High-order Phase Transition in Random Hypergraphs

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Abstract

In this paper, we study the high-order phase transition in random r-uniform hypergraphs. For a positive integer n and a real \( p \in [0,1] \), let \( H := H^r(n,p) \) be the random r-uniform hypergraph on the vertex set \([n]\), where each r-set is an edge with probability \( p \) independently. For \( 1 \leq s \leq r-1 \) and two s-sets \( S \) and \( S' \), we say \( S \) is connected to \( S' \) if there is a sequence of alternating s-sets and edges \( S_0, F_1, S_1, F_2, \ldots, F_k, S_k \) which satisfies \( S_0, S_1, \ldots, S_k \) are s-sets, \( S_0 = S, S_k = S' \), \( F_1, F_2, \ldots, F_k \) are edges of \( H \), and \( S_{i-1} \cup S_i \subset F_i \) for each \( 1 \leq i \leq k \). This is an equivalence relation over the family of all s-sets \( \binom{[n]}{s} \) and results a partition: \( (V_s) = \bigcup_i C_i \). Each \( C_i \) is called an s-th-order connected component and a component \( C_i \) is a giant s-th-order connected component if \( |C_i| = \Theta(n^s) \). We prove that the sharp threshold of the existence of the s-th-order giant connected components in \( H^r(n,p) \) is

\[
1 + o(1) \left( \frac{(r-1)!}{(r-s-1)!} \right) \left( \frac{1}{p} \right)^{r-s}.
\]

1 Introduction

The theory of random graphs was born when Erdős and Rényi wrote a series of remarkable papers on the evolution of random graphs around 1960 [10, 11]. In their original papers, Erdős and Rényi considered the uniform model \( G_{n,m} \) where a random graph \( G \) is selected uniformly among all graphs with \( n \) vertices and \( m \) edges. Later, the binomial model \( G(n,p) \) became the de facto random graph model, which is also referred as Erdős–Rényi random graph model. In \( G(n,p) \), each pair of vertices becomes an edge with probability \( p \) independently. It is remarkable that the random graph \( G(n,p) \) experiences phase transition as \( p \) passes through the threshold \( \frac{1}{n} \). When \( p < \frac{1}{2n} \), almost surely all connected components of \( G(n,p) \) are of order \( O(\ln n) \); when \( p \approx \frac{1}{n} \), almost surely the largest connected component has size \( \Theta(n^{2/3}) \) [3, 12, 15]; when \( p > \frac{1}{2n} \), almost surely there is a unique giant component of size \( \Theta(n) \) while all other connected components have size \( O(\ln n) \). For a more detailed description of the phase transition phenomenon in \( G(n,p) \), see Chapter 11 of [2].

There are a lot of literature on the phase transition of other random graphs. Aiello, Chung, and Lu [1] introduced a general random graph model—the random graph with given expected degree sequence. Chung and Lu [7] studied the connected components in this random graph model and determined the size of the giant connected components in the supercritical phase [8]. Bollobás, Janson, and Riordan [4] investigated the phase transition phenomenon in inhomogeneous random graphs with a given kernel function.

There are some attempts on generalizing the Erdős and Rényi’s classical work on random graphs to random hypergraphs. Let \( H_{n,m}^r \) be the random r-uniform hypergraph such that

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This author was supported in part by NSF grant DMS 1000475, NSF grant DMS 1300547 and ONR grant N00014-13-1-0717.

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This author was supported in part by ONR MURI N000140810747, and AFOSR AF/SURF 552082.
each r-uniform hypergraph with n vertices and m edges is selected with the same probability. The components structure of the random hypergraph \( H^r(n, m) \) was first analyzed by Schmidt-Pruzan and Shamir [19]. Namely, they proved that for \( m < \frac{n}{r(r-1)} \), almost surely the largest connected component in \( H^r(n, m) \) is of size \( O(\ln n) \); for \( m > \frac{n}{r(r-1)} \), almost surely the largest connected component is of order \( \Theta(n^{2/3}) \); for \( m = \frac{n}{r(r-1)} \), almost surely there is a unique giant connected component with \( \Theta(n) \) vertices. Karoński and Łuczak [14] took a closer look at the connected components of \( H^r(n, m) \) when \( m \) is near \( \frac{n}{r(r-1)} \). The connected components in all known results on random hypergraphs are in the following sense. Let \( H \) be a uniform hypergraph, two vertices \( u \) and \( u' \) are connected if there is an (alternating) sequence of vertices and edges \( v_0, F_1, v_1, F_2, \ldots, F_k, v_k \) such that \( u = v_0, u' = v_k, \) and \( \{v_{i-1}, v_i\} \subseteq F_i \in E(H) \) for \( 1 \leq i \leq k \).

In a hypergraph \( H = (V, E) \), high-order connections exist besides the vertex-to-vertex connection. For \( 1 \leq s \leq r - 1 \) and two s-sets \( S \) and \( S' \), \( S \) and \( S' \) are connected if there is a sequence of alternating s-sets and edges \( S_0, F_1, S_1, F_2, \ldots, F_k, S_k \) such that \( S_0, S_1, \ldots, S_k \) are s-sets, \( S_0 = S, S_k = S' ; F_1, F_2, \ldots, F_k \) are edges of \( H \), and \( S_{i-1} \cup S_i \subseteq F_i \) for \( 1 \leq i \leq k \). It is easy to verify this is an equivalence relation over the family of all s-sets \( \binom{V}{s} \) and then results a partition \( \binom{V}{s} = \bigcup_i C_i \). Each \( C_i \) is called an \( s \)-th-order connected component. We say a component \( C_i \) is a giant \( s \)-th-order component if \( |C_i| = \Theta(n^s) \). Alternately, we construct an auxiliary graph \( H^s \) with the vertex \( \binom{V}{s} \) and a pair of two \( s \)-sets \( S, S' \) forms an edge in \( H^s \) if \( S \cup S' \subseteq F \) for some \( F \in E(H) \). The \( s \)-th-order connected components of \( H \) are exactly the connected components of the auxiliary graph \( H^s \).

In this paper, we consider the “binomial model” of random \( r \)-uniform hypergraphs \( H^r(n, p) \). Namely, \( H^r(n, p) \) has the vertex set \( [n] \) and each \( r \)-set of \( [n] \) becomes an edge with probability \( p \) independently. For additional information on random hypergraphs, the reader is referred to the survey wrote by Karoński and Łuczak [14]. Our goal is to study the phase transition of the \( s \)-th-order connected components in \( H^r(n, p) \) as \( p \) increases from 0 to 1. Bollobás and Riordan (see page 442 of [14]) claimed the following. The branching arguments in [14] can show that the threshold for the emergence of an \( s \)-th-order giant component is at \( p \sim \frac{1}{\binom{n}{s}} \left( \frac{c}{r-1} \right)^{\binom{s}{2}} \) for each \( 1 \leq s \leq r - 1 \). We confirm this by using a different approach.

In what follows we will use the following asymptotic notation. For two functions \( f(x) \) and \( g(x) \) taking nonnegative values, we say \( f(x) = O(g(x)) \) (or \( g(x) = \Omega(f(x)) \)) if \( f(x) \leq Cg(x) \) for some positive constant \( C \) and \( f(x) = o(g(x)) \) if \( \lim_{x \to \infty} \frac{f(x)}{g(x)} = 0 \). If \( f(x) = O(g(x)) \) and \( g(x) = \Omega(f(x)) \), then we write \( f(x) = \Theta(g(x)) \).

We will prove the following main theorem.

**Theorem 1** Let \( H^r(n, p) \) be the random \( r \)-uniform hypergraph. Let \( \epsilon \) be a small positive constant. For \( 1 \leq s \leq r - 1 \), the following statements hold.

1. **Subcritical phase:** If \( p < \frac{1 - \epsilon}{\binom{n}{s-1}} \), then almost surely all \( s \)-th-order connected components in \( H^r(n, p) \) have size \( O(\ln n) \).

2. **Supercritical phase:** If \( p > \frac{1 + \epsilon}{\binom{n}{s-1}} \), then almost surely there is a unique giant \( s \)-th-order connected component of size \( \Theta\left(\binom{n}{s}\right) \). All other \( s \)-th-order connected components in \( H^r(n, p) \) have size \( O(\ln n) \). Moreover, if \( p = \frac{\epsilon}{\binom{n}{s-2}} \) with \( \epsilon = \frac{1 + \epsilon}{\binom{n}{s-1}} \). Almost surely the size of the giant \( s \)-th-order connected component is \( (z + o(1)) \binom{n}{s} \), where \( z \) satisfies \( 0 < z < 1 \) and is a solution of the following equation

\[
1 - x = e^{\epsilon (1-x) \binom{s}{1}}.
\]
The case $s = 1$ corresponds to the vertex-to-vertex connection. The threshold of giant connected component is $p = (1+o(1))^{1\over (r-1)(r-1)}$. Equivalently, the number of edges at this critical point is about $(1+o(1))^{1\over (r-1)} n$. This is consistent to Schmidt-Pruzan and Shamir’s result on $H_{n,m}^r$.

Our method is similar to the classical approach for studying the phase transition of Erdős-Rényi random graph model. We will couple the branching process in hypergraphs with a multi-fold Poisson branching process. The detail is explained in section 2. In section 3, we will prove the main theorem. There are variations of the $s$-th-order connection and the phase transition is similar. Those results are discussed in the last section.

2 Multi-fold Poisson branching process

Let $Z$ be a distribution over the nonnegative integers. Let $Z_1, Z_2, \ldots$, be a countable sequence of independent identically distributed random variables, each having distribution $Z$.

A Galton-Watson process is a stochastic process \( \{Y_t\}_{t=0}^{\infty} \) which evolves according to the recursive formula

\[
Y_t = Y_{t-1} - 1 + Z_t, \quad \text{for } t \geq 1,
\]

where $Y_0 = 1$. The process can be interpreted as follows. We think the children being born in a Depth-first Search manner. Starting from a single root node, we can call her Eve. Eve has $Z$ children and her children are kept in the last-in first-out order. Now Eve’s last child has $Z$ children and all of them are also stacked in the last-in first-out order. A node is dead if its children have been exposed, and is live otherwise. Each step, we explore Eve’s last live descendant and its children are added to the live descendants with last-in first-out order. Note that $Y_t$ is the number of live descendants after $t$ nodes have been explored. Equation (2) shows the recursive formula for $Y_t$. Let $T$ be the total number of nodes (including Eve herself) created in the process. If the process goes on forever, then we write $T = \infty$.

Let $f(x) = \sum_{k=0}^{\infty} \Pr(Z = k)x^k$ be the generating function of the probability distribution $Z$. Then the extinction probability $\Pr(T < \infty)$ is one of the roots of $x = f(x)$. Furthermore, the extinction probability is equal to one if $E(Z) \leq 1$ and is strictly less than one if $E(Z) > 1$. For the proof, see Chapter 11 in [2].

For a fixed integer $m \geq 1$, an $m$-fold Poisson branching process, denoted by $T_{m,c}^{po}$, is a special Galton-Watson process with the ensemble $Z_t$

\[
\Pr(Z_t = k) = \begin{cases} 
    e^{-c} \frac{c^k}{(k/m)!}, & \text{if } m|k \\
    0, & \text{otherwise}.
\end{cases}
\]

Here $c$ is some constant. Each $Z_t$ is called an $m$-fold Poisson random variable. The random variable $Z_t$ can be interpreted as the Poisson distribution $\text{Po}(c)$ duplicated $m$ times. Note the generating function for $Z_t$ is

\[
f(x) = \sum_{k=0}^{\infty} e^{-c} \frac{c^k}{(k/m)!} x^k = \sum_{n=0}^{\infty} e^{-c} \frac{c^m}{n!} x^{nm} = e^{c(x^m-1)},
\]

We have the following lemma.

**Lemma 1** Let $T_{m,c}^{po}$ be the $m$-fold Poisson branching process defined above. We have

1. If $c \leq \frac{1}{m}$, then the extinction probability is 1.
2. If \( c > \frac{1}{m} \), then the extinction probability is in \((0, 1)\) and satisfies the equation

\[
x = e^{c(x^m-1)}.
\]

The proof of this lemma goes the same line as the one for proving Lemma 11.4.1 in [2] and is omitted here. To estimate the population in the \( m \)-fold Poisson branching process, we need the following lemma from [9].

**Lemma 2** If \( X_1, X_2, \ldots, X_n \) are nonnegative independent random variables, then we have the following bound for the sum \( X = \sum_{i=1}^{n} X_i \):

\[
\Pr(X \leq E(X) - \lambda) \leq e^{-\frac{\lambda^2}{2 \sum_{i=1}^{n} E(X_i)}}.
\]

We have the following lemma.

**Lemma 3** For fixed \( m \) and \( c > \frac{1}{m} \), let \( T \) be the total population of the \( m \)-fold Poisson branching process \( T_{m,c}^{po} \). Then there exists a constant \( C = C(m, c) \) satisfying for any positive integer \( K \)

\[
\Pr(T = K) < e^{-CK}.
\]

**Proof:** Let \( \{Z_i\}_{i=1}^\infty \) be a sequence of independent identically distributed random variables. Each of them have the \( m \)-fold Poisson distribution. We observe

\[
\Pr(T = K) = \Pr(Z_1 + Z_2 + \ldots + Z_K = K - 1).
\]

Let \( X = \sum_{i=1}^{K} Z_i \). Then \( E(X) = \sum_{i=1}^{K} E(Z_i) = mcK \) and \( \sum_{i=1}^{K} E(Z_i^2) = m^2(c + c^2)K \). Applying Lemma 2 we have

\[
\Pr(X = K) = \Pr(X - E(X) = -(mc - 1)K - 1)
< \Pr(X - E(X) < -(mc - 1)K)
\leq e^{-\frac{(mc-1)^2 K^2}{2m^2(c+c^2)K}}
= e^{-CK}.
\]

Here \( C = \frac{(mc-1)^2}{2m^2(c+c^2)} \).

**□**

3 **Proof of the main theorem**

3.1 **The graph branching process**

Let \( H^r(n, p) \) be the random \( r \)-uniform hypergraph. We will consider the graph branching process of \( s \)-sets described as follows. We will maintain four families of sets \( D, L, N, R \):

1. \( D \): the family of all dead \( s \)-sets (i.e., the \( s \)-sets whose neighborhoods have been explored.)
2. \( L \): the family of all live \( s \)-sets (i.e., the \( s \)-sets in the queue and whose neighborhoods are ready to be explored.)
3. \( N \): the family of all neutral \( s \)-sets (i.e., the \( s \)-sets which have not been visited.)
4. \( R \): the family of all \( r \)-sets which has not been queried to be an edge or not.

The following is the pseudo-code for the graph branching process.
Algorithm Graph branching process:
Initially $D \leftarrow \emptyset$, $L \leftarrow \emptyset$, $N \leftarrow \binom{V}{s}$, and $R \leftarrow \binom{V}{r}$.

while $N \neq \emptyset$

Select an arbitrary $S_0 \in N$ and move $S_0$ from $N$ to $L$.

while $L \neq \emptyset$

Select the last $S \in L$ and move $S$ from $L$ to $D$.

for each $r$-set $F \in R$ containing $S$

Delete $F$ from $R$ and query whether $F$ is an edge of $H^r(n, p)$.

If $F$ is an edge, then add all $(\binom{F}{s})$ to the end of $L$ with an arbitrary order.

end for each

end while

Write out the component consisting of all $s$-sets in $D$ and reset $D$ to $\emptyset$.

end while

end algorithm

Although the output of components does not depend on the way how to select $S_0$ and $S$, the analysis often prefers one way to another. Many papers studying the phase transition in random graphs use Breadth-first Search and treat $L$ as a queue. Here we treat $L$ as a stack (in last-in first-out order). This is a variation of Depth-first Search. It is slightly different from the Depth-first Search used by Krivelevich and Sudakov in [15], where the live set always forms a path. Here we still have a unique path $P$ from $S_0$ to the last visited $s$-set $S$ in the Depth-first Search tree. The neighbors of every node in $P$ are already explored; thus the nodes in $P$ are in $D$. The live nodes (elements in $L$) are children of some node in $P$.

Let $T^r_{r,s,p}$ be the size of the first component being explored. Immediately, we can couple the Graph branching process $T^r_{r,s,p}$ and the $m$-fold Poisson branching process $T^p_{m,c}$ as follows. Assume $p = \frac{c}{\binom{n-s}{r-s}}$. For each $s$-set, the number of $r$-set $F$ containing $S$ is $\binom{n-s}{r-s}$. When $n$ tends to infinity, the number of edges explored can be approximated by the Poisson distribution Po$(c)$. When each edge $F$ is discovered, there are at most $m := \binom{r}{s} - 1$ new $s$-sets added into $L$. Thus, the size of component explored is dominated by the $m$-fold Poisson branching process $T^p_{m,c}$. In other words, for any integer $K$,

$$\Pr(T^r_{r,s,p} > K) \leq \Pr(T^p_{m,c} > K).$$

This can be used to prove the statement for the subcritical phase.

3.2 Subcritical phase

Before proving the subcritical phase of the main theorem, we need to prove the following quite simple lemma.

**Lemma 4** Let $Z$ be the $m$-fold Poisson random variable with mean $c > 0$. Then for any positive integer $k$,

$$\Pr(Z \geq mk) \leq \frac{e^k}{k!}.$$

**Proof:** By the definition, we have

$$\Pr(Z \geq mk) = \sum_{n=k}^{\infty} \frac{e^{-c} c^n}{n!} = e^{-c} R_{k-1}(c).$$

Here $R_{k-1}(c) = \sum_{n=k}^{\infty} \frac{c^n}{n!}$ is the remainder of the $(k-1)$-th degree Taylor series of the function $f(c) = e^c$. By a well-known fact of calculus, we have

$$|R_{k-1}(c)| \leq \frac{e^c}{k!}.$$
When \( p \leq \frac{1}{m} \), by Lemma 4, we have for each \( 1 \) of independent identically distributed random variables, each having the distribution. We have

\[
\Pr(|C(S)| > K) \leq \Pr(T_{m,c}^{po} > K).
\]

Recall the definition of the \( m \)-fold Poisson distribution. Let \( \{Z_i\}_{i=1}^{\infty} \) be a countable sequence of independent identically distributed random variables, each having the \( m \)-fold Poisson distribution. We have

\[
\Pr(T_{m,c}^{po} > K) \leq \Pr(Z_1 + Z_2 + \ldots + Z_K \geq K).
\]

By Lemma 2 we have for each \( 1 \leq t \leq K \),

\[
\Pr(Z_t \geq K) \leq \frac{e^{[K/m]}}{([K/m])!}.
\]

We get the probability that there is some \( 1 \leq t \leq K \) such that \( Z_t \geq K \) is at most \( \frac{K^c[K/m]}{([K/m])!} \). Condition on \( Z_t \leq K \) for each \( 1 \leq t \leq K \), the Chernoff’s inequality gives

\[
\Pr(Z_1 + Z_2 + \ldots + Z_K \geq K) \leq e^{-cK} \text{ for some positive constant } c'.
\]

We get

\[
\Pr(Z_1 + Z_2 + \ldots + Z_K \geq K) \leq e^{-cK} + \frac{K^c[K/m]}{([K/m])!}.
\]

Now, if \( K = K' \ln n \) for \( K' \) large enough, we get

\[
\Pr(|C(S)| > K' \ln n) \leq \frac{1}{n^{s+1}}.
\]

Therefore, the probability that there is some \( S \) such that \( |C(S)| > K' \ln n \) is at most \( 1/n \). Equivalently, with high probability each component has size \( O(\ln n) \) in this case.

### 3.3 Supercritical phase

We assume \( p = \frac{c}{(s-1)} \) with a constant \( c \) satisfying \( c > \frac{1}{(r-1)} \).

At the beginning of the graph branching process, most edges are not revealed. For each \( S \), the number of edges containing \( S \) follows the binomial distribution \( \text{bin}((1+o(1))(\binom{n}{r-1}), p) \). When \( p = \frac{c}{(r-1)} \), this is roughly a Poisson distribution with mean \( c \). Each edge brings in \( \binom{r}{s} \) new \( s \)-sets. This suggests us to couple the graph branching process with the \( m \)-fold Poisson branching process \( T_{m,c}^{po} \) where \( m = \binom{r}{s} - 1 \). However, there are two major obstacles:

1. It is possible that there is some \( r \)-set \( F \) containing \( S \) has already been queried earlier. Thus we do not need to query it any more, which decreases the number of \( r \)-sets containing \( S \).

2. When a new edge \( F \) is found, not every \( s \)-sets of \( \binom{F}{s} \) (other than \( S \)) is new. Some of them may be already in \( L \cup D \). This is a serious problem since it affects the number of new \( s \)-sets added in \( L \) when we have a new edge.

To overcome the difficulty, for each \( s \)-set \( S \), we define a new family of \( r \)-sets \( R_S \) as follows,

\[
R_S = \{ F \in R : S \subset F \text{ and } S' \not\subset F \text{ for each } S' \in L \cup D \}.
\]

With this new definition, we modify the graph branching process algorithm. For a fixed \( \epsilon > 0 \), if \( |R_S| \geq (1 - \epsilon) \binom{n}{s} \), then we query all \( r \)-sets in \( R_S \); otherwise, we halt the process. Here is the pseudo-code for the new search algorithm.
Algorithm New Search:
Select an arbitrary \( S_0 \in \mathcal{N} \) and move \( S_0 \) from \( \mathcal{N} \) to \( \mathcal{L} \).

while \( \mathcal{L} \neq \emptyset \)
   Select the last element \( S \in \mathcal{L} \)
   if \( |\mathcal{R}_S| < (1 - \epsilon) \binom{n}{r-s} \)
      Halt the process.
   else
      Move \( S \) from \( \mathcal{L} \) to \( \mathcal{D} \).
      for each \( r \)-set \( F \in \mathcal{R}_S \)
         Delete \( F \) from \( \mathcal{R} \) and query whether \( F \) is an edge of \( H^r(n, p) \).
         If \( F \) is an edge, then add all \( \binom{F}{s} \) to the end of \( \mathcal{L} \) with an arbitrary order.
      end for each
   end if
end while
Write out all \( s \)-sets in \( \mathcal{D} \cup \mathcal{L} \).

end algorithm

Note that each new edge \( F \) brings in \( \binom{r}{s} - 1 \) new \( s \)-sets to \( \mathcal{L} \) as the definition of \( \mathcal{R}_S \).

When we run the New Search algorithm, there are two possibilities. One is that the condition \( |\mathcal{R}_S| < (1 - \epsilon) \binom{n}{r-s} \) is never satisfied. In this case, the process will continue until \( \mathcal{L} = \emptyset \). The output is a connected component. We have the following lemma.

Lemma 5 The probability that the New Search outputs a connected component is \( 1 - z + O(\epsilon) \), where \( z \) satisfies
\[
z = e^{c(z^m - 1)}.
\]
With probability at least \( 1 - \frac{1}{n} \), the size of this connected component is at most \( O(\ln n) \).

Proof: Couple the graph branching process with the \( m \)-fold Poisson branching process \( T_{m,c}^{(\mathcal{R})} \) and \( T_{m,(1-\epsilon)c} \). By Lemma 3 we get the first half part.

For the second part, let \( C(S) \) be the connected component containing the \( s \)-set \( S \). Similar to the subcritical case, we have \( \Pr(|C(S)| \geq K) \leq \Pr(T_{m,c}^{(\mathcal{R})} \geq K) \). By Lemma 5 we have \( \Pr(T_{m,c}^{(\mathcal{R})} = K) \leq e^{-CK} \) for some positive constant \( C \). Thus we get
\[
\Pr(|C(S)| \geq K' \ln n) \leq \Pr(T_{m,c}^{(\mathcal{R})} \geq K' \ln n) \leq \sum_{K=K' \ln n}^{\infty} e^{-CK} \leq \frac{1}{n^{s+1}} \tag{3}
\]
for \( K' \) large enough. The union bound gives us the second part of the lemma. \( \square \)

In general, the output is not an \( s \)-th-order connected component; but it is a partial component (i.e., a connected piece). We say a component is small if its size is at most \( O(\ln n) \); it is large if its size is \( \Omega\left(\binom{n}{s}/\ln^s n\right) \).

We keep running the New Search algorithm until we find a large partial component. Almost surely we will end up with a large partial component as the following theorem. Actually, we will prove there is a path with size \( \Omega\left(\binom{n}{s}/\ln^s n\right) \). Note that we can couple this process with the \( m \)-fold Poisson branching process \( T_{m,c}^{(\mathcal{R})} \), with \( m = \binom{r}{s} - 1 \) and \( c' = (1 - \epsilon)c \).

Lemma 6 When the modified branching process halts, with probability \( 1 - O\left(\frac{1}{n^{s+1}}\right) \), the partial component contains a path of length \( \Omega\left(\binom{n}{s}/\ln^s n\right) \).

Proof: If the lemma holds for \( H^r(n, p_1) \), then it holds for any \( H^r(n, p_2) \) with \( p_2 > p_1 \). Without loss of generality, we assume
\[
\frac{1}{\binom{r}{s}} - 1 < c < \frac{1}{\binom{r-1}{s-1} - 1}. \tag{4}
\]
Since \( c > \frac{1}{m} \), we can assume \( c' = (1 - \epsilon)c > \frac{1}{m} \) by choosing \( \epsilon \) small enough. The extinction probability \( z \) of \( T_{m,c}^{po} \) is strictly less than 1. By Lemma 5 the probability that the New Search algorithm finds a small component is about \( (1 - z) \). If we repeat the New Search \( K' \ln n \) times for a large \( K' \), then the probability that the algorithm does not halt is at most

\[ (1 - z)^{K' \ln n} = O(n^{-\Omega(1)}). \]

Therefore, with high probability the modified branching process halts after we repeat the algorithm at most \( O(n \ln n) \) times. When the process halts, it stops at an \( s \)-set \( S \) such that \( |R_S| \leq (1 - \epsilon) \binom{n}{r} \). In other words,

\[ \sum_{S'} \left( \frac{n}{r - s - 1} \ln^2 n \right) \geq \epsilon \left( \frac{n}{r - s} \right), \tag{5} \]

where the summation is over all \( s \)-sets which have been visited. We make convention \( \left( \frac{n}{r - |S \cup S'|} \right) = 0 \) if \( |S \cup S'| > r \).

For those \( S' \) in the small components, we have \( S' \neq S \). Thus each \( S' \) can contribute at most \( \left( \frac{n}{r - s - 1} \right) \) to the summation of the left hand side of (5). Since each small component has size at most \( O(\ln n) \) and the number of small components found is at most \( O(\ln n) \), the contribution from small components in total is at most

\[ O \left( \left( \frac{n}{r - s - 1} \right) \ln^2 n \right) = o \left( \epsilon \left( \frac{n}{r - s} \right) \right). \]

Thus, the major contribution of the summation in equation (5) comes from \( S' \) in the partial component.

Let \( P \) be the path from \( S_0 \) to \( S \) in the Depth-first Search tree (DFS tree, for short), where \( S_0 \) is the first \( s \)-set entering the component and containing \( S \). If we delete \( P \) from the DFS tree, the family of \( s \)-set \( D - P \) falls into several small subtrees. Each subtree has a root. By coupling with \( T_{m,c}^{po} \), we get that each subtree has size at most \( O(\ln n) \) and there are at most \( O(\ln n) \) of them. A similar argument shows that the total contribution to inequality (5) is at most

\[ O \left( \left( \frac{n}{r - s - 1} \right) \ln^2 n \right) = o \left( \epsilon \left( \frac{n}{r - s} \right) \right). \]

Recall that all live \( s \)-sets are children of a particular \( s \)-set in \( P \). Lemma 4 implies with high probability we have \( O(\ln n) \) live \( s \)-sets. Similarly, we can ignore the contribution from the live \( s \)-sets. Inequality (5) implies that there exists \( T \subseteq S \) and a new constant \( \epsilon_1 \) satisfying

\[ |\{ S' \in P : S \cap S' = T \} | \geq \epsilon_1 \left( \frac{n}{s - |T|} \right). \]

If \( T = \emptyset \), then we are done. Otherwise, let \( S_T := \{ S' \in P : T \subseteq S' \} \). If we restrict \( H_r^x(n,p) \) to \( S_T \), then the new random hypergraph can be viewed a subgraph of \( H_r^{s - |T|}(n - |T|, p) \). By induction on \( s \), every connected component has size at most \( O(\ln n) \) since

\[ p < \frac{1}{\left( \frac{r - |T|}{s - |T|} - 1 \right) \left( \frac{n - |T|}{r - s} \right)} \]

by the assumption of \( p \), see [4]. Two adjacent nodes on the path \( P \) are \( T \)-connected if they both contain \( T \). The result above implies that the length of a \( T \)-connected segment is at most \( O(\ln n) \). Thus, there are at least \( \epsilon_2 (s - |T|) / \ln n \) nodes \( S' \) of \( P \) containing \( T \) but its parent node \( S'' \) (in \( P \)) does not contain \( T \).
To understand the transition from a node $S''$ not containing $T$ to a node $S'$ containing $T$, we introduce a jump move. Fix two disjoint subsets $A$ and $B$ satisfying $|A| + |B| \leq s$, an edge $(S'', S')$ on $\mathcal{P}$ is called an $(A, B)$-jump if $S' \cap S'' = A$, $B \cap S'' = \emptyset$, and $A \cup B \subset S'$. Here $A$ could be $\emptyset$ and $B \neq \emptyset$. A necessary condition for $(A, B)$-jump occurring at $S''$ is $A \subset S''$, $B \cap S'' = \emptyset$, and an $r$-set $F$ containing $A \cup B$ is revealed to be an edge of $H^r(n, p)$.

We observe the $(A, B)$ jump happens at $S''$ with probability at most

$$
\left(\frac{n - s}{r - s - |B|}\right)^p \leq \frac{c(r - s)!}{(r - s - |B|)! n^{|B|}} = \frac{c_1}{n^{|B|}}.
$$

(6)

Here $c_1$ is a new constant $\frac{c(r-s)!}{(r-s-|A|)!n!}$.

Intuitively, we may expect that if the number of $s$-sets in $\mathcal{P}$ (like $S''$) containing $A$ and having empty intersection with $B$, denoted by $d_A$, is small, then the number of $(A, B)$-jump is small. We have the following claim:

**Claim a:** For any function $g(n) \geq 8s \ln n$, suppose $d_A < g(n) \frac{n^{|B|}}{2c_1}$. Then it has probability at most $n^{-3s}$ that there are at least $g(n)$ $(A, B)$-jumps.

We assume the $s$-sets containing $A$ and having empty intersection with $B$ are $S_1, \ldots, S_{d_A}$. For each $1 \leq i \leq d_A$, let $X_i$ be the random indicator variable for the $(A, B)$-jump occurring at $S_i$. Let $X = \sum_{i=1}^{d_A} X_i$ be the number of $(A, B)$-jumps. For $1 \leq i \neq j \leq d_A$, the collection of $r$-sets we query for $S_i$ is disjoint with the one for $S_j$ by our New Search algorithm, so $X_i$'s are independent random variables. Recalling (6), we have

$$
E(X) \leq d_A \frac{c_1}{n^{|B|}} < \frac{g(n)}{2}.
$$

Applying Chernoff’s inequality, we have

$$
\Pr(X \geq g(n)) \leq \Pr \left( X - E(X) \geq \frac{g(n)}{2} \right) \\
\leq e^{-3g(n)/8} \\
< \frac{1}{n^{3s}}.
$$

Recall there are at least $c_2 \left( \frac{n}{s - |T|} \right) / \ln n$ nodes $S'$ of $\mathcal{P}$ containing $T$ but their parent node $S''$ (in $\mathcal{P}$) does not contain $T$. Each pair is an $(A, B)$-jump for some partition of $T$. Here it is possible $A = \emptyset$. By the average argument, there exists a pair of $(A, B)$ such that the number of $(A, B)$-jumps on $\mathcal{P}$ is at least

$$
g(n) = \frac{1}{2|T|} c_2 \left( \frac{n}{s - |T|} \right) / \ln n.
$$

As $|T| < s$, we have $g(n) \geq 8s \ln n$. By Claim a, we have

$$
|d_A| > g(n) \frac{n^{|B|}}{2c_1} = \epsilon_3 \left( \frac{n}{s - |A|} \right) / \ln n
$$

with a new absolute constant $\epsilon_3$. Here we used the fact $A \cup B = T$ and $A \cap B = \emptyset$. Since $B \neq \emptyset$, we have $A \subset T$. This is a progress with a constant multiplicative factor $\Theta(1/ \ln n)$.

View $A$ as a new $T$ and iterate this process above until we get $T = \emptyset$ which gives

$$
|\mathcal{P}| = \Omega \left( \frac{n^3}{\ln^3 n} \right).
$$
After the algorithm first halts, we query all \( r \)-sets which we have not checked. Finally, we output all components. We already proved that there exists at least one large component. We next show the large components is unique.

**Lemma 7** With high probability, the large component is unique.

**Proof:** Let \( C \) be a component such that \( |C| = \eta(n)/\ln^s n \) for some \( \eta > 0 \). We first prove that \( C \) must have some nice properties. For each \( U \subset [n] \) such that \( 1 \leq |U| \leq s - 1 \), let

\[
\Gamma_C(U) = \{ S \in C \text{ such that } U \subset S \} \text{ and } d_C(U) = |\Gamma_C(U)|.
\]

We next show the following claim.

**Claim b:** For each \( 1 \leq i \leq s - 1 \), with high probability \( \binom{n}{i} \subset \cup_{S \in C} \binom{S}{i} \) and there exists some \( \delta_i > 0 \) such that \( d_C(U) \geq \delta_in^{s-i}/\ln^s n \) for each \( U \in \binom{n}{i} \).

We prove the claim by induction on \( i \). For the case \( i = 1 \), we suppose there is some \( 1 \leq i \leq n \) such that \( i \not\in \cup_{S \in C} S \). For each \( S \in C \), let \( T \) be an arbitrary subset of \( S \) with size \( s - 1 \) and \( S' = T \cup \{ i \} \). Then the number of \( r \)-sets containing \( S \cup S' \) is \( \binom{n}{s-1} \). Let \( S \) runs over all \( s \)-sets of \( C \). Note that each \( r \)-set counts at most \( \binom{r}{s-1} \) times. The expected number of \( r \)-sets containing \( S \cup S' \) for some \( S \in C \) is at least \( p\binom{n}{s-1}|C|/\binom{r}{s-1} = \Omega(n^{s-1}/\ln^s n) \).

The Chernoff’s inequality shows that with high probability there are at least \( \Omega(n^{s-1}/\ln^s n) \) \( r \)-sets containing \( S \cup S' \) for some \( S \in C \), i.e., \( i \in \cup_{S \in C} S \). We can use the same argument to show for each \( i \in [n] \), we have \( d_C(i) \geq \delta_i n^{s-1}/\ln^s n \) for some positive constant \( \delta_i \).

For the inductive step, we suppose there is some \( U \in \binom{n}{i} \) such that \( U \not\in \cup_{S \in C} \binom{S}{i} \). Let \( U' \) be an arbitrary subset of \( U \) with size \( i - 1 \). By the inductive hypothesis, we have \( d_C(U') \geq \delta_i n^{s-i}/\ln^s n \). For each \( S \in \Gamma_C(U') \), let \( T \) be a subset of \( S \) such that \( |T| = s - 1 \) and \( U' \subset T \). We define a new \( s \)-set \( S' = U \cup T \). The remaining argument goes the same line as the base case and it is omitted here.

We are now ready to prove the lemma. Let \( C_1 \) and \( C_2 \) be two components with size at least \( \eta(n)/\ln^s n \) for some \( \eta > 0 \). Let \( U \) be an arbitrary \((s-1)\)-subset of \([n] \). By Claim b, we have \( d_{C_1}(U) \geq \delta_i n^{s-i}/\ln^s n \) for each \( i \in [1, 2] \). For each \( S_1 \in \Gamma_{C_1}(U) \) and \( S_2 \in \Gamma_{C_2}(U) \), the number of \( r \)-sets containing \( S_1 \cup S_2 \) is \( \binom{n}{s-1} \). With the lower bounds on \( d_{C_1}(U) \) and \( d_{C_2}(U) \), the expected number of \( r \)-sets containing \( S_1 \cup S_2 \) for some \( S_1 \in \Gamma_{C_1}(U) \) and \( S_2 \in \Gamma_{C_2}(U) \) is \( \Omega(n/\ln^{2s}n) \). Now the Chernoff’s inequality gives with high probability \( S_1 \) is adjacent to \( S_2 \) for some \( S_1 \in \Gamma_{C_1}(U) \) and \( S_2 \in \Gamma_{C_2}(U) \), i.e., with high probability \( C_1 \) and \( C_2 \) are in the same component. We proved the lemma.

We need to show the size of the largest component.

**Lemma 8** With high probability, the largest \( i \)-th-order connected component of \( H^r(n, p) \) is of size \((z + o(1))\binom{n}{i}\), where \( z \) satisfies \( 0 < z < 1 \) and is a solution of the following equation

\[
1 - x = e^{c((1-x)^{(i-1)})^{-1}},
\]

(7)

**Proof:** To prove this lemma, we need a auxiliary Galton-Watson branching process-the \( m \)-fold binomial branching process, where we have a countable collection of independent identically distributed variables \( Z_1, Z_2, \ldots \), such that

\[
\Pr(Z_i = k) = \begin{cases} \binom{n}{k/m}p^{k/m}(1-p)^{n-k/m} & \text{if } m|k; \\ 0 & \text{otherwise.} \end{cases}
\]

Here \( Z_i \) can be understood as the binomial distribution \( \text{bin}(n, p) \) duplicated \( m \) times.
Recall \( m = \binom{r}{2} - 1 \) and \( p = \frac{c}{(c-1)} \), where \( c = \frac{1+\epsilon}{(c-1)} \). Let \( T_{n,p}^{\text{bin},m} \) denote the \( m \)-fold binomial branching process. For an \( s \)-set \( S \), let \( C(S) \) be the \( s \)-th-order connected component containing \( S \) and \( z = \Pr(|C(S)| \text{ is not small}) \). For a fixed \( t \), we observe

\[
\Pr \left( T_{\binom{n-s-st}{r-s}}^{\text{bin},m}, p \geq t \right) \leq z \leq \Pr \left( T_{\binom{n-s-st}{r-s}}^{\text{bin},m}, p \geq t \right). \tag{8}
\]

In the graph branching process, when we explore the neighbors of the current \( s \)-set \( S \), we query all \( r \)-sets (have not checked) containing \( S \). To get the lower bound in (8), we can consider only the \( r \)-set \( F \) such that

\[
S \subset F \text{ and } F \cap (\bigcup_{S' \in L \cup D} (S' \setminus S)) = \emptyset.
\]

Note that this kind of \( F \) brings in \( \binom{n}{s} - 1 \) new \( s \)-sets if \( F \) is an edge in \( H^r(n,p) \). We observe the number of \( F \) satisfying the condition above is at least \( \binom{n-s-st}{r-s} \). Then we get the lower bound. The upper bound follows similarly. We have \( \Pr \left( T_{\binom{n-s-st}{r-s}}^{\text{bin},m}, p \geq t \right) \) and \( \Pr \left( T_{\binom{n-s-st}{r-s}}^{\text{bin},m}, p \geq t \right) \) are approximately \( \Pr(T_{m,c}^{\text{po}} \geq t) \). Thus \( z \sim \Pr(T_{m,c}^{\text{po}} \geq t) \). As \( c \) is fixed and \( t \to \infty \), we get

\[
z \sim \Pr(T_{m,c}^{\text{po}} = \infty),
\]

where \( z \) is a solution of the equation (7) in (0, 1), see also Lemma 1. Now, each \( s \)-set has probability \( z \) to be in the large connected component. The expected number of \( s \)-sets in the large connected component is \( z\binom{n}{s} \). Since we already showed the large connected component is unique, we proved the lemma.

\[\square\]

### 4 Variations of the \( s \)-th-order connection

Given an \( r \)-uniform hypergraph, there are some variation of the \( s \)-th-order connection for each \( 1 \leq s \leq r - 1 \). The definition of the \( s \)-th-order connection will be split into two cases.

The loose case is when \( 1 \leq s \leq r/2 \) and the tight case is when \( r/2 < s \leq r - 1 \).

The first variation was defined in [17] when the authors were studying the Laplacian spectra of random \( r \)-uniform hypergraphs. Let \( H = (V, E) \) be an \( r \)-uniform hypergraph. For each \( 1 \leq s \leq r - 1 \), let \( S, S' \in \binom{V}{s} \).

**Loose case:** We say \( S \) is connected to \( S' \) if there is an alternating sequence of \( s \)-sets and edges \( S_0, F_1, S_1, F_2, S_2, \ldots, S_{k-1}, F_k, S_k \) such that \( S_i \in \binom{V}{s} \), \( S_0 = S, S_k = S' \), \( F_i \in E(H) \), \( S_i \cap S_{i+1} = \emptyset \) and \( S_i \cup S_{i+1} \subseteq F_i \) for \( 0 \leq i \leq k - 1 \).

**Tight case:** We say \( S \) is connected to \( S' \) if there is an alternating sequence of \( s \)-sets and edges \( S_0, F_1, S_1, F_2, S_2, \ldots, S_{k-1}, F_k, S_k \) such that \( S_i \in \binom{V}{s} \), \( S_0 = S, S_k = S' \), \( F_i \in E(H) \), and \( S_i \cup S_{i+1} = F_i \) for \( 0 \leq i \leq k - 1 \).

For this definition of the \( s \)-th-order connection, we observe the following fact. We assume \( r \neq 2s \) for a moment. If there is some edge \( F \) containing an \( s \)-set \( S \), then \( S \) is connected to each \( s \)-set \( S' \in \binom{V}{s} \setminus \{S\} \). The argument is that we can find an alternating sequence of \( s \)-sets and edges, where all edges are \( F \) and each \( s \)-set is in \( \binom{V}{s} \), satisfying the condition in the definition of the \( s \)-th connection. Let us move to the random \( r \)-uniform graph. In the graph branching process, we choose a live \( s \)-set \( S \) and query all \( r \)-sets (not queried yet) containing \( S \). If we get one edge \( F \), then we can add a connected component (all \( s \)-sets of \( F \) but \( S \)) to the original connected component. Since we care only the size of connected components.
when we study the phase transition of the random uniform hypergraph, we can view all $s$-sets in $\binom{[n]}{s} \setminus \{S\}$ as neighbors of $S$ in the graph branching process when we find an edge $F$. Thus, the definition above of the $s$-th-order connection and the definition in the introduction give the same size of connected components in the random uniform hypergraph. Note that when $r = 2s$, each edge containing $S$ can only give one more neighbor of $S$ if we consider the $s$-th-order connection as described above. We have the following theorem on the size of $s$-th-order connected components (in the sense above) in $H^r(n, p)$.

**Theorem 2** Let $H^r(n, p)$ be the random $r$-uniform hypergraph. For $r \neq 2s$, we have the following.

**Subcritical phase:** If $p \leq \frac{1-\epsilon}{\binom{n}{2}}$ for some $\epsilon > 0$, then almost surely each $s$-th-order connected component is with size $O(\ln n)$.

**Supercritical phase:** If $p > \frac{1+\epsilon}{\binom{n}{2}}$, then almost surely there is a unique giant $s$-th-order connected component of size $\Theta(\binom{n}{s})$. All other $s$-th-order connected components in $H^r(n, p)$ have size $O(\ln n)$.

Moreover, if $p = \frac{c}{\binom{n}{2}}$ with $c = \frac{1+\epsilon}{\binom{n}{2} - 1}$. Almost surely the size of the giant $s$-th-order connected component is $(z + o(1))\binom{n}{s}$, where $z \in (0, 1)$ is a solution of the following equation

$$1 - x = e^{c(1-x)(\binom{n}{2} - 1)}.$$ 

For $r = 2s$, we have the following.

**Subcritical phase:** If $p \leq \frac{1-\epsilon}{\binom{n}{s}}$ for some $\epsilon > 0$, then almost surely each $s$-th-order connected component is with size $O(\ln n)$.

**Supercritical phase:** If $p > \frac{1+\epsilon}{\binom{n}{s}}$, then almost surely there is a unique giant $s$-th-order connected component of size $\Theta(\binom{n}{s})$. All other $s$-th-order connected components in $H^r(n, p)$ have size $O(\ln n)$.

Moreover, if $p = \frac{c}{\binom{n}{s}}$ with $c > \frac{1+\epsilon}{\binom{n}{s} - 1}$. Almost surely the size of the giant $s$-th connected component is $(z + o(1))\binom{n}{s}$, where $z \in (0, 1)$ is a solution of the following equation

$$1 - x = e^{-\epsilon x}.$$ 

There is one more variation of the $s$-th-order connection from [16] where the authors were studying the random walks on hypergraphs. Given a $r$-uniform hypergraph $H = (V, E)$, let $V^s$ be the set of all $s$-tuples with distinct coordinates. For $x = (x_1, x_2, \ldots, x_s) \in V^s$, let $[x] = \{x_1, x_2, \ldots, x_s\}$. Basically, $[x]$ is the set of all coordinates of $x$. For positive integers $n$ and $j$, let $(n)_j$ be the falling factorial $u(n - 1) \cdots (n - j + 1)$. The definition of a new $s$-th connection is following. For each $1 \leq s \leq r - 1$, Let $x, x' \in V^s$.

**Loose case:** We say $x$ is connected to $x'$ if there is an alternating sequence of $s$-tuples and edges $x_0, F_1, x_1, F_2, x_2, \ldots, x_{k-1}, F_k, x_k$ such that $x_i \in V^s, x_0 = x, x_k = x', F_i \in E(H), [x_i] \cap [x_{i+1}] = \emptyset$ and $[x_i] \cup [x_{i+1}] \subseteq F_i$ for $0 \leq i \leq k - 1$.

**Tight case:** We say $x$ is connected to $x'$ if there is an alternating sequence of $s$-tuples and edges $x_0, F_1, x_1, F_2, x_2, \ldots, x_{k-1}, F_k, x_k$ such that $x_i \in V^s, x_0 = x, x_k = x', F_i \in E(H)$, and $[x_i] \cup [x_{i+1}] = F_i$ for $0 \leq i \leq k - 1$. Moreover, assume $x_i = (x_i^1, x_i^2, \ldots, x_i^s)$ for each $0 \leq i \leq k$. Then either $x_i^{r-s+j} = x_{i+1}^j$ for each $0 \leq i \leq k - 1$ and $1 \leq j \leq 2s - r$; or $x_{i+1}^{r-s+j} = x_i^j$ for each $0 \leq i \leq k - 1$ and $1 \leq j \leq 2s - r$. 


With this new definition of the $s$-th-order connection, we have the same observation as the first variation in the loose case. Namely, if we get an edge $F$ containing an $s$-tuple $x$, then we get a component consists of all $s$-tuples with coordinates from $F$. We have a similar theorem on the size of the $s$-th-order connected component in $H'(n, p)$ with critical probability $p = 1/(r_s) - 1\binom{n}{r_s}$ in the loose case. We do not state the theorem here. For the tight case, the analysis is more complicated and we leave it for future study.

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