MAXIMAL SUBALGEBRAS OF THE CLASSICAL LINEAR LIE SUPERA LGEbras

IRINA SHCHEPOCHKINA

Abstract. Dynkin’s classification of maximal subalgebras of simple finite dimensional complex Lie algebras is generalized to linear Lie superalgebras. Namely, the maximal non-simple irreducible subalgebras of $\mathfrak{gl}(p|q)$, $\mathfrak{q}(n)$, $\mathfrak{s}l(p|q)$, $\mathfrak{osp}(m|2n)$, $\mathfrak{pe}(n)$, and $\mathfrak{spe}(n)$ are classified.

Introduction

0.1. Dynkin’s result. In 1951, Dynkin published two remarkable papers somewhat interlaced in their theme: classification of semi-simple [D1] and maximal [D2] subalgebras of simple (finite dimensional) Lie algebras. These classifications are of interest per se; they also proved to be useful in the studies of integrable systems and in representation theory.

A. L. Onishchik and D. Leites asked me to generalize Dynkin’s results to “classical” Lie superalgebras. In this paper (partly preprinted in [Sh1], [Sh3]) I try to give the answer in a form similar to that of Dynkin’s result. Let me first remind Dynkin’s result. In what follows the adjective “linear” describes a Lie (super)algebra $\mathfrak{g}$ whose elements are realized as linear operators in a linear space $V$.

Let $\mathfrak{g}$ be a simple complex linear Lie algebra, e.g., $\mathfrak{g} = \mathfrak{sl}(V)$, $\mathfrak{o}(V)$, or $\mathfrak{sp}(V)$, $\mathfrak{h} \subset \mathfrak{g}$ its maximal subalgebra. Then only the following 3 cases can occur:

1) the representation of $\mathfrak{h}$ in $V$ is irreducible, $\mathfrak{h}$ is not simple. Dynkin’s list is as follows (here the cases $\dim V_2 = 4$ in the second and fourth lines, as well as $\dim V_2 = \dim V_1 = 2$ in the third one, are exceptional because $\mathfrak{o}(4) \cong \mathfrak{sp}(2) \oplus \mathfrak{sp}(2)$):

| $\mathfrak{h}$ | $\mathfrak{g}$ | condition |
|---------------|---------------|------------|
| $\mathfrak{sl}(V_1) \oplus \mathfrak{sl}(V_2)$ | $\mathfrak{sl}(V_1 \otimes V_2)$ | $\dim V_2 \geq \dim V_1 \geq 2$ |
| $\mathfrak{sp}(V_1) \oplus \mathfrak{o}(V_2)$ | $\mathfrak{sp}(V_1 \otimes V_2)$ | $\dim V_1 \geq 2$, $\dim V_2 \geq 3$, $\dim V_2 \neq 4$ or $\dim V_1 = 2$ and $\dim V_2 = 4$; |
| $\mathfrak{sp}(V_1) \oplus \mathfrak{sp}(V_2)$ | $\mathfrak{o}(V_1 \otimes V_2)$ | $\dim V_2 \geq \dim V_1 \geq 2$ except $\dim V_1 = \dim V_2 = 2$, |
| $\mathfrak{o}(V_1) \oplus \mathfrak{o}(V_2)$ | $\mathfrak{o}(V_1 \otimes V_2)$ | $\dim V_2 \geq \dim V_1 \geq 3$ and $\dim V_1$, $\dim V_2 \neq 4$. |

AMS Subject classification: 17A70

Keywords and phrases: maximal subalgebra, Lie algebra, Lie superalgebra

*I am thankful to D. Leites for help; RFBR grants 99-01-00245 and 01-01-00460a; NFR for partial financial support and Stockholm University, where most results of this paper were preprinted in 1988 (no. 32/1988-15), for hospitality.*
2) \( \mathfrak{h} \) is simple and irreducible (i.e., \( \mathfrak{h} \) irreducibly acts on \( V \)). For practically every irreducible representation of a simple \( \mathfrak{h} \) in a linear space \( V \) the image is a maximal subalgebra in one of the three classical simple linear algebras \( \mathfrak{sl}(V), \mathfrak{o}(V) \) or \( \mathfrak{sp}(V) \). Dynkin proved this and listed the exceptional cases.

3) \( \mathfrak{h} \) is reducible. Then \( \mathfrak{h} \) can be described as the collection of all operators from \( \mathfrak{g} \) that preserve a subspace \( W \subset V \). Here \( W \) can be arbitrary for \( \mathfrak{g} = \mathfrak{sl} \) whereas for \( \mathfrak{g} = \mathfrak{o}(n) \) and \( \mathfrak{sp}(2n) \) the bilinear form \( \omega \) on \( V \) preserved by \( \mathfrak{g} \) must either be non-degenerate or identically vanish on \( W \). Such algebras \( \mathfrak{h} \) are certain parabolic subalgebras of \( \mathfrak{g} \).

Passing to subsuperalgebras we encounter the same cases. Difficulties in their superization range widely:

**Superization of case 3)** to Lie superalgebras with Cartan matrix and linear ones is more or less straightforward, see [ZZO].

**Superization of case 2)** requires a more or less explicit description of finite dimensional irreducible modules over simple Lie superalgebras. There are two types of such modules typical and atypical ones, cf. [K1], [K2], [PS]. Observe that Lemma 4.4 demonstrates that, unlike Lie algebra case, the images of Lie superalgebras \( \mathfrak{sl}(m|n), \mathfrak{osp}(2|2n), \mathfrak{pc}(n), \mathfrak{pe}(n), \mathfrak{vect}(0|n), \mathfrak{svect}(0|n), \mathfrak{h}(0|n), \mathfrak{h}^\prime(0|n) \) in the typical modules are not maximal, except for the case considered in Lemma 3.4.1.

The case of other algebras and atypical modules constitutes an open problem. For a partial result see [II]. With a general character formula, even conjectural, [PS], [Se] one can now hope to be able to derive the complete result.

**Superization of case 1)** is what is done in this paper: the description of the irreducible non-simple maximal subsuperalgebras of linear complex Lie superalgebras either simple or “classical”, i.e., certain algebras closely related to simple ones.

Regrettably, a precise description of the maximal Lie superalgebras of type 1) is more involved than Dynkin’s description (0.1) above. Indeed, there are too many exceptional cases occasioned by small dimensions. Nevertheless, in Main Theorem, I distinguish four main types of subalgebras such that any type 1) linear Lie superalgebra is contained in one of these four types of Lie superalgebras. Thus, the subalgebras distinguished in Main Theorem are the main candidates for the roles of maximal subalgebras.

Observe that two of these four types are similar to Dynkin’s types whereas the other two are of totally different nature, and the picture is similar to that over fields of prime characteristic.

In §1 I describe the main constructions and give a precise formulation of Main Theorem. Statements describing when the subalgebras from Main Theorem are indeed maximal are collected in Tables 1–3. §2 is devoted to a proof of maximality of Dynkin-type subalgebras, in §3 the other two types are considered. In §4 I prove Main Theorem.

In what follows, \( \mathfrak{i} \in \mathfrak{a} \) designates a semi-direct sum of algebras of which \( \mathfrak{i} \) is an ideal; same notation is used for indecomposable modules with a submodule \( \mathfrak{i} \).

### 0.2. Comparison with the case of prime characteristic.

Our results resemble Ten’s result in prime characteristic [II]. To formulate it, recall that a subalgebra of a (finite dimensional) Lie algebra is called regular if it is invariant with respect to a maximal torus.

**Theorem.** Let \( \mathbb{K} \) be an algebraically closed field of characteristic \( p > 3 \).

1) Any non-semi-simple maximal subalgebra of \( \mathfrak{sl}(n) \) for \( n \neq 0 \mod p, \mathfrak{o}(n) \) or \( \mathfrak{sp}(2n) \) is regular.

2a) Let \( \dim V = np^m, (n,p) = 1, n > 1 \). If \( \mathfrak{m} \) is an irreducible maximal subalgebra in \( \mathfrak{sl}(V) \) such that \( \mathfrak{pm} = \mathfrak{m}/\mathbb{K}1_V \) is not semi-simple, then \( V = U \otimes \mathfrak{O}_m \), where \( \mathfrak{O}_m = \mathbb{K}[x_1,\ldots,x_m]/(x_1^p,\ldots,x_m^p) \) and \( \mathfrak{m} = \mathfrak{gl}(U) \otimes \mathfrak{O}_m \in \mathfrak{vect}(\mathfrak{m}) \).
2b) If \( n = 1 \), then in addition to the above examples 2a) the algebra \( \mathfrak{m} = \mathfrak{sp}(2m) \oplus \mathfrak{hei}(2m|0) \) (where \( \mathfrak{hei}(2m|0) \) is an even version of the Heisenberg Lie algebra, see sec. 2.7) is also maximal in \( \mathfrak{sl}(V) \).

3) Any maximal subalgebra in \( \mathfrak{g}_2 \) is regular except \( \text{vect}(1) \) for \( p = 7 \) and \( \mathfrak{sl}(2) \) for \( p > 7 \).

0.3. Related results. Maximal solvable Lie subsuperalgebras. Such subalgebras for \( \mathfrak{gl}(m|n) \) and \( \mathfrak{sl}(m|n) \) are classified in [Sh2]. A bizarre series of subalgebras was discovered. Maximal solvable subalgebras — Borel subalgebras — of simple Lie superalgebras are important in representation theory (e.g., for construction of Verma modules). In super setting, the maximal solvable subalgebras can be larger than what is used for construction of Verma modules and what Penkov justly suggested to call Borel subalgebras. Conjecturally, these larger algebras are related with atypical representations.

Superization: case 4) A nonhomogeneous with respect to parity subalgebra \( \mathfrak{h} \) of the Lie superalgebra \( \mathfrak{g} \) is called Volichenko algebra. A list of simple finite dimensional Volichenko subalgebras in simple Lie superalgebras is obtained under a technical condition by Serganova [S]. (For motivations and infinite dimensional case see [LS], [KL].) Simple Volichenko algebras are one more, new, type of maximal subalgebras of simple Lie superalgebras: Volichenko subalgebras are not Lie subsuperalgebras.

1. Notation, Background and Main Statements

Before we formulate our result, we need to fix several notations and constructions. Formulas of linear algebra are generalized to linear superalgebra by means of linearity and Sign Rule, such are, for example, definitions of supercommutator, Lie superalgebra. There are, however, notions, e.g., supertrace, which though follow from Sign Rule, are not obvious direct corollaries, cf. [D]. Some facts, like existence of (at least) two analogs of the general linear algebra, or two types of bilinear forms (even and odd) are even less familiar.

1.1. Basics. Throughout the paper the ground field is \( \mathbb{C} \). A superspace is a \( \mathbb{Z}/2 \)-graded linear space \( V = V_0 \oplus V_1 \), where \( \mathbb{Z}/2 = \{0, 1\} \) and \( p(v) = \bar{1} \) if \( v \in V_1 \). The superdimension of \( V \) is a pair \( N = m|n \), where \( m = \dim V_0, n = \dim V_1 \). The usual formula \( \dim V \otimes W = \dim V \cdot \dim W \) becomes manifest if we introduce a formal symbol \( \varepsilon \) such that \( \varepsilon^2 = 1 \) and set \( \dim V = \dim V_0 + \dim V_1 \varepsilon \).

For a superspace \( V = V_0 \oplus V_1 \) denote by \( \Pi(V) \) another copy of the same superspace: with the shifted parity, i.e., \( \Pi(V_i) = V_{i+1} \). The subsuperspace \( U \subset V \) is a subspace such that \( U = U \cap V_0 \oplus U \cap V_1 \). A superspace structure in \( V \) induces the superspace structure in the space \( \text{End}(V) \).

The Lie superalgebra of all linear operators in \( V \) is called the general Lie superalgebra. It is denoted by \( \mathfrak{gl}(V) \) or \( \mathfrak{gl}(\dim V) \). Having selected a homogeneous basis of \( V \), we can represent operators by supermatrices; in this paper I only need supermatrices in the standard format, i.e., when the even basis vectors of \( V \) are collected together and come first.

The space of operators with zero supertrace constitutes the special linear Lie subsuperalgebra \( \mathfrak{sl}(V) \) also denoted \( \mathfrak{sl}(\dim V) \).

There are, however, at least two super versions of \( \mathfrak{gl}(V) \), not one. Another version is called the queer Lie superalgebra and is defined as the one that preserves the complex structure given by an odd operator \( J \), i.e., is the centralizer \( C(J) \) of \( J \):

\[
\mathfrak{q}(V) = C(J) = \{ X \in \mathfrak{gl}(V) \mid [X, J] = 0 \}, \text{ where } J^2 = -\text{id}.
\]
It is clear that by a change of basis we can reduce $J$ to the form $J_{2n} = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$. Then in matrix form we have

$$q(n) = \left\{ X \in \mathfrak{gl}(n|n) \mid X = \begin{pmatrix} A & B \\ B & A \end{pmatrix} \right\}.$$  

On $q(n)$, the queer trace is defined: $qtr : \begin{pmatrix} A & B \\ B & A \end{pmatrix} \mapsto \text{tr } B$. Denote by $sq(n)$ the Lie superalgebra of queertraceless operators.

Observe that the identity representations of $\mathfrak{g} = \mathfrak{q}(V)$ and $\mathfrak{sq}(V)$ in $V$, though irreducible in super setting, are not irreducible in the non-graded sense: take linearly independent vectors $v_1, \ldots, v_n$ from $V_0$; then $\text{Span}(v_1 + J(v_1), \ldots, v_n + J(v_n))$ is a $\mathfrak{g}$-invariant subspace of $V$ which is not a subsuperspace.

A representation is called irreducible of $G$-type if it has no invariant subspace; it is called irreducible of $Q$-type if it has no invariant subsuperspace, but has an invariant subspace.

1.2. The action of $\mathfrak{g}_1(V_1) \oplus \mathfrak{g}_2(V_2)$ in $V_1 \otimes V_2$. Given two irreducible (of $G$- or $Q$-type) linear Lie superalgebras $\mathfrak{g}_i \subset \mathfrak{gl}(V_i)$, we obtain a representation of the Lie superalgebra $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ in the superspace $V = V_1 \otimes V_2$, the tensor product of the given representations:

$$X_1 + X_2 \mapsto X_1 \otimes 1 + 1 \otimes X_2 \quad \text{for } X_1 \in \mathfrak{g}_1, \ X_2 \in \mathfrak{g}_2. \quad (1.1)$$

**G-construction:** not both the $\mathfrak{g}_i$ are of $Q$-type. If both $\mathfrak{g}_1$ and $\mathfrak{g}_2$ contain the identity operators, the representation (1.1) has a 1-dimensional kernel. By $\mathfrak{g}_1 \circlearrowleft \mathfrak{g}_2$ we will mean the image of the direct sum $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ under the representation (1.1). Observe that $\mathfrak{g}_1 \circlearrowleft \mathfrak{g}_2$ is irreducible in this case.

It is convenient to retain the notation $\mathfrak{g}_1 \circlearrowleft \mathfrak{g}_2$ even in the absence of the kernel, i.e., when $\mathfrak{g}_1 \circlearrowleft \mathfrak{g}_2 \cong \mathfrak{g}_1 \oplus \mathfrak{g}_2$.

**Q-construction:** both the $\mathfrak{g}_i$ are irreducible of $Q$-type. In this case the standard action of $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ in $V_1 \otimes V_2$ is reducible. That is why the construction of $\mathfrak{g}_1 \circlearrowleft \mathfrak{g}_2$ is different, namely, as follows.

Let $U$ be a fixed $(1|1)$-dimensional superspace. The operators $J_U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $I_U = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \in \text{End}(U)$ supercommute and $J_U^2 = I_U^2 = -1$. Let us represent $n_1|n_1$-dimensional superspace $V_1$ as a tensor product $V_1 = (V_1)_0 \otimes U$ and $n_2|n_2$-dimensional superspace $V_2$ as a tensor product $V_2 = U \otimes (V_2)_0$. Set $J = 1 \otimes J_U \in \text{End}(V_1)$ and $I = I_U \otimes 1 \in \text{End}(V_2)$. It is clear that

$$C(J) = \text{End}(V_1)_0 \otimes \text{Span}(1, I_U) \cong q(V_1);$$

$$C(I) = \text{Span}(1, J_U) \otimes \text{End}(V_2)_0 \cong q(V_2).$$

Define the $Q$-tensor product by setting

$$V_1 \otimes^Q V_2 := (V_1)_0 \otimes U \otimes (V_2)_0.$$  

The map

$$X_1 + X_2 \mapsto X_1 \otimes 1_{(V_2)_0} + 1_{(V_1)_0} \otimes X_2 \quad \text{for } X_1 \in \mathfrak{g}_1, \ X_2 \in \mathfrak{g}_2$$

determines an irreducible representation of the Lie superalgebra $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ in the space $V_1 \otimes^Q V_2$.

We will denote the image of $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ in this superspace by $\mathfrak{g}_1 \circlearrowleft \mathfrak{g}_2$. The usual tensor product $V_1 \otimes V_2$ considered as a $\mathfrak{g}_1 \oplus \mathfrak{g}_2$-module is a direct sum of two submodules equivalent to $V_1 \otimes^Q V_2$. (When $\mathfrak{g}_1 = \mathfrak{g}_2 = \mathfrak{g}$ and, consequently, $V_1 = V_2$, Sergeev denotes the diagonal action of $\mathfrak{g}$ in $V \otimes^Q V$ by $2^{-1}(V \otimes V)$, cf. [Ser2].)
1.3. Projectivization. If $g \subset \mathfrak{gl}(n|n)$ is a Lie sub-superalgebra containing scalar operators then the projective Lie superalgebra of type $g$ is $\mathfrak{pg} = g / \mathbb{C}$. Lie superalgebras $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ described in sec. 1.2 are projective.

Projectivization sometimes leads to new Lie superalgebras, for example: $\mathfrak{pgl}(n|n)$, $\mathfrak{psl}(n|n)$, $\mathfrak{pq}(n)$, $\mathfrak{psq}(n)$; whereas $\mathfrak{pgl}(p|q) \cong \mathfrak{sl}(p|q)$ if $p \neq q$.

1.4. Lie superalgebras that preserve bilinear forms. We will often use a general notation $\mathfrak{aut}(\omega)$ for the Lie superalgebra that preserves the non-degenerate bilinear form $\omega$ in the superspace $V$, i.e., $\mathfrak{aut}(\omega) = \{X \in \mathfrak{gl}(V) \mid \omega(Xv_1, v_2) + (-1)^{p(X)p(v)}\omega(v_1, Xv_2) = 0 \text{ for any } v_1, v_2 \in V\}$.

If the form $\omega$ is even and supersymmetric, then the Lie superalgebra $\mathfrak{aut}(\omega)$ is called orthosymplectic and denoted $\mathfrak{osp}(V) = \mathfrak{osp}(\dim V)$. Observe that the passage from $V$ to $\Pi(V)$ sends the supersymmetric forms to superanti-symmetric ones. That is why we use the notation $\mathfrak{osp}^k$ for the Lie superalgebra that preserves the superanti-symmetric form. We have an isomorphism $\mathfrak{osp}(V) \cong \mathfrak{osp}^k(\Pi(V))$, but matrix representations of elements from $\mathfrak{osp}(V)$ and $\mathfrak{osp}^k(\Pi(V))$ are different.

If the form $\omega$ is odd, then the Lie superalgebra $\mathfrak{aut}(\omega)$ is called, as A. Weil suggested, periplectic and denoted $\mathfrak{pe}^{sy}(n)$ or $\mathfrak{pe}^{sk}(n)$, in accordance with symmetry of $\omega$. The passage from $V$ to $\Pi(V)$ sends the supersymmetric forms to superanti-symmetric ones and establishes an isomorphism $\mathfrak{pe}^{sy}(n) \cong \mathfrak{pe}^{sk}(n) := \mathfrak{pe}(n)$.

The special periplectic superalgebra is $\mathfrak{spe}(n) = \{X \in \mathfrak{pe}(n) \mid \text{str } X = 0\}$.

Observe that the map $\chi_\lambda : X \mapsto \lambda \cdot \text{str } X$, where $\lambda \in \mathbb{C}$, $\lambda \neq 0$, determines a nontrivial character of $\mathfrak{pe}(n)$. Denote by $\mathfrak{pe}_\chi(n)$ the image of $\mathfrak{pe}(n)$ in the representation id $\otimes \chi_\lambda$.

1.5. Sergeev Lie superalgebra. A. Sergeev proved that there is just one nontrivial central extension of $\mathfrak{spe}(n)$. Let us represent an arbitrary element $X \in \mathfrak{as}$ as a pair $X = x + d \cdot z$, where $x = \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \in \mathfrak{spe}(4)$ ($\text{tr } A = 0, B = B^t, C = -C^t$), $d \in \mathbb{C}$ and $z$ is the central element. The bracket in $\mathfrak{as}$ in the matrix form is

$$[x + d \cdot z, x' + d' \cdot z] = [x, x'] + \text{tr } CC' \cdot z.$$

1.6. Heisenberg Lie superalgebra. Denote by $\mathfrak{hei}(0|m)$ the Heisenberg Lie superalgebra with $m$ odd generators of creation and annihilation, i.e., the Lie superalgebra with odd generators $\xi_1, \ldots, \xi_n; \eta_1, \ldots, \eta_m$ if $m = 2n$ or $\xi_1, \ldots, \xi_{n-1}; \eta_1, \ldots, \eta_{n-1}$ and $\theta$ if $m = 2n - 1$ and an even generator $z$ satisfying the relations

$$[\xi_i, \eta_j] = \delta_{i,j} \cdot z; \quad [\xi_i, \xi_j] = [\eta_i, \eta_j] = [z, \mathfrak{hei}(0|2n)] = 0;$$

$$[\theta, \theta] = z; \quad [\theta, \xi_i] = [\theta, \eta_j] = 0.$$

Irreducible finite dimensional representation of $\mathfrak{hei}(0|m)$ were first described in [K], see also [Ser1].

Each irreducible finite dimensional representation of $\mathfrak{hei}(0|m)$ is scalar on $z$ and the only one that sends $z$ to the identity operator is realized in the superspace $\Lambda(n) = \Lambda(\xi)$ for $m = 2n$ or $\Lambda(n) = \Lambda(\xi, \theta)$ for $m = 2n - 1$ by the formulas:

$$z \mapsto 1, \quad \xi_i \mapsto \xi_i, \quad \eta_i \mapsto \partial\xi_i; \quad \theta \mapsto \theta + \partial\theta.$$

This representation is irreducible of $G$-type if $m = 2n$ and irreducible of $Q$-type if $m = 2n - 1$. 

The normalizer of $\mathfrak{hei}(0|m)$ in $\mathfrak{gl}(\Lambda(n))$ is $\mathfrak{g} = \mathfrak{hei}(0|m) \oplus \mathfrak{o}(m)$; it acts in the spinor representation of $\mathfrak{o}(m)$, or, in terms of differential operators: $\mathfrak{g} = \text{Span}(1, \xi, \partial, \xi \partial, \partial \xi)$ for $m = 2n$ and $\mathfrak{g} = \text{Span}(1, \xi, \partial, \theta + \partial_b, \xi(\theta + \partial_b), \partial(\theta + \partial_b), \xi \partial, \partial \xi)$ for $m = 2n - 1$.

Let $n = 3$. Then $\mathfrak{g}$ is contained in $\tilde{\mathfrak{g}} = (\mathfrak{hei}(0|6) \oplus V) \oplus \mathfrak{o}(6)$ (sum as $\mathfrak{o}(6)$-modules), where the highest weight of the $\mathfrak{o}(6) \simeq \mathfrak{sl}(4)$-module $V$ is $(2,0,0)$, i.e., $\tilde{\mathfrak{g}}$ is isomorphic to the nontrivial central extension as of $\mathfrak{spo}(4)$. (Observe, that $\mathfrak{sl}(4)$-module $\mathfrak{hei}(0|6)$ is the direct sum of the trivial module and the exterior square of the dual to the standard 4-dimensional $\mathfrak{sl}(4)$-module.)

1.7. **Densities.** Let $\mathfrak{vect}(0|n) = \mathfrak{der}(\Lambda(n))$ be the Lie superalgebra of vector fields on $(0|n)$-dimensional superspace. Irreducible representations of $\mathfrak{vect}(0|n)$ are described in [BL]. The most important for us will be a one-parameter family of representations $T^\lambda$ of $\mathfrak{vect}(0|n)$ in the superspace $\text{Vol}^\lambda = \Lambda(\xi) \text{vol}^\lambda(\xi)$ of $\lambda$-densities. We define it by the formula

$$T^\lambda(D)(f(\xi) \text{ vol}^\lambda) = (D(f) + \lambda \text{ div } D \cdot f) \text{ vol}^\lambda.$$ 

The representations $T^\lambda$ are irreducible if $\lambda \neq 0, 1$. The representation $T^0$ determines the action of the Lie superalgebra of vector fields $\mathfrak{vect}(0|n)$ in the space of functions; the constants form an invariant 1-dimensional subspace. The representation $T^1$ is the dual representation in the space of volume forms, $\text{Vol}$, it contains an irreducible subspace of codimension $\varepsilon^n$ spanned by the volume element $\text{vol}$. Therefore, $T^0 \cong (\Pi^\circ(T^1))^\ast$.

Define the form $\omega_{1/2}$ on $\sqrt{\text{vol}}$ by the formula

$$\omega_{1/2}(f \sqrt{\text{vol}}, g \sqrt{\text{vol}}) = \int fg \cdot \text{vol}.$$ 

It is clear that $T^{1/2}(\mathfrak{vect}(0|n))$ preserves $\omega_{1/2}$.

1.8. **The Poisson superalgebra.** On $\Lambda(m) = \Lambda(\Theta_1, \ldots, \Theta_m)$, define a Lie superalgebra structure by setting (the extra minus is convenient for calculations with weights)

$$\{f, g\}_{\text{P.b.}} = -(-1)^{p(f)} \sum_{j \leq m} \frac{\partial f}{\partial \Theta_j} \frac{\partial g}{\partial \Theta_j}.$$ 

Sometimes it is more convenient to re-denote the $\Theta$’s and set (here $i^2 = -1$ and $[m + 1/2]$ is the integer part):

$$\xi_j = \frac{1}{\sqrt{2}}(\Theta_j - i \Theta_{n+j}) ; \; \eta_j = \frac{1}{\sqrt{2}}(\Theta_j + i \Theta_{n+j}) \text{ for } j \leq n = [m + 1/2] \; \theta = \Theta_{2n+1}.$$ 

In new indeterminates the Poisson bracket is defined by formula (the summand with $\theta$ only exists for $m$ odd):

$$\{f, g\} = -(-1)^{p(f)} \left( \sum_{i \leq n} \left( \frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial \eta_i} + \frac{\partial f}{\partial \eta_i} \frac{\partial g}{\partial \xi_i} \right) + \frac{\partial f}{\partial \theta} \frac{\partial g}{\partial \theta} \right).$$ 

This Lie superalgebra is denoted $\mathfrak{po}(0|m)$; it is a finite dimensional analog of the Poisson algebra. It turns into the general matrix superalgebra under quantization, see [LSH].

On $\mathfrak{po}(0|m)$, define the grading to be $\text{gr}(f) = \deg f - 2$, where $\deg f$ is the degree of $f$ as an element of $\Lambda(m)$: $\deg \Theta_i = 1$ for any $i$. Thus, $\mathfrak{po}(0|m) = \bigoplus_{i \leq -2} \mathfrak{po}_i$, where $\mathfrak{po}_0 \cong \mathfrak{o}(m)$ and where $\bigoplus_{i < 0} \mathfrak{po}_i$ is isomorphic to $\mathfrak{hei}(0|m)$.

Since trace and queer trace are quantum versions of the integral, $\mathfrak{po}(0|m)$ possesses an ideal $\mathfrak{spo}(0|m)$, the special Poisson superalgebra, of codimension $\varepsilon^m$, and a 1-dimensional center, the space of constant functions.
The classical isomorphisms of Lie algebras \( \mathfrak{sl}(4) \cong \mathfrak{o}(6) \) and their modules \( \Lambda^2(\text{id}_{\mathfrak{sl}(4)}) \cong \text{id}_{\mathfrak{o}(6)} \) show that as described in sec. 1.5 can be embedded into \( \mathfrak{po}(0|6) \). The embedding sends the central element \( z \in \mathfrak{as} \) into \( 1 \in \mathfrak{po}(0|6) \).

1.9. **Simplicity.** The Lie superalgebras \( \mathfrak{sl}(m|n) \) for \( m > n \geq 1 \), \( \mathfrak{psl}(n|n) \) for \( n > 1 \), \( \mathfrak{psq}(n) \) for \( n > 2 \), \( \mathfrak{osp}(m|2n) \) for \( mn \neq 0 \) and \( \mathfrak{spe}(n) \) for \( n > 2 \) are simple, see [K] (as well as [Kap], [FK], [SNR]).

1.10. **Almost simplicity.** We say that a Lie superalgebra \( \mathfrak{g} \) is almost simple if it can be included (non-strictly) between a simple Lie superalgebra \( \mathfrak{s} \) and the Lie superalgebra \( \mathfrak{ders} \) of the derivations of the latter: \( \mathfrak{s} \subset \mathfrak{g} \subset \mathfrak{ders} \).

1.11. **Theorem** (Main Theorem). 1° Let \( \mathfrak{g} \) be an irreducible linear Lie superalgebra which is neither almost simple nor a central extension of an almost simple Lie superalgebra.

Then \( \mathfrak{g} \) is contained in one of the following four major types of Lie superalgebras:

1) \( \mathfrak{gl}(V_1) \bigodot \mathfrak{gl}(V_2) \);
2) \( \mathfrak{q}(V_1) \bigodot \mathfrak{q}(V_2) \);
3) \( \mathfrak{gl}(V) \otimes \Lambda(n) \in \mathfrak{vect}(0|n) \);
4) \( \mathfrak{hei}(0|2n) \in \mathfrak{o}(2n) \).

2° Let, in addition to conditions of 1°, \( \mathfrak{g} \) be a subalgebra of \( \mathfrak{q}(V) = C(J) \) for some \( J \). Then \( \mathfrak{g} \) is contained in one of the following Lie superalgebras (numbered as in 1°):

1q) \( \mathfrak{q}(V_1) \bigodot \mathfrak{gl}(V_2) \) and \( J = J_1 \otimes 1 \), where \( \mathfrak{q}(V_1) = C(J_1) \);
3q) \( \mathfrak{q}(V_1) \otimes \Lambda(n) \in \mathfrak{vect}(0|n) \) and \( J = J_1 \otimes 1 \), where \( \mathfrak{q}(V_1) = C(J_1) \);
4q) \( \mathfrak{hei}(0|2n) \in \mathfrak{o}(2n-1) \).

3° Let, in addition to conditions of 1°, \( \mathfrak{g} \) preserve a non-degenerate homogeneous form \( \omega \), either symmetric or skew-symmetric. Then \( \mathfrak{g} \) is contained in one of the following Lie superalgebras (numbered as in 1°):

1ω) \( \mathfrak{aut}(\omega_1) \bigodot \mathfrak{aut}(\omega_2) \) and \( \omega = \omega_1 \otimes \omega_2 \);
3ω) \( \mathfrak{aut}(\omega_1) \otimes \Lambda(n) \in T^{1/2} \mathfrak{vect}(0|n) \) and \( \omega = \omega_1 \otimes \omega_{1/2} \).

1.12. **Maximal subalgebras from Main Theorem.** Tables 1–2 describe when subalgebras of the form \( \mathfrak{g}_1 \bigodot \mathfrak{g}_2 \) are maximal in a linear Lie superalgebra \( \mathfrak{g} \). These subalgebras are similar to those that Dynkin described.

1.13. **Table 1.** In this table we assume that \( \mathfrak{g}_i \subset \mathfrak{gl}(V_i) \), and \( \dim V_i = N_i = m_i + n_i \varepsilon \neq 1 \) or \( \varepsilon \).

| \( \mathfrak{g}_1 \) | \( \mathfrak{g}_2 \) | \( \mathfrak{g} \) | conditions |
|---|---|---|---|
| \( \mathfrak{gl}(N_1) \) | \( \mathfrak{gl}(N_2) \) | \( \mathfrak{gl}(N_1N_2) \) | \( N_i \neq 1 + \varepsilon; \ m_1 \neq n_1 \) or \( m_2 \neq n_2 \) |
| \( \mathfrak{gl}(N_1) \) | \( \mathfrak{gl}(N_2) \) | \( \mathfrak{sl}(N_1N_2) \) | \( N_i \neq 1 + \varepsilon; \ m_1 = n_1 \) and \( m_2 = n_2 \) |
| \( \mathfrak{sl}(N_1) \) | \( \mathfrak{sl}(N_2) \) | \( \mathfrak{sl}(N_1N_2) \) | \( N_i \neq 1 + \varepsilon; \ m_1 \neq n_1 \) or \( m_2 \neq n_2 \) |
| \( \mathfrak{q}(n_1) \) | \( \mathfrak{q}(n_2) \) | \( \mathfrak{sl}(n_1n_2|n_1n_2) \) | \( n_1n_2 > 1 \) |
| \( \mathfrak{q}(n_1) \) | \( \mathfrak{gl}(m_2 + n_2) \) | \( \mathfrak{q}(n_1(m_2 + n_2)) \) | \( n_1 \geq 1; m_2 \neq n_2 \) |
| \( \mathfrak{q}(n_1) \) | \( \mathfrak{gl}(m_2 + n_2\varepsilon) \) | \( \mathfrak{sq}(n_1(m_2 + n_2)) \) | \( n_1 \geq 1; m_2 = n_2 > 1 \) |
The case $N_i = 1 + \varepsilon$ is exceptional because if we identify the $(1 + \varepsilon)$-dimensional superspace $V$ with $\Lambda(1)$, then $\mathfrak{gl}(V) \cong \Lambda(1) \in \mathfrak{vect}(0|1)$. Therefore,

$$\mathfrak{g}_1 \circ \mathfrak{gl}(V) \subset \mathfrak{g}_1 \otimes \Lambda(1) \in \mathfrak{vect}(0|1)$$

for any Lie superalgebra $\mathfrak{g}_1$. The case $n_1n_2 = 1$ in the fourth line is exceptional due to the fact that $q(1) \circ q(1) \cong \mathfrak{sl}(1|1)$.

1.14. Table 2. Table 2 describes maximal subalgebras of the form $\mathfrak{aut}(\omega_1) \oplus \mathfrak{aut}(\omega_2)$ of the Lie superalgebra $\mathfrak{g}$ that preserves a non-degenerate bilinear form $\omega = \omega_1 \otimes \omega_2$. It is clear that if both forms $\omega_1$ and $\omega_2$ are supersymmetric or skew then $\omega$ is symmetric while if one of the forms $\omega_1$ or $\omega_2$ is symmetric and the other one is skew then $\omega$ is skew. We take into account isomorphisms $\mathfrak{osp}(V) \cong \mathfrak{osp}^{sk}(\Pi(V))$ and $\mathfrak{pe}^{sy}(V) \cong \mathfrak{pe}^{sk}(\Pi(V))$, see sec. 1.4, and skip the types of symmetry just to save space.

In this table $N_i = n_i + 2m_i\varepsilon$, and $n_im_i \neq 0$. The conditions are occasioned by the fact that the identity representations of $\mathfrak{osp}(2)$ and $\mathfrak{pe}(2)$ are reducible; $\mathfrak{pe}(2)$ is an exception because

$$\mathfrak{pe}(2) \cong \mathfrak{sp}(2) \otimes \Lambda(1) \in T^{1/2}(\mathfrak{vect}(0|1)).$$

If we identify the 2|2-dimensional superspace of the identity representation of $\mathfrak{pe}(2)$ with $\mathbb{C}^2 \otimes \Lambda(1)$, we obtain an embedding

$$\mathfrak{aut}(\omega_1) \circ \mathfrak{pe}(2) \subset \mathfrak{aut}(\omega_1 \otimes \omega_2) \otimes \Lambda(1) \in T^{1/2}(\mathfrak{vect}(0|1)),$n_1, n_2 > 2

where $\omega_2$ is the standard form in $\mathbb{C}^2$ preserved by $\mathfrak{sp}(2)$.

| $\mathfrak{g}_1$ | $\mathfrak{g}_2$ | $\mathfrak{g}$ | conditions |
|-----------------|-----------------|-----------------|------------|
| $\mathfrak{osp}(N_1)$ | $\mathfrak{osp}(N_2)$ | $\mathfrak{osp}(N_1N_2)$ | $-$ $-$ $-$ |
| $\mathfrak{o}(n)$ | $\mathfrak{osp}(N_2)$ | $\mathfrak{osp}(nN_2)$ | $n > 2, n \neq 4$ |
| $\mathfrak{sp}(2n)$ | $\mathfrak{osp}(N_2)$ | $\mathfrak{osp}(2nN_2)$ | $n \geq 1$ |
| $\mathfrak{pe}(n_1)$ | $\mathfrak{pe}(n_2)$ | $\mathfrak{osp}(2n_1n_2|2n_1n_2)$ | $n_1, n_2 > 2$ |
| $\mathfrak{osp}(n_1|2m_1)$ | $\mathfrak{pe}(n_2)$ | $\mathfrak{pe}(n_1n_2 + 2m_1n_2)$ | $n_2 > 2, n_1 \neq 2m_1$ |
| $\mathfrak{osp}(2m_1|2m_1)$ | $\mathfrak{pe}(n)$ | $\mathfrak{spe}(4nn)$ | $n > 2$ |
| $\mathfrak{osp}(n_1|2m_1)$ | $\mathfrak{pe}_\lambda(n_2)$ | $\mathfrak{pe}_\mu(n_1n_2 + 2m_1n_2)$, for $\mu = \frac{\lambda}{n_1 - 2m_1}$ | $n_2 > 2, n_1 \neq 2m_1$ |
| $\mathfrak{o}(n)$ | $\mathfrak{pe}(m)$ | $\mathfrak{pe}(nm)$ | $n, m > 2, n \neq 4$ |
| $\mathfrak{sp}(2n)$ | $\mathfrak{pe}(m)$ | $\mathfrak{pe}(2nm)$ | $m > 2, n \geq 1$ |

1.15. Table 3: Non-Dynkin type of subalgebras. The maximal subalgebras considered in Tables 1–2 are similar to those considered by Dynkin. There are, however, maximal subalgebras of linear superalgebras of totally different nature. We represent them in Table 3: $\mathfrak{g}_1$ is a maximal subalgebra in $\mathfrak{g}$; we set $\dim V_i = N_i = m_i + n_i\varepsilon$. In this case we allow $m_1n_1 = 0$ but, of course, exclude $m_1 = n_1 = 0$. In lines 5–7 we assume that $n_1$ is even and in lines 8–9 we assume that $m_1 = n_1$. As in Table 2 we do not mention types of symmetry of bilinear forms.
The exceptional cases:

1) If \( \dim V_1 = 1 \) or \( \varepsilon \), then \( \mathfrak{gl}(V_1) \otimes \Lambda(1) \in \mathbf{vect}(0|1) \cong \mathfrak{gl}(1|1) \).

2) If \( \dim V_1 = 1 + \varepsilon \), then having identified \( V_1 \) with \( \Lambda(1) \), we obtain

\[
\mathfrak{gl}(V_1) \otimes \Lambda(n) \in \mathbf{vect}(0|n) \cong (\Lambda(1) \in \mathbf{vect}(1)) \otimes \Lambda(n) \in \mathbf{vect}(0|n) = \\
\Lambda(1) \otimes \Lambda(n) \in (\mathbf{vect}(1) \otimes \Lambda(n) \in \mathbf{vect}(0|n)) \subset \Lambda(n + 1) \in \mathbf{vect}(0|n + 1).
\]

3) The isomorphism (1.2) induces the inclusion

\[
\mathfrak{pe}(2) \otimes \Lambda(n) \in T^{1/2}(\mathbf{vect}(0|n)) \subset \mathfrak{sp}(2) \otimes \Lambda(n + 1) \in T^{1/2}(\mathbf{vect}(0|n + 1)).
\]

4) \( n = 3 \) in line 10; for motivation see sec. 1.6.

2. **Irreducible Maximal Subalgebras of the Form \( \mathfrak{g}_1 \odot \mathfrak{g}_2 \)**

In this section we will prove theorems summarized in Tables 1 and 2.

**2.1. Theorem.** Let \( \dim V_i = N_i = m_i + n_i \varepsilon \) and \( N_i \neq 1 \) or \( \varepsilon \) or \( 1 + \varepsilon \). Let \( V = V_1 \otimes V_2 \). Then

1) \( \mathfrak{gl}(V_1) \odot \mathfrak{gl}(V_2) \) is a maximal subalgebra in \( \mathfrak{gl}(V) \) if \( n_1 \neq m_1 \) or \( n_2 \neq m_2 \), otherwise it is maximal in \( \mathfrak{sl}(V) \).

2) The following subalgebras are maximal in \( \mathfrak{sl}(V) \):

i) \( \mathfrak{sl}(V_1) \oplus \mathfrak{sl}(V_2) \) if \( n_1 \neq m_1 \) or \( n_2 \neq m_2 \);

ii) \( \mathfrak{gl}(V_1) \ominus \mathfrak{gl}(V_2) \) if \( n_1 = m_1 \) and \( n_2 = m_2 \).

**Proof.** 1) Let \( g = \mathfrak{gl}(V_1) \odot \mathfrak{gl}(V_2) \). Consider \( \mathfrak{gl}(V) \) as \( g \)-module. Then \( \mathfrak{gl}(V) \cong \mathfrak{gl}(V_1) \otimes \mathfrak{gl}(V_2) \) and the bracket in \( \mathfrak{gl}(V) \) is defined via

\[
[A \otimes B, C \otimes D] = (-1)^{p(B)p(C)}[A, C] \otimes BD + (-1)^{p(A \otimes B)p(C)}CA \otimes [B, D].
\]

(2.1)

Observe that, as \( \mathfrak{gl}(V_i) \)-module, \( \mathfrak{gl}(V_i) \) contains only two nontrivial submodules: \( \mathbb{C} \cdot \text{id} \) and \( \mathfrak{sl}(V_i) \). Thus, the minimal \( g \)-submodule \( W \) of \( \mathfrak{gl}(V) \) larger than \( g \) is of the form

\[
W = g + \mathfrak{sl}(V_1) \otimes \mathfrak{sl}(V_2).
\]

(2.2)

If \( m_1 \neq n_1 \) and \( m_2 \neq n_2 \), then the sum in (2.2) is direct and \( W = \mathfrak{gl}(V) \), i.e., \( g \) is maximal in \( \mathfrak{gl}(V) \).
Let $\mathbb{C} \cdot \text{id}_2 \cong \mathfrak{gl}(V_i)/\mathfrak{sl}(V_i)$ be a trivial $\mathfrak{gl}(V_i)$-module. If $m_1 \not= n_1$ but $m_2 = n_2$, then $\mathfrak{gl}(V)/W \cong \mathfrak{sl}(V_1) \otimes \mathbb{C} \cdot \text{id}_2$. If $m_1 = n_1$ and $m_2 = n_2$, then

$$\mathfrak{gl}(V)/W \cong \mathfrak{sl}(V_1) \otimes \mathbb{C} \cdot \text{id}_2 + \mathbb{C} \cdot \text{id}_1 \otimes \mathfrak{sl}(V_2) + \mathbb{C} \cdot \text{id}_1 \otimes \mathbb{C} \cdot \text{id}_2.$$

Since the spaces $\mathfrak{sl}(V_i)$ are not closed with respect to the operator product, formula (2.1) demonstrates that in both cases any subalgebra strictly containing $g$ must contain $\mathfrak{sl}(V_i) \otimes \mathfrak{gl}(V_2) + \mathfrak{gl}(V_1) \otimes \mathfrak{sl}(V_2) = \mathfrak{sl}(V)$. To complete the proof it suffices to observe that in the first case the subalgebra $g$ is not contained in $\mathfrak{sl}(V)$.

2.2. Theorem. Let $\dim V_i = N_i = m_i + n_i \varepsilon$, where $m_1 = n_1 \geq 1$ and $N_2 \not= 1$, or $\varepsilon$, or $1 + \varepsilon$. Let $V = V_1 \otimes V_2$. Then the Lie subsuperalgebra $\mathfrak{g} = q(V_1) \oplus \mathfrak{gl}(V_2)$ is maximal in $q(V)$ if $m_2 \not= n_2$; it is maximal in $\mathfrak{sq}(V)$ if $m_2 = n_2$.

Proof. The proof of this theorem largely repeats that of the previous one. Consider $q(V)$ as $g$-module. Then $q(V) \cong q(V_1) \otimes \mathfrak{gl}(V_2)$ and the bracket in $\mathfrak{gl}(V)$ is defined via (2.1).

Note that for $m_1 > 2$ the space $q(V_1)$ considered as a $q(V_1)$-module, contains two nontrivial submodules, $\mathbb{C} \cdot \text{id}$ and $\mathfrak{sq}(V_1)$. Therefore, the minimal $g$-submodule $W \subset q(V)$, is of the form $W = g + \mathfrak{sq}(V_1) \otimes \mathfrak{sl}(V_2)$. Since $\mathfrak{sq}(V_1)$ is not closed with respect to the operator product and taking into account formula (2.1), we see that any subalgebra $h \subset q(V)$ strictly containing $g$ should satisfy

$$h \supset \mathfrak{sq}(V_1) \otimes \mathfrak{gl}(V_2) + q(V_1) \otimes \mathfrak{sl}(V_2) = \mathfrak{sq}(V).$$

(2.3)

For $m_1 = 2$ the space $q(V_1) = q(2)$, considered as a $q(2)$-module, contains one more nontrivial submodule, $\mathbb{C} \cdot \text{id} \oplus \mathfrak{sq}(2)$. Therefore, the minimal $g$-module $W \subset q(V)$ is in this case of the form $W = g + \mathfrak{sq}(V_1) \otimes \mathfrak{sl}(V_2)$. Nevertheless, even in this case formula (2.1) leads to inclusion (2.3). To complete the proof for $m_1 \geq 2$ it only remains to observe that for $m_2 \not= n_2$ the Lie superalgebra $g$ is not contained in $\mathfrak{sq}(V)$.

The case $m_1 = 1$ is even simpler: $q(V)/g \cong p\mathfrak{gl}(V_2)$.

2.3. Embedding $q_1 \oplus q_2 \subset \mathfrak{sl}(V_1 \otimes q_2)$. First, describe $\mathfrak{gl}(V_i)$ as a $q(V_i)$-module.

As in sec. 1.2, introduce a $(1|1)$-dimensional superspace $U$ and consider the following basis of $\text{End}(U)$:

$$1_U, \quad J_U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad I_U = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad D_U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$ Let us realize the $n_1|n_1$-dimensional superspace $V_1$ as a tensor product $V_1 = (V_1)_0 \otimes U$ and $n_2|n_2$-dimensional superspace $V_2$ as a tensor product $V_2 = U \otimes (V_2)_0$. Set $J = 1 \otimes J_U$ and $I = I_U \otimes 1$. It is clear that

$$C(J) = \text{End}(V_1)_0 \otimes \text{Span}(1, I_U) \cong q(V_1);$$
$$C(I) = \text{Span}(1, J_U) \otimes \text{End}(V_2)_0 \cong q(V_2).$$

Then $\mathfrak{gl}(V_i)$, as a $q(V_i)$-module is a direct sum of two reducible but indecomposable modules:

$$\mathfrak{gl}(V_1) = \mathfrak{gl}(V_1)_0 \otimes \text{Span}(1, I_U) \oplus \mathfrak{gl}(V_1)_0 \otimes \text{Span}(D_U, J_U);$$
(2.4)
$$\mathfrak{gl}(V_2) = \text{Span}(1, J_U) \otimes \mathfrak{gl}(V_2)_0 \oplus \text{Span}(D_U, I_U) \otimes \mathfrak{gl}(V_2)_0.$$ (2.5)

Observe that the first summands in (2.4) and (2.5) are isomorphic to $q(V_i)$, whereas the second ones, as $q(V_i)$-modules, are isomorphic to $\Pi(q(V_i))$.

2.3.1. Theorem. If $n_1 n_2 > 1$, then $g = q(V_1) \oplus q(V_2)$ is maximal in $\mathfrak{sl}(V_1 \otimes q V_2)$. 
Proof. Formulas (2.4)—(2.5) show that the minimal \( g \)-submodule \( W \subset \mathfrak{sl}(V_1 \otimes Q V_2) \) containing \( g \) is of the form
\[
W = g + \mathfrak{sl}(V_1) \otimes \mathfrak{gl}(U) \otimes \mathfrak{sl}(V_2).
\]
When we close \( W \) with respect to the bracketing we obtain \( \mathfrak{sl}(V_1 \otimes Q V_2) \).

2.4. The case with the bilinear form. Let a non-degenerate homogeneous (with respect to parity) supersymmetric or superanti-symmetric bilinear form \( \omega \) be given in a superspace \( V \). Consider two objects associated with \( \omega \):
1) The Lie superalgebra \( \mathfrak{aut}(\omega) \) of operators preserving \( \omega \):
\[
\mathfrak{aut}(\omega) = \{ A \in \mathfrak{gl}(V) \mid \omega(Ax, y) + (-1)^{p(A)p(x)} \omega(x, Ay) = 0 \}.
\]
2) The space \( \text{sym}(\omega) \) of operators supersymmetric with respect to \( \omega \):
\[
\text{sym}(\omega) = \{ A \in \mathfrak{gl}(V) \mid \omega(Ax, y) = (-1)^{p(A)p(x)} \omega(x, Ay) \}.
\]
It is clear that, as a linear space, \( \mathfrak{gl}(V) = \mathfrak{aut}(\omega) \oplus \text{sym}(\omega) \).

Set
\[
\mathfrak{saut}(\omega) = \mathfrak{aut}(\omega) \cap \mathfrak{sl}(V), \quad \text{sym}(\omega) = \text{sym}(\omega) \cap \mathfrak{sl}(V).
\]
Note that if \( p(\omega) = 0 \), then \( \mathfrak{aut}(\omega) = \mathfrak{saut}(\omega) \), and if \( p(\omega) = 1 \), then \( \text{sym}(\omega) = \text{sym}(\omega) \).

If \( \omega \) is even, it determines a canonical isomorphism \( V \cong V^* \). In this case,
\[
g = \mathfrak{aut}(\omega) = \begin{cases} 
\mathfrak{osp}(V) \cong \Lambda^2(V) \text{ and } \text{sym}(\omega) = S^2(V) & \text{if } \omega \text{ is symmetric} \\
\mathfrak{osp}^k(V) \cong S^2(V) \text{ and } \text{sym}(\omega) = \Lambda^2(V) & \text{if } \omega \text{ is skew}.
\end{cases}
\]
In both cases the space \( \text{sym}(\omega) \) contains a 1-dimensional subspace \( \mathbb{C} \omega \) corresponding to scalar operators.

If, moreover, \( \dim V_0 \neq \dim V_1 \), then \( \text{sym}(\omega) = \mathbb{C} \omega \oplus \text{sym}(\omega) \), and the \( g \)-module \( \text{sym}(\omega) \) is irreducible.

If \( \dim V_0 = \dim V_1 \), then \( \mathbb{C} \omega \subset \text{sym}(\omega) \) and \( \dim(\text{sym}(\omega)/\text{sym}(\omega)) = 1 \) and \( \text{sym}(\omega)/\mathbb{C} \omega \) is irreducible.

If \( \omega \) is odd, a canonical isomorphism \( V^* \cong \Pi(V) \) implies \( V \otimes V^* \cong V \otimes \Pi(V) \cong \Pi(V \otimes V) \). In this case,
\[
g = \mathfrak{aut}(\omega) = \begin{cases} 
\mathfrak{pe}^{sy}(V) \cong \Pi(\Lambda^2(V)) \supset \mathfrak{saut}(\omega) = \mathfrak{pe}^{sy}(V) & \text{if } \omega \text{ is symmetric} \\
\mathfrak{pe}^{sk}(V) \cong \Pi(S^2(V)) \supset \mathfrak{saut}(\omega) = \mathfrak{pe}^{sk}(V) & \text{if } \omega \text{ is skew}
\end{cases}
\]
and
\[
\text{sym}(\omega) = \text{ssym}(\omega) = \begin{cases} 
\Pi(S^2(V)) \supset \mathbb{C} \omega & \text{if } \omega \text{ is symmetric} \\
\Pi(\Lambda^2(V)) \supset \mathbb{C} \omega & \text{if } \omega \text{ is skew}.
\end{cases}
\]
In both cases the space \( \text{sym}(\omega)/\mathbb{C} \omega \) is an irreducible \( g \)-module if \( \mathfrak{saut}(\omega) \) is simple, i.e., if \( \dim V = n|n \) and \( n > 2 \).

2.4.1. Lemma. 1) \( [\text{sym}(\omega), \text{sym}(\omega)] \subset \mathfrak{aut}(\omega) \), but \( [\text{sym}(\omega), \text{sym}(\omega)] \) is not contained in \( \mathfrak{saut}(\omega) \) if \( p(\omega) = 1 \).

2) Set \( \{A, B\} = AB + (-1)^{p(A)p(B)} BA \). Then
\[
\{\mathfrak{aut}(\omega), \mathfrak{aut}(\omega)\}, \{\text{sym}(\omega), \text{sym}(\omega)\} \subset \text{sym}(\omega);
\]
\[
\{\mathfrak{aut}(\omega), \text{sym}(\omega)\} \subset \mathfrak{aut}(\omega);
\]
\[
\mathfrak{aut}(\omega_1 \otimes \omega_2) = \mathfrak{aut}(\omega_1) \otimes \text{sym}(\omega_2) + \text{sym}(\omega_1) \otimes \mathfrak{aut}(\omega_2);
\]
\[
\text{sym}(\omega_1 \otimes \omega_2) = \mathfrak{aut}(\omega_1) \otimes \mathfrak{aut}(\omega_2) + \text{sym}(\omega_1) \otimes \text{sym}(\omega_2).
\]
If \( p(\omega) = 0 \), then the subspace \( \text{ssym}(\omega) \) is not closed with respect to \( \{\cdot, \cdot\} \).
3) \[ [A_1 \otimes B_1, A_2 \otimes B_2] = \frac{1}{2}((-1)^{p(A_2)p(B_1)}[A_1, A_2] \otimes \{B_1, B_2\} + (-1)^{p(A_2)(p(A_1) + p(B_1))}\{A_2, A_1\} \otimes [B_1, B_2]). \tag{2.7} \]

Proof: direct calculations. \(\square\)

2.5. Theorem. (Cf. Table 2) Let Lie superalgebras \(\mathfrak{saut}(\omega_1)\) and \(\mathfrak{saut}(\omega_2)\) be simple. Then Lie subalgebra \(\mathfrak{g} = \aut(\omega_1) \oplus \aut(\omega_2)\) is maximal in \(\aut(\omega_1 \otimes \omega_2)\) if either \(p(\omega_1) + p(\omega_2) = 0\) or if \(p(\omega_i) = 0, p(\omega_j) = 1\) and \(\dim(V_i)_0 \neq \dim(V_i)_1\) for \((i, j) = (1, 2)\). If \(p(\omega_i) = 0, p(\omega_j) = 1\) and \(\dim(V_i)_0 = \dim(V_i)_1\), then \(\mathfrak{g}\) is maximal in \(\mathfrak{saut}(\omega_1 \otimes \omega_2)\).

Proof. Formula (2.6) and the description of \(\aut(\omega_i)\) and \(\sym(\omega_i)\) as \(\aut(\omega_i)\)-modules immediately imply that any subalgebra of \(\mathfrak{h} \subset \aut(\omega_1 \otimes \omega_2)\) containing \(\mathfrak{g}\) must also contain at least one of the submodules \(\mathfrak{saut}(\omega_1) \otimes \sym(\omega_2)\) or \(\sym(\omega_1) \otimes \mathfrak{saut}(\omega_2)\). But then, by (2.7) we see that \(\mathfrak{h}\) must contain both of these modules, hence,

\[ \mathfrak{h} \supset \mathfrak{saut}(\omega_1) \otimes \sym(\omega_2) + \sym(\omega_1) \otimes \mathfrak{saut}(\omega_2). \tag{2.8} \]

If \(p(\omega_i) = 0\) for \(i = 1, 2\), then the rhs of (2.8) coincides with \(\aut(\omega_1 \otimes \omega_2)\). If \(p(\omega_i) = 1\) for \(i = 1, 2\), then by bracketing the elements from distinct summands and taking into account that \(\{\mathfrak{saut}(\omega_i), \sym(\omega_i)\} = \aut(\omega_i)\) we again obtain that \(\mathfrak{h} = \aut(\omega_1 \otimes \omega_2)\).

Finally, if \(p(\omega_2) = 0\) and \(p(\omega_2) = 1\), then the first summand in (2.8) coincides with \(\aut(\omega_1) \otimes \sym(\omega_2)\). By bracketing the elements from distinct summands we see that apart from inclusion (2.8) there is an inclusion \(\mathfrak{h} \supset \sym(\omega_1) \otimes \aut(\omega_2)\), i.e., \(\mathfrak{h} \supset \mathfrak{saut}(\omega_1 \otimes \omega_2)\).

This completes the proof when \(\dim(V_i)_0 = \dim(V_i)_1\). For \(\dim(V_i)_0 \neq \dim(V_i)_1\) it suffices to observe that \(\mathfrak{g}\) is not contained in \(\mathfrak{sl}(V)\). \(\square\)

3. Irreducible linear maximal subalgebras of non-Dynkin’s form

3.1. Theorem. If \(m = 2n, \) where \(n \geq 2, n \neq 3,\) then \(\mathfrak{g} = \mathfrak{hei}(0|m) \oplus \mathfrak{o}(m)\) is a maximal subalgebra in \(\mathfrak{sl}(\Lambda(n))\).

If \(n = 3,\) then \(\mathfrak{g} \subset \mathfrak{as} \subset \mathfrak{sl}(\Lambda(3))\), and \(\mathfrak{as}\) is maximal in \(\mathfrak{sl}(\Lambda(3))\).

If \(m = 2n - 1, n > 1,\) then \(\mathfrak{g}\) is a maximal subalgebra in \(\mathfrak{sq}(\Lambda(n))\).

Proof. Let \(\mathfrak{hei} = \mathfrak{hei}(0|m)\). Consider the image of the universal enveloping algebra \(U(\mathfrak{hei})\) in the irreducible representation from sec. 1.6. Due to irreducibility this image, considered as a Lie superalgebra, coincides with \(\mathfrak{gl}(\Lambda(n))\) for \(m = 2n\) and with \(\mathfrak{q}(\Lambda(n))\) for \(m = 2n - 1\).

Consider \(U(\mathfrak{hei})_L\) as a filtered Lie superalgebra with respect to the filtration induced by the natural filtration of the enveloping algebra, and consider the associated graded one, \(\gr(U(\mathfrak{hei})_L)\). As is known (LSH), \(\gr(U(\mathfrak{hei})_L)\) is isomorphic to the Poisson superalgebra with its standard grading described in sec. 1.8, \(\mathfrak{po}(0|m) = \bigoplus_{i=-2}^{m-2} \mathfrak{po}_i\). The graded image \(\gr(\mathfrak{g})\) of the Lie superalgebra \(\mathfrak{g}\) coincides with the non-positive part of \(\mathfrak{po}\):

\[ \gr(\mathfrak{g}) = \bigoplus_{-2 \leq i \leq 0} \mathfrak{po}_i. \]

Since \(\mathfrak{po}(0|m)\) is transitive, \(\mathfrak{po}_1\) is (for \(m \neq 6\)) an irreducible \(\mathfrak{o}(m) \cong \mathfrak{po}_0\)-module and generates \(\bigoplus_{i=1}^{m-3} \mathfrak{po}_i\), the positive part of the special Poisson subalgebra, \(\mathfrak{spo}(0|m)\), we see that \(\gr(\mathfrak{g})\) is maximal in \(\mathfrak{spo}(0|m)\) for \(m \neq 6\).

To complete the proof, it suffices to observe that \(\mathfrak{spo}(0|2n) = \gr(\mathfrak{sl}(\Lambda(n)))\) and \(\mathfrak{g} \subset \mathfrak{sl}(\Lambda(n))\), whereas \(\mathfrak{spo}(0|2n - 1) = \gr(\mathfrak{sq}(\Lambda(n)))\) and \(\mathfrak{g} \subset \mathfrak{sq}(\Lambda(n))\). \(\square\)
3.2. \( V = V_1 \otimes \Lambda(n) \). Let \( \dim V_1 = (m_1, n_1) \), \( n = \mathfrak{gl}(V_1) \otimes \Lambda(n) \) and \( g \) the semidirect sum of the ideal \( n \) and the subalgebra \( \text{vect}(0|n) \) with the natural action on the ideal. The Lie superalgebra \( g \) has a natural faithful representation \( \rho \) in \( V = V_1 \otimes \Lambda(n) \) defined by the formulas for any \( A \otimes \varphi \in n, \ D \in \text{vect}(0|n) \), and \( v \otimes \psi \in V \) we have
\[
\rho(A \otimes \varphi)(v \otimes \psi) = (-1)^{\rho(\varphi)\rho(v)} Av \otimes \varphi \psi,
\]
\[
\rho(D)(v \otimes \psi) = (-1)^{\rho(D)\rho(v)} v \otimes D(\psi).
\]
In the sequel, we will always identify elements of \( g \) with their images under \( \rho \). Therefore, we will consider \( g \) embedded in \( \mathfrak{gl}(V) \) which coincides, as a linear space, with \( \text{End}(V) \cong \text{End}(V_1) \otimes \text{End}(\Lambda(n)) \).

3.3. Theorem. 1) The Lie superalgebra \( g = \mathfrak{gl}(V_1) \otimes \Lambda(n) \oplus \text{vect}(0|n) \) is a maximal Lie subsuperalgebra of \( \mathfrak{sl}(V) \), where \( V = V_1 \otimes \Lambda(n) \), in all cases except
a) \( \dim V_1 = 1 + \varepsilon \) or
b) \( n = 1 \) and \( \dim V_1 = m_1 + n_1\varepsilon \) with \( m_1 \neq n_1 \).

2) If \( n = 1 \) and \( \dim V_1 = m_1 + n_1\varepsilon \), where \( m_1 \neq n_1 \) and \( m_1 + n_1 > 1 \), then \( g \) is maximal Lie subsuperalgebra of \( \mathfrak{gl}(V) \). In this case the Lie superalgebra \( \mathfrak{sl}(V) \) is maximal in \( \mathfrak{sl}(V) \).

Let us first prove a particular case of Theorem.

3.3.1. Lemma. If \( n > 1 \), then \( g = \Lambda(n) \oplus \text{vect}(0|n) \) is maximal in \( \mathfrak{sl}(\Lambda(n)) \).

Proof. We will make use of the arguments used in the proof of Theorem 3.1. Observe that \( g \) contains a \( G \)-type irreducible subalgebra \( \mathfrak{hei} = \mathfrak{hei}(0|2n) = \text{Span}(1, \xi, \partial) \). This means that the image of \( U(\mathfrak{hei}) \) in \( \text{End}(\Lambda(n)) \) coincides with \( \text{End}(\Lambda(n)) \), and the graded Lie superalgebra associated with \( U(\mathfrak{hei})_L \), is isomorphic to \( \mathfrak{po}(0|2n) \). Clearly, subalgebra \( \mathfrak{sl}(\Lambda(n)) \) corresponds to \( \mathfrak{spo}(0|2n) \).

Let us realize the elements of \( \mathfrak{po}(0|2n) \) by means of generating functions in \( \xi, \eta \):
\[
\xi_i \mapsto \xi_i, \quad \partial_i \mapsto \eta_i.
\]
The image \( \text{gr}(g) = \bigoplus_{i \geq -2} g_i \) of \( g \) in \( \mathfrak{po}(0|2n) \) is the linear space of functions of degree \( \leq 1 \) in \( \eta \)'s. In particular, \( g_0 \) consists of the elements of \( \mathfrak{po}(0|2n)_0 \cong \mathfrak{o}(2n) \) that preserve the space \( \text{Span}(\xi_1, \ldots, \xi_n) \). By Dynkin's theorem \( g_0 \) is a maximal subalgebra in \( \mathfrak{o}(2n) \).

On the other hand, it is clear that
\[
g_i = \{ X \in g_{i-1} \mid [X, g_{-1}] \subset g_{i-1} \} \quad \text{for all } i \geq 1,
\]
i.e., \( g \) is a maximal subalgebra in \( \mathfrak{po}(0|2n) \) with a given non-positive part. Therefore, any subalgebra \( \mathfrak{h} \subset \mathfrak{po}(0|2n) \) containing \( g \) must contain the whole of \( \mathfrak{po}_0 \). Since for \( n \neq 3 \) the \( \mathfrak{o}(2n) \)-action in \( \mathfrak{po}_1 \) is irreducible, \( \mathfrak{h} \) must contain the whole component \( \mathfrak{po}_1 \), hence, the whole of \( \mathfrak{spo} \).

If \( n = 3 \), then \( g_1 \) does not lie in any of the irreducible \( \mathfrak{o}(6) \)-submodules of \( \mathfrak{po}_1 \). Therefore, again, \( \mathfrak{h} \supset \mathfrak{spo} \).

In both cases we see that \( g \) is maximal in \( \mathfrak{sl}(\Lambda(n)) \). \( \square \)

3.4. Proof of Theorem 3.2.1. Let \( \dim V_1 \neq 1 \) or \( \varepsilon \) or \( 1 + \varepsilon \) and let \( \mathfrak{h} \subset \mathfrak{gl}(V) \) be a subalgebra containing \( g \). Clearly, to prove both headings of Theorem, it suffices to show that \( \mathfrak{h} \supset \mathfrak{sl}(V) \).

Observe that \( g \) contains two subalgebras, \( g_1 = \mathfrak{gl}(V_1) \otimes 1 \) and \( g_2 = 1 \otimes \Lambda(n) \oplus \text{vect}(0|n) \). Since \( \text{End}(V_1) \) only contains two nontrivial \( \mathfrak{gl}(V_1) \)-submodules, \( \mathbb{C} \cdot 1 \) and \( \mathfrak{sl}(V_1) \), we deduce that \( \mathfrak{h} \) must contain a subspace of one of the two types: either \( 1 \otimes W \) or \( \mathfrak{sl}(V_1) \otimes W \) for a \( g_2 \)-invariant subspace \( W \subset \mathfrak{gl}(\Lambda(n)) \).
In the first case, by Lemma 3.2.2, \( \mathfrak{h} \supseteq 1 \otimes \mathfrak{sl}(\Lambda(n)) \) and, by Theorem 2.1, \( \mathfrak{h} \supseteq \mathfrak{sl}(V_1 \otimes \Lambda(n)) \).

In the second case, let us realize the elements of \( \mathfrak{gl}(\Lambda(n)) \) by differential operators acting on \( \Lambda(n) \). It is clear that by bracketing with \( \xi \) and \( \partial \), we can reduce any differential operator to the form \( \partial_j \), i.e., \( W \supseteq \partial_1, \ldots, \partial_n \). Applying formula (2.1) to elements of the form \( A \otimes \partial_j \) and \( B \otimes D \), where \( D \in \Lambda(\partial) \) is a differential operator with constant coefficients, we see that \( W \supseteq \Lambda(\partial) \). Therefore, by \( \mathfrak{g} \)-invariance, \( W \supseteq \mathfrak{sl}(\Lambda(n)) \) and, therefore, \( \mathfrak{h} \supseteq \mathfrak{sl}(V_1 \otimes \Lambda(n)) \). □

3.5. **Theorem.** The Lie superalgebra \( \mathfrak{g} = q(V_1) \otimes \Lambda(n) \subseteq \mathfrak{vect}(0|n) \) is a maximal Lie subsuperalgebra of \( \mathfrak{sq}(V) \), where \( V = V_1 \otimes \Lambda(n) \).

Proof is similar to that of Theorem 3.2.1.

3.6. **Lie superalgebras preserving a non-degenerate bilinear form.** We use notations from sec. 1.4 and 1.7. Observe that for \( n \leq 2 \) we have isomorphisms:

\[
T^{1/2}(\mathfrak{vect}(0|1)) \cong \mathfrak{pe}(1); \quad T^{1/2}(\mathfrak{vect}(0|2)) \cong \mathfrak{osp}(2|2).
\]

3.6.1. **Lemma.** For \( n > 2 \) the Lie superalgebra \( \mathfrak{g} = T^{1/2}(\mathfrak{vect}(0|n)) \) is maximal in \( \mathfrak{saut}(\omega_{1/2}) \).

**Proof.** Let us realize the space \( \mathfrak{gl}(\Lambda(\xi)) \) by differential operators. Observe that any function \( \varphi \in \Lambda(\xi) \) is an element from the space \( \text{sym}(\omega_{1/2}) \), and any differential operator with constant coefficients \( D \in \Lambda^k(\partial) \) belongs to \( \mathfrak{aut}(\omega_{1/2}) \) for \( k \) odd and to \( \text{sym}(\omega_{1/2}) \) for \( k \) even. Since \( \{A, B\} = 2AB - (-1)^{p(A)p(B)}[A, B] \), it follows that

\[
\mathfrak{aut}(\omega_{1/2}) = \text{Span}(\varphi, D \mid \varphi \in \Lambda(\xi), D \in \bigoplus_k \Lambda^{2k+1}(\partial));
\]

\[
\text{sym}(\omega_{1/2}) = \text{Span}(\varphi, D \mid \varphi \in \Lambda(\xi), D \in \bigoplus_k \Lambda^{2k}(\partial)).
\]

Set \( V_k = \bigoplus_{k \leq l, k \equiv l \mod 2} \{\Lambda(\xi), \Lambda^k(\partial)\} \). Clearly, each \( V_k \) is \( \mathfrak{g} \)-invariant, and as follows from [BL], the \( \mathfrak{g} \)-action in \( V_l/V_{l-2} \) is irreducible for \( l \neq 0, n \) (we assume that \( V_{-2} = V_{-1} = 0 \)).

The explicit formula for the bracket of \( A = \{\xi_1\xi_2, \partial_1\} \in \mathfrak{g} \) and \( B = \{\xi_1\xi_2, \partial_1\partial_2\partial_3 \ldots \partial_k\} \in V_k \) is

\[
[A, B] = -\xi_2\partial_3 \ldots \partial_k \in V_{k-2};
\]

it shows that the representation of \( \mathfrak{g} \) in \( V_k \) is not completely reducible and the minimal \( \mathfrak{g} \)-invariant subspace \( W \) such that \( \mathfrak{g} \subset W \subset \mathfrak{aut}(\omega_{1/2}) \) is \( V_3 \) for \( n > 3 \) and \( \mathfrak{saut}(\omega_{1/2}) \cong \mathfrak{spe}(4) \) for \( n = 3 \).

Finally, the formula

\[
[\partial_1\partial_2\partial_3, \xi_1\partial_1\partial_2 \ldots \partial_k] = \partial_2\partial_3\partial_1 \ldots \partial_k \in V_{k+2}
\]

shows that \( V_3 \) generates the whole Lie superalgebra \( \mathfrak{saut}(\omega_{1/2}) \). □

A corollary from the proof of Lemma 3.4.1 is the following statement: the minimal \( \mathfrak{g} \)-invariant subspace \( W \) such that \( \Lambda(n) \subset W \subset \text{sym}(\omega_{1/2}) \) is \( W = V_2 \).

3.7. **Theorem** (Cf. Table 3). Let \( \omega_1 \) be a non-degenerate supersymmetric or skew bilinear form in \( V_1 \) of dimension \( N_1 = m_1 + n_1 \varepsilon \), where \( N_1 \neq 1, 2 \) and \( n_1 \) is even if \( p(\omega_1) = 0 \), and where \( m_1 = n_1 > 2 \) if \( p(\omega_1) = \bar{1} \).

Then \( \mathfrak{g} = \mathfrak{aut}(\omega_1) \otimes \Lambda(n) \subseteq T^{1/2}(\mathfrak{vect}(0|n)) \) is a maximal subalgebra in \( \mathfrak{saut}(\omega_1 \otimes \omega_{1/2}) \) except for the case when \( n = 1, p(\omega_1) = 0 \) and \( m_1 \neq n_1 \); in this case \( \mathfrak{g} \) is maximal in \( \mathfrak{aut}(\omega_1 \otimes \omega_{1/2}) \).

Lemma 3.4.1 and its Corollary make it possible, essentially, to combine proofs of Theorem 3.2.1 and 2.4.2.
4. Proof of Main Theorem

4.1. Superization of a Proposition by Dixmier. In this section \( g \) is an irreducible linear Lie superalgebra, \( \rho \) its standard representation in a finite dimensional superspace \( V \) (in particular \( \rho \) is faithful). Let \( i \) be an ideal of \( g \). We will assume that \( \dim V > 1 \) (because the case \( \dim V = 1 \) is trivial). Our proof of Main Theorem is largely based on the following constructions and statements (Theorems 4.1.1 and 4.1.2), a superization of statements well-known to the reader from Dixmier’s book \([Di]\) (Proposition 5.5.1).

Let \( \tau \) be an irreducible subrepresentation of the restriction \( \rho|_i \) in the subspace \( U \subset V \) and let \( V_1 \subset V \) be the sum of all \( i \)-submodules of \( \rho|i \), isomorphic to either \( \tau \) or \( \Pi(\tau) \).

The stabilizer \( \text{st}(\tau) \) of \( \tau \) is the set of \( Y \in g \) such that there exists an \( A \in \text{End}(U) \) for which \( \tau([Y,X]) = [A,\tau(X)] \) for any \( X \in i \).

4.1.1. Theorem. 1) \( h = \text{st}(\tau) \supset g_0 \); 2) \( V_1 \) is \( h \)-invariant; 3) If \( h \neq g \), then \( \rho \equiv \text{Ind}_h^g \sigma \), where \( \sigma \) is a representation of \( h \) in \( V_1 \).

4.1.2. Theorem. Let \( g \) be an irreducible linear Lie superalgebra, \( \rho \) its standard representation (in particular, \( \rho \) is faithful). Let \( i \) be a nontrivial ideal of \( g \). Then 3 cases are possible: A) \( \rho|_i \) is irreducible; B) \( \rho|_i \) is a multiple of an irreducible \( i \)-module \( \tau \) and the multiplicity of \( \tau \) is \( > 1 \); C) there exists a proper subalgebra \( h \subset g \) such that \( i \subset h \) and \( \rho \equiv \text{ind}_h^g \sigma \) for an irreducible \( h \)-module \( \sigma \).

Proof largely follows the lines of \([Di]\) with one novel case: irreducible modules of \( Q \)-type might occur. Fortunately, their treatment is rather straightforward and I would rather save paper by omitting the verification.

We will say that the representation \( \rho \) is of type \( A \), \( B \), or \( C \) with respect to the ideal \( i \) if the corresponding case holds. Lemmas 4.5, 4.2 and 4.4 deal with types \( A \), \( B \) and \( C \), respectively.

4.2. Lemma. B) 1) If \( \rho \) is of type \( B \) with respect to the ideal \( i \), then either \( g \subset gl(V_1) \odot gl(V_2) \) or \( g \subset q(V_1) \odot q(V_2) \) for some \( V_1 \) and \( V_2 \). 2) If, moreover, \( g \subset q(V) \), then \( g \subset q(V_1) \odot q(V_2) \) for some \( V_1 \), \( V_2 \); 3) If \( \omega \) is a non-degenerate 2-form on \( V \) and \( g \subset \text{aut}(\omega) \), then \( g \subset \text{aut}(\omega_1) \odot \text{aut}(\omega_2) \), where \( \omega = \omega_1 \otimes \omega_2 \).

Proof of Lemma 4.2 follows, except statements involving \( q \), the standard scheme of the proof of a similar statement for Lie algebras, and the exception is also easy to consider, so I skip it.

4.3. Remark. If \( \rho \) is of type \( B \) with respect to the (nontrivial) ideal \( i \) and \( \dim \tau = 1 \) or \( \varepsilon \) then due to the faithfulness of \( \rho \) the ideal \( i \) should be a 1-dimensional center of \( g \).

4.4. Lemma. C) 1) If \( \rho \) is of type \( C \) with respect to the ideal \( i \), then \( g \subset gl(V_1) \otimes \Lambda(n) \in \text{vect}(0|n) \) for some \( V_1 \) and \( n \) and \( V = V_1 \otimes \Lambda(n) \). 2) If, moreover, \( g \subset q(n) \), then \( g \subset q(V_1) \otimes \Lambda(n) \in \text{vect}(0|n) \). 3) If \( \rho \) is of type \( C \) with respect to the ideal \( i \) and \( g \subset \text{aut}(\omega) \) for a non-degenerate form \( \omega \), then \( g \subset \text{aut}(\omega_1) \otimes \Lambda(n) \in T^{1/2}(\text{vect}(0|n)) \), where \( \omega = \omega_1 \otimes \omega_{1/2} \) and the representation \( T^{1/2} \) of \( \text{vect}(0|n) \) in the superspace \( \sqrt{\text{vol}} \) of half-densities is defined in sec. 1.7.

Proof. By definition of type \( C \), \( \rho = \text{ind}_h^g \sigma \). Let

\[
h_1 = \{ X \in \text{Ker} \sigma \mid [X,g] \subset h \}.
\]
Clearly, $\mathfrak{h}_1$ is a subalgebra in $\mathfrak{g}$ and ideal in $\mathfrak{h}$. For each integer $i > 1$ define inductively a subalgebra

$$\mathfrak{h}_i = \{ X \in \mathfrak{h}_{i-1} \mid [X, \mathfrak{g}] \subset \mathfrak{h}_{i-1} \}.$$ 

We obtain a decreasing filtration in $\mathfrak{g}$:

$$\mathfrak{h}_{-1} = \mathfrak{g} \supset \mathfrak{h}_0 = \mathfrak{h} \supset \mathfrak{h}_1 \supset \ldots$$

Since $\dim \mathfrak{g} < \infty$, the filtration stabilizes, i.e., $\mathfrak{h}_k = \mathfrak{h}_{k+1} = \ldots$ for some $k$. This means that $\mathfrak{h}_k$ is an ideal in $\mathfrak{g}$ lying in the kernel of $\sigma$. But then $\mathfrak{h}_k \subset \text{Ker} \rho$ and, therefore, $\mathfrak{h}_k = 0$ because $\rho$ is faithful.

Consider the associated graded Lie superalgebra

$$\text{gr}(\mathfrak{g}) = \bigoplus_{i=-1}^{k-1} \mathfrak{g}_i,$$

where $\mathfrak{g}_i = \mathfrak{h}_i/\mathfrak{h}_{i+1}$.

Observe that we have two homomorphisms of $\mathfrak{h}$:

$$\sigma : \mathfrak{h} \longrightarrow \text{gl}(V_1) \quad \text{and} \quad \text{ad}_{\mathfrak{g}/\mathfrak{h}} : \mathfrak{h} \longrightarrow \text{gl}(\mathfrak{g}/\mathfrak{h}),$$

and $\mathfrak{h}_1 = \text{Ker} \sigma \cap \text{Ker}(\text{ad}_{\mathfrak{g}/\mathfrak{h}})$. Set $W = \mathfrak{g}/\mathfrak{h}$, and let $\dim W = 0/n$. We have obtained an embedding

$$\mathfrak{g}_0 \subset \text{gl}(V_1) \oplus \text{gl}(W) \cong \text{gl}(V_1) \oplus W^* \otimes W,$$

which for every $i > 0$ induces an embedding

$$\mathfrak{g}_i \subset \text{gl}(V_1) \otimes \Lambda^i(W^*) \oplus \Lambda^{i+1}(W^*) \otimes W,$$

which add up to an embedding of the whole $\mathfrak{g}$:

$$\text{gr}(\mathfrak{g}) \subset \mathfrak{gl}(V_1) \otimes \Lambda(W^*) \in \text{vect}(W^*).$$

It remains to observe that since $\text{vect}(W^*)$ contains a grading operator, the embedding exists not only for $\text{gr}(\mathfrak{g})$, but for $\mathfrak{g}$ itself. Under the embedding the space $V_1$ is identified with $V_1 \otimes \Lambda(W) \cong V_1 \otimes \Lambda(W^*)^*$. As we observed in sec. 1.7, the space $\Lambda(W^*)^*$, as a $\text{vect}(W^*)$-module is isomorphic to $\text{II}^*(T^1)$. Since we are interested not in the representation itself but only in the image of $\mathfrak{g}$ in $\mathfrak{gl}(V)$, we can replace for convenience $V_1 \otimes \Lambda(W)$ with $\tilde{V}_1 \otimes \Lambda(W)$ for some $\tilde{V}_1$. This completes the proof of the first heading of Lemma.

To prove the second heading, observe that if $\mathfrak{g} \subset \mathfrak{q}(V) = C(J)$, then subspace $V_1$ is $J$-invariant and, therefore, $\sigma(\mathfrak{h}) \subset \mathfrak{q}(V_1) = C(J|_{V_1})$.

Finally, if $\mathfrak{g}$ preserves a non-degenerate bilinear form $\omega$ in $V$, consider its restriction $\tilde{\omega}$ on $V_1$. Since $\sigma$ is irreducible, $\tilde{\omega}$ is either non-degenerate or vanishes identically. Denote by $V_1^\perp$ the subspace orthogonal to $V_1$ with respect to $\omega$. Clearly, $V_1^\perp$ is $\mathfrak{h}$-invariant.

If $\tilde{\omega}$ is non-degenerate, then $V = V_1 \oplus V_1^\perp$. This means that $\mathfrak{g} \subset \text{aut}(\tilde{\omega}) \otimes 1 \oplus 1 \otimes \text{vect}(0/n)$. Therefore, the $\mathfrak{g}$-action in $V$ is reducible; contradiction.

If $\tilde{\omega} \equiv 0$, then by $\mathfrak{g}$-invariance of $\omega$, the space $V_1^\perp$ must contain the image of $V$ under $\rho(\text{Ker} \sigma)$. Hence, the $\mathfrak{h}$-module $V/V_1^\perp \cong V_1^*$ should be of the form $V_1 \otimes \Lambda^n(W)$. This implies that $\omega = \omega_1 \otimes \omega_{1/2}$ for some $\omega_1$ and $\mathfrak{g} \subset \text{aut}(\omega_1) \otimes \Lambda(n) \in T^{1/2}(\text{vect}(0/n))$. □

To complete the proof of Main Theorem, it suffices to consider the case when $\rho$ possesses the following property: for any ideal $i \subset \mathfrak{g}$ either $\rho$ is of type $A$ with respect to $i$ or $\rho|_i$ is the multiple of a character. Due to Remark 4.3 the second possibility means that $i$ is a 1-dimensional center of $\mathfrak{g}$. In particular, any nontrivial commutative ideal of $\mathfrak{g}$ coincides with its 1-dimensional center.

Let $\mathfrak{r}$ be the radical of the linear Lie superalgebras $\mathfrak{g}$. We see that either $\dim \mathfrak{r} \leq 1$ or $\mathfrak{r}$ is not commutative.
In the case when \( \mathfrak{r} \) is not commutative, consider the derived series of \( \mathfrak{r} \):
\[
\mathfrak{r} \supset \mathfrak{r}_1 \supset \cdots \supset \mathfrak{r}_k \supset \mathfrak{r}_{k+1} = 0, \text{ where } \mathfrak{r}_{i+1} = [\mathfrak{r}_i, \mathfrak{r}_i].
\]
Clearly, each \( \mathfrak{r}_k \) is an ideal in \( \mathfrak{g} \) and the last ideal, \( \mathfrak{r}_k \) is commutative. Hence, \( \dim \mathfrak{r}_k = 1 \) and \( \mathfrak{r}_k \) is the center of \( \mathfrak{g} \).

4.5. Lemma. A) 1) If \( \rho \) is of type A with respect to \( \mathfrak{r}_{k-1} \) and \( \rho|_{\mathfrak{r}_k} \) is scalar, then either \( \mathfrak{r}_{k-1} \cong \mathfrak{hei}(0|2n) \) or \( \mathfrak{r}_{k-1} \cong \mathfrak{hei}(0|2n-1) \) and \( \mathfrak{g} \cong \Lambda(n) \) or \( \Pi(\Lambda(n)) \) and \( \mathfrak{g} \subset \mathfrak{hei}(0|2n) \notin \mathfrak{o}(2n) \).
2) If additionally \( \mathfrak{g} \subset \mathfrak{q}(V) \), then \( \mathfrak{r}_{k-1} \cong \mathfrak{hei}(0|2n-1) \) and \( \mathfrak{g} \subset \mathfrak{hei}(0|2n-1) \notin \mathfrak{o}(2n-1) \).
3) Under assumptions of heading 1) \( \mathfrak{g} \) does not preserve any non-degenerate bilinear form on \( V \).

Proof. We will prove headings 1 and 2 simultaneously.

1) Since \( \mathfrak{r}_k \) is the center of \( \mathfrak{g} \) and \( \dim \mathfrak{r}_k = 1 \) we have \( \mathfrak{r}_{k-1} = \mathfrak{hei}(0|m) \) for some \( m \);
2) \( \rho|_{\mathfrak{r}_{k-1}} \) is irreducible and faithful, so it can be realized in the superspace of functions, \( \Lambda(n) \), or in \( \Pi(\Lambda(n)) \), where \( n = [\frac{m+1}{2}] \); observe that \( \rho|_{\mathfrak{r}_{k-1}} \) is irreducible of \( G \)-type for \( m = 2n \) and it is irreducible of \( Q \)-type for \( m = 2n - 1 \), see sec. 1.6;
3) \( \mathfrak{g} \) is contained in the normalizer of \( \mathfrak{hei}(0|m) \) in \( \mathfrak{gl}(\Lambda(n)) \), i.e., \( \mathfrak{g} \subset \mathfrak{hei}(0|m) \notin \mathfrak{o}(m) \). \( \square \)

4.6. Lemma. Let \( \dim \mathfrak{r} \leq 1 \), i.e., \( \mathfrak{g} \) is either semi-simple or a nontrivial central extension of a semi-simple Lie superalgebra but NOT an almost simple or a central extension of an almost simple Lie superalgebra. Then we can always choose an ideal \( \mathfrak{i} \neq \mathfrak{r} \) such that \( \rho \) is of type B or C with respect to \( \mathfrak{i} \).

4.7. Proof of Lemma 4.6 for semi-simple Lie superalgebras. By definition \( \mathfrak{g} \) is semi-simple if its radical is zero. By analogy with description of semi-simple Lie algebras over fields of prime characteristic, V. Kac [K] described semi-simple finite dimensional Lie superalgebras as follows. Let \( \mathfrak{s}_1, \ldots, \mathfrak{s}_k \) be simple Lie superalgebras, let \( n_1, \ldots, n_k \) be nonnegative integers, \( \Lambda(n_j) \) be the supercommutative Grassmann superalgebra, and \( \mathfrak{s} = \bigoplus \mathfrak{s}_j \otimes \Lambda(n_j) \). Then \( \mathfrak{der} = \bigoplus ((\mathfrak{der} \mathfrak{s}_j) \otimes \Lambda(n_j)) \subseteq 1 \otimes \mathfrak{vect}(n_j) \). Let \( \mathfrak{g} \) be a subalgebra of \( \mathfrak{der} \mathfrak{s} \) containing \( \mathfrak{s} \).

1) If the projection of \( \mathfrak{g} \) on \( 1 \otimes \mathfrak{vect}(n_j) \) coincides with \( \mathfrak{vect}(n_j) \) for each \( j = 1, \ldots, k \), then \( \mathfrak{g} \) is semi-simple.
2) All semi-simple Lie superalgebras arise in the manner indicated.

Let \( \dim \mathfrak{r} = 0 \), i.e., \( \mathfrak{g} \) is semi-simple. Since \( \mathfrak{g} \) is not almost simple, then, due to Kac’s description, the alternative arises: either \( \mathfrak{g} \) contains an ideal \( \mathfrak{i} \) of the form
\[
\mathfrak{i} = \mathfrak{s} \otimes \Lambda(n) \quad \text{with simple } \mathfrak{s} \text{ and } n > 0
\]
or \( \mathfrak{g} = \bigoplus_{j \leq k} \mathfrak{s}_j \), where each \( \mathfrak{s}_j \) is almost simple and \( k > 1 \).

4.7.1. Lemma. If \( \mathfrak{g} = \bigoplus_{j \leq k} \mathfrak{s}_j \) and \( k \geq 2 \), then any irreducible faithful representation of \( \mathfrak{g} \) is of type B with respect to any its ideal \( \mathfrak{s}_j \).

Proof. Since the stabilizer of any irreducible representation of \( \mathfrak{s}_j \) is the whole \( \mathfrak{g} \), the type of any irreducible representation of \( \mathfrak{g} \) with respect to \( \mathfrak{s}_j \) can be either A or B. Due to faithfulness case A is ruled out. \( \square \)

4.7.2. Lemma. Let \( \mathfrak{s} \) be a simple Lie superalgebra and \( \mathfrak{i} = \mathfrak{s} \otimes \Lambda(n), n > 0 \). Then \( \mathfrak{i} \) has no faithful irreducible finite dimensional representations.

For proof see sec. 4.7.4.

4.7.2a. Corollary. If \( \mathfrak{g} \) contains an ideal \( \mathfrak{i} \) of the form (4.1), then \( \mathfrak{g} \) can not have any faithful irreducible finite dimensional representation of type A with respect to the ideal \( \mathfrak{i} \).
4.7.2b. Corollary. Lemmas 4.7.1, 4.7.2 and Corollary 4.7.2.1 prove Lemma 4.6 for semi-simple Lie superalgebras.

Recall the following well-known and simple statement.

4.7.3. Lemma. If \( \mathfrak{s} \) is a simple Lie superalgebra, then

\[
[\mathfrak{s}_1, \mathfrak{s}_1] = \mathfrak{s}_0 \quad \text{and} \quad [\mathfrak{s}_0, \mathfrak{s}_1] = \mathfrak{s}_1.
\]

4.8. Proof of Lemma 4.7.2. For \( n > 0 \) the Lie superalgebra \( \mathfrak{i} \) contains supercommutative ideal

\[
\mathfrak{n} = \mathfrak{s} \otimes \xi_1 \ldots \xi_n.
\]

Let us show that \( \mathfrak{n} \) is contained in the kernel of any irreducible representation \( \rho \) of \( \mathfrak{i} \). Consider a nilpotent ideal

\[
\mathfrak{m} = \mathfrak{s} \otimes \bigoplus_{i \geq 1} \Lambda^i(\xi) \subset \mathfrak{i}.
\]

As follows from \([K], [Ser1]\), any irreducible finite dimensional representation of \( \mathfrak{m} \) is given by a character \( \lambda \in \mathfrak{m}^* \) that vanishes on \( [\mathfrak{m}_0, \mathfrak{m}_0] \oplus \mathfrak{m}_1 \).

For \( n = 2k \) we have

\[
[\mathfrak{m}_0, \mathfrak{m}_0] \supset [\mathfrak{s}_1 \otimes \Lambda^1(\xi), \mathfrak{s}_1 \otimes \Lambda^{2k-1}(\xi)] = [\mathfrak{s}_1, \mathfrak{s}_1] \otimes \Lambda^{2k}(\xi) = \mathfrak{n}_0
\]

and \( \mathfrak{m}_1 \supset \mathfrak{s}_1 \otimes \Lambda^{2k}(\xi) = \mathfrak{n}_1 \). For \( n = 2k + 1 > 1 \) we have

\[
[\mathfrak{m}_0, \mathfrak{m}_0] \supset [\mathfrak{s}_1 \otimes \Lambda^1(\xi), \mathfrak{s}_0 \otimes \Lambda^{2k}(\xi)] = [\mathfrak{s}_1, \mathfrak{s}_0] \otimes \Lambda^{2k+1}(\xi) = \mathfrak{n}_0
\]

and \( \mathfrak{m}_1 \supset \mathfrak{s}_0 \otimes \Lambda^{2k+1}(\xi) = \mathfrak{n}_1 \). Thus, \( \mathfrak{n} \) is contained in the kernel of any irreducible finite dimensional representation of the ideal \( \mathfrak{m} \), hence, in the kernel of any irreducible finite dimensional representation of \( \mathfrak{i} \).

If \( n = 1 \) and \( \tau \) is an arbitrary irreducible subrepresentation of \( \rho|_{\mathfrak{n}} \), then \( \dim \tau = 1 \) or \( \varepsilon \) because \( \mathfrak{n} \) is supercommutative. Hence, \( \tau|_{\mathfrak{n}_i} = 0 \). Heading 1) of Theorem 4.1.1 implies, that the restriction of \( \tau \) onto \( [\mathfrak{g}_0 \otimes \mathfrak{1}, \mathfrak{n}] \supset [\mathfrak{s}_0, \mathfrak{s}_1] \otimes \xi = \mathfrak{s}_1 \otimes \xi = \mathfrak{n}_0 \) must vanish. Thus, \( \tau|_{\mathfrak{n}} = 0 \) and by heading 3) of Theorem 4.1.1, \( \mathfrak{n} \subset \ker \rho \).

Lemma 4.7.2 is proved. \( \square \)

4.9. Proof of Lemma 4.6 for central extensions of semi-simple Lie superalgebras. In this section we assume that \( \dim \mathfrak{r} = 1 \), hence (see Remark 4.3), \( \mathfrak{r} \) is the center of \( \mathfrak{g} \), and \( \tilde{\mathfrak{g}} = \mathfrak{g}/\mathfrak{r} \) is semi-simple.

First, consider the case when \( \tilde{\mathfrak{g}} = \bigoplus_{i \leq k} \mathfrak{s}_i \), where the \( \mathfrak{s}_i \) are almost simple and \( k > 1 \). Let

\[
\pi : \mathfrak{g} \longrightarrow \tilde{\mathfrak{g}} \quad \text{be the natural projection. The Lie superalgebra} \quad \pi^{-1}(\mathfrak{s}_1) = \mathfrak{i} \quad \text{is an ideal in} \quad \mathfrak{g} \quad \text{and} \quad \dim \mathfrak{i} > 1.
\]

4.9.1. Lemma. If \( k > 1 \), then \( \rho \) can not be irreducible of type A with respect to \( \mathfrak{i} \).

Proof. Assume the contrary, let \( \rho|_{\mathfrak{i}} \) be irreducible. Set \( \mathfrak{g}_+ = \bigoplus_{i \geq 1} (\mathfrak{s}_i)_0 \) and \( \mathfrak{g}_+ \) is a Lie algebra.

Then \( [\mathfrak{s}_1, \mathfrak{g}_+] = 0 \). Since \( \rho(\mathfrak{r}) \) acts by scalar operators and \( \rho \) is a finite dimensional representation, it follows that \( [\pi^{-1}(\mathfrak{g}_+), \mathfrak{i}] = 0 \). Since \( \rho|_{\mathfrak{i}} \) is irreducible by the hypothesis, this means that \( \rho(\pi^{-1}(\mathfrak{g}_+)) = \mathbb{C} \cdot 1 \) and, since \( \rho \) is faithful, this implies \( \mathfrak{g}_+ = 0 \). \( \square \)

4.10. \( \tilde{\mathfrak{g}} \) contains an ideal \( \mathfrak{i} \) of the form (4.1). The central extension is defined by a cocycle \( c : \tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}} \longrightarrow \mathbb{C} \). The cocycle condition is

\[
c(f, [g, h]) = c([f, g], h) + (-1)^{p(f)p(g)} c(g, [f, h]) \quad \text{for any} \quad f, g, h \in \tilde{\mathfrak{g}}.
\]
As earlier, we assume that $\mathfrak{g}$ has a faithful finite dimensional representation; so the restriction of $c$ to $\hat{\mathfrak{g}}_0 \times \hat{\mathfrak{g}}_0$ is trivial. Besides, $c|_{\hat{\mathfrak{g}}_0 \times \hat{\mathfrak{g}}_1} = 0$ by parity considerations. Therefore, nonzero values of the cocycle $c$ are only possible on $\hat{\mathfrak{g}}_1 \times \hat{\mathfrak{g}}_1$.

4.10.1. Lemma. Let, as above, $n = s \otimes \Lambda^n(\xi)$. Then $c|_{n \times \hat{\mathfrak{g}}} = 0$.

Proof. First, let us prove that $c|_{n_1 \times n_1} = 0$. If $n = 2k + 1$, it suffices to check condition (4.2) for the triple $f = f_0 \otimes \xi_1 \cdots \xi_n$, $g = g_1 \otimes \xi_1 \cdots \xi_n$ and $h = h_1 \otimes 1$, where $f_0 \in s_0$ and $g_1, h_1 \in s_1$. With Lemma 4.7.3, the left hand side of (4.2) gives us the values of $c$ on an arbitrary pair of elements from $\mathfrak{n}_1$, whereas the right hand side vanishes because $[f, g] = 0$ and $c(g, [f, h]) = 0$ since $p(g) = 0$.

If $n = 2k$, similar arguments are applicable to the triple $f = f_1 \otimes \xi_1 \cdots \xi_n$, $g = g_0 \otimes \xi_1 \cdots \xi_n$ and $h = h_1 \otimes 1$, where $f_1, h_1 \in s_1$ and $g_0 \in s_0$.

Now set $\mathfrak{L}^k = s \otimes \Lambda^k(\xi_1, \ldots, \xi_n)$ and let us verify that $c|_{n \times \mathfrak{L}^k} = 0$. We will perform the inverse induction on $k$. For $k = n$ we have already verified the fact.

Let the statement be true for all $k > k_0$. Let us show that it is true for $k = k_0$ as well. Observe that due to description of semi-simple Lie superalgebras, $\hat{\mathfrak{g}}$ contains $n$ elements $\eta_i$ such that $\text{ad} \eta_k|_{\mathfrak{L}(n)} = \partial_{\xi_i} + D_i + X_i \otimes \alpha_i$, where $D_i \in \mathfrak{vect}(0|n)$ and $D_i(0) = 0$, $X_i \in \mathfrak{ders}$, $\alpha_i \in \Lambda(\xi)$. Let $\varphi \in \Lambda^{k_0+1}(\xi_1, \ldots, \xi_n)$, $\psi \in \Lambda^n(\xi_1, \ldots, \xi_n)$; $g, h \in \mathfrak{s}$ and $p(g \otimes \varphi) = 0$; $p(h \otimes \psi) = 1$.

Then we have

$$c([\eta, g \otimes \varphi], h \otimes \psi) = (-1)^{p(g)} \left( c(g \otimes \frac{\partial \varphi}{\partial \xi_i}, h \otimes \psi) + c(g \otimes D_i \varphi, h \otimes \psi) \right) +$$

$$(-1)^{p(g)+p(\alpha_i)} c([X_i, g] \otimes \alpha_i \varphi, h \otimes \psi).$$

As $D_i \varphi$, $\alpha_i \varphi \in \bigoplus_{l \geq k_0+1} \Lambda^l(\xi)$, the last two summands vanish by the inductive hypothesis.

On the other hand, due to (4.2) we have

$$c([\eta, g \otimes \varphi], h \otimes \psi) = c(\eta, [g \otimes \varphi, h \otimes \psi]) + c([\eta, h \otimes \psi], g \otimes \varphi).$$

As $\deg \varphi + \deg \psi = k_0 + 1 + n > n$, the bracket in the first summand above vanishes. The second summand vanishes by parity considerations. So $c(g \otimes \frac{\partial \varphi}{\partial \xi_i}, h \otimes \psi) = 0$, for arbitrary $g \otimes \frac{\partial \varphi}{\partial \xi_i} \in \mathfrak{L}^{k_0+1}_{1}$ and $h \otimes \psi \in \mathfrak{L}^{k_0}_{1} = \mathfrak{n}_1$.

4.10.2. Lemma. i has no faithful irreducible finite dimensional representations.

Proof. Word-for-word proof of Lemma 4.7.2 with the help of Lemma 4.8.2. 

4.11. Corollary. $\rho$ can not be of type $A$ with respect to the ideal $i$. 

4.12. Summing up. Lemmas 4.2 (Lemma B), 4.4 (Lemma C), 4.5 (Lemma A), sec. 4.4 and Lemma 4.6 put together prove Main Theorem. 

References

[BL] Bernstein J., Leites D., Invariant differential operators and irreducible representations of Lie superalgebras of vector fields. Sel. Math. Sov., v. 1, N 2, 1981, 143–160

[D] Deligne P. et al (eds.) Quantum fields and strings: a course for mathematicians. Vol. 1, 2. Material from the Special Year on Quantum Field Theory held at the Institute for Advanced Study, Princeton, NJ, 1996–1997. AMS, Providence, RI; Institute for Advanced Study (IAS), Princeton, NJ, 1999. Vol. 1: xxii+723 pp.; Vol. 2: pp. i–xxiv and 727–1501

[Di] Dixmier J. Algèbres enveloppantes, Gautier-Villars, Paris, 1974; Enveloping algebras. Revised reprint of the 1977 translation. Graduate Studies in Mathematics, 11. American Mathematical Society, Providence, RI, 1996. xx+379 pp.
[D1] Dynkin E.B. Semisimple subalgebras of semi-simple Lie algebras. Matem. Sbornik, 1952, v.30 (72), 349–462 (Russian) = English transl. in Moscow Math. Soc. Translations Ser. 2, v.6, 111–244

[D2] Dynkin E.B. Maximal subgroups of classical groups. Trudy Moskovskogo Matem. obshchestva, 1952, v.1 (Russian) = English transl. in: Moscow Math. Soc. Translations Ser. 2, v.6, 245–378

[FK] Freund P., Kaplansky I., Simple supersymmetries. J. Math. Phys. 17 (1976), no. 2, 228–231

[J1] Jeugt J. van der. Principal five-dimensional subalgebras of Lie superalgebras, J. Math. Phys., 27 (12), 1986, 2842–2847

[K] Kac V. G., Lie superalgebras. Adv. Math, 1976, 26, 8–96

[K1] Kac V.G., Characters of typical representations of classical Lie superalgebras. Comm. Algebra 5 (1977), no. 8, 889–897

[K2] Kac V.G., Representations of classical Lie superalgebras. Differential geometrical methods in mathematical physics, II (Proc. Conf., Univ. Bonn, Bonn, 1977), pp. 597–626, Lecture Notes in Math., 676, Springer, Berlin, 1978.

[Kap] Kaplansky I., Superalgebras. Pacific J. Math. 86 (1980), no. 1, 93–98

[KL] Kotchetkov Yu., Leites D., Simple Lie algebras in characteristic 2 recovered from superalgebras and on the notion of a simple finite group. In: Kegel O. et. al. (eds.) Proc. Intnat. algebraic conference, Novosibirsk, August 1989, Contemporary Math. AMS, 1992, (Part 2), v. 131, 59–67

[LS] Leites D., Serganova V. Metasymmetry and Volichenko algebras. Phys. Lett. B v. 252, n.1, 1990, 91–96; id., Symmetries wider than supersymmetries. In: S. Duplij and J. Wess (eds.) Noncommutative structures in mathematics and physics, Proc. NATO Advanced Research Workshop, Kiev, 2000. Kluwer, 2001, 13–30

[LSh] Leites D., Shchepochkina I., Howe’s duality and Lie superalgebras. In: S. Duplij and J. Wess (eds.) Noncommutative structures in mathematics and physics, Proc. NATO Advanced Research Workshop, Kiev, 2000. Kluwer, 2001, 93–112

[PS] Penkov I., Serganova V., Generic irreducible representations of finite dimensional Lie superalgebras. Internat. J. Math. 5, 1994, 389–419

[SNR] Scheunert M., Nahm W., Rittenberg V., Classification of all simple graded Lie algebras whose Lie algebra is reductive. I, II. Construction of the exceptional algebras. J. Mathematical Phys. 17 (1976), no. 9, 1626–1639, 1640–1644

[Se] Serganova V., Characters of irreducible representations of simple Lie superalgebras. Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998). Doc. Math. 1998, Extra Vol. II, 583–593 (electronic).

[S] Serganova V., Simple Volichenko algebras. In: Kegel O. et. al. (eds.) Proc. Intnat. algebraic conference, Novosibirsk, August 1989, Contemporary Math. AMS, 1992, (Part 2), v. 131, 50–58

[Ser1] Sergeev A., Irreducible representations of solvable Lie superalgebras. In: Leites D. (ed.) Seminar on supermanifolds, n. 22. Reports of Dept. of Math. of Stockholm Univ. 1988, n. 4, 1–12; arXiv:math.RT/9810109 and Represent. Theory 3 (1999), 435–443

[Ser2] Sergeev A., An analog of the classical invariant theory for Lie superalgebras. I, II. Michigan J. Math, v. 48, 2001, ???: arXiv:math.RT/9810113; math.RT/9904079

[Sh1] Shchepochkina I., On maximal and maximal solvable subalgebra of simple Lie superalgebras. In: Onishchik A. (ed.), Questions of group theory and homological algebra, Yaroslavl, 1987, 156–161 (in Russian)

[Sh2] Shchepochkina I., Maximal solvable subalgebra of the Lie superalgebras $gl(m|n)$ and $sl(m|n)$. Funktsional. Anal. i Prilozhen. 28 (1994), no. 2, 92–94; English translation in Functional Anal. Appl. 28 (1994), no. 2, 147–149

[Sh3] Shchepochkina I., Maximal subalgebras of matrix Lie superalgebras In: Leites D.(ed.) Seminar on Supermanifolds v. 32, Reports of Stockholm University, 1992, 1–43, 1997, 18 pp.; arXiv:hep-th/9702122

[T] Ten O.K. Semisimple maximal subalgebras of Lie $p$-algebras of classical type. (Russian) Vestnik Moskov. Univ. Ser. 1 Mat. Mekh. 1987, no. 4, 69–71, 103. id., Letter to the editor, Vestnik Moskov. Univ. Ser. 1 Mat. Mekh. 1988, no. 2, 103. 17B50; id., On nonsemi-simple maximal subalgebras of Lie $p$-algebras of classical type. (Russian) Vestnik Moskov. Univ. Ser. 1 Mat. Mekh. 1987, no. 2, 65–67, 103.

[ZZO] Zharov O., Zolotaryev B., Onishchik A., Reducible maximal subalgebra of classical Lie superalgebras. In: Onishchik A. (ed.), Questions of group theory and homological algebra, Yaroslavl Univ., Yaroslavl, 1987, 161–163 (in Russian)
INDEPENDENT UNIVERSITY OF MOSCOW, BOLSHOJ VLASIEVSKY PER, DOM 11, RU-119 002 MOSCOW, RUSSIA; IRINA@MCCME.RU