Extended superconformal symmetry and Calogero-Marchioro model

Pijush K. Ghosh

Department of Physics, Ochanomizu University,
2-1-1 Ohtsuka, Bunkyo-ku, Tokyo 112-8610, Japan.

Abstract

We show that the two dimensional Calogero-Marchioro Model (CMM) without the harmonic confinement can naturally be embedded into an extended $SU(1,1|2)$ superconformal Hamiltonian. We study the quantum evolution of the superconformal Hamiltonian in terms of suitable compact operators of the $\mathcal{N}=2$ extended de Sitter superalgebra with central charge and discuss the pattern of supersymmetry breaking. We also study the arbitrary $D$ dimensional CMM having dynamical $OSp(2|2)$ supersymmetry and point out the relevance of this model in the context of the low energy effective action of the dimensionally reduced Yang-Mills theory.

*E-mail address: pijush@degway.phys.ocha.ac.jp
I. INTRODUCTION

The Calogero-Moser-Sutherland (CMS) system is a class of exactly solvable models in one dimension \([1,3]\). These models have been studied extensively from the time of its inception more than thirty years ago and are well understood. There are many higher dimensional generalizations of these models \([3,3]\). Unfortunately, not a single of these models are known to be exactly solvable or integrable. Among all these systems, the two dimensional Calogero-Marchioro model (CMM) deserves a special attention for several reasons. First of all, for a certain value of the coupling constant, different \(n\)-point correlation functions can be calculated analytically by mapping this model to a complex Random Matrix Theory \([7,13,14]\). This model also has been studied extensively \([3,3,16]\) in connection with several condensed matter systems like, Quantum Hall effect, Quantum Dots, two dimensional Bose systems etc., revealing many interesting features.

The purpose of this paper is to unveil one more new feature of this model. We first study the \(D\) dimensional \(N\)-particle super-CMM with \(N D\) bosonic and \(N D\) fermionic degrees of freedom. We show how infinitely many exact eigenstates can be constructed, both in supersymmetry-breaking and supersymmetry-preserving phases, using the dynamical \(OSp(2|2)\) symmetry of the model. We then show, within the specific formalism, only the two dimensional CMM without the harmonic confinement can naturally be embedded into an extended superconformal Hamiltonian. In other words, we construct an extended \(\mathcal{N} = 2\) superconformal version of the two dimensional CMM. This construction is valid for arbitrary values of the coupling constant and also for arbitrary \(N\) number of particles. We study the quantum evolution in terms of suitable compact operators of the extended \(\mathcal{N} = 2\) de Sitter superalgebra. Though we are able to find an infinite number of exact eigenstates of these super-operators, the set is not complete and we
are unable to find the complete spectrum. We also discuss the supersymmetry breaking pattern of the extended $\mathcal{N} = 2$ de Sitter supersymmetry with central charge and show how the half or the complete breakdown of supersymmetry occurs. Finally, we point out the relevance of our findings in the context of super-Yang-Mills theory.

We organize the paper in the following way. We first introduce the conformal CMM model in arbitrary dimensions in the next section. An infinite number of excited eigenstates corresponding to the radial excitations are constructed algebraically using the underlying $SU(1, 1)$ symmetry. We construct the superconformal CMM in arbitrary dimensions in Sec. III. We also obtain infinitely many exact eigenstates using the dynamical $OSp(2|2)$ symmetry of the model. The extended $\mathcal{N} = 2$ superconformal CMM in $D = 2$ is constructed in Sec. IV. The symmetry algebra of the model and the supersymmetry-breaking pattern is discussed. Finally, in Sec. V, we summarize our findings and discuss the relevance of our results. We point out a possible relation between the $D$ dimensional CMM considered in this paper and the low energy effective action of the $D + 1$ dimensional Yang-Mills theory dimensionally reduced to $0 + 1$ dimension.

**II. CONFORMAL CMM**

We first consider the three operators $h$, $D$ and $K$ given by,

\[
    h = \frac{1}{2} \sum_{i,\mu} p_{i,\mu}^2 + \frac{g}{2} (g + D - 2) \sum_{i \neq j} \vec{r}_{ij}^2 + \frac{g^2}{2} \sum_{i \neq j \neq k} (\vec{r}_{ij} \cdot \vec{r}_{ik}) \vec{r}_{ij} \vec{r}_{ik},
\]

\[
    D = -\frac{1}{4} \sum_{i,\mu} \{x_{i,\mu}, p_{i,\mu}\}, \quad K = \frac{1}{2} \sum_{i,\mu} x_{i,\mu}^2, \quad p_{i,\mu} = -i \frac{\partial}{\partial x_{i,\mu}}, \quad \vec{r}_{ij} = \vec{r}_i - \vec{r}_j,
\]

where $\vec{r}_i$ is the $D$ dimensional position vector of the $i$th particle with $x_{i,\mu}$'s as the components and $g$ is the coupling constant. We fix the convention that the Roman indices run from 1 to $N$, while the Greek indices run from 1 to $D$. These three operators admit the $O(2, 1)$ algebra,
\[ [h, \mathcal{D}] = i h, \quad [h, K] = 2i \mathcal{D}, \quad [K, \mathcal{D}] = -i K. \] (2)

For the general conformal Hamiltonian, the many-body interaction of \( h \) (the last two terms) should be replaced by a degree \(-2\) homogeneous function of the coordinates. The Hamiltonian \( h \) describes the CMM without the harmonic confinement. However, the ground-state of \( h \) with the ground-state energy \( E = 0 \) is not even plane-wave normalizable. Following the prescription suggested by de Alfaro, Fubini and Furlan \([17]\) for such quantum mechanical model with conformal symmetry, the quantum evolution can be described by an appropriate compact operator. This compact operator can be constructed from the linear combination of the Hamiltonian \( h \), the Dilatation generator \( \mathcal{D} \) and the conformal generator \( K \). Following \([17]\), we choose this compact operator \( H \) as, \( H = h + K \). The introduction of \( K \) breaks the scale invariance. The operator \( H \) is the \( D \) dimensional CMM.

In one dimension, \( H \) is exactly solvable and known as the rational CMS Hamiltonian. In \( D \geq 2 \), though an infinitely many exact eigenstates of this Hamiltonian can be found, the complete eigen-spectrum is still not known. The ground-state wave-function is determined as \([8,9]\),

\[ \psi_0 = \prod_{i<j} | \vec{r}_i - \vec{r}_j |^g e^{-\frac{1}{2} \sum_i \vec{r}_i^2}, \] (3)

with the ground-state energy \( E_0 = \frac{ND}{2} + gN(N-1)/2 \). Using the underlying \( SU(1,1) \) symmetry,

\[ B_2^\pm = -\frac{1}{2} (h - K \mp 2i \mathcal{D}), \quad [H, B_2^\pm] = \pm 2B_2^\pm, \quad [B_2^-, B_2^+] = H, \] (4)

one can construct infinitely many exact eigenstates of this Hamiltonian. In particular,

\[ \psi_n = (B_2^\pm)^n \psi_0, \] (5)
are exact eigenstates of $H$ with $E_n = E_0 + 2n$. For $D = 3$, these exact eigenstates corresponding to the radial excitations were first obtained in [1] by directly solving the Schrödinger equation. Following the same method, these eigenstates were constructed for arbitrary $D$ in [2]. However, we provide here an algebraic construction of these radial excitations in Eq. (5), using the underlying $SU(1,1)$ symmetry. Unfortunately, the complete spectrum of $H$ is still not known. The incompleteness of the spectrum can be understood in the following way. In the limit $g \to 0$, the Hamiltonian $H$ reduces to that of a system of $N$ free harmonic oscillators in $D$ dimensions. Thus, in this limit, the complete spectrum of a system of $N$ free oscillators in $D$ dimensions should be reproduced. This is not the case, as can be seen from the expressions $\psi_n$ and $E_n$ given above.

III. $\mathcal{N} = 1$ SUPERCONFORMAL CMM : $OSP(2|2)$

We now construct the supersymmetric version of $h$ and $H$. The supercharge $q$ and its conjugate $q^\dagger$ are defined as,

$$q = \sum_{i,\mu} \psi_{i,\mu}^\dagger a_{i,\mu}, \quad q^\dagger = \sum_{i,\mu} \psi_{i,\mu} a_{i,\mu}^\dagger, \quad (6)$$

where the $ND$ fermionic variables $\psi_{i,\mu}$'s satisfy the Clifford algebra,

$$\{\psi_{i,\mu}, \psi_{j,\nu}\} = 0 = \{\psi_{i,\mu}^\dagger, \psi_{j,\nu}^\dagger\}, \quad \{\psi_{i,\mu}, \psi_{j,\nu}^\dagger\} = \delta_{ij} \delta_{\mu,\nu}. \quad (7)$$

The operators $a_i(a_i^\dagger)$'s are analogous to bosonic annihilation (creation) operators. They are defined in terms of the momentum operators $p_{i,\mu}$ and the superpotential $W(x_{1,1}, x_{1,2}, \ldots, x_{1,D}, x_{2,1}, \ldots, x_{N,D-1}, x_{N,D})$ as,

$$a_{i,\mu} = p_{i,\mu} - iW_{i,\mu}, \quad a_{i,\mu}^\dagger = p_{i,\mu} + iW_{i,\mu}, \quad W_{i,\mu} = \frac{\partial W}{\partial x_{i,\mu}}. \quad (8)$$
For the general superconformal quantum mechanics, the superpotential should have the following form,

\[ W = -\ln G, \quad \sum_{i,\mu} x_{i,\mu} \frac{\partial G}{\partial x_{i,\mu}} = d G, \]  

(9)

where \( d \) is any arbitrary constant. We choose the superpotential \( W \) as,

\[ G = \prod_{i<j} | \vec{r}_{ij} |^g, \]  

(10)

which results in the following Hamiltonian,

\[ h_s = \frac{1}{2} \{ q, q^\dagger \} \]

\[ = h + g \sum_{i \neq j,\mu} \left( 2 (x_{i,\mu} - x_{j,\mu})^2 \vec{r}_{ij}^{-2} - 1 \right) \vec{r}_{ij}^{-2} \left( \psi_{i,\mu}^\dagger \psi_{i,\mu} - \psi_{i,\mu}^\dagger \psi_{j,\mu} \right) \]

\[ + 2g \sum_{i \neq j,\mu \neq \nu} (x_{i,\mu} - x_{j,\mu}) (x_{i,\nu} - x_{j,\nu}) \vec{r}_{ij}^{-4} \left( \psi_{i,\mu}^\dagger \psi_{i,\nu} - \psi_{i,\mu}^\dagger \psi_{j,\nu} \right). \]

(11)

The super-Hamiltonian \( h_s \) is the supersymmetric generalization of \( h \). This can be checked by projecting \( h_s \) in the zero-fermion sector ( \( \psi_{i,\mu} |0 >= 0 \) ) of the \( 2^{DN} \) dimensional fermionic Fock space.

The super-Hamiltonian \( h_s \) does not have a normalizable ground-state. Following the standard procedure in the literature [17-19], the quantum evolution can be described by the operator \( R \) or \( H_s \) defined as,

\[ H_s = R + B - c, \quad R = h_s + K, \quad B = \frac{1}{2} \sum_{i,\mu} [\psi_{i,\mu}^\dagger, \psi_{i,\mu}], \quad c = \frac{g}{2} N(N - 1). \]

(12)

The new operator \( H_s \) is the supersymmetric generalization of the \( D \) dimensional CMM \( H \). The complete eigen-spectrum of this operator is known [20,21] for \( D = 1 \), both in supersymmetry preserving ( \( g > 0 \) ) as well as supersymmetry breaking ( \( g < 0 \) ) phases. No attempt has been made so far to study \( H_s \) with its full generality for \( D \geq 2 \). We find that the ground-state of \( H_s \) in the supersymmetric phase (\( g > 0 \)) is determined as,
\[ \psi_s^0 = \psi_0 |0 >. \] A comment is in order at this point. The ground-state wave-function \( \psi_s^0 \) is normalizable for \( g > -\frac{1}{2} \). However, a stronger criteria that each momentum operator \( p_{i,\mu} \) is self-adjoint for the wave-functions of the form \( \psi_s^0 \) requires \( g > 0 \). The supersymmetry is preserved for \( g > 0 \), while it is broken for \( g < 0 \) \([18,21]\). Let us now define the following operators,

\[
Q_1 = q - iS, \quad Q_2 = q^\dagger - iS^\dagger, \quad S = \sum_{i,\mu} \psi_{i,\mu} x_{i,\mu}, \\
Q_1^\dagger = q^\dagger + iS, \quad Q_2^\dagger = q + iS, \quad S^\dagger = \sum_{i,\mu} \psi_{i,\mu} x_{i,\mu}.
\]

(13)

Note that the super-Hamiltonian \( H_s = \frac{1}{2} \{ Q_1, Q_1^\dagger \} \). One can define bosonic and fermionic creation operators \([18,21]\),

\[
B_2^\dagger = -\frac{1}{4} \{ Q_1^\dagger, Q_2^\dagger \}, \quad F_2^\dagger = Q_2^\dagger.
\]

(14)

It can be checked easily,

\[
[H_s, B_2^\dagger] = 2B_2^\dagger, \quad [H_s, F_2^\dagger] = 2F_2^\dagger.
\]

(15)

We construct a set of exact eigenstates with the help of these operators. In particular,

\[ \psi_{n,\nu} = B_2^{\dagger n} F_2^{\dagger \nu} \psi_s^0, \]

(16)

are the exact eigenstates of \( H_s \) with the energy \( E_{n,\nu} = 2(n + \nu) \). The bosonic quantum number \( n \) can take any non-negative integer values, while the fermionic quantum number \( \nu = 0, 1 \). The super-Hamiltonian \( H_s \) reduces to that of \( N \) free super-oscillators in \( D \) dimensions in the limit \( g \to 0 \). In the same limit, one would thus expect to obtain the complete eigen-spectrum of \( N \) free super-oscillators in \( D \) dimensions from \( \psi_{n,\nu} \) and \( E_{n,\nu} \). Unfortunately, \( E_{n,\nu} \) and \( \psi_{n,\nu} \) describe only a small part of the complete spectrum of the free super-oscillator Hamiltonian. Thus, the set of exact eigenstates (16) is not complete and we are unable to find the complete spectrum.
The supersymmetry breaking phase of $H_s$ is characterized by $g < 0$. A set of exact eigenstates in this phase can also be constructed by using a duality property of this Hamiltonian. Consider a dual-Hamiltonian $\tilde{H}_s$ constructed in terms of $Q_2$ and $Q_2^\dagger$ as,

$$\tilde{H}_s = \frac{1}{2}\{Q_2, Q_2^\dagger\}.$$  This Hamiltonian can also be obtained from $H_s$ by making $g \rightarrow -g$ and $\psi_{i,\mu} \leftrightarrow \psi_{i,\mu}^\dagger$ [21]. We determine the ground-state of $\tilde{H}_s$ in its own supersymmetric phase ($g < 0$) as,

$$\tilde{\psi}_0 = \prod_{i<j} |\vec{r}_i - \vec{r}_j|^{-g} e^{-\frac{1}{2}\sum_i r_i^2} |ND>, \quad \psi_{i,\mu}^\dagger |ND > = 0.$$  (17)

Note that $\tilde{H}_s$ is related to $H_s$ by the following relation,

$$H_s = \tilde{H}_s + B - 2c.$$  (18)

Thus, $\tilde{\psi}_0$ is also an exact eigenstate of $H_s$ with the ground-state energy $E_0 = B - 2c$, which is positive definite for $g < 0$. This is in fact the ground-state wave-function of $H_s$ in the supersymmetry breaking phase. A comment is in order at this point. Usually, there are no general methods to find eigenstates in supersymmetry-breaking phase of a model. However, the duality symmetry of $H_s$ plays an important role to understand the supersymmetry-breaking phase of the model. Firstly, the wave-function $\tilde{\psi}_0$ is guaranteed to be the ground-state of $H_s$ for $g < 0$, because of the relation [18] and the fact that $\tilde{\psi}_0$ is the ground state of the dual-Hamiltonian $\tilde{H}_s$ in its own supersymmetry-preserving phase $g < 0$. Further, an algebraic construction of excited states of $H_s$ for $g < 0$ is possible using the duality symmetry. In particular, a set of excited states can be obtained by acting different powers of the bosonic creation operator $\tilde{B}_2^\dagger$ and the fermionic creation operator $\tilde{F}_2^\dagger$ on $\tilde{\psi}_0$, where these operators are obtained from [14] by making $g \rightarrow -g$ and $\psi_{i,\mu} \leftrightarrow \psi_{i,\mu}^\dagger$. In particular, the eigenstates and the corresponding eigenvalues are,

$$\tilde{\psi}_{n,\nu} = \tilde{B}_2^{i_n} \tilde{F}_2^{j_{n,\nu}} \tilde{\psi}_0, \quad \tilde{E}_{n,\nu} = E_0 + 2(n + \nu).$$  (19)

This set of exact eigenstates is not again complete.
IV. $\mathcal{N} = 2$ SUPERCONFORMAL CMM : $SU(1,1|2)$

After the center of mass separation, the super-Hamiltonian $h_s$ for $D = 2$ and $N = 2$ reduces to the model considered in [18]. This model has been shown to have extended $SU(1,1|2)$ superconformal symmetry [18]. We generalize the work of [18] for arbitrary two dimensional $N$ particle systems and find the criteria for having $SU(1,1|2)$ superconformal symmetry in the following. The superpotential (9) with the further constraint,

$$G = f(z_1, z_2, \ldots, z_N) \, g(z_1^*, z_2^*, \ldots, z_N^*), \quad z_k = x_{k,1} + i x_{k,2}, \quad z_k^* = x_{k,1} - i x_{k,2};$$

always gives rise to $\mathcal{N} = 2$ superconformal Hamiltonian. The homogeneity condition on $G$ implies that the (anti-)holomorphic function $(g)f$ should also be homogeneous. Note that except for the two dimensional CMM and a nearest-neighbor variant of this model [11], none of the other two dimensional model [8–10] satisfies the above criteria. Thus, the two dimensional CMM enjoys a special status over all other models. We specialize to $D = 2$ and CMM in rest of the discussions.

Let us define an operator $Y$ and its conjugate $Y^\dagger$ as,

$$Y = \frac{1}{2} \sum_i \epsilon_{\mu\nu} \psi_{i,\mu} \psi_{i,\nu}, \quad Y^\dagger = -\frac{1}{2} \sum_i \epsilon_{\mu\nu} \psi_{i,\mu}^\dagger \psi_{i,\nu}^\dagger, \quad (21)$$

where $\epsilon_{\mu\nu}$ is the two dimensional Levi-Civita pseudo-tensor. We follow the convention that the repeated indices of the Levi-Civita pseudo-tensor are always summed over. The operators $Y$, $Y^\dagger$ and $B$ constitute a $SU(2)$ algebra,

$$[Y, Y^\dagger] = -B, \quad [B, Y] = -2Y, \quad [B, Y^\dagger] = 2Y^\dagger. \quad (22)$$

Further, we have the following commutation relations,

$$[Y^\dagger, \psi_{i,\mu}] = \epsilon_{\mu\nu} \psi_{i,\nu}^\dagger = \bar{\psi}_{i,\mu}, \quad [Y, \psi_{i,\mu}^\dagger] = -\epsilon_{\mu\nu} \psi_{i,\nu} = -\bar{\psi}_{i,\mu}. \quad (23)$$
Following [18], it can be shown that the unitary transformation $U$, which represents a $180^\circ$ rotation around the second axis in the internal space, performs the following transformation,

$$U^{-1}\psi_{i,\mu}U = \bar{\psi}_{i,\mu}, \quad U^{-1}\psi^\dagger_{i,\mu}U = \bar{\psi}^\dagger_{i,\mu}. \quad (24)$$

The $SU(2)$ generators $Y, Y^\dagger$ and $B$ commute with the Hamiltonian $h_s$. The Hamiltonian $h_s$ has the internal $SU(2)$ symmetry and is invariant under the unitary transformation $U$.

The extended $\mathcal{N} = 2$ supersymmetry can be constructed by combining together the $SU(2)$ generators, the operators $Q_1, Q_2, S$ and their conjugates and a set of new operator $\bar{A} = U^{-1}AU$ corresponding to each odd operator $A$. Define the new supercharges $\bar{q}$ and $\bar{q}^\dagger$ following this prescription as [18],

$$\bar{q} = \sum_{i,\mu} \bar{\psi}^\dagger_{i,\mu}a_{i,\mu} = \sum_i \epsilon_{\mu,\nu}\psi_{i,\nu}a_{i,\mu}, \quad \bar{q}^\dagger = \sum_{i,\mu} \bar{\psi}_{i,\mu}a^\dagger_{i,\mu} = \sum_i \epsilon_{\mu,\nu}\psi^\dagger_{i,\nu}a^\dagger_{i,\mu}. \quad (25)$$

These supercharges satisfy the following anti-commutation relations [18],

$$\frac{1}{2}\{q, q^\dagger\} = h_s, \quad \frac{1}{2}\{\bar{q}, \bar{q}^\dagger\} = h_s. \quad (26)$$

All other anticommutators among themselves vanish. The super-Hamiltonian now will have a quartet structure. However, as noted earlier, $h_s$ does not have a normalizable ground-state. The quantum evolution can be described by $R = h_s + K$ or $H_s$. We now explore the full $SU(1,1|2)$ symmetry. Define [18],

$$\bar{Q}_1 = \bar{q} - i\bar{S}, \quad \bar{Q}_2 = \bar{q}^\dagger - i\bar{S}^\dagger, \quad \bar{S} = \sum_{i,\mu} \bar{\psi}^\dagger_{i,\mu}x_{i,\mu} = \sum_i \epsilon_{\mu,\nu}\psi_{i,\nu}x_{i,\mu},$$

$$\bar{Q}_1^\dagger = \bar{q}^\dagger + i\bar{S}^\dagger, \quad \bar{Q}_2^\dagger = \bar{q} + i\bar{S}, \quad \bar{S}^\dagger = \sum_{i,\mu} \bar{\psi}_{i,\mu}x_{i,\mu} = \sum_i \epsilon_{\mu,\nu}\psi^\dagger_{i,\nu}x_{i,\mu}. \quad (27)$$

The operators $Q_1, Q_2$ and their conjugates have the following anti-commutator algebra,
\[
\frac{1}{2}\{Q_1, Q_\dagger_1\} = R + B - c = H_s, \\
\frac{1}{2}\{\bar{Q}_2, Q_\dagger_2\} = R + B + c = H_s + 2c, \\
\frac{1}{2}\{Q_1, Q_\dagger_2\} = -\frac{1}{2}\{Q_1, \bar{Q}_2\} = -iJ,
\]

(28)

where the angular momentum operator is defined as [18],

\[
J = \sum_i \epsilon_{\mu\nu} (x_{i,\nu} p_{i,\mu} + i\psi_{i,\mu}^\dagger \psi_{i,\nu}).
\]

(29)

Similarly the only non-vanishing anti-commutators among \(\bar{Q}_1, Q_2\) and their conjugates are,

\[
\frac{1}{2}\{Q_2, Q_\dagger_2\} = R - B + c = \tilde{H}_s, \\
\frac{1}{2}\{Q_1, Q_\dagger_1\} = R - B - c = \tilde{H}_s - 2c, \\
\frac{1}{2}\{Q_2, Q_\dagger_1\} = -\frac{1}{2}\{Q_2, \bar{Q}_1\} = -iJ.
\]

(30)

All other non-vanishing anti-commutators are given by,

\[
-\frac{1}{2}\{Q_1, Q_\dagger_1\} = \frac{1}{2}\{Q_2, Q_\dagger_2\} = 2Y^\dagger, \quad -\frac{1}{2}\{Q_1, Q_\dagger_1\} = \frac{1}{2}\{Q_2, Q_\dagger_2\} = 2Y, \\
\frac{1}{4}\{Q_1, Q_2\} = \frac{1}{4}\{\bar{Q}_1, \bar{Q}_2\} = -B_2, \quad \frac{1}{4}\{Q_\dagger_1, Q_\dagger_2\} = \frac{1}{4}\{\bar{Q}_\dagger_1, \bar{Q}_\dagger_2\} = -B_\dagger_2.
\]

(31)

The evolution can be described either by \(H_s\) or \(\tilde{H}_s\).

The supercharges \(Q_1\) and \(\bar{Q}_2\) are the generators of an extended \(N = 2\) de Sitter supersymmetry with the central charge \(c\). It is amusing to note that the central charge \(c\) is precisely the energy of the classical minimum equilibrium configurations of the bosonic part of \(H_s\). However, we do not find any topological origin of \(c\), as in the case of field theories admitting soliton solutions in the Bogomol’nyi-Prasad- Sommerfeld limit. As mentioned earlier, \(\psi_0^s\) is the ground state of \(H_s\) in the supersymmetric phase. This essentially implies that the supersymmetry associated with the generator \(Q_2\) has broken.
Thus, this is the case corresponding to the spontaneous breakdown of supersymmetry from $\mathcal{N} = 2 \to \mathcal{N} = 1$. For $g < 0$, the supersymmetry spontaneously breaks down completely. The eigen-spectrum of $H_s$ in this supersymmetry breaking phase can be constructed from $\tilde{H}_s$.

The anti-commutator algebra (28) is not in diagonal form because of the last equation. The eigenstates of $H_s$ correspond to the angular momentum eigenvalue $j = 0$. Following [18] exactly, let us define,

$$\mu = \cos \theta \, Q_1 + i \sin \theta \, \bar{Q}_2, \quad \nu = i \sin \theta \, Q_1 + \cos \theta \, \bar{Q}_2, \quad \tan(2\theta) = j/c.$$ (32)

It can be checked easily that,

$$\frac{1}{2}\{\mu, \mu^\dagger\} = R + B - \sqrt{c^2 + j^2}, \quad \frac{1}{2}\{\nu, \nu^\dagger\} = R + B + \sqrt{c^2 + j^2}, \quad \{\mu, \nu^\dagger\} = 0.$$ (33)

The condition that the supersymmetric ground-state is annihilated by both $\mu$ and $\mu^\dagger$ gives,

$$\psi_{s0}^s(j) = \prod_{i<j} (z_i - z_j)^{g^+} \left( z_i^* - z_j^* \right)^{g^-} e^{-\frac{1}{2} \sum_i z_i z_i^*} |0 >,$$

$$g^+ = \frac{1}{N(N-1)} \left[ (j^2 + c^2)^{\frac{1}{2}} \mp j \right].$$ (34)

Note that for $j = 0$, $g^+ = g^- = \frac{j}{2}$ and $\psi_{s0}^s(j = 0)$ reduces to $\psi_{0}^s$. The eigenstates in (34) carry an angular momentum,

$$j = \frac{1}{2} (g_+ - g_-) N(N-1).$$ (35)

Note that $j$ receives contribution only from the bosonic part of $\psi_{s0}^s(j)$. Rest of the analysis can be carried out in a straightforward way. In particular, one can verify easily,

$$H_s(j) = \frac{1}{2}\{\mu, \mu^\dagger\}, \quad [H_s(j), B_2^\dagger] = 2B_2^\dagger, \quad [H_s(j), F_2^\dagger] = 2F_2^\dagger.$$ (36)

Thus, we construct the excited states as,
\[ \psi_{n,\nu}(j) = B_2^n F_2^{j\nu} \psi_0(j), \]  

(37)

where the bosonic quantum number \( n \) can take any non-negative integer values, while the fermionic quantum number \( \nu = 0, 1 \). Note that all these eigenstates have the same angular momentum.

\textbf{V. SUMMARY & DISCUSSIONS}

We have constructed and studied the \( D \) dimensional superconformal CMM having dynamical \( OSp(2|2) \) symmetry. Though we have obtained an infinite number of exact states corresponding to the bosonic and the fermionic excitations, the complete spectrum is still not known. Further, we have shown that the two dimensional CMM can naturally be embedded into an extended \( SU(1,1|2) \) superconformal Hamiltonian. This construction of extended \( N = 2 \) superconformal many-particle Hamiltonian is valid for arbitrary number of particles and also for arbitrary values of the coupling constant. This is the central result of our paper. We have also studied the evolution of this system in terms of operators of the extended \( N = 2 \) de Sitter supersymmetry and discussed the supersymmetry-breaking pattern.

It may be worth mentioning here, recently, attempt to construct one dimensional CMS Hamiltonian with extended superconformal symmetry has been made \cite{22}. It is found that within the specific formalism, the \( SU(1,1|2) \) superconformal CMS model in one dimension can be constructed only for a certain value of the coupling constant. Further, though a general formulation of the multidimensional supersymmetric quantum mechanics with \( N = 2 \) was given in \cite{23}, no nontrivial many-particle systems of CMS-type have been shown yet to result from such formulation. To the best of our knowledge, we are not aware of any other work discussing \( SU(1,1|2) \) superconformal Hamiltonian.
of CMM-type with its full generality. Within this background, the extended $\mathcal{N} = 2$ superconformal CMM presented in this letter appears to be the first such example in the literature. The space-time dimensionality plays an obvious role in our analysis. However, we would like to stress it again that only the CMM and a nearest-neighbour variant of this model [11], among several other interesting many-particle two dimensional models [8–10], are amenable for such a construction.

The history of studying supersymmetric quantum mechanical model with higher number of supercharges [24] is long. One of the major reason for the renewed interest in the (Super-)conformal Quantum Mechanics is its relevance in the study of adS/CFT correspondence and black holes [25]. Though a direct connection between the CMM and the black hole physics can not be established at this point, we observe a possible relation between the $D$ dimensional CMM and the low energy effective action of $D+1$ dimensional Yang-Mills theory dimensionally reduced to $0 + 1$ dimension. This observation is based on the existing results on this subject in the literature [13,12].

It is known [13,7,14] that the Hamiltonian $h$ for $D = 2$ and $g = \frac{1}{2}$ describes the dynamics of a Gaussian ensemble of $N \times N$ normal matrices in the limit $N \to \infty$. The Gaussian action of the normal matrices is given by,

$$A(M, M^\dagger) = \frac{1}{4} \int dt \ Tr \left( \frac{\partial M^\dagger}{\partial t} \frac{\partial M}{\partial t} \right), \quad [M, M^\dagger] = 0. \quad (38)$$

The second equation defines $M$ to be normal matrices. The action $A$ with $M$ as normal matrices is the low energy effective action of $2 + 1$ dimensional Yang-Mills(YM) theory dimensionally reduced to $0 + 1$ dimension with the choice of gauge $A_0 = 0$ [12]. A term of the form $[M, M^\dagger]^2$ drops out in the low energy limit giving rise to the constraint on $M$ to be normal matrices. Thus, for the first time in the literature, we observe the relation between the two dimensional CMM with $g = \frac{1}{2}$ and the low energy effective action of $2 + 1$ dimensional YM theory dimensionally reduced to $0 + 1$ dimension. It is desirable
to extend this result for arbitrary value of $g$, much akin to the one dimensional CMS system.

It is worth recalling that an attempt to construct higher dimensional generalizations of the one dimensional CMS system from many-matrix models has been made in [12]. At the classical level, the resulting Hamiltonian contains only a two-body interaction term of the form $\sum_{i \neq j} \vec{r}_{ij}^2$. No trace of a three body term as in $h$ has been found. However, for $D = 2$, the many-matrix model considered in [12] is identical to $A$ with $M$ as normal matrix which reduces to CMM with $g = \frac{1}{2}$ in the quantum mechanical treatment [13]. Thus, it is expected that the highly constrained classical models considered in [12] should give rise to the CMM upon quantization for $D = 2$. We also expect that this will provide us a connection between the low energy effective action of $2 + 1$ dimensional YM theory dimensionally reduced to $0 + 1$ dimension and the two dimensional CMM for arbitrary value of $g$. Based on this observation, we believe that the $D$ dimensional super-CMM considered in this paper is in fact related to the low energy effective action of the $D + 1$ dimensional super-YM theory dimensionally reduced to $0 + 1$ dimension. Since the dimensionally reduced super-YM theory appears in many areas of recent research activity like M-theory, D0-branes etc. [29], it is of immense interest to put our belief relating CMM and super-YM on a firm footing.

ACKNOWLEDGMENTS

I would like to thank Tetsuo Deguchi for a careful reading of the manuscript and valuable comments. This work is supported by a fellowship (P99231) of the Japan Society for the Promotion of Science. I would also like to acknowledge support in terms of a fellowship from the Institute of Mathematical Sciences, Chennai, where a part of this work has been carried out.
REFERENCES

[1] F. Calogero, J. Math. Phys. (N.Y.) 10 (1969) 2191; 10 (1969) 2197.

[2] B. Sutherland, J. Math. Phys.(N.Y.) 12 (1971) 246; 12 (1971) 251; Phys. Rev. A 4 (1971) 2019.

[3] M. A. Olshanetsky and A. M. Perelomov, Phys. Rep. 71 (1981) 314; 94 (1983) 6.

[4] A. Polychronakos, Les Houches Lectures 1998, hep-th/9902157.

[5] E. D’Hoker and D. H. Phong, hep-th/9912271; A. Gorsky and A. Mironov, hep-th/0011197; A. J. Bordner, E. Corrigan and R. Sasaki, Prog. Theor. Phys. 102 (1999) 499, hep-th/9905101; S. P. Khastgir, A. J. Pocklington and R. Sasaki, J. Phys. A 33 (2000) 9033, hep-th/0005277.

[6] F. Calogero and C. Marchioro, J. Math. Phys. (N.Y.) 14 (1973) 182.

[7] A. Khare and K. Ray, Phys. Lett. A230 (1997) 139, hep-th/9609023.

[8] M. V. N. Murthy, R. K. Bhadury and D. Sen, Phys. Rev. Lett. 76 (1996) 4103, cond-mat/9603153; R. K. Bhadury, A. Khare, J. Law, M. V. N. Murthy and D. Sen, J. Phys. A 30 (1997) 2557, cond-mat/9609012.

[9] P. K. Ghosh, Phys. Lett. A229 (1997) 203, cond-mat/9610024, cond-mat/9607009.

[10] A. Khare, Phys. Lett. A245 (1998) 14, cond-mat/9804212; R. K. Ghosh and S. Rao, Phys. Lett. A238 (1998) 213, hep-th/9705141; B. Sutherland, Behavior of an interacting three-dimensional quantum fluid in a time-dependent trap, University of Utah Preprint.

[11] G. Auberson, S. R. Jain and A. Khare, J. Phys. A34 (2001) 695, cond-mat/0004012.
[12] A. Polychronakos, Phys. Lett. \textbf{B408} (1997) 117, \texttt{hep-th/9705047}.

[13] M. V. Feigel'man and M. A. Skvortsov, Nucl. Phys. \textbf{B506[FS]} (1997) 665, \texttt{cond-mat/9703213}.

[14] G. Oas, Phys. Rev. \textbf{E55} (1997) 205, \texttt{cond-mat/9610073}.

[15] C. Kane, S. Kivelson, D.-H. Lee and S. C. Zhang, Phys. Rev. \textbf{B43} (1991) 3255.

[16] G. Date, P. K. Ghosh, M. V. N. Murthy, Phys. Rev. Lett. \textbf{81} (1998) 3051, \texttt{cond-mat/9802302}; G. Date, M. V. N. Murthy, R. Vathsan, Jr. of Phys. : Condensed Matter \textbf{10} (1998) 5876, \texttt{cond-mat/9802034}.

[17] V. de Alfaro, S. Fubini, and G. Furlan, Nuovo Cim. \textbf{34a} (1976) 569.

[18] S. Fubini and E. Rabinovici, Nucl. Phys. \textbf{B245} (1984) 17.

[19] V. P. Akulov and I. A. Pashnev, Theor. Math. Phys. \textbf{56} (1983) 862.

[20] P. K. Ghosh, Nucl. Phys. B \textbf{595} (2001) 519, \texttt{hep-th/0007208}.

[21] D. Z. Freedman and P. F. Mende, Nucl. Phys. \textbf{B344} (1990) 317; L. Brink, A. Turbiner and N. Wyllard, J. Math. Phys. \textbf{39} (1998) 1285, \texttt{hep-th/9705219}; L. Brink, T. H. Hansson, S. Konstein and M. A. Vasiliev, Nucl. Phys. \textbf{B401} (1993) 591, \texttt{hep-th/9302023}; A. J. Bordner, N. S. Manton and R. Sasaki, Prog. Theor. Phys. \textbf{103} (2000) 463, \texttt{hep-th/9910033}; M. V. Ioffe and A. I. Neelov, J. Phys. \textbf{A33} (2000) 1581, \texttt{quant-ph/0001063}.

[22] N. Wyllard, J. Math. Phys. \textbf{41} (2000) 2826, \texttt{hep-th/9910160}.

[23] E. E. Donets, A. Pashnev, J. Juan Rosales and M. M. Tsulaia, Phys. Rev. \textbf{D61} (2000) 043512, \texttt{hep-th/9907224}.

17
[24] M. de Crombrugghe and V. Rittenberg, Ann. of Phys. 151 (1983) 99; E. A. Ivanov, S. O. Krivonos and V. M. Leviant, J. Phys. A22 (1989) 4201; A. I. Pashnev, Theor. Math. Phys. 69 (1986) 1172; V. Akulov and M. Kudinov, Phys. Lett. B460 (1999) 365, hep-th/9905070.

[25] P. Claus et. al. Phys. Rev. Lett. 81 (1998) 4553, hep-th/9804177; G. W. Gibbons and P. K. Townsend, Phys. Lett. B454 (1999) 187, hep-th/9812034.

[26] W. Taylor, hep-th/0002016; L. Susskind, hep-th/0101029.