Delay Compensation for Regular Linear Systems

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Abstract

This is the third part of four series papers, aiming at the delay compensation for the abstract linear system \((A, B, C)\). Both the input delay and output delay are investigated. We first propose a full state feedback control to stabilize the system \((A, B)\) with input delay and then design a Luenberger-like observer for the system \((A, C)\) in terms of the delayed output. We formulate the delay compensation in the framework of regular linear systems. The developed approach builds upon an upper-block-triangle transform that is associated with a Sylvester operator equation. It is found that the controllability/observability map of system \((-A, B)/(−A, −C)\) happens to be the solution of the corresponding Sylvester equation. As an immediate consequence, both the feedback law and the state observer can be expressed explicitly in the operator form. The exponential stability of the resulting closed-loop system and the exponential convergence of the observation error are established without using the Lyapunov functional approach. The theoretical results are validated through the delay compensation for a benchmark one-dimensional wave equation.

Keywords: Delay, Luenberger-like observer, regular linear system, observer, stabilization.

1 Introduction

It is well known that the time-delay is ubiquitous in engineering practices. Since the Smith predictor was introduced in [23], a fair amount of research results about the delay compensation have been done for finite-dimensional systems. However, the control of infinite-dimensional systems with time-delay is still a challenging problem and the corresponding results are much less than that for finite-dimensional ones. In [13], [22] and [27], the input delay is compensated for the reaction-diffusion equation by the method of partial differential equation (PDE) backstepping. These results can be considered as more or less the extensions of delay compensation for the ordinary differential equations (ODEs) discussed in [12] and [15]. When there are only finite unstable modes in the
open-loop system, the input delay can be compensated by the finite-dimensional spectral truncation technique. See for instance [21] and [16].

Although arbitrarily small delay in the feedback may destroy the stability of the system [3], some delays are still helpful to the system stability. When the output delay happens to be the propagation time, the wave equation can be stabilized by a delayed non-collocated boundary displacement feedback [4]. When the output delay equals even multiples of the propagation time, a direct feedback can stabilize the wave equation exponentially [26]. Even the wave equation with nonlinear boundary condition can be stabilized by the positive effect of the delay [5].

Stabilizations for one-dimensional wave and beam equations with arbitrarily long output delays are discussed in [9] and [8] where the problem is solved by both observer and predictor: The state is estimated in the time span where the observation is available; and the state is predicted in the time interval where the observation is not available. Very recently, the idea used in [9] and [8] has been extended to an abstract linear systems in [10, 19] and [18]. However, the systems considered in [10] are only limited to the conservative system and even the common unstable finite-dimensional linear systems do not belong to such class. Although the systems studied in [18] can be unstable, the bounded control operator must be required.

Since the delay dynamics are usually dominated by a transport equation [32], the problem of input or output delay compensation for infinite-dimensional systems can be described by a PDE-PDE cascade system. In contrast with the ODE-PDE or PDE-ODE cascade, the control of PDE-PDE cascade is much more complicated and the corresponding results are still fairly scarce. Some results about this topic can be found in the monograph [14].

In this paper, we consider the delay compensation for general abstract linear systems. Both the input delay and output delay are considered systematically. Let \((A, B, C)\) be a linear system with the state space \(Z\), input space \(U\) and the output space \(Y\). The problem is described by

\[
\dot{z}(t) = Az(t) + Bu(t-\tau), \quad y(t) = Cz(t-\mu), \tag{1.1}
\]

where \(y(t)\) is the measured output, \(u(t)\) is the control input, both of them are delayed by \(\mu\) and \(\tau\) units of time, respectively. We will study the input and output delay compensation separately. There are two key issues. The first one is about the stabilization of system (1.1) by the state feedback, and the second one is on the design of state observer for system (1.1) in terms of the delayed output \(y(t)\). Thanks to the separation principle of the linear systems, the output feedback law of system (1.1) is almost trivial once these two key issues are addressed. The developed approach is systematic and can be applied to the general regular linear systems which cover the common transport equations, reaction-diffusion equations, wave equations and the Euler-Bernoulli beam equations.

By writing the delay dynamics as a transport equation, the delay compensation for system (1.1) then amount to controlling or observing a PDE-PDE cascade. In this paper, the main idea of the PDE-PDE cascade treatment comes from the well known fact that the upper-block-triangle matrix can be decoupled as a block-diagonal matrix by an upper-block-triangle transformation that
is associated with a Sylvester matrix equation. More precisely,

\[
\begin{pmatrix} I & S \\ 0 & I \end{pmatrix} \begin{pmatrix} A_1 & Q \\ 0 & A_2 \end{pmatrix} \begin{pmatrix} I & S \\ 0 & I \end{pmatrix}^{-1} = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},
\]

(1.2)

where \( A_1 \), \( A_2 \) and \( Q \) are matrices with appropriate dimensions, \( I \) is the identity matrix on appropriate dimensional spaces and \( S \) is the solution of the Sylvester equation \( A_1 S - S A_2 = Q \). Owing to the block-diagonal structure, either stabilization or observer design of the transformed system \( \text{diag}(A_1, A_2) \) is much simpler than the original upper-block-triangle matrix. We will treat the delay compensation for general regular linear systems by following this idea. In our previous studies [6] and [7], this idea has been used to compensate the actuator dynamics and sensor dynamics for abstract linear systems. Different from [6] and [7] where at least one of \( A_1 \) and \( A_2 \) is required to be bounded, the delay compensation for infinite-dimensional systems considered in this paper always leads to a PDE-PDE cascade system. Generally speaking, the Sylvester operator equation with unbounded operators is hard to be solved. Fortunately, we find that the controllability/observability map of system \((−A,B)/(−A,−C)\) happens to be the solution of corresponding Sylvester operator equation. As a result, the upper-block-triangle transformation that decouples the PDE-PDE cascade system can be obtained explicitly.

The paper is organized as follows. In Section 2, we present some preliminaries on the regular linear systems. Section 3 investigates the vanishing shift semigroup which is used to describe the delay dynamics. Section 4 considers a Sylvester operator equation that is crucial to the input delay compensation. The state feedback is proposed to stabilize system \((A,B)\) with the input delay in Section 5. Sections 6 and 7 are devoted to the sensor delay compensation. The Sylvester operator equation that is used to output delay compensation is considered in Section 6 and the Luenberger-like observer is designed in terms of the delayed output in Section 7 where the exponentially convergence of the observer is also proved. In Section 8, the developed approaches are applied to a one-dimensional wave equation to validate the theoretical results. For easy readability, some results that are less relevant to the delay compensator design are arranged in the Appendix.

## 2 Background on regular linear systems

This section presents a brief overview of the regular linear system theory. We only summarize the results that will be used in the sections thereafter. We refer the interested reader to the references [25, 28, 29, 30] and [31] for more details. We first introduce the definition of dual space with respect to a pivot space that has been discussed extensively in [25] and is crucial in the theory of unbounded control and observation.

Suppose that \( X \) is a Hilbert space and \( A : D(A) \subset X \to X \) is a densely defined operator with \( \rho(A) \neq \emptyset \). The operator \( A \) can determine two Hilbert spaces: \((D(A), \| \cdot \|_1)\) and \((D(A^*))^\prime, \| \cdot \|_{-1}\), where \([D(A^*)]^\prime\) is the dual space of \( D(A^*) \) with respect to the pivot space \( X \), and the norms \( \| \cdot \|_1 \)
and \( \| \cdot \|_1 \) are defined by
\[
\begin{align*}
\| x \|_1 &= \| (\beta - A)x \|_X, \quad \forall \ x \in D(A), \\
\| x \|_{-1} &= \| (\beta - A)^{-1}x \|_X, \quad \forall \ x \in X,
\end{align*}
\]
These two spaces are independent of the choice of \( \beta \in \rho(A) \) since different choices of \( \beta \) lead to equivalent norms. For brevity, we denote the two spaces as \( D(A) \) and \( [D(A^*)]' \) in the sequel. The adjoint of \( A^* \in \mathcal{L}(D(A^*), X) \), denoted by \( \tilde{A} \), is defined as
\[
\langle \tilde{A} x, y \rangle_{[D(A^*)]'}, D(A^*) = \langle x, A^* y \rangle_X, \quad \forall \ x \in X, \ y \in D(A^*).
\]
It is evident that \( \tilde{A} x = A x \) for any \( x \in D(A) \). So \( \tilde{A} \in \mathcal{L}(X, [D(A^*)]') \) is an extension of \( A \). Since \( A \) is densely defined, such an extension is unique. By [25, Proposition 2.10.3], we have \( (\beta - \tilde{A}) \in \mathcal{L}(X, [D(A^*)]') \) and \( (\beta - \tilde{A})^{-1} \in \mathcal{L}([D(A^*)]', X) \) which imply that \( \beta - \tilde{A} \) is an isomorphism from \( X \) to \( [D(A^*)]' \).

Suppose that \( Y \) is the output Hilbert space and \( C \in \mathcal{L}(D(A), Y) \). The \( \Lambda \)-extension of \( C \) with respect to \( A \) is defined by
\[
C_A x = \lim_{\lambda \to +\infty} C \lambda (\lambda - A)^{-1} x, \quad \forall \ x \in D(C_A) = \{ x \in X \mid \text{the limit exists} \}.
\]
Define the norm
\[
\| x \|_{D(C_A)} = \| x \|_X + \sup_{\lambda \geq \lambda_0} \| C \lambda (\lambda - A)^{-1} x \|_Y, \quad \forall \ x \in D(C_A),
\]
where \( \lambda_0 \in \mathbb{R} \) such that \( [\lambda_0, \infty) \subset \rho(A) \). Then, it follows from [29, Proposition 5.3] that \( D(C_A) \) with norm \( \| \cdot \|_{D(C_A)} \) is a Banach space and \( C_A \in \mathcal{L}(D(C_A), Y) \). Moreover, we have the continuous embeddings:
\[
D(A) \hookrightarrow D(C_A) \hookrightarrow X \hookrightarrow [D(A^*)]' .
\]

The following results are brought from [31]:

**Proposition 2.1.** Let \( X, U \) and \( Y \) be the state space, input space and the output space, respectively. The triple \( (A, B, C) \) is said to be a regular linear system if and only if the following assertions hold true:

(i) \( A \) generates a \( C_0 \)-semigroup \( e^{At} \) on \( X \);

(ii) \( B \in \mathcal{L}(U, [D(A^*)]') \) and \( C \in \mathcal{L}(D(A), Y) \) are admissible for the \( C_0 \)-semigroup \( e^{At} \);

(iii) \( C_A(s - \tilde{A})^{-1}B \) exists for some (hence, for every) \( s \in \rho(A) \);

(iv) \( s \to \| C_A(s - \tilde{A})^{-1}B \| \) is bounded on some right half-plane.

**Definition 1.** Let \( X \) and \( U \) be Hilbert spaces, let \( A \) be the generator of a \( C_0 \)-semigroup \( e^{At} \) on \( X \) and let \( B \in \mathcal{L}(U, [D(A^*)]') \). Then, \( F \in \mathcal{L}(D(A), U) \) stabilizes system \( (A, B) \) exponentially if the following assertions hold:

(i) \( (A, B, F) \) is a regular triple;

(ii) there exists an \( s \in \rho(A) \) such that \( I \) is an admissible feedback operator for \( F_A(s - \tilde{A})^{-1}B \);

(iii) \( A + BF_A \) generates an exponentially stable \( C_0 \)-semigroup \( e^{(A+BF_A)t} \) on \( X \).
Definition 2. Let $X$ and $Y$ be the Hilbert spaces, and let $A$ be the generator of a $C_0$-semigroup $e^{At}$ on $X$ and $C \in \mathcal{L}(D(A), Y)$. Then, $L \in \mathcal{L}(Y, [D(A^*)]'')$ detects system $(A, C)$ exponentially if the following assertions hold true:

(i) $(A, L, C)$ is a regular triple;
(ii) there exists an $s \in \rho(A)$ such that $I$ is an admissible feedback operator for $C_{\lambda}(s - A)^{-1}L$;
(iii) $A + LC_{\lambda}$ generates an exponentially stable $C_0$-semigroup $e^{(A + LC_{\lambda})t}$ on $X$.

3 Vanishing shift semigroup

It is well known that the time-delay dynamics can be modeled as a transport equation which is usually associated with a vanishing shift semigroup. In this section, we introduce some preliminaries on shift semigroup that is useful for delay compensations.

Let $U$ be a Hilbert space with norm $||\cdot||_U$. For any $\alpha > 0$, we denote by $L^2([0, \alpha]; U)$ the Hilbert space of the measurable and square integrable functions from $[0, \alpha]$ to $U$. The inner product is

$$\langle \phi_1, \phi_2 \rangle_{L^2([0, \alpha]; U)} = \int_0^\alpha \langle \phi_1(x), \phi_2(x) \rangle_U dx, \ \forall \ \phi_1, \phi_2 \in L^2([0, \alpha]; U). \quad (3.1)$$

Define the operator $G_\alpha : D(G_\alpha) \subset L^2([0, \alpha]; U) \to L^2([0, \alpha]; U)$ by

$$(G_\alpha f)(x) = -\frac{d}{dx} f(x), \ \forall \ f \in D(G_\alpha) = \{ f \in H^1([0, \alpha]; U) \ | \ f(0) = 0 \}. \quad (3.2)$$

Then, $G_\alpha$ generates a vanishing right shift semigroup $e^{G_\alpha t}$ on $L^2([0, \alpha]; U)$, given by

$$\left( e^{G_\alpha t} f \right)(x) = \begin{cases} f(x-t), & x-t \geq 0, \\ 0, & x-t < 0, \end{cases} \ \forall \ f \in L^2([0, \alpha]; U). \quad (3.3)$$

The adjoint of $G_\alpha$ is

$$(G_\alpha^* f)(x) = \frac{d}{dx} f(x), \ \forall \ f \in D(G_\alpha^*) = \{ f \in H^1([0, \alpha]; U) \ | \ f(\alpha) = 0 \}, \quad (3.4)$$

which generates a vanishing left shift semigroup

$$\left( e^{G_\alpha^* t} f \right)(x) = \begin{cases} f(x+t), & x+t \leq \alpha, \\ 0, & x+t > \alpha, \end{cases} \ \forall \ f \in L^2([0, \alpha]; U). \quad (3.5)$$

Obviously, both $e^{G_\alpha t}$ and $e^{G_\alpha^* t}$ are exponentially stable on $L^2([0, \alpha]; U)$.

Let $[H^1([0, \alpha]; U)]'$ be the dual space of $H^1([0, \alpha]; U)$ with respect to the pivot space $L^2([0, \alpha]; U)$. Define the operator $B_\alpha : U \to [H^1([0, \alpha]; U)]'$ by

$$\langle B_\alpha u, f \rangle_{[H^1([0, \alpha]; U)]',[H^1([0, \alpha]; U)]} = \langle u, f(0) \rangle_U, \ \forall \ f \in H^1([0, \alpha]; U), \ \forall \ u \in U \quad (3.6)$$

and the operator $C_\alpha : D(C_\alpha) \subset L^2([0, \alpha]; U) \to U$ by

$$C_\alpha f = f(\alpha), \ \forall \ f \in D(C_\alpha) = H^1([0, \alpha]; U). \quad (3.7)$$
Lemma 3.1. For any $\alpha > 0$, let $G_\alpha$, $B_\alpha$ and $C_\alpha$ be defined by (3.2), (3.6) and (3.7), respectively. Then, both $B_\alpha \in \mathcal{L}(U, [D(G_\alpha^*)]')$ and $C_\alpha \in \mathcal{L}(D(G_\alpha), U)$ are admissible for the vanishing right shift semigroup $e^{G_\alpha t}$ and

$$
(\lambda - \tilde{G}_\alpha)^{-1}B_\alpha = E_\lambda, \quad \lambda \in \mathbb{C},
$$

(3.8)

where the operator $E_\lambda \in \mathcal{L}(U, D(C_\alpha))$ is given by

$$
E_\lambda u = e^{-\lambda x}u, \quad x \in [0, \alpha], \quad \forall \ u \in U.
$$

(3.9)

Moreover, $(G_\alpha, B_\alpha, C_\alpha)$ is a regular linear system.

Proof. It follows from (3.6) that the adjoint of $B_\alpha$ satisfies $B_\alpha^* \in \mathcal{L}(D(G_\alpha^*), U)$ and $B_\alpha^* f = f(0)$ for any $f \in D(G_\alpha^*)$. From (3.5), we deduce that

$$
\int_0^\alpha \|B_\alpha^* e^{G_\alpha t} f\|_U^2 dt = \int_0^\alpha \|f(t)\|_U^2 dt = \|f\|_{L^2([0,\alpha]; U)}^2, \quad f \in D(G_\alpha^*),
$$

(3.10)

which implies that $B_\alpha^*$ is admissible for $e^{G_\alpha t}$ and thus, $B_\alpha$ is admissible for $e^{G_\alpha t}$. Similarly, it follows from (3.3) and (3.7) that $C_\alpha \in \mathcal{L}(D(G_\alpha), U)$ is admissible for $e^{G_\alpha t}$.

By a straightforward computation, it follows that $\rho(G_\alpha) = \rho(G_\alpha^*) = \mathbb{C}$ and

$$
\left< (\lambda - \tilde{G}_\alpha)E_\lambda u, \phi \right>_{[D(G_\alpha^*)]', D(G_\alpha^*)} = \left< E_\lambda u, (\lambda - G_\alpha^*) \phi \right>_{L^2([0,\alpha]; U)}
$$

$$
= \int_0^\alpha \left< e^{-\lambda x} u, \overline{\phi(x)} \right>_U dx - \int_0^\alpha \left< e^{-\lambda x} u, \frac{d}{dx} \phi(x) \right>_U dx
$$

$$
= \lambda \int_0^\alpha \left< e^{-\lambda x} u, \phi(x) \right>_U dx + \left< u, \phi(0) \right>_U - \left< e^{-\lambda x} u, \phi(x) \right>_U dx
$$

$$
= \left< u, \phi(0) \right>_U, \quad \forall \ u \in U, \ \phi \in D(G_\alpha^*), \ \lambda \in \mathbb{C},
$$

(3.11)

which, together with (3.6), leads to $(\lambda - \tilde{G}_\alpha)E_\lambda = B_\alpha$. This means that (3.8) holds. By (3.7), (3.8) and (3.9), we conclude that $C_\alpha(\lambda - \tilde{G}_\alpha)^{-1}B_\alpha u = C_\alpha E_\lambda u = e^{-\lambda x} u$ for any $u \in U$. This implies that $C_\alpha(\lambda - \tilde{G}_\alpha)^{-1}B_\alpha \in \mathcal{L}(U)$ and $\lambda \rightarrow \|C_\alpha(\lambda - \tilde{G}_\alpha)^{-1}B_\alpha\|$ is bounded on some right half-plane. Hence, $(G_\alpha, B_\alpha, C_\alpha)$ is a regular linear system. \hfill \Box

As in [24, Section 2.2], we define a subspace of $L^2([0, \alpha]; U)$ by

$$
G_{B_\alpha} = \left\{ f \in L^2([0, \alpha]; U) \mid \tilde{G}_\alpha f + B_\alpha u \in L^2([0, \alpha]; U), u \in U \right\}.
$$

(3.12)

For any $f \in G_{B_\alpha}$, there exists a $u_f \in U$ such that $\tilde{G}_\alpha f + B_\alpha u_f \in L^2([0, \alpha]; U)$. This shows that $\tilde{G}_\alpha^{-1}(\tilde{G}_\alpha f + B_\alpha u_f) = f + \tilde{G}_\alpha^{-1}B_\alpha u_f \in D(G_\alpha)$ and hence $f \in D(G_\alpha) + \tilde{G}_\alpha^{-1}B_\alpha U$. So we obtain $G_{B_\alpha} \subset D(G_\alpha) + \tilde{G}_\alpha^{-1}B_\alpha U$. For any $g = g_1 + \tilde{G}_\alpha^{-1}B_\alpha u_g \in D(G_\alpha) + \tilde{G}_\alpha^{-1}B_\alpha U$ with $g_1 \in D(G_\alpha)$ and $u_g \in U$, a simple computation shows that $\tilde{G}_\alpha g = B_\alpha(-u_g) = \tilde{G}_\alpha g_1 \in L^2([0, \alpha]; U)$, which means that $g \in G_{B_\alpha}$ and hence $D(G_\alpha) + \tilde{G}_\alpha^{-1}B_\alpha U \subset G_{B_\alpha}$. Therefore,

$$
G_{B_\alpha} = D(G_\alpha) + \tilde{G}_\alpha^{-1}B_\alpha U.
$$

(3.13)
By [24, Section 2.2], \( G_{B^\alpha} \) with inner product
\[
\|f\|_{G_{B^\alpha}}^2 = \|f\|^2_{L^2([0,\alpha];U)} + \|u_f\|^2_{\tilde{G}_\alpha} + \|\tilde{G}_\alpha f + B_\alpha u_f\|^2_{L^2([0,\alpha];U)}
\] (3.14)
is a Hilbert space, where \( u_f \in U \) such that \( \tilde{G}_\alpha f + B_\alpha u_f \in L^2([0,\alpha];U) \). It follows from Lemma 3.1 that 
\(-\tilde{G}^{-1}_\alpha B_\alpha U = \{c_u \mid c_u(x) \equiv u, x \in [0,\alpha], u \in U\} \) which, together with (3.2), (3.13) and (3.7), gives
\[
G_{B^\alpha} = H^1([0,\alpha];U) = D(C_\alpha).
\] (3.15)

4 Sylvester equation associated with input delay

In this section, we consider a Sylvester equation that is closely related to the input delay compensation. Let \( A \) be a generator of the \( C_0 \)-semigroup \( e^{At} \) on \( Z \). Suppose that \( B \in \mathcal{L}(U,[D(A^*)])' \) is admissible for \( e^{-At} \). Then, \( B^* \in \mathcal{L}(D(A^*),U) \) is admissible for \( e^{-A^*t} \). By exploiting [25, Proposition 4.3.4, p.124], \( B^* e^{A^*(t-\tau)} h = B^* e^{A^*(\tau-\tau)} h \in H^1([0,\tau];U) \) for any \( h \in D(A^*) \) and \( \tau > 0 \). As a consequence, we can define the operator \( S_\tau : H^1([0,\tau];U) \to [D(A^*)]' \) by
\[
\langle S_\tau f,z \rangle_{D(A^*)}',D(A^*) = \left< f, B^* e^{A^*(t-\tau)} z \right>_{H^1([0,\tau];U)}
\] (4.1)
for any \( z \in D(A^*) \) and \( f \in [H^1([0,\tau];U)]' \). Suppose that \( g \in L^2([0,\tau];U) \). Then
\[
\langle S_\tau g,z \rangle_{D(A^*)}',D(A^*) = \left< g, B^* e^{A^*(t-\tau)} z \right>_{L^2([0,\tau];U)} = \int_0^\tau \left< g(x), B^* e^{A^*(x-\tau)} z \right>_U \, dx
\]
\[
= \left< \int_0^\tau e^{\tilde{A}(x-\tau)} B g(x) \, dx, z \right>_{D(A^*)}',D(A^*)}, \quad \forall \, z \in D(A^*) \subset Z.
\] (4.2)

Since \( B \) is admissible for \( e^{-At} \), we have \( \int_0^\tau e^{\tilde{A}(x-\tau)} B g(x) \, dx \in Z \) which, together with (4.2), implies that \( S_\tau \in \mathcal{L}(L^2([0,\tau];U),Z) \) and
\[
S_\tau g = \int_0^\tau e^{\tilde{A}(x-\tau)} B g(x) \, dx, \quad \forall \, g \in L^2([0,\tau];U).
\] (4.3)

This means that \( S_\tau \) is an extension of the controllability map of system \((-A,B)\), as defined in [2, Definition 4.1.3, p.143].

**Lemma 4.1.** Let \((A,B,K)\) be a regular linear system with the state space \( Z \), input space \( U \) and the output space \( U \). Suppose that \( G_\tau, B_\tau \) and \( C_\tau \) are defined by (3.2), (3.6) and (3.7) with \( \alpha = \tau \), respectively. Then, the operator \( S_\tau \in \mathcal{L}(L^2([0,\tau];U),Z) \) defined by (4.1) satisfies:
\[
S_\tau B_\tau = e^{-\tilde{A}_\tau} B \in \mathcal{L}(U,[D(A^*)])'
\] (4.4)
and
\[
\left\{
\begin{array}{l}
\tilde{A} S_\tau f - S_\tau \tilde{G}_\tau f = B C_\tau f, \\
K A e^{A^*} S_\tau f \in U,
\end{array}
\right. \quad \forall \, f \in G_{B_\tau},
\] (4.5)
where \( G_{B_\tau} \) is defined by (3.12) with \( \alpha = \tau \).
Proof. By (4.1) and (3.6), we deduce
\[
\langle S_T B_T u, h \rangle_{[D(A^*)]'}, D(A^*) = \left\langle B_T u, B^* e^{A^*(\tau-h)} \right\rangle_{[H^1([0,\tau];U)]'}, H^1([0,\tau];U) = \left\langle u, B^* e^{-A^*\tau} h \right\rangle_U
\]
which leads to (4.4) easily. It follows from (3.2), (3.7) and (4.3) that
\[
S_T G_T g = - \int_0^\tau e^{A(x-\tau)} B g(x) dx = -B g(\tau) + \tilde{A} \int_0^\tau e^{\tilde{A}(x-\tau)} B g(x) dx
\]
which implies that the Sylvester equation \( \tilde{A} S_T - S_T G_T = B C_T \) holds on \( D(G_T) \). For any \( f \in G_{B_T} \), by (3.13), \( f \) can be divided into two parts \( f = g_f + \tilde{G}_a^{-1} B_\alpha u_f \), where \( g_f \in D(G_T) \) and \( u_f \in U \). By Lemma 3.1, \( \tilde{G}_T^{-1} B_T u_f = -E_0 u_f \equiv u_f \). Owing to (3.13), (4.7) and \( g_f \in D(G_T) \), the first equation of (4.5) holds if we can prove that
\[
\tilde{A} S_T E_0 u_f - S_T \tilde{G}_T E_0 u_f = B C_T E_0 u_f.
\]
Actually, it follows from (4.4) that
\[
-S_T \tilde{G}_T E_0 u_f = S_T \tilde{G}_T (\tilde{G}_T^{-1} B_T u_f) = S_T B_T u_f = e^{-A_T} B u_f.
\]
By (3.7) and (4.3), it follows that
\[
B C_T E_0 u_f - \tilde{A} S_T E_0 u_f = B u_f - \tilde{A} \int_0^\tau e^{\tilde{A}(x-\tau)} B u_f dx = e^{-A_T} B u_f,
\]
which, together with (4.9), leads to (4.8) easily. Therefore, the first equation of (4.5) holds.

Now, we prove the remaining part of (4.5). Since \((A, B, K)\) is regular, we have
\[
K \int_0^\tau e^{\tilde{A}(s)} B g(s) ds \in H^1_{loc}([0, +\infty); U), \ \forall \ g \in H^1_{loc}([0, \infty); U).
\]
In particular,
\[
K \int_0^\tau e^{\tilde{A}(\tau-s)} B g(s) ds \in U, \ \forall \ g \in H^1([0, \tau]; U).
\]
For any \( f \in G_{B_T} \), it follows from (3.15) that \( f(\tau - \cdot) \in H^1([0, \tau]; U) \). Since \( B \) is admissible for \( e^{-A_T} \),
\[
e^{A_T} S_T f = e^{A_T} \int_0^\tau e^{\tilde{A}(x-\tau)} B f(x) dx = \int_0^\tau e^{\tilde{A}x} B f(x) dx = \int_0^\tau e^{\tilde{A}(\tau-x)} B f(\tau - x) dx,
\]
which, together with (4.12), leads to \( K_A e^{A_T} S_T f \in U \). The proof is complete.

\[\square\]

**Lemma 4.2.** Suppose that \( A \) is the generator of a \( C_0 \)-semigroup \( e^{A_T} \) on \( Z \), \( B \in \mathcal{L}(U, [D(A^*)]') \) is admissible for \( e^{A_T} \) and \( K \in \mathcal{L}(D(A), U) \) is admissible for \( e^{A_T} \). Then, for any \( \tau > 0 \), \( K \in \mathcal{L}(D(A), U) \) stabilizes system \((A, B)\) exponentially if and only if \( Ke^{A_T} \in \mathcal{L}(D(A), U) \) stabilizes system \((A, e^{-A_T} B)\) exponentially.
Proof. By Lemma 9.1 in Appendix, both \( e^{-\hat{A}\tau} \) and \( Ke^{A\tau} \) are admissible for \( e^{At} \). For any \( \lambda \in \rho(A) \), a simple computation shows that

\[
K_{\lambda}(\lambda - \hat{A})^{-1}B = K_{\lambda} [e^{A\tau}(\lambda - \hat{A})^{-1}e^{-\hat{A}\tau}]B = (Ke^{A\tau})_{\lambda}(\lambda - \hat{A})^{-1}e^{-\hat{A}\tau}B,
\]

where \((Ke^{A\tau})_{\lambda}\) is the \( \lambda \)-extension of \( Ke^{A\tau} \) with respect to \( A \). By Proposition 2.1, \((A,B,K)\) is a regular triple if and only if \((A,e^{-\hat{A}\tau}B,Ke^{A\tau})\) is a regular triple. Moreover, \( I \) is an admissible feedback operator for \( K_{\lambda}(\lambda - \hat{A})^{-1}B \) is equivalent to that \( I \) is an admissible feedback operator for \((Ke^{A\tau})_{\lambda}(\lambda - \hat{A})^{-1}e^{-\hat{A}\tau}B \). Since \( e^{\hat{A}\tau} \in \mathcal{L}(Z) \) and

\[
\left( A + e^{-\hat{A}\tau}BK_{\lambda}e^{A\tau} \right)z = \left( A + e^{-\hat{A}\tau}BK_{\lambda}e^{A\tau} \right)z = e^{-\hat{A}\tau}(A + BK_{\lambda})e^{A\tau}z, \quad \forall \ z \in Z,
\]

\( A + e^{-\hat{A}\tau}BK_{\lambda}e^{A\tau} \) is exponentially stable in \( Z \) if and only if \( A + BK_{\lambda} \) is exponentially stable in \( Z \). Finally, the proof is completed by Definition 1.

\[\square\]

5 Input delays compensator design

This section is devoted to the input delay compensation. Let \( Z \) and \( U \) be Hilbert spaces. Suppose that the operator \( A : D(A) \subset Z \to Z \) generates a \( C_{0} \)-semigroup \( e^{At} \) on \( Z \) and \( B \in \mathcal{L}(U,[D(A^{*})]') \) is admissible for \( e^{At} \). Consider the following linear system:

\[
\dot{z}(t) = Az(t) + Bu(t - \tau), \quad \tau > 0,
\]

where \( z(t) \) is the state and \( u : [-\tau, \infty) \to U \) is the control that is delayed by \( \tau \) units of time. If we let

\[
\phi(x,t) = u(t - x), \quad x \in [0, \tau], \ t \geq 0,
\]

then, system (5.1) can be written as the delay free form:

\[
\begin{cases}
\dot{z}(t) = Az(t) + B\phi(\tau,t), \\
\phi_{t}(x,t) + \phi_{x}(x,t) = 0 \quad \text{in} \ U, \ x \in (0,\tau), \\
\phi(0,t) = u(t).
\end{cases}
\]

We consider system (5.3) in the state space \( Z_{\tau}(U) = Z \times L^{2}([0,\tau];U) \) with the inner product

\[
\langle (z_{1},f_{1})^{T},(z_{2},f_{2})^{T} \rangle_{Z_{\tau}(U)} = \langle z_{1},z_{2} \rangle Z + \langle f_{1},f_{2} \rangle_{L^{2}([0,\tau];U)}, \quad \forall \ (z_{j},f_{j})^{T} \in Z_{\tau}(U), \ j = 1,2,
\]

where \( \langle \cdot, \cdot \rangle_{L^{2}([0,\tau];U)} \) is given by (3.1) with \( \alpha = \tau \). In terms of \( G_{\tau}, B_{\tau} \) and \( C_{\tau} \) defined by (3.2), (3.6) and (3.7) with \( \alpha = \tau \), respectively, system (5.3) can be written as the abstract form

\[
\begin{cases}
\dot{z}(t) = \hat{A}z(t) + BC_{\tau A}\phi(\cdot,t), \\
\phi_{t}(\cdot,t) = \hat{G}_{\tau}\phi(\cdot,t) + B_{\tau}u(t).
\end{cases}
\]
Let $S_{\tau} : [H^1([0, \tau]; U)]' \to [D(A^\tau)]'$ be defined by (4.1) and

$$S(z, f)^\top = (z + S_{\tau} f, f)^\top, \quad \forall (z, f)^\top \in \mathcal{Z}_{\tau}(U).$$

(5.6)

By Lemma 4.1, $S \in \mathcal{L}(Z_{\tau}(U))$ is invertible and its inverse is given by

$$S^{-1}(z, f)^\top = (z - S_{\tau} f, f)^\top, \quad \forall (z, f)^\top \in \mathcal{Z}_{\tau}(U).$$

(5.7)

Suppose that $(z, \phi) \in C([0, \infty); \mathcal{Z}_{\tau}(U))$ is a solution of system (5.5). Inspired by [6], we introduce the transformation

$$(\tilde{z}(t), \tilde{\phi}(\cdot, t))^\top = S(z(t), \phi(\cdot, t))^\top.$$  

(5.8)

By (4.4) and (4.5), the transformation (5.8) converts system (5.5) into

$$
\begin{align*}
\dot{\tilde{z}}(t) &= \tilde{A}\tilde{z}(t) + e^{-\tilde{A}\tau} B u(t), \\
\dot{\tilde{\phi}}(\cdot, t) &= \tilde{G}_{\tau}\tilde{\phi}(\cdot, t) + B_{\tau} u(t),
\end{align*}
$$

(5.9)

provided $\phi(\cdot, t) \in G_{B_{\tau}}$. Since $G_{\tau}$ is already stable, the stabilization of system (5.9) amounts to the stabilization of the pair $(A, e^{-\tilde{A}\tau} B)$. By Lemma 4.2, the stabilizer of (5.9) can be designed as

$$u(t) = K_{A} e^{A\tau} \tilde{z}(t), \quad t \geq 0,$$

(5.10)

where $K \in \mathcal{L}(D(A), U)$ stabilizes system $(A, B)$ exponentially. Under the feedback (5.10), we get the closed-loop system of (5.9)

$$
\begin{align*}
\dot{\tilde{z}}(t) &= \tilde{A}\tilde{z}(t) + e^{-\tilde{A}\tau} B K_{A} e^{A\tau} \tilde{z}(t), \\
\dot{\tilde{\phi}}(\cdot, t) &= \tilde{G}_{\tau}\tilde{\phi}(\cdot, t) + B_{\tau} K_{A} e^{A\tau} \tilde{z}(t).
\end{align*}
$$

(5.11)

This transformed system can be written abstractly as

$$
\frac{d}{dt}(\tilde{z}(t), \tilde{\phi}(\cdot, t))^\top = \mathcal{A}_{\tilde{S}}(\tilde{z}(t), \tilde{\phi}(\cdot, t))^\top,
$$

(5.12)

where $\mathcal{A}_{\tilde{S}}$ is given by

$$\mathcal{A}_{\tilde{S}} = \begin{pmatrix} \tilde{A} + e^{-\tilde{A}\tau} B K_{A} e^{A\tau} & 0 \\ B_{\tau} K_{A} e^{A\tau} & \tilde{G}_{\tau} \end{pmatrix},$$

(5.13)

with

$$D(\mathcal{A}_{\tilde{S}}) = \left\{ \begin{pmatrix} \tilde{z} \\ f \end{pmatrix} \in \mathcal{Z}_{\tau}(U) \mid \tilde{A}\tilde{z} + e^{-\tilde{A}\tau} B K_{A} e^{A\tau} \tilde{z} \in \mathcal{Z} \right\}. $$

(5.14)

Combining (5.8), (5.10) and (4.3), the stabilizer of the original system (5.5) is

$$u(t) = (K_{A} e^{A\tau}, 0) (\tilde{z}(t), \tilde{\phi}(\cdot, t))^\top = (K_{A} e^{A\tau}, 0) S(z(t), \phi(\cdot, t))^\top = K_{A} e^{A\tau} (\tilde{z}(t) + S_{\tau} \phi(\cdot, t) = K_{A} e^{A\tau} z(t) + K_{A} \int_{0}^{\tau} e^{\tilde{A}x} B \phi(x, t) dx,$$

(5.15)
under which the closed-loop system of (5.3) is:

\[
\begin{aligned}
\dot{z}(t) &= Az(t) + B\phi(\tau, t), \\
\phi_t(x, t) + \phi_x(x, t) &= 0 \quad \text{in } U, \ x \in (0, \tau), \\
\phi(0, t) &= K_\Lambda \int_0^\tau e^{\tilde{A}r}B\phi(x, t)dx + K_\Lambda e^{A\tau}z(t).
\end{aligned}
\]  

(5.16)

Define

\[
\mathcal{A} = \begin{pmatrix}
\tilde{A} & BC_\tau L \\
B_\tau K_\Lambda e^{A\tau} & \tilde{G}_\tau + \tilde{B}_\tau K_\Lambda e^{A\tau}S_\tau
\end{pmatrix}
\]  

(5.17)

with

\[
D(\mathcal{A}) = \left\{ \begin{pmatrix} z \\ f \end{pmatrix} \in Z_\tau(U) \mid \begin{array}{c}
\tilde{A}z + BC_\tau L f \in Z \\
\tilde{G}_\tau f + B_\tau K_\Lambda e^{A\tau} (S_\tau f + z) \in L^2([0, \tau]; U)
\end{array} \right\}.
\]  

(5.18)

Then, the closed-loop system (5.16) can be written abstractly as

\[
\frac{d}{dt}(z(t), \phi(\cdot, t))^T = \mathcal{A}(z(t), \phi(\cdot, t))^T, \quad t \geq 0.
\]  

(5.19)

**Theorem 5.1.** Let \(G_\tau, B_\tau, C_\tau\) be given by (3.2), (3.6) and (3.7) with \(\alpha = \tau > 0\), respectively. Suppose that \(S_\tau\) is given by (4.1) and \(K \in L(D(A), U)\) stabilizes system \((A, B)\) exponentially. Then, the operator \(\mathcal{A}\) defined by (5.17) generates an exponentially stable \(C_0\)-semigroup \(e^{\mathcal{A}t}\) on \(Z_\tau(U)\). As a result, for any \((z(0), \phi(\cdot, 0))^T \in Z_\tau(U)\), system (5.16) admits a unique solution \((z, \phi)^T \in C([0, \infty); Z_\tau(U))\) that decays to zero exponentially in \(Z_\tau(U)\) as \(t \to \infty\).

**Proof.** By Lemma 4.1, the operator \(S_\tau\) satisfies (4.3), (4.4) and (4.5). We first claim that \(\mathcal{A}\) is similar to \(\mathcal{A}_S\), i.e.,

\[
S\mathcal{A}S^{-1} = \mathcal{A}_S \quad \text{and} \quad D(\mathcal{A}_S) = S D(\mathcal{A}),
\]  

(5.20)

where \(S\) is given by (5.6).

For any \((z, f)^T \in D(\mathcal{A}_S), (5.14)\) and (3.12) imply that \(f \in G_{B_\tau}\). Moreover, \(K_\Lambda e^{A\tau}S_\tau f \in U\) due to Lemma 4.1. Hence, it follows from (5.7) and (5.14) that

\[
(\tilde{G}_\tau + B_\tau K_\Lambda e^{A\tau}S_\tau)f + B_\tau K_\Lambda e^{A\tau}(z - S_\tau f) = \tilde{G}_\tau f + B_\tau K_\Lambda e^{A\tau}z \in L^2([0, \tau]; U).
\]  

(5.21)

Combine (4.5), (4.4), (5.14) and the fact \(S_\tau \in L(L^2([0, \tau]; U), Z)\) to get

\[
\tilde{A}(z - S_\tau f) + BC_\tau f = \tilde{A}z - S_\tau \tilde{G}_\tau f = \tilde{A}z + e^{-\tilde{A}\tau}BK_\Lambda e^{A\tau}z - S_\tau B_\tau K_\Lambda e^{A\tau}z + S_\tau \tilde{G}_\tau f
\]  

(5.22)

\[
= (\tilde{A}z + e^{-\tilde{A}\tau}BK_\Lambda e^{A\tau}z) - S_\tau (\tilde{G}_\tau f + B_\tau K_\Lambda e^{A\tau}z) \in Z,
\]

which, together with (5.21), (5.18) and (5.7), yields \(S^{-1}(z, f)^T \in D(\mathcal{A})\). Hence \(D(\mathcal{A}_S) \subset S D(\mathcal{A})\) due to the arbitrariness of \((f, z)^T \in D(\mathcal{A}_S)\).

On the other hand, for any \((z, f)^T \in D(\mathcal{A}), (5.18)\) and (3.12) imply that \(f \in G_{B_\tau}\) and

\[
\tilde{G}_\tau f + B_\tau K_\Lambda e^{A\tau} (S_\tau f + z) \in L^2([0, \tau]; U).
\]  

(5.23)
It follows from (4.4), (4.5), (5.23) and the fact $S_{\tau} \in \mathcal{L}(L^2([0, \tau]; U), Z)$ that
\[
\dot{A}(z + S_{\tau}f) + e^{-\bar{A}\tau}BK_{\Lambda}e^{\bar{A}\tau}(z + S_{\tau}f) = \dot{A}z + S_{\tau}G_{\tau}f + BC_{\tau}f + S_{\tau}B_{\tau}K_{\Lambda}e^{\bar{A}\tau}(z + S_{\tau}f)
\]
\[
= (\dot{A}z + BC_{\tau}f) + S_{\tau}[\bar{G}_{\tau}f + B_{\tau}K_{\Lambda}e^{\bar{A}\tau}(S_{\tau}f + z)] \in Z. \tag{5.24}
\]
We combine (5.23), (5.24), (5.6) and (5.14) to get $S(z, f)^{\top} \in D(\mathcal{A}_2)$ and hence $SD(\mathcal{A}) \subset D(\mathcal{A}_2)$. In summary, we arrive at $SD(\mathcal{A}) = D(\mathcal{A}_2)$. Moreover, for any $(z, f)^{\top} \in D(\mathcal{A}_2)$, it follows from (5.14) and (3.12) that $f \in G_{B_{\tau}}$. By virtue of (4.5), a simple computation shows that $S\mathcal{A}S^{-1}(z, f)^{\top} = \mathcal{A}_2(z, f)^{\top}$ for any $(z, f)^{\top} \in D(\mathcal{A}_2)$. Hence, the similarity (5.20) holds.

Since $K \in \mathcal{L}(D(A), U)$ stabilizes $(A, B)$ exponentially, it follows from Lemma 4.2 that $Ke^{\bar{A}\tau} \in \mathcal{L}(D(A), U)$ stabilizes $(A, e^{-\bar{A}\tau}B)$ exponentially. In particular, $A + e^{-\bar{A}\tau}BK_{\Lambda}e^{\bar{A}\tau}$ generates an exponentially stable $C_0$-semigroup $e^{(A + e^{-\bar{A}\tau}BK_{\Lambda}e^{\bar{A}\tau})t}$ on $Z$ and $Ke^{\bar{A}\tau}$ is admissible for $e^{(A + e^{-\bar{A}\tau}BK_{\Lambda}e^{\bar{A}\tau})t}$. By Lemma 9.2 in Appendix, the operator $\mathcal{A}_2$ generates an exponentially stable $C_0$-semigroup $e^{\mathcal{A}_2t}$ on $Z_{\tau}(U)$. Owing to the similarity of $\mathcal{A}_2$ and $\mathcal{A}$, the operator $\mathcal{A}$ generates an exponentially stable $C_0$-semigroup $e^{\mathcal{A}t}$ on $Z_{\tau}(U)$ as well. The proof is complete.

\begin{remark}
\textbf{Remark 5.1.} When $A$ is a matrix, it follows from (5.2) that the controller (5.15) takes form:
\[
u(t) = K\int_{t-\tau}^{t} e^{A(t-s)}Bu(s)ds + Ke^{\bar{A}\tau}z(t), \quad t \geq \tau, \tag{5.25}
\]
which is the same as those obtained from the spectrum assignment approach in [11], the “reduction approach” in [1] and the PDE backstepping method in [12]. We point out that the Lyapunov function has not been used in the stability analysis of our method. This avoids the difficulty about the Lyapunov-based technique for stabilization of PDEs with delay. Another advantage of the proposed approach is that we never need the target system as that by the backstepping approach. This avoids the possibility that when the target system is not chosen properly, there is no state feedback control and even if the target system is good enough, there is difficulty in solving PDE kernel equation for the backstepping transformation.

\section{Output delays and Sylvester equation}

In this and next sections, we consider the output delay compensation, which is the most common dynamic phenomena arising in control engineering practice. Consider the following system in the state space $Z$, input space $U$ and the output space $Y$:
\[
\dot{z}(t) = Az(t) + Bu(t), \quad y(t) = C_\Lambda z(t - \mu), \quad \mu > 0, \tag{6.1}
\]
where $A : D(A) \subset Z \rightarrow Z$ is the system operator, $B \in \mathcal{L}(U, [D(A^*)]'')$ is the control operator, $C \in \mathcal{L}(D(A), Y)$ is the observation operator, $u(t)$ is the control input, and $y(t)$ is the measurement that is delayed by $\mu$ units of time. Let $\psi(x, t) = C_\Lambda z(t - x)$ for $x \in [0, \mu]$ and $t \geq \mu$. Then, system
(6.1) can be written as
\[
\begin{align*}
\dot{z}(t) &= A z(t) + B u(t), \\
\psi_t(x,t) + \psi_x(x,t) &= 0 \quad \text{in } Y, \quad x \in [0,\mu], \\
\psi(0,t) &= C_\Lambda z(t), \\
y(t) &= \psi(\mu,t).
\end{align*}
\] (6.2)

We consider system (6.2) in state space $Z_\mu(Y) = Z \times L^2([0,\mu];Y)$. The inner product of $Z_\mu(Y)$ is given by (5.4) with $\alpha = \mu$ and $U = Y$. In terms of the operators $G_\mu$, $B_\mu$ and $C_\mu$, which are given by (3.2), (3.6) and (3.7) with $\alpha = \mu$ and $U = Y$, respectively, system (6.2) can be written as
\[
\begin{align*}
\dot{z}(t) &= \tilde{A} z(t) + B u(t), \\
\psi_t(\cdot,t) &= \tilde{G}_\mu \psi(\cdot,t) + B_\mu C_\Lambda z(t), \\
y(t) &= C_\mu \Lambda \psi(\cdot,t).
\end{align*}
\] (6.3)

The following Theorem guarantees that the mapping from each initial data and control input signal to the state and the output observation signal is continuous.

**Theorem 6.1.** Let $G_\mu$, $B_\mu$ and $C_\mu$ be given by (3.2), (3.6) and (3.7) with $\alpha = \mu$ and $U = Y$, respectively. Suppose that $(A,B,C)$ is a well-posed linear system in the sense of Salamon in [24]. Then, system (6.3) is also well-posed: For any $(z(0),\psi(\cdot,0))^\top \in Z_\mu(Y)$ and $u \in L^2_{\text{loc}}([0,\infty);U)$, system (6.3) admits a unique solution $(z,\psi)^\top \in C([0,\infty);Z_\mu(Y))$ that satisfies, for any $T > 0$, there exists a positive constant $C_T$ such that
\[
\int_0^T \|y(s)\|_Y^2 \, ds + \|(z(T),\psi(\cdot,T))^\top\|_{Z_\mu(Y)} \leq C_T \left[ \int_0^T \|u(s)\|_U^2 \, ds + \|(z(0),\psi(\cdot,0))^\top\|_{Z_\mu(Y)} \right].
\] (6.4)

**Proof.** Since $z$-subsystem is independent of $\psi$-subsystem, the solution of (6.3) can be expressed explicitly:
\[
z(t) = e^{At}z(0) + \int_0^t e^{A(t-s)}Bu(s) \, ds, \quad \psi(x,t) = \begin{cases} C_\Lambda z(t-x), & t-x \geq 0, \\ \psi(x-t,0), & t-x < 0, \end{cases}
\] (6.5)

where $x \in [0,\mu]$. Moreover,
\[
y(t) = C_\mu \Lambda \psi(\cdot,t) = \psi(\mu,t) = \begin{cases} C_\Lambda z(t-\mu), & t-\mu \geq 0, \\ \psi(\mu-t,0), & t-\mu < 0 \end{cases}
\] (6.6)

and
\[
\|\psi(\cdot,t)\|_{L^2([0,\mu];Y)}^2 = \begin{cases} \int_0^t \|C_\Lambda z(t-x)\|_Y^2 \, dx + \int_0^\mu \|\psi(x-t,0)\|_Y^2 \, dx, & 0 \leq t < \mu \\ \int_0^\mu \|C_\Lambda z(t-x)\|_Y^2 \, dx, & t \geq \mu \\ \int_0^t \|C_\Lambda z(x)\|_Y^2 \, dx + \int_0^{\mu-t} \|\psi(x,0)\|_Y^2 \, dx, & 0 \leq t < \mu \\ \int_{t-\mu}^t \|C_\Lambda z(x)\|_Y^2 \, dx, & t \geq \mu. \end{cases}
\] (6.7)
Since \((A,B,C)\) is well-posed, for any \(t > 0\), there exists a \(C_t > 0\) such that
\[
\int_0^t \|C_A z(s)\|^2_Y ds + \|z(t)\|^2_Z \leq C_t \left[ \|z(0)\|^2_Z + \int_0^t \|u(s)\|^2_Z ds \right],
\]
which, together with (6.5), (6.6) and (6.7), leads to (6.4) easily. The proof is complete. \(\square\)

Let \((A,C)\) be an observation system with the state space \(Z\) and output space \(Y\). Suppose that \(A\) generates a \(C_0\)-semigroup \(e^{At}\) on \(Z\) and \(C \in \mathcal{L}(D(A), Y)\) is admissible for \(e^{At}\). As defined in [2, Definition 4.1.12, p.154], for any \(\mu > 0\), the observability map of system \((-A,-C)\) is
\[
\Psi_\mu : Z \to L^2([0,\mu]; Y)
\]
\[
z \to -C_A e^{-At} z, \quad x \in [0,\mu], \quad \forall z \in Z.
\]
Since \(C\) is admissible for the \(C_0\)-semigroup \(e^{-At}\), \(\Psi_\mu \in \mathcal{L}(Z, L^2([0,\mu]; Y))\). For any \(f \in D(G^*_\mu) \subset H^1([0,\mu]; Y)\), it follows from (2.2), (3.4), (3.6) and the fact \([H^1([0,\mu]; Y)]' \subset [D(G^*_\mu)]'\) that
\[
\left\langle \hat{G}_\mu \Psi_\mu z, f \right\rangle_{[D(G^*_\mu)]', D(G^*_\mu)} = \left\langle -C_A e^{-At} z, G^*_\mu f \right\rangle_{L^2([0,\mu]; Y)}
\]
\[
= -\int_0^\mu \left\langle C_A e^{-As} z, \frac{d}{d\sigma} f(\sigma) \right\rangle_Y d\sigma = \langle Cz, f(0) \rangle_Y - \int_0^\mu \left\langle C_A e^{-As} A z, f(\sigma) \right\rangle_Y d\sigma
\]
\[
= \langle B_\mu Cz, f \rangle_{[D(G^*_\mu)]', D(G^*_\mu)} + \left\langle \Psi_\mu A z, f \right\rangle_{L^2([0,\mu]; Y)}, \quad \forall z \in D(A).
\]
Owing to the arbitrariness of \(f \in D(G^*_\mu)\), (6.10) implies that the following Sylvester equation holds in \([D(G^*_\mu)]'\):
\[
\tilde{G}_\mu \Psi_\mu z - \Psi_\mu A z = B_\mu C_A z, \quad \forall z \in D(A).
\]

Suppose that \((A,F_1,C)\) is a linear system with the state space \(Z\), input space \(Y\) and the output space \(Y\). We define a subspace of \(Z\) by
\[
Z_{F_1} = \left\{ z \in Z \mid \tilde{A}z + F_1 y \in Z, y \in Y \right\}.
\]
As in [24, Section 2.2], \(Z_{F_1}\) with inner product
\[
\|z\|^2_{Z_{F_1}} = \|z\|^2_Z + \|y_z\|^2_Y + \|\tilde{A}z + F_1 y_z\|^2_Z
\]
is a Hilbert space, where \(y_z \in Y\) such that \(\tilde{A}z + F_1 y_z \in Z\).

**Lemma 6.1.** Let \(Z\) and \(Y\) be Hilbert spaces. Suppose that \(A\) generates a \(C_0\)-semigroup \(e^{At}\) on \(Z\), \(C \in \mathcal{L}(D(A), Y)\) is admissible for \(e^{At}\) and \(F_1 \in \mathcal{L}(Y, [D(A^*)]'\). Then, \(Z_{F_1}\) defined by (6.12) satisfies
\[
Z_{F_1} = D(A) + (\lambda - \tilde{A})^{-1} F_1 Y, \quad \lambda \in \rho(A).
\]

Suppose that \(Z_{F_1} \subset D(C)\) and \(G_\mu\), \(B_\mu\) and \(C_\mu\) are defined by (3.2), (3.6) and (3.7) with \(\alpha = \mu\) and \(U = Y\), respectively. Define the operator \(P_\mu : (Z + F_1 Y) \subset [D(A^*)]' \to [D(G^*_\mu)]'\) by
\[
P_\mu z = \left[ B_\mu C_A + (\lambda - G_\mu) \Psi_\mu \right] (\lambda - \tilde{A})^{-1} z, \quad \forall z \in (Z + F_1 Y),
\]
where $\lambda \in \rho(A)$. Then, the following assertions hold true:

(i) $P_{\mu}$ is independent of $\lambda$ and is an extension of $\Psi_{\mu}$, i.e.,

$$P_{\mu}z = \Psi_{\mu}z, \ \forall \ z \in Z; \quad (6.16)$$

(ii) $P_{\mu}$ is satisfied by the following Sylvester equation on $Z_{F_1}$:

$$\tilde{G}_{\mu}P_{\mu}z - P_{\mu}\tilde{A}z = B_{\mu}C_{\Lambda}z, \ \forall \ z \in Z_{F_1}; \quad (6.17)$$

(iii) $P_{\mu}$ and $C_{\mu\Lambda}$ satisfy:

$$C_{\mu\Lambda}P_{\mu} = -C_{\lambda}e^{-\lambda\mu} \in \mathcal{L}(D(A), Y). \quad (6.18)$$

Proof. For any $z \in Z_{F_1} \subset Z$, there exists a $y \in Y$ such that $\tilde{A}z + F_1y \in Z$ and hence $(\lambda - \tilde{A})z - F_1y \in Z$ for any $\lambda \in \rho(A)$. As a result, $z - (\lambda - \tilde{A})^{-1}F_1y \in D(A)$ and $z \in D(A) + (\lambda - \tilde{A})^{-1}F_1Y$. So $Z_{F_1} \subset D(A) + (\lambda - \tilde{A})^{-1}F_1Y$. For any $z = z_1 + (\lambda - \tilde{A})^{-1}F_1yz \in D(A) + (\lambda - \tilde{A})^{-1}F_1Y$, where $z_1 \in D(A)$ and $y_z \in Y$, a simple computation shows that $(\lambda - \tilde{A})z + F_1(-yz) = (\lambda - \tilde{A})z_1 \in Z$ and hence $\tilde{A}z + F_1yz \in Z$. By (6.12), $z \in Z_{F_1}$ and hence $D(A) + (\lambda - \tilde{A})^{-1}F_1Y \subset Z_{F_1}$. Therefore, (6.14) holds.

Proof of (i). Since $(\lambda - \tilde{A})^{-1}z = (\lambda - A)^{-1}z \in D(A)$ for any $z \in Z$, it follows from (6.11) that

$$P_{\mu}z = \left[ B_{\mu}C_{\lambda} + (\lambda - \tilde{G}_{\mu})\Psi_{\mu} \right] (\lambda - A)^{-1}z$$

$$= -\Psi_{\mu}A(\lambda - A)^{-1}z + \lambda\Psi_{\mu}(\lambda - A)^{-1}z \quad (6.19)$$

$$= \Psi_{\mu}(\lambda - A)(\lambda - A)^{-1}z = \Psi_{\mu}z, \ \forall \ z \in Z.$$  Hence, $P_{\mu}$ is an extension of $\Psi_{\mu}$.

Proof of (ii). For any $z_{F_1} \in Z_{F_1}$, by (6.14), there exist $z \in D(A)$ and $y \in Y$ such that $z_{F_1} = z + (\lambda - \tilde{A})^{-1}F_1y$ for some $\lambda \in \rho(A)$. Thanks to (6.11) and (6.16), it suffices to prove

$$\tilde{G}_{\mu}P_{\mu}[(-\tilde{A})^{-1}F_1y] - B_{\mu}C_{\lambda}[(-\tilde{A})^{-1}F_1y] = P_{\mu}\tilde{A}[(-\tilde{A})^{-1}F_1y]. \quad (6.20)$$

Actually, it follows (6.15) and (6.16) that

$$P_{\mu}F_1y = B_{\mu}C_{\lambda}(\lambda - \tilde{A})^{-1}F_1y + \lambda P_{\mu}(\lambda - \tilde{A})^{-1}F_1y - \tilde{G}_{\mu}P_{\mu}(\lambda - \tilde{A})^{-1}F_1y, \quad (6.21)$$

which yields

$$\tilde{G}_{\mu}P_{\mu}[(-\tilde{A})^{-1}F_1y] - B_{\mu}C_{\lambda}[(-\tilde{A})^{-1}F_1y] = -P_{\mu}F_1y + \lambda P_{\mu}(\lambda - \tilde{A})^{-1}F_1y$$

$$= -P_{\mu}(\lambda - \tilde{A})(\lambda - A)^{-1}F_1y + \lambda P_{\mu}(\lambda - \tilde{A})^{-1}F_1y = P_{\mu}\tilde{A}[(-\tilde{A})^{-1}F_1y]. \quad (6.22)$$

Hence, (6.17) can be obtained by (6.11), (6.22) and the fact $z_{F_1} = z + (\lambda - \tilde{A})^{-1}F_1y$ easily.

Proof of (iii). It follows from (6.11) that

$$\tilde{G}_{\mu}\Psi_{\mu}z - B_{\mu}C_{\lambda}z = \Psi_{\mu}Az \in L^2([0, \mu]; Y), \ \forall \ z \in D(A), \quad (6.23)$$

which, together with (3.12), (3.15) and (6.16), yields $P_{\mu}z = \Psi_{\mu}z \in G_{B_{\mu}} = H^1([0, \mu]; Y) = D(C_{\mu})$, where $G_{B_{\mu}}$ is defined by (3.12) with $\alpha = \mu$. Owing to (6.9) and (3.7), we arrive at $C_{\mu\Lambda}P_{\mu}z = C_{\mu\Lambda}\Psi_{\mu}z = -C_{\lambda}e^{-\lambda\mu}z \in Y$. So (6.18) holds. The proof is complete. \qed
Remark 6.1. We claim that $P_\mu$ is independent of the choice of $\lambda$. So the notation $P_\mu$ that is absent of $\lambda$ does not cause any confusion. Indeed, for any $\lambda_1, \lambda_2 \in \rho(A)$ and $\lambda_1 \neq \lambda_2$, a simple computation shows that

$$B_\mu C_A \left[ (\lambda_1 - \tilde{A})^{-1} - (\lambda_2 - \tilde{A})^{-1} \right] = (\lambda_2 - \lambda_1)B_\mu C_A(\lambda_1 - \tilde{A})^{-1}(\lambda_2 - \tilde{A})^{-1} \quad (6.24)$$

and

$$-\tilde{G}_\mu \Psi_\mu \left[ (\lambda_1 - \tilde{A})^{-1} - (\lambda_2 - \tilde{A})^{-1} \right] = -(\lambda_2 - \lambda_1)\tilde{G}_\mu \Psi_\mu(\lambda_1 - \tilde{A})^{-1}(\lambda_2 - \tilde{A})^{-1}. \quad (6.25)$$

Notice that $(\lambda_1 - \tilde{A})^{-1}(\lambda_2 - \tilde{A})^{-1}z \in D(A)$ for any $z \in Z + F_1 Y \subset [D(A^*)]'$, it follows from (6.11) that

$$\tilde{G}_\mu \Psi_\mu \tilde{z} - \Psi_\mu \Lambda z = B_\mu C_A \tilde{z}, \quad \tilde{z} = (\lambda_1 - \tilde{A})^{-1}(\lambda_2 - \tilde{A})^{-1}z \in D(A). \quad (6.26)$$

Combining (6.11), (6.16), (6.24), (6.25) and (6.26), for any $z \in Z + F_1 Y$, we obtain

$$\left[ B_\mu C_A + (\lambda_1 - \tilde{G}_\mu)\Psi_\mu \right](\lambda_1 - \tilde{A})^{-1}z - \left[ B_\mu C_A + (\lambda_2 - \tilde{G}_\mu)\Psi_\mu \right](\lambda_2 - \tilde{A})^{-1}z$$

$$= (\lambda_2 - \lambda_1) \left[ B_\mu C_A \tilde{z} - \tilde{G}_\mu \Psi_\mu \tilde{z} \right] + \lambda_1 \Psi_\mu (\lambda_1 - \tilde{A})^{-1}z - \lambda_2 \Psi_\mu (\lambda_2 - \tilde{A})^{-1}z$$

$$= -(\lambda_2 - \lambda_1)\Psi_\mu A\tilde{z} + \lambda_1 \Psi_\mu (\lambda_1 - \tilde{A})^{-1}z - \lambda_2 \Psi_\mu (\lambda_2 - \tilde{A})^{-1}z \quad (6.27)$$

$$= -P_\mu \tilde{A} \left[ (\lambda_1 - \tilde{A})^{-1}z - (\lambda_2 - \tilde{A})^{-1}z \right] + \lambda_1 P_\mu (\lambda_1 - \tilde{A})^{-1}z - \lambda_2 P_\mu (\lambda_2 - \tilde{A})^{-1}z$$

$$= -P_\mu (\lambda_1 - \tilde{A})(\lambda_1 - \tilde{A})^{-1}z - P_\mu (\lambda_2 - \tilde{A})(\lambda_2 - \tilde{A})^{-1}z = P_\mu z - P_\mu z = 0.$$ 

Therefore, $P_\mu$ is independent of the choice of $\lambda$.

Lemma 6.2. Suppose that $A$ is the generator of the $C_0$-semigroup $e^{At}$ acting on $Z$ and $C \in \mathcal{L}(D(A), Y)$ is admissible for $e^{At}$. Then for any $\mu > 0$, $F \in \mathcal{L}(Y, [D(A^*)]')$ detects system $(A, C)$ exponentially if and only if $e^{\tilde{A}\mu}F$ detects system $(A, C_A e^{-A\mu})$ exponentially.

Proof. By Lemma 9.1 in Appendix, both $C_A e^{-A\mu}$ and $e^{\tilde{A}\mu}F$ are admissible for $e^{At}$. Similarly to (4.14), a simple computation shows that

$$C_A e^{-A\mu}(\lambda - \tilde{A})^{-1}e^{\tilde{A}\mu}F = C_A (\lambda - \tilde{A})^{-1}F, \quad \forall \lambda \in \rho(A), \quad (6.28)$$

which implies that $(A, F, C)$ is a regular triple if and only if $(A, e^{\tilde{A}\mu}F, C_A e^{-A\mu})$ is a regular triple. Moreover, $I$ is an admissible feedback operator for $C_A(s - \tilde{A})^{-1}F$ is equivalent to that $I$ is an admissible feedback operator for $C_A e^{-A\mu}(\lambda - \tilde{A})^{-1}e^{\tilde{A}\mu}F$. Since $e^{\tilde{A}\mu} \in \mathcal{L}(Z)$ and

$$A + e^{\tilde{A}\mu}FC_A e^{-A\mu} = A + e^{\tilde{A}\mu}FC_A e^{\tilde{A}\mu} = A + FC_A, \quad (6.29)$$

$A + FC_A$ is exponentially stable if and only if $A + e^{-\tilde{A}\mu}FC_A e^{\tilde{A}\mu}$ is exponentially stable. The proof is complete due to Definition 2.

\[ \Box \]
7 Luenberger-like observer

In this section, we will design the observer for system (6.3) and prove the well-posedness. We begin with the following infinite-dimensional Luenberger-like observer:

\[
\begin{align*}
\dot{z}(t) &= A\tilde{z}(t) - F_1[y(t) - C_\mu \hat{\psi}(\cdot, t)] + Bu(t), \\
\dot{\hat{\psi}}(\cdot, t) &= G_\mu \hat{\psi}(\cdot, t) + B_\mu C_\Lambda \hat{z}(t) - F_2[y(t) - C_\mu \hat{\psi}(\cdot, t)],
\end{align*}
\]

where \( F_1 \in \mathcal{L}(Y, [D(A^*)]') \) and \( F_2 \in \mathcal{L}(Y, [D(G_\mu^*)]') \) are tuning operators to be determined. If we let the errors be

\[
\tilde{z}(t) = z(t) - \hat{z}(t), \quad \tilde{\psi}(\cdot, t) = \psi(\cdot, t) - \hat{\psi}(\cdot, t),
\]

then they are governed by

\[
\begin{align*}
\dot{z}(t) &= A\tilde{z}(t) + F_1 C_\mu A \tilde{\psi}(\cdot, t), \\
\dot{\hat{\psi}}(\cdot, t) &= (G_\mu + F_2 C_\mu A)\tilde{\psi}(\cdot, t) + B_\mu C_\Lambda \tilde{z}(t).
\end{align*}
\]

Similarly to (5.6) and (5.7), if \( Z_{F_1} \subset D(C_\Lambda) \), we can define the transformation

\[
\mathbb{P}(z, f)^\top = (z, f + P_\mu z)^\top, \quad \forall (z, f)^\top \in \mathcal{Z}_\mu(Y),
\]

where the operator \( P_\mu \) is given by (6.15). It is easy to see that \( \mathbb{P} \in \mathcal{L}(\mathcal{Z}_\mu(Y)) \) is invertible and its inverse is given by

\[
\mathbb{P}^{-1}(z, f)^\top = (z, f - P_\mu z)^\top, \quad \forall (z, f)^\top \in \mathcal{Z}_\mu(Y).
\]

Let

\[
(\tilde{z}(t), \tilde{\psi}(\cdot, t))^\top = \mathbb{P}(\tilde{z}(t), \tilde{\psi}(\cdot, t))^\top, \quad t \geq 0.
\]

By (6.17), the transformation (7.6) can convert system (7.3) into

\[
\begin{align*}
\dot{\tilde{z}}(t) &= (A - F_1 C_\mu A P_\mu)\tilde{z}(t) + F_1 C_\mu A \tilde{\psi}(\cdot, t), \\
\dot{\tilde{\psi}}(\cdot, t) &= (G_\mu + F_2 C_\mu A P_\mu + P_\mu F_1 C_\mu A P_\mu)\tilde{\psi}(\cdot, t) - (P_\mu F_1 C_\mu A P_\mu + F_2 C_\mu A P_\mu)\tilde{z}(t),
\end{align*}
\]

provided \( \tilde{z}(t) \in Z_{F_1} \). Choosing specially \( F_2 = -P_\mu F_1 \), system (7.7) is reduced to

\[
\begin{align*}
\dot{\tilde{z}}(t) &= (A - F_1 C_\mu A P_\mu)\tilde{z}(t) + F_1 C_\mu A \tilde{\psi}(\cdot, t), \\
\dot{\tilde{\psi}}(\cdot, t) &= \tilde{G}_\mu \tilde{\psi}(\cdot, t),
\end{align*}
\]

which is a simple cascade system and can be written as

\[
\frac{d}{dt} (\tilde{z}(t), \tilde{\psi}(\cdot, t))^\top = \mathcal{A}_P (\tilde{z}(t), \tilde{\psi}(\cdot, t))^\top,
\]

where

\[
\mathcal{A}_P = \begin{pmatrix}
\tilde{A} - F_1 C_\mu A P_\mu & F_1 C_\mu A \\
0 & \tilde{G}_\mu
\end{pmatrix},
\]

\[
D(\mathcal{A}_P) = \left\{ \begin{pmatrix} z \\ \psi \end{pmatrix} \in \mathcal{Z}_\mu(Y) \mid (\tilde{A} - F_1 C_\mu A P_\mu)z + F_1 C_\mu A \psi \in Z, \quad \tilde{G}_\mu \psi \in L^2([0, \mu]; Y) \right\}.
\]
With the setting \( F_2 = -P_\mu F_1 \), the observer (7.1) is reduced to be
\[
\begin{cases}
\dot{z}(t) = \dot{A}z(t) - F_1[y(t) - C_\mu A\psi(\cdot, t)] + Bu(t), \\
\dot{\psi}_t(\cdot, t) = \dot{G}_\mu \psi(\cdot, t) + B_\mu C_\Lambda \dot{z}(t) + P_\mu F_1[y(t) - C_\mu A\psi(\cdot, t)].
\end{cases}
\tag{7.11}
\]
System (7.11) can be written as
\[
\frac{d}{dt}(\dot{z}(t), \dot{\psi}(\cdot, t))^\top = \mathcal{A}(\dot{z}(t), \dot{\psi}(\cdot, t))^\top + \mathcal{F}y(t) + (B, 0)^\top u(t),
\tag{7.12}
\]
where
\[
\mathcal{A} = \begin{pmatrix} \dot{A} & F_1 C_\mu A \\ B_\mu C_\Lambda & \dot{G}_\mu - P_\mu F_1 C_\mu A \end{pmatrix}
\quad \text{and} \quad
\mathcal{F} = \begin{pmatrix} -F_1 \\ P_\mu F_1 \end{pmatrix}
\tag{7.13}
\]
with
\[
D(\mathcal{A}) = \left\{ \begin{pmatrix} z \\ \psi \end{pmatrix} \in Z_\mu(Y) \mid \begin{array}{l}
\dot{A}z + F_1 C_\mu A \psi \in Z \\
B_\mu C_\Lambda z + (\dot{G}_\mu - P_\mu F_1 C_\mu A) \psi \in L^2([0, \mu]; Y)
\end{array} \right\}.
\tag{7.14}
\]

**Lemma 7.1.** Let \( Z \) and \( Y \) be Hilbert spaces. Suppose that \( A \) is a generator of the \( C_0 \)-semigroup \( e^{At} \) acting on \( Z \), \( C \in \mathcal{L}(D(A), Y) \) is admissible for \( e^{At} \), \( F_1 \in \mathcal{L}(Y, [D(A^*)]^*) \) and \( Z_{F_1} \) defined by (6.12) satisfies \( Z_{F_1} \subset D(C_\Lambda) \). Suppose that \( G_\mu, B_\mu \) and \( C_\mu \) are defined by (3.2), (3.6) and (3.7) with \( \alpha = \mu \), respectively. Let \( \mathcal{A} \) and \( \mathcal{A}_\mu \) be given by (7.13) and (7.10), respectively. Then,
\[
P_\mathcal{A}P^{-1} = \mathcal{A}_\mu \quad \text{and} \quad D(\mathcal{A}_\mu) = P D(\mathcal{A}),
\tag{7.15}
\]
where \( P \) is given by (7.4).

**Proof.** By Lemma 6.1, the operator \( P_\mu \) is well defined via (6.15). For any \((z, \psi)^\top \in D(\mathcal{A}_\mu)\), it follows from (7.10) and (6.12) that \( z \in Z_{F_1} \) and
\[
\dot{A}z + F_1 C_\mu A(\psi - P_\mu z) \in Z.
\tag{7.16}
\]
By (7.10), (6.16), (6.17) and (6.9), it follows that
\[
B_\mu C_\Lambda z + (\dot{G}_\mu - P_\mu F_1 C_\mu A)(\psi - P_\mu z) = -P_\mu \dot{A}z + \dot{G}_\mu \psi - P_\mu F_1 C_\mu A(\psi - P_\mu z)
\]
\[
= \dot{G}_\mu \psi - P_\mu \left[ \dot{A}z + F_1 C_\mu A(\psi - P_\mu z) \right]
\tag{7.17}
\]
\[
= \dot{G}_\mu \psi - \Psi_\mu \left[ \dot{A}z + F_1 C_\mu A(\psi - P_\mu z) \right] \in L^2([0, \mu]; Y).
\]
We combine (7.14), (7.16) and (7.17) to get \( P^{-1}(z, \psi)^\top \in D(\mathcal{A}) \). Consequently, \( D(\mathcal{A}_\mu) \subset P D(\mathcal{A}) \) due to the arbitrariness of \((z, \psi)^\top \in D(\mathcal{A}_\mu)\).

On the other hand, for any \((z, \psi)^\top \in D(\mathcal{A}), (7.14) \) and (6.12) imply that \( z \in Z_{F_1} \). Furthermore, it follows from (6.17) that
\[
\dot{G}_\mu (\psi + P_\mu z) = \dot{G}_\mu \psi + P_\mu \dot{A}z + B_\mu C_\Lambda z
\]
\[
= \dot{G}_\mu \psi - P_\mu F_1 C_\mu A \psi + B_\mu C_\Lambda z + P_\mu (\dot{A}z + F_1 C_\mu A \psi),
\tag{7.18}
\]

which, together with (7.14), (6.9) and (6.16), leads to
\[ \tilde{G}_\mu(\psi + P_\mu z) \in L^2([0, \mu]; Y). \]  
(7.19)
This implies that \((\psi + P_\mu z) \in D(C_{\mu A}). \) Since \(\psi \in D(C_{\mu A})\), we have \(P_\mu z \in D(C_{\mu A})\). As a result, it follows from (7.14) that
\[ (\tilde{A} - F_1 C_{\mu A} P_\mu) z + F_1 C_{\mu A}(\psi + P_\mu z) = \tilde{A} z + F_1 C_{\mu A} \psi \in Z. \]  
(7.20)
Combining (7.10), (7.19) and (7.20), we arrive at \(\mathbb{P}(z, \psi)^\top \in D(\mathcal{A}_F)\) and hence \(\mathbb{P}D(\mathcal{A}) \subset D(\mathcal{A}_F)\).

To sum up, we thus obtain \(\mathbb{P}D(\mathcal{A}) = D(\mathcal{A}_F)\). For any \((z, \psi)^\top \in D(\mathcal{A}_F)\), it follows from (7.10) and (6.12) that \(z \in Z_{F_1}\). By virtue of (6.17), a simple computation shows that \(\mathbb{P}_F \mathbb{P}^{-1}(z, \psi)^\top = \mathcal{A}_F(z, \psi)^\top \) for any \((z, \psi)^\top \in D(\mathcal{A}_F)\). Hence, the similarity (7.15) holds. The proof is complete.

By Lemma 7.1, the observer (7.11) is convergent provided \(\mathcal{A}_F\) is stable. Owing to the upper-block-triangular structure of \(\mathcal{A}_F\) and since \(\tilde{G}_\mu\) is exponentially stable already, we only need to choose \(F_1\) such that \(\tilde{A} - F_1 C_{\mu A} P_\mu\) stable. By Lemma 6.2 and (6.18), we can choose \(F_1 \in \mathcal{L}[Y, [D(A^*)]'\) by the following scheme:

\[
\begin{aligned}
\{ 
& (i) \quad \text{choose } F \in \mathcal{L}(Y, [D(A^*)]') \text{ to detects } (A, C) \text{ exponentially;} \\
& (ii) \quad \text{let } F_1 = e^{\tilde{A}_\mu} F.
\end{aligned}
\]  
(7.21)
Under (7.21), the observer (7.11) is found to be
\[
\begin{align*}
\dot{z}(t) &= \tilde{A} \hat{z}(t) - e^{\tilde{A}_\mu} F[y(t) - C_{\mu A} \hat{\psi}(\cdot, t)] + Bu(t), \\
\dot{\psi}(t, \cdot) &= \tilde{G}_\mu \hat{\psi}(\cdot, \cdot) + B_\mu C_\Lambda \hat{z}(t) + P_\mu e^{\tilde{A}_\mu} F[y(t) - C_{\mu A} \hat{\psi}(\cdot, t)],
\end{align*}
\]  
(7.22)
or equivalently,
\[
\begin{align*}
\dot{z}(t) &= A \hat{z}(t) - e^{\tilde{A}_\mu} F[y(t) - \hat{\psi}(\mu, t)] + Bu(t), \\
\dot{\psi}(t, x, \cdot) &= P_\mu e^{\tilde{A}_\mu} F[y(t) - \hat{\psi}(\mu, t)], \\
\dot{\psi}(0, t) &= C_\Lambda \hat{z}(t),
\end{align*}
\]  
(7.23)
where \(P_\mu\) is given by (6.15) and \(G_\mu, B_\mu, \text{ and } C_\mu\) are defined by (3.2), (3.6) and (3.7) with \(\alpha = \mu\), respectively.

**Theorem 7.1.** Let \((A, B, C)\) be a regular linear system with the state space \(Z\), input space \(U\) and output space \(Y\). Suppose that \(\mu > 0, F \in \mathcal{L}(Y, [D(A^*)]'\) detects system \((A, C)\) exponentially and
\[ (s - \tilde{A})^{-1} e^{\tilde{A}_\mu} F \subset D(C_{\mu A}) \text{ for some } s \in \rho(A). \]  
(7.24)
Then, the observer (7.23) of system (6.2) is well-posed: For any \((\hat{z}(0), \hat{\psi}(\cdot, 0))^\top \in Z_\mu(Y)\) and \(u \in L^2_{\text{loc}}([0, \infty); U)\), the observer (7.23) admits a unique solution \((\hat{z}, \hat{\psi})^\top \in C([0, \infty); Z_\mu(Y))\) such that
\[ e^{\omega t} \| (\hat{z}(t) - \hat{z}(t), \psi(\cdot, t) - \hat{\psi}(\cdot, t))^\top \|_{Z_\mu(Y)} \to 0 \text{ as } t \to \infty, \]  
(7.25)
where \(\omega\) is a positive constant that is independent of \(t\).
Proof. Let $F_1 = e^{\hat{A}t}F$. Then (7.24) implies that $Z_{F_1} \subset D(C_A)$, where $Z_{F_1}$ is defined by (6.12). By Lemma 6.1, the operator $P_\mu$ in (7.23) is well defined.

Since $F$ detects system $(A,C)$ exponentially, it follows from Lemma 6.2 that $e^{\hat{A}t}F$ detects $(A,C_Ae^{-\Lambda t})$ exponentially. As a result, the operator $\tilde{A} - F_1C_{\Lambda}P_\mu$ generates an exponentially stable $C_0$-semigroup $e^{(\tilde{A} - F_1C_{\Lambda}P_\mu)t}$ on $Z$ and moreover, $F_1$ is admissible for $e^{(\tilde{A} - F_1C_{\Lambda}P_\mu)t}$. Since $\tilde{G}_\mu$ is exponentially stable already and $C_\mu$ is admissible for $e^{\tilde{G}_\mu t}$, it follows from Lemma 7.1 of Appendix that the operator $\mathcal{A}_\mu$ defined by (7.10) generates an exponentially stable $C_0$-semigroup $e^{\mathcal{A}_\mu t}$ on $Z_\mu(Y)$. By Lemma 7.1, $\mathcal{A}$ and $\mathcal{A}_\mu$ are similar each other. Therefore, the operator $\mathcal{A}$ defined by (7.13) generates an exponentially stable $C_0$-semigroup $e^{\mathcal{A}t}$ on $Z_\mu(Y)$. As a result, the following system

$$
\begin{align*}
\dot{z}(t) &= A\hat{z}(t) + e^{\hat{A}t}FC_{\Lambda}\psi(\cdot,t), \\
\dot{\psi}(t) &= (G_\mu - P_\mu e^{\hat{A}t}FC_{\Lambda})\psi(\cdot,t) + B_\mu C_{\Lambda}\hat{z}(t)
\end{align*}
$$

with initial state

$$
\begin{align*}
\hat{z}(0) &= z(0) - \hat{z}(0), \\
\psi(\cdot,0) &= \psi(\cdot,0) - \hat{\psi}(\cdot,0)
\end{align*}
$$

admits a unique solution $(\hat{z},\hat{\psi})^\top \in C([0,\infty);Z_\mu(Y))$ that decays exponentially to zero in $Z_\mu(Y)$ as $t \to \infty$.

By Theorem 6.1, system (6.3) admits a unique solution $(z,\psi)^\top \in C([0,\infty);Z_\mu(Y))$ for any $(z(0),\psi(\cdot,0))^\top \in Z_\mu(Y)$ and $u \in L^2_{loc}([0,\infty);U)$. Let $\hat{z}(t) = z(t) - \hat{z}(t)$ and $\hat{\psi}(t) = \psi(t) - \hat{\psi}(t)$. Then, such a defined $(\hat{z}(t),\hat{\psi}(t))^\top$ is a solution of observer (7.23). Owing to the linearity of (7.23), the solution is unique. Since system (7.26) happens to be the error system between system (6.2) and its observer (7.23), the convergence (7.25) holds. The proof is complete.

The assumption (7.24) seems a bit awkward. When $F$ or $C$ is a bounded operator, this assumption can be deduced from that $F \in \mathcal{L}(Y,[D(A^*)]')$ detects system $(A,C)$ exponentially. When $F$ and $C$ are unbounded, we have the following Corollary:

**Corollary 7.1.** Let $(A,B,C)$ be a regular linear system with the state space $Z$, input space $U$ and output space $Y$. Suppose that $F \in \mathcal{L}(Y,[D(A^*)]')$ detects system $(A,C)$ exponentially. Then, the observer (7.23) of system (6.2) is well-posed for almost every $\mu > 0$. That is, for any $(\hat{z}(0),\hat{\psi}(\cdot,0))^\top \in Z_\mu(Y)$ and $u \in L^2_{loc}([0,\infty);U)$, the observer (7.23) admits a unique solution $(\hat{z},\hat{\psi})^\top \in C([0,\infty);Z_\mu(Y))$ such that (7.25) holds for some positive constant $\omega$.

**Proof.** Since $(A,F,C)$ is a regular linear system, $C$ is admissible for $e^{At}$. This implies that $e^{At}C \subset D(C_A)$ for almost every $\mu > 0$. Consequently,

$$
(s - \hat{A})^{-1}e^{\hat{A}t}FY = e^{At}(s - A)^{-1}FY \subset e^{At}Z \subset D(C_A), \quad \forall \ s \in \rho(A),
$$

which, together with Theorem 7.1, completes the proof. 

\[\square\]
Remark 7.1. When $A$ is a matrix, it follows from (6.9) and (6.16) that the observer (7.23) takes form

$$
\begin{align*}
\dot{\hat{z}}(t) &= A\hat{z}(t) - e^{A\mu}F[y(t) - \hat{\psi}(\mu, t)] + Bu(t), \\
\hat{\psi}_1(x, t) + \hat{\psi}_2(x, t) &= -Ce^{A(\mu-x)}F[y(t) - \hat{\psi}(\mu, t)], \\
\hat{\psi}(0, t) &= C\hat{z}(t),
\end{align*}
$$

(7.29)

which is the same as the observer in [12]. In contrast to the PDE backstepping method used in [12], we never need the target system. Moreover, the Lyapunov function has not used in the proof of observer convergence. Once again as Remark 5.1, this gives a way to avoid the difficulties in construction of the Lyapunov functional for PDEs with delay.

8 Application to 1-D wave equation

To show the effectiveness of the developed approach, we apply the abstract results to the benchmark wave equation:

$$
\begin{align*}
\begin{cases}
z_{tt}(\sigma, t) &= z_{\sigma\sigma}(\sigma, t), \quad \sigma \in (0, 1), \\
z(0, t) &= 0, \quad z(1, t) = u(t - \tau),
\end{cases}
\end{align*}
$$

(8.1)

where $u(t)$ is the control input which suffers from a time-delay $\tau > 0$. The input space is $\mathbb{R}$ and the state space is $Z = \{(f, g) \in H^1(0, 1) \times L^2(0, 1) \mid f(0) = 0\}$ with the inner product

$$
\langle (f_1, g_1), (f_2, g_2) \rangle_Z = \int_0^1 f_1'(x)f_2'(x) + g_1(x)g_2(x)dx, \quad \forall (f_i, g_i) \in Z, \quad i = 1, 2.
$$

(8.2)

Define the operator $A : D(A) \subset Z \to Z$ by

$$
\begin{align*}
A(f, g)^\top &= (g, f''(x))^\top, \quad \forall (f, g)^\top \in D(A), \\
D(A) &= \{(f, g) \in H^2(0, 1) \times H^1(0, 1) \mid f(0) = g(0) = 0, f'(1) = 0\}.
\end{align*}
$$

(8.3)

In view of (5.2), system (8.1) can be written as the form

$$
\begin{align*}
&\frac{d}{dt}(z(\cdot, t), z_t(\cdot, t))^\top = A(z(\cdot, t), z_t(\cdot, t))^\top + B\phi(\tau, t), \\
&\phi_t(x, t) + \phi_x(x, t) = 0, \quad x \in (0, \tau), \\
&\phi(0, t) = u(t),
\end{align*}
$$

(8.4)

where the control operator $B = (0, \delta(\cdot - 1))^\top$ and $\delta(\cdot)$ is the Dirac distribution. Let

$$
K(f, g)^\top = -k_1g(1) \quad \text{for any } (f, g)^\top \in D(A), \quad k_1 > 0.
$$

(8.5)

Then, $K = -k_1B^*$ and it is well known that $K$ stabilizes system $(A, B)$ exponentially. In view of (5.16), we obtain the feedback

$$
u(t) = K_\Lambda \int_0^t e^{A_x}B\phi(x, t)dx + K_\Lambda e^{A_x}(z(\cdot, t), z_t(\cdot, t))^\top.
$$

(8.6)
We next seek the analytic form of the feedback. Let \( z_\delta = (\sigma, 0)^T \) with \( \sigma \in [0, 1] \). A simple computation shows that \( \tilde{A}z_\delta = B \) and

\[
e^{Ax}z_\delta = \left( \sum_{n=0}^{\infty} \frac{(-1)^n 2 \cos \omega_n x \sin \omega_n \sigma}{\omega_n^2} + \sum_{n=0}^{\infty} \frac{(-1)^n+1 2 \sin \omega_n x \sin \omega_n \sigma}{\omega_n} \right)^T, \tag{8.7}
\]

where

\[
\omega_n = \frac{2n+1}{2}, \quad \sigma \in [0, 1], \ x \in [0, \tau], \ n = 0, 1, 2, \ldots.
\tag{8.8}
\]

Moreover, it follows from (8.7) that

\[
\int_0^\tau e^{Ax}z_\delta(x,t)dx = \left( \sum_{n=0}^{\infty} \frac{(-1)^n 2\alpha_n(t) \sin \omega_n \sigma}{\omega_n^2} + \sum_{n=0}^{\infty} (-1)^n+1 \frac{2\beta_n(t) \sin \omega_n \sigma}{\omega_n} \right)^T,
\tag{8.9}
\]

where

\[
\alpha_n(t) = \int_0^\tau \cos \omega_n x \phi(x,t)dx \quad \beta_n(t) = \int_0^\tau \sin \omega_n x \phi(x,t)dx.
\tag{8.10}
\]

Since \( B \) is admissible for \( e^{At} \) and \( \phi(\cdot, t) \in L^2(0, \tau) \), we have

\[
\int_0^\tau e^{Ax}B\phi(x,t)dx = \int_0^\tau e^{Ax}\tilde{A}\tilde{A}^{-1}B\phi(x,t)dx = \int_0^\tau e^{Ax}z_\delta\phi(x,t)dx = \int_0^\tau A e^{Ax}z_\delta\phi(x,t)dx = 0.
\tag{8.11}
\]

which implies that

\[
\int_0^\tau e^{Ax}z_\delta\phi(x,t)dx \in D(A).
\tag{8.12}
\]

Combining (8.3), (8.5), (8.12), (8.11), (8.9) and (8.10), we arrive at

\[
K_A \int_0^\tau e^{Ax}z_\delta\phi(x,t)dx = K_A e^{Ax}z_\delta\phi(x,t)dx = -2k1 \sum_{n=0}^{\infty} \alpha_n(t).
\tag{8.13}
\]

By a straightforward computation, we have

\[
K_A e^{Ax}(z(\cdot, t), z_t(\cdot, t))^T = -k1 \sum_{n=0}^{\infty} (-1)^n \omega_n [z_n(t) \cos \omega_n \tau - \gamma_n(t) \sin \omega_n \tau],
\tag{8.14}
\]

where

\[
\gamma_n(t) = 2 \int_0^1 z(\sigma, t) \sin \omega_n \sigma d\sigma, \quad \zeta_n(t) = \frac{2}{\omega_n} \int_0^1 (\sigma, t) \sin \omega_n \sigma d\sigma, \quad n = 0, 1, 2, \ldots.
\tag{8.15}
\]

By (8.6), (8.13) and (8.14), we get the closed-loop system

\[
\begin{cases}
  z_{tt}(\sigma, t) = z_{\sigma\sigma}(\sigma, t), \quad \sigma \in (0, 1), \\
  z(0, t) = 0, \quad z(1, t) = \phi(\tau, t), \\
  \phi_t(x, t) + \phi_x(x, t) = 0, \quad x \in (0, \tau), \\
  \phi(0, t) = -2k1 \sum_{n=0}^{\infty} \alpha_n(t) - k1 \sum_{n=0}^{\infty} (-1)^n \omega_n [z_n(t) \cos \omega_n \tau - \gamma_n(t) \sin \omega_n \tau],
\end{cases}
\tag{8.16}
\]

where \( k1 > 0, \alpha_n(t) \) is given by (8.10) and \( \zeta_n(t), \gamma_n(t) \) are given by (8.15). By Theorem 5.1, the solution of closed-loop system (8.16) is well posed and decays to zero exponentially as \( t \to \infty \).
Remark 8.1. The infinite series in the closed-loop system (8.16) can also be written as a dynamic form. Actually, a simple computation shows that

\[ v_1(\cdot, \tau; t) = \int_0^\tau e^{\lambda s} B\phi(s, t)ds, \]  

(8.17)

where

\[
\begin{align*}
  v_{1x}(\sigma, x; t) &= v_{1\sigma}(\sigma, x; t), \quad \sigma \in (0, 1), \quad 0 < x \leq \tau, \\
  v_1(0, x; t) &= 0, \quad v_1(1, x; t) = \phi(\tau - x, t), \\
  (v_1(\sigma, 0; t), v_{1x}(\sigma, 0; t)) &\equiv (0, 0), \quad \sigma \in [0, 1].
\end{align*}
\]

(8.18)

The notation \(v_1(\cdot, \cdot; t)\) means that the function \(v_1\) depends on the time \(t\). If we let

\[ v_2(\cdot, x; t) = e^{Ax}(z(\cdot, t), z_t(\cdot, t))^\top, \]

(8.19)

then it is governed by

\[
\begin{align*}
  v_{2x}(\sigma, x; t) &= v_{2\sigma}(\sigma, x; t), \quad \sigma \in (0, 1), \quad x > 0, \\
  v_2(0, x; t) &= v_2(1, x; t) = 0, \\
  (v_2(\sigma, 0; t), v_{2x}(\sigma, 0; t)) &= (z(\sigma, t), z_t(\sigma, t)), \quad \sigma \in [0, 1].
\end{align*}
\]

(8.20)

Combining (8.17), (8.18), (8.20), (8.6) and (8.5), we obtain the following closed-loop system:

\[
\begin{align*}
  z_{tt}(\sigma, t) &= z_{\sigma\sigma}(\sigma, t), \quad \sigma \in (0, 1), \\
  z(0, t) &= 0, \quad z_\sigma(1, t) = \phi(\tau, t), \\
  \phi_t(x, t) + \phi_x(x, t) &= 0, \quad x \in (0, \tau), \\
  \phi(0, t) &= -k_1v_{1x}(1, \tau, t) - k_1v_{2x}(1, \tau, t), \quad k_1 > 0, \\
  v_1(\cdot, \cdot; t), v_2(\cdot, \cdot; t) \text{ are given by (8.18) and (8.20).}
\end{align*}
\]

(8.21)

Now we consider the output delay compensation for the wave equation in (8.1). Suppose that we can measure the average velocity around \(\sigma_0 \in (0, 1)\) and the output is

\[ y(t) = \int_0^1 m(\sigma)z_\tau(\sigma, t - \mu)d\sigma, \quad \mu > 0, \]

(8.22)

where \(m \in L^2(0, 1)\) is the shaping function around the sensing point \(\sigma_0\). System (8.1) with output (8.22) can be written as

\[
\begin{align*}
  z_{tt}(\sigma, t) &= z_{\sigma\sigma}(\sigma, t), \quad \sigma \in (0, 1), \\
  z(0, t) &= 0, \quad z_\sigma(1, t) = u(t - \tau), \quad \tau \geq 0, \\
  \psi_t(x, t) + \psi_x(x, t) &= 0, \quad x \in [0, \mu], \quad \mu > 0, \\
  \psi(0, t) &= \int_0^1 m(s)z_\tau(s, t)ds, \\
  y(t) &= \psi(\mu, t).
\end{align*}
\]

(8.23)
The observation operator $C$ is given by

$$ C : (f, g) \rightarrow \int_{0}^{1} m(\sigma) g(\sigma) d\sigma, \quad (f, g) \in \mathbb{Z}. \quad (8.24) $$

It is evident that $C$ is bounded. We choose $m$ such that $(A, C)$ is exactly observable. If we let $F = -k_2 C^*$, $k_2 > 0$, then $F \in \mathcal{L}(\mathbb{R}, \mathbb{Z})$ is given by $F q = -k_2 q(0, m(t))$ for any $q \in \mathbb{R}$. Since $F$ detects system $(A, C)$ exponentially [17], by (7.23), the observer of system (8.23) is

$$
\begin{align*}
\frac{d}{dt}(\hat{z}(\cdot, t), \hat{\sigma}(\cdot, t))^T &= A(\hat{z}(\cdot, t), \hat{\sigma}(\cdot, t))^T - e^{A\mu} F[y(t) - \hat{\psi}(\mu, t)] + B u(t - \tau), \\
\hat{\psi}(x, t) + \hat{\psi}_x(x, t) &= P_\mu e^{A\mu} F[y(t) - \hat{\psi}(\mu, t)], \\
\hat{\psi}(0, t) &= \int_{0}^{1} m(\sigma) \hat{z}_t(\sigma, t) d\sigma.
\end{align*}
$$

(8.25)

Since

$$ e^{A \mu} F = e^{A \mu} F = -k_2 \left( \sum_{n=0}^{\infty} f_n \sin \omega_n x \sin \omega_n \sigma, \sum_{n=0}^{\infty} f_n \omega_n \cos \omega_n x \sin \omega_n \sigma \right)^T, \quad (8.26) $$

where $0 \leq x \leq \mu$, $0 \leq \sigma \leq 1$, $\omega_n$ is given by (8.8) and

$$ f_n = \frac{2}{\omega_n} \int_{0}^{1} m(\sigma) \sin \omega_n \sigma d\sigma, \quad n = 0, 1, 2, \ldots, \quad (8.27) $$

it follows from (6.16) and (6.9) that

$$ P_\mu e^{A \mu} F = -C e^{A(\mu - x)} F = k_2 \int_{0}^{1} \left( \sum_{n=0}^{\infty} f_n \omega_n \cos \omega_n (\mu - x) \sin \omega_n \sigma \right) m(\sigma) d\sigma \quad (8.28) $$

$$ = k_2 \sum_{n=0}^{\infty} \frac{f_n^2 \omega_n^2}{2} \cos \omega_n (\mu - x). $$

Since

$$ \sum_{n=0}^{\infty} \frac{f_n^2 \omega_n^2}{2} = 2 \sum_{n=0}^{\infty} \left| \int_{0}^{1} m(\sigma) \sin \omega_n \sigma d\sigma \right|^2 < +\infty, \quad (8.29) $$

the series in (8.28) is convergent. Combining (8.26) and (8.28), the observer (8.25) becomes

$$
\begin{align*}
\hat{z}_{1t}(\sigma, t) &= \hat{z}_{2}(\sigma, t) + k_2 \left( \sum_{n=0}^{\infty} f_n \sin \omega_n \mu \sin \omega_n \sigma \right) [y(t) - \hat{\psi}(\mu, t)], \\
\hat{z}_{2t}(\sigma, t) &= \hat{z}_{1\sigma}(\sigma, t) + k_2 \left( \sum_{n=0}^{\infty} f_n \omega_n \cos \omega_n \mu \sin \omega_n \sigma \right) [y(t) - \hat{\psi}(\mu, t)] + u(t - \tau), \\
\hat{\psi}(x, t) + \hat{\psi}_x(x, t) &= k_2 \left[ \sum_{n=0}^{\infty} \frac{f_n^2 \omega_n^2}{2} \cos \omega_n (\mu - x) \right] [y(t) - \hat{\psi}(\mu, t)], \\
\hat{\psi}(0, t) &= \int_{0}^{1} m(\sigma) \hat{z}_t(\sigma, t) d\sigma,
\end{align*}
$$

(8.30)

where $k_2 > 0$, $\hat{z}_{1}(\sigma, t) = \hat{z}(\sigma, t)$ and $\hat{z}_{2}(\sigma, t) = \hat{z}_t(\sigma, t)$ for any $\sigma \in [0, 1]$ and $t \geq 0$. By Theorem 7.1, $(\hat{z}_{1}(\cdot, t), \hat{z}_{2}(\cdot, t))$ converges to $(z(\cdot, t), z_t(\cdot, t))$ exponentially in $\mathbb{Z}$ as $t \rightarrow \infty$. 

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Remark 8.2. By our abstract theory, the proposed approach is still working for the delayed boundary output $y(t) = z(1, t - \mu)$. In this case, we can choose $F = -k_2(0, \delta(\cdot - 1))^\top$ with $k_2 > 0$. However, since $F$ is unbounded now, it is not easy to obtain the analytic forms of the gain operators $e^{\hat{A}_\mu}F$ and $P_\mu e^{\hat{A}_\mu}F$ in the state space. Therefore, a further effort is still needed for the observer design of infinite-dimensional systems with unbounded delayed output.

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multiples of the wave propagation time, *SIAM J.Control Optim.*, 49(2011), 517-554.
Lemma 9.1. Suppose that \((A, B, C)\) is a linear system with the state space \(Z\), input space \(U\) and the output space \(Y\). For any \(0 \neq q \in \mathbb{R}\), the following assertions are true:

(i). If \(C \in \mathcal{L}(D(A), Y)\) is admissible for \(e^{At}\), then \(Ce^{Aq} \in \mathcal{L}(D(A), Y)\) is admissible for \(e^{At}\) as well;

(ii). If \(B \in \mathcal{L}(U, [D(A)]')\) is admissible for \(e^{At}\), then \(e^{\tilde{A}q}B \in \mathcal{L}(U, [D(A)]')\) is admissible for \(e^{At}\) as well.

Proof. For any \(z_0 \in D(A)\) and \(\tau > 0\), we have

\[
\int_{0}^{\tau} \|Ce^{Aq}e^{At}z_0\|^2_Y dt = \int_{q}^{q+\tau} \|Ce^{As}z_0\|^2_Y ds \leq \int_{0}^{q+\tau} \|Ce^{As}z_0\|^2_Y ds. \tag{9.1}
\]

Since \(C \in \mathcal{L}(D(A), Y)\) is admissible for \(e^{At}\), there exists an \(M > 0\) such that

\[
\int_{0}^{\tau+q} \|Ce^{As}z_0\|^2_Y ds \leq M \|z_0\|^2_Z, \tag{9.2}
\]

which, together with (9.1), shows that \(Ce^{Aq} \in \mathcal{L}(D(A), Y)\) is admissible for \(e^{At}\).

Since \(B\) is admissible for \(e^{At}\), \(B^*\) is admissible for \(e^{A*t}\). By (i) just proved, \(B^* e^{A*t-q}\) is admissible for \(e^{A*t}\). Hence, \(e^{\tilde{A}q}B\) is admissible for \(e^{At}\). This completes the proof. \(\square\)

Lemma 9.2. Let \((A, C)\) be a linear system with the state space \(Z\) and output space \(U\). Suppose that \(C\) is admissible for \(e^{At}\). Let \(G_\alpha\) and \(B_\alpha\) be given by (3.2) and (3.6), respectively. Define \(Z_\tau(U) = Z \times L^2([0, \tau]; U)\) and

\[
A = \left( \begin{array}{cc} \tilde{A} & 0 \\ B_\alpha C_\Lambda & \tilde{G}_\alpha \end{array} \right), \quad D(A) = \left\{ \begin{pmatrix} z \\ g \end{pmatrix} \in Z_\tau(U) \left| \begin{array}{c} \tilde{A}z \in Z \\ \tilde{G}_\alpha g + B_\alpha C_\Lambda z \in L^2([0, \alpha]; U) \end{array} \right. \right\}. \tag{9.3}
\]
Then, the operator \( A \) generates a \( C_0 \)-semigroup \( e^{At} \) on \( Z_T(U) \). Moreover, if we suppose further that \( e^{At} \) is exponentially stable in \( Z \), then \( e^{At} \) is exponentially stable in \( Z_T(U) \).

**Proof.** The operator \( A \) is associated with the following system:

\[
\begin{align*}
\dot{z}(t) &= Az(t), \\
\phi_t(\cdot, t) + \phi_x(\cdot, t) &= 0 \quad \text{in} \ U, \quad \phi(0, t) = C_A z(t).
\end{align*}
\] (9.4)

Since \( C \) is admissible for the semigroup \( e^{At} \), for any \((z(0), \phi(0, 0))^\top \in D(A)\), we have \( z(0) \in D(A) \), \( z \in C^1([0, \infty); Z) \), \( C_A z \in H_{loc}^1([0, \infty); U) \) and

\[
\phi(x, t) = \begin{cases} 
C_A (t - x), & t - x \geq 0, \\
\phi(x - t, 0), & t - x < 0,
\end{cases} \quad x \in [0, \alpha].
\] (9.5)

Therefore, system (9.4) admits a unique continuously differentiable solution \((z, \phi) \in C^1([0, \infty); Z_T(U))\) for any \((z(0), \phi(0, 0))^\top \in D(A)\). By [20, Theorem 1.3, p.102], the operator \( A \) generates a \( C_0 \)-semigroup \( e^{At} \) on \( Z_T(U) \).

Finally, we show the exponential stability. Suppose that \((z, \phi)^\top \in C([0, \infty); Z_T(U))\) is a solution of system (9.4). Since \( e^{At} \) is exponentially stable on \( Z \) and \( \phi \)-subsystem is independent of the \( z \)-subsystem, there exist two positive constants \( \omega_A \) and \( L_A \) such that

\[
\|z(t)\|_Z \leq L_A e^{-\omega_A t} \|z(0)\|_Z, \quad \forall \ t \geq 0.
\] (9.6)

Moreover, it follows from [25, Proposition 4.3.6, p.124] that

\[
v_\omega \in L^2([0, \infty); U), \quad v_\omega(t) = e^{At}C_A z(t), \quad 0 < \omega < \omega_A,
\] (9.7)

which, together with (9.5), implies that \( \phi(\cdot, t) \) decays to zero exponentially as \( t \to \infty \). So \((z, \phi)\) decays to zero exponentially in \( Z_T(U) \). The proof is complete.

**Lemma 9.3.** Let \((A, B)\) be a linear system with the state space \( Z \) and input space \( U \). Suppose that \( B \) is admissible for \( e^{At} \). Let \( G_\alpha \) and \( C_\alpha \) be given by (3.2) and (3.7), respectively. Define \( Z_T(U) = Z \times L^2([0, \tau]; U) \) and

\[
A = \begin{pmatrix} \tilde{A} & BC_\alpha \lambda \\ 0 & \tilde{G}_\alpha \end{pmatrix}, \quad D(A) = \left\{ \begin{pmatrix} z \\ g \end{pmatrix} \in Z_T(U) \mid \tilde{A} z + BC_\alpha \lambda g \in Z, \quad \tilde{G}_\alpha g \in L^2([0, \alpha]; U) \right\}.
\] (9.8)

Then, \( A \) generates a \( C_0 \)-semigroup \( e^{At} \) on \( Z_T(U) \). Moreover, if we suppose further that \( e^{At} \) is exponentially stable in \( Z \), then \( e^{At} \) is exponentially stable in \( Z_T(U) \).

**Proof.** Almost the same as Lemma 9.2, we can prove that the operator \( A \) generates a \( C_0 \)-semigroup \( e^{At} \) on \( Z_T(U) \). It suffices to prove the exponential stability. Consider the classical solution of the following system:

\[
\begin{align*}
\dot{z}(t) &= Az(t) + B\phi(\alpha, t), \\
\phi_t(\cdot, t) + \phi_x(\cdot, t) &= 0 \quad \text{in} \ U, \quad \phi(0, t) = 0.
\end{align*}
\] (9.9)

Since \( e^{\tilde{G}_\alpha t} \) vanishes after time \( \alpha \) and \( e^{At} \) is exponentially stable in \( Z \), the solution \((z, \phi)\) decays to zero exponentially in \( Z_T(U) \) as \( t \to \infty \). The proof is complete.