Collision model for non-Markovian quantum dynamics

Silvan Kretschmer, Kimmo Luoma, and Walter T. Strunz
Institut für Theoretische Physik, Technische Universität Dresden, D-01062, Dresden, Germany

We study the applicability of collisional models for non-Markovian dynamics of open quantum systems. By allowing interactions between the separate environmental degrees of freedom in between collisions we are able to construct a collision model that allows to study quantum memory effects in open system dynamics. We also discuss the possibility to embed non-Markovian collision model dynamics into Markovian collision model dynamics in an extended state space. As a concrete example we show how using the proposed class of collision models we can discretely model non-Markovian amplitude damping of a qubit. In the time-continuous limit, we obtain the well-known results for spontaneous decay of a two level system in to a structured zero-temperature reservoir.

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I. INTRODUCTION

An open quantum system consists of a system of interest (\(S\), the open system) and an environment (\(\mathcal{E}\)). Usually, the effects of the environment onto the open system dynamics are incorporated into an effective description in terms of master equations [1, 2] for the reduced density operator of the open system. In general, the validity of the master equation approach is guaranteed in a Markovian setting, where the master equation takes the famous Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) form and is a generator of a dynamical semigroup [3–4]. Often, when starting from the microscopic model for \(S + \mathcal{E}\), a series of approximations has to be made in order to be able to describe the open system dynamics in terms of the GKSL-master equation [2, 5]. Naturally, these approximations are not generally valid and the dynamical semigroup is not an exact description for all possible physical situations of interest [6].

Collision models offer an alternative route to the description of open quantum system dynamics [7–20]. This description is appealing because the reduced system can be obtained in many cases without any approximations, hence the complete positivity (CP) of the dynamical map is guaranteed. Collision models also provide a physically transparent way to introduce indirect measurements and conditional states for the open system [13, 14]. One can look at collision models as an approximation of continuous in time quantum dynamics but the study of discrete-time dynamics is interesting in its own right as we will show in this article.

A collision model consists of a system that interacts locally in time with different environmental degrees of freedom [9]. As such, the description gives the dynamics discretely in time in terms of maps \(\Phi_n : \mathcal{S}(\mathcal{H}_S) \rightarrow \mathcal{S}(\mathcal{H}_S)\), where \(\mathcal{S}(\mathcal{H}_S)\) is the state space of the open system. If, at each collision, the open system interacts with a new uncorrelated environmental degree of freedom then the dynamical map satisfies the semigroup property: \(\Phi_{n+m} = \Phi_n \Phi_m\) [8,11,13,16], see also Fig. 1. Sometimes we call separate environmental degree of freedoms sub-environments.

During recent years there has been considerable interest to generalize the collision model description beyond Markovian quantum processes, i.e. beyond the dynamical semigroup [12, 14, 17–20]. One way to do this is to introduce correlated localized bath states [17, 19]. For our purposes the pre-correlated bath particles are not suitable because we want to introduce the memory as a part of the collision model dynamics. A second way to introduce memory effects to the open system dynamics is to consider a system that sequentially collides with the same local environment but the evolution of the local environment is given by a quantum channel between the collisions. This type of model was considered in [12] for the purpose of processing quantum information. A third, and the most relevant way for our purposes, is to allow an interaction between the separate environmental degrees of freedom in between collisions. In [18] the authors considered a partial swap interaction between the separate environmental degrees of freedom that allows to propagate correlations from earlier collisions forward in time. The authors were able to obtain a general form for the master equation in the continuous time limit. However, a link to the exactly solvable model for spontaneous decay of a two level system into a structured reservoir [6] is still missing and will be established based on our novel collision model. Therefore, our work is in close analogy of that in [18] since we also want to introduce tunable memory effects by incorporating interactions between the environmental degrees of freedom, but we take this interaction to be unitary.

The structure of the paper is the following. In Sec. II we introduce the concept of a collision model in detail. First, we discuss a construction that leads always to a CP-divisible discrete dynamical map. Then we shift our focus to a generalized model that allows to study indivisible quantum dynamical maps and non-Markovian dynamics. Section III contains an application of the collision model, we show how time continuous amplitude damping can be simulated using our discrete collision model. In Sec. IV we discuss in detail the divisibility and non-Markovian properties of the example system. In Sec. V we show how possibly non-Markovian coll-
sion model dynamics can be embedded into Markovian collision dynamics for an extended system and lastly in Sec. VI we conclude.

II. COLLISION MODELS

In a collision model the system of interest (the open system) interacts locally in time in a discrete way with separate degrees of freedom of the environment. A suitable Hilbert space for the collision model is \( \mathcal{H} = \mathcal{H}_S \otimes \bigotimes_{k=1}^N \mathcal{H}_{e,k} \), where \( \mathcal{H}_S \) is the Hilbert space of the system and \( \mathcal{H}_{e,k} \) is the Hilbert space for \( k \)-th environment particle. There is in general no restriction on the possible initial state \( |\Psi_0\rangle \in \mathcal{H} \) of the collision model. However, we aim to connect the open system dynamics of the collision model in the continuous limit to a dynamics given by a dynamical map \( \Phi \), and therefore we assume that the initial state is of the form \( |\Psi_0\rangle = |\varphi\rangle \otimes |\omega\rangle \). Now, state \( |\omega\rangle \in \bigotimes_k \mathcal{H}_{e,k} \) could be generally correlated, and it is sometimes advantageous to consider correlated initial states \([17]\) but we aim in this work to model quantum memory effects dynamically and therefore we choose the initial state to be \( |\omega\rangle = \bigotimes_k |0_k\rangle \). In general we focus on such collision models that might have local dynamics on the open system only, given by unitary operator \( U_0 \), interaction of the open system and \( i \)-th sub-environment, given by unitary operator \( W_i \), and possibly a unitary coupling between the sub-environments \( i \) and \( j \) given by \( V_{ij} \).

Before going into details of the collision models we define for further discussion the concept of divisibility. We call the dynamical map \( \Phi_n \equiv \Phi_{n,0} \) divisible if the dynamical map can be composed as \( \Phi_{n+m} = \Phi_{m+n,0} \Phi_m \), where \( \Phi_{n+m,0} \) is a completely positive and trace preserving (CPT) map. If the “two times” map \( \Phi_{k,l} \) is not CPT for some pair \((k,l)\) then the dynamics is said to be indivisible. The two times map is defined as \( \Phi_{n+m,n} = \Phi_{n+m} \circ \Phi_n^{-1} \) \([21]\]. These notions generalize to continuous families of maps in a trivial way.

1. Collision model for divisible quantum dynamics

The simplest possible model fitting to our above described scenario emerges when we choose \( V_{ij} = I \). The first two steps of this type of model is presented in Fig. 1. The open system dynamics for this type of collisional model is given by \( \rho_0 = |\varphi\rangle\langle\varphi| \rightarrow \rho_n = \mathcal{E}_n(\rho_0) = \text{tr}_E[(W_n U_0 \cdots W_U U_0) \rho_0 \otimes |\omega\rangle\langle\omega|(U_0^\dagger W_1^\dagger \cdots U_0^\dagger W_1)] \). It is clear that the dynamical map \( \mathcal{E} \) satisfies discrete semigroup property \( \mathcal{E}_{n+m} = \mathcal{E}_n \mathcal{E}_m \), where \( m,n \in \mathbb{Z}_+ \). Also \( \mathcal{E}_n = \mathcal{E}_n^n \) clearly holds. The continuous limit of a collision model can be thought to emerge when \( \mathcal{E}_t \approx \mathcal{E}^t \), where \( n = [t/g^2] \) (where \([x]\) is a nearest integer to \(x\)) and \( g \) is some parameter of the collision model ultimately to be related to the small time interval during which single collisions take place \([17]\). A heuristic way to obtain the GKSL master equation, ie. the continuous limit for this collision model is to use the relation \( \dot{\rho}_t = \frac{d}{dt} \mathcal{E}_t(\rho_t) = \mathcal{L}\rho_t \) valid for continuous dynamical maps, whenever the inverse exists. We would also like to point out the trivial observation that the \((i-1)\)th environment particle does not participate after the \((i-1)\)th collision to the dynamics anymore.

As an example we consider a collision model where system and environment consists of qubits and initially all the environment qubits are set to their ground state \( |0\rangle \). We take the interaction between the system and the \( i \)-th sub-environment during a collision to be

\[ W_i = e^{-i g (\sigma_+ \otimes \sigma^{(i)}_+ + \sigma_- \otimes \sigma^{(i)}_+)} , \]

(1)

where \( g \) describes the coupling strength and we set \( U_0 = I \) \([22]\]. If we then set \( g = \sqrt{\gamma_0 t} \), take the derivative with respect to \( \gamma_0 \) and find the inverse map we can conclude after expanding the resulting expression to first order in \( \gamma_0 \) that in the limit \( \gamma_0 \to 0 \) the collision model approximates the amplitude damping dynamical semigroup generated by

\[ \mathcal{L}\rho = \gamma \sigma_- \rho \sigma_+ - \gamma \frac{1}{2} (\sigma_+ \sigma_- - \rho) . \]

(2)

2. Collision model for indivisible quantum dynamics

Our main goal is to describe indivisible quantum dynamics, using collisional models where we allow for interaction between the separate environmental degrees of freedom. The simplest possible way to do this to allow only nearest neighbor interactions. The dynamical map for the open system is

\[ \rho_n \equiv \Phi_n(|\varphi\rangle\langle\varphi|) = \text{tr}_E[K_n(|\varphi\rangle\langle\varphi| \otimes |\omega\rangle\langle\omega|) K_n^\dagger] , \]

(3)

where \( K_1 = W_1 U_0 \) and \( K_n \) is defined iteratively as \( K_n = W_n U_0 V_{n-1} K_{n-1} \) for \( n \geq 2 \). A schematic presentation of such model can be found in Fig. 2. Clearly, the state of the \( i \)-th sub-environment is generally not \( |0_i\rangle \) when \( W_i \) is applied because before the \( i \)-th collision the uncorrelated \( i \)-th sub-environment first interacts with \((i-1)\)th sub-environment via \( V_{i,i-1} \) and after that the system interacts
with $i$th environmental degree of freedom via $W_i$.

The question of constructing indivisible quantum dynamics is thus a question of choosing suitable two qubit unitary $V_{j+1,j}$. Obviously, if we choose $V_{j+1,j} = U_A \otimes U_B$ the construction is essentially equivalent to the one presented in Fig. 1 since $U_A$ can be absorbed to $W_j$ and $U_B$ is just a rotation of the initial state. Therefore the unitary operator $V_{i+1,i}$, has to be at least non-separable.

Intuitively it is clear that by introducing the coupling $V_{i+1,i}$, we can propagate information from the earlier collisions through the correlations created between $\mathcal{H}_{e,i+1} \otimes \mathcal{H}_{e,i}$ and these correlations might have influence on the nature of the dynamics. To verify this intuition we first tested the collision model construction by sampling pairs of two qubit unitaries $Z_1$ and $Z_2$ from the uniform Haar measure [24] and using those as a building block for the collision model. Specifically we constructed the maps $\Phi_2, \Phi_1$ and $\Phi_{2,1} = \Phi_2 \circ \Phi_1^{-1}$ according to Eq. 3.

By sampling $10^6$ pairs $(Z_1, Z_2)$ we observed that approximately $17\%$ of the maps $\Phi_2$ were indivisible, i.e. $\Phi_{2,1}$ was not completely positive.

As a second test we chose randomly pairs of two qubit unitaries $V = e^{-i \sum_{j=1}^3 \alpha_j |\sigma_j \otimes \sigma_j\rangle\langle \sigma_j |}$ such that $\pi/4 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3 \geq 0$, $\alpha_1 + \alpha_2 \geq \pi/4$ and $\alpha_2 + \alpha_3 \leq \pi/4$. These correspond to maximally entangling unitaries [24]. In this case by sampling $10^6$ pairs we obtained that $35\%$ of samples resulted in indivisible dynamical maps. From these two tests we can conclude that the proposed collision model (see Eq. 3 and Fig. 2) is suitable for the study of general open system dynamics.

### III. Collision model description for time continuous amplitude damping

The process of spontaneous emission of a two level system to a structured reservoir can be described with the following exact master equation (in a suitable interaction picture)

$$\dot{\rho}_t = -\frac{i}{2} s_i [\sigma_+ \sigma_- , \rho_t] + \gamma_t \left( \sigma_- \rho \sigma_+ - \frac{1}{2} \{\sigma_+ \sigma_- , \rho\} \right),$$

where $s_i$ is a time dependent shift on the transition energy of the two level system and $\gamma_t$ is a time dependent decay rate which may be temporarily negative. Functions $s_i$ and $\gamma_t$ are related to the zero temperature bath correlation function $\chi_{ss}[3,25]$. The solution of the master equation can be given as

$$\rho_t = \Phi_t^{(c)} \rho \Phi_t^{(c)\dagger} = \rho_{t=0} + \rho_{t=0} \lambda \int_0^t ds \chi_{\lambda s - \lambda t} \eta_s,$$

where $\eta_s$ satisfies the following differential equation $\dot{\eta}_s = -\lambda \int_0^s ds \chi_{s-t} \eta_t$ and initial condition $\eta_0 = 1$. $\lambda$ is an additional parameter for the coupling strength between system and the environment. The resulting dynamical map $\Phi_t$ describes amplitude damping with a time dependent parameter $|\eta_t|^2$ [26].

The two times map $\Phi_{s+t,t}^{(c)} = \Phi_t^{(c)} \circ (\Phi_t^{(c)})^{-1}$ can be
written as
\[
\Phi^{(c)}_{t+s,t} = \rho_{11} \frac{\eta^{t+s}}{\eta_t} |2\rangle\langle 1| + (1 - \rho_{11}) \frac{\eta^{t+s}}{\eta_t} |0\rangle\langle 0| + \rho_{10} \frac{\eta^{t+s}}{\eta_t} |1\rangle\langle 0| + \rho_{01} \frac{\eta^{t+s}}{\eta_t} |0\rangle\langle 1|.
\]
One sees that this is again an amplitude damping channel iff \( |\eta^{t+s}|^2 \in [0, 1] \). We denote by \( M_{t+s,t} \) the Choi matrix of the map \( \Phi^{(c)}_{t+s,t} \). From \( M_{t+s,t} \) we see that \( \Phi^{(c)}_{t+s,t} \) is completely positive whenever
\[
|\eta^{t+s}|^2 \leq |\eta_t|^2.
\]
Thus the dynamical map is indivisible whenever the above condition is violated. This corresponds to a temporarily negative 
\( \eta_t \) in the master equation (4) which corresponds to the initial condition \( g, \chi, \phi \), arbitrary but fixed phase which is to be determined later. To fully specify the considered model, we take the spectral density of the reservoir to be a single Lorentzian leading to an exponential bath correlation function \( \chi_t = \frac{1}{\lambda} e^{-\Gamma t - i\Omega t}, t \geq 0 \). From now on, with the exception of Sec. V A 1, we will use dimensionless units
\[
\Gamma t = \tau, \tilde{\eta} = \eta / \Gamma, \tilde{\lambda} = \lambda / \Gamma,
\]
which allow us to write
\[
\dot{\eta}_t = -\frac{\tilde{\lambda}}{2} \int_0^\tau d\tau' e^{-(\tau - \tau') - i\tilde{\Omega}(\tau - \tau')} \eta_{\tau'}.
\]
The well known analytical solution in these units is
\[
\eta_t = e^{-\tau - i\tilde{\Omega} \tau} \left( \frac{1 + i\tilde{\Omega}}{b} \sinh \left( \frac{\tau}{2} b \right) + \cosh \left( \frac{\tau}{2} b \right) \right),
\]
where \( b = \sqrt{(1 + i\tilde{\Omega})^2 - 2\tilde{\lambda}} \).
To use collision model (3) to simulate discontinuously the map (4), we require that at the \( n \)th step the discrete collision model dynamics approximates the time-continuous dynamics given by Eq. (5), i.e., \( \Phi_n \approx \Phi^{(c)}_{n\delta \tau} \).
First of all, the environment is taken to consist of qubits, i.e., \( \mathcal{H}_e = \mathbb{C}^2 \), and all environment qubits are initially prepared to be in their ground state, \( |0_i\rangle \). As before, we take the interaction between the system and environment qubits to be \( W_i \), see Eq. (1). We also choose the interaction between the environment qubits to be similar to \( W_i \) but with independent parameterization
\[
V_{n,n-1} = e^{-iG(e^{i\phi} \sigma^{(n-1)} \otimes \sigma^{(n)} + e^{-i\phi} \sigma^{(n-1)} \otimes \sigma^{(n)})},
\]
where \( G \) is a real valued coupling constant and \( \phi \) is an arbitrary but fixed phase which is to be determined later.
Lastly, we choose \( U_0 = \mathbb{1} \). The collision model is conveniently parametrized with parameters \( (g, G, \phi) \).
The excitation number operator for the collision model is \( N = \sigma^+ \sigma^- + \sum_k \sigma^+_k \sigma^-_k \). Since
\[
|V_{n,n-1}, N\rangle = |W_i, N\rangle = 0,
\]
the excitation number is a conserved quantity. From the initial condition \( |0\rangle\langle 0| \) for the environment qubits and from the excitation number conservation it follows that there can be only one excitation in the collision model if the initial state of the system qubit contains it. We set \( \rho \) to be the initial state of the system qubit. By plugging in \( V_{i-1}, W_i \) and \( U_0 \) to Eq. (7) we find that the state of the system qubit, \( \rho_n \) after \( n \) collisions is
\[
\rho_n = \Phi_n(\rho) = \rho_{11} |\eta_n|^2 |1\rangle\langle 1| + (1 - \rho_{11}) |\eta_n|^2 |0\rangle\langle 0| + \rho_{10} \eta_n |1\rangle\langle 0| + \rho_{01} \eta_n^* |0\rangle\langle 1|,
\]
which shows that the collision model dynamics for the system qubit is of amplitude damping type. The discrete function \( \eta_n \) satisfies the following equation
\[
\eta_n = \cos g \left( \eta_{n-1} - \tan^2 g \sum_{j=0}^{n-2} \eta_j \left( -ie^{i\phi} \cos g \sin G \right)^{n-1-j} \right),
\]
see Sec. V for the derivation. \( \eta_n \) depends on the past values \( \eta_j, j \leq n - 2 \), through a discrete memory kernel which emerges from the “single step memory” \( V_{i-1} \) in the collision model that propagates information from the past collisions.
The only question that is left is if the time-continuous limit of the collision model dynamics given by Eqs. (3), (11), and (12) corresponds to the time continuous amplitude damping dynamics of Eqs. (5), (6). Clearly, if \( \eta_n \) coincides with \( \eta_t \) in some limit the answer is affirmative. We find that by choosing the parameters \( (g, G, \phi) \) of the collision model as
\[
G = \arcsin(e^{-\delta \tau}), \quad \phi = \pi / 2 - \tilde{\Omega} \delta \tau,
\]
and then demanding that \( \frac{2n+1}{\delta \tau} \) converges to Eq. (8) in the limit \( \delta \tau \to 0 \). For the proof see Appendix A.

IV. COMPARISON OF TIME CONTINUOUS AND COLLISION MODEL DYNAMICS

In this section we compare the continuous in time and collision model with each other in the case of amplitude damping dynamics for the system qubit. We discuss their differences in terms of divisibility or non-Markovianity. For simplicity we study only the resonant case, i.e., we set \( \tilde{\Omega} = 0 \) for the rest of this section. From Eq. (14) follows that \( \phi = \pi / 2 \).
Amplitude damping in the collision model is determined by the function \( \eta_n \). From Eq. (12) we see that we reach all possible values of the function \( \eta_n \) when
$g,G \in [0, 2\pi)$. The absolute value $|\eta_n|$ is symmetric with respect to reflections of both $g,G$ with respect to $\pi/2$. Therefore we define the domain $S_c$ of the collision model parameters to be

$$(g,G) \in S_c = [0 + \epsilon, \pi/2 - \epsilon) \times [0 + \epsilon, \pi/2 - \epsilon),$$

(16)

where $\epsilon \ll 1$. Equations (13)-(15) give the transformation between the parameters $(\Omega, \lambda, \delta\tau)$ and $(G, g, \phi)$. The inverse transformation is

$$\delta\tau = \ln(1/\sin G),$$

$$\lambda = \frac{2g^2}{\ln(1/\sin G)^2}. $$

(17)

(18)

The inverse function theorem states that the transformation $(g,G,\phi) \mapsto (\delta\tau, \lambda, \Omega)$ is invertible apart from some neighborhood of the singular point $\delta\tau = 0$. Since the parameters of the collision model and the time continuous model are in one-to-one correspondence, we can re-parametrize $\eta_n$ in terms of $\lambda, \delta\tau$ and identify time step $n$ with a time $n\delta\tau = \tau$ in the time continuous model. We can thus write $\eta_n = \eta_n^{(g,G)} = \eta_n^{(\delta\tau, \lambda)} = \eta_{n\delta\tau} = \eta_\tau$.

A. Divisibility

Open system dynamics is called indivisible in the time-continuous case if the two times map $\Phi_{\tau+\tau',\tau}$ is not completely positive, where $\tau, \tau' > 0$. In the time-discrete case the definition is that the map is indivisible if $\Phi_{m+n,n}$ is not completely positive. Both maps are indivisible if the associated Choi matrix is not positive. We denote the time-continuous and the time-discrete Choi matrices with $M_{\tau+\tau',\tau}$ and $M_{m+n,n}$, respectively.

The negativity condition of the Choi matrices $M_{n+1,n}$ in the time-discrete and $M_{\tau+\delta\tau,\tau}$ in the time-continuous cases is

$$|\eta_{n+1}|^2 > |\eta_n|^2, $$

(19)

$$|\eta_{\tau+\delta\tau}|^2 > |\eta_\tau|^2. $$

(20)

These are the conditions for the non-complete positivity of the associated two times maps. We know that the time-continuous model is indivisible when $\lambda > 1/2$ [28]. In Fig. 4 we have plotted the boundary $\lambda = 1/2$ on set $S_{c=0.01}$ with a solid gray line. Above this line the collision model dynamics is indivisible. In Fig. 5 we have plotted the image of $S_{c=0.01}$ under the mapping (17) (18) with dark gray non-solid lines. The dotted dark gray line is the image of the boundary $(g,G)|_{g=\epsilon}$. Solid gray line is the boundary $\lambda = 1/2$ between the divisible and indivisible dynamics. Light gray non-solid lines are the boundaries of the set $S_{c=0.005}$. One sees how the continuous limit $\delta\tau \to 0$ is obtained by letting $g \to 0$ for a fixed $\lambda$. Colored data points in both figures correspond to non-Markovianity of the dynamics for $n = 30$ collisions, which we will discuss next.

B. Non-Markovianity

Various different criteria for non-Markovian open system dynamics have been proposed in the literature recently, [16, 29–35]. For further detail see a recent review article [36]. In this work we use the measure proposed in [37] which is based on contractivity of the trace distance under positive maps [38]. For an application of this non-Markovianity measure to time-discrete dynamics, see for example [39].

Let $D = \frac{1}{2} |\rho^{(1)} - \rho^{(2)}|$ be the trace distance between two quantum states. In the time continuous case the measure for non-Markovianity is given as an optimization over initial state pairs that maximize the time intervals of trace distance increase during some evolution interval $\tau' \in [\tau_0, \tau]$. It can be expressed as

$$N_\tau = \max_{\{\rho^{(i)}\}} \int_{D(\tau') \geq 0} \dot{D}(\tau') d\tau'. $$

(21)

It has been shown that the optimal state pair is orthogonal [39]. Crucial difference between time-continuous dynamics and collision model dynamics is that in the latter case the derivative of the trace distance does not exist. Therefore the measure in the time-discrete setting for $k \in [0, n]$ is given by

$$N_n = \max_{\{\rho^{(i)}\}} \sum_{D(k+1) - D(k) > 0} D(k+1) - D(k). $$

(22)

For the special case studied here we can express the trace distance between two initial states $\rho^{(1)}, \rho^{(2)}$ under...
either discrete or continuous amplitude damping as
\[
D(n) = \frac{1}{2} \sqrt{|\eta_n|^2(d_x^2 + d_y^2) + |\eta_n|^4d_z^2},
\]
where \(d_{\alpha,n} = \text{tr} \left[ \sigma_\alpha \left( \rho^{(1)} - \rho^{(2)} \right) \right] \), \(\sigma_\alpha, \alpha = 1, 2, 3\) are the Pauli matrices. The time-continues case is obtained by replacing the discrete index \(n\) with a continuous parameter \(\tau\). We know that for this case the optimal pair of initial states is any pair of states on the equator of the Bloch sphere \([10]\). We choose the initial state pair to be \(|\varphi_\pm\rangle = \frac{1}{\sqrt{2}} (|0\rangle \pm |1\rangle)\) in order to obtain simple expression for the trace distance evolution between the optimal pair: \(D^*(n) = |\eta_n|\) in the discrete case and \(D^*(\tau) = |\eta_\tau|\) in the time continuous case. Since \(D^*(\tau + \delta\tau) - D^*(\tau) = |\eta_{\tau+\delta\tau}| - |\eta_{\tau}|\) we can deduce that whenever \(|\eta_{\tau}|\) behaves non-monotonically between subsequent steps it is a signature of non-Markovianity of the dynamics. This observation explains why the definition \(S_\epsilon\) in Eq. \([16]\) is reasonable. By recalling Eq. \([19]\) we see that for this specific system the indivisibility of the two times map \(\Phi_{n+1,n}\) implies the non-Markovianity of the dynamics. This is seen also in Figs. 3 and 5 where non-Markovianity occurs in the regions of indivisibility.

In Fig. 4 we chose 14 400 pairs of values for \((g, G) \in S_\epsilon\), where \(\epsilon = 0.01\) and calculated the non-Markovianity of the collision model dynamics using the optimal initial pair for \(n = 30\) collisions. In this figure we also plotted the boundary between divisible and indivisible dynamics, \(\lambda = 1/2\) of the continuous in time model. We see that the non-Markovianity of the collision model dynamics occurs in the region of indivisibility of the time-continuous model. Small inconsistencies are due to numerical inaccuracies. Nonzero value for \(\epsilon\) is chosen to avoid divergences. From the figure we see that by increasing the value of \(G\), the strength of the environment particle coupling \(V_{i,i-1}\), the dynamics becomes more non-Markovian.

In Fig. 5 we mapped the non-Markovianity values to \((\delta\tau, \lambda)\) coordinates using Eqs. \([17]\) and \([18]\). We also mapped the boundaries of the region \(S_\epsilon\) for \(\epsilon = 0.01\) with dark gray non-solid lines and for \(\epsilon = 0.005\) with light-gray non-solid lines. Solid gray line is the boundary \(\lambda = 1/2\). There are regions the image of \(S_{\epsilon=0.01}\) such that \(\lambda > 1/2\) that do not show non-Markovianity. These emerge because the number of collisions \(n = 30\) is too small for the first revival of \(|\eta_{n0}\rangle\) to occur.

Interestingly, in the case of amplitude damping dynamics, we can construct the sub-optimal non-Markovianity of the collision model dynamics to the indivisibility of the map \(\Phi^{(c)}_{\tau+\delta\tau}\) of the time-continuous case. It is based on the following observations. The only possibly negative eigenvalue of the Choi matrix \(M_{\tau+\delta\tau}\) is \(\lambda^{(M)}_{\tau+\delta\tau,\tau} = |\eta_\tau|^2 - |\eta_{\tau+\delta\tau}|^2\) Evolution of the trace distance in the collision model for the sub-optimal initial state pair \(|0\rangle \langle 1|\) (\(dz = 2, dx = dy = 0\) in Eq. \([23]\) is \(\tilde{D}(k+1) - \tilde{D}(k) = |\eta_{k+1}|^2 - |\eta_k|^2\). This just the possibly negative eigenvalue of the Choi matrix \(M_{k+1,k}\). Now, by Eqs. \([17]\), \([18]\) and the discussion in the beginning of the

\[\text{Figure 4. (Color online) Non-Markovianity of the collision model after } n = 30 \text{ steps and using the optimal initial state pair } |\varphi_\pm\rangle = \frac{1}{\sqrt{2}} (|0\rangle \pm |1\rangle). \text{ We chose uniformly 14 400 values } g, G \in S_\epsilon, \text{ where } \epsilon = 0.01. \text{ We have chosen } \epsilon \neq 0 \text{ in order to avoid divergences. The black line corresponds to the boundary between divisible and indivisible dynamics of the continuous in time model, } \lambda = 1/2.\]

By defining the sub-optimal non-Markovianity as usual \(\mathcal{M}_n = \sum D(k)\), we can write
\[
\tilde{D}(k+1) - \tilde{D}(k) = |\eta_{k+1}|^2 - |\eta_k|^2 = \lambda^{(M)}_{\tau+\delta\tau,\tau}. (24)
\]

By defining the sub-optimal non-Markovianity as usual \(\mathcal{M}_n = \sum D(k)\), we can write
\[
\mathcal{M}_n = - \sum_{\lambda^{(M)}_{\tau+\delta\tau,\tau} < 0} \lambda^{(M)}_{\tau+\delta\tau,\tau}. (25)
\]

We would like to point out that \(\mathcal{M}_n > 0 \implies \mathcal{N}_n > 0\). We have thus shown that in this case the discrete non-Markovian dynamics provides us information about the indivisibility properties of the time-continuous dynamical map.

\[\text{V. MARKOVIAN EMBEDDING OF NON-MARKOVIAN DYNAMICS}\]

It is well known in the classical case that non-Markovian processes can be embedded into Markovian processes in an extended state space. This can also be done in the quantum case, a prime example being the pseudomode method \([6]\) where the indivisible dynamics of the system of interest is obtained by partial trace from a state of a larger system that obeys the GKS master equation. In this section we study the possibility to embed the class of collisional models presented schematically in Fig. 2 when the open system and the environment consists of qubits.

By taking a partial trace over the Hilbert space \(\bigotimes_{i=1}^{n-1} \mathcal{H}_{e,i}\) of the total state \(\mathcal{H}_n(\varphi \otimes \omega)K_\omega^n\) after \(n\) collisions we obtain a two qubit state \(\kappa_n \in \mathcal{S} (\mathcal{H}_S \otimes \mathcal{H}_E)\)
\[\text{Figure 5. (Color online) Data from Fig. 4 is mapped to (}\delta \tau, \tilde{\lambda})\text{-coordinates using Eqs. (17), (18). Solid line is the boundary } \tilde{\lambda} = 1/2 \text{ between divisible and indivisible dynamics. Area confined by dark gray (non-solid) lines correspond to the image of } S_\epsilon \text{ where } \epsilon = 0.01 \text{ under the same mapping. Light gray (non-solid) lines indicate the image of } S_{\epsilon'} \text{ with } \epsilon' = 0.005. \text{ The dotted lines correspond to the image of the boundary } g = \epsilon \text{ of the set } S_\epsilon. \text{ We thus see how the time continuous limit } \delta \tau \to 0 \text{ is obtained for fixed } \tilde{\lambda} \text{ by letting } g \to 0.\]

\[\text{Figure 6. Qubit+ancilla embedding of the non-Markovian collision model, first three steps. We have left out single qubit unitaries } U_0 \text{ for the system qubit for simplicity.}\]

The reduced state of the system qubit can be obtained from state } \kappa_n \text{ by tracing out the } n\text{th environment qubit}

\[\rho_n = \text{tr}_n \{\kappa_n\}. \tag{26}\]

It turns out that the two qubit state then evolves according to the map } \Lambda \text{ defined by}

\[\kappa_n = \Lambda \kappa_n \equiv \text{tr}_{n-1} \left\{ W_n V_{n,n-1} (\kappa_{n-1} \otimes |0_n\rangle\langle 0_n|) V_{n,n-1}^\dagger W_n^\dagger \right\}, \tag{27}\]

with an initial condition

\[\kappa_1 = W_1 \rho_0 \otimes |0_1\rangle\langle 0_1| W_1^\dagger. \tag{28}\]

The map } \Lambda \text{ does not depend on the step numbers since we assume that the collision model is homogeneous. Then clearly the state } \kappa_n \text{ can be written as } \kappa_n = \Lambda^{n-1} \kappa_1, \text{ where } \Lambda^k = \Lambda \circ \Lambda^{k-1}. \text{ The structure of the collision model however gives a very special form for the initial state } \kappa_1 \text{ of the enlarged dynamics, especially it might be an entangled state between the system and the first environment qubit.}

From this follows our first non-trivial observation. The map } \Lambda \text{ is not necessarily completely positive since it is obtained from tracing over } (n - 1)\text{th qubit which might be entangled with the system qubit. However it is easy to see that the } \kappa_n \text{ is a valid state since the map } \Lambda \text{ is positive and trace preserving. Thus we have at least a model for positively divisible dynamics.}

The relevant Hilbert space for the bipartite state after } n \text{ collisions is } H_S \otimes H_{\epsilon,n}. \text{ This Hilbert space is isomorphic to } H_S \otimes H_{\epsilon,1}. \text{ We can use swap operators } S_{k,l} \text{ that swap the states between the environment particle } k \text{ and } l \text{ to implement this isomorphism}

\[W_n V_{n,n-1} = (S_{n,1})^2 W_n (S_{n,1})^2 V_{n,n-1} = S_{n,1} W_1 S_{n,1} V_{n,n-1}, \tag{29}\]

where the identity follows from } S_{k,l}^{-1} = S_{k,l}. \text{ The benefit of writing the collision model this way is that now the system qubit always interacts with the first environment particle which we will call } \text{ancilla} \text{ from now on. Non-Markovian dynamics of the system qubit might be possible to embed into Markovian dynamics for the enlarged qubit+ancilla system. Circuit diagram for the collision model with ancilla qubit is presented in Fig. 6. Clearly, the map } \Lambda \text{ can be written also as } \Lambda \kappa_n = \text{tr}_{n-1} \{ S_{n,1} W_1 S_{n,1} V_{n,n-1} \kappa_{n-1} (|\phi\rangle\langle \phi| \otimes |0\rangle\langle 0|) V_{n,n-1}^\dagger S_{n,1} W_1^\dagger \}, \text{ using the isomorphism between the original and qubit+ancilla collision models. In the next subsection we illustrate the embedding construction with a specific example.}

### A. Embedding collisional amplitude damping dynamics

We take } W_1 \text{ from Eq. (1) and } V_{i,i-1} \text{ from Eq. (10). We assume that initially the system qubit can be in an arbitrary pure state and ancilla qubit is in its ground state. Equation (28) then gives the initial state } \kappa_1, \text{ the explicit form for the initial state can be found in Appendix 13.}

From the initial condition and from the conservation of the excitation number we can deduce that } (11| \kappa_n | 11) = (11| \kappa_1 | 11) = (11| \kappa_1 | 01) = (11| \kappa_1 | 00) = (00| \kappa_n | 11) = (01| \kappa_n | 11) = (10| \kappa_n | 01) = 0. \text{ Thus the dynamics is confined in to a subspace } \{|0\rangle, |01\rangle, |00\rangle\} \text{ where the first index is for the system and the second for the ancilla.}

The map } \Lambda \text{ is not necessarily completely positive as discussed earlier since the state } \kappa_1 \text{ is entangled. The reader may find the map } \Lambda \text{ written explicitly in the ba-}
we see that the parameters with write the state $\kappa_n$ continuing this way we deduce that for arbitrary $i$ and where $\kappa$ the mixed state produces a mixed state $\phi$ is of the superposition form. Namely, $\kappa_1 = |\psi\rangle\langle\psi|$, where $|\psi\rangle = c_0|00\rangle + c_1|\cos g|01\rangle - i|\sin g|01\rangle$, see Appendix [B]. For future reference we write $|\tilde{\psi}_1\rangle = g_1|00\rangle + \alpha_1|01\rangle + \beta_1|10\rangle \equiv |\psi\rangle$. Symbol $|\tilde{\varphi}\rangle$ is used to denote that vector $\varphi$ is not necessarily normalized to unity.

Starting from a state $\kappa_1$ the dynamical map $\Lambda$ produces a mixed state $\kappa_2 = \sum B_i \kappa_1 B_i^\dagger$, where $B_1 \kappa_1 B_1^\dagger$ can be written as $|\xi_2|^2|00\rangle\langle00|$. The other component of the mixed state $\kappa_2$ is $B_2 \kappa_1 B_2^\dagger$ which can be written as $|\tilde{\psi}_2\rangle\langle\tilde{\psi}_2|$, where $|\tilde{\psi}_2\rangle = g_2|00\rangle + \alpha_2|01\rangle + \beta_2|10\rangle$. Continuing this way we deduce that for arbitrary $n$ we can write the state $\kappa_n$ as

$$\kappa_n = |\tilde{\psi}_n\rangle\langle\tilde{\psi}_n| + |\xi_n|^2|00\rangle\langle00|,$$

with $|\xi_1|^2 = 0$ and $|\tilde{\psi}_1\rangle = |\psi\rangle$.

By comparing the coefficients of states $\kappa_{n+1}$ and $\kappa_n$ we see that the parameters $\alpha_n$, $\beta_n$ and $\xi_n$ satisfy the following recursion relation

$$\alpha_{n+1} = (\alpha_n c - \beta_n e^{i\phi} s) \kappa_n,$$

$$\beta_{n+1} = -(\alpha_n is + \beta_n e^{i\phi} s),$$

$$\xi_{n+1} = C \beta_n + \xi_n,$$

with initial conditions $\xi_1 = c_0$, $\alpha_1 = c_1 \cos g$ and $\beta_1 = -i c_1 \sin g$.

By tracing out the environment qubit from $\kappa_n$ and by using the amplitude damping channel structure of the map $\Phi_n$ we can write the function $\eta_n$ in terms of collision model as $\eta_n = \alpha_n/(c_1|^2)$. Then from the recursion relations (34), (35) we obtain Eq. (12) for the function $\eta_n$.

### 1. Continuous limit of $\Lambda_n$

Note that in this section we will use physical units, eg. $(t, \Gamma, \Omega, \lambda)$, since the discussion in this section becomes most natural by this choice. Map $\Lambda$ describes discrete Markovian dynamics. We expect that the continuous limit is given by a dynamical semigroup $e^{\mathcal{L}t}$. For a small time interval $\delta t$ we can use the dynamical semigroup to write $\kappa_{n+1} \approx \kappa_n + \delta t \mathcal{L} \kappa_n$. Then we demand that to first order in $\delta t$ we have

$$\mathcal{I} + \delta t \mathcal{L} \kappa_n = \Lambda \kappa_n.$$

It turns out that the continuous limit of $\Lambda$ is given by the following Lindblad master equation

$$\mathcal{L} \kappa = -i \left\{ \alpha (\sigma_+ \sigma_- \mathcal{I}\sigma + \beta \mathcal{I} \sigma_\sigma_\sigma \kappa) \right\}
+ \delta \left\{ \sigma_+ \sigma_- \sigma + \sigma_- \sigma_\sigma + \kappa \right\}
- \gamma \left\{ \mathcal{I} \sigma_+ \sigma_- \kappa + \mathcal{I} \sigma_- \kappa \mathcal{I} \sigma_+ + \kappa \right\},$$

where $\alpha$, $\beta$, $\delta$ and $\gamma$ are free parameters.

By choosing the $\delta t$ dependence of the collision model parameters as in Eqs. (13) - (15) and then by expanding by expanding the state $\Lambda \kappa$ to first order in $\delta t$ and comparing to $(\mathcal{I} + \delta t \mathcal{L}) \kappa$ we obtain the following conditions for the free parameters: $\alpha = 0$, $\beta = \Omega$, $\delta = \sqrt{\Delta} / 2$ and $\gamma = \Gamma$. What we have obtained is that in the continuous limit the generator of the two qubit dynamics is given by

$$\mathcal{L} \kappa = -i \left( \Omega \mathcal{I} \sigma_+ \sigma_- \kappa \right)
+ \sqrt{\Delta} \left( \sigma_+ \sigma_- \sigma + \sigma_- \sigma_+ + \kappa \right)
- \Gamma \left( \mathcal{I} \sigma_+ \sigma_- \kappa + \mathcal{I} \sigma_- \kappa \mathcal{I} \sigma_+ + \kappa \right).$$

We can identify this as the pseudomode master equation for the spontaneously decaying two level system when the environment is initially in the zero temperature vacuum state $|0\rangle$. 

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sis $\{ |10\rangle, |01\rangle, |00\rangle \}$ in Appendix [C]. However we know from the Kraus theorem that if we can write the map in the form $\Lambda \kappa = \sum B_i \kappa B_i^\dagger$, where $\sum B_i^\dagger B_i = \mathcal{I}$ then the map is completely positive and trace preserving [26]. The search of the Kraus operators is non-trivial because of the correlated initial state [11], [42]. However, the fact that the Kraus operators can not be found using the standard approach does not mean that they do not necessarily exist for this map [43].
VI. CONCLUSIONS

In this work we have studied the possibility to use collisional models for studying indivisible quantum dynamics and non-Markovianity. We proposed in Sec. [II](12) a collisional model that shows potential for constructing indivisible dynamical maps.

We also showed in Sec. [III](12) that our model can be used to simulate the process of spontaneous decay of a two level system to a structured reservoir, which leads to amplitude damping, exactly. We discussed in Sec. [IV](12) the non-Markovianity of the collision model dynamics which is discrete in time and its relation to the indivisibility of the time-continuous dynamical map.

In Sec. [V](12) we showed how one can distinguish a particular environment particle and interpret it as a pseudomode. We also showed how this leads to positively divisible dynamics for the combined bipartite system. We studied in detail the case of amplitude damping and found out that in the time continuous limit we obtain the well known pseudomode master equation.

We can conclude that the collision model approach shows great potential for studying quantum memory effects in open quantum system dynamics. They might open a way to study continuous monitoring of the open system in the case where the dynamical map is indivisible. Clearly, a very wide open question is what properties are generally required for the operators W_{ij} and V_{i,j-1} so that the collision model dynamics is indeed indivisible.

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Appendix A: Proof of Eqs. (13)-(15)

Here we show how the continuous-in-time limit for the collisional model dynamics given by Eqs. (12) can be written explicitly as

\[ \kappa_1 = \rho_{11} (\cos^2 g |10\rangle \langle 10| + \sin^2 g |01\rangle \langle 01|) + \rho_{00} |00\rangle \langle 00| + i \rho_{11} \cos g \sin g (|10\rangle \langle 01| + |01\rangle \langle 10|) + \rho_{01} (\cos g |10\rangle \langle 01| - i \sin g |01\rangle \langle 00|) + \rho_{00} (\cos g |00\rangle \langle 10| + i \sin g |00\rangle \langle 01|). \]

\[ \kappa_1 \]

Appendix B: Initial condition

The initial state for the open system+ancilla evolution in Sec. [V](12) is given by \( \kappa_1 = W_1 \rho_0 \otimes |01\rangle \langle 01| W_1^\dagger \) which can be written explicitly as

\[ \kappa_1 = \rho_{11} (\cos^2 g |10\rangle \langle 10| + \sin^2 g |01\rangle \langle 01|) + \rho_{00} |00\rangle \langle 00| + i \rho_{11} \cos g \sin g (|10\rangle \langle 01| + |01\rangle \langle 10|) + \rho_{01} (\cos g |10\rangle \langle 01| - i \sin g |01\rangle \langle 00|) + \rho_{00} (\cos g |00\rangle \langle 10| + i \sin g |00\rangle \langle 01|). \]

\[ \kappa_1 \]

Appendix C: Map \( \kappa_{n+1} = \Lambda \kappa_n \)

Here we write explicitly the dynamical map studied in Sec. [V](12). The map \( \Lambda : \kappa_n \rightarrow \kappa_{n+1} \) can be written in the basis \{ |10\rangle, |01\rangle, |00\rangle \} as

\[ \begin{bmatrix}
\kappa_{10,10} e^{2S^2} S_{00,01} - |c|^2 S (e^{-i\phi} \kappa_{10,01} + e^{i\phi} \kappa_{01,10}), \\
- i (c \kappa_{10,10} e^{-i\phi} \kappa_{01,10} - e^{i\phi} S^2 \kappa_{10,01} - e^{-i\phi} S^2 \kappa_{01,10}), \\
- i S \kappa_{00,10} e^{-i\phi} S \kappa_{01,01}
\end{bmatrix}
\begin{bmatrix}
c \kappa_{00,00} e^{-i\phi} S \kappa_{01,00} - e^{i\phi} S^2 \kappa_{01,01}, \\
- i (c \kappa_{00,00} e^{i\phi} S \kappa_{01,00} + e^{-i\phi} S^2 \kappa_{01,01}), \\
- i (c \kappa_{00,00} e^{-i\phi} S \kappa_{01,00})
\end{bmatrix}
\]

where \( c = \cos g, \ s = \sin g, \ S = \sin G \) and \( \kappa_{ij,kl} = \langle ij | \kappa_n | kl \rangle \). Validity of the Kraus operators is obtained. We begin by assuming that \( g \propto \delta \tau \). Then we assume that \( \delta \tau \ll 1 \) and in Eq. (12) we use \( \cos^2 g \approx 1, \ tan^2 g \approx g^2 \). With these the difference \( \Delta \eta_n = \eta_n - \eta_{n-1} \) can be written as \( \Delta \eta_n = -g^2 \sum_{j=0}^{n-2} \eta_j (\cos g \sin G)^{n-1-j} \). When we substitute to the above equation \( g \approx 1 \) and Eqs. (13)-(15) we obtain \( \Delta \eta_n = -\lambda_2 \delta \tau \sum_{j=0}^{n-2} \delta \tau \eta_j e^{-j(1+\Omega)\delta \tau}, \) where we are allowed to replace \( (n-2) \delta \tau \approx (n-1) \delta \tau \approx \tau \), and we obtain

\[ \eta_T = \lim_{\delta \tau \to 0} \frac{\Delta \eta_n}{\delta \tau} = -\frac{1}{2} \int_0^\tau d\tau' e^{-j(1+\Omega)(\tau-\tau')} \eta_{\tau'}, \] (A1)

which is the defining equation for \( \eta_T \).

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