COMPUTING EQUATIONS FOR RESIDUALLY FREE GROUPS

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ABSTRACT. We show that there is no algorithm deciding whether the maximal residually free quotient of a given finitely presented group is finitely presentable or not.

Given a finitely generated subgroup \( G \) of a finite product of limit groups, we discuss the possibility of finding an explicit set of defining equations (i.e., of expressing \( G \) as the maximal residually free quotient of an explicit finitely presented group).

1. Introduction

Any countable group \( G \) has a largest residually free quotient \( \text{RF}(G) \), equal to \( G/\bigcap_{f \in \mathcal{H}} \ker f \) where \( \mathcal{H} \) is the set of all homomorphisms from \( G \) to a non-Abelian free group \( \mathbb{F} \). Since any two countably generated non-Abelian free groups can be embedded in each other, this notion does not depend on the rank of the free group \( \mathbb{F} \) considered.

In the language of \([BMR99]\), if \( R \) is a finite set of group equations on a finite set of variables \( S \), then \( G = \text{RF}(\langle S \mid R \rangle) \) is the coordinate group of the variety defined by the system of equations \( R \). We say that \( R \) is a set of defining equations of \( G \) over \( S \). Equational noetherianness of free groups implies that any finitely generated residually free group \( G \) has a (finite) set of defining equations \([BMR99]\).

On the other hand, any finitely generated residually free group embeds into a finite product of limit groups (also known as finitely generated fully residually free groups), which correspond to the irreducible components of the variety defined by \( R \) \([BMR99, KM98, Sel01]\). Conversely, any subgroup of a finite product of limit groups is residually free.

This gives three possibilities to define a finitely generated residually free group \( G \) in an explicit way:

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(1) give a finite presentation of $G$ (if $G$ is finitely presented);
(2) give a set of defining equations of $G$: write $G = \text{RF}((S \mid R))$, with $S$ and $R$ finite;
(3) write $G$ as the subgroup of $L_1 \times \cdots \times L_n$ generated by a finite subset $S$, where $L_1, \ldots, L_n$ are limit groups given by some finite presentations.

We investigate the algorithmic possibility to go back and forth between these ways of defining $G$.

One can go from 2 to 3: given a set of defining equations of $G$, one can find an explicit embedding into some product of limit groups [KM98, KM05, BHMS09, GW09].

Conversely, if $G$ is given as a subgroup of a product of limit groups, and if one knows that $G$ is finitely presented, one can compute a presentation of $G$ [BHMS09]. Obviously, a finite presentation is a set of defining equations.

Since residually free groups are not always finitely presented, we investigate the following question:

**Question.** Let $L = L_1 \times \cdots \times L_n$ be a product of limit groups. Let $G$ be the subgroup generated by a finite subset $S \subset L$. Can one algorithmically find a finite set of defining equations for $G$, that is, find a finite presentation $\langle S \mid R \rangle$ such that $G = \text{RF}((S \mid R))$?

We will prove that this question has a negative answer. On the other hand, we introduce a closely related notion which has better algorithmic properties.

Let $\text{RF}_{na}(G)$ be the quotient $G/\bigcap_{f \in \mathcal{H}_{na}} \ker f$ where $\mathcal{H}_{na}$ is the set of all homomorphisms from $G$ to $\mathbb{F}$ with non-Abelian image. Of course, $\text{RF}_{na}(G)$ is a quotient of $\text{RF}(G)$, which forgets the information about morphisms to $\mathbb{Z}$. In fact (Lemma 2.2), it is the quotient of $\text{RF}(G)$ by its center.

We say that $G$ is a residually non-Abelian free group if $G = \text{RF}_{na}(G)$, i.e., if every non-trivial element of $G$ survives in a non-Abelian free quotient of $G$; equivalently, $G$ is residually non-Abelian free if and only if $G$ is residually free and has trivial center. Given a residually non-Abelian free group $G$, we say that $R$ is a set of na-equations of $G$ over $S$ if $G = \text{RF}_{na}((S \mid R))$.

We write $Z(G)$ for the center of $G$, and $b_1(G)$ for the torsion-free rank of $H_1(G, \mathbb{Z})$.

**Theorem 1.**

- There is an algorithm which takes as input presentations of limit groups $L_1, \ldots, L_n$, and a finite subset $S \subset L_1 \times \cdots \times L_n$, and which computes a finite set of na-equations for $G/Z(G) = \text{RF}_{na}(G)$, where $G = \langle S \rangle$.
- One can compute a finite set of defining equations for $G = \langle S \rangle$ if and only if one can compute $b_1(G)$.

Since there is no algorithm computing $b_1(\langle S \rangle)$ from $S \subset \mathbb{F}_2 \times \mathbb{F}_2$ [BM09], we deduce the following corollary.
Corollary 1. There is no algorithm which takes as an input a finite subset $S \subseteq \mathbb{F}_2 \times \mathbb{F}_2$ and computes a finite set of equations for $\langle S \rangle$.

We also investigate the possibility to decide whether a residually free quotient is finitely presented. Using Theorem 1 and [Gru78], we prove the following theorem.

Theorem 2. There is no algorithm which takes as an input a finite group presentation $\langle S | R \rangle$, and which decides whether $\text{RF}(\langle S | R \rangle)$ is finitely presented.

2. The residually non-Abelian free quotient $\text{RF}_{na}$

We always denote by $G$ a finitely generated group, and by $F$ a non-Abelian free group.

Definition 2.1. $\text{RF}(G)$ is the quotient of $G$ by the intersection of the kernels of all morphisms $G \to F$.

$\text{RF}_{na}(G)$ is the quotient of $G$ by the intersection of the kernels of all morphisms $G \to F$ with non-Abelian image.

One may view $\text{RF}(G)$ as the image of $G$ in $\mathbb{F}^\mathcal{H}$, where $\mathcal{H}$ is the set of all morphisms $G \to F$, and $\text{RF}_{na}(G)$ as the image in $\mathbb{F}^{\mathcal{H}_{na}}$, where $\mathcal{H}_{na}$ is the set of all morphisms with non-Abelian image.

Every homomorphism $G \to F$ factors through $\text{RF}(G)$ (through $\text{RF}_{na}(G)$ if its image is not Abelian). By definition, $G$ is residually free if and only if $G = \text{RF}(G)$, residually non-Abelian free if and only if $G = \text{RF}_{na}(G)$.

Lemma 2.2. There is an exact sequence

$$1 \to Z(\text{RF}(G)) \to \text{RF}(G) \to \text{RF}_{na}(G) \to 1.$$  

In particular, $G$ is residually non-Abelian free if and only if $G$ is residually free and $Z(G) = 1$. If $G$ is a non-Abelian limit group, it has trivial center and $\text{RF}_{na}(G) = \text{RF}(G) = G$.

Proof of Lemma 2.2. Recall that $F$ is commutative transitive, that is, that centralizers of nontrivial elements are Abelian (i.e., cyclic) [LS01]. Let $H = \text{RF}(G)$. Consider $a \in Z(H)$ and $f : H \to F$ with $f(a) \neq 1$. The image of $f$ centralizes $f(a)$, so is Abelian by commutative transitivity of $F$. Thus, $a$ has trivial image in $\text{RF}_{na}(H) = \text{RF}_{na}(G)$.

Conversely, consider $a \in H \setminus Z(H)$, and $b \in H$ with $[a, b] \neq 1$. There exists $f : H \to F$ such that $f([a, b]) \neq 1$. Then $f(H)$ is non-Abelian, and $f(a) \neq 1$. This means that the image of $a$ in $\text{RF}_{na}(G)$ is nontrivial.

Any epimorphism $f : G \to H$ induces epimorphisms $f_{RF} : \text{RF}(G) \to \text{RF}(H)$ and $f_{na} : \text{RF}_{na}(G) \to \text{RF}_{na}(H)$.
Lemma 2.3. Let $f : G \to H$ be an epimorphism. Then $f_{RF} : RF(G) \to RF(H)$ is an isomorphism if and only if $f_{na} : RF_{na}(G) \to RF_{na}(H)$ is an isomorphism and $b_1(G) = b_1(H)$.

Proof. Note that $f_{RF}$ (resp., $f_{na}$) is an isomorphism if and only if any morphism $G \to \mathbb{F}$ (resp., any such morphism with non-Abelian image) factors through $f$. The lemma then follows from the fact that the embedding $\text{Hom}(H, \mathbb{Z}) \to \text{Hom}(G, \mathbb{Z})$ induced by $f$ is onto if and only if $b_1(G) = b_1(H)$. □

Given a product $L_1 \times \cdots \times L_n$, we denote by $p_i$ the projection onto $L_i$.

Lemma 2.4. Let $G \subset L = L_1 \times \cdots \times L_n$ with $L_i$ a limit group. Let $I \subset \{1, \ldots, n\}$ be the set of indices such that $p_i(G)$ is Abelian. Then $RF_{na}(G)$ is the image of $G$ in $L' = \prod_{i \notin I} L_i$ (viewed as a quotient of $L_1 \times \cdots \times L_n$).

Proof. Note that $G = RF(G)$. An element $(x_1, \ldots, x_n) \in G$ is in $Z(G)$ if and only if $x_i$ is central in $p_i(G)$ for every $i$. Since $p_i(G)$ is Abelian or has trivial center, $Z(G)$ is the kernel of the natural projection $L \to L'$. The result follows from Lemma 2.2. □

Lemma 2.5. $RF(G)$ is finitely presented if and only if $RF_{na}(G)$ is.

Proof. If $H$ is any residually free group, the abelianization map $H \to H_{ab}$ is injective on $Z(H)$ since any element of $Z(H)$ survives in some free quotient of $H$, which has to be cyclic (see [BHMS09, Lemma 6.2]). In particular, $Z(H)$ is finitely generated if $H$ is. Applying this to $H = R(G)$, the exact sequence of Lemma 2.2 gives the required result. □

3. Proof of the theorems

Let $S$ be a finite set of elements in a group. We define $S_0 = S \cup \{1\}$. If $R, R'$ are sets of words on $S \cup S^{-1}$, then $R^{S_0}$ is the set of all words obtained by conjugating elements of $R$ by elements of $S_0$, and $[R^{S_0}, R']$ is the set of all words obtained as commutators of words in $R^{S_0}$ and words in $R'$.

Proposition 3.1. Let $A_1, \ldots, A_n$ be arbitrary groups, with $n \geq 2$. Let $G \subset A_1 \times \cdots \times A_n$ be generated by $S = \{s_1, \ldots, s_k\}$. Let $p_i : G \to A_i$ be the projection. Assume that $p_i(G) = RF_{na}(\langle S \mid R_i \rangle)$ for some finite set of relators $R_i$.

Then the set

$$
\tilde{R} = [R_{n}^{S_0}, [R_{n-1}^{S_0}, \ldots [R_{3}^{S_0}, [R_{2}^{S_0}, R_1]] \ldots]
$$

is a finite set of $na$-equations of $RF_{na}(G)$ over $S$, i.e., $RF_{na}(G) = RF_{na}(\langle S \mid \tilde{R} \rangle)$. 

An equality such as $p_i(G) = RF_{na}(\langle S \mid R_i \rangle)$ means that there is an isomorphism commuting with the natural projections $F(S) \to p_i(G)$ and $F(S) \to RF_{na}(\langle S \mid R_i \rangle)$, where $F(S)$ denotes the free group on $S$.

**Proof of Proposition 3.1.** Recall that a free group $\mathbb{F}$ is CSA: commutation is transitive on $\mathbb{F} \setminus \{1\}$, and maximal Abelian subgroups are malnormal [MR96]. In particular, if two nontrivial subgroups commute, then both are Abelian. If $A, B$ are nontrivial subgroups of $\mathbb{F}$, and if $A$ commutes with $B, B^{x_1}, \ldots, B^{x_p}$ for elements $x_1, \ldots, x_p \in \mathbb{F}$, then $\langle A, B, x_1, \ldots, x_p \rangle$ is Abelian.

We write $\tilde{G} = \langle S \mid \tilde{R} \rangle = \langle S \mid [R_{n_0}^0, [R_{n-1}^0, \ldots, [R_{2}^0, R_1] \ldots] \rangle$. We always denote by $\varphi : F(S) \to \mathbb{F}$ a morphism with non-Abelian image. We shall show that such a $\varphi$ factors through $G$ if and only if it factors through $\tilde{G}$. This implies the desired result $RF_{na}(G) = RF_{na}(\tilde{G})$: both groups are equal to the image of $F(S)$ in $\mathbb{F}^{\mathcal{H}_{na}}$, where $\mathcal{H}_{na}$ is the set of all $\varphi$’s which factor through $G$ and $\tilde{G}$.

We proceed by induction on $n$. We first claim that $\varphi$ is trivial on $\tilde{R}$ if and only if it is trivial on some $R_i$. The if direction is clear. For the only if direction, observe that the image of $[R_{n_0}^0, [R_{n-1}^0, \ldots, [R_{2}^0, R_1] \ldots]$ commutes with all conjugates of $\varphi(R_n)$ by elements of $\varphi(F(S))$, so $R_n$ or $[R_{n-1}^0, \ldots, [R_{2}^0, R_1] \ldots]$ has trivial image. The claim follows by induction.

Now suppose that $\varphi$ factors through $\tilde{G}$. Then $\varphi$ kills $\tilde{R}$, hence some $R_i$. It follows that $\varphi$ factors through $p_i(G)$, hence through $G$.

Conversely, suppose that $\varphi$ factors through $f : G \to \mathbb{F}$. Consider the intersection of $G$ with the kernel of $p_n : G \to A_n$ and the kernel of $p_{1, \ldots, n-1} : G \to A_1 \times \cdots \times A_{n-1}$. These are commuting normal subgroups of $G$. If both have nontrivial image in $\mathbb{F}$, the CSA property implies that the image of $f$ is Abelian, a contradiction. We deduce that $f$ factors through $p_n$ or through $p_{1, \ldots, n-1}$, and by induction that it factors through some $p_i$. Thus, $\varphi$ kills $R_i$, hence $\tilde{R}$ as required. \hfill \Box

**Proof of Theorem 1.** Given a finite subset $S \subset L_1 \times \cdots \times L_n$, where each $L_i$ is a limit group, we want to find a finite set of na-equations for $G/Z(G) = RF_{na}(G)$, where $G = \langle S \rangle$.

Using a solution of the word problem in a limit group, one can find the indices $i$ for which $p_i(G) \subset L_i$ is Abelian (this amounts to checking whether the elements of $p_i(S)$ commute).

First, assume that no $p_i(G)$ is Abelian. As pointed out in [GW09] or [BHMS09, Lemma 7.5], one deduces from [Wil08] an algorithm yielding a finite presentation $\langle S \mid R_i \rangle$ of $p_i(G)$. Since $p_i(G)$ is not Abelian, one has $p_i(G) = RF_{na}(\langle S \mid R_i \rangle)$, and Proposition 3.1 yields a finite set of na-equations for $RF_{na}(G)$ over $S$ (if $n = 1$, then $RF_{na}(G) = p_1(G)$). If some $p_i(G)$’s are
Abelian, we simply replace $G$ by its image in $L’$ as in Lemma 2.4. This proves
the first assertion of the theorem.

We now prove that one can find a finite set of defining equations if and
only if one can compute $b_1(G)$. Suppose that $b_1(G)$ is known. We want a
finite set $R$ such that $RF(G) = RF(\langle S \mid R \rangle)$. If $n = 1$, then $G$ is a subgroup of
the limit group $L_1$, and one can find a finite presentation of $G$ as explained
above. So assume $n \geq 2$. Consider the finite presentation $\tilde{G} = \langle S \mid \tilde{R} \rangle$ given
by Proposition 3.1, so that $RF_{na}(\tilde{G}) = RF_{na}(G)$.

We claim that $G$ is a quotient of $\tilde{G}$. To see this, we consider an
$x \in F(S)$ which is trivial in $\tilde{G}$ and we prove that it is trivial in $G$. If not, residual
freeness of $G$ implies that $x$ survives under a morphism $\varphi : F(S) \to F$ which
factors through $G$. If $\varphi$ has non-Abelian image, it factors through $RF_{na}(G) =
RF_{na}(\tilde{G})$, hence through $\tilde{G}$, contradicting the triviality of $x$ in $\tilde{G}$. If the image
is Abelian, $\varphi$ also factors through $\tilde{G}$ because all relators in $\tilde{R}$ are commutators.

Since $\tilde{R}$ is finite, we can compute $b_1(\tilde{G})$. If $b_1(\tilde{G}) = b_1(G)$, we are done by
Lemma 2.3 since $G$ is a quotient of $\tilde{G}$. If $b_1(\tilde{G}) > b_1(G)$, we enumerate all
trivial words of $G$ (using an enumeration of trivial words in each $p_i(G)$), and
we add them to the presentation of $\tilde{G}$ one by one. We compute $b_1$ after each
addition, and we stop when we reach the known value $b_1(G)$.

Conversely, if we have a finite set of defining equations for $G$, so that $G =
RF(\langle S \mid \tilde{R} \rangle)$, we can compute $b_1(\langle S \mid \tilde{R} \rangle)$, which equals $b_1(G)$ by Lemma 2.3.

\textbf{Theorem 3.} There is no algorithm which takes as input a finite group pre-
sentation $\langle S \mid \tilde{R} \rangle$, and which decides whether $RF(\langle S \mid \tilde{R} \rangle)$ is finitely presented.

\textbf{Proof.} Given a finite set $S \subset F_2 \times F_2$, Theorem 1 provides a finite set $\tilde{R}$
such that $RF_{na}(\langle S \rangle) = RF_{na}(\langle S \mid \tilde{R} \rangle)$. Using Lemma 2.5, we see that finite
presentability of $RF(\langle S \mid \tilde{R} \rangle)$ is equivalent to that of $RF_{na}(\langle S \mid \tilde{R} \rangle)$, hence to
that of $RF(\langle S \rangle) = \langle S \rangle$. But it follows from [Gru78] that there is no algorithm
which decides, given a finite set $S \subset F_2 \times F_2$, whether $\langle S \rangle$ is finitely presented.

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