Convergence analysis of a finite volume scheme for solving non-linear aggregation-breakage population balance equations

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Abstract

This paper presents stability and convergence analysis of a finite volume scheme (FVS) for solving aggregation, breakage and the combined processes by showing Lipschitz continuity of the numerical fluxes. It is shown that the FVS is second order convergent independently of the meshes for pure breakage problem while for pure aggregation and coupled equations, it shows second order convergent on uniform and non-uniform smooth meshes. Furthermore, it gives only first order convergence on non-uniform grids. The mathematical results of convergence analysis are also demonstrated numerically for several test problems.

Keywords: Aggregation, breakage, finite volume, consistency, convergence.

1 Introduction

The aggregation-breakage population balance equations (PBEs) are the models for the growth of particles by combined effect of aggregation and breakage. Each particle is identified here by its size, i.e. volume or mass. The equations we consider in this paper describe the time evolution of the particle size distribution (PSD) under the simultaneous effect of binary aggregation and multiple breakage. In binary aggregation, two particles combine together to form a bigger one whereas in breakage process, a big particle breaks into two or many fragments. There are many engineering applications, including aerosol physics, high shear granulation, highly demanding nano-particles and pharmaceutical industries etc., see Sommer et al. [28], Gokhale et al. [6] and references therein. Binary breakage is not sufficient for some of these applications, therefore, multiple fragmentation is preferred. The temporal change of the particle number density, \( f(t, x) \geq 0 \), of particles of volume \( x \in \mathbb{R}_{>0} \) at time \( t \in \mathbb{R}_{>0} \) in a spatially homogeneous physical system undergoing an aggregation-breakage process is described by the following well
known PBEs, see \[23, 32\]

\[
\frac{\partial f(t, x)}{\partial t} = \frac{1}{2} \int_0^x \beta(x - u, u) f(t, x - u)f(t, u) du - \int_0^\infty \beta(x, u) f(t, u)f(t, x) du \\
+ \int_x^\infty b(x, \epsilon)S(\epsilon) f(t, \epsilon) d\epsilon - S(x) f(t, x),
\]

(1)

with initial data

\[
f(0, x) = f^{in}(x) \geq 0, \quad x \in [0, \infty].
\]

The first two terms on the right-hand side (rhs) are due to aggregation while the third and fourth terms model the breakage process. The two positive terms describe the creation of particles of size $x$ and are called the birth terms for aggregation respectively breakage. The two negative terms describe the disappearance of particles of size $x$ and are commonly called the death terms.

The aggregation kernel $\beta(x, y) \geq 0$ characterizes the rate at which two particles of volumes $x$ and $y$ combine together. It also satisfies the symmetry condition $\beta(x, y) = \beta(y, x)$. The selection function $S(\epsilon)$ describes the rate at which particles of size $\epsilon$ are selected to break. The breakage function $b(x, \epsilon)$ for a given $\epsilon > 0$ gives the size distribution of particle sizes $x \in [0, \epsilon]$ resulting from the breakage of a particle of size $\epsilon$. For the particular case of $b(x, \epsilon) = 2/\epsilon$, the multiple breakage PBE turns into the binary breakage PBE. The breakage function has the following important properties

\[
\int_0^x b(u, x) du = \bar{N}(x), \quad \int_0^x ub(u, x) du = x.
\]

(3)

The function $\bar{N}(x)$, which may be infinite, denotes the number of fragments obtained from the breakage of particle of size $x$. The second integral shows that the total mass created from the breakage of a particle of size $x$ is again $x$. In aggregation-breakage processes the total number of particles varies in time while the total mass of particles remains conserved. In terms of $f$, the total number of particles and the total mass of particles at time $t \geq 0$ are respectively given by

\[
M_0(t) := \int_0^\infty f(t, x) dx, \quad M_1(t) := \int_0^\infty xf(t, x) dx.
\]

It is easy to show that the total number of particles $M_0(t)$ decreases by aggregation and increases by breakage processes while the total mass $M_1(t)$ does not vary during these events. For the total mass conservation

\[
\int_0^\infty xf(t, x) dx = \int_0^\infty xf^{in}(x) dx, \quad t \geq 0,
\]

holds. However, for some special cases of $\beta$ when it is sufficiently large compared to the selection function $S$, a phenomenon called gelation occurs. In this case the total mass of particles is not conserved, see Escobedo et al. [4] and further citations for details.

Mathematical results on existence and uniqueness of solutions of the equation (1) and further citations can be found in McLaughlin et al. [22] and W. Lamb [14] for rather general aggregation kernels, breakage and selection functions. In our analysis we consider them to be twice continuously differentiable functions. The PBEs (1) can only be solved analytically for a limited
number of simplified problems, see Ziff [32], Dubovskii et al. [8] and the references therein. This
certainly leads to the necessity of using numerical methods for solving general PBEs. Several
numerical methods have been introduced to solve the PBEs. Stochastic methods (Monte-Carlo)
have been developed, see Lee and Matsoukas [15] for solving equations of binary breakage. Finite
element techniques can be found in Mahoney and Ramkrishna [19] and the references therein for
the equations of simultaneous aggregation, growth and nucleation. Some other numerical
techniques are available in the literature such as the method of successive approximations by D.
Ramkrishna [26], method of moments [18, 21], finite volume methods [24, 10] and sectional methods [8, 12, 30] to solve such PBEs.

A completely different numerical approach was proposed by Filbet and Laurencot [5] for solv-
ing aggregation PBEs by discretizing a well known mass balance formulation. They thereby
introduced an application of the FVS to solve the aggregation problem. Further, Bourgade and
Filbet [1] have extended their scheme to solve the case of binary aggregation and binary breakage
PBEs and gave a convergence proof of approximate solutions in the space $L^\infty(0, T; L^1(0, R))$.
For a special case of a uniform mesh they have shown error estimates of first order. The scheme
has also been extended to two-dimensional aggregation problems by Qamar and Warnecke [25].
Finally it has been observed that the FVS is a good alternative to the methods mentioned above
for solving the PBEs due to its automatic mass conservation property.

Since Bourgade and Filbet have considered aggregation with binary breakage problems on uni-
form meshes only. The objective here is to analyze such a FVS to solve the aggregation with
multiple breakage PBEs on general meshes. We also demonstrate mathematically the missing
stability and the convergence analysis of the FVS for simultaneous aggregation-breakage PBEs
by following Hundsdorfer and Verwer [7] and Linz [17]. The mathematical results are verified
numerically for several test problems on four different types of uniform and non-uniform grids.

This paper is organized as follows. First, we derive the FVS to solve aggregation-breakage PBEs.
Then in Section 3 some useful definitions and theorems are reviewed from [7, 17] which are used
in further analysis of the method. Here we also discuss the consistency and prove the Lipschitz
continuity of the numerical fluxes to get the convergence results. Later on the convergence
analysis is numerically tested for several problems in Section 4. Further, Section 5 summarizes
some conclusions. At the end of the paper one Appendix is provided which gives a bound on
total number of particles for the aggregation-breakage terms.

## 2 Finite volume scheme

In this section a FVS for solving aggregation-breakage PBEs is discussed. Following Filbet and
Laurencot [5] for aggregation, a new form of the breakage PBE is presented in order to apply
the FVS efficiently. Then stability and convergence analysis will be discussed for the method.

### 2.1 Aggregation-breakage PBE in a conservative form

Writing the aggregation and breakage terms in divergence form enable us to get a precise amount
of mass dissipation or conservation. It can be written in a conservative form of mass density
\[ \frac{\partial [xf(t, x)]}{\partial t} + \frac{\partial}{\partial x} \left( F^{\text{agg}}(t, x) + F^{\text{brk}}(t, x) \right) = 0. \]  

(4)

The abbreviations \textit{agg} and \textit{brk} are used for aggregation and breakage terms respectively. The flux functions \( F^{\text{agg}} \) and \( F^{\text{brk}} \) are given by

\[ F^{\text{agg}}(t, x) = \int_{0}^{x} \int_{x-u}^{x} u\beta(u, v)f(t, u)f(t, v)dvdu, \quad \text{and} \]

(5)

\[ F^{\text{brk}}(t, x) = -\int_{x}^{\infty} \int_{x}^{\infty} ub(u, v)S(v)n(t, v)dvdu. \]  

(6)

It should be noted that both forms of aggregation-breakage PBEs (1) and (4) are interchangeable by using the Leibniz integration rule. The concept of this conservative formulation of the PBE has been used in Tanaka et al. [29] and Makino et al. [20]. It should also be mentioned that the equation (4) reduces into the case of pure aggregation or pure breakage process when \( F^{\text{brk}}(t, x) \) or \( F^{\text{agg}}(t, x) \) is zero, respectively.

In the PBE (4) the volume variable \( x \) ranges from 0 to \( \infty \). In order to apply a numerical scheme for the solution of the equation a first step is to fix a finite computational domain \( \Omega := [0, x_{\text{max}}] \) for an \( 0 < x_{\text{max}} < \infty \). Hence, for \( x \in \Omega \) and time \( t \in (0, T] \) where \( T < \infty \), the aggregation and the breakage fluxes for the truncated conservation law for \( n \), i.e. for

\[ \frac{\partial [xn(t, x)]}{\partial t} + \frac{\partial}{\partial x} \left( F^{\text{agg}}(t, x) + F^{\text{brk}}(t, x) \right) = 0 \]  

(7)

are given as

\[ F^{\text{agg}}(t, x) = \int_{0}^{x} \int_{x-u}^{x_{\text{max}}} u\beta(u, v)n(t, u)n(t, v)dvdu, \quad \text{and} \]

(8)

\[ F^{\text{brk}}(t, x) = -\int_{x}^{x_{\text{max}}} \int_{0}^{x} ub(u, v)S(v)n(t, v)dvdu. \]  

(9)

Here the variable \( n(t, x) \) denotes the solution to the truncated equation. We are given with initial data

\[ n(0, x) = f^{\text{in}}(x), \quad x \in \Omega. \]  

(10)

For further analysis, all the kinetic parameters \( \beta, S \) and \( b \) are considered to be two times continuously differentiable function, i.e.

\[ \beta, b \in C^2([0, x_{\text{max}}] \times [0, x_{\text{max}}]) \quad \text{and} \quad S \in C^2([0, x_{\text{max}}]). \]  

(11)

From (11), there exists some non-negative constants \( Q \) and \( Q_1 \) depending on \( x_{\text{max}} \) such that

\[ \beta(x, y) \leq Q \quad \text{and} \quad b(x, y)S(y) \leq Q_1 \quad \text{for} \quad x, y \in [0, x_{\text{max}}]. \]  

(12)
Remark 2.1. The formulation we use here is a non-conservative truncation for the pure aggregation operator as $F_{\text{agg}}(t, x_{\text{max}}) \geq 0$ while it is mass conserving for the pure breakage equation, i.e. $F_{\text{brk}}(t, x_{\text{max}}) = 0$. Hence, the combined formulation \(^7\) is a non-conservative truncation as used by Bourgade and Filbet \(^1\). One could make a conservative truncation by replacing $x_{\text{max}}$ by $x_{\text{max}} - u$ in \(^3\). This would give $F_{\text{agg}}(t, x_{\text{max}}) = 0$. But it describes an artificial interruption of the aggregation process without a real physical justification. With our truncation particles that are too large leave the system.

2.2 Numerical discretization

Finite volume methods are a class of discretization schemes used to solve mainly conservation laws, see LeVeque \(^{16}\). For a semi-discrete scheme, the interval $[0, x_{\text{max}}]$ is discretized into small cells

$$\Lambda_i := ]x_{i-1/2}, x_{i+1/2}], \quad i = 1, ..., I,$$

with

$$x_{1/2} = 0, \quad x_{I+1/2} = x_{\text{max}}, \quad \Delta x_i = x_{i+1/2} - x_{i-1/2} \leq \Delta x,$$

where $\Delta x$ is the maximum mesh size. The representative of each size, usually the center of each cell $x_i = (x_{i-1/2} + x_{i+1/2})/2$, is called pivot or grid point. The FVS has been carried over to the discretization of such equations by instead of interpreting $\hat{n}(t)$ as an approximation to a point value at a grid point, i.e. $n(t, x_i)$, rather taking an approximation of the cell average of the solution on cell $i$ at time $t$

$$\hat{n}_i(t) \approx n_i = \frac{1}{\Delta x_i} \int_{x_{i-1/2}}^{x_{i+1/2}} n(t, x) dx. \quad (13)$$

Integrating the conservation law on a cell in space $\Lambda_i$, the FVS is given as \(^{16}\)

$$\frac{x_i d\hat{n}_i(t)}{dt} = \frac{1}{\Delta x_i} \left[ J_{\text{agg}}^{i+1/2} - J_{\text{agg}}^{i-1/2} + J_{\text{brk}}^{i+1/2} - J_{\text{brk}}^{i-1/2} \right]. \quad (14)$$

The term $J_{\text{agg}}^{i+1/2}$ is called the numerical flux which is an appropriate approximation of the truncated continuous flux function $F_{\text{agg}}$ and/or $F_{\text{brk}}$ depending upon the processes under consideration.

In case of a breakage process, the numerical flux may be approximated from the mass flux $F_{\text{brk}}$ as follows

$$F_{\text{brk}}^{i+1/2} = - \int_{x_{i+1/2}}^{x_{i+1/2}} \int_0^{x_{i+1/2}} ub(u, \epsilon) S(\epsilon) n(t, \epsilon) du d\epsilon$$

$$= - \sum_{k=i+1}^J \int_{\Lambda_k} S(\epsilon) n(t, \epsilon) \sum_{j=1}^i \int_{\Lambda_j} ub(u, \epsilon) du d\epsilon. \quad (15)$$

Using our assumptions that $S \in C^2([0, x_{\text{max}}])$, $b \in C^2([0, x_{\text{max}}] \times [0, x_{\text{max}}])$ and applying the midpoint rule we can rewrite \(^{15}\) as

$$F_{\text{brk}}^{i+1/2} = - \sum_{k=i+1}^J n_k(t) S(x_k) \Delta x_k \sum_{j=1}^i x_j b(x_j, x_k) \Delta x_j + \mathcal{O}(\Delta x^2)$$

$$= J_{\text{brk}}^{i+1/2}(n) \quad (16)$$
Similarly for the aggregation problem,

\[ F^{agg}(x_{i+1/2}) = \int_0^{x_{i+1/2}} \int_{x_{i+1/2}}^{x_{i+2}} u\beta(u, v)n(t, u)n(t, v)dvdu. \]  

(17)

From Filbet and Laurençot [5], the above equation can be written as

\[ F^{agg}(x_{i+1/2}) = \sum_{k=1}^{i} (xn)_k \Delta x_k \left( \sum_{j=\alpha_{i,k}}^{I} n_j \beta_{j,k} \Delta x_j + n_{\alpha_{i,k}-1} \beta_{\alpha_{i,k}-1,k} \left( x_{\alpha_{i,k}-1/2} - (x_{i+1/2} - x_k) \right) \right) \]

\[ + O(\Delta x^2). \]

(19)

Here, the parameter \( I \) denotes the number of cells. The integer \( \alpha_{i,k} \) corresponds to the index of each cell such that

\[ x_{i+1/2} - x_k \in \Lambda_{\alpha_{i,k}} - 1. \]  

(18)

Applying mid point approximation for the first term and Taylor series expansion of the second term about the point \( x_{\alpha_{i,k}-1} \) give with \( (xn)_k = x_k n_k \)

\[ F^{agg}(x_{i+1/2}) = \sum_{k=1}^{i} x_k n_k \Delta x_k \left( \sum_{j=\alpha_{i,k}}^{I} n_j \beta_{j,k} \Delta x_j + n_{\alpha_{i,k}-1} \beta_{\alpha_{i,k}-1,k} \left( x_{\alpha_{i,k}-1/2} - (x_{i+1/2} - x_k) \right) \right) \]

\[ = J^{agg}_{i+1/2}(n) \]

(19)

Let us denote the vector \( n := [n_1, \ldots, n_I] \) obtained by \( L^2 \) projection of the exact solution \( n \) into the space of step functions constant on each cell. It is worth to mention that this projection error can easily be shown of second order, see remark 3.3.3 in [11]. We also define the vectors

\[ \Delta J^{agg}(n) := [\Delta J_{1}^{agg}(n), \ldots, \Delta J_{I}^{agg}(n)] \quad \text{and} \quad \Delta J^{brk}(n) := [\Delta J_{1}^{brk}(n), \ldots, \Delta J_{I}^{brk}(n)] \]

where

\[ \Delta J_{i}^{agg}(n) = \frac{1}{x_i \Delta x_i} \left[ J_{i+1/2}^{agg}(n) - J_{i-1/2}^{agg}(n) \right], \quad \Delta J_{i}^{brk}(n) = \frac{1}{x_i \Delta x_i} \left[ J_{i+1/2}^{brk}(n) - J_{i-1/2}^{brk}(n) \right]. \]

(20)

Substituting the values of \( J_{i+1/2}^{agg} \) and \( J_{i+1/2}^{brk} \) from equations (19) and (16), respectively to get

\[ \Delta x_i \Delta J_{i}^{agg}(n) = \sum_{k=1}^{i-1} \frac{x_k}{x_i} n_k \Delta x_k \left( - \sum_{j=\alpha_{i-1,k}}^{\alpha_{i,k}-1} n_j \beta_{j,k} \Delta x_j + \beta_{\alpha_{i-1,k}-1,k} n_{\alpha_{i-1,k}-1} (x_{\alpha_{i-1,k}-1/2} - (x_{i+1/2} - x_k)) \right. \]

\[ \left. - \beta_{\alpha_{i-1,k}-1,k} n_{\alpha_{i-1,k}-1} (x_{\alpha_{i-1,k}-1/2} - (x_{i+1/2} - x_k)) \right) + n_i \Delta x_i \left( \sum_{j=\alpha_{i,i}}^{I} n_j \beta_{j,i} \Delta x_j \right. \]

\[ + n_{\alpha_{i,i}-1} \beta_{\alpha_{i,i}-1,i} (x_{\alpha_{i,i}-1/2} - (x_{i+1/2} - x_i)) \]

(21)
and
\[ \Delta_i \Delta_J^{\text{agg}}(n) = - \sum_{k=i+1}^I S(x_k) n_k \Delta_x b(x_i, x_k) \Delta x_i + S(x_i) n_i \Delta x_i \sum_{j=1}^{i-1} \frac{x_j}{x_i} b(x_j, x_i) \Delta x_j. \]  
(22)

By denoting the vector \( \hat{n} := [\hat{n}_1, \ldots, \hat{n}_I] \) for the numerical approximations of the average values of \( n(t, x) \), the equation (14) can be rewritten as
\[
\frac{d\hat{n}(t)}{dt} = -\left[ \Delta J^{\text{agg}}(\hat{n}) + \Delta J^{\text{brk}}(\hat{n}) \right] = J(\hat{n}).
\]  
(23)

In order to retain the overall high accuracy, the semi-discrete scheme (23) can be combined with any higher order time integration method. It is worth to mention here that dealing with the pure cases of aggregation or breakage is easy by setting one of the two numerical fluxes is zero.

3 Convergence analysis

Before discussing the convergence of the semi-discrete scheme, let us review some useful definitions and theorems from [7, 17] that will be used in the subsequent analysis. Let \( \| \cdot \| \) denote the discrete \( L^1 \) norm on \( \mathbb{R}^I \) that is defined as
\[
\| \hat{n}(t) \| = \sum_{i=1}^I |\hat{n}_i(t)| \Delta x_i.
\]  
(24)

In this work, we deal with this norm by interpreting the discrete data as step functions.

**Definition 3.1.** The *spatial truncation error* is defined by the residual left by substituting the exact solution \( n(t) = [n_1(t), \ldots, n_I(t)] \) into equation (23) as
\[
\sigma(t) = \frac{d\hat{n}(t)}{dt} + (\Delta J^{\text{agg}}(\hat{n}) + \Delta J^{\text{brk}}(\hat{n})).
\]  
(25)

The scheme (23) is called consistent of order \( p \) if, for \( \Delta x \to 0 \),
\[
\| \sigma(t) \| = O(\Delta x^p), \quad \text{uniformly for all } t, \quad 0 \leq t \leq T.
\]

**Definition 3.2.** The *global discretization error* is defined by \( \epsilon(t) = n(t) - \hat{n}(t) \). The scheme (23) is called convergent of order \( p \) if, for \( \Delta x \to 0 \),
\[
\| \epsilon(t) \| = O(\Delta x^p), \quad \text{uniformly for all } t, \quad 0 \leq t \leq T.
\]

It is important that our numerical solution remains non-negative for all times. This is guaranteed by the next well known theorem where we have \( \hat{M} \geq 0 \) for a vector \( \hat{M} \in \mathbb{R}^I \) iff all its components are non-negative.

**Theorem 3.3.** (Hundsdorfer and Verwer [7, Chap. 1, Theorem 7.1]). Suppose that \( \Delta J^{\text{agg}}(\hat{n}) \) and \( \Delta J^{\text{brk}}(\hat{n}) \) are continuous and satisfy the Lipschitz conditions
\[
\| \Delta J^{\text{agg}}(\hat{n}) - \Delta J^{\text{agg}}(\hat{m}) \| \leq L_1 \| \hat{n} - \hat{m} \| \quad \text{for all } \hat{n}, \hat{m} \in \mathbb{R}^I.
\]
and
\[ \| \Delta J^{brk}(\hat{n}) - \Delta J^{brk}(\hat{m}) \| \leq L_2 \| \hat{n} - \hat{m} \| \quad \text{for all } \hat{n}, \hat{m} \in \mathbb{R}^I. \]

Then the solution of the semi-discrete system (14) is non-negative if and only if for any vector \( \hat{n} \in \mathbb{R}^I \) and all \( i = 1, \ldots, I \) and \( t \geq 0 \),
\[ \hat{n} \geq 0, \quad \hat{n}_i = 0 \implies J_i(\hat{n}) \geq 0. \]

Now we state a useful theorem from Linz [17] which we use to show that the FVS is convergent.

**Theorem 3.4.** Let us assume that a Lipschitz condition on \( J(n) \) is satisfied for \( 0 \leq t \leq T \) and for all \( n, \hat{n} \in \mathbb{R}^I \) where \( n \) and \( \hat{n} \) are the projected exact and numerical solutions defined in (7) and (23), respectively. More precisely there exists a Lipschitz constant \( L < \infty \) such that
\[ \| J(n) - J(\hat{n}) \| \leq L \| n - \hat{n} \|, \]
holds. Then a consistent discretization method is also convergent and the convergence is of the same order as the consistency.

**Proof.** A more general result is proven in Linz [17].

Due to Theorem 3.4 for the convergence of our scheme it remains to show that the method is consistent and the Lipschitz condition (26) is satisfied by the fluxes.

### 3.1 Consistency

The following lemma gives the consistency order of the FVS for aggregation-breakage PBEs.

**Lemma 3.5.** Consider the function \( S \in C^2([0, x_{max}]) \) and \( b, \beta \in C^2([0, x_{max}] \times [0, x_{max}]) \). Then, for any family of meshes, the consistency of the semi-discrete scheme (23) is of second order for the pure breakage process, i.e. with \( \Delta J^{agg}(\hat{n}) = 0 \). For the aggregation and coupled processes, the scheme is second order consistent on uniform and non-uniform smooth meshes while on oscillatory and random meshes it is first order consistent.

**Proof.** The spatial truncation error (25) is given by
\[ \sigma_i(t) = \frac{dn_i(t)}{dt} + (\Delta J_i^{agg}(n) + \Delta J_i^{brk}(n)). \] (27)

Integrating (7) over \( \Lambda_i \) and applying the mid-point rule in the time derivative term, we interpret
\[ \frac{dn_i(t)}{dt} = \frac{-1}{x_i \Delta x_i} \left[ F^{agg}(x_{i+1/2}) - F^{agg}(x_{i-1/2}) + F^{brk}(x_{i+1/2}) - F^{brk}(x_{i-1/2}) \right] + \mathcal{O}(\Delta x^2). \]

Substituting this into the equation (27) and using (20) give the following form
\[ \sigma_i(t) = \frac{-1}{x_i \Delta x_i} \left[ F^{agg}(x_{i+1/2}) - F^{agg}(x_{i-1/2}) - J_{i+1/2}^{agg}(n) + J_{i-1/2}^{agg}(n) ight. \\
+ F^{brk}(x_{i+1/2}) - F^{brk}(x_{i-1/2}) - J_{i+1/2}^{brk}(n) + J_{i-1/2}^{brk}(n) \right] + \mathcal{O}(\Delta x^2) \\
= \sigma_i^{agg}(t) + \sigma_i^{brk}(t) + \mathcal{O}(\Delta x^2). \] (28)
Let us now begin with
\[ F^{\text{brk}}(x_{i+1/2}) - F^{\text{brk}}(x_{i-1/2}) = \left( \sum_{k=i+1}^j \int_{\Lambda_k} S(\epsilon)n(t, \epsilon) \int_0^{x_{i+1/2}} u_b(u, \epsilon) \, du \, d\epsilon \right) - \left( \sum_{k=i-1}^{i-2} \int_{\Lambda_k} S(\epsilon)n(t, \epsilon) \int_0^{x_{i-1/2}} u_b(u, \epsilon) \, du \, d\epsilon \right). \]

We now use Taylor series expansion of the functions \( K_{x_{i+1/2}}(\epsilon) := n(t, \epsilon) \int_0^{x_{i+1/2}} u_b(u, \epsilon) \, du \) about \( x_k \) and further rearrangement of terms yield \( \sigma_i^{\text{brk}}(t) \) as
\[ \sigma_i^{\text{brk}}(t) = \frac{1}{x_i \Delta x_i} \left( \sum_{k=i+1}^j \left[ K'_{x_{i+1/2}}(x_k) - K'_{x_{i-1/2}}(x_k) \right] \int_{\Lambda_k} S(\epsilon)(\epsilon - x_k) \, d\epsilon \right) - K'_{x_{i-1/2}}(x_i) \int_{\Lambda_i} S(\epsilon)(\epsilon - x_i) \, d\epsilon + O(\Delta x^3). \]

Applying the mid-point rule, it should be noted that
\[ \int_{\Lambda_k} S(\epsilon)(\epsilon - x_k) \, d\epsilon = O(\Delta x^3) \quad \text{and} \quad K'_{x_{i+1/2}}(x_k) - K'_{x_{i-1/2}}(x_k) = O(\Delta x). \]

Thus we obtain \( \sigma_i^{\text{brk}}(t) = O(\Delta x^2) \). Hence, for the pure breakage process, the consistency of the semi-discrete scheme (23) is two which is determined by using (24) as
\[ ||\sigma(t)|| = \sum_{i=1}^f |\sigma_i^{\text{brk}}(t)| \Delta x_i = O(\Delta x^2), \]

independently of the type of meshes.

Due to the non-linearity of the aggregation problem, it is not easy to determine the consistency order on general meshes and therefore, we evaluate it on various meshes separately. The results can be combined to the results of breakage process to give the consistency of the coupled processes. We know from (17)
\[ F^{\text{agg}}(x_{i+1/2}) - F^{\text{agg}}(x_{i-1/2}) = \left( \sum_{j=1}^i \int_{\Lambda_j} u n(t, u) \int_{x_{i+1/2} - u}^{x_{\max}} \beta(u, v)n(t, v) \, dv \, du \right) - \left( \sum_{j=1}^{i-1} \int_{\Lambda_j} u n(t, u) \int_{x_{i-1/2} - u}^{x_{\max}} \beta(u, v)n(t, v) \, dv \, du \right). \]

Define \( L_{x_{i+1/2}}(u) := n(t, u) \int_{x_{i+1/2} - u}^{x_{\max}} \beta(u, v)n(t, v) \, dv \). Taylor series expansion of the functions \( L_{x_{i+1/2}}(u) \) about \( x_j \) gives
\[ F^{\text{agg}}(x_{i+1/2}) - F^{\text{agg}}(x_{i-1/2}) = \left( \sum_{j=1}^i \int_{\Lambda_j} u \left( L_{x_{i+1/2}}(x_j) + (u - x_j)L'_{x_{i+1/2}}(x_j) \right) \, du \right) - \left( \sum_{j=1}^{i-1} \int_{\Lambda_j} u \left( L_{x_{i-1/2}}(x_j) + (u - x_j)L'_{x_{i-1/2}}(x_j) \right) \, du \right) + O(\Delta x^3). \] (29)
Applying the mid-point rule, it should again be noted that

\[ \int_{\Lambda_j} u(u - x_j) \, du = \mathcal{O}(\Delta x^3) \quad \text{and} \quad \mathcal{L}'_{x+1/2}(x_j) - \mathcal{L}'_{x-1/2}(x_j) = \mathcal{O}(\Delta x). \]

Therefore, by defining \( LHS := F^{agg}_{x+1/2} - F^{agg}_{x-1/2} \), the equation (29) reduces to

\[ LHS = \left( \sum_{j=1}^{i} \int_{\Lambda_j} u \mathcal{L}_{x+1/2}(x_j) \, du - \sum_{j=1}^{i-1} \int_{\Lambda_j} u \mathcal{L}_{x-1/2}(x_j) \, du \right) + \mathcal{O}(\Delta x^3). \]

Substituting the values of \( \mathcal{L}_{x+1/2}(x_j) \) yield (leaving the third order terms)

\[ LHS = \left( \sum_{j=1}^{i} \int_{\Lambda_j} u \int_{x_{i+1/2} - x_j}^{x_{i+1/2}} \beta(x_j, v)n(t, v) dv du - \sum_{j=1}^{i-1} \int_{\Lambda_j} u \int_{x_{i-1/2} - x_j}^{x_{i-1/2}} \beta(x_j, v)n(t, v) dv du \right). \]

Now, \( I_1 \) is equivalent to

\[ I_1 = \sum_{j=1}^{i} \int_{\Lambda_j} u_n \int_{x_{i+1/2} - x_j}^{x_{i+1/2}} \beta(x_j, v)n(t, v) dv du. \]

Applying the mid-point approximation for the second term, we figure out

\[ I_1 = \sum_{j=1}^{i} x_j n_j \Delta x_j \left[ \int_{x_{i+1/2} - x_j}^{x_{i+1/2} - x_j} \beta(x_j, v)n(t, v) dv \right. \]

\[ + \left. \sum_{k=\alpha_{i,j}}^{I} \beta_{j,k} n_k \Delta x_k + \sum_{k=\alpha_{i,j}}^{I} \int_{\Lambda_k} (v - x_k)^2 / 2(\beta(x_j, v)n(t, v))'' dv \right] + \mathcal{O}(\Delta x^3). \]

Similarly, we estimate

\[ I_2 = \sum_{j=1}^{i-1} x_j n_j \Delta x_j \left[ \int_{x_{i+1/2} - x_j}^{x_{i+1/2} - x_j} \beta(x_j, v)n(t, v) dv \right. \]

\[ + \left. \sum_{k=\alpha_{i-1,j}}^{I} \beta_{j,k} n_k \Delta x_k + \sum_{k=\alpha_{i-1,j}}^{I} \int_{\Lambda_k} (v - x_k)^2 / 2(\beta(x_j, v)n(t, v))'' dv \right] + \mathcal{O}(\Delta x^3). \]

Subtracting the third term from \( I_2 \) to \( I_1 \) gives

\[ \left[ \sum_{j=1}^{i} \sum_{k=\alpha_{i,j}}^{I} - \sum_{j=1}^{i-1} \sum_{k=\alpha_{i-1,j}}^{I} \right] x_j n_j \Delta x_j \int_{\Lambda_k} (v - x_k)^2 / 2(\beta(x_j, v)n(t, v))'' dv = \]

\[ \left[ - \sum_{j=1}^{i-1} \sum_{k=\alpha_{i-1,j}}^{I-1} x_j n_j \Delta x_j \int_{\Lambda_k} (v - x_k)^2 / 2(\beta(x_j, v)n(t, v))'' dv + \mathcal{O}(\Delta x^3). \]
By using Lemma 3.6 which is stated in the next section, the summation over \( k \) is finite in this term. Hence, the rhs of this equation becomes of order \( O(\Delta x^3) \) and can be omitted. Therefore,

\[
LHS = \sum_{j=1}^{i} x_j n_j \Delta x_j \left[ \int_{x_{i+1/2}-x_j}^{x_{i+1/2}} \beta(x_j, v) n(t, v) dv \right] + \sum_{k=\alpha_{i,j}}^{l} \beta_{j,k} \Delta x_k + O(\Delta x^3).
\]

Open the Taylor series about the points \( x_{\alpha_{i,j}} \) in \( I_3 \) and \( x_{\alpha_{i,j}}-1 \) in \( I_4 \) as well as by using the relation (19), we finally obtain

\[
LHS = \left( J_{x_{i+1/2}}^{agg} + \sum_{j=1}^{i} x_j n_j \Delta x_j \int_{x_{i+1/2}-x_j}^{x_{i+1/2}} \beta(x_j, v) n(t, v) dv \right) + O(\Delta x^3).
\]

Let \( f(x_j, v) = \beta(x_j, v) n(t, v) \) and \( \frac{\partial f}{\partial x_{\alpha_{i,j}}} = f'(x_j, x_{\alpha_{i,j}}) \). This implies that

\[
\sigma_{i,j}^{agg}(t) = \frac{1}{x_i \Delta x_i} \left[ \sum_{j=1}^{i} x_j n_j \Delta x_j \int_{x_{i+1/2}-x_j}^{x_{i+1/2}} (v - x_{\alpha_{i,j}}) f'(x_j, x_{\alpha_{i,j}}) dv \right] - \sum_{j=1}^{i-1} x_j n_j \Delta x_j \int_{x_{i+1/2}-x_j}^{x_{i+1/2}} (v - x_{\alpha_{i-1,j}}) f'(x_j, x_{\alpha_{i-1,j}}) dv + O(\Delta x^2). \tag{30}
\]

Now the consistency order on four different types of meshes are evaluated:

### 3.1.1 Uniform mesh

Let us assume that the first mesh is uniform, i.e. \( \Delta x_i = \Delta x \) for all \( i \). In this case \( x_{i+1/2} - x_j \) and \( x_{\alpha_{i,j}}-1 \) become the same and are equal to the pivot point \( x_{i-j+1} \). Similarly,

\[
x_{i+1/2} - x_j = x_{\alpha_{i,j}}-1 = x_{i-j}.
\tag{31}
\]

Applying the Taylor series expansion of the function \( f'(x_j, x_{\alpha_{i,j}}-1 + (x_{\alpha_{i,j}}-1 - x_{\alpha_{i,j}}-1)) \) about the point \( x_{\alpha_{i,j}}-1 \) in the first term on the rhs of the equation (30) to get

\[
\sigma_{i,j}^{agg}(t) = \frac{1}{x_i \Delta x_i} \left[ \sum_{j=1}^{i} x_j n_j \Delta x_j f'(x_j, x_{\alpha_{i,j}}-1) \int_{x_{i+1/2}-x_j}^{x_{i+1/2}} (v - x_{\alpha_{i,j}}) dv 
- \int_{x_{i+1/2}-x_j}^{x_{i+1/2}} (v - x_{\alpha_{i,j}}) dv \right] + O(\Delta x^2).
\]
Further by facilitating the integrals and using the relation (31), we have

$$\sigma^{agg}_i(t) = \frac{1}{x_i \Delta x_i} \left[ \sum_{j=1}^{i-1} x_j n_j \Delta x_j f'(x_j, x_{\alpha_{i,j-1}}) \left( \frac{\Delta x_{\alpha_{i,j-1}}^2}{8} - \frac{\Delta x_{\alpha_{i-1,j-1}}^2}{8} \right) \right] + O(\Delta x^2).$$

Hence, $\sigma^{agg}_i(t) = O(\Delta x^2)$ and so the order of consistency is given by using (24) as

$$\|\sigma(t)\| = \sum_{i=1}^{I} |\sigma^{agg}_i(t)| \Delta x_i = O(\Delta x^2).$$

Therefore, the scheme is second order consistent on uniform grids.

### 3.1.2 Non-uniform smooth mesh

A smooth transformation from uniform grids leads to such meshes. In this case grids are assumed to be smooth in the sense that $\Delta x_i - \Delta x_{i-1} = O(\Delta x^2)$ and $2\Delta x_i - (\Delta x_{i-1} + \Delta x_{i+1}) = O(\Delta x^3)$, where $\Delta x$ is the maximum mesh width. For example, let us consider a variable $\xi$ with uniform mesh and a smooth transformation $x = g(\xi)$ to get non-uniform smooth mesh, see Figure 1.

For the analysis here, we have considered the exponential transformation as $x = \exp(\bar{h})$. The term $\bar{h}$ is the width of the uniform grid. Here again we achieve second order consistency.

Equation (30) can be rewritten by setting $j = j - 1$ in second term as

$$\sigma^{agg}_i(t) = \frac{1}{x_i \Delta x_i} \left[ \sum_{j=1}^{i} x_j n_j \Delta x_j \int_{x_{i,j-1/2}}^{x_{i,j-1/2}} \left( v - x_{\alpha_{i,j-1}} \right) f'(x_j, x_{\alpha_{i,j-1}}) dv \right] + O(\Delta x^2).$$
Now we simplify \( A - B \) as

\[
A - B = \sum_{j=2}^{i} x_{j-1} n_{j-1} \Delta x_j \int_{x_{i+1/2}-x_j}^{x_{\alpha_{i,j}-1/2}} (v - x_{\alpha_{i,j}-1}) f'(x_{j-1}, x_{\alpha_{i-1,j-1}-1}) dv \\
- \sum_{j=2}^{i} x_{j-1} n_{j-1} \Delta x_{j-1} \int_{x_{i+1/2}-x_{j-1}}^{x_{\alpha_{i-1,j}-1/2}} (v - x_{\alpha_{i-1,j}-1}) f'(x_{j-1}, x_{\alpha_{i-1,j-1}-1}) dv + O(\Delta x^3).
\]

Further it can be rewritten as

\[
A - B = \sum_{j=2}^{i} x_{j-1} n_{j-1} (\Delta x_j - \Delta x_{j-1}) \int_{x_{i+1/2}-x_j}^{x_{\alpha_{i,j}-1/2}} (v - x_{\alpha_{i,j}-1}) f'(x_{j-1}, x_{\alpha_{i-1,j-1}-1}) dv \\
+ \sum_{j=2}^{i} x_{j-1} n_{j-1} \Delta x_{j-1} \int_{x_{i+1/2}-x_{j-1}}^{x_{\alpha_{i-1,j}-1/2}} (v - x_{\alpha_{i-1,j}-1}) f'(x_{j-1}, x_{\alpha_{i-1,j-1}-1}) dv \\
- \sum_{j=2}^{i} x_{j-1} n_{j-1} \Delta x_{j-1} \int_{x_{i+1/2}-x_{j-1}}^{x_{\alpha_{i-1,j-1}-1/2}} (v - x_{\alpha_{i-1,j-1}-1}) f'(x_{j-1}, x_{\alpha_{i-1,j-1}-1}) dv + O(\Delta x^3).
\]

For such smooth meshes, \( \Delta x_j - \Delta x_{j-1} = O(\Delta x^2) \) holds. Setting \( \alpha_{i,j} - 1 = \alpha_1, \alpha_{i-1,j-1} - 1 = \alpha_2 \) and \( g_{i,j} = x_{j-1} n_{j-1} \Delta x_{j-1} f'(x_{j-1}, x_{\alpha_{i-1,j-1}-1}) \) yield

\[
A - B = \sum_{j=2}^{i} g_{i,j} \left( \int_{x_{i+1/2}-x_j}^{x_{\alpha_{2,j}+1/2}} (v - x_{\alpha_1}) dv - \int_{x_{i-1/2}-x_{j-1}}^{x_{\alpha_{2,j}+1/2}} (v - x_{\alpha_2}) dv \right) + O(\Delta x^3).
\]

It can further be simplified as

\[
A - B = \sum_{j=2}^{i} g_{i,j} \left( \frac{\Delta x_{\alpha_1}^2}{4} - \Delta x_{\alpha_2}^2 + [(x_{i-1/2} - x_{j-1}) - x_{\alpha_2}]^2 - [(x_{i+1/2} - x_j) - x_{\alpha_1}]^2 \right) + O(\Delta x^3).
\]

Since \( x_{i+1/2} - x_j \in \Lambda_{\alpha_1,j-1} \), thus \( x_{i-1/2} - x_{j-1} \in \Lambda_{\alpha_1,j-1} \). Further notice that \( x_{i+1/2} - x_j = r(x_{i-1/2} - x_{j-1}) \) and therefore \( \alpha_1 = \alpha_2 + 1 \). Again by using the condition \( \Delta x_j - \Delta x_{j-1} = O(\Delta x^2) \), we determine \( \Delta x_{\alpha_1}^2 - \Delta x_{\alpha_2}^2 = O(\Delta x^3) \). Now, to get a second order consistency of the scheme, it is remained to show that

\[
[(x_{i-1/2} - x_{j-1}) - x_{\alpha_2}]^2 - [(x_{i+1/2} - x_j) - x_{\alpha_1}]^2 = O(\Delta x^3)
\]

or equivalently,

\[
[(x_{i-1/2} - x_{j-1}) - x_{\alpha_2}] - [(x_{i+1/2} - x_j) - x_{\alpha_1}] = O(\Delta x^2).
\]

Let us consider \( \xi_1, \xi_2 \) are corresponding points in the uniform mesh for \( x_{\alpha_2} \) and \( x_{i-1/2} - x_{j-1} \), respectively. Consider \( h_1 = \xi_2 - \xi_1 \) which is given as

\[
h_1 = \xi_2 - \xi_1 = \log (x_{i-1/2} - x_{j-1}) - \log (x_{\alpha_2}) = \log \left( \frac{x_{i-1/2} - x_{j-1}}{x_{\alpha_2}} \right).
\]
Similarly, taking $h_2 = \xi_4 - \xi_3$ where $\xi_3$ and $\xi_4$ are the points in the uniform mesh corresponding to the points $x_{\alpha_1}$ and $x_{i+1/2} - x_j$, respectively, we evaluate

$$h_2 = \xi_4 - \xi_3 = \log (x_{i+1/2} - x_j) - \log (x_{\alpha_1}) = \log \left( \frac{x_{i+1/2} - x_j}{x_{\alpha_1}} \right) = h_1.$$ Setting $h = h_1 = h_2$. Further

$$\xi_3 - \xi_1 = \log (x_{\alpha_1}) - \log (x_{\alpha_2}) = \log \left( \frac{x_{\alpha_1}}{x_{\alpha_2}} \right) = \log (r) = \bar{h}.$$ Finally, the equation (32) can be estimated by using Taylor series expansion as

$$\left[ (x_{i-1/2} - x_{j-1}) - x_{\alpha_2} \right] - \left[ (x_{i+1/2} - x_j) - x_{\alpha_1} \right] = \left[ g(\xi_2) - g(\xi_1) \right] - \left[ g(\xi_4) - g(\xi_3) \right]$$

$$= h g' (\xi_1) - h g' (\xi_3) + O(h^2)$$

$$= h (g' (\xi_1) - g' (\xi_1 + \bar{h})) + O(h^2)$$

$$= -h h g'' (\xi_1) + O(h^2) = O(h^2).$$

Hence, by using (28) and (24) the order of consistency for the pure aggregation process is two for the smooth meshes $x_{i+1/2} = rx_{i-1/2}$.

### 3.1.3 Oscillatory and random meshes

A mesh is known to be an oscillatory mesh, if for $r > 0 (r \neq 1)$ it is given as

$$(33)$$

$$\Delta x_{i+1} := \begin{cases} r \Delta x_i & \text{if } i \text{ is odd,} \\ \frac{1}{r} \Delta x_i & \text{if } i \text{ is even.} \end{cases}$$

From the equation (33), it is clear that the first two terms on the rhs can not be cancel out for an oscillatory or a random mesh. Therefore, $\sigma_{agg}^i (t) = O(\Delta x)$ and so the accuracy of the semi-discrete scheme (23) is one by using the relation (24) on such meshes.

Now for the coupled aggregation and breakage problems, the local truncation error of each process can be combined and give second order consistency on uniform and non-uniform smooth meshes whereas it is of first order on the other two types of grids.

### 3.2 Lipschitz continuity of the fluxes

To prove the Lipschitz continuity of the numerical flux $J(\hat{n})$ in (23), the following three lemmas are used.

**Lemma 3.6.** Let us assume that the points $x_{i+1/2} - x_k$ for given $i, k$ and $j = 1, 2, \ldots, p$ where $p \geq 2$ lie in the same cell $\Lambda_{\alpha}$ for some index $\alpha$. We also assume that our grid satisfies the quasi-uniformity condition

$$\frac{\Delta x_{i+1}}{\Delta x_i} \leq C$$

for some constant $C$ (independent of the mesh size). Then $p$ is bounded by $C + 1$. 

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Proof. Our assumption on the points implies that by (18), we have
\[ \alpha_{i,k} - 1 = \alpha_{i+1,k} - 1 = \ldots = \alpha_{i+p-1,k} - 1 = \alpha. \]
Clearly, \( \Delta x_\alpha \geq \Delta x_{i+1} + \Delta x_{i+2} + \ldots + \Delta x_{i+p-1} \). This implies that
\[ \frac{\Delta x_\alpha}{\Delta x_l} \leq \frac{\Delta x_{\max}}{\Delta x_{\min}} \leq C \quad \Rightarrow \quad \frac{\Delta x_\alpha}{C} \leq \Delta x_l \quad \text{for} \quad l = i + 1, \ldots, i + p - 1. \]
Therefore, \( \Delta x_\alpha \geq (p-1)\frac{\Delta x_\alpha}{C} \), giving \( p \leq (C+1) \). \( \square \)
In the next two lemmas the boundedness of the total number of particles for the aggregation and multiple breakage equations are discussed.

**Lemma 3.7.** Let us assume that the kernels \( \beta, S \) and \( b \) satisfy the boundedness condition (12). Then the total number of particles for the continuous aggregation-breakage equation (7) is bounded by a constant \( C_{T,x_{\max}} > 0 \) depending on \( T \) and \( x_{\max} \), namely
\[ \int_0^{x_{\max}} n(t,x)dx = N(t) = \sum_{i=1}^{l} N_i(t) \leq N(0) \exp(x_{\max}Q_1T) = C_{T,x_{\max}}. \]
Proof. The proof can be found in Appendix A. \( \square \)

**Lemma 3.8.** Under the same assumptions on \( \beta, S \) and \( b \) considered in the previous lemma, we have boundedness of the total number of particles for the discrete aggregation-breakage equation (14) by using the finite volume scheme. The bound in this case is again \( C_{T,x_{\max}} \) as before, i.e.
\[ \sum_{i=1}^{l} \hat{n}_i \Delta x_i = \hat{N}(t) = \sum_{i=1}^{l} \hat{N}_i(t) \leq \hat{N}(0) \exp(x_{\max}Q_1T) = C_{T,x_{\max}} \]
provided that the initial data \( \hat{N}(0) \) and \( N(0) \) are the same.
Proof. The proof has been given in Appendix A. \( \square \)

Now, the Lipschitz continuity of the numerical flux \( J(n) \) defined as in (23) is shown.

**Lemma 3.9.** Let us assume that our grid satisfies the quasi-uniformity condition (34). We also assume that the kernels \( \beta, S \) and \( b \) satisfy the bounds (12) which are \( \beta \leq Q \) and \( bS \leq Q_1 \). Then there exists a Lipschitz constant \( L := (4C + 6)QC_{T,x_{\max}} + 2Q_1x_{\max} < \infty \) for some constants \( C, C_{T,x_{\max}} > 0 \) such that
\[ \| J(n) - J(\hat{n}) \| \leq L \| n - \hat{n} \|, \]
holds.
Proof. From (23), we have the following discretized form of the equation
\[ \frac{d\hat{n}(t)}{dt} = - \left[ \Delta J^{agg}(\hat{n}) + \Delta J^{brk}(\hat{n}) \right] = J(\hat{n}). \]
To prove the Lipschitz conditions on $\mathbf{J}(\hat{n})$, it is sufficient to find the Lipschitz conditions on $\Delta \mathbf{J}^{agg}(\hat{n})$ and $\Delta \mathbf{J}^{brk}(\hat{n})$ separately. For the aggregation,

$$\| \Delta \mathbf{J}^{agg}(n) - \Delta \mathbf{J}^{agg}(\hat{n}) \| = \sum_{i=1}^{l} \Delta x_i | \Delta \mathbf{J}^{agg}_i(n) - \Delta \mathbf{J}^{agg}_i(\hat{n}) |.$$ 

Substituting the value of $\Delta \mathbf{J}^{agg}_i(n)$ from the equation (21) yields

$$\| \Delta \mathbf{J}^{agg}(n) - \Delta \mathbf{J}^{agg}(\hat{n}) \| \leq \sum_{i=1}^{l} \left| \sum_{k=1}^{i-1} \frac{x_k}{x_i} \Delta x_k \sum_{j=\alpha_{i-1,k}}^{\alpha_{i,k}-1} \beta_{j,k} \Delta x_j (-n_j n_k + \hat{n}_j \hat{n}_k) \right|$$

$$+ \sum_{i=1}^{l} \left| \sum_{k=1}^{i-1} \frac{x_k}{x_i} \beta_{\alpha_{i,k}-1,k} \Delta x_k (x_{\alpha_{i,k}-1/2} - (x_{i+1/2} - x_k))(n_k n_{\alpha_{i,k}-1} - \hat{n}_k \hat{n}_{\alpha_{i,k}-1}) \right|$$

$$+ \sum_{i=1}^{l} \left| \sum_{k=1}^{i-1} \frac{x_k}{x_i} \beta_{\alpha_{i-1,k}-1,k} \Delta x_k (x_{\alpha_{i-1,k}-1/2} - (x_{i-1/2} - x_k))(n_k n_{\alpha_{i-1,k}-1} - \hat{n}_k \hat{n}_{\alpha_{i-1,k}-1}) \right|$$

$$+ \sum_{i=1}^{l} \left( \sum_{j=\alpha_{i,i}}^{\alpha_{i,i}-1} \beta_{j,i} \Delta x_i \Delta x_j (n_i n_{\alpha_{i,i}-1} - \hat{n}_i \hat{n}_{\alpha_{i,i}-1}) \right)$$

$$\leq S_1 + S_2 + S_3 + S_4. \quad (38)$$

Now the terms $S_i, i = 1, \ldots, 4$ in (38) are evaluated one by one. First the term $S_1$ is simplified which may be estimated

$$S_1 \leq \sum_{i=1}^{l} \left( \sum_{k=1}^{i-1} \frac{x_k}{x_i} \Delta x_k \sum_{j=\alpha_{i-1,k}}^{\alpha_{i,k}-1} \beta_{j,k} \Delta x_j | n_j n_k - \hat{n}_j \hat{n}_k | \right).$$

Since $k < i$ implies that $x_k < x_i$. Using the relation $xy - \hat{x} \hat{y} = 1/2[(x - \hat{x})(y + \hat{y}) + (x + \hat{x})(y - \hat{y})]$, bound $\beta(x, y) \leq Q$ and setting $N_i = n_i \Delta x_i$ give

$$S_1 \leq \frac{Q}{2} \sum_{i=1}^{l} \left( \sum_{k=1}^{i-1} \Delta x_k | n_k - \hat{n}_k | \sum_{j=\alpha_{i-1,k}}^{\alpha_{i,k}-1} (N_j + \hat{N}_j) + \sum_{k=1}^{i} (N_k + \hat{N}_k) \sum_{j=\alpha_{i-1,k}}^{\alpha_{i,k}-1} \Delta x_j | n_j - \hat{n}_j | \right).$$

Open the summation for each $i$, we obtain

$$S_1 \leq \frac{Q}{2} \sum_{k=1}^{l} \Delta x_k | n_k - \hat{n}_k | \sum_{j=\alpha_{0,k}}^{\alpha_{1,k}-1} (N_j + \hat{N}_j) + \frac{Q}{2} \sum_{k=1}^{l} (N_k + \hat{N}_k) \sum_{j=\alpha_{0,k}}^{\alpha_{1,k}-1} \Delta x_j | n_j - \hat{n}_j |.$$

Having Lemmas 3.7 and 3.8 which say that the total number of particles is bounded by a constant $CT_{T,x_{\text{max}}}$, $S_1$ is further simplified as $S_1 \leq 2QC_{T,x_{\text{max}}} \| n - \hat{n} \|$.

Now the term $S_2$ is calculated from (38) which is taken as

$$S_2 \leq \sum_{i=1}^{l} \left( \sum_{k=1}^{i-1} \frac{x_k}{x_i} \beta_{\alpha_{i,k}-1,k} \Delta x_k (x_{\alpha_{i,k}-1/2} - (x_{i+1/2} - x_k))(n_k n_{\alpha_{i,k}-1} - \hat{n}_k \hat{n}_{\alpha_{i,k}-1}) \right).$$
Similarly, for the breakage problem, we have

\[
S_2 \leq \sum_{i=1}^{I} \sum_{k=1}^{I-1} \frac{Q}{2} \Delta x_k \Delta x_{\alpha_{i,k}-1} \left( \left( n_k - \hat{n}_k \right) \left( n_{\alpha_{i,k}-1} + \hat{n}_{\alpha_{i,k}-1} \right) + \left( n_k + \hat{n}_k \right) \left( n_{\alpha_{i,k}-1} - \hat{n}_{\alpha_{i,k}-1} \right) \right)
\]

\[
\leq \frac{Q}{2} \sum_{i=1}^{I} \sum_{k=1}^{I-1} \Delta x_k |n_k - \hat{n}_k| (N_{\alpha_{i,k}-1} + \hat{N}_{\alpha_{i,k}-1}) + \frac{Q}{2} \sum_{i=1}^{I} \sum_{k=1}^{I-1} \Delta x_{\alpha_{i,k}-1} |n_{\alpha_{i,k}-1} - \hat{n}_{\alpha_{i,k}-1}| (N_k + \hat{N}_k).
\]

Changing the order of summation gives

\[
S_2 \leq \frac{Q}{2} \sum_{k=1}^{I} \left( \sum_{i=k+1}^{I} \Delta x_k |n_k - \hat{n}_k| \left( N_{\alpha_{i,k}-1} + \hat{N}_{\alpha_{i,k}-1} \right) \right) + \frac{Q}{2} \sum_{k=1}^{I} \left( N_k + \hat{N}_k \right) \sum_{i=k+1}^{I} \Delta x_{\alpha_{i,k}-1} |n_{\alpha_{i,k}-1} - \hat{n}_{\alpha_{i,k}-1}|.
\]

By using the Lemma 3.6 which shows that the number of repetition of index in a cell is finite and bounded by some constant \(C\), we obtain \(S_2 \leq 2CQC_{T,x_{\text{max}}} \| n - \bar{\hat{n}} \|.\) The same bound on \(S_3\) is achieved because the only difference is that the index \(i - 1\) is used instead of \(i\).

Finally the expression \(S_4\) from (58) can be written as

\[
S_4 \leq \sum_{i=1}^{I} \left( \sum_{j=\alpha_{i,i}}^{I} \beta_{j,i} \Delta x_j \Delta x_j |n_j n_j - \hat{n}_i \hat{n}_j| + \beta_{\alpha_{i,i}-1,i} \Delta x_i (x_{\alpha_{i,i}-1/2} - (x_{i+1/2} - x_i)) |n_i n_{\alpha_{i,i}-1} - \hat{n}_i \hat{n}_{\alpha_{i,i}-1}| \right)
\]

\[
\leq \frac{Q}{2} \sum_{i=1}^{I} \sum_{j=1}^{I} (N_i + \hat{N}_j) \Delta x_j |n_j - \hat{n}_j| + \frac{Q}{2} \sum_{i=1}^{I} \sum_{j=1}^{I} (N_j + \hat{N}_j) \Delta x_i |n_i - \hat{n}_i|
\]

\[
+ \frac{Q}{2} \sum_{i=1}^{I} \sum_{j=1}^{I} \Delta x_i |n_i - \hat{n}_i| (N_{\alpha_{i,i}-1} + \hat{N}_{\alpha_{i,i}-1}) + \frac{Q}{2} \sum_{i=1}^{I} \sum_{j=1}^{I} (N_i + \hat{N}_j) \Delta x_{\alpha_{i,i}-1} |n_{\alpha_{i,i}-1} - \hat{n}_{\alpha_{i,i}-1}|.
\]

Further simplification gives \(S_4 \leq 4QC_{T,x_{\text{max}}} \| n - \bar{\hat{n}} \|.\) Adding all the results from \(S_1, S_2, S_3\) and \(S_4\) yields

\[
\| \Delta J_{\text{agg}}(n) - \Delta J_{\text{agg}}(\hat{n}) \| \leq (4C + 6)QC_{T,x_{\text{max}}} \| n - \bar{\hat{n}} \|, \quad (39)
\]

with a Lipschitz constant \(L_1 = (4C + 6)QC_{T,x_{\text{max}}}.\)

Similarly, for the breakage problem, we have

\[
\| \Delta J_{\text{brk}}(n) - \Delta J_{\text{brk}}(\hat{n}) \| = \sum_{i=1}^{I} \Delta x_i \left| \Delta J_{\text{brk}}(n) - \Delta J_{\text{brk}}(\hat{n}) \right|.
\]

By using the equation (22), the above equation reduces to

\[
\| \Delta J_{\text{brk}}(n) - \Delta J_{\text{brk}}(\hat{n}) \| \leq \sum_{i=1}^{I} \left| \sum_{k=1}^{I} S_k (n_k - \hat{n}_k) \Delta x_k \Delta x_{b_{i,k}} - S_i (n_i - \hat{n}_i) \sum_{j=1}^{I} \frac{x_j}{x_i} b_{j,i} \Delta x_j \Delta x_i \right|.
\]
Since \( x_j < x_i \) for \( j < i \) and having \( bS \leq Q_1 \) from (12), the above can be simplified as

\[
\| \Delta J^{\text{brk}}(n) - \Delta J^{\text{brk}}(\hat{n}) \| \leq Q_1 \sum_{i=1}^{I} \Delta x_i \sum_{k=1}^{I} \Delta x_k | n_k - \hat{n}_k | + Q_1 \sum_{i=1}^{I} | n_i - \hat{n}_i | \Delta x_i \sum_{j=1}^{I} \Delta x_j. \quad (40)
\]

Therefore, the following is obtained

\[
\| \Delta J^{\text{brk}}(n) - \Delta J^{\text{brk}}(\hat{n}) \| \leq 2Q_1 x_{\text{max}} \| n - \hat{n} \|,
\]

with a Lipschitz constant \( L_2 = 2Q_1 x_{\text{max}} \). Hence, the Lipschitz conditions for \( J(\hat{n}) \) with a Lipschitz constant \( L = (4C + 6)QC_{T,x_{\text{max}}} + 2Q_1 x_{\text{max}} \) is shown.

Hence, by Theorem 3.4 the order of convergence of the FVS for the aggregation or breakage or coupled processes is same as the order of consistency which we have seen before in Lemma 3.5.

### 4 Numerical Results

The mathematical results on convergence analysis are verified numerically for pure aggregation, breakage and also for the combined processes considering several test problems. All numerical simulations below were carried out to investigate the experimental order of convergence (EOC) on four different types of meshes discussed in the next subsection.

If the problem has analytical solutions, the following formula is used to calculate the EOC

\[
\text{EOC} = \frac{\ln(E_I/E_{2I})}{\ln(2)}. \quad (41)
\]

Here \( E_I \) and \( E_{2I} \) are the discrete relative error norms calculated by dividing the error \( \| N - \hat{N} \| \) by \( \| N \| \) where \( N, \hat{N} \) are the number of particles obtained mathematically and numerically, respectively. The symbols \( I \) and \( 2I \) correspond to the number of degrees of freedom.

Now, in case of unavailability of the analytical solutions, the EOC can be computed as

\[
\text{EOC} = \ln\left( \frac{\| \hat{N}_I - \hat{N}_{2I} \|}{\| \hat{N}_{2I} - \hat{N}_{4I} \|} \right) / \ln(2),
\]

(42)

where \( \hat{N}_I \) is obtained by the numerical scheme using a mesh with \( I \) degrees of freedom.

Before going into the details of the test cases, in the following subsection we discuss briefly four different types of uniform and non-uniform meshes where global truncation errors are obtained numerically. These meshes have also been used in J. Kumar and Warnecke [9].

#### 4.1 Meshes

**Uniform mesh**: A uniform mesh is obtained when \( \Delta x_i = \Delta x \) for all \( i \).

**Non-uniform smooth mesh**: We are familiar with such a mesh from the previous section and Figure 1. For the numerical computations, a geometric mesh is considered.
**Oscillatory mesh:** The numerical verification has been done on an oscillatory mesh by taking \( r = 2 \) in the equation (33). In this case, the EOC is evaluated numerically by dividing the computation domain into 30 uniform mesh points initially. Then each cell is divided by a 1:2 ratio on further levels of computation.

**Random mesh:** Similar to the previous case, we started again with a geometric mesh with 30 grid points but then each cell is divided into two parts of random width in the further refined levels of computation. Here, we performed ten runs on different random grids and the relative errors are measured. The average of these errors over ten runs is used to calculate the EOC.

### 4.2 Numerical examples

#### 4.2.1 Pure aggregation

**Test case 1:**

The numerical verification of the EOC of the FVS for aggregation is discussed by taking two problems, namely the case of sum and product aggregation kernels. The analytical solutions for both problems taking the negative exponential \( n(0, x) = \exp(-\alpha x) \) as initial condition has been given in Scott [27]. Hence, the EOC is computed by using the relation (41). Table 1 shows that the EOC is 2 on uniform and non-uniform smooth meshes and is 1 on oscillatory and random grids in both cases. The computational domain in this case is taken as \([1E−6, 1000]\) which corresponds to the \( \xi \) domain \([\ln(1E−6), \ln(1000)]\) for the exponential transformation \( x = \exp(\xi) \) for the geometric mesh. The parameter \( \alpha = 10 \) was taken in the initial condition. The simulation result is presented at time \( t = 0.5 \) and \( t = 0.3 \) respectively for the sum and the product aggregation kernels corresponding to the aggregation extent \( \hat{N}(t)/\hat{N}(0) \approx 0.80 \).

#### 4.2.2 Pure breakage

**Test case 2:**

Here, the EOC is calculated for the binary breakage \( b(x, y) = 2/y \) together with the linear and quadratic selection functions, i.e. \( S(x) = x \) and \( S(x) = x^2 \). The analytical solutions for such problems have been given in Ziff and McGrady [31] for a mono-disperse initial condition of size unity, i.e. \( n(0, x) = \delta(x − 1) \). Hence, by using the formula (41), we observe from the Table 2 that the FVS is second order convergent on all the grids. The computational domain in this case is taken as \([1E−3, 1]\). Since the rate of breaking particles taking quadratic selection function is less than that of linear selection function, we take \( t = 100, 200 \) for linear and quadratic selection functions, respectively. The time has been chosen differently for both the selection functions to have the same extent of breakage \( \hat{N}(t)/\hat{N}(0) \approx 22 \).

**Test case 3:**

Now the case of multiple breakage with the quadratic selection function \( S(x) = x^2 \) is considered where an analytical solution is not known. Therefore, the EOC is calculated using (42). For the
Table 1: EOC (41) of the numerical schemes for Test case 1.

(a) Uniform mesh

| Grid points | $\beta(x, y) = x + y$ | Error | EOC       | $\beta(x, y) = xy$ | Error | EOC       |
|-------------|-----------------------|-------|-----------|---------------------|-------|-----------|
| 60          | 0.24E-3               | -     | 0.0177    | 1.95                | 0.0045| 1.96      |
| 120         | 0.11E-3               | 1.93  | 0.0012    | 1.94                | 0.0003| 1.96      |
| 240         | 0.01E-3               | 1.94  | 0.0003    | 1.92                | 0.0001| 1.99      |

(b) Non-uniform smooth mesh

| Grid points | $\beta(x, y) = x + y$ | Error | EOC       | $\beta(x, y) = xy$ | Error | EOC       |
|-------------|-----------------------|-------|-----------|---------------------|-------|-----------|
| 60          | 0.0047                | -     | 0.0086    | 1.99                | 0.0023| 1.90      |
| 120         | 0.0012                | 1.98  | 0.0006    | 1.96                | 0.0006| 1.96      |
| 240         | 0.0003                | 1.96  | 0.0001    | 1.99                | 0.0001| 1.99      |

(c) Oscillatory mesh

| Grid points | $\beta(x, y) = x + y$ | Error | EOC       | $\beta(x, y) = xy$ | Error | EOC       |
|-------------|-----------------------|-------|-----------|---------------------|-------|-----------|
| 60          | 0.0029                | -     | 0.0048    | 1.29                | 0.0019| 1.29      |
| 120         | 0.014                 | 1.24  | 7.66E-4   | 1.31                | 2.8E-4| 1.21      |
| 240         | 2.20E-4               | 1.31  | 3.52E-4   | 1.12                | 1.5E-4| 1.02      |

(d) Random mesh

| Grid points | $\beta(x, y) = x + y$ | Error | EOC       | $\beta(x, y) = xy$ | Error | EOC       |
|-------------|-----------------------|-------|-----------|---------------------|-------|-----------|
| 60          | 0.79E-3               | -     | 0.0017    | 1.06                | 8.2E-4| 1.06      |
| 120         | 0.42E-3               | 0.98  | 2.8E-4    | 1.21                | 1.5E-4| 1.02      |
| 240         | 0.22E-3               | 1.02  | 2.8E-4    | 1.21                | 1.5E-4| 1.02      |

In numerical simulations, the following normal distribution as an initial condition is taken

$$n(0, x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$  \hfill (43)

The computations are made for two breakage functions considered by Diemer and Olson [2] and Ziff [32], respectively

- case(i): $b(x, y) = \frac{px^c(y-x)^{c+(p-2)}[c+(c+1)(p-1)]!}{y^{pc+p-1}c!(c+(c+1)(p-2))!}$, $p \in \mathbb{N}, p \geq 2$

- case(ii): $b(x, y) = \frac{12x^p}{y^{2p-1}} \left(1 - \frac{x}{y}\right)$.

In case(i) the relation $\int_0^y b(x, y)dx = p$ holds where $p$ gives the total number of fragments per breakage event. The parameter $c \geq 0$ is responsible for the shape of the daughter particle distribution, see also [28]. The numerical solutions are obtained using $p = 4, c = 2$. The second breakage function gives ternary breakage. For the numerical simulation the minimum and maximum values of $x$ are taken as $1E-3$ and 1 respectively. The time $t = 100$ is set to get the breakage extent $\hat{N}(t)/\hat{N}(0) \approx 22$ in case(i) while $t = 150$ is used for case(ii). As expected from the mathematical analysis, we again observe from the Table 3 that the FVS shows convergence of second order on all the meshes. The computations for higher values of $p$ up to 19 are also tested and observed that there is no marked difference in the EOC.

4.2.3 Coupled aggregation-breakage

Test case 4:
Table 2: EOC (41) of the numerical schemes for Test case 2.

(a) Uniform smooth mesh

| Grid points | $S(x) = x$ | Error | EOC | $S(x) = x^2$ | Error | EOC |
|-------------|------------|-------|-----|--------------|-------|-----|
| 60          | 0.3312     | -     | 0.1870 | -             | 0.1870 | -   |
| 120         | 0.0829     | 1.99  | 0.0482 | 1.95         | 0.0482 | 1.95|
| 240         | 0.0207     | 2.00  | 0.0126 | 1.94         | 0.0126 | 1.94|
| 480         | 0.0052     | 2.00  | 0.0034 | 1.90         | 0.0034 | 1.90|

(b) Non-uniform smooth mesh

| Grid points | $S(x) = x$ | Error | EOC | $S(x) = x^2$ | Error | EOC |
|-------------|------------|-------|-----|--------------|-------|-----|
| 60          | 0.0526     | -     | 0.1638 | -             | 0.1638 | -   |
| 120         | 0.0136     | 1.95  | 0.0423 | 1.95         | 0.0423 | 1.95|
| 240         | 0.0034     | 1.99  | 0.0112 | 1.92         | 0.0112 | 1.92|
| 480         | 0.0009     | 2.00  | 0.0031 | 1.85         | 0.0031 | 1.85|

(c) Oscillatory mesh

| Grid points | $S(x) = x$ | Error | EOC | $S(x) = x^2$ | Error | EOC |
|-------------|------------|-------|-----|--------------|-------|-----|
| 60          | 0.0577     | -     | 0.1310 | -             | 0.1310 | -   |
| 120         | 0.0157     | 1.88  | 0.0376 | 1.80         | 0.0376 | 1.80|
| 240         | 0.0042     | 1.91  | 0.0105 | 1.84         | 0.0105 | 1.84|
| 480         | 0.0011     | 1.91  | 0.0030 | 1.82         | 0.0030 | 1.82|

(d) Random mesh

| Grid points | $S(x) = x$ | Error | EOC | $S(x) = x^2$ | Error | EOC |
|-------------|------------|-------|-----|--------------|-------|-----|
| 60          | 0.3516     | -     | 1.1106 | -             | 1.1106 | -   |
| 120         | 0.1001     | 1.81  | 0.3301 | 1.75         | 0.3301 | 1.75|
| 240         | 0.0282     | 1.83  | 0.0944 | 1.81         | 0.0944 | 1.81|
| 480         | 0.0078     | 1.85  | 0.0268 | 1.82         | 0.0268 | 1.82|

Table 3: EOC (42) of the numerical schemes for Test case 3.

(a) Uniform smooth mesh

| Grid points | case(i) | Error | EOC | case(ii) | Error | EOC |
|-------------|---------|-------|-----|----------|-------|-----|
| 60          | -       | -     | -   | -        | -     | -   |
| 120         | 2.0655  | -     | 4.7916 | -         | -     | -   |
| 240         | 0.6548  | 1.75  | 2.5829 | 2.16      | 2.5829 | 2.16|
| 480         | 0.1789  | 1.93  | 0.4364 | 1.91      | 0.4364 | 1.91|
| 960         | 0.0441  | 2.10  | 0.1792 | 1.67      | 0.1792 | 1.67|

(b) Non-uniform smooth mesh

| Grid points | case(i) | Error | EOC | case(ii) | Error | EOC |
|-------------|---------|-------|-----|----------|-------|-----|
| 60          | -       | -     | -   | -        | -     | -   |
| 120         | 0.0244  | -     | 0.0113 | -         | -     | -   |
| 240         | 0.0060  | -     | 0.0028 | 2.01      | 0.0028 | 2.01|
| 480         | 0.0015  | 1.98  | 0.0007 | 2.00      | 0.0007 | 2.00|
| 960         | 0.0004  | 2.02  | 0.0002 | 2.00      | 0.0002 | 2.00|

(c) Oscillatory mesh

| Grid points | case(i) | Error | EOC | case(ii) | Error | EOC |
|-------------|---------|-------|-----|----------|-------|-----|
| 60          | -       | -     | -   | -        | -     | -   |
| 120         | 0.78E-3 | -     | 0.91E-3 | -         | -     | -   |
| 240         | 0.21E-3 | 1.74  | 0.28E-3 | 1.84      | 0.28E-3 | 1.84|
| 480         | 0.06E-3 | 1.93  | 0.09E-3 | 1.92      | 0.09E-3 | 1.92|
| 960         | 0.01E-3 | 2.02  | 0.02E-3 | 1.95      | 0.02E-3 | 1.95|

(d) Random mesh

| Grid points | case(i) | Error | EOC | case(ii) | Error | EOC |
|-------------|---------|-------|-----|----------|-------|-----|
| 60          | -       | -     | -   | -        | -     | -   |
| 120         | 0.92E-3 | -     | 0.89E-3 | -         | -     | -   |
| 240         | 0.18E-3 | 1.71  | 0.14E-3 | 1.82      | 0.14E-3 | 1.82|
| 480         | 0.05E-3 | 1.82  | 0.02E-3 | 1.90      | 0.02E-3 | 1.90|
| 960         | 0.02E-3 | 1.91  | 0.01E-3 | 1.92      | 0.01E-3 | 1.92|
Finally, the EOC is evaluated for the simultaneous aggregation-breakage problem considering a constant aggregation kernel $\beta(x, y) = \beta_0$ and breakage kinetics $b(x, y) = 2/y, S(x) = x$. The analytical solutions for this problem are given by Lage [13] for the following two different initial conditions

- case(i): $n(0, x) = N_0 \left( \frac{2N_0}{x_0} \right)^2 x \exp \left(-2x \frac{N_0}{x_0} \right)$
- case(ii): $n(0, x) = N_0 \left( \frac{N_0}{x_0} \right) \exp \left(-x \frac{N_0}{x_0} \right)$.

This is a special case where the number of particles stays constant. The later initial condition is a steady state solution. For the simulation the computational domain $[1E-2, 10]$ with $N_0 = x_0 = 1$ and time $t = 0.3$ is taken. From Table 4, we find that the FVS is second order convergent on uniform and non-uniform smooth meshes and it gives first order on oscillatory and random meshes using (41). It should be mentioned that the computation has also been done for the product aggregation kernel $\beta(x, y) = xy$ and the linear selection function $S(x) = x$ taken together with two different general breakage functions as stated in the previous section. Analytical solutions are not available for such problems and so the EOC was calculated using (42). We observed again that the FVS shows similar results of convergence for these meshes.

5 Conclusions

In this article the convergence analysis of the finite volume techniques was studied for the non-linear aggregation and multiple breakage equations. We showed the consistency and then
proved the Lipschitz continuity of the numerical fluxes to complete the convergence results. This investigation was based on the basic existing theorems and definitions from the book of Hundsdorfer and Verwer [7] and the paper of Linz [17]. It was noticed that the scheme was second order convergent for a family of meshes for the pure breakage problem. For the aggregation and combined processes, it was not straightforward to evaluate the consistency and the convergence error on general meshes. This depended upon the type of grids chosen for the computations. Moreover, in these cases the method gave second order convergence on uniform and non-uniform smooth meshes while on non-uniform grids it showed only first order. The mathematical results of convergence analysis were verified numerically on several meshes by taking various examples of pure aggregation, pure breakage and the combined problems.

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A Bound on total number of particles

We give the proof of Lemmas 3.7 and 3.8 in Appendices A.1 and A.2, respectively.

A.1 Continuous aggregation and multiple breakage equation

*Proof. [Lemma 3.7]*

Integrating the equation (7) with respect to \( x \) from 0 to \( x_{\text{max}} \) gives

\[
\frac{d}{dt} \int_0^{x_{\text{max}}} n(t, x) dx = \int_0^{x_{\text{max}}} \frac{1}{x} \frac{\partial}{\partial x} (F^{agg} + F^{brk}) dx.
\]  

(44)

From the equations (8) and (9), we know that

\[
\frac{\partial}{\partial x} (F^{agg}(t, x)) = \frac{\partial}{\partial x} \int_0^x \int_{x-u}^{x_u} u\beta(u, v)n(t, u)n(t, v)dvdu
\]

and

\[
\frac{\partial}{\partial x} (F^{brk}(t, x)) = -\frac{\partial}{\partial x} \int_x^{x_{\text{max}}} \int_0^x ub(u, v)S(v)n(t, v)dudv.
\]

Applying the Leibniz integration rule on each of the flux separately ensures

\[
\frac{\partial}{\partial x} (F^{agg}(t, x)) = \int_0^{x_{\text{max}}} x\beta(x, v)n(t, x)n(t, v)dv - \int_0^x u\beta(u, x-u)n(t, u)n(t, x-u)du
\]

(45) and

\[
\frac{\partial}{\partial x} (F^{brk}(t, x)) = -\int_x^{x_{\text{max}}} x\beta(x, v)n(t, x)n(t, v)dv + \int_0^x ub(u, x)S(x)n(t, x)du.
\]

(46)

Inserting (45) and (46) into (44) to get

\[
\frac{dN(t)}{dt} = \int_0^{x_{\text{max}}} \int_0^x u\beta(u, x-u)n(t, u)n(t, x-u)dudx - \int_0^{x_{\text{max}}} \int_0^{x_{\text{max}}} \beta(x, v)n(t, x)n(t, v)dvdx
\]

\[
+ \int_0^{x_{\text{max}}} \int_x^{x_{\text{max}}} b(x, v)S(v)n(t, v)dvdx - \int_0^{x_{\text{max}}} \int_0^x \frac{u}{x} b(u, x)S(x)n(t, x)du dx.
\]

(47)

Changing the order of integration for the first and third integrals on the rhs of (47) yields

\[
\frac{dN(t)}{dt} = \int_0^{x_{\text{max}}} \int_0^x \beta(u, x-u)n(t, u)n(t, x-u)dudx - \int_0^{x_{\text{max}}} \int_0^{x_{\text{max}}} \beta(x, v)n(t, x)n(t, v)dvdx
\]

\[
+ \int_0^{x_{\text{max}}} \int_0^v b(x, v)S(v)n(t, v)dvdx - \int_0^{x_{\text{max}}} \int_0^x \frac{u}{x} b(u, x)S(x)n(t, x)du dx.
\]

(48)

Since \( x \geq u \) for the first integral, this implies that \( u/x \leq 1 \). Substituting \( x = z + u \) such that \( dx = dz \), the above can be rewritten as

\[
\frac{dN(t)}{dt} \leq \int_0^{x_{\text{max}}} \int_0^z \beta(u, z)n(t, u)n(t, z)dzdu - \int_0^{x_{\text{max}}} \int_0^{x_{\text{max}}} \beta(x, v)n(t, x)n(t, v)dvdx
\]

\[
+ \int_0^{x_{\text{max}}} \int_0^v S(v)n(t, v)dvdx - \int_0^{x_{\text{max}}} \int_0^z \frac{S(x)n(t, x)}{x} dzdu.
\]

Notice that the first two integrals combined give a negative value. Using the relation (3) of the breakage function in the last integral and due to negativity

\[
\frac{dN(t)}{dt} \leq \int_0^{x_{\text{max}}} S(v)n(t, v) \int_0^v b(x, v)dv dx.
\]

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From the bounds \(12\) we know that \(bS \leq Q_1\). Estimating \(v \leq x_{\text{max}}\) leads to
\[
\frac{dN(t)}{dt} \leq Q_1 x_{\text{max}} N(t).
\]
Therefore, the total number of particles is bounded and the bound is given as
\[
N(t) \leq N(0) \exp(x_{\text{max}} Q_1 t) \leq N(0) \exp(x_{\text{max}} Q_1 T) = C_{T,x_{\text{max}}},
\]
\[\square\]

A.2 Discrete aggregation and multiple breakage equation

**Proof.** [Lemma 3.8] by \(\Delta x_i/x_i\) and summing with respect to \(i\) gives
\[
\frac{d}{dt}(\sum_{i=1}^{l} \tilde{n}_i(t) \Delta x_i) = -\sum_{i=1}^{l} \frac{1}{x_i} \left[ J_{i+1/2}^{\text{agg}} - J_{i-1/2}^{\text{agg}} + J_{i+1/2}^{\text{brk}} - J_{i-1/2}^{\text{brk}} \right].
\]
We write out the summation over \(i\) of the aggregation fluxes \(J_{i\pm 1/2}^{\text{agg}}\) to get
\[
-\sum_{i=1}^{l} \frac{1}{x_i} \left[ J_{i+1/2}^{\text{agg}} - J_{i-1/2}^{\text{agg}} \right] = \frac{1}{x_1} J_{1/2}^{\text{agg}} - J_{1+1/2}^{\text{agg}} \left( \frac{1}{x_1} - \frac{1}{x_2} \right) - \cdots - J_{I-1/2}^{\text{agg}} \left( \frac{1}{x_{I-1}} - \frac{1}{x_I} \right) - \frac{1}{x_I} J_{I+1/2}^{\text{agg}}.
\]
For the breakage fluxes \(J_{i\pm 1/2}^{\text{brk}}\) in \(49\) we substitute the definition \(16\). Introducing the notations \(\tilde{N}_i(t) = \tilde{n}_i(t) \Delta x_i\) and \(\tilde{N}(t) = \sum_{i=1}^{l} \tilde{N}_i(t)\) ensures
\[
\frac{d\tilde{N}(t)}{dt} = \frac{1}{x_1} J_{1/2}^{\text{agg}} - \sum_{i=1}^{l-1} J_{i+1/2}^{\text{agg}} \left( \frac{1}{x_i} - \frac{1}{x_{i+1}} \right) - \frac{1}{x_I} J_{I+1/2}^{\text{agg}} + \sum_{i=1}^{l} \sum_{j=1}^{l} \tilde{N}_i(t) S(x_i) b(x_i, x_j) \Delta x_i - \sum_{i=1}^{l} \tilde{N}_i(t) S(x_i) \sum_{j=1}^{i} \frac{x_j}{x_i} b(x_j, x_i) \Delta x_j.
\]
Due to positivity of \(J_{i+1/2}^{\text{agg}}\) for all \(i\) and \(J_{1/2}^{\text{agg}} = 0\), we estimate
\[
\frac{d\tilde{N}(t)}{dt} \leq \sum_{i=1}^{l} \sum_{j=1}^{l} \tilde{N}_i(t) S(x_i) b(x_i, x_j) \Delta x_i - \sum_{i=1}^{l} \tilde{N}_i(t) S(x_i) \sum_{j=1}^{i} \frac{x_j}{x_i} b(x_j, x_i) \Delta x_j.
\]
Changing the order of summation for the first term and the summation indices in the second term yield
\[
\frac{d\tilde{N}(t)}{dt} \leq \sum_{k=1}^{l} \tilde{N}_k(t) S(x_k) \left[ \sum_{i=1}^{k-1} b(x_i, x_k) \Delta x_i (1 - x_i/x_k) \right].
\]
Since \(i < k\) implies that \(1 - x_i/x_k < 1\). Having the bound \(bS \leq Q_1\) gives \(d\tilde{N}(t)/dt \leq x_{\text{max}} Q_1 \tilde{N}(t)\). Therefore, the following bound is obtained on the total number of particles by using the FVS as
\[
\tilde{N}(t) \leq \tilde{N}(0) \exp(x_{\text{max}} Q_1 t) \leq \tilde{N}(0) \exp(x_{\text{max}} Q_1 T) = C_{T,x_{\text{max}}},
\]
which is the same bound as explained in the previous lemma, provided \(\tilde{N}(0) = N(0)\). \[\square\]