Dirac’s Legacy: Light-Cone Quantization

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Abstract

In recent years light-cone quantization of quantum field theory has emerged as a promising method for solving problems in the strong coupling regime. This approach has a number of unique features that make it particularly appealing, most notably, the ground state of the free theory is also a ground state of the full theory.
I. INTRODUCTION

One of the central problems in particle physics is to determine the structure of hadrons such as the proton and neutron in terms of their fundamental quark and gluon degrees of freedom. Over the past twenty years two fundamentally different pictures of hadronic matter have developed. One, the constituent quark model (CQM), or the quark parton model is closely related to experimental observation. The other, quantum chromodynamics (QCD) is based on an elegant non-abelian quantum field theory. The light-front formulation of QCD appears to be the only practical hope of reconciling QCD with the CQM. This elegant approach to quantum field theory is a Hamiltonian gauge fixed formulation that avoids many of the most difficult problems in the equal time formulation of the theory. The idea of deriving a null-plane constituent model from QCD actually dates from the early seventies, and there is a rich literature on the subject [11,9,24]. The main thrust of this talk will be to discuss the complexities of vacuum that are unique to the light-front formulation of field theories.

An intuitive approach for solving relativistic bound-state problems would be to solve the gauge fixed, Hamiltonian eigenvalue problem. One imagines that there is an expansion in multi-particle occupation number Fock states. It is clearly a formidable task to calculate the structure of hadrons in terms of their fundamental degrees of freedom in QCD. Even in the case of abelian quantum electrodynamics, very little is known about the nature of the bound state solutions in the strong-coupling domain. A calculation of bound state structure in QCD has to deal with many complicated aspects of the theory simultaneously: confinement, vacuum structure, spontaneous breaking of chiral symmetry (for massless quarks), while describing a relativistic many-body system which apparently has unbounded particle number. The analytic problem of describing QCD bound states is compounded not only by the physics of confinement, but also by the fact that the wavefunction of a composite of relativistic constituents has to describe systems of an arbitrary number of quanta with arbitrary momenta and helicities. The conventional Fock state expansion based on equal-time quantization quickly becomes intractable because of the complexity of the vacuum in a relativistic quantum field theory. Furthermore, boosting such a wavefunction from the hadron’s rest frame to a moving frame is as complex a problem as solving the bound state problem itself.

Fortunately, “light-front” quantization, which can be formulated independent of the Lorentz frame, offers an elegant avenue of escape. There are, in fact, many reasons to quantize relativistic field theories at fixed light-front time. Dirac [12], in 1949, showed that a maximum number of Poincaré generators become independent of the dynamics in the “front form” formulation, including certain Lorentz boosts. Unlike the equal-time Hamiltonian formalism, quantization on a plane tangential to the light-front can be formulated without reference to the choice of a specific Lorentz frame. The eigen solutions of the light-front Hamiltonian have Lorentz scalars $M^2$ as eigenvalues, and describe bound states of arbitrary four-momentum and invariant mass $M$, allowing the computation of scattering amplitudes and other dynamical quantities.

However, the most remarkable feature of this formalism is the apparent simplicity of the light-front vacuum. In many theories the vacuum state of the free Hamiltonian is an eigenstate of the total light-front Hamiltonian. This means that all constituents in a physical
eigenstate are directly related to that state, and not disconnected vacuum fluctuations. The Fock expansion constructed on this vacuum state provides a complete relativistic many-particle basis for diagonalizing the full theory. The natural gauge for light-cone Hamiltonian theories is the light-cone gauge $A^+ = 0$. In this physical gauge the gluons have only two physical transverse degrees of freedom, and thus it is well matched to perturbative QCD calculations. The simplicity of the light-cone Fock representation relative to that in equal-time quantization arises from the fact that the physical vacuum state has a much simpler structure on the light cone. Indeed, kinematical arguments suggest that the light-cone Fock vacuum is the physical vacuum state.

The success of the CQM or the Feynman parton model is a powerful for a light-front formulation of QCD. The ideas of the parton model seem more easily formulated in the light-front picture of quantum field theory than in the equal-time formulation. This is a highly desirable feature if one wishes to have a constituent picture of relativistic bound states and describe, for example, a baryon as primarily a three-quark state plus a few higher Fock states à la Tamm and Dancoff.

Studies of model light-front field theories have shown that the zero modes can in fact support certain kinds of vacuum structure. The long range phenomena of spontaneous symmetry breaking [18,5,42,22,46] as well as the topological structure [26,43] can in fact be reproduced with a careful treatment of the zero mode(s) of the fields in a quantum field theory defined in a finite spatial volume and quantized at equal light-front time. These phenomena are realized in quite different ways. For example, spontaneous breaking of $Z_2$ symmetry in $\phi^{1+1}_{1+1}$ occurs via a constrained zero mode of the scalar field [19]. There the zero mode satisfies a nonlinear equation that relates it to the dynamical modes in the problem. At the critical coupling a bifurcation of the solution occurs [19]. One must choose one solution to use in formulating the theory. This choice is analogous to what in the conventional language one would call the choice of vacuum state. These solutions lead to new operators in the Hamiltonian which break the $Z_2$ symmetry at and beyond the critical coupling. The various solutions contain $c$-number pieces which produce the possible vacuum expectation values of $\phi$. The properties of the strong-coupling phase transition in this model are reproduced, including its second-order nature and a reasonable value for the critical coupling [3,42].

Apart from the question of whether or not VEVs arise, solving the constraint equations really amounts to determining the Hamiltonian (and other Poincaré generators). In general, $P^-$ becomes very complicated when the zero mode contributions are included; this is in some sense the price one pays to achieve a formulation with a simple vacuum. It may be possible to think of the discretization as a cutoff which removes states with $0 < p^+ < \pi/L$, and the zero mode contributions to the Hamiltonian as effective interactions that restore the discarded physics. In the light-cone power counting analysis of Wilson [50] it is clear that there will be a huge number of allowed operators.

Quite separately, a dynamical zero mode was shown in Ref. [26] to arise in pure $SU(2)$ Yang-Mills theory in $1+1$ dimensions. A complete fixing of the gauge leaves the theory with one degree of freedom, the zero mode of the vector potential $A^+$. The theory has a discrete spectrum of zero-$P^+$ states corresponding to modes of the flux loop around the finite space. Only one state has a zero eigenvalue of the energy $P^-$, and is the true ground state of the theory. The nonzero eigenvalues are proportional to the length of the spatial box, consistent
with the flux loop picture. This is a direct result of the topology of the space. As the theory considered there was a purely topological field theory the exact solution was identical to that in the conventional equal-time approach on the analogous spatial topology [21,29,45]. In the present context, the difficulty is that the zero mode in \( A^+ \) is in fact gauge-invariant, so that the light-cone gauge \( A^+ = 0 \) cannot be reached. Thus one has a pair of interconnected problems: first, a practical choice of gauge; and second, the presence of constrained zero modes of the gauge field. In several recent papers [24,26,41] these problems were separated and consistent gauge fixing conditions were introduced to allow isolation of the dynamical and constrained fields.

The study of these low dimensional theories is part of a long-term program to attack QCD\(_{3+1}\) through the zero mode sectors starting with studies of lower dimensional theories which are themselves zero mode sectors of higher dimensional theories [25,26]. A complete gauge fixing has recently been given for discrete light-cone quantized QED\(_{3+1}\) which further supports this program [27] and one sees how zero modes naturally arise and the special role that they play. In appears that the central problem in light-front QCD will be to disentangle and solve the constraints for the dependent zero modes in terms of the independent fields in the context of a particular gauge fixing.

**A. Constrained Zero Modes**

As mentioned previously, the light-front vacuum state is simple; it contains no particles in a massive theory. However, one commonly associates important long range properties of a field theory with the vacuum: spontaneous symmetry breaking, the Goldstone pion, and color confinement. How do these complicated phenomena manifest themselves in light-front field theory?

If one cannot associate long range phenomena with the vacuum state itself, then the only alternative is the zero momentum components or “zero modes” of the field (long range ↔ zero momentum). In some cases, the zero mode operator is not an independent degree of freedom but obeys a constraint equation. Consequently, it is a complicated operator-valued function of all the other modes of the field [35]. This problem has recently been attacked from several directions. The question of whether boundary conditions can be consistently defined in light-front quantization has been discussed by McCartor and Robertson [38] and Lenz [30,31]. They have shown that for massive theories the energy and momentum derived from light-front quantization are conserved and are equivalent to the energy and momentum one would normally write down in an equal-time theory. Heinzl and Werner et al. [19,18] considered \( \phi^4 \) theory in (1+1)–dimensions and solved the zero mode constraint equation by truncating the equation to one particle and retaining all modes. They implicitly retain a two particle contribution in order to obtain finite results. Other authors [17] find that, for theories allowing spontaneous symmetry breaking, there is a degeneracy of light-front vacua and the true vacuum state can differ from the perturbative vacuum through the addition of zero mode quanta. In addition to these approaches there are many others [6].

The definitive analysis by Pinsky, van de Sande, Bender and Hiller [18,22] of the zero mode constraint equation for (1+1)–dimensional \( \phi^4 \) field theory with symmetric boundary conditions shows how spontaneous symmetry breaking occurs within the context of this
model. This theory has a $Z_2$ symmetry $\phi \rightarrow -\phi$ which is spontaneously broken for some values of the mass and coupling. Their approach is to apply a Tamm-Dancoff truncation to the Fock space. Thus operators are finite matrices and the operator valued constraint equation for the zero mode can be solved numerically. The truncation assumes that states with a large number of particles or large momentum do not have an important contribution to the zero mode.

One finds the following general behavior: for small coupling (large $g$, where $g \propto 1/coupling$) the constraint equation has a single solution and the field has no vacuum expectation value (VEV). As one increase the coupling (decrease $g$) to the “critical coupling” $g_{\text{critical}}$, two additional solutions which give the field a nonzero VEV appear. These solutions differ only infinitesimally from the first solution near the critical coupling, indicating the presence of a second order phase transition. Above the critical coupling ($g < g_{\text{critical}}$), there are three solutions: one with zero VEV, the “unbroken phase,” and two with nonzero VEV, the “broken phase”. The “critical curves” shown in Figure 1, is a plot the VEV as a function of $g$.

Since the vacuum in this theory is trivial, all of the long range properties must occur in the operator structure of the Hamiltonian. Above the critical coupling ($g < g_{\text{critical}}$) quantum oscillations spontaneously break the $Z_2$ symmetry of the theory. In a loose analogy with a symmetric double well potential, one has two new Hamiltonians for the broken phase, each producing states localized in one of the wells. The structure of the two Hamiltonians is determined from the broken phase solutions of the zero mode constraint equation. One finds that the two Hamiltonians have equivalent spectra. In a discrete theory without zero modes it is well known that, if one increases the coupling sufficiently, quantum correction will generate tachyons causing the theory to break down near the critical coupling. Here the zero mode generates new interactions that prevent tachyons from developing. In effect what happens is that, while quantum corrections attempt to drive the mass negative, they also change the vacuum energy through the zero mode and the diverging mass eigenvalue can never catch the vacuum eigenvalue. Thus, tachyons never appear in the spectra.

In the weak coupling limit ($g$ large) the solution to the constraint equation can be obtained in perturbation theory. This solution does not break the $Z_2$ symmetry and is believed to simply insert the missing zero momentum contributions into internal propagators. This must happen if light-front perturbation theory is to agree with equal-time perturbation
theory.

Another way to investigate the zero mode is to study the spectrum of the field operator $\phi$. Here one finds a picture that agrees with the symmetric double well potential analogy. In the broken phase, the field is localized in one of the minima of the potential and there is tunneling to the other minimum.

### B. Canonical Quantization

The details of the Dirac-Bergmann prescription and its application to the system considered here are discussed elsewhere in the literature [33,49]. For a classical field the $(\phi^4)_{1+1}$ Lagrange density is

$$L = \partial_+ \phi \partial_- \phi - \frac{\mu^2}{2} \phi^2 - \frac{\lambda}{4!} \phi^4.$$  \hspace{1cm} (1.1)

One puts the system in a box of length $d$ and impose periodic boundary conditions. Then

$$\phi(x) = \frac{1}{\sqrt{d}} \sum_n q_n(x^+) e^{ik_n^+x^-},$$  \hspace{1cm} (1.2)

where $k_n^+ = 2\pi n/d$ and summations run over all integers unless otherwise noted.

The $\int dx^- \phi(x)^n$ minus the zero mode part is

$$\Sigma_n = \frac{1}{n!} \sum_{i_1,i_2,\ldots,i_n \neq 0} q_{i_1} q_{i_2} \ldots q_{i_n} \delta_{i_1+i_2+\ldots+i_n,0}$$  \hspace{1cm} (1.3)

and the canonical Hamiltonian is

$$P^- = \frac{\mu_0^2 q_0^2}{2} + \mu^2 \Sigma_2 + \frac{\lambda q_0^4}{4!d} + \frac{\lambda q_0^2 \Sigma_2}{2!d} + \frac{\lambda q_0 \Sigma_3}{d} + \frac{\lambda \Sigma_4}{d}.$$  \hspace{1cm} (1.4)

Following the Dirac-Bergmann prescription, one identify first-class constraints which define the conjugate momenta

$$0 = p_n - ik_n^+ q_n,$$  \hspace{1cm} (1.5)

where

$$[q_m, p_n] = \frac{\delta_{m,n}}{2}, \quad m, n \neq 0.$$  \hspace{1cm} (1.6)

The secondary constraint is,

$$0 = \mu^2 q_0 + \frac{\lambda q_0^3}{3!d} + \frac{\lambda q_0 \Sigma_2}{d} + \frac{\lambda \Sigma_3}{d},$$  \hspace{1cm} (1.7)

which determines the zero mode $q_0$. This result can also be obtained by integrating the equations of motion.
To quantize the system one replaces the classical fields with the corresponding field operators, and the Dirac bracket by $i$ times a commutator. One must choose a regularization and an operator-ordering prescription in order to make the system well-defined.

One begin by defining creation and annihilation operators $a_k^\dagger$ and $a_k$, 

$$q_k = \sqrt{\frac{d}{4\pi |k|}} a_k, \quad a_k = a_k^\dagger, \quad k \neq 0,$$

where $\sqrt{d/4\pi |k|}$ is the Planck constant. The operators satisfy the usual commutation relations 

$$[a_k, a_l] = 0, \quad [a_k^\dagger, a_l] = 0, \quad [a_k, a_l^\dagger] = \delta_{k,l}, \quad k, l > 0.$$

Likewise, one defines the zero mode operator 

$$q_0 = \sqrt{\frac{d}{4\pi}} a_0.$$

In the quantum case, one normal orders $\Sigma_n$. General arguments suggest that the Hamiltonian should be symmetric ordered \[4\]. However, it is not clear how one should treat the zero mode since it is not a dynamical field. As an ansatz one treats $a_0$ as an ordinary field operator when symmetric ordering the Hamiltonian. The tadpoles are removed from the symmetric ordered Hamiltonian by normal ordering the terms having no zero mode factors and by subtracting, 

$$\frac{3}{2} a_0^2 \sum_{n \neq 0} \frac{1}{|n|}.$$ 

In addition, one subtract a constant so that the VEV of $H$ is zero. Note that this renormalization prescription is equivalent to a conventional mass renormalization and does not introduce any new operators into the Hamiltonian. The constraint equation for the zero mode can be obtained by taking a derivative of $P^-$ with respect to $a_0$. One finds, 

$$0 = ga_0 + a_0^3 + \sum_{n \neq 0} \frac{1}{|n|}(a_0 a_n a_{-n} + a_n a_{-n} a_0 + a_n a_0 a_{-n} - \frac{3a_0}{2}) + 6\Sigma_3$$

where $g = 24\pi \mu^2/\lambda$. It is clear from the general structure of constraint equation that $a_0$ as a function of the other modes is not necessarily odd under the transform $a_k \rightarrow -a_k, \ k \neq 0$ associated with the $\mathbb{Z}_2$ symmetry of the system. Consequently, the zero mode can induce $\mathbb{Z}_2$ symmetry breaking in the Hamiltonian.

In order to render the problem tractable, one can impose a Tamm-Dancoff truncation on the Fock space. Define $M$ to be the number of nonzero modes and $N$ to be the maximum number of allowed particles. Thus, each state in the truncated Fock space can be represented by a vector of length $S = (M + N)!/(M!N!)$ and operators can be represented by $S \times S$ matrices. One can define the usual Fock space basis, $|n_1, n_2, \ldots, n_M\rangle$, where $n_1 + n_2 + \ldots + n_M \leq N$. In matrix form, $a_0$ is real and symmetric. Moreover, it is block diagonal in states of equal $P^+ \text{ eigenvalue.}$
FIG. 2. Convergence to the large $d$ limit of $1 \to 1$ setting $E = g/p$ and dropping any constant terms.

C. Perturbative Solution of the Constraints

In the limit of large $g$, one can solve the constraint equation perturbatively. Then one substitutes the solution back into the Hamiltonian and calculates various amplitudes to arbitrary order in $1/g$ using Hamiltonian perturbation theory.

It can be shown that the solutions of the constraint equation and the resulting Hamiltonian are divergence free to all orders in perturbation theory for both the broken and unbroken phases. The perturbative solution for the unbroken phase is

$$a_0 = -\frac{6}{g} \Sigma_3 + \frac{6}{g^2} \left(2 \Sigma_2 \Sigma_3 + 2 \Sigma_3 \Sigma_2 + \sum_{k=1}^{M} \frac{a_k \Sigma_3 a_k^\dagger + a_k^\dagger \Sigma_3 a_k - \Sigma_3}{k} \right) + O\left(1/g^3\right). \quad (1.13)$$

Substituting this into the Hamiltonian, one obtains a complicated but well defined expression. The finite volume box acts as an infra-red regulator and the only possible divergences are ultra-violet. Using diagrammatic language, any loop of momentum $k$ with $\ell$ internal lines has asymptotic form $k^{-\ell}$. Only the case of tadpoles $\ell = 1$ is divergent. If there are multiple loops, the effect is to put factors of $\ln(k)$ in the numerator and the divergence structure is unchanged. Looking at Eq. (1.13), the only possible tadpole is from the contraction in the term

$$\frac{a_k \Sigma_3 a_{-k}}{k} \quad (1.14)$$

which is canceled by the $\Sigma_3/k$ term. This happens to all orders in perturbation theory: each tadpole has an associated term which cancels it. Likewise, in the Hamiltonian one has similar cancellation. As with the zero mode, such cancellations occur to all orders in perturbation theory. For the unbroken phase, the effect of the zero mode should vanish in the
infinite volume limit, giving a “measure zero” contribution to the continuum Hamiltonian. However, for finite box volume the zero mode does contribute, compensating for the fact that the longest wavelength mode has been removed from the system. Thus, inclusion of the zero mode improves convergence to the infinite volume limit; it acts as a form of infra-red renormalization. In addition, one can use the perturbative expansion of the zero mode to study the operator ordering problem. One can directly compare our operator ordering ansatz with a truly Weyl ordered Hamiltonian and with Maeno’s operator ordering ansatz \[33\]. As an example, let us examine \[O(\lambda^2)\] contributions to the processes \(1 \rightarrow 1\) which as shown in Figure 2 including the zero mode greatly improves convergence to the large volume limit. The zero mode compensates, in an optimal manner, for the fact that one has removed the longest wavelength mode from the system.

D. Non-Perturbative Solution: One Mode, Many Particles

Consider the case of one mode \(M = 1\) and many particles. In this case, the zero-mode is diagonal and can be written as

\[ a_0 = f_0 |0\rangle \langle 0| + \sum_{k=1}^{N} f_k |k\rangle \langle k|. \]  

(1.15)

Note that \(a_0\) in (1.13) is even under \(a_k \rightarrow -a_k, k \neq 0\) and any non-zero solution breaks the \(Z_2\) symmetry of the original Hamiltonian. The VEV is given by

\[ \langle 0|\phi|0\rangle = \frac{1}{\sqrt{4\pi}} \frac{\langle 0|a_0|0\rangle}{\langle 0|f_0|0\rangle}. \]  

(1.16)

Substituting (1.15) into the constraint equation and sandwiching the constraint equation between Fock states, one get a recursion relation for \(\{f_n\}\):

\[ 0 = gf_n + f_n^3 + (4N - 1)f_n + (n + 1)f_{n+1} + nf_{n-1} \]  

(1.17)

where \(n \leq N\), and one define \(f_{N+1}\) to be unknown. Thus, \(\{f_1, f_2, \ldots, f_{N+1}\}\) is uniquely determined by a given choice of \(g\) and \(f_0\). In particular, if \(f_0 = 0\) all the \(f_k\)'s are zero independent of \(g\). This is the unbroken phase.

Consider the asymptotic behavior for large \(n\). If \(f_n \gg 1\) in this limit, then the \(f_n^3\) term will dominate and

\[ f_{n+1} \sim \frac{f_n^3}{n}, \]  

(1.18)

thus,

\[ \lim_{n \rightarrow \infty} f_n \sim (-1)^n \exp(3^n \text{constant}) . \]  

(1.19)

One must reject this rapidly growing solution. Hence, one only seek solutions where \(f_n\) is small for large \(n\). For large \(n\), the terms linear in \(n\) dominate and Eq. (1.17) becomes

\[ f_{n+1} + 4f_n + f_{n-1} = 0 . \]  

(1.20)
There are two solutions to this equation:
\[ f_n \propto (\sqrt{3} \pm 2)^n. \]  
(1.21)

One must reject the plus solution because it grows with \( n \). This gives the condition
\[ -\frac{\sqrt{3} - 3 + g}{2\sqrt{3}} = K, \quad K = 0, 1, 2, \ldots \]  
(1.22)

Concentrating on the \( K = 0 \) case, one finds a critical coupling
\[ g_{\text{critical}} = 3 - \sqrt{3} \]  
(1.23)
or
\[ \lambda_{\text{critical}} = 4\pi \left( 3 + \sqrt{3} \right) \mu^2 \approx 60 \mu^2. \]  
(1.24)

In comparison, values of \( \lambda_{\text{critical}} \) from 22\( \mu^2 \) to 55\( \mu^2 \) have been reported for equal-time quantized calculations \([4,14,28]\). The solution to the linearized equation is an approximate solution to the full Eq. \((1.17)\) for \( f_0 \) sufficiently small. Next, one needs to determine solutions of the full nonlinear equation which converge for large \( n \).

One can study the critical curves by looking for numerical solutions to Eq. \((1.17)\). The method used here is to find values of \( f_0 \) and \( g \) such that \( f_{N+1} = 0 \). Since one seeks a solution where \( f_n \) is decreasing with \( n \), this is a good approximation. One finds that for \( g > 3 - \sqrt{3} \) the only real solution is \( f_n = 0 \) for all \( n \). For \( g \) less than \( 3 - \sqrt{3} \) there are two additional solutions. Near the critical point \( |f_0| \) is small and
\[ f_n \approx f_0 \left( 2 - \sqrt{3} \right)^n. \]  
(1.25)

The critical curves are shown in Figure 1. These solutions converge quite rapidly with \( N \). The critical curve for the broken phase is approximately parabolic in shape:
\[ g \approx 3 - \sqrt{3} - 0.9177 f_0^2. \]  
(1.26)

One can also study the eigenvalues of the Hamiltonian for the one mode case. The Hamiltonian is diagonal for this Fock space truncation and,
\[ \langle n | H | n \rangle = \frac{3}{2} n(n-1) + n g - \frac{f_n^4}{4} - \frac{2n+1}{4} f_n^2 + \frac{n+1}{4} f_{n+1}^2 + \frac{n}{4} f_{n-1}^2 - C. \]  
(1.27)

The invariant mass eigenvalues are given by
\[ P^2 |n\rangle = 2P^+ P^- |n\rangle = \frac{n\lambda \langle n | H | n \rangle}{24\pi} |n\rangle \]  
(1.28)

In Figure 2 the dashed lines show the first few eigenvalues as a function of \( g \) without the zero-mode. When one include the broken phase of the zero mode, the energy levels shift as shown by the solid curves. For \( g < g_{\text{critical}} \) the energy levels increase above the value they
FIG. 3. The lowest three energy eigenvalues for the one mode case as a function of $g$ from the numerical solution of Eq. (1.27) with $N = 10$. The dashed lines are for the unbroken phase $f_0 = 0$ and the solid lines are for the broken phase $f_0 \neq 0$. 
FIG. 4. Probability distribution of eigenvalues of $\sqrt{4\pi}\phi$ for the vacuum with $M = 1$, $N = 10$, and no zero mode. Also shown is the infinite $N$ limit from Eq. (1.31).

had without the zero mode. The higher levels change very because $f_n$ is small for large $n$. In the more general case of many modes and many particles many of the features that were seen in the one mode and one particle cases remain.

One can also investigate the shape of the critical curve near the critical coupling as a function of the cutoff $K$. In scalar field theory, $\langle 0|\phi|0 \rangle$ acts as the order parameter of the theory. Near the critical coupling, one can fit the VEV to some power of $g - g_{\text{critical}}$; this will give us the associated critical exponent $\beta$, $\langle 0|a_0|0 \rangle \propto (g_{\text{critical}} - g)^\beta$. (1.29)

They have calculated this as a function of cutoff and found a result consistent with $\beta = 1/2$, independent of cutoff $K$. The theory $(\phi^4)_{1+1}$ is in the same universality class as the Ising model in 2 dimensions and the correct critical exponent for this universality class is $\beta = 1/8$. If one were to use the mean field approximation to calculate the critical exponent, the result would be $\beta = 1/2$. This is what was obtained in this calculation. Usually, the presence of a mean field result indicates that one is not probing all length scales properly. If one had a cutoff $K$ large enough to include many length scales, then the critical exponent should approach the correct value. However, one cannot be certain that this is the correct explanation of our result since no evidence that $\beta$ decreases with increase $K$ is seen.

E. Spectrum of the Field Operator

How does the zero mode affect the field itself? Since $\phi$ is a Hermitian operator it is an observable of the system and one can measure $\phi$ for a given state $|\alpha \rangle$. $\tilde{\phi}_i$ and $|\chi_i \rangle$ are the eigenvalue and eigenvector respectively of $\sqrt{4\pi}\phi$:

$\sqrt{4\pi}\phi |\chi_i \rangle = \tilde{\phi}_i |\chi_i \rangle$, \hspace{1cm} $\langle \chi_i | \chi_j \rangle = \delta_{i,j}$ . (1.30)

The expectation value of $\sqrt{4\pi}\phi$ in the state $|\alpha \rangle$ is $\sum_i \tilde{\phi}_i |\langle \chi_i | \alpha \rangle|^2$. In the limit of large $N$, the
probability distribution becomes continuous. If one ignores the zero mode, the probability of obtaining \( \tilde{\phi} \) as the result of a measurement of \( \sqrt{4\pi\phi} \) for the vacuum state is

\[
P(\tilde{\phi}) = \frac{1}{\sqrt{2\pi\tau}} \exp \left( -\frac{\tilde{\phi}^2}{2\tau} \right) d\tilde{\phi}
\]

(1.31)

where \( \tau = \sum_{k=1}^{M} 1/k \). The probability distribution comes from the ground state wave function of the Harmonic oscillator where one identifies \( \phi \) with the position operator. This is just the Gaussian fluctuation of a free field. When \( N \) is finite, the distribution becomes discrete as shown in Figure 4. In general, there are \( N + 1 \) eigenvalues such that \( \langle \chi_i | 0 \rangle \neq 0 \), independent of \( M \). Thus if one wants to examine the spectrum of the field operator for the vacuum state, it is better to choose Fock space truncations where \( N \) is large. With this in mind, one examines the \( N = 50 \) and \( M = 1 \) case as a function of \( g \) in Figure 5. Note that near the critical point, Figure 5a, the distribution is approximately equal to the free field case shown in Figure 4. As one moves away from the critical point, Figures 5b–d, the distribution becomes increasingly narrow with a peak located at the VEV of what would be the minimum of the symmetric double well potential in the equal-time paradigm. In addition, there is a small peak corresponding to minus the VEV. In the language of the equal-time paradigm, there is tunneling between the two minima of the potential. The spectrum of \( \phi \) has been examined for other values of \( M \) and \( N \); the results are consistent with the example discussed here.

F. Physical Picture and Classification of Zero Modes in Gauge Theories

When considering a gauge theory, there is a “zero mode” problem associated with the choice of gauge in the compactified case. This subtlety, however, is not particular to the light cone; indeed, its occurrence is quite familiar in equal-time quantization on a torus [34,39,29]. In the present context, the difficulty is that the zero mode in \( A^+ \) is in fact gauge-invariant, so that the light-cone gauge \( A^+ = 0 \) cannot be reached. Thus one has a pair of interconnected problems: first, a practical choice of gauge; and second, the presence of constrained zero modes of the gauge field. In ref. [27], the generalize gauge fixing in a discrete formalism is described by Kalloniatis and Robertson.

One defines, for a periodic quantity \( f \), its longitudinal zero mode

\[
\langle f \rangle_o \equiv \frac{1}{2L} \int_{-L}^{L} dx^- f(x^-, x_\perp)
\]

(1.32)

and the corresponding normal mode part

\[
\langle f \rangle_n \equiv f - \langle f \rangle_o.
\]

(1.33)

The “global zero mode”—the mode independent of all the spatial coordinates is denoted by \( \langle f \rangle \):

\[
\langle f \rangle \equiv \frac{1}{\Omega} \int_{-L}^{L} dx^- \int_{-L_\perp}^{L_\perp} d^2 x_\perp f(x^-, x_\perp).
\]

(1.34)
FIG. 5. Probability distribution of eigenvalues of $\sqrt{4\pi\phi}$ for the vacuum with couplings (a) $g = 1$, (b) $g = 0$, (c) $g = -1$, and (d) $g = -2$. $M = 1$, $N = 50$, and the positive VEV solution to the constraint equation is used.
Finally, the quantity which will be of most interest to us is the “proper zero mode,” defined by

\[ f_0 \equiv \langle f \rangle_0 - \langle f \rangle. \tag{1.35} \]

By integrating over the appropriate direction(s) of space, one can project the equations of motion onto the various sectors. The global zero mode sector requires some special treatment, and will not be discussed here. Consider the proper zero mode sector of the equations of motion

\[ - \partial^2_\perp A^+_0 = g J^+_0 \tag{1.36} \]

\[ -2(\partial_\perp)^2 A^+_0 - \partial^2_\perp A^-_0 - 2\partial_i \partial_\perp A^+_0 = g J^-_0 \tag{1.37} \]

\[ - \partial^2_\perp A^+_0 + \partial_i \partial_\perp A^+_0 + \partial_i \partial_j A^+_j = g J^+_i. \tag{1.38} \]

One first observe that Eq. (1.36), the projection of Gauss’ law, is a constraint which determines the proper zero mode of \( A^+ \) in terms of the current \( J^+ \):

\[ A^+_0 = -g \frac{1}{\partial^2_\perp} J^+_0. \tag{1.39} \]

The equations (1.37) and (1.38) then determine the zero modes \( A^-_0 \) and \( A^+_i \). Eq. (1.39) is clearly incompatible with the strict light-cone gauge \( A^+ = 0 \), which is most natural in light-cone analyses of gauge theories. Here one encounter a common problem in treating axial gauges on compact spaces. It is not possible to bring an arbitrary gauge field configuration to one satisfying \( A^+ = 0 \) via a gauge transformation, and the light-cone gauge is incompatible with the chosen boundary conditions. The closest one can come is to set the normal mode part of \( A^+ \) to zero, which is equivalent to

\[ \partial_\perp A^+ = 0. \tag{1.40} \]

This condition does not, however, completely fix the gauge—one is free to make arbitrary \( x^- \)-independent gauge transformations without undoing Eq. (1.40). One may therefore impose further conditions on \( A^+ \) in the zero mode sector of the theory.

Acting on Eq. (1.38) with \( \partial_\perp \). The transverse field \( A^+_i \) then drops out and one obtain an expression for the time derivative of \( A^+_0 \):

\[ \partial_\perp A^+_0 = g \frac{1}{\partial^2_\perp} \partial_\perp J^+_0. \tag{1.41} \]

Inserting this back into Eq. (1.38) one then find, after some rearrangement,

\[ - \partial^2_\perp (\delta^j_j - \partial_\perp \partial_j) A^+_0 = g \left( \delta^j_j - \partial_\perp \partial_j \right) J^+_0. \tag{1.42} \]
Now the operator \((\delta^i_j - \partial_i \partial_j / \partial^2_\perp)\) is nothing more than the projector of the two-dimensional transverse part of the vector fields \(A_0^i\) and \(J_0^i\). No trace remains of the longitudinal projection of the field \((\partial_i \partial_j / \partial^2_\perp)A_0^j\) in Eq. (1.42). This reflects precisely the residual gauge freedom with respect to \(x^-\)-independent transformations. To determine the longitudinal part, an additional condition is required.

The general solution to Eq. (1.42) is

\[
A_0^i = -g \frac{1}{\partial^2_\perp} J_0^i + \partial_i \varphi(x^+, x_\perp),
\]

where \(\varphi\) must be independent of \(x^-\) but is otherwise arbitrary. Imposing a condition on, say, \(\partial_i A_0^i\) will uniquely determine \(\varphi\). In ref. [26], for example, the condition \(\partial_i A_0^i = 0\) was proposed as being particularly natural. This choice, taken with the other gauge conditions has been called the “compactification gauge.” In this case

\[
\varphi = g \frac{1}{\partial^2_\perp^2} \partial_i J_0^i.
\]

Of course, other choices are also possible. For example, one can generalize Eq. (1.44) to

\[
\varphi = \alpha g \frac{1}{\partial^2_\perp^2} \partial_i J_0^i,
\]

with \(\alpha\) a real parameter. Then the “generalized compactification gauge.” condition corresponding to this solution is

\[
\partial_i A_0^i = -g(1 - \alpha) \frac{1}{\partial^2_\perp} \partial_i J_0^i.
\]

An arbitrary gauge field configuration \(B^\mu\) can be brought to one satisfying Eq. (1.46) via the gauge function

\[
\Lambda(x_\perp) = -\frac{1}{\partial^2_\perp^2} \left[ g(1 - \alpha) \frac{1}{\partial^2_\perp} \partial_i J_0^i + \partial_i B_0^i \right].
\]

This is somewhat unusual in that \(\Lambda(x_\perp)\) involves the sources as well as the initial field configuration, but this is perfectly acceptable. More generally, \(\varphi\) can be any (dimensionless) function of gauge invariants constructed from the fields in the theory, including the currents \(J^\pm\). For our purposes Eq. (1.46) suffices.

One now has relations defining the proper zero modes of \(A^i\),

\[
A_0^i = -g \frac{1}{\partial^2_\perp} \left( \delta^i_j - \alpha \frac{\partial \partial_j}{\partial^2_\perp} \right) J_0^j,
\]

as well as \(A_0^+\) Eq. (1.39). All that remains is to use the final constraint Eq. (1.37) to determine \(A_0^-\). Using eqs. (1.41) and (1.46), one finds that Eq. (1.37) can be written as

\[
\partial^2_\perp A_0^- = -g J_0^- - 2\alpha g \frac{1}{\partial^2_\perp} \partial_+ \partial_i J_0^i.
\]
After using the equations of motion to express $\partial_+ J_0^i$ in terms of the dynamical fields at $x^+ = 0$, this may be straightforwardly solved for $A_0^-$ by inverting the $\partial_+^2$. In what follows, however, one has no need of $A_0^-$. It does not enter the Hamiltonian, for example; as usual, it plays the role of a multiplier to Gauss’ law Eq. (1.38), which one is able to implement as an operator identity.

The extension of the present work to the case of QCD is complicated by the fact that the constraint relations for the gluonic zero modes are nonlinear, as in the $\phi^4$ theory. A perturbative solution of the constraints is of course still possible, but in this case, since the effective coupling at the relevant (hadronic) scale is large, it is clearly desirable to go beyond perturbation theory. In addition, because of the central role played by gauge fixing in the present work, one may expect complications due to the Gribov ambiguity [15], which prevents the selection of unique representatives on gauge orbits in nonperturbative treatments of Yang-Mills theory. Preliminary step in this direction on the pure glue theory in 2+1 dimensions is found in ref. [26]. There one finds that some of the nonperturbative techniques used recently in 1+1 dimensions [5,42] can be applied.

G. Dynamical Zero Modes

Our concern in this section is with those zero modes that are true dynamical independent fields. They can arise due to the boundary conditions in gauge theory preventing one from fully implement the traditional light-cone gauge $A^+ = 0$. The development of the understanding of this problem in DLCQ can be traced in Refs. [36,18,25,26]. It has its analogue in instant form approaches to gauge theory [34,21].

Consider the zero mode subsector of the pure glue theory in (1+1) dimension, namely where only zero mode external sources excite only zero mode gluons. This is not an approximation but rather a consistent solution, a sub-regime within the complete theory. A similar framing of the problem lies behind the work of Lüscher [32] and van Baal [17] using the instant form Hamiltonian approach to pure glue gauge theory in 3+1 dimensions. The beauty of this reduction in the 1+1 dimensional theory is two-fold. First, it yields a theory which is exactly soluble. This is useful given the dearth of soluble models in field theory. Secondly, the zero mode theory represents a paring down to the point where the front and instant forms are manifestly identical, which is nice to know indeed.

Consider an SU(2) non-Abelian gauge theory in 1+1 dimensions with classical sources coupled to the gluons. The Lagrangian density is

$$\mathcal{L} = \frac{1}{2} \text{Tr} (F_{\mu\nu} F^{\mu\nu}) + 2 \text{Tr} (J_\mu A^\mu)$$

(1.50)

where $F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu - g[A_\mu, A_\nu]$. With a finite interval in $x^-$ from $-L$ to $L$, one imposes periodic boundary conditions on all gauge potentials $A_\mu$.

One cannot eliminate the zero mode of the gauge potential. The reason is evident: it is invariant under periodic gauge transformations. But of course one can always perform a rotation in color space. In line with other authors [3,14,13], one chooses this so that $A_3^+$ is the only non-zero element, since in our representation only $\sigma^3$ is diagonal. In addition, one can impose the subsidiary gauge condition $A_3^- = 0$ The reason is that there still remains
freedom to perform gauge transformations that depend only on light-cone time $x^+$ and the color matrix $\sigma^3$.

The above procedure would appear to have enabled complete fixing of the gauge. This is still not so. Gauge transformations

$$V = \exp\{ix^-\left(\frac{n\pi}{2L}\right)\sigma^3\}$$

(1.51)

generate shifts, according to Eq.(1.47), in the zero mode component

$$A_3^+ \to A_3^+ + \frac{n\pi}{gL}.$$  

(1.52)

All of these possibilities, labeled by the integer $n$, of course still satisfy $\partial_- A^+ = 0$, but as one sees $n = 0$ should not really be included. One can verify that the transformations $V$ also preserve the subsidiary condition. One notes that the transformation is $x^-$-dependent and $Z_2$ periodic. It is thus a simple example of a Gribov copy in 1+1 dimensions. Following the conventional procedure one demands

$$A_3^+ \neq \frac{n\pi}{gL}, \quad n = \pm 1, \pm 2, \ldots.$$ 

(1.53)

This eliminates singular points at the Gribov ‘horizons’ which in turn correspond to a vanishing Faddeev-Popov determinant.

For convenience we henceforth use the notation

$$A_3^+ = v, \quad x^+ = t, \quad w^2 = \frac{\not{J^+} \cdot \not{J^+}}{g^2} \quad \text{and} \quad \not{J^+} = \frac{B}{2}.$$ 

(1.54)

The only conjugate momentum is

$$p \equiv \Pi^+ = \partial^- A_3^+ = \partial^- v.$$ 

(1.55)

The Hamiltonian density $T^{+-} = \partial^- A_3^+ \Pi^+ - \mathcal{L}$ leads to the Hamiltonian

$$H = \frac{1}{2}[p^2 + \frac{w^2}{v^2} + Bv](2L).$$ 

(1.56)

Quantization is achieved by imposing a commutation relation at equal light-cone time on the dynamical degree of freedom. Introducing the variable $q = 2Lv$, the appropriate commutation relation is $[q(x^+), p(x^+)] = i$. The field theoretic problem reduces to quantum mechanics of a single particle as in Manton’s treatment of the Schwinger model in Ref. [34]. One thus has to solve the Schrödinger equation

$$\frac{1}{2}\left(-\frac{d^2}{dq^2} + \frac{(2Lw)^2}{q^2} + \frac{Bq}{2L}\right)\psi = \mathcal{E}\psi,$$ 

(1.57)

with the eigenvalue $\mathcal{E} = E/(2L)$ actually being an energy density.
All eigenstates $\psi$ have the quantum numbers of the naive vacuum adopted in standard front form field theory: all of them are eigenstates of the light-cone momentum operator $P^+$ with zero eigenvalue. The true vacuum is now that state with lowest $P^-$ eigenvalue. In order to get an exactly soluble system one eliminates the source $2B = J_3^o$.

The boundary condition that is to be imposed comes from the treatment of the Gribov problem. Since the wave function vanishes at $q = 0$ one must demand that the wavefunctions vanish at the first Gribov horizon $q = \pm 2\pi/g$. The overall constant $R$ is then fixed by normalization. This leads to the energy density only assuming the discrete values

$$\mathcal{E}_m^{(\nu)} = \frac{g^2}{8\pi^2}(X_m^{(\nu)})^2, \quad m = 1, 2, \ldots,$$

(1.58)

where $X_m^{(\nu)}$ denotes the m-th zero of the $\nu$-th Bessel function $J_\nu$. In general, these zeroes can only be obtained numerically. Thus

$$\psi_m(q) = R \sqrt{q} J_\nu(\sqrt{2\mathcal{E}_m^{(\nu)}} q)$$

(1.59)

is the complete solution. The true vacuum is the state of lowest energy namely with $m = 1$.

The exact solution is genuinely non-perturbative in character. It describes vacuum-like states since for all of these states $P^+ = 0$. Consequently, they all have zero invariant mass $M^2 = P^+ P^-$. The states are labeled by the eigenvalues of the operator $P^-$. The linear dependence on $L$ in the result for the discrete energy levels is also consistent with what one would expect from a loop of color flux running around the cylinder.

In the source-free equal time case Hetrick [21] uses a wave function that is symmetric about $q = 0$. For our problem this corresponds to

$$\psi_m(q) = N \cos(\sqrt{2\epsilon_m} q)$$

(1.60)

where $N$ is fixed by normalization. At the boundary of the fundamental modular region $q = 2\pi/g$ and $\psi_m = (-1)^m N$, thus $\sqrt{2\epsilon_m} 2\pi / g = m\pi$ and

$$\epsilon = \frac{g^2 (m^2 - 1)}{8}.$$

(1.61)

Note that $m = 1$ is the lowest energy state and has as expected one node in the allowed region $0 \leq g \leq 2\pi/g$. Hetrick [21] discusses the connection to the results of Rajeev [45] but it amounts to a shift in $\epsilon$ and a redefining of $m \to m/2$. It has been argued by van Baal that the correct boundary condition at $q = 0$ is $\psi(0) = 0$. This would give a sine which matches smoothly with the Bessel function solution. This calculation offers the lesson that even in a front form approach, the vacuum might not be just the simple Fock vacuum. Dynamical zero modes do imbue the vacuum with a rich structure.

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