SLOW-FAST DYNAMICS AND NONLINEAR OSCILLATIONS IN TRANSMISSION OF MOSQUITO-BORNE DISEASES

CHUNHUA SHAN
Department of Mathematics and Statistics
The University of Toledo, Toledo, OH 43606, USA

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Abstract. Disease transmission can present significantly different cyclic patterns including small fluctuations, regular oscillations, and singular oscillations with short endemic period and long inter-epidemic period. In this paper we consider the slow-fast dynamics and nonlinear oscillations during the transmission of mosquito-borne diseases. Under the assumption that the host population has a small natural death rate, we prove the existence of relaxation oscillation cycles by geometric singular perturbation techniques and the delay of stability loss. Generation and annihilation of periodic orbits are investigated through local, semi-local bifurcations and bifurcation of slow-fast cycles. It turns out that relaxation oscillation cycles occur only if the basic reproduction number $R_0$ is greater than 1, while small fluctuations and regular oscillations exist under much less restrictive conditions. Our results here provide a sound explanation for different cyclic patterns exhibited in the transmission of mosquito-borne diseases.

1. Mosquito-borne disease models. Recurrence of infectious diseases has been observed from empirical data, such as measles, pertussis, malaria, dengue fever, West Nile virus and ect [2, 4, 25, 27]. Mechanisms for the periodic oscillations have been extensively studied in the mathematical epidemiology such as the delay effect [13, 25], nonlinear incidence rate [15, 24, 23, 28], time-dependent parameters and seasonality [12, 25] and reference therein. Recently by introducing small parameters, slow-fast dynamics and relaxation oscillations were investigated in an SIS model [19] and an SIR compartment model [20].

Disease transmission can exhibit quite different cyclic patterns, which include small fluctuations, regular oscillations, and “singular” oscillations with short endemic period and long inter-epidemic period. Investigation of these cyclic patterns is important for diseases prevention, control and prediction. However, there is no much work on the mechanisms of recurrence and cyclic patterns for vector-borne diseases. In this paper, we will formulate a slow-fast mosquito-borne disease model with two time scales, explore these different cyclic phenomena, and interpret them as periodic solutions perturbed from local bifurcation, semi-local bifurcation, and global bifurcation of slow-fast cycles. In particular, we consider the human natural
death rate as a small parameter, and prove the existence of relaxation oscillation cycle by geometric singular perturbation theory [10, 11] and delay of stability loss [3, 21, 22, 9, 26, 29]. Our results extend the results on relaxation oscillations for an SIR compartment models [20] to a vector-borne diseases model. To the best of our knowledge, this is the first time to study relaxation oscillations in vector-borne diseases.

Let $S_h$, $I_h$ and $R_h$ be the numbers of susceptible, infectious and recovered humans, respectively. $S_v$ and $I_v$ are numbers of the susceptible and infectious mosquitoes, respectively. By extending the SIR model incorporated with limited health resources studied by Shan and Zhu [30] and the vector-borne disease model studied by Abdelrazec, Belair et al. [1], we consider the mosquito-borne disease model as following

$$\begin{cases}
\frac{dS_h}{dt} = g_1(S_h, I_h, R_h) - \frac{\beta_1 S_h I_v}{N} - d_1 S_h, \\
\frac{dI_h}{dt} = \frac{\beta_1 S_h I_v}{N} - (d_1 + \nu_1) I_h - \mu(I_h, b) I_h, \\
\frac{dR_h}{dt} = \mu(I_h, b) I_h - d_1 I_h, \\
\frac{dS_v}{dt} = g_2(S_v, I_v) - \frac{\beta_2 S_v I_h}{N} - d_2 S_v, \\
\frac{dI_v}{dt} = \frac{\beta_2 S_v I_h}{N} - d_2 I_v,
\end{cases} \quad (1)$$

where $N = S_h + I_h + R_h$ and $M = S_v + I_v$, which satisfy

$$\begin{align*}
\frac{dN}{dt} &= g_1(S_h, I_h, R_h) - d_1 N - \nu_1 I_h, \\
\frac{dM}{dt} &= g_2(S_v, I_v) - d_2 M. \quad (2)
\end{align*}$$

In system (1) the standard incidence rate is adopted to model the cross-infection between humans and mosquitoes. The function $g_1(S_h, I_h, R_h)$ is the birth rate of humans, and function $g_2(S_v, I_v)$ is the birth rate of mosquitoes. The parameters $\beta_1$ and $\beta_2$ are contact transmission rates from mosquitoes to humans and from humans to mosquitoes, respectively. $\nu_1$ is the disease induced rate and $d_2$ is the mosquito death rate.

The per capita recovery rate of humans generally depends on the three ingredients: the specific mosquito-borne disease with which the humans are infected, the number of infectious humans seeking treatment and available health resources. The available health resources including health care workers, medical equipments and number of hospital beds have been proved as a very important factor in the disease transmission such as the spread of COVID-19 over the world. A simple form of rational function

$$\mu(I_h, b) = \mu_0 + \frac{(\mu_1 - \mu_0)b}{I_h + b}$$

were used to formulate the per capita recovery rate in [30], in which parameter $b$ characterizes the availability of health resources, and it is also referred to as the half-saturation rate. Parameters $\mu_0$ and $\mu_1$ denote the minimum and maximum per capita recovery rates, respectively. For detailed formulation of $\mu(I_h, b)$, the readers are referred to the work [30, 31]. Here, we choose this form for our study.

Compared to the time period of disease transmission, the lifespan of humans are quite long, so we have the first assumption.

Assumption 1. $d_1 > 0$ small, and we write $d_1 = \varepsilon$. 

For mosquito-borne diseases occurring in certain region during a certain time period, it is reasonable to assume that both humans' and mosquitoes' birth and death rates balance. We also assume there is no human disease-induced death. Mathematically, we have the following assumption.

Assumption 2. $g_1(S_h, I_h, R_h) = \varepsilon N$, $g_2(S_v, I_v) = d_2 M$, $\nu_1 = 0$.

Since $g_2(S_v, I_v) = d_2 M$, total mosquitoes population is a constant, say, $M(t) \equiv M^0$. The total population of humans is also a constant and suppose that $N(t) \equiv N^0$.

Consequently, system (1) reduces to the equivalent system

$$\begin{aligned}
\frac{dS_h}{dt} &= \varepsilon N^0 - \frac{\beta_1 S_h I_v}{N^0} - \varepsilon S_h, \\
\frac{dI_h}{dt} &= \frac{\beta_1 S_h I_v}{N^0} - \varepsilon I_h - \mu(I_h) b I_h, \\
\frac{dI_v}{dt} &= \frac{\beta_2 (M^0 - I_v)}{N^0} - d_2 I_v,
\end{aligned}$$

and we only consider the dynamics of slow-fast system (3) in $D$, where

$$D = \{(S_h, I_h, I_v)|S_h \geq 0, I_h \geq 0, S_h + I_h \leq N^0, 0 \leq I_v \leq M^0\}.$$

One can easily verify that region $D$ is an attracting set with respect to the flow of system (3).

There are two objectives in this paper. Firstly we aim to investigate different cyclic phenomena of mosquito-borne diseases. Secondly we want to explain why some mosquito-borne diseases exhibit long inter-epidemic period and short endemic period. We will provide biological interpretations in the last section.

The paper is organized as following. In section 2, the threshold dynamics of system (3) is carried out. In section 3, we first show that system (3) has a two-dimensional center manifold which consists of a family of heteroclinic loops for $\varepsilon = 0$. Due to the persistence of the center manifold for $0 < \varepsilon \ll 1$, we work on the perturbed center manifold, and prove the existence of relaxation oscillation cycle by geometric singular perturbation theory and delay of stability loss. In section 4, we consider limit cycles bifurcated from local and semi-local bifurcations, including Hopf bifurcation and Bogdanov-Takens bifurcation. Generation and annihilation of limit cycles are explored. Biological interpretations are given in section 5.

2. Threshold dynamics. In this section we consider the threshold dynamics of system (3) for $\varepsilon > 0$.

System (3) has a unique disease free equilibrium $E_0(N^0, 0, 0)$. The stability of $E_0$ depends on $R_0$, the well-known basic reproduction number, which can be computed in terms of next generation matrix as below

$$R_0 = \rho \left[ \begin{array}{ccc}
0 & \frac{\beta_1}{N^0} & 0 \\
\frac{\beta_2 M^0}{N^0} & 0 & \varepsilon + \mu_1 \\
0 & 0 & d_2
\end{array} \right]^{-1} = \sqrt{\frac{\beta_1 \beta_2 M^0}{(\varepsilon + \mu_1)d_2 N^0}},$$

where $\rho$ is the spectral radius of a matrix.

Proposition 1.

(1) If $R_0 < 1$, $E_0$ is a stable node and $\dim W^s_{loc}(E_0) = 3$.

(2) If $R_0 > 1$, $E_0$ is a saddle and $\dim W^u_{loc}(E_0) = 2, \dim W^s_{loc}(E_0) = 1$.

Proof. The characteristic equation of the linearization of system (3) at $E_0$ is

$$(\lambda + \varepsilon)(\lambda^2 + \delta_1 \lambda + \delta_0) = 0,$$
generality, we will change $R$ to $W$ for simplicity.

When $\beta < 0$, system (3) undergoes a pitchfork bifurcation if

$$\delta_0 = (\varepsilon + \mu_1)d_2(1 - R_0^2) \text{ and } \delta_1^2 - 4\delta_0 > 0,$$

so the desired results follow from the stable manifold theorem and Hartman-Grobman theorem.

From Proposition 1, it is apparent that $R_0 = 1$ is a threshold for the dynamics of system (3). The study of $R_0 = 1$ is of importance to understand qualitative behaviors of system (3) as the biological parameters vary, which may be due to the environmental changes or human intervention.

**Theorem 2.1.** If $R_0 = 1$, then $\dim W_{loc}^*(E_0) = 2$, $\dim W_{loc}^c(E_0) = 1$. Moreover,

(a) if $b \neq \hat{b}$, $E_0$ is a saddle-node point; System (3) undergoes a backward (resp. forward) bifurcation as $R_0$ passing through 1 for $b < \hat{b}$ (resp. $b > \hat{b}$).

(b) if $b = \hat{b}$, $E_0$ is an attracting node; System (3) undergoes a pitchfork bifurcation as $R_0$ passing through 1.

Here, we have $\hat{b} = \frac{\varepsilon d_2(\mu_1 - \mu_0)N_0^0}{(\varepsilon + \mu_1)(\varepsilon \beta_2 + \varepsilon d_2 + \mu_1 d_2)}$.

**Proof.** When $R_0 = 1$, from Proposition 1, $E_0$ has a zero eigenvalue and two negative eigenvalues, so $\dim W_{loc}^*(E_0) = 2$, $\dim W_{loc}^c(E_0) = 1$. In order to study related bifurcations involving $E_0$, we consider the dynamics on $W_{loc}^c(E_0)$. Without loss of generality, we will change $R_0$ by perturbing the parameter $\beta_1$.

To solve $R_0 = 1$ for $\beta_1$, and we obtain $\hat{\beta}_1 = \frac{(\varepsilon + \mu_1)d_2N_0^0}{\beta_2M_0^0}$. Let $\beta_1 = \hat{\beta}_1 + \rho$ where $\rho$ is small. Applying the center manifold theorem with parameters, we can reduce system (3) on the local center manifold $W_{loc}^c(E_0)$, on which the dynamics is governed by the scalar equation

$$\dot{u} = \left[\frac{\beta_2M_0^0\rho}{(\varepsilon + \mu_1 + d_2)N_0^0} + O(\rho^2)\right] u
- \left[\frac{(\varepsilon + \mu_1)(\varepsilon \beta_2 + \varepsilon d_2 + \mu_1 d_2)(b - \hat{b})}{(\varepsilon + \mu_1 + d_2)\beta_2\varepsilon M_0^0 N_0^0} + O(\rho)\right] u^2 + O(u^3).$$

(4)

Let the right hand side of system (4) be $h(u, \varepsilon)$. Then

$$h(0, 0) = h_u(0, 0) = h_{\rho}(0, 0) = 0, h_{uu}(0, 0) > 0, \ sgn(h_{uu}(0, 0)) = sgn(\hat{b} - b).$$

Therefore, if $b \neq \hat{b}$ system (3) undergoes a transcritical bifurcation when $\rho = 0$. Since $R_0$ is increasing in $\beta_1$, system (3) undergoes a forward bifurcation if $b > \hat{b}$ and undergoes a backward bifurcation if $b < \hat{b}$.

If $b = \hat{b}$, the reduced system (4) on $W_{loc}^c(E_0)$ becomes

$$\dot{u} = \left[\frac{\beta_2M_0^0\rho}{(\varepsilon + \mu_1 + d_2)N_0^0} + O(\rho^2)\right] u + O(\rho)u^2
- \left[\frac{d_2(\varepsilon + \mu_0)(\varepsilon + \mu_1)(\varepsilon \beta_2 + \varepsilon d_2 + \mu_1 d_2)^2}{(\mu_1 - \mu_0)(\varepsilon + \mu_1 + d_2)^2\beta_2^2(M_0^0)^2} + O(\rho)\right] u^3 + O(u^4),$$

and $h_{uu}(0, 0) > 0, \ h_{uu}(0, 0) = 0, \ sgn(h_{uu}(0, 0)) < 0$.

Hence, system (3) undergoes a pitchfork bifurcation if $b = \hat{b}$.

**Remark 1.** When $R_0 = 1$, $\hat{b}$ not only determines the type of disease free equilibrium, but also distinguishes two types of transcritical bifurcations: the forward
bifurcation and the backward bifurcation as shown in Theorem 2.1. Furthermore, \( \hat{b} \) is the upper bound for the disease outbreak when \( R_0 < 1 \). See Fig. 1 (a).

**Theorem 2.2.** Let \( \mathbb{R} = \sqrt{\frac{\beta_1 \beta_2 M^0}{(\varepsilon + \mu_0)d_2 N^0}} \). If \( \mathbb{R} \leq 1 \), then the disease free equilibrium \( E_0 \) constitutes the only equilibrium point, and it is globally asymptotically stable. 

**Proof.** Consider the Lyapunov function \( V = I_h/\beta_1 + I_v/d_2 \) in \( R^3^+ \). The total derivative of \( V \) with respect to system (3) is

\[
V' = \left[ \frac{S_h}{N^0} I_v - \frac{(\varepsilon + \mu_0)}{\beta_1} I_h - \frac{(\mu_1 - \mu_0) b I_h}{(I_h + b) \beta_1} \right] + \left[ \frac{\beta_2 M^0}{d_2 N^0} I_h - \frac{\beta_2 I_h I_v}{d_2 N^0} - I_v \right]
\]

\[
\leq - \left[ (1 - \frac{S_h}{N^0}) I_v + \frac{(\varepsilon + \mu_0)}{\beta_1} (1 - \mathbb{R}^2) I_h + \frac{(\mu_1 - \mu_0) b I_h}{(I_h + b) \beta_1} + \frac{\beta_2 I_h I_v}{d_2 N^0} \right].
\]

Then \( V' \leq 0 \) for \( \mathbb{R} \leq 1 \). According to Lasalle’s Invariance Principle [18], all orbit of system (3) approach the largest positively invariant subset of the set \( V' = 0 \), the singleton \( E_0 \). So \( E_0 \) is globally asymptotically stable. \( \square \)

For any endemic equilibrium \( E(S_h, I_h, I_v) \), a straightforward calculation yields that its coordinates satisfy

\[
S_h = \frac{\varepsilon N^0 - (\varepsilon + \mu(I_h)) I_h}{\varepsilon}, \quad I_v = \frac{\beta_2 M^0 I_h}{\beta_2 I_h + d_2 N^0},
\]

and \( I_h \) is a root of

\[
f(I_h) := A_2 I_h^2 + A_1 I_h + A_0,
\]

where

\[
A_2 = (\varepsilon + \mu_0)((\beta_1 M^0 + \varepsilon N^0) \beta_2),
A_1 = \beta_2 (\varepsilon + \mu_1)(\beta_1 M^0 + \varepsilon N^0) b - \varepsilon N^0[\beta_1 \beta_2 M^0 - (\varepsilon + \mu_0) d_2 N^0],
A_0 = \varepsilon b d_2 (\varepsilon + \mu_1)(N^0)^2(1 - \mathbb{R}^2).
\]

System (3) has at most two endemic equilibria, say, \( E_1 \) and \( E_2 \). Suppose \( I_{h1} \) and \( I_{h2} \) are two possible real roots with \( I_{h1} \leq I_{h2} \), and let \( I_{h1} \) and \( I_{h2} \) be the \( I_h \) coordinates of \( E_1 \) and \( E_2 \), respectively. Then we have the following results.

**Theorem 2.3.** System (3) can have up to two endemic equilibria. More precisely,

(i) if \( R_0 > 1 \), there exists a unique endemic equilibrium \( E_2 \).
(ii) if \( R_0 = 1 \), system (3) has a unique endemic equilibrium \( E_2 \) if and only if \( A_1 < 0 \); otherwise there is no endemic equilibrium.
(iii) if \( R_0 < 1 \), system (3) has two endemic equilibria \( E_1 \) and \( E_2 \) if and only if \( \Delta > 0 \) and \( A_1 < 0 \). These two equilibria coalesce if and only if \( \Delta = 0 \) and \( A_1 < 0 \), and system (3) has a unique equilibrium \( E^* \); Otherwise, there is no endemic equilibrium. Here \( \Delta = A_1^2 - 4 A_2 A_0 \).

It is difficult to biologically interpret the conditions in Theorem 2.3 in terms of coefficients \( A_i \) (\( i = 0, 1, 2 \)). Notice that for different mosquito-borne diseases, contact transmission rates between humans and mosquitoes are different. Also we want to explore the impact of limited health resources on the disease transmission. Hence, \( \beta_1 \) and \( b \) are chosen as bifurcation parameters. Firstly, the threshold \( R_0 = 1 \) defines a straight line \( C_0 \) in \( (\beta_1, b) \) plane,

\[
C_0 : \beta_1 = g_0(b) = \frac{(\varepsilon + \mu_1)d_2 N^0}{\beta_2 M^0}.
\]

The condition \( A_1 = 0 \) defines a hyperbola
For simplicity, denote $\beta$ tangent to $C$ intersect $H$ the arc $s$ saddle-node bifurcation occurs.

The forward bifurcation will occur when crossing $C$ into three subregions $V_0$, $V_1$, and $V_2$, in which there are 0, 1 and 2 simple endemic equilibria, respectively. The forward bifurcation will occur when crossing $C^+_0$, backward bifurcation will occur when crossing $C^-_0$, and the pitchfork bifurcation will occur when crossing $C_0$ at $p_1$ transversally. See Fig. 1 (b). On the arc $p_1p_3$, two equilibria $E_1$ and $E_2$ coalesce into an equilibrium $E^*$, and a saddle-node bifurcation occurs.

For $\beta_1 = \hat{\beta}_1$ if $b = \hat{b}$.

For simplicity, denote $\hat{\beta}_1 = \frac{(\epsilon + \mu_1)d_2N^0}{\beta_1}$. The branch of $C_1$ in the first quadrant intersect $C_0$ at $p_1(\hat{\beta}_1, \hat{b})$ and $\beta_1$-axis at $p_3(\hat{\beta}_1, 0)$.

Let $C^\pm_0 = \{ (\beta_1, b) | (\beta_1, b) \in C_0, sgn(b - b) = \pm 1 \}$, then $C_0 = C^+_0 \cup C^-_0 \cup \{ p_1 \}$. Here $C^+_0$ and $C^-_0$ are two parts of $C_0$ separated by the point $p_1$.

The condition $\Delta = 0$ defines a quadratic curve

$$C_\Delta : \Delta(\beta_1, b) := (\epsilon + \mu_1)^2(\beta_1M^0 + \epsilon N^0)^2\beta_2^2b^2$$

$$-2\epsilon\beta_2N^0(\beta_1M^0 + \epsilon N^0)[(\mu_1 - \epsilon - 2\mu_0)\beta_1\beta_2M^0 + (\epsilon + \mu_0)(\epsilon + \mu_1)d_2N^0]b$$

$$+ [\beta_1\beta_2M^0 - (\epsilon + \mu_0)d_2N^0]^2 \epsilon^2(N^0)^2 = 0.$$  

$C_\Delta$ has two branches $C^\pm_\Delta : b = g^\pm_\Delta(\beta_1)$ which can be easily obtained by solving $\Delta_0(\beta_1, b) = 0$ for $b$, where $g^\pm_\Delta(\beta_1)$ are well defined only for $\beta_1 \in [0, \hat{\beta}_1]$ with $g^-_\Delta(\beta_1) \leq g^+\Delta(\beta_1)$, and $g^\pm_\Delta(\beta_1)$ if and only if $\beta_1 = 0$. Furthermore, $C^-_\Delta$ is tangent to $\beta_1$-axis at the point $p_3$. By the implicit function theorem, one can check that $C^-_\Delta$ is decreasing for $\beta_1 \in [0, \hat{\beta}_1]$ and increasing for $\beta_1 \in [\hat{\beta}_1, \hat{\beta}_1]$.

For the relative positions of curves $C_1$ and $C_\Delta$, indeed we have

$$sgn(\Delta(\beta_1, g_1(\beta_1))) = sgn(\hat{\beta}_1 - \beta_1).$$

Hence, we sketch curves $C_0$, $C_1$ and $C_\Delta$ as shown in Fig. 1 (a), which separate the first quadrant into three subregions $V_0$, $V_1$ and $V_2$, in which there are 0, 1 and 2 simple endemic equilibria, respectively. The forward bifurcation will occur when crossing $C^+_0$, backward bifurcation will occur when crossing $C^-_0$, and the pitchfork bifurcation will occur when crossing $C_0$ at $p_1$ transversally. See Fig. 1 (b). On the arc $p_1p_3$, two equilibria $E_1$ and $E_2$ coalesce into an equilibrium $E^*$, and a saddle-node bifurcation occurs.

\[ \begin{align*}
C_1 & : b = g_1(\beta_1) = \frac{[\beta_1\beta_2M^0 - (\epsilon + \mu_0)d_2N^0]\epsilon N^0}{(\epsilon + \mu_1)(\beta_1M^0 + \epsilon N^0)}.
\end{align*} \]
3. Slow-fast dynamics and relaxation oscillation cycles.

3.1. Slow-fast dynamics. Let $\varepsilon \to 0$ in system (3), then we obtain the layer equations

$$
\begin{aligned}
\frac{dS_h}{dt} &= -\beta_1 S_h I_v, \\
\frac{dI_h}{dt} &= \beta_1 S_h N^0 - \mu(I_h, b) I_h, \\
\frac{dI_v}{dt} &= \frac{\beta_2 (M^0 - I_v) I_h}{N^0} - d_2 I_v,
\end{aligned}
$$

(7)

which has a line of equilibria

$$
Z_0 = \{(S_h, I_h, I_v)|0 \leq S_h \leq N^0, I_h = I_v = 0\}.
$$

It is called the slow manifold. By switching to the slow time scale $\tau = \varepsilon t$, the slow dynamics on the slow manifold $Z_0$ are governed by reduced equations

$$
\frac{dS_h}{d\tau} = N^0 - S_h, \quad I_h = I_v = 0.
$$

(8)

Note that the movements of system (3), for $\varepsilon > 0$ small, in the vicinity of $Z_0$ will be slow with speed of order $O(\varepsilon)$, while movements not close to $Z_0$ will be fast of order $O(1)$.

Consider the layer problem (7). The linearization at each point $(S_h, 0, 0) \in Z_0$ is

$$
\begin{pmatrix}
0 & 0 & -\beta_1 S_h \\
0 & -\mu_1 & \beta_1 N^0 \\
0 & \beta_2 M^0 & -d_2
\end{pmatrix},
$$

which has three eigenvalues, where $\lambda_1 = 0$, and $\lambda_2$ and $\lambda_3$ are roots of the equation

$$
\lambda^2 + \delta_1 \lambda + \delta_0 = 0,
$$

where

$$
\delta_1 = \mu_1 + d_2, \quad \delta_0 = \mu_1 d_2 - \frac{\beta_1 \beta_2 S_h M^0}{(N^0)^2}.
$$

Figure 2. Dynamics of layer problem (7) with $S_h^0 < N^0$. 
Note that the invariant set in these two planes are set of $\gamma$ if $\delta$ is positive eigenvalue ($\langle \mu, d \rangle$). For the types of equilibria in $Z_0$, we have three cases:

- If $S_h > N^0$, all the points in $Z_0$ are normally attracting.
- If $S_h = N^0$, all the points in $Z_0$ are normally attracting except the boundary point $(N^0, 0)$ with a double zero eigenvalue.
- If $S_h < N^0$, the points in $Z_0$ with $S_h < S_h^0$ are normally attracting; the points in $Z_0$ with $S_h^0 < S_h < N^0$ are normally hyperbolic saddles.

For the first two cases, slow manifold $Z_0$ (without boundary point $(N^0, 0, 0)$ of case 2) is normally hyperbolic. By geometric singular perturbation theory [10, 11], there exists a one-dimensional manifold $Z_c$ lies in the vicinity of $Z_0$ and is diffeomorphic to $Z_0$. The restriction of system (3) on $Z_c$ is a small smooth perturbation of the reduced problem (8). Furthermore, there exists a two-dimensional stable invariant foliation with base $Z_c$, along which the dynamics of system (3) is a small smooth perturbation of those of layer problem (7).

We will assume $S_h^0 < N^0$. Let $p_0 = (S_h^0, 0, 0)$. Along the flow of reduce problem (8) from the origin, i.e., the direction of increasing $S_h$, eigenvalue $\lambda_3$ changes its sign from negative to positive across the point $p_0$, and $p_0$ is a turning point which has a double zero eigenvalue. The following theorem characterizes the global dynamics of layer problem (7).

**Theorem 3.1.** Consider layer problem (7).

(a) The slow manifold $Z_0$ is the global attractor.

(b) For each equilibrium $(S_h, 0, 0)$ with $S_h > S_h^0$, from which there is a heteroclinic orbit to an equilibrium $(\bar{S}_h, 0, 0)$ with $0 < \bar{S}_h < S_h^0$. Moreover, the ordering $0 < S_h^0 < S_h^0 < S_h^0$ holds if $S_h^0 < S_h^0 < S_h^0$.

**Proof.** (a) Let $\gamma = \{(S_h(t), I_h(t), I_v(t))| t \geq 0\}$ be any orbit of system (7) with initial condition $(S_h(0), I_h(0), I_v(0))$ in $D$. Since $D$ is an attracting compact set with respect to the flow of system (7), then the $\omega$-limit set of $\gamma$ is a nonempty, connected compact set. If $(S_h(0), I_h(0), I_v(0)) \in Z_0$, then $\omega$-limit set of $\gamma$ is the singleton $\{(S_h(0), I_h(0), I_v(0))\}$. If $(S_h(0), I_h(0), I_v(0)) \in D \setminus Z_0$, consider the Lyapunov function $S_h(t)$, which satisfies $S_h'(t) = -\frac{\mu S_h(t)}{\beta_1 S_h(t)}$, and $S_h'(t) = 0$ if and only of $S_h = 0$ or $I_c = 0$. The Lasalle’s Invariance Principle [18] implies that the $\omega$-limit set of $\gamma$ is a subset of the largest invariant set in the planes $S_h = 0$ and $I_c = 0$.

Note that the invariant set in these two planes are $Z_0$ or its compact sets. However, $S_h'(t) \leq 0$ and $S_h(t)$ is bounded from below, then $S_h(t)$ monotonically decreases and $S_h(t) \to \bar{S}_h$ as $t \to \infty$ for some $\bar{S}_h \geq 0$ depending on the initial condition. Hence, the $\omega$-limit set of $\gamma$ is a singleton in $Z_0$, and $Z_0$ is the global attractor.

(b) For the orbit $\gamma$ with initial condition $(S_h(0), I_h(0), I_v(0)) \in D$ where $S_h(0) > 0$, we claim that $0 < S_h < S_h^0$ where $S_h = \lim_{t \to \infty} S_h(t)$. Indeed, the equilibrium $(S_h, 0, 0)$ with $S_h > S_h^0$ has one zero eigenvalue, one negative eigenvalue $\lambda_2$ and one positive eigenvalue $\lambda_3$. The eigenvectors associated with the eigenvalues $\lambda_2$ and $\lambda_3$ are

$v_2 = \langle -\lambda_2 + \mu_1, \lambda_2, \frac{N_0}{\beta_1 S_h} \lambda_2 (\lambda_2 + \mu_1) \rangle$, $v_3 = \langle -\lambda_3 + \mu_1, \lambda_3, \frac{N_0}{\beta_1 S_h} \lambda_3 (\lambda_3 + \mu_1) \rangle$,
the exterior of the region $D$. Therefore, the local one-dimensional stable manifold $W^s(S_h,0,0)$ except $(S_h,0,0)$ stays outside of $D$. Hence, the $\omega$-limit set of $\gamma$ is a singleton in $\mathcal{Z}_0$ with $S_h < S_h^0$, i.e., $\bar{S}_h < S_h^0$.

For the equilibrium $(S_h,0,0)$ with $S_h > S_h^0$, there exists a one-dimensional unstable local manifold $W^u(S_h,0,0)$, by (a), which extends and converges to $(\bar{S}_h,0,0) \in \mathcal{Z}_0$ with $\bar{S}_h < S_h^0$ as $t \to \infty$. Therefore, the unstable manifold $W^u(S_h^0,0,0)$, whose $\omega$- and $\alpha$-limit sets are $(\bar{S}_h,0,0)$ and $(S_h,0,0)$, respectively, is a heteroclinic orbit connecting the two points $(\bar{S}_h,0,0)$ and $(S_h,0,0)$. It remains to show that $\bar{S}_h > 0$.

Since the two-dimensional stable manifold of the origin is the invariant plane $S_h = 0$.

Therefore, for any $(S_h,0,0) \in \mathcal{Z}_0$ with $S_h > S_h^0$, there does not exist a heteroclinic orbit which connects this point to the origin. Hence, $\bar{S}_h > 0$.

Regarding to ordering of the heteroclinic orbits, we define the transition map

$$H : (S_h^0, \infty) \to (0, S_h^0), \quad H(S_h) = \bar{S}_h,$$

where $\bar{S}_h$ is defined by the heteroclinic orbits. Note that the equilibrium $(S_h^0,0,0)$ has double zero eigenvalues and one negative eigenvalue $\lambda_2$. Locally, there is a two-dimensional center manifold $W^c(S_h^0,0,0)$, which can be taken to consist of heteroclinic orbits from $(S_h,0,0) \in \mathcal{Z}_0$ with $S_h^0 < S_h < S_h^0 + \delta_0$ for some $\delta_0 > 0$ small to a point $(\bar{S}_h,0,0) \in \mathcal{Z}_0$. For $S_h^0 < S_h^1 < S_h^2 < S_h^0 + \delta_0$, it is clear that $S_h^2 < S_h^1 < S_h^0$. By the smoothness of vector fields of system (7) and uniqueness of solutions, $H$ is a smooth function. Therefore, the monotone decreasing property of $H$ holds globally on $(S_h^0, \infty)$.

A heteroclinic orbit combining two points $(S_h,0,0)$ and $(\bar{S}_h,0,0)$ and the part of slow manifold $\mathcal{Z}_0$ between such two points (with compatible orientation) forms a loop, say $\Gamma$. Here $\Gamma$ is called a limit periodic set and it is indeed a slow-fast cycle of canard type of system (7). See Definition 5 of [8]. A relaxation oscillation cycle $\gamma^\varepsilon$ bifurcating from $\Gamma$ is a limit cycle of system (3), which is in a small neighborhood of $\Gamma$, and $\gamma^\varepsilon \to \Gamma$ (in Hausdorff sense) as $\varepsilon \to 0$. The study of relaxation oscillation cycles is mathematically and biologically interesting and important.

### 3.2. Center manifold

Due to the existence of turning point, the normal hyperbolicity of $\mathcal{Z}_0$ fails, and the layer problem (7) is not able to fully characterize the dynamics of the full model (3) for $\varepsilon > 0$ small [10, 11]. Thanks to center manifold theorem developed by Chow, Liu and Yi [5, 6], we can study the slow-fast dynamics of system (3) on the center manifold $M(\mathcal{Z}_0)$ defined as below.

Let $M(\mathcal{Z}_0)$ be the union of the family of heteroclinic orbits in Theorem 3.1. By differentiability of solutions of system (7) on $t$ and initial conditions, $M(\mathcal{Z}_0)$ is two-dimensional smooth invariant manifold.

**Theorem 3.2.** The invariant manifold $M(\mathcal{Z}_0)$ is persistent for $\varepsilon > 0$ small, i.e., there is an invariant manifold $M^\varepsilon$ for system (3) so that $M^\varepsilon \to M(\mathcal{Z}_0)$ as $\varepsilon \to 0$.

**Proof.** For each point $p = (S_h,0,0) \in \mathcal{Z}_0$, the eigenvalues of the linearization at $p$ are $\lambda_1 = 0$ and $\lambda_{2,3} = -\frac{\delta_0 \pm \sqrt{\delta_0^2 + 4\mu}}{2}$, which satisfy $\lambda_2 < \lambda_1$ and $\lambda_2 < \lambda_3$ for all $S_h \in [0, N^0]$. By the center manifold theorem developed by Chow, Liu and Yi [5, 6], for the invariant set $\mathcal{Z}_0$, there exits a two-dimensional center manifold $W^c(\mathcal{Z}_0)$, and it is persistent for $\varepsilon > 0$ small. At each point $p \in \mathcal{Z}_0$, the tangent space $T_p(W^c(\mathcal{Z}_0))$ is spanned by the eigenvectors associated with $\lambda_1$ and $\lambda_3$. Furthermore, the center manifold $W^c(\mathcal{Z}_0)$ is invariant with respect to the flow of system (7), and contains
$Z_0$ and all orbits bounded in the vicinity of $Z_0$. Hence, $W^c(Z_0)$ coincides with $M(Z_0)$. □

In order to study the transition map $H$ defined in Theorem 3.1, we need the information of the invariant manifold $M(Z_0)$, which is not easy to obtain. Note that $W^c(Z_0)$ coincides with $M(Z_0)$, so we can probe the local information of $M(Z_0)$ near the turning point $(S_h^0, 0, 0)$ by center manifold $W^c(S_h^0, 0, 0)$.

We derive an approximation of center manifold $W^c(S_h^0, 0, 0)$. Since $S_h$-axis is invariant and is a subset of $M(Z_0)$, we consider it as a graph of

$$I_v = \alpha(S_h, I_h)I_h = \alpha_0(S_h)I_h + \alpha_1(S_h)I_h^2 + \alpha_2(S_h, I_h)I_h^3,$$

where $\alpha_1(S_h)$, $\alpha_2(S_h)$ and $\alpha_3(S_h)$ are to be determined.

Taking the derivative of the above equation with respect to $t$ yields to

$$I'_v = \alpha_0'(S_h)S_h' I_h + \alpha_0(S_h)I_h' + 2\alpha_1(S_h)I_hI_h' + O(I_h^3).$$

By system (7), we have

$$\frac{\beta_2}{N^0} \left[ M^0 - \left( \alpha_0 I_h + \alpha_1 I_h^2 + O(I_h^3) \right) \right] I_h - d_2 \left[ \alpha_0 I_h + \alpha_1 I_h^2 + O(I_h^3) \right]$$

$$= \left( \alpha_0 + 2\alpha_1 I_h \right) \left\{ \frac{\beta_1 S_h}{N^0} \left( \alpha_0 I_h + \alpha_1 I_h^2 + O(I_h^3) \right) - \left( \mu_1 - \frac{\mu_0}{b} \right) I_h + O(I_h^3) \right\}$$

$$- d_0 \left( \frac{\beta_1 S_h}{N^0} \left( \alpha_0 I_h + \alpha_1 I_h^2 + O(I_h^3) \right) \right) I_h + O(I_h^3),$$

where $\alpha_0 = \alpha_0(S_h)$, and $\alpha_1 = \alpha_1(S_h)$.

Equating coefficients of $I_h', I_h^2$ and $I_h^3$, we obtain that

$$0 = 0, \alpha_0(S_h) = \frac{-\left( d_2 - \mu_1 \right) + \sqrt{d_2^2 - 4d_0}}{2}, \alpha_1(S_h) = \frac{\alpha_0 \alpha_0' \beta_1 S_h - \alpha_0 \beta_2 S_h - \mu_1 - \mu_0}{3 \beta_1 S_h - \mu_1 - \mu_0}.$$

Since $\frac{3 \beta_1 S_h}{N^0} \alpha_0(S_h^0) = 3\mu_1$, then the denominator of $\alpha_1(S_h)$ equals $\mu_1 + d_2 \neq 0$ at $S_h = S_h^0$.

Therefore, the center manifold $W^c(S_h^0, 0, 0)$ is

$$I_v = \frac{-\left( d_2 - \mu_1 \right) + \sqrt{d_2^2 - 4d_0}}{2} I_h + \frac{\alpha_0 \alpha_0' \beta_1 S_h - \alpha_0 \beta_2 S_h - \mu_1 - \mu_0}{3 \beta_1 S_h - \mu_1 - \mu_0} I_h^2 + O(I_h^3),$$

on which system (7) reduces to a two-dimensional system

$$\begin{align*}
\frac{dS_h}{dt} &= -\beta_1 \left( \alpha_0(S_h)I_h + \alpha_1(S_h)I_h^2 + O(I_h^3) \right) S_h, \\
\frac{dI_h}{dt} &= \beta_1 \left( \alpha_0(S_h)I_h + \alpha_1(S_h)I_h^2 + O(I_h^3) \right) S_h - \mu(I_h, b) I_h.
\end{align*}$$

The dynamics of system (7) on $M(Z_0)$ have been sketched in Fig. 3.

**Theorem 3.3.** The transition map $H$ satisfies $H'(S_h^0) = -1$ and

$$H''(S_h^0) = -\frac{2N^0}{\beta_1 S_h^0} \left( \frac{\alpha_1(S_h^0) \mu_1}{\alpha_0(S_h^0) b} \right) + \frac{\mu_1 - \mu_0}{\alpha_0(S_h^0) b}.$$

**Proof.** Let $u = S_h - S_h^0$. Then

$$\begin{align*}
\frac{du}{dt} &= -\beta_1 \left( \alpha_0(u + S_h^0)I_h + \alpha_1(u + S_h^0)I_h^2 + O(I_h^3) \right) (u + S_h^0), \\
\frac{dI_h}{dt} &= \beta_1 \left( \alpha_0(u + S_h^0)I_h + \alpha_1(u + S_h^0)I_h^2 + O(I_h^3) \right) (u + S_h^0) - \mu(I_h, b) I_h,
\end{align*}$$

(11)
Figure 3. Dynamics of system (7) on $M(Z_0)$, in which double arrow indicates the fast movement along the regular orbits, and single arrow indicates the slow moment on the slow manifold $Z_0$. The blue line is the one of a family of slow-fast cycles.

and

$$\frac{dI_h}{du} = -1 + \frac{N^0\beta(I_h, b)}{\beta_1[a_0(u + S_0^h) + a_1(u + S_0^h)]}.$$  

The Taylor’s expansion of the right hand side of the above equation leads to

$$\frac{dI_h}{du} = L_0 + L_1u + L_2u^2 + O(u^3) + (Q_0 + Q_1u + O(u^2))I_h + O(I_h^2),\quad (12)$$

where

$$L_0 = 0, \quad L_1 = -\frac{d_2}{(d_2 + \mu_1)^2}, \quad L_2 = \frac{(d_2 + \mu_1)d_2^2}{(d_2 + \mu_1)^3(S_0^h)^2},$$

$$Q_0 = -\frac{N^0}{\beta_1 S_0^h} \left( \frac{\alpha_1 \alpha_1}{\alpha_0^2} + \frac{\alpha_1 - \alpha_0}{\alpha_0 b} \right),$$

$$Q_1 = \frac{N^0}{\beta_1 S_0^h} \left( \frac{\alpha_1 \alpha_1}{\alpha_0^2} + \frac{\alpha_1 - \alpha_0}{\alpha_0 b} \right) - \frac{N^0}{\beta_1 S_0^h} \left( \frac{\alpha_1 \alpha_0 - 2\alpha_0 \alpha_1 \alpha_0^2}{\alpha_0^3} \mu_1 + \frac{(\mu_1 - \mu_0)\alpha_0^2}{\alpha_0^3 b} \right),$$

and $\alpha_0$, $\alpha_1$ and $\alpha_1$ are evaluated at $S_0^h$.

By the existence and smooth dependence of solutions on parameters, for $u$ small, we assume $I_h(u) = \zeta_0 + \zeta_1 u + \zeta_2 u^2 + O(u^3)$. Substituting $I(u)$ into (12) and equating terms of like powers of $u$, we obtain

$$\zeta_1 = Q_0\zeta_0 + O(\zeta_0^2),$$

$$\zeta_2 = \frac{1}{2} L_1 + \frac{1}{2}(Q_0^2 + Q_1)\zeta_0 + O(\zeta_0^3).$$

Then the series solution of equation (11) is

$$I_h(u) = \zeta_0 + \zeta_0 Q_0 u + \frac{1}{2} L_1^2 u^2 + O(\zeta_0 u^2).$$

Notice that $I_h(u)$ satisfies $I_h(u) = 0$ for $0 \leq u \ll 1$, by which we have

$$\zeta_0 + \zeta_0 Q_0 u + \frac{1}{2} L_1^2 u^2 + O(\zeta_0 u^2) = 0.$$

Hence,

$$\zeta_0 = -\frac{1}{2} L_1 u^2 + \frac{1}{2} L_1 Q_0 u^3 + O(u^4).$$
For the transition map $H$, it is easy to see that
\[ H(S_h) - S_h^0 = H'(S_h^0)(S_h - S_h^0) + \frac{1}{2} H''(S_h^0)(S_h - S_h^0)^2 + O((S_h - S_h^0)^3), \]
and $I_h(H(S_h) - S_h^0) = 0$, which leads to
\[ |(H'(S_h^0))^2 - 1|(S_h - S_h^0)^2 + |Q_0 - H'(S_h^0)Q_0 + H'(S_h^0)H''(S_h^0)[(S_h - S_h^0)^3 + O((S_h - S_h^0)^4)] = 0. \]
Equating the coefficients of like powers of $(S_h - S_h^0)$, we have
\[ (H'(S_h^0))^2 - 1 = 0, \quad Q_0 - H'(S_h^0)Q_0 + H'(S_h^0)H''(S_h^0) = 0. \]
Since $H$ is monotone decreasing, $H'(S_h^0) = -1$ and $H''(S_h^0) = 2Q_0$.

3.3. **Existence of relaxation oscillation cycles.** Due to the fact that $S_h$-axis is the only invariant set on $\{I_v = 0\}$ for all $\varepsilon > 0$, and the persistence of $M(Z_0)$ for $\varepsilon > 0$ small, we may assume $M^\varepsilon$ as a graph of
\[ I_h^\varepsilon(S_h, I_h) = \tilde{a}_0(S_h, \varepsilon)I_h + \tilde{a}_1(S_h, \varepsilon)I_h^2 + O(I_h^3), \]
where $\tilde{a}_0(S_h, 0) = a_0(S_h)$ and $\tilde{a}_1(S_h, 0) = a_1(S_h)$. System (3) reduced on the center manifold $M^\varepsilon$ is
\[ \begin{cases} \frac{dS_h}{dt} = \varepsilon(N^0 - S_h) - \beta_1S_h[\tilde{a}_0(S_h, \varepsilon) + \tilde{a}_1(S_h, \varepsilon)I_h + O(I_h^2)]I_h, \\ \frac{dI_h}{dt} = \beta_1S_h[\tilde{a}_0(S_h, \varepsilon) + \tilde{a}_1(S_h, \varepsilon) + O(I_h^2)]I_h - \mu(I_h, b)I_h. \end{cases} \tag{13} \]
For $\varepsilon = 0$, system (13) reduces to system (9), for which the $S_h$-axis consists of equilibria. See Fig. 3. The equilibria are normally attracting for $0 \leq S_h < S_h^0$, and normally repelling for $S_h > S_h^0$. For $\varepsilon > 0$ small, it follows from system (13) that the $S_h$-axis remains invariant on which the flow moves to the right and tends to $(N^0, 0)$ as $t \to \infty$. Let $\gamma^\varepsilon$ be an orbit of system (13) that starts at $(S_h, \delta_0)$ with $0 < S_h < S_h^0$ and $\delta_0 > 0$ small. Then it is attracted exponentially fast toward the $S_h$-axis, drifts to the right along the $S_h$-axis, surpasses the turning point $(S_h^0, 0)$, travels for a fair amount of time and finally is repelled from $S_h$-axis. In other words, the orbit $\gamma^\varepsilon$ does not leave the $S_h$-axis as soon as it surpasses the turning point. Whereas it stays in the vicinity of $S_h$-axis until a repulsion has built up to balance the attraction accumulated before the turning point. This phenomenon is called the **delay of stability loss** [3, 21, 22, 9, 26, 29].

As $\varepsilon \to 0$, the limiting position at which $\gamma^\varepsilon$ is repelled from $S_h$-axis can be characterized by the so called entry-exit function. The entry-exit function of slow-fast systems has been studied by many researchers, We refer the reader to [3, 14, 21, 22, 7, 9, 26, 16, 17, 29, 32] and references therein. The entry-exit function of system (13) is defined as below. Suppose that the orbit $\gamma^\varepsilon$ intersects the line $I_h = \delta_0$ ($\delta_0 > 0$ small) at the point $(P_\varepsilon(S_h), \delta_0)$ after surpassing the turning point. As $\varepsilon \to 0$, the map $P_\varepsilon(S_h)$ approaches the entry-exit function $P : (0, S_h^0) \to [S_h^0, N^0)$, which is given by
\[ \int_{S_h}^{P(S_h)} \frac{\beta_1}{N^0 - \xi} \frac{\alpha_1(\xi)\xi - \mu_1}{N^0 - \xi} d\xi = 0 \tag{14} \]
and $P(S_h^0) = S_h^0$.

The left hand of equation (14) is called the slow divergence integral, which is a crucial tool to study the cyclicity of a slow-fast cycle. The roots of slow divergence integral and its derivatives characterize the number and stability of limit cycles.
(which are relaxation oscillations) bifurcated from slow-fast cycle. For our model, we do not have the global information of the center manifold $M(Z_0)$. What we have is the local information near the turning point from the local center manifold $W^c(S^0_h, 0, 0)$, by which the map $P(S_h)$ can be studied near $S_h = S^0_h$.

For simplicity, let

$$
\phi(x) = \int_{S^0_h}^{x} \frac{\beta_1 \alpha_0(\xi) \xi - \mu_1}{N^0 - \xi} d\xi, \quad (15)
$$

Then by fundamental theorem of calculus,

$$
\phi(S^0_h) = 0, \quad \phi'(S^0_h) = \frac{\beta_1 S^0_h \alpha_0(S^0_h) - \mu_1}{N^0 - S^0_h} = 0,
$$

$$
\phi''(S^0_h) = \frac{\beta_1 \beta_2 M^0}{(d_2 + \mu_1)(N^0)^2(N^0 - S^0_h)} \neq 0,
$$

and equation (14) becomes

$$
\int_{S^0_h}^{P(S_h)} \frac{\beta_1 \alpha_0(\xi) \xi - \mu_1}{N^0 - \xi} d\xi - \int_{S^0_h}^{S_h} \frac{\beta_1 \alpha_0(\xi) \xi - \mu_1}{N^0 - \xi} d\xi = 0,
$$

equivalently,

$$
\phi(P(S_h)) - \phi(S_h) = 0. \quad (16)
$$

Taking the derivative with respect to $S_h$ of equation (16) twice and three times, we obtain

$$
\phi''(P(S_h))(P'(S_h))^2 + \phi'(P(S_h))P''(S_h) - \phi''(S_h) = 0,
$$

$$
\phi'''(P(S_h))(P'(S_h))^3 + 3\phi''(P(S_h))P'(S_h)P''(S_h) + \phi'(P(S_h))P'''(S_h) - \phi'''(S_h) = 0.
$$

Evaluate the above equations at $S_h = S^0_h$ and notice that $P(S^0_h) = S^0_h$, we have

$$
\phi''(S^0_h)((P'(S^0_h))^2 - 1) = 0 \implies P'(S^0_h) = -1,
$$

which is by the monotone decreasing property of $P$, and we also have

$$
2\phi'''(S^0_h) + 3\phi''(S^0_h)P''(S^0_h) = 0 \implies P''(S^0_h) = -\frac{2\phi'''(S^0_h)}{3\phi''(S^0_h)}.
$$

By (15), we have

$$
\phi''(S^0_h) = -\frac{2\beta_1 \beta_2 M^0[\beta_1 \beta_2 M^0(N^0 - S^0_h) - (d_2 + \mu_1)N^0]}{(d_2 + \mu_1)^3(N^0)^2(N^0 - S^0_h)^2},
$$

and then

$$
P''(S^0_h) = \frac{4[\beta_1 \beta_2 M^0(N^0 - S^0_h) - (d_2 + \mu_1)N^0]}{3(d_2 + \mu_1)^2(N^0)^2(N^0 - S^0_h)}. \quad (17)
$$

**Remark 2.** Indeed $\phi(x)$ can be calculated explicitly as

$$
\phi(x) = \frac{1}{2}(d_2 + \mu_1) \ln(N^0 - x) - \sqrt{(d_2 - \mu_1)^2 + \frac{4\beta_1 \beta_2 M x}{N^2}}
$$

$$
+ \sqrt{(d_2 - \mu_1)^2 + \frac{4\beta_1 \beta_2 M}{N^2}} \arctanh \left( \frac{\sqrt{(d_2 - \mu_1)^2 + \frac{4\beta_1 \beta_2 M x}{N^2}}}{\sqrt{(d_2 - \mu_1)^2 + \frac{4\beta_1 \beta_2 M}{N^2}}} \right) + C,
$$

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where $C$ is a constant such that $\varphi(S_h^0) = 0$, from which derivatives $\varphi(x)$ at $S_h^0$ can be obtained as well.

**Theorem 3.4.** Let $G(S_h) : [S_h^0, \infty) \to R$ be a displacement map, and $G(S_h) = P(H(S_h)) - S_h$. If

$$G''(S_h^0) = P''(S_h^0) - H''(S_h^0) > 0,$$

where $H''(S_h^0)$ and $P''(S_h^0)$ are given in formulas (10) and (17), then system (3) has at least one stable relaxation oscillation cycle for $\varepsilon > 0$ small.

**Proof.** For $\varepsilon > 0$ small, the roots of the displacement map $G(S_h) = P(H(S_h)) - S_h$ characterize the number of relaxation oscillation cycles of system (3) perturbed from the slow-fast cycle $\Gamma$. By invariance of $D$, we have $G(N^0) = P(H(N^0)) - N^0 < 0$. Since $G(S_h^0) = 0$ and $G''(S_h^0) > 0$, by the geometry of $G(S_h)$ on $[S_h^0, N^0)$, $G(x)$ has at least one root of $G(x)$, say $S_h^1 \in (S_h^0, N^0)$, and $G(S_h) > 0$ for $S_h \in (S_h^0, S_h^1)$, $G(S_h) < 0$ for $S_h \in (S_h^1, S_h^* + \rho_0)$ with some $\rho > 0$ small. Hence, system (3) has at least one relaxation oscillation cycle for $\varepsilon > 0$ small, and it is stable.

**Remark 3.** System (7) has a turning point if $S_h^0 < N^0$. For $\varepsilon > 0$ small, by formula of $R_0$ in Section 2, we have $R_0 > 1$. Hence, relaxation oscillation cycles exist only for $R_0 > 1$.

![Figure 4](image.png)

**Figure 4.** Two limit cycles are on the center manifold $M^\ast$, and the outer one is a relaxation oscillation cycle.

**Example of relaxation oscillation cycle.** Fix $d_2 = 0.9$, $\beta_1 = 0.09$, $\beta_2 = 0.1$, $\mu_1 = 0.052$, $\mu_0 = 0.05$, $M = 10000$, $N = 1000$, $b = 0.1$. System (3) has a turning point $(520, 0, 0)$ for $\varepsilon = 0$. If we choose $\varepsilon = 1 \times 10^{-4}$, then $R_0 \approx 1.39$, and system (3) has a unique endemic equilibrium $E_2(502.89, 0.99, 1.10)$, which is stable. By formulas (10) and (17), $P''(S_h^0) \approx -0.73$, $P''(S_h^0) \approx 0.00012$, and $G''(S_h^0) \approx 0.73$. By Theorem 3.4, system (3) has a stable relaxation oscillation cycle on $M^\ast$. In Fig. 4, the orbit starting at $(520, 13, 13)$ (gold curve) spirals inward, and the orbit starting at $(520, 6.6)$ (blue curve) spirals outward. Hence, these two orbits tend a stable relaxation oscillation cycle. The orbit near $E_2$ starting at $(520, 2.2)$ (red curve) spirals inward and converges to $E_2$. By Poincaré-Bendixson theorem, there exits another unstable limit cycles surrounding $E_2$, which may or may not be bifurcated from the slow-fast cycle.
4. Local and semi-local nonlinear oscillations. In this section, we consider nonlinear oscillations bifurcated from local and semi-local bifurcations which may also explain small fluctuations and regular oscillations of mosquito-borne diseases. Stability loss of an equilibrium is a common mechanism to bring about periodic solutions [13], so we start with Hopf bifurcation.

4.1. Hopf bifurcation. The variational matrix of system (3) at any endemic equilibrium \( E(S_h, I_h, I_v) \) is

\[
J(E) = \begin{pmatrix}
-\frac{\beta_2 S_h}{N^0} - \varepsilon & 0 & -\frac{\beta_2 S_h}{N^0} \\
\frac{\beta_1 f_v}{N^0} & -\frac{\beta_1 S_h}{N^0} & \frac{\beta_1}{N^0} \\
0 & \frac{\beta_2 S_h}{N^0} & -\frac{\beta_2 I_h}{N^0} - d_2
\end{pmatrix},
\]

where \( S_h \) and \( I_v \) are given in Eq. (5). The characteristic equation at \( E(S_h, I_h, I_v) \) is

\[
\lambda^3 + B_2 \lambda^2 + B_1 \lambda + B_0 = 0,
\]

where

\[
B_2(I_h) = -\text{tr}(J(E)), \quad B_0(I_h) = -\text{det}(J(E))
\]

\[
B_1(I_h) = \left| \begin{array}{ccc}
\frac{\beta_1 f_v}{N^0} + \varepsilon & 0 & -\varepsilon - \frac{\beta_1 S_h}{N^0} \\
\frac{\varepsilon}{N^0} & -\varepsilon - \frac{\beta_1 S_h}{N^0} & \frac{\beta_1}{N^0} \\
0 & \frac{\beta_2 S_h}{N^0} & -\frac{\beta_2 I_h}{N^0} - d_2
\end{array} \right|.
\]

\[\text{Lemma 4.1.} \ E_1 \text{ is unstable if it exists, and } \dim W^s_{loc}(E_1) = 2, \dim W^u_{loc}(E_1) = 1.\]

\[\text{Proof.} \text{ Firstly we prove the assertion: for any positive equilibrium } E(S_h, I_h, I_v),
\]

\[
B_0 = -\text{det}(J(E)) = \frac{f'(I_h)I_h}{(I_h + b)(N^0)^2}.
\]

Let \( \tilde{f}_1(S_h, I_h, I_v) = \varepsilon(N^0 - S_h - I_h) - \mu(I_h, b)I_h \), \( \tilde{f}_2(S_h, I_h, I_v) \) and \( \tilde{f}_3(S_h, I_h, I_v) \) be the right hand side of the last two equations of system (3). Solve system

\[
\tilde{f}_1(S_h, I_h, I_v) = 0, \quad \tilde{f}_3(S_h, I_h, I_v) = 0
\]

for \( S_h \) and \( I_v \) in terms of \( I_h \), and we obtain equation (5). Substituting (5) into \( \tilde{f}_2 \) yields the function

\[
F(I_h) = \tilde{f}_2(S_h(I_h), I_h, I_v(I_h)). \tag{18}
\]

By the implicit function theorem, we have

\[
F'(I_h) = \frac{\partial \tilde{f}_2}{\partial S_h} \frac{dS_h}{dI_h} + \frac{\partial \tilde{f}_2}{\partial I_h} + \frac{\partial \tilde{f}_2}{\partial I_v} \frac{dI_v}{dI_h} = \frac{\partial \tilde{f}_2}{\partial S_h} \left( \frac{\partial f_1, f_3}{\partial (S_h, I_h, I_v)} \right) + \frac{\partial \tilde{f}_2}{\partial I_h} - \frac{\partial \tilde{f}_2}{\partial I_v} \left( \frac{\partial f_1, f_3}{\partial (S_h, I_h, I_v)} \right)
\]

\[
= \frac{\partial (f_1, f_3)}{\partial (S_h, I_h, I_v)} + \frac{\partial (f_1 - f_2, f_3)}{\partial (S_h, I_h, I_v)} = \frac{\partial \tilde{f}_2}{\partial (S_h, I_h, I_v)} \left( d_2 + \frac{\beta_2 I_h}{N^0} \right) \varepsilon. \tag{19}
\]

Note that

\[
F(I_h) = \frac{-I_h f(I_h)}{(I_h + b)(d_2 N^0 + \beta_2 I_h) \varepsilon N^0}, \quad F'(I_h) = \frac{-I_h f'(I_h)}{(I_h + b)(d_2 N^0 + \beta_2 I_h) \varepsilon N^0}. \tag{20}
\]
The last equality holds because \( f(I_h) = 0 \), where \( f(I_h) \) is defined in Eq. (6). Combining equation (19) and the second equation of (20), we obtain the desired assertion.

Since \( \mu I_h' = \mu I_h + \mu' I_h I_h = \mu_0 + \frac{(\mu_1 - \mu_2)\theta^2}{(1 + b\theta)^2} > 0 \), then \( B_2 > 0 \). Notice that \( f'(I_h) < 0 \) when \( I_h = I_{h1} \), by the assertion, so we have \( B_0 = -\det(J(E_1)) < 0 \), and \( J(E_1) \) has at least one positive eigenvalue, say \( \lambda_1 > 0 \). Let \( \lambda_i (i = 2, 3) \) be the other two eigenvalues of \( J(E_1) \). Because \( \lambda_1 \lambda_2 \lambda_3 = -B_0 > 0 \), \( \lambda_2 \) and \( \lambda_3 \) are either real with same nonzero sign or complex conjugate. Note that \( \lambda_1 + \lambda_2 + \lambda_3 = -B_2 < 0 \), so \( \lambda_2 \) and \( \lambda_3 \) are either negative or complex conjugate with negative real parts. By the stable manifold theorem, we have \( \dim W^s_{loc}(E_1) = 2 \), \( \dim W^u_{loc}(E_1) = 1 \), and \( E_1 \) is unstable.

By the assertion in Lemma 4.1, one has \( B_0 > 0 \) at \( E_2 \). Hence, \( J(E_2) \) always has at least one real negative eigenvalue, and the other two eigenvalues either real with same nonzero sign or complex conjugate. By Routh-Hurwitz criteria, \( E_2 \) is stable if \( B_2 B_1 - B_0 > 0 \); Otherwise, it is unstable. The change of stability of \( E_2 \) is related to the Hopf bifurcation. Define

\[
D := D(I_h) = B_2(I_h)B_1(I_h) - B_0(I_h).
\] (21)

By substituting \( I_h = I_{h2} \) into \( D(I_h) \), we consider \( D = D(\varepsilon, b) \) dependent of \( \varepsilon \) and \( b \).

**Theorem 4.2.** Assume that \( S_h^0 < N^0 \) and \( 0 < \varepsilon \ll 1 \).

(a) \( E_2 \rightarrow (S_h^0, 0, 0) \) as \( \varepsilon \rightarrow 0 \).

(b) There exist a \( C^\infty \) function \( b = \phi(\varepsilon) \) such that \( \tilde{b} = \phi(0) > 0 \) and \( D(\varepsilon, \phi(\varepsilon)) = 0 \) for \( 0 < \varepsilon \ll 1 \).

(c) A Hopf bifurcation occurs for \( b = \phi(\varepsilon) \) at \( E_2 \). Furthermore, \( E_2 \) is stable (resp. unstable) if \( b < \phi(\varepsilon) \) (resp. \( b > \phi(\varepsilon) \)) in a small neighborhood of \( (\varepsilon, b) = (0, \tilde{b}) \).

**Proof.** (a). System (7) has a turning point if \( S_h^0 < N^0 \). For \( \varepsilon > 0 \) sufficiently small, by Remark 3 and Theorem 2.3, we have \( R_0 > 1 \). Hence, system (3) has a unique endemic equilibrium \( E_2 \). By direct calculation where L’Hopital’s rule is applied, \( E_2 \) tends to the turning point \((S_h^0, 0, 0)\) as \( t \rightarrow \infty \).

(b) and (c). For the sake of simplicity, let \( \xi_1 = \beta_1 \beta_2 M^0, \xi_2 = \mu_1 d_2 N^0 \). Then we have \( \xi_1 > \xi_2 \) due to \( S_h^0 < N^0 \). We rewrite asymptotic expansion of \( D(\varepsilon, b) \) in \( \varepsilon \) as below:

\[
D(\varepsilon, b) = \Delta_1(b)\varepsilon + O(\varepsilon^2) = (\Delta_1(b) + O(\varepsilon))\varepsilon,
\]

where

\[
\Delta_1(b) = (d_2^2 + \mu_1^2)M^0 + \frac{(d_2 + \mu_1)(\xi_1 - \xi_2)}{\xi_1} + \frac{d_2 \mu_1 - \mu_0 (\xi_1 + \xi_2)}{\xi_1 \xi_2} \cdot \frac{(\mu_1 - \mu_0)(d_2 + \mu_1)d_2^2(N^0)^2(\xi_1 - \xi_2)}{\xi_1 \xi_2 b}.
\]

It is clear that \( \Delta_1(b) = 0 \) has a unique root, say \( \tilde{b} \), which is positive, and

\[
\frac{\partial \Delta_1}{\partial b}(\tilde{b}) = \frac{(\mu_1 - \mu_0)(d_2 + \mu_1)d_2^2(N^0)^2(\xi_1 - \xi_2)}{\xi_1 \xi_2} \cdot \frac{1}{\tilde{b}^2} \neq 0.
\]

By implicit function theorem, there exist a \( C^\infty \) function \( b = \phi(\varepsilon) \) such that \( b = \tilde{b}(0) \) and \( \Delta_1(\phi(\varepsilon)) + O(\varepsilon) = 0 \) for \( 0 \leq \varepsilon \ll 1 \). The real part of complex conjugate eigenvalues \( \lambda_2 \) and \( \lambda_3 \) of \( J(E_2) \) vanishes for \( b = \phi(\varepsilon) \), and the transversality condition
holds as \( b \) goes through \( b = \phi(\varepsilon) \). \( E_2 \) has different stability on the two sides of the curve of \( b = \phi(\varepsilon) \). Hence, we complete the proof.

**Remark 4.** The Hopf bifurcation and the transition of small-amplitude periodic orbits to canard cycles through the canard explosion near the canard point were studied in [17]. The expression of Hopf bifurcation was derived in Eq. (4.15) of [17] is just the function \( b = \phi(\varepsilon) \) in Theorem 4.2. As an example of Hopf bifurcation curve, we choose \( d_2 = 0.9, \beta_1 = 0.09, \beta_2 = 0.1, \mu_1 = 0.052, \mu_0 = 0.05, M = 10000, N = 1000, \) and plot \( b = \phi(\varepsilon) \) in Fig. 5 (a).

\[ \begin{align*}
\text{Figure 5.} & \quad \text{(a) Hopf bifurcation curve in (\( \varepsilon, b \))-plane.} \\
& \quad \text{(b) Hopf bifurcation curve in (\( \beta, b \))-plane, where} \quad \varepsilon = 4 \times 10^{-5}, d_2 = 0.02, \mu_0 = 0.03, \mu_1 = 0.0305, N = 10000, M = 250000, \beta_2 = 0.025. \\
\end{align*} \]

In Theorem 4.2, we considered Hopf bifurcation in the limiting process \( \varepsilon \to 0 \). If we hold \( \varepsilon > 0 \) fixed, Hopf bifurcation may still exists. As an illustration, we choose \( \beta_1 \) and \( b \) as parameters, fix \( \varepsilon = 4 \times 10^{-5} \) and other parameters as indicated in Fig. 5 (b). Then equation \( D(I_{h_2}) = 0 \) defines a curve \( C_h \) of Hopf bifurcation in \((\beta_1, b)\) parameter space. \( E_2 \) has different stability on the two sides of \( C_h \), and Hopf bifurcation occurs when crossing \( C_h \) transversally at which \( E_2 \) has a pair of pure imaginary roots and the transversality condition holds. The Hopf bifurcation curve terminates at two points \( BT^- \) and \( BT^+ \) at which \( E_1 \) and \( E_2 \) coalesce into \( E^* \), and \( J(E^*) \) has a double zero eigenvalues. There is a point \( H^2 \) on \( C_h \) at which the degenerate Hopf bifurcation occurs. The point \( H^2 \) separate \( C_h \) into two arcs \( H^2 BT^- \) and \( H^2 BT^+ \) on which supercritical and subcritical Hopf bifurcation occur, respectively. It is well know that there is a curve of saddle-node bifurcation of limit cycles near point \( H^2 \), and two hyperbolic limit cycles exist in a cone-shaped neighborhood of \( H^2 \).

**Multiple limit cycles.** In Fig. 6 (a), \( E_2 \) is unstable. The orbit starting at the point \((8200, 2.07, 64.8)\) (blue curve) spirals inward, and the orbit starting at the point \((8350, 2.07, 64.8)\) (green curve) spirals outwards. So one stable limit cycle exists, which is bifurcated from the supercritical Hopf bifurcation. In Fig. 6 (b), \( E_2 \) is stable. The orbit starting at \((9836, 0.14, 4.5)\) (blue curve) spirals inward, the orbit starting at \((9846, 0.14, 4.5)\) (green curve) spirals outwards, and the orbit starting at the point \((9860, 0.14, 4.5)\) (pink curve) converges to \( E_2 \). So two limit cycles exist near \( E_2 \), in which the inner one is unstable and the outer one is stable. They are bifurcated from the supercritical and subcritical Hopf bifurcation, respectively.
4.2. Cusp singularity of codimension 2.

**Theorem 4.3.** Assume that \( \varepsilon > 0 \) and
\[
\begin{align*}
f(I_h^*) = f'(I_h^*) = f''(I_h^*) = 0, B_1(I_h^*) &= 0,
\end{align*}
\]
then \( E^* = (S_h^*, I_h^*, I_v^*) \) is a cusp singularity. If \( D'(I_h^*) \neq 0 \), system (3) localized at \( E^* \) is \( C^\infty \) topologically equivalent to
\[
\begin{align*}
\dot{x} &= y + O(|x, y|^3), \\
\dot{y} &= -x^2 - \text{sgn}(D'(I_h^*))xy + O(|x, y|^3), \\
\dot{z} &= \text{tr}(J(E^*))z + O(z^2).
\end{align*}
\]

Hence, \( E^* \) is a cusp of codimension 2. Moreover,
- If \( D'(I_h^*) < 0 \), \( E^* \) is a supercritical Bogdanov-Takens bifurcation point;
- If \( D'(I_h^*) > 0 \), \( E^* \) is a subcritical Bogdanov-Takens bifurcation point.

If \( D'(I_h^*) = 0 \), \( E^* \) is a cusp with codimension at least 3.

**Proof.** From the assertion in Lemma 4.1, \( f(I_h^*) = f'(I_h^*) = 0 \) implies that \( E^* \) emerges from coalescence of \( E_1 \) and \( E_2 \), and \( B_0(I_h^*) = 0 \). If \( B_1(I_h^*) = 0 \), \( J(E^*) \) has three eigenvalues
\[
\lambda_{1,2} = 0 \quad \text{and} \quad \lambda_3 = \text{tr}(J(E^*)) < 0.
\]
From linear algebra, there exists a nonsingular \( 3 \times 3 \) matrix \( T \) such that the affine mapping
\[
(x, y, z) = (S_h - S_h^*, I_h - I_h^*, I_v - I_v^*)T
\]
brings system (3) to
\[
\begin{align*}
\frac{dx}{dt} &= y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + O(xz, yz, |x, y, z|^3) \\
\frac{dy}{dt} &= b_{20}x^2 + b_{11}xy + b_{02}y^2 + O(xz, yz, |x, y, z|^3), \\
\frac{dz}{dt} &= \text{tr}(J(E^*))z + O(|x, y, z|^2).
\end{align*}
\]
The expressions of coefficients $a_{ij}$ and $b_{ij}$ are complicated. In order to reduce the text we explain the method and present certain expressions if necessary. By the center manifold theorem, there exists a two dimensional local center manifold

$$W^c_{loc}(E^*) = \{(x, y, z)| z = O(|x, y|^2)\}.$$ 

Then on the center manifold, system (23) reduces to

$$\begin{cases}
\dot{x} = y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + O(|x, y|^3), \\
\dot{y} = b_{20}x^2 + b_{11}xy + b_{02}y^2 + O(|x, y|^3).
\end{cases}$$

(24)

For $(x, y)$ near $(0, 0)$ the diffeomorphism

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x + \frac{1}{2}(a_{11} + b_{20})x^2 + a_{02}xy + O(|x, y|^3) \\
y - a_{20}x^2 + b_{02}xy + O(|x, y|^3) \end{pmatrix}$$

eliminates the non-resonant quadratic terms in system (24), and we obtain that

$$\begin{cases}
\dot{x} = y + O(|x, y|^3), \\
\dot{y} = \sigma_1 x^2 + \sigma_2 xy + O(|x, y|^3),
\end{cases}$$

(25)

where $\sigma_1 = b_{20}$, $\sigma_2 = b_{11} + 2a_{20}$. More explicitly,

$$\sigma_1 = \frac{-\frac{\beta_1 l_v}{N^0} - \varepsilon \frac{\beta_1 S_h}{N^0} - \frac{\beta_2 l_v}{N^0} - d_2}{\frac{\beta_1 l_v}{N^0} - \frac{\beta_1 S_h}{N^0} - \frac{\beta_2 l_v}{N^0} - d_2},$$

$$\sigma_2 = \frac{-\frac{\beta_1 l_v}{N^0} - \varepsilon \frac{\beta_1 S_h}{N^0} - \frac{\beta_2 l_v}{N^0} - d_2}{\frac{\beta_1 l_v}{N^0} - \frac{\beta_1 S_h}{N^0} - \frac{\beta_2 l_v}{N^0} - d_2} \frac{2(trJ(E^*))}{(trJ(E^*))^2} D'(I^*_h).$$

Here, $F(I_h)$ and $D(I_h)$ are defined by Eq. (18) and Eq. (21). Note that

$$\left| \begin{array}{cc}
-\frac{\beta_1 l_v}{N^0} - \varepsilon & -\frac{\beta_1 S_h}{N^0} - \frac{\beta_2 l_v}{N^0} - d_2 \\
0 & -\frac{\beta_1 l_v}{N^0} - \frac{\beta_1 S_h}{N^0} - \frac{\beta_2 l_v}{N^0} - d_2
\end{array} \right| > 0, \quad \left| \begin{array}{cc}
-\frac{\beta_1 l_v}{N^0} - \varepsilon & -\frac{\beta_1 S_h}{N^0} - \frac{\beta_2 l_v}{N^0} - d_2 \\
0 & -\frac{\beta_1 l_v}{N^0} - \frac{\beta_1 S_h}{N^0} - \frac{\beta_2 l_v}{N^0} - d_2
\end{array} \right| < 0,$$

and

$$F''(I^*_h) = \frac{-f'(I^*_h)f''(I^*_h)}{(f'(I^*_h))^2} < 0.$$

Hence, $\sigma_1 < 0$. The rescaling

$$(x, y, t) \to (\sigma_1 \frac{\sigma_2}{\sigma_1} \text{sgn}(\sigma_2)x, \sigma_2 \frac{\sigma_2}{\sigma_1} \text{sgn}(\sigma_2)y, \frac{\sigma_2}{\sigma_1} \text{sgn}(\sigma_2)t)$$

brings system (25) to

$$\begin{cases}
\dot{x} = y + O(|x, y|^3), \\
\dot{y} = -x^2 - \text{sgn}(D'(I^*_h))xy + O(|x, y|^3).
\end{cases}$$

Hence, system (3) localized at $E^*$ is $C^\infty$ topologically equivalent to system (22). □

The existence of cusp singularity of codimension 2 implies that saddle-node bifurcation, homoclinic loop bifurcation and Hopf bifurcation may occur when perturbing system (3) by two parameters. For system (3), it is technically difficult to develop an universal unfolding of the cusp singularity in terms of model parameters. The reason is that any two parameters cannot be solved explicitly from the conditions $f(I^*_h) = f'(I^*_h) = 0$. Nevertheless, we synthesize Fig. 1 (a), Fig. 5 (b), Theorem 4.3, we sketch the bifurcation diagram in the $(\beta_1, b)$-plane. The bifurcation diagram presented in Fig. 7 is the simplest and compatible with all constrains. Phase portraits on $W^c(E^*)$ in generic regions and
on bifurcation curves are also plotted. Here, $\beta_1 = \hat{\beta}_1$ corresponds to the threshold $R_0 = 1$.

Table: Description of the bifurcation curves of Fig. 7.

| Notation | Description                                                                 |
|----------|------------------------------------------------------------------------------|
| $SN^{\pm}$ | repelling (attracting) saddle-node bifurcation                               |
| $H^{\pm}$ | subcritical (supercritical) Hopf bifurcation                                 |
| $HO^{\pm}$ | repelling (attracting) homoclinic loop bifurcation                           |
| $SN_{lc}$ | saddle-node bifurcation of limit cycles                                     |
| $BT^{\pm}$ | subcritical (supercritical) Bogdanov-Taken bifurcation                       |
| $H^2$    | degenerate Hopf bifurcation                                                 |
| $HO^2$   | degenerate homoclinic loop bifurcation                                       |
| $C$      | a point where a homoclinic loop and a weak focus of order 1 coexist         |

It is easy to see from the Fig. 7 that limit cycles can be generated and annihilated through Hopf bifurcation, homoclinic bifurcation, saddle-node bifurcation of limit cycles. In Fig. 8 we choose different values for $\beta_1$, and plot variables $S_h$ and $I_h$ of period solutions against the parameter $b$. For other parameters, they are held fixed as those in Fig. 5 (b). In Fig. 8 (a), the stable limit cycles appear and disappear through two supercritical Hopf bifurcations. In Fig. 8 (b), an unstable limit cycle (pink curve) appear from a subcritical Hopf bifurcation and a stable limit cycle appears from an attracting homoclinic loop bifurcation, and they coalesce into a semi-stable limit cycles and then disappear through a saddle-node bifurcation of limit cycles. In Fig. 8 (c) a stable limit cycle is generated through a supercritical Hopf bifurcation and annihilated through an attracting homoclinic loop bifurcation. In Fig. 8 (d) an unstable limit cycle is generated through a subcritical Hopf bifurcation and annihilated through a repelling homoclinic loop bifurcation. Here, $\beta_1 = 0.00115$ in Fig. 8 (a), $\beta_1 = 0.0009760249$ in (b) and (c), and $\beta_1 = 0.00097$ in (d).

5. **Biological interpretations.** We will explain the relations between the recurrence of diseases and oscillations of system (3) in this section.

*Threshold dynamics and oscillations.* $R_0$ serves as a threshold for the dynamics of system (3). The relaxation oscillation cycles exist only when $R_0 > 1$. The
homoclinic loop bifurcations occurs only when $R_0 < 1$. While for Hopf bifurcation and saddle-node bifurcation of limit cycles, they may occur for $R_0$ smaller, equal to and greater than 1.

Cyclic phenomena. Different cyclic phenomena can be interpreted as periodic solutions perturbed. For slow-fast system (3), relaxation oscillation cycles may be generated from the bifurcation of slow-fast cycle, which is global. In addition, there are other nonlinear oscillations can be generated through local bifurcation such as Hopf bifurcation, and semi-local bifurcation such as homoclinic bifurcation and saddle-node bifurcation of limit cycles. Vice verse, limit cycles including relaxation oscillation cycles can be annihilated through these local, semi-local or global bifurcations.

Inter-epidemic period and endemic period. The existence of relaxation oscillation cycles was proven for $\varepsilon > 0$ small by the geometric singular perturbation theory and delay of stability loss. Relaxation oscillation cycles have clear biological interpretations. Notice that periodic solutions (relaxation oscillation cycles) in the vicinity of the slow manifold move with the speed of $O(\varepsilon)$, and the slow manifold $S_h$-axis is the disease-free region, so the time period a relaxation oscillation cycle spends in the vicinity of the $S_h$-axis corresponds to the inter-epidemic period with low disease incidence, which is of $O(1/\varepsilon)$. The time period a relaxation oscillation cycle spends

Figure 8. Generation and annihilation of limit cycles. Green curves signify stable limit cycles and pink curves signify unstable limit cycles.
along the fast orbits is the endemic period, which is of $O(1)$. See Fig. 9. For the limit cycle generated through local bifurcations, it could be interpreted as small fluctuation of endemic due to its small amplitude. For the limit cycle generated through semi-local bifurcations such as homoclinic loop bifurcation or double limit cycle bifurcation, it could be interpreted as regular oscillations of endemic because lengths of inter-epidemic period and endemic period are of the same scale.

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REFERENCES

[1] A. Abdelrazec, J. Belair, C. Shan and H. Zhu, Modeling the spread and control of dengue with limited public health resources, Math. Biosci., 271 (2016), 136–145.
[2] R. M. Anderson and R. M. May, Infectious Diseases of Humans: Dynamics and Control, Oxford Science Publications, Oxford University Press, Oxford, UK, 1992.
[3] E. Benoît, Linear dynamic bifurcation with noise, in: E. Benoît (Ed.), Dynamic Bifurcations, Luminy, 1990, in: Lecture Notes in Math., vol.1493, Springer, Berlin, 1991, 131–150.
[4] CDC, West Nile virus final annual maps & data for 1999-2018, https://www.cdc.gov/westnile/statsmaps/finalmapsdata/index.html.
[5] S.-N. Chow, W. Liu and Y. Yi, Center manifold theory for smooth invariant manifolds, Trans. Amer. Math. Soc., 352 (2000), 5179–5211.
[6] S.-N. Chow, W. Liu and Y. Yi, Center manifold theory for invariant sets, J. Differential Equations, 168 (2000), 355–385.
[7] P. De Maesschalck and F. Dumortier, Time analysis and entry-exit relation near planar turning points, J. Differential Equations, 215 (2005), 225–267.
[8] P. De Maesschalck, F. Dumortier and R. Roussarie, Cyclicity of common slow-fast cycles, Indag. Math., 22 (2011), 165–206.
[9] P. De Maesschalck and S. Schecter, The entry-exit function and geometric singular perturbation theory, J. Differential Equations, 260 (2016), 6697–6715.
[10] N. Fenichel, Persistence and smoothness of invariant manifolds for flows, Indiana Univ. Math. J., 21 (1971), 193–226.
[11] N. Fenichel, Geometric singular perturbation theory for ordinary differential equations, J. Differential Equations, 31 (1979), 53–98.
[12] H. W. Hethcote, Asymptotic behavior in a deterministic epidemic model, Bull. Math. Biol., 35 (1973), 607–614.
[13] H. W. Hethcote, H. W. Stech and P. Van Den Driessche, Nonlinear oscillations in epidemic models, *SIAM J. Appl. Math.*, 40 (1981), 1–9.

[14] T.-H. Hsu, Number and stability of relaxation oscillations for predator-prey systems with small death rates, *SIAM J. Appl. Dyn. Syst.*, 18 (2019), 33–67.

[15] J. Huang, S. Ruan, P. Yu and Y. Zhang, Bifurcation analysis of a mosquito population model with a saturated release rate of sterile mosquitoes, *SIAM J. Appl. Dyn. Syst.*, 18 (2019), 939–972.

[16] M. Krupa and P. Szumolyn, Extending geometric singular perturbation theory to nonhyperbolic points-fold and canard points in two dimensions, *SIAM J. Math. Anal.*, 33 (2001), 286–314.

[17] M. Krupa and P. Szumolyn, Relaxation oscillation and canard explosion, *J. Differential Equations*, 174 (2001), 312–368.

[18] J. P. LaSalle, *The Stability of Dynamical Systems*, Regional Conference Series in Appl. Math., 25, SIAM, Philadelphia, 1976.

[19] C. Li, J. Li, Z. Ma and H. Zhu, Canard phenomenon for an SIS epidemic model with nonlinear incidence, *J. Math. Anal. Appl.*, 420 (2014), 987–1004.

[20] M. Li, W. Liu, C. Shan and Y. Yi, Turning points and relaxation oscillation cycles in epidemic models, *SIAM J. Appl. Math.*, 76 (2016), 663–687.

[21] W. Liu, Exchange lemmas for singularly perturbation problems with certain turning points, *J. Differential Equations*, 167 (2000), 134–180.

[22] W. Liu, Geometric singular perturbations for multiple turning points: Invariant manifolds and exchange lemmas, *J. Dynam. Differential Equations*, 18 (2006), 667–691.

[23] W. Liu, Simon A. Levin and Y. Iwasa, Influence of nonlinear incidence rates upon the behavior of SIRS epidemiological models, *J. Math. Biol.*, 23 (1986), 187–204.

[24] M. Lu, J. Huang, S. Ruan and P. Yu, Bifurcation analysis of an SIRS epidemic model with a generalized nonmonotone and saturated incidence rate, *J. Differential Equations*, 267 (2019), 1859–1898.

[25] W. P. London and J. A. Yorke, Recurrent outbreaks of measles, chickenpox and mumps. I. Seasonal variation in contact rates, *Am. J. Epidemiol.*, 98 (1973), 453–468.

[26] E. F. Mishchenko, Yu. S. Kolesov, A. Yu. Kolesov and N. Kh. Rozov, *Asymptotic Methods in Singularly Perturbed Systems*, translated from the Russian by I. Aleksanova, Monographs in Contemporary Mathematics, Consultants Bureau, New York, 1994.

[27] N. G. Reich, et al., Interactions between serotypes of dengue highlight epidemiological impact of cross-immunity, *J. R. Soc. Interface.*, 10 (2013), art. no. 0414.

[28] S. Ruan and W. Wang, Dynamical behavior of an epidemic model with a nonlinear incidence rate, *J. Differential Equations*, 188 (2003), 135–163.

[29] S. Schecter, Persistent unstable equilibria and closed orbits of a singularly perturbed equation, *J. Differential Equations*, 60 (1985), 131–141.

[30] C. Shan and H. Zhu, Bifurcations and complex dynamics of an SIR model with the impact of the number of hospital beds, *J. Differential Equations*, 257 (2014), 1662–1688.

[31] C. Shan, Y. Yi and H. Zhu, Nilpotent singularities and dynamics in an SIR type of compartmental model with hospital resources, *J. Differential Equations*, 260 (2016), 4339–4365.

[32] M. Wechselberger, *Geometric Singular Perturbation Theory Beyond the Standard Form*, Springer Nature Switzerland AG, 2020.

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E-mail address: chunhua.shan@utoledo.edu