Acoustic mirror Chern insulator with projective parity-time symmetry

Xiao Xiang1,*, Feng Gao1,*, Yu-Gui Peng1,†, Qi-Li Sun1, Jie Zhu2,†, Xue-Feng Zhu1,†

1School of Physics and Innovation Institute, Huazhong University of Science and Technology, Wuhan 430074, China
2Institute of Acoustics, School of Physics Science and Engineering, Tongji University, Shanghai 200092, China

*X. Xiang and F. Gao contributed equally to this work.
†Corresponding authors: ygpeng@hust.edu.cn (Y.-G.P.); jiezhu@tongji.edu.cn (J. Z.); xfzhu@hust.edu.cn (X.-F. Z.).
In condensed matter physics, symmetry profoundly governs the fundamentals of topological matter. The emergence of new topological phase is typically linked to the enrichment of symmetries. Different parity-time symmetry relations \((PT)^2=\pm 1\) distinguish between spinless and spinful physical systems. In spinless systems, creating pseudo-spins can realize fragile topological phase but not break the time-reversal symmetry. Therefore, growing attentions were recently focused on the strong topological phase in spinless systems. Here we break the framework of crystallographic symmetry groups by utilizing the projective symmetry \((PM)^2=-1\) in a \(Z_2\) gauge field, to realize the Mirror Chern Insulator (MCI) in a bilayer twisted Hofstadter model. We discover that MCI is actually a new topological phase in classical wave systems, contradicting the popular belief that a “static” spinless system is unlikely to support Chern Insulators. In experiments, the edge modes were unambiguously observed with odd-shaped boundaries, confirming the strong topological features. The clockwise and anti-clockwise edge states with opposite group velocities were completely separated via an energy drain. In addition, we demonstrate that MCI has robust topological whispering gallery modes. Our work establishes a strong foundation for investigating exotic topological effects arising from the interplays between artificial gauge fields and wave systems.
The hunt for exotic matter has long been a priority in condensed matter physics, with the appearance of several unique quantum matters\textsuperscript{1,2}. Since the discovery of topological insulators, topological matters have piqued the interest of many people due to the interesting counter-intuitive properties\textsuperscript{3-5} beyond natural materials, such as the robust electron transport despite disorders or defects\textsuperscript{1,6}. Due to differing symmetries, electronic topological insulators may have distinct quantum phases such as the quantum Hall phase and quantum spin Hall phase\textsuperscript{1,6}. In a spinful system, the Chern numbers for quantum Hall phase and quantum spin Hall phase are nonzero and zero, depending on whether the Berry connection is an even or odd function in the Brillouin zone. For the spinless systems such as the photonic and phononic systems, recent studies have concentrated on the judicious construction of topological phases\textsuperscript{7-10}. For example, in similarity to the quantum spin Hall and valley Hall effects, fragile topological phases with one-way wave transport can be implemented in classical wave systems using additional degrees of freedom such as pseudo-spin and valley-spin, without breaking the time-reversal symmetry\textsuperscript{11-18}. However, to realize the strong topology of quantum Hall effect in spinless systems, time-reversal symmetry breaking is deemed necessary to ensure the Chern numbers nonzero, which is very difficult in spinless systems due to the lack of spin-orbit couplings\textsuperscript{19-20}. To solve this problem, researchers have introduced the magneto-optic effect, flow circulation, and gyroscopic rotation in electromagnetics, acoustics, and mechanics respectively to mimic the pseudo-magnetic field\textsuperscript{21-27}.

However, the pursuit of new mechanisms to construct fresh topological matters is a recurrent impetus for physics community. Introducing an equivalent vector potential is a reasonable way to analogize the bizarre quantum topological effects. Researchers attempted to build an artificial gauge field based on the vector potential and explored the topological states with spin-like features in spinless systems. The inclusion of gauge degrees of freedom in $Z_2$ gauge field, for example, allows the system symmetry to be represented projectively\textsuperscript{28}. In this case, the parity-time (PT) operator satisfies an algebraic relation of $(PMT)^2 = (-1)^{s+1}$, where $PM$ is the mirror parity operator and $s$ is the spin in the system. Since $(PM)^2 = -1$, the $Z_2$ gauge field can transition the physical systems between spinless and spinful, where real hopping amplitudes in one system are transformed into complex ones in the other system via the unitary transformation. This
theory has been effectively extended to the spinless wave systems, resulting in the discovery of a few new topological phases such as the Bloch Klein bottle and Möbius insulators\textsuperscript{29-33}. Now the obvious issue is whether we can use analogies with condensed states to revolutionize the existing strategy of constructing topological phases. Can we, for example, beat the impossible, and realize strong topological insulators in time-invariant spinless systems without any biased settings.

In this work, we propose a new topological phase of Mirror Chern Insulator (MCI) in time-invariant spinless systems using projective algebra in the $\mathbb{Z}_2$ gauge field. The essence of MCI is that the rigorous condition of pseudo-magnetic field $(T)^2=-1$ for the strong topology can be removed by employing the mirror parity operator $P_M$ with $(P_M)^2=-1$. In spinless systems, the $\mathbb{Z}_2$ gauge field can manipulate the symmetries of operators $P_M$ and $T$ to satisfy $(P_M T)^2=-1$. We realize the MCI by judiciously twisting the inter-layer couplings between two acoustic crystal layers and creating effective $\pi$-flux blocks that resemble the Hofstadter model in each layer. For the experimental implementation, we develop distinctive positive and negative couplings between the two crystal layers to successfully generate desired couplings in each layer, resulting in a twisted Hofstadter lattice. Our findings reveal that MCI represents a new topological phase in classical spinless wave systems. The strong topology of edge modes in MCI is confirmed, where the clockwise and anti-clockwise boundary states can be entirely separated by introducing an energy drain. Furthermore, we observe the substantial and robust whispering gallery modes in MCI with strong topological protection. Our results disprove the long-held belief that CI cannot be created in a spinless system without breaking the time reversal symmetry, paving the way for new topological insulators or other phases with projective PT symmetry.

**Construction of MCI with projective PT symmetry**

In a two-dimensional (2D) electron system, strong magnetic fields perpendicular to the 2D system can induce the quantum Hall effect and, further, CI with a chiral edge state under strong topological protection. The existence of magnetic field disrupts the time-reversal symmetry in CI, where the transport of electrons is unidirectional. In Fig.
we show that a CI’s mirror counterpart is actually the CI under the time-reversal operation with a flipped magnetic field and reversed electron orbits. Chiral edge states propagate clockwise and anti-clockwise in a spinful CI and its mirror counterpart, respectively. Recently, researchers discovered that in the presence of gauge degrees of freedom, system symmetries can be projectively expressed and effectively achieved in the Z2 gauge field. The projective symmetry has far-reaching ramifications for the topological framework, allowing us to suggest a plethora of new topological phases of MCI in spinless systems.

Here we begin with a 2D Hofstadter CI model. The Hofstadter Hamiltonian is

\[ \hat{H} = -J \sum (a_{m+1,n}^\dagger a_{m,n} + e^{i(k_x-m\phi)} a_{m,n+1}^\dagger a_{m,n} + h.c.) , \]

where \( J \) is the hopping in \( x \) direction, \( J e^{i\phi} \) is the hopping in \( y \) direction, and \( \phi=2\pi/5 \). As depicted in Fig. 1b, the lattice structures of CI(\( \phi \)) and CI(\( -\phi \)) exhibit mirror symmetries, and the half-cell band structures of CI(\( \pm\phi \)) are calculated. The red and blue bands in the gap indicate that the mirrored one-way edge states transporting at the upper boundaries of CI(\( \phi \)) and CI(\( -\phi \)) have positive and negative group velocities, respectively. When we directly couple two Hofstadter models with opposite fluxes, the zero-flux system will have \((PT)^2=1\) rather than \((PT)^2=-1\). By linking a \( \mathbb{Z}_2 \) gauge field to the general gauge configuration that satisfies \((P)^2=1\), indicated by the left block in Fig. 1c, we may create a twisted block that obeys \((P_M)^2=-1\). The gauge relation in this case is \( P_M=GP \). \( G=\sigma_3 \otimes \rho_3 \) is the gauge transformation to recover the gauge configuration, with \( \sigma \) and \( \rho \) denoting the Pauli matrices acting on the row and column of Hamiltonian matrix. The Hamiltonian of the twisted block in Fig. 1c is

\[ H = J \cos(n\phi)\sigma_0 \otimes \rho_1 + J \sin(n\phi)\sigma_2 \otimes \rho_2 . \]

When we consider the eigen-spaces of \( P_M \), the Hamiltonian of the twisted block can be expressed as

\[ H = (J \cos(n\phi)\sigma_1 + J \sin(n\phi)\sigma_2) \otimes (J \cos(n\phi)\sigma_1 - J \sin(n\phi)\sigma_2) , \]

(1)

as a result of the unitary transformation \( S=e^{-i\phi\sigma_3\otimes\rho_3/4} \). According to Eq. (1), the two terms of the Hamiltonian’s direct product correspond to the complex hopping of \( Je^{i\phi} \) and \( Je^{-i\phi} \), respectively. We note that the unitary transformation transforms the real
couplings into complex ones, generating a nonzero effective magnetic field in each layer and showcasing the potential to realize the effects of spinful systems in spinless systems. We define the Hamiltonians of CI(±ϕ) by $H_{\text{CI}(\pm\phi)}$, and substitute the complex hopping in CI(±ϕ) with the twisted blocks in Fig. 1c. By coupling the two CIs with opposite magnetic fluxes, a time-invariant spinless MCI can be created to satisfy $(P_M T)^2 = -1$. The momentum space Hamiltonian of MCI is

$$H_{\text{MCI}}(k) = \begin{bmatrix} H_{k_+}(k, \phi) & -iH_{k_-}(k, \phi) \\ iH_{k_-}(k, \phi) & H_{k_+}(k, \phi) \end{bmatrix},$$

where $H_{k_\pm}(k, \phi) = \frac{1}{2}[H_{\text{CI}}(k, \phi) \pm H_{\text{CI}}(k, -\phi)]$ and * denotes the conjugate operation.

The comprehensive derivative processes of the theory can be found in Supplemental Materials. We subsequently build the quasi-2D ball-stick MCI model in Fig. 1d, in which the complicated hoppings in the Hofstadter models are replaced by positive and negative hoppings. The half-cell band structure is calculated and presented in Fig. 1d, showing an elegant combination of CI(ϕ) and CI(−ϕ). It reveals that the mirrored edge states in the gap are bounded at the upper boundary of MCI with positive and negative group velocities, respectively.

**Fig. 1. | MCI with projective PT symmetry.** a, CI and its mirrored counterpart under opposite magnetic fields. Electrons on the edge are transporting in anti-clockwise and
clockwise directions, respectively. b, The Hofstadter model and its mirror image with opposite magnetic fluxes (\(\phi\) and \(-\phi\)). The half-cell band structures are also calculated to demonstrate the existence of one-way edge states at one boundary of CI(\(\phi\)) and CI(\(-\phi\)), with opposite propagation directions. c, The transformation of parity operator \((P)^2=1\) into mirror parity operator \((P_M)^2=-1\) by twisting the hopping block. The twisted block satisfying \((P_M)^2=-1\) is identical to the complex hopping block under the unitary transformation. d, Construction of the bilayer Hofstadter model MCI by coupling the spinful CI(\(\phi\)) and CI(\(-\phi\)) to form a spinless system. The occurrence of mirrored edge states at the same boundary with opposite propagation directions is demonstrated by the half-cell band structure.

**Design of the twisted bilayer Hofstadter model**

For the MCI demonstration, we create the twisted bilayer Hofstadter model using acoustic resonators and coupling tubes. We can accurately manipulate the hopping amplitude which is completely determined by the geometries of resonators and tubes and is a real value. As depicted in the left panel of Fig. 2a, the bilayer Hofstadter model is a square lattice connected by coupling tubes. The red dashed box in Fig. 2a represents a primitive cell, and the inset illustrates the cell’s detailed architecture. In our design, the air cavities in a primitive cell have a height of \(h=50\) mm, an interlayer spacing of \(l_s=40\) mm, and horizontal spacings in \(x\) and \(y\) directions \(l_x=l_y=30\) mm. The lattice constants are \(a_x=300\) mm and \(a_y=60\) mm. In one primitive cell, there are 7 distinct sets of coupling tubes. The detailed geometric parameters of each coupling tube can be found in Supplemental Materials. The top layer and bottom layer respectively resemble the typical Hofstadter model with gauge fluxes of \(\phi=2\pi/5\) and \(-\phi=-2\pi/5\), as marked by the red and blue arrows. To help comprehending the intricate structure, we present the top view of the bilayer Hofstadter model, where the solid lattice and semi-transparent lattice correspond to the top layer and bottom layer, respectively. To generate positive and negative hopping, we connect the in-layer cavities with straight-link and cross-link tubes, and connect the inter-layer cavities via tubes with and without space-coiling, as schematically illustrated in Fig. 2b. The intensity fields of topologically protected edge
states in MCI at 3448 Hz are simulated by a finite element solver, where the frequency is selected in the band gap of MCI, as shown in Fig. 2c. In simulations, the source is set at one edge of the lattice, and the pressure field on the edge is extracted to calculate the boundary-mode dispersion curves of MCI in the momentum space via the Fourier transformation. The calculation result (shown by the thermal diagram in Fig. 2d) agrees well with the band structures (the white lines). The Chern numbers of the five bulk bands are respectively \(-1, -1, 4, -1, -1\) from the bottom for the top-layer lattice, which is discussed in details in Supplemental Materials. The mirrored edge modes are gapless and locate in the band gaps. Because in MCI there are a pair of spinless edge states with positive and negative group velocities on one boundary, the feature is substantially different from the CI with only one spinful edge state on the boundary.

Fig. 2 | Acoustic realization of MCI. a, A twisted bilayer Hofstadter lattice comprising acoustic resonators and coupling tubes. Inset: the configuration of one primitive cell. The tight-binding model is shown on the right, with the colored spheres and sticks representing acoustic cavities and the waveguides with different coupling strengths. b, Schematic drawing of the negative and positive hopping realization for in-layer and interlayer couplings. c, The intensity field in MCI at 3448 Hz, indicating the presence of strong topological protection, where the frequency is selected in the band gap of MCI. d, Band structure of MCI. The white lines depict MCI’s Bloch band structure. The
thermal diagram displays the spectrum amplitudes of dispersion curves, calculated by Fourier transforming the simulated pressure fields of boundary modes.

**Paired boundary states in MCI with strong topological protection**

To experimentally demonstrate the strong topology in MCI, we have fabricated a sample using 3D printing. As shown by the lattice structure in Fig. 3a, the sample adopts an irregular shape, to fairly demonstrate the strong robustness of edge states in MCI. In the experiments, sound source is set at the positions $S_1$, $S_2$, and $S_3$, respectively. We measure the acoustic wave pressure in the cavities on the source boundary (shown in the red dashed box in Fig. 3a) and conduct the Fourier transform to calculate the spectrum amplitudes of edge states dispersion curves. The transmission spectra are also examined at Ports $P_1$ and $P_2$, as indicated by the circles in Fig. 3a, to reveal the band locations of edges states. An energy drain is introduced on the far side of the lattice to interrupt the periodicity of edge-mode circulation. In Figs. 3b-d, we present the spectrum amplitudes of dispersion curves when the sound source is respectively set at $S_1$, $S_2$, and $S_3$. The experimental results confirm that the mirrored edge states with time-reversal symmetry can be completely separated with the energy drain. When the source is placed at $S_1$, the edge state on the source boundary has only positive group velocity (propagating from bottom up) and no back-reflection from the sharp corners. When the source is set to $S_2$, the edge state has just negative group velocity (propagating from top to bottom) on the source boundary, also with no back-reflection at the sharp corners. When the source is positioned at $S_3$, the edge state on the source boundary, however, has both positive and negative group velocities, with sound waves travelling from the center to the top and bottom, respectively. In Fig. 3e, we show the measured sound pressure amplitude spectra at $P_1$ and $P_2$, with the source at $S_2$. It agrees well with the Bloch band structure in Figs. 3b-d. The edge states locate in the bands of 3316–3340 Hz, 3349–3373 Hz and 3425–3475 Hz, as highlighted by the pastel ribbons. It is also worth mentioning that between 3390 Hz and 3405 Hz (dark ribbon), the edge states intermingle with the bulk states, so that the transmissions at $P_1$ and $P_2$ are comparable. The simulated and measured acoustic intensity distributions in MCI at 3438 Hz with
the source at $S_1$ are shown in Fig. 3f, and they agree with each other well. The result proves that there exist two types of chiral edge modes that propagate clockwise and anti-clockwise. The two chiral edge modes are launched at the source and absorbed at the far-side drain, each with excellent topological protection against sharp corners or boundary defects.

Fig. 3 | Demonstration of strong topology in MCI. a, The MCI sample fabricated by 3D printing. Sharp corners are introduced to demonstrate the robustness of chiral edge states in MCI. In experiments, the sound source is set at positions $S_1$, $S_2$, and $S_3$, respectively. A drain is set at the far side of the lattice to absorb all the incident sound energy. b-d, Spectrum amplitudes of dispersion curves calculated by Fourier transform on sound pressure at the source boundary, as marked by the red dashed box in a. e. The transmission spectra measured at Ports $P_1$ and $P_2$, as marked by the circles in a, with the source at $S_2$. f. The simulated and experimentally measured acoustic intensity field distributions in MCI at 3438 Hz, with the source at $S_1$.

Whispering gallery of strong topology in MCI

Finally, we demonstrate the detection of strong topological whispering gallery modes in MCI. Whispering gallery is a significant wave physics phenomenon that can be utilized to create high-quality lasers and sensors. In contrast to the prior topological whispering gallery of fragile topology, the MCI is expected to support the
whispering gallery mode of strong topology. In principle it should be robust against any defects. For the proof experimental demonstration, to reduce the negative effect of viscous damping, we fabricate a small-size sample of $1 \times 4$ primitive cells. The sample shown in Fig. 4a has 48 acoustic cavities in total. We put the sound source in one cavity on the boundary and measure the pressure amplitude spectrum in another cavity on the boundary. The results in Fig. 4b reveal sound pressure amplification peaks at 3430 Hz and 3455 Hz in simulations, and at 3431 Hz and 3455 Hz in experiments. In Fig. 4c, the intensity field distributions present the well-agreed simulation and experimental observation of whispering gallery modes in MCI at 3431 Hz and 3455 Hz, respectively. The whispering gallery modes here render a clear standing wave pattern. The mode at 3431 Hz, for example, has three nodes and three antinodes, while the mode at 3455 Hz has four nodes and four antinodes. In addition, the observed whispering gallery modes in MCI are shown to be robust against the four sharp corners of rectangular MCI, indicating the presence of strong topological protection.
Fig. 4 | Topological whispering gallery in MCI. a, A small MCI sample with 1×4 primitive cells. b, Sound pressure amplitude spectrum measured at the boundary of MCI. The amplitude amplification peaks indicate the existence of whispering gallery modes. c, Sound intensity field distributions in MCI at 3431 Hz and 3455 Hz, showing the distinctive standing wave patterns and strong topological protection.

Conclusion and outlook

We propose a new topological phase of Mirror Chern insulator to realize the strong topology in a time-invariant spinless system. The fundamental algebra of system symmetry group is changed by using the projective parity-time symmetry of $(PMT)^2 = -1$ in the $Z_2$ gauge field. We construct the MCI using a twisted bilayer Hofstadter lattice and realize it in acoustic systems. The distinguished properties of MCI, such as mirrored chiral edge states protected by strong topology, are experimentally demonstrated. Using an energy drain to break the periodicity of edge-mode circulation, we can successfully separate the two chiral edge states with opposite group velocities, each of which propagates under strong topological protection. In addition, we discover the topological whispery gallery modes in MCI. Our work paves the way towards the implementation of various spinful effects in spinless classical wave systems. The enrichment of system symmetries with the projective parity-time symmetry will help the quest for additional intriguing topological phases in an artificial gauge field. It may further lead the study on topological matters into a new direction.

Methods

Numerical simulations. All full-wave numerical simulations were performed by using commercial finite-element solver. In the simulation, the structures of acoustic cavities and coupling tubes are rigid, due to the huge acoustic impedance mismatch between solids (photosensitive resin) and air. The density and speed of sound in air are 1.3 kg/m$^3$ and 343 m/s (at room temperature), respectively. The detailed simulations of positive/negative hopping between cavities and the onsite energies in cavities can be referred to the Supplemental Materials.
**Experiments.** The MCI samples were fabricated by the 3D printing. The fabrication precision was ~100 μm. In experiments, we perforated small holes (radius: 3 mm) on each cavity to facilitate the insertion of small speakers and microphones for sound wave generation and detection. Two microphone probes (Brüel & Kjær 4138 A 15; diameter: 1.5 mm) were used to measure the phase and amplitude of sound waves. The acoustic signals were automatically recorded by a network analyzer (Brüel & Kjær 3160-A-042) and then processed to map out the acoustic field.

**Data availability**
The data that support the plots within this work and other findings of this study are available from the corresponding authors upon reasonable request.

**Code availability**
The codes that support the findings of this study are available from the corresponding author upon reasonable request.

**Acknowledgments**
The authors acknowledge financial support by the National Natural Science Foundation of China (Grant Nos. 11690030 and 11690032).

**Author contributions**
X. Xiang and F. Gao contributed equally to this work. X. Xiang, Q. Sun and Y. Peng developed the theory and did the simulations. X. Xiang designed the experiments and fabricated the samples. X. Xiang and F. Gao performed the experiments. All authors analyzed the data and wrote the manuscript. X. Zhu, Y. Peng and J. Zhu supervised the project.

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