Shock waves in superconducting cosmic strings: instability to extrinsic perturbations

Ernst Trojan and George V. Vlasov
Moscow Institute of Physics and Technology
PO Box 3, Moscow, 125080, Russia
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Abstract

Superconducting cosmic string may admit shock-like discontinuities of the current when the latter is spacelike ("magnetic" regime), while no shock at timelike current ("electric" regime) was discovered in numerical simulations. We find that the necessary and enough conditions for existence of stable shocks and show that the shock can be unstable in the presence of infinitesimal extrinsic perturbations of the string worldsheet. The shocks in the "magnetic" regime are not vulnerable to this instability but the shocks in the "electric" regime do not survive.

1 Introduction

Cosmic strings are 2-dimensional topological defects that are believed to be formed at a phase transition in the early universe [1, 2]. When the core of the string is small with respect to its radius, the behavior is determined by the Goto-Nambu action

\[ S = \int \Lambda \sqrt{\det h_{ab}} d\sigma^1 d\sigma^2 \]  \hspace{1cm} (1)

with the surface metric \( h_{ab} \) and surface Lagrangian \( \Lambda \). The cosmic strings can be endowed with internal structure [3], and their Lagrangian is dependent
on the magnitude of the current $\chi$, giving $\Lambda = -m^2$ in the limit $\chi \to 0$. Such superconducting strings admit a large variety of stable loop solutions (or, vortons) and their intercommutation may frequently take place. The problem of vorton stability and evolution of superconducting string networks attract the interest of researchers because such string loops can accumulate significant part of the universe mass and the expected dominant effect of their reconnection is particle radiation (when some particles are expelled away).

The equations of motion admit solutions in the form of infinitesimal perturbations of two types $^{[4]}$: extrinsic perturbations of the world sheet which concern the string geometry, and sound type longitudinal perturbations within the world sheet which concern the current $\chi$. Nonlinear effects become dominant rapidly in longitudinal perturbations of finite amplitude $\Delta \chi = \chi_+ - \chi_- \neq 0$ and they may either disappear or form stable discontinuities (magnitude $\chi$ becomes discontinuous) similar to shock waves propagating along the string. Such strings loops may tend to fold on themselves and make contact points of self-intersection so that it will be energetically favored to some of the trapped particles to move out of the string, thus, generating emission, associated with possible visible consequences $^{[1, 2, 5, 6]}$.

First, the shocks they were predicted $^{[7]}$ at spacelike currents $\chi > 0$ ("magnetic" regime) and no shock was expected at timelike currents ("electric" regime) that was referred to the evolutionary condition $^{[7]}$. However, the latter imposes no restriction except the growth of the current $\chi_+ - \chi_- > 0$ that can be satisfied in the "electric" regime as well $^{[8]}$. Nevertheless, according to numerical simulations $^{[9, 10]}$, the dynamical evolution of a bosonic current carrier can develop shocks in the "magnetic" regime and no discontinuous solution was discovered in the "electric" regime. As a matter of fact, the "electric" shock waves are forbidden, and the problem of shock stability at timelike currents remains unresolved.

In the present paper we analyze the phenomenon which is responsible for the absence of shocks in the "electric" regime. It is the shock instability caused by perturbations of the string worldsheet.

2 Intrinsic and extrinsic perturbations

The dynamics of current-carrying cosmos strings is determined by "extrinsic" equations of motion $^{[4]}$

$$\perp_\sigma (\perp_\mu u^\nu \nabla_\nu u^\sigma - T v^\nu \nabla_\nu v^\sigma) = 0 \quad \perp_\sigma (u^\nu \nabla_\nu v^\sigma - v^\nu \nabla_\nu u^\sigma) = 0$$

(2)
and "intrinsic" equations of motion
\[ \eta_{\mu}^{\nu} \nabla_{\nu} (\mu v^\mu) = 0 \quad \eta_{\mu}^{\nu} \nabla_{\nu} (nu^\mu) = 0 \]

(3)

where projective tensors are
\[ \perp_{\sigma}^{\mu} = g_{\sigma}^{\mu} - \eta_{\sigma}^{\mu} \quad \eta_{\mu \nu} = v^\mu v^\nu - u^\mu u^\nu \]

(4)

and \( u^\mu \) and \( v^\mu \) are unit vectors
\[ u^\mu u^\mu = -1 = -v^\mu v^\mu \quad u^\mu v^\mu = 0 \]

(5)

Parameters \( U, T, \mu, n \) are determined by the EOS, and
\[ \mu^2 = \chi \quad n^2 = K^2 \chi \quad \chi > 0 \]

(6)

at spacelike currents, while
\[ \mu^2 = -K^2 \chi \quad n^2 = -\chi \quad \chi > 0 \]

(7)

at timelike currents, where function \( K(\chi) \) is ultimately defined as
\[ K = -\left( \frac{2d\Lambda}{d\chi} \right)^{-1} \]

(8)

The extrinsic equations of motion (2) admit solutions in the form of infinitesimal perturbations of the worldsheet ("wiggles"), which propagate at velocity \([4]\)
\[ c^{1,2} = \frac{T}{U} = \left( \frac{\Lambda + \chi/K}{\Lambda} \right)^{\text{sign}\chi} \]

(9)

The 'intrinsic' equations of motion (3) admit infinitesimal longitudinal perturbations ("woggles" or sound waves) which propagate within the worldsheet at velocity
\[ c^2 = -\frac{dT}{dU} = \frac{n}{\mu} \frac{d\mu}{dn} = \left( 1 + 2 \frac{K'\chi}{K} \right)^{-\text{sign}\chi} \]

(10)

where
\[ K' = \frac{dK}{d\chi} \]

(11)
The Lagrangian $\Lambda$ is obtained by numerical integration over the coordinates orthogonal to the worldsheet. However, there are derived a few explicit analytical models. The linear model [13, 14, 15]

$$\Lambda = -m^2 - \frac{\chi}{2}$$

is applied to the cosmic strings, carrying fermionic currents. The following models are applied to the cosmic strings which carry bosonic currents:

$$\Lambda = -m\sqrt{m^2 + \chi} \quad \text{Ref. [16]}$$

$$\Lambda = -m^2 - \frac{\chi}{2} \left(1 - \frac{\chi}{m^2_*}\right) \quad \text{Ref. [12]}$$

$$\Lambda = -m^2 - \frac{\chi}{2} \left(1 + \frac{\chi}{m^2_*}\right)^{-1} \quad \text{Ref. [11]}$$

$$\Lambda = -m^2 - m^2_* \ln \left(1 + \frac{\chi}{m^2_*}\right) \quad \text{Ref. [11, 17]}$$

The linear EOS [12], is not enough to discover discontinuities (shock waves) because the sound speed is constant and equal to the speed of light

$$c = 1$$

However, shock waves are observed when more complicated bosonic EOS [13] is taken.

A typical behavior of $c^\perp$ vs $\chi$ is shown in Fig. [1]. The speed $c^\perp(\chi)$ increases at $\chi < 0$, and decreases at $\chi > 0$. In the "transonic" model (13) the speed of transversal perturbations coincides with the sound speed

$$c^\perp = c$$

while for all other models (14)-(16) it is always [11, 12, 17]:

$$c^\perp > c$$

3 Shock waves

The sound perturbations [10] are involved in the "intrinsic" equations of motion [3] and these perturbations can transform into discontinuities similar
to relativistic shock waves \[7, 9, 10\]. The shock-wave solution must satisfy
the stability criterion or evolutionary condition \[18, 19\], that for strings is
formulated as \[7\]:
\[
    \mathcal{w} - \mathcal{c} + \mathcal{w} < \mathcal{c} - \mathcal{w} + (20)
\]
where labels ”−” and ”+” correspond to the state before and behind the
shock, respectively. This stability criterion \(20\) results in inequality \(8\)
\[
    \mathcal{w} < \mathcal{c} - \mathcal{w} + (21)
\]
for all string models \(13)-(16)\) in the ”magnetic” regime. In the ”electric”
regime another inequality takes place
\[
    \mathcal{c} + > \mathcal{w} + > \mathcal{w} - > \mathcal{c} - (22)
\]
The evolutionary condition \(20\) also results in the growth of the current
\(8\)
\[
    \mathcal{\chi} + > \mathcal{\chi} - (23)
\]
However, the magnitude of shock wave in the ”electric” regime (at \(\mathcal{\chi} - < 0\))
cannot exceed \(|\mathcal{\chi} -|\), so that a transition to the ”magnetic” regime \(\mathcal{\chi} - < 0 \rightarrow
\mathcal{\chi} + > 0\) is impossible. As for the magnitude of ”magnetic” shock wave (at
\(\mathcal{\chi} - > 0\)), it can be arbitrary.

Our preliminary analysis \[7\] was based on intuitive statement: if arbi-
trary ”electric” shock wave is not admitted, then, no ”electric” shock is
possible. However, the inequality \(23\) does not disqualify small-amplitude
shock waves in the ”electric” regime. Nevertheless, no shock wave was dis-
covered in numerical simulations in the ”electric” regime \[9, 10\]. The regime
of timelike currents \(\mathcal{\chi} < 0\) can admit only smooth solution instead of shock-
wave discontinuity. Hence, there is another physical mechanism which is
making ”electric” shocks impossible. The puzzle is hidden in the shock wave
instability caused by perturbations of the string worldsheet.

### 4 Instability to perturbations of worldsheet

What happens with the string geometry when the intrinsic equations of mo-
tion \[3\] admit discontinuous solution \(\mathcal{\chi} + \neq \mathcal{\chi} -\)? It is clear that the speed
of transversal perturbations \(9\) is subject to change, hence, \(c^\perp_+ \neq c^\perp_-\). The
growth of current \(23\), in the view of Fig. \[1\] will always result in
\[
    c^\perp_- > c^\perp_+ (24)
\]
in the "magnetic" regime (χ > 0) of all models (13)-(16). The growth of current (23) in the "electric" regime (χ < 0) results in

\[ c_{-} < c_{+} \]

Of course, there is no transversal discontinuity in the sense of shock wave, but rather the string geometry is changed \[9, 10\]. However, perturbations of the worldsheet may lead to instability of the shock front which is called as corrugation instability in the mechanics of continuous media \[20\].

Consider a shock wave which propagates along the string at velocity \( D_- \) (see Fig. 2a), and there is finite flow \( D_+ \neq 0 \) behind the shock front (while the shock is reduced to a sound wave in the limit \( D_+ \to 0 \)). Let us consider the problem in the reference frame, co-moving the shock where the shock front is at rest, and the flow before the front has velocity \( w_- = -D_- \), while the flow behind the front has velocity \( w_+ \) (positive direction is taken from the right to the left). Extrinsic perturbations can appear before and behind the shock hypersurface and they can propagate in two different directions with respect to the shock front (see Fig. 2b).

A perturbation, coming from infinity behind the front (see Fig. 3), runs at velocity

\[ C_+ = w_+ \oplus c_+^\perp = \frac{w_+ + c_+^\perp}{1 + w_+ c_+^\perp} \]

and before the front its velocity is

\[ C_- = w_- \oplus c_-^\perp = \frac{w_- + c_-^\perp}{1 + w_- c_-^\perp} \]

If \( C_- < C_+ \), then, the perturbation behind the front becomes fully independent because no perturbation from the domain before the shock will interfere with it. The perturbations before the shock have no influence on the perturbations behind the shock, and the shock wave hypersurface divides the string into two independent domains. To avoid it, we must request

\[ \frac{w_- + c_-^\perp}{1 + w_- c_-^\perp} > \frac{w_+ + c_+^\perp}{1 + c_+^\perp w_+} \]

A perturbation, coming from infinity before the front (see Fig. 4), runs at velocity

\[ \tilde{C}_- = w_- \ominus c_+^\perp = \frac{w_- - c_-^\perp}{1 - w_- c_-^\perp} \]
and a perturbation behind the shock runs at velocity

$$\bar{C}_+ = w_+ \oplus c_+^\perp = \frac{w_+ - c_+^\perp}{1 - w_+c_+^\perp}$$

(30)

All possible relations between $\bar{C}_-$ and $\bar{C}_+$ are plotted in Fig. 4. Without regard of the particular sign of velocities, the inequality $\bar{C}_- < \bar{C}_+$ always implies that there will be no link between the perturbations before and behind the shock. To avoid this situation, we must request

$$\bar{C}_- > \bar{C}_+ \quad \Leftrightarrow \quad \frac{w_- - c_-^\perp}{1 - w_-c_-^\perp} > \frac{w_+ - c_+^\perp}{1 - w_+c_+^\perp}$$

(31)

It may occur $\bar{C}_- < 0$ and $\bar{C}_+ < 0$ (both perturbations propagate from the left to the right and collinear to the shock velocity $D_-$), however, as soon as $\bar{C}_- > \bar{C}_+$ or $|\bar{C}_-| < |\bar{C}_+|$, then, a link between perturbations at opposite sides of the front is established.

An extrinsic perturbation can be emitted by the shock in the direction of its propagation, and it runs at velocity (29). When an extrinsic perturbation is emitted by the shock in the direction opposite to its propagation, it runs at velocity (26). Such perturbations are also shown in Fig. 3 and 4. If they propagate independent from the shock flow, the stable shock structure will will be corrupted.

When either of inequalities (31) or (28) is borken, the energy will be scattered beyond the self-consistent shock-wave regime that implies instability and decay of the shock [20]. The constraints (31) and (28) are enough for existence of stable shock waves.

In the light of (21) and (24), inequality (28) is always satisfied in the "magnetic" regime. Inequality (31) is automatically satisfied in the "magnetic" regime of the "transonic" model (13), as it follows from (18) and (21). As for other three models (14)-(16), the inequality (31) is also satisfied but it can be clarified in the direct calculation (see Fig. 5, 6, 7).

In the light of (22) and (25), the inequality (28) is never satisfied in the "electric" regime, and it is the physical reason why no "electric" shock was found in the numerical simulations [9, 10].
5 Conclusion

A superconducting cosmic string may admit a stable shock-like discontinuity of the current when the latter is spacelike \( \chi > 0 \). A discontinuity of timelike current \( \chi < 0 \) cannot not exist. In the present paper we have explained why the "electric" shocks are impossible: it is due to the shock instability to extrinsic perturbations of the string worldsheet. As soon as this instability takes place, i.e. when either of inequality \( 28 \) or \( 31 \) is broken, the energy of the shock wave will dissipate. If an arbitrary discontinuity \( \Delta \chi \neq 0 \) is created in the "electric" regime, e.g. during intercommutation of distinct string loops, this discontinuity will be unstable, its energy will be converted without restriction into extrinsic vibrations of the string worldsheet, and the discontinuity will decay until it becomes a smooth transition \( \chi_- \Rightarrow \chi_+ \).

As for the stable shock waves in the "magnetic" regime they do exist \([7, 9, 10]\), and no restriction of their existence was found. However, a discontinuity with initial state \( \chi_- = 0 \) will be unstable because the formula \( 10 \) yields \( c_- (\chi = 0) = 1 \), and the shock wave velocity \( w_- = 1 \) does not satisfy the evolutionary condition \( 20 \). We can expect some minimal amplitude \( \chi_{\text{min}} > 0 \) to trigger a discontinuity, and special analysis of "magnetic" shock wave at \( \chi_- \to 0^+ \) is desirable. It is the subject for further research.

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Figure 1: The speed of extrinsic perturbations $c^\perp$ vs current $\chi$ (parameters $m = 1$, $m_* = 0.5$)

Solid line – model (16), dashed line – model (15), dotted line – model (14)
Figure 2: Shock wave velocities and velocities of extrinsic perturbations in the reference frame co-moving the shock wave front.
Figure 3: Possible relations between velocities of perturbations $w_- \oplus c_-$ and $w_+ \oplus c_+$ before and behind the front. Perturbations can be emitted from the shock hypersurface (two bottom graphs).
Figure 4: Possible relations between velocities of perturbations $w_- \ominus c_-$ and $w_+ \ominus c_+$ before and behind the front. Perturbations emitted from the shock hypersurface are depicted in the bottom).
Figure 5: Velocities $\bar{C}_-$ (29) [solid] and $\bar{C}_+$ (30) [dashed] for EOS (14) at $m = m_* = 1$ and various initial current $\chi_-$ and increment $\Delta \chi$.
Figure 6: The same plots as in Fig. 5 but for EOS (15)
Figure 7: The same plots as in Fig. 5 but for EOS (16)