Stability Analysis for the Discrete-Time T-S Fuzzy System with Stochastic Disturbance and State Delay

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ABSTRACT
In this paper, we study the stability for discrete-time Takagi and Sugeno (T-S) fuzzy systems with perturbation disturbance and time delay. Stochastic delay-dependent stability criteria are derived for stochastic T-S fuzzy systems with time-invariant and time-varying delays, respectively. For the time-varying delay case, a novel fuzzy Lyapunov–Krasovskii functional (LKF) without requiring all the involved symmetric matrices to be positive definite is constructed to reduce the conservatism. These stability conditions are then represented in terms of finite linear matrix inequalities (LMIs), which can be solved efficiently by using standard LMIS optimisation techniques. Two numerical examples are given to illustrate the feasibility of the proposed method.

1. Introduction
In 1985, Takagi and Sugeno first proposed the method to use fuzzy systems to approximate nonlinear systems [1]. Since then, T-S fuzzy systems have attracted great attention of a wide range of scholars because they can provide effective measures for the control of nonlinear systems. Rich results are presented in [2–5]; $H_{\infty}$ control designs are studied in [5–8]; fault detection has been investigated in [9,10] and sliding mode control based on fuzzy model can be found in [11,12].

The traditional T-S fuzzy dynamic mode is described by a family of fuzzy IF-THEN rules that represent local linear input-output relations of a linear system. While the good local linearity may be broken by the stochastic noise. Stochastic noise is an ideal signal to simulate irregular internal and external interference. Around the problems of stochastic systems, lots of efficient approaches have been proposed by scholars. Asynchronous output feedback control based on the stochastic T-S fuzzy model is presented in [13]. Su et al. [14] studies $H_{\infty}$ model reduction of T-S fuzzy stochastic systems. For more results about the stochastic T-S fuzzy system, one may refer to [15,16] and the reference therein.

As is well-known, time delay may cause the instability of the systems. Stability results can be classified into two types: delay-independent stability and delay-dependent stability. Most researchers concentrate on studying delay-dependent stability because of its less
Various approaches have been proposed for presenting the stability conditions of the discrete systems with time delay, see in [17–21] and the references therein. Although it is feasible to use a full Lyapunov matrix to analyse the stability of the discrete-time delay systems, the computational complexity caused by this method is high. It makes the use of Lyapunov–Krasovskii functional become popular, which provides an effective way in obtaining delay-dependent stability results for the discrete time-delay systems. In the case of time-varying delays, delay-dependent stability conditions of discrete-time T-S fuzzy systems with stochastic disturbance are given in [22], which is derived by the construction of a fuzzy Lyapunov–Krasovskii functional. It is worth noting that they need requiring all the involved symmetric matrices in a chosen fuzzy Lyapunov–Krasovskii functional to be positive definite. Such a requirement can lead to the conservatism in the stability criteria.

Motivation by the aforementioned analysis, we intend to investigate the stochastic stability for discrete-time Takagi and Sugeno (T-S) fuzzy systems with stochastic perturbation and time delay. Stochastic delay-dependent stability criteria are derived for stochastic T-S fuzzy systems with time-invariant and time-varying delays, respectively. In this paper, we mainly analyse the stability of the discrete T-S fuzzy systems with stochastic disturbance and time-varying delays. Different fuzzy Lyapunov–Krasovskii functions are constructed for constant time delay and time-varying delay, respectively.

For the time-varying delay case, a novel fuzzy Lyapunov–Krasovskii functional without requiring all the involved symmetric matrices to be positive definite is constructed to reduce the conservatism. These stability conditions are then represented in terms of finite linear matrix inequalities (LMIs), which can be solved efficiently by using standard LMI optimisation techniques. Finally, two numerical examples are given to illustrate the feasibility of the proposed method.

The remaining parts of this paper are organised as follows. In the second part, the formation and preparation of discrete-time T-S fuzzy systems with stochastic disturbance and time-varying delays are introduced. The delay-dependent stability analysis is given in the section there. In the fourth part, we provide some simulation results to verify the effectiveness of the method. The last section draws the conclusion of this paper.

Notation: The notations that are used throughout this paper are fairly standard. The superscript ‘T’ stands for matrix transposition; \( \mathbb{R}^n \) denotes the \( n \)-dimensional Euclidean space; the notation \( P > 0 (\geq 0) \) means that \( P \) is real symmetric and positive definite (semidefinite); and \( \mathbb{R}^{m\times n} \) is the set of all real matrices of dimension \( m \times n \); and in symmetric block matrices or long matrix expressions, we use an asterisk (*) to represent a term that is induced by symmetry; \( \text{diag}\{ \cdots \} \) stands for a block-diagonal matrix; Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

2. Problem Formulation and Preliminaries

Consider a discrete nonlinear time-delay system that is represented by the following T-S fuzzy stochastic model with delay:

**Plant rule i:**

\[
\text{IF } \theta_1(k) \text{ is } M_{i1} \text{ and } \theta_2(k) \text{ is } M_{i2} \text{ and } \ldots, \text{ and } \theta_g(k) \text{ is } M_{ig}
\]
\[ x(k + 1) = A_i x(k) + A_{di} x(k - h(k)) + [E_i x(k) + E_{di} x(k - h(k))] \omega(k), \]
\[ x(l) = \psi(l), \quad l = -h_2, -h_2 + 1, \ldots, 0, \]

where \( i \in S \triangleq \{1, 2, \ldots, r\} \), \( r \) is the number of IF-THEN rules, \( M_{ij} \) is the fuzzy set, \( \theta(k) = [\theta_1(k), \theta_2(k), \ldots, \theta_g(k)] \) are the premise variables, \( x(k) \in \mathbb{R}^n \) is the state vector, \( \omega(k) \) is a 1-D, zero mean Gaussian white noise sequence on a probability space \((\Omega_1, \mathcal{F}, \mathbb{P})\) with \( \mathbb{E}\{\omega(k)\} = 0 \); and \( \mathbb{E}\{\omega^2(k)\} = 1 \); \( \psi(l) = -h_2, -h_2 + 1, \ldots, 0 \) is the given initial condition sequence, \( h(k) \) is the time delay, which is a positive integer and satisfies \( 1 \leq h_1 \leq h(k) \leq h_2 \), where \( h_1 \) and \( h_2 \) are constant positive scalars that represent the minimum and maximum delay, respectively. \( A_i, A_{di}, E_i \) and \( E_{di} \) are known constant matrices with appropriate dimensions. The fuzzy basis functions are given by

\[
\lambda_i(\theta(k)) = \frac{\prod_{j=1}^g M_{ij}(\theta_j(k))}{\sum_{i=1}^r \prod_{j=1}^g M_{ij}(\theta_j(k))}, \quad i \in S
\]

with \( M_{ij}(\theta_j(k)) \) representing the grade of membership of \( \theta_j(k) \) in \( M_{ij} \). For simplicity, we will replace \( \lambda_i(\theta(k)) \) by \( \lambda_i \) in some places. By definition, the fuzzy basis functions satisfy \( \lambda_i \geq 0(\forall i \in S) \) and \( \sum_{i=1}^r \lambda_i = 1 \).

Then, the defuzzified output of the T-S fuzzy system (1) can be represented as

\[
x(k + 1) = \sum_{i=1}^r \lambda_i [A_i x(k) + A_{di} x(k - h(k))] + \sum_{i=1}^r \lambda_i [E_i x(k) + E_{di} x(k - h(k))] \omega(k). \tag{3}
\]

Before proceeding, the following lemmas will be used to derive our main results.

**Lemma 2.1 ([23]):** Assume that \( a \in \mathbb{R}^n_a, b \in \mathbb{R}^n_b, \) and \( \mathcal{N} \in \mathbb{R}^{n_a \times n_b} \). Then for any matrices \( X \in \mathbb{R}^{n_a \times n_a}, Y \in \mathbb{R}^{n_a \times n_b}, \) and \( Z \in \mathbb{R}^{n_b \times n_b} \) satisfying \( \begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} \geq 0 \), the following inequality holds, i.e.

\[
-2a^T \mathcal{N} b \leq \begin{bmatrix} a \\ b \end{bmatrix}^T \begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.
\]

**Lemma 2.2 ([2]):** For any constant matrix \( \mathcal{M} \in \mathbb{R}^{m \times m} \) with \( \mathcal{M} > 0 \), integers \( l_2 > h_1 \), vector function \( \omega : \{h_1, \ldots, l_2\} \rightarrow \mathbb{R}^m \), then

\[
(l_2 - h_1 + 1) \sum_{i=h_1}^{l_2} \omega^T(i) \mathcal{M} \omega(i) \geq \left( \sum_{i=h_1}^{l_2} \omega(i) \right)^T \mathcal{M} \left( \sum_{i=h_1}^{l_2} \omega(i) \right).
\]
Lemma 2.3 ([4]): There exists a matrix $X$ such that

$$
\begin{bmatrix}
P & Q & X \\
Q^T & R & V \\
X^T & V^T & S
\end{bmatrix} > 0
$$

(4)

if and only if

$$
\begin{bmatrix}
P & Q \\
Q^T & R
\end{bmatrix} > 0, \quad (5)
$$

$$
\begin{bmatrix}
R & V \\
V^T & S
\end{bmatrix} > 0. \quad (6)
$$

Lemma 2.4 ([24]): Given a 2 × 2-symmetric matrix

$$
P = \begin{bmatrix} p_{11} & p_{12} \\
p_{12} & p_{22} \end{bmatrix}, \quad p_{ij} \in \mathbb{R}
$$

one has

$$
x^T P x < 0 \quad \forall x_i \geq 0, \quad x = (x_1, x_2) \neq 0
$$

(7)

if and only if there is $q_{12} \in \mathbb{R}$ such that

$$
\begin{bmatrix} p_{11} & q_{12} \\
q_{12} & p_{22} \end{bmatrix} < 0, \quad p_{12} \leq q_{12}.
$$

(8)

A sufficient condition for (7) and (8) is

$$
p_{ii} < 0, \quad p_{ii} + p_{ij} < 0, \quad i, j = 1, 2.
$$

(9)

3. Delay-Dependent Stability Analysis

In this section, we analyse the stability for the fuzzy time-delay system. Firstly, we analyse the case of constant time delay. The system of (3) can be transformed into the following compact form:

$$
x(k + 1) = \overline{A}(k)x(k) + \overline{A}_d(k)x(k - \tau) + [\overline{E}(k)x(k) + \overline{E}_d(k)x(k - \tau)]\omega(k),
$$

(10)

where

$$
\overline{A}(k) \triangleq \sum_{i=1}^{r} \lambda_i A_{ii}, \quad \overline{A}_d(k) \triangleq \sum_{i=1}^{r} \lambda_i A_{di},
$$

$$
\overline{E}(k) \triangleq \sum_{i=1}^{r} \lambda_i E_{ii}, \quad \overline{E}_d(k) \triangleq \sum_{i=1}^{r} \lambda_i E_{di}.
$$

The following result on the bounding of cross products of vectors will be used in the proof of Theorem 3.1.
Denote \( \eta(l) \triangleq x(l + 1) - x(l) \). Then for the fuzzy time-delay system (10), we have

\[
x(k - \tau) = x(k) - \sum_{l=k-\tau}^{k-1} \eta(l).
\]

(11)

For convenience of notations, we make the following definitions:

\[
\begin{align*}
\tilde{P}(k) & \triangleq \sum_{i=1}^{r} \lambda_i P_i, \\
\tilde{Z}(k) & \triangleq \sum_{i=1}^{r} \lambda_i Z_i, \\
\tilde{Q}(l) & \triangleq \sum_{i=1}^{r} \lambda_i Q_i, \\
G(k) & \triangleq \sum_{i=1}^{r} \lambda_i G_i, \\
F(k) & \triangleq \sum_{i=1}^{r} \lambda_i F_i
\end{align*}
\]

in which the matrices \( P_i, Z_i, Q_i, \) and \( X_i, i \in S \), are \( n \times n \), symmetric, and positive definite, \( Y_i, G_i, \) and \( F_i, i \in S \), are \( n \times n \). Then we have the following theorem.

**Theorem 3.1:** The system in (10) is stochastically stable if there exist matrices \( P_i > 0, Q_i > 0, X_i > 0, Z_i > 0, Y_i, i \in S, \) and matrices \( G_i, F_i, i \in S, \) which ensure that \( G^{-1}(k) \) and \( F^{-1}(k) \) exist, such that the following inequalities hold:

\[
\begin{bmatrix}
\frac{1}{2} P_{111}(k) & * & * & * \\
-\tilde{Y}^T(k) & -\frac{1}{2} \tilde{Q}(k - \tau) & * & * \\
\tilde{X}(k) & \tilde{A}(k) \tilde{G}^T(k) & \tilde{A}_d(k) \tilde{F}^T(k) & \Pi_{33}(k + 1) \\
\tau (\tilde{A}(k) - I) & \tilde{G}^T(k) & \tau \tilde{A}_d(k) & \tau \Pi_{44}(k)
\end{bmatrix}
< 0,
\]

(13)

\[
\begin{bmatrix}
\frac{1}{2} P_{111}(k) & * & * & * \\
-\tilde{Y}^T(k) & -\frac{1}{2} \tilde{Q}(k - \tau) & * & * \\
\tilde{X}(k) & \tilde{E}(k) \tilde{G}(k) & \tilde{E}_d(k) \tilde{F}(k) & \Pi_{33}(k + 1) \\
\tau \tilde{E}(k) & \tau \tilde{G}(k) & \tau \tilde{E}_d(k) & \tau \Pi_{44}(k)
\end{bmatrix}
< 0,
\]

(14)

\[
\begin{bmatrix}
\tilde{X}(k) & * \\
\tilde{Y}^T(k) & Z(l)
\end{bmatrix}
\geq 0,
\]

(15)
Thus, it follows from (13) and (14) that
\[
\tilde{\Pi}_{11}(k) = -\bar{P}(k) + 2\tau \bar{X}(k) + \bar{G}(k)\tilde{Q}(k,k)\bar{G}^T(k) + 2\bar{G}(k)\bar{F}^{-1}(k)\bar{Y}^T(k) + 2\bar{Y}(k)\bar{F}^{-1}(k)\bar{G}^T(k),
\]
\[
\tilde{\Pi}_{33}(k + 1) = -\bar{G}(k + 1) - \bar{G}^T(k + 1) + \bar{P}(k + 1),
\]
\[
\tilde{\Pi}_{44}(k) = -\frac{\tau}{2} (\bar{F}(k) + \bar{F}^T(k) - \bar{Z}(k)).
\]

**Proof:** From the facts
\[ P_i > 0, \quad Z_i > 0, \]
we have
\[
\{
\tilde{P}(k + 1) - \bar{G}(k + 1)\}\tilde{P}^{-1}(k + 1)\{\tilde{P}(k + 1) - \bar{G}(k + 1)\}^T \geq 0,
\]
\[
\{\bar{Z}(k) - \bar{F}(k)\}^T\bar{Z}^{-1}(k)\{\bar{Z}(k) - \bar{F}(k)\} \geq 0,
\]
then
\[
-\tilde{P}^{-1}(k + 1) \leq -\bar{G}(k + 1) - \bar{G}^T(k + 1) + \bar{P}(k + 1),
\]
\[
\bar{F}^T(k)\bar{Z}^{-1}(k)\bar{F}(k) \geq \bar{F}(k) + \bar{F}^T(k) - \bar{Z}(k).
\]

Thus, it follows from (13) and (14) that
\[
\begin{bmatrix}
\frac{1}{2} \tilde{\Pi}_{11}(k) & * & * & * \\
-\bar{Y}^T(k) & -\frac{1}{2} \bar{Q}(k - \tau) & * & * \\
\bar{A}(k)\bar{G}^T(k) & \bar{A}_d(k)\bar{F}^T(k) & -\tilde{P}^{-1}(k + 1) & * \\
\tau(\bar{A}(k) - I)\bar{G}^T(k) & \tau\bar{A}_d(k)\bar{F}^T(k) & 0 & -\frac{\tau}{2} F^T(k)Z^{-1}(k)F(k)
\end{bmatrix} < 0, \quad (18)
\]
\[
\begin{bmatrix}
\frac{1}{2} \tilde{\Pi}_{11}(k) & * & * & * \\
-\bar{Y}^T(k) & -\frac{1}{2} \bar{Q}(k - \tau) & * & * \\
\bar{E}(k)\bar{G}^T(k) & \bar{E}_d(k)\bar{F}^T(k) & -\tilde{P}^{-1}(k + 1) & * \\
\tau(\bar{A}(k) - I)\bar{G}^T(k) & \tau\bar{A}_d(k)\bar{F}^T(k) & 0 & -\frac{\tau}{2} F^T(k)Z^{-1}(k)F(k)
\end{bmatrix} < 0. \quad (19)
\]

Define matrices \( T_1 \triangleq \text{diag}\{\bar{G}^{-1}(k), \bar{F}^{-1}(k), I, I\} \) and \( T_2 \triangleq \text{diag}\{\bar{G}^{-1}(k), \bar{F}^{-1}(k)\} \). Premultiplying and postmultiplying (18) by \( T_1 \) and \( T_1^T \), respectively, (19) by \( T_1 \) and \( T_1^T \), respectively, (15) by \( T_2 \) and \( T_2^T \), respectively, and considering (12), we have
\[
\begin{bmatrix}
\frac{1}{2} \tilde{\Pi}_{11}(k) & * & * & * \\
-\tilde{Y}^T(k) & -\frac{1}{2} \tilde{Q}(k, k - \tau) & * & * \\
\tilde{A}(k) & \tilde{A}_d(k) & -\tilde{P}^{-1}(k + 1) & * \\
\tau(\tilde{A}(k) - I) & \tau\tilde{A}_d(k) & 0 & -\frac{\tau}{2} \tilde{Z}^{-1}(k)
\end{bmatrix} < 0, \quad (20)
\]
where
\[ \tilde{\Pi}_{11}(k) = -\tilde{P}(k) + 2\tau \tilde{X}(k) + \tilde{Q}(k, k) + 2\tilde{Y}^T(k) + 2\tilde{Y}(k). \]

By the Schur complement theorem, it follows that (20) and (21) are equivalent to
\[ \mathcal{M}_1 \triangleq \begin{bmatrix} \frac{1}{2} \tilde{\Pi}_{11}(k) + \Gamma_1 & * \\ -\tilde{Y}^T(k) + \Gamma_2 & \frac{1}{2} \tilde{Q}(k, k - \tau) + \Gamma_3 \end{bmatrix} < 0, \] (23)
\[ \mathcal{M}_2 \triangleq \begin{bmatrix} \frac{1}{2} \tilde{\Pi}_{11}(k) + \Lambda_1 & * \\ -\tilde{Y}^T(k) + \Lambda_2 & \frac{1}{2} \tilde{Q}(k, k - \tau) + \Lambda_3 \end{bmatrix} < 0, \] (24)

where
\[ \Gamma_1 = \tilde{A}^T(k)\tilde{P}(k + 1)\tilde{A}(k) + 2\tau [\tilde{A}(k) - \tilde{l}]^T \tilde{Z}(k) \tilde{A}(k) - \tilde{l}, \]
\[ \Gamma_2 = \tilde{A}_d^T(k)\tilde{P}(k + 1)\tilde{A}(k) + 2\tau \tilde{A}_d^T(k) \tilde{Z}(k) \tilde{A}(k) - \tilde{l}, \]
\[ \Gamma_3 = \tilde{A}_d^T(k)\tilde{P}(k + 1)\tilde{A}_d(k) + 2\tau \tilde{A}_d^T(k) \tilde{Z}(k) \tilde{A}_d(k), \]
\[ \Lambda_1 = \tilde{E}^T(k)\tilde{P}(k + 1)\tilde{E}(k) + 2\tau \tilde{E}^T(k) \tilde{Z}(k) \tilde{E}(k), \]
\[ \Lambda_2 = \tilde{E}_d^T(k)\tilde{P}(k + 1)\tilde{E}(k) + 2\tau \tilde{E}_d^T(k) \tilde{Z}(k) \tilde{E}(k), \]
\[ \Lambda_3 = \tilde{E}_d^T(k)\tilde{P}(k + 1)\tilde{E}_d(k) + 2\tau \tilde{E}_d^T(k) \tilde{Z}(k) \tilde{E}_d(k). \]

Summing up (23) and (24), we have
\[ \mathcal{M} \triangleq \begin{bmatrix} \tilde{\Pi}_{11}(k) + \Sigma_1 & * \\ -2\tilde{Y}^T(k) + \Sigma_2 & -\tilde{Q}(k, k - \tau) + \Sigma_3 \end{bmatrix} < 0, \] (25)

where
\[ \Sigma_1 = \tilde{A}_d^T(k)\tilde{P}(k + 1)\tilde{A}(k) + \tilde{E}^T(k)\tilde{P}(k + 1)\tilde{E}(k) + 2\tau (\tilde{A}(k) - \tilde{l})^T \tilde{Z}(k) (\tilde{A}(k) - \tilde{l}) \]
\[ + 2\tau \tilde{E}^T(k) \tilde{Z}(k) \tilde{E}(k), \]
\[ \Sigma_2 = \tilde{A}_d^T(k)\tilde{P}(k + 1)\tilde{A}(k) + \tilde{E}_d^T(k)\tilde{P}(k + 1)\tilde{E}(k) + 2\tau \tilde{A}_d^T(k) \tilde{Z}(k) (\tilde{A}(k) - \tilde{l}) \]
\[ + 2\tau \tilde{E}_d^T(k) \tilde{Z}(k) \tilde{E}(k), \]
\[ \Sigma_3 = \tilde{A}_d^T(k)\tilde{P}(k + 1)\tilde{A}_d(k) + \tilde{E}_d^T(k)\tilde{P}(k + 1)\tilde{E}_d(k) + 2\tau \tilde{A}_d^T(k) \tilde{Z}(k) \tilde{A}_d(k) \]
\[ + 2\tau \tilde{E}_d^T(k) \tilde{Z}(k) \tilde{E}_d(k). \]
Substituting (11) into (10), we obtain
\[ x(k + 1) = \overline{A}_v(k)x(k) - \overline{A}_d(k) \sum_{l=k-\tau}^{k-1} \eta(l) + [\overline{E}_v(k)x(k) - \overline{E}_d(k) \sum_{l=k-\tau}^{k-1} \eta(l)]\omega(k), \] (26)

where
\[ \overline{A}_v(k) = \overline{A}(k) + \overline{A}_d(k), \quad \overline{E}_v(k) = \overline{E}(k) + \overline{E}_d(k). \]

Then, we construct the fuzzy LKF
\[ V(k) = V_1(k) + V_2(k) + V_3(k), \] (27)

where
\[ V_1(k) = x^T(k)\overline{P}(k)x(k), \]
\[ V_2(k) = 2 \sum_{s=-\tau}^{1} \sum_{l=k+s}^{k-1} \eta(l)\overline{Z}(l)\eta(l), \]
\[ V_3(k) = \sum_{l=k-\tau}^{k-1} x^T(l)\overline{Q}(k,l)x(l). \]

Along the trajectory of system (26) and taking expectation, we have
\[ \mathbb{E}\{\Delta V_1\} \triangleq \mathbb{E}\{V_1(k + 1) - V_1(k)\} \]
\[ = x^T(k)\overline{A}_v^T(k)\overline{P}(k + 1)\overline{A}_v(k)x(k) - x^T(k)\overline{A}_v^T(k)\overline{P}(k + 1)\overline{A}_d(k) \sum_{l=k-\tau}^{k-1} \eta(l) \]
\[ - \left( \sum_{l=k-\tau}^{k-1} \eta(l) \right)^T \overline{A}_d^T(k)\overline{P}(k + 1)\overline{A}_v(k)x(k) \]
\[ + \left( \sum_{l=k-\tau}^{k-1} \eta(l) \right)^T \overline{A}_d^T(k)\overline{P}(k + 1)\overline{A}_d(k) \sum_{l=k-\tau}^{k-1} \eta(l) \]
\[ + x^T(k)\overline{E}_v^T(k)\overline{P}(k + 1)\overline{E}_v(k)x(k) - x^T(k)\overline{E}_v^T(k)\overline{P}(k + 1)\overline{E}_d(k) \sum_{l=k-\tau}^{k-1} \eta(l) \]
\[ - \left( \sum_{l=k-\tau}^{k-1} \eta(l) \right)^T \overline{E}_d^T(k)\overline{P}(k + 1)\overline{E}_v(k)x(k) \]
\[ + \left( \sum_{l=k-\tau}^{k-1} \eta(l) \right)^T \overline{E}_d^T(k)\overline{P}(k + 1)\overline{E}_d(k) \sum_{l=k-\tau}^{k-1} \eta(l) - x^T(k)\overline{P}(k)x(k) \]
\[ = x^T(k)[\overline{A}_v^T(k)\overline{P}(k + 1)\overline{A}_v(k) + \overline{E}_v^T(k)\overline{P}(k + 1)\overline{E}_v(k) - \overline{P}(k)]x(k) \]
By using Lemma 2.1, we have

\[ + \mu^T(k)[\tilde{A}_d^T(k)\tilde{P}(k + 1)\tilde{A}_d(k) + \tilde{E}_d^T(k)\tilde{P}(k + 1)\tilde{E}_d(k)]\mu(k) \]

\[ - 2 \sum_{l=k-\tau}^{k-1} x^T(k)\Theta_1\eta(l) - 2 \sum_{l=k-\tau}^{k-1} x^T(k)\Theta_2\eta(l) \]

\[ = x^T(k)[\tilde{A}(k) + \tilde{A}_d(k)]^T\tilde{P}(k + 1)[\tilde{A}(k) + \tilde{A}_d(k)] \]

\[ + [\tilde{E}(k) + \tilde{E}_d(k)]^T\tilde{P}(k + 1)[\tilde{E}(k) + \tilde{E}_d(k)] - \tilde{P}(k)x(k) \]

\[ + [x(k) - x(k - \tau)]^T[\tilde{A}_d^T(k)\tilde{P}(k + 1)\tilde{A}_d(k) + \tilde{E}_d^T(k)\tilde{P}(k + 1)\tilde{E}_d(k)] \]

\[ \times [x(k) - x(k - \tau)] - 2 \sum_{l=k-\tau}^{k-1} x^T(k)\Theta_1\eta(l) - 2 \sum_{l=k-\tau}^{k-1} x^T(k)\Theta_2\eta(l) \]

\[ = x^T(k)\tilde{A}_d^T(k)\tilde{P}(k + 1)\tilde{A}_d(k) + \tilde{A}_d^T(k)\tilde{P}(k + 1)\tilde{A}_d(k) + \tilde{A}_d^T(k)\tilde{P}(k + 1)\tilde{A}_d(k) \]

\[ + \tilde{A}_d^T(k)\tilde{P}(k + 1)\tilde{A}_d(k) + \tilde{E}_d^T(k)\tilde{P}(k + 1)\tilde{E}_d(k) + \tilde{E}_d^T(k)\tilde{P}(k + 1)\tilde{E}_d(k) \]

\[ + x^T(k)\tilde{A}_d^T(k)\tilde{P}(k + 1)\tilde{A}_d(k)x(k) + x^T(k)\tilde{E}_d^T(k)\tilde{P}(k + 1)\tilde{E}_d(k)x(k) \]

\[ - x^T(k)\tilde{A}_d^T(k)\tilde{P}(k + 1)\tilde{A}_d(k)x(k - \tau) \]

\[ - x^T(k)\tilde{E}_d^T(k)\tilde{P}(k + 1)\tilde{E}_d(k)x(k - \tau) \]

\[ - x^T(k - \tau)\tilde{A}_d^T(k)\tilde{P}(k + 1)\tilde{A}_d(k)x(k) \]

\[ - x^T(k - \tau)\tilde{E}_d^T(k)\tilde{P}(k + 1)\tilde{E}_d(k)x(k) \]

\[ - x^T(k - \tau)\tilde{A}_d^T(k)\tilde{P}(k + 1)\tilde{A}_d(k)x(k - \tau) \]

\[ - x^T(k - \tau)\tilde{E}_d^T(k)\tilde{P}(k + 1)\tilde{E}_d(k)x(k - \tau) \]

\[ - 2 \sum_{l=k-\tau}^{k-1} x^T(k)\Theta_1\eta(l) - 2 \sum_{l=k-\tau}^{k-1} x^T(k)\Theta_2\eta(l), \]

where

\[ \mu(k) \triangleq \sum_{l=k-\tau}^{k-1} \eta(l) = x(k) - x(k - \tau), \]

\[ \Theta_1 \triangleq \bar{A}_v^T(k)\tilde{P}(k + 1)\bar{A}_d(k), \]

\[ \Theta_2 \triangleq \bar{E}_v^T(k)\tilde{P}(k + 1)\bar{E}_d(k). \]

By using Lemma 2.1, we have

\[-2x^T(k)\Theta_1\eta(l) \leq \left[x(k)\eta(l)\right]^T\left[\bar{X}(k)\eta(l)\right] \leq 2x^T(k)(-\Theta_1)\eta(l) + \eta^T(l)\bar{Z}(l)\eta(l), \]

\[ x^T(k)\tilde{X}(k)x(k) + 2x^T(k)(-\Theta_1)\eta(l) + \eta^T(l)\bar{Z}(l)\eta(l), \quad (29) \]
Considering (29) and (30), we have

\[-2x^T(k)\Theta_2 \eta(l) \leq \left[ x(k) \right]^T \begin{bmatrix} \tilde{X}(k) \\
\tilde{Y}(k) - \Theta_2^T \tilde{Z}(l) \end{bmatrix} \left[ x(k) \right] \]

\[= x^T(k)\tilde{X}(k)x(k) + 2x^T(k)(\tilde{Y}(k) - \Theta_2)\eta(l) + \eta^T(l)\tilde{Z}(l)\eta(l) \]

(30)

with \(\tilde{X}(k), \tilde{Y}(k),\) and \(\tilde{Z}(l)\) satisfying (22).

Considering (29) and (30), we have

\[-2 \sum_{l=k-\tau}^{k-1} x^T(k)\Theta_1 \eta(l) = -2x^T(k) \left[ [\tilde{A}(k) + \tilde{A}_d(k)]^T \tilde{P}(k + 1)\tilde{A}_d(k) \right] [x(k) - x(k - \tau)] \]

\[= -2x^T(k)\tilde{X}(k)\tilde{P}(k + 1)\tilde{A}_d(k)x(k) \]

\[-x^T(k)\tilde{A}_d^T(k)\tilde{P}(k + 1)\tilde{A}_d(k)x(k) \]

\[+ x^T(k)\tilde{X}(k)\tilde{P}(k + 1)\tilde{A}_d(k)x(k) \]

\[-x^T(k)\tilde{A}_d^T(k)\tilde{P}(k + 1)\tilde{A}_d(k)x(k - \tau)] \]

(31)

\[-2 \sum_{l=k-\tau}^{k-1} x^T(k)\Theta_2 \eta(l) = -2x^T(k) \left[ [\tilde{E}(k) + \tilde{E}_d(k)]^T \tilde{P}(k + 1)\tilde{E}_d(k) \right] [x(k) - x(k - \tau)] \]

\[= -2x^T(k)\tilde{E}(k)\tilde{P}(k + 1)\tilde{E}_d(k)x(k) \]

\[-x^T(k)\tilde{E}_d^T(k)\tilde{P}(k + 1)\tilde{E}_d(k)x(k - \tau) \]

\[+ x^T(k)\tilde{E}_d^T(k)\tilde{P}(k + 1)\tilde{E}_d(k)x(k) \]

\[+ x^T(k)\tilde{E}(k)\tilde{P}(k + 1)\tilde{E}_d(k)x(k - \tau)] \]

(32)

\[\mathbb{E}\{\Delta V_2\} \triangleq \mathbb{E}\{V_2(k + 1) - V_2(k)\} \]

\[= 2\tau \eta^T(k)\tilde{Z}(k)\eta(k) - 2 \sum_{l=k-\tau}^{k-1} \eta^T(l)\tilde{Z}(l)\eta(l) \]

\[= 2\tau [x^T(k)[\tilde{A}(k) - \eta] \tilde{Z}(k)[\tilde{A}(k) - \eta]x(k) \]

\[+ x^T(k)[\tilde{A}(k) - \eta] \tilde{Z}(k)\tilde{E}_d(k)x(k - \tau) \]

\[+ x^T(k-\tau)\tilde{A}_d^T(k)\tilde{Z}(k)[\tilde{A}(k) - \eta]x(k - \tau) \]

\[+ x^T(k-\tau)\tilde{A}_d^T(k)\tilde{Z}(k)\tilde{E}_d(k)x(k - \tau) \]

\[+ x^T(k-\tau)\tilde{E}(k)\tilde{E}_d(k)x(k - \tau) \]

\[+ x^T(k-\tau)\tilde{E}_d^T(k)\tilde{Z}(k)\tilde{E}_d(k)x(k - \tau)] \]

(33)

\[\mathbb{E}\{\Delta V_3\} \triangleq \mathbb{E}\{V_3(k + 1) - V_3(k)\} \]

\[= x^T(k)\tilde{Q}(k,k)x(k) - x^T(k-\tau)\tilde{Q}(k,k-\tau)x(k - \tau) \]

(34)
Considering \( \eta(k) = (\overline{A}(k) - I)x(k) + \overline{A}_d(k)x(k - \tau) + [\overline{E}(k)x(k) + \overline{E}_d(k)x(k - \tau)]\omega(k) \) and summing up (28)–(34), we have

\[
\mathbb{E}\{\Delta V\} \triangleq \mathbb{E}\{\Delta V_1\} + \mathbb{E}\{\Delta V_2\} + \mathbb{E}\{\Delta V_3\} \leq \xi^T(k)\mathcal{M}\xi(k),
\]  

(35)

where \( \xi(k) \triangleq \begin{bmatrix} x(k) \\ x(k-\tau) \end{bmatrix} \). Obviously, it follows from (25) that \( \mathbb{E}\{\Delta V\} < 0 \) for all \( \xi(k) \neq 0 \), which concludes from the stability theory that the system in (10) is stochastically stable.

The inequalities we got in Theorem 3.1 contain time-delay parameters. Those parameters are merely available online, therefore, it is impossible for us to check the feasibility of those inequalities. We need to transform those PLMIs [24] into strict LMIs, and then check their feasibility by computer software. Thus, we restrict ourselves to the case of

\[
\overline{F}(k) = \varepsilon G(k).
\]  

(36)

Then, we have the following theorem.

**Theorem 3.2:** The system in (10) is stochastically stable if for some scalar \( \varepsilon > 0 \), there exist matrices \( P_i > 0 \), \( Q_i > 0 \), \( X_i > 0 \), \( Z_i > 0 \), \( Y_i \), and \( G_i \), \( i \in S \), satisfying the LMIs:

\[
\Phi_{stii} < 0, \quad s, t, i \in S,
\]

(37)

\[
\frac{1}{r-1} \Phi_{stii} + \frac{1}{2} (\Phi_{stij} + \Phi_{stji}) < 0, \quad s, t, i, j \in S, i \neq j,
\]

(38)

\[
\Omega_{stii} < 0, \quad s, t, i \in S,
\]

(39)

\[
\frac{1}{r-1} \Omega_{stii} + \frac{1}{2} (\Omega_{stij} + \Omega_{stji}) < 0, \quad s, t, i, j \in S, i \neq j,
\]

(40)

\[
\Psi_{si} \geq 0, \quad s, i \in S,
\]

(41)

where

\[
\Phi_{stij} = \begin{bmatrix}
\frac{1}{2} \Pi_{11,i} & * & * & *
\\
-Y_i^T & -\frac{1}{2} Q_s & * & *
\\
A_i G_i^T & \varepsilon A_i d_i G_i^T & \Pi_{33,t} & *
\\
\tau (A_i - I) G_i^T & \tau \varepsilon A_i d_i G_i^T & 0 & \Pi_{44,i}
\end{bmatrix},
\]

(42)

\[
\Omega_{stij} = \begin{bmatrix}
\frac{1}{2} \Pi_{11,i} & * & * & *
\\
-Y_i^T & -\frac{1}{2} Q_s & * & *
\\
E_i G_i^T & \varepsilon E_i d_i G_i^T & \Pi_{33,t} & *
\\
\tau E_i G_i^T & \tau \varepsilon E_i d_i G_i^T & 0 & \Pi_{44,i}
\end{bmatrix},
\]

(43)

\[
\Psi_{si} = \begin{bmatrix}
X_i \\
Y_i^T \\
Z_s
\end{bmatrix}
\]

(44)

with

\[
\Pi_{11,i} = -P_i + 2\tau X_i + \varepsilon^{-2} Q_i + 2\varepsilon^{-1} Y_i^T + 2\varepsilon^{-1} Y_i,
\]

\[
\Pi_{33,t} = -G_t - G_t^T + P_t,
\]

\[
\Pi_{44,i} = -\frac{\tau}{2} (F_i + F_i^T - Z_i).
\]
**Proof:** Note that the matrices in inequality (13) of Theorem 3.1 can be unfolded as

\[
\begin{bmatrix}
\frac{1}{2} \Pi_{11}(k) & * & * & * \\
-\nu^T(k) & -\frac{1}{2} Q(k - \tau) & * & * \\
A(k)G^T & A_d(k)F^T(k) & \Pi_{33}(k + 1) & * \\
\tau (A(k) - I)G^T & \tau A_d(k)F^T(k) & 0 & \Pi_{44}(k)
\end{bmatrix}
\]

\[
= \sum_{s=1}^{r} \sum_{t=1}^{s} \sum_{1 \leq i < j \leq r} h_s(\theta(k - \tau)) h_t(\theta(k + 1)) \left[ \frac{1}{r-1} h_i^2 \Phi_{stii} + \frac{1}{r-1} h_j^2 \Phi_{stjj} 
+ h_i h_j (\Phi_{stij} + \Phi_{stji}) \right].
\] (45)

So we just need to satisfy

\[
\sum_{1 \leq i < j \leq r} \left[ \frac{1}{r-1} h_i^2 \Phi_{stii} + \frac{1}{r-1} h_j^2 \Phi_{stjj} + h_i h_j (\Phi_{stij} + \Phi_{stji}) \right] < 0
\] (46)

and thus a sufficient condition for (46) is

\[
x^T \left[ \frac{1}{r-1} h_i^2 \Phi_{stii} + \frac{1}{r-1} h_j^2 \Phi_{stjj} \right] x + h_i h_j x^T (\Phi_{stij} + \Phi_{stji}) x < 0, \quad \forall x \neq 0.
\] (47)

By Lemma 2.4, if conditions (37) and (38) hold, then (47) is fulfilled. To sum up, if conditions (37) and (38) hold, then (13) is fulfilled. By the similar line of the proof, we can get that if conditions (39) and (40) hold, then (14) is fulfilled; if conditions (41) hold, then (15) is satisfied. Therefore, it follows from Theorem 3.1 that the time-delay fuzzy system (10) is stochastically stable.

It is noted that we can reduce the number of LMIs by selecting a specific matrix. For example, if we take \( P_1 = P, Q_1 = Q, X_i = X, Z_i = Z, Y_i = Y, \) and \( G_i = G, i \in S, \) then the number would be greatly reduced. In this case, the fuzzy LKF (27) becomes a non-fuzzy one. Then, we can get a corollary as follows.

**Corollary 3.3:** The system in (10) is stochastically stable if for some scalar \( \varepsilon > 0, \) there exist matrices \( P > 0, Q > 0, X > 0, Z > 0, Y, \) and \( G, \) satisfying the LMIs:

\[
\begin{bmatrix}
\frac{1}{2} \Pi_{11} & * & * & * \\
-\nu^T & -\frac{1}{2} Q & * & * \\
A_iG^T & \varepsilon A_dG^T & \Pi_{33} & * \\
\tau (A_i - I)G^T & \tau \varepsilon A_dG^T & 0 & \Pi_{44}
\end{bmatrix}
< 0, \quad i \in S,
\] (48)

\[
\begin{bmatrix}
\frac{1}{2} \Pi_{11} & * & * & * \\
-\nu^T & -\frac{1}{2} Q & * & * \\
E_iG^T & \varepsilon E_dG^T & \Pi_{33} & * \\
\tau E_iG^T & \tau \varepsilon E_dG^T & 0 & \Pi_{44}
\end{bmatrix}
< 0, \quad i \in S,
\] (49)

\[
\begin{bmatrix}
X & * \\
Y^T & Z
\end{bmatrix}
\geq 0,
\] (50)
where

\[
\begin{align*}
\Pi_{11} &= -P + 2\tau X + \epsilon^{-2}Q + 2\epsilon^{-1}Y^T + 2\epsilon^{-1}Y, \\
\Pi_{33} &= -G - G^T + P, \\
\Pi_{44} &= -\frac{\tau}{2}(F + F^T - Z).
\end{align*}
\]

**Proof:** The result in this corollary is a special case of Theorem 3.2; therefore, we omit the proof here. \(\blacksquare\)

Next, we will analyse the time-varying delay, we give the open-loop system of (3) in a compact form

\[
x(k + 1) = \overline{A}(k)x(k) + \overline{A}_d(k)x(k - h(k)) + [\overline{E}(k)x(k) + \overline{E}_d(k)x(k - h(k))]\omega(k),
\]

where

\[
\overline{A}(k) \triangleq \sum_{i=1}^{r} \lambda_i A_i, \quad \overline{A}(k) \triangleq \sum_{i=1}^{r} \lambda_i A_{di},
\]

\[
\overline{E}(k) \triangleq \sum_{i=1}^{r} \lambda_i E_i, \quad \overline{E}_d(k) \triangleq \sum_{i=1}^{r} \lambda_i E_{di}.
\]

**Theorem 3.4:** The system in (51) is stochastically stable if there exist matrices \(P = P^T, Q_1 = Q_1^T, Q_2 = Q_2^T, Q_3 > 0, Z_1 > 0, Z_2 > 0, X, \overline{Y} = [Y_1^T \ Y_2^T \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]^T, \overline{W} = [W_1^T \ W_2^T \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]^T\), such that the following LMIs hold:

\[
\begin{bmatrix}
-Z_2 & \overline{Y}^T & X \\
\overline{Y} & \overline{W} & \overline{W}
\end{bmatrix} < 0,
\]

\[
\begin{bmatrix}
\frac{1}{h_2}P + Z_1 & -Z_1 \\
-Z_1 & \Omega
\end{bmatrix} > 0,
\]

\[
\begin{bmatrix}
P + h_{12}Z_2 & -h_{12}Z_2 \\
-h_{12}Z_2 & h_2Q_2 + h_{12}Z_2
\end{bmatrix} > 0,
\]

where

\[
h_{12} = h_2 - h_1,
\]

\[
\Omega = Q_1 + Q_2 + (h_{12} + 1)Q_3 + Z_1,
\]

\[
\gamma_{11} = \overline{A}_d^T(k)P\overline{A}(k) - P + Q_1 + Q_2 + (h_{12} + 1)Q_3 - Z_1 + \overline{E}_d^T(k)P\overline{E}(k),
\]

\[
\gamma_{12} = \overline{A}_d^T(k)P\overline{A}_d(k) + \overline{E}_d^T(k)P\overline{E}_d(k),
\]

\[
\gamma_{22} = \overline{A}_d^T(k)P\overline{A}_d(k) + \overline{E}_d^T(k)P\overline{E}_d(k) - Q_3,
\]
\[
\Psi = \begin{pmatrix}
\gamma_{11} & \gamma_{12} & Z_1 & 0 & h_1(\bar{A}(k) - I)^TZ_1 & h_{12}(\bar{A}(k) - I)^TZ_2 & h_1E^T(k)Z_1 & h_{12}E^T(k)Z_2 \\
* & \gamma_{22} & 0 & 0 & h_1\bar{A}_d^T(k)Z_1 & h_{12}\bar{A}_d^T(k)Z_2 & h_1\bar{E}_d^T(k)Z_1 & h_{12}\bar{E}_d^T(k)Z_2 \\
* & * & * & -Q_1 - Z_1 & 0 & 0 & 0 & 0 \\
* & * & * & -Q_2 & 0 & 0 & 0 & 0 \\
* & * & * & -Z_1 & 0 & 0 & 0 & 0 \\
* & * & * & * & -Z_2 & 0 & 0 & 0 \\
* & * & * & * & * & -Z_1 & 0 & 0 \\
* & * & * & * & * & * & -Z_2 & \\
\end{pmatrix} + [0 - Y + W] T - [0 - Y + W] T. \tag{55}
\]

**Proof:** Define a Lyapunov functional as

\[
V(k) = V_1(k) + V_2(k) + V_3(k) + V_4(k), \tag{56}
\]

where

\[
V_1(k) = x^T(k)P x(k),
\]

\[
V_2(k) = 2 \sum_{j=1}^{k-1} \sum_{i=k-h_j}^{k-1} x^T(i)Q_j x(i),
\]

\[
V_3(k) = \sum_{i=k-h(k)}^{k-1} x^T(i)Q_3 x(i) + \sum_{j=-h_2+1}^{-h_1} \sum_{i=k+j}^{k-1} x^T(i)Q_3 x(i),
\]

\[
V_4(k) = -1 \sum_{j=-h_1}^{-1} \sum_{i=k+j}^{k-1} h_1 \Delta x^T(i)Z_1 \Delta x(i) + \sum_{j=-h_2}^{-h_1-1} \sum_{i=k+j}^{k-1} h_{12} \Delta x^T(i)Z_2 \Delta x(i),
\]

where

\[
\Delta x(i) = x(i + 1) - x(i). \tag{57}
\]

Under the condition of the theorem, we first show that there exists a scalar \(\delta_1 > 0\), such that

\[
V(k) \geq \delta_1 |x(k)|^2. \tag{58}
\]

For this purpose, we note that

\[
V_1(k) = x^T(k)P x(k) = \sum_{i=k-h_2}^{k-1} \frac{1}{h_2} x^T(k)P x(k) = \sum_{i=k-h_2}^{k-h_1-1} \frac{1}{h_2} x^T(k)P x(k) + \sum_{i=k-h_1}^{k-1} \frac{1}{h_2} x^T(k)P x(k) \tag{59}
\]
and
\[ \sum_{i=k-h_2}^{k-1} x^T(i)Q_2x(i) = \sum_{i=k-h_2}^{k-h_1-1} x^T(i)Q_2x(i) + \sum_{i=k-h_1}^{k-1} x^T(i)Q_2x(i), \]
so, we have
\[ V_2(k) = \sum_{i=k-h_1}^{k-1} x^T(i)Q_1x(i) + \sum_{i=k-h_2}^{k-h_1-1} x^T(i)Q_2x(i) + \sum_{i=k-h_1}^{k-1} x^T(i)Q_2x(i). \] (60)

Furthermore, \( Q_3 > 0 \) and \( h_1 \leq h(k) \leq h_2 \) together imply that
\[ \sum_{i=k-h_k}^{k-1} x^T(i)Q_3x(i) \geq \sum_{i=k-h_1}^{k-1} x^T(i)Q_3x(i) \]
and
\[ \sum_{j=-h_2+1}^{-h_1} \sum_{i=k+j}^{k-1} x^T(i)Q_3x(i) = \sum_{j=k-h_2+1}^{k-h_1-1} (i - k + h_2)x^T(i)Q_3x(i) \]
\[ + \sum_{i=k-h_1}^{k-1} (h_2 - h_1)x^T(i)Q_3x(i) \]
\[ \geq h_{12} \sum_{i=k-h_2}^{k-1} x^T(i)Q_3x(i). \]

Thus, we have
\[ V_3(k) \geq \sum_{i=k-h_1}^{k-1} x^T(i)Q_3x(i) + h_{12} \sum_{i=k-h_2}^{k-1} x^T(i)Q_3x(i). \] (61)

Applying Lemma 2.2 and using the relations in (57), we obtain
\[ \sum_{j=-h_1}^{-1} \sum_{i=k+j}^{k-1} h_1 \Delta x^T(i)Z_1 \Delta x(i) \geq h_1 \sum_{j=-h_1}^{-1} \frac{1}{j} \left( \sum_{i=k+j}^{k-1} \Delta x(i) \right)^T Z_1 \left( \sum_{i=k+j}^{k-1} \Delta x(i) \right) \]
\[ = h_1 \sum_{j=-h_1}^{-1} \frac{1}{j} [x(k) - x(k + j)]^T Z_1 [x(k) - x(k + j)] \]
\[ \geq \sum_{j=-h_1}^{-1} [x(k) - x(k + j)]^T Z_1 [x(k) - x(k + j)] \]
\[ = \sum_{i=k-h_1}^{k-1} [x(k) - x(i)]^T Z_1 [x(k) - x(i)] \]
and
\[
-h_{1}^{-1} \sum_{j=-h_2}^{k-1} h_{12} \Delta x^T(i) Z_2 \Delta x(i) \geq h_{12} \sum_{j=-h_2}^{-h_1-1} \frac{1}{j} \left( \sum_{i=k+j}^{k-1} \Delta x(i) \right)^T Z_2 \left( \sum_{i=k+j}^{k-1} \Delta x(i) \right)
\]
\[
= h_{12} \sum_{j=-h_2}^{-h_1-1} \frac{1}{j} [x(k) - x(k+j)]^T Z_2 [x(k) - x(k+j)]
\]
\[
\geq h_{12} \sum_{j=-h_2}^{-h_1-1} [x(k) - x(k+j)]^T Z_2 [x(k) - x(k+j)]
\]
\[
= h_{12} \sum_{i=k-h_2}^{k-h_1-1} [x(k) - x(i)]^T Z_2 [x(k) - x(i)],
\]
so, we have
\[
V_4(k) \geq \sum_{i=k-h_1}^{k-1} [x(k) - x(i)]^T Z_1 [x(k) - x(i)] + \frac{h_{12}}{h_2} \sum_{i=k-h_2}^{k-h_1-1} [x(k) - x(i)]^T Z_2 [x(k) - x(i)]. \tag{62}
\]

Then, it follows from (59)–(62) that
\[
V(k) \geq \sum_{i=k-h_1}^{k-1} \left\{ \frac{1}{h_2} x^T(k) P x(k) + x^T(i) [Q_1 + Q_2 + \frac{h_{12}}{h_2} Q_3] x(i) \right\}
\]
\[
+ \sum_{i=k-h_1}^{k-1} [x(k) - x(i)]^T Z_1 [x(k) - x(i)]
\]
\[
+ \sum_{i=k-h_2}^{k-h_1-1} \left\{ \frac{1}{h_2} x^T(k) P x(k) + x^T(i) Q_2 x(i) + \frac{h_{12}}{h_2} [x(k) - x(i)]^T Z_2 [x(k) - x(i)] \right\}
\]
\[
= \sum_{i=k-h_1}^{k-1} \left[ x^T(k) \right. x^T(i) \left[ \begin{array}{cc} \frac{1}{h_2} P + Z_1 & -Z_1 \\ -Z_1 & \Omega \end{array} \right] \]
\[
+ \frac{1}{h_2} \sum_{i=k-h_2}^{k-h_1-1} k^{-1} x^T(k) \left[ \begin{array}{cc} P + h_{12} Z_2 & -h_{12} Z_2 \\ -h_{12} Z_2 & h_2 Q_2 + h_{12} Z_2 \end{array} \right] x(i) \right]. \tag{63}
\]

This, together with (53) and (54), imply that there exists a scalar $\delta_1 > 0$, such that (58) holds.

Now, we show that there exists a scalar $\delta_2 > 0$, such that
\[
\Delta V(k) \leq -\delta_2 |x(k)|^2. \tag{64}
\]

We have
\[
\mathbb{E} \{ \Delta V_1(k) \} \triangleq \mathbb{E} \{ V_1(k+1) - V_1(k) \}
\]
\[
= x^T(k+1) P x(k+1) - x^T(k) P x(k)
\]
\[ \begin{align*}
&= [\overline{A}(k)x(k) + \overline{A}_d(k)x(k - h(k))] + [\overline{E}(k)x(k) + \overline{E}_d(k)x(k - h(k))]^{\omega(k)}]^{T} \\
&\times P[\overline{A}(k)x(k) + \overline{A}_d(k)x(k - h(k))] + [\overline{E}(k)x(k) + \overline{E}_d(k)x(k - h(k))]
- x^T(k)Px(k) \\
&= x^T(k)\overline{A}^T(k)P\overline{A}(k)x(k) + x^T(k)\overline{A}^T(k)P\overline{A}_d(k)x(k - h(k)) \\
&\quad + x^T(k - h(k))\overline{A}_d^T(k)P\overline{A}(k)x(k) + x^T(k)\overline{E}^T(k)P\overline{E}(k)x(k) \\
&\quad + x^T(k - h(k))\overline{A}_d^T(k)P\overline{A}_d(k)x(k - h(k)) \\
&\quad + x^T(k)\overline{E}_d^T(k)P\overline{E}_d(k)x(k - h(k)) + x^T(k - h(k))\overline{E}_d^T(k)P\overline{E}_d(k)x(k) \\
&\quad + x^T(k - h(k))\overline{E}_{d}^T(k)P\overline{E}_d(k)x(k - h(k)) - x^T(k)Px(k), \\
&\text{where} \quad \overline{A}, \overline{A}_d, \overline{E}, \overline{E}_d, \omega, P, \text{ and } \overline{E}_d \text{ are defined.}
\end{align*} \]

\[ \begin{align*}
\mathbb{E}\{\Delta V_2(k)\} &\triangleq \mathbb{E}\{V_2(k + 1) - V_2(k)\} \\
&= \sum_{i=k-h_1+1}^{k} x^T(i)Q_1x(i) + \sum_{i=k-h_2+1}^{k} x^T(i)Q_2x(i) - \sum_{i=k-h_1}^{k-1} x^T(i)Q_1x(i) \\
&\quad - \sum_{i=k-h_2}^{k-1} x^T(i)Q_2x(i), \quad (65)
\end{align*} \]

\[ \begin{align*}
\mathbb{E}\{\Delta V_3(k)\} &\triangleq \mathbb{E}\{V_3(k + 1) - V_3(k)\} \\
&= \sum_{i=k-h(k+1)+1}^{k} x^T(i)Q_3x(i) + \sum_{j=-h_2+1}^{h_1} \sum_{i=k+j+1}^{k} x^T(i)Q_3x(i) \\
&\quad - \sum_{i=k-h(k)}^{k-1} x^T(i)Q_2x(i) - \sum_{j=-h_2+1}^{h_1} \sum_{i=k+j}^{k-1} x^T(i)Q_3x(i) \\
&= \sum_{i=k-h(k+1)+1}^{k-1} x^T(i)Q_3x(i) - \sum_{i=k-h(k)+1}^{k-1} x^T(i)Q_2x(i) - \sum_{i=k-h(k)}^{k-1} x^T(k)Q_3x(k) \\
&\quad - x^T(k - h(k))Q_3x(k - h(k)) + h_{12}x^T(k)Q_3x(k) \\
&\quad - \sum_{i=k-h_1}^{k-h_2+1} x^T(i)Q_3x(i) \\
&\leq \sum_{i=k-h_2+1}^{k-h_1} x^T(i)Q_3x(i) + x^T(k)Q_3x(k) - x^T(k - h(k))Q_3x(k - h(k)) \\
&\quad + h_{12}x^T(k)Q_3x(k) - \sum_{i=k-h_2+1}^{k-h_1} x^T(i)Q_3x(i), \quad (67)
\end{align*} \]
Next, we introduce several slack matrices to further reduce conservatism. According to the definition of $\Delta x(i)$, for any matrices $Y = [Y_1^T \ Y_2^T \ 0 \ 0]^T$ and $W = [W_1^T \ W_2^T \ 0 \ 0]^T$, we have

\[
0 = 2\zeta^T(k)Y \begin{bmatrix} x(k - h_1) - x(k - h(k)) - \sum_{i=k-h(k)}^{k-h_1-1} \Delta x(i) \end{bmatrix},
\]

\[
0 = 2\zeta^T(k)W \begin{bmatrix} x(k - h(k)) - x(k - h_2) - \sum_{i=k-h_2}^{k-h(k)-1} \Delta x(i) \end{bmatrix},
\]

where

\[
\zeta(k) = [x^T(k) x^T(k - h(k)) x^T(k - h_1) x^T(k - h_2)]^T.
\]

Note that

\[
\Delta x(k) = x(k + 1) - x(k) = [\bar{A}(k) - I]x(k) + \bar{A}_d(k)x(k - h(k)) + [\bar{E}(k)x(k) + \bar{E}_d(k)x(k - h(k))]\omega(k).
\]

Let

\[
\Psi = \begin{bmatrix}
\gamma_{11} & \bar{A}_d^T(k)P\bar{A}_d(k) + \bar{E}_d^T(k)P\bar{E}_d(k) & Z_1 & 0 \\
* & \bar{A}_d^T(k)P\bar{A}_d(k) + \bar{E}_d^T(k)P\bar{E}_d(k) - Q_3 & 0 & 0 \\
* & * & -Q_1 - Z_1 & 0 \\
* & * & * & -Q_2
\end{bmatrix}
\]
Then it is derived from (65)–(72) that
\[
\begin{align*}
\Delta V(k) &\leq \xi^T(k) \Psi \xi(k) - 2\xi^T(k) Y \sum_{i=k-h(k)}^{k-h_1-1} \Delta x(i) - 2\xi^T(k) W \sum_{i=k-h_2}^{k-h(k)-1} \Delta x(i) \\
&- \sum_{i=k-h(k)}^{k-h_1-1} \Delta x^T(i) h_{12} Z_2 \Delta x(i) - \sum_{i=k-h_2}^{k-h(k)-1} \Delta x^T(i) h_{12} Z_2 \Delta x(i).
\end{align*}
\]

Rewriting \( h_{12} = h_2 - h_1 \) as \( h_{12} = h_2 - h(k) + h(k) - h_1 \), we have
\[
\begin{align*}
\Delta V(k) &\leq \frac{1}{h_{12}} \sum_{i=k-h(k)}^{k-h_1-1} \left[ \xi^T(k) \right] \left[ \begin{array}{c}
\Psi \\
Y^T
\end{array} \right] \left[ \begin{array}{c}
Y \\
-Z_2
\end{array} \right] \left[ \xi(k) \right] \\
&+ \frac{1}{h_{12}} \sum_{i=k-h_2}^{k-h(k)-1} \left[ \xi^T(k) \right] \left[ \begin{array}{c}
\Psi \\
W^T
\end{array} \right] \left[ \begin{array}{c}
W \\
-Z_2
\end{array} \right] \left[ \xi(k) \right] .
\end{align*}
\]

On the other hand, by Lemma 2.3, there exists an \( X \) of appropriate dimensions such that (52) holds if and only if
\[
\begin{bmatrix}
\Psi & Y \\
Y^T & -Z_2
\end{bmatrix} < 0, \quad \begin{bmatrix}
\Psi & W \\
W^T & -Z_2
\end{bmatrix} < 0.
\]

According to the Schur complement theorem, the system (75) is equivalent to
\[
\begin{bmatrix}
\Psi & Y \\
Y^T & -Z_2
\end{bmatrix} < 0, \quad \begin{bmatrix}
\Psi & W \\
W^T & -Z_2
\end{bmatrix} < 0.
\]

Therefore, if the condition (52) is satisfied, so does the condition (76). By (74), there exists a scalar \( \delta_2 > 0 \) such that \( \Delta V(k) \leq -\delta_2 \| x(k) \|^2 < 0 \) for \( x(k) \neq 0 \), which is concluded that the system in (51) is stochastically stable.

\[\blacksquare\]

4. Illustrative Examples

In this section, two examples are employed to illustrate the method developed in Sections 3.

Example 4.1 shows that this method is effective and the stability condition based on the new fuzzy LKF is less conservative than that based on non-fuzzy LKF. Example 4.2 shows that this method reduces the conservatism.
Example 4.1 ([22]): Consider the discrete-time delay fuzzy system (10) with
\[ r = 2, \quad \tau = 1, \quad \varepsilon = 50, \]
\[ A_1 = \begin{bmatrix} 0.4 & 0 \\ 0.01 & -0.3 \end{bmatrix}, \quad A_{d1} = \begin{bmatrix} 0.2 & 0 \\ 0 & -0.2 \end{bmatrix}, \]
\[ E_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad E_{d1} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \]
\[ A_2 = \begin{bmatrix} -0.4 & 0 \\ 0.02 & 0.2 \end{bmatrix}, \quad A_{d2} = \begin{bmatrix} -0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \]
\[ E_2 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad E_{d2} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}. \]

By using the Matlab LMI Toolbox, it can be found that the LMIs of Theorem 3.2 have the feasible solution
\[ P_1 = \begin{bmatrix} 0.0102 & 0.0000 \\ 0.0000 & 0.0111 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0.0096 & -0.0002 \\ -0.0002 & 0.0126 \end{bmatrix}, \]
\[ Q_1 = \begin{bmatrix} 4.5957 & -0.0094 \\ -0.0094 & 4.1561 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 4.5421 & -0.0167 \\ -0.0167 & 4.1895 \end{bmatrix}, \]
\[ X_1 = 10^{-3} \cdot \begin{bmatrix} 0.4758 & 0.0189 \\ 0.0189 & 0.5181 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0.0002 & -0.0000 \\ -0.0000 & 0.0015 \end{bmatrix}, \]
\[ Z_1 = \begin{bmatrix} 0.3935 & 0.0024 \\ 0.0024 & 0.3238 \end{bmatrix}, \quad Z_2 = \begin{bmatrix} 0.1836 & -0.0028 \\ -0.0028 & 0.6364 \end{bmatrix}, \]
\[ Y_1 = \begin{bmatrix} -0.0043 & -0.0002 \\ 0.0000 & -0.0026 \end{bmatrix}, \quad Y_2 = \begin{bmatrix} -0.0023 & 0.0003 \\ -0.0001 & -0.0058 \end{bmatrix}, \]
\[ G_1 = \begin{bmatrix} 0.0117 & -0.0000 \\ 0.0001 & 0.0154 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 0.0111 & -0.0001 \\ -0.0002 & 0.0173 \end{bmatrix}. \]

However, we can prove that the LMIs of Corollary 3.3 are infeasible. This shows that with \( \varepsilon = 50 \), Theorem 3.2 guarantees the stability of the system while Corollary 3.3 cannot. As we expected, this method is effective and the proposed fuzzy LKF-based stability condition is less conservative than the non-fuzzy LKF-based one.

Example 4.2: Consider the discrete-time delay fuzzy system (51) with
\[ r = 2, \]
\[ A_1 = \begin{bmatrix} 0.4 & 0 \\ 0.01 & -0.3 \end{bmatrix}, \quad A_{d1} = \begin{bmatrix} 0.2 & 0 \\ 0 & -0.2 \end{bmatrix}, \]
\[ E_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad E_{d1} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}. \]
\[
A_2 = \begin{bmatrix} -0.4 & 0 \\ 0.02 & 0.2 \end{bmatrix}, \quad A_{d2} = \begin{bmatrix} -0.2 & 0 \\ 0 & 0.2 \end{bmatrix},
\]
\[
E_2 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad E_{d2} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}.
\]

For different value of \( h_1 \), the admissible upper bound \( h_2 \) of the delay is listed in Table 1. Compared with the results in [21], the proposed method is less conservative. The corresponding feasible solutions are as follows:

\[
h_1 = 1,
\]
\[
P = \begin{bmatrix} 9.1618 & 0.0034 \\ 0.0034 & 5.2445 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 0.0020 & 0.0005 \\ 0.0005 & 0.0585 \end{bmatrix},
\]
\[
Q_2 = \begin{bmatrix} 0.0040 & 0.0007 \\ 0.0007 & 0.0875 \end{bmatrix}, \quad Q_3 = \begin{bmatrix} 0.7708 & 0.0037 \\ 0.0037 & 0.4498 \end{bmatrix},
\]
\[
Z_1 = \begin{bmatrix} 0.0025 & 0.0004 \\ 0.0004 & 0.0733 \end{bmatrix}, \quad Z_2 = \begin{bmatrix} 0.0001 & 0.0000 \\ 0.0000 & 0.0038 \end{bmatrix},
\]

\[
h_1 = 2
\]
\[
P = \begin{bmatrix} 4.8501 & 0.0016 \\ 0.0016 & 2.8847 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 0.0013 & 0.0003 \\ 0.0003 & 0.0387 \end{bmatrix},
\]
\[
Q_2 = \begin{bmatrix} 0.0021 & 0.0003 \\ 0.0003 & 0.0445 \end{bmatrix}, \quad Q_3 = \begin{bmatrix} 0.4080 & 0.0021 \\ 0.0021 & 0.2468 \end{bmatrix},
\]
\[
Z_1 = \begin{bmatrix} 0.0003 & 0.0001 \\ 0.0001 & 0.0098 \end{bmatrix}, \quad Z_2 = \begin{bmatrix} 0.0001 & 0.0000 \\ 0.0000 & 0.0021 \end{bmatrix},
\]

\[
h_1 = 3
\]
\[
P = \begin{bmatrix} 6.6283 & 0.0021 \\ 0.0021 & 3.8574 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 0.0019 & 0.0004 \\ 0.0004 & 0.0558 \end{bmatrix},
\]
\[
Q_2 = \begin{bmatrix} 0.0028 & 0.0004 \\ 0.0004 & 0.0571 \end{bmatrix}, \quad Q_3 = \begin{bmatrix} 0.5576 & 0.0027 \\ 0.0027 & 0.3295 \end{bmatrix},
\]
\[
Z_1 = \begin{bmatrix} 0.0002 & 0.0000 \\ 0.0000 & 0.0057 \end{bmatrix}, \quad Z_2 = \begin{bmatrix} 0.0001 & 0.0000 \\ 0.0000 & 0.0028 \end{bmatrix},
\]

\[
h_1 = 4
\]
\[
P = \begin{bmatrix} 5.2816 & 0.0017 \\ 0.0017 & 2.9213 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 0.0014 & 0.0003 \\ 0.0003 & 0.0440 \end{bmatrix},
\]
\[
Q_2 = \begin{bmatrix} 0.0021 & 0.0003 \\ 0.0003 & 0.0419 \end{bmatrix}, \quad Q_3 = \begin{bmatrix} 0.4443 & 0.0021 \\ 0.0021 & 0.2492 \end{bmatrix},
\]
\[
Z_1 = \begin{bmatrix} 0.0001 & 0.0000 \\ 0.0000 & 0.0024 \end{bmatrix}, \quad Z_2 = \begin{bmatrix} 0.0001 & 0.0000 \\ 0.0000 & 0.0021 \end{bmatrix},
\]
Table 1. Admissible upper bound $h_2$ for various $h_1$.

| $h_1$ | 1   | 2   | 3   | 4   | 5   |
|-------|-----|-----|-----|-----|-----|
| [21]  | 4   | 5   | 6   | 7   | 8   |
| Theorem 3.4 | 7   | 8   | 9   | 10  | 11  |

$h_1 = 5$,

\[ P = \begin{bmatrix} 15.6528 & 0.0050 \\ 0.0050 & 9.3339 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 0.0046 & 0.0010 \\ 0.0010 & 1.0410 \end{bmatrix}, \]

\[ Q_2 = \begin{bmatrix} 0.0068 & 0.0009 \\ 0.0009 & 0.1283 \end{bmatrix}, \quad Q_3 = \begin{bmatrix} 1.3167 & 0.0066 \\ 0.0066 & 0.7975 \end{bmatrix}, \]

\[ Z_1 = \begin{bmatrix} 0.0001 & 0.0000 \\ 0.0000 & 0.0048 \end{bmatrix}, \quad Z_2 = \begin{bmatrix} 0.0002 & 0.0000 \\ 0.0000 & 0.0067 \end{bmatrix}. \]

5. Conclusion

The stability of Discrete T-S fuzzy stochastic system with time delay is studied. The fuzzy stochastic turbulence considered in the new system has broadened the applications in the more complicated irregular internal and external interference cases. The symmetric matrices involved in the novel Lyapunov–Krasovskii functional get rid of the positive definiteness restrictions. Numerical experiments show that the stability condition, obtained by this new Lyapunov–Krasovskii functional, is less conservative.

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No potential conflict of interest was reported by the author(s).

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