The down/up crossing properties of Markov branching processes

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Abstract

It is well-known that 0 is the absorbing state for a branching system. Each particle in the system lives a random long time and gives a random number of new particles at its death time. It stops when the system has no particle. This paper is devoted to studying the fixed range crossing numbers until any time \( t \). The joint probability distribution of fixed range crossing numbers of such processes until time \( t \) is obtained by using a new method. In particular, the probability distribution of total death number is given for Markov branching processes until time \( t \).

Keywords: Markov branching process; Down crossing; Up crossing; Joint probability distribution.

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1. Introduction

The ordinary Markov branching processes (MBPs) play an important role in the classical field of stochastic processes. Some related references are Harris[10], Athreya and Ney[5], Asmussen and Hering[3].

The basic property governing the evolution of an MBP is the branching property, different particles act independently when giving birth or death. Each particle in the system lives a random long time and gives a random number of new particles at its death time. The system stops when there is no particle in it. It is well-known that 0 is the absorbing state for a branching system. Markov branching processes are well studied and there are many references such as, Harris[10], Athreya and Ney[5], Asmussen and Hering[3]. Furthermore, some generalized branching systems are studied, Sevast’yanov[18], Vatutin[20], Li & Chen[11] and Li, Chen & Pakes[13] considered the interacting branching processes and branching processes with state-independent immigration. Moreover, Li & Liu[14] added state-independent migration to the above branching process. Yamazato[21] investigated a branching process with immigration which only occurs at state zero. Being viewed as an extension of Yamazato’s model, Chen[17] discussed a more general branching process with or without resurrection. For the further discussion of this model, see Chen[6,7], Chen, Li & Ramesh[8] and Chen, Pollett, Zhang & Li[9] considered weighted Markov branching processes.
process. Within this structure, Chen, Li and Ramesh [8] considered the uniqueness and extinction of weighted Markov branching processes, which is the further consideration of branching models discussed in Chen[7].

However, for Markov branching processes, there are some important problems remained open. Such as, how many particles died until time $t$? what is the $m$-birth number of particles until time $t$ (here $m \neq 0$ is fixed)? how many particles who ever lived in the system (i.e., the total death number) until its extinction? Such problems are important and interesting. For convenience, such number is called fixed range crossing number henceforth. For example, the $-1$-range crossing number (down crossing number) is just the total death number for the process and if $m > 0$, then the $m$-range crossing (up crossing number) is just the total number of times that a particle in the system gave $m$ new particles at its death time. Since the down/up crossing numbers are random variables, therefore, it needs to discuss the probability distribution of $m$-range crossing number for the process until time $t$ or until its extinction. The main purpose of this paper is to consider such problems for Markov branching processes.

In order to begin our discussion, we first define our model by specifying the infinitesimal generator, i.e., the so-called $Q$-matrix. Throughout this paper, let $\mathbb{Z}^+ = \{0, 1, 2, \cdots \}$.

**Definition 1.1.** A $Q$-matrix $Q = (q_{ij}; i, j \in \mathbb{Z}^+)$ is called a weighted branching $Q$-matrix (henceforth referred to as a MB $Q$-matrix), if

$$q_{ij} = \begin{cases} ib_{j-i-1}, & \text{if } i \geq 1, j \geq i - 1, \\ 0, & \text{otherwise}, \end{cases} \quad (1.1)$$

where

$$b_j \geq 0 \ (j \neq 1), \ 0 < -b_1 = \sum_{j \neq 1} b_j < \infty. \quad (1.2)$$

**Definition 1.2.** A Markov branching process (henceforth referred to as MBP) is a continuous-time Markov chain with state space $\mathbb{Z}^+$ whose transition function $P(t) = (p_{ij}(t); i, j \in \mathbb{Z}^+)$ satisfies

$$p'_{ij}(t) = \sum_{k=0}^{\infty} p_{ik}(t)q_{kj}, \ i \geq 0, \ j \geq 1, \ t \geq 0, \quad (1.3)$$

where $Q = (q_{ij}; i, j \in \mathbb{Z}^+)$ is defined in (1.1)-(1.2).

Harris [10] derived the regularity criteria for MBPs in terms of the death rate $b_0$ and birth rates $\{b_k; k \geq 2\}$. Therefore, we assume the process is regular in the following.

2. Preliminaries

Let $\mathbb{N} \subset \mathbb{Z}^+$ be a finite subset with $1 \notin \mathbb{N}$ and $b_k > 0$ for all $k \in \mathbb{N}$. The number of elements in $\mathbb{N}$ is denoted by $N$, i.e., $N = |\mathbb{N}|$. We will consider the $(N-1)$-range crossing number of the process until time $t$, i.e., the joint probability distribution of the $N$-dimensional random
vector $Y(t) = (Y_i(t); \ i \in \mathbb{N})$, where $Y_i(t)$ denotes the $(i-1)$-range crossing number of the process until time $t$.

In order to begin our discussion, define

$$B(u) = \sum_{j=0}^{\infty} b_j u^j$$

and

$$B_N(u, v) = \sum_{j \in \mathbb{N}} b_j v_j u^j, \quad \bar{B}_N(u) = \sum_{j \in \mathbb{N}^c} b_j u^j,$$

where $v = (v_j; \ j \in \mathbb{N})$. It is obvious that $B(u)$, $\bar{B}_N(u)$ are well defined at least on $[0, 1]$, and $B_N(u, v)$ is well defined at least on $[0, 1]^{N+1}$.

The following lemma is due to mathematical analysis and thus the proof is omitted here.

**Lemma 2.1.** Suppose that $\{f_k; \ k \in \mathbb{Z}_+^N\}$ is a sequence on $\mathbb{Z}_+^N$, $F(v) = \sum_{k \in \mathbb{Z}_+^N} v^k$ is the generating function of $\{f_k; \ k \in \mathbb{Z}_+^N\}$. Then for any $j \in \mathbb{Z}_+$,

$$F^j(v) = \sum_{l \in \mathbb{Z}_+^N} f_l^* v^l$$

where

$$f_0^* = 1, \ f_1^* = 0 (l \neq 0), \ f_i^* = \sum_{k^{(1)} + \cdots + k^{(j)} = l} f_{k^{(1)}} \cdots f_{k^{(j)}}, \ (j \geq 1)$$

is the $j$'th convolution of $\{f_k; \ k \in \mathbb{Z}_+^N\}$.

The function $\bar{B}_N(u) + B_N(u, v)$ will play an important role in our discussion. The following theorem reveals its properties.

**Theorem 2.1.** (i) For any $v \in [0, 1]^{N+1}$,

$$\bar{B}_N(u) + B_N(u, v) = 0$$

possesses at most 2 roots in $[0, 1]$. The minimal nonnegative root of $\bar{B}_N(u) + B_N(u, v) = 0$ is denoted by $\rho(v)$, then $\rho(v) \leq \rho$, where $\rho$ is the minimal nonnegative root of $B(u) = 0$.

(ii) $\rho(v) \in C\infty([0, 1)^N)$ and $\rho(v)$ can be expanded as a multivariate Taylor series

$$\rho(v) = \sum_{k \in \mathbb{Z}_+^N} \rho_k v^k.$$  

where $\rho_k \geq 0, \forall \ k \in \mathbb{Z}_+^N$.

**Proof.** Note that $0 \leq B_N(u, 0) \leq B_N(u, v) \leq B_N(u, 1)$, we know that

$$\bar{B}_N(u) + B_N(u, v) \leq B(u).$$
(i) follows from Li and Chen [12]. Next to prove (ii). It follows from Y. Li, J. Li and Chen ?? that \( \rho(v_2, v_3) \in C^\infty([0, 1]^2) \).

Suppose that

\[
\rho(v) = \sum_{k \in \mathbb{Z}_+^N} \rho_k v^k.
\]

Substituting the above expression of \( \rho(v) \) into (2.4) yields

\[
0 \equiv \bar{B}_N(\rho(v)) + B_N(\rho(v), v)
= \sum_{j \in \mathbb{N}^c} b_j (\rho(v))^j + \sum_{j \in \mathbb{N}} b_j (\rho(v))^j v_j
= \sum_{j \in \mathbb{N}^c} b_j \sum_{l \geq 0} \rho_l^{*j} v^j + \sum_{j \in \mathbb{N}} b_j \sum_{l \geq 0} \rho_l^{*(j)} v^{l+e_j}
= \sum_{l \geq 0} (\sum_{j \in \mathbb{N}^c} b_j \rho_l^{*j}) v^l + \sum_{j \in \mathbb{N}} b_j \sum_{l \geq 0} \rho_l^{*(j)} v^{l+e_j}.
\]

We next prove \( \rho_l \geq 0 \) using mathematical induction respect to \( l \cdot 1 \). If \( l \cdot 1 = 0 \), then \( \rho_0 = \rho(0) \geq 0 \) since it is the minimal nonnegative root of \( \bar{B}_N(u) + B_N(u, 0) = 0 \). If \( l \cdot 1 = 1 \), i.e., \( l = e_k \) for some \( k \in \mathbb{N} \). Then,

\[
\sum_{j \in \mathbb{N}^c} b_j j \rho_0^{j-1} \rho e_k + b_k \rho_0^k = 0,
\]
i.e.,

\[
\sum_{j \in \mathbb{N}^c} b_j j \rho_0^{j-1} \rho e_k + b_k \rho_0^k = 0.
\]

Hence

\[
\rho e_k = -\frac{b_k \rho_0^k}{\bar{B}_N(\rho_0)} \geq 0, \quad k \in \mathbb{N},
\]
since \( \bar{B}_N(\rho_0) < 0 \).

Assume \( \rho_l \geq 0 \) for \( l \) satisfying \( l \cdot 1 \leq m \), then for \( \bar{l} \cdot 1 = m + 1 \), there exists \( l \) and \( k \in \mathbb{N} \) such that \( \bar{l} = l + e_k \) and \( l \cdot 1 \leq m \), therefore,

\[
\sum_{j \in \mathbb{N}^c} b_j j \rho_l^{*j} + b_k \rho_l^k = 0,
\]
i.e.,

\[
\sum_{j \in \mathbb{N}^c} b_j j \rho_0^{j-1} \rho_{l+e_k} + \sum_{j \in \mathbb{N}^c \setminus \{1\}} b_j \sum_{l^{(1)} + \ldots + l^{(j)} = l+e_k} \sum_{l^{(1)}, \ldots, l^{(j)} \leq m} \rho_{l^{(1)}} \cdots \rho_{l^{(j)}} + b_k \rho_l^k = 0.
\]

Hence

\[
\rho_l = \rho_{l+e_k} = -\sum_{j \in \mathbb{N}^c \setminus \{1\}} b_j \sum_{l^{(1)} + \ldots + l^{(j)} = l+e_k} \sum_{l^{(1)}, \ldots, l^{(j)} \leq m} \rho_{l^{(1)}} \cdots \rho_{l^{(j)}} + b_k \rho_l^k
\geq \frac{\rho_l}{\bar{B}_N(\rho_0)} \geq 0,
\]
since \( \bar{B}_N(\rho_0) < 0 \). By mathematical induction, we know that \( \rho_l \geq 0, \forall l \in \mathbb{Z}_+^N \). The proof is complete. \( \square \)
3. Down/up crossing property

In this section, we consider the down/up crossing properties of Markov branching processes.

Let \( \mathbb{N} \subset \mathbb{Z}_+ \) be a finite subset with \( 1 \notin \mathbb{N} \) and \( b_k > 0 \) for all \( k \in \mathbb{N} \). \( N = |\mathbb{N}| \) denotes the number of elements in \( \mathbb{N} \).

The main purpose of this paper is to count the \((N-1)\)-range crossing numbers. However, the MBP itself cannot reveal such crossing numbers directly. Therefore, we need to find a new method to discuss the property of such crossing numbers. For this purpose, we construct a new \( Q \)-matrix \( \tilde{Q} = (q(i,k),(j,l); \ (i, k), (j, l) \in \mathbb{Z}_+^{N+1}) \).

\[
q(i,k),(j,l) = \begin{cases} w_i b_{j-i+1}, & \text{if } i \geq 1, j-i+1 \in \mathbb{N}^c, l = k \\ w_i b_{j-i+1}, & \text{if } i \geq 1, j-i+1 \in \mathbb{N}, l = k + e_{j-i+1} \\ 0, & \text{otherwise.} \end{cases} \tag{3.1}
\]

Therefore, \( \tilde{Q} \) determines a \((N+1)\)-dimensional Markov chain \((X(t), Y(t))\), where \( X(t) \) is the weighted Markov branching process, \( Y(t) = \{Y_k(t); k \in \mathbb{N}\} \) (assume \( Y_k(0) = 0 \ (k \in \mathbb{N}) \)) counts the \((N-1)\)-range crossing numbers until time \( t \). In particular,

(i) if \( \mathbb{N} = \{0\} \) then \( Y_0(t) \) counts the down crossing number (i.e., the death number) of \( \{X(t) : t \geq 0\} \) until time \( t \).

(ii) If \( \mathbb{N} = \{m\} \ (m \geq 2) \), then \( Y_m(t) \) counts the \((m-1)\)-range up crossing number of \( \{X(t) : t \geq 0\} \) until time \( t \).

(iii) If \( \mathbb{N} = \{0, m\} \ (m \geq 2) \), then \( Y(t) = \{Y_0(t), Y_m(t)\} \) counts the death number and the \((m-1)\)-range up crossing number of \( \{X(t) : t \geq 0\} \) until time \( t \).

Let \( P(t) = (p(i,k),(j,l); (i, k), (j, l) \in \mathbb{Z}_+^{N+1}) \) denote the transition probability of \((X(t), Y(t))\).

**Lemma 3.1.** Suppose that \( P(t) \) is the transition probability of \((X(t), Y(t))\). Then

(i) for any \( (u, v) \in [0, 1]^{N+1} \),

\[
\sum_{(j,l) \in \mathbb{Z}_+^{N+1}} p(i,o),(j,l)(t) u^j v^l = [\tilde{B}_{N}(u) + B_{N}(u, v)] \cdot \sum_{j \geq 1, k \in \mathbb{Z}_+^{N}} p(i,o),(j,k)(t) j u^{j-1} v^k \tag{3.2}
\]

where \( B_{N}(u), B_{N}(u, v) \) are defined in (2.2), \( v^l = \prod_{k \in \mathbb{N}} v_k^l \). Moreover,

\[
\sum_{(j,l) \in \mathbb{Z}_+^{N+1}} p(i,o),(j,l)(t) u^j v^l - u^i \\
= [\tilde{B}_{N}(u) + B_{N}(u, v)] \cdot \sum_{j \geq 1, k \in \mathbb{Z}_+^{N}} (\int_0^t p(i,o),(j,k)(s) ds) \cdot j u^{j-1} v^k. \tag{3.3}
\]

(ii) for any \( (u, v) \in [0, 1]^{N+1} \) and \( (i, m) \in \mathbb{Z}_+^{N+1} \),

\[
\sum_{(j,l) \in \mathbb{Z}_+^{N+1}} p(i,m),(j,l)(t) u^j v^l = \left[ \sum_{(j,l) \in \mathbb{Z}_+^{N+1}} p(i,o),(j,l)(t) u^j v^l \right]^i \cdot v^m. \tag{3.4}
\]
Lemma 3.2. Suppose that which achieves (3.4). The proof is complete.

Proof. (i) follows from Kolmogorov forward equation and some algebra. We only need to prove (ii).

For any \(i \geq 0\), \(m \in \mathbb{Z}_+^N\), let \(X_k(t)\) be the offspring number at time \(t\) of \(k\)th particle, \(Y_k(t)\) be the \((N-1)\) crossing number of \(X_k(t)\). Then, \(\{(X_k(t), Y_k(t)); k = 1, \ldots, i\}\) are independent and identically distributed with the same distribution as \((X(t), Y(t))\) starting at \((X(0), Y(0)) = (1, 0)\). Therefore,

\[
E[u^{X(t)} \cdot v^{Y(t)}|(X(0), Y(0)) = (i, m)] = E[u^{\sum_{k=1}^i X_k(t)} \cdot v^{m+\sum_{k=1}^i Y_k(t)}] = E[\prod_{k=1}^i u^{X_k(t)} \cdot v^{Y_k(t)}] \cdot v^m
\]

which achieves (3.4). The proof is complete. \(\square\)

Lemma 3.2. Suppose that \(v \in [0, 1]^N\) and \(u \in [0, 1]\).

(i) The differential equation

\[
\begin{align*}
\frac{\partial y}{\partial t} &= B_N(y, v) + \bar{B}_N(y) \\
y|_{t=0} &= u
\end{align*}
\]

has unique solution \(y = G(t, u, v)\).

(ii) If \(u \in [0, \rho(v)]\), then \(y = G(t, u, v)\) is increasing to \(\rho(v)\) as \(t \uparrow \infty\). If \(u \in (\rho(v), 1]\), then \(y = G(t, u, v)\) is decreasing to \(\rho(v)\) as \(t \uparrow \infty\). If \(u = \rho(v)\), then \(G(t, u, v) \equiv \rho(v)\).

Proof. We first prove the existence of solution to (3.5). Denote \(H(x) = B_N(x, v) + \bar{B}_N(x) - b_1 x\). Take \(y_0(t) \equiv u\) and let

\[
y_n(t) = e^{b_1 t} \cdot \left(u + \int_0^t e^{-b_1 s} H(y_{n-1}(s)) ds\right), \quad n \geq 1.
\]

(a) If \(u \in (\rho(v), 1]\), it can be proved that \(y_n(t) > \rho(v)\) and \(y_n(t) \leq y_{n-1}(t)\) \((n \geq 1)\). Indeed, obviously, \(y_0(t) \equiv u > \rho(v)\). Assume \(y_n(t) > \rho(v)\), then

\[
y_{n+1}(t) = e^{b_1 t} \cdot \left[u + \int_0^t e^{-b_1 s} H(y_n(s)) ds\right] > e^{b_1 t} \cdot \left[u + \int_0^t e^{-b_1 s} H(\rho(v)) ds\right] = e^{b_1 t} \cdot \left[u - b_1 \rho(v) \int_0^t e^{-b_1 s} ds\right] = e^{b_1 t} \cdot \left[u + \rho(v) e^{-b_1 t} - \rho(v)\right] > \rho(v).
\]
On the other hand, \( y_1(t) = e^{b_1 t} \cdot [u + \int_0^t e^{-b_1 s} H(u) ds] < e^{b_1 t} \cdot [u - b_1 u \int_0^t e^{-b_1 s} ds] = u = y_0(t) \).

Assume \( y_n(t) \leq y_{n-1}(t) \), then,

\[
y_{n+1}(t) = e^{b_1 t} \cdot [u + \int_0^t e^{-b_1 s} H(y_n(s)) ds] \\
\leq e^{b_1 t} \cdot [u + \int_0^t e^{-b_1 s} H(y_{n-1}(s)) ds] \\
= y_n(t).
\]

Therefore, it follows from monotone convergence theorem that \( G(t, u, v) = \lim_{n \to \infty} y_n(t) \) exists and satisfies

\[
G(t, u, v) = e^{b_1 t} \cdot [u + \int_0^t e^{-b_1 s} H(G(s, u, v)) ds].
\]

Hence, \( y(t) = G(t, u, v) \) is a solution of (3.5). Since \( B_N(y, v) + \bar{B}_N(y) < 0 \) for all \( y \in (\rho(v), 1] \), we know that \( G(t, u, v) \) is decreasing and it is easy to see that \( \lim_{t \to \infty} G(t, u, v) = \rho(v) \).

(b) If \( u \in [0, \rho(v)) \), it can be proved that \( y_n(t) < \rho(v) \) and \( y_n(t) \geq y_{n-1}(t) \) \( (n \geq 1) \). Indeed, obviously, \( y_0(t) \equiv u < \rho(v) \). Assume \( y_n(t) < \rho(v) \), similar as in (a),

\[
y_{n+1}(t) < e^{b_1 t} \cdot [u + \int_0^t e^{-b_1 s} H(\rho(v)) ds] < \rho(v).
\]

On the other hand, \( y_1(t) = e^{b_1 t} \cdot [u + \int_0^t e^{-b_1 s} H(u) ds] > e^{b_1 t} \cdot [u - b_1 u \int_0^t e^{-b_1 s} ds] = u = y_0(t) \).

Assume \( y_n(t) \geq y_{n-1}(t) \), then,

\[
y_{n+1}(t) = e^{b_1 t} \cdot [u + \int_0^t e^{-b_1 s} H(y_n(s)) ds] \\
\geq e^{b_1 t} \cdot [u + \int_0^t e^{-b_1 s} H(y_{n-1}(s)) ds] \\
= y_n(t).
\]

Therefore, it follows from monotone convergence theorem that \( G(t, u, v) = \lim_{n \to \infty} y_n(t) \) exists and satisfies

\[
G(t, u, v) = e^{b_1 t} \cdot [u + \int_0^t e^{-b_1 s} H(G(s, u, v)) ds].
\]

Hence, \( y(t) = G(t, u, v) \) is a solution of (3.5). Since \( B_N(y, v) + \bar{B}_N(y) > 0 \) for all \( y \in [0, \rho(v)) \), we know that \( G(t, u, v) \) is increasing and it is easy to see that \( \lim_{t \to \infty} G(t, u, v) = \rho(v) \).

(c) If \( u = \rho(v) \) then it is obvious that \( G(t, u, v) \equiv \rho(v) \).

Now we prove uniqueness of the solution. Since \( B_N(y, v) + \bar{B}_N(y) \) satisfies Lipschitz condition with respect to \( y \) for any fixed \( (u, v) \in [0, 1) \times [0, 1]^N \), by ordinary differential equation theory, we know that (3.5) has unique solution \( y = G(t, u, v) \) for any fixed \( (u, v) \in [0, 1) \times [0, 1)^N \).
For \( u = 1 \), assume \( \tilde{y}(t) \) is another solution of (3.5). Since \( \tilde{y}'(0) = B_{\mathbb{N}}(1) + \hat{B}_{\mathbb{N}}(1) < 0 \), we know that \( \tilde{y}(t) \uparrow 1 \) as \( t \downarrow 0 \). Hence, for any sufficient small \( \varepsilon > 0 \), \( \tilde{y}(\varepsilon) \in (\rho(\mathbf{v}), 1) \). It is easy to see that \( \tilde{y}(t) = \tilde{y}(\varepsilon + t) \) is a solution of (3.5) with the initial condition \( \tilde{y}(0) = \tilde{y}(\varepsilon) \). Let

\[
\delta_\varepsilon = \inf\{t \geq 0; G(t, 1, \mathbf{v}) = \tilde{y}(\varepsilon)\}. 
\]

Then, \( \delta_\varepsilon \downarrow 0 \) as \( \varepsilon \downarrow 0 \). Similarly, \( \tilde{y}(t) = G(\delta_\varepsilon + t, 1, \mathbf{v}) \) is also a solution of (3.5) with the initial condition \( \tilde{y}(0) = \tilde{y}(\varepsilon) \). However, (3.5) has unique solution with initial condition \( \tilde{y}(\varepsilon) \in [0, 1] \). Therefore,

\[
\tilde{y}(\varepsilon + t) = G(\delta_\varepsilon + t, 1, \mathbf{v}), \ \forall t \geq 0,
\]

and hence, \( \tilde{y}(t) \equiv G(t, 1, \mathbf{v}) \). The proof is complete. \( \square \) \( \square \)

The following theorem gives the joint probability generating function of \((\mathbb{N} - 1)\)-crossing numbers until time \( t \), i.e., the joint probability generating function of \( \mathbf{Y}(t) \).

**Theorem 3.1.** Suppose that \( \{X(t); t \geq 0\} \) is a Markov branching process with \( X(0) = 1 \). Then the joint probability generating function of \( \mathbf{Y}(t) \) is given by

\[
E[\mathbf{v}^{\mathbf{Y}(t)}|X(0) = 1] = G(t, 1, \mathbf{v}), \ \mathbf{v} \in [0, 1]^N,
\]

where \( y = G(t, u, \mathbf{v}) \) is the unique solution of (3.5). Furthermore,

\[
P(\mathbf{Y}(t) = k|X(0) = 1) = g_k(t), \ k \in \mathbb{Z}_+^N,
\]

where

\[
\begin{align*}
g_0(t) &= G(t, 1, 0), \\
g_k(t) &= \hat{B}_{\mathbb{N}}(g_0(t)) \cdot \int_0^t \frac{F_k(s)}{B_\mathbb{N}(g_0(s))} ds, \ k \neq 0
\end{align*}
\]

(3.8)

with

\[
F_k(t) = \sum_{i \in \mathbb{N}} b_i g_{k - e_i}(t) + \sum_{i \in \mathbb{N}^c} b_i \sum_{l_1, \ldots, l_i \neq k} \sum_{l_1 + \cdots + l_i = k} g_{l_1}(t) \cdots g_{l_i}(t)
\]

and \( \{g_k^{(i)}(t); k \in \mathbb{Z}_+^N\} \) is the \( i \)th convolution of \( \{g_k(t); k \in \mathbb{Z}_+^N\} \).

**Proof.** Let \( P(t) = (p_{(i,k),(j,l)}(t); (i, k), (j, l) \in \mathbb{Z}_+^{N+1}) \) be the transition probability of \((X(t), \mathbf{Y}(t))\). We first prove that

\[
G(t, u, \mathbf{v}) = \sum_{(j, \ell) \in \mathbb{Z}_+^{N+1}} p_{(1, 0),(j, \ell)}(t) u^j v^\ell, \ (u, \mathbf{v}) \in [0, 1] \times [0, 1]^N.
\]

(3.9)

i.e., it suffices to prove that

\[
y(t, u, \mathbf{v}) = \sum_{(j, \ell) \in \mathbb{Z}_+^{N+1}} p_{(1, 0),(j, \ell)}(t) u^j v^\ell
\]
is the solution of (3.5). Indeed, it follows from Kolmogorov backward equation that
\[ p'_{(1,0),(j,l)}(t) = \sum_{n \geq 0, k \in \mathbb{Z}_+^N} q_{(1,0),(n,k)} \cdot p_{(n,k),(j,l)}(t) \]
\[ = \sum_{n \in \mathbb{N}} b_n \cdot p_{(n,e_n),(j,l)}(t) + \sum_{n \in \mathbb{N}^c} b_n \cdot p_{(n,o),(j,l)}(t), \quad \forall t \geq 0. \]

Multiplying \( u^j v^l \) on both sides of the above equality, then summing over \( j \) and \( l \) and using Lemma 3.1 yield that
\[
\sum_{(j,l) \in \mathbb{Z}_+^{N+1}} p'_{(1,0),(j,l)}(t) \cdot u^j v^l = \sum_{n \in \mathbb{N}} b_n \cdot \sum_{(j,l) \in \mathbb{Z}_+^{N+1}} p_{(n,e_n),(j,l)}(t) \cdot u^j v^l + \sum_{n \in \mathbb{N}^c} b_n \cdot \sum_{(j,l) \in \mathbb{Z}_+^{N+1}} p_{(n,o),(j,l)}(t) \cdot u^j v^l
\]
\[
= \sum_{n \in \mathbb{N}} b_n \left( \sum_{(j,l) \in \mathbb{Z}_+^{N+1}} p_{(1,0),(j,l)}(t) \cdot u^j v^l \right)^n \cdot v^{\epsilon_n} + \sum_{n \in \mathbb{N}^c} b_n \cdot \left( \sum_{(j,l) \in \mathbb{Z}_+^{N+1}} p_{(1,0),(j,l)}(t) \cdot u^j v^l \right)^n
\]
\[
= B_N \left( \sum_{(j,l) \in \mathbb{Z}_+^{N+1}} p_{(1,0),(j,l)}(t) \cdot u^j v^l, v \right) + \bar{B}_N \left( \sum_{(j,l) \in \mathbb{Z}_+^{N+1}} p_{(1,0),(j,l)}(t) \cdot u^j v^l \right)
\]
which implies that \( y(t, u, v) = \sum_{(j,l) \in \mathbb{Z}_+^{N+1}} p_{(1,0),(j,l)}(t) u^j v^l \) satisfies
\[
\frac{\partial y}{\partial t} = B_N(y, v) + \bar{B}_N(y).
\]

Finally, it is easy to see that
\[ y(0, u, v) = u. \]
Therefore (3.9) is proved. Hence, it follows from (3.9) and \( Y(0) = 0 \) that
\[
E[v^{X(t)}|X(0) = 1] = \sum_{t \in \mathbb{N}_t} P(Y(t) = 1|X(0) = 1, Y(0) = 0) \cdot v^t
\]
\[
= \sum_{t \in \mathbb{N}_t} \sum_{j=0}^{\infty} p_{(1,0),(j,l)}(t) \cdot v^t
\]
\[
= G(t, 1, v).
\]

Finally, it follows from the above proof that \( G(t, u, v) \) can be expanded as a multivariate nonnegative Taylor series. Suppose that
\[
G(t, 1, v) = \sum_{k \in \mathbb{Z}_+^N} g_k(t) v^k.
\]
By (3.9),

\[
\sum_{k \in \mathbb{Z}_+^N} g'_k(t) v^k = \sum_{i \in \mathbb{N}} b_i \left( \sum_{k \in \mathbb{Z}_+^N} g_k(t) v^k \right)^i v^{e_i} + \sum_{i \in \mathbb{N}} b_i \left( \sum_{k \in \mathbb{Z}_+^N} g_k(t) v^k \right)^i v^{e_i}.
\]

\[
= \sum_{i \in \mathbb{N}} b_i \left( \sum_{k \in \mathbb{Z}_+^N} g_{k}^{* (i)}(t) v^{k+e_i} + \sum_{i \in \mathbb{N}} b_i \sum_{k \in \mathbb{Z}_+^N} g_{k}^{* (i)}(t) v^k
\]

\[
= \sum_{k \in \mathbb{Z}_+^N} \sum_{i \in \mathbb{N}} b_i g_{k}^{* (i)}(t) v^{k+e_i} + \sum_{i \in \mathbb{N}} b_i g_{k}^{* (i)}(t) v^k.
\]

where \(\{g_k^{* (i)}(t); \ k \in \mathbb{Z}_+^N\}\) is the \(i\)th convolution of \(\{g_k(t); \ k \in \mathbb{Z}_+^N\}\) and here we have used the notation \(g_k^{* (i)}(t) = 0\) if \(k \notin \mathbb{Z}_+^N\). Comparing the coefficients on the both sides of the above equality yields that

\[
\begin{cases}
g'_0(t) = \sum_{i \in \mathbb{N}} b_i g_0^{* (i)}(t) = B_N(g_0(t)), \\
g'_k(t) = \sum_{i \in \mathbb{N}} b_i g_{k-e_i}^{* (i)}(t) + \sum_{i \in \mathbb{N}} b_i g_{k}^{* (i)}(t), \quad k \neq 0.
\end{cases}
\]

(3.10)

It is easy to see that \(g_k(0) = P(Y(0) = k|X(0) = 1) = \delta_{0,k}\) and hence \(g_0(t) = G(t, 1, 0)\). On the other hand, by the second equation of (3.10),

\[
g'_k(t) - g_k(t) \tilde{B}_N(g_0(t)) = F_k(t),
\]

where

\[
F_k(t) = \sum_{i \in \mathbb{N}} b_i g_{k-e_i}^{* (i)}(t) + \sum_{i \in \mathbb{N}} b_i \sum_{l_1, \ldots, l_i \neq k, l_1 + \ldots + l_i = k} g_{l_1}(t) \cdots g_{l_i}(t)
\]

Therefore, note that \(g_k(0) = P(Y(0) = k|X(0) = 1) = 0\) for all \(k \neq 0\), we have

\[
g_k(t) e^{-\int_0^t B_0'(g_0(s))ds} = \int_0^t F_k(s) \cdot e^{-\int_0^s B_0'(g_0(x))dx} ds.
\]

Hence,

\[
g_k(t) = \tilde{B}_N(g_0(t)) \cdot \int_0^t \frac{F_k(s)}{\tilde{B}_N(g_0(s))} ds, \quad k \neq 0.
\]

The proof is complete. \(\Box\)

**Remark 3.1.** (i) Generally, if \(X(t)\) starts from \(X(0) = i(> 1)\), then the joint probability generating function of \((\mathbb{N} - 1)\)-crossing numbers until time \(t\) is

\[
E[v^{Y(t)}|X(0) = i] = [G(t, 1, v)]^i.
\]
(ii) By carefully checking the proof of Theorem 3.1, we see that $G(t, u, v)$ can be expanded as a nonnegative multivariate Taylor series

$$G(t, u, v) = \sum_{(j, l) \in \mathbb{Z}^N_+} g_{j,l}(t) u^j v^l, \quad (u, v) \in [0, 1] \times [0, 1]^N,$$

(3.11)

where $g_{j,l}(t) = p_{(1,0),(j,l)}(t)$ for any $(j, l) \in \mathbb{Z}^N_+$. Therefore, if the solution $G(t, u, v)$ is known, then we can obtain $\{p_{(1,0),(j,l)}(t); (j, l) \in \mathbb{Z}^N_+\}$. Hence, by (3.4), the transition probability function $P(t) = (p_{(i,k),(j,l)}(t); (i, k), (j, l) \in \mathbb{Z}^N_+)$ can be obtained.

The following theorem gives a recursive algorithm of $g_{j,l}(t)$.

**Theorem 3.2.** Suppose that $\{X(t); t \geq 0\}$ is a Markov branching process with $X(0) = 1$.

(i) If $0 \notin \mathbb{N}$, then $g_{j,k}(t)$ is given by

$$
\begin{aligned}
\begin{cases}
g_{00}(t) = G(t, 0, 0) \\
g_{j,k}(t) = B_{ij}(g_{00}(t)) \cdot [\delta_{j,1} \delta_{k,0} e_0^{-1} + \int_0^t \frac{F_{j,k}(s)}{B_{ij}(g_{00}(s))} ds],
\end{cases}
\end{aligned}
$$

(3.12)

where

$$
F_{j,k}(t) = \sum_{i \in \mathbb{N}} b_i g_{j,-e_i}(t) + \sum_{i \in \mathbb{N}^c} b_i \sum_{(l_1,k_1), \ldots, (l_m,k_m) \neq (j,k)} g_{l_1,k_1}(t) \cdots g_{l_m,k_m}(t).
$$

and $\{g_{j,-e_i}(t); (j, k) \in \mathbb{Z}^N_+\}$ is the $i$th convolution of $\{g_{j,k}(t); (j, k) \in \mathbb{Z}^N_+\}$. Here $g_{j,k}(t) = 0$ if $k \notin \mathbb{Z}^N_+$.

(ii) If $0 \in \mathbb{N}$, then $g_{j,k}(t)$ is given by

$$
\begin{aligned}
\begin{cases}
\quad g_{00}(t) = 0 \\
g_{j,k}(t) = e^{bt} \delta_{j,1} \delta_{k,0} + \int_0^t F_{j,k}(s) e^{-b_i s} ds,
\end{cases}
\end{aligned}
$$

(3.13)

Proof. Suppose that

$$
G(t, u, v) = \sum_{(j,k) \in \mathbb{Z}^N_+} g_{j,k}(t) u^j v^k.
$$

By (3.5),

$$
\begin{aligned}
\sum_{(j,k) \in \mathbb{Z}^N_+} g'_{j,k}(t) u^j v^k &= \sum_{i \in \mathbb{N}} b_i \left( \sum_{(j,k) \in \mathbb{Z}^N_+} g_{j,k}(t) u^j v^k \right)^i u^{e_i} + \sum_{i \in \mathbb{N}^c} b_i \left( \sum_{(j,k) \in \mathbb{Z}^N_+} g_{j,k}(t) u^j v^k \right)^i \\
&= \sum_{i \in \mathbb{N}} b_i \sum_{(j,k) \in \mathbb{Z}^N_+} g_{j,k}(t) u^j v^{k+e_i} + \sum_{i \in \mathbb{N}^c} b_i \sum_{(j,k) \in \mathbb{Z}^N_+} g_{j,k}(t) u^j v^k \\
&= \sum_{(j,k) \in \mathbb{Z}^N_+} \sum_{i \in \mathbb{N}} b_i g_{j,k}(t) u^j v^{k+e_i} + \sum_{(j,k) \in \mathbb{Z}^N_+} \sum_{i \in \mathbb{N}^c} b_i g_{j,k}(t) u^j v^k \\
&= \sum_{(j,k) \in \mathbb{Z}^N_+ \setminus \{0\}} \sum_{i \in \mathbb{N}} b_i g_{j,k-e_i}(t) u^j v^k + \sum_{(j,k) \in \mathbb{Z}^N_+ \setminus \{0\}} \sum_{i \in \mathbb{N}^c} b_i g_{j,k}(t) u^j v^k.
\end{aligned}
$$
where \( \{g^{(i)}_{jk}(t); (j, k) \in \mathbb{Z}^{N+1}_+\} \) is the \( i \)th convolution of \( \{g_{jk}(t); (j, k) \in \mathbb{Z}^{N+1}_+\} \) and here we have used the notation \( g^{*}_{jk}(t) = 0 \) if \( k \notin \mathbb{Z}^N_+ \). Comparing the coefficients on the both sides of the above equality yields that

\[
g'_{jk}(t) = \sum_{i \in \mathbb{N}} b_i g^*_{j-k-e_i}(t) + \sum_{i \in \mathbb{N}^c} b_i g^*_{j}(t), \quad (j, k) \in \mathbb{Z}^{N+1}_+.
\]  

(3.14)

It is easy to see that

\[
g_{jo}(0) = P(X(0) = j, Y(0) = 0|X(0) = 1) = \delta_{j1}.
\]

For \((j, k) = (0, 0), \) by (3.14),

\[
g_{00}(t) = \sum_{i \in \mathbb{N}^c} b_i g_i^0(0) = \bar{B}_N(g_{00}(0)),
\]

which implies

\[
g_{00}(t) = G(t, 0, 0).
\]

For \((j, k) \neq (0, 0), \) by (3.14),

\[
g'_{jk}(t) = \sum_{i \in \mathbb{N}} b_i g^*_{j-k-e_i}(t) + \sum_{i \in \mathbb{N}^c} b_i g^*_{j}(t)
\]

\[
= g_{jk}(t) \sum_{i \in \mathbb{N}^c} ib_i g_{00}^{i-1}(t) + \sum_{i \in \mathbb{N}} b_i g^*_{j-k-e_i}(t) + \sum_{i \in \mathbb{N}^c} g_{l_1 k_1}(t) \cdots g_{l_i k_i}(t)
\]

\[
= g_{jk}(t) \bar{B}'_N(g_{00}(t)) + F_{j,k}(t)
\]

(3.15)

where

\[
F_{j,k}(t) = \sum_{i \in \mathbb{N}} b_i g^*_{j-k-e_i}(t) + \sum_{i \in \mathbb{N}^c} g_{l_1 k_1}(t) \cdots g_{l_i k_i}(t).
\]

If \( 0 \notin \mathbb{N}, \) then

\[
\int_0^t \bar{B}'_N(g_{00}(s)) ds = \int_0^t \bar{B}'_N(g_{00}(s)) \frac{g'_{00}(s)}{B'_N(g_{00}(s))} ds = \frac{\bar{B}_N(g_{00}(t))}{b_0}
\]

Hence,

\[
g_{jk}(t) = \bar{B}_N(g_{00}(t)) \cdot [\delta_{j1} \delta_{k0} b_0^{-1} + \int_0^t \frac{F_{j,k}(s)}{B'_N(g_{00}(s))} ds], \quad (j, k) \neq (0, 0).
\]

If \( 0 \in \mathbb{N}, \) then by (3.14), it is easy to see that

\[
g_{00}(t) = 0,
\]
and

\[ B_{\mathbb{N}}'(g_{00}(t)) = b_1. \]

By (3.15),

\[ g'_{jk}(t) = b_1 g_{jk}(t) + F_{j,k}(t). \]

Hence,

\[ g_{jk}(t) = e^{b_1 t} [\delta_{j,1} \delta_{k,0} + \int_0^t F_{j,k}(s) e^{-b_1 s} ds]. \]

The proof is complete. □ □

As direct consequences of Theorem 3.1 and Remark 3.1, the following corollaries give the probability distributions of death number and \((m-1)\)-range up crossing number until time \(t\) for fixed \(m > 1\).

**Corollary 3.1.** Suppose that \(\{X(t); t \geq 0\}\) is a Markov branching process with \(X(0) = 1\). Then the probability generating function of death number until time \(t\) is given by

\[ E[v^{Y_0(t)}|X(0) = 1] = G(t, 1, v), \quad v \in [0, 1], \tag{3.16} \]

where \(G(t, u, v)\) is the unique solution of the equation

\[ \left\{ \begin{array}{l}
\frac{\partial u}{\partial t} = B(y) - b_0(1 - v), \\
y|_{t=0} = u,
\end{array} \right. \quad u, v \in [0, 1]. \]

**Proof.** Note that \(\mathbb{N} = \{0\}\) and

\[ B_{\mathbb{N}}(y, v) + \bar{B}_{\mathbb{N}}(y) = B(y) - b_0(1 - v). \]

By Theorem 3.1, we immediately obtain the result. The proof is complete. □ □

**Corollary 3.2.** Suppose that \(\{X(t); t \geq 0\}\) is a Markov branching process with \(X(0) = 1\) and \(m(> 1)\) is fixed. Then the probability generating function of \((m-1)\)-range up-crossing number until time \(t\) is given by

\[ E[v^{Y_m(t)}|X(0) = 1] = G(t, 1, v), \quad v \in [0, 1], \tag{3.17} \]

where \(G(t, u, v)\) is the unique solution of the equation

\[ \left\{ \begin{array}{l}
\frac{\partial u}{\partial t} = B(y) - b_m(1 - v)y^m, \\
y|_{t=0} = u,
\end{array} \right. \quad u, v \in [0, 1]. \]

**Proof.** Note that \(\mathbb{N} = \{m\}\) and

\[ B_{\mathbb{N}}(y, v) + \bar{B}_{\mathbb{N}}(y) = B(y) - b_m(1 - v)y^m. \]

By Theorem 3.1, we immediately obtain the result. The proof is complete. □ □
Let
\[ \tau = \inf\{t \geq 0; \ X(t) = 0\} \]  
be the extinction time of \( X(t) \).

By Theorem 3.1, we can get the following result which is due to Li Y. and Li J. [15].

**Theorem 3.3.** Suppose that \( \{X(t); t \geq 0\} \) is a Markov branching process with \( X(0) = 1 \). Then the probability generating function \( G(v) \) of \((N-1)\)-range crossing numbers conditioned on \( \tau < \infty \) is given by
\[ G(v) = \rho^{-1} \cdot G(\infty, 1, v) = \rho^{-1} \cdot \rho(v), \quad v \in [0,1]^N, \]  
where \( \rho \) is the minimal nonnegative root of \( B(u) = 0 \).

Furthermore, if \( \rho < 1 \) then
\[ P(Y(\infty) = \infty | \tau = \infty) = 1. \]  

**Proof.** It follows from Lemma 3.2 and Theorem 3.1 that
\[ G(t,1,v) = \sum_{l \in \mathbb{Z}_+^N} p_{(1,0),(0,l)}(t) v^l + \sum_{j=1}^{\infty} \left( \sum_{l \in \mathbb{Z}_+^N} p_{(1,0),(j,l)}(t) v^l \right) \rho(v)^j, \quad \forall t \geq 0. \]  
By (3.23) with \( i = 1 \) and \( u = \rho(v) \),
\[ \rho(v) = \sum_{l \in \mathbb{Z}_+^N} p_{(1,0),(0,l)}(t) v^l + \sum_{j=1}^{\infty} \left( \sum_{l \in \mathbb{Z}_+^N} p_{(1,0),(j,l)}(t) \rho(v)^j \right) v^l, \quad \forall t \geq 0. \]  
Letting \( t \to \infty \) in (3.21) and (3.22) yield that
\[ G(\infty, 1, v) = \sum_{l \in \mathbb{Z}_+^N} p_{(1,0),(0,l)}(\infty) v^l + \lim_{t \to \infty} \sum_{l \in \mathbb{Z}_+^N} \left( \sum_{j=1}^{\infty} p_{(1,0),(j,l)}(t) \right) v^l. \]  
and
\[ \rho(v) = \sum_{l \in \mathbb{Z}_+^N} p_{(1,0),(0,l)}(\infty) v^l. \]  
By (3.23) and (3.24),
\[ \sum_{l \in \mathbb{Z}_+^N} p_{(1,0),(0,l)}(\infty) v^l = G(\infty, 1, v) = \rho(v), \]  
Therefore,
\[ G(v) = \sum_{l \in \mathbb{Z}_+^N} P(Y(\tau) = l | \tau < \infty) \cdot v^l \]
\[ = \rho^{-1} \cdot \sum_{l \in \mathbb{Z}_+^N} p_{(1,0),(0,l)}(\infty) v^l \]
\[ = \rho^{-1} \cdot \rho(v) \]
\[ = \rho^{-1} \cdot \rho(v) \]
Again by (3.23) and (3.24),

\[
P(Y(\tau) = l | \tau = \infty) \\
\leq P(Y(\tau) \leq l | \tau = \infty) \\
= (1 - \rho)^{-1} \cdot P(Y(\tau) \leq l, \tau = \infty) \\
= (1 - \rho)^{-1} \cdot \lim_{t \to \infty} P(Y(t) \leq l, \tau > t) \\
= (1 - \rho)^{-1} \cdot \lim_{t \to \infty} \sum_{m \leq l} \sum_{j=1}^{\infty} P(Y(t) \leq l, \tau > t) \\
= 0,
\]

which implies (3.20), where \( m \leq l \) means every component of \( m \) is not bigger than \( l \). The proof is complete. \( \square \)

It can be seen from Theorems 3.1 and 3.3, in order to obtain the joint probability generating function of \( Y(t) \), the key point is to find the solution of (3.5). Therefore, we now consider how to find the solution of (3.5) and Taylor expansion of the solution for some special and important cases.

**Theorem 3.4.** Suppose that \( X(t) \) is a birth-death type branching process with death rate \( pb \) and birth rate \( qb \) (here, \( b > 0, p \in (0, 1), p + q = 1 \), \( X(0) = 1 \)). \( Y(t) \) is the death number of \( X(t) \) until time \( t \). Then the probability generating function of \( Y(t) \) is given by

\[
G(t, 1, v) = \beta(v) + \frac{\alpha(v) - \beta(v)}{1 + \frac{\alpha(v) - \beta(v)}{1 - \beta(v)} \cdot e^{(\alpha(v) - \beta(v))bt}}.
\]  

(3.25)

where

\[
\alpha(v) = \frac{1 + \sqrt{1 - 4pqv}}{2q}, \quad \beta(v) = \frac{1 - \sqrt{1 - 4pqv}}{2q}.
\]

More specifically, \( g_n(t) = P(Y(t) = n) \) \((n \geq 0)\) is given by

\[
\begin{aligned}
g_0(t) &= \frac{1}{q + p \cdot e^{bt}}, \\
g_n(t) &= \frac{1}{(q + p \cdot e^{bt})^2} \cdot \int_0^t (q + p \cdot e^{bs})^2 e^{-bs} F_n(s) ds, \quad n \geq 1
\end{aligned}
\]

(3.26)

with

\[
F_n(t) = bp\delta_{1,n} + bq \sum_{k=1}^{n-1} g_k(t) g_{n-k}(t).
\]

**Proof.** Note that \( \mathbb{N} = \{0\} \), \( B(y) = b(p - y + qy^2) \) and

\[
B_{\mathbb{N}}(y, v) + \bar{B}_{\mathbb{N}}(y) = b(pv - y + qy^2).
\]

It is obvious that for any \( v \in [0, 1] \),

\[
B_{\mathbb{N}}(y, v) + \bar{B}_{\mathbb{N}}(y) = b(pv - y + qy^2) = bq(y - \alpha(v))(y - \beta(v))
\]
where
\[
\alpha(v) = \frac{1 + \sqrt{1 - 4pqv}}{2q}, \quad \beta(v) = \frac{1 - \sqrt{1 - 4pqv}}{2q}.
\]

Therefore, (3.5) becomes
\[
\left\{ \begin{array}{l}
\frac{dy}{(y-\alpha(v))(y-\beta(v))} = bqdt, \\
y(0) = u.
\end{array} \right. \tag{3.27}
\]

Note that \(\alpha(v) > \beta(v)\) for \(v \in [0, 1)\). Solve (3.29), one get
\[
y(t) = G(t, u, v) = \beta(v) + \alpha(v) - \beta(v) - \frac{1}{1 - \beta(v)} e^{(\alpha(v) - \beta(v))btq}.
\]

Hence,
\[
G(t, 1, v) = \beta(v) + \frac{\alpha(v) - \beta(v)}{1 + \frac{\alpha(v) - \beta(v)}{1 - \beta(v)} e^{(\alpha(v) - \beta(v))btq}}.
\]

Therefore, by Theorem 3.1,
\[
\left\{ \begin{array}{l}
go_0(t) = G(t, 1, 0) = \frac{1}{q + p e^{bt}} \\
g_n(t) = \frac{e^{bt}}{(q + p e^{bt})^2} \int_0^t (q + p \cdot e^{bs})^2 e^{-bs} F_n(s) ds, \quad n \geq 1
\end{array} \right.
\]

with
\[
F_n(t) = bp\delta_{1,n} + bq \sum_{k=1}^{n-1} g_k(t) g_{n-k}(t), \quad n \geq 1.
\]

The proof is complete. \(\square\) \(\square\)

The following result is a direct consequence of Theorem 3.4.

**Corollary 3.3.** Suppose that \(X(t)\) is a birth-death type branching process with death rate \(pb\) and birth rate \(qb\) (here, \(b > 0, \ p \in (0, 1), \ p + q = 1\), \(X(0) = 1\). Then the probability generating function of death number conditioned on \(\tau < \infty\) is given by
\[
E[v^{Y(\tau)}|\tau < \infty] = \beta(v),
\]

where
\[
\beta(v) = p[v + \sum_{n=2}^{\infty} \frac{(2n-3)!!2^{n-1}(pq)^{n-1}}{n!} v^n].
\]

Finally, we give another example.
Theorem 3.5. Suppose that $X(t)$ is a Markov branching process with $b_0 = pb$, $b_3 = qb$ (here, $b > 0$, $p \in (0, 1), p + q = 1$), $X(0) = 1$. $Y(t)$ is the death number of $X(\cdot)$ until time $t$. Then the probability generating function of $Y(t)$ is given by

$$E[v^{Y(t)}|X(t) = 1] = G(t, 1, v) = \sum_{n=0}^{\infty} g_n(t) v^n,$$

where

$$\begin{cases} g_0(t) = G(t, 1, 0) = (q + pe^{2bt})^{-1/2} \\ g_n(t) = e^{2bt} \cdot (q + p \cdot e^{2bt})^{-3/2} \cdot \int_0^t e^{-2bs}(q + p \cdot e^{2bs})^{3/2} F_n(s) ds, \quad n \geq 1 \end{cases}$$

with

$$F_n(t) = b p \delta_{1,n} + b q \cdot \sum_{k_1+k_2+k_3<n, k_1+k_2+k_3=n} g_{k_1}(t) g_{k_2}(t) g_{k_3}(t).$$

Hence, the probability generating function of death number conditioned on $\tau < \infty$ is given by

$$E[v^{Y(\tau)}|\tau < \infty] = \beta(v) = \sum_{n=0}^{\infty} g_n v^n,$$

where

$$\begin{cases} g_0 = 0, \\ g_1 = p, \\ g_n = q \sum_{i,j,k<n, i+j+k=n} g_i g_j g_k, \quad n \geq 2. \end{cases}$$

Proof. Note that $N = \{0\}$, \(B(y) = b(p - y + qy^3)\) and

$$B_N(y, v) + \bar{B}_N(y) = b(pv - y + qy^3).$$

Let $y(t) = G(t, 1, v) = \sum_{n=0}^{\infty} g_n(t) v^n$ be the solution of (3.5) with $u = 1$, then

$$\begin{cases} g'_0(t) = b(-g_0(t) + qg_3^2(t)), \\ g_0(0) = 1. \end{cases}$$

Solving the above equation yields

$$g_0(t) = (q + pe^{2bt})^{-1/2}.$$

Therefore, the first result follows from Theorem 3.4 and taking limit yields the second result. The proof is complete. $\Box$ $\Box$

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References

[1] Anderson W. (1991). Continuous-Time Markov Chains: An Applications-Oriented Approach. Springer-Verlag, New York.

[2] Asmussen S. and Jagers P. (1997). Classical and Modern Branching Processes. Spinger, Berlin.

[3] Asmussen S. and Hering H. (1983). Branching Processes. Birkhauser, Boston.

[4] Athreya K.B. (1994). Large Deviation Rates for Branching Processes–I. Single Type Case. The Annals of Appl. Probab., 4(3):779-790.

[5] Athreya K.B. and Ney P.E. (1972). Branching Processes. Springer, Berlin.

[6] Chen A.Y. (2002). Uniqueness and extinction properties of generalised Markov branching processes. J. Math. Anal. Appl., 274(2):482-494

[7] Chen A.Y. (2002). Ergodicity and stability of generalised Markov branching processes with resurrection. J. Appl. Probab., 39(4):786-803

[8] Chen A.Y. and Li J.P. and Ramesh N. (2005). Uniqueness and extinction of weighted Markov branching processes. Methodol. Comput. Appl. Probab., 7(4):489-516

[9] Chen A.Y. and Pollett P. and Li J.P. and Zhang H.J. (2007). A remark on the uniqueness of weighted Markov branching processes. J. Appl. Probab., 44(1):279-283

[10] Harris T.E. (1963). The theory of branching processes. Springer, Berlin and Newyork.

[11] Li J.P. and Chen A.Y. (2006). Markov branching processes with immigration and resurrection. Markov Process. Related Fields, 12(1):139-168

[12] Li J.P. and Chen A.Y. (2008). Decay property of stopped Markovian Bulk-arriving queues. Adv. Appl. Probab., 40(1):95-121.

[13] Li J.P., Chen A.Y. and Pakes A.G. (2012) Asymptotic properties of the Markov branching process with immigration. J. Theoret. Probab., 25(1):122-143.

[14] Li J.P. and Liu Z.M. (2011). Markov branching processes with immigration-migration and resurrection. Sci. China Math., 54(1):1043-1062.

[15] Li Y.Y., Li J.P. and Chen A.Y. (2020) The down/up crossing properties of weighted Markov branching processes manuscript.

[16] Liu J.N. and Zhang M. (2016). Large deviation for supercritical branching processes with immigration. Acta Mathematica Sinica, English Series. 32(8):893-900.

[17] Renshaw E. and Chen A.Y. (1997). Birth-death processes with mass annihilation and state-dependent immigration. Comm. Statist. Stochastic Models, 13(2):239-253.
[18] Sevastyanov B.A. (1949). *On certain types of Markov processes* (in Russian). *Uspehi Mat. Nauk*, 4, 194.

[19] Sun Q. and Zhang M. (2017). Harmonic moments and large deviations for supercritical branching processes with immigration. *Frontiers of Mathematics in China*, 12(5):1201–1220.

[20] Vatutin V.A. (1974). Asymptotic behavior of the probability of the first degeneration for branching processes with immigration. *Teor. Verojatnost. i Primenen*, 19(1):26-35.

[21] Yamazato M. (1975). Some results on continuous time branching processes with state-dependent immigration. *J. Math. Soc. Japan*, 27(3):479-496.