A SUBEXPONENTIAL VECTOR-VALUED BOHNENBLUST-HILLE TYPE INEQUALITY

N. ALBUQUERQUE, D. NÚÑEZ-ALARCÓN, AND D. M. SERRANO-RODRÍGUEZ

ABSTRACT. Bayart, Pellegrino and Seoane recently proved that the polynomial Bohnenblust–Hille inequality for complex scalars is subexponential. We show that a vector valued polynomial Bohnenblust-Hille inequality on complex Banach lattices is also subexponential for some special cases. Our main result result recovers the best known constants of the classical polynomial inequality provided in [5].

1. INTRODUCTION AND PRELIMINARIES

The Bohnenblust–Hille inequality for complex homogeneous polynomials ([7], 1931) asserts that there is a function \( D : \mathbb{N} \to [1, \infty) \) such that no matter how we select a positive integer \( N \) and an \( m \)-homogeneous polynomial \( P \) on \( \mathbb{C}^N \), the \( \ell_{2 \infty} \)-norm of the set of coefficients of \( P \) is bounded above by \( D(m) \) times the supremum norm of \( P \) on the unit polydisc. Having good estimates for \( D(m) \) is crucial for applications (for instance to the determination of the exact asymptotic growth of the Bohr radius). The best known estimates for \( D(m) \) are due to F. Bayart, D. Pellegrino and J. Seoane ([5]) and show that the growth of \( D(m) \) is subexponential. More precisely, in [5] it is shown that for any \( \varepsilon > 0 \), there is \( \kappa > 0 \) such that

\[
D(m) \leq \kappa (1 + \varepsilon)^m,
\]

for all positive integers \( m \). In this paper we obtain similar estimates on some special Banach lattices. Our approach is inspired in ideas and results from [5].

Let us recall some classical notions from Banach Space Theory. A Banach space \( X \) has cotype \( q \), for \( 2 \leq q < \infty \), if there is a constant \( C > 0 \) such that

\[
\left( \sum_{i=1}^{n} \|x_i\|^q \right)^{\frac{1}{q}} \leq C \left( \int_{I} \left\| \sum_{j=1}^{n} r_j(t)x_j \right\|^2 dt \right)^{\frac{1}{2}},
\]

no matter how we select finitely many vectors \( x_1, \ldots, x_n \in X \), where \( I := [0, 1] \) and \( r_i \) denotes the \( i \)-th Rademacher function. The smallest of all these constants is denoted by \( C_q(X) \) and it is called the cotype \( q \) constant of \( X \). The infimum of the cotypes assumed by \( X \) is denoted by \( \text{cot} X \). A result due to Kahane generalizes Khinchine’s inequality to arbitrary Banach spaces and provides estimates between \( \ell_p \) norms of Rademacher means:

**Theorem 1.1** (Kahane Inequality). Let \( 0 < p, q < \infty \). Then there is a constant \( K_{p,q} > 0 \) for which

\[
\left( \int_{I} \left\| \sum_{k=1}^{n} r_k(t)x_k \right\|^q dt \right)^{\frac{1}{q}} \leq K_{p,q} \left( \int_{I} \left\| \sum_{k=1}^{n} r_k(t)x_k \right\|^p dt \right)^{\frac{1}{p}}
\]

for all Banach spaces \( X \) and \( x_1, \ldots, x_n \in X \).

For \( p \in [1, \infty) \), the weak \( \ell_p \)-norm of \( x_1, \ldots, x_N \) in a Banach space \( X \) is defined by

\[
\| (x_i)_{i=1}^{N} \|_{w,p} := \sup_{\|x'\|_{X'} \leq 1} \left( \sum_{i=1}^{N} |x'(x_i)|^p \right)^{\frac{1}{p}}.
\]

Throughout this paper we will only consider complex Banach spaces. From now on \( X, X_1, \ldots, X_m, Y \) stands for Banach spaces and \( \mathcal{L}(X_1, \ldots, X_m; Y) \) denotes the Banach space of all (bounded) \( m \)-linear operators \( U : X_1 \times \cdots \times X_m \to Y \). A linear operator \( u : X \to Y \) is absolutely summing if \( \|(u(x_j))\|_{\ell_1} < \infty \), whenever \( \| (x_j)_{j=1}^{\infty} \|_{w,1} < \infty \). The concept of absolutely summing operator was extended to a general notion of absolutely \((p; q)\)-summing operators in the 1960’s by B. Mitiagin and A. Pełczyński [14] and A. Pietsch [20]. A linear operator \( u : X \to Y \) is absolutely \((p; q)\)-summing if \( \|(u(x_j))\|_{\ell_q} < \infty \), whenever \( \| (x_j)_{j=1}^{\infty} \|_{w,q} < \infty \).

Key words and phrases. Bohnenblust–Hille inequality.

N. Albuquerque was supported by CAPES.
A multilinear approach to absolutely \((p,q)\)-summing operators is the concept of multiple \((q,p)\)-summing operators (see, e.g., \cite{13,19}). For \(p_1,\ldots,p_m \in \{1,\infty\}, U \in \mathcal{L}(X_1,\ldots,X_m;Y)\) is called multiple \((p_1,\ldots,p_m)\)-summing if there exists a constant \(C > 0\) such that
\[
\left( \sum_{i_1,\ldots,i_m=1}^{N} \left\| U \left( x^{(1)}_{i_1}, \ldots, x^{(m)}_{i_m} \right) \right\|_{Y}^{q} \right)^{\frac{1}{q}} \leq C \prod_{k=1}^{m} \left\| \left( x^{(k)}_{i_k} \right) \right\|_{w_{p_k}}^{N}
\]
holds for each finite choice of vectors \(x^{(k)}_{i_k} \in X_k, 1 \leq i \leq N, 1 \leq k \leq m\). The vector space of all multiple \((q,p_1,\ldots,p_m)\)-summing operators is denoted by \(\Pi_{(q,p_1,\ldots,p_m)}^m(X_1,\ldots,X_m;Y)\). The infimum, \(\pi_{(q,p_1,\ldots,p_m)}^m(U)\), taken over all possible constants \(C\) satisfying the previous inequality defines a complete norm in \(\Pi_{(q,p_1,\ldots,p_m)}^m(X_1,\ldots,X_m;Y)\). When \(p_1 = \cdots = p_m = p\) we say that \(U\) is multiple \((q,p)\)-summing and the notation is simplified to \(\Pi_{(q,p)}^m(X_1,\ldots,X_m;Y)\) and \(\pi_{(q,p)}^m(\cdot)\). When \(p = q\) we say that \(U\) is multiple \(p\)-summing, and the notation is simplified to \(\Pi_{p}^m(X_1,\ldots,X_m;Y)\) and \(\pi_{p}^m(\cdot)\).

A vector lattice \(X\) is a Banach lattice when it is a Banach space and has the property that \(\|x\| < \|y\|\), whenever \(x,y \in X\) fulfills \(|x| < |y|\). Let us recall some notions from the theory of Banach lattices. A Banach lattice \(X\) is \(p\)-concave, \(1 \leq q < \infty\), if there is a constant \(C > 0\) such that, regardless of the choice of finitely many \(x_1,\ldots,x_N \in X\), we have
\[
\left( \sum_{n=1}^{N} \|x_n\|^{q} \right)^{\frac{1}{q}} \leq C \left( \sum_{n=1}^{N} |x_n|^{q} \right)^{\frac{1}{q}},
\]
and the optimal constant \(C\) is denoted by \(M_q(X)\). On the other hand, a Banach lattice \(X\) is called \(p\)-convex, \(1 \leq p < \infty\), if there is a constant \(C > 0\) such that for every \(x_1,\ldots,x_N \in X\) we have
\[
\left( \sum_{n=1}^{N} |x_n|^{p} \right)^{\frac{1}{p}} \leq C \left( \sum_{n=1}^{N} \|x_n\|^{p} \right)^{\frac{1}{p}},
\]
and the optimal constant \(C\) is denoted by \(M_p(X)\). For more details see, e.g., \cite{12,17}.

A special case of Banach lattices are the Banach function spaces. Throughout this paper we fix a positive, finite measure space \((\Omega, \Sigma, \mu)\). Let \(\text{sim} \Sigma\) denote the vector space of all complex-valued \(\Sigma\)-simple functions and \(L^0(\mu)\) be the space of all (equivalence classes of) complex-valued \(\Sigma\)-measurable functions modulo \(\mu\)-null functions. The space \(L^0(\mu)\) is a complex vector lattice \cite{17} p. 18. An order ideal \(X(\mu) \subset L^0(\mu)\) is called a Banach function space (B.f.s. for short) based on the measure space \((\Omega, \Sigma, \mu)\) if sim \(\Sigma \subset X(\mu)\) and \(X(\mu)\) is equipped with a lattice complete norm \(\|\cdot\|_{X(\mu)}\).

Now we remind the notion of a \(p\)-th power of a Banach function space as presented in \cite{17}. Suppose that \((X(\mu), \|\cdot\|_{X(\mu)})\) is a \(p\)-convex B.f.s. with \(p\)-convexity constant equals to \(1\) (if \(M_p(X(\mu)) \neq 1\), one may renorm the space in order to obtain it \cite[Proposition 1.d.8]{12}), for some \(1 \leq p \leq 2\). The \(p\)-th power is defined as
\[
X(\mu)[p] := \left\{ f \in L^0(\mu) : |f|^p \in X(\mu) \right\}
\]
and
\[
\|f\|_{X(\mu)[p]} := \| |f|^p \|_{X(\mu)}^{\frac{1}{p}}, \quad f \in X(\mu)[p].
\]
It is well-known that \(\|\cdot\|_{X(\mu)[p]}\) defines a lattice complete norm on the order ideal \(X(\mu)[p]\) of the vector lattice \(L^0(\mu)\), with sim \(\Sigma \subset X(\mu)[p]\). In other words, \((X(\mu)[p], \|\cdot\|_{X(\mu)[p]})\) is a B.f.s.

We also need some useful multi-index notation: for positive integers \(m, n\), we set
\[
\mathcal{M}(m,n) := \{ i = (i_1, \ldots, i_m) : i_1, \ldots, i_m \in \{1,\ldots,n\} \},
\]
and for \(k = 1,\ldots,m\), \(\mathcal{P}_k(m)\) denotes the set of the subsets of \(\{1,\ldots,m\}\) with cardinality \(k\). For \(S = \{s_1,\ldots,s_k\} \in \mathcal{P}_k(m)\), its complement will be \(\hat{S} := \{1,\ldots,m\} \setminus S\), and \(s_k\) shall mean \((i_{s_1},\ldots,i_{s_k}) \in \mathcal{M}(k,n)\). For a multi-indexes \(i_1 \in \mathcal{M}(m,n)\), we denote by \(|i_1|\) the cardinality of the set of multi-index \(j \in \mathcal{M}(m,n)\) such that there is a permutation \(\sigma\) of \(\{1,\ldots,m\}\) such that \(i_{\sigma(k)} = j_k\), for every \(k = 1,\ldots,m\).

The following powerful generalization of the Béla inequality will be crucial for our estimates.

**Lemma 1.2** (Bayart, Pellegrino, Seoane). Let \(m, n\) positive integers, \(1 \leq k \leq m\) and \(1 \leq s \leq q\). Then for all scalar matrix \((a_i)_{i \in \mathcal{M}(m,n)}\),
2. A subpolynomial multilinear Bohnenblust-Hille inequality on Hilbert lattices

The multilinear Bohnenblust-Hille inequality was originally proved in 1931 with constants with exponential growth. It was only in 2012 that Pellegrino and Seoane, based on previous work of Defant, Popa and Schwarting, provided (surprising) constants with subpolynomial growth. In this section we show that for some vector-valued cases the Bohnenblust-Hille type inequality also has constants with subpolynomial growth. Our results improve some estimates from [2], in some special cases.

Thanks to Grothendieck we know that every continuous linear operator from $\ell_1$ to any Hilbert space is absolutely summing. This result is one of the several consequences of the famous Grothendieck’s inequality, called “the fundamental theorem the metric theory of tensor products”. A Banach space $X$ is called “the fundamental theorem the metric theory of tensor products”. A Banach space $X$ is said $GT$ space if every continuous linear operator from $X$ to $H$ is absolutely summing (thus is $p$-summing for $1 \leq p < \infty$), where $H$ stands for a Hilbert space.

The next result from [3, Theorem 4.3] improves the estimates on variations of Bohnenblust-Hille inequality introduced in [15] and [16, Appendix A].

**Theorem 2.1.** Let $t \in [1, 2)$ and $m > 1$. Then

$$
\left( \sum_{i_1, \ldots, i_m = 1}^{\infty} |U(e_{i_1}, \ldots, e_{i_m})|^{2t(m-1) - 2t} \right)^{\frac{1}{2t(m-1) - 2t}} \leq C_{m,t}^{\infty} \|U\|
$$

for all $m$-linear forms $U : c_0 \times \cdots \times c_0 \to \mathbb{C}$, with

$$
C_{m,t}^{\infty} \leq \prod_{j=2}^{m} \Gamma \left( 2 - \frac{2 - t}{j(t - 2t + 2)^{\frac{j-2}{j}}} \right).
$$

Theses estimates combined with the Grothendieck’s Theorem lead us to improve the constants of some special cases from [2, Theorem 6.1]:

**Theorem 2.2.** Let $X$ be a $GT$ space and $H$ be a Hilbert space, $v \in [1, 2)$ and $v : X \to H$ a linear operator. Then

$$
\left( \sum_{i_1, \ldots, i_m = 1}^{\infty} \|vT(e_{i_1}, \ldots, e_{i_m})\|_{H}^{2t(m-1) - 2t} \right)^{\frac{1}{2t(m-1) - 2t}} \leq C_{m,r}^{\infty} \prod_{j=2}^{m} \frac{\pi}{2r(m-1)^{\frac{j-2}{j}}} \left( \|v\| \|T\| \right),
$$

for all $m$-linear operators $T : c_0 \times \cdots \times c_0 \to X$, with $C_{m,r}^{\infty}$ as in Theorem 2.1.

**Proof.** By Grothendieck’s theorem, $v$ is $p$-summing, with $p := \frac{2rm}{2r(m-1)^{\frac{j-2}{j}}}$. So,

$$
\left( \sum_{i_1, \ldots, i_m = 1}^{\infty} \|vT(e_{i_1}, \ldots, e_{i_m})\|_{H}^{p} \right)^{\frac{1}{p}} \leq \pi_p(v) \sup_{\phi \in B_X} \left( \sum_{i_1, \ldots, i_m = 1}^{\infty} \|\phi T(e_{i_1}, \ldots, e_{i_m})\|_{H}^{p} \right)^{\frac{1}{p}}.
$$

Since the operator $\phi T$ is an $m$-linear form, Theorem 2.1 lead us to

$$
\sup_{\phi \in B_X} \left( \sum_{i_1, \ldots, i_m = 1}^{\infty} \|\phi T(e_{i_1}, \ldots, e_{i_m})\|_{H}^{p} \right)^{\frac{1}{p}} \leq C_{m,r}^{\infty} \|T\|.
$$

Therefore,

$$
\left( \sum_{i_1, \ldots, i_m = 1}^{\infty} \|vT(e_{i_1}, \ldots, e_{i_m})\|_{H}^{p} \right)^{\frac{1}{p}} \leq C_{m,r}^{\infty} \pi_p(v) \|T\|.
$$

□
It is worth noting that, since the optimal estimates for the constants of Kahane’s inequality are unknown (up to particular situations), the constants in the multilinear vector-valued Bohnenblust-Hille inequality from [2] Theorem 6.1 are not necessarily subpolynomial. As mentioned in [3], an adaptation of the procedure presented in [5] shows that the constants of Theorem 6.1 are subpolynomial and, therefore, we conclude that the estimates of Theorem 6.1 are indeed subpolynomial. More precisely, for all \( t \in [1, 2) \), there exists a constant \( \kappa_{t,C} > 0 \) such that, for each \( m > 1 \),

\[
C_{m,t}^C \leq \kappa_{t,C} \cdot m^{\frac{(s-1)(t-2)}{t}},
\]

where \( \gamma \) stands for the Euler constant (see [3, Theorem 4.4]).

3. A subexponential polynomial Bohnenblust-Hille inequality on Banach lattices

In this section we present a vector-valued polynomial Bohnenblust-Hille inequality on Banach lattices with estimates that recover the one presented in [5].

To simplify the notation, we denote the B.f.s. \( \left( X(\mu), \| \cdot \|_{X(\mu)} \right) \) by \( (X, \| \cdot \|_X) \). Let us denote by \( \mu^n \) the normalized Lebesgue finite measure on the \( n \)-dimensional torus \( T^n := \{ (z_1, \ldots, z_n) \in \mathbb{C}^n; |z_i| = 1 \} \). The following inequality is a combination of Krivine’s calculus (as presented, e.g., in [21, p. 61]) and a result due to Bayart [4, Theorem 9] (see, for instance, [12, p.40-42]) and a result due to Bayart [4, Theorem 9] (see, for instance, [21, p. 61]).

**Theorem 3.1** (Bayart’s inequality). Let \( 0 < p < q < \infty \). For every \( M \)-homogeneous polynomial \( P(z) = \sum_{|\alpha|=m} c_\alpha z^\alpha \) on \( \mathbb{C}^n \) with values in a Banach lattice \( Y \),

\[
\left( \int_{T^n} |P(z)|^q \, d\mu^n(z) \right)^\frac{1}{q} \leq \left( \frac{q}{p} \right)^\frac{1}{q} M_q(X) \left( \int_{T^n} \|P(z)\|_X^p \, d\mu^n(z) \right)^{\frac{1}{p}}.
\]

The following variant will be useful to obtain our main result.

**Lemma 3.2.** Let \( 1 \leq p \leq 2 \leq q < \infty \), and \( (X, \| \cdot \|_X) \) a \( p \)-convex and \( q \)-concave B.f.s.. For every \( m \)-homogeneous polynomial \( P(z) = \sum_{|\alpha|=m} c_\alpha z^\alpha \) on \( \mathbb{C}^n \) with values in \( X \), we have

\[
\left( \sum_{|\alpha|=m} \|c_\alpha\|_{X}^q \right)^{\frac{1}{q}} \leq M_q(X) \left( \int_{\mathbb{T}^n} \|P(z)\|_X^p \, d\mu^n(z) \right)^{\frac{1}{p}}.
\]

**Proof.** Using the orthonormality of the monomials \( z^\alpha \) in \( L^2 \) and the \( q \)-concavity of \( X \), with \( 2 \leq q \), we get

\[
\left( \sum_{|\alpha|=m} \|c_\alpha\|_{X}^q \right)^{\frac{1}{q}} \leq M_q(X) \left( \int_{\mathbb{T}^n} \|P(z)\|_X^p \, d\mu^n(z) \right)^{\frac{1}{p}}.
\]

By Theorem 3.1, we have

\[
\left( \sum_{|\alpha|=m} \|c_\alpha\|_{X}^q \right)^{\frac{1}{q}} \leq M_q(X) \left( \int_{\mathbb{T}^n} |P(z)|^q \, d\mu^n(z) \right)^{\frac{1}{q}} \|P(z)\|_X^p \, d\mu^n(z) \right)^{\frac{1}{p}} \leq M_q(X) \left( \int_{\mathbb{T}^n} |P(z)|^p \, d\mu^n(z) \right)^{\frac{1}{p}} \|P(z)\|_X^p \, d\mu^n(z) \right)^{\frac{1}{p}} 
\]

Now we present the main result of this paper.

**Theorem 3.3.** Let \( Y \) a Banach space, \( X \) a \( 2 \)-concave and \( 2 \)-convex B.f.s., and \( v : Y \to X \) an \( (r,1) \)-summing operator with \( 1 \leq r < 2 \). Let us fix \( 1 \leq k \leq m \) and define

\[
\rho := \frac{2mr}{2 + (m-1)r}.
\]
For every $m$-homogeneous polynomial $P : \ell^p_{\infty} \to Y$, defined by $P(z) = \sum_{|\alpha|=m} c_\alpha z^\alpha$, we have

$$
\left( \sum_{|\alpha|=m} \|vc_\alpha\|^p_X \right)^{\frac{1}{p}} \leq \left( \frac{2}{s_k} \right)^{\frac{m}{(m-k)!}} \left( C_2(X) \right)^{k-1} \left( \prod_{j=1}^{k-1} K \frac{2^j}{(m-j)!} \right) C_{m,k} M_2(X) \pi_1(\rho) \|P\|_{\ell^p_{\infty},}
$$

where $K_{p,2}$ is the constant of the Kahane inequality, $s_k := \frac{2kr}{2r(k-1)r}$ and $C_{m,k} := \frac{m^m}{(m-k)!} \sqrt{\frac{(m-k)!}{m!}}$.

Proof. Let us fix $r < 2$ and define $s_k < 2$ by $s_k = \frac{2kr}{2r(k-1)r}$. Let $L : \ell^p_{\infty} \times \cdots \times \ell^p_{\infty} \to Y$ be the unique $m$-linear map associated to $P$. Thus, $c_\alpha = |i_\alpha| a_{i_\alpha}$, with $a_{i_\alpha} := L e_{i_\alpha}$, for some $i_\alpha \in J(m,n)$. Since $|i|^\rho - 1 \leq (\rho)^\rho$, using Lemma 1.2 with $s = s_k < 2$, we get

$$
\left( \sum_{|\alpha|=m} \|vc_\alpha\|^p_X \right)^{\frac{1}{p}} = \left( \sum_{i \in J(m,n)} \|i| va_i\|^p_X \right)^{\frac{1}{p}} \\
= \left( \sum_{i \in M(m,n)} \frac{1}{|i|} \|i| va_i\|^p_X \right)^{\frac{1}{p}} \\
\leq \left( \sum_{i \in M(m,n)} \|va_i\|^p_X \right)^{\frac{1}{p}} \\
\leq \left[ \prod_{S \subseteq P_k} \left( \sum_{i_S \in M(k,n)} \left( \sum_{i_{\bar{S}} \in M(m-k,n)} \|va_{i_{\bar{S}}}\|^2_X \right)^{\frac{1}{2}} \right) \right]^{\frac{1}{2}}.
$$

Note that $|i| \leq |i|_{m'} \frac{m!}{(m-k)!}$. Thus

$$
\left( \sum_{|\alpha|=m} \|vc_\alpha\|^p_X \right)^{\frac{1}{p}} \leq \sqrt{\frac{m!}{(m-k)!}} \left[ \prod_{S \subseteq P_k} \left( \sum_{i_S \in M(k,n)} \left( \sum_{i_{\bar{S}} \in M(m-k,n)} \|va_{i_{\bar{S}}}\|^2_X \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \right]^{\frac{1}{2}}.
$$

Let us fix $S \in P_k(m)$. Notice that

$$
P_{i_S}(z) := \sum_{i_{\bar{S}} \in J(m-k,n)} |i_{\bar{S}}| va_{i_{\bar{S}}} z_{i_{\bar{S}}},
$$

is a $(m-k)$-homogeneous polynomial on $X$, related with the symmetric $m$-linear operator $vL$, and

$$
vL (e_{i_S}, z_{i_{\bar{S}}}) = P_{i_S}(z),
$$
where \( z_{i\xi} \) denotes \( z \) on the coordinates \( i\xi \). Since \( X \) is 2-convex and \( s_k < 2 \), it also is \( s_k \)-convex. Then Lemma 3.2 lead us to
\[
\left( \sum_{I\in M(k,n)} \left( \sum_{I\in M(m-k,n)} \left\| i\xi \right\|^2 v a_i \right)^{\frac{1}{2}} \right)^{\frac{1}{2}}
\]
\[
\leq \left( \sum_{I\in M(k,n)} \left( \sum_{I\in J(m-k,n)} \left\| i\xi \right\|^2 v a_i \right)^{\frac{1}{2}} \right)^{\frac{1}{2}}
\]
\[
\leq \left( \frac{2}{s_k} \right)^{m-k} M_2(X) \left( \sum_{I\in M(k,n)} \left( \int_{T^n} \left( \sum_{I\in J(m-k,n)} \left\| i\xi \right\|^2 v a_i z_{i\xi} \right) \, dz \right) \right)^{\frac{1}{2}}
\]
\[
= \left( \frac{2}{s_k} \right)^{m-k} M_2(X) \left( \int_{T^n} \left( \sum_{I\in M(k,n)} \left\| vL (e_{i\xi}, z_{i\xi}) \right\|_{X} \right) \, dz \right)^{\frac{1}{2}}
\]
\[
= \left( \frac{2}{s_k} \right)^{m-k} M_2(X) \left( \int_{T^n} \left( \sum_{I\in M(k,n)} \left\| vL (e_{i\xi}, z_{i\xi}) \right\|_{X} \right) \, dz \right)^{\frac{1}{2}}
\]

The operator \( v \) is \((r,1)\)-summing and \( X \) has cotype 2. For each \( z \in T^n \), \( L (\cdot, z_{i\xi}) \) is a \( k \)-linear operator, thus, combining [2] Theorem 6.1 and Corollary 6.3 and Harris’s Polarization Formula we get
\[
\left( \sum_{I\in M(k,n)} \left\| vL (e_{i\xi}, z_{i\xi}) \right\|_{X} \right)^{\frac{1}{2}}
\]
\[
\leq (C_2(X))^{k-1} \left( \prod_{j=1}^{k-1} K_{x_j \in (1,2)^k} \right) \left( \prod_{j=1}^{k-1} K_{x_j \in (1,2)^k} \right) \pi(r,1)(v) \sup_{x^{(1)}, \ldots, x^{(k)} \in T^n} \left\| L (x^{(1)}, \ldots, x^{(k)}, z_{i\xi}) \right\|,
\]
\[
\leq (C_2(X))^{k-1} \left( \prod_{j=1}^{k-1} K_{x_j \in (1,2)^k} \right) \pi(r,1)(v) \left( \frac{(m-k)!}{m!} \right)^{m-k} \left( \frac{m^m}{m!} \right)^m \left\| P \right\|_{\Delta^n}.
\]

Therefore, we obtain
\[
\left( \sum_{|\alpha|=m} \left\| v c_{\alpha} \right\|_{X} \right)^{\frac{1}{2}}
\]
\[
\leq \left( \frac{2}{s_k} \right)^{m-k} \left( \frac{(m-k)!}{m!} \right)^{m-k} M_2(X) (C_2(X))^{k-1} \left( \prod_{j=1}^{k-1} K_{x_j \in (1,2)^k} \right) \pi(r,1)(v) \left\| P \right\|_{\Delta^n}.
\]

Notice that, \( \frac{2}{s_k} < 1 + \varepsilon \) for a given \( \varepsilon > 0 \) and \( k \) large enough. Thus, proceeding as in [3] we obtain the subexponential constants in Theorem 3.3.

**Corollary 3.4.** Under the same hypotheses of the previous theorem, for any \( \varepsilon > 0 \), there is \( \kappa > 0 \), such that
\[
\left( \sum_{|\alpha|=m} \left\| v c_{\alpha} \right\|_{X}^{2(m-1)r} \right)^{\frac{2(m-1)r}{2(m-1)r}} \leq \kappa (1 + \varepsilon)^m \left\| P \right\|_{\Delta^n},
\]

holds for all positive integer \( n \) and every \( m \)-homogeneous polynomial \( P : \ell^p_{\infty} \rightarrow Y \) defined by \( P(z) = \sum_{|\alpha|=m} c\alpha z^\alpha \).

Since \( M_2(C) = 1 \) and since the identity operator \( \text{id} : C \rightarrow C \) is \((1,1)\)-summing with \( \pi_{(1,1)}(\text{id}) = 1 \), taking \( X = Y = C \) on theorem 3.3 we recover the polynomial Bohnenblust-Hille inequality with estimates presented in [3] Theorem 5.2. More precisely, if \( m \geq 1 \) and \( k \in \{1, \ldots, m-1\} \), then for any \( n \geq 1 \) and any \( m \)-homogeneous polynomial \( P(z) = \sum_{|\alpha|=m} c_{\alpha} z^\alpha \) on \( C^n \),
\[
\left( \sum_{|\alpha|=m} \left| c_{\alpha} \right| \right)^{\frac{1}{m-k}} \leq \left( 1 + \frac{1}{k} \right)^{\frac{m-k}{2}} \left( \frac{(m-k)!}{m!} \right)^{\frac{1}{2}} \left( \frac{m^m}{(m-k)^m} \right)^{m-k} C_{\alpha}^2 \left\| P \right\|_{\Delta^n},
\]
where \( C_k^m \) denotes the \( k \)-th multilinear Bohnenblust-Hille constant.

Combining Theorem 2.2 and Theorem 3.3, we have the following result.

**Theorem 3.5.** Let \( m \geq 1, r \in [1,2] \), \( Y \) be a GT space, \( H \) a Hilbert function space, and \( v: Y \to H \) a linear operator. Let us fix \( k \in \{1, \ldots, m\} \). For all positive integers \( n \) and every \( m \)-homogeneous polynomial \( P: \ell^\infty_n \to H \) defined by \( P(z) = \sum_{|\alpha|=m} c_\alpha z^\alpha \),

\[
\left( \sum_{|\alpha|=m} \| v c_\alpha \|_H \right)^{2(r-1)} \leq \left( \frac{2}{s_k} \right)^{2m} C_{m,k} C_k^m \pi^{\frac{2(r-1)}{2r-(m+1)}} \left( \frac{2}{s_k} \right)^{2r} \pi^{\frac{2r}{2r-(m+1)}} \| P \|_{\ell^\infty_n},
\]

where \( s_k := \frac{2kr}{2\pi (k-1)^r} \) and \( C_{m,k} := \left( \frac{(m-k)!}{m!} \right)^{\frac{1}{2} - \frac{m}{(m-k)^{m-k}}} \).

**Acknowledgment.** The authors would like to thank Prof. E. A. Sánchez Pérez for his fruitful comments regarding Banach lattices.

**References**

[1] N. Albuquerque, F. Bayart, D. Pellegrino and J. B. Seoane-Sepúlveda, *Sharp generalizations of the multilinear Bohnenblust–Hille inequality*, Journal of Functional Analysis, DOI:10.1016/j.jfa.2013.08.013 (2014).

[2] N. Albuquerque, F. Bayart, D. Pellegrino and J. B. Seoane-Sepúlveda, *Optimal Hardy-Littlewood type inequalities for polynomials and multilinear operators*, arXiv:1311.3177 [math.FA], 7Feb2014.

[3] N. Albuquerque, D. Núñez-Alarcón, J. Santos and D. Serrano-Rodríguez *Absolutely summing multilinear operators via interpolation*, arXiv:1404.4949v1 [math.FA], 22Apr2014.

[4] F. Bayart, *Hardy spaces of Dirichlet series and their composition operators*, Monatsh. Math., 136(3):203-236, 2002.

[5] F. Bayart, D. Pellegrino and J. B. Seoane-Sepúlveda, *The Bohnenblust–Hille inequality for homogeneous polynomials is hypercontractive*, Ann. of Math. (2) 174 (2011), no. 1, 485-497.

[6] A. Defant, M. Maestre, U. Schwarting, *Bohr radii of vector valued holomorphic functions*, Advances in Mathematics 231 (2012) 2837-2857.

[7] A. Defant, D. Popa, U. Schwarting, *Coordinatewise multiple summing operators*, J. Funct. Anal. 259 (2010), 220–242.

[8] D. Diniz, G. Muñoz-Fernandez, D. Pellegrino, J. Seoane-Sepulveda, *The asymptotic growth of the constants in the Bohnenblust–Hille inequality is optimal*, J. Funct. Anal. 263 (2012), 415–428.

[9] H. P. Boas and D. Khavinson, *Bohr’s power series theorem in several variables*, Proc. Amer. Math. Soc. 125 (1997), no. 10, 2975-2979.

[10] H. F. Bohnenblust and E. Hille, *On the absolute convergence of Dirichlet series*, Ann. of Math. (2) 32 (1931), 600-622.

[11] A. Defant, L. Frerick, J. Ortega-Cerdà, M. Ounaïes and K. Seip, *The Bohnenblust-Hille inequality for homogeneous polynomials is hypercontractive*, Ann. of Math. (2) 174 (2011), no. 1, 485-497.

[12] A. Defant, M. Maestre, U. Schwarting, *Bohr radius of vector valued holomorphic functions*, Advances in Mathematics 231 (2012) 2837-2857.

[13] A. Defant, D. Popa, U. Schwarting, *Coordinatewise multiple summing operators*, J. Funct. Anal. 259 (2010), 220–242.

[14] D. Diniz, G. Muñoz-Fernandez, D. Pellegrino, J. Seoane-Sepulveda, *The asymptotic growth of the constants in the Bohnenblust–Hille inequality is optimal*, J. Funct. Anal. 263 (2012), 415–428.

[15] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces II: Function Spaces*, Springer-Verlag, Germany, 1991.

[16] M. Matos, *Fully absolutely summing and Hilbert-Schmidt multilinear mappings*, Collect. Math. 54, 2 (2003), 111-136.

[17] B. Mitjašin, A. Pelczyński, *Nuclear operators and approximative dimension*, Proc. of ICM, Moscow. (1966), 366–372.

[18] D. Núñez-Alarcón, D. Pellegrino, J.B. Seoane-Sepúlveda, *On the Bohnenblust-Hille inequality and a variant of Littlewood’s 4/3-Inequality*, J. Funct. Anal. 264 (2013), 326–336.

[19] D. Núñez-Alarcón, D. Pellegrino, J.B. Seoane-Sepúlveda, D.M. Serrano-Rodríguez, *There exist multilinear Bohnenblust–Hille constants \( C_{m,n} \) with \( \lim_{m \to \infty} (C_{m,1} - C_m) = 0 \), J. Funct. Anal. 264 (2013).

[20] S. Okada, W. Ricker and E. Sánchez-Pérez, *Optimal Domain and Integral Extension of Operators: Acting in Function Spaces*, Birkhäuser-Verlag, Germany, 2008.

[21] D. Pellegrino, J. Seoane-Sepúlveda, *New upper bounds for the constants in the Bohnenblust-Hille inequality*, J. Math. Anal. Appl. 386 (2012), 300–307.

[22] D. Pérez-García, *Operadores multilineales absolutamente sumantes*. PhD Thesis, Univ. Complut. de Madrid, (2003).

[23] A. Pietsch, *Absolut p-sammierende Abbildungen in normierten Räumen*, Studia Math. 27 (1967), 333–353.

[24] U. C. Schwarting, *Vector Valued Bohnenblust-Hille Inequalities*, Diss. Universität Oldenburg (2013).