Pfaffian-type Sugawara operators

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Abstract

We show that the Pfaffian of a generator matrix for the affine Kac–Moody algebra \( \hat{\mathfrak{o}}_{2n} \) is a Segal–Sugawara vector. Together with our earlier construction involving the symmetrizer in the Brauer algebra, this gives a complete set of Segal–Sugawara vectors in type \( D \).

1 Introduction

For each affine Kac–Moody algebra \( \hat{\mathfrak{g}} \) associated with a simple Lie algebra \( \mathfrak{g} \), the corresponding vacuum module \( V(\mathfrak{g}) \) at the critical level is a vertex algebra. The structure of the center \( z(\hat{\mathfrak{g}}) \) of this vertex algebra was described by a remarkable theorem of Feigin and Frenkel in [3], which states that \( z(\hat{\mathfrak{g}}) \) is the algebra of polynomials in infinitely many variables which are associated with generators of the algebra of \( \mathfrak{g} \)-invariants in the symmetric algebra \( S(\mathfrak{g}) \). For a detailed proof of the theorem, its extensions and significance for the representation theory of the affine Kac–Moody algebras see [4].

In a recent paper [5] we used the symmetrizer in the Brauer algebra to construct families of elements of \( z(\hat{\mathfrak{g}}) \) (Segal–Sugawara vectors) for the Lie algebras \( \mathfrak{g} \) of types \( B \), \( C \) and \( D \) in an explicit form. In types \( B \) and \( C \) they include complete sets of Segal–Sugawara vectors generating the center \( z(\hat{\mathfrak{g}}) \), while in type \( D \) one vector in the complete set of [5] was not given explicitly. The aim of this note is to produce this Segal–Sugawara vector in \( z(\hat{\mathfrak{o}}_{2n}) \) which is associated with the Pfaffian invariant in \( S(\mathfrak{o}_{2n}) \).

Simple explicit formulas for generators of the Feigin–Frenkel center \( z(\hat{\mathfrak{g}}(n)) \) were given recently in [1], [2], following Talalaev’s construction of higher Gaudin Hamiltonians [6]. So, together with the results of [5] we get a construction of generators of the Feigin–Frenkel centers for all classical types.
2 Pfaffian-type generators

Denote by \( E_{ij}, 1 \leq i, j \leq 2n \), the standard basis vectors of the Lie algebra \( \mathfrak{gl}_{2n} \). Introduce the elements \( F_{ij} \) of \( \mathfrak{gl}_{2n} \) by the formulas

\[
F_{ij} = E_{ij} - E_{ji}. \tag{2.1}
\]

The Lie subalgebra of \( \mathfrak{gl}_{2n} \) spanned by the elements \( F_{ij} \) is isomorphic to the even orthogonal Lie algebra \( \mathfrak{o}_{2n} \). The elements of \( \mathfrak{o}_{2n} \) are skew-symmetric matrices. Introduce the standard normalized invariant bilinear form on \( \mathfrak{o}_{2n} \) by

\[
\langle X, Y \rangle = \frac{1}{2} \text{tr} XY, \quad X, Y \in \mathfrak{o}_{2n}.
\]

Now consider the affine Kac–Moody algebra \( \widehat{\mathfrak{o}}_{2n} = \mathfrak{o}_{2n}[t, t^{-1}] \oplus \mathbb{C} K \) and set \( X[r] = X t^r \) for any \( r \in \mathbb{Z} \) and \( X \in \mathfrak{o}_{2n} \). The element \( K \) is central in \( \widehat{\mathfrak{o}}_{2n} \) and

\[
[X[r], Y[s]] = [X, Y][r + s] + r \delta_{r, -s} \langle X, Y \rangle K.
\]

Therefore, for the generators we have

\[
[F_{ij}[r], F_{kl}[s]] = \delta_{kj} F_{il}[r + s] - \delta_{il} F_{kj}[r + s] - \delta_{ki} F_{jl}[r + s] + \delta_{jl} F_{ki}[r + s] + r \delta_{r, -s} \langle \delta_{kj} \delta_{il} - \delta_{ki} \delta_{jl} \rangle K.
\]

The **vacuum module at the critical level** \( V(\widehat{\mathfrak{o}}_{2n}) \) can be defined as the quotient of the universal enveloping algebra \( U(\widehat{\mathfrak{o}}_{2n}) \) by the left ideal generated by \( \mathfrak{o}_{2n}[t] \) and \( K + 2n - 2 \) (note that the dual Coxeter number in type \( D_n \) is \( h' = 2n - 2 \)). The Feigin–Frenkel center \( \mathfrak{z}(\widehat{\mathfrak{o}}_{2n}) \) is defined by

\[
\mathfrak{z}(\widehat{\mathfrak{o}}_{2n}) = \{ v \in V(\widehat{\mathfrak{o}}_{2n}) \mid \mathfrak{o}_{2n}[t] v = 0 \}.
\]

Any element of \( \mathfrak{z}(\widehat{\mathfrak{o}}_{2n}) \) is called a **Segal–Sugawara vector**. A complete set of Segal–Sugawara vectors \( \phi_{22}, \phi_{44}, \ldots, \phi_{2n-22n-2}, \phi'_{n} \) was produced in [3], where all of them, except for \( \phi'_{n} \), were given explicitly. We will produce \( \phi'_{n} \) in Theorem \( [2.1] \) below.

Combine the generators \( F_{ij}[-1] \) into the skew-symmetric matrix \( F[-1] = [F_{ij}[-1]] \) and define its Pfaffian by

\[
Pf F[-1] = \frac{1}{2^n n!} \sum_{\sigma \in \mathfrak{S}_{2n}} \text{sgn} \sigma \cdot F_{\sigma(1) \sigma(2)}[-1] \cdots F_{\sigma(2n-1) \sigma(2n)}[-1].
\]

Note that the elements \( F_{ij}[-1] \) and \( F_{kl}[-1] \) of \( \widehat{\mathfrak{o}}_{2n} \) commute, if the indices \( i, j, k, l \) are distinct. Therefore, we can write the formula for the Pfaffian in the form

\[
Pf F[-1] = \sum_{\sigma} \text{sgn} \sigma \cdot F_{\sigma(1) \sigma(2)}[-1] \cdots F_{\sigma(2n-1) \sigma(2n)}[-1], \tag{2.2}
\]

summed over the elements \( \sigma \) of the subset \( \mathfrak{B}_{2n} \subset \mathfrak{S}_{2n} \) which consists of the permutations with the properties \( \sigma(2k-1) < \sigma(2k) \) for all \( k = 1, \ldots, n \) and \( \sigma(1) < \sigma(3) < \cdots < \sigma(2n-1) \).
**Theorem 2.1.** The element \( \phi'_n = \text{Pf} \ F[-1] \) is a Segal–Sugawara vector for \( \widehat{\mathfrak{o}}_{2n} \).

**Proof.** We need to show that \( \mathfrak{o}_{2n} \{ \} \phi'_n = 0 \) in the vacuum module \( V(\widehat{\mathfrak{o}}_{2n}) \). It suffices to verify that for all \( i, j \),

\[
F_{ij}[0] \text{Pf} \ F[-1] = F_{ij}[1] \text{Pf} \ F[-1] = 0. \tag{2.3}
\]

Note that for any permutation \( \pi \in \mathfrak{S}_{2n} \) the mapping

\[
F_{ij}[\sigma] \mapsto F_{\pi(i)\pi(j)}[\sigma], \quad K \mapsto \hat{K}
\]
defines an automorphism of the Lie algebra \( \widehat{\mathfrak{o}}_{2n} \). Moreover, the image of \( \text{Pf} \ F[-1] \) under its extension to \( U(\widehat{\mathfrak{o}}_{2n}) \) coincides with \( \text{sgn} \pi \cdot \text{Pf} \ F[-1] \). Hence, it is enough to verify (2.3) for \( i = 1 \) and \( j = 2 \).

Observe that \( F_{12}[0] \) commutes with all summands in (2.2) with \( \sigma(1) = 1 \) and \( \sigma(2) = 2 \). Suppose now that \( \sigma \in \mathcal{B}_{2n} \) is such that \( \sigma(2) > 2 \). Then \( \sigma(3) = 2 \) and \( \sigma(4) > 2 \). In \( V(\widehat{\mathfrak{o}}_{2n}) \) we have

\[
F_{12}[1] F_{1\sigma(2)}[1] F_{\sigma(4)}[1] \cdots F_{\sigma(2n-1)} \sigma(2n)[1] = -F_{2\sigma(2)}[1] F_{2\sigma(4)}[1] \cdots F_{\sigma(2n-1)} \sigma(2n)[1] + F_{1\sigma(2)}[1] F_{1\sigma(4)}[1] \cdots F_{\sigma(2n-1)} \sigma(2n)[1].
\]

Set \( i = \sigma(2) \) and \( j = \sigma(4) \). Note that the permutation \( \sigma' = \sigma(24) \) also belongs to the subset \( \mathcal{B}_{2n} \), and \( \text{sgn} \sigma' = -\text{sgn} \sigma \). We have

\[
- F_{2i}[1] F_{2j}[1] + F_{1i}[1] F_{1j}[1] + F_{2i}[1] F_{2j}[1] - F_{1i}[1] F_{1j}[1] = F_{ij}[2] - F_{ij}[2] = 0.
\]

This implies that the terms in the expansion of \( F_{12}[0] \text{Pf} \ F[-1] \) corresponding to pairs of the form \( (\sigma, \sigma') \) cancel pairwise. Thus, \( F_{12}[0] \text{Pf} \ F[-1] = 0 \).

Now we verify that

\[
F_{12}[1] \text{Pf} \ F[-1] = 0. \tag{2.4}
\]

Consider first the summands in (2.2) with \( \sigma(1) = 1 \) and \( \sigma(2) = 2 \). In \( V(\widehat{\mathfrak{o}}_{2n}) \) we have

\[
F_{12}[1] F_{1\sigma(3)} \sigma(4)[1] \cdots F_{\sigma(2n-1)} \sigma(2n)[1] = -K F_{\sigma(3)} \sigma(4)[1] \cdots F_{\sigma(2n-1)} \sigma(2n)[1].
\]

Furthermore, let \( \tau \in \mathcal{B}_{2n} \) with \( \tau(2) > 2 \). Then \( \tau(3) = 2 \) and \( \tau(4) > 2 \). We have

\[
F_{12}[1] F_{1\tau(2)}[1] F_{2\tau(4)}[1] \cdots F_{\tau(2n-1)} \tau(2n)[1] = -F_{2\tau(2)}[0] F_{2\tau(4)}[1] \cdots F_{\tau(2n-1)} \tau(2n)[1] + F_{\tau(2)} \tau(4)[1] \cdots F_{\tau(2n-1)} \tau(2n)[-1].
\]

Note that for any given \( \sigma \), the number of elements \( \tau \) such that the product

\[
F_{\tau(2)} \tau(4)[1] \cdots F_{\tau(2n-1)} \tau(2n)[-1]
\]
coincides, up to a sign, with the product

\[ F_{\sigma(3)} \sigma(4)[-1] \cdots F_{\sigma(2n-1)} \sigma(2n)[-1] \]
equals 2n - 2. Indeed, for each \( k = 2, \ldots, n \) the unordered pair \( \{\tau(2), \tau(4)\} \) can coincide with the pair \( \{\sigma(2k-1), \sigma(2k)\} \). Hence, taking the signs of permutations into account, we find that

\[ F_{12}[1] Pf F[-1] = (-K - 2n + 2) \sum_{\sigma} \text{sgn} \sigma \cdot F_{\sigma(3)} \sigma(4)[-1] \cdots F_{\sigma(2n-1)} \sigma(2n)[-1], \]

summed over \( \sigma \in \mathcal{B}_{2n} \) with \( \sigma(1) = 1 \) and \( \sigma(2) = 2 \). Since \(-K - 2n + 2 = 0\) at the critical level, we get (2.4).

Introduce formal Laurent series

\[ F_{ij}(z) = \sum_{r \in \mathbb{Z}} F_{ij}[r] z^{-r-1} \quad \text{and} \quad F_{ij}(z)_+ = \sum_{r < 0} F_{ij}[r] z^{-r-1} \]

and expand the Pfaffians of the matrices \( F(z) = [F_{ij}(z)] \) and \( F(z)_+ = [F_{ij}(z)_+] \) by

\[ \text{Pf} F(z) = \sum_{p \in \mathbb{Z}} S_p z^{-p-1} \quad \text{and} \quad \text{Pf} F(z)_+ = \sum_{p < 0} S_p^+ z^{-p-1}. \]

Invoking the vertex algebra structure on the vacuum module \( V(\mathfrak{o}_{2n}) \) (see [4]), we derive from Theorem 2.1 that the coefficients \( S_p \) are Sugawara operators for \( \hat{\mathfrak{o}}_{2n} \); they commute with the elements of \( \hat{\mathfrak{h}}_{2n} \) (note that normal ordering is irrelevant here, as the coefficients of the series pairwise commute). Moreover, the coefficients \( S_p^+ \) are elements of the Feigin–Frenkel center \( \mathfrak{z}(\hat{\mathfrak{o}}_{2n}) \).

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