Phenomenological aspects of possible vacua of a neutrino flavor model

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Abstract: We discuss a supersymmetric model with discrete flavor symmetry \(A_4 \times Z_3\). The additional scalar fields which contribute masses of leptons in the Yukawa terms are introduced in this model. We analyze their scalar potential and find that they have various vacuum structures. We show the relations among 24 different vacua and classify them into two types. We derive expressions of the lepton mixing angles, Dirac \(CP\) violating phase and Majorana phases for the two types. The model parameters which are allowed by the experimental data of the lepton mixing angles are different for each type. We also study the constraints on the model parameters which are related to Majorana phases. The different allowed regions of the model parameters for the two types are shown numerically for a given region of two combinations of the \(CP\) violating phases.

Keywords: flavor symmetry, non-Abelian discrete group, neutrino flavor model

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1 Introduction

Although all the elementary particles in the standard model (SM) have now been discovered, with the discovery of the Higgs boson, there still exist phenomena which cannot be explained in the framework of the SM. One of these is the neutrino oscillation phenomenon, which implies two non-zero neutrino mass squared differences and two large lepton mixing angles. In order to explain this, many authors propose a neutrino flavor model with non-Abelian discrete flavor symmetry in the lepton sector [1–4]. Even before the discovery of the non-zero \(\theta_{13}\) [5–7], a few authors suggested a tiny mixing angle \(\theta_{13}\) based on non-Abelian discrete flavor symmetry [8]. Recent results from the T2K and NO\(\nu\)A experiments [9, 10] imply \(CP\) violation through the Dirac \(CP\) phase. They studied electron neutrino appearance in a muon neutrino beam. The Majorana phases are also sources of the \(CP\) violating phases if neutrinos are Majorana particles. The KamLAND-Zen experiment [11] is searching for neutrinoless double beta (0\(\nu\beta\beta\)) decay to check the Majorana nature of neutrinos. Therefore, it is important to predict not only mixing angles but also \(CP\) phases with the non-Abelian discrete flavor model.

The non-Abelian discrete flavor symmetry can easily explain large lepton mixing angles, e. g. tri-bimaximal mixing (TBM) [12, 13], which is a simple framework for the lepton mixing angles. Indeed, Altarelli and Feruglio (AF) proposed a simple flavor model and predicted TBM by using \(A_4\) discrete flavor symmetry [14, 15]. They introduced \(SU(2)\) gauge singlet scalar fields, so-called “flavons”, and derived the TBM in the lepton sector. The non-zero \(\theta_{13}\) can be realized by another \(A_4\) non-trivial singlet flavon [8] in addition to the flavons introduced by AF. The origin of non-vanishing \(\theta_{13}\) is related to a new contribution to the mass matrices. Matrices which have the same structure as that in Ref. [8] also appear in extra-dimensional models with the \(S_3\) and \(S_4\)
flavor symmetries [16, 17]. The $\Delta(27)$ model also includes these matrices [18].

In this paper, we study phenomenological aspects of a supersymmetric model with $A_4 \times Z_3$ symmetries. The three generations of the left-handed leptons are expressed as the $A_4$ triplet, $l = (l_\mu, l_\tau, l_\nu)$, while the right-handed charged leptons $e_R, \mu_R$, and $\tau_R$ are $A_4$ singlets denoted as $1, 1''$, and $1'$ respectively. Three right-handed neutrinos are also described as the triplet of $A_4$. We introduce the $SU(2)$ gauge singlet flavons of $A_4$ triplets, $\phi_T = (\phi_{T1}, \phi_{T2}, \phi_{T3})$ and $\phi_S = (\phi_{S1}, \phi_{S2}, \phi_{S3})$. In addition, $\xi$ and $\xi'$ are also introduced as the $SU(2)$ gauge singlet flavons with the two kinds of singlet representations of $A_4$, 1 and 1' respectively.

We focus on the vacuum structure of the flavor model. The scalar sectors of this model consist of many flavons in addition to the SM Higgs boson. Then, we analyze the scalar potential and show the 24 different sets of VEVs which come from 24 combinations of 4 (6) possible VEVs of the flavon $\phi_T$ ($\phi_S$). The 24 different vacua are classified into two types which are not related to each other under the transformations $A_4$. Therefore, we expect that the two types of vacua have different expressions for the physical observables in terms of the model parameters such as Yukawa couplings. We ask the following question: whether these different vacua are physically distinct from each other. The purpose of this paper is to clarify the differences and relations among the VEVs and their physical consequences. In particular, we investigate the mixing angles, CP violating phase, and effective mass for neutrinoless double beta ($0\nu\beta\beta$) decay.

This paper is organized as follows. In Section 2, we introduce the supersymmetric model with $A_4 \times Z_3$ symmetry. In Section 3, we study the classification of vacua and derive the formulae for the mixing angles and CP phases. In Section 4, we discuss the phenomenological aspects for mixing angles and CP violating phases. The numerical analyses for the effective mass of $0\nu\beta\beta$ decay are presented. Section 5 is devoted to a summary. In Appendix, we show the multiplication rule of the $A_4$ group.

2 Supersymmetric model with $A_4 \times Z_3$ symmetry

In this section, we introduce a supersymmetric model with $A_4 \times Z_3$ symmetry. We analyze the scalar potential and derive the mass matrices of the lepton sector.

2.1 Model

We introduce three heavy right-handed Majorana neutrinos. The leptons and scalars in our model are listed in Table 1.

Table 1. The representations of $SU(2)_L$ and $A_4$, and the charge assignment of $Z_3$ and $U(1)_R$ for leptons and scalars: $l_{e,\mu,\tau}$, $\{e, \mu, \tau\}_R$, $\{\nu_e, \nu_\mu, \nu_\tau\}_R$, and $h_{u,d}$ denote left-handed leptons, right-handed charged leptons, right-handed neutrinos, and Higgs fields, respectively. The other scalars are gauge singlet flavons and denoted as $\phi_T$, $\phi_S$, $\xi$, and $\xi'$.

| $SU(2)_L$ | $A_4$ | $Z_3$ | $U(1)_R$ |
|-----------|-------|-------|----------|
| $l_{e,\mu,\tau}$ | $\phi_T$ | $\phi_S$ | $\xi$, $\xi'$ |
| $\{e, \mu, \tau\}_R$ | $\{\nu_e, \nu_\mu, \nu_\tau\}_R$ | $h_{u,d}$ | |
We have introduced the additional $SU(2)$ gauge singlet $\xi$, which we call the “driving fields”.

The charge assignments of these fields are summarized in Table 2.

### Table 2. The driving fields and their representations and charge assignment.

|         | $\phi_0^T$ | $\phi_0^S$ | $\xi$ |
|---------|------------|------------|-------|
| $SU(2)$ | 1          | 1          | 1     |
| $A_1$   | 3          | 3          | 1     |
| $Z_3$   | 1          | $\omega^2$ | $\omega^2$ |
| $U(1)_R$| 2          | 2          | 2     |

We have introduced the additional $SU(2)$ gauge singlet $\xi$, which we call the “driving fields”. The charge assignments of these fields are summarized in Table 2.

### 2.2 Potential analysis

In this subsection, we derive the VEVs for the scalar fields $\phi, \phi^s, \xi, \xi'$, $\phi^T_0, \phi^S_0$. One can derive the scalar potential from the superpotentials in Eqs. (6) and (7) as

$$V = V_T + V_S,$$

where

$$V_T = \sum_x \left| \frac{\partial w_T}{\partial X} \right|^2 = -M\phi_{T1} + \frac{2}{3} g(\phi^{2}_{T1} - \phi_{T2}\phi_{T3})^2 + -M\phi_{T3} + \frac{2}{3} g(\phi^{2}_{T2} - \phi_{T3}\phi_{T1})^2 + -M\phi_{T2} + \frac{2}{3} g(\phi^{2}_{T1} - \phi_{T3}\phi_{T2})^2,$$

and

$$V_S = \sum_y \left| \frac{\partial w_S}{\partial Y} \right|^2 = \frac{2}{3} g_1(\phi^{2}_{S1} - \phi_{S2}\phi_{S3}) - g_2\phi_{S1}\xi + g_2\phi_{S3}\xi' - g_2\phi_{S2}\xi + g_2\phi_{S3}\xi' + g_2\phi_{S2}\phi_{S3}.$$

The sum for $X, Y$ runs over all the scalar fields:

$$X = \{ \phi_{T1}, \phi_{T2}, \phi_{T3}, \phi^{T}_{01}, \phi^{T}_{02}, \phi^{T}_{03} \},$$

$$Y = \{ \phi_{S1}, \phi_{S2}, \phi_{S3}, \phi^{S}_{01}, \phi^{S}_{02}, \phi^{S}_{03}, \xi, \xi' \}.$$

The scalar potential $V$ is minimized at $V = V_T = V_S = 0$.

There are several solutions for the minimization condition. We obtain sets of solutions denoted as $\eta_1$ and $\lambda_\eta$ ($m = 1-4, n = 1-3$), where $\eta_m$ and $\lambda_\eta$ are the solutions of $V_T = 0$ and $V_S = 0$ respectively. Hereafter, we call them the set of VEV alignments and show them explicitly as follows:

$$\eta_1 \equiv \left\{ \langle \phi_T \rangle = \frac{v_T}{3}, \langle \phi_T^T \rangle = \frac{2}{3} \right\},$$

$$\eta_2 \equiv \left\{ \langle \phi_T \rangle = \frac{v_T}{3}, \langle \phi_T^T \rangle = \frac{2}{3} \right\},$$

$$\eta_3 \equiv \left\{ \langle \phi_T \rangle = \frac{v_T}{3}, \langle \phi_T^T \rangle = \frac{2}{3} \right\},$$

$$\eta_4 \equiv \left\{ \langle \phi_T \rangle = \frac{v_T}{3}, \langle \phi_T^T \rangle = \frac{2}{3} \right\}.$$
\[ \lambda_{+}^{\pm} \equiv \left\{ \langle \phi_{S} \rangle = \pm v_{S} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \langle \xi' \rangle = u', \langle \phi_{h}^{c} \rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}, \quad (15) \]

\[ \lambda_{2}^{\pm} \equiv \left\{ \langle \phi_{S} \rangle = \pm v_{S} \begin{pmatrix} 1 \\ \omega \\ \omega^{2} \end{pmatrix}, \langle \xi' \rangle = \omega u', \langle \phi_{h}^{c} \rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}, \quad (16) \]

\[ \lambda_{3}^{\pm} \equiv \left\{ \langle \phi_{S} \rangle = \pm v_{S} \begin{pmatrix} 1 \\ \omega^{2} \\ \omega^{3} \end{pmatrix}, \langle \xi' \rangle = \omega^{2} u', \langle \phi_{h}^{c} \rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}, \quad (17) \]

where \( v_{T} = \frac{\sqrt{2} M_{W}}{2 g} \), \( v_{S} = \frac{\sqrt{2} M_{Z}}{g_{S}} \), \( u' = \frac{2}{g_{S}} u \) and \( u \) is the VEV of \( \xi \), \( \langle \xi \rangle = u \). The superscript of \( \lambda^{\pm} \) denotes the overall sign of the VEV \( \langle \phi_{S} \rangle \). In total, we obtain 24 sets of vacua, since there are four sets of alignment for \( \eta_{m} \) and six sets for \( \lambda_{m}^{\pm} \).

### 2.3 Mass matrix for charged leptons and neutrinos

We derive charged lepton mass matrices and neutrino mass matrices from the Yukawa interactions in Eqs. (2), (3), and (4). These matrices are expressed in various forms corresponding to the VEV alignments. The charged lepton mass matrices \( M_{l}^{(m)} \) for Eqs. (11)–(14) are

\[ M_{l}^{(1)} = \frac{\nu_{d} v_{T}}{\Lambda} \begin{pmatrix} y_{e} & 0 & 0 \\ 0 & y_{\mu} & 0 \\ 0 & 0 & y_{\tau} \end{pmatrix}, \quad (18) \]

\[ M_{l}^{(2)} = \frac{\nu_{d} v_{T}}{3 \Lambda} \begin{pmatrix} -y_{e} & 2y_{\mu} & 2y_{\tau} \\ 2y_{\mu} & -y_{\mu} & 2y_{\tau} \\ 2y_{\tau} & 2y_{\mu} & -y_{\tau} \end{pmatrix} = S M_{l}^{(1)}, \quad (19) \]

\[ M_{l}^{(3)} = \frac{\nu_{d} v_{T}}{3 \Lambda} \begin{pmatrix} -y_{e} & 2\omega y_{\mu} & 2\omega^{2} y_{\tau} \\ 2\omega^{2} y_{\mu} & -y_{\mu} & 2\omega y_{\tau} \\ 2\omega y_{\tau} & 2\omega^{2} y_{\mu} & -y_{\tau} \end{pmatrix} = T^{\dagger} S M_{l}^{(1)}, \quad (20) \]

\[ M_{l}^{(4)} = \frac{\nu_{d} v_{T}}{3 \Lambda} \begin{pmatrix} -y_{e} & 2\omega^{2} y_{\mu} & 2\omega y_{\tau} \\ 2\omega^{2} y_{\mu} & -y_{\mu} & 2\omega y_{\tau} \\ 2\omega y_{\tau} & 2\omega^{2} y_{\mu} & -y_{\tau} \end{pmatrix} = T S T^{\dagger} M_{l}^{(1)}, \quad (21) \]

respectively, where the matrices \( S \) and \( T \) are

\[ S = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^{2} \end{pmatrix}. \quad (22) \]

The Dirac mass matrix for neutrinos obtained from Eq. (3) is

\[ M_{\nu} = y_{D} v_{\nu} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \quad (23) \]

It is noted that the Dirac mass matrix is determined independently of the VEV alignments. The Majorana mass matrices \( M_{R}^{(m)} \) for the corresponding set of solutions Eqs. (15)–(17) are given as follows:

\[ M_{R}^{(1)} = \frac{1}{3} y_{\phi_{S}} v_{S} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} + y_{\nu} u \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad (24) \]

\[ M_{R}^{(2)} = \frac{1}{3} y_{\phi_{S}} v_{S} \begin{pmatrix} 2 & -\omega^{2} & -\omega \\ -\omega^{2} & 2\omega & -1 \\ -\omega & 1 & 2\omega^{2} \end{pmatrix} + y_{\nu} u \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad (25) \]

\[ M_{R}^{(3)} = \frac{1}{3} y_{\phi_{S}} v_{S} \begin{pmatrix} 2 & -\omega & -\omega^{2} \\ -\omega & 2\omega^{2} & -1 \\ -\omega^{2} & 1 & 2\omega \end{pmatrix} + y_{\nu} u \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad (26) \]

In order to generate the light neutrino mass matrices, we adopt the seesaw mechanism [19–21]. The effective neutrino mass matrices are given by the well-known formula, \( M_{\nu} = -M_{D} M_{R}^{-1} M_{D}^{T} \), through the seesaw mechanism. We obtain the 6 different effective neutrino mass.

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1) There are still other solutions for \( V = 0 \), including the trivial solution which makes all the VEVs vanish. It leads to the vanishing of all the lepton masses and mixing angles. In addition to the trivial solution, there are solutions with non-zero VEVs of the driving fields. This case leads to the breakdown of \( U(1)_{R} \) symmetry. In this paper, we only discuss the vacua where \( U(1)_{R} \) symmetry is conserved.
matrices from Eqs. (23)-(26) as follows:

\[
M^{(1)\pm}_\nu = \pm a \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + b^\pm \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + c \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

\[
M^{(2)\pm}_\nu = T^\dagger M^{(1)\pm}_\nu T,
\]

\[
M^{(3)\pm}_\nu = TM^{(1)\pm}_\nu T,
\]

where

\[
a = k y_\alpha v_S,
\]

\[
c = k (y_\alpha u' - y_\nu u),
\]

\[
d = k y_\nu u',
\]

\[
b^\pm = \frac{\alpha^2}{2} \pm \frac{\alpha^2}{2d - c} \left( \frac{1}{3} - \frac{d^2}{a^2} \right),
\]

\[
k = \frac{y_\nu b^\pm v_u}{y_\nu^2 u^2 + y_\nu^2 u'^2 - (y_\nu y_\mu y_u' + y_\nu y_\mu y_u')}.
\]

### 3 Classification of vacua and PMNS mixing matrix

In this section, we classify the 24 different vacua and derive the lepton mixing matrix $U_{\text{PMNS}}$, called the **Pontecorvo-Maki-Nakagawa-Sakata (PMNS) mixing matrix**. In order to classify the vacua, we discuss the relations among the VEV alignments with the transformations of $A_4$. We show that the 24 vacua are classified into two types in the following subsection. Then, one finds the two different PMNS matrices with diagonalizing matrices for the charged lepton and effective neutrino mass matrices Eqs. (18)-(21), and (27)-(29).

#### 3.1 Relations among sets of VEV alignments

The generators of $A_4$ are expressed as the following forms for the representations $1, 1', 1''$ and $3$,

\[
S(1) = S(1') = S(1'') = 1,
\]

\[
S(3) = \begin{pmatrix} 1 & 2 & -1 \\ -2 & 1 & 2 \\ -2 & 2 & 1 \end{pmatrix},
\]

\[
T(1) = 1, \quad T(1') = \omega, \quad T(1'') = \omega^2,
\]

\[
T(3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}.
\]

The sets of VEV alignment $\eta_n, \lambda^s_n$ are associated through the transformations of these generators. As an example, we show the $T$ transformation on $\lambda^+_1$.

\[
T[\lambda^+_1] = T(3) v_S \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \langle \xi' \rangle = T(1') u', \quad \langle \phi_0^s \rangle = T(3) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \lambda^+_2.
\]

Fig. 1. (color online) Map of the transitions among the VEV alignments under the transformations $S$ and $T$: The solid arrow corresponds to the transition due to $T$ transformation and the dashed two headed arrow shows the transition due to $S$ transformation. In the map, $\eta_1$ is invariant under $T$ transformation while $\lambda^+_1$ are invariant under $S$ transformation.
3.2 Classification of 24 vacua

In this subsection, we show the relations among the 24 different Lagrangians derived from the 24 different combinations of VEV alignments in Eqs. (11)–(17). We find the two sets of 12 equivalent Lagrangians with the appropriate field redefinitions. Then, the 24 Lagrangians are classified into two types. For simplicity, we write the Lagrangian of this model in a short form:

\[ \mathcal{L}(\psi, \phi_1, \phi_2), \]

where \( \psi \) represents the fermion fields such as \( l \) and \( \nu_R \), \( \phi_1 \) and \( \phi_2 \) represent the scalar fields, which should have their VEVs written as \( \eta_m \) and \( \lambda_n^\pm \) respectively. We write the Lagrangian in the broken phase for the VEV alignment \( (\eta_m, \lambda_n^\pm) \) with fluctuations \( h_1 \) and \( h_2 \) as

\[ \mathcal{L}_{\eta_m}^{\pm}(\psi, h_1, h_2) = \mathcal{L}(\eta_m + h_1, \lambda_n^\pm + h_2). \]

Then, we prove the following equation:

\[ \mathcal{L}(\psi', \eta_m + h_1', \lambda_n^\pm + h_2') = \mathcal{L}(\psi, G^{-1}\eta_m + h_1, G^{-1}\lambda_n^\pm + h_2), \]

where \( G \) denotes the transformation composed of \( S \) and \( T \) in Eqs. (30) and (31). There are 12 independent transformations including the identity element:

\[ G_1; (e, T, T^2, S, T^2S, ST, ST^2, TST, TST^2, T^2ST^2). \]

The redefined fields are written as follows,

\[ \psi' = G \psi, \quad h_i' = Gh_i \quad (i = 1, 2). \]

The right-hand side of Eq. (39) corresponds to the Lagrangian for the vacuum \( (G^{-1}\eta_m, G^{-1}\lambda_n^\pm) \) while the left-hand side is the Lagrangian for the vacuum \( (\eta_m, \lambda_n^\pm) \) in terms of the redefined fields. In the symmetric phase, the Lagrangian \( \mathcal{L}(\psi, \phi_1, \phi_2) \) is invariant under the \( G \) transformation,

\[ \mathcal{L}(G \psi, G \phi_1, G \phi_2) = \mathcal{L}(\psi, \phi_1, \phi_2). \]

One obtains the following equation from Eq. (42) for the vacuum \( (G^{-1}\eta_m, G^{-1}\lambda_n^\pm) \),

\[ \mathcal{L}(G \psi, \eta_m + G h_1, \lambda_n^\pm + G h_2) = \mathcal{L}(\psi, G^{-1}\eta_m + h_1, G^{-1}\lambda_n^\pm + h_2). \]

Finally, one obtains the relation Eq. (39) by applying the field definition Eq. (41) to the left-hand side of Eq. (43). The relation Eq. (39) implies the equality of the Lagrangians for the two vacua \( (\eta_m, \lambda_n^\pm) \) and \( (G^{-1}\eta_m, G^{-1}\lambda_n^\pm) \).

Here, we briefly show how to find the equivalent vacua with Fig. 1. For example, let us consider the \( T \) transformation in terms of the vacuum of \( (\eta_1, \lambda_1^+) \). One finds that \( \eta_1 \) is invariant and \( \lambda_1^+ \) transfers to \( \lambda_2^+ \) under the \( T \) transformation. Therefore, \( L_{11}^1 \) and \( L_{12}^1 \) are equivalent. One can find 12 equivalent vacua by applying 12 independent transformations in Eq. (40) to the vacuum \( (\eta_1, \lambda_1^+) \). Then, we classify the 24 Lagrangians into two types:

Type I:

\[ \{L_{11}^1, L_{12}^1, L_{13}^1, L_{21}^1, L_{22}^1, L_{23}^1, L_{31}^1, L_{33}^1, L_{41}^1, L_{43}^1\}, \]

Type II:

\[ \{L_{11}^2, L_{12}^2, L_{13}^2, L_{21}^2, L_{22}^2, L_{23}^2, L_{31}^2, L_{33}^2, L_{41}^2, L_{43}^2\}. \]

Type I and type II are disconnected because of the absence of a transformation which relates one type to the other. Since all the Lagrangians which belong to the same type lead to the same physical consequences, we consider only \( L_{11}^1 \) and \( L_{11}^2 \) as the representatives of their types:

\[ L_1^1 \equiv L_{11}^1, \quad L_1^2 \equiv L_{11}^2. \]

We also define the representative mass matrices for charged leptons and neutrinos as

\[ M_l \equiv M_l^{(1)}, \quad M_\nu \equiv M_\nu^{(1)}, \quad M_\nu^+ \equiv M_\nu^{(1)^*}. \]

It is noted that the charged lepton mass matrix \( M_l^{(1)} \) is diagonal.

3.3 PMNS matrices for two types

In this subsection, we construct the PMNS matrices for the two types, \( L_1^1 \) and \( L_1^2 \). Since the charged lepton mass matrix \( M_l \) is diagonal, the PMNS matrix is determined so that it diagonalizes the neutrino mass matrices in Eq. (27):

\[ (U_{\text{PMNS}}^1)^\dagger M_{\nu}^{(1)}(U_{\text{PMNS}}^1)^* = (U_{\text{PMNS}}^2)^\dagger M_{\nu}^{(1)}(U_{\text{PMNS}}^2)^* \]

\[ = \left( \begin{array}{ccc} m_1 & m_2 & m_3 \\ m_1 & m_2 & m_3 \\ m_1 & m_2 & m_3 \end{array} \right), \]

where the left-handed neutrino masses \( m_1, m_2 \) and \( m_3 \) are positive. The PMNS matrices are expressed as the following forms for the two types:

\[ U_{\text{PMNS}}^1 = U_{\text{THM}} U_{13} (\theta, \sigma) \left( \begin{array}{c} e^{i\phi_1} \\ e^{i\phi_2} \end{array} \right), \]

\[ U_{\text{PMNS}}^2 = U_{\text{THM}} \left( \begin{array}{c} 1 \\ i \end{array} \right) U_{13} (\theta, \sigma) \left( \begin{array}{c} e^{i\phi_1} \\ e^{i\phi_2} \end{array} \right), \]

\[ = U_{\text{THM}} U_{13} (\theta + \pi/2, \sigma) \left( \begin{array}{c} e^{i\phi_1 + \sigma} \\ e^{i\phi_2} \end{array} \right), \]

\[ = U_{\text{THM}} U_{13} (\theta + \pi/2, \sigma) \left( \begin{array}{c} e^{i\phi_1 + \sigma} \\ e^{i\phi_2} \end{array} \right), \]
The unitary matrix $U_{TBM}$ is the tri-bimaximal mixing matrix and $U_{tB}^{\pm}(\theta,\sigma)$ denotes the unitary rotation matrix:

$$U_{TBM} = \begin{pmatrix} 2/\sqrt{6} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix}, \quad (51)$$

$$U_{13}(\theta,\sigma) = \begin{pmatrix} \cos \theta & 0 & e^{i\sigma} \sin \theta \\ 0 & 1 & 0 \\ -e^{i\sigma} \sin \theta & 0 & \cos \theta \end{pmatrix}. \quad (52)$$

We have introduced the parameters $\theta, \sigma$ and $\phi$, ($i=1,2,3$). They are written in terms of the complex parameters of the neutrino mass matrix, $a, b, c$ and $d$, in Eq. (27)\(^1\).

In the rest of this subsection, we derive the explicit forms of the parameters $\theta, \sigma$ and $\phi$, in terms of the model parameters $a, b, c$ and $d$. In the first step, one rotates $M_r, M^\ell_r$ with the tri-bimaximal mixing matrix.

$$U_{TBM}^T M_1^r (M^\ell_1)^T U_{TBM} = \begin{pmatrix} A & 0 & B \\ 0 & C & 0 \\ B^* & 0 & D \end{pmatrix}, \quad (53)$$

$$U_{TBM}^T M_3^\ell (M^\ell_3)^T U_{TBM} = \begin{pmatrix} D & 0 & -B^* \\ 0 & C & 0 \\ -B & 0 & A \end{pmatrix}. \quad (54)$$

where

$$A = \left| a + c - \frac{d^2}{2} \right|^2 + \left| \frac{\sqrt{3}}{2} d \right|^2, \quad (55)$$

$$B = \left( a + c - \frac{d^2}{2} \right) \frac{\sqrt{3}}{2} d^* + \frac{\sqrt{3}}{2} d \left( a - c + \frac{d^2}{2} \right)^* \equiv |B| e^{i\phi_n}, \quad (56)$$

Finally, the other parameters $\phi_i$ are determined as follows,

$$\phi_1 = \frac{1}{2} \tan^{-1} \left[ \frac{\text{Im}[a] + \text{Im}[c - \frac{d^2}{2}] \cos \theta - \text{Re}[a] \cos \theta + \text{Re}[c - \frac{d^2}{2}] \sin \theta - \frac{\sqrt{3}}{2} \text{Im}[d] \sin \theta}{\text{Re}[a] \cos \theta + \text{Re}[c - \frac{d^2}{2}] \sin \theta - \frac{\sqrt{3}}{2} \text{Re}[d] \sin \theta} \right] - \sigma, \quad (65)$$

$$\phi_2 = \frac{1}{2} \tan^{-1} \left[ \frac{\text{Im}[a^2 - (c^2 - cd + d^2)] \text{Re}[2d - c] - \text{Re}[a^2 - (c^2 - cd + d^2)] \text{Im}[2d - c]}{\text{Re}[a^2 - (c^2 - cd + d^2)] \text{Re}[2d - c] + \text{Im}[a^2 - (c^2 - cd + d^2)] \text{Im}[2d - c]} \right], \quad (66)$$

$$\phi_3 = \frac{1}{2} \tan^{-1} \left[ \frac{\text{Im}[a] - \text{Im}[c - \frac{d^2}{2}] \cos \theta - \text{Re}[a] \cos \theta - \text{Re}[c - \frac{d^2}{2}] \sin \theta + \frac{\sqrt{3}}{2} \text{Re}[d] \sin \theta}{\text{Re}[a] - \text{Re}[c - \frac{d^2}{2}] \cos \theta + \text{Im}[a] \cos \theta - \text{Im}[c - \frac{d^2}{2}] \sin \theta + \frac{\sqrt{3}}{2} \text{Im}[d] \sin \theta} \right] + \sigma. \quad (67)$$

\(^1\) There are six real parameters since $b$ is written by using $a, c, d$.  

\[ C = \left| a^2 - (c^2 - cd + d^2) \right|^2 / 2d - c, \quad (57) \]

\[ D = \left| a + c - \frac{d^2}{2} \right|^2 + \left| \frac{\sqrt{3}}{2} d \right|^2. \quad (58) \]

The mass eigenvalues are determined as

$$m_1^2 = \frac{A + D}{2} \pm \frac{1}{2} \sqrt{(A - D)^2 + 4 |B|^2}, \quad (59)$$

$$m_2^2 = C, \quad (60)$$

$$m_3^2 = \frac{A + D}{2} \pm \frac{1}{2} \sqrt{(A - D)^2 + 4 |B|^2}, \quad (61)$$

where the upper and lower signs in these mass eigenvalues correspond to the normal hierarchy (NH) and the inverted hierarchy (IH). Next, we diagonalize the rotated mass matrices, Eqs. (53) and (54), with $U_{13}(\theta, \sigma)$ and $U_{13}(\theta, c + d, \sigma)$ respectively.
We briefly explain the derivation of \( \phi_i \) for the mass matrix \( M_i^j \). We first diagonalize \( M_i^j \) with the unitary matrices \( U_{\text{RBM}} \) and \( U_{13}(\theta, \sigma) \) according to Eq. (48). However, the diagonalized neutrino mass matrix consists of complex elements. Then, the parameters \( \phi_i \) are determined so that all the elements of the diagonalized matrix are real and positive.

4 Phenomenological aspects

We study the phenomenological aspects of this model and show the differences between the two types of vacua.

\[
U_{\text{PDG}}^{\text{PMNS}} = \begin{pmatrix}
    c_{12}c_{13} & s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{CP}} & c_{12}s_{23} - c_{12}s_{23}s_{13}e^{i\delta_{CP}} \\
    -s_{12}c_{13} & s_{12}c_{23} + s_{12}s_{23}s_{13}e^{i\delta_{CP}} & -s_{12}s_{23} - s_{12}s_{23}s_{13}e^{i\delta_{CP}} \\
    s_{13} & s_{23}c_{13} & c_{23}
\end{pmatrix}
\]

where \( s_{ij} \) and \( c_{ij} \) denote the lepton mixing angles \( \sin \theta_{ij} \) and \( \cos \theta_{ij} \), respectively. They are written in terms of the PMNS matrix elements:

\[
\sin^2 \theta_{12} = \frac{|U_{e1}|^2}{1 - |U_{e3}|^2}, \quad \sin^2 \theta_{23} = \frac{|U_{e2}|^2}{1 - |U_{e3}|^2}, \quad \sin^2 \theta_{13} = |U_{e3}|^2,
\]

(69)

where \( U_{\alpha i} \) denote the PMNS matrix elements. The Dirac CP violating phase \( \delta_{CP} \) can be obtained with the Jarlskog invariant

\[
\sin \delta_{CP} = \frac{J_{CP}}{s_{23}c_{23}s_{12}c_{13}c_{13}},
\]

\[
J_{CP} = \text{Im}[U_{e1}U_{\mu2}U_{\mu1}^*U_{\mu2}^*].
\]

(70)

(71)

In order to obtain these parameters from our model, we substitute the PMNS matrix elements in Eqs. (49) and (50). The observables, such as mixing angles and CP violating phases, are described with the model parameters in different forms for the two types. In the following subsections, we discuss the relation between the observables and model parameters. The numerical analyses are also shown in this section.

4.1 Mixing angles and CP violating phases

In this subsection, we discuss the lepton mixing angles, CP violating phases and the effective mass for \( 0\nu\beta\beta \) decay. At first, we introduce the PDG parametrization of the PMNS matrix:

\[
U_{\text{PDG}}^{\text{PMNS}} = \begin{pmatrix}
    c_{12}c_{13} & s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{CP}} & c_{12}s_{23} - c_{12}s_{23}s_{13}e^{i\delta_{CP}} \\
    -s_{12}c_{13} & s_{12}c_{23} + s_{12}s_{23}s_{13}e^{i\delta_{CP}} & -s_{12}s_{23} - s_{12}s_{23}s_{13}e^{i\delta_{CP}} \\
    s_{13} & s_{23}c_{13} & c_{23}
\end{pmatrix}
\]

(68)

(50). For the type I case, the matrix elements are given as follows:

\[
U_{e1} = \frac{2}{\sqrt{6}} e^{i\phi_1} \cos \theta_1,
\]

\[
U_{e2} = U_{\mu2} = \frac{1}{\sqrt{3}} e^{i\phi_2},
\]

\[
U_{e3} = \frac{2}{\sqrt{6}} e^{-i(\sigma - \phi_3)} \sin \theta_1,
\]

\[
U_{\mu1} = \left(-\frac{1}{\sqrt{6}} \cos \theta + \frac{1}{\sqrt{2}} e^{i\sigma} \sin \theta \right) e^{i\phi_1},
\]

\[
U_{\mu3} = \left(-\frac{1}{\sqrt{6}} e^{-i\sigma} \sin \theta - \frac{1}{\sqrt{2}} \cos \theta \right) e^{i\phi_3}.
\]

(72)

(73)

(74)

(75)

(76)

The mixing angles, Dirac CP violating phase and Majorana phases for both types are listed in Table 3.

| \( \sin^2 \theta_{12} \) | \( \frac{1}{2 + \cos 2\theta} \) | \( \frac{1}{2 - \cos 2\theta} \) |
| \( \sin^2 \theta_{23} \) | \( \frac{1}{2} \left(1 + \frac{\sqrt{3} \sin 2\theta}{\cos \sigma} \right) \) | \( \frac{1}{2} \left(1 - \frac{\sqrt{3} \sin 2\theta}{\cos \sigma} \right) \) |
| \( \sin^2 \theta_{13} \) | \( \frac{1}{3} (1 - \cos 2\theta) \) | \( \frac{1}{3} (1 + \cos 2\theta) \) |
| \( \sin \delta_{CP} \) | \( \frac{\sin 2\theta}{|\sin 2\theta|} \left(\frac{2 + \cos 2\theta \sin \sigma}{\sin 2\theta \sin \sigma} \right) \) | \( \frac{\sin 2\theta}{|\sin 2\theta|} \left(\frac{2 - \cos 2\theta \sin \sigma}{\sin 2\theta \sin \sigma} \right) \) |
| \( \alpha + \delta_{CP} \) | \( \phi_1 - \phi_1 + \sigma \) | \( \phi_1 - \phi_1 + \sigma \) |
| \( \beta + \delta_{CP} \) | \( \phi_2 - \phi_1 + \sigma \) | \( \phi_2 - \phi_1 + \frac{\pi}{2} \) |

Table 3. Mixing angles, Dirac CP phase and Majorana phases for the two types of vacua.

One can adopt either of the two types to predict the mixing angles and the Dirac CP violating phases, since both types give the same predictions. However, we note the following two facts. First, if one fixes \( \cos 2\theta \approx 1 \) to obtain small \( \sin^2 \theta_{13} \) in type I, \( \sin^2 \theta_{13} \) in type II reaches 2/3, which is dis favored in the experiments. Second, as shown in Subsection 3.3, the model parameters \( \theta, \sigma \) and \( \phi_i \) are expressed in the same forms for the two types with \( a, b, \)
of the model parameters $\theta$ and $\sigma$ in different forms for the two types.

The experimental data for $\sin^2 \theta_{13}$ in Table 4 is realized by the following value of $\theta$ with NH or IH:

Type I: 9.81° $\leq |\theta| \leq 10.9°$ (NH), 9.86° $\leq |\theta| \leq 11.0°$ (IH),
(84)

Type II: 79.1° $\leq |\theta| \leq 80.2°$ (NH), 79.0° $\leq |\theta| \leq 80.1°$ (IH),
(85)

The value of $\sigma$ is allowed in $-180° \leq \sigma \leq 180°$ for both of the two types, since the error of $\sin^2 \theta_{23}$ from the experiments is large.

Next, we discuss the parameters $\phi_i$ in the expressions of the Majorana phases of Eqs. (82) and (83). The effective mass $|m_{ee}|$ in Eq. (81) depends on the two combinations of Dirac and Majorana phases, $2(\alpha+\delta_{CP})$ and $2(\beta+\delta_{CP})$. If we determine both $|m_{ee}|$ and the lightest neutrino mass, we obtain the constraints on these two combinations. In order to find how the numerical constraints on $\phi_i$ are different in the two types, we consider a specific situation. As an example, we assume that $|m_{ee}|$ is predicted in the region as shown in Fig. 3. We note that the lightest neutrino mass is constrained from the cosmological upper bound for the neutrino mass sum, $\sum_\nu m_\nu < 0.16$ eV [23]. This plot is obtained when the Dirac and Majorana phases are randomly chosen from the region A1 in Fig. 2,

$$0<\alpha+\delta_{CP}<\pi/4 , \ 0<\beta+\delta_{CP}<\pi/4 \quad (86)$$

In this situation, the phase differences $\phi_1-\phi_3$ and $\phi_2-\phi_3$ for one type can be distinguished from those for the other type. The constraints on the phase differences are shown in Fig. 4. For type I, the phase difference $\phi_2-\phi_3$ is proportional to $\phi_1-\phi_3$. However, for type II, $\phi_2-\phi_3$ is independent of the value of $\phi_1-\phi_3$ because $\sigma$ is absent in the expression of $\phi_2-\phi_3$ in Eq. (83).

$$2(\beta+\delta_{CP})$$

$$2(\alpha+\delta_{CP})$$

As we have shown in the previous subsection, the mixing angles and Dirac $CP$ phase are expressed in terms

| $\Delta m_{31}^2/eV^2$ | Normal hierarchy (NH) | Inverted hierarchy (IH) |
|--------------------------|----------------------|----------------------|
| $\Delta m_{31}^2/eV^2$   | $(7.03\sim 8.09)\times 10^{-3}$ | $(7.03\sim 8.09)\times 10^{-5}$ |
| $\Delta m_{21}^2/eV^2$   | $(2.467\sim 2.643)\times 10^{-3}$ | $-(2.565\sim 2.318)\times 10^{-3}$ |
| $\sin^2\theta_{12}$     | 0.271~0.345          | 0.271~0.345          |
| $\sin^2\theta_{13}$     | 0.385~0.635          | 0.393~0.640          |
| $\sin^2\theta_{13}$     | 0.01934~0.02392      | 0.01953~0.02408      |

Table 4. The experimental data for the mass squared differences and mixing angles with $3\sigma$ range [22].
5 Summary

We have studied phenomenological aspects of a supersymmetric model with $A_4 \times Z_3$ symmetry. We found 24 degenerate vacua at the 24 minima of the scalar potential. Then, we discussed the relations among the 24 different vacua and classified them into two types. Both types consist of 12 vacua which are related to each other by transformations of $A_4$. We proved that the 12 vacua are equivalent and lead to the same physical consequences. However, we found that we obtain different physical consequences from the vacua of different types. Therefore, we analyzed the two types of vacua to find the different phenomenological consequences of the two types. In particular, we investigated observables such as mixing angles, Dirac $CP$ phase, Majorana phases and effective mass for $0\nu\beta\beta$ decay.

These observables are expressed in terms of the model parameters $\theta$, $\sigma$ and $\phi_i$. The angle $\theta$ and phase $\sigma$ are determined by the deviation from the tri-bimaximal mixing matrix. The two types lead to different expressions for the mixing angles and Dirac $CP$ violating phase in terms of $\theta$ and $\sigma$. Therefore, one should take different model parameters in each type in order to realize the experimental results. Although one can adopt both of the two types to predict the observable parameters, the two types cannot realize the current experimental data simultaneously. The Majorana phases $\alpha$ and $\beta$ are parametrized in the different expressions for each type by the model parameters $\phi_i$ in addition to $\theta$ and $\sigma$. In order to find numerical differences between the two types of Majorana phase, we considered the specific situation where the lightest mass and effective mass for the $0\nu\beta\beta$ decay are determined in a certain region. We showed the allowed regions of the phase differences, $\phi_1-\phi_3$ and $\phi_2-\phi_3$. The regions are quite different for the two types: the phase differences for type I are proportional to each other, while those for type II are not.

The VEVs $\eta_m$ and $\lambda_n^\pm$ transfer to the different VEVs by transformations of $A_4$. However, the transformations for $\eta_m$ and $\lambda_n^\pm$ are closed differently since they have the $Z_3$ and $Z_2$ residual symmetries from $A_4$ respectively. We have pointed out that some combinations of the VEVs can lead to different physical consequences. When we consider models with two or more flavons, we should take account of the combination of VEVs.
Appendix A

Multiplication rule of $A_4$ group

In this appendix, we show the multiplication of the $A_4$ group. The multiplication rule of the triplets is written as follows:

$$
\begin{pmatrix}
a_1 \\
a_2 \\
a_3
\end{pmatrix}
\otimes
\begin{pmatrix}
b_1 \\
b_2 \\
b_3
\end{pmatrix}
= (a_1 b_1 + a_2 b_3 + a_3 b_2) \begin{pmatrix}1 \oplus (a_3 b_1 + a_1 b_2 + a_2 b_1) \oplus (a_2 b_1 + a_3 b_2 + a_1 b_3) \oplus \frac{1}{3} (2 a_1 b_1 - a_2 b_3 - a_3 b_2) \\
2 a_3 b_1 - a_1 b_2 - a_2 b_1 \\
2 a_2 b_1 - a_3 b_2 - a_1 b_3
\end{pmatrix}
\begin{pmatrix}a_1, a_2, a_3
\end{pmatrix},
$$

while that for singlets is,

$$1' \otimes 1'' = 1.'$$

In order to derive the $A_4$ invariant superpotential in Eq. (1), we have used the multiplication rules. Their derivation is shown in the reviews in Refs. [1–4].

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