Gamma factors of level zero supercuspidal representations

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Abstract. We give an explicit formula for the twisted gamma factor for a pair of irreducible supercuspidal representations of level zero. We also obtain an explicit formula for the unramified base change of level zero supercuspidal representations.

0. Introduction

0.1. Main result and Motivation. Throughout this note, let $F$ be a $p$-adic field with residue field of $q$ elements. Denote by $\mathfrak{o}$ and $\mathfrak{p}$ the ring of integers of $F$ and the maximal ideal of $\mathfrak{o}$.

Let $\pi$ and $\tau$ be irreducible smooth representations of $\text{GL}_n(F)$ and $\text{GL}_m(F)$. Let $\psi$ be a nontrivial character of the additive group of $F$. The gamma factor $\gamma(s, \pi \times \tau, \psi)$ was first introduced in [JPSS], along with an $L$-function $L(s, \pi \times \tau)$ and a local constant $\varepsilon(s, \pi \times \tau, \psi)$.

It is desirable to have explicit formulae for these gamma factors. By [JPSS], the computation can be reduced to the case of supercuspidal representations. When $\tau = 1_F$ is the trivial representation of $\text{GL}_1(F)$, then $\gamma(s, \pi \times 1_F, \psi)$ is nothing but the gamma factor $\gamma(s, \pi, \psi)$ in the sense of Godement-Jacquet [GJ]. There are explicit formulae for $\gamma(s, \pi, \psi)$ in terms of “non abelian Gauss sums” (See [Bus], [BF]; also [BH2] for the $\text{GL}_2$ case). On the other hand, in the twisted case, one has only an explicit formula for the conductor of the local constants $\varepsilon(s, \pi \times \tau, \psi)$ (See [BHK2]).

In this short note, we confine ourselves to irreducible supercuspidal representations of level zero. A smooth representation $\pi$ of $\text{GL}_n(F)$ is called of level zero if it contains a nontrivial fixed vector for the subgroup $1 + \mathfrak{p} \mathfrak{M}_n(\mathfrak{o})$ of $\text{GL}_n(\mathfrak{o})$. Our main result is

Theorem 0.1. Let $\pi$ resp. $\tau$ be an irreducible supercuspidal representation of level zero of $\text{GL}_n(F)$ resp. $\text{GL}_m(F)$, with $n > m$. Let $\psi$ be a character of $F$ of level one, that is, $\psi$ is trivial on $\mathfrak{p}$ but not on $\mathfrak{o}$. Then

$$\gamma(s, \pi \times \tau, \psi) = (-1)^{nm-(n,m)} q^{-nm/2} \prod_{i=0}^{(n,m)-1} G(\tilde{\eta}_\pi^q \circ \mathcal{N}_{[n,m],n} \cdot \tilde{\eta}_\tau \circ \mathcal{N}_{[n,m],m}, \tilde{\psi}).$$

(0.1)

Here

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• \((n, m)\) is the greatest common divisor of \(n\) and \(m\);  
• \(\tilde{\eta}_n\) resp. \(\tilde{\eta}_r\) is the regular character of \(\mathbb{F}_q^\times\) resp. \(\mathbb{F}_q^\times\) that corresponds to \(\tilde{\eta}\) resp. \(\tilde{\tau}\) via Green’s parameterization (see \((3.2)\) \((3.3)\)), while \(\tilde{\eta}\) resp. \(\tilde{\tau}\) is the cuspidal representation of \(\text{GL}_n(\mathbb{F}_q)\) resp. \(\text{GL}_m(\mathbb{F}_q)\) which is uniquely determined by \(\eta\) resp. \(\tau\) (see \((3.3)\) \((3.6)\));  
• \(N_{[n,m]} r\) resp. \(N_{[n,m]} m\) is the norm map from \(\mathbb{F}_q^{[n,m]}\) to \(\mathbb{F}_q^\times\) resp. \(\mathbb{F}_q^\times\);  
• \(\psi\) is the character on \(\kappa_F \cong \mathbb{F}_q\) which is induced by \(\psi\), \(\tilde{\psi}(x) = \psi(x)\) for all \(x \in \mathfrak{o}\).

and

\[
G(\beta, \varphi) := \sum_{a \in \mathbb{F}_q^\times} \beta(a)\varphi(\text{Tr}a^{-1}) = \sum_{a \in \mathbb{F}_q^\times} \beta^{-1}(a)\varphi(\text{Tr}a)
\]

is the Gauss sum for a multiplicative character \(\beta\) of \(\mathbb{F}_q^\times\) and an additive character \(\varphi\) of \(\mathbb{F}_q\), where \(\text{Tr}\) is the trace map from \(\mathbb{F}_q^\times \to \mathbb{F}_q\).

The motivation of our work arises from a question on gamma factors over finite fields that was proposed in [NZ Conjecture 2.2]. A formula similar to \((0.1)\) for gamma factors over finite fields is given in Theorem \((6.3)\). Let us add some words on the strategy of our method. We first use a result in [BHK1] to reduce the general case to the case \(m = 1\). The main point of our work is then to determine the base change, in the sense of [AC], of a level zero supercuspidal representation. We use the explicit local Langlands correspondence for level zero representations, developed in [BH3], to transfer the question to the Galois side. When \(m = 1\), the computation is a consequence of a result on gamma factors over finite fields [Nic2] and a connection formula between gamma factors for level zero supercuspidal representations and gamma factors over finite fields [NZ].

\section*{0.2. Notations and Conventions.} (1) For a smooth representation \(\pi\) of \(\text{GL}_n(F)\), denote by \(\pi^\vee\) the smooth dual of \(\pi\). If \(\pi\) is irreducible, denote by \(\omega_n\) the central character of \(\pi\). For representations \(\pi_i\) of \(\text{GL}_{n_i}(F), i = 1, \ldots, r\), denote by \(\pi_1 \times \cdots \times \pi_r\) the representation of \(\text{GL}_{n_1+\cdots+n_r}(F)\) obtained from \(\pi_1 \otimes \cdots \otimes \pi_r\) by normalized parabolic induction. For a representation \(\pi\) and a character \(\chi\) of \(\text{GL}_n(F)\), denote \(\chi\pi\) to be the representation on the space of \(\pi\) given by \((\chi\pi)(g) = \chi(g)\pi(g)\). Let \(|\cdot|\) denote the normalized absolute value on \(F\) and \(\nu\) denote the character \(|\cdot|\circ \det\) of \(\text{GL}_n(F)\).

(2) Denote by \(U\) the group of units in \(\mathfrak{o}\), by \(\kappa = \mathfrak{o}/\mathfrak{v}\) the residue field of \(F\). We fix a uniformizer \(\sigma\) in \(\mathfrak{v}\).

If \(E/F\) is a finite field extension, we use the analogous notations \(\mathfrak{v}_E, \mathfrak{p}_E, U_E, \) etc. The norm map \(E^\times \to F^\times\) is denoted \(N_{E/F}\), and the trace \(E \to F\) is \(\text{Tr}_{E/F}\).

We fix a separable algebraic closure \(\bar{F}\) of \(F\). All finite extension of \(F\) are supposed to be contained in \(\bar{F}\). Hence an unramified extension over \(F\) of a fixed degree is unique. For \(E/F\) a finite unramified extension, we still use \(\sigma\) as a uniformizer in \(\mathfrak{p}_E\).

For a finite field extension \(K/F\), denote by \(\mathcal{W}_K\) the Weil group of \(K\). We will view \(\mathcal{W}_K\) as a subgroup of \(\mathcal{W}_F\) as in [BH2] \S 28.5]. Let \(a_F : \mathcal{W}_F \to F\) denote the Artin
reciprocity map ([BH2] §29.1). We fix a geometric Frobenius element \( \Phi \) in \( \mathcal{W}_F \) and an arithmetic Frobenius element \( \phi = \Phi^{-1} \).

If \( E/F \) is a finite extension, we write \( \text{Ind}_{E/F} \) rather than \( \text{Ind}^{W_F}_{W_E} \) for the functor of smooth induction from \( \mathcal{W}_E \) to \( \mathcal{W}_F \). Also, if \( \rho \) is a smooth representation of \( \mathcal{W}_F \), we put \( \text{Res}_{E/F} \rho = \rho|_{\mathcal{W}_E} \).

1. Preliminaries

1.1. Gamma factors for Weil representations. Let \( \mathcal{G}_n^{ss}(F) \) denote the set of isomorphism classes of semisimple smooth representations of \( \mathcal{W}_F \) of dimension \( n \) and write \( \mathcal{G}_n^{ss}(F) = \bigcup_{n \geq 1} \mathcal{G}_n^{ss}(F) \). For \( \sigma \in \mathcal{G}_n^{ss}(F) \) and a nontrivial character \( \psi_F \) of \( F \), let \( L(s, \sigma) \) be the Artin \( L \)-function and \( \varepsilon(s, \sigma, \psi_F) \) be the local constant defined by Langlands and Deligne (see [Del]). Denote by \( \sigma^\vee \) the contragradient representation of \( \sigma \). Similar to the definition of local gamma factors in [JPSS], we define

\[
\gamma(s, \sigma, \psi_F) = \frac{\varepsilon(s, \sigma, \psi_F) L(1 - s, \sigma^\vee)}{L(s, \sigma)}.
\]

Suppose that \( \sigma_1, \sigma_2 \in \mathcal{G}_n^{ss}(F) \). By the additive properties of \( L \) and \( \varepsilon \) ([BH2] §29.3,29.4), we have

\[
\gamma(s, \sigma_1 \oplus \sigma_2, \psi_F) = \gamma(s, \sigma_1, \psi_F) \gamma(s, \sigma_2, \psi_F).
\]

Let \( K/F \) be a finite field extension and \( \rho \in \mathcal{G}_n^{ss}(K) \). Note that \( (\text{Ind}_{K/F} \rho)^\vee \equiv \text{Ind}_{K/F} \rho^{\vee} \).

By the inductive properties of \( L \) and \( \varepsilon \) ([BH2] §29.3,29.4), we have

\[
\frac{\gamma(s, \text{Ind}_{K/F} \rho, \psi_F)}{\gamma(s, \rho, \psi_K)} = \frac{\gamma(s, \text{Ind}_{K/F} 1_K, \psi_F)^n}{\gamma(s, 1_K, \psi_K)^n},
\]

where \( 1_K \) stands for the trivial representation of \( \mathcal{W}_K \) and \( \psi_K = \psi_F \circ \text{Tr}_{K/F} \).

Remark 1.1. The functions \( L \) and \( \varepsilon \) can be defined for Weil-Deligne representations of \( F \) (see [BH2] §31.3), so is \( \gamma \) by (1.1). It turns out that \( \gamma(s, (\sigma, n), \psi_F) = \gamma(s, \sigma, \psi_F) \), where we denote by \( (\sigma, n) \) a Weil-Deligne representation and \( n \) is the nilpotent part.

1.2. Local Langlands correspondence for \( \text{GL}_n(F) \). Let \( \mathcal{A}_n(F) \) be the set of isomorphism classes of irreducible smooth representations of \( \text{GL}_n(F) \) and \( \mathcal{A}_n^0(F) \) its subset consisting of supercuspidal representations. Let \( \mathcal{G}_n(F) \) be the set of isomorphism classes of semisimple Weil-Deligne representations of \( F \) of dimension \( n \). Denote by \( \mathcal{G}_n^0(F) \) the set of isomorphism classes of irreducible smooth representations of \( \mathcal{W}_F \) of dimension \( n \). We identify \( \mathcal{G}_n^0(F) \) as a subset of \( \mathcal{G}_n(F) \) by letting the nilpotent part be 0.

The Langlands correspondence for \( \text{GL}_n \) over \( F \), proved by Harris-Taylor [HT] and by Henniart [Hen2], is the unique collection of bijections

\[
I^F_n : \mathcal{A}_n(F) \rightarrow \mathcal{G}_n(F)
\]

[1] In this note, we do not need the nilpotent part in Weil-Deligne representations.
such that the map $I_{n}^{F}$ is given by the local class field theory, and that for $\pi \in \mathcal{A}_{n}(F)$, $\pi' \in \mathcal{A}_{m}(F)$, we have
\[
L(s, \pi \times \pi') = L(s, I_{n}^{F}(\pi) \otimes I_{m}^{F}(\pi')),
\]
\[
e(s, \pi \times \pi', \psi) = e(s, I_{n}^{F}(\pi) \otimes I_{m}^{F}(\pi'), \psi).
\]
Hence
\[
(1.5) \quad \gamma(s, \pi \times \pi', \psi) = \gamma(s, I_{n}^{F}(\pi) \otimes I_{m}^{F}(\pi'), \psi).
\]

The maps $\{I_{n}^{F}\}$ are first constructed for $\mathcal{A}_{0}^{G}(F)$ and then extended to $\mathcal{A}_{n}(F)$, using the classification of Langlands, Bernstein and Zelevinsky. We recall briefly this process (see [Hen1 Chapter 2]). According to [Zel 9.3], each essentially square-integrable representation $\pi$ is of the form $St_{k}(\rho)$, for some $k \geq 1$ and $\rho \in \mathcal{A}_{0}^{G}(F)$, where $St_{k}(\rho)$ denotes the unique irreducible quotient of $\rho \times \nu \times \cdots \times \nu^{k-1}\rho$. We have
\[
I_{n}^{F}(St_{k}(\rho)) = I_{n}^{F}(\rho) \otimes \text{Sp}_{k},
\]
where $\text{Sp}_{k}$ denotes the special Weil-Deligne representation of $\mathcal{W}_{F}$ of dimension $k$ ([BH2 §31.1]). Denote by $\mathcal{A}_{n}^{G}(F)$ the isomorphism classes of essentially square-integrable representations. For every $\pi \in \mathcal{A}_{n}^{G}(F)$, there is a unique $\alpha(\pi) \in \mathbb{R}$ such that $\nu^{-\alpha}\pi$ is unitary and square-integrable. Let $r \geq 1$ and $\pi_{1}, \cdots, \pi_{r}$ be elements of $\mathcal{A}_{n}^{G}(F)$. Suppose that
\[
(\ast) \quad \alpha(\pi_{1}) \geq \cdots \geq \alpha(\pi_{r}).
\]
Then the representation $\pi_{1} \times \cdots \times \pi_{r}$ has a unique irreducible quotient $J(\pi_{1}, \cdots, \pi_{r})$. By the Langlands classification ([Sil]), each $\pi \in \mathcal{A}_{n}(F)$ is of the form $J(\pi_{1}, \cdots, \pi_{r})$ for some $r \geq 1$ and $\pi_{1}, \cdots, \pi_{r} \in \mathcal{A}_{n}^{G}(F)$. The isomorphism class of $J(\pi_{1}, \cdots, \pi_{r})$ does not depend on the order of $\pi_{i}$ provided that the condition (3) is satisfied. We have
\[
I_{n}^{F}(J(\pi_{1}, \cdots, \pi_{r})) = I_{n_{1}}^{F}(\pi_{1}) \oplus \cdots \oplus I_{n_{r}}^{F}(\pi_{r}),
\]

2. Base change and automorphic induction

A key ingredient in our method is to use a formula on the twisted local gamma factors which compares base change and automorphic induction.

Let $K/F$ be a cyclic extension. Recall that, in [AC §1.6], base change associates to every $\pi \in \mathcal{A}_{n}^{G}(F)$ an irreducible smooth representation $\pi_{K/F}$ of $\text{GL}_{n}(K)$. Via the Langlands correspondence, the base change from $F$ to $K$ corresponds to the restriction to $\mathcal{W}_{K}$ from representations of $\mathcal{W}_{F}$ ([BH1 §3.1]). So, we have
\[
(2.1) \quad \text{Res}_{K/F}(I_{n}^{F}(\pi)) = I_{n}^{K}(\pi_{K/F}).
\]

Assume that $K/F$ is cyclic of degree $d$. According to [HH], automorphic induction sends any generic $\rho$ in $\mathcal{A}_{n}(K)$ to a unique $\rho^{K/F} \in \mathcal{A}_{nd}(F)$ (see also [BH1 §2.5]). The automorphic induction from $K$ to $F$ corresponds, again via the Langlands correspondence,

\footnote{The definition of $\pi_{K/F}$ was extended to all $\pi \in \mathcal{A}_{n}(F)$ in [AC pp. 59-60], but we do not need this fact in our context.}
to the induction to $W_F$ from representations of $W_K$, i.e.,
\begin{equation}
\text{Ind}_{K/F}(I^K_F(\rho)) = I^F_n(\rho^{K/F}),
\end{equation}
where $\rho$ is generic ([BHK1 §3.9]).

The following result can be found in [BHK1 A.8]. We supply here a “proof” different than that in loc.cit.

**Proposition 2.1.** Assume $K/F$ to be a cyclic extension. Let $\pi \in \mathcal{A}_n^0(F)$ and $\rho \in \mathcal{A}_n^0(K)$ with $\rho$ generic. Let $\psi_F$ be a nontrivial character of $F$. Then
\begin{equation}
\gamma(s, \pi \times \rho^{K/F}, \psi_F) = \gamma(s, \pi_{K/F} \times \rho, \psi_K)
\end{equation}
where $\Xi$ is the group of characters of $F^\times$ that is trivial on $N_{K/F}(K^\times)$ and $\psi_K := \psi_F \circ \text{Tr}_{K/F}$.

**Proof.** We omit the subscripts in the $I^n_F$.
\begin{equation}
\gamma(s, \pi \times \rho^{K/F}, \psi_F) = \gamma(s, I(\pi) \otimes I(\rho^{K/F}), \psi_F)
= \gamma(s, I(\pi) \otimes \text{Ind}_{K/F} I(\rho), \psi_F)
= \gamma(s, \text{Ind}_{K/F}(\text{Res}_{K/F} I(\pi) \otimes I(\rho)), \psi_F).
\end{equation}
In view of (1.3), we have
\begin{equation}
\gamma(s, \text{Ind}_{K/F}(\text{Res}_{K/F} I(\pi) \otimes I(\rho)), \psi_F)
= \gamma(s, \text{Res}_{K/F} I(\pi) \otimes I(\rho), \psi_K).
\end{equation}

Note that, by local class field theory, $\Xi$ corresponds to the group of characters of $W_F$ that is trivial on $W_K$. The equality (2.3) follows then from the local Langlands correspondence (1.5) and the simple fact that
\begin{equation}
\text{Ind}_H^G 1_H = \bigoplus_{\chi \in G/H} \chi,
\end{equation}
where $H$ is a normal subgroup of $G$ of finite index such that $G/H$ is abelian, $1_H$ is the trivial representation of $H$ and $\overline{G/H}$ is the group of characters of $G/H$. Q.E.D.

**Remark 2.2.** Strictly speaking, this is not really a proof, as the formula (2.3) predates and is used to establish the local Langlands correspondence for $\text{GL}_n(F)$, while we use the Langlands correspondence in the “proof”. The original proof in [BHK1] involves a global method.

3. Gamma factors over finite fields

3.1. Basic facts. In her thesis [Rod], Roditty considered a finite field analogue of Rankin-Selberg convolution in the same way as in [JPSS]. There she defined the gamma
factor (a complex number in this case, instead of a function!) as the ratio appeared in certain functional equations.

We fix a nontrivial character $\psi$ of $\mathbb{F}_q$ in this subsection. Let $U_n(\mathbb{F}_q)$ be the group of upper triangular unipotent matrices in $GL_n(\mathbb{F}_q)$. Let

$$\psi_n(u) = \psi\left(\sum_{i=1}^{n-1} u_{i,i+1}\right), \quad \text{for } u = (u_{ij}) \in U_n(\mathbb{F}_q).$$

If $\pi$ is a generic representation of $GL_n(\mathbb{F}_q)$, denote by $W(\pi, \psi_n)$ the Whittaker model of $\pi$ with respect to $\psi_n$. We refer to [Nie1] for the undefined notations in this subsection.

**Theorem 3.1** ([Rod], Theorem 5.1, 5.4 or [Nie1], Theorem 2.10). Let $\pi$ be an irreducible cuspidal representation of $GL_n(\mathbb{F}_q)$ and $\tau$ an irreducible generic representation of $GL_m(\mathbb{F}_q)$, with $n > m$. Then there exists a complex number $\gamma(\pi \times \tau, \psi)$ such that

$$\gamma(\pi \times \tau, \psi) q^m \sum_{g \in U_n(\mathbb{GL}_m(\mathbb{F}_q))} \sum_{x \in M_{n-m-1-i,m}} W_\pi\left(\begin{pmatrix} g & 0 & 0 \\ x & I_{n-m-1-k} & 0 \\ 0 & 0 & I_{k+1}\end{pmatrix}\right) W_\tau(g)$$

$$= \sum_{g \in U_n(\mathbb{GL}(\mathbb{F}_q))} \sum_{y \in M_{n-k}} W_\pi\left(\begin{pmatrix} 0 & I_{n-m-k} & 0 \\ 0 & 0 & I_k \\ g & 0 & y\end{pmatrix}\right) W_\tau(g),$$

for all $0 \leq k \leq n - m - 1$, $W_\pi \in W(\pi, \psi_n)$ and $W_\tau \in W(\tau, \psi_m^{-1})$.

There is a distinguished element in the Whittaker model that can be used to compute the gamma factors.

**Proposition 3.2** ([Gel], Proposition 4.5). Let $\pi$ be an irreducible generic representation of $GL_n(\mathbb{F}_q)$ with $\chi_\pi$ its (trace) character. The function

$$J_{\pi,\psi_n}(g) = |U_n(\mathbb{F}_q)|^{-1} \sum_{u \in U_n(\mathbb{F}_q)} \chi_\pi(gu) \psi_n(u^{-1})$$

lies in $W(\pi, \psi_n)$ and satisfies

$$J_{\pi,\psi_n}(u_1gu_2) = \psi_n(u_1u_2) J_{\pi,\psi_n}(g), \quad u_i \in U_n(\mathbb{F}_q) \quad \text{and} \quad J_{\pi,\psi_n}(I_n) = 1.$$

The function $J_{\pi,\psi_n}$ is called the (normalized) Bessel function of $\pi$ with respect to $\psi_n$. In terms of Bessel functions, we have the following

**Proposition 3.3** ([Rod], Lemma 6.1.4). Let $\pi$ be an irreducible cuspidal representation of $GL_n(\mathbb{F}_q)$ and $\tau$ an irreducible generic representation of $GL_m(\mathbb{F}_q)$, $n > m$. Then

$$\gamma(\pi \times \tau, \psi) = \sum_{g \in U_n(\mathbb{GL}_m(\mathbb{F}_q))} J_{\pi,\psi_n}\left(\begin{pmatrix} I_{n-m} & 0 \\ g & 0\end{pmatrix}\right) J_{\tau,\psi_n^{-1}}(g).$$

3.2. $n \times 1$ gamma factors. We recall Green’s parameterization [Gre] for irreducible cuspidal representations of $GL_n(\mathbb{F}_q)$. A character $\eta$ of $\mathbb{F}_q^\times$ is called $\mathbb{F}_q$-regular if the conjugates $\eta^s$, $s \in \text{Gal}(\mathbb{F}_q^s/\mathbb{F}_q)$ are distinct. Two characters $\eta_1$ and $\eta_2$ are called equivalent if $\eta_1 = \eta_2^j$ for some integer $j$, i.e., they are in the same Frobenius orbit.
Remark 3.4. A character $\eta$ is $\mathbb{F}_q$-regular if and only if $\eta$ cannot factor through $N_{\mathbb{F}_q}/\mathbb{F}_q$ for some subextension $\mathbb{F}_{q'}/\mathbb{F}_q$.

Let $\Lambda_n(\mathbb{F}_q)$ denote the set of equivalence classes of $\mathbb{F}_q$-regular characters of $\mathbb{F}_q^\times$. Let $\mathcal{R}_n^\theta(\mathbb{F}_q)$ denote the set of isomorphism classes of irreducible representations of $GL_n(\mathbb{F}_q)$.

Green’s parameterization gives a bijection
\[
\mathcal{R}_n^\theta(\mathbb{F}_q) \leftrightarrow \Lambda_n(\mathbb{F}_q)
\]
\[
\pi \leftrightarrow \eta_n
\]
The (trace) character $\chi_\pi$ of $\pi$ is connected with $\eta_\pi$ in an explicit way (see [Gel] §6). So, by (3.1), the Bessel function $J_\pi(\phi_\chi)$ can be expressed in terms of $\eta_\pi$.

In view of (3.2), Nien computes the $n \times 1$ gamma factor as an “abelian Gauss sum” using special values of Bessel functions. Denote by $\mathbb{F}_{q'}$ the set of characters of $\mathbb{F}_q^\times$.

Proposition 3.5 ([Nie2], Theorem 1.1). Let $\pi$ be an irreducible cuspidal representation of $GL_n(\mathbb{F}_q)$, $n \geq 2$ and $\tau \in \mathbb{F}_q^\times$. Then
\[
\gamma(\pi \times \tau, \psi) = (-q^{-1} \tau(-1))^{n-1} G(\eta_\pi \cdot \tau \circ N_{\mathbb{F}_q}, \psi).
\]
where $\eta_\pi$ is the regular character of $\mathbb{F}_q^\times$, which corresponds to $\pi$ by Green’s parameterization; the Gauss sum $G$ is as defined in (0.2).

3.3. Connection with local gamma factors over $p$-adic fields. We recall a result of Nien and Zhang which shows that gamma factors over finite fields and gamma factors for level zero supercuspidal representations over $p$-adic fields are closely related.

A representation $\pi \in \mathcal{R}_n(\mathbb{F}_q)$ is called of level zero if it contains a nontrivial fixed vector for the subgroup $1 + pM_n(0)$ of $GL_n(0)$. According to [BK] Theorem 8.4.1, every level zero supercuspidal representation is of the form
\[
\pi \cong c\text{-Ind}_{F^\times GL_n(0)}^{GL_n(F)} \chi \sigma,
\]
where $\sigma$ is a representation of $GL_n(0)$ that is inflated from an irreducible cuspidal representation $\tilde{\sigma}$ of $GL_n(\mathbb{F}_q) = GL_n(0)/p$, $\chi$ is a character of $F^\times$ such that $\chi|_{U_q}$ equals to the central character of $\sigma$ and $c\text{-Ind}$ is the compact induction.

The character $\chi$ and the representation $\tilde{\sigma}$ are uniquely determined by $\pi$. Let $\mathcal{R}_n^\theta(F)_0$ denote the set of isomorphism classes of level zero supercuspidal representations of $GL_n(F)$. Theorem 8.4.1 in [BK] then gives a bijection
\[
\mathcal{R}_n^\theta(F)_0 \leftrightarrow C^\times \times \mathcal{R}_n^\theta(\mathbb{F}_q)
\]
\[
\pi \leftrightarrow (\omega_\pi(\pi), \tilde{\sigma}).
\]
Convention: For the rest of this note, if $\pi \in \mathcal{R}_n^\theta(F)_0$, we shall always denote by $\tilde{\pi}$ the second component of the image of $\pi$ under the bijection (3.6).

Theorem 3.6 ([NZ], Theorem 3.11). Let $\pi \in \mathcal{R}_n^\theta(F)_0$ and $\tau \in \mathcal{R}_m^\theta(F)_0$, $n > m$. Suppose that $\psi$ has level one. Then
\[
\gamma(s, \pi \times \tau, \psi) = \omega_{\tau}(-1)^{n-1} q^{(n-m-1)/2} \gamma(\tilde{\pi} \times \tilde{\tau}, \tilde{\psi}),
\]
where $\tilde{\psi}$ is the character of $\kappa_F \cong \mathbb{F}_q$ that is induced by $\psi$.

4. Explicit Local Langlands Correspondence for level zero supercuspidal representations

To determine the base change of a representation as indicated in Section 2, we need an explicit knowledge of the local Langlands correspondence. The explicit correspondence for level zero representations of $GL_n(F)$ was worked out by Bushnell and Henniart in [BH3]. We recall it here.

4.1. Admissible tame pairs. Recall that a tame pair over $F$ consists of a finite unramified field extension $E/F$ and a character $\theta$ of $E^\times$ that is trivial on $U_E^1 = 1 + p_E$. A tame pair $(E/F, \theta)$ is called admissible if the conjugates $\theta^s, s \in \text{Gal}(E/F)$, are distinct. The following lemma is easy and can be found in [BH2 §19.1].

**Lemma 4.1.** Let $(E/F, \theta)$ be a tame pair. The following statements are equivalent:

(i) The pair $(E/F, \theta)$ is admissible;

(ii) The restrictions $\theta^s|_{U_E}, s \in \text{Gal}(E/F)$ are distinct;

(iii) The character $\theta$ cannot factor through $N_{E/K}$ for any subextension $K/F$.

Two admissible tame pairs $(E_i/F, \theta_i), i = 1, 2$, are called $F$-isomorphic if there is an $F$-isomorphism $\alpha : E_1 \to E_2$ such that $\theta_1 = \theta_2 \circ \alpha$. The degree of $(E/F, \theta)$ is the degree $[E : F]$ of the extension $E/F$. Denote by $\mathcal{T}_n(F)$ the $F$-isomorphism classes of admissible tame pairs of degree $n$.

Let $(E/F, \theta)$ be a tame pair of degree $n$. As $\theta$ is trivial on $U_E^1$, it reduces to a character $\tilde{\theta}$ of $U_E/U_E^1 \cong \kappa_E^\times \cong \mathbb{F}_q^\times$. The pair $(E/F, \theta)$ is admissible if and only if $\tilde{\theta}$ is $\mathbb{F}_q$-regular.

Recall that $\Lambda_n(\mathbb{F}_q)$ is denoted as the set of equivalence classes of $\mathbb{F}_q$-regular characters of $\mathbb{F}_q$. Then we have a bijection

$$
\begin{align*}
\mathcal{T}_n(F) & \leftrightarrow \mathbb{C}^\times \times \Lambda_n(\mathbb{F}_q) \\
(E/F, \theta) & \leftrightarrow (\theta(\sigma), \tilde{\theta}).
\end{align*}
$$

4.2. Parameterization of level zero representations. Firstly, composing (4.1) with Green’s parameterization (3.3) and then the description of level zero supercuspidal representations (3.6), we therefore have a bijective map

$$
\pi_n : \mathcal{T}_n(F) \to \mathcal{P}_n^0(F),
$$

$$
(E/F, \theta) \mapsto \pi_n(\theta).
$$

Secondly, let $(E/F, \theta) \in \mathcal{T}_n(F)$. We view $\theta$ as a character of $\mathcal{W}_E$ via Artin’s reciprocity map $a_E$, and form the smooth induced representation

$$
\sigma_n(\theta) = \text{Ind}_{\mathcal{W}_E}^{\mathcal{W}_F} \theta = \text{Ind}_{E/F} \theta.
$$

Recall that $\sigma \in \mathcal{G}_n^0(F)$ is called of level zero if $\sigma$ is trivial on the wild inertia subgroup of $\mathcal{W}_F$ (see [BH3]). Denote by $\mathcal{G}_n^0(F)_0$ the subset of $\mathcal{G}_n^0(F)$ consisting of classes of representations of level zero. The following result is contained in [BH1] A2.A3.
Proposition 4.2. Let \((E/F, \theta)\) be an admissible tame pair of degree \(n\). Then

1. The representation \(\text{Ind}_{E/F} \theta\) is irreducible and of level zero. The equivalence class of \(\text{Ind}_{E/F} \theta\) depends only on that of \((E/F, \theta)\);
2. The map
   \[
   \sigma_n: \mathcal{T}_n(F) \longrightarrow \mathcal{G}_n^0(F) \quad (E/F, \theta) \longrightarrow \text{Ind}_{E/F} \theta
   \]
   is a bijection.

4.3. Explicit correspondence for level zero supercuspidal representations. It turns out that \(\sigma_n \circ \pi^{-1}\) is not the local Langlands correspondence for level zero supercuspidal representations. Some modifications have to be made.

Proposition 4.3 ([BH3], Theorem 2). Let \((E/F, \theta)\) be an admissible tame pair of degree \(n\). Define \(\Delta_E\) to be the unique unramified character of \(E^\wedge\) of order 2. Then
\[
\iota_F^n(\pi_n(\theta)) = \sigma_n(\Delta_E^{n-1} \theta).
\]

5. Unramified base change

We determine the unramified base change of a level zero supercuspidal representation in this section.

For two unramified extensions \(K\) and \(E\) over \(F\), we will write \(KE\) as the compositum of \(K\) and \(E\), which is still an unramified extension. Recall that \(\phi\) is a fixed arithmetic Frobenius element in \(W_F\). For two nonnegative integers \(n\) and \(m\), let \((n, m)\) denote as usual the greatest common divisor of \(n\) and \(m\).

Proposition 5.1. Let \((E/F, \theta)\) be an admissible tame pair of degree \(m\). Assume that \(K/F\) is an unramified field extension of degree \(m\). Then
\[
\pi_{K/F} = \pi_{n/\langle n, m \rangle}(\varsigma_1) \times \cdots \times \pi_{n/\langle n, m \rangle}(\varsigma_{\langle n, m \rangle}),
\]
where, for each \(i\),
\[
\varsigma_i = \Delta_{K/E}^{n/\langle n, m \rangle - 1} \cdot (\Delta_E^{n-1} \theta) \circ \phi_i |_{E} \circ N_{KE/E}
\]
is a character of \((KE)^\wedge\) and \((KE/K, \varsigma_i)\) is an admissible tame pair. The representation on the right hand side of (5.1) does not depend on the order of these \(\pi_{n/\langle n, m \rangle}(\varsigma_i)\).

Proof. By (2.1) and (4.4), we have
\[
\iota_n^K(\pi_{K/F}) = \text{Res}_{K/F} \iota_n^F(\pi) = \text{Res}_{K/F} \text{Ind}_{E/F} \Delta_E^{n-1} \theta.
\]
As \(W_E\) is a normal subgroup of \(W_F\) of finite index, we can apply Mackey’s restriction formula (A.1). Note that, as \(E/F, K/F\) are unramified extensions, the inertia groups \(I_F, I_E\) and \(I_K\) are all the same. Recall that \(\Phi\) is a fixed geometric Frobenius element in \(W_F\), then \(\Phi^n\) (resp. \(\Phi^m\)) is a geometric Frobenius element of \(W_E\) (resp. \(W_K\)). Therefore we can take \(\{\Phi, \cdots, \Phi^{\langle n, m \rangle}\}\) to be a set of representatives of double cosets \(W_K \backslash W_F / W_E\).
Note that $EK$ is the unramified extension of $F$ of degree $[n,m]$, where $[n,m]$ is the least common multiple of $n$ and $m$, and that $W_E \cap W_K = W_{KE}$. Applying (5.1), we get

\[(5.4) \quad \text{Res}_{KE/F} \text{Ind}_{E/F}(\Delta_E^{n-1}\theta) = \sum_{i=1}^{[n,m]} \text{Ind}_{KE/K}(\Phi_i^{(\Delta_E^{n-1}\theta)})|_{W_{KE}}.\]

It follows from [Ser] Proposition 11, §4, Chap XIII] that the character $\Phi_i^{(\Delta_E^{n-1}\theta)}$ of $W_E$ corresponds exactly to the character $(\Delta_E^{n-1}\theta)|_{E} \circ \phi|_E$ of $E$ via local class field theory, where $\phi = \Phi_i^{-1}$ is an arithmetic Frobenius element of $W_F$. Hence the restriction of $\Phi_i^{(\Delta_E^{n-1}\theta)}$ to $W_{KE}$ corresponds to the character $(\Delta_E^{n-1}\theta) \circ \phi|_E \circ N_{KE/E}$ of $(KE)^\times$. So we rewrite (5.4) as

\[(5.5) \quad \text{Res}_{KE/F} \text{Ind}_{E/F}(\Delta_E^{n-1}\theta) = \sum_{i=1}^{[n,m]} \text{Ind}_{KE/K}(\Delta_E^{n-1}\theta) \circ \phi|_E \circ N_{KE/E}.\]

**Claim:** $(KE/K, (\Delta_E^{n-1}\theta) \circ \phi|_E \circ N_{KE/E})$ is an admissible tame pair of degree $n/\langle n,m \rangle$.

In fact, $KE/K$ is an unramified field extension of degree $|n, m|/m = n/(n,m)$. Set $\xi = (\Delta_E^{n-1}\theta) \circ \phi|_E \circ N_{KE/E}$. As $N_{KE/E}(U_{KE}^1) \subset U_E^1$ and $\theta$ is trivial on $U_{KE}^1$, $\xi$ is trivial on $U_E^1$. So $(KE/K, \xi)$ is by definition a tame pair of degree $n/(n,m)$. We are then left to show that $(KE/K, \xi)$ is admissible. Suppose that $\xi$ is not admissible, then it is fixed by a subgroup $G' \subset \text{Gal}(KE/K)$, where $\text{Gal}(KE/K)$ is isomorphic to $\text{Gal}(KE/E)$. As the orders of $\text{Gal}(KE/K)$ and $\text{Gal}(KE/E)$ are coprime, we infer that $\xi$ is fixed by a subgroup of $\text{Gal}(KE/F)$ that is strictly larger than $\text{Gal}(KE/E)$. By the arguments in the proof of Lemma (5.1), this means that $\xi$ factors through $N_{KE/L}$, where $L/F$ is a subextension of $E/F$. By the transitivity of the norm map and the fact that $N_{KE/E}$ is surjective, we conclude that $(\Delta_E^{n-1}\theta) \circ \phi|_E$ factors through $N_{E/L}$. This implies easily that $\theta$ also factors through $N_{E/L}$, which contradicts the assumption that $(E/F, \theta)$ is an admissible pair.

Therefore, by (5.4), we have

\[(5.6) \quad \text{Ind}_{KE/K}((\Delta_E^{n-1}\theta) \circ \phi|_E \circ N_{KE/E} = L_{n/(n,m)}(\pi_{n/(n,m)}(\xi) \times \cdots \times \pi_{n/(n,m)}(\xi))\]

with $\xi$ defined in (5.2). Note that no two of the representations $\pi_{n/(n,m)}(\xi)$ and $\pi_{n/(n,m)}(\xi)$ are linked in the sense of [Zel] and that the condition (22) is satisfied, hence the representation

\[\pi_{n/(n,m)}(\xi) \times \cdots \times \pi_{n/(n,m)}(\xi)\]

is irreducible ([Zel] Theorem 4.2]) and does not depend on the order of these $\pi_{n/(n,m)}(\xi)$.

By (7.7), together with (5.6), (5.5) and (5.3), we get

\[L_{n}(\pi_{KE/F}) = \sum_{i=1}^{[n,m]} L_{n/(n,m)}(\pi_{n/(n,m)}(\xi)) = L_{n}(\pi_{n/(n,m)}(\xi) \times \cdots \times \pi_{n/(n,m)}(\xi)).\]

So, by the Langlands correspondence, we are done.

Q.E.D.

In particular, we see that an unramified base change of $\pi \in \mathcal{A}^0_0(F)$ is again supercuspidal if and only if the degree of the base field extension is coprime with $n$. 

---

[127x-73] This text appears to be a continuation of a proof or discussion in a mathematical paper. It involves advanced topics in algebraic geometry, specifically dealing with the irreducibility of certain representations and the Langlands correspondence. The text references various sources, such as [Ser] and [Zel]. The goal is to establish a claim about the irreducibility of a certain set of representations under given conditions, and it makes use of various lemmas and propositions to arrive at this conclusion. The notation and concepts are quite technical and are typical of research-level mathematics in this field.
To make use of (2.3) later in our computation, we need one simple observation. Any level zero supercuspidal representation is automorphically induced by a character.

**Lemma 5.2.** Let \((K/F, \lambda)\) be an admissible tame pair of degree \(m\) and \(\pi_m(\lambda)\) be the representation as in (4.2). Then

\[
\pi_m(\lambda) = (\Delta_{K}^{m-1})^{K/F}.
\]

**Proof.** This follows immediately from (2.2) and (4.4). Q.E.D.

**6. Main result**

Before the proof of our main result, we note that

**Lemma 6.1.** Let \(K/F\) be an unramified field extension of degree \(m\). Let \(\Xi\) be the group of characters of \(F^\times\) that is trivial on \(N_{K/F}(K^\times)\). Suppose that \(\psi\) has level one. Then

\[
\prod_{\chi \in \Xi} \gamma(s, \chi, \psi) = (-1)^{m-1} \gamma(s, 1_K, \psi_K).
\]

**Proof.** The characters in \(\Xi\) are unramified as \(K/F\) is unramified and the norm map \(N_{K/F}\) is surjective. Then by [BH1] §23.4, §23.5, for \(\chi \in \Xi\), we have that

\[
\gamma(s, \chi, \psi) = \frac{q^{s-1/2} - \chi(\sigma)}{1 - \chi(\sigma)} = \frac{q^{s-1/2} - \chi(\sigma)}{q^{s-1} - \chi(\sigma)}.
\]

When \(\chi\) runs over \(\Xi\), the value \(\chi(\sigma)\) runs over the \(m\)-th roots of unity, as the group \(\Xi\) is cyclic of order \(m\). Hence

\[
\prod_{\chi \in \Xi} \gamma(s, \chi, \psi) = (-1)^{m} \frac{q^{m-1/2} - 1}{q^{m-1} - 1} = (-1)^{m-1} \gamma(s, 1_K, \psi_K).
\]

Q.E.D.

**Theorem 6.2.** Let \(\pi \in \mathcal{R}_0^0(F)_0\) and \(\tau \in \mathcal{R}_0^0(F)_0\) with \(n > m\) and \(\psi\) of level one. Then

\[
\gamma(s, \pi \times \tau, \psi) = (-1)^{m-1} \frac{q^{mn/2} \prod_{i=0}^{(n,m)-1} G(\tilde{\eta}_\tau \tilde{q}_i \circ N_{[n,m],n} \cdot \tilde{\eta}_\tau \circ N_{[n,m],m}, \tilde{\psi})}.
\]

where \(\tilde{\eta}_\tau\) resp. \(\tilde{\tau}\) is the regular character of \(\mathbb{F}_{q^n}^\times\) resp. \(\mathbb{F}_{q^m}^\times\) that corresponds to \(\tilde{\eta}\) resp. \(\tilde{\tau}\) via Green’s parameterization and \(G\) is the Gauss sum as defined by (2.2).

**Proof.** We split the proof in the following steps.

Step (1). Suppose that \((K/F, \eta)\) is the admissible tame pair that corresponds to \(\tau\) by \(\pi_m\) (see (4.2)). By Lemma 5.2, we have

\[
\tau = \pi_m(\eta) = (\Delta_{K}^{m-1})^{K/F}.
\]

Applying (2.3), we get

\[
\gamma(s, \pi \times \tau, \psi) = \gamma(s, \pi \times (\Delta_{K}^{m-1})^{K/F}, \psi)
\]

\[
= \gamma(s, \pi_{K/F} \times (\Delta_{K}^{m-1})^{K/F}, \psi) \prod_{\chi \in \Xi} \gamma(s, \chi, \psi_F) \frac{\gamma(s, 1_K, \psi_K)}{\gamma(s, 1_K, \psi_K)}
\]

where 

In view of (3.4), we get (6.6), the regular character which corresponds to the gamma factor on the right hand side of (6.8) is a gamma factor over finite fields. From (6.6), we can also reformulate a formula for gamma factors over finite fields. We have, for each $i$, that
\[
\gamma(s, \pi_{\eta_K}(\chi_i) \times \Delta_{K_i}^{m-1} \eta_K, \psi_K) = \eta_K(-1)^{n/(n,m)-1} q_{[n,m]}^{2-m} G(\eta_K^{q_{[n,m]}^m} \circ N_{[n,m]} \circ \eta_K, \psi_K).
\]
Finally, combining (6.5), (6.7), (6.8) and (6.9) together, we get (6.4).

Q.E.D.

We can also reformulate a formula for gamma factors over finite fields.

**Theorem 6.3.** Let $\pi$ and $\tau$ be irreducible cuspidal representations of $\text{GL}_n(F_q)$ and $\text{GL}_m(F_q)$ respectively, $n > m$. Let $\eta_\pi$ and $\eta_\tau$ be the corresponding $F_q$-regular characters of $F_q^\times$ and of $F_q^\times$ via Green’s parameterization. Then
\[
\gamma(\pi \times \tau, \psi) = (-1)^{n-(n,m)} \eta_\pi(-1)^{n-1} q^{-mn} \prod_{i=0}^{(n,m)-1} G(\eta_\pi^{q_{[n,m]}^m} \circ N_{[n,m]} \circ \eta_\pi, \psi_K).
\]

**Appendix A. Mackey’s restriction formula**

**Proposition A.1.** Let $G$ be a locally profinite group. Let $H$ be an open subgroup, and let $(\sigma, W)$ be a smooth representation of $H$. Let $K$ be a closed subgroup of $G$. There is a
natural isomorphism
\[
\text{Res}_K^G c\text{-Ind}_H^G \sigma \cong \bigoplus_{\sigma' \in \mathcal{K}\setminus G/H} c\text{-Ind}_K^{G\cap \sigma' H} \text{Res}_{K\cap \sigma' H}^G \sigma'
\]
where $\mathcal{K}\setminus G/H$ is the set of representative of cosets $K\setminus G \cap g_i H g_i^{-1}$. Then $\{k^{(j)}_i g_i\}$ is a set of representative of cosets $G/H$. For $w \in \mathcal{W}$, let $f_w \in c\text{-Ind}_H^G \sigma$ such that $f_w$ is supported in $H$ and $f_w(h) = \sigma(h)w$. Suppose $\mathcal{W}$ is a $C$-basis of $W$. Then, by the lemma in [BH2] 2.5, $\{k^{(j)}_i g_i f_w \mid w \in \mathcal{W}\}$ is a $C$-basis of $c\text{-Ind}_H^G \sigma$. For a fixed $g_i = g$, we show that $\{k_i g f_w \mid w \in \mathcal{W}\}$ is a $C$-basis of a $K$-representation that is isomorphic to $c\text{-Ind}_K^{G\cap \sigma' H} \text{Res}_{K\cap \sigma' H}^G \sigma$. This latter representation has a $C$-basis $\{k_i \phi_w \mid w \in \mathcal{W}\}$, where $\phi_w$ is supported in $K\cap \sigma' H$ and $\phi_w(k) = \sigma(g^{-1} k g)w$. Then the map $k_i g f_w \rightarrow k_i \phi_w$ extends linearly and gives the required $K$-isomorphism. In fact, $k_i \phi_w = \lambda(g) k_i g f_w |_{K}$, where $\lambda(g)$ is the left translation by $g$, so the $K$-equivalence follows immediately. Q.E.D.

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