Blow-up of a stable stochastic differential equation

Matti Leimbach, Michael Scheutzow
Technische Universität Berlin

Abstract

We examine a 2-dimensional ODE which exhibits explosion in finite time. Considered as an SDE with additive white noise, it is known to be complete - in the sense that for each initial condition there is almost surely no explosion. Furthermore, the associated Markov process even admits an invariant probability measure. On the other hand, as we will show, the corresponding local stochastic flow will almost surely not be strongly complete, i.e. there exist (random) initial conditions for which the solutions explode in finite time.

1 Introduction

Consider the complex-valued Itô-type stochastic differential equation (SDE)
\[ dZ_t = (Z_t^n + F(Z_t)) \, dt + \sigma dB_t, \quad (1) \]
where \( n \geq 2 \), \( \sigma \geq 0 \), \( F \in O(|z|^{n-1}) \) as \( |z| \to \infty \) is locally Lipschitz and \( B = W^{(1)} + iW^{(2)} \) is a complex Brownian motion on a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})\) satisfying the usual conditions.

Under the additional assumption that \( F \) is a polynomial of \( z \) and \( \overline{z} \) of degree at most \( n - 1 \) it is known (see [HM14a] and [HM14b]) that for every fixed initial condition \((X_0, Y_0) = (x_0, y_0)\) the one-point motion, i.e. the process which solves this equation and starts in \((x_0, y_0)\), exhibits non-explosion almost surely if \( \sigma > 0 \) and, moreover, the associated Markov process admits a (unique) invariant probability measure. This is a remarkable fact, since there is explosion in finite time for some initial conditions in the deterministic case (i.e. \( \sigma = 0 \)). This is obvious in the particular case \( F = 0 \) (take an initial condition on the positive real line) and will follow from our main result for general \( F \). Turning an explosive ODE into a non-explosive SDE with an invariant distribution by adding noise is often called noise-induced stability and was also studied in [Sch93] and more recently in [BHW12], [AKM12].

Now, we would like to know if the noise induces an even stronger kind of stability, namely the existence of a random attractor. In this paper, we show that the corresponding local stochastic flow will explode (or blow up) almost surely and therefore there cannot be a random attractor (for the definition and basic properties of random attractors, see [CP94]). SDEs which have a unique global solution for each initial condition are called complete. Since the local stochastic flow associated to (1) explodes, it is – by definition – not strongly complete.
So far there are only a few examples which are known to be complete but not strongly complete, see for instance [Elw78], [LS11].

2 Transformation into Cartesian coordinates

For our purpose it is convenient to transform equation (1) into Cartesian coordinates. The rest of this paper deals only with equation (2) below.

Denote the real and imaginary part of \( F \) by \( F_1 \) and \( F_2 \), i.e.

\[ F = F_1 + iF_2, \]

Further, there are functions \( \hat{F}_1, \hat{F}_2: \mathbb{R}^2 \to \mathbb{R} \), such that \( F_j(x + iy) = \hat{F}_j(x, y), \)

\( j = 1, 2. \)

If we rewrite \( Z_t = X_t + iY_t \), SDE (1) is equivalent to

\[
\begin{align*}
\frac{dX_t}{dt} &= \left( \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^j \binom{n}{2j} x^{n-2j} y^{2j} + \hat{F}_1(X_t, Y_t) \right) dt + \sigma dW_t^{(1)}, \\
\frac{dY_t}{dt} &= \left( \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} (-1)^j \binom{n}{2j+1} x^{n-2j-1} y^{2j+1} + \hat{F}_2(X_t, Y_t) \right) dt + \sigma dW_t^{(2)}.
\end{align*}
\]

Abbreviate

\[
\begin{align*}
b_1(x, y) &:= \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^j \binom{n}{2j} x^{n-2j} y^{2j}, \\
b_2(x, y) &:= \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} (-1)^j \binom{n}{2j+1} x^{n-2j-1} y^{2j+1}.
\end{align*}
\]

At first sight these drift terms look quite unhandy, but the following lemma yields convenient expressions.

Lemma 2.1

For \( x > 0, y \in \mathbb{R} \) we have

\[
\begin{align*}
b_1(x, y) &= (x^2 + y^2)^{\frac{n}{2}} \cos\left( n \arctan \left( \frac{y}{x} \right) \right), \\
b_2(x, y) &= (x^2 + y^2)^{\frac{n}{2}} \sin\left( n \arctan \left( \frac{y}{x} \right) \right).
\end{align*}
\]

Proof. Write \( z \) in Cartesian and polar coordinates, i.e. \( z = x + iy = re^{i\phi} \). For \( x > 0 \) polar coordinates can be expressed in terms of cartesian coordinates via \( r = \sqrt{x^2 + y^2}, \phi = \arctan(y/x). \) Therefore,

\[
\begin{align*}
z^n &= (x + iy)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} (iy)^j \\
&= \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^j \binom{n}{2j} x^{n-2j} y^{2j} + i \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} (-1)^j \binom{n}{2j+1} x^{n-2j-1} y^{2j+1}, \\
&= r^n e^{n i \phi} = r^n \cos(n\phi) + ir^n \sin(n\phi) \\
&= (x^2 + y^2)^{\frac{n}{2}} \cos\left( n \arctan \left( \frac{y}{x} \right) \right) + i (x^2 + y^2)^{\frac{n}{2}} \sin\left( n \arctan \left( \frac{y}{x} \right) \right).
\end{align*}
\]
The lemma follows by comparing the real and imaginary parts of both expressions. □

3 Defining the problem and main result

First, we introduce local stochastic flows on $\mathbb{R}^d$, $d \geq 1$.

Definition 3.1
Let $\epsilon(s, x)$, $s \geq 0$, $x \in \mathbb{R}^d$ be a random field with values in $(s, \infty)$, such that $\epsilon(s, x)$ is lower semicontinuous in $s$ and $x$. Set $\mathcal{D}_{s,t}(\omega) := \{ x \in \mathbb{R}^d : \epsilon(s, x, \omega) > t \}$ and let $\phi_{s,t}(x, \omega), x \in \mathbb{R}^d, 0 \leq s \leq t < \epsilon(s, x)$ be a continuous $\mathbb{R}^d$-valued random field defined on the random domain of parameters $(s, t, x)$ for which $x \in \mathcal{D}_{s,t}(\omega)$.

Denote the range of $\phi_{s,t}(\cdot, \omega)$ on $\mathcal{D}_{s,t}(\omega)$ by $\mathcal{R}_{s,t}(\omega)$.

$\phi$ (or $\phi_{s,t}$) is called a local stochastic flow, if for almost all $\omega \in \Omega$

i) $\phi_{s,s}(\cdot, \omega) = \text{Id}_{\mathbb{R}^d}$ for all $s \geq 0$,

ii) $\phi_{s,t}(\cdot, \omega) : \mathcal{D}_{s,t}(\omega) \to \mathcal{R}_{s,t}(\omega)$ is a homeomorphism for all $0 \leq s < t$ and the inverse is continuous in $(s, t, x)$,

iii) $\phi_{s,u}(\cdot, \omega) = \phi_{t,u}(\phi_{s,t}(\cdot, \omega), \omega)$ holds on $\mathcal{D}_{s,u}(\omega)$ for all $0 \leq s \leq t \leq u$
holds true.

A local stochastic flow is called stochastic flow if for all $0 \leq s \leq t$ and $\omega \in \Omega$

$\mathcal{D}_{s,t}(\omega) = \mathcal{R}_{s,t}(\omega) = \mathbb{R}^d$.

According to [Kun90, Theorem 4.7.1] there exists a local stochastic flow $\phi_{s,t}(x, \omega), x \in \mathbb{R}^2, 0 \leq s \leq t < \epsilon(s, x)$, which is the maximal solution to equation (1) starting at time $s$ in $x$, where $\epsilon(s, x)$ is the explosion time.

In the following, we write $\phi_t$ instead of $\phi_{0,t}$ and denote the $i$th component of $\phi_t$ by $\phi_t^{(i)}$, $i = 1, 2$. We use $\phi_t^{(1)}(z)$ and $X_t$ respectively $\phi_t^{(2)}(z)$ and $Y_t$ interchangeably, whenever the initial condition $z$ is not of importance or clear from the context.

Our main result is the explosion (or blow up or lack of strong completeness) of the local stochastic flow $\phi$.

Theorem 3.2
Let $\phi$ be the local stochastic flow associated to $\mathcal{I}$, then there exists $T \in (0, \infty)$ such that

$$\lim_{x_0 \to \infty} \mathbb{P} \left( \sup_{z \in \mathcal{I}} \sup_{t \leq T} \phi_t^{(1)}(z) = \infty \right) = 1,$$

where the initial set is given by $\mathcal{I} := \{ x_0 \} \times \left[ -\tan \left( \frac{\pi}{4n} \right) x_0, \tan \left( \frac{\pi}{4n} \right) x_0 \right].$

Remark 3.3
The theorem shows that we have almost sure blow-up:

$$\mathbb{P} \left( \exists z \in \mathbb{R}^2 : \sup_{t \leq T} \phi_t^{(1)}(z) = \infty \right) \geq \lim_{x_0 \to \infty} \mathbb{P} \left( \sup_{z \in \mathcal{I}} \sup_{t \leq T} \phi_t^{(1)}(z) = \infty \right) = 1.$$
4 Heuristic idea

For the rest of this paper fix $\alpha \in (0, \tan(\frac{\pi}{2n}))$, and define the cone

$$C := \{(x, y) \in \mathbb{R}^2 : x \geq x^*, |y| \leq \alpha x\},$$

where we will choose $x^* > 0$ sufficiently large later on (depending only on $n$ and $F$).

We know that for every initial condition in $C$, the solution of the SDE will almost surely eventually leave $C$. Some trajectories leave this region via the upper boundary and some via the lower boundary. Due to the continuity of the map $z \mapsto \phi_t(z)$, one may hope to be able to show that there will be (random) initial conditions in between these two kinds of points for which the trajectories will actually remain inside $C$ forever. In the following section we will see that if such trajectories exist, then they will explode within time $T$ (which is small provided the initial condition has a large $x$-component) provided the noise in the $x$-direction is not too large up to time $T$.

It then remains to show that there actually exist trajectories which stay inside $C$ forever (until they blow up). Let us sketch the idea of the proof in case $W^{(1)} \equiv 0$: Figure 1 shows the image of the set of initial conditions $\{(x_0, y) \mid |y| \leq \tan(\frac{\pi}{2n}) x_0\}$ under the map $\phi_t$ for some $x_0 > x^* > 0$ and some $t > 0$. The idea of the proof is to show that, for large $x_0$, it is very unlikely that any trajectory whose $y$-coordinate happens to be above level $\alpha x_0/2$ at some time will hit the level $y = \alpha x_0/4$ before leaving the cone $C$ through its upper boundary (Lemma 5.1). This will then allow us to show the existence of points which stay inside $C$ forever (until explosion).

![Figure 1: Bounds away from the x-axis](attachment:figure1.png)
5 Auxiliary results and proof of Theorem 3.2

First, we establish a lower bound for the $x$-component as long as the trajectory stays inside the cone $C$. Then we formalize what is shown in Figure 1.

Define
\[
\begin{align*}
\tau(z) &:= \inf\{t \geq 0 : \phi^{(2)}_t(z) \geq \alpha \phi^{(1)}_t(z)\}, \\
\tau(z) &:= \inf\{t \geq 0 : \phi^{(2)}_t(z) \leq -\alpha \phi^{(1)}_t(z)\}, \\
\tau(z) &:= \tau(z) \land \tau(z).
\end{align*}
\]

5.1 Lower bound

Note, that we have a lower bound $\varepsilon > 0$ of the following term uniformly for all $(x, y) \in C$

\[
\cos\left(n \arctan\left(\frac{y}{x}\right)\right) \geq \varepsilon > 0.
\]

Because of $F \in \mathcal{O}(|z|^{-1}) \subset o(|z|^n)$ as $|z| \to \infty$, there exists $x^* > 0$, such that

\[
\frac{|\tilde{F}_1(x, y)|}{(x^2 + y^2)^{\frac{1}{2}}} \leq \frac{\varepsilon}{2}
\]

holds for all $x \geq x^*, y \in \mathbb{R}$.

Fix $c > 0$, $x_0 \geq x^* + c$ and $z \in \mathcal{J} = \{x_0\} \times [-\tan\left(\frac{\pi n}{2}\right) x_0, \tan\left(\frac{\pi n}{2}\right) x_0]$. Then on the event

\[
\{\tau(z) > T\} \cap \{\sup_{t \in [0, T]} |W^{(1)}_t| \leq c\},
\]

we have for all $t \in [0, T] \cap D$ ($D$ is the maximal domain on which $X_t$ is defined)

\[
X_t = x_0 + \int_0^t b_1(X_s, Y_s) + \tilde{F}_1(X_s, Y_s)ds + \sigma W^{(1)}_t
\]

\[
\geq x_0 - c + \int_0^t \left(X_s^2 + Y_s^2\right) \left(\cos\left(n \arctan\left(\frac{Y_s}{X_s}\right)\right) - \frac{|\tilde{F}_1(X_s, Y_s)|}{(X_s^2 + Y_s^2)^{\frac{1}{2}}}ight)ds
\]

\[
\geq x_0 - c + \frac{\varepsilon}{2} \int_0^t X_s^p ds.
\]

Applying a (reversed) Gronwall type argument (similar to [Bih56 page 83f]), we see that for all $t \in [0, T] \cap D$

\[
X_t \geq \frac{x_0 - c}{\left(1 - \frac{\varepsilon}{2} (n - 1)(x_0 - c)^{n-1} t\right)^{\frac{1}{n-1}}},
\]

where $T := \frac{1}{(n-1)(x_0 - c)^{n-1}}$, which is an upper bound for the explosion time, i.e. $(X_t)$ blows up up to time $T$ on the set $\{\tau(z) > T\} \cap \{\sup_{t \in [0, T]} |W^{(1)}_t| \leq c\}$. Observe that the heuristic ideas remain valid on $\{\sup_{t \in [0, T]} |W^{(1)}_t| \leq c\}$ when replacing $x_0$ by $x_0 - c$: if the event $\{\sup_{t \in [0, T]} |W^{(1)}_t| \leq c\}$ occurs, then any trajectory starting in $\mathcal{J}$ which does not leave the cone $C$ up to $T$ blows up before (or at) time $T$. 

5.2 Bounds away from the $x$-axis

Throughout the rest of the paper, $c > 0$ will be fixed and $x_0 > x^* + c$ is a number which will later be sent to infinity.

Because of $F \in \mathcal{O}(|z|^{n-1})$ there is a $C > 0$ such that for all $(x, y) \in \mathbb{R}^2$ with $|(x, y)| \geq x^*$, where $x^* > 0$ is sufficiently large,

$$\frac{|\hat{F}_2(x, y)|}{|(x, y)|^{n-1}} \leq C$$

holds true. Further, we define $x_1 := x_0 - c$ and $T := \frac{1}{2(n-1)x_1}$ as before. Observe that $x_1$ tends to $\infty$ and $T$ tends to 0 as $x_0 \to \infty$.

**Lemma 5.1**

For $x^*$ sufficiently large, the following holds. Let $(X_t, Y_t)_{t \in [0, T]}$ solve equation (2) with initial condition $(X_0, Y_0) = z \in \{x_0\} \times [-\tan\left(\frac{n}{2}\right) x_0, \tan\left(\frac{n}{2}\right) x_0]$. Define $\nu^+ := \inf\{t \geq 0: Y_t \geq \frac{b}{2}x_1\}$. Then for all $t \in [\nu^+, \tau(z)] \cap D$, where, again, $D$ is the maximal domain on which $X_t$ is defined, we have

$$Y_t \geq \frac{\alpha}{4} x_1 \quad \text{on} \quad \{\sup_{t \in [0, T]} \sigma|W_t^{(2)}| \leq \frac{\alpha}{8} x_1\} \cap \{\inf_{t \in [0, T] \cap D} X_t \geq x_1\} =: B.$$

**Proof.** Define $\tau := \inf\{t > \nu^+: Y_t \leq \frac{b}{4}x_1\} \cup \tau(z)$. We will show $\tau = \tau(z)$ on $B$, which proves the statement. For $t \geq 0$, such that $\nu^+ + t \in D$ we have

$$Y_{(\nu^++t)\wedge \tau} = Y_{\nu^+} + \sigma(W_{(\nu^++t)\wedge \tau}^{(2)} - W_{\nu^+}^{(2)})$$

$$+ \int_{\nu^+}^{(\nu^++t)\wedge \tau} b_2(X_s, Y_s) + \hat{F}_2(X_s, Y_s)ds$$

$$\geq \frac{\alpha}{4} x_1 + \int_{\nu^+}^{(\nu^++t)\wedge \tau} |(X_s, Y_s)|^{n-1} \times$$

$$\left[ \sqrt{X_s^2 + Y_s^2 \sin \left( n \arctan \left( \frac{Y_s}{X_s} \right) \right)} - \frac{|\hat{F}_2(X_s, Y_s)|}{|(X_s, Y_s)|^{n-1}} \right] ds$$

$$\geq \frac{\alpha}{4} x_1 + \int_{\nu^+}^{(\nu^++t)\wedge \tau} |(X_s, Y_s)|^{n-1} \times$$

$$\left[ \sqrt{X_s^2 + Y_s^2 \sin \left( n \arctan \left( \frac{Y_s}{X_s} \right) \right) } - C \right] ds$$

$$\geq \frac{\alpha}{4} x_1.$$

We justify the last step by showing $I \geq 0$. First, note that there exists a $b_n > 0$, such that for all $z \in [0, b_n)$

$$\sin(n \arctan(z)) \geq z.$$

Second, define $a_n := \sin(n \arctan(b_n))$. Recall, that for $s \in [\nu^+, \tau]$, we have $\alpha X_s \geq Y_s \geq \frac{b}{4}x_1$, which implies

$$\sin \left( n \arctan \left( \frac{Y_s}{X_s} \right) \right) \geq a_n \wedge \frac{Y_s}{X_s}.$$
Finally,
\[ I \geq \left( a_n \wedge \frac{Y_s}{X_s} \right) \sqrt{X_s^2 + Y_s^2} - C \geq \left( a_n X_s \right) \wedge Y_s - C \geq \left( a_n x_1 \right) \wedge \frac{\alpha}{4} x_1 - C \]
is non-negative if we choose \( x^* \) (and therefore also \( x_1 \)) sufficiently large. \( \square \)

**Remark 5.2**
If \( \nu^+ \) is replaced by \( \nu^- := \inf \{ t \geq 0 : Y_t \leq -\frac{\alpha}{2} x_1 \} \) then we obtain in the same way
\[ Y_t \leq -\frac{\alpha}{4} x_1 \]
for \( t \in [\nu^-, \tau(z)] \cap D \).

The previous lemma and remark are a formal description of what was explained in Section 4, see also Figure 1. It will be very useful to show the existence of points which stay inside \( \mathcal{C} \) until explosion (Lemma 5.3).

**Lemma 5.3**
\[ \{ \sup_{z \in \mathcal{J}} \tau(z) > T \} \supset \{ \sup_{t \in [0,T]} \sigma |W_t^{(2)}| \leq \frac{\alpha}{8} x_1 \} \cap \{ \sup_{t \in [0,T]} \sigma |W_t^{(1)}| \leq c \}. \]

**Proof.** Define the random sets
\[
R := \{ z \in \mathcal{J} : \tau(z) \leq \tau(z) \wedge T \}, \\
B := \{ z \in \mathcal{J} : \tau(z) \leq \tau(z) \wedge T \}, \\
G := \mathcal{J} \setminus (R \cup B).
\]
Note that \( R \) and \( B \) are disjoint and
\[ \{ \sup_{z \in \mathcal{J}} \tau(z) > T \} = \{ G \neq \emptyset \}. \]

For ease of notation we define
\[ B_1 := \{ \sup_{t \in [0,T]} \sigma |W_t^{(1)}| \leq c \} \quad B_2 := \{ \sup_{t \in [0,T]} \sigma |W_t^{(2)}| \leq \frac{\alpha}{8} x_1 \}. \]
Let \( \omega \in B_1 \cap B_2 \).
Since \( \omega \in B_1 \) there is a minimal drift in the \( x \)-component for all trajectories starting in \( \mathcal{J} \) as long as they stay inside \( \mathcal{C} \). Furthermore, there is a lower bound in the \( x \)-coordinate for those trajectories, namely \( x_1 = x_0 - c \).
Obviously \( R(\omega) \) and \( B(\omega) \) are not empty since \( (x_0, x_0) \in R(\omega) \) and \( (x_0, -x_0) \in B(\omega) \).
Assume now that \( \omega \in \{ G = \emptyset \} \) which is equivalent to \( \omega \in \{ J = R \cup B \} \). We show that \( R(\omega) \) and \( B(\omega) \) are (non-empty) closed subsets of \( \mathcal{J} \), whose disjoint union is equal to the connected set \( \mathcal{J} \), which is a contradiction.
Take a converging sequence \( z_n \to z \) with \( z_n \in R(\omega) \) for all \( n \in \mathbb{N} \) and assume that \( z \in B(\omega) \). Then, thanks to the continuity of \( \phi_t(z, \omega) \) in \( (t,z) \), there is a (random) \( n \in \mathbb{N} \) such that
\[ \sup_{t \in [0,\tau(z)]} |\phi_t^{(2)}(z, \omega) - \phi_t^{(2)}(z_n, \omega)| \leq \frac{\alpha}{3} x_1. \]
Due to Lemma 5.1, we can conclude that $\phi_2(z, \omega)$ was never above $\alpha x_1/2$ before time $\tau(z)$, and therefore $\phi_2(z_n, \omega)$ was never above $5\alpha x_1/6$. Because of $z \in B(\omega)$, there is a time $\tau(z) < T$ such that $\phi_2(z, \omega) \leq -\alpha x_1$. Again, due to Lemma 5.1, $z_n$ cannot be in $R(\omega)$. Since this is a contradiction, we have $z \not\in B(\omega)$ and therefore $z \in R(\omega)$. Thus, $R(\omega)$ is closed and, by symmetry, so is $B(\omega)$. Therefore the proof of the lemma is complete. \(\square\)

5.3 Proof of Theorem 3.2

Note that with the lower bound on the $x$-component (see (3)) we have the following inclusion

$$\{\sup_{z \in I} \phi_1^{(1)}(z) = \infty\} \supset \{\sup_{z \in I} \tau(z) > T\} \cap \{\sup_{t \in [0, T]} \sigma|W_t^{(1)}| \leq c\} =: A.$$

We show that the probability of $A$ already tends to 1 as $x_0 \to \infty$.

$$P(A) \geq P\left(A, \sup_{t \in [0, T]} \sigma|W_t^{(2)}| \leq \frac{\alpha}{8} x_1\right) \geq 1 - P\left(\sup_{t \in [0, T]} \sigma|W_t^{(1)}| > c\right) - P\left(\sup_{t \in [0, T]} \sigma|W_t^{(2)}| > \frac{\alpha}{8} x_1\right)$$

which converges to 1 as $x_0 \to \infty$ (which implies $x_1 \to \infty$ and $T \to 0$). This completes the proof.

Remark 5.4

We never used any specific properties of the Brownian motions $W^{(1)}, W^{(2)}$, apart from the fact that both are processes which start in 0 and have continuous paths. Note that in this case the SDE (2) written in integral form can be solved pathwise for each $\omega \in \Omega$ and the local maximal solutions depend continuously upon the initial condition, so all arguments above remain valid in this case. Depending on the nature of the noise, the equation may or may not be complete.

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M. Leimbach, Technische Universität, Berlin, MA 7-5, Str. des 17.Juni 135, 10623 Berlin, Germany, leimbach@math.tu-berlin.de

M. Scheutzow, Technische Universität, Berlin, MA 7-5, Str. des 17.Juni 135, 10623 Berlin, Germany, ms@math.tu-berlin.de