A Tail Estimate with Exponential Decay for the Randomized Incremental Construction of Search Structures

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Abstract

The Randomized Incremental Construction (RIC) of search DAGs for point location in planar subdivisions, nearest-neighbor search in 2D points, and extreme point search in 3D convex hulls, are well known to take $O(n \log n)$ expected time for structures of $O(n)$ expected size. Moreover, searching takes w.h.p. $O(\log n)$ comparisons in the first and w.h.p. $O(\log^2 n)$ comparisons in the latter two DAGs. However, the expected depth of the DAGs and high probability bounds for their size are unknown.

Using a novel analysis technique, we show that the three DAGs have w.h.p. i) a size of $O(n)$, ii) a depth of $O(\log n)$, and iii) a construction time of $O(n \log n)$. One application of these new and improved results are remarkably simple Las Vegas verifiers to obtain search DAGs with optimal worst-case bounds. This positively answers the conjectured logarithmic search cost in the DAG of Delaunay triangulations [Guibas et al.; ICALP 1990] and a conjecture on the depth of the DAG of Trapezoidal subdivisions [Hemmer et al.; ESA 2012]. It also shows that history-based RIC circumvents a lower bound on runtime tail estimates of conflict-graph RICs [Sen; STACS 2019].

Keywords
Randomized Incremental Construction, Data Structures, Tail Bound, Las Vegas Algorithm

1 Introduction

The Randomized Incremental Construction (RIC) is one of the most successful and influential paradigms in the design of algorithms and data structures. Its simplicity makes the method particularly useful for many, seemingly different problems that ask to compute a defined structure for a given set of objects. The idea is to first permute all $n$ objects, uniformly at random, before inserting them, one at a time, in an initially empty structure under this order. Treaps [SA96, Vui80] are a 1D example of history-based RIC that also demonstrates the algorithmic use of high probability bounds and rebuilding to maintain worst-case guarantees. (See Appendix A for background on RIC and Backward Analysis.)

A landmark problem for RIC is computing a planar subdivision, called trapezoidation, that is induced by a set of $n$ line segments [dBCvKO08, Mul94]. Every trapezoidation contains $O(n + k)$ faces, where $k$ is the number of intersections. Clarkson and Shor [CS89] gave the conflict-graph RIC, Mulumley [Mul90] gave the conflict-list RIC, and Seidel [Sci91] gave the history-based RIC that builds the Trapezoidal Search DAG (TSD) online, each taking $O(n \log n + k)$ expected time.

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The TSD is the history of trapezoidations that are created during the RIC and allows to find the trapezoid, of the current subdivision, that contains a query point. TSDs have a worst-case size of $\Omega(n^2)$, but their expected size is $O(n + k)$. Searching takes w.h.p. $O(\log n)$ comparisons and the longest search path (search depth) is also w.h.p. $O(\log n)$, since there are only $O(n^2)$ different search paths (e.g., [dBCvKO08, Chapter 6.4]). Similar to Treaps, TSDs allow fully-dynamic updates such that, after each update, the underlying random permutation is uniformly from those over the current set of segments. Early algorithms generalize search tree rotations to abstract, complex structures in order to reuse the point location search and leaf level insertion algorithms [Mul91]. Simpler search and recursive top-down algorithms were described recently [BGvRS20]. The bounds on expected insertion and deletion time of both methods however require that the update entails a non-adversarial, random object.

In contrast to Treaps, fundamental questions about the reliability of RIC runtime, maintenance of small DAGs, certifying logarithmic search costs, and avenues to de-randomization, are still not completely understood. High probability bounds for the TSD construction time are only known under additional assumptions (cf. Section 1.1) and high probability bounds for space and logarithmic search in the DAG of 2D Delaunay triangulations and of 3D convex hulls [GKS92] are unknown.

### 1.1 Related Work

Guibas et al. [GKS92] showed that history-based RIC for 2D Delaunay triangulations and 3D convex hulls takes $O(n \log n)$ expected time. Their analysis however reveals nothing about the search comparisons in the query phase (see also [dBCvKO08, Section 9.5]) and the query bound is w.h.p. $O(\log^2 n)$ [GKS92, Theorem 5.4]. The authors state “We believe that this query time is actually $O(\log n)$” in several remarks throughout the paper (e.g. p384, p401, p411). Works on (dynamization of) the two problems [Mul91, Cha10] also have the same $O(\log^2 n)$ query bound.

The work of Hemmer et al. [HKH16] shows how to turn the TSDs expected query time into a worst-case bound. They give two, Las Vegas verifier, algorithms to estimate the search depth. Their exact algorithm runs in $O(n \log n)$ expected time and their $O(1)$-approximation runs in $O(n \log n)$ time. Their CGAL implementation [WBF+20] however refrains from these verifiers and simply uses the TSD depth to trigger rebuilds, which is a readily available in RIC. Clearly the TSD depth is an upper bound, since the (combinatorial) paths are a super set of the search paths. However, the ratio between depth and search depth is $\Omega(n/\log n)$. The authors conjecture that TSD depth is $O(\log n)$ with at least constant probability (see Conjecture 1 in [HKH12a]). To the best of our knowledge, the expected value of this quantity is still unknown.

The theory developed for RICs lead to a tail bound technique [MSW92, CMS93] that holds as soon as the actual geometric problem under consideration provides a certain boundedness property. The strongest known tail bound is from Clarkson et al. [CMS93, Corollary 26], which states the following. Given a function $M$ such that $M(j)$ upper bounds the size of the structure on $j$ objects. If $M(j)/j$ is non-decreasing, then, for all $\beta > 1$, the probability that the history size exceeds $\beta M(n)$ is at most $(e/\beta)^{\beta/e}$. This includes the TSD size for non-crossing segments ($k = 0$), but also the DAGs for 3D convex hulls and 2D Delaunay triangulations. Assuming intersecting segments, Matoušek and Seidel [MS92] show how to use an isoperimetric inequality for permutations to derive a tail bound of $O(n^{-c})$, given there are at least $k \geq C n \log^{15} n$ many intersections in the input (both constants $c$ and $C$ depend on the deviation threshold $\beta$). Mehlhorn et al. [MSW92] show that the general approach can yield a tail bound of at most $1/e^{\Omega(k/n \log n)}$, given there are at least $k \geq n \log n \log \log \log n$ intersections in the input segments.

Recently, Sen [Sen19] gave tail estimates for conflict-graph RICs (cf. Chapter 3.4 in [Mul94])
| Technique          | Bound                  | With Prob. ≥                     | Condition                                      |
|--------------------|------------------------|----------------------------------|------------------------------------------------|
| Isoperimetric      | \(O(n + k)\)           | \(1 - O(1/n^c)\)                | \(k \geq Cn \log^{10} n\)                    |
| Hoeffding          | \(O(n + k)\)           | \(1 - 1/e^{C(n \log n)}\)       | \(k \geq n \log n \log \log n\)             |
| Freedman Events    | \(O(n + k)\)           | \(1 - 1/e^{n+k}\)               | \(k \geq n \log n\)                         |
| Hoeffding          | \(O(\beta n)\)         | \(1 - (e/\beta)^3/e\)           |                                                 |
| Pairwise Events    | \(O(n)\)               | \(1 - 1/e^n\)                   |                                                 |

Table 1: Tail bounds for the history size of TSDs on \(n\) segments. \(k\) denotes the number of intersections and \(\alpha(n)\) the inverse of Ackermann’s function.

using Freedman’s inequality for Martingales. The work also shows a lower bound on tail estimates for the runtime, i.e. the total number of conflict-graph modifications, for computing the trapezoidation of non-crossing segments that rules out high probability tail bounds [Sen19, Section 6]. In conflict-graph RIC, not only one endpoint per segment is maintained in conflict lists, but edges in a bipartite graph, over existing trapezoids and uninserted segments, that contain an edge if and only if the geometric objects intersect (see Appendix and Figure 4 in [Sen19]). Hence the lower bound construction only applies to conflict-graph RIC and does not translate to the history-based RIC.

1.2 Contribution

We introduce a new and direct technique to analyze the history size that fully abstracts from the geometric problem to Pairwise Events of object adjacency. Using a matrix property of the events enables an inductive Chernoff argument, despite the lack of full independence. The main result in Section 3 is a much sharper tail estimate for the TSD size (see Table 1). This complements the known high probability bound for the point location cost and shows that TSD construction takes w.h.p. \(O(n \log n)\) time. Moreover, maintaining a TSD size of \(O(n)\), in the static and dynamic setting, merely adds an expected rebuild cost of \(O(1)\).

Unlike geometric Backward Analysis of the query cost, we deal with union bounds over exponential domains, inherent to our combinatorial approach, in Section 4. We derive (inverse) polynomial probability bounds and show that the exponential union bound adds up to a high probability bound. The main result in this section is that the TSD has w.h.p. \(O(\log n)\) depth, and thus confirms the conjecture of Hemmer et al. [IKH12a, IKH16] with a substantially stronger bound.

In Section 5.2, we show that our technique allows to obtain identical bounds for size, depth, and runtime for the search DAGs of 2D Delaunay triangulations and 3D convex hulls. This improvement answers the conjectured query time bound of Guibas et al. [GKS92] affirmatively. Additional bookkeeping during RIC allows us to track our (high probability) bound on the maximum cost of edge weighted root-to-leaf paths in the DAGs. Hence space can be made \(O(n)\) and query time can be made \(O(\log n)\) worst-case bounds with rebuilding.

In Section 5.1, we extend the technique to yield improved bounds for a history-based RIC with \(\omega(n^2)\) worst-case size. We show that the TSD size is \(O(n + k)\) with probability at least \(1 - 1/e^{n+k}\), where \(k \geq 0\) is the number of intersections of the segments (see Table 1).

2 Recap: Trapezoidal Search DAGs

For a set \(S\) of \(n\) segments in the plane, we denote by \(\mathcal{K}(S) \subseteq \binom{S}{2}\) the set of crossings and \(k = |\mathcal{K}(S)|\). We identify the permutations over \(S\) with the set of bijective mappings to \(\{1, \ldots, n\}\), i.e. \(\mathbf{P}(S) = \)
\[ \pi : S \to \{1, \ldots, n\} \mid \pi \text{ bijective}. \]

An implicit, infinitesimal shear transformation allows to assume, without loss of generality, that all distinct end and intersection points have different \( x \)-coordinates (e.g. Chapter 6.3 in [dBCvKO08]). Trapezoidation \( T(S) \) is defined by emitting two vertical rays (in negative and positive \( y \)-direction) from each end and intersection point until the ray meets the first segment or the bounding rectangle (see Figure 1). To simplify presentation, we also implicitly move common end and intersection points infinitesimally along their segment, towards their interior. This gives that segments have no points in common, though there may exist additional spatially empty trapezoids in \( T(S) \). We identify \( T(S) \) with the set of faces in this decomposition of the plane. Elements in \( T(S) \) are trapezoids with four boundaries that are defined by at least one and at most four segments of \( S \) (see Figure 1). Note that boundaries of the trapezoids in \( T(S) \) are solely determined by the set of segments \( S \), irrespective of the permutation. We will need the following notations. Let \( \gamma > 1 \) be the smallest constant\(^1\) such that \( |T(S)| \leq (n+k)\gamma \) holds for any \( S \). For a segment \( s \in S \), let \( f(s, S) = \{ \Delta \in T(S) : \Delta \text{ is bounded by } s \} \) denote the set of faces that are bounded by \( s \) (i.e. top, bot, left, or right). Let \( s_i = \pi^{-1}(i) \) be the priority \( i \) segment and let \( S_{\leq k} = \{s_1, \ldots, s_k\} \).

The expected size of the TSD is typically analyzed by considering \( \sum_{j=1}^n D_j \) where the random variable \( D_j := |f(s_j, S_{\leq j})| \) denotes the number of faces that are created by inserting \( s_j \) into trapezoidation \( T(S_{\leq j-1}) \), equivalently that are removed by deleting \( s_j \) from \( T(S_{\leq j}) \) (see Figure 2).

Classic Backward Analysis [dBCvKO08, p. 136] in this context is the following argument. Let \( S' \subseteq S \) be a fixed subset of \( j \) segments, then

\[
\mathbb{E}_{\mathbf{P}(S)} \left[ D_j \mid S_{\leq j} = S' \right] = \frac{1}{j} \sum_{s \in S'} \sum_{\Delta \in T(S')} \chi(\Delta \in f(s, S')) \leq 4 \frac{|T(S')|}{j}
\]

where the binary indicator variable \( \chi(\Delta \in f(s, S')) = 1 \) if and only if the trapezoid \( \Delta \) is bounded.

\(^1\)Counting trapezoids in a \( x \)-sweep shows \( |T(S)| = 1 + 3(n+k) \) (see [dBCvKO08, p.127] and [Sei93, Section 3]).
by segment $s$. The equality is due to that every segment in $S'$ is equally likely to be picked for $s_j$. For $k = 0$, we have that $|T(S')| \leq \gamma j$, regardless of the actual set $S'$, and $\mathbb{E}[D_j] \leq 4\gamma$ holds unconditionally for each step $j$. For $k > 0$ one observes that any given crossing in $K(S)$ is present in $S_{\leq j}$ with probability $\frac{j(j-1)}{n(n-1)}$, hence summing over the $k$ crossings gives that $\mathbb{E}[T(S_{\leq j})] \leq \gamma(j + kj^2/n^2)$ and $\mathbb{E}[D_j] \leq 4\gamma(1 + kj/n^2)$ and thus $\sum_j \mathbb{E}[D_j] \leq 4\gamma(n + k)$. Replacing $k$ in this bound with the number of intersection points incurs a more technical argument (see [Sei93, p. 46]).

Since the destruction of a face (of a leaf node) creates at most three DAG nodes, the expected number of TSD nodes is at most $12\gamma(n + k)$.

### 3 Stronger tail bounds using Pairwise Events

Let $S$ be a set of $n$ non-crossing segments throughout this section. Segments $s$ and $s'$ are called adjacent in $S$ if there is a face $\Delta \in T(S)$ that is incident to both, i.e. both define some part of the boundary of $\Delta$. We define for each $1 \leq i < j \leq n$ an event, i.e. a binary random variable, $X_{i,j} : \mathcal{P}(S) \to \{0,1\}$ that occurs if and only if $s_i$ and $s_j$ are adjacent in $T(S_{\leq j})$. That is

$$X_{i,j} = \begin{cases} 1 & \text{if } f(s_j, S_{\leq j}) \text{ contains a trapezoid bounded by } s_i \\ 0 & \text{otherwise} \end{cases}.$$ 

To simplify presentation, we place the events in a lower triangle matrix and call the set $r(j) := \{X_{i,j} : 1 \leq i < j\}$ the events of row $j$ and the set $c(i) := \{X_{i,j} : i < j \leq n\}$ the events of column $i$.

Imagine that the random permutation is built backwards, i.e. by successively choosing one of the remaining elements uniformly at random to assign the largest available priority value. For every step $j$ at least one of the row events occurs, i.e. $0 < \sum_{i<j} X_{i,j} < j$, since at least one trapezoid is destroyed in step $j$ and the exact probability of the events $r(j)$ depends on the geometry of the segments $S_{\leq j}$.

Consider the events in row $j$. Conditioned on the random permutation starting with set $S' = S \setminus \{s_{j+1}, \ldots, s_n\}$, the experiment chooses $s \in S'$ uniformly at random and assigns the priority value $j$ to it. Clearly the number of occurring (row) events depends on which segment of $S'$ is picked as $s_j$, as this determines the value $f(s_j, S')$. Note that the choice of $s_j$ also fixes a partition of $S' = \{s_j\} \cup A \cup N$ into those segments that are and aren’t adjacent to $s_j$, the sets $A$ and $N$ respectively. This defines a partition $S_{\leq j} = \{s_j\} \cup A_j \cup N_j$ in every backward step $j$. Eventually $s_i$ is picked from $S_{\leq i}$, which determines the outcomes of all events in $c(i)$. i.e. when $s_i$ is picked from $S_{\leq i}$, the objects in the set are multicolored ($A_j$ or $N_j$ for each $j > i$) and $X_{i,j}$ occurs if and only if the pick has the respective color $A_j$.

This shows for the event probability that

$$\mathbb{E}[X_{i,j}|s_{i+1}, \ldots, s_n] = \mathbb{E}[X_{i,j}|Y, s_{i+1}, \ldots, s_n]$$

for every $Y \in c(i')$ with $i' > i$. See Table 2 for an example.

| $X(\pi)$ | $A_2 = \{a\}$ | $A_3 = \{a, b\}$ | $A_4 = \{c\}$ | $D_1 = 4$ |
|-----------|---------------|----------------|---------------|--------|
| 1         | 1             | 1             | 0             | $D_2 = 5$ |
| 1         | 1             | 0             | 1             | $D_3 = 6$ |
| 0         | 0             | 1             | 0             | $D_4 = 4$ |

Table 2: Outcome of the pairwise events and the partitions for segments $S = \{a, b, c, d\}$ and order $\pi$ from Figures 1 and 2.
Moreover, we have, for every \( t > 0 \), the two equations
\[
\begin{align*}
\mathbb{E}\left[ \prod_{j>1} \exp(tX_{1,j}) \mid s_2, \ldots, s_n \right] &= \prod_{j>1} \mathbb{E}\left[ \exp(tX_{1,j}) \mid s_2, \ldots, s_n \right] \\
\mathbb{E}\left[ \prod_{j>i} \exp(tX_{i,j}) \mid s_i, \ldots, s_n \right] &= \prod_{j>i} \mathbb{E}\left[ \exp(tX_{i,j}) \mid s_i, \ldots, s_n \right],
\end{align*}
\]
where \((s_1, \ldots, s_n)\) denotes the condition that the random permutations have this suffix. Note that for a set of events \( \{B_i\} \) that are either certain or impossible, i.e. \( \mathbb{E}[B_i] = \Pr[B_i] \in \{0,1\} \), we have that the outcome of each event is identical to its probability and thus
\[
\mathbb{E}\left[ \prod_i \exp(tB_i) \right] = \prod_i \mathbb{E} \left[ \exp(tB_i) \right].
\]

There is a close relation between the row events \( r(j) \) and the random variable \( D_j \).

**Lemma 1.** For each \( \pi \in \mathcal{P}(S) \) and \( j \geq 2 \), we have \( D_j(\pi)/6 \leq \sum_{i<j} X_{i,j}(\pi) \leq 3D_j(\pi) \).

**Proof.** Let \( S^\prime := S_{\leq j}(\pi) \) be the segments with priority at most \( j \) in \( \pi \). Clearly every trapezoid that is incident to \( s_j \) is bounded by at most three other segments, which gives the upper bound. For the lower bound, we first count those trapezoids of \( f(s_j, S^\prime) \) that have \( s_j \) as top or bottom boundary. Let \( P \) be the set of endpoints that define the vertical boundaries of these trapezoids, excluding the endpoints of \( s_j = (q_l, q_r) \). Partition \( P \) into points \( P^+ \) above and \( P^- \) below \( s_j \), which blocks their vertical rays in trapezoidation \( T(S^\prime) \). Consider the two sets \( P^+ \) and \( P^- \) sorted by their \( x \)-coordinates.

Between the endpoints of \( s_j \), the vertical boundaries of points in \( P^+ \) can define at most \( |P^+| + 1 \) trapezoids. Hence \( f(s_j, S^\prime) \) contains at most \( |P| + 2 \) trapezoids that have \( s_j \) on their top or bottom boundary. The remaining trapezoids of \( f(s_j, S^\prime) \) are either bounded by \( q_l \) or by \( q_r \). There is at most one trapezoid in \( T(S^\prime) \) that has endpoint \( q_l \) as right vertical boundary but not \( s_j \) as bottom or top segment. The argument for \( q_r \) is symmetric.

Putting the bounds for all cases of trapezoids in \( f(s_j, S^\prime) \) together and using that \(|P| \leq 2 \sum_{i<j} X_{i,j}(\pi)\), we have
\[
D_j(\pi) \leq (2 + |P|) + 2 \leq \left( 2 + 2 \sum_{i<j} X_{i,j}(\pi) \right) + 2 \leq 6 \sum_{i<j} X_{i,j}(\pi).
\]
In the last step we used the fact that \( 1 \leq \sum_{i<j} X_{i,j}(\pi) \). \( \Box \)

Note that the respective upper and lower bounds hold for every permutation \( \pi \in \mathcal{P}(S) \). This shows that the expected number of events that occur in row \( j \) is at most \( 3 \mathbb{E}[D_j] \leq 12 \gamma \) (and at least 1). Thus \( \mathbb{E}[\sum_{i,j} X_{i,j}] \) is in the interval \([n - 1, 12 \gamma n]\).

Furthermore, consider the isolated event \( X_{i,j} \) in row \( j \). Since the element \( s_i \) is picked uniformly at random from the set \( S_{\leq j-1} \), we have that its event probability is within the range
\[
\frac{1}{j-1} \leq \mathbb{E}[X_{i,j}] \leq \frac{3 \mathbb{E}[D_j]}{j-1} \leq \frac{12 \gamma}{j-1}.
\]
Hence the events have roughly Harmonic distribution, i.e. up to bounded multiplicative distortions.

We find it noteworthy that our technique completely captures, with only one lemma, the entire nature of the geometric problem within these (constant) distortion factors of the pairwise events and thus generalizes easily to other RICs (see Section 5).

However, due to the nature of the incremental selection process, there is a dependence between the events in \( r(j) \), e.g. between \( X_{i,j} \) and \( \{X_{i+1,j}, \ldots, X_{j-1,j}\} \). We circumnavigate this obstacle using conditional expectations in the proof of our tail bound.
Theorem 1. The random variable \( X := \sum_{i,j} X_{i,j} \) has an exponential upper tail, i.e., we have \( \Pr[X \geq T] \leq \exp(\mathbb{E}[X] - T \ln 2) \) for all \( T > 0 \).

Proof. To leverage Equation (2) and (3) for our events, we regroup the summation terms by column index. Let \( t := \ln 2 \) and \( C_i := \sum_{Y \in c(i)} Y \) for each \( 1 \leq i \leq n \). Markov’s inequality gives that

\[
\Pr \left[ \sum_{i<n} C_i \geq T \right] = \Pr \left[ \exp \left( t \sum_{i<n} C_i \right) \geq e^{tT} \right] \leq \mathbb{E} \left[ \exp \left( t \sum_{i<n} C_i \right) \right] / e^{tT}. \tag{5}
\]

Defining \( Q_i := \exp(tC_1 + \ldots + tC_i) \), we will show by induction that \( \mathbb{E}[Q_{n-1}|s_n] \leq \exp(\mathbb{E}[\sum_{i<n} C_i|s_n]) \) for each \( s_n \in S \). The condition \((s_1, \ldots, s_n)\) denotes that the permutations \( \mathcal{P}(S) \) are restricted to those that have this suffix of elements.

For \( i = 1 \) and each suffix condition \((s_2, \ldots, s_n)\), we have

\[
\mathbb{E}[e^{tC_1}|s_2, \ldots, s_n] = \mathbb{E} \prod_{j=2}^{n} e^{tX_{1,j}|s_2, \ldots, s_n} \\
= \prod_{j=2}^{n} \mathbb{E}[e^{tX_{1,j}}|s_2, \ldots, s_n] \\
= \prod_{j=2}^{n} (1 - \mathbb{E}[X_{1,j}|s_2, \ldots, s_n])e^0 + \mathbb{E}[X_{1,j}|s_2, \ldots, s_n]e^t \\
= \prod_{j=2}^{n} (1 + (e^t - 1) \mathbb{E}[X_{1,j}|s_2, \ldots, s_n]) \\
\leq \exp \left( \mathbb{E} \left[ \sum_{j=2}^{n} X_{1,j}|s_2, \ldots, s_n \right] \right) = \exp \left( \mathbb{E}[C_1|s_2, \ldots, s_n] \right),
\]

where the second equality is due to Equation (2) under the given suffix condition. The third equality is due to the definition of expected values, the fourth due to the distributive rule, and the fifth equality due to our choice of \( t \). The inequality is due to the well known inequality \( 1 + x \leq e^x \).

For \( i > 1 \) and each condition \((s_{i+1}, \ldots, s_n)\), let \( S' = S \setminus \{s_{i+1}, \ldots, s_n\} \) and we have

\[
\mathbb{E} \left[ Q_i | s_{i+1}, \ldots, s_n \right] = \frac{1}{i} \sum_{s_i \in S'} \mathbb{E} \left[ Q_{i-1} \cdot e^{tC_i} | s_i, s_{i+1}, \ldots, s_n \right] \\
= \frac{1}{i} \sum_{s_i \in S'} \mathbb{E} \left[ \mathbb{E}[Q_{i-1}|c(i), s_{i+1}, \ldots, s_n] e^{tC_i} | s_i, \ldots, s_n \right] \\
\leq \frac{1}{i} \sum_{s_i \in S'} \exp(\mathbb{E}[C_1 + \ldots + C_{i-1}|s_i, \ldots, s_n]) \cdot \mathbb{E}[e^{tC_i}|s_i, \ldots, s_n] \\
\leq \exp(\mathbb{E}[C_1 + \ldots + C_i|s_{i+1}, \ldots, s_n]).
\]

The first equality is due to that every element of \( S' \) is equally likely to be picked for \( s_i \). The second equality is due to the ‘law of total expectation’. The third equality is due to a property of our events, see Equation (1). The resulting terms are bounded by the induction hypothesis and
analogously to the case \( i = 1 \), but using Equation (3) for the events \( c(i) \) instead. This concludes the induction.

Since \( \mathbb{E}[Q_{n-1}] = \frac{1}{n} \sum_{s_n \in S} \mathbb{E}[Q_{n-1}|s_n] \), we have that \( \mathbb{E}[Q_{n-1}] \leq \exp\left( \mathbb{E}[\sum_{i<n} C_i] \right) \) and the result follows from (5). \( \square \)

Using the upper bound from Lemma 1, we have \( \mathbb{E}[\sum_{i,j} X_{i,j}] \leq 12 \gamma n \). Since the lower bound of the lemma holds for every permutation, we have for all \( T > 0 \) that \( \Pr[T \leq \sum_j D_j] \leq \Pr[T \leq 6 \sum_{i,j} X_{i,j}] \). Hence, choosing \( T = \beta n \) with a sufficiently large constant \( \beta \) gives the following result.

**Corollary 1.** The TSD size is \( \mathcal{O}(n) \) with probability at least \( 1 - 1/e^n \).

This complements the known high probability bound for the point location cost\(^2\) and shows that the TSD of non-crossing segments has, with very high probability, size \( \mathcal{O}(j) \) after every insertion step \( j \). Since the RIC time for the TSD of non-crossing segments solely entails point location costs and search node creations, we have shown the following statements.

**Corollary 2.** The Randomized Incremental Construction of a TSD for \( n \) non-crossing segments takes w.h.p. \( \mathcal{O}(n \log n) \) time.

**Corollary 3.** The TSD size for non-crossing segments can be made deterministic \( \mathcal{O}(n) \) with ‘rebuild if too large’ by merely increasing the expected construction time by an additive constant.

### 4 Depth in the History DAG is w.h.p. logarithmic

For the TSD and other RICs (cf. Section 5), the ubiquitous argument shows that the search path to an arbitrary, but fixed, point has with high probability logarithmic length. Since there are only \( \mathcal{O}(n^2) \) search paths, the high probability bound is strong enough to address each of them in a union bound (see e.g. [dBCvKO08, Chapter 6]). However, a DAG on \( n \) vertices of degree at most two may well contain \( \Omega(3^{n/2}) \) different combinatorial paths (see Figure 3), hence the same argument cannot be used to obtain a high probability bound. Our technique allows to derive high probability bounds for this problem and thus gives remarkably simple Las Vegas verifiers using the length of a longest combinatorial path. We first introduce the method for the TSD of non-crossing segments and discuss modifications for the RIC of Delaunay Triangulations and Convex Hull in Section 5.

Each root-to-leaf path in the TSD visits a sequence of ‘full region’ nodes \( (u_1, \ldots, u_m) \), i.e. those nodes whose associated trapezoids \( \Delta(u_i) \) are actual faces of the trapezoidation \( \mathcal{T}(S_{\leq j}) \) for some step \( j \in \{1, \ldots, n\} \) (see Figure 2). The length of the sequence of face transitions, is within a factor of three of the path length since a face destruction inserts at most three edges in the TSD to connect a trapezoid of \( \mathcal{T}(S_{\leq i-1}) \) with one in \( \mathcal{T}(S_{\leq i}) \). We are interested in an upper bound on the number of face-transitions that lead from the trapezoidation \( \mathcal{T}(S_{\leq 1}) \) to a face of \( \mathcal{T}(S_{\leq n}) \).

Given such a trapezoid \( \Delta \), we call \( b(\Delta) := \min\{\pi(s) : s \text{ is a boundary of } \Delta\} \) the boundary priority of the trapezoid. For a sequence of trapezoids \( \Delta_1, \Delta_2, \ldots, \Delta_m \) on a root-to-leaf path, the respective sequence of boundary priority values \( b(\Delta_i) \) is monotonously increasing. Let \( 1 = b_0 < b_1 < b_2 < \ldots < b_{\ell} \) be their sequence of distinct values. The number of trapezoids on this path that have the boundary priority value \( b_i \) is at most

\[
\left| \{\Delta_\eta : b(\Delta_\eta) = b_i\} \right| \leq \sum_{b_i < j \leq b_{i+1}} X_{b_i,j},
\]

\(^2\)Cf. [dBCvKO08, Chapter 6.4] and [Mul94, Lemma 3.1.5 and Theorem 3.1.4].
Figure 3: Insertion order for non-crossing segments that results in a TSD with depth \( \Omega(n) \) and \( \Omega(3^{n/2}) \) root-to-leaf paths.

since the destruction of a trapezoid necessitates that the segment with priority \( b_i \) is adjacent to the segment \( s_j \) that causes the destruction. Moreover, event \( X_{b_i, b_{i+1}} \) needs to occur if the sequence of priority values stems from a sequence of face transitions in the TSD, i.e. only if \( X_{b_i, b_{i+1}} \) occurs we can have a face transition from one with boundary priority \( b_i \) to \( b_{i+1} \). We call a sequence of indices \((geometrically)\) feasible on the permutation \( \pi \in P(S) \) if those specified events occur (e.g. Figure 4).

In the example of Table 2, only the sequences \((1, 2)\), \((1, 2, 3)\), \((1, 2, 3, 4)\), \((1, 3)\) and \((1, 3, 4)\) are feasible on the permutation. Note that this also means that any feasible sequence \((b_0, \ldots, b_\ell)\), always has at least \( \ell \) occurring events in its truncated column sums, that is

\[
\ell \leq m \leq \sum_{i=1}^\ell \sum_{b_{i-1} < j \leq b_i} X_{b_{i-1}, j} + \sum_{b_i < j \leq n} X_{b_i, j}.
\]

To simplify notation in this section, let \( \delta \) be the smallest integer such that Equation (4) turns into \( E[X_{i,j}] \leq \frac{12\gamma}{\delta} \leq \delta/j \) for all \( j > 1 \) (i.e. \( \delta := \lceil 12\gamma/2 \rceil \)). Thus the expected value of the truncated column sum \( \mu_{x,y} := E[\sum_{x < j \leq y} X_{x,j}] \) is at most \( \delta (H_y - H_x) \), where \( H_n \) denotes the \( n \)-th Harmonic number. Summing over the expectation bounds of the sequence’s parts, we have for the number of trapezoids \( m \) along this sequence that \( m/\delta \) is at most

\[
\sum_{i=1}^\ell \left( H_{b_i} - H_{b_{i-1}} \right) + \left( H_n - H_{b_0} \right) = H_n - H_{b_0}.
\]

Note that this bound does not depend on the actual sequence of boundary priority values \( \{b_i\} \), hence the expected number of face transitions of \emph{any} sequence is \( O(\ln n) \).

4.1 How likely are long combinatorial paths?

We draw the random matrix \( C_{i,k} := \sum_{i<j \leq k} X_{i,j} \) from the probability space. From the \( n \times n \) elements of \( C \), the diagonal and upper triangle entries are all zero. Note that in \( C \), the values within a column heavily depend on those of the rows above. If we find that all matrix elements of \( C \) are within a constant factor, say \( \beta > 1 \), of their expected value, we immediately have that the bound in Equation (6) holds for any sequence of column indices, i.e. \emph{all} TSD paths regardless...
of their number. Establishing an upper bound on \( \Pr[\exists 1 \leq i < k \leq n : C_{i,k} > \beta \mu_{i,k}] \) which, if at most constant, would bound any sequence of column indices and resolve the conjecture of Hemmer et al. [HKH16]. However, since the events are binary, it is impossible to show strong concentration within a constant factor of their expected values for each entry of \( C \) (e.g. \( 1/\mathbb{E}[X_{i,i+1}] = \Omega(i) \)). Instead, our direct approach uses a special independence property for those sets of events that are given by a fixed sequence and that long index sequences are very unlikely feasible over \( \mathbb{P}(S) \).

Let \( I(n) \) be the set of all monotonous index sequences over \( \{1, \ldots, n\} \) that start with 1, i.e.

\[
I(n) := \left\{ (b_0, b_1, \ldots, b_\ell) : 1 = b_0 < b_1 < b_2 < \ldots < b_\ell \leq n \right\}, \text{ thus } |I(n)| = 2^{n-1}.
\]

For each sequence \( \sigma = (b_0, b_1, \ldots, b_\ell) \in I(n) \), let \( X(\sigma) := \{X_{b_0,2}, \ldots, X_{b_0,b_1}, X_{b_1,b_1+1}, \ldots, X_{b_\ell,n}\} \) be the \( n-1 \) events of the sequence. Each sequence \( \sigma = (b_0, b_1, \ldots, b_\ell) \) is associated to the two random variables \( \sigma, \sigma' : \mathbb{P}(S) \to \mathbb{N} \) that map permutation \( \pi \) to

\[
\sigma(\pi) := \sum_{Z \in X(\sigma)} Z(\pi) \quad \text{ and } \quad \sigma'(\pi) := \sigma(\pi) - \sum_{i=1}^\ell X_{b_i-1,b_i}(\pi).
\]

Note that \( \mathbb{E}[\sigma] \leq \delta H_n \) for any \( \sigma \in I(n) \), due to Equation (6). Let \( \beta > 1 \) be a sufficiently large constant, we define for each sequence \( \sigma \in I(n) \) three events

\[
F_\sigma(\pi) \iff \bigwedge_{(b_i,b_{i+1}) \in \sigma} X_{b_i,b_{i+1}}(\pi) \quad \text{ and } \quad B_\sigma(\pi) \iff \sigma(\pi) \geq \beta \ln n \quad \text{ and } \quad B'_\sigma(\pi) \iff \sigma'(\pi) \geq \beta \ln n.
\]

That is, \( F_\sigma \) occurs if sequence \( \sigma \) is feasible on the permutation \( \pi \). Note that \( \mathbb{E}[\sigma|F_\sigma] \geq |\sigma| \), for a sequence \( \sigma = (b_0, b_1, \ldots, b_\ell) \) of length \( |\sigma| : = \ell \). To circumnavigate this dependency, we also use the events \( B'_\sigma \) that drop the feasibility events from the summation and thus have \( \Pr[B'_\sigma] \leq \Pr[B_\sigma] \).

**Lemma 2.** For each sequence \( \sigma \in I(n) \), the events in \( X(\sigma) \) are independent, i.e. for each \( Y \subseteq X(\sigma) \) we have \( \Pr[\bigwedge_{Z \in Y} Z] = \prod_{Z \in Y} \Pr[Z] \).

**Proof.** Let \( X_{i,j} \in Y \) with \( j \) maximal and \( Y' = Y \setminus \{X_{i,j}\} \). To prove \( \Pr[Y', X_{i,j}] = \Pr[Y'] \Pr[X_{i,j}] \), we partition the permutations \( \mathbb{P}(S) \) and show that the equation holds in each class, thus globally and consequently \( \Pr[\bigwedge_{Z \in Y} Z] = \prod_{Z \in Y} \Pr[Z] \). Let \( (s_1, \ldots, s_n) \) be an arbitrary permutation suffix of \( n+1-i \) elements from \( S \). Conditioned on the permutations ending with this suffix, we have

\[
\Pr[Y', X_{i,j} | s_1, \ldots, s_n] = \Pr[Y' | X_{i,j}, s_1, \ldots, s_n] \cdot \Pr[X_{i,j} | s_1, \ldots, s_n]
\]

where the last equation is due to \( X_{i,j} \) being completely determined by \( S_{\leq j}, s_j \) and \( s_i \). \( \square \)

From this lemma, we have for each sequence \( \sigma = (b_0, \ldots, b_\ell) \in I(n) \) that

\[
\Pr[F_{(b_0, \ldots, b_\ell)}] = \prod_{i=1}^\ell \Pr[X_{b_{i-1},b_i}]
\]

\[
\Pr[B'_\sigma \land F_\sigma] = \Pr[B'_\sigma] \cdot \Pr[F_\sigma]
\]

\[
\Pr[B_\sigma] \leq n^{\delta - \beta \ln 2},
\]

where the last inequality is a standard application of Chernoff’s technique.

**Lemma 3.** The expected number of feasible sequences is at most polynomial in \( n \).
Proof. Let \( g(b_0, \ldots, b_\ell) = b_\ell \) denote the last element of a sequence. We group the summation terms into those that have and haven’t \( g(\sigma) = n \), yielding

\[
\sum_{\sigma \in I(n)} \Pr[F_\sigma] = \sum_{\sigma \in I(n) : g(\sigma) = n} \Pr[F_\sigma] + \sum_{\sigma \in I(n) : g(\sigma) < n} \Pr[F_\sigma] = \sum_{\sigma \in I(n-1)} \Pr[F_\sigma] \cdot \Pr[X_{g(\sigma),n}] + \sum_{\sigma \in I(n-1)} \Pr[F_\sigma] \leq \left( \frac{\delta}{n} + 1 \right) \sum_{\sigma \in I(n-1)} \Pr[F_\sigma] \leq \ldots \leq \text{Pr}[\sigma_1, \ldots, \sigma_\ell] = \mathcal{O}(n^\delta),
\]

where the second equality uses Eq. (7) and the inequality uses that \( \Pr[X_{i,j}] \leq \delta/j \).

**Observation 1.** For summation over the integers \( \{1, \ldots, n\} \), we have \( \sum_{i_1 < i_2 < \ldots < i_\ell} \frac{1}{i_1 i_2 \ldots i_\ell} < \mathcal{H}_n^\ell / \ell! \).

Proof. To see that

\[
\sum_{i_1 < i_2 < \ldots < i_\ell} \frac{1}{i_1 i_2 \ldots i_\ell} = \sum_{i_1} \sum_{i_2 \neq i_1} \sum_{i_3 \notin \{i_1, i_2\}} \ldots \sum_{i_\ell \notin \{i_1, i_2, \ldots, i_{\ell-1}\}} \frac{1}{i_1 i_2 \ldots i_\ell} / \ell!
\]

one considers a fixed set \( A = \{i_1, \ldots, i_\ell\} \) that appears in the expression on the left side and observes that \( A \) appears exactly \( \ell! \) times as assignment to the \( \ell \) indices on the right side. The inequality follows, since the right side is less than \( \frac{1}{\ell!} \sum_{i_1} \sum_{i_2} \ldots \sum_{i_\ell} \frac{1}{i_1 i_2 \ldots i_\ell} = \mathcal{H}_n^\ell / \ell! \).

We are now ready to show our second main result.

**Theorem 2.** There is a constant \( \beta > 1 \), such that the probability that any feasible column sequence exceeds \( \beta \ln n \) is less than \( 1/n^\Omega(\beta) \).

Proof. Using the union bound, we have \( \Pr\left[ \bigcup_{\sigma \in I(n)} B_\sigma \land F_\sigma \right] \leq \sum_{\ell=1}^{n-1} \sum_{|\sigma| = \ell} \Pr[B_\sigma \land F_\sigma] \).

For long sequences, i.e. \( \ell > \beta' H_n \), we use the following bound

\[
\sum_{\sigma : |\sigma| = \ell} \Pr[B_\sigma \land F_\sigma] \leq \sum_{\sigma : |\sigma| = \ell} \Pr[F_\sigma] = \frac{\delta}{\ell} \prod_{\sigma : |\sigma| = \ell} \Pr\left[ \frac{\delta}{b_i} \right] \leq \sum_{i_1 < i_2 < \ldots < i_\ell} \frac{\delta^\ell}{i_1 i_2 \ldots i_\ell} < \delta^\ell \mathcal{H}_n^\ell / \ell!
\]

which suffices for a union bound over the (less than \( n \)) possible values of \( \ell \).

For short sequences, i.e. \( \ell \leq \beta' H_n \), we show that a \( \beta > \beta' \) is sufficient in the definition of the events \( B_\sigma \). Let random variable \( Z := \sigma - \sigma' \leq \ell \). We have that

\[
\Pr[\beta \ln n \leq \sigma, F_\sigma] = \Pr[\beta \ln n - Z \leq \sigma', F_\sigma]
\]

where the second equality uses Eq. (7) and the inequality uses that \( \Pr[X_{i,j}] \leq \delta/j \).

Therefore, we have

\[
\Pr[\beta \ln n \leq \sigma, F_\sigma] = \Pr[\beta \ln n - Z \leq \sigma', F_\sigma] = \mathcal{O}(n^\delta).
\]
probability bound over all short sequences is at most $|n^{d/2} / n| \leq O(n^\beta)$ where the last inequality is due to Lemma 3. This is $1 - 1/e^{n+k}$, since $O(n) \neq O(1)$.

This shortcoming points out an interesting property of our Pairwise Event technique that we will strengthen in this section. The underlying problem in obtaining a stronger bound is that the worst-case size of the TSD on $n$ segments is $\Omega(\alpha(n)^2 n^2)$, which exceeds the event count $\binom{n}{2}$ by more than a constant factor. We overcome this obstacle by instead using $\binom{n}{2}$ events to analyze the total number of structural changes of the RIC on crossing segments.

Recall that $|T(S)| = 1 + 3n + 3k$ (see Section 2). We partition set of trapezoidal faces $T(S) = T_\circ(S) \cup T_\circ(S)$ into two classes, those where $leftp(\Delta)$ is a crossing and those where $leftp(\Delta)$ is a segment endpoint (or the left point of the domain boundary). This partition extends to the set of incident faces $f(s_j, S_{\leq j}) =: f_\circ(s_j, S_{\leq j}) \cup f_\circ(s_j, S_{\leq j})$ and we define the random variables $E_j, O_j, C_j : P(S) \to \mathbb{N}$ such that

$$D_j = |f_\circ(s_j, S_{\leq j})| + |f_\circ(s_j, S_{\leq j})|,$$

where $C_j$ is the number of trapezoids whose $leftp$ is due to a crossing of $s_j$ and $O_j$ the remainder of this class. Note that $3k = \sum_j C_j(\pi)$ regardless of the permutation $\pi \in P(S)$, and we have $3k + \mathbb{E}[\sum_j O_j] = O(n+k)$. We define the events

$$X_{i,j} \iff f_\circ(s_j, S_{\leq j}) \text{ contains a trapezoid bounded by } s_i, \quad 1 \leq i < j \leq n$$

$$Y_{h,i,j} \iff f_\circ(s_j, S_{\leq j}) \text{ contains a trapezoid with crossing } (s_h, s_i), \quad 1 \leq h < i < j \leq n.$$
Lemma 4. For each $\pi \in P(S)$ and $j > 1$, we have $O_j(\pi) = \sum_{Y \in r(j)} Y(\pi)$.

Proof. Let $S' = S_{\leq j}$. From the definition, $O_j = |f_\circ(s_j, S')| - C_j$ counts only those trapezoids of $T_\circ(S')$ that are incident to $s_j$ but due to a crossing of some $\{s, s'\} \subseteq S' \setminus \{s_j\}$. For “$\leq$”, observe that only one of the three trapezoids due to a crossing $\{s, s'\}$ can be incident to $s_j$ and each such trapezoid has one occurring row event. For “$\geq$”, we use that crossing line segments $\{s, s'\}$ that are adjacent to $s_j$ have at most one intersection point that is either above or below $s_j$ in $T(S')$, each of these trapezoids is counted exactly once in $O_j$.

Theorem 3. The upper-tail $\Pr[\sum_{h \leq i < j} Y_{h,i,j} \geq T] \leq \exp(\mathbb{E}[\sum_{h \leq i < j} Y_{h,i,j}] - T \ln 2)$ for all $T > 0$.

Proof. To use the proof technique of Theorem 1, it is sufficient to have the analogues of Eq. (1), (2), and (3). We have, for each event $Y \in c(h)$ and event $Y' \in c(h')$ with $h' > h$, that

$$\mathbb{E}[Y | s_{h+1}, \ldots, s_n] = \mathbb{E}[Y | Y', s_{h+1}, \ldots, s_n].$$

To see this, we consider the crossings of $s_j$ in $S_{\leq j}$ and (again) think of the backward process that builds the random permutation by successively choosing one of the remaining elements. Picking $s_j$ fixes a partition of $S_{\leq j} = \{s_j\} \cup A_j \cup N_j$, into those segments that are adjacent to a crossing of $s_j$ and those segments that are not adjacent. Hence picking $s_h$ from given $S \setminus \{s_{h+1}, \ldots, s_n\}$ determines the outcome of all events in $c(h)$, i.e. $Y_{h,i,j}$ occurs if and only if $s_h$ has color $A_j$ and $A_j$.

Hence for given suffix condition, the events in $c(h)$ are either certain or impossible and thus

$$\mathbb{E}[\prod_{Y \in c(1)} \exp(tY)|s_2, \ldots, s_n] = \prod_{Y \in c(1)} \mathbb{E}[\exp(tY)|s_2, \ldots, s_n]$$

$$\mathbb{E}[\prod_{Y \in c(h)} \exp(tY)|s_h, \ldots, s_n] = \prod_{Y \in c(h)} \mathbb{E}[\exp(tY)|s_h, \ldots, s_n] \quad \forall \ 2 \leq h \leq n - 2.$$

The remaining arguments of the proof of Theorem 1 require no modification, and consequently this proves the theorem.

We conclude

$$\Pr\left[\sum_j E_j + O_j + C_j \geq 2T\right] \leq \Pr\left[\sum_j E_j \geq T\right] + \Pr\left[\sum_j B_j + C_j \geq T\right]$$

$$\leq \exp\left(12\gamma_n - T \ln \frac{n}{6}\right) + \Pr\left[\sum_j B_j \geq T - 3k\right]$$

$$\leq \exp\left(12\gamma_n - T \ln \frac{n}{6}\right) + \exp\left(\mathbb{E}[\sum Y_{h,i,j}] - (T - 3k) \ln 2\right)$$

$$\leq 1/e^{n+k},$$

where the last inequality follows from $\mathbb{E}[\sum Y_{h,i,j}] = \mathbb{E}[\sum_j O_j] = O(n+k)$ and choosing $T = \beta(n+k)$, with a sufficiently large constant $\beta > 1$. We have shown the following:

Corollary 4. The TSD size is $O(n+k)$ with probability at least $1 - 1/e^{n+k}$.

5.2 History DAGs for 2D Delaunay Triangulation and 3D Convex Hulls

We briefly outline the RIC to define the necessary terminology. For the input set $S$ of $n$ sites, we compute the Delaunay triangulation $T(S)$ by inserting the sites, one at a time, to derive $T(S_j)$ from $T(S_{j-1})$. The initially empty triangulation $T(S_0)$ only contains the bounding triangle $(-\infty, -\infty) (+\infty, -\infty)(0, +\infty)$. Given the triangle $pqr$ in $T(S_{j-1})$ that contains the next site
\( s_j \), we split the face into three triangles, i.e. \( pqs_j, qrs_j \) and \( rps_j \), and scan the list of incident triangles of \( s_j \), in CCW order. If one such triangle is not (local) Delaunay, we flip the edge and replace its former entry with both, now incident, triangles in the CCW list until all incident triangles are Delaunay. The work is proportional to the degree of \( s_j \) in the triangulation of \( \mathcal{T}(S_j) \), which is expected \( O(1) \) due to standard Backward Analysis.

For locating the face that contains \( s_j \), there are two well known search DAG variants that use the triangulation history \( \mathcal{T}(S_0), \mathcal{T}(S_1), \ldots, \mathcal{T}(S_{j-1}) \). The first method [GKS92, Section 3] keeps record of all intermediary (non-Delaunay) triangles to search for next site \( s_j \) (e.g. [dBCvKO08, Chapter 9]). Instead of keeping intermediary triangles, the second method [GKS92, Section 5] simply keeps the final CCW lists, referenced by the deleted triangles, which allows point location searches to descend in the history using (radial) binary searches (e.g. [Mul94, Chapter 3.3]). To simplify the presentation, we assume that the final CCW lists are stored as array for the binary search. (Skip lists can be used for the pointer machine model.)

It is well known that the radial-search method can also be used for the RIC of 3D Convex Hulls, with minor changes of the algorithm and analysis (see Figure 5; cf. [Mul94]). The history size of either method is expected \( O(n) \) and the first method has expected \( O(n \log n) \) runtime (see e.g. [dBCvKO08, Chapter 9.5]). However, high probability bounds for the history size are unknown, the best bound is Corollary 26 in [CMS93], as discussed in Section 1.1. Moreover, the first method does not yield any bound for general point location queries (only for the input sites). This is overcome by the second method, that has w.h.p. \( O(\log^2 n) \) search cost for all query points (see [Mul94, Theorem 3.3.2]).

We now improve the tail bounds for the DAG size and show that point location cost is w.h.p. \( O(\log n) \), using the proposed technique of Pairwise Events.

### 5.2.1 Improved Space Bounds

To simplify presentation, we use the ordinary non-degeneracy assumptions that no four points are co-circular (Delaunay) and no four points are co-planar (i.e. the faces of the 3D convex hull are triangles). Let the random variable \( D_j \) be the number of triangles in \( \mathcal{T}(S_j) \) that are incident to \( s_j \). Recall that \( \mathbb{E}[D_j] = O(1) \) and \( \sum_j \mathbb{E}[D_j] = O(n) \). We define event \( X_{i,j} \) to occur if and only if \( s_i \) and \( s_j \) are adjacent in \( \mathcal{T}(S_j) \), i.e. they form an edge of the triangulation. The analogue of Lemma 1 is even simpler, since \( D_j = \sum_{i<j} X_{i,j} \) holds with equality (i.e. \( D_j \) is both upper and lower bound for the number of occurring row events). Using Theorem 1 with these bounds gives the following result.

**Corollary 5.** In the Randomized Incremental Construction of 2D Delaunay Triangulations and
3D Convex Hulls, the history size is $O(n)$ with probability at least $1 - 1/e^n$.

5.2.2 Improved Point Location Bound

To improve the query time bound of the point location we need to modify our arguments of Section 4. We discuss the, more technical, radial-search method. The boundary priority of a triangle $\Delta = pqr$ is $b(\Delta) := \max\{\pi(p), \pi(q), \pi(r)\}$. Since a sequence $\Delta_i$ of (Delaunay) triangles on a root-to-leaf path of the history DAG has strictly increasing boundary index sequences. Let $\sigma = (b_0, \ldots, b_\ell)$ be a monotonous sequence of index values. The search cost for a DAG path with index sequence $\sigma$ is at most

$$\sum_{i=1}^\ell X_{b_{i-1},b_i} \cdot \left[1 + \log_2 \left( D_{b_i} \right) \right].$$

We have $E[X_{i,j} \cdot D_j] = E[X_{i,j}] E[D_j | X_{i,j}] = E[X_{i,j}] E[D_j]$, since the choice of $s_i$ from $S_j \setminus \{s_j\}$ does not depend on which elements are ‘adjacent’ to $s_j$. Since the function $-\log(x)$ is convex, Jensen’s inequality gives that $E[X_{i,j}] O(\log E[D_j])$ is at most $E[X_{i,j}] O(\log E[D_j]) = O(E[X_{i,j}])$. Using the Harmonic Distribution of the events, i.e. $E[X_{i,j}] = O(1/j)$, we have that the expected cost for the sequence $\sigma$ is at most $O(\ln n)$.

The property $E[X_{i,j} \cdot D_j] = E[X_{i,j}] E[D_j]$ allows us to prove a high probability bound, analogue to the proof of Equation (9), for the upper tail of the search cost for any given sequence $\sigma$. The remaining arguments in Section 4 require no modification, beside using an appropriate constant $\gamma$, such that $|T(S)| \leq \gamma n$, for triangulations. As a result we get:

**Corollary 6.** Worst-case point location cost in the History of 2D Delaunay Triangulations and 3D Convex Hulls is w.h.p. $O(\log n)$.

To obtain a Las Vegas verifier, we now describe a simple addition to the RIC that tracks this upper bound (on the radial search costs) as weighted path lengths in the History DAG over the course of the construction, i.e. from $T(S_{j-1})$ to $T(S_j)$. We associate the weight $[\log_2 D_j] + 1$ to every outgoing pointer of the CCW list of $s_j$ to the incident triangles of $T(S_j)$.

For computing the maximum cost of an edge weighted path to the new triangles in $T(S_j)$, we simply propagate the maximum cost from those triangles $\Delta$ that are deleted from $T(S_{j-1})$ in the course of triangle replacements that create the CCW list, i.e. the replacements $(\Delta''', \Delta'')$ of the entry $\Delta'$ store the maximum of $\Delta$ and $\Delta'$. Eventually, aforementioned weight of $[\log_2 D_j] + 1$ is added to value of each of the triangles in the final CCW list of $s_j$.

**Corollary 7.** Point location in the History DAGs of 2D Delaunay Triangulations and 3D Convex Hulls can be made worst-case $O(\log n)$ by merely increasing the expected construction time by an additive constant.

**Conclusion and Future Work**

We introduced a simple analysis technique that gives improved tail estimates for the size of the search structures from RIC. High probability bounds, though of general interest for dynamic maintenance, were unknown. Consequently, we provided the insight that history-based RIC of the trapezoidization gives with high probability a search DAG of $O(n)$ size and $O(\log n)$ query time in $O(n \log n)$ time. It has been shown recently that conflict-graph RIC cannot achieve the time bound in the worst-case.
Our technique also gives novel, high probability bounds for the combinatorial path length, which eluded the geometric backward analysis technique. This allowed us to prove a recent conjecture on the depth of the search DAG for point location in planar subdivisions and a long-standing conjecture on the query time in the search DAGs for 2D nearest-neighbor search and extreme point search in 3D convex hulls. Consequently, we provided the insight that identical high probability bounds on size, query time, and construction time, hold for these search DAGs.

The new algorithms we obtain from this are remarkably simple Las Vegas verifiers for history-based RIC that give worst-case optimal DAGs, i.e. linear size and logarithmic search cost.

For future work, we are interested in a finer calibration of the constants (e.g. if only polynomial decay is needed) and extensions of the technique for other history-based RICs with super-quadratic worst-case size (e.g. Delaunay tessellations and convex hulls in higher dimensions).

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A Background on RIC of Search Structures

Mulmuley’s book [Mul94] gives an excellent introduction to the paradigm. A simple 1D geometric problem that can be solved by RIC is to compute the intervals induced by a given set of points on a line (e.g. the x-axis). In this case, the structure for the empty set of points is the interval (−∞, +∞) and, at every point insertion, the interval that contains the point is split into two open intervals (left and right of the point). There are two well known methods to identify the interval that needs to be split for the next point, that are called maintaining conflict lists and keeping a searchable history of all structures created in the process. Using conflict lists, all points are placed in the initial interval (e.g. they have a pointer to their interval) and every time an interval is split, its points are partitioned in the left and right interval (cf. partitions in quicksort). Using the history structures, one starts with the initial interval and every time an interval is split, the split point is stored therein together with two pointers to the respective left and right result intervals (cf. binary search trees).

Though the RIC seems unguided, the resulting search structures have surprisingly good expected performance measures on any input, i.e. the expectation is over the random permutations of the objects. The randomized binary search trees, for example, have a worst case size of \( O(n) \) and every leaf has expected depth \( O(\log n) \). A beautiful simple argument for this is due to Backward Analysis [Sei93]. One fixes an arbitrary search point \( q \) (within one leaf) and counts how often \( q \) changes its interval during the construction, or equivalently during deleting all objects in the reverse order. This leads to an expected search cost of \( O(\log n) \) for \( q \). Moreover, Chernoff’s method shows that deviations of more than a constant factor from the expected value are very unlikely, i.e. no more than inverse proportional to a polynomial in \( n \) whose degree can be made arbitrary large by increasing the constant. That is, the search path to \( q \) has \( O(\log n) \) length with high probability (w.h.p.). Since there are only \( n + 1 \) different search paths, the longest of them is w.h.p. within a constant factor of the expected value. As a result the tree height is w.h.p. within a constant factor of the optimum.

Tail bounds have immediate algorithmic applications, e.g. when the expected performance measure of a search structure needs to be made a worst-case property (within a constant factor). Treaps [SA96, Vui80], for example, are a fully-dynamic version of randomized binary search trees with expected logarithmic update time whose shape is, after each update, dictated by a random permutation that is uniformly from those over the current set of objects. Since it is easy to maintain the height of the root in Treaps, one can simply rebuild a degraded tree entirely (with a fresh permutation) until the data structure is again within a constant factor of optimum. Given a high probability tail bound for a performance measure of interest, the simple rebuild strategy (to attain worst-case guarantees) for dynamic search structures only adds an expected rebuild cost to the update time that is at most a constant. This demonstrates that tail bounds with polynomial or exponential decay, rather than constant, are of general interest for maintaining dynamic data structures.
B Experiments on the TSD size and depth

The work of Hemmer et al. provides an extensive experimental discussion of the differences between search depth and depth (see Appendix A in [HKH12b]). The experimental data on their TSD implementation in CGAL [WBF20] does however not comprise the final and intermediary sizes of the structure during the construction. This section provides additional experimental data, derived from their TSD implementation in CGAL 5.1.1, that focuses to exhibit how our proposed tail estimate on the TSD size compares against the known high probability bound for the search depth. Since the computation of the search depth for the intermediary structures entails considerable work, we only compute the search depth for the first 1000 segment insertions. Beside this we also recorded the depth, which is constant time accessible throughout the entire construction.

Our experiments comprise one real (NC) and one synthetic (rnd-hor-10K) data set. The NC data set is from the OpenStreetMap project and contains all 188,082 line segments that are associated to streets in New Caledonia, as of June 1st 2020. The rnd-hor-10K data set contains horizontal segments with the y-coordinates \(\{1, \ldots, 10^4\}\) and x-coordinates that are chosen uniformly at random from \((0, 100.0)\).

Figures 6 and 7 show the absolute values of size, depth, and search depth during the TSD construction with two different random permutations on the three data sets. The figures also contain a plot of the TSD size relative to \(n\) and TSD depth relative to \(\log n\) to make relative deviations from optimum visually better accessible. In our experiment, the depth and search depth are very closely related and the largest discrepancy is observed on the rnd-hor-10K data set (see also Figure 3). Fluctuations of the TSD sizes between the two runs are visually barely distinguishable, as suggested by our exponential tail bound. The depth and search depth show more fluctuations during the construction, yet within a small constant factor of \(\log(n)\) as suggested by our high probability bound.
Figure 6: TSD size (absolute and relative, top) and depth vs search depth (absolute and relative, bottom) on the NC data set.

Figure 7: TSD size (absolute and relative, top) and depth vs search depth (absolute and relative, bottom) on the rnd-hor-10K data set.