Copula Modeling for Data with Ties

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Received: December 2016 / Accepted: date

Abstract Copula modeling has gained much attention in many fields recently with the advantage of separating dependence structure from marginal distributions. In real data, however, serious ties are often present in one or multiple margins, which cause problems to many rank-based statistical methods developed under the assumption of continuous data with no ties. Simple methods such as breaking the ties at random or using average rank introduce independence into the data and, hence, lead to biased estimation. We propose an estimation method that treats the ranks of tied data as being interval censored and maximizes a pseudo-likelihood based on interval censored pseudo-observations. A parametric bootstrap procedure that preserves the observed tied ranks in the data is adapted to assess the estimation uncertainty and perform goodness-of-fit tests. The proposed approach is shown to be very competitive in comparison to the simple treatments in a large scale simulation study. Application to a bivariate insurance data illustrates the methodology.

Keywords interval censoring · multivariate distribution · pseudo-observations · rank

1 Introduction

Multivariate modeling based on copulas has been extensively applied in many fields such as finance (e.g., Mackenzie and Spears 2014), actuarial science (e.g., You and Li 2014), hydrology (e.g., Parent et al. 2014), public health (e.g., Hu and Liang 2014), and so on. An important advantage of such models is that the dependence structure of a multivariate distribution is separated from its marginal distributions. The most popular approach to copula modeling is rank-based, which does not specify the parametric form of the marginal distributions (e.g., Genest et al. 1995, 2007). Under the assumption of continuous marginal distributions, no ties are expected from the observed data so the ranks are unique. In many applications, however, ties are often present in one or multiple margins due to precision limit and rounding in observed data. For example, two variables, loss and expenses, in an insurance application (Frees and Valdez 1998) from 1466 uncensored claims have only 541 and 1401 unique values, respectively. Presence of ties may have significant effect on the accuracy of parameter estimation and statistical testing for copulas due to the rank-based method (Kojadinovic and Yan 2010, Genest et al. 2011, Kojadinovic 2016). Ties may occur in practice due to two major reasons: precision/rounding issue and the discontinuity of true marginal models. We assume that the true marginal distributions are continuous, so we only consider the first situation, where ties cause information loss.

Handling data with ties in copula modeling has not been fully studied. Discarding the ties is obviously not desirable because it throws data away.
In rank-based methods, naïve approaches are to use average rank or to break the ties at random multiple times and summarize the multi-data results. Kojadinovic and Yan (2010) compared the two naïve methods using the bivariate insurance data from Frees and Valdez (1998): both methods give similar parameter estimates, but in goodness-of-fit test, using average rank rejects the Gumbel copula which fits well the data as indicated by results from 100 replicates from breaking ties at random. Conceptually, both naïve methods introduce independence into the data. Neither of them accounts for the dependence information hidden in the tied data, their estimation may be biased, especially when the dependence is strong, and goodness-of-fit tests will not hold their sizes by overly rejecting the null hypothesis when the null hypothesis is true. Pappada et al. (2016) proposed two randomization strategies beyond the naïve independence randomization: co-monotone and mixed randomization (which mixes the co-monotonicity and the independence via some weight). Nonetheless, the co-monotone randomization introduces perfect dependence into the data, and the mixed randomization alters the distribution of the data, albeit less severely.

We propose to handle tied data by treating their ranks as being interval censored and using ideas for interval censored data from survival analysis (e.g., Sun 2007; Chen et al. 2012). For bivariate data, each pair of observation falls into four categories: both observed, exactly one or the other observed, or both censored. Interval censored pseudo-observations can be used to construct a pseudo-likelihood, which can be maximized to obtain point estimates. To make inferences, the standard parametric bootstrap would not capture the variation in the estimation because bootstrap samples contain no ties. We propose a parametric bootstrap procedure that preserves the ties in the observed data in each bootstrap sample inspired by Bücher and Kojadinovic (2015). The same bootstrap procedure can be used in goodness-of-fit tests to assess the significance of a wide class of testing statistics constructed from the goodness-of-fit empirical process (Genest et al. 2009; Kojadinovic et al. 2011). In a large scale simulation study, the point and interval estimation were shown to be unbiased and provide valid uncertainty measures, respectively; the goodness-of-fit tests maintained their sizes and have substantial power.

The rest of this article is organized as follows. The proposed method is described with detail in Section 2. A large scale numerical study is reported in Section 3. The insurance data is used to illustrate the method in Section 4. A discussion concludes in Section 6.

2 Methodology

2.1 Interval Censored Pseudo-Observations

Let $(X, Y)$ be a continuous random vector with marginal distribution functions $F$ and $G$, and joint distribution function $H$. By Sklar’s theorem (Sklar 1959), there is a unique copula $C : [0, 1]^2 \to [0, 1]$ such that

$$H(x, y) = C(F(x), G(y)).$$

The copula $C$ completely characterizes the dependence structure in $H$. This representation suggests that the dependence structure can be separated from the marginal distributions in multivariate modeling. Let $(X_i, Y_i), i = 1, \ldots, n$ be a random sample from $H$. Often, the marginal distributions are modeled by their empirical distributions and the copula is modeled parametrically, leading to a semiparametric inference in multivariate modeling (Genest et al. 1995). This approach avoids the bias in copula estimation caused by misspecified marginal distributions (Kim et al. 2007).

Continuous data have no ties and no ambiguity in ranks. Let $\hat{F}_n$ and $\hat{G}_n$ be the empirical distribution functions of $F$ and $G$, respectively. Pseudo-observations $U_i$ and $V_i$ are simply $\hat{F}_n(X_i)$ and $\hat{G}_n(Y_i)$ rescaled by a constant $n/(n+1)$ to avoid evaluation of the copula density on the edges of unit square ending at $(1, 1)$. That is,

$$(U_i, V_i) = \left(\frac{n}{n+1} \hat{F}_n(X_i), \frac{n}{n+1} \hat{G}_n(Y_i)\right), \quad (1)$$

for $i = 1, \ldots, n$. Without ties, the pseudo-observations at each margin have jumps of size $1/(n+1)$.

The pseudo-likelihood estimator of $\theta$ is constructed from the margin-free pseudo-observations (Genest et al. 1995):

$$\hat{\theta}_n = \arg\max_{\theta \in \Theta} \sum_{i=1}^n \log c(U_i, V_i; \theta),$$

where $c(\cdot, \cdot; \theta)$ is the density of $C$ with parameter vector $\theta$ and parameter space $\Theta$.

In practice, ties are commonly observed due to rounding or lack of precision in measurements, which makes ranks and pseudo-observations not fully observed but interval censored. An interval
censored observation is a data point that is known to be somewhere between two values but the exact value is unknown. For illustration, consider a toy example of 9 observations where the sorted pseudo-observations of \( X \) from (1) are

\[
(U_1, \ldots, U_9) = (1, 2, 5, 5, 6, 8, 8, 9)/10.
\]  

(2)

In this example, there are ties in the 3rd, 4th, and 5th pseudo-observations and in the 7th and 8th pseudo-observations. If average ranks (also known as mid-ranks) are used, they will be 4 and 7.5. Handling ties by their average ranks invalidates the parametric bootstrap method because no ties would be in bootstrap samples, and the distribution of the many test statistics is not well approximated \cite{Kojadinovic2016}. Breaking the ties at random gives many possibilities of untied data, whose results could be summarized \cite{Kojadinovic2016, Yan2010}. As shown in our simulation study, however, breaking the ties at random can lead to bias in copula estimation when the dependence is high, which is expected because it introduces independence into the data, ignoring the dependence among the interval censored pseudo-observations.

We propose to use the concept of interval censored data from survival analysis to handle tied data in copula estimation. In particular, we define upper and lower boundaries of pseudo-observations, respectively, as

\[
(U_i, \nu_i) = \left( \frac{n \hat{F}_n(X_i) + 1}{n + 1}, \frac{n \hat{G}_n(Y_i) + 1}{n + 1} \right),
\]

\[
(U_i, \nu_i) = \left( \frac{n \hat{F}_n(X_i)}{n + 1}, \frac{n \hat{G}_n(Y_i)}{n + 1} \right),
\]

where \( \hat{F}_n(x-) \) and \( \hat{G}_n(y-) \) are the left limit of \( \hat{F}_n \) and \( \hat{G}_n \) at \( x \) and \( y \), respectively. Note that the upper bounds are the same as \((U_i, V_i)\). If \( X_i \) (or \( Y_i \)) is a tied observation, then its pseudo observation \( U_i \) (or \( V_i \)) is interval censored by \([U_i, \bar{U}_i]\) (or \([V_i, \bar{V}_i]\)). If \( X_i \) (or \( Y_i \)) is not a tied observation, the interval reduces to a single value, i.e., \( U_i = \bar{U}_i = U_i \) (or \( V_i = \bar{V}_i = V_i \)).

2.2 Pseudo-Likelihood Estimator

The observation \((U_i, V_i)\)’s contribution to the pseudo likelihood, \(L_i(\theta)\), depends on the censoring pattern on the two margins. There are four cases.

(1) If \( U_i < \bar{U}_i \) and \( V_i < \bar{V}_i \) (i.e., the observation is tied observation in both margins), then \( L_i(\theta) \) is

\[
L_i(\theta) = C_{\theta}(U_i, V_i) - C\left(U_i, \bar{V}_i\right) - C\left(\bar{U}_i, V_i\right) + C\left(\bar{U}_i, \bar{V}_i\right).
\]

(2) If \( U_i < \bar{U}_i \) and \( V_i = \bar{V}_i \) (i.e., the observation is a tied observation only in \( X \)), then \( L_i(\theta) \) is

\[
\frac{\partial C_{\theta}(u, v)}{\partial u} \bigg|_{u=U_i, v=V_i} - \frac{\partial C_{\theta}(u, v)}{\partial v} \bigg|_{u=U_i, v=V_i}.
\]

(3) If \( U_i = \bar{U}_i = \bar{U}_i \) and \( V_i < \bar{V}_i \) (i.e., the observation is a tied observation only in \( Y \)), then \( L_i(\theta) \) is

\[
\frac{\partial C_{\theta}(u, v)}{\partial u} \bigg|_{u=U_i, v=V_i} - \frac{\partial C_{\theta}(u, v)}{\partial v} \bigg|_{u=U_i, v=V_i}.
\]

(4) If \( U_i = \bar{U}_i = \bar{U}_i \) and \( V_i = \bar{V}_i \) (i.e., the observation is not tied in either margin), then \( L_i(\theta) = c(U_i, V_i; \theta) \).

The adjusted pseudo-likelihood function under interval censoring is

\[
\mathcal{L}(\theta) = \sum_{i=1}^{n} \log L_i(\theta).
\]

The maximum pseudo-likelihood estimation (MPLE) of \( \theta \) is then

\[
\hat{\theta}_n = \arg \max_{\theta \in \Theta} \mathcal{L}(\theta).
\]

This estimator reduces to the traditional MPLE when neither margin has tied observations. For implementation, we need partial derivatives of the copula in addition to the distribution and density functions. Expressions of these partial derivatives for commonly used copulas are available from R package \textit{copula} \cite{Hofert2016}.

2.3 Confidence Interval Estimation

The asymptotic properties of the pseudo-likelihood estimator are challenging to establish due to the inclusion of interval censored pseudo-observations. We resort to bootstrap for confidence intervals, but a plain vanilla parametric bootstrap procedure would not work in this case because no ties would be present if bootstrap samples are generated from the fitted copulas. The parametric bootstrap procedure needs to be modified so that the ties in the observed data are somehow preserved in each of the bootstrap samples in order to sufficiently capture the uncertainty in parameter estimation.
Given a sample generated from the fitted copula, which contains no ties, we introduce ties into the sample such that at each margin the ties in the observed data are reproduced in the bootstrap sample. Let $\hat{F}_n$ and $\hat{G}_n$ be the empirical distributions of the observed pseudo-observations $U_i$'s and $V_i$'s, respectively, i.e., $\hat{F}_n(u) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}(U_i \leq u)$ and $\hat{G}_n(v) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}(V_i \leq v)$. When ties are present, $\hat{F}_n$ and $\hat{G}_n$ have jumps of sizes greater than $1/n$. Let $U^{(b)}_i$'s and $V^{(b)}_i$'s be the pseudo-observations from a bootstrap sample, which have no ties, generated from the fitted copula. Ties are introduced into to $U^{(b)}_i$'s and $V^{(b)}_i$'s by applying the corresponding quantile functions $\hat{F}_n^{-1}$ and $\hat{G}_n^{-1}$ of $\hat{F}_n$ and $\hat{G}_n$ to $U^{(b)}_i$'s and $V^{(b)}_i$'s, respectively:

$$
\left( U^{(b)}_i, V^{(b)}_i \right) = \left( \hat{F}_n^{-1}(U^{(b)}_i), \hat{G}_n^{-1}(V^{(b)}_i) \right),
$$

where $\hat{F}_n^{-1}(y) = \inf\{u : \hat{F}_n(u) \geq y\}$. After this transformation, $U^{(b)}_i$'s and $V^{(b)}_i$'s are tie-adjusted bootstrap pseudo-observations whose marginal empirical distributions are the same as those of $U_i$'s and $V_i$'s, respectively.\cite{Buercher2015}. Note that the joint empirical distribution of $(U^{(b)}_i, V^{(b)}_i)$, however, is not the same as that of $(U_i, V_i)$, which is the source of variation of the bootstrap sample.

After ties are introduced, we can further obtain the upper and lower boundaries of the pseudo-observations of $U^{(b)}_i$'s and $V^{(b)}_i$'s,

$$
\left( U^{(b)}_{\alpha/2}, V^{(b)}_{\alpha/2} \right) = \left( U^{(b)}_i, V^{(b)}_i \right),
$$

$$
\left( U^{(b)}_{1 - \alpha/2}, V^{(b)}_{1 - \alpha/2} \right) = \left( \hat{F}_n^{-1}(U^{(b)}_i) - \frac{1}{n + 1}, \hat{G}_n^{-1}(V^{(b)}_i) - \frac{1}{n + 1} \right),
$$

where $\hat{F}_n$ and $\hat{G}_n$ are the empirical distribution functions of $U^{(b)}_i$ and $V^{(b)}_i$ (and also of $U_i$ and $V_i$). Note that

$$
U^{(b)}_{\alpha/2} = U_{n : \alpha/2}, \quad U^{(b)}_{1 - \alpha/2} = U_{n : 1 - \alpha/2},
$$

$$
V^{(b)}_{\alpha/2} = V_{n : \alpha/2}, \quad V^{(b)}_{1 - \alpha/2} = V_{n : 1 - \alpha/2},
$$

where the subscript of $A_{n : i}$ represents the $i$th order statistics (i.e., $i$th smallest number) of the sequence $\{A_i\}_{i=1}^n$.

We illustrate the tie-preserving procedure using the same toy example with pseudo-observations in Section 2.3. The bootstrap pseudo-observations (without ties) after being sorted are always

$$
(U^{(b)}_{9,1}, \ldots, U^{(b)}_{9,9}) = (1, 2, 3, 4, 5, 6, 7, 8, 9)/10.
$$

By applying \cite{Buercher2015}, we obtain the tie-adjusted bootstrap pseudo-observations

$$
(U^{(b)}_{9,1}, \ldots, U^{(b)}_{9,9}) = (1, 2, 5, 5, 6, 8, 8, 9)/10,
$$

where we have changed 3/10 and 4/10 to 5/10, and 7/10 to 8/10 to match the ties in the observed pseudo-observations. Consequently, the lower and upper boundaries of pseudo-observations of $(U^{(b)}_i, V^{(b)}_i, \ldots, U^{(b)}_9)$ are

$$
(U^{(b)}_{9,1}, \ldots, U^{(b)}_{9,9}) = (1, 2, 3, 3, 6, 7, 7, 9)/10.
$$

The same procedure can be applied to the other margin $V_i$.

In summary, the tie-preserving parametric bootstrap procedure given the MPLE $\hat{\theta}_n$ to construct a $1 - \alpha$ confidence interval runs as follows. For some large integer $B$, repeat the following steps (1) to (3) for every $b \in \{1, \ldots, B\}$:

1. Generate bootstrap pseudo-observations with no ties from the fitted copula $C_{\hat{\theta}}$.
2. Obtain tie-adjusted pseudo-observations via \cite{Buercher2015}.
3. Obtain the MPLE $\hat{\theta}^{(b)}_n$ using the tie-adjusted pseudo-observations.

A bootstrap sample $(\hat{\theta}^{(1)}_n, \ldots, \hat{\theta}^{(B)}_n)$ is formed to approximate the sampling distribution of $\hat{\theta}_n$. The sample $\alpha/2$ and $1 - \alpha/2$ quantiles can then be used to form a confidence interval of level $1 - \alpha$.

The computing cost of the tie-preserving parametric bootstrap procedure is similar to that of the standard parametric bootstrap procedure. The only extra part is the tie-preserving step, which is minimal compared to the optimization in the fitting for each bootstrap sample.

2.4 Goodness-of-Fit Test

Goodness-of-fit tests with standard parametric bootstrap are known to be vulnerable to ties in keeping their sizes.\cite{Kojadinovic2010}. This is because goodness-of-fit test statistics (usually distance-based) tend to be bigger when ties are present. However, when a standard parametric bootstrap generates tie-free samples, it leads to under-estimation of the magnitude of the null sampling distribution of the testing statistic. Consequently, the tests would not hold their sizes with over rejection. From our numerical studies, the empirical size of a 5%-level test could be 100% when even a moderate amount of ties are present. Therefore,
preserving ties in parametric bootstrap is crucial [Kojadinovic 2016].

We propose to adapt the standard bootstrap procedure for goodness-of-fit [Genest and Rémillard 2008] with observed ties- preserved [Kojadinovic 2016]. The null hypothesis is

\[ H_0 : C \in \mathcal{C} = \{C_\theta : \theta \in \Theta\} \quad \text{versus} \quad H_1 : C \notin \mathcal{C}. \]

Consider goodness-of-fit tests based on the goodness-of-fit empirical process

\[ C_n(u, v) = \sqrt{n}(C_n(u, v) - C_{\hat{\theta}_n}(u, v)), \]

\[ (u, v) \in [0,1]^2, \]

where the \( C_n \) is the empirical copula defined as

\[ C_n(u, v) = \frac{1}{n} \sum_{i=1}^n \mathbb{1}(U_i \leq u, V_i \leq v), \]

and \( \hat{\theta}_n \) is a parametric estimator of \( \theta \) (which could be the MPLE from [3] or other estimator) under the null hypothesis \( H_0 \). Statistics of goodness-of-fit tests can be formed as \( F(C_n) \), where \( F \) is a functional of \( C_n \). Examples are Kolmogorov–Smirnov, Anderson–Darling, and Cramer-von Mises (CvM) distance [Genest et al. 2009; Kojadinovic and Yan 2010]. We use the CvM statistic, which has been known to have a good power [Kojadinovic and Yan 2010], to illustrate the procedure. The CvM statistic is defined as

\[ D_n = \int_{[0,1]^2} C_n^2(u, v) \, dC_n(u, v) \]

\[ = \sum_{i=1}^n \left( C_n(U_i, V_i) - C_{\hat{\theta}_n}(U_i, V_i) \right). \]  

(5)

After \( D_n \) is obtained, we use the following bootstrap procedure to draw samples from the distribution of \( D_n \) under \( H_0 \). For some large integer \( B \), repeat the following steps for each \( b \in \{1, ..., B\} \):

1. Generate bootstrap pseudo-observations with no ties from the fitted copula \( C_{\hat{\theta}_n} \).
2. Obtain tie-adjusted pseudo-observations via [4].
3. Obtain the MPLE \( \hat{\theta}^{(b)}_n \) using the tie-adjusted pseudo-observations.
4. Obtain the empirical copula \( C^{(b)}_n \) based on the tied-adjusted pseudo-observations.
5. Obtain test statistic (CvM distance) \( D^{(b)}_n \) using [5].

An approximated p-value of the observed test statistic is then

\[ \sum_{b=1}^B \mathbb{1}(D^{(b)}_n \geq D_n)/B. \]

Again, this tie-preserving bootstrap procedure has similar computing cost compared to the standard parametric bootstrap procedure. The difference from the procedure of [Kojadinovic 2016] is that, after each tie-preserving bootstrap sample is obtained, we use the interval censoring approach for estimation instead of average ranks.

3 Numerical Studies

A large-scale simulation study was carried out to assess the performance of proposed methods in point estimation, interval estimation, and goodness-of-fit.

3.1 Point Estimation

We first study the accuracy of the point estimation of the proposed method (denoted as “censoring”) and compare it with two existing methods, breaking ties at random (denoted as “random”) and using the average of ties (denoted as “average”). For the random method, we use the mean of 100 randomizations. Data were generated from three one-parameter copulas parameterized by Kendall’s \( \tau \), Clayton (C), Gumbel (G), and normal (N), with \( \tau \in \{0.1, \ldots, 0.9\} \) to control the dependence level. Ties were introduced by rounding the first margin to the first decimal place. Three sample sizes were considered \( n \in \{100, 200, 400\} \).

The estimation error of the MPLE estimator \( \hat{\tau}^n, \hat{\tau}^n - \tau \), from 1000 replicates are summarized in Figure [1]. It is clear that, as expected, the estimates from the average method and the random method have little bias when the dependence is weak (lower \( \tau \)), but as \( \tau \) increases, they become more biased. The estimate from the censoring method remains unbiased in all settings. Variances of all three methods are comparable across all settings. Therefore, the mean squared error (MSE) of the censoring method is smaller. Furthermore, as the sample size increases, the variance of the estimate from the censoring method reduces accordingly.

We then study the effect of the severity of ties on the estimation accuracy. Data were generated from the three copulas with \( \tau = 0.75 \) and \( n = 200 \). The first margin is rounded to the first decimal place if its value is smaller than \( \lambda \), which controls the percentage of ties. We use the three methods to estimate \( \tau \) and obtain their corresponding root mean square errors (RMSEs) from 1000 replications. These RMSEs are displayed in Figure [2].
Fig. 1: Boxplots of estimation error for Kendall’s $\tau$ using three methods (i.e., random, average, and censoring) for three types of copulas (i.e., Clayton, Gumbel, and normal). Sample size is $n \in \{100, 200, 400\}$. Ties were introduced by rounding the first margin to the first decimal place.

Fig. 2: Comparison of RMSEs of Kendall’s $\tau$ for different methods (i.e., random, average, and censoring) under three copulas (i.e., Clayton, Gumbel and normal) with different percentages of ties. Sample size is $n = 200$. Ties were introduced by rounding the first margin to the first decimal place.
censoring method has the smallest RMSE among the three methods, and its RMSE remains stable regardless of the changes in the severity of ties. The RMSEs of the average method and the random method increase as the percentage of ties increases, with a faster rate for data generated from the Gumbel copula.

3.2 Interval Estimation

To assess the coverage properties of the bootstrap confidence intervals, we generated data from the three copulas (C, G, and N) with Kendall’s $\tau \in \{0.25, 0.50, 0.75\}$ with sample size $n \in \{50, 100, 200\}$. Ties were introduced by rounding the first margin to the first decimal place. The 95% confidence intervals of the censoring method were constructed with the tie-preserving bootstrap procedure with bootstrap sample size $B = 1000$.

The empirical coverage rates of the confidence intervals based on 500 replicates are summarized in Table 1. All the empirical coverage rates are close to the nominal level except that in the setting with $n = 50$ and $\tau = 0.25$, the coverage rate is about 90%. The results suggest that the tie-preserving bootstrap procedure provides confidence intervals that are valid for inferences for sample size over 100 or Kendall’s $\tau$ over 0.50.

3.3 Goodness-of-Fit Test

The finite-sample performance of goodness-of-fit tests using the censoring method in estimation was assessed. Data were generated from three copulas (C, G, and N) with Kendall’s $\tau \in \{0.25, 0.5, 0.75\}$ and sample size $n = 100$. Three patterns of ties were considered: no ties or ties were introduced by rounding one margin or both margins to the first decimal place. For each configuration, we ran 500 replicates, for each replicate, goodness-of-fit tests were performed with each of the three families of copulas (C, G, and N) serving as the hypothesized copula. The parametric bootstrap sample size was $B = 200$. In the bootstrap procedure, two methods of preserving ties were considered: matching the observed ranks as proposed in Section 2.4, and the rounding the margins with ties to the first decimal place. Note that the rounding method is under that assumption of known tie-introducing mechanism, which is unavailable in general. We included this method as a benchmark only to investigate whether knowing the tie-introducing mechanism helps to improve the performance of the tests.

The empirical rejection percentages of the goodness-of-fit tests with level 5% are summarized in Table 2. When the hypothesized copula is the same as the data generating copula, the reported percentages are put in bold, representing the empirical sizes. The empirical sizes are close to the nominal size of 5% in most cases. The two methods of preserving ties showed little difference, except that the test is conservative for the Clayton copula with $\tau = 0.75$, with empirical rejection percentage 1.6 and 0.4, respectively, for one and two side ties. When the hypothesized copula is not the data generating copula, the empirical powers of the tests are lower than those obtained when no ties are present. This is expected due to the information loss in ties. Between the two tie-preserving methods, the rounding approach seems to have slightly higher power, but the advantage seems quite limited. Note that, however, the rounding approach may not be applicable in practice because we may not know the true tie-introducing mechanism.

Now that the difference between the two tie-preserving methods is little, we focus on the matching ties method and investigate sample sizes 50 and 200. The results are summarized in Table 3. As expected, the test holds its size better at sample size 200, and the power increases as the sample size increases in all settings.

4 Real Data Example

The bivariate insurance data considered in Frees and Valdez (1998) has often been used as illustration in copula modeling (Kojadinovic and Yan 2010). The two variables are indemnity payment and allocated loss adjustment expense, observed from 1466 uncensored claims of an insurance company. A lot of ties are present in indemnity payment, with only 541 unique values. Ties are much less in allocated loss adjustment expense (1401 unique values). Existing works have demonstrated that it is necessary to account for ties to analyze this data set (Kojadinovic 2016).

We performed goodness-of-fit tests for four copulas, Clayton, survival Clayton, Gumbel, and normal, using the censoring method with the tie-preserving bootstrap procedure with bootstrap sample size $B = 1000$. The $p$-values for Clayton, survival Clayton, Gumbel, and normal copulas are 0.000, 0.000, 0.168, and 0.000, respectively. Only the Gumbel copula is not rejected at the 5% level, which is con-
Table 1: Empirical coverage rate (in percentage) of the 95% confidence interval of the censoring method for different types of copulas (i.e., C=Clayton, G=Gumbel, N=normal), different levels of Kendall’s \( \tau \in \{0.25, 0.5, 0.75\} \), and different sample sizes \( n \in \{50, 100, 200\} \). Results are based on 500 replicates, each with bootstrap sample size \( B = 1000 \).

\[
\begin{array}{ccccccc}
\tau = 0.25 & \tau = 0.5 & \tau = 0.75 \\
n & C & G & N & C & G & N & C & G & N \\
50 & 89.9 & 89.7 & 89.5 & 93.2 & 93.6 & 91.0 & 93.6 & 93.8 & 95.4 \\
100 & 92.6 & 92.4 & 93.8 & 95.4 & 93.6 & 92.6 & 94.8 & 96.2 & 97.6 \\
200 & 94.0 & 93.4 & 91.6 & 96.0 & 94.8 & 93.0 & 93.2 & 94.8 & 95.4 \\
\end{array}
\]

Table 2: Empirical rejection percentage of the goodness-of-fit tests with sample size \( n = 100 \) for three types of copulas (C = Clayton, G = Gumbel, and N = Normal) based on 500 replicates, each with bootstrap sample size \( B = 200 \). Ties were introduced by rounding data from the first margin to first decimal place.

| Ties pattern | Kendall’s \( \tau \) | True copula | Hypothesized copula |
|--------------|----------------------|-------------|---------------------|
|              |                      |             | C                   | G                   | N                   |
|              |                      | Match       | Round               | Match               | Round               |
| No ties      | 0.25                 | C           | 5.5                  | 62.0                | 15.9                |
|              |                      | G           | 79.1                | 5.3                 | 16.8                |
|              |                      | N           | 47.7                | 13.6                | 5.2                 |
| 0.5          | C                    | 7.0         | 96.8                | 60.0                |
|              | G                    | 99.0        | 6.2                 | 28.6                |
|              | N                    | 88.2        | 24.0                | 5.0                 |
| 0.75         | C                    | 4.0         | 100.0               | 85.8                |
|              | G                    | 100.0       | 3.0                 | 31.0                |
|              | N                    | 98.2        | 21.6                | 3.0                 |
| One side     | 0.25                 | C           | 4.4                  | 4.2                 | 57.6                | 58.9                | 3.8                  | 18.2                |
|              |                      | G           | 76.5                | 75.9                | 4.2                 | 3.4                 | 18.3                | 18.6                |
|              |                      | N           | 44.9                | 45.5                | 12.7                | 12.1                | 3.5                 | 6.4                 |
| 0.5          | C                    | 4.6         | 5.4                 | 95.4                | 95.6                | 52.2                | 55.2                |
|              | G                    | 99.8        | 99.6                | 6.2                 | 6.6                 | 32.4                | 33.6                |
|              | N                    | 87.0        | 88.2                | 22.2                | 21.4                | 3.6                 | 4.2                 |
| 0.75         | C                    | 1.6         | 3.8                 | 99.6                | 99.6                | 79.4                | 79.6                |
|              | G                    | 100.0       | 4.0                 | 4.2                 | 25.6                | 26.6                |
|              | N                    | 96.6        | 97.0                | 14.4                | 15.4                | 3.4                 | 3.8                 |
| Two sides    | 0.25                 | C           | 6.4                  | 4.4                 | 51.4                | 55.6                | 13.8                | 15.4                |
|              |                      | G           | 71.7                | 73.5                | 4.7                 | 3.7                 | 19.4                | 18.8                |
|              |                      | N           | 40.2                | 38.6                | 8.8                 | 11.8                | 5.0                 | 4.2                 |
| 0.5          | C                    | 4.6         | 3.8                 | 96.0                | 96.8                | 53.2                | 54.6                |
|              | G                    | 98.6        | 99.0                | 4.2                 | 5.8                 | 28.6                | 31.8                |
|              | N                    | 82.8        | 85.0                | 19.2                | 18.6                | 5.6                 | 4.8                 |
| 0.75         | C                    | 0.4         | 4.2                 | 97.8                | 98.0                | 75.0                | 83.2                |
|              | G                    | 99.6        | 100.0               | 4.4                 | 5.4                 | 26.6                | 30.6                |
|              | N                    | 93.6        | 94.8                | 10.6                | 15.0                | 4.6                 | 4.4                 |

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Table 3: Empirical rejection percentages of the goodness-of-fit tests with sample size \( n \in \{50, 200\} \) for three types of copulas (C = Clayton, G = Gumbel, and N = Normal) based on 500 replicates, each with bootstrap sample size \( B = 200 \). Ties were introduced by rounding data from the first margin to first decimal place. Matching rank was used to preserve ties in bootstrap sample.

| Ties pattern | Hypothesized copula | \( n = 50 \) | \( n = 200 \) | \( n = 50 \) | \( n = 200 \) |
|--------------|---------------------|-------------|-------------|-------------|-------------|
| One side 0.25 | C                   | 4.3         | 6.4         | 32.5        | 88.8        | 6.0         | 41.8        |
|              | G                   | 50.7        | 94.2        | 6.9         | 5.4         | 9.4         | 30.4        |
|              | N                   | 24.1        | 69.4        | 10.7        | 25.6        | 2.6         | 6.8         |
| 0.5          | C                   | 3.2         | 4.0         | 72.9        | 100.0       | 21.9        | 93.8        |
|              | G                   | 89.2        | 100.0       | 4.7         | 5.2         | 21.8        | 42.2        |
|              | N                   | 59.2        | 99.8        | 11.6        | 38.6        | 5.0         | 4.6         |
| 0.75         | C                   | 1.1         | 5.2         | 83.7        | 100.0       | 35.1        | 99.0        |
|              | G                   | 94.6        | 100.0       | 2.9         | 5.8         | 17.1        | 47.6        |
|              | N                   | 73.3        | 100.0       | 7.4         | 32.2        | 3.9         | 4.0         |
| Two sides 0.25| C                   | 4.6         | 4.4         | 11.9        | 55.6        | 1.5         | 18.0        |
|              | G                   | 46.5        | 73.5        | 3.3         | 3.7         | 10.8        | 19.2        |
|              | N                   | 24.2        | 38.6        | 6.8         | 11.8        | 3.9         | 4.8         |
| 0.5          | C                   | 2.9         | 3.8         | 53.2        | 96.8        | 8.4         | 55.0        |
|              | G                   | 86.0        | 99.0        | 2.5         | 5.8         | 16.1        | 26.4        |
|              | N                   | 57.2        | 85.0        | 10.3        | 18.6        | 3.4         | 4.2         |
| 0.75         | C                   | 0.7         | 4.2         | 75.9        | 98.0        | 34.6        | 78.4        |
|              | G                   | 89.8        | 100.0       | 1.0         | 5.4         | 13.9        | 30.6        |
|              | N                   | 60.2        | 94.8        | 2.8         | 15.0        | 1.8         | 5.0         |

Discussion

Unlike the average rank approach, independence randomization (Kojadinovic and Yan, 2010), or co-monotone/mixed randomization (Pappada et al., 2016), the interval censoring approach does not distort the features of the observed data. Consequently, it does not have the bias that other approaches may have introduced, especially when the dependence is strong. When the dependence is weak, although the point estimate may not be very different from the point estimate with the average rank method, the small difference might still propagate to become important when estimation is repetitively needed as in the case of parametric bootstrap procedures. The interval censoring method can be applied to model discrete data, in which case it has the same spirit as Nikoloulopoulos and Karlis (2009). The limiting distribution of the MPLE using the interval censored pseudo-observations is a challenging problem for two rea-
sons. First, likelihood estimator from interval censored data do not achieve the standard \( n^{1/2} \)-rate (Wellner, 1995; van der Vaart and Wellner, 2000). Second, the interval censored data used in the estimation are pseudo-observations resulting from the probability integral transform with marginal empirical distribution functions, instead of the observations. Establishing the asymptotic properties of the MPLE from interval-censored pseudo-observations would be a contribution of strong interest.

The tie-preserving parametric bootstrap procedure provides valid finite sample inferences for the estimator from the interval censoring method. The procedure can be applied to many inference problems for copula modeling with tied data (Kojadinovic, 2016). The parameter estimation step in the procedure for bootstrap sample with ties could use the average rank method as in Kojadinovic (2016), which would, however, leads to biased estimation with strong dependence. A combination of the interval censoring method for estimation and the tie-preserving bootstrap procedure for inference appears to be a practical approach to rank-based copula modeling for data with ties. Applications to inferences such as tests for exchangeability, extreme-value dependence, and radial symmetry merits further research.

Acknowledgements J. Yan’s research was partially supported by an NSF grant (DMS 1521730). Yang Li’s research was partially supported by the Fundamental Research Funds for the Central Universities, and the Research Funds (15XNI011) of Renmin University of China.

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