LOCAL DERIVATIONS OF FINITARY INCIDENCE ALGEBRAS

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Abstract. Let $P$ be a partially ordered set, $R$ a commutative ring with identity and $FI(P, R)$ the finitary incidence algebra of $P$ over $R$. In this note we prove that each $R$-linear local derivation of $FI(P, R)$ is a derivation, which partially generalizes Theorem 3 of [21].

Introduction

Local derivations appeared in the early 90’s in the works by Kadison [13] and Larson-Sourour [18]. Kadison proved in [13, Theorem A] that each local derivation of a von Neumann algebra with values in its dual bimodule is a derivation. Brešar showed in [6] that Theorem A by Kadison remains valid for any normed bimodule. The main result of Larson and Sourour [18] says that the algebra of all bounded operators on a complex infinite-dimensional Banach space has no proper local derivations. An alternative proof of this fact (which also works in the real case) was given in [7]. In the case of 2-local derivations one can even drop the linearity and continuity as was shown by Šemrl in [23].

The incidence algebra $I(P, R)$ of a locally finite preordered set $P$ over a commutative ring $R$ is a classical object in the area of derivations and their generalizations. When $|P| = n < \infty$, the algebra $I(P, R)$ can be seen as a subalgebra of the full matrix algebra $M_n(R)$, and by this reason $I(P, R)$ is sometimes called a structural matrix algebra. We would like to note that $M_n(R)$, as well as its subalgebra $T_n(R)$ of upper triangular matrices over $R$, are particular cases of $I(P, R)$. On the other hand, if $P$ is finite and connected with $|P| \geq 2$, then $I(P, R)$ is a triangular algebra [25] (when $P$ is finite, but not necessarily connected, one has $I(P, R) = \bigoplus_{j=1}^k I(P_j, R)$, where $P_1, \ldots, P_k$ are the connected components of $P$, so if each $P_j$ has at least 2 elements, then $I(P, R)$ is a direct sum of triangular algebras). The case of finite $P$ is easier to deal with, since $I(P, R)$ possesses the natural basis formed by matrix units, and it only suffices to study the behavior of a derivation on the elements of the basis (see [19, 20, 9, 12, 21, 4, 5, 11, 8, 27, 1]). In the infinite case the latter does not work (unless one imposes some extra restrictions as in [24]), and some other technique is needed (see [3, 22, 17, 14, 16, 26]).
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Based on an earlier work by Nowicki [19], Nowicki and Nowosad proved in [21, Theorem 3] that each \( R \)-linear local derivation of \( I(P, R) \) is a derivation, provided that \( P \) is a finite preordered set and \( R \) is a commutative ring. Alizadeh and Bitarafan improved a particular case of [21, Theorem 3] by showing in [1, Theorem 3.7] that \( M_n(R) \) has no proper (additive, but not necessarily \( R \)-linear) local derivations with values in \( M_n(M) \), where \( M \) is 2-torsion free central \( R \)-bimodule and \( n \geq 3 \). Applying arguments similar to those used by Nowicki and Nowosad [21], Zhao, Yao and Wang proved in [27, Theorem 2.1] that each local Jordan derivation of \( T_n(R) \) is a derivation.

In this short note, which was inspired by the recent preprint [10] by Courtemanche, Dugas and Herden, we adapt the ideas from [21] to the infinite case using the technique elaborated in [14, 16]. More precisely, we show that each \( R \)-linear local derivation of the finitary incidence algebra \( FI(P, R) \) of an arbitrary poset \( P \) over a commutative unital ring \( R \) is a derivation, giving thus another partial generalization of [21, Theorem 3].

1. Preliminaries

Let \( R \) be a ring. An additive map \( d : R \rightarrow R \) is called a derivation of \( R \), if it satisfies

\[
d(rs) = d(r)s + rd(s)
\]

for all \( r, s \in R \). Each \( a \in R \) defines the derivation \( \text{ad}_a \), given by \( \text{ad}_a(r) = ar - ra \).

A derivation of such a form is called inner. A local derivation [13, 18] of \( R \) is an additive map \( d : R \rightarrow R \), such that for any \( r \in R \) there is a derivation \( d_r \) of \( R \) with \( d(r) = d_r(r) \). Obviously, each derivation of \( R \) is a local derivation of \( R \). Observe also that for any local derivation \( d \) of \( R \) and any idempotent \( e \) of \( R \) one has

\[
d(e) = d_e(e) = d_e(e)e + ed_e(e) = d(e)e + ed(e).
\]

Let \((P, \leq)\) be a partially ordered set and \( R \) a commutative ring with identity. With any pair of \( x \leq y \) from \( P \) associate a symbol \( e_{xy} \) and denote by \( I(P, R) \) the \( R \)-module of formal sums

\[
\alpha = \sum_{x \leq y} \alpha(x, y) e_{xy},
\]

where \( \alpha(x, y) \in R \). If \( x \) and \( y \) run through a subset \( X \) of the ordered pairs \( x \leq y \) in the sum \( (2) \), then it is meant that \( \alpha(x, y) = 0 \) for any pair \( x \leq y \) which does not belong to \( X \).

The sum \( (2) \) is called a finitary series [15], whenever for any pair of \( x, y \in P \) with \( x < y \) there exists only a finite number of \( u, v \in P \), such that \( x \leq u < v \leq y \) and \( \alpha(u, v) \neq 0 \). The set of finitary series, denoted by \( FI(P, R) \), is an \( R \)-submodule of \( I(P, R) \) which is closed under the convolution of the series:

\[
\alpha \beta = \sum_{x \leq y} \left( \sum_{x \leq z \leq y} \alpha(x, z) \beta(z, y) \right) e_{xy}
\]

for \( \alpha, \beta \in FI(P, R) \). Thus, \( FI(P, R) \) is an \( R \)-algebra, called the finitary incidence algebra of \( P \) over \( R \). Moreover, \( I(P, R) \) is a bimodule over \( FI(P, R) \) under \( (3) \).
2. LOCAL DERIVATIONS OF $FI(P, R)$

Given $x \leq y$, we identify $e_{xy}$ with the series $1_R e_{xy} \in FI(P, R)$. Note that
\[ e_{xy} e_{uv} = \delta_{yu} e_{xv}, \]
where $\delta$ is the Kronecker delta. In particular, the elements $e_x := e_{xx}$ are orthogonal idempotents of $FI(P, R)$, and for any $\alpha \in FI(P, R)$ one has
\[ e_x \alpha e_y = \begin{cases} \alpha(x, y) e_{xy}, & x \leq y, \\ 0, & x \not\leq y. \end{cases} \]
We shall also consider the idempotents $e_X := \sum_{x \in X} 1_R e_{xx} \in FI(P, R)$, where $X \subseteq P$.

For any $\alpha \in FI(P, R)$ and $x \leq y$ we define
\[ \alpha^p|_x = \alpha(x, y) e_{xy} + \sum_{x \leq v < y} \alpha(x, v) e_{xv} + \sum_{x < u \leq y} \alpha(u, y) e_{uy}. \]

Observe that the sums in (6) are finite, so $\alpha \mapsto \alpha^p|_x$ is a well-defined map $FI(P, R) \rightarrow FI(P, R)$. Moreover, it is $R$-linear and satisfies
\[ \begin{align*} 
(\alpha^p|_x)^p|_x &= \alpha^p|_x, \\
(e_X)^p|_x &= e_X \cap \{x, y\}. 
\end{align*} \]

The next result is a partial generalization of [14] Lemma 8].

**Lemma 2.1.** For each $R$-linear local derivation $d$ of $FI(P, R)$ and $x \leq y$ one has
\[ d(\alpha)(x, y) = d(\alpha^p|_x)(x, y). \]

**Proof.** We first assume that $d$ is an $R$-linear derivation of $FI(P, R)$. By (5)
\[ d(\alpha(x, y)e_{xy}) = d(e_x \alpha e_y) = d(e_x) \alpha e_y + e_x d(\alpha) e_y + e_x d(e_y), \]
whence
\[ d(\alpha)(x, y) = d(\alpha(x, y)e_{xy})(x, y) - (d(e_x) \alpha)(x, y) - (d(e_y))(x, y). \]

In view of (3) and (9) the right-hand side of (10) is
\[ d((\alpha|_x^p)(x, y)e_{xy})(x, y) - (d(e_x) \alpha|_x^p)(x, y) - (d(\alpha|_x^p)(x, y))(x, y), \]
which is $d(\alpha|_x^p)(x, y)$ by the same (11), whence (9).

Now let $d$ be an $R$-linear local derivation of $FI(P, R)$. Then using the result of the previous case and (7)
\[ d(\alpha)(x, y) = d(\alpha - \alpha|_x^p)(x, y) + d(\alpha|_x^p)(x, y) \\
= d_{\alpha - \alpha|_x^p}((\alpha - \alpha|_x^p)|_x^p)(x, y) + d(\alpha|_x^p)(x, y) \\
= d_{\alpha - \alpha|_x^p}(\alpha|_x^p - (\alpha|_x^p)|_x^p)(x, y) + d(\alpha|_x^p)(x, y) \\
= d(\alpha|_x^p)(x, y), \]
which proves (9). \qed

We shall also need the following lemma which partially generalizes Lemma 1 from [14].
Lemma 2.2. Let \( d \) be an \( R \)-linear local derivation of \( FI(P, R) \) and \( X \subseteq P \). Then for all \( u \leq v \) one has

\[
d(e_X)(u, v) = \begin{cases} 
    d(e_u)(u, v), & \text{if } u \in X \text{ and } v \notin X, \\
    d(e_v)(u, v), & \text{if } u \notin X \text{ and } v \in X, \\
    0, & \text{otherwise.}
\end{cases}
\]  

(11)

Proof. The first two cases of (11), as well as the case \( u, v \notin X \), are immediate consequences of [3] and Lemma 2.4. Now let \( u, v \in X \). Then \( d(e_X)(u, v) = d_{e_X}(e_X)(u, v), \) the latter being zero by [14, Lemma 1]. \( \square \)

Corollary 2.3. Let \( d \) be an \( R \)-linear local derivation of \( FI(P, R) \) and \( x \leq y \). Then

\[
d(e_x)(x, y) = -d(e_y)(x, y).
\]  

(12)

Indeed, if \( x = y \), then \( d(e_x)(x, x) = 0 \) thanks to Lemma 2.2 and if \( x < y \), then \( d(e_x + e_y)(x, y) = d(e_{\{x,y\}})(x, y) = 0 \) by the same reason.

The following fact is a partial generalization of [14, Lemma 2].

Lemma 2.4. Let \( d \) be an \( R \)-linear local derivation of \( FI(P, R) \). Then there exists \( \alpha \in FI(P, R) \) such that \( d(e_x) = ad_{\alpha}(e_x) \) for all \( x \in P \).

Proof. Define

\[
\alpha = \sum_{x \leq y} d(e_y)(x, y)e_{xy} \in I(P, R).
\]

Then \( \alpha e_x = d(e_x)e_x \), and since by (12)

\[
\alpha = -\sum_{x \leq y} d(e_x)(x, y)e_{xy},
\]

one similarly has \( e_x\alpha = -e_xd(e_x) \). So, by (11)

\[
d(e_x) = d(e_x)e_x + e_xd(e_x) = \alpha e_x - e_x\alpha = ad_{\alpha}(e_x).
\]

It remains to prove that \( \alpha \in FI(P, R) \). Suppose that there is an infinite set \( S \) of pairs \((x_i, y_i)\), such that \( x \leq x_i < y_i \leq y \) and \( \alpha(x_i, y_i) \neq 0 \). For each fixed \( u \) there is only a finite number of \( i \) such that \( x_i = u \), as \( d(e_u)(u, y_i) = -\alpha(x_i, y_i) \neq 0 \) for such \( u \) and \( d(e_u) \) is a finitary series. Similarly for each \( v \) there is only a finite number of \( j \) such that \( y_j = v \). Using this observation, similarly to what was done in the proof of [14, Lemma 2], we may construct an infinite \( S' \subseteq S \), such that for any two pairs \((x_i, y_i)\) and \((x_j, y_j)\) from \( S' \) one has \( x_i \neq y_j \). Let \( X = \{x_i \mid (x_i, y_i) \in S'\} \). Note that \( y_i \notin X \) for any \( (x_i, y_i) \in S' \). So, using Lemma 2.2, we have for all \( (x_i, y_i) \in S' \)

\[
d(e_X)(x_i, y_i) = d(e_{X \setminus \{x_i\}} + e_{x_i})(x_i, y_i)
\]

\[
= d(e_{X \setminus \{x_i\}})(x_i, y_i) + d(e_{x_i})(x_i, y_i)
\]

\[
= d(e_{x_i})(x_i, y_i) = -\alpha(x_i, y_i) \neq 0.
\]

This contradicts the fact that \( d(e_X) \in FI(P, R) \). \( \square \)

It follows from Lemma 2.4 that it suffices to describe the local derivations of \( FI(P, R) \) which satisfy

\[
d(e_x) = 0 \tag{13}
\]

for all \( x \in P \).
Lemma 2.5. Let \( d \) be an \( R \)-linear local derivation of \( FI(P,R) \) satisfying (13) for all \( x \in P \). Then there exists \( \sigma \in I(P,R) \), such that
\[
d(\alpha)(x,y) = \sigma(x,y)\alpha(x,y)
\] (14)
for all \( \alpha \in FI(P,R) \) and \( x \leq y \).

Proof. We first show that
\[
d(e_{xy})(u,v) = 0 \text{ for } (u,v) \neq (x,y).
\] (15)
In view of (13), equality (15) is trivial, when \( x = y \). For \( x < y \) observe by Lemma 2.4 that
\[
d(e_{xy})(u,v) = d((e_{xy})^u)(u,v).
\] (16)
The latter may be non-zero in the following two cases:

(i) \( u = x < y \leq v \);
(ii) \( u \leq x < y = v \).

(i) Let \( u = x < y < v \). Notice from (1) that \( e_y + e_{xy} \) is an idempotent of \( FI(P,R) \), so by (1), (13) and (16)
\[
d(e_y + e_{xy})(u,v) = (d(e_y + e_{xy})(e_y + e_{xy}) + (e_y + e_{xy})d(e_y + e_{xy}))(x,v)
\]
\[
= d(e_y + e_{xy})(y,v) = d(e_{xy})(y,v) = 0.
\]

(ii) Let \( u < x < y = v \). Considering the idempotent \( e_x + e_{xy} \in FI(P,R) \), as above we get
\[
d(e_x + e_{xy})(u,v) = (d(e_x + e_{xy})(e_x + e_{xy}) + (e_x + e_{xy})d(e_x + e_{xy}))(u,y)
\]
\[
= d(e_x + e_{xy})(u,x) = d(e_{xy})(u,x) = 0,
\]
completing the proof of (15).

Define
\[
\sigma = \sum_{x \leq y} d(e_{xy})(x,y)e_{xy} \in I(P,R).
\] (17)
Using Lemma 2.4 and (15) and linearity of \( d \) we conclude that
\[
d(\alpha)(x,y) = d(\alpha^u)(x,y) = \alpha(x,y)d(e_{xy})(x,y) = \sigma(x,y)\alpha(x,y).
\]

Lemma 2.6. Let \( d \) be as in Lemma 2.5. Then the corresponding element \( \sigma \in I(P,R) \) given by (17) satisfies
\[
\sigma(x,y) + \sigma(y,z) = \sigma(x,z)
\] (18)
for all \( x \leq y \leq z \).

Proof. Clearly, (18) holds, when \( x = y \) or \( y = z \), thanks to (13) and (17). Suppose that \( x < y < z \) and take
\[
\alpha = e_{xy} + e_{yz} - e_{xz} - e_y.
\] (19)
Then by (19) and (18) and Lemma 2.4 we have
\[
\sigma(x,y) = d(\alpha)(x,y) = d_\alpha(\alpha)(x,y) = d_\alpha(\alpha^u)(x,y) = d_\alpha(e_{xy} - e_y)(x,y),
\]
\[
\sigma(y,z) = d(\alpha)(y,z) = d_\alpha(\alpha)(y,z) = d_\alpha(\alpha^z)(y,z) = d_\alpha(e_{yz} - e_y)(y,z),
\]
\[
-\sigma(x,z) = d(\alpha)(x,z) = d_\alpha(\alpha)(x,z) = d_\alpha(\alpha^z)(x,z) = d_\alpha(e_{xy} + e_{yz} - e_{xz})(x,z).
\]
Adding these equalities, we get
\[
\sigma(x, y) + \sigma(y, z) - \sigma(x, z) = \Delta_\alpha(e_{xy})(x, y) + \Delta_\alpha(e_{yz})(y, z) - \Delta_\alpha(e_{xz})(x, z) 
\] (20)
\[
- \Delta_\alpha(e_y)(x, y) + \Delta_\alpha(e_{yz})(x, z) 
\] (21)
\[
- \Delta_\alpha(e_y)(y, z) + \Delta_\alpha(e_{xy})(x, z). 
\] (22)

Observe that the right-hand side of (20) is zero by [16, Lemma 4]. To show that (21) and (22) are also zero, write
\[
\Delta_\alpha(e_{yz})(x, z) = \Delta_\alpha(e_y e_{yz})(x, z) = (\Delta_\alpha(e_y)e_{yz} + e_y \Delta_\alpha(e_{yz}))(x, z) = \Delta_\alpha(e_y)(x, y),
\]
\[
\Delta_\alpha(e_{xy})(x, z) = \Delta_\alpha(e_{xy}e_y)(x, z) = (\Delta_\alpha(e_{xy})e_y + e_{xy} \Delta_\alpha(e_y))(x, z) = \Delta_\alpha(e_y)(y, z).
\]

\[\square\]

Theorem 2.7. Each $R$-linear local derivation of $FI(P, R)$ is a derivation.

Proof. By Lemmas 2.4–2.6 each $R$-linear local derivation of $FI(P, R)$ is a sum of an inner derivation and a map of the form (14) with $\sigma$ satisfying (18). It is readily checked by a direct application of (3) that such a map (14) is a derivation (see also [14, Lemma 3] for a similar construction).

\[\square\]

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