Locally convex surfaces immersed
in a Killing submersion

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Abstract

We generalize Hadamard-Stoker-Currier Theorems for surfaces immersed in a Killing submersion over a strictly Hadamard surface whose fibers are the trajectories of a unit Killing field. We prove that every complete surface whose principal curvatures are greater than a certain function (depending on the ambient manifold) at each point, must be properly embedded, homeomorphic to the sphere or to the plane and, in the latter case, we study the behavior of the end.

1 Introduction

J. Stoker [S] generalized the result of J. Hadamard [H] that a compact strictly locally convex surface in the Euclidean 3-space $\mathbb{R}^3$ is embedded and homeomorphic to the sphere. Later on, J. Stoker showed that a complete strictly locally convex immersed surface in $\mathbb{R}^3$ must be embedded and homeomorphic to the sphere if it is closed or to the plane if it is open. In the latter case,
the surface is a graph over a planar domain. This result is known currently as the Hadamard-Stoker Theorem.

Also M. Do Carmo and F. Warner [CW] extended Hadamard’s Theorem to the hyperbolic 3-space $\mathbb{H}^3$ assuming the surface is compact and has positive extrinsic curvature. The complete case in $\mathbb{H}^3$ was treated by R. J. Currier in [C]. Currier’s Theorem says that a complete immersed surface in $\mathbb{H}^3$ whose principal curvatures are greater than or equal to one, is embedded and homeomorphic to the sphere if it is closed or to the plane if it is open.

Recently, J. Espinar, J. Gálvez and H. Rosenberg [EGR] extended the Hadamard-Stoker theorem for immersed surfaces in $\mathbb{H}^2 \times \mathbb{R}$ assuming that such a surface is connected, complete and whose extrinsic curvature is positive. More precisely they showed that if $\Sigma$ is a complete connected immersed surface in $\mathbb{H}^2 \times \mathbb{R}$ with positive extrinsic curvature, $K_e > 0$, then $\Sigma$ is properly embedded. Moreover, $\Sigma$ is homeomorphic to $\mathbb{S}^2$ if it is closed or to $\mathbb{R}^2$ if it is open. In the latter case, $\Sigma$ is a graph over a convex domain of $\mathbb{H}^2 \times \{0\}$ or $\Sigma$ has a simple end (we will make explicit the definition of a simple end in Section 3).

We work in this paper on immersed surfaces in Riemannian 3-manifold which fiber over a Riemmanian surface and whose fibers are the trajectories of a unit Killing vector field. The study of immersed surfaces in such a manifold is a topic of increasing interest (see [RST] or [LR] and references therein). In particular, they include the metric product spaces $\mathbb{M}^2 \times \mathbb{R}$ for any Riemmanian surface $\mathbb{M}^2$, the Heisenberg spaces, the Berger spheres or $\widetilde{\text{PSL}}(2, \mathbb{R})$.

As far as we know, locally convex surfaces in $\widetilde{\text{PSL}}(2, \mathbb{R})$ have not yet been studied. We prove a Hadamard-Stoker-Currier type theorem in these spaces (more in the sense of Currier Theorem, i.e., giving conditions on the principal curvatures of the surfaces instead of the extrinsic curvature). Actually, basic problems on locally convex surface remain open, for example, it is not yet known the classification of complete surfaces with positive extrinsic curvature in $\widetilde{\text{PSL}}(2, \mathbb{R})$. Indeed, we do not know the parametrization of the rotational spheres with constant extrinsic curvature in these spaces.

We start by establishing the notation and preliminaries results in Section 2. Section 2 is divided in three parts: the first one focused on Hadamard surfaces, where we set up the basic notation and some Lemmas on foliations by geodesics. Afterwards, we study Riemannian submersions over Riemannian surfaces whose fibers are the trajectories of a unit Killing vector field.
We parametrize such manifolds by two functions κ and τ, where κ is the curvature of the base. τ depends on ξ (see Proposition 2.6). In the last part of Section 2, we study the geometry of vertical cylinders and we set up the necessary concepts we use later on.

Section 3 is devoted to our main results, specifically

**Theorem 3.3** Let Σ ⊂ M(κ, τ) be a complete connected immersed surface so that k_ι(p) > |τ(p)| for all p ∈ Σ, where M(κ, τ) is a strict Hadamard-Killing submersion. Then Σ is properly embedded. Moreover, Σ is homeomorphic to S^2 or to R^2. In the later case, Σ has a simple end (see Definition 3.2) or Σ is a Killing graph over a convex domain of M^2.

The above Theorem 3.3 generalizes [EGR, Theorem 3.1]. Moreover, it can be applied to surfaces in PSL(2, R).

# 2 Preliminaries

## 2.1 On Hadamard surfaces

Here we will remind some definitions and results that we will need later. For more details on Hadamard manifolds with non positive sectional curvature see [E].

Let M^2 be a Hadamard surface, that is, M^2 is a complete, simply connected surface with Gaussian curvature κ ≤ 0.

It is well known that given two points p, q ∈ M^2, there exists an unique geodesic γpq joining p and q. We say that two geodesics γ, β in M^2 are asymptotic if there exists a constant C > 0 such that d(γ(t), β(t)) ≤ C for all t > 0. To be asymptotic is an equivalence relation on the oriented unit speed geodesics or on the set of unit vectors of M^2. We will denote by γ(+∞) and γ(−∞) the equivalence classes of the geodesics t → γ(t) and t → γ(−t) respectively. Moreover, an equivalence class is called point at infinity. M^2(∞) denotes the set of all points at infinity for M^2 and M^2* = M^2 ∪ M^2(∞).

The set M^2* = M^2 ∪ M^2(∞) admits a natural topology, called the cone topology, which makes M^2* homeomorphic to the closed 2−disk in R^2.

Let p, q, r ∈ M^2 so that q and r are distinct from p. Then ∠p(q, r) denotes the angle at p subtended by q and r, that is, ∠p(q, r) is defined to be the
angle between $\gamma'_{pq}(0)$ and $\gamma'_{pr}(0)$, where $\gamma_{pq}$ and $\gamma_{pr}$ are the geodesics joining $p$ to $q$ and $p$ to $r$ respectively. Now, we recall two important results on geodesic triangles.

- **Law of cosines:** Let $p$, $q$ and $r$ be distinct points at $M^2$, and let $a$, $b$, $c$ be the lengths of the sides of the geodesic triangle with vertices $p$, $q$ and $r$. Let $\alpha$, $\beta$ and $\gamma$ denote the angles opposite to the sides of lengths $a$, $b$ and $c$ respectively. Then:
  
  1. $c^2 \geq a^2 + b^2 - 2ab \cos \gamma$;
  2. (Double law of cosines) $c \leq b \cos \alpha + a \cos \beta$.

- **Angle sum theorem:** The sum of the interior angles of a geodesic triangle in any simply connected manifold $M^2$ with $\kappa \leq 0$ is at most $\pi$. Actually, this follows from the Gauss-Bonnet Theorem.

When $M^2$ is a Hadamard surface with sectional curvature bounded above by a negative constant then any two asymptotic geodesics $\gamma, \beta$ satisfy that the distance between the two curves $\gamma_{(t,+\infty)}, \beta_{(t,+\infty)}$ is zero for any $t \in \mathbb{R}$. For each point $p \in M^2$ and $x \in M^2(\infty)$, there is an unique geodesic $\gamma_{px}$ with initial condition $\gamma_{px}(0) = p$ and it is in the equivalence class of $x$. For each point $p \in M^2$ we may identify $M^2(\infty)$ with the circle $S^1$ of unit vectors in $T_pM^2$ by means of the bijection

$$G_p : S^1 \subset T_pM^2 \rightarrow M^2(\infty), \quad v \mapsto \lim_{t \rightarrow +\infty} \gamma_{p,v}(t)$$

where $\gamma_{p,v}$ is the geodesic with initial conditions $\gamma_{p,v}(0) = p$ and $\gamma'_{p,v}(0) = v$. In addition the hypothesis on the sectional curvature (it is bounded above by a negative constant) yields there is an unique geodesic joining two points of $M^2(\infty)$.

Given a set $\Omega \subseteq M^2$, we denote by $\partial_\infty \Omega$ the set $\partial \Omega \cap M^2(\infty)$, where $\partial \Omega$ is the boundary of $\Omega$ for the cone topology. We orient $M^2$ so that its boundary at infinity is oriented counter-clockwise.

Let $\alpha$ be a complete oriented geodesic in $M^2$, then

$$\partial_\infty \alpha = \{\alpha^-\,\alpha^+\}$$

where $\alpha^- = \lim_{t \rightarrow -\infty} \alpha(t)$ and $\alpha^+ = \lim_{t \rightarrow +\infty} \alpha(t)$. Here $t$ is arc length along $\alpha$. We identify $\alpha$ with its boundary at infinity, writing $\alpha = \{\alpha^-, \alpha^+\}$. 4
Definition 2.1 Let $\theta_1$ and $\theta_2 \in M^2(\infty)$, we define the oriented geodesic joining $\theta_1$ and $\theta_2$, $\alpha(\theta_1, \theta_2)$, as the oriented geodesic from $\theta_1 \in M^2(\infty)$ to $\theta_2 \in M^2(\infty)$.

Definition 2.2 Let $\alpha$ a oriented complete geodesic in $\mathbb{M}^2$. Let $J$ be the standard counter-clockwise rotation operator. We call exterior set of $\alpha$ in $\mathbb{M}^2$, $\text{ext}_{\mathbb{M}^2}(\alpha)$, the connected component of $\mathbb{M}^2 \setminus \alpha$ towards which $J\alpha'$ points. The other connected component of $\mathbb{M}^2 \setminus \alpha$ is called the interior set of $\alpha$ in $\mathbb{M}^2$ and denoted by $\text{int}_{\mathbb{M}^2}(\alpha)$.

Now, we establish a Lemma that will be used later.

Lemma 2.3 Let $\mathbb{M}^2$ be a Hadamard surface. Let $p \in \mathbb{M}^2 \setminus \alpha$, where $\alpha$ is a complete geodesic in $\mathbb{M}^2$ and $q \in \alpha$ such that $d(p, q) = d(p, \alpha)$. Let $\beta$ be a complete geodesic joining $p$ to $q$, then $\beta$ intersects orthogonally $\alpha$ in exactly one point. Here, $d$ denotes the distance function.

Proof. Let $r \in \alpha$ and let $\gamma_{pr}$ be the geodesic joining $p$ to $r$. Consider the geodesic triangle of vertices $p, q, r$. Set $\varphi := \angle q(p, r)$, $\theta := \angle p(q, r)$ and $\phi := \angle r(p, q)$ with lengths of opposite sides $a, b, c$ respectively.

Suppose that $\phi \neq \frac{\pi}{2}$ and suppose, for example, $\phi < \frac{\pi}{2}$. On the one hand, from the double law of cosines, we have inequality:

$$c \leq a \cos \theta + b \cos \varphi$$

Moving $r$ towards $p$, we get that $\theta \to 0$ and $\varphi \to \pi - \phi$ as $r \to q$. Therefore we conclude using the above inequality that

$$c \leq a + bm < a,$$

where $m = \cos \varphi < 0$

but is a contradiction, since $c := d(p, q)$ is the minimal distance. \qed

Our next step is to use the above Lemma for proving the following,

Lemma 2.4 Let $\mathbb{M}^2$ be a Hadamard surface with Gaussian curvature $\kappa \leq 0$ and $\alpha$ a complete geodesic in $\mathbb{M}^2$. Let $s$ be the arc length parameter along $\alpha$. Set $S = \bigcup_{s \in \mathbb{R}} \beta_s$, where $\beta$ is the complete geodesic in $\mathbb{M}^2$ orthogonal to $\alpha(s)$ and $\beta_s(0) = \alpha(s)$ for all $s \in \mathbb{R}$. Then, $S$ is a foliation of $\mathbb{M}^2$. 

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Proof. First of all, from Lemma 2.3, we have that $M^2 \subseteq S$. So, we only have to prove that $\beta_{s_0} \cap \beta_{s_1} = \emptyset$ for $s_0 \neq s_1$. Actually, this follows from the Gauss-Bonnet formula.

Assume there exist $p \in \beta_{s_0} \cap \beta_{s_1}$. First, note that $p \notin \alpha$ since $\beta(s_0) \neq \beta(s_1)$. Now, consider the geodesic triangle of vertices $p$, $\beta_{s_0}(0)$ and $\beta_{s_1}(0)$. Since $\beta_{s_0}$ and $\beta_{s_1}$ meet orthogonally to $\alpha$, then the angles $\varphi$ and $\phi$ subtended at $\beta_{s_0}(0)$ and $\beta_{s_1}(0)$ are equal to $\pi/2$ respectively. Moreover, the angle $\theta$ subtended at $p$ is positive. Thus $\varphi + \phi + \theta > \pi$, which contradicts the Angle Sum Theorem. 

Next, we establish another result about foliations on Hadamard surfaces.

**Lemma 2.5** Let $M^2$ be a Hadamard surface with Gaussian curvature $\kappa \leq c < 0$. Let $S = \bigcup_y \alpha(x_0, y)$, where $x_0$ is a fixed point of $\partial_\infty M^2$ and $y \in \partial_\infty M^2 \setminus \{x_0\}$. Then $S$ is a foliation of $M^2$.

**Proof.** It is clear that $M^2 \subseteq S$, so we only need to prove that $\alpha(x_0, y_1) \cap \alpha(x_0, y_2) = \emptyset$ for $y_1 \neq y_2$.

Assume that $r \in \alpha(x_0, y_1) \cap \alpha(x, y_2)$. In this case, we have two distinct geodesics arcs, $\alpha(x_0, y_1)$ and $\alpha(x_0, y_2)$ joining $r$ to $x_0$, a contradiction. 

2.2 On Killing submersions

Now, we establish the definitions and properties of a Riemannian 3-manifold which fiber over a Riemannian surface and whose fibers are the trajectories of a unit Killing vector field.

Let $\mathcal{M}$ be a 3-dimensional Riemannian manifold so that it is a Riemannian submersion $\pi : \mathcal{M} \to M^2$ over a surface $(M^2, g)$ with Gauss curvature $\kappa$, and the fibers, i.e. the inverse image of a point at $M^2$ by $\pi$, are the trajectories of a unit Killing vector field $\xi$, and hence geodesics. Denote by $\langle . , \rangle$, $\nabla$, $\wedge$, $\bar{R}$ and $[ , ]$ the metric, Levi-Civita connection, exterior product, Riemann curvature tensor and Lie bracket in $\mathcal{M}$, respectively. Moreover, associated to $\xi$, we consider the operator $J : \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$ given by

$$JX := X \wedge \xi, \ X \in \mathfrak{X}(\mathcal{M}).$$

Given $X \in \mathfrak{X}(\mathcal{M})$, $X$ is *vertical* if it is always tangent to fibers, and *horizontal* if always orthogonal to fibers. Moreover, if $X \in \mathfrak{X}(\mathcal{M})$, we denote
by \(X^v\) and \(X^h\) the projections onto the subspaces of vertical and horizontal vectors respectively.

Now, we remind the definition of two tensors that appear naturally (see [O] for details) when we have a submersion. Given \(X, Y \in \mathfrak{X}(\mathcal{M})\) we define

\[
\mathcal{T}_X Y = (\nabla_{X^v} Y^v)^h + (\nabla_{X^v} Y^h)^v, \tag{1}
\]

and

\[
\mathcal{A}_X Y = (\nabla_{X^h} Y^h)^v + (\nabla_{X^h} Y^v)^h. \tag{2}
\]

We will not recall the properties of these tensors, we refer the reader to [O] for the properties we will make use.

First of all, we will see how we can associate a function to the ambient manifold \(\mathcal{M}\).

**Proposition 2.6** Let \(\mathcal{M}\) be as above. There exists a function \(\tau : \mathcal{M} \to \mathbb{R}\) so that

\[
\nabla_X \xi = \tau X \wedge \xi, \tag{3}
\]

*Proof.* Set \(X \in \mathfrak{X}(\mathcal{M})\). Since \(\xi\) is a unit Killing field, we have

\[
\langle \nabla_X \xi, X \rangle = 0,
\]

and

\[
\langle \nabla_X \xi, \xi \rangle = \frac{1}{2} X \langle \xi, \xi \rangle = 0.
\]

Thus, for any horizontal \(X \in \mathfrak{X}(\mathcal{M})\), there exist \(\tau_X : \mathcal{M} \to \mathbb{R}\) so that

\[
\nabla_X \xi = \tau_X X \wedge \xi.
\]

Hence, let \(\{X, Y\} \in \mathfrak{X}(\mathcal{M})\) be an orthonormal basis of horizontal vector fields so that \(\det(X, Y, \xi) = 1\), we have

\[
\nabla_X \xi = \tau_X X \wedge \xi \tag{4}
\]

\[
\nabla_Y \xi = \tau_Y Y \wedge \xi. \tag{5}
\]

Hence, it is enough to prove that \(\tau_X = \tau_Y\). Take the scalar product of (4) and \(Y\); and the scalar product of (5) and \(X\), then

\[
\langle \nabla_X \xi, Y \rangle = \tau_X \langle X \wedge \xi, Y \rangle = \tau_X \det(X, \xi, Y)
\]

\[
= - \tau_X \det(X, Y, \xi) = - \tau_X,
\]

\[
\langle \nabla_Y \xi, X \rangle = \tau_Y \langle Y \wedge \xi, X \rangle = \tau_Y \det(Y, \xi, X)
\]

\[
= \tau_Y \det(X, Y, \xi) = \tau_Y.
\]
Since $\xi$ is a Killing vector field, the above equation yields
\[ 0 = \langle \nabla_X \xi, Y \rangle + \langle \nabla_Y \xi, X \rangle = -\tau_X + \tau_Y, \]
thus $\tau_X = \tau_Y$.

Proposition 2.6 makes natural to introduce the following definition:

**Definition 2.7** A Riemannian submersion over a surface $M^2$ whose fibers are the trajectories of a unit Killing vector field $\xi$ will be called Killing submersion and denoted by $M(\kappa, \tau)$, where $\kappa$ is the Gauss curvature of $M^2$ and $\tau$ is given in Proposition 2.6.

Our first task is to compute the sectional curvature $\bar{K}(X,Y)$ of any plane generated by $X, Y \in \mathfrak{X}(M(\kappa, \tau))$. Let $\{X, Y\} \subseteq TM(\kappa, \tau)$ be an orthonormal basis of horizontal vector fields so that $\{X, Y, \xi\}$ is positively oriented. Then
\[ \bar{K}(X,Y) = \kappa - 3\tau^2, \tag{6} \]
\[ \bar{K}(X,\xi) = \tau^2. \tag{7} \]

**Proof.** From [O, Corollary 1], we have
\[ \bar{K}(X,Y) = \kappa - 3\|A_X Y\|^2, \]
and using [O, Lemma 2] we know that $A_X Y = \frac{1}{2}[X,Y]^v$. Thus,
\[
\begin{align*}
\langle A_X Y, \xi \rangle &= \frac{1}{2}\langle [X,Y]^v, \xi \rangle = \frac{1}{2}\langle [X,Y], \xi \rangle = \frac{1}{2}\langle \nabla_X Y, \xi \rangle - \frac{1}{2}\langle \nabla_Y X, \xi \rangle = \\
&= -\frac{1}{2}\langle Y, \nabla_X \xi \rangle + \frac{1}{2}\langle X, \nabla_Y \xi \rangle = -\frac{1}{2}\langle Y, -\tau Y \rangle + \frac{1}{2}\langle X, \tau X \rangle = \\
&= \tau
\end{align*}
\]
where we have used that $\{X,Y,\xi\}$ is positively oriented, i.e., $\nabla_X \xi = -\tau Y$ and $\nabla_Y \xi = \tau X$. So,
\[ A_X Y = \tau \xi, \]
since $A_X Y$ is vertical. Hence, we obtain (6).
Again, [O, Corollary 1] gives
\[ K(X, \xi) = \langle (\nabla_X T) \xi, X \rangle + \|A_X \xi\|^2 - \|T_\xi X\|^2. \]

On one hand, \( A_X \xi = (\nabla_X \xi)^h \), i.e., it is a horizontal vector field. Then,
\[
\langle A_X \xi, X \rangle = \langle \nabla_X \xi, X \rangle = -\tau \langle Y, X \rangle = 0,
\]
\[
\langle A_X \xi, Y \rangle = \langle \nabla_X \xi, Y \rangle = -\tau \langle Y, Y \rangle = -\tau,
\]
that is, \( A_X \xi = -\tau Y \), thus \( \|A_X \xi\|^2 = \tau^2 \).

On the other hand, \( T_\xi X = (\nabla_\xi X)^v \) is vertical, so
\[
\langle T_\xi X, \xi \rangle = \langle \nabla_\xi X, \xi \rangle = -\langle X, \nabla_\xi \xi \rangle = 0,
\]
which implies \( T_\xi X = 0 \), hence \( \|T_\xi X\|^2 = 0 \).

Finally, since \( \nabla_\xi \xi = 0 \) and \( T_Y \xi = 0 \),
\[
(\nabla_X T) \xi, X \rangle = \nabla_X T_\xi \xi - T_\xi \nabla_X \xi - T_\xi \nabla_X \xi
= \nabla_X (\nabla_\xi \xi)^h + \tau T_\xi Y + T_\xi (\tau Y)
= (\nabla_\xi (\tau Y))^v = (\xi(\tau)Y + \tau \nabla_\xi Y)^v
= \tau (\nabla_\xi Y)^v,
\]
we obtain
\[
\langle (\nabla_X T) \xi, X \rangle = 0.
\]

Summarizing, \( \|A_X \xi\|^2 = \tau^2 \), \( \|T_\xi X\|^2 = 0 \), and \( \langle (\nabla_X T) \xi, X \rangle = 0 \), thus (7) follows from the expression of \( \bar{K}(X, \xi) \).

### 2.3 On surfaces in Killing submersions

Let \( \Sigma \subset \mathcal{M}(\kappa, \tau) \) be an oriented immersed connected surface. We endow \( \Sigma \) with the induced metric (First Fundamental Form), \( \langle , \rangle_{|\Sigma} \), in \( \mathcal{M}(\kappa, \tau) \), which we still denote by \( \langle , \rangle \). Denote by \( \nabla \) and \( R \) the Levi-Civita connection and the Riemann curvature tensor of \( \Sigma \) respectively, and \( S \) the shape operator, i.e., \( SX = -\nabla_X N \) for all \( X \in \mathfrak{X}(\Sigma) \) where \( N \) is the unit normal vector field along the surface. Then \( II(X, Y) = \langle SX, Y \rangle \) is the Second Fundamental Form of \( \Sigma \). Moreover, we denote by \( J \) the (oriented) rotation of angle \( \pi/2 \) on \( T\Sigma \).
Set $\nu = \langle N, \xi \rangle$ and $T = \xi - \nu N$, i.e., $\nu$ is the normal component of the vertical field $\xi$, called the angle function, and $T$ is the tangent component of the vertical field.

We now study some particular surfaces in $\mathcal{M}(\kappa, \tau)$. To do so, we will require some definitions.

**Definition 2.9** We say that $\Sigma \subset \mathcal{M}(\kappa, \tau)$ is a vertical cylinder over $\alpha$ if $\Sigma := \pi^{-1}(\alpha)$, where $\alpha$ is a curve on $(\mathbb{M}^2, g)$. If $\alpha$ is a geodesic, $\Sigma := \pi^{-1}(\alpha)$ is called a vertical plane.

Let us start by studying the geometry of a vertical cylinder:

**Proposition 2.10** Let $\Sigma \subset \mathcal{M}(\kappa, \tau)$ be a vertical cylinder over $\alpha$. Then, the mean, Gaussian and extrinsic curvature are respectively

$$H = \frac{k_g}{2}, \quad K = 0, \quad K_e = -\tau^2,$$

where $k_g$ is the geodesic curvature of $\alpha$ with respect to $g$. Moreover, these cylinders are characterized by $\nu \equiv 0$. In particular, a complete vertical cylinder is isometric to $\mathbb{R}^2$. Also, when $\tau \equiv 0$, a vertical plane in $\mathbb{M}^2 \times \mathbb{R}$ is totally geodesic.

**Proof.** Let us parametrize $\alpha \subset \mathbb{M}^2$ by arc-length. Let $\vec{t}$ and $\vec{n}$ be the tangent and normal vector fields along $\alpha$. Denote by $\vec{T}$ and $N$ the unique horizontal lifts to $\mathcal{M}(\kappa, \tau)$. Note that $\{\vec{T}, \xi\} \in \mathfrak{X}(\Sigma)$ is an orthonormal basis and $N$ is the unit normal vector field along $\Sigma$, in particular $\nu \equiv 0$. Moreover, it is clear that $\Sigma$ is flat, i.e., $K \equiv 0$. We choose $N$ so that $\{\vec{T}, N, \xi\}$ is positively oriented.

The second fundamental form applied to a pair of vector fields $X, Y \in \mathfrak{X}(\Sigma)$ is given by $II(X, Y) = \langle \nabla_X Y, N \rangle$. We want to compute the second fundamental form of $\Sigma$ in the basis $\{\vec{T}, \xi\}$. Then,

$$\nabla_{\xi} \xi = 0, \quad \text{since it is a unit Killing vector field.}$$

$$\langle \nabla_{\vec{T}} \xi, N \rangle = \tau \langle \vec{T} \wedge \xi, N \rangle = \tau \det(\vec{T}, \xi, N) = -\tau.$$ 

$$\langle \nabla_{\vec{T}} \vec{T}, N \rangle = g(\nabla^\mathbb{M}^2_{\vec{T}} \vec{T}, \vec{n}) = k_g.$$
Thus, if we set $X := x^1\xi + x^2\vec{T}$ and $Y := y^1\xi + y^2\vec{T}$, we have

$$II(X, Y) = (x^1, x^2) \begin{pmatrix} 0 & -\tau \\ -\tau & k_g \end{pmatrix} \begin{pmatrix} y^1 \\ y^2 \end{pmatrix}.$$ 

So, since the mean and extrinsic curvatures are the trace and determinant respectively of the second form, we obtain the result. 

Henceforth, most of the results are proven under the assumption that $\mathcal{M}(\kappa, \tau)$ fibers over a strict Hadamard surface $\mathbb{M}^2$, that is, the Gaussian curvature $\kappa$ of $\mathbb{M}^2$ is bounded above by a negative constant. Therefore, we define:

**Definition 2.11** We say that $\mathcal{M}(\kappa, \tau)$ is a strict Hadamard-Killing submersion if it fibers over a strict Hadamard surface $\mathbb{M}^2$, i.e., $\mathbb{M}^2$ has Gaussian curvature $\kappa$ bounded above by a negative constant.

We will introduce a definition according to that given for complete geodesics in a Hadamard surface since the notions of interior and exterior domains of a horizontal oriented geodesic extend naturally to vertical planes.

**Definition 2.12** Let $\mathcal{M}(\kappa, \tau)$ be a Hadamard-Killing submersion. For a complete oriented geodesic $\alpha$ in $\mathbb{M}^2$ we call, respectively, interior and exterior of the vertical plane $P = \pi^{-1}(\alpha)$ the sets

$$int_{\mathcal{M}(\kappa, \tau)}(P) = \pi^{-1}(int_{\mathbb{M}^2}(\alpha)), \quad ext_{\mathcal{M}(\kappa, \tau)}(P) = \pi^{-1}(ext_{\mathbb{M}^2}(\alpha))$$

Moreover, we will often use foliations by vertical planes of $\mathcal{M}(\kappa, \tau)$. We now make this precise.

**Definition 2.13** Let $\mathcal{M}(\kappa, \tau)$ be a Hadamard-Killing submersion. Let $P$ be a vertical plane in $\mathcal{M}(\kappa, \tau)$, and let $\beta(t)$ be an oriented horizontal geodesic in $\mathbb{M}^2$, with $t$ arc length along $\beta$, $\beta(0) = p_0 \in P$, $\beta'(0)$ orthogonal to $P$ at $p_0$ and $\beta(t) \in ext_{\mathcal{M}(\kappa, \tau)}(P)$ for $t > 0$. We define the oriented foliation of vertical planes along $\beta$, denoted by $P_\beta(t)$, to be the vertical planes orthogonal to $\beta(t)$ with $P = P_\beta(0)$.

**Remark 2.14** The Definition 2.13 is actually a foliation by Lemma 2.4.
To finish, we will give the definition of a particular type of curve in a vertical plane. To do so, we recall a few concepts about Killing graphs in a Killing submersion (see [RST]).

Under the assumption that the fibers are complete geodesics of infinite length, it can be shown (see [St]) that such a fibration is topologically trivial. Moreover, there always exists a global section
\[ s : \mathbb{M}^2 \to \mathcal{M}(\kappa, \tau), \]
so, considering the flow \( \phi_t \) of \( \xi \), a trivialization of the fibration is given by the diffeomorphism
\[ \mathbb{M}^2 \times \mathbb{R} \to \mathcal{M}(\kappa, \tau), \quad (p, t) \mapsto \phi_t(s(p)) \]

**Definition 2.15** Let \( \pi : \mathcal{M}(\kappa, \tau) \to \mathbb{M}^2 \) be a Killing submersion. Let \( \Omega \subset \mathbb{M}^2 \) be a domain. A Killing graph over \( \Omega \) is a surface \( \Sigma \subset \mathcal{M}(\kappa, \tau) \) which is the image of a section \( s : \overline{\Omega} \to \mathcal{M}(\kappa, \tau) \), with \( s \in C^2(\Omega) \cap C^0(\overline{\Omega}) \). We may also consider graphs, \( \Sigma \subset \mathcal{M}(\kappa, \tau) \), without boundary.

Now, we can establish the announced definition.

**Definition 2.16** Let \( P \) be a vertical plane in \( \mathcal{M}(\kappa, \tau) \) and \( \alpha \) a complete embedded convex curve in \( P \). We say that \( \alpha \) is an untilted curve in \( P \) if there exists a point \( p \in \alpha \) so that \( \phi_t(p) \) is contained in the convex body bounded by \( \alpha \) in \( P \) for all \( t > 0 \) (or \( t < 0 \)). Otherwise, we say that \( \alpha \) is tilted.

### 3 Hadamard-Stoker-Currier type theorems

We devote this section to the proof of a Hadamard-Stoker-Currier type theorem in a strict Hadamard-Killing submersion. First, note that if \( \Sigma \subset \mathcal{M}(\kappa, \tau) \) is an immersed surface with positive extrinsic curvature, then we can choose a globally defined unit normal vector field \( N \) so that the principal curvatures, i.e., the eigenvalues of the shape operator, are positive. We denote them by \( k_i \) for \( i = 1, 2 \).

We start with the following elementary result.

**Proposition 3.1** Let \( \Sigma \subset \mathcal{M}(\kappa, \tau) \) be an immersed surface whose principal curvatures satisfy \( k_i(p) > |\tau(p)| \) for all \( p \in \Sigma \). Let \( P \) be a vertical plane. If \( \Sigma \) and \( P \) intersect transversally then each connected component \( C \) of \( \Sigma \cap P \) is a strictly convex curve in \( P \).
Proof. Let us parametrize $C$ as $\gamma(t)$ where $t$ is arc length. Then

$$\nabla^P_{\gamma'}\gamma' + II_P(\gamma', \gamma')N_P = \nabla_{\gamma'}\gamma' = \nabla_{\gamma'}\gamma' + II(\gamma', \gamma')N$$

where $\nabla^P$, $\nabla$ and $\nabla$ are the connections on $P$, $\mathcal{M}(\kappa, \tau)$ and $\Sigma$ respectively, $II_P$ and $II$ are the second fundamental forms of $P$ and $\Sigma$ respectively, and $N_P$ and $N$ are the unit normal vector fields along $P$ and $\Sigma$, respectively.

Taking inner product in the above equality,

$$\|\nabla^P_{\gamma'}\gamma'\|^2 + II_P(\gamma', \gamma')^2 = \|\nabla_{\gamma'}\gamma'\|^2 + II(\gamma', \gamma')^2.$$

Thus,

$$\|\nabla^P_{\gamma'}\gamma'\|^2 = \|\nabla_{\gamma'}\gamma'\|^2 + II(\gamma', \gamma')^2 - II_P(\gamma', \gamma')^2 \geq II(\gamma', \gamma')^2 - II_P(\gamma', \gamma')^2 > 0,$$

since $k_i > |\tau|$. Thus $\nabla^P_{\gamma'}\gamma' \neq 0$, that is, the geodesic curvature of $C$ vanishes nowhere on $P$.

**Definition 3.2** Let $\mathcal{M}(\kappa, \tau)$ be a strict Hadamard-Killing submersion. Let $\Sigma \subset \mathcal{M}(\kappa, \tau)$ be a surface. We say that $\Sigma$ has a simple end if the boundary at infinity of $\pi(\Sigma)$ is a unique point $\theta_0 \in \mathbb{M}(\infty)$ and, in addition, for all $\theta_1, \theta_2 \in \mathbb{M}(\infty) \setminus \{\theta_0\}$ the intersection of the vertical plane $\pi^{-1}(\alpha)$ and $\Sigma$ is empty or a compact set, where $\alpha$ is a geodesic joining $\theta_1$ to $\theta_2$.

At this point we have enough information to prove our main result.

**Theorem 3.3** Let $\Sigma \subset \mathcal{M}(\kappa, \tau)$ be a complete connected immersed surface so that $k_i(p) > |\tau(p)|$ for all $p \in \Sigma$, where $\mathcal{M}(\kappa, \tau)$ is a strict Hadamard-Killing submersion. Then $\Sigma$ is properly embedded. Moreover, $\Sigma$ is homeomorphic to $\mathbb{S}^2$ or to $\mathbb{R}^2$. In the later case, $\Sigma$ has a simple end or $\Sigma$ is a Killing graph over a convex domain of $\mathbb{M}^2$.

**Proof.** The proof follows the ideas in [EGR, Theorem 3.1]. We distinguish two cases depending on the existence of a point $p$ at $\Sigma$ so that $N(p)$ is horizontal, that is, $N(p)$ is orthogonal to the fiber $\xi$.

**Case 1:** Suppose there is no point $p \in \Sigma$ so that $N(p)$ is horizontal. Then, $\Sigma$ is embedded and homeomorphic to the plane. Moreover, it is a Killing graph over a convex domain in $\mathbb{M}^2$. 

Proof of Case 1: For proving this case, we first show the following

Assertion 1: Let $P$ be a vertical plane that meets transversally $\Sigma$, then each connected component of $\Sigma \cap P$ is an open embedded strictly convex curve. Moreover, each connected component is a Killing graph over an open interval in $\alpha$, where $\alpha$ is the complete geodesic in $M^2$ so that $P := \pi^{-1}(\alpha)$.

Proof of the Assertion 1: We only need to show that each connected component is embedded, since we already know that it is strictly convex by Proposition 3.1.

Let $C$ be a connected component of $\Sigma \cap P$ that is not embedded, then it has a loop $L \subset C$ homeomorphic to a circle, i.e., there exists a homeomorphism $c : S^1 \to L$ and $c' \neq 0$ except at one point. Clearly, there is a point $q \in c(S^1)$ where $c'$ is vertical. Then $\nu(q) = 0$ which is a contradiction.

This argument also proves that a connected component can not be compact. Also, it proves that each connected component is a Killing graph over $\alpha$, where $\alpha$ is the complete geodesic in $M^2$ so that $P := \pi^{-1}(\alpha)$. This proves the Assertion 1.

Now, let $P$ be a vertical plane which meets $\Sigma$ transversally and $P_\beta(t)$ be the oriented foliation of vertical planes along $\beta$ (see Definition 2.13). From Assertion 1, each connected component is an open embedded strictly convex curve. Let $C(0)$ be an embedded component of $P_\beta \cap \Sigma$. Let us consider how $C(0)$ varies as $t$ increases to $+\infty$. First, note that no two components of $P_\beta(0) \cap \Sigma$ can join to the component $C(t_0)$ associated to $C(0)$ at some $t_0 > 0$. Otherwise, the unit normal vector field $N$ should point up in a component and down in the other for $t_0 - \epsilon < t < t_0$ ($\epsilon$ small enough), since $N$ is globally defined. Thus, by continuity, this would produce a point where $N$ is horizontal, a contradiction. Hence, from Assertion 1, the component $C(0) \subset P_\beta(0) \cap \Sigma$ varies continuously to one open embedded curve $C(t) \subset P_\beta(t) \cap \Sigma$ as $t$ increases. The only change possible is that $C(t)$ goes to infinity as $t$ converges to some $t_1$ and disappears in $P_\beta(t_1)$. Similarly $C(0)$ varies continuously to one embedded curve of $P_\beta(t) \cap \Sigma$ as $t \to -\infty$. Hence $\Sigma$ connected yields $P_\beta(t) \cap \Sigma$ is at most one component for all $t$. So, we observe that $P_\beta(t) \cap \Sigma$ is empty or homeomorphic to $\mathbb{R}$ for each $t$, hence $\Sigma$ is topologically $\mathbb{R}^2$. To finish, again from Assertion 1, we conclude $\Sigma$ is a Killing graph. The fact that $\Sigma$ is a Killing graph over a convex domain follows from Proposition 3.1.

This completes the proof of Case 1.
Case 2: Suppose there is a point \( p_0 \in \Sigma \) so that \( N(p_0) \) is horizontal. Then, \( \Sigma \) is embedded and homeomorphic to the sphere or to the plane, in which case, \( \Sigma \) has a simple end.

By assumption \( N \) is horizontal at \( p_0 \) and so, the tangent plane \( T_{p_0} \Sigma \) is spanned by \( \{\xi(p_0), X(p_0)\} \), where \( X(p_0) \) is horizontal. Set \( \bar{p}_0 := \pi(p_0) \) and \( v := d\pi_{p_0}(X(p_0)) \). Let \( \alpha \) be the complete geodesic in \( M^2 \) with initial conditions \( \alpha(0) = \bar{p}_0 \) and \( \alpha'(0) = v \). Set \( P := \pi^{-1}(\alpha) \). Note that \( p_0 \in P \cap \Sigma \) and the principal curvatures of \( \Sigma \) at \( p_0 \) are greater than the principal curvatures of \( P \) at \( p_0 \), thus \( \Sigma \) lies (locally around \( p_0 \)) on one side of \( P \).

Without loss of generality we can assume that \( N(p_0) \) points to \( \text{ext} \ M(\kappa, \tau)(P) \) (see Definition 2.2), therefore, \( \Sigma \) lies (locally around \( p_0 \)) in \( \text{ext} \ M(\kappa, \tau)(P) \).

Moreover, we parametrize the boundary at infinity by \( B : [0, 2\pi] \to M^2(\infty) \) so that \( B(0) = \alpha^-, B(\pi) = \alpha^+ \) and \( \partial_{\infty} \text{ext} M(\kappa, \tau)(P) = B([0, \pi]) \). Also, from now on, we identify the points at infinity with the points of the interval \([0, 2\pi]\).

Let \( N_P \) be the unit normal vector field along \( P \) pointing into \( \text{ext} M(\kappa, \tau)(P) \). Then, there exists neighborhoods \( V \subset P \) and \( U \subset \Sigma \) so that

\[
U := \{\exp_q(f(q)N_P(q)) : q \in V\},
\]

where \( f : V \to \mathbb{R} \) is a smooth function and \( \exp \) is the exponential map in \( M(\kappa, \tau) \).

Let \( P_\beta(t) \) be the foliation of vertical planes along \( \beta \) (see Definition 2.13). From Proposition 3.1 and the fact that locally \( \Sigma \) is (in exponential coordinates) a graph, there is \( \epsilon > 0 \) such that the curves \( P_\beta(t) \cap U \) are embedded strictly convex curves (in \( P_\beta(t) \)) for \( 0 < t < \epsilon \). Perhaps, \( P_\beta(t) \cap \Sigma \) has other components distinct from \( C(t) \) for each \( 0 < t < \epsilon \), but we only care how \( C(t) \) varies as \( t \) increases. We also denote by \( C(t) \) the continuous variation of the curves \( P_\beta(t) \cap \Sigma \) when \( t > \epsilon \).

We distinguish two cases:

Case A: If \( C(t) \) remains compact as \( t \) increases, then \( \Sigma \) is properly embedded and homeomorphic to the sphere or to the plane. In the later case, \( \Sigma \) has a simple end.

**Proof of Case A:** By topological arguments, if \( C(t) \) remains compact and non-empty as \( t \) increases, then the \( C(t) \) remains embedded. So, \( C(t) \) is either embedded compact strictly convex curves for all \( t \), or embedded compact strictly convex curves until \( \tilde{t} \) and this component either it becomes a point, or it drifts off to infinity.
If \( C(t) \) remains compact and non-empty as \( t \to +\infty \), then since \( \Sigma \) is connected, \( \Sigma \) must be embedded. In addition, because \( C(0) \) is a point and \( C(t) \) is homeomorphic to a circle for every positive \( t \), \( \Sigma \) is homeomorphic to \( \mathbb{R}^2 \).

Now, from the fact that \( C(t) \) remains compact, then

\[
\partial_{\infty} \pi(\Sigma) = \{ B(\theta_0) \},
\]

where \( B(\theta_0) = \beta^+ \), and \( \Sigma \) has a simple end.

If there exists \( \bar{t} > 0 \) such that \( C(t) \) are compact for all \( 0 < t < \bar{t} \) and the component \( C(t) \) disappears for \( t > \bar{t} \), then, \( \Sigma \) connected yields that it is either compact, embedded and topologically \( S^2 \) or non compact, embedded and topologically \( \mathbb{R}^2 \). That is, if the \( C(t) \) converge to a non empty compact set as \( t \) converges to \( \bar{t} \) then \( C(\bar{t}) \) must be a point (because our surface has no boundary) and \( \Sigma \) is a sphere. Otherwise the \( C(t) \) drift off to infinity as \( t \) converges to \( \bar{t} \) and \( \Sigma \) is topologically a plane.

We now show that in the latter case, the vertical projection \( \pi \) of \( \Sigma \) has asymptotic boundary one of the two points at infinity of \( \pi(P_\beta(\bar{t})) \).

Without lost of generality we can assume that \( P_\beta(\bar{t}) = \pi^{-1}(\gamma) \), \( \gamma = \{ \gamma^-, \gamma^+ \} \) where \( B(\theta^-) = \gamma^- \) and \( B(\theta^+) = \gamma^+ \). Note that \( \theta^- \in (0, \theta_0) \) and \( \theta^+ \in (\theta_0, \pi) \). Consider the vertical plane \( Q = \pi^{-1}(\beta) \). Let \( \tilde{C} \) be the component of \( Q \cap \Sigma \) containing \( p_0 \). First observe that \( \tilde{C} \) is compact, otherwise it would intersect \( \pi^{-1}(r) \), where \( r := \pi(Q) \cap \pi(P_\beta(\bar{t})) \in \mathbb{M}^2 \), in two points, which is not the case. Thus, we can consider the disk \( \tilde{D} \) bounded by \( \tilde{C} \) on \( \Sigma \).

Let \( Q_\gamma(t) \) denote the foliation by vertical planes along \( \gamma \), \( Q_\gamma(0) = Q \). There exists \( t_0 \) (we can assume \( t_0 < 0 \)) satisfying \( Q_\gamma(t_0) \) touches \( \tilde{D} \) on one side of \( \tilde{D} \) by compactness. Let \( q_0 \in \tilde{D} \cap Q_\gamma(t_0) \) be the point where they touch. Consider the variation \( \tilde{C}(t) \) of \( q_0 \) on \( \Sigma \cap Q_\gamma(t) \) from \( t = t_0 \) to infinity. Then, \( \tilde{C}(t) \) is a convex embedded curve for \( t \) in a maximal interval \( (t_0, t_0) \) with \( 0 < t_0 \leq \infty \). Hence, \( \Sigma \) is foliated by the \( \tilde{C}(t) \), \( \tilde{C} = \tilde{C}(0) = Q \cap \Sigma \) and \( \theta^- \notin \partial_{\infty} \pi(\Sigma) \) because \( \Sigma \) is on one side of \( Q_\gamma(t_0) \).

Now, we will show that \( \partial_{\infty} \pi(\Sigma) = \{ B(\theta^+) \} \). Let \( \gamma(\theta) := \gamma(\theta^+ \theta) \) where \( B(\theta^+) = \beta^- \) (see Definition 2.1), for \( \theta \in [0, \pi] \). Let \( \theta \) be the value of \( \theta \) such that \( \gamma(\theta) \) is asymptotic to \( \gamma^+ \). Let \( Q(\theta) = \pi^{-1}(\gamma(\theta)) \). For each \( \theta, \theta^+ \theta \leq \pi \), we have \( \Sigma \cap Q(\theta) \) is one connected embedded compact curve \( C'(\theta) \). The proof of this is the same as the previous one for \( \tilde{C} \). Notice that each \( C'(\theta) \) is non empty since \( p_0 \in C'(\theta) \).
Now \( C'(\bar{t}) \) cannot be compact, otherwise \( \Sigma \) could not be asymptotic to the plane \( P_\beta(\bar{t}) \), a contradiction.

In order to complete the proof, we show that \( \Sigma \) has a simple end. Observe that \( C'(\theta) \) is compact, \( \bar{t} < \theta < \theta_0 \) because \( \Sigma = \cup_{0 \leq t < \bar{t}} C(t) \). Moreover, \( C''(\theta) \subset \bar{D}, 0 < \theta < \theta_0 \), and \( \bar{D} \) is compact. Thus, it is easy to conclude that \( \Sigma \) has a simple end.

Thus we have proved that \( \Sigma \) is either a properly embedded sphere or \( \Sigma \) is a properly embedded plane with a simple end. This proves Case A.

**Case B:** If \( C(t) \) becomes non-compact, then \( \Sigma \) is a properly embedded plane with a simple end.

**Proof of Case B:** Let \( \bar{t} > 0 \) be the smallest \( t \) with \( C(\bar{t}) \) non-compact, \( C(\bar{t}) \) the limit of the \( C(t) \) as \( t \rightarrow \bar{t} \), \( C(\bar{t}) \) is an embedded strictly convex curve in \( P_\beta(\bar{t}) \).

**Claim 1:** \( C(\bar{t}) \) is tilted (see Definition 2.16).

**Proof of Claim 1:** Let us assume that \( C(\bar{t}) \) is untilted, then there is point \( q \in C(\bar{t}) \) so that \( \pi^{-1}(\bar{q}) \) touches once to \( C(\bar{t}) \), where \( \bar{q} := \pi(q) \). First of all, note that \( \bar{\Sigma} = \bigcup_{0 \leq t \leq \bar{t}} C(t) \subset \Sigma \) is embedded. Let \( \Gamma_{\bar{p}\bar{q}} \) be the complete horizontal geodesic (in \( \mathbb{M}^2 \)) joining \( \bar{p} \) and \( \bar{q} \). Let \( Q = \pi^{-1}(\Gamma_{\bar{p}\bar{q}}) \), and consider \( \pi^{-1}(r_0) \) where \( r_0 := \pi(Q) \cap \pi(P_\beta(0)) \) and \( \pi^{-1}(r_{\bar{t}}) \) where \( r_{\bar{t}} := \pi(Q) \cap \pi(P_\beta(\bar{t})) \). Note that \( \pi^{-1}(r_0) \) and \( \pi^{-1}(r_{\bar{t}}) \) are parallel lines in \( Q \). Also, \( \alpha_Q = Q \cap \bar{\Sigma} \) is a non-compact embedded strictly convex curve in \( Q \) such that \( \pi^{-1}(r_0) \) is tangent to \( \alpha_Q \) at \( p_0 \in \alpha_Q \) and \( \alpha_Q \cap \pi^{-1}(r_{\bar{t}}) \) is exactly one point, since \( C(\bar{t}) \) is untilted. But this is a contradiction because \( \alpha_Q \) is a strictly convex curve in \( Q \), which is isometric to \( \mathbb{R}^2 \), and it must intersect \( \pi^{-1}(r_{\bar{t}}) \) twice. Thus, \( C(\bar{t}) \) is tilted.

And we claim that

**Claim 2:** \( \partial_\infty \pi(C(\bar{t})) \) is one point.

**Proof of Claim 2:** Let us denote by \( D(t) \) the convex body bounded by \( C(t) \) in \( P_\beta(t) \) for each \( 0 < t < \bar{t} \). Thus, the limit, \( D(\bar{t}) \), of \( D(t) \) as \( t \) increases to \( \bar{t} \) is an open convex body bounded by \( C(\bar{t}) \) in \( P_\beta(\bar{t}) \) which is isometrically \( \mathbb{R}^2 \). If \( \partial_\infty \pi(C(\bar{t})) \) has two points, the only possibility is that \( C(\bar{t}) \) is untilted, which is impossible by Claim 1.

Set \( P_\beta(\bar{t}) = \pi^{-1}(\gamma), \gamma = \{ \gamma^-, \gamma^+ \} \) where \( B(\theta^-) = \gamma^- \) and \( B(\theta^+) = \gamma^+ \). Note that \( \theta^- \in (0, \theta_0) \) and \( \theta^+ \in (\theta_0, \pi) \). From Claim 2, we may assume that \( \partial_\infty \pi(C(\bar{t})) = \{ B(\theta^-) \} \).
Let $\delta_0 > 0$ and $t_{\delta_0} < \bar{t}$ such that $P_\beta(t_{\delta_0}) = \pi^{-1}(\Gamma(\delta_0))$ where $\Gamma(\delta_0) := \{B(\theta^- - \delta_0), B(\theta^+ + \delta_0)\}$ (we may assume this by choosing $B$ in the right way). We denote by $\tilde{\Sigma} = \cup_{0 \leq t \leq t_{\delta_0}} C(t) \subset \Sigma$ and note that $\tilde{\Sigma}$ is connected and embedded.

Let us consider the complete horizontal geodesic given by $\Gamma(\delta_0, s) := \{B(\theta^- - \delta_0 + s), B(\theta^+ + \delta_0)\}$ and the vertical plane $Q(s) = \pi^{-1}(\Gamma(\delta_0, s))$ for each $0 \leq s \leq \theta^+ - \theta^-$ (note that $Q(s)$, for $0 < s < \theta^+ - \theta^-$ is a foliation of $ext_{M(\kappa, \tau)}(Q(0))$). So, $Q(0) = P_\gamma(t_{\delta_0})$ and $Q(0) \cap \tilde{\Sigma} = C(t_{\delta_0})$ is an embedded compact strictly convex curve. Let us consider how $\alpha(s) = Q(s) \cap \Sigma$ varies as $s$ increases to $\theta^+ - \theta^- - \delta_0$. At this point, we have two cases:

1. If $\alpha(s)$ remains compact for all $0 \leq s < \theta^+ - \theta^- - \delta_0$, then $\Sigma$ is properly embedded, homeomorphic to the plane and has a simple end.

   In this case, letting $\delta_0 \to 0$, falls into Case A. So, it is easy to realize that $\Sigma$ is properly embedded, homeomorphic to the plane and has a simple end at $B(\theta^+) \in \mathbb{M}^2(\infty)$.

2. $\alpha(s)$ can not become non-compact.

   Let us assume that $\alpha(s)$ becomes non-compact. Let $0 < \bar{s} \leq \theta^+ - \theta^- - \delta_0$ be the smallest $s$ with $\alpha(\bar{s})$ non-compact, $\alpha(\bar{s})$ is the limit of the $\alpha(s)$ as $s \to \bar{s}$. Also,

   \[ \partial_{\infty}\pi(\alpha(\bar{s})) = \{B(\theta^- - \delta_0 + \bar{s})\}, \]

   otherwise it must be $\{B(\theta^+ + \delta_0)\}$ which contradicts that $C(t_{\delta_0})$ is compact.

   Clearly $\delta_0 < \bar{s}$. For each $\delta \leq \delta_0$ we consider the complete horizontal geodesic given by $\sigma(\delta) = \{B(\theta^- - \delta_0 + \bar{s} - \delta, B(\theta^+ + \delta)\}$ and the vertical plane $T(\delta) = \pi^{-1}(\sigma(\delta))$. Let us denote by $\tilde{\Sigma}_2 = \cup_{0 \leq s \leq \bar{s} - 2\delta_0} \alpha(s) \subset \Sigma$ and note that $\tilde{\Sigma}_2$ is connected and embedded, so, $\tilde{\Sigma} = \tilde{\Sigma}_1 \cup \tilde{\Sigma}_2 \subset \Sigma$ is connected and embedded. For each $\delta$, $0 < \delta \leq \delta_0$, $E(\delta) = T(\delta) \cap \tilde{\Sigma}$ is a strictly convex compact embedded curve in $T(\delta)$. As $\delta \to 0$, these curves converge to a convex curve in $T(0)$ with $\partial_{\infty}\pi(E(0))$ the two points $\{B(\theta^- - \delta_0 + \bar{s}), B(\theta^+)\}$. This contradicts Claim 2. Hence $\alpha(s)$ can not become non-compact.

   This proves Claim B, and so Theorem 3.3. \qed

In particular, we can recover (and generalize) the afore mentioned [EGR, Theorem 3.1].
Corollary 3.4 Let \( \Sigma \) be a complete connected immersed surface in \( \mathbb{M}^2 \times \mathbb{R} \) with positive extrinsic curvature, where \( \mathbb{M}^2 \) is a Hadamard surface with Gaussian curvature bounded above by a negative constant. Then \( \Sigma \) is properly embedded and bounds a strictly convex domain in \( \mathbb{M}^2 \times \mathbb{R} \). Moreover, \( \Sigma \) is homeomorphic either to \( \mathbb{S}^2 \) or to \( \mathbb{R}^2 \). In the later case, \( \Sigma \) is either a graph over a convex domain in \( \mathbb{M}^2 \) or \( \Sigma \) has a simple end.

In a product space \( \tau = 0 \), and so, since the extrinsic curvature is the product of the principal curvatures, it is enough to ask that the extrinsic curvature is positive. Moreover, the assertion about that \( \Sigma \) bounds a strictly convex domain follows from the fact that, in a product space, the vertical planes are totally geodesics and Proposition 3.1.

Remark 3.5 Corollary 3.4 is sharp in the sense that there exists complete embedded surfaces with positive extrinsic curvature and a simple end in \( \mathbb{H}^2 \times \mathbb{R} \) (see [EGR, Section 4]). Moreover, vertical cylinder in a product space has zero extrinsic curvature.

Remark 3.6 Moreover, Theorem 3.3 can be applied to surfaces in \( \widetilde{\text{PSL}}(2, \mathbb{R}) \) whose principal curvatures are greater than the curvature of the fiber \( \tau \). It would be interesting to investigate the existence of examples in \( \widetilde{\text{PSL}}(2, \mathbb{R}) \) with principal curvatures greater that the curvature of the fiber \( \tau \) and a simple end.

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