On the computation of the Apéry set of numerical monoids and affine semigroups

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Let $A = k[x_1, \ldots, x_n]$. 
Introduction to Groebner bases.

Let $A = k[x_1, \ldots, x_n]$. There is a division procedure in $A$ which parallels the Euclidean division (after fixing a monomial ordering).

A Groebner basis of an ideal $I$ of $A$ is essentially a set of generators of the ideal, which is very useful for obtaining or checking fundamental properties.

Definition: Buchberger’s criteria, 1965

A set of polynomials $B = \{f_1, \ldots, f_s\} \subset I$ is a Groebner basis of $I$ if and only if for all $f, g \in B$, the remainder of dividing the polynomial $S(f, g)$ (Syzygy) by $B$ is zero.

On the computation of the Apéry set.

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- the Groebner basis of $I$

The normal form of $f \in k[x_1, ..., x_n]$ w.r.t. $B$ is $\bar{f} = f \mod B$.

$p_i = \exp(LT(g_i))$, being $LT(f)$ the leading monomial,

"copy" of the positive quadrant with origin $q_i$

$k_{q_i} = q_i + \mathbb{Z}^n_{\geq 0} \subset \mathbb{Z}^n_{\geq 0}$.

Using this notation, define the stair of the Groebner basis as:

$E = \bigcup_i k_{q_i} \subset \mathbb{Z}^n_{\geq 0}$.

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Numerical monoids. Basic concepts

Let $S = \{\lambda_1 a_1 + \lambda_2 a_2 + \ldots + \lambda_n a_n | \lambda_i \in \mathbb{Z} \geq 0\}$ be a numerical monoid.

Notation:
The set $G(S) = \mathbb{Z} \geq 0 \setminus S$ is finite → "gaps".

$\# G(S) = g(S)$.

Frobenius number $f(S) = \max \mathbb{Z} \geq 0 (G(S))$.

The conductor: $c(S) = f(S) + 1$.

The multiplicity: $m(S) = \min S \setminus \{0\}$.

The embedding dimension: $e(S)$ is the minimal number of generators of $S$.

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and let \( B = \{ g_1, \ldots, g_r \} \) a reduced basis, with \( E = \bigcup K_{q_i} \) its staircase. 

Numerical monoid,
\[ S = \{ y_1 a_1 + \ldots + y_n a_n | y_i \in \mathbb{Z}_{\geq 0} \} \]
\[ \leftrightarrow \mathbb{E} \cap \{ x = 0 \} \]
Established by the following:
\[ G : S \rightarrow \mathbb{E} \cap \{ x = 0 \} \subset \mathbb{Z}_{k+1}^+ \]
\[ m \rightarrow \exp \left( N_B(x m) \right) \]
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Established by the following:

\[ G : S \rightarrow \overline{E} \cap \{x = 0\} \subset \mathbb{Z}_{\geq 0}^{k+1} \]

\[ m \mapsto \exp(N_B(x^m)) \]
Bijection between the Groebner and numerical monoids.

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In both cases
\[ (\sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_k) = \exp(N_B(x^m)) \rightarrow N = \sigma_0 + a_1\sigma_1 + a_2\sigma_2 + \ldots + a_k\sigma_k \]
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The Apéry set.

Definition

Let $a \in S$ be a numerical monoid, the Apéry set of $S$ with respect to an element $s \in S$ can be defined as $\text{Ap}(S, a) = \{0, w_0, \ldots, w_{a-1}\}$ where $w_i$ is the smallest element in $S$ congruent with $i \mod a$.

Lemma

With the previous notation, for all $s \in S$ $\text{Ap}(S, a) = \{s \in S | s - a/\in S\}$. 

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**Lemma**

With the previous notation, for all $s \in S$

$$Ap(S, a) = \{s \in S \mid s - a \not\in S\}.$$
Algorithm. Numerical case.

Let $S = \langle a_1, \ldots, a_k \rangle$ be a numerical monoid in a polynomial ring $\mathbb{Q}[x, y_1, \ldots, y_k]$, $a_i \neq 0$.

1: We define the binomial ideal:

$$I_S = \langle y_1 - x a_1, y_2 - x a_2, y_3 - x a_3, \ldots, y_k - x a_k \rangle \subset \mathbb{Q}[x, y_1, \ldots, y_k].$$

2: Let us define an elimination ordering for $x$, noted $\sigma_j$:

$$
\begin{pmatrix}
1 & 0 & 0 & \cdots & (j+1) & 0 & \cdots & 0 \\
0 & \cdots & 0 & a_1 & a_2 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & 1 & \cdots & 0 & \cdots & 0 \\
\end{pmatrix}
$$

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  1 & 0 & 0 & \cdots & 0 & \cdots & 0 \\
  0 & a_1 & a_2 & \cdots & 0 & \cdots & a_k \\
  0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & \cdots & 0 & \cdots & 1 \\
  0 & 0 & 0 & \cdots & 1 & \cdots & 0 \\
\end{pmatrix}
$$
3: Let $\mathcal{B}_j$ be the reduced Groebner basis of $I_S$ with respect to $\sigma_j$. 

Theorem
Under the previous conditions $A_p(S, a_j) = \Delta_{\sigma_j}(S, a_j)$. 

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3: Let $B_j$ be the reduced Groebner basis of $I_S$ with respect to $\sigma_j$.

4: We define the following set.
3: Let $\mathcal{B}_j$ be the reduced Groebner basis of $I_S$ with respect to $\sigma_j$.

4: We define the following set.

$$\Delta_{\sigma_j}(S, a_j) := \left\{ N \in \mathbb{Z}_{\geq 0} \mid \exp \left( \mathcal{N}_j \left( x^N \right) \right) \in \{ x = y_j = 0 \} \cap \overline{E(I)} \right\}$$
3: Let $\mathcal{B}_j$ be the reduced Groebner basis of $I_S$ with respect to $\sigma_j$.

4: We define the following set.

$$\Delta_{\sigma_j}(S, a_j) := \left\{N \in \mathbb{Z}_{\geq 0} \mid \exp \left( N_j \left( x^N \right) \right) \in \{ x = y_j = 0 \} \cap \overline{E(I)} \right\}$$
3: Let $B_j$ be the reduced Groebner basis of $I_S$ with respect to $\sigma_j$.

4: We define the following set.

$$\Delta_{\sigma_j}(S, a_j) := \left\{ N \in \mathbb{Z}_{\geq 0} \mid \exp\left(Nj_\sigma(x^N)\right) \in \{x = y_j = 0\} \bigcap \overline{E(I)} \right\}$$

Theorem

Under the previous conditions $Ap(S, a_j) = \Delta_{\sigma_j}(S, a_j)$. 
Proof.

Let $n \in \text{Ap} (S, a_j) = \Rightarrow n \in S = \Rightarrow$ there are $x_1, \ldots, x_k \in \mathbb{Z} \geq 0$ such that $n = k \sum_{i=1}^{n} a_i x_i$.

Being in the Apéry set, we already know $n - a_j / \in S$. We want to prove $\exp (N_j (x_n)) \in \{y_j = 0\}$.
Proof.

\[ Ap(S, a_j) \subseteq \Delta_{\sigma_j}(S, a_j). \]
Proof.

- \( Ap(S, a_j) \subseteq \Delta_{\sigma_j}(S, a_j) \).
  Let \( n \in Ap(S, a_j) \implies n \in S y n > a_j \)
Proof.

- \( Ap(S, a_j) \subseteq \Delta_{\sigma_j}(S, a_j) \).

Let \( n \in Ap(S, a_j) \implies n \in S \) y \( n > a_j \) which implies \( n \in S \implies \) there are \( x_1, ..., x_k \in \mathbb{Z}_{\geq 0} \) such that

\[
n = \sum_{i=1}^{k} a_i x_i.
\]
Proof.

- $Ap(S, a_j) \subseteq \Delta_{\sigma_j}(S, a_j)$.
  
  Let $n \in Ap(S, a_j) \implies n \in S$ y $n > a_j$ which implies $n \in S \implies$ there are $x_1, ..., x_k \in \mathbb{Z}_{\geq 0}$ such that

  $$n = \sum_{i=1}^{k} a_i x_i.$$  

  Being in the Apéry set, we already know $n - a_j \notin S$. 

Proof.

- \( Ap(S, a_j) \subseteq \Delta_{\sigma_j}(S, a_j) \).

  Let \( n \in Ap(S, a_j) \implies n \in S \) and \( n > a_j \) which implies \( n \in S \implies \) there are \( x_1, \ldots, x_k \in \mathbb{Z}_{\geq 0} \) such that

  \[
  n = \sum_{i=1}^{k} a_i x_i.
  \]

  Being in the Apéry set, we already know \( n - a_j \notin S \). We want to prove

  \[
  \exp(N_j(x^n)) \in \{x = 0\} \cap \{y_j = 0\}.
  \]
Proof.

- \( Ap(S, a_j) \subseteq \Delta_{\sigma_j}(S, a_j) \).
  Let \( n \in Ap(S, a_j) \implies n \in S \) y \( n > a_j \) which implies \( n \in S \implies \) there are \( x_1, \ldots, x_k \in \mathbb{Z}_{\geq 0} \) such that

\[
    n = \sum_{i=1}^{k} a_i x_i.
\]

Being in the Apéry set, we already know \( n - a_j \notin S \). We want to prove

\[
    \exp (N_j(x^n)) \in \{x = 0\} \cap \{y_j = 0\}. \quad \text{for } n \in S
\]
Proof.

- \( Ap(S, a_j) \subseteq \Delta_{\sigma_j}(S, a_j) \).

Let \( n \in Ap(S, a_j) \implies n \in S \) y \( n > a_j \) which implies \( n \in S \implies \) there are \( x_1, ..., x_k \in \mathbb{Z}_{\geq 0} \) such that

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\]

We have to prove

\[
    \exp(N_j(x^n)) \in \{y_j = 0\}.
\]
Let us write $\exp(N_j(x^n)) = (\gamma_1, \gamma_2, \ldots, \gamma_k)$.
Proof

Let us write \( \exp(N_j(x^n)) = (\gamma_1, \gamma_2, \ldots, \gamma_k) \).

From the expression above

\[
n - a_j = a_1 \gamma_1 + \ldots + a_j(\gamma_j - 1) + \ldots + a_k \gamma_k. \quad (\star)
\]
Proof

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n - a_j = a_1\gamma_1 + \ldots + a_j(\gamma_j - 1) + \ldots + a_k\gamma_k. \quad (\star)
\]

As \( n - a_j \notin S \),
Proof

Let us write $exp(N_j(x^n)) = (\gamma_1, \gamma_2, ..., \gamma_k)$.

From the expression above

$$n - a_j = a_1\gamma_1 + ... + a_j(\gamma_j - 1) + ... + a_k\gamma_k. \quad (\star)$$

As $n - a_j \not\in S$, this expression $(\star)$ must have a strictly negative coefficient.
Proof

Let us write \( \exp(N_j(x^n)) = (\gamma_1, \gamma_2, \ldots, \gamma_k) \).

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\]

As \( n - a_j \notin S \), this expression \((\star)\) must have a strictly negative coefficient.

As \( \gamma_i \in \mathbb{Z}_{\geq 0} \quad \forall \ i = 1, \ldots, k \)
Proof

Let us write \( \exp(N_j(x^n)) = (\gamma_1, \gamma_2, \ldots, \gamma_k) \).

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n - a_j = a_1\gamma_1 + \ldots + a_j(\gamma_j - 1) + \ldots + a_k\gamma_k. \tag{\star}
\]

As \( n - a_j \notin S \), this expression (\( \star \)) must have a strictly negative coefficient.

As \( \gamma_i \in \mathbb{Z}_{\geq 0} \) \( \forall \ i = 1, \ldots, k \) it must be \( (\gamma_j - 1) \notin \mathbb{Z}_{\geq 0} \).
Proof

Let us write \( \exp(\mathcal{N}_j(x^n)) = (\gamma_1, \gamma_2, \ldots, \gamma_k) \).

From the expression above

\[
n - a_j = a_1 \gamma_1 + \ldots + a_j(\gamma_j - 1) + \ldots + a_k \gamma_k. \quad (\ast)
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As \( n - a_j \notin S \), this expression (\( \ast \)) must have a strictly negative coefficient.

As \( \gamma_i \in \mathbb{Z}_{\geq 0} \quad \forall \ i = 1, \ldots, k \) it must be \( (\gamma_j - 1) \notin \mathbb{Z}_{\geq 0} \).

And therefore

\[
\gamma_j = 0
\]

as we wanted to show.
\[ \Delta_{\sigma_j}(S, a_j) \subseteq Ap(S, a_j). \]
Proof

- $\Delta_{\sigma_j}(S, a_j) \subseteq Ap(S, a_j)$.

  Take $n \in \Delta_{\sigma_j}(S, a_j) \implies n \in S$ with

  $exp(N_j(x^n)) \in \{y_j = 0\} \cap \{x = 0\}$
Proof

- $\Delta_{\sigma_j}(S, a_j) \subseteq \text{Ap}(S, a_j)$.
  
  Take $n \in \Delta_{\sigma_j}(S, a_j) \implies n \in S$ with
  
  $\exp(N_j(x^n)) \in \{y_j = 0\} \cap \{x = 0\}$

  \[\Downarrow \quad \exists \gamma_1, \ldots, \gamma_{j-1}, \gamma_{j+1}, \ldots, \gamma_k \in \mathbb{Z}_{\geq 0}, \]

  \[N_j(x^n) = y_1^{\gamma_1} \cdots y_{j-1}^{\gamma_{j-1}} \cdot y_{j+1}^{\gamma_{j+1}} \cdots y_k^{\gamma_k}.\]
Proof

- $\Delta_{\sigma_j}(S, a_j) \subseteq Ap(S, a_j)$.

Take $n \in \Delta_{\sigma_j}(S, a_j) \implies n \in S$ with

$\exp(N_j(x^n)) \in \{y_j = 0\} \cap \{x = 0\}$

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\[ N_j(x^n) = y_1^{\gamma_1} \cdots y_{j-1}^{\gamma_{j-1}} \cdots y_{j+1}^{\gamma_{j+1}} \cdots y_k^{\gamma_k}. \]

$n \in Ap(S, a_j)$?
\begin{itemize}
\item \( \Delta_{\sigma_j}(S, a_j) \subseteq \text{Ap}(S, a_j) \).
\end{itemize}

Take \( n \in \Delta_{\sigma_j}(S, a_j) \implies n \in S \) with 
\( \exp(N_j(x^n)) \in \{y_j = 0\} \cap \{x = 0\} \)

\[
\Downarrow \quad \exists \gamma_1, \ldots, \gamma_{j-1}, \gamma_{j+1}, \ldots, \gamma_k \in \mathbb{Z}_{\geq 0},
\]

\[
N_j(x^n) = y_1^{\gamma_1} \cdots y_{j-1}^{\gamma_{j-1}} \cdot y_{j+1}^{\gamma_{j+1}} \cdots y_k^{\gamma_k}.
\]

\( n \in \text{Ap}(S, a_j) \)?

R.A:

Let us assume \( n \notin \text{Ap}(S, a_j) \). So either \( n \notin S \) or \( n > a_j \) and 
\( n - a_j \in S \).
\[ \Delta_{\sigma_j}(S, a_j) \subseteq Ap(S, a_j). \]

Take \( n \in \Delta_{\sigma_j}(S, a_j) \) \( \implies n \in S \) with
\[ \exp(N_j(x^n)) \in \{y_j = 0\} \cap \{x = 0\} \]

\[ \Downarrow \exists \gamma_1, \ldots, \gamma_{j-1}, \gamma_{j+1}, \ldots, \gamma_k \in \mathbb{Z}_{\geq 0}, \]

\[ N_j(x^n) = y_1^{\gamma_1} \cdots y_{j-1}^{\gamma_{j-1}} \cdot y_{j+1}^{\gamma_{j+1}} \cdots y_k^{\gamma_k}. \]

\( n \in Ap(S, a_j)? \)

R.A:
Let us assume \( n \notin Ap(S, a_j) \). So either \( n \notin S \) or \( n > a_j \) and \( n - a_j \in S \).
\( \implies \exists \alpha_1, \ldots, \alpha_k \in \mathbb{Z}_{\geq 0} \) such that

\[ n = \sum_{i=1, i\neq j}^{k} a_i \alpha_i + a_j(\alpha_j + 1) \implies (\alpha_1, \ldots, (\alpha_j + 1), \ldots, \alpha_k) \in \mathbb{Z}_{\geq 0}^k. \]
So we have two expressions for $n$, 

$$\gamma_1, \ldots, (j), \ldots, \gamma_k$$ 

$$n - a_j: \alpha_1, \ldots, (\alpha_j + 1), \ldots, \alpha_k$$ 

which yields, 

$$\sum_{i=1, i \neq j}^k a_i (\alpha_i - \gamma_i) + a_j (\alpha_j + 1) = 0 \, .$$  

(∗)
Proof

So we have two expressions for $n$,

1. Normal form:

$\left(\gamma_1, \ldots, \gamma_k\right)$

$$\left(\gamma_1, \ldots, \begin{array}{c}(j) \\ 0 \end{array}, \ldots, \gamma_k\right)$$
So we have two expressions for $n$,

1. **Normal form:**

\[
\left(\gamma_1, \ldots, \underbrace{0}_{(j)}, \ldots, \gamma_k\right)
\]

2. $n - a_j$:

\[
\left(\alpha_1, \ldots, (\alpha_j + 1), \ldots, \alpha_k\right)
\]
Proof

So we have two expressions for $n$,

1. **Normal form:**

   $$(\gamma_1, \ldots, (\gamma_j, \ldots, 0, \ldots, \gamma_k))$$

2. **$n - a_j$:**

   $$(\alpha_1, \ldots, (\alpha_j + 1), \ldots, \alpha_k)$$

which yields,

$$\sum_{i=1, i \neq j}^{k} a_i(\alpha_i - \gamma_i) + a_j(\alpha_j + 1) = 0. \quad (*)$$
Proof

So by the definition of $\sigma_j$, we know that

$$(0, \gamma_1, \ldots, 0, \ldots, \gamma_k) <_{\sigma_j} (0, \alpha_1, \alpha_2, \ldots, \alpha_k)$$
Proof

So by the definition of $\sigma_j$, we know that

\[
(0, \gamma_1, \ldots, 0, \ldots, \gamma_k) <_{\sigma_j} (0, \alpha_1, \alpha_2, \ldots, \alpha_k)
\]

\[
\begin{align*}
\sum_{i=1, i \neq j}^{k} \gamma_i a_i &\leq \sum_{i=1, i \neq j}^{k} \alpha_i a_i \\
\sum_{i=1, i \neq j}^{k} a_i (\alpha_i - \gamma_i) &\geq 0.
\end{align*}
\]
Proof

So by the definition of $\sigma_j$, we know that

$$
(0, \gamma_1, \ldots, 0, \ldots, \gamma_k) <_{\sigma_j} (0, \alpha_1, \alpha_2, \ldots, \alpha_k)
$$

$$
\uparrow
$$

\begin{align*}
\sum_{i=1, i \neq j}^k \gamma_i a_i & \leq \sum_{i=1, i \neq j}^k \alpha_i a_i \quad \Rightarrow \quad \sum_{i=1, i \neq j}^k a_i (\alpha_i - \gamma_i) \geq 0. \\
\sum_{i=1, i \neq j}^k a_i (\alpha_i - \gamma_i) + a_j (\alpha_j + 1) & \geq 0
\end{align*}

\Rightarrow a_j = 0 \rightarrow QED
Proof

So by the definition of $\sigma_j$, we know that

$$(0, \gamma_1, \ldots, 0, \ldots, \gamma_k) <_{\sigma_j} (0, \alpha_1, \alpha_2, \ldots, \alpha_k)$$

$$\uparrow$$

$$\sum_{i=1,i\neq j}^k \gamma_i a_i \leq \sum_{i=1,i\neq j}^k \alpha_i a_i \implies \sum_{i=1,i\neq j}^k a_i (\alpha_i - \gamma_i) \geq 0.$$
Proof

So by the definition of $\sigma_j$, we know that

$$(0, \gamma_1, \ldots, 0, \ldots, \gamma_k) <_{\sigma_j} (0, \alpha_1, \alpha_2, \ldots, \alpha_k)$$

$$\Updownarrow$$

$$\sum_{i=1, i \neq j}^k \gamma_i a_i \leq \sum_{i=1, i \neq j}^k \alpha_i a_i \implies \sum_{i=1, i \neq j}^k a_i (\alpha_i - \gamma_i) \geq 0.$$

$$\sum_{i=1, i \neq j}^k a_i (\alpha_i - \gamma_i) + a_j (\alpha_j + 1) = 0 \implies a_j = 0 \rightarrow \leftarrow$$
Proof

So by the definition of $\sigma_j$, we know that

$$(0, \gamma_1, \ldots, 0, \ldots, \gamma_k) <_{\sigma_j} (0, \alpha_1, \alpha_2, \ldots, \alpha_k)$$

$$\uparrow$$

$$\sum_{i=1, i \neq j}^k \gamma_i a_i \leq \sum_{i=1, i \neq j}^k \alpha_i a_i \implies \sum_{i=1, i \neq j}^k a_i (\alpha_i - \gamma_i) \geq 0.$$ 

$$\sum_{i=1, i \neq j}^k a_i (\alpha_i - \gamma_i) + a_j (\alpha_j + 1) \geq 0 \implies a_j = 0 \implies QED.$$
Corollary

The set $\Delta_{\sigma_j}(S, a_j)$ does not depend on the choice of $\sigma_j$. 
Corollary

The set $\Delta_{\sigma_j}(S, a_j)$ does not depend on the choice of $\sigma_j$.

Let $S = \langle 7, 8, 9, 13 \rangle$. 
Algorithm. Example

Corollary

The set $\Delta_{\sigma_j}(S, a_j)$ does not depend on the choice of $\sigma_j$.

Let $S = \langle 7, 8, 9, 13 \rangle$. We have.

$$I_S = \langle y_1 - x^7, y_2 - x^8, y_3 - x^9, y_4 - x^{13} \rangle \subset \mathbb{Q}[x, y_1, y_2, y_3],$$
The set $\Delta_{\sigma_j}(S, a_j)$ does not depend on the choice of $\sigma_j$.

Let $S = \langle 7, 8, 9, 13 \rangle$. We have.

$$I_S = \langle y_1 - x^7, y_2 - x^8, y_3 - x^9, y_4 - x^{13} \rangle \subset \mathbb{Q}[x, y_1, y_2, y_3],$$

For this monoid we have

$$Ap(S, 13) = \{0, 7, 8, 9, 14, 15, 16, 17, 18, 23, 24, 25, 32\}.$$
A possible picture of $Ap(S,13)$ (for a chosen monomial ordering) is
A possible picture of $A_p(S, 13)$ (for a chosen monomial ordering) is
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A possible picture of $Ap(S, 13)$ (for a chosen monomial ordering) is
Another possible picture of \( Ap(S, 13) \) is

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
x & y_4 & y_3 & y_1 & 0 & 1 & 2 & 3 & 4 \\
\hline
0 & 0 & 0 & 0 & 21 & 28 & & & \\
1 & 1 & 8 & 15 & 22 & 29 & 36 & & \\
2 & 2 & 16 & 23 & 30 & 37 & 44 & & \\
3 & 3 & 24 & 31 & 38 & 45 & 52 & & \\
4 & 4 & 32 & 39 & 46 & 53 & 60 & & \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
x & y_4 & y_3 & y_1 & 0 & 1 & 2 & 3 & 4 \\
\hline
0 & 1 & 9 & 16 & 23 & 30 & 37 & & \\
1 & 1 & 17 & 24 & 31 & 38 & 45 & & \\
2 & 2 & 25 & 32 & 39 & 46 & 53 & & \\
3 & 3 & 33 & 40 & 47 & 54 & 61 & & \\
4 & 4 & 41 & 48 & 55 & 62 & 69 & & \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
y_3 & y_1 & 0 & 1 & 2 & 3 \\
\hline
0 & 18 & 25 & 32 & 39 & & \\
1 & 26 & 33 & 40 & 47 & & \\
2 & 34 & 41 & 48 & 55 & & \\
3 & 42 & 49 & 56 & 63 & & \\
\hline
\end{array}
\]
Another possible picture of $Ap(S, 13)$ is

$$\begin{array}{cccccc}
\hline
x & y_4 & y_3 & y_1 & y_1 & y_1 & y_1 \\
\hline
0 & 0 & 0 & 0 & 1 & 2 & 3 \\
1 & 0 & 7 & 14 & 21 & 28 \\
2 & 8 & 15 & 22 & 29 & 36 \\
3 & 16 & 23 & 30 & 37 & 44 \\
4 & 24 & 31 & 38 & 45 & 52 \\
\hline
\end{array}$$

$$\begin{array}{cccccc}
\hline
x & y_4 & y_3 & y_1 & y_1 & y_1 & y_1 \\
\hline
0 & 1 & 2 & 3 & 4 & 4 & 4 \\
1 & 0 & 7 & 14 & 21 & 28 & 28 \\
2 & 8 & 15 & 22 & 29 & 36 & 36 \\
3 & 16 & 23 & 30 & 37 & 44 & 44 \\
4 & 24 & 31 & 38 & 45 & 52 & 52 \\
\hline
\end{array}$$

In both cases $Ap(S, 13) = \Delta_{\sigma_j}(S, 13)$. 
Algorithm: Affine case.

An affine monoid is a finitely generated monoid that is isomorphic to a submonoid of $\mathbb{Z}^d$, $d \geq 0$.

We will say that a monoid $S$ is pointed if $S \cap (-S) = \{0\}$. Equivalently, if the rational cone, $\text{pos}(S) := \{\lambda_1 a_1 + \ldots + \lambda_k a_k | \lambda_i \in \mathbb{Q} \geq 0\}$, is pointed.

Let $\Lambda \subseteq \{a_1, \ldots, a_k\}$ such that $\text{pos}(S) = \text{pos}(\Lambda)$. The Apéry set of $S$ with respect to $\Lambda$ is defined as follows: $\text{Ap}(S, \Lambda) = \{a \in S | a - b \not\in S, \forall b \in \Lambda\}$.
An affine monoid is a finitely generated monoid that is isomorphic to a submonoid of $\mathbb{Z}^d$, $d \geq 0$. 

A point is said to be pointed if $S \cap (-S) = \{0\}$. Equivalently, if the rational cone, $\text{pos}(S) := \{\lambda_1 a_1 + \ldots + \lambda_k a_k | \lambda_i \in \mathbb{Q} \geq 0\}$, is pointed.

Let $\Lambda \subseteq \{a_1, \ldots, a_k\}$ such that $\text{pos}(S) = \text{pos}(\Lambda)$. The Apéry set of $S$ with respect to $\Lambda$ is defined as follows: $\text{Ap}(S, \Lambda) = \{a \in S | a - b \not\in S, \forall b \in \Lambda\}$. 

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On the computation of the Apéry set.

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We will say that a monoid $S$ is pointed if $S \cap (-S) = \{0\}$.

Equivalently, if the rational cone, $\text{pos}(S) := \{\lambda_1 a_1 + \ldots + \lambda_k a_k | \lambda_i \in \mathbb{Q} \geq 0\}$ is pointed.

Let $\Lambda \subseteq \{a_1, \ldots, a_k\}$ such that $\text{pos}(S) = \text{pos}(\Lambda)$. The Apéry set of $S$ with respect to $\Lambda$ is defined as follows: $\text{Ap}(S, \Lambda) = \{a \in S | a - b \not\in S, \forall b \in \Lambda\}$.
Algorithm: Affine case.

An affine monoid is a finitely generated monoid that is isomorphic to a submonoid of $\mathbb{Z}^d$, $d \geq 0$. We will say that a monoid $S$ is pointed if $S \cap (-S) = \{0\}$. (0 is the only invertible element of $S$)
An affine monoid is a finitely generated monoid that is isomorphic to a submonoid of $\mathbb{Z}^d$, $d \geq 0$. We will say that a monoid $S$ is pointed if $S \cap (-S) = \{0\}$. (0 is the only invertible element of $S$)

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An affine monoid is a finitely generated monoid that is isomorphic to a submonoid of $\mathbb{Z}^d$, $d \geq 0$.

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is pointed
Algorithm: Affine case.

An affine monoid is a finitely generated monoid that is isomorphic to a submonoid of \( \mathbb{Z}^d, d \geq 0 \).

We will say that a monoid \( S \) is pointed if \( S \cap (-S) = \{0\} \).

(0 is the only invertible element of \( S \))

Equivalently, if the rational cone,

\[
pos(S) := \{ \lambda_1 a_1 + \ldots + \lambda_k a_k \mid \lambda_i \in \mathbb{Q}_{\geq 0} \}
\]

is pointed. Let \( \Lambda \subseteq \{ a_1, \ldots, a_k \} \) such that \( pos(S) = pos(\Lambda) \). The Apéry set of \( S \) with respect to a \( \Lambda \) is defined as follows:
Algorithm. Affine case.

An affine monoid is a finitely generated monoid that is isomorphic to a submonoid of $\mathbb{Z}^d$, $d \geq 0$.

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Equivalently, if the rational cone,

$$pos(S) := \{ \lambda_1 a_1 + \ldots + \lambda_k a_k \mid \lambda_i \in \mathbb{Q}_{\geq 0} \}$$

is pointed. Let $\Lambda \subseteq \{a_1, \ldots, a_k\}$ such that $pos(S) = pos(\Lambda)$. The Apéry set of $S$ with respect to a $\Lambda$ is defined as follows:

$$Ap(S, \Lambda) = \{ a \in S \mid a - b \not\in S, \forall b \in \Lambda \}.$$
Algorithm: Affine case.

Suppose \( \Lambda = \{ a_k - n + 1, \ldots, a_k \} \), \( n \leq k \).

Consider \( Q = [x_1, \ldots, x_d, y_1, \ldots, y_k] \).

And let \( \prec \Lambda \) be a block–ordering over \( \Lambda \) such that:

\[ \begin{align*}
\prec \Lambda &= \begin{cases}
\text{arbitrary order} & \text{for } x \\
\text{S–graded reversed lex} & \text{for } y
\end{cases}
\end{align*} \]

\( \forall j \in \{1, \ldots, k-n\} \)

\( \forall j \in \{k-n+1, \ldots, k\} \).
Algorithm: Affine case.

Suppose $\Lambda = \{a_{k-n+1}, \ldots, a_k\}$, $n \leq k$. 
Algorithm: Affine case.

Suppose $\Lambda = \{a_{k-n+1}, \ldots, a_k\}$, $n \leq k$. Consider $\mathbb{Q}[x_1, \ldots, x_d, y_1, \ldots, y_k]$. 
Algorithm: Affine case.

Suppose \( \Lambda = \{ a_{k-n+1}, \ldots, a_k \} \), \( n \leq k \).
Consider \( \mathbb{Q}[x_1, \ldots, x_d, y_1, \ldots, y_k] \). And let \( \prec_{\Lambda} \) be a block-ordering over \( A \) such that...
Algorithm: Affine case.

Suppose $\Lambda = \{a_{k-n+1}, \ldots, a_k\}$, $n \leq k$.
Consider $\mathbb{Q}[x_1, \ldots, x_d, y_1, \ldots, y_k]$. And let $\prec_{\Lambda}$ be a block–ordering over $A$ such that

$$\prec_{\Lambda} = \begin{cases} \text{monomial arbitrary order } \prec_x & \text{for } x \\ S\text{–graded reversed lex } \prec_y & \text{for } y \mid y_j \prec_y y_i \end{cases}$$

$\forall j \in \{1, \ldots, k - n\}$ $y$ $\forall j \in \{k - n + 1, \ldots, k\}$. 
Algorithm: Affine case.

\[ I \subseteq S \quad \text{the kernel of the ring homomorphism} \quad \tilde{\phi} : \mathbb{Q}[x_1, \ldots, x_d, y_1, \ldots, y_k] \to \mathbb{Q}[x_1, \ldots, x_d] \]

\[ y_j \mapsto x_{a_j} = x_{a_{j1}} \cdots x_{a_{jd}} \quad x_i \mapsto x_{a_i} \]

\[ B \Lambda, \quad \text{the reduced Groebner basis of} \quad I \quad \text{w.r.t.} \quad \Lambda. \]

\[ N \Lambda, \quad \text{for the normal form operator with respect to this basis.} \]

\[ \begin{array}{c}
\{ z_1, \ldots, z_n \} \leftarrow \{ y_k - n + 1, \ldots, y_k \}
\end{array} \]

\[ \text{We define the set} \quad Q \prec \Lambda(S) = \left\{ a \in \mathbb{Z}^d_{\geq 0} \mid \exp(N \Lambda(x^a)) \in \{ x_1 = \ldots = x_d = \ldots = z_i = 0 \} \cap E(I_S) \right\} \]

**Theorem**

\[ Q \prec \Lambda(S) = \text{Ap}(S, \Lambda). \]
Algorithm: Affine case.

Let $I_\mathcal{S}$ the kernel of the ring homomorphism

\[ \phi: \mathbb{Q}[x_1, \ldots, x_d, y_1, \ldots, y_k] \rightarrow \mathbb{Q}[x_1, \ldots, x_d], \quad y_j \mapsto x_{a_j} := x_{a_1} \cdots x_{a_d} \]

$B \Lambda$, the reduced Groebner basis of $I_\mathcal{S}$ w.r.t. $\prec \Lambda$.

Noted \( \{z_1, \ldots, z_n\} \leftarrow \{y_k - n + 1, \ldots, y_k\} \).

We define the set

\[ Q_\prec \Lambda(\mathcal{S}) = \{ a \in \mathbb{Z}^d \geq 0 | \exp(N_\Lambda(x^a)) \in \{x_1 = \ldots = x_d = \ldots = z_i = 0\} \cap E(I_\mathcal{S}) \} \]

Theorem $Q_\prec \Lambda(\mathcal{S}) = Ap(\mathcal{S}, \Lambda)$.
Algorithm: Affine case.

Let $I_S$ the kernel of the ring homomorphism

$$
\tilde{\phi} : \mathbb{Q}[x_1, \ldots, x_d, y_1, \ldots, y_k] \rightarrow \mathbb{Q}[x_1, \ldots, x_d]
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$$
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$$x_i \longmapsto x_i$$

$B_\Lambda$. 

Algorithm: Affine case.
Algorithm: Affine case.

Let $I_S$ the kernel of the ring homomorphism

$$
\bar{\phi} : \mathbb{Q}[x_1, \ldots, x_d, y_1, \ldots, y_k] \longrightarrow \mathbb{Q}[x_1, \ldots, x_d]
$$

$$
y_j \longmapsto x^{a_j} := x_1^{a_{1j}} \cdots x_d^{a_{dj}}
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$\mathcal{B}_\Lambda$, the reduced Groebner basis of $I_S$ w.r.t. $\prec_\Lambda$. 
Algorithm: Affine case.

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$$y_j \mapsto x^{a_j} := x_1^{a_1j} \cdots x_d^{a_dj}$$

$$x_i \mapsto x_i$$

$B_\Lambda$, the reduced Groebner basis of $I_S$ w.r.t. $\prec_\Lambda$.

$N_\Lambda$. 
Algorithm: Affine case.

Let $I_\Sigma$ the kernel of the ring homomorphism

\[ \widetilde{\phi} : \mathbb{Q}[x_1, \ldots, x_d, y_1, \ldots, y_k] \rightarrow \mathbb{Q}[x_1, \ldots, x_d] \]

\[ y_j \mapsto x^{a_j} := x_1^{a_{1j}} \cdots x_d^{a_{dj}} \]

\[ x_i \mapsto x_i \]

$\mathcal{B}_\Lambda$, the reduced Groebner basis of $I_\Sigma$ w.r.t. $\prec_\Lambda$.

$N_\Lambda$, for the normal form operator with respect to this basis.
Algorithm: Affine case.

Let $I_S$ the kernel of the ring homomorphism

$\tilde{\phi} : \mathbb{Q}[x_1, \ldots, x_d, y_1, \ldots, y_k] \rightarrow \mathbb{Q}[x_1, \ldots, x_d]$

$y_j \mapsto \mathbf{x}^{a_j} := x_1^{a_{1j}} \cdots x_d^{a_{dj}}$

$x_i \mapsto x_i$

$B_\Lambda$, the reduced Groebner basis of $I_S$ w.r.t. $\vartriangleleft_\Lambda$.

$N_\Lambda$, for the normal form operator with respect to this basis.

Noted $\{z_1, \ldots, z_n\} \leftarrow \{y_{k-n+1}, \ldots, y_k\}$.
Algorithm: Affine case.

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\bar{\phi} : \mathbb{Q}[x_1, \ldots, x_d, y_1, \ldots, y_k] \rightarrow \mathbb{Q}[x_1, \ldots, x_d]
$$

$$
y_j \mapsto x^{a_j} := x_1^{a_{1j}} \cdots x_d^{a_{dj}}
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x_i \mapsto x_i
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Algorithm: Affine case.

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Noted \( \{z_1, \ldots, z_n\} \leftarrow \{y_{k-n+1}, \ldots, y_k\} \).

We define the set

$$Q_{\prec_\Lambda}(S) = \left\{ a \in \mathbb{Z}_{\geq 0}^d \mid \exp(N_\Lambda(x^a)) \in \{x_1 = \ldots x_d = \ldots = z_i = 0\} \cap E(I_S) \right\}$$
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Noted $\{z_1, \ldots, z_n\} \leftarrow \{y_{k-n+1}, \ldots, y_k\}$.

We define the set

\[ Q_{\prec_\Lambda}(S) = \left\{ a \in \mathbb{Z}_{\geq 0}^d \mid \exp(N_\Lambda(x^a)) \in \{x_1 = \ldots x_d = \ldots = z_i = 0\} \cap \overline{E(I_S)} \right\} \]

**Theorem**

$Q_{\prec_\Lambda}(S) = Ap(S, \Lambda)$. 
Index

1. Introduction to Groebner bases.
2. Numerical monoids.
3. The Groebner correspondence.
4. An algorithm to the computation of the Apéry set.
5. Computation of the type set.
Let $S$ be a numerical monoid.
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- We define the partial order $\leq_S$ in $S$ as $x \leq_S y \iff y - x \in S$.
Gorenstein condition

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- We define the partial order $\leq_S$ in $S$ as $x \leq_S y \iff y - x \in S$
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  - $x + s \in S$ for all $s \in S \setminus \{0\}$.

\[ t(S) = \# \bigcup_{n \in S} \{ \max_{\leq_S} A_p(S, n) \} \]
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$$S \text{ is symmetric } \iff t(S) = 1$$
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- An integer
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- $x \in \mathbb{Z}$ is a pseudo-Frobenius number of $S$ if:
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The set of pseudo-Frobenius numbers: $\rightarrow PF(S)$.

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The set of pseudo–Frobenius numbers: $\rightarrow PF(S)$.

**Type**: $t(S) = \#PF(S)$

- $S$ is symmetric $\iff t(S) = 1 \iff PF(S) = \{f(S)\}$

An integer $g \in PF(S)$ if and only if for any $n \in S$, $g + n$ is a maximal element in $Ap(S, n)$ with respect to the ordering $\leq_S$.

$$t(S) = \# \bigcup_{n \in S} \left\{ \max_{\leq_S} Ap(S, n) \right\}.$$
Gorenstein condition

Nijenhuis and Wilf studied the property:

\[ g(S) = \frac{1}{2}(f(S) + 1) \]
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Case \( n = 2, S = \langle a_1, a_2 \rangle \Rightarrow g(S) = \frac{1}{2}(f(S) + 1). \) (Sylvester)

Let \( S = \langle a_1, \ldots, a_k \rangle \) and let us consider the set

\[ T(S) = \{ m \in Ap(S, a_k) \mid m + a_i \notin Ap(S, a_k), \forall i = 1, \ldots, k \} . \]
Gorenstein condition

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Let \( S = \langle a_1, \ldots, a_k \rangle \) and let us consider the set

\[ T(S) = \{ m \in Ap(S, a_k) \mid m + a_i \notin Ap(S, a_k), \forall i = 1, \ldots, k \} . \]

Then

\[ g(S) = \frac{1}{2}(f(S) + 1) \iff \#T(S) = 1. \]
The Gorenstein set and $PF(S)$

Proposition $PF(S) = \{m - a_k | m \in T(S)\}$.

Corollary Under the previous assumptions: $\#T(S) = t(S)$.

$T(S)$: Type set.
The Gorenstein set and $PF(S)$

**Proposition**

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The Gorenstein set and $PF(S)$

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\[ PF(S) = \{ m - a_k \mid m \in T(S) \} . \]

Corollary

Under the previous assumptions:

- \( \# T(S) = t(S) \).
- \( S \) verifies the Gorenstein condition if and only if it is symmetric.
The Gorenstein set and $PF(S)$

**Proposition**

$$PF(S) = \{m - a_k \mid m \in T(S)\}.$$  

**Corollary**

Under the previous assumptions:

- $\#T(S) = t(S)$.
- $S$ verifies the Gorenstein condition if and only if it is symmetric.
- $T(S) = \{\max_{\leq S} Ap(S, a_k)\}$. 

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Proposition

\[ PF(S) = \{ m - a_k \mid m \in T(S) \}. \]

Corollary

Under the previous assumptions:

- \( \#T(S) = t(S). \)
- \( S \) verifies the Gorenstein condition if and only if it is symmetric.
- \( T(S) = \{ \text{max}_{\leq_S} Ap(S, a_k) \}. \)

\( T(S) \): Type set.
Computation of the type set.

If we want to compute the set $T(S)$ of the monoid $S = \langle 3, 7, 11 \rangle$.

$Ap(S, 11) = \{0, 3, 6, 7, 9, 10, 12, 13, 15, 16, 19\}$.

We can use the partial ordering $\leq_S$:

$0 \rightarrow \rightarrow 3 \rightarrow \rightarrow 6 \rightarrow \rightarrow 9 \rightarrow \rightarrow 12 \rightarrow \rightarrow 15 \rightarrow \rightarrow 19 \rightarrow \rightarrow 16 \rightarrow \rightarrow 13 \rightarrow \rightarrow 10 \rightarrow \rightarrow 7 \rightarrow \rightarrow 4 \rightarrow \rightarrow 8$.

Then $T(S) = \{15, 19\}$, $PF(S) = \{4, 8\}$.
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We can use the partial ordering $\leq_S$:

$$\begin{align*}
0 & \rightarrow 3 \rightarrow 6 \rightarrow 9 \rightarrow 12 \rightarrow 15 \\
7 & \rightarrow 10 \rightarrow 13 \rightarrow 16 \rightarrow 19
\end{align*}$$
Computation of the type set.

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$$ Ap(S, 11) = \{0, 3, 6, 7, 9, 10, 12, 13, 15, 16, 19\}. $$

We can use the partial ordering $\leq_S$:

$$ T(S) = \{15, 19\}, \; PF(S) = \{4, 8\} $$
Computation of the type set.

With the previous algorithm we compute $A_p(S, a_k)$ using an order $\sigma_j$ whose normal form will be denoted by $N_k(\cdot)$.

Let $N \in A_p(S, a_k)$, and therefore let us write $\exp(N_k(xN)) = (0, \gamma_1, \ldots, \gamma_{k-1}, 0)$.

**Definition**

We will say $N$ is an extremal element of $A_p(S, a_k)$ for $\sigma_j$ if, for all $i = 1, \ldots, k$ we have $(0, \gamma_1, \ldots, \gamma_i + 1, \ldots, \gamma_k - 1, 0) / \in E(IS) \cap \{x = y_k = 0\}$.

We will denote:

$\partial \sigma_j(S, a_k) = \{\text{Extremal elements of } A_p(S, a_k) \text{ for } \sigma_j\}$.

$\partial \sigma_j(S, a_k)$ depends on the $\sigma_j$ order chosen.
Computation of the type set.

With the previous algorithm we compute $A_p(S, a_k)$ using an order $\sigma_j$.
Computation of the type set.

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$$\exp \left( N_k \left( x^N \right) \right) = (0, \gamma_1, ..., \gamma_{k-1}, 0).$$

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We will say $N$ is an extremal element of $\text{Ap}(S, a_k)$ for $\sigma_j$ if, for all $i = 1, ..., k$ we have

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We will denote:

$$\partial_{\sigma_j}(S, a_k) = \{ \text{Extremal elements of } \text{Ap}(S, a_k) \text{ for } \sigma_j \}.$$

$\partial_{\sigma_j}(S, a_k)$ depends on the $\sigma_j$ order chosen.
Computation of the type set.

Lemma.

$T(S) \subset \partial \sigma_j(S, a_k)$ for any Apéry ordering $\sigma_j$.

Proof:

RA: Assume it is not so, for some $\exists N \in T(S)$ with $\exp(N_k(x_N)) = (0, \gamma_1, ..., \gamma_{k-1}, 0)$, $\Rightarrow \exists i \in \{1, ..., k-1\}$ such that $(0, \gamma_1, ..., \gamma_i+1, ..., \gamma_{k-1}, 0) \in E(IS) \cap \{x = y_{k-1} = 0\}$ $\Rightarrow N + a_i \in \text{Ap}(S, a_k)$.
Lemma.

\( T(S) \subset \partial_{\sigma_j}(S, a_k) \) for any Apéry ordering \( \sigma_j \).

Proof:

Assume it is not so, for some

\[ \exists \ N \in T(S) \text{ with } \exp(N_k(x_N)) = (0, \gamma_1, \ldots, \gamma_{k-1}, 0), \]

\[ \Rightarrow \exists i \in \{1, \ldots, k-1\} \text{ such that } (0, \gamma_1, \ldots, \gamma_i+1, \ldots, \gamma_{k-1}, 0) \in E(I_S) \cap \{x = y\}, \]

\[ \Rightarrow N + a_i \in Ap(S, a_k). \]
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\[ T(S) \subset \partial_{\sigma_j}(S, a_k) \] for any Apéry ordering \( \sigma_j \).

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RA: Assume it is not so, for some $\exists N \in T(S)$ with $\exp(N_k(x^N)) = (0, \gamma_1, ..., \gamma_{k-1}, 0)$,
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\( T(S) \subset \partial_{\sigma_j}(S, a_k) \) for any Apéry ordering \( \sigma_j \).

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Lemma.

\[ T(S) \subset \partial_{\sigma_j}(S, a_k) \] for any Apéry ordering \( \sigma_j \).

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\( \implies N + a_i \in Ap(S, a_k) \).
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Theorem.

If $O$ is the set of Apéry orderings with respect to $a_k$, then

\[
\bigcap_{\sigma_k \in O} \partial \sigma_k(S, a_k) = T(S).
\]

Proof:

By the previous lemma, we only need to prove the following:

If $N/ \in T(S)$, then there exists an Apéry ordering $\sigma_k \in O$ such that $N/ \in \partial \sigma_k(S, a_k)$.

If $N/ \in T(S)$, there must exist a generator $a_i$ such that $N/ + a_i \in Ap(S, a_k)$.

Let us take $\sigma_k \in O$ with the reverse lexicographic ordering in $\{y_1, y_2, \ldots, y_i - 1, y_i + 1, \ldots, y_k - 1, y_i\}$.

If $\exp(N \sigma_k(x_N)) = (0, \beta_1, \ldots, \beta_i, \ldots, \beta_{k-1}, 0)$.

Then it must hold that,

$\exp(N \sigma_k(x_N + a_i)) = (0, \beta_1, \ldots, \beta_i + 1, \ldots, \beta_{k-1}, 0)$,
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If $\mathcal{O}$ is the set of Apéry orderings with respect to $a_k$, then

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Theorem.
If $\mathcal{O}$ is the set of Apéry orderings with respect to $a_k$, then

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Proof: By the previous lemma, we only need to proof the following: $N \notin T(S)$, then there exists an Apéry ordering $\sigma_k \in \mathcal{O}$ such that $N \notin \partial_{\sigma_k}(S, a_k)$. 

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Theorem.

If $\mathcal{O}$ is the set of Apéry orderings with respect to $a_k$, then

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- If $N \notin T(S)$, then there exists an Apéry ordering $\sigma_k \in \mathcal{O}$ such that $N \notin \partial_{\sigma_k}(S, a_k)$.
- If $N \notin T(S)$ there must exist a generator $a_i$ such that
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**Theorem.**
If $\mathcal{O}$ is the set of Apéry orderings with respect to $a_k$, then
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If $N \notin T(S)$, then there exists an Apéry ordering $\sigma_k \in \mathcal{O}$ such that $N \notin \partial_{\sigma_k}(S, a_k)$.
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Theorem.  
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If $N \notin T(S)$ there must exist a generator $a_i$ such that $N + a_i \in Ap(S, a_k)$.  
Let us take $\sigma_k \in \mathcal{O}$ with the reverse lexicographic ordering in  
$\{y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_{k-1}, y_i\}$. 

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Computation of the type set.

**Theorem.**

If $\mathcal{O}$ is the set of Apéry orderings with respect to $a_k$, then

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If $N \notin T(S)$, then there exists an Apéry ordering $\sigma_k \in \mathcal{O}$ such that $N \notin \partial_{\sigma_k}(S, a_k)$.

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If $\exp\left(N_{\sigma_k}\left(x^N\right)\right) = (0, \beta_1, \ldots, \beta_i, \ldots, \beta_{k-1}, 0)$. 
Computation of the type set.

**Theorem.**

If \( \mathcal{O} \) is the set of Apéry orderings with respect to \( a_k \), then

\[
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\]

**Proof:** By the previous lemma, we only need to prove the following:

If \( N \notin T(S) \), then there exists an Apéry ordering \( \sigma_k \in \mathcal{O} \) such that \( N \notin \partial_{\sigma_k}(S, a_k) \).

If \( N \notin T(S) \) there must exist a generator \( a_i \) such that \( N + a_i \in Ap(S, a_k) \).

Let us take \( \sigma_k \in \mathcal{O} \) with the reverse lexicographic ordering in \( \{y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_{k-1}, y_i\} \).

If \( \exp\left(N_{\sigma_k}(x^N)\right) = (0, \beta_1, \ldots, \beta_i, \ldots, \beta_{k-1}, 0) \).

Then it must hold that,
Computation of the type set.

**Theorem.**
If $\mathcal{O}$ is the set of Apéry orderings with respect to $a_k$, then

$$\bigcap_{\sigma_k \in \mathcal{O}} \partial_{\sigma_k}(S, a_k) = T(S).$$

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- If $N \notin T(S)$, then there exists an Apéry ordering $\sigma_k \in \mathcal{O}$ such that $N \notin \partial_{\sigma_k}(S, a_k)$.
- If $N \notin T(S)$ there must exist a generator $a_i$ such that $N + a_i \in Ap(S, a_k)$.

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If $\exp\left( N_{\sigma_k} \left( x^N \right) \right) = (0, \beta_1, \ldots, \beta_i, \ldots, \beta_{k-1}, 0)$. 

Then it must hold that,

$$\exp\left( N_{\sigma_k} \left( x^{N+a_i} \right) \right) = (0, \beta_1, \ldots, \beta_i + 1, \ldots, \beta_{k-1}, 0),$$

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This comes straightforwardly, if we had otherwise

$$\exp \left(N_{\sigma_k} \left(x^{N+a_i}\right)\right) = (0, \alpha_1, ..., \alpha_i, ..., \alpha_{k-1}, 0)$$

Then we have

$$(0, \alpha_1, ..., \alpha_i, ..., \alpha_{k-1}, 0) <_{\sigma_k} (0, \beta_1, ..., \beta_i + 1, ..., \beta_{k-1}, 0)$$
Computation of the type set.

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Then we have

\[ (0, \alpha_1, \ldots, \alpha_i, \ldots, \alpha_{k-1}, 0) <_{\sigma_k} (0, \beta_1, \ldots, \beta_i + 1, \ldots, \beta_{k-1}, 0) \]

But this is not possible by the definition of Apéry order,
This comes straightforwardly, if we had otherwise

\[ \exp \left( N_{\sigma_k} \left( x^{N+a_i} \right) \right) = (0, \alpha_1, \ldots, \alpha_i, \ldots, \alpha_{k-1}, 0) \]

Then we have

\[ (0, \alpha_1, \ldots, \alpha_i, \ldots, \alpha_{k-1}, 0) \prec_{\sigma_k} (0, \beta_1, \ldots, \beta_i + 1, \ldots, \beta_{k-1}, 0) \]

But this is not possible by the definition of Apéry order, therefore

\[ N \not\in \partial_{\sigma_k}(S, a_k). \]
Computation of the type set. Example.

Let $S = \langle 7, 8, 9, 13 \rangle$ the previous example.

$A_1(S, 13) = \{0, 7, 8, 9, 14, 15, 16, 17, 18, 23, 24, 25, 32\}$.

$T(S) = \{32\}$, $S$ is symmetrical.
Let $S = \langle 7, 8, 9, 13 \rangle$ the previous example.
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Let $S = \langle 7, 8, 9, 13 \rangle$ the previous example.

\[ Ap(S, 13) = \{0, 7, 8, 9, 14, 15, 16, 17, 18, 23, 24, 25, 32\}. \]

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Algorithm. Example.

Set $\Delta_{\sigma_1}(S, 13)$:
Set $\Delta_{\sigma_1}(S, 13)$:

| x | y4 | y3 | y1 |
|---|----|----|----|
| 0 | 0  | 0  | 0  |
| 0 | 1  | 2  | 3  | 4  |
| 0 | 7  | 14 | 21 | 28 |
| 1 | 15 | 22 | 29 | 36 |
| 2 | 23 | 30 | 37 | 44 |
| 3 | 24 | 31 | 38 | 45 | 52 |
| 4 | 32 | 39 | 46 | 53 | 60 |
| 5 | 40 | 47 | 54 | 61 | 68 |
Algorithm. Example.

Set $\Delta_{\sigma_1}(S, 13)$:
Algorithm. Example.

Set $\Delta_{\sigma_1}(S, 13)$:

| $x$ | $y_4$ | $y_3$ | $y_1$ |
|-----|-------|-------|-------|
| 0   | 0     | 0     | 1     |
| 0   | 7     | 14    | 21    |
| 1   | 8     | 15    | 22    |
| 2   | 16    | 23    | 30    |
| 3   | 24    | 31    | 38    |
| 4   | 32    | 39    | 46    |
| 5   | 40    | 47    | 54    |

| $x$ | $y_4$ | $y_3$ | $y_1$ |
|-----|-------|-------|-------|
| 0   | 0     | 1     | 2     |
| 1   | 9     | 16    | 23    |
| 1   | 17    | 24    | 31    |
| 2   | 25    | 32    | 39    |
| 3   | 33    | 40    | 47    |
| 4   | 41    | 48    | 55    |
| 5   | 49    | 56    | 63    |

| $y_3$ | $y_1$ |
|-------|-------|
| 2     | 0     |
| 1     | 1     |

| $y_3$ | $y_1$ |
|-------|-------|
| 0     | 1     |
| 1     | 2     |

| $y_3$ | $y_1$ |
|-------|-------|
| 2     | 0     |
| 1     | 1     |

| $y_3$ | $y_1$ |
|-------|-------|
| 0     | 1     |
| 1     | 2     |

| $y_3$ | $y_1$ |
|-------|-------|
| 2     | 0     |
| 1     | 1     |

| $y_2$ | $y_1$ |
|-------|-------|
| 0     | 18    |
| 1     | 26    |
| 2     | 34    |
| 3     | 42    |
Algorithm. Example.

Set $\Delta_{\sigma_1}(S, 13)$:

\[
\partial_{\sigma_1}(S, 13) = \{ 14, 18, 23, 25, 32 \}.
\]
Algorithm. Example.

Set $\Delta_{\sigma_3}(S, 13)$:

| $x$ | $y_4$ | $y_3$ | $y_1$ |
|-----|-------|-------|-------|
| 0   | 0     | 0     | 0     |
| 0   | 7     | 14    | 21    |
| 1   | 15    | 22    | 29    |
| 2   | 23    | 30    | 37    |
| 3   | 31    | 38    | 45    |
| 4   | 39    | 46    | 53    |

$\Delta_{\sigma_1}(S, 13) = \{24, 32\}$ implies $\Delta_{\sigma_1}(S, 13) \cap \Delta_{\sigma_3}(S, 13) = \{32\}$.
Algorithm. Example.

Set $\Delta_{\sigma_3}(S, 13)$:

\[
\partial_{\sigma_1}(S, 13) = \{ 24, 32 \}
\]
Algorithm. Example.

Set $\Delta_{\sigma_3}(S, 13)$:

| $x$ | $y_4$ | $y_3$ | $y_1$ |
|-----|-------|-------|-------|
| 0   | 0     | 0     | 0     |
|     | 0     | 1     | 2     |
| y2->| 0     | 7     | 14    |
|     | 1     | 15    | 22    |
|     | 2     | 23    | 30    |
|     | 3     | 31    | 38    |
|     | 4     | 39    | 46    |
|     |       |       | 53    |
|     |       |       | 60    |

| $x$ | $y_4$ | $y_3$ | $y_1$ |
|-----|-------|-------|-------|
| 0   | 1     | 0     | 1     |
|     | 1     | 9     | 16    |
| y2->| 0     | 23    | 30    |
|     | 1     | 24    | 31    |
|     | 2     | 32    | 39    |
|     | 3     | 40    | 47    |
|     | 4     | 48    | 55    |
|     |       |       | 62    |
|     |       |       | 69    |

| $y_3$ | $y_1$ |
|-------|-------|
| 2     | 0     |
| y2->  | 1     |
| 0     | 18    |
| 1     | 26    |
| 2     | 34    |
| 3     | 42    |

$\partial_{\sigma_1}(S, 13) = \{24, 32\}$
Algorithm. Example.

Set $\Delta_{\sigma_3}(S, 13)$:

\[
\begin{array}{cccccccc}
\hline
x & y_4 & y_3 & y_1 \\
\hline
0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 \\
\hline
y_2 \rightarrow & 0 & 0 & 7 & 14 & 21 & 28 \\
& 1 & 8 & 15 & 22 & 29 & 36 \\
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\end{array}
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\end{array}
\]

\[
\begin{array}{cccccccc}
\hline
y_3 & y_1 \\
\hline
2 & 0 & 1 & 2 & 3 \\
\hline
y_2 \rightarrow & 0 & 18 & 25 & 32 & 39 \\
& 1 & 26 & 33 & 40 & 47 \\
& 2 & 34 & 41 & 48 & 55 \\
& 3 & 42 & 49 & 56 & 63 \\
\hline
\end{array}
\]

$\partial_{\sigma_1}(S, 13) = \{24, 32\} \implies \partial_{\sigma_1}(S, 13) \cap \partial_{\sigma_3}(S, 13) = \{32\}.$
Thank you very much!
Grazie tante!