Competitive Control with Delayed Imperfect Information

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Abstract

This paper studies the impact of imperfect information in online control with adversarial disturbances. In particular, we consider both delayed state feedback and inexact predictions of future disturbances. We introduce a greedy, myopic policy that yields a constant competitive ratio against the offline optimal policy with delayed feedback and inexact predictions. A special case of our result is a constant competitive policy for the case of exact predictions and no delay, a previously open problem. We also analyze the fundamental limits of online control with limited information by showing that our competitive ratio bounds for the greedy, myopic policy in the adversarial setting match (up to lower-order terms) lower bounds in the stochastic setting.

1 INTRODUCTION

The design and analysis of controllers with imperfect information is a long-standing challenge for the fields of online learning and robust control. This paper studies the impact of imperfect information in the context of a disturbed linear dynamical system with feedback delay. Specifically, we consider a disturbed online Linear Quadratic Regulator (LQR) optimal control problem with state feedback delay and inexact predictions of future disturbances, governed by $x_{t+1} = Ax_t + Bu_t + w_t$, where $x_t$, $u_t$, and $w_t$ are the state, control, and disturbance respectively.

A growing literature at the interface of learning and control has emerged in recent years with the goal of designing controllers under various criteria, such as regret\textsuperscript{1} \cite{Dean2018,Agarwal2019a}, dynamic regret\textsuperscript{1} \cite{Li2019,Yu2020}, and competitive ratio\textsuperscript{1} \cite{Shi2020,Goel2019}. However, this line of work has made little progress when it comes to using imperfect prediction information, and has not approached the challenge of delayed feedback at all. In fact, for even the case of perfect information and no delay, the question of whether there exists a constant competitive policy for general LQR systems is unresolved.

Contributions.

In this paper, we show that a simple,
myopic, and practical policy that generalizes model predictive control (MPC) obtains a constant competitive ratio bound in the case of delayed imperfect information (Theorem 3). We also show that the competitive ratio bound exponentially increases (decreases) as the amount of delay (prediction) increases, which highlights the cost associated with delay and the power of predictions, even when they are inexact. To the best of our knowledge, this result represents the first constant competitive bound for general LQR control with adversarial disturbance, even in the case of exact predictions with no delay. Further, it represents the first finite-time performance bounds (regret or competitive ratio) for either the case of inexact predictions or delayed feedback. Additionally, we prove that this myopic policy is near-optimal in terms of competitive ratio, by showing that our competitive ratio bounds for the myopic policy in the adversarial setting match the lower bounds in the stochastic setting.

We would like to emphasize the generality of our result. The model we consider is the general LQR setting with bounded adversarial disturbance in the dynamics, where only stabilizability is assumed. Further, the prediction errors are assumed to be adversarial. Additionally, our results compare to the globally optimal policies without any constraints, rather than compare to the optimal linear or static policy.

Our result adds further evidence that the structure of LQR allows simple algorithmic ideas to be effective: Simchowitz and Foster (2020) recently proved that the naive exploration is optimal in online LQR adaptive control problem with unknown \((A, B)\), and Yu et al. (2020) proved the classic MPC is near-optimal in online LQR control with exact future predictions. Combined with the current paper, there is growing evidence that simple, myopic policies that build on MPC are constant-competitive and near-optimal, even in adversarial settings with delayed imperfect information, which sheds light on key algorithmic principles and fundamental limits in continuous control.

**Related work.** There is a growing literature of papers that approach the control of linear dynamical systems with tools and concepts from machine learning. Within this literature, most work focuses on the design of controllers with low regret (Dean et al., 2018; Agarwal et al., 2019; Simchowitz and Foster, 2020). While the study of competitive ratio has received some attention (Goel and Wierman, 2019; Shi et al., 2020), only special LQR systems are covered and results have been more difficult to obtain.

In these lines of work, very few papers focus on settings where the controller has access to predictions of future disturbances. The few examples of results in the case of predictions, focus on settings where predictions are exact (Yu et al., 2020). For example, Li et al. (2019) analyzed dynamic regret with predictions of future cost functions, but the predictions are exact and there is no disturbance in the dynamics. Even outside of control in the related area of online optimization, when predictions are considered, they are typically assumed to be exact (Lin et al., 2019). One exception is (Chen et al., 2015), which uses a less general model of prediction error than the current paper, but the connection to control is unclear.

In contrast to the literature on predictions, there is no work studying the regret or competitive ratio of policies subject to delayed feedback. The issue has received considerable attention in the control community (Kim and Park, 1999; Bejczy et al., 1990), but the focus is typically on stability and no finite-time performance bounds exist to this point to the best of our knowledge.

**2 MODEL**

We consider an online Linear Quadratic Regulator (LQR) optimal control problem with adversarial disturbances in the dynamics. In particular, we consider a linear system initialized with \(x_0 \in \mathbb{R}^n\) and controlled by \(u_t \in \mathbb{R}^m\) at each step \(t \in \{0, 1, \ldots, T-1\}\) where \(T\) is the total length of the problem. The system dynamics is governed by:

\[
x_{t+1} = Ax_t + Bu_t + w_t,
\]

where \(w_t\) is the disturbance. We assume that \(w_t\) is bounded, i.e., \(\|w_t\| \leq r\). The goal of the controller is to minimize the following cost:

\[
J = \sum_{t=0}^{T-1} (x_t^\top Q x_t + u_t^\top R u_t) + x_T^\top Q_f x_T,
\]

given matrices \(A, B, Q, R, Q_f\). We consider an online setting where an adversary selects \(\{w_t\}_{t=0}^{T-1}\) in an adaptive manner, and the controller makes the decision \(u_t\) at every time step \(t\), potentially based on delayed imperfect information (see Section 2.1).

We make the standard assumptions that \(Q, Q_f \succeq 0, R \succ 0\), and \((A, B)\) is stabilizable (Goel and Hassibi, 2020; Yu et al., 2020), i.e., \(\exists K_0 \in \mathbb{R}^{m \times n}\) such that \(\rho(A - BK_0) < 1\). We further assume \((A^\top, Q)\) is stabilizable to guarantee the stability of the closed-loop (Anderson and Moore, 2007). This assumption is more general than the standard assumption that \(Q \succ 0\), i.e., \(Q \succ 0\) implies the stabilizability of \((A^\top, Q)\).

Throughout this paper, we use \(\rho(\cdot)\) to denote the spectral radius of a matrix and \(\|\cdot\|\) to denote the 2-norm of a vector or the spectral norm of a matrix.
Note that many important problems can be seen to be special cases of the model described above. Two motivating examples are input-disturbed systems and the Linear Quadratic (LQ) tracking problem (Anderson and Moore, 2007).

**Example: Linear Quadratic (LQ) Tracking.** The LQ tracking problem is defined via dynamics $x_{t+1} = Ax_t + Bu_t + \tilde{w}_t$ and cost function:

$$J = \sum_{t=0}^{T-1} (x_{t+1} - y_{t+1})^T Q(x_{t+1} - y_{t+1}) + u_t^T R u_t,$$

where $\{y_t\}^{T=1}$ is the desired trajectory to track. To fit this into our model, let $\hat{x}_t = x_t - \hat{y}_t$. Then, we get $J = \sum_{t=0}^{T-1} \hat{x}_{t+1}^T Q \hat{x}_{t+1} + u_t^T R u_t$ and $\hat{x}_{t+1} = Ax_t + Bu_t + \tilde{w}_t$, which is a LQR control problem with disturbance $\tilde{w}_t = \hat{w}_t + Ay_t - \hat{y}_{t+1}$ in the dynamics. Note that in many LQ tracking problems, delayed observations and imperfect predictions are fundamental challenges (Anderson and Moore, 2007).

### 2.1 Delayed Imperfect Information

The LQR optimal control problem introduced above is typically studied without predictions or delays. Classically, at each time $t \geq 0$, the controller observes $x_t$ and then decides $u_t$ without knowing $w_t$. In other words, the controller is given $x_0$ to start and then, at each time, it decides $u_t$ before obtaining $w_t$. Thus, $u_t$ is a function of all the previous information: $u_t = \pi_t(x_0, u_0, u_1, \ldots, u_{t-1}, w_1, \ldots, w_{t-1})$, or equivalently, $u_t = \pi_t(x_0, u_0, u_1, \ldots, u_{t-1}, x_1, \ldots, x_t)$.

The motivation for the work in this paper is that in many real-world problems (Shi et al., 2019; Lazic et al., 2018), predictions of future information are available and using them is crucially important, even though they are typically noisy. For example, data-driven model-based control is a prominent and successful approach where it is crucial to consider model mismatch due to statistical learning error. Further, in many situations, there is feedback delay in the system which means the controller must make decisions $u_t$ before the current state $x_t$ is observed. The addition of feedback delay leads to considerable additional difficulty.

Formally, to model the delayed inexact predictions that are common in many applications, we model the revealed information at time $t$ as follows. At time step $t$, the revealed information is:

$$x_0, u_0, \ldots, u_{t-1}, \tilde{w}_t, \ldots, \tilde{w}_T,$$

where $d \geq 0$ is the feedback delay, and $\tilde{w}_{t+1}$ is the potentially inexact prediction of $w_t$ at time $t$.

We use $e_{t-d+i|t} = w_{t-d+i} - \tilde{w}_{t-d+i|t}$ ($i \geq 0$) to represent the estimation error and we assume the predictor satisfies $|e_{t-d+i|t}| \leq \epsilon_i |w_{t-d+i|t}|$ for all $i$ and $t$. Thus, $\epsilon_i$ measures the quality of the predictor. Note that we may assume $0 \leq \epsilon_i \leq 1$ for all $i$ since if $\epsilon_i > 1$ for some $i$, we can simply let $\tilde{w}_{t-d+i|t} = 0$ and then $\epsilon_i = 1$ (if $\tilde{w}_{t-d+i|t} = w_{t-d+i|t} = 0$ we define $\epsilon_i = 0$). Additionally, note that, although predictions are available for every time step, predictions that are far future may have bad quality, i.e., $\epsilon_i$ is typically large for large $i$. Therefore, a good control policy may not use all predictions in the same way, i.e., only using the predictions with smaller estimation error may yield better performance.

The **delayed imperfect information** setting we consider generalizes many existing settings for studying LQR control. The classic setting is the special case where $d = 0$ and $\tilde{w}_{t+i|t} = 0$ for all $i$ and $t$. The offline optimal setting considered by Goel and Hassibi (2020) is the special case where $d = 0$ and $\epsilon_i = 0$ for all $i$. The setting considered by Yu et al. (2020), where exact predictions without delay are available, corresponds to $d = 0$, $\epsilon_i = 0$ for $0 \leq i \leq k - 1$ and $w_{t+i|t} = 0$ for all $t$ and $i \geq k$.

### 2.2 A Myopic Policy

In this paper, we study a myopic policy that extends the classic model predictive control (MPC) approach to the setting of delayed imperfect information.

When there are predictions, but no delays, MPC is a popular and successful approach (Lazic et al., 2018; Camacho and Alba, 2013). In fact, Yu et al. (2020) recently showed that MPC has a near-optimal dynamic regret in the case of exact predictions and no delay. To define MPC formally, suppose the controller uses $k$ predictions. At each time step $t$, the controller optimizes based on $x_t, \tilde{w}_t, \ldots, \tilde{w}_{t+k-1}$:

$$(u_t, \ldots, u_{t+k-1}) = \arg \min_u \left( \sum_{i=t}^{t+k-1} (x_i^T Q x_i + u_i^T R u_i) + x_{t+k}^T \tilde{Q} x_{t+k} \right),$$

s.t. $x_{t+1} = Ax_t + Bu_t + \tilde{w}_{t|t}$, $t \leq i \leq t + k - 1$. (2)

This optimization is myopic in the sense that it assumes that the length of the problem is $k$ instead of $T$ and only uses predicted future disturbances within those $k$ steps. The terminal cost matrix $\tilde{Q}_t$ in Equation (2) may or may not be the same as the terminal cost matrix $Q_f$ of the original problem (Equation (1)).
and can be viewed as a hyper-parameter of the algorithm. Similarly, \( k \) is also a hyper-parameter. Larger \( k \) is not necessarily better because the predictions in the far future may have very large errors. In this paper, we let \( \hat{Q}_t = P \), where \( P \) is the solution of the discrete algebraic Riccati equation (DARE):

\[
P = Q + A^T PA - A^T PB(R + B^T PB)^{-1}B^T PA. \tag{3}
\]

The output of Equation (2) is \( k \) control actions corresponding to time \( t, t + 1, \ldots, t + k - 1 \), respectively, but only the first (\( u_t \)) is applied to the system. The rest (i.e., \( u_{t+1}, \ldots, u_{t+k-1} \)) are discarded. The explicit solution of the MPC optimization Equation (2) can be computed and is given below (Yu et al. 2020).

**Proposition 1.** The MPC policy at time \( t \) is:

\[
u_t = -(R + B^T PB)^{-1}B^T \left( PAx_t + \sum_{i=0}^{k-1} F_i^T P \hat{w}_{t+i|t} \right),
\]

where \( F = A - B(R + B^T PB)^{-1}B^T PA =: A - BK \).

Note that \( \rho(F) < 1 \), i.e., the closed loop is stable (Yu et al. 2020). As stated above, MPC does not directly apply to the case of delayed imperfect information. To adapt it, we consider two cases: (i) when the number of predictions available is longer than the feedback delay, i.e., \( k \geq d \), and (ii) when the delay is longer than the number of predictions available, i.e., \( k < d \).

When \( k \geq d \), the extension is perhaps straightforward. Here, although the controller does not know the current state \( x_t \), it knows \( x_{t-d} \) and \( \hat{w}_{t-d|t}, \ldots, \hat{w}_{t-1|t} \). Thus, it can estimate the current state. This means that it is possible to simply use this estimation, \( \hat{x}_{t|t} \), as a replacement for \( x_t \) in the algorithm, which yields the following:

\[
u_t = -(R + B^T PB)^{-1}B^T \left( PA\hat{x}_{t|t} \right)
\]

When \( k < d \), the extension is not as obvious. In this setting, the quality of the predictions is poor enough that it is better not to use the predictions to estimate the current state. Thus, one cannot simply estimate the current state and run classic MPC. In this case, the key is to view (classic) MPC from a different perspective: MPC locally solves an optimal control problem by treating known disturbances (using predictions) as exact, and treating unknown disturbances as zero (Yu et al. 2020; Camacho and Alba 2013). This view highlights the fact that, underlying MPC is the assumption that predictions are exact. Following this philosophy, in the case when predictions are not enough to be used to estimate the current state, we can instead assume that unknown disturbances are exactly zero. The following theorem derives the optimal policy under this “optimistic” assumption.

**Theorem 2.** Suppose there are \( d \) delays and \( k \) exact predictions with \( k < d \). Assume all used predictions are exact and other disturbances (with unused predictions) are zero. The optimal policy at time \( t \) is:

\[
u_t = -(R + B^T PB)^{-1}B^T PA \left( A^{d-k} \hat{x}_{t-d+k|t} \right.
\]

\[
+ \sum_{i=0}^{d-k-1} A^i B u_{t-1-i} \left). \tag{5}
\]

In other words, the policy in Equation (5) first obtains the greedy estimation \( \hat{x}_{t-d+k|t} \) using predictions \( \hat{w}_{t-d|t}, \ldots, \hat{w}_{t-d+k-1|t} \), and then estimates the current state by treating \( u_{t-d+k} = \cdots = u_{t-1} = 0 \). In fact, instead of treating them as zero, we can impose other values or distributions on those disturbances. This would generalize Theorem 2 to a broader class of policies.

To summarize the two cases above, the myopic generalization of MPC we study in this paper is described as follows. Suppose we want to use \( k \) predictions. If \( k \geq d \), then we estimate the current state \( x_t \) and apply Equation (1). If \( k < d \), then we estimate the state at time \( t - d + k \) and apply Equation (5). If fact, the two cases coincide when \( k = d \).

### 2.3 Performance Metric

To study the performance of the policy described above, we analyze its competitive ratio, which bounds the worst-case ratio of the cost of the online algorithm (Alg) compared to the optimal offline cost (Opt) with the full exact knowledge of \( \{w_t\}_{t=0}^{T-1} \). Formally, we study the so-called weak competitive ratio, which allows for an additive constant factor. In particular, we say that the policy is (weakly) \( c \)-competitive when \( \text{Alg} \leq c \text{Opt} + O(1) \) for any fixed \( A, B, Q, R, r \), and for any adversarially and adaptively chosen disturbances \( \{w_t\}_{t=0}^{T-1} \). When the competitive ratio \( c \) is a constant, independent of \( T \) we say that the algorithm is constant competitive.

While there has been considerable success in recent years designing control policies that are no-regret (including dynamic regret), e.g., Dean et al. (2018); Agarwal et al. (2019); Yu et al. (2020), there have been very few examples of constant competitive controllers. The few results that exist, e.g., Shi et al. (2020); Goel and Wierman (2019), tend to have restrictive assumptions on the dynamics and/or disturbances. This is...
because the study of competitive ratio adds difficulty in a few dimensions. In particular, results studying regret tend to focus on regret compared to the offline optimal static linear policy [Agarwal et al., 2019], while the competitive ratio directly compares to the optimal offline (potentially non-linear and non-static) policy (Shi et al., 2020). Characterizing the optimal offline policy is known to be difficult. In fact, the optimal static linear policy can have cost that is an order-of-magnitude larger than the optimal offline cost, which makes achieving a constant competitive ratio more challenging (Shi et al., 2020).

### 3 RESULTS

Our main result provides bounds on the competitive ratio for a generalized form of MPC (Section 2.2) in the case of inexact delayed predictions. To our knowledge, this represents the first constant-competitive bound for a policy in the general online LQR setting, even without considering delayed feedback or inexact predictions. We present our general result below and then discuss the special cases of (i) exact predictions and no delay, (ii) inexact predictions and no delay, and (iii) delay but no access to predictions. The special cases illustrate the contrast between inexact and exact predictions as well as the impact of delay.

**Theorem 3** (Main result). Let $c = \|P\|P^{-1}\|F\|\|1 + F\|$ and $H = B(R + B^\top PB)^{-1}B^\top$. Suppose there are $d$ steps of delays and the controller uses $k$ predictions. When $k \geq d$,

$$
\text{Alg} \leq \left( \frac{c \sum_{i=0}^{d-1} \epsilon_i \|A^{d-i}\| + c \sum_{i=d}^{k-1} \epsilon_i \|F^{i-d}\| + \|F^{k-d}\|}{\|H\|^{-1} \lambda_{\min}(P^{-1} - FP^{-1}F^\top - H)} \right)^2 + 1 \text{Opt} + O(1).
$$

When $k \leq d$,

$$
\text{Alg} \leq \left( \frac{\sum_{i=0}^{k-1} \epsilon_i \|A^{d-i}\| + \sum_{i=k}^{d-1} \|A^{d-i}\| + 1}{\|H\|^{-1} \lambda_{\min}(P^{-1} - FP^{-1}F^\top - H)} \right)^2 + 1 \text{Opt} + O(1).
$$

The $O(1)$ is with respect to $T$. It may depend on the system parameters $A, B, Q, R, Q_f$ and the range of disturbances $r$, but not on $T$. When $Q_f = P$ the $O(1)$ is zero.

The two cases in the theorem correspond to the two cases in the algorithm: when predictions are of high enough quality to allow estimation of the current state and when they are not. Note that the closed-loop dynamics is stable, i.e., $\rho(F) = \rho(A - BK) < 1$. Therefore, there exists a constant $\gamma$ such that $\|F^i\| \leq \gamma^{(\rho(F)+1)i}$ for all $i \geq 1$ from Gelfand’s formula.

In the first case, we see that the quality of predictions in the near future has more impact, especially when $\rho(A) > 1$. In the second case, we see that the amount of delay $d$ exponentially increases the bound if $\epsilon_i > 0$ and $\rho(A) > 1$. We explore the insights that follow from the bound further by looking at special cases in the subsections that follow but, before moving to the special cases, we provide an overview of the proof of Theorem 3.

**Proof Sketch.** We first prove Theorem 3 in the case $Q_f = P$. In this case, the $O(1)$ is not needed. Then, we analyze the impact of the terminal cost $Q_f$ and show that it introduces at most an $O(1)$ additional cost.

**Lemma 4.** Suppose $Q_f = P$. Then, the conclusion of Theorem 3 holds.

The proof of the result in the case of $Q_f = P$ can be found in Appendix and follows from a novel difference analysis of the quadratic cost-to-go functions. Here, we focus on the second part of the proof, i.e., reducing the case of $Q_f \neq P$ to the case of $Q_f = P$. To that end, let $\text{Alg}(X)$ be the cost of our algorithm when the terminal cost is $X$ and, similarly, let $\text{Opt}(Y)$ be the cost of the policy that is optimal for terminal cost $Y$ when the terminal cost is actually $X$.

Our analysis proceeds by first bounding the impact of the terminal cost on the gap between the algorithm and the optimal cost.

**Lemma 5.** For any algorithm,

$$
\text{Alg}(Q_f) - \text{Opt}^P(Q_f) \leq \text{Alg}(P) - \text{Opt}^P(P) + O(1).
$$

Then, we prove that the terminal cost only has an $O(1)$ impact on the optimal cost.

**Lemma 6.** The followings are equal up to $O(1)$ difference: $\text{Opt}^P(P), \text{Opt}^P(Q_f), \text{Opt}^Q(Q_f)$.

Together, these imply that

$$
\text{Alg}(Q_f) - \text{Opt}^Q(Q_f) \\
\leq \text{Alg}(Q_f) - \text{Opt}^P(Q_f) + O(1) \\
\leq \text{Alg}(P) - \text{Opt}^P(P) + O(1) \\
\leq \frac{\text{Alg}(P) - \text{Opt}^P(P)}{\text{Opt}^P(P)} \text{Opt}^P(P) + O(1) \\
\leq \frac{\text{Alg}(P) - \text{Opt}^P(P)}{\text{Opt}^P(P)} \text{Opt}^Q(Q_f) + O(1).
$$
Thus, we can complete the proof by concluding that
\[ \text{Alg}(Q_f) \leq \frac{\text{Alg}(P)}{\text{Opt}(P)} \text{Opt}(Q_f) + O(1). \]

### 3.1 Exact Predictions Without Delay

In the sections that follow, we explore special cases of Theorem 3 in order to highlight the impact of inexact predictions and delay. First, we present the special case of exact accurate, exact predictions and no feedback delays. Formally, we have \( d = 0, \epsilon_i = 0 \) for \( 0 \leq i \leq k-1 \) and \( \hat{w}_{t+i} = 0 \) for all \( t \) and \( i \geq k \).

The main result for this setting is given below. It directly follows from the \( k \geq d \) case of Theorem 3.

**Theorem 7.** Suppose there are \( k \) exact predictions and no feedback delay. Then:
\[ \text{Alg} \leq \left[ 1 + \frac{\|F^k\|^2H}{\lambda_{\min}(P^{-1} - FP^{-1}F^\top - H)} \right] \text{Opt} + O(1). \]

Thus, the competitive ratio exponentially decreases as \( k \) goes up. To illustrate how the primary parameters \( A, B, Q \), and \( R \) affect the competitive ratio in this case, it is useful to consider the case when \( n = m = 1 \), as shown in both Corollary 7.1 and Figure 1.

**Corollary 7.1.** Assume there are \( k \) exact predictions and no feedback delay, and let \( n = m = 1 \) and \( Q_f = P \).

- If \( A^2 \gg B^2Q/R + 1 \),
  \[ \frac{\text{Alg}}{\text{Opt}} \leq 1 + \frac{2A^{4-2k}}{B^2Q/R}. \]

- If \( B^2Q/R \gg A^2 + 1 \),
  \[ \frac{\text{Alg}}{\text{Opt}} \leq 1 + \frac{A^{2k}}{(B^2Q/R)^{2k-1}}. \]

Interestingly, in this case, the competitive bound only depends on \( A^2 \) and \( B^2Q/R \). It does not depend on the sign of \( A \), nor on \( B, Q \) or \( R \) as long as \( B^2Q/R \) is fixed. Further, when \( k \geq 3 \), we see that the competitive ratio is small if \( B, Q \) are small, \( R \) is large, or \( A \) is either very large or very small. However, when \( k = 0 \) or \( 1 \), a large \( A \) can result in a large competitive ratio. When \( k = 0 \), a large value of \( B^2Q/R \) also results in a large competitive ratio. We see below that this phenomenon is similar to the case of delay (see Section 3.3).

Theorem 7 is tight in the sense that there exist systems where the competitive ratio is \( 1 + \Theta(\|F^k\|^2) \) (see Appendix).

### 3.2 Inexact Predictions Without Delay

We next consider the case where predictions are inexact, but there is no feedback delay. The contrast with the previous section highlights the impact of prediction error.

As discussed in Section 2.1, the controller should optimize \( k \) to utilize predictions with smaller estimation errors while avoiding the use of those with larger errors. The following directly follows from the \( d = 0 \) case of Theorem 3 and reduces to the exact case when \( \epsilon_i = 0 \).

**Theorem 8.** Suppose there are \( k \) inexact predictions and no feedback delay.
\[ \text{Alg} \leq \left[ \|H\| \left( \sum_{i=0}^{k-1} \epsilon_i \|F^i\| + \|F^k\| \right)^2 \right] \left[ \lambda_{\min}(P^{-1} - FP^{-1}F^\top - H) \right] + 1 \] \[ \text{Opt} + O(1). \]

This subsection differs from the previous one in that the controller can minimize the bound in Theorem 8 with respect to \( k \). We characterize this optimization in the following result in 1-d systems, and also provide simulation evidence in Section 4 (see Figure 1).

**Corollary 8.1.** Suppose there are \( k \) inexact predictions and no feedback delay. Assume \( n = m = 1 \). Given non-decreasing \( \{\epsilon_i\} \), to minimize the competitive ratio bound in Theorem 3, the optimal number \( k \) of predictions to use is such that:
\[ \epsilon_{k-1} < \frac{1 - |F|}{1 + |F|} < \epsilon_k. \]

As in the previous section, the one-dimensional setting highlights the dependence of the competitive ratio on the system parameters.

**Corollary 8.2.** Assume there are \( k \) inexact predictions and no feedback delay, and let \( n = m = 1 \) and \( Q_f = P \).

- If \( A^2 \gg B^2Q/R + 1 \),
  \[ \frac{\text{Alg}}{\text{Opt}} \leq 1 + \frac{2A^4}{B^2Q/R} \left( \sum_{i=0}^{k-1} \epsilon_i |A|^i + \frac{1}{|A|^k} \right)^2. \]

- If \( B^2Q/R \gg A^2 + 1 \),
  \[ \frac{\text{Alg}}{\text{Opt}} \leq 1 + \frac{B^2Q}{R} \left( \sum_{i=0}^{k-1} \epsilon_i |B|^i + \frac{1}{|B|^k} \right)^2. \]

The dependence of the competitive ratio on \( A, B, Q, R \) is similar to the case of exact predictions. In particular, we find that the prediction quality in the near future is (exponentially) more important than further into the future, which is consistent with the robust MPC literature (Cannon and Kouvaritakis 2005).

In the exact prediction case we show that Theorem 7 is tight with respect to \( \|F^k\| \). In contrast, in the inexact case the tightness of \( \epsilon_i \) and \( \|F^i\| \) in Theorem 8 remains as an open question.
3.3 Delay Without Predictions

The last special case we consider is the case with delays but no (usable) predictions. This case separates the impact of delay from that of predictions. Here, \( \bar{w}_{t-d+i+it} = 0 \) for all \( t \) and \( i \geq 0 \). When \( k \leq d \), via Theorem \( 9 \) we have that:

\[
\text{Alg} \leq \left[ \frac{\|H\|}{\lambda_{\min}(P^{-1} - FP^{-1}F^T - H)} \right] + 1 \text{ Opt} + O(1).
\]

Depending on whether the spectral radius \( \rho(A) < 1 \), there are two simplifications one can make: (i) if \( \rho(A) < 1 \), then \( \|A^i\| \leq \kappa a^i \) for \( a = (\rho(A) + 1)/2 < 1 \) for a constant \( \kappa \), and (ii) if \( \rho(A) > 1 \), \( \|A^i\| \leq \|A^i\| \). These yield the following result.

**Theorem 9.** Suppose there are \( d \) delays and no predictions are available. If \( \rho(A) < 1 \), then the competitive ratio is bounded by a constant no matter how many delays there are:

\[
\text{Alg} \leq \left[ \frac{\|H\|}{\lambda_{\min}(P^{-1} - FP^{-1}F^T - H)} \right] + 1 \text{ Opt} + O(1) \leq \left[ \frac{\|H\|}{1 - \rho(A)} \right] + 1 \text{ Opt} + O(1).
\]

If \( \rho(A) > 1 \), then the competitive ratio bound grows exponentially with the number of delay steps:

\[
\text{Alg} \leq \left[ \frac{\|H\|}{\lambda_{\min}(P^{-1} - FP^{-1}F^T - H)} \right] + 1 \text{ Opt} + O(1).
\]

As in the previous subsections, it is useful to consider the one-dimensional case to get insights about the impact of the system parameters.

**Corollary 9.1.** Assume there are \( d \) delays and no predictions are available, and let \( n = m = 1 \) and \( Q_f = 1 \).

If \( A^2 \gg B^2Q/R + 1 \),

\[
\frac{\text{Alg}}{\text{Opt}} \leq 1 + \frac{2A^2B^2Q/R}{B^2Q/R}.
\]

If \( B^2Q/R \gg A^2 + 1 \), then if \( |A| > 1 \),

\[
\frac{\text{Alg}}{\text{Opt}} \leq 1 + \frac{A^2B^2Q/R}{(|A| - 1)^2}.
\]

If \( |A| < 1 \),

\[
\frac{\text{Alg}}{\text{Opt}} \leq 1 + \frac{(1 - |A|)^2B^2Q/R}{(1 - |A|)^2} \leq 1 + \frac{B^2Q/R}{(1 - |A|)^2}.
\]

In contrast to the case with \( k \geq 3 \) inexact predictions, when there are delays, a large value of \( B^2Q/R \) or \( A \) does not lead to a small competitive ratio. Instead, in the case of feedback delay, this results in a large competitive ratio. This is consistent with results from robust control theory, i.e., the less stable the open loop is (\( |A| \) is larger), the more impact delay has (Zhou and Doyle, 1998).

Theorem 9 is tight in the sense that there exist systems such that the competitive ratio is \( 1 + \Theta(A^{2d}) \) (see Appendix).

4 NUMERICAL EXAMPLES

To illustrate our results, we end the paper by presenting numerical examples that highlight the impact of delayed, inexact predictions. To that end, we consider a 2-d tracking problem with the following trajectory (Li et al. (2019), illustrated in Figure 2):

\[
y_i = \left[ 13 \cos \left( \frac{i}{2} \right) - 5 \cos \left( \frac{2i}{7} \right) - 2 \cos \left( \frac{3i}{4} \right) - \cos \left( \frac{i}{4} \right) \right].
\]
We consider following double integrator dynamics:

\[ p_{t+1} = p_t + 0.2v_t + h_t, \quad v_{t+1} = v_t + 0.2u_t + \eta_t, \]

where \( p_t \in \mathbb{R}^2 \) is the position, \( v_t \) is the velocity, \( u_t \) is the control, and \( h_t, \eta_t \sim U[-1,1]^2 \) are i.i.d. noises. The objective is to minimize

\[
\sum_{t=0}^{T-1} \| p_t - y_i \|^2 + 0.0016\| u_i \|^2,
\]

where we let \( T = 200 \). This problem can be converted to the standard LQR with disturbance \( w_t \) by letting \( x_t = [p_t \ v_t] \) and \( \tilde{w}_t = [h_t \ \eta_t] \) and then using the reduction in the LQ tracking example in Section 2.

Note that the disturbances are the combination of a deterministic trajectory and i.i.d. noises. In contrast, our theoretical results focus on more challenging adversarial disturbances. Nonetheless, the numerical results are consistent with our theorems.

In our first experiment, we study the effect of the number of delays or predictions. For simplicity, we exclude the effect of inexactness of the predictions — a prediction is either exact (\( \epsilon_i = 0 \)) or uninformative (\( \epsilon_i = d_{i+|i|} = 0 \)). In this case, each exact prediction cancels a step of delay so, delays can be viewed as “negative” predictions. Figure 2 shows the performance of the proposed myopic policy in Section 2.2 with different numbers of predictions or delays. We see that the cost exponentially decreases (increases) as the number of predictions (delays) increases.

In the second experiment, we study the effect of inexact predictions and show that the controller needs to optimize how many predictions are used — it is better to use only a few predictions and ignore those that are too noisy. Specifically, we let \( \epsilon_i = \frac{1}{2}i^2 \), i.e., the noise level of predictions grows quadratically fast with the number of steps into the future. Each estimation error \( \epsilon_{t+i|i} = w_{t+i} - \tilde{w}_{t+i|i} \) is independently sampled from \( U[-\epsilon_i \| w_{t+i} \|, \epsilon_i \| w_{t+i} \|] \). This process is repeated 8 times, with each instance depicted by an orange line and their maximum represented by a red line in Figure 3. Figure 3 summarizes the results, suggesting that (i) with exact predictions, the cost will decrease as the number of predictions increase (the blue line); and (ii) with inexact predictions, using fewer predictions may yield better performance.

5 CONCLUDING REMARKS

Our result presents the first constant-competitive policy for general LQR control with adversarial disturbances and delayed imperfect information. We also show that in the case of exact predictions with no delay, or in the case of delay with no predictions, the competitive ratio bounds of the proposed myopic policy match the lower bound. However, in the inexact prediction case, the tightness of \( \epsilon_i \) in our bounds remains as an open question. Other important extensions include nonlinear dynamics and time-variant linear systems, which can also lead to studying online learning of robust controllers under model mismatch.
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A COST CHARACTERIZATION LEMMA

Before we start our proofs, we first present a technical lemma that is used in many of the proofs below. This lemma connects the control cost of a policy to its difference from the offline optimal policy.

Lemma 10. Suppose at each time $t$, the controller applies the following policy:

$$u_t = -(R + B^TPB)^{-1}B^T(PAx_t + \sum_{i=0}^{T-t-1} F^{T-i}Pw_{t+i} - \eta_t).$$

(6)

If $Q_f = P$, then the control cost is given by:

$$\text{Alg} = \sum_{t=0}^{T-1} \left( x_t^TPw_t + 2w_t^T \sum_{i=1}^{T-t-1} F^{T-i}Pw_{t+i} - \left( \sum_{i=0}^{T-t-1} F^{T-i}Pw_{t+i} \right)^T H \left( \sum_{i=0}^{T-t-1} F^{T-i}Pw_{t+i} \right) \right) + \sum_{t=0}^{T-1} \eta_t^T H \eta_t + x_0^TPx_0 + 2x_0^T \sum_{i=0}^{T-1} F^{T+i}Pw_i.$$  

(7)

Note that the optimal offline policy has $\eta_t = 0$ for all $t$ in Equation (7) (as derived by Goel and Hassibi [2020]), and as a result, the extra cost of $\text{Alg}$ is given by

$$\text{Alg} - \text{Opt} = \sum_{t=0}^{T-1} \eta_t^T H \eta_t.$$  

(8)

In Equation (6), $\eta_t$ can be regarded as the difference between the applied policy and the offline optimal policy.

We also present below a lemma that has appeared in the body, as we will prove the two lemmas at one time.

Lemma 5. For any algorithm, $\text{Alg}(Q_f) - \text{Opt}^P(Q_f) \leq \text{Alg}(P) - \text{Opt}^P(P) + O(1)$, where the $O(1)$ term is with respect to $T$ and it is zero when $Q_f = P$.

Proof of Lemmas 5 and 10. Given a disturbance sequence $w$, we define the cost-to-go function of a policy described by Equation (6):

$$V^\text{Alg}_t(x_t; w) := \sum_{i=t}^{T-1} (x_i^TQx_i + u_i^TRu_i) + x_i^TP_Tx_T = x_t^TQx_t + u_t^TRu_t + V^\text{Alg}_{t+1}(x_{t+1}; w).$$

We will show by backward induction that $V^\text{Alg}_t(x_t; w) = x_t^TP_tx_t + x_t^Tv_t + q_t$ for some $P_t$, $v_t$ and $q_t$. Let $\Delta_t = P_t - P$, where $P$ is the solution of DARE (Equation [3]). When $t = T$, we have $P_T = Q_f$, $v_T = 0$ and $q_T = 0$. Assume this is true at $t + 1$. Then,

$$V^\text{Alg}_t(x_t; w) = x_t^TQx_t + u_t^TRu_t + (Ax_t + Bu_t + w_t)^T \Delta_{t+1}(Ax_t + Bu_t + w_t) + (Ax_t + Bu_t + w_t)^T v_{t+1} + q_{t+1}$$

$$= u_t^T(R + B^TP_{t+1}B)u_t + 2u_t^TB^T(P_{t+1}Ax_t + P_{t+1}w_t + v_{t+1}/2)$$

$$+ x_t^TQx_t + (Ax_t + w_t)^T \Delta_{t+1}(Ax_t + w_t) + (Ax_t + w_t)^T v_{t+1} + q_{t+1}$$

$$= u_t^T(R + B^TP_{t+1}B)u_t + u_t^TB^T \Delta_{t+1}Bu_t$$

$$+ 2u_t^TB^T(PAx_t + Pw_t + v_{t+1}/2) + 2u_t^TB^T(\Delta_{t+1}Ax_t + \Delta_{t+1}w_t + v_{t+1}/2)$$

$$+ x_t^TQx_t + (Ax_t + w_t)^T P(Ax_t + w_t) + (Ax_t + w_t)^T \Delta_{t+1}(Ax_t + w_t) + (Ax_t + w_t)^T v_{t+1} + q_{t+1}$$

$$= (PAx_t + \sum_{i=0}^{T-t-1} F^{T-i}Pw_{t+i} - \eta_t)^T H (PAx_t + \sum_{i=0}^{T-t-1} F^{T-i}Pw_{t+i} - \eta_t)$$

$$+ (PAx_t + \sum_{i=0}^{T-t-1} F^{T-i}Pw_{t+i} - \eta_t)^T H \Delta_{t+1}H (PAx_t + \sum_{i=0}^{T-t-1} F^{T-i}Pw_{t+i} - \eta_t).$$
\[-2 \left( P A x_t + \sum_{i=0}^{T-t-1} F^{\top i} P w_{t+i} - \eta_t \right)^\top H(P A x_t + P w_t + v_{t+1}/2)\]
\[-2 \left( P A x_t + \sum_{i=0}^{T-t-1} F^{\top i} P w_{t+i} - \eta_t \right)^\top H(\Delta_{t+1} A x_t + \Delta_{t+1} w_t + v_{t+1}/2)\]
\[+ x_t^\top Q x_t + (A x_t + w_t)^\top P(A x_t + w_t) + (A x_t + w_t)^\top \Delta_{t+1} (A x_t + w_t) + (A x_t + w_t)^\top v_{t+1} + q_{t+1}\]
\[
= (P A x_t)^\top H(P A x_t) + 2(P A x_t)^\top H \left( \sum_{i=0}^{T-t-1} F^{\top i} P w_{t+i} - \eta_t \right) + \\
\left( \sum_{i=0}^{T-t-1} F^{\top i} P w_{t+i} - \eta_t \right)^\top H \left( \sum_{i=0}^{T-t-1} F^{\top i} P w_{t+i} - \eta_t \right) + \\
(P A x_t)^\top H(\Delta_{t+1} H(P A x_t)) + 2(P A x_t)^\top H \Delta_{t+1} H \left( \sum_{i=0}^{T-t-1} F^{\top i} P w_{t+i} - \eta_t \right) + \\
\left( \sum_{i=0}^{T-t-1} F^{\top i} P w_{t+i} - \eta_t \right)^\top H \Delta_{t+1} H \left( \sum_{i=0}^{T-t-1} F^{\top i} P w_{t+i} - \eta_t \right) + \\
2(P A x_t)^\top H(\Delta_{t+1}(A x_t + w_t)) + 2 \left( \sum_{i=0}^{T-t-1} F^{\top i} P w_{t+i} - \eta_t \right)^\top H \Delta_{t+1}(A x_t + w_t)
\]
\[+ x_t^\top Q x_t + (A x_t + w_t)^\top P(A x_t + w_t) + (A x_t + w_t)^\top \Delta_{t+1}(A x_t + w_t) + (A x_t + w_t)^\top v_{t+1} + q_{t+1}\]
\[
= x_t^\top (Q + A^\top P A - A^\top P H P A + F^\top \Delta_{t+1} F) x_t
\]
\[+ 2 x_t^\top F^\top P w_{t+1} x_t + x_t^\top F^\top v_{t+1} - x_t^\top F^\top H \left( \sum_{i=0}^{T-t-1} F^{\top i} P w_{t+i} - \eta_t \right)
\]
\[+ \left( \sum_{i=0}^{T-t-1} F^{\top i} P w_{t+i} - \eta_t \right)^\top H \left( \sum_{i=0}^{T-t-1} F^{\top i} P w_{t+i} - \eta_t \right) - 2 \left( \sum_{i=0}^{T-t-1} F^{\top i} P w_{t+i} - \eta_t \right)^\top H(P w_t + v_{t+1}/2)
\]
\[+ w_t^\top P w_t + w_t^\top v_{t+1} + q_{t+1} + O(\Delta_{t+1}).\]

Thus, \( P + \Delta_t = P_t = Q + A^\top P A - A^\top P H P A + F^\top \Delta_{t+1} F = P + F^\top \Delta_{t+1} F \) and thus \( \Delta_t = F^\top \Delta_{t+1} F = O(\lambda^{2(T-t)}) \), where \( \lambda = \frac{1+\rho(F)}{2} \). The recursive formulae for \( v_t \) and \( q_t \) are given by:

\[v_t = 2F^\top P w_t + F^\top v_{t+1} + O(\lambda^{2(T-t)}) = 2 \sum_{i=0}^{T-t-1} F^{\top i+1} P w_{t+i} + O(\lambda^{T-t})\]
\[v_{t+1} = 2 \sum_{i=1}^{T-t-1} F^{\top i} P w_{t+i} + O(\lambda^{T-t}),\]
\[q_t = q_{t+1} + w_t^\top P w_t + 2 w_t^\top \sum_{i=1}^{T-t-1} F^{\top i} P w_{t+i} + \left( \sum_{i=0}^{T-t-1} F^{\top i} P w_{t+i} - \eta_t \right)^\top H \left( \sum_{i=0}^{T-t-1} F^{\top i} P w_{t+i} - \eta_t \right)
\]
\[-2 \left( \sum_{i=0}^{T-t-1} F^{\top i} P w_{t+i} - \eta_t \right)^\top H \left( \sum_{i=0}^{T-t-1} F^{\top i} P w_{t+i} + O(\lambda^{T-t}) \right) + O(\lambda^{T-t})\]
\[= q_{t+1} + w_t^\top P w_t + 2 w_t^\top \sum_{i=1}^{T-t-1} F^{\top i} P w_{t+i} - \left( \sum_{i=0}^{T-t-1} F^{\top i} P w_{t+i} \right)^\top H \left( \sum_{i=0}^{T-t-1} F^{\top i} P w_{t+i} + \eta_t H \eta_t \right)
\]
\[+ O(\lambda^{T-t}).\]
As such, we obtain Theorem 2.

Replacing disturbances are zero, and (ii) Proof.

Lemma 10 implies that when \( k < d \) and the controller uses

Therefore, \( (\text{Alg}(Q_f) - \text{Opt}^P(Q_f)) - (\text{Alg}(P) - \text{Opt}^P(P)) = O(1) \).

\( \square \)

B PROOF OF THEOREM 2

Theorem 2. Suppose there are \( d \) delays and \( k \) exact predictions with \( k < d \). Assume all used predictions are exact and other disturbances (with unused predictions) are zero. The optimal policy at time \( t \) is:

\[
\text{Alg}(Q_f) - \text{Opt}^P(Q_f) = x_0^\top P \bigg( \sum_{i=0}^{T-1} \lambda_i H \eta_i + O(1) \bigg).
\]

Proof. Lemma 10 implies that when \( Q_f = P \), the offline optimal policy is given by

\[
u_t = - (R + B^\top PB)^{-1} B^\top PA \bigg( A^{d-k} \hat{x}_{t-d+k|t} + \sum_{i=0}^{d-k-1} A^i Bu_{t-1-i} \bigg).\]

Replacing \( w_{t+i} \) by zero and \( x_t \) by \( \hat{x}_{t|t} \) in the above policy, we obtain:

\[
u_t = -(R + B^\top PB)^{-1} B^\top PA \hat{x}_{t|t}.\]

As such, we obtain Theorem 2. \( \square \)

C PROOF OF THEOREM 3

Theorem 3. Let \( c = \|P\| \|P^{-1}\| (1 + \|F\|) \) and \( H = B(R + B^\top PB)^{-1} B^\top \). Suppose there are \( d \) steps of delays and the controller uses \( k \) predictions. When \( k \geq d \),

\[
\text{Alg} \leq \left( \frac{c \sum_{i=0}^{d-k-1} \epsilon_i \|A^{d-i}\| + c \sum_{i=d}^{k-1} \epsilon_i \|F^{i-d}\| + \|F^{k-d}\|}{\|H\|^{-1}\lambda_{\min}(P^{-1} - FP^{-1}F^\top - H)} + 1 \right) \text{Opt} + O(1).
\]
When $k \leq d$,
\[
\text{Alg} \leq \left[ \frac{c \sum_{i=0}^{k-1} \| A^{d-i} \| + c \sum_{i=k}^{d-1} \| A^{d-i} \| + 1}{\| H \| \lambda_{\min}^{-1}(P^{-1} - F P^{-1} F^\top - H)} \right]^2 + 1 \text{Opt} + O(1).
\]

The $O(1)$ is with respect to $T$. It may depend on the system parameters $A, B, Q, R, Q_f$ and the range of disturbances $r$, but not on $T$. When $Q_f = P$ the $O(1)$ is zero.

The proof outline provided in the body lays out a set of lemmas that, together, prove Theorem 3. Here, we provide proofs for each of them.

C.1 Proof of Lemma 4

Lemma 4 Suppose $Q_f = P$. Then, the conclusion of Theorem 3 holds.

Proof. This lemma considers the case of $Q_f = P$. Lemma 10 implies that when $Q_f = P$, the cost of the offline optimal policy is:
\[
\text{Opt} = \sum_{t=0}^{T-1} \left( w_t^T P w_t + 2 w_t^T \sum_{i=1}^{T-t-1} F^\top_i P w_{t+i} - \left( \sum_{i=0}^{T-t-1} F^\top_i P w_{t+i} \right)^\top H \left( \sum_{i=0}^{T-t-1} F^\top_i P w_{t+i} \right) \right) + x_0^T P x_0 + 2 x_0^T \sum_{i=0}^{T-t-1} F^\top_i P w_i.
\]

We consider the following substitution:
\[
\psi_t = \sum_{i=0}^{T-t-1} F^\top_i P w_{t+i}, \quad w_t = P^{-1}(\psi_t - F^\top \psi_{t+1}). \tag{10}
\]

Then, the offline optimal cost can be lower bounded:
\[
\text{Opt} \geq \sum_{t=0}^{T-1} \left( \psi_t^T P \psi_t + 2 \psi_t^T H \psi_t \right) + x_0^T P x_0 + 2 x_0^T F^\top \psi_0
\]
\[
= \sum_{t=0}^{T-1} \left( \psi_t^T P^{-1} \psi_t - \psi_{t+1}^T F P^{-1} F^\top \psi_{t+1} - \psi_t^T H \psi_t \right) + x_0^T P x_0 + 2 x_0^T F^\top \psi_0
\]
\[
= \sum_{t=0}^{T-1} \left( \psi_t^T P^{-1} \psi_t - \psi_t^T F P^{-1} F^\top \psi_t - \psi_t^T H \psi_t \right) + \psi_0^T P^{-1} F^\top \psi_0 + x_0^T P x_0 + 2 x_0^T F^\top \psi_0 \tag{11}
\]
\[
= \sum_{t=0}^{T-1} \left( (P^{-1} - F P^{-1} F^\top - H) \psi_t + (F^\top \psi_0 + P x_0)^\top P^{-1} (F^\top \psi_0 + P x_0) \right)
\]
\[
\geq \lambda_{\min}(P^{-1} - F P^{-1} F^\top - H) \sum_{t=0}^{T-1} \| \psi_t \|^2.
\]

The myopic policy has two cases and we analyze each of them below.

Case 1: $k \geq d$. In this case, the controller estimates $x_t$ using $x_{t-d}$ and $\hat{w}_{t-d|t}, \ldots, \hat{w}_{t-1|t}$.
\[
x_t - \hat{x}_t|t = (A x_{t-1} + B u_{t-1} + w_{t-1}) - (A \hat{x}_{t-1|t} + B u_{t-1} + \hat{w}_{t-1|t}) = A(x_{t-1} - \hat{x}_{t-1|t}) + e_{t-1|t}.
\]

Applying similar procedures repetitively, we obtain:
\[
x_t - \hat{x}_t|t = e_{t-1|t} + A e_{t-2|t} + \cdots + A^{d-1} e_{t-d|t} = \sum_{i=1}^{d} A^{i-1} e_{t-i|t}.
\]

Case 2: $k < d$. In this case, the controller estimates $x_t$ using $x_{t-k}$ and $\hat{w}_{t-k|t}, \ldots, \hat{w}_{t-1|t}$.
Comparing Equations (4) and (6), we have
\[
\eta_t = \sum_{i=1}^{d} PA^i \epsilon_{t-i} + \sum_{i=0}^{k-d-1} F_i^{\top} P \epsilon_{t+i} + \sum_{i=k-d}^{T-t-1} F_i^{\top} P \omega_{t+i}.
\] (12)

Using the substitution in Equation (10), we bound Equation (12) as follows.
\[
\|\eta_t\| = \left\| \sum_{i=1}^{d} PA^i \epsilon_{t-i} + \sum_{i=0}^{k-d-1} F_i^{\top} P \epsilon_{t+i} + F_i^{\top} \psi_{t+k-d} \right\|
\leq \sum_{i=1}^{d} \|P\| \|A^i\| \left\| \epsilon_{d-i} \|w_{t-i}\| \right\| + \sum_{i=0}^{k-d-1} \|F^i\| \|P\| \left\| \epsilon_{i-d} \|w_{t+i}\| \right\| + \|F^{k-d}\| \|\psi_{t+k-d}\|.
\] (13)

Note that when \( t < d \), some terms in Equation (13) have negative subscripts. Those terms do not actually exist and should be regarded as zero. However, for the clarity of the proof, we keep them in the formula. In the later derivations, although we treat them as potentially non-zero, they do not affect our result because we are looking for an upper bound. Let \( \eta = (\|\eta_0\|, \ldots, \|\eta_{T-1}\|) \in \mathbb{R}^T \) and \( \psi = (\|\psi_0\|, \ldots, \|\psi_{T-1}\|) \). Equation (13) provides a linear inequality relationship between \( \eta \) and \( \psi \). We define matrix \( M = \{M_t,s\}_{t,s=0}^{T-1} \in \mathbb{R}^{T \times T} \) such that \( M_{t,s} \) is the coefficient of \( \|\psi_s\| \) in the bound of \( \|\eta_t\| \) in Equation (13). Then, \( \eta \leq M \psi \).

\[
\sum_{t=0}^{T-1} \eta_t^\top H \eta_t \leq \|H\| \|\eta\| \|\psi^\top M^T M \psi\| \leq \lambda_{\max}(M^T M) \|H\| \|\psi\|^2.
\] (14)

**Proposition 11** (Gershgorin circle theorem). Let \( A \in \mathbb{C}^{n \times n} \). Let \( D(A_{i,i}, R_i) \subseteq \mathbb{C} \) be a closed disc centered at \( A_{i,i} \) with radius \( R_i = \sum_{j \neq i} |A_{i,j}| \). Then, every eigenvalue of \( A \) lies within at least one of the discs \( D(A_{i,i}, R_i) \).

We use Proposition 11 to bound the eigenvalues of \( M^T M \):
\[
\lambda_{\max}(M^T M) \leq \max_{i} \sum_{j=0}^{T-1} (M^T M)_{i,j} = \|M^T M\|_{\infty}.
\] (15)

Plugging \( \|\psi_s\| = 1 \) for all \( s \) into Equation (13), we have:
\[
M \mathbf{1} \leq \left( \|P\| \|P^{-1}\| (1 + \|F\|) \left( \sum_{i=1}^{d} \|A^i\| \|\epsilon_{d-i}\| + \sum_{i=0}^{k-d-1} \|F^i\| \|\epsilon_{i-d}\| + \|F^{k-d}\| \right) \right) \mathbf{1}.
\]

Thus, Equation (15) can be further bounded:
\[
\lambda_{\max}(M^T M) \leq \left( c \left( \sum_{i=1}^{d} \|A^i\| \|\epsilon_{d-i}\| + \sum_{i=0}^{k-d-1} \|F^i\| \|\epsilon_{i-d}\| + \|F^{k-d}\| \right) \right)^2,
\]
where \( c = \|P\| \|P^{-1}\| (1 + \|F\|) \). Together with Equations (8) and (14), this implies that
\[
\text{Alg} - \text{Opt} = \sum_{t=0}^{T-1} \eta_t^\top H \eta_t \leq \left( c \left( \sum_{i=1}^{d} \|A^i\| \|\epsilon_{d-i}\| + \sum_{i=0}^{k-d-1} \|F^i\| \|\epsilon_{i-d}\| + \|F^{k-d}\| \right) \right)^2 \|H\| \|\psi\|^2.
\]

Together with Equation (11), we have
\[
\frac{\text{Alg} - \text{Opt}}{\text{Opt}} \leq \left( c \left( \sum_{i=1}^{d} \|A^i\| \|\epsilon_{d-i}\| + \sum_{i=0}^{k-d-1} \|F^i\| \|\epsilon_{i-d}\| + \|F^{k-d}\| \right) \right)^2 \|H\| \lambda_{\min}^{-1}(P^{-1} - FP^{-1}F^\top - H).
\]
Case 2: $k < d$. We start from Equation (9). In this case, we have the following equations.

\[
x_t - \hat{x}_{t|t} = A(x_{t-1} - \hat{x}_{t-1|t}) + w_{t-1} - 0.
\]

\[
\vdots
\]

\[
x_{t-d+k+1} - \hat{x}_{t-d+k+1|t} = A(x_{t-d+k} - \hat{x}_{t-d+k|t}) + w_{t-d+k} - 0.
\]

\[
x_{t-d+k} - \hat{x}_{t-d+k|t} = A(x_{t-d+k-1} - \hat{x}_{t-d+k-1|t}) + w_{t-d+k-1} - \hat{w}_{t-d+k-1|t}.
\]

\[
\vdots
\]

\[
x_{t-d+1} - \hat{x}_{t-d+1|t} = A(x_{t-d} - \hat{x}_{t-d|t}) + w_{t-d} - \hat{w}_{t-d|t}.
\]

Note that in the last line, $x_{t-d} = \hat{x}_{t-d|t}$. Thus, all of the above equations can be combined into the following:

\[
x_t - \hat{x}_{t|t} = \sum_{i=0}^{k-1} A^{d-i-1} e_{t-d+i|t} + \sum_{i=k}^{d-1} A^{d-i-1} w_{t-d+i}.
\]

Therefore, the policy can be written as:

\[
u_t = -(R + B^T PB)^{-1} B^T PA \hat{x}_{t|t}
\]

\[
= -(R + B^T PB)^{-1} B^T PA \left( x_t - \sum_{i=0}^{k-1} A^{d-i-1} e_{t-d+i|t} - \sum_{i=k}^{d-1} A^{d-i-1} w_{t-d+i} \right).
\]

We compare this with Equation (6) to get

\[
\eta_t = \sum_{i=0}^{k-1} PA^{d-i} e_{t-d+i|t} + \sum_{i=k}^{d-1} PA^{d-i} w_{t-d+i} + \sum_{i=0}^{T-t-1} F^{i} P w_{t+i}.
\]

With the substitution in Equation (10),

\[
\|\eta_t\| = \left\| \sum_{i=0}^{k-1} P A^{d-i} e_{t-d+i|t} + \sum_{i=k}^{d-1} P A^{d-i} w_{t-d+i} + \psi_t \right\|
\]

\[
\leq \sum_{i=0}^{k-1} \|P\| \|A^{d-i}\| \|e_i\| \|w_{t-d+i}\| + \sum_{i=k}^{d-1} \|P\| \|A^{d-i}\| \|w_{t-d+i}\| + \|\psi_t\|
\]

\[
\leq \sum_{i=0}^{k-1} \|P\| \|A^{d-i}\| \|\psi_{t-d+i}\| + \|F\| \|\psi_{t-d+i+1}\|
\]

\[
+ \sum_{i=k}^{d-1} \|P\| \|A^{d-i}\| \|P^{-1}\| \|\psi_{t-d+i}\| + \|F\| \|\psi_{t-d+i+1}\| + \|\psi_t\|.
\]

(16)

Similar to the previous case, we define matrix $M = \{M_{t,s}\}_{t,s=0}^{T-1} \in \mathbb{R}^{T \times T}$ such that $M_{t,s}$ is the coefficient of $\|\psi_s\|$ in the bound of $\|\eta_t\|$ in Equation (16). Then, by Proposition 11

\[
\lambda_{\text{max}}(M^T M) \leq \left( c \sum_{i=0}^{k-1} \|A^{d-i}\| \|e_i\| + c \sum_{i=k}^{d-1} \|A^{d-i}\| + \|\psi_t\| \right)^2.
\]

\[
\frac{\text{Alg} - \text{Opt}}{\text{Opt}} \leq \sum_{t=0}^{T-1} \eta_t^T H \eta_t \leq \|H\| \|\psi^T M \psi\| \leq \lambda_{\text{max}}(M^T M) \|H\| \|\psi\|^2.
\]

\[
\frac{\text{Alg} - \text{Opt}}{\text{Opt}} \leq \left( c \sum_{i=0}^{k-1} \|A^{d-i}\| \|e_i\| + c \sum_{i=k}^{d-1} \|A^{d-i}\| + \|\psi_t\| \right) \frac{ \|H\| \lambda_{\text{min}}^{-1}(P^{-1} - FP^{-1} F^T - H) }{\text{Opt}}.
\]
C.2 Proof of Lemma 5

See Appendix A.

C.3 Proof of Lemma 6

Lemma 6. The followings are equal up to $O(1)$ difference: $\text{Opt}^P(P)$, $\text{Opt}^P(Q_f)$, $\text{Opt}^{Q_f}(Q_f)$.

Proof. By definition,

$$\text{Opt}^P(P) = \min_X \text{Opt}^X(P) \leq \text{Opt}^{Q_f}(P),$$

$$\text{Opt}^{Q_f}(Q_f) = \min_X \text{Opt}^X(Q_f) \leq \text{Opt}^P(Q_f).$$

Moreover, for any $X$,

$$\text{Opt}^X(Q_f) - \text{Opt}^X(P) = x_T^T(Q_f - P)x_T = O(1),$$

where $x_T$ is the final state obtained by the policy that is optimal assuming the terminal cost is $X$. Therefore,

$$\text{Opt}^P(P) \leq \text{Opt}^{Q_f}(P) = \text{Opt}^{Q_f}(Q_f) + O(1) \leq \text{Opt}^P(Q_f) + O(1) = \text{Opt}^P(P) + O(1).$$

As a result, $\text{Opt}^P(P)$, $\text{Opt}^P(Q_f)$, $\text{Opt}^{Q_f}(Q_f)$, $\text{Opt}^{Q_f}(P)$ are all equal up to a difference of $O(1)$. □

D TIGHTNESS OF THEOREM 7

For the case of $k$ exact predictions and no delays, we give an example where the competitive ratio is $1 + \Omega(\|F^k\|^2)$, where the asymptotic notation is with respect to $k$.

Let $n = m = 1$, $Q_f = P$, $x_0 = 0$ and $A, B, Q, R$ are such that $F > 0$. Using the notation in Equation (6), we have

$$\eta_t = \sum_{i=k}^{T-t-1} F^i P w_{t+i}.$$

Let $w_t = r$ be a constant for all $t$. Then,

$$\text{Alg} - \text{Opt} = \sum_{t=0}^{T-1} H \eta_t^2 \geq \sum_{t=0}^{T-1} H (F^k Pr)^2 = \Omega(F^{2k}).$$

As a result,

$$\frac{\text{Alg}}{\text{Opt}} = 1 + \Omega(F^{2k}).$$

This example directly generalizes to higher dimensions by stacking independent one-dimensional systems together, so that all matrices are diagonal.

E TIGHTNESS OF THEOREM 9

For the case of $d$ steps of delay and no usable predictions, we give an example where the competitive ratio is $1 + \Omega(\|A^d\|^2)$, where the asymptotic notation is with respect to $d$.

Let $n = m = 1$, $Q_f = P$, $x_0 = 0$ and $w_t = r$ for all $t$.

$$\eta_t = \sum_{i=0}^{d-1} P A^{d-i} w_{t-i} + \sum_{i=0}^{T-t-1} F^i P w_{t+i}.$$

$$\text{Alg} - \text{Opt} = \sum_{t=0}^{T-1} H \eta_t^2 \geq \sum_{t=0}^{T-1} H (PA^d r)^2 = \Omega(A^{2d}).$$
\[
\frac{\text{Alg}}{\text{Opt}} \geq 1 + \Omega(A^{2d}).
\]

Similar to the previous example, this one-dimensional example generalizes to high dimensions by simply stacking 1-d systems together.