Magnon drag induced by magnon-magnon interactions characteristic of noncollinear magnets

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A noncollinear magnet consists of the magnetic moments forming a noncollinear spin structure. Because of this structure, the Hamiltonian of magnons acquires the cubic terms. Although the cubic terms are the magnon-magnon interactions characteristic of noncollinear magnets, their effects on magnon transport have not been clarified yet. Here we show that in a canted antiferromagnet the cubic terms cause a magnon drag that magnons drag magnon spin current and heat current, which can be used to enhance these currents by tuning a magnetic field. For a strong magnetic field, we find that the cubic terms induce low-temperature peaks of a spin-Seebeck coefficient, a magnon conductivity, and a magnon thermal conductivity, and that each value is one order of magnitude larger than the noninteracting value. This enhancement is mainly due to the magnetic field dependence of the coupling constant of the cubic terms through the magnetic-field dependent canting angle. Our magnon drag offers a way for controlling the magnon currents of noncollinear magnets via the many-body effect.

I. INTRODUCTION

Drag effects are nonequilibrium many-body effects. In contrast to electronic and magnetic properties, transport properties are essentially nonequilibrium because a current makes a system out of equilibrium. Then, transport properties are often described by a theory without interactions, but they are drastically changed by the effects of interactions, many-body effects. One of such examples is the phonon drag. The total momentum of electrons or phonons is not conserved with the electron-phonon interaction. As a result, phonons drag an electron charge or phonons is not conserved with the electron-phonon interaction. Many-body effects. One of such examples is the spin-Coulomb drag. As a result, phonons drag an electron charge or phonons is not conserved with the electron-phonon interaction. Many-body effects. One of such examples is the spin-Coulomb drag.

II. MODEL

A. Magnon Hamiltonian of the canted antiferromagnet

Our noncollinear magnet is described by the spin Hamiltonian,

\[ H = 2J \sum_\langle i,j \rangle S_i \cdot S_j - h \sum_{i=1}^{N/2} S_i^z - h \sum_{j=1}^{N/2} S_j^z. \]  

where \( J \) is the exchange interaction, \( h \) is the magnetic field, and \( N \) is the number of magnons.
Here the first term is the antiferromagnetic Heisenberg interaction between nearest-neighbor spins, and the other terms are the couplings with the magnetic field \( h = -g \mu_B B \), where \( g \) and \( \mu_B \) are the \( g \)-factor and Bohr magneton, respectively. We have omitted the dipolar interaction because it may be negligible for MnF\(_2\) (see Appendix A). We consider a three-dimensional case on the body-centered cubic lattice [Fig. 2(a)]; \( i \)'s and \( j \)'s in Eq. (1) are site indices for sublattices \( A \) and \( B \), respectively. In the range of \( 0 < h < 4JzS \), where \( z = 8 \), the canted state for \( S_i = \frac{1}{2}(S \sin \phi \ 0 \ S \cos \phi) \) and \( S_j = \frac{1}{2}(S \sin \phi \ 0 \ -S \cos \phi) \) is stabilized. For \( h = 0 \) or \( h > 4JzS = h_c \), the stabilized state becomes the Néel or the ferromagnetic state, respectively. (Note that the energy of the canted, the Néel, or the ferromagnetic state divided by \( N/2 \) is given in the mean-field approximation by \( \epsilon_{\text{AF}} = -2JzS^2 - \frac{h^2}{4Jz} \), \( \epsilon_{\text{AF}} = -2JzS^2 \), or \( \epsilon_{\text{FM}} = 2JzS^2 - 2Sh \), respectively.) Therefore, we choose the magnetic field to be \( 0 < h < h_c \), in the range of which low-energy excitations can be described by magnons for the canting antiferromagnet. Hereafter we set \( k_B = 1 \), \( h = 1 \), and \( a = 1 \), where \( a \) is the lattice constant.

To describe magnon properties, we rewrite Eq. (1) using the Holstein-Primakoff transformation for non-collinear magnets.\(^{109,109}\) As derived in Appendix B, the magnon Hamiltonian of our canted antiferromagnet is written as

\[
H = H_0 + H_{\text{int}},
\]

where the noninteracting part \( H_0 \) consists of the quadratic terms,

\[
H_0 = \sum_{q} \left( a_q^\dagger b_q + a_q b_q^\dagger \right) \left( A_q B_q \right) + \left( A_q^\dagger B_q^\dagger \right) \left( a_q^\dagger b_q^\dagger \right)
\]

and the interaction part \( H_{\text{int}} \) consists of the cubic terms,

\[
H_{\text{int}} = \sum_{q,q',q''q'} \delta_{q+q''q'} J_3(q) (b_q a_{q'}^\dagger a_{q''} - a_{q'} b_q^\dagger b_{q''}) + (\text{H.c.}).
\]

We have omitted the constant terms and quartic terms for simplicity. In Eq. (4), \( a_q \) and \( b_q \) are the Fourier coefficients of the magnon operators, the \( 2 \times 2 \) matrices \( A_q \) and \( B_q \) are given by \( (A_q)_{11} = (A_q)_{22} = \frac{1}{2}(2Jz \cos 2\phi S + h \sin \phi) = A \), \( (A_q)_{12} = (A_q)_{21} = -\frac{1}{2}J^{(+)}(q)S = A'(q) \), \( (B_q)_{12} = (B_q)_{21} = -\frac{1}{2}J^{(+)}(q)S = B'(q) \), and \( (B_q)_{11} = (B_q)_{22} = 0 \), \( (J^{(+)}(q) = (\cos 2\phi \mp 1)J(q) \), and \( J(q) = 8J \cos \frac{2\pi}{2} \cos \frac{2\pi}{2} \). In Eq. (4),

\[
J_3(q) = \frac{4S}{N} \sin 2\phi J(q).
\]

Equation (4) is similar to that of the electron-phonon interaction because the former and latter describe the creation and annihilation processes for three magnons and for two electrons and a phonon, respectively.

The coupling constant of the cubic terms depends on the magnetic field though the magnetic field dependence of the canting angle \( \phi \). Since \( \sin 2\phi = \frac{2\sin(\sqrt{(4JzS)^2 - h^2})}{4JzS} \) in our canted antiferromagnet, \( J_3(q) \) depends on the magnetic field. Figure 2(b) shows the \( h/J \) dependence of \( |J_3(q)/J(q)|^2 \) for \( S = \frac{5}{2} \) or \( \frac{3}{2} \). (Note that \( h_c = 4JzS \) for \( S = \frac{5}{2} \) or \( \frac{3}{2} \) is 80J or 48J, respectively.) We see the coupling constant of the cubic terms for \( S = \frac{5}{2} \) or \( \frac{3}{2} \) is maximum at \( h \sim 57J \) or \( 34J \), respectively. In addition, the coupling constant for \( S = \frac{5}{2} \) at \( h = 65J \) is much larger than that at \( h = 20J \). This suggests that the effects of the cubic terms are more considerable for strong magnetic fields than those for weak magnetic fields. (In fact, we will show in Sec. III B that the cubic terms cause the huge enhancement of the magnon-transport coefficients at \( h = 65J \) compared with that at \( h = 20J \).) We emphasize that the magnetic-field dependent coupling constant is characteristic of canted antiferromagnets. (Such a dependence is absent in the case of the phonon drag.)
FIG. 2. (a) The spin structure of the canted antiferromagnet on the body-centered cubic lattice with the $x$, $y$, and $z$ axes. The case for $S = \tfrac{3}{2}$ corresponds to MnF$_2$. The circles on the corners of the cube represent the sites on sublattice $A$, whereas that on the center represents the site on sublattice $B$. The arrows in the cube represent the canting spins. The magnetic field $h = -g\mu_B B$ is applied along the $x$ axis, where $g$ is the $g$-factor and $\mu_B$ is the Bohr magneton. The temperature gradient $\nabla T$ or the non-thermal external field $E_S$ is applied along the $z$ axis; as a result, the magnon spin current and heat current along it are induced. (b) The $h/J$ dependence of $|J_2(q)/J(q)|^2$ for $S = \tfrac{3}{2}$ or $\tfrac{5}{2}$ with $N = 20\hbar$ and $J = 1$. Here $J_2(q)$ is the coupling constant of the cubic terms. The red or blue curve represents that dependence for $S = \tfrac{3}{2}$ or $\tfrac{5}{2}$, respectively. The magnon-band dispersion relations along the symmetric lines in the momentum space at (c) $h = 20J$, (d) $40J$, and (e) $60J$ for $S = \tfrac{3}{2}$ with $N = 20\hbar$. The blue and red curves represent the energies divided by $J$ for the $\beta$-band and $\alpha$-band magnon $[i.e., \epsilon_\beta(q)/J]$ and $\epsilon_\alpha(q)/J$, respectively. The vertical dashed lines correspond to the values of $h$.

### B. Noninteracting magnon bands

We diagonalize Eq. (3) using the Bogoliubov transformation,

\[
\begin{pmatrix}
 a_q \\
 b_q \\
 a^\dagger_{-q} \\
 b^\dagger_{-q}
\end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix}
 c_q & c'_q & s_q & s'_q \\
 c_q - c'_q & s_q & -s'_q & 0 \\
 s_q & c'_q & c_q & s'_q \\
 s_q - s'_q & c_q & -c'_q & 0
\end{pmatrix} \begin{pmatrix}
 a_q \\
 b_q \\
 a^\dagger_{-q} \\
 b^\dagger_{-q}
\end{pmatrix},
\]

where $c_q = \cosh \theta_q$, $s_q = \sinh \theta_q$, $c'_q = \cosh \theta'_q$, and $s'_q = \sinh \theta'_q$. By substituting Eq. (6) into Eq. (3) and setting $\tanh 2\theta_q = -\frac{B'(q)}{A + A'(q)}$ and $\tanh 2\theta'_q = \frac{B'(q)}{A - A'(q)}$, we obtain

\[
H_0 = \sum_q [\epsilon_\alpha(q)a^\dagger_qa_q + \epsilon_\beta(q)b^\dagger_qb_q],
\]

where $\epsilon_\alpha(q) = 2\sqrt{|A + A'(q)|^2 - B'(q)^2}$ and $\epsilon_\beta(q) = 2\sqrt{|A - A'(q)|^2 - B'(q)^2}$. (Those choices of the hyperbolic functions are necessary to make the off-diagonal terms zero.) Figures 2(c) and 2(e) show the magnon-band dispersion for $S = \tfrac{3}{2}$ at $h = 20J$, 40J, and 60J. The band splitting energy at $q = 0$ is equal to $h$ and larger than those at the other $q$'s. This property is distinct from the property of a two-sublattice ferrimagnet in which the band splitting energies at $q = 0$ and the others are the same. Moreover, it indicates that even for $T < h$, the upper-branch magnons can contribute to transport properties. (This is true, as shown in Fig. 3.)

Note that we do not study the interacting magnon-band dispersion in this paper because the magnon-band energies appearing in the magnon-transport coefficients are the noninteracting ones [see Eqs. (15) and (16)].

### III. MAGNON-TRANSPORT COEFFICIENTS

#### A. Magnon-drag terms of $S_m$, $\sigma_m$, and $\kappa_m$

The magnon-transport coefficients $S_m$, $\sigma_m$, and $\kappa_m$ are connected with $j_S$ and $j_Q$, magnon spin and heat current densities:

\[
\begin{pmatrix}
 j_S \\
 j_Q
\end{pmatrix} = \begin{pmatrix}
 L_{11} & L_{12} \\
 L_{21} & L_{22}
\end{pmatrix} \begin{pmatrix}
 E_S \\
 -\nabla T
\end{pmatrix},
\]

where $L_{11} = \sigma_m$, $L_{12}(= L_{21}) = S_m$, $L_{22} = \kappa_m$, $E_S$ is a non-thermal external field, such as a magnetic-field gradient, and $\nabla T$ is a temperature gradient. (Note that our definition of $\kappa_m$ is enough to analyze its property at low temperatures at which the magnon picture remains valid.) Due to zero magnon chemical potential in equilibrium, $j_Q = j_E$ holds, where $j_E$ is a magnon energy current density. By using the continuity equations, we can express $J_k = Nj_k (k = S, E)$ as follows (see Appendix C):

\[
J_k = \sum_q \sum_{l,l'=1}^4 v_{kl}(q)x_{ql}^lx_{q'l'},
\]
where \( x_{q_1} = a_q \), \( x_{q_2} = b_q \), \( x_{q_3} = a^\dagger_q \), \( x_{q_4} = b^\dagger_q \),
\( \psi^0_q(q) = \psi^0(q) = \psi^0_1(q) \), and \( \psi^{11}_q(q) = e^{11}_q \); the finite terms of \( \psi^0_2(q) \) and \( e^{11}_q \) are given by
\( \psi^{11}_2(q) = -\psi_2(q) = \frac{\partial B'(q)}{\partial q}, \psi_2(q) = -\frac{\partial A'(q)}{\partial q} \), \( e^{11}_2(q) = -e^{11}_2(q) = -2A'(q) \partial A'(q) \), \( e_2(q) = e_2(q) = -e^{11}_2(q) = -2B'(q) \partial B'(q) - 2A'(q) \partial A'(q) \).
Hereafter we concentrate on the magnon transport with \( E_S \) or \( -\mathbf{v}/T \) applied along the \( z \) axis [Fig. 2(a)].

Since the magnon lifetime \( \tau \) is supposed to be long enough to regard magnons as quasiparticles, we derive \( L_{12}, L_{11}, \) and \( L_{22} \) using the linear-response theory [9–11, 13–15, 17] in the limit \( \tau \to \infty \). In the linear-response theory, \( L_{\mu\eta} (\mu, \eta = 1, 2) \) is given by
\[
L_{\mu\eta} = \lim_{\omega \to 0} \frac{\Phi_{\mu\eta}^R(\omega) - \Phi_{\mu\eta}^R(0)}{\omega^2} ,
\]
where \( \Phi_{\mu\eta}^R(\omega) = \Phi_{\mu\eta}(i\Omega_n \rightarrow \omega + i\delta) (\delta = 0+) \), \( \Omega_n = 2\pi Tn (n > 0) \),
\[
\Phi_{12}(i\Omega_n) = \int_0^{T^{-1}} d\tau e^{-i\Omega_n \tau} \frac{1}{N} (T_{\tau}J_5(\tau)J_6),
\]
\[
\Phi_{11}(i\Omega_n) = \int_0^{T^{-1}} d\tau e^{-i\Omega_n \tau} \frac{1}{N} (T_{\tau}J_6^*(\tau)J_5^*),
\]
\[
\Phi_{22}(i\Omega_n) = \int_0^{T^{-1}} d\tau e^{-i\Omega_n \tau} \frac{1}{N} (T_{\tau}J_6^*(\tau)J_6^*),
\]
and \( T_{\tau} \) is the time-ordering operator [9]. Since \( J_{\hat{z}, 5} \) and \( J_{\hat{z}, 6} \) are written as Eq. (9), we can calculate Eqs. (11)–(13) by using a method of Green’s functions [9, 13, 15, 17, 18]. In their calculations, we treat \( H_{\text{int}} \) in the second-order perturbation theory. As derived in Appendix D, \( L_{\mu\eta} \) can be written as follows:
\[
L_{\mu\eta} = L_{\mu\eta}^0 + L_{\mu\eta}^L ,
\]
where \( L_{\mu\eta}^0(\mu, \eta = 1, 2) \) is the noninteracting term,
\[
L_{\mu\eta}^0 = -\frac{2}{N} \sum_{q} \sum_{\nu, \nu'} \sum_{\alpha, \beta} \tilde{j}_{\mu\nu\nu'}(q) \tilde{j}_{\alpha\beta\nu'}(q) \tau \frac{\partial n[\epsilon_\nu(q)]}{\partial \epsilon_\nu(q)} ,
\]
and \( L_{\mu\eta}^L \) is the magnon-drug term due to the cubic term,
\[
L_{\mu\eta}^L = \frac{\pi}{2} \sum_{q, q', \nu, \nu', \nu''} \sum_{\alpha, \beta} \tilde{j}_{\mu\nu\nu'}(q) \tilde{j}_{\alpha\beta\nu'}(q) \gamma^2 \frac{\partial n[\epsilon_\nu(q)]}{\partial \epsilon_\nu(q)} \times S \sin^2 \phi \sum_{p=1,2,3} F_{p\nu\nu'}(q, q').
\]
In Eq. (15), \( n(x) = 1/(e^{x/T} - 1), \)
\( \tilde{j}_{\mu\nu\nu'}(q) = v_{\nu\nu'}(q), \)
and \( \tilde{j}_{2\nu\nu'}(q) = e_{\nu\nu'}(q), \)
where \( v_{\alpha\alpha}(q) = -v_{\beta\beta}(q) = 2v_{\nu\nu}(q), \)
\( e_{\alpha\alpha}(q) = 2e_{\nu\nu}(q) + e_{11}(q), \)
and \( e_{\beta\beta}(q) = 2[-e_{11}(q) + \epsilon_{11}(q)] \).
In Eq. (16),
\[
F_{p\nu\nu'}(q, q') = \left\{ n[\epsilon_{\nu'}(q - q')] - n[\epsilon_{\nu'}(q')] \right\}
\]
\[
\times \Delta[\epsilon_{\nu}(q) - \epsilon_{\nu}(q') + \epsilon_{\nu'}(q' - q') - \epsilon_{\nu'}(q') - \epsilon_{\nu'}(q)]
\]
\[
\times \Delta[\epsilon_{\nu}(q) - \epsilon_{\nu}(q') - \epsilon_{\nu'}(q') + \epsilon_{\nu'}(q)]
\]
and the finite components of \( v_{p\nu\nu'}(q, q') \)'s are given by those for \( (\nu, \nu', \nu'') = (\beta, \beta, \beta), (\beta, \alpha, \alpha), (\alpha, \beta, \alpha), \) (\( \alpha, \alpha, \beta \)) (for their expressions, see Appendix D).

B. Magnon-drug induced enhancement and low-temperature peaks of \( S_m, \sigma_m \), and \( \kappa_m \)

To determine the effects of the cubic terms quantitatively, we evaluate \( S_m, \sigma_m \), and \( \kappa_m \) numerically. We set \( J = 1 \) and \( S = \frac{1}{2} \). (In the case of \( S = \frac{1}{2} \), the magnon picture for the canted antiferromagnet is valid in the range of \( 0 < h < 0.5J \).)

The transition temperature \( T_c = \frac{\pi^2}{3} S(S + 1) \) is consistent with the Néel temperature \( T_N \) of MnF2 \( (S = \frac{1}{2}) \) if \( J \approx 1.5 K \approx 0.13 \text{meV} \). Note that \( h = 20J \) and \( 65J \) correspond to \( |B| \approx 20 \) and \( 65 \), respectively, using \( h = -\mu g B, \) with \( g = 2 \) and \( J = 0.13 \text{meV} \).

We believe such magnetic fields could be experimentally realized because the magnetic field of the order of 1000T is experimentally accessible [22].

We perform the momentum summations by dividing the first Brillouin zone into a \( N_q \)-point mesh [25] and setting \( N_q = 20^3 (= N/2) \). We consider the temperature range \( 0 < T \leq 28J(\approx 0.67T) \) because the perturbation theory with magnon-magnon interactions can reproduce the perpendicular susceptibility of MnF2 up to about 0.67T [25].

For simplicity, \( \tau \) is chosen to be \( \tau = \gamma_0 + \gamma_1 T + \gamma_2 T^2 \), where \( \gamma_0 = 10^{-2} J, \gamma_1 = 10^{-4}, \) and \( \gamma_2 = 10^{-3} \). We replace the delta functions in Eqs. (17)–(19) by the Lorentzian ones using \( \tau(x) \sim \frac{1}{\pi^2} \frac{\beta x}{(x + \gamma)} \), where \( \gamma = 1/2\tau \).

Figures 3(a)–3(c) show the temperature dependences of \( S_m, \sigma_m \), and \( \kappa_m \) at a weak magnetic field \( h = 20J \). The contributions from the upper-branch magnons are
non-negligible even at sufficiently low temperatures in the absence of the cubic terms (compare the red and yellow curves of these figures). Even in the presence of the cubic terms, the upper-branch magnons give the non-negligible contributions (compare the light blue and blue curves). Furthermore, the magnon-drag terms enhance $S_m$, $\sigma_m$, and $\kappa_m$. For example, the ratios $L_{12}/L_{12}^0$, $L_{11}/L_{11}^0$, and $L_{22}/L_{22}^0$ at $T = 7.5J$ are about 1.4, 1.1, and 1.2, respectively. (As we will show below, these ratios become much larger for $h = 65J$.) The broad peak of $S_m$ is consistent with the experimental result of MnF$_2$ because the voltage observed in the spin-Seebeck effect is proportional to $S_m$.

We turn to the results for a strong magnetic field $h = 65J$. Figure 3(d) shows that the magnon-drag term causes a peak at a low temperature $T = 7.5J \sim 0.16T_c$, at which the ratio $L_{12}/L_{12}^0$ reaches about 22. This low-temperature peak is similar to that induced by the phonondrag. In contrast to the phonon drag, our magnon drag induces a low-temperature peak of $\sigma_m$, as shown in Fig. 3(e). Thus, our magnon drag could explain a peak observed in $\sigma_m$ [52] if a noncollinear state is stabilized. A similar peak is observed also in $\kappa_m$ [Fig. 3(f)]. The ratios $L_{11}/L_{11}^0$ and $L_{22}/L_{22}^0$ at $T = 7.5J$ are about 23 and 20, respectively. These results suggest that our magnon drag can be used to enhance the magnon spin current and heat current by tuning the magnetic field. The contributions from the upper-branch magnons are non-negligible also for $h = 65J$ in the absence and presence of the cubic terms. Note that the larger enhancement of the magnon-transport coefficients for $h = 65J$ than for $h = 20J$ comes mainly from the magnetic field dependence of the coupling constant of the cubic terms.

We emphasize that our magnon drag can induce a similar peak for any transport coefficient described by magnon currents. This is an important difference between our magnon drag and the other drag effects. Therefore, our magnon drag provides a mechanism for a low-temperature peak of a transport coefficient.

IV. DISCUSSION

We discuss the generality of our magnon drag. The mechanism for our magnon drag will work as long as the magnon Hamiltonian contains the cubic terms. This is because the second-order perturbation of the cubic terms leads to the similar magnon-drag term. Thus, the simi-
lar enhancement of magnon-transport coefficients may be expected to occur in other noncollinear magnets, such as those with the Dzyaloshinsky-Moriya interaction or the dipolar interaction. We should note that our magnon drag does not necessarily occur in any noncollinear magnets because there is a noncollinear magnet in which the cubic terms are zero. The cubic terms in the magnon Hamiltonian are vital for our magnon drag.

We comment on two ways to reduce the critical magnetic field at which a low-temperature peak appears. One is to make $S$ smaller; in our model for $S = 3/2$, the similar peaks of $S_m$, $\sigma_m$, and $\kappa_m$ are obtained at $h = 40 J$ (see Appendix E). The other is to reduce the dimension; for example, in a two-dimensional canted antiferromagnet, a low-temperature peak could be realized at smaller $h$’s. Thus, the low-temperature peaks due to the magnon drag induced by the cubic terms could be realized in various noncollinear magnets.

Our results suggest a similar drag for phonons or photons. For example, a phonon drag could be realized in the presence of the anharmonicity of lattice forces, which leads to the cubic terms in the phonon Hamiltonian. Our theory is useful to study transport properties for other Bose quasiparticles.

Finally, we discuss the differences between the present magnon drag and another one induced by the quartic terms. The first-order perturbation of the quartic terms causes another magnon drag. In contrast to the present magnon drag, its effect is described by the drag terms proportional to $\tau$. Thus, the effects of the magnon drag induced by the quartic terms are to modify the values of the magnon-transport coefficients. More importantly, it does not cause any peak, and its effects are negligible at low temperatures. Meanwhile, the present magnon drag causes the enhancement of $S_m$, $\sigma_m$, and $\kappa_m$ even at low temperatures and their low-temperature peaks for the strong magnetic fields. Since many-body effects are usually negligible at low temperatures, the enhancement and low-temperature peaks shown in the present paper may be unusual many-body effects. Note that since the Holstein-Primakoff method is based on the $1/S$ expansion, the effects of the second-order perturbation due to the cubic terms should be compared with those of the first-order perturbation due to the quartic terms. [The second-order terms of the cubic terms and the first-order terms of the quartic terms are both $O(S^0)$.

V. CONCLUSION

In summary, we showed the magnon drag induced by the cubic terms. Its effects on $S_m$, $\sigma_m$, and $\kappa_m$ are described by the terms proportional to $\tau^2$, whereas the non-interacting terms are proportional to $\tau$. Our magnon drag enhances $S_m$, $\sigma_m$, and $\kappa_m$ even at low temperatures and induces their low-temperature peaks for the strong magnetic field. It provides a mechanism for explaining a peak observed in a transport coefficient. The broad peak of $S_m$ for the weak magnetic field agrees with the experimental result of MnF$_2$. Our results open a way to control the magnon spin current and heat current of noncollinear magnets by tuning the magnetic field.

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Appendix A: Estimate of the dipolar interaction energy

We estimate the dipolar interaction energy for MnF$_2$. According to the argument of Ref 56, the dipolar interaction energy $U_{\text{dip}}$ will be estimated from $U_{\text{dip}} \approx \frac{g\mu_B}{r}$, where $r$ is the distance between two magnetic dipoles. This equation can be written as $U_{\text{dip}} \approx \frac{e^2}{a_0 (137)^2} (\frac{a_0}{r})^3 \approx 27.2 \frac{1}{(137)^2} (\frac{a_0}{r})^3 (eV)$, where $a_0 \approx 0.53 \AA$. For MnF$_2$, the lattice constant along the $a$ or $b$ axis is $a \approx 4.9 \AA$, and that along the $c$ axis is $c \approx 3.3 \AA$ (This difference in the lattice constant has been neglected in our model for simplicity.) Setting $r = a$ or $c$ in the above relation, we get $U_{\text{dip}} \approx 1.4$ or $5.8 \mu eV$, respectively. Since these values are much smaller than the antiferromagnetic Heisenberg interaction, the dipolar interaction may be negligible for MnF$_2$.

Appendix B: Holstein-Primakoff transformation for a noncollinear magnet

Before performing the Holstein-Primakoff transformation, we need to rewrite the spin Hamiltonian in terms of rotated spin operators. In general, magnons describe spin fluctuations, the deviations from the ground-state magnetic moments. Since their directions are site-dependent in noncollinear magnets, we need to perform a rotation of the spin at each site.

In our case, the ground-state magnetic moments are characterized by $S_i = \frac{i}{2} (S \sin \phi 0 S \cos \phi)$ and $S_j = \frac{i}{2} (S \sin \phi 0 - S \cos \phi)$ when $i$ and $j$ belong to sublattices $A$ and $B$, respectively. Thus, we introduce the following rotated spin operators:

$$S_i' = R(-\phi)S_i,$$
$$S_j' = R(\pi + \phi)S_j,$$

where the rotation matrix $R(\theta)$ is given by $[R(\theta)]_{ij} = \cos \theta (\delta_{ij} - \sin \theta)$, and $[R(\theta)]_{ij} = \sin \theta \cos \theta (\delta_{ij} - \sin \theta)$. The rotation angles have been chosen in order that $S_i'$ and $S_j'$ satisfy $S_i' = (S \sin \phi 0 S \cos \phi)$ and $S_j' = (S \sin \phi 0 - S \cos \phi)$.

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Because of this property, we can apply the Holstein-Primakoff transformation similar to that for ferromagnets to the spin Hamiltonian expressed in terms of $S_i^x$ and $S_j^x$. Combining Eqs. (B1) and (B2) with Eq. (B3), we obtain

$$H = 2J \sum_{i,j} [\cos 2\phi (S_i^x S_j^x + S_i^y S_j^y) + S_i^z S_j^z]$$

$$+ 2J \sin 2\phi \sum_{i,j} (S_i^x S_j^x - S_i^y S_j^y)$$

$$- \hbar \sum_i \cos \phi S_i^x + \sin \phi S_i^y$$

$$- \hbar \sum_j \cos \phi S_j^x + \sin \phi S_j^y). \quad (B3)$$

We now apply the Holstein-Primakoff transformation,

$$S_i^z = S - a_i^\dagger a_i, \quad S_i^{\pm} = \sqrt{2S} - a_i^\dagger a_i, \quad S_i^{-} = (S_i^+)^\dagger, \quad (B4)$$

$$S_j^z = S - b_j^\dagger b_j, \quad S_j^{\pm} = \sqrt{2S} - b_j^\dagger b_j, \quad S_j^{-} = (S_j^+)^\dagger. \quad (B5)$$

to Eq. (B3). To consider magnon-magnon interactions, we apply a $1/S$ expansion to the above equations of $S_i^{\pm}$ and $S_j^{\pm}$; the result is

$$S_i^{\pm} \sim \sqrt{2S}a_i - \frac{1}{2\sqrt{2S}} a_i^\dagger a_i, \quad (B6)$$

$$S_i^{-} \sim \sqrt{2S}a_i^\dagger - \frac{1}{2\sqrt{2S}} a_i a_i^\dagger, \quad (B7)$$

$$S_j^{\pm} \sim \sqrt{2S}b_j - \frac{1}{2\sqrt{2S}} b_j^\dagger b_j, \quad (B8)$$

$$S_j^{-} \sim \sqrt{2S}b_j^\dagger - \frac{1}{2\sqrt{2S}} b_j b_j^\dagger. \quad (B9)$$

Substituting Eqs. (B6)–(B9) and the first equations of Eqs. (B4) and (B5) into Eq. (B3), we obtain Eq. (B3) with Eqs. (B4) and (B5).

**Appendix C: Derivation of Eq. (9)**

We derive Eq. (9) using the continuity equations. This derivation can be performed in a way similar to those for another noncollinear magnet and for a collinear magnet.

First, we derive the spin current operator $J^S$, the $k = S$ component of Eq. (9). We suppose that the $z$ components of $S_i^z$ and $S_j^z$ satisfy the following continuity equation:

$$\frac{dS_i^z}{dt} + \nabla \cdot j_i^S = 0, \quad (C1)$$

where $j_i^S$ is a spin current operator at site $m$. Using Eq. (C1), we have

$$\frac{d}{dt} \left( \sum_m R_m s_m^z \right) = - \sum_m R_m \nabla \cdot j_m^S = \sum_m j_m^S = J_m^S, \quad (C2)$$

where $l = A$ or $B$ for $m \in A$ or $B$, respectively. (Note that $m \in A$ or $B$ means that site $m$ belongs to sublattice $A$ or $B$, respectively.) In deriving this equation, we have omitted the surface contributions. Equation (C2) can be rewritten as follows:

$$J_A^S = i[H, \sum_i R_i S_i^z], \quad (C3)$$

$$J_B^S = i[H, \sum_j R_j S_j^z]. \quad (C4)$$

Then, the spin current operator $J^S$ is given by

$$J^S = J_A^S + J_B^S. \quad (C5)$$

Since we consider the magnon system described by $H = H_0 + H_{\text{int}}$, we replace the $H$'s, $S_i^z$'s, and $S_j^z$'s in Eqs. (C3) and (C4) by the $H_0$'s, $S - a_i^\dagger a_i$, and $S - b_j^\dagger b_j$, respectively. As a result, we have

$$J_A^S = \sum_i \{iR_i[a_i^\dagger a_i, H_0] = \sum_i iR_i[a_i^\dagger a_i, H_{AB}], \quad (C6)$$

$$J_B^S = \sum_j \{iR_j[b_j^\dagger b_j, H_0] = \sum_j iR_j[b_j^\dagger b_j, H_{AB}], \quad (C7)$$

where

$$H_{AB} = - j^{(+)} S \sum_{i,j} (a_i b_j + a_i^\dagger b_j^\dagger) - j^{(-)} S \sum_{i,j} (a_i^\dagger b_j + a_i b_j^\dagger), \quad (C8)$$

and $j^{(+)} = (\cos 2\phi \pm 1)J$. After some algebra, Eqs. (C6) and (C7) reduce to

$$J_A^S = \sum_{i,j} \{iR_i S \left[ j^{(+)} (a_i b_j - a_i^\dagger b_j^\dagger) + j^{(-)} (a_i^\dagger b_j - a_i b_j^\dagger) \right], \quad (C9)$$

$$J_B^S = \sum_{i,j} \{iR_j S \left[ j^{(+)} S (a_i b_j - a_i^\dagger b_j^\dagger) - j^{(-)} (a_i^\dagger b_j - a_i b_j^\dagger) \right]. \quad (C10)$$

Combining these equations with Eq. (C5), we have

$$J_S = i \sum_{i,j} (R_i + R_j, j^{(+)} S (a_i b_j - a_i^\dagger b_j^\dagger)$$

$$+ i \sum_{i,j} (R_i - R_j, j^{(-)} S (a_i^\dagger b_j - a_i b_j^\dagger). \quad (C11)$$

By using the Fourier coefficients of the magnon operators,

$$a_i = \sqrt{\frac{2}{N}} \sum_q a_q e^{-iqR_i}, b_j = \sqrt{\frac{2}{N}} \sum_q b_q e^{-iqR_j}, \quad (C12)$$
we can express Eq. [C11] as follows:

\[
J_S = - \sum_q \frac{\partial \tilde{j}^{(+))(q)}}{\partial q} S(a_{-q}b_q + a_{-q}^{\dagger}b_q) - \sum_q \frac{\partial \tilde{j}^{(-)(q)}}{\partial q} S(a_qb_q + a_q^{\dagger}b_q) = - \frac{S}{2} \sum_q \left[ \frac{\partial \tilde{j}^{(+))(q)}}{\partial q} (a_{-q}b_q + a_{-q}^{\dagger}b_q - a_qb_q - a_q^{\dagger}b_q) + \frac{\partial \tilde{j}^{(-)(q)}}{\partial q} (a_qb_q + a_q^{\dagger}b_q - a_{-q}b_q - a_{-q}^{\dagger}b_q) \right],
\]

(C13)

where \( \tilde{j}^{(\pm)}(q) = (\cos 2\phi \pm 1)J(q) \). This is equivalent to the \( k = S \) component of Eq. [9].

Then, we derive the energy current operator \( J_E \), the \( k = E \) component of Eq. [9]. We suppose that the Hamiltonian at site \( m \), \( h_m \), satisfies the following continuity equation:

\[
\frac{dh_m}{dt} + \nabla \cdot j_m^{(E)} = 0,
\]

(C14)

where \( j_m^{(E)} \) is an energy current operator at site \( m \). In a way similar to the derivation of \( J_S \), we can determine the energy current operator \( J_E \) from

\[
J_E = i[H_0, \sum_n R_n h_n] = i \sum_{m,n} R_n [h_m, h_n],
\]

(C15)

where \( \sum_{n=1}^{N/2} h_i + \sum_{j=1}^{N/2} h_j = H_0 \), \( h_i = h_{iAA} + h_{iAB} \), and \( h_j = h_{jBB} + h_{jBA} \). Here \( h_{iAA}, h_{iAB}, h_{jBB}, \) and \( h_{jBA} \) are given by

\[
h_{iAA} = (2Jz \cos 2\phi S + h \sin \phi) a_i^{\dagger} a_i,
\]

(h16)

\[
h_{iAB} = - \frac{1}{2} S \sum_n \left[ \tilde{j}^{(+)m}_i (a_i b_n + a_i^{\dagger} b_n) + \tilde{j}^{(-)m}_i (a_i b_n + a_i^{\dagger} b_n) \right],
\]

(C17)

\[
h_{jBB} = (2Jz \cos 2\phi S + h \sin \phi) b_j^{\dagger} b_j,
\]

(C18)

\[
h_{jBA} = - \frac{1}{2} S \sum_m \left[ \tilde{j}^{(+)m}_i (a_m b_j + a_m^{\dagger} b_j) + \tilde{j}^{(-)m}_i (a_m b_j + a_m^{\dagger} b_j) \right],
\]

(C19)

where \( \tilde{j}^{(\pm)}_{ij} = (\cos 2\phi \pm 1)J_{ij} \), and \( J_{ij} = J \) for nearest-neighbor \( i \) and \( j \). Combining these equations with Eq. [C15], we have

\[
J_E = \sum_{m,n} (R_n - R_m) \left[ [h_{mAA}, h_{nAB}] + [h_{mAB}, h_{nAA}] + [h_{mAB}, h_{nBA}] + [h_{mBB}, h_{nBA}] \right] + \sum_{m,n} R_n \left[ [h_{mAB}, h_{nAA}] + [h_{mBA}, h_{nBA}] \right].
\]

(C20)

After some calculations, Eq. (C20) reduces to

\[
J_E = \sum_{m,n,i,j} i(R_i - R_j) S(2Jz \cos 2\phi S + h \sin \phi) \tilde{j}^{(-)}_{ij}(a_i b_j^{\dagger} + a_j b_i^{\dagger}) + \sum_{m,n,i,j} S^2 \frac{i}{2} (R_m - R_n) \left[ \tilde{j}^{(+)m}_i (J^{(+)m}_i - J^{(-)m}_i) b_j^{\dagger} b_n + \tilde{j}^{(-)m}_i (J^{(+)}m_{-j} - J^{(-)m}_{-j}) a_m^{\dagger} a_n \right].
\]

(C21)

By using the Fourier coefficients of the magnon operators [Eq. (C12)], Eq. (C21) can be written as follows:

\[
J_E = \sum_{m,n} \left\{ (2Jz \cos 2\phi S + h \sin \phi) \frac{\partial \tilde{j}^{(-)}(q)}{\partial q} S(a_q b_q + a_q^{\dagger} b_q) + S^2 \left[ \frac{\partial \tilde{j}^{(+)m}_i (q)}{\partial q} b_j^{\dagger} b_n + \frac{\partial \tilde{j}^{(-)m}_i (q)}{\partial q} a_m^{\dagger} a_n \right] \right\}
\]

\[
= \sum_{m,n} \left[ (2Jz \cos 2\phi S + h \sin \phi) \frac{\partial \tilde{j}^{(-)}(q)}{\partial q} S \times (a_q b_q + a_q^{\dagger} b_q - a_{-q} b_q^{\dagger} - a_{-q}^{\dagger} b_q) \right.
\]

\[
+ \left. \sum_{m,n} \left[ \frac{\partial \tilde{j}^{(+)m}_i (q)}{\partial q} b_j^{\dagger} b_n + \frac{\partial \tilde{j}^{(-)m}_i (q)}{\partial q} a_m^{\dagger} a_n \right] \times (a_q b_q + a_q^{\dagger} b_q - a_{-q} b_q^{\dagger} - a_{-q}^{\dagger} b_q) \right].
\]

(C22)

This gives the \( k = E \) component of Eq. [9].

### Appendix D: Derivations of Eqs. (15) and (16) with the expressions of \( \psi^{(p)}_{\mu \nu \mu \nu} (q, q^\prime) \)’s appearing in Eqs. (17)–(19)

We derive Eqs. (15) and (16), \( L^0_{\mu \nu} \) and \( L^1_{\nu \mu} \) \( (\mu, \nu = 1, 2) \) in the limit \( \tau \to \infty \), and show the expressions of \( \psi^{(p)}_{\mu \nu \mu \nu} (q, q^\prime) \)’s \( (p = 1, 2, 3) \) appearing in Eqs. (17)–(19). Since we can derive \( L_{11}, L_{22}, L_{11}, \) and \( L_{22} \) in a way similar to the derivation of \( L_{11}^0 \) and \( L_{22}^1 \), we explain the derivations of \( L_{12}^0 \) and \( L_{12}^1 \) below. Their derivations can be performed in a way similar to those of the spin-Seebeck coefficient of a collinear magnet[25] and of the Seebeck coefficient of a metal[24]. The \( \psi^{(p)}_{\mu \nu \mu \nu} (q, q^\prime) \)’s are given by Eqs. (D64)–(D75) with Eqs. (D76)–(D92).

First, we derive Eq. (15), the expression of \( L^0_{\mu \nu} \) in the limit \( \tau \to \infty \). After deriving the general expression of \( L^0_{12} \) [Eq. (D22)], we derive its expression in the limit \( \tau \to \infty \) [Eq. (D23)]. Then, we explain how \( L^0_{11} \) and \( L^0_{22} \) are obtained
we have

\[ \text{L} \]

respectively.

Eqs. (D41)–(D43) by the integrals. The most important information is about the imaginary parts; in the actual calculations, we consider them correctly. The \( C \) in these panels, we neglect the horizontal shifts due to the noninteracting energies, such as \( -\), used to derive Eq. (D17); its contributions from the region for \( -\Omega \) in Eqs. (D41)–(D43) by the integrals. The \( C' \) or \( C'' \) is used to replace the sum over \( m \) in Eqs. (D41) or (D42) or in Eq. (D43), respectively.

from \( L_{12}^0 \) and show their expressions in the limit \( \tau \to \infty \) [Eqs. (D29) and (D30)]. Substituting Eq. (9) into Eq. (11), we have

\[
\Phi_{12}(i\Omega_n) = \frac{1}{N} \sum_{q, q'} \sum_{l_1, l_2, l_3, l_4} v_{l_1 l_2}^*(q) e_{l_3 l_4}^*(q') G_{l_1 l_2 l_3 l_4}^{II}(q, q'; i\Omega_n),
\]

(D1)

where

\[
G_{l_1 l_2 l_3 l_4}^{II}(q, q'; i\Omega_n) = \int_0^{T-1} d\tau e^{i\Omega_n \tau} (T_\tau x_{ql_1}^\dagger(\tau) x_{ql_2}(\tau) x_{ql_3}^\dagger x_{ql_4}).
\]

(D2)

The expectation value in Eq. (D2) can be calculated by using the method of Green’s function. Equation (D1) provides a starting point to derive \( L_{12}^0 \) and \( L_{12}^1 \). To derive \( L_{12}^0 \), we evaluate Eq. (D2) without the effects of \( H_{\text{int}} \) using the Wick’s theorem; the result is

\[
G_{l_1 l_2 l_3 l_4}^{II}(q, q'; i\Omega_n) = \delta_{q, q'} T \sum_m G_{l_1 l_1}(q, i\Omega_m) G_{l_2 l_2}(q, i\Omega_n + m),
\]

(D3)

where \( G_{l_{l'}}(q, i\Omega_m) \) is the magnon Green’s function in the Matsubara-frequency representation,

\[
G_{l_{l'}}(q, i\Omega_m) = \int_0^{T-1} d\tau e^{i\Omega_m \tau} G_{l_{l'}}(q, \tau) = -\int_0^{T-1} d\tau e^{i\Omega_m \tau} (T_\tau x_{ql}^\dagger(\tau) x_{ql'}^\dagger),
\]

(D4)

and \( \Omega_m = 2\pi T m \). Substituting Eq. (D3) into Eq. (D1), we obtain

\[
\Phi_{12}^{(0)}(i\Omega_n) = \frac{1}{N} \sum_q \sum_{l_1, l_2, l_3, l_4} v_{l_1 l_2}^*(q) e_{l_3 l_4}^*(q) T \sum_m G_{l_1 l_1}(q, i\Omega_m) G_{l_2 l_2}(q, i\Omega_n + m).
\]

(D5)

By using the Bogoliubov transformation [Eq. (9)],

\[
x_{ql} = \sum_{\nu = \alpha_1, \beta_1, \alpha_2, \beta_2} (P_{q})_{l_0} x_{q l}'
\]

(D6)
where

\[
x'_{q\alpha_1} = \alpha_q, \ x'_{q\beta_1} = \beta_q, \ x'_{q\alpha_2} = \alpha_q^* = \beta_q^*, \ x'_{q\beta_2} = \beta_q^*,
\]

\[
(P_q)_{1\alpha_1} = (P_q)_{2\alpha_1} = (P_q)_{3\alpha_2} = (P_q)_{4\alpha_2} = \frac{1}{\sqrt{2}} \cosh \theta_q,
\]

\[
(P_q)_{3\alpha_1} = (P_q)_{4\alpha_1} = (P_q)_{1\alpha_2} = (P_q)_{2\alpha_2} = \frac{1}{\sqrt{2}} \sinh \theta_q,
\]

\[
(P_q)_{1\beta_1} = -(P_q)_{2\beta_1} = -(P_q)_{3\beta_2} = -(P_q)_{4\beta_2} = \frac{1}{\sqrt{2}} \cosh \theta_q',
\]

\[
(P_q)_{3\beta_1} = -(P_q)_{4\beta_1} = (P_q)_{1\beta_2} = -(P_q)_{2\beta_2} = \frac{1}{\sqrt{2}} \sinh \theta_q',
\]

we can rewrite Eq. (D5) as follows:

\[
\Phi^{(0)}_{12}(i\Omega_n) = \frac{1}{N} \sum_q \sum_{\nu,\nu'=\alpha_1,\alpha_2,\beta_2} v^z_{\nu'\nu}(q) e^{z}_{\nu'\nu}(q) T \sum_m G_{\nu'}(q, i\Omega_m) G_{\nu}(q, i\Omega_{m+n}),
\]

where

\[
v^z_{\nu'\nu}(q) = \sum_{l_1, l_2=1}^4 (P_q)_{l_1\nu'}(P_q)_{l_2\nu} v^z_{l_1l_2}(q),
\]

\[
e^{z}_{\nu'\nu}(q) = \sum_{l_1, l_2=1}^4 (P_q)_{l_1\nu'}(P_q)_{l_2\nu} e^{z}_{l_1l_2}(q),
\]

\[
G_{\alpha_1}(q, i\Omega_m) = \frac{1}{i\Omega_m - \epsilon_\alpha(q)}, \ G_{\beta_1}(q, i\Omega_m) = \frac{1}{i\Omega_m - \epsilon_\beta(q)},
\]

\[
G_{\alpha_2}(q, i\Omega_m) = -\frac{1}{i\Omega_m + \epsilon_\alpha(q)}, \ G_{\beta_2}(q, i\Omega_m) = -\frac{1}{i\Omega_m + \epsilon_\beta(q)}.
\]

Then, to perform the analytic continuation, we replace the Matsubara-frequency summation in Eq. (D12) by the corresponding integral, and the result is

\[
T \sum_m G_{\nu'}(q, i\Omega_m) G_{\nu}(q, i\Omega_{m+n}) = \int_C \frac{dz}{2\pi i} n(z) G_{\nu'}(q, z) G_{\nu}(q, z + i\Omega_n) + T[G_{\nu'}(q, 0) G_{\nu}(q, i\Omega_n) + G_{\nu'}(q, -i\Omega_n) G_{\nu}(q, 0)]
\]

\[
= \int_{-\infty}^\infty \frac{dz}{2\pi i} n(z) \left\{ G^{(R)}_{\nu'}(q, z + i\Omega_n) G^{(R)}_{\nu}(q, z) - G^{(A)}_{\nu'}(q, z) \right\}
\]

\[
+ \left\{ G^{(R)}_{\nu'}(q, z) - G^{(A)}_{\nu'}(q, z) \right\} G^{(A)}_{\nu}(q, z - i\Omega_n) \right\},
\]

where the contour \( C \) is shown in Fig. 4(a), \( n(z) \) is the Bose distribution function \( n(z) = 1/(e^{z/T} - 1) \), \( G^{(R)}_{\nu'}(q, z) \) and \( G^{(A)}_{\nu'}(q, z) \) are the retarded and advanced magnon Green’s functions, respectively,

\[
G^{(R)}_{\alpha_1}(q, z) = \frac{1}{z + i\gamma - \epsilon_\alpha(q)}, \ G^{(R)}_{\beta_1}(q, z) = \frac{1}{z - i\gamma - \epsilon_\beta(q)},
\]

\[
G^{(R)}_{\alpha_2}(q, z) = -\frac{1}{z + i\gamma + \epsilon_\alpha(q)}, \ G^{(R)}_{\beta_2}(q, z) = -\frac{1}{z + i\gamma + \epsilon_\beta(q)},
\]

and \( \gamma = 1/2\tau \) is the magnon damping. By combining Eq. (D17) with Eq. (D12) and performing the analytic continuation \( i\Omega_n \to \omega + i\delta \) [i.e., \( \Phi^{(0)}_{12}(\omega) = \Phi^{(0)}_{12}(i\Omega_n \to \omega + i\delta) \)], we obtain

\[
\Phi^{(R)}_{12}(\omega) = \frac{1}{N} \sum_q \sum_{\nu,\nu'=\alpha_1,\alpha_2,\beta_2} v^z_{\nu'\nu}(q) e^{z}_{\nu'\nu}(q) \int_{-\infty}^\infty \frac{dz}{2\pi i} n(z)
\]

\[
\times \left\{ G^{(R)}_{\nu'}(q, z + \omega) G^{(R)}_{\nu}(q, z) - G^{(A)}_{\nu'}(q, z) \right\} + \left\{ G^{(R)}_{\nu'}(q, z) - G^{(A)}_{\nu'}(q, z) \right\} G^{(A)}_{\nu}(q, z - \omega) \right\}.
\]
After some calculations, Eq. \((\text{D20})\) reduces to
\[
\Phi^{R(0)}_{12}(\omega) \sim \Phi^{R(0)}_{12}(0) - \frac{\omega}{2N} \sum_q \sum_{\nu,\nu' = \alpha_1, \alpha_2, \beta_1, \beta_2} v_{\nu \nu'}(q)e_{\nu \nu'}(q) \int_{-\infty}^{\infty} \frac{dz}{\pi} \frac{\partial n(z)}{\partial z} [-4 \text{Im} G^{(R)}_\nu(q, z) \text{Im} G^{(R)}_{\nu'}(q, z)].
\]  
\(\text{(D21)}\)

In deriving this equation, we have used \(f(z \pm \omega) = f(z) \pm \omega \frac{\partial f(z)}{\partial z} + O(\omega^2)\), \(v_{\nu \nu'}(q) = v_{\nu \nu'}(q, 0)\), and \(e_{\nu \nu'}(q) = e_{\nu \nu'}(q, 0)\). Combining Eq. \((\text{D21})\) with Eq. \((10)\), we have
\[
L_{12}^0 = \lim_{\omega \to 0} \frac{\Phi^{R(0)}_{12}(\omega) - \Phi^{R(0)}_{12}(0)}{i \omega} = -\frac{1}{N} \sum_q \sum_{\nu, \nu' = \alpha_1, \alpha_2, \beta_1, \beta_2} v_{\nu \nu'}^2(q)e_{\nu \nu'}(q) \int_{-\infty}^{\infty} \frac{dz}{\pi} \frac{\partial n(z)}{\partial z} \text{Im} G^{(R)}_\nu(q, z) \text{Im} G^{(R)}_{\nu'}(q, z).
\]

Then, we take the limit \(\tau = 1/2\gamma \to \infty\). In this limit, the integral part in Eq. \((\text{D22})\) reduces to
\[
I_{\nu \nu'}(q) = \int_{-\infty}^{\infty} \frac{dz}{\pi} \frac{\partial n(z)}{\partial z} \text{Im} G^{(R)}_\nu(q, z) \text{Im} G^{(R)}_{\nu'}(q, z) \sim \begin{cases} 1 & (\nu = \nu' = \alpha_1, \alpha_2) \\ \frac{1}{2\gamma} \frac{\partial n[\epsilon_\alpha(q)]}{\partial \epsilon_\nu(q)} & (\nu = \nu' = \beta_1, \beta_2) \\ 0 & (\nu \neq \nu') \end{cases}.
\]
\(\text{(D23)}\)

These limiting expressions can be obtained by using Eqs. \((\text{D18})\) and \((\text{D19})\) and doing the integral \(\text{Im} G^{(R)}_\nu(q, z) \text{Im} G^{(R)}_{\nu'}(q, z)\). Combining Eq. \((\text{D23})\) with Eq. \((\text{D22})\), we obtain the expression of \(L_{12}^0\) in the limit \(\tau = 1/2\gamma \to \infty\),
\[
L_{12}^0 \sim -\frac{1}{N} \sum_q \sum_{\nu, \nu' = \alpha_1, \alpha_2, \beta_1, \beta_2} v_{\nu \nu'}^2(q)e_{\nu \nu'}(q)\frac{\partial n[\epsilon_\nu(q)]}{\partial \epsilon_\nu(q)}.
\]
\(\text{(D24)}\)

where \(\epsilon_{\alpha_1}(q) = \epsilon_{\alpha_2}(q) = \epsilon_\alpha(q)\) and \(\epsilon_{\beta_1}(q) = \epsilon_{\beta_2}(q) = \epsilon_\beta(q)\). In addition, using Eqs. \((\text{D13})\) and \((\text{D14})\) and Eqs. \((\text{D8})-\,(\text{D11})\), we have
\[
v_{\alpha_1 \alpha_1}(q) = -v_{\beta_1 \beta_1}(q) = -v_{\alpha_2 \alpha_2}(q) = -v_{\beta_2 \beta_2}(q) = 2v_{12}(q),
\]
\(\text{(D25)}\)
\[
e_{\alpha_1 \alpha_1}(q) = e_{\alpha_2 \alpha_2}(q) = 2[e_{12}(q) + e_{11}(q)],
\]
\(\text{(D26)}\)
\[
e_{\beta_1 \beta_1}(q) = e_{\beta_2 \beta_2}(q) = -2[e_{12}(q) + e_{11}(q)],
\]
\(\text{(D27)}\)

where \(v_{12}(q), e_{12}(q),\) and \(e_{11}(q)\) are defined below Eq. \((9)\). Thus, Eq. \((\text{D24})\) reduces to
\[
L_{12}^0 \sim -\frac{2}{N} \sum_q \sum_{\nu, \nu' = \alpha, \beta} v_{\nu \nu'}^2(q)e_{\nu \nu'}(q)\frac{\partial n[\epsilon_\nu(q)]}{\partial \epsilon_\nu(q)},
\]
\(\text{(D28)}\)

where \(v_{\alpha}(q) = -v_{\beta}(q) = v_{\alpha_1}(q), e_{\alpha}(q) = e_{\alpha_1}(q),\) and \(e_{\beta}(q) = e_{\beta_1}(q)\). Then, Eqs. \((9)\) and \((11)\) show that \(L_{11}^0\) and \(L_{22}^0\) are obtained by replacing \(e_{\nu}(q)\) in Eq. \((\text{D28})\) by \(e_{\nu}(q)\) and by replacing \(v_{\nu}(q)\) in Eq. \((\text{D28})\) by \(e_{\nu}(q)\), respectively. Therefore, \(L_{11}^0\) and \(L_{22}^0\) in the limit \(\tau \to \infty\) are given by
\[
L_{11}^0 \sim -\frac{2}{N} \sum_q \sum_{\nu, \nu' = \alpha, \beta} v_{\nu \nu'}(q)v_{\nu \nu'}(q)\frac{\partial n[\epsilon_\nu(q)]}{\partial \epsilon_\nu(q)},
\]
\(\text{(D29)}\)
\[
L_{22}^0 \sim -\frac{2}{N} \sum_q \sum_{\nu, \nu' = \alpha, \beta} e_{\nu \nu'}(q)e_{\nu \nu'}(q)\frac{\partial n[\epsilon_\nu(q)]}{\partial \epsilon_\nu(q)}.
\]
\(\text{(D30)}\)

Equations \((\text{D28})-\,(\text{D30})\) give Eq. \((15)\).

Next, we derive Eq. \((\text{16})\), the expression of \(H_{\text{int}}\) in the limit \(\tau \to \infty\) \([\text{Eqs. (D93)}, \text{ (D97)}, \text{ and (D98)}]\), and show the explicit expressions of \(v_{\nu \nu'}^{R(0)}(q, q')\) [\text{Eqs. (D64)-(D75)}]. (This derivation can be done in a way similar to that of the phonon-drag term of a metal.) Before evaluating Eq. \((\text{D2})\) with the effects of \(H_{\text{int}}\), we express \(H_{\text{int}}\) in terms of the operators \(x_{q1}\) and \(x_{q1}^\dagger\). Since \(H_{\text{int}}\) is defined as Eq. \((4)\), we have
\[
H_{\text{int}} = \sum_{q, q', q''} \sum_{q, q', q''} \delta_{q+q', q''-q'} J_3(q)(b_q a_{q'}^\dagger a_{q''} - a_{q'}^\dagger b_q a_{q''} + b_{-q} a_{-q'}^\dagger a_{-q''} - a_{-q} b_{-q'}^\dagger a_{-q''}) + (\text{H.c.})
\]
\[
= \sum_{q, q', q''} \sum_{q, q', q''} \delta_{q+q', q''-q'} J_3(q) \left[ \sum_{l=1}^{2} \text{sgn}(l)(x_{q1} x_{q1}^\dagger x_{q1} x_{q1}^\dagger + x_{q1}^\dagger x_{q1} x_{q1}^\dagger x_{q1}^\dagger) + \sum_{l=3}^{4} \text{sgn}(l)(x_{q1} x_{q1}^\dagger x_{q1} x_{q1}^\dagger + x_{q1}^\dagger x_{q1} x_{q1}^\dagger x_{q1}^\dagger) \right],
\]
\(\text{(D31)}\)
where

\[
\text{sgn}(l) = \begin{cases} 
-1 & (l = 1, 3) \\
1 & (l = 2, 4)
\end{cases}, \quad \bar{l} = \begin{cases}
2 & (l = 1) \\
1 & (l = 2) \\
4 & (l = 3) \\
3 & (l = 4)
\end{cases}.
\] (D32)

To derive \( L_{12} \), we evaluate Eq. (D2) in the second-order perturbation theory\(^{[23,31]}\) using the Wick’s theorem and Eqs. (D6) and (D31); the result is

\[
\Delta G_{12}^{(11)}(\mathbf{q}, \mathbf{q}'; i\Omega_n) = \int_0^1 d\tau e^{-i\Omega_n \tau} \int_0^1 d\tau_1 \int_0^1 d\tau_2 \frac{1}{2} \langle T_x x_{q_1}^\dagger(\tau) x_{\bar{q} q_2}^\dagger(\tau) x_q x_{q^*} H_{\text{int}}(\tau_1) H_{\text{int}}(\tau_2) \rangle \\
= \int_0^1 d\tau e^{-i\Omega_n \tau} \int_0^1 d\tau_1 \int_0^1 d\tau_2 \sum_{\nu_1, \nu_2, \nu_3, \nu_4 = a_1, a_2, \bar{a}_2} (P_{q_1}) \nu_1 (P_{q_2}) \nu_2 (P_{q^*}) \nu_3 (P_{q^*}) \nu_4 \\
\times \sum_{k=a, b, c} \tilde{V}^{(k)}_{\nu_1 \nu_2 \nu_3 \nu_4 \nu_5}(\mathbf{q}, \mathbf{q}'; \tau, \tau_1, \tau_2),
\] (D33)

where

\[
\tilde{V}^{(a)}_{\nu_1 \nu_2 \nu_3 \nu_4 \nu_5}(\mathbf{q}, \mathbf{q}') = -\frac{1}{4} \sum_{l, l' = 1}^4 \text{sgn}(l) \text{sgn}(l') [J_3(\mathbf{q})^2 (P_{q_1}) \nu_1 (P_{q_2}) \nu_2 (P_{q^*}) \nu_3 (P_{q^*}) \nu_4 (P_{q^*}) \nu_5 (P_{q^*}) \nu_5 \\
+ J_5(\mathbf{q}) J_3(\mathbf{q} - \mathbf{q}')(P_{q_1}) \nu_1 (P_{q_2}) \nu_2 (P_{q^*}) \nu_3 (P_{q^*}) \nu_4 (P_{q^*}) \nu_5 (P_{q^*}) \nu_5 \\
+ J_5(\mathbf{q} - \mathbf{q}') J_3(\mathbf{q})(P_{q_1}) \nu_1 (P_{q_2}) \nu_2 (P_{q^*}) \nu_3 (P_{q^*}) \nu_4 (P_{q^*}) \nu_5 (P_{q^*}) \nu_5 \\
+ J_5(\mathbf{q} - \mathbf{q}')^2 (P_{q_1}) \nu_1 (P_{q_2}) \nu_2 (P_{q^*}) \nu_3 (P_{q^*}) \nu_4 (P_{q^*}) \nu_5 (P_{q^*}) \nu_5],
\] (D34)

\[
\tilde{V}^{(b)}_{\nu_1 \nu_2 \nu_3 \nu_4 \nu_5}(\mathbf{q}, \mathbf{q}') = -\frac{1}{4} \sum_{l, l' = 1}^4 \text{sgn}(l) \text{sgn}(l') [J_3(\mathbf{q})^2 (P_{q_1}) \nu_1 (P_{q_2}) \nu_2 (P_{q^*}) \nu_3 (P_{q^*}) \nu_4 (P_{q^*}) \nu_5 (P_{q^*}) \nu_5 \\
+ J_5(\mathbf{q}') J_3(\mathbf{q} - \mathbf{q}')(P_{q_1}) \nu_1 (P_{q_2}) \nu_2 (P_{q^*}) \nu_3 (P_{q^*}) \nu_4 (P_{q^*}) \nu_5 (P_{q^*}) \nu_5 \\
+ J_5(\mathbf{q} - \mathbf{q}') J_3(\mathbf{q})(P_{q_1}) \nu_1 (P_{q_2}) \nu_2 (P_{q^*}) \nu_3 (P_{q^*}) \nu_4 (P_{q^*}) \nu_5 (P_{q^*}) \nu_5 \\
+ J_5(\mathbf{q} - \mathbf{q}')^2 (P_{q_1}) \nu_1 (P_{q_2}) \nu_2 (P_{q^*}) \nu_3 (P_{q^*}) \nu_4 (P_{q^*}) \nu_5 (P_{q^*}) \nu_5],
\] (D35)

\[
\tilde{V}^{(c)}_{\nu_1 \nu_2 \nu_3 \nu_4 \nu_5}(\mathbf{q}, \mathbf{q}') = -\frac{1}{4} \sum_{l, l' = 1}^4 \text{sgn}(l) \text{sgn}(l') [J_3(\mathbf{q})^2 (P_{q_1}) \nu_1 (P_{q_2}) \nu_2 (P_{q^*}) \nu_3 (P_{q^*}) \nu_4 (P_{q^*}) \nu_5 (P_{q^*}) \nu_5 \\
+ J_5(\mathbf{q}') J_3(\mathbf{q} - \mathbf{q}')(P_{q_1}) \nu_1 (P_{q_2}) \nu_2 (P_{q^*}) \nu_3 (P_{q^*}) \nu_4 (P_{q^*}) \nu_5 (P_{q^*}) \nu_5 \\
+ J_5(\mathbf{q} - \mathbf{q}') J_3(\mathbf{q})(P_{q_1}) \nu_1 (P_{q_2}) \nu_2 (P_{q^*}) \nu_3 (P_{q^*}) \nu_4 (P_{q^*}) \nu_5 (P_{q^*}) \nu_5 \\
+ J_5(\mathbf{q} - \mathbf{q}')^2 (P_{q_1}) \nu_1 (P_{q_2}) \nu_2 (P_{q^*}) \nu_3 (P_{q^*}) \nu_4 (P_{q^*}) \nu_5 (P_{q^*}) \nu_5],
\] (D36)

and

\[
f_{(a)}^{(a)}(\mathbf{q}, \mathbf{q}'; \tau, \tau_1, \tau_2) = G_{\nu_1} (\mathbf{q}, \tau_1 - \tau) G_{\nu_2} (\mathbf{q}, \tau - \tau_2) G_{\nu_3} (\mathbf{q}', \tau_2) G_{\nu_4} (\mathbf{q}', -\tau_1) G_{\nu_5} (\mathbf{q}' - \mathbf{q}, \tau_1 - \tau_2),
\] (D37)

\[
f_{(b)}^{(b)}(\mathbf{q}, \mathbf{q}'; \tau, \tau_1, \tau_2) = G_{\nu_1} (\mathbf{q}, \tau_2 - \tau) G_{\nu_2} (\mathbf{q}, \tau - \tau_1) G_{\nu_3} (\mathbf{q}', \tau_1) G_{\nu_4} (\mathbf{q}', -\tau_2) G_{\nu_5} (\mathbf{q}' - \mathbf{q}, \tau_1 - \tau_2),
\] (D38)

\[
f_{(c)}^{(c)}(\mathbf{q}, \mathbf{q}'; \tau, \tau_1, \tau_2) = G_{\nu_1} (\mathbf{q}, \tau_1 - \tau) G_{\nu_2} (\mathbf{q}, \tau - \tau_2) G_{\nu_3} (\mathbf{q}', \tau_2) G_{\nu_4} (\mathbf{q}', -\tau_1) G_{\nu_5} (\mathbf{q} + \mathbf{q}', \tau_1 - \tau_2).
\] (D39)

By combining Eqs. (D33)–(D39) with Eq. (D41) and doing the integrals about \( \tau, \tau_1, \) and \( \tau_2 \) in Eq. (D33), we obtain

\[
\Delta \Phi_{12}(i\Omega_n) = \frac{1}{N} \sum_{\mathbf{q} : \mathbf{q}'} \sum_{\nu_1, \nu_2, \nu_3, \nu_4, \nu_5 = a_1, \bar{a}_2, a_2, \bar{a}_2} v_{\nu_1 \nu_2 \nu_3 \nu_4 \nu_5}^z(\mathbf{q}) v_{\nu_1 \nu_2 \nu_3 \nu_4 \nu_5}^z(\mathbf{q}') \sum_{k=a, b, c} \tilde{V}_{\nu_1 \nu_2 \nu_3 \nu_4 \nu_5}^{(k)}(\mathbf{q}, \mathbf{q}') \bar{l}^{(k)}_{\nu_1 \nu_2 \nu_3 \nu_4 \nu_5}(\mathbf{q}, \mathbf{q}'; i\Omega_n),
\] (D40)
where

\[ I_{v_1v_2v_3v_4v_5}^{(a)}(q, q'; i\Omega_n) = T^2 \sum_{m, m'} G_{v_1}(q, i\Omega_m) G_{v_2}(q, i\Omega_{n+m}) G_{v_3}(q', i\Omega_{m'}) G_{v_4}(q', i\Omega_{m'-m}) G_{v_5}(q' - q, i\Omega_{m'-m}), \]  

(D41)

\[ I_{v_1v_2v_3v_4v_5}^{(b)}(q, q'; i\Omega_n) = T^2 \sum_{m, m'} G_{v_1}(q, i\Omega_m) G_{v_2}(q, i\Omega_{n+m}) G_{v_3}(q', i\Omega_{m'}) G_{v_4}(q', i\Omega_{m'}) G_{v_5}(q - q', i\Omega_{m'-m}), \]  

(D42)

\[ I_{v_1v_2v_3v_4v_5}^{(c)}(q, q'; i\Omega_n) = T^2 \sum_{m, m'} G_{v_1}(q, i\Omega_m) G_{v_2}(q, i\Omega_{n+m}) G_{v_3}(q', i\Omega_{m'}) G_{v_4}(q', i\Omega_{m'-n}) G_{v_5}(q + q', i\Omega_{m'+m'}). \]  

(D43)

Then, to perform the analytic continuation, we replace the Matsubara-frequency summations in Eqs. (D41)–(D43) by the corresponding integrals in a way similar to that for metals [50]. Namely, since an intraband pair of the retarded and advanced Green’s functions, such as \( G_{v_4}^{(R)}(q, z) G_{v_5}^{(A)}(q, z) \), gives the leading contribution in the limit \( \tau \to \infty \) [50], we can express Eqs. (D41)–(D43) in this limit as follows:

\[
\tilde{I}_{v_1v_2v_3v_4v_5}^{(a)}(q, q'; i\Omega_n) \sim \delta_{v_1v_2} \delta_{v_3v_4} \int_{-\infty}^{\infty} \frac{dz}{2\pi i} n(z) \left[ -G_{v_1}^{(A)}(q, z) G_{v_2}^{(R)}(q, z + i\Omega_n) \right] \\
\times \int_{-\infty}^{\infty} \frac{dz'}{2\pi i} n(z') \left[ -G_{v_3}^{(R)}(q', z' + i\Omega_n) G_{v_4}^{(A)}(q', z') G_{v_5}^{(R)}(q' - q, z' - z) \right] \\
+ G_{v_3}^{(R)}(q', z' + z + i\Omega_n) G_{v_4}^{(A)}(q', z' + z) [G_{v_5}^{(R)}(q' - q, z') - G_{v_5}^{(A)}(q' - q, z')] \\
+ G_{v_3}^{(A)}(q', z') G_{v_5}^{(A)}(q', z' - i\Omega_n) G_{v_5}^{(R)}(q' - q, z' - z - i\Omega_n) \\
+ \delta_{v_1v_2} \delta_{v_3v_4} \int_{-\infty}^{\infty} \frac{dz}{2\pi i} n(z) G_{v_1}^{(A)}(q, z - i\Omega_n) G_{v_2}^{(R)}(q, z) \right], 
\]  

(D44)

\[
\tilde{I}_{v_1v_2v_3v_4v_5}^{(b)}(q, q'; i\Omega_n) \sim \delta_{v_1v_2} \delta_{v_3v_4} \int_{-\infty}^{\infty} \frac{dz}{2\pi i} n(z) \left[ -G_{v_1}^{(A)}(q, z) G_{v_2}^{(R)}(q, z + i\Omega_n) \right] \\
\times \int_{-\infty}^{\infty} \frac{dz'}{2\pi i} n(z') \left[ -G_{v_3}^{(R)}(q', z' + i\Omega_n) G_{v_4}^{(A)}(q', z') G_{v_5}^{(A)}(q' - q, z' + z + i\Omega_n) \right] \\
- G_{v_3}^{(R)}(q', z' + z + i\Omega_n) G_{v_4}^{(A)}(q', z' + z) [G_{v_5}^{(R)}(q' - q, z') - G_{v_5}^{(A)}(q' - q, z')] \\
+ G_{v_3}^{(R)}(q', z') G_{v_5}^{(A)}(q', z' - i\Omega_n) G_{v_5}^{(R)}(q' - q, z' + z + i\Omega_n) \\
+ \delta_{v_1v_2} \delta_{v_3v_4} \int_{-\infty}^{\infty} \frac{dz}{2\pi i} n(z) G_{v_1}^{(A)}(q, z - i\Omega_n) G_{v_2}^{(R)}(q, z) \right], 
\]  

(D45)
\[ I_{\nu_1\nu_2\nu_3\nu_4}(q, q'; i\Omega_n) \sim \delta_{\nu_1\nu_2}\delta_{\nu_3\nu_4} \int_{-\infty}^{\infty} \frac{dz}{2\pi i} n(z) [-G^{(A)}_{\nu_1}(q, z)G^{(R)}_{\nu_2}(q, z + i\Omega_n)] \]
\[ \times \int_{-\infty}^{\infty} \frac{dz'}{2\pi i} n(z') [-G^{(R)}_{\nu_3}(q', z' + i\Omega_n)G^{(A)}_{\nu_4}(q', z')G^{(R)}_{\nu_5}(q' + q, z' + z + i\Omega_n) \]
\[ + G^{(R)}_{\nu_3}(q', z' - z)G^{(A)}_{\nu_4}(q', z' - z - i\Omega_n)G^{(R)}_{\nu_5}(q' + q, z' - z)] \]
\[ + G^{(R)}_{\nu_3}(q', z')G^{(A)}_{\nu_4}(q', z' - i\Omega_n)G^{(A)}_{\nu_5}(q + q, z' + z + i\Omega_n)] \]
\[ + \delta_{\nu_1\nu_2}\delta_{\nu_3\nu_4} \int_{-\infty}^{\infty} \frac{dz'}{2\pi i} n(z) G^{(A)}_{\nu_1}(q, z - i\Omega_n)G^{(R)}_{\nu_2}(q, z) \]
\[ \times \int_{-\infty}^{\infty} \frac{dz'}{2\pi i} n(z') [-G^{(R)}_{\nu_3}(q', z' + i\Omega_n)G^{(A)}_{\nu_4}(q', z')G^{(R)}_{\nu_5}(q' + q, z' + z) \]
\[ + G^{(R)}_{\nu_3}(q', z' - z + i\Omega_n)G^{(A)}_{\nu_4}(q', z' - z - i\Omega_n)G^{(R)}_{\nu_5}(q' + q, z' - z)] \]
\[ + G^{(R)}_{\nu_3}(q', z', z' - i\Omega_n)G^{(A)}_{\nu_5}(q' + q, z' + z - i\Omega_n)]. \]  

In replacing the sums over \( m \) in Eqs. (D41)–(D43) by the contour integrals, we have considered the contributions only from the region for \(-\Omega_n < \text{Im} \omega < 0\) in the contour \( C \) shown in Fig. [a] because they include the pair of the retarded and advanced Green’s functions. Furthermore, in replacing the sums over \( m' \) in Eqs. (D41), (D42), and (D43) by the integrals, we have used the contours \( C' \), \( C'' \), and \( C''' \), respectively; the \( C' \) and \( C'' \) are shown in Figs. [b] and [c]. We now perform the analytic continuation of Eqs. (D44)–(D46) using the replacement \( i\Omega_n \to \omega + i\delta \); the results are

\[ \Delta I^{R(a)}_{\nu\nu'
u''}(q, q'; \omega) = \tilde{I}^{R(a)}_{\nu\nu\nu''}(q, q'; \omega) - \tilde{I}^{R(a)}_{\nu\nu\nu''}(q, q'; 0) \]
\[ \sim i\omega \int_{-\infty}^{\infty} \frac{dz}{2\pi} \frac{\partial n(z)}{\partial z} g_{\nu}(q, z) \int_{-\infty}^{\infty} \frac{dz'}{2\pi} \frac{\partial n(z')}{\partial z'} [n(z') - n(z') - n(z') - n(z')] g_{\nu'}(q', z') \text{Im} G^{(R)}_{\nu''}(q' - q, z' - z), \]  

\[ \Delta I^{R(b)}_{\nu\nu'
u''}(q, q'; \omega) = \tilde{I}^{R(b)}_{\nu\nu\nu''}(q, q'; \omega) - \tilde{I}^{R(b)}_{\nu\nu\nu''}(q, q'; 0) \]
\[ \sim i\omega \int_{-\infty}^{\infty} \frac{dz}{2\pi} \frac{\partial n(z)}{\partial z} g_{\nu}(q, z) \int_{-\infty}^{\infty} \frac{dz'}{2\pi} \frac{\partial n(z')}{\partial z'} [n(z') - n(z') - n(z') - n(z')] g_{\nu'}(q', z') \text{Im} G^{(R)}_{\nu''}(q' - q, z' - z'), \]  

\[ \Delta I^{R(c)}_{\nu\nu'
u''}(q, q'; \omega) = \tilde{I}^{R(c)}_{\nu\nu\nu''}(q, q'; \omega) - \tilde{I}^{R(c)}_{\nu\nu\nu''}(q, q'; 0) \]
\[ \sim i\omega \int_{-\infty}^{\infty} \frac{dz}{2\pi} \frac{\partial n(z)}{\partial z} g_{\nu}(q, z) \int_{-\infty}^{\infty} \frac{dz'}{2\pi} \frac{\partial n(z')}{\partial z'} [n(z') - n(z') - n(z') - n(z')] g_{\nu'}(q', z') \text{Im} G^{(R)}_{\nu''}(q' + q, z' + z + z'), \]

where we have introduced \( g_{\nu}(q, z) = G^{(A)}_{\nu}(q, z)G^{(R)}_{\nu}(q, z) \), used \( n(z) - n(z + \omega) \sim -\omega \frac{\partial n(z)}{\partial z} \), and neglected the \( O(\omega^2) \) terms. Combining Eqs. (D47)–(D49) with Eq. (D40) and \( \Delta \Phi^{R(2)}_{12} = \Delta \Phi_{12}(i\Omega_n \to \omega + i\delta) \), we obtain

\[ L^{(1)}_{12} = \lim_{\omega \to 0} \frac{\Delta \Phi^{R(2)}_{12}(\omega) - \Delta \Phi^{R(0)}_{12}(0)}{i\omega} = \frac{1}{N} \sum_{q, q', \nu, \nu', \nu''} \sum_{n=\alpha_1,\alpha_2} \sum_{\beta_1, \beta_2} \nu^{2}_{\nu\nu'}(q) \nu^{2}_{\nu\nu'}(q') \sum_{k=a,b,c} V^{(k)}_{\nu\nu\nu'}(q, q') f^{(k)}_{\nu\nu\nu'}(q, q'), \]

where

\[ V^{(k)}_{\nu\nu\nu'}(q, q') = \tilde{V}^{(k)}_{\nu\nu\nu'}(q, q'), \]

\[ f^{(k)}_{\nu\nu\nu'}(q, q') = \int_{-\infty}^{\infty} \frac{dz}{2\pi} \frac{\partial n(z)}{\partial z} g_{\nu}(q, z) \int_{-\infty}^{\infty} \frac{dz'}{2\pi} \frac{\partial n(z')}{\partial z'} [n(z') - n(z') - n(z') - n(z')] g_{\nu'}(q', z') \text{Im} G^{(R)}_{\nu''}(q' - q, z' - z), \]

\[ f^{(k)}_{\nu\nu\nu'}(q, q') = \int_{-\infty}^{\infty} \frac{dz}{2\pi} \frac{\partial n(z)}{\partial z} g_{\nu}(q, z) \int_{-\infty}^{\infty} \frac{dz'}{2\pi} \frac{\partial n(z')}{\partial z'} [n(z') - n(z') - n(z') - n(z')] g_{\nu'}(q', z') \text{Im} G^{(R)}_{\nu''}(q' - q, z' - z'), \]

\[ f^{(k)}_{\nu\nu\nu'}(q, q') = \int_{-\infty}^{\infty} \frac{dz}{2\pi} \frac{\partial n(z)}{\partial z} g_{\nu}(q, z) \int_{-\infty}^{\infty} \frac{dz'}{2\pi} \frac{\partial n(z')}{\partial z'} [n(z') - n(z') - n(z') - n(z')] g_{\nu'}(q', z') \text{Im} G^{(R)}_{\nu''}(q' + q, z' + z + z'). \]

Note that \( \tilde{V}^{(k)}_{\nu\nu\nu'}(q, q') \)'s have been given by Eqs. (D34)–(D36). In the limit \( \tau = 1/2\gamma \to \infty \), we can easily do the integrals in Eqs. (D52)–(D54) by using the approximate relations,

\[ g_{\nu}(q, z) = G^{(A)}_{\nu}(q, z)G^{(R)}_{\nu}(q, z) = \frac{1}{\left[z + (-1)^\nu \epsilon_{\nu}(q)\right]^2 + \gamma^2} \sim \frac{\pi}{\gamma} [z + (-1)^\nu \epsilon_{\nu}(q)], \]

\[ \text{Im} G^{(R)}_{\nu}(q, z) = (-1)^\nu \frac{\gamma}{\left[z + (-1)^\nu \epsilon_{\nu}(q)\right]^2 + \gamma^2} \sim (-1)^\nu \pi \delta[z + (-1)^\nu \epsilon_{\nu}(q)], \]
where \((-1)^\nu = -1\) for \(\nu = \alpha_1, \beta_1\) and 1 for \(\nu = \alpha_2, \beta_2\). Combining these equations with Eqs. (D52)–(D54), we obtain

\[
I^{(a)}_{\nu'\nu''}(q, q') \sim \frac{\pi}{2\gamma^2} \frac{\partial n_{\nu}(q)}{\partial \nu}(q) \{ n\{(-1)^{\nu' + 1} \epsilon_{\nu'}(q') - n\{(-1)^{\nu'' + 1} \epsilon_{\nu''}(q' - q)\}\}(-1)^\nu
\]

\[
\times \delta\{(-1)^{\nu'} \epsilon_{\nu}(q) - (-1)^{\nu'} \epsilon_{\nu'}(q') + (-1)^{\nu'} \epsilon_{\nu''}(q' - q)\},
\]

\[
I^{(b)}_{\nu'\nu''}(q, q') \sim \frac{\pi}{2\gamma^2} \frac{\partial n_{\nu}(q)}{\partial \nu}(q) \{ n\{(-1)^{\nu'} \epsilon_{\nu'}(q' - q)\} - n\{(-1)^{\nu'' + 1} \epsilon_{\nu''}(q')\}\}(-1)^\nu
\]

\[
\times \delta\{(-1)^{\nu'} \epsilon_{\nu}(q) - (-1)^{\nu''} \epsilon_{\nu'}(q' - q)\},
\]

\[
I^{(c)}_{\nu'\nu''}(q, q') \sim \frac{\pi}{2\gamma^2} \frac{\partial n_{\nu}(q)}{\partial \nu}(q) \{ n\{(-1)^{\nu'} \epsilon_{\nu'}(q')\} - n\{(-1)^{\nu'' + 1} \epsilon_{\nu''}(q' + q)\}\}(-1)^\nu
\]

\[
\times \delta\{(-1)^{\nu'} \epsilon_{\nu}(q) + (-1)^{\nu''} \epsilon_{\nu'}(q') - (-1)^{\nu''} \epsilon_{\nu'}(q' + q)\},
\]

where the delta functions represent the energy conservation relations in the scattering processes due to the second-order \(H_{\text{int}}\). These equations can be obtained also by using Eqs. (D18) and (D19) and the relation \(\frac{1}{\gamma + 1\gamma^2} \sim \frac{1}{\gamma} \delta(x)\), instead of Eqs. (D55) and (D56), and doing the integrals in Eqs. (D52)–(D54). This is the reason why we have used that relation about the Lorentz function in the numerical evaluations of \(S_m, m\), and \(\kappa_m\). Then, performing some calculations using Eqs. (D51), (D34)–(D36), and (D8)–(D11), we find that the finite terms of \(V^{(p)}_{\nu'\nu''}(q, q')\)'s \((p = 1, 2, 3)\) are given by those for \((\nu, \nu', \nu'') = (\beta, \beta, \beta), (\beta, \alpha, \alpha), (\alpha, \alpha, \beta), (\alpha, \alpha, \beta)\), which are expressed as follows:

\[
V^{(1)}_{\nu'\nu''}(q, q') = V^{(a)}_{\nu'\nu''}(q, q') + V^{(a)}_{\nu'\nu''}(q, q') + V^{(b)}_{\nu'\nu''}(q, q') + V^{(c)}_{\nu'\nu''}(q, q') + V^{(c)}_{\nu'\nu''}(q, q'),
\]

\[
V^{(2)}_{\nu'\nu''}(q, q') = V^{(a)}_{\nu'\nu''}(q, q') + V^{(a)}_{\nu'\nu''}(q, q') + V^{(b)}_{\nu'\nu''}(q, q') + V^{(b)}_{\nu'\nu''}(q, q') + V^{(c)}_{\nu'\nu''}(q, q') + V^{(c)}_{\nu'\nu''}(q, q'),
\]

\[
V^{(3)}_{\nu'\nu''}(q, q') = V^{(a)}_{\nu'\nu''}(q, q') + V^{(a)}_{\nu'\nu''}(q, q') + V^{(b)}_{\nu'\nu''}(q, q') + V^{(b)}_{\nu'\nu''}(q, q') + V^{(c)}_{\nu'\nu''}(q, q') + V^{(c)}_{\nu'\nu''}(q, q'),
\]

[Note that if \((\nu, \nu', \nu'') = (\beta, \beta, \beta)\), we have \((\nu_1, \nu'_1, \nu''_1) = (\beta_1, \alpha_1, \alpha_2), (\nu_2, \nu'_2, \nu''_2) = (\beta_2, \alpha_2, \alpha_1), \text{etc.}\) Since \(V^{(k)}_{\nu'\nu''}(q, q')\)'s \((k = a, b, c)\) include the square of the coupling constant of \(H_{\text{int}}\) [see Eqs. (D34)–(D36) with Eq. (D51)] and \(J_3(q) = \sqrt{\frac{45}{N}} \sin 2\phi J(q)\), we can write the finite terms of \(V^{(p)}_{\nu'\nu''}(q, q')\)'s \((p = 1, 2, 3)\) as follows:

\[
V^{(p)}_{\nu'\nu''}(q, q') = v^{(p)}_{\nu'\nu''}(q, q') \frac{S}{2N} \sin^2 2\phi,
\]

where

\[
v^{(1)}_{\beta\beta}(q, q') = +v_{a0}(q, q') C'_q - v_{a0}(q, q') C'_q - v_{c0}(q, q') C'_q - v_{a0}(q, q') C'_q C'_q C'q - q + S'_q S'_q S'_q q - q',
\]

\[
v^{(2)}_{\beta\beta}(q, q') = -v_{a0}(q, q') C'_q + v_{a0}(q, q') C'_q - v_{c0}(q, q') C'_q - v_{a0}(q, q') C'_q C'_q C'q - q + S'_q S'_q S'_q q - q',
\]

\[
v^{(3)}_{\beta\beta}(q, q') = -v_{a0}(q, q') C'_q - v_{a0}(q, q') C'_q + v_{c0}(q, q') C'_q - v_{a0}(q, q') C'_q C'_q C'q - q + S'_q S'_q S'_q q - q',
\]

\[
v^{(1)}_{\beta\alpha}(q, q') = +v_{a1}(q, q') C'_q - v_{a1}(q, q') C'_q - v_{c1}(q, q') C'_q - v_{a1}(q, q') C'_q C'_q C'q - q + S'_q S'_q S'_q q - q',
\]

\[
v^{(2)}_{\beta\alpha}(q, q') = -v_{a1}(q, q') C'_q + v_{a1}(q, q') C'_q - v_{c1}(q, q') C'_q - v_{a1}(q, q') C'_q C'_q C'q - q + S'_q S'_q S'_q q - q',
\]

\[
v^{(3)}_{\beta\alpha}(q, q') = -v_{a1}(q, q') C'_q - v_{a1}(q, q') C'_q + v_{c1}(q, q') C'_q - v_{a1}(q, q') C'_q C'_q C'q - q + S'_q S'_q S'_q q - q',
\]

\[
v^{(1)}_{\alpha\beta}(q, q') = +v_{a2}(q, q') C'_q - v_{a2}(q, q') C'_q - v_{c2}(q, q') C'_q - v_{a2}(q, q') C'_q C'_q C'q - q + S'_q S'_q S'_q q - q',
\]

\[
v^{(2)}_{\alpha\beta}(q, q') = -v_{a2}(q, q') C'_q + v_{a2}(q, q') C'_q - v_{c2}(q, q') C'_q - v_{a2}(q, q') C'_q C'_q C'q - q + S'_q S'_q S'_q q - q',
\]

\[
v^{(3)}_{\alpha\beta}(q, q') = -v_{a2}(q, q') C'_q - v_{a2}(q, q') C'_q + v_{c2}(q, q') C'_q - v_{a2}(q, q') C'_q C'_q C'q - q + S'_q S'_q S'_q q - q',
\]

\[
v^{(1)}_{\alpha\alpha}(q, q') = +v_{a3}(q, q') C'_q - v_{a3}(q, q') C'_q - v_{c3}(q, q') C'_q - v_{a3}(q, q') C'_q C'_q C'q - q + S'_q S'_q S'_q q - q',
\]

\[
v^{(2)}_{\alpha\alpha}(q, q') = -v_{a3}(q, q') C'_q + v_{a3}(q, q') C'_q - v_{c3}(q, q') C'_q - v_{a3}(q, q') C'_q C'_q C'q - q + S'_q S'_q S'_q q - q',
\]

\[
v^{(3)}_{\alpha\alpha}(q, q') = -v_{a3}(q, q') C'_q - v_{a3}(q, q') C'_q + v_{c3}(q, q') C'_q - v_{a3}(q, q') C'_q C'_q C'q - q + S'_q S'_q S'_q q - q',
\]
In deriving them, we have used the identity appearing in Eqs. (17)–(19). By combining Eqs. (D63)–(D92), (D57)–(D59), and (D25)–(D27) with Eq. (D50), we respectively, we can express \( m \) in the limit \( \nu \to \infty \) becomes

\[
\sum_{q, q', \nu, \nu'} J^{(p)}_{\nu, \nu'}(q, q') \left[ 1 + n(\epsilon_{\nu'}(q - q')) \right] n(\epsilon_{\nu'}(q')) \delta(\epsilon_{\nu}(q) - \epsilon_{\nu'}(q') - \epsilon_{\nu'}(q - q'))
\]

Equations (D93), (D97), and (D98) yield Eq. (16).

**Appendix E: Additional numerical results of \( S_m, \sigma_m, \) and \( \kappa_m \)**

We present the additional results of the numerically evaluated \( S_m, \sigma_m, \) and \( \kappa_m \) for \( S = \frac{3}{2} \) with \( \frac{J}{k_B T_c} = 20 \) and \( J = 1. \) (In the case of \( S = \frac{3}{2} \), the magnon picture for the canted antiferromagnet is valid in the range of \( 0 < h < 48J \).) Since the transition temperature for \( S = \frac{3}{2} \) becomes \( T_c = 20J \), we choose the temperature range to be \( 0 < T \leq 12J(= 0.6T_c) \). Figures (E1)–(E3)
show the temperature dependences of $S_m$, $\sigma_m$, and $\kappa_m$ for $S = \frac{3}{2}$ at $h = 40J$. For $S = \frac{3}{2}$, the low-temperature peaks are observed at the $h$ lower than $65J$. Then, the ratios $L_{12}/L_{11}$, $L_{11}/L_{11}$, and $L_{22}/L_{22}$ at $T = 5J (= 0.25T_c)$ reach about 60, 66, and 52, respectively. The larger enhancement for $S = \frac{3}{2}$ than that for $S = \frac{3}{2}$ comes from the property that the smaller the $S$ is, the more considerable the effects of magnon-magnon interactions become. This general property is due to the difference between the $S$ dependences of $H_0$ and $H_{\text{int}}$. 

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