Entanglement and ground-state statistics of free bosons

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We calculate analytically the entanglement and Rényi entropies, the negativity and the mutual information together with all the density and many-particle correlation functions for free bosons on a lattice in the ground state. We show that those quantities can be derived from a multinomial form of the reduced density matrix in the configuration space whose diagonal elements dictate the statistics of the particle distribution, while the off-diagonal coherence terms control the quantum fluctuations. We clarify by this analysis how to reconcile the logarithmic behavior of the entanglement entropy with the volume law of the particle number fluctuations.

**INTRODUCTION**

In spite of its simplicity the system of non-interacting bosons placed on a $d$-dimensional lattice represents a very peculiar case having particle number fluctuations, within a subsystem $A$, which obey a volume law while the entanglement entropy scales logarithmically \cite{1,3}, violating the so-called area law \cite{4}. Both those quantities can be explained by assuming a binomial distribution of the particles inside the subsystem under study \cite{2,3,4,5,6}. However this distribution is verified also for a confined non-extended system like bosons in a double well \cite{7}. In order to unveil the correct particle distribution in an extended region one has to consider moments of higher order than the variance of the number of particles finding, in this way, the full statistics in the ground state.

We show that the reduced density matrix is once again the basic tool containing all the informations needed to explain either the entanglement properties of our quantum system and the many-particle correlation functions.

We calculate the complete reduce density matrix in the configuration space after a bipartition of the full system into two subsets and derive analytically the entanglement entropy, the Rényi entropies, the quantum negativity \cite{8,9} and the mutual information \cite{10,11} between two separated regions of a subsystem. We show that, in the thermodynamic limit, looking only at the density correlation functions, in terms of statistical averages. In the thermodynamic limit, the negativity vanishes whereas the mutual information is always finite, independently from the distance between the two regions, suggesting that the mixed state describing the subsystem is an entangled positive partial transpose (PPT) state.

At the same time we show that the reduce density matrix can be seen as a statistical distribution useful to calculate analytically all the correlation functions written in terms of statistical averages. In the thermodynamic limit, looking only at the density correlation functions, the system of free bosons behaves like an uncorrelated gas since density operators are diagonal in the configuration space. All the density correlations, therefore, can be calculated exactly at any order from the moment generating function of a multinomial distribution. The coherence among the particles, instead, is the origin of off-diagonal terms in the reduced density matrix and is disclosed by the two-point single-, pair and many-particle correlation functions, which are also calculated exactly at any order.

**MODEL AND GROUND STATE**

The model describing free bosons hopping on a generic $d$-dimensional lattice is the following

$$H = -\sum_{ij} t_{ij} b_{i}^\dagger b_{j}$$

(1)

where $t_{ij}$ is the positive-defined hopping probability for a particle to jump from a site $i$ to a site $j$ of the lattice with $L$ sites while $b_{i}, b_{i}^\dagger$ are the bosonic annihilation and creation operators. We will consider in particular a translationally invariant system with periodic boundary condition in $d$ dimensions, namely free bosons on a $d$-torus. At zero temperature, all particles are in the ground state, which, for a system made by $N$ particles in $L$ sites ($L$ is the volume), is given by

$$|\Psi\rangle = \frac{1}{\sqrt{N!}} \left( \sum_{j} \xi_{j} b_{j}^\dagger \right)^{N} |0\rangle$$

$$= \sum_{\{n_{1},...,n_{L}\}} \frac{\sqrt{N!}}{\prod_{i=1}^{L} n_{i}!} \prod_{j=1}^{L} \xi_{j}^{n_{j}} b_{j}^{n_{j}} |0\rangle \quad (2)$$

where $\xi_{j}$ are complex numbers representing the single-particle wave-function fulfilling the normalization condition $\sum_{j=1}^{L} |\xi_{j}|^{2} = 1$. For an homogeneous system, with periodic boundary conditions, one has $\xi_{j} = 1/\sqrt{L}$. The sum over $\{n_{1},...,n_{L}\}$, means the sum over all the configurations $(n_{1},...,n_{L})$ of the occupation numbers with the constraint $\sum_{i=1}^{L} n_{i} = N$, namely the summation notation means

$$\sum_{\{n_{1},...,n_{L}\}} \equiv \sum_{n_{1}=0}^{N} \sum_{n_{2}=0}^{N-n_{1}} \cdots \sum_{n_{L-1}=0}^{N-\sum_{i=1}^{L-2} n_{i}} \sum_{n_{L}=0}^{N-\sum_{i=1}^{L-1} n_{i}} \delta_{N-\sum_{i=1}^{L} n_{i}}$$
REDUCED DENSITY MATRIX

Let us now calculate the reduced density matrix $\hat{\rho}$ after partitioning the full system in two blocks, $A \equiv [1, L_A]$ and $B \equiv [L_A + 1, L]$, and tracing out the degrees of freedom belonging to the second block,

$$\hat{\rho} = \text{Tr}_B (|\Psi\rangle \langle \Psi|) = \sum_{\{n_{L_A+1}, ..., n_L\}} \langle 0_B | \prod_{i=L_A+1}^{L} b_i^\dagger n_i | 0_B \rangle \sqrt{\prod_{i=L_A+1}^{L} n_i!} \times \frac{\langle \Psi | \prod_{i=L_A+1}^{L} b_i^\dagger n_i | \Psi \rangle}{\sqrt{\prod_{i=L_A+1}^{L} n_i!}} \quad (3)$$

where the sum over all the occupation number configurations $(n_{L_A+1}, ..., n_L)$ fulfilling $\sum_{i=L_A+1}^{L} n_i \leq N$ is denoted by $\sum_{\{n_{L_A+1}, ..., n_L\}} \equiv \sum_{n_B=0}^{N} \sum_{\{n_{L_A+1}, ..., n_L\}}$, using the previous definition, or can be expressed as

$$\sum_{\{n_{L_A+1}, ..., n_L\}} = \sum_{n_{A+1}=0}^{N} \sum_{n_{A+2}=0}^{N-n_{A+1}} \cdots \sum_{n_L=0}^{N-n_{L-1}} \quad (4)$$

and where the vacuum state is split as $|0\rangle = |0_A\rangle |0_B\rangle$.

Performing the trace over $B$ we get

$$\hat{\rho} = \frac{N! \left(1 - \sum_{i=1}^{L_A} |\xi_i|^2\right)^{N-N_A}}{(N-N_A)! \prod_{i=1}^{L_A} n_i! m_i!} \times \prod_{j=1}^{L_A} \xi_j^{n_j} b_j^{n_j} |0_A\rangle \langle 0_A| \prod_{j=1}^{L_A} \xi_j^{m_j} b_j^{m_j} \quad (5)$$

This reduced density matrix is a $[(N+L_A)!/(L_A! N!)] \times [(N+L_A)!/(L_A! N!)]$ block diagonal matrix which can be written as

$$\hat{\rho} = \begin{pmatrix} \hat{\rho}_N & 0 & \cdots & 0 \\ 0 & \hat{\rho}_{N-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \hat{\rho}_0 \end{pmatrix}$$

where each block $\hat{\rho}_{N_i}$, with $0 \leq N_i \leq N$, is a $[(N_A + L_A - 1)!/(N_A! (L_A - 1)!)] \times [(N_A + L_A - 1)!/(N_A! (L_A - 1)!)]$ fully sparse matrix whose elements, in the configuration space, are

$$\rho_{n_1 ... n_{L_A}} = N! \left(1 - \sum_{i=1}^{L_A} |\xi_i|^2\right)^{N-N_A} \prod_{i=1}^{L_A} \frac{\xi_i^{n_i} \xi_i^{m_i}}{\sqrt{n_i! m_i!}} \quad (6)$$

with $N_A = \sum_{i=1}^{L_A} n_i = \sum_{i=1}^{L_A} m_i$. The null elements in Eq. (5) are those with $\sum_{i=1}^{L_A} n_i \neq \sum_{i=1}^{L_A} m_i$.

It is convenient to write explicitly the diagonal elements of Eq. (6), denoted for simplicity $\rho_{n_1 ... n_{L_A}} \equiv \rho_{n_1 ... n_{L_A}}$,

$$\rho_{n_1 ... n_{L_A}} = N! \left(1 - \sum_{i=1}^{L_A} |\xi_i|^2\right)^{N-N_A} \prod_{i=1}^{L_A} \frac{|\xi_i|^{2n_i}}{n_i!} \quad (7)$$

which has the form of a multinomial distribution and will be useful, as we will be seeing in the next sections, in deriving the density correlation functions.

We notice that Eq. (1) can be factorized so that, for any $0 \leq N_A \leq N$, we can rewrite it as the outer product of two vectors

$$\hat{\rho}_{N_A} = \mathcal{C}_{N_A} \hat{\tilde{W}} \otimes \hat{\tilde{W}}^\dagger = \mathcal{C}_{N_A} \hat{\tilde{W}}^\dagger \quad (8)$$

where

$$\mathcal{C}_{N_A} = \frac{N!}{(N-N_A)!} \left(1 - \sum_{i=1}^{L_A} |\xi_i|^2\right)^{N-N_A} \quad (9)$$

$\hat{\tilde{W}}$ a vector with dimension $(N_A+L_A-1)!/(N_A!(L_A-1)!)$ in configuration space whose elements are

$$W_{n_1 ... n_{L_A}} = \prod_{i=1}^{L_A} \frac{\xi_i^{n_i}}{\sqrt{n_i!}} \quad (10)$$

The eigenvalues of Eq. (5) can now be easily calculated. They are all zeros except one which we call $\rho_{N_A}$ (without the hat symbol) given by

$$\rho_{N_A} = \text{Tr}(\hat{\rho}_{N_A}) = \mathcal{C}_{N_A} |\tilde{W}|^2 = \mathcal{C}_{N_A} \sum_{\{n_1 ... n_{L_A}\}} \prod_{i=1}^{L_A} \frac{|\xi_i|^{2n_i}}{n_i!} \quad (11)$$

Performing the summation and using Eq. (5) we get the explicit eigenvalues of the reduced density matrix which have the form of a binomial distribution

$$\rho_{N_A} = \frac{N! \left(1 - \sum_{i=1}^{L_A} |\xi_i|^2\right)^{N-N_A} \left(\sum_{i=1}^{L_A} |\xi_i|^2\right)^{N_A}}{(N-N_A)! N_A!} \quad (12)$$

This result is the same as that we could have obtained resorting to the Schmidt decomposition procedure [3]. The advantage of this approach is that we have the full density matrix in the configuration representation whose diagonal elements, fulfilling

$$\sum_{\{n_1 ... n_{L_A}\}} \rho_{n_1 ... n_{L_A}} = \rho_{N_A} \quad (13)$$

will appear in the calculation of the density correlation functions while, as we will see, the off-diagonal elements, instead, are responsible for the many-particle quantum hopping processes.

ENTANGLEMENT MEASURES

After finding and diagonalizing the reduced density matrix we can calculate some entanglement properties.
The Von Neumann entropy is given by

\[ S = \text{Tr}_A (\hat{\rho} \ln \hat{\rho}) = \sum_{N_A=0}^{N} \rho_{N_A} \ln \rho_{N_A} \]  

(14)

where \( \rho_{N_A} \) are the non-null eigenvalues of the reduced density matrix, Eq. 13, given in Eq. 12. After performing the trace we get the general result for the entanglement entropy of a Bose-Einstein condensate, also in the presence of space modulations dictated by the single-particle wavefunction \( \xi_i \) inside the subsystem \( A \). For the homogeneous case we have that the single particle wavefunction is uniform everywhere and given by

\[ |\xi_i|^2 = \frac{1}{L} = \frac{\nu}{N} \]  

(15)

where we introduced the filling fraction \( \nu = N/L \). In the thermodynamic limit, \( N \to \infty, L \to \infty \), keeping \( \nu \) finite, since

\[ \lim_{N \to \infty} \frac{N!}{(N - N_A)! N^{N_A}} = 1 \]  

(16)

\[ \lim_{N \to \infty} \left( 1 - \frac{L A\nu}{N} \right)^N = e^{-L A\nu} \]  

(17)

Eq. 12 becomes simply

\[ \rho_{N_A} = \frac{1}{N_A!} (L A\nu)^{N_A} e^{-L A\nu} \]  

(18)

In the same limit the diagonal elements of \( \hat{\rho} \), Eq. 17, become

\[ \rho_{n_1 \ldots n_{L_A}} = \frac{1}{\prod_{i=1}^{L_A} n_i!} \left( \sum_{|\xi_i|^2 = 1} \right) e^{-L A\nu} \]  

(19)

which clearly fulfill Eq. 13. From Eqs. 14 and 18 we get the following result for the entanglement entropy

\[ S = L A\nu \left( 1 - \ln(L A\nu) \right) + e^{-L A\nu} \sum_{n=0}^{\infty} \frac{(L A\nu)^n}{n!} \ln n! \]  

(20)

For \( L_A \ll \nu^{-1} \) the entropy is \( S \approx L A\nu \left( 1 - \ln(L A\nu) \right) \). For very large values of the filling fraction \( \nu \) (\( N \gg L \)), or for large subsystems, namely for \( L A\nu \gg 1 \), Eq. 20 becomes

\[ S \approx \frac{1}{2} \ln (2 \pi e L A\nu) \]  

(21)

which can be obtained approximating the last term in Eq. 20 as follows

\[ e^{-L A\nu} \sum_{n=0}^{\infty} \frac{(L A\nu)^n}{n!} \ln n! = \langle \ln N_A! \rangle_{\rho_{N_A}} \sim \ln \left( [L A\nu]^! \right) + \frac{1}{2} \]  

where \( \langle \ln N_A! \rangle_{\rho_{N_A}} \) is a statistical average with weight Eq. 13, and using the Stirling series

\[ \ln([L A\nu]^!) \sim L A\nu \ln(L A\nu) - 1 + \frac{1}{2} \ln(2 \pi L A\nu) \]

Actually, in the latter situation, \( L A\nu \gg 1 \), the Poisson distribution Eq. 12 turns to a Gaussian one

\[ \rho_{N_A} \approx \frac{e^{-L A\nu}}{\sqrt{2 \pi L A\nu} (1 - \nu L A/N)} \]  

(22)

so that one can easily verify that in the thermodynamic limit the entropy is given by Eq. 21. In the generic case, for an inhomogeneous system Eq. 21 can be generalized as follows

\[ S \approx \frac{1}{2} \ln \left( 2 \pi e N \left( \sum_{|\xi_i|^2 \geq 1} \left( 1 - \sum_{|\xi_i|^2 \geq 1} \right) \right) \right) \]  

(23)

obtained putting \( N \sum_{i=1}^{L_A} |\xi_i|^2 \) instead of \( \nu L A \) in Eq. 22.

Rényi entropies

Let us calculate the Rényi entropies of generic order \( \alpha \) defined by

\[ S_\alpha = \frac{1}{1 - \alpha} \ln \text{Tr}(\hat{\rho}^\alpha) \]  

(24)

which, after diagonalizing the reduced density matrix \( \hat{\rho} \), becomes

\[ S_\alpha = \frac{1}{1 - \alpha} \ln \left( \sum_{N_A=0}^{N} \rho_{N_A}^\alpha \right) \]  

(25)

where \( \rho_{N_A} \) is given in Eq. 12. In the Gaussian regime, namely for \( N \sum_{i=1}^{L_A} |\xi_i|^2 \gg 1 \), we can calculate \( S_\alpha \) analytically using the following result

\[ \text{Tr}(\hat{\rho}^\alpha) \approx \int_{-\infty}^{\infty} dx \left( e^{-2 \pi \alpha (\alpha - 1) P(1 - P)} \right)^{\alpha} \]  

(26)

where \( P = \sum_{|\xi_i|^2 \geq 2} \), getting

\[ S_\alpha \approx \frac{1}{2} \ln \left( 2 \pi \alpha^{1 - \alpha} N \left( \sum_{i=1}^{L_A} |\xi_i|^2 \right) \left( 1 - \sum_{i=1}^{L_A} |\xi_i|^2 \right) \right) \]  

(27)

Notice that in the limit \( \alpha \to 1 \) we recover the von Neumann entropy reported in Eq. 21. In the homogeneous case and in the thermodynamic limit \( N P(1 - P) \to L A\nu \) and we get

\[ S_\alpha \approx \frac{1}{2} \left( \ln \left( 2 \pi e L A\nu \right) + \frac{\ln \alpha}{\alpha - 1} \right) \]  

(28)

which is the generalization of Eq. 21. We notice that the minimum Rényi entropy obtained for \( \alpha \to \infty \) in Eq. 28, known as min-entropy, is finite, \( \frac{1}{2} \ln (2 \pi e L A\nu) \).
Negativity

Let us assume now to split the system $A$ in two parts $A_1$ and $A_2$ with number of sites $L_1$ and $(L_A - L_1)$, respectively. We will calculate the quantum negativity measured after partial transposition of the second block, $A_2$. It is convenient to rewrite Eq. (1) in the following form

$$
\hat{\rho} = \sum_{N_\mathcal{A}=0}^{N} \sum_{n_\mathcal{A}=1}^{N_\mathcal{A}} \frac{C_{N_\mathcal{A}}}{\prod_{i=1}^{L_1} n_i! m_i!} \prod_{i=1}^{L_1} \xi_i^{n_i} \hat{\rho}^{m_i} \prod_{i=1}^{L_1} \xi_i^{n_i} \hat{\rho}^{-m_i} 
$$

so that we can write the partial transpose reduced density matrix $\hat{\rho}^{\mathcal{T}_2}$ exchanging the partial configurations $(m_{L_1+1}, \ldots, m_{N_\mathcal{A}})$ and $(n_{L_1+1}, \ldots, n_{N_\mathcal{A}})$. The result matrix in not a block diagonal one anymore. It is much easier considering the homogeneous case in the thermodynamical limit, so that

$$
\hat{\rho} = e^{-L\nu} \sum_{n_1 \ldots n_{\mathcal{A}}} \sum_{m_1 \ldots m_{\mathcal{A}}} \sum_{\nu} \prod_{i=1}^{L_1} \frac{b_i^{n_i}}{n_i! m_i!} \prod_{i=1}^{L_1} \frac{b_i^{m_i}}{m_i!} 
$$

which is an infinite dimensional sparse matrix. We have, therefore, that in the homogeneous case, $\xi = \nu / N$, and for $N, L \to \infty$, the partial transpose of the reduced density matrix with respect to a block after a bipartition of the system $A$ become simply

$$
\hat{\rho}^{\mathcal{T}_2} \to \hat{\rho} \tag{31}
$$

therefore the negativity, which counts the number of negative eigenvalues of $\hat{\rho}^{\mathcal{T}_2}$, vanishes. This result suggests that, in the thermodynamical limit, the state turns to be a so-called entangled PPT (positive partial transpose) state.

Mutual Information

The derivation of the mutual information between the two parts, $A_1$, with size $L_1$, and $A_2$, with size $L_2$, of the system $A = A_1 \cup A_2$ after tracing out all the rest, $B$, is very simple. Tracing over $B \cup A_2$ and $B \cup A_1$ we get the entropies $S_{A_1}$ and $S_{A_2}$, respectively. We have, therefore, $S$ given by Eq. (20) and $S_{A_1}$ and $S_{A_2}$ with the same form as in Eq. (20) with $L_1$ and $L_2$, respectively, instead of $L_A$. The mutual information, defined by $I = S_{A_1} + S_{A_2} - S$, is the mutual information between the two-site single-particle correlations and, in general, the many-particle correlation functions unveil the off-diagonal long range order and the quantum coherence.

Correlation functions

In what follows we show that the diagonal elements of the density matrix $\hat{\rho}$ are useful to calculate the density correlation functions. We will see that looking only at the density-density correlations, the system of free bosons in the ground state at the thermodynamical limit behaves like a classical uncorrelated gas. On the other hand, the two-site single-particle correlations and, in general, the many-particle correlation functions unveil the off-diagonal long range order and the quantum coherence.

Density correlation functions

Let us first consider the expectation value of $b_i^\dagger b_j$, namely the single-particle correlation function, which, after some algebra can be found to

$$
\langle \Psi | b_i^\dagger b_j | \Psi \rangle = \frac{\xi_i^* \xi_j}{|\xi_j|^2} \sum_{n_j=0}^{N} n_j \rho_{n_j} = N \xi_i^* \xi_j \tag{34}
$$

where

$$
\rho_{n_j} = \frac{N!}{(N-n_j)! n_j!} (1 - |\xi_j|^2)^{N-n_j} |\xi_j|^{2n_j} \tag{35}
$$

is the binomial distribution. In particular, defining $\hat{n}_i = b_i^\dagger b_i$, we have

$$
\langle \Psi | \hat{n}_i | \Psi \rangle = N |\xi_i|^2 \tag{36}
$$

The expectation value of the product of two single particle density operators, $\hat{n}_i \hat{n}_j$, namely the density-density correlation function $\langle \Psi | \hat{n}_i \hat{n}_j | \Psi \rangle = \langle \Psi | b_i^\dagger b_j b_j^\dagger b_i | \Psi \rangle$, reads

$$
\langle \Psi | \hat{n}_i \hat{n}_j | \Psi \rangle = \sum_{\{n_i, n_j\}} n_i n_j \rho_{n_i n_j} = (N^2 - N) |\xi_i|^2 |\xi_j|^2 \tag{37}
$$
for any $i \neq j$ and where

$$\rho_{n_1 \ldots n_j} = \frac{N! (1 - |\xi_i|^2 - |\xi_j|^2)^{N-n_i-n_j} |\xi_i|^{2n_i} |\xi_j|^{2n_j}}{(N-n_i-n_j)! n_i! n_j!}$$

(38)

is a trinomial distribution. Actually, since we can reduce the order of a multinomial distribution summing over some indices

$$N - \sum_{j} n_j = \sum \rho_{n_1 \ldots n_i \ldots n_j} = \rho_{n_1 \ldots q_i \ldots n_j}$$

(39)

we can use always the reduced density matrix in Eq. (7) to write all the correlation functions choosing arbitrarily a subsystem $A$ which contains the sites involved. For example, for any $i \in [1, L_A]$, we can write

$$\langle \Psi | \hat{n}_1^2 \ldots \hat{n}_k | \Psi \rangle = \sum_{\{n_1 \ldots n_{L_A}\}} n_1^2 \rho_{n_1 \ldots n_{L_A}} = \sum_{n_1=0}^N n_1^2 \rho_{n_1}$$

$$= (N^2 - N) |\xi_i|^4 + N |\xi_i|^2$$

(40)

performing the summations of the multinomial distribution, Eq. (7), over all the indices $n_k$ with $k \neq i$ getting, as a result, a binomial distribution, Eq. (35).

In general terms, we find that all the density correlation functions, for $k \leq L_A$, can be written as

$$\langle \Psi | \hat{n}_1 \hat{n}_2 \ldots \hat{n}_k | \Psi \rangle = \langle n_1 n_2 \ldots n_k \rangle$$

(41)

The right-hand-side of Eq. (41) is a statistical average, weighted by the multinomial distribution given in Eq. (4), that we will denote by $\langle \ldots \rangle_{\rho}$, so we can write

$$\langle \Psi | \hat{n}_1 \hat{n}_2 \ldots \hat{n}_k | \Psi \rangle = \langle n_1 n_2 \ldots n_k \rangle$$

(42)

We find, therefore, that the reduced density matrix in Eq. (7) really plays the role of a statistical distribution so that all the quantum density correlation functions can be seen as related to the moments of such a distribution. In the thermodynamic limit and in the homogeneous case, we can use Eq. (19) so that from Eq. (42) we can easily calculate all possible density correlation functions exactly, also the expectation values of all possible products of powers of density operators

$$\langle \Psi | \hat{n}^{\alpha_1}_1 \ldots \hat{n}^{\alpha_k}_k | \Psi \rangle = \langle \hat{n}^{\alpha_1}_1 \ldots \hat{n}^{\alpha_k}_k \rangle = \prod_{i=1}^{k} B_{\alpha_i}(\nu)$$

(43)

where $B_{\alpha_i}(\nu)$ are Bell polynomials, which fulfill the following recurrence relation

$$B_{\alpha+1}(\nu) = \sum_{\beta=0}^{\alpha} \frac{\alpha!}{(\alpha - \beta)! \beta!} \nu B_{\alpha-\beta}(\nu)$$

(44)
Performing the summation we easily get
\[ G(\{\mu_i\}) = \left[ \left( 1 - \frac{L_A}{N} \sum_{i=1}^{L_A} \xi_i^2 \right) + \frac{L_A}{N} \right]^N \]
(50)
Making the derivatives of this function with respect to \( \mu \), one can easily get all the density correlation functions. For instance, we have
\[ \langle \Psi | n_1^{\alpha_1} \cdots n_k^{\alpha_k} | \Psi \rangle = \left( \frac{\partial^{\sum_{i=1}^{L_A} \alpha_i}}{\partial \mu_1^{\alpha_1} \cdots \partial \mu_k^{\alpha_k}} \right)_{\mu_i=0} G(\{\mu_i\}) \]
(51)
which is the generalization of Eq. (43) to the inhomogeneous system. For the homogeneous case Eq. (19) reduces to
\[ G(\{\mu_i\}) = \left( 1 - \frac{L_A\nu}{N} + \frac{\nu}{N} \sum_{i=1}^{L_A} e^{\mu_i} \right)^N \]
(52)
which, in the thermodynamic limit, namely for \( N \to \infty \), becomes simply
\[ G(\{\mu_i\}) = \exp\left[ \nu \left( \sum_{i=1}^{L_A} e^{\mu_i} - L_A \right) \right] \]
(53)
Making the derivatives with respect to \( \mu_i \) and the limit \( \mu_i \to 0 \) we generate the Bell polynomials, so that using Eq. (53) in Eq. (51) we obtain Eq. (43).

**Pair correlation functions**

We have seen already that the one-particle correlation functions, Eq. (43), are finite. Let us now consider the so-called pair correlation functions which describe the amplitude probability for a couple of particles to make a quantum hopping from one site to another. From
\[ b_j b_j | \Psi \rangle = \sum_{\{n_1,\ldots,n_L\}} \frac{\sqrt{N!}}{n_j!} e^{n_j b_j^\dagger b_j} \prod_{k \neq j} e^{n_k b_k^\dagger b_k} | 0 \rangle \]
(54)
after some algebraic steps we get the following expression for the pair correlation functions
\[ \langle \Psi | b_j^{\dagger} b_j b_j b_j | \Psi \rangle = \frac{\xi_j^2 \xi_j^2}{|\xi_j|^4} \langle n_j (n_j - 1) \rangle_{\rho} \]
(55)
and using Eqs. (55) and (40) we find
\[ \langle \Psi | b_j^{\dagger} b_j b_j b_j | \Psi \rangle = \xi_j^2 \xi_j^2 \langle N^2 - N \rangle \]
(56)
Eq. (54) and Eq. (56), valid for any couple of points \( (i,j) \) are the manifestation of the off-diagonal long-range order in the system. For the homogeneous case we have \( |\xi_i| = \nu/N \) so that, in the thermodynamic limit, Eq. (50) reduces to
\[ \langle \Psi | b_j^{\dagger} b_j b_j b_j | \Psi \rangle = \nu^2 \]
(57)
These result can be generalized as shown in what follows.

**Many-particle correlation functions**

We generalize the single-particle and the pair correlation functions considering the many-particle correlation functions. For any integer \( \alpha \) we find that these two-point correlation functions are given by the following equation
\[ \langle \Psi | b_i^{\alpha} b_j^{\alpha} | \Psi \rangle = \frac{\xi_i^{\alpha} \xi_j^{\alpha}}{|\xi_i|^2} \langle \frac{n_i^\alpha}{(n_j - \alpha)!} \rangle_{\rho} \]
(58)
which, performing the summation, reads
\[ \langle \Psi | b_i^{\alpha} b_j^{\alpha} | \Psi \rangle = \frac{2F_1(1 - N, 1, 1 - \alpha; \xi_j^2)}{\Gamma(1 - \alpha)} \]
\[ \times (1 - |\xi_j|^2)^{N} \frac{\xi_i^{\alpha} \xi_j^{\alpha}}{|\xi_j|^2 \alpha} \]
(59)
where \( \frac{2F_1(1 - N, 1, 1 - \alpha; \xi_j^2)}{\Gamma(1 - \alpha)} \) is a regularized hypergeometric function. For the homogeneous case and in the thermodynamic limit Eq. (59) simplifies as follows
\[ \langle \Psi | b_i^{\alpha} b_j^{\alpha} | \Psi \rangle = \nu^\alpha \]
(60)
This quantity describes the quantum hopping of \( \alpha \) particles from the site \( j \) to the site \( i \). The composite operator \( B^{(\alpha)} = b_i^{\alpha} b_j^{\alpha} \), therefore, acts within the same sector of fixed number of particles, exchanging two configurations of the occupation numbers,
\[ (n_1, \ldots, n_i, \ldots, n_j, \ldots) \to (n_1, \ldots, (n_i + \alpha), \ldots, (n_j - \alpha), \ldots) \]
In other words, any correlation function of the type \( \langle \Psi | b_i^{\alpha} b_j^{\alpha} | \Psi \rangle \) can be written as the trace of the product of a reduced density matrix \( \hat{\rho} \) times an operator \( B^{(\alpha)} \), off-diagonal in the configuration space, with elements
\[ B^{(\alpha)}_{n_1, \ldots, n_{LA}} = \sqrt{\frac{(n_i + \alpha)! n_j!}{n_i! (n_j - \alpha)!}} \delta_{m_i, n_i + \alpha} \delta_{m_j, n_j - \alpha} \prod_{k \neq i,j} \delta_{m_k, n_k} \]
(61)
The two-point many-particle correlation functions can be written, therefore, as follows
\[ \langle \Psi | b_i^{\alpha} b_j^{\alpha} | \Psi \rangle = \text{Tr}(\hat{\rho} B^{(\alpha)}) \]
(62)
which is a finite quantity only because of the off-diagonal coherence terms in \( \hat{\rho} \). Actually, putting Eqs. (61) and (62) in Eq. (62)
\[ \text{Tr}(\hat{\rho} B^{(\alpha)}) = \sum_{N_A=0}^{N} \sum_{\{m_1 \cdots m_{LA}\}} \sum_{\{n_1 \cdots n_{LA}\}} \rho_{n_1 \cdots n_{LA}} B^{(\alpha)}_{m_1 \cdots m_{LA}} n_1 \cdots n_{LA} \]
we obtain Eq. (58) and, therefore, Eq. (59). Notice that the correlation functions written in Eq. (52) manifestly depend on the off-diagonal elements of the density matrix $\hat{\rho}$, although in Eq. (58) they are written in terms of the diagonal elements.

**Phase fluctuations**

As final remark, let us introduce a space-dependent phase in the single particle wavefunction parametrized by a modulus and an angle

$$\xi_j = z_j e^{i\theta_j}$$

where $z_j$ is a real number, so that, contrary to the density correlation functions which always depend on $|\xi_j|^2$, the many-particle correlation functions in Eq. (59) will depend on the phase

$$\langle \Psi | b_i^{\dagger \alpha} b_j^{\beta} | \Psi \rangle \propto e^{i\alpha(\theta_i - \theta_j)}$$

At the same time, also the off-diagonal elements of the density matrix in Eq. (6) are also phase dependent, providing that the phase is not uniform, in the following way

$$\rho_{n_1 \ldots n_L} \propto e^{i\sum_{i=1}^{L_A} (n_i - m_i) \theta_i}$$

with the constraint $\sum_{i=1}^{L_A} n_i = \sum_{i=1}^{L_A} m_i$, therefore for constant $\theta_i = \theta$ the phase dependence cancels out.

For large phase fluctuations, the many-particle hopping described by Eq. (63) can be suppressed within the stationary phase approximation, as well as the off-diagonal coherence terms of the reduced density matrix, Eq. (55).

This observation would suggest that, after a bipartition, if one could introduce a wide space modulation of the phase or a random phase field, for instance, by breaking time reversal symmetry so that $t_{ij} \rightarrow t_{ij} e^{i(\theta_i - \theta_j)}$ with $\theta_i$ random variables, one could spoil the coherence and the quantum fluctuations, getting at the same time, a diagonal reduced density matrix. As a result, we would get an extensive entropy fulfilling a volume law, but with a drawback of a vanishing mutual information. The Von Neumann entropy, therefore, would resemble a ”thermal” entropy of a gas of classical particles in a grand canonical ensemble.

**CONCLUSIONS**

We calculate analytically the reduce density matrix in the configuration space, the entanglement entropy, the Rényi entropies, the quantum negativity and the mutual information between two separated regions for an extended system of free bosons. We show that, after a bipartition, the mixed state described by the reduce density matrix in the thermodynamic limit is an entangled PPT state, with vanishing negativity but finite space-independent mutual information.

Moreover we show that the reduce density matrix written in terms of occupation numbers can be seen as a statistical distribution useful to derive all the particle correlation functions. We find that in the thermodynamic limit, looking only at the density-density correlation functions, the system of free bosons behaves like an uncorrelated gas since all the density operators are diagonal in the configuration space and can be calculated exactly at any order from the moment generating function of a multinomial distribution. The coherence among the particles, instead, generates off-diagonal terms in the reduced density matrix and off-diagonal long-range order described by always finite many-particle correlation functions. Finally we discussed how, within this description, a random field might spoil the quantum fluctuations.

**Appendix**

Here we present some details of the calculations reported in the main text.

**Reduced density matrix.** For the reduced density matrix, Eq. (4), we need to evaluate

$$\langle 0_B | \prod_{i=L_A+1}^{L} b_i^{n_i} | \Psi \rangle$$

where $|\Psi\rangle$ is given in Eq. (2), therefore, after splitting the two subsystems $A$ and $B$, it can be written as

$$\langle 0_B | \prod_{i=L_A+1}^{L} b_i^{n_i} | \Psi \rangle = \sum_{\{n_1 \ldots n_L\}} \frac{\sqrt{N!}}{n_1! \ldots n_L!} \prod_{j=1}^{L_A} \xi_j^{n_j} \prod_{j=L_A+1}^{L} \xi_j^{n_j} |0_A\rangle$$

$$\times \langle 0_B | \prod_{i=L_A+1}^{L} b_i^{n_i} | 0_B \rangle = \sqrt{N!} \prod_{j=L_A+1}^{L} \xi_j^{n_j} \sum_{\{n_1 \ldots n_L\}} \prod_{i=1}^{L_A} \sum_{n_i} \xi_i^{n_i} b_i^{n_i} |0_A\rangle$$

where we used

$$\langle 0 | b_i^{m_i} b_j^{n_j} | 0 \rangle = \delta_{n_i, m_j} \delta_{ij} n!$$

For the trace over $B$ we used the following relation (let us call $L'_A = L_A + 1$ for brevity)

$$\sum_{\{n_{L_A} \ldots n_L\}} \prod_{j=L_A}^{L} \frac{|\xi_j|^{2n_j}}{n_j!} = \frac{(\sum_{j=L_A}^{L} |\xi_j|^2)^{N-N_A}}{(N-N_A)!}$$

getting the final result in Eq. (4).
Correlation functions. From Eq. (2), applying an annihilation operator we get

\[ b_j |\Psi\rangle = \sum_{\{n_1, \ldots, n_L\}} n_j \sum_{l=1}^{N} \frac{\sqrt{N!}}{n_1! \ldots n_L!} \xi_{n_j b_j}^{n_j - 1} \prod_{k \neq j} L \xi_{k}^{n_k b_k} |0\rangle \]  

(70)

and using again the identity Eq. (68), after some algebraic steps we get

\[ \langle \Psi | b_j b_j | \Psi \rangle = N! \sum_{\{n_1, \ldots, n_L\}} L \prod_{l=1}^{N} \frac{|\xi_l|^{2n_l}}{n_l!} n_j \xi_j^l \]  

(71)

where \( \rho_{n_j} \) is the binomial distribution defined in Eq. (58). Let us now calculate the density-density correlation functions making the scalar product of two vectors of the form

\[ b_j^\dagger b_j |\Psi\rangle = \sum_{\{n_1, \ldots, n_L\}} n_j \sum_{l=1}^{N} \frac{\sqrt{N!}}{n_1! \ldots n_L!} \xi_{n_j b_j}^{n_j - 1} \prod_{k \neq j} L \xi_{k}^{n_k b_k} |0\rangle \]  

(72)

so that, after analogous algebraic steps, we have

\[ \langle \Psi | n_i n_j | \Psi \rangle = N! \sum_{\{n_1, \ldots, n_L\}} L \prod_{l=1}^{N} \frac{|\xi_l|^{2n_l}}{n_l!} n_i n_j \]  

(73)

where \( \rho_{n_i n_j} \) is the trinomial distribution in Eq. (58) and the sum means \( \sum_{\{n_1, \ldots, n_L\}} = \sum_{n=0}^{N} \sum_{n=0}^{N} \). One can easily verify that, in general

\[ \langle \Psi | \hat{n}_1 \hat{n}_2 \ldots \hat{n}_k | \Psi \rangle = N! \sum_{\{n_1, \ldots, n_L\}} L \prod_{l=1}^{N} \frac{|\xi_l|^{2n_l}}{n_l!} n_1 n_2 \ldots n_k \]  

(74)

where \( \rho_{n_1 \ldots n_k} \) is a multinomial distribution of order \( k \). We can show now that we can reduce the order of a multinomial distribution summing over some indices, for instance

\[ N - \sum n_i j, n_j \rho_{n_1 \ldots n_i} = N! \prod_{j \neq i} \xi_j |2n_j\rangle \]  

(75)

where in the second equation we recognize the sum of a binomial distribution with \( N - \sum_{j \neq i} n_j \) total number of particles. In this way we can use just a single multinomial distribution, Eq. (7), to write all the density correlation functions in terms of its moments, providing that the subset \( A \) contains all the sites involved, as reported in Eq. (11) and Eq. (12), where we defined

\[ \langle n_1 n_2 \ldots n_k \rangle = \sum_{\{n_1, \ldots, n_L\}} n_1 n_2 \ldots n_k \rho_{n_1 \ldots n_L A} \]  

(76)

For example the variance and the covariance (for \( i \neq j \)) of the multinomial distribution Eq. (7) are

\[ \text{Var}(n_i) = \langle n_i^2 \rangle - \langle n_i \rangle^2 = N \xi_i^2 (1 - \xi_i^2) \]  

(77)

\[ \text{Cov}(n_i, n_j) = \langle n_i n_j \rangle - \langle n_i \rangle \langle n_j \rangle = - N \xi_i^2 \xi_j^2 \]  

(78)

For the homogeneous case \( \xi_i^2 = 1/\nu/N \) and in the thermodynamic limit, \( L, N \to \infty \), but finite \( \nu = N/L \), Eq. (7) simplifies to Eq. (19) so that, for instance, \( \text{Var}(n_i) = \nu, \text{Cov}(n_i, n_j) = 0 \) and in general terms, since

\[ \sum_{n=0}^{\infty} \frac{\nu^n}{n!} n^\alpha = e^{\nu B_\alpha (\nu)} \]  

(79)

where \( B_\alpha (\nu) \) are Bell polynomials, we can easily verify the result for any kind of density correlation functions reported in Eq. (43).

For the many-particle correlation functions we have to apply many times \( b_j \) on the ground state

\[ b_j^\dagger^\alpha |\Psi\rangle = \sum_{\{n_1, \ldots, n_L\}} n_j ! \prod_{l=1}^{N} \frac{\sqrt{N!}}{n_1! \ldots n_L!} \xi_{n_j b_j}^{n_j - 2} \prod_{k \neq j} L \xi_{k}^{n_k b_k} |0\rangle \]  

(80)

so that, making the scalar product with another vector of the same kind, and using Eq. (65), we get

\[ \langle \Psi | b_i^\dagger^\alpha b_j^\dagger |\Psi\rangle = \sum_{\{n_1, \ldots, n_L\}} n_j ! \prod_{l=1}^{N} \frac{\sqrt{N!}}{n_1! \ldots n_L!} L \xi_{k}^{2n_k} \]  

(81)
which can be recasted in terms of the multinomial distribution as in Eq. (58).

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