MICROLOCAL APPROACH TO LUSZTIG’S SYMMETRIES

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To Victor Ginzburg on his 60th birthday

ABSTRACT. We reformulate the De Concini – Toledano Laredo conjecture about the monodromy of the Casimir connection in terms of a relation between Lusztig’s symmetries of quantum group modules and the monodromy in the vanishing cycles of factorizable sheaves.

1. INTRODUCTION

1.1. Let \( g \) be a semisimple Lie algebra, and \( \mathfrak{h} \) a Cartan subalgebra. Let \( \mathfrak{h}^{\text{reg}} \subset \mathfrak{h} \) be the complement to the root hyperplanes arrangement. For an integrable \( g \)-module \( V \), C. De Concini and C. Procesi [7] have introduced an integrable Casimir connection with coefficients in the trivial vector bundle \( V \otimes \mathcal{O}_{\mathfrak{h}^{\text{reg}}} \) (it was later rediscovered by J. Millson, V. Toledano Laredo [17] and J. Felder, Y. Markov, V. Tarasov, A. Varchenko [8]) and conjectured that its monodromy can be expressed in terms of the action of the quantum Weyl group [16], [20] on the corresponding Weyl module \( W_V \) over the corresponding quantum group \( U_v(g) \). This conjecture was later independently formulated and proved by V. Toledano Laredo for \( v \) in the formal neighbourhood of 1. The key notion introduced in his proof was the notion of a (quasi-)Coxeter category. The original definition of this notion is of combinatorial nature. We suggest a more topological version of this definition in Section 3. It is a collection of local systems of restriction functors on the open strata of hyperplane arrangements arising from the root hyperplanes of a root system, compatible under Verdier specialization. This approach makes it clear for example that the category of representations of a rational Cherednik algebra carries a Coxeter structure, see [4].

One of the main examples of a Coxeter category is a category of integrable representations of a quantum group. We consider the category \( \mathcal{C} \) of representations of Lusztig’s small quantum group. It has a geometric realization as the category \( \mathcal{FS} \) of factorizable sheaves [3]. This is the category of certain compatible collections of perverse sheaves on the configuration spaces of a Riemann surface. One of our key observations is that the category \( \mathcal{FS} \) carries a natural Coxeter structure (in our topological definition).

We conjecture that the equivalence \( \Phi: \mathcal{FS} \xrightarrow{\sim} \mathcal{C} \) of [3] takes the Coxeter structure on \( \mathcal{FS} \) to the Coxeter structure on \( \mathcal{C} \). This is essentially a reformulation of De Concini – Toledano Laredo conjecture (hence it follows from the results of V. Toledano-Laredo for \( v \) in the formal neighbourhood of 1). Roughly, it says that the monodromy in the vanishing cycles of factorizable sheaves acts by Lusztig’s symmetries.

1.2. Let us formulate the last point of Section 1.1 more precisely. We choose a Borel subalgebra \( \mathfrak{b} \subset \mathfrak{b} \subset \mathfrak{g} \). The corresponding set of simple roots is denoted by \( I \); for \( i \in I \) the
corresponding simple root is denoted \( \alpha_j \). We fix a Weyl group invariant symmetric bilinear form \( \langle \cdot, \cdot \rangle \) on \( \mathfrak{h}^* \) such that the square length of a short root is \( \alpha_j \cdot \alpha_j = 2 \).

We fix a primitive root of unity \( \zeta \) of degree \( \ell \); for simplicity in this introduction, we assume that \( \ell \) is not divisible by 2 and 3. We consider an integral dominant weight \( \lambda \in \mathfrak{h}^* \) such that \( 0 \leq \langle \lambda, \alpha_i \rangle < \ell \) for any \( i \in I \) (pairings with simple coroots).

For \( \beta = \sum_{i \in I} b_i \alpha_i \in \mathbb{N}[I] \) we consider the configuration space \( \mathbb{A}^\beta \) of colored divisors on the complex affine line \( \mathbb{A}^1 \). The open subspace \( \mathfrak{H}^\beta \subset \mathbb{A}^\beta \) of multiplicity free divisors on \( \mathbb{A}^1 \setminus \{0\} \) carries a 1-dimensional local system \( \mathfrak{H}_\lambda^\beta \) with the following monodromies: \( \zeta^{-2\alpha_i} \alpha_j \) when a point of colour \( i \) goes counterclockwise around a point of colour \( j \neq i \); \( -\zeta^{-\alpha_i} \alpha_i \) when two points of colour \( i \) trade their positions going around a halfcircle counterclockwise; and \( \zeta^{2\alpha_i} \alpha_i \) when a point of colour \( i \) goes around \( 0 \) counterclockwise. We denote by \( \mathfrak{H}_\lambda^\beta \) the Goresky-MacPherson extension of \( \mathfrak{H}_\lambda^\beta \) to \( \mathbb{A}^\beta \) (a perverse sheaf).

We have a pairing \( \langle \cdot, \cdot \rangle: \mathfrak{h} \times \mathbb{A}^\beta \rightarrow \mathbb{A}^1 \) given in the coordinates \( (t_{i,r})_{i \in I} \) on \( \mathbb{A}^\beta \), and \( (z_j)_{j \in I} \) in the basis of fundamental coweights on \( \mathfrak{h} \), as follows: \( \langle (z_j), (t_{i,r}) \rangle = \sum_{i \in I} z_i \sum_{r=1}^{b_i} t_{i,r} \). The vanishing cycles \( \Phi_\lambda^\beta := \Phi(\cdot, \cdot)^p \mathfrak{H}_\lambda^\beta \) of the pullback of \( \mathfrak{H}_\lambda^\beta \) to \( \mathfrak{h} \times \mathbb{A}^\beta \) is a perverse sheaf supported on \( \mathfrak{h} \simeq \mathfrak{h} \times \{ \beta \cdot 0 \} \).

We conjecture that \( \Phi_\lambda^\beta \) is smooth along \( \mathfrak{h}^{\text{reg}} \subset \mathfrak{h} \). In order to describe its monodromy on \( \mathfrak{h}^{\text{reg}} \), recall that one of the main results of [3] is a canonical identification of the stalk \( \Phi_\lambda^\beta_{\mathfrak{c}_0} \) at the fundamental Weyl chamber in \( \mathfrak{h}^{\text{reg}} \) with the weight space \( L_\lambda \) of the irreducible module with highest weight \( \lambda \) over the Lusztig big quantum group \( \tilde{U}_\zeta \) (note that the restriction of \( L_\lambda \) to the Lusztig small quantum group \( \tilde{u}_\zeta \) remains irreducible since \( \lambda \) is an \( \ell \)-restricted weight). We conjecture that the local system \( \Phi_\lambda^\beta|_{\mathfrak{h}^{\text{reg}}} \) is given by the following representation of the fundamental groupoid of \( \mathfrak{h}^{\text{reg}} \): the stalk at a Weyl chamber \( wC_0 \) in \( \mathfrak{h}^{\text{reg}} \) is \( L_{w(\lambda-\beta)} \) (\( w \) runs through the Weyl group \( W \)), and the half monodromies around the walls are given by the Lusztig symmetries \( T_{i,\pm} \) and \( T_{i,\pm}' \) (see Section 4 for details).

1.3. Here is the outline of the paper.

In Section 2 we consider an elementary example of type \( A_2 \). We compare the action of Lusztig’s symmetries in the “almost extremal” weight spaces of integrable modules over the corresponding quantum group (i.e. the weights obtained from the extremal ones by subtracting a root) with the monodromy action in the vanishing cycles of related perverse sheaves on the plane.

In Section 3 we propose a topological reformulation of Toledano Laredo’s notion of Coxeter category. It is very similar to Deligne’s topological reformulation [4] of the notion of braided tensor category.

In Section 4 we recall the (algebraic) construction of the Coxeter structure on the category of integrable modules over Lusztig’s big quantum group, in terms of Lusztig’s symmetries. It gives rise to the Coxeter structure on the category \( \mathcal{C} \) of representations of Lusztig’s small quantum group.

In Section 5 we recall very concisely the category of factorizable sheaves \( \mathcal{F}_S \) introduced in [3] and the equivalence \( \Phi: \mathcal{F}_S \rightarrow \mathcal{C} \) (see Section 5.2 and Section 5.3). We also take an opportunity to correct a confusion in [3] between Langlands dual types (see Section 5.1).
Then we go on to construct a Coxeter structure on $\mathcal{FS}$. The construction goes through the De Rham realization of $\mathcal{FS}$, and works only for $v$ sufficiently close to 1, but we expect it to work for arbitrary $v$. The construction also uses some results on iterated specialization and microlocalization over hyperplane arrangements presented in Section 6 which might be of independent interest. A more systematic approach to these questions is developed in [9].

Finally, in Section 7 we formulate the main conjecture that the equivalence $\Phi: \mathcal{FS} \xrightarrow{\sim} \mathcal{C}$ intertwines the Coxeter structures on $\mathcal{FS}$ and on $\mathcal{C}$.

1.4. Acknowledgments. We are grateful to R. Fedorov, S. Khoroshkin, B. Feigin, A. Postnikov, G. Rybnikov, L. Rybnikov, V. Toledano Laredo, D. Gaitsgory, D. Kazhdan, M. Kapranov, A. Braverman, R. Bezrukavnikov, A. Varchenko for the inspiring discussions, and M. Kashiwara for an important reference.

In fact, this note arose from a question asked by R. Fedorov in the summer 2012.

The research of M.F. has been funded within the framework of the HSE University Basic Research Program and the Russian Academic Excellence Project '5-100'.

2. An example

2.1. Algebra. We follow the notations of [16]. Let $U$ be the quantum universal enveloping algebra of type $A_2$, over the ring $A = \mathbb{Z}[v^{\pm 1}]$. The positive (resp. negative) subalgebra $U^+$ (resp. $U^-$) is generated by the divided powers $E_1^{(r)}$ (resp $F_1^{(r)}$), $i = 1, 2$, $r \in \mathbb{N}$. Let $\Lambda = (\mu_1, \mu_2) \in \mathbb{N}^2$ be a dominant highest weight such that $\mu_1 \geq 1 \leq \mu_2$, and $L(\Lambda)$ the corresponding integrable $U$-module with the highest vector $v$. We will be interested in the weight spaces $L(\Lambda)(\mu_{1-1}, \mu_{2-1})$, $L(\Lambda)(-\mu_{1+1}, \mu_{1+2})$, $L(\Lambda)(\mu_{1+2}, -\mu_{2+2})$, $L(\Lambda)(-\mu_{1+2}, -\mu_{2+1})$, $L(\Lambda)(-\mu_{1+2}, -\mu_{2+1})$ (these weights form a single Weyl orbit).

They have canonical bases $(F_1 F_2 v, F_2 F_1 v, (F_1^{(\mu_1)} F_2 v, F_2 F_1^{(\mu_1)} v), (F_2^{(\mu_2)} F_1 v, F_1 F_2^{(\mu_2)} v), (F_1^{(\mu_1+\mu_2-1)} F_1^{(\mu_1-1)} v, F_2 F_1^{(\mu_1+\mu_2-1)} F_1^{(\mu_1-1)} v), (F_2^{(\mu_2+1)} F_1^{(\mu_2)} v, F_1 F_2^{(\mu_2+1)} F_1^{(\mu_2)} v), (F_1^{(\mu_1+\mu_2-1)} F_2 v, F_2^{(\mu_1+\mu_2-1)} F_2 v), (F_2^{(\mu_2+1)} F_1^{(\mu_2)} v, F_1 F_2^{(\mu_2+1)} F_2 v), F_1^{(\mu_1+\mu_2-1)} F_1^{(\mu_1-1)} v, F_2^{(\mu_1+\mu_2-1)} F_2^{(\mu_1-1)} v)$, respectively.

We are interested in the action of Lusztig’s symmetries $T_{1,2,\pm}^i$ on the above weight spaces.

Lemma 2.2. $T_{1,2}^i (F_1 F_2 v) = -v^{\pm(\mu_1+1)} F_1^{(\mu_1+1)} F_1 v + F_2^{(\mu_1+1)} F_2 v$;

$T_{1,2}^i (F_2 F_1 v) = -v^{\pm(\mu_2+1)} F_2^{(\mu_2+1)} F_1 v + F_1^{(\mu_2+1)} F_2 v$;

$T_{1,2}^i (F_1^{(\mu_1)} F_2 v) = -v^{\pm\mu_2} F_2 F_1^{(\mu_1)} F_1 v$;

$T_{1,2}^i (F_2^{(\mu_2)} F_1 v) = -v^{\pm\mu_1} F_1 F_2^{(\mu_2)} F_2 v$;
Proof. We consider two subalgebras $U_1, U_2 \subset U$ of type $A_1$: the first one is generated by $E_1^{(r)}, F_1^{(r)}$, $r \in \mathbb{N}$, the second one is generated by $E_2^{(r)}, F_2^{(r)}$, $r \in \mathbb{N}$. To prove the first formula, we consider the $U_1$-submodule $M_1$ of $L(\Lambda)$ with the highest vector $F_2v$ and canonical base $F_2v, F_1F_2v, \ldots, F_1^{(1)}F_2v, F_1^{(1)(1)+1}F_2v$. We also consider another $U_1$-submodule $M'_1$ of $L(\Lambda)$ with the highest vector $w^+ := (v^{(1)} - v^{(-1)})F_1F_2v - (v^{(1)} - v^{(-1)})F_2F_1v$ and the lowest vector (in the same canonical base) $w^- = (v - v^{-1})F_1^{(1)}F_2v - (v^{(1)} + v^{(-1)})F_2F_1^{(1)}v$. In effect, it is straightforward that $E_1w^+ = 0$, and it follows from \cite{16} Lemma 42.1.2. (d) that $F_1w^- = 0$; hence $F_1^{(1)(1)-1}w^+ = aw^-$ for some $a \in A$. The fact that $a = 1$ follows by comparing the coefficients of $F_1F_2v$ in $w^+$ and of $F_1^{(1)}F_2v$ in $w^-$. Now according to \cite{16} Proposition 5.2.2. (a)], $T_{1,2}w^+ = w^-$, $T_{1,2}F_1F_2v = -v^{(1)+1}F_1^{(1)}F_2v$. From this we deduce the first two formulas. The $\ell$-formulas are proved similarly. Say, to prove the 5th and 6th formulas we consider the $U_2$-submodule $M_2$ of $L(\Lambda)$ with the highest vector $F_1^{(1)}v$ and canonical base $F_1^{(1)}v, F_2F_1^{(1)}v, \ldots, F_2^{(1)+1}F_1^{(1)}v, F_2^{(1)+2}F_1^{(1)}v$. We also consider another $U_2$-submodule $M'_2$ of $L(\Lambda)$ with the highest vector $x^+ := (v^{(1)} + v^{(2)} - v^{(1)} - v^{(2)})F_1^{(1)}v - (v^{(1)} - v^{(2)})F_2F_1^{(1)}v$ and the lowest vector (in the same canonical base) $x^- = (v^{(1)} + v^{(2)} - v^{(1)} - v^{(2)})F_1^{(1)}v - (v^{(1)} - v^{(2)})F_2^{(1)+2}F_1^{(1)}v$. Then $T_{2,2}x^+ = x^-$, $T_{2,2}F_2F_1^{(1)}v = -v^{(1)+1+2}F_2^{(1)+2}F_1^{(1)}v$. From this we deduce the 5th and 6th formulas. And so on.

2.3. Topology. Let $A_\mathbb{R}$ be a 2-dimensional real vector space and let $A$ be its complexification with coordinates $(t_1, t_2)$ stratified by 3 lines: $t_1 = 0$, $t_2 = 0$, $t_1 - t_2 = 0$. Let $\mathcal{L}$ be the shriek extension of the one-dimensional local system on the complement of the 3 lines with monodromies $v^{2\mu_1}, v^{2\mu_2}, v^{2\mu_3}$. In applications to algebra, $2\mu_3 = 3$. The dual vector space $A^*$ has coordinates $(z_1, z_2)$, and the dual stratification consists of the lines $z_1 = 0$, $z_2 = 0$, $z_1 + z_2 = 0$. This is the root hyperplane arrangement of type $A_2$. There are 6 real chambers of this arrangement: $C_0$ is the dominant chamber containing an interior point $z^{(e)} = (1, 1)$; the other chambers with interior points $z^{(1)} = (-1, 2), z^{(21)} = (-2, 1), z^{(12)} = z^{(21)}, z^{(1)} = (-1, -1), z^{(12)} = (1, -2), z^{(2)} = (2, -1)$. The chambers are naturally numbered by the Weyl group $W$ of type $A_2$ generated by simple reflections $s_1, s_2$. For $w \in W$ we have $C_w \ni z^{(w)}$, say $C_{s_1s_2s_1}(C_{121}$ for short) contains $z^{(12)}$. We have 6 real affine lines $\ell_w$, $w \in W$, in $A_\mathbb{R}$ given by equations $z^{(w)} = 1$. For example $\ell_\varepsilon$, $\ell_1$ are given by the equations $t_1 + t_2 = 1$, $-t_1 + 2t_2 = 1$ respectively. More generally, for $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, let us denote by $\ell_{w, \varepsilon}$ the real straight line given by the equation $z^{(w)} = \varepsilon$.

The microlocalization (Fourier transform) $\mu L$ is a certain constructible complex on $A^*$. We will be interested only in its restriction to the complement of the 3 lines in $A^*$, which is a 2-dimensional local system. Let us describe this local system explicitly.

The stalk $\mu^{(w)}L$ at $z^{(w)}$ equals the vanishing cycles $\Phi_{z^{(w)}}L$. Let $i_w$ denote the inclusion $\ell_w \hookrightarrow A$. Then $\Phi_{z^{(w)}}L$ may be identified with $H^1(\ell_w, i_w^*L)$. It is a 2-dimensional vector space with the base dual to the basis $\ell_w, \ell_w'$ of 1-cycles with coefficients in $i_w^*L^*$. The 1-cycles are defined as follows: $\ell_w'$ is the interval between the points $(1, 0)$ and $(1/2, 1/2)$; $\ell_w''$ is the interval between the points $(1/2, 1/2)$ and $(0, 1)$; $\ell_1'$ is the interval between the points $(1, 1)$ and $(0, 1/2)$; $\ell_1''$ is the interval between the points $(0, 1/2)$ and
is the interval between the points \((0,1)\) and \((-1/2,0)\); \(\ell_{21}'\) is the interval between the points \((-1/2,0)\) and \((-1,1)\); \(\ell_{12}'\) is the interval between the points \((-1,0)\) and \((-1/2,-1/2)\); \(\ell_{21}''\) is the interval between the points \((-1/2,-1/2)\) and \((0,-1)\); \(\ell_{2}'\) is the interval between the points \((0,-1)\) and \((1/2,0)\); \(\ell_2''\) is the interval between the points \((1/2,0)\) and \((1,1)\); \(\ell_{12}''\) is the interval between the points \((-1,-1)\) and \((0,-1/2)\); \(\ell_1''\) is the interval between the points \((0,-1/2)\) and \((1,0)\); \(\ell_{12}''\) is the interval between the points \((-1,0)\) and \((-1/2,-1/2)\); \(\ell_{212}'\) is the interval between the points \((-1/2,-1/2)\) and \((0,-1)\).

More generally, for any \(\varepsilon > 0\) we have canonical isomorphisms

\[
\Phi_{x(w)}L = H^1(\ell_{w,e}, i_w^*L) \overset{\sim}{\rightarrow} H^1(\mathbf{L}_{w,e}, i_{w,e}^*L)
\]

where \(i_{w,e} : \ell_{w,e} \rightarrow \mathbf{A}\), and we can define similar parallelly transported bases in \(H_1(\mathbf{L}_{w,e}, i_{w,e}^*L)\).

For two neighbouring chambers \(C_y, C_w, y, w \in W\), let \(\gamma_{y,w}^\pm\) be a path from \(C_y\) to \(C_w\) obtained from a straight line interval modified near the wall between these two chambers by going around it in the positive (resp. negative) imaginary halfspace. We will keep the same notation for the induced operator (half monodromy along \(\gamma_{y,w}^\pm\) from \(\Phi_{x(y)}L\) to \(\Phi_{x(w)}L\)).

**Lemma 2.4.** \(\gamma_{e,121}^\pm \phi_e' = -v^{\pm(\mu_1+\mu_3)} \phi_e''\), \(\gamma_{e,121}^\pm \phi_1'' = -v^{\pm\mu_1} \phi_1'' + \phi_1'\);
\(\gamma_{e,2}^\pm \phi_e'' = -v^{\pm(\mu_2+\mu_3)} \phi_e'\), \(\gamma_{e,2}^\pm \phi_e' = -v^{\pm\mu_2} \phi_e'' + \phi_e'\);
\(\gamma_{1,21}^\pm \phi_1' = -v^{\pm(\mu_2+\mu_1)} \phi_1''\), \(\gamma_{1,21}^\pm \phi_1'' = -v^{\pm\mu_2} \phi_1'' + \phi_1'\);
\(\gamma_{2,12}^\pm \phi_2' = -v^{\pm(\mu_1+\mu_2)} \phi_2''\), \(\gamma_{2,12}^\pm \phi_2'' = -v^{\pm\mu_1} \phi_2'' + \phi_2'\);
\(\gamma_{21,121}^\pm \phi_1'' = -v^{\pm(\mu_3+\mu_2)} \phi_1''\), \(\gamma_{21,121}^\pm \phi_1'' = -v^{\pm\mu_3} \phi_1'' + \phi_1'\);
\(\gamma_{12,212}^\pm \phi_2'' = -v^{\pm(\mu_3+\mu_1)} \phi_2''\), \(\gamma_{12,212}^\pm \phi_2'' = -v^{\pm\mu_3} \phi_2'' + \phi_2'\).

**Proof.** All the formulas being similar, we prove the first two. For the transposed map between dual spaces we must check that

\[\gamma_{e,1}' = \ell_{e,1}''\), \(\gamma_{1,21}' = \ell_{1,21}''\), \(\gamma_{e,2}' = \ell_{e,2}''\), \(\gamma_{2,12}' = \ell_{2,12}''\).

(Note that the second equality is equivalent to \(\gamma_1^\pm (\ell_1' + v^{\pm\mu_1} \ell_1'') = -v^{\pm\mu_3} \ell_1''\).) To prove it, we rotate the line \(\ell_1\) clockwise in \(\mathbb{R}^6\) with the point \((0,1/2)\) fixed and observe what happens with the real cycles \(\ell_1\) and \(\ell_1''\). At some critical moment the rotated line becomes parallel to the \(t_1\)-axis, at this moment we must pass for a short time into the complex upper (or lower) halfspace, and at the end we get the line parallel to \(\ell_e\). We see that at the end of this rotation \(\ell_1\) turns into \(\ell_e''\), whereas \(\ell_1''\) stretches and after the critical moment turns into the necessary linear combination of \(\ell_e''\) and \(\ell_e''\).

**Remark 2.5.** Writing down the composition \(\gamma_{21,121}^\pm \gamma_{1,21}^\pm \gamma_{e,1}^\pm\) in our bases as the product of matrices we find
\[
\begin{pmatrix}
0 & 1 \\
-v^{\pm(\mu_3+\mu_2)} & -v^{\pm\mu_3}
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
-v^{\pm(\mu_2+\mu_1)} & -v^{\pm\mu_2}
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
-v^{\pm(\mu_1+\mu_3)} & -v^{\pm\mu_1}
\end{pmatrix}
= \begin{pmatrix}
v^{\pm(\mu_1+\mu_2+\mu_3)} & 0 \\
0 & v^{\pm(\mu_1+\mu_2+\mu_3)}
\end{pmatrix},
\]

cf. [15 Corollary 5.9].
Remark 2.6. In case $\mu_1 = \mu_2 = \mu_3 = 1$ all the six weight spaces considered in Section 2.1 coincide with $L(1,1)(0,0)$ with the base $F_1F_2v$, $F_2F_1v$. In this base the operator $T'_1\pm$ of the first line of Lemma 2.2 corresponding to the operator $\gamma^\pm_{e,1}$ of Lemma 2.4 has the matrix
\[
\begin{pmatrix}
-v^{\pm 2} & -v^{\pm 1} \\
0 & 1
\end{pmatrix},
\]
while the operator $T'_{2\pm}$ of the second line of Lemma 2.2 corresponding to the operator $\gamma^\pm_{e,2}$ of Lemma 2.4 has the matrix
\[
\begin{pmatrix}
1 & 0 \\
-v^{\pm 1} & -v^{\pm 2}
\end{pmatrix}. \]
Note that $(\gamma^-_{e,1})^{-1} = \gamma^+_{e,1}$, and $(\gamma^-_{e,2})^{-1} = \gamma^+_{e,2}$.

2.7. Discussion. We set $\mu_3 = 1$. The theory of factorizable sheaves \cite{[3]} provides a canonical isomorphism $\Phi_2(e)\mathcal{L} \cong L(\Lambda)(\mu_1^{-1},\mu_2^{-1})$. The stalks of microlocalization at the other chambers $\Phi_2(w)\mathcal{L}$ do not have an algebraic interpretation in the framework of this theory. However, the comparison of Lemma 2.2 and Lemma 2.4 shows that the monodromy of the local system $\mu\mathcal{L}$ (as the automorphism group of $\Phi_2(e)\mathcal{L} \cong L(\Lambda)(\mu_1^{-1},\mu_2^{-1})$) can be expressed in terms of Lusztig’s symmetries $T'_{1,2\pm}$, $T''_{1,2\pm}$. In fact, the comparison of Lemma 2.2 and Lemma 2.4 suggests a much more precise relation, in particular, between a natural topological basis in $\Phi_2(e)\mathcal{L}$ and the canonical basis on the algebraic side. Unfortunately, we have no clue how to define such a topological basis in general. However, the relation between the monodromy and Lusztig’s symmetries seems to generalize. This is the subject of the main body of the note.

3. Coxeter categories

3.1. Notations. Let us set up a few notations related to a simple Lie algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}$ and Borel subalgebra $\mathfrak{b} \subset \mathfrak{b} \subset \mathfrak{g}$. The set of simple coroots is denoted by $I$; for $i \in I$ the corresponding simple coroot is denoted $\check{\alpha}_i$ or sometimes simply $i$. The corresponding simple root is denoted $\alpha_i$ or sometimes $i'$. We fix a Weyl group invariant symmetric bilinear form $\langle \cdot , \cdot \rangle$ on $\mathfrak{h}^*$ such that the square length of a short root is $\alpha_i \cdot \check{\alpha}_i = 2$. This bilinear form gives rise to an isomorphism $\mathfrak{h} \cong \mathfrak{h}^*$ so that the coroot lattice $Y$ generated by $\{\check{\alpha}_i\}_{i \in I}$ embeds into $\mathfrak{h}^*$. We then have $\check{\alpha}_j \cdot \check{\alpha}_i \in \{2,1,\frac{2}{3}\}$, and $\alpha_i \cdot \alpha_i \in \{2,4,6\}$. We set $d_i = \alpha_i \cdot \check{\alpha}_i/2$. Let $d$ be the ratio of the square lengths of the long and short roots, so that $d \in \{1,2,3\}$. We set $d_i = d/d_i$. Then
\[
\langle \alpha_i, \check{\alpha}_j \rangle = \frac{\alpha_i \cdot \check{\alpha}_j}{d_j} = d_i \check{\alpha}_i \cdot \check{\alpha}_j = \frac{d \check{\alpha}_i \cdot \check{\alpha}_j}{d_i}.
\]

3.2. The fundamental groupoid of $\mathfrak{h}_D^\text{reg}$. We follow the notations of \cite{[4]}. Let $D$ be the Dynkin diagram of the simple Lie algebra $\mathfrak{g}$ with Cartan $\mathfrak{h}$ (so that $I$ is the set of vertices of $D$). The root system of $\mathfrak{h} \subset \mathfrak{g}$ is $R_D \subset \mathfrak{h}^*$. The complement in $\mathfrak{h}$ to the root hyperplanes is the open subset $\mathfrak{h}_D^\text{reg}$.

Given a subset $D'$ of the set of vertices of $D$, we denote by $\mathfrak{h}_{D'}$ the quotient of $\mathfrak{h}$ by the center of the corresponding Levi subalgebra $L_{D'} \subset \mathfrak{g}$. In other words $\mathfrak{h}_{D'}^\text{reg} \subset \mathfrak{h}_{D}^\text{reg}$ is spanned by the simple roots corresponding to the vertices from $D'$. We denote by $\mathfrak{h}_{D'}^\text{reg}$ the complement in $\mathfrak{h}_{D'}$ to the root hyperplanes of the root subsystem $R_{D'}$ corresponding to $D' \subset D$.

\footnote{see Conjecture \cite{[7]} however.}
We recall the Salvetti presentation of the fundamental groupoid of $\mathfrak{h}^\text{reg}_{D'}$, cf. [19]. Let $\mathfrak{h}^\text{reg}_{D',\mathbb{R}}$ denote the set of real points of $\mathfrak{h}^\text{reg}_{D'}$. It is a union of the connected components called chambers. We fix a chamber $C_0^{D'}$ formed by the points with positive coordinates (in the basis of fundamental coweights). The Weyl group $W_{D'}$ acts on the set $C_{D'}$ of chambers simply transitively on the left. The choice of $C_0^{D'}$ identifies $C_{D'}$ with $W_{D'}$, and defines the right action of $W_{D'}$ on $C_{D'}$ (transferred from the right action of $W_{D'}$ on itself). The set of walls of $C_0^{D'}$ is canonically identified with the set of vertices of $D'$. The left action of $W_{D'}$ on $C_{D'}$ extends this identification to any chamber. For $i \in D'$ the right action of a simple reflection works as follows: $C \cdot s_i$ is a unique neighbouring chamber $C'$ having the $s_i$-wall in common with $C$.

The set of objects of the fundamental groupoid $\Pi(\mathfrak{h}^\text{reg}_{D'})$ is $C_{D'}$. Given a straight line interval $\gamma$ connecting the endpoints $\gamma_1 \in C_1$ and $\gamma_2 \in C_2$ and intersecting only one wall at a time, we define the morphisms $\gamma^\pm \in \text{Mor}_{\Pi(\mathfrak{h}^\text{reg}_{D'})}(C_1, C_2)$ as follows. The path $\gamma^+$ (resp. $\gamma^-$) coincides with $\gamma$ away from the small neighbourhoods of its intersection with walls, where $\gamma^+$ (resp. $\gamma^-$) goes around the intersection in the positive (resp. negative) imaginary direction in $\mathfrak{h}^\text{reg}_{D'}$. According to Salvetti, $\Pi(\mathfrak{h}^\text{reg}_{D'})$ is generated by the set of morphisms $\gamma^\pm$ with relations $\beta^\pm = \gamma^\pm$ provided $\gamma_1, \beta_1$ lie in the same chamber $C_1$, and $\gamma_2, \beta_2$ lie in the same chamber $C_2$.

### 3.3. The fundamental groupoid of $N^\text{reg}_{D'/D'',D'}$

Given a third subdiagram $D'' \subset D' \subset D$, we consider the linear subspace $\mathfrak{h}_{D'/D''} \subset \mathfrak{h}_{D'}$ spanned by the fundamental coweights in $D' - D'' \subset D'$. For example, $\mathfrak{h}_{D/D''} \subset \mathfrak{h}_{D}$ is the center of the Levi $L_{D''} \subset g$. We have an exact sequence

$$0 \longrightarrow \mathfrak{h}_{D'/D''} \longrightarrow \mathfrak{h}_{D'} \longrightarrow \mathfrak{h}_{D''} \longrightarrow 0$$

which may serve as another definition of $\mathfrak{h}_{D'/D''}$.

We denote by $\mathfrak{h}_{D'/D'',D'}^\text{reg}$ the complement in $\mathfrak{h}_{D'/D''}$ to the root hyperplanes (roots in $R_{D''}$) not containing $\mathfrak{h}_{D'/D''}$. The connected components of the real part $\mathfrak{h}_{D'/D'',\mathbb{R}}^\text{reg}$ are called chambers; the set of chambers is denoted $C_{D'/D''}$. It is naturally isomorphic to the set of parabolics in $g_{D''}$ containing the standard Levi $L_{D''}$, see e.g. [18, I.1.10]. We say that a chamber $C \in C_{D'}$ is adjacent to $\mathfrak{h}_{D'/D''} \subset \mathfrak{h}_{D'}$ if the intersection of the closure $\overline{C}$ with $\mathfrak{h}_{D'/D''}$ has the maximal (real) dimension $\dim \mathfrak{h}_{D'/D''}$; then this intersection is the closure of a chamber in $C_{D'/D''}$ to be denoted $\pi(C)$. The set of chambers adjacent to $\mathfrak{h}_{D'/D''}$ is denoted $A_{D'/D''}$. Thus we have a projection $\pi: A_{D'/D''} \rightarrow C_{D'/D''}$.

The natural projection $\text{pr}: \mathfrak{h}_{D'} \rightarrow \mathfrak{h}_{D''}$ (see Section 3.2) works in the bases of fundamental coweights as follows: $\text{pr} \omega_i = 0$ if $i \in D' - D'' \subset D'$; and if $i \in D'' \subset D'$, then $\text{pr} \omega_i$ goes to the corresponding fundamental coweight in $\mathfrak{h}_{D''}$. Given a chamber $C \in A_{D'/D''}$, its projection $\pi(C)$ is a chamber in $C_{D''}$. Thus we have a projection $\text{pr}: A_{D'/D''} \rightarrow C_{D''}$.

**Lemma 3.4.** The product $\text{pr} \times \pi: A_{D'/D''} \times A_{D'/D''} \rightarrow C_{D''} \times C_{D'/D''}$ establishes a one-to-one correspondence. □

**Definition 3.5.** (a) For a chamber $C \in C_{D''}$, we define a subgroupoid $\Pi_{\text{pr}^{-1}(C)}(\mathfrak{h}^\text{reg}_{D'}) \subset \Pi(\mathfrak{h}^\text{reg}_{D'})$ as follows: the objects are $\text{pr}^{-1}(C) \subset A_{D'/D''} \subset C_{D'}$, and the morphisms are
generated by $\gamma^\pm$ where $\gamma$ is a straight line interval parallel to $h_{D'/D''}$ (i.e. such that $\text{pr}\gamma$ is a point).

(b) For a chamber $C \in C^{D'/D''}$, we define a subgroupoid $\Pi_{\pi^{-1}(C)}(h_{D'}^{\text{reg}}) \subset \Pi(h_{D'}^{\text{reg}})$ as follows: the objects are $\pi^{-1}(C) \subset A^{D'/D''} \subset C^{D'}$ and the morphisms are generated by $\delta^\pm$ where $\delta$ is a straight line interval inside the union of closures of chambers adjacent to $C$.

(c) A subgroupoid $\Pi_{A^{D'/D''}}(h_{D'}^{\text{reg}}) \subset \Pi(h_{D'}^{\text{reg}})$ is generated by all the groupoids in (a,b) above. That is, its objects are $A^{D'/D''}$, and the morphisms are all the possible products of morphisms in (a,b) above (see Figure 1).

**Lemma 3.6.** (a) For any $C \in C^{D''}$, $\pi$ induces an equivalence $\Pi_{\text{pr}^{-1}(C)}(h_{D'}^{\text{reg}}) \sim \Pi(h_{D'}^{\text{reg}})$.

(b) For any $C \in C^{D'/D''}$, $\text{pr}$ induces an equivalence $\Pi_{\pi^{-1}(C)}(h_{D'}^{\text{reg}}) \sim \Pi(h_{D'}^{\text{reg}})$.

(c) The natural projection morphisms $\pi: \Pi_{A^{D'/D''}}(h_{D'}^{\text{reg}}) \to \Pi(h_{D'}^{\text{reg}})$ and $\text{pr}: \Pi_{A^{D'/D''}}(h_{D'}^{\text{reg}}) \to \Pi(h_{D'}^{\text{reg}})$ give rise to an equivalence $\Pi_{A^{D'/D''}}(h_{D'}^{\text{reg}}) \sim \Pi(h_{D'}^{\text{reg}}) \times \Pi(h_{D''}^{\text{reg}})$.

**Proof.** (M. Kapranov) The relations in the Salvetti complex [19] follow from a cell decomposition of the complement which is glued out of intervals, $2n$-gons (for any codim 2 cell where $n$ hyperplanes meet) and so on, and the relations in the fundamental groupoid are obtained from the 2-skeleton i.e., from these $2n$-gons.
So the 2-dimensional case implies the general one. See example with \( n = 3 \) in Figure [1]. We keep “vertical” and “horizontal” 2\( k \)-gons (contributing to \( \Pi_{\reg}(\mathfrak{h}_{D'}) \) or \( \Pi_{\reg}(\mathfrak{h}_{D''}) \)) intact, and replace the remaining 2\( n \)-gons with rectangles like the dotted one in Figure [1]. It follows that the “horizontal” and “vertical” morphisms commute. This produces a 2-dimensional CW-subcomplex of the complement which is (the 2-skeleton of) the product of two separate 2-dimensional subcomplexes.

3.7. Specialization. We consider the normal bundle \( N_{h_{D'D''}/D'} \sim \mathfrak{h}_{D'D''} \times \mathfrak{h}_{D''} \), and its open subspace \( \mathfrak{h}_{D'D''/D'}^\reg = \mathfrak{h}_{D'D''}^\reg \times \mathfrak{h}_{D''}^\reg \subset N_{h_{D'D''}/D'} \) with Poincaré groupoid \( \Pi_{A_{D'D''}}(\mathfrak{h}_{D'}^\reg) \). Then the Verdier specialization of a local system on \( \mathfrak{h}_{D'}^\reg \) along \( \mathfrak{h}_{D'D''} \) will be a well defined local system on \( \mathfrak{h}_{D'D''/D'}^\reg \). At the level of representations of Poincaré groupoids, the specialization is nothing but restriction to \( \Pi_{A_{D'D''}}(\mathfrak{h}_{D'}^\reg) \).

**Definition 3.8.** (A) A pure Coxeter category of type \( D \) is the collection of the following data:

(a) A category \( \mathcal{C}_{D'} \) for any subset \( D' \subset D \);

(b) For \( D'' \subset D' \) a local system of restriction functors \( F_{D'D''}: \mathcal{C}_{D'} \to \mathcal{C}_{D''} \) on \( \mathfrak{h}_{D'D''}^\reg \);

(c) For \( D''' \subset D'' \subset D' \) an isomorphism of local systems of functors:

\[
\phi_{D'D''D'''}: \text{Sp}_{h_{D'D''/D'''}} F_{D'D''} \sim \to F_{D''D'''} \circ F_{D'D''} \]

on \( \mathfrak{h}_{D'D''}^\reg \times \mathfrak{h}_{D''}^\reg \) which satisfy the natural “cocycle” or “pentagon” identity associated with \( D''' \subset D'' \subset D'' \subset D' \).

(d) In case \( D''' \) is disjoint from \( D' \) (i.e. no vertex of \( D''' \) is connected by an edge to a vertex of \( D' \), and \( D'' \cap D' = \emptyset \)) we have a canonical isomorphism \( \mathfrak{h}_{D'D''}^\reg = \mathfrak{h}_{D''D'''}^\reg = \mathfrak{h}_{D''D'''}^\reg \), and we are given a homomorphism of local systems of endomorphism algebras \( \eta: \text{End}(F_{D'D''}) \to \text{End}(F_{D''D'''}(\mathfrak{h}_{D''D'''})) \).

(B) A tensor Coxeter category of type \( D \) is the additional datum of braided balanced tensor structures on \( \mathcal{C}_{D'} \) such that

(i) The pullback of \( F_{D'D''} \) to the universal cover \( \tilde{\mathfrak{h}}_{D'D''}^\reg \) is a (trivial) local system of functors equipped with tensor structures \( \tilde{F}_{D'D''}: \mathcal{C}_{D'} \to \mathcal{C}_{D''} \).

(But we do not require them to respect the balance and braiding. Also, the monodromy isomorphisms of stalks \( \gamma_x: (F_{D'D''})_x \sim \to (F_{D'D''})_y \) where \( \gamma: x \sim \to y \) is a path in \( \Pi(\mathfrak{h}_{D'D''}^\reg) \) are required to be isomorphisms of tensor functors. Neither are the isomorphisms \( (\text{Ad}) \) above required to respect the tensor structure.);

(ii) The isomorphisms of (c) above pulled back to \( \mathfrak{h}_{D'D''}^\reg \times \mathfrak{h}_{D''D'''}^\reg \) are isomorphisms of tensor functors;

(iii) Let \( \gamma_0 \in \pi_1(\mathfrak{h}_{D'D''}^\reg) \) be the generator of the centre; geometrically it is a loop \( \exp(2\pi i \theta) \cdot x \), \( 0 \leq \theta \leq 1 \), \( x \in \mathfrak{h}_{D'D''}^\reg \). The automorphism \( \gamma_0^*: F_{D'D''} \to F_{D'D''} \) (it is the automorphism induced by the \( \mathbb{C}^* \)-monodromic structure on the sheaf \( F_{D'D''} \)) is inverse to the ratio of the auto-bimorphisms of the identity functors of \( \mathcal{C}_{D'} \) and \( \mathcal{C}_{D''} \).}

\(^2\)We thank A. Appel and V. Toledano Laredo for correcting mistakes in the original version of the definition.
3.9. Comparison with the Appel-Toledano-Laredo Coxeter braided tensor categories. If we impose an additional assumption that the local systems in Definition 3.8(Ab) are lifted from \( \Pi_{D,D^\prime}^{\text{reg}}/W_{D,D^\prime}/D_{D^\prime} \) (quotient with respect to the free action of the finite group \( W_{D,D^\prime} := \text{Norm}_{L_{D^\prime}}(L_{D^\prime})/L_{D^\prime} \) (normalizer of the Levi subgroup \( L_{D^\prime} \) in \( L_{D^\prime} \subset G \), modulo \( L_{D^\prime} \)), then we get an equivalent version of definitions 3.10, 4.1.

In the example of factorizable sheaves \( \mathcal{F}_S_D \) (Section 5.4 below), the balance on an irreducible sheaf \( \mathcal{L}(\lambda) \) is multiplication by \( \zeta^{\lambda-(\lambda+2\rho)} \). Factorizable sheaves \( \mathcal{F}_{\mathcal{D}_C} \) for Levi=Cartan also have a nontrivial braiding and balance; namely, on an irreducible sheaf \( \mathcal{L}(\lambda) \) the balance is multiplication by \( \zeta^{\mu-(\mu+2\rho)} \). The ratio of these two balances on a weight component \( \mathcal{L}(\lambda) \) of \( \mathcal{L}(\lambda) \) is \( \zeta^{\lambda-(\lambda+2\rho)-(\lambda-\alpha+2\rho)} \) and coincides with the monodromy automorphism of the monodromic sheaf \( \mathcal{L}(\lambda) \).

The identity \( \Delta_i(T_i) = R_{i}^{21} \cdot (T_i \otimes T_i) \), and more generally, for any \( D^\prime \subset D, \Delta_{D^\prime}(T_{w_0}^{D^\prime}) = R_{D^\prime}^{21} \cdot (T_{w_0}^{D^\prime} \otimes T_{w_0}^{D^\prime}) \), implies \( \Delta_{D^\prime}(T_{w_0}^{D^\prime})^2(T_{w_0}^{D^\prime} \otimes T_{w_0}^{D^\prime})^{-1} = R_{D^\prime}^{12} \circ R_{D^\prime}^{21} \), which in view of Definition 3.8(Biii) is nothing but the usual relation between the braiding and the balance.

4. Algebra

4.1. The Lusztig symmetries. Given \( \zeta \in \mathbb{C}, \zeta^6 \neq 1 \), we consider the Lusztig small quantum group \( \mathfrak{u}_{D^\prime} \) (see e.g. [3, 0.2.12]). We extend it by the projectors to the weight spaces \( 1_\lambda, \lambda \in X \), to obtain the algebra \( \hat{\mathfrak{u}}_{D^\prime} \) such that \( \text{Rep}(\hat{\mathfrak{u}}_{D^\prime}) = \mathcal{C}_{D^\prime} \) (notations of [3, 0.2.11-0.2.13]). The algebra \( \hat{\mathfrak{u}}_{D^\prime} \) is a subalgebra of the Lusztig big quantum group \( R\hat{\mathfrak{u}}_{D^\prime} \) Chapter 31] (where \( R: \mathbb{Z}[v^\pm 1] \rightarrow \mathbb{C}, v \mapsto \zeta \), generated by \( E_i = E_i^{(1)}, F_i = F_i^{(1)}, i \in D^\prime \), and \( 1_\lambda, \lambda \in X \). According to [16] Chapters 33,35, if \( \zeta \) is a root of unity (primitive of order \( \ell \)), there is a reductive algebraic group \( \hat{G}_{D^\prime,\zeta} \) with Cartan torus \( \hat{T}_{D^\prime,\zeta} \subset \hat{G}_{D^\prime,\zeta} \) and a tensor functor \( \text{Fr}^\zeta: \text{Rep}(\hat{G}_{D^\prime,\zeta}) \rightarrow \text{Rep}(R\hat{\mathfrak{u}}_{D^\prime}) \) (pullback with respect to the quantum Frobenius homomorphism). Note that the character lattice \( X^*(\hat{T}_{D^\prime,\zeta}) \) is naturally a sublattice of the weight lattice \( X \).

The Lusztig symmetries \( T_{i,e}^\prime, T_{i,e}'' \), \( i \in D^\prime, e = \pm 1 \), of \( R\hat{\mathfrak{u}}_{D^\prime} \) clearly preserve the subalgebra \( \hat{\mathfrak{u}}_{D^\prime} \) and restrict to the same named symmetries of this subalgebra. We define a functor \( T_u \) from \( \Pi(\mathfrak{h}_{D^\prime}^{\text{reg}}) \) to the category of \( \mathbb{C} \)-algebras on generators as follows: \( T_u(C) = \hat{\mathfrak{u}}_{D^\prime} \) for any \( C \in \mathbb{C} \); for \( \gamma \) a straight line interval connecting the endpoints in two neighbouring chambers \( C_1, C_2 \) with the common wall of type \( s_i, i \in D' \), we set \( T_u(\gamma^+) = T_{i,1}^\prime \) (resp. \( T_{i-1,1}^\prime \)) and \( T_u(\gamma^-) = T_{i-1,1}^\prime \) (resp. \( T_{i,1}^\prime \)), if \( \gamma \) goes from a Bruhat smaller chamber to the bigger one (resp. from a Bruhat bigger chamber to the smaller one). According to [16] Theorem 39.4.3, \( T_u \) is well defined.

Given an integrable \( R\hat{\mathfrak{u}}_{D^\prime} \)-module \( M \) with the Lusztig symmetries \( T_{i,e}^\prime, T_{i,e}'' \), \( M \rightarrow M \) 41.2.3], we define a functor \( T_M \) from \( \Pi(\mathfrak{h}_{D^\prime}^{\text{reg}}) \) to the category of \( \mathbb{C} \)-vector spaces on generators as follows: \( T_M(C) = \hat{\mathfrak{u}}_{D^\prime} \) for any \( C \in \mathbb{C} \); for \( \gamma \) a straight line interval connecting the endpoints in two neighbouring chambers \( C_1, C_2 \) with the common wall of type \( s_i, i \in D' \), we set \( T_M(\gamma^+) = T_{i,1}^\prime \) (resp. \( T_{i-1,1}^\prime \)) and \( T_M(\gamma^-) = T_{i-1,1}^\prime \) (resp. \( T_{i,1}^\prime \)), if \( \gamma \) goes from a Bruhat smaller chamber to the bigger one (resp. from a Bruhat bigger chamber to the smaller one). According to [16] Proposition 41.2.4, \( T_M \) is well defined.
Let $\mathcal{C}_{D'}$ (resp. $\mathcal{C}'_{D'}$) denote the category of integrable $R\hat{U}_{D'}$-modules (resp. $\mathfrak{u}_{D'}$-modules), and let $\Upsilon: R\mathcal{C}_{D'} \to \mathcal{C}_{D'}$ stand for the restriction functor. In the previous paragraph we have defined the local system $RF_{D',\emptyset}$ on $\mathfrak{h}^\text{reg}_{D'}$ of restriction functors $R\mathcal{C}_{D'} \to \text{Vect}_X = R\mathcal{C}_0$ to the category of $X$-graded $\mathbb{C}$-vector spaces.

**Proposition 4.2.** There exists a unique local system $F^c_{D',\emptyset}$ on $\mathfrak{h}^\text{reg}_{D'}$ of restriction functors $\mathcal{C}_{D'} \to \text{Vect}_X = \mathcal{C}_0$ such that $RF_{D',\emptyset} = F^c_{D',\emptyset} \circ \Upsilon$.

**Proof.** According to [2, Theorem 4.7], we view $\mathcal{C}_{D'}$ as the category of Hecke-eigen-objects in $R\mathcal{C}_{D'}$. That is, an object of $\mathcal{C}_{D'}$ is an object $M$ of $R\mathcal{C}_{D'}$ endowed with a collection of isomorphisms $\alpha_V: \text{Fr}(V) \otimes M \xrightarrow{\sim} \text{Res}_{\mathfrak{g}_{D',\emptyset}}(V) \otimes M$, $V \in \text{Rep}(\mathfrak{g}_{D',\emptyset})$. Since the Lusztig symmetries act on $\text{Fr}(V)$ and on $\text{Res}_{\mathfrak{g}_{D',\emptyset}}(V)$, a Hecke-eigen-object $(M, \alpha)$ gives rise to a representation $T_{(M, \alpha)}$ of $\Pi(\mathfrak{h}^\text{reg}_{D'})$. Hence the action of $\Pi(\mathfrak{h}^\text{reg}_{D'})$ on $RF_{D',\emptyset}$ canonically extends to the action of $\Pi(\mathfrak{h}^\text{reg}_{D'})$ on $F^c_{D',\emptyset}$.

**Remark 4.3.** For example, if $\lambda \in X$ is a dominant $\ell$-restricted weight (recall that $\ell$ is the order of $\zeta$), then the irreducible $\mathfrak{u}_{D'}$-module $L_{\lambda}'$ with highest weight $\lambda$ is the restriction of the irreducible $R\hat{U}_{D'}$-module $L_{\lambda}'$ with highest weight $\lambda$, and $T_{L_{\lambda}'} = T_{L_{\lambda}}$.

### 4.4. A Coxeter structure on $R\mathcal{C}, \mathcal{C}$

We need to define the local systems of restriction functors $F^c_{D',D''}$, not just $F^c_{D',\emptyset}$, as in the previous subsection. To this end we restrict the action of the fundamental groupoid $\Pi(\mathfrak{h}^\text{reg}_{D'})$ defined in Proposition 3.2 to the subgroupoid $\Pi(\mathfrak{h}^\text{reg}_{D'})$ of $\Pi(\mathfrak{h}^\text{reg}_{D'})$, see Definition 3.5(a). More precisely, we consider a wall between two neighbouring chambers $c_1, c_2$ of $\mathfrak{h}^\text{reg}_{D'/D''}$. Let $C_1$ (resp. $C_2$) be a (unique) chamber of $\mathfrak{h}^\text{reg}_{D'/D''}$ adjacent to $c_1$ (resp. $c_2$) such that $\text{pr}(C_1) = \text{pr}(C_2) = C_0^{D''}$ (the fundamental chamber of $\mathfrak{h}^\text{reg}_{D'/D''}$). Let $\gamma$ be a straight line interval going from $c_1$ to $c_2$, and let $\Gamma$ be its lifting from $C_1$ to $C_2$ parallelly to $\mathfrak{h}^\text{reg}_{D'/D''}$. In the notations of Section 4.1, we set $T^{-1}_M(\Gamma^+) = \text{Id}_M$, $T^{-1}_M(\Gamma^-) = T^{-1}_M(\Gamma^-) \circ T^{-1}_M(\Gamma^-) \circ T^{-1}_M(\Gamma^-) \circ T^{-1}_M(\Gamma^-)$. It suffices to prove that $T^{-1}_U(\Gamma^-) \circ T^{-1}_U(\Gamma^-) = \text{Id}_{R^U_{D'/D''}}$. According to [13, Corollary 5.9], we have $T^{-1}_U(\Gamma^-) \circ T^{-1}_U(\Gamma^-) = K_i^{-2}E_i$, $T^{-1}_U(\Gamma^-) \circ T^{-1}_U(\Gamma^-)(F_i) = F_iK_i^2$ for any $i \in D'' \cup d$, and $T^{-1}_U(\Gamma^-) \circ T^{-1}_U(\Gamma^-)(F_i) = F_iK_i^2$ for any $i \in D''$.

**Lemma 4.5.** The action of $\Pi(\mathfrak{h}^\text{reg}_{D'/D''})$ on a $R\hat{U}_{D'}$-module $M$ commutes with the action of $R\hat{U}_{D''}$. 

**Proof.** Note that $\text{(c)}$ and $\text{(d)}$ are neighbours of types of walls intersected by $\Gamma$ are all in $D''$ except for exactly one $d \in D' - D''$. Let $C_1'$ be the chamber adjacent to $c_1$ such that $\text{pr}(C_1') = w_0^{D''}C_0^{D''}$. Then $C_2' = w_0^{D''}C_1'$. Let $\Delta$ be a straight line interval going from $C_1'$ to $c_1$, and ending at the starting point of $\Gamma$, and let $\Gamma\Delta$ be the concatenation of $\Gamma$ and $\Delta$. Then $T^{-1}_M(\Gamma^-) \circ T^{-1}_M(\Gamma^+) = T^{-1}_M(\Gamma^+) \circ T^{-1}_M(\Gamma^-) \circ T^{-1}_M(\Gamma^-) \circ T^{-1}_M(\Gamma^-) \circ T^{-1}_M(\Gamma^-) \circ T^{-1}_M(\Gamma^-) \circ T^{-1}_M(\Gamma^-) \circ T^{-1}_M(\Gamma^-) \circ T^{-1}_M(\Gamma^-) \circ T^{-1}_M(\Gamma^-)$.

Now by Lemma 3.6(a) (and Proposition 4.2) we obtain the desired local system of restriction functors $RF_{D',D''}: R\mathcal{C}_{D'} \to R\mathcal{C}_{D''}$ (resp. $F^c_{D'/D''}: \mathcal{C}_{D'} \to \mathcal{C}_{D''}$) on $\mathfrak{h}^\text{reg}_{D'/D''}$. The
Remark 4.6. The Coxeter structure on \( R \) studied in [21] differs from ours by the twist by an invertible local system. More precisely, for a weight component \( M \subset M \), in the setup of Section 4.1 the Coxeter structure of [21, 4.1.3] \( T_{M^+}^T(\gamma^+) = \zeta_{d_i(\alpha_i, \lambda)}/4T_{i+1}^T = (-1)^{\langle \alpha_i, \lambda \rangle} \zeta_{d_i(\alpha_i, \lambda)}^2/4T_{i+1}^T \) for \( \gamma \) going through an \( s_i \)-wall from a Bruhat bigger chamber to a Bruhat smaller one; \( T_{M^+}^T(\gamma^-) = \zeta_{d_i(\alpha_i, \lambda)}/4T_{i+1}^T = \zeta_{d_i(\alpha_i, \lambda)}^2/4T_{M^+}(\gamma^-) \) for \( \gamma \) going through an \( s_i \)-wall from a Bruhat bigger chamber to a Bruhat smaller one; the remaining two half-monodromies are the inverses of the above two.

Note that if \( s_i \lambda = \lambda \) then the scalar factors above are identically equal to one. We define \( M_{W^+} := \bigoplus_{\mu \in W^+} M_{\mu} \), the direct sum over the Weyl group orbit of \( \lambda \). Since \( T_{M^+}^T(\gamma^+) \) arises from a local system on \( \text{bs} \) [21] (i.e. \( R_{FD}^T \) possesses a \( W \)-equivariant structure), it follows that \( R_{FD}^T \) also possesses a \( W \)-equivariant structure.

5. Topology

5.1. Erratum to [3]. We take this opportunity to correct a blunder pertaining to the non-simply laced case of [3]. Let us define the quantities \( i' \cdot j' \) as \( \alpha_i \cdot \alpha_j \) in the sense of Section 3.1. Then throughout [3] in all formulas the occurrences of \( i \cdot j \) should be replaced by \( i' \cdot j' \).

For example:
- in [3] Part 0.2.1: \( \langle i, j' \rangle = 2i' \cdot j'/i' \cdot i' \), and \( d_i = i' \cdot i'/2 \);
- in the relation [3] Part 0.2.7(d) one should replace \( \zeta_{i'j'} \) by \( \zeta^{i'j'} \).

Thus, in [3] Part 0.2.7 \( K_i = K_i^{d_i} \) as before, and \( \zeta_i = \zeta^{d_i} \), but the meaning of \( d_i \) should be changed: \( d_i \) is defined not as half the square length of the coroot \( \alpha_i \) but as half the square length of the root \( \alpha_i \).

On the geometric side, the monodromy of the cohesive local system corresponding to a full counterclockwise turn of a point \( i \) around \( j \) should be \( \zeta^{-2i'j'} \), cf. [3] Part 0, 3.10], and Section 5.2 below.

To summarize, the main assertion of [3] (reviewed below in more details) consists of two parts: first, an equivalence of the geometric category \( FS \) with a category of graded modules over the algebra \( u \) defined in [3] Part 0, 2.7]. This assertion is true, and our correction just replaces the root system by the dual one on both sides. The second assertion is an identification of \( u \) with the Lusztig’s small quantum group. This identification is described in [3] Part 0, 2.12, or Part II, 12.5] and should be corrected: the “geometric” algebra \( u \) is isomorphic to the “Langlands dual” Lusztig’s algebra connected with the dual root system.

This replacement of the root system by its dual is a rather subtle point. Its origin lies in the definition of the braiding in [16], cf. the proof of [16] Lemma 32.2.3].

Also, there is a misprint in the definition of a balance in [3] IV.6.6]: \( n(\lambda) \) must be replaced by \( 2n(\lambda) = \lambda \cdot (\lambda + 2\rho) \).

5.2. A review of [3]: cohesive system and algebra \( u^- \). For \( \beta \in \mathbb{N}[I] \) we consider the configuration space \( \mathbb{A}^{\beta} \) of colored divisors on the complex affine line \( \mathbb{A}^1 \). The open subspace
\( \tilde{\mathcal{A}}^\beta \subset \mathcal{A}^\beta \) of multiplicity free divisors carries a 1-dimensional cohesive local system \( \mathcal{G}^\beta \) with the following monodromies: \( \zeta^{-2 \alpha_i - \alpha_j} \) when a point of colour \( i \) goes counterclockwise around a point of colour \( j \neq i \), and \( -\zeta^{-\alpha_i + \alpha_j} \) when two points of colour \( i \) trade their positions going around a half-circle counterclockwise. We denote by \( \mathcal{G}^\beta \) the Goresky-MacPherson extension of \( \mathcal{G}^\beta \) to \( \tilde{\mathcal{A}}^\beta \) (a perverse sheaf). Given two disjoint open discs \( \mathcal{A}^1 \supset D(p_i, \varepsilon_i), i = 1, 2 \), with centers in \( p_i \) of radii \( \varepsilon_i \), and \( \beta_{1,2} \in \mathbb{N}[I] \), we have an open embedding \( m: D(p_1, \varepsilon_1)^{\beta_1} \times D(p_2, \varepsilon_2)^{\beta_2} \hookrightarrow \mathcal{A}^{\beta_1 + \beta_2} \) and a canonical isomorphism \( \psi: m^*\mathcal{G}^{\beta_1 + \beta_2} \cong \mathcal{G}^{\beta_1} \boxtimes \mathcal{G}^{\beta_2} \). We denote by \( r \) the closed embedding \( \tilde{\mathcal{A}}^\beta_R \hookrightarrow \mathcal{A}^\beta \); we keep the same notation for \( D(p, \varepsilon)^{\beta} \rightarrow D(p, \varepsilon)^{\beta} \) in case \( p \in \mathbb{R} \). We consider the real hyperbolic stalk \( \Phi_R(\mathcal{G}^\beta) := H^*_c(\mathcal{A}^\beta_R, r^*\mathcal{G}^\beta) \). According to [5, Theorem I.3.9], \( \Phi_R(\mathcal{G}^\beta) \) lives in cohomological degree 0. According to [5, Theorem I.3.5], we have a canonical isomorphism \( \Phi_R(\mathcal{G}^\beta)^* \simeq \Phi_R(D\mathcal{G}^\beta) \) where \( D \) stands for the Verdier duality.

We have canonical isomorphisms \( \Phi_R(\mathcal{G}^\beta) \simeq H^*_c(D(p, \varepsilon)^{\beta}, r^*\mathcal{G}^\beta) \) for arbitrary \( p \in \mathbb{R} \), \( \varepsilon \in \mathbb{R}_{>0} \).

The isomorphism \( \psi^{-1} \) above gives rise to the multiplication map \( \Phi_R(\mathcal{G}^{\beta_1}) \otimes \Phi_R(\mathcal{G}^{\beta_2}) \simeq H^*_c(D(1, \varepsilon)^{\beta_1}, r^*\mathcal{G}^{\beta_1}) \otimes H^*_c(D(0, \varepsilon)^{\beta_2}, r^*\mathcal{G}^{\beta_2}) \rightarrow H^*_c(\mathcal{A}^{\beta_1 + \beta_2}, r^*\mathcal{G}^{\beta_1 + \beta_2}) = \Phi_R(\mathcal{G}^{\beta_1 + \beta_2}) \). The above selfduality gives rise to the comultiplication map \( \Phi_R(\mathcal{G}^{\beta_1 + \beta_2}) \rightarrow \Phi_R(\mathcal{G}^{\beta_1}) \otimes \Phi_R(\mathcal{G}^{\beta_2}) \).

According to [5, I.II], the twisted graded Hopf algebra \( \Phi_R(\mathcal{G}) := \bigoplus_{\beta \in \mathbb{N}[I]} \Phi_R(\mathcal{G}^\beta) \) is naturally isomorphic to \( u^- \), the negative part of the small quantum group at \( v = \zeta \).

5.3. A review of [5]: factorizable sheaves. We have an open subset \( \mathcal{A}^\lambda \subset \mathcal{A}^\lambda \) of configurations of distinct coloured points in \( \mathcal{A}^1 \setminus \{0\} \). It carries a 1-dimensional cohesive local system \( \mathcal{G}^\lambda \) with the monodromies around diagonals same as the ones of \( \mathcal{G}^\beta \), and also the monodromy \( \zeta^{2 \lambda - \alpha} \) when a point of colour \( i \) goes around 0 counterclockwise (here \( \lambda \) is a weight). We denote by \( \mathcal{G}^\lambda \) the Goresky-MacPherson extension of \( \mathcal{G}^\lambda \) to \( \mathcal{A}^\lambda \) (a perverse sheaf). Denoting by \( A(p, \varepsilon) \) the complement in \( \mathcal{A}^1 \) to the closure of \( D(p, \varepsilon) \) (an open annulus), we have an open embedding \( m: A(0, \varepsilon)^{\beta_1} \times D(0, \varepsilon)^{\beta_2} \hookrightarrow \mathcal{A}^{\beta_1 + \beta_2} \) and a canonical isomorphism \( \psi: m^*\mathcal{G}^{\beta_1 + \beta_2} \cong \mathcal{G}^{\beta_1}_{\lambda - \beta_2} \boxtimes \mathcal{G}^{\beta_2} \). A factorizable sheaf of highest weight \( \lambda \) is a collection of perverse sheaves \( \mathcal{M}^\beta \) on \( \mathcal{A}^\beta \) equipped with factorization isomorphisms \( m^*\mathcal{M}^{\beta_1 + \beta_2} \cong \mathcal{G}^{\beta_1}_{\lambda - \beta_2} \boxtimes \mathcal{M}^{\beta_2} \). In particular, since for \( p_1 \in \mathbb{R} \) big enough, and \( \varepsilon_1 \) small enough, \( D(p_1, \varepsilon_1) \subset A(0, \varepsilon) \), and the restriction of \( \mathcal{G}^{\beta_1}_{\lambda - \beta_2} \) to \( A(p_1, \varepsilon)^{\beta_1} \) is canonically isomorphic to \( \mathcal{G}^{\beta_1} \), we obtain isomorphisms

\[
\mathcal{M}^{\beta_1 + \beta_2}|_{D(p_1, \varepsilon)^{\beta_1}} \times D(0, \varepsilon)^{\beta_2} \cong \mathcal{G}^{\beta_1} \boxtimes \mathcal{M}^{\beta_2} \tag{5.1}
\]

Let \( a: \mathcal{A}^\beta \rightarrow \mathcal{A}^1 \) be the addition, and \( \Phi_a(\mathcal{M}^\beta) \) the corresponding vanishing cycles. It is a perverse sheaf on the hypersurface \( a = 0 \), but since \( \mathcal{M}^\beta \) is smooth along coordinate-diagonal stratification, \( \Phi_a(\mathcal{M}^\beta) \) is supported at the origin \( \{\beta: 0\} \subset \mathcal{A}^\beta \), so we will view \( \Phi_a(\mathcal{M}^\beta) \) just as a vector space. It is canonically isomorphic to \( H^*_c(\mathcal{A}^{\beta_1}_{R^+}, r^+\mathcal{M}^{\beta_1}) \) where \( r_+: \mathcal{A}^{\beta_1}_{R^+} \hookrightarrow \mathcal{A}^{\beta_1} \) is the closed embedding of the “real halfspace” formed by the real configurations in the preimage \( a^{-1}(\mathbb{R}_{>0}) \). Since the vanishing cycles commute with duality, we have a canonical isomorphism \( \Phi_a(\mathcal{M}^\beta)^* \simeq \Phi_a(D\mathcal{M}^\beta) \) (see e.g. [5, Theorem 0.6.3]). The isomorphism \( \Phi_R(\mathcal{M}^\beta)^* \otimes \Phi_a(\mathcal{M}^\beta) \simeq H^*_c(D(p_1, \varepsilon)^{\beta_1}, r^*\mathcal{M}^{\beta_1}) \otimes H^*_c(D(0, \varepsilon)^{\beta_2}, r^*\mathcal{M}^{\beta_2}) \rightarrow H^*_c(\mathcal{A}^{\beta_1 + \beta_2}_{R^+}, r^+\mathcal{M}^{\beta_1 + \beta_2}) = \Phi_a(\mathcal{M}^{\beta_1 + \beta_2}), \) i.e. to the action of
\[ u^+ \simeq \Phi_R(J) \text{ on } \Phi_a(M) := \bigoplus_{\beta \in \mathbb{N}[I]} \Phi_a(M^\beta). \] The above selfduality gives rise to the coaction \( \Phi_a(M^{\beta_1+\beta_2}) \to \Phi_R(J^{\beta_1}) \otimes \Phi_a(M^{\beta_2}); \) equivalently, \( \Phi_R(J^{\beta_1}) \cong \Phi_a(M^{\beta_1+\beta_2}) \to \Phi_a(M^{\beta_2}). \) Taking into account the isomorphism \( \Phi_R(J^{\beta_1}) \cong (u_{-\beta_1})^* \cong u_{\beta_1}^+ \), we obtain an action of \( u^\pm \) on \( \Phi_a(M) \). We assign to \( \Phi_a(M^\beta) \) the weight \( \lambda - \beta \). This, together with the action of \( u^\pm \), defines the action of \( \hat{u} \) on \( \Phi_a(M) \) (an isomorphism of \( \hat{u} \) and the Lusztig small quantum group \( \hat{u} = \hat{u}_D \) is established in \([3\text{, Theorem 2.13}], \text{cf. Section 5.1}\). The resulting functor from the category \( \mathcal{FS} \) of factorizable sheaves to the category \( \mathcal{C} \) of \( \hat{u} \)-modules (to be denoted \( \Phi \)) is an equivalence of categories.

### 5.4. A Coxeter structure on \( \mathcal{FS} \)

The diagram \( D \) is the Dynkin diagram of an irreducible Cartan datum \((I, \cdot)\) of finite type. The category of factorizable sheaves introduced in \([3\text{, 0.4.6}]\) will be denoted by \( \mathcal{FS}_D \). For a subdiagram \( D' \subset D \) we denote by \( \mathcal{FS}_{D'} \) a similarly defined category with grading by the weight lattice \( X \). That is, compared to the definition of \( \mathcal{FS}_D \), the lattice of weights is always \( X \), while the set of colors is \( D' \subset D = I \). The braided balanced tensor structure on \( \mathcal{FS}_{D'} \) is introduced in \([3\text{, 0.5.9}, 0.5.10, IV.6.6]\).

In order to construct the local systems \( F_{\mathcal{FS}_{D/D'}}^{\mathcal{FS}} \) of restriction functors \( \mathcal{FS}_{D'} \to \mathcal{FS}_{D''} \) we vary the definition \([3\text{, 0.6.7, 0.6.8}]\) of the vanishing cycles functor \( \Phi \) in the following way. Let \( \mathbb{N}[D'] \ni \beta = \sum_{j \in D'} b_j \alpha_j \) and let \( \mathbb{A}^\beta = \prod_{j \in D'} (\mathbb{A}^1)^{(b_j)} \) be the configuration space of \( D' \)-colored effective divisors on the affine line \( \mathbb{A}^1 \) with coordinate \( t \), of degree \( \nu \). For \( D'' \subset D' \) we have a pairing \( \langle \cdot, \cdot \rangle : \mathfrak{h}_{D'/D''} \times \mathbb{A}^\beta \to \mathbb{A}^1 \) given in the coordinates \((t_{j,s})_{j \in D', 1 \leq s \leq b_j} \) on \( \mathbb{A}^\beta \), and \((z_j)_{j \in D'' - D'} \) in the basis of fundamental coweights on \( \mathfrak{h}_{D'/D''} \) as follows: \( \langle (z_j), (t_{j,s}) \rangle := \sum_{j \in D'' - D'} z_j \sum_{s=1}^{b_j} t_{j,s} \). The decomposition \( \beta = \beta'' + \iota \beta := \sum_{j \in D''} b_j \alpha_j + \sum_{k \in D'' - D'} b_k \alpha_k \) gives rise to the direct product decomposition \( \mathbb{A}^\beta = \mathbb{A}^{\beta''} \times \mathbb{A}^{\iota \beta} \). Clearly, given a perverse sheaf \( M^\beta \) on \( \mathbb{A}^\beta \), the vanishing cycles \( \Phi_{\langle \cdot, \cdot \rangle} M^\beta \) is a perverse sheaf supported on \( \mathfrak{h}_{D'/D''} \times \mathbb{A}^{\beta''} \times 0^{\iota \beta} \simeq \mathfrak{h}_{D'/D''} \times \mathbb{A}^{\beta''} \).

Let us write \( \zeta \) in the form \( \zeta = \exp(\pi i \varkappa) \).

**Theorem 5.5.** If \( M^\beta \) is a part of data of a factorizable sheaf \( M \) and \( \varkappa \) is sufficiently close to 0, then \( \Phi_{\langle \cdot, \cdot \rangle} M^\beta \big|_{\mathfrak{h}_{D'/D''} \times \mathbb{A}^{\beta''}} \) is smooth along \( \mathfrak{h}_{D'/D''}^{\text{reg}} \).

**Proof.** It suffices to consider an irreducible \( M \), and hence an irreducible \( M^\beta \). We may and will assume \( D' = D \). Then \( M^\beta \) is isomorphic to the Goresky-MacPherson sheaf \( J^\beta \) of Section 5.3 for a certain weight \( \lambda \). For \( \beta = \sum_{i \in I} b_i \alpha_i \), we consider an unfolding \( \pi : J \to I \) such that for any \( i \in I \), \( \pi^{-1}(i) = b_i \). Then the product of symmetric groups \( \Sigma_\pi := \prod_{i \in I} \mathfrak{S}_{b_i} \) acts on the affine space \( \mathbb{A}^J \), and \( \mathbb{A}^\beta = \mathbb{A}^J / \Sigma_\pi \). We denote the natural projection \( \mathbb{A}^J \to \mathbb{A}^\beta \) by \( \pi \) as well. We denote by \( \hat{\mathbb{A}}^J \subset \mathbb{A}^J \) the complement to the diagonals in \((\mathbb{A}^1 \setminus \{0\})^J \). We consider the one-dimensional local system \( J_\lambda^J \) on \( \hat{\mathbb{A}}^J \) with the following monodromies: \( \zeta^{-2 \alpha_i} \cdot \alpha_j \) when a point of colour \( i \) goes counterclockwise around a point of colour \( j \neq i \), and \( \zeta^{2 \alpha_i} \cdot \alpha_i \) when a point of colour \( i \) goes around 0 counterclockwise. We denote by \( J_\lambda^I \) the Goresky-MacPherson extension of \( J_\lambda^J \) to \( \mathbb{A}^J \). Then \( J^J_\lambda \) carries an evident \( \Sigma_\pi \)-equivariant structure, so \( \Sigma_\pi \) acts on the perverse sheaf \( \pi_\ast J^I_\lambda \), and \( J^J_\lambda \) is nothing but the subsheaf \((\pi_\ast J^I_\lambda)_{\Sigma_\pi}^{-} \) of \( \Sigma_\pi \)-antiinvariants in \( \pi_\ast J^I_\lambda \) (see \([3\text{, II.6.13}]\)).
For $D'' \subset D = I$, let $J'' := \pi^{-1}(D'') \subset J$, and $J = J \setminus J''$, so that $\mathbb{A}^{J''} \to \mathbb{A}^J$ (we set the remaining coordinates to be all zeros). The construction of Section 5.4 gives rise to a linear map $m : \mathfrak{h}_{D''/D'} \times \mathbb{A}^{J''} \to \mathbb{T}^*_{\mathbb{A}^{J''/\mathbb{A}^{J''}}} = (\mathbb{A}^J)^*_{\mathbb{A}^{J''}}$, and we have $\Phi_{\langle \cdot \rangle} \mathcal{M}^\beta = (\pi_* \mathcal{M}_{\mathbb{A}^{J''/\mathbb{A}^{J''}}}^\beta)^*_{\mathbb{A}^{J''}}$ where $\mathcal{M}$ is the microlocalization functor $\mathfrak{I}$.

Now by Kashiwara-Schapira theorem identifying the microlocalization and the Fourier transform (see [11, Proposition 8.6.3], [5], [12]), we have $\mu_{\mathbb{A}^{J''/\mathbb{A}^{J''}}}^{\pi} \mathcal{M}_{\mathbb{A}^{J''/\mathbb{A}^{J''}}} = \mathfrak{I}^{\mathbb{A}_{\mathbb{A}^{J''}}} \mathcal{M}_{\mathbb{A}^{J''/\mathbb{A}^{J''}}}$. The specialization $\mathcal{S}_{\mathbb{A}^{J''/\mathbb{A}^{J''}}}^{\pi} \mathcal{M}_{\mathbb{A}^{J''/\mathbb{A}^{J''}}} = \mathcal{M}_{\mathbb{A}^{J''/\mathbb{A}^{J''}}} (\mathcal{M}_{\mathbb{A}^{J''/\mathbb{A}^{J''}}} = \mathfrak{I}^{\mathbb{A}_{\mathbb{A}^{J''}}} \mathcal{M}_{\mathbb{A}^{J''/\mathbb{A}^{J''}}} = \mathcal{M}_{\mathbb{A}^{J''/\mathbb{A}^{J''}}})$. This provides the desired construction of the local systems $\mathcal{M}_{\mathbb{A}^{J''/\mathbb{A}^{J''}}}$.

Proof of Theorem 5.3. The conclusion of Theorem 5.5 holds true for an arbitrary $\zeta \in \mathbb{C}^*$. The collection of perverse sheaves $\Phi_{\langle \cdot \rangle} \mathcal{M}_{\mathbb{A}^{J''/\mathbb{A}^{J''}}}^{\pi} |_{\mathfrak{h}_{D''/D'}^{\mathbb{A}^{J''} \times \mathbb{A}^{J''}}} : \nu \in Y^+$, enjoys the $D''$-factorization property, i.e., may be viewed as a local system over $\mathfrak{h}_{D''/D'}^{\mathbb{A}^{J''} \times \mathbb{A}^{J''}}$ of objects of $\mathcal{F}_{D''}^\mathfrak{I}$. This provides the desired construction of the local systems $\mathcal{F}_{D''}^\mathfrak{I}$ of restriction functors $\mathcal{F}_{D''}^\mathfrak{I}$. The isomorphisms of Definition 3.8(Ac) are a particular case of the following construction. Let $\langle \cdot, \cdot \rangle : W \times V \to \mathbb{A}^1$ be a bilinear pairing between two complex vector spaces. Let $U \subset W$ be a linear subspace. We denote the restriction of $\langle \cdot, \cdot \rangle$ to $U \times V$ by $\langle \cdot, \cdot \rangle_U$. Let $\mathcal{M}$ be a perverse sheaf on $V$ smooth along a central hyperplane arrangement. We will view $\Phi_{\langle \cdot, \cdot \rangle} \mathcal{M}$ as a perverse sheaf on $W \times W^\perp$ (where $W^\perp \subset V$ is the annihilator of $W$). Note that the pairing $\langle \cdot, \cdot \rangle_U$ descends to the well defined pairing $\langle \cdot, \cdot \rangle_{W/U}$ between $W/U$ and $U^\perp$.

Theorem 5.9. There is a canonical isomorphism $\Phi_{\langle \cdot, \cdot \rangle} \mathcal{M} \cong \Phi_{\langle \cdot, \cdot \rangle} \mathcal{M} \to \Phi_{\langle \cdot, \cdot \rangle} \mathcal{M}$ of perverse sheaves on $U \times W/U \times W^\perp$.

Proof of Theorem 5.9. Let $V^\perp \subset W$ be the kernel of the bilinear pairing $\langle \cdot, \cdot \rangle : W \times V \to \mathbb{A}^1$. The smooth base change for the projection $W \to W/V^\perp$ reduces the claim.
to a construction of a canonical isomorphism
\[ \text{Sp}_Y \times X^\perp/Y \times Y^\perp \mathcal{M} \to \mathcal{M} \]
\[ \text{Sp}_Y \times X^\perp/Y \times X^\perp \mathcal{M} \]
(5.3)
of perverse sheaves on \( Y \times (X/Y)^\perp \times (V/X)^\perp \), where \( Y := W^\perp \subset X := U^\perp \subset V \), and \( \mathcal{M} \) is the microlocalization functor \([11]\). This isomorphism will be proved in the next Section, see Theorem 6.4.

6. Iterated specialization and microlocalization

6.1. Iterated specialization. Fix a complex or real vector space \( V \) equipped with a finite central hyperplane arrangement \( \mathcal{H} = \{ H_i \} \). A linear subspace \( V' \subset V \) is called a flat if it is an intersection of some hyperplanes from \( \mathcal{H} \). A filtration
\[ \ldots \subset V_{i+1} \subset V_i \subset \ldots \subset V_0 = V \]
is called admissible if all \( V_i \) are flats or 0.

Let \( V'' \subset V' \subset V \) be an admissible filtration. \( \mathcal{H} \) induces a central arrangement on \( V'/V'' \). Let \( Sh_{\mathcal{H}'}(V'/V'') \) denote the category of constructible sheaves smooth along the corresponding stratification, and let \( D^b_{\mathcal{H}'}(V'/V'') \) denote the bounded derived category of complexes whose cohomology belongs to \( Sh_{\mathcal{H}'}(V'/V'') \).

If \( V \) is complex and \( \mathcal{H} \) is real, which means that all \( H_i \) are given by real equations, then the abelian subcategory \( Perv_{\mathcal{H}'}(V) \subset D^b_{\mathcal{H}'}(V) \) admits a description in terms of linear algebra (quiver) data, cf. \([10]\).

Given a flat \( W \subset V \), we have the specialization functor \([22], [11]\)
\[ \text{Sp}_W : D^b_{\mathcal{H}'}(V) \to D^b_{\mathcal{H}'}(W \oplus V/W) \]
whose value on \( \mathcal{M} \in D^b_{\mathcal{H}'}(V) \) can be described as follows \([22]\) Section 9]. We fix hermitian metrics on \( V \) and on \( W \). Let \( \rho \) denote the natural projection \( V \to W \). Let \( \xi = (w, u) \in W \oplus V/W \). We fix a sufficiently small \( \varepsilon > 0 \) and \( 0 < \rho < \varepsilon \). We set \( U_{\varepsilon, \rho} = \{ v \in V : \rho \| w - v \| + \| \rho u - \rho (v) \| < \varepsilon \rho \} \): an open subset of \( V \). Then the stalk \( \text{Sp}_W(\mathcal{M})_\xi = R\Gamma(U_{\varepsilon, \rho}, \mathcal{M}) \).

The following lemma is a consequence of this description of specialization, cf. \([9]\) Theorem 3.17.

**Lemma 6.2.**
(a) Let \( V_2 \subset V_1 \subset V \) be an admissible filtration. We have a natural isomorphism of functors \( D^b_{\mathcal{H}'}(V) \to D^b_{\mathcal{H}'}(V_2 \oplus V_1 / V_2 \oplus V/V_1) \),
\[ \phi_{12} : \text{Sp}_{V_1} \circ \text{Sp}_{V_2} \to \text{Sp}_{V_2} \circ \text{Sp}_{V_1} \]
Let us abbreviate the notation as \( \phi_{12} : \text{Sp}_1 \text{Sp}_2 \to \text{Sp}_2 \text{Sp}_1 \).

(b) Let \( V_3 \subset V_2 \subset V_1 \subset V \) be an admissible filtration. The various isomorphisms \( \phi \) from (a) satisfy the pentagon relation \( \phi_{23} \circ \phi_{13} \circ \phi_{12} = \phi_{12} \circ \phi_{23} : \text{Sp}_1 \text{Sp}_2 \text{Sp}_3 \to \text{Sp}_3 \text{Sp}_2 \text{Sp}_1 \). \( \Box \)

6.2.1. Let us explain the relation (b). We have eight categories related to the subquotients of \( V \):
\[ D^b_{\mathcal{H}'}(V) \]
\[ D^b_{\mathcal{H}'}(V_3 \oplus V_3) \]
\[ D^b_{\mathcal{H}'}(V_2 \oplus V_2) \]
\[ D^b_{\mathcal{H}'}(V_1 \oplus V_1) \]
\[ D^b_{\mathcal{H}'}(V_3 \oplus V_2 / V_3 \oplus V/V_2) \]
\[ D^b_{\mathcal{H}'}(V_3 \oplus V_3 / V_3 \oplus V/V_3) \]
\[ D^b_{\mathcal{H}'}(V_2 \oplus V_2) \]
\[ D^b_{\mathcal{H}'}(V_1 \oplus V_1 / V_1 \oplus V/V_1) \]
\[ D^b_{\mathcal{H}'}(V_3 \oplus V_2 / V_3 \oplus V_2) \]
\[ D^b_{\mathcal{H}'}(V_3 \oplus V_3 / V_3 \oplus V_3) \]
\[ D^b_{\mathcal{H}'}(V_2 \oplus V_2 / V_2 \oplus V/V_2) \]
\[ D^b_{\mathcal{H}'}(V_1 \oplus V_1 / V_1 \oplus V/V_1) \]
\[ D^b_{\mathcal{H}'}(V_3 \oplus V_2 / V_3 \oplus V/V_2) \]
\[ D^b_{\mathcal{H}'}(V_3 \oplus V_3 / V_3 \oplus V/V_3) \]
\[ D^b_{\mathcal{H}'}(V_2 \oplus V_2 / V_2 \oplus V/V_2) \]
\[ D^b_{\mathcal{H}'}(V_1 \oplus V_1 / V_1 \oplus V/V_1) \]
\[ D^b_{\mathcal{H}'}(V_3 \oplus V_2 / V_3 \oplus V/V_2) \]
\[ D^b_{\mathcal{H}'}(V_3 \oplus V_3 / V_3 \oplus V/V_3) \]
\[ D^b_{\mathcal{H}'}(V_2 \oplus V_2 / V_2 \oplus V/V_2) \]
\[ D^b_{\mathcal{H}'}(V_1 \oplus V_1 / V_1 \oplus V/V_1) \]
they are in bijection with the vertices of a cube. The functors $\text{Sp}$ act from a category to all the categories one level below it. There are six longest paths from the category $D^b_{\mathfrak{S}}(V)$ to the category $D^b_{\mathfrak{S}}((V_3 \oplus V_2/V_3 \oplus V_1/V_2 \oplus V/V_1)$; these paths are in bijection with the symmetric group $S_3$ (this is an instance of a wellknown geometric fact: the longest paths on an $n$-cube are in bijection with the symmetric group $S_n$).

The paths are connected by homotopies arising from the natural transformations $\phi$; this is a weak Bruhat order on $S_3$. Among these six paths there are two which are equal. The pentagon (b) above is a hexagon where one of the natural transformations on the right is the identity.

6.3. Iterated microlocalization. We also have the microlocalization functor which may be defined as the composition

$$\mu_{W/V}: D^b_{\mathfrak{S}}(V) \xrightarrow{\text{Sp}^W} D^b_{\mathfrak{S}}(W \oplus W/W) \xrightarrow{\text{F}_T^W} D^b_{\mathfrak{S}}(W \oplus (V/W)^*)$$

where $\text{F}_T^W$ is the Fourier-Sato transformation $[11]$. In order to keep track of the ambient space, from now on we will use another notation for the specialization: $\text{Sp}^W = \text{Sp}_{W/V}$.

**Theorem 6.4.** Let $V$ be a complex vector space and let $Y \subset X \subset V$ be an admissible filtration. For $M \in \text{Perv}_{\mathfrak{S}}(V)$ there exists a canonical isomorphism

$$\psi_{XY}: \text{Sp}_Y \times X \rightarrow Y \xrightarrow{\mu_Y \otimes \mu_M} \mu_Y \times X \xrightarrow{\mu_X \otimes \mu_Y \otimes \mu_M} \mu_X \otimes \mu_Y \otimes \mu_M$$

(6.1)

of perverse sheaves on $Y \times (X/Y)^* \times (V/X)^*$.

These isomorphisms satisfy the pentagon relation connected with an admissible filtration $Y \subset X \subset Z \subset V$.

6.5. Proof of Theorem 6.4. The rest of this Section is devoted to the proof of this theorem.

The following properties are consequences of the definition.

6.5.1. For a linear subspace $Y \subset V$ and a perverse sheaf $M$ on $V$ we have a canonical isomorphism $\mu_{Y/V}^* \mu_Y \otimes \mu_M \xrightarrow{\sim} \mu_{Y/(V/Y)} \otimes \mu_Y \otimes \mu_M$.

6.5.2. For a perverse sheaf $M$ on the product of two vector spaces $Y \times Z$, monodromic along the projection $Y \times Z \rightarrow Y$, we have a canonical isomorphism $\mu_{Y/(V \times Z)} \otimes \mu_Y \otimes \mu_M \xrightarrow{\sim} F \text{Sp}_{Y \times Z} \otimes \mu_Y \otimes \mu_M$ between the microlocalization and the Fourier-Sato transform on the vector bundle $Y \times Z \rightarrow Y$.

6.5.3. We will have to work on the $D$-module side of the Riemann-Hilbert correspondence, so we recall the definition of the specialization functor in this context. Given a linear subspace $Y \subset V$ we choose a complementary subspace $Z \subset V$, $V \simeq Y \oplus Z$, with linear coordinates $z_1, \ldots, z_d$. Then the ring of differential operators $D_V$ has a grading such that $\deg z_i = 1$, $\deg \partial_y = -1$, $\deg y = \deg \partial_y = 0$ for any $i = 1, \ldots, d$, and any linear coordinates $y$ on $Y$. Let $F^*D_V$ be the corresponding descending filtration. Note that $\text{gr}^F D_V$ is canonically isomorphic to $D_{Y \times (V/Y)}$. Let $M$ be a regular holonomic $D$-module on $V$. It possesses a unique (descending) Malgrange-Kashiwara filtration $\ldots F^{-1}M \supset F^0M \supset F^1M \supset \ldots$ compatible with the filtration on $D_V$ such that (a) for $j > 0$ and $k \gg 0$, we have $F^{\pm k}M = F^{\pm j}D_VF^{\pm k}M$; (b) the generalized eigenvalues of the Euler vector field $\sum_{1 \leq i \leq d} z_i \partial_{z_i}$ on $F^kM/F^{k+1}M$ have real parts in $[k, k+1)$ for any $k \in \mathbb{Z}$.
The specialization of $M$ is defined as a $D_{\mathcal{Y} \times (V/Y)}$-module $\text{Sp}_{Y/V}M := \text{gr}^F M$. We say that $M$ is potentially monodromic along $Y$ if the Malgrange-Kashiwara filtration $F^* M$ is compatible with some grading $G^* M$ compatible with the grading on $D_V$. Equivalently, we can choose a complementary subspace $Z \subset V$ such that $M$ is monodromic along the corresponding projection $V \to Y$. Any such choice defines the isomorphisms $Y \times Z \xrightarrow{\sim} V$ and $\text{Sp}_{Y/V} M \xrightarrow{\sim} M$.

6.5.4. For a regular holonomic $D$-module on the product of two vector spaces $Y \times Z$, monodromic along the projection $Y \times Z \to Y$, its microlocalization $\mu_{Y/Y \times Z} M \xrightarrow{\sim} FT_{Y/Y \times Z}$ is the following regular holonomic $D_{Y \times Z}$-module. We choose some linear coordinates $y_1, \ldots, y_s$ on $Y$, and $z_1, \ldots, z_d$ on $Z$. Let $\xi_1, \ldots, \xi_d$ be the dual coordinates on $Z^\ast$. Then we have an isomorphism $D_{Y \times Z^\ast} \xrightarrow{\sim} D_{Y \times Z^\ast}: y_i \mapsto y_i$, $\partial_{y_i} \mapsto \partial_{y_i}$, $\xi_j \mapsto \partial_{z_j}$, $\partial_{\xi_j} \mapsto -z_j$ (independent of the choice of coordinates), and $\mu_{Y/Y \times Z} M$ is nothing but $M$ viewed as a $D_{Y \times Z}$-module via this isomorphism.

Now we proceed to the construction of isomorphism (6.1) in the $D$-module setting. All the $D_Y$-modules below are assumed to be regular holonomic and smooth along some central hyperplane arrangement in $V$.

**Lemma 6.6.** If a regular holonomic $D_Y$-module $M$ is potentially monodromic along both $Y$ and $X$, then there is a canonical isomorphism (6.1).

**Proof.** The microlocalization $\mu_{Y/V} M$ is monodromic along the projection $Y \times Y^\perp \to Y$ and potentially monodromic along $Y \times X^\perp$. Any choice of the direct complement $Z \subset X: X = Y \oplus Z$ as in the definition of potential monodromicity, gives rise to the isomorphisms $Y \times Z^\ast \times X^\perp \xrightarrow{\sim} Y \times Y^\perp$ and $\text{Sp}_{Y \times Z^\ast} \mu_{Y/Y \times Z} M \xrightarrow{\sim} \mu_{Y/V} M$. So it remains to construct an isomorphism $\mu_{Y/V} M \xrightarrow{\sim} \mu_{Y \times V^\perp /V} \mu_{X/V} M$. We choose a direct complement $S \subset V: V = X \oplus S$ as in the definition of potential monodromicity. Then the required isomorphism follows from the explicit formulas in Section 6.5.3. One can check that it does not depend on the choices of the complements $Z$ and $S$. □

**Lemma 6.7.** If a regular holonomic $D_Y$-module $M$ is potentially monodromic along $Y$, then there is a canonical isomorphism (6.1).

**Proof.** By Section 6.5.4 we can replace the RHS of (6.1) by $\mu_{Y \times X^\perp /X \times X^\perp} \mu_{X/V} \text{Sp}_{X/V} M$. But $\text{Sp}_{X/V} M$ is potentially monodromic along both $Y$ and $X$. So it remains to use Lemma 6.6 and construct an isomorphism

$$\text{Sp}_{Y \times X^\perp /Y \times Y^\perp} \mu_{Y/V} M \xrightarrow{\sim} \mu_{Y \times (V/X)} \text{Sp}_{X/V} M \quad (6.2)$$

We choose a direct complement $Z \subset X: X = Y \oplus Z$, and a direct complement $S \subset V: V = X \oplus S$ such that $Z \oplus S$ is as in the definition of potential monodromicity. We choose the linear coordinates $z_1, \ldots, z_d$ in $Z$, and $s_1, \ldots, s_e$ in $S$, and the dual coordinates $\xi_1, \ldots, \xi_d$ in $Z^\ast$, and $\eta_1, \ldots, \eta_k$ in $S^\ast$. The $D_Y$-module $M$ has a grading $G^* M$ compatible with the grading on $D_V$ such that deg $z_i = \deg s_j = 1$, deg $\partial_{z_i} = \deg \partial_{s_j} = -1$, $\deg y_k = \deg \partial_{y_k} = 0$. According to Section 6.5.3, the microlocalization $\mu_{Y/V} M$ amounts to the substitution $\xi_i \mapsto \partial_{z_i}$, $\eta_j \mapsto \partial_{s_j}$, $\partial_{\xi_i} \mapsto -z_i$, $\partial_{\eta_j} \mapsto -s_j$, $y_k \mapsto y_k$, $\partial_{y_k} \mapsto \partial_{y_k}$. To compute the specialization $\text{Sp}_{X/V} M$ we use the Malgrange-Kashiwara filtration $F^* M$ as in Section 6.5.3. To compute the LHS of (6.2) we use the unique filtration $F^* M$ compatible as in Section 6.5.3 with the
grading on $D_V$ such that $\deg z_i = -1$, $\deg s_j = 1$, $\deg s_j = \deg y_k = \deg \partial y_k = 0$ (note that it is not the Malgrange-Kashiwara grading of $D_V$, but rather the Fourier image of one).

The construction of the isomorphism $\Phi$ amounts to the construction of the isomorphism

$$\text{gr}^F M \xrightarrow{\sim} \text{gr}^F M$$

(6.3)

Note that both $F^i M$ and $'F^i M$ are compatible with the grading $G^i M$, that is $F^i M = \oplus_j (F^i M \cap G^j M)$ and $'F^i M = \oplus_j ( 'F^i M \cap G^j M)$ for any $i \in \mathbb{Z}$. From the uniqueness of $'F^i M$ and $F^i M$ it follows that $'F^k M \cap G^j M = F^{k+j} M \cap G^j M$. Thus the desired isomorphism $\Phi$ is the direct sum of natural isomorphisms $( 'F^k M \cap G^j M)/( 'F^{k+1} M \cap G^j M) = (F^{k+j} M \cap G^j M)/(F^{k+j+1} M \cap G^j M)$.

\[ \square \]

6.8. The end of the proof. Now we can finish the construction of isomorphism $\Phi$ for an arbitrary $D_V$-module $M$. It suffices to construct the isomorphism $\Phi$ for arbitrary $M$ (not necessarily potentially monodromic along $Y$). By Section 6.5.1 we can replace the RHS of (6.2) by $\mu_y/X \times (V/X) \text{Sp}_{Y/V} X \times (V/X) \text{Sp}_{Y/V} M$. By Lemma 6.2 we can replace this by $\mu_y/X \times (V/X) \text{Sp}_{Y/V} Y \times (V/Y) \text{Sp}_{Y/V} M$. However, $\text{Sp}_{Y/V} M$ is already potentially monodromic along $Y$, so by Lemma 6.7 we have $\text{Sp}_{Y/V} \times (Y \times Y^\perp) \mu_{Y/V} \text{Sp}_{Y/V} M \xrightarrow{\sim} \text{Sp}_{Y/V} \times (Y \times Y^\perp) \mu_{Y/V} \text{Sp}_{Y/V} M$. One last application of Section 6.5.1 allows to replace the LHS by $\text{Sp}_{Y/V} \times (Y \times Y^\perp) \mu_{Y/V} \text{Sp}_{Y/V} M$.

This completes the proof of Theorem 6.4 and hence that of Theorem 5.9.

Remark 6.9. The statements of Lemma 6.2 and Theorem 6.4 look similar, but the proofs are very different: one is topological, another is De Rham (via $D$-modules). Let us comment on this discrepancy. On the one hand, Lemma 6.2 has an easy proof in De Rham setting as well. On the other hand, let $\text{Perv} \mathfrak{H}(V) \subset D_b^c \mathfrak{H}(V)$ denote the subcategory of perverse sheaves. If all $H_i \in \mathfrak{H}$ are given by real equations then $\text{Perv} \mathfrak{H}(V)$ admits an explicit description in terms of linear algebra (quiver) data, cf. [10]. The specialization and microlocalization functors can be described in this language, and a topological proof of Lemma 6.2 (resp. Theorem 6.4) is given in [9, Theorem 3.17] (resp. [9, Theorem 5.6]).

7. Discussion

7.1. Desiderata. We have gone to all this trouble just to conjecture that the functor $\Phi$ of [8] takes the Coxeter structure of Section 5.4 to the one of Section 4.1. By [8, Theorem 3.2] and Fourier-microlocalization this would imply that the monodromy of the Casimir connection is given by the Lusztig symmetries for any $\zeta$. Note that the functor $\Phi$ of [8] is nothing but the stalk at the fundamental chamber $C_0^D$ of the local system of the restriction functors $F_D^{38}; \mathfrak{S}_D \rightarrow \text{Vect}_X$ of Section 5.3.

7.2. Tilted functors $\Phi$. Recall the setup of Section 5.3. Let us choose a point $z^{(w)}$ in a chamber $C_w \subset \mathfrak{h}_{reg}^\beta$, $w \in W$. Instead of $\Phi_a(M^\beta)$ let us consider the spaces of “tilted” vanishing cycles $\Phi_a(M^\beta) := \Phi_a(M^\beta) \simeq H^*_\beta(A_{\mathbb{R}}^\beta, r_* M^\beta)$ where $r: \mathbb{A}^\beta_{\mathbb{R}} \hookrightarrow A^\beta \subset \mathbb{A}^\beta_{\mathbb{R}} \rightarrow \mathbb{R}_{\geq 0}$. In particular, $\Phi_a \simeq \Phi_e$. 

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For a real $p_1$ such that $\langle z^{(w)}, \beta \cdot p_1 \rangle$ is positive and big enough, similarly to Section 5.3 we obtain the map

$$
\Phi_{\mathbb{R}}(\mathbb{C}) \otimes \Phi_w(M^{(\beta)}) \simeq H^*(D(p_1, \varepsilon)\mathbb{R}^{\mathbb{R}}, r^*\mathbb{C}) \otimes H^*(D(0, \varepsilon), r_*M^{(\beta)}) \rightarrow H^*(A_{\mathbb{R}}^{\mathbb{R}}, r_*M^{(\beta)}) = \Phi_w(M^{(\beta)}),
$$

i.e. the action of $u^-$ on $\Phi_w(M)$ similarly to Section 5.3. We assign to $\Phi_w$ the weight $w(\lambda - \beta)$, and using the isomorphisms $T_{w^+}^{w^-} : u^+ \rightarrow T_{w^+}^{w^-}(u^-) \subset u$, we obtain the action of $T_{w^+}^{w^-}(u^+), T_{w^+}^{w^-}(u^-)$ on $\Phi_w(M)$. This, together with the above grading, defines an action of $\hat{u}$ on $\Phi_w(M)$, i.e. gives rise to two functors $\Phi_{w^\pm} : \mathcal{F} \rightarrow \mathcal{C}$.

Given a straight line interval $\gamma_w$ from $z^{(e)}$ to $z^{(w)}$ we obtain the corresponding “halfmonodromy” transformations $\gamma_w^{\pm} \Phi_w(M) : \Phi_w(M^{(\beta)}) \rightarrow \Phi_w(M^{(\beta)})$ (independent of the choice of $\gamma_w$).

The following conjecture is a reformulation of Section 7.1.

**Conjecture 7.3.** The maps $\{\gamma^+_w, \gamma^-_w, M \in \mathcal{F}\}$ define two natural transformations of functors $\gamma_w^+ : \Phi \rightarrow \Phi^+_w$ and $\gamma_w^- : \Phi \rightarrow \Phi^-_w$.

As we have already mentioned, the main theorem of [21] implies this conjecture for $\zeta = \exp(h)$, $h$ being a formal parameter.

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