Discounting, beyond utilitarianism*

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Abstract

Discounted utilitarianism and the Ramsey equation prevail in the debate on the discount rate on consumption. The utility discount rate is assumed to be constant and to reflect either the uncertainty about the existence of future generations or a pure preference for the present. We question the unique status of discounted utilitarianism and discuss the implications of alternative criteria addressing the key issues of equity in risky situations and variable population. To do so, we first characterize a class of intertemporal social objectives, named Expected Equally Distributed Equivalent (EEDE) criteria, which embody reasonable ethical principles. The class is more flexible in terms of population ethics and it disentangles risk aversion and inequality aversion. We show that these social objectives imply interesting modifications of the Ramsey formula, and shed new light on Weitzman’s “dismal theorem”.

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1 Introduction

The current debate on the economics of climate change has focused primarily on the choice of the social discount rate, which is crucial because of the long term impacts of most greenhouse gases. The starting point of most analyses, despite their differences, remains the seminal work by Ramsey (1928), who showed in a standard discounted utilitarian model that the optimal consumption discount rate is given by the following equation, known as the Ramsey formula:

\[ r = \delta + \eta g, \]  

where \( r \) is the consumption discount rate, \( \delta \) the utility discount rate, \( \eta \) the elasticity of the utility of consumption, and \( g \) the growth rate of consumption.

Most of the debate following the publication of the Stern review (Stern, 2006) has been about the appropriate value of \( \delta \). Some authors (Schelling, 1995; Stern, 2006) endorsed an ‘ethical’ approach supporting a low value of \( \delta \) in line with the probability of human extinction, while others (Nordhaus, 2007, 2008; Weitzman, 2007) called for larger values based on the preferences revealed on financial markets. Several papers (Atkinson et al., 2009; Dasgupta, 2008; Anthoff, Tol and Yohe, 2009) have also stressed the key role of \( \eta \) in the analysis. It has been interpreted in at least three different ways: in terms of risk aversion; in terms of inequality aversion; in terms of the intertemporal elasticity of substitution. Unfortunately, the workhorse model of welfare economics behind the Ramsey formula, namely, the discounted expected utility model, is unable to distinguish between these three different notions.

Another difficulty behind the Ramsey formula, in addition to the ambiguous meaning of its key components, is that it leaves aside a key issue of the climate change problem, namely population size. There is a significant literature on population ethics discussing how to compare populations of different sizes (classical references in that field include Broome, 1991, 2004; Ng, 1989; Blackorby, Bossert and Donaldson, 2005). However, the literature on climate change has hardly taken stock of these contributions and to the best of our knowledge the impact of (uncertain) population size on social discounting has hardly been studied.\(^1\) One reason is that population size does not affect the social discount rate if one

\(^1\)A recent exception is Millner (2013), but the paper only considers variants of the utilitarian criterion.
remains within the limited scope of discounted utilitarianism.

The aim of this paper is to address the above difficulty of the Ramsey formula in a framework involving risk on future welfare and population size. We characterize a specific class of welfare models, the Expected Equally Distributed Equivalent (EEDE) criteria, which distinguishes risk aversion and inequality aversion, and is explicit about how populations of different sizes are compared. The model admits the discounted utilitarian model as a special case. It is also explicit about the meaning of the different parameters, so that they can be precisely discussed. Doing so, we endorse a normative approach. We believe that this approach is appropriate for the problem of climate change which imply future generations whose interests cannot be taken into account by market mechanisms. At the very least, we think the approach is worth being pursued.

From the social welfare model, we can derive a social discount rate. We show that it involves three elements in addition to the standards components of the Ramsey formula: a term related to population size; a term related to the relative priority of the welfare of generations concerned by the investment; a covariance term. The term related to population size is likely to increase the social discount rate because future generations live in more populated societies on average so that their welfare has less social importance. We provide an exemple showing that the term can be substantial. The effect of relative priority and covariance is less clear; we however indicate that they may reverse the conclusions of Weitzman’s dismal theorem (Weitzman, 2009).

The present paper relates two distincts lines of research. The first one tries to incorporate equity considerations in frameworks involving risks. As discussed above, the standard utilitarian approach cannot distinguish risk aversion and inequality aversion. Actually, the utilitarian criterion has been advocated in a seminal paper by Harsanyi (1955), who argued that, in the presence of risk, the concavity of the utility function should represent the population’s attitude about risk-taking. Harsanyi’s result has been criticized for being unable to take into account equity considerations both ex ante and ex post (Diamond, 1967; Broome, 2004; Ben Porath, Gilboa and Schmeidler, 1997; Gajdos and Maurin, 2004). In the present paper, we adopt an ex post approach that makes it possible for the concavity of the utility function to depend on equity considerations (Fleurbaey, 2010; Grant, Kajii, Polak and Safra, 2012), while remaining in the realm of the expected utility paradigm to ensure social
rationality, e.g., statewise dominance and time consistency.\textsuperscript{2}

The second line of research on which the paper builds bears on the evaluation of policies involving a variable population. Classical references in this field are Broome (1991), Broome (2004), Ng (1989) and Blackorby, Bossert and Donaldson (2005). We particularly build on Blackorby, Bossert and Donaldson (2007), which addresses both issues of variable population and uncertainty. This paper is formally the closest to our work and inspires our approach to the variable population problem, but they introduce strong Pareto and separability principles which impose the additive structure of utilitarianism, whereas we consider more general possibilities. Bommier and Zuber (2008) address a similar question, but they focus on the risk on population size. They rely on a weaker Pareto principle than Blackorby, Bossert and Donaldson (2007) but even this version may not be satisfactory according to arguments in Fleurbaey (2010). Asheim and Zuber (2014) also address the issue of variable population and risk, but their criterion is not an expected utility.

The paper is organized as follows. Section 2 introduces a family of social objectives that generalize the utilitarian criterion. This family makes it possible to introduce equity considerations while leaving some role for individual judgements in the evaluation of aggregate risks. Section 2 also discusses properties of this family in terms of the evaluation of situations involving populations of different sizes. Section 3 derives the implications of this family for the social discount rate and shows that the Ramsey formula (1) needs to be supplemented with additional terms involving population size, the global social welfare and the correlation between the different components of the social discount rate. Section 4 derives the implications of these new criteria for the question of catastrophic risks which has been studied in the utilitarian context by Weitzman (2009). Section 5 concludes. An Appendix contains the proofs of the results of Section 2.

\textsuperscript{2}An ex ante approach has been suggested by Diamond (1967) and investigated by Epstein and Segal (1992) but, as the latter contribution acknowledges, the ex ante approach may imply time consistency problems.
2 Social evaluation

2.1 The framework

We let $\mathbb{N}$ denote the set of positive integers, $\mathbb{N}_3$ the set of integers starting from 3, $\mathfrak{N}_3$ the set of subsets of $\mathbb{N}$ with cardinality at least 3, $\mathbb{R}$ the set of real numbers, and $\mathbb{R}_+$ the set of non-negative real numbers. For a set $S$ and any $n \in \mathbb{N}$, $S^n$ is the $n$-fold Cartesian product of $S$.

Our framework is adapted from Blackorby, Bossert and Donaldson (2007). The set of potential individuals (who may or may not exist) is $\mathbb{N}$. In the definition of a person, we include all his or her relevant characteristics (gender, birthplace, and so forth) and in particular the generation it belongs to. Hence there exists a mapping $T : \mathbb{N} \rightarrow \mathbb{N}$ that associates to each individual $i$ the date he or she will be born provided he or she comes to life, $T(i)$.

In contrast with Blackorby, Bossert and Donaldson (2007), we work directly with utility numbers. Hence an alternative $u$ is a collection of utility numbers, one for each individual alive in the alternative. Let $X$ be an interval in $\mathbb{R}$. We let $U = \bigcup_{n \in \mathbb{N}_3} X^n$ denote the set of possible alternatives — an alternative is a vector of utility numbers, one for each individual living in that particular alternative. Note that we restrict attention to situations in which the population has at least three members. In a variable-population framework, the size of the population may vary from one alternative to another. It is important to keep track of the population in an alternative. For any $u \in U$, we let $N(u)$ be the set of individuals in the alternative and $n(u) = |N(u)|$ be the number of individuals in the alternative.

Uncertainty is described by $m \in \mathbb{N} \setminus \{1\}$ states of the world. A prospect is a vector belonging to the set $U = U^m$ with typical element $u = (u_1, \cdots, u_m)$. A lottery is the combination of a probability vector $p = (p_1, \cdots, p_m) \in \Sigma^{m-1}$ — where $\Sigma^{m-1}$ denotes the closed $(m-1)$-simplex — and a prospect $u$. The set of lotteries is denoted

$$L = \{(p, u); p \in \Sigma^{m-1}, u \in U\}.$$ 

We choose to work with lotteries rather than prospects even though, in principle, all lotteries can be reformulated as prospects for a suitable partition of states of the world.\footnote{If we consider a framework à la Savage, with a infinite number of states of the world, it is possible to...}
The reason is that it is convenient in applications to be able to describe possible scenarios not only in terms of varying consequences, but also in terms of different probabilities over a small set of identified states of the world. This makes our analysis more amenable to applications. In particular, the role of probabilities in the determination of the discount rate will be much more transparent in this way.

For a prospect \( u \), whenever \( i \in N(u_s) \) for \( s \in \{1, \cdots, m\} \), \( u^i_s \) denotes the utility of individual \( i \) in state of the world \( s \). For a subpopulation \( \mathcal{N} \subset \mathbb{N} \), we denote by \( U_{\mathcal{N}} \) the set of prospects such that, for every \( u \in U_{\mathcal{N}} \), and every \( s \in \{1, \cdots, m\} \), \( N(u_s) = \mathcal{N} \). These are the prospects such that the same individuals are present in all states of the world. In this case, \( u^i = (u^i_1, \cdots, u^i_m) \) represents the prospect of individual \( i \). We let \( 1_m \) be the unit vector of \( \mathbb{R}^m \).

Our problem is to define a social ordering, i.e., a transitive and complete binary relation \( R \) on \( L \). The expression \( (p, u)R(q, v) \) will mean that \( (p, u) \) is at least as good as \( (q, v) \). We let \( P \) and \( I \) denote the corresponding strict preference and indifference relations.

More precisely, our problem is to select reasonable orderings among the myriad possible orderings of this set. The standard way of making such a selection is to list basic requirements (axioms) that embody appealing ethical principles, and to seek the orderings that satisfy such requirements.

2.2 Principles

We first want \( R \) to be as rational as one could be, given that it serves for a reasoned evaluation of social situations. The expected utility criterion, in spite of many criticisms, remains the benchmark of rational decision-making under risk and the following axiom requires \( R \) to take the form of expected (social) utility.

**Axiom 1 (Social expected utility hypothesis)** There exists a continuous function \( V : U \to \mathbb{R} \) such that, for all \( (p, u), (q, v) \in L \):

\[
(p, u)R(q, v) \iff \sum_{s=1}^{m} p_s V(u_s) \geq \sum_{s=1}^{m} q_s V(v_s)
\]

partition the set of states of the world in \( m \) subsets such that the subjective probabilities are \( (p_1, \cdots, p_m) \).

Restricting attention to acts which are constant on each subset, we obtain situations formally similar to the one we describe.
One limitation implied by this axiom is that it prevents \( R \) from evaluating what happens in one state of the world taking into account what would have happened in other states. In this fashion, ex ante fairness in lotteries (Diamond, 1967) is ignored, unless the utility numbers \( u_s \) in any given state do incorporate a measure of the chances that individuals had in other states. It is formally easy to generalize the criterion and rewrite it as \( \sum_{s=1}^{m} p_s V_s(p, u) \), but it is then difficult to come up with a precise proposal for the state-specific functions \( V_s \) that would evaluate the consequences in state \( s \) as a function of the whole lottery \( (p, u) \) (see Fleurbaey, Gajdos and Zuber, 2014).

The next axiom is a standard anonymity requirement.

**Axiom 2 (Anonymity)** For all \( \mathcal{N}, \mathcal{M} \in \mathcal{M}_3 \), for all \( p \in \Sigma_{m-1} \), for all \( u \in U_{\mathcal{N}} \) and \( v \in U_{\mathcal{M}} \), if \( |\mathcal{N}| = |\mathcal{M}| \) and if there exists a bijection \( \pi : \mathcal{N} \to \mathcal{M} \) such that \( u^i = v^{\pi(i)} \) for all \( i \in \mathcal{N} \), then \( (p, u)I(p, v) \).

In the context of intertemporal decision making and social discounting, Anonymity has been much discussed as a principle of intergenerational equity. Many economists in the utilitarian tradition have denounced the discounted utilitarian criterion because it deviates from the ideal of equal treatment of all individuals. For instance, Frank Ramsey famously described discounting as a “practice which is ethically indefensible and arises merely from the weakness of the imagination” (Ramsey, 1928, p. 543). The Stern review (Stern, 2006) also emphasized this ethical flaw of the standard approach. Drawing on these criticisms, a prolific literature has studied whether it would be possible to combine Anonymity with the Pareto principle in the context of infinite consumption streams. Although some positive results have been obtained, most of this literature stemming from Diamond (1965) has reached negative conclusions (Basu and Mitra, 2003; Zame, 2007; Lauwers, 2010). One way out of this dilemma is of course to consider a variable population framework and a risk on the population size, as proposed by Dasgupta and Heal (1979), Bommier and Zuber (2008) and Roemer (2011).

The Pareto principle is the hallmark of social evaluation, but the principle of consumer sovereignty is normally invoked when the individuals are fully informed about the options. In the presence of risk, by definition the individuals do not know what will ultimately happen if they choose such or such option, so that respecting their ex ante preferences is
less compelling than under full information. In particular, there are situations in which
the distribution of final situations across individuals is known ex ante, while it is only the
identity of winners and losers that is not known. In such situations, the ignorant individuals
may all be willing to take a risk, but everyone knows that it is not in the interest of the
ultimate losers and everyone knows that this ex ante unanimous preference for a risky lottery
will break down as soon as uncertainty is resolved. In view of such considerations, we restrict
the application of the Pareto principle to situations in which such a breakdown of unanimity
with greater information cannot occur. Two cases are retained here. There is first the case
of risk-free prospects, in which full information about final utilities prevails.

Axiom 3 (Pareto for no risk) For all \( N \in \mathbb{N}_3 \), for all \( p, q \in \Sigma^{m-1} \) and \( u, v \in U_N \) such
that for all \( i \in N \), there is \( u(i), v(i) \in X \) such that \( u^i = u(i)1_m \) and \( v^i = v(i)1_m \),

\[
u(i) \geq v(i) \text{ for all } i \in N \implies (p, u)R(q, v),
\]

and

\[
u(i) \geq v(i) \text{ for all } i \in N \\
v(j) > v(j) \text{ for some } j \in N \implies (p, u)P(q, v).
\]

Second, there is the case in which all individuals share exactly the same fate in all states
of the world. They may ultimately regret\(^4\) having taken a risk if they are unlucky, but they
will unanimously do so.

Axiom 4 (Pareto for equal risk) For all \( N \in \mathbb{N}_3 \), for all \( p, q \in \Sigma^{m-1} \), for all \( u, v \in U_N \)
such that for all \( s \in \{1, \cdots, m\} \) there is \( u(s), v(s) \in X \) such that \( u^i_s = u(s) \) and \( v^i_s = v(s) \)
for all \( i \in N \),

\[
(p, u)R(q, v) \iff \sum_{s=1}^{m} p_s u(s) \geq \sum_{s=1}^{m} q_s v(s).
\]

We also introduce a requirement of subpopulation separability. The motivation for such
axioms is primarily a matter of simplicity. Under separability it is possible to perform the
evaluation of a certain change affecting a particular population (e.g., the present and future

\(^4\)The notion of regret used here corresponds to a comparison with the decision that would have been made
under full information about the final state of the world. It does not mean that individuals would want to
change their decisions if they had to do it again under the same informational circumstances.
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generations) independently of the rest of the population that is not concerned (e.g., the past generations). The following axiom however only applies to riskless prospects in which utility is the same in all states.

**Axiom 5 (Separability for sure prospects)** For all $\mathcal{N} \in \mathbb{N}_3$, for all $p, q \in \Sigma^{m-1}$, for all $u, v, \tilde{u}, \tilde{v} \in U_{\mathcal{N}}$, if there exists $\mathcal{M}$ such that $\mathcal{M} \subset \mathcal{N}$ and

\[
\begin{align*}
    u^i &= v^i = u(i)1_m \text{ for all } i \in \mathcal{M}, \\
    \tilde{u}^i &= \tilde{v}^i = \tilde{u}(i)1_m \text{ for all } i \in \mathcal{M}, \\
    u^j &= \tilde{u}(j)1_m \text{ for all } j \in \mathcal{N} \setminus \mathcal{M}, \\
    v^j &= \tilde{v}(j)1_m \text{ for all } j \in \mathcal{N} \setminus \mathcal{M},
\end{align*}
\]

then $(p, u) R (q, v) \iff (p, \tilde{u}) R (q, \tilde{v})$.

We do not impose separability on risky prospects. One reason is that the separability principle may not be so attractive in that case. Consider the following prospects, described by matrices in which a cell gives the utility of an individual in a particular state of the world (rows are for two individuals, columns for two equiprobable states). It seems natural that the social ordering satisfies

\[
\begin{pmatrix}
    0 & 1 \\
    0 & 1 \\
\end{pmatrix}
\]

is preferred to

\[
\begin{pmatrix}
    1 & 0 \\
    0 & 1 \\
\end{pmatrix}
\]

because individual expected utilities are the same and less inequality ex post is obtained in the preferred prospect. The second individual faces the same personal prospect in both social prospects. Separability in risky situations would imply that

\[
\begin{pmatrix}
    0 & 1 \\
    1 & 0 \\
\end{pmatrix}
\]

is preferred to

\[
\begin{pmatrix}
    1 & 0 \\
    1 & 0 \\
\end{pmatrix}
\]

This conclusion is not appealing, because the two individuals still have the same expected utility in the two lotteries, but now the preferred lottery is one where inequality prevails ex post.

Our last axiom deals with the comparison of populations with different sizes. We want to be as flexible as possible and simply require the comparison to be possible in a certain systematic way. This is the role of the “critical-level” function.
Axiom 6 (Critical-level consistency) There exists a function $C : \mathbb{R} \times \mathbb{N} \to X$ such that for all $p \in \Sigma^{m-1}$, for all $u, v \in U$, for all $s \in \{1, \cdots, m\}$ and for all $k \in \mathbb{N} \setminus N(u_s)$, if $u_{s'} = v_{s'}$ for all $s' \in \{1, \cdots, m\} \setminus \{s\}$, $u_i^s = v_i^s$ for all $i \in N(u_s)$, $N(v_s) = N(u_s) \cup \{k\}$, $v_k^s = C(V(u_s), n(u_s))$, then $(p, u)I(p, v)$.

Compared to the Critical-Level Consistency Axiom by Blackorby, Bossert and Donaldson (2007), Axiom 6 imposes that the critical level depends on the welfare of the society without the additional individual rather than the whole vector $u_s$. This is more restrictive but it seems reasonable to argue that if we replace $u_s$ with another vector $\tilde{u}_s$ such that $V(\tilde{u}_s) = V(u_s)$ and $n(\tilde{u}_s) = n(u_s)$, there is no reason to change the critical level.

2.3 A family of equitable social objectives

The following proposition identifies a first family of social objectives.

Proposition 1 The social ordering $R$ satisfies Axioms 1, 2, 3, 4, 5 and 6 iff there exists a continuous increasing function $\phi$ and a sequence $(\alpha_n, \beta_n) \in \mathbb{R}_{++} \times \mathbb{R}$ such that for all $(p, u), (q, v) \in L$:

$$(p, u)R(q, v) \iff$$

$$\sum_{s=1}^{m} p_s \left[ \frac{1}{n(u_s)} \sum_{i \in N(u_s)} \phi(u_i^s) + \beta_{n(u_s)} \right] \geq \sum_{s=1}^{m} q_s \left[ \frac{1}{n(v_s)} \sum_{i \in N(v_s)} \phi(v_i^s) + \beta_{n(v_s)} \right]$$

and the sequence $(\alpha_n, \beta_n), n \geq 3$ satisfies the recursive property: for all $x \in V(U)$,

$$(n + 1)\phi \left( \frac{x - \beta_{n+1}}{\alpha_{n+1}} \right) = \phi(C(x, n)) + n\phi \left( \frac{x - \beta_n}{\alpha_n} \right),$$  \quad (2)

where $C$ is the critical-level function of Axiom 6.

Proof. See Appendix A. ■
Proposition 1 extends previous results by Fleurbaey (2010) to the variable population framework. The criteria axiomatized indeed involve taking the expected value of the equally-distributed equivalent (EDE) welfare, and we therefore use the label Expected Equally Distributed Equivalent (EEDE). Recall that for any given distribution of utilities across individuals, its equally-distributed equivalent is the utility level that, were it equally enjoyed by all individuals, would generate the same level of social welfare as the contemplated distribution.

Compared to the initial proposition in Fleurbaey (2010), Proposition 1 provides a specific (namely additively separable) form for the EDE function

\[ e(u_s) = \phi^{-1} \left( \frac{1}{n(u_s)} \sum_{i \in N(u_s)} \phi(u^i_s) \right). \]  

Proposition 1 also shows how the EDE relates to the VNM function \( V \) for different population sizes.

The class of criteria axiomatized in Proposition 1 clearly disentangles risk preferences and preferences for redistribution. As mentioned in the introduction, the discounted utilitarian model cannot distinguish between these two notions. On the contrary, in the EEDE criteria, individual risk attitudes are embodied in the utility indices \( u^i_s \), while social preferences for (welfare) redistribution are embodied in the functions \( \phi \). Note that it would be natural to assume that \( \phi \) is a concave function if the society wants to redistribute welfare ex post. This conclusion could be ensured by using a Pigou-Dalton principle of welfare redistribution.

Another feature of the EEDE criteria is that they explicitly deal with the population issue. In particular, Proposition 1 provides specific results on the form of the critical-level function. Indeed, the recursive property (2) identified in this Proposition is quite constraining. If for all \( n \), \( \alpha_n = \alpha \) and \( \beta_n = \beta \), Equation (2) implies

\[ C(x, n) = \frac{x - \beta}{\alpha}, \]

which means that the critical level is equal to the EDE. When the social ordering exhibits a strong aversion to inequality, the EDE is close to the lowest utility, which implies that it is then considered acceptable to add new members to society when their utility level is above the minimum.
The critical level can also be independent of \(V(u_s)\), but here again the constraints are substantial, as stated in the following result.

**Corollary 1** If the critical-level function \(C(x,n)\) associated with the social ordering defined in Proposition 1 is independent of \(x\) and if the social ordering is averse to inequality (\(\phi\) strictly concave), then \(C\) satisfies the following properties:

1. Either \(C(x,n) = \min X\) for all \(x \in X\) and for all \(n \in \mathbb{N}_3\); or \(C(x,n) = \max X\) for all \(x \in X\) and for all \(n \in \mathbb{N}_3\).
2. If \(X\) is bounded below and unbounded above, then \(C(x,n) = \min X\) for all \(x \in X\) and for all \(n \in \mathbb{N}_3\) and there exists \(\epsilon\) such that \(0 < \epsilon < 1\) and for all \((p,u),(q,v) \in L:\)

\[
(p,u) \succsim (q,v) \iff \sum_{s=1}^{m} p_s \left( \sum_{i \in N(u_s)} (u_s^i - \min X)^{1-\epsilon} \right)^{1/(1-\epsilon)} \geq \sum_{s=1}^{m} q_s \left( \sum_{i \in N(v_s)} (v_s^i - \min X)^{1-\epsilon} \right)^{1/(1-\epsilon)}.
\]

(In this case, \(\alpha_n = n^{1-\epsilon}, \beta_n = -n^{1-\epsilon} \min X\).)

**Proof.** See Appendix B. ■

The case \(C(x,n) = \max X\) is not palatable, as it implies a strong form of anti-populationism: it is never worth adding people to a population. It is noteworthy that the social ordering highlighted in point 2 has limited inequality aversion, as \(0 < \epsilon < 1\).

It is possible to have \(C(x,n) = c > \min X\) when \(\phi\) is no longer required to be concave everywhere and the social ordering is inequality averse above the critical level but inequality prone below it, i.e., with a formula like

\[
V(u_s) = \phi^{-1} \left( \frac{1}{n(u_s)} \sum_{i \in N(u_s)} \phi(u_s^i - c) \right),
\]

where \(\phi(z) = z^{1-\epsilon}\) when \(z > 0\), \(\phi(z) = -(-z)^{1-\epsilon}\) when \(z < 0\), and \(0 < \epsilon < 1\). Although this is a controversial form of social ordering, a possible justification for it is that it focuses on raising individuals above the critical level \(c\), even if this means sacrificing those who cannot make it (this is a triage approach discussed in Roemer, 2009).
For the analysis of the discount rate in the next section, what is important is the behavior of $\alpha_n/n$. In this respect, we have the following result:

**Corollary 2** If $\phi$ is concave and differentiable on $X$ and $\phi'(0) > 0$, $C(x,n)$ is non-decreasing in $x$, and for all $n \in \mathbb{N}$, $C(\alpha_n \min X + \beta_n, n) \geq \min X$, then $\alpha_n/n$ is non-increasing in $n$.

**Proof.** See Appendix C. ■

Corollary 2 may seem a general result. The assumption that $\phi$ is differentiable and positive at $\min X$ is however not innocuous. Observe that this condition is not satisfied in the case highlighted in point 2 of Corollary 1, and that indeed $\alpha_n/n$ is increasing in $n$ in that case.

## 3 Implications for the discount rate

In this section we derive the social discount rate for the family of EEDE social welfare functions. In the discounted utilitarian approach yielding Equation (1), the discount rate on consumption is the simple addition of a discount rate on utility and a specific term relative to consumption, combining the growth rate of consumption and the rate of decrease of marginal utility.

With the alternative criteria proposed here, the expression of the social discount rate is substantially modified. First, in addition to the utility discount rate, a term related to population size appears. Second, the consumption term generally differs from the utilitarian approach due to equity concerns. Two additional terms appear: one has to do with global social welfare; the other is related to the covariance between the different components of the social discount rate.

### 3.1 Social and person-to-person discount rates

The vast majority of the research on the social discount rate considers generations rather than individuals. It computes the social discount rate to be applied to assess transfer from one generation to another. In the present paper which allows for inequalities within generations, we must consider individuals rather than generations. We therefore follow the
approach that we have developed in a more general framework in Fleurbaey and Zuber (2014), and first compute person-to-person discount rates.

To define this concept, we first need to further specify our framework. Recall that we consider a social welfare function taking the general form
\[
W(p, u) = \sum_{s=1}^{m} p_s V(u_s).
\]
In this formula, we now assume that in each state \( s \in \{1, \cdots, m\} \)
\[
\begin{align*}
    u_s &= (u(c^i_s))_{i \in N(u_s)}, \\
    c^i_s &\in X_c
\end{align*}
\]
where \( c^i_s \in X_c \) is the consumption of individual \( i \) in state \( s \), and \( X_c \) is an interval of real numbers. We therefore assume here that all individuals have the same utility function \( u \). Extending the analysis to the case of heterogeneous utility functions is cumbersome but straightforward.

When individuals, not generations, are the constitutive elements of social welfare, the discount rate must be computed primarily between two individuals.

**Definition 1** The person-to-person discount rate from an individual \( i \) in period 0 to an individual \( j \) in period \( t \), denoted \( \rho_{i,j}^t \), is:
\[
\rho_{i,j}^t = \left( \frac{\sum_{s=1}^{m} p_s \frac{\partial V(u_s)}{\partial u^i_s} u'(c^i_s)}{\sum_{s=1}^{m} p_s \frac{\partial V(u_s)}{\partial u^j_s} u'(c^j_s)} \right)^{\frac{1}{t}} - 1. ~ \tag{4}
\]
To understand this definition, imagine that today (period 0) individual \( i \) can make an investment whose sure rate of return is \( r \) for the benefit of individual \( j \) living in period \( t \). In the margin, such an investment has no effect on social welfare if:
\[
\sum_{s=1}^{m} p_s \frac{\partial V(u_s)}{\partial u^j_s} u'(c^j_s) (1 + r)^t = \sum_{s=1}^{m} p_s \frac{\partial V(u_s)}{\partial u^i_s} u'(c^i_s),
\]
with the convention that \( \frac{\partial V(u_s)}{\partial u^j_s} = 0 \) if \( j \) does not exist in state \( s \). Observe that, while the existence and consumption of \( i \) in period 0 is certain, the social marginal utility of her consumption \( \frac{\partial V(u_s)}{\partial u^i_s} \) may vary across states of the world.

When many individuals from period 0 make an investment that benefits many individuals in period \( t \), one can evaluate the investment with a social discount rate that aggregates the
person-to-person discount rates, provided the shares of the individuals in the investment (either as investors or as beneficiaries) are fixed. Suppose that each donor $i$ in period 0 bears a fraction $\sigma^i_0$ of the marginal investment $\varepsilon$, and that each recipient $j$ in period $t$ receives a fraction $\sigma^j_t$, with $\sum_i \sigma^i_0 = \sum_j \sigma^j_t = 1$. The social discount rate is again the sure rate of return on the marginal investment that leaves social welfare unchanged.

**Definition 2** The social discount rate from period 0 to period $t$, denoted $\rho_t$, is:

$$\rho_t = \left( \frac{\sum_{s=1}^m p_s \sum_{i:T(i)=0} \sigma^i_0 \frac{\partial V(u_s)}{\partial u^i_s} u' \left( c^i_s \right)}{\sum_{s=1}^m p_s \sum_{j:T(j)=t} \sigma^j_t \frac{\partial V(u_s)}{\partial u^j_s} u' \left( c^j_s \right)} \right)^{\frac{1}{t}} - 1$$

$$= \left( \sum_i \sigma^i_0 \left[ \sum_j \sigma^j_t \left( 1 + \rho^i,j_t \right)^{-1} \right] \right)^{\frac{1}{t}} - 1. \tag{5}$$

In the sequel we shall focus on the computation of the person-to-person discount rate. But it is important to bear in mind that such person-to-person discount rates are only elements of the more general social discount rate, which are aggregated through the formula displayed in Equation (5). Such an aggregation may give rise to interesting intra-generational equity issues that are explored in Fleurbaey and Zuber (2014) but not in the present paper.

### 3.2 The Ramsey formula revisited

Let us first briefly examine how Equation (4) applies in the utilitarian case, before examining the alternative criteria characterized in the previous section. Note that the standard critical level utilitarian case is actually a special case of the criterion axiomatized in Proposition 1, namely when $\phi(u) = u$, $\alpha_n = n$ and $\beta_n = -nu_c$, where $u_c$ is a critical utility level. In that case, $V(u_s) = \sum_{i \in N(u_s)} (u^i_s - u_c)$, and $\partial V/\partial u^i_s = \partial V/\partial u^j_t = 1$ for the individuals of period 0 and for the individuals of period $t$ in the states in which they exist.

This considerably simplifies formula (4). Let the probability of $j$ coming to existence be $p(j) = \sum_{s,j \in N(u_s)} p_s$, and let $\pi^j = -\ln(p(j))$. Also, for a random variable $X_s$, denote $EX_s = \sum_{s=1}^m p_s X_s$ its expected value and $E^j X_s = \sum_{s,j \in N(u_s)} (p_s/p(j)) X_s$ its expected
discounting beyond utilitarianism

value conditional on \( j \) existing. In the critical-level utilitarian case, one obtains

\[
\rho_{t}^{i,j} = \left( \frac{u'(c^i_j)}{\sum_{s,j \in N(u_s)} p_s u'(c^i_s)} \right)^{\frac{1}{t}} - 1
\]

\[
\simeq \frac{\pi^j_t}{t} - \frac{1}{t} \ln \left( \frac{E^j u'(c^i_s)}{u'(c^i)} \right).
\]

This expression of the discount rate is clearly very close to the usual Ramsey formula. Indeed, if we denote \( \delta = \frac{\pi^j_t}{t} \) and assume that \( u(c) = c^{1-\eta}/(1 - \eta) \) and \( c^i_j = e^\eta c^i \) we directly obtain Equation (1). This provides a foundation of the rate of pure time preferences \( \delta \) based on the risk on the existence of future generations, as proposed by Dasgupta and Heal (1979) and Stern (2006). Another noticeable feature of the above expression is that the critical level plays no role in the value of the discount rate: population ethics views do not affect discounting.

Let us now examine the social discount rate for the more general family of social welfare functions introduced in the Section 2, namely

\[
W(p, u) = \sum_{s=1}^{m} p_s \left[ \alpha_{n(u_s)} \phi^{-1} \left( \frac{1}{n(u_s)} \sum_{i \in N(u_s)} \phi(u(c^i_s)) \right) + \beta_{n(u_s)} \right]. \tag{6}
\]

To do so, we need to introduce additional notation. We first introduce the terms \( \mu^i = u'(c^i) \phi'(u(c^i)) \) and \( \mu^s = u'(c^s) \phi'(u(c^s)) \) that only concerns individual consumption levels.

Then let us recall and adapt the definition of the equally distributed equivalent welfare level, as expressed in Equation (3):

\[
e(u_s) = \phi^{-1} \left( \frac{1}{n(u_s)} \sum_{i \in N(u_s)} \phi(u(c^i_s)) \right).
\]

The social welfare level in a given state of the world \( s \) is positively related to the following magnitude:

\[
\xi_s = \left[ \phi'(e(u_s)) \right]^{-1}.
\]

We also need the following term, introduced in Corollary 2, that relates to the value of population size in a given state of the world: \( \nu_s = \alpha_{n(u_s)}/n(u_s) \).

Lastly, we need notation for covariance between variables. For two random variables \( X_s \) and \( Y_s \), \( Cov(X_s, Y_s) = E(X_s - EX_s)(Y_s - EY_s) \), with similar notation for the covariance
Proof. For the EEDE family of social welfare functions, we have:

\[
\begin{align*}
\text{Cov}(X_s, Y_s, Z_s) &= \frac{1}{3} \left( \text{Cov}(X_s, Y_s) E Z_s + \text{Cov}(Z_s, Y_s) E X_s + \text{Cov}(X_s, Z_s) E Y_s \right) + \\
&= \frac{1}{3} \left( \text{Cov}(X_s Z_s, Y_s) + \text{Cov}(X_s Y_s, Z_s) + \text{Cov}(Y_s Z_s, X_s) \right),
\end{align*}
\]

with similar notation for the covariance conditional on the existence of \( j \).

We are now able to state the following result:

**Proposition 2** For the EEDE family of social welfare functions (6), the person-to-person discount rate can be approximated in the following way:

\[
\rho_{t,j}^{ij} \approx \frac{\pi^j}{t} - \frac{1}{t} \ln \left( \frac{E^j_{t,i} \mu_s^j}{\mu^i} \right) - \frac{1}{t} \ln \left( \frac{E^j_{t,i} \nu_s^j}{E \nu_s} \right) - \frac{1}{t} \ln \left( \frac{E^j_{t,i} \xi_s^j}{E \xi_s} \right) - \frac{1}{t} \ln \left( 1 + \frac{\text{Cov}^j(\mu_s^j, \xi_s^j, \nu_s^j)}{E \mu_s^j E \xi_s^j E \nu_s} \right) - \frac{1}{t} \ln \left( 1 + \frac{\text{Cov}^j(\nu_s^j, \xi_s^j, \mu_s^j)}{E \nu_s^j E \xi_s^j E \mu_s} \right).
\] (7)

**Proof.** For the EEDE family of social welfare functions, we have:

\[
1 + \rho_{t,j}^{ij} = \left( \frac{\sum_{s:1 \in N(u_a)} p_s \left[ \frac{\alpha_{n(u_a)} \phi(u(c^t)) u(c^t)}{n(u_a)} \right]}{\sum_{s:1 \in N(u_a)} p_s \left[ \frac{\alpha_{n(u_a)} \phi(u(c^t)) u(c^t)}{n(u_a)} \right]} \right)^{\frac{1}{t}} = \left( \frac{1}{p(j)} \right)^{\frac{1}{t}} \left( \frac{\mu^i E^j_{t,i} \nu_s^j}{E^j \nu_s^j E^j_{t,i} \xi_s^j} \right)^{\frac{1}{t}}.
\]

Therefore,

\[
\rho_{t,j}^{ij} \approx \ln(1 + \rho_{t,j}^{ij}) = \frac{\pi^j}{t} - \frac{1}{t} \ln \left( \frac{E^j_{t,i} \nu_s^j \mu_s^j}{E^j \nu_s^j E^j_{t,i} \xi_s^j} \right)
\]

Then we use the fact that, by definition, \( E[\nu_s \xi_s^j] = E \nu_s E \xi_s^j + \text{Cov}(\nu_s, \xi_s^j) \), and \( E^j[\nu_s \mu_s^j \xi_s^j] = E^j \nu_s E^j \mu_s^j E^j \xi_s^j + \text{Cov}^j(\mu_s^j, \nu_s, \xi_s^j) \), so that:

\[
\rho_{t,j}^{ij} \approx \frac{\pi^j}{t} - \frac{1}{t} \ln \left( \frac{E^j \mu_s^j E^j \nu_s E^j \xi_s^j \left( 1 + \frac{\text{Cov}^j(\mu_s^j, \xi_s^j, \nu_s^j)}{E \mu_s^j E \xi_s^j E \nu_s} \right)}{\mu^i E \nu_s E \xi_s^j \left( 1 + \frac{\text{Cov}(\xi_s^j, \nu_s^j)}{E \xi_s^j E \nu_s} \right)} \right).
\]

We can compare formula (7) with the discounting formula obtained in the critical-level utilitarian case. The first term \( \frac{\pi^j}{t} \) was already present in the utilitarian case and embodies the risk on the existence of future generations.

The second term of formula (7) is clearly related to the term \(- \frac{1}{t} \ln \left( \frac{E^j u(c^t)}{u(c^t)} \right)\) of the Utilitarian formula in the sense that it compares the social priority of the individuals based
on their levels of consumption. The difference is that the new formula adds an equity term $\phi'(u)$, which represent the social priority of individual welfare. This terms introduces society’s equity preferences, while the marginal utility of consumption in the standard Utilitarian formula should be measured using individuals VNM utilities that represent their risk preferences. This extra term will provide extra reasons to discount future consumption if the future individual is richer. A simple illustration can be provided using the functional forms $u(c) = \frac{c^{1-\eta}}{1-\eta}$ and $\phi(u) = -\left(\frac{-u}{1+u}\right)^{1+\epsilon}$ if $\eta > 1$ and $\phi(u) = \frac{u^{1-\epsilon}}{1-\epsilon}$ if $0 < \eta < 1$. We then obtain:

$$\phi'(u(c))u'(c) = \begin{cases} (\eta - 1)^{-\epsilon}c^{-(\eta + \epsilon(\eta - 1))} & \text{if } \eta > 1, \\ (1 - \eta)^{\epsilon}c^{-(\eta + \epsilon(1 - \eta))} & \text{if } \eta < 1. \end{cases}$$

In the case where $c_s = e^g c^i$, we readily obtain

$$-\frac{1}{t} \ln \left( \frac{E^j \mu_{s-j}}{\mu^i} \right) = \begin{cases} (\eta + \epsilon(\eta - 1))g & \text{if } \eta > 1, \\ (\eta + \epsilon(1 - \eta))g & \text{if } \eta < 1, \end{cases}$$

so that we discount the future more, the higher the social preference for redistribution $\epsilon$ (with the utilitarian case corresponding to $\epsilon = 0$), provided $g > 0$. The situation is more tricky when there is risk. Indeed, the introduction of $\epsilon$ can be seen as increasing in $\eta$, the individual coefficient of relative risk aversion. Gollier (2002) has showed that such an increase may have an ambiguous effect on discounting when $g$ is normally distributed, depending on the mean and variance of the distribution of $g$.

The third term involving $\nu_s = \frac{\alpha_n(u_s)}{n(u_s)}$ captures the additional role of the risk on population size. One way to look at this term is that it compares the correlation between the existence of the concerned individuals and a ‘value’ of population size measured by $\nu_s$. Indeed one can write $E^j \nu_s = Ev_s + Cov(1^j_s, \nu_s)/p(j)$ where $1^j_s = 1$ if $j \in N(u_s)$ and 0 otherwise (one has $E1^j_s = p(j)$), so that $-\frac{1}{t} \ln \left( E^j \nu_s / Ev_s \right) = -\frac{1}{t} \ln \left( 1 + \frac{Cov(1^j_s, \nu_s)}{p(j)Ev_s} \right)$. The direction of this term will depend on the monotonicity of the sequence $(\alpha_n/n)_{n \in \mathbb{N}}$: if it is increasing, the term will negative; if it is decreasing, the term will be negative. Indeed, the future individual belongs on average to larger (intertemporal) populations; if each individual of a larger population brings lower social value (as measured by $\alpha_n/n$), then there is an additional reason to discount future consumption.

Note that views regarding the critical level function will have strong implications for this term. Indeed, as shown in Corollary 1, if the critical level is constant (and $X$ is
bounded below but not above) then \(\frac{\alpha_n}{n} = n^{\frac{\kappa}{\tau}}\), so that the population term will tend to decrease the discount rate. On the contrary, as shown in Corollary 2, if the critical level function is increasing in its first component and \(\phi\) is concave, everywhere differentiable with positive derivative, then \(\frac{\alpha_n}{n}\) will be non-increasing, so that the population term will tend to increase the discount rate.

To illustrate these implications, consider a case in which there is a risk on the existence of future generations. Each period, with probability \(p\) the world survives to the next period, and with probability \(1-p\) the human species (and any species relevant for welfare) disappears.\(^5\) We also consider that potential population (i.e. absent the extinction risk) grows, so that total (intertemporal) population when the world exists until generation \(t\) is \(n_t = (1 + n)^t n_0\). The real number \(n\) can be interpreted as a long-run population growth rate.\(^6\) A last simplifying assumption is that \(\frac{\alpha_n}{n} = \kappa\), so that \(\nu = \frac{\alpha_n}{n} = n^{\kappa-1}\). The case 2 of Corollary 1 corresponds to \(\kappa > 1\); Corollary 2, on the contrary, would suggest \(\kappa < 1\). In this simple case, we obtain that

\[
E_\nu_s = \sum_{T=0}^{+\infty} p(1-p) T n_0^{\kappa-1} (1+n)^{(\kappa-1)T} = n_0^{\kappa-1} \frac{1}{1 - (1-p)(1+n)^{\kappa-1}},
\]

provided that \(0 < (1-p)(1+n)^{\kappa-1} < 1\). Also,

\[
E^j_\nu_s = \sum_{T=t}^{+\infty} p(1-p) T^{t-1} n_0^{\kappa-1} (1+n)^{(\kappa-1)t} = n_0^{\kappa-1} (1+n)^{(\kappa-1)t} \frac{1}{1 - (1-p)(1+n)^{\kappa-1}}.
\]

Hence \(-\frac{1}{T} \ln \left(\frac{E^j_\nu_s}{E_\nu_s}\right) = (1-\kappa) \ln(1+n)\). This highlights that this term can be a significant component of the discount rate, and that \(\kappa > 1\) (Corollary 1) or \(\kappa < 1\) (Corollary 2) makes a difference.

\(^5\)In this example, we consider a countably infinite number of states of the world. All our formulas can be extended to that case.

\(^6\)Indeed, assume that we take past populations into account, and that from generation \(-\tau\) on population grows at rate \(n\). Hence population in period \(T > 0\) is \(n_T = N_{-\tau}(1+n)^{T+\tau}\) so that total population when the world exist until generation \(T\) is \(n_T = n_{-\tau-1} + N_{-\tau} \sum_{s=-\tau}^{T} (1+n)^s = n_{-\tau-1} + N_{-\tau} \frac{1}{n} (1+n)^{T+1} - 1 = (1+n)^{T+1} N_{-\tau} \frac{1}{n} \left(1 - \frac{1}{(1+n)^{\tau+1}} + \frac{\tau}{(1+n)^{\tau+1} N_{-\tau}} \right)\). If \(\tau\) is sufficiently large, we have \(n_0 \approx \frac{(1+n)^{\tau+1} N_{-\tau}}{n}\) and \(n_t \approx (1+n)^T n_0\) for all \(t > 0\).
The next term, $-\frac{1}{t} \ln \left( \frac{E^j \xi_s}{E \xi_s} \right)$, looks at expected social welfare levels in states where the individuals exist. Much like the second term, it can be viewed as measures of the association between the existence of the concerned individuals and the social welfare level. Indeed one can write $E^j \xi_s = E \xi_s + \text{cov}(1^j_s, \xi_s)/p(j)$. The intuition is the following: if the future individual belongs on average to populations where social welfare is higher, we should further discount her consumption, because social welfare would be increased in states where general welfare is already higher if we were to transfer money to her. Note that this term, like the first and third term is zero if the future person exists in all states of the world.

This is not true of the last term which is a general covariance term. This term is more complicated to account for. Note however that if the risk on population is independent of the risk on consumption, the expression simplifies to

$$- \ln \left( 1 + \frac{\text{Cov}^j(\mu^j_s, \xi_s)}{E^j \mu^j_s E^j \xi_s} \right),$$

because in such a case, $\text{Cov}(\xi_s, \nu_s) = 0$ and $\text{Cov}^j(\mu^j_s, \xi_s, \nu_s) = E^j \nu_s \text{cov}^j(\mu^j_s, \xi_s)$. In this case, the covariance term transparently complements the previous term: in addition to the covariance between the existence of the individual and general welfare, the covariance between her social priority and general welfare matters. When this covariance is high for the future person, we want to discount her consumption less because she has a high priority in good social states, i.e., in states in which the value of the EDE is sensitive to $j$’s fate because it is sensitive to the less well-off.

Another interesting possibility is when $\xi_s$ is almost the same in all states of the world. This may occur when the society has strong preferences for redistribution and there is a significant number of poor people in all state of the world (perhaps because these are many poor past people, or because we are not able to completely alleviate poverty in the future). Then $\text{cov}(\xi_s, \nu_s) = 0$ and $\text{cov}^j(\mu^j_s, \xi_s, \nu_s) = E^j \xi_s \text{cov}^j(\mu^j_s, \nu_s)$, so that the term becomes

$$- \ln \left( 1 + \frac{\text{Cov}^j(\mu^j_s, \nu_s)}{E^j \mu^j_s E^j \nu_s} \right).$$

---

7This may occur when we think that the extinction risk is purely exogenous and that fertility is not influenced by economic conditions. The assumption seems less sensible if we consider endogenous fertility, and the possibility that consumption and population size may be influenced by a common phenomenon, e.g. climate change.
If in addition population size and consumption are independent, this term disappears and only the first three terms of (7) remain, providing a formula for the discount rate which would also be obtained with the following number-dependent prioritarian social criterion:

\[ V(\mu_s) = \frac{\alpha_n(\mu_s)}{n(\mu_s)} \left( \sum_{i \in N(\mu_s)} \phi(\mu_s^i) \right) + \beta_n(\mu_s). \]

4 Catastrophic risk

In an influential recent paper, Weitzman (2009) suggests that, in the presence of a fat tail in the distribution of risk, the discount rate can approach \(-1\), implying an absolute priority to future consumption (the “dismal theorem”). His argument relies on the utilitarian criterion. In this section we reexamine it in the context of the EEDE criteria introduced in Section 2.

Weitzman’s basic line of reasoning is as follows. The utilitarian discount rate, without approximation, satisfies the equation:

\[
\ln \left( 1 + \rho^{i,j}_{i,t} \right) = -\ln p(j) - \frac{1}{t} \ln \left( \frac{E^j u'(c^j)}{u'(c^i)} \right).
\]

The critical term for the argument is \(E^j u'(c^j)\), which, in the case of a CRRA function \(u(c^j) = \frac{1}{1-\eta}(c^j)^{1-\eta}, \eta > 1\), and a continuous distribution of \(c^j\) depicted by a probability distribution function (henceforth PDF) \(f\), is equal to \(\int (c^j)^{-\eta} f(c^j)dc^j\). If one changes variables so as to refer to a growth rate, \(c^j = c_0 e^{gt}\), the expression becomes

\[
e^{-\eta \gamma} \int e^{-\eta \gamma f(g)}dg,
\]

where \(\hat{f}\) is the PDF of the growth rate of consumption. Expression (8) is essentially the moment-generating function of \(\hat{f}\), and it is infinite if \(\hat{f}\) has a fat tail in the negative values representing catastrophic risks.

A fat tail means that \(\hat{f}(g) \propto (-g)^{-k}\) for some \(k > 0\) when \(g \to -\infty\). The PDF \(f\) cannot have a fat tail in the low values of \(c^j\) because \(c^j\) is bounded from below. However, one has \(f(c^j) = \hat{f} \left( \frac{1}{t} \ln \frac{c^j}{c_0} \right) \propto (-\ln c^j)^{-k}\) when \(c^j \to 0\). Such a PDF, for instance, has the property that, conditional on \(c^j < q\), the probability of \(c^j < q/2\) remains around 50% when \(q \to 0\).

Weitzman (2009) motivated the fat tail assumption by the example of climate change, arguing that scientific uncertainty about the functioning of the climate system may give
rise to a fat tailed distribution of future temperature. In fact, we do not need a fat tailed
distribution of \( \text{temperatures} \) to support an argument in favor of giving an absolute priority
to the future. It is sufficient that: 1) whenever temperature \( T \) is above a certain threshold \( T^* \), consumption is 0; 2) \( u'(0) = +\infty \); and 3) there is a positive probability that \( T > T^* \).
The result is true more generally if there is a subsistence level \( c_{\text{min}} \) such that \( u'(c_{\text{min}}) = +\infty, \)
and future generations are back to the subsistence level when \( T > T^* \).

Given the frightening worst-case scenarios involving temperature increase above 10\(^\circ\)C
or 20\(^\circ\)C, it is not unreasonable to assign a positive probability to the event of having a
substantial part of the population at the subsistence level in future generations —in fact,
extinction would be more accurate description of the situation.\(^8\) The weakness of the argu-
ment in the preceding paragraph is rather the assumption of an infinite marginal utility at
the subsistence level so that it is an absolute priority to raise \( c \) above \( c_{\text{min}} \) at any period.
A typical form for the utility function could be \( u(c) = \frac{1}{1-\eta} \left(c^{1-\eta} - (c_{\text{min}})^{1-\eta}\right) \), which has
a finite marginal utility at \( c_{\text{min}} > 0 \). With such a function, the utilitarian discount rate
remains finite even when the probability of \( c = c_{\text{min}} \) is positive.\(^9\)

Let us now examine how Weitzman’s result change when using the EEDE family of social
welfare functions introduced in the Section 2, namely
\[
W(p, u) = \sum_{s=1}^{m} p_s \left[ \alpha n(u_s) \phi^{-1} \left( \frac{1}{n(u_s)} \sum_{i \in N(u_s)} \phi(u(c^i_s)) \right) + \beta n(u_s) \right].
\]

To simplify things, first assume that the risk on population is independent of the risk
on consumption and that the equally distributed equivalent is almost the same in all states
of the world. In that case, we have indicated at the end of Section 3.2 that
\[
\rho_{t}^{i,j} \simeq \frac{\pi_{t}^{j}}{t} - \frac{1}{t} \ln \left( \frac{E^j \phi'(u(c^j_s))u'(c^j_s)}{\phi'(u^j)u'(u^j)} \right) - \frac{1}{t} \ln \left( \frac{E^j u^j}{E u^j} \right).
\]
The critical term here is \( E^j \phi'(u(c^j_s))u'(c^j_s) \). And we may well obtain the dismal result that
\( E^j \phi'(u^j)u'(c^j_s) \to +\infty \), even if \( u'(c_{\text{min}}) \neq 0 \). We only need that \( \phi'(u_{\text{min}}) = +\infty \), so that it

\(^8\)One should not forget that widespread premature deaths and bare survival for large numbers of the
population is already the case today, for reasons having little to do with the climate. As Schelling (1995)
argued, if the possible poverty of future generations is the reason to give them priority, we should give a
stronger priority to the poor who exist today with certainty.

\(^9\)Actually, Weitzman (2009) considers the utility function \( u(c) = \frac{1}{1-\eta} \left(c^{1-\eta} - (c_{\text{min}})^{1-\eta}\right) \) in his paper, but
he obtains a “dismal” discount rate by letting \( c_{\text{min}} \to 0 \).
is an absolute priority to raise $c$ above $c_{\text{min}}$. The difference with the Weitzman result is that the absolute priority does not come from an infinite marginal utility of consumption but from an absolute social priority to raise people’s welfare above the minimal level of welfare.

This line of argument, however, no longer works when the equally distributed equivalent utility level varies between states of the world. Consider the following example. Assume that there is no risk on population and no intra-generational inequality. The world exists for $T$ generations, each generation has $N$ individuals consuming $c_t^s$. Assume that $c_t^s = e^{g_t^s}c_0$, so that uncertainty is only about the consumption growth rate. Assume also that $u(c) = c^{1-\eta}$, with $0 < \eta < 1$, and $\phi(u) = \ln u$. The EDE utility level in state $s$ is therefore:

\[
e(u_s) = \exp \left( \frac{1}{T} \sum_{t=0}^{T} \ln \left( (e^{g_t^s}c_0)^{1-\eta} \right) \right) = c_0^{1-\eta} \exp \left( \frac{1}{T} \sum_{t=0}^{T} (1-\eta)g_t^s \right) = c_0^{1-\eta} \exp \left( \frac{(T+1)(1-\eta)g_s}{2} \right).
\]

Noting that $(\phi'(u))^{-1} = u$ when $\phi(u) = \ln u$, we have that $(\phi'(c(u_s)))^{-1} = c_0^{1-\eta} \exp \left( \frac{(T+1)(1-\eta)}{2} g_s \right)$.

Also, $\phi(u(c)) = (1-\eta)\ln c$ so that $\phi'(u(c))u'(c) = (1-\eta)c^{-1}$. Thus the discount rate is defined by:

\[
1 + \rho_{t,j}^i = \left( \frac{E_{\phi'(u(c_i^j))u'(c_i^j)}^s}{E_{\phi'(u(c_i^j))u'(c_i^j)}^s} \right)^{-\frac{1}{\mu}} = \left( \frac{E(1-\eta)(c_i^j)^{-1}c_0^{1-\eta} \exp \left( \frac{(T+1)(1-\eta)}{2} g_s \right)}{E(1-\eta)(c_0)^{-1}c_0^{1-\eta} \exp \left( \frac{(T+1)(1-\eta)}{2} g_s \right)} \right)^{-\frac{1}{\mu}}
\]

\[
= \left( \frac{E \exp \left( \left( \frac{(T+1)(1-\eta)}{2} - t \right) g_s \right)}{E \exp \left( \frac{(T+1)(1-\eta)}{2} g_s \right)} \right)^{-\frac{1}{\mu}}.
\]

If $\frac{(T+1)(1-\eta)}{2} > t$, that is if the planning horizon is sufficiently long compared to the distance with the future individual, the expectation $E \exp \left( \left( \frac{(T+1)(1-\eta)}{2} - t \right) g_s \right)$ may converge even in the presence of a fat (left) tail in the distribution of the negative values of $g$. In that case, what would be problematic is rather a fat (right) tail in the distribution of the positive values of $g$. Note that the expectation $E \exp \left( \frac{(T+1)(1-\eta)}{2} g_s \right)$ would also converge, so that the discount rate would have a finite value, in contrast to Weitzman’s result.

More strikingly, in this case, society may focus mainly on the good outcomes, rather than on catastrophic states. To see that, consider that $g$ is uniformly distributed on the
discounting beyond utilitarianism

\[ \text{interval } [g, \bar{g}], \text{ with } g < 0 < \bar{g}. \text{ Then,} \]

\[
E \exp \left( \left( \frac{(T+1)(1-\eta)}{2} - t \right) g_s \right) = e^{\frac{(T+1)(1-\eta)}{2} \bar{g}} - e^{\frac{(T+1)(1-\eta)}{2} g}
\]

and

\[
E \exp \left( \left( \frac{(T+1)(1-\eta)}{2} \right) g_s \right) = e^{\frac{(T+1)(1-\eta)}{2} \bar{g}} - e^{\frac{(T+1)(1-\eta)}{2} g}.
\]

Hence,

\[ 1 + \rho_t^{i,j} = e^{\bar{g}} \left( 1 - \frac{t}{(T+1)(1-\eta)} \right)^\frac{1}{t} \left( 1 - e^{\frac{(T+1)(1-\eta)}{2}(\bar{g}-g)} \right)^\frac{1}{t}. \]

If \( g \to -\infty \) and \( T \to \infty \), then:

\[ \rho_t^{i,j} \approx \ln(1 + \rho_t^{i,j}) \approx \bar{g}. \]

The main intuition behind this result is the following. In the EEDE approach, the social priority of the consumption of both the current and the future generation is nil when the growth rate of consumption is negative. This is so because their relative social priority embodied in the term \( \phi'(u_j(s)) \phi'(e(u_j)) \) is zero, and this is sufficient to overcome their differences in terms of marginal utility of consumption.

5 Conclusion

In this paper, we have introduced a general framework in which the horizon is finite but uncertain, and uncertainty bears on future utility as well as on the composition of the future population. Doing so, we have characterized non-utilitarian criteria which embody a greater concern for equity than utilitarianism, at the cost of weakening the Pareto principle. We have thus been able to explicitly introduce concerns for population size and to disentangle risk aversion and inequality aversion.

We showed that the expression of the social discount rate should then be modified in several respects. First, the consumption term should be augmented to take into account equity concerns. Second, a population term appears, which will depend on ethical views regarding population size. Third, two correlation terms emerge: one between the existence of future generations and intertemporal welfare; another between individual priority, population size and intertemporal welfare.
The role of correlations between individual and social well-being as an important factor in evaluations is a key contribution to the refinement of the Ramsey formula, that may reverse the conclusions of Weitzman’s dismal theorem. Benefitting an individual who is badly off when the population is well off has a greater impact on social welfare, on average, than benefitting an individual who is badly off when the population is also badly off. This may seem disturbing because it seems to give a bonus to the states of the world in which the population is relatively well off. This occurs, however, only in the very special trade-off between helping a poor with a positive correlation with social welfare and a poor with a negative correlation. But most policy issues affect broader populations. Suppose one invests in a public good that is useful mostly in bad states (e.g., flood protection). When a bad state occurs, the investment benefits more individuals who are badly off. Even if the correlation between their well-being and social welfare is high, the fact that the investment benefits many badly off individuals may be sufficient to give it a greater social value than a similar investment that would create a public good suited to good states (e.g., a new transportation infrastructure).

Concerning the effect of inequality aversion on social discounting, it is known that inequality aversion increases discounting when future generations are better-off. It is also known that when growth is uncertain, and there is a substantial risk of future generations being less well-off, a higher inequality aversion can on the contrary decrease the discount rate. Our more general approach adds that, if the investment helps the most vulnerable in future generations, inequality aversion further decreases the discount rate. In addition, inequality aversion magnifies the effect of the correlation on discounting when future consumption is uncertain.

In the end, this paper provides reasons to think that the specific features of climate policies may justify evaluating them with a lower discount rate than other policies. Indeed, they protect the vulnerable, whose fate may be inversely correlated to that of the rich, and they provide more benefits in states of the world in which damages hit the poorest. Further research is however needed to substantiate those intuitions. It would require a more precise description of the uncertainty (on consumption and the existence of future generations) as well as good scenarios describing the costs and benefits. Moreover, in order to assess climate policies, one may also go beyond the discount rate and evaluate the changes in the risks
they induce, their non-marginal effects and their precise impact.

Another direction of research that we intend to pursue is to enrich the framework further so as to make it possible to discuss the measurement of individual well-being. In this paper the measurement of utility has been treated as exogenous. A more concrete description of the economic allocations would enable us to further specify the social evaluation criteria in relation to principles of fairness, and to provide more concrete indications for applications to the assessment of integrated scenarios describing the long-term evolution of the climate and the economy. In particular, the relative prices of different commodities (environmental goods vs consumption goods) change with time, yielding different discount rates (Gollier, 2010). It may be important to take into account the relative scarcity of some goods when evaluating the welfare of future generations.

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Appendix

Appendix A Proof of Proposition 1

We begin with the following lemma:

Lemma 1 If the social ordering \( R \) satisfies Axioms 1, 2, 3, and 5 then, for all \( n \in \mathbb{N}_3 \), there exist continuous and increasing functions \( \Psi_n \) and \( \phi_n \) such that, for all \( u_s \) such that \( n(u_s) = n \),

\[
V(u_s) = \Psi_n \left( \sum_{i \in N(u_s)} \phi_n(u_i^s) \right).
\]
Proof. Take any $\mathcal{N} \in \mathfrak{N}_3$. Define the ordering $\bar{R}_N$ on $X^{\lvert \mathcal{N} \rvert}$ as follows:

$$(u(i))_{i \in \mathcal{N}} \bar{R}_N (v(i))_{i \in \mathcal{N}} \iff (p, (u(i)1_m)_{i \in \mathcal{N}}) \bar{R} (p, (v(i)1_m)_{i \in \mathcal{N}}).$$

By Axiom 1, the relation $\bar{R}_N$ is transitive, reflexive, complete and continuous. By Axiom 3, the relation is monotonic. By Axiom 5, any subset of $\mathcal{N}$ is separable. Therefore, as $\lvert \mathcal{N} \rvert \geq 3$, there exist continuous and increasing function $\phi_i^N$ such that $\sum_{i \in \mathcal{N}} \phi_i^N(u(i))$ represents $\bar{R}_N$. By Axiom 2 the representation must be symmetric and can be written $\sum_{i \in \mathcal{N}} \phi_i^N(u(i))$. By the definition of the relation $\bar{R}_N$, we also obtain that, whenever $N(u_s) = N(v_s) = N$:

$$V(u_s) \geq V(v_s) \iff \sum_{i \in \mathcal{N}} \phi_i^N(u_s^i) \geq \sum_{i \in \mathcal{N}} \phi_i^N(v_s^i).$$

Thus there must exist a continuous and increasing function $\Psi_N$ such that, for all $u_s$ such that $N(u_s) = N$,

$$V(u_s) = \Psi_N \left( \sum_{i \in \mathcal{N}} \phi_i^N(u_s^i) \right).$$

Note that Axiom 2 imposes the Anonymity requirement for subpopulations of the same size but that may differ. In particular, it implies that, whenever $\lvert \mathcal{N} \rvert = \lvert \mathcal{M} \rvert$ and there is a bijection $\pi : \mathcal{N} \rightarrow \mathcal{M}$ such that $u_s^i = v_{\pi(i)}^s$ for all $i \in \mathcal{N}$,

$$\Psi_N \left( \sum_{i \in \mathcal{N}} \phi_i^N(u_s^i) \right) = \Psi_M \left( \sum_{i \in \mathcal{M}} \phi_i^M(v_s^i) \right).$$

We can therefore take $\Psi_N = \Psi_M = \Psi_{\lvert \mathcal{N} \rvert}$ and $\phi_N = \phi_M = \phi_{\lvert \mathcal{N} \rvert}$. $lacksquare$

Take any $\mathcal{N} \in \mathfrak{N}_3$. For every $u \in X^{\lvert \mathcal{N} \rvert}$, define the equally distributed equivalent (EDE) of $u$ as the scalar $e(u) \in X$ such that

$$(p, (u,...,u)) \bar{R}_m (p, e(u)1_{m,n}),$$

where $1_{m,n}$ is the $m \times n$ unit matrix (all its components equal one). By Axiom 3 and the continuity of $V$, $e(u)$ exists for every $u \in X^{\lvert \mathcal{N} \rvert}$. By the representation in Lemma 1, we obtain that

$$e(u_s) = \phi_{\lvert \mathcal{N} \rvert}^{-1} \left( \frac{1}{\lvert \mathcal{N} \rvert} \sum_{i \in \mathcal{N}} \phi_{\lvert \mathcal{N} \rvert}(u_s^i) \right).$$
By Axiom 1, the fact that \((p, (u, ..., u))I(p, e(u)1_{m,n})\) implies that \(V(u) = V(e(u)1_n)\), where \(n = n(u_s)\) and \(1_n\) is the unit vector of \(\mathbb{R}^n\). Therefore, for all \(u, v \in X_N\) and all \(p, q \in \Sigma^{m-1}\),

\[(p, u)R(q, v) \iff (p, (e(u)s)_{s=1, ..., m})R(q, (e(v)s)_{s=1, ..., m})\]

By Axiom 4,

\[(p, (e(u)s)_{s=1, ..., m})R(q, (e(v)s)_{s=1, ..., m}) \iff \sum_{s=1}^{m} p_se(u_s) \geq \sum_{s=1}^{m} q_se(v_s).\]

Summarizing, for all \(u, v \in X_N\) and all \(p, q \in \Sigma^{m-1}\),

\[(p, u)R(q, v) \iff \sum_{s=1}^{m} p_s\phi^{-1}_{|N|}\left(\frac{1}{|N|}\sum_{i \in N} \phi_{|N|}(u^i_s)\right) \geq \sum_{s=1}^{m} q_s\phi^{-1}_{|N|}\left(\frac{1}{|N|}\sum_{i \in N} \phi_{|N|}(v^i_s)\right).\]

The representation \(\phi^{-1}_{|N|}\left(\frac{1}{|N|}\sum_{i \in N} \phi_{|N|}(u^i_s)\right)\) is a vNM utility function for the society. Hence there exist a positive real number \(\alpha_{|N|}\) and a real number \(\beta_{|N|}\) such that

\[V(u_s) = \alpha_{|N|}\phi^{-1}_{|N|}\left(\frac{1}{|N|}\sum_{i \in N} \phi_{|N|}(u^i_s)\right) + \beta_{|N|}\]

Consider the allocations \(u, v\) described in Axiom 6. Next consider \(w, z \in U\) such that:

- \(w_s' = z_s' = v_s'\) for all \(s' \in \{1, ..., m\} \setminus \{s\}\)
- \(N(w_s) = N(u_s) = N\) and \(N(z_s) = N(u_s) \cup \{k\}\)
- \(w_s^i = z_s^i = e(u_s)\) for all \(i \in N(u_s)\)
- \(z_s^k = C(V(w_s), n(w_s))\).

The prospect \(w\) is similar to \(u\) except that individuals get the EDE welfare level in state \(s\). Therefore, we have \(V(w_s) = V(u_s)\). By Axiom 6, we also have \(V(v_s) = V(u_s)\) and
\[ V(z_s) = V(w_s), \] so that \( V(z_s) = V(v_s) \). Using the representation of \( V \) and the expression for the EDE, we obtain:

\[
\alpha_{|N|+1} \phi_{|N|+1}^{-1} \left( \frac{1}{|N|+1} \sum_{i \in N} \phi_{|N|+1} \circ \phi_{|N|}^{-1} \left( \frac{1}{|N|} \sum_{i \in N} \phi_{|N|}(u^i_s) \right) + \frac{1}{|N|+1} \phi_{|N|+1}(z^k_s) \right) \\
+ \beta_{|N|+1} = \alpha_{|N|+1} \phi_{|N|+1}^{-1} \left( \frac{1}{|N|+1} \sum_{i \in N} \phi_{|N|+1}(u^i_s) + \frac{1}{|N|+1} \phi_{|N|+1}(v^k_s) \right) + \beta_{|N|+1}
\]

Because \( v^k_s = C(V(u_s), n(u_s)) = C(V(w_s), n(w_s)) = z^k_s \), the equality reduces to:

\[ |N| \phi_{|N|+1} \circ \phi_{|N|}^{-1} \left( \frac{1}{|N|} \sum_{i \in N} \phi_{|N|}(u^i_s) \right) = \sum_{i \in N} \phi_{|N|+1}(u^i_s) \]

Denoting \( x_i = \frac{\phi_{|N|}(u^i_s)}{|N|} \), \( F(x) = |N| \phi_{|N|+1} \circ \phi_{|N|}^{-1}(x) \) and \( G(x) = \phi_{|N|+1} \circ \phi_{|N|}^{-1}(|N|x) \), this yields the functional equation

\[ F \left( \sum_{i \in N} x_i \right) = \sum_{i \in N} G(x_i) \]

The solution of this Pexider equation is \( F(x) = ax + b \) and \( G(x) = ax + b/|N| \) for some \( a > 0 \) and \( b \in \mathbb{R} \). Letting \( y = \phi_{|N|}^{-1}(x) \), the equation

\[ |N| \phi_{|N|+1} \circ \phi_{|N|}^{-1}(x) = ax + b \]

is equivalent to

\[ |N| \phi_{|N|+1}(y) = a \phi_{|N|}(y) + b. \]

In other words, \( \phi_{|N|+1}(x) = A_{|N|} \phi_{|N|}(x) + B_{|N|} \) for some \( A_{|N|} > 0 \) and \( B_{|N|} \in \mathbb{R} \). Reasoning by recurrence, it can be shown that, for all \( l \in \mathbb{N} \), there exist a positive real number \( a_l \) and a real number \( b_l \) such that \( \phi_l = a_l \phi + b_l \), for a continuous increasing function \( \phi \), so that:

\[ V(u_s) = \alpha_{|N|} \phi_{|N|}^{-1} \left( \frac{1}{|N|} \sum_{i \in N} \phi(u^i_s) \right) + \beta_{|N|}. \]
With this formula, we can compute the critical level $C (V (u_s), n(u_s))$ used to construct the prospect $v$ from $u$ in Axiom 6:

$$
\phi (C (V (u_s), |N|)) = (|N| + 1)\phi \left( \frac{V (u_s) - \beta |N|+1}{\alpha |N|+1} \right) - \sum_{i \in N} \phi (u^i_s)
$$

$$
= (|N| + 1)\phi \left( \frac{V (u_s) - \beta |N|+1}{\alpha |N|+1} \right) - |N|\phi \left( \frac{V (u_s) - \beta |N|}{\alpha |N|} \right).
$$

**Appendix B  Proof of Corollary 1**

Let $c_n = C(x, n)$. By a simple change of variable, $z = (x - \beta_n)/\alpha_n$, we obtain the functional equation:

$$
(n + 1)\phi \left( \frac{\alpha_n}{\alpha_{n+1}} z + \frac{\beta_n - \beta_{n+1}}{\alpha_{n+1}} \right) = n\phi (z) + \phi (c_n),
$$

where $z \in X$. Letting $a = \frac{\alpha_n}{\alpha_{n+1}} > 0$, $b = \frac{\beta_n - \beta_{n+1}}{\alpha_{n+1}}$, this equation reads

$$
\phi (az + b) = \frac{n}{n + 1} \phi (z) + \frac{1}{n + 1} \phi (c_n).
$$

The equation implies $\phi (ac_n + b) = \phi (c_n)$, so that $ac_n + b = c_n$.

Let $f(z) = \phi (z + c_n) - \phi (c_n)$ for $z \in X$. One obtains:

$$
f (az) = \phi (az + c_n) - \phi (c_n)
$$

$$
= \phi (a (z + c_n) + b) - \phi (c_n)
$$

$$
= \frac{n}{n + 1} \phi (z + c_n) + \frac{1}{n + 1} \phi (c_n) - \phi (c_n)
$$

$$
= \frac{n}{n + 1} f(z).
$$

Note that it is impossible to have $a = 1$ because that would mean $f(z) = \frac{n}{n+1} f(z)$ for all $z$.

The general solution to $f (az) = \frac{n}{n+1} f(z)$ is (Polyanin and Manzhirov, 2007): $f(z) = \Theta(z) |z|^\omega$ and $a^\omega = \frac{n}{n+1}$, where $\Theta(z)$ is an arbitrary periodic continuous (except possibly at 0) solution to the functional equation $\Theta(az) = \Theta(z)$. 

One therefore has $\phi(z) = f(z - c_n) + \phi(c_n) = \Theta(z - c_n) |z - c_n|^{\omega} + \phi(c_n)$. The case $z = c_n$ requires that $\omega > 0$. Therefore, the fact that $\phi$ is increasing implies that for all $z \in X$, $\Theta(z - c_n) > 0$ if $z > c_n$ and $\Theta(z - c_n) < 0$ if $z < c_n$.

If there is $z \in X$ such that $z > c_n$, the strict concavity of $\phi$ imposes $\omega < 1$. If there is $z \in X$ such that $z < c_n$, the strict concavity of $\phi$ imposes $\omega > 1$. Therefore, only two cases are possible: either for all $z \in X$, $z \geq c_n$, $\Theta(z - c_n) > 0$ (except possibly at $z = c_n$), and $c_n = \min X$, or for all $z \in X$, $z \leq c_n$, $\Theta(z - c_n) < 0$ (except possibly at $z = c_n$), and $c_n = \max X$.

If $X$ is not bounded above, the latter case is excluded, and the fact that $f$ is increasing also implies that $\Theta$ must be a constant (positive) function. Let $\epsilon = 1 - \omega$, and note that $\alpha_{n+1} = ((n + 1)/n)^{1/\epsilon} \alpha_n = (n + 1)^{1/\epsilon} \chi$ where $\chi = \alpha_3/(3^{1/\epsilon})$. We also know that $ac_n + b = c_n$ so that $\beta_n - \beta_{n+1} = (\alpha_{n+1} - \alpha_n)c_n = (\alpha_{n+1} - \alpha_n)\min X$. A sum of such expressions yields $\beta_3 - \beta_{n+1} = (\alpha_{n+1} - \alpha_3)\min X$ so that $\beta_{n+1} = -\alpha_{n+1}\min X + \zeta$ where $\zeta = \beta_3 + \alpha_3\min X$. Hence we obtain:

$$
\alpha_n \phi^{-1} \left( \frac{1}{n} \sum_{i \in N(u_s)} \phi(u_s^i) \right) + \beta_n = \alpha_n \left( \frac{1}{n} \left( \sum_{i \in N(u_s)} (u_s^i - \min X)^{1-\epsilon} \right) \right)^{\frac{1}{1-\epsilon}} + \min X
$$

$$
- \alpha_n \min X + \zeta
$$

$$
= \chi n^{1/\epsilon} \left( \frac{1}{n} \sum_{i \in N(u_s)} (u_s^i - \min X)^{1-\epsilon} \right)^{\frac{1}{1-\epsilon}} + \zeta
$$

The proof is completed.

**Appendix C  Proof of Corollary 2**

$C (\alpha_n \min X + \beta_n, n) \geq \min X$ means that

$$(n + 1) \phi \left( \frac{x - \beta_{n+1}}{\alpha_{n+1}} \right) = \phi (C (x, n)) + n \phi \left( \frac{x - \beta_n}{\alpha_n} \right)$$

reads

$$(n + 1) \phi \left( \frac{\alpha_n \min X + \beta_n - \beta_{n+1}}{\alpha_{n+1}} \right) \geq \phi (\min X) + n \phi (\min X),$$
implying
\[ \alpha_n \min X + \beta_n \geq \alpha_{n+1} \min X + \beta_{n+1}. \]

If \( C(x, n) \) is non-decreasing in \( x \), so is the expression
\[ (n + 1)\phi \left( \frac{x - \beta_{n+1}}{\alpha_{n+1}} \right) - n\phi \left( \frac{x - \beta_n}{\alpha_n} \right). \]
This implies
\[ \frac{n + 1}{\alpha_{n+1}} \phi' \left( \frac{x - \beta_{n+1}}{\alpha_{n+1}} \right) \geq \frac{n}{\alpha_n} \phi' \left( \frac{x - \beta_n}{\alpha_n} \right). \]

For \( x = \alpha_n \min X + \beta_n \), this reads
\[ \frac{n + 1}{\alpha_{n+1}} \phi' \left( \frac{\alpha_n \min X + \beta_n - \beta_{n+1}}{\alpha_{n+1}} \right) \geq \frac{n}{\alpha_n} \phi' \left( \min X \right). \]

As \( \frac{\alpha_n \min X + \beta_n - \beta_{n+1}}{\alpha_{n+1}} \geq \min X \), by concavity \( \phi' \left( \frac{\alpha_n \min X + \beta_n - \beta_{n+1}}{\alpha_{n+1}} \right) \leq \phi' \left( \min X \right) \), and therefore
\[ \frac{n + 1}{\alpha_{n+1}} \geq \frac{n}{\alpha_n}. \]