Heat kernel for the elliptic system of linear elasticity with boundary conditions

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Abstract

We consider the elliptic system of linear elasticity with bounded measurable coefficients in a domain where the second Korn inequality holds. We construct heat kernel of the system subject to Dirichlet, Neumann, or mixed boundary condition under the assumption that weak solutions of the elliptic system are Hölder continuous in the interior. Moreover, we show that if weak solutions of the mixed problem are Hölder continuous up to the boundary, then the corresponding heat kernel has a Gaussian bound. In particular, if the domain is a two dimensional Lipschitz domain satisfying a corkscrew or non-tangential accessibility condition on the set where we specify Dirichlet boundary condition, then we show that the heat kernel has a Gaussian bound. As an application, we construct Green’s function for elliptic mixed problem in such a domain.

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1 Introduction

In a domain (i.e. a connected open set) \( \Omega \subset \mathbb{R}^n \) \((n \geq 2)\), we consider the differential operator

\[
Lu = \frac{\partial}{\partial x_i} \left( A_{ij}^{\alpha\beta}(x) \frac{\partial u^i}{\partial x_j} \right),
\]

where \( u \) is a column vector with components \( u^1, \ldots, u^n \), \( A_{ij}^{\alpha\beta}(x) \) are \( n \times n \) matrices whose elements \( a_{ij}^{\alpha\beta}(x) \) are bounded measurable functions which satisfy the symmetry condition

\[
a_{ij}^{\alpha\beta}(x) = a_{ji}^{\beta\alpha}(x) = a_{ij}^{\alpha\beta}(x),
\]

where \( \xi \) is an arbitrary \( n \times n \) matrix with real entries \( \xi^\alpha_i, \kappa_1, \kappa_2 > 0 \), and \(|\xi| = (\xi, \xi)^{1/2}\). Here and below, we will follow the convention that we sum over repeated indices and notation \((\xi, \eta) = \xi^\alpha_i \eta_i^\alpha\) for \( n \times n \) matrices \( \xi = (\xi^\alpha_i) \) and \( \eta = (\eta_i^\alpha)\). The operator \( L \) defined by (1.1) can also be written in coordinate form as follows.

\[
(Lu)_i = \sum_{a,\beta=1}^{n} \frac{\partial}{\partial x_a} \left( a_{ij}^{\alpha\beta}(x) \frac{\partial u^i}{\partial x_a} \right), \quad i = 1, \ldots, n.
\]

Our assumptions on the coefficients \( a_{ij}^{\alpha\beta} \) will include the equations of linear elasticity and in this case, the coefficients \( a_{ij}^{\alpha\beta}(x) \) are usually referred to as the elasticity tensor; see e.g. [2, 18]. In the classical theory of linear elasticity, the elasticity tensor for a homogeneous isotropic body is given by the formula

\[
a_{ij}^{\alpha\beta} = \lambda \delta_{ij} \delta_{\alpha\beta} + \mu (\delta_{ij} \delta_{\alpha\beta} + \delta_{ij} \delta_{\alpha\beta}),
\]

where \( \lambda > 0, \mu > 0 \) are the Lamé constants, \( \delta_{ij} \) is the Kronecker symbol. In this case, the conditions (1.2) and (1.3) are satisfied with \( \kappa_1 = 2\mu \) and \( \kappa_2 = 2\mu + n\lambda \). We define the traction \( \tau = \tau(u) \) by the formula

\[
\tau(u) = \nu_a A_{ij}^{\alpha\beta}(x) \frac{\partial u^i}{\partial x_a},
\]

where \( \nu = (\nu_1, \ldots, \nu_n) \) is the unit outward normal to \( \partial \Omega \).

We consider following boundary value problems, which are the ones most frequently considered in the theory of linear elasticity.

1. Dirichlet (displacement) problem

\[
Lu = f \text{ in } \Omega, \quad u = \Phi \text{ on } \partial \Omega. \quad \text{(DP)}
\]

2. Neumann (traction) problem

\[
Lu = f \text{ in } \Omega, \quad \tau(u) = \nu \text{ on } \partial \Omega. \quad \text{(NP)}
\]
3. Mixed problem

\[ Lu = f \text{ in } \Omega, \quad u = \Phi \text{ on } D, \quad \tau(u) = \varphi \text{ on } N, \]

where \( D \) and \( N \) are subsets of \( \partial \Omega \) such that \( D \cup N = \partial \Omega \) and \( D \cap N = \emptyset \).

In the above, the equation as well as the boundary condition should be interpreted in a weak sense; see Section 2 for a precise formulation.

In this article, we are concerned with the heat kernel associated with the mixed problem (MP). By allowing \( D = \partial \Omega \), \( N = \emptyset \) and \( D = \emptyset \), \( N = \partial \Omega \) in (MP), we may regard (DP) and (NP) as extreme cases of (MP) and this is why we focus on (MP). By the heat kernel for (MP), we mean an \( n \times n \) matrix valued function \( K(x, y, t) \) satisfying

\[
\begin{cases}
\partial_t K(x, y, t) - L_x K(x, y, t) = 0 & \text{in } \Omega \times (0, \infty), \\
K(x, y, t) = 0 & \text{on } D \times (0, \infty), \\
\tau_x (K(x, y, t)) = 0 & \text{on } N \times (0, \infty), \\
K(x, y, 0) = \delta_y I & \text{on } \Omega,
\end{cases}
\]

where \( \delta_y (\cdot) \) is Dirac delta function concentrated at \( y \), \( I \) is the \( n \times n \) identity matrix, and the equation as well as the boundary condition should be interpreted in some weak sense. The heat kernels for (DP) and (NP) are similarly defined and they are frequently referred to as Dirichlet and Neumann heat kernels.

We assume that \( \Omega \) is a bounded \((\varepsilon, \delta)\)-domain of Jones [15] and if \( D \neq \emptyset \) and \( D \neq \partial \Omega \), we assume further that it has a Lipschitz portion; if \( D = \partial \Omega \), then we require none of these conditions (see H1 in Section 2.5). We prove that if weak solutions of the system \( Lu = 0 \) are locally Hölder continuous (see H2 in Section 2.5), then the heat kernel for (MP) exists and satisfies a natural growth estimate near the pole; see Theorem 3.1. It is known that weak solutions of \( L \) are Hölder continuous if \( n = 2 \) or if the coefficients are uniformly continuous. We also prove that if the gradient of weak solutions of the system \( Lu = 0 \) satisfy the growth condition called Dirichlet property (see H3 in Section 2.5), then the heat kernel has a Gaussian upper bound; see Theorem 3.10. The Dirichlet property is known to hold in the case when \( \Omega \) is a Lipschitz domain in \( \mathbb{R}^2 \) and \( D \) satisfies the corkscrew condition (see [25]) or when the coefficients and domains are sufficiently smooth and \( D \cap N = \emptyset \) (see [29]). As an application, we construct Green’s function for the elliptic system from the heat kernel and in the presence of Dirichlet property, we show that the Green’s function has the usual bound of \( C|x - y|^{2-n} \) (or logarithmic bound if \( n = 2 \)); see Theorem 4.3.

A few remarks are in order. The Dirichlet or Neumann heat kernels for elliptic equations are studied by many authors; see Davies [6], Robinson [25], Varopoulos et al. [27], and references therein. For the heat kernel for second-order elliptic operators in divergence form satisfying Robin-type boundary conditions, we mention Gesztesy et al. [10]. Dirichlet and Neumann Green’s functions for strongly parabolic systems are studied in Cho et al. [3, 4] and Choi and Kim [5]. We should mention that our paper, though technically more involved, is an extension of their method. The elliptic Green function for (MP) in two dimensional domains is constructed in Taylor et al. [25] by a different method not involving the heat kernel.
Previously, Taylor et al. [26] give a construction of the Green function for a class of mixed problems for the Laplacian in a Lipschitz domain in dimensions two and higher.

In recent years, there has been increasing interest in the study of elliptic equations under mixed boundary conditions from a variety of viewpoints. Haller-Dintelmann et al. [14] consider Hölder continuity of solutions of a single equation with an interest in applications in control. Mazzucato and Nistor [20] study the mixed problem for the elliptic system of elasticity in polyhedral domains under mixed boundary conditions. Finally, a recent monograph of Maz’ya and Rossmann [19] treats mixed problems for systems in polyhedral domains. We refer to the above works and their references for background on the study of mixed problems for systems. It would be interesting to see if the well-posedness results in these works can be used to obtain fundamental estimates $H^2$ and $H^3$ that we use to obtain further estimates for the heat kernel.

The organization of the paper is as follows. In Section 2, we introduce some notation and definitions including the precise definition of the heat kernel for (MP). In Section 3, we state our main theorems (Theorems 3.1 and 3.10), which we briefly described above. In Section 4, we construct, as an application, the Green’s function for (MP) and obtain the usual bounds. We give the proofs for our main results in Section 5 and some technical lemmas are proved in Appendix.

2 Preliminaries

2.1 Notation and definition

Throughout the article, we let $\Omega$ denote a domain in $\mathbb{R}^n$ and let $D$ and $N$ be fixed subsets of $\partial \Omega$ such that $D \cup N = \partial \Omega$ and $D \cap N = \emptyset$. We use $X = (x, t)$ to denote a point in $\mathbb{R}^{n+1}$; $x = (x_1, \ldots, x_n)$ will always be a point in $\mathbb{R}^n$. We also write $Y = (y, s)$, $X_0 = (x_0, t_0)$, and reserve notation $\hat{Y} = (y, 0) = (y_1, \ldots, y_n, 0)$.

We define the parabolic distance in $\mathbb{R}^{n+1}$ by

$$|X - Y|_{\mathscr{P}} = \max(|x - y|, \sqrt{|t - s|}),$$

where $|\cdot|$ denotes the usual Euclidean norm, and write $|X|_{\mathscr{P}} = |X - 0|_{\mathscr{P}}$. We use the following notation for basic cylinders in $\mathbb{R}^{n+1}$.

$$Q(X, r) = \{Y \in \mathbb{R}^{n+1} : |Y - X|_{\mathscr{P}} < r\},$$

$$Q_-(X, r) = \{Y = (y, s) \in \mathbb{R}^{n+1} : |Y - X|_{\mathscr{P}} < r, s < t\},$$

$$Q_+(X, r) = \{Y = (y, s) \in \mathbb{R}^{n+1} : |Y - X|_{\mathscr{P}} < r, s > t\}.$$ We also use $B(x, r) = \{y \in \mathbb{R}^n : |y - x| < r\}$ to denote a ball in $\mathbb{R}^n$. For the vector valued function $u = (u^1, \ldots, u^n)^T$, we denote by $\varepsilon(u)$ (called the strain tensor) the matrix whose elements are

$$\varepsilon_{ij}(u) = \frac{1}{2} \left( \frac{\partial u^i}{\partial x_j} + \frac{\partial u^j}{\partial x_i} \right).$$
We denote by $\mathcal{R}$ the linear space of rigid displacements of $\mathbb{R}^n$; i.e.
\[ \mathcal{R} = \{ Ax + b : A \text{ is real } n \times n \text{ skew-symmetric matrix and } b \in \mathbb{R}^n \} \] (2.1)
Note that $\mathcal{R}$ is a real vector space of dimension $N = n(n + 1)/2$ and that
\[ \langle u, v \rangle = \int_{\Omega} u \cdot v \, dx = \int_{\Omega} v^T u \, dx \]
defines an inner product on $\mathcal{R}$. Let us fix an orthonormal basis $\{ \omega_i \}_{i=1}^N$ in $\mathcal{R}$
and define the projection operator $\pi_{\mathcal{R}} : L^2(\Omega)^n \to \mathcal{R}$ by
\[ \pi_{\mathcal{R}}(u) = \sum_{i=1}^N \langle u, \omega_i \rangle \omega_i. \] (2.3)
The above formula still makes sense for $u = \delta_y e_j$, where $e_j$ is the $k$th unit (column) vector in $\mathbb{R}^n$. For $y \in \Omega$, we denote by $T_y = T_y(x)$ an $n \times n$ matrix valued function such that
\[ T_y e_i = \begin{cases} \pi_{\mathcal{R}}(\delta_y e_j) & \text{if } D = \emptyset, \\ 0 & \text{if } D \neq \emptyset. \end{cases} \] (2.4)
Roughly speaking, $T_y$ is an orthogonal projection of $\delta_y I$ on $\mathcal{R}$ if $D = \emptyset$ and $0$ otherwise. We set
\[ \mathcal{B}(u, v) = a_{ij} \frac{\partial u^i}{\partial x_j} \frac{\partial v^j}{\partial x_i}. \]
It follows from (1.2) the form $\mathcal{B}$ is symmetric (i.e., $\mathcal{B}(u, v) = \mathcal{B}(v, u)$) and from (1.3) that
\[ \kappa_2^{-1} \mathcal{B}(u, u) \leq |v|^2 \leq \kappa_1 \mathcal{B}(u, u). \] (2.5)
It is easy to verify that (1.2), (1.3) imply (see [22, Lemma 3.1, p. 30])
\[ a_{ij} \xi^i \eta^j \leq \frac{1}{4} \kappa_2 |\xi + \eta|^2 \leq \kappa_2 |\xi||\eta|. \]
Finally, we denote $d(x) = d(x) = \text{dist}(x, \Omega^c)$ when $\Omega$ is clear from the context
and write $a \wedge b = \min(a, b)$ and $a \vee b = \max(a, b)$ for $a, b \in \mathbb{R}$.

2.2 Function spaces
We use the notation in Gilbarg & Trudinger [13] for the standard functions spaces defined on $U \subset \mathbb{R}^n$ such as $L^p(U), W^{k,p}(U), C^k(U)$, etc. For $\Gamma \subset \partial U$, let $W^{1,2}(U; \Gamma)$ be the subspace obtained by taking the closure in $W^{1,2}(U)$ of smooth functions in $\bar{U}$ which vanish in a neighborhood of $\Gamma$. Note that we have $W^{1,2}(U; \partial U) = W^{1,2}_{0}(U)$. We shall denote
\[ W^{1,2}(U; \Gamma) := \begin{cases} W^{1,2}(U; \Gamma) : \int_{\Gamma} u \, dx = 0 & \text{if } \Gamma \neq \emptyset, \\ \{ u \in W^{1,2}(U) : \int_{\Gamma} u \, dx = 0 \} & \text{if } \Gamma = \emptyset. \end{cases} \]
For $\Omega$ and $D$ as above, we define
\[ V := \begin{cases} W^{1,2}(\Omega; D)^n : \langle u, v \rangle = 0, \forall v \in \mathcal{R} & \text{if } D \neq \emptyset, \\ \{ u \in W^{1,2}(\Omega)^n : \langle u, v \rangle = 0, \forall v \in \mathcal{R} \} & \text{if } D = \emptyset. \end{cases} \] (2.6)
where $\mathcal{R}$ and $(u, v)$ are as in \[2.1\] and \[2.2\]. Notice that $V \subset \tilde{W}^{1,2}(\Omega; D)$. For $\mu \in (0, 1]$, we denote

$$
[u]_{\mu,U} = [u]_{\mu,U} + |u|_{\mu,U} = \sup_{x,y \in U} \frac{|u(x) - u(y)|}{|x - y|^\mu} + \sup_{x \in U} |u(x)|.
$$

For spaces of functions defined on $Q \subset \mathbb{R}^{n+1}$, we borrow notation mainly from Ladyzhenskaya et al. \[17\]. To avoid confusion, spaces of functions defined on $Q \subset \mathbb{R}^{n+1}$ shall be always written in script letters. For $q \geq 1$, we let $\mathcal{L}_q(Q)$ denote the Banach space consisting of measurable functions on $Q$ that are $q$-integrable. For $Q = \Omega \times (a, b)$, we denote by $\mathcal{L}_{q,r}(Q)$ the Banach space consisting of all measurable functions on $Q$ with a finite norm

$$
||u||_{\mathcal{L}_{q,r}(Q)} = \left( \int_\Omega \left( \int_a^b |u(x,t)|^q \, dt \right)^{\frac{r}{q}} \, dx \right)^{\frac{1}{r}},
$$

where $q \geq 1$ and $r \geq 1$. Thus $\mathcal{L}_q(Q)$ is the space $L_q(Q)$. By $C^{0,\alpha}(\tilde{Q})$ we denote the set of all bounded measurable functions $u$ on $Q$ for which $|u|_{C^{0,\alpha}(\tilde{Q})}$ is finite, where we define the parabolic Hölder norm as follows:

$$
|u|_{C^{0,\alpha}(\tilde{Q})} = |u|_{C^{\alpha}(\tilde{Q})} = \sup_{x,y \in \tilde{Q}} \frac{|u(x) - u(y)|}{|x - y|^\alpha} + \sup_{x \in \tilde{Q}} |u(x)|, \quad \alpha \in (0, 1].
$$

We write $u \in \mathcal{E}_C^{\infty}(Q)$ (resp. $\mathcal{E}_C^{\infty}(\tilde{Q})$) if $u$ is an infinitely differentiable function on $\mathbb{R}^{n+1}$ with compact support in $Q$ (resp. $\tilde{Q}$). We write $D_i u = \partial u/\partial x_i$ ($i = 1, \ldots, n$) and $D_t u = \partial u/\partial t$. We also write $Du = D_i u = (D_1 u, \ldots, D_n u)$. We write $Q(t)$ for the set of all points $(x, t)$ in $Q$ and $I(Q)$ for the set of all $t$ such that $Q(t)$ is nonempty. We denote

$$
||u||_{\mathcal{E}_C^{\infty}(Q)} = \int_Q |D_i u|^2 \, dx \, dt + \text{ess sup} \sup_{t \in (0, T)} \int_{Q(t)} |u(x,t)|^2 \, dx.
$$

The space $\mathcal{W}_q^{1,0}(Q)$ denotes the Banach space consisting of functions $u \in \mathcal{L}_q(Q)$ with weak derivatives $D_i u \in \mathcal{L}_q(Q)$ ($i = 1, \ldots, n$) with the norm

$$
||u||_{\mathcal{W}_q^{1,0}(Q)} = ||u||_{\mathcal{L}_q(Q)} + ||D_i u||_{\mathcal{L}_q(Q)}
$$

and by $\mathcal{W}_q^{1,1}(Q)$ the Banach space with the norm

$$
||u||_{\mathcal{W}_q^{1,1}(Q)} = ||u||_{\mathcal{L}_q(Q)} + ||D_i u||_{\mathcal{L}_q(Q)} + ||D_t u||_{\mathcal{L}_q(Q)}.
$$

In the case when $Q$ has a finite height (i.e., $Q \subset \mathbb{R}^{n} \times (-T, T)$ for some $T < \infty$), we define $\mathcal{V}_2(Q)$ as the Banach space consisting of all elements of $\mathcal{W}_2^{1,1}(Q)$ having a finite norm $||u||_{\mathcal{V}_2(Q)} := ||u||_{\mathcal{L}_2(Q)}$ and the space $\mathcal{V}_2^{1,0}(Q)$ is obtained by completing the set $\mathcal{W}_2^{1,1}(Q)$ in the norm of $\mathcal{V}_2(Q)$. When $Q$ does not have a finite height, we say that $u \in \mathcal{V}_2(Q)$ (resp. $\mathcal{V}_2^{1,0}(Q)$) if $u \in \mathcal{V}_2(Q_T)$ (resp. $\mathcal{V}_2^{1,0}(Q_T)$) for all $T > 0$, where $Q_T = Q \cap \{|t| < T\}$, and $||u||_{\mathcal{V}_2(Q)} < \infty$. Note that this definition allows that $1 \in \mathcal{V}_2^{1,0}(\Omega \times (0, \infty))$. Finally, we write $u \in \mathcal{L}_{q,\text{loc}}(Q)$ if $u \in \mathcal{L}_{q}(Q')$ for all $Q' \Subset Q$ and similarly define $\mathcal{W}_q^{1,0}(Q')$, etc.
2.3 Weak solutions

For $f, g_a \in L^2(U)^n$, where $\alpha = 1, \ldots, n$, we say that $u$ is a weak solution of $Lu = f + D_\alpha g_a$ in $U$ if $u \in W^{1,2}(U)^n$ and for any $v \in W^{1,2}_0(U)^n$ satisfies the identity

$$\int_U \mathcal{B}(u, v) = -\int_U f \cdot v + \int_U g_a \cdot D_\alpha v. \quad (2.7)$$

Let $\Gamma_D, \Gamma_N$ be disjoint subsets of $\partial U$. Recall that the traction $\tau(u)$ is defined by the formula (1.4). We say that $u$ is a weak solution of

$$Lu = f + D_\alpha g_a \quad \text{in} \quad U, \quad u = 0 \quad \text{on} \quad \Gamma_D, \quad \tau(u) = g_a \cdot n_a \quad \text{on} \quad \Gamma_N$$

if $u \in W^{1,2}(U; \Gamma_D)^n$ and for any $v \in W^{1,2}(U; \partial U \setminus \Gamma_N)^n$ satisfies the identity (2.7). Let $\Omega, D, N$ be as above. For $f \in L^2(\Omega)^n$, we say that $u$ is a weak solution of the mixed problem

$$Lu = f \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad D, \quad \tau(u) = 0 \quad \text{on} \quad N$$

if $u \in W^{1,2}(\Omega; D)^n$ and satisfies for any $v \in W^{1,2}(\Omega; D)^n$ the identity

$$\int_\Omega \mathcal{B}(u, v) = -\int_\Omega f \cdot v. \quad (2.8)$$

Let $Q$ be a cylinder $U \times (a, b)$, where $U \subset \mathbb{R}^n$ and $-\infty < a < b < \infty$. For $f \in \mathcal{L}_2(\Omega)^n$ and $g_a \in \mathcal{L}_2(\Omega)^n$, we say that $u$ is a weak solution of

$$u_t - Lu = f + D_\alpha g_a$$

if $u \in \mathcal{V}_{2}(Q)^n$ and satisfies for all $\phi \in \mathcal{C}^\infty_c(Q)^n$ the identity

$$-\int_Q u \cdot \phi_t + \int_Q \mathcal{B}(u, \phi) = \int_Q f \cdot \phi - \int_Q g_a \cdot D_\alpha \phi. \quad (2.8)$$

Let $\Gamma_D, \Gamma_N$ be disjoint subsets of $\partial U$. We say that $u$ is a weak solution of

$$\begin{cases}
  u_t - Lu = f + D_\alpha g_a & \text{in} \quad U \times (a, b) =: Q \\
  u = 0 & \text{on} \quad \Gamma_D \times (a, b) \\
  \tau(u) = -g_a \cdot n_a & \text{on} \quad \Gamma_N \times (a, b) =: S
\end{cases}$$

if $u \in \mathcal{V}_{2}(Q)^n$, $u(\cdot, t) \in W^{1,2}(U; \Gamma_D)^n$ for a.e. $t \in (a, b)$, and it satisfies the identity (2.8) for all $\phi \in \mathcal{C}^\infty_c(Q \cup S)^n$. Next, denote $Q = \Omega \times (a, b)$, and let $f \in \mathcal{L}_2(\Omega)^n$, $g_a \in \mathcal{L}_2(\Omega)^n$, and $\psi \in L^2(\Omega)^n$. By a weak solution in $\mathcal{V}_2(Q)^n$ (resp. $\mathcal{V}_{2, \partial}(Q)^n$) of the problem

$$\begin{cases}
  u_t - Lu = f + D_\alpha g_a & \text{in} \quad \Omega \times (a, b) \\
  u = 0 & \text{on} \quad D \times (a, b) \\
  \tau(u) = -g_a \cdot n_a & \text{on} \quad N \times (a, b) \\
  u(\cdot, a) = \psi & \text{on} \quad \Omega
\end{cases} \quad (2.9)$$

we mean $u(x, t)$ in $\mathcal{V}_2(Q)^n$ (resp. $\mathcal{V}_{2, \partial}(Q)^n$) such that $u(\cdot, t) \in W^{1,2}(\Omega; D)^n$ for a.e. $t \in (a, b)$ and satisfying the identity

$$-\int_Q u \cdot \phi_t + \int_Q \mathcal{B}(u, \phi) - \int_Q f \cdot \phi + \int_Q g_a \cdot D_\alpha \phi = \int_\Omega \psi(x) \cdot \phi(x, a) \, dx$$

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for any \( \phi(x, t) \in \mathcal{C}_c^\infty(\Omega) \) that vanishes on \( D \times (a, b) \) and equals to zero for \( t = a \). We may identify \( f \in L^2_{2,1}(Q)^n \) as an element in \( L^1(a, b; V') \) in the sense

\[
[f(\cdot, t)](v) = \int_{\Omega} f(x, t) \cdot v(x) \, dx, \quad v \in V,
\]

and consider the problem

\[
\begin{align*}
   u_t - Lu &= f & \text{in } \Omega \times (a, b) \\
   u &= 0 & \text{on } D \times (a, b) \\
   \tau(u) &= 0 & \text{on } N \times (a, b) \\
   u(\cdot, a) &= \psi & \text{on } \Omega \\
   u(\cdot, t) &\in V & \text{for a.e. } t \in (a, b). 
\end{align*}
\tag{2.10}
\]

In the above, we impose the compatibility condition for \( \psi \) that \( (\psi, v) = 0 \) for all \( v \in \mathcal{R} \) in the case when \( D = \emptyset \). We shall say that \( u \) is a weak solution of the problem (2.10) if \( u \in \mathcal{Y}_2^{1,0}(Q)^n \), \( u(\cdot, t) \in V \) for a.e. \( t \in (a, b) \), and satisfies the identity

\[
- \int_{\Omega} u \cdot \phi_t + \int_{\Omega} \mathcal{B}(u, \phi) - \int_{\Omega} f \cdot \phi = \int_{\Omega} \psi(x) \cdot \phi(x, a) \, dx
\]

for any \( \phi(x,t) \in \mathcal{C}_c^\infty(\Omega) \) that vanishes on \( D \times (a, b) \), satisfies \( (\phi(\cdot, t), v) = 0 \) for all \( t \in [a,b] \) and any \( v \in \mathcal{R} \), and equals to zero for \( t = a \). Note that if \( D \neq 0 \), then a weak solution of the problem (2.10) is also a weak solution in \( \mathcal{Y}_2^{1,0}(Q)^n \) of the problem (2.9) with \( g_\alpha = 0 \) for all \( \alpha = 1, \ldots, n \), and vice versa. However, when \( D = \emptyset \), they are not the same in general; note that if \( f(\cdot, t) = \tilde{f}(\cdot, t) \in \mathcal{R} \), then they are the same as elements in \( L^1(a, b; V') \).

### 2.4 Heat kernel for the system of linear elasticity

We say that an \( n \times n \) matrix valued function \( K(x, y, t) \), with measurable entries \( K_{ij} : \Omega \times \Omega \times [0, \infty) \rightarrow \mathbb{R} \), is the heat kernel for \( (\text{MP}) \) if it satisfies the following properties, where we denote \( \Omega = \Omega \times [0, \infty) \).

a) For all \( y \in \Omega \), elements of \( K(\cdot, y, \cdot) \) belong to \( \mathcal{W}^{1,0}_{\text{loc}}(\Omega) \cap \mathcal{W}_2(\Omega \setminus \Omega, (\tilde{Y}, r)) \) for any \( r > 0 \).

b) For all \( y \in \Omega \), \( K(\cdot, y, \cdot) \) is a generalized solution of the problem (1.5) in the sense that \( K(\cdot, y, t) \in W^{1,2}(\Omega; \mathcal{D})^n \) for a.e. \( t > 0 \) and for any \( \phi = (\phi^1, \ldots, \phi^n)^T \in \mathcal{C}_c^\infty(\Omega)^n \) that vanishes on \( N \times [0, \infty) \), we have the identity

\[
- \int_{\Omega} K(x, y, t) \frac{\partial}{\partial t} \phi(x, t) \, dx \, dt + \int_{\Omega} a_{\alpha\beta}(x, y, t) \frac{\partial}{\partial x_i} K_{ij}(x, y, t) \frac{\partial}{\partial x_j} \phi(x, t) \, dx \, dt = \phi^i(y, 0). \tag{2.11}
\]

For any \( f = (f^1, \ldots, f^n)^T \in \mathcal{C}_c^\infty(\Omega)^n \), the function \( u \) given by

\[
u(x, t) := \int_{\Omega} K(y, x, t - s)^T f(y, s) \, dy \, ds
\]

...
is, for any $T > 0$, a unique weak solution in $\mathcal{Y}_2^{1,0}(\Omega \times (0, T))^n$ of the problem

\[
\begin{align*}
  u_t - Lu &= f & \text{in } \Omega \times (0, T), \\
  u &= 0 & \text{on } D \times (0, T), \\
  \tau(u) &= 0 & \text{on } N \times (0, T), \\
  u(\cdot, 0) &= 0 & \text{on } \Omega.
\end{align*}
\]  

(2.12)

We note that part c) of the above definition gives the uniqueness of the heat kernel for $\mathcal{H}$.  

2.5 Basic Assumptions and their consequences  

H1. We assume $\Omega \subset \mathbb{R}^n$, $n \geq 2$ is a bounded domain. If $D = \partial \Omega$, we do not make any further assumption. Otherwise, we assume the $(\varepsilon, \delta)$-condition of Jones [15] for some $\varepsilon, \delta > 0$. For any $x, y \in \Omega$ such that $|x - y| < \delta$, there is a rectifiable arc $\gamma$ joining $x$ to $y$ and satisfying

\[
|\gamma| \leq \frac{1}{\varepsilon}|x - y|, \quad d(z) \geq \frac{c|x - z||y - x|}{|x - y|} \text{ for all } z \text{ on } \gamma,
\]

where $|\gamma|$ denotes the arc length of $\gamma$ and $d(z)$ is the distance from $z$ to the complement of $\Omega$. If $D \neq \partial \Omega$ and $D \neq \emptyset$, we assume further that $D$ has a portion of Lipschitz boundary; i.e. there exist $x_0 \in D$ and a neighborhood $V$ of $x_0$ in $\mathbb{R}^n$ and new orthogonal coordinates $\{y_1, \ldots, y_n\}$ such that $V$ is a hypercube in the new coordinates:

\[
V = \{(y_1, \ldots, y_n) : -a_j < y_j < a_j, 1 \leq j \leq n\};
\]

there exists a Lipschitz continuous function $\phi$ defined in

\[
V' = \{(y_1, \ldots, y_{n-1}) : -a_j < y_j < a_j, 1 \leq j \leq n - 1\};
\]

and such that

\[
|\phi(y')| \leq \frac{a_n}{2} \text{ for every } y' = (y_1, \ldots, y_{n-1}) \in V',
\]

$\Omega \cap V = \{y = (y', y_n) \in V : y_n < \phi(y')\}$,

$D \cap V = \{y = (y', y_n) \in V : y_n = \phi(y')\}$.

In other words, in a neighborhood $V$ of $x_0$, $\Omega$ is below the graph of $\phi$ and $D$ is the graph of $\phi$.

Basically, we introduce the assumption H1 is to guarantee the multiplicative inequality (2.15) and the second Korn inequality (2.18) are available to us. We recall that the following multiplicative inequality holds for any $u$ in $W^{1,2}(\mathbb{R}^n)$ with $n \geq 1$; see [17] Theorem 2.2, p. 62].

\[
||u||_{L^{2+2/(n-2)}(\mathbb{R}^n)} \leq C(n)||Du||_{L^{2/(n-2)}(\mathbb{R}^n)^n},
\]

(2.13)

If we assume H1, then there is an extension operator $E : W^{1,2}(\Omega) \to W^{1,2}(\mathbb{R}^n)$ such that the following holds; see [24] Theorem 8.

\[
||Eu||_{L^{2}(\mathbb{R}^n)} \leq C||u||_{L^{2}(\mathbb{R}^n)}, \quad ||Eu||_{W^{1,2}(\mathbb{R}^n)} \leq C||u||_{W^{1,2}(\Omega)}.
\]

(2.14)
Then by combining (2.14) and (2.13), for any \( u \in \tilde{W}^{1,2}(\Omega; D) \), we obtain
\[
\|u\|_{L^2(\Omega; \mathbb{R}^n)} \leq C\|D\epsilon(u)\|_{L^2(\Omega; \mathbb{R}^n)}^{\alpha/(\alpha+1)}\|\epsilon(u)\|_{L^2(\Omega)}^{2/(\alpha+1)}
\leq C\|\epsilon(u)\|_{L^2(\Omega)}^{\alpha/(\alpha+1)}\|\epsilon(u)\|_{L^2(\Omega)}^{2/(\alpha+1)},
\]
where in the last step we used H1 to apply the Friedrichs inequality (or Poincaré's inequality if \( D = \emptyset \)):
\[
\|u\|_{L^2(\Omega; \mathbb{R}^n)} \leq C\|D\epsilon(u)\|_{L^2(\Omega)}, \quad \forall u \in \tilde{W}^{1,2}(\Omega; D).
\]
We have proved that H1 implies that there is \( \gamma = \gamma(n, \Omega, D) \) such that for any \( u \in \tilde{W}^{1,2}(\Omega; D) \), we have
\[
\|u\|_{L^2(\Omega; \mathbb{R}^n)} \leq \gamma\|D\epsilon(u)\|_{L^2(\Omega)}^{\alpha/(\alpha+1)}\|\epsilon(u)\|_{L^2(\Omega)}^{2/(\alpha+1)}.
\]
If \( u \in \mathcal{F}_2(\Omega \times (a, b)) \) is such that \( u(\cdot, t) \in \tilde{W}^{1,2}(\Omega; D) \) for a.e. \( t \in (a, b) \), where \(-\infty < a < b \leq \infty \), then by (2.15) we have (see [17] pp. 74–75)
\[
\|u\|_{L^2(\Omega; \mathbb{R}^n)} \leq \gamma\|D\epsilon(u)\|_{L^2(\Omega)}. \quad \text{(2.16)}
\]
Another important consequence of the inequality (2.15) is the following: For any \( u \in \mathcal{F}_2(\Omega \times (a, b)) \) with \( b - a < \infty \), we have
\[
\|u\|_{L^2(\Omega; \mathbb{R}^n)} \leq \left(2\gamma + (b-a)^{1/2}\Omega^{-1}\right)^{1/2}\|D\epsilon(u)\|_{L^2(\Omega)}. \quad \text{(2.17)}
\]
We refer to [17] Eq. (3.8), p. 77 for the proof of (2.17). Moreover, H1 implies the following second Korn inequality: (see [8] for the proof)
\[
\|u\|_{W^{1,2}(\Omega)} \leq C\left\{\|u\|_{L^2(\Omega)} + \|\epsilon(u)\|_{L^2(\Omega)}\right\}. \quad \text{(2.18)}
\]
In fact, if \( u \in W^{1,2}(\Omega; \partial\Omega)^n = W^{1,2}_0(\Omega)^n \), we have the first Korn inequality
\[
\|D\epsilon(u)\|_{L^2(\Omega)} \leq 2\|\epsilon(u)\|_{L^2(\Omega)}. \quad \text{(2.21)}
\]
Also, we have the following inequalities for any \( u \in V \):
\[
\|u\|_{W^{1,2}(\Omega)} \leq C\|\epsilon(u)\|_{L^2(\Omega)}. \quad \text{(2.19)}
\]
The inequality (2.19) is obtained by utilizing (2.18) in the proof of [14] Theorem 2.7, p. 21. By (2.19) and (2.21), for any \( u \in V \), we have
\[
\int_{\Omega} \mathcal{R}(u, u) \, dx \geq c\int_{\Omega} |\nabla u|^2 \, dx. \quad \text{(2.20)}
\]

**Lemma 2.21.** Assume H1 and let \( \psi \in L^2(\Omega)^n \) and \( f \in \mathcal{L}_{2,1}(\Omega)^n \), where \( \Omega = \Omega \times (a, b) \) and \(-\infty < a < b < \infty \). Then, there exists a unique weak solution \( u \) in \( \mathcal{F}_{2,1}^{1/2}(\Omega)^n \) of the problem (2.25). Moreover, if we assume that \( \langle \psi, v \rangle = 0 \) for all \( v \in \mathcal{R} \) in the case when \( D = \emptyset \), then there also exists a unique weak solution of the problem (2.10). If \( \|f\|_{L^2(\Omega; \partial\Omega)^n} < \infty \), then the weak solution \( u \) of the problem (2.10) satisfies an energy inequality
\[
\|u\|_{L^2(\Omega; \mathbb{R}^n)} \leq C\left\{\|f\|_{L^2(\Omega; \partial\Omega)^n} + \|\psi\|_{L^2(\Omega)}\right\}, \quad \text{(2.22)}
\]
where \( C \) depends only on \( n, \kappa_1, \kappa_2 \) and the constants appearing in (2.16) and (2.19).
Proof. With the aid of the second Korn inequalities \(2.18\) or \(2.19\), it follows from the standard Galerkin’s method and the energy inequality.

**H2.** There exist \(\mu_0 \in (0, 1)\) and \(A_0 > 0\) such that if \(u\) is a weak solution of \(Lu = 0\) in \(B = B(x_0, r)\), where \(x_0 \in \Omega\) and \(0 < r \leq d(x_0)\), then \(u\) is Hölder continuous in \(\frac{1}{2}B = B(x_0, r/2)\) with an estimate

\[
[u]_{\frac{1}{2}B} \leq A_0 r^{-\mu_0} \left( \int_B |u(y)|^2 \, dy \right)^{1/2}.
\]

Here, we use the notation \(\int_B u \, dx = \frac{1}{|B|} \int_B u \, dx\).

It follows from H2 that a weak solution of \(u_t - Lu\) is also locally Hölder continuous.

**Lemma 2.24.** H2 implies that there exist \(\mu_1 \in (0, \mu_0)\) and \(A_1 > 0\) such that whenever \(u\) is a weak solution in \(\mathcal{V}(Q)^n\) of \(u_t - Lu = 0\) in \(Q = Q_{(x_0, r)}\), where \(x_0 = (x_0, t_0)\in Q\) and \(0 < r \leq d(x_0)\), \(u\) is Hölder continuous in \(\frac{1}{2}Q = Q_{(x_0, r/2)}\) and we have an estimate

\[
[u]_{\frac{1}{2}Q} \leq A_1 r^{-\mu_1} (n+2/2) \|u\|_{L^2(Q)}.
\]

Proof. With the second Korn inequality available to us, the proof is essentially the same as that of \[16\] Theorem 3.3.

Finally, we introduce a condition that was originally considered by Auscher and Tchamitchian \[3\] and is referred to as the Dirichlet property.

**H3.** There exist \(\mu_0 \in (0, 1)\) and \(A_0 > 0\) such that if \(u\) is a weak solution of

\[
\begin{cases}
Lu = 0 & \text{in } B \cap \Omega, \\
u = 0 & \text{on } B \cap \Omega \cap N,
\end{cases}
\]

where \(B = B(x_0, r)\) with \(x_0 \in \Omega\) and \(0 < r \leq \text{diam } \Omega\), then for any \(0 < \rho < r\), we have

\[
\int_{B(x_0, \rho) \cap \Omega} |Du|^2 \, dx \leq A_0 \left( \frac{\rho}{r} \right)^{n+2/2-\mu_0} \int_{B(x_0, \rho) \cap \Omega} |Du|^2 \, dx.
\]

The following lemma says that H3 implies H2. Moreover, it shows that if there is \(\beta > 0\) such that for all \(x_0 \in \Omega\) and \(0 < r \leq \text{diam } \Omega\), we have

\[
|\Omega \cap B(x_0, r)| \geq \beta r^d,
\]

then, weak solutions of \(u_t - Lu = 0\) with homogeneous boundary data are Hölder continuous up to the boundary. It is not hard to check that \((\varepsilon, \delta)\)-domains satisfy condition \(2.26\), so that domains that satisfy H1 also satisfy \(2.26\).

**Lemma 2.27.** Let \(Q = \Omega \times [0, \infty), D = D \times [0, \infty), \) and \(N = N \times [0, \infty).\) If \(\Omega\) satisfies the condition \(2.26\), then H3 implies that there exist \(\mu_1 \in (0, \mu_0)\) and \(A_1 > 0\) such that if \(u\) is a weak solution in \(\mathcal{V}_2(Q \cap Q)^n\) of

\[
\begin{cases}
u_t - Lu = 0 & \text{in } Q \cap Q, \\
u = 0 & \text{on } Q \cap D, \\
\tau(u) = 0 & \text{on } Q \cap N,
\end{cases}
\]

(2.28)
\[ \rho^{\theta_1} u_{\rho^{\theta_1} Q_q} + |u|_{\rho^{\theta_1} Q_q} \leq A_1 r^{-\theta_1} \|u\|_{L^2(Q_q)}. \] (2.29)

**Proof.** See Appendix 6.1

3 **Main theorems**

**Theorem 3.1.** Assume the conditions \( H1 \) and \( H2 \). Then there exists a unique heat kernel \( K(x,y,t) \) for (MP). It satisfies the symmetry relation

\[ K(x,y,t) = K(y,x,t)^T \] (3.2)

and thus by (2.12), for any \( f \in \mathcal{C}_c^\infty(\Omega \times [0,\infty))^n \), we have

\[ u(x,t) = \int_0^t \int_{\Omega} K(x,y,t-s)f(y,s) \, dy \, ds \] (3.3)

is, for any \( T > 0 \), a unique weak solution in \( \mathcal{X}_2^{1,0}(\Omega \times (0,T))^n \) of the problem (2.12).

Also, for \( \psi \in L^2(\Omega)^n \), the function \( u \) given by

\[ u(x,t) = \int_{\Omega} K(x,y,t)\psi(y) \, dy \] (3.4)

is, for any \( T > 0 \), a unique weak solution in \( \mathcal{X}_2^{1,0}(\Omega \times (0,T))^n \) of the problem

\[
\begin{cases}
  u_t - Lu = 0 & \text{in } \Omega \times (0,T), \\
  u = 0 & \text{on } D \times (0,T), \\
  \tau(u) = 0 & \text{on } N \times (0,T), \\
  u(\cdot,0) = \psi & \text{on } \Omega
\end{cases}
\] (3.5)

and if \( \psi \) is continuous at \( x_0 \in \Omega \) in addition, then

\[
\lim_{(x_0,t_0) \to (x,y,0)} \int_{\Omega} K(x,y,t)\psi(y) \, dy = \psi(x_0). \] (3.6)

Moreover, the following estimates holds for all \( y \in \Omega \), where we use notation \( Q = \Omega \times [0,\infty), \, d_y = d(y), \text{ and } \tilde{Y} = (y,0) \).

1) \( \|K(x,y)\|_{L^p(\Omega \times \{y\})} \leq C_p r^{-\nu(y,t)/p}, \forall r \in (0,d_y), \forall p \in \left[ 1, \frac{n+2}{n-2} \right] \).

2) \( \|(x,t) \in Q : |K(x,y,t)| > \lambda \| \leq C \lambda^{-(2+n)/\alpha}, \forall \lambda > d_y^{\alpha} \).

3) \( \|D_y K(x,y,\cdot)\|_{L^p(\Omega \times \{y\})} \leq C_p r^{-\nu(y,t)/p}, \forall r \in (0,d_y), \forall p \in \left[ 1, \frac{n+2}{n-2} \right] \).

4) \( \|(x,t) \in Q : |D_y K(x,y,t)| > \lambda \| \leq C \lambda^{-(2+n)/\alpha}, \forall \lambda > d_y^{\alpha-1} \).

5) For \( X = (x,t) \in Q \) satisfying \( |X - \tilde{Y}|_\rho < d_y/2 \), we have

\[ |K(x,y,t)| \leq C |X - \tilde{Y}|_{\rho}. \] (3.7)
6) For $X = (x, t)$ and $X' = (x', t')$ in $Q$ satisfying
\[ 2|X' - X|_\infty < |X - \hat{Y}|_\infty < d_y/2, \]
we have
\[ |K(x', y, t') - K(x, y, t)| \leq C|X' - X|_\infty \sup |X - \hat{Y}|_\infty^{\mu_1}, \quad (3.8) \]

In the above, $C$ are constants depending only on $n, \kappa_1, \kappa_2, \Omega, D, \mu_0, A_0$ and $C_p$ depend on $p$ in addition.

**Remark 3.9.** It will be clear from the proof that besides the estimates 1) - 6) in Theorem 3.10 we also have
7) $\|K(x, y, t)\|_{\mathcal{L}^p(Q; Q(y, t))} \leq Cr^{-n/2}, \forall r \in (0, d_y]$.
8) $\|\hat{K}(x, y, t)\|_{\mathcal{L}^p(Q; Q(y, t))} \leq Cr^{-n/2}, \forall r \in (0, d_y]$.

Here, we use the notation
\[ K(x, y, t) = \begin{cases} K(x, y, t) & \text{if } D \neq \emptyset, \\ K(x, y, t) - \pi_R(\delta_t I)(x) & \text{if } D = \emptyset. \end{cases} \]

where $\pi_R$ is as defined in (3.9), see also (5.28). Moreover, if $\Omega$ is such that it admits a bounded linear trace operator from $W^{1,2}(\Omega)$ to $L^2(\partial \Omega)$, then it can be shown that for $f \in L^p_{\eta_1, \eta_2}(\Omega \times (0, T))$ and $g \in L^p_{\eta_3, \eta_4}(N \times (0, T))$, where $\eta_k$ and $\eta_k$ ($k = 1, 2$) are subject to the conditions of [17] Theorem 5.1, p. 170, the function $u$ defined by
\[ u(x, t) = \int_0^t \int_{\Omega} K(x, y, t-s)f(y, s)\,dy\,ds + \int_0^\infty \int_{\Omega} K(x, y, t-s)g(y, s)\,dS_x\,ds \]

is a unique weak solution in $\gamma^1_{2,0}(\Omega \times (0, T))$ of the problem
\[ \begin{cases} \mathcal{L}u = f & \text{in } \Omega \times (0, T) \\ u = 0 & \text{on } D \times (0, T) \\ \tau(u) = g & \text{on } N \times (0, T) \\ u(\cdot, 0) = 0 & \text{on } \Omega. \end{cases} \]

The proof is similar to that of the representation formula (3.9) and is omitted.

**Theorem 3.10.** Assume $H_1$ and $H_3$. There exists the heat kernel $K(x, y, t)$ for $\hat{M}$ and it satisfies all the properties stated in Theorem 3.7. Moreover, for $x, y \in \Omega$ and $t > 0$, we have the Gaussian bound
\[ |K(x, y, t)| \leq \frac{C}{(\sqrt{t} \wedge \text{diam } \Omega)} \exp\left(-\frac{\delta|x - y|^2}{t}\right), \quad (3.11) \]

where $C = C(n, \kappa_1, \kappa_2, \Omega, D, \mu_0, A_0)$ and $\delta = \delta(n, \kappa_2) > 0$. Furthermore, for $X = (x, t)$ and $X' = (x', t')$ in $Q$ satisfying $|X' - X|_\infty < \frac{1}{2} (|X - \hat{Y}|_\infty \wedge \text{diam } \Omega)$, we have
\[ |K(x', y, t') - K(x, y, t)| \leq C \left\{ \frac{|X' - X|_\infty}{|X - \hat{Y}|_\infty \wedge \text{diam } \Omega} \right\}^{\mu_1} \times \frac{1}{(\sqrt{t} \wedge \text{diam } \Omega)^n} \exp\left(-\frac{\delta|x - y|^2}{4t}\right), \quad (3.12) \]

where $\mu_1$ is as in Lemma 2.27.
4 Applications

4.1 Some examples

1. If the coefficients are constant, then it is well known that H2 holds with $\mu_0 = 1$ and $A_0 = A_0(n, \kappa_1, \kappa_2)$. In fact, it is also known that if the coefficients belong to the VMO class, then H2 holds with $\mu_0$ and $A_0$ depending on the BMO modulus of the coefficients as well as on $n, \kappa_1, \kappa_2$. Therefore, the conclusions of Theorem 3.1 are valid in these cases.

2. If the coefficients belong to the VMO class, the domain $\Omega$ is of class $C^1$, and $D \cap \bar{N} = \emptyset$, then it is known that H3 holds. Therefore, the conclusions of Theorem 3.10 are valid in this case.

3. If $n = 2$, then it is well known that H2 holds with $\mu_0 = \mu_0(\kappa_1, \kappa_2)$ and $A_0 = A_0(\kappa_1, \kappa_2)$. Therefore, the conclusions of Theorem 3.1 are valid. In fact, if $\Omega$ is a Lipschitz domain and $D$ is a (possibly empty) set satisfying the corkscrew condition, i.e., for each $x \in \partial D$ (where the boundary is taken with respect to $\partial \Omega$) and $r \in (0, r_0)$, we may find $x_r \in D$ so that $|x - x_r| \leq r$ and $\text{dist}(x_r, \partial \Omega \setminus D) \geq M^{-1}r$, where $r_0 > 0$ and $M > 0$ are constants, then it is known that H3 holds; see [25]. Therefore, the conclusions of Theorem 3.10 are valid in this case.

4.2 Green’s function for the elliptic system

We say that an $n \times n$ matrix valued function $G(x, y)$ is the Green’s function of $L$ for (MP) if it satisfies the following properties:

i) $G(\cdot, y) \in W^{1,1}_{\text{loc}}(\Omega)$ and $G(\cdot, y) \in W^{1,2}(\Omega \setminus B(y, r))$ for all $y \in \Omega$ and $r > 0$. In the case when $D = \emptyset$, we require $\int_\Omega v(x)^T G(x, y) dx = 0$ for any $v \in \mathbb{R}$.

ii) $G(\cdot, y)$ is a weak solution of

\[-L G(\cdot, y) = \delta_y I - T_y \text{ in } \Omega, \quad G(\cdot, y) = 0 \text{ on } D, \quad \tau(G(\cdot, y)) = 0 \text{ on } N\]

in the sense that we have the identity

\[\int_\Omega \sum_{i,j,k} a^{ij}_{ik} \frac{\partial}{\partial x_k} G_{jk}(\cdot, y) \frac{\partial \phi^i}{\partial x_j} dx = \phi(y)\]

for any $\phi = (\phi^1, \ldots, \phi^n)^T \in C^0(\bar{\Omega})^n \cap V$; see (4.4) and (2.6) for the definition of $T_y$ and $V$.

iii) For any $f \in C^\infty_c(\bar{\Omega})^n \cap V$, the function $u$ defined by

\[u(x) = \int_\Omega G(y, x)^T f(y) dy\]

(4.1)

is the weak solution in $V$ of the problem

\[
\begin{align*}
Lu &= f \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } D, \\
\tau(u) &= 0 \quad \text{on } N.
\end{align*}
\]

(4.2)
We note that with aid of the second Korn inequality (2.19), which is valid for any $u \in V$, the unique solvability of the problem (4.2) in the space $V$ is an immediate consequence of Lax-Milgram lemma. We also note that the property iii) of the above definition together with the requirement
\[
\int_{\Omega} v(x)^T G(x, y) \, dx = 0, \quad \forall v \in \mathcal{R}
\]
gives the uniqueness of the Green’s function.

**Theorem 4.3.** Assume the conditions $H1$ and $H2$. Then, there exists a unique Green’s function $G(x, y)$ for (MP). We have
\[
G(y, x) = G(x, y)^T
\]
and thus by (4.1), for any $f \in C^\infty(\overline{\Omega}) \cap V$, we find
\[
u(x) = \int_{\Omega} G(x, y) f(y) \, dy
\]
is a unique weak solution in $V$ of the problem (4.2). If we assume $H3$ instead of $H2$, then for any $x, y \in \Omega$, we have
\[i) \quad n = 2 \quad |G(x, y)| \leq C \left\{ 1 + \ln \left| \frac{\text{diam } \Omega}{|x - y|} \right| \right\}, \quad (4.5)\]
\[ii) \quad n \geq 3 \quad |G(x, y)| \leq C|x - y|^{2-n}, \quad (4.6)\]
and moreover, for any $x, y \in \Omega$ satisfying $|x - x'| < \frac{1}{2}|x - y|$, we have
\[
|G(x', y) - G(x, y)| \leq C|x' - x|^{\mu_1/2}|x - y|^{2-n-\mu_1}. \quad (4.7)
\]
In the above, $C$ is a constant depending on the prescribed parameters and diam $\Omega$ and $\mu_1 \in (0, 1)$ is as in Lemma 2.27.

**Corollary 4.8.** Let $n = 2$ and assume $H1$. Then, there exists the Green’s function for (MP) that satisfies (4.3). Moreover, if $\Omega$ is a Lipschitz domain and $D$ satisfies the corkscrew condition described in Section 4.1 then the estimates 4.5, 4.7 hold, and $\mu_1$ and $C$ are constants determined by $\kappa_1$, $\kappa_2$, $\Omega$, and $D$.

**Proof.** Follows from Example 3 in Section 4.1 and Theorem 4.3.

---

5 Proofs of main theorems

5.1 Proof of Theorem 3.1

In the proof, we denote by $C$ a constant depending on the prescribed parameters $n, \kappa_1, \kappa_2, I_0, A_0$ as well as on $\Omega$ and $D$; if it depends also on some other parameters such as $p$, it will be written as $C_p$, etc.

We fix a $\Phi \in C^\infty(\mathbb{R}^n)$ such that $\Phi$ is supported in $B(0, 1)$, $0 \leq \Phi \leq 2$, and $\int_{\mathbb{R}^n} \Phi = 1$. Let $y \in \Omega$ be fixed but arbitrary. For $\epsilon > 0$, we define
\[
\Phi_{\epsilon, \nu}(x) = e^{-\nu} \Phi((x - y)/\epsilon)
\]
and let \( v_c = v_{c,j,k} \) be a unique weak solution in \( \mathcal{F}^{2,0}_2(\Omega \times (0,T))^n \) of the problem
\[
\begin{aligned}
&u_t - Lu = 0 \quad \text{in } \Omega \times (0,T), \\
u = 0 \quad \text{on } D \times (0,T), \\
\tau(u) = 0 \quad \text{on } N \times (0,T), \\
u(\cdot, 0) = \Phi_{y,j,k} e_k \quad \text{on } \Omega,
\end{aligned}
\tag{5.1}
\]
where \( e_k \) is the \( k \)-th unit column vector in \( \mathbb{R}^n \); see Lemma 2.21. By the uniqueness, we find that \( v_c \) does not depend on a particular choice of \( T \) and thus by setting \( v_c(x, t) = 0 \) for \( t < 0 \) and letting \( T \to \infty \), we may assume the \( v_c \) is defined on the entire \( \Omega \times (-\infty, \infty) \). We define the mollified heat kernel \( K^c(x, y, t) \) to be an \( n \times n \) matrix valued function whose \( k \)-th column is \( v_{c,j,k}(x,t) \); i.e.,
\[
K^c_k(x, y, t) = v^c_k(x,t) = v_{c,j,k}(x,t).
\]
For \( f \in C^\infty(\bar{\Omega} \times (-\infty, \infty))^n \), fix \( a, b \) so that \( a < 0 < b \) and \( \text{supp } f \subset \bar{\Omega} \times (a,b) \). Let \( u \) be a weak solution in \( \mathcal{F}^{2,0}_2(\Omega \times (a,b))^n \) of the backward problem
\[
\begin{aligned}
&-u_t - Lu = f \quad \text{in } \Omega \times (a,b), \\
u = 0 \quad \text{on } D \times (a,b), \\
\tau(u) = 0 \quad \text{on } N \times (a,b), \\
u(\cdot, b) = 0 \quad \text{on } \Omega.
\end{aligned}
\tag{5.2}
\]
Then, it is easy to see that we have
\[
\int_\Omega \Phi_{y,j}(x) u^k(x,0) \, dx = \int_0^b \int_\Omega K^c_k(x,y,t) f(x,t) \, dx \, dt.
\tag{5.3}
\]
Next, we define \( K^c(x, y, t) \) by
\[
K^c(x, y, t) := \begin{cases} 
K^c(x, y, t) & \text{if } D \neq \emptyset, \\
K^c(x, y, t) - 1_{(0,\infty)}(t) \pi_\emptyset(\Phi_{y,j}) & \text{if } D = \emptyset,
\end{cases}
\tag{5.4}
\]
where \( \pi_\emptyset(\Phi_{y,j})(x) \) is an \( n \times n \) matrix whose \( k \)-th column is \( \pi_\emptyset(\Phi_{y,j}) e_k(x) \). We set \( \check{v}_{c,j,k} = v_{c,j,k} \) to be the \( k \)-th column of \( \check{K}^c(x, y, j) \). It is easy to verify that \( \check{v}_{c,j,k} \in V \) for a.e. \( t > 0 \). Therefore, for any \( T > 0 \), it is the weak solution of the problem (see Section 2.3 and Lemma 2.21)
\[
\begin{aligned}
&u_t - Lu = 0 \quad \text{in } \Omega \times (0,T), \\
u = 0 \quad \text{on } D \times (0,T), \\
\tau(u) = 0 \quad \text{on } N \times (0,T), \\
u(\cdot, 0) = \psi_{c,j,k} \quad \text{on } \Omega, \\
u(\cdot, t) = 0 \quad \text{in } V \quad \text{for a.e. } t \in (0,T),
\end{aligned}
\tag{5.5}
\]
where we denote
\[
\psi_{c,j,k} := \begin{cases} 
\Phi_{y,j} e_k & \text{if } D \neq \emptyset, \\
\Phi_{y,j} e_k - \pi_\emptyset(\Phi_{y,j}) e_k & \text{if } D = \emptyset.
\end{cases}
\]
By the energy inequality, we get (see Lemma 2.21)
\[
\|v_{c,j,k}\|_{L^2(\Omega)} \leq C\|\psi_{c,j,k}\|_{L^2(\Omega)} \leq Ce^{-n/2}.
\tag{5.5}
\]
Lemma 5.9. $H_2$ implies that $\tilde{u}$ is a continuous function supported in $Q_*(X_0, R) \subset \Omega \times (-\infty, \infty)$. Fix $b > t_0 + R^2$ and let $\tilde{u}$ be the weak solution of the backward problem

\[
\begin{align*}
-\pi - Lu &= f & &\text{in } \Omega \times (a, b) \\
u &= 0 & &\text{on } D \times (a, b) \\
\tau(u) &= 0 & &\text{on } \gamma \times (a, b) \\
u(t, b) &= 0 & &\text{on } \Gamma \\
u(t, t) &\in V \quad & &\text{for all } t \in (a, b).
\end{align*}
\] (5.6)

The unique solvability of the above problem is similar to Lemma 5.2 and by setting $\tilde{u}(x, t) = 0$ for $t > b$ and letting $a \to -\infty$, we may again assume that $\tilde{u}$ is defined on $\Omega \times (-\infty, \infty)$. Then, similar to (2.22), we have

\[
|\tilde{u}|_{L^2(-\infty, \infty)} \leq C|f|_{L^2(\Omega)}.
\] (5.7)

By using Hölder’s inequality, we derive from (5.7) that

\[
|\tilde{u}|_{L^2(Q_*(X_0, R))} \leq CR^{3n/2}|f|_{L^\infty(Q_*(X_0, R))}.
\]

Then by Lemma 5.9 below and the above estimate, we obtain

\[
|\tilde{u}|_{L^2(Q_*(X_0, R/2))} \leq CR^2|f|_{L^\infty(Q_*(X_0, R))}.
\] (5.8)

**Lemma 5.9.** $H_2$ implies that $\tilde{u}$ is a continuous function supported in $Q_*(X_0, R/2)$ and satisfies the estimate

\[
|\tilde{u}|_{L^2(Q_*(X_0, R/2))} \leq C\left(R^{-(n+2)/2}|\tilde{u}|_{L^2(\Omega)} + R^3|f|_{L^\infty(\Omega)}\right),
\] (5.10)

where we denote $\alpha Q = Q_*(X_0, aR)$. In fact, the same conclusion is true if $\tilde{u}$ is a weak solution in $\mathcal{V}_2(Q)$ of $-u_t - Lu = f$ (or $u_t - Lu = f$) with $f \in L^\infty(Q)$.\]

**Proof.** See Appendix 6.2

Note that similar to (5.2), we have the identity

\[
\int_{\Omega} \Phi_{x, \tau}(u) \tilde{u}^2(x, 0) \, dx = \int_{-\infty}^{\infty} \int_{\Omega} \tilde{K}_\tau^\varepsilon(\cdot, y, \cdot) \tilde{u} \, dX.
\] (5.11)

If $B(y, \varepsilon) \times \{0\} \subset Q_*(X_0, R/2)$, then (5.11) together with (5.2) yields

\[
\left| \int_{Q_*(X_0, R)} \tilde{K}_\tau^\varepsilon(\cdot, y, \cdot) \tilde{u} \, dX \right| \leq |\tilde{u}|_{L^2(Q_*(X_0, R/2))} \leq CR^2 |f|_{L^\infty(Q_*(X_0, R))}.
\]

Therefore, by duality, it follows that we have

\[
|\tilde{K}(\cdot, y, \cdot)|_{L^1(Q_*(X_0, R))} \leq CR^2
\] (5.12)

provided $0 < R < d_y$ and $B(y, \varepsilon) \times \{0\} \subset Q_*(X_0, R/2)$. For $X \in Q$ such that $0 < d := |X - \tilde{X}|_{\partial Q} < d_y/6$, if we set $r = d/3$, $X_0 = (y, -2d^2)$, and $R = 6d$, then it is easy to see that for $\varepsilon < d/3$, we have

\[
B(y, \varepsilon) \times \{0\} \subset Q_*(X_0, R/2), \quad Q_*(X, r) \subset Q_*(X_0, R),
\]

and also that $\tilde{v}_\varepsilon = \tilde{v}_{\varepsilon, \varepsilon, \varepsilon}$ is a weak solution in $\mathcal{V}_2(Q_*(X, r))$ of $u_t - Lu = 0$.\]
\textbf{Lemma 5.13.} H2 implies that for any \( p > 0 \), we have
\[
|u|_{\mathcal{B}_Q} \leq C_p r^{-\frac{n+2}{2}} \|u\|_{\mathcal{E}_Q}.
\]

\textit{Proof.} It follows from the estimate (5.10) in Lemma 5.9 together with a standard argument described in [12, pp. 80–82]. \( \blacksquare \)

Note that by Lemma 5.13 and (5.12), we obtain
\[
|\vartheta_j(x)| \leq C r^{-n/2} \|\vartheta_j\|_{\mathcal{E}_Q} \leq C r^{-n/2} \|\vartheta_j\|_{\mathcal{E}_Q(\Omega, 0, 0)} \leq C d^{-n/2},
\]
That is, for \( X = (x, t) \in Q \) satisfying \( 0 < |X - \hat{Y}|_{G, \rho} < d/6 \), we have
\[
|K^{(c)}(x, y, t)| \leq C|X - \hat{Y}|^{-n/2}_{G, \rho}, \quad \forall \varepsilon \leq \frac{1}{6}|X - \hat{Y}|_{G, \rho}. \quad (5.14)
\]
Next, we claim that for \( 0 < R \leq d/6 \), we have
\[
|K^{(c)}(\cdot, y, \cdot)|_{\mathcal{B}_Q(\hat{Y}, R)} \leq C R^{-n/2}, \quad \forall \varepsilon > 0. \quad (5.15)
\]
To prove (5.15), we only need to consider the case when \( R > 6\varepsilon \). Indeed, if \( R \leq 6\varepsilon \), then (5.5) yields
\[
|K^{(c)}(\cdot, y, \cdot)|_{\mathcal{B}_Q(\hat{Y}, R)} \leq |K^{(c)}(\cdot, y, \cdot)|_{\mathcal{B}_Q(\hat{Y}, 6\varepsilon)} \leq C\varepsilon^{-n/2} \leq C R^{-n/2}.
\]

Fix a cut-off function \( \zeta \in C^\infty_0(Q(\hat{Y}, R)) \) such that
\[
\zeta \equiv 1 \text{ on } Q(\hat{Y}, R/2), \quad 0 \leq \zeta \leq 1, \quad |D_\alpha \zeta| \leq 4R^{-1}, \quad |\zeta| \leq 16R^{-2}. \quad (5.16)
\]
By using the second Korn inequality (2.20) and (5.14), we derive from (5.1) that
\[
\sup_{t \geq 0} \int_{\Omega} |(1 - \zeta)\vartheta_j(x, t)|^2 \, dx + \int_Q |D_j((1 - \zeta)\vartheta_j)|^2 \, dx \, dt
\leq C \int_Q \left( |D_j \zeta|^2 + |(1 - \zeta)\zeta| \right) |\vartheta_j|^2 \, dx \, dt
\leq CR^{-2} \int_{|X - \hat{Y}|_{G, \rho} < R} |X - \hat{Y}|^{-n/2} \, dX \leq CR^{-n}, \quad (5.17)
\]
which implies the desired estimate (5.15). In fact, we obtain from (5.17) that
\[
\|K^{(c)}(\cdot, y, \cdot)\|_{\mathcal{B}_Q} \leq CR^{-n/2}, \quad \forall \varepsilon > 0. \quad (5.18)
\]

We claim that for \( 0 < R \leq d/6 \), we have
\[
\|K^{(c)}(\cdot, y, \cdot)\|_{\mathcal{B}_Q(Q(\hat{Y}, R))} \leq CR^{-n/2}. \quad (5.19)
\]

Indeed, set \( Q_{(1)} := \Omega \times (R^2, \infty) \) and \( Q_{(2)} := (\Omega \setminus \hat{B}(y, R)) \times (0, R^2) \) and note that by (2.16) and (5.13) we have
\[
\|K^{(c)}(\cdot, y, \cdot)\|_{\mathcal{E}_Q(Q_{(1)}, (0, R^2))} \leq C_{\gamma} \|K^{(c)}(\cdot, y, \cdot)\|_{Q_{(1)}} \leq C_{\gamma} R^{-n/2}
\]
and similarly, by (2.17) and (5.13), we have
\[
\|K^{(c)}(\cdot, y, \cdot)\|_{\mathcal{E}_Q(Q_{(2)}, (0, R^2))} \leq C_{\gamma}(2^\gamma + 1)^{\frac{1}{2}} R^{-n/2}.
\]

By combining the above two inequalities, we get (5.19). 

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Lemma 5.20. For any $y \in \Omega$ and $\epsilon > 0$, we have

\[
||X = (x, t) \in Q; ||K^\epsilon(x, y, t) > \lambda || \leq C \lambda^{-\frac{2}{n+2}}, \quad \forall \lambda > d_y^{\frac{n}{2}},
\]
(5.21)

\[
||X = (x, t) \in Q; ||D_x K^\epsilon(x, y, t) > \lambda || \leq C \lambda^{-\frac{n}{n+2}}, \quad \forall \lambda > d_y^{\frac{n+1}{n}}.
\]
(5.22)

Also, for any $y \in \Omega$, $0 < R \leq d_y$, and $\epsilon > 0$, we have

\[
||K^\epsilon(\cdot, y, \cdot)||_{L^p(Q_0, (\tilde{\gamma}, R))] \leq C_p R^{-n+\frac{n}{n+2}} \epsilon, \quad \forall p \in [1, \frac{2n}{n+2}),
\]
(5.23)

\[
||D_x K^\epsilon(\cdot, y, \cdot)||_{L^p(Q_0, (\tilde{\gamma}, R))] \leq C_p R^{-n+\frac{n}{n+2}} \epsilon, \quad \forall p \in [1, \frac{2n}{n+2}).
\]
(5.24)

Proof. We derive (5.23) and (5.24), respectively, from (5.21) and (5.22), which in turn follow from (5.19) and (5.15), respectively; see [3] Lemmas 3.3 and 3.4.

Lemma 5.25. Suppose $[u_i]_{i=0}^\infty$ is a sequence in $L^2(Q)$ such that $\sup_i |u_i|_{Q} \leq A < \infty$, then exists $u \in L^2(Q)$ satisfying $|u|_{Q} \leq A$ and a subsequence $u_{i_j}$ that converges to $u$ weakly in $L^2(Q \times (0, T))$ for any $T > 0$. Moreover, if each $u_i(t, \cdot) \in V$ for a.e. $t \in (0, \infty)$, then we also have $u(t, \cdot) \in V$ for a.e. $t \in (0, \infty)$.

Proof. See [3] Lemma A.1.

The above two lemmas contain all the ingredients for the construction of a function $K(\cdot, y, \cdot)$ such that for a sequence $e_i$, tending to zero, we have

\[
\tilde{K}^\epsilon(\cdot, y, \cdot) \rightharpoonup K(\cdot, y, \cdot) \text{ weakly in } L^1(Q) \times (\tilde{\gamma}, d_y),
\]
(5.26)

where $1 < q \leq 2$ and $\zeta$ is as in (5.16) with $R = \tilde{d}_y/2$, and $T > 0$ is arbitrary. It is routine to check that $K(\cdot, y, \cdot)$ satisfies the same estimates as in Lemma 5.20 as well as (5.15) and (5.19); see [3] Section 4.2. Note that by Lemma 5.25 we have $K(t, y, \cdot) \in V$ for a.e. $t > 0$. We define $K(x, y, t)$ by

\[
K(x, y, t) := \begin{cases} 
\tilde{K}(x, y, t) & \text{if } D \neq \emptyset, \\
\tilde{K}(x, y, t) + 1_{[0, \omega_0]}(t) \pi_R(\tilde{\delta}_y I)(x) & \text{if } D = \emptyset.
\end{cases}
\]
(5.27)

where $\pi_R(\tilde{\delta}_y I)$ is an $n \times n$ matrix valued function whose $k$-th column is

\[
\pi_R(\tilde{\delta}_y I) = \sum_{i=1}^N \alpha^k_i(y) \omega_i,
\]

where $\omega_i = (\alpha^1_i, \ldots, \alpha^n_i)^T \in \mathcal{R}$ and $\int_{\Omega} \alpha_i : \omega_i dx = \delta_{ij}$.

Then, it is easy to see that $K(x, y, t)$ satisfies the estimates 1) - 4) in Theorem 3.1 because $\tilde{K}(x, y, t)$ satisfies all of them as we noted above, and $\mathcal{R}$ is a finite dimensional vector space so that all norms over $\mathcal{R}$ are equivalent. For example, to see the estimate 3) in the theorem holds, observe that

\[
||D_x \omega||_{L^2(Q_0, (\tilde{\gamma}, R))] \leq r||\omega||_{L^2(B(0, R))}||x||_{L^2(B(0, R))}^{\frac{1}{2} - \frac{1}{2}} \leq C r^{-n/2(n+2)} ||\omega||_{L^2(B(0, R))} \leq C(d\Omega)^{1/2} r^{-n-1+(n+2)/p}.
\]
Since \( \tau_R(\partial_x e_k) \in Y \), it is a weak solution of \( u_l - Lu = 0 \). Therefore, by repeating the proof for (5.14), we find \( K(x, y, t) \) satisfies the estimate (3.7), while the estimate (3.8) is obtained from (3.7) and Lemma 2.24. We have thus shown that \( K(x, y, t) \) satisfies all the estimates 1) - 6) in Theorem 3.1.

We now prove that \( K(x, y, t) \) satisfies all the properties stated in Section 2.4 so that it is indeed the heat kernel for \( MP \). First, note that the property a) is clear from 1), 3) in the theorem and 8) in Remark 3.9. To verify the property b), first note that by (5.3) together with (5.26) and (5.27), we have

\[
K^n(\cdot, y, \cdot) \rightarrow K(\cdot, y, \cdot) \text{ weakly in } \mathcal{Y}_0^{1,0}(\hat{\Omega}, (d\tilde{y}, d\epsilon))^{\mathbb{R}^2},
\]

(1 - \( \zeta \))\( K^n(\cdot, y, \cdot) \rightarrow (1 - \zeta)K(\cdot, y, \cdot) \text{ weakly in } \mathcal{Y}_2^{1,0}(\Omega \times (0, T))^{\mathbb{R}^2}, \quad (5.29)

for any \( T > 0 \). Next, suppose \( \phi = (\phi^1, \ldots, \phi^n)^T \) is supported in \( \Omega \times (0, T) \) and note that by (5.1) we have

\[
\int_\Omega \Phi_{\epsilon x}(x) \phi^i(x, 0) \, dx = \int_0^T \int_\Omega -K^\epsilon_{ij}(x, y, t) \frac{\partial}{\partial t} \phi^i(x, t) \, dx \, dt + \int_0^T \int_\Omega \phi^i(x, t) \frac{\partial}{\partial x_j} K^\epsilon_{ij}(x, y, t) \frac{\partial}{\partial x_i} \phi^j(x, t) \, dx \, dt.
\]

By writing \( \phi = \eta \phi + (1 - \eta)\phi_0 \) where \( \eta \in C_c(\Omega) \) satisfying \( \eta = 1 \) on \( Q(\hat{\Omega}, d\epsilon/2) \) and using (5.26) and taking \( \mu \rightarrow \infty \) in the above, we get the identity (2.11); see [3, p. 1662] for the details. To verify the property c), let us denote \( f(x, t) = f(x, -t) \) and let \( \hat{u} \) be a unique weak solution in \( \mathcal{Y}_2^{1,0}(\Omega \times (-T, 0))^{\mathbb{R}^n} \) of the backward problem

\[
\begin{align*}
-u_l - Lu &= \hat{f} & \text{in } \Omega \times (-T, 0) \\
u &= 0 & \text{on } D \times (-T, 0) \\
\tau(u) &= 0 & \text{on } N \times (-T, 0) \\
u(x, 0) &= 0 & \text{on } \Omega.
\end{align*}
\]

By letting \( T \rightarrow \infty \), we may assume that \( \hat{u} \) is defined on \( \Omega \times (-\infty, 0) \). Then, similar to (5.2), for \( t > 0 \), we have

\[
\int_\Omega \Phi_{\epsilon x}(y) \hat{u}^i(y, -t) \, dy = \int_{-t}^0 \int_\Omega K^\epsilon_{ik}(y, s + t) \hat{f}^k(y, s) \, dy \, ds.
\]

We note \( H2 \) implies, similar to Lemma 5.5, that \( \hat{u} \) is continuous in \( \Omega \times (-\infty, 0) \). By writing \( f = \zeta f + (1 - \zeta)f \) and use (5.29) to get

\[
\hat{u}^i(x, -t) = \int_0^t \int_\Omega K^\epsilon_{ik}(y, x, t - s) \hat{f}^k(y, s) \, dx \, ds.
\]

If we set \( u(x, t) = \hat{u}(x, -t) \), then it becomes a weak solution in \( \mathcal{Y}_2(\Omega \times (0, T))^{\mathbb{R}^n} \) of the problem (2.12), and thus by the uniqueness the property c) is confirmed. Therefore, we have shown that \( K(x, y, t) \) is indeed the heat kernel for \( MP \).

Now, we prove the identity (3.2). Let

\[
\hat{K}^\epsilon_{ik}(y, x, s) = \hat{v}^i_{\delta k} (y, s) = \hat{v}^i_{\delta k} (y, t - s) = K^\epsilon_{ik}(y, x, t - s) \quad (5.30)
\]
and
\[ \hat{\phi}_{\varepsilon, \delta}(y, s) = \Phi_{\varepsilon, \delta}(y, t - s), \]
where \( \psi \) and \( \Phi_{\varepsilon, \delta} \) are as above. Observe that \( \hat{\phi}_{\varepsilon, \delta}(y, s) \) is, for any \( -T < t \), a unique weak solution in \( \gamma^{1,2}_0(\Omega \times (-T, t)) \) of the problem
\[
\begin{aligned}
-\nu - Lv &= 0 \quad \text{in } \Omega \times (-T, t) \\
v &= 0 \quad \text{on } D \times (-T, t) \\
\tau(v) &= 0 \quad \text{on } N \times (-T, t) \\
v(x, t) &= \hat{\phi}_{\varepsilon, \delta} e_i \quad \text{on } \Omega.
\end{aligned}
\] (5.31)

Then, similar to (5.2), we have
\[
\int_\Omega \hat{K}^\varepsilon(x, x, t) \Phi_{\varepsilon, x} = \int_\Omega K^\varepsilon(x, y, t) \hat{\phi}_{\varepsilon, x}. \tag{5.32}
\]

By repeating the proof of [3, Lemma 3.5], we obtain (5.32) from (5.31) as well as the following representation of the mollified heat kernel:
\[
K^\varepsilon(x, y, t) = \int_\Omega K(z, x, t) \Phi_{\varepsilon, x}(z) dz.
\]

In particular, by continuity of \( K(x, y, t) \) and (3.2), we have
\[
\lim_{\varepsilon \to 0} K^\varepsilon(x, y, t) = K(x, y, t). \tag{5.33}
\]

Now, we turn to the proof of the formula (5.34). Let \( u \) be the weak solution in \( \gamma^{1,0}_2(\Omega \times (0, T)) \) of the problem (3.5). Let \( X = (x, t) \in \Omega \times (0, T) \) be fixed but arbitrary and let \( \hat{\psi}_\varepsilon = \hat{\phi}_{\varepsilon, \delta} \) be as in (5.30). Then, it follows from the equations (3.5) and (5.34) that for sufficiently small \( \delta \), we have
\[
\int_\Omega \psi(y) \hat{\phi}_{\varepsilon, \delta}(y) dy = \int_\Omega u(y) \hat{\phi}_{\varepsilon, \delta}(y) dy.
\]

Therefore, by using (5.30), we obtain
\[
\int_\Omega K^\varepsilon(y, x, t) \psi(y) dy = \int_\Omega u(y) \hat{\phi}_{\varepsilon, \delta}(y) dy. \tag{5.34}
\]

By (5.15) and (5.33) with \( x \) in place of \( y \), we find by the dominated convergence theorem that
\[
\lim_{\mu \to \infty} \int_\Omega K^\varepsilon(y, x, t) \psi(y) dy = \int_\Omega K_0(y, x, t) \psi(y) dy.
\]

By Lemma (5.9) we find that \( u \) is continuous at \( X = (x, t) \). Therefore, by taking the limit \( \mu \to \infty \) in (5.34) and using (3.2) we obtain (5.4).

Finally, let \( u \) be a weak solution \( \gamma^{1,0}_2(\Omega \times (0, T)) \) of the problem (3.5) and \( \phi \) be a Lipschitz function on \( \Omega \) satisfying \( |\nabla \phi| \leq K \) a.e. for some \( K > 0 \). Denote
\[
l(t) := \int_\Omega \phi^2|u(x, t)|^2 dx.
\]
Then $I(t)$ satisfies for $a.e. \ t > 0$ the differential inequality
\[
I'(t) = -2 \int_\Omega \left\{ 2\phi \mathcal{L}(u, u) + 2e^{2r} |D u| \frac{\partial \phi}{\partial x_a} \right\} dx
\]
\[
\leq \int_\Omega \left\{ -2k_1 e^{2r} |e(u)|^2 + 4k_2 e^{2r} |e(u)||\nabla \phi||u| \right\} dx
\]
\[
\leq \int_\Omega \left\{ -2k_1 e^{2r} |e(u)|^2 + 2k_2 e^{2r} |e(u)|^2 + 2(\kappa_1) K^2 e^{2r} |u|^2 \right\} dx
\]
\[
\leq 2(k_1^2 / \kappa_1) K^2 I(t). \tag{5.35}
\]

Then, by repeating the argument in [3, Section 4.4], we obtain the formula
\[
\left(3.12\right).
\]
The theorem is proved.

### 5.2 Proof of Theorem 3.10

By Lemma 2.27 and a remark preceding it, we observe that the conditions
H1 and H3 imply the condition H2 and the condition (LB) in [4]. Then,
by Theorem 3.1, the heat kernel $K(x, y, t)$ exists and by using (5.35) and
repeating the proof of [4, Theorem 3.1] with $R = \text{diam} \Omega$, we obtain
the Gaussian bound (5.11). Also, by (2.29), for $X = (x, t)$ and $X' = (x', t')$ in $Q$
satisfying
\[
2|X' - X| < r \leq r_0 := |X - \hat{Y}| \wedge \text{diam} \Omega,
\]
we have
\[
|K(x', y, t') - K(x, y, t)| \leq C |X' - X|^\mu_1 r^{-\mu_1 - (n+2)/2} |K(t, y, t)| \| \partial_2 (Q_0 \cap \Omega) \| \tag{5.36}
\]
Note that the estimate (5.11) implies that for $0 < s \leq \text{(diam} \Omega)^2$, we have
\[
|K(z, y, s)| \leq C \left[ |z - y| \wedge \sqrt{s} \right]^{-\mu} . \tag{5.37}
\]
From the above estimate, we obtain the estimate (5.12) by repeating
the proof of [3, Theorem 3.7]. More precisely, we consider the following three
possible cases.

i) Case $|x - y| \leq \sqrt{t} < \text{diam} \Omega$: In this case, we have
\[
r_0 = \sqrt{t} = |X - \hat{Y}| \wedge \frac{|x - y|^2}{t} \leq 1.
\]
If $|X' - X| < r_0 / 8$, then we take $r = r_0 / 4$ in (5.36) and use (5.37) to get
\[
|K(x', y, t') - K(x, y, t)| \leq C |X' - X|^\mu_1 r^{-\mu_1 - (n+2)/2} |K(t, y, t)|
\]
which implies (5.12). If $r_0 / 8 \leq |X' - X| \leq r_0 / 2$, then we have
\[
|x' - y| \leq 3r_0 / 2 \quad \text{and} \quad r_0 / 2 \leq \sqrt{t} \leq r_0 < \text{diam} \Omega
\]
and thus, by (5.11) we get
\[
|K(x', y, t') - K(x, y, t)| \leq |K(x', y, t')| + |K(x, y, t)| \leq Cr_0^n,
\]
which also implies (5.12).
ii) Case $\sqrt{t} < |x - y|$: In this case, $r_0 = |x - y| < \text{diam } \Omega$. Similar to \[4\] Eq. (5.22), for all $(z, s) \in Q(X, r_0/2) \cap Q$, we have

$$|K(z, y, s)| \leq C r^{-n/2} \exp \left( -\theta |x - y|^2 / 4t \right). \quad (5.38)$$

If $|X' - X|_{\mathcal{P}} < r_0/4$, then we take $r = r_0/2$ in (5.36) and use (5.38) to get

$$|K(x', y, t') - K(x, y, t)| \leq C |X' - X|_{\mathcal{P}} r^{-n/2} t^{-n/2} \exp \left( -\theta |x - y|^2 / 4t \right). \quad (5.39)$$

Since $t \geq d^2$, for all $(z, s) \in Q(X, r_0/2) \cap Q$, we have

$$\exp \left( -\theta |z - y|^2 / s \right) \leq e^{\theta / 4} \exp \left( -\theta |x - y|^2 / 2t \right). \quad (5.40)$$

If $|X' - X|_{\mathcal{P}} < r_0/4$, then we take $r = r_0/2$ in (5.36) and use (5.40) to obtain (5.39). If $r_0/4 \leq |X' - X|_{\mathcal{P}} \leq r_0/2$, then by (3.11) and (5.40) to get

$$|K(x', y, t') - K(x, y, t)| \leq C d^{-n} \exp \left( -\theta |x - y|^2 / 2t \right).$$

Therefore, we also obtain (3.12) in this case.

Theorem is proved. \[\Box\]

### 5.3 Proof of Theorem 4.3

Assuming H1 and H2, we construct the Green’s function $G(x, y)$ for MP as follows. Note that (2.19) implies that there is a constant $\theta$ such that for any $u \in V$, we have

$$\|u\|_{L^2(\Omega)} \leq \theta \|u\|_{L^2(\Omega)}. \quad (5.41)$$

By utilizing (5.41) and following the proof of \[7\] Lemma 3.2, we get that for $x, y \in \Omega$ with $x \neq y$, we have

$$\int_0^\infty |K(x, y, t)| dt < \infty,$$

where $K(\cdot, y, \cdot)$ is as in the proof of Theorem 3.1. We then define

$$G(x, y) := \int_0^\infty K(x, y, t) dt. \quad (5.42)$$

Then the symmetry relation (4.4) is an immediate consequence of (3.2) once we show that $G(x, y)$ is the Green’s function. We shall prove below that $G(x, y)$ indeed enjoys the properties stated in Section 4.2. Denote

$$\tilde{K}(x, y, s) = \int_0^s \hat{K}(x, y, t) \, ds$$

so that we have

$$G(x, y) = \lim_{t \to \infty} \tilde{K}(x, y, t).$$
Therefore, we have $f$ by (5.27), the assumption that $f \in L^p(\Omega \times (0,T))$ for any $p \in (1, \frac{4}{d+y})$. The integral (4.1) is absolutely convergent and thus, we also have $f$ show that the property iii) in Section 4.2 also holds. Let $\mathbf{G}$ be defined by the formula (4.1). Recall that columns of $\hat{\mathbf{K}}(t, y, t)$ are members of $V$; see Lemma 5.25. Therefore, in the case when $D = \emptyset$, for any $v \in \mathcal{R}$, we have

$$\int_{\Omega} v(x)^T \mathbf{K}(x, y, t) dx = 0$$

and thus, we also have

$$\int_{\Omega} v(x)^T \mathbf{G}(x, y) dx = 0.$$ 

We have shown that $\mathbf{G}(x, y)$ satisfies the property i) in Section 4.2. For the proof of the property ii) in Section 4.2, we refer to [7, Section 3.2]. Finally, we show that the property iii) in Section 4.2 also holds. Let $f \in C^\infty(\Omega)^n \cap V$ and $u$ be defined by the formula (4.1). The integral (4.1) is absolutely convergent by the property i) of section 4.2. Similarly, Lemma 5.43 implies that

$$v(x, t) := \int_{\Omega} \hat{\mathbf{K}}(x, y, t) f(y) dy$$

is well defined. Observe that

$$v(x, t) = \int_0^t \int_{\Omega} \hat{\mathbf{K}}(x, y, s) f(y) dy ds = \int_0^t \int_{\Omega} \mathbf{K}(x, y, t-s) f(y) dy ds. \quad (5.44)$$

Therefore, we have

$$\lim_{l \to +\infty} v(x, t) = \int_{\Omega} \mathbf{G}(x, y) f(y) dy = u(x) \quad (5.45)$$

$$v(x, t) = \int_{\Omega} \hat{\mathbf{K}}(x, y, t) f(y) dy. \quad (5.46)$$

By [5.27], the assumption that $f \in V$, and (5.44), we find from (5.44) that $v$ is, for any $T > 0$, the weak solution in $L^1(\Omega \times (0,T))^n$ of the problem

$$\begin{cases}
    v_t - L v = f & \text{in } \Omega \times (0,T) \\
    v = 0 & \text{on } D \times (0,T) \\
    \tau(v) = 0 & \text{on } N \times (0,T) \\
    v(\cdot, 0) = 0 & \text{on } \Omega.
\end{cases}$$
Then, by setting \( \phi \) and the representation formula \((3.4)\) implies that \( v \) is, for any \( T > 0 \), the weak solution in \( \mathcal{W}^{1,2}((-\infty, 0) \times \Omega, T) \) of the problem \((3.5)\) with \( \psi = f \). Then, we have (see [17, Eq. (3.42)])

\[
\|v(t, \cdot)\|_{L^2(\Omega)} \leq C e^{-\epsilon t} \|f\|_{L^2(\Omega)}, \quad \forall t > 0.
\] (5.47)

Observe that by \((5.44), (5.46)\), and the assumption that \( f \in V \), we have

\[
\forall (v, t) \in V \quad \text{and} \quad \forall (v, t) \in V \quad \text{for a.e.} \quad t > 0.
\]

Also, note that for any \( \phi = (\phi_1, \ldots, \phi_n) \in V \) and for a.e. \( t > 0 \), we have

\[
\int_{\Omega} a_{ij}^{\alpha \beta} \frac{\partial \phi^i}{\partial x_p} (\cdot, t) \frac{\partial \phi^j}{\partial x_q} \, dx = \int_{\Omega} f^i (\cdot, t) \, dx - \int_{\Omega} \psi (\cdot, t) \phi^i \, dx.
\] (5.48)

Then, by setting \( \phi = v(t, \cdot) \) in \((5.48)\) and using \((5.41)\), for a.e. \( t > 0 \), we have

\[
\|v(t, \cdot)\|_{L^2(\Omega)}^2 \leq C \left( \|f\|_{L^2(\Omega)} C \|v(t, \cdot)\|_{L^2(\Omega)} \right) \leq C \|f\|_{L^2(\Omega)} \|v(t, \cdot)\|_{L^2(\Omega)},
\]

where we have used \((5.41)\). Therefore, by \((2.19)\), for a.e. \( t > 0 \), we have

\[
\|v(t, \cdot)\|_{L^2(\Omega)} \leq C \|f\|_{L^2(\Omega)}.
\]

Then, by the weak compactness of the space \( W^{1,2}(\Omega)^n \) together with the fact that \( V \) is weakly closed in \( W^{1,2}(\Omega)^n \), we find that there is a sequence \( \{v_n\}_{n=1}^\infty \) tending to infinity and \( \tilde{u} \in V \) such that

\[
v(t_n, \cdot) \rightharpoonup \tilde{u} \quad \text{weakly in} \quad W^{1,2}(\Omega)^n.
\]

By \((5.45)\), we must have \( u = \tilde{u} \in V \) and thus, for all \( \phi \in V \), we get

\[
\lim_{n \to \infty} \int_{\Omega} a_{ij}^{\alpha \beta} \frac{\partial \phi^i}{\partial x_p} (t_n, \cdot) \frac{\partial \phi^j}{\partial x_q} \, dx = \int_{\Omega} a_{ij}^{\alpha \beta} \frac{\partial \phi^i}{\partial x_p} \frac{\partial \phi^j}{\partial x_q} \, dx.
\] (5.49)

Then, by \((5.49), (5.47), \) and \((5.48)\), for any \( \phi \in V \), we obtain

\[
\int_{\Omega} a_{ij}^{\alpha \beta} \frac{\partial \phi^i}{\partial x_p} \frac{\partial \phi^j}{\partial x_q} \, dx = \int_{\Omega} f^i \, dx,
\]

which shows \( u \) is a weak solution in \( V \) of the problem \((4.2)\); see the remark that appears above Theorem \((4.3)\). Therefore, we verified that \( G(x, y) \) defined by the formula \((5.42)\) also satisfies the property iii) in Section \((4.2)\) and thus it is indeed the Green’s function for \((MP)\).

Next, we assume \( H3 \) instead of \( H2 \) and proceed to prove the second part of the theorem. In the rest of the proof, we shall denote

\[
d := \text{diam} \, \Omega.
\]

By Theorem \((5.10)\), we have the Gaussian bound \((5.11)\). In particular, for \( X = (x, t) \in Q \) satisfying \( \sqrt{T} \leq \text{diam} \, \Omega \), we have

\[
|K(x, y, t)| \leq C|X - \tilde{Y}|_{\phi}^n.
\] (5.50)
Similar to [4] Eq. (6.17), we have
\[ |\mathbf{K}(x, y, t)| \leq C r^{-n} e^{-\kappa_1 r^2 (0-2t)} \quad t \geq 2r^2, \quad 0 < r \leq d. \]  
(5.51)

We set \( r := \frac{1}{2} \min(\rho, d) \). If \( 0 < |x - y| \leq r \), then by (5.42), we have
\[ |G(x, y)| \leq \int_{0}^{x-y} + \int_{1/2}^{2r} + \int_{2r}^{\infty} |\mathbf{K}(x, y, t)| dt =: I_1 + I_2 + I_3. \]  
(5.52)

It then follows from (5.50) and (5.51) that
\[ I_1 \leq C \int_{0}^{x-y} |x - y|^{-\alpha} dt \leq \mathcal{C}|x - y|^{2-n}, \]
\[ I_2 \leq C \int_{1/2}^{2r} r^{-\alpha} \frac{1}{t^{\alpha/2}} \, dt \leq \begin{cases} \mathcal{C} + C \ln(r/|x - y|) & \text{if } n = 2, \\ \mathcal{C}|x - y|^{4-n} & \text{if } n \geq 3. \end{cases} \]
\[ I_3 \leq C \int_{2r}^{\infty} r^{-n} e^{-\kappa_1 r^2 (0-2t)} \, dt \leq C r^2 r^{-n}. \]

Combining all together we get that if \( 0 < |x - y| \leq r \), then
\[ |G(x, y)| \leq \begin{cases} (1 + (\rho/r)^2 + \ln(r/d) + \ln(d/|x - y|)) & \text{if } n = 2, \\ (1 + (\rho/r)^2)|x - y|^{2-n} & \text{if } n \geq 3. \end{cases} \]  
(5.53)

In the case when \( |x - y| \geq r \), we estimate by (5.50) and (5.51) that
\[
|G(x, y)| \leq \int_{0}^{2r} |\mathbf{K}(x, y, t)| dt + \int_{2r}^{\infty} |\mathbf{K}(x, y, t)| dt \\
\leq C \int_{0}^{2r} r^{-\alpha} \, dt + C \int_{2r}^{\infty} r^{-n} e^{-\kappa_1 r^2 (0-2t)} \, dt \leq Cr^{2-n} + Cr^2 r^{-n}. \]  
(5.54)

By (5.53) and (5.54), we get (4.5) and (4.6). Finally, we turn to the proof of the estimate (4.7). Because we assume H3, the conclusions of Theorem (3.10) are valid. By (3.14) and the definition (5.22), if \( |X - \tilde{Y}|_{\mathcal{D}} \leq d \), then we have
\[ |\mathbf{K}(x', y, t) - \tilde{K}(x, y, t)| \leq |\mathbf{K}(x', y, t) - K(x, y, t)| + C|x - x'| \\
\leq C|x' - x| |X - \tilde{Y}|_{\mathcal{D}}^{\alpha-\mu_1} \quad \text{whenever } |x - x'| < \frac{1}{2} |x - y|. \]  
(5.55)

We claim that for \( 0 < r \leq d \) and \( t > 3r^2 \), we have
\[ |\mathbf{K}(x', y, t) - \tilde{K}(x, y, t)| \leq C|x' - x|^{\alpha-\mu_1} r^{-n-\mu_1} e^{-\kappa_1 r^2 (0-2t^2)} \]  
(5.56)

whenever \( |x - x'| < \frac{1}{2} |x - y| \). Assume the claim (5.55) for the moment. Similar to (5.52), in the case when \( 0 < |x - y| \leq r := \frac{1}{2} \min(\rho, d) \), we get
\[
|G(x', y) - G(x, y)| \leq \\
\int_{0}^{x-y} + \int_{1/2}^{2r} + \int_{2r}^{\infty} |\mathbf{K}(x', y, t) - \tilde{K}(x, y, t)| \, dt =: I_1 + I_2 + I_3. \]
It follows from (5.55) that
\[
I_1 \leq C|x'| - x||^p_1 \int_0^{\infty} |x - y| - n/2|1/2 dt \leq C|x'| - x||^p_1 |x - y|2 - n/2|1/2 .
\]
\[
I_2 \leq C|x'| - x||^p_1 \int_0^{\infty} t - n/2|1/2 dt \leq C|x'| - x||^p_1 |x - y|2 - n/2|1/2 .
\]
Also, by (5.56), we obtain
\[
I_3 \leq C|x'| - x||^p_1 \int_{2r}^{\infty} e^{-y(x', y, t)} dt \leq C|x'| - x||^p_1 \leq C \left( \frac{2r}{t} \right) |x' - x||^p_1 |x - y|2 - n/2|1/2 .
\]
Combining the above estimates together, we obtain (4.7) when \(|x - y| \leq r.
In the case when \(|x - y| \geq r,
by using (5.55) and (5.56), we estimate
\[
|G(x', y) - G(x, y)| \leq \int_0^{\infty} + \int_{y}^{\infty} |\mathcal{K}(x', y, t) - \mathcal{K}(x, y, t)| dt \leq C|x'| - x||^p_1 |x - y|2 - n/2|1/2 + C|x'| - x||^p_1 |x - y|2 - n/2|1/2 .
\]
Therefore, we also obtain (4.7) when \(|x - y| \geq r.
It only remains for us to prove the claim (5.51). The strategy is similar to the proof of (4.12). Note that each column of \(\mathcal{K}(x, y, t)\) is a weak solution in \(\mathcal{L}(Q)\) of \(u_t - Lu = 0\) provided that \(Q \in Q \setminus \{y\} \). Therefore, similar to (5.36), for \(0 < r \leq d\) and \(t > \frac{3r^2}{2}\), we have
\[
|\mathcal{K}(x', y, t) - \mathcal{K}(x, y, t)| \leq C|x'| - x||^p_1 |x - y|2 - n/2|1/2 |\mathcal{K}(x, y, t)|_{L^2(Q \setminus (x, y) \subset Q)}
\]
whenever \(|x - x'| < r/2\). Then by (5.51), we obtain (5.55).  

6 Appendix

6.1 Proof of Lemma 2.27
We first show H3 implies H2. Suppose \(u\) is a weak solution of \(Lu = 0\) in \(B(x_0, r) \subset \Omega\). By a well-known theorem of Morrey [21] Theorem 3.5.2], we have
\[
|u|_{L^2(B(x_0, r/2))}^2 \leq Cr^{2 - 2\omega} \|Du\|_{L^2(B(x_0, r/4))}^2 .
\]
By using the second Korn inequality, we get Caccioppoli’s inequality for \(u\), that is, for any \(\lambda \in \mathbb{R}^n\) and \(0 < \rho \leq r/2\), we have
\[
\int_{B(\rho, \rho)} |Du|^2 dx \leq C\rho^{-2} \left( 1 + \frac{\text{diam } \Omega}{\rho} \right) \int_{B(\rho, 2\rho)} |u - \lambda|^2 dx .
\]
Then, we get the estimate (2.26) from (2.25).
Next, we prove the estimate \( \text{(2.29)} \). We first establish a global version of \([16, \text{Lemma } 4.3]\). For \( \lambda \geq 0 \), we denote by \( L^2(\mathcal{U}) \) the linear space of functions \( u \in L^2(\mathcal{U}) \) such that

\[
\|u\|_{L^2(\mathcal{U})} = \left\{ \sup_{0 < r < \text{diam } \mathcal{U}} r^{-1} \int_{B(x,r) \cap \mathcal{U}} |u|^2 \, dx \right\}^{1/2} < \infty.
\]

**Lemma 6.1.** Let \( f \in L^{2,1}(B \cap \Omega) \), where \( \lambda \geq 0 \) and \( B = B(x_0, R) \) with \( x_0 \in \tilde{\Omega} \) and \( 0 < R < \text{diam } \Omega \). Suppose \( u \) is a weak solution of

\[
\begin{align*}
Lu &= f & \text{in } B \cap \Omega, \\
u &= 0 & \text{on } B \cap D, \\
\tau(u) &= 0 & \text{on } B \cap N.
\end{align*}
\]

If we assume H3, then, for \( 0 \leq \gamma < \gamma_0 := \min(\lambda + 4, n + 2\mu_0) \), we have

\[
\int_{B(x,r) \cap \Omega} |Du|^2 \, dx \leq C \left( \int_{B(x,r) \cap \Omega} |f|^2 \, dx + |\nabla f|^2_{L^2(\Omega)} \right) \quad \text{(6.2)}
\]

uniformly for all \( x \in \frac{1}{2}B \cap \tilde{\Omega} \) and \( 0 < r \leq R/2 \). Here, \( \frac{1}{2}B = B(x_0, R/2) \) and \( C \) is a constant depending only on \( n, \kappa_1, \kappa_2, \mu_0, A_0, \lambda, \gamma, \) and \( \Omega \). We may take \( \gamma = \gamma_0 \) in (6.2) if \( \gamma_0 < n \). Moreover, if \( \gamma < n \), then \( u \in L^{2,1}(\frac{1}{2}B \cap \Omega) \) and

\[
\|u\|_{L^{2,1}(\frac{1}{2}B \cap \Omega)} \leq C \left( \|u\|_{L^2(\Omega)} + \|Du\|_{L^2(\Omega)} + |f|_{L^2(\Omega)} \right). \quad \text{(6.3)}
\]

**Proof.** Let \( x \in B(x_0, R/2) \cap \partial \Omega \) and \( 0 < r \leq r_0 \wedge (R/2) \), where \( r_0 > 0 \) is such that

\[
\Omega \setminus \bigcup_{x_0 \in \mathcal{U}} B(x, r_0) \supset B(y, \delta_0) \quad \text{for some } y \in \Omega \text{ and } \delta_0 > 0. \quad \text{(6.4)}
\]

Denote

\[
\tilde{N} := N \cap B(x, r), \quad \tilde{D} := \partial(B(x, r) \cap \Omega) \setminus \tilde{N}
\]

and let \( \nu \) be a unique weak solution of the problem

\[
Lv = f \quad \text{in } B(x, r) \cap \Omega, \quad \nu = 0 \quad \text{on } \tilde{D}, \quad \tau(\nu) = 0 \quad \text{on } \tilde{N}.
\]

By testing with \( \nu \) and using the Sobolev inequality, we get

\[
\int_{B(x,r) \cap \Omega} |\nu|^2 \, dy \leq C \left( \int_{B(x,r) \cap \Omega} |f|^p \, dy \right)^{1/q} \left( \int_{B(x,r) \cap \Omega} |\nu|^p \, dy \right)^{1/p},
\]

where \( 1/p + 1/q = 1 \) and \( q = 2n/(n + 2) \) if \( n \geq 3 \) and \( q = 2/(\alpha + 1) \) if \( n = 2 \), where \( \alpha \in [\mu_0, 1) \). We extend \( \nu \) to \( \Omega \) by setting \( \nu = 0 \) in \( \Omega \setminus B(x, r) \). Note that \( \nu \in W^{1,2}(\Omega) \) and \( D\nu = 0 \) on \( \Omega \setminus B(x, r) \). By the Sobolev inequality and Hölder’s inequality, we then obtain

\[
\|\nu\|_{L^2(\Omega)} \leq C_{\text{r}} \|f\|_{L^2(\Omega)} \|\nu\|_{W^{1,2}(\Omega)}.
\]

By the assumption \( \text{(6.4)} \), we have

\[
c \|D\nu\|_{L^2(\Omega)} \leq \|\nu\|_{L^2(\Omega)}, \quad \|\nu\|_{W^{1,2}(\Omega)} \leq C \|D\nu\|_{L^2(\Omega)},
\]

where \( C \) is a constant.
for some constants $c, C > 0$ independent of $v, x,$ and $r$. Therefore, we have
\[
\int_{B(x, r) \cap \Omega} |Dv|^2 \, dy \leq Cr^{1+2\alpha} ||f||_{L^2(\Omega)}^{2}. \]

Let $w := u - v$. Then, $w$ is a weak solution of $Lw = 0$ in $B(x, r) \cap \Omega$. For $0 < \rho < r$, we have
\[
\int_{B(x, \rho) \cap \Omega} |Du|^2 \, dy \leq 2 \int_{B(x, \rho) \cap \Omega} |Dv|^2 \, dy + 2 \int_{B(x, \rho) \cap \Omega} |Dw|^2 \, dy \leq Cr^{1+2\alpha} ||f||_{L^2(\Omega)}^{2} + C \frac{2^\alpha}{\rho^{2}} \int_{B(x, \rho) \cap \Omega} |Du|^2 \, dy.
\]
Thus, by [11] Lemma 2.1, p. 86, we obtain (6.2) for $x \in B(x_0, R/2) \cap \partial \Omega$ and $0 < r \leq r_0 \wedge (R/2)$. The general case of the estimate (6.2) is obtained by combining this case and the interior case, which is already covered by [16] Lemma 4.3. The estimate (6.3) is an easy consequence of the estimate (6.2) and the Poincaré's inequality; see [11] Proposition 1.2, p. 68.

Let $u$ be a weak solution in $\mathcal{Y}_2(Q \cap Q^e)$ of (2.28), where $Q = Q_{(X_0, r)}$ with $X_0 = (x_0, t_0) \in Q$ and $0 < r \leq \sqrt{t_0} \wedge \text{diam} \Omega$. Then, we have (see [16] Lemma 4.2)
\[
\text{ess sup}_{b_0 - (r/2)^2 \leq r \leq b_0} \int_{B(x_0, r^2)^e \cap \Omega} \|u(t, x)\|^2 \, dx \leq Cr^{-6} \int_{Q \cap \Omega} \|u\|^2 \, dX,
\]
\[
\text{ess sup}_{b_0 - (r/2)^2 \leq r \leq b_0} \int_{B(x_0, r^2)^e \cap \Omega} \|D_t u(t, x)\|^2 \, dx \leq Cr^{-4} \int_{Q \cap \Omega} \|D_t u\|^2 \, dX.
\]

Also, we have (cf. [4] Lemma 8.6)
\[
\int_{Q \cap \Omega} |u - A|^2 \, dX \leq Cr^2 \int_{Q \cap \Omega} |D_t u|^2 \, dX; \quad A := \frac{1}{f} \int_{Q \cap \Omega} u \, dX.
\]

By using Lemma 6.1 and the above inequalities, we repeat the proof of [16] Theorem 3.3] with obvious modifications to get
\[
\|u\|_{L^2_{1/4 Q \cap Q^e}} \leq A_1 r^{-\alpha/2} \|u\|_{L^2_{1/2 Q \cap Q^e}}. \tag{6.5}
\]

Finally, we show that the above estimate and the condition (2.26) implies
\[
\|u\|_{L^2_{1/4 Q \cap Q^e}} \leq A_1 r^{-(1/2)} \|u\|_{L^2_{1/2 Q \cap Q^e}}. \tag{6.6}
\]

For $Y \in \frac{1}{4} Q \cap Q$ and $Z \in Q$ satisfying $|Z - Y|_{\partial Q} < r/4$, we have
\[
|u(Y)|^2 \leq 2^{1-4\alpha} r^{2\mu_2} \|u\|^2_{L^2_{1/4 Q \cap Q^e}} + 2|u(Z)|^2.
\]

By taking the average over $Z$ and using (6.3), we get
\[
|u(Y)|^2 \leq \left(2^{1-4\alpha} A_1^2 + \beta^{-1} 2^{2\alpha+5} \right) r^{-\alpha-2} \|u\|^2_{L^2_{1/2 Q \cap Q^e}}.
\]

By adopting a covering argument, we obtain (6.6), and thus (2.29).
6.2 Proof of Lemma 5.9

Recall that \( \tilde{u} \) is the weak solution of the problem (5.6), where \( f \) is understood as an element of \( L^1((a, b); V') \). We define

\[
\tilde{f}(x, t) := f(x, t) - \pi_R[f(t)](x), \quad (x, t) \in Q := \Omega \times (a, b).
\]

Note that \( f = \tilde{f} \) in \( L^1((a, b); V') \). Also, for all \( t \in (a, b) \), we have

\[
\|\pi_R f(t)\|_{L^\infty(\Omega)} \leq C \|\pi_R f(t)\|_{L^2(\Omega)} \leq C\|f(t)\|_{L^2(\Omega)} \leq C\|\Omega\|^{1/2} \|f\|_{L^\infty(Q)},
\]

where we used the fact that all norms in \( R \) are equivalent and that \( \pi_R \) is an orthogonal projection. Therefore, we have

\[
\|\tilde{f}\|_{L^\infty(Q)} \leq C \|f\|_{L^\infty(Q)}.
\]

Let \( u \) be the weak solution in \( \mathcal{V}^{1,0}_2(Q) \) of the problem

\[
\begin{aligned}
-ut - Lu &= \tilde{f} & \text{in } \Omega \times (a, b) \\
u &= 0 & \text{on } D \times (a, b) \\
\tau(u) &= 0 & \text{on } N \times (a, b) \\
u(\cdot, b) &= 0 & \text{on } \Omega.
\end{aligned}
\]

Then it is easy to see that \( u(\cdot, t) \in \mathcal{V} \) for a.e. \( t \in (a, b) \) and thus \( u \) is a weak solution of the problem (5.6). Therefore, by the uniqueness, we conclude that \( \tilde{u} = u \). In particular, \( \tilde{u} \) is a weak solution in \( \mathcal{V}^{1,0}_2(Q) \) of \( -ut + Lu = f \). Because of the inequality (6.7), it is then enough to prove the second part of the lemma and we refer to [3, Section 3.2] for its proof.

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