ALGEBRAIC STRUCTURE OF MULTI-PARAMETER QUANTUM GROUPS

TIMOTHY J. HODGES, THIERRY LEVASSEUR, AND MARGARITA TORO

Introduction

Let \( G \) be a connected semi-simple complex Lie group. We define and study the multi-parameter quantum group \( C_{q,p}[G] \) in the case where \( q \) is a complex parameter that is not a root of unity. Using a method of twisting bigraded Hopf algebras by a cocycle, \( 2 \), we develop a unified approach to the construction of \( C_{q,p}[G] \) and of the multi-parameter Drinfeld double \( D_{q,p} \). Using a general method of deforming bigraded pairs of Hopf algebras, we construct a Hopf pairing between these algebras from which we deduce a Peter-Weyl-type theorem for \( C_{q,p}[G] \). We then describe the prime and primitive spectra of \( C_{q,p}[G] \), generalizing a result of Joseph. In the one-parameter case this description was conjectured, and established in the \( SL(n) \)-case, by the first and second authors in \( 13, 14 \). It was proved in the general case by Joseph in \( 18, 19 \). In particular the orbits in \( \text{Prim} C_{q,p}[G] \) under the natural action of the maximal torus \( H \) are indexed, as in the one-parameter case by the elements of the double Weyl group \( W \times W \). Unlike the one-parameter case there is not in general a bijection between \( \text{Symplectic} G \) and \( \text{Prim} C_{q,p}[G] \). However in the case when the symplectic leaves are algebraic such a bijection does exist since the orbits corresponding to a given \( w \in W \times W \) have the same dimension.

In the first section we discuss the Poisson structures on \( G \) defined by classical \( r \)-matrices of the form \( r = a - u \) where \( a = \sum_{\alpha \in R^+_+} e_{\alpha} \wedge e_{-\alpha} \in \wedge^2 g \) and \( u \in \wedge^2 h \). Given such an \( r \) one constructs a Manin triple of Lie groups \( (G \times G, G, G_r) \). Unlike the one-parameter case (where \( u = 0 \)), the dual group \( G_r \) will generally not be an algebraic subgroup of \( G \times G \). In fact this happens if and only if \( u \in \wedge^2 h_Q \). Since the quantized universal enveloping algebra \( U_q(g) \) is a deformation of the algebra of functions on the algebraic group \( G_r \) \( 1 \), this explains the difficulty in constructing multi-parameter versions of \( U_q(g) \). From \( 22, 30 \), one has that the symplectic leaves are the connected components of \( G \cap G_r x G_r \) where \( x \in G \). Since \( r \) is \( H \)-invariant, the symplectic leaves are permuted by \( H \) with the orbits being contained in Bruhat cells in \( G \times G \) indexed by \( W \times W \). In the case where \( G_r \) is algebraic, the symplectic leaves are also algebraic and an explicit formula is given for their dimension.

The philosophy of \( 13, 14 \) was that, as in the case of enveloping algebras of algebraic solvable Lie algebras, the primitive ideals of \( C_q[G] \) should be in bijection with the symplectic leaves of \( G \) (in the case \( u = 0 \)). Indeed, since the Lie bracket on \( g_r = \text{Lie}(G_r) \) is the linearization of the Poisson structure on \( G \), \( \text{Prim} C_{q,p}[G] \) should resemble \( \text{Prim} U(g_r) \). The study of the multi-parameter versions \( C_{q,p}[G] \) is similar to the case of enveloping algebras of general solvable Lie algebras. In the general case \( \text{Prim} U(g_r) \) is in bijection with the co-adjoint orbits in \( g_r^* \) under the
action of the ‘adjoint algebraic group’ of \( \mathfrak{g} \). It is therefore natural that, only in the case where the symplectic leaves are algebraic, does one expect and obtain a bijection between the symplectic leaves and the primitive ideals.

In section 2 we define the notion of an \( \textbf{L} \)-bigraded Hopf \( \mathbb{K} \)-algebra, where \( \textbf{L} \) is an abelian group. When \( A \) is finitely generated such bigradings correspond bijectively to morphisms from the algebraic group \( \textbf{L}^\vee \) to the (algebraic) group \( R(A) \) of one-dimensional representations of \( A \). For any antisymmetric bicharacter \( p \) on \( \textbf{L} \), the multiplication in \( A \) may be twisted to give a new Hopf algebra \( A_p \). Moreover, given a pair of \( \textbf{L} \)-bigraded Hopf algebras \( A \) and \( U \) equipped with an \( \textbf{L} \)-compatible Hopf pairing \( A \times U \to \mathbb{K} \), one can deform the pairing to get a new Hopf pairing between \( A_{p^{-1}} \) and \( U_p \). This deformation commutes with the formation of the Drinfeld double in the following sense. Suppose that \( A \) and \( U \) are bigraded Hopf algebras equipped with a compatible Hopf pairing \( A^{op} \times U \to \mathbb{K} \). Then the Drinfeld double \( A \bowtie U \) inherits a grading such that \(( A \bowtie U)_p \cong A_p \bowtie U_p \).

Let \( \mathbb{C}[G] \) denote the usual one-parameter quantum group (or quantum function algebra) and let \( U_q(\mathfrak{g}) \) be the quantized enveloping algebra associated to the lattice \( \textbf{L} \) of weights of \( G \). Let \( U_q(\mathfrak{g}^+) \) and \( U_q(\mathfrak{g}^-) \) be the usual sub-Hopf algebras of \( U_q(\mathfrak{g}) \) corresponding to the Borel subalgebras \( \mathfrak{b}^+ \) and \( \mathfrak{b}^- \) respectively. Let \( D_q(\mathfrak{g}) = U_q(\mathfrak{g}^+) \bowtie U_q(\mathfrak{g}^-) \) be the Drinfeld double. Since the groups of one-dimensional representations of \( U_q(\mathfrak{g}^+) \), \( U_q(\mathfrak{g}^-) \), \( D_q(\mathfrak{g}) \) and \( \mathbb{C}[G] \) are all isomorphic to \( H = L^\vee \), these algebras are all equipped with \( \textbf{L} \)-bigradings. Moreover the Rosso-Tanisaki pairing is compatible with the bigradings on \( U_q(\mathfrak{g}^+) \) and \( U_q(\mathfrak{g}^-) \). For any anti-symmetric bicharacter \( p \) on \( \textbf{L} \) one may therefore twist simultaneously the Hopf algebras \( U_q(\mathfrak{g}^+) \), \( U_q(\mathfrak{g}^-) \) and \( D_q(\mathfrak{g}) \) in such a way that \( D_{q,p}(\mathfrak{g}) \cong U_{q,p}(\mathfrak{g}^+) \bowtie U_{q,p}(\mathfrak{g}^-) \). The algebra \( D_{q,p}(\mathfrak{g}) \) is the ‘multi-parameter quantized universal enveloping algebra’ constructed by Okado and Yamane and previously in special cases in \( [5, 22] \). The canonical pairing between \( \mathbb{C}[G] \) and \( U_q(\mathfrak{g}) \) induces a \( \textbf{L} \)-compatible pairing between \( \mathbb{C}[G] \) and \( D_q(\mathfrak{g}) \). Thus there is an induced pairing between the multi-parameter quantum group \( \mathbb{C}_{q,p}[G] \) and the multi-parameter double \( D_{q,p^{-1}}(\mathfrak{g}) \). Recall that the Hopf algebra \( \mathbb{C}_{q,p}[G] \) is defined as the restricted dual of \( U_q(\mathfrak{g}) \) with respect to a certain category \( C \) of modules over \( U_q(\mathfrak{g}) \). There is a natural deformation functor from this category to a category \( C_p \) of modules over \( D_{q,p^{-1}}(\mathfrak{g}) \) and \( \mathbb{C}_{q,p}[G] \) turns out to be the restricted dual of \( D_{q,p^{-1}}(\mathfrak{g}) \) with respect to this category. This Peter-Weyl theorem for \( \mathbb{C}_{q,p}[G] \) was also found by Andruskiewitsch and Enriquez in \( [8] \) using a different construction of the quantized universal enveloping algebra and in special cases in \( [3, 14] \).

The main theorem describing the primitive spectrum of \( \mathbb{C}_{q,p}[G] \) is proved in the final section. Since \( \mathbb{C}_{q,p}[G] \) inherits an \( \textbf{L} \)-bigrading, there is a natural action of \( H \) as automorphisms of \( \mathbb{C}_{q,p}[G] \). For each \( w \in W \times W \), we construct an algebra \( A_w = (\mathbb{C}_{q,p}[G]/I_w)_{\mathcal{E}_w} \) which is a localization of a quotient of \( \mathbb{C}_{q,p}[G] \). For each prime \( P \in \text{Spec} \mathbb{C}_{q,p}[G] \) there is a unique \( w \in W \times W \) such that \( P \supset I_w \) and \( PA_w \) is proper. Thus \( \text{Spec} \mathbb{C}_{q,p}[G] \cong \bigsqcup_{w \in W \times W} \text{Spec} w \mathbb{C}_{q,p}[G] \) where \( \text{Spec} w \mathbb{C}_{q,p}[G] \cong \text{Spec} A_w \) is the set of primes of type \( w \). The key results are then Theorems \( [1, 14] \) and \( [1, 15] \) which state that an ideal of \( A_w \) is generated by its intersection with the center and that \( H \) acts transitively on the maximal ideals of the center. From this it follows that the primitive ideals of \( \mathbb{C}_{q,p}[G] \) of type \( w \) form an orbit under the action of \( H \).

An earlier version of our approach to the proof of Joseph’s theorem is contained in the unpublished article \( [17] \). The approach presented here is a generalization of this proof to the multi-parameter case.
Hence $d_i$ can be chosen in order to induce a perfect pairing $P$ inside $D$ form on $g$ analytic subgroup of $D$ denote a Lie subalgebra of $d$ left and right translation by $g$ Poisson Lie group structure on $G$. We may write \( \pi \) and that therefore the tensor $r$ Then it is well known that $u = a - u$. It then is known that $r$ satisfies the modified Yang-Baxter equation $[3, 20]$ and that therefore the tensor $\pi(g) = (l_g)_* r - (r_g)_* r$ furnishes $G$ with the structure of a Poisson Lie group, see $[13, 22, 30]$ $(l_g)_*$ and $(r_g)_*$ are the differentials of the left and right translation by $g \in G$.

We may write $u = \sum_{1 \leq i, j \leq n} u_{ij} h_i \otimes h_j$ for a skew-symmetric $n \times n$ matrix $[u_{ij}]$. The element $u$ can be considered either as an alternating form on $\mathfrak{h}^*$ or a linear map $u \in \text{End} \mathfrak{h}$ by the formula

$$\forall x \in \mathfrak{h}, \quad u(x) = \sum_{i, j} u_{i,j}(x, h_i) h_j.$$ 

The Manin triple associated to the Poisson Lie structure on $G$ given by $r$ is described as follows. Set $u_\pm = u \pm I \in \text{End} \mathfrak{h}$ and define

$$\vartheta: \mathfrak{h} \rightarrow \mathfrak{t}, \quad \vartheta(x) = -(u_-(x), u_+(x)),$$

$$a = \vartheta(\mathfrak{h}), \quad \mathfrak{g}_r = a \oplus u^+.$$

1.1. **Notation.** Let $\mathfrak{g}$ be a complex semi-simple Lie algebra associated to a Cartan matrix $[a_{ij}]_{1 \leq i, j \leq n}$. Let $\{d_i\}_{1 \leq i \leq n}$ be relatively prime positive integers such that $[d_i]_{a_{ij}}$ are relatively prime positive integers such that $[d_i, a_{ij}]_{1 \leq i, j \leq n}$ is symmetric positive definite.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, $\mathfrak{R}$ the associated root system, $\mathfrak{B} = \{\alpha_1, \ldots, \alpha_n\}$ a basis of $\mathfrak{R}$, $\mathfrak{R}_+$ the set of positive roots and $W$ the Weyl group. We denote by $\mathfrak{P}$ and $\mathfrak{Q}$ the lattices of weights and roots respectively. The fundamental weights are denoted by $\varpi_1, \ldots, \varpi_n$ and the set of dominant integral weights by $\mathfrak{P}^+ = \sum_{i=1}^n \mathbb{N} \varpi_i$. Let $(-, -)$ be a non-degenerate $\mathfrak{g}$-invariant symmetric bilinear form on $\mathfrak{g}$; it will identify $\mathfrak{g}$, resp. $\mathfrak{h}$, with its dual $\mathfrak{g}^*$, resp. $\mathfrak{h}^*$. The form $(-, -)$ can be chosen in order to induce a perfect pairing $\mathfrak{P}_g \times \mathfrak{Q}_g \rightarrow \mathbb{Z}$ such that

$$\langle \varpi_i, \alpha_j \rangle = \delta_{ij} d_i, \quad \langle \alpha_i, \alpha_j \rangle = d_i a_{ij}.$$ 

Hence $d_i = (\alpha_i, \alpha_i)/2$ and $(\alpha, \alpha) \in 2\mathbb{Z}$ for all $\alpha \in \mathfrak{R}$. For each $\alpha_j$ we denote by $h_j \in \mathfrak{h}$ the corresponding coroot: $\varpi_i(h_j) = \delta_{ij}$. We also set

$$n^\pm = \oplus_{\alpha \in \mathfrak{R}_+} \mathfrak{g}_\alpha, \quad \mathfrak{h}^\pm = \mathfrak{h} \oplus n^\pm, \quad \mathfrak{d} = \mathfrak{g} \times \mathfrak{g}, \quad \mathfrak{t} = \mathfrak{h} \times \mathfrak{h}, \quad \mathfrak{u}^\pm = n^\pm \times n^\mp.$$

Let $G$ be a connected complex semi-simple algebraic group such that $\text{Lie}(G) = \mathfrak{g}$ and set $D = G \times G$. We identify $G$ (and its subgroups) with the diagonal copy inside $D$. We denote by $\text{exp}$ the exponential map from $\mathfrak{d}$ to $D$. We shall in general denote a Lie subalgebra of $\mathfrak{d}$ by a gothic symbol and the corresponding connected analytic subgroup of $D$ by a capital letter.

1.2. **Poisson Lie group structure on $G$.** Let $a = \sum_{\alpha \in \mathfrak{R}^+} e_\alpha \wedge e_{-\alpha} \in \wedge^2 \mathfrak{g}$ where the $e_\alpha$ are root vectors such that $\langle e_\alpha, e_\beta \rangle = \delta_{\alpha, -\beta}$. Let $u \in \wedge^2 \mathfrak{h}$ and set $r = a - u$. Then it is well known that $r$ satisfies the modified Yang-Baxter equation $[3, 20]$ and that therefore the tensor $\pi(g) = (l_g)_* r - (r_g)_* r$ furnishes $G$ with the structure of a Poisson Lie group, see $[13, 22, 30]$ $(l_g)_*$ and $(r_g)_*$ are the differentials of the left and right translation by $g \in G$.

We may write $u = \sum_{1 \leq i, j \leq n} u_{ij} h_i \otimes h_j$ for a skew-symmetric $n \times n$ matrix $[u_{ij}]$. The element $u$ can be considered either as an alternating form on $\mathfrak{h}^*$ or a linear map $u \in \text{End} \mathfrak{h}$ by the formula

$$\forall x \in \mathfrak{h}, \quad u(x) = \sum_{i, j} u_{i,j}(x, h_i) h_j.$$ 

These results are algebraic analogs of results of Levendorskii $[20, 21]$ on the irreducible representations of multi-parameter function algebras and compact quantum groups. The bijection between symplectic leaves of the compact Poisson group and irreducible *-representations of the compact quantum group found by Soibelman in the one-parameter case, breaks down in the multi-parameter case.

After this work was completed, the authors became aware of the work of Constantini and Varagnolo $[4, 8]$ which has some overlap with the results in this paper.
Following [31] one sees easily that the associated Manin triple is \((\mathfrak{d}, \mathfrak{g}, \mathfrak{g}_r)\) where \(\mathfrak{g}\) is identified with the diagonal copy inside \(\mathfrak{d}\). Then the corresponding triple of Lie groups is \((D, G, G_r)\), where \(D = \text{exp}(\mathfrak{a})\) is an analytic torus and \(G_r = AU^+\). Notice that \(\mathfrak{g}_r\) is a solvable, but not in general algebraic, Lie subalgebra of \(\mathfrak{d}\).

The following is an easy consequence of the definition of \(\mathfrak{a}\) and the identities 
\[
 u_+ + u_- = 2u, u_+ - u_- = 2I:
\]
\[
 \mathfrak{a} = \{(x, y) \in \mathfrak{t} \mid x + y = u(y - x)\} = \{(x, y) \in \mathfrak{t} \mid u_+(x) = u_-(y)\}.
\]

(1.1)

Recall that \(\exp : \mathfrak{h} \to H\) is surjective; let \(L_H\) be its kernel. We shall denote by
\[
 X(K)
\]
The group of characters of an algebraic torus \(K\). Any \(\chi \in X(H)\) is given by \(\chi(\exp x) = \exp d\chi(x), x \in \mathfrak{h}\), where \(d\chi \in \mathfrak{h}^*\) is the differential of \(\chi\). Then
\[
 X(H) \cong L = L_H = \{\xi \in \mathfrak{h}^* \mid \xi(L_H) \subset 2i\pi\mathbb{Z}\}.
\]
One can show that \(L\) has a basis consisting of dominant weights.

Recall that if \(G\) is a connected simply connected algebraic group with Lie algebra \(\mathfrak{g}\) and maximal torus \(\tilde{H}\), we have
\[
 L_{\tilde{H}} = \mathfrak{P} = \oplus_{j=1}^n 2i\pi\mathbb{Z}h_j, \quad X(\tilde{H}) \cong \mathfrak{P},
\]
\[
 Q \subset L \subset \mathfrak{P}, \quad \pi_1(G) = L_H/\mathfrak{P} \cong \mathfrak{P}/L.
\]
Notice that \(L_H/\mathfrak{P}\) is a finite group and \(\exp u(L_H)\) is a subgroup of \(H\). We set
\[
 \Gamma_0 = \{(a, a) \in T \mid a^2 = 1\}, \quad \Delta = \{(a, a) \in T \mid a^2 \in \exp u(L_H)\},
\]
\[
 \Gamma = \Gamma_0 \cap H = \{(a, a) \in T \mid a = \exp x = \exp y, x + y = u(y - x)\}.
\]

It is easily seen that \(\Gamma = G \cap G_r\).

**Proposition 1.1.** We have \(\Delta = \Gamma \Gamma_0\).

**Proof.** We obviously have \(\Gamma_0 \subset \Delta\). Let \((\exp h, \exp h) \in \Gamma, h \in \mathfrak{h}\). By definition there exist \((x, y) \in \mathfrak{a}, \ell_1, \ell_2 \in L_H\) such that
\[
 x = h + \ell_1, \quad y = h + \ell_2, \quad y + x = u(y - x).
\]

Hence \(y + x = 2h + \ell_1 + \ell_2 = u(\ell_2 - \ell_1)\) and \((\exp h)^2 = \exp 2h = \exp u(\ell_2 - \ell_1)\).

This shows \((\exp h, \exp h) \in \Delta\). Thus \(\Gamma \Gamma_0 \subset \Delta\).

Let \((a, a) \in \Delta, a = \exp h, h \in \mathfrak{h}\). From \(a^2 \in \exp u(L_H)\) we get \(\ell, \ell' \in L_H\) such that \(2h = u(\ell_2) + \ell\). Set \(x = h - \ell/2 - \ell'/2\), \(y = h + \ell'/2 - \ell/2\). Then \(y + x = u(y - x)\) and we have \(\exp(-\ell/2 + \ell'/2) = \exp(\ell'/2 - \ell/2)\), since \(\ell' \in L_H\). If \(b = \exp(-\ell'/2 + \ell/2)\) we obtain \(\exp x = \exp y = ab^{-1}\), hence \((a, a) = (\exp x, \exp y)(b, b) \in \Gamma \Gamma_0\).

Therefore \(\Gamma \Gamma_0 = \Delta\).

**Remark.** When \(u\) is “generic” \(\Gamma_0\) is not contained in \(\Gamma\). For example, take \(G\) to be \(SL(3, \mathbb{C})\) and \(u = \alpha(h_1 \otimes h_2 - h_2 \otimes h_1)\) with \(\alpha \notin \mathbb{Q}\).

Considered as a Poisson variety, \(G\) decomposes as a disjoint union of symplectic leaves. Denote by \(\text{Symp} G\) the set of these symplectic leaves. Since \(r\) is \(H\)-invariant, translation by an element of \(H\) is a Poisson morphism and hence there is an induced action of \(H\) on \(\text{Symp} G\). The key to classifying the symplectic leaves is the following result, cf. [22, 31].

**Theorem 1.2.** The symplectic leaves of \(G\) are exactly the connected components of \(G \cap G_r x G_r\) for \(x \in G\).
Remark that $A$, $\Gamma$ and $G_r$ are in general not closed subgroups of $D$. This has for consequence that the analysis of $\text{Symp} \, G$ made in \cite{15} Appendix A in the case $u = 0$ does not apply in the general case.

Set $Q = HG_r = TU^+$. Then $Q$ is a Borel subgroup of $D$ and, recalling that the Weyl group associated to the pair $(G, T)$ is $W \times W$, the corresponding Bruhat decomposition yields $D = \sqcup_{w \in W \times W} QwQ = \sqcup_{w \in W \times W} QwG_r$. Therefore any symplectic leaf is contained in a Bruhat cell $QwQ$ for some $w \in W \times W$.

**Definition.** A leaf $A$ is said to be of type $w$ if $A \subset QwQ$. The set of leaves of type $w$ is denoted by $\text{Symp}_w(\mathfrak{g})$.

For each $w \in W \times W$ set $w = (w_+, w_-)$, $w_+ \in W$, and fix a representative $\dot{w}$ in the normaliser of $T$. One shows as in \cite{15} Appendix A that $G \cap Q\dot{w}G_r \neq \emptyset$, for all $w \in W \times W$; hence $\text{Symp}_w(\mathfrak{g}) \neq \emptyset$ and $G \cap G_r \dot{w}G_r \neq \emptyset$, since $QwQ = \cup_{h \in H} hG_r \dot{w}G_r$.

The adjoint action of $D$ on itself is denoted by $\text{Ad}$. Set

$$
U_w^- = \text{Ad} \, w(U) \cap U^+, \quad A'_w = \{ a \in A \mid awG_r = \dot{w}G_r \},
$$

$$
T'_w = \{ t \in T \mid tG_r \dot{w}G_r = G_r \dot{w}G_r \}, \quad H'_w = H \cap T'_w.
$$

Recall that $U_w^-$ is isomorphic to $\mathbb{C}^{l(w)}$ where $l(w) = l(w_+) + l(w_-)$ is the length of $w$. We set $s(w) = \dim H'_w$.

**Lemma 1.3.** (i) $A'_w = \text{Ad} \, w(A) \cap A$ and $T'_w = A \cdot \text{Ad} \, w(A) = AH'_w$.

(ii) We have $\text{Lie}(A'_w) = \mathfrak{a}'_w = \{ \vartheta(x) \mid x \in \text{Ker}(u_-, w_-^{-1}u_+ - u_+ w_-^{-1} u_-) \}$ and $\dim \mathfrak{a}'_w = n - s(w)$.

**Proof.** (i) The first equality is obvious and the second is an easy consequence of the bijection, induced by multiplication, between $U^- w \times T \times U^+$ and $QwQ = QwG_r$.

(ii) By definition we have $\mathfrak{a}'_w = \{ \vartheta(x) \mid x \in \mathfrak{h}, w^{-1}(\vartheta(x)) \in \mathfrak{a} \}$. From \cite{13} we deduce that $\vartheta(x) \in \mathfrak{a}'_w$ if and only if $u_+ w_+^{-1}(-u_-(x)) = u_- w_-^{-1}(-u_+(x))$.

It follows from (i) that $\dim T'_w = n + \dim H'_w = 2n - \dim A'_w$, hence $\dim \mathfrak{a}'_w = n - s(w)$.

Recall that $u \in \text{End} \, \mathfrak{h}$ is an alternating bilinear form on $\mathfrak{h}^*$. It is easily seen that $\forall \lambda, \mu \in \mathfrak{h}^*, u(\lambda, \mu) = -\langle t(u(\lambda), \mu) \rangle$, where $t(u) \in \text{End} \, \mathfrak{h}^*$ is the transpose of $u$.

**Notation.** Set $t^u = -\Phi, \Phi_{\pm} = \Phi \pm I, \sigma(w) = \Phi_{-w} \Phi_+ - \Phi_+ w \Phi_-$, where $w \in W$ is considered as an element of $\text{End} \, \mathfrak{h}^*$.

Observe that $t^u_{\pm} = -\Phi_{\mp}$ and that

$$
u(\lambda, \mu) = (\Phi \lambda, \mu), \quad \text{for all } \lambda, \mu \in \mathfrak{h}^*.
$$

Furthermore, since the transpose of $w \in \text{End} \, \mathfrak{h}^*$ is $w^{-1} \in \text{End} \, \mathfrak{h}$, we have $t^\sigma(w) = u_- w_-^{-1} u_+ - u_+ w_+^{-1} u_-$. Hence by Lemma \cite{13}

$$
s(w) = \dim Ker_{\mathfrak{h}^*} \sigma(w), \quad \dim A'_w = \dim Ker_{\mathfrak{h}^*} \sigma(w).
$$
1.3. The algebraic case. As explained in [4.3] the Lie algebra $\mathfrak{g}_r$ is in general not algebraic. We now describe its algebraic closure. Recall that a Lie subalgebra $\mathfrak{m}$ of $\mathfrak{g}$ is said to be algebraic if $\mathfrak{m}$ is the Lie algebra of a closed (connected) algebraic subgroup of $D$.

**Definition.** Let $\mathfrak{m}$ be a Lie subalgebra of $\mathfrak{g}$. The smallest algebraic Lie subalgebra of $\mathfrak{g}$ containing $\mathfrak{m}$ is called the algebraic closure of $\mathfrak{m}$ and will be denoted by $\tilde{\mathfrak{m}}$.

Recall that $\mathfrak{g}_r = \mathfrak{a} \oplus \mathfrak{u}^+$. Notice that $\mathfrak{u}^+$ is an algebraic Lie subalgebra of $\mathfrak{d}$; hence it follows from [4, Corollary II.7.7] that $\tilde{\mathfrak{g}}_r = \tilde{\mathfrak{a}} \oplus \mathfrak{u}^+$. Thus we only need to describe $\tilde{\mathfrak{a}}$. Since $\mathfrak{t}$ is algebraic we have $\tilde{\mathfrak{a}} \subseteq \mathfrak{t}$ and we are reduced to characterize the algebraic closure of a Lie subalgebra of $\mathfrak{t} = \text{Lie}(\mathcal{T})$.

The group $T = H \times H$ is an algebraic torus (of rank $2n$). The map $\chi \mapsto d\chi$ identifies $X(T)$ with $\mathbb{L} \times \mathbb{L}$.

Let $\mathfrak{h} \subset \mathfrak{t}$ be a subalgebra. We set $\mathfrak{h}^\perp = \{ \theta \in X(T) \mid \mathfrak{h} \subseteq \text{Ker}_t \theta \}$.

The following proposition is well known. It can for instance be deduced from the results in [4, II. 8].

**Proposition 1.4.** Let $\mathfrak{k}$ be a subalgebra of $\mathfrak{t}$. Then $\tilde{\mathfrak{k}} = \cap_{\theta \in \mathfrak{h}^\perp} \text{Ker}_t \theta$ and $\tilde{\mathfrak{k}}$ is the Lie algebra of the closed connected algebraic subgroup $\tilde{K} = \cap_{\theta \in \mathfrak{h}^\perp} \text{Ker}_T \theta$.

**Corollary 1.5.** We have

$$\tilde{\mathfrak{a}} = \cap_{(\lambda, \mu) \in \mathfrak{a}^+} \text{Ker}_t (\lambda, \mu), \quad \tilde{\mathfrak{A}} = \cap_{(\lambda, \mu) \in \mathfrak{a}^+} \text{Ker}_T (\lambda, \mu).$$

**Proof.** From the definition of $\mathfrak{a} = \partial(\mathfrak{h})$ we obtain

$$(\lambda, \mu) \in \mathfrak{a}^\perp \iff \forall x \in \mathfrak{h}, \lambda(-u_-(x)) + \mu(-u_+(x)) = 0.$$  

The first equality then follows from $\iota u_\pm = -\Phi_\mp$. The remaining assertions are consequences of Proposition 1.4. \(\square\)

Set

$$\mathfrak{h}_Q = \mathbb{Q} \otimes_\mathbb{Z} \mathbf{P}^\circ = \bigoplus_{i=1}^n \mathbb{Q} h_i, \quad \mathfrak{h}^*_Q = \mathbb{Q} \otimes_\mathbb{Z} \mathbf{P} = \bigoplus_{i=1}^n \mathbb{Q} \omega_i,$$

$$\mathfrak{a}_Q^\perp = \mathfrak{h}_Q \otimes_\mathbb{Z} \mathfrak{a}^\perp = \{ (\lambda, \mu) \in \mathfrak{h}_Q^* \times \mathfrak{h}_Q^* \mid \Phi_+ \lambda + \Phi_- \mu = 0 \}.$$  

Observe that $\dim_{\mathbb{Q}} \mathfrak{a}_Q^\perp = \text{rk}_{\mathbb{Q}} \mathfrak{a}^\perp$ and that, by Corollary 1.5,

$$\dim \tilde{\mathfrak{a}} = 2n - \dim_{\mathbb{Q}} \mathfrak{a}_Q^\perp. \quad (1.4)$$

**Lemma 1.6.** $\mathfrak{a}_Q^\perp \cong \{ \nu \in \mathfrak{h}_Q^* \mid \Phi \nu \in \mathfrak{h}_Q^* \}$.

**Proof.** Define a $\mathbb{Q}$-linear map

$$\{ \nu \in \mathfrak{h}_Q^* \mid \Phi \nu \in \mathfrak{h}_Q^* \} \rightarrow \mathfrak{a}_Q^\perp, \quad \nu \mapsto (-\Phi_- \nu, \Phi_+ \nu).$$

It is easily seen that this provides the desired isomorphism. \(\square\)
Theorem 1.7. The following assertions are equivalent:
(i) \( g_w \) is an algebraic Lie subalgebra of \( \mathfrak{g} \);
(ii) \( u(P \times P) \subset \mathbb{Q} \);
(iii) \( \exists m \in \mathbb{N}^* \), \( \Phi(mP) \subset P \);
(iv) \( \Gamma \) is a finite subgroup of \( T \).

Proof. Recall that \( g_w \) is algebraic if and only if \( a = \tilde{a} \), i.e. \( n = \dim a = \dim \tilde{a} \). By (1.4) and Lemma 1.6 this is equivalent to \( \Phi(P) \subset H_q^* = \mathbb{Q} \otimes \mathbb{Z} P \). The equivalence of (i) to (iii) then follows from the definitions, (1.2) and the fact that \( ^t u = -\Phi \).

To prove the equivalence with (iv) we first observe that, by Proposition 1.1, \( \Gamma \) is finite if and only if \( \exp u(L_H) \) is finite. Since \( L_H/P^0 \) is finite this is also equivalent to \( \exp u(P^0) \) being finite. This holds if and only if \( u(mP^0) \subset P^0 \) for some \( m \in \mathbb{N}^* \). Hence the result.

When the equivalent assertions of Theorem 1.7 hold, we shall say that we are in the algebraic case or that \( u \) is algebraic. In this case all the subgroups previously introduced are closed algebraic subgroups of \( D \) and we may define the algebraic quotient varieties \( D/G \), and \( \bar{G} = G/\Gamma \). Let \( p \) be the projection \( G \rightarrow \bar{G} \). Observe that \( \bar{G} \) is open in \( D/G \), and that the Poisson bracket of \( G \) passes to \( \bar{G} \). We set

\[
C_w = G_r w G_r / G_r, \quad C_w = Q w G_r / G_r = \bigcup_{h \in H} h C_{w h},
\]

\[
\mathcal{B}_w = C_w \cap \bar{G}, \quad B_w = C_w \cap \bar{G}, \quad A_w = p^{-1}(B_w).
\]

The next theorem summarizes the description of the symplectic leaves in the algebraic case.

Theorem 1.8. 1. \( \text{Symp}_w G \neq \emptyset \) for all \( w \in W \times W \), \( \text{Symp} \bar{G} = \bigcup_{w \in W \times W} \text{Symp}_w G \).

2. Each symplectic leaf of \( G \), resp. \( \bar{G} \), is of the form \( h \mathcal{B}_w \), resp. \( h A_w \), for some \( h \in H \) and \( w \in W \times W \), where \( \mathcal{B}_w \) denotes a fixed connected component of \( p^{-1}(B_w) \).

3. \( C_w \cong A_w \times U_w^- \) where \( A_w = A/A'_w \) is a torus of rank \( s(w) \). Hence \( \dim C_w = \dim B_w = \dim A_w = \ell(w) + s(w) \) and \( H/\text{Stab}_{H} A_w \) is a torus of rank \( n - s(w) \).

Proof. The proofs are similar to those given in [15] Appendix A] for the case \( u = 0 \). \( \square \)

2. Deformations of Bigraded Hopf Algebras

2.1. Bigraded Hopf Algebras and their deformations. Let \( L \) be an (additive) abelian group. We will say that a Hopf algebra \( (A, i, m, \epsilon, \Delta, S) \) over a field \( K \) is an \( L \)-bigraded Hopf algebra if it is equipped with an \( L \times L \) grading

\[
A = \bigoplus_{(\lambda, \mu) \in L \times L} A_{\lambda, \mu}
\]

such that

1. \( K \subset A_{0,0} \), \( A_{\lambda, \mu} A_{\lambda', \mu'} \subset A_{\lambda+\lambda', \mu+\mu'} \) (i.e. \( A \) is a graded algebra)
2. \( \Delta(A_{\lambda, \mu}) \subset \sum_{\nu \in L} A_{\lambda, \nu} \otimes A_{-\nu, \mu} \)
3. \( \lambda \neq -\mu \) implies \( \epsilon(A_{\lambda, \mu}) = 0 \)
4. \( S(A_{\lambda, \mu}) \subset A_{\mu, \lambda} \).
A similar calculation shows that $\sum \text{antipode}$. So $\Delta$ is also a homomorphism. Finally notice that

$$\lambda, \mu$$

are multiplicative in both entries and that, for all $\lambda, \mu \in L$,

$$\sum \Delta(x) = (\sum \epsilon, \mu) = (\lambda, -\lambda).$$

Then the map $\tilde{p} : (L \times L) \times (L \times L) \to \mathbb{K}^*$ given by

$$\tilde{p}(\lambda, \mu, (\lambda', \mu')) = p(\lambda, \lambda') p(\mu, \mu')^{-1}$$

is a 2-cocycle on $L \times L$ such that $\tilde{p}(0, 0) = 1$.

One may then define a new multiplication, $m_p$, on $A$ by

$$\forall a \in A_{\lambda, \mu}, b \in A_{\lambda', \mu'}, \quad a \cdot b = p(\lambda, \lambda') p(\mu, \mu')^{-1} ab. \quad (2.1)$$

**Theorem 2.1.** $A_p := (A, i, m_p, \epsilon, \Delta, S)$ is an $L$-bigraded Hopf algebra.

**Proof.** The proof is a slight generalization of that given in [2]. It is well known that $A_p = (A, i, m_p)$ is an associative algebra. Since $\Delta$ and $\epsilon$ are unchanged, $(A, \Delta, \epsilon)$ is still a coalgebra. Thus it remains to check that $\epsilon, \Delta$ are algebra morphisms and that $S$ is an antipode.

Let $x \in A_{\lambda, \mu}$ and $y \in A_{\lambda', \mu'}$. Then

$$\epsilon(x \cdot y) = p(\lambda, \lambda') p(\mu, \mu')^{-1} \epsilon(xy)$$

$$= p(\lambda, \lambda') p(\mu, \mu')^{-1} \delta_{\lambda, -\lambda} \delta_{\mu, -\mu} \epsilon(x) \epsilon(y)$$

$$= p(\lambda, \lambda') \delta_{\lambda, -\lambda} \delta_{\mu, -\mu} \epsilon(x) \epsilon(y)$$

$$= \epsilon(x) \epsilon(y)$$

So $\epsilon$ is a homomorphism. Now suppose that $\Delta(x) = \sum x_{\lambda, \nu} \otimes x_{-\nu, \mu}$ and $\Delta(y) = \sum y_{\lambda', \nu'} \otimes y_{-\nu', \mu'}$. Then

$$\Delta(x) \cdot \Delta(y) = \left(\sum x_{\lambda, \nu} \otimes x_{-\nu, \mu}\right) \cdot \left(\sum y_{\lambda', \nu'} \otimes y_{-\nu', \mu'}\right)$$

$$= \sum x_{\lambda, \nu} \cdot y_{\lambda', \nu'} \otimes x_{-\nu, \mu} \cdot y_{-\nu', \mu'}$$

$$= p(\lambda, \lambda') p(\mu, \mu')^{-1} \sum p(\nu, \nu')^{-1} p(-\nu, -\nu') x_{\lambda, \nu} y_{\lambda', \nu'} \otimes x_{-\nu, \mu} y_{-\nu', \mu'}$$

$$= p(\lambda, \lambda') p(\mu, \mu')^{-1} \Delta(xy)$$

$$= \Delta(x \cdot y)$$

So $\Delta$ is also a homomorphism. Finally notice that

$$\sum S(x_{(1)}) \cdot x_{(2)} = \sum S(x_{\lambda, \nu}) \cdot x_{-\nu, \mu}$$

$$= \sum p(\nu, -\nu) p(\lambda, \mu)^{-1} S(x_{\lambda, \nu}) x_{-\nu, \mu}$$

$$= p(\lambda, \mu)^{-1} \sum S(x_{\lambda, \nu}) \cdot x_{-\nu, \mu}$$

$$= p(\lambda, \mu)^{-1} \epsilon(x)$$

$$= \epsilon(x)$$

A similar calculation shows that $\sum x_{(1)} \cdot S(x_{(2)}) = \epsilon(x)$. Hence $S$ is indeed an antipode. \qed
Remark. The isomorphism class of the algebra $A_p$ depends only on the cohomology class \([\overline{p}] \in H^2(L \times L, \mathbb{K}^*)\), [8] §3.

Remark. Theorem 2.1 is a particular case of the following more general construction. Let \((A, i, m)\) be a \(\mathbb{K}\)-algebra. Assume that $F \in GL_K(A \otimes A)$ is given such that (with the usual notation)

1. $F(m \otimes 1) = (m \otimes 1)F_{23}F_{13}$ ; $F(1 \otimes m) = (1 \otimes m)F_{13}F_{12}$
2. $F(i \otimes 1) = i \otimes 1$ ; $F(1 \otimes i) = 1 \otimes i$
3. $F_{12}F_{13}F_{23} = F_{23}F_{13}F_{12}$, i.e. $F$ satisfies the Quantum Yang-Baxter Equation.

Set $m_F = m \circ F$. Then \((A, i, m_F)\) is a \(\mathbb{K}\)-algebra.

Assume furthermore that \((A, i, m, \epsilon, \Delta, S)\) is a Hopf algebra and that

4. $F : A \otimes A \rightarrow A \otimes A$ is morphism of coalgebras
5. $mF(S \otimes 1)\Delta = m(S \otimes 1)\Delta$ ; $mF(1 \otimes S)\Delta = m(1 \otimes S)\Delta$.

Then $A_F := (A, i, m_F, \epsilon, \Delta, S)$ is a Hopf algebra. The proofs are straightforward verifications and are left to the interested reader.

When $A$ is an $L$-bigraded Hopf algebra and $p$ is an antisymmetric bicharacter as above, we may define $F \in GL_K(A \otimes A)$ by

$$\forall a \in A_{\lambda, \mu}, \forall b \in A_{\lambda', \mu'}, \quad F(a \otimes b) = p(\lambda, \lambda')p(\mu, \mu')^{-1}a \otimes b.$$ 

It is easily checked that $F$ satisfies the conditions (1) to (5) and that the Hopf algebras $A_F$ and $A_p$ coincide.

A related construction of the quantization of a monoidal category is given in [24].

### 2.2. Diagonalizable subgroups of $R(A)$.

In the case where $L$ is a finitely generated group and $A$ is a finitely generated algebra (which is the case for the multi-parameter quantum groups considered here), there is a simple geometric interpretation of $L$-bigradings. They correspond to algebraic group maps from the diagonalizable group $L'$ to the group of one dimensional representations of $A$.

Assume that $\mathbb{K}$ is algebraically closed. Let \((A, i, m, \epsilon, \Delta, S)\) be a Hopf $\mathbb{K}$-algebra. Denote by $R(A)$ the multiplicative group of one dimensional representations of $A$, i.e. the character group of the algebra $A$. Notice that when $A$ is a finitely generated $\mathbb{K}$-algebra, $R(A)$ has the structure of an affine algebraic group over $\mathbb{K}$, with algebra of regular functions given by $\mathbb{K}[R(A)] = A/J$ where $J$ is the semiprime ideal $\cap_{h \in R(A)} \ker h$. Recall that there are two natural group homomorphisms $l, r : R(A) \rightarrow Aut_{\mathbb{K}}(A)$ given by

$$l_h(x) = \sum h(S(x_{(1)}))x_{(2)} = \sum h^{-1}(x_{(1)})x_{(2)}$$
$$r_h(x) = \sum x_{(1)}h(x_{(2)}).$$

**Theorem 2.2.** Let $A$ be a finitely generated Hopf algebra and let $L$ be a finitely generated abelian group. Then there is a natural bijection between:

1. $L$-bigradings on $A$;
2. Hopf algebra maps $A \rightarrow \mathbb{K}L$ (where $\mathbb{K}L$ denotes the group algebra);
3. morphisms of algebraic groups $L' \rightarrow R(A)$.

**Proof.** The bijection of the last two sets of maps is well-known. Given an $L$-bigrading on $A$, we may define a map $\phi : A \rightarrow \mathbb{K}L$ by $\phi(a_{\lambda, \mu}) = \epsilon(a)u_{\lambda}$. It is easily verified that this is a Hopf algebra map. Conversely, given a map $L' \rightarrow R(A)$ we may construct an $L$ bigrading using the following result.
Theorem 2.3. Let $(A, i, m, \epsilon, \Delta, S)$ be a finitely generated Hopf algebra over $\mathbb{K}$. Let $H$ be a closed diagonalizable algebraic subgroup of $R(A)$. Denote by $L$ the (additive) group of characters of $H$ and by $\langle - , - \rangle : L \times H \to \mathbb{K}^*$ the natural pairing. For $(\lambda, \mu) \in L \times L$ set

$$A_{\lambda, \mu} = \{ x \in A \mid \forall h \in H, l_h(x) = \langle \lambda, h \rangle x, r_h(x) = \langle \mu, h \rangle x \}.$$ 

Then $(A, i, m, \epsilon, \Delta, S)$ is an $L$-bigraded Hopf algebra.

Proof. Recall that any element of $A$ is contained in a finite dimensional subcoalgebra of $A$. Therefore the actions of $H$ via $r$ and $l$ are locally finite. Since they commute and $H$ is diagonalizable, $A$ is $L \times L$ diagonalizable. Thus the decomposition $A = \bigoplus_{(\lambda, \mu) \in L \times L} A_{\lambda, \mu}$ is a grading.

Now let $C$ be a finite dimensional subcoalgebra of $A$ and let $\{c_1, \ldots, c_n\}$ be a basis of $H \times H$ weight vectors. Suppose that $\Delta(c_i) = \sum t_{ij} \otimes c_j$. Then since $c_i = \sum t_{ij} c_j$, the $t_{ij}$ span $C$ and it is easily checked that $\Delta(t_{ij}) = \sum t_{ik} \otimes t_{kj}$. Since $l_h(c_i) = h^{-1}(t_{ij}) c_j$ for all $h \in H$ and the $c_i$ are weight vectors, we must have that $h(t_{ij}) = 0$ for $i \neq j$. This implies that

$$l_h(t_{ij}) = h^{-1}(t_{ij}) t_{ij}, \quad r_h(t_{ij}) = h(t_{jj}) t_{ij}$$

and that the map $\lambda_i(h) = h(t_{ii})$ is a character of $H$. Thus $t_{ij} \in A_{-\lambda_i, \lambda_j}$ and hence

$$\Delta(t_{ij}) = \sum t_{ik} \otimes t_{kj} \in \sum A_{-\lambda_i, \lambda_k} \otimes A_{-\lambda_k, \lambda_j}.$$ 

This gives the required condition on $\Delta$. If $\lambda + \mu \neq 0$ then there exists an $h \in H$ such that $\langle - , h \rangle \neq (\mu, h)$. Let $x \in A_{\lambda, \mu}$. Then

$$\langle (\mu, h) \epsilon(x) = \epsilon(r_h(x)) = h(x) = \epsilon(l_h^{-1}(x)) = \langle - , h \rangle \epsilon(x).$$

Hence $\epsilon(x) = 0$. The assertion on $S$ follows similarly. \qed

Remark. In particular, if $G$ is any algebraic group and $H$ is a diagonalizable subgroup with character group $L$, then we may deform the Hopf algebra $\mathbb{K}[G]$ using an antisymmetric bicharacter on $L$. Such deformations are algebraic analogs of the deformations studied by Rieffel in [27].

2.3. Deformations of dual pairs. Let $A$ and $U$ be a dual pair of Hopf algebras. That is, there exists a bilinear pairing $\langle \ | \rangle : A \times U \to \mathbb{K}$ such that:

1. $\langle a | 1 \rangle = \epsilon(a) \ ; \ \langle 1 | u \rangle = \epsilon(u)$
2. $\langle a_1 u_1 u_2 \rangle = \sum \langle a_{(1)} | u_1 \rangle \langle a_{(2)} | u_2 \rangle$
3. $\langle a_1 a_2 | u \rangle = \sum \langle a_{(1)} | u_{(1)} \rangle \langle a_{(2)} | u_{(2)} \rangle$
4. $\langle S(a) | u \rangle = \langle a | S(u) \rangle$

Assume that $A$ is bigraded by $L$, $U$ is bigraded by an abelian group $Q$ and that there is a homomorphism $\gamma : Q \to L$ such that

$$\langle A_{\lambda, \mu} | U_{\gamma, \delta} \rangle \neq 0 \text{ only if } \lambda + \mu = \bar{\gamma} + \bar{\delta}.$$ 

(2.2)

In this case we will call the pair $\{A, U\}$ an $L$-bigraded dual pair. We shall be interested in $\S 3$ and $\S 4$ in the case where $Q = L$ and $^* = Id$. 


Remark. Suppose that the bigradings above are induced from subgroups $H$ and $\hat{H}$ of $R(A)$ and $R(U)$ respectively and that the map $Q \to L$ is induced from a map $h \mapsto \hat{h}$ from $H$ to $\hat{H}$. Then the condition on the pairing may be restated as the fact that the form is ad-invariant in the sense that for all $a \in A$, $u \in U$ and $h \in H$,  

$$\langle a, u | h \rangle = \langle a | Ad_h u \rangle$$

where $Ad_h a = r_h l_h(a)$.

Theorem 2.4. Let $\{A, U\}$ be the bigraded dual pair as described above. Let $p$ be an antisymmetric bicharacter on $L$ and let $p$ be the induced bicharacter on $Q$. Define a bilinear form $\langle \cdot \mid \cdot \rangle_p: A_{p^{-1}} \times U_p \to \mathbb{K}$ by:

$$\langle a_{\lambda, \mu} | u_{\gamma, \delta} \rangle_p = p(\lambda, \gamma)^{-1} p(\mu, \delta)^{-1} \langle a_{\lambda, \mu} \mid u_{\gamma, \delta} \rangle.$$

Then $\langle \cdot \mid \cdot \rangle_p$ is a Hopf pairing and $\{A_{p^{-1}}, U_p\}$ is an $L$-bigraded dual pair.

Proof. Let $a \in A_{\lambda, \mu}$ and let $u_i \in U_{\gamma_i, \delta_i}$, $i = 1, 2$. Then

$$\langle a | u_1 u_2 \rangle_p = p(\gamma_1, \gamma_2) p(\delta_1, \delta_2)^{-1} p(\lambda, \gamma_1 + \gamma_2)^{-1} p(\mu, \delta_1 + \delta_2)^{-1} \langle a \mid u_1 u_2 \rangle.$$

Suppose that $\Delta(a) = \sum_{\nu} a_{\lambda, \nu} \otimes a_{-\nu, \mu}$. Then by the assumption on the pairing, the only possible value of $\nu$ for which $\langle a_{\lambda, \nu} | u_1 \rangle \langle a_{-\nu, \mu} | u_2 \rangle$ is non-zero is $\nu = \gamma_1 + \delta_1 - \lambda = \mu - \gamma_2 - \delta_2$. Therefore

$$\langle a(1) | u_1 \rangle \langle a(2) | u_2 \rangle_p = p(\lambda, \gamma_1)^{-1} p(\nu, \delta_1)^{-1} p(\mu, \delta_2)^{-1} \langle a(1) | u_1 \rangle \langle a(2) | u_2 \rangle$$

$$= p(\lambda, \gamma_1)^{-1} p(\mu - \gamma_2 - \delta_2, \delta_1)^{-1} \langle a(1) | u_1 \rangle \langle a(2) | u_2 \rangle$$

$$= p(\gamma_1, \gamma_2) p(\delta_1, \delta_2)^{-1} p(\lambda, \gamma_1 + \gamma_2)^{-1} p(\mu, \delta_1 + \delta_2)^{-1} \langle a | u_1 u_2 \rangle = \langle a | u_1 u_2 \rangle_p.$$

This proves the first axiom. The others are verified similarly.\hfill \square

Corollary 2.5. Let $\{A, U, p\}$ be as in Theorem 2.4. Let $M$ be a right $A$-comodule with structure map $\rho : M \to M \otimes A$. Then $M$ is naturally endowed with $U$ and $U_p$ left module structures, denoted by $(u, x) \mapsto ux$ and $(u, x) \mapsto u \cdot x$ respectively. Assume that $M = \bigoplus_{\lambda \in \Gamma} M_{\lambda}$ for some $\mathbb{K}$-subspaces such that $p(M_{\lambda}) \subset \sum_{\nu} M_{-\nu} \otimes A_{\nu, \lambda}$. Then we have $U_{\gamma, \delta} M_{\lambda} \subset M_{\lambda - \gamma - \delta}$ and the two structures are related by

$$\forall u \in U_{\gamma, \delta}, \forall x \in M_{\lambda}, \quad u \cdot x = p(\lambda, \gamma - \delta)p(\gamma, \delta)ux.$$

Proof. Notice that the coalgebras $A$ and $A_{p^{-1}}$ are the same. Set $\rho(x) = \sum x(0) \otimes x(1)$ for all $x \in M$. Then it is easily checked that the following formulas define the desired $U$ and $U_p$ module structures:

$$\forall u \in U, \quad ux = \sum x(0) \langle x(1) | u \rangle, \quad u \cdot x = \sum x(0) \langle x(1) | u \rangle_p.$$

When $x \in M_{\lambda}$ and $u \in U_{\gamma, \delta}$ the additional condition yields

$$\forall u \in U, \quad u \cdot x = \sum x(0) \rho(\nu, -\gamma)p(\lambda, -\delta)x(1) \mid u \rangle.$$

But $\langle x(1) | u \rangle \neq 0$ forces $-\nu = \lambda - \gamma - \delta$, hence $\langle x(1) | u \rangle = p(\lambda, \gamma - \delta)p(\gamma, \delta)\sum x(0) \langle x(1) | u \rangle = p(\lambda, \gamma - \delta)p(\gamma, \delta)ux.$ \hfill \square
Denote by $A^{\text{op}}$ the opposite algebra of the $K$-algebra $A$. Let $\{A^{\text{op}}, U, \langle | \rangle\}$ be a dual pair of Hopf algebras. The double $A \otimes U$ is defined as follows, \[**3.3**]. Let $I$ be the ideal of the tensor algebra $T(A \otimes U)$ generated by elements of type

$$1 - 1_A, \quad 1 - 1_U$$

(a)

$$xx' - x \otimes x', \quad x, x' \in A, \quad yy' - y \otimes y', \quad y, y' \in U$$

(b)

$$x_1 \otimes y_1 \langle x_2 | y_2 \rangle - \langle x_1 | y_1(x_2 \otimes y_2), \quad x \in A, \quad y \in U$$

(c)

Then the algebra $A \otimes U := T(A \otimes U)/I$ is called the Drinfeld double of $\{A, U\}$. It is a Hopf algebra in a natural way:

$$\Delta(a \otimes u) = (a(1) \otimes u(1)) \otimes (a(2) \otimes u(2)),$$

$$\epsilon(a \otimes u) = \epsilon(a) \epsilon(u), \quad S(a \otimes u) = (S(a) \otimes 1)(1 \otimes S(u)).$$

Notice for further use that $A \otimes U$ can equally be defined by relations of type (a), (b), $(c_{x,y})$ or (a), (b), $(c_{y,x})$, where we set

$$x \otimes y = (x_1 | y_1) \langle x_2 | y(3) \rangle S(y(3)) y(2) \otimes x_2, \quad x \in A, \quad y \in U$$

$(c_{x,y})$

$$y \otimes x = (x_1 | S(y(1))) \langle x_2 | y(3) \rangle y(2) \otimes y(3), \quad x \in A, \quad y \in U$$

$(c_{y,x})$

**Theorem 2.6.** Let $\{A^{\text{op}}, U\}$ be an $L$-bigraded dual pair, $p$ be an antisymmetric bicharacter on $L$ and $\tilde{p}$ be the induced bicharacter on $Q$. Then $A \otimes U$ inherits an $L$-bigrading and there is a natural isomorphism of $L$-bigraded Hopf algebras:

$$(A \otimes U)_{p} \cong A_{p} \otimes U_{\tilde{p}}.$$  

**Proof.** Recall that as a $K$-vector space $A \otimes U$ identifies with $A \otimes U$. Define an $L$-bigrading on $A \otimes U$ by

$$\forall \lambda, \mu \in L, \quad (A \otimes U)_{\lambda, \mu} = \sum_{\gamma, \delta \in \lambda, \mu} A_{\lambda, \mu} \otimes U_{\gamma, \delta}.$$  

To verify that this yields a structure of graded algebra on $A \otimes U$ it suffices to check that the defining relations of $A \otimes U$ are homogeneous. This is clear for relations of type (a) or (b). Let $x_{\lambda, \mu} \in A_{\lambda, \mu}$ and $y_{\gamma, \delta} \in U_{\gamma, \delta}$. Then the corresponding relation of type (c) becomes

$$\sum_{\nu, \xi} x_{\lambda, \mu} y_{\gamma, \delta} \langle x_{\nu, \mu} | y_{\xi, \lambda} \rangle - \langle x_{\lambda, \mu} | y_{\gamma, \delta} \rangle y_{\xi, \lambda} x_{\nu, \mu}.$$  

$(\star)$

When a term of this sum is non-zero we obtain $-\nu + \mu = -\xi + \delta$, $\lambda + \nu = \gamma + \xi$. Hence $\lambda - \gamma = -\nu + \delta = -\mu + \delta$, which shows that the relation $(\star)$ is homogeneous. It is easy to see that the conditions (2), (3), (4) of \[**2.1**\] hold. Hence $A \otimes U$ is an $L$-bigraded Hopf algebra.

Notice that $(A_{p})^{\text{op}} \cong (A^{\text{op}})_{p^{-1}}$, so that Theorem \[**2.4**\] defines a suitable pairing between $(A_{p})^{\text{op}}$ and $U_{\tilde{p}}$. Thus $A_{p} \otimes U_{\tilde{p}}$ is defined. Let $\phi$ be the natural surjective homomorphism from $T(A \otimes U)$ onto $A_{p} \otimes U_{\tilde{p}}$. To check that $\phi$ induces an isomorphism it again suffices to check that $\phi$ vanishes on the defining relations of $(A \otimes U)_{p}$. Again, this is easy for relations of type (a) and (b). The relation $(\star)$ says that

$$p(\lambda, \gamma)p(\nu, \xi)p(\nu, \mu | y_{\xi, \lambda} \cdot y_{\gamma, \delta} - p(\xi, \nu)p(\delta, \mu | y_{\gamma, \xi}y_{\xi, \lambda} \cdot x_{\nu, \mu} = 0.$$
in \((A \rtimes U)_p\). Multiply the left hand side of this equation by \(p(\lambda, -\gamma)p(\mu, -\delta)\) and apply \(\phi\). We obtain the following expression in \(A_p \rtimes U_p\):

\[
p(-\nu, \xi)\phi(\mu, -\delta)(x_{-\nu, \mu} | y_{-\xi, \delta})x_{\lambda, \nu}y_{\gamma, \xi} - p(\lambda, -\gamma)p(\nu, -\xi)\phi(\lambda, \gamma)p(y_{\gamma, \xi}y_{-\xi, \delta}x_{-\nu, \mu})
\]

which is equal to

\[
(x_{-\nu, \mu} | y_{-\xi, \delta})p(x_{\lambda, \nu}y_{\gamma, \xi} - \phi(\mu, -\delta)) - \phi(\lambda, \gamma)p(y_{\gamma, \xi}y_{-\xi, \delta}x_{-\nu, \mu}).
\]

But this is a defining relation of type (c) in \(A_p \rtimes U_p\), hence zero.

It remains to see that \(\phi\) induces an isomorphism of Hopf algebras, which is a straightforward consequence of the definitions.

\[\square\]

### 2.4. Cocycles

Let \(L\) be, in this section, an arbitrary free abelian group with basis \(\{\omega_1, \ldots, \omega_n\}\) and set \(h^* = \mathbb{C} \otimes_{\mathbb{Z}} L\). We freely use the terminology of \([2]\). Recall that \(H^2(L, \mathbb{C}^*)\) is in bijection with the set \(\mathcal{H}\) of multiplicatively antisymmetric \(n \times n\)-matrices \(\gamma = [\gamma_{ij}]\). This bijection maps the class \([c]\) onto the matrix defined by \(\gamma_{ij} = c(\omega_i, \omega_j)/c(\omega_j, \omega_i)\). Furthermore it is an isomorphism of groups with respect to component-wise multiplication of matrices.

**Remark.** The notation is as in \([2,1]\). We recalled that the isomorphism class of the algebra \(A_p\) depends only on the cohomology class \([\hat{p}] \in H^2(L \times L, \mathbb{K}^*)\). Let \(\gamma \in \mathcal{H}\) be the matrix associated to \(p\) and \(\gamma^{-1}\) its inverse in \(\mathcal{H}\). Notice that the multiplicative matrix associated to \(\hat{p}\) is then \(\hat{\gamma} = [\gamma_{ij}^{-1}]\) in the basis given by the \((\omega_i, 0), (0, \omega_i) \in L \times L\). Therefore the isomorphism class of the algebra \(A_p\) depends only on the cohomology class \([p] \in H^2(L, \mathbb{K}^*)\).

Let \(h \in \mathbb{C}^*\). If \(x \in \mathbb{C}\) we set \(q^x = \exp(-xh/2)\). In particular \(q = \exp(-h/2)\). Let \(u : L \times L \to \mathbb{C}\) be a complex alternating \(Z\)-bilinear form. Define

\[
p : L \times L \to \mathbb{C}^*, \quad p(\lambda, \mu) = \exp \left( -\frac{h}{4} u(\lambda, \mu) \right) = \exp \left( \frac{h}{2} u(\lambda, \mu) \right) = q^{\frac{1}{2}u(\lambda, \mu)}.
\]

Then it is clear that \(p\) is an antisymmetric bicharacter on \(L\).

Observe that, since \(h^* = \mathbb{C} \otimes_{\mathbb{Z}} L\), there is a natural isomorphism of additive groups between \(\wedge^2 h\) and the group of complex alternating \(Z\)-bilinear forms on \(L\), where \(h\) is the \(\mathbb{C}\)-dual of \(h^*\). Set \(Z_h = \{ u \in \wedge^2 h \mid u(L \times L) \subset \frac{4i\pi}{h} \mathbb{Z} \}\).

**Theorem 2.7.** There are isomorphisms of abelian groups:

\[
H^2(L, \mathbb{C}^*) \cong \mathcal{H} \cong \wedge^2 h / Z_h.
\]

**Proof.** The first isomorphism has been described above. Let \(\gamma = [\gamma_{ij}] \in \mathcal{H}\) and choose \(u_{ij}, 1 \leq i < j \leq n\) such that \(\gamma_{ij} = \exp(-\frac{h}{2} u_{ij})\). We can define \(u \in \wedge^2 h\) by setting \(u(\omega_i, \omega_j) = u_{ij}, 1 \leq i < j \leq n\). It is then easily seen that one can define an injective morphism of abelian groups

\[
\varphi : H^2(L, \mathbb{C}^*) \cong \mathcal{H} \longrightarrow \wedge^2 h / Z_h, \quad \varphi(\gamma) = [u]
\]

where \([u]\) is the class of \(u\). If \(u \in \wedge^2 h\), define a 2-cocycle \(p\) by the formula (2.3). Then the multiplicative matrix associated to \([p] \in H^2(L, \mathbb{C}^*)\) is given by

\[
\gamma_{ij} = p(\omega_i, \omega_j)/p(\omega_j, \omega_i) = p(\omega_i, \omega_j)^2 = \exp \left( \frac{h}{2} u(\omega_i, \omega_j) \right).
\]

This shows that \([u] = \varphi([\gamma_{ij}]])\); thus \(\varphi\) is an isomorphism. \[\square\]
We list some consequences of Theorem 2.7. We denote by \([u]\) an element of \(\wedge^2 h/\mathbb{Z}\) and we set \([p]=\varphi^{-1}([u])\). We have seen that we can define a representative \(p\) by the formula (2.3).

1. \([p]\) of finite order in \(H^2(L, \mathbb{C}^*) \Leftrightarrow u(L \times L) \subset i\pi \mathbb{Q}\), and \(q\) root of unity \(\Leftrightarrow \mathbb{Q}\). Notice that \(u=0\) is algebraic, whether \(q\) is a root of unity or not. Assume that \(q\) is a root of unity; then we get from 1 that \([p]\) of finite order \(\Leftrightarrow u\) is algebraic.

2. Assume that \(q\) is not a root of unity and that \(u\neq 0\). Then \([p]\) of finite order implies \((0) \neq u(L \times L) \subset i\pi \mathbb{Q}\). This shows that \(0 \neq u\) algebraic \(\Rightarrow [p]\) is not of finite order.

**Definition.** The bicharacter \(p: (\lambda, \mu) \mapsto q^{\frac{1}{2}u(\lambda, \mu)}\) is called \(q\)-rational if \(u\in \wedge^2 h\) is algebraic.

The following technical result will be used in the next section. Recall the definition of \(\Phi_-=\Phi-I\) given in the Section 1.

**Proposition 2.8.** Let \(K = \{\lambda \in L : (\Phi_-, \lambda, L) \subset \frac{4i\pi}{\hbar} \mathbb{Q}\}\). If \(\lambda\) is not a root of unity, then \(K = 0\).

**Proof.** Let \(\lambda \in K\). We can define \(z: h^*_Q \to \mathbb{Q}\), by

\[
\forall \mu \in h^*_Q, \quad (\Phi_-, \lambda, \mu) = \frac{4i\pi}{\hbar} z(\mu).
\]

The map \(z\) is clearly \(\mathbb{Q}\)-linear. It follows, since \((\ , \ )\) is non-degenerate on \(h^*_Q\), that there exists \(\nu \in h^*_Q\) such that \(z(\mu) = (\nu, \mu)\) for all \(\mu \in h^*_Q\). Therefore \(\Phi_-\lambda = \frac{4i\pi}{\hbar} \nu\), and so \(\Phi\lambda = \lambda + \frac{4i\pi}{\hbar} \nu\).

Now, \((\Phi\lambda, \lambda) = u(\lambda, \lambda) = 0\) implies that

\[
\frac{4i\pi}{\hbar}(\nu, \lambda) = -(\lambda, \lambda)
\]

If \((\lambda, \lambda) \neq 0\) then \(\hbar \in i\pi \mathbb{Q}\), contradicting the assumption that \(q\) is not a root of unity. Hence \((\lambda, \lambda) = 0\), which forces \(\lambda = 0\) since \(\lambda \in L \subset h^*_Q\).

\[
3. \text{Multiparameter Quantum Groups}
\]

**3.1. One-parameter quantized enveloping algebras.** The notation is as in sections 1 and 2. In particular we fix a lattice \(L\) such that \(\mathbb{Q} \subset L \subset \mathbb{P}\) and we denote by \(G\) the connected semi-simple algebraic group with maximal torus \(H\) such that \(\text{Lie}(G) = \mathfrak{g}\) and \(X(H) \cong L\).

Let \(q \in \mathbb{C}^*\) and assume that \(q\) is not a root of unity. Let \(\hbar \in \mathbb{C} \setminus i\pi \mathbb{Q}\) such that \(q = \exp(-\hbar/2)\) as in 2.4. We set

\[
q_i = q^{d_i}, \quad \hat{q}_i = (q_i - q_i^{-1})^{-1}, \quad 1 \leq i \leq n.
\]

Denote by \(U^0\) the group algebra of \(X(H)\), hence

\[
U^0 = \mathbb{C}[k_\lambda : \lambda \in L], \quad k_0 = 1, \quad k_\lambda k_\mu = k_{\lambda+\mu}.
\]
Set $k_i = k_{\alpha_i}$, $1 \leq i \leq n$. The one parameter quantized enveloping algebra associated to this data, cf. \[33\], is the Hopf algebra

$$U_q(\mathfrak{g}) = U^0[e_i, f_i; 1 \leq i \leq n]$$

with defining relations:

$$k_{ij}e_i k_i^{-1} = q^{(\lambda_i, \alpha_j)}e_i, \quad k_{ij}f_i k_i^{-1} = q^{-(\lambda_i, \alpha_j)}f_i$$

$$e_i f_j - f_j e_i = \delta_{ij}q_i(k_i - k_i^{-1})$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \left[ \frac{1-a_{ij}}{k} \right] q_i^{1-a_{ij}-k} e_i^{1-a_{ij}-k} e_j e_i^k = 0, \text{ if } i \neq j$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \left[ \frac{1-a_{ij}}{k} \right] q_i^{1-a_{ij}-k} f_i^{1-a_{ij}-k} f_j e_i^k = 0, \text{ if } i \neq j$$

where $[m]_k = (t - t^{-1}) \ldots (t^m - t^{-m})$ and $[m]_k = \frac{[m]}{[m]_k}$. The Hopf algebra structure is given by

$$\Delta(k_i) = k_i \otimes k_i, \quad \epsilon(k_i) = 1, \quad S(k_i) = k_i^{-1}$$

$$\Delta(e_i) = e_i \otimes 1 + k_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes k_i^{-1} + 1 \otimes f_i$$

$$\epsilon(e_i) = \epsilon(f_i) = 0, \quad S(e_i) = -k_i^{-1}e_i, \quad S(f_i) = -f_i k_i.$$

We define subalgebras of $U_q(\mathfrak{g})$ as follows

$$U_q(n^+) = \mathbb{C}[e_i; 1 \leq i \leq n], \quad U_q(n^-) = \mathbb{C}[f_i; 1 \leq i \leq n]$$

$$U_q(b^+) = U^0[e_i; 1 \leq i \leq n], \quad U_q(b^-) = U^0[f_i; 1 \leq i \leq n].$$

For simplicity we shall set $U^\pm = U_q(n^\pm)$. Notice that $U^0$ and $U_q(b^\pm)$ are Hopf subalgebras of $U_q(\mathfrak{g})$. Recall \[23\] that the multiplication in $U_q(\mathfrak{g})$ induces isomorphisms of vector spaces

$$U_q(\mathfrak{g}) \cong U^- \otimes U^0 \otimes U^+ \cong U^+ \otimes U^0 \otimes U^-.$$

Set $Q^+_+ = \oplus_{i=1}^n \mathbb{N}q_i$ and

$$\forall \beta \in Q^+_+, \quad U^+_\beta = \{u \in U^\pm \mid \forall \lambda \in L, \ k_{\lambda} u k_{\lambda}^{-1} = q^{(\lambda, \pm \beta)} u\}.$$

Then one gets: $U^\pm = \oplus_{\beta \in Q^+_+} U^\pm_{\beta\beta}$.

### 3.2. The Rosso-Tanisaki-Killing form

Recall the following result, \[23\].

**Theorem 3.1.** 1. There exists a unique non degenerate Hopf pairing

$$\langle \cdot | \cdot \rangle : U_q(b^+)^{op} \otimes U_q(b^-) \rightarrow \mathbb{C}$$

satisfying the following conditions:

(i) $\langle k_{\lambda} | k_{\mu} \rangle = q^{-(\lambda, \mu)}$;

(ii) $\forall \lambda \in L, 1 \leq i \leq n, \langle k_{\lambda} | f_i \rangle = \langle e_i | k_{\lambda} \rangle = 0$;

(iii) $\forall i \leq j \leq n, \langle e_i | f_j \rangle = -\delta_{ij}q_i$.

2. If $\gamma, \eta \in Q^+_+, \ (U^+_\gamma | U^-_{\eta}) \neq 0$ implies $\gamma = \eta$.

The results of \[23\] then apply and we may define the associated double:

$$D_q(\mathfrak{g}) = U_q(b^+) \bowtie U_q(b^-).$$
It is well known, e.g. [11], that
\[
D_q(\mathfrak{g}) = \mathbb{C}[s_\lambda, t_\lambda, e_i, f_i; \lambda \in \mathbb{L}, 1 \leq i \leq n]
\]
where \(s_\lambda = k_\lambda \otimes 1, t_\lambda = 1 \otimes k_\lambda, e_i = e_i \otimes 1, f_i = 1 \otimes f_i\). The defining relations of the double given in [2.3] imply that
\[
s_\lambda t_\mu = t_\mu s_\lambda, \quad e_i f_j - f_j e_i = \delta_{ij} q_i (s_\alpha_i - t_\alpha_i^{-1})
\]
\[
s_\lambda e_j t_\lambda^{-1} = q^{(\lambda, \alpha_j)} e_j, \quad t_\lambda e_j t_\lambda^{-1} = q^{(\lambda, \alpha_j)} e_j, \quad s_\lambda f_j s_\lambda^{-1} = q^{- (\lambda, \alpha_j)} f_j, \quad t_\lambda f_j t_\lambda^{-1} = q^{- (\lambda, \alpha_j)} f_j.
\]
It follows that
\[
D_q(\mathfrak{g})/(s_\lambda - t_\lambda; \lambda \in \mathbb{L}) \cong U_q(\mathfrak{b}^\pm), \quad e_i \mapsto e_i, f_i \mapsto f_i, s_\lambda \mapsto k_\lambda, t_\lambda \mapsto k_\lambda.
\]
Observe that this yields an isomorphism of Hopf algebras. The next proposition collects some well known elementary facts.

**Proposition 3.2.** 1. Any finite dimensional simple \(U_q(\mathfrak{b}^\pm)\)-module is one dimensional and \(R(U_q(\mathfrak{b}^\pm))\) identifies with \(H\) via
\[
\forall h \in H, \quad h(k_\lambda) = \langle \lambda, h \rangle, \quad h(e_i) = 0, \quad h(f_i) = 0.
\]
2. \(R(D_q(\mathfrak{g}))\) identifies with \(H\) via
\[
\forall h \in H, \quad h(s_\lambda) = \langle \lambda, h \rangle, \quad h(t_\lambda) = \langle \lambda, h \rangle^{-1}, \quad h(e_i) = h(f_i) = 0.
\]

**Corollary 3.3.** 1. \(\{U_q(\mathfrak{b}^\pm)^{op}, U_q(\mathfrak{b}^-)\}\) is an \(\mathbb{L}\)-bigraded dual pair. We have
\[
k_\lambda \in U_q(\mathfrak{b}^\pm)-\lambda, \quad e_i \in U_q(\mathfrak{b}^+)-\alpha_i, \quad f_i \in U_q(\mathfrak{b}^-)_{0, -\alpha_i}.
\]
2. \(D_q(\mathfrak{g})\) is an \(\mathbb{L}\)-bigraded Hopf algebra where
\[
s_\lambda \in D_q(\mathfrak{g})-\lambda, \quad t_\lambda \in D_q(\mathfrak{g})-\lambda, \quad e_i \in D_q(\mathfrak{g})-\alpha_i, \quad f_i \in D_q(\mathfrak{g})_{0, \alpha_i}.
\]

**Proof.** 1. Observe that for all \(h \in H\),
\[
l_h (k_\lambda) = h^{-1}(k_\lambda) = \langle -\lambda, h \rangle k_\lambda, \quad r_h (k_\lambda) = h(k_\lambda) = \langle \lambda, h \rangle k_\lambda,
\]
\[
l_h (e_i) = h^{-1}(e_i) e_i = \langle -\alpha_i, h \rangle e_i, \quad r_h (e_i) = e_i,
\]
\[
l_h (f_i) = f_i, \quad r_h (f_i) = h(k_i^{-1}) f_i = \langle -\alpha_i, h \rangle f_i.
\]
It is then clear that \(U_q^+, 0 = U_q^+\) and \(U_q^-, \gamma = U_q^-\), for all \(\gamma \in \mathbb{Q}_+\). The claims then follow from these formulas, Theorem 2.3, Theorem 3.1 and the definitions.

2. The fact that \(D_q(\mathfrak{g})\) is an \(\mathbb{L}\)-bigraded Hopf algebra follows from Theorem 2.3. The assertions about the \(\mathbb{L} \times \mathbb{L}\) degree of the generators is proved by direct computation using Proposition 3.3.

**Remark.** We have shown in Theorem 2.6 that, as a double, \(D_q(\mathfrak{g})\) inherits an \(\mathbb{L}\)-bigrading given by:
\[
D_q(\mathfrak{g})_{\alpha, \beta} = \sum_{\lambda - \gamma = \alpha, \mu - \delta = \beta} U_q(\mathfrak{b}^\pm)_{\lambda, \mu} \otimes U_q(\mathfrak{b}^-)_{\gamma, \delta}.
\]
It is easily checked that this bigrading coincides with the bigrading obtained in the above corollary by means of Theorem 2.3.
3.3. One-parameter quantized function algebras. Let $M$ be a left $D_q(\mathfrak{g})$-module. The dual $M^*$ will be considered in the usual way as a left $D_q(\mathfrak{g})$-module by the rule: $(uf)(x) = f(S(u)x)$, $x \in M$, $f \in M^*$, $u \in D_q(\mathfrak{g})$. Assume that $M$ is an $U_q(\mathfrak{g})$-module. An element $x \in M$ is said to have weight $\mu \in \mathbf{L}$ if $k_\lambda x = q^{(\lambda,\mu)}x$ for all $\lambda \in \mathbf{L}$; we denote by $M_\mu$ the subspace of elements of weight $\mu$.

It is known, [3], that the category of finite dimensional (left) $U_q(\mathfrak{g})$-modules is a completely reducible braided rigid monoidal category. Set $L^* = \mathbf{L} \cap \mathbf{P}^+$ and recall that for each $\Lambda \in \mathbf{L}^+$ there exists a finite dimensional simple module of highest weight $\Lambda$, denoted by $L(\Lambda)$, cf. [23] for instance. One has $L(\Lambda)^* \cong L(w_0\Lambda)$, where $w_0$ is the longest element of $W$. (Notice that the results quoted usually cover the case where $\mathbf{L} = \mathbf{Q}$.

One defines the modules $L(\lambda)$ in the general case in the following way. Let us denote temporarily the algebra $U_q(\mathfrak{g})$ by $U_q(\mathfrak{g}(\lambda))$. Given a module $L(\lambda)$ defined on $U_q(\mathfrak{g}(\lambda))$ we may define an action of $U_q(\mathfrak{g}(\lambda))$ on an object $x$ of weight $\mu$, where $q^{(\lambda,\mu)}$ is as defined in section 2.4.)

Let $C_q$ be the subcategory of finite dimensional $U_q(\mathfrak{g})$-modules consisting of finite direct sums of $L(\lambda)$, $\Lambda \in \mathbf{L}^+$. The category $C_q$ is closed under tensor products and the formation of duals. Notice that $C_q$ can be considered as a braided rigid monoidal category of $D_q(\mathfrak{g})$-modules where $s_\lambda, t_\lambda$ act as $k_\lambda$ on an object of $C_q$.

Let $M \in \text{obj}(C_q)$, then $M = \bigoplus_{\mu \in \mathbf{L}} M_\mu$. For $f \in M^*$, $v \in M$ we define the coordinate function $c_{f,v} \in U_q(\mathfrak{g})^*$ by

$$c_{f,v}(u) = \langle f, uv \rangle$$

where $\langle , \rangle$ is the duality pairing. Using the standard isomorphism $(M \otimes N)^* \cong N^* \otimes M^*$ one has the following formula for multiplication,

$$c_{f,v}c_{f',v'} = c_{f \otimes f', v \otimes v'}.$$  

**Definition.** The quantized function algebra $C_q[G]$ is the restricted dual of $C_q$; that is to say

$$C_q[G] = \mathbb{C}[c_{f,v} : v \in M, f \in M^*, M \in \text{obj}(C_q)].$$

The algebra $C_q[G]$ is a Hopf algebra; we denote by $\Delta, \epsilon, S$ the comultiplication, counit and antipode on $C_q[G]$. If $\{v_1, \ldots, v_s ; f_1, \ldots, f_s\}$ is a dual basis for $M \in \text{obj}(C_q)$ one has

$$\Delta(c_{f,v}) = \sum_i c_{f,v_i} \otimes c_{f_i,v}, \quad \epsilon(c_{f,v}) = \langle f, v \rangle, \quad S(c_{f,v}) = c_{v,f}. \quad (3.1)$$

Notice that we may assume that $v_j \in M_{v_j}$, $f_j \in M_{v_j}^*$. We set

$$C(M) = \mathbb{C}[c_{f,v} : f \in M^*, v \in M], \quad C(M)_{\lambda,\mu} = \mathbb{C}[c_{f,v} : f \in M_{\lambda}^*, v \in M_{\mu}].$$

Then $C(M)$ is a subcoalgebra of $C_q[G]$ such that $C(M) = \bigoplus_{\lambda,\mu \in \mathbf{L}} C(M)_{\lambda,\mu}$. When $M = L(\lambda)$ we abbreviate the notation to $C(M) = C(\lambda)$. It is then classical that

$$C_q[G] = \bigoplus_{\Lambda \in \mathbf{L}^+} C(\Lambda).$$

Since $C_q[G] \subset U_q(\mathfrak{g})^*$ we have a duality pairing

$$\langle , \rangle : C_q[G] \times D_q(\mathfrak{g}) \rightarrow \mathbb{C}.$$
Observe that there is a natural injective morphism of algebraic groups
\[ H \longrightarrow R(C_q[G]), \quad h(c_{f,v}) = \langle \mu, h \rangle c_{f,v} \text{ for all } v \in M_\mu, M \in \text{obj}(C_q). \]
The associated automorphisms \( r_h, l_h \in \text{Aut}(C_q[G]) \) are then described by
\[ \forall c_{f,v} \in C(M)_{\lambda,\mu}, \quad r_h(c_{f,v}) = \langle \mu, h \rangle c_{f,v}, \quad l_h(c_{f,v}) = \langle \lambda, h \rangle c_{f,v}. \]
Define
\[ \forall (\lambda, \mu) \in L \times L, \quad C_q[G]_{\lambda,\mu} = \{ a \in C_q[G] \mid r_h(a) = \langle \mu, h \rangle a, l_h(a) = \langle \lambda, h \rangle a \}. \]

**Theorem 3.4.** The pair of Hopf algebras \( \{ C_q[G], D_q(\mathfrak{g}) \} \) is an \( L \)-bigraded dual pair.

**Proof.** It follows from (3.1) that \( C_q[G] \) is an \( L \)-bigraded Hopf algebra. The axioms (1) to (4) of 2.3 are satisfied by definition of the Hopf algebra \( C_q[G] \). We take \( \hat{\cdot} \) to be the identity map of \( L \). The condition (2.3) is consequence of \( D_q(\mathfrak{g})_{\gamma,\delta} M_\mu \subseteq M_{\mu-\gamma-\delta} \) for all \( M \in C_q \). To verify this inclusion, notice that

\[ e_j \in D_q(\mathfrak{g})_{-\alpha_j,0}, \quad f_j \in D_q(\mathfrak{g})_{0,\alpha_j}, \quad e_j M_\mu \subseteq M_{\mu+\alpha_j}, f_j M_\mu \subseteq M_{\mu-\alpha_j}. \]

The result then follows easily.

Consider the algebras \( D_q^{-1}(\mathfrak{g}) \) and \( C_{q^{-1}}[G] \) and use \( \hat{\cdot} \) to distinguish elements, subalgebras, etc. of \( D_q^{-1}(\mathfrak{g}) \) and \( C_{q^{-1}}[G] \). It is easily verified that the map \( \sigma : D_q(\mathfrak{g}) \rightarrow D_q^{-1}(\mathfrak{g}) \) given by

\[ s_\lambda \mapsto \hat{s}_\lambda, \quad t_\lambda \mapsto \hat{t}_\lambda, \quad e_i \mapsto q_i^{1/2} \hat{f}_i \hat{t}_{i\alpha_i}, \quad f_i \mapsto q_i^{-1/2} \hat{e}_i \hat{s}_{\alpha_i}^{-1} \]
is an isomorphism of Hopf algebras.

For each \( \Lambda \in \mathbf{L}^+ \), \( \sigma \) gives a bijection \( \sigma : L(-w_0 \Lambda) \rightarrow \mathcal{L}(\Lambda) \) which sends \( v \in L(-w_0 \Lambda)_{\mu} \) onto \( \hat{v} \in \mathcal{L}(\Lambda)_{-\mu} \). Therefore we obtain an isomorphism \( \sigma : C_{q^{-1}}[G] \rightarrow C_q[G] \) such that

\[ \forall f \in L(-w_0 \Lambda)_{-\lambda}, v \in L(-w_0 \Lambda)_{\mu}, \quad \sigma(\hat{f}_v) = c_{f,v}. \quad (3.2) \]

Notice that

\[ \sigma(D_q(\mathfrak{g})_{\gamma,\delta}) = D_q^{-1}(\mathfrak{g})_{-\gamma,\delta} \text{ and } \sigma(C_{q^{-1}}[G]_{\lambda,\mu}) = C_q[G]_{-\lambda,-\mu}. \quad (3.3) \]

**3.4. Deformation of one-parameter quantum groups.** We continue with the same notation. Let \( [p] \in H^2(L, \mathbb{C}^+) \). As seen in §2.4 we can, and we do, choose \( p \) to be an antisymmetric bicharacter such that

\[ \forall \lambda, \mu \in \mathbf{L}, \quad p(\lambda, \mu) = q^u(\lambda, \mu) \]

for some \( u \in \wedge^2 \mathfrak{h} \). Recall that \( \tilde{p} \in Z^2(L \times L, \mathbb{C}^+) \), cf. [2.1].

We now apply the results of §2.4 to \( D_q(\mathfrak{g}) \) and \( C_q[G] \). Using Theorem 2.3 we can twist \( D_q(\mathfrak{g}) \) by \( \tilde{p}^{-1} \) and \( C_q[G] \) by \( \hat{p} \). The resulting \( L \)-bigraded Hopf algebras will be denoted by \( D_{q,\hat{p}}^{-1}(\mathfrak{g}) \) and \( C_{q,\hat{p}}[G] \). The algebra \( C_{q,\hat{p}}[G] \) will be referred to as the multi-parameter quantized function algebra. Versions of \( D_{q,\hat{p}}^{-1}(\mathfrak{g}) \) are referred to by some authors as the multi-parameter quantized enveloping algebra. Alternatively, this name can be applied to the quotient of \( D_{q,\hat{p}}^{-1}(\mathfrak{g}) \) by the radical of the pairing with \( C_{q,\hat{p}}[G] \).
**Theorem 3.5.** Let $U_{q,p}^{-1}(b^+)$ and $U_{q,p}^{-1}(b^-)$ be the deformations by $p^{-1}$ of $U_q(b^+)$ and $U_q(b^-)$ respectively. Then the deformed pairing

$$\langle \quad \rangle_{p^{-1}} : U_{q,p}^{-1}(b^+) \otimes U_{q,p}^{-1}(b^-) \to \mathbb{C}$$

is a non-degenerate Hopf pairing satisfying:

$$\forall x \in U^+,\ y \in U^-,\ \lambda,\mu \in L, \quad \langle x \cdot k_\lambda | y \cdot k_\mu \rangle_{p^{-1}} = q^\langle \Phi-\lambda,\mu \rangle (x \cdot y). \quad (3.4)$$

Moreover,

$$U_{q,p}^{-1}(b^+) \otimes U_{q,p}^{-1}(b^-) \cong (U_q(b^+) \otimes U_q(b^-))_{p^{-1}} = D_{q,p^{-1}}(g).$$

**Proof.** By Theorem 2.4 the deformed pairing is given by

$$\langle a,\mu | u,\delta \rangle_{p^{-1}} = p(\lambda,\mu)p(\mu,\delta)\langle a,\mu | u,\delta \rangle.$$ 

To prove (3.4) we can assume that $x \in U_{\gamma}^+,\ y \in U_{\mu}^-$. Then we obtain

$$\langle x \cdot k_\lambda | y \cdot k_\mu \rangle_{p^{-1}} = p(\lambda + \gamma,\mu)p(\lambda,\mu - \nu)\langle x \cdot k_\lambda | y \cdot k_\mu \rangle$$

by the definition of the product $\cdot$ and [23] 2.1.3. But $\langle x | y \rangle_{p^{-1}} = \langle x | y \rangle$. Therefore [23] 2.1.4 shows that $\langle \quad \rangle_{p^{-1}}$ is non-degenerate on $U_+^+ \times U_-^-$. It then follows from (3.4) and Proposition 2.8 that $\langle \quad \rangle_{p^{-1}}$ is non-degenerate. The remaining isomorphism follows from Theorem 2.6. \[\square\]

Many authors have defined multi-parameter quantized enveloping algebras. In [14, 23] a definition is given using explicit generators and relations, and in [1] the construction is made by twisting the comultiplication, following [26]. It can be easily verified that these algebras and the algebras $D_{q,p^{-1}}(g)$ coincide. The construction of a multi-parameter quantized function algebra by twisting the multiplication was first performed in the $GL(n)$-case in [1].

The fact that $D_{q,p^{-1}}(g)$ and $\mathbb{C}[G]$ form a Hopf dual pair has already been observed in particular cases, see e.g. [14]. We will now deduce from the previous results that this phenomenon holds for an arbitrary semi-simple group.

**Theorem 3.6.** $\{\mathbb{C}_{q,p}[G], D_{q,p^{-1}}(g)\}$ is an $L$-bigraded dual pair. The associated pairing is given by

$$\forall a \in \mathbb{C}_{q,p}[G]_{\lambda,\mu},\ \forall u \in D_{q,p^{-1}}(g)_{\gamma,\delta}, \quad \langle a, u \rangle_{p} = p(\lambda,\gamma)p(\mu,\delta)\langle a, u \rangle.$$ 

**Proof.** This follows from Theorem 2.4 applied to the pair $\{A, U\} = \{\mathbb{C}_q[G], D_q(g)\}$ and the bicharacter $p^{-1}$ (recall that the map $^*$ is the identity). \[\square\]

Let $M \in \text{obj}(\mathbb{C}_q)$. The left $D_q(g)$-module structure on $M$ yields a right $\mathbb{C}_q[G]$-comodule structure in the usual way. Let $\{v_1, \ldots, v_n; f_1, \ldots, f_s\}$ be a dual basis for $M$. The structure map $\rho : M \to M \otimes \mathbb{C}_q[G]$, is given by $\rho(x) = \sum_j v_j \otimes c_{f_j,x}$ for $x \in M$. Using this comodule structure on $M$, one can check that

$$M_\mu = \{x \in M \mid \forall h \in H, \ r_h(x) = \langle \mu, h \rangle x\}.$$
Proposition 3.7. Let $M \in \text{obj} (\mathcal{C}_q)$. Then $M$ has a natural structure of left $D_{q,p^{-1}}(g)$ module. Denote by $M^*$ this module and by $(u,x) \mapsto u \cdot x$ the action of $D_{q,p^{-1}}(g)$. Then

$$\forall u \in D_q(g)_{\gamma,\delta}, \forall x \in M_\lambda, \quad u \cdot x = p(\lambda, \delta - \gamma)p(\delta, \gamma)ux.$$  

Proof. The proposition is a translation in this particular setting of Corollary 2.5.

Denote by $\mathcal{C}_{q,p}$ the subcategory of finite dimensional left $D_{q,p^{-1}}(g)$-modules whose objects are the $M^*$, $M \in \text{obj}(\mathcal{C}_q)$. It follows from Proposition 3.7 that if $M \in \text{obj}(\mathcal{C}_q)$, then $M^* = \oplus_{\mu \in \Lambda} M^*_\mu$, where $M^*_\mu = \{ x \in M \mid \forall \alpha \in \mathbb{L}, \quad s_\alpha \cdot x = p(\mu, 2\alpha)q^{(\mu,\alpha)} x, \quad t_\alpha \cdot x = p(\mu, -2\alpha)q^{(\mu,\alpha)} x \}$. Notice that $p(\mu, \pm 2\alpha)q^{(\mu,\alpha)} = q^{\pm(\Phi_2,\mu,\alpha)}$.

Theorem 3.8. 1. The functor $M \mapsto M^*$ from $\mathcal{C}_q$ to $\mathcal{C}_{q,p}$ is an equivalence of rigid monoidal categories.

2. The Hopf pairing $(,)_p$ identifies the Hopf algebra $C_{q,p}[G]$ with the restricted dual of $\mathcal{C}_{q,p}$, i.e. the Hopf algebra of coordinate functions on the objects of $\mathcal{C}_{q,p}$.

Proof. 1. One needs in particular to prove that, for $M, N \in \text{obj}(\mathcal{C}_q)$, there are natural isomorphisms of $D_{q,p^{-1}}(g)$-modules: $\varphi_{M,N} : (M \otimes N)^* \rightarrow M^* \otimes N^*$. These isomorphisms are given by $x \otimes y \mapsto p(\lambda, \mu)x \otimes y$ for all $x \in M_\lambda, y \in N_\mu$. The other verifications are elementary.

2. We have to show that if $M \in \text{obj}(\mathcal{C}_q)$, $f \in M^*$, $v \in M$ and $u \in D_{q,p^{-1}}(g)$, then $(c_{f,v}, u)_p = (f, u \cdot v)$. It suffices to prove the result in the case where $f \in M^*_\lambda$, $v \in M_\mu$ and $u \in D_{q,p^{-1}}(g)_{\gamma,\delta}$. Then

$$\begin{align*}
(f, u \cdot v) &= p(\mu, \delta - \gamma)p(\delta, \gamma)(f, uv) \\
&= \delta_{-\lambda + \gamma + \delta, \mu}p(-\lambda + \gamma + \delta, \delta - \gamma)p(\delta, \gamma)(f, uv) \\
&= p(\lambda, \gamma)p(\mu, \delta)(f, uv) \\
&= (c_{f,v}, u)_p
\end{align*}$$

by Theorem 3.4. 

Recall that we introduced in §3.2 isomorphisms $\sigma : D_q(g) \rightarrow D_{q^{-1}}(g)$ and $\sigma : \mathbb{C}_q[G] \rightarrow \mathbb{C}_{q^{-1}}[G]$. From (3.3) it follows that, after twisting by $\tilde{p}^{-1}$ or $\tilde{p}$, $\sigma$ induces isomorphisms

$$D_{q,p^{-1}}(g) \xrightarrow{\sim} D_{q^{-1},p^{-1}}(g), \quad \mathbb{C}_{q^{-1},p}[G] \xrightarrow{\sim} \mathbb{C}_{q,p}[G]$$

which satisfy (3.2).

3.5. Braiding isomorphisms. We remarked above that the categories $\mathcal{C}_{q,p}$ are braided. In the one parameter case this braiding is well-known. Let $M$ and $N$ be objects of $\mathcal{C}_q$. Let $E : M \otimes N \rightarrow M \otimes N$ be the operator given by

$$E(m \otimes n) = q^{(\lambda,\mu)}m \otimes n$$

for $m \in M_\lambda$ and $n \in N_\mu$. Let $\tau : M \otimes N \rightarrow N \otimes M$ be the usual twist operator. Finally let $C$ be the operator given by left multiplication by

$$C = \sum_{\beta \in \mathbb{Q}^+} C_\beta$$
where $C_\beta$ is the canonical element of $D_q(g)$ associated to the non-degenerate pairing $U^+_{\beta} \otimes U^-_{\beta} \to \mathbb{C}$ described above. Then one deduces from [3, 4.3] that the operators

$$\theta_{M,N} = \tau \circ C \circ E^{-1} : M \otimes N \to N \otimes M$$

define the braiding on $C_q$.

As mentioned above, the category $C_{q,p}$ inherits a braiding given by

$$\psi_{M,N} = \varphi_{N,M} \circ \theta_{M,N} \circ \varphi_{M,N}^{-1}$$

where $\varphi_{M,N}$ is the isomorphism $(M \otimes N)^* \to M^* \otimes N^*$ introduced in the proof of Theorem 3.8 (the same formula can be found in [3, §10] and in a more general situation in [24]). We now note that these general operators are of the same form as those in the one parameter case. Let $M$ and $N$ be objects of $C_{q,p}$ and let

$$E : M \otimes N \to M \otimes N$$

be the operator given by

$$E(m \otimes n) = q^{(\Phi_{+,\lambda})(m)} m \otimes n$$

for $m \in M_\lambda$ and $n \in N_\mu$. Denote by $C_\beta$ the canonical element of $D_{q,p^{-1}}(g)$ associated to the nondegenerate pairing $U_{q,p^{-1}}(b^+)_{-\beta,0} \otimes U_{q,p^{-1}}(b^-)_{0,-\beta} \to \mathbb{C}$ and let $C : M \otimes N \to M \otimes N$ be the operator given by left multiplication by

$$C = \sum_{\beta \in Q_+} C_\beta.$$

**Theorem 3.9.** The braiding operators $\psi_{M,N}$ are given by

$$\psi_{M,N} = \tau \circ C \circ E^{-1}.$$

Moreover $(\psi_{M,N})^* = \psi_{M^*,N^*}$.

**Proof.** The assertions follow easily from the analogous assertions for $\theta_{M,N}$. \hfill $\square$

The following commutation relations are well known [31, 21, 4.2.2]. We include a proof for completeness.

**Corollary 3.10.** Let $\Lambda, \Lambda' \in L^+$, let $g \in L(\Lambda')^*_{-\eta}$ and $f \in L(\Lambda)^*_{-\mu}$ and let $v_\Lambda \in L(\Lambda)_\lambda$. Then for any $v \in L(\Lambda')_\gamma$, $c_{g,v} \cdot c_{f,v_\Lambda} = q^{(\Phi_{+,\lambda})(-\gamma)} c_{f,v_\Lambda} \cdot c_{g,v} + q^{(\Phi_{+,\lambda})(-\gamma)} \sum_{\nu \in Q_+} c_{f_{v_\Lambda} \cdot c_{g_{v_\Lambda},v}}$

where $f_{\nu} \in (U_{q,p^{-1}}(b^+))_{-\nu+\mu}$ and $g_{\nu} \in (U_{q,p^{-1}}(b^-))_{-\nu+\mu}$ are such that $\sum f_{\nu} \otimes g_{\nu} = \sum_{\beta \in Q^+ \setminus \{0\}} C_\beta(f \otimes g)$.

**Proof.** Let $\psi = \psi_{L(\Lambda),L(\Lambda')}$. Notice that

$$c_{f \otimes g, \psi(v_\Lambda \otimes v)} = c_{\psi(f \otimes g), v_\Lambda \otimes v}.$$ 

Using the theorem above we obtain

$$\psi^*(f \otimes g) = q^{-(-\mu,\eta)}(g \otimes f + \sum g_{\nu} \otimes f_{\nu})$$

and

$$\psi(v_\Lambda \otimes v) = q^{-(-\lambda,\gamma)}(v \otimes v_\Lambda). \quad (3.5)$$

Combining these formulae yields the required relations. \hfill $\square$
4. Prime and Primitive Spectrum of $\mathbb{C}_{q,p}[G]$ 

In this section we prove our main result on the primitive spectrum of $\mathbb{C}_{q,p}[G]$; namely that the $H$ orbits inside $\text{Prim}_w \mathbb{C}_{q,p}[G]$ are parameterized by the double Weyl group. For completeness we have attempted to make the proof more or less self-contained. The overall structure of the proof is similar to that used in [16] except that the proof of the key 3.12 (and the lemmas leading up to it) form a modified and abbreviated version of Joseph’s proof of this result in the one-parameter case [18]. One of the main differences with the approach of [18] is the use of the Rosso-Tanisaki form introduced in 3.2 which simplifies the analysis of the adjoint action of $\mathbb{C}_{q,p}[G]$. The ideas behind the first few results of this section go back to Soibelman’s work in the one-parameter ‘compact’ case [33]. These ideas were adapted to the multi-parameter case by Levendorskii [20].

4.1. Parameterization of the prime spectrum. Let $q, p$ be as in 3.4. For simplicity we set

$$A = \mathbb{C}_{q,p}[G]$$

and the product $a \cdot b$ as defined in (3.1) will be denoted by $ab$.

For each $\Lambda \in \mathbf{L}^+$ choose weight vectors

$$v_{\Lambda} \in L(\Lambda)_{\Lambda}, \quad v_{w_0\Lambda} \in L(\Lambda)_{w_0\Lambda}, \quad f_{-\Lambda} \in L(\Lambda)_{-\Lambda}^{\ast}, \quad f_{-w_0\Lambda} \in L(\Lambda)_{-w_0\Lambda}^{\ast}$$

such that $\langle f_{-\Lambda}, v_{\Lambda} \rangle = \langle f_{-w_0\Lambda}, v_{w_0\Lambda} \rangle = 1$. Set

$$A^+ = \sum_{\mu \in \mathbf{L}^+} \sum_{f \in L(\mu)^*} \mathbb{C}c_{f,v_{\mu}}, \quad A^- = \sum_{\mu \in \mathbf{L}^+} \sum_{f \in L(\mu)^*} \mathbb{C}c_{f,v_{w_0\mu}}.$$  

Recall the following result.

**Theorem 4.1.** The multiplication map $A^+ \otimes A^- \to A$ is surjective.

**Proof.** Clearly it is enough to prove the theorem in the one-parameter case. When $\mathbf{L} = \mathbf{P}$ the result is proved in [23, 3.1] and [18, Theorem 3.7].

The general case can be deduced from the simply-connected case as follows. One first observes that $\mathbb{C}_{q}[G] \subset \mathbb{C}_{q}[\tilde{G}] = \bigoplus_{\Lambda \in \mathbf{P}^+} C(\Lambda)$. Therefore any $a \in \mathbb{C}_{q}[G]$ can be written in the form $a = \sum_{\Lambda' \Lambda''} c_{\Lambda' \Lambda''} g_{v_{\Lambda'}} g_{v_{-\Lambda''}}$ where $\Lambda' - \Lambda'' \in \mathbf{L}$. Let $\Lambda \in \mathbf{P}$ and $\{v_i; f_i\}$ be a dual basis of $L(\Lambda)$. Then we have

$$1 = \epsilon(c_{v_\Lambda, f_{-\Lambda}}) = \sum_i c_{f_i, v_{\Lambda}} c_{v_i, f_{-\Lambda}}.$$  

Let $\Lambda'$ be as above and choose $\Lambda$ such that $\Lambda + \Lambda' \in \mathbf{L}^+$. Then, for all $i$, $c_{f_i, v_{\Lambda'}} c_{f_i, v_{\Lambda}} \in C(\Lambda + \Lambda') \cap A^+$ and $c_{v_i, f_{-\Lambda'}} c_{g_{v_{-\Lambda''}}, v_{\Lambda}} \in C(-w_0(\Lambda + \Lambda'')) \cap A^-$. The result then follows by inserting 1 between the terms $c_{f_i, v_{\Lambda'}}$ and $c_{g_{v_{-\Lambda''}}, v_{\Lambda}}$.

**Remark.** The algebra $A$ is a Noetherian domain (this result will not be used in the sequel). The fact that $A$ is a domain follows from the same result in [18, Lemma 3.1]. The fact that $A$ is Noetherian is consequence of [18, Proposition 4.1] and [33, Theorem 3.7].

For each $y \in W$ define the following ideals of $A$

$$I_y^+ = \langle c_{f,v_{\Lambda}} \mid f \in (U_{q,p^{-1}}(b^+)L(\Lambda)y\Lambda)^\perp, \Lambda \in \mathbf{L}^+ \rangle,$$

$$I_y^- = \langle c_{f,v_{w_0\Lambda}} \mid f \in (U_{q,p^{-1}}(b^-)L(\Lambda)y_{w_0\Lambda})^\perp, \Lambda \in \mathbf{L}^+ \rangle.$$
Lemma 4.2. Let second identity can be deduced from the first one by applying \( \sigma \).

Proof. The first identity follows from Corollary 3.10 and the definition of \( f \).

Proposition 4.3. For all \( w \in W \times W \), \( A_w \neq (0) \).

Proof. Notice first that since the generators of \( A_w \) and the elements of \( E_w \) are \( L \times L \) homogeneous, it suffices to work in the one-parameter case. The proof is then similar to that of [13, Theorem 2.2.3] (written in the \( SL(n) \)-case). For completeness we recall the steps of this proof. The technical details are straightforward generalizations to the general case of [13, loc. cit.].

For \( 1 \leq i \leq n \) denote by \( U_q(\mathfrak{sl}_2) \) the Hopf subalgebra of \( U_q(\mathfrak{g}) \) generated by \( e_i, f_i, k_i^{\pm 1} \). The associated quantized function algebra \( A_i \cong C_q[SL(2)] \) is naturally a quotient of \( A \). Let \( \sigma_i \) be the reflection associated to the root \( \alpha_i \). It is easily seen that there exist \( M_+^i \) and \( M_-^i \), non-zero \( (A_i)_{(\sigma,e)} \) and \( (A_i)_{(e,\sigma)} \) modules respectively. These modules can then be viewed as non-zero \( A \)-modules.

Let \( w_+ = \sigma_{i_1} \ldots \sigma_{i_k} \) and \( w_- = \sigma_{j_1} \ldots \sigma_{j_m} \) be reduced expressions for \( w_{\pm} \). Then

\[
M_{i_1}^+ \otimes \cdots \otimes M_{i_k}^+ \otimes M_{j_1}^- \otimes \cdots \otimes M_{j_m}^-
\]

is a non-zero \( A_w \)-module.

In the one-parameter case the proof of the following result was found independently by the authors in [10, 1.2] and Joseph in [13, 6.2].

Theorem 4.4. Let \( P \in \text{Spec} \mathbb{C}_{q,p}[G] \). There exists a unique \( w \in W \times W \) such that \( P \supset I_w \) and \( (P/I_w) \cap E_w = \emptyset \).

Proof. Fix a dominant weight \( \Lambda \). Define an ordering on the weight vectors of \( L(\Lambda)^* \) by \( f \leq f' \) if \( f' \in U_{q,p^{-1}}(b^+)f \). This is a preordering which induces a partial ordering on the set of one dimensional weight spaces. Consider the set:

\[
\mathcal{F}(\Lambda) = \{ f \in L(\Lambda)_p^* \mid c_{f,\nu A} \notin P \}.
\]
Let $f$ be an element of $\mathcal{F}(\Lambda)$ which is maximal for the above ordering. Suppose that $f'$ has the same property and that $f$ and $f'$ have weights $\mu$ and $\mu'$ respectively. By Corollary 3.10 the two elements $c_{f,v_\lambda}$ and $c_{f',v_\lambda}$ are normal modulo $P$. Therefore we have, modulo $P$,

\[ c_{f,v_\lambda}c_{f',v_\lambda} = q^{(\Phi_+\Lambda,\Lambda)}(\Phi_+\mu,\mu')c_{f',v_\lambda}c_{f,v_\lambda} = q^{2(\Phi_+\Lambda,\Lambda)}(\Phi_+\mu,\mu')c_{f,v_\lambda}c_{f',v_\lambda}. \]

But, since $u$ is alternating, $2(\Phi_+\Lambda,\Lambda) - (\Phi_+\mu,\mu') - (\Phi_+\mu',\mu) = 2(\Lambda,\Lambda) - (\mu,\mu')$. Since $P$ is prime and $q$ is not a root of unity we can deduce that $(\Lambda,\Lambda) = (\mu,\mu')$. This forces $\mu = \mu' \in W(\Lambda)$. In conclusion, we have shown that for all dominant $\Lambda$ there exists a unique (up to scalar multiplication) maximal element $g_\lambda = \pi_1(\Lambda)$ with weight $-w_\Lambda \Lambda, w_\Lambda \in W$. Applying the argument above to a pair of such elements, $c_{g_\lambda,v_\lambda}$ and $c_{g_\lambda,v_{\lambda'}}$, yields that $(w_\Lambda, w_\Lambda, \Lambda') = (\Lambda, \Lambda')$ for all $\Lambda, \Lambda' \in L^+$. Then it is not difficult to show that this furnishes a unique $w_+ \in W$ such that $w_+ \Lambda = w_+ \Lambda$ for all $\Lambda \in L^+$. Thus for each $\Lambda \in L^+$,

\[ c_{g_\lambda,v_\lambda} \in P \iff g \not\in f_{-\lambda} \Lambda. \]

Hence $P \supset I_{w_+}$ and $P \cap \mathcal{E}_{w_+} = \emptyset$. It is easily checked that such a $w_+$ must be unique. Using $\sigma$ one deduces the existence and uniqueness of $w_-$. 

\begin{definition}
A prime ideal $P$ such that $P \supset I_w$ and $P \cap \mathcal{E}_w = \emptyset$ will be called a prime ideal of type $w$. We denote by Spec$_w \mathbb{C}_{q,p}[G]$, resp. Prim$_w \mathbb{C}_{q,p}[G]$, the subset of Spec $\mathbb{C}_{q,p}[G]$ consisting of prime, resp. primitive, ideals of type $w$.
\end{definition}

Clearly Spec$_w \mathbb{C}_{q,p}[G] \cong$ Spec $A_w$ and $\sigma(\text{Spec}_w \mathbb{C}_{q^{-1},p}[G]) = \text{Spec}_w \mathbb{C}_{q,p}[G]$. The following corollary is therefore clear.

\begin{corollary}
One has
\begin{align*}
\text{Spec} \mathbb{C}_{q,p}[G] &= \sqcup_{w \in W \times W} \text{Spec}_w \mathbb{C}_{q,p}[G], \\
\text{Prim} \mathbb{C}_{q,p}[G] &= \sqcup_{w \in W \times W} \text{Prim}_w \mathbb{C}_{q,p}[G].
\end{align*}
\end{corollary}

We end this section by a result which is the key idea in \cite{18} for analyzing the adjoint action of $A$ on $A_w$. It says that in the one parameter case the quantized function algebra $\mathbb{C}_q[\mathcal{B}^{-}]$ identifies with $U_q(\mathcal{L}^+)$ through the Rosso-Tanisaki-Killing form, \cite{4}. Evidently this continues to hold in the multi-parameter case. For completeness we include a proof of that result.

Set $\mathbb{C}_q[\mathcal{B}^{-}] = A/I_{w_0,e}$. The embedding $U_{q,p^{-1}}(\mathcal{B}^{-}) \to D_{q,p^{-1}}(g)$ induces a Hopf algebra map $\phi : A \to U_{q,p^{-1}}(\mathcal{B}^{-})$, where $U_{q,p^{-1}}(\mathcal{B}^{-})$ denotes the cofinite dual. On the other hand the non-degenerate Hopf algebra pairing $\langle \ , \ \rangle_{p^{-1}}$ furnishes an injective morphism $\theta : U_{q,p^{-1}}(\mathcal{B}^{-})^{op} \to U_{q,p^{-1}}(\mathcal{B}^{-})^*$. 

\begin{proposition}
1. $\mathbb{C}_q[\mathcal{B}^{-}]$ is an $L$-bigraded Hopf algebra.

2. The map $\gamma = \theta^{-1}\phi : \mathbb{C}_q[\mathcal{B}^{-}] \to U_{q,p^{-1}}(\mathcal{B}^{-})^{op}$ is an isomorphism of Hopf algebras.
\end{proposition}

\begin{proof}
1. It is easy to check that $I_{w_0,e}$ is an $L \times L$ graded bi-ideal of the bialgebra $A$. Let $\mu \in L^+$ and fix a dual basis $\{v_\nu ; f_{-\nu}\}^\mu$ of $L(\mu)$ (with the usual abuse of notation). Then

\[ \sum_\nu c_{v_\nu,f_{-\gamma}}c_{f_{-\nu},v_\gamma} = \sum_\nu S(c_{f_{-\nu},v_\nu})c_{f_{-\nu},v_\gamma} = \epsilon(c_{f_{-\nu},v_\gamma}). \]

Taking $\gamma = \eta = \mu$ yields $\overline{c}_\mu c_\mu = 1$ modulo $I_{w_0,e}$. If $\gamma = w_0\mu$ and $\eta \neq w_0\mu$, the above relation shows that $S(c_{f_{-\nu},v_{w_0\mu}})\overline{c}_{-w_0\mu} \in I_{w_0,e}$. Thus $I_{w_0,e}$ is a Hopf ideal.
2. We first show that
\[ \forall \Lambda \in L^+, c_f, v_A \in C(\Lambda)_{-\lambda, \Lambda}, \exists! x_\lambda \in U_{\Lambda+}^+ \quad \phi(c_f, v_A) = \theta(x_\lambda \cdot k_\Lambda). \] (4.1)

Set \( c = c_f, v_A \). Then \( c(U_{-\eta}) = 0 \) unless \( \eta = \Lambda - \lambda \); denote by \( \bar{c} \) the restriction of \( c \) to \( U^- \). By the non-degeneracy of the pairing on \( U_{\Lambda-\lambda}^+ \times U_{-\Lambda}^- \) we know that there exists a unique \( x_\lambda \in U_{\Lambda-\lambda}^+ \) such that \( \bar{c} = \theta(x_\lambda) \). Then, for all \( y \in U_{-\Lambda}^- \), we have
\[ c(y \cdot k_\mu) = \langle f, y \cdot k_\mu \cdot v_\Lambda \rangle = q^{-\langle \Phi_{-\lambda, \mu} \rangle} \bar{c}(y) = q^{-\langle \Phi_{-\lambda, \mu} \rangle} \langle x_\lambda \mid y \rangle \]
by (3.4). This proves (4.1).

We now show that \( \phi \) is injective on \( A^+ \). Suppose that \( c = c_f, v_A \in C(\Lambda)_{-\lambda, \Lambda} \cap \text{Ker} \phi \), hence \( c = 0 \) on \( U_{q,p^{-1}}(b^-) \). Since \( L(\Lambda) = U_{q,p^{-1}}(b^-) v_\Lambda = D_{q,p^{-1}}(g) v_\Lambda \) it follows that \( c = 0 \). An easy weight argument using (4.1) shows then that \( \phi \) is injective on \( A^+ \).

It is clear that \( \text{Ker} \phi \supset I_{(w, e)} \), and that \( A^+ A^- = A \) implies \( \phi(A) = \phi(A^+[\bar{c}_\mu; \mu \in L^+]) \). Since \( \bar{c}_\mu = c_\mu^{-1} \) modulo \( I_{(w, e)} \) by part 1, if \( a \in A \) there exists \( \nu \in L^+ \) such that \( \phi(c_\nu)\phi(a) = \phi(A^+ \nu) \). The inclusion \( \text{Ker} \phi \subset I_{(w, e)} \) follows easily. Therefore \( \gamma \) is a well defined Hopf algebra morphism.

If \( \alpha_j \in B \), there exists \( \Lambda \in L^+ \) such that \( L(\Lambda)_{-\lambda, \alpha_j} \neq (0) \). Pick \( 0 \neq f \in L(\Lambda)_{-\lambda, \alpha_j}^+ \). Then (4.1) shows that, up to some scalar, \( \phi(c_f, v_A) = \theta(\alpha_j \cdot k_\Lambda) \). If \( \lambda \in L \), there exists \( \Lambda \in W \lambda \cap \Lambda^+ \); in particular \( L(\Lambda)_\lambda \neq (0) \). Let \( v \in L(\Lambda)_\lambda \) and \( f \in L(\Lambda)_{-\lambda}^+ \) such that \( \langle f, v \rangle = 1 \). Then it is easily verified that \( \phi(c_f, v) = \theta(\kappa_\lambda) \).

This proves that \( \gamma \) is surjective, and the proposition.

4.2. The adjoint action. Recall that if \( M \) is an arbitrary \( A \)-bimodule one defines the adjoint action of \( A \) on \( M \) by
\[ \forall a \in A, x \in M, \quad \text{ad}(a) x = \sum a_{(1)} x S(a_{(2)}) . \]

Then it is well known that the subspace of ad-invariant elements \( M^{ad} = \{ x \in M \mid \forall a \in A, \text{ad}(a) x = c(a) x \} \) is equal to \( \{ x \in M \mid \forall a \in A, ax = xa \} \).

Henceforth we fix \( w \in W \times W \) and work inside \( A_w \). For \( \Lambda \in L^+, f \in L(\Lambda)^+ \) and \( v \in L(\Lambda) \) we set
\[ z_f^+ = c_{w_\Lambda}^{-1} c_{f, v_A}, \quad z_v^- = \tilde{c}_{w_\Lambda}^{-1} c_{v, f_{-\Lambda}}. \]

Let \( \{ \omega_1, \ldots, \omega_n \} \) be a basis of \( L \) such that \( \omega_i \in L^+ \) for all \( i \). Observe that \( c_{w_\Lambda} c_{w_{\Lambda'}} \) and \( c_{w_{\Lambda'}} c_{w_\Lambda} \) differ by a non-zero scalar (similarly for \( \tilde{c}_{w_\Lambda} \tilde{c}_{w_{\Lambda'}} \)). For each \( \lambda = \sum_i \epsilon_i \omega_i \in L \) we define normal elements of \( A_w \) by
\[ c_{w_\Lambda} = \prod_{i=1}^n c_{\epsilon_i \omega_i}^i, \quad \tilde{c}_{w_\Lambda} = \prod_{i=1}^n \tilde{c}_{\epsilon_i \omega_i}^i, \quad d_\lambda = (\tilde{c}_{w_\Lambda} c_{w_\Lambda})^{-1}. \]

Notice then that, for \( \Lambda \in L^+ \), the “new” \( c_{w_\Lambda} \) belongs to \( \mathbb{C} c_{f_{-\lambda}, v_A} \) (similarly for \( \tilde{c}_{w_\Lambda} \)). Define subalgebras of \( A_w \) by
\[ C_w^+ = \mathbb{C}[z_f^+, z_v^-, c_{w_\Lambda}; f \in L(\Lambda)^+, v \in L(\Lambda), \Lambda \in L^+, \lambda \in L], \quad C_w^- = \mathbb{C}[z_v^-, c_{w_\Lambda}; v \in L(\Lambda), \Lambda \in L^+]. \]
Recall that the torus $H$ acts on $A_{\Lambda,\mu}$ by $r_h(a) = \mu(h)a$, where $\mu(h) = \langle \mu, h \rangle$. Since the generators of $I_w$ and the elements of $E_w$ are eigenvectors for $H$, the action of $H$ extends to an action on $A_w$. The algebras $C_w$ and $C^+_w$ are obviously $H$-stable.

**Theorem 4.7.** 1. $C^H_w = \mathbb{C}[z^+_f, z^-_f; f \in L(\Lambda)^*, \Lambda \in \mathbf{L^+}]$.

2. The set $D = \{d_\Lambda; \Lambda \in \mathbf{L^+}\}$ is an Ore subset of $C^H_w$. Furthermore $A_w = (C_w)_D$ and $A^H_w = (C^H_w)_D$.

3. For each $\Lambda \in \mathbf{L}$, let $(A_w)_\Lambda = \{a \in A_w \mid r_h(a) = \lambda(h)a\}$. Then $A_w = \bigoplus_{\Lambda \in \mathbf{L}} (A_w)_\Lambda$ and $(A_w)_\Lambda = A^H_w c_w\Lambda$. Moreover each $(A_w)_\Lambda$ is an ad-invariant subspace.

**Proof.** Assertion 1 follows from

$$\forall h \in H, \quad r_h(z^+_f) = z^+_f, \quad r_h(c_{w\Lambda}) = \lambda(h)c_{w\Lambda}, \quad r_h(c_{w^{-1}}) = \lambda(h)\lambda^{-1}c_{w^{-1}}.$$ 

Let $\{v_{i}; f_{i}\_i$ be a dual basis for $L(\Lambda)$. Then

$$1 = \epsilon(c_{v_{-1}}, v_{\Lambda}) = \sum_i S(c_{f_{-1}}, v_i)c_{f_i, v_{\Lambda}} = \sum_i c_{v_i, f_{-1}}c_{f_i, v_{\Lambda}}.$$ 

Multiplying both sides of the equation by $d_{\Lambda}$ and using the normality of $c_{w\Lambda}$ and $c_{w^{-1}}$ yields $d_{\Lambda} = \sum_i a_iz^{-a_i}z^a_i$ for some $a_i \in \mathbb{C}$. Thus $D \subset C^H_w$. Now by Theorem 4.1 any element of $A_w$ can be written in the form $c_{f^1, v_1}c_{f_2, v_2}d_{\Lambda}^{-1}$ where $v_1 = v_{\Lambda_1}$, $v_2 = v_{\Lambda_2}$ and $\Lambda_1, \Lambda_2, \Lambda \in \mathbf{L^+}$. This element lies in $(A_w)_\Lambda$ if and only if $\Lambda_1 - \Lambda_2 = \Lambda$. In this case $c_{f_1, v_1}c_{f_2, v_2}d_{\Lambda}^{-1}$ is equal, up to a scalar, to the element $z^+_{f_1, \Lambda_1}z^-_{f_2, \Lambda_2}c_{w\Lambda} \in (C^H_w)_D c_{w\Lambda}$. Since the adjoint action commutes with the right action of $H$, $(A_w)_\Lambda$ is an ad-invariant subspace. The remaining assertions then follow.

We now study the adjoint action of $C_{q,p}[G]$ on $A_w$. The key result is Theorem 4.12.

**Lemma 4.8.** Let $T_\Lambda = \{z^+_f \mid f \in L(\Lambda)^*\}$. Then $C^+_{w} = \bigcup_{\Lambda \in \mathbf{L}} T_\Lambda$.

**Proof.** It suffices to prove that if $\Lambda, \Lambda' \in \mathbf{L^+}$ and $f \in L(\Lambda)^*$, then there exists a $g \in L(\Lambda + \Lambda')^*$ such that $z^+_f = z^+_g$. Clearly we may assume that $f$ is a weight vector. Let $\iota : L(\Lambda + \Lambda') \to L(\Lambda) \otimes L(\Lambda')$ be the canonical map. Then

$$c_{f, v_{\Lambda}}c_{f_{-1}, v_{w\Lambda}, v_{\Lambda'}} = c_{f_{-1}, v_{w\Lambda}, v_{\Lambda'}}c_{f, v_{\Lambda}} = c_{g, v_{\Lambda}}.$$ 

where $g = \iota(f_{-1}, v_{w\Lambda}, v_{\Lambda'})$. Multiplying the images of these elements in $A_w$ by the inverse of $c_{w(\Lambda + \Lambda')}$ in $C^*c_{w\Lambda}c_{w\Lambda'}$ yields the desired result.

**Proposition 4.9.** Let $E$ be an object of $C_{q,p}$ and let $\Lambda \in \mathbf{L^+}$. Let $\sigma : L(\Lambda) \to E \otimes L(\Lambda) \otimes E^*$ be the map $(1 \otimes \psi^{-1})(\iota \otimes 1)$ where $\iota : \mathbb{C} \to E \otimes E^*$ is the canonical embedding and $\psi^{-1} : E^* \otimes L(\Lambda) \to L(\Lambda) \otimes E^*$ is the inverse of the braiding map described in $\S 3.5$. Then for any $c \in C_{q,p}[G]_{\eta, \gamma} \in L(\Lambda)^*$

$$\text{ad}(c)z^+_f = q^{(\Phi_{w, \Lambda}, \sigma)}z^+_\sigma(f \otimes g)^*.$$ 

In particular $C^+_{w}$ is a locally finite $C_{q,p}[G]$-module for the adjoint action.

**Proof.** Let $\{v_i, g_i\}$ be a dual basis of $E$ where $v_i \in E_{v_i}, g_i \in E_{-v_i}$. Then $\iota(1) = \sum v_i \otimes g_i$. By (3.3) we have

$$\psi^{-1}(g_i \otimes v_\Lambda) = a_i(v_\Lambda \otimes g_i).$$
where $a_i = q^{- \langle \Phi + \Lambda, \nu_i \rangle} = q^{\langle \Phi - \nu_i, \Lambda \rangle}$. On the other hand the commutation relations given in Corollary 3.10 imply that $c_{g,v}c_{w\Lambda} = ba_tc_{w\Lambda}c_{g,v}$, where $b = q^{\langle \Phi + w\Lambda, \eta \rangle}$.

Therefore

$$ad(c).z_j^+ = \sum ba(tc_{w_1}c_{g,v}c_{f,v})c_{e,g} = bc_{w_1}c_{e,g}f_{g\otimes g}\sum a_{i,v_1\otimes v_2} = bc_{w_1}c_{e,g}f_{g\otimes g}.\sigma(v_3).$$

Since the map $\sigma$ is a morphism of $D_{q,p^{-1}}(g)$-modules it is easy to see that $c_v\otimes g,\sigma(v_3) = c_{\sigma^*(v\otimes g),v_3}$.

Let $\gamma : \mathbb{C}_q[G] \rightarrow U_{q,p^{-1}}(b^+)$ be the algebra anti-isomorphism given in Proposition 1.6.

**Lemma 4.10.** Let $c = c_{g,v} \in \mathbb{C}_q[G]_{-\eta,\gamma}$, $f \in L(\Lambda)^*$ be as in the previous theorem and $x \in U_{q,p^{-1}}(b^+)$ be such that $\gamma(c) = x$. Then

$$c_{S^{-1}(x),f,v_3} = c_{\sigma^*(v\otimes g),v_3}.$$

**Proof.** Notice that it suffices to show that

$$c_{S^{-1}(x),f,v_3}(y) = c_{\sigma^*(v\otimes g),v_3}(y)$$

for all $y \in U_{q,p^{-1}}(b^-)$. Denote by $\langle \cdot | \cdot \rangle$ the Hopf pairing $\langle \cdot | \cdot \rangle_{p^{-1}}$ between $U_{q,p^{-1}}(b^+)^{op}$ and $U_{q,p^{-1}}(b^-)$ as in §3.4. Let $\chi$ be the one dimensional representation of $U_{q,p^{-1}}(b^+)$ associated to $v_3$ and let $\tilde{\chi} = \chi \cdot \gamma$. Notice that $\chi(x) = \langle x | t_{-\Lambda} \rangle$; so $\tilde{\chi}(c) = c(t_{-\Lambda})$. Recalling that $\gamma$ is a morphism of coalgebras and using the relation $(c_{xy})$ of §2.3 in the double $U_{q,p^{-1}}(b^+) \times U_{q,p^{-1}}(b^-)$, we obtain

$$c_{S^{-1}(x),f,v_3}(y) = f(xyv_3) = \sum \langle x(1) | y(1) \rangle \langle y(3) | S(y(3)) \rangle f(y(2)x(2)v_3)$$

$$= \sum \langle x(1) | y(1) \rangle \langle x(3) | S(y(3)) \rangle \chi(x(2)) f(y(2)v_3)$$

$$= \sum \langle x(1) | \chi(x(2)) \rangle \langle y(1) | x(3) \rangle \langle S(y(3)) | f(y(2)v_3) \rangle$$

$$= \sum \langle \chi(c(2)) | y(1) \rangle \langle y(3) | S(y(3)) \rangle f(y(2)v_3)$$

$$= \sum r_{\chi}(c(1)) \chi(y(1)) c_{f,v_3}(y(2)) S(c_2(y(3))).$$

Since $r_{\chi}(c_{g,v}) = q^{\langle \Phi - \nu_i, \Lambda \rangle}c_{g,v}$, one shows as in the proof of Proposition 1.3 that

$$c_{S^{-1}(x),f,v_3}(y) = \sum r_{\chi}(c(1)) \chi(y(1)) c_{f,v_3}(y(2)) S(c_2(y(3)))$$

$$= \sum q^{\langle \Phi - \nu_i, \Lambda \rangle} c_{g,v} c_{f,v_3} c_{v,g_i}(y)$$

$$= c_{\sigma^*(v\otimes g),v_3}(y),$$

as required.

**Theorem 4.11.** Consider $C_{w}^{+}$ as a $\mathbb{C}_q[G]$-module via the adjoint action. Then

1. $\text{Soc } C_{w}^{+} = \mathbb{C}$.
2. $\text{Ann } C_{w}^{+} \supset I_{(w_0,e)}$.
3. The elements $c_{f_{\mu},v}$, $\mu \in L^+$, act diagonalizeably on $C_{w}^{+}$.
4. $\text{Soc } C_{w}^{+} = \{ z \in C_{w}^{+} \mid \text{Ann } z \supset I_{(e,e)} \}$. 
Proof. For \( \Lambda \in L^+ \), define a \( U_{q,p^{-1}}(b^+) \)-module by
\[
S_{\Lambda} = (U_{q,p^{-1}}(b^+)v_{w+\Lambda})^* = L(\Lambda)^*/(U_{q,p^{-1}}(b^+)v_{w+\Lambda})^+. 
\]
It is easily checked that \( \text{Soc} S_{\Lambda} = C\mathcal{F}_{w+\Lambda} \) (see [18, 7.3]). Let \( \delta : S_{\Lambda} \to T_{\Lambda} \) be the linear map given by \( f \mapsto z^+_f \). Denote by \( \zeta \) the one-dimensional representation of \( \mathbb{C}_{q,p}[G] \) given by \( \zeta(c) = c(t_{w+\Lambda}) \). Let \( c \in C(E) - \eta, \gamma \). Then \( l_\zeta(c) = q^{-}(\Phi_{-}(\eta, w+\Lambda))c = q^{-}(\Phi_{+}(w+\Lambda, \eta))c \). Then, using Proposition 4.8 and Lemma 4.10 we obtain,
\[
\text{ad}(l_\zeta(c))\delta(f) = q^{-}(\Phi_{+}(w+\Lambda, \eta))\text{ad}(c)\cdot z^+_f = z^+_{\text{Soc}(c)}f = \delta(S^{-1}(\gamma(c))f).
\]
Hence, \( \text{ad}(l_\zeta(c))\delta(f) = \delta(S^{-1}(\gamma(c))f) \) for all \( c \in A \). This immediately implies part (2) since \( \text{Ker} \gamma \supset I_{(w_0, e)} \) and \( l_\zeta(I_{(w_0, e)}) = I_{(w_0, e)} \). If \( S_{\Lambda} \) is given the structure of an \( A \)-module via \( S^{-1}_A \), then \( \delta \) is a homomorphism from \( S_{\Lambda} \) to the module \( T_{\Lambda} \) twisted by the automorphism \( l_\zeta \). Since \( \delta(f_{w+\Lambda}) = 1 \) it follows that \( \delta \) is bijective and that \( \text{Soc} T_{\Lambda} = \delta(\text{Soc} S_{\Lambda}) = \mathbb{C} \). Part (1) then follows from Lemma 4.8. Part (3) follows from the above formula and the fact that \( \gamma(c_{f_{-\mu}, v_{\mu}}) = s_{-\mu} \). Since \( A/I_{(e, e)} \) is generated by the images of the elements \( c_{f_{-\mu}, v_{\mu}} \), (4) is a consequence of the definitions.

**Theorem 4.12.** Consider \( C^H_w \) as a \( \mathbb{C}_{q,p}[G] \)-module via the adjoint action. Then
\[
\text{Soc} C^H_w = \mathbb{C}.
\]

**Proof.** By Theorem 4.11 we have that \( \text{Soc} C^+_w = \mathbb{C} \). Using the map \( \sigma \), one obtains analogous results for \( C^-_w \). The map \( C^+_w \otimes C^-_w \to C^H_w \) is a module map for the adjoint action which is surjective by Theorem 4.1. So it suffices to show that \( \text{Soc} C^+_w \otimes C^-_w = \mathbb{C} \). The following argument is taken from [18].

By the analog of Theorem 4.11 for \( C^-_w \) we have that the elements \( c_{f_{-\Lambda}, v_{\Lambda}} \) act as commuting diagonalizable operators on \( C^-_w \). Therefore an element of \( C^+_w \otimes C^-_w \) may be written as \( \sum a_i \otimes b_i \) where the \( b_i \) are linearly independent weight vectors. Let \( c_{f_{\Lambda}, v_{\Lambda}} \) be a generator of \( I^+_w \). Suppose that \( \sum a_i \otimes b_i \in \text{Soc}(C^+_w \otimes C^-_w) \). Then
\[
0 = \text{ad}(c_{f_{\Lambda}, v_{\Lambda}})(\sum a_i \otimes b_i) = \sum_{i,j} \text{ad}(c_{f_{v_j}, v_{j}}) a_i \otimes \text{ad}(c_{f_{v_j}, v_{\Lambda}}) b_i = \sum_i \text{ad}(c_{f_{v_{\Lambda}}}) a_i \otimes \text{ad}(c_{f_{-\Lambda}, v_{\Lambda}}) b_i = \sum_i \text{ad}(c_{f_{v_{\Lambda}}}) a_i \otimes \alpha_i b_i
\]
for some \( \alpha_i \in \mathbb{C}^+ \). Thus \( \text{ad}(c_{f_{v_{\Lambda}}}) a_i = 0 \) for all \( i \). Thus the \( a_i \) are annihilated by the left ideal generated by the \( c_{f_{v_{\Lambda}}} \). But this left ideal is two-sided modulo \( I_{(w_0, e)} \) and \( \text{Ann} C^+_w \supset I_{(w_0, e)} \). Thus the \( a_i \) are annihilated by \( I_{(e, e)} \) and so lie in \( \text{Soc} C^+_w \) by Theorem 4.11. Thus \( \sum a_i \otimes b_i \in \text{Soc}(C \otimes C^-_w) = \mathbb{C} \otimes \mathbb{C} \). \( \square \)

**Corollary 4.13.** The algebra \( A^H_w \) contains no nontrivial \( \text{ad} \)-invariant ideals. Furthermore, \( (A^H_w)^{\text{ad}} = \mathbb{C} \).

**Proof.** Notice that Theorem 4.12 implies that \( A^H_w \) contains no nontrivial \( \text{ad} \)-invariant ideals. Since \( A^H_w \) is a localization of \( C^H_w \) the same must be true for \( A^H_w \). Let \( a \in (A^H_w)^{\text{ad}} \setminus \mathbb{C} \). Then \( a \) is central and so for any \( \alpha \in \mathbb{C} \), \( (a - \alpha) \) is a non-zero
ad-invariant ideal of $A^H_w$. This implies that $a - \alpha$ is invertible in $A^H_w$ for any $\alpha \in \mathbb{C}$. This contradicts the fact that $A^H_w$ has countable dimension over $\mathbb{C}$.

\begin{theorem}
Let $Z_w$ be the center of $A_w$. Then
\begin{enumerate}
\item $Z_w = A^ad_w$;
\item $Z_w = \bigoplus_{\lambda \in L} Z_\lambda$ where $Z_\lambda = Z_w \cap A^H_w c_{w\lambda}$;
\item If $Z_\lambda \neq (0)$, then $Z_\lambda = C u_\lambda$ for some unit $u_\lambda$;
\item The group $H$ acts transitively on the maximal ideals of $Z_w$.
\end{enumerate}
\end{theorem}

\begin{proof}
The proof of (1) is standard. Assertion (2) follows from Theorem 1.3. Let $u_\lambda$ be a non-zero element of $Z_\lambda$. Then $u_\lambda = ac_{w\lambda}$, for some $a \in A^H_w$. This implies that $a$ is normal and hence $a$ generates an ad-invariant ideal of $A^H_w$. Thus $a$ (and hence also $u_\lambda$) is a unit by Theorem 1.14. Since $Z_0 = \mathbb{C}$, it follows that $Z_\lambda = C u_\lambda$. Since the action of $H$ is given by $r_h(u_\lambda) = \lambda(h) u_\lambda$, it is clear that $H$ acts transitively on the maximal ideals of $Z_w$. 
\end{proof}

\begin{theorem}
The ideals of $A_w$ are generated by their intersection with the center, $Z_w$.
\end{theorem}

\begin{proof}
Any element $f \in A_w$ may be written uniquely in the form $f = \sum a_\lambda c_{w\lambda}$ where $a_\lambda \in A^H_w$. Define $\pi : A_w \to A^H_w$ to be the projection given by $\pi(\sum a_\lambda c_{w\lambda}) = a_0$ and notice that $\pi$ is a module map for the adjoint action. Define the support of $f$ to be $\text{Supp}(f) = \{ \lambda \in L \mid a_\lambda \neq 0 \}$. Let $I$ be an ideal of $A_w$. For any set $Y \subseteq L$ such that $0 \in Y$ define $I_Y = \{ b \in A^H_w \mid b = \pi(f) \text{ for some } f \in I \text{ such that } \text{Supp}(f) \subseteq Y \}$.

If $I$ is ad-invariant then $I_Y$ is an ad-invariant ideal of $A^H_w$ and hence is either $(0)$ or $A^H_w$.

Now let $I' = (I \cap Z_w)A_w$ and suppose that $I \neq I'$. Choose an element $f = \sum a_\lambda c_{w\lambda} \in I \setminus I'$ whose support $S$ has the smallest cardinality. We may assume without loss of generality that $0 \in S$. Suppose that there exists $g \in I'$ with $\text{Supp}(g) \subseteq S$. Then there exists a $g' \in I'$ with $\text{Supp}(g') \subseteq S$ and $\pi(g') = 1$. But then $f - a_0 g'$ is an element of $I'$ with smaller support than $F$. Thus there can be no elements in $I'$ whose support is contained in $S$. So we may assume that $\pi(f) = a_0 = 1$. For any $c \in \mathbb{C}_{q,p}[G]$, set $f_c = \text{ad}(c).f - \epsilon(c)f$. Since $\pi(f_c) = 0$ it follows that $|\text{Supp}(f_c)| < |\text{Supp}(f)|$ and hence that $f_c = 0$. Thus $f \in I \cap A^ad_w = I \cap Z_w$, a contradiction.

Putting these results together yields the main theorem of this section, which completes Corollary 1.5 by describing the set of primitive ideals of type $w$.

\begin{theorem}
For $w \in W \times W$ the subsets $\text{Prim}_w \mathbb{C}_{q,p}[G]$ are precisely the $H$-orbits inside $\text{Prim} \mathbb{C}_{q,p}[G]$.
\end{theorem}

Finally we calculate the size of these orbits in the algebraic case. Set $L_w = \{ \lambda \in L \mid Z_\lambda \neq (0) \}$. Recall the definition of $s(w)$ from [1.3] and that $p$ is called $q$-rational if $u$ is algebraic. In this case we know by Theorem 1.7 that there exists $m \in \mathbb{N}$ such that $\Phi(mL) \subset L$.

\begin{proposition}
Suppose that $p$ is $q$-rational. Let $\lambda \in L$ and $y_\lambda = c_w \Phi_{m \lambda} \tilde{c}_w \Phi_{m \lambda}$.
\end{proposition}

\begin{then}
1. $y_\lambda$ is ad-semi-invariant. In fact, for any $c \in A_{-\eta,\gamma}$,

$$\text{ad}(c) y_\lambda = q^{(m \sigma(w) \lambda, \eta)} \epsilon(c) y_\lambda,$$

where $\sigma(w) = \Phi_- w \Phi_+ - \Phi_+ w \Phi_-$. 

2. $L_w \cap 2mL = 2 \text{Ker} \sigma(w) \cap mL$

3. $\dim Z_w = n - s(w)$

Proof. Using Lemma 4.2, we have that for $c \in A_{-\eta,\gamma}$

$$cy_\lambda = q^{(\Phi_+ w_+ \Phi_- m \lambda, -\eta)} q^{(\Phi_+ w_+ m \lambda, \gamma)} q^{(\Phi_- w_- m \lambda, \eta)} q^{(\Phi_- w_- \Phi_+ m \lambda, -\gamma)} y_\lambda c$$

From this it follows easily that

$$\text{ad}(c) y_\lambda = q^{(m \sigma(w) \lambda, \eta)} \epsilon(c) y_\lambda.$$

Since (up to some scalar) $y_\lambda = d^{-1} d^{-1} = 2^{-2}$ it follows from Theorem 4.7 that $y_\lambda \in (A_w)^{2m\lambda}$. However, as a $C_{q,p}[G]$-module via the adjoint action, $A^\text{H}_w y_\lambda \cong A^\text{H}_w \otimes C_{q,p} \lambda$ and hence $\text{Soc} A^\text{H}_w y_\lambda = \mathcal{C}y_\lambda$. Thus $Z_{2m\lambda} \neq (0)$ if and only if $y_\lambda$ is ad-invariant; that is, if and only if $m \sigma(w) = 0$. Hence

$$\dim Z_w = \text{rk} L_w = \text{rk}(L_w \cap 2mL) = \text{rk} \text{Ker} m\sigma(w)$$

$$= \dim \text{Ker}_H \sigma(w) = n - s(w)$$

as required.

Finally, we may deduce that in the algebraic case the size of the of the $H$-orbits $\text{Sym}_w G$ and $\text{Prim}_w C_{q,p}[G]$ are the same, cf. Theorem 1.8.

**Theorem 4.18.** Suppose that $p$ is $q$-rational and let $w \in W \times W$. Then

$$\forall P \in \text{Prim}_w C_{q,p}[G], \quad \dim(H/\text{Stab}_H P) = n - s(w).$$

Proof. This follows easily from theorems 4.15, 4.16 and Proposition 4.17. 

**References**

[1] N. Andruskiewitsch and B. Enriquez, Examples of compact matrix pseudogroups arising from the twisting operation, *Comm. Math. Phys.*, 149 (1992), 195-207.

[2] M. Artin, W. Schelter and J. Tate, Quantum deformations of $GL_n$, *Comm. Pure Appl. Math.*, 44 (1991), 879-895.

[3] A. A. Belavin and V. G. Drinfeld, Triangle equations and simple Lie algebras, Mathematical Physics reviews (S. P. Novikov, ed.), Harwood, New York 1984, 93-166.

[4] A. Borel, *Linear algebraic groups*, W.A. Benjamin, New York 1969.

[5] W. Chin and I. Musson, Multi-parameter quantum enveloping algebras, preprint 1994.

[6] W. Chin and D. Quinn, Rings graded by polycyclic-by-finite groups, *Proc. Amer. Math. Soc.*, 102 (1988), 235-241.

[7] M. Constantini and M. Varagnolo, Quantum double and multi-parameter quantum group, *Comm. Alg.*, 22 (1994), 6305-6321.

[8] M. Constantini and M. Varagnolo, Multi-parameter quantum function algebra at roots of 1, preprint.

[9] P. Cotta-Ramusino and M. Rinaldi, Multi-parameter quantum groups related to link diagrams, *Comm. Math. Phys.* 142 (1991), 589-604.

[10] C. De Concini and V. Lyubashenko, Quantum function algebra at roots of 1, *Adv. Math.*, 108 (1994), 205-262.

[11] C. De Concini and C. Procesi, Quantum groups, in D-modules, representation theory and quantum groups, Lecture Notes in Mathematics 1565, Springer 1994, 31-140.

[12] J. Dixmier, *Algèbres Enveloppantes*, Gauthier-Villars, Paris 1974.

[13] V.G.Drinfeld, Quantum groups, in Proceedings ICM, Berkeley, AMS, 1987, 798-820.
[14] T. Hayashi, Quantum groups and quantum determinants, J. Algebra, 152 (1992), 146-165.
[15] T. J. Hodges and T. Levasseur, Primitive ideals of $C_q[SL(3)]$, Comm. Math. Phys., 156 (1993), 581-605.
[16] T. J. Hodges and T. Levasseur, Primitive ideals of $C_q[SL(n)]$, J. Algebra, 168 (1994), 455-468.
[17] T. J. Hodges and T. Levasseur, Primitive ideals of $C_q[G]$, University of Cincinnati 1993 (unpublished).
[18] A. Joseph, On the prime and primitive spectrum of the algebra of functions on a quantum group, J. Algebra, 169 (1994), 441-511.
[19] A. Joseph, Idéaux premiers et primitifs de l’algèbre des fonctions sur un groupe quantique, C. R. Acad. Sci. Paris, 316 I (1993), 1139-1142.
[20] S. Z. Levendorskii, Twisted function algebras on a compact quantum group and their representations, St. Petersburg Math J. 3 (1992), 405-423.
[21] S. Z. Levendorskii and Y. S. Soibelman, Algebras of functions on compact quantum groups, Schubert cells and quantum tori, Comm. Math. Phys. 139 (1991), 141-170.
[22] J.-H. Lu and A. Weinstein, Poisson Lie groups, dressing transformations and Bruhat decompositions, J. Differential Geometry, 31 (1990), 501–526.
[23] G. Lusztig, Quantum deformations of certain simple modules over enveloping algebras, Advances in Math., 70 (1988), 237-249.
[24] V. Lychagin, Braided differential operators and quantizations in ABC-categories, C. R. Acad. Sci. Paris, 318 I (1994), 857-862.
[25] M. Okado and H. Yamane, $R$-matrices with gauge parameters and multi-parameter quantized enveloping algebras, in Special functions (Olayama 1990), ICM-90 Satell. Conf. Proc., Springer 1991, 289-293.
[26] N. Y. Reshetikhin, Multi-parameter quantum groups and twisted quasi-triangular Hopf algebras, Lett. Math. Phys, 20 (1990), 331-335.
[27] M. Rieffel, Compact quantum groups associated with toral subgroups, in Representation Theory of Groups and Algebras, Contemporary Math., 145 (1993), 465-491.
[28] M. Rosso, Analogues de la forme de Killing et du théorème de Harish-Chandra pour les groupes quantiques, Ann. Sci. Ec. Norm. Sup., 23 (1990), 445-467.
[29] M. Rosso, Représentations des groupes quantiques, in Séminaire Bourbaki, Exposé 744, Soc. Math. France 1991, 443-483.
[30] M. A. Semenov-Tian-Shansky, Dressing transformations and Poisson group actions, Publ. RIMS, Kyoto Univ., 21 (1985), 1237–1260.
[31] Ya. S. Soibelman, The algebra of functions on a compact quantum group, and its representations, Leningrad Math. J. 2 (1991), 161–178.
[32] A. Sudberry, Consistent multi-parameter quantization of $GL(n)$, J. Phys., A 23 (1990), L697-L704.
[33] T. Tanisaki, Killing forms, Harish-Chandra isomorphisms, and universal $R$-Matrices for Quantum Algebras, in Infinite Analysis, Proceedings of the RIMS research project 1991, Part B, World Scientific, 1992, 941-961.

University of Cincinnati, Cincinnati, OH 45221-0025, U.S.A.
E-mail address: timothy.hodges@uc.edu

Université de Poitiers, 86022 Poitiers, France
E-mail address: thierry.levasseur@mapts.univ-poitiers.fr

Universidad Nacional de Colombia, Apartado Aéreo, Medellín, Colombia
E-mail address: mmtoro@perseus.unalmed.edu.co