**Dispersive estimates for Klein-Gordon equations via a physical space approach**

*Willie Wong*

Based on commit adbfa2c of 2019-09-12 16:48

**Abstract** Building on the hyperboloidal foliation approach of Lefloch and Ma, we extend Klainerman’s physical-space approach to dispersive estimates to recover the frequency-restricted $L^1 - L^\infty$ dispersive estimates for Klein-Gordon equations. The hyperboloidal foliation approach naturally only provide estimates within a fixed forward light-cone, and is based on an initial data norm that is not translation invariant. Both of these problems can be handled with frequency-dependent physical-space truncations. To handle the lack of scaling symmetry for the Klein-Gordon equation and complete the argument, we also need to keep track of the effectiveness of our estimates in the vanishing mass limit.

1. Introduction

Fundamental to the study of dispersive equations are the dispersive estimates they enjoy, such estimates (for e.g. the Airy, Schrödinger, and wave equations) typically deriving from oscillatory integration control of the explicit Fourier representations of the corresponding fundamental solutions (see [5] and references therein). An alternative physical-space method relying weighted energy estimates was first introduced by Klainerman [1] for the wave equation in dimension $d \geq 3$, and extended to the Schrödinger equation by the author [7]. In this short note we present a physical-space method for studying the dispersive estimates enjoyed by the Klein-Gordon equation. Unlike the other model dispersive equations (Airy, Schrödinger, and wave), the Klein-Gordon equation does not enjoy scaling homogeneity. Its low frequency components are known to behave more akin to solutions to the Schrödinger equation, while its high frequency components behave more like waves, which complicates the study of its dispersive estimates; see [4] for a detailed discussion using the oscillatory integration method. Therefore it seems worthwhile to indicate how the same frequency-restricted results can be derived using a physical-space argument.

Our approach will be a combination of Klainerman’s physical space approach

*Michigan State University, East Lansing, USA; [wongwwy@math.msu.edu](mailto:wongwwy@math.msu.edu)*
for studying dispersive estimates of the wave equation, which is based on weighted energy estimates and the Klainerman-Sobolev inequalities adapted to constant time hypersurfaces, with global-Sobolev inequalities and energy estimates adapted to hyperboloidal foliations introduced in [3, 2]. The weighted energy estimates used in the hyperboloidal foliation method are spatially inhomogeneous, and we will paste together frequency-adjusted cut-offs to localize the dependence on initial data.

2. Preliminaries

For convenience, all functions are assumed to be smooth. Throughout we will denote by $\Box_m$, for $m \geq 0$, the differential operator

\begin{equation}
\Box_m \equiv -\partial_{tt}^2 + \sum_{i=1}^{d} \partial_{ii}^2 - m^2
\end{equation}

where as indicate we will be working on the space-time $\mathbb{R}^{1,d} \equiv \mathbb{R} \times \mathbb{R}^d \ni (t,x)$. The homogeneous Klein-Gordon equation with mass parameter $m$ is the equation

\begin{equation}
\Box_m \phi = 0.
\end{equation}

For convenience we will typically prescribe our initial data at $t = 2$:

\begin{align}
\phi(2,x) &= f(x), \\
\partial_t(2,x) &= g(x).
\end{align}

In the hyperboloidal method (see [3]; also [6]) we consider the level-sets $\Sigma_\tau$ of the function

\begin{equation}
\tau \equiv \sqrt{t^2 - |x|^2}
\end{equation}

defined on the subset $\{(t,x) \in \mathbb{R}^{1,d} \mid t > |x|\}$. These level sets are hyperboloids asymptotic to the cone $\{t = |x|\}$. The Minkowski metric on $\mathbb{R}^{1,d}$ induces Riemannian metrics on $\Sigma_\tau$, and we write $\text{dvol}_\tau$ for the corresponding induced volume form. We also define the $\Sigma_\tau$-tangential vector fields $L^i$, where $i$ ranges from 1,\ldots,d

\begin{equation}
L^i \equiv x^i \partial_t + t \partial_{x^i}.
\end{equation}

The fundamental energy identity for the Klein-Gordon equation, adapted to $\Sigma_\tau$ hypersurfaces, read [3,6]
2.6 Lemma
Let \( \phi \) solve (2.2) with initial data (2.3), then for every \( \tau > 0 \) the following weighted energy inequality holds, provided \( f \in H^1 \) and \( g \in L^2 \):

\[
E_m(\phi, \tau) \overset{\text{def}}{=} \int_{\Sigma_\tau} \frac{1}{t\tau} \sum_{i=1}^d \left( L^i \phi \right)^2 + \frac{\tau}{t} \left( \partial_t \phi \right)^2 + t\tau m^2 \phi^2 \, d\text{vol}_\tau \leq \int_{\mathbb{R}^d} g^2 + |\nabla f|^2 + m^2 f^2 \, dx.
\]

Furthermore, if the supports \( \text{supp}(f), \text{supp}(g) \subset B(0,2) \), then the equality is achieved. ■

The basic point-wise estimate is the following global Sobolev inequality [7]:

2.7 Proposition
Given any \( \ell \in \mathbb{R} \), there exists a constant \( C \) depending only on the dimension \( d \) and the number \( \ell \), such that for any \( \tau > 0 \)

\[
\left\| t^{1-\ell} t^{d+\ell-1} \phi \right\|_{L^\infty(\Sigma_\tau)} \leq C \sum_{k=0}^{\lfloor \frac{\ell}{2} \rfloor} \sum_{i_1, \ldots, i_k=1}^d \int_{\Sigma_\tau} t^\ell |L^{i_1} L^{i_2} \cdots L^{i_k} \phi|^2 \, d\text{vol}_\tau.
\]

Note that the constant in the above proposition is independent of \( \tau \). The right hand side of the global Sobolev inequality can be controlled by the energies \( E_m \) of higher order derivatives, and immediately we have

\[
\tag{2.8}
m^2 \left\| t^d \phi \right\|^2_{L^\infty(\Sigma_\tau)} + \left\| t^{2d-2} (\partial_t \phi)^2 \right\|^2_{L^\infty(\Sigma_\tau)} + \sum_{i=1}^d \left\| t^{d-1} (L^i \phi)^2 \right\|_{L^\infty(\Sigma_\tau)} \leq C \sum_{k=0}^{\lfloor \frac{\ell}{2} \rfloor} \sum_{i_1, \ldots, i_k=1}^d E_m(\phi, \tau).
\]

Combining Lemma 2.6 and (2.8) we have the following decay estimate for spatially localized initial data.

2.9 Proposition
Fix \( m_0 > 0 \). There exists a constant \( C \) depending only on the dimension \( d \) and \( m_0 \) such that whenever \( \phi \) solves (2.2) for \( m \in (0, m_0] \) with initial data (2.3) satisfying \( f, g \in C_0^\infty(\mathbb{R}, B(0,1)) \), the following decay estimates hold on the region \( \{ t \geq 2 \} \)

\[
m^2 t^d |\phi|^2 + t^{d-1} |\partial_t \phi|^2 + t^{d-1} |\nabla \phi|^2 \leq \left\| f \right\|^2_{H^\infty(\mathbb{R}, L^{d/2})} + \left\| g \right\|^2_{H^\infty(\mathbb{R}, L^{d/2})}.
\]

■
Proof. To control the right hand side of Proposition 2.7 we need to consider the equation satisfied by $L^1 \cdots L^k \phi$. For convenience in the course of the argument we will use $A \lesssim B$ to denote that $A$ is bounded by $B$ up to a universal factor that depends only on $m_0$ and $d$. The Lorentz boosts $L^i$ are well-known to commute with the operator $\Box$, hence we can easily check that if $\phi$ solves (2.2) with initial data (2.3), then $L^i \phi$ also solves (2.2) this time with initial data

$$L^i \phi(2, x) = f'(x) \overset{\text{def}}{=} 2 \partial_x f(x) + x^i g(x),$$

$$\partial_t L^i \phi(2, x) = g'(x) \overset{\text{def}}{=} \partial_x f(x) + 2 \partial_x g(x) + x^i \nabla f(x) - x^i m^2 f(x).$$

In particular, since $f, g \in C^\infty_0(B(0,1))$, and $m \leq m_0$,

$$\|\nabla f'\|_{L^2} \lesssim \|\nabla^2 f\|_{L^2} + \|g\|_{H^1},$$

$$\|f'\|_{L^2} \lesssim \|\nabla f\|_{L^2} + \|g\|_{L^2},$$

$$\|g'\|_{L^2} \lesssim \|f\|_{H^2} + \|\nabla g\|_{L^2}.$$

Hence for $k \leq \lfloor \frac{d}{2} \rfloor + 1$, corresponding the conserved energy

$$\mathcal{E}_m(L^1 \cdots L^k \phi, \tau) \lesssim \|f\|^2_{H^{\lfloor \frac{d}{2} \rfloor + 2}} + \|g\|^2_{H^\frac{d}{2}}.$$

And by (2.8) we obtain

$$m^2 \|t^d \phi\|^2_{L^\infty(\Sigma_t)} + \|t^2 t^{d-2} (\partial_t \phi)^2\|_{L^\infty(\Sigma_t)} + \sum_{i=1}^d \|t^{d-2} (L^i \phi)^2\|_{L^\infty(\Sigma_t)}$$

$$\lesssim \|f\|^2_{H^{\lfloor \frac{d}{2} \rfloor + 2}} + \|g\|^2_{H^{\lfloor \frac{d}{2} \rfloor + 1}}.$$

Next, denote by $D = \{t \geq 2, |x| \leq t-1\}$. By finite speed of propagation, the support of the $\phi$ when $t \geq 2$ is contained in $D$. We note that $D$ is covered by the $\cup_{\tau \geq \sqrt{3}} \Sigma_\tau$. Furthermore, on $D$ we have that

$$\tau^2 = (t + |x|)(t - |x|) \geq t + |x| \geq t.$$

Additionally, we have

$$t \partial_{x^i} = L^i - x^i \partial_t$$

and $x^i \leq t$ on $D$, meaning that

$$t^{d-1} |\partial_{x^i} \phi|^2 \leq 2 t^{d-3} |L^i \phi|^2 + t^{d-1} |\partial_t \phi|^2$$

and proving the theorem. \qed
3. Dispersive estimate

For \( k \geq 0 \) denote by \( P_k \) the Littlewood-Paley projector on \( \mathbb{R}^d \) to frequency \( \approx 2^k \). By \( P_{-1} \) we denote the low-frequency projector to frequencies \( \lesssim 1 \), such that \( u = \sum_{j=-1}^{\infty} P_j u \).

Let \( B_i \) be an enumeration of the balls of radius 1 centered at the lattice points \( \left( \frac{1}{\sqrt{d}} \mathbb{Z} \right)^d \). The family \( \{B_i\} \) is an open cover of \( \mathbb{R}^d \), and every point is contained in no more than \( (16d)^{d/2} \) balls. Define \( \chi_i \) to be a partition of unity subordinate to \( \{B_i\} \); we can require the first \( d+2 \) derivatives of the family \( \chi_i \) to be uniformly bounded. Under these conditions, we have that, for any function \( u \in W^{k,1}(\mathbb{R}^d) \) with \( k \leq d+2 \), that

\[
\|u\|_{W^{k,1}} \leq \sum_i \|\chi_i u\|_{W^{k,1}} \leq \sum_i \|u\|_{W^{k,1}(B_i)} \leq \|u\|_{W^{k,1}}
\]

with the implicit constants depending on the dimension \( d \) and the uniform bound on the family \( \chi_i \).

Consider \( m_0 \) a constant fixed once and for all for the remainder of this manuscript. We will now describe the dispersive estimates satisfied by solutions to \( \Box_{m_0} \phi = 0 \) with initial condition \((2.3)\). To do so, we will first decompose the initial data as

\[
f = \sum_{j=-1}^{\infty} P_j f, \quad g = \sum_{j=-1}^{\infty} P_j g
\]

and observe that we can write

\[
\phi = \sum_{j=-1}^{\infty} \phi_j
\]

and \( \phi_j \) solves \( \Box_{m_0} \phi_j = 0 \) with initial data \( P_j f, P_j g \) prescribed on \( \{t = 2\} \).

For convenience we will write

\[
s_d = \lfloor \frac{d}{2} \rfloor + 1.
\]

Low frequency estimate for \( \phi_{-1} \)— Let \( \psi_k \) solve \( \Box m_0 \psi_k = 0 \) with initial data \( (\chi_k P_{-1} f, \chi_k P_{-1} g) \). By linearity we have \( \phi_{-1} = \sum_k \psi_k \). Proposition \((2.9)\) guarantees that

\[
(m_0)^2 t^d |\psi_k|^2 + t^{d-1} |\partial \psi_k|^2 \leq \|\chi_k P_{-1} f\|_{H^{s_d+1}_t}^2 + \|\chi_k P_{-1} g\|_{H^{s_d+1}_t}^2.
\]

Sobolev embedding \( W^{\frac{d}{2},1} \subset L^2 \) implies

\[
(m_0)^2 t^d |\psi_k|^2 + t^{d-1} |\partial \psi_k|^2 \leq \|\chi_k P_{-1} f\|_{W^{s_d+1}}^2 + \|\chi_k P_{-1} g\|_{W^{s_d+1}}^2.
\]
Taking square roots and summing over $k$ and using the pointwise estimate (3.1) we get the point-wise estimate
\[
m_0 t^{d/2} |\phi_{-1}| + t^{(d-1)/2} |\partial \phi_{-1}| \lesssim \|P_{-1}f\|_{W^{1/2+1/2}} + \|P_{-1}g\|_{W^{1/2+1/2}}.
\]
and finally using the fact we have the frequency bound due to the projector $P_{-1}$, we conclude
\[
(3.2) \quad m_0 t^{d/2} |\phi_{-1}| + t^{(d-1)/2} |\partial \phi_{-1}| \lesssim \|P_{-1}f\|_{L^1} + \|P_{-1}g\|_{L^1}.
\]

**High frequency estimates for $\phi_k$, $k \geq 0$** — Given that $\phi_k$ solves $\Box_{m_0} \phi_k = 0$, with initial data $(P_k f, P_k g)$ prescribed at $t = 2$, let us consider the function
\[
(3.3) \quad \tilde{\phi}_k(t, x) \overset{\text{def}}{=} \phi_k(2 + t - \frac{2}{2^k}, \frac{x}{2^k}).
\]
Observe that $\tilde{\phi}_k$ solves $\Box_{2+\frac{t}{m_0}} \tilde{\phi}_k = 0$, with initial data $(\tilde{f}_k, \tilde{g}_k)$, prescribed at $t = 2$ given by
\[
\tilde{f}_k(x) = (P_k f)(2^{-k} x), \quad \tilde{g}_k(x) = 2^{-k}(P_k g)(2^{k-1} x).
\]
In particular $\tilde{f}_k(x)$ and $\tilde{g}_k(x)$ have frequency support $\approx 1$.

Let $\psi_{k,j}$ be the solution to $\Box_{2+\frac{t}{m_0}} \psi_{k,j} = 0$, with initial data $(\chi f \tilde{f}_k, \chi \tilde{g}_k)$. Since $k \geq 0$ we can apply Proposition 2.9 to obtain estimates on $\psi_{k,j}$, noting that the mass parameter is now $2^{-k} m_0 \leq m_0$. Then summing in analogous fashion to the previous subsubsection, we arrive at
\[
(3.4) \quad 2^{-k} m_0 t^{d/2} |\tilde{\phi}_k| + t^{(d-1)/2} |\partial \tilde{\phi}_k| \lesssim \|\tilde{f}_k\|_{L^1} + \|\tilde{g}_k\|_{L^1}.
\]
We can expand the left hand side and perform a change of variable $s = 2 + \frac{t-z}{2^k}$ to obtain
\[
2^{-k} m_0 2^{kd/2} (s-2)^{d/2} |\phi_k(s, y)| + 2^{-k} 2^{k(d-1)/2} (s-2)^{(d-1)/2} |\partial \phi_k(s, y)| \lesssim \|\tilde{f}_k\|_{L^1} + \|\tilde{g}_k\|_{L^1}.
\]
Finally reversing the change of variables for $\tilde{f}_k$ and $\tilde{g}_k$ we obtain that, for $t > 2$, that
\[
(3.5) \quad m_0 (t-2)^{d/2} |\phi_k| \lesssim 2^{kd/2+k} \|P_k f\|_{L^1} + 2^{kd/2} \|P_k g\|_{L^1},
\]
\[
(3.6) \quad (t-2)^{(d-1)/2} |\partial \phi_k| \lesssim 2^{k(d-1)/2+2k} \|P_k f\|_{L^1} + 2^{k(d-1)/2+k} \|P_k g\|_{L^1}.
\]
4. Discussion

First, gathering up the estimates in the previous section we obtain the following dispersive estimate. Here we place the initial data at \( t = 0 \) to make it more comparable to the standard presentations.

4.1 Theorem (Klein-Gordon dispersive estimate)

Let \( m_0 \in [0, M] \) be fixed, and let \( d \in \mathbb{Z}_+ \). Consider the initial value problem

\[
\begin{align*}
\Box_{m_0} \phi &= 0 \\
\phi(0, x) &= f(x) \\
\partial_t \phi(0, x) &= g(x)
\end{align*}
\]

Then there exists a universal constant \( C \) depending only on \( M \) and \( d \), such that the following dispersive estimates hold:

\[
\begin{align*}
(4.3a) \quad m_0 |P_{-1} \phi| &\leq \frac{C}{(1 + t)^{d/2}} (\|P_{-1}f\|_{L^1} + \|P_{-1}g\|_{L^1}) \\
(4.3b) \quad m_0 |P_k \phi|_{k \geq 0} &\leq \frac{C \cdot 2^{kd/2}}{t^{d/2}} (2^k \|P_k f\|_{L^1} + \|P_k g\|_{L^1}) \\
(4.3c) \quad |\partial P_k \phi|_{k \geq 0} &\leq \frac{C \cdot 2^{k(d-1)/2}}{t^{(d-1)/2}} (2^k \|P_k f\|_{L^1} + 2^k \|P_k g\|_{L^1}).
\end{align*}
\]

4.4 Remark

1. The theorem makes clear the heuristic that “high frequency parts behave like solutions to the wave equation”. Indeed, the decay estimate (4.3c) is the only one that will survive in the vanishing mass limit \( m_0 \to 0 \).

2. The latter two estimates can be combined. By replacing \( P_k f \mapsto \partial \Delta^{-1} P_k f \), we see that (4.3c) implies

\[
|P_k \phi| \leq \frac{C \cdot 2^{k(d-1)/2}}{t^{(d-1)/2}} (2^k \|P_k f\|_{L^1} + \|P_k g\|_{L^1}).
\]

Indeed, interpolating between the two we find in fact that, for any \( s \in \left[ \frac{d-1}{2}, \frac{d}{2} \right] \),

\[
|P_k \phi| \leq \frac{C \cdot 2^{ks}}{t^s} (2^k \|P_k f\|_{L^1} + \|P_k g\|_{L^1}),
\]

showing the trade off between regularity and decay for the Klein-Gordon equation.

7
References

[1] S. Klainerman. “A commuting vectorfields approach to Strichartz-type inequalities and applications to quasi-linear wave equations”. In: Int. Math. Res. Not. IMRN 2001.5 (2001), pp. 221–274. issn: 1073-7928. doi: 10.1155/S1073792801000137. url: http://dx.doi.org/10.1155/S1073792801000137.

[2] Sergiu Klainerman. “Global existence of small amplitude solutions to non-linear Klein-Gordon equations in four space-time dimensions”. In: Comm. Pure Appl. Math. 38.5 (1985), pp. 631–641. issn: 0010-3640. doi: 10.1002/cpa.3160380512. url: http://dx.doi.org/10.1002/cpa.3160380512.

[3] Philippe G. LeFloch and Yue Ma. The Hyperboloidal Foliation Method. World Sci. Publishing, 2015. 160 pp. isbn: 978-981-4641-62-3.

[4] Kenji Nakanishi and Wilhelm Schlag. Invariant Manifolds and Dispersive Hamiltonian Evolution Equations. Zurich Lectures in Advanced Mathematics. European Mathematical Society, 2011. doi: 10.4171/095.

[5] Terence Tao. Nonlinear dispersive equations: local and global analysis. CBMS Regional Conference Series in Mathematics. American Mathematical Society and Conference Board of the Mathematical Sciences, 2006.

[6] W. W. Y. Wong. “Small data global existence and decay for two dimensional wave maps”. In: arXiv pre-print (2017). eprint: arXiv: 1712.07684.

[7] Willie Wai Yeung Wong. “A commuting-vector-field approach to some dispersive estimates”. In: Arch. Math. (Basel) 110.3 (2018), pp. 273–289. issn: 0003-003X. doi: 10.1007/s00013-017-1114-4. url: https://doi.org/10.1007/s00013-017-1114-4.