A Functional Approach to the
Heat Kernel in Curved Space

L. Martin
D.G.C. McKeon
Department of Applied Mathematics
University of Western Ontario
London
CANADA
N6A 5B7

e-mail: LCM@APMATHS.UWO.CA
TMLEAFS@APMATHS.UWO.CA
Tel: (519)679-2111, ext. 8789
Fax: (519)661-3523
PACS No.: 11.20Dj
1 Abstract

The heat kernel $M_{xy} = \langle x | \exp \left[ \frac{1}{\sqrt{g}} \partial_{\mu} g^{\mu\nu} \sqrt{g} \partial_{\nu} \right] t | y \rangle$ is of central importance when studying the propagation of a scalar particle in curved space. It is quite convenient to analyze this quantity in terms of classical variables by use of the quantum mechanical path integral; regrettably it is not entirely clear how this path integral can be mathematically well defined in curved space. An alternate approach to studying the heat kernel in terms of classical variables was introduced by Onofri. This technique is shown to be applicable to problems in curved space; an unambiguous expression for $M_{xy}$ is obtained which involves functional derivatives of a classical quantity. We illustrate how this can be used by computing $M_{xx}$ to lowest order in the curvature scalar $R$.

2 Introduction

The heat kernel $\langle x | e^{-Ht} | y \rangle$ for an elliptic operator $H$ is a fundamental quantity in quantum field theory [1,2]. Its expansion in powers of $t$ can be used to analyze the divergence structure of a theory [2,3]; one can also expand in powers of the background field on which $H$ depends to obtain Green’s functions [1,4].

In both cases, the quantum mechanical path integral is a useful means of representing the heat kernel, for then only classical variables need to be manipulated in performing calculations [5,6,7]. At a practical level, when computing Green’s functions using this technique, loop-momentum integrals are avoided and algebraically complicated vertices in gauge theories
are not encountered. The application of the quantum mechanical path integral to problems in curved space [8,9] is unfortunately hampered by the difficulties encountered in defining unambiguously the quantum mechanical path integral in curved space [10-13].

Fortunately, an alternate way of representing the heat kernel in terms of classical variables can be developed using an approach initiated by Onofri [14]. This technique has been shown to be a viable way of computing Green’s functions in flat space by using it to evaluate the one-loop vacuum polarization in scalar electrodynamics [15]. It retains all the advantages of the quantum mechanical path integral. In this paper, it is shown that one can use Onofri’s method to handle the heat kernel in $D$ dimensional curved space that arises when one considers the propagation of scalars in which case

$$H = \frac{1}{2} \frac{1}{\sqrt{g}} p_\mu (g^{\mu \nu} \sqrt{g}) p_\nu . \quad (p = -i \partial) \quad (1)$$

We use a normal coordinate expansion [16] of the metric $g_{\mu \nu}$, following the approach initially used in the non-linear sigma model [17]. The general formalism is developed, and the diagonal elements of the heat kernel is computed to leading order in the Riemann scalar $R$.

3 The Heat Kernel

We begin with the normal coordinate expansion of the metric tensor $g_{\mu \nu}$ about a point $x_0$ where the metric is flat [16]:

$$g_{\mu \nu} (x_0 + \pi (q)) = \eta_{\mu \nu} + \frac{1}{3} R_{\mu \alpha \beta \nu} (x_0) q^\alpha q^\beta + \frac{1}{6} R_{\mu \alpha \beta \nu \gamma} (x_0) q^\alpha q^\beta q^\gamma$$

$$+ \left( \frac{1}{20} R_{\mu \alpha \beta \nu \gamma \delta} (x_0) + \frac{2}{45} R_{\mu \alpha \beta \sigma} (x_0) R_{\gamma \delta \nu} (x_0) \right) q^\alpha q^\beta q^\gamma q^\delta + \ldots \quad (2)$$
so that \( H \) defined in (1) becomes, to at most fourth order in \( y^\alpha \),

\[
H = \frac{1}{2} \left[ 1 + \frac{1}{6} R_{\alpha \beta} q^\alpha q^\beta + \frac{1}{12} R_{\alpha \beta \gamma \delta} q^\alpha q^\beta q^\gamma + \left( \frac{1}{72} R_{\alpha \beta} R_{\gamma \delta} + \frac{1}{40} R_{\alpha \beta \gamma \delta} \right) \right. \\
+ \frac{1}{180} R^\lambda_{\alpha \beta \sigma} R_{\lambda \gamma \delta} q^\alpha q^\beta q^\gamma q^\delta \left]_p \eta^{\mu \nu} + \left( -\frac{1}{6} R_{\alpha \beta} \eta^{\mu \nu} \\
+ \frac{1}{3} R^\mu_{\alpha} \nu_\beta \right) q^\alpha q^\beta \right. \\
+ \left( -\frac{1}{12} \eta^{\mu \nu} R_{\alpha \beta \gamma} + \frac{1}{6} R^\mu_{\alpha} \nu_\beta \gamma \right) q^\alpha q^\beta q^\gamma \\
+ \left( \eta^{\mu \nu} \left( \frac{1}{72} R_{\alpha \beta} R_{\gamma \delta} - \frac{1}{20} R_{\alpha \beta \gamma \delta} - \frac{1}{90} R^\lambda_{\alpha \beta \kappa} R_{\lambda \gamma \delta} \right) \right. \\
\left. + \frac{1}{20} R^\mu_{\alpha} \nu_\beta \gamma \delta + \frac{1}{15} R^\mu_{\alpha} \nu_\beta \gamma \delta \right) q^\alpha q^\beta q^\gamma q^\delta \left]_{p \nu} . \tag{3}
\]

(We define \( R \equiv R^\alpha_{\alpha} \) where \( R_{\alpha \beta} \equiv R^\mu_{\alpha \mu \beta} \).) If we now express \( H \) in the form

\[
H = \frac{1}{2} \left[ p^2 + M(q) p^2 + p_\mu N^{\mu \nu}(q) p_\nu + M(q) p_\mu N^{\mu \nu}(q) p_\nu \right] , \tag{4}
\]

then it is possible to employ the techniques of [14], as used in [15]. We first use the relation

\[
e^{A+B} = e^A T \exp \int_0^1 d\tau \left[ e^{-A \tau} B e^{A \tau} \right] \tag{5}
\]

\((T - \text{path ordering})\)

to write the matrix element \( M_{xy} \) as

\[
<x \mid e^{-Ht} \mid y > = <x \mid e^{-\frac{1}{2} p^2 T} \exp \left( -\frac{t}{2} \int_0^1 d\tau \left[ e^{\frac{1}{2} p^2 \tau} \left( Mp^2 + p_\mu N^{\mu \nu} p_\nu \right) + Mp_\mu N^{\mu \nu} p_\nu e^{-\frac{1}{2} p^2 \tau} \right] \right) \mid y > . \tag{6}
\]

Since we have the identity

\[
e^A f(B)e^{-A} = f(e^A, B) = f(B + \frac{1}{1!}[A, B] + \frac{1}{2!}[A, [A, B]] + \ldots) , \tag{7}
\]
we can rewrite $M_{xy}$ in (6) as

$$M_{xy} = \langle x | e^{-\frac{t}{2}p^2 T} \exp - \frac{t}{2} \int_0^1 d\tau \left[ M(q - it\tau p^2 + p_\mu N^{\mu\nu}(q - it\tau p)p_\nu \\
+ M(q - it\tau p)p_\mu N^{\mu\nu}(q - it\tau p)p_\nu \right] | y \rangle$$

(8)

since

$$[q, p] = i.$$ (9)

Insertion of appropriate complete sets of states into (8) now gives us

$$M_{xy} = \frac{1}{(2\pi t)^{D/2}} \int dx' < x' | e^{-(x-q)^2/2t} T \exp - \frac{t}{2} \int_0^1 d\tau \left[ M(q - it\tau p^2 \\
+ p_\mu N^{\mu\nu}(q - it\tau p)p_\nu + M(q - it\tau p)p_\mu N^{\mu\nu}(q - it\tau p)p_\nu \right] \right. e^{(x-q)^2/2t} | y >.$$ (10)

Again employing (7), we rewrite (10) in the form

$$M_{xy} = \frac{e^{-(x-y)^2/2t}}{(2\pi t)^{D/2}} \int dx' < x' | T \exp - \frac{t}{2} \int_0^1 d\tau \left[ M(q(1 - \tau) - it\tau p + \tau x) \left( p + \frac{i(x-q)}{t} \right)^2 \\
+ \ldots \right] | y >.$$ (11)

$$= \frac{e^{-(x-y)^2/2t}}{(2\pi t)^{D/2}} < p = 0 | e^{ip\cdot y} T \exp - \frac{t}{2} \int_0^1 d\tau \left[ M(q(1 - \tau) - it\tau p + \tau x) \left( p + \frac{i(x-q)}{t} \right)^2 \\
+ \ldots \right] e^{-ip\cdot y} | q = 0 >.$$ (12)

A last application of (7) converts (12) to

$$M_{xy} = \frac{e^{-(x-y)^2/2t}}{(2\pi t)^{D/2}} < p = 0 | T \exp - \frac{t}{2} \int_0^1 d\tau \left[ M(y(\tau) + q(1 - \tau) - it\tau p) \\
\left( \Delta + p - \frac{iq}{t} \right)^2 + \left( \Delta + p - \frac{iq}{t} \right) N^{\mu\nu}(y(\tau) + q(1 - \tau) - it\tau p) \right]$$
\[
\left(\Delta + p - \frac{iq}{t}\right)_{\nu} + M \left(y(\tau) + q(1 - \tau) - it\tau p\right) \left(\Delta + p - \frac{iq}{t}\right)_{\mu}
\]

\[
N_{\mu\nu} \left(y(\tau) + q(1 - \tau) - it\tau p\right) \left(\Delta + p - \frac{iq}{t}\right)_{\nu} \bigg| q = 0 >
\]

where \(y(\tau) = y + (x - y)\tau\) and \(\Delta = \frac{i}{\tau}(x - y)\). An expansion in (13) of the functions \(M\) and \(N_{\mu\nu}\) about \(y(\tau)\) followed by an expansion of the path-ordered exponential along the lines of [14] leads to

\[
M_{xy} = \frac{e^{-(x-y)^2/2t}}{(2\pi t)^{D/2}} \sum_{N=0}^{\infty} \left(-\frac{t}{2}\right)^N \int_0^1 d\tau_1 \int_0^{\tau_1} d\tau_2 \ldots \int_0^{\tau_{N-1}} d\tau_N
\]

\[
\left\{ \left[ M \left(\frac{\delta y(\tau_1)}{\delta \alpha(\tau_1)}\right) \left(\Delta + \frac{\delta}{\delta \xi(\tau_1)}\right) \cdot \left(\Delta + \frac{\delta}{\delta \xi(\tau_1)}\right) \right] + \left(\Delta + \frac{\delta}{\delta \xi(\tau_1)}\right) N_{\mu\nu} \left(\frac{\delta y(\tau_1)}{\delta \beta(\tau_1)}\right) \left(\Delta + \frac{\delta}{\delta \xi(\tau_1)}\right)_{\nu}\right\}
\]

\[
\ldots \left[ M \left(\frac{\delta y(\tau_N)}{\delta \alpha(\tau_N)}\right) \left(\Delta + \frac{\delta}{\delta \xi(\tau_N)}\right) \cdot \left(\Delta + \frac{\delta}{\delta \xi(\tau_N)}\right) \right] + \left(\Delta + \frac{\delta}{\delta \xi(\tau_N)}\right) N_{\mu\nu} \left(\frac{\delta y(\tau_N)}{\delta \beta(\tau_N)}\right) \left(\Delta + \frac{\delta}{\delta \xi(\tau_N)}\right)_{\nu}\right\}
\]

\[
+ M \left(\frac{\delta y(\tau_N)}{\delta \alpha(\tau_N)}\right) \left(\Delta + \frac{\delta}{\delta \xi(\tau_N)}\right) N_{\mu\nu} \left(\frac{\delta y(\tau_N)}{\delta \beta(\tau_N)}\right) \left(\Delta + \frac{\delta}{\delta \xi(\tau_N)}\right)_{\nu}\right\}
\]

\[
< p = 0 | e^{\alpha(\tau_1)-(P_1^{a} + Q_1^{b})} e^{\xi(\tau_1)-(P_1^{a} + Q_1^{b})} e^{\beta(\tau_1)-(P_1^{a} + Q_1^{b})} e^{\zeta(\tau_1)-(P_1^{a} + Q_1^{b})}
\]

\[
\ldots e^{\alpha(\tau_N)-(P_N^{a} + Q_N^{b})} e^{\xi(\tau_N)-(P_N^{a} + Q_N^{b})} e^{\beta(\tau_N)-(P_N^{a} + Q_N^{b})} e^{\zeta(\tau_N)-(P_N^{a} + Q_N^{b})} | q = 0 >
\]

when \(\alpha = \beta = \xi = \zeta = 0\). We have defined

\[
Q_1^{a} = q(1 - \tau_1)
\]

\[
P_1^{a} = -it\tau p
\]
\( Q^b_i = -iq/t \) \hspace{1cm} (17)

and

\[ I^b_P = p \] \hspace{1cm} (18)

If now \( [Q^c_i, I^d_P] = C^{(c,d)}_{(i,j)} \) with \( C^{(s)}_{(c,d)} = C^{(c,d)}_{(i,j)} \) \( (i < j) \) and \( C^{(s)}_{(c,d)} = C^{(d,c)}_{(j,i)} \), then we can use the result from (14);

\[
< p = 0 | e^{\lambda_1(P_1+Q_1)} \ldots e^{\lambda_N(P_N+Q_N)} | q = 0 > = \exp \frac{1}{2} \sum_{i,j} \left( C^{(s)}_{ij} \lambda_i \lambda_j \right) .
\] \hspace{1cm} (19)

This leaves us with

\[
M_{xy} = \frac{e^{-|x-y|^2/4t}}{(2\pi t)^{D/2}} \exp \frac{1}{2} \int_0^1 dt_1 \int_0^1 dt_2 \left[ \right.
\left. \frac{\delta}{\delta \alpha(t_1)} \cdot \frac{\delta}{\delta \alpha(t_2)} + 2 \frac{\delta}{\delta \beta(t_1)} \cdot \frac{\delta}{\delta \beta(t_2)} \right]
\left. + \frac{\delta}{\delta \xi(t_1)} \cdot \frac{\delta}{\delta \xi(t_2)} + 2i \frac{\delta}{\delta \xi(t_1)} \cdot \frac{\delta}{\delta \xi(t_2)} \right]
\left. + \frac{\delta}{\delta \zeta(t_1)} \cdot \frac{\delta}{\delta \zeta(t_2)} + 2i \left( \theta_1(t_1 - t_2) - t_1 \right) \right]
\exp \left. - \frac{t}{2} \int_0^1 d\tau \left[ M(y(\tau) + \alpha(\tau))(\Delta + \xi(\tau)) \cdot (\Delta + \zeta(\tau)) + \frac{\delta}{\delta \beta(t_1)} \cdot \frac{\delta}{\delta \beta(t_2)} \right]
\left. + \Delta + \xi(\tau) \right)_\mu N^{\mu\nu}(y(\tau) + \beta(\tau))(\Delta + \zeta(\tau))_\nu
\left. + M(y(\tau) + \alpha(\tau))(\Delta + \xi(\tau))_\mu N^{\mu\nu}(y(\tau) + \beta(\tau))(\Delta + \zeta(\tau))_\nu \right],
\] \hspace{1cm} (20)

where \( t_\prec = \min(t_1, t_2), t_\succ = \max(t_1, t_2) \) and \( \theta_a(\tau) = 1(\tau > 0), = 0(\tau < 0), = a(\tau = 0) \). In (20), we have an unambiguous closed form expression for the heat kernel in terms of classical variables, a feature it shares with the path integral representation of the heat kernel.
For purposes of illustration, let us compute the contribution to $M_{xx}$ that is linear in the Riemann scalar $R$. Those terms in (20) needed for this are

$$M^{(R)}_{xx} = \frac{1}{(2\pi t)^{D/2}} \left( \frac{1}{2!} \right) \int_0^1 dt_1 dt_2 \int_0^1 dt'_1 dt'_2 \left\{ \left[ t \langle 1 - t \rangle \left( \frac{\delta}{\delta \alpha(t_1)} \cdot \frac{\delta}{\delta \alpha(t_2)} + 2 \frac{\delta}{\delta \alpha(t_1)} \cdot \frac{\delta}{\delta \beta(t_2)} + \frac{\delta}{\delta \beta(t_1)} \cdot \frac{\delta}{\delta \beta(t_2)} \right) \right] 
\left[ \frac{1}{t} \left( \frac{\delta}{\delta \xi(t'_1)} \cdot \frac{\delta}{\delta \xi(t'_2)} + 2 \frac{\delta}{\delta \xi(t'_1)} \cdot \frac{\delta}{\delta \zeta(t'_2)} + \frac{\delta}{\delta \zeta(t'_1)} \cdot \frac{\delta}{\delta \zeta(t'_2)} \right) \right] 
\right. 
\left. + (2i)^2 \left[ (\theta_1(t_1 - t_2) - t_1) \left( \frac{\delta}{\delta \alpha(t_1)} \cdot \frac{\delta}{\delta \alpha(t_2)} \cdot \frac{\delta}{\delta \alpha(t_1)} \cdot \frac{\delta}{\delta \beta(t_2)} + \frac{\delta}{\delta \beta(t_1)} \cdot \frac{\delta}{\delta \beta(t_2)} \right) 
\right. 
\left. + (\theta_0(t_1 - t_2) - t_1) \left( \frac{\delta}{\delta \beta(t_1)} \cdot \frac{\delta}{\delta \xi(t_2)} \right) \right] 
\left[ (\theta_1(t'_1 - t'_2) - t'_1) \left( \frac{\delta}{\delta \alpha(t'_1)} \cdot \frac{\delta}{\delta \alpha(t'_2)} \cdot \frac{\delta}{\delta \alpha(t'_1)} \cdot \frac{\delta}{\delta \beta(t'_2)} + \frac{\delta}{\delta \beta(t'_1)} \cdot \frac{\delta}{\delta \beta(t'_2)} \right) 
\right. 
\left. + (\theta_0(t'_1 - t'_2) - t'_1) \left( \frac{\delta}{\delta \beta(t'_1)} \cdot \frac{\delta}{\delta \xi(t'_2)} \right) \right] \right\} \right\}.$$

It is a straightforward exercise to compute the functional derivatives in (21) and to then evaluate all remaining integrals to obtain

$$M^{(R)}_{xx} = \frac{1}{(2\pi t)^{D/2}} \left( \frac{Rt}{12} \right)$$

in agreement with [2,9]. Further contributions can be similarly evaluated.
4 Discussion

The heat kernel associated with the propagation of a scalar particle in curved space can be expressed in terms of classical variables in several ways. One involves writing the heat kernel in terms of the quantum mechanical path integral; this unfortunately is not a straightforward exercise and a number [10-13] of different effective actions for the path integral have been proposed. (Those effective actions which are derived by a non-covariant time-slicing are in fact not covariant themselves.) An alternate approach to expressing the heat kernel in curved space in terms of classical variables has been developed in the preceding section. It is unambiguous and easily used in practical calculations as it only involves evaluating functional derivatives of an effective action and computing some elementary integrals. We shall endeavour to illustrate how this approach can be used to facilitate higher loop corrections in quantum gravity.

5 Acknowledgements

We would like to thank F. Dilkes for useful discussion. NSERC provided financial support. R. and D. MacKenzie made a useful suggestion and M. Harris provided motivation for this work.

References

[1] J. Schwinger, Phys. Rev. 82 (1951) 664.
[2] B. DeWitt, Phys. Rev. 162 (1967) 1239.

[3] R.T. Seeley, Amer. Math. Soc. 10 (1967) 228.

[4] D.G.C. McKeon and T.N. Sherry, Phys. Rev. Lett. 59 (1987) 532.

[5] A. Polyakov, “Gauge Fields and Strings” Horwood Academic Publishers, Chur (1987).

[6] M. Strassler, Nucl. Phys. B385 (1992) 145

M. Schmidt and C. Schubert, Phys. Lett. B318 (1993) 438.

[7] D.G.C. McKeon, Can. J. Phys. 70 (1992) 652.

[8] D.G.C. McKeon and S.K. Wong, J. Math. Phys. 36 (1995) 1691.

[9] D.G.C. McKeon and F.A. Dilkes, Phys. Rev. D (to be published).

[10] B. DeWitt, Rev. Mod. Phys. 29 (157) 377.

[11] M. Omate, Nucl. Phys. B120 (1977) 325

K.S. Chen, J. Math. Phys. 13 (1972) 1723

J.S. Dowker, J. Phys. A7 (1974) 1256

G.A. Ringwood, J. Phys. A9 (1976) 1253.

[12] F. Bastianelli, Nucl. Phys. B376 (1992) 113.

[13] J. de Boer, B. Peters, K. Skenderis and P. van Niewenhuizen, Nucl. Phys. B446 (1995) 211.

[14] E. Onofri, Amer. J. Phys. 46 (1978) 379.
[15] D.G.C. McKeon, Can. J. Phys. (in press).

[16] V.I. Petrov, “Einstein Spaces” Pergamon, Oxford (1969).

[17] J. Honerkamp, Nucl. Phys. B36 (1972) 130

L. Alvarez-Gaume, D.Z. Freedman and S. Mukhi, Ann. Phys. 134 (1981) 85.