Phase Transition of the Ising model on a Hyperbolic Lattice

Takatsugu Iharagi\textsuperscript{1)}, Andrej Gendiar\textsuperscript{2,3)}, Hiroshi Ueda\textsuperscript{4)}, and Tomotoshi Nishino\textsuperscript{1)}

\textsuperscript{1}Department of Physics, Graduate School of Science, Kobe University, Kobe 657-8501, Japan
\textsuperscript{2}Institute of Electrical Engineering, Slovak Academy of Sciences, Dúbravská cesta 9, SK-841 04, Bratislava, Slovakia
\textsuperscript{3}Institute of Physics, Slovak Academy of Sciences, Dúbravská cesta 9, SK-845 11, Bratislava, Slovakia
\textsuperscript{4}Department of Material Engineering Science, Graduate School of Engineering Science, Osaka University, Osaka 560-8531, Japan

The matrix product structure is considered on a regular lattice in the hyperbolic plane. The phase transition of the Ising model is observed on the hyperbolic (5, 4) lattice by means of the corner-transfer-matrix renormalization group (CTMRG) method. Calculated correlation length is always finite even at the transition temperature, where mean-field like behavior is observed. The entanglement entropy is also always finite.

KEYWORDS: DMRG, CTMRG, Hyperbolic, Entanglement

1. Introduction

Classification of phase transitions is one of the central issue in the study of lattice models in statistical mechanics. When a system exhibits the second-order transition, normally the correlation length diverges at the transition point. As a result of scale invariance at the criticality, the transitions are characterized by scaling indices, where their values are completely classified in two-dimension by means of the conformal field theory.

The mean-field like 2nd-order transition is exceptional in the point that the correlation length may not play an important role, in particular when the transition is described by the Landau free energy that is expressed as a simple polynomial of the order parameter.\textsuperscript{1,1)} In this article we focus on the mean-field like transition observed for the Ising model on the hyperbolic lattices,\textsuperscript{2–11)} the regular lattice in two-dimensional (2D) plane with constant negative curvature.\textsuperscript{12)} Among the hyperbolic (p, q)-lattices, which are the tessellations of the regular p-gons with the coordination number q,\textsuperscript{13)} we consider the (5, 4)-lattice shown in Fig. 1 as an example. We calculate the correlation length \( \xi \) and entanglement entropy \( S \) in the neighborhood of the second-order transition temperature \( T_0 \), and judge whether or not the system is critical at this temperature.

In the next section we explain the matrix product structure of the Ising model on the (5, 4)-lattice. We employ the corner transfer matrix renormalization group (CTMRG) method,\textsuperscript{14,15)} a variant of the density matrix renormalization group (DMRG) method\textsuperscript{16–18)} applied to 2D classical models,\textsuperscript{19)} to obtain the thermodynamic properties of the model. We show the calculated results on \( \xi \) and \( S \) in \S 3. Conclusions are summarized in the last section.

2. Matrix Product Structure on the Hyperbolic Lattice

Consider the ferromagnetic Ising model on the (5, 4)-lattice shown in Fig. 1. Each pair of neighboring sites is on a geodesic, which is drawn either by a line that passes through the center of the disk or by an arc. When there is no external magnetic field, the Hamiltonian is given by

\begin{equation}
H = -J \sum_{ij} \sigma_i \sigma_j, \tag{2.1}
\end{equation}

where \( \sigma_i = \pm 1 \) denotes the Ising spin variable at the \( i \)-th site, and where \( J > 0 \) is the coupling strength between neighboring pair of sites denoted by \( \langle ij \rangle \). It is convenient to introduce the interaction-round-a-face (IRF) Boltzmann weight

\begin{equation}
W_{ijklm} = \exp \left[ -\beta J \left( \sigma_i \sigma_j + \sigma_j \sigma_k + \sigma_k \sigma_l + \sigma_l \sigma_m + \sigma_m \sigma_i \right) \right] \tag{2.2}
\end{equation}

for each pentagon, where \( i,j,k,l,m \) denote the sites around it, and where \( \beta \) represents the inverse temperature. The partition function of the system is then ex-
where the product is taken over all the pentagons denoted by the group of sites \( \langle i j k l m \rangle \), and where the sum is taken over all the possible spin configurations. We assume that the system is sufficiently large, and we consider the thermal equilibrium state deep inside the system, where the part is far from the system boundary.

Let us observe the structure of the \((5,4)\)-lattice. The thick straight \((horizontal)\) line drawn in Fig. 1 divides the whole lattice into upper and lower halves. Let us introduce the new labeling

\[
\ldots, \sigma^{(0)}_{\ell + 1}, \sigma^{(0)}_{\ell}, \sigma^{(0)}_{\ell - 1}, \sigma^{(0)}_{\ell - 2}, \ldots
\]

from left to right for those spins on this line. We use the notation \( \sigma^{(0)} \) when we refer to the whole part of this \textit{horizontal} row of spins.

We next observe the thick arc that is perpendicular to the horizontal line we have considered, and that passes through the \( \ell \)-th site \( \sigma^{(0)}_{\ell} \). We label those spins on this \textit{vertical} geodesic as

\[
\ldots, \sigma^{(-1)}_{\ell}, \sigma^{(0)}_{\ell}, \sigma^{(1)}_{\ell}, \sigma^{(2)}_{\ell}, \ldots
\]

from downward to upward. We use the notation \( \sigma^{j}_{\ell} \) when we refer to the whole part of this \textit{vertical} column of spins. For the latter convenience, we introduce the half-infinite spin columns

\[
\begin{align*}
\sigma^{D}_{\ell} &= \ldots, \sigma_{\ell}^{(-3)}, \sigma_{\ell}^{(-2)}, \sigma_{\ell}^{(-1)} \\
\sigma^{U}_{\ell} &= \sigma_{\ell}^{(1)}, \sigma_{\ell}^{(2)}, \sigma_{\ell}^{(3)}, \ldots
\end{align*}
\]

for the lower and the upper part of the column spin \( \sigma^{j}_{\ell} \). It should be noted that the spins

\[
\ldots, \sigma^{(j)}_{\ell + 1}, \sigma^{(j)}_{\ell}, \sigma^{(j)}_{\ell - 1}, \sigma^{(j)}_{\ell - 2}, \ldots
\]

for \( j \neq 0 \) are not on a geodesic, and therefore they cannot be regarded as a row spin. We have not explicitly shown the system boundary, and have not given any special label to the boundary spins. When the system is off critical, we do not have to consider about the system boundary so strictly.

We introduce the column-to-column transfer matrix

\[
T_{\ell + 1, \ell} = T(\sigma^{D}_{\ell + 1}, \sigma^{j}_{\ell})
\]

which is created by multiplying all the IRF weights inside the \textit{stripes} between \( \ell + 1 \)-th and \( \ell \)-th \textit{vertical} geodesics, and taking spin configuration sum except for those column spins \( \sigma^{D}_{\ell + 1} \) and \( \sigma^{j}_{\ell} \). According to the division of the column spin

\[
\sigma^{j}_{\ell} = \sigma^{D}_{\ell}, \sigma^{(0)}_{\ell}, \sigma^{U}_{\ell}
\]

we can also express the column to column transfer matrix as a product of its upper and lower parts

\[
T_{\ell + 1, \ell} = T(\sigma^{D}_{\ell + 1}, \sigma_{\ell + 1}^{D}, \sigma_{\ell}^{U}) = P_{\ell + 1, \ell}^{D} P_{\ell + 1, \ell}^{U}
\]

where \( P_{\ell + 1, \ell}^{D} \) and \( P_{\ell + 1, \ell}^{U} \) are half-column transfer matrices (HCTMs)

\[
P_{\ell + 1, \ell}^{D} = P(\sigma_{\ell + 1}^{D}, \sigma_{\ell}^{D}, \sigma^{(0)}_{\ell})
\]

\[
P_{\ell + 1, \ell}^{U} = P(\sigma^{(0)}_{\ell}, \sigma^{U}_{\ell})
\]

Using these notations the lower and the upper halves of the system are represented by the IRF-type matrix products

\[
\Psi^{D}(\sigma^{(0)}_{\ell}) = \sum_{\sigma^{(0)}_{\ell}} P(\sigma^{D}_{\ell + 1}, \sigma^{(0)}_{\ell + 1}) \Psi^{D}(\sigma^{(0)}_{\ell + 1})
\]

\[
\Psi^{U}(\sigma^{(0)}_{\ell}) = \sum_{\sigma^{(0)}_{\ell}} P(\sigma^{(0)}_{\ell}, \sigma^{U}_{\ell}) \Psi^{U}(\sigma^{(0)}_{\ell}),
\]

respectively, where configuration sums are taken for all the \( \sigma^{D}_{\ell} \) and \( \sigma^{U}_{\ell} \). Note that the partition function \( Z \) in Eq. (2.3) is expressed as the inner product between \( \Psi^{D} \) and \( \Psi^{U} \)

\[
Z = \sum_{\sigma^{(0)}_{\ell}} \Psi^{D}(\sigma^{(0)}_{\ell}) \Psi^{U}(\sigma^{(0)}_{\ell}),
\]

where the configuration sum is taken over the horizontal row spins.

Statistical property of the left half of the system can be represented by the successive product of the column-to-column transfer matrices

\[
\Phi^{L}(\sigma^{D}_{0}, \sigma^{(0)}_{0}, \sigma^{U}_{0}) = \Phi^{L}(\prod_{\ell \geq 0} T_{\ell + 1, \ell})
\]

where \( \Phi^{L} \) is a vector of the system boundary that specifies the boundary condition at the left border of the system. In the same manner the property of the right half of the system is represented as

\[
\Phi^{R}(\sigma^{D}_{0}, \sigma^{(0)}_{0}, \sigma^{U}_{0}) = \left( \prod_{\ell \leq 0} T_{\ell, \ell - 1} \right)^{V^{R}}
\]

We assume that a very weak symmetry breaking field is imposed to the boundary spins, though we do not explicitly refer to this condition in the following. Since the lattice structure shown in Fig. 1 is isotropic, it is also possible to represent \( \Phi^{L}(\sigma^{D}_{0}) \) and \( \Phi^{R}(\sigma^{D}_{0}) \) as products of the half-row transfer matrices in the same manner as we have expressed \( \Psi^{D}(\sigma^{(0)}_{0}) \) and \( \Psi^{U}(\sigma^{(0)}_{0}) \) in Eq. (2.12).

Taking a partial contraction between \( \Phi^{L}(\sigma^{D}_{0}) \) and \( \Phi^{R}(\sigma^{D}_{0}) \), we obtain the density matrix

\[
\rho^{D} = \rho(\sigma^{D}_{0} | \sigma^{D}_{0})
\]

\[
= \sum_{\sigma^{D}_{0}, \sigma^{D}_{0}} \Phi^{L}(\sigma^{D}_{0}, \sigma^{D}_{0}, \sigma^{D}_{0}) \Phi^{R}(\sigma^{D}_{0}, \sigma^{D}_{0}, \sigma^{D}_{0})
\]

that is treated in the context of the bipartite entanglement to the vertical direction, or the block diagonal density matrix

\[
\rho(\sigma^{D}_{0} | \sigma^{D}_{0})
\]

(2.17)
Baxter’s variational formulation for the square lattice Ising model. Thus we can apply the CTMRG method, or the DMRG method, to obtain the correlation length \( \xi \) and the entanglement entropy \( S \) of the Ising model on the \((5,4)\) lattice.

The CTMRG method maps the half-column spins \( \sigma^D_\ell \) and \( \sigma^U_\ell \), respectively, to block spins \( \zeta^D_\ell \) and \( \zeta^U_\ell \) by means of the the renormalization group (RG) transformation, which is obtained by the diagonalization of the density matrix in Eq. (2.17), or that in Eq. (2.16). Through this RG transformation, the HCTMs are mapped to the renormalized ones

\[
\tilde{P}^D_{\ell+1,\ell} = \tilde{P}(\zeta^D_{\ell+1}, \sigma^D_{\ell+1}, \sigma^D_{\ell}, \sigma^D_0),
\]

\[
\tilde{P}^U_{\ell+1,\ell} = \tilde{P}(\sigma^D_{\ell+1}, \zeta^U_{\ell+1}, \sigma^U_{\ell}, \zeta^U_0),
\]

where we put \( \sim \) marks on top of renormalized matrices. The column-to-column transfer matrix is renormalized in the same manner

\[
\tilde{T}_{\ell+1,\ell} = \tilde{T}(\zeta^D_{\ell+1}, \zeta^U_{\ell+1}, \zeta^D_{\ell}, \zeta^U_0).
\]

In the previous studies we have shown that the system exhibits the mean-field like second-order phase transition, where the spontaneous magnetization

\[
M = \frac{\sum \rho(\sigma^D_0, \sigma^D_0, \sigma^D_0) \zeta^D_0}{\sum \rho(\sigma^D_0, \sigma^D_0, \sigma^D_0) \zeta^D_0}
\]

below the transition temperature \( T_0 \) is proportional to \( \sqrt{T_0 - T} \).

3. Correlation Length Obtained by CTMRG

The matrix product representation of \( \Psi^D(\sigma^D_0) \) and \( \Psi^U(\sigma^U_0) \) in Eq. (2.12) has the same form as those for the square lattice Ising model. Thus we can apply Baxter’s variational formulation\(^{1,22-24}\) or the DMRG method\(^{16-19}\) for the calculation of the free energy and other thermodynamic functions. In this article we employ the CTMRG method,\(^{14,15}\) a variant of the DMRG method, to obtain the correlation length \( \xi \) and the entanglement entropy \( S \), by means of the the renormalization group (RG) transformation, which is obtained by the diagonalization of the density matrix in Eq. (2.17), or that in Eq. (2.16).

3.1. Correlation Length Obtained by CTMRG

The CTMRG method maps the half-column spins \( \sigma^D_\ell \) and \( \sigma^U_\ell \), respectively, to block spins \( \zeta^D_\ell \) and \( \zeta^U_\ell \) by means of the the renormalization group (RG) transformation, which is obtained by the diagonalization of the density matrix in Eq. (2.17), or that in Eq. (2.16). Through this RG transformation, the HCTMs are mapped to the renormalized ones

\[
\tilde{P}^D_{\ell+1,\ell} = \tilde{P}(\zeta^D_{\ell+1}, \sigma^D_{\ell+1}, \sigma^D_{\ell}, \sigma^D_0),
\]

\[
\tilde{P}^U_{\ell+1,\ell} = \tilde{P}(\sigma^D_{\ell+1}, \zeta^U_{\ell+1}, \sigma^U_{\ell}, \zeta^U_0),
\]

where we put \( \sim \) marks on top of renormalized matrices. The column-to-column transfer matrix is renormalized in the same manner

\[
\tilde{T}_{\ell+1,\ell} = \tilde{T}(\zeta^D_{\ell+1}, \zeta^U_{\ell+1}, \zeta^D_{\ell}, \zeta^U_0).
\]

In the previous studies we have shown that the system exhibits the mean-field like second-order phase transition,\(^{9-11}\) where the spontaneous magnetization

\[
M = \frac{\sum \rho(\sigma^D_0, \sigma^D_0, \sigma^D_0) \zeta^D_0}{\sum \rho(\sigma^D_0, \sigma^D_0, \sigma^D_0) \zeta^D_0}
\]

below the transition temperature \( T_0 \) is proportional to \( \sqrt{T_0 - T} \).

3.2. Correlation Length Obtained by CTMRG

Figure 2 shows the correlation length

\[
\xi = \frac{1}{\log \lambda_0 - \log \lambda_1}
\]

calculated from the largest eigenvalue \( \lambda_0 \) and the second largest one \( \lambda_1 \) of \( T_{\ell+1,\ell} \). We have regarded the interaction parameter \( J \) as the unit of energy, and set the lattice constant as the unit of length. We keep at most \( m = 40 \) states for block spins, and actually \( m = 5 \) is sufficient enough to draw the figure. It is clear that \( \xi \) is of the order of the lattice constant even at the second order transition point \( T_0 = 2.799 \). The fact shows that the system is always off-critical, and thus the transition point cannot be called as the critical point. The entanglement entropy

\[
S = -\sum_i \omega_i \log \omega_i ,
\]

where \( \omega_i \) is the \( i \)-th eigenvalue of the reduced density matrix \( \tilde{\rho}^D = \tilde{\rho}^D(\zeta^D_0 | \zeta^D_0) \) in Eq. (2.16), shown in Fig. 3 is also finite for any temperature \( T \). The fact coincides that a very small number of states \( m = 5 \) is sufficient for getting thermodynamic quantities precisely.\(^{9-11}\)

4. Conclusions and Discussions

We have calculated the correlation function \( \xi \) and the entanglement entropy \( S \) of the Ising model on the hyperbolic \((5,4)\) lattice. Both of them remains finite at the transition temperature \( T_0 \). Therefore the mean-field like second-order phase transition observed for this system is not related to critical phenomena with diverging \( \xi \). This off-critical behavior is common to phase transitions observed in the tensor product formulations,\(^{25-29}\) where the trial state is finitely correlated due to the state number limitation for the block spin variables.

The mean-field nature of the phase transition might be explained by the path integral representation of the Green function on the lattice, if dominant contribution comes from the shortest path between two points. It should be noted that in the hyperbolic plane any deviation from the shortest path causes increase of path length more than that on the flat plane. Such an increase reduces entanglement between two points. It should be noted that the Hausdorff dimension of the \((5,4)\) lattice, or more general the hyperbolic \((p, q)\) lattices, is infinite.

From the path integral picture on the hyperbolic plane,
one reaches a deformation to 1D quantum Hamiltonians, so called the hyperbolic deformation.\textsuperscript{30,31} We conjecture that mean-field behavior also appears in ground-state phase transitions of deformed 1D quantum systems, when $N\lambda$ is sufficiently larger than unity, where $N$ is the system size.

It is known that Kasteleyn formalism is applicable for dimer models on the hyperbolic lattices.\textsuperscript{32} Thus there is a mathematical interest on the hyperbolic lattices to find out a commutable row-to-row or column-to-column transfer matrices.

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