Distributed Stochastic Approximation Algorithm With Expanding Truncations

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Abstract

The distributed stochastic approximation (DSA) is used to seek the roots of a function being the sum of local functions, each of which is assigned to an agent from multiple agents connected in a network. Each agent obtains the local information composed of the observation of its local function and the information shared from its neighboring agents. Based on the local information, a DSA algorithm with expanding truncations (DSAAWET) is proposed for each agent in the paper. Here a network expanding truncation mechanism is introduced, by which the conditions required for the local functions and for the observation noises have greatly been weakened in comparison with the existing results. It is also shown that all estimates approach to a consensus value belonging to the root set of the sum function under some weak assumptions. A numerical example on the distributed optimization is given to demonstrate the theoretic results given in the paper.

Index Terms

Distributed stochastic approximation algorithm, multi-agent system, expanding truncations, strong consistency.

I. INTRODUCTION

A. Background and Motivations

For recent years, various distributed algorithms have been proposed in connection with the problems arising from sensor networks and networked systems, for example, the consensus problem [1]-[4], the

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distributed parameter estimation [5] [6], the distributed adaptive filter [7], the sensor localization [8], the distributed optimization [9] [12], the distributed stochastic approximation [13], the distributed control [14] [15] and so on. In contrast to the centralized algorithms where all information is transferred to a fusion center and the collected information is processed there, the distributed algorithm works in the situation where the goal is cooperatively accomplished by a network of agents with computation and communication abilities allocated in the distributed environment.

DSA solves the distributed root-seeking problem, while SA solves the problem in a centralized setting. SA is important because many problems from systems and control can be reduced to estimating unknown parameters, which can always be treated as roots of some regression functions that can be observed but the observations may be corrupted by noises. So, the original systems and control problem is transformed to searching roots of some regression function based on its noisy observations. The root-seeking problem with noisy observations was first solved in [16], proposing the first SA algorithm, which is now called as the Robbins-Monro (RM) algorithm. Attempt to weaken conditions for convergence of the RM algorithm has been made not only with respect to the regression function but also red to the observation noise [17], [18], [19]. Among various modifications the stochastic approximation algorithm with expanding truncations (SAWET) [18], [20] requires possibly the weakest conditions on the regression function and on the observation noise for its convergence. Applying the RM algorithm and SAWET to solve systems and control problems is well addressed in [18], [19], [20], and in many papers as well, e.g., [21], [22], [23] among others.

The distributed SAWET (DSAWET) is proposed in the paper to estimate the roots of a function, being the sum of local functions. A local function is assigned to each agent from a network of multiple agents. For each agent an SAWET is proposed on the basis of its local information, which is composed of 1) the observation of its local function possibly corrupted by noise and 2) the information shared from its neighboring agents. The totality of all these algorithms composes DSAWET.

The key issue of designing DSAWET is to propose a network expanding truncation mechanism, which is an extension of the expanding truncation technique proposed in [18], but to prove that the truncation ceases in a finite number of steps is much more complicated in comparison with the centralized situation.

The estimates given by DSAWET proposed in the paper are shown to approach to a consensus value belonging to the root set of the sum function under some weak assumptions. In comparison with the existing results (e.g., [13]), except the network topology, the conditions imposed in the paper on the local functions and on the observation noise are considerably weaker than those used in other papers. To be more precise, in the present paper the observation noise is allowed to be state-dependent and the local
functions have no growth rate restriction.

The remainder of this paper is organized as follows. In Section II, the distributed root-seeking problem is formulated, and DSAAWET is proposed to estimate the root of the sum function. The convergence result for the proposed algorithm is formulated in Section III, while its proof is presented in Section IV with some details placed in Appendices. A numerical example is demonstrated in Section V, and some concluding remarks are given in Section VI.

B. Graph Theory Notations

Consider a network of $N$ agents. The communication relationship between agents is described by the digraph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}_G, \mathcal{A}_G\}$, where $\mathcal{V} = \{1, \cdots, N\}$ is the node set, and the node $i$ represents the agent $i$; $\mathcal{E}_G \subset \mathcal{V} \times \mathcal{V}$ is the edge set, and $(j, i) \in \mathcal{E}_G$ if and only if $i$ can obtain the information from $j$ by assuming $(i, i) \in \mathcal{E}_G$; $\mathcal{A}_G = [a_{ij}] \in \mathbb{R}^{N \times N}$ is the adjacent matrix of $\mathcal{G}$, where $a_{ij} > 0$ if $(j, i) \in \mathcal{E}_G$, and $a_{ij} = 0$, otherwise. If for any $i \in \mathcal{V}$ such that $\sum_{j=1}^{N} a_{ji} = \sum_{j=1}^{N} a_{ij}$, then $\mathcal{G}$ is called the balanced digraph.

For a given pair $i, j \in \mathcal{V}$, if there exists a sequence of nodes $i_1, \cdots, i_p$ such that $(i, i_1) \in \mathcal{E}_G$, $(i_1, i_2) \in \mathcal{E}_G, \cdots, (i_p, j) \in \mathcal{E}_G$, then $(i, i_1, \cdots, i_p, j)$ is called the directed path from $i$ to $j$, and of which the length is $p + 1$. Denote by $d_{i,j}$ the length of the shortest directed path from $i$ to $j$. If there exists a directed path from $i$ to $j$ for any $i, j \in \mathcal{V}$, then $\mathcal{G}$ is called strongly connected.

A matrix $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ is called nonnegative and is denoted by $A \geq 0$, if $a_{i,j} \geq 0 \forall i, j = 1, \cdots, n$. A nonnegative square matrix $A$ is called doubly stochastic if $A1 = 1$ and $1^T A = 1^T$, where $1$ denotes the vectors of appropriate dimensions with all entries equal to 1, and $X^T$ denotes the transpose of $X$.

For a given nonnegative matrix $\Lambda = [\lambda_{ij}] \in \mathbb{R}^{n \times n}$ with positive diagonal entries, denote by $\mathcal{G}_\Lambda = \{\mathcal{V}, \mathcal{E}_{\mathcal{G}_\Lambda}, \mathcal{A}_{\mathcal{G}_\Lambda}\}$ the digraph generated by $\Lambda$, where $\mathcal{V} = \{1, \cdots, n\}$; $\mathcal{A}_{\mathcal{G}_\Lambda} = [\bar{a}_{ij}] \in \mathbb{R}^{n \times n}$ with $\bar{a}_{ij} = a_{i,j} \ \forall i, j \in \mathcal{V}; (j, i) \in \mathcal{E}_{\mathcal{G}_\Lambda}$ if $\bar{a}_{ij} > 0$.

II. DSAAWET

In this section, we first formulate the distributed root-seeking problem, which many practical problems are reduced to, and then define DSAAWET.

A. Problem formulation

Consider the case where all agents in a network cooperatively search the root of the sum function:

$$f(x) = \frac{1}{N} \sum_{i=1}^{N} f_i(x),$$  (1)
where $f_i(\cdot) : \mathbb{R}^l \to \mathbb{R}^l$ is the local function assigned to the agent $i$ and can only be observed by the agent $i$. In addition to observing its local function, each agent obtains the shared information from its neighboring agents. These compose the local information of the agent. It is required to design an algorithm to seek the roots of $f(\cdot)$ on the basis of the local information. The root set of $f(\cdot)$ is denoted by $J \triangleq \{ x \in \mathbb{R}^l : f(x) = 0 \}$.

B. Examples of distributed root-seeking

Many distributed problems from systems and control can be transformed to distributed root-seeking. Let us consider two simple examples.

In [9]-[12] the goal of the network is to cooperatively solve the following unconstrained optimization problem

$$
\min_x c(x) = \sum_{i=1}^{N} c_i(x),
$$

(2)

where $c_i : \mathbb{R}^l \to \mathbb{R}$ is the local objective function of $i$, and $c_i$ can only be observed by $i$ itself. If the cost functions $c_i, i = q, \ldots, N$ are differentiable, then this unconstrained distributed optimization problem (2) becomes collectively finding root of the function $f(x) = \sum_{i=1}^{N} f_i(x)$ with $f_i = -\nabla c_i$, where $\nabla c_i$ is the gradient of $c_i$.

In the distributed adaptive filter considered by [7] the goal of the network is to estimate the $M \times 1$ unknown signal $s^0$ from measurements collected by $N$ agents. Each agent $i$, $i = 1, \ldots, N$ at time $k$ obtains an autoregressive matrix $H_{i,k} \in \mathbb{R}^{d_i \times M}$ and an observation $Y_{i,k} \in \mathbb{R}^{d_i \times 1}$. The objective of the network is to cooperatively find $s \in \mathbb{R}^{M \times 1}$ such that

$$
E \| Y_k - H_k s \|_2^2 = \min_{s^0} E \| Y_k - H_k s^0 \|_2^2,
$$

(3)

where $Y_k \triangleq \text{col}\{Y_{1,k}, \ldots, Y_{N,k}\} = (Y_{1,k}^T, \ldots, Y_{N,k}^T)^T \in \mathbb{R}^{d \times 1}$, $H_k \triangleq \text{col}\{H_{1,k}, \ldots, H_{N,k}\} \in \mathbb{R}^{d \times M}$ with $d = \sum_{i=1}^{N} d_i$.

If for $i = 1, \ldots, N$

$$
EH_{i,k}^T H_{i,k} \triangleq R_{i,h}, \quad EH_{i,k}^T Y_{i,k} \triangleq R_{i,hy} \quad \forall k \geq 0,
$$

then solving (3) is reduced to seeking roots of $f(\cdot)$ defined by (1) with $f_i(x) = -R_{i,h}x + R_{i,hy}, i = 1, \ldots, N$. 

C. Definition of DSAAWET

We use a nonnegative matrix \( W(k) = [\omega_{ij}(k)]_{i,j=1}^{N} \) with positive diagonal entries to describe the communication network at time \( k \), and the corresponding digraph is denoted by \( \mathcal{G}_W(k) = \{\mathcal{V}, E_{\mathcal{G}_W(k)}, A_{\mathcal{G}_W(k)}\} \). Denote by \( N_i(k) = \{j \in \mathcal{V} : \omega_{ij}(k) > 0\} \) the neighboring agents of the agent \( i \) at time \( k \).

Denote by \( x_{i,k} \in \mathbb{R}^l \) the estimate for the root of \( f(\cdot) \) given by the agent \( i \) at time \( k \) for any \( i \in \mathcal{V} \).

While obtaining the information shared from its neighboring agents, the agent \( i \) has its local observation

\[
O_{i,k+1} = f_i(x_{i,k}) + \epsilon_{i,k+1},
\]

where \( \epsilon_{i,k+1} \) is the observation noise.

In the centralized environment \( f(\cdot) \) is observed at \( x_k \), the estimate of its root at time \( k \), and SAAWET [18], [20] is applied to generate \( x_{k+1} \), while in the distributed environment each agent can only observe the value of its local function at its local estimate \( x_{i,k} \) as shown in (4). In general, \( x_{i,k}, i = 1, \cdots, N \) are not the same.

DSAAWET to be defined consists of i) the consensus part making all local estimates to tend to the same value and ii) the innovation part assuring the consensus value to be in the root set \( J \). The network expanding truncation mechanism is designed to guarantee the boundedness of the estimates without imposing restrictive conditions on \( f_i(\cdot), i = 1, \cdots N \).

Let \( x^* \) be a fixed point in \( \mathbb{R}^l \), and let \( \{M_k\} \) be a sequence of positive numbers increasingly diverging to infinity with \( M_1 \geq ||x^*|| \). The estimate sequence \( \{x_{i,k}\}_{k \geq 1} \) of agent \( i \) is produced by

\[
\sigma_{i,0} = 0, \quad \hat{\sigma}_{i,k} \overset{\Delta}{=} \max_{j \in N_i(k)} \sigma_{j,k},
\]

\[
x'_{i,k+1} = \left( \sum_{j \in N_i(k)} \omega_{ij}(k)\left(x_{j,k}I_{[\sigma_j,k = \hat{\sigma}_{i,k}]} + x^*I_{[\sigma_j,k < \hat{\sigma}_{i,k}} \right) + \gamma_k O_{i,k+1} \right)I_{[\sigma_{i,k} = \hat{\sigma}_{i,k}]} + x^*I_{[\sigma_{i,k} < \hat{\sigma}_{i,k}]}),
\]

\[
x_{i,k+1} = x^*I_{||x'_{i,k+1}|| > M_{\sigma_{i,k}}} + x'_{i,k+1}I_{||x'_{i,k+1}|| \leq M_{\sigma_{i,k}}},
\]

\[
\sigma_{i,k+1} = \hat{\sigma}_{i,k} + I_{||x'_{i,k+1}|| > M_{\sigma_{i,k}}},
\]

where \( O_{i,k+1} \) is defined by (4), \( \gamma_k > 0 \) is the step size, and \( I_A \) is the indicator function of a random event \( A \), i.e.,

\[
I_A(\omega) \overset{\Delta}{=} \begin{cases} 
1, & \text{if } \omega \in A, \\
0, & \text{if } \omega \notin A.
\end{cases}
\]

We call \( \sigma_{i,k} \) the truncation number of the agent \( i \) up-to-time \( k \). From (5)-(8), we see that \( x_{i,k+1} \) is generated by the following three steps:
1) Agent $i$ sets its truncation number to be the largest one, denoted by $\hat{\sigma}_{i,k}$, among the truncation numbers of its neighbors.

2) Agent $i$ produces a new intermediate estimate $x'_{i,k+1}$ which is simply set to be the fixed point $x^*$ whenever $\sigma_{i,k} < \hat{\sigma}_{i,k}$; and is a combination of the consensus part being a weighted average of the estimates obtained at its neighboring agents with the innovation part processing the information contained in the local current observation $O_{i,k+1}$, if $\sigma_{i,k} = \hat{\sigma}_{i,k}$.

3) Agent $i$ defines its estimate $x_{i,k+1}$ at time $k + 1$: If $\|x'_{i,k+1}\|$ remains in the truncation bound, i.e., $\|x'_{i,k+1}\| \leq M\hat{\sigma}_{i,k}$, then $x_{i,k+1} = x'_{i,k+1}$; Otherwise, if $x'_{i,k+1}$ exits from the truncation bound $M\hat{\sigma}_{i,k}$, then set $x_{i,k+1} = x^*$ and the truncation bound is enlarged from $M\hat{\sigma}_{i,k}$ to $M\hat{\sigma}_{i,k+1}$.

Remark 2.1: First, it is noticed that $\sigma_{i,k+1} \geq \hat{\sigma}_{i,k} \geq \sigma_{i,k} \forall k \geq 0$ by (5) and (8). Further, it is concluded that $x_{i,k+1} = x^*$ if $\sigma_{i,k+1} > \sigma_{i,k}$. This can be seen from the following consideration:

i) If $\sigma_{j,k} \leq \sigma_{i,k} \forall j \in N_i(k)$, then from (5) we derive $\hat{\sigma}_{i,k} = \sigma_{i,k}$. From the assumption $\sigma_{i,k+1} > \sigma_{i,k}$, by (8) it follows that $\|x'_{i,k+1}\| > M\hat{\sigma}_{i,k}$, and hence from (7) we derive $x_{i,k+1} = x^*$. ii) If there exists $j \in N_i(k)$ such that $\sigma_{j,k} > \sigma_{i,k}$, then from (5) we derive $\hat{\sigma}_{i,k} = \max_{j \in N_i} \sigma_{j,k} > \sigma_{i,k}$, and from (6) we have $x'_{i,k+1} = x^*$. Consequently, by (7) we have $x_{i,k+1} = x^*$.

### III. Convergent Result

In this section, we give the assumptions and show the corresponding convergence property of the estimates $\{x_{i,k}\}$ generated by (5)-(8).

#### A. Assumptions

We list the assumptions to be used.

A1 $\gamma_k > 0$, $\gamma_k \xrightarrow[k \to \infty]{} 0$, and $\sum_{k=1}^{\infty} \gamma_k = \infty$.

A2 There exists a continuously differentiable function $v(\cdot) : \mathbb{R}^l \to \mathbb{R}$ such that

$$\sup_{\Delta \leq d(x,J) \leq \Delta} f^T v(x) < 0$$

for any $\Delta \geq \delta > 0$, where $d(x,J) = \inf_y \{\|x - y\| : y \in J\}$ and $v_x(\cdot)$ denotes the gradient of $v(\cdot)$. Further, $v(J) \triangleq \{v(x) : x \in J\}$ is nowhere dense and there exists a constant $c_0 > 0$ such that $x^*$ used in (5) satisfies

$$\|x^*\| < c_0, \text{ and } v(x^*) < \inf_{\|x\| = c_0} v(x).$$

A3 The functions $f_i(\cdot)$ $\forall i \in V$ are continuous.

A4 i) $W(k) \forall k \geq 0$ are doubly stochastic matrices with positive diagonal entries;
ii) There exists a constant \( 0 < \eta < 1 \) such that
\[
\omega_{ij}(k) \geq \eta \quad \forall j \in N_i(k) \quad \forall i \in \mathcal{V} \quad \forall k \geq 0;
\]

iii) The digraph \( \mathcal{G}_\infty = \{\mathcal{V}, \mathcal{E}_\infty\} \) is strongly connected, where
\[
\mathcal{E}_\infty = \{(j, i) : (j, i) \in \mathcal{E}_{W(k)} \text{ for infinitely many indices } k\};
\]

iv) There exists a positive integer \( B \) such that
\[
(j, i) \in \mathcal{E}_{W(k)} \cup \mathcal{E}_{W(k+1)} \cup \cdots \cup \mathcal{E}_{W(k+B-1)} \quad \forall (j, i) \in \mathcal{E}_\infty \quad \forall k \geq 0.
\]

A5 Along indices \( \{n_k\} \) of any convergent subsequence \( x_{i,n_k} \)
\[
\lim_{T \to 0} \limsup_{k \to \infty} \frac{1}{T} \left\| \sum_{m=n_k}^{m(n_k,t_k)} \gamma_m \varepsilon_{i,m+1} I[\|x_{i,m}\| \leq K] \right\| = 0 \quad \forall t_k \in [0, T], \quad \text{and} \quad \gamma_k \varepsilon_{i,k+1} \xrightarrow[k \to \infty]{} 0
\]
for sufficiently large \( K \) and any \( i \in \mathcal{V} \), where \( m(n_k,t_k) = \max\{m : \sum_{s=n_k}^{m} \gamma_s \leq t_k\} \).

**Remark 3.1:** If \( f(\cdot) \) is the gradient of some function \( c(\cdot) \), i.e., \( f(x) = c_x(x) \), then in A2 one may take \( v(x) = -f(x) \) to satisfy (9).

Conditions A4 describes the connectivity property of the communication graph. Refer to [9] for more detailed explanations about this communication topology assumption. Set
\[
\Phi(k, k+1) = I, \quad \Phi(k, s) = W(k) \cdots W(s) \quad \forall k \geq s.
\]
If A4 holds, then by Proposition 1 [9] there exist constants \( c > 0 \) and \( 0 < \rho < 1 \) such that
\[
\| \Phi(k, s) - \frac{1}{N} 11^T \| \leq c\rho^{k-s+1} \quad \forall k \geq s.
\]

For a given undirected graph \( \mathcal{G} = \{\mathcal{V}, \mathcal{E}_G\} \), the agent \( i \) is always assumed to be a neighboring agent of itself. If the entries of \( A_G = [a_{ij}]_{i,j=1}^N \) are Metropolis weights [24], then \( A_G \) is a doubly stochastic matrix with positive diagonal entries.

**B. Convergence Theorem**

Introduce the following notations: \( D_\bot \triangleq (I_N - \frac{11^T}{N}) \otimes I \), where \( I_N \in \mathbb{R}^{N \times N} \) is the identity matrix, and \( \otimes \) is the Kronecker product,
\[
X_k \triangleq \text{col}\{x_{1,k}, \cdots, x_{N,k}\}, \quad \varepsilon_k \triangleq \text{col}\{\varepsilon_{1,k}, \cdots, \varepsilon_{N,k}\}, \quad F(X_k) \triangleq \text{col}\{f_1(x_{1,k}), \cdots, f_N(x_{N,k})\}.
\]

Set \( X_{\bot,k} = D_\bot X_k \) to denote the disagreement vector of \( X_k \), and \( x_k = \frac{1}{N} \sum_{i=1}^N x_{i,k} \) the average of the estimates for all the agents at time \( k \).
**Theorem 3.2:** Let \( \{x_{i,k}\} \) be produced by (5)-(8) for any initial value \( x_{i,0} \). Assume A1-A4 hold. Then
\[
X_{\perp,k} \xrightarrow[k \to \infty]{\text{\( \to \)}} 0 \quad \text{and} \quad d(x_k, J) \xrightarrow[k \to \infty]{\text{\( \to \)}} 0
\]
for the sample paths where A5 holds for all agents, where \( 0 \) denotes matrices or vectors of compatible dimensions with all entries equal to zero.

**Remark 3.3:** The DSA algorithm proposed in [10] is based on the RM algorithm:
\[
\tilde{\theta}_{i,k+1} = \theta_{i,k} + \gamma_k Y_{i,k+1},
\]
\[
\theta_{i,k+1} = \sum_{j=1}^{N} w_{k+1}(i, j) \tilde{\theta}_{j,k+1},
\]
where \( \theta_{i,k+1} \) is the estimate of the root derived at the agent \( i \) at time \( k+1 \), \( Y_{i,k+1} \) is the local observation of the agent \( i \), \( \gamma_k \) is a deterministic step size, and \( W_k := [w_k(i, j)] \in \mathbb{R}^{N \times N} \) is a nonnegative matrix with \( W(k)1 = 1 \).

Convergence analysis given in [10] is for the case where the growth rate of the functions is not faster than linear and the noise is of conditional zero-mean. In addition, a certain condition is imposed on the spectral radius of \( E(W_k^T (I_N - \frac{1}{N}) W_k) \).

Different from [10] and [13], the algorithm (5)-(8), as that given in [18] for the centralized environment, involves expanding truncations, which guarantee the boundedness of the estimates without requiring restrictive conditions on the functions by showing that the truncation ceases in a finite number of steps. As noise concerns, as shown in [18] and [20], A5 is probably the weakest requirement on the noise since it is also necessary for convergence whenever the root \( x^0 \) of \( f(\cdot) \) is a singleton and \( f(\cdot) \) is continuous at \( x^0 \).

**IV. PROOF OF THEOREM 3.2**

Assume A1-A4 hold. For any fixed sample path \( \omega \), where A5 holds for all the agents, the proof of Theorem 3.2 is carried out in the following way:

1) First, two auxiliary sequences \( \tilde{X}_k \) and \( \tilde{\varepsilon}_{k+1} \) are defined and their properties are clarified.
2) Second, the boundedness of \( \{\tilde{X}_k\} \) is proved and \( \{X_k\} \) is shown to differ from \( \tilde{X}_k \) only by finite number of steps.
3) Finally, it is proved that all estimates converge to the same value belonging to the root set \( J \).
A. Auxiliary Sequences

Set \( \tau_{i,m} \triangleq \inf\{k : \sigma_{i,k} = m\} \), the smallest time when the truncation number of agent \( i \) has reached \( m \), where \( \inf\{\emptyset\} \triangleq \infty \), and \( \tau_m \triangleq \min_{i \in \mathcal{V}} \tau_{i,m} \), the smallest time when at least one of agents has its truncation number reached \( m \).

**Lemma 4.1:** Set \( \tilde{\tau}_{j,m} \triangleq \tau_{j,m} \land \tau_{m+1} \), where \( a \land b = \min\{a, b\} \). If A4 holds, then for \( m \geq 1 \)

\[
\tilde{\tau}_{j,m} \leq \tau_m + BD \quad \forall j \in \mathcal{V},
\]

where \( D \triangleq \max_{i,j \in \mathcal{V}} d_{i,j} \) is the diameter of the digraph \( G_{\infty} \), and \( B \) is the positive integer given in A4 iv).

**Proof:** For any \( m \geq 1 \) let \( \tau_m = k_1 \). Then, there is an \( i \) such that \( \tau_{i,m} = k_1 \). Since \( G_{\infty} \) is strongly connected by A4 iii), for any \( j \in \mathcal{V} \) there exists a sequence of nodes \( i_1, i_2, \ldots, i_{d_{i,j} - 1} \) such that \((i, i_1, \ldots, i_{d_{i,j} - 1}, j)\) is the shortest path from \( i \) to \( j \) with length \( d_{i,j} \).

Noticing that \((i, i_1) \in \mathcal{E}_{G_{\infty}}\), by A4 iv) we have

\[
(i, i_1) \in \mathcal{E}_{G_{W(k_1)}} \cup \mathcal{E}_{G_{W(k_1+1)}} \cup \cdots \cup \mathcal{E}_{G_{W(k_1+B-1)}}.
\]

Therefore, there exists a positive integer \( k'_1 \in [k_1, k_1 + B - 1] \) such that

\[
(i, i_1) \in \mathcal{E}_{G_{W(k'_1)}}.
\]

So, \( i \in N_{i_1}(k'_1) \), and hence by (5) and (8) we derive

\[
\sigma_{i_1,k_1+B} \geq \sigma_{i_1,k_1+1} \geq \tilde{\sigma}_{i_1,k'_1} = \max_{p \in N_{i_1}(k'_1)} \sigma_{p,k'_1} \geq \sigma_{i_1,k'_1} \geq \sigma_{i_1,k_1} = m.
\]

Similarly, there exists a positive integer \( k'_2 \in [k_1 + B, k_1 + 2B - 1] \) such that

\[
(i_1, i_2) \in \mathcal{E}_{G_{W(k'_2)}}.
\]

Similar to (17) we derive

\[
\sigma_{i_2,k_1+2B} \geq \sigma_{i_2,k_2+1} \geq \tilde{\sigma}_{i_2,k'_2} = \max_{p \in N_{i_2}(k'_2)} \sigma_{p,k'_2} \geq \sigma_{i_1,k'_2} \geq \sigma_{i_1,k_1+B} \geq m.
\]

Continuing this estimation procedure, we finally reach the following inequality

\[
\sigma_{j,k_1+BD_{i,j}} \geq m \quad \forall j \in \mathcal{V}.
\]

For the case where \( \sigma_{j,k_1+BD_{i,j}} = m \) \( \forall j \in \mathcal{V} \), we have \( \tau_{j,m} \leq k_1 + BD_{i,j} \) \( \forall j \in \mathcal{V} \). By noticing \( \tau_m = k_1 \), from here we obtain (16):

\[
\tilde{\tau}_{j,m} \leq \tau_{j,m} \leq \tau_m + BD_{i,j} \leq \tau_m + BD \quad \forall j \in \mathcal{V}.
\]
For the case where \( \sigma_{j,k_1+Bd_{i,j}} > m \) for some \( j \in \mathcal{V} \), we must have

\[
\tau_{m+1} \leq k_1 + Bd_{i,j} \text{ for some } j \in \mathcal{V},
\]

(19)

because the converse event \( \{ \tau_{m+1} > k_1 + Bd_{i,j} \} \forall j \in \mathcal{V} \) leads to \( \sigma_{j,k_1+Bd_{i,j}} \leq m \) \( \forall j \in \mathcal{V} \), which contradicts with the assumption that \( \sigma_{j,k_1+Bd_{i,j}} > m \) for some \( j \in \mathcal{V} \).

Again, by noticing \( \tau_m = k_1 \), from (19) we obtain (16):

\[
\tilde{\tau}_{j,m} \leq \tau_{m+1} \leq \tau_m + d_{i,j} \leq \tau_m + BD \quad \forall j \in \mathcal{V}.
\]

Remark 4.2: If \( \tau_{m+1} > \tau_m + BD \) for some \( m \), then by (16) we must have \( \tilde{\tau}_{j,m} = \tau_{j,m} \), and again by (16) \( \tilde{\tau}_{j,m} \leq \tau_m + BD \). This means that the algorithm for any agent must have its \( m \)-th truncation after \( \tau_m \) in no more than BD steps. In other words, the truncation numbers of algorithms for all agents are all equal to \( m \) in the interval \( [\tau_m, \tau_{m+1}) \) with the possible exception at no more than \( BD \) steps.

Define two auxiliary sequences \( \{\tilde{x}_{i,k}\}_{k \geq 0} \) and \( \{\tilde{\varepsilon}_{i,k+1}\}_{k \geq 0} \) as follows:

\[
\tilde{x}_{i,k} \triangleq x^*, \quad \tilde{\varepsilon}_{i,k+1} \triangleq -f_i(x^*) \quad \forall k : \tau_m \leq k < \tilde{\tau}_{i,m}, \quad \tilde{x}_{i,k} \triangleq x_{i,k}, \quad \tilde{\varepsilon}_{i,k+1} \triangleq \varepsilon_{i,k+1} \quad \forall k : \tilde{\tau}_{i,m} \leq k < \tau_{m+1},
\]

(20)

(21)

where \( m \) is a nonnegative integer.

Notice that for a fixed sample path \( \omega \) there exists a unique nonnegative integer \( m \) corresponding to any nonnegative integer \( k \geq 0 \) such that \( \tau_m \leq k < \tau_{m+1} \) and by definition \( \tilde{\tau}_{i,m} \leq \tau_{m+1} \forall i \in \mathcal{V} \). So, \( \{\tilde{x}_{i,k}\}_{k \geq 0} \) and \( \{\tilde{\varepsilon}_{i,k+1}\}_{k \geq 0} \) are uniquely determined by the sequences \( \{x_{i,k}\}_{k \geq 0} \) and \( \{\varepsilon_{i,k+1}\}_{k \geq 0} \).

We now clarify properties of the sequences \( \{\tilde{x}_{i,k}\}_{k \geq 0} \) and \( \{\tilde{\varepsilon}_{i,k+1}\}_{k \geq 0} \).

Lemma 4.3: The sequences \( \{\tilde{x}_{i,k}\}, \{\tilde{\varepsilon}_{i,k+1}\} \) defined by (20) (21) satisfy the following recursive formulas

\[
\tilde{x}_{i,k+1} \triangleq \sum_{j \in N_i(k)} \omega_{ij}(k)\tilde{x}_{j,k} + \gamma_k(f_i(\tilde{x}_{i,k}) + \tilde{\varepsilon}_{i,k+1}),
\]

(22)

\[
\tilde{x}_{i,k+1} = \tilde{x}_{i,k+1}I_{\sum_{j \in \mathcal{V}}\|\tilde{x}_{j,k+1}\| \leq M_{\sigma_k}} + x^*I_{\sum_{j \in \mathcal{V}}\|\tilde{x}_{j,k+1}\| > M_{\sigma_k}},
\]

(23)

\[
\sigma_{k+1} = \sigma_k + I_{\sum_{j \in \mathcal{V}}\|\tilde{x}_{j,k+1}\| > M_{\sigma_k}}, \quad \sigma_0 = 0,
\]

(24)

where \( \sigma_k = \max_{i \in \mathcal{V}} \sigma_{i,k} \).

Proof: See Appendix A.

Remark 4.4: Write

\[
\tilde{X}_k \triangleq \text{col}\{\tilde{x}_{1,k}, \ldots, \tilde{x}_{N,k}\}, \quad \tilde{\varepsilon}_{k+1} \triangleq \text{col}\{\tilde{\varepsilon}_{1,k+1}, \ldots, \tilde{\varepsilon}_{N,k+1}\}.
\]

(25)
It is worth noting that in the case \( \lim_{k \to \infty} \sigma_k = \sigma < \infty \) the sequence \( \{\tilde{X}_k\}_{k \geq 1} \) coincides with the sequence \( \{X_k\}_{k \geq 1} \) in a finite number of steps. This is because \( \tau_{\sigma+1} = \infty \) by the definition of \( \tau_{\sigma+1} \), and hence \( \tilde{\tau}_{i,\sigma} = \tau_{i,\sigma} \wedge \tau_{\sigma+1} = \tau_{i,\sigma} \) and by (21) it follows that

\[
\tilde{x}_{i,k} = x_{i,k}, \quad \tilde{\varepsilon}_{i,k+1} = \varepsilon_{i,k+1} \quad \forall k \geq \tau_{i,\sigma}.
\]

By noticing \( \tau_{i,\sigma} \leq \tau_\sigma + BD \) by (16), this implies

\[
\tilde{X}_k = X_k, \quad \tilde{\varepsilon}_{k+1} = \varepsilon_{k+1} \quad \forall k \geq \tau_\sigma + BD.
\]  

**Lemma 4.5:** If A5 holds, then for any \( \{n_k\} \) such that \( \tilde{x}_{i,n_k} \) converges, and for sufficiently large \( K > 0 \) the following takes place

\[
\lim_{T \to 0} \limsup_{k \to \infty} \frac{1}{T} \left\| \sum_{s = n_k}^{m(n_k, t_k) \wedge (\tau_{n_k} + 1 - 1)} \gamma_s \tilde{\varepsilon}_{i,s} I[\|\tilde{x}_{i,s}\| \leq K] \right\| = 0 \quad \forall t_k \in [0, T].
\]

**Proof:** See Appendix B.

### B. Boundedness of Truncation Numbers

**Lemma 4.6:** Assume A1, A3-A5 hold. If \( \{\tilde{X}_{n_k}\} \) is a convergent subsequence, then there exist some constants \( c_0 > 0, c_1 > 0, M'_0 > 0, \) and \( T > 0 \) such that for sufficiently large \( k \)

\[
\|\tilde{X}_{m+1} - \tilde{X}_{n_k}\| \leq c_1 T_k + M'_0, \\
\|\tilde{x}_{m+1} - \tilde{x}_{n_k}\| \leq c_2 T_k \quad \forall m : n_k \leq m \leq m(n_k, t_k) \quad \forall T_k \in [0, T],
\]

where \( \tilde{x}_k \) is defined by

\[
\tilde{x}_k \triangleq \frac{1}{N} (1^T \otimes I) \tilde{X}_k = \frac{1}{N} \sum_{i=1}^{N} \tilde{x}_{i,k}.
\]

**Proof:** See Appendix C.

**Lemma 4.7:** Assume A1-A5 hold. Then any nonempty interval \( [\delta_1, \delta_2] \) with \( d([\delta_1, \delta_2], v(J)) > 0 \) cannot be crossed by infinitely many subsequences \( \{v(\tilde{x}_{n_k}), \cdots, v(\tilde{x}_{m_k})\} \) with \( \|\tilde{X}_{n_k}\| \) bounded, where by “crossing \([\delta_1, \delta_2]\) by \( \{v(\tilde{x}_{n_k}), \cdots, v(\tilde{x}_{m_k})\} \)” we mean that \( v(\tilde{x}_{n_k}) \leq \delta_1, v(\tilde{x}_{m_k}) \geq \delta_2 \), and \( \delta_1 < v(\tilde{x}_s) < \delta_2 \) \( \forall s : n_k < s < m_k \).

**Proof:** See Appendix D.

**Lemma 4.8:** Assume A1-A5 hold. Then \( \sigma_k \) produced by (22)-(24) is bounded, i.e.,

\[
\lim_{k \to \infty} \sigma_k = \sigma < \infty.
\]

**Proof:** See Appendix E.
C. Proof of Theorem 3.2

First, we show the boundedness of \{X_k\}. Since \(\lim_{k \to \infty} \sigma_k = \sigma < \infty\), from (24) it is seen that \{\hat{x}_{i,k} \forall i \in V\}_{k \geq 1} are bounded and by (23) \{\tilde{x}_{i,k} \forall i \in V\}_{k \geq 1} are also bounded.

By Remark 4.4 it is seen that the sequence \{\tilde{X}_k\}_{k \geq 1} may differ from the sequence \{X_k\}_{k \geq 1} only for a finite number of first steps. Since \{\tilde{X}_k\}_{k \geq 1} are bounded, we know that \{X_k\}_{k \geq 1} are also bounded.

Second, we show the convergence of \(v(\bar{x}_k)\). Since \{\bar{x}_k\} are bounded, let us set

\[ v_1 \triangleq \lim_{k \to \infty} \inf v(\bar{x}_k) \leq \lim_{k \to \infty} \sup v(\bar{x}_k) \triangleq v_2. \]

We want to prove \(v_1 = v_2\).

Assume the converse: \(v_1 < v_2\). Since \(v(J)\) is nowhere dense, there exists a nonnegative interval \([\delta_1, \delta_2] \in (v_1, v_2)\) such that \(d([\delta_1, \delta_2], v(J)) > 0\). Then \(v(\bar{x}_k)\) crosses the interval \([\delta_1, \delta_2]\) infinitely many times. This contradicts Lemma 4.7. Therefore, \(v_1 = v_2\), which implies the convergence of \(v(\bar{x}_k)\).

Third, we prove \(d(\bar{x}_k, J) \to 0\).

Assume the converse. Then by the boundedness of \{\bar{x}_k\} there exists a convergent subsequence \(\bar{x}_{n_k} \to \bar{x}\) with \(d(\bar{x}, J) \geq \vartheta > 0\). From (29) it follows that for sufficiently small \(t > 0\) and large \(k\)

\[ d(\bar{x}_s, J) > \frac{\vartheta}{2} \quad \forall s : n_k \leq s \leq m(n_k, t), \]

and hence from (9) there exists a constant \(b > 0\) such that

\[ v_x(\bar{x}_s)^T f(\bar{x}_s) < -b \quad \forall s : n_k \leq s \leq m(n_k, t). \]

As shown in Appendix D for obtaining (D.21), from here it follows that

\[ \lim_{k \to \infty} \sup v(\bar{x}_{m(n_k, t)+1}) \leq v(\bar{x}) - \frac{b}{2} \]

for sufficiently small \(t > 0\). This contradicts with the convergence of \(v(\bar{x}_k)\).

Finally, we show \(d(x_k, J) \to 0\) and \(X_{\perp,k} \to 0\)

Since \{\bar{x}_k\} may differ from \{\tilde{X}_k\} only by a finite number of first steps, \(x_k = \frac{1}{N} I_L X_k\) may differ from \(\bar{x}_k = \frac{1}{N} I_L \tilde{X}_k\) at most by a finite number of first steps. Therefore, \(d(x_k, J) \to 0\).

We now prove \(X_{\perp,k} \to 0\).

Noticing \(\lim_{k \to \infty} \sigma_k = \sigma < \infty\), we have

\[ \| \hat{x}_{i,k+1} \| \leq M_\sigma \quad \forall k \geq \tau_\sigma \quad \forall i \in V, \]  

(32)
because if there were a \( k_1 \geq \tau_\sigma \) and an \( i \in \mathcal{V} \) such that \( \| \hat{x}_{i,k_1} \| > M_\sigma \), then from (24) and \( \sigma_{k_1} \geq \sigma \) it would follow that \( \sigma_{k_1+1} = \sigma_{k_1} + 1 > \sigma \). This contradicts with \( \lim_{k \to \infty} \sigma_k = \sigma \). Thereby from \( \sigma_k \equiv \sigma \ \forall k \geq \tau_\sigma \) and (22), (23), and (32) it is seen that for all \( k \geq \tau_\sigma \)

\[
\hat{x}_{i,k+1} = \hat{x}_{i,k+1} = \sum_{j \in N_i(k)} \omega_{ij} (k) \hat{x}_{j,k} + \gamma_k (f_i (\hat{x}_{i,k}) + \hat{\varepsilon}_{i,k+1}) \ \forall i \in \mathcal{V}. 
\]

Then from (26) it follows that

\[
X_{k+1} = (W(k) \otimes I) X_k + \gamma_k (F(X_k) + \varepsilon_{k+1}) \quad \forall k \geq k_0
\]

with \( k_0 \triangleq \tau_\sigma + BD \), and hence similar to (C.38) we derive

\[
X_{\perp,s+1} = \left[ (\Phi (k, k_0) - \frac{1}{N} 11^T) \otimes I \right] X_{k_0} + \sum_{m = k_0}^{k} \gamma_m \left[ (\Phi (k-1,m) - \frac{1}{N} 11^T) \otimes I \right] F(X_m)
\]

\[
+ \sum_{m = k_0}^{k} \gamma_m \left[ (\Phi (s-1,m) - \frac{1}{N} 11^T) \otimes I \right] \varepsilon_{m+1}.
\]

Therefore, from (13) by the continuity of \( F(\cdot) \) and the boundedness of \( \{X_s\} \), we ensure that there exist positive constants \( c_1', c_2', c_3' \) such that

\[
\| X_{\perp,k+1} \| \leq c_1' \rho^{k+1-k_0} + c_2' \sum_{m = k_0}^{k} \gamma_m \rho^{k-m}
\]

\[
+ c_3' \sum_{m = k_0}^{k} \gamma_m \rho^{k-m} \| \varepsilon_{m+1} \| \quad \forall k \geq k_0.
\]

By noticing \( \gamma_k \xrightarrow[k \to \infty]{} 0 \) and \( \gamma_k \varepsilon_{k+1} \xrightarrow[k \to \infty]{} 0 \), for any \( \epsilon > 0 \) there exists a positive integer \( k_\epsilon > k_0 \) such that

\[
\| \gamma_k \varepsilon_{k+1} \| \leq \epsilon, \quad \sup_{k \geq k_\epsilon} \gamma_k \leq \epsilon.
\]

By noticing \( \gamma_k \xrightarrow[k \to \infty]{} 0 \) and \( \gamma_k \varepsilon_{k+1} \xrightarrow[k \to \infty]{} 0 \), for any \( \epsilon > 0 \) there exists a positive integer \( k_\epsilon > k_0 \) such that

\[
\| \gamma_k \varepsilon_{k+1} \| \leq \epsilon, \quad \sup_{k \geq k_\epsilon} \gamma_k \leq \epsilon.
\]

By noticing \( \gamma_k \xrightarrow[k \to \infty]{} 0 \) and \( \gamma_k \varepsilon_{k+1} \xrightarrow[k \to \infty]{} 0 \), for any \( \epsilon > 0 \) there exists a positive integer \( k_\epsilon > k_0 \) such that

\[
\| \gamma_k \varepsilon_{k+1} \| \leq \epsilon, \quad \sup_{k \geq k_\epsilon} \gamma_k \leq \epsilon.
\]

Considering the second part at the right-hand side of (33), from (34) we know

\[
\sum_{m = k_0}^{k} \gamma_m \rho^{k-m} = \sum_{m = k_0}^{k_{k-1}} \gamma_m \rho^{k-m} + \sum_{m = k_{k-1}}^{k-1} \gamma_m \rho^{k-m} \leq \sum_{m = k_0}^{k_{k-1}} \gamma_m \rho^{k-m} + \frac{\epsilon}{1 - \rho} \xrightarrow[k \to \infty]{} \frac{\epsilon}{1 - \rho}.
\]

Similarly, for the third part at the right-hand side of (33) we obtain

\[
\sum_{m = k_0}^{k} \gamma_m \rho^{k-m} \| \varepsilon_{m+1} \| = \sum_{m = k_0}^{k_{k-1}} \gamma_m \rho^{k-m} \| \varepsilon_{m+1} \|
\]

\[
+ \epsilon \sum_{m = k_{k-1}}^{k} \rho^{k-m} \leq \epsilon \rho^{k-k_0} \sum_{m = k_0}^{k_{k-1}} \gamma_m \| \varepsilon_{m+1} \| + \frac{\epsilon \epsilon}{1 - \rho} \xrightarrow[k \to \infty]{} \frac{\epsilon \epsilon}{1 - \rho}.
\]

Letting \( \epsilon \to 0 \), then from (33), (35) and (36) we conclude that

\[
X_{\perp,k} \xrightarrow[k \to \infty]{} 0.
\]

The proof is completed.
V. NUMERICAL EXAMPLE

The numerical example is on distributed optimization.

Consider a network of three agents with local cost functions given by

\[
L_1(x, y) = x^2 + y^2 + 10 \sin(x) + 12 \cos(y);
\]
\[
L_2(x, y) = (x - 4)^2 + (y - 1)^2 - 10 \sin(x) - 12 \cos(y) - 0.1y^3;
\]
\[
L_3(x, y) = 0.01(x - 2)^4 + (y - 2)^2 + 0.1y^3.
\]

(37)

The task is to find the minimum value of the cost function \( L(x, y) \) given by

\[
L(x, y) = \sum_{i=1}^{3} L_i(x, y) = x^2 + y^2 + (x - 4)^2 + (y - 1)^2 + 0.01(x - 2)^4 + (y - 2)^2.
\]

Though each local cost function is non-convex, the total cost function \( L(x, y) \) is convex. The minimum of \( L(x, y) \) is achieved at \( (x^0, y^0) \) with \( x^0 = 2, y^0 = 1 \)

Since the cost function is differentiable, the unconstraint distributed optimization consists in seeking the root of the function \( f(x, y) = \sum_{i=1}^{3} f_i(x, y) \), where \( f_i(x, y) \) is the gradient of \( L_i(x, y) \) given by

\[
f_1(x, y) = \begin{pmatrix} 2x + 10 \cos(x) \\ 2y - 12 \sin(y) \end{pmatrix},
\]
\[
f_2(x, y) = \begin{pmatrix} 2(x - 4) - 10 \cos(x) \\ 2(y - 1) + 12 \sin(y) - 0.3y^2 \end{pmatrix},
\]
\[
f_3(x, y) = \begin{pmatrix} 0.04(x - 2)^3 \\ 2(y - 2) + 0.3y^2 \end{pmatrix}.
\]

(38)

These functions are continuous, and the growth rates of \( f_2(x, y) \) and \( f_3(x, y) \) are faster than linear.

Let the communication relationship of the network be described by the matrix

\[
W = \begin{pmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 1/3 & 1/6 \\ 0 & 1/6 & 5/6 \end{pmatrix}.
\]

(39)

Assume that the observation noise of each agent is a sequence of iid random vectors \( \in N(0, I) \). Set \( x^* = (-1, 4)^T \), \( \gamma_k = \frac{1}{k} \), and \( M_k = 2^k \). The first and second components of the initial values for all agents are set to be mutually independent and uniformly distributed over the intervals \([-2, 6]\) and \([-2, 4]\), respectively.

Since A1-A5 hold for all agents for almost all sample paths, by Theorem 3.2 the estimates given by (5)-(8) converge a.s. to the same point belonging to the set \( J \).

The estimates of \( x^0 \) and \( y^0 \) for the three agents are demonstrated in Fig. 1 and Fig. 2 respectively.
VI. CONCLUDING REMARKS

In this paper a distributed stochastic approximation algorithm with expanding truncations (DSAAWET) is proposed. In comparison with the centralized SAAWET, here the algorithm is based on a network expanding truncation mechanism. The estimates generated by the algorithm is proved to converge to the same value of the root set under some weak conditions. A numerical example is provided to demonstrate the theoretic results.

For further research it is of interest to consider DSAAWET over random networks taking into account the possible package loss in communication. It is also of interest to consider the possibility of removing the continuity assumption on \( f_i(\cdot) \), since for the centralized SAAWET the function is only required be measurable and locally bounded.

APPENDIX A

PROOF OF LEMMA 4.3

For a given \( k \), there exists a unique \( m \) such that \( \tau_m \leq k < \tau_{m+1} \) for a fixed sample path \( \omega \). Then the truncation number of all agents is no larger than \( m \) at time \( k \), i.e., \( \sigma_{i,k} \leq m \ \forall i \in \mathcal{V} \). For a fixed \( k \in [\tau_m, \tau_{m+1}) \) define the following sets:

\[
\mathcal{V}_m(k) \triangleq \{ i \in \mathcal{V} : \sigma_{i,k} = m \}, \quad \mathcal{V}^c_m(k) \triangleq \{ i \in \mathcal{V} : \sigma_{i,k} < m \}, \\
\mathcal{V}^1_{m+1}(k+1) \triangleq \{ i \in \mathcal{V} : \sigma_{i,k+1} = m + 1 \}.
\]  

(A.1)

For proving Lemma 4.3 we need the following auxiliary results.

Lemma A.1: If \( \tau_m \leq k < \tau_{m+1} \), then

\[
a) \quad \mathcal{V}^1_{m+1}(k+1) \cap \mathcal{V}^c_m(k) = \emptyset, \quad \mathcal{V}^1_{m+1}(k+1) \subset \mathcal{V}_m(k),
\]  

(A.2)
\[ b) \quad \tilde{x}_{j,k} = x^* \quad \forall j \in N_i(k) \quad \forall i \in \mathcal{V}_m^c(k), \quad \text{(A.3)} \]

\[ c) \quad \tilde{x}_{j,k} = \begin{cases} x_{j,k}, & \text{if } j \in N_i(k) \text{ and } j \in \mathcal{V}_m(k) \\ x^*, & \text{if } j \in N_i(k) \text{ and } j \in \mathcal{V}_m^c(k) \end{cases} \quad \forall i \in \mathcal{V}_m(k), \quad \text{(A.4)} \]

\[ d) \quad \tilde{x}_{i,k+1} = \begin{cases} x_{i,k+1}, & \text{if } \mathcal{V}_{m+1}^1(k+1) = \emptyset, \\ x^*, & \text{if } \mathcal{V}_{m+1}^1(k+1) \neq \emptyset \end{cases} \quad \forall i \in \mathcal{V}, \quad \text{(A.5)} \]

\[ e) \quad \tilde{x}_{i,k+1} = x^* \quad \forall i \in \mathcal{V}, \quad \text{if } \mathcal{V}_{m+1}^1(k+1) \neq \emptyset. \quad \text{(A.6)} \]

**Proof:**

a) We first prove that for any \( i \in \mathcal{V}_m^c(k) \)

\[ \sigma_{i,k+1} \leq m. \quad \text{(A.7)} \]

For any \( i \in \mathcal{V}_m^c(k) \), (A.7) is proved as follows:

1) If \( \sigma_{j,k} < m \) \( \forall j \in N_i(k) \), then \( \hat{\sigma}_{i,k} < m \) by (5), and hence \( \sigma_{i,k+1} \leq \hat{\sigma}_{i,k} + 1 < m + 1 \) by (8). This implies (A.7).

2) If there exists \( j \in N_i(k) \) such that \( \sigma_{j,k} = m \), then \( \hat{\sigma}_{i,k} \geq m \). However, by assumption \( k < \tau_{m+1} \) we know \( \sigma_{i,k} < m + 1 \) \( \forall i \in \mathcal{V} \), and hence by (5) we derive \( \hat{\sigma}_{i,k} \leq m \). This combining with \( \hat{\sigma}_{i,k} \geq m \) implies \( \hat{\sigma}_{i,k} = m \), and hence \( \sigma_{i,k} < \hat{\sigma}_{i,k} \). By (6) we see \( x_{i,k+1}' = x^* \). Noticing \( \| x^* \| \leq M_1 \) and \( M_m \geq M_1 \), we have \( \| x^* \| \leq M_m \), and by (8) we derive \( \sigma_{i,k+1} = \hat{\sigma}_{i,k} = m \).

Therefore, we conclude that (A.7) holds for any \( i \in \mathcal{V}_m^c(k) \), and hence \( \mathcal{V}_{m+1}^1(k+1) \cap \mathcal{V}_m^c(k) = \emptyset \). Note that \( \mathcal{V}_m^c(k) \cup \mathcal{V}_m(k) = \mathcal{V} \) and \( \mathcal{V}_{m+1}^1(k+1) \subset \mathcal{V} \) for \( k \in [\tau_m, \tau_{m+1}) \). So, from \( \mathcal{V}_{m+1}^1(k+1) \cap \mathcal{V}_m^c(k) = \emptyset \) it follows that \( \mathcal{V}_{m+1}^1(k+1) \subset \mathcal{V}_m(k) \). Consequently, (A.2) holds.

b) For any fixed \( i \in \mathcal{V}_m^c(k) \), we have \( \sigma_{j,k-1} < m \) \( \forall j \in N_i(k) \).

This is because, otherwise, there would exist a \( j_1 \in N_i(k) \) such that \( \sigma_{j_1,k-1} \geq m \). Then, by (5) and (8) we would have \( \sigma_{i,k} \geq \hat{\sigma}_{i,k-1} \geq \sigma_{j_1,k-1} \geq m \), which is contradictory with \( i \in \mathcal{V}_m^c(k) \).

We now show (A.3).

1) If \( j \in N_i(k) \) and \( j \in \mathcal{V}_m^c(k) \), then by the definition of \( \tau_{j,m} \) we derive \( \tau_{j,m} > k \), and hence \( \tilde{\tau}_{j,m} = \tau_{m+1} \cap \tau_{j,m} > k \). This together with (20) shows that \( \tilde{x}_{j,k} = x^* \).

2) If \( j \in N_i(k) \) and \( j \in \mathcal{V}_m(k) \), then \( \sigma_{j,k} = m \) and \( \sigma_{j,k} > \sigma_{j,k-1} \), since we have shown \( \sigma_{j,k-1} < m \) \( \forall j \in N_i(k) \). By Remark 2.1 it follows that \( x_{j,k} = x^* \). From \( \sigma_{j,k-1} < m \) and by the definition of \( \tau_{j,m} \), we derive \( \tau_{j,m} = k \), and \( \tilde{\tau}_{j,m} = \tau_{m+1} \cap \tau_{j,m} = k \), which combining (21) leads to \( \tilde{x}_{j,k} = \bar{x}_{j,k} = x^* \).

Thus, the proof of (A.3) is completed.
c) We now show (A.4).

1) If \( j \in N_i(k) \) and \( j \in V^c_m(k) \), then \( \tau_{j,m} > k \) by the definition of \( \tau_{j,m} \), and hence \( \tilde{\tau}_{j,m} = \tau_{m+1} \\land \tau_{j,m} > k \). Then, \( \tilde{x}_{j,k} = x^* \) by (20).

2) If \( j \in N_i(k) \) and \( j \in V_m(k) \), then, again by definition, \( \tau_{j,m} \leq k \), and hence \( \tilde{\tau}_{j,m} = \tau_{m+1} \\land \tau_{j,m} \leq k \).

Then \( \tilde{x}_{j,k} = x_{j,k} \) follows from (21).

Thus, (A.4) has been proved.

d) The condition \( V^1_{m+1}(k+1) = \emptyset \) means that the truncation number of all agents is smaller than \( m + 1 \) at time \( k + 1 \). Then, \( \tau_{m+1} > k + 1 \) by its definition.

We now prove the second assertion in (A.5) by considering the following two cases:

1) For the case where \( i \in V^c_m(k) \) and \( \sigma_{i,k+1} < m \), by definition \( \tau_{i,m} > k + 1 \), and hence \( \tilde{\tau}_{i,m} = \tau_{m+1} \\land \tau_{i,m} > k + 1 \). Therefore, \( \tilde{x}_{i,k+1} = x^* \) by (20);

2) For the case where \( i \in V^c_m(k) \) and \( \sigma_{i,k+1} = m \), from \( \sigma_{i,k} < m \) and by the definition of \( \tau_{i,m} \) we derive \( \tilde{\tau}_{i,m} = \tau_{m+1} \\land \tau_{i,m} = k + 1 \). From \( \sigma_{i,k+1} > \sigma_{i,k} \) and Remark 2.1 it follows that \( x_{i,k+1} = x^* \). On the other hand, by \( \tilde{\tau}_{i,m} = k + 1 \) from (21) it follows that \( \tilde{x}_{i,k+1} = x_{i,k+1} \). Therefore, \( \tilde{x}_{i,k+1} = x^* \).

To prove the first assertion in (A.5) we note that \( \tau_{i,m} \leq k \) for any \( i \in V_m(k) \). Hence, \( \tilde{\tau}_{i,m} = \tau_{m+1} \\land \tau_{j,m} \leq k \), and by (21) we see \( \tilde{x}_{i,k+1} = x_{i,k+1} \). Thus, (A.5) is proved.

e) The inequality \( V^1_{m+1}(k+1) \neq \emptyset \) means that the truncation number of some agent is \( m + 1 \) at time \( k + 1 \), and hence \( \tau_{m+1} = k + 1 \) by definition.

We now prove (A.6) by considering the following cases:

1) For \( i \in V^1_{m+1}(k+1) \), i.e., \( \sigma_{i,k+1} = m + 1 \), we have \( \sigma_{i,k} \leq m < \sigma_{i,k+1} \), and hence from Remark 2.1 we derive \( x_{i,k+1} = x^* \). By the definition of \( \tau_{i,m+1} \) we obtain \( \tau_{i,m+1} = k + 1 \), and hence

\[
\tilde{\tau}_{i,m+1} = \tau_{i,m+1} \\land \tau_{m+2} = k + 1,
\]

which by (21) implies

\[
\tilde{x}_{i,k+1} = x_{i,k+1} = x^*. \tag{A.8}
\]

2) For \( i \notin V^1_{m+1}(k+1) \), we have \( \sigma_{i,k+1} < m + 1 \) and hence \( \tau_{i,m+1} > k + 1 \). Therefore,

\[
\tilde{\tau}_{i,m+1} = \tau_{i,m+1} \\land \tau_{m+2} > k + 1,
\]

which by (20) implies

\[
\tilde{x}_{i,k+1} = x^*. \tag{A.9}
\]
Combining (A.8) and (A.9) proves (A.6).

Proof of Lemma 4.3: Set \( \tilde{x}_{i,0}' = \tilde{x}_{i,0}, \sigma_0' = 0 \), and let \( \tilde{x}_{i,k}' \) and \( \sigma_k' \) be given by the following recursion:

\[
\tilde{x}_{i,k+1}' = \sum_{j \in N_i(k)} \omega_{ij}(k)\tilde{x}_{j,k}' + \gamma_k(f_i(\tilde{x}_{i,k}') + \tilde{\varepsilon}_{i,k+1}), \tag{A.10}
\]

\[
\tilde{x}_{i,k+1}' = \tilde{x}_{i,k+1}' I_{\cap \{ x_i \in V \mid \| \tilde{x}_{i,k+1}' \| \leq M_{\sigma_k'} \}} + x^* I_{\cup \{ x_i \in V \mid \| \tilde{x}_{i,k+1}' \| > M_{\sigma_k'} \}}, \tag{A.11}
\]

\[
\sigma_{k+1}' = \sigma_k' + I_{\cup \{ x_i \in V \mid \| \tilde{x}_{i,k+1}' \| > M_{\sigma_k'} \}}. \tag{A.12}
\]

In order to prove Lemma 4.3, it suffices to show that for any \( p \geq 1 \)

\[
\tilde{x}_{i,p}' = \tilde{x}_i, \quad \forall i \in V, \tag{A.13}
\]

\[
\sigma_p' = \max_{i \in V} \sigma_{i,p} = \sigma_p. \tag{A.14}
\]

We prove (A.13) and (A.14) by induction.

Step 1: We first prove that (A.13) and (A.14) hold for \( p = 1 \).

Since \( \sigma_{i,0} = 0 \) \( \forall i \in V \), from (5) we know \( \tilde{\sigma}_{i,0} = 0 \) \( \forall i \in V \). From (6) we then derive

\[
x_{i,1}' = \sum_{j \in N_i(0)} \omega_{ij}(0)x_{j,0} + \gamma_0(f_i(x_{i,0}) + \varepsilon_{i,1}). \tag{A.15}
\]

Notice that \( \tau_{i,0} = 0 \) \( \forall i \in V \) and \( \tau_1 \geq 1 \), and hence \( \tilde{\tau}_{i,0} = 0 < \tau_1 \). From here by (21) we derive

\[
\tilde{x}_{i,0}' = x_{i,0}, \quad \tilde{\varepsilon}_{i,1} = \varepsilon_{i,1} \quad \forall i \in V. \tag{A.16}
\]

From (A.10) and (A.16) it follows that

\[
\tilde{x}_{i,1}' = \sum_{j \in N_i(0)} \omega_{ij}(0)x_{j,0} + \gamma_0(f_i(x_{i,0}) + \varepsilon_{i,1}),
\]

which incorporating with (A.15) leads to

\[
\tilde{x}_{i,1}' = x_{i,1}' \quad \forall i \in V. \tag{A.17}
\]

We now prove (A.13) and (A.14) for \( p = 1 \) by considering the following two cases:

Case 1: \( V_1^1 (1) \) is nonempty, i.e., there exists \( i_0 \in V \) such that \( \| x_{i_0,1}' \| > M_0 \). Then by (A.17) we have

\[
\| \tilde{x}_{i_0,1}' \| = \| x_{i_0,1}' \| > M_0.
\]

From (A.11) and (A.12) we conclude that \( \tilde{x}_{i_1}' = x^* \) \( \forall i \in V \) and \( \sigma_1' = 1 = \max_{i \in V} \sigma_{i,1} \). By \( \tilde{x}_{i,1}' = x^* \) \( \forall i \in V \) and (A.6) we have \( \tilde{x}_{i,1} = x_{i,1} \) \( \forall i \in V \).

Consequently, (A.13) (A.14) hold for \( p = 1 \) when \( V_1^1 (1) \neq \emptyset \).
Case 2: $V_1^1(1) = \emptyset$, i.e., $\sigma_{i,1} = 0 \ \forall i \in \mathcal{V}$. By the definition of $\tau_1$ we have $\tau_1 > 1$. From $\sigma_{i,1} = 0$ and (8) it follows that
\[ \| x_{i,1}' \| \leq M_0 \ \forall i \in \mathcal{V}. \tag{A.18} \]
From (7) we have $x_{i,1} = x_{i,1}' \ \forall i \in \mathcal{V}$, and by (21)
\[ \tilde{x}_{i,1} = x_{i,1} = x_{i,1}' \ \forall i \in \mathcal{V}. \]
This combining with (A.17) leads to
\[ \tilde{x}_{i,1} = \tilde{x}_{i,1}' \ \forall i \in \mathcal{V}. \tag{A.19} \]
From (A.17) and (A.18) we obtain
\[ \| \tilde{x}_{i,1}' \| = \| x_{i,1}' \| \leq M_0 \ \forall i \in \mathcal{V}, \]
and hence from (A.11) (A.12) it follows that
\[ \tilde{x}_{i,1} = \tilde{x}_{i,1}' \ \forall i \in \mathcal{V}, \tag{A.20} \]
\[ \sigma_1' = 0 = \max_{i \in \mathcal{V}} \sigma_{i,1}. \tag{A.21} \]
Combining (A.19) (A.20), we obtain
\[ \tilde{x}_{i,1} = \tilde{x}_{i,1}' \ \forall i \in \mathcal{V}. \tag{A.22} \]
From (A.22) and (A.21) we see that (A.13) (A.14) hold for $p = 1$ when $V_1^1(1) = \emptyset$.

Thus, (A.13) (A.14) hold for $p = 1$.

Step 2: Inductively, we assume (A.13) (A.14) hold for $p = 1, 2, \cdots, k$. We intend to show that (A.13) (A.14) also hold for $k + 1$. At a fixed sample path $\omega$ for a given integer $k$ there exists a unique integer $m$ such that $\tau_m \leq k < \tau_{m+1}$.

i) We first prove (A.13) holds for $k + 1$ by considering the following cases:

Case 1: If $i \in \mathcal{V}_m^c(k)$, then $\tau_{i,m} > k$ by definition, and hence $\tilde{\tau}_{i,m} = \tau_{i,m} \land \tau_{m+1} > k$. Then, from (20) we have
\[ \tilde{x}_{i,k} = x^* \quad \tilde{e}_{i,k+1} = -f_i(x^*). \tag{A.23} \]
From (A.3) in Lemma A.1 we know
\[ \tilde{x}_{j,k} = x^* \ \forall j \in \mathcal{N}_i(k) \ \forall i \in \mathcal{V}_m^c(k). \]
Then by the inductive assumption we have
\[ \tilde{x}_{j,k}' = x^* \ \forall j \in \mathcal{N}_i(k). \]
This incorporating with (A.23) and (A.10) implies
\[ \hat{\mathbf{x}}_{i,k+1} = \mathbf{x}^* \quad \forall i \in \mathcal{V}_m^c(k), \] (A.24)
and hence by (A.11) we derive
\[ \tilde{\mathbf{x}}_{i,k+1} = \mathbf{x}^* \quad \forall i \in \mathcal{V}_m^c(k). \] (A.25)

If \( \mathcal{V}_{m+1}(k+1) = \emptyset \), then by the second statement in (A.5) we derive \( \tilde{x}_{i,k+1} = x^* \quad \forall i \in \mathcal{V}_m^c(k) \).
If \( \mathcal{V}_{m+1}(k+1) \neq \emptyset \), then by (A.6) we derive \( \tilde{x}_{i,k+1} = x^* \quad \forall i \in \mathcal{V}_m^c(k) \).

In summary, whether \( \mathcal{V}_{m+1}(k+1) \) is empty or not, we have the following equality
\[ \tilde{x}_{i,k+1} = x^* \quad \forall i \in \mathcal{V}_m^c(k). \]

This together with (A.25) leads to
\[ \hat{x}_{i,k+1} = \tilde{x}_{i,k+1} \quad \forall i \in \mathcal{V}_m^c(k). \] (A.26)

**Case 2**: If \( i \in \mathcal{V}_m(k) \), then noticing \( k \in [\tau_m, \tau_{m+1}) \) by (5) we have \( \hat{\sigma}_{i,k} = m \). Hence from (6) it follows that
\[ x'_{i,k+1} = \sum_{j \in N_i(k)} \omega_{ij}(k)(x_{j,k}I_{\sigma_{j,k}=m} + x^*I_{\sigma_{j,k}<m}) + \gamma_k(f_i(x_{i,k}) + \hat{\varepsilon}_{i,k+1}). \] (A.27)

Since \( \sigma_{i,k} = m \), we have \( \tau_{i,m} \leq k \) and
\[ \tilde{\tau}_{i,m} = \tau_{m+1} \land \tau_{i,m} \leq k. \]

By (21) it is clear that
\[ \tilde{x}_{i,k} = x_{i,k}, \quad \tilde{\varepsilon}_{i,k+1} = \varepsilon_{i,k+1} \quad \forall i \in \mathcal{V}_m(k). \] (A.28)

By (A.10) and the inductive assumption we have
\[ \hat{x}_{i,k+1} = \sum_{j \in N_i} \omega_{ij}(k)(x_{j,k}I_{\sigma_{j,k}=m} + x^*I_{\sigma_{j,k}<m}) + \gamma_k(f_i(x_{i,k}) + \hat{\varepsilon}_{i,k+1}). \]

Further, by (A.4) and (A.28) it follows that for any \( i \in \mathcal{V}_m(k) \)
\[ \hat{x}_{i,k+1} = \sum_{j \in N_i(k)} \omega_{ij}(k)(x_{j,k}I_{\sigma_{j,k}=m} + x^*I_{\sigma_{j,k}<m}) + \gamma_k(f_i(x_{i,k}) + \varepsilon_{i,k+1}). \]

Comparing this with (A.27) we see
\[ \hat{x}_{i,k+1} = x'_{i,k+1} \quad \forall i \in \mathcal{V}_m(k). \] (A.29)

We now prove (A.13) for \( k + 1 \):
\[ \tilde{x}_{i,k+1} = \hat{x}_{i,k+1} \quad \forall i \in \mathcal{V}_m(k) \] (A.30)
by considering the following two cases.

Case 2a: The case where $\mathcal{V}_{m+1}^{1}(k+1) \neq \emptyset$ implies that there exists $i_0 \in \mathcal{V}$ such that $\| x_{i_0,k+1}^\prime \| > M_m$, and hence by (A.29) we derive $\| \hat{x}_{i_0,k+1}^\prime \| = \| x_{i_0,k+1}^\prime \| > M_m$. From (A.11) we then have $\hat{x}_{i,k+1} = x^* \quad \forall i \in \mathcal{V}$, which combining with (A.6) leads to (A.30).

Case 2b: The case $\mathcal{V}_{m+1}^{1}(k+1) = \emptyset$ implies that

$$
\| x_{i,k+1}^\prime \| \leq M_m \quad \forall i \in \mathcal{V}(k),
$$

(A.31)

which incorporating with (7) (8) leads to

$$
x_{i,k+1} = x_{i,k+1}^\prime \quad \text{and} \quad \sigma_{i,k+1} = m \quad \forall i \in \mathcal{V}(k).
$$

(A.32)

Then by (A.32) and the first statement in (A.5) it follows that

$$
\hat{x}_{i,k+1} = x_{i,k+1}^\prime \quad \forall i \in \mathcal{V}(k),
$$

which combining with (A.29) implies

$$
\hat{x}_{i,k+1} = \hat{x}_{i,k+1}^\prime \quad \forall i \in \mathcal{V}(k).
$$

(A.33)

By (A.29) and (A.31) we have

$$
\| \hat{x}_{i,k+1}^\prime \| = \| x_{i,k+1} \| \leq M_m \quad \forall i \in \mathcal{V}(k).
$$

(A.34)

From $\| x^* \| \leq M_1$, $M_m \geq M_1$ and (A.24) we derive

$$
\| \hat{x}_{i,k+1}^\prime \| \leq M_m \quad \forall i \in \mathcal{V}_m^c(k),
$$

(A.35)

which combining with (A.34) yields

$$
\| \hat{x}_{i,k+1} \| \leq M_m \quad \forall i \in \mathcal{V}.
$$

(A.36)

By (A.11) (A.36) we have

$$
\hat{x}_{i,k+1} = \hat{x}_{i,k+1}^\prime, \quad \forall i \in \mathcal{V}(k),
$$

which combining with (A.33) leads to (A.30).

Thus, (A.30) has been proved for both Cases 2a and 2b.

Combining (A.26) for Case 1 and (A.30) for Case 2, we assure that (A.13) holds for $k + 1$.

ii) We now prove that (A.14) holds for $k + 1$ by considering the following cases:

1) The case $\mathcal{V}_{m+1}^{1}(k+1) = \emptyset$, i.e., $\sigma_{i,k+1} < m + 1$, implies $\max_{i \in \mathcal{V}} \sigma_{i,k+1} = m$. By (A.36) and (A.12), we derive $\sigma_{k+1}^\prime = m$, and hence (A.14) holds for $k + 1$. 
2) The case $V_{m+1}^k(k+1) \neq \emptyset$ and the second statement in (A.2) imply that there exists an $i_1 \in V_m(k)$ such that $\|x_{i_1,k+1}'\| > M_m$, and hence by (A.29) we obtain

$$\|x_{i_1,k+1}'\| \geq M_m.$$  \hspace{1cm} (A.37)

From here by (A.12) we derive

$$\sigma_{k+1}' = m + 1 = \max_{i \in V} \sigma_{i,k+1}.$$ Thus for both cases 1) and 2) we know that (A.14) holds for $k+1$.

In summary, we have proven that (A.13) (A.14) hold for $\forall k \geq 1$ by induction. Therefore, $\tilde{x}_{i,k}$ and $\tilde{\varepsilon}_{i,k+1}$ satisfy (22)-(24), and the proof of Lemma 4.3 is completed.  \hspace{1cm} $\blacksquare$

APPENDIX B

PROOF OF LEMMA 4.5

Proof: We now prove (27) by considering the following two cases:

Case 1: $\lim_{k \to \infty} \sigma_k = \sigma < \infty$.

From Remark 4.4 we know that there exists $k_0 \geq 0$ such that for any $k \geq n_{k_0}$

$$\varepsilon_{k+1} = \tilde{\varepsilon}_{k+1}, \quad X_k = \tilde{X}_k, \quad \tau_{\sigma_{k+1}} = \infty,$$

and hence for all $k \geq k_0$

$$m(n_k,t_k) \wedge (\tau_{\sigma_{k+1}} + 1 - 1) \leq \sum_{s=n_k}^{m(n_k,t_k)} \gamma_s \tilde{\varepsilon}_{i,s+1} I[\|\tilde{x}_{i,s}\| \leq K] \leq \sum_{s=n_k}^{m(n_k,t_k)} \gamma_s \varepsilon_{i,s+1} I[\|x_{i,s}\| \leq K] \quad \forall t_k \in [0,T].$$  \hspace{1cm} (B.1)

Since $\{x_{i,n_k}\}_{k \geq 1}$ is a convergent subsequence, from A5 and (B.1) it follows that (27) holds for sufficiently large $K > 0$.

Case 2: $\lim_{k \to \infty} \sigma_k = \infty$.

We partition the index set $\{n_k\}$ into three disjoint subsets as follows:

$$\{n_k^{(1)}\} = \{p \in \{n_k\} : \tilde{\tau}_{i,p} \leq p\},$$

$$\{n_k^{(2)}\} = \{p \in \{n_k\} : \tilde{\tau}_{i,p} > p, \tilde{\tau}_{i,p} = \tau_{\sigma_{p+1}}\},$$

$$\{n_k^{(3)}\} = \{p \in \{n_k\} : \tilde{\tau}_{i,p} > p, \tilde{\tau}_{i,p} < \tau_{\sigma_{p+1}}\}.$$ 

Since $\tilde{\tau}_{i,p} = \tau_{i,p} \wedge \tau_{\sigma_{p+1}} \leq \tau_{\sigma_{p+1}}$, we have $\{n_k\} = \{n_k^{(1)}\} \cup \{n_k^{(2)}\} \cup \{n_k^{(3)}\}$. Denote by $n_p^{(i)}$ the $p$th index in the index set $\{n_k^{(i)}\}$, $i = 1, 2, 3$.

i) For the $p$th index in the index set $\{n_k^{(1)}\}$, we have $\tilde{\tau}_{i_{n_p^{(1)}},p} \leq n_p^{(1)}$. Then from (21) we derive

$$\tilde{x}_{i,s} = x_{i,s}, \quad \tilde{\varepsilon}_{i,s+1} = \varepsilon_{i,s+1} \quad \forall s : n_p^{(1)} \leq s < \tau_{\sigma_{n_p^{(1)}} + 1}.$$  \hspace{1cm} (B.2)
From (B.2) we have

\[
\frac{1}{T} \left\| \sum_{n=n_p^{(1)}} m(n_p^{(1)}, t_p)_{\tau_{\sigma_{n_p^{(1)}}}+1-1} \gamma_s \hat{e}_{i,s} + 1I[\|\hat{x}_{i,s}\| \leq K] \right\| = \frac{1}{T} \left\| \sum_{s=n_p^{(1)}} m(n_p^{(1)}, t_p)_{\tau_{\sigma_{n_p^{(1)}}}+1-1} \gamma_s \hat{e}_{i,s} + 1I[\|x_{i,s}\| \leq K] \right\| \quad \forall t_p \in [0, T]. \tag{B.3}
\]

If there exist infinitely many indices in the index set \( \{n_k^{(1)}\} \), then \( \{x_{i,n_p^{(1)}}\}_{p \geq 1} \) is a convergent subsequence. Therefore, by A5 we know that for sufficiently large \( K \),

\[
\lim_{T \to 0} \limsup_{p \to \infty} \frac{1}{T} \left\| \sum_{m=n_p^{(1)}} m(n_p^{(1)}, t_p)_{\tau_{\sigma_{n_p^{(1)}}}+1-1} \gamma_{m} \hat{e}_{i,m} + 1I[\|x_{i,m}\| \leq K] \right\| = 0 \quad \forall t_p \in [0, T]. \tag{B.4}
\]

For any \( t_p \in [0, T] \) with a given \( T > 0 \), we set

\[
m(n_p^{(1)}, t_p)_{\tau_{\sigma_{n_p^{(1)}}}+1-1} t'_p \triangleq \sum_{s=n_p^{(1)}} \gamma_s. \tag{B.5}
\]

Then \( t'_p \leq \sum_{s=n_p^{(1)}} m(n_p^{(1)}, t_p) \gamma_s \leq t_p \leq T \). From \( m(n_p^{(1)}, t'_p) = \max\{m : \sum_{s=n_p^{(1)}} \gamma_s \leq t'_p \} \) and (B.5) it follows that

\[
m(n_p^{(1)}, t'_p) \geq m(n_p^{(1)}, t_p)_{\tau_{\sigma_{n_p^{(1)}}}+1-1}. \tag{B.6}
\]

Noting

\[
\sum_{s=n_p^{(1)}} m(n_p^{(1)}, t_p)_{\tau_{\sigma_{n_p^{(1)}}}+1-1} \gamma_s = t'_p + \gamma_{1+m(n_p^{(1)}, t_p)_{\tau_{\sigma_{n_p^{(1)}}}+1-1}} > t'_p,
\]

we have \( m(n_p^{(1)}, t_p) \leq m(n_p^{(1)}, t_p)_{\tau_{\sigma_{n_p^{(1)}}}+1-1} \). By (B.6) we then have

\[
m(n_p^{(1)}, t'_p) = m(n_p^{(1)}, t_p)_{\tau_{\sigma_{n_p^{(1)}}}+1-1}. \tag{B.7}
\]

Putting \( t'_p \) defined by (B.5) into (B.4), by (B.7) we obtain

\[
\lim_{T \to 0} \limsup_{p \to \infty} \frac{1}{T} \left\| \sum_{m=n_p^{(1)}} m(n_p^{(1)}, t_p)_{\tau_{\sigma_{n_p^{(1)}}}+1-1} \gamma_{m} \hat{e}_{i,m} + 1I[\|x_{i,m}\| \leq K] \right\| = 0 \quad \forall t_p \in [0, T]
\]

for sufficiently large \( K > 0 \). Then, from (B.3) we derive

\[
\lim_{T \to 0} \limsup_{p \to \infty} \frac{1}{T} \left\| \sum_{m=n_p^{(1)}} m(n_p^{(1)}, t_p)_{\tau_{\sigma_{n_p^{(1)}}}+1-1} \gamma_{m} \hat{e}_{i,m} + 1I[\|\hat{x}_{i,m}\| \leq K] \right\| = 0 \quad \forall t_p \in [0, T]. \tag{B.8}
\]
ii) For the $p$th index $n_p^{(2)}$ in $\{n_k^{(2)}\}$, we have $\tilde{\tau}_{i,\sigma_{n_p^{(2)}}} > n_p^{(2)}$ and $\tilde{\tau}_{i,\sigma_{n_p^{(2)}}} = \tau_{\sigma_{n_p^{(2)}}}$. By the definition of $\tau_m$ we have $\tau_{\sigma_{n_p^{(2)}}} \leq n_p^{(2)}$, and hence from (16) and (20) we derive

$$\tilde{x}_{i,s} = x^*, \quad \tilde{\varepsilon}_{i,s+1} = -f_i(x^*) \quad \forall s : n_p^{(2)} \leq s < \tau_{\sigma_{n_p^{(2)}}} + 1,$$

$$\tau_{\sigma_{n_p^{(2)}}} + 1 \leq \tau_{\sigma_{n_p^{(2)}}} + D \leq n_p^{(2)} + D.$$  \hfill (B.9)

If there are infinitely many indices in $\{n_k^{(2)}\}$, then by (B.9) we know that

$$m(n_p^{(2)}, t_p) \land (\tau_{\sigma_{n_p^{(2)}}} + 1) \leq \sum_{s=n_p^{(2)}} \gamma_s \tilde{\varepsilon}_{i,s+1} I_{[\|\tilde{x}_{i,s}\| \leq K]} \leq \sum_{s=n_p^{(2)}} \gamma_s \|f_i(x^*)\|,$$

and hence by (B.10) and by that $\gamma_k \to 0$ it follows that

$$\frac{m(n_p^{(2)}, t_p) \land (\tau_{\sigma_{n_p^{(2)}}} + 1)}{1/T} \sum_{s=n_p^{(2)}} \gamma_s \tilde{\varepsilon}_{i,s+1} I_{[\|\tilde{x}_{i,s}\| \leq K]} \leq \frac{D}{T} \|f_i(x^*)\| \sup_{s \geq n_p^{(2)}} \gamma_s \to 0 \quad \forall t_p \in [0, T].$$  \hfill (B.11)

iii) For the $p$th index $n_p^{(3)}$ in $\{n_k^{(3)}\}$, we have $\tilde{\tau}_{i,\sigma_{n_p^{(3)}}} > n_p^{(3)}$ and $\tilde{\tau}_{i,\sigma_{n_p^{(3)}}} < \tau_{\sigma_{n_p^{(3)}}}$. So, we derive

$${\tilde{\tau}}_{i,\sigma_{n_p^{(3)}}} = {\tau}_{i,\sigma_{n_p^{(3)}}} \land {\tau}_{\sigma_{n_p^{(3)}}} + 1 = {\tau}_{i,\sigma_{n_p^{(3)}}}.$$  

Then by (16) we obtain

$$n_p^{(3)} \leq {\tau}_{i,\sigma_{n_p^{(3)}}} + D \leq n_p^{(3)} + D, \quad \text{and hence}$$

$$n_p^{(3)} < {\tau}_{i,\sigma_{n_p^{(3)}}} = {\tilde{\tau}}_{i,\sigma_{n_p^{(3)}}} \leq n_p^{(3)} + D.$$  \hfill (B.12)

Then, by (20) (21) we have

$$\tilde{x}_{i,s} = x^*, \quad \tilde{\varepsilon}_{i,s+1} = -f_i(x^*) \quad \forall s : n_p^{(3)} \leq s < {\tau}_{i,\sigma_{n_p^{(3)}}},$$

$$\tilde{x}_{i,s} = x_i, s, \quad \tilde{\varepsilon}_{i,s+1} = \varepsilon_{i,s+1} \quad \forall s : {\tau}_{i,\sigma_{n_p^{(3)}}} \leq s < {\tau}_{\sigma_{n_p^{(3)}}} + 1.$$  \hfill (B.13)
For any sequence \( \{y_k\}_{k \geq 0} \), define \( \sum_{s=k_1}^{k_2} y_s \triangleq 0 \) for \( k_2 < k_1 \). Combining (B.12) and (B.13), we derive

\[
\frac{1}{T} \left\| \sum_{s=n_p^{(3)}}^{m(n_p^{(3)}, t_p) \wedge (\tau_{\sigma_{n_p^{(3)}}^{(3)}} + 1)} \gamma_s \mathcal{E}_{i, s+1} I_{[\|x_{i, s}\| \leq K]} \right\| \leq \frac{1}{T} \left\| \sum_{s=n_p^{(3)}}^{m(n_p^{(3)}, t_p) \wedge (\tau_{\sigma_{n_p^{(3)}}^{(3)}} + 1)} \gamma_s f_i(x^*) I_{[\|x_{i, s}\| \leq s < \tau_{\sigma_{n_p^{(3)}}}]} \right\|
\]

\[
\frac{1}{T} \left\| \sum_{s=n_p^{(3)}}^{m(n_p^{(3)}, t_p) \wedge (\tau_{\sigma_{n_p^{(3)}}^{(3)}} + 1)} \gamma_s \mathcal{E}_{i, s+1} I_{[\|x_{i, s}\| \leq K]} \right\| \leq \frac{1}{T} \left\| \gamma_s \mathcal{E}_{i, s+1} I_{[\|x_{i, s}\| \leq K]} \right\| \quad \forall t_p \in [0, T].
\]

(B.14)

By the definition of \( \tau_{i, \sigma_{n_p^{(3)}}} \), we know that the truncation number for the agent \( i \) at time \( \tau_{i, \sigma_{n_p^{(3)}}} \) is \( \sigma_{n_p^{(3)}} \), and it is smaller than \( \sigma_{n_p^{(3)}} + 1 \). Consequently, from Remark 2.1 we have \( x_{i, \tau_{i, \sigma_{n_p^{(3)}}}} = x^* \). If there are infinitely many indices in \( \{n_k^{(3)}\} \), then \( \{x_{i, \tau_{i, \sigma_{n_p^{(3)}}}}\}_{p \geq 1} \) is a convergent subsequence, and hence from A5 we have

\[
\lim_{T \to 0} \limsup_{p \to \infty} \frac{1}{T} \left\| \sum_{s=\tau_{i, \sigma_{n_p^{(3)}}}}^{m(\tau_{i, \sigma_{n_p^{(3)}}, t_p})} \gamma_s \mathcal{E}_{i, s+1} I_{[\|x_{i, s}\| \leq K]} \right\| = 0 \quad \forall t_p \in [0, T]
\]

(B.15)

for sufficiently large \( K > 0 \).

Set \( t_p^* = \sum_{s=n_p^{(3)}}^{m(\tau_{i, \sigma_{n_p^{(3)}}, t_p})} \gamma_s \). By noticing \( m(n_p^{(3)}, t_p) = \max\{m : \sum_{s=n_p^{(3)}}^{m} \gamma_s \leq t_p\} \) we know \( m(n_p^{(3)}, t_p) < \tau_{i, \sigma_{n_p^{(3)}}} \) when \( t_p < t_p^* \). Therefore,

\[
\frac{1}{T} \left\| \sum_{s=\tau_{i, \sigma_{n_p^{(3)}}}}^{m(n_p^{(3)}, t_p) \wedge (\tau_{\sigma_{n_p^{(3)}}^{(3)}} + 1)} \gamma_s \mathcal{E}_{i, s+1} I_{[\|x_{i, s}\| \leq K]} \right\| = 0 \quad \forall t_p \in [0, t_p^*].
\]

(B.16)

For any \( t_p \in [t_p^*, T] \) with \( T > 0 \) being a given constant, set

\[
t_p' = \sum_{s=\tau_{i, \sigma_{n_p^{(3)}}}}^{m(n_p^{(3)}, t_p) \wedge (\tau_{\sigma_{n_p^{(3)}}^{(3)}} + 1)} \gamma_s.
\]

(B.17)

Since \( \tau_{i, \sigma_{n_p^{(3)}}} \leq \tau_{\sigma_{n_p^{(3)}}^{(3)}} + 1 \), by the definition of \( m(n_p^{(3)}, t_p) \) we derive \( m(n_p^{(3)}, t_p) \geq \tau_{i, \sigma_{n_p^{(3)}}} \) when
\[ t_p \geq t^*_p \]. Thus we have

\[ 0 < \gamma_{i, \sigma_{n_p}}^{(3)} \leq t'_p \leq \sum_{s=\tau_{i, \sigma_{n_p}}^{(3)}}^{m(n_p^{(3)}, t_p))} \gamma_s < \sum_{s=n_p^{(3)}}^{m(n_p^{(3)}, t_p))} \gamma_s \leq t_p. \]

Noting that \( t'_p \) defined by (B.17) is similar to that defined by (B.5), by a discussion similar to (B.7) we derive

\[ m(\tau_{i, \sigma_{n_p}}^{(3)}, t'_p) = m(n_p^{(3)}, t_p) \land (\tau_{\sigma_{n_p}}^{(3)} + 1 - 1). \]  

(B.18)

By putting \( t'_p \) defined by (B.17) into (B.15), from (B.18) it follows that for sufficiently large \( K > 0 \)

\[ \lim_{T \to 0} \limsup_{p \to \infty} \frac{1}{T} \left\| \sum_{s=\tau_{i, \sigma_{n_p}}^{(3)}}^{m(n_p^{(3)}, t_p) \land (\tau_{\sigma_{n_p}}^{(3)} + 1 - 1)} \gamma_s \tilde{e}_{i,s+1} I_\left[ \| \tilde{x}_{i,s} \| \leq K \right] \right\| = 0 \quad \forall t_p \in [t^*_p, T]. \]  

(B.19)

Combining (B.16) and (B.19), we conclude that for sufficiently large \( K > 0 \)

\[ \lim_{T \to 0} \limsup_{p \to \infty} \frac{1}{T} \left\| \sum_{s=n_p^{(3)}}^{m(n_p^{(3)}, t_p) \land (\tau_{\sigma_{n_p}}^{(3)} + 1 - 1)} \gamma_s \tilde{e}_{i,s+1} I_\left[ \| \tilde{x}_{i,s} \| \leq K \right] \right\| = 0 \quad \forall t_p \in [0, T]. \]  

(B.20)

At the right hand side of (B.14) the first term tends to zero as \( p \) goes to infinity, while the second term satisfies (B.20). Consequently,

\[ \lim_{T \to 0} \limsup_{p \to \infty} \frac{1}{T} \left\| \sum_{s=n_p^{(3)}}^{m(n_p^{(3)}, t_p) \land (\tau_{\sigma_{n_p}}^{(3)} + 1 - 1)} \gamma_s \tilde{e}_{i,s+1} I_\left[ \| \tilde{x}_{i,s} \| \leq K \right] \right\| = 0 \quad \forall t_p \in [0, T]. \]  

(B.21)

From (B.8) (B.11) and (B.21) we know that (27) holds for the index set \( \{n_k\} \) when \( K > 0 \) is sufficiently large.

Combining Case 1 and Case 2 discussed above, we conclude that (27) holds for sufficiently large \( K > 0 \).

\[ \square \]

**APPENDIX C**

**PROOF LEMMA 4.6**

Prior to proving the lemma we first clarify properties of the products of doubly stochastic matrices \( \{W(k)\} \), and give a preliminary result to be used in the proof of the lemma.

Let \( \{W(k)\} \) be a sequence of doubly stochastic matrices. It is clear that

\[ W(k) \frac{11^T}{N} = \frac{11^T}{N}, \quad \frac{11^T}{N} W(k) = \frac{11^T}{N} \quad \forall k \geq 0. \]
By the definition of \( \Phi(k, s) \) given in \((1.2)\), we know that
\[
\Phi(k, s) \frac{11^T_N}{N} = \Phi(k, s + 1)W(s) \frac{11^T_N}{N} = \Phi(k, s + 1) \frac{11^T_N}{N} = \cdots = \frac{11^T_N}{N} \quad \forall k \geq s - 1, \tag{C.1}
\]
and
\[
\frac{11^T_N}{N} \Phi(k, s) = \frac{11^T_N}{N} \quad \forall k \geq s - 1. \tag{C.2}
\]
From \((C.2)\) it follows that for \( k \geq s - 1 \)
\[
D_\perp(\Phi(k, s) \otimes I_l) = [(I_N - \frac{11^T_N}{N}) \Phi(k, s)] \otimes I_l = (\Phi(k, s) - \frac{11^T_N}{N}) \otimes I_l. \tag{C.3}
\]
Noting \((A \otimes B)(C \otimes D) = AC \otimes BD\), we derive
\[
[(\Phi(k, s) - \frac{11^T_N}{N}) \otimes I_l][(I_N - \frac{11^T_N}{N}) \otimes I_l] = [(\Phi(k, s) - \frac{11^T_N}{N})(I_N - \frac{11^T_N}{N})] \otimes I_l,
\]
and hence from \((C.1)\) and \((C.3)\)
\[
D_\perp(\Phi(k, s) \otimes I_l)D_\perp = (\Phi(k, s) - \frac{11^T_N}{N}) \otimes I_l - [(\Phi(k, s) - \frac{11^T_N}{N}) \frac{11^T_N}{N}] \otimes I_l
\]
\[
= (\Phi(k, s) - \frac{11^T_N}{N}) \otimes I_l = D_\perp(\Phi(k, s) \otimes I_l) \quad \forall k \geq s - 1. \tag{C.4}
\]
Set
\[
\Psi(k - 1, k) \triangleq I_{N^1}, \quad \Psi(k, s) \triangleq [D_\perp(W(k) \otimes I_l)][D_\perp(W(k - 1) \otimes I_l)] \cdots [D_\perp(W(s) \otimes I_l)] \quad \forall k \geq s. \tag{C.5}
\]
From \((C.4)\) with \( s = k \) it follows that
\[
\Psi(k, k - 1) = [D_\perp(W(k) \otimes I_l)]D_\perp[W(k - 1) \otimes I_l]
\]
\[
= D_\perp(W(k) \otimes I_l)[W(k - 1) \otimes I_l] = D_\perp[\Phi(k, k - 1) \otimes I_l] \quad \forall k \geq 1. \tag{C.6}
\]
Therefore, by \((C.3)\) and \((C.6)\) we know that
\[
\Psi(k, s) = \Psi(k, k - 1)\Psi(k - 2, s) = D_\perp[\Phi(k, k - 1) \otimes I_l]\Psi(k - 2, s) = \cdots
\]
\[
= D_\perp(\Phi(k, s) \otimes I_l) = (\Phi(k, s) - \frac{1}{N} 11^T) \otimes I_l \quad \forall k \geq s, \tag{C.7}
\]
and hence by \((C.1), (C.7)\) and the fact \((A \otimes B)(C \otimes D) = AC \otimes BD\) we conclude that
\[
\Psi(k, s)D_\perp = [(\Phi(k, s) - \frac{1}{N} 11^T) \otimes I_l][(I_N - \frac{11^T_N}{N}) \otimes I_l]
\]
\[
= [(\Phi(k, s) - \frac{1}{N} 11^T)(I_N - \frac{11^T_N}{N})] \otimes I_l = (\Phi(k, s) - \frac{1}{N} 11^T) \otimes I_l \quad \forall k \geq s.
\]
By noticing \( \Psi(k - 1, k) = I_{N^1} \) and \( \Phi(k - 1, k) = I_N \), from here we conclude that
\[
\Psi(k, s)D_\perp = (\Phi(k, s) - \frac{1}{N} 11^T) \otimes I_l \quad \forall k \geq s - 1. \tag{C.8}
\]
Lemma C.1: For a given $T > 0$ if for sufficiently large $k$ and $\forall T_k \in [0, T]$

$$\| \sum_{m=n_k}^{s} \gamma_m \bar{\epsilon}_{m+1} \| \leq T_k \quad \forall s : n_k \leq s \leq s_k,$$

where $s_k \leq m(n_k, T_k)$, then

$$\| \sum_{m=n_k}^{s} \gamma_m (\Phi(s - 1, m) - \frac{1}{N} 11^T) \otimes I_l) \bar{\epsilon}_{m+1} \| \leq (2 + \frac{c(\rho + 1)}{1 - \rho})T_k \quad \forall s : n_k \leq s \leq s_k.$$  \hspace{1cm} \text{(C.10)}

Proof: Setting $\Gamma_n \triangleq \sum_{m=1}^{n} \gamma_m \bar{\epsilon}_{m+1}$, we have

$$\sum_{m=n_k}^{s} \gamma_m (\Phi(s - 1, m) \otimes I_l) \bar{\epsilon}_{m+1} = \sum_{m=n_k}^{s} (\Phi(s - 1, m) \otimes I_l)(\Gamma_m - \Gamma_{n_k - 1} - (\Gamma_{m-1} - \Gamma_{n_k - 1}))$$

$$= (\Gamma_s - \Gamma_{n_k - 1}) + \sum_{m=n_k}^{s-1} (\Phi(s - 1, m) - \Phi(s - 1, m + 1)) \otimes I_l)(\Gamma_m - \Gamma_{n_k - 1}).$$

\hspace{1cm} \text{(C.11)}

From \text{(C.9)} it follows that for sufficiently large $k$ and $\forall T_k \in [0, T]$

$$\| \Gamma_s - \Gamma_{n_k - 1} \| \leq T_k \quad \forall s : n_k \leq s \leq s_k,$$

which combining with \text{(13)} \text{(C.11)} assures that for sufficiently large $k$

$$\| \sum_{m=n_k}^{s} \gamma_m (\Phi(s - 1, m) \otimes I_l) \bar{\epsilon}_{m+1} \| \leq \| \Gamma_s - \Gamma_{n_k - 1} \|$$

$$+ \sum_{m=n_k}^{s-1} \| \Phi(s - 1, m) - \Phi(s - 1, m + 1) \| \| \Gamma_m - \Gamma_{n_k - 1} \|$$

$$\leq T_k + \sum_{m=n_k}^{s-1} \| \Phi(s - 1, m) - \frac{1}{N} 11^T \| + \| \Phi(s - 1, m + 1) - \frac{1}{N} 11^T \| T_k$$

$$\leq T_k + c(\rho + 1) \sum_{m=n_k}^{s-1} T_k \leq T_k + \frac{c(\rho + 1)}{1 - \rho} T_k \quad \forall s : n_k \leq s \leq s_k.$$  \hspace{1cm} \text{(C.13)}

From \text{(C.12)} it follows that for sufficiently large $k$

$$\| \frac{1}{N} 11^T \otimes I_l \sum_{m=n_k}^{s} \gamma_m \bar{\epsilon}_{m+1} \| = \| \frac{1}{N} 11^T \otimes I_l(\Gamma_s - \Gamma_{n_k - 1}) \|$$

$$\leq \| \frac{1}{N} 11^T \otimes I_l \| \| \Gamma_s - \Gamma_{n_k - 1} \| \leq T_k \quad \forall s : n_k \leq s \leq s_k.$$  \hspace{1cm} \text{(C.14)}

Note that

$$\| \sum_{m=n_k}^{s} \gamma_m (\Phi(s - 1, m) - \frac{1}{N} 11^T) \otimes I_l) \bar{\epsilon}_{m+1} \|$$

$$\leq \| \sum_{m=n_k}^{s} \gamma_m \frac{1}{N} 11^T \otimes I_l \bar{\epsilon}_{m+1} \| + \| \sum_{m=n_k}^{s} \gamma_m \Phi(s - 1, m) \otimes I_l \bar{\epsilon}_{m+1} \|.$$
This implies (C.10) by (C.13) and (C.14).

**Proof of Lemma 4.6:** If there is no truncation at \( k = m + 1 \) for \( \tilde{X}_k \), then (22)-(24) can be rewritten as

\[
\tilde{X}_{m+1} = (W(m) \otimes I_l)\tilde{X}_m + \gamma_m(F(\tilde{X}_m) + \tilde{\varepsilon}_{m+1}),
\]

where \( \tilde{\varepsilon}_{m+1} \) and \( \tilde{X}_m \) are defined by (25), while \( F(\tilde{X}_m) \) by (14).

Assume \( \tilde{X}_{n_k} \xrightarrow{k \to \infty} \tilde{X} \). Let \( C > \| \tilde{X} \| \). There exists an integer \( k_C > 0 \) such that

\[
\| \tilde{X}_{n_k} \| \leq (C + \| \tilde{X} \| )/2 \quad \forall k \geq k_C.
\]

By Lemma 4.5 we know that there exist a constant \( T_1 > 0 \) and a positive integer \( k_0 \geq k_C \) such that for sufficiently large \( K > 0 \)

\[
\frac{1}{T_0} \sum_{s=n_{k_0}}^{m(n_k,t_k)\wedge(\tau_{n_k+1}-1)} \gamma_s \tilde{\varepsilon}_{i,s+1} I_{[\| \tilde{z}_i \| \leq k]} \leq \frac{1}{\sqrt{N}} \quad \forall t_k \in [0,T_0] \quad \forall T_0 \in [0,T_1] \quad \forall k \geq k_0 \quad \forall i \in V.
\]

Define

\[
M' \triangleq 1 + \frac{C + \| \tilde{X} \|}{2}(c_\rho + 2),
\]

\[
H_1 \triangleq \max_{\tilde{X}} \{ \| F(X) \| : \| X \| \leq M' + 1 + \frac{C + \| \tilde{X} \|}{2} \},
\]

\[
c_1 \triangleq H_1 + 3 + \frac{c(\rho + 1)}{1 - \rho}, \quad \text{and} \quad c_2 \triangleq \frac{H_1 + 1}{\sqrt{N}},
\]

where \( c \) and \( \rho \) are given by (13). Select \( T > 0 \) such that

\[
0 < T \leq T_1 \quad \text{and} \quad c_1 T < 1.
\]

For any \( k \geq k_0 \) and any \( T_k \in [0,T] \) define

\[
s_k \triangleq \sup \{ s \geq n_k : \| \tilde{X}_j - \tilde{X}_{n_k} \| \leq c_1 T_k + M' \quad \forall j : n_k \leq j \leq s \}.
\]

From (C.16) and (C.21) it follows that

\[
\| \tilde{X}_s \| \leq c_1 T_k + \| \tilde{X}_{n_k} \| + M' \leq M' + 1 + \frac{C + \| \tilde{X} \|}{2} \quad \forall s : n_k \leq s \leq s_k.
\]

If \( \lim_{k \to \infty} \sigma_k = \sigma < \infty \), then there exists a positive integer \( k_1 > k_0 \) such that \( \sigma_{n_k} = \sigma \) for all \( k \geq k_1 \), and hence \( \tau_{\sigma+1} = \infty \) by definition. Therefore, we have \( s_k < \tau_{\sigma_{n_k}+1} = \infty \) for all \( k \geq k_1 \) and any \( T_k \in [0,T] \).

If \( \lim_{k \to \infty} \sigma_k = \infty \), then there exists a positive integer \( k_2 > k_0 \) such that \( M_{\sigma_{n_k}} > M' + 1 + \frac{C + \| \tilde{X} \|}{2} \) for all \( k \geq k_2 \). Hence, from (C.23) we know that \( s_k < \tau_{\sigma_{n_k}+1} = \infty \) for all \( k \geq k_1 \) and any \( T_k \in [0,T] \).
In summary, whether \( \sigma_k \) diverges or not, there exists a positive integer \( k_1 > k_0 \) such that

\[
s_k < \tau_{\sigma_{k+1}} \quad \forall k \geq k_1 \quad \forall T_k \in [0, T]. \tag{C.24}
\]

We now prove \( s_k > m(n_k, T_k) \).

Assume the converse that for sufficiently large \( k \geq k_1 \) and any \( T_k \in [0, T] \)

\[
s_k \leq m(n_k, T_k). \tag{C.25}
\]

From (C.17) it follows that for sufficiently large \( K > 0 \)

\[
\frac{1}{T_k} \left\| \sum_{s=n_k}^{m(n_k, t'_k) \wedge (\tau_{\sigma_{k+1}} - 1)} \gamma_s \tilde{z}_{i, s+1} I[||\tilde{z}_{i, s}|| \leq K] \right\| \leq \frac{1}{\sqrt{N}} \quad \forall k \geq k_1 \quad \forall T_k \in [0, T] \quad \forall i \in \mathcal{V}. \tag{C.26}
\]

Set \( t_k(s) \triangleq \sum_{m=n_k}^{s} \gamma_m \) for sufficiently large \( k \geq k_1 \quad \forall T_k \in [0, T] \) and \( \forall s : n_k \leq s \leq s_k \). Then from (C.25) it follows that \( t_k(s) \leq \sum_{m=n_k}^{s_k} \gamma_m \leq T_k \). Similar to (B.7), we derive \( m(n_k, t_k(s)) = s \leq s_k \), and hence by (C.24)

\[
m(n_k, t_k(s)) \wedge (\tau_{\sigma_{k+1}} - 1) = s \wedge (\tau_{\sigma_{k+1}} - 1) = s. \tag{C.27}
\]

By (C.23), we have \( \| \tilde{X}_s \| \leq K_0 \quad \forall s : n_k \leq s \leq s_k \), where \( K_0 \triangleq M_0 + 1 + \frac{C + \| \tilde{X} \|}{2} \). Therefore, by replacing \( t'_k \) in (C.26) with \( t_k(s) \), from (C.27) it follows that

\[
\frac{1}{T_k} \left\| \sum_{m=n_k}^{s} \gamma_m \tilde{z}_{i, m+1} \right\| \leq \frac{1}{\sqrt{N}} \quad \forall s : n_k \leq s \leq s_k \quad \forall i \in \mathcal{V}
\]

for sufficiently large \( k \geq k_1 \quad \forall T_k \in [0, T] \) and \( K = K_0 \), and hence

\[
\| \sum_{m=n_k}^{s} \gamma_m \tilde{z}_{m+1} \| = \left\| \begin{pmatrix} \sum_{m=n_k}^{s} \gamma_m \tilde{z}_{1, m+1} \\ \vdots \\ \sum_{m=n_k}^{s} \gamma_m \tilde{z}_{N, m+1} \end{pmatrix} \right\| \leq \sqrt{N} \max_{i \in \mathcal{V}} \| \sum_{m=n_k}^{s} \gamma_m \tilde{z}_{i, m+1} \| \leq T_k \quad \forall s : n_k \leq s \leq s_k.
\]

Let us consider the following algorithm starting from \( n_k \) without truncation

\[
Z_{m+1} = (W(m) \otimes I_t)Z_m + \gamma_m(F(Z_m) + \tilde{e}_{m+1}), \quad Z_{n_k} = \tilde{X}_{n_k}. \tag{C.29}
\]

From (C.24) we know that (C.15) holds for \( m = n_k, \ldots, s_k - 1 \) for all \( k \geq k_1 \) and \( \forall T_k \in [0, T] \).

Hence, from (C.29) it follows that

\[
Z_m = \tilde{X}_m \quad \forall m : n_k \leq m \leq s_k \quad \forall k \geq k_1 \quad \forall T_k \in [0, T]. \tag{C.30}
\]
Consequently, by (C.23) and (C.30) we know that for all \( k \geq k_1 \) and any \( T_k \in [0, T] \)

\[
\| Z_m \| \leq M'_0 + 1 + \frac{C+ \| \bar{X} \|}{2} \quad \forall m : n_k \leq m \leq s_k.
\]

(C.31)

Hence from (C.19) it follows that

\[
\| F(Z_m) \| \leq H_1 \quad \forall m : n_k \leq m \leq s_k \quad \forall k \geq k_1 \quad \forall T_k \in [0, T].
\]

(C.32)

By setting \( z_k = \frac{1}{T} \otimes I_l Z_k \) from (C.29) and \( \frac{1}{N} (W(s) \otimes I_l) = \frac{1}{T} \otimes I_l \) it follows that

\[
z_{s+1} = z_s + \frac{1}{N} \sum_{m=n_k}^{s} \gamma_m (F(Z_m) + \tilde{\varepsilon}_{m+1}),
\]

(C.33)

and hence

\[
\| z_{s+1} - z_{n_k} \| = \left\| \frac{1}{N} \sum_{m=n_k}^{s} \gamma_m (F(Z_m) + \tilde{\varepsilon}_{m+1}) \right\| \leq \left\| \frac{1}{N} \sum_{m=n_k}^{s} \gamma_m (F(Z_m) + \tilde{\varepsilon}_{m+1}) \right\|
\]

\[
\leq \frac{1}{\sqrt{N}} \sum_{m=n_k}^{s} \gamma_m \| F(Z_m) \| + \frac{1}{\sqrt{N}} \sum_{m=n_k}^{s} \gamma_m \tilde{\varepsilon}_{m+1} \| \quad \forall s : n_k \leq s \leq s_k.
\]

(C.34)

From (C.25) we note that for sufficiently large \( k \geq k_1 \) and any \( T_k \in [0, T] \)

\[
\sum_{m=n_k}^{s} \gamma_m \leq \sum_{m=n_k}^{s_k} \gamma_m \leq T_k \quad \forall s : n_k \leq s \leq s_k,
\]

which incorporating with (C.28) (C.32) (C.34) implies that for sufficiently large \( k \geq k_1 \) and any \( T_k \in [0, T] \)

\[
\| z_{s+1} - z_{n_k} \| \leq \frac{H_1 + 1}{\sqrt{N}} T_k = c_2 T_k \quad \forall s : n_k \leq s \leq s_k.
\]

(C.35)

Denote by \( Z_{\perp,s} \triangleq D_{\perp} Z_s \) the disagreement vector of \( Z_s \). Then from \( Z_{n_k} = \bar{X}_{n_k} \) and (C.16) it follows that for all \( k \geq k_1 \)

\[
\| Z_{\perp,n_k} \| \leq \| (I_N - \frac{11}{N}) \otimes I_l \| \| \bar{X}_{n_k} \| \leq 2 \| \bar{X}_{n_{k}} \| \leq C + \| \bar{X} \|.
\]

(C.36)

From (C.29) we see that

\[
Z_{\perp,s+1} = D_{\perp} Z_{s+1} = D_{\perp} (W(s) \otimes I_l) Z_s + \gamma_s D_{\perp} (F(Z_s) + \tilde{\varepsilon}_{s+1}),
\]

and inductively

\[
Z_{\perp,s+1} = \Psi(s, n_k) Z_{n_k} + \sum_{m=n_k}^{s} \gamma_m \Psi(s - 1, m) D_{\perp} (F(Z_m) + \tilde{\varepsilon}_{m+1}) \quad \forall s \geq n_k,
\]

(C.37)

where \( \Psi(s, n_k) \) is defined by (C.5).
Then by (C.35), (C.36) and (C.41) we conclude that for sufficiently large $k$ we have
\[
Z_{\perp,s+1} = [(\Phi(s, n_k) - \frac{1}{N}11^T) \otimes I_l]Z_{n_k} + \sum_{m=n_k}^s \gamma_m[(\Phi(s - 1, m) - \frac{1}{N}11^T) \otimes I_l]F(Z_m)
\] (C.38)
\[
+ \sum_{m=n_k}^s \gamma_m[(\Phi(s - 1, m) - \frac{1}{N}11^T) \otimes I_l]\varepsilon_{m+1}.
\]

Consequently, by (C.28), Lemma C.1, and (C.39) it is seen that for sufficiently large $k$ and any $T_k \in [0, T]$ we have
\[
\|Z_{\perp,s+1}\| \leq \frac{C+\|\bar{X}\|}{2} \rho^{s+1-n_k} + \sum_{m=n_k}^s \gamma_m H_1 \rho^{s-m}
\] (C.39)
\[
+ \sum_{m=n_k}^s \gamma_m[(\Phi(s - 1, m) - \frac{1}{N}11^T) \otimes I_l]\varepsilon_{m+1} \quad \forall s : n_k \leq s \leq s_k.
\]

By noting that
\[
\sum_{m=n_k}^s \gamma_m H_1 \rho^{s-m} \leq \sup_{m \geq n_k} \gamma_m \sum_{m=n_k}^s H_1 \rho^{s-m} \leq \frac{cH_1}{1-\rho} \sup_{m \geq n_k} \gamma_m,
\]
from (C.28), Lemma C.1 and (C.39) it is seen that for sufficiently large $k \geq k_1$ and any $T_k \in [0, T]$ we have
\[
\|Z_{\perp,s+1}\| \leq \frac{C+\|\bar{X}\|}{2} \rho^{s+1-n_k} + \frac{cH_1}{1-\rho} \sup_{m \geq n_k} \gamma_m + (2 + \frac{c(\rho+1)}{1-\rho})T_k \quad \forall s : n_k \leq s \leq s_k.
\] (C.40)

Consequently, by $\gamma_k \xrightarrow{k \to \infty} 0$ and $0 < \rho < 1$ it follows that for sufficiently large $k \geq k_1$ and any $T_k \in [0, T]$ we have
\[
\|Z_{\perp,s+1}\| \leq \frac{C+\|\bar{X}\|}{2} \rho + 1 + (2 + \frac{c(\rho+1)}{1-\rho})T_k \quad \forall s : n_k \leq s \leq s_k.
\] (C.41)

Noticing $Z_{\perp,s} = D_{\perp} Z_s = Z_s - (1 \otimes I_l)z_s$, we derive
\[
Z_s = Z_{\perp,s} + (1 \otimes I_l)z_s,
\] (C.42)
and hence
\[
\|Z_{s+1} - Z_{n_k}\| = \| (1 \otimes I_l)z_{s+1} + Z_{\perp,s+1} - Z_{\perp,n_k} - (1 \otimes I_l)z_{n_k}\|
\]
\[
\leq \|Z_{\perp,s+1}\| + \|Z_{n_k}\| + \|(1 \otimes I_l)(z_{s+1} - z_{n_k})\|
\]
\[
\leq \|Z_{\perp,s+1}\| + \|Z_{\perp,n_k}\| + \sqrt{N} \|z_{s+1} - z_{n_k}\|.
\]

Then by (C.35), (C.36) and (C.41) we conclude that for sufficiently large $k \geq k_1$ and any $T_k \in [0, T]$ we have
\[
\|Z_{s+1} - Z_{n_k}\| \leq \frac{C+\|\bar{X}\|}{2}(c\rho + 2) + 1 + (3 + H_1 + \frac{c(\rho+1)}{1-\rho})T_k = M_0' + c_1 T_k \quad \forall s : n_k \leq s \leq s_k.
\] (C.43)
where $M'_0$ and $c_1$ are defined by (C.18) and (C.20), respectively.

Therefore, from (C.43) it is seen that for sufficiently large $k \geq k_1$ and any $T_k \in [0, T]$

$$\| Z_{s_k+1} \| \leq \| Z_{n_k} \| + M'_0 + c_1 T_k \leq M'_0 + 1 + \frac{C+\|X\|}{2}.$$  \hfill (C.44)

Then, by $Z_{s_k+1} = [W(s_k) \otimes I_l] \tilde{X}_{s_k} + \gamma_{s_k}(F(\tilde{X}_{s_k}) + \tilde{\varepsilon}_{s_k+1})$ and (C.15) we derive

$$\| \tilde{X}_{s_k+1} \| \leq M'_0 + 1 + \frac{C+\|X\|}{2}.$$  \hfill (C.45)

By noting $M_{\sigma_{s_k}} > M'_0 + 1 + \frac{C+\|X\|}{2} \forall k \geq k_1$, we conclude that for sufficiently large $k \geq k_1$ and any $T_k \in [0, T]$

$$s_k + 1 < \tau_{\sigma_{s_k+1}},$$

which implies that (C.15) holds for $m = s_k$, thereby $\tilde{X}_{s_k+1} = Z_{s_k+1}$. From (C.43) and $Z_{n_k} = \tilde{X}_{n_k}$ it follows that for sufficiently large $k \geq k_1$ and any $T_k \in [0, T]$

$$\| \tilde{X}_{s_k+1} - \tilde{X}_{n_k} \| \leq M'_0 + c_1 T_k,$$

which contradicts with the definition of $s_k$ given by (C.22). Consequently, (C.25) does not hold. Therefore, by the definition of $s_k$ given in (C.22), we know that (28) holds for sufficiently large $k$ and any $T_k \in [0, T]$.

From (C.24) and $s_k > m(n_k, T_k)$ it is seen that $m(n_k, T_k) < \tau_{\sigma_{s_k+1}}$. Hence from (C.26) we conclude that for sufficiently large $k$ and any $T_k \in [0, T]$

$$\frac{1}{T_k} \| \sum_{s=n_k}^{m(n_k, T_k)} \gamma_s \tilde{\varepsilon}_{i,s}+1 \| \leq \frac{1}{\sqrt{N}} \forall T_k \in [0, T_k] \forall i \in \mathcal{V}.$$  \hfill (C.46)

By (C.45), similar to (C.28), we derive that for sufficiently large $k$ and any $T_k \in [0, T]$

$$\| \sum_{p=n_k}^{m(n_k, T_k)} \gamma_p \tilde{\varepsilon}_{i,p+1} \| \leq \sqrt{N} \max_{i \in \mathcal{V}} \| \sum_{p=n_k}^{m(n_k, T_k)} \gamma_p \tilde{\varepsilon}_{i,p+1} \| \leq T_k \forall m : n_k \leq m \leq m(n_k, T_k).$$  \hfill (C.47)

From (C.23) and $s_k > m(n_k, T_k)$ it follows that for sufficiently large $k$ and any $T_k \in [0, T]$

$$\| \tilde{X}_m \| \leq M'_0 + 1 + \frac{\|X\|}{2} \forall m : n_k \leq m \leq m(n_k, T_k),$$

and hence by (C.19) we obtain

$$\| F(\tilde{X}_m) \| \leq H_1 \forall m : n_k \leq m \leq m(n_k, T_k).$$  \hfill (C.48)

Note that

$$\tilde{x}_{m+1} = \tilde{x}_m + \frac{1^T \otimes I_l}{N} \gamma_m (F(\tilde{X}_m) + \tilde{\varepsilon}_{m+1}).$$  \hfill (C.49)
by (30) (C.15) and $1^T \otimes L_N (W(s) \otimes I) = \frac{1^T \otimes L}{N}$. Since (C.15) holds for all $m : n_k \leq m \leq m(n_k, T_k)$ for sufficiently large $k$ and any $T_k \in [0, T]$, we conclude that (C.48) holds for all $m : n_k \leq m \leq m(n_k, T_k)$.

Then by a discussion similar to (C.34) it is seen that for sufficiently large $k$ and any $T_k \in [0, T]$

$$
\| \bar{x}_{m+1} - \bar{x}_{n_k} \| \\
\leq \frac{1}{\sqrt{N}} \sum_{p=n_k}^{m} \gamma_p \| F(\bar{x}_p) \| + \frac{1}{\sqrt{N}} \sum_{p=n_k}^{m} \gamma_p \bar{e}_{p+1} \| \quad \forall m : n_k \leq m \leq m(n_k, T_k).
$$

(C.49)

Consequently, from (C.46) (C.47) and (C.49) it follows that for sufficiently large $k$ and any $T_k \in [0, T]$

$$
\| \bar{x}_{m+1} - \bar{x}_{n_k} \| \leq \frac{H_1}{\sqrt{N}} + \frac{1}{\sqrt{N}} T_k = c_2 T_k \quad \forall m : n_k \leq m \leq m(n_k, T_k).
$$

(C.50)

So, (29) holds for sufficiently large $k$ and $\forall T_k \in [0, T]$.

The proof of Lemma 4.6 is completed.

\[\square\]

APPENDIX D

PROOF OF LEMMA 4.7

Proof of Lemma 4.7: By the boundedness of $\| \bar{X}_{n_k} \|$, without loss of generality, we may assume $\bar{X}_{n_k} \xrightarrow{k \to \infty} \bar{X}$. Setting $\bar{x} \triangleq \frac{1^T \otimes L}{N} \bar{X}$, we have $\bar{x}_{n_k} \xrightarrow{k \to \infty} \bar{x}$.

Inequality (28) in Lemma 4.6 assures that there exists a $T \in (0, 1)$ such that $m(n_k, T) < \tau_{\sigma_{n_k} + 1}$ and $\{ \bar{X}_s : n_k \leq s \leq m(n_k, T) + 1 \}$ are bounded for sufficiently large $k$.

Since $1^T W(s) = 1^T$, from (C.15) it follows that for $s : n_k \leq s \leq m(n_k, T)$

$$
\bar{x}_{s+1} = \bar{x}_s + \gamma_s f(\bar{x}_s) + \frac{1^T \otimes L}{N} \gamma_s \bar{e}_{s+1} + \gamma_s \sum_{i=1}^{N} \frac{f_i(\bar{x}_{i,s}) - f_i(\bar{x}_s)}{N}.
$$

(D.1)

Setting $e_{i,s+1} \triangleq \frac{f_i(\bar{x}_{i,s}) - f_i(\bar{x}_s)}{N}$, $e_{s+1} \triangleq \sum_{i=1}^{N} e_{i,s+1}$, and

$$
\zeta_{s+1} \triangleq \frac{1^T \otimes L}{N} \bar{e}_{s+1} + e_{s+1}
$$

(D.2)

we rewrite (D.1) as follows:

$$
\bar{x}_{s+1} = \bar{x}_s + \gamma_s (f(\bar{x}_s) + \zeta_{s+1}) \quad \forall s : n_k \leq s \leq m(n_k, T).
$$

(D.3)

We prove the lemma by two steps.

Step 1: We show that $\{ \zeta_{k+1} \}_{k \geq 0}$ defined by (D.2) satisfies

$$
\lim_{T \to 0} \lim_{k \to \infty} \frac{1}{T} \left\| \sum_{s=n_k}^{m(n_k, T_k)} \gamma_s \zeta_{s+1} \right\| = 0 \quad \forall T_k \in [0, T].
$$

(D.4)
Note that for any $T_k \in [0, T]$, $m(n_k, T_k) < \tau_{n_k+1}$ for sufficiently large $k$ and $\{\tilde{X}_s : n_k \leq s \leq m(n_k, T_k) + 1\}$ are bounded. So, for sufficiently large $K$ we have

$$m(n_k, T_k) \wedge (\tau_{n_k+1} - 1)$$

and hence by (27) we derive

$$\lim \limsup_{T \to 0} \frac{1}{T} \left\| \sum_{s=n_k}^{m(n_k, T_k)} \gamma_s \tilde{e}_{i,s+1} I[\|\tilde{e}_{i,s}\| \leq K] \right\| = \lim \limsup_{T \to 0} \frac{1}{T} \left\| \sum_{s=n_k}^{m(n_k, T_k)} \gamma_s \tilde{e}_{i,s+1} \right\| \quad \forall T_k \in [0, T],$$

which implies

$$\lim \limsup_{T \to 0} \frac{1}{T} \left\| \sum_{s=n_k}^{m(n_k, T_k)} \gamma_s \tilde{e}_{i,s+1} \right\| = 0 \quad \forall T_k \in [0, T].$$

So for (D.4) it suffices to show

$$\lim \limsup_{T \to 0} \frac{1}{T} \left\| \sum_{s=n_k}^{m(n_k, T_k)} \gamma_s e_{s+1} \right\| = 0 \quad \forall T_k \in [0, T].$$

(D.5)

Similar to (C.40), we can show that there exist positive constants $c_3, c_4, c_5$ such that for sufficiently large $k$

$$\left\| \tilde{X}_{i,s+1} \right\| \leq c_3 \rho^{s+1-n_k} + c_4 \sup_{m \geq n_k} \gamma_m + c_5 T \quad \forall s : n_k \leq s \leq m(n_k, T).$$

(D.7)

Since $0 < \rho < 1$, there exists a positive integer $m'$ such that $\rho^{m'} < T$. By noticing $\gamma_k \xrightarrow{k \to \infty} 0$ we see $\sum_{m=n_k}^{n_k+m'} \gamma_m \xrightarrow{k \to \infty} 0$. Thus, for sufficiently large $k$ we have $n_k + m' < m(n_k, T)$.

From (D.7) and $\gamma_k \xrightarrow{k \to \infty} 0$ it follows that for sufficiently large $k$

$$\left\| \tilde{X}_{i,s+1} \right\| \leq o(1) + (c_3 + c_5) T \quad \forall s : n_k + m' \leq s \leq m(n_k, T),$$

(D.8)

where $o(1) \to 0$ as $k \to \infty$.

Since $\tilde{x}_{n_k} \xrightarrow{k \to \infty} \tilde{x}$, from (29) (D.8) it follows that for sufficiently large $k$

$$\left\| \tilde{x}_s - \tilde{x} \right\| \leq \left\| \tilde{x}_s - \tilde{x}_{n_k} \right\| + \left\| \tilde{x}_{n_k} - \tilde{x} \right\| = o(1) + O(T),$$

$$\left\| \tilde{x}_{i,s} - \tilde{x} \right\| \leq \left\| \tilde{x}_{i,s} - \tilde{x}_{s} \right\| + \left\| \tilde{x}_s - \tilde{x} \right\| = o(1) + O(T) \quad \forall s : n_k + m' \leq s \leq m(n_k, T),$$

(D.9)

where $O(T) \to 0$ as $T \to 0$. By the continuity of $f_i(\cdot)$ we derive

$$\left\| f_i(\tilde{x}_{i,s}) - f_i(\tilde{x}_s) \right\| \leq \left\| f_i(\tilde{x}_{i,s}) - f_i(\tilde{x}) \right\| + \left\| f_i(\tilde{x}_s) - f_i(\tilde{x}) \right\| = o(1) + O(T).$$

Consequently,

$$\left\| e_{i,s+1} \right\| = \frac{1}{N} \left\| f_i(\tilde{x}_{i,s}) - f_i(\tilde{x}_s) \right\| = o(1) + O(T).$$
By the definition of $e_{s+1}$ it can be seen that for sufficiently large $k$

$$\|e_{s+1}\| = o(1) + O(T) \quad \forall s : n_k + m' \leq s \leq m(n_k, T). \quad (D.10)$$

By the boundedness of $\{\bar{X}_s : n_k \leq s \leq m(n_k, T)\}$ and by the continuity of $f_i(\cdot)$ we know that there exists a constant $c_e > 0$ such that $\|e_{s+1}\| \leq c_e$. Then from $(D.10)$ it follows that for sufficiently large $k$:

$$\frac{1}{T} \| \sum_{s=n_k}^{n_k+m'} \gamma_s e_{s+1} \| \leq \frac{1}{T} (\sum_{s=n_k}^{m(n_k, T)} \| e_{s+1} \| + \sum_{s=n_k+m'+1}^{m(n_k, T)} \| e_{s+1} \|)
\leq \frac{1}{T} \left( \sum_{s=n_k}^{n_k+m'} \gamma_s c_e + (o(1) + O(T)) \sum_{s=n_k+m'+1}^{m(n_k, T)} \gamma_s \right)
\leq \left( \frac{c_e m'}{T} \sup_{s \geq n_k} \gamma_s + \frac{T_k(O(T) + o(1))}{T} \right) \leq \frac{c_e m'}{T} \sup_{s \geq n_k} \gamma_s + O(T) + o(1) \quad \forall T_k \in [0, T].$$

This combining with $\gamma_k \xrightarrow{k \to \infty} 0$ yields

$$\limsup_{k \to \infty} \frac{1}{T} \| \sum_{s=n_k}^{m(n_k, T)} \gamma_s e_{s+1} \| = O(T) \quad \forall T_k \in [0, T]. \quad (D.11)$$

By letting $T \to 0$, we know that $(D.6)$ holds, which incorporating with $(D.5)$ leads $(D.4)$.

**Step 2:** Assume the converse: for some nonempty interval $[\delta_1, \delta_2]$ with $d((\delta_1, \delta_2), v(J)) > 0$, there are infinitely many crossings $\{v(\bar{x}_{n_k}), \cdots, v(\bar{x}_{m_k})\}$ with $\|\bar{X}_{n_k}\|$ bounded.

By setting $T_k = \gamma_{n_k}$ in $(29)$, we derive

$$\|\bar{x}_{n_k+1} - \bar{x}_{n_k}\| \leq c_2 \gamma_{n_k} \xrightarrow{k \to \infty} 0.$$ 

By the definition of crossings $v(\bar{x}_{n_k}) \leq \delta_1 < v(\bar{x}_{n_k+1})$, so we obtain

$$v(\bar{x}_{n_k}) \xrightarrow{k \to \infty} \delta_1 = v(\bar{x}), \quad d(\bar{x}, J) \overset{\triangle}{=} \vartheta > 0. \quad (D.12)$$

Then by $(29)$ it can be seen that for sufficiently small $t > 0$ and large $k$:

$$d(\bar{x}, J) > \frac{\vartheta}{2} \quad \forall s : n_k \leq s \leq m(n_k, t) + 1. \quad (D.13)$$

From $(D.3)$ we obtain

$$v(\bar{x}_{m(n_k, t)+1}) = v(\bar{x}_{n_k}) + \sum_{s=n_k}^{m(n_k, t)} \gamma_s (f(\bar{x}_s) + \zeta_{s+1})
= v(\bar{x}_{n_k}) + v_x(\xi_k)^T \sum_{s=n_k}^{m(n_k, t)} \gamma_s (f(\bar{x}_s) + \zeta_{s+1}), \quad (D.14)$$
where $\xi_k$ is in-between $\bar{x}_{n_k}$ and $\bar{x}_{m(n_k,t)+1}$. We rewrite (D.14) as follows:

$$v(\bar{x}_{m(n_k,t)+1}) - v(\bar{x}_{n_k}) = \sum_{s=n_k}^{m(n_k,t)} \gamma_s v_x(\bar{x}_s)^T f(\bar{x}_s)$$

(D.15)

$$+ \sum_{s=n_k}^{m(n_k,t)} \gamma_s (v_x(\xi_k) - v_x(\bar{x}_s))^T f(\bar{x}_s) + v_x(\xi_k)^T \sum_{s=n_k}^{m(n_k,t)} \gamma_s \xi_{s+1}.$$  

By noticing the boundeness of $\{\bar{x}_s : n_k \leq s \leq m(n_k,t)\}$ for sufficiently small $t > 0$ and large $k$, and by the continuity of $f(\cdot)$ there exists a constant $c_6 > 0$ such that

$$\sum_{s=n_k}^{m(n_k,t)} \gamma_s || f(\bar{x}_s) || \leq c_6 t.$$  

(D.17)

Note that $\xi_k$ is in-between $\bar{x}_{n_k}$ and $\bar{x}_{m(n_k,t)+1}$. Then by the continuity of $v_x(\cdot)$ and (29) we know that for sufficiently small $t > 0$ and large $k$

$$v_x(\xi_k) - v_x(\bar{x}_s) = O(t) \quad \forall s : n_k \leq s \leq m(n_k,t),$$  

(D.18)

which incorporating with (D.17) yields

$$\sum_{s=n_k}^{m(n_k,t)} \gamma_s (v_x(\xi_k) - v_x(\bar{x}_s))^T f(\bar{x}_s) \leq O(t) \sum_{s=n_k}^{m(n_k,t)} \gamma_s || f(\bar{x}_s) || \leq O(t) t.$$  

(D.19)

Since $\bar{x}_{n_k} \xrightarrow{k \to \infty} \bar{x}$, by the continuity of $v_x(\cdot)$ and (29) it follows that

$$v(\bar{x}_s) - v_x(\bar{x}) = o(1) + O(t) \quad \forall s : n_k \leq s \leq m(n_k,t)$$

for sufficiently small $t > 0$ and large $k$. Then by (D.18) we derive

$$v_x(\xi_k) - v_x(\bar{x}) = o(1) + O(t) \quad \forall s : n_k \leq s \leq m(n_k,t).$$

Consequently, for sufficiently small $t > 0$ and large $k$

$$v_x(\xi_k)^T \sum_{s=n_k}^{m(n_k,t)} \gamma_s \xi_{s+1} = [(v_x(\xi_k) - v_x(\bar{x})) + v_x(\bar{x})]^T \sum_{s=n_k}^{m(n_k,t)} \gamma_s \xi_{s+1}$$

(D.20)

\leq (o(1) + O(t) + \| v_x(\bar{x}) \| \| \sum_{s=n_k}^{m(n_k,t)} \gamma_s \xi_{s+1} \| \forall s : n_k \leq s \leq m(n_k,t).$$
Therefore, by putting inequalities (D.16) (D.19) (D.20) into (D.15) we obtain
\[
v(\bar{x}_{m(n_k,t)+1}) - v(\bar{x}_{n_k}) \leq -\alpha_1 t + O(t) t + (o(1) + O(t) + \| v_x(\bar{x}) \|) \sum_{s=n_k}^{m(n_k,t)} \gamma_s \zeta_{s+1} \| .
\]
Hence by (D.12) it follows that
\[
\limsup_{k \to \infty} v(\bar{x}_{m(n_k,t)+1}) \leq \delta_1 - \alpha_1 t + O(t) t + (o(1) + O(t) + \| v_x(\bar{x}) \|) \sum_{s=n_k}^{m(n_k,t)} \gamma_s \zeta_{s+1} ,
\]
for sufficiently small \( t \).

However, by the continuity of \( v_x(\cdot) \) and (29) we know that
\[
\lim_{t \to 0} \max_{m(n_k,t) \leq m \leq m(n_k,t)+1} \| v(\bar{x}_{m+1}) - v(\bar{x}_{n_k}) \| = 0,
\]
which implies that \( m(n_k,t)+1 < m_k \) for sufficiently small \( t \). Therefore, we conclude that \( v(\bar{x}_{m(n_k,t)+1}) \in (\delta_1, \delta_2) \), which is contradictory with (D.21). Consequently, the converse assumption is not true. The proof is completed.

\[\square\]

**APPENDIX E**

**PROOF OF LEMMA 4.8**

**Proof of Lemma 4.8** We prove the boundedness of the truncation numbers. Assume the converse:

\[
\lim_{k \to \infty} \sigma_k = \infty.
\]

Then, for any \( k \geq 1 \) there exists a positive integer \( n_k \) such that \( \sigma_{n_k} = k \) and \( \sigma_{n_k-1} = k-1 \). If \( \sigma_{i,n_k} = k \), then by Remark 2.1 we have \( x_{i,n_k} = x^* \), and hence from (A.4) it follows that \( \bar{x}_{i,n_k} = x_{i,n_k} = x^* \); If \( \sigma_{i,n_k} < k \), then from (A.3) it follows that \( \bar{x}_{i,n_k} = x^* \). So, in both cases \( \bar{x}_{i,n_k} = x^* \) for any \( i \in \mathcal{V} \). Then \( \bar{X}_{n_k} = (1 \otimes I_l)x^* \), thereby \( \{ \bar{X}_{n_k} \}_{k \geq 1} \) is a convergent subsequence, and \( \bar{x}_{n_k} = x^* \).

Since \( \{ M_k \} \) is a sequence of positive numbers increasingly diverging to infinity, there exists a positive integer \( k_0 \) such that

\[
M_k \geq 2\sqrt{N}c_0 + 2 + M'_1 \quad \forall k \geq k_0,
\]

where

\[
M'_1 = 2 + (2\sqrt{N}c_0 + 2)(c_\rho + 2).
\]
In what follows, we prove that under the converse assumption \( \{\bar{x}_{n_k}\} \) starting from \( x^* \) crosses \( \|x\| = c_0 \) infinitely many times.

Define

\[
m_k \triangleq \inf \{s > n_k : \|\bar{X}_s\| \geq 2\sqrt{Nc_0} + 2 + M'_1\}, \tag{E.4}
\]

\[
l_k \triangleq \sup \{s < m_k : \|\bar{X}_s\| \leq 2\sqrt{Nc_0} + 2\}. \tag{E.5}
\]

By noticing \( \|\bar{X}_{n_k}\| = \sqrt{N} \|x^*\| \) and \( \|x^*\| < c_0 \), we derive \( \|\bar{X}_{n_k}\| < \sqrt{Nc_0} \). Hence from (E.4) (E.5) it is seen that \( n_k \leq l_k < m_k \).

By the definition of \( l_k \) we know that \( \bar{X}_{l_k} \) is bounded, then there exists a convergent subsequence, denoted still by \( \bar{X}_{l_k} \). By denoting \( \bar{X} \) the limiting point of \( \bar{X}_{l_k} \), from \( \|\bar{X}_{l_k}\| \leq 2\sqrt{Nc_0} + 2 \) it follows that \( \|\bar{X}\| \leq 2\sqrt{Nc_0} + 2 \).

By Lemma 4.6 there exist constants \( c_1 > 0, c_2 > 0, M'_0 > 0, 0 < T < 1 \) with \( c_1T \leq 1 \) such that for sufficiently large \( k \geq k_0 \)

\[
\|\bar{X}_{m+1} - \bar{X}_{l_k}\| \leq c_1T + M'_0 \quad \forall m : l_k \leq m \leq m(l_k, T). \tag{E.6}
\]

Then we derive

\[
\|\bar{X}_{m+1}\| \leq \|\bar{X}_{l_k}\| + c_1T + M'_0 \quad \forall m : l_k \leq m \leq m(l_k, T),
\]

where \( M'_0 > 0 \) is defined by (C.18) with \( C = 2\sqrt{Nc_0} + 2 \).

Consequently, for sufficiently large \( k \geq k_0 \)

\[
\|\bar{X}_{m+1}\| \leq 2\sqrt{Nc_0} + 2 + 1 + 1 + (2\sqrt{Nc_0} + 2)(c\rho + 2) = 2\sqrt{Nc_0} + 2 + M'_1 \quad \forall m : l_k \leq m \leq m(l_k, T), \tag{E.7}
\]

where \( M'_1 \) is defined by (E.3).

From (E.7) and by the definition of \( m_k \) defined in (E.4), we conclude \( m(l_k, T) + 1 \leq m_k \) for sufficiently large \( k \geq k_0 \). Then by (E.5) we know that for sufficiently large \( k \geq k_0 \)

\[
2\sqrt{Nc_0} + 2 < \|\bar{X}_{m+1}\| \leq 2\sqrt{Nc_0} + 2 + M'_1 \quad \forall m : l_k \leq m \leq m(l_k, T). \tag{E.8}
\]

Noticing that \( \{\bar{X}_{m+1} : l_k \leq m \leq m(l_k, T)\} \) are bounded, similarly to (C.40) we know that for sufficiently large \( k \geq k_0 \)

\[
\|\bar{X}_{\perp, m+1}\| \leq (2\sqrt{Nc_0} + 2)c\rho^{m+1-l_k} + \frac{cH_1}{1 - \rho} \sup_{m \geq l_k} \gamma_m + 2T + \frac{c(\rho + 1)}{1 - \rho}T \quad \forall m : l_k \leq m \leq m(l_k, T). \tag{E.9}
\]
Since $0 < \rho < 1$, there exists a positive integer $m_0$ such that $4c\rho^{m_0} < 1$. By $\gamma_k \rightarrow 0$ it is clear that $\sum_{m=l_k}^{l_k+m_0} \gamma_m \rightarrow 0$. Then for sufficiently large $k \geq k_0$ we have $l_k + m_0 < m(l_k, T)$. Consequently, from (E.8) it is seen that for sufficiently large $k \geq k_0$

$$\| \tilde{X}_{l_k+m_0} \| > 2\sqrt{Nc_0} + 2.$$  \hspace{1cm} (E.10)

By (E.9), $c_1 T < 1$, and $\gamma_k \rightarrow 0$ it follows that for sufficiently large $k \geq k_0$

$$\| \tilde{X}_{l_k,m_0} \| \leq 2c\rho^{m_0}(\sqrt{Nc_0} + 1) + \frac{1}{2} + c_1 T \leq \frac{1}{2}(\sqrt{Nc_0} + 1) + \frac{1}{2} + 1 = \frac{\sqrt{Nc_0}}{2} + 2,$$  \hspace{1cm} (E.11)

where $c_1$ is defined by (C.20). By noticing that

$$(1 \otimes I_l)\tilde{x}_{l_k+m_0} = \tilde{X}_{l_k+m_0} - \tilde{X}_{l_k,m_0},$$

and by (E.10) and (E.11) we derive

$$\sqrt{N} \| \tilde{x}_{l_k+m_0} \| = \| \tilde{X}_{l_k+m_0} - \tilde{X}_{l_k,m_0} \| \geq \| \tilde{X}_{l_k+m_0} \| - \| \tilde{X}_{l_k,m_0} \| > \frac{3}{2} \sqrt{Nc_0}.$$  \hspace{1cm} (E.12)

Hence $\| \tilde{x}_{l_k+m_0} \| > c_0$.

In summary, we have shown that for sufficiently large $k \geq k_0$, starting from $x^*$, $\{\tilde{x}_{n_k}\}$ crosses the ball $\| x \| = c_0$ before time $n_{k+1}$. Hence $\{\tilde{x}_{n_k}\}$ starting from $x^*$ crosses the ball $\| x \| = c_0$ infinitely many times, since $\lim_{k \rightarrow \infty} \sigma_k = \infty$.

Since $v(J)$ is nowhere dense, there exists a nonempty interval $[\delta_1, \delta_2] \in (v(x^*), \inf_{\| x \| = c_0} v(x))$ with $d([\delta_1, \delta_2], v(J)) > 0$. Therefore, under the converse assumption $\{v(\tilde{x}_k)\}$ crosses $[\delta_1, \delta_2]$ infinitely many times while $\{\tilde{X}_{n_k}\}$ are bounded. This contradicts Lemma 4.7. Therefore, (E.1) is impossible and (31) holds.

\[\square\]

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