Proof of Moll’s Minimum Conjecture

William Y. C. Chen and Ernest X. W. Xia

Center for Combinatorics, LPMC-TJKLC
Nankai University, Tianjin 300071, P. R. China
emails: 1chen@nankai.edu.cn, 2xxw@cfc.nankai.edu.cn

Abstract. Let $d_i(m)$ denote the coefficients of the Boros-Moll polynomials. Moll’s minimum conjecture states that the sequence $\{i(i+1)(d_{i+1}^2(m) - d_{i-1}(m)d_{i+1}(m))\}_{1 \leq i \leq m}$ attains its minimum at $i = m$ with $2^{-2m}m(m+1)(\binom{2m}{m})^2$. This conjecture is stronger than the log-concavity conjecture proved by Kauers and Paule. We give a proof of Moll’s conjecture by utilizing the spiral property of the sequence $\{d_i(m)\}_{0 \leq i \leq m}$, and the log-concavity of the sequence $\{i!d_i(m)\}_{0 \leq i \leq m}$.

Keywords: ratio monotonicity, log-concavity, Boros-Moll polynomials.

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1 Introduction

The objective of this note is to give a proof of Moll’s conjecture on the minimum value of a sequence involving the coefficients of the Boros-Moll polynomials which arise in the evaluation of the following quartic integral, see, [1–6, 11]. It has been shown that for any $a > -1$ and any nonnegative integer $m$,

$$\int_0^\infty \frac{1}{(x^4 + 2ax^2 + 1)^{m+1}}dx = \frac{\pi}{2^{m+3/2}(a + 1)^{m+1/2}}P_m(a),$$

where

$$P_m(a) = 2^{-2m} \sum_k 2^k \binom{2m-2k}{m-k} \binom{m+k}{k}(a+1)^k.$$ (1.1)
Write $P_m(a)$ as
\[ P_m(a) = \sum_{i=0}^{m} d_i(m) a^i. \]  
(1.2)

The polynomials $P_m(a)$ are called the Boros-Moll polynomials. By (1.2), $d_{i}(m)$ can be expressed as
\[ d_{i}(m) = 2^{-2m} \sum_{k=i}^{m} 2^k \binom{2m-2k}{m-k} \binom{m+k}{k} \binom{k}{i}. \]  
(1.3)

From the above formula (1.3) one sees that the coefficients $d_{i}(m)$ are positive. Boros and Moll [3,4] have proved that for $m \geq 2$ the sequence $\{d_{i}(m)\}_{0 \leq i \leq m}$ is unimodal and the maximum entry appears in the middle, that is,
\[ d_0(m) < d_1(m) < \cdots < d_{\left[\frac{m}{2}\right]}(m) < d_{\left[\frac{m}{2}\right]+1}(m) > \cdots > d_m(m). \]

Moll [11] conjectured that the sequence $\{d_{i}(m)\}_{0 \leq i \leq m}$ is log-concave for $m \geq 2$. Kauers and Paule [9] have proved this conjecture by using a computer algebra approach. Chen and Xia [8] have shown that the sequence $\{d_{i}(m)\}_{0 \leq i \leq m}$ satisfies the strongly ratio monotone property which implies the log-concavity and the spiral property. Chen and Gu [7] have proved that the sequence $\{i!d_{i}(m)\}_{0 \leq i \leq m}$ is log-concave.

In fact, Moll [10,12] proposed a stronger conjecture than the log-concavity conjecture. He formulated his conjecture in terms of the numbers $b_{i}(m)$ as defined by
\[ b_{i}(m) = \sum_{k=i}^{m} 2^k \binom{2m-2k}{m-k} \binom{m+k}{k} \binom{k}{i}. \]  
(1.4)

Clearly, $b_{i}(m) = 2^{2m}d_{i}(m)$ and the log-concavity of $d_{i}(m)$ is equivalent to that of $b_{i}(m)$.

**Conjecture 1.1.** Given $m \geq 2$, for $1 \leq i \leq m$,
\[ (m+i)(m+1-i)b_{i-1}^2(m) + i(i+1)b_{i}^2(m) - i(2m+1)b_{i-1}(m)b_{i}(m), \]
attains its minimum at $i = m$ with $2^{2m}m(m+1)\binom{2m}{m}^2$.

We will give a proof of the above conjecture by using the spiral property of $\{d_{i}(m)\}_{0 \leq i \leq m}$ and the log-concavity of $\{i!d_{i}(m)\}_{0 \leq i \leq m}$.
2 Proof of Moll’s Minimum Conjecture

As pointed out by Moll [12], his conjecture implies that \( \{d_i(m)\}_{0 \leq i \leq m} \) is log-concave for \( m \geq 2 \). To see this, we may employ a recurrence relation to reformulate his conjecture by using the three terms \( d_{i-1}(m), d_i(m) \) and \( d_{i+1}(m) \). Recall that Kauers and Paule [9] and Moll [12] have independently derived the following recurrence relation for \( 1 \leq i \leq m \),

\[
i(i-1)d_i(m) = (i-1)(2m+1)d_{i-1}(m) - (m+2-i)(m+i-1)d_{i-2}(m).
\]  

(2.1)

Note that we have adopted the convention that \( d_i(m) = 0 \) for \( i < 0 \) or \( i > m \). From (2.1) and the relation \( d_i(m) = 2^{-2m}b_i(m) \), it follows that

\[
(m + i)(m + 1 - i)b_{i-1}(m) + i(i + 1)b_i^2(m) - i(2m + 1)b_{i-1}(m)b_i(m) = i(i + 1)\left(b_i^2(m) - b_{i+1}(m)b_{i-1}(m)\right).
\]

Thus, Moll’s conjecture can be restated as follows.

**Theorem 2.1.** Given \( m \geq 2 \), for \( 1 \leq i \leq m \), \( i(i+1)(d_i^2(m) - d_{i+1}(m)d_{i-1}(m)) \) attains its minimum at \( i = m \) with \( 2^{-2m}m(m+1)(\binom{2m}{m})^2 \).

Chen and Xia [8] have shown that the Boros-Moll polynomials satisfy the ratio monotone property which implies the log-concavity and the spiral property.

**Theorem 2.2.** Let \( m \geq 2 \) be an integer. The sequence \( \{d_i(m)\}_{0 \leq i \leq m} \) is strictly ratio monotone, that is,

\[
\frac{d_m(m)}{d_0(m)} < \frac{d_{m-1}(m)}{d_1(m)} < \cdots < \frac{d_{m-i}(m)}{d_i(m)} < \frac{d_{m-i-1}(m)}{d_{i+1}(m)} < \cdots < \frac{d_{m-[\frac{m-1}{2}]}(m)}{d_{m-[\frac{m+1}{2}]}(m)} < 1,
\]

\[
\frac{d_0(m)}{d_{m-1}(m)} < \frac{d_1(m)}{d_{m-2}(m)} < \cdots < \frac{d_{i-1}(m)}{d_{m-i}(m)} < \frac{d_i(m)}{d_{m-i-1}(m)} < \cdots < \frac{d_{m-[\frac{m}{2}]-1}(m)}{d_{m-[\frac{m}{2}]}(m)} < 1.
\]

As a consequence of Theorem 2.2, the spiral property of \( \{d_i(m)\}_{0 \leq i \leq m} \) can be stated as follows.

**Corollary 2.3.** (Chen and Xia [8]) For \( m \geq 2 \), the sequence \( \{d_i(m)\}_{0 \leq i \leq m} \) is spiral, that is,

\[
d_m(m) < d_0(m) < d_{m-1}(m) < d_1(m) < d_{m-2}(m) < \cdots < d_{m-[\frac{m}{2}]}(m).
\]  

(2.2)
Chen and Gu [7] have shown that \( \{ild_i(m)\}_{0 \leq i \leq m} \) is log-concave. This property can be recast in the following form.

**Theorem 2.4.** For \( m \geq 2 \) and \( 1 \leq i \leq m - 1 \),

\[
id_i^2(m) > (i + 1)d_{i+1}(m)d_{i-1}(m). \tag{2.3}
\]

We are now ready to present a proof of Theorem 2.1.

**Proof.** First, it follows from (1.3) that

\[
m(m + 1)d_m^2(m) = 2^{-2m}m(m + 1)\left(\frac{2m}{m}\right)^2. \tag{2.4}
\]

We now proceed to show that for \( 1 \leq i \leq m - 1 \),

\[
i(i + 1) \left( d_i^2(m) - d_{i+1}(m)d_{i-1}(m) \right) > m(m + 1)d_m^2(m). \tag{2.5}
\]

We first consider the case \( 1 \leq i \leq m - 2 \). By (2.3), we find that

\[
i(i + 1) \left( d_i^2(m) - d_{i+1}(m)d_{i-1}(m) \right) > i(i + 1)d_i^2(m) - i^2d_i^2(m) = id_i^2(m). \tag{2.6}
\]

Using the spiral property (2.2), we see that for \( 1 \leq i \leq m - 2 \),

\[
id_i^2(m) \geq d_1^2(m) > d_{m-1}^2(m). \tag{2.7}
\]

Combining (2.6) and (2.7), we get

\[
i(i + 1) \left( d_i^2(m) - d_{i+1}(m)d_{i-1}(m) \right) > d_i^2(m) > d_{m-1}^2(m). \tag{2.8}
\]

On the other hand, by direct computation we may deduce from (1.3) that

\[
d_{m-1}(m) = \frac{2m + 1}{2}d_m(m). \tag{2.9}
\]

By (2.8) and (2.9), we have for \( 1 \leq i \leq m - 2 \),

\[
i(i + 1) \left( d_i^2(m) - d_{i+1}(m)d_{i-1}(m) \right)
> \left(\frac{2m + 1}{2}\right)^2 d_m^2(m) > m(m + 1)d_m^2(m), \tag{2.10}
\]

and hence (2.5) is true for \( 1 \leq i \leq m - 2 \). It remains to consider the case \( i = m - 1 \). Again, by (1.3) we find that

\[
d_{m-1}(m) = 2^{-m-1}(2m + 1)\left(\frac{2m}{m}\right), \tag{2.11}
\]

\[
d_{m-2}(m) = 2^{-m-1}(m - 1)(4m^2 + 2m + 1)\left(\frac{2m}{m}\right). \tag{2.12}
\]
From (2.4), (2.11) and (2.12), we deduce that
\[
m(m-1)\left(d_{m-1}^2(m) - d_m(m)d_{m-2}(m)\right) \\
= m(m-1)2^{-2m}\left(\frac{2m}{m}\right)^2\left(\frac{(2m+1)^2}{4} - \frac{(m-1)(4m^2 + 2m + 1)}{4(2m-1)}\right) \\
= \frac{m(4m^2 + 6m - 1)}{4(2m-1)}m(m-1)2^{-2m}\left(\frac{2m}{m}\right)^2 \\
> m(m+1)2^{-2m}\left(\frac{2m}{m}\right)^2 = m(m+1)d_m^2(m). \tag{2.13}
\]

Thus (2.5) holds for \(i = m - 1\), and so it holds for \(1 \leq i \leq m - 1\). This completes the proof.

We conclude with the following ratio monotonicity conjecture. If it is true, it would imply that the sequence \(\{i(i+1)(d_i^2(m) - d_{i+1}(m)d_{i-1}(m))\}_{1 \leq i \leq m}\) is both spiral and log-concave for \(m \geq 2\).

**Conjecture 2.5.** The sequence \(\{i(i+1)(d_i^2(m) - d_{i+1}(m)d_{i-1}(m))\}_{1 \leq i \leq m}\) is strongly ratio monotone.

For example, for \(m = 8\), we have
\[
P_8(a) = \frac{4023459}{32768} + \frac{3283533}{4096}a + \frac{9804465}{4096}a^2 + \frac{8625375}{2048}a^3 + \frac{9695565}{2048}a^4 \\
+ \frac{1772199}{512}a^5 + \frac{819819}{512}a^6 + \frac{109395}{256}a^7 + \frac{6435}{128}a^8.
\]

Let \(c_i = i(i+1)(d_i^2(8) - d_{i+1}(8)d_{i-1}(8))\) for \(1 \leq i \leq 8\). One can verify that
\[
\frac{c_8}{c_1} < \frac{c_7}{c_2} < \frac{c_6}{c_3} < \frac{c_5}{c_4} < 1 \quad \text{and} \quad \frac{c_1}{c_7} < \frac{c_2}{c_6} < \frac{c_3}{c_5} < 1.
\]

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