stable ones. We show that the probability of finding such a transition is exponentially small. This indicates that, in a sequence of quasi equilibrium configurations, the system will remain in the metastable states till it approaches very closely the critical point beyond which no metastable configuration exists. Near that point, we relate the divergence of the local temperature fluctuations to the approach of the instability of the whole system, thereby generalizing the usual fluctuations analysis in the cases where long range forces are present. When angular momentum is added to the cavity, the above picture is slightly modified. Nevertheless, at high angular momentum, the black hole loses most of its mass before it reaches the critical point at which it evaporates completely.

\S Present address: Laboratoire de Physique Theorique de l' Ecole Normale Superieure, 24 Rue Lhomond, 75005 Paris, France (E-mail: parenta@physique.ens.fr)

† Permanent address: The Racah Institute of Physics, Jerusalem 91904, Israel (E-mail: jkatz@hujivms.bitnet)
1. Introduction and summary

A black hole in an isolated cavity which is bigger than its Schwarzschild radius can be in equilibrium with surrounding radiation only if the total energy $E$ of the system is greater than a critical value $E_C$, which depends on the volume of the cavity and the number of fields in the radiation (Hawking 1976). While local equilibrium configurations exist for energies greater than $E_C$, it is generally assumed that black holes must evaporate if $E_C < E < E_B \approx 1.3E_C$ since for $E < E_B$ pure radiation has more entropy than the composite system (Gibbons and Perry 1978).

This holds only if the system is given unlimited time to relax. If one considers instead a sequence of configurations starting, say, from an equilibrium at $E = E_B$ and in which the total energy is decreased quasi-statically, the system reaches new local equilibrium configurations in a finite time (Zurek 1980). During that time, the probability of finding a fluctuation high enough that the black hole completely evaporates is exponentially small. So small that the black hole is most likely to survive in a metastable superheated state up to energies very close indeed to $E_C$ (Okamoto, Katz and Parentani 1994). This is true for cavities whose radius $L^*$ is greater than $10^6$ Planck lengths $l_P = (\hbar G/c^3)^{1/2} \approx 10^{-33}$ cm ($L \equiv L^*/l_P > 10^6$). For those cavities, back-reaction due to quantum matter effects (York 1985), quantum gravity and spontaneous nucleation due to thermal fluctuations (Piran and Wald 1982) may be neglected since the maximum temperature, reached by equilibrium configurations, is always smaller than $T_D \simeq 0.37L^{-1/2}$ (in Planck units). Thus more than three orders of magnitude separate the equilibrium temperature from the Planck one. This indicates that a thermodynamic analysis may be safely performed. On the contrary, for smaller boxes the validity of this analysis becomes dubious.

Self-gravitating thermal radiation exhibits a similar behavior in an evolutionary sequence in which the total energy increases quasi-statically. Indeed, there is now an upper limit of energy $E_A \simeq 0.25L$ (here $E$ is measured in Planck units) above which no equilibrium configuration exists (Klein 1947). Thus for energies $E_B < E < E_A$ ($E_B \approx 0.20L^{3/5} \ll E_A \approx 0.25L$ for $L > 10^6$) pure radiation is in metastable superheated states. We shall prove that, as in the black hole situation, pure radiation is most likely to remain in those metastable states almost up to $E_A$. At that point, a thermodynamic as well as dynamical instability (Sorkin, Wald and Zhang 1980) develops and most of the radiation

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* Throughout the text numerical values are given with two significant digits
collapses into a black hole leading to a composite state of a black hole in equilibrium with surrounding left-over radiation. The spontaneous evaporation of the black hole into pure radiation near $E_C$ and the collapse of the radiation near $E_A$ provides thus a unified picture relating the different phases of a black hole and self-gravitating radiation in a cavity.

Uniform rotation added to the cavity gives a total angular momentum $J$ shared between the radiation and the black hole. The equilibrium configurations change in the following way. The critical energy $E_C(J)$ at which the black hole evaporates is now a function of $J$. It increases with increasing angular momentum while the fraction of black hole energy at that point $E_{bhC}/E_C$ decreases from $4/5$ at $J = 0$ to almost zero in the limiting rotating case: $J = L^2/2$. Hence for fast rotating cavities, at fixed $J$ and with slowly decreasing total energy, superheated Kerr black holes will evaporate with little latent heat, since black holes survive almost down to the critical point $E_C(J)$.

All of this concerns microcanonical situations in which the total energy is the control parameter. In canonical situations, with the temperature kept fixed, almost all the configurations wherein a black hole coexists with radiation are unstable. There exits nevertheless a narrow range of energies for very fast rotating cavities in which the canonical ensemble is stable. One thus recovers in this narrow range the already noticed flip of the heat capacity for Kerr black holes (Davies 1981, Kaburaki et al. 1993; cf. Katz et al. 1993 for Kerr-Newman black holes). When the radiation is included, this happens, however, when gravitational effects (neglected in the present analysis) are important.

In this paper, we analyze the stability limits, the different degrees of instability and the fluctuations of each of the two states of the system (with or without black hole) in evolutionary sequences of equilibrium configurations, with and without angular momentum. The analysis is purely thermodynamic. We assume there exists a state function whose stationary value is a function of the total mass-energy $E$, total angular momentum $J$ and the volume of the cavity $V = \lambda L^3$. We consider separately the microcanonical and the canonical ensembles since they are not equivalent. In a microcanonical ensemble, $E$ is the main control parameter and the inverse temperature

$$\frac{1}{kT} = \frac{\partial(S/k)}{\partial E} = \beta(E, J, L)$$

is the conjugate parameter of $E$ with respect to the total entropy $S$. In a canonical ensemble, the Massieu function is not $S$ but $S/k - \beta E = -\beta F$ ($F$ is the free energy), the main control parameter is
\( \beta \) and the conjugate parameter of \( \beta \) with respect to \(-\beta F\) is

\[
\frac{\partial(-\beta F)}{\partial \beta} = -E(\beta, J, L).
\]  

(2)

The linear series of equilibrium configurations in both ensembles, \( \beta(E) \) and \(-E(\beta)\) at fixed \( J \) and \( L \), are identical but the stability limits are not. Stability limits are obtained by applying the Poincaré method to linear series (see Ledoux 1958). We shall consider the linear series of conjugate parameters (Katz 1978, 1979; Thompson 1979) since this is the most appropriate way to observe changes of stability.

While state functions out of equilibrium, \( \Omega \), are never used, some assumptions must be made about \( \Omega \) which are stated in section 2 where the extremization of the entropy is related to the Einstein equations when self-gravity is taken into account. Stability analysis in terms of Poincaré’s method is then briefly reviewed. We shall see that we need to know \( \beta(E) \) only and not \( \Omega \) itself, a useful feature when analyzing the stability of self-bound radiation in a box. Furthermore, having determined \( \beta(E) \) from the solution of the Einstein equations, we do not need to attribute \textit{a priori} an entropy to the black hole. Instead, we recover it from the equilibrium configurations.

The theory of fluctuations is then considered. Fluctuation theory as given by Landau and Lifshitz (1980) and Callen (1985) is not entirely applicable to self-gravitating systems for the reason that when gravitational interactions are important one cannot anymore split the system into a little subsystem and the rest which behaves like a heat reservoir for the little one. Nevertheless, we shall see that the mean quadratic fluctuations of the inverse temperature, in a small subsystem, are simply related to its own thermodynamic parameters as well as the parameters of the whole system.

Section 4 gives details about \( \beta(E) \) in non-rotating and rotating cavities. Fluctuations near turning points are calculated in section 5 and a summary of stability conditions and phase transitions in slowly evolving systems through a succession of quasi-equilibrium states is described in section 6.

2. The stability of equilibrium configurations in mean field theory

In a stationary axially symmetric distribution of matter in local thermodynamic equilibrium, with fixed total mass-energy and angular momentum, and with Einstein constraint equations given, the
total entropy $S$ is stationary, $\delta S = 0$, if Einstein “dynamical” equations are satisfied. Then Tolman’s (1934) thermodynamic equilibrium conditions for local temperature hold and the angular velocity is uniform [Katz and Manor (1975); for non-rotating, spherical configurations see Cocke (1965)]. Here we shall use the partition function (Hawking 1978, Horwitz and Weil 1982, York 1988 and Brown et al. 1990) to sketch a formal deduction of both equilibrium and stability conditions for spherical distributions with zero angular momentum. This procedure will naturally exhibit the relations between the Einstein equations, quantum field theory in curved space and the extremization of the entropy.

Consider, for definiteness, the state function $\Omega$ which encodes the total number of states of a scalar field in a curved spacetime with total mass-energy $E$ as measured from infinity. The field is confined to a spherical box of “radius” $L$ (i.e. the surface of the cavity is $4\pi L^2$). Then $\Omega(E, L)$ is

$$\Omega(E, L) = e^{S(E, L)} = Tr\phi(E - H) = \int \frac{dt}{2\pi} e^{iEt} \int \frac{Dg_{\mu\nu}}{Diff} \frac{D\phi}{exp [iS_{Ein} + iS_{\phi}].}$$

(3)

The denominator “$Diff$” indicates that one should not integrate over geometries related by a diffeomorphism; $S_{Ein} = \int d^4x \sqrt{-g} R$, and $S_{\phi} = \int d^4x \sqrt{-g} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$. Since there is no unique definition of a local matter Hamiltonian in general relativity, we have used the path integral formalism. We refer to Regge and Teitelboim (1974) for a definition of energy in asymptotically flat spacetimes. Integrating in (3) over all periodic matter configurations (of period $t$), one obtains

$$\Omega(E, L) = \int \frac{dt}{2\pi} \int \frac{Dg_{\mu\nu}}{Diff} \exp [iEt + iS_{Ein}] Z\{g_{\mu\nu}\}; it]$$

(4)

where $Z$ is the partition function at fixed Lorentzian time $t$ in the background geometry $\{g_{\mu\nu}\}$.

Calculating (4) at the stationary configuration of $iEt + iS_{Ein} + \ln Z\{g_{\mu\nu}\}; it]$ gives the dominant contribution to $\Omega$. At the stationary point, $g_{\mu\nu}$ satisfies the time independent Einstein equations with an energy tensor of a thermal bath if at the saddle point $t = -i\beta$. In the absence of horizons, the matter field configurations are defined for $0 \leq r \leq L$ and one recovers, up to quantum matter corrections, the equations of a self-gravitating perfect fluid (Klein 1947, Gibbons and Hawking 1978). In the presence of horizons (see Carlip and Teitelboim 1993 for the treatment of the boundary term at the horizon), the regularity of the stress energy-momentum tensor, needed to satisfy Einstein equations, fixes the temperature of the matter (Hawking 1978). Then by integrating the time-time Einstein constraint
In the Hartle-Hawking vacuum (Howard 1984). Since \( E = m(L, M_{bh}) \), one may invert this relation and express \( M_{bh} \) in terms of \( E \) and \( L \). Therefore, one obtains the sought-for \( \beta = \beta(E) \) law (with \( L \) held fixed). Notice that Einstein equations with back-reaction taken into account provide an alternative way to determine the entropy of the black hole. Indeed, by integrating \( \beta(E)dE \) one obtains the total entropy \( S(E, L) \), which reduces to the black hole entropy when the radiation energy is negligible. One may also subtract from \( S(E, L) \) the entropy of the radiation, but there is an arbitrariness in this subtraction — since there is no unique definition of the entropy density nearby the hole where the Tolman relation breaks down — which indicates that the concept of the entropy of an isolated black hole is probably meaningless.

In order to illustrate the Poincaré method, we perform the \( \ell \)-integration in (3) (Horwitz and Katz 1978) which gives an \( \Omega \) of the form

\[
\Omega(E, L) = \int \frac{Dg_{\mu\nu}}{D\text{iff}} e^{-w(E, L, \{g_{\mu\nu}\})}.
\]

We do not need to be specific about the function \( w(E, L, \{g_{\mu\nu}\}) \) precisely because Poincaré analysis deals with the succession of extrema of \( w(E, L, \{g_{\mu\nu}\}) \) and not the function itself. A specific example is given in Sorkin et al. (1980) in the case of pure radiation without horizon, wherein \( w \) depends on \( g_{rr} \) only.

The stationary value of \( w \) is the classical equilibrium entropy \( S \). The local stability of equilibrium configurations is controlled by the quadratic fluctuations of \( w \). Imagine we make a Fourier decomposition of the \( g_{\mu\nu} \)'s. Since the domain of existence is finite \((0 \leq r \leq L)\), the \( g_{\mu\nu} \)'s are replaced by a denumerable set of variables, say, \( x^i \) \((i = 1, 2, ...)\). To order two, \( w \) may thus be written

\[
w \approx S + \frac{1}{2} \left( \frac{\partial^2 w}{\partial x^i \partial x^j} \right)_e (x^i - X^i)(x^j - X^j)
\]

where an index \( e \) means ‘at equilibrium’ [thus \( S = w_e \)] and \( X^i(E, L) \) are equilibrium values of \( x^i \).

Let \( \lambda_i \)'s denote the \((ordered: \lambda_1 \leq \lambda_2 \leq ...)\) elements of the diagonalized matrix \(-\left( \partial^2 w/\partial x^i \partial x^j \right)\).
The $\lambda_i$’s are known as the Poincaré coefficients of stability (Lyttleton 1953). The equilibrium is thermodynamically (locally) stable, or stable for arbitrary small fluctuations, if and only if all the $\lambda_i$’s are positive i.e. if $\lambda_1 > 0$. In unstable situations, the number of $\lambda_i < 0$ characterize the degree of instability.

The matrix $-(\partial^2 w/\partial x^i \partial x^j)_e$ is a second order differential operator of dimension one over a finite domain. We may assume that the spectrum of eigenvalues is non-degenerate: $\lambda_1 < \lambda_2 < \ldots$. Indeed, the slightest asymmetric perturbation in a system would lift the degeneracy (Thompson and Hut 1973). Having then a non-degenerate spectrum of Poincaré coefficients, the following properties hold (Katz 1978, 1979):

(a) Consider the linear series $\beta(E)$. Changes of stability along the linear series can only occur at vertical tangents, $\partial \beta / \partial E = \pm \infty$, like point A or C in figure 1. Such points where $E$ is locally minimum or maximum are called turning points. Between two vertical tangents all equilibrium configurations have the same degree of instability.

(b) In the neighborhood of a vertical tangent, when the linear series turns clockwise (its tangent goes from negative to positive values through infinity), one Poincaré coefficient changes sign from negative to positive value. That is, the system becomes stable or less unstable. If the linear series turns counter-clockwise, the changes of sign are reversed and the system becomes unstable or more unstable. It is thus enough to know the degree of stability of one configuration to know the degree of stability of all configurations.

(c) Of particular interest are linear series with multiple turning points spiraling inwards against the clock. If we follow the spiral towards the limit point, we meet a succession of vertical tangents and beyond each one, an additional Poincaré coefficient becomes negative. Counter-clockwise spirals represent thus a succession of equilibrium configurations that are more and more unstable.

Upon considering different ensembles, the following properties hold (Parentani 1994):

(d) The most stable ensemble is always the most isolated one. The degree of stability of equilibrium configurations in any ensemble related by a Legendre transformation (which expresses the contact with a reservoir) to the most stable is immediately known if one knows the degree of stability in the
most stable ensemble.

3. Fluctuations in gravitating systems

In classical thermodynamics, it is well known (Landau and Lifshitz 1980, Callen 1985 and Landsberg 1990) that the mean square fluctuations of the fundamental thermodynamic quantities pertaining to any small part of a system (or to the system as a whole) are related to the specific heat. For instance, in a canonical ensemble of total volume $V$, the mean quadratic fluctuations $(\Delta E)^2$ of the energy $E$ induced by the contact with the reservoir are given by

$$\frac{(\Delta E)^2}{\beta^2} = \frac{\partial (-E)}{\partial \beta} = C_V$$

(8)

where $C_V$ is the heat capacity at constant volume $V$. The mean square fluctuations $(\Delta \beta)^2$ of the inverse temperature which result from those energy fluctuations are given by

$$\frac{(\Delta \beta)^2}{\beta^2} = -\frac{\partial \beta}{\partial E} = \frac{\beta^2}{C_V}.$$  

(9)

since $\beta$ is a function of $E$ only.

In a microcanonical ensemble, these fluctuations vanish since the total energy $E$ is kept fixed. Nevertheless, within a small subsystem of volume $V'$, the temperature fluctuates and the mean square fluctuations of $\beta'$ are given by

$$\frac{(\Delta \beta')^2}{\beta'^2} = -\frac{\partial \beta'}{\partial E'} = \frac{\beta'^2}{C_{V'}}.$$  

(10)

where $C_{V'}$ is the specific heat of the subsystem and $E'$ its energy. This equation is valid only if the specific heat of the rest of the system is much bigger that the one of the subsystem (for homogeneous systems it requires $V' \ll V$). The equivalence of ensembles (the equality of the fluctuations (10) whether one works in the microcanonical or the canonical ensemble) means therefore that the rest of the system can be correctly treated as a heat reservoir for the little subsystem. Finally, we recall that the 'true' fluctuations are given by these estimates only if the characteristic dynamical time for the fluctuations to evolve is much bigger than $\beta$ itself.

In gravitating systems, there are long range forces. Therefore, when gravitational interactions are important, one cannot treat the rest of the system as a passive reservoir. Indeed the existence of stable
microcanonical ensembles with negative specific heat indicates the decisive role of the energy constraint between the little subsystem and the rest. Furthermore, we stress the fact that when a microcanonical ensemble approaches instability, its heat capacity given in equation (8) is always negative (thus the canonical ensemble is already unstable) for stable states and positive in unstable states (for which the canonical ensemble is still unstable) see section 4 and, for instance, figure 1.

The relation between mean square fluctuations and thermodynamic coefficients in gravitating systems is thus more complicated than equation (10) and we shall display it in two steps. We shall first see that, since the vicinity of a turning point is dominated by a single eigenmode, one can relate the fluctuations of the least stable mode $x^1$ to the heat capacity of the whole system $C_V$. Then, we shall relate the fluctuations of the inverse temperature within a small subsystem to $C_V$ itself. To prove the first point, consider a microcanonical ensemble (with $J = 0$) in which we define a temperature function $\tilde{\beta}(x^i; E, V)$ away from equilibrium (see also Okamoto et al. 1994)

$$\tilde{\beta} = \frac{\partial w}{\partial E}.$$  

(11)

At equilibrium one has $\tilde{\beta}(X^i(E, V); E, V) = \beta(E, V)$ where the equilibrium values $x^i = X^i(E, V)$ are solutions of

$$\frac{\partial w}{\partial x^i} = 0.$$  

(12)

Therefore, the slope of the linear series and the derivatives of $\tilde{\beta}$ are related by

$$\frac{\partial \beta(E, V)}{\partial E} = \left(\frac{\partial \tilde{\beta}(x^i; E, V)}{\partial E}\right)_e + \sum_i \left(\frac{\partial \tilde{\beta}}{\partial x^i}\right)_e \frac{\partial X^i(E, V)}{\partial E}.$$  

(13)

The derivative $\partial X^i/\partial E$ can be obtained from equation (12) by the following identity

$$\left(\frac{\partial}{\partial E} \left[\left(\frac{\partial w}{\partial x^i}\right)_e\right]\right) = \left(\frac{\partial \tilde{\beta}}{\partial x^i}\right)_e - \lambda_i \frac{\partial X^i}{\partial E} \equiv 0$$  

(14)

where there is no summation on $i$. Thus, from equation (14) we deduce $\partial X^i/\partial E$ in terms of $(\partial \tilde{\beta}/\partial x^i)_e$ which we may insert into (13) to obtain

$$\frac{\partial \beta}{\partial E} = \left(\frac{\partial \tilde{\beta}}{\partial E}\right)_e + \sum_i \frac{(\partial \tilde{\beta}/\partial x^i)^2_e}{\lambda_i}.$$  

(15)
Consider now a linear series of stable configurations near a turning point. At that point the first eigenvalue $\lambda_1$ changes sign. As a result the right hand side of equation (15) is entirely dominated by the first term

$$\frac{\partial \beta}{\partial E} \approx \frac{1}{\lambda_1} \left( \frac{\partial \tilde{\beta}}{\partial x^1} \right)_e.$$  \hspace{1cm} (16)

The changes of $w$ are also dominated by the fluctuations of $x^1$ and given by

$$w \approx S - \frac{1}{2} \lambda_1 (\Delta x^1)^2$$  \hspace{1cm} (17)

which, with equation (16), becomes

$$w \approx S - \frac{1}{2} \frac{(\Delta \tilde{\beta})^2}{\partial \beta/\partial E}$$  \hspace{1cm} (18)

where

$$\Delta \tilde{\beta} = \left( \frac{\partial \tilde{\beta}}{\partial x^1} \right)_e \Delta x^1.$$  \hspace{1cm} (19)

This $\Delta \tilde{\beta}$ represents fluctuations in the inverse-temperature function induced, near the critical point, by the fluctuations of the least stable mode $x^1$. We can now use the standard arguments of fluctuation theory (Landau and Lifshitz 1980) and say that the probability $dW$ for a fluctuation of $\tilde{\beta}$ in the range $\tilde{\beta} + \Delta \tilde{\beta}$ and $\tilde{\beta} + \Delta \tilde{\beta} + d\tilde{\beta}$ is proportional to $\exp(w - S)$ and therefore,

$$dW = \frac{1}{\sqrt{2\pi}} \frac{d\tilde{\beta}}{\sqrt{(\partial \beta/\partial E)}} \exp \left[ -\frac{1}{2} \frac{(\Delta \tilde{\beta})^2}{\partial \beta/\partial E} \right]$$  \hspace{1cm} (20)

Thus the mean square fluctuations of $\tilde{\beta}$ are given by

$$\overline{(\Delta \tilde{\beta})^2} = \frac{\partial \beta}{\partial E} = -\frac{\beta^2}{C_V}.$$  \hspace{1cm} (21)

They are bounded when the microcanonical ensemble is stable (i.e. when $\lambda_1 > 0$ or as we already point out when $C_V < 0$. See equation (16)). It is convenient to express $dW$ in terms of $C_V$ and of the dimensionless ratio

$$\frac{\tilde{\beta}(x^1; E, L)}{\beta} \equiv \tilde{u}.$$  \hspace{1cm} (22)

Then

$$dW = \sqrt{-\frac{C_V}{2\pi}} \exp \left[ +\frac{1}{2} C_V (\Delta \tilde{u})^2 \right] d\tilde{u}$$  \hspace{1cm} (23)
and, following (21), the mean square fluctuations of $\tilde{u}$ are

$$\langle (\Delta \tilde{u})^2 \rangle = -C_V^{-1}. \quad (24)$$

Our task now is to relate those rather formal fluctuations of $\tilde{\beta}$ to the fluctuations of the temperature within a small part of the system as well as to understand the origin of the unusual sign in equation (21). As we have already said, in the presence of long range forces, one cannot exactly split a system into two parts since there is no more a local definition of energy. If nevertheless some small part is less coupled gravitationally to the rest of the system, one may use it as the "small" system in which one can compute the fluctuations of $\beta$. When this is not the case, we shall see that one can still consider the fluctuations within the outermost layer of radiation even though the energy into that layer is not well defined. We designate by the subscript 2 the little subsystem which is confined for radii $L - l < r < L$ (where $l \ll L$). The subscript 1 refers to the rest of the system which is thus either the black hole surrounded by radiation up to that last layer, or pure radiation. Instead of the energy, we shall use the Schwarzschild mass (since it is a local quantity) and we thus split it into $E_1$ and $E - E_1$. We shall see that the partition mass $E_1$ will play a role very similar to the least stable mode $x^1$. This is because $\partial x^1 E_1 \neq 0$.

Let $w(E_1; E, V)$ be the entropy out of equilibrium by which we now mean that the only variable out of equilibrium is $E_1$. All other variables have been replaced by their equilibrium values. Then

$$w(E_1; E, V) = S_1(E_1, V_1) + S_2(E - E_1, V - V_1; E_1) \quad (25)$$

where $S_1$ and $S_2$ are the entropies of the two parts. We emphasize the double dependence of $E_1$ in $S_2$. When $E_1$ varies, the mass left over in the system 2 is $E - E_1$ but the gravitational potential in which 2 evolves is changed as well. This later dependence is nevertheless parametric if $l \ll L$. By parametric we mean that upon taking derivatives with respect to $E_1$, this later dependence gives an additional term which is $O(l/L)$ smaller than the usual term and which may be safely neglected. Equilibrium between the two parts requires, as usual, the equality of the temperatures:

$$\partial_{E_1} w(E_1; E, V) = \partial_{E_1} S_1 + \partial_{E_1} S_2 \equiv \beta_1 - \beta_2 = 0 \quad (26)$$

where $\beta_1$ and $\beta_2$ are the inverse temperatures of the two parts and where we have neglected the additional term (This does not mean that we neglect completely the extra dependence in $E_1$, for
instance, \( \beta_2 \) depends explicitly on \( E_1 \) through the Tolman dependence (see the Appendix equation (A.3)). Then the lowest order variations of \( w \) near equilibrium due to an exchange of energy \( \Delta E_1 \) is:

\[
w - S = \frac{1}{2}(\partial_{E_1}^2 w)(\Delta E_1)^2 = \frac{1}{2}[\partial_{E_1} \beta_1 + \partial_{E_2} \beta_2](\Delta E_2)^2
\]

This energy fluctuation induces, in turn, a fluctuation of \( \beta_2 \) in the small part given by

\[
\Delta \beta_2 = \Delta E_2(\partial_{E_2} \beta_2)
\]

Then the entropy fluctuation induced by the latter one is

\[
w - S = \frac{1}{2}(\partial_{E_2} \beta_2) \left[ 1 + (\partial_{E_1} \beta_1)(\partial_{E_2} \beta_2) \right](\Delta \beta_2)^2 = \frac{1}{2}(\partial_{E_2} \beta_2) \frac{dE}{d\beta} (\partial_{E_1} \beta_1)(\Delta \beta_2)^2
\]

since the slope of the \( \beta(E) \) curve of the entire system is given by

\[
\frac{d\beta}{dE} = (\partial_{E_2} \beta_2) \left[ 1 - \frac{dE_1}{dE} \right] = (\partial_{E_1} \beta_1) \left[ 1 + (\partial_{E_1} \beta_1) \partial_{E_2} \beta_2 \right]^{-1}
\]

because

\[
dE = dE_1 \left[ 1 + (\partial_{E_2} \beta_2)(\partial_{E_1} \beta_1) \right]
\]

Thus the mean square fluctuations of \( \beta_2 \) given by the inverse coefficient appearing in equation (29) are

\[
(\Delta \beta_2)^2 = -(\partial_{E_2} \beta_2) \frac{d\beta}{dE} (\partial_{E_1} \beta_1).
\]

This is the generalization of equation (10) when one cannot treat the rest of the system as a passive reservoir. Indeed, one does recover (10) when the subsystem 1 can be approximated by the whole system in which case the coefficient of \( \partial_{E_2} \beta_2 \) is 1. One sees also that the point wherein the fluctuations of \( \beta_2 \) will diverge is controlled entirely by the divergence of \( d\beta/dE \) when the denominator in equation (30) vanishes. That is, one probes locally, through the fluctuations in the small subsystem the stability of the whole system 1 + 2. Furthermore, near the critical point, the mean fluctuations of \( \beta_2 \) are equal to the fluctuations of \( \tilde{\beta} \) (and therefore independent of \( l \)) since \( (\partial_{E_2} \beta_2)(\partial_{E_1} \beta_1) = -1 \) at the critical point as seen in equation (30). This later factor of \(-1\) explains the unusual sign in equation (21).

For the interested reader, we also point out the analogy of equation (32) which gives the fluctuations when the rest of the system cannot be treated as a reservoir and the expression which relates the
fluctuations in one ensemble to the fluctuations in another ensemble related to the first one by a Legendre transformation (Parentani 1994). In both cases when the ensembles are nonequivalent, the fluctuations are controlled by the correction factor: $\frac{d\beta}{dE}(\partial \beta_1 E_1)$, the coefficient of $\partial E_1 \beta_2$ in equation (32). And it is only when the ensembles are equivalent that this factor reduces to 1. Another common feature is the fact that when $d\beta/dE = 0$ it does not imply that the fluctuations of $\beta_2$ vanishes because $\partial E_1 \beta_1$ vanishes as well, see equation (30).

For a canonical ensemble, when one consider configurations which are stable microcanonically, the necessary and sufficient condition for stability is the positivity of $C_V$. For those configurations only one may safely apply the analysis of Landau and Lifshitz and find that the fluctuations of the total energy are indeed given by equation (8). Finally let us mention that equivalent ensembles would have vertical and horizontal slopes in $\beta(E)$ at practically the same point since both ensembles become simultaneously unstable. Thus the curve $\beta(E)$ would make an angle bigger than $90^\circ$. Nothing like that is happening in the following situations.

4. The linear series of conjugate parameters

In Planck units, $E$, $J$ and $\beta$ are respectively given by

$$E = \frac{E^*}{E_P}, \quad J = \frac{J^*}{\hbar}, \quad \beta = \beta^* E_P$$

(33)

in which $E_P = \sqrt{\hbar c^5/G} \approx 10^{16}$ erg and asterisks denote quantities in cgs units. Instead of using Planck units for $E$, $J$ and $\beta$, we shall use dimensionless quantities expressed in terms of $G$, $c$ and $L^*$ as is common in classical general relativity. One thus define

$$\mathcal{E} = \frac{G E^*}{c^4 L^*}, \quad \mathcal{J} = \frac{G J^*}{c^5 L^{*2}}, \quad b = \frac{\hbar c \beta^*}{8 \pi L^*}.$$ 

(34)

Then $\mathcal{E}$ varies between 0 and 1/2 (the Schwarzschild limit) and since $J^* < E^* L^*/c$ (the centrifugal limit), $\mathcal{J}$ is always smaller than $\mathcal{E}$. Thus,

$$0 \leq \mathcal{J} \leq \mathcal{E} \leq \frac{1}{2}.$$ 

(35)
We shall see nevertheless the appearance of a fourth dimensionless quantity $L = L^*/l_P$ which will reflect the fact that the Hawking temperature has a quantum origin. Finally, the relation between $E$, $J$, $\beta$ and $\mathcal{E}$, $\mathcal{J}$, $b$ is

$$\mathcal{E} = \frac{E}{L}, \quad \mathcal{J} = \frac{J}{L^2}, \quad b = \frac{\beta}{8\pi L}$$

(36)

(i) Schwarzschild Black Holes.

Consider first a Schwarzschild black hole of mass $M_{bh}^*$ in equilibrium with radiation enclosed in a spherical cavity with fixed radius $L$. At sufficiently low temperature (high $\beta$ or $b$), when most of the mass-energy is in the black hole, the total energy of the black hole and the radiation is, to a good approximation (see the Appendix) given by the sum of mass-energies as in flat spacetime (Hawking 1976); in Planck units

$$E = E_{bh} + E_{rad} = \frac{\beta}{8\pi} + \frac{\pi^2}{15} n V \beta^{-4}$$

(37)

where $n$ is the sum over the helicity states of the massless fields. Translated into our classical units, (37) becomes

$$\mathcal{E} = \mathcal{E}_{bh} + \mathcal{E}_{rad} = b + \sigma L^{-2} b^{-4}, \quad \sigma = \frac{4\pi^3}{45} \left( \frac{1}{8\pi} \right)^4 n \simeq 6.91 \cdot 10^{-6} n.$$  

(38)

From now on we shall take $n = 1$. The function $\mathcal{E}(b)$ becomes eventful when $\mathcal{E}_{bh}$ and $\mathcal{E}_{rad}$ are of the same order of magnitude. Indeed the lower bound for $\mathcal{E}$ is reached when

$$\frac{d\mathcal{E}}{db} = 0$$

(39)

at which point (C in figure 1a) one has

$$\mathcal{E}_{bh} = 4\mathcal{E}_{rad}, \quad \mathcal{E} = \mathcal{E}_C = \frac{5}{4} b_C, \quad b_C = (4\sigma L^{-2})^{1/5} = 0.123 L^{-5/7}.$$  

(40)

When $\mathcal{E} > \mathcal{E}_C$, the system’s energy is dominated either by the black hole or by the radiation. Indeed for $b > b_C$, when $\mathcal{E} > 1.6\mathcal{E}_C$, less than 1% of the energy is in the radiation. For $b < b_C$, when $\mathcal{E} > 25\mathcal{E}_C$, less than 1% is in the black hole mass. At higher energies, along this later series, when $b \approx L^{-1/2}$, one has to take into account the radiation self-gravitating effects and solve Einstein equations [Klein (1947) in the case of pure radiation, see also the Appendix]. One finds that there is a
maximal temperature where $b_D \simeq 0.11 L^{-1/2}$ and $\mathcal{E}_D \simeq 0.123$ (see figure 1b). For still higher energies, $b$ increases again till one finds a maximal energy $\mathcal{E}_A \simeq 0.246$ where $b_A \simeq 0.135 L^{-1/2}$ beyond which there are no more equilibrium configurations.

Both curves, $b(\mathcal{E})$ and $b(\mathcal{E}_{\text{rad}})$ are drawn in figure 1a and 1b. In figure 1b the two lines are almost on top of each other for $\mathcal{E} \gtrsim \mathcal{E}_D$ (due to the small mass of the hole: $\mathcal{E}_{\text{bh}}/\mathcal{E}_{\text{rad}} \approx b \approx L^{-1/2}$) and they spiral inwards counterclockwise with an almost common limit point $Z$ at $\mathcal{E}_Z = 3/14$ and $b_Z \simeq 1.23 b_D \simeq 0.132 L^{-1/2}$.

Such counterclockwise spirals appear in Newtonian theory as well as in general relativity for equations of state of the form $P = K \rho$ where $P$ is the pressure, $\rho$ the density and $K$ a number (Chandrasekhar 1972); here $K = 1/3$. The inward spiral will not differ very much whether we keep $L$ or the proper volume of the sphere fixed. Therefore, stability limits will not differ significantly either.

The two linear series $b(\mathcal{E})$ and $b(\mathcal{E}_{\text{rad}})$ coexist between two vertical lines for $\mathcal{E}_C \leq \mathcal{E} \leq \mathcal{E}_A$. Thus between these lines there will be always one metastable state. We emphasis the origin of $\mathcal{E}_C < \mathcal{E}_A$ which is due to the different nature of the instability: with or without a black hole at the Hawking temperature. This leads to the scaling

$$\frac{E_C}{E_A} = 0.65 L^{-2/5}. \quad (41)$$

Very important also is the fact that there is a maximal temperature $T_D \simeq 0.37 L^{-1/2}$. Thus for $L > 10^6$, $T$ is always at least three orders of magnitude below the Planck temperature. We may thus safely neglect quantum matter effects (York 1985), quantum gravity and spontaneous nucleation (Piran and Wald 1984). For smaller boxes, not only should one take into account the quantum matter effects induced by the presence of the hot black hole, but one has presumably to abandon the thermodynamic analysis all together since the fluctuations become important and are not governed anymore by their thermodynamics estimates. Indeed, for those configurations, the thermal and the dynamical characteristic times come to coincide.

Finally, we mention that Balbinot and Barletta (1988) computed the change of the black hole entropy $S_{\text{bh}}$ due to some quantum matter back-reaction. This provides a model for black hole remnants since the tiny black hole may now be in equilibrium with radiation for arbitrary large (or small)
temperatures. One easily shows that for cavities with \( L > 10^6 \) one finds the linear series to be almost unmodified and thus stability limits unaffected (contrary to what was suggested) since the modifications are dimensionalized by the Planck mass and since \( T \lesssim 0.37 \times 10^{-3} \).

(ii) Rotating Black Holes.

Consider now a Kerr black hole at mid-height on the z axis of a cylindrical cavity of radius \( L^* \). When thermodynamical equilibrium is achieved, the radiation has the same temperature as the hole and rotates with the same angular velocity: \( \Omega_{bh} = \Omega_{rad} \). The peripheral velocity of the cavity, in units of \( c \), is

\[
\nu = \frac{\Omega_{rad} L^*}{c}, \quad 0 \leq \nu \leq 1.
\]

(Schumacher et al. 1992) have calculated what becomes of equation (38) when the gravitational pull of the hole and the self-gravitational effects of the radiation are neglected. This amounts to add to the energy of a rotating black hole \( \mathcal{E}_{bh}(b, \nu) \) the energy of the radiation calculated (in special relativity) in rotating coordinates. We shall push the analysis of that approximation beyond its presumed limits of validity in order to gain some insight of what may still happen at very high angular momentum (\( \nu \to 1 \)) when general relativity has to be taken into account.

Introducing a non-dimensional parameter \( h \equiv J/M_{bh}r_H \) where \( r_H \) is the radius of the horizon (Okamoto and Kaburaki 1992), the mass-energy, the angular momentum and the angular velocity of the hole can be written

\[
\mathcal{E}_{bh} = b(1 - h^2), \quad \mathcal{J}_{bh} = \frac{2h}{1 + h^2}(1 - h^2)b^2, \quad 0 \leq h \leq 1
\]

\[
\Omega_{bh} = \frac{4\pi}{\beta} \frac{h}{1 - h^2}.
\]

At equilibrium, \( \Omega_{bh} = \Omega_{cav} \), thus equations (42) and (44) give

\[
\nu(h, b) = \frac{h}{2b(1 - h^2)}.
\]

For a cylinder of height \( H^* = 2L^* \), Schumacher et al. found

\[
\mathcal{E}(b, h) = b(1 - h^2) + \frac{3}{2} \sigma L^{-2} b^{-4} \frac{1 - \nu^2/3}{(1 - \nu^2)^2}
\]
and

$$J(b, h) = \frac{2h}{1+h^2}(1-h^2)^2b^2 + \sigma L^{-2}b^{-4}\frac{\nu}{(1-\nu^2)^2}. \quad (47)$$

Equations (47) with (45) immediately indicate that most of the angular momentum is in the radiation except when $b$ is close to $1/2$, i.e. when the black hole fills up the entire cavity. For a given value of $J$ held fixed, equation (47) gives $h(b)$ which, substituted in equation (46) gives $b(\mathcal{E})$. The $b(\mathcal{E})$ function can only be written in parametric form and we have to solve it numerically. We have drawn $b(\mathcal{E})$ for $J = 1/40$ in figure 2a and for $1/8$ in figure 2b.

A common feature of the two linear series of figures 1 and 2 is that black holes and radiation coexist to the right hand side of a vertical line $CC_{rad}$ since there is always a minimum energy $\mathcal{E}_C(J)$ required to find a black hole in equilibrium with the radiation. There is no maximal energy, the equivalent of $\mathcal{E}_A$, on the right hand side because we have not take into account the self-gravity of the rotating radiation. There are also two important modifications in the behavior of $b(\mathcal{E})$ at fixed $J$ when one compares linear series at different $J$. Both are shown in figure 3. First, the turning point $C$ moves in the plane $(\mathcal{E}, b)$; $\mathcal{E}_C$ increases with $J$. Second, the energy of the black hole at $C$, $\mathcal{E}_{bh}C(J)$ — the energy gained by the radiation between $C$ and $C_{rad}$ — divided by the total energy $\mathcal{E}_C(J)$ decreases with $J$.

Another novelty appears for $J \gtrsim 1/40$. We remind the reader that in the absence of radiation, it was noted that for $J_{bh}/M_{bh}^2 \gtrsim 0.68$ the slope $\partial \beta/\partial M_{bh}$ becomes negative, going through 0 (Davies 1977). When the radiation is taken into account, there is no change of sign in $\partial \beta/\partial E$ since the radiation dominates the equilibrium configuration before one approaches the critical ratio 0.68 of $J_{bh}/M_{bh}^2$. For $J \gtrsim 1/40$ one does recover this phenomenon in a small interval of energy (see figure 2b). There are now two changes of sign upon decreasing the energy, before the black hole starts to evaporate. At lower energies the radiation dominates the equilibrium again and $\partial \beta/\partial E$ returns positive.

5. Fluctuations near the turning points

† This limit is obtained by substituting $\nu$ from (45) into (47). For a given $J$, one finds 3 values of $0 \leq h \leq 1$ corresponding to 3 points on $b(\mathcal{E})$. Two of the points are on the left and right of the local maximum, point $X$ in figure 2b, when $J$ is high enough, say $1/8$. As $J$ decreases the 2 points come closer to each other and point $X$ goes down the line $b(\mathcal{E}_{bh})$. At about $J \approx 1/40$ the points coincide and $X$ is at the bottom of $b(\mathcal{E}_{bh})$. For $J \lesssim 1/40$, there is no local maximum anymore.
We shall now compute the mean square fluctuations of the rescaled variable $\tilde{u}$, equation (32), near the turning points C and A. We shall also find the probabilities $dW$, equation (23), that the fluctuations be big enough for the system to jump from metastable to stable configurations. Finally we shall say a few words about characteristic times and for the probability rates.

At the turning point $C_V = 0$. Thus, for an equilibrium configuration $P$ in the vicinity of the turning point, to lowest order in $(\beta_P - \beta_0)$ where $\beta_0$ stands for either $\beta_C$ or $\beta_A$ one has

\[
-C_V = \beta_P^2 \left( \frac{\partial E}{\partial \beta} \right)_P \simeq \beta_0^2 \left( \frac{\partial^2 E}{\partial \beta^2} \right)_0 (\beta_P - \beta_0) = \left( \beta^3 \frac{\partial^2 E}{\partial \beta^2} \right)_0 \Delta u
\]

where

\[
\Delta u = \frac{\beta_P - \beta_0}{\beta_0}, \quad |\Delta u| \ll 1.
\]

$\Delta u$ parametrises the equilibrium configurations $P$ and should not be confused with the fluctuation out of equilibrium $\Delta \tilde{u}$ defined in equation (22). In terms of $E$ and $b$, one has thus

\[
-C_V = \beta^{-2} \left( \frac{\partial \beta}{\partial E} \right) \simeq 8\pi L^2 \left( \frac{b^3 \partial^2 E}{\partial b^2} \right)_0 \Delta u.
\]

Near point C, equation (38) and (40) give

\[
-(C_V)_C \simeq 1.9L^{6/5} \Delta u.
\]

Near point A, one finds

\[
-(C_V)_A \simeq 23L^{3/2}(-\Delta u)
\]

(see the end of the Appendix for the numerical factor 23). The scaling law $L^{3/2}$ itself is easily found: $b(E_{\text{rad}})$ scales like $L^{-1/2}$ and the radiation entropy scales like $b^{-3} \propto L^{3/2}$.

Near points A and C, the mean fluctuations of $\tilde{u}$, given in equation (24), are extremely small (and therefore the temperature fluctuations as well) owing to the presence of the positive power of $L$ in $C_V$. Nevertheless very close to the turning point, the fluctuations diverge and thus will be high enough to induce a transition towards stable configurations. We therefore estimate at which distance from the turning points, that is, for which $\Delta u$, will the mean square fluctuations be high enough. To this end, we note that the minimal size that a fluctuation should possess in order to provoke the transition is $2\Delta u$. Indeed smaller fluctuations have not reached the minimal entropy configuration which lies onto
the unstable branch of the linear series and thus which is as well at a distance $\Delta u$ from the turning point. Hence the system will most likely return towards the initial metastable configuration. The minimal size of $(\Delta \tilde{u})^2$ is thus reached when $(\Delta \tilde{u})^2 \simeq (2\Delta u)^2$, that is when

$$- C_V^{-1} = 4(\Delta u)^2. \quad (53)$$

Using equation (51) we see that near point C this happens when $(\Delta u)_C = 0.51L^{-2/5}$. At point A, following (52), $(\Delta u)_A \simeq 0.22L^{-1/2}$. If $L > 10^6$, one finds $(\Delta u)_C < 2.0 \cdot 10^{-3}$ and $(\Delta u)_A < 2.2 \cdot 10^{-4}$. Thus only very nearby the turning point will the mean fluctuations cause a phase transition. For $\Delta u$'s bigger than $(\Delta u)_C$ or $(\Delta u)_A$ the probabilities decrease drastically like $\exp(2(\Delta u)^2C_V))$. This shows that when $\Delta u$ is only a few times bigger than $(\Delta u)_C$ or $(\Delta u)_A$ the metastable black hole near C and the metastable radiation near A are perfectly stable.

Having found $dW$, we now estimate probabilities of fluctuations per unit time. The rate at which a particular fluctuation of energy $\Delta E$ occurs is given, by virtue of the fluctuation dissipation theorem, by the inverse time it takes a cavity to return to equilibrium after the addition of the energy $\Delta E$. This time depends on the peculiar dynamics of the system. For a black hole in equilibrium with radiation, the time, following Zurek (1980), is of the order of $\beta^4 \Delta E$. For pure radiation, the time is of the order of $\beta^2 \Delta E$. Hence both times are proportional to powers of $L$. Thus probabilities per unit time behave essentially like $dW$ itself, i.e., the negative exponential of powers of $L$ dominates completely the probability rates.

6. Stability and phase transitions in evolutionary sequences through quasi-equilibrium states: A summary

(i) The Microcanonical Ensembles

(a) Non-rotating Black Holes and Radiation.

Consider an equilibrium configuration in the $E-b$ plane, say at point F higher than point B in figure 1a. Let us remove energy by small amounts so that the system stays practically in equilibrium and evolution takes place along the linear series $b = b(E)$. At energies $E < E_B$, the entropy of pure
radiation $S_{\text{rad}}$ is greater than the entropy of the composite system $S$. This is true only if one does not add a constant to the black hole $S_{\text{bh}}$ which will shift the point B. We emphasis that the addition of this constant, on the contrary, does not affect our computations of the mean fluctuations between B and C nor the fact that the black hole will evaporate at C. For $\mathcal{E} < \mathcal{E}_B$, the temperature is higher than the temperature at $\mathcal{E}_B$, the black hole is thus superheated (Gibbons and Perry 1978). We have just shown in section 5 that the probability for a fluctuation to lead to the total evaporation is completely negligible as long as $\Delta u \gtrsim (\Delta u)_C \simeq 0.51 L^{-2/5}$. Thus the system will evolve almost down to $\mathcal{E}_C$ staying in these superheated states. If one remove energy below $\mathcal{E}_C$, the black hole cannot survive in equilibrium anymore and will evaporate into radiation to a $b = b_{C,\text{rad}} = 0.082 L^{-2/5}$, that is, with a temperature given by $5^{1/4} T_C$, since $4/5$ of the energy was in the black hole. Removing more energy will simply cool the radiation down.

Now if one reverses the process and starts to add energy to the cavity, say from point G in figure 1a, the evolution takes place along the linear series $b(\mathcal{E}_{\text{rad}})$. The radiation heats up and reaches the point $B_{\text{rad}}$ where $\mathcal{E}_{B,\text{rad}} = \mathcal{E}_B$. Beyond this point the radiation finds itself also in a superheated state. The chances to form a black hole at those energies are exponentially small (see also Piran and Wald 1982). The radiation will thus continue to evolve (from figure 1a to figure 1b) in that superheated state as long as $|\Delta u| \gtrsim |\Delta u|_A \simeq 0.22 L^{-1/2}$, that is, almost up to point A. It will never become supercooled because $b_A \simeq 0.14 L^{-1/2} < b_{B,\text{rad}} \simeq 0.20 L^{-2/5}$ for $L > 10^6$. Near point A, the radiation will collapse and form, near point $A_{\text{bh}}$, a black hole in equilibrium with the left over radiation.

One has thus a closed circuit of equilibrium configurations which can be experienced counterclockwisely only.

(b) Black-Holes in Rotating Cavities — Self-Gravity Neglected.

If we compare figures 1 and 2, we see that rotation does not modify very much the above picture. Even for fast rotation, significant differences occur only in the late stages of evolution before evaporation. As one approaches point C in figure 2, the black hole has already lost most of its energy (see figure 3). The higher the angular momentum, the smaller this energy. In evolutionary sequences with decreasing energy of fast rotating cavities, evaporation of black holes will go almost unnoticed, the first order phase transition being very mild. Since we have not taken into account the self-gravity of
the radiation, one does not find the equivalent of the point A nor the closed circuit.

(ii) Canonical Ensembles

(a) Non-rotating Black Hole in a Cavity

If one assume that one may control the temperature at infinity instead of letting the system be isolated, stability conditions can be read from the same \( b(\mathcal{E}) \) diagram but rotated \( 90^\circ \) clockwise (or equivalently one looks for horizontal tangents). Looking at figure 1, we observe the following situation. At low temperature, high \( b \), with very little radiation, the lonely black hole cannot be stable since it has negative heat capacity. Then the \(-\mathcal{E}(b)\) curve rotates counterclockwise and the sequence of inwards spiraling configurations will become more unstable each time one encounters a vertical tangent. Thus black holes in a cavity with radiation at fixed \( b \) are always unstable. We mention that York (1985) has considered cavities with fixed temperature at the boundary rather than fixed temperature as measured from infinity (see equation (A.3) in the Appendix). This leads to a different Legendre transformation which defines another free energy. Therefore, it is not surprising that this other ensemble has different stability limits when the energy approaches the Schwarzschild limit.

One should, however, question the physical relevance of canonical configurations in general wherein self-gravitating effects are important. This is because one cannot ignore the self-gravity of the reservoir needed to fix the temperature. Indeed, in order to maintain properly the temperature fixed, the reservoir has to be large compared to the system itself. Thus the reservoir would be within its own Schwarzschild radius. It appears therefore that the canonical situations have hardly any physical relevance.

Pure radiation behaves quite differently. Black body radiation in a cavity at low temperature (and hence small mass) is stable. If we slowly heat the cavity, the sequence of evolution is the \(-\mathcal{E}_{\text{rad}}(b)\) curve of figure 1. Equilibrium configurations are all stable up to a temperature \( T_D \) or down to point \( D \) where \( C_V \) becomes negative. For \( b \) below \( b_D \) there is no equilibrium configuration and at higher energies the configurations are all unstable, since \(-\mathcal{E}_{\text{rad}}(b)\) spirals inwards counterclockwisely.

(b) Black Holes in Rotating Cavities — Self-Gravity Neglected

Black holes in rotating cavities and in a heat bath are as dull as non-rotating black holes. They
are all unstable. The situation changes, however, somewhat when $J \gtrsim 1/40$. Then, as can be seen on figure 2b rotated clockwise $90^\circ$, two vertical tangents appear at point $X$ and $Y$ enclosing a narrow range of energies. At high $b$, i.e. $b > b_X$, the canonical ensemble is certainly unstable, the specific heat being negative. Since we know that the microcanonical ensemble is stable for those configurations, the number of negative Poincaré coefficients is only one (see point (d) of section 2). From the point $X$, the system will stay stable if we slowly change the temperature as to decreasing the energy from $\mathcal{E}_X$ to $\mathcal{E}_Y$. At point $Y$ we reach again a vertical tangent. The counterclockwise turn of the line signals that instability is back. All equilibrium configurations at higher temperatures are thus unstable. Notice that the “stabilizing” effect of the angular momentum appears in a domain where general relativity begins to be important.

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Appendix A: On self-gravitating radiation in a box

The properties of global equilibrium configurations of a self-gravitating, finite-sized, non-rotating, spherical symmetric fluid with \textit{no black hole in the center} has been studied for various reasons by Klein (1947), Chandrasekhar (1972), Sorkin \textit{et al}. (1982) and Page (1992). The equation of state of a gas of photons ($n = 1$) is

$$P = \frac{1}{3}\rho$$

(A.1)

where $\rho$ is the mass-energy density. Under local thermodynamic equilibrium conditions [see equation (38)], $\rho$ is given by

$$\rho = \frac{\pi^2}{15}T^4(r)$$

(A.2)

in Planck units where $T(r)$ is the local temperature function of $r$, $T(r) = T_{900}(r)^{-1/2}$ where $g_{900}$ is the
$r$-dependent lapse function which relates the local proper time to the asymptotic one. In particular, on the surface of the cavity, one has

$$T(L)\sqrt{1 - \frac{2E_{\text{rad}}(L)}{L}} = T(L)\sqrt{1 - 2E_{\text{rad}}(L)} = T = \frac{1}{\beta}. \quad (A.3)$$

where $T$ is the “temperature as measured at infinity”. The distribution $\rho(r)$ is given by the Tolman-Oppenheimer-Volkoff equation which can be solved as follows (Sorkin et al. 1981). Set

$$q = 4\pi r^2 \rho, \quad \mu = \frac{4\pi}{r} \int_0^r dr' r'^2 \rho'. \quad (A.4)$$

Then the TOV equation reads

$$\frac{dq}{d\mu} = \frac{2q(1 - 4\mu - (2/3)q)}{(1 - 2\mu)(q - \mu)}. \quad (A.5)$$

Initial conditions are $\mu = q = 0$, this leads to a family of solutions parametrized by the density at $r = 0$. On the boundary $r = L$

$$q(L) = 4\pi L^2 \rho(L), \quad \mu(L) = E_{\text{rad}}. \quad (A.6)$$

The second equation is used to express the density at $r = 0$ in terms of $E_{\text{rad}}$. Thus, the solution of equation (A.5) defines, with (A.3) and (A.6) the function $b(E_{\text{rad}})$

$$b^4 = \frac{3\sigma L^{-2}}{q(1 - 2E_{\text{rad}})^2}. \quad (A.8)$$

where $q$ is a function of $E_{\text{rad}}$.

When self gravity is negligible ($q \simeq 3\mu \ll 1$), $b(E_{\text{rad}})$ reduces to the expression valid in flat space time $E_{\text{rad}} = \sigma VT^4$. When $E_{\text{rad}} > 0.1$, self gravity is important and $E_{\text{rad}}(b)$ is given by (A.8). Special points are given in the table (A.9); the limit point $Z$ of figure 1 is where the density at $r = 0$ is infinite.

| Special points* | D  | A  | E  | Z  |
|----------------|----|----|----|----|
| $E_{\text{rad}}$ | 0.123 | 0.246 | 0.235 | 0.214 |
| $b \simeq L^{-1/2} \times$ | 0.107 | 0.135 | 0.140 | 0.132 |
| $T \simeq L^{-1/2} \times$ | 0.372 | 0.295 | 0.284 | 0.301 |

*See figure 1. For comparison, $E_C \simeq 0.154 L^{-2/5}$ and $b_{E_{\text{rad}}} \simeq 0.082 L^{-2/5}$.

When a black hole is present in the center of the cavity, (A.1) and (A.2) are no more valid. One has to take into account the quantum behavior of the mass energy density since the Tolman relations certainly break down near the horizon. The quantum version of the density is provided by
the expectation value of the time-time component of the stress energy tensor in the so-called Hartle-Hawking “vacuum” (Howard 1984). One finds first that the contribution to the gravitational mass for $2M_{bh} < r < 6M_{bh}$ is of the order of $M_{bh}^{-1}$ (in our units, it means $b^{-1}$) and secondly that for $r > 6M_{bh}$ one may approximate the mass energy density by (A.2). Thus one may approximate (3) by

$$m(r, M_{bh}) = M_{bh} + \int_{6M_{bh}}^{r} dr' r'^2 M_{bh}^4 (1 - 2M_{bh}/r')^{-2}.$$  \hfill (A.10)

(again for cavities with $L > 10^6$). We have also neglected in (A.10) the self-gravity of the radiation.

For $b > b_C$, this is certainly valid since $E_{rad} < E/5$ and since $E_C \ll L/2$. For smaller black holes ($E_{bh} < (4/5)E$), the configurations are unstable and it is therefore useless to calculate the corrections. From (A.10) and for $b > b_C$, it is easy to verify that the corrections to $b(E)$ are small and have no effect on thermodynamic stability limits. The reader might consult Page (1992) to find an explicit evaluation of corrections to the entropy.

We now give the derivation of $(C_V)_A$ appearing in equation (52). We use equation (A.8) with equation (A.6) and obtain the following expression for $d\mathcal{E}/db$ (we now drop the index $rad$)

$$\frac{3\sigma L^{-2}}{4b^3} \frac{d\mathcal{E}}{db} = \frac{q(1 - 2\mathcal{E})^3(q - \mathcal{E})}{2(\frac{8}{3}q - 1 + 2\mathcal{E})}. \hfill (A.11)$$

Using again (A.6) with (49), we also obtain for $dq/db$

$$\frac{3\sigma L^{-2}}{4b^3} \frac{dq}{db} = \frac{q^2(1 - 2\mathcal{E})^2(1 - 4\mathcal{E} - \frac{2}{3}q)}{(\frac{8}{3}q - 1 + 2\mathcal{E})}. \hfill (A.12)$$

These two equations give us a means to calculate the second derivative of $d^2\mathcal{E}/db^2$

$$\left(\frac{3\sigma L^{-2}}{4b^3} \frac{d^2q}{db^2}\right) = \frac{\partial}{\partial q} \left[\frac{q(1 - 2\mathcal{E})^3(q - \mathcal{E})}{2(\frac{8}{3}q - 1 + 2\mathcal{E})}\right] \left(\frac{dq}{db}\right)_A. \hfill (A.13)$$

At point A, the first derivative is zero

$$(−C_V)_A = 64\pi L^2 b_A \frac{E_A(1 - 2\mathcal{E}_A)}{1 - \frac{3}{4}\mathcal{E}_A} \Delta u. \hfill (A.14)$$

With $b_A$ given in the above table, one has thus

$$(−C_V)_A \simeq 23 L^{3/2} (−\Delta u). \hfill (A.15)$$
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Figure captions

**Figure 1.** $b(\mathcal{E}_{bh})$, $b(\mathcal{E}_{rad})$ and $b(\mathcal{E})$ are drawn for $L = 10^6$, in figure 1a for $\mathcal{E} \lesssim 10^{-3}$ and in figure 1b for $\mathcal{E} \gtrsim 10^{-1}$. Between $\mathcal{E} = 10^{-3}$ and $10^{-1}$, $b(\mathcal{E}_{rad})$ and $b(\mathcal{E})$ come closer and closer to the point that in figure 1b the two lines are indistinguishable. The line $b(\mathcal{E})$ through points FCQDAZ is a counterclock inward spiral. The dotted line in the lower left hand corner of figure 1b is the non-relativistic (no selfgravity) $b(\mathcal{E}_{rad})$ for comparison.

**Figure 2.** $b(\mathcal{E}_{bh}, \mathcal{J})$, $b(\mathcal{E}_{rad}, \mathcal{J})$ and $b(\mathcal{E}, \mathcal{J})$ curves for constant angular momentum $\mathcal{J}$ and $L = 2.65 \cdot 10^4$. The low value of $L$ is to make clear figures. Figure 2a is for $\mathcal{J} = 1/40$ and figure 2b is for $\mathcal{J} = 1/8$. Both lines are drawn with the same limits of $b$ and $\mathcal{E}$, showing the displacement to right for increasing $\mathcal{J}$ of the linear series $b(\mathcal{E}_{rad}, \mathcal{J})$, $b(\mathcal{E}, \mathcal{J})$ and of the point C. Once the equilibrium configurations leave the linear series $b(\mathcal{E}_{bh}, \mathcal{J})$, the black hole starts to loose mass with respect to the radiation, see figure 3.

**Figure 3.** This figure displays two lines: (a) The minimum of energy $\mathcal{E}_C(\mathcal{J})$ as a function of increasing angular momentum. The scale of $\mathcal{E}_C$ displayed on the left is the same as in figure 2. The curve is parametrized in real values of $\mathcal{E}_C$. Notice that beyond $\mathcal{J} = 3/16 \approx 0.2$, the energy $\mathcal{E}_C$ is highly relativistic and the non-relativistic curve is likely to be different. (b) The ratio $\mathcal{E}_{bh,C}/\mathcal{E}_C = b_C/\mathcal{E}_C$ with its scale displayed on the right. For $\mathcal{J} = 0$, $\mathcal{E}_{bh,C}/\mathcal{E}_C = 4/5$; this is well above the limits of the drawing.
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