High Energy Scattering and D-Pair Creation in Matrix String Theory

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Abstract

In this paper we use the matrix string approach to begin a study of high energy scattering processes in M-theory. In particular we exhibit an instanton-type configuration in 1+1 super-Yang-Mills theory that can be interpreted as a non-perturbative description of a string interaction. This solution is used to describe high energy processes with non-zero longitudinal momentum exchange, in which an arbitrary number of eigenvalues get transferred between the two scattering states. We describe a direct correspondence between these semi-classical SYM configurations and the Gross-Mende saddle points. We also study in detail the pair production of D-particles via a one-loop calculation which in the 1+1D gauge theory language is described by the (perturbative) transition between states with different electric flux. Finally, we discuss a possible connection between these calculations in which D-particle production gives important corrections to the Gross-Mende process.

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1. Introduction and Summary of Results

High energy processes in string theory were first considered from the point of view of conventional string perturbation theory by Gross and Mende [1] in the regime of fixed angle scattering and in the near forward regime by Amati et al in [2]. The recent insights from M-theory, however, have provided a large number of new non-perturbative tools which can now be used to put these works into a new perspective, and extend the results into new directions. For instance, it was long believed that the string length $\ell_s$ marks the minimal distance that can be probed via scattering processes in string theory. This belief was based on the fact that fundamental strings tend to increase in size when boosted to high energies, and thus appear to be incapable of penetrating substringy distance scales. Since the discovery of D-particles as non-perturbative solitons of the IIA theory, however, we know that there exists small scale structure that, at least for weak string coupling, extends well below the string length [3-6]. This particular realization provided important motivation for the Matrix theory conjecture of [7] that all localized excitations of M-theory (including the fundamental strings) are representable as multi-D-particle bound states[11,12].

In this paper we begin a study of high energy processes in type IIA string theory, by making use of this Matrix theory formalism. We focus on the four graviton scattering amplitude, and in particular we will present a detailed calculation of the pair production rate of D-particles via this process. Our aim is to probe in this way the transition region between the conventional perturbative string regime and the strong coupling regime described by 11-dimensional M-theory. (See fig. 1.)

From the ten dimensional perspective of IIA string theory, D-pair production is an inelastic scattering process, in which two strings exchange one unit of D-particle charge. It is inherently nonperturbative and thus inaccessible to conventional perturbative methods. It is also inaccessible in the traditional Matrix theory approach since the anti-D particles are boosted to infinite energy. From the eleven dimensional perspective, on the other hand, the D-pair creation process can simply be thought of as the elastic scattering of two particles in which one unit of Kaluza-Klein momentum in the 11 direction is exchanged. Via this interpretation, one can rather straightforwardly obtain a tree level estimate of the probability amplitude. This estimate should be reliable for large values for the $S^1$ compactification radius $R_{11}$ and for collision energies sufficiently below the 11-dimensional

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1 For reviews see [8-10] and references therein.
Planck energy. At high energies and/or small values for $R_{11}$, on the other hand, we expect the physics of the scattering process to be quite different from (semi-)classical supergravity. In the following we will attempt to gain more insight into this regime via the Matrix string approach.

Matrix string theory arises from the original Matrix theory proposal [7] via compactification on a circle, and starts from the action of 1+1-dimensional maximally supersymmetric Yang-Mills theory with gauge group $U(N)$,

$$S = \int d\tau \int_0^1 d\sigma \mathrm{Tr} \left\{ -\frac{g_s^2}{4} F_{\mu \nu}^2 - \frac{1}{2} (D_\mu X)^2 + \frac{1}{4g_s^2} [X, X]^2 + i \bar{\psi} \gamma^\mu \partial_\mu \psi - \frac{1}{g_s} \bar{\psi} \Gamma^i [X_i, \psi] \right\},$$

defined on the circle $0 \leq \sigma < 1$. The derivation of this action is reviewed in the Appendix; here and henceforth we work in string units, $l_s = 1$. Via the identification of the eigenvalues of the matrices $X^I$ with the transverse location of type IIA supersymmetric strings, this
SYM model can be reinterpreted as a non-perturbative formulation of light-cone gauge IIA string [13-15]. In this correspondence, the string coupling constant $g_s$ is inversely proportional to the Yang-Mills coupling $g_{YM}$ and the free string limit therefore arises in the strong coupling limit of the Yang-Mills model. This correspondence has been worked out in some detail in [15]. More generally, however, all regimes of the $S^1$ compactification of M-theory, as indicated in fig 1, should according to the Matrix string conjecture of [7,13,14,15] via the above identifications be described by particular regimes of (the large $N$ limit) of the 1+1D supersymmetric gauge theory.

In particular, it is expected that in the weak coupling, moderate energy limit of the SYM theory it effectively reduces to the Matrix quantum mechanics description of 11-dimensional supergravity. Indeed, a new feature of matrix string theory (relative to standard light-cone string theory) is that via the electric flux of the gauge field, the string states can be adorned with an extra quantum number, identified with the D-particle charge [15]. In a small $g_s$ expansion, these flux sectors energetically decouple, corresponding to the fact that D-particles can not be produced via perturbative string interactions. Nonetheless, electric flux can get created in the gauge theory: it is easy to see that electric flux creation is a simple one-loop effect that takes place whenever a virtual pair of charged particles gets created and annihilated, after forming a loop that winds one or more times around the $\sigma$ cylinder.

In the following we will develop a new method for studying high energy scattering and D-pair production in Matrix string theory, which will be based on a semi-classical expansion from the SYM perspective. An important novelty of this method is that it applies to processes with arbitrary longitudinal momentum exchange. In the gauge theory language, this means that the transitions between the initial and final states that we will consider will involve a non-perturbative tunneling process in which an arbitrary number of eigenvalues get transferred between the two scattering states. Most previous calculations in Matrix theory relied on perturbative SYM corrections and thus were necessarily restricted to zero $p^+$ transfer.\footnote{In [16] Polchinski and Pouliot analyzed graviton scattering with non-zero M-momentum transfer in Matrix theory. In their case, the M-momentum was identified with the magnetic flux of the SYM gauge theory, and the corresponding instanton was a magnetic monopole. Here we will consider different kind of momentum transfer, namely of longitudinal momentum represented by the size $N$ of the Matrix bound states, \textit{i.e.} the number of D-particles in the original Matrix dictionary of [7]. This will require a different, less familiar type of instanton process.}
Concretely, we will construct SYM saddle point configurations that will allow us to interpolate between ingoing matrix configurations of the form

\[
\vec{X}_{\text{in}}(\tau) = \frac{1}{2} \begin{pmatrix}
\frac{\vec{p}_1}{N_1} \tau + \vec{b} I_1 & 0 \\
0 & \frac{\vec{p}_2}{N_2} \tau - \vec{b} I_2
\end{pmatrix}
\]

and outgoing configurations of the form

\[
\vec{X}_{\text{out}}(\tau) = \begin{pmatrix}
\frac{\vec{p}_3}{N_3} \tau + \vec{b} I_3 & 0 \\
0 & \frac{\vec{p}_4}{N_4} \tau - \vec{b} I_4
\end{pmatrix}
\]

where \( I_i \) are \( N_i \times N_i \) identity matrices, where all \( N_i \)'s are different (but subject to the momentum constraint \( N_1 + N_2 = N_3 + N_4 \)). These in and out configurations each describe two widely separated gravitons with different light-cone momenta \( p^+_{(i)} = N_{(i)}/R \) and transverse momenta \( \vec{p}_{(i)} \), and with relative impact parameter \( \vec{b} \).

The interpolating solutions that we will construct, essentially look like an appropriate matrix generalization of perturbative string worldsheets. The importance of these solutions is not entirely obvious, however, since a priori one would expect that the range of validity of the semi-classical Yang-Mills approximation has no overlap with that of perturbative string theory. Indeed, as emphasized in [15] the two regimes appear related via a strong/weak coupling duality. However, as we will argue in the following, even at small or moderate string coupling \( g_s \), at sufficiently high collision energies and/or impact parameters one enters a regime in which the semi-classical SYM methods may provide an accurate description of the scattering process.

Just like string/M-theory, the 1+1 SYM model contains various length scales: (i) the circumference of the cylinder (set equal to 1), (ii) the scale set by the Yang-Mills coupling \( \ell_{YM} = 1/g_{YM} \)

\[
\ell_{YM} \simeq g_s
\]

(iii) the typical mass scale set by the Higgs expectation values of the SYM model. The latter length scale is inversely proportional to the impact parameter \( b \) of the string/M-theory scattering process:

\[
\ell_b \simeq \frac{g_s}{b}.
\]

Finally, (iv) there is also the length scale \( \ell_E \) determined by the typical size of the SYM energy \( E \), which is related to the relative space-time momenta via \( E \simeq p^2/N \).

The existence of these scales allows us to find small dimensionless ratios that may parameterize the strength of the SYM processes taking place at that scale. For example,
while $g_{YM} = 1/g_s$ defines the effective coupling of SYM processes that take place at the scale of the YM cylinder, we also have

$$g_{YM}^{\text{eff}}(b) \simeq \ell_b/\ell_{YM} \simeq 1/b,$$

as the dimensionless coupling at the scale $\ell_b$. Similarly, we can also associate an effective coupling $g_{YM}^{\text{eff}}(E)$ with the scale set by the SYM energy $E$. This suggests the possibility that even if $g_s$ is small or of order 1, processes at these other 2D length scales can be accurately described by perturbative and/or semi-classical SYM methods. This will require however that we consider the limit of high collision energies and sufficiently large impact parameters.\[3\]

\[3\] In this context it may be of relevance that in classical 10-dimensional DLCQ supergravity, the impact parameter $b$ scales with the transverse relative momentum $p$ via $b \simeq \left( g_s^2 p^2 / N \sin \theta \right)^{1/6}$ with $N$ the DLCQ $p_+$-momentum. Hence, at least in this classical context, and for fixed scattering angles $\theta$ and $g_s$ of order 1, the condition that $b$ is large is automatically satisfied in limit of large $p^2 \gg N$.

The two types of processes that we will consider, high energy scattering with non-zero $\Delta p_+$ and the D-pair production, may at first sight seem quite unrelated. However, there are several connections between these two types of processes. First of all, it is worth pointing out that in both cases the scattering process involves (depending on which duality frame one chooses) the transfer of D-particle charge and/or momentum between the two scattering particles. Indeed, the rank $N$ started out as identified with D-particle charge,
and only after the duality it and the electric flux $E$ are mapped onto each other under an 11-9 flip: i.e. the interchange of the 11-th and 9-th direction. (See fig 2.) Hence quantitative understanding both types of processes will have a direct bearing on the Lorentz invariance of the Matrix formalism.

Another connection between the two calculations is related to a fundamental puzzle in the original calculation of [1], namely the apparently dominant contribution of arbitrarily high genus to scattering amplitudes. The saddle point trajectory at loop order $G$ typically describes a process as depicted in fig. 3: two incoming strings, that are wound $N = G + 1$ times, interact and then propagate as $N$ intermediate short strings. The $N$ strings then join together again, producing a final state of two different $N$ times wound outgoing strings (see fig. 3). It was found in [1] that the contributions of these higher order interactions grows larger with the genus $G$. This instability appears to signal a fundamental breakdown of conventional string perturbation theory in the high energy regime.

Fig. 3: This figure depicts a typical saddle point trajectory that contributes to the high energy scattering amplitude of fundamental strings, according to the perturbative physical picture proposed in [1].

On the other hand, the fragmented form of the intermediate state in fig. 3 gives a strong hint of some underlying non-perturbative structure that looks quite similar to that of the multi-D-particle bound state dynamics of Matrix theory. This suggests that the Matrix treatment may provide a rather natural stabilizing mechanism for a cutoff on the genus. Furthermore, our study will show that D-particle pair production becomes relevant at this cutoff – when the strings become maximally fragmented. This leads us to suspect a deeper relation between Gross and Mende’s high energy, fixed angle scattering and the non-perturbative process of D-pair creation.

We’ll begin this paper with a quick review of some properties of Matrix strings. We then turn to fixed angle scattering. Section three will recall aspects of fixed angle scattering
in the traditional string framework, with particular emphasis on its description in light-
cone gauge. This will be followed by a discussion of string interactions in the Matrix string
formulation. In particular, we will find a local instanton solution in the two-dimensional
theory that describes the splitting/joining interaction. Furthermore, the condition for this
instanton to be matched to the incoming/outgoing states is precisely that we be working
at the world-sheet moduli corresponding to the saddlepoint surfaces of \([1]\).

We then turn to D-pair production, which we consider both from the supergravity
perspective as well as via a one loop Yang Mills calculation valid for arbitrary \(N\). This
is followed by a discussion of the ranges of validity of our calculations. In particular, we
suggest that the ranges of validity of the two calculations overlap, and allow the picture
we’ve mentioned in which they complement each other. This connection is further discussed
in the final section, together with some other observations and speculations. An appendix
contains discussion of the derivation of the Matrix string formalism following the approach
of \([17,18]\).

2. Matrix String Theory

In this section we recall some basic features of the matrix string approach. This
approach arises from DLCQ matrix theory quantized on a circle of “radius” \(R\) by com-
pactifying on another circle, and interpreting this additional compactification as the route
by which M-theory descends to type IIA string theory. For example, D0 branes are states
with non-zero momentum in this extra direction.

2.1. Summary of Matrix Strings

The Matrix string lagrangian is given in eq. (1.1). Its form can be derived from M-
theory on \(S^1\), by combining Sen and Seiberg’s arguments\([17,18]\) with the compactification
prescription of \([19]\). (These arguments, and a more complete discussion of the relation
among the parameters of the theory, are presented in the Appendix.) The basic dynamical
variables of the theory are \(N \times N\) hermitian matrices and include the 8 scalar fields \(X^I\)
and the 8 fermion fields \(\psi^a_L\) and \(\psi^\dot{a}_R\). The Yang-Mills time variable \(\tau\) is related to target
space time by \(\tau = tR\) in units where \(l_{st} = 1\).

The \(\sigma\) direction corresponds to the T-dual of the compact direction (herein called \(x^9\))
of M-theory. The target-space coordinate in this circle direction is identified with the zero
mode of the gauge field \(A_\sigma\). The gauge equivalence under quantized shifts

\[
A_\sigma \equiv A_\sigma + 2\pi m
\]  

(2.1)
with \( m \in \mathbb{Z} \) translates via the identification of \( A_{\sigma} \) with the compact \( X^9 \) (see Appendix) into the periodic boundary condition along the M-theory circle.

We can turn off the string interactions by considering only SYM configurations that describe widely separated strings, \textit{i.e.} matrix configurations \( X_i \) such that all differences between their eigenvalues are large. In this situation all charged fields (relative to the Higgs scalars) become very massive and effectively decouple from the dynamics. Hence in this limit all matrix fields take the form

\[
\tilde{X}^I = \begin{pmatrix}
X_1^I \\
X_2^I \\
\vdots \\
\emptyset \\
\emptyset \\
X_N^I
\end{pmatrix}
\]  

(2.2)

and the corresponding effective SYM Hamiltonian reduces to the free field form

\[
H_0 = \frac{1}{2} \sum_{i=1}^{N} \int d\sigma \left[ \frac{1}{g_s^2} E_i^2 + (\Pi_i^I)^2 + (\partial_\sigma X_i^I)^2 + \text{fermions} \right]
\]  

(2.3)

where \( E_i \) denotes the (diagonal part of the) electric flux.

For example, later in the paper we will consider matrices of the form

\[
\tilde{X} = \frac{\vec{p}_\tau + \vec{b}}{2} \begin{pmatrix}
I_1/N_1 & 0 \\
0 & -I_2/N_2
\end{pmatrix}
\]  

(2.4)

where \( I_1 \) and \( I_2 \) are \( N_1 \times N_1 \) and \( N_2 \times N_2 \) identity matrices. This corresponds (see Appendix) to two widely separated configurations with momenta \( p^+_i = N_i/R \) and \( \vec{p}_i = \pm \vec{p}/2 \), and with relative impact parameter \( \vec{b}/2N_1 + \vec{b}/2N_2 \).

In the SYM/IIA string dictionary, the electric flux \( E_i \) gets translated into D-particle number. Indeed, in a canonical formalism the electric flux is conjugate to the (zero mode of the) gauge field \( A_{\sigma} \), and thus represents the quantized momentum in the compactified direction. The quantum ground states of the SYM theory, corresponding to asymptotic particle states, are thus labeled by their transverse momentum \( p_\perp \), their light-cone momenta \( p^+ \) and \( p^- \), and their D-particle charge.

Depending on their topology, the eigenvalue fields \( X_i^I(\sigma) \) in (2.2) combine into one or more separate strings. For example, the trivial boundary condition

\[
X_i^I(\sigma + 1) = X_i^I(\sigma)
\]  

(2.5)
corresponds to a collection of elementary quanta (which may be thought of as minimal length strings), whereas the string of maximal length $N$ is described by the periodicity condition

$$X_i^{I}(\sigma + 1) = X_{i+1}^{I}(\sigma), \quad i \in (1, \ldots, N).$$  \hfill (2.6)

We can write this condition as the $N \times N$ matrix equation

$$X_i^{I}(\sigma + 1) = VX_{i}^{I}(\sigma)V^{-1} \quad (2.7)$$

with $V$ the cyclic permutation matrix on the $N$ eigenvalues,

$$V = \begin{pmatrix} 0 & 1 & \emptyset \\ 0 & 1 & \ddots \\ \emptyset & \ddots & 1 \\ 1 & \emptyset & 0 \end{pmatrix}. \quad (2.8)$$

As a result of this cyclic boundary condition, we can glue the $N$ eigenvalue fields $X_i^{I}(\sigma)$ together into one single scalar field $X^{I}(\sigma)$ defined on a long interval $0 \leq \sigma < N$. Hence, when we expand this field in the usual way in oscillation modes, the frequency spacing between these modes is $N$ times smaller than for a single valued field.

In general the total matrix (2.2) will satisfy a periodicity condition of the form (2.7) with $V$ a block diagonal matrix consisting of (say) $s$ blocks of order $N_{(i)}$, such that each block can be taken of the form (2.8), and thus (as described above) defines a string of length $N_{(i)}$. As above, the space-time interpretation of this length is as the light-cone momentum

$$p_{(i)}^{+} = \frac{N_{(i)}}{R}. \quad (2.9)$$

This free string gas provides a good description of the SYM Hilbert space in the limit where all strings are far apart, i.e. if the eigenvalues in the matrix $X^{I}$ are well separated between the different blocks.

To avoid possible confusion later on, it is important to point out that the matrix $V$ that specifies the periodicity condition on the eigenvalues is not equal to the Wilson line of the gauge field $A_\sigma$ around the $S^1$. This identification would arise only if we insist on minimizing the potential energy term $(D_\sigma X)^2$ with respect to $A_\sigma$.

A related point is that for finite string coupling $g_s$, and certainly in the large $g_s$ limit, the bound states with total light-cone momentum $N$ are clearly no longer necessarily described by means of a single long string. More generally, one would expect that the
bound state wave function will have support on more subtle bound state configurations consisting of several (up to $N$) short(er) strings. In other words, it seems reasonable to expect that for large $g_s$ the long strings will tend to “fractionate” into many smaller constituents.

2.2. D-particles and Electric Flux Sectors

As we have indicated, an important new feature of the matrix string formalism (relative to standard light-cone string theory) is that via the electric flux, string states can also be adorned with a non-vanishing D-particle charge. In this subsection we will describe this correspondence in somewhat more detail.

To add to this interpretation, let us first show that each separate string can carry only one type of electric flux. Consider again the single string with length $N$. Define the $U(N)$ matrix $U$ such that

$$UV = VUe^{\frac{2\pi i}{N}}$$

(2.10) with $V$ as in (2.8). Hence we can take

$$U = \begin{pmatrix}
1 & 0 \\
e^{\frac{2\pi i}{N}} & 0 \\
0 & \ldots & 0 \\
0 & 0 & 0 & \ldots & e^{\frac{2(N-1)\pi i}{N}}
\end{pmatrix}$$

(2.11)

The $SU(N)$ part of the electric flux in this sector is defined as

$$\hat{U}|\psi_e\rangle = \exp\left(\frac{2\pi ie}{N}\right)|\psi_e\rangle$$

(2.12) with $e \in \mathbb{Z}_N$ and $\hat{U}$ the quantum operator that implements the constant gauge rotation

$$(A, X) \rightarrow (UAU^{-1}, UXU^{-1}).$$

(2.13)

Since diagonal matrices are inert under this gauge rotation, we conclude that the $SU(N)$ part of the electric flux dynamically decouples from the diagonal configurations (2.2) that describe the separate freely propagating strings. Now recall that in $U(N)$ SYM theory, the overall $U(1)$ part of the electric flux is related to the $SU(N)$ part $e$ via

$$\text{tr}E = e \pmod N.$$
Supersymmetry ensures that the ground state in the $SU(N)$ sector has zero energy even for $e \neq 0$. Hence the total ground state energy receives only a contribution from the overall $U(1)$ flux. In the following we will thus identify $e$ with the total $U(1)$ electric flux. From the above description it is further clear that we can indeed turn on only one electric flux per long string, as is appropriate for its identification with D-particle charge.

The energy (2.3) of the ground state in this electric flux sector is equal to

$$H_0 = \frac{e^2}{2Ng_s^2}. \quad (2.15)$$

General ground state configurations

$$|N^{(i)}, p_{\perp}^{(i)}, e^{(i)}\rangle \quad (2.16)$$

of $s$ separate strings of individual length $N^{(i)}$, transverse momenta $p_{\perp}^{(i)}$, and D-particle charge $e^{(i)}$ have a SYM energy equal to

$$H_0 = \sum_{i=1}^{s} \frac{1}{2N^{(i)}} \left[ (p_{\perp}^{(i)})^2 + (e^{(i)}/g_s)^2 \right] \quad (2.17)$$

(recall this is defined with respect to the time $\tau = tR$) which, when rescaled by $R$, is the sum of the $p_-$ light-cone momenta of the corresponding collection of string ground states

$$\sum_{i=1}^{s} \frac{1}{2p_{+}^{(i)}} \left[ (p_{\perp}^{(i)})^2 + (M^{(i)})^2 \right]. \quad (2.18)$$

In particular, we read off from (2.17) that the states with D-particle charge $e^{(i)}$ each have mass

$$M^{(i)} = \frac{e^{(i)}}{g_s} = \frac{e^{(i)}}{Rg} \quad (2.19)$$

in accordance with their identification as graviton states with non-zero KK momentum in the compact direction.

2.3. IR limit and String Interactions

In the IR limit, $g_s \to 0$, the SYM dynamics effectively reduces to a free orbifold sigma model on the symmetric product space $(\mathbb{R}^8)^N/S_N$. The interacting theory for $g_s > 0$ arises by relaxing this IR limit. Correspondingly, one may view the interacting string theory as
obtained via a perturbation of the $S_N$-orbifold conformal field theory. In first order, this perturbation is described via a modification of the CFT Hamiltonian

$$H = H_0 + \lambda \int d\sigma V_{\text{twist}}$$  \hspace{1cm} (2.20)

Here $V_{\text{twist}}$ is an appropriate CFT twist operator that generates simple transpositions of two string coordinates $[15]$. In terms of the bosonic twist-operators $\tau$ and fermionic spin fields $\Sigma$ it takes the form

$$V_{\text{twist}} = \sum_{i<j} (\tau^I \Sigma_J \otimes \tau^J \Sigma_I)_{ij}.$$

This is a weight $(\frac{3}{2}, \frac{3}{2})$ conformal field. The above twist field operator is an intertwiner between different topological sectors of the orbifold model that are related by a basic splitting and joining interaction between two strings. Hence if we use the above Hamiltonian for computing scattering amplitudes via standard perturbation theory, we will indeed reproduce the conventional perturbation expansion of type IIA string theory $[20]$. This weak string coupling expansion is a strong coupling expansion from the SYM perspective.

3. Fixed Angle Scattering of Strings

High energy, fixed angle processes in superstring theory were first studied in detail from the point of view of conventional string perturbation theory by Gross and Mende $[1]$. Central to their approach is the observation that in the limit of large external momenta, the Polyakov path integral at each given perturbative order is dominated by a finite number of saddle point configurations. Furthermore, it was proposed that all these saddle points essentially describe the same preferred worldsheet trajectory, up to an overall factor depending on the loop order.

In the subsequent sections we will find independent evidence from the point of view of matrix string theory that supports this physical picture. In this section we recall some of the main results of $[1]$. In addition we will give a useful characterization of the Gross-Mende saddle points in terms of the light-cone gauge formulation of string perturbation theory.
Fig. 4: This figure indicates the kinematics of the transverse momenta $p_i$.

It will be useful to first recall a few geometric facts about the light-cone gauge formulation of string perturbation theory. Consider a tree level string diagram that describes the scattering of four external massless particles with light-cone momenta $p_i^+ = N_i/R$ and transverse momenta $p_i$. For definiteness, we will describe this process in the center of mass frame in the transversal direction

$$\vec{p}_1 + \vec{p}_2 = 0$$
$$\vec{p}_3 + \vec{p}_4 = 0$$

(3.1)

The four transversal momenta $p_i$ can all be chosen to lie within one given plane. We can thus write all $p_i$ as complex numbers. In addition, longitudinal momentum and energy conservation imply that

$$N_1 + N_2 = N_3 + N_4$$

(3.2)

$$\frac{|p_1|^2}{N_1} + \frac{|p_2|^2}{N_2} = \frac{|p_3|^2}{N_3} + \frac{|p_4|^2}{N_4}$$

(3.3)

For a given set of locations $z_i$ of the corresponding vertex operators, the classical location of the worldsheet is described by

$$X^+(z, \overline{z}) = \frac{1}{2} \sum_i \epsilon_i N_i \log |z - z_i|^2$$

(3.4)

$$X(z, \overline{z}) = \frac{1}{2} \sum_i \epsilon_i p_i \log |z - z_i|^2$$

(3.5)

where $\epsilon_i = 1$ for the incoming and $-1$ for the outgoing particles. Here $(z, \overline{z})$ denotes a general conformal parameterization of the string world sheet.

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4 For a review of some salient features, see [21].
In the light-cone gauge, one chooses a fixed world-sheet parameterization by identifying \( X^+ \) with the world-sheet time \( \tau \)

\[ X^+(z, \bar{z}) = \tau \]  

(3.6)

which via (3.4) amounts to setting

\[ w \equiv \tau + i\sigma = \frac{1}{2\pi} \sum_{i} \epsilon_i N_i \log(z - z_i) . \]  

(3.7)

The differential \( \omega = dw \) is a specific globally defined holomorphic differential on the world-surface; existence and uniqueness of such a differential at arbitrary genus \([22,21]\) generalizes the construction to higher loop amplitudes. Notice that (due to the branch cuts in the logarithm) the coordinate \( \sigma \) in (3.7) is defined on an interval \( 0 \leq \sigma < (N_1 + N_2) \).

The light-cone coordinate system (3.7) specifies a particular time-slicing of the string world-sheet. In this coordinate frame, there are therefore specific points on the world-sheet at which strings split or join. These interactions take place at zeros of \( \omega \), that is critical points \( z = z_0 \) of the light-cone coordinate \( X^+ \), at which

\[ dX^+ \big|_{z=z_0} = 0 . \]  

(3.8)

Inserting the explicit form (3.4) for \( X^+ \) gives

\[ \sum_{i=1}^{S} \frac{\epsilon_i N_i}{z_0 - z_i} = 0 . \]  

(3.9)

In the specific case of the four-point scattering amplitude, this condition can be reduced to an equation relating the interaction point and the cross ratio

\[ \lambda = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_2)(z_3 - z_4)} \]  

(3.10)

via

\[ \frac{N_1}{z_0} + \frac{N_2}{z_0 - 1} = \frac{N_3}{z_0 - \lambda} . \]  

(3.11)

For given \( \lambda \), this is a quadratic equation for \( z_0 \) with in general two solutions \( z_0^+ \) and \( z_0^- \), representing the simple splitting and joining interaction respectively.

Now we are ready to discuss the Gross-Mende saddle point. In the covariant formulation used in \([I]\), it is characterized by the condition that it minimizes the Polyakov
action for the classical trajectory specified by (3.4)-(3.5). This action takes the form of a two-dimensional “Coulomb energy” of four light-like “charges” given by the momenta $p_i$:

$$E_C = \frac{1}{2} \sum_{i<j} p_i \cdot p_j \log |z_i - z_j|^2 .$$  \hfill (3.12)

Due to conformal invariance, this energy $E_C$ depends on the locations $z_i$ of the vertex-operators only by means of the cross ratio $\lambda$. The variation of $E_C$ with respect to $\lambda$ reads

$$\partial_\lambda E_C(\lambda) = \frac{p_1 \cdot p_3}{\lambda} + \frac{p_2 \cdot p_3}{\lambda - 1} .$$  \hfill (3.13)

The saddle point equation $\partial_\lambda E_C = 0$ is solved by

$$\lambda = \frac{p_1 \cdot p_3}{p_1 \cdot p_2} = \frac{t}{s} .$$  \hfill (3.14)

This saddle point corresponds to a particular classical world-sheet trajectory which at high energies gives the dominant contribution to the scattering amplitude.

For later reference, it will be useful to translate the above description of the GM saddle point into the light-cone gauge language. To begin with, in the complex parameterization for the $p_i$, the Mandelstam parameters $s$ and $t$ are expressed as

$$s = -2p_1 \cdot p_2 = (N_1 + N_2)^2 \frac{|p_1|^2}{N_1 N_2}$$

$$t = -2p_1 \cdot p_3 = \frac{|N_3 p_1 - N_1 p_3|^2}{N_1 N_3}$$  \hfill (3.15)

so that (3.14) takes the form

$$\lambda = \frac{N_2}{N_3} \frac{|N_3 p_1 - N_1 p_3|^2}{(N_1 + N_2)^2 |p_1|^2} .$$  \hfill (3.16)

Together with (3.11), this saddle point specifies a particular set of locations for the two interaction points $z_0^\pm$ of the light-cone string diagram. We now claim that this preferred location of the interaction points $z = z_0^\pm$ is singled out by the requirement that, in the immediate neighborhood of $z = z_0^\pm$, the transverse coordinate fields $X(z)$ are (anti-)analytic functions of $z$

$$\partial_z X |_{z = z_0} = 0$$

$$\partial_{\bar{z}} \bar{X} |_{z = z_0} = 0 .$$  \hfill (3.17)
To verify this claim, let us compute the cross ratio $\lambda$ from (3.17). The result should be equal to (3.14). Inserting the solution (3.5) into (3.17) gives

$$\sum_i \frac{\epsilon_i p_i}{z_0^+ - z_i} = 0$$  \hspace{1cm} (3.18)

In terms of the cross-ratio $\lambda$ defined in (3.10) this reads

$$\frac{p_1}{z_0^+ (z_0^+ - 1)} = -\frac{p_3}{z_0^+ - \lambda}$$  \hspace{1cm} (3.19)

where we used that $p_1 + p_2 = 0$. When combined with the equation (3.11), which relates $\lambda$ with the location of the interaction points $z_0$, this equation can indeed be used to compute $\lambda$ in terms of the scattering data. If we subtract $N_3$ times (3.19) from $p_3$ times (3.11), we obtain a linear equation for $z_0$, solved by

$$z_0^+ = \frac{N_1 p_3 - N_3 p_1}{(N_1 + N_2)p_3}.$$  \hspace{1cm} (3.20)

Further, from (3.19) we find that

$$\lambda = z_0^+ \left(1 + \frac{p_3}{p_1} (z_0^+ - 1)\right).$$  \hspace{1cm} (3.21)

After inserting (3.20) into (3.21), it’s a simple calculation to verify that the resulting expression for $\lambda$ indeed coincides with the high energy saddle point (3.14). Note that for the saddle-point configuration, $\lambda$ is in fact real. The interaction points $z_0^+$ and $z_0^-$ are in this case each others complex conjugate.

In [1], Gross and Mende propose the following attractive generalization of the saddle-point to higher orders in the string perturbation expansion. They assume that the dominant saddle points at genus $G$ take the form of a $G+1$ fold cover of the same four-punctured sphere as described above, branched over the four locations $z_i$ of the vertex operators. The resulting surfaces are known as $Z_N$ curves with $N = G + 1$. The classical trajectory of these higher order saddle points has the same shape as the tree level trajectory, but its size is $N$ times smaller. (The intuitive reason is that they describe multiple wound strings, so that the effective string tension is $N$ times bigger than usual) . Correspondingly, since the different trajectories are weighted by the world sheet area, the higher order trajectories give contributions proportional to $e^{-E_C/N}$ (with $E_C$ given in (3.12)). The higher genus contributions are thus quite strongly enhanced at high energy. We refer to [1] for a more detailed description.
It is worth pointing out that the structure of the $Z_N$ curves and the corresponding space-time trajectories, as depicted in fig. 3, are quite reminiscent of the description of the “long string” boundary conditions in section 2. In our view, this (proposed) structure of the higher order interactions is one of several indications that the Gross-Mende approach to high energy string scattering may have a natural implementation in the Matrix string context. The above light-cone characterization of the GM saddle point in terms of the holomorphicity conditions (3.17) will be critical in establishing this relation!

4. Matrix String Interactions

In this section we will prepare the ingredients for the semi-classical study of high energy scattering in the Matrix string framework. To begin with, we notice that the above light-cone gauge description of the dominant string world sheet trajectories can rather easily be put into a matrix form, as follows. Starting from equations (3.4) and (3.5), we represent the classical string trajectory by means of a diagonal $N \times N$ matrix (with $N = N_1 + N_2$) by first writing the transversal coordinates $\vec{X}$ as a function of $\omega$ defined in (3.7), and then “roll up” the spatial interval $0 \leq \sigma < N$ onto the short interval $0 \leq \sigma < 1$. Concretely, we define the diagonal matrix elements of $X(\sigma)$ via $X_{kk}(\sigma) = X(\sigma + k)$, and in this way we indeed create matrix configurations that, away from the interaction times, satisfy the long string boundary condition (2.7) and (2.8).

These diagonal matrix configurations represent particular solutions to the SYM equations of motion, that are regular everywhere except at the interaction points. If at some point in the $(\sigma, \tau)$ plane two eigenvalues $X_I$ and $X_J$ coincide, we enter a phase where locally the gauge symmetry is restored to $U(2)$. In general we should thus expect that in this local region the semi-classical SYM solution will need to become truly non-abelian.

It is readily seen that the diagonal matrix configurations constructed via the above procedure from the CFT solution (3.5) is not single valued around the interaction points. Instead, as explained in [15], in going around the interaction point, the matrix $X$ undergoes a simple transposition of the two degenerating eigenvalues. In the gauge theory language, this means that the diagonal CFT solution (3.5) in fact hides a delta-function Yang-Mills curvature at the interaction point, such that the infinitesimal Wilson line around it coincides with this permutation group element [23]. In this section we will describe how the Yang-Mills dynamics in fact smoothes out this singularity.
Concretely, we will now exhibit a smooth and single-valued Yang-Mills configuration that describes the local splitting or joining of one or two matrix strings. Ultimately, we will be interested in obtaining global classical solutions to the SYM equations of motion that minimize the Yang-Mills action for given asymptotic conditions on the matrix fields $X$, as written in eqns (1.2) and (1.3) in the Introduction.

4.1. SYM Solution near Interaction Point

It seems reasonable to assume that, at least in the immediate neighborhood of the interaction point, these minimal action configurations of the SYM model are described by supersymmetric configurations. Hence, instead of trying to solve the full Yang-Mills equations, we will restrict ourselves to the special class of solutions satisfying a dimensionally reduced version of the self-duality equations from four to two dimensions. We will choose to work with variables

$$X = \frac{1}{2}(X^1 + iX^2), \quad \overline{X} = \frac{1}{2}(X^1 - iX^2),$$

setting the remaining $X^i$’s to zero. The self-duality conditions then become

$$F_w = -\frac{i}{g_s^2} [X, \overline{X}]$$
$$D_w X = 0$$
$$D_w \overline{X} = 0.$$

The above equations are most conveniently analyzed by writing

$$A_w(w, \overline{w}) = -iG\partial_w G^{-1}$$
$$A_{\overline{w}}(w, \overline{w}) = i(\partial_{\overline{w}} G^{-1})\overline{G}.$$
where \( G(w, \overline{w}) \) denotes an element of the complexified \((\overline{G} \neq G^{-1})\) gauge group. This parametrization of \( A_\alpha \) allows one to solve the second and third equation of (4.2), via
\[
X(w, \overline{w}) = G\tilde{X}(\overline{w})G^{-1}.
\]
(4.4)
The first equation in (4.2) then takes the following form
\[
\partial_w(\Omega\partial_w\Omega^{-1}) = -\frac{1}{g_s^2}[\Omega\tilde{X}(\overline{w})\Omega^{-1}, \tilde{X}(w)]
\]
(4.5)
with
\[
\Omega = GG.
\]
(4.6)
Let us now look at the local neighborhood of an interaction point. For convenience, we choose coordinates such that it is located at \( w = 0 \). Since the interaction involves only two eigenvalues, it is sufficient to consider only the corresponding \( SU(2) \) part of the matrices. The matrix \( \tilde{X} \), which parametrizes the local coordinate distance between the two interacting strings, can be chosen of the following form
\[
\tilde{X}(\overline{w}) \simeq \pm B\sqrt{w}\tau_3
\]
(4.7)
for some constant \( B \). The \( \pm \) indicates that the interaction point \( w = 0 \) represents a square root branch point for the diagonal matrix \( \tilde{X} \) in (4.7), which therefore is multi-valued.

The diagonal matrix \( \tilde{X}(\overline{w}) \), together with \( A = 0 \), represents a valid solution of the SYM equations (4.2) except at the interaction point, where analyticity fails. Therefore we’ll look for a true solution of the form (4.4), where \( G \to 1 \) asymptotically far from \( w = 0 \). A helpful Ansatz for \( G(w, \overline{w}) \) is
\[
G = e^{\frac{1}{2}\alpha\tau_1}
\]
(4.8)
where for \( \alpha(w, \overline{w}) \) we choose a real function (so that \( G = \overline{G} \) and \( \Omega = \exp(\alpha\tau_1) \)) that tends to zero far away from the interaction point. We now compute
\[
\Omega\tilde{X}\Omega^{-1} = B\sqrt{w}e^{\alpha\tau_1}\tau_3e^{-\alpha\tau_1} = B\sqrt{w}\left(\begin{array}{cc}
\cosh 2\alpha & -\sinh 2\alpha \\
\sinh 2\alpha & -\cosh 2\alpha
\end{array}\right).
\]
(4.9)
Hence
\[
\left[\Omega\tilde{X}\Omega^{-1}, \tilde{X}\right] = 2|B|^2|w|\sinh 2\alpha \tau_1
\]
(4.10)
and thus we find that under the present Ansatz the equation of motion (4.11) reduces to
\[ \partial_w \partial_{\overline{w}} \alpha = \frac{2}{g_s^2} |B|^2 |w| \sinh 2\alpha \] (4.11)
which is essentially the familiar sinh-Gordon equation. (It can be transformed to the exact sinh-Gordon equation after a (multi-valued) coordinate transformation \( w \to \tilde{w} = w^{3/2} \).)

The boundary condition that we must impose on \( \alpha(w, \overline{w}) \) at \( w = 0 \) follows from the requirement that the Yang-Mills configuration be regular. This condition is most easily understood in the gauge where \( X \) is single-valued near \( w = 0 \); in this gauge the YM curvature \( F_{\overline{w}w} \) should be a regular function at \( w = 0 \). The configuration (4.4)-(4.8), however, is (for single-valued and real \( \alpha \)) multi-valued. We can make \( X \) single-valued by applying the singular gauge transformation
\[ X \to UXU^{-1}, \quad A_w \to -iUD_wU^{-1} \] (4.12)
with gauge parameter
\[ U = e^{\pm i\theta \tau_1/4} \] (4.13)
with \( \theta = \frac{1}{2i} \log(w/\overline{w}) \) the azimuthal angle around \( w = 0 \). In this gauge
\[ A_w = i \left[ \frac{1}{2} \partial_w \alpha \pm \frac{1}{8w} \right] \tau_1 \]
\[ A_{\overline{w}} = -i \left[ \frac{1}{2} \partial_{\overline{w}} \alpha \pm \frac{1}{8\overline{w}} \right] \tau_1 \] (4.14)
Using that \( \partial_w \frac{1}{w} = \pi \delta^{(2)}(w) \), this gives
\[ F_{\overline{w}w} = -i\tau_1 \left( \partial_w \partial_{\overline{w}} \alpha \pm \frac{\pi}{4} \delta^{(2)}(w) \right). \] (4.15)
The regularity requirement at \( w = 0 \) is therefore that \( \partial_w \partial_{\overline{w}} \alpha \simeq \mp \frac{\pi}{4} \delta^{(2)}(w) \). We thus deduce that the solution to equation (4.11) that we want must satisfy the following asymptotic condition
\[ \alpha(w, \overline{w}) \simeq \mp \frac{1}{2} \log |w| + \text{const.} \quad w \to 0 \] (4.16)
while at large distances from the interaction point \( \alpha \) must tend to zero.

Now let us write \( \alpha = \alpha(r) \) with \( r = |w| \). The equation of motion (4.11) reduces to the ordinary non-linear differential equation
\[ (\partial_r^2 + \frac{1}{r} \partial_r)\alpha = \frac{8}{g_s^2} |B|^2 r \sinh 2\alpha. \] (4.17)
A numerical solution to this equation is depicted in fig. 6. For large \( r = |w| \) the solution looks like
\[ \alpha(r) \sim \mp \frac{1}{|w|^{3/4}} \exp \left( -\frac{8|B|}{3g_s} |w|^{3/2} \right). \] (4.18)
5. High Energy Scattering of Matrix Strings

The above matrix solution of the string interaction should be viewed as a local description in the immediate neighborhood of the interaction point. In general, it must therefore be glued via an appropriate patching procedure into a complementary CFT type solution (e.g. as described in section 3) that matches with the asymptotic scattering data at the far past and future. The idea here is that (as we will see shortly) at sufficiently high collision energies, the size of the interaction regions are small compared to the rest of the matrix string world sheet. Hence, while the behavior (4.7) provides the asymptotics for large |w| at the UV scale of the matrix solution, it also provides the local boundary condition near the interaction point for the CFT solution for X that describes the IR part of the saddle point.

The solution (4.2) that we described is not the most general SYM description of a string splitting and joining event, but rather the most symmetric one, with smallest action. This means therefore that there is a non-trivial matching condition on the corresponding matrix string world sheet: from (4.7) we see that we must require that the transverse string coordinates X behave holomorphically near the interaction point. Remarkably, this
is exactly the same condition as (3.17), which tells us that the shape of the string world sheet must be precisely that of the Gross-Mende saddle point! Therefore these solutions indeed seem appropriate to a YM generalization of the high-energy scattering of \[1\]. In this section, we fill in a few more details of this connection.

5.1. Evaluation of the classical action

In order to estimate scattering amplitudes via the instanton processes, one must calculate the instanton action. The bosonic part of the SYM action (with only two \(X\)-fields non-vanishing) can be written as

\[
S = \int d^2 w \left\{ -g_s^2 \left( F_{w\bar{w}} + \frac{i}{g_s^2} [X, \bar{X}] \right)^2 + 4D_w XD_{\bar{w}} \bar{X} \right\} + \oint (\bar{X} D X + X D \bar{X} - \bar{X} D X - X D \bar{X})
\] (5.1)

and thus for the supersymmetric configuration that satisfy (4.2), the total classical action reduces to a boundary term

\[
S_{cl} = \oint (\bar{X} \partial X + X \partial \bar{X})
\] (5.2)

identical to the boundary term needed to glue the non-abelian matrix solution described in the previous section into the CFT type solution. Hence we claim that, in the limit that the matrix interaction points become sufficiently small, the SYM action for the above saddle point configurations coincides with the CFT action, \(i.e.,\) for the case of a tree level string diagram it equals the “Coulomb energy” (3.12), where me must insert the saddle-point value for locations \(z_i\). It is perhaps worth pointing out that this saddle point actions is indeed fully Lorentz invariant, as it should be. While this is not surprising once we’ve established the connection with the GM saddle point, it does seem to represent a rather non-trivial statement from the SYM point of view!

More generally we see that from (5.1) we can derive (as usual) an inequality, which suggests that whenever the interaction does not take place at a holomorphic point for the \(X\)-fields, the SYM action is always larger than the corresponding CFT action. This provides additional evidence for the conjecture that the above type of configurations indeed represent dominant saddle-points, that minimize the SYM action.

Obviously, there exist a large number of CFT-type solutions for which \(X\) varies (anti-) holomorphically near all interaction points. In particular, there are the higher genus \(Z_N\)-curves of \([1]\). In addition it is also possible to write down SYM solutions that describe
multiple string world sheets, but nonetheless still satisfy the appropriate boundary conditions, as specified in eqns (1.2) and (1.3) in the Introduction. Ideally, one would like to know which (sub-class) of these solutions provide the truly dominant contribution to the scattering amplitude. We will not attempt to answer this question here, and will restrict ourselves to some qualitative remarks in the concluding section.

5.2. Minimal distance

The parameter $|B|$ that governs the size of the interaction vertex, as seen in (1.18), can be straightforwardly determined in terms of the momenta of the external states. The coordinate system $(w, \overline{w})$ on the Yang-Mills cylinder that we used in the analysis of the self-dual Yang-Mills equation (4.2), coincides with the light-cone coordinates defined in (3.7). From this and (4.7) we immediately find

$$|B|^2 = 2\frac{|\partial z X(z_+^0)|^2}{|\partial^2 z X(z_0^+)|^2}. \quad (5.3)$$

A straightforward calculation then gives

$$|B|^2 = \frac{|p_1 \overline{p}_3 - \overline{p}_1 p_3|(N_1 + N_2)}{\sqrt{N_1 N_2 N_3 N_4}}. \quad (5.4)$$

It is interesting to note that for this solution, even though the eigenvalues of the complex coordinate matrix $X$ indeed vanish at the interaction point, the full matrix coordinate $X$ in fact does not! Instead, near $w = 0$ it approaches the constant non-diagonalizable matrix

$$X(w, \overline{w}) \simeq \text{const. } g_s^{1/3} B^{2/3} \begin{pmatrix} 1 & \mp 1 & \pm 1 \\ \pm 1 & -1 & -1 \\ \end{pmatrix} \quad w \to 0 \quad (5.5)$$

(The value of the overall constant can quite easily be determined numerically.) From this we read off that the minimal “distance” between the two interacting strings is in fact non-zero! Instead, we have

$$d_{\text{min}} = \sqrt{\text{tr}(X(0)\overline{X}(0))} \sim g_s^{1/3}|B|^{2/3}. \quad (5.6)$$

Although it is tempting to speculate (as indeed we will do in the concluding section), the precise physical significance of this result is as yet unclear to us. We do notice, however, that the typical world sheet size $\ell_{\text{inst}}$ of the matrix interaction region, as can be read off from (1.18), is naturally expressed in terms of this minimal relative distance as $\ell_{\text{inst}} = (g_s/|B|)^{2/3} = g_s/d_{\text{min}}$. 

23
5.3. Fluctuation Determinant

In principle it should be possible to compute the one-loop determinant of the quantum fluctuations around these semi-classical configurations. An important motivation for performing such an analysis is to obtain a semi-classical estimate for the absolute strength of the splitting and joining interactions in matrix string theory. Duality symmetries of M-theory gives the precise prediction that this strength should be governed by the string coupling $g_s$. To verify this, one would need to compare the SYM one-loop determinant with the Gross-Mende fluctuation determinant, coming from the gaussian integration over the Riemann surface moduli around the saddle-point. We leave this for future work.

It seems even more worthwhile to look for true new physical effects that might arise from the one-loop corrections. Compared to the conventional perturbative string description, the new degrees of freedom in matrix string theory are the charged components of the $X$-fields, as well as the extra gauge potential $A_\alpha$. These new degrees of freedom are non-perturbative from the string perspective, and their quantum fluctuations could thus potentially lead to new physics. As we will show in the next section, there is indeed such a new effect: the pair creation of D-particles.

6. D-Particle Pair Production

In this section we turn to the process of pair creation of D charge, which is in our description $x^9$ momentum or equivalently (under the Matrix string duality) electric flux. This can be viewed as a contribution to the fluctuations about the high-energy scattering processes of the preceding sections, or as a process worthy of interest in its own right in the context of graviton scattering. There are several viewpoints from which this can be investigated. In the limit where $x^9$ decompactifies, this simply matches onto the standard supergravity calculation\[24\]. In fact, we can work backwards from this, using the method of images, and compute the amplitude at large finite $R_9$, in the special situation with source and probe particles, $N_1 \gg N_2$. We will discuss this calculation first. Alternately, one can study this process directly in the matrix string approach, and derive the pair-production rate via a one-loop Yang Mills calculation. This latter approach gives a leading order result valid for arbitrary $N_1$ and $N_2$, and also more readily makes connection with the other results of this paper. Furthermore, the Yang-Mills calculation also apparently extends beyond the region where supergravity is a valid approximation.
6.1. Supergravity calculation

Consider 11-dimensional supergravity compactified on $S^1 \times S^1$, where one $S^1$ a lightlike circle of “radius” $R$

$$x^- \equiv x^- + 4\pi R$$

(6.1)

and the other $S^1$ denotes a space-like circle of radius $R_9$. As we’ve seen, in the M-theory/matrix string correspondence this second radius $R_9$ is expressed as $R_9 = g_s$ in string units.

Consider in this set-up the scattering process of two massless particles of light-cone momenta $p_i^+ = N_i/R$ and transverse momenta $p_i$. Let us first consider the probe situation $N_2 \ll N_1$. Then one can already get quite useful information about the scattering process from considering the classical gravitational force between the two particles. The boosted particle with $p_+ = N_1/R$ produces via its stress-energy a non-trivial gravitational background, described by the generalized Aichelburg-Sexl shockwave geometry of the form \cite{25,26}

$$ds^2 = -dx^- (dx^+ + f(r, a) dx^-) + dx^2 + g_s^2 da^2.$$  (6.2)

with

$$f(r, a) = \sum_k \frac{15 N_1 g_s^3}{2R^2 (r^2 + g_s^2 (2\pi k + a)^2)^{7/2}}$$

(6.3)

Here $a$ denotes the coordinate distance from the gravitational source in the compact $x^9$ direction, and the sum over $k$ arises from the image points in this direction.

The momentum four vector of the second massless particle moving in this background geometry will satisfy a dispersion relation of the form

$$2p^- (p^+ + f(r, a)p^-) = p^2 + e^2 / g_s^2$$

(6.4)

where $e$ denotes the quantized momentum in the $x^9$ direction. We can solve for the light-cone hamiltonian $p^-$ of the particle and obtain

$$p^- = \frac{p^+}{2f(r, a)} \left\{ \sqrt{\left(1 - \frac{2f(r, a)}{(p^+)^2} \frac{p^2 + e^2 / g_s^2}{(p^+)^2}\right)} - 1 \right\}$$

(6.5)

Substituting $p^+ = N_2/R$, (and rescaling the light-cone time by a factor of $R$) we can write this as

$$H = H_0 + H_{int}$$

(6.6)
where
\[ H_0 = \frac{1}{2N_2}(p^2 + e^2/g_s^2) \]  \hspace{1cm} (6.7)
and
\[ H_{\text{int}} \simeq -\frac{15N_1g_s^2}{8N_2^3} \sum_k \frac{(p^2 + e^2/g_s^2)^2}{\left(r^2 + g_s^2(2\pi k + a)^2\right)^{7/2}} + \cdots \]  \hspace{1cm} (6.8)

Hence the motion of the second particle in terms of the light-cone time \( x^+ \) looks like that of a particle with mass \( N_2 \) moving in \( R^8 \times S^1 \) under the influence of an interaction potential given by (6.8).

From this description we can now quite easily extract a low energy prediction for the D-pair production rate. To this end, it is useful to rewrite the interaction Hamiltonian via a Poisson resummation as
\[ H_{\text{int}} \simeq -\frac{15N_1g_s^2}{2\pi N_2} \sum_n \exp(ina) \int dT T^2 \exp(-Tr^2) \exp(-n^2/4g_s^2T) \]  \hspace{1cm} (6.9)

The \( n = 1 \) term in this series is the term that corresponds to changing the compact momentum by one unit, \( i.e. \) to D-charge production. Working to first order in perturbation theory, we can then compute the corresponding phase shift, using
\[ \delta = -\int d\tau H_{\text{int}}(b^2 + p^2\tau^2) \]  \hspace{1cm} (6.10)

6.2. D-pair production via electric flux creation

We now study this problem of D-charge creation in the Matrix string framework. More generally, we consider scattering states which asymptotically have momenta of the form
\[ p^\mu = (p^-, \vec{p}, p_9 = n/R_9, p^+ = N/R) \]  \hspace{1cm} (6.11)

These include both gravitons (\( n = 0 \)) and D0-branes – or anti-branes – (\( n = \pm 1 \)). The case of current interest begins with an initial state of two gravitons, and pair produces a D particle pair. This process is intrinsically non-perturbative from the point of view of string theory. It is also a process not accessible in the standard Matrix theory approach, where the anti-branes are boosted away to infinite energy.

In principle (for example on a sufficiently large computer) it appears possible to calculate such amplitudes to arbitrary order in the coupling \( g = g_{YM} = 1/g_s \), and calculate the D-pair production rate even for small \( g_s \). In this section we will work to leading non-trivial order (one-loop), and leave further calculations to other work. Similar calculations have been performed in the context of Matrix theory in [27].
6.3. One-loop calculation, \( N = 2 \)

For simplicity we begin with the case where the incoming and outgoing particles all have \( N = 1 \). The next subsection will generalize to arbitrary \( N \). The asymptotic states thus take the form (2.4); specifically,

\[
\bar{X}^1 = \frac{1}{2} \begin{pmatrix} p \tau & 0 \\ 0 & -p \tau \end{pmatrix}, \quad \bar{X}^2 = \frac{1}{2} \begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix},
\]

(6.12)
corresponding to two particles with center of mass momentum \( p \) and impact parameter \( b \) (measured in string units). It will also be useful work with a non-trivial gauge background

\[
\bar{A}_\sigma = \begin{pmatrix} a/2 + e \tau / g_s^2 & 0 \\ 0 & -a/2 - e \tau / g_s^2 \end{pmatrix}, \quad \bar{E} = \begin{pmatrix} e & 0 \\ 0 & -e \end{pmatrix}.
\]

(6.13)
The constant electric field corresponds to a non-zero D-charge for the incoming and outgoing particles, with quantization

\[
e \in \mathbb{Z}.
\]

(6.14)
The prototypical example of production of D-charge is in processes where this changes by one unit,

\[
\Delta E = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}.
\]

(6.15)
Introducing the constant background potential \( a \) will help keep track of such changes.

In the large string coupling/small Yang-Mills coupling limit, the leading contribution to D-charge producing processes is easily computed via a one-loop super-Yang Mills calculation. Calculations at higher loop order then give subleading corrections in \( g = 1/g_s \).

Our starting point is the Yang-Mills action (1.1), although it will be simpler to begin with it written in its un-dimensionally reduced form in terms of the gauge field

\[
A^M = (A^\mu, g X^i),
\]

(6.16)
with \( M = 0, \ldots, 9, \mu = 0, 9 \) and \( i = 1, \ldots, 8 \). We will decompose the gauge field into background and fluctuation pieces,

\[
A_M = \bar{A}_M + g \tilde{A}_M.
\]

(6.17)
The Feynman background gauge-fixed lagrangian is

\[
\mathcal{L} = -\text{Tr} \left\{ \frac{1}{4g^2} \text{Tr} (F_{MN}^2) + \frac{1}{2g^2} \text{Tr} (\tilde{D}^M A_M)^2 - i \bar{\psi} \tilde{D} \psi \right\}
\]

(6.18)
where $\bar{D}_M = \partial_M + i\bar{A}_M$. Using the decomposition (6.17) we find

$$
\mathcal{L} = -Tr \left\{ \frac{\bar{F}^2}{4g^2} + \frac{1}{2} (\bar{D}_M \bar{A}_N)^2 + i\bar{F}^{MN} [\bar{A}_M, \bar{A}_N] - i\bar{\psi} \bar{D}\psi \\
+ g\bar{\psi} \bar{A}\psi + ig\bar{D}_M \bar{A}_N [\bar{A}_M, \bar{A}_N] - \frac{g^2}{4} [\bar{A}_M, \bar{A}_N]^2 \right\}. 
$$

(6.19)

The amplitude in question is given by

$$
\mathcal{A}(a, e) = \int D\bar{A}_\mu D\bar{X}^i D\psi e^{iS} 
$$

(6.20)

where the boundary conditions on the functional integral are chosen to correspond to the asymptotic behavior given in (6.12), (6.13).

If we write

$$
A_M = \frac{1}{2} A_M a \sigma^a = \frac{1}{2} A_M^+ \sigma^+ + \frac{1}{2} A_M^- \sigma^- + \frac{1}{2} A_M^3 \sigma^3, 
$$

(6.21)

with

$$
\sigma^\pm = \frac{\sigma^1 \pm i \sigma^2}{\sqrt{2}}, 
$$

(6.22)

then the couplings in (6.19) include the standard charged minimal couplings of $A_+, A_-, \psi_+$, and $\psi_-$ to the $U(1)$ field $\bar{A}_\mu 3$. The amplitude to create unit electric flux is therefore given by summing the loop contributions to (6.20) in which one of these charged particles circulates once around the $\sigma$-direction; higher encirclings yield more flux. Therefore we need the contribution of the charged state windings to the one-loop amplitude.

This immediately follows by reading off the spectrum from the second through fourth terms of (6.19) in the backgrounds (6.12) and (6.13). We begin by defining

$$
\tilde{p}^2 = g^2 p^2 + 4g^4 e^2 .
$$

(6.23)

In the bosonic sector we find the neutral fields

$$
\bar{X}^3_i, \ i = 1, \cdots, 8; \quad m^2 = 0
$$

(6.24)

and the charged fields

$$
\bar{X}^\pm_i, \ i = 2, \cdots, 8; \quad m^2 = r^2 \equiv \tilde{p}^2 - 2 + g^2 b^2
$$

$$
\frac{1}{\tilde{p}}(g\bar{A}^9 \bar{X}^\pm + 2eg^2 \bar{A}^1 \bar{X}^\pm); \quad m^2 = r^2
$$

$$
\bar{A}^0 \pm + \frac{i}{\tilde{p}}(2eg^2 \bar{A}^9 \bar{X}^0 + g\bar{A}^1 \bar{X}^0); \quad m^2 = r^2 + 2 \tilde{p}
$$

(6.25)

$$
\bar{A}^0 \pm - \frac{i}{\tilde{p}}(2eg^2 \bar{A}^9 \bar{X}^0 + g\bar{A}^1 \bar{X}^0); \quad m^2 = r^2 - 2 \tilde{p}.
$$
For the fermions, we have 16 massless uncharged states, 8 charged states with masses $m^2 = r^2 + \hat{p}$, and 8 charged states with masses $m^2 = r^2 - \hat{p}$, as in \[27\]. Finally, including the ghosts gives one complex, uncharged field $C^3$ with $m^2 = 0$ and one complex charged field $C^\pm$ with $m^2 = r^2$.

All of the charged fields are minimally coupled to the background field $\bar{A}_9 \equiv \bar{A}_\sigma$. At one loop level, we have

$$ A_1(a, e) = \int \prod_I D\Phi_I e^{iS^{(2)}[\Phi_I]} $$

(6.26)

where I labels the charged fields enumerated above (the uncharged contributions cancel), arbitrary winding is allowed, and where $S^{(2)}$ is the quadratic part of the action \((6.19)\) including the coupling to $\bar{A}_3^\sigma$ through $\bar{D}$. Working with phase shifts, we then have

$$ i\delta_1 = \ln A_1 = \sum_I \ln \sum_n \int \prod_I D\Phi_I e^{iS^{(2)}[\Phi_I]} $$

$$ = -\sum_I (-1)^{F_I} \sum_n \int_0^\infty dS \int_n^\infty D\tau D\sigma e^{i \int_0^S ds \left[ \dot{\sigma}_\mu^2/2 - g^2 (p^2 \tau^2 + b^2)/2 - \bar{A}_3^\sigma \dot{\sigma}(s) \right] - m^2(r)} \Delta(p, e, S) $$

(6.27)

Here we have used the functional integral representation in terms of the first-quantized trajectory $\sigma^\nu(s) = (\tau(s), \sigma(s))$, $n$ is the winding number about the cylinder, and $F_I$ denotes fermion number of the field.

For general winding $n$ the functional integrals in \((6.27)\) are readily rewritten in terms of functional determinants. For example, with $m^2 = r^2$ we have

$$ \int_n^\infty D\tau D\sigma e^{i \int_0^S ds \left[ \dot{\sigma}_\mu^2/2 - g^2 (p^2 \tau^2 + b^2)/2 - \bar{A}_3^\sigma \dot{\sigma}(s) \right]} $$

$$ = e^{-ina + in^2/2S - ig^2b^2S/2} \Delta(p, e, S) $$

(6.28)

where

$$ \Delta(p, e, S) = \det^{-1/2} \left( \begin{array}{cc} \partial_s^2 - g^2p^2 & -2g^2e\partial_s \\ 2g^2e\partial_s & -\partial_s^2 \end{array} \right) $$

Combining such expressions and defining $S = 2T$ then gives

$$ i\delta_1(a, p, e) = \sum_n e^{-ina} \int_0^\infty dT/T e^{in^2/4T - ig^2b^2T} \Delta(p, e, T) \left[ -6 - 2\cos(2gpT) + 8\cos(gpT) \right] $$

(6.30)
Recalling the quantization rule (6.14), we see that as long as \( gp \gg 1 \) the electric background only contributes at higher order in \( g \); neglecting this, the determinant is readily evaluated using

\[
\det \frac{i}{2}(-\partial_s^2 + g^2 p^2) = gp \prod_{n=1}^{\infty} \left[ \left( \frac{2n\pi}{S} \right)^2 + g^2 p^2 \right] = 2\sinh(gpT)
\]

\[
\det \frac{i}{2}(-\partial_s^2) = \sqrt{\frac{2\pi T}{i}}.
\]

The phase shift then becomes

\[
i\delta_1(a, p, e) = \frac{1}{2} \sqrt{\frac{i}{2\pi}} \sum_n e^{-ina} \int_0^\infty \frac{dT}{T} e^{in^2/4T} e^{-i g^2 b^2 T} \frac{1}{\sqrt{T} \sinh(gpT)} \left[ -6 - 2\cos(2gpT) + 8\cos(gpT) \right].
\]

In the full amplitude \( A_1 = \exp\{i\delta_1\} \), the coefficient of the term \( e^{-ina} \) is the amplitude to make a transition from a state with electric flux \( e \) to \( e + n \):

\[
A_1(a, e) = \sum_n e^{-ina} \langle e + n|e \rangle.
\]

We have found that this is independent of \( e \) to order \( g^2 \). The amplitude for a change by one unit of charge (e.g. two gravitons to \( D\bar{D} \) pair), as well as the effective interaction Hamiltonian, can be derived from these expressions in the range \( p \ll b^2 \). There the integrand in (6.32) can be expanded in \( pT \) to find

\[
i\delta_1(a, p, e) \approx -\frac{g^3 p^3}{2} \sqrt{\frac{i}{\pi}} \sum_n e^{-ina} \int dT T^{3/2} e^{-in^2/4T - i g^2 b^2 T};
\]

the leading order \( D\bar{D} \) production amplitude is just the coefficient of \( e^{-ia} \) in the series. From (6.34) and (6.10) we can also work backwards to extract the effective Hamiltonian. We find

\[
H_{int} = -\frac{15}{8} g^{-3} p^4 \sum_k \frac{1}{[r^2 + (a + 2\pi k)^2 / g^2]^{7/2}},
\]

in agreement with (6.8).
6.4. Generalization to arbitrary $N$

The one-loop calculation of the preceding subsection is readily generalized to the case where the incoming and outgoing particles have arbitrary $p_{11}$, or equivalently, $N$. In this case there are a variety of different boundary conditions that may be placed on the $N \times N$ blocks. Two that we will consider are the trivial boundary condition,

$$X(\sigma + 1) = X(\sigma),$$  \hspace{1cm} (6.36)

and the single long string boundary condition,

$$X(\sigma + 1) = V^{-1}X(\sigma)V$$  \hspace{1cm} (6.37)

where $V$ is given in (2.8).

In the case of two incoming states with momenta $N_1, N_2$, we write

$$X^i = \tilde{X}^i + \hat{X}^i$$  \hspace{1cm} (6.38)

where $X^i$ is an $(N_1 + N_2) \times (N_1 + N_2)$ matrix. In particular, the background is taken to be

$$\tilde{X}^1 = \frac{p\tau}{2} \begin{pmatrix} I/N_1 & 0 \\ 0 & -I/N_2 \end{pmatrix} \equiv \frac{p\tau}{2} T_D$$

$$\tilde{X}^2 = \frac{b}{2} T_D$$  \hspace{1cm} (6.39)

where we have split the matrix into $N_1 \times N_1$ and $N_2 \times N_2$ blocks corresponding to two “clusters,” and $I$ represents the corresponding identity matrices.

A useful decomposition of the fluctuations $\hat{X}^i$ is in terms of the matrices

$$T^{a_1} = \begin{pmatrix} t^{a_1} & 0 \\ 0 & 0 \end{pmatrix}, \quad T^{a_2} = \begin{pmatrix} 0 & 0 \\ 0 & t^{a_2} \end{pmatrix},$$  \hspace{1cm} (6.40)

where $t^{a_i}$ are hermitian generators of $SU(N_i); T^{a_1 a_2}_+, T^{a_1 a_2}_-$, which have matrix elements

$$(T^{a_1 a_2})_{\beta_1 \beta_2} = \sqrt{2} \delta_{a_1 \beta_1} \delta_{N_1 + \alpha_2 \beta_2}$$

$$(T^{a_1 a_2})_{\beta_1 \beta_2} = \sqrt{2} \delta_{N_1 + \alpha_1 \beta_1} \delta_{\alpha_2 \beta_2};$$  \hspace{1cm} (6.41)

and $T_D$:

$$\hat{X}^i = \frac{\tilde{X}_D}{2} T_D + \frac{\tilde{X}_{a_1}}{2} T^{a_1} + \frac{\tilde{X}_{a_2}}{2} T^{a_2} + \frac{\tilde{X}_{+ a_1 a_2}}{2} T^{+ a_1 a_2} + \frac{\tilde{X}_{- a_1 a_2}}{2} T^{- a_1 a_2}.$$  \hspace{1cm} (6.42)
Following the preceding subsection (and working with $e = 0$ for simplicity) we find that the charged states now have an extra $N_1 N_2$ in their multiplicities, and have masses

\[
\begin{align*}
\tilde{X}^i_{\pm \alpha_1 \alpha_2} & : m^2 = \frac{r^2}{4\nu^2}, \\
\tilde{A}^0_{\pm \alpha_1 \alpha_2} & : m^2 = \frac{r^2}{4\nu^2}, \\
\tilde{A}^0_{\pm \alpha_1 \alpha_2} + \tilde{X}^1_{\pm \alpha_1 \alpha_2} & : m^2 = \frac{r^2}{4\nu^2} + \frac{gp}{\nu}, \\
\tilde{A}^0_{\pm \alpha_1 \alpha_2} - \tilde{X}^1_{\pm \alpha_1 \alpha_2} & : m^2 = \frac{r^2}{4\nu^2} - \frac{gp}{\nu}
\end{align*}
\] (6.43)

where

\[
\frac{1}{\nu} = \frac{1}{N_1} + \frac{1}{N_2}.
\] (6.44)

Likewise, the charged fermions and ghosts have masses as in the $N = 2$ case with the trivial rescalings to

\[
\tilde{p} = \frac{p}{2\nu}, \quad \tilde{b} = \frac{b}{2\nu}
\] (6.45)

Therefore, in the case of trivial boundary conditions the amplitude (and hamiltonian) is exactly as computed in (6.32) with the only difference being multiplication by $N_1 N_2$ and replacement of $p$ and $b$ as in (6.45). Note that $2\tilde{p} = \frac{p}{N_1} + \frac{p}{N_2}$ is simply relative velocity of the two clusters, and $\tilde{b} = \frac{1}{2}(\frac{b}{N_1} + \frac{b}{N_2})$ is precisely the impact parameter between the clusters.

In the case of long-string boundary conditions, this result is modified. Now

\[
X(\sigma + 2\pi) = \left( \begin{array}{cc} V^{-1}_1 & 0 \\ 0 & V^{-1}_2 \end{array} \right) X(\sigma) \left( \begin{array}{cc} V_1 & 0 \\ 0 & V_2 \end{array} \right),
\] (6.46)

and in particular the charged off-diagonal blocks satisfy twisted boundary conditions

\[
\begin{align*}
X_+(\sigma + 1) &= V_1^{-1}X_+(\sigma)V_2 \\
X_- (\sigma + 1) &= V_2^{-1}X_- (\sigma)V_1.
\end{align*}
\] (6.47)

The matrices $V$ can be diagonalized by working on basis vectors

\[
w_k = \frac{1}{\sqrt{N}} \left( \begin{array}{c}
\lambda^k \\
\lambda^{2k} \\
\vdots \\
\lambda^{(N-1)k}
\end{array} \right), \quad \lambda = e^{2\pi i/\nu}, \quad k \in \mathbb{Z}
\] (6.48)
and in this basis simply give phases $\lambda^k$. Thus the amplitude (6.32) is modified to

$$i \delta_1 = \frac{1}{2} \sqrt{\frac{i}{2\pi}} \sum_{\alpha_1=1}^{N_1} \sum_n e^{-i n [2\pi (\alpha_1/N_1 - \alpha_2/N_2) + a]} \int_0^\infty \frac{dT}{T} e^{in^2/4T - ig\bar{b}^2 T} \frac{1}{\sqrt{T \sin(g\bar{p}T)}} \left[ -6 - 2 \cos(2g\bar{p}T) + 8 \cos(g\bar{p}T) \right],$$

(6.49)

and the interaction hamiltonian takes the form

$$H_{\text{int}} \approx g^4 p^4 \sum_{n, \alpha_1, \alpha_2} e^{-i n [2\pi (\alpha_1/N_1 - \alpha_2/N_2) + a]} \int dTT^2 e^{in^2/4T - ir^2 T}.$$  

(6.50)

For non-zero $n$, the supergravity correspondence no longer holds when $N > 2$: the matrix string then yields a different result. In (6.50) the expression in the summation only gives a non-zero contribution when the integer $n$ is a multiple of both $N_1$ and $N_2$. Hence we see that the minimal exchanged D-particle number between two long strings of length $N_1$ and $N_2$ must be proportional to $N_1 N_2$ (if the lengths are relatively prime), else the amplitude will be simply zero.

This leads us to the conclusion that the long strings do not give an effective means of creating D-particles. For two strings to create a minimally charge D-pair, the strings apparently first need to each emit a minimal length string such that two short strings of both collections can exchange a single D-particle. In the SYM language, this last process is effectively an $SU(2)$ process, where correspondence with 11-dimensional supergravity is found. It is important to note, however, that the electric flux thus created must then subsequently spread out over the complete $U(N_i)$ gauge group, since otherwise it would not carry the SYM energy appropriate for the massive D-particle with $p^+ = N_i/R$.

Furthermore, in the sector with a fixed $p^+$ momentum, we now have an improved idea of what state contributes most to D-charge production: it is the state with trivial boundary conditions, (6.36), corresponding to a collection of minimal length strings. Since this is the state that yields amplitudes agreeing with low-energy supergravity in the limit $g \to 0$, it is apparently this state (or a bound version of it when finite $g$ effects are taken into account) that dominates the wavefunction of the graviton in the small $g$ region, rather than the state with the long string boundary conditions (6.37).
7. Ranges of validity

In this section we will give a preliminary discussion of the relevant scales and ranges of validity of the calculations of the preceding sections. This analysis is preliminary in that the systematics of the perturbation theory for the Yang-Mills lagrangian (1.1) has not been performed at the level of that for pure Matrix theory[26] and additional subtleties are possible. We leave such analysis for future work. For simplicity we will consider the case where the $p^+$ momentum of the two incoming states are comparable, $N_1 \sim N_2 \sim N$. Our arguments readily generalize to the probe situation $N_1 \gg N_2$.

We begin by considering the expansion of the action about a classical background as in (6.19); such a treatment is relevant both for corrections to the saddlepoint solutions of section four as well as for the systematic treatment of D-pair production.

This expansion is governed by the Yang-Mills coupling $g_{YM}$, and naively one expects the condition $g_{YM} \ll 1$ for corrections to be small. However, as mentioned in the introduction, the Yang-Mills coupling is scale dependent and one expects the relevant scale to be set by the physics one is considering.

For example, in the scattering with background (6.12), loops of the charged, massive states of the YM theory play a central role. One either has a loop localized on the cylinder, whose calculation leads to the $\mathcal{O}(v^4/r^6)$ supergravity potential, or the loop can encircle the cylinder leading to the D-pair production that we have computed. These massive states receive masses of minimum size $b/g_s$ through the Higgs effect, setting the length scale $\ell_b \simeq g_s/b$. At this scale, we expect the relevant dimensionless parameter to be

$$g_{YM} \ell_b \simeq \frac{1}{b} .$$

(7.1)

Smallness of this parameter thus requires

$$b \gg 1 .$$

(7.2)

For the case of pair creation, there is another requirement arising from the condition that the backreaction due to the created electric field be small. One way of stating this is to require that the YM energy be large as compared to the energy stored in the electric field,

$$p^2 \gg g_{YM}^2 .$$

(7.3)
Finally, in the case of the string interactions of section four, we see from (4.18) that the relevant scale is set by the parameter $|B|$, and is given by

$$\ell_{\text{inst}} \simeq \left( \frac{g_s}{|B|} \right)^{2/3} = \left( \frac{g_s^2 N}{p^2 \sin \theta} \right)^{1/3}. \quad (7.4)$$

At this scale the dimensionless coupling is given by

$$g_{YM} \ell_{\text{inst}} \simeq \left( \frac{N}{g_s p^2 \sin \theta} \right)^{1/3}. \quad (7.5)$$

Another condition to apply the methods of section four is that the size of the instanton be small as compared to the size of the cylinder, $\ell_{\text{inst}} \ll 1$, or

$$p^2 \gg \frac{g_s^2 N}{\sin \theta}. \quad (7.6)$$

It is certainly possible to simultaneously satisfy the conditions (7.2), (7.3), and (7.6), as well as the more stringent condition $g_{YM} \ll 1$, for finite $N$ and large $s \sim p^2$. If all important corrections are governed by expansions in the parameters of (7.1) and (7.5), then it appears possible to even push the calculations into the range $g_s \lesssim 1$.

A more complete analysis can be performed in the large $g_s$ (large $R_9$) case in the restricted energy range

$$\frac{1}{g_s} \ll E \ll g_s. \quad (7.7)$$

The lower bound corresponds to the energy threshold to create D charge, and the upper bound is the energy to create winding states wrapping $x^9$. In between these bounds the theory can be effectively described by Matrix theory DLCQ quantized in 10 dimensions.

As explained in [26], the Matrix expansion is an expansion in terms of the form

$$\left( \frac{N}{r^3} \right)^L \left( \frac{u^2}{R^2 r^4} \right)^n \quad (7.8)$$

where $L$ counts loops. The terms with $L = n$ are readily identified with terms in the corresponding supergravity expansion, and the small parameter justifying this expansion is

$$\frac{N v^2}{R^2 r^7} \ll 1. \quad (7.9)$$

This has a simple physical interpretation, which is easily seen by estimating the net transverse momentum transfer due to the potential

$$\frac{N^2 u^2}{R^3 r^7}; \quad (7.10)$$
this gives
\[ \Delta p_\perp = \frac{N^2 v^3}{R^3 b^7} . \] (7.11)

The condition (7.9) is then easily seen to be \( \Delta p_\perp \ll p \), or equivalently \( \theta \ll 1 \) where \( \theta \) is the scattering angle. Combining this with the threshold condition (7.7) then yields
\[ g_s p \gg 1 , \] (7.12)
in correspondence with (7.7).

Expansion terms with \( n > L \) are then suppressed for
\[ p^2 \ll N^2 r^4 , \] (7.13)
and terms with \( n < L \) for
\[ r \gg (N g_s)^{1/3} . \] (7.14)

It is unclear whether the latter condition is strictly necessary; the first term in this expansion vanishes [27,26], and the other terms have been conjectured to vanish in [28].

To better understand these conditions, we convert them into statements relating the Mandelstam parameter \( s \sim p^2 \) and the angle \( \theta \). It is easily seen that condition (7.13) becomes
\[ s \ll N^2 \left( \frac{N}{\theta} \right)^{4/3} \] (7.15)
and the condition (7.14) becomes
\[ s \gg g_s^{7/3} N^{10/3} \theta . \] (7.16)

Comparing (7.6) with (7.15) and (7.16), we see that within the energy range given by (7.7) the instanton and D-particle production calculations are not obviously simultaneously valid. However, outside of this range, we appeal to the preceding (less rigorous) analysis which suggests that these calculations are indeed simultaneously valid at large \( s \), and may even be extendable to \( g_s \lesssim 1 \). It is partly on this basis that we will, in the next section, consider the consequences of combining these two calculations.
8. Discussion and Conclusions

We begin this section by recalling several observations from our preceding discussion. The first is that, as pointed out in section 6.4, string scattering only efficiently produces D charge if the strings break off at least one minimal length string. Furthermore, sections four and five discussed saddlepoint configurations that are expected to make important contributions to high energy, fixed angle scattering. Combining these yields a picture of how the important non-perturbative process of D-charge production can arise in high-energy string scattering.

The analysis of Gross and Mende\[1\] found saddlepoints believed to dominate scattering at high energy. These saddlepoints have a common structure at arbitrary genus, and the contributions of these saddlepoints grows with the genus suggesting the relevance of non-perturbative effects. We have found a new version of their analysis in which a mechanism appears that can cut off this growth. The cutoff originates from the minimal string length, which is in our language the minimal $p^\pm$. String fragmentation is stopped when the string breaks into the maximal number of minimal-length strings.\[5\]

It is precisely in the context of minimal length string scattering that we have found that D-charge pair production can become an important effect. We therefore have a very nice picture in which the instantons of section four and five lead to maximal fragmentation of the strings, and this is followed by the production of D-charge via the process of section six. Here we expect that the size (4.18) of the instanton, as well as the corresponding minimal distance (5.6), may be an important ingredient in determining the size of both these effects.

From the stringy viewpoint this is an intrinsically non-perturbative process. This is suggestive that there is in fact a basic connection between these two processes, and in particular that the non-perturbative production of D-charge is an important correction to the high-energy scattering analysis of [1]. While we believe that, by combining the various ingredients presented in this paper, it may be possible to obtain definite quantitative estimates of these corrections, we leave further analysis of this connection for future work.

Next we turn to several other observations and connections.

\[5\] In this sense, the matrix string formalism behaves very similar to the discretized models of string theory, advocated in particular by C. Thorn. We thank S. Shenker for drawing our attention to this similarity.
First, recall that Banks and Susskind [29] previously considered the $D\bar{D}$ system in the context of perturbative string theory. There they found an instability with unknown outcome. In the present framework we have been able to treat the same system analytically, at least in the large $g_s$ limit, without signs of pathology. In principle, the Matrix string calculations appear to extend to arbitrary $g_s$. One might hope that some extrapolation of our approach could shed further light on the discussion of [29].

It is an interesting conceptual question under which circumstances one needs to include the virtual effects of D-particles propagating in loops. Although in the literal sense of an expansion about $g_s = 0$ they do not contribute, since they have infinite mass there, there is clearly a strong sense in which D-particles can be found in intermediate states when $g_s$ is finite. Indeed, intermediate states with D-charge are distinguished from other intermediate states only by the presence of electric flux, and there is no apparent reason why these should be suppressed at finite $g_s$. In fact, looking at the results of section 6, leads one to suspect that it may be possible to extend the matrix string interactions as summarized in section 2.3 to include the possibility of electric flux “pair” creation. The eleven dimensional symmetry of M-theory, in particular, suggests as a possible generalization of the DVV string-interaction vertex, an expression of the form $V_{int} = V_{twist}\delta(A_{12})$ (with $A_{12}$ the difference between the U(1) gauge fields on the two strings that are created). With this choice of vertex, the couplings between string and all $n$-D-particle bound states are all of the same strength. This would suggest that there may possibly exist a systematic semi-classical expansion in string theory – generalizing the standard perturbative expansion – in which the D-particle-loop contributions play the role of instanton-like corrections. Indeed, in other recent studies of non-perturbative contributions to string scattering amplitudes [30-33] it was suggested that D-particle loops are related via T-duality to D-instanton contributions in IIB string theory. It clearly would be interesting to see if the suggestive formulas obtained in these works can possibly be reproduced via the Matrix string methods developed in this paper.

To conclude, we have succeeded in using the Matrix String approach to begin an investigation of aspects of high energy string scattering, and in particular to begin to explore the role of important non-perturbative (from the string viewpoint) processes such as D-charge production. Further investigation along these lines is expected to unravel a rich structure at substringy scales, and may shed further light on the short distance structure and fundamental degrees of freedom and dynamics of M-theory.
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While we were writing up our results, we received [34], which has some overlap with the discussion of section six.

Appendix A. The Matrix String Approach at Finite N.

Most of our calculations in the main text are performed for type IIA string theory in the DLCQ approach, in other words for the IIA string compactified on a lightlike circle of “radius” \( R \)

\[
x^− \equiv x^− + 4\pi R
\]  \hspace{1cm} (A.1)

This means that from the M-theory point of view we have compactified on \( S^1 \times S^1 \), with the second compactification on a circle whose radius \( R_9 \) is expressed in terms of the corresponding string length as \( R_9 = g_s \ell_s \). Through most of the paper, we work in string units \( \ell_s = 1 \), so that \( R_9 = g_s \).

In this Appendix we will rederive the 1+1 dimensional \( U(N) \) SYM lagrangian as the appropriate Matrix formulation of M-theory in this discrete light-cone limit. This is useful in understanding the precise scalings and relations between parameters. We will follow closely the reasoning of [18,17,19].

To begin with, consider M-theory compactified on a circle with radius \( R_{11} \ll \tilde{l}_{pl} \), where \( \tilde{l}_{pl} \) is the eleven-dimensional Planck scale. Suppose we restrict attention to the sector with momentum \( p_{11} = N/R_{11} \). The lowest excitations in this sector occur at energies \( E \sim R_{11} \tilde{M}^2_{pl} \), and are described by the action (here for simplicity we omit fermions)[14,13,10]

\[
S = \int \frac{dt}{2R_{11}} \text{Tr} \left( \dot{X}^2 + \frac{1}{2} R_{11}^2 \tilde{M}^6_{pl} [X, X]^2 \right). \hspace{1cm} (A.2)
\]
$X$ is an $N \times N$ matrix; the diagonal terms in $X$ correspond to the coordinates of the branes, with kinetic energies $M_{D0} \dot{X}^2/2 = \dot{X}^2/2R_{11}$ and the off-diagonal components describe the creation of straight strings stretched between the branes, with characteristic energies $\sim |x_i - x_j|R_{11}\tilde{M}^3_{pl}$. This action neglects string oscillations and higher energy excitations (Planck scale excitations, brane creation). The lowest of these begin at the string scale (defined with respect to the compactification on $R_{11}$)

$$\tilde{M}_s = \sqrt{R_{11}\tilde{M}^3_{pl}}.$$ (A.3)

Next we compactify on a circle of radius $\tilde{R}_9$, through the identification $X_9 \simeq X_9 + 2\pi \tilde{R}_9$. In the matrix context, Taylor [13] has argued that this is most easily described by passing to the dual circle of radius $R'_9 = 1/2\pi \tilde{R}_9\tilde{M}_s^2$. The $X_9$ variables dualize to a gauge field through the identification

$$A_x = \tilde{M}^2_s \sum_n e^{inx/R'_9} X^9_0 n$$ (A.4)

and the wrapped strings combine into a new definition of the remaining fields,

$$\tilde{X}_i = \sum_n e^{inx/R'_9} X^i_0 n,$$ (A.5)

with the resulting action

$$S = \int \frac{dt}{2R_{11}} \int_0^{2\pi R'_9} dx \frac{1}{2\pi R'_9} \text{Tr} \left\{ \dot{\tilde{X}}^2 + \frac{1}{\tilde{M}^2_s} \dot{A}^2_x - \left( \partial_x \tilde{X} + i[A_x, \tilde{X}] \right)^2 + \frac{1}{2} \tilde{M}^4_s [\tilde{X}, \tilde{X}]^2 \right\}.$$ (A.6)

It’s then convenient to redefine the spatial coordinate,

$$x = 2\pi R'_9 \sigma; \quad A_x = \frac{A_\sigma}{2\pi R'_9},$$ (A.7)

giving

$$S = \int \frac{dt}{2R_{11}} \int_0^1 d\sigma \text{Tr} \left\{ \dot{\tilde{X}}^2 + \tilde{R}_9^2 \dot{A}^2_\sigma - \frac{\tilde{R}_9^2 R^2_{11}}{l^6_{pl}} (D_\sigma \tilde{X})^2 + \frac{1}{2} \frac{R^2_{11}}{l^6_{pl}} [\tilde{X}, \tilde{X}]^2 \right\}.$$ (A.8)

The action (A.8) so far corresponds to the low-energy action of the system of N D0 branes on the circle of radius $R_{11}$. The next step is to take the limit

$$R_{11} \rightarrow 0, \tilde{M}_{pl} \rightarrow \infty$$ (A.9)
holding
\[ \tilde{M}_{pl}^2 R_{11} \equiv M_{pl}^2 R \] (A.10)

fixed, following [18,17]. This gives the infinite momentum limit, \( p_{11} \to \infty \). At the same
time, one defines a rescaled \( X \) and \( R_9 \),
\[ X = \frac{\tilde{M}_{pl}}{M_{pl}} \tilde{X} ; \quad R_9 = \frac{\tilde{M}_{pl}}{M_{pl}} \tilde{R}_9 . \] (A.11)

In the limit (A.9),(A.10) all the higher excitations mentioned above (string, Planck, and
\( D_0 \)) decouple and the lagrangian (A.8) becomes exact. Written in terms of the rescaled
variables,
\[ S = \int dt \int_0^1 d\sigma \text{Tr} \left\{ \dot{X}^2 + R_9^2 \dot{A}_\sigma^2 - \frac{R_9^2 R^2}{\tilde{M}_{pl}} (D_\sigma X)^2 + \frac{1}{2} \frac{R^2}{\tilde{M}_{pl}^2} [X, X]^2 \right\} . \] (A.12)

Eq. (A.12) therefore describes DLCQ IIA string theory lightlike compactified on a
circle of “radius” \( R \). The physical string coupling and string scale are defined by the usual
relations
\[ R_9 = g_s l_{st} , \quad l_{pl} = g_1^{1/3} l_{st} . \] (A.13)

Finally, as a matter of convenience it is easier to work in terms of a rescaled time
\[ \tau = tR/l_{st}^2 , \] (A.14)
and to redefine
\[ X \to l_{st} X , \] (A.15)
corresponding to measuring \( X \) in string units. With these redefinitions the action indeed
takes the form as given in the introduction. In the main text we will take \( l_{st} = 1 \).

In order to describe physical scattering processes, we will also need to parameterize
the external states. We will separate off the two compact momenta from the transverse
momentum \( \vec{p} \), and write
\[ p^\mu = (p^-, \vec{p}, p_9 = n/R_9, p^+ = N/R) . \] (A.16)

For example, when we compute scattering of two gravitons into two \( D0 \) particles (now
defined with respect to the \( R_9 \) compactification), initially \( n = 0 \) and finally \( n = \pm 1 \).
Notice that scaling the momentum according to (A.11), finite \( p \) corresponds to infinite
\( \tilde{p} = \tilde{M}_p p / M_{pl} \). However, the corresponding energy scale in the original variables of (A.8) is \( E \sim \tilde{p}^2 R_{11} = R^2 p^2 \) which remains finite as the other excitations decouple in the limit \( R_{11} \to 0 \). In the case of two particles in the center of mass frame each with one unit of DLCQ momentum, and with relative momentum \( \vec{p} \), the asymptotic state is described by

\[
X = \frac{1}{2} \left( \begin{array}{cc}
R \tilde{p} t & \vec{b} \\
0 & -R \tilde{p} t - \vec{b}
\end{array} \right). \tag{A.17}
\]

In the rescaled units defined by (A.14), (A.15), and \( l_{st} = 1 \), this becomes

\[
X = \frac{1}{2} \left( \begin{array}{cc}
\tilde{p} \tau & \vec{b} \\
0 & \tilde{p} \tau - \vec{b}
\end{array} \right), \tag{A.18}
\]

with the understanding that the external momentum and impact parameter are measured in string units.
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