Rodrigues Formula for the Nonsymmetric Multivariable Laguerre Polynomial

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Extending a method developed by Takamura and Takano, we present the Rodrigues formula for the nonsymmetric multivariable Laguerre polynomials which form the orthogonal basis for the $B_N$-type Calogero model with distinguishable particles. Our construction makes it possible for the first time to algebraically generate all the nonsymmetric multivariable Laguerre polynomials with different parities for each variable.

KEYWORDS: nonsymmetric multivariable Laguerre polynomial, $B_N$-Calogero model, Dunkl-Cherednik operator, Rodrigues formula

§1. Introduction

Quantum many-body problems have played an important role in various fields of physics. In particular, exactly solvable models have been extensively studied by many physicists and mathematicians. Since exactly solvable models in quantum mechanics have the same number of mutually commuting conserved operators as their degrees of freedom, they are usually called quantum integrable systems. A good example of them is the Calogero-Sutherland type model (C-S model) which has inverse-square long-range interactions between particles in a one-dimensional space. The Dunkl-Cherednik operator formulation is a very powerful tool for systematic construction of the conserved operators for many kinds of C-S models. Their simultaneous eigenfunctions, which form the orthogonal bases of their Hilbert spaces, are written by products of the Jastrow type ground state wave function and multivariable orthogonal polynomials that are generalized versions of classical orthogonal polynomials.

The Sutherland model describing the particles on a circle is a representative of the family of the C-S models. The orthogonal basis for the Sutherland model has been known as the Jack polynomials. Richness of its analytic properties enables us to calculate the Green function, the density-density correlation function and so on [15,14]. That is why studies on the multivariable orthogonal polynomials have interested many physicists.

On the other hand, the Calogero model describing the particles in the harmonic potential has been treated by the Dunkl-Cherednik operator formulation as well as the Sutherland model. The orthogonal basis of the Calogero model is a one-parameter deformation of the Jack polynomial, i.e. the Hi-Jack polynomial [3,4] which is called the multivariable Hermite polynomial from the viewpoint of generalization of the classical orthogonal polynomial. The orthogonal bases for other C-S models were also constructed [14].

Under the stimulus of the Haldane-Shastry model, the C-S models with spin degrees of freedom appeared [21,22,23]. To investigate the orbital part of the eigenfunction, the C-S models with distinguishable particles were introduced. The Hamiltonians for these C-S models have the coordinate exchange operator $K_{jk}$ acting on an arbitrary $N$-variable function as follows,

$$K_{jk} f(x_1, \cdots, x_j, \cdots, x_k, \cdots) = f(x_1, \cdots, x_k, \cdots, x_j, \cdots).$$

To obtain eigenfunctions of the C-S models with distinguishable particles, it is enough to consider nonsymmetric polynomials diagonalizing their Cherednik operators. Knop and Sahi recursively constructed the nonsymmetric Jack polynomial for the Sutherland model with distinguishable particles [24]. Baker and Forrester translated it to other nonsymmetric polynomials [25].

Recently, Takamura and Takano algebraically constructed the nonsymmetric Jack polynomial [26]. The essential point of their work is introduction of two types of operators, i.e. the braid-exclusion and the raising operators. In the previous paper [14], we considered the Calogero model with distinguishable particles,

$$\hat{H}_C^{(A)} = \frac{1}{2} \sum_{j=1}^{N} (p_j^2 + \omega^2 x_j^2) + \frac{1}{2} \sum_{j,k=1}^{N} \frac{a(K_{jk})}{(x_j - x_k)^2},$$

where the constants $a$ and $\omega$ are the coupling parameter and the strength of the external harmonic well, respectively, and $p_j = \frac{1}{i} \frac{\partial}{\partial x_j}$. Generalizing the method developed by Takamura and Takano, we provided the Rodrigues formula for the nonsymmetric multivariable Hermite polynomial. While it is difficult to compute the coefficient of the top term of the polynomial, it is easy to give simple expressions for the nonsymmetric eigenfunctions and to calculate their norms by the Takamura-Takano method.

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The above Calogero Hamiltonian (2.1) is invariant under the action of the $A_{N-1}$-type Weyl group, i.e. under $S_N$, on the indices of the particle. Thus the model is sometimes called the $A_{N-1}$-Calogero model. Similarly, the Calogero models associated with other Weyl groups exist. The system, which has the additional interactions with the boundary at the origin and the mirror image particles, is called the $B_N$-calogero model. The orthogonal basis for the $B_N$-calogero model is known to be the multivariable Laguerre polynomial which is even in each variable. For the $B_N$-calogero model with distinguishable particles, i.e. including both the coordinate exchange and reflection operators, the non-symmetric eigenfunctions of the Cherednik operators have already been constructed by the Knop-Sahi method. However they are also restricted to functions which are even in each variable. In this paper, extending the Takamura-Takano method, we shall present the Rodrigues formula for the non-symmetric multivariable Laguerre polynomial. Our method enables an algebraic construction of the non-symmetric multivariable Laguerre polynomial which holds different parities for each variable.

The plan of the paper is the following. Section 2 is devoted to a summary of the Dunkl-Cherednik operator formulation for the $B_N$-calogero model. In 3, we shall present the Rodrigues formula for the non-symmetric multivariable Laguerre polynomial which is allowed to hold different parities for each variable. We shall calculate their norms in 4. The final section is devoted to concluding remarks.

§2. Dunkl-Cherednik Operator Formulation for the $B_N$-Calogero Model

We briefly summarize the Dunkl-Cherednik operator formulation for the $B_N$-Calogero model. The $B_N$-Calogero model with both the coordinate exchange and reflection operators is defined by

\[
H_C^{(B)} = \frac{1}{2} \sum_{j=1}^{N} \left( p_j^2 + \omega^2 x_j^2 + \frac{b - t_j}{x_j^2} \right) + \frac{1}{2} \sum_{j,k=1 \atop j \neq k}^{N} \left( \frac{a(a - K_{jk})}{(x_j - x_k)^2} + \frac{a(a - t_j t_k K_{jk})}{(x_j + x_k)^2} \right)
\]

(2.1)

where $b$ is another coupling parameter and $t_j$ is the reflection operator,

\[ t_j f(\cdots, x_j, \cdots) = f(\cdots, -x_j, \cdots) \]

(2.2)

Hereafter we call eq. (2.1) the $B_N$-calogero model with distinguishable particles. The ground state wave function and the ground state energy are

\[
\phi_k^{(B)}(x) = \prod_{1 \leq j < k \leq N} |x_j - x_k|^a \prod_{l=1}^{N} |x_l|^b \exp \left( -\frac{1}{2} \omega \sum_{m=1}^{N} x_m^2 \right),
\]

\[
E_k^{(B)} = \frac{1}{2} \omega N(1 + 2(N - 1)a + 2b),
\]

(2.3)

respectively. For the sake of simplicity, we write $f(x)$ for a function with $N$ variables, $f(x_1, x_2, \cdots, x_N)$. To investigate the polynomial part of the eigenfunction, we carry out a similarity transformation on the Hamiltonian (2.1) with the ground state (2.3),

\[
H_C^{(B)} \equiv (\phi_k^{(B)}(x))^{-1} (H_C^{(B)} - E_k^{(B)}) \phi_k^{(B)}(x)
\]

\[
= \frac{1}{2} \sum_{j=1}^{N} \left( \omega x_j \frac{\partial^2}{\partial x_j^2} + \frac{b}{2x_j} \frac{\partial}{\partial x_j} + \frac{b(t_j - 1)}{2x_j^2} \right) + \frac{1}{2} \delta a \sum_{j \neq k} \left( \frac{2}{x_j^2 - x_k^2} \left( x_j \frac{\partial}{\partial x_j} - x_k \frac{\partial}{\partial x_k} \right) + K_{jk} - 1 \right) \frac{1}{(x_j - x_k)^2} + \frac{1}{(x_j + x_k)^2}
\]

(2.4)

For the transformed Hamiltonian, the ground state wave function is unity and the ground state energy is zero. In what follows, we sometimes call the operator (2.4) the $B_N$-calogero Hamiltonian.

The Dunkl operators associated with the $B_N$-type Weyl group are defined as

\[
\nabla_j \equiv \frac{\partial}{\partial x_j} + \frac{b(1 - t_j)}{x_j} + a \sum_{k(l \neq j)} \left( 1 - \frac{K_{jk}}{x_j - x_k} + \frac{1 - t_j t_k K_{jk}}{x_j + x_k} \right)
\]

(2.5)

The commutation relation with coordinates $x_j$ is given by

\[
[\nabla_j, x_k] = \delta_{jk} \left( 1 + a \sum_{l(\neq j)} (1 + t_j t_l) K_{jl} + 2b t_j \right) - (1 - \delta_{jk}) a(1 - t_j t_k) K_{jk}
\]

(2.6)

We introduce the creation- and annihilation-like operators and Cherednik operators for the $B_N$-Calogero model,

\[
\alpha_j \equiv x_j - \frac{1}{2} \omega \nabla_j, \quad \alpha_j^\dagger \equiv \nabla_j,
\]

(2.7)

\[
d_j \equiv \alpha_j^\dagger \alpha_j + a \sum_{k=j+1}^{N} (1 + t_j t_k) K_{jk} + b t_j
\]

(2.8)

Strictly, the operators $\alpha_j^\dagger$ and $\alpha_j$ are not Hermitian conjugate each other. But we still use the notation $\dagger$ since they have the relation $\phi_k^{(B)} \alpha_j^\dagger (\phi_k^{(B)})^{-1} = (\phi_k^{(B)} \alpha_j (\phi_k^{(B)})^{-1})$ after recovering the effect of the ground state (2.3). Although there exist different definitions for the Cherednik operator, we select the one (2.8) for convenience of later discussions. Commutation relations among these Dunkl-Cherednik operators are given as follows,

\[
[\alpha_j^\dagger, \alpha_k^\dagger] = [\alpha_j, \alpha_k] = [d_j, d_k] = 0,
\]

\[
[\alpha_j, \alpha_k^\dagger] = \delta_{jk} \left( 1 + a \sum_{l(\neq j)} (1 + t_j t_l) K_{jl} + 2b t_j \right) - (1 - \delta_{jk}) a(1 - t_j t_k) K_{jk},
\]

\[
[d_j, \alpha_k^\dagger] = \delta_{jk} \left( \alpha_j^\dagger + a \sum_{l=1}^{j-1} \alpha_l^\dagger (1 + t_j t_l) K_{jl} + a \sum_{l=j+1}^{N} \alpha_l^\dagger (1 - t_j t_l) K_{jl} \right)
\]
where $\Theta(x)$ is the Heaviside function,
\[
\Theta(x) = \begin{cases} 
0, & (x \leq 0), \\
1, & (x > 0).
\end{cases}
\]
The Cherednik operators obey the following relations with the coordinate exchange and reflection operators,
\[
\begin{align*}
d_j K_{j,j+1} - K_{j,j+1}d_j+1 &= a(1 + t_j t_{j+1}), \\
d_{j+1} K_{j,j+1} - K_{j,j+1}d_j &= -a(1 + t_j t_{j+1}), \\
[d_j, K_{j,k+1}] &= 0, \quad (j \neq k, k + 1), \\
[d_j, t_k] &= 0.
\end{align*}
\]
In terms of the Cherednik operator $d_j$, the Hamiltonian $H_{C}^{(B)}$ is rewritten as
\[
H_{C}^{(B)} = \omega \sum_{j=1}^{N} \left( d_j - (N - 1)a - b \right). \tag{2.11}
\]
It is well-known that the power sums of the Cherednik operators provide the mutually commuting conserved operators for the $B_N$-Calogero model since $\{d_j\}$ mutually commute, $[d_j, d_k] = 0$. Equation (2.11) shows that the Hamiltonian (2.4), which is one of the conserved operators, is written by the commuting Cherednik operators. Thus the Cherednik operators $\{d_j\}$ themselves are the conserved operators for the $B_N$-Calogero model with distinguishable particles. Because the Cherednik operators and the reflection operators mutually commute (2.10), the simultaneous eigenfunction of the Cherednik operators is, at the same time, the simultaneous eigenfunction of the reflection operators. In other words, the parities for each variable are good quantum numbers. As we shall see shortly in the definition of the simultaneous eigenfunction (2.12a), this property appears as a restriction on a parity for a variable.

All the eigenfunctions are labeled by the symbol $\lambda_\sigma$ (composition) consisting of a partition $\lambda$ and a distinct permutation $\sigma \in S_N$. A partition $\lambda$ is a sequence of $N$ nonnegative integers,
\[
\lambda = \{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0\}.
\]
Distinct permutations $\sigma$ and $\tau$ must satisfy $\lambda_{\sigma(j)} \neq \lambda_{\tau(j)}$ for some $j \in \{1, 2, \cdots, N\}$. To define the nonsymmetric multivariable Laguerre polynomials, we introduce the Bruhat order $\prec$,
\[
\mu \prec \lambda \iff \begin{cases} 
\mu^D = 1 & (\mu < \lambda), \\
\mu = \lambda \text{ when } \mu = \lambda \text{ then the first non-vanishing difference } \\
\tau(j) - \sigma(j) > 0, 
\end{cases}
\]
where the symbol $\prec$ is the dominance order,
\[
\mu \prec \lambda \iff \mu \neq \lambda, |\mu| = |\lambda| \text{ and } \sum_{k=1}^{l} \mu_k \leq \sum_{k=1}^{l} \lambda_k,
\]
for all $l = 1, 2, \cdots, N$. A set of indistinct permutations for a partition $\lambda$,
\[
\{\sigma\} \equiv \{\tau \in S_N | \lambda_\tau = \lambda_\sigma \text{ and } \lambda_\tau \prec \lambda_\sigma \text{ for } \tau \neq \sigma\},
\]
is represented by the permutation $\sigma$ which gives the minimum in the sense of the Bruhat order $\prec$.

The nonsymmetric multivariable Laguerre polynomial, $l_{\lambda_\sigma}(x; 1/a, 1/b, \omega)$, is the nondegenerate simultaneous eigenfunction of the Cherednik operators $\{d_j\}$ with the coefficient of its top term $x^{\lambda_\sigma} \equiv x_1^{\lambda_{\sigma(1)}} x_2^{\lambda_{\sigma(2)}} \cdots x_N^{\lambda_{\sigma(N)}}$ conventionally taken to be unity,
\[
l_{\lambda_\sigma}(x; 1/a, 1/b, \omega) = x^{\lambda_\sigma} + \sum_{\mu \prec \lambda} w_{\lambda_\sigma, \mu} (a, b, 1) x^{\mu},
\]
and $^B\mu \prec \lambda_\sigma (\nu \neq \lambda_\sigma(k) \mod 2)$ when $\mu_{\tau(k)} < \lambda_{\sigma(k)}$ (2.12a)
\[
d_j l_{\lambda_\sigma}(x; 1/a, 1/b, \omega) = \lambda_{\sigma(j)} l_{\lambda_\sigma}(x; 1/a, 1/b, \omega), \tag{2.12b}
\]
where
\[
\tilde{\lambda}_{\sigma(j)} \equiv \lambda_{\sigma(j)} + 2a \left( \# \{ 1 \leq k < j | \lambda_{\sigma(k)} < \lambda_{\sigma(j)} \} \right) + b.
\]
From eq. (2.12b), we see that each monomial, $x^{\lambda_\sigma}$ and $x^{\mu}$, in the nonsymmetric multivariable Laguerre polynomial has the same parity for a variable. The energy eigenvalue of the eigenfunction $l_{\lambda_\sigma}$ is given by
\[
H_{C}^{(B)} l_{\lambda_\sigma} = \sum_{k=1}^{N} \lambda_{\sigma(k)} l_{\lambda_\sigma}. \tag{3.1}
\]
The nonsymmetric multivariable Laguerre polynomials span the orthogonal basis for the $B_N$-Calogero model with distinguishable particles.

\section{3. Rodrigues Formula for the Nonsymmetric Multivariable Laguerre Polynomial}

Generalizing the method developed by Takamatsu and Takano for the nonsymmetric Jack polynomial, $\mathcal{E}^{(B)}$ we shall present the Rodrigues formula for the nonsymmetric multivariable Laguerre polynomial.

We introduce two types of operators. The first type is the braid-exclusion operator for the $B_N$-Calogero model,
\[
X_{j,j+1} \equiv \left[ d_j, K_{j,j+1} \right], \tag{3.1}
\]
which satisfies the following relations,
\[
\begin{align*}
d_j X_{j,j+1} - X_{j,j+1}d_j &= 0, \\
d_{j+1} X_{j,j+1} - X_{j+1,j}d_j &= 0, \\
[d_j, X_{k,k+1}] &= 0, \quad (j \neq k, k + 1), \\
t_j X_{j,j+1} - X_{j+1,j}t_j &= 0, \\
t_{j+1} X_{j,j+1} - X_{j,j+1}t_{j+1} &= 0,
\end{align*}
\]
where the symbol $\prec$ is the dominance order,
\[
\mu \prec \lambda \iff \begin{cases} 
\mu^D = 1 & (\mu < \lambda), \\
\mu = \lambda \text{ when } \mu = \lambda \text{ then the first non-vanishing difference } \\
\tau(j) - \sigma(j) > 0, 
\end{cases}
\]
where the symbol $\prec$ is the dominance order,
\[
\mu \prec \lambda \iff \begin{cases} 
\mu^D = 1 & (\mu < \lambda), \\
\mu = \lambda \text{ when } \mu = \lambda \text{ then the first non-vanishing difference } \\
\tau(j) - \sigma(j) > 0, 
\end{cases}
\]
where the symbol $\prec$ is the dominance order,
With the help of eqs. (3.5), we have the relations, respectively. The operator 
be even in each variable 
be introduced by Baker and Forrester. 
The above definitions (3.4) are different from those introduced by Baker and Forrester. 
We do not impose the restriction to eigenfunctions such that they should be even in each variable \( x_j \). The Knop-Sahi operator is related to eqs. (2.8), (2.1) and (2.2),

\[
\begin{align*}
    d_j e^t &= e^t d_j + 1, \\
    d_N e^t &= e^t (d_1 + 1), \\
    X_{j,j+1} e^t &= e^t X_{j+1,j} + 2, \\
    X_{N-1,N} e^t &= (e^t)^2 X_{1,2}, \\
    t_j e^t &= e^t t_j + 1, \\
    t_N e^t &= -e^t t_1.
\end{align*}
\]

Next, using the braid-exclusion and Knop-Sahi operators, we define the constituent operators,

\[
\begin{align*}
    b_j^\dagger &= X_{j,j+1} \cdots X_{N-1,N} e^t, & (j = 1, \cdots, N - 1), \\
    b_N^\dagger &= e^t, \\
    b_j^\dagger &= (b_j^\dagger)^t = e X_{N-1} \cdots X_{j,j+1}, & (j = 1, \cdots, N - 1), \\
    b_N^\dagger &= (b_N^\dagger)^t = e.
\end{align*}
\]

With the help of eqs. (2.5), we have the relations,

\[
\begin{align*}
    d_j b_k^\dagger &= \begin{cases} 
        b_k^\dagger d_j + 1, & (1 \leq j \leq k - 1), \\
        b_k^\dagger (d_1 + 1), & (j = k), \\
        b_k^\dagger d_j, & (k + 1 \leq j \leq N),
    \end{cases} \\
    X_{j,j+1} b_k^\dagger &= \begin{cases} 
        b_k^\dagger X_{j,j+1}, & (1 \leq k \leq j - 1), \\
        b_k^\dagger (d_1 + d_j - 2a^2 (1 - t_j t_{j+1})), & (k = j), \\
        b_k^\dagger, & (k = j + 1), \\
        b_k^\dagger X_{j+1,j+2}, & (j + 2 \leq k \leq N),
    \end{cases}
\end{align*}
\]

In the end, we define the raising and lowering operators for the \( B_N \)-Calogero model as follows,

\[
\begin{align*}
    a_j^\dagger &\equiv (b_j^\dagger)^t, \\
    a_j &\equiv (b_j)^t.
\end{align*}
\]

The commutation relations with the Cherednik operator \( d_j \) are simply expressed by

\[
\begin{align*}
    [d_j, a_k^\dagger] &= \begin{cases} 
        0, & (j > k), \\
        a_k^\dagger, & (j \leq k),
    \end{cases} \\
    [a_j, a_k^\dagger] &= [a_j, a_k] = 0.
\end{align*}
\]

Comparing eq. (3.11) with eq. (2.12b), we identify \( \tilde{\sigma}(x) \) as the eigenstate with the composition \( \lambda_{1d} \).

We consider the eigenstates with the general composition \( \lambda_\sigma \). The first and second relations of eqs. (3.2) show that the braid-exclusion operator \( X_{j,j+1} \) acts on an eigenstate to exchange the eigenvalue of \( d_j \) (or \( d_{j+1} \)) for that of \( d_{j+1} \) (or \( d_j \)). For a composition \( \lambda_\sigma \) where the distinct permutation \( \sigma \) is expressed by the product of transpositions as

\[
\sigma = (k_1, k_1 + 1) \cdots (k_2, k_2 + 1) (k_1, k_1 + 1),
\]

the Rodrigues formula for the nonsymmetric eigenfunction with the composition \( \lambda_\sigma \) is presented as follows,

\[
\tilde{\sigma}_\lambda (x) = X_{k_1,k_1+1} X_{k_2,k_2+1} \cdots X_{k_i,k_i+1} \times (a_1^\dagger)^{\lambda_1} - (a_2^\dagger)^{\lambda_2} - \cdots - (a_N^\dagger)^{\lambda_N} \langle 0 |, (3.12)
\]

and the eigenvalue is

\[
\lambda_{\sigma(j)} \tilde{\sigma}_\lambda. (3.13)
\]

Since the nondegenerate eigenfunction \( \tilde{\sigma}_\lambda \) have the same eigenvalue of the nonsymmetric multivariable Laguerre polynomial \( l_{\lambda\sigma} (2.12a) \), we conclude that \( \tilde{\sigma}_{\lambda_\sigma} (x) \) is identified with \( \tilde{\sigma}_\lambda (x) \) up to normalization. \( \tilde{\sigma}_\lambda (x) \propto \tilde{\sigma}_{\lambda_\sigma} (x) \).

We note that the eigenstate (3.13) is also the simultaneous eigenstate of the reflection operators \( \{ t_j \} \) since the Cherednik operator \( d_j \) commutes with all \( t_k \), i.e. \([d_j, t_k] = 0\). So the parity for a variable \( x_j \) of the nonsymmetric multivariable Laguerre polynomial is restricted to even or odd as we have explained in the previous section. It is possible that the parity for a variable \( x_j \) is different from the one for a variable \( x_k \) (\( k \neq j \)) this means that the eigenstate we have constructed is...
allowed to be the functions whose parities are different for each variable. After symmetrization on the variables, the eigenstate including different parities for each variable vanishes and the eigenstate with the single parity for all variables is left. The result reproduces the one by Baker and Forrester [3].

§4. Norm Formula

In this section we shall calculate the norm of the eigenfunction obtained in the previous section. The Knop-Sahi operators for the $B_N$-Calogero model have the following properties,

$$e^\dagger e = d_N - bt_N, \quad ee^\dagger = d_1 + bt_1 + 1.$$ 

From the definition (3.8) and the above relations, it is easy to verify

$$b_j^\dagger b_j = (d_j - bt_j) \prod_{k=j+1}^N ((d_j - dk)^2 - 2a^2(1 + tjtk)), $$

$$b_j b_j^\dagger = (d_1 + bt_j + 1) \prod_{k=j+1}^N ((d_k - d_1 - 1)^2 - 2a^2(1 - tk_1)). $$

Owing to the definition of the raising and lowering operators (3.8), we have the following expressions of the number-like operator,

$$a_j^\dagger a_j = \prod_{k=1}^j (d_k - bt_k) \prod_{l=j+1}^N ((d_k - dl)^2 - 2a^2(1 + tktl)), $$

$$a_j a_j^\dagger = \prod_{k=1}^j (d_k + bt_k + 1) \prod_{l=j+1}^N ((d_k - dl + 1)^2 - 2a^2(1 - tktl)). $$

We recall the definitions of the inner product and the norm for the $B_N$-Calogero model,

$$\langle f, g \rangle \overset{\text{def}}{=} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dx_1 \cdots dx_N |\phi_g^{(B)}| \phi_f^{(B)}(x_1, \ldots, x_N), $$

$$|f|^2 \overset{\text{def}}{=} \langle f, f \rangle. $$

The norm of the eigenfunction $\tilde{l}_\lambda$ with the composition $\lambda = \lambda_{id}$ is calculated as follows,

$$\langle \tilde{l}_\lambda, \tilde{l}_\lambda \rangle = \prod_{l=1}^N \prod_{m=1}^{\lambda_l} \prod_{r=1}^{l - \lambda_l + 1} \left(\lambda_m + (-1)^{\lambda_m - r} b - r + 1\right) \times \prod_{k=l+1}^N \left(\lambda_m - \lambda_k - r + 1\right)^2 - 2a^2(1 - (-1)^{\lambda_m + \lambda_k - r})\right]. $$

Thus we can recursively calculate the norms for all the eigenstates.

§5. Concluding Remarks

Generalizing the Takamura-Takano method, we have investigated the $B_N$-Calogero model with distinguishable particles and algebraically constructed the nonsymmetric eigenfunctions for the model. In other words, we have presented the Rodrigues formula for the nonsymmetric multivariable Laguerre polynomial. The formula provides us with not only known eigenfunctions which are even in all variables, but also those which have different parities for each variable.

So far, only nonsymmetric multivariable Laguerre polynomials with even parities for all variables have been considered. Under the restriction on parities, the Cherednik operators for the $B_N$-Calogero model can be mapped to those for the $A_{N-1}$-Sutherland model. That is, only the nonsymmetric multivariable Laguerre polynomials which can be mapped to the well-investigated $(A_{N-1})$-Jack polynomials have been studied. In this paper, we apply an algebraic method to the whole nonsymmetric multivariable Laguerre polynomials with no restriction on parities, which cannot generally be mapped to the nonsymmetric $(A_{N-1})$-Jack polynomials, for the first time.

The Calogero model with distinguishable particles we have considered in this paper is mapped to the spin Calogero model by replacing the coordinate exchange operator $K_{jk}$ with the spin exchange operator $P_{jk}$ and identifying the reflection operators as unity. This mapping is possible when the whole eigenfunction is symmetric (or anti-symmetric) under the exchange of particles, i.e. $K_{jk} P_{jk} = 1$ (or $-1$), and when the parities for each coordinate variable are even. The eigenfunction we have constructed corresponds to the orbital part of the whole one. To obtain the whole eigenfunction of the spin Calogero model, we have to take an appropriate linear combination of products of the orbital and spin parts. We note that the particle exchange operators $K_{jk} P_{jk}$ and the reflection operators $t_j$ do not generally com-
mate, \( t_j K_{jk} P_{jk} = K_{jk} P_{jk} t_k \), which is the reason why the nonsymmetric multivariable Laguerre polynomials with different parities for each coordinate variable have not been studied. However the abandoned eigenstate is necessary when we solve the diagonalization problem for a more general \( B_N \)-Calogero model explicitly including the reflection operators in the Hamiltonian.

In the limit \( \omega \rightarrow \infty \), the nonsymmetric multivariable Laguerre polynomial reduces to the nonsymmetric \( B_N \)-Jack polynomial discussed in previous papers.\(^{30, 31}\) We should note that the Rodrigues formula and an algebraic calculation of the norm of the nonsymmetric \( B_N \)-Jack polynomials are given in a parallel way to those of the nonsymmetric multivariable Laguerre polynomial. The nonsymmetric \( B_N \)-Jack polynomials diagonalize the Cherednik operators

\[
D_j = x_j \nabla_j + a \sum_{k=j+1}^{N} (1 + t_j t_k) K_{jk} + b t_j,
\]

whose physical significance is not clear at the moment.

An algebraic approach is very useful for construction of eigenstates and calculation of physical quantities, because of the connection with the concept of field theory. It is interesting to verify whether the symmetrized Takamura-Takano raising operators agree with the ones constructed by the Lapointe-Vinet method\(^{8, 9}\) which is left for future study.

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