A Complexity Trichotomy for the Six-Vertex Model

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Abstract

We prove a complexity classification theorem that divides the six-vertex model on graphs into exactly three types. For every setting of the parameters of the model, the computation of the partition function is precisely: Either (1) solvable in polynomial time for every graph, or (2) #P-hard for general graphs but solvable in polynomial time for planar graphs, or (3) #P-hard even for planar graphs. The classification has an explicit criterion. In addition to matchgates and matchgates-transformable signatures, we discover previously unknown families of planar-tractable partition functions by a non-local connection to #CSP, defined in terms of a “loop space”. For the proof of #P-hardness, we introduce the use of Möbius transformations as a powerful new tool to prove that certain complexity reductions succeed in polynomial time.

1 Introduction

Partition functions are Sum-of-Product computations. In physics, one considers a set of particles connected by some bonds. Then physical considerations impose various local constraints, each with a suitable weight. Given a configuration satisfying all the local constraints, the product of local weights is the weight of the configuration, and its sum over all configurations is the value of the partition function. It encodes much information about the physical system.

This is essentially the same set-up as counting constraint satisfaction problems (#CSP). Fix a set of constraint functions $F$, the problem #CSP($F$) is as follows: The input is a bipartite graph $G = (U, V, E)$, where $U$ are the variables (spins), $V$ is labeled by constraint functions from $F$, and $E$ describes how the constraints are applied on the variables. The output is the sum, over all assignments to variables in $U$, of the product of constraint function evaluations in $V$. Note that each function in $F$ has a fixed arity, and in general takes values in $\mathbb{C}$ (not just $\{0, 1\}$). A spin system is the special case of #CSP where the constraints are binary functions (in which case each $v \in V$ has degree 2 and can be replaced by an edge), and possibly some unary functions (representing external fields).

By definition, a partition function is an exponential sized sum. But in some cases, clever algorithms exist that can compute it in polynomial time. Well-known examples of partition functions from physics that have been investigated intensively in complexity theory include the Ising model, Potts model, hardcore gas and Ice model [14, 10, 11, 18, 25]. Most of these are spin systems. If
particles take (+) or (−) spins, each can be modeled by a Boolean variable, and local constraints are expressed by edge (binary) constraint functions. These are nicely modeled by the #CSP framework. Some physical systems are more naturally described as orientation problems, and these can be modeled by Holant problems, of which #CSP is a special case. Roughly speaking, Holant problems [7] (see Section 2 for definitions) are tensor networks where edges of a graph are variables while vertices are local constraint functions. Spin systems can be simulated easily as Holant problems, but Freedman, Lovász and Schrijver proved that simulation in the reverse direction is generally not possible [9]. In this paper we study a family of partition functions that fit the Holant problems naturally, but not as a spin system. This is the six-vertex model.

In physics, the six-vertex model concerns crystal lattices with hydrogen bonds. Remarkably it can be expressed perfectly as a family of Holant problems with 6 parameters, although in physics people are more focused on regular planar structures such as lattice graphs, and asymptotic limit. Previously, without being able to account for the planar restriction, it has been proved [5] that there is a complexity dichotomy where computing the partition function \( Z_{\text{Six}} \) is either in P or #P-hard. However the more interesting problem is what happens on planar structures where physicists had discovered some remarkable algorithms, such as Kasteleyn’s algorithm for planar perfect matchings [21, 15, 16]. Concomitantly, and also probably because of that, to achieve a complete complexity classification in the planar case is more challenging. It must isolate precisely those problems that are #P-hard in general graphs but P-time computable on planar graphs.

In this paper we prove a complexity trichotomy theorem for the six-vertex models: According to the 6 parameters from \( \mathbb{C} \), the partition function \( Z_{\text{Six}} \) is either computable in P-time, or #P-hard on general graphs but computable in P-time on planar graphs, or remains #P-hard on planar graphs. The classification has an explicit criterion. In addition to matchgates and matchgates-transformable signatures, we discover previously unknown families of planar-tractable \( Z_{\text{Six}} \) by a non-local connection to #CSP, defined in terms of a “loop space”.

The six-vertex model has a long history in physics. Linus Pauling in 1935 first introduced the six-vertex models to account for the residual entropy of water ice [20]. Consider a large number of oxygen and hydrogen atoms in a 1 to 2 ratio. Each oxygen atom (O) is connected by a bond to four other neighboring oxygen atoms (O), and each bond is occupied by one hydrogen atom (H). Physical constraint requires that each (H) is closer to either one or the other of the two neighboring (O), but never in the middle of the bond. Pauling argued [20] that, furthermore, the allowed configurations are such that at each oxygen (O) site, exactly two hydrogen (H) are closer to it, and the other two are farther away. The placement of oxygen and hydrogen atoms can be naturally represented by vertices and edges of a 4-regular graph. The constraint on the placement of hydrogen atoms (H) can be represented by an orientation of the edges of the graph, such that at every vertex (O), exactly two edges are oriented toward the vertex, and exactly two edges are oriented away from it. In other words, this is an Eulerian orientation. Since there are \( \binom{4}{2} = 6 \) local valid configurations, this is called the six-vertex model. In addition to water ice, potassium dihydrogen phosphate \( \text{KH}_2\text{PO}_4 \) (KDP) also satisfies this model.

The valid local configurations of the six-vertex model are illustrated in Figure 1. The energy \( E \) of the system is determined by six parameters \( \epsilon_1, \epsilon_2, \ldots, \epsilon_6 \) associated with each type of the local configuration. If there are \( n_i \) sites in local configurations of type \( i \), then \( E = n_1\epsilon_1 + n_2\epsilon_2 + \ldots + n_6\epsilon_6 \). Then the partition function is \( Z_{\text{Six}} = \sum e^{-E/k_B T} \), where the sum is over all valid configurations, \( k_B \) is Boltzmann’s constant, and \( T \) is the system’s temperature. Mathematically, this is a sum-of-product computation where the sum is over all Eulerian orientations of the graph, and the product
Figure 1: Valid configurations of the six-vertex model

is over all vertices where each vertex contributes a factor $c_i = c^\epsilon_i$ if it is in configuration $i$ ($1 \leq i \leq 6$) for some constant $c$.

Some choices of the parameters are well-studied. On the square lattice graph, when modeling ice one takes $\epsilon_1 = \epsilon_2 = \ldots = \epsilon_6 = 0$. In 1967, Elliott Lieb [19] famously showed that, as the number of vertices $N \to \infty$, the value of the “partition function per vertex” $W = Z^{1/N}$ approaches $(\frac{4}{3})^{3/2} \approx 1.5396007 \ldots$ (Lieb’s square ice constant). This matched experimental data $1.540 \pm 0.001$ so well that it is considered a triumph. Other well-known six-vertex models include: the KDP model of a ferroelectric ($\epsilon_1 = \epsilon_2 = 0$, and $\epsilon_3 = \epsilon_4 = \epsilon_5 = \epsilon_6 > 0$), the Rys $F$ model of an antiferroelectric ($\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 > 0$, and $\epsilon_5 = \epsilon_6 = 0$). Historically these are widely considered among the most significant applications ever made of statistical mechanics to real substances. In classical statistical mechanics the parameters are all real numbers while in quantum theory the parameters are complex numbers in general.

Disregarding the planarity restriction, [5] proved that computing the partition function $Z_{\text{Six}}$ is either in P or $\#P$-hard. However known cases of planar P-time computable $Z_{\text{Six}}$ (but $\#P$-hard on general graphs) are all $\#P$-hard in this classification. In this paper we tackle the more difficult planar case, and prove a complexity trichotomy theorem. The most interesting part is the classification of those $Z_{\text{Six}}$ which are $\#P$-hard in general but P-time computable on planar structures. The classification is valid for all parameter values $c_1, c_2, \ldots, c_6 \in \mathbb{C}$. (To state our theorem in the strict Turing machine model, we take $c_1, c_2, \ldots, c_6$ to be algebraic numbers.) The dependence of this trichotomy on the values $c_1, c_2, \ldots, c_6$ is explicit.

We show that constraints that are expressible as matchgates or those that are transformable by a holographic transformation to matchgates do constitute a family of $Z_{\text{Six}}$ which are $\#P$-hard in general but P-time computable on planar structures. This is as expected and is derivable from known results (Kasteleyn’s algorithm for planar perfect matchings, and Valiant’s holographic algorithms based on matchgates [23, 24]). However we also discover an additional planar tractable family of $Z_{\text{Six}}$ which is not transformable to matchgates. This polynomial time tractability on planar graphs is via a non-local transformation to tractable $\#\text{CSP}$, where the variables in $\#\text{CSP}$ correspond to certain loops in the six-vertex model graph. The transformation produces instances of $\#\text{CSP}$ that are not necessarily planar; however it crucially depends on the global planar topology of the $Z_{\text{Six}}$ instance, that the constraint functions defining these $\#\text{CSP}$ instances belong to tractable (P-time computable) families (even for non-planar $\#\text{CSP}$.)

After carving out this last tractable family, we set about to prove that everything else is $\#P$-hard, even for the planar case. A powerful tool in such proofs is the interpolation technique [22]. Typically an interpolation proof can succeed when certain quantities (such as eigenvalues) are not roots of unity, lest the iteration repeat after a bounded number of steps. A sufficient condition is that these quantities have complex norm $\neq 1$. However for some constraint functions, we can show that all constructions necessarily produce only relevant quantities of unit norm. One main contribution
of this work is to introduce the use of Möbius transformations \( z \mapsto \frac{a z + b}{c z + d} \) as a tool to deal with such difficult cases. We show that in this case we can define a natural Möbius transformation that maps unit circle to unit circle on \( \mathbb{C} \). By exploiting the conformal mapping property we can obtain a suitable Möbius transformation which generates a group of infinite order. This allows us to show that our interpolation proof succeeds.

2 Preliminaries and Notations

In this paper, \( i \) denotes \( \sqrt{-1} \), a square root of \(-1\).

2.1 Definitions and Notations

A constraint function \( f \), or a signature, of arity \( k \) is a map \( \{0, 1\}^k \to \mathbb{C} \). Fix a set \( \mathcal{F} \) of constraint functions. A signature grid \( \Omega = (G, \pi) \) is a tuple, where \( G = (V, E) \) is a graph, \( \pi \) labels each \( v \in V \) with a function \( f_v \in \mathcal{F} \) of arity \( \deg(v) \), and the incident edges \( E(v) \) at \( v \) with input variables of \( f_v \). We consider all 0-1 edge assignments \( \sigma \), each gives an evaluation \( \prod_{v \in V} f_v(\sigma|_{E(v)}) \), where \( \sigma|_{E(v)} \) denotes the restriction of \( \sigma \) to \( E(v) \). The counting problem on the instance \( \Omega \) is to compute

\[
\text{Holant}_\Omega = \text{Holant}(\Omega; \mathcal{F}) = \sum_{\sigma: E \to \{0, 1\}} \prod_{v \in V} f_v(\sigma|_{E(v)}).
\]

The Holant problem parameterized by the set \( \mathcal{F} \) is denoted by \( \text{Holant}(\mathcal{F}) \). If \( \mathcal{F} = \{ f \} \) is a single set, for simplicity, we write \( \{ f \} \) as \( f \) directly, and also we write \( \{ f, g \} \) as \( f, g \). When \( G \) is a planar graph, the corresponding signature grid is called a planar signature grid. We use \( \text{Holant}(\mathcal{F} | \mathcal{G}) \) to denote the Holant problem over signature grids with a bipartite graph \( H = (U, V, E) \), where each vertex in \( U \) or \( V \) is assigned a signature in \( \mathcal{F} \) or \( \mathcal{G} \) respectively. We list the values of a signature \( f : \{0, 1\}^k \to \mathbb{C} \) as a vector of dimension \( 2^k \) in lexicographic order. Signatures in \( \mathcal{F} \) are considered as row vectors (or contravariant tensors); signatures in \( \mathcal{G} \) are considered as column vectors (or covariant tensors). Similarly, \( \text{Pl-Holant}(\mathcal{F} | \mathcal{G}) \) denotes the Holant problem over signature grids with a planar bipartite graph.

A signature \( f \) of arity 4 has the signature matrix \( M(f) = M_{x_1x_2,x_4x_3}(f) = \begin{bmatrix} f_{0000} & f_{0010} & f_{0001} & f_{0011} \\ f_{1000} & f_{1010} & f_{1001} & f_{1011} \\ f_{0100} & f_{0110} & f_{0101} & f_{0111} \\ f_{1100} & f_{1110} & f_{1101} & f_{1111} \end{bmatrix} \). Notice the order reversal \( x_4x_3 \); this is for the convenience of composing these signatures in a planar fashion. If \( (i, j, k, \ell) \) is a permutation of \( (1, 2, 3, 4) \), then the \( 4 \times 4 \) matrix \( M_{x_1x_j,x_\ell x_k}(f) \) lists the 16 values with row index \( x_i x_j \in \{0, 1\}^2 \) and column index \( x_\ell x_k \in \{0, 1\}^2 \) in lexicographic order.

The planar six-vertex model is \( \text{Pl-Holant}(\neq 2) \), where \( M(f) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & z & y & 0 \\ z & 0 & 0 & 0 \end{bmatrix} \). The outer matrix of \( M(f) \) is the submatrix \( \begin{bmatrix} M(f)_{1,1} & M(f)_{1,4} \\ M(f)_{4,1} & M(f)_{4,4} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix} \), and is denoted by \( M_{\text{Out}}(f) \). The inner matrix of \( M(f) \) is \( \begin{bmatrix} M(f)_{2,2} & M(f)_{2,3} \\ M(f)_{3,2} & M(f)_{3,3} \end{bmatrix} = \begin{bmatrix} b & c \\ z & y \end{bmatrix} \), and is denoted by \( M_{\text{In}}(f) \). A binary signature \( g \) has the signature matrix \( M(g) = M_{x_1x_2}(g) = \begin{bmatrix} g_{00} & g_{01} \\ g_{01} & g_{11} \end{bmatrix} \). Switching the order, \( M_{x_2x_1}(g) = \begin{bmatrix} g_{00} & g_{01} \\ g_{01} & g_{11} \end{bmatrix} \). We use \( (\neq 2) \) to denote binary \( \text{DISEQUALITY} \) signature \( (0, 1, 1, 0)^T \). It has the signature matrix \( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \). Let \( N = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \). Note that \( N \) is the double \( \text{DISEQUALITY} \) \( (x_1 \neq x_4) \land (x_2 \neq x_3) \), which is the function of connecting two pairs of edges by \( (\neq 2) \). A function is symmetric if its value
depends only on the Hamming weight of its input. A symmetric function \( f \) on \( k \) Boolean variables can be expressed as \([f_0, f_1, \ldots, f_k]\), where \( f_w \) is the value of \( f \) on inputs of Hamming weight \( w \). For example, \((=_k)\) is the EQUALITY signature \([1, 0, \ldots, 0, 1]\) (with \( k - 1 \) many 0’s) of arity \( k \). The support of a function \( f \) is the set of inputs on which \( f \) is nonzero.

Counting constraint satisfaction problems (\#CSP) can be defined as a special case of Holant problems. An instance of \#CSP(\( F \)) is presented as a bipartite graph. There is one node for each variable and for each occurrence of constraint functions respectively. Connect a constraint node to a variable node if the variable appears in that occurrence of constraint, with a labeling on the edges for the order of these variables. This bipartite graph is also known as the constraint graph. If we attach each variable node with an EQUALITY function, and consider every edge as a variable, then the \#CSP is just the Holant problem on this bipartite graph. Thus \#CSP(\( F \)) \( \equiv_T \) Holant (\( \mathcal{E}Q \mid F \)), where \( \mathcal{E}Q = \{=,=,=,\ldots\} \) is the set of EQUALITY signatures of all arities. By restricting to planar constraint graphs, we have the planar \#CSP framework, which we denote by Pl-\#CSP. The construction above also shows that Pl-\#CSP(\( F \)) \( \equiv_T \) Pl-Holant (\( \mathcal{E}Q \mid F \)).

### 2.2 Gadget Construction

One basic notion used throughout the paper is gadget construction. We say a signature \( f \) is constructible or realizable from a signature set \( F \) if there is a gadget with some dangling edges such that each vertex is assigned a signature from \( F \), and the resulting graph, when viewed as a black-box signature with inputs on the dangling edges, is exactly \( f \). If \( f \) is realizable from a set \( F \), then we can freely add \( f \) into \( F \) while preserving the complexity.

![Figure 2: An \( F \)-gate with 5 dangling edges.](image)

Formally, this notion is defined by an \( F \)-gate. An \( F \)-gate is similar to a signature grid \((G, \pi)\) for Holant(\( F \)) except that \( G = (V, E, D) \) is a graph with some dangling edges \( D \). The dangling edges define external variables for the \( F \)-gate (See Figure 2 for an example). We denote the regular edges in \( E \) by 1, 2, \ldots, \( m \) and the dangling edges in \( D \) by \( m+1, \ldots, m+n \). Then we can define a function \( f \) for this \( F \)-gate as

\[
f(y_1, \ldots, y_n) = \sum_{x_1, \ldots, x_m \in \{0,1\}} H(x_1, \ldots, x_m, y_1, \ldots, y_n),
\]

where \((y_1, \ldots, y_n) \in \{0,1\}^n\) is an assignment on the dangling edges and \( H(x_1, \ldots, x_m, y_1, \ldots, y_n) \) is the value of the signature grid on an assignment of all edges in \( G \), which is the product of evaluations at all vertices in \( V \). We also call this function \( f \) the signature of the \( F \)-gate.

An \( F \)-gate is planar if the underlying graph \( G \) is a planar graph, and the dangling edges, ordered counterclockwise corresponding to the order of the input variables, are in the outer face in a planar
embedding. A planar $F$-gate can be used in a planar signature grid as if it is just a single vertex with the particular signature.

Using planar $F$-gates, we can reduce one planar Holant problem to another. Suppose $g$ is the signature of some planar $F$-gate. Then Pl-Holant$(F, g) \leq_T$ Pl-Holant$(F)$. The reduction is simple. Given an instance of Pl-Holant$(F, g)$, by replacing every occurrence of $g$ by the $F$-gate, we get an instance of Pl-Holant$(F)$. Since the signature of the $F$-gate is $g$, the Holant values for these two signature grids are identical.

In this paper, we focus on planar graphs, and we assume the edges incident to a vertex are ordered counterclockwise. When connecting two signatures, we need to keep the counterclockwise signature grids are identical. A planar $F$-gate can be used in a planar signature grid as if it is just a single vertex ordered counterclockwise. When connecting two signatures, we need to keep the counterclockwise signature grids are identical.

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There are three common gadgets we will use in this paper. The first gadget construction is as follows. Suppose $f_1$ and $f_2$ have signature matrices $M_{x_1,x_2,x_3,x_4}(f_1)$ and $M_{x_1,x_2,x_3,x_4}(f_2)$, where $(i, j, k, \ell)$ and $(s, t, u, v)$ are permutations of $(1, 2, 3, 4)$. By connecting $x_\ell$ with $x_s$, $x_k$ with $x_t$, both using Disequality $(\neq 2)$, we get a signature of arity 4 with the signature matrix $M_{x_1,x_2,x_3,x_4}(f_1)N M_{x_1,x_2,x_3,x_4}(f_2)$ by matrix product with row index $x_i x_j$ and column index $x_u x_v$ (See Figure 4).

![Figure 3](image-url)
A binary signature $g$ has the signature vector $g(x_1, x_2) = (g_{00}, g_{01}, g_{10}, g_{11})^T$, and also $g(x_2, x_1) = (g_{00}, g_{10}, g_{01}, g_{11})^T$. Without other specification, $g$ denotes $g(x_1, x_2)$. Let $f$ be a signature of arity 4 with the signature matrix $M_{x_1, x_2, x_3, x_4}(f)$ and $(s, t)$ be a permutation of $(1, 2)$. The second gadget construction is as follows. By connecting $x_1$ with $x_s$ and $x_k$ with $x_t$, both using DISEQUALITY $(\neq 2)$, we get a binary signature with the signature matrix $M_{x_1, x_2, x_3, x_4}N_g(x_s, x_t)$ as a matrix product with index $x_1x_2$ (See Figure 5). If $g_{00} = g_{11}$, then $N(g_{00}, g_{01}, g_{10}, g_{11})^T = (g_{11}, g_{10}, g_{01}, g_{00})^T = (g_{00}, g_{10}, g_{01}, g_{11})^T$, and similarly, $N(g_{00}, g_{10}, g_{01}, g_{11})^T = (g_{00}, g_{10}, g_{01}, g_{11})^T$. Therefore, $M_{x_1, x_2, x_3, x_4}N_g(x_s, x_t) = M_{x_1, x_2, x_3, x_4}g(x_i, x_s)$, which means that connecting variables $x_i$, $x_k$ of $f$ with, respectively, variables $x_s$, $x_t$ of $g$ using $N$ is equivalent to connecting them directly without $N$. Hence, in the setting Pl-Holant $(\neq 2) f, g$ we can form $M_{x_1, x_2, x_3, x_4}g(x_i, x_s)$, which is technically $M_{x_1, x_2, x_3, x_4}N_g(x_s, x_t)$, provided that $g_{00} = g_{11}$. Note that for a binary signature $g$, we can rotate it by $180^\circ$ without violating planarity, and so both $g(x_s, x_t)$ and $g(x_t, x_s)$ can be freely used once we get one of them.

A signature $f$ of arity 4 also has the $2 \times 8$ signature matrix

$$M_{x_1, x_2, x_3, x_4}(f) = \begin{bmatrix}
    f_{0000} & f_{0010} & f_{0011} & f_{0012} & f_{0100} & f_{0101} & f_{0102} & f_{0111} \\
    f_{1000} & f_{1010} & f_{1011} & f_{1012} & f_{1100} & f_{1101} & f_{1102} & f_{1111}
\end{bmatrix}.$$

Suppose the signature matrix of $g$ is $M_{x_s, x_t}(g)$ and the signature matrix of $f$ is $M_{x_1, x_2, x_3, x_4}(f)$. Our third gadget construction is as follows. By connecting $x_i$ with $x_i$ using DISEQUALITY $(\neq 2)$, we get a signature $h$ of arity 4 with the signature matrix $M_{x_1, x_2, x_3, x_4}(g)M(\neq 2)M_{x_1, x_2, x_3, x_4}(f)$ by matrix product with row index $x_i$ and column index $x_i$. In particular, if $M_{y_1, y_2}(g) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$,
then connecting $y_2$ with $x_1$ via $(\neq 2)$ gives

$$M_{y_1,x_2,x_4,x_3}(h) = M_{y_1,y_2}(g)M(\neq 2)M_{x_1,x_2,x_4,x_3}(f)$$

$$= \begin{bmatrix} 0 & 1 \\ t & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} f_{0000} & f_{0010} & f_{0001} & f_{0011} \\ f_{0100} & f_{0110} & f_{0101} & f_{0111} \\ f_{1000} & f_{1010} & f_{1001} & f_{1011} \\ f_{1100} & f_{1110} & f_{1101} & f_{1111} \end{bmatrix}$$

$$= \begin{bmatrix} f_{0000} & f_{0010} & f_{0001} & f_{0011} \\ f_{0100} & f_{0110} & f_{0101} & f_{0111} \\ tf_{1000} & tf_{1010} & tf_{1001} & tf_{1011} \\ tf_{1100} & tf_{1110} & tf_{1101} & tf_{1111} \end{bmatrix}.$$

If we rename the variable $y_1$ by $x_1$, then $M_{x_1,x_2,x_4,x_3}(h) = \begin{bmatrix} f_{0000} & f_{0010} & f_{0001} & f_{0011} \\ f_{0100} & f_{0110} & f_{0101} & f_{0111} \\ tf_{1000} & tf_{1010} & tf_{1001} & tf_{1011} \\ tf_{1100} & tf_{1110} & tf_{1101} & tf_{1111} \end{bmatrix}$. That is, the new signature has the matrix obtained from multiplying $t$ to the last two rows of $M_{x_1,x_2,x_4,x_3}(f)$ corresponding to $x_1 = 1$. Similarly we can modify the last two columns of $M_{x_1,x_2,x_4,x_3}(f)$. Given $g = (0,1,t,0)^T$, we call the modification from $M_{x_1,x_2,x_4,x_3}(f)$ to

$$\begin{bmatrix} f_{0000} & f_{0010} & f_{0001} & f_{0011} \\ f_{0100} & f_{0110} & f_{0101} & f_{0111} \\ tf_{1000} & tf_{1010} & tf_{1001} & tf_{1011} \\ tf_{1100} & tf_{1110} & tf_{1101} & tf_{1111} \end{bmatrix}$$

the operation of $t$ scaling on $x_1 = 1$. Similarly we call the modification from $M_{x_1,x_2,x_4,x_3}(f)$ to

$$\begin{bmatrix} f_{0000} & f_{0010} & tf_{0001} & tf_{0011} \\ f_{0100} & f_{0110} & tf_{0101} & tf_{0111} \\ f_{1000} & f_{1010} & tf_{1001} & tf_{1011} \\ f_{1100} & f_{1110} & tf_{1101} & tf_{1111} \end{bmatrix}$$

the operation of $t$ scaling on $x_4 = 1$.

For any scalar $c \neq 0$ and any set of signatures $\mathcal{F}$, we have $\text{Holant}(\mathcal{F} \cup \{f\}) \equiv_T \text{Holant}(\mathcal{F} \cup \{cf\})$, and $\text{Pl-Holant}(\mathcal{F} \cup \{f\}) \equiv_T \text{Pl-Holant}(\mathcal{F} \cup \{cf\})$. Thus a scalar $c \neq 0$ does not change the complexity of a Holant problem. Hence we can normalize any particular nonzero signature entry to be 1.

### 2.3 Holographic Transformation

To introduce the idea of holographic transformation, it is convenient to consider bipartite graphs. For a general graph, we can always transform it into a bipartite graph while preserving the Holant value, as follows. For each edge in the graph, we replace it by a path of length two. (This operation
is called the 2-stretch of the graph and yields the edge-vertex incidence graph.) Each new vertex is assigned the binary Equality signature \(=2 = [1, 0, 1]\).

For an invertible 2-by-2 matrix \(T \in \text{GL}_2(\mathbb{C})\) and a signature \(f\) of arity \(n\), written as a column vector (contravariant tensor) \(f \in \mathbb{C}^2\), we denote by \(T^{-1}f = (T^{-1})^\otimes_n f\) the transformed signature. For signatures written as row vectors (covariant tensors) we define \(fT\) and \(FT\) similarly.

For a signature set \(F\), define \(T^{-1}F = \{T^{-1}f \mid f \in F\}\) the set of transformed signatures. For signatures written as row vectors (covariant tensors) we define \(FT\) and \(F\) similarly. Whenever we write \(T^{-1}f\) or \(T^{-1}F\), we view the signatures as column vectors; similarly for \(fT\) or \(FT\) as row vectors. In the special case of the Hadamard matrix \(H_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\), we also define \(\hat{F} = H_2F\).

Note that \(H_2\) is orthogonal. Since constant factors are immaterial, for convenience we sometime drop the factor \(\frac{1}{\sqrt{2}}\) when using \(H_2\).

Let \(T \in \text{GL}_2(\mathbb{C})\). The holographic transformation defined by \(T\) is the following operation: given a signature grid \(\Omega = (H, \pi)\) of Holant \((F | G)\), for the same bipartite graph \(H\), we get a new signature grid \(\Omega' = (H, \pi')\) of Holant \((FT | T^{-1}G)\) by replacing each signature in \(F\) or \(G\) with the corresponding signature in \(F\) or \(T^{-1}G\).

**Theorem 2.1** (Valiant’s Holant Theorem [25]). For any \(T \in \text{GL}_2(\mathbb{C})\),

\[
\text{Holant}(\Omega; F | G) = \text{Holant}(\Omega'; FT | T^{-1}G).
\]

Therefore, a holographic transformation does not change the complexity of the Holant problem in the bipartite setting. This theorem also holds for planar instances.

**Definition 2.2.** We say a signature set \(F\) is \(C\)-transformable if there exists a \(T \in \text{GL}_2(\mathbb{C})\) such that \((0, 1, 1, 0)^T \otimes 2 \in C\) and \(T^{-1}F \subseteq C\).

This definition is important because if \(\text{Pl-Holant}(C)\) is tractable, then \(\text{Pl-Holant}(\neq 2 | F)\) is tractable for any \(C\)-transformable set \(F\).

### 2.4 Polynomial Interpolation

Polynomial interpolation is a powerful technique to prove \#P-hardness for counting problems. We use polynomial interpolation to prove the following lemmas.

**Lemma 2.3.** Let \(f\) be a 4-ary signature with the signature matrix \(M(f) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}\), where \(b \neq 0\)

is not a root of unity. Let \(\chi_1\) be a 4-ary signature with the signature matrix \(M(\chi_1) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\).

Then for any signature set \(F\) containing \(f\), we have

\[
\text{Pl-Holant}(\neq 2 | F \cup \{\chi_1\}) \leq_T \text{Pl-Holant}(\neq 2 | F).
\]

**Proof.** We construct a series of gadgets \(f_{2s+1}\) by a chain of \(2s + 1\) many copies of \(f\) linked by the double Disequality \(N\) (See Figure 7). Clearly \(f_{2s+1}\) has the following signature matrix

\[
M(f_{2s+1}) = M(f)(NM(f))^{2s} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & b^{2s+1} & 0 & 0 \\ 0 & 0 & b^{2s+1} & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.
\]
The matrix $M(f_{2s+1})$ has a good form for polynomial interpolation. Suppose $\chi_1$ appears $m$ times in an instance $\Omega$ of Pl-Holant($\not= 2$, $\mathcal{F} \cup \{\chi_1\}$). We replace each appearance of $\chi_1$ by a copy of the gadget $f_{2s+1}$ to get an instance $\Omega_{2s+1}$ of Pl-Holant($\not= 2$, $\mathcal{F} \cup \{f_{2s+1}\}$), which is also an instance of Pl-Holant($\not= 2$, $\mathcal{F}$). We divide $\Omega_{2s+1}$ into two parts. One part consists of $m$ signatures $f_{2s+1}$ and its signature is represented by $(M(f_{2s+1}))^\otimes m$. Here we rewrite $(M(f_{2s+1}))^\otimes m$ as a column vector. The other part is the rest of $\Omega_{2s+1}$ and its signature is represented by $A$ which is a tensor expressed as a row vector. Then, the Holant value of $\Omega_{2s+1}$ is the dot product $\langle A, (M(f_{2s+1}))^\otimes m \rangle$, which is a summation over $4m$ bits. That is, a sum over all 0, 1 values for the $4m$ edges connecting the two parts. We can stratify all 0, 1 assignments of these $4m$ bits having a nonzero evaluation of a term in Pl-Holant$\Omega_{2s+1}$ into the following categories:

- There are $i$ many copies of $f_{2s+1}$ receiving inputs 0011 or 1100;
- There are $j$ many copies of $f_{2s+1}$ receiving inputs 0110 or 1001;

where $i + j = m$.

For any assignment in the category with parameter $(i, j)$, the evaluation of $(M(f_{2s+1}))^\otimes m$ is clearly $b^{(2s+1)j}$. Let $a_{ij}$ be the summation of values of the part $A$ over all assignments in the category $(i, j)$. Note that $a_{ij}$ is independent from the value of $s$ since we view the gadget $f_{2s+1}$ as a block. Since $i + j = m$, we can denote $a_{ij}$ by $aj$. Then, we rewrite the dot product summation and get

$$\text{Pl-Holant}_{\Omega_{2s+1}} = \langle A, (M(f_{2s+1}))^\otimes m \rangle = \sum_{0 \leq j \leq m} a_{jr}b^{(2s+1)j}.$$ 

Under this stratification, the Holant value of Pl-Holant($\Omega$, $\not= 2$, $\mathcal{F} \cup \{\chi_1\}$) can be represented as

$$\text{Pl-Holant}_\Omega = \langle A, (M(\chi_1))^\otimes m \rangle = \sum_{0 \leq j \leq m} a_{jr}.$$

Since $b \neq 0$ is not a root of unity, the linear equation system has a nonsingular Vandermonde matrix

$$
\begin{bmatrix}
   b^0 & b^1 & \cdots & b^m \\
   (b^3)^0 & (b^3)^1 & \cdots & (b^3)^m \\
   \vdots & \vdots & \ddots & \vdots \\
   (b^{2m+1})^0 & (b^{2m+1})^1 & \cdots & (b^{2m+1})^m 
\end{bmatrix}.
$$

By oracle querying the values of Pl-Holant$\Omega_{2s+1}$, we can solve the coefficients $a_{jr}$ in polynomial time and obtain the value of $p(x) = \sum_{0 \leq j \leq m} a_{jr}x^j$ for any $x$. Let $x = 1$, we get Pl-Holant$\Omega$. Therefore, we have Pl-Holant($\not= 2$, $\mathcal{F} \cup \{\chi_1\}$) $\ll_T$ Pl-Holant($\not= 2$, $\mathcal{F}$).
Corollary 2.4. Let \( f \) be a 4-ary signature with the signature matrix \( M(f) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \), where \( b \neq 0 \) is not a root of unity. Let \( \chi_2 \) be a 4-ary signature with the signature matrix \( M(\chi_2) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \). Then for any signature set \( F \) containing \( f \), we have

\[
\text{Pl-Holant}(\neq 2| F \cup \{\chi_2\}) \leq_T \text{Pl-Holant}(\neq 2| F).
\]

**Proof.** We still construct a series of gadgets \( f_{2s+1} \) by a chain of odd many copies of \( f \) linked by the double \textsc{Disequality} \( N \). Clearly \( f_{2s+1} \) has the following signature matrix

\[
M(f_{2s+1}) = M(f) (NM(f))^{2^s} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & b^{2s+1} & 0 & 0 \\ 0 & 0 & b^{2s+1} & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}.
\]

Suppose \( \chi_2 \) appears \( m \) times in an instance \( \Omega \) of \( \text{Pl-Holant}(\neq 2| f \cup \chi_2) \). We replace each appearance of \( \chi_2 \) by a copy of the gadget \( f_{2s+1} \) to get an instance \( \Omega_{2s+1} \) of \( \text{Pl-Holant}(\neq 2| F \cup \{f_{2s+1}\}) \). In the same way as in the proof of Lemma 2.3, we divide \( \Omega_{2s+1} \) into two parts. One part is represented by \( (M(f_{2s+1}))^\otimes m \) and the other part is represented by \( A \). Then, the Holant value of \( \Omega_{2s+1} \) is the dot product \( \langle A, (M(f_{2s+1}))^\otimes m \rangle \). We can stratify all 0, 1 assignments of these \( 4m \) bits having a nonzero evaluation of a term in \( \text{Pl-Holant}_{\Omega_{2s+1}} \) into the following categories:

- There are \( i \) many copies of \( f_{2s+1} \) receiving inputs 0011;
- There are \( j \) many copies of \( f_{2s+1} \) receiving inputs 0110 or 1001;
- There are \( k \) many copies of \( f_{2s+1} \) receiving inputs 1100;

where \( i + j + k = m \).

For any assignment in those categories with parameters \( (i, j, k) \) where \( k \equiv 0 \pmod{2} \), the evaluation of \( (M(f_{2s+1}))^\otimes m \) is clearly \((-1)^i b^{(2s+1)j} = b^{(2s+1)j} \). And for any assignment in those categories with parameters \( (i, j, k) \) where \( k \equiv 1 \pmod{2} \), the evaluation of \( (M(f_{2s+1}))^\otimes m \) is clearly \((-1)^i b^{(2s+1)j} = -b^{(2s+1)j} \). Since \( i + j + k = m \), the index \( i \) is determined by \( j \) and \( k \). Let \( a_{j0} \) be the summation of values of the part \( A \) over all assignments in those categories \( (i, j, k) \) where \( k \equiv 0 \pmod{2} \), and \( a_{j1} \) be the summation of values of the part \( A \) over all assignments in those categories \( (i, j, k) \) where \( k \equiv 1 \pmod{2} \). Note that \( a_{j0} \) and \( a_{j1} \) are independent from the value of \( s \). Let \( a_j = a_{j0} - a_{j1} \). Then, we rewrite the dot product summation and get

\[
\text{Pl-Holant}_{\Omega_{2s+1}} = \langle A, (M(f_{2s+1}))^\otimes m \rangle = \sum_{0 \leq j \leq m} (a_{j0} b^{(2s+1)j} - a_{j1} b^{(2s+1)j}) = \sum_{0 \leq j \leq m} a_j b^{(2s+1)j}.
\]

Under this stratification, the Holant value of \( \text{Pl-Holant}(\Omega| F \cup \chi_2) \) can be represented as

\[
\text{Pl-Holant}_{\Omega} = \langle A, (M(\chi_2))^\otimes m \rangle = \sum_{0 \leq j \leq m} (a_{j0} - a_{j1}) = \sum_{0 \leq j \leq m} a_j.
\]

Since \( b \neq 0 \) is not a root of unity, the Vandermonde coefficient matrix has full rank. Hence we can solve for all the values \( a_j \) in polynomial time and obtain the value \( \sum_{0 \leq j \leq m} a_j \), and so we get \( \text{Pl-Holant}_{\Omega} \). Therefore, we have \( \text{Pl-Holant}(\neq 2| F \cup \{\chi_2\}) \leq_T \text{Pl-Holant}(\neq 2| F) \). \( \square \)
**Lemma 2.5.** Let \( g = (0,1,t,0)^T \) be a binary signature, where \( t \neq 0 \) is not a root of unity. Then for any binary signature \( g' \) of the form \((0,1,t',0)^T\) and any signature set \( F \) containing \( g \), we have

\[
\text{Pl-Holant} \left( \neq_2 \mid F \cup \{g'\} \right) \leq_T \text{Pl-Holant} \left( \neq_2 \mid F \right).
\]

Inductively, for any finite signature set \( B \) consisting of binary signatures of the form \((0,1,t',0)^T\) and any signature set \( F \) containing \( g \), we have

\[
\text{Pl-Holant} \left( \neq_2 \mid F \cup B \right) \leq_T \text{Pl-Holant} \left( \neq_2 \mid F \right).
\]

**Proof.** Note that \( M(g) = \begin{bmatrix} 0 & 1 \\ t & 0 \end{bmatrix} \). Connecting the variable \( x_2 \) of a copy of \( g \) with the variable \( x_1 \) of another copy of \( g \) using \((\neq_2)\), we get a signature \( g_2 \) with the signature matrix

\[
M(g_2) = M_{x_1,x_2}(g)M(\neq_2)M_{x_1,x_2}(g) = \begin{bmatrix} 0 & 1 \\ t & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ t & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ t^2 & 0 \end{bmatrix}.
\]

That is, \( g_2 = (0,1,t^2,0)^T \). Recursively, we can construct \( g_s = (0,1,t^s,0)^T \) for \( s \geq 1 \). Here, \( g_1 \) denotes \( g \). Given an instance \( \Omega' \) of \( \text{Pl-Holant} \left( \neq_2 \mid F \cup \{g'\} \right) \), in the same way as in the proof of Lemma 2.3, we can replace each appearance of \( g' \) by \( g_s \) and get an instance \( \Omega_s \) of \( \text{Pl-Holant} \left( \neq_2 \mid F \right) \). Similarly, the Holant value of \( \Omega_s \) can be represented as

\[
\text{Pl-Holant}_{\Omega_s} = \sum_{0 \leq j \leq m} a_j (t^s)^j,
\]

while the Holant value of \( \Omega' \) can be represented as

\[
\text{Pl-Holant}_{\Omega'} = \sum_{0 \leq j \leq m} a_j (t')^j.
\]

Since \( t \neq 0 \) is not a root of unity, all \( t^s \) are distinct, and so the Vandermonde coefficient matrix has full rank. Hence, we can solve for all \( a_j \), and then compute \( \sum_{0 \leq j \leq m} a_j (t')^j \). So we get \( \text{Pl-Holant}_{\Omega'} \). Therefore, we have \( \text{Pl-Holant} \left( \neq_2 \mid F \cup \{g'\} \right) \leq_T \text{Pl-Holant} \left( \neq_2 \mid F \right) \). The second part of this lemma follows directly by the first part. \(\square\)

**Remark:** Note that the reason why the interpolation can succeed is that we can construct polynomially many binary signatures \( g_s \) of the form \((0,1,t_s,0)^T\), where all \( t_s \) are distinct such that the Vandermonde coefficient matrix has full rank. According to this, we have the following corollary.

**Corollary 2.6.** Given a signature set \( F \), if we can use \( F \) to construct polynomially many distinct binary signatures \( g_s = (0,1,t_s,0)^T \), then for any finite signature set \( B \) consisting of binary signatures of the form \((0,1,t',0)^T\), we have

\[
\text{Pl-Holant} \left( \neq_2 \mid F \cup B \right) \leq_T \text{Pl-Holant} \left( \neq_2 \mid F \right).
\]

In Lemma 6.4, we will show how to construct polynomially many distinct binary signatures \( g_s = (0,1,t_s,0)^T \) using Möbius transformations [1]. A Möbius transformation of the extended complex plane \( \mathbb{C} = \mathbb{C} \cup \{\infty\} \), the complex plane plus a point at infinity, is a rational function of the form \( z \mapsto \frac{az + b}{cz + d} \) of a complex variable \( z \), where the coefficients \( a, b, c, d \) are complex numbers.
satisfying \( \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc \neq 0 \). It is a bijective conformal map. In particular, a Möbius transformation mapping the unit circle \( S^1 = \{ z \mid |z| = 1 \} \) to itself is of the form \( \varphi(z) = e^{i\theta} \frac{(z + \alpha)}{1 + \alpha \overline{z}} \) denoted by \( M(\alpha, e^{i\theta}) \), where \( |\alpha| \neq 1 \), or \( \varphi(z) = e^{i\theta}/z \). When \( |\alpha| < 1 \), it maps the interior of \( S^1 \) to the interior, while when \( |\alpha| > 1 \), it maps the interior of \( S^1 \) to the exterior. A Möbius transformation is completely determined by its values on any 3 distinct points of \( \overline{C} \).

An interpolation proof based on a lattice structure will be given in Lemma 6.1, where the following lemma is used.

**Lemma 2.7.** [5] Suppose \( \alpha, \beta \in \mathbb{C} - \{0\} \), and the lattice \( L = \{(j, k) \in \mathbb{Z}^2 \mid \alpha^j \beta^k = 1\} \) has the form \( L = \{(ns, nt) \mid n \in \mathbb{Z}\} \), where \( s, t \in \mathbb{Z} \) and \( (s, t) \neq (0, 0) \). Let \( \phi \) and \( \psi \) be any numbers satisfying \( \phi^s \psi^t = 1 \). If we are given the values \( N_\ell = \sum_{j,k \geq 0, j+k \leq m}(\alpha^j \beta^k)^\ell x_{j,k} \) for \( \ell = 1, 2, \ldots \left( \frac{m+2}{2} \right) \), then we can compute \( \sum_{j,k \geq 0, j+k \leq m} \phi^j \psi^k x_{j,k} \) in polynomial time.

### 2.5 Tractable Signature Sets

We give some sets of signatures that are known to define tractable counting problems. These are called tractable signatures. There are three families: affine signatures, product-type signatures, and matchgate signatures [4].

**Affine Signatures**

**Definition 2.8.** For a signature \( f \) of arity \( n \), the support of \( f \) is

\[
\text{supp}(f) = \{(x_1, x_2, \ldots, x_n) \in \mathbb{Z}_2^n \mid f(x_1, x_2, \ldots, x_n) \neq 0 \}.
\]

**Definition 2.9.** A signature \( f(x_1, \ldots, x_n) \) of arity \( n \) is affine if it has the form

\[
\lambda \cdot \chi_{AX=0} \cdot i^{Q(X)},
\]

where \( \lambda \in \mathbb{C} \), \( X = (x_1, x_2, \ldots, x_n, 1) \), \( A \) is a matrix over \( \mathbb{Z}_2 \), \( Q(x_1, x_2, \ldots, x_n) \in \mathbb{Z}_4[x_1, x_2, \ldots, x_n] \) is a quadratic (total degree at most 2) multilinear polynomial with the additional requirement that the coefficients of all cross terms are even, i.e., \( Q \) has the form

\[
Q(x_1, x_2, \ldots, x_n) = a_0 + \sum_{k=1}^{n} a_k x_k + \sum_{1 \leq i < j \leq n} 2b_{ij} x_i x_j,
\]

and \( \chi \) is a 0-1 indicator function such that \( \chi_{AX=0} \) is 1 iff \( AX = 0 \). We use \( \mathcal{A} \) to denote the set of all affine signatures.

The following two lemmas follow directly from the definition.

**Lemma 2.10.** Let \( g \) be a binary signature with support of size 4. Then, \( g \in \mathcal{A} \) iff \( g \) has the signature matrix \( M(g) = \lambda \begin{bmatrix} a & \psi \\ \psi & b \end{bmatrix} \), for some nonzero \( \lambda \in \mathbb{C} \), \( a, b \in \mathbb{N} \) and \( a + b + c + d \equiv 0 \) (mod 2).

**Lemma 2.11.** Let \( h \) be a unary signature with support of size 2. Then, \( h \in \mathcal{A} \) iff \( h \) has the form \( M(h) = \lambda \begin{bmatrix} a \\ \psi \end{bmatrix} \), for some nonzero \( \lambda \in \mathbb{C} \), and \( a, b \in \mathbb{N} \).
Product-Type Signatures

Definition 2.12. A signature on a set of variables $X$ is of product type if it can be expressed as a product of unary functions, binary equality functions $([1,0,1])$, and binary disequality functions $([0,1,0])$, each on one or two variables of $X$. We use $\mathcal{P}$ to denote the set of product-type functions.

Note that the variables of the functions in the product need not be distinct. E.g., let $f(x,y,z)$ be listed as $\begin{bmatrix} f_{000} & 0 & 0 & f_{001} \\ 0 & 0 & 0 & 0 \\ 0 & f_{010} & f_{011} & 0 \\ f_{100} & 0 & 0 & f_{111} \end{bmatrix}$. $f$ is the product of $(=2)(x,y)$, $(\neq2)(x,z)$ and $[a,b](x)$. Let $g = \begin{bmatrix} g_{00} & g_{01} \\ g_{10} & g_{11} \end{bmatrix} = \begin{bmatrix} 0 \\ c \\ a \\ d \end{bmatrix}$, and $h(x,y,z,w) = f(x,y,z)g(z,w)$, sharing a variable $z$. Then $f,g,h \in \mathcal{P}$.

The following theorem is known [8], since $(\neq2) \in \mathcal{A} \cap \mathcal{P}$.

Theorem 2.13. Let $\mathcal{F}$ be any set of complex-valued signatures in Boolean variables. If $\mathcal{F} \subseteq \mathcal{A}$ or $\mathcal{F} \subseteq \mathcal{P}$, then Holant$(\neq2|\mathcal{F})$ is tractable.

Problems defined by $\mathcal{A}$ are tractable essentially by Gauss sums [2]. Problems defined by $\mathcal{P}$ are tractable by a propagation algorithm.

Matchgate Signatures

Matchgates were introduced by Valiant [23, 24] to give polynomial-time algorithms for a collection of counting problems over planar graphs. As the name suggests, problems expressible by matchgates can be reduced to computing a weighted sum of perfect matchings. The latter problem is tractable over planar graphs by Kasteleyn’s algorithm [16], a.k.a. the FKT algorithm [21, 15]. These counting problems are naturally expressed in the Holant framework using matchgate signatures. We use $\mathcal{M}$ to denote the set of all matchgate signatures; thus Pl-Holant$(\neq2|\mathcal{M})$ is tractable, as well as Pl-Holant$(\neq2|\mathcal{M})$.

The parity of a signature is even (resp. odd) if its support is on entries of even (resp. odd) Hamming weight. We say a signature satisfies the even (resp. odd) Parity Condition if all entries of odd (resp. even) weight are zero. For signatures of arity at most 4, the matchgate signatures are characterized by the following lemma.

Lemma 2.14. (cf. Lemma 2.3, Lemma 2.4 in [3]) If $f$ has arity $\leq 3$, then $f \in \mathcal{M}$ iff $f$ satisfies the Parity Condition.

If $f$ has arity 4 and $f$ satisfies the even Parity Condition, i.e.,

$$M_{x_1,x_2,x_4,x_3}(f) = \begin{bmatrix} f_{0000} & 0 & 0 & f_{0011} \\ f_{1000} & 0 & 0 & f_{1011} \\ 0 & f_{0110} & f_{0101} & 0 \\ 0 & f_{1100} & 0 & f_{1111} \end{bmatrix},$$

then $f \in \mathcal{M}$ iff

$$\det M_{\text{Out}}(f) = \det M_{\text{In}}(f).$$

Holographic transformations extend the reach of the FKT algorithm further, as stated below.

Theorem 2.15. Let $\mathcal{F}$ be any set of complex-valued signatures in Boolean variables. If $\mathcal{F}$ is $\mathcal{M}$-transformable, then Pl-Holant$(\neq2|\mathcal{F})$ is tractable.

Recall the signature class $\hat{\mathcal{M}} = H_2 \mathcal{M}$, where $H_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$. The following lemmas can be proved easily.
Lemma 2.16. A signature $f$ with the signature matrix $M(f) = \begin{bmatrix} 0 & 0 & 0 & a \\ 0 & b & c & 0 \\ 0 & z & y & 0 \\ x & 0 & 0 & 0 \end{bmatrix}$ is $\mathcal{M}$-transformable iff $f \in \hat{\mathcal{M}}$.

Lemma 2.17. A binary signature $g$ with the signature matrix $M(g) = \begin{bmatrix} g_{00} & g_{01} \\ g_{10} & g_{11} \end{bmatrix}$ is in $\hat{\mathcal{M}}$ iff $g_{00} = \epsilon g_{11}$ and $g_{01} = \epsilon g_{10}$, where $\epsilon = \pm 1$.

Lemma 2.18. A signature $f$ with the signature matrix $M(f) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & z & y & 0 \\ x & 0 & 0 & 0 \end{bmatrix}$ is in $\hat{\mathcal{M}}$ iff $b = \epsilon y$ and $c = \epsilon z$, where $\epsilon = \pm 1$.

Lemma 2.19. If $f$ has the signature matrix $M(f) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & z & y & 0 \\ x & 0 & 0 & 0 \end{bmatrix}$, where $abxy \neq 0$, then $f \notin \hat{\mathcal{M}}$.

2.6 Known Dichotomies and Hardness Results

Definition 2.20. A 4-ary signature is non-singular redundant iff in one of its four $4 \times 4$ signature matrices, the middle two rows are identical and the middle two columns are identical, and the determinant

$$
\det \begin{bmatrix} f_{0000} & f_{0010} & f_{0011} \\ f_{0100} & f_{0110} & f_{0111} \\ f_{1100} & f_{1110} & f_{1111} \end{bmatrix} \neq 0.
$$

Theorem 2.21. [6] If $f$ is a non-singular redundant signature, then $\text{Pl-Holant}(\neq_2 | f)$ is #P-hard.

Theorem 2.22. [17] Let $G$ be a connected plane graph and $\mathcal{EO}(H)$ be the set of all Eulerian orientations of the medial graph $H = H(G)$. Then

$$
\sum_{O \in \mathcal{EO}(H)} 2^{\beta(O)} = 2T(G; 3, 3),
$$

where $T$ is the Tutte polynomial, and $\beta(O)$ is the number of saddle vertices in the orientation $O$, i.e., vertices in which the edges are oriented “in, out, in, out” in cyclic order.

Remark: Note that $\sum_{O \in \mathcal{EO}(H)} 2^{\beta(O)}$ can be expressed as $\text{Pl-Holant}(\neq_2 | f)$ on $H$, where $f$ has the signature matrix $M(f) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & z & y & 0 \\ x & 0 & 0 & 0 \end{bmatrix}$. Therefore, $\text{Pl-Holant}(\neq_2 | f)$ is #P-hard.

Theorem 2.23. [4] Let $\mathcal{F}$ be any set of complex-valued signatures in Boolean variables. Then $\text{Pl-\#CSP}(\mathcal{F})$ is #P-hard unless $\mathcal{F} \subseteq \mathcal{A}$, $\mathcal{F} \subseteq \mathcal{P}$, or $\mathcal{F} \subseteq \hat{\mathcal{M}}$, in which case the problem is computable in polynomial time. If $\mathcal{F} \subseteq \mathcal{A}$ or $\mathcal{F} \subseteq \mathcal{P}$, then $\#\text{CSP}(\mathcal{F})$ is computable in polynomial time without planarity; otherwise $\#\text{CSP}(\mathcal{F})$ is #P-hard.

Theorem 2.24. [5] Let $f$ be a 4-ary signature with the signature matrix $M(f) = \begin{bmatrix} 0 & 0 & 0 & a \\ 0 & b & c & 0 \\ 0 & z & y & 0 \\ x & 0 & 0 & 0 \end{bmatrix}$, then $\text{Holant}(\neq_2 | f)$ is #P-hard except for the following cases:

- $f \in \mathcal{P}$;
- $f \in \mathcal{A}$;
- there is a zero in each pair $(a, x), (b, y), (c, z)$;

in which cases $\text{Holant}(\neq_2 | f)$ is computable in polynomial time.
3 Main Theorem and Proof Outline

**Theorem 3.1.** Let \( f \) be a signature with the signature matrix \( M(f) = \begin{bmatrix} 0 & 0 & 0 & a \\ 0 & b & c & 0 \\ 0 & z & y & 0 \\ x & 0 & 0 & 0 \end{bmatrix} \), where \( a, b, c, x, y, z \in \mathbb{C} \). Then Pl-Holant(\( \neq 2 \mid f \)) is polynomial time computable in the following cases, and \#P-hard otherwise:

1. \( f \in \mathcal{P} \) or \( \mathcal{A} \);
2. There is a zero in each pair \((a, x), (b, y), (c, z)\);
3. \( f \in \mathcal{M} \) or \( \widehat{\mathcal{M}} \);
4. \( c = z = 0 \) and
   (i). \((ax)^2 = (by)^2\), or
   (ii). \(x = a\alpha, b = a\sqrt{\gamma}, \) and \( y = a\sqrt{\gamma}, \) where \( \alpha, \beta, \gamma \in \mathbb{N}, \) and \( \beta \equiv \gamma \pmod{2} \).

If \( f \) satisfies condition 1 or 2, then Holant(\( \neq 2 \mid f \)) is computable in polynomial time without the planarity restriction; otherwise (the non-planar) Holant(\( \neq 2 \mid f \)) is \#P-hard.

Let \( N \) be the number of zeros among \( a, b, c, x, y, z \). The following division of all cases into Cases I, II, III and IV may not appear to be the most obvious, but it is done to simplify the organization of the proof. We define:

Case I: There is exactly one zero in each pair.
Case II: There is a zero pair.
Case III: \( N = 2 \) and having no zero pair, or \( N = 1 \) and the zero is in an outer pair.
Case IV: \( N = 1 \) and the zero is in an inner pair, or \( N = 0 \).

Cases I, II, III and IV are clearly disjoint. To see that they cover all cases, note that if \( N \geq 3 \), then either there is a zero pair (in Case II), or \( N = 3 \) and each pair has exactly one zero (in Case I). If \( N = 2 \), then either it has a zero pair (in Case II), or it has no zero pair (in Case III). If \( N = 1 \), then either the single zero is in an outer pair (in Case III), or the single zero is in an inner pair (Case IV). If \( N = 0 \) it is in Case IV.

Also note that if \( N = 2 \) and it has no zero pair, then the two zeros are in different pairs, which implies that there is a zero in an outer pair. So in Case III, there is a zero in an outer pair regardless \( N = 1 \) or \( N = 2 \). In Case III an outer pair has exactly one zero, and the other two pairs together have at most one zero.

In Case II, depending on whether the zero pair is inner or outer we have two different connections to \#CSP. A previously established connection to \#CSP (see [5]) can be adapted in the planar setting to handle the case with a zero outer pair. This connection is a local transformation, and we observe that it preserves planarity. A significantly more involved non-local connection to \#CSP is discovered in this paper when the inner pair is zero (and no outer pair is zero). We show that by the support structure of the signature we can define a set of circuits, which forms a partition of the edge set. There are exactly two valid configurations along each such circuit, corresponding to its two cyclic orientations. These circuits may intersect in complicated ways, including self-intersections. But we can define a \#CSP problem, where the variables are these circuits, and their edge functions exactly account for the intersections. We show that Pl-Holant(\( \neq 2 \mid f \)) is equivalent to these \#CSP problems, which are non-planar in general. However, crucially, because Pl-Holant(\( \neq 2 \mid f \)) is planar, every two such circuits must intersect an even number of times. Due to the planarity of Pl-Holant(\( \neq 2 \mid f \)) we can exactly carve out a new class of tractable problems via this non-local \#CSP connection, by the kind of constraint functions they produce in the \#CSP problems.

For the proof of \#P-hardness in this paper, one particularly difficult case is in Lemma 6.4. This
is where we introduce Möbius transformations to prove dichotomy theorems for counting problems. In this case, all constructible binary signatures correspond to points on the unit circle $S^1$, and any iteration of the construction amounts to mapping this point by a Möbius transformation which preserves $S^1$.

The following is an outline on how Case I to Case IV are handled.

I. There is exactly one zero in each pair. In this case, Holant ($\neq 2 | f$) is tractable, proved in [5].

II. There is a zero pair:
   1. An outer pair $(a, x)$ or $(b, y)$ is a zero pair. We prove that Pl-Holant ($\neq 2 | f$) is tractable if $f \in \mathcal{P}, \mathcal{M}, \hat{\mathcal{M}}$, and is #P-hard otherwise.
      In this Case II.1, we can rotate the signature $f$ such that the matrix $M_{Out}(f)$ is the zero matrix. Let $M(f_{In}) = M_{In}(f) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. We reduce Pl-#CSP($f_{In}$) to Pl-Holant ($\neq 2 | f$) via a local replacement (Lemma 4.2). We apply the dichotomy of Pl-#CSP to get #P-hardness (Theorem 4.3). Tractability of Pl-Holant ($\neq 2 | f$) follows from known tractable signatures.
   2. The inner pair $(c, z)$ is a zero pair and no outer pair is a zero pair. We prove that Pl-Holant ($\neq 2 | f$) is #P-hard unless $f$ satisfies condition 4, in which case it is tractable. This is the non-local reduction described above. The tractable condition 4 is previously unknown. (Curiously, in Case II.2, condition 4 subsumes $f \in \mathcal{M}$.)

III. 1. There are exactly two zeros and they are in different pairs;
   2. There is exactly one zero and it is in an outer pair.
      We prove that Pl-Holant ($\neq 2 | f$) is #P-hard unless $f \in \mathcal{M}$, in which case it is tractable.
      In Case III, there exists an outer pair which contains a single zero. By connecting two copies of the signature $f$, we can construct a 4-ary signature $f_1$ such that one outer pair is a zero pair. When $f \notin \mathcal{M}$, we can realize a signature $M(g) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ by interpolation using $f_1$ (Lemma 5.1). This $g$ can help us “extract” the inner matrix of $M(f)$. By connecting $f$ and $g$, we can construct a signature that belongs to Case II. We then prove #P-hardness using the result of Case II (Theorem 5.2).

IV. 1. There is exactly one zero and it is in the inner pair;
   2. All values in $\{a, x, b, y, c, z\}$ are nonzero.
      We prove that Pl-Holant ($\neq 2 | f$) is #P-hard unless $f \in \mathcal{M}$, in which case it is tractable.
      Assume $f \notin \mathcal{M}$. The main idea is to use Möbius transformations. However, there are some settings where we cannot do so, either because we don’t have the initial signature to start the process, or the matrix that would define the Möbius transformation is singular. So we first treat the following two special cases.
      • If $a = \epsilon x$, $b = \epsilon y$ and $c = \epsilon z$, where $\epsilon = \pm 1$, by interpolation based on a lattice structure, either we can realize a non-singular redundant signature or reduce from the evaluation of the Tutte polynomial at $(3, 3)$, both of which are #P-hard (Lemma 6.1).
      • If $\det \begin{bmatrix} b & c \\ z & y \end{bmatrix} = 0$ or $\det \begin{bmatrix} z & 0 \\ b & c \end{bmatrix} = 0$, then either we can realize a non-singular redundant signature or a signature that is #P-hard by Lemma 6.1 (Lemma 6.2).

If $f$ does not belong to the above two cases, we want to realize binary signatures of the form $(0, 1, t, 0)^T$, for arbitrary values of $t$. If this can be done, by carefully choosing the values of $t$, we can construct a signature that belongs to Case III and it is #P-hard when $f \notin \mathcal{M}$ (Lemma 6.3). We realize binary signatures by connecting $f$ with ($\neq 2$). This corresponds naturally to a Möbius transformation. By discussing the following different forms of binary signatures we
get, we can either realize arbitrary \((0, 1, t, 0)^T\) or a signature belonging to Case II.2 that does not satisfy condition 4, therefore is \#P-hard (Theorem 6.8).

- If we can get a signature of the form \(g = (0, 1, t, 0)^T\) where \(t \neq 0\) is not a root of unity, then by connecting a chain of \(g\), we can get polynomially many distinct binary signatures \(g_i = (0, 1, t^i, 0)^T\). Then, by interpolation, we can realize arbitrary binary signatures of the form \((0, 1, t^i, 0)^T\).

- Suppose we can get a signature of the form \((0, 1, t, 0)^T\), where \(t \neq 0\) is an \(n\)-th primitive root of unity \((n \geq 5)\). Now, we only have \(n\) many distinct signatures \(g_i = (0, 1, t^i, 0)^T\). But we can relate \(f\) to two Möbius transformations due to \(\det \begin{bmatrix} b & c \\ z & y \end{bmatrix} \neq 0\) and \(\det \begin{bmatrix} a & z \\ x & y \end{bmatrix} \neq 0\). For each Möbius transformation \(\varphi\), we can realize the signatures \(g = (0, 1, \varphi(t^i), 0)^T\). If \(|\varphi(t^i)| \neq 0, 1\) or \(\infty\) for some \(i\), then this is treated above, as this \(\varphi(t^i)\) is nonzero and not a root of unity. Otherwise, since \(\varphi\) is a bijection on the extended complex plane \(\hat{\mathbb{C}}\), it can map at most two points of \(S^1\) to 0 or \(\infty\). Hence, \(|\varphi(t^i)| = 1\) for at least three distinct \(t^i\).

  But a Möbius transformation is determined by any three distinct points. This implies that \(\varphi\) maps \(S^1\) to itself. Such mappings \(\varphi\) have a known special form \(e^{i\theta} \frac{\alpha + \frac{\alpha}{3}}{1 + \frac{\alpha}{3}}\) (or \(e^{i\theta}/3\), but the latter form actually cannot occur in our context.) By exploiting its property we can construct a signature \(f^t\) such that its corresponding Möbius transformation \(\varphi^t\) defines an infinite group. This implies that \(\varphi^k(t)\) are all distinct. Then, we can get polynomially many distinct binary signatures \((0, 1, \varphi^k(t), 0)\), and realize arbitrary binary signatures of the form \((0, 1, t^i, 0)^T\) (Lemma 6.4).

- Suppose we can get a signature of the form \((0, 1, t, 0)^T\) where \(t \neq 0\) is an \(n\)-th primitive root of unity \((n = 3, 4)\). Then we can either relate it to two Möbius transformations mapping the unit circle to itself, or realize a double pinning \((0, 1, 0, 0)^T = (1, 0)^T \otimes (0, 1)^T\) (Corollary 6.5).

- Suppose we can get a signature of the form \((0, 1, 0, 0)^T\). By connecting \(f\) with it, we can get new signatures of the form \((0, 1, t, 0)^T\). Similarly, by analyzing the value of \(t\), we can either realize arbitrary binary signatures of the form \((0, 1, s, 0)^T\), or realize a signature that belongs to Case II.2, which is \#P-hard (Lemma 6.6).

- Suppose we can only get signatures of the form \((0, 1, \pm 1, 0)\). That implies \(a = \epsilon x\), \(b = \epsilon y\) and \(c = \epsilon z\), where \(\epsilon = \pm 1\). This has been treated before.

As Case I has already been proved tractable in [5], we only deal with Cases II, III and IV, and they are each dealt with in the next three sections.

### 4 Case II: One Zero Pair

If an outer pair is a zero pair, by rotational symmetry, we may assume \((a, x)\) is a zero pair.

**Definition 4.1.** Given a 4-ary signature \(f\) with the signature matrix

\[
M(f) = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & b & c & 0 \\
0 & z & y & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
\]

we denote by \(\tilde{f}_{\text{In}}\) the binary signature with \(M(\tilde{f}_{\text{In}}) = M_{\text{In}}(f) \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ y & z \end{bmatrix}\). Given a set \(\mathcal{F}\) consisting of signatures of the form (4.1), we define \(\tilde{f}_{\text{In}} = \{\tilde{f}_{\text{In}} \mid f \in \mathcal{F}\}\).
Lemma 4.2. For any set $F$ of signatures of the form (4.1),
\[
\text{Pl-}\#\text{CSP}(\tilde{F}_{\text{In}}) \leq_T \text{Pl-Holant}(\neq_2|F).
\]

**Proof.** We adapt a proof from [5], making sure that the reduction preserves planarity. This need to preserve planarity necessitates the twist introduced in the definition of $\tilde{f}_{\text{In}}$ and $\tilde{F}_{\text{In}}$. We prove this reduction in two steps. In each step, we begin with a signature grid and end with a new signature grid such that the Holant values of both signature grids are the same.

For step one, let $G = (U, V, E)$ be a planar bipartite graph representing an instance of $\text{Pl-}\#\text{CSP}(\tilde{F}_{\text{In}}) = \text{Pl-Holant}(\tilde{\mathcal{Q}}|\tilde{F}_{\text{In}})$, where each $u \in U$ is a variable, and each $v \in V$ has degree two and is labeled by some $\tilde{f}_{\text{In}} \in \tilde{F}_{\text{In}}$. We define a cyclic order of the edges incident to each vertex $u \in U$, and split $u$ into $k = \text{deg}(u)$ vertices. Then we connect the $k$ edges originally incident to $u$ to these $k$ new vertices so that each vertex is incident to exactly one edge. We also connect these $k$ new vertices in a cycle according to the cyclic order (see Figure 8b). Thus, in effect we have replaced $u$ by a cycle of length $k = \text{deg}(u)$.

(If $k = 1$ then there is a self-loop. If $k = 2$ then the cycle consists of two parallel edges.) Each of $k$ vertices has degree 3, and we label them by $(\neq_3)$. This defines a signature grid for a planar holant problem, since the construction preserves planarity. Also clearly this does not change the value of the partition function. The resulting graph has the following properties: (1) every vertex has either degree 2 or degree 3; (2) each degree 2 vertex is connected to degree 3 vertices; (3) each degree 3 vertex is connected to exactly one degree 2 vertex.

Now step two. For every $v \in V$, $v$ has degree 2 and is labeled by some $\tilde{f}_{\text{In}} \in \tilde{F}_{\text{In}}$. We contract the two edges incident to $v$ to produce a new vertex $v'$. The resulting graph $G' = (V', E')$ is 4-regular and planar. We put a node on every edge of $G'$ (these are all edges of the cycles created in step one) and label it by $(\neq_2)$ (see Figure 8c). Next, we assign a copy of the corresponding $f$ to every $v' \in V'$. The input variables $x_1, x_2, x_3, x_4$ are carefully assigned at each copy of $f$ (as illustrated in Figure 9) such that there are exactly two configurations to each original cycle, which correspond to cyclic orientations, due to the $(\neq_2)$ on it and the support set of $f$. These cyclic
Figure 9: Assign input variables of $f$: Suppose the binary signature $g$ is applied to (the ordered pair) $(u, u')$. The variables $u$ and $u'$ have been replaced by cycles of length $\deg(u)$ and $\deg(u')$ respectively. For the cycle $C_u$ representing a variable $u$, we associate the value $u = 0$ with a clockwise orientation, and $u = 1$ with a counterclockwise orientation. Then by the support of $f$, which is contained in $(x_1 \neq x_2) \land (x_3 \neq x_4)$, $(x_1, x_2)$ can only take assignment $(0, 1)$ or $(1, 0)$, and similarly $(x_3, x_4)$ can only take assignment $(0, 1)$ or $(1, 0)$. We associate $(x_1, x_2) = (0, 1)$ to $u = 0$ (clockwise orientation), and $(x_1, x_2) = (1, 0)$ to $u = 1$ (counterclockwise orientation). Consistently, $(x_3, x_4) = (0, 1)$ when $u' = 0$, and $(x_3, x_4) = (1, 0)$ when $u' = 1$.

orientations correspond to the $\{0, 1\}$ assignments at the original variable $u \in U$. Under this one-to-one correspondence, the value of $\tilde{f}_{in}$ is perfectly mirrored by the value of $f$. Therefore, we have 

$$\text{Pl-}\#\text{CSP}(\tilde{F}_{in}) \leq_T \text{Pl-Holant}(\neq_2|F).$$

Figure 10: A self-loop on the cycle representing variable $w$ is created for each constraint $\tilde{f}_{in}(w, w)$. This creates a degree 4 vertex labeled by $f$, with four input variables $(x_1, x_2, x_3, x_4)$ as described. Note that the self-loop is created locally on the cycle such that it does not affect anything having to do with other cycles. Base on the support of $f$, the values $x_1 \neq x_2$ and $x_3 \neq x_4$. By the $\neq$ on the loop, we also have $x_1 \neq x_4$. Hence $(x_1, x_2) = (x_3, x_4) = (0, 1)$ or $(1, 0)$. It is clear that the former corresponds to $w = 0$ (clockwise orientation), and the latter corresponds to $w = 1$ (counterclockwise orientation). This is consistent with the association in Figure 9.
There is also the possibility that the binary constraint \( \overline{f}_{\text{in}} \) is applied to a single variable, say \( w \), resulting in a unary constraint that takes value \( \overline{f}_{\text{in}}(0,0) = c \) if \( w = 0 \) and \( \overline{f}_{\text{in}}(1,1) = z \) if \( w = 1 \). To reflect that, we simply introduce a self-loop on the cycle representing the variable \( w \) for every such occurrence, as illustrated in Figure 10. It is clear that the values \( c \) and \( z \) are perfectly mirrored by the values that the local copy \( f \) takes under the two orientations for the cycle corresponding to \( w = 0 \) and 1.

**Theorem 4.3.** Let \( f \) be a 4-ary signature of the form (4.1). Then Pl-Holant(\( \neq 2 \mid f \)) is #P-hard unless \( f \in \mathcal{P} \), \( f \in \mathcal{A} \), or \( f \in \mathcal{M} \), in which cases the problem is tractable.

**Proof.** Tractability follows from Theorems 2.13 and 2.15. For any \( f \) of the form (4.1), note that the support of \( f \) is contained in \( (x_1 \neq x_2) \wedge (x_3 \neq x_4) \). We have

\[
f(x_1, x_2, x_3, x_4) = \overline{f}_{\text{in}}(x_1, x_3) \cdot \chi_{x_1 \neq x_2} \cdot \chi_{x_3 \neq x_4},
\]

where \( \chi \) is the 0-1 indicator function. Thus, \( \overline{f}_{\text{in}} \in \mathcal{P} \) or \( \mathcal{A} \) is equivalent to \( f \in \mathcal{P} \) or \( \mathcal{A} \). In addition, by Lemmas 2.17 and 2.18, \( \overline{f}_{\text{in}} \in \mathcal{M} \) is equivalent to \( f \in \mathcal{M} \). Therefore, if \( f \notin \mathcal{P}, \mathcal{A} \) or \( \mathcal{M} \), then \( \overline{f}_{\text{in}} \notin \mathcal{P}, \mathcal{A} \) or \( \mathcal{M} \). By Theorem 2.23, Pl-#CSP(\( \overline{f}_{\text{in}} \)) is #P-hard, and then by Lemma 4.2, Pl-Holant(\( \neq 2 \mid f \)) is #P-hard.

Remark: One may observe that if \( f \in \mathcal{M} \), then Pl-Holant(\( \neq 2 \mid f \)) is also tractable as \( f \) and \( (=) \) are both realized by matchgates. However, Theorem 4.3 already accounted for this case because for signature \( f \) of the form (4.1), \( f \in \mathcal{M} \) implies \( f \in \mathcal{P} \).

Now, we consider the case that the inner pair is a zero pair and no outer pair is a zero pair. Note that a signature in the form (4.2) has support contained in \( (x_1 \neq x_3) \wedge (x_2 \neq x_4) \).

**Definition 4.4.** Given a 4-ary signature \( f \) with the signature matrix

\[
M(f) = \begin{bmatrix}
0 & 0 & 0 & a \\
0 & b & 0 & 0 \\
0 & 0 & y & 0 \\
x & 0 & 0 & 0
\end{bmatrix},
\]

(4.2)

where \( (a, x) \neq (0, 0) \) and \( (b, y) \neq (0, 0) \), let \( \mathcal{G}_f \) denote the set of all binary signatures \( g_f \) of the form

\[
M(g_f) = \begin{bmatrix}
a^{k_1 + \ell_1} y^{k_2 + \ell_2} x^{k_3 + \ell_3} y^{k_4 + \ell_4} & a^{k_2 + \ell_4} y^{k_3 + \ell_3} x^{k_4 + \ell_4} y^{k_1 + \ell_1} \\
a^{k_3 + \ell_3} y^{k_1 + \ell_1} x^{k_2 + \ell_2} y^{k_4 + \ell_4} & a^{k_4 + \ell_4} y^{k_3 + \ell_3} x^{k_2 + \ell_2} y^{k_1 + \ell_1}
\end{bmatrix},
\]

satisfying \( k = \ell \), where \( k = \sum_{i=1}^4 k_i, \ell = \sum_{i=1}^4 \ell_i \) and \( k_1, k_2, k_3, k_4, \ell_1, \ell_2, \ell_3, \ell_4 \in \mathbb{N} \). Let \( \mathcal{H}_f \) denote the set of all unary signatures \( h_f \) of the form

\[
M(h_f) = \begin{bmatrix}
a^{m_1} y^{m_2} x^{m_3} y^{m_4} & a^{m_3} y^{m_4} x^{m_1} y^{m_2}
\end{bmatrix},
\]

where \( m_1, m_2, m_3, m_4 \in \mathbb{N} \).

Let \( k = k_1 = \ell_1 = \ell = 1 \), we get a specific signature \( g_{1f} \in \mathcal{G}_f \), with \( M(g_{1f}) = \begin{bmatrix}
a^2 & b y \\
b y & a x
\end{bmatrix} \). Let \( k = k_1 = \ell_3 = \ell = 1 \), we get another specific signature \( g_{2f} \in \mathcal{G}_f \), with \( M(g_{2f}) = \begin{bmatrix}
a x & b y \\
y^2 & a x
\end{bmatrix} \).
Remark: For any $i, j \in \{1, 2, 3, 4\}$, let $k = k_i = \ell_j = \ell = 1$, we can get 16 signatures in $G_f$ that have similar signature matrices to $M(g_{1j})$ and $M(g_{2j})$. For example, Choosing $k = k_3 = \ell_1 = \ell = 1$, we get $g'_{2j}(x_1, x_2)$ with the signature matrix $M(g'_{2j}) = \begin{bmatrix} ax & y^2 \\ t^2 & ax \end{bmatrix}$. Indeed $g'_{2j}(x_1, x_2) = g_{2j}(x_2, x_1)$. In fact, $G_f$ is the closure by the Hadamard product (entry-wise product) of these 16 basic signature matrices.

Lemma 4.5. Let $f$ be a signature of the form (4.2). Then,

$$\text{Pl-Holant}(\neq 2 | f) \leq_T \#\text{CSP}(G_f \cup H_f),$$

(4.3)

If $a^2 = x^2 \neq 0$, $b^2 = y^2 \neq 0$ and $(\frac{b}{a})^8 \neq 1$, then

$$\#\text{CSP}(g_{1j}, g_{2j}) \leq_T \text{Pl-Holant}(\neq 2 | f).$$

(4.4)

Proof. We divide the proof into two parts: We show the reduction (4.3) in Part I, and the reduction (4.4) in Part II.

Part I: Suppose $\Omega = (G, \pi)$ is a given instance of $\text{Pl-Holant}(\neq 2 | f)$, where $G = (U, V, E)$ is a plane bipartite graph. Every vertex $v \in V$ has degree 4, and we list its incident four edges in counterclockwise order. Two edges both incident to a vertex $v \in V$ are called adjacent if they are adjacent in this cyclic order, and non-adjacent otherwise. Two edges in $G$ are called 2-ary edge twins if they are both incident to a vertex $u \in U$ (of degree 2), and 4-ary edge twins if they are non-adjacent but both incident to a vertex $v \in V$ (of degree 4). Both 2-ary edge twins and 4-ary edge twins are called edge twins.

Each edge has a unique 2-ary edge twin at its endpoint in $U$ of degree 2 and a unique 4-ary edge twin at its endpoint in $V$ of degree 4. The reflexive and transitive closure of the symmetric binary relation edge twin forms a partition of $E$ as an edge disjoint union of circuits: $C_1, C_2, \ldots, C_k$. Note that $C_i$ may include repeated vertices called self-intersection vertices, but no repeated edges. We arbitrarily pick an edge $e_i$ of $C_i$ to be the leader edge of $C_i$. Given the leader edge $e_i = (u, v)$ of $C_i$, with $u \in U$ and $v \in V$, the direction from $u$ to $v$ defines an orientation of the circuit $C_i$. * For any edge twins $\{e, e'\}$, this orientation defines one edge, say $e'$, as the successor of the other if $e'$ comes right after $e$ in the orientation. When we list the assignments of edges in a circuit, we list successive values of successors, starting with the leader edge.

Thus, any nonzero term in the sum

$$\text{Pl-Holant}_\Omega = \sum_{\sigma : E \to \{0, 1\}} \prod_{w \in U \cup V} f_w(\sigma |_{E(w)}),$$

the assignment of all edges $\sigma : E \to \{0, 1\}$ can be uniquely extended from its restriction on leader edges $\sigma' : \{e_1, e_2, \ldots, e_k\} \to \{0, 1\}$. This is because the support of $f$ is contained in $(x_1 \neq x_3) \land (x_2 \neq x_4)$. Thus, at each vertex $v \in V$, $f_v(\sigma |_{E(v)}) \neq 0$ only if each pair of edge twins in $E(v)$ is assigned value $(0, 1)$ or $(1, 0)$. The same is true for any vertex $u \in U$ of degree 2, which is labeled $(\neq 2)$. Thus, if the leader edge $e_i$ in $C_i$ takes value 0 or 1 respectively, then all edges on $C_i$ must take values $(0, 1, 0, 1) \ldots, (0, 1, 0) \ldots, (1, 0, 1, 0)$. This default orientation should not be confused with the orientation in the proof of Lemma 4.2.

*This default orientation should not be confused with the orientation in the proof of Lemma 4.2.
with \( e_i \). In particular, all pairs of 4-ary edge twins in \( C_i \) take assignment \((0,1)\) when \( e_i = 0 \) and \((1,0)\) when \( e_i = 1 \) (listing the value of the successor second). Then, we have

\[
\text{Pl-Holant}_\Omega = \sum_{\sigma':\{e_1,\ldots,e_k\} \to \{0,1\}} \prod_{v\in V} f_v(\sigma'|_{E(v)}),
\]

where \( \sigma' \) denotes the unique extension of \( \sigma' \).

For all \( 1 \leq i < j \leq k \), let \( V_{i,j} = C_i \cap C_j \) denote the set of all intersection vertices between \( C_i \) and \( C_j \). Denote by \( \sigma'_{(e_i,e_j)} \) an assignment \( \{e_i, e_j\} \to \{0,1\} \). Define a binary function \( g_{i,j} \) on \( e_i \) and \( e_j \) as follows: For any \( b, b' \in \{0,1\} \), let

\[
g_{i,j}(b,b') = \prod_{v\in V_{i,j}} f_v(\sigma'_{(e_i,e_j)}|_{E(v)}),
\]

where \( \sigma'_{(e_i,e_j)} \) is the unique extension of \( \sigma'_{(e_i,e_j)} \) on the union of edge sets of \( C_i \) and \( C_j \) as described above, and \( \sigma'_{(e_i,e_j)} \) is the unique assignment on \( \{e_i, e_j\} \) such that \( e_i \mapsto b \) and \( e_j \mapsto b' \). Since all edges incident to vertices in \( V_{i,j} \) are either in \( C_i \) or \( C_j \), the assignment values of these edges are determined by \( \sigma'_{(e_i,e_j)} \). Hence, \( g_{i,j} \) is well-defined.

We show that \( g_{i,j} \in G_f \) by induction on the number \( n \) of self-intersection vertices in \( C_i \). Note that in this proof, \( i \) and \( j \) (with \( i < j \)) are not treated symmetrically.

For each vertex \( v \in V_{i,j} \), consider the two pairs of edge twins incident to it. We label the edge twins in \( C_i \) by the variables \((x_{1}, x_{3})\) such that \( x_{3} \) is the successor of \( x_{1} \) in the orientation of \( C_i \). Hence, for all \( v \in V_{i,j} \), these variables \((x_{1}, x_{3})\) take the same assignment \((0,1)\) when \( e_i = 0 \) and \((1,0)\) when \( e_i = 1 \). Then, label the edge twins in \( C_j \) at \( v \) by \((x_{2}, x_{4})\) so that the 4 edges at \( v \) are ordered \((x_{1}, x_{2}, x_{3}, x_{4})\) in counterclockwise order. This choice of \((x_{2}, x_{4})\) is unique given the labeling \((x_{1}, x_{3})\).

As we traverse \( C_i \) according to the orientation of \( C_i \), locally there is a notion of the left side of \( C_i \). At any vertex \( v \in C_i \cap C_j \), if we take the traversal of \( C_j \) according to the orientation of \( C_j \), it either comes into or goes out of the left side of \( C_i \). We call \( v \in C_i \cap C_j \) of the former kind “entry-vertices”, and the latter kind “exit-vertices” (see Figure 11).

![Figure 11: Intersection vertices between \( C_i \) and \( C_j \)](image)

At any entry-vertex \( v \in V_{i,j} \), the variable \( x_{4} \) is the successor of \( x_{2} \), while at any exit-vertex \( x_{2} \) is the successor of \( x_{4} \). Therefore, at entry-vertices, variables \((x_{2}, x_{4})\) take assignment \((0,1)\)
when \( e_j = 0 \) and \((1,0)\) when \( e_j = 1 \), while at exit-vertices they take assignment \((1,0)\) and \((0,1)\) respectively instead.

| \((e_i, e_j)\) | entry-vertices | exit-vertices |
|----------------|----------------|---------------|
| \((x_1, x_2, x_3, x_4)\) | \(f\) | \(f^{\pi}\) | \(f^{2\pi}\) | \((x_1, x_2, x_3, x_4)\) | \(f\) | \(f^{\pi}\) | \(f^{2\pi}\) |
| \((0,0)\) | \((0,0,1,1)\) a | y | x | b | \((0,1,1,0)\) b | a | y | x |
| \((0,1)\) | \((0,1,1,0)\) b | a | y | x | \((0,0,1,1)\) a | y | x | b |
| \((1,1)\) | \((1,1,0,0)\) x | b | a | y | \((1,0,0,1)\) y | x | b | a |
| \((1,0)\) | \((1,0,0,1)\) y | x | b | a | \((1,1,0,0)\) x | b | a | y |

Table 1: The values of \( f \) and its rotated copies at intersection vertices

Table 1 summarizes the values of \( f \) and its rotated copies at intersection vertices \( V_i,j \). According to the 4 different assignments of \((e_i, e_j)\) as listed in column 1 of the table, column 2 and column 7 (indexed by \((x_1, x_2, x_3, x_4)\)) list the assignments of \((x_1, x_2, x_3, x_4)\) at entry-vertices and exit-vertices separately. With respect to this local labeling of \((x_1, x_2, x_3, x_4)\), the signature \( f \) has four rotated forms:

\[
M(f) = \begin{bmatrix} 0 & 0 & 0 & a \\ 0 & 0 & y & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & 0 & x \end{bmatrix}, \quad M(f^{\pi}) = \begin{bmatrix} 0 & 0 & 0 & b \\ 0 & 0 & b & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & 0 & a \end{bmatrix}, \quad M(f^{2\pi}) = \begin{bmatrix} 0 & 0 & b & 0 \\ 0 & 0 & a & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 0 & y \end{bmatrix}, \quad M(f^{3\pi}) = \begin{bmatrix} 0 & 0 & 0 & b \\ 0 & 0 & b & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & 0 & a \end{bmatrix}.
\]

columns 3, 4, 6 and columns 8, 9, 10, 11 list the corresponding values of the signature \( f \) in four forms \( f, f^{\pi}, f^{2\pi}\) and \( f^{3\pi}\) respectively.

Suppose there are \( k_1, k_2, k_3 \) and \( k_4 \) many entry-vertices assigned \( f, f^{\pi}, f^{2\pi}, \) and \( f^{3\pi}\), respectively, and there are \( \ell_1, \ell_2, \ell_3 \) and \( \ell_4 \) many exit-vertices assigned \( f^{\pi}, f^{2\pi}, f^{3\pi}, f^{4\pi}\) respectively. Then, according to the assignments of \((e_i, e_j)\), the values of \( g_{i,j} \) are listed in Table 2, and its signature matrix is given below:

\[
M(g_{i,j}) = \begin{bmatrix} a^{k_1+\ell_1}y^{k_2+\ell_2}x^{k_3+\ell_3}b^{k_4+\ell_4} & a^{k_2}b^{k_3}c^{k_4}d^{\ell_1} \\ a^{k_1+\ell_1}y^{k_2+\ell_2}x^{k_3+\ell_3}b^{k_4+\ell_4} & a^{k_2}b^{k_3}c^{k_4}d^{\ell_1} \\ a^{k_1}y^{k_2}x^{k_3}b^{k_4}d^{\ell_1} & a^{k_1}y^{k_2}x^{k_3}b^{k_4}d^{\ell_1} \\ a^{k_1}y^{k_2}x^{k_3}b^{k_4}d^{\ell_1} & a^{k_1}y^{k_2}x^{k_3}b^{k_4}d^{\ell_1} \end{bmatrix}.
\]

Table 2: The values of \( g_{i,j} \)

Our proof that \( g_{i,j} \in \mathcal{G}_f \) is based on the assertion that the number of “entry-vertices” and “exit-vertices” are equal, namely \( \sum_{i=1}^4 k_i = \sum_{i=1}^4 \ell_i \).

- First, consider the base case \( n = 0 \). That is, \( C_i \) is a simple cycle without self-intersection. By the Jordan Curve Theorem, \( C_i \) divides the plane into two regions, an interior region and an exterior region. In this case, as we traverse \( C_i \) according to the orientation of \( C_i \), the left side of the traversal is always the same region; we call it \( L_i \) (which could be either the interior or
the exterior region, depending on the choice of the leader edge $e_i$). If we traverse $C_j$ according to the orientation of $C_j$, we enter and exit the region $L_i$ an equal number of times. Therefore there is an equal number of “entry-vertices” and “exit-vertices”. Hence $\sum_{i=1}^{4} k_i = \sum_{i=1}^{4} \ell_i$. It follows that $g_{i,j} \in G_f$ by the definition of $G_f$.

- Inductively, suppose $g_{i,j} \in G_f$ holds for any circuit $C_i$ with at most $n$ self-intersections. Let $C_i$ have $n + 1$ self-intersections. We decompose $C_i$ into two edge-disjoint circuits, each of which has at most $n$ self-intersections (See Figure 12). Take any self-intersection vertex $v^*$ of $C_i$. There are two pairs of 4-ary edge twins {$e, e'$} and {$e, e'$}, where $e'$ is the successor of $e$ and $e'$ is the successor of $e$. Note that $e$ and $e'$ are oriented toward $v^*$, and $e'$ and $e'$ are oriented away from $v^*$. By the definition of edge twins, {$e, e'$} are adjacent, and {$e', e'$} are adjacent. We can break $C_i$ into two oriented circuits $C_i^1$ and $C_i^2$, by splitting $v^*$ into two vertices, and let $e'$ follow $e$ and let $e'$ follow $e$. Let the mapping $\gamma : [0, 1) \to \mathbb{R}^2$, such that $\gamma(0) = \gamma(1/2) = \gamma(1) = v^*$, represent the traversal of $C_i$. Then we can define two mappings $\gamma_1, \gamma_2 : [0, 1) \to \mathbb{R}^2$, such that $\gamma_1(t) = \gamma(t/2)$ and $\gamma_2(t) = \gamma((t+1)/2)$. Then $\gamma_1, \gamma_2$ represent $\{C_i^1, C_i^2\}$ respectively. It follows that $C_i$ is the edge disjoint union of $C_i^1$ and $C_i^2$ and they both inherit the same orientation from that of $C_i$. Any vertex in $V_{i,j}$ is distinct from a self intersection point of $C_i$ and thus $V_{i,j}$ is a disjoint union $V_{i,j}^1 \cup V_{i,j}^2$, where $V_{i,j}^1 = C_i^1 \cap C_j$ and $V_{i,j}^2 = C_i^2 \cap C_j$.

Since $C_i^1$ inherits the orientation from $C_i$, the orientation on $C_i^1$ is consistent with the orientation starting by choosing a leader edge on $C_i^1$. The same is true for the orientation on $C_i^2$.

Thus, by induction, on each $C_i^1 \cap C_j$ and $C_i^2 \cap C_j$ there are an equal number of “entry-vertices” and “exit-vertices”. Hence $\sum_{i=1}^{4} k_i = \sum_{i=1}^{4} \ell_i$, and so $g_{i,j} \in G_f$, completing the induction.

Let $V_i$ be the set of all self-intersections of $C_i$. Let $\sigma'_{(e_i)}$ denote the restriction of $\sigma'$ on $\{e_i\}$. Define a unary function $h_i$ on $e_i$ as follows: For any $b \in \{0, 1\}$, let

$$h_i(b) = \prod_{v \in V_i} f_i(\sigma'_{(e_i)}) |_{E(v)},$$

where $\sigma'_{(e_i)}$ is the unique extension of $\sigma'_{(e_i)}$ on the edge set of $C_i$, and $\sigma'_{(e_i)}$ is the unique assignment on $\{e_i\}$ such that $e_i \mapsto b$. The assignment of those edges incident to vertices in $V_i$ can be uniquely extended from the assignment $\sigma'_{(e_i)}$. Hence, $h_i$ is well-defined. We show that $h_i \in \mathcal{H}_f$.

For each vertex in $V_i$, since it is a self-intersection vertex, the two pairs of edge twins incident to it are both in $C_i$. We still first label each pair of edge twins by a pair of variables $(x_1, x_3)$ obeying the orientation of $C_i$. That is, $x_3$ is always the successor of $x_1$. Now by the definition of 4-ary edge twins, the two edges labeled $x_1$ are adjacent. Hence at each vertex in $V_i$, starting from one $x_1$, the four incident edges are labeled by $(x_1, x_1, x_3, x_3)$ in counterclockwise order. We pick the pair of variables $(x_1, x_3)$ that appear in the second and fourth positions in this listing and change them to

Figure 12: Decompose $C_i$ into $C_i^1$ and $C_i^2$. 

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(x_2, x_4), so that the four edges are now labeled by (x_1, x_2, x_3, x_4) in counterclockwise order. Clearly, (x_2, x_4) and (x_1, x_3) take the same assignment. That is, at each vertex in V_i, the assignment of (x_1, x_2, x_3, x_4) is (0, 0, 1, 1) when \( e_i = 0 \), and (1, 1, 0, 0) when \( e_i = 1 \). Under this labeling, the signature f still has four rotated forms. The values of these four forms are listed in Table 3.

| \( e_i \) | (x_1, x_2, x_3, x_4) | f | \( f_{\pi/2} \) | \( f_{\pi} \) | \( f_{3\pi/2} \) |
|---|---|---|---|---|
| 0 | (0, 0, 1, 1) | a | y | x | b |
| 1 | (1, 1, 0, 0) | x | b | a | y |

Table 3: The values of f and its rotated forms at self-intersection vertices

Suppose on \( V_i \) there are \( m_1, m_2, m_3 \) and \( m_4 \) many vertices assigned \( f, f_{\pi/2}, f_{\pi} \) and \( f_{3\pi/2} \) respectively. Then, we have

\[
M(h_i) = [a^{m_1}y^{m_2}x^{m_3}b^{m_4} a^{m_3}y^{m_4}x^{m_1}b^{m_2}].
\]

It follows that \( h_i \in \mathcal{H}_f \).

For any vertex \( v \in V \), it is either in some \( V_{i,j} \) or some \( V_{i} \). Thus,

\[
\text{Pl-Holant}_\Omega = \sum_{\sigma': \{e_1, \ldots, e_k\} \rightarrow \{0, 1\}} \left( \prod_{v \in V_{i,j}, 1 \leq i < j \leq k} f_v(\sigma'|E(v)) \right) \left( \prod_{v \in V_i, 1 \leq i \leq k} f_v(\sigma'|E(v)) \right)
\]

\[
= \sum_{\sigma': \{e_1, \ldots, e_k\} \rightarrow \{0, 1\}} \left( \prod_{1 \leq i < j \leq k} g_{i,j}(\sigma'(e_i), \sigma'(e_j)) \right) \left( \prod_{1 \leq i \leq k} h_i(\sigma'(e_i)) \right),
\]

where \( g_{i,j} \in \mathcal{G}_f \) and \( h_i \in \mathcal{H}_f \). Therefore, Pl-Holant(\( \not\equiv 2 \mid f \)) \( \leq_T \text{CSP}(\mathcal{G}_f \cup \mathcal{H}_f) \).

Here, we give an example for the reduction (4.3).

**Example.** The signature grid \( \Omega = (G, \pi) \) for Pl-Holant(\( \not\equiv 2 \mid f \)) in Figure 13 has two circuits \( C_1 \) (the SQUARE) and \( C_2 \) (the HORIZONTAL EIGHT) in \( G \). We have chosen (arbitrarily) a leader edge \( e_i \) for each circuit \( C_i \). In Figure 13 they are near the top left corner. Given the leader, the direction from its endpoint of degree 2 to the endpoint of degree 4 gives a default orientation of the circuit. Given a nonzero term in the sum Pl-Holant_\( \Omega \), as a consequence of the support of \( f \), the assignment of edges in each circuit is uniquely determined by the assignment of its leader. That is, any assignment of the leaders \( \sigma': \{e_1, e_2\} \rightarrow \{0, 1\} \) can be uniquely extended to an assignment of all edges \( \sigma : E \rightarrow \{0, 1\} \) such that on each circuit the values of 0, 1 alternate.

![Figure 13: An example for the reduction (4.3)](image-url)
Consider the signatures \(f_{v_1}, f_{v_2}, f_{v_3}\) and \(f_{v_4}\) on the intersection vertices between \(C_1\) and \(C_2\). Assume \(C_1\) does not have self-intersection (as is The SQUARE); otherwise, we will decompose \(C_1\) further and reason inductively. Without self-intersection, \(C_1\) has an interior and exterior region by the Jordan Curve Theorem. For the chosen orientation of \(C_1\), its left side happens to be the interior region. With respect to \(C_1\), the circuit \(C_2\) enters and exits the interior of \(C_1\) alternately. Thus, we can divide the intersection vertices into an equal number of “entry-vertices” and “exit-vertices”. In this example, \(f_{v_1}\) and \(f_{v_4}\) are on “entry-vertices”, while \(f_{v_2}\) and \(f_{v_3}\) are on “exit-vertices”. By analyzing the values of each \(f\) when \(e_1\) and \(e_2\) take assignment 0 or 1, we can view each \(f\) as a binary constraint on \((C_1, C_2)\). Depending on the 4 different rotation forms of \(f\) and whether \(f\) is on “entry-vertices” or “exit-vertices”, the resulting binary constraint has 8 different forms (See Table 1). By multiplying these constraints, we get the binary constraint \(g_{1,2}\). This can be viewed as a binary edge function on the circuits \(C_1\) and \(C_2\). The property of \(g_{1,2}\) crucially depends on there are an equal number of “entry-vertices” and “exit-vertices”. For any \(b, b' \in \{0, 1\}\),

\[
g_{1,2}(b, b') = \prod_{1 \leq i < 4} f_{v_i}(\sigma'_{(e_1, e_2)} | E_{(v_i)}),
\]

where \(\sigma'_{(e_1, e_2)}\) uniquely extends to \(C_1\) and \(C_2\) the assignment \(\sigma'_{(e_1, e_2)}(e_1) = b\) and \(\sigma'_{(e_1, e_2)}(e_2) = b'\).

If the placement of \(f_{v_1}\) were to be rotated clockwise \(\frac{\pi}{2}\), then \(f_{v_1}\) will be changed to \(f_{v_1}^*\) in the above formula, where \(M_{x_1, x_2, x_3, x_4}(f_{v_1}^*) = M_{x_1, x_2, x_3, x_4}(f_{v_1})\).

For the self-intersection vertex \(f_{v_5}\), the notions of “entry-vertex” and “exit-vertex” do not apply. \(f_{v_5}\) gives rise to a unary constraint \(h_2\) on \(e_2\). Depending on the 4 different rotation forms of \(f\), \(h_2\) has 4 different forms (see Table 3). For any \(b \in \{0, 1\}\),

\[
h_2(b) = f_{v_5}(\sigma'_{(e_2)} | E_{(v_5)}),
\]

where \(\sigma'_{(e_2)}\) uniquely extends to \(C_2\) the assignment \(\sigma'_{(e_2)}(e_2) = b\).

Therefore, we have

\[
\text{Pl-Holant}_\Omega = \sum_{\sigma : E \rightarrow \{0, 1\}} \prod_{v \in V(G)} f_v(\sigma | E_{(v)})
\]

\[
= \sum_{\sigma' : \{e_1, e_2\} \rightarrow \{0, 1\}} \left( \prod_{1 \leq i < 4} f_{v_i}(\sigma'| E_{(v_i)}) \right) f_{v_5}(\sigma'| E_{(v_5)})
\]

\[
= \sum_{\sigma' : \{e_1, e_2\} \rightarrow \{0, 1\}} g_{1,2}(\sigma'(e_1), \sigma'(e_2)) h_2(\sigma'(e_2)).
\]

**Part II:** Suppose \(I\) is a given instance of \#CSP\((g_{1,f}, g_{2,f})\). Each constraint \(g_{1,f}\) and \(g_{2,f}\) is applied on certain pairs of variables. It is possible that they are applied to a single variable, resulting in two unary constraints. We will deal with such constraints later. We first consider the case that every constraint is applied on two distinct variables.

For any pair \(i < j\), consider all binary constraints on variables \(x_i\) and \(x_j\) \((i < j)\). Note that \(g_{1,f}\) is symmetric, that is, \(g_{1,f}(x_i, x_j) = g_{1,f}(x_j, x_i)\). We assume all the constraints between \(x_i\) and \(x_j\) are: \(s_{i,j}\) many constraints \(g_{1,f}(x_i, x_j)\), \(t_{i,j}\) many constraints \(g_{2,f}(x_i, x_j)\) and \(t'_{i,j}\) many constraints \(g_{2,f}'(x_j, x_i)\). Let \(g_{i,j}(x_i, x_j)\) be the function product of these constraints. That is,

\[
g_{i,j}(x_i, x_j) = g_{1,f}^{s_{i,j}}(x_i, x_j) g_{2,f}^{t_{i,j}}(x_i, x_j) g_{2,f}'^{t'_{i,j}}(x_j, x_i).
\]
Then, we have

\[
\#\text{CSP}(I) = \sum_{\sigma: \{x_1, \ldots, x_k\} \to \{0,1\}} \prod_{1 \leq i < j \leq n} g_{i,j}(\sigma(x_i), \sigma(x_j)).
\]

We prove the reduction (4.4) in two steps. We first reduce \#\text{CSP}(I) to both instances \(\Omega_i\) (for \(i = 1, 2\)) of Pl-Holant \((\not= 2| f, \chi_i)\) respectively, where \(\chi_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\) and \(\chi_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}\). The instance \(\Omega_i\) is constructed as follows:

1. Draw a cycle \(D_1\), i.e., a homeomorphic image of \(S^1\), on the plane. For \(2 \leq j \leq k\) successively draw cycles \(D_j\), and for all \(1 \leq i < j\) let \(D_j\) intersect transversally with \(D_i\) at least \(2(s_{i,j} + t_{i,j} + t'_{i,j})\) many times. This can be done since we can let \(D_j\) enter and exit the interiors of \(D_i\) successively. A concrete realization is as follows: Place \(k\) vertices \(D_i\) on a semi-circle in the order of \(i = 1, \ldots, k\). For \(1 \leq i < j \leq k\), connect \(D_i\) and \(D_j\) by a straight line segment \(L_{ij}\). Now thicken each vertex \(D_i\) into a small disk, and deform the boundary circle of \(D_j\) so that, for every \(1 \leq i < j\), it reaches across to \(D_i\) along the line segment \(L_{ij}\), and intersects the boundary circle of \(D_i\) exactly \(2(s_{i,j} + t_{i,j} + t'_{i,j})\) many times. (There are also other intersections between these cycles \(D_i\)'s due to crossing intersections between those line segments. This is why we say “at least” this many intersections in the overall description. We will deal with those extra intersection vertices later.) We can draw these cycles to satisfy the following conditions:
   a. There is no self-intersection for each \(D_i\).
   b. Every intersection point is between exactly two cycles. They intersect transversally.
      Each intersection creates a vertex of degree 4.

These intersecting cycles define a planar 4-regular graph \(G'\), where intersection points are the vertices.

2. Replace each edge of \(G'\) by a path of length two. We get a planar bipartite graph \(G = (V,E)\).
   On one side, all vertices have degree 2, and on the other side, all vertices have degree 4. We can still define edge twins as in Part I. Moreover, we still divide the graph into some circuits \(C_1, \ldots, C_k\). In fact, \(C_i\) is just the cycle \(D_i\) after the replacement of each edge by a path of length two.
   Let \(V_{i,j} = C_i \cap C_j\) \(i < j\) be the intersection vertices between \(C_i\) and \(C_j\). Clearly, \(|V_{i,j}|\) is even and at least \(2(s_{i,j} + t_{i,j} + t'_{i,j})\). Since there is no self-intersection, each circuit is a simple cycle. As we did in Part I, we pick an edge \(e_i\) as the leader edge of \(C_i\) and this gives an orientation of \(C_i\).
   We can define “entry-vertices” and “exit-vertices” as in Part I. Among \(V_{i,j}\), half are entry-vertices and the other half are exit-vertices. (This notion is defined in terms of \(C_j\) with respect to \(C_i\); the roles of \(i\) and \(j\) are not symmetric.) List the edges in \(C_i\) according to the orientation of \(C_i\) starting with the leader edge \(e_i\). After we place copies of \(f\) on each vertex, the support of \(f\), which is contained in \((x_1 \neq x_3) \land (x_2 \neq x_4)\), ensures that every 4-ary twins can only take values \((0, 1)\) or \((1, 0)\), since the 4-ary twin edges are non-adjacent. Then all edges in \(C_i\) can only take assignment \((0, 1, 0, 1, \cdots, 0, 1)\) when \(e_i = 0\) and \((1, 0, 1, 0, \cdots, 1, 0)\) when \(e_i = 1\).

3. Label all vertices of degree 2 by \((\not= 2)\). For any vertex in \(V_{i,j}\) \(i < j\), as we showed in Part I, we can label the four edges incident to it by variables \((x_1, x_2, x_3, x_4)\) in a way such that when \(\sigma' : (e_i, e_j) \mapsto (b, b') \in \{0,1\}^2\), we have \((x_1, x_2, x_3, x_4) = (b, b', 1-b, 1-b')\) at any entry-vertex, and \((x_1, x_2, x_3, x_4) = (b, 1-b', 1-b, b')\) at any exit-vertex (See Table 1). Note that \(f\) has four rotation forms under this labeling. We have (at least) \(s_{i,j} + t_{i,j} + t'_{i,j}\) many entry-vertices
and as many exit-vertices. Let $V'_{i,j}$ be the set of these $2(s_{i,j} + t_{i,j} + t'_{i,j})$ vertices. For vertices in $V'_{i,j}$, we label $s_{i,j}$ many entry-vertices by $f$ and $s_{i,j}$ many exit-vertices by $f^\pi$, $t_{i,j}$ many entry-vertices by $f$ and $t_{i,j}$ many exit-vertices by $f^\pi$, and $t'_{i,j}$ many entry-vertices by $f$ and $t'_{i,j}$ many exit-vertices by $f^\pi$. Refer to Table 2, this choice amounts to taking

$$k_1 = s_{i,j} + t_{i,j}, \quad k_3 = t'_{i,j}, \quad \ell_1 = s_{i,j} + t'_{i,j}, \quad \ell_3 = t_{i,j},$$

and all other $k_i, \ell_i$'s equal to 0. Recall that $g_{1,1}(x_1, x_2)$ corresponds to choosing $k_1 = \ell_1 = 1$ and the others all 0, $g_{2,1}(x_1, x_2)$ corresponds to choosing $k_1 = \ell_3 = 1$ and the others all 0, and $g_{2,1}(x_2, x_1)$ corresponds to choosing $k_3 = \ell_1 = 1$ and the others all 0, then we have

$$\prod_{v \in V'_{i,j}} f_v(\sigma'_{(e_i, e_j)} \mid E(v)) = g_{1,1}^{s_{i,j}}(e_i, e_j) g_{2,1}^{t_{i,j}}(e_i, e_j) g_{2,1}^{t'_{i,j}}(e_j, e_i) = g_{i,j}(e_i, e_j).$$

For all vertices in $V_{i,j} \setminus V'_{i,j}$, if we label them by an auxiliary signature $\chi_1 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$, then, referring to Table 2 (Here $a = x = b = y = 1$), we have

$$\prod_{v \in V_{i,j} \setminus V'_{i,j}} \chi_1(\sigma'_{(e_i, e_j)} \mid E(v)) = 1,$$

for all assignments $\sigma'$ on $\{e_i, e_j\}$. We can also label the vertices in $V_{i,j} \setminus V'_{i,j}$ by an auxiliary signature $\chi_2 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. By our (semi-circle) construction, in $V_{i,j} \setminus V'_{i,j}$, the number of entry-vertices is equal to the number of exit-vertices. We label all entry-vertices by $\chi_2$ and label all exit-vertices by its rotated form $\chi_2^\pi = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$. Refer to Table 2 (here $a = b = y = 1, x = -1$, and $k = k_1 = \ell_1 = \ell$, and the crucial equation is $g_{i,j}(1, 1) = x^{k_1 + \ell_1} = (-1)^2 = 1$), we have

$$\prod_{v \in V_{i,j} \setminus V'_{i,j}} \chi_2(\sigma'_{(e_i, e_j)} \mid E(v)) = 1,$$

for all assignments $\sigma'$ on $\{e_i, e_j\}$.

Figure 14: Creating self-loop locally on cycle $C_w$
Then, consider the case that \( g_1 \) and \( g_2 \) are applied to the pair variables \((w, w)\), in which case \( g_1 \) and \( g_2 \) effectively become unary constraints \([a^2, x^2]\) and \([ax, ax]\) on the variable \( x \). The latter is a constant multiple of \([1, 1]\) and can be ignored. The unary constraint \([a, x]\), and hence also \([a^2, x^2]\), can be easily realized by \( f \) in \( \text{Pl-Holant}(\neq 2 \mid f, \chi_1) \), by creating a self-loop for the cycle representing the variable \( w \), denoted by \( C_w \) (See Figure 14). Note that the self-loop is created locally on the cycle \( C_w \) such that it does not affect other cycles. As we did in Part I, we label the four edges incident to a self-intersection vertex by \((x_1, x_2, x_3, x_4)\) such that \( x_3 \) is the successor of \( x_1 \) and \( x_4 \) is the successor of \( x_2 \) depending on the default orientation of \( C_w \), and \((x_1, x_2, x_3, x_4)\) are labeled in counterclockwise order. Then, we have \((x_1, x_3) = (x_2, x_4) = (0, 1)\) when \( w = 1 \) and \((0, 1)\) when \( w = 0 \). That is, \( g_1(0, 0) = a^2 = f_{0011}^2 \) and \( g_1(1, 1) = x^2 = f_{1100}^2 \).

Now, we get an instance \( \Omega_s \) \((s = 1, 2)\) for each problem \( \text{Pl-Holant}(\neq 2 \mid f, \chi_s) \) respectively. Note that \( \chi_s \) has the support \((x_1 \neq x_3) \wedge (x_2 \neq x_4)\) as \( f \). As we have showed in Part I, for any nonzero term in the sum \( \text{Pl-Holant}_{\Omega_s} \), the assignment of all edges \( \sigma : E \rightarrow \{0, 1\} \) can be uniquely extended from the assignment of all leader edges \( \sigma' : \{e_1, e_2, \ldots, e_k\} \rightarrow \{0, 1\} \). Therefore, we have

\[
\#\text{CSP}(I) = \sum_{\sigma' : \{e_1, \ldots, e_k\} \rightarrow \{0, 1\}} \prod_{1 \leq i < j \leq n} g_{i,j}(\sigma'(e_i), \sigma'(e_j)) = \sum_{\sigma' : \{e_1, \ldots, e_k\} \rightarrow \{0, 1\}} \left( \prod_{v \in V_{ij}} f_v(\sigma'|_{E(v)}) \right) \left( \prod_{v \in V_{ij}, V_{ij}'} \chi_{sv}(\sigma'|_{E(v)}) \right) = \text{Pl-Holant}_{\Omega_s}
\]

for \( s = 1, 2 \). That is, \( \#\text{CSP}(g_1, g_2) \leq_T \text{Pl-Holant}(\neq 2 \mid f, \chi_s) \), \((s = 1, 2)\).

From the hypothesis of the reduction (4.4), we have \( a = \pm x \neq 0, b = \pm y \neq 0, \) and \((b/a)^8 \neq 1 \). We show by interpolation

\[
\text{Pl-Holant}(\neq 2 \mid f, \chi_1) \leq_T \text{Pl-Holant}(\neq 2 \mid f)
\]

when \( a = \epsilon x, b = \epsilon y \), and

\[
\text{Pl-Holant}(\neq 2 \mid f, \chi_2) \leq_T \text{Pl-Holant}(\neq 2 \mid f)
\]

when \( a = \epsilon x, b = -\epsilon y \), where \( \epsilon = \pm 1 \).

- If \( a = x \) and \( b = y \), since they are all nonzero, and \((b/a)^8 \neq 1 \), by normalization we may assume
  \[
  M(f) = \begin{bmatrix}
  0 & 0 & 0 & 0 \\
  0 & b & 0 & 0 \\
  0 & 0 & b & 0 \\
  0 & 0 & 0 & b
  \end{bmatrix},
  \]
  where \( b \neq 0 \) and \( b^8 \neq 1 \).

If \( b \) is not a root of unity, by Lemma 2.3, we have \( \text{Pl-Holant}(\neq 2 \mid f, \chi_1) \leq_T \text{Pl-Holant}(\neq 2 \mid f) \). Otherwise, \( b \) is a root of unity. Construct a gadget \( f_{\Sigma} \) as shown in Figure 15. Given an assignment \((x_1, x_2, x_3, x_4) \) to \( f_{\Sigma} \), and suppose \( f_{\Sigma}(x_1, x_2, x_3, x_4) \neq 0 \). Then because of the support of \( f_{v_1}, f_{v_2} \) and \( f_{v_3} \) we must have \( x_1 \neq x_3 \). Similarly \( x_2 \neq x_4 \). Also \( f_{v_5} \) receives the same input as \( f_{\Sigma} \). Hence the support of \( f_{\Sigma} \) is contained in \((x_1 \neq x_3) \wedge (x_2 \neq x_4) \), i.e., contained in \\{(0, 0, 1, 1), (1, 1, 0, 0), (0, 1, 1, 0), (1, 0, 0, 0)\}\. In particular, the edges on each DIAGONAL LINE of this gadget can only take assignments \((0, 1, 0, 1, 0, 1) \) or \((1, 0, 1, 0, 1, 0) \), otherwise the we get zero. On the other hand, the SQUARE cycle in this gadget is a circuit itself, so that the edges in it can only take two assignments \((0, 1, 0, 1, 0, 1) \) or \((1, 0, 1, 0, 1, 0) \). We simplify the notation to \((0, 1) \) and \((1, 0) \) respectively. On \((x_1 \neq x_3) \wedge (x_2 \neq x_4) \), the value of \( f_{\Sigma} \) is the sum over these two terms. 

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For the signature $f$, if one pair of its edge twins flips its assignment between $(0, 1)$ and $(1, 0)$, then the value of $f$ changes from 1 to $b$, or from $b$ to 1. If both pairs of edge twins flip their assignments, then the value of $f$ does not change. According to this property, we give the Table 4. Here, we place a suitably rotated copy of $f$ at vertices $v_i$ to get $f_{v_i}$ (for $1 \leq i \leq 5$)

| $(x_1, x_2, x_3, x_4)$ | SQUARE | $f_{v_1}$ | $f_{v_2}$ | $f_{v_3}$ | $f_{v_4}$ | $f_{v_5}$ | $f_{\Box}$ |
|------------------------|--------|----------|----------|----------|----------|----------|---------|
| $(0, 0, 1, 1)$         | (0, 1) | 1        | 1        | 1        | 1        | 1        | $1 + b^4$ |
|                        | (1, 0) | $b$      | $b$      | $b$      | $b$      | 1        | $1 + b^4$ |
| $(1, 1, 0, 0)$         | (0, 1) | $b$      | $b$      | $b$      | $b$      | 1        | $1 + b^4$ |
|                        | (1, 0) | 1        | 1        | 1        | 1        | 1        | $1 + b^4$ |
| $(0, 1, 1, 0)$         | (0, 1) | 1        | $b$      | 1        | $b$      | 1        | $2b^3$   |
|                        | (1, 0) | $b$      | 1        | 1        | 1        | 1        | $2b^3$   |
| $(1, 0, 0, 1)$         | (0, 1) | $b$      | 1        | $b$      | 1        | $b$      | $2b^3$   |
|                        | (1, 0) | 1        | $b$      | 1        | $b$      | 1        | $2b^3$   |

Table 4: The values of gadget $f_{\Box}$ when $a = x = 1$ and $b = y$

so that the values of $f_{v_i}$ are all 1 under the assignment $(x_1, x_2, x_3, x_4) = (0, 0, 1, 1)$ and the SQUARE is assigned $= (0, 1)$ (row 2 of Table 4). When the assignment of SQUARE flips from $(0, 1)$ to $(1, 0)$, one pair of edge twins of each vertex except $v_5$ flips its assignment. So the values of $f$ on these vertices except $v_5$ change from 1 to $b$ (row 3). When $(x_1, x_3)$ flips its assignment, one pair of edge twins of $v_1, v_3$ and $v_5$ flip their assignments. When $(x_2, x_4)$ flips its assignment, one pair of edge twins of $v_2, v_4$ and $v_5$ flip their assignments. Using this fact, we get other rows correspondingly.

Hence, $f_{\Box}$ has the signature matrix $M(f_{\Box}) = \begin{bmatrix} 0 & 0 & 0 & 1+b^4 \\ 0 & 2b^3 & 0 & 0 \\ 0 & 0 & 2b^3 & 0 \\ 1+b^4 & 0 & 0 & 0 \end{bmatrix}$. Since $b^8 \neq 1$, we have

$1 + b^4 \neq 0$, by normalization we can write $M(f_{\Box}) = \begin{bmatrix} 0 & 0 & 0 & 1+b^4 \\ 0 & 2b^3 & 0 & 0 \\ 0 & 0 & 2b^3 & 0 \\ 1+b^4 & 0 & 0 & 0 \end{bmatrix}$. Since $|b| = 1$ and $b^4 \neq 1$,

we have $|1 + b^4| < 2$. Then $|\frac{2b^3}{1+b^4}| > |b^3| = 1$, which means $\frac{2b^3}{1+b^4}$ is not a root of unity. By Lemma 2.3, we have Pl-Holant($\not\equiv 2| f, \chi_1$) $\leq_T$ Pl-Holant($\not\equiv 2| f, f_{\Box}$). Since $f_{\Box}$ is constructed by $f$, we have Pl-Holant($\not\equiv 2| f, \chi_1$) $\leq_T$ Pl-Holant($\not\equiv 2| f$).

Figure 15: The SQUARE gadget
• If $a = -x$ and $b = -y$, then $M(f) = \begin{bmatrix} 0 & 0 & a \\ 0 & b & 0 \\ -a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Connect the variable $x_4$ with $x_3$ of $f$ using $(\neq 2)$, and we get a binary signature $g'$, where
\[
g' = M_{x_1x_2x_4x_3}(0, 1, 1, 0)^T = (0, b, -b, 0)^T.
\]
Since $b \neq 0$, $g'$ can be normalized as $(0, 1, -1, 0)^T$. Modifying $x_1 = 1$ of $f$ by $-1$ scaling, we get a signature $f'$ with the signature matrix $M(f') = \begin{bmatrix} 0 & 0 & a \\ 0 & b & 0 \\ -a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. As we have proved above, \[ \text{Pl-Holant}(\neq 2\mid f, \chi_1) \leq_T \text{Pl-Holant}(\neq 2\mid f, f'). \]
Since $f'$ is constructed by $f$, we have \[ \text{Pl-Holant}(\neq 2\mid f, \chi_1) \leq_T \text{Pl-Holant}(\neq 2\mid f). \]
• If $a = -x$, $b = y$ or $a = x$, $b = -y$, by normalization and rotational symmetry, we may assume $M(f) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$, where $b \neq 0$ and $b^8 \neq 1$. If $b$ is not a root of unity, by Corollary 2.4, we have \[ \text{Pl-Holant}(\neq 2\mid f, \chi_2) \leq_T \text{Pl-Holant}(\neq 2\mid f). \] Otherwise, $b$ is a root of unity. Our discussion on the support of $f_{32}$ still holds: It is contained in $(x_1 \neq x_3) \land (x_2 \neq x_4)$; on $(x_1, x_2, x_3, x_4)$ with $(x_1 \neq x_3) \land (x_2 \neq x_4)$, $v_5$ receives the same input, and the value of $f_{32}$ is the sum over two assignments $(0, 1)$ and $(1, 0)$ for the SQUARE.

For the signature $f$, if one pair of its edge twins flips its assignment between $(0, 1)$ and $(1, 0)$, then the value of $f$ changes from $\pm 1$ to $b$, or $b$ to $\mp 1$. If two pairs of edge twins both flip their assignments, then the value of $f$ does not change if the value is $b$, or changes its sign if the value is $\pm 1$. According to this property, we have the following Table 5. Here, we place a suitably rotated copy of $f$ at vertices $v_i$ to get $f_{v_i}$ (for $1 \leq i \leq 5$) so that the values of $f_{v_i}$ are all 1 under the assignment $(x_1, x_2, x_3, x_4) = (0, 0, 1, 1)$ and the SQUARE is assigned $= (0, 1)$ (row 2 of Table 5). When the assignment of SQUARE flips from $(0, 1)$ to $(1, 0)$, one pair of edge twins at each vertex except $v_5$ flips its assignment. So the values of $f$ at these vertices except $v_5$ change from 1 to $b$ (row 3). When $(x_1, x_3)$ flips its assignment, one pair of edge twins at $v_1$, $v_3$ and $v_5$ flips their assignments. When $(x_2, x_4)$ flips its assignment, one pair of edge twins at $v_2$, $v_4$ and $v_5$ flips their assignments. Using this fact, we get other rows correspondingly.

| $(x_1, x_2, x_3, x_4)$ | SQUARE | $f_{v_1}$ | $f_{v_2}$ | $f_{v_3}$ | $f_{v_4}$ | $f_{v_5}$ | $f_{32}$ |
|------------------------|--------|----------|----------|----------|----------|----------|--------|
| $(0, 0, 1, 1)$          | (0, 1) | 1        | 1        | 1        | 1        | 1        | $1 + b^4$ |
| (0, 1, 0)              | (0, 1) | $b$      | $b$      | $b$      | $b$      | 1        | $-(1 + b^4)$ |
| (1, 1, 0, 0)           | (0, 1) | $-1$     | $-1$     | $-1$     | $-1$     | 1        | $2b^3$ |
| (0, 1, 1, 0)           | (0, 1) | 1        | $b$      | 1        | $b$      | $b$      | $2b^3$ |
| (1, 1, 0, 1)           | (0, 1) | $b$      | 1        | $b$      | 1        | $b$      | $2b^3$ |
| (1, 0, 0, 1)           | (0, 1) | $-1$     | $b$      | 1        | $b$      | $b$      | $2b^3$ |

Table 5: The values of gadget $f_{32}$ when $a = -x = 1$ and $b = y$
Hence, \( f_{\mathbb{R}} \) has the signature matrix
\[
\begin{bmatrix}
0 & 0 & 0 & 1+b^4 \\
0 & 2b^3 & 0 & 0 \\
0 & 0 & 2b^3 & 0 \\
-(1+b^4) & 0 & 0 & 0
\end{bmatrix}.
\]
Since \(|b| = 1\) and \(b^8 \neq 1\), we have \(b^4 \neq \pm 1\), therefore \(0 < |1 + b^4| < 2\), and so \(\frac{2b^3}{1+b^4}\) is not a root of unity. By Corollary 2.4, \(\text{Pl-Holant}(\neq 2| f, \chi_2) \leq_T \text{Pl-Holant}(\neq 2| f, f_{\mathbb{R}})\), and hence \(\text{Pl-Holant}(\neq 2| f, \chi_2) \leq_T \text{Pl-Holant}(\neq 2| f)\).

In summary, we have
\[
\text{Pl-Holant}(\neq 2| f, \chi_1) \leq_T \text{#CSP}(g_1, g_2) \leq_T \text{Pl-Holant}(\neq 2| f, \chi_2)
\]
\[
\text{Pl-Holant}(\neq 2| f) \leq_T \text{Pl-Holant}(\neq 2| f)
\]
Therefore, we have \(\text{#CSP}(g_1, g_2) \leq_T \text{Pl-Holant}(\neq 2| f)\) when \(a^2 = x^2 \neq 0\), \(b^2 = y^2 \neq 0\) and \((\frac{b}{a})^8 \neq 1\).

**Remark:** A crucial point in the reduction (4.3) is the fact that the given instance graph \(G\) of \(\text{Pl-Holant}(\neq 2| f)\) is planar so that \(\sum_i k_i = \sum_i \ell_i\). Otherwise this does not hold in general; for example the latitudinal and longitudinal closed cycles on a torus intersect at a single point. The equation \(\sum_i k_i = \sum_i \ell_i\) is crucial to obtain tractability in the following theorem.

**Theorem 4.6.** Let \(f\) be a 4-ary signature of the form (4.2), where \((a, x) \neq (0, 0)\) and \((b, y) \neq (0, 0)\). Then \(\text{Pl-Holant}(\neq 2| f)\) is \#P-hard unless

(1) \((ax)^2 = (by)^2\), or

(ii) \(x = a^\alpha x, b = a \sqrt[\beta]{y}, y = a \sqrt[\gamma]{\beta}\), where \(\alpha, \beta, \gamma \in \mathbb{N}\), and \(\beta \equiv \gamma \pmod{2}\),

in which cases, the problem is tractable in polynomial time.

**Proof of Tractability:**

- In case (i), if \(ax = by = 0\), then \(f\) has support of size at most 2. So we have \(f \in \mathcal{P}\), and hence \(\text{Pl-Holant}(\neq 2| f)\) is tractable by Theorem 2.13. Otherwise, \((ax)^2 = (by)^2 \neq 0\).

For any signature \(g\) in \(\mathcal{G}_f\), we have \(g_{00} \cdot g_{11} = (ax)^{k_1 + \ell_1 + k_3 + \ell_3} (by)^{k_2 + \ell_2 + k_4 + \ell_4}\) and \(g_{01} \cdot g_{10} = (ax)^{k_2 + \ell_2 + k_4 + \ell_4} (by)^{k_1 + \ell_1 + k_3 + \ell_3}\). Since \((k_1 + \ell_1 + k_3 + \ell_3) - (k_2 + \ell_2 + k_4 + \ell_4) \equiv k + \ell \equiv 0 \pmod{2}\), we have

\[\frac{g_{00} \cdot g_{11}}{g_{01} \cdot g_{10}} = \frac{(ax)^{k_1 + \ell_1 + k_3 + \ell_3} (by)^{k_2 + \ell_2 + k_4 + \ell_4}}{(ax)^{k_2 + \ell_2 + k_4 + \ell_4} (by)^{k_1 + \ell_1 + k_3 + \ell_3}} = 1.\]

That is, \(g \in \mathcal{P}\). Since any signature \(h\) in \(\mathcal{H}_f\) is unary, \(h \in \mathcal{P}\). Hence, we have \(\mathcal{G}_f \cup \mathcal{H}_f \subseteq \mathcal{P}\). By Theorem 2.23, \(#\text{CSP}(\mathcal{G}_f \cup \mathcal{H}_f)\) is tractable. By reduction (4.3) of Lemma 4.5, we have \(\text{Pl-Holant}(\neq 2| f)\) is tractable.

- In case (ii), for any signature \(g \in \mathcal{G}_f\) defined in Definition 4.4, \(M(g)\) is of the form

\[a^{k+\ell} \left[ \sqrt[4]{1} \sum_{i} \left( \frac{1}{\sqrt[4]{1}} \right) \frac{\beta^2(k_4 + \ell_4) + \gamma(k_2 + \ell_2) + 2\alpha(k_3 + \ell_3)}{\beta^2(k_1 + \ell_1) + \gamma(k_1 + \ell_1) + 2\alpha(k_2 + \ell_2)} \right] \]

where

\[a^{k+\ell} \left[ \sqrt[4]{1} \sum_{i} \left( \frac{1}{\sqrt[4]{1}} \right) \frac{\beta^2(k_4 + \ell_4) + \gamma(k_2 + \ell_2) + 2\alpha(k_3 + \ell_3)}{\beta^2(k_1 + \ell_1) + \gamma(k_1 + \ell_1) + 2\alpha(k_2 + \ell_2)} \right] = a^{k+\ell} \left[ \frac{1}{\sqrt[4]{1}} \sum_{i} \left( \frac{1}{\sqrt[4]{1}} \right) \frac{\beta^2(k_4 + \ell_4) + \gamma(k_2 + \ell_2) + 2\alpha(k_3 + \ell_3)}{\beta^2(k_1 + \ell_1) + \gamma(k_1 + \ell_1) + 2\alpha(k_2 + \ell_2)} \right].\]
where \( p_{00}, p_{01}, p_{10} \) and \( p_{11} \) denote the integer exponents of \( \sqrt{i} \) in the respective entries of \( g \). Since \( \beta \equiv \gamma \pmod{2} \), if they are both even, then \( p_{00} \equiv p_{01} \equiv p_{10} \equiv p_{11} \equiv 0 \pmod{2} \); if they are both odd, then \( p_{00} \equiv p_{11} \equiv k_{2} + \ell_{2} + k_{4} + \ell_{4} \equiv k_{1} + \ell_{1} + k_{3} + \ell_{3} \equiv p_{10} \equiv p_{11} \pmod{2} \). If these exponents are all odd, we can take out a \( \sqrt{i} \). Hence, \( g \) is of the form \( a' [v_{00}, v_{01}, v_{10}, v_{11}]^{T} \), where \( a' = a^{k+\ell} \) or \( a^{k+\ell} \sqrt{i} \), and either \( q_{ij} = \frac{p_{ij}}{2} \) for all \( i, j \in \{0, 1\} \) are integers, or \( q_{ij} = \frac{p_{ij}-1}{2} \) for all \( i, j \in \{0, 1\} \) are integers. Thus,

\[
q_{00} + q_{01} + q_{10} + q_{11} \equiv (p_{00} + p_{01} + p_{10} + p_{11})/2 \pmod{2} .
\]

Moreover, since \( p_{00} + p_{01} + p_{10} + p_{11} = (k + \ell)(\beta + \gamma + 2\alpha) \equiv 0 \pmod{4} \), using the assumption that \( \beta \equiv \gamma \pmod{2} \) and \( k \equiv \ell \pmod{2} \), we conclude that \( q_{00} + q_{01} + q_{10} + q_{11} \equiv 0 \pmod{2} \). Therefore, \( g \in \mathcal{A} \) by Lemma 2.10.

In this case, for any signature \( h \in \mathcal{H}_{f} \), \( M(h) \) is of the form

\[
a^{m} \left[ \sqrt{\beta m_{4} + \gamma m_{2} + 2am_{3}} \quad \sqrt{\beta m_{2} + \gamma m_{4} + 2am_{1}} \right].
\]

Since \( \beta \equiv \gamma \pmod{2} \), we have \( \beta m_{4} + \gamma m_{2} \equiv \beta m_{2} + \gamma m_{4} \pmod{2} \). Hence, \( h \) is of the form \( a^{m}[v_{00}, v_{01}] \), for some integers \( q_{0}, q_{1} \), where \( a' = a^{m} \) or \( a^{m} \sqrt{i} \). That is, \( h \in \mathcal{A} \) by Lemma 2.11.

Hence, \( \mathcal{G}_{f} \cup \mathcal{H}_{f} \subseteq \mathcal{A} \). By Theorem 2.23, \( \#\text{CSP}(\mathcal{G}_{f} \cup \mathcal{H}_{f}) \) is tractable. By reduction (4.3) of Lemma 4.5, we have \( \text{Pl-Holant}(\neq 2, f) \) is tractable.

**Proof of Hardness:** We are given that \( f \) does not belong to case (i) or case (ii). Note that \( M_{x_{4}, x_{3}, x_{2}}(f) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & 0 & 0 \\ y & 0 & 0 & 0 \end{bmatrix} \) and \( M_{x_{2}, x_{3}, x_{1}}(f) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \end{bmatrix} \). Connect variables \( x_{3}, x_{2} \) of a copy of the signature \( f \) with variables \( x_{2}, x_{3} \) of another copy of signature \( f \) both using \( (\neq 2) \). We get a signature \( f_{1} \) with the signature matrix

\[
M(f_{1}) = M_{x_{4}, x_{1}}(f)NM_{x_{2}, x_{3}, x_{1}}(f) = \begin{bmatrix} 0 & 0 & 0 & by \\ 0 & x & 0 & 0 \\ 0 & a & 0 & 0 \\ by & 0 & 0 & 0 \end{bmatrix}.
\]

Similarly, connect \( x_{3}, x_{2} \) of a copy of signature \( f \) with \( x_{4}, x_{1} \) of another copy of signature \( f \) both using \( (\neq 2) \). We get a signature \( f_{2} \) with the signature matrix

\[
M(f_{2}) = M_{x_{4}, x_{1}}(f)NM_{x_{3}, x_{2}}(f) = \begin{bmatrix} 0 & 0 & 0 & b \sqrt{i} \\ 0 & 0 & ax & 0 \\ 0 & ax & 0 & 0 \\ y \sqrt{i} & 0 & 0 & 0 \end{bmatrix}.
\]

Notice that \( M(f_{1}) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & by & 0 & 0 \\ 0 & 0 & by & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \), \( M(f_{2}) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & b \sqrt{i} & ax & 0 \\ 0 & ax & y \sqrt{i} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \), \( M(g_{1,f}) = \begin{bmatrix} 0 & 0 & by & b \sqrt{i} \\ 0 & ax & 0 & 0 \\ 0 & ax & 0 & 0 \\ ax & 0 & 0 & 0 \end{bmatrix} \) and \( M(g_{2,f}) = \begin{bmatrix} ax & b \sqrt{i} \\ by & 0 & 0 & 0 \\ 0 & ax & 0 & 0 \\ 0 & 0 & ax & 0 \end{bmatrix} \).

Recall that \( M(f_{1}) = M_{[1]}^{(f_{1})}M_{[1]}^{(f_{1})} \). We have \( \hat{g}_{f_{i}} = f_{i}^{\sqrt{i}}M_{[1]}^{(f_{i})}M_{[1]}^{(f_{i})} \). We get \( g_{1,f} = \hat{g}_{f_{i}} \). That is, \( f_{i}(x_{1}, x_{2}, x_{3}, x_{4}) = g_{1,f}(x_{2}, x_{4}) \cdot \chi_{x_{1} \neq x_{4}} \cdot \chi_{x_{2} \neq x_{3}} \). Now, we analyze \( g_{1,f} \) and \( g_{2,f} \).

- If \( \{g_{1,f}, g_{2,f}\} \subseteq \mathcal{P} \), then either \((ax)^{2} = (by)^{2}\) if either signature is degenerate, or \( g_{1,f} \) and \( g_{2,f} \) are each generalized EQUALITY or generalized INEQUALITY respectively. In the latter case, since \((a, x) \neq (0, 0)\) and \((b, y) \neq (0, 0)\), it forces that \( ax = by = 0 \). So we still have \((ax)^{2} = (by)^{2}\). That is, \( \{a, b, x, y\} \) belongs to case (i). A contradiction.

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If \( \{g_1, g_2\} \subseteq \mathcal{A} \), there are two subcases. Note that the support of a function in \( \mathcal{A} \) has size a power of 2.

- If both \( g_1 \) and \( g_2 \) have support of size at most 2, then we have \( ax = by = 0 \) due to \( (a, x) \neq (0, 0) \) and \( (b, y) \neq (0, 0) \). This belongs to case (i). A contradiction.

- Otherwise, at least one of \( g_1 \), or \( g_2 \), has support of size 4. Then \( abxy \neq 0 \) and therefore both \( g_1 \) and \( g_2 \) have support of size 4. Let \( x' = \frac{x}{a}, b' = \frac{b}{a} \) and \( y' = \frac{y}{a} \). By normalization, we have

\[
M(g_{1,2}) = a^2 \begin{bmatrix} 1 & b'y' \\ b'y' & x'^2 \end{bmatrix}.
\]

Since \( g_1, g_2 \in \mathcal{A} \), by Lemma 2.10, \( x'^2 \) and \( b'y' \) are both powers of \( i \), and the sum of all exponents is even. It forces that \( x'^2 = i^{2\alpha} \) for some \( \alpha \in \mathbb{N} \). Then, we can choose \( \alpha \) such \( x' = i^\alpha \). Also, we have

\[
M(g_{2,2}) = a^2 \begin{bmatrix} x' & b'^2 \\ y'^2 & x' \end{bmatrix}.
\]

Since \( g_2 \in \mathcal{A} \) and \( x' \) is already a power of \( i \), \( y'^2 \) and \( b'^2 \) are both powers of \( i \). That is, \( b' = \sqrt{i^\beta} \) and \( y' = \sqrt{i^\gamma} \). Also, since \( g_1, g_2 \in \mathcal{A} \), \( b'y' = \sqrt{i^{\beta+\gamma}} \) is a power of \( i \), which means \( \beta \equiv \gamma \pmod{2} \). That is, \( \{a, b, x, y\} \) belongs to case (ii). A contradiction.

If \( \{g_1, g_2\} \subseteq \mathcal{M} \), then by Lemma 2.17, we have both \( a^2 = ex^2, by = eyb \) and \( ax = e'ax, y^2 = e'b'^2 \), for some \( e, e' \in \{1, -1\} \). If \( e = -1 \) then \( by = 0 \), and then by the second set of equations \( b = y = 0 \), contrary to assumption that \( (b, y) \neq (0, 0) \). So \( e = 1 \). Similarly \( e' = 1 \). Hence

\[
a^2 = x^2 \quad b^2 = y^2,
\]

and it also follows that all 4 entries are nonzero.

Therefore, if \( \{a, b, x, y\} \) does not satisfy (4.5) then \( \{g_1, g_2\} \notin \mathcal{P}, \mathcal{A} \) or \( \mathcal{M} \). By Theorem 2.23, \( \text{Pl-}\#\text{CSP}(g_1, g_2) \) is \#P-hard. Then by Lemma 4.2, \( \text{Pl-Holant}(\neq 2 | f^x_i, f^x_j) \) is \#P-hard, and hence \( \text{Pl-Holant}(\neq 2 | f) \) is \#P-hard.

Otherwise, the 4 nonzero entries \( \{a, b, x, y\} \) satisfy (4.5). If \( \left(\frac{b}{a}\right)^8 = 1 \), i.e., \( b = a \sqrt{i^\beta} \) for some \( \gamma \in \mathbb{N} \), then \( x = \pm a = ai^\alpha \), and \( y = \pm b = a \sqrt{i^{\beta+4\delta}} \) for some \( \alpha, \delta \in \mathbb{N} \). It follows that \( \{a, b, x, y\} \) satisfies (ii), a contradiction.

So \( \left(\frac{b}{a}\right)^8 \neq 1 \), and we can apply reduction (4.4) of Lemma 4.5. By the reduction (4.4), we have \( \#\text{CSP}(g_1, g_2) \leq_T \text{Pl-Holant}(\neq 2 | f) \). Moreover, since \( \{a, b, x, y\} \) does not belong to case (i) or case (ii), we have \( \{g_1, g_2\} \notin \mathcal{P} \) or \( \mathcal{A} \). By Theorem 2.23, \( \#\text{CSP}(g_1, g_2) \) is \#P-hard. Therefore, we have \( \text{Pl-Holant}(\neq 2 | f) \) is \#P-hard.

\[\square\]

**Corollary 4.7.** Let \( f \) be a 4-ary signature of the form (4.2), where \( (a, x) \neq (0, 0) \) and \( (b, y) \neq (0, 0) \). If \( |ax| \neq |by| \) then \( \text{Pl-Holant}(\neq 2 | f) \) is \#P-hard.

5 Case III: \( N = 2 \) with No Zero Pair or \( N = 1 \) with Zero in an Outer Pair

If there are exactly two zeros \( N = 2 \) with no zero pair, then the two zeros are in different pairs, at least one of them must be in an outer pair. So in Case III there is a zero in an outer pair regardless \( N = 1 \) or \( N = 2 \). By rotational symmetry, we may assume \( a = 0 \), and we prove this case in Theorem 5.2. We first give the following lemma.
**Lemma 5.1.** Let \( f \) be a 4-ary signature with the signature matrix \( M(f) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & z & y & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \), where \( \Lambda = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \). \( \det M_{\text{in}}(f) = by - cz \neq 0 \). Let \( g \) be a 4-ary signature with the signature matrix \( M(g) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \). Then for any signature set \( \mathcal{F} \) containing \( f \), we have

\[
\text{Pl-Holant}(\#_2|\mathcal{F} \cup \{g\}) \leq_T \text{Pl-Holant}(\#_2|\mathcal{F}).
\]

**Proof.** We construct a series of gadgets \( f_s \) by a chain of \( s \) copies of \( f \) linked by double DISEQUALITY \( N \). \( f_s \) has the signature matrix

\[
M(f_s) = M(f)(NM(f))^{s-1} = N(NM(f))^s = N \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & z & y & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
\]

The inner matrix of \( NM(f) \) is \( N_{\text{in}} M_{\text{in}}(f) = \begin{bmatrix} z & y \\ 0 & c \end{bmatrix} \). Suppose its spectral decomposition is \( Q^{-1} \Lambda \), where \( \Lambda = \begin{bmatrix} \bar{\lambda}_1 & \mu \\ 0 & \bar{\lambda}_2 \end{bmatrix} \) is the Jordan Canonical Form. Note that \( \lambda_1 \lambda_2 = \det \Lambda = \det(N_{\text{in}} M_{\text{in}}(f)) \neq 0 \). We have \( M(f_s) = NP^{-1} \Lambda_s P \), where

\[
P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \Lambda_s = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \bar{\lambda}_1 & \mu \end{bmatrix}^s \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]

1. Suppose \( \mu = 0 \) and \( \frac{\bar{\lambda}_2}{\bar{\lambda}_1} \) is a root of unity, with \( (\frac{\bar{\lambda}_2}{\bar{\lambda}_1})^n = 1 \). Then \( \Lambda_n = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \).

2. Suppose \( \mu = 0 \) and \( \frac{\bar{\lambda}_2}{\bar{\lambda}_1} \) is not a root of unity. The matrix \( \Lambda_s = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \) has a good form for interpolation. Suppose \( g \) appears \( m \) times in an instance \( \Omega \) of \( \text{Pl-Holant}(\#_2|\mathcal{F} \cup \{g\}) \). Replace each appearance of \( g \) by a copy of the gadget \( f_s \) to get an instance \( \Omega_s \) of \( \text{Pl-Holant}(\#_2|\mathcal{F} \cup \{f_s\}) \), which is also an instance of \( \text{Pl-Holant}(\#_2|\mathcal{F}) \). We can treat each of the \( m \) appearances of \( f_s \) as a new gadget composed of four functions in sequence \( N, P^{-1}, \Lambda_s \) and \( P \), and define this new instance by \( \Omega' \). We divide \( \Omega_s \) into two parts. One part consists of \( m \) signatures \( \Lambda_s^{\otimes m} \). Here \( \Lambda_s^{\otimes m} \) is expressed as a column vector. The other part is the rest of \( \Omega_s \) and its signature is represented by \( A \) which is a tensor expressed as a row vector. Then the Holant value of \( \Omega_s \) is the dot product \( \langle A, \Lambda_s^{\otimes m} \rangle \), which is a summation over \( 4m \) bits. That is, the value of the \( 4m \) edges connecting the two parts. We can stratify all \( 0, 1 \) assignments of these \( 4m \) bits having a nonzero evaluation of a term in \( \text{Pl-Holant}_{\Omega_s} \) into the following categories:

- There are \( i \) many copies of \( \Lambda_s \) receiving inputs 0110;
- There are \( j \) many copies of \( \Lambda_s \) receiving inputs 1001;
where $i + j = m$.

For any assignment in the category with parameter $(i, j)$, the evaluation of $\Lambda_s \otimes^m$ is clearly

$$\lambda^i \lambda^j = \lambda^m \left( \frac{s}{\lambda} \right)^j.$$

Let $a_{ij}$ be the summation of values of the part $A$ over all assignments in the category $(i, j)$. Note that $a_{ij}$ is independent from the value of $s$ since we view the gadget $\Lambda_s$ as a block. Since $i + j = m$, we can denote $a_{i j}$ by $a_j$. Then we rewrite the dot product summation and get

$$\text{Pl-Holant}_{\Omega_s} = \langle A, \Lambda_s \otimes^m \rangle = \lambda^m \sum_{0 \leq j \leq m} a_j \left( \frac{s}{\lambda} \right)^j.$$

Note that $M(g) = NP^{-1}(NM(g))P$, where $NM(g) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Similarly, divide $\Omega$ into two parts. Under this stratification, we have

$$\text{Pl-Holant}_{\Omega} = \langle A, (NM(g)) \otimes^m \rangle = \sum_{0 \leq j \leq m} a_j.$$

Since $\frac{\lambda^s}{\lambda^1}$ is not a root of unity, the Vandermonde coefficient matrix

$$\begin{bmatrix}
\rho^0 & \rho^1 & \cdots & \rho^m \\
\rho^0 & \rho^2 & \cdots & \rho^{2m} \\
\vdots & \vdots & \ddots & \vdots \\
\rho^0 & \rho^{m+1} & \cdots & \rho^{(m+1)m}
\end{bmatrix},$$

has full rank, where $\rho = \frac{\lambda^s}{\lambda^1}$. Hence, by oracle querying the values of $\text{Pl-Holant}_{\Omega_s}$, we can solve for $a_j$, and thus obtain the value of $\text{Pl-Holant}_{\Omega}$ in polynomial time.

3. Suppose $\mu = 1$, and $\lambda_1 = \lambda_2$ denoted by $\lambda$. Then $\Lambda_s = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \lambda^s & s\lambda^{s-1} & 0 \\ 0 & 0 & \lambda^s & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. We use this form to give a polynomial interpolation. As in the case above, we can stratify the assignments of $\Lambda_s \otimes^m$ of these $4m$ bits having a nonzero evaluation of a term in $\text{Pl-Holant}_{\Omega_s}$ into the following categories:

- There are $i$ many copies of $\Lambda_s$ receiving inputs 0110 or 1001;
- There are $j$ many copies of $\Lambda_s$ receiving inputs 0101;

where $i + j = m$.

For any assignment in the category with parameter $(i, j)$, the evaluation of $\Lambda_s \otimes^m$ is clearly

$$\lambda^i (s \lambda^{s-1})^j = \lambda^{sk} \left( \frac{s}{\lambda} \right)^j.$$

Let $a_{ij}$ be the summation of values of the part $A$ over all assignments in the category $(i, j)$. $a_{ij}$ is independent from $s$. Since $i + j = m$, we can denote $a_{ij}$ by $a_j$. Then, we rewrite the dot product summation and get

$$\text{Pl-Holant}_{\Omega_s} = \langle A, \Lambda_s \otimes^m \rangle = \lambda^m \sum_{0 \leq j \leq m} a_j \left( \frac{s}{\lambda} \right)^j,$$

for $s \geq 1$. We consider this as a linear system for $1 \leq s \leq m + 1$. Similarly, divide $\Omega$ into two parts. Under this stratification, we have

$$\text{Pl-Holant}_{\Omega} = \langle A, (NM(g)) \otimes^m \rangle = a_0.$$
The Vandermonde coefficient matrix
\[
\begin{bmatrix}
\rho_0 & \rho_1^1 & \cdots & \rho_1^m \\
\rho_0 & \rho_2^1 & \cdots & \rho_2^m \\
\vdots & \vdots & \ddots & \vdots \\
\rho_0 & \rho_{m+1}^1 & \cdots & \rho_m^m
\end{bmatrix},
\]
has full rank, where \(\rho_s = s/\lambda\) are all distinct. Hence, we can solve \(a_0\) in polynomial time and it is the value of \(\text{Pl-Holant}_\Omega\). Therefore, we have \(\text{Pl-Holant}(\neq_2 | \mathcal{F} \cup \{g\}) \leq_T \text{Pl-Holant}(\neq_2 | \mathcal{F})\). Theorem 5.2 gives a classification for Case III.

**Theorem 5.2.** Let \(f\) be a 4-ary signature with the signature matrix
\[
M(f) = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & b & c & 0 \\
0 & z & y & 0 \\
x & 0 & 0 & 0
\end{bmatrix},
\]
where \(x \neq 0\) and there is at most one number in \(\{b, c, y, z\}\) that is 0. Then \(\text{Pl-Holant}(\neq_2 | f)\) is \(\#P\)-hard unless \(f \in \mathcal{M}\), in which case the problem is tractable.

**Proof.** Tractability follows from Theorem 2.15.

Suppose \(f \notin \mathcal{M}\). By Lemma 2.14, \(\det M_{\text{In}}(f) \neq \det M_{\text{Out}}(f) = 0\), that is \(\begin{bmatrix} b & c \\ z & y \end{bmatrix} = by - cz \neq 0\).

Note that \(M_{x_1,x_2,x_4,x_3}(f) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & z & y & 0 \\ x & 0 & 0 & 0 \end{bmatrix}\), \(M_{x_3,x_4,x_2,x_1}(f) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z \\ 0 & 0 & y & 0 \\ 0 & z & b & 0 \end{bmatrix}\), and \(M_{x_2,x_3,x_1,x_4}(f) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & y \\ 0 & 0 & z & 0 \\ 0 & 0 & c & 0 \end{bmatrix}\).

Connect variables \(x_4, x_3\) of a copy of signature \(f\) with variables \(x_3, x_4\) of another copy of signature \(f\) both using \((\neq_2)\). We get a signature \(f_1\) with the signature matrix
\[
M(f_1) = M_{x_1,x_2,x_4,x_3}(f)NM_{x_3,x_4,x_2,x_1}(f) = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & b_1 & c_1 & 0 \\
0 & z_1 & y_1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
\]
where \(\begin{bmatrix} b_1 & c_1 \\ z_1 & y_1 \end{bmatrix} = \begin{bmatrix} b & c \\ z & y \end{bmatrix} \cdot \begin{bmatrix} z & b \\ y & c \end{bmatrix}\). This \(f_1\) has the form in Lemma 5.1. Here, \(\det \begin{bmatrix} b_1 & c_1 \\ z_1 & y_1 \end{bmatrix} = -(by - cz)^2 \neq 0\). By Lemma 5.1, we have
\[
\text{Pl-Holant}(\neq_2 | f, g) \leq_T \text{Pl-Holant}(\neq_2 | f, f_1),
\]
where \(g\) has the signature matrix \(M(g) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\).

- If \(bcyz \neq 0\), connect variables \(x_1, x_4\) of signature \(f\) with variables \(x_1, x_2\) of signature \(g\) both using \((\neq_2)\). We get a signature \(f_2\) with the signature matrix
\[
M(f_2) = M_{x_2,x_3,x_1,x_4}(f)NM_{x_1,x_2,x_4,x_3}(g) = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & z & 0 \\
0 & c & x & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]
• Otherwise, connect variables \( x_4, x_3 \) of signature \( f \) with variables \( x_1, x_2 \) of signature \( g \) both using \((\neq 2)\). We get a signature \( f_2 \) with the signature matrix

\[
M(f_2) = M_{x_1 x_2 x_4 x_3}(f)NM_{x_1 x_2 x_4 x_3}(g) = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & b & c & 0 \\
0 & z & y & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
\]

and there is exactly one entry in \( \{b, c, y, z\} \) that is zero.

In both cases, the support of \( f_2 \) has size 3, which means \( f_2 \notin \mathcal{P}, \mathcal{A} \) or \( \hat{\mathcal{M}} \). By Theorem 4.3, \( \text{Pl-Holant}(\neq 2 \mid f_2) \) is \#P-hard. Since

\[
\text{Pl-Holant}(\neq 2 \mid f_2) \leq_T \text{Pl-Holant}(\neq 2 \mid f, g) \leq_T \text{Pl-Holant}(\neq 2 \mid f, f_1) \leq_T \text{Pl-Holant}(\neq 2 \mid f),
\]

we have \( \text{Pl-Holant}(\neq 2 \mid f) \) is \#P-hard.

\[\square\]

6 Case IV: \( N = 1 \) with Zero in the Inner Pair or \( N = 0 \)

By rotational symmetry, if there is one zero in the inner pair, we may assume it is \( c = 0 \), and \( abxyz \neq 0 \). We first consider the case that \( x = \epsilon a, y = \epsilon b \) and \( z = \epsilon c \), where \( \epsilon = \pm 1 \).

**Lemma 6.1.** Let \( f \) be a 4-ary signature with the signature matrix

\[
M(f) = \begin{bmatrix}
0 & 0 & 0 & a \\
0 & b & c & 0 \\
0 & \epsilon c & \epsilon b & 0 \\
\epsilon a & 0 & 0 & 0
\end{bmatrix}, \quad \text{where } \epsilon = \pm 1 \text{ and } abc \neq 0.
\]

Then \( \text{Pl-Holant}(\neq 2 \mid f) \) is \#P-hard if \( f \notin \mathcal{M} \).

**Proof.** If \( \epsilon = -1 \) we first transform the case to \( \epsilon = 1 \) as follows. Connecting the variable \( x_4 \) with \( x_3 \) of \( f \) using \((\neq 2)\) we get a binary signature \( g_1 \), where

\[
g_1 = M_{x_1 x_2 x_4 x_3}(f)(0, 1, 1, 0)^T = (0, b + c, -(b + c), 0)^T.
\]

Also connecting the variable \( x_1 \) with \( x_2 \) of \( f \) using \((\neq 2)\) we get a binary signature \( g_2 \), where

\[
g_2 = ((0, 1, 1, 0)M_{x_1 x_2 x_4 x_3}(f))^T = (0, b - c, -(b - c), 0)^T.
\]

Since \( bc \neq 0, b + c \) and \( b - c \) cannot be both zero. Without loss of generality, suppose \( b + c \neq 0 \). By normalization, we have \( g_1 = (0, 1, -1, 0)^T \). Then, modifying \( x_1 = 1 \) of \( f \) with \(-1\) scaling we get a signature with the signature matrix \( \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & b & c & 0 \\
0 & c & b & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \). Therefore, it suffices to show \#P-hardness for the case \( \epsilon = 1 \).

Since \( f \notin \mathcal{M} \), by Lemma 2.14, \( c^2 - b^2 \neq a^2 \). We prove \#P-hardness in the following three Cases depending on the values of \( a, b \) and \( c \).

**Case 1:** Either \( c^2 - b^2 \neq 0 \) and \( |c + b| \neq |c - b| \), or \( c^2 - a^2 \neq 0 \) and \( |c + a| \neq |c - a| \). By rotational symmetry, we may assume \( c^2 - b^2 \neq 0 \) and \( |c + b| \neq |c - b| \). We may normalize \( a = 1 \) and assume \( M(f) = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & b & 0 & 0 \\
0 & c & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix} \), where \( c^2 - b^2 \neq 0 \) or 1.
We construct a series of gadgets $f_s$ by a chain of $s$ copies of $f$ linked by double DISEQUALITY $N$. $f_s$ has the signature matrix

$$M(f_s) = M(f)(NM(f))^{s-1} = N(NM(f))^s = N\begin{bmatrix} 1 & 0 & 0 \\ 0 & c & b \\ 0 & 0 & 1 \end{bmatrix}.$$ 

We diagonalize $[c\ b\ c]^s$ using $H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ (note that $H^{-1} = H$), and get $M(f_s) = NPA_sP$, where

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & H & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad \Lambda_s = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & (c+b)^s & 0 & 0 \\ 0 & 0 & (c-b)^s & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$ 

The signature matrix $\Lambda_s$ has a good form for polynomial interpolation. In the following, we will reduce Pl-Holant$(\neq 2|\hat{f})$ to Pl-Holant$(\neq 2| f)$, for suitably chosen $M(\hat{f}) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & b & \hat{c} & 0 \\ 0 & \hat{c} & b & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, and use that to prove that Pl-Holant$(\neq 2| f)$ is $\#P$-hard.

Suppose $\hat{f}$ appears $m$ times in an instance $\hat{\Omega}$ of Pl-Holant$(\neq 2| \hat{f})$. We replace each appearance of $\hat{f}$ by a copy of the gadget $f_s$ to get an instance $\Omega_s$ of Pl-Holant$(\neq 2| f)$. We can treat each of the $m$ appearances of $f_s$ as a new gadget composed of four functions in sequence $N$, $P$, $\Lambda_s$ and $P$, and denote this new instance by $\Omega_s$. We divide $\Omega_s$ into two parts. One part consists of $m$ occurrences of $\Lambda_s$, which is $\Lambda_s^\otimes 2^m$, and is written as a column vector of dimension $2^{4m}$. The other part is the rest of $\Omega'$ and its signature is expressed by a tensor $A$, written as a row vector of dimension $2^{4m}$. Then the Holant value of $\Omega'$ is the dot product $\langle A, \Lambda_s^\otimes 2^m \rangle$, which is a summation over $4m$ bits, i.e., the values of the $4m$ edges connecting the two parts. We can stratify all 0,1 assignments of these $4m$ bits having a nonzero evaluation of a term in Pl-Holant$_{\Omega_s}$ into the following categories:

- There are $i$ many copies of $\Lambda_s$ receiving inputs 0000 or 1111;
- There are $j$ many copies of $\Lambda_s$ receiving inputs 0110;
- There are $k$ many copies of $\Lambda_s$ receiving inputs 1001;

where $i + j + k = m$.

For any assignment in the category with parameter $(i, j, k)$, the evaluation of $\Lambda_s^\otimes 2^m$ is clearly $(c+b)^s(c-b)^s$. Let $a_{ijk}$ be the summation of values of the part $A$ over all assignments in the category $(i, j, k)$. Note that $a_{ijk}$ is independent of the value of $s$. Since $i + j + k = m$, we can denote $a_{ijk}$ by $a_{jk}$. Then we rewrite the dot product summation and get

$$\text{Pl-Holant}_{\Omega_s} = \text{Pl-Holant}_{\hat{\Omega}} = \langle A, \hat{\Lambda}^\otimes 2^m \rangle = \sum_{0 \leq j+k \leq m} a_{jk}(c+b)^j(c-b)^k.$$ 

Under this stratification, correspondingly we can define $\hat{\Omega}'$ and $\hat{\Lambda}$ from $\hat{\Omega}$. Then we have

$$\text{Pl-Holant}_{\hat{\Omega}} = \text{Pl-Holant}_{\hat{\Omega}} = \langle A, \hat{\Lambda}^\otimes 2^m \rangle = \sum_{0 \leq j+k \leq m} a_{jk}(\hat{c}+\hat{b})^j(\hat{c}-\hat{b})^k,$$

where the same set of values $a_{jk}$ appear. Let $\phi = \hat{c}+\hat{b}$ and $\psi = \hat{c}-\hat{b}$. If we can obtain the value of $p(\phi, \psi) = \sum_{0 \leq j+k \leq m} a_{jk}\phi^j\psi^k$ from oracle queries to Pl-Holant$_{\Omega_s}$ (for $s \geq 1$) in polynomial time, then
we will have proved

$$\text{Pl-Holant}(\neq 2 | \hat{f}) \leq_T \text{Pl-Holant}(\neq 2 | f).$$

Let $\alpha = c + b$ and $\beta = c - b$. Since $c^2 - b^2 \neq 0$ or 1, we have $\alpha \neq 0$, $\beta \neq 0$ and $\alpha \beta \neq 1$. Also, by assumption $|c + b| \neq |c - b|$, we have $|\alpha| \neq |\beta|$. Define $L = \{(j,k) \in \mathbb{Z}^2 \mid \alpha^j \beta^k = 1\}$. This is a sublattice of $\mathbb{Z}^2$. Every lattice has a basis. There are 3 cases depending on the rank of $L$.

- $L = \{(0,0)\}$. All $\alpha^j \beta^k$ are distinct. It is an interpolation reduction in full power. That is, we can interpolate $p(\phi, \psi)$ for any $\phi$ and $\psi$ in polynomial time. Let $\phi = 4$ and $\psi = 0$, that is $\hat{b} = 2$ and $\hat{c} = 2$, and hence $M(\hat{f}) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$. That is, $\hat{f}$ is non-singular redundant. By Theorem 2.21, Pl-Holant($\neq 2 | \hat{f}$) is $\#P$-hard, and hence Pl-Holant($\neq 2 | f$) is $\#P$-hard.

- $L$ contains two independent vectors $(j_1, k_1)$ and $(j_2, k_2)$ over $\mathbb{Q}$. Then the nonzero vectors $j_2(j_1, k_1) - j_1(j_2, k_2) = (0, j_2k_1 - j_1k_2)$ and $k_2(j_1, k_1) - k_1(j_2, k_2) = (k_2j_1 - k_1j_2, 0)$ are in $L$. Hence, both $\alpha$ and $\beta$ are roots of unity. This implies that $|\alpha| = |\beta| = 1$, a contradiction.

- $L = \{(ns, nt) \mid n \in \mathbb{Z}\}$, where $s, t \in \mathbb{Z}$ and $(s, t) \neq (0, 0)$. Without loss of generality, we may assume $t \geq 0$, and $s > 0$ when $t = 0$. Also, we have $s + t \neq 0$, otherwise $|\alpha| = |\beta|$, a contradiction. By Lemma 2.7, for any numbers $\phi$ and $\psi$ satisfying $\phi^s \psi^t = 1$, we can obtain $p(\phi, \psi)$ in polynomial time. Since $\phi = \hat{c} + \hat{b}$ and $\psi = \hat{c} - \hat{b}$, we have $\hat{b} = {\hat{c} + \hat{b}}$ and $\hat{c} = {\hat{c} + \hat{b}}$.

That is $M(\hat{f}) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & \phi^s \psi^t & \phi^t \psi^s & 0 \\ 0 & \phi^s \psi^t & \phi^t \psi^s & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$. There are three cases depending on the values of $s$ and $t$.

- $s \geq 0$ and $s + t \geq 2$. Consider the function $q(x) = (2 - x)^x - 1$. Since $s \geq 0$ and $t \geq 0$, $q(x)$ is a polynomial. Clearly, 1 is a root and 0 is not a root. If $q(x)$ has no other roots, then for some constant $\lambda \neq 0$,

$$q(x) = \lambda(x - 1)^{s+t} = (-1)^{s+t} \lambda((2 - x) - 1)^{s+t}.$$

(In fact by comparing leading coefficients, $\lambda = (-1)^s$.) Notice that $x^t | q(x) + 1$, while $x^t \not| \lambda(x - 1)^{s+t} + 1$ for $t \geq 2$. Also, notice that $(2 - x)^s | q(x) + 1$, while $(2 - x)^s \not| (-1)^{s+t} \lambda((2 - x) - 1)^{s+t}$ for $s \geq 2$. Hence, $t = s = 1$, which means $\alpha \beta = 1$. Contradiction. Therefore, $q(x)$ has a root $x_0$, with $x_0 \neq 1$ or 0. Let $\psi = x_0$ and $\phi = 2 - x_0$. Then $\phi^s \psi^t = 1$ and $M(\hat{f}) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & x_0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$. Note that $M_{x_0 x_0 x_1 x_4}(\hat{f}) = \begin{bmatrix} 0 & 0 & 0 & 1-x_0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1-x_0 & 0 & 0 & 0 \end{bmatrix}$. Since $1 - x_0 \neq 0$, $\hat{f}$ is non-singular redundant. By Theorem 2.21, Pl-Holant($\neq 2 | \hat{f}$) is $\#P$-hard and hence Pl-Holant($\neq 2 | f$) is $\#P$-hard.

- $s < 0$ and $t > 0$. (Note that $s < 0$ implies $t > 0$.) Consider the function $q(x) = x^t - (2 - x)^{-s}$. Since $t > 0$ and $-s > 0$, it is a polynomial. Clearly, 1 is a root, but neither 0 nor 2 is a root. Since $t + s \neq 0$, the highest order term of $q(x)$ is either $x^t$ or $-(-x)^{-s}$, which means the coefficient of the highest order term is $\pm 1$. While the constant term of $q(x)$ is $-2^{-s} \neq \pm 1$. Hence, $q(x)$ cannot be of the form $\lambda(x - 1)^{\max(t, -s)}$ for some constant $\lambda \neq 0$. Moreover, since $t + s \neq 0$, $\max(t, -s) \geq 2$, which means $q(x)$ has a root $x_0$, where $x_0 \neq 0, 1, 2$. Dividing by the nonzero term $(2 - x_0)^{-s}$ we have $(2 - x_0)^{s}x_0^t = 1$. Now we let $\psi = x_0$ and $\phi = 2 - x_0$, and we have Pl-Holant ($\neq 2 | f$) is $\#P$-hard by the same proof as above.

- $s \geq 0$ and $s + t = 1$. In this case, we have $(s, t) = (0, 1)$ or $(1, 0)$ since $t \geq 0$.

* $(s, t) = (1, 0)$. Let $\phi = 1$ and $\psi = \frac{1}{2}$. Then we have $\phi^s \psi^0 = 1$ and $M(\hat{f}) =$
\[
\begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & \frac{1}{2} & 0 & 0 \\
0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]
Let \( M(f') = 4M_{x_2x_3,x_1x_4}(f) = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 3 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix} \). Clearly, \( \text{Pl-Holant}(\neq 2 | f') \leq \text{Pl-Holant}(\neq 2 | f) \). For \( M(f') \), correspondingly we define \( \alpha' = 3 + 4 = 7 \) and \( \beta' = 3 - 4 = -1 \). Obviously, \( \alpha' \neq 0 \), \( \beta' \neq 0 \), \( \alpha'\beta' \neq 1 \), and \( |\alpha'| \neq |\beta'| \). Let \( L' = \{(j, k) \in \mathbb{Z}^2 : \alpha'j\beta'^k = 1\} \). Then we have \( L' = \{(ns', nt') : n \in \mathbb{Z} \} \), where \( s' = 0 \) and \( t' = 2 \). Therefore, \( s' \neq 0 \) and \( s' + t' \neq 2 \). As we have showed above, we have \( \text{Pl-Holant}(\neq 2 | f') \) is #P-hard, and hence \( \text{Pl-Holant}(\neq 2 | f) \) is #P-hard.

* \((s, t) = (0, 1)\). Let \( \phi = 3 \) and \( \psi = 1 \). Then we have \( \phi^0\psi^1 = 1 \) and \( M(\hat{f}) = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 1 & 2 & 0 \\
0 & 2 & 1 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix} \).

By Theorem 2.22, \( \text{Pl-Holant}(\neq 2 | \hat{f}) \) is #P-hard, and hence \( \text{Pl-Holant}(\neq 2 | f) \) is #P-hard.

**Case 2:** If \( c^2 - b^2 \neq 0 \) and \(|c + b| = |c - b|\), or \( c^2 - a^2 \neq 0 \) and \(|c + a| = |c - a|\). By rotational symmetry, we may assume \( c^2 - b^2 \neq 0 \) and \(|c + b| = |c - b|\). Normalizing \( f \) by assuming \( c = 1 \), we have \( M(f) = \begin{bmatrix}
0 & 0 & a & 0 \\
0 & 0 & 0 & b \\
0 & 1 & b & 0 \\
a & 0 & 0 & 0
\end{bmatrix} \), where \( 1^2 - b^2 \neq 0 \) and \( 1^2 - b^2 \neq a^2 \) due to \( f \notin \mathcal{M} \). Since \(|1 + b| = |1 - b|\), \( b \) is a pure imaginary number (as \( b \neq 0 \)).

Connect variables \( x_4, x_3 \) of a copy of signature \( f \) with variables \( x_1, x_2 \) of another copy of signature \( f \) both using \((\neq 2)\). We get a signature \( f_1 \) with the signature matrix

\[
M(f_1) = M_{x_1x_2,x_4x_3}(f)NM_{x_1x_2,x_4x_3}(f) = \begin{bmatrix}
0 & 0 & 0 & a^2 \\
0 & 2b & b^2 + 1 & 0 \\
0 & b^2 + 1 & 2b & 0 \\
a^2 & 0 & 0 & 0
\end{bmatrix}.
\]

- **a.** If \( c^2 - a^2 = 0 \), that is \( a^2 = 1 \), and then \( M(f_1) = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 2b & b^2 + 1 & 0 \\
0 & 0 & 2b & 0 \\
1 & 0 & 0 & 0
\end{bmatrix} \). Since \( b^2 < 0 \), we have \((b^2 + 1)^2 - (2b)^2 = b^2 - 1 > 1 = (a^2)^2 \), which means \( f_1 \notin \mathcal{M} \).
  - If \( b^2 = -1 \), then \( M(f_1) = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix} \). By Corollary 4.7, \( \text{Pl-Holant}(\neq 2 | f_1) \) is #P-hard, and hence \( \text{Pl-Holant}(\neq 2 | f) \) is #P-hard.
  - If \( b^2 = -2 \), then \( M(f_1) = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix} \). Connect two copies of \( f_1 \), and we have a signature \( f_2 \) with the signature matrix

\[
M(f_2) = M_{x_1x_2,x_4x_3}(f_1)NM_{x_1x_2,x_4x_3}(f_1) = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}.
\]

It is easy to check that \( f_2 \notin \mathcal{M} \), by Lemma 2.14. Then, \( f_2 \) belongs to Case 1. Therefore, \( \text{Pl-Holant}(\neq 2 | f_2) \) is #P-hard, and hence \( \text{Pl-Holant}(\neq 2 | f) \) is #P-hard.

- If \( b^2 \neq -1 \) or \( -2 \), then \( b^2 + 1 \neq \pm 1 \) due to \( b \neq 0 \), hence \( 1^2 - (b^2 + 1)^2 \neq 0 \). Also, since \( b^2 + 1 \) is a real number and \( b^2 + 1 \neq 0 \), we have \(|(b^2 + 1) + 1| \neq |(b^2 + 1) - 1| \). Then \( f_1 \), which is not in \( \mathcal{M} \) as shown above, has a signature matrix of the form \( \begin{bmatrix}
0 & 0 & 0 & a_1 \\
0 & b_1 & c_1 & 0 \\
a_1 & 0 & 0 & 0
\end{bmatrix} \).
where $a_1 = a^2 = 1$, $b_1 = 2b$, and $c_1 = b^2 + 1$, and $a_1 b_1 c_1 \neq 0$, $c_1^2 - a_1^2 \neq 0$ and $|c_1 + a_1| \neq |c_1 - a_1|$. That is, $f_1$ belongs to Case 1. Therefore, $\text{Pl-Holant}(\neq 2 \mid f_1)$ is #P-hard, and hence $\text{Pl-Holant}(\neq 2 \mid f)$ is #P-hard.

b. If $c^2 - a^2 \neq 0$ and $|c + a| = |c - a|$, i.e., $|1 + a| = |1 - a|$, then $a \neq 0$ is also a pure imaginary number. Connect variables $x_1, x_4$ of a copy of signature $f$ with variables $x_2, x_3$ of another copy of signature $f$. We get a signature $f_3$ with the signature matrix

$$M(f_3) = M_{x_2x_3,x_1x_4}(f) M_{x_2x_3,x_1x_4}(f) = \begin{bmatrix} 0 & 0 & 0 & b^2 \\ 0 & 2a & a^2 + 1 & 0 \\ 0 & a^2 + 1 & 2a & 0 \\ b^2 & 0 & 0 & 0 \end{bmatrix}.$$ 

Note that $f_3 \in \mathcal{M}$ implies $(a^2 - 1)^2 = (b^2)^2$. Since $f \notin \mathcal{M}$, $1 - a^2 \neq b^2$. Hence, $f_3 \in \mathcal{M}$ implies $a^2 - 1 = b^2$. Similarly, $f_1 \in \mathcal{M}$ implies $b^2 - 1 = a^2$. Clearly, $f_1$ and $f_3$ cannot both be in $\mathcal{M}$. Without loss of generality, we may assume $f_3 \notin \mathcal{M}$.

• If $a^2 \neq -1$, then there are two subcases.
  
  - $(a^2 + 1)^2 - (b^2)^2 = 0$. Since $a$ is a pure imaginary number, $|a^2 + 1 + 2a| = |a + 1|^2 = |a - 1|^2 = |a^2 + 1 - 2a|$. Then $f_3$ has the signature matrix of the form

$$\begin{bmatrix} 0 & 0 & 0 & a_3 \\ 0 & b_3 & c_3 & 0 \\ 0 & c_3 & b_3 & 0 \\ a_3 & 0 & 0 & 0 \end{bmatrix},$$

does not belong to Case 1. Therefore, $\text{Pl-Holant}(\neq 2 \mid f_3)$ is #P-hard, and hence $\text{Pl-Holant}(\neq 2 \mid f)$ is #P-hard.

- $(a^2 + 1)^2 - (b^2)^2 = 0$. Since $a^2 + 1$ and $b^2$ are both nonzero real numbers due to $a$ and $b$ are both pure imaginary numbers, we have $|a^2 + 1 + 2b| \neq |a^2 + 1 + b^2|$. Then $f_3$ has the signature matrix of the form

$$\begin{bmatrix} 0 & 0 & 0 & a_3 \\ 0 & b_3 & c_3 & 0 \\ 0 & c_3 & b_3 & 0 \\ a_3 & 0 & 0 & 0 \end{bmatrix},$$

and $|c_3 + a_3| \neq |c_3 - a_3|$. That is, $f_3$ belongs to Case 1. Therefore, $\text{Pl-Holant}(\neq 2 \mid f_3)$ is #P-hard, and hence $\text{Pl-Holant}(\neq 2 \mid f)$ is #P-hard.

• If $a^2 = -1$ and $b^2 = -2$, then $M(f_3) = \begin{bmatrix} 0 & 0 & 0 & b^2 \\ 0 & 2a & 0 & 0 \\ b^2 & 0 & 0 & 0 \end{bmatrix}$, where $|2a| = 2 \neq |b^2|$. By Corollary 4.7, $\text{Pl-Holant}(\neq 2 \mid f_3)$ is #P-hard, and hence $\text{Pl-Holant}(\neq 2 \mid f)$ is #P-hard.

• If $a^2 = -1$ and $b^2 = -2$, then $M(f_1) = \begin{bmatrix} 0 & 0 & 0 & \pm 2\sqrt{2a} \\ 0 & \pm 2\sqrt{2a} & -1 & 0 \\ 0 & 0 & 1 & \\ -1 & 0 & \pm 2\sqrt{2a} & 0 \end{bmatrix}$, which means $f_1$ is non-singular redundant. Therefore, we have $\text{Pl-Holant}(\neq 2 \mid f_1)$ is #P-hard, and hence $\text{Pl-Holant}(\neq 2 \mid f)$ is #P-hard.

c. If $c^2 - a^2 \neq 0$ and $|c + a| \neq |c - a|$. This is Case 1. Done.

Case 3: $c^2 - b^2 = 0$ and $c^2 - a^2 = 0$. If $c = b$ or $c = a$, then $f$ is non-singular redundant, and hence $\text{Pl-Holant}(\neq 2 \mid f)$ is #P-hard. Otherwise, $a = b = -c$. By normalization, we have

$$M(f) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \text{ and then } M(f_1) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & 0 \\ 0 & 2 & -2 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$$ 

Notice that $2^2 - 1^2 \neq 0$ and $|2 + 1| \neq |2 - 1|$. That is, $f_1$ belongs to Case 1. Therefore, $\text{Pl-Holant}(\neq 2 \mid f_1)$ is #P-hard, and hence $\text{Pl-Holant}(\neq 2 \mid f)$ is #P-hard.

Case 1 to Case 3 cover all cases for $(a, b, c)$: Suppose $(a, b, c)$ does not satisfy Case 3. Then
Lemma 6.2. Let \( f \) be a 4-ary signature with the signature matrix

\[
M(f) = \begin{bmatrix}
0 & 0 & 0 & a \\
0 & b & c & 0 \\
0 & z & y & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad \text{where} \quad abcxyz \neq 0.
\]

If \( by - cz = 0 \) or \( ax - cz = 0 \), then \( \text{Pl-Holant}(\neq_2| f) \) is \#P-hard.

**Proof.** By rotational symmetry, we assume \( by - cz = 0 \). By normalization, we assume \( b = 1 \), and then \( y = cz \). That is, \( M_{x_1x_2x_4x_3}(f) = \begin{bmatrix} 0 & 0 & 0 & a \\ 0 & 1 & c & 0 \\ 0 & z & y & 0 \\ x & 0 & 0 & 0 \end{bmatrix} \).

- If \( 1 + c \neq 0 \). Connecting the variables \( x_4 \) with \( x_3 \) of \( f \) using \( (\neq_2) \) we get a binary signature \( g_1 \), where

\[
g_1 = M_{x_1x_2x_4x_3}(f)(0, 1, 1, 0)^T = (0, 1 + c, (1 + c)z, 0)^T.
\]

Note that \( g_1(x_1, x_2) \) can be normalized as \( (0, z^{-1}, 1, 0)^T \). That is \( g(x_2, x_1) = (0, 1, z^{-1}, 0)^T \).

Modifying \( x_1 = 1 \) of \( f \) with \( z^{-1} \) scaling we get a signature \( f_1 \) with the signature matrix \[
\begin{bmatrix}
0 & 0 & 0 & a \\
0 & 0 & 0 & c \\
0 & 0 & 0 & 1 \\
x/z & 0 & 0 & 0
\end{bmatrix}
\]. Connecting the variable \( x_1 \) with \( x_2 \) of \( f_1 \) using \( (\neq_2) \) we get a binary signature \( g_2 \), where

\[
g_2 = ((0, 1, 1, 0)M_{x_1x_2x_4x_3}(f))^T = (0, 2, 2c, 0)^T,
\]

and \( g_2(x_1, x_2) \) can be normalized to \( g_2(x_2, x_1) = (0, 1, c^{-1}, 0)^T \). Modifying \( x_4 = 1 \) of \( f_1 \) with \( c^{-1} \) scaling we get a signature \( f_2 \) with the signature matrix \[
\begin{bmatrix}
0 & 0 & 0 & a/c \\
0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
x/z & 0 & 0 & 0
\end{bmatrix}
\]. It is non-singular redundant. By Lemma 2.21, we have \( \text{Pl-Holant}(\neq_2| f_2) \) is \#P-hard, and hence \( \text{Pl-Holant}(\neq_2| f) \) is \#P-hard.

- If \( 1 + z \neq 0 \), then connecting the variable \( x_1 \) with \( x_2 \) of \( f \) using \( (\neq_2) \) we get a binary signature \( g'_1 \), where

\[
g'_1 = ((0, 1, 1, 0)M_{x_1x_2x_4x_3})(f)^T = (0, 1 + z, (1 + z)c, 0)^T.
\]

\( g'_1(x_1, x_2) \) can be normalized to \( (0, c^{-1}, 1, 0)^T \). By the same analysis as in the case \( 1 + c \neq 0 \), we have \( \text{Pl-Holant}(\neq_2| f) \) is \#P-hard.

- Otherwise, \( 1 + c = 0 \) and \( 1 + z = 0 \), that is \( c = z = -1 \). Then \( M_{x_1x_2x_4x_3}(f) = \begin{bmatrix} 0 & 0 & 0 & a \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ x & 0 & 0 & 0 \end{bmatrix} \),

and \( M_{x_3x_4x_2x_1}(f) = \begin{bmatrix} 0 & 0 & 0 & z \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ a & 0 & 0 & 0 \end{bmatrix} \). Connecting variables \( x_4, x_3 \) of a copy of signature \( f \) with variables \( x_3, x_4 \) of another copy of signature \( f \), we get a signature \( f_3 \) with the signature matrix

\[
M(f_3) = M_{x_1x_2x_4x_3}(f)NM_{x_3x_4x_2x_1}(f) = \begin{bmatrix} 0 & 0 & 0 & ax \\ 0 & -2 & 2 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},
\]

Clearly, \( ax \neq 0 \) and so \( f_3 \notin \mathcal{M} \) by Lemma 2.14. By Lemma 6.1, \( \text{Pl-Holant}(\neq_2| f_3) \) is \#P-hard and hence \( \text{Pl-Holant}(\neq_2| f) \) is \#P-hard. \qed
In the following Lemmas 6.3, 6.4, 6.6 and Corollaries 6.5, 6.7, let \( f \) be a 4-ary signature with the signature matrix

\[
M(f) = \begin{bmatrix}
0 & 0 & 0 & a \\
0 & b & c & 0 \\
0 & z & y & 0 \\
x & 0 & 0 & 0
\end{bmatrix},
\]

where \( abxyz \neq 0 \), \( \det \begin{bmatrix} b & c \\ z & y \end{bmatrix} = by - cz \neq 0 \) and \( \det \begin{bmatrix} a & z \\ x & y \end{bmatrix} = ax - cz \neq 0 \). Moreover \( f \notin \mathcal{M} \), that is \( cz - by \neq ax \). These lemmas handle “generic” cases of this section and will culminate in Theorem 6.8, which is a classification for Case IV. It is here we will use Möbius transformations to handle interpolations where it is particularly difficult to get desired signatures of “infinite order”.

**Lemma 6.3.** Let \( g = (0,1,t,0)^T \) be a binary signature, where \( t \neq 0 \) is not a root of unity. Then \( \text{Pl-Holant}(\neq 2, f, g) \) is \#P-hard.

**Proof.** Let \( \mathcal{B} = \{g_1,g_2,g_3\} \) be a set of three binary signatures \( g_i = (0,1,t_i,0)^T \), for some \( t_i \in \mathbb{C} \). By Lemma 2.5, we have \( \text{Pl-Holant}(\neq 2, \mathcal{B}) \leq \text{Pl-Holant}(\neq 2, f, g) \). We will show that \( \text{Pl-Holant}(\neq 2, \mathcal{B}) \) is \#P-hard and it follows that \( \text{Pl-Holant}(\neq 2, f, g) \) is \#P-hard.

Modifying \( x_1 = 1 \) of \( f \) with \( t_i \) \((i = 1,2)\) scaling separately, we get two signatures \( f_{t_i} \) with the signature matrices \( M(f_{t_i}) = \begin{bmatrix} 0 & 0 & 0 & a \\ 0 & b & c & 0 \\ 0 & z & y & 0 \\ t_i & 0 & 0 & 0 \end{bmatrix}. \) Note that \( \det M_{\text{In}}(f_{t_i}) = t_i \det M_{\text{In}}(f) \) and \( \det M_{\text{Out}}(f_{t_i}) = t_i \det M_{\text{Out}}(f) \).

Connecting variables \( x_4, x_3 \) of \( f \) with variables \( x_1, x_2 \) of \( f_{t_1} \) both using \((\neq 2)\), we get a signature \( f_1 \) with the signature matrix

\[
M(f_1) = \begin{bmatrix}
0 & 0 & 0 & a_1 \\
0 & b_1 & c_1 & 0 \\
0 & z_1 & y_1 & 0 \\
x_1 & 0 & 0 & 0
\end{bmatrix} = M(f)NM(f_{t_1}) = \begin{bmatrix}
0 & 0 & 0 & a^2 \\
0 & t_1bz + bc & t_1by + c^2 & 0 \\
0 & t_1z^2 + yb & t_1yz + yc & 0 \\
t_1 & t_1x^2 & 0 & 0
\end{bmatrix}.
\]

We first show that there is a \( t_1 \neq 0 \) such that \( b_1y_1c_1z_1 \neq 0 \) and \((b_1z)(y_1c) - (c_1b)(z_1y) \neq 0 \). Consider the quadratic polynomial

\[
p(t) = (tbz + bc)(tzy + yc)cz - (tb + c^2)(t^2z^2 + yb)by.
\]

We have \( p(t_1) = (b_1z)(y_1c) - (c_1b)(z_1y) \). Notice that the coefficient of the quadratic term in \( p(t) \) is \( byz^2(cz - by) \). It is not equal to zero since \( byz^2 \neq 0 \) and \( cz - by \neq 0 \). That is, \( p(t) \) has degree 2, and hence it has at most two roots. Also we have the following three implications by the definitions of \( b_1, y_1, c_1, z_1 \): \( b_1y_1 = 0 \implies t_1 = -\frac{c}{z} \), \( c_1 = 0 \implies t_1 = -\frac{c^2}{by} \), and \( z_1 = 0 \implies t_1 = -\frac{by}{cz} \). Therefore we can choose such a \( t_1 \) that does not take these values \( 0, -\frac{c}{z}, -\frac{c^2}{by} \) and \( -\frac{by}{cz} \), and \( t_1 \) is not a root of \( p(t) \). Then, we have \( t_1 \neq 0, b_1y_1c_1z_1 \neq 0 \) and \((b_1z)(y_1c) - (c_1b)(z_1y) \neq 0 \).

Connecting variables \( x_4, x_3 \) of \( f_1 \) with variables \( x_1, x_2 \) of \( f_{t_2} \) both using \((\neq 2)\), we get a signature \( f_2 \) with the signature matrix

\[
M(f_2) = \begin{bmatrix}
0 & 0 & 0 & a_2 \\
0 & b_2 & c_2 & 0 \\
0 & z_2 & y_2 & 0 \\
x_2 & 0 & 0 & 0
\end{bmatrix} = M(f_1)NM(f_{t_2}) = \begin{bmatrix}
0 & 0 & 0 & a_1a \\
0 & t_2b_1z + c_1b & t_2b_1y + c_1c & 0 \\
0 & t_2z_1z + y_1b & t_2z_1y + y_1c & 0 \\
t_2 & t_2x_1x & 0 & 0
\end{bmatrix}.
\]

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Since \( b_1z \neq 0 \) and \( c_1b \neq 0 \), we can let \( t_2 = -\frac{c_1b}{b_1z} \) and \( t_2 \neq 0 \). Then \( b_2 = t_2b_1z + c_1b = 0 \). Since \((b_1z)(y_1c) - (c_1b)(z_1y) \neq 0\), we have \( y_2 = t_2z_1y + y_1c \neq 0 \). Notice that

\[
\det M_{\text{In}}(f_2) = \det M_{\text{In}}(f_1) \cdot (-1) \cdot \det M_{\text{In}}(f_{t_2}) = \det M_{\text{In}}(f) \cdot (-1) \cdot \det M_{\text{In}}(f_{t_1}) \cdot (-1) \cdot \det M_{\text{In}}(f_{t_2}) = t_1t_2 \det M_{\text{In}}(f)^3 \\
\neq 0.
\]

We have \( \det M_{\text{In}}(f_2) = b_2y_2 - c_2z_2 = -c_2z_2 \neq 0 \). Similarly, we have \( \det M_{\text{Out}}(f_2) = -a_2x_2 = t_1t_2 \det M_{\text{Out}}(f)^3 \neq 0 \). Therefore, \( M(f_2) \) is of the form \[
\begin{bmatrix}
0 & 0 & a_2 \\
0 & 0 & c_2 \\
z_2 & 0 & 0
\end{bmatrix}
\]
, where \( a_2x_2y_2c_2z_2 \neq 0 \). That is, \( f_2 \) is a signature in Case III. If \( f_2 \notin \mathcal{M} \), then Pl-Holant (\( \neq 2 \), \( f_2 \)) is \#P-hard by Theorem 5.2, and hence Pl-Holant (\( \neq 2 \), \( \{f\} \cup B \)) is \#P-hard.

Otherwise, \( f_2 \in \mathcal{M} \), which means \( \frac{\det M_{\text{In}}(f_2)}{\det M_{\text{Out}}(f_2)} = 1 \). Thus \( \frac{\det M_{\text{In}}(f)^3}{\det M_{\text{Out}}(f)^3} = 1 \). Since \( f \notin \mathcal{M} \), \( \frac{\det M_{\text{In}}(f)}{\det M_{\text{Out}}(f)} \neq 1 \), and hence \( \frac{\det M_{\text{In}}(f)^7}{\det M_{\text{Out}}(f)^7} \neq 1 \). Similar to the construction of \( f_1 \), we construct \( f_3 \). First, modify \( x_1 = 1 \) of \( f_1 \) with \( t_3 \) scaling. We get a signature \( f_{t_3} \) with the signature matrix \( M(f_{t_3}) = \begin{bmatrix}
0 & 0 & a_3 \\
b_3 & c_3 & 0 \\
z_3 & y_3 & 0
\end{bmatrix} \]
. Note that \( \det M_{\text{In}}(f_{t_3}) = t_3 \det M_{\text{In}}(f_1) \) and \( \det M_{\text{Out}}(f_{t_3}) = t_3 \det M_{\text{Out}}(f_1) \). Then connect variables \( x_4, x_3 \) of \( f_1 \) with variables \( x_1, x_2 \) of \( f_{t_3} \) both using \( \neq 2 \). We get a signature \( f_3 \) with the signature matrix

\[
M(f_3) = \begin{bmatrix}
0 & 0 & a_3 \\
b_3 & c_3 & 0 \\
z_3 & y_3 & 0
\end{bmatrix} = M(f_1)NM(f_{t_3}) = \begin{bmatrix}
0 & 0 & 0 \\
t_3b_1z_1 + b_1c_1 & t_3b_1y_1 + c_1^2 & 0 \\
t_3z_1^2 + y_1b_1 & t_3y_1z_1 + y_1c_1 & 0
\end{bmatrix}.
\]

Since \( c_1 \neq 0 \) and \( z_1 \neq 0 \), we can define \( t_3 = -\frac{a_1}{z_1} \) and \( t_3 \neq 0 \). Then \( b_3 = b_1(t_3z_1 + c_1) = 0 \) and \( y_3 = y_1(t_3z_1 + c_1) = 0 \). Notice that

\[
\det M_{\text{In}}(f_3) = \det M_{\text{In}}(f_1) \cdot (-1) \cdot \det M_{\text{In}}(f_{t_3}) = -\det M_{\text{In}}(f_1) \cdot t_3 \det M_{\text{In}}(f_1) = -t_3^2 \det M_{\text{In}}(f)^4 \neq 0.
\]

We have \( \det M_{\text{In}}(f_3) = -c_3z_3 \neq 0 \) and similarly, \( \det M_{\text{Out}}(f_3) = -a_3x_3 = -t_3^2 \det M_{\text{Out}}(f)^4 \neq 0 \). That is, \( M(f_3) \) is of the form \[
\begin{bmatrix}
0 & 0 & a_3 \\
0 & 0 & c_3 \\
x_3 & 0 & 0
\end{bmatrix}
\]
where \( a_3x_3c_3z_3 \neq 0 \).

Connect variables \( x_4, x_3 \) of \( f_2 \) with variables \( x_1, x_2 \) of \( f_3 \) both using \( \neq 2 \). We get a signature \( f_4 \) with the signature matrix

\[
M(f_4) = \begin{bmatrix}
0 & 0 & a_4 \\
b_4 & c_4 & 0 \\
z_4 & y_4 & 0
\end{bmatrix} = M(f_2)NM(f_3) = \begin{bmatrix}
0 & 0 & a_2a_3 \\
0 & 0 & c_2c_3 \\
z_2z_3 & y_2c_3 & 0
\end{bmatrix}.
\]

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Clearly, $f_4$ is a signature in Case III. Also, notice that
\[
\det M_{\text{in}}(f_4) = \det M_{\text{in}}(f_2) \cdot (-1) \cdot \det M_{\text{in}}(f_3) = t_1 t_2 \det M_{\text{in}}(f)^3 \cdot t_3 t_1^2 \det M_{\text{in}}(f)^4 = t_3 t_2 t_1^3 \det M_{\text{in}}(f)^7.
\]
and
\[
\det M_{\text{out}}(f_4) = t_3 t_2 t_1^3 \det M_{\text{out}}(f)^7.
\]
We have
\[
\frac{\det M_{\text{in}}(f_4)}{\det M_{\text{out}}(f_4)} = \frac{\det M_{\text{in}}(f)^7}{\det M_{\text{out}}(f)^7} \neq 1,
\]
which means $f_4 \notin \mathcal{M}$. By Theorem 5.2, Pl-Holant $(\neq 2 | f_4)$ is #P-hard, and hence Pl-Holant $(\neq 2 | \{f\} \cup \mathcal{B})$ is #P-hard.

**Lemma 6.4.** Let $g = (0, 1, t, 0)^T$ be a binary signature where $t$ is an $n$-th primitive root of unity, and $n \geq 5$. Then Pl-Holant $(\neq 2 | f, g)$ is #P-hard.

**Proof.** Note that $M_{x_1, x_2}(g) = [0 \ 1 \ t \ 0]$. Connect the variable $x_2$ of a copy of signature $g$ with the variable $x_1$ of another copy of signature $g$ using $(\neq 2)$. We get a signature $g_2$ with the signature matrix
\[
M_{x_1, x_2}(g_2) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ t & 0 & 1 & 0 \\ 0 & t & 0 & 0 \\ 0 & t^2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ t^2 & 0 \\ 0 & 0 \end{bmatrix}.
\]
That is, $g_2 = (0, 1, t^2, 0)^T$. Similarly, we can construct $g_i = (0, 1, t^i, 0)^T$ for $1 \leq i \leq 5$. Here, $g_1$ denotes $g$. Since the order $n \geq 5$, $g_i$ are distinct $(1 \leq i \leq 5)$.

Connect variables $x_4, x_3$ of signature $f$ with variables $x_1, x_2$ of $g_i$ for $1 \leq i \leq 5$ respectively. We get binary signatures $h_i$, where
\[
h_i = M_{x_1 x_2, x_4 x_3}(f) g_i = \begin{bmatrix} 0 & 0 & 0 & a \\ 0 & b & c & 0 \\ 0 & z & y & 0 \\ x & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ b + ct & 0 \\ 0 & z + yt & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ b + ct & 0 \\ 0 & z + yt & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
\]
Let $\varphi(z) = \frac{z + yb}{b + cz}$. Since $\det \begin{bmatrix} b & c \\ z & y \end{bmatrix} = by - cz \neq 0$, $\varphi(z)$ is a Möbius transformation of the extended complex plane $\hat{\mathbb{C}}$. We rewrite $h_i$ in the form of $(b + ct^i)(0, 1, \varphi(t^i), 0)^T$, with the understanding that if $b + ct^i = 0$, then $\varphi(t^i) = \infty$, and we define $(b + ct^i)(0, 1, \varphi(t^i), 0)^T$ to be $(0, 1, z + yt^i, 0)^T$. If there is a $t^i$ such that $\varphi(t^i)$ is not a root of unity, and $\varphi(t^i) \neq 0$ and $\varphi(t^i) \neq \infty$, by Lemma 6.3, we have Pl-Holant $(\neq 2 | f, h_i)$ is #P-hard, and hence Pl-Holant $(\neq 2 | f, g_1)$ is #P-hard. Otherwise, $\varphi(t^i)$ is 0, $\infty$ or a root of unity for $1 \leq i \leq 5$. Since $\varphi(z)$ is a bijection of $\hat{\mathbb{C}}$, there is at most one $t^i$ such that $\varphi(t^i) = 0$ and at most one $t^i$ such that $\varphi(t^i) = \infty$. That means, there are at least three $t^i$ such that $|\varphi(t^i)| = 1$. Since a Möbius transformation is determined by any 3 distinct points, mapping 3 distinct points from $S^1$ to $S^1$ implies that this $\varphi(z)$ maps $S^1$ homeomorphically onto $S^1$ (so in fact there is no $t^i$ such that $\varphi(t^i) = 0$ or $\infty$). Such a Möbius transformation has a special form:
\[
\mathcal{M}(\alpha, e^{i\theta}) = e^{i\theta} \frac{\beta + \alpha \gamma}{1 + \alpha \beta}, \text{ where } |\alpha| \neq 1. \text{ (It cannot be of the form } e^{i\theta}/\beta, \text{ since } b \neq 0.)
\]
By normalization in signature \( f \), we may assume \( b = 1 \). Compare the coefficients, we have \( c = \bar{\alpha}, y = e^{i\theta} \) and \( z = \alpha e^{i\theta} \). Here \( \alpha \neq 0 \) due to \( z \neq 0 \). Also, since \( M_{x_2x_1x_1x_4}(f) = \begin{bmatrix} 0 & 0 & 0 & y \\ 0 & a & z & 0 \\ 0 & c & x & 0 \\ b & 0 & 0 & 0 \end{bmatrix} \) and \( \det \begin{bmatrix} a & z \\ c & x \end{bmatrix} = ax - cz \neq 0 \), we have another Möbius transformation \( \psi(z) = \frac{c + x_3}{a + z_3} \). Plug in \( c = \bar{\alpha} \) and \( z = \alpha e^{i\theta} \), we have

\[
\psi(z) = \frac{\bar{\alpha} + x_3}{a + \alpha e^{i\theta}} = \frac{\bar{\alpha}}{1 + \alpha e^{i\theta}} = \frac{\bar{\alpha} + x_3}{a + \bar{\alpha} e^{i\theta}}.
\]

By the same proof for \( \varphi(z) \), we get Pl-Holant \( (\neq 2) f, g \) is \#P-hard, unless \( \psi(z) \) also maps \( S^1 \) to \( S^1 \). Hence, we can assume \( \psi(z) \) has the form \( \mathcal{M}(\beta, e^{i\theta}) = e^{i\theta}(\bar{\beta} + \beta) \), where \( |\beta| \neq 1 \). (It is clearly not of the form \( e^{i\theta}/\beta \).) Compare the coefficients, we have

\[
\begin{cases}
\alpha e^{i\theta}/a = \bar{\beta} \\
\bar{\alpha}/a = e^{i\theta} \beta.
\end{cases}
\]

Solving these equations, we get \( a = e^{i\theta} \alpha/\bar{\beta} \) and \( x = \bar{\alpha}/\beta \). Let \( \gamma = \alpha/\bar{\beta} \), and we have \( a = \gamma e^{i\theta} \) and \( x = \gamma \), where \( |\gamma| \neq |\alpha| \) since \( |\beta| \neq 1 \) and \( \gamma \neq 0 \) since \( x \neq 0 \). Then, we have signature matrices

\[
\begin{align*}
M_{x_2x_2x_4x_3}(f) &= \begin{bmatrix} 0 & 0 & 0 & \gamma e^{i\theta} \\ 0 & 0 & 0 & \gamma e^{i\theta} \\ 0 & \gamma e^{i\theta} & 0 & 0 \\ 0 & \gamma e^{i\theta} & 0 & 0 \end{bmatrix}, \\
M_{x_2x_3x_1x_4}(f) &= \begin{bmatrix} 0 & 0 & 0 & \gamma e^{i\theta} \\ 0 & 0 & 0 & \gamma e^{i\theta} \\ 0 & \gamma e^{i\theta} & 0 & 0 \\ 0 & \gamma e^{i\theta} & 0 & 0 \end{bmatrix}, \\
M_{x_3x_4x_2x_1}(f) &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
\end{align*}
\]

and \( M_{x_4x_1x_3x_2}(f) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \). Connect variables \( x_4, x_3 \) of a copy of signature \( f \) with variables \( x_3, x_4 \) of another copy of signature \( f \) using \( (\neq 2) \). We get a signature \( f_1 \) with the signature matrix

\[
M(f_1) = M_{x_2x_2x_4x_3}(f)NM_{x_3x_4x_2x_1}(f) = \begin{bmatrix} 0 & 0 & 0 & \gamma \gamma e^{i\theta} \\ 0 & (\alpha + \bar{\alpha}) e^{i\theta} & 1 + \alpha^2 & 0 \\ 0 & (1 + \alpha^2) e^{i2\theta} & (\alpha + \bar{\alpha}) e^{i\theta} & 0 \\ \gamma \gamma e^{i\theta} & 0 & 0 & 0 \end{bmatrix}.
\]

- If \( \alpha + \bar{\alpha} \neq 0 \), normalizing \( M_{x_1x_2x_4x_3}(f_1) \) by dividing by \( (\alpha + \bar{\alpha}) e^{i\theta} \), we have

\[
M(f_1) = \begin{bmatrix} 0 & 0 & 0 & \gamma \gamma (\alpha + \bar{\alpha}) \\ 0 & 1 & (1 + \alpha^2) e^{-i\theta} & 0 \\ 0 & (1 + \alpha^2) e^{i\theta} & 1 & 0 \\ \gamma \gamma (\alpha + \bar{\alpha}) & 0 & 0 & 0 \end{bmatrix}.
\]

Note that \( (1 + \alpha^2) e^{i\theta} \) and \( (1 + \alpha^2) e^{-i\theta} \) are conjugates. Let \( \delta = (1 + \alpha^2) e^{i\theta} \), and then \( \bar{\delta} = (1 + \alpha^2) e^{-i\theta} \). We have \( |\delta|^2 = \delta \bar{\delta} = (1 + \alpha^2)^2 (\alpha + \bar{\alpha})^2 \neq 1 \) due to \( \det M_{x_1x_2x_4x_3}(f_1) \neq 0 \), and \( \delta \neq 0 \).
due to $|\alpha| \neq 1$. Consider the inner matrix of $M(f_1)$, we have $M_{in}(f_1) = \begin{bmatrix} 1 & \frac{\delta}{\delta} \\ 0 & 1 \end{bmatrix}$. Notice that the two eigenvalues of $M_{in}(f_1)$ are $1 + |\delta|$ and $1 - |\delta|$, and obviously $\frac{1 - |\delta|}{1 + |\delta|} \neq 1$, which means there is no integer $n > 0$ and complex number $C$ such that $M_{in}^n(f_1) = CI$. Note that $\varphi_1(z) = \frac{\delta + z}{1 + \delta z}$ is a Möbius transformation of the form $\mathcal{M}(\delta, 1)$ mapping $S^1$ to $S^1$. Connect variables $x_1, x_3$ of signature $f_1$ with variables $x_1, x_2$ of signatures $g_i$. We get binary signatures $g_{(i, \varphi_1)}$, where

$$g_{(i, \varphi_1)} = M_{x_1x_2x_4x_3}(f_1)g_i = \begin{bmatrix} 0 & 0 & 0 & * \\ 0 & 1 & \delta & 0 \\ 0 & \delta & 1 & 0 \\ * & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ 1 + \tilde{t}^i \\ \delta + t^i \\ 0 \end{pmatrix} = (1 + \tilde{t}^i) \begin{pmatrix} 0 \\ 1 \\ \varphi_1(t^i) \\ 0 \end{pmatrix}. $$

Since $\varphi_1$ is a Möbius transformation mapping $S^1$ to $S^1$ and $|t^i| = 1$, we have $|\varphi_1(t^i)| = 1$, which means $1 + \delta \tilde{t}^i \neq 0$. Hence, $g_{(i, \varphi_1)}$ can be normalized as $(0, 1, \varphi_1(t^i), 1)^T$. Successively construct binary signatures $g_{(i, \varphi_1^n)}$ by connecting $f_1$ with $g_{(i, \varphi_1^{n-1})}$. We have

$$g_{(i, \varphi_1^n)} = M(f_1)g_{(i, \varphi_1^{n-1})} = M^n(f_1)g_i = C_{(i,n)}(0, 1, \varphi_1^n(t^i), 1)^T,$$

where $C_{(i,n)} = \prod_{0 \leq k \leq n-1} (1 + \delta \varphi_1^k(t^i))$. We know $C_{(i,n)} \neq 0$, because for any $k$, $1 + \delta \varphi_1^k(t^i) \neq 0$ due to $|\varphi_1^k(t^i)| = \left| \frac{\delta + \varphi_1^{k-1}(t^i)}{1 + \delta \varphi_1^{k-1}(t^i)} \right| = 1.$ Hence, $g_{(i, \varphi_1^n)}$ can be normalized as $(0, 1, \varphi_1^n(t^i), 1)^T$.

Notice that the nonzero entries $(1, \varphi_1^n(t^i))^T$ of $g_{(i, \varphi_1^n)}$ are completely decided by the inner matrix $M_{in}(f_1)$. That is

$$M_{in}^n(f_1) \begin{pmatrix} 1 \\ t^i \end{pmatrix} = C_{(i,n)} \left( \begin{pmatrix} 1 \\ \varphi_1^n(t^i) \end{pmatrix} \right).$$

If for each $i \in \{1, 2, 3\}$, there is some $n_i \geq 1$ such that $(1, \varphi_1^{n_i}(t^i))^T = (1, t^i)^T$, then $\varphi_1^{n_0}(t^i) = t^i$, where $n_0 = n_1n_2n_3$ for $1 \leq i \leq 3$, i.e., the Möbius transformation $\varphi_1^{n_0}$ fixes three distinct complex numbers $t, t^2, t^3$. So the Möbius transformation is the identity map, i.e., $\varphi_1^{n_0}(z) = z$ for all $z \in \mathbb{C}$. This implies that $M_{in}^{n_0}(f_1) = C \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ for some constant $C \neq 0$. This contradicts the fact that the ratio of the eigenvalues of $M_{in}$ is not a root of unity. Therefore, there is an $i$ such that $(1, \varphi_1^n(t^i))^T$ are all distinct for $n \in \mathbb{N}$. Then, we can realize polynomially many distinct binary signatures of the form $(0, 1, \varphi_1^n(t^i), 1)^T$. By Lemma 2.6, we have Pl-Holant$(\neq 2| f, g)$ is #P-hard.

- Otherwise $\alpha + \bar{\alpha} = 0$, which means $\alpha$ is a pure imaginary number. We already have $\alpha \neq 0$ due to $z \neq 0$. Also $|\alpha| \neq 1$ from the form of $\mathcal{M}(\alpha, e^{\theta})$. Let $\alpha = ri$, where $r \in \mathbb{R}$ and $|r| \neq 0$ or $1$. Connect variables $x_4, x_1$ of a copy of signature $f$ with variables $x_4, x_1$ of another copy.
of signature $f$, we get a signature $f_2$ with the signature matrix

$$M(f_2) = M_{x_2x_3x_4}(f)NM_{x_4x_1x_3x_2}(f)$$

$$= \begin{bmatrix}
0 & 0 & 0 & e^{i\theta}
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 & 1
\end{bmatrix}$$

$$= \begin{bmatrix}
0 & 0 & 0 & e^{i\theta}
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 & 1
\end{bmatrix}$$

- If $-\gamma + \bar{\gamma} \neq 0$, normalizing $M(f_2)$ by dividing the quantity $(-\gamma + \bar{\gamma})rie^{i\theta}$, we have

$$M_{\text{In}}(f_2) = \begin{bmatrix}
1 & (\gamma^2 - r^2)e^{-i\theta} & (\gamma^2 - r^2)e^{i\theta}
\end{bmatrix}
\begin{bmatrix}
-\gamma + \bar{\gamma} & 0 & rie^{i\theta}
\end{bmatrix}$$

Note that $(\gamma^2 - r^2)e^{i\theta}$ and $(\gamma^2 - r^2)e^{-i\theta}$ are conjugates. Let $\zeta = \frac{(\gamma^2 - r^2)e^{-i\theta}}{(-\gamma + \bar{\gamma})ri}$, and then $|\zeta| \neq 1$ due to $\det M_{\text{In}}(f_2) \neq 0$, and $\zeta \neq 1$ due to $|\gamma| \neq |\alpha| = |r|$ (as $|\beta| \neq 1$). With the same analysis as for $M_{\text{In}}(f_1)$ in the case $\alpha + \bar{\alpha} \neq 0$, the ratio of the two eigenvalues of $M_{\text{In}}(f_2) = \begin{bmatrix}
1 & \zeta & \gamma
\end{bmatrix}$ is also not equal to 1, which means there is no integer $n$ and complex number $C$ such that $M_{\text{In}}^n(f_2) = CI$. Notice that $\varphi_2(3') = \frac{\zeta + \bar{\zeta}}{1 + \zeta \bar{\zeta}}$ is also a Möbius transformation of the form $M(\zeta, 1)$ mapping $S^1$ to $S^1$. Similarly, we can realize polynomially many distinct binary signatures, and hence $\text{Pl-Holant}(\neq 2, f, g)$ is $\#P$-hard.

- Otherwise, $-\gamma + \bar{\gamma} = 0$, which means $\gamma$ is a real number. We have $\gamma \in \mathbb{R}, |\gamma| \neq 0$ or $|r|$. Connect variables $x_4, x_3$ of a copy of signature $f$ with variables $x_1, x_2$ of another copy of signature $f$, we get a signature $f'$ with the signature matrix

$$M(f') = M_{x_1x_2x_4x_3}(f)NM_{x_1x_2x_4x_3}(f)$$

$$= \begin{bmatrix}
0 & 0 & 0 & \gamma e^{i\theta}
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 & 1
\end{bmatrix}$$

$$= \begin{bmatrix}
0 & 0 & 0 & \gamma e^{i\theta}
\end{bmatrix}$$

* If $e^{i\theta} = 1$, then $M(f) = \begin{bmatrix}
0 & 0 & 0 & \gamma
\end{bmatrix}$, and $M_{\text{In}}(f) = \begin{bmatrix}
1 & \alpha
\end{bmatrix}$. Since $|\alpha| \neq 1$, same as the analysis of $M_{\text{In}}(f_1)$, we can realize polynomially many binary signatures, and hence $\text{Pl-Holant}(\neq 2, f, g)$ is $\#P$-hard.

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* Otherwise $e^{i\theta} \neq 1$, normalizing $M(f')$ by dividing by $(e^{i\theta} - 1)ri$, we have

$$M(f') = \begin{bmatrix}
0 & 0 & 0 & \gamma^2 e^{i\theta}
0 & 1 & e^{i\theta} - r^2 & \frac{\gamma^2 e^{i\theta}}{(e^{i\theta} - 1)ri}
0 & \frac{1 - e^{i\theta}r^2}{(e^{i\theta} - 1)ri} & e^{i\theta} & 0
\gamma^2 & 0 & 0 & 0
\end{bmatrix}. $$

Note that $\frac{1 - e^{i\theta}r^2}{(e^{i\theta} - 1)ri}$ and $\frac{e^{i\theta} - r^2}{(e^{i\theta} - 1)ri}$ are conjugates. Let $\alpha' = \frac{1 - e^{i\theta}r^2}{(e^{i\theta} - 1)ri}$ and $\gamma' = \frac{\gamma^2 e^{i\theta}}{(e^{i\theta} - 1)ri}$. Then

$$M(f') = \begin{bmatrix}
0 & 0 & 0 & \gamma' e^{i\theta}
0 & 1 & \alpha' & 0
0 & \alpha' e^{i\theta} & e^{i\theta} & 0
\gamma' & 0 & 0 & 0
\end{bmatrix}. $$

Notice that $M(f')$ and $M(f)$ have the same form. Similar to the construction of $f_2$, we can construct a signature $f'_2$ using $f'$ instead of $f$. Since $-\gamma' + \bar{\gamma'} = -\frac{\gamma^2 e^{i\theta}}{(e^{i\theta} - 1)ri} + \frac{\gamma^2}{(e^{i\theta} - 1)ri} = -\frac{\gamma^2}{ri} \neq 0$, by the analysis of $f_2$, we can still realize polynomially many binary signatures and hence Pl-Holant($\neq 2 | f, g$) is #P-hard. \qed

**Remark:** The order $n \geq 5$ promises that there are at least three points mapped to points on $S^1$, since at most one point can be mapped to 0 and at most one can be mapped to $\infty$. When the order $n$ is 3 or 4, if no point is mapped to 0 or $\infty$, then there are still at least three points mapped to points on $S^1$. So, we have the following corollary.

**Corollary 6.5.** Let $g = (0, 1, t, 0)^T$ be a binary signature where $t$ is an $n$-th primitive root of unity, and $n = 3$ or 4. Let $g_m$ denote $(0, 1, t^m, 0)^T$. For any cyclic permutation $(i, j, k, \ell)$ of $(1, 2, 3, 4)$, if there is no $g_m$ such that $M_{x_i, x_j, x_k, x_\ell}(f)g_m = d_1(0, 1, 0, 0)^T$ or $d_2(0, 0, 1, 0)^T$, where $d_1, d_2 \in \mathbb{C}$, then Pl-Holant($\neq 2 | f, g$) is #P-hard.

We normalize $f$ by setting $b = 1$ in (6.6).

**Lemma 6.6.** Let $g = (0, 1, 0, 0)^T$ be a binary signature. Then Pl-Holant($\neq 2 | f, g$) is #P-hard.

**Proof.** Connecting variables $x_4$, $x_3$ of the signature $f$ with variables $x_2$ and $x_1$ of $g$ both using $(\neq 2)$ we get a binary signature $g_1$, where

$$g_1 = M_{x_1, x_2, x_3}(f)(0, 1, 0, 0)^T = (0, 1, z, 0)^T.$$

$g_1(x_1, x_2)$ can be normalized to $(0, z^{-1}, 1, 0)^T$ since $z \neq 0$. So we have $g_1(x_2, x_1) = (0, 1, z^{-1}, 0)$. Then, modifying $x_1 = 1$ of $f$ with $z^{-1}$ scaling, we get a signature $f_1$ with the signature matrix

$$M(f_1) = \begin{bmatrix}
0 & 0 & 0 & a \\
0 & 1 & c & 0 \\
0 & 1 & y & 0 \\
x/z & 0 & 0 & 0
\end{bmatrix}. $$

We denote it by $\begin{bmatrix}
0 & 0 & 0 & a \\
0 & 0 & 1 & c \\
0 & 1 & y & 0 \\
x & 1 & 0 & 0
\end{bmatrix}$, where $x_1 y_1 \neq 0$.  

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- If \( c = 0 \), connecting variables \( x_4, x_3 \) of \( f_1 \) with variables \( x_1, x_2 \) of \( g \) both using \((\neq 2)\) we get a binary signature \( h_1 \), where
  \[
h_1 = M_{x_1x_2,x_4x_3}(f_1)(0,0,1,0)^T = (0,1,y_1,0)^T.
  \]
  Also, connecting the variable \( x_4 \) with \( x_3 \) of \( f_1 \) using \((\neq 2)\) we get a binary signature \( h_2 \), where
  \[
h_2 = M_{x_1x_2,x_4x_3}(f_1)(0,1,1,0)^T = (0,2,y_1,0)^T.
  \]
  \( h_2 \) can be normalized to \((0,1,\frac{y_1}{2},0)^T \). Clearly, \(|y_1| \neq \frac{|y_2|}{2}\), so they cannot both be roots of unity. By Lemma 6.3, Pl-Holant \((\neq 2| f, h_1, h_2)\) is \#P-hard, and we conclude that Pl-Holant \((\neq 2| f, g)\) is \#P-hard.

- Otherwise \( c \neq 0 \). Connecting variables \( x_2, x_1 \) of \( g \) with variables \( x_1, x_2 \) of \( f \) both using \((\neq 2)\) we get a binary signature \( g_2 \), where
  \[
g_2 = ((0,1,0,0)M_{x_1x_2,x_4x_3}(f_1))^T = (0,1,c,0)^T.
  \]
  which can be normalized to \( g_2(x_2,x_1) = (0,1,c^{-1},0)^T \). Then, modifying \( x_4 = 1 \) of \( f_1 \) with \( c^{-1} \) scaling, we get a signature \( f_2 \) with the signature matrix \( M(f_2) = \begin{bmatrix} 0 & 0 & 0 & a_2 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & y_2 & 0 \\ x_2 & 0 & 0 & 0 \end{bmatrix} \) which we denote by \( \begin{bmatrix} 0 & 0 & 0 & a_2 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & y_2 & 0 \\ x_2 & 0 & 0 & 0 \end{bmatrix} \), where \( a_2x_2y_2 \neq 0 \). Notice that \( M_{x_2x_3,x_1x_4}(f_2) = \begin{bmatrix} 0 & 0 & 0 & y_2 \\ 0 & a_2 & 1 & 0 \\ 0 & 1 & x_2 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \). Connect variables \( x_1, x_4 \) of signature \( f_2 \) with variables \( x_2, x_1 \) of \( g \) both using \((\neq 2)\). We get a binary signature \( h_3 \), where
  \[
h_3 = M_{x_2x_3,x_1x_4}(f_2)(0,1,0,0)^T = (0,a_2,1,0)^T.
  \]
  \( h_3 \) can be normalized as \((0,1,\frac{1}{a_2},0)^T \). Also connect variables \( x_1, x_4 \) of signature \( f_2 \) with variables \( x_1, x_2 \) of \( g \) both using \((\neq 2)\). We get a binary signature \( h_4 \), where
  \[
h_4 = M_{x_2x_3,x_1x_4}(f_2)(0,0,1,0)^T = (0,1,x_2,0)^T.
  \]
  If \(|a_2| \neq 1 \) or \(|x_2| \neq 1 \), then \( a_2 \) or \( x_2 \) is not a root of unity. By Lemma 6.3, Pl-Holant \((\neq 2| f, h_3, h_4)\) is \#P-hard, and hence Pl-Holant \((\neq 2| f, g)\) is \#P-hard. Otherwise, \(|a_2| = |x_2| = 1 \). Same as the construction of \( h_1 \) and \( h_2 \), construct binary signatures \( h'_1 \) and \( h'_2 \) using \( f_2 \) instead of \( f_1 \). We get
  \[
h'_1 = M_{x_1x_2,x_4x_3}(f_2)(0,0,1,0)^T = (0,1,y_2,0)^T,
  \]
  and
  \[
h'_2 = M_{x_1x_2,x_4x_3}(f_2)(0,1,1,0)^T = (0,2,1+y_2,0)^T.
  \]
  Note that \( h'_2 \) can be normalized as \((0,1,\frac{1+y_2}{2},0)^T \).
  - If \( y_2 \) is not a root of unity, then by Lemma 6.3, Pl-Holant \((\neq 2| f, h'_1)\) is \#P-hard, and hence Pl-Holant \((\neq 2| f, g)\) is \#P-hard.
  - If \( y_2 \) is an \( n \)-th primitive root of unity and \( n \geq 5 \), then by Lemma 6.4, Pl-Holant \((\neq 2| f, h'_1)\) is \#P-hard, and hence Pl-Holant \((\neq 2| f, g)\) is \#P-hard.
  - If \( y_2 = \frac{-1+\sqrt{3}}{2} \) or \( \pm i \), then \( 0 < |\frac{1+y_2}{2}| < 1 \), which means it is not zero neither a root of unity. By Lemma 6.3, Pl-Holant \((\neq 2| f, h'_2)\) is \#P-hard, and hence Pl-Holant \((\neq 2| f, g)\) is \#P-hard.

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- If \( y_2 = 1 \), then \( f_2 \) is non-singular redundant and hence \( \text{Pl-Holant}(\neq 2 | f, g) \) is \#P-hard.
- If \( y_2 = -1 \). Connect two copies of \( f_2 \), we get a signature \( f_3 \) with the signature matrix

\[
M(f_3) = M_{x_1x_2x_4x_3}(f_2)NM_{x_1x_2x_4x_3}(f_2) = \begin{bmatrix}
0 & 0 & 0 & a^2_2 \\
0 & 2 & 0 & 0 \\
0 & 0 & -2 & 0 \\
2^2 & 0 & 0 & 0
\end{bmatrix}.
\]

Since \( |a_2| = |x_2| = 1, |a^2_2x_2^2| = 1 \neq 4 \). Therefore, applying Corollary 4.7 to \( \{a^2_2, 2, x_2^2, -2\} \), we get \( \text{Pl-Holant}(\neq 2 | f_3) \) is \#P-hard, and hence \( \text{Pl-Holant}(\neq 2 | f, g) \) is \#P-hard. \( \Box \)

Combining Lemma 6.4, Corollary 6.5 and Lemma 6.6, we have the following corollary.

**Corollary 6.7.** Let \( g = (0, 1, t, 0)^T \) be a binary signature where \( t \) is an \( n \)-th primitive root of unity, and \( n \geq 3 \). Then \( \text{Pl-Holant}(\neq 2 | f, g) \) is \#P-hard.

Now, we are able to prove the following theorem for Case IV.

**Theorem 6.8.** Let \( f \) be a 4-ary signature with the signature matrix

\[
M(f) = \begin{bmatrix}
0 & 0 & 0 & a \\
0 & b & c & 0 \\
0 & z & y & 0 \\
x & 0 & 0 & 0
\end{bmatrix},
\]

where \( abxyz \neq 0 \). \( \text{Pl-Holant}(\neq 2 | f) \) is \#P-hard unless \( f \in \mathcal{M} \), in which case, \( \text{Pl-Holant}(\neq 2 | f) \) is tractable.

**Proof.** Tractability follows by 2.15.

Now suppose \( f \notin \mathcal{M} \). Connect the variable \( x_4 \) with \( x_3 \) of \( f \) using \( (\neq 2) \), and we get a binary signature \( g_1 \), where

\[
g_1 = M_{x_1x_2x_4x_3}(0, 1, 1, 0)^T = (0, b + c, z + y, 0)^T.
\]

Connect the variable \( x_1 \) with \( x_2 \) of \( f \) using \( (\neq 2) \), and we get a binary signature \( g_2 \), where

\[
g_2 = ((0, 1, 1, 0)M_{x_1x_2x_4x_3})^T = (0, b + z, c + y, 0)^T.
\]

- If one of \( g_1 \) and \( g_2 \) is of the form \((0, 0, 0, 0)^T\), then \( by = (-c)(-z) = cz \). That is \( by - cz = 0 \). Here \( c \neq 0 \) due to \( by \neq 0 \). By Lemma 6.2, \( \text{Pl-Holant}(\neq 2 | f) \) is \#P-hard.
- If one of \( g_1 \) and \( g_2 \) can be normalized as \((0, 1, 0, 0)\) or \((0, 0, 1, 0)\). By Lemma 6.6, \( \text{Pl-Holant}(\neq 2 | f) \) is \#P-hard.
- If one of \( g_1 \) and \( g_2 \) can be normalized as \((0, 1, t, 0)^T\), where \( t \neq 0 \) is not a root of unity, then by Lemma 6.3, \( \text{Pl-Holant}(\neq 2 | f) \) is \#P-hard.
- If one of \( g_1 \) and \( g_2 \) can be normalized as \((0, 1, t, 0)^T\), where \( t \) is an \( n \)-th primitive root of unity and \( n \geq 3 \), then by Corollary 6.7, \( \text{Pl-Holant}(\neq 2 | f) \) is \#P-hard.
- Otherwise, \( g_1 \) and \( g_2 \) do not belong to those cases above, which means both \( g_1 \) and \( g_2 \) both can be normalized as \((0, 1, \epsilon_1, 0)\) and \((0, 1, \epsilon_2, 0)\), where \( \epsilon_1 = \pm 1 \) and \( \epsilon_2 = \pm 1 \). That is, \( b + c = \epsilon_1(z + y) \neq 0 \) and \( b + z = \epsilon_2(c + y) \neq 0 \).
  - If \( b + c = z + y \) and \( b + z = c + y \), then \( b = y \) and \( c = z \). This case will be proved below.
  - If \( b + c = -(z + y) \) and \( b + z = c + y \), then \( b + z = c + y = 0 \), so \( g_2 = (0, 0, 0, 0)^T \), a contradiction.

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7 Proof of the Main Theorem

Now we are ready to prove the main theorem, Theorem 3.1.

**Proof of Tractability:**

- If \( f \) satisfies condition 1 or 2, then by Theorem 2.24, Holant(\( \neq 2 \mid f \)) is tractable without the planarity restriction. Obviously, Pl-Holant(\( \neq 2 \mid f \)) is tractable.
- If \( f \) satisfies condition 3, then by Theorem 2.15, Pl-Holant(\( \neq 2 \mid f \)) is tractable.
- If \( f \) satisfies condition 4, then by Theorem 4.6, Pl-Holant(\( \neq 2 \mid f \)) is tractable.

**Proof of Hardness:**

Since \( f \) does not satisfy condition 2, \( f \) does not belong to Case I. Therefore it belongs to Cases II, III, or IV.

- Suppose \( f \) belongs to Case II.
  - If an outer pair is a zero pair, since \( f \) does not satisfy condition 1 or condition 3, then by Theorem 4.3, Pl-Holant(\( \neq 2 \mid f \)) is \#P-hard.
  - If the inner pair is a zero pair and no outer pair is zero, since \( f \) does not satisfy condition 4, then by Theorem 4.6, Pl-Holant(\( \neq 2 \mid f \)) is \#P-hard.

- Suppose \( f \) belongs to Case III. Since \( f \) does not satisfy condition 3, then by Theorem 5.2, Pl-Holant(\( \neq 2 \mid f \)) is \#P-hard.

- Suppose \( f \) belongs to Case IV. Since \( f \) does not satisfy condition 3, then by Theorem 6.8, Pl-Holant(\( \neq 2 \mid f \)) is \#P-hard.

\( \square \)
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