Wave Functions and Energies of Magnetopolarons in Semiconductor Quantum Wells

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The classification of magnetopolarons in semiconductor quantum wells (QW) is represented. Magnetopolarons appear due to the Johnson - Larsen effect. The wave functions of usual and combined magnetopolarons are obtained by the dianalization of the Schrödinger equation.

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I. INTRODUCTION

The Johnson - Larsen effect [1-3] arises at a condition

$$\omega_{LO} = j\omega_{e(h)}H,$$

where $$\omega_{LO}$$ is the frequency of a longitudinal optical (LO) phonon,

$$\omega_{e(h)}H = \frac{|e|H}{e_m c_{e(h)}}$$

is the cyclotron frequency, $$m_{e(h)}$$ is the electron (hole) effective mass, $$H$$ is the magnetic field, $$j$$ is some number

The Johnson - Larsen effect is called also as a magnetopolaron resonance, and the statets which are being formed under condition (1) in semiconductors - by magnetopolarons. At magnetic fields appropriate to the condition (1) the resonant connection between the Landau bands with different quantum numbers $$n$$ (see Fig. 1) arises. The electron-phonon interaction results into removing of a degeneration in crossing points of energy levels, what influences magneto-optical effects. For the first time the magnetopolaron states were discovered in a bulk InSb in the interband light absorption [1-3].

After the pioneer Johnson-Larsen works the magnetopolaron effect has attracted the attention of theoretical and experimental groups. Magnetopolaron features in transport and optical phenomena were intensively investigated. During last years the new wave of interest to the Johnson-Larsen effect was stimulated by appearance of low-dimensional semiconductor objects, in which the effect amplifies due to the size-quantization of electronic excitations.

The formation of polaron states takes place in three-dimensional (3D), and in quasi-two-dimensional (2D) systems. The distinction between these systems consists in energy spectra of electrons (holes) at presence of a quantizing magnetic field: in 3D systems there appear one-dimensional Landau bands, in 2D - discrete energy levels. This distinction results into different splitting of energy levels of an electron-phonon system.

In both 3D and 2D systems magnetopolaron states play an important role in formation of frequency dependences of magneto-optical effects, such as interband absorption of light, cyclotron resonance and Raman scattering of light (see reviews [4-7]).

In [8] Korovin and Pavlov have shown that in bulk semiconductors the magnetopolaron splitting is proportional to $$\alpha^{2/3}\hbar\omega_{LO}$$, where $$\alpha$$ is the Fröhlich dimensionless electron-phonon coupling constant [9] ($$\alpha < < 1$$).

In quasi-2D systems (in particular in semiconductor quantum wells), effect amplifies, and the distance between components of splitted peaks (for example, of interband light absorption) becomes proportional to $$\alpha^{1/2}\hbar\omega_{LO}$$ [10-17].

II. CLASSIFICATION OF MAGNETOPOLARONS.

In Fig. 1 the continuous lines represent terms of an electron-phonon system concerning to the size-quantization quantum numbers $$l$$. The model is used in which all phonons have the disperionless frequency $$\omega_{LO}$$. On the abscissa axis the relation $$\omega_{e(h)H}/\omega_{LO}$$ is represented, on the ordinate axis - the relation $$E/\omega_{LO}$$, where $$E$$ is the energy counted from the energy $$\epsilon_l^{e(h)}$$, appropriate to $$l$$-th energy level of the size-quantization (values $$\epsilon_l^{e(h)}$$ for QWs of a finite depth are given, for example, in [18]).

The polaron states correspond to crossing points of terms. By black circles "twofold" polarons are designated appropriate to crossing only of two terms. Let us consider a certain crossing point to which number $$j$$ (see (1)) corresponds. Let $$n$$ is the number of a Landau level crossing the given point at $$N = 0$$. Then for existence of a twofold polaron the condition to be carried

$$2j > n \geq j.$$  

(2)

It is easy to see that only one twofold polaron corresponds to $$j = 1$$ (designated by the letter A). To $$j = 2$$, i.e. $$\omega_H/\omega_{LO} = 1/2$$, two twofold polarons ($D$ and $E$) correspond, to $$j = 3$$, i.e. $$\omega_H/\omega_{LO} = 1/3$$, three twofold polarons ($F$, $K$ and $L$) correspond, etc.

In Fig. 1 polaron, located more to the left of $$\omega_H/\omega_{LO} = 1/3$$ are not shown. Let us notice that for
experiment the existence of polarons, appropriate to integers $j > 1$ is extremely important. Really, the resonant $H_{resj} = \omega_{LO}mc/j|e|$ decreases in $j$ times in comparison with $H_{res1}$ for the polaron $A$.

Threefold polarons, appropriate to crossing of three terms, are located above twofold polarons, higher - fourfold polarons, etc. In Fig. 1 threefold polarons are designated by black triangles, fourfold - by black squares. The number of polarons of each grade at given $j$ is equal to $j$. Threefold polarons in bulk crystals for the first time were considered in [19], in QWs - in [20-22].

Let us note that for crossing of three and more terms in one point it is necessary an equidistance of Landau levels. In order the theory [22] of threefold polarons would be applicable, it is necessary, that amendments to energies $\Delta \varepsilon_j$ of size-quantization and, hence, on QW's depth and width. Really, with the help of Fig. 1 it is easy to obtain

$$ (\omega_H/\omega_{LO})_P = X_P = 1 - \frac{\Delta \varepsilon}{\hbar \omega_{LO}}, $$

$$ (\omega_H/\omega_{LO})_Q = X_Q = 1 + \frac{\Delta \varepsilon}{\hbar \omega_{LO}}. $$

In Fig. 1 the case $\Delta \varepsilon < \hbar \omega_{LO}$ is represented. If $\Delta \varepsilon > \hbar \omega_{LO}$, there is only one combined polaron, to which there corresponds the second equality of (3). It is essential that the magnetopolaron $P$ in Fig. 1 corresponds to much smaller $H$, than the polaron $A$. It should facilitate its experimental observation.

One more kind combined polaron [2, 4] is not represented in Fig. 1, as it exists only under condition

$$ \Delta \varepsilon = \hbar \omega_{LO}, $$

when the terms $l', n, N = 0$ and $l, n, N = 1$ coincide at any magnetic fields. For performance of a resonant condition (4) a certain distance between levels $l$ and $l'$ is required, that is reached only by selection of QW width and depth. Magnetic field is required only for formation of Landau levels and can be chosen rather small. Under condition (4)”a special polaron state ” arises.

In order the picture represented in Fig. 1 would be applicable, it is necessary that the distances between the next levels $l, l - 1, l + 1$ were much greater, than $\Delta E$ of

FIG. 1: Energy levels of an electron (hole) - phonon system as functions of a magnetic field. Points of crossings of lines correspond to polaron states. Black circles are the twofold polarons, triangles - threefold polarons, squares - fourfold polarons. Empty circles are weak polarons, empty squares - polarons, triangles - threefold polarons, squares - fourfold polarons. The number of phonons; $l, l'$ are the size-quantization quantum numbers. 

$\Delta \varepsilon = \varepsilon_f - \varepsilon_g$ is the Landau quantum number, $N$ is the number of phonons; $l, l'$ are the size-quantization quantum numbers.
polaron splittings. As distance between levels decreases with the growth of QW width \( d \), restrictions from above on \( d \) are imposed (numerical estimations see in [26] (Fig. 2,3)).

III. THE HAMILTONIAN OF A SYSTEM.

The energy spectrum of twofold magnetopolarons - usual (classical), and combined - was determined by two ways giving identical results. The first way was used in [8] and it consists in definition of poles of one-particle Green function of an electron. It also is applied in [25, 26]. Other way is described in [18] and devoted to the polaron \( A \). The polaron wave functions are represented as a superposition of wave functions of the unperturbed states (in a case of the polaron \( A \) these are states \( n = 1,N = 0 \) and \( n = 0,N = 1 \)) with unknown factors. The Schrödinger equation is reduced to a system of two equations for two factors. Equating determinant to 0, we obtain square-law equation for polaron energies of states \( p = a \) and \( p = b \). Advantage in comparison with the first way is that simultaneously with the energy calculation we calculate the magnetopolaron wave functions. And these functions are necessary for the description of many magneto-optical effects.

In present work we generalize results of [18] for the polaron \( A \) on case of any twofold polarons, including usual, combined and "special polaron state". We pay the special attention to wave functions, before unknown. The theory is not extended on weak, threefold, fourfold etc. polarons.

Let us consider a semiconductor QW of type \( I \) with the energy gap \( E_g \) and barrier \( \Delta E_c \) for electrons. For definiteness we investigate magnetopolarons with participation of electrons. Results can be easily used for description of magnetopolarons with participation of holes.

The magnetic field is directed along an axis \( z \) perpendicularly to the QW plane. The vector potential is chosen as \( A = A(0,xH,0) \). The Schrödinger equation for electrons, interacting with LO phonons, looks like

\[
\mathcal{H}_\Theta = E\Theta, \mathcal{H} = \mathcal{H}_0 + V, \mathcal{H}_0 = \mathcal{H}_e + \mathcal{H}_{ph},
\]

and

\[
\mathcal{H}_e \Psi_{n,k_y,l} = [(n + 1/2)\hbar \omega_{eH} + \varepsilon_l] \Psi_{n,k_y,l},
\]

where

\[
\Psi_{n,k_y,l} = \Phi_n(x + a_H^2 k_y) \frac{1}{\sqrt{\pi/2^n n! a_H}} H_n(x/a_H) e^{-x^2/(2a_H^2)},
\]

\[
a_H = \sqrt{\hbar/|e|H} \text{ is the magnetic length, } H_n(t) \text{ is the Hermitian polynomial. The functions } \phi_l(z) \text{ and levels } \varepsilon_l \text{ of energy of size-quantization of electrons in a QW of finite depth are determined, for example, in [18], } \mathcal{H}_{ph} \text{ is the Hamiltonian of the phonon system, } V \text{ is the electron-phonon interaction. In a case of an indefinitely deep QW, when } \Delta E_c \to \infty
\]

\[
\phi_l(z) = \begin{cases} \sqrt{\pi} \sin(\frac{n \pi}{d} + \frac{m \pi}{d}), & |z| \leq \frac{d}{2}, \\ 0, & |z| \geq \frac{d}{2}. \end{cases}
\]

\[
\varepsilon_l(z) = \frac{m^2 \hbar^2 l^2}{2m_e d^2},
\]

\( m_e \) is the electron effective mass.

Let us designate as \( \Psi_{ph0}(y) \) and \( \Psi_{ph\nu}(y) \) the wave functions of the phonon system appropriate to absence of phonons and to presence of one phonon with indexes \( \nu \equiv (q_1, \mu) \), where \( q_1 \) is the phonon wave vector in a plane \( xy, \mu \) are other indexes [27], \( Y \) are the coordinates of the phonon subsystems. Let us assume

\[
\mathcal{H}_{ph} \Psi_{ph0} = 0, \mathcal{H}_{ph} \Psi_{ph\nu} = \hbar \omega_{\nu} \Psi_{ph\nu}.
\]

Functions with the large number of phonons will be unnecessary, as (it is visible in a Fig. 1) in formation of classical twofold polarons the appropriate states do not participate, the same as in formation of combined polarons \( P \) and \( Q \).

We use model, in which the dispersion of \( LO \) phonons is not taken into account, i.e. we believe

\[
\omega_{\nu} = \omega_{LO}.
\]

The influence of the phonon dispersion on a magnetopolaron spectrum is discussed in [23].

The electron-phonon interaction looks like

\[
V = \sum_\nu [C_\nu(r_1,z)b_\nu + C_\nu^+(r_1,z)b_\nu^+],
\]

where \( b_\nu^+(b_\nu) \) is the phonon creation (annihilation) operator,

\[
C_\nu(r_1,z) = Ce^{iq\cdot r_1} \xi_\nu(z),
\]

and \( \xi_\nu(z) \) is chosen so, that \( \xi_\nu(z = 0) = 1 \).

In a single QW instead of bulk \( LO \) phonons there are three types of phonons [27]. First, it is so-called phonons semi-space, not penetrating in a QW. Besides interface phonons are available, which damp at outside of QW. At last, confined phonons exist in a QW material. These fluctuations will not penetrate into a barrier, their amplitude equals 0 on QW borders. In a case of confined phonons [27] a set of indexes \( \nu \) includes \( q_1 \) and discrete indexes \( \mu \), and the interaction (11) is determined as

\[
\xi_\nu(z) = \xi_\mu(\mu, z) = \begin{cases} \cos(\frac{\pi \nu}{2}), \mu = 1,3,..., & |z| \leq \frac{d}{2}, \\ \sin(\frac{\pi \nu}{2}), \mu = 2,4,..., & |z| \leq \frac{d}{2}, \\ 0, & |z| \geq \frac{d}{2}, \end{cases}
\]

\[
C_\nu = C_{q_1,\mu} = -\hbar \omega_{LO} \sqrt{\frac{8\pi \alpha l}{\text{Sh}d[q_1^2 + (\mu \pi/d)^2]^2}}.
\]
where $\alpha$ is the Fröhlich constant [9], $l = \sqrt{\hbar/2m_0\omega_{LO}}$, $S_0$ is the normalization area.

In many theoretical calculations of magnetopolaron spectra in a QW the Fröhlich interaction with $LO$ phonons [9] is used. Thus, $j = q_2\xi_0(z) = e^{i\varphi}q_2$,

$$C_{\nu} = C_{q_2} = -\hbar\omega_{LO}\sqrt{\frac{4\pi\alpha^3}{V_0}q_2^3}.$$  

$$\alpha = \frac{e^2}{2\hbar\omega_{LO}l}(\varepsilon_\infty - \varepsilon_0^{-1}),$$  

(13)

$V_0$ is the normalization volume, $\varepsilon_\infty(\varepsilon_0)$ is the high-frequency (static) dielectric permeability of QW.

In [26] it is investigated, when use of (13) is lawful for the description of magnetopolaron spectra in a QW. It is shown that at a large size of QW, it is possible to neglect by interaction of electrons with interface phonons, and the interaction with confined phonons (12) results in the same results as (13).

We do not concretize the form of the interaction (11) below.

IV. WAVE FUNCTIONS AND ENERGIES OF MAGNETOPOLARONS.

Let us consider a polaron, arising due to crossing of terms $n_0,t_0,N = 1$ and $n_1,t_1,N = 0$, where $n$ is the Landau quantum number, $l$ is the quantum number of the size-quantization, $N$ is the number of phonons. We search for the wave function as a superposition

$$\Theta(x,y,z,Y) = \sum_{k_y} a_0(k_y) \Psi_{n_1,k_y,l_1}(x,y,z)\psi_{ph0}(Y)$$

$$+ \sum_{k_y,\nu} a_1(k_y) \Psi_{n_0,k_y,\nu}(x,y,z)\psi_{ph\nu}(Y).$$  

(14)

The indexes 0 and 1 at factors $a_0(k_y)$ and $a_1(k_y)$ designate the number of phonons. For convenience of the further calculations we introduce designations

$$\Psi_{n_1,k_y,l_1}(x,y,z) = \Psi_{1,k_y}(x,y,z),$$

$$\Psi_{n_0,k_y,\nu}(x,y,z) = \Psi_{0,k_y}(x,y,z)$$  

(15)

and also

$$\Sigma_1 = (n_1 + 1/2)\hbar\omega_e + e_{l_1},$$

$$\Sigma_0 = (n_0 + 1/2)\hbar\omega_e + e_{l_0}.$$  

(16)

Then the Schrödinger equation may be written down as

$$(E - \Sigma_1)\psi_{ph0} \sum_{k_y} a_0(k_y) \Psi_{1,k_y}$$

$$+ (E - \Sigma_0 - \hbar\omega_{LO}) \sum_{k_y} a_1(k_y,\nu) \psi_{ph\nu}$$

$$- \psi_{ph0} \sum_{k_y} \Psi_{0,k_y} \sum_{\nu} C_\nu(r,\perp,z) a_1(k_y,\nu)$$

$$- \sum_{k_y} a_0(k_y) \Psi_{1,k_y} \sum_{\nu} C_\nu^*(r,\perp,z) \psi_{ph\nu} = 0.$$  

(17)

In (17) we used an approximation

$$V\psi_{ph\nu}(Y) \simeq C_\nu(r,\perp,z)\psi_{ph0}(Y),$$

because we consider the interaction only between states with $N = 0$ and $N = 1$. All other possible transitions result in the amendments of higher order on $\alpha$.

Let us multiply (17) on $\psi_{ph\nu}^*(Y)$ and $\psi_{ph0}^*(Y)$ and integrate on $Y$. Using the properties of the orthogonality and normalization of phonon functions, we obtain two equations

$$(E - \Sigma_1) \sum_{k_y} a_0(k_y) \Psi_{1,k_y}$$

$$- \sum_{k_y,\nu} \Psi_{0,k_y} \sum_{\nu} C_\nu(r,\perp,z) a_1(k_y,\nu) = 0,$$

$$(E - \Sigma_0 - \hbar\omega_{LO}) \sum_{k_y} a_1(k_y,\nu) \Psi_{0,k_y}$$

$$- \sum_{k_y} \Psi_{1,k_y} C_\nu^*(r,\perp,z) a_0(k_y) = 0.$$  

(18)

First of the equations (18) we multiply on $\Psi_{1,k_y}^*(x,y,z)$, second - on $\Psi_{0,k_y}^*(x,y,z)$ and we integrate on $x,y,z$. We obtain

$$(E - \Sigma_1) \sum_{k_y} a_0(k_y) \delta_{k_y,k_y'}$$

$$- \sum_{k_y,\nu} a_1(k_y,\nu) M^*(k_y,k_y',\nu) = 0,$$

$$(E - \Sigma_0 - \hbar\omega_{LO}) \sum_{k_y} a_1(k_y,\nu) \delta_{k_y,k_y'}$$

$$- \sum_{k_y} a_0(k_y) M(k_y,k_y',\nu) = 0,$$  

(19)

where the designation for a matrix element is introduced

$$M(k_y,k_y',\nu) = \int dx dy dz \Psi_{0,k_y}^*(x,y,z)$$

$$\times C_\nu(r,\perp,z) \Psi_{1,k_y}(x,y,z).$$  

(20)

Using designations (11) and (15) we obtain

$$M(k_y,k_y',\nu) = \delta_{k_y,k_y'} e^{ia_0^*k_y q_y} (k_y + q_y/2),$$  

(21)

where

$$U^*(\nu) = C_\nu^* \mathcal{K}_{n_1,n_0} (a_{Hq_y} - a_{Hq_x}) M^*(\nu),$$  

(22)

$$\mathcal{K}_{nm}(p_x,p_y) = \mathcal{K}_{nm}(p_x,p_y) = \left[\frac{\text{min}(n_l,m_l)}{\text{max}(n_l,m_l)}\right]^{1/2}$$

$$\times |n-m| \left(\frac{p}{\sqrt{2}}\right)^{|n-m|} e^{-p^2/4} e^{i(\phi - \pi/2)(n-m)}$$

$$\times L_{\text{min}(n,m)}(p^2/2),$$  

(23)
\[ p = \sqrt{p_x^2 + p_y^2}, \phi = \arctan(p_y/p_x), I_{\alpha}^n(t) \] is the Laguerre polynomial,

\[ \mathcal{M} (\nu) = \int dz \varphi_{10}(z) \varphi_{11}(z) \xi_{\nu}(z). \] (24)

In (22) we used the integral

\[ \mathcal{K}_{nm}(x,y) = e^{ixy/2} \int dt f_m(t) f_n(t + x)e^{ity}, \] (25)

where

\[ f_n(t) = \frac{1}{\sqrt{2 2^n n!}} e^{-t^2/2} H_n(t). \]

Having substituted (21) in (18) and having executed summation on \( k_y \), we obtain

\[
(E - \Sigma_1) a_0(k_y) - \sum_{\nu} a_1(k_y - q_y, \nu) \times e^{-ia_y^2 q_x(k_y - q_y)/2} U(\nu) = 0,
\]

\[
(E - \Sigma_2 - \hbar \omega_{LO}) a_1(k_y, \nu) - a_0(k_y + q_y) \times e^{-ia_y^2 q_x(k_y + q_y)/2} U^*(\nu) = 0. \] (26)

We have from the second equation

\[
a_1(k_y, \nu) = a_0(k_y + q_y) e^{ia_y^2 q_x(k_y + q_y)/2} \times \frac{U^*(\nu)}{E - \Sigma_2 - \hbar \omega_{LO}}. \] (27)

Having substituted (27) in the first equation, we obtain the square-law equation for energy \( E \)

\[
(E - \Sigma_1)(E - \Sigma_0 - \hbar \omega_{LO}) \times \sum_{\nu} |U(\nu)|^2 = 0. \] (28)

Let us introduce a designation

\[ w(n_o, n_1, l_0, l_1) = \sum_{\nu} |U(\nu)|^2. \] (29)

Then it follows from (22)

\[
w(n_o, n_1, l_0, l_1) = \sum_{\nu} |C_\nu|^2 B_{n_o n_1}(a_H^2 q_i^2/2)|\mathcal{M}_{l_0 l_1}(\nu)|^2, \] (30)

where

\[
B_{n_o n_1}(u) = \frac{\min(n_o!, n_1!)}{\max(n_o!, n_1!)} |n_o - n_1|! \times e^{-u} |f_{\min(n_o, n_1)}(u)|^2. \] (31)

The equation (28) has two solutions

\[
E_{a,b} = \frac{1}{2}(\Sigma_0 + \Sigma_1 + \hbar \omega_{LO}) \pm \sqrt{(\Sigma_1 - \Sigma_0 - \hbar \omega_{LO})^2 + 4w(n_o, n_1, l_0, l_1)}, \] (32)

where the indexes \( a \) and \( b \) correspond to + and −. The energy distance between two magnetopolaron states is equal

\[ \Delta E = \sqrt{\lambda^2 + 4w(n_o, n_1, l_0, l_1)}, \] (33)

where

\[ \lambda = (n_1 - n_0) \hbar \omega_{eH} - \hbar \omega_{LO} + \varepsilon_{l_1} - \varepsilon_{l_2} \]

describes a deviation from an exact resonance. An energy spectrum of anyone twofold polaron (classical or combined) is schematically represented in Fig. 2.

Using (14), (27), (32) and designation (15), we obtain the magnetopolaron wave functions for states \( p = a \) and \( p = b \)

\[
\Theta_p(x, y, z, Y) = \sum_{k_y} a_{0p}(k_y) \left[ \Psi_{1,k_y}(x, y, z) \psi_{ph_0}(Y) + (E_p - \Sigma_0 - \hbar \omega_{LO})^{-1} \sum_{\nu} \exp[i a_H^2 q_x(k_y - q_y/2)] \times U_0^* \Psi_{0,k_y-q_y}(x, y, z) \psi_{ph_\nu}(Y) \right]. \] (34)

The direct calculation shows that functions with indexes \( p = a \) and \( p = b \) are orhtogonal, i.e.

\[ \int dY d^3r \Theta_p^* \Theta_a = 0, \] (35)

and from the normalization condition

\[ \int dY d^3r \Theta_p^* \Theta_p = 1 \] (36)
we obtain the requirement
\[ \sum_{k_y} |a_{op}(k_y)|^2 = \left[ 1 + \frac{w(n_0, n_1, l_0, l_1)}{(E_p - \Sigma_0 - \hbar \omega_{LO})^2} \right]^{-1}. \quad (37) \]

Let us choose factors \( a_{op}(k_y) \) as
\[ a_{op}(k_y) = \delta_{k_y, k'_y} \left[ 1 + \frac{w(n_0, n_1, l_0, l_1)}{(E_p - \Sigma_0 - \hbar \omega_{LO})^2} \right]^{-1/2}. \quad (38) \]

Then the polaron wave functions are characterized by indexes \( p \) and \( k_y \) in designations (15) and finally
\[ \Theta_{p, k_y}(x, y, z, Y) = \left[ 1 + \frac{w(n_0, n_1, l_0, l_1)}{(E_p - \Sigma_0 - \hbar \omega_{LO})^2} \right]^{-1/2} \times \left[ \Psi_{1, k_y}(x, y, z) \psi_{ph0}(Y) \right. \times \sum_{\nu} \exp[i a_H q_x (y - q_y/2)] U^* (\nu) \times \left. \Psi_{0, k_y - q_y}(x, y, z) \psi_{ph \nu}(Y) \right]. \quad (39) \]

The conditions of orthogonality and normalization are carried out
\[ \int dY d^3 \Theta^*_{p', k_y} \Theta_{p, k_y} = \delta_{p, p'} \delta_{k_y, k'_y}. \quad (40) \]

With the help of wave functions (39) we determine the probability to find a system in a state with zero phonons and with one \( LO \) phonon with an index \( \nu \). We have
\[ Q_{op} = \left[ 1 + \frac{w(n_0, n_1, l_0, l_1)}{(E_p - \Sigma_0 - \hbar \omega_{LO})^2} \right]^{-1} = \frac{1}{2} \left[ 1 \pm \frac{\lambda}{\sqrt{\lambda^2 + 4w^2}} \right], \quad (41) \]
\[ Q_{\nu p} = \left[ \frac{|U(\nu)|^2}{(E_p - \Sigma_0 - \hbar \omega_{LO})^2} \right] \times \left[ 1 + \frac{w(n_0, n_1, l_0, l_1)}{(E_p - \Sigma_0 - \hbar \omega_{LO})^2} \right]^{-1}. \quad (42) \]

Summarizing \( Q_{\nu p} \) on \( \nu \) we obtain the total probability to find our system in a state with one phonon
\[ Q_{1p} = \sum_{\nu} Q_{\nu p} = \frac{1}{2} \left[ 1 \pm \frac{\lambda}{\sqrt{\lambda^2 + 4w^2}} \right] = 1 - Q_{op}. \quad (43) \]

In all formulas the top sign corresponds to \( p = a \), and bottom - to \( p = b \). In an exact resonance, when \( \lambda = 0 \) or
\[ \hbar \omega_{eH} n_1 + \varepsilon_{l_1} = \hbar \omega_{eH} n_0 + \varepsilon_{l_1} + \hbar \omega_{LO}, \quad (44) \]
which is reached at the resonant \( H_{res} \) magnetic field, the energies of polaron states are equal
\[ E_{a, b}^{res} = \Sigma_1 \pm \sqrt{w(n_0, n_1, l_0, l_1)}, \quad (45) \]
and the polaron splitting is as follows
\[ \Delta E^{res} = 2 \sqrt{w(n_0, n_1, l_0, l_1)}. \quad (46) \]

Numerical calculations of \( \Delta E^{res} \) for some polarons are given in [25, 26].

In the exact resonance probabilities of states without phonons and with one phonon are equal, i. e.
\[ Q_{op} = Q_{1p} = 1/2. \quad (47) \]

Let us consider a situation far away from the resonance, when
\[ |\Sigma_1 - \Sigma_0 - \hbar \omega_{LO}| >> \Delta E^{res}. \quad (48) \]
The results are different for cases \( \Sigma_1 - \Sigma_0 - \hbar \omega_{LO} < 0 \) and \( \Sigma_1 - \Sigma_0 - \hbar \omega_{LO} > 0 \). In Fig. 2 the left part from a point \( \Sigma_1 - \Sigma_0 - \hbar \omega_{LO} = 0 \) corresponds to the first case, right - to the second case. Introducing the indexes \( left \) and \( right \) we obtain
\[ E_{a, b}^{left} = E_{a, b}^{right} = \Sigma_0 + \frac{w}{\Sigma_1 - \Sigma_0 + \hbar \omega_{LO}}. \quad (49) \]
\[ E_{a, b}^{left} = E_{a, b}^{right} = \Sigma_0 + \frac{w}{\Sigma_1 - \Sigma_0 + \hbar \omega_{LO}}. \quad (50) \]
\[ \Theta_{a, k_y, left} = \Theta_{b, k_y, right} = \Psi_{1, k_y} \psi_{ph0} + \frac{1}{\Sigma_1 - \Sigma_0 + \hbar \omega_{LO}} \sum_{\nu} \exp[i a_H q_x (y - q_y/2)] \times U^*(\nu) \Psi_{0, k_y - q_y} \psi_{ph \nu}, \quad (51) \]
\[ \Theta_{a, k_y, left} = \Theta_{b, k_y, right} = \Psi_{1, k_y} \psi_{ph0} + \frac{1}{\sqrt{w}} \sum_{\nu} \exp[i a_H q_x (y - q_y/2)] U^*(\nu) \times \Psi_{0, k_y - q_y} \psi_{ph \nu}. \quad (52) \]
\[ \Theta_{a, k_y, left} = \Theta_{b, k_y, right} = \Psi_{1, k_y} \psi_{ph0}, \quad (53) \]

The results (49) - (53) are in accordance with formulas of the perturbation theory (see, for example, [28, page 165]) if to take into account only two states of the system with indexes \( n_0, l_0, N = 1 \) and \( n_1, l_1, N = 0 \). Amendments to the energy are proportional \( \alpha \), amendments to the wave functions are proportional to \( \alpha^{1/2} \). Far away from the resonance in a point \( \Sigma_1 = \Sigma_0 + \hbar \omega_{LO} \) we have to take into account possible transitions in other states of our system.

Thus, the energy spectra and wave functions of usual (classical) and combined magneto-polarons in semiconductor QWs are calculated. These functions are necessary for theoretical consideration of the optical phenomena, in which the Johnson - Larsen effect is essential.
