Abstract

We introduce a general framework for continuous-time betting markets, in which a bookmaker can dynamically control the prices of bets on outcomes of random events. In turn, the prices set by the bookmaker affect the rate or intensity of bets placed by gamblers. The bookmaker seeks a price process that maximizes his expected (utility of) terminal wealth. We obtain explicit solutions or characterizations to the bookmaker’s optimal bookmaking problem in various interesting models.

Keywords: HJB equation; optimization; Poisson process; sports betting; stochastic control; utility

1 Introduction

Sports gambling is a large and fast-growing industry. According to a recent report by Zion (2018), the global sports betting market was valued at around 104.31 billion USD in 2017 and is estimated to reach approximately 155.49 billion by 2024. As noted by Street and Smith (2018), growth within the United States is expected to be particularly strong due to a May 2018 ruling by the Supreme Court, which deemed the Professional and Amateur Sports Protection Act (PASPA) unconstitutional and thereby paved the way for states to rule individually on the legality of sports gambling. Less than a year after the PASPA ruling, eleven states had passed bills legalizing gambling on sports compared to just one (Nevada) prior to the PASPA ruling. At the time of the writing of this paper, several additional states have drafted bills to legalize sports betting within their borders.

In this paper, we analyze sports betting markets from the perspective of a bookmaker. A bookmaker sets prices for bets placed on different outcomes of sporting events, collects revenue from these bets, and pays out winning bets. Often the outcomes are binary (team A wins or team B wins). But, some events may have more than two outcomes (e.g., in an association football match, a team may win, lose or tie). Moreover, the outcomes need not be mutually exclusive (e.g., separate bets could be placed on team A winning and player X scoring). Ideally, a bookmaker strives to set prices in such a way that, no matter what the outcome of a particular event, he collects sufficient betting revenue to pay out all winning bets while also retaining some
revenue as profit. However, because a bookmaker only controls the prices of bets – not the number of bets placed on different outcomes – he may lose money on a sporting event if a particular outcome occurs.

Consider, for example, a sporting event with only two mutually exclusive outcomes: A and B. Suppose the bookmaker has received many more bets on outcome A than on outcome B. If outcome A occurs, then the bookmaker would lose money unless the revenue he has collected from bets placed on outcome B is sufficient to cover the payout of bets placed on outcome A. Thus, if a bookmaker finds himself in a situation in which he has collected many more bets on outcome A than on outcome B, he may raise the price of a bet placed on outcome A and/or lower the price of a bet placed on outcome B. This strategy would lower the demand for bets placed on outcome A and increase the demand for bets placed on outcome B. As a result, the bookmakers would reduce the risk that he incurs a large loss, but he would do so at the cost of sacrificing expected profits.

Traditionally, bookmakers would only take bets prior to the start of a sporting event. In such cases, the probabilities of particular outcomes would remain fairly static as the bookmaker received bets. More recently, however, bookmakers have begun to take bets on sporting events as the events occur. In these cases, the probabilities of particular outcomes evolve stochastically in time as the bookmaker takes bets. This further complicates the task of a bookmaker who, in addition to considering the number of bets he has collected on particular outcomes, must also consider the dynamics of the sporting event in play. For example, scoring a point near the end of a basketball game when the score is tied would have a much larger effect on the outcome of the game than scoring a point in the first quarter. And, the effect of scoring a goal in the first half of an association football match would be greater than the effect of scoring a goal early in a game of basketball.

In this paper, we provide a general framework for studying optimal bookmaking, which takes into account the situations described above. In our framework, the probabilities of outcomes are allowed to either be static or to evolve stochastically in time, and bets on any particular outcomes may arrive either at a deterministic rate or at a stochastic intensity. This rate or intensity is a decreasing function of the price the bookmaker sets and an increasing function of the probability that the outcome occurs. In this setting a bookmaker may seek to maximize either expected wealth or expected utility from wealth. We provide numerous examples of in-game dynamics. And we analyze specific optimization problems in which the bookmaker’s optimal strategy can be obtained explicitly.

Existing literature on optimal bookmaking in the context of stochastic control appears to be rather scarce. In one of the few existing papers on the topic, Hodges et al. (2013) consider the bookmaker of a horse race. In their paper, the probability that a given horse wins the race is fixed and the number of bets placed on a given horse is a normally distributed random variable with a mean that is proportional to the probability that the horse will win and inversely proportional to the price set by the bookmaker. In this one-period setting, the bookmaker seeks to set prices in order to maximize his utility from terminal wealth. Unlike our paper, Hodges et al. (2013) do not allow the bookmaker to dynamically adjust prices as bets come in nor do they find an analytic solution for the bookmaker’s optimal control. As far as we are aware, our paper is the first to provide a general framework for studying optimal bookmaking in a dynamic setting.

Similar to our work, Divos et al. (2018) consider dynamic betting during an association football match.
However, rather than approach bookmaking as an optimal control problem, they use a no-arbitrage replication argument to determine the value of a bet whose payoff is a function of the goals scored by each of the two teams. As "hedging assets" they consider bets that pay the final goal tally of the two teams. By contrast, in our work, the bookmaker has no underlying assets to use as hedging instruments.

In certain respects, the optimal bookmaking problem we consider is similar to the optimal market making problem considered by Ho and Stoll (1981); Avellaneda and Stoikov (2008); Guéant and Pham (2013); Guéant (2017); Adrian et al. (2019) and the optimal execution problem analyzed in Guéant et al. (2012); Gatheral and Schied (2013); Bayraktar and Ludkovski (2014); Cartea and Jaimungal (2015); Kratz and Schöneborn (2015). In these papers, a market maker offers limit orders to buy and sell a risky asset whose reference price is a stochastic process. The intensity at which limit orders are filled is a decreasing function of how far below (above) the reference price the limit order to buy (sell) is. In this setting, a market maker seeks to maximize either his expected wealth or his expected utility from wealth, which includes both cash generated from filled limit orders and the value of any holdings in the risky asset. In some cases, the market maker also seeks to minimize his holdings in the risky asset. In a sense, a market maker offering limit orders to buy and sell a risky asset is akin to a bookmaker taking bets on mutually exclusive outcomes during a sporting event. And, a market maker seeking to minimize holdings in a risky asset is similar to a bookmaker seeking to have equal money bet on mutually exclusive outcomes.

The rest of this paper proceeds as follows. In Section 2, we present a general framework for continuous-time betting markets, state the bookmaker’s optimization problem and define his value function. In Section 3, we provide a characterization of the bookmaker’s value function as the solution of a partial (integro-)differential equation (PDE). In Section 4, we study the bookmaker’s optimization problem in a semi-static setting. In this section, the probabilities of outcomes remain constant in time, but bets arrive at rates that depend on the prices set by the bookmaker. In Section 5, we study the wealth maximization problem for a risk-neutral bookmaker. In Section 6, we focus on the bookmaker’s optimization problem when his preferences are characterized by a utility function of exponential form. Lastly, in Section 7, we offer some concluding remarks.

## 2 A General Framework for Continuous Time Betting Markets

Let us fix a probability space \((\Omega, \mathcal{F}, P)\) and a filtration \(\mathcal{F} = (\mathcal{F}_t)_{0 \leq t \leq T}\) completed by \(P\)-null sets, where \(T < \infty\) is a finite time horizon. We shall suppose that all the stochastic processes and random variables introduced in this paper are well-defined and adapted under the given filtered probability space, which may be further characterized for specific examples or models in the sequel. We will think of \(P\) as the real world or physical probability measure. Consider a finite number of subsets \((A_i)_{i \in \mathbb{N}}\) of \(\Omega\), each of which is \(\mathcal{F}_T\)-measurable (i.e., \(A_i \in \mathcal{F}_T\) for all \(i\)). We can think of \(A_i\) as a particular set of outcomes of a sporting event, which finishes at time \(T\). To avoid trivial cases, we suppose \(P(A_i) \in (0,1)\) for all \(i\). Note that the sets \((A_i)_{i \in \mathbb{N}}\) need not be a partition of \(\Omega\). In particular, there may be outcomes \(\omega\) such that \(\omega \notin \bigcup_{i=1}^{n} A_i\). Moreover, sets may overlap (i.e., we may have \(A_i \cap A_j \neq \emptyset\) where \(i \neq j\)).
We will denote by $P_i^t = (P_i^t)_{0 \leq t \leq T}$ the $\mathcal{F}_t$-conditional probability of $A_i$. We have

$$P_i^t = \mathbb{E}_t 1_{A_i},$$

where $\mathbb{E}_t \cdot := \mathbb{E}(\cdot | \mathcal{F}_t)$ denotes conditional expectation. Note that, by the tower property of conditional expectation, $P_i^t$ is a $(\mathbb{P}, \mathcal{F})$-martingale. We will denote by $P = (P_1^t, P_2^t, \cdots, P_n^t)$ the vector of conditional probabilities. In general, the conditional probability $P_i^t$ is a stochastic process. However, we will also consider scenarios where it is reasonable to assume that $P_i^t$ is a fixed constant $p_i$ for all $t < T$. Let us take a look at some examples of sporting events and show how we can describe them probabilistically.

**Example 2.1.** Consider bets taken on sets of outcomes of a sporting event (e.g., football match, horse race, etc.) prior to the start of the event. Suppose that information leading up to the event is constant over time. In this case, the conditional probability that any particular set of outcomes occurs should be constant over time. We can model this situation as follows. Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P}) = ([0,1], \mathcal{B}([0,1]), \mu)$ and a filtration $\mathcal{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$, where $\mu$ is the Lebesgue measure and (up to $\mu$-null sets) $\mathcal{F}_t$ is given by $\mathcal{F}_t = \{\emptyset, \Omega\}$ for $t < T$ with $\mathcal{F}_T = \mathcal{B}([0,1])$. Sets of the form $A_i \in \mathcal{B}([0,1])$ have constant probability for all $t < T$ because $P_i^t = \mathbb{E}_t 1_{A_i} = \mu(A_i)$ for $t < T$. We also have $P_i^T = 1 1_{\{A_i\}}$.

**Example 2.2.** Suppose the number of goals scored by player X in an association football match is a Poisson process with intensity $\mu$. Consider an in-game bet on a set of outcomes of the form $A_i = \text{"player X will finish the match with exactly } i \text{ goals."}$ We can model the conditional probability that $A_i$ occurs as follows. Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ be the augmented filtration generated by the Poisson process $N^{\mu} = (N^{\mu}_t)_{0 \leq t \leq T}$ with intensity $\mu$. Then we have

$$P_i^t = \sum_{n=0}^{i} P(N^{\mu}_T - N^{\mu}_t = i - n) 1_{\{N^{\mu}_t = n\}} = \sum_{n=0}^{i} \frac{e^{-\mu(T-t)}(\mu(T-t))^{i-n}}{(i-n)!} 1_{\{N^{\mu}_t = n\}}. \quad (2.1)$$

From (2.1) we can easily construct the conditional probabilities of outcomes of the form \text{"player X will finish the match with between $i$ and $j$ goals (inclusive)."} For example, if $A_j = \text{"player X will score at least one goal"}$ then we have

$$P_j^t = 1_{\{N^{\mu}_t \geq 1\}} + 1_{\{N^{\mu}_t = 0\}} (1 - e^{-\mu(T-t)}).$$

The dynamics of $P_j^t$ can be easily deduced

$$dP_j^t = 1_{\{P_j^t < 1\}} e^{-\mu(T-t)} (dN^{\mu}_t - \mu dt), \quad P_j^0 = 1 - e^{-\mu T}. \quad (2.2)$$

Observe that $P_j^t$ in (2.2) is a martingale, as it must be.

**Example 2.3.** Consider an National Basketball Association (NBA) game. Although points in NBA games are integer-valued, as the number of points in an NBA game is on the order of 100, it is reasonable to approximate the number of points scored as a $\mathbb{R}$-valued process. To this end, let us consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\mathcal{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ for a Brownian motion $W = (W_t)_{0 \leq t \leq T}$. We can model the point differential $X = (X_t)_{0 \leq t \leq T}$ between team A and team B as a Brownian motion with drift
\(X_t = \mu t + \sigma W_t\) where the sign and size of \(\mu\) captures how much team A is favored by. Now, consider a bet on a set of outcomes of the form \(A_i = \text{"team A will win by i points or more."}\) Then we have

\[
P^i_t = P(X_T \geq i|X_t) = P(X_T - X_t \geq i - X_t) = P\left(W_T - W_t \geq \frac{i - X_t - \mu(T-t)}{\sigma}\right)
\]

\[
= 1 - \Phi\left(\frac{i - X_t - \mu(T-t)}{\sigma \sqrt{T-t}}\right) = \Phi\left(\frac{X_t + \mu(T-t) - i}{\sigma \sqrt{T-t}}\right),
\]

where \(\Phi\) is the cumulative distribution function (c.d.f.) of a standard normal random variable. We can easily compute the dynamics of \(P^i_t\) using Itô’s Lemma. We have

\[
dP^i_t = \Phi'(\Phi^{-1}(P^i_t)) \cdot \frac{1}{\sqrt{T-t}} dW_t, \quad P^i_0 = \Phi\left(\frac{\mu T - i}{\sigma \sqrt{T}}\right).
\]

Observe that \(P^i\) is a martingale, as it must be.

Having seen a few examples of how we can describe sporting events probabilistically, let us now focus on the payoff structure of bets. Throughout this paper, we will assume that a bet placed on a set of outcomes \(A_i\) pays one unit of currency at time \(T\) if and only if \(\omega \in A_i\) (i.e., if \(A_i\) occurs). Thus, we have

\[
\text{"payoff of a bet placed on } A_i \" = 1_{A_i} = P^i_T. \quad (2.3)
\]

**Remark 2.4.** In the US, if a bookmaker quotes odds of +120 on outcome A, then a $120 bet on outcome A would pay $120 + $100 = $220 if A occurs. If a bookmaker quotes odds of -110 on outcome B, then a $100 bet on outcome B would pay $100 + $110 = $210 if B occurs. The payoff in (2.3) is simply a convenient normalization, which can be applied to any bet that pays a fixed (i.e., non-random) amount if a particular outcome or set of outcomes occurs.

In our framework, the bookmaker cannot control the number of bets that the public places on a set of outcomes \(A_i\) directly. However, the bookmaker can set the price of a bet placed on \(A_i\) and this will affect the rate or intensity at which bets on \(A_i\) are placed. We will denote by \(u^i = (u^i_t)_{0 \leq t < T}\) the price set by the bookmaker of a bet placed on \(A_i\). The vector of prices will be denoted as \(u = (u^1, u^2, \cdots, u^n)\). It will be helpful at this stage to introduce a set of admissible pricing strategies \(A(t, T)\), which we define as follows

\[
A(t, T) := \{u = (u_s)_{s \in [t, T]} : u \text{ is progressively measure w.r.t. } \mathcal{F} \text{ and } u_s \in A_s, \text{ for all } s \in [0, T)\}, \quad A := [0, 1]^n, \quad (2.4)
\]

with \(t \in [0, T)\). Note that we do not include the control \(u_T\) in the definition of \(A(t, T)\), as the case at time \(T\) is trivial. Without loss of generality, we set \(u^i_T = 1_{A_i}\) for all \(i \in \mathbb{N}_n\).

Let us denote by \(X^u = (X^u_t)_{0 \leq t \leq T}\) the total revenue generated by the bookmaker and by \(Q^u, i = (Q^u_t, i)_{0 \leq t \leq T}\) the total number of bets placed on a set of outcomes \(A_i\). Note that we have indicated with a superscript the dependencies of \(X^u\) and \(Q^u\) on bookmaker’s pricing policy \(u\). The relationship between \(X^u\), \(Q^u\) and \(u\) is

\[
dX^u_t = \sum_{i=1}^n u^i_t dQ^u_t, i.
\]
Observe that $X^u$ and $Q^{u,i}$ for all $i \in \mathbb{N}$ are non-decreasing processes. Typically, we will have $X_0^u \equiv X_0 = 0$ and $Q_0^{u,i} \equiv Q_0^i = 0$ for all $i \in \mathbb{N}_n$. However, we do not require this (to account for the fact that the bookmaker may have taken the bets before time 0).

In this paper, we consider two models for $Q^{u,i}$. In one model, bets on a set of outcomes $A_i$ arrive at a rate per unit time, which is a function $\lambda_i : \mathbb{A} \times \mathbb{A} \to \mathbb{R}_+$ of the vectors of conditional probabilities $P$ of outcomes and prices $u$ set by the bookmaker. Under this model, we have

$$
\text{Continuous arrivals : } Q^{u,i}_t = \int_0^t \lambda_i(P_s, u_s)ds + Q^i_0.
$$

In another model, bets on a set of outcomes $A_i$ arrive as a state-dependent Poisson process $N^{u,i} = (N^{u,i}_t)_{0 \leq t \leq T}$ whose instantaneous intensity is a function $\lambda_i : \mathbb{A} \times \mathbb{A} \to \mathbb{R}_+$ of the vectors of conditional probabilities $P$ of outcomes and prices $u$ set by the bookmaker. Under this model, we have

$$
\text{Poisson arrivals : } Q^{u,i}_t = \int_0^t dN^{u,i}_t + Q^i_0, \quad E_t dN^{u,i}_t = \lambda_i(P_t, u_t)dt.
$$

Throughout the paper, we will refer to the function $\lambda_i$ as the rate function when bets arrive continuously and the intensity function when bets arrive as a state-dependent Poisson process.

**Remark 2.5.** For sporting events with a large betting interest (e.g., the Super Bowl, the UEFA Champions League final, etc.), the continuous arrivals model given by (2.5) is sufficient to capture the dynamics of bet arrivals. However, for sporting events with limited betting interest (e.g., curling in the Winter Olympics, the Westminster Dog Show, etc.), the dynamics of bet arrivals are better captured by the Poisson arrivals model in (2.6).

Although our framework is sufficiently general to allow for the rate/intensity function $\lambda_i$ to depend on the entire vector of conditional probabilities $P$ and vector of prices $u$, the case in which $\lambda_i$ depends only on the conditional probability $P^i$ and price $u^i$ will be of special interest. In cases for which

$$
\lambda_i(P_t, u_t) \equiv \lambda_i(P^i_t, u^i_t)
$$

for all $i \in \mathbb{N}$ and $t \in [0, T]$,

we expect $\lambda_i$ to be an increasing function of the conditional probability $P^i$ of outcome $A_i$ and a decreasing function of the price $u^i$ set by the bookmaker. Examples of rate/intensity functions $\lambda_i : [0,1] \times [0,1] \to \mathbb{R}_+$ that satisfy those analytical properties include

$$
\lambda_i(p_i, u_i) := \frac{p_i}{1-p_i} \frac{1-u_i}{u_i},
$$

$$
\lambda_i(p_i, u_i) := \frac{\log u_i}{\log p_i}.
$$

The functions (2.7) and (2.8) have reasonable qualitative behavior because (i) as the price $u_i^t$ of a bet on outcome $A_i$ goes to zero, the intensity of bets goes to infinity

$$
\lim_{u_i \to 0} \lambda_i(p_i, u_i) = \infty,
$$
(ii) as the price $u_i^t$ of a bet on outcome $A_i$ goes to one, the intensity of bets goes to zero

$$\lim_{u_i \to 1} \lambda_i(p, u_i) = 0,$$

and (iii) all fair bets $u_i^T = P_i^T$ have the same intensity

$$\lambda_i(p_i, p_i) = \lambda_i(q_i, q_i).$$

Another rate/intensity function $\lambda_i$ we shall consider in this paper is

$$\lambda_i(p, u_i) := xe^{-\beta(u_i - p_i)}, \quad x, \beta > 0.$$  (2.11)

As we shall see, the form of $\lambda_i$ in (2.11) facilitates analytic computation of optimal pricing strategies. Note, however, that $\lambda_i$ in (2.11) does not satisfy (2.9) or (2.10).

Let us denote by $Y_T^u$ the total wealth of the bookmaker just after paying out all winning bets, assuming he follows pricing policy $u$. Then we have

$$Y_T^u = X_T^0 - \sum_{i=1}^n P_T^i Q_t^i = X_T^0 - \sum_{i=1}^n P_T^i Q_0^i + \sum_{i=1}^n \int_0^T (u_i^t - P_T^i) \, dQ_t^u,$$  (2.12)

where $Q_t^u$ is given by either (2.5) or (2.6).

We will denote by $J$ the bookmaker's objective function. We shall assume $J$ is of the form

$$J(t, x, p, q; u) = E_{t, x, p, q} U(Y_T^u),$$

where $U : \mathbb{R} \to \mathbb{R}$ is either the identity function or a utility function. Here, we have introduced the notation $E_{t, x, p, q} \cdot = E(\cdot | X_t^0 = x, P_t = p, Q_t^0 = q)$. When we take $U$ to be the identity function, the bookmaker is risk-neutral. When we take $U$ to be an increasing and concave utility function, the bookmaker is risk-averse.

The bookmaker seeks an optimal (or $\varepsilon$-optimal) control or pricing policy $u^*$ (or $u^\varepsilon$) to the problem:

$$V(t, x, p, q) := \sup_{u \in A(t, T)} J(t, x, p, q; u),$$  (2.13)

where the admissible set $A(t, T)$ is defined by (2.4). We shall refer to the function $V$ as the bookmaker's value function.

### 3 PDE Characterization of the Value Function

Let us denote by $M$ be the infinitesimal generator of the process $P$, and by $\partial_t, \partial_x$ and $\partial_q_i$ the partial derivative operators with respect to the corresponding arguments. For any $u \in A_i$, define operator $L^u$ by either of the following

$$L^u := \sum_{i=1}^n \lambda_i(p, u)(u_i \partial_x - \partial_q_i) + M,$$  (3.1)

$$L^u := \sum_{i=1}^n \lambda_i(p, u)(\partial_{q_i}^u a_i - 1) + M,$$  (3.2)
where \( \theta^z_i \) is a shift operator of size \( z \) in the variable \( q_i \), that is, \( \theta^z_i f(q) := f(q_1, \ldots, q_i + z, \ldots, q_n) \). Suppose the bookmaker were to fix the prices of bets at a constant \( u_t = u \). Then \( L^u \) as defined in (3.1) is the generator of \( (X^u, P, Q^u) \) assuming the dynamics of \( Q^u \) are described by the continuous arrivals model (2.5), and \( L^u \) as defined in (3.2) is the generator of \( (X^u, P, Q^u) \) assuming the dynamics of \( Q^u \) are described by the Poisson arrivals model (2.6).

As is standard in stochastic control theory, we provide a verification theorem to Problem (2.13) in a general setting. We refer the reader to (Fleming and Soner, 2006, Section III.8) and (Yong and Zhou, 1999, Section 4.3) for proofs.

**Theorem 3.1.** Let \( v : [0, T] \times \mathbb{R}_+ \times A \times \mathbb{R}^n_+ \to \mathbb{R} \) be a real-valued function which is at least once differentiable with respect to all arguments and satisfies

\[
\partial_x v > 0 \quad \text{and} \quad \partial_{q_i} v < 0, \quad \forall i \in \mathbb{N}_n.
\]

Suppose the function \( v \) satisfies the Hamilton-Jacobi-Bellman (HJB) equation

\[
\partial_t v + \sup_{\hat{u} \in A} L^\hat{u} v = 0, \quad v(T, x, p, q) = E_U \left( x - \sum_{i=1}^n q_i I_{A_i} \right), \tag{3.3}
\]

where \( L^\hat{u} \) is given by either (3.1) or (3.2). Then \( v(t, x, p, q) = V(t, x, p, q) \) is the value function to Problem (2.13) and the optimal price process \( u^* = (u^*_s)_{s \in [t, T]} \) is given by

\[
u^*_s = \arg \max_{u \in A} L^u v(s, X^*_s, P_s, Q^*_s). \tag{3.4}
\]

**Remark 3.2.** The PDE characterization in (3.3) to the value function and the optimal price process given by (3.4) are obtained without any assumptions on function \( U \), the arrival rate/intensity function \( \lambda_i \), or the conditional probabilities \( P \). With additional assumptions, we may simplify (3.3) and (3.4) to more tractable forms. For instance, when \( Q^u \) is defined by the continuous arrivals model (2.5), we can simplify (3.4) and obtain for all \( i \in \mathbb{N}_n \) that

\[
\lambda_i(P_s, u^*_s) + \sum_{j=1}^n u^*_j \cdot \partial_{u_j} \lambda_i(P_s, u^*_s) \cdot \partial_{q_i} V = 0. \tag{3.5}
\]

Equation (3.5) reduces the problem of finding the optimal price process \( u^* \) in feedback form to solving a system of \( n \) equations. We shall apply the results of Theorem 3.1 to obtain the value function and/or the optimal price process in closed forms in the subsequent sections.

### 4 Analysis of the Semi-static Setting

In this section, we solve the main problem (2.13) in a semi-static setting. The standing assumptions of this section are as follows.

**Assumption 4.1.** The arrivals process \( Q^{a,i} \) is given by the continuous arrivals model (2.5). The vector of conditional probabilities is a vector of constants, namely, \( P_t \equiv p \in (0, 1)^n \) for all \( t \in [0, T) \). The utility
function U is continuous and strictly increasing. The rate function \( \lambda_i = \lambda_i(u^i) \) is a continuous and decreasing function of \( u^i \) for all \( i \in \mathbb{N}_n \).

For notational simplicity, we write the rate function as \( \lambda_i(u^i) \) in the rest of this section because the conditional probabilities \( P_t \equiv p \) are fixed constants. Under Assumption 4.1, we have from (2.12) that the bookmaker’s terminal wealth \( Y_u^T \) is given by

\[
Y_u^T = X_t - \sum_{i=1}^{n} Q_i \cdot I_{A_i} + \sum_{i=1}^{n} \int_t^T \lambda_i(u^i_s) u^i_s \, ds - \sum_{i=1}^{n} \int_t^T \lambda_i(u^i_s) ds \cdot I_{A_i}. \tag{4.1}
\]

As \( \lambda_i \) is decreasing by Assumption 4.1, its inverse \( \lambda_i^{-1} \) exists. Defining the function \( f_i \) by

\[
f_i(x) := x \cdot \lambda_i^{-1}(x), \quad x > 0, \quad i \in \mathbb{N}_n,
\]

we are able to rewrite \( Y_u^T \) in (4.1) as

\[
Y_u^T = X_t - \sum_{i=1}^{n} Q_i \cdot I_{A_i} + \sum_{i=1}^{n} \int_t^T f_i(\lambda_i(u^i_s)) ds - \sum_{i=1}^{n} \int_t^T \lambda_i(u^i_s) ds \cdot I_{A_i}.
\]

Denote by \( \hat{f}_i \) the concave envelope of \( f_i \) and define \( \hat{Y}_u^T \) by

\[
\hat{Y}_u^T := X_t - \sum_{i=1}^{n} Q_i \cdot I_{A_i} + \sum_{i=1}^{n} \int_t^T \hat{f}_i(\lambda_i(u^i_s)) ds - \sum_{i=1}^{n} \int_t^T \lambda_i(u^i_s) ds \cdot I_{A_i}. \tag{4.2}
\]

It is obvious that \( \hat{Y}_u^T \geq Y_u^T \) for any pricing policy \( u \). We are now ready to present the main results of this section.

**Theorem 4.2.** Let Assumption 4.1 hold, we have

\[
V(t,x,p,q) = \sup_{u \in A} E_{t,x,p,q} U \left( x - \sum_{i=1}^{n} q_i I_{A_i} + (T-t) \sum_{i=1}^{n} \hat{f}_i(\lambda_i(u^i)) \right) := \hat{V}, \tag{4.3}
\]

where

\[
A = [0,1]^n \quad \text{and} \quad \mathcal{D}_\lambda = (\mathcal{D}_{\lambda_1}, \mathcal{D}_{\lambda_2}, \ldots, \mathcal{D}_{\lambda_n}),
\]

with \( \mathcal{D}_{\lambda_i} \) being the range of \( \lambda_i \) for all \( i \in \mathbb{N}_n \).

**Remark 4.3.** The results in Theorem 4.2 help us reduce the original problem, which is a dynamic optimization problem over an \( n \)-dimensional stochastic process, into a static optimization problem over an \( n \)-dimensional constant vector.

The proof of Theorem 4.2 relies on the following two lemmas.

**Lemma 4.4.** Let Assumption 4.1 hold, we have

\[
\sup_{u \in A} E_{t,x,p,q} U \left( \hat{Y}_u^T \right) = \sup_{u \in A(t,T)} E_{t,x,p,q} U \left( \hat{Y}_u^T \right),
\]

where \( \hat{Y}_u^T \) is defined by (4.2) and set \( A(t,T) \) is defined by (2.4).
Proof. As the “≤” is obvious, we proceed to show the converse inequality is also true. Let u be any pricing policy adopted by the bookmaker. As $\lambda_i$ is continuous, there exists a constant $u_i^*$ such that

$$\lambda_i(u_i^*) = \frac{1}{T-t} \int_t^T \lambda_i(u_i^*) \, ds.$$  

By the concavity of $\hat{f}_i$, we obtain

$$\frac{1}{T-t} \int_t^T \hat{f}_i(\lambda_i(u_i^*)) \, ds \leq \hat{f}_i \left( \frac{1}{T-t} \int_t^T \lambda_i(u_i^*) \, ds \right) = \hat{f}_i(\lambda_i(u_i^*)).$$

As a result, we have

$$\mathbb{E}_{r,x,p,q} \mathbb{U} \left( \hat{\mathbb{Y}}^{u} \right) \leq \mathbb{E}_{r,x,p,q} \mathbb{U} \left( \hat{\mathbb{Y}}^{u} \right) \leq \sup_{\hat{u} \in \mathcal{A}} \mathbb{E}_{r,x,p,q} \mathbb{U} (\hat{\mathbb{Y}}^{\hat{u}}).$$

Then the result follows from the arbitrariness of u. 

Lemma 4.5. Let Assumption 4.1 hold. Then we have

$$\mathbb{V}(t,x,p,q) \geq \sup_{u \in \mathcal{A}} \mathbb{E}_{r,x,p,q} \mathbb{U} \left( \hat{\mathbb{Y}}^{u} \right).$$

Proof. For any $\varepsilon > 0$, let $u^\varepsilon = (u_1^\varepsilon, \ldots, u_n^\varepsilon)$ be an $\varepsilon/2$-optimizer for $\hat{\mathbb{V}}$ in (4.3). Let $\delta > 0$ be small enough so that

$$\mathbb{E}_{r,x,p,q} \mathbb{U} \left( \hat{\mathbb{Y}}^{u^\varepsilon} - \delta \right) > \mathbb{E}_{r,x,p,q} \mathbb{U} \left( \hat{\mathbb{Y}}^{u^\varepsilon} \right) - \varepsilon/2 > \mathbb{V} - \varepsilon.$$

Next choose $v_i, w_i, \rho_i \in [0,1]$ for each $i \in \mathbb{N}_n$ so that the following conditions are met:

$$\rho_i \cdot \lambda_i(v_i) + (1 - \rho_i) \cdot \lambda_i(w_i) = \lambda_i(u_i^\varepsilon),$$

$$\rho_i \cdot f_i(\lambda_i(v_i)) + (1 - \rho_i) \cdot f_i(\lambda_i(w_i)) > \hat{f}_i(\lambda_i(u_i^\varepsilon)) - \frac{\delta}{n(T-t)}.$$  

We construct a pricing strategy $\hat{u}$ by

$$\hat{u}_i = v_i \cdot 1_{\{t \leq s < t + \rho_i(T-t)\}} + w_i \cdot 1_{\{t + \rho_i(T-t) \leq s \leq T\}}. \quad (4.4)$$

It is easy to verify that

$$\mathbb{Y}_{T}^{\hat{u}} > \hat{\mathbb{Y}}^{u^\varepsilon} - \delta,$$

which in turn implies

$$\mathbb{V}(t,x,p,q) \geq \mathbb{E}_{r,x,p,q} \mathbb{U} \left( \mathbb{Y}_{T}^{\hat{u}} \right) \geq \mathbb{E}_{r,x,p,q} \mathbb{U} \left( \mathbb{Y}_{T}^{u^\varepsilon} - \delta \right) > \mathbb{V} - \varepsilon.$$  

Hence, the desired result follows. \qed
Proof of Theorem 4.2. The result follows from Lemmas 4.4 and 4.5 and the fact that \( f_i \leq \hat{f}_i \).

Corollary 4.6. If \( \hat{u}^* = (\hat{u}_1^*, \hat{u}_2^*, \cdots, \hat{u}_n^*) \) is an optimizer for \( \hat{V} \) and

\[
f_i(\hat{\lambda}_i(\hat{u}_i^*))) = \hat{f}_i(\hat{\lambda}_i(\hat{u}_i^*)), \quad i \in \mathbb{N}_n,
\]

then \( \hat{u}^* \) is an optimizer for \( V \). In general, the pricing policy \( \hat{u} \) defined in (4.4) is an \( \varepsilon \)-optimizer for \( V \).

Remark 4.7. If \( f_i \) is concave, (4.5) is satisfied. For example, this is the case if \( \lambda_i \) is given by (2.7). However, if \( \lambda_i \) is given by (2.8), \( f_i \) in general is not a concave function. In fact, we have

\[
f''_i(x) < 0 \quad \text{if} \quad 0 < x < -\frac{2}{\log p_i}, \quad \text{and} \quad f''_i(x) > 0 \quad \text{if} \quad x > -\frac{2}{\log p_i}.
\]

As mentioned in Remark 4.3, Theorem 4.2 allows us to reduce the target optimization problem into a much simpler static optimization one, but the general characterization (either explicit or numerical solutions) of the optimal constant vector \( \hat{u}^* \) to \( \hat{V} \) is still not available. Below we provide an example where the optimizer is given by a system of equations.

Corollary 4.8. Let Assumption 4.1 hold. Assume further that (i) the sets \( (A_i)_{i \in \mathbb{N}_n} \) form a partition of \( \Omega \), (ii) the rate function \( \lambda_i \) is given by (2.7), and (iii) \( U \) is given by

\[
U(y) = -e^{-\gamma y}, \quad \gamma > 0.
\]

Then the optimizer \( \hat{u}^* \) of \( \hat{V} \), defined in (4.3), solves the following equation

\[
p_i \cdot \left( \frac{1}{\hat{u}_i^*} - 1 \right) \cdot g(t, p_i, q_i; \hat{u}_i^*) = \sum_{j \neq i} p_j \cdot g(t, p_j, q_j; \hat{u}_j^*), \quad \forall i \in \mathbb{N}_n,
\]

where function \( g \) is defined by

\[
g(t, p_i, q_i; u_i) := \exp \left( \gamma q_i + \gamma (T-t) \frac{p_i}{1-p_i} \left( \frac{1}{\hat{u}_i^*} - 1 \right) \right),
\]

for all \( t \in [0, T) \), \( p_i \in (0, 1) \), \( q_i \in \mathbb{R}_+ \), and \( u_i \in (0, 1) \). Moreover, if we replace assumption (i) with (i') the sets \( (A_i)_{i \in \mathbb{N}_n} \) are independent, then \( \hat{u}^* \) solves the following equation

\[
p_i \cdot \left( \frac{1}{\hat{u}_i^*} - 1 \right) \cdot \exp \left( \gamma (T-t) \frac{p_i}{1-p_i} \left( \frac{1}{\hat{u}_i^*} - 1 \right) \right) = 1 - p_i, \quad \forall i \in \mathbb{N}_n.
\]

Proof. Equations (4.6) and (4.7) are the first-order conditions of the corresponding optimization problems under the given assumptions. For instance, with assumptions (i'), (ii) and (iii), the optimization problem becomes

\[
\min_{u_i \in (0, 1)^n} \gamma (T-t) \sum_{i=1}^n p_i \cdot u_i + \sum_{i=1}^n \log \left( 1 - p_i + p_i \cdot \exp \left( \gamma (T-t) \frac{p_i}{1-p_i} \left( \frac{1}{\hat{u}_i^*} - 1 \right) \right) \right),
\]

which has the first-order condition (4.7).
Remark 4.9. From (4.6), we obtain
\[
\frac{\partial \hat{u}_i^*}{\partial q_i} > 0.
\]
This result is consistent with our intuition that, if the bookmaker has received many (few) bets on the set of outcomes $A_i$, the bookmaker should increase (decrease) the price of a bet on $A_i$ to balance the books.

In Figure 1, we plot $\hat{u}^*$, given by (4.7), as a function of $p$, with $\gamma = 2$ and $T - t = 1$. As expected, the optimizer is an increasing function of the conditional probability. In addition, $\hat{u}^*$ is a concave function of $p$.

![Figure 1: A plot of $\hat{u}^*$ as a function of $p$ with $\gamma = 2$ and $T - t = 1$](image)

5 Wealth Maximization

In this section, we consider Problem (2.13) when $U(y) = y$ (i.e., the objective of the bookmaker is to maximize his expected terminal wealth). Henceforth, we shall refer to this problem as the wealth maximization problem. We solve the wealth maximization problem using three different methods, and obtain the solutions Problem (2.13) in Theorem 5.1 and Corollaries 5.3, 5.5 and 5.6.

5.1 Method I

The following Theorem transforms the bookmaker’s dynamic optimization problem into a static optimization problem.
Theorem 5.1. Assume \( U(y) = y \), and the bet arrival process \( Q^{u,i} \) is given by either (2.5) or (2.6) for all \( i \in \mathbb{N}_n \). Then we have

\[
V(t,x,p,q) = x - p \cdot q + E_t x p q \int_t^T \sup_{u \in A} \sum_{i=1}^n \lambda_i(P_s, \hat{u}) \cdot (\hat{u}_i - P_s^i) ds, \tag{5.1}
\]

where \( p \cdot q = \sum_{i=1}^n p_i q_i \).

**Proof.** As \( Q^{u,i} \) is given by either (2.5) or (2.6), we have

\[
E_t dQ^{u,i}_t = \lambda_i(P_t, u_t) dt, \quad E_t 1_{A_i} = P_t^i,
\]

and, as a result,

\[
V(t,x,p,q) = x - p \cdot q + \sup_{u \in A(t,T)} \sum_{i=1}^n \left( \int_t^T \lambda_i(P_s, u_s) \cdot (u_s^i - P_s^i) ds \right)
\]

\[
= x - p \cdot q + \sup_{u \in A} \sum_{i=1}^n \lambda_i(P_s, \hat{u}) \cdot (\hat{u}_i - P_s^i) ds,
\]

where the last equality can be shown by a measurable selection argument (see Wagner (1977)).

Remark 5.2. For any \( \varepsilon > 0 \), \( u^\varepsilon = (u^\varepsilon_t(p))_{t \leq s < T} = (u^\varepsilon_t^1(p), \ldots, u^\varepsilon_t^n(p))_{t \leq s < T} \) measurably selected such that

\[
\sum_{i=1}^n \lambda_i(p, u^\varepsilon_t(p)) \cdot (u^\varepsilon_t^i(p) - p_i) \geq \sup_{u \in A} \sum_{i=1}^n \lambda_i(p, \hat{u}) (\hat{u}_i - p_i) - \varepsilon, \quad \forall p \in A
\]

is an \( \varepsilon \)-optimizer of the value function \( V \).

In light of Theorem 5.1, we are able to reduce the complexity of the wealth maximization problem significantly. As mentioned above, Theorem 5.1 allows us to transform a dynamic optimization problem (over \( u \in A(t,P) \)) into a static optimization problem (over \( \hat{u} \in A \)), and shows that the value function are the same for these two problems. Under the general setup, the characterization in (5.1) is not enough for us to obtain the optimal price process \( u^* \) explicitly. However, when the rate or intensity function \( \lambda_i \) is given by (2.7) or (2.8), we are able to find \( u^* \) in closed forms, see the corollary below.

Corollary 5.3. Assume \( U(y) = y \), and the arrival process \( Q^{u,i} \) is given by either (2.5) or (2.6) for all \( i \in \mathbb{N}_n \).

(i) If the rate or intensity function \( \lambda_i \) is given by (2.7), then the optimal price process to the wealth maximization problem is \( u^* = (u^*_1, u^*_2, \ldots, u^*_n)_{t \in [0,T]} \), where \( u^*_i \) is given by

\[
u_t^i = \frac{\hat{p}_t^i}{\hat{p}_t^{n+1}}, \quad \forall i \in \mathbb{N}_n.
\tag{5.2}
If we assume further that \( P_t \equiv p \in (0,1)^n \) for all \( t \in [0,T) \), then the optimal price process \( u^* \) is
\[
\begin{align*}
u^*_{i,t} & = \sqrt{p_i}, & \forall i \in \mathbb{N}_n, \tag{5.3}
\end{align*}
\]
and the value function \( V \) is given by
\[
V(t,x,p,q) = x - p \cdot q + (T-t) \sum_{i=1}^{n} \frac{P_i}{1-P_i} (1-\sqrt{p_i})^2.
\]

(ii) If the rate or intensity function \( \lambda_i \) is given by (2.8), then the optimal price process to the wealth maximization problem is \( u^* = (u^*_1, u^*_2, \ldots, u^*_n)^T \), where \( u^*_i \) is the unique solution on \((e^{-1}, 1)\) to the equation
\[
r(1 + \log r) = P_i^t, \quad \forall i \in \mathbb{N}_n. \tag{5.4}
\]

Proof. According to Theorem 5.1, we have
\[
u^*_{i,t} = \arg \max_{\hat{u}_i \in [0,1]} \frac{1-\hat{u}_i}{\hat{u}_i} \left( \frac{\hat{u}_i-P_i^t}{P_i^t} \right),
\]
when \( \lambda_i \) is given by (2.7); and
\[
u^*_{i,t} = \arg \max_{\hat{u}_i \in [0,1]} \log(\hat{u}_i)(\hat{u}_i-P_i^t),
\]
when \( \lambda_i \) is given by (2.8). The rest follows naturally. \( \square \)

Let \( \hat{l} : (0,1) \rightarrow (e^{-1}, 1) \) be the solution function to (5.4), namely, \( u^*_{i,t} = \hat{l}(P_i^t) \). We easily verify that
\[
0 < \frac{\partial \hat{l}(z)}{\partial z} = \frac{1}{2 + \log \hat{l}(z)} \in \left( \frac{1}{2}, 1 \right) \quad \text{and} \quad \frac{\partial^2 \hat{l}(z)}{\partial z^2} = \frac{-1}{\hat{l}(z)(2 + \log \hat{l}(z))^3} < 0.
\]

Thus, the function \( \hat{l} \) is increasing and concave. As such, the optimal price \( u^*_{i,t} \) on set of outcomes \( A_i \) increases when the conditional probability \( P_i^t \) increases and the rate of increase is larger when \( P_i^t \) is small.

5.2 Method II (Dynamic Programming Method)

Now, we use the PDE characterization of the bookmaker's value function (Theorem 3.1) to solve the wealth maximization problem under the continuous arrivals model (2.5). We begin the analysis by establishing some analytical properties for the value function \( V \).

Proposition 5.4. Let \( U(y) = y \) for all \( y \in \mathbb{R} \). If the value function \( V(t,x,p,q) \) is differentiable with respect to \( t, x \) and \( q \), we have
\[
\begin{align*}
\partial_t V(t,x,p,q) & < 0, \quad \partial_x V(t,x,p,q) = 1, \quad \text{and} \quad \partial_q V(t,x,p,q) = -p_i, \forall i \in \mathbb{N}_n.
\end{align*}
\]
Proof. Recalling the definition of $Y^u_T$ in (2.12), for any price process $u \in \mathcal{A}(t,T)$, we obtain

$$E_{t,x,p,q} Y^u_T = E_{t,x,p,q} \left[ x + \sum_{i=1}^{n} \int_{t}^{T} u^i_t dQ^i - \sum_{i=1}^{n} P^i_T q_i - \sum_{i=1}^{n} P^i_T \int_{t}^{T} dQ^i \right]$$

$$= x - \sum_{i=1}^{n} p_i q_i + E_{t,x,p,q} \sum_{i=1}^{n} \int_{t}^{T} (u^i_t - P^i_t) \lambda(P^i_t, u^i_t) ds.$$

It is clear that, for the optimal price process $u^*$, we have $u^i_t - P^i_t > 0$ for all $s \in [t,T]$ and $i \in \mathbb{N}_n$. The desired results are then obvious. 

We now present the explicit solutions to the wealth maximization problem.

Corollary 5.5. Assume $U(y) = y$ and the bets arrive according to the continuous arrivals model (2.5).

(i) If the rate function $\lambda_i$ is given by (2.7), then the optimal price process $u^*$ is given by (5.2).

(ii) If the rate function $\lambda_i$ is given by (2.8), then the optimal price process $u^*$ is the solution to (5.4).

Proof. (i) Under the continuous arrivals model (2.5) with rate function (2.7), we derive from (3.4) that

$$\partial_x V - \left( u^i_t \right)^{-2} \partial_q V = 0, \quad \forall i \in \mathbb{N}_n.$$

Together with the results from Proposition 5.4, we obtain, for all $s \in [t,T]$, that

$$u^i_t = \sqrt{-\partial_q V(s, X^*_s, P^*_s, Q^*_s)} = \sqrt{p_i} \in (0,1), \quad \forall i \in \mathbb{N}_n.$$

(ii) Under the continuous arrivals model (2.5) with rate function (2.8), we derive from (3.4) that

$$\left(1 + \log u^i_t\right) \cdot \partial_x V + \frac{1}{u^i_t} \cdot \partial_q V = 0, \quad \forall i \in \mathbb{N}_n.$$

The desired results are then obtained. 

5.3 Method III

Lastly, we use the results from Theorem 4.2 to solve the wealth maximization problem.

Corollary 5.6. Let Assumption 4.1 hold and suppose $U(y) = y$. If an interior optimizer $\Lambda^*$ in Theorem 4.2 exists, then we have

$$\hat{f}_i'(\Lambda^*_i) = p_i, \quad \forall i \in \mathbb{N}_n.$$

In particular, if $\lambda_i$ is given by (2.7), we obtain

$$\Lambda^*_i = \frac{p_i}{1 - p_i} \left(\frac{1}{\sqrt{p_i}} - 1\right) \quad \text{and} \quad \hat{u}^i_*(1 + \log \hat{u}^i_*) = p_i, \quad \forall i \in \mathbb{N}_n$$

and if $\lambda_i$ is given by (2.8), then

$$p_i^{\Lambda^*_i} (1 + \Lambda^*_i \log p_i) = p_i \quad \text{and} \quad \hat{u}^i_*(1 + \log \hat{u}^i_*) = p_i, \quad \forall i \in \mathbb{N}_n.$$

Proof. The first general result is immediate thanks to Theorem 4.2 and the assumption of $U(y) = y$. As pointed out in Remark 4.7, if $\lambda_i$ is given by (2.7), we have $f_i = \hat{f}_i$. If $\lambda_i$ is given by (2.8), we calculate $\hat{f}_i(x) = f'_i(x) > 0$ for $0 < x < -\frac{1}{\log p_i}$. 

15
5.4 Comparison of the three methods

In terms of the model generality, Theorem 5.1 obtained via Method I is the most general one, since (i) the conditional probabilities \( P \) can be modeled by any stochastic process taking values in \((0,1)\) and (ii) the arrival process is given by either the continuous arrival model (2.5) or the Poisson arrivals model (2.6). Corollary 5.5 of Method II holds when the arrival process is given by the continuous arrival model (2.5). Corollary 5.6 of Method III is further restricted to \( P \) being constants.

In terms of application scope, Method I is the most restricted one, as it only applies to the wealth maximization problem. Both Methods II and III are developed to solve the main problem for a general function \( U \), and hence can be applied to solve concave utility maximization problem (see for instance Corollary 4.8).

Both Theorem 5.1 of Method I and Theorem 4.2 of Method III provide an explicit characterization to the value function, but do not provide an explicit solution to the optimal price process. Method II (dynamic programming method) provides characterizations to both the value function and the optimal price process. However, solving the HJB equation (3.3) in Method II is often a challenging task.

From a computational point of view, the static optimization problem in Method I (see (5.1)) or in Method III (see (4.3)) is easy to solve, while finding the numerical solutions to the HJB equation (3.3) under the feedback strategy (3.4) may be difficult computationally.

5.5 Probability the Bookmaker Makes a Profit

In this subsection, we analyze the probability that the bookmaker makes profits when he follows the optimal price process, denoted by \( \mathbb{P}(Y^*_T > 0) \), when the rate or intensity function \( \lambda_i \) is given by (2.7).

We shall assume the vector of conditional probabilities \( P \) is a constant and that the bookmaker has taken zero bets at time \( t = 0 \) and has zero initial wealth

\[
\mathbb{P}_t = (p_1, p_2, \ldots, p_n), \quad \forall \ t < T, \quad X_0 = 0, \quad Q^i_0 = 0, \quad \forall \ i \in \mathbb{N}_n.
\]

Under these conditions we study the bookmaker's terminal wealth \( Y^*_T \) under the optimal policy \( u^* \) given by (5.3). For simplicity, we consider betting on the result of coin toss. The two possible outcomes are \( A_1 = \{\text{Heads}\} \) and \( A_2 = \{\text{Tails}\} \). Suppose \( P^1_t = \hat{p} \in (0,1) \) for all \( t \in [0,T) \). We carry out the analysis based on the two models of the arrival process \( Q^{u^*,j} \) – the continuous arrivals model (2.5) and the Poisson arrivals model (2.6).

Case 1: Continuous arrivals model (2.5) with \( \lambda_i \) given by (2.7). In this case, the bookmaker's terminal wealth \( Y^*_T \) is given by

\[
Y^*_T(\text{Heads}) = \psi_1(\hat{p}) \cdot T \quad \text{and} \quad Y^*_T(\text{Tails}) = \psi_1(1 - \hat{p}) \cdot T,
\]

where function \( \psi_1 \) is defined by

\[
\psi_1(\hat{p}) := \frac{\hat{p}}{1 - \hat{p}} \left( 2 - \sqrt{\hat{p}} - \frac{1}{\sqrt{\hat{p}}} \right) + \frac{1 - \hat{p}}{\hat{p}} (1 - \sqrt{1 - \hat{p}}), \quad \hat{p} \in (0,1).
\]
As seen in Figure 2, $\psi_1$ is a decreasing function over $(0,1)$ with

$$\lim_{\hat{p} \to 0} \phi_1(\hat{p}) = \frac{1}{2} \quad \text{and} \quad \lim_{\hat{p} \to 1} \phi_1(\hat{p}) = 0.$$  

Hence, regardless of the probability that the coin toss results in a heads, the bookmaker is guaranteed to make a profit by following the optimal price policy given by (5.3). If $\hat{p} = 0.5$ (the coin is fair), the bookmaker’s profit is a constant given by

$$Y_T^* = \phi_1(0.5) \cdot T \approx 0.171573 \cdot T.$$
\[
\hat{p} \sum_{j=1}^{\infty} e^{-\lambda_2^* T_j} \left( \frac{\lambda_2^* T_j}{j!} \right)^j + (1 - \hat{p}) \sum_{i=0}^{\infty} e^{-\lambda_1^* T_i} \left( \frac{\lambda_1^* T_i}{i!} \right)^i \sum_{j=0}^{\mathcal{M}_2(i)} e^{-\lambda_2^* T_j} \left( \frac{\lambda_2^* T_j}{j!} \right)^j,
\]

where \(\mathcal{M}_1(k)\) is the largest integer less than \(\sqrt{1 - \hat{p}} \cdot k\) and \(\mathcal{M}_2(k)\) is the largest integer less than \(\sqrt{\hat{p}} \cdot 1 - \sqrt{1 - \hat{p}} \cdot k\) for all \(k = 0, 1, \ldots\). The expression (5.5) allows us to compute \(\mathbb{P}(Y_T^* > 0)\) numerically in an efficient way. For instance, given \(\hat{p} = 0.5\), we compute

\[
\mathbb{P}(Y_T^* > 0) = \begin{cases} 
33.6747\% & T = 1, \\
54.4348\% & T = 2, \\
76.8231\% & T = 5, \\
86.4919\% & T = 10.
\end{cases}
\]

### 6 Exponential Utility Maximization

In this section, we study Problem (2.13) when \(U(y) = -e^{-\gamma y}\), with \(\gamma > 0\), henceforth called the exponential utility maximization problem. We solve this problem assuming \(Q^u\) is described by the Poisson arrivals model (2.6).\(^1\) We summarize the key results in Theorem 6.2. Let us begin by stating some assumptions, which are assumed to hold throughout the analysis that follows.

**Assumption 6.1.** The utility function \(U\) is given by \(U(y) = -e^{-\gamma y}\), where \(\gamma > 0\). The vector of conditional probabilities is a vector of constants, namely, \(p_t = p \in (0, 1)^n\) for all \(t \in [0, T)\). The bet arrivals process \(Q_{u,i}\) is given by the Poisson arrivals model (2.6). The rate function \(\lambda_i\) is given by (2.11), i.e., \(\lambda_i(p_i, u_i) = \kappa e^{-\beta (u_i - p_i)}\), where \(\kappa, \beta > 0\).

As in the previous section, because the conditional probability of set \(A_i\) is assumed to be a fixed constant \(p_i\), in order to simplify the notation, we write the intensity function as \(\lambda_i(u_i)\) for all \(i \in \mathbb{N}_n\). Under Assumption 6.1, with \(X_t = x, P_t = p\) and \(Q_t = q\), the bookmaker’s terminal wealth \(Y_T^u\) is given by

\[
Y_T^u = \left( x + \sum_{i=1}^{n} \int_{t}^{T} u_i^1 \text{d}N_{t}^{u,i} - \sum_{i=1}^{n} \left( q_i + \int_{t}^{T} \text{d}N_{t}^{u,i} \right) 1_{A_i} \right).
\]

The standard method of solving a stochastic control problem is to develop a verification theorem first, solve the associated HJB PDE with appropriate boundary conditions, and verify all the conditions in the verification theorem are met. We have followed exactly this standard method previously; see Theorem 3.1 and Corollary 5.5. It is clear that the general verification theorem obtained in Theorem 3.1 also applies to the problem we are considering in this section, with operator \(\mathcal{L}^u\) given by (3.2) and \(\mathcal{M} = 0\). Please refer to (Øksendal and Sulem, 2005, Chapter 3) for standard stochastic control theory with jumps.

The HJB associated with the exponential utility maximization problem is

\[
\partial_t V(t, x, p, q) + \sum_{i=1}^{n} \sup_{u_i} (\lambda_i(u_i)[V(t, x + u_i, p, q + e_i) - V(t, x, p, q)]) = 0,
\]

\(^1\)Notice Corollary 4.8 solves the exponential utility maximization problem under the continuous arrivals model (2.5), among other model assumptions.
and the boundary conditions are

\[ V(T, x, p, q) = -e^{-\gamma x} \cdot a(q), \]  

(6.2)

where \( e_i \in \mathbb{N}^n \) is the vector whose \( i \)-th component is 1 and other components are 0, and

\[ a(q) := E \exp \left( \gamma \sum_{i=1}^n q_i A_i \right). \]

To better present the solutions in Theorem 6.2, we define constants \( c, b_i \) and \( h_i \) by

\[ c := \frac{\beta}{\gamma}, \quad b_i := x e^p \cdot \frac{\gamma}{\beta + \gamma} \cdot \left( \frac{\beta}{\beta + \gamma} \right)^{\frac{\beta}{\gamma}}, \quad \text{and} \quad h_i := cb_i, \quad i \in \mathbb{N}_n, \]  

(6.3)

and functions \( d(q) \) and \( \alpha_k(q), k = 0, 1, 2, \ldots, \) by

\[ d(q) := [a(q)]^{-c}, \]

\[ \alpha_0(q) := d(q), \]

\[ \alpha_k(q) := \frac{1}{k!} \sum_{|j| = k} \left( \prod_{i=1}^n h_i \right) \cdot d(q + j), \]

where \( j = (j_1, \ldots, j_n) \in \mathbb{N}^n \) and \( |j| := j_1 + \cdots + j_n \). For instance, when \( k = 1 \), we have

\[ \alpha_1(q) := \sum_{i=1}^n h_i \cdot d(q + e_i). \]

**Theorem 6.2.** Let Assumption 6.1 hold. The value function \( V \) of Problem (2.13) is given by

\[ V(t, x, p, q) = -e^{-\gamma x} \cdot [G(t, q)]^{-1/c}, \]  

(6.4)

where function \( G \) is defined by

\[ G(T, q) = d(q), \quad G(t, q) = \sum_{k=0}^\infty \alpha_k(q) \cdot (T - t)^k, \quad t \in [0, T). \]  

(6.5)

The optimal price process \( u^* = (u^*_s)_{s \in [t, T]} \) to Problem (2.13) is given by

\[ u^*_s = -\frac{1}{\gamma} \log \left( \frac{\beta \cdot H(s, Q^*_s)}{(\beta + \gamma) \cdot H(s, Q^*_s + e_i)} \right), \quad i \in \mathbb{N}_n, \]  

(6.6)

where \( H(t, q) := [G(t, q)]^{-1/c} \).

**Proof.** As the utility function is of exponential form, we make the following Ansatz

\[ V(t, x, p, q) = -e^{-\gamma x} \cdot H(t, q). \]

Inserting the Ansatz into the HJB equation (6.1) and the boundary condition (6.2), we obtain

\[ \partial_t H(t, q) + \sum_{i=1}^n \inf_{u_i} (\lambda_i(u_i)|e^{-\gamma u_i}H(t, q + e_i) - H(t, q)) = 0, \quad \text{and} \quad H(T, q) = a(q). \]  

(6.7)
Solving the infimum problems in (6.7) gives
\[ u^*_i = -\frac{1}{\gamma} \log \left( \frac{\beta \cdot H(t, q)}{(\beta + \gamma) \cdot H(t, q + e_i)} \right) > 0, \] (6.8)
which proves the optimal price process in (6.6) once H is found. Next, substituting \( u^*_i \) in (6.8) back into (6.7), we obtain
\[ \partial_t H(t, q) - \sum_{i=1}^{n} b_i \cdot \frac{[H(t, q)]^c + 1}{[H(t, q + e_i)]^c} = 0, \] (6.9)
where constants \( b_i \) and \( c \) are defined by (6.3). Now let
\[ G(t, q) := [H(t, q)]^{-c}. \]
We establish the equation for \( G \) as follows
\[ \partial_t G(t, q) + \sum_{i=1}^{n} h_i \cdot G(t, q + e_i) = 0, \quad \text{and} \quad G(T, q) = d(q). \] (6.10)
where \( h_i = cb_i \) as in (6.3). The results from Lemma 6.3 verify that \( G \) given by (6.5) solves (6.10), and therefore the value function to Problem (2.13) is given by (6.4).

**Lemma 6.3.** We have

1. The series \( x \mapsto \sum_{k=0}^{\infty} \alpha_k(q)x^k \) is convergent with radius \(+\infty\), and hence, \( G(t, q) \) given by (6.5) is well defined and is in fact analytic.

2. \( G(t, q) \) given by (6.5) is a solution to (6.10).

**Proof.** Denote by \( b := \max\{b_1, \ldots, b_n\} \). For \( k \) large enough, we have
\[ |\alpha_k(q)| \leq \frac{1}{k!} \sum_{j \in \mathbb{N}^n : |j| = k} b^k = \frac{1}{k!} \left( \frac{k + n - 1}{n - 1} \right) b^k \sim \frac{k^{n-1}}{k! (n-1)!} b^k < \frac{k^n}{k!} b^k \sim \frac{b^{k-n}}{(k-n)!} b^n, \]
which proves the first result. To show the second result, we obtain
\[ \partial_t G(t, q) = -\sum_{k=1}^{\infty} k \cdot \alpha_k(q) (T-t)^{k-1} = -\sum_{k=0}^{\infty} (k+1) \alpha_{k+1}(q) \cdot (T-t)^k, \]
\[ \sum_{i=1}^{n} h_i G(t, q + e_i) = \sum_{k=0}^{\infty} \left( \sum_{i=1}^{n} h_i \cdot \alpha_k(q + e_i) \right) (T-t)^k. \]
Since we have
\[ \sum_{i=1}^{n} h_i \cdot \alpha_k(q + e_i) = \sum_{i=1}^{n} h_i \frac{1}{k!} \sum_{j \in \mathbb{N}^n : |j| = k} \left( h_1^{j_1} \cdots h_n^{j_n} \right) \cdot d(q + e_i + j) \]
\[ = \frac{1}{k!} \sum_{i=1}^{n} \sum_{j \in \mathbb{N}^n : |j| = k} h_i \cdot \left( h_1^{j_1} \cdots h_n^{j_n} \right) \cdot d(q + e_i + j) \]
\[
\begin{align*}
&= \frac{1}{k!} \sum_{j' \in \mathbb{N}^n : |j'| = k+1} \left( h_{i_1}^{j'_1} \cdots h_{i_n}^{j'_n} \right) \cdot d(q + j') \\
&= (k + 1) \alpha_{k+1}(q),
\end{align*}
\]
the second result follows.

In the analysis above, we do not impose any upper bound on the number of bets the bookmaker takes. However, if the bookmaker sets an upper for each betting event, we need to modify the results in Theorem 6.2 as described in the corollary below.

**Corollary 6.4.** Let Assumption 6.1 hold. Assume the total number of bets placed on set of outcomes \( A_i \) is at most \( m_i \), where \( i \in \mathbb{N}_n \). The value function to Problem (2.13) is given by

\[
V(t, x, p, q) = -e^{-\gamma x} \left[ \hat{G}(t, q) \right]^{-1/c},
\]

where \( \hat{G} \) is defined by

\[
\begin{align*}
\hat{G}(T, q) &= d(q), \\
\hat{G}(t, q) &= \sum_{k=0}^{|m|-|q|} \hat{\alpha}_k(q) \cdot (T-t)^k, \quad t \in [0, T),
\end{align*}
\]

with functions \( \hat{\alpha}_k(q) \) given by

\[
\hat{\alpha}_0(q) := d(q), \quad \hat{\alpha}_k(q) := \frac{1}{k!} \sum_{j \in I(k, q)} \left( \prod_{i=1}^n h_i^{j_i} \right) \cdot d(q + j),
\]

for all \( k = 1, 2, \cdots, \sum_{i=1}^n m_i \) and

\[
I(k, q) := \{ j \in \mathbb{N}^n : |j| = k, q + j \leq m := (m_1, m_2, \cdots, m_n) \}.
\]

The optimal price process \( u^* = (u^* s)_{s \in [t, T]} \) to Problem (2.13) is given by

\[
u^* s = -\frac{1}{\gamma} \log \left[ \frac{\beta \cdot \hat{H}(s, Q^* s)}{(\beta + \gamma) \cdot \hat{H}(s, Q^* s + e_i)} \right],
\]

for all \( i \in I(q) \), where \( \hat{H}(t, q) := [\hat{G}(t, q)]^{-1/c} \) and

\[
I(q) := \{ i : q_i < m_i \} \subset \mathbb{N}_n = \{1, \ldots, n\}. \quad (6.11)
\]

**Proof.** With the extra upper bound assumption, the bets on \( A_i \) will arrive according to a Poisson process at intensity \( \lambda_i(u_i) \) if the total number is less than \( m_i \); and 0 if otherwise. Notice that all the equations (6.1), (6.7), (6.9), and (6.10) still hold, except that the summation over index \( i \) in these equations will be restricted to the set \( I(q) \), defined by (6.11). All the results follow naturally by Theorem 6.2. \( \square \)
7 Conclusion

In this paper, we introduce a general framework for continuous-time betting markets. A bookmaker takes bets on random (sporting) events and pays off the winning bets at the terminal time T. The conditional probability of a set of outcomes is exogenous and may be stochastic, and the bets placed on a set of outcomes may arrive either at a continuous rate or as a state-dependent Poisson process. The bookmaker controls (updates) the prices of bets dynamically. In turn, the prices set by the bookmaker affect the rates or intensities of bet arrivals. The bookmaker seeks to set prices in order to maximize his expected terminal wealth or expected utility of terminal wealth. We are able to obtain either explicit solutions or characterizations of such optimal bookmaking problems in a number of distinct settings.

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