Analytical expressions of 3 and 4-loop sunrise Feynman integrals and 4-dimensional lattice integrals

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Abstract

In this paper we continue a work started some years ago concerning the identification of the analytical expression of Feynman integrals whose calculation involves multiple elliptic integrals. We rewrite and simplify the analytical expression of an on-shell 3-loop equal-mass self-mass integral found previously. We collect and analyze a number of results on double and triple elliptic integrals. Therefore, we are able, for the first time, to identify a very compact analytical expression for the on-shell 4-loop self-mass integral which is one master integral of the 4-loop electron $g$-2. Moreover, we identify the analytical expressions of some integrals which appear in lattice perturbation theory, and in particular the 4-dimensional generalized Watson integral.

Keywords: Feynman diagram, master integral, elliptic integral, lattice green function, Watson integral.

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1 Introduction

Analytical values of many Feynman diagrams are expressed in the four dimension limit in terms of polylogarithmic functions of various flavours (Nielsen polylogarithms, Harmonic polylogarithms[1], Harmonic sums[2][3], multiple zeta values, Euler sums[4], etc...). But some Feynman integrals cannot be described only in terms of polylogarithms. At two-loop, the discontinuity of the off-shell massive “sunrise” diagram with all different masses is expressed by elliptic functions. At three or more loop the situation worsens. These diagrams contains nested multiple elliptic integrals, for which the current mathematical knowledge is scarce or missing. At this preliminary stage, “experimental mathematics” is the best tool. In other words, high-precision numerical values of the integrals are fitted[5] with various candidate analytical expression until an agreement is found. The equalities are then checked up to hundreds or thousands of digits. In this way the analytical expression is found beyond any doubt. A rigorous proof of the results so obtained may follow later on [6, 7].

In Ref.[5] we calculated a very high-precision value of the 3-loop scalar master integral of the diagram $S_3$ of Fig.1; we were able to fit this value with products of elliptic integrals, verifying the equality with a precision of thousand of digits.

In this paper we update the work on the 3-loop integral, and we rewrite that result in much more compact way by using some identities between elliptic integrals. We consider this 3-loop result as introductory to the real important problem: finding the analytical expression of the 4-loop on-shell “sunrise” diagram $S_4$ of Fig.1.

This is of physical importance, because this diagrams gives the simpler non-trivial master integral of the 4-loop electron $g$-2, of which a whole high-precision numerical calculation is under way[8]. The analytical constants which appear in this master integral very likely will appear in the analytical expression of 4-loop $g$-2.

In this paper, we review and develop the reasoning used to identify the 3-loop integral and we apply them to the 4-loop integrals, identifying, for the first time, its analytical expression. We apply also this procedure to the values of some 4-dimensional lattice integrals, and we found their analytical expression; surprisingly, we discover that they contain the same analytical constants of the 4-loop integrals.

The plan of the paper is the following: In section 2 we simplify the analytical expressions found in Ref.[5] by using identities between elliptic integrals. In section 3 we study the 4-loop “sunrise” integral. We collect a number of results on a “simplified” version of integrals involved. Then we use these results as guide for identifying the analytical expressions suitable for fitting the 4-loop results. In section 4 we show the analytical results found for 4-loop integrals. In section 5 we are able to fit the values of some 4-dimensional lattice integrals with the same analytical constants discovered in the 4-loop integrals. In section 6 we give our conclusions.

2 Three-loop single-scale self-mass integral

2.1 The results of Ref.[5]

In Ref.[5] we considered the Feynman diagram $S_3(p^2, m_1^2, m_2^2, m_3^2, m_4^2, D)$ for equal masses $m_j = 1$, and on shell external momentum (see Fig.1)

$$S_3(-1, 1, 1, 1, 1, D) = \int \frac{[d^D q_1] [d^D q_2] [d^D q_3]}{(q_1^2 + 1)(q_2^2 + 1)(q_3^2 + 1)((p - q_1 - q_2 - q_3)^2 + 1)} , \quad p^2 = -1 ,$$

(1)
Figure 1: Three-loop and four-loop self-mass diagrams.

where

\[ [d^D q] = \frac{d^D q}{\pi^{D/2} \Gamma \left( 3 - \frac{D}{2} \right)}. \]  

(2)

By using an hyperspherical representation for the integral, we found that the value of \( S_3 \) could be expressed as a sum of various double elliptic integrals, whose the simplest was

\[
A_3 = \int_0^\infty \frac{dl}{R(l, -1, -1)} \int_0^\infty \frac{dm}{R(m, l, -1)R(m, -1, -1)} = 2.641 \, 379 \, 476 \, 689 \, 431 \, 349 \ldots ,
\]

(3)

\[ R(x, y, z) = \sqrt{x^2 + y^2 + z^2 - 2xy - 2xz - 2yz}. \]

(4)

We were not able to calculate \( A_3 \) integral in analytical form. Therefore, we evaluated it at very high precision and we tried to fit the numerical value with various kinds of analytical expressions. In Ref.[5] we found that

\[ A_3 = K(w_-)K(w_+), \quad w_\pm = \frac{z_\pm}{z_\pm - 1}, \quad z_\pm = -(2 - \sqrt{3})^4(4 \pm \sqrt{15})^2, \]

(5)

where \( K \) is the first of the two elliptic integrals

\[
K(m) = \int_0^1 \frac{dt}{\sqrt{1 - t^2} \sqrt{1 - mt^2}}, \quad E(m) = \int_0^1 \frac{dt}{\sqrt{1 - t^2}}.
\]

(6)

We were able to fit the values of \( S_3 \) in 2 and 4 dimensions:\footnote{The two-dimensional value (7) was not published in [5].}

\[
S_3(-1, 1, 1, 1, 1, D = 2) = \frac{4\pi}{\sqrt{15}} K(w_-)K(w_+),
\]

(7)

\[
S_3(-1, 1, 1, 1, 1, D = 4 - 2\epsilon) = 2\epsilon^3 + \frac{22}{3}\epsilon^2 + \frac{577}{36}\epsilon - 1 + \frac{4\pi}{\sqrt{15}} \left( \frac{35}{8}\pi + \frac{131}{12} K(w_-)K(w_+) \right)
\]

\[
- \frac{7}{2} \left( E(1 - w_-)E(1 - w_+) + 5E(w_-)E(w_+) \right) + \frac{6191}{216} + O(\epsilon).
\]

(8)

Eq.(5), Eq.(7) and Eq.(8) were checked up to 30000, 40000 and 1200 digits, respectively. Here ends the summary of the results of [5].
2.2 New relations between elliptic integrals

Now, we note that the arguments $w_{\pm}$ of the elliptic integrals are *singular values*. Quite in general, $k_r$ is a called singular value if

$$
\frac{K(1-k_r)}{K(k_r)} = \sqrt{r},
$$

(9)

where $r$ is an integer or rational number. We find that the arguments $w_{\pm}$ of elliptic integrals are

$$
w_- = k_1, \quad w_+ = k_{5/3},
$$

that is

$$
\frac{K(1-w_-)}{K(w_-)} = \sqrt{15}, \quad \frac{K(1-w_+)}{K(w_+)} = \sqrt{\frac{5}{3}},
$$

(10)

$$
\frac{K(w_+)}{K(w_-)} = \sqrt{\frac{15 - \sqrt{5}}{2}}.
$$

(11)

The elliptic integrals of second kind with singular values as arguments of Eq.(8) are reduced with [9]

$$
E(k_r) = \frac{\pi}{4\sqrt{r}K(k_r)} + K(k_r) \left(1 - \frac{\alpha_r}{\sqrt{r}}\right),
$$

(12)

$$
E(1-k_r) = \frac{\pi}{4K(k_r)} + K(k_r)\alpha_r.
$$

(13)

By using the values [9]

$$
\alpha_{15} = \frac{\sqrt{15} - \sqrt{5} - 1}{2}, \quad \alpha_{5/3} = \frac{\sqrt{15} - \sqrt{5} + 1}{6},
$$

(15)

$$
K(k_{15}) = \sqrt{\frac{(\sqrt{5} + 1)P}{240\pi}},
$$

(16)

where

$$
P \equiv \Gamma\left(\frac{1}{15}\right) \Gamma\left(\frac{2}{15}\right) \Gamma\left(\frac{4}{15}\right) \Gamma\left(\frac{8}{15}\right),
$$

(17)

we are able to rewrite Eq.(7) and Eq.(8) in the very compact form

$$
S_3(-1, 1, 1, 1, D = 2) = \frac{P}{40\sqrt{3}}
$$

(18)

$$
S_3(-1, 1, 1, 1, D = 4 - 2\epsilon) = 2\epsilon^{-3} + \frac{22}{3}\epsilon^{-2} + \frac{577}{36}\epsilon^{-1} + \frac{6191}{216} - \frac{14\sqrt{5}\pi^4}{P} - \frac{\sqrt{5}}{900}P + O(\epsilon).
$$

(19)

Eqs.(10)-(15) were also shown by David Broadhurst in his beautiful talk given in Bielefeld[10]. Our unpublished results Eqs.(17)-(19) were shown to David Broadhurst after the talk.
2.3 The path to Eq.(5)

We recall here the points that suggested us the form of Eq.(5); this will useful in the next section. First, we calculated analytically the simplest double elliptic integral:

\[
\int_0^1 \frac{dx}{\sqrt{1-x^2}} \int_0^1 \frac{dy}{\sqrt{1-y^2}\sqrt{1-x^2y^2}} = K^2 \left( \frac{1}{2} \right) = 3.437 592 909 \ldots ,
\]

which factorizes in a square of an elliptic integral.

Then, we observed that in the diagram \( S_3(-1,1,1,1,1) \) the value of \( p^2 \) is not a threshold \((p^2 = -16)\) or pseudothresholds \((p^2 = -4,0)\). We expect that the analytical expression of off-threshold diagrams is somewhat more complicated than that of on-threshold diagrams. So we considered the above graph with a mass changed: \( S_3(-1,1,1,1,2) \). The value of \( p^2 = -1 \) is now on a pseudothreshold (which are \( p^2 = -1,-9,-25 \)). The integral corresponding to Eq.(3) for \( S_3(-1,1,1,1,2) \) is

\[
A_3' = \int_0^\infty \frac{dl}{R(l,-1,-1)} \int_0^\infty \frac{dm}{R(m,l,-4)R(m,-1,-1)} = 1.474 585 992 \ldots .
\]

We were able to fit the numerical value of \( A_3' \) with

\[
A_3' = \frac{1}{\sqrt{3}} \left( \frac{\pi}{2} \right)^2 \sqrt{3} \Gamma^2 \left( \frac{2-\sqrt{3}}{4} \right) = \frac{\Gamma^6 (\frac{7}{8})}{2^{14}\pi^2}.
\]

Subsequently, as we expected the form of \( A_3 \) be more complicated than \( A_3' \), we tried also products of \( K \) with different arguments. Then we found Eq.(5).

3 Four-loop single-scale self-mass integral

Now we consider the single-scale 4-loop self-mass diagram \( S_4(p^2,m_1^2,m_2^2,m_3^2,m_4^2,D) \) of Fig.1, in the case of equal masses \( m_j = 1 \) and on shell external momentum \( p^2 = -1 \). This diagram has 3 master integrals, whose the simplest is

\[
S_4(-1,1,1,1,1,1,D) = \int \frac{[d^D q_1] [d^D q_2] [d^D q_3] [d^D q_4]}{(q_1^2 + 1)(q_2^2 + 1)(q_3^2 + 1)(q_4^2 + 1)((p - q_1 - q_2 - q_3 - q_4)^2 + 1)} .
\]

This is one of the several master integrals which appear in the calculation of 4-loop \( g-2 \).

We note immediately that the value of \( p^2 = -1 \) is already on a pseudothreshold \((p^2 = -1,-9,-25)\) so that we cannot use the trick of mass change used in section 2.3 to obtain a diagram with a simpler analytical structure.

3.1 High-precision numerical values

For a meaningful fit we need an high-precision numerical value of this integral. This can be obtained by using the difference equation method presented in [12][13]. We raise to \( n \) one denominator of Eq.(23)

\[
X_4(n) = \int \frac{[d^D q_1] [d^D q_2] [d^D q_3] [d^D q_4]}{(q_1^2 + 1)^n(q_2^2 + 1)(q_3^2 + 1)(q_4^2 + 1)((p - q_1 - q_2 - q_3 - q_4)^2 + 1)} .
\]
The function \( X_4(n) \) satisfies the fourth-order difference equation

\[
p_1X_4(n+3) + p_2X_4(n+2) + p_3X_4(n+1) + p_4X_4(n) + p_5X_4(n-1) = 24(D-2)^4J^3(1)J(n) ,
\]

\[
p_1 = -768n(n+1)(n+2)(n-2D+5) ,
\]

\[
p_2 = 128n(n+1)(11n^2 + (63 - 26D)n + 11D^2 - 57D + 76) ,
\]

\[
p_3 = 4n\left(-129n^3 + (294D - 588)n^2 + (-148D^2 + 592D - 567)n + 8D^3 - 48D^2 + 46D + 36\right) ,
\]

\[
p_4 = 2\left(-60n^4 + (294D - 468)n^3 + (-485D^2 + 1499D - 1164)n^2
\]

\[
\quad + (308D^3 - 1363D^2 + 2015D - 996)n - 60D^4 + 326D^3 - 658D^2 + 584D - 192\right) ,
\]

\[
p_5 = -(n - D + 1)(n-2D+3)(2n - 3D + 4)(2n - 5D + 8) ,
\]

\[
J(n) = \int \frac{[d^Dq_1]}{(q_1^2 + 1)^n} .
\]

It contains in the r.h.s. the integral obtained from \( S_4 \) by contracting a line, which factorizes into a product of 4 one-loop tadpoles. The solution of Eq.(25) compatible with the large-\( n \) boundary condition \( X_4(n) \propto n^{-D/2} \) can be written as \( X_4(n) = C_1X_1^{HO} + C_2X_2^{HO} + X^{NH} \), where \( X_1^{HO}, X_2^{HO} \) and \( X^{NH} \) are the two solutions of the homogeneous equation compatible with the above behaviour and one particular solution of the nonhomogeneous equation Eq.(25). The constant \( C_1 \) and \( C_2 \) are calculated from the 3-loop self-mass integrals obtained from \( S_4 \) by deleting a line, that is \( S_3 \). The amount of calculations needed to work out and solve the systems of difference equations is high, so that the calculations have been performed by means of an automatic tool, the program SYS described in Ref.[12].

In two dimension one finds

\[
S_4(-1,1,1,1,1,D = 2) = 40.2451219019305821264798187417\ldots
\]

and in the limit \( D \to 4 \)

\[
S_4(-1,1,1,1,1,1,D = 4 - 2\epsilon) = -\frac{5}{2\epsilon^4} - \frac{45}{4\epsilon^3} - \frac{4255}{144\epsilon^2} - \frac{106147}{1728\epsilon}
\]

\[
- 141.72215618664768694996791 - 521.14654568600250441775466\epsilon
\]

\[
- 3347.9933650782886117865341\epsilon^2 - 17951.3774774809944931097622\epsilon^3
\]

\[
- 101753.8165331173182139560386\epsilon^4 + O(\epsilon^5) .
\]

### 3.2 Triple elliptic integrals

By using an hyperspherical representation for the integral, \( S_4(-1,1,1,1,1,1,D = 2) \) and the finite part of \( S_4(-1,1,1,1,1,1,D = 4 - 2\epsilon) \) contains triple elliptic integrals, whose the simplest is

\[
A_4 = \int_0^\infty \frac{dl}{R(l, -1, -1)} \int_0^\infty \frac{dm}{R(m, l, -1)} \int_0^\infty \frac{dr}{R(m, r, -1)R(r, -1, -1)} = 8.749361490\ldots
\]

We prefer to redistribute the arguments of \( R \) functions and to consider the similar integral

\[
\int_0^\infty \frac{dl}{R(l, -1, -1)} \int_0^\infty \frac{dm}{R(m, -1, -1)} \int_0^\infty \frac{dr}{R(r, l, m)R(r, -1, -1)} = i8.749361490\ldots = iA_4 ,
\]
and the companion integral

\[ B_4 = \int_0^\infty \frac{dl}{R(l,-1,-1)} \int_0^\infty \frac{dm}{R(m,-1,-1)} \int_1^{\infty} \frac{dr}{R(r,l,m)R(r,-1,-1)} = 9.607\,815\,129\ldots, \quad (33) \]

where \( r = (\sqrt{l} \pm \sqrt{m})^2 \) are the two zeroes of \( R(r,l,m) \). \( A_4 \) and \( B_4 \) are the 4-loop analogous of the 3-loop constant \( A_3 \), and we have to find their analytical expressions.

### 3.3 Simplifying the problem

First of all we consider the simplest triple elliptic integral which structure similar to Eq.(32)

\[ A = \int_0^1 dx \int_0^1 dy \int_0^1 \frac{dz}{\sqrt{1-x^2} \sqrt{1-y^2} \sqrt{1-z^2} \sqrt{1-x^2 y^2 z^2}} = 4.335\,593\,665\ldots. \quad (34) \]

Changing the limits of integration over \( z \) to the zero of \( 1 - x^2 y^2 z^2 \) we obtain a companion integral analogous to Eq.(33)

\[ B = \int_0^1 dx \int_0^1 dy \int_0^{1/xy} \frac{dz}{\sqrt{z^2 - \frac{1}{x^2} \sqrt{1-x^2 y^2 z^2}}} = 6.997\,563\,016\ldots. \quad (35) \]

We expect that the study of the simpler constants \( A \) and \( B \) can help us to understand the analytical expressions of \( A_4 \) and \( B_4 \). Integrating over \( z \)

\[ A = \int_0^1 dx \int_0^1 dy \frac{d}{\sqrt{1-x^2}} K(x^2 y^2), \quad (36) \]

\[ B = \int_0^1 dx \int_0^1 dy \frac{d}{\sqrt{1-y^2}} K(1-x^2 y^2). \quad (37) \]

Integrating over \( y \) and \( x \)

\[ A = \int_0^1 dx \frac{d}{\sqrt{1-x^2}} K^2 \left( \frac{1-\sqrt{1-x^2}}{2} \right) = \left( \frac{\pi}{2} \right)^3 \, _4F_3 \left( \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}; 1 \right), \quad (38) \]

\[ B = \int_0^1 dx \frac{d}{\sqrt{1-x^2}} K \left( \frac{1-\sqrt{1-x^2}}{2} \right) K \left( \frac{1+\sqrt{1-x^2}}{2} \right). \quad (39) \]

No analytical expression is known for \( \, _4F_3 \left( \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}; 1 \right) \). Trying to understand the reason of that, we study the following family of integrals:
\[ \int_0^1 \! dt \, K^m(t) \, K^n(1-t) \, \left( \frac{1}{\sqrt{t}} \right)^\alpha \left( \frac{1}{\sqrt{1-t}} \right)^\beta . \] (40)

We consider here only the integrals which have results containing elliptic constants. At level \( m+n = 1 \) there is the integral

\[ \int_0^1 \! dt \, \frac{K(t)}{\sqrt{t(1-t)}} = 2K^2 \left( \frac{1}{2} \right) , \] (41)

equivalent to Eq.(20); we note that it factorizes in a square of \( K(1/2) = \Gamma^2(1/4)/(4\sqrt{\pi}) \).

At level \( m+n = 2 \) we find numerically that the six integrals

\[ \int_0^1 \! dt \, \frac{K^2(t)}{\sqrt{t}} = B , \] (42)

\[ \int_0^1 \! dt \, \frac{K^2(t)}{\sqrt{1-t}} = 2B , \] (43)

\[ \int_0^1 \! dt \, \frac{K^2(t)}{\sqrt{t(1-t)}} = 4A , \] (44)

\[ \int_0^1 \! dt \, \frac{K(t)K(1-t)}{\sqrt{t}} = \int_0^1 \! dt \, \frac{K(t)K(1-t)}{\sqrt{1-t}} = 2A , \] (45)

\[ \int_0^1 \! dt \, \frac{K(t)K(1-t)}{\sqrt{t(1-t)}} = 2B , \] (46)

can be expressed in terms of \( A \) and \( B \). At level \( m+n = 4 \) we find a surprise: six integrals factorizes into the fourth power of \( K(1/2) \).

\[ \int_0^1 \! dt \, K^3(t) = \frac{4}{5} K^4 \left( \frac{1}{2} \right) , \] (47)

\[ \int_0^1 \! dt \, \frac{K^3(t)}{\sqrt{t}} = \frac{6}{5} K^4 \left( \frac{1}{2} \right) , \] (48)

\[ \int_0^1 \! dt \, \frac{K^3(t)}{\sqrt{1-t}} = 4K^4 \left( \frac{1}{2} \right) , \] (49)
\[\int_0^1 dt \, K^2(t)K(1 - t) = \frac{2}{3} K^4 \left( \frac{1}{2} \right), \]  
(50)

\[\int_0^1 dt \, \frac{K^2(t)K(1 - t)}{\sqrt{t}} = \frac{4}{3} K^4 \left( \frac{1}{2} \right), \]  
(51)

\[\int_0^1 dt \, \frac{K^2(t)K(1 - t)}{\sqrt{1 - t}} = 2K^4 \left( \frac{1}{2} \right). \]  
(52)

The analytical results Eqs.(42)-(52) have been fitted numerically and checked up to 200 digits of precision. We find results factorized at level 2 and 4, but not at level 3 (odd). That reminds us of the non-factorization of values of Riemann \(\zeta\)-function at odd integers, and suggest us to consider the constants \(A\) and \(B\) as irreducible objects. We can relate them to a multidimensional \(\zeta\)-like quadruple series. Let us consider the integral

\[I_m = \int_0^1 dt \, \frac{K^2(t)}{\sqrt{1 - t}} \left( \frac{K(1 - t)}{K(t)} \right)^m. \]  
(53)

This integral corresponds to the integrals (43), (45), (42), for \(m = 0, 1, 2\), respectively, and evaluates to \(I_0 = 2B\), \(I_1 = 2A\), \(I_2 = B\). Applying the change of variable from the theory of elliptic functions

\[q = \exp(-\pi K(1 - t)/K(t)) \quad \text{or equivalently} \quad 1 - t = (\theta_4(q)/\theta_3(q))^4\]  
(54)

then

\[I_m = \pi^{2-m} \int_0^1 dq \left( \theta_4^2(q)\theta_3(q) \frac{d}{dq}\theta_3(q) - \theta_3^2(q)\theta_4(q) \frac{d}{dq}\theta_4(q) \right) (-\log q)^m. \]  
(55)

Expanding in series the \(\theta\) functions, and integrating over \(q\) term-by-term, \(I_m\) becomes a quadruple series

\[I_m = m! \pi^{2-m} \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \frac{(-1)^{i+j}(k^2 - i^2)}{(i^2 + j^2 + k^2 + l^2)^{m+1}}; \]  
(56)

for \(m = 2\) the series converges, so that

\[B = 2 \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \frac{(-1)^{i+j}(k^2 - i^2)}{(i^2 + j^2 + k^2 + l^2)^3}; \]  
(57)

where the prime means that the origin \(i = j = k = l = 0\) must be excluded in the summation.

### 3.4 Integrals of products of homogeneous solutions

We come back to the 4-loop integral. If we close the 4-loop self-mass diagram \(S_4\) by connecting together the two external lines, we obtain a 5-loop vacuum diagram. In \(D = 2\) dimensions its value is given by the integral of the square of the 2-loop self-mass diagram \(\int d^2p \, S_2^2(p^2, 1, 1, 1)\). The vacuum
The diagram is expected to have the same analytical structure of $S_4$, but with higher transcendentality. $S_2(u)$ satisfies a second order nonhomogeneous differential equation (see [11] for more details). The corresponding homogeneous differential equation has two solutions $J_1(u)$ and $J_2(u)$. In order to reduce the transcendentality we may substitute $S_2(u)$ with $J_1(u)$ or $J_2(u)$. Their analytical expressions are given in Ref.[11]; for example, if $0 \leq u \leq 1$ they are (see the Appendix of [11]),

\[
J_1(u) = \frac{1}{\sqrt{(1 + \sqrt{u})^3(3 - \sqrt{u})}} K(a(u)) ,
\]

\[
J_2(u) = \frac{1}{\sqrt{(1 + \sqrt{u})^3(3 - \sqrt{u})}} K(1 - a(u)) ,
\]

\[
a(u) = \frac{(1 - \sqrt{u})^3(3 + \sqrt{u})}{(1 + \sqrt{u})^3(3 - \sqrt{u})} .
\] (58)

In Ref.[11] we found that

\[
\int_0^1 du J_1(u) = Cl_2(\pi/3) .
\] (59)

We consider here the integrals of products of $J_i$: we have found up to 2400-digits precision that

\[
\int_0^1 du J_1^2(u) = \frac{1}{8}A_4 ,
\] (60)

\[
\int_0^1 du J_2^2(u) = \frac{3}{4}A_4 ,
\] (61)

\[
\int_0^1 du J_1(u)J_2(u) = \frac{1}{4}B_4 ,
\] (62)

and

\[
\int_1^9 du J_1^2(u) = \frac{3}{8}A_4 ,
\] (63)

\[
\int_1^9 du J_2^2(u) = \frac{9}{8}A_4 ,
\] (64)

\[
\int_1^9 du J_1(u)J_2(u) = \frac{1}{2}B_4 .
\] (65)

Therefore we have identified an one-dimensional integral representations for the numerical constants $A_4$ and $B_4$. 
3.5 The key observation

Eqs. (60)-(65) are not satisfactory as “elementary” definitions of $A_4$ and $B_4$, because of the complexity of the arguments of the $K$ function in Eq. (58). We make a comparison between the integrals of products of $K$ of the family (40) and the integrals of products of $J_i$. Eq. (59) has to be related to

$$\int_0^1 \frac{K(t)}{\sqrt{1-t}} = \frac{\pi^2}{2}$$  (66)

We would like to modify in some way the integral (66) so that the result contains the constant $C_2(\pi/3)$. Our many year experience with the analytical calculation of 3-loop $g$-2 suggests that such a constant appears usually in integrals containing the polynomial $1 + t + t^2$ in the denominator. This is a factor of $1 - t^3$. Therefore we try to consider a “cubic” modification of the usual elliptic integral $K(m)$. One fruitful choice is

$$K_c(m) = \int_0^1 \frac{dt}{\sqrt{(1-t^3)(1-mt^3)}}\text{,} \quad E_c(m) = \int_0^1 \frac{dt}{\sqrt{1-mt^3}}\text{,}$$  (67)

or, equivalently, expressing them in terms of the hypergeometric function

$$K_c(m) = \frac{2\pi}{\sqrt{27}} \, 2F_1\left(\frac{1}{3}, \frac{2}{3}; m\right)\text{,} \quad E_c(m) = \frac{2\pi}{\sqrt{27}} \, 2F_1\left(\frac{1}{3}, -\frac{1}{3}; m\right)\text{.}$$  (68)

Now we may calculate numerical values of the integrals similar to Eq. (40) but with $K$ replaced by $K_c$, and look for relations with the constants $A_4$ and $B_4$. Luckily, we find

$$A_4 = \frac{9}{5} \int_0^1 dx \frac{K_c(x)K_c(1-x)}{\sqrt{1-x}}\text{,}$$  (69)

$$B_4 = \frac{3\sqrt{3}}{4} \int_0^1 dx \frac{K_c^2(x)}{\sqrt{1-x}}\text{.}$$  (70)

We stress the tremendous simplification obtained going from the usual description with elliptic integrals Eqs. (58)-(62) to the “cubic” version Eqs. (68)-(70).

4 Four-loop results

For the sake of brevity we define the following constants

$$C = \int_0^1 dx \frac{K_c^2(x)}{\sqrt{1-x}} = 7.396 \, 099 \, 534 \, 768 \, 919 \, 553 \, 449 \, 114 \, 417 \, 961 \, 526 \, 519 \, 642 \ldots\text{,}$$  (71)

$$D = \int_0^1 dx \frac{K_c(x)K_c(1-x)}{\sqrt{1-x}} = 4.860 \, 756 \, 383 \, 778 \, 595 \, 063 \, 430 \, 474 \, 772 \, 965 \, 586 \, 029 \, 529 \ldots\text{,}$$  (72)

$$E = \int_0^1 dx \frac{E_c^2(x)}{\sqrt{1-x}} = 2.376 \, 887 \, 326 \, 184 \, 666 \, 003 \, 152 \, 855 \, 958 \, 761 \, 330 \, 926 \, 023 \ldots\text{.}$$  (73)
Now we look for relations between the numerical values of the above constants $C$, $D$ and $E$ and the numerical values of $S_4(D = 2)$ Eq.(28) and $S_4(D = 4 - 2\epsilon)$ Eq.(29). We find that

$$S_4(-1, 1, 1, 1, 1, D = 2) = \pi \sqrt{3} C,$$  \hfill (74)

or, alternatively,

$$S_4(-1, 1, 1, 1, 1, D = 2) = \frac{4}{3} \pi B_4;$$  \hfill (75)

we note the appearance of the factor $4\pi /3$, similar to the appearance of $4\pi /\sqrt{15}$ in Eq.(7). By using the integer-relation search program PSLQ[17] we have been able to fit the numerical result of Eq.(29) with the analytical expression

$$S_4(-1, 1, 1, 1, 1, D = 4 - 2\epsilon) = -\frac{5}{2\epsilon^4} - \frac{45}{4\epsilon^3} - \frac{4255}{144\epsilon^2} - \frac{106147}{1728\epsilon} + c_0 + O(\epsilon),$$

$$c_0 = \frac{\pi \sqrt{3}}{240} (297C - 1477E) - \frac{2320981}{20736}.$$  \hfill (76)

The equalities Eq.(74) and Eq.(76) are the main result of this paper; they have been checked up to 2400 digits of precision.

5 Four-dimensional lattice integrals

Considering lattice perturbation theory, at one loop level one finds these integrals[14]

$$Z_0 = \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \frac{1}{\sum_{\lambda=1}^{4} \sin^2(k\lambda/2)} = 0.154 \, 933 \, 390 \, 231 \, 060 \, 214 \ldots,$$  \hfill (77)

and

$$Z_1 = \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \frac{\sin^2(k_1/2) \sin^2(k_2/2)}{\sum_{\lambda=1}^{4} \sin^2(k\lambda/2)} = 0.107 \, 781 \, 313 \, 539 \, 874 \, 001 \ldots,$$  \hfill (78)

Values of $Z_0$ and $Z_1$ with 400 digits of precision are provided in [14]. Some years ago, while visiting the Department of Physics of Parma, York Schröder pointed out to the author that, whether 3-loop $g$-2 was known in analytical form, no analytical result was known for the “simple” lattice 1-loop tadpole $Z_0$. Puzzled by this fact, and noting that $Z_0$ can be reduced to a triple elliptic integral, we have tried to relate the numerical values of $Z_0$ and $Z_1$ to the new constants $C$, $D$ and $E$. Working with only 10-digits precision numbers we have discovered that

$$\frac{S_4(-1, 1, 1, 1, 1, 1, D = 2)}{\pi^4 Z_0} \approx \frac{8}{3}.$$  \hfill (79)

That is

$$Z_0 \pi^3 = \frac{3\sqrt{3}}{8} C.$$  \hfill (80)
By using again PSLQ, we have also found that

\[ Z_1 \pi^3 = -\frac{\sqrt{3}}{20} (3C + 7E) + \frac{\pi^3}{4} - \frac{\pi}{3}. \]  

(81)

We have checked Eq.(80) and Eq.(81) up to 400 digits of precision, the maximum precision of the available numerical values of \( Z_0 \) and \( Z_1 \).

Moreover, the integral (77) can be rewritten into the so-called Watson integral in 4-dimensions (see [15, 16])

\[ u(4) = \frac{4}{(2\pi)^3} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{dk_1 \, dk_2 \, dk_3 \, dk_4}{4 - \cos k_1 - \cos k_2 - \cos k_3 - \cos k_4} = 8 \, Z_0. \]  

(82)

From Eq.(80) we have

\[ u(4) \pi^3 = 3\sqrt{3}C. \]  

(83)

6 Conclusions

In this paper we have at last identified beyond any reasonable doubt the analytical constants which appears in the simplest non-trivial 4-loop \( g^2 \) master integral. We also discovered that the same constants appear in some 4-dimensional lattice integrals. Clearly we still do not have a rigorous proof of these relations, but, once the form of the results is known, we hope that proofs will be easier to find (see the very recent papers[6, 7]).

6.1 Note on Ref.[6]

While we were completing this paper, kindly David Broadhurst sent us a copy of his new paper[6]. In that paper, several elliptic evaluations of Bessel moments are done. In particular, a proof of our Eq.(5) and Eq.(22) is given, as well as of Eq.(44) and Eq.(46). Constants analogous to our \( A \) and \( B \) are found, \( c_{4,0} = 2\pi A \) and \( s_{4,0} = B \), as well as \( t_{6,1} = A_4 / 8 \) and \( s_{6,1} = S_4(D = 2)/16 \). In addition, several relations between Bessel moments are found, and some evaluations of double elliptic integrals are done.

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References

[1] E. Remiddi and J. A. M. Vermaseren, Int. J. Mod. Phys. A15 (2000) 725.

[2] J. A. M. Vermaseren, Int. J. Mod. Phys. A14 (1999) 2037.

[3] J. Blumlein and S. Kurth, Phys. Rev. D60 (1999) 014018.
[4] D. J. Broadhurst and D. Kreimer, *Phys. Lett.* **B393** (1997) 403.

[5] S. Laporta, *Phys. Lett.* **B549** (2002) 115.

[6] D. H. Bailey, J. M. Borwein, D. Broadhurst, M. L. Glasser, preprint arXiv:0801.0891v2 [hep-th].

[7] D. Broadhurst, preprint arXiv:0801.4813v3 [hep-th].

[8] “Numerical calculation of electron g-2 at 4 loops in QED”, talk presented by S. Laporta at 45th course of the International School of Subnuclear Physics, Erice, Sicily, 30 Aug 2007, http://www.ccsem.infn.it/issp2007/docs_NT/laporta_erice.pdf

[9] J. M. Borwein and P. B. Borwein, *Pi and the AGM: A study in analytic number theory and computational complexity*, New York, Wiley, 1987. http://mathworld.wolfram.com/EllipticIntegralSingularValue.html

[10] “Reciprocal PSLQ and the tiny nome of Bologna”, talk given by D. Broadhurst at the International Workshop “Frontiers in perturbative quantum field theory”, Bielefeld, 14 June 2007.

[11] S. Laporta and E. Remiddi, *Nucl. Phys.* **B704** (2005) 349.

[12] S. Laporta, *Int. J. Mod. Phys.* **A15** (2000) 5087.

[13] S. Laporta, *Phys. Lett.* **B504** (2001) 188.

[14] S. Capitani, *Phys. Rept.* **382** (2003) 113.

[15] G. S. Joyce, *J. Phys. A: Math.Gen.* **36** (2003) 911.

[16] M. L. Glasser and A. J. Guttmann, *J. Phys. A: Math. Gen.* **27** (1994) 7011.

[17] H. R. P. Ferguson and D. H. Bailey, RNR Technical Report RNR-91-032.