Hybrid deformed algebra

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Abstract

By considering $p, q$-deformed and $\mu$-deformed algebras we propose an association of them to form a hybrid deformed algebra. The increased number of available parameters can provide us with a richer tool to investigate new scenarios within hybrid deformed statistics.

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I. INTRODUCTION

In conducting a study on basic hypergeometric series at the beginning of the last century, the mathematician F.H. Jackson developed what we call \( q \)-calculus and its famous Jackson derivatives (JD) \[1\]. The emergence of deformed algebra played a central role in the representation of quantum groups \[2–5\]. A concrete realization very widespread in the literature is the deformation of the Heisenberg algebra with the introduction of the parameter of deformation \( q \) through \( q \)-oscillators \[6–28\], which obey the commutation relations that define the Lie algebra, and that in the limit \( q \to 1 \) we recover the original algebra. This new mathematical structure immediately attracted interest due to its relevance to various problems in theoretical physics which includes, e.g., cosmology and condensed matter physics.

The statistical mechanics of bosons and fermions are strongly linked to microscopic thermodynamics, that in macroscopic level we can describe these systems through macroscopic variables such as pressure and internal energy. The generalization comes naturally through intermediate statistics \[29–37\] and non-standard quantum statistics \[38–41\] as well as non-extensive models \[42–48\]. In addition to the \( q \)-deformation parameter, the insertion of two parameters \( p, q \) \[49–58\] the \( q_1 \) and \( q_2 \)-deformed Fibonacci oscillators \[59–62\], as well as multiple parameters \[63–71\] were proposed in the literature.

We also emphasize another type of deformation which, as we shall see, presents the same structure of the \( q \)-algebra, but with the insertion of a new \( \mu \) parameter that leads to the \( \mu \)-deformation \[72, 73\] which presents a new definition of the so-called basic number. As we shall show, all these proposals allow us to generalize the polynomials and the Fibonacci sequence. Quantum algebra with two (or more) deformation parameters may have greater flexibility when dealing with realistic (phenomenological) applications in physical models.

In this paper, we describe the deformed oscillator models intensively investigated in literature by Jackson, Biedenharn-Macfarlane, Arik-Coon and \( p, q \)-oscillators \[1–3, 49, 59\], to show that all deformed algebras satisfy Fibonacci relations. For this, we will start with the generalized definition of the basic number for the Fibonacci oscillators, such that when we take the appropriate values for \( q_1 \) and \( q_2 \) we describe the aforementioned models.

For the sake of generality, we are led to propose a new model, where we associate the well-known \( q \)-deformed models with the \( \mu \)-deformation \[72, 73\]. This will produces a hybrid model that allows more flexibility in handling it, that is, we can insert at the same time one, two or three
deformation parameters to a system, and by taking the proper limits the model returns to standard ones. We can apply the hybrid model in various known areas of physics, such as in recent works using standard models of deformation, e.g., in the black holes physics \[74\], dark energy and dark matter \[75–78\].

The paper is organized as follows. In Section \[\text{II}\] we generalize the Heisenberg algebra by initially using the Fibonacci oscillators and in the sequel obtaining the other models of \((q \text{ or } p, q)\)-deformation. In Section \[\text{III}\] we define our hybrid deformation model and finally, in Section \[\text{IV}\] we make our final comments.

\section{DEFORMED HEISENBERG ALGEBRA}

We will start with using the so-called Fibonacci oscillators, however, it is worth noting that the oscillators that will be described below are all accommodated within the same mathematical structure of the deformed generalized oscillator, which is defined by Heisenberg algebra in terms of operators of annihilation and creation \((c \text{ and } c^\dagger, \text{ respectively})\), of the operator number \(N\) and structure function \(\Phi(x)\), satisfying the relations

\[
c_i c_i^\dagger - q_1^2 c_i c_i^\dagger = q_2^{2n_i} \quad \text{or} \quad c_i c_i^\dagger - q_2^2 c_i^\dagger c_i = q_1^{2n_i},
\]

\[
[N, c] = c^\dagger, \quad [N, c^\dagger] = -c.
\]

\[
c_i^\dagger c_i = \Phi(N) = [N], \quad c_i c_i^\dagger = \Phi(N + 1) = [N + 1],
\]

where \(\Phi(x)\) is a characteristic positive analytical function for the deformation regime, with \(\Phi(0) = 0\). We conclude from Eq. (3) that for \(N = \Phi^{-1}(c_i^\dagger c_i)\), the commutation and anti-commutation relations are satisfied, i.e.,

\[
[c_i, c_i^\dagger] = [N + 1] - [N], \quad \{c_i, c_i^\dagger\} = [N + 1] + [N].
\]

The commutation relations (1) are valid for the definition of the Fibonacci \textit{basic number or n-bracket} [59]

\[
[n_{i, q_1, q_2}] = c_i^\dagger c_i = \frac{q_2^{2n_i} - q_1^{2n_i}}{q_1^2 - q_2^2},
\]

where \(q_1\) and \(q_2\) are parameters of deformation that are real, positive, independent and at the limit
$q_1 = 1$ and $q_2 \to 1$ (or vice versa) we have to $[n_{i,q_1,q_2}] = n_i$.

The Fock space spanned by the orthonormalized eigenstates $|n\rangle$ is constructed according to

$$|n\rangle = \frac{(c^\dagger)^n}{\sqrt{n!}}|0\rangle, \quad c|0\rangle = 0,$$

and the actions of $c$, $c^\dagger$ and $N$ on the states $|n\rangle$ in the Fock space are known to be

$$c^\dagger|n\rangle = [n + 1]^{1/2}|n + 1\rangle,$$

$$c|n\rangle = [n]^{1/2}|n - 1\rangle,$$

$$N|n\rangle = n|n\rangle.$$

We can rewrite the general commutation relations through the following general relation [60]

$$[N + 2] = \alpha[N + 1] + \beta[N],$$

where $\alpha$ and $\beta$ depend on the $n$-bracket definition and initial conditions: $[0] = 0$ and $[1] = 1$. In the case of definition (5), we have $\alpha = q_1^2 + q_2^2$ and $\beta = -q_1^2 q_2^2$. On the other hand, regardless of the definition of the basic number the relation (10) satisfies the Fibonacci relation. We can write a sequence in the form

$$[0] = 0,$$

$$[1] = 1,$$

$$[2] = \alpha,$$

$$[3] = \alpha^2 + \beta,$$

$$[4] = \alpha^3 + 2\alpha\beta,$$

$$[5] = \alpha^4 + \beta^2 + 3\alpha^2\beta,$$

$$[6] = \alpha^5 + 4\alpha^3\beta + 3\alpha^2\beta^2,$$

$$\vdots$$

such that when $\alpha = \beta = 1$ we obtain the famous Fibonacci sequence $0, 1, 1, 2, 3, 5, 8, \ldots$. 

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We will demonstrate from definition (5) that it is possible to describe the deformation models for $q$ or $p, q$. We can write the basic numbers and their respective commutation relations by determining suitable values for $q_1$ and $q_2$:

1. For $q_1 = \sqrt{q}$ and $q_2 = 1$,

$$[n]_q = \frac{q^n - 1}{q - 1}, \quad cc^\dagger - qc^\dagger c = 1,$$  \hspace{1cm} (12)

that implies, $\alpha = 1 + q$ and $\beta = -q$.

2. For $q_1 = \sqrt{q}$ and $q_2 = \frac{1}{\sqrt{q}}$, we have symmetry

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}},$$  \hspace{1cm} (13)

$$cc^\dagger - qc^\dagger = q^{-n} \quad \text{or} \quad cc^\dagger - q^{-1}c^\dagger c = q^n,$$  \hspace{1cm} (14)

then, $\alpha = q + q^{-1}$ and $\beta = -1$.

3. When $q_1 = q$ and $q_2 = 1$,

$$[n]_{q^2} = \frac{q^{2n} - 1}{q^2 - 1}, \quad cc^\dagger - q^{2n}c^\dagger c = 1,$$  \hspace{1cm} (15)

we have, $\alpha = 1 + q^2$ and $\beta = -q^2$.

4. When $q_1 = q$ and $q_2 = \frac{1}{q}$, we again have a symmetry

$$[n]_{q^2} = \frac{q^{2n} - q^{-2n}}{q^2 - q^{-2}},$$  \hspace{1cm} (16)

$$cc^\dagger - q^{2}c^\dagger c = q^{-2n}, \quad \text{or} \quad cc^\dagger - q^{-2}c^\dagger c = q^{2n},$$  \hspace{1cm} (17)

and we get, $\alpha = q^2 + q^{-2}$ and $\beta = -1$.

5. For the model $p, q$-deformed we take $q_1 = \sqrt{q}$ and $q_2 = \sqrt{p}$,

$$[n]_{p,q} = \frac{q^n - p^n}{q - p},$$  \hspace{1cm} (18)

$$cc^\dagger - qc^\dagger c = p^n, \quad \text{or} \quad cc^\dagger - pc^\dagger c = q^n,$$  \hspace{1cm} (19)
with $\alpha = q + p$ and $\beta = -pq$.

All models presented above belong to the Fibonacci class of oscillators. In the literature we find various applications of these models. Depending on the chosen definition of the basic number, we get different generalizations of thermostatistics.

III. HYBRID DEFORMATION MODEL

A. $\mu$-Deformation

The $\mu$-oscillator definition [72, 73] presents the same structure as the models presented in the previous section. Of course, the insertion of the parameter $\mu$ presents us with a definition of the basic number structurally different from the previous definitions

$$c^\dagger c = \Phi_\mu(N) = \frac{N}{1 + \mu N}, \quad cc^\dagger = \Phi_\mu(N + 1) = \frac{N + 1}{1 + \mu(N + 1)}. \quad (20)$$

$$c^\dagger c = \Phi_\mu(N) = [n]^\mu = \frac{n}{1 + n\mu}, \quad (21)$$

which in the limit $\mu \to 0$ we have $[n]^\mu = n$. The basic commutation relations for the $\mu$-oscillator are the same as for the $q$-oscillator as we can see in (2). We have the following different initial conditions [73]

$$[0]^\mu = 0, \quad [1]^\mu = \frac{1}{1 + \mu} \neq 1. \quad (22)$$

We can also write the Fibonacci sequence in the same way as in (11), as long as

$$\lim_{\mu \to 0} [1]^\mu = 1. \quad (23)$$

B. Association of $q$-deformation with $\mu$-deformation

Let us now associate $q$-deformation with $\mu$-deformation. In order to provide this we set our basic number as follows

$$[n]_{q_1, q_2}^\mu = \frac{q_1^{2[n]^\mu} - q_2^{2[n]^\mu}}{q_1^2 - q_2^2}, \quad (24)$$

$$\lim_{\mu \to 0} [n]_{q_1, q_2}^\mu = [n]_{q_1, q_2} = \frac{q_1^{2n} - q_2^{2n}}{q_1^2 - q_2^2}, \quad \text{and} \quad \lim_{q_1 = q_2 \to 1} [n]_{q_1, q_2}^\mu = [n]^\mu = \frac{n}{1 + n\mu}. \quad (25)$$
Of course, the definition given by (24) with the respective values adopted for \(q_1\) and \(q_2\) can recover the definitions written for the models we have studied in the previous section.

The commutation relations of the hybrid model and the initial conditions are given by

\[
cc^\dagger - q_1^2 c^\dagger c = q_2^{2[\mu]}_{q_1,q_2}, \quad \text{or} \quad cc^\dagger - q_2^2 c^\dagger c = q_1^{2[\mu]}_{q_1,q_2},
\]

(26)

\[
[0]_{q_1,q_2}^\mu = 0, \quad [1]_{q_1,q_2}^\mu = \frac{q_1^{2[\mu]} - q_2^{2[\mu]}}{q_1^2 - q_2^2}.
\]

(27)

We have

\[
\alpha = \frac{q_1^{1+2\mu} - q_2^{1+2\mu}}{q_1^2 - q_2^2}, \quad \text{and} \quad \beta = \frac{q_1^{6+3\mu} - q_2^{6+3\mu}}{q_1^2 - q_2^2} - \alpha^2.
\]

(28)

Our proposal presents an association between two different types of deformation, which increases our power of application to describe a system, because we have inserted a new disturbance factor (or disorder) whose role can be played by the \(\mu\)-deformation in addition to the usual \(q\)-deformation.

Since the deformation parameters \(q_1, q_2\) or \(\mu\) are independent, we further increase our manipulation power to address impurities and disorders of a physical system. More specifically, we can imagine, e.g., a nano film deposition system (sputtering or cathodic evaporation is used extensively in industry and in semiconductor research for the deposition of thin films of various materials) [79], in which several parameters are controllable, such as the temperature of the substrate where the material is deposited, the power of the DC source, the working pressure, the insertion of a gas during the process, among others. It is possible to associate the adjustable parameters with the deformation parameters to calculate the thermodynamic quantities of the films, e.g., thermal and electrical conductivity.

IV. CONCLUSIONS

In this work we show that through the Fibonacci oscillators, we can write the models for one \((q)\) or two \((p, q)\) deformation parameters. And we can generically write the Fibonacci sequence, which as we know is a linear combination that encompasses arithmetic and geometric progressions, that is a well-known way to describe a sequence of integers. The choice of which model to use depends greatly on the physical system being investigated.

We have proposed a new model of deformation, the hybrid deformation, where we associate
$(q_1, q_2)$-deformation with $\mu$-deformation. This means that at the same time we insert three parameters of deformation, and that at the limit of $q_1 = q_2 \to 1$ we have only the parameter $\mu$ playing the role of deformation in the system. On the other hand, at the limit of $\mu \to 0$ we recover the usual Fibonacci oscillators.

The search for generalized statistics makes possible to solve problems where the standard statistic [80, 81] does not work. The results obtained with the insertion of the $\mu$-deformation enlarge our field of investigation, such as investigating factors of disorder or impurities in a thermodynamic system.

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