“Integrating in” and Effective Lagrangian for Non-Supersymmetric Yang-Mills Theory

Igor Halperin and Ariel Zhitnitsky

Physics and Astronomy Department
University of British Columbia
6224 Agricultural Road, Vancouver, BC V6T 1Z1, Canada
e-mail: higor@physics.ubc.ca
arz@physics.ubc.ca

PACS numbers: 12.38.Aw, 11.15.Tk, 11.30.-j.
Keywords: Anomalous Ward identities, holomorphy, vacuum sectors.

Abstract:

Recently a non-supersymmetric analog of Veneziano-Yankielowicz (VY) effective Lagrangian has been proposed and applied for the analysis of the $\theta$ dependence in pure Yang-Mills theory. This effective Lagrangian is similar in many respects to the VY construction and, in particular, exhibits a kind of low energy holomorphy which is absent in the full YM theory. Here we incorporate a heavy fermion into this effective theory by using the “integrating in” technique. We find that, in terms of this extended theory, holomorphy of the effective Lagrangian for pure YM theory naturally implies a holomorphic dependence on the heavy fermion mass. It is shown that this analysis fixes, under certain assumptions, a dimensionless parameter which enters the effective Lagrangian and determines the number of nondegenerate vacuum sectors in pure YM theory. We also compare our results for the vacuum structure and $\theta$ dependence to those obtained recently by Witten on the basis of AdS/CFT correspondence.
1 Introduction

The remarkable progress made over the past years in understanding the nonperturbative properties of supersymmetric (SUSY) N=1 and N=2 theories owes a lot to holomorphy. Supersymmetry requires a superpotential $W_{\text{eff}}(X_i, g_i)$ of the effective low energy theory to be a holomorphic function of the light fields $X_i$ and the coupling constants $g_i$, and thus powerfully constrains the large distance dynamics of SUSY theories [1] (see e.g. [2] for a review). The famous Veneziano-Yankielowicz (VY) effective Lagrangian [3], obtained from the anomalous Ward identities of SUSY YM theory, is holomorphic in its fields and parameters. Moreover, the effective VY potentials for theories with different numbers of matter fields (e.g. ones with $N_f$ and $N_f - 1$ fermion flavors) are related to each other by holomorphic decoupling relations [3]. More recently, an operation inverse to integrating out of a heavy fermion was suggested under the name of “integrating in” procedure [4]. The integrating in technique provides a powerful method to obtain relations between parameters and functional forms of superpotentials for theories with different matter contents. In particular, it was shown in [4] that Affleck-Dine-Seiberg superpotential [5] can be obtained from VY Lagrangian for SUSY gluodynamics by the integrating in/out procedure. The integration in technique was used to study the phase structure of SUSY theories [6], and to calculate numerical parameters in different models [7].

Recently a non-supersymmetric analog of the VY effective Lagrangian has been proposed [8] using an infinite series of anomalous Ward identities for YM theory. This construction has some striking similarities to its supersymmetric counterpart. Specifically, we have found that this effective Lagrangian (more precisely, effective potential) possesses both a “dynamical” part, which is similar to the original VY form [3], and a “topological” part, which is analogous to an improvement of the VY Lagrangian, suggested recently [4]. The picture of the physical $\theta$ dependence in pure YM theory, following from the analysis of this effective Lagrangian, is rather similar to the one found for SUSY YM theory [10, 9]: the correct $2\pi$ periodicity in $\theta$ is recovered when a set of disconnected vacua is taken into account. The difference from the supersymmetric case is in the absence of degeneracy between the vacua. As a result, all vacua but one with a lowest (for a fixed $\theta$) energy contribute zero to the partition function in the thermodynamic limit [8]. The number of vacua can be found from explicit dynamical calculations only. For the SUSY case, such calculations can be done [10], while no similar technique is known in the non-supersymmetric case. The information on the number of vacua was coded in the effective Lagrangian of Ref.[8] in a dimensionless number $\xi \sim N^{-1}c$ (see Eq.(3) below). (On general grounds, the number of vacua in YM theory should be proportional to the number of colors $N_c$.)

Another remarkable property of the effective potential of Ref.[8] is its holomorphic structure. At first sight, a claim of holomorphy of an effective Lagrangian for YM theory may sound suspicious, as the full theory is not seen to possess any holomorphy, in contrast to supersymmetric models. As will be discussed in detail below, there is no contradiction here: holomorphy is the property of a nonperturbative effective potential (defined within a particular regularization scheme) which is fixed in this approach by the anomalous 

\footnote{It is well known that the VY effective Lagrangian is not a genuine Wilsonian effective Lagrangian for light degrees of freedom, but rather has a different meaning, see Sect.2.}
Ward identities and some additional arguments, and describes the large distance physics only. We do not expect that kinetic terms and/or perturbative contributions would be holomorphic as well, but they are irrelevant anyway for our purposes.

In this paper we analyse holomorphy of the effective Lagrangian of Ref.[8] using the integration in technique [4]. By the integrating in procedure we obtain an effective Lagrangian for the YM field interacting with a heavy fermion. It will be shown that holomorphy of the effective Lagrangian for pure YM theory corresponds to a holomorphic dependence on a fermion mass in an extended theory including the heavy fermion. When gluodynamics is defined as a low energy limit of the latter theory, the integrating in procedure fixes (under certain plausible assumptions specified below) the aforementioned parameter $\xi$, and thus allows one to find the number of different nondegenerate $\theta$ vacuum sectors [8] for pure YM theory. As will be shown below, this approach results in the same value $\xi = 4/(3b)$ which was obtained in [11] with a different method which implies, however, the same regularization prescription.

Our presentation is organized as follows. In Sect.2 we recall the construction [8] of the effective Lagrangian for pure gluodynamics. In Sect.3 a new effective Lagrangian for the theory including a heavy fermion is obtained by the integration in technique, assuming the standard form of the fermion mass term and preservation of holomorphy under the procedure of integrating in/out. We show that the parameter $\xi$ can be fixed by comparing the holomorphic properties of two effective Lagrangians. The matching of two effective theories is further discussed in Sect.4 from the point of view of their global, “topological”, properties. We also study in this section a connection of the present approach with the previous analysis of Ref.[11]. In Sect.5 we discuss our results and compare them with different approaches to the problems of interest. In particular, we argue that our picture of the vacuum structure and $\theta$ dependence is in qualitative agreement with that recently obtained by Witten [12] in the limit $N_c \to \infty$ within a different approach. The final Sect.6 contains our conclusions.

2 Effective Lagrangian for gluodynamics

The purpose of this section is to describe an effective Lagrangian for YM theory, which was constructed in our previous paper [8]. Before proceeding with the presentation, we would like to pause for a comment on the meaning of this effective Lagrangian. As there exist no Goldstone bosons in pure YM theory, no Wilsonian effective Lagrangian, which would correspond to integrating out heavy modes, can be constructed for gluodynamics. Instead, one speaks in this case of an effective Lagrangian as a generating functional for vertex functions of the composite fields $G^2$ and $\bar{G}G$. Moreover, only the potential part of this Lagrangian can be found as it corresponds to zero momentum n-point functions of $G^2, \bar{G}G$, fixed by arguments appealing to renormalizability of the theory. The kinetic part is not fixed in this way. Thus, such an effective Lagrangian is not very useful for calculating the S-matrix, but is perfectly suitable for addressing the vacuum properties. Specifically, space-time independent fields are amenable to a study within this framework.

2 For SUSY theories, Veneziano-Yankielowicz effective Lagrangian [3] has the same meaning, see [1].
Analogously to the case of SUSY theories, an effective Lagrangian for pure YM theory is constructed using an infinite series of anomalous Ward identities which serve as matching conditions ensuring consistency of the large distance properties of the theory with its small distance behavior fixed by renormalizability and asymptotic freedom. The complete set of two-point correlation functions is

\[
\lim_{q \to 0} i \int dx e^{iqx} \langle 0 \mid T \left\{ \frac{\beta(\alpha_s)}{4\alpha_s} G^2(x) \frac{\beta(\alpha_s)}{4\alpha_s} G^2(0) \right\} \mid 0 \rangle = -4\left( \frac{\beta(\alpha_s)}{4\alpha_s} \right)^2 G^2, \tag{1}
\]

\[
\lim_{q \to 0} i \int dx e^{iqx} \langle 0 \mid T \left\{ \frac{\beta(\alpha_s)}{4\alpha_s} G^2(x) \frac{\alpha_s}{8\pi} G G(0) \right\} \mid 0 \rangle = -4\left( \frac{\alpha_s}{8\pi} G G \right), \tag{2}
\]

\[
\lim_{q \to 0} i \int dx e^{iqx} \langle 0 \mid T \left\{ \frac{\alpha_s}{8\pi} G G(x) \frac{\alpha_s}{8\pi} G G(0) \right\} \mid 0 \rangle = \xi^2 \left( \frac{\beta(\alpha_s)}{4\alpha_s} \right)^2 G^2. \tag{3}
\]

Here \( \xi \) is a dimensionless parameter which will be assumed to be a rational number. (An irrational value of \( \xi \) would presumably produce a non-differentiable \( \theta \) dependence for YM theory.) Multi-point correlation functions of the fields \( G^2_{\mu\nu} \) and \( G_{\mu\nu} \bar{G}_{\mu\nu} \) are obtained by differentiating Eqs.(1-3) with respect to \( \theta \) and \( 1/g_0^2 \), see below. In what follows we use the one-loop \( \beta \)-function \( \beta(\alpha_s) = -b\alpha_s^2/(2\pi) \) where \( b = (11/3)N_c \), though most of the discussion below can also be formulated with formally keeping the full \( \beta \)-function.

A few comments on these Ward identities are in order. All correlation functions (1)-(3) are defined via Wick type of the T-product, i.e. by the differentiation of the path integral in respect to corresponding parameters, see below. Perturbative contributions to the conformal anomaly matrix element \( \langle -\beta\alpha_s/(8\pi) G^2 \rangle \) in Eqs. (1) and (3) are subtracted to any finite order in \( \alpha_s \) by definition. In this case, its dependence on the bare coupling constant \( g_0 \) corresponding to the cut-off scale \( M_R \) is fixed by the dimensional transmutation formula

\[
\langle -\frac{b\alpha_s}{8\pi} G^2 \rangle = c_1 \left[ M_R \exp \left( -\frac{8\pi^2}{bg_0^2} \right) \right]^4 \equiv c_1 \Lambda_{YM}^4.
\]

Analogously, \( \langle \alpha_s/(8\pi) G G \rangle = c_2 \Lambda_{YM}^4 \). Here numerical constants \( c_1, c_2 \) are independent of \( g_0 \), but depend on the vacuum angle \( \theta \) which is allowed to be non-zero in Eqs.(1-3). We note that the constants \( c_1, c_2 \) depend on a particular regularization scheme used to define the nonperturbative vacuum condensates. However, once specified, the VEV (4) determines all zero momentum correlation functions of \( \beta(\alpha_s)/(4\alpha_s) G^2 \), with perturbative tails subtracted. The Ward identities (1)-(3) then follow by the differentiation of the above expressions with respect to \( 1/g_0^2 \). They were derived long ago by Novikov, Shifman, Vainshtein and Zakharov (NSVZ) [13]. By derivation, the two-point functions (1-3) do not contain perturbative contributions\(^3\). As was discussed in detail in Ref.[5], the zero

\(^3\) Although we will occasionally call Eqs. (1)-(3) the Ward identities, it should be mentioned that Eq. (3) is not precisely a Ward identity, but rather is a scheme dependent relation involving an unspecified at this stage parameter \( \xi \).

\(^4\) The same result (1) was obtained in [13] using canonical methods with Pauli-Villars regularization. To one loop order in regulator fields, it was found that perturbative contributions add an identity to both sides of Eq.(1). The absence of perturbative contributions to the correlation functions (2), (3) is obvious.
momentum correlation functions (1-3) and their n-point generalizations are generated by the differentiation of log\((Z/Z_{PT})\) with respect to \(1/g_0^2\) and \(\theta\), where

\[
Z(\theta) = Z_{PT} \exp \{-iV E_v(\theta)\} = Z_{PT} \exp \left\{-iV \langle 0 | -\frac{b\alpha_s}{32\pi} G^2 | 0 \rangle_\theta \right\}.
\] (5)

Here the perturbatively defined partition function \(Z_{PT}\) does not depend on \(\theta\) and absorbs perturbative contributions to the conformal anomaly. The dimensionless parameter \(\xi\) in Eq.(3) is related to the \(\theta\) dependence of the vacuum energy \(E_v(\theta)\) in Eq.(5):

\[
E_v(\theta) = E_v(0) f(\theta) = 1 - 2\xi^2 \theta^2 + \cdots.
\] (6)

Let us introduce complex linear combinations of the composite fields

\[
H = \frac{b\alpha_s}{16\pi} \left(-G^2 + i\frac{2}{b\xi} G \tilde{G}\right), \quad \bar{H} = \frac{b\alpha_s}{16\pi} \left(-G^2 - i\frac{2}{b\xi} G \tilde{G}\right).
\] (7)

In terms of these combinations, the Ward identities (1-3) take particularly simple forms (for an arbitrary value of the vacuum angle \(\theta\)):

\[
\lim_{q \to 0} i \int dx e^{iqx} \langle 0 | T\{H(x) H(0)\} | 0 \rangle = -4 \langle H \rangle,
\]

\[
\lim_{q \to 0} i \int dx e^{iqx} \langle 0 | T\{\bar{H}(x) \bar{H}(0)\} | 0 \rangle = -4 \langle \bar{H} \rangle,
\]

\[
\lim_{q \to 0} i \int dx e^{iqx} \langle 0 | T\{\bar{H}(x) H(0)\} | 0 \rangle = 0.
\] (8)

It can be seen that the n-point zero momentum correlation function of the operator \(H\) equals \((-4)^n - 1 \langle H \rangle\). Multi-point correlation functions of the operator \(\bar{H}\) are analogously expressed in terms of its vacuum expectation value \(\langle \bar{H} \rangle\). At the same time, it is easy to check that the decoupling of the fields \(H\) and \(\bar{H}\) holds for arbitrary n-point functions of \(H, \bar{H}\). This is the origin of holomorphy of an effective Lagrangian for YM theory.

An effective low energy Lagrangian (more precisely, effective potential) is now constructed as the (Legendre transform of) generating functional for zero momentum correlation functions of the marginal operators \(G_{\mu\nu} \tilde{G}_{\mu\nu}\) and \(G_{\mu\nu} G_{\mu\nu}\), such as Eq.(8) and their n-point generalizations. It is a function of effective zero momentum fields \(h, \bar{h}\) which describe the vacuum expectation values of the fields \(H, \bar{H}\):

\[
\int dx h = \langle \int dx H \rangle, \quad \int dx \bar{h} = \langle \int dx \bar{H} \rangle .
\] (9)

\footnote{To avoid possible misunderstanding, we note that the right hand side of the last equation in (8) does contain perturbative contributions proportional to regular powers of \(\alpha_s\). However, they are irrelevant for our purposes. The decoupling of the fields \(H\) and \(\bar{H}\) holds at the level of nonperturbative \(O(e^{-1/\alpha_s})\) effects. Holomorphy of an effective potential for YM theory has the same status. Thus, in contrast to the supersymmetric case where holomorphy is an exact property of the effective superpotential, in the present case it only refers to a “nonperturbative” effective potential which does not include perturbative effects to any finite order in \(\alpha_s\). We assume that perturbative and nonperturbative effects can be separated, at least in principle or/and by some suitable convention.}
Omitting details of the derivation, which can be found in [8], we give the final answer for the improved effective potential $F(h, \bar{h})$:

$$e^{-iV F(h, \bar{h})} = \sum_{n=-\infty}^{+\infty} \sum_{k=0}^{q-1} \exp \left\{ -\frac{iV}{4} \left( h \log \frac{h}{C} + \bar{h} \log \frac{\bar{h}}{\bar{C}} \right) + i\pi V \left( k + \frac{q}{p} \frac{\theta + 2\pi n}{2\pi} \right) \frac{h - \bar{h}}{2i} \right\},$$

(10)

where the constants $C, \bar{C}$ can be set real and expressed in terms of the vacuum energy at $\theta = 0$, $C = \bar{C} = -2eE_v(0)$, see Eq.(3), and $V$ is the 4-volume. The holomorphic structure of the effective potential is explicit in Eq.(10). The integer numbers $p$ and $q$ are relatively prime. As was shown in Ref.[8], they are related to the parameter $\xi$ introduced in Eq.(3), $2\xi = q/p$. The symbol $\text{Log}$ in Eq.(10) stands for the principal branch of the logarithm. The effective potential (10) produces an infinite series of anomalous WI’s. By construction, it is a periodic function of the vacuum angle $\theta$. The effective potential (10) is suitable for a study of the YM vacuum as described above.

The double sum over the integers $n, k$ in Eq.(3) appears as a resolution of an ambiguity of the effective potential as defined from the anomalous WI’s. As was discussed in [8], this ambiguity is due to the fact that any particular branch of the multi-valued function

$$\log \frac{p}{q} \text{Log} z = \frac{p}{q} \text{Log} z + 2\pi i (n + k\frac{p}{q}), \quad n = 0, \pm 1, \ldots; \quad k = 0, 1, \ldots, q - 1$$

(11)

corresponding to some fixed values of $n, k$, satisfies the anomalous WI’s. However, without the summation over the integers $n, k$ in Eq.(10), the effective potential would be multi-valued and unbounded from below. An analogous problem arises with the original VY effective Lagrangian. It was cured by Kovner and Shifman in [9] by a similar prescription of summation over all branches of the multi-valued VY superpotential. Moreover, the whole structure of Eq.(10) is rather similar to that of the (amended) VY effective potential. Namely, it contains both the “dynamical” and “topological” parts (the first and the second terms in the exponent, respectively). The “dynamical” part of the effective potential (10) is similar to the VY effective potential $\sim S \log (S/\Lambda)^N$ (here $S$ is an anomaly superfield), while the “topological” part is akin to the improvement [9] of the VY effective potential. Similarly to the supersymmetric case, the infinite sum over $n$ reflects the summation over all integer topological charges in the original YM theory. The difference of our case from that of supersymmetric YM theory is that an effective potential of the form $(1/N)\phi \log (\phi/\Lambda)^N$, as in the SUSY case, implies a simpler form of the “topological” term $\sim 2\pi in/N (\phi - \bar{\phi})$ with only one “topological number” $n$ which specifies the particular branch of the multi-valued logarithm. In our case, we allow for a more general situation when the parameter $N$ is a rational number $N = p/q$. In this case we have two integer valued “topological numbers” $n$ and $k$, specifying the branches of the logarithm and rational function, respectively. Our choice is related to the fact that some proposals to fix the values of $p, q$ suggest that $q \neq 1$. One may expect the integers $p$ and $q$ are related to a discrete symmetry surviving the anomaly, which may not be directly visible in the original fundamental Lagrangian. Our purpose in this paper is to try to find these numbers by analysing the properties of the effective potential (10).
It should be stressed that the improved effective potential (10) contains more information in comparison to that present in the anomalous Ward identities just due to the presence of the “topological” part in Eq. (10). Without this term Eq. (10) would merely be a kinematical reformulation of the content of anomalous Ward identities for YM theory. The reason is that the latter refer, as usual, to the infinite volume (thermodynamic) limit of the theory, where only one state of a lowest energy (for $\theta$ fixed) survives. This state corresponds to one particular branch of the multivalued effective potential in Eq.(10). At the same time, the very fact of multi-valuedness of the effective potential implies that there are other vacua which should all be taken into consideration when $\theta$ is varied. When summing over the integers $n, k$, we keep track of all (including excited) vacua of the theory, and simultaneously solve the problems of multi-valuedness and unboundedness from below of the “one-branch theory”. The most attractive feature of the proposed structure of the effective potential (10) is that the same summation over $n, k$ reproduces the topological charge quantization and $2\pi$ periodicity in $\theta$ of the original YM theory.

3 Integrating in the heavy fermion

In this section the effective Lagrangian (10) for pure YM theory will be analysed from a different standpoint. Considering pure gluodynamics as a low energy limit of a theory describing the YM field interacting with a heavy fermion, we now wish to construct an effective Lagrangian for the latter theory starting from the effective Lagrangian (10). Our purpose here is to try to understand, in this framework, holomorphy of the effective potential (10) and the constraints imposed by it. As will be argued below and in Sect.4, a relation between the holomorphic and “topological” properties of two Lagrangians is non-trivial, and allows one to fix the crucial parameter $\xi = q/(2p)$ entering Eq.(10), under two plausible assumptions, see below.

The task of constructing such an effective Lagrangian for the theory with a heavy fermion is achieved by using the “integrating in” technique, developed in the content of SUSY theories in Ref.[4] and reviewed by Intriligator and Seiberg in [2]. The integrating in procedure can be viewed as a method of introducing an auxiliary field into the effective Lagrangian for pure YM theory. Using the renormalization group properties of the YM effective Lagrangian, the latter is extended to include the auxiliary field $U$, which will be later on identified with the chiral combination $\bar{\Psi}_L \Psi_R$ of a heavy fermion.

To conform with the notation and terminology of Ref.[4], we will call pure YM theory and the theory with a heavy fermion the d-theory (from “downstairs”) and the u-theory (from “upstairs”), respectively. The effective potential of the d-theory is then $W_d + W_d^+$ with

$$W_d = \frac{1}{4} \frac{q}{p} \hbar \log \left( \frac{\hbar}{c} \right)^{p/q} = \frac{1}{4} \frac{q}{p} \hbar \log \frac{\hbar^{p/q}}{(c\Lambda_{YM})^{p/q}}$$

(12)

(here $c$ is a dimensionless numerical coefficient), and the summation over all branches of the logarithm in the partition function is implied. In this section Eq.(12) will be understood as representing a branch (section) of the multi-valued effective potential, which corresponds to a lowest energy state for small $\theta \ll \pi$. As was shown in [8], this section corresponds to the principal branch of the rational function in the logarithm in Eq.(12).
In this case the vacuum expectation value $\langle H \rangle$ depends on $\theta$ as follows:

$$\langle H \rangle_\theta = \langle H \rangle_0 e^{2i\xi\theta}$$

(13)

We now want to relate [10, 4] the dimensional transmutation parameter $\Lambda_{YM}$ of pure YM theory to the scale parameter $\Lambda_{QCD}$ of the u-theory including a heavy quark of mass $m \gg \Lambda_{QCD}, \Lambda_{YM}$. We assume both parameters to be defined in the $\overline{MS}$ scheme, in which no threshold factors arise in corresponding matching conditions. The matching condition then follows from the standard one-loop relations

$$\Lambda_{YM} = M_0 \exp \left(-\frac{8\pi^2}{b_{YM} g^2(M_0)}\right), \quad b_{YM} \equiv b = \frac{11}{3} N_c,$$

$$\Lambda_{QCD} = M_0 \exp \left(-\frac{8\pi^2}{b_{QCD} g^2(M_0)}\right), \quad b_{QCD} = \frac{11}{3} N_c - \frac{2}{3},$$

(14)

and the requirement that the coupling constants of the d- and u-theories coincide at the decoupling scale $M_0 = m$. We obtain

$$\Lambda_{YM}^4 = \Lambda_{QCD}^4 \left(\frac{m^2}{\Lambda_{QCD}^2}\right)^{4/(3b)},$$

(15)

As was explained in Refs.[10, 4, 2], Eq.(15) reflects the fact that, for fixed $\Lambda_{QCD}$, the scale parameter $\Lambda_{YM}$ characterizes the low energy theory surviving below the scale $m$, and thus depends on $m$. In this sense, the constant in the logarithm in Eq.(12) also depends on $m$.

$$c \Lambda_{YM}^{p/q} = (c \Lambda_{QCD}^{p/q}) \left(\frac{m}{\Lambda_{QCD}}\right)^{8p/(3bq)}.$$

(16)

Following Ref.[4], we now wish to consider (a particular branch of) the effective potential (12) as the result of integrating out the auxiliary field $U$ in the new effective potential $W \equiv W_u - mU$ which corresponds to the u-theory: $W_d(h, m) = W(h, m, \langle U \rangle)$, or

$$W_d = [W_u - mU]_{\langle U \rangle},$$

(17)

where $\langle U \rangle$ is a solution of the classical equation of motion for the auxiliary field $U$:

$$\frac{\partial W_u}{\partial U} - m = 0.$$

(A) The stage, Eq.(17) is merely a definition; later on, the term $mU$ will be identified with the fermion mass term in the effective Lagrangian.) Let us note that, according to Eq.(17), $W_d$ should depend holomorphically on $\langle U \rangle$. Our assumption is that this is only possible if an effective potential $W$ of the u-theory is itself holomorphic in the field $U$. In our opinion, this assumption appears to be quite reasonable[4]. Furthermore, one can see that Eqs. (17), (18) actually define the potential $W_d$ as the Legendre transform of $W_u$.

We are not aware of any counter-example where a holomorphic d-potential would be obtained from a non-holomorphic u-potential by the integrating out procedure.
Therefore we can find the unknown function $W_u$ from the known potential $W_d$ by the inverse Legendre transform:

$$W_u = [W_d + mU]_{(m)} \, ,$$

where $\langle m \rangle$ solves the equation

$$\frac{\partial}{\partial m} (W_d + mU) = 0 \, .$$

Eq. (20) can be considered as an equation of motion for the auxiliary “field” $m$. It is important to note that Eqs. (17 - 20) imply that $m$ should be treated as a complex parameter to preserve the holomorphic structure of Eq. (13). When substituted in Eq. (19), a solution $\langle m \rangle$ of Eq. (20) defines the potential $W_u(h, U, \langle m \rangle)$. When this function is found, the effective potential $W$ of the u-theory is defined by the relation

$$W(h, U, m) = W_u(h, U, \langle m \rangle) - mU \, ,$$

in accord with Eq. (17).

The solution of Eq. (20) is easy to find using Eqs. (12), (16):

$$\langle m \rangle = \frac{2}{3b} \frac{h}{U} \, .$$

Thus, Eq. (19) yields

$$W_u = -\frac{1}{4} \frac{q}{p} h \log \left( \frac{\Lambda_{QCD}^4}{h} \right)^{p/q} \left( \frac{2}{3b} \frac{h}{\Lambda_{QCD} U} \right)^{8p/(3bp)} + \frac{2}{3b} h \, .$$

Finally, Eq. (21) results in the effective potential of the u-theory

$$W = \frac{1}{4} \frac{q}{p} h \log \left( \frac{h}{\Lambda_{QCD}^4} \right)^{(1-8/3b)p/q} \left( \frac{3b}{2} \frac{U}{\Lambda_{QCD}^4} \right)^{8p/(3bp)} + \frac{2}{3b} h - mU \, .$$

We expect this effective Lagrangian to describe YM theory coupled to a heavy quark, corresponding to the field $U$, such that integrating out $U$ brings us back to the effective Lagrangian (12) for pure gluodynamics. Indeed the equation of motion for the field $U$ stemming from the effective potential (24) reads

$$m \langle U \rangle = \frac{2}{3b} h \, .$$

Inserting this classical vacuum expectation value (VEV) back to Eq. (24) (i.e. integrating out the field $U$), we reproduce the effective potential of the d-theory, Eq. (12). Note that as Eq. (25) should preserve the $N_c$ counting rule, we obtain $\langle h \rangle \sim N_c^2$, $b \sim N_c$, $\langle U \rangle \sim N_c$. The $N_c$ dependence of the VEV $\langle U \rangle$ is consistent with the identification $\langle U \rangle \sim \langle \Psi_L \Psi_R \rangle$ which will be suggested below.

Analogously to the effective potential (12) of the d-theory, the new potential (24) is not a single-valued function. The single-valuedness should be imposed, as was done in Eq. (10), by the summation over all branches of $W$ in the partition function. This procedure will be
considered in detail in the next section, while here we would like to identify the field $U$ of the effective theory with a corresponding operator of the fundamental theory. As is seen from (25), $U$ has dimension 3, and thus should describe the VEV of an operator bilinear in the heavy quark fields. Furthermore, as long as $m$ is effectively considered as a complex parameter, this operator can only be $\bar{\Psi}_L \Psi_R$ or $\bar{\Psi}_R \Psi_L$, in accord with the structure of the mass term in the underlying fundamental theory. (Here the second assumption of the present approach is implicit: we assume the standard form of the fermion mass term in the effective potential (24).) To find the exact correspondence, we note that, when the VEV of the field $h$ is chosen to correspond to a lowest energy state for $\theta < \pi$, Eq.(25) implies

$$\langle mU + \bar{m}\bar{U} \rangle_\theta = -\left(\frac{\alpha_s}{12\pi}G^2\right)_0 \cos (2\xi\theta) ,$$

$$\langle mU - \bar{m}\bar{U} \rangle_\theta = -i\left(\frac{\alpha_s}{12\pi}G^2\right)_0 \sin (2\xi\theta) ,$$  \hspace{1cm} (26)

which can be compared with the relations\footnote{The equations (27) follow from the operator product expansions $\langle m\bar{\Psi}\Psi \rangle = -\langle \alpha_s/(12\pi)G^2 \rangle + O(1/m^2)$, $\langle m\bar{\Psi}\gamma_5\Psi \rangle = \langle \alpha_s/(8\pi)GG \rangle + O(1/m^2)$, and Eq.(13).} between the VEV's in the underlying theory:

$$\langle m\bar{\Psi}_L \Psi_R + m\bar{\Psi}_R \Psi_L \rangle_\theta = -\left(\frac{\alpha_s}{12\pi}G^2\right)_0 \cos (2\xi\theta) + O \left(\frac{1}{m^2}\right) ,$$

$$\langle m\bar{\Psi}_L \Psi_R - m\bar{\Psi}_R \Psi_L \rangle_\theta = -\frac{3b}{4} \xi \left(\frac{\alpha_s}{12\pi}G^2\right)_0 \sin (2\xi\theta) + O \left(\frac{1}{m^2}\right) .$$  \hspace{1cm} (27)

Comparing Eqs. (26) and (27) and using the relation $2\xi = q/p$ \footnote{The equations (27) follow from the operator product expansions $\langle m\bar{\Psi}\Psi \rangle = -\langle \alpha_s/(12\pi)G^2 \rangle + O(1/m^2)$, $\langle m\bar{\Psi}\gamma_5\Psi \rangle = \langle \alpha_s/(8\pi)GG \rangle + O(1/m^2)$, and Eq.(13).}, we conclude that

$$\xi = \frac{4}{3b} , \quad \frac{q}{p} = \frac{8}{3b} ; \quad \langle U \rangle = \langle \bar{\Psi}_L \Psi_R \rangle .$$  \hspace{1cm} (28)

We thus see that the introduction of the heavy quark into the effective theory fixes the parameter $\xi$ which enters the effective Lagrangian (12) for pure YM theory. The value obtained coincides with the one suggested by us previously within a different method \footnote{The equations (27) follow from the operator product expansions $\langle m\bar{\Psi}\Psi \rangle = -\langle \alpha_s/(12\pi)G^2 \rangle + O(1/m^2)$, $\langle m\bar{\Psi}\gamma_5\Psi \rangle = \langle \alpha_s/(8\pi)GG \rangle + O(1/m^2)$, and Eq.(13).}, based on a different assumption. (The correspondence between the two approaches to calculate the crucial parameter $\xi$ will be further discussed in the next section.) Moreover, the comparison of Eqs. (26) and (27) shows that the chiral field $U$ corresponds to the chiral fermion bilinear of the fundamental Lagrangian:

$$mU \leftrightarrow m\bar{\Psi}_L \Psi_R , \quad \bar{m}\bar{U} \leftrightarrow m\bar{\Psi}_R \Psi_L .$$  \hspace{1cm} (29)

This correspondence between the operators of the effective and underlying theories has the same meaning as Eqs.(13), i.e. the classical field $U$ describes the VEV of the chiral combination $\bar{\Psi}_L \Psi_R$ of the full theory.

It may be instructive to discuss the result (28) in a slightly different and more intuitive way. Using Eqs.(27), we can write

$$\langle mU \rangle \sim \langle m\bar{\Psi}_L \Psi_R \rangle = -\frac{1}{2} \left(1 + \frac{3b}{4} \xi\right) \left(\frac{\alpha_s}{12\pi}G^2\right)e^{2i\xi\theta} - \frac{1}{2} \left(1 - \frac{3b}{4} \xi\right) \left(\frac{\alpha_s}{12\pi}G^2\right)e^{-2i\xi\theta} ,$$

\hspace{1cm} (30)
which is a superposition of the holomorphic ($\sim \exp(2i\xi \theta)$) and anti-holomorphic ($\sim \exp(-2i\xi \theta)$) functions. At the same time, the equation of motion (25) requires $\langle U \rangle$ to be a holomorphic function, as $\langle h \rangle$ is such a function. This is only possible, as is seen from Eq. (30), when $1 - (3b/4)\xi = 0$ which is equivalent to (28). In other words, the requirement of holomorphy for the u-theory singles out parameters of the d-theory.

We would like to pause here to discuss the following issue. The main result of our calculations, eq. (28) is the direct consequence of two fundamental principles. The first principle is a modified definition of the path integral when the prescription of summation over all topological sectors is introduced both at the fundamental and effective Lagrangian level, see Eq.(10). As was discussed earlier, this definition does not change any local properties of the theory (in particular, it does not alter the WI’s) but drastically changes the global properties of the theory. The prescription of summation over the topological classes will be further discussed in Sects. 4 and 5. The second fundamental principle we adopted from the SUSY theories is the property of holomorphy in the form of a natural requirement to have a holomorphic u-theory if a d-theory satisfies this property (and vice versa). However, in the SUSY case these two principles are argued to lead to a suspicious new chiral invariant vacuum, whose analog certainly can not exist in QCD as we know from experiment. The question is: How is it possible that a similar derivation and prescription adopted for QCD does not lead to the new chiral invariant vacuum advocated in the SUSY case?\footnote{We thank the Referee for bringing our attention to this potential problem.}

Our view of this problem is that we believe that in both cases (supersymmetric and non-supersymmetric) this suspicious state does not appear as a well-defined vacuum state; however in the SUSY case the situation might be less clear than in the ordinary QCD, see below. The argument is the following. Formally, such a state seemingly does appear as a solution of the equation of motion stemming from the effective Lagrangian. However, it does not appear to be a state stable against quantum fluctuations. Indeed, the modulus of the “order parameter” $\phi = \langle \lambda \lambda \rangle$ vanishes in this candidate “vacuum”. However, the phase of $\phi$ is ill-defined at this point, and the matrix of second derivatives describing quantum fluctuations is also ill-defined. An additional source of ambiguity of the matrix of second derivatives is a freedom of redefinitions $|\phi| \to |\phi|^n$ or even $|\phi| \to \log(|\phi|)$ in the effective Lagrangian, which are able to change the sign of $W''_{|\phi|}$ for the candidate “vacuum” $|\phi| = 0$. This freedom of redefinitions of $|\phi|$ is due to our lack of knowledge of the kinetic term in the effective SUSY Lagrangian, which otherwise would fix the correct definition of canonical field. Such a behavior of the effective potential is an indication that this state is not a genuine vacuum state stable against quantum fluctuations. In fact, the situation for the SUSY theories is a little bit more controversial because one could argue that in the SUSY case the vacuum energy is zero and thus all quantum fluctuations should cancel out, no matter what the eigenvalues for specific fluctuations are. Nevertheless, we believe that an accurate analysis of quantum fluctuations in the SUSY case in the background of the chiral invariant vacuum state within some suitable regularization would demonstrate that this state should be dismissed as an inappropriate candidate for the vacuum state.

To end up this section, we would like to discuss an apparent problem related to the effective potential (24). Proceeding by analogy with SUSY theories, we might naively
expect that the effective potential, obtained for large $m \to \infty$, could be continued to small masses $m \leq \Lambda$, where the heavy “glueball” fields $h, \bar{h}$ could be integrated out. (This is e.g. how the Affleck-Dine-Seiberg superpotential \[5\] was obtained in Ref.[4] from the VY effective potential for SUSY gluodynamics.) However, such a procedure gives correct results for supersymmetric theories just due to specific Ward identities which allow one to prove that the dependence of the gluino condensate on the chiral superfield mass $m$, viz. $\langle \lambda \lambda \rangle \sim \sqrt{m}$ for SU(2) gauge group, obtained for SQCD for small $m \ll \Lambda$, is actually exact and valid also for large $m \to \infty$ \[10\]. As no such relation holds in the non-supersymmetric case, we actually have no reason to continue the above formulas to the region of small $m \simeq \Lambda$. If we still do so, one can easily see that we do not reproduce in this way the anomalous term in the effective chiral Lagrangian of Ref.[14]. This seeming problem with Eq.(24) is resolved by the fact that the “QCD limit” $m \to 0$ and the “YM limit” $m \to \infty$ are actually separated by a kind of phase transition which changes the number of vacua in the theory. As the number of vacua is determined by the $\beta$-function (see below in Sect. 4), it should not change when a very heavy fermion is added to pure YM theory, and the effective theory below the decoupling scale $M_0 \simeq m$ is considered. (This requirement will be given a formal content in the next section.) On the contrary, when the light quarks are present, the $\beta$-function changes, and so does the number of vacua. Thus, analytic continuation of the above formulas to the small $m \ll \Lambda$ region leads to a theory other than the one described by the effective chiral Lagrangian of Ref.[14]. As is shown in \[13\], the latter is correctly reproduced by a different procedure. Namely, one should start directly from an effective Lagrangian for QCD with light quarks, which realizes at the tree level the anomalous conformal and chiral symmetries of QCD. The “glueball” part of this effective Lagrangian for QCD \[15\] is similar to Eq.(10).

4 Global quantization, holomorphy, and Ward identities in the effective Lagrangian approach

In this section we would like to discuss a few related topics. First, we analyse the construction of the effective potential \[24\] of the u-theory while keeping track of multi-valuedness of effective potentials for both d- and u-theories. As we will see shortly, this analysis is consistent with the result \[28\]. Another purpose of this section is to discuss holomorphy of the effective theory in the fermion mass $m$ from the point of view of the underlying QCD Lagrangian. This will allow us to establish the correspondence with an alternate method which was suggested by us earlier in Ref.[11] to find the parameter $\xi$, where the same value $\xi = 4/(3b)$ was obtained. It will be shown how the results of Ref.[11] follow from the effective Lagrangian approach.

The analysis of the previous section was based on dealing with a fixed branch of the effective potential. As was discussed for SUSY theories in \[3\] and for pure YM theory in \[8\], the multi-valuedness of the effective potential necessitates the summation over all branches of a multi-valued action in the path integral (this is the prescription for constructing an improved effective potential). This procedure enforces some global quantization rules in the effective theory \[3\] \[8\] which reflect quantization of the topological charge in the fundamental theory. Technically, the global quantization in the d-theory
arises due to the Poisson formula

$$\sum_n \exp \left( 2\pi i n V \frac{q}{p} \frac{\bar{h} - h}{4i} \right) = \sum_m \delta \left( V \frac{q}{p} \frac{h - \bar{h}}{4i} - m \right).$$

Let us now consider the problem of matching the d- and u- theories from the point of view of the integrating out procedure in the u-theory with account for the fact of multi-valuedness of the resulting potential. Given any branch of a multi-valued action, which is compatible with the Ward identities of the d-theory and the renormalization group, one achieves the single-valuedness of the partition function for the u-theory by the summation over all branches of the action in the path integral. In other words, we impose a universal rule for defining the partition function (improved effective potential) for both the d- and u-theories. Furthermore, it is natural to require that, after integrating out the field $U$ in such a partition function for the u-theory, we should come back to the correct partition function of the d-theory. This requirement has a non-trivial content, since generically the logarithms in Eqs. (12) and (24) imply different global structures in the complex $h$ plane, i.e. different global quantization rules for the d- and u-theories. In this way we extend the integrating in/out procedure to match not only the perturbative scale redefinition in the low energy d-theory, but also the global, topological, properties of the latter. The meaning of this requirement is that **turning on the heavy quark should not change the number of $\theta$ vacuum sectors**. As we will see shortly, this “topological” matching of the two theories agrees with the result (28) for the parameter $\xi$.

Let us define the improved effective potential $\tilde{F}$ for the u-theory in the same way as was done in Eq.(10). Using Eq.(24), we obtain

$$e^{-iV\tilde{F}} = \sum_{n,m=-\infty}^{+\infty} \sum_{k,l} \exp \left\{ -iV \left[ \frac{1}{4} (1 - \frac{8}{3b}) h \log \frac{h}{cA^4_{QCD}} + \frac{2}{3b} h \log \frac{3b}{2 cA^3_{QCD}} + U \right] \right\}$$

where the sums over $k, l$ are finite. It is clear from Eq.(32) that the minimization condition $\tilde{F}_U = 0$ leads to the same Eq.(25). The reason for that is the absence of global quantization rules for the field $U$ which is thus unconstrained. It can be readily seen that substituting this solution back to Eq.(32), we reproduce exactly the “dynamical” term in Eq.(10). However, as we stated above, the “topological” term for the u-theory should also match the one of the d-theory. Let us now compare these two expressions. For the u-theory, the “topological” term is

$$i V \frac{q}{p} \left[ k' \frac{p}{q} (1 - \frac{8}{3b}) + \frac{8}{q} \frac{8}{3b} + n + m + \frac{\theta}{2\pi} \right] \frac{h - \bar{h}}{2i},$$

where the integers $n, m$ are ranged from $(-\infty)$ to $(+\infty)$, while the integers $k, l$ reside on finite intervals. On the other hand, the “topological” term of the d-theory is

$$i V \frac{q}{p} \left( k' \frac{p}{q} + n' + \frac{\theta}{2\pi} \right) \frac{h - \bar{h}}{2i}, \quad n' = 0, \pm 1, \ldots, \quad k' = 0, 1, \ldots, q - 1.$$
One can see that expression (33) has the same form as (34) only if

\[ \frac{p}{q} \frac{8}{3b} = r = \text{integer} , \]  

(35)

for which expression (33) becomes

\[ iV \pi \frac{8}{3br} \left( k \frac{3b}{8} + n + m + r(l - k) + \frac{\theta}{2\pi} \right) \frac{h - \bar{h}}{2i} . \]  

(36)

Therefore, unless the constraint (35) is satisfied, integrating out the field \( U \) in the \( u \)-theory does not lead back to the correct \( d \)-theory, but instead yields a theory with a different global quantization rule for the field \( h \) (i.e. different quantum mechanically).

It is remarkable that these “topological” arguments are consistent with the value of \( \xi \) which was arrived upon in Eq. (28) (still, they are less restrictive, as it usually happens for considerations based on topological arguments, than Eq. (28) above). On the other hand, in contrast to the argument of the previous section, no reference to the underlying theory was made above; only the internal consistency of the method just developed was used. The agreement between these two different lines of reasoning thus supports both the reliability of the calculated value of \( \xi \), and self-consistency of the effective Lagrangian constructed.

The next topic we would like to discuss is holomorphy of the effective Lagrangians. We have seen in the previous section that the holomorphic structure of the effective potential for the \( d \)-theory implies holomorphy in the fermion mass \( m \) of the effective potential for the \( u \)-theory. This means that \( m \) should be effectively treated as a complex parameter, with the correspondence rule (29) between the operators of the effective and fundamental theories. Here we would like to discuss what such kind of holomorphy implies in terms of the fundamental QCD Lagrangian.

Let us first note that in our case holomorphy refers to a physical heavy fermion with mass \( m \to \infty \). In the fundamental theory, the introduction of a heavy fermion of mass \( m \) requires a regularization on yet much higher ultra-violet scale \( M_R \), which can be thought of as the mass of a Pauli-Villars regulator. In the infinite mass limit \( m, M_R \to \infty \) the properties of the theory in respect to the physical and regulator fermions are identical (up to some sign differences) by definition. Thus, we may expect the fundamental theory to be holomorphic, in a sense, in the Pauli-Villars regulator mass \( M_R \). The analysis of this paper shows what kind of holomorphy we may expect: it should be holomorphy of nonperturbative vacuum condensates or zero momentum correlation functions with perturbative tails subtracted, because these are the objects generated by the effective Lagrangian. Precisely this kind of holomorphy for QCD with light quarks in the Pauli-Villars fermion mass \( M_R \) was suggested some time ago by Kühn and Zakharov in a somewhat different context. Working in the chiral limit, these authors have related, using analyticity in \( M_R \), the proton matrix element of the topological density \( \langle p | G \tilde{G} | p \rangle \) to the matrix element \( \langle p | G^2 | p \rangle \) which is fixed by the conformal anomaly. Recently, we

\[ ^{9} \text{As was mentioned in the Introduction, we do not expect that kinetic terms in the effective Lagrangian possess analogous holomorphic properties. Similarly, holomorphy is apparently of no use for correlation functions with non-zero momentum in the full theory.} \]
have applied a similar idea to relate the zero momentum two-point function of $G\tilde{G}$ to that of $G^2$. The statement of holomorphy in the fermion mass $m \to \infty$ was rephrased there as a method to evaluate zero momentum correlation functions of the chiral fermion bilinears $\bar{\Psi}_L\Psi_R$ and $\bar{\Psi}_R\Psi_L$ with the operators $G^2, G\tilde{G}$ from the known Ward identities involving the gluon operators only. As will be clear below, the value of $\xi$, the parameter of interest, follows from these relations. It is our purpose here to show that these results of Ref. follow from the effective Lagrangian constructed in Sect.3. Therefore, it comes as no surprise that in the effective Lagrangian approach we end up with precisely the same value of $\xi$ as that obtained in . (Again, we recall that the same regularization scheme is implied in the approach of and the present paper.)

Performing the inverse Legendre transform of the effective potential (24) with respect to $H$ and $U$, we obtain the generating functional $\tilde{W}_m(J, j)$ of connected Green functions:

$$\tilde{W}_m(J, j) = -\frac{1}{4}c\Lambda_{QCD}^4 \left(\frac{m + j}{\Lambda_{QCD}}\right)^{8/(3b)} e^{-4J}.$$  \hspace{1cm} (37)

It is clear from the above analysis that the sources $j, J$ are holomorphic: the differentiation in respect to $j$ produces insertions of the chiral operator $-\bar{\Psi}_L\Psi_R$, while the derivative $\partial/\partial J$ produces insertions of the operator $H$. As is seen from Eq.(37), the differentiation with respect to $j$ can be substituted by the differentiation with respect to $m$. In this way we find the correlation function

$$i \int dx\langle 0\mid T\{H(x) m\bar{\Psi}_L\Psi_R(0)\}\mid 0\rangle = -m \frac{\partial}{\partial m} \frac{\partial}{\partial J} \tilde{W}_m(J, 0) = -\frac{8}{3b} \langle H \rangle. \hspace{1cm} (38)$$

Analogously we can find further correlation functions:

$$i \int dx\langle 0\mid T\{\tilde{H}(x) m\bar{\Psi}_R\Psi_L(0)\}\mid 0\rangle = 0 \hspace{1cm} (39)$$

$$i \int dx\langle 0\mid T\{\tilde{H}(x) m\bar{\Psi}_L\Psi_R(0)\}\mid 0\rangle = 0 \hspace{1cm} (40)$$

$$i \int dx\langle 0\mid T\{H(x) m\bar{\Psi}_R\Psi_L(0)\}\mid 0\rangle = 0 \hspace{1cm} (41)$$

The relations proposed in Ref. now follow if we take linear combinations of Eqs. (38-41). Taking the sum of Eqs. (38) and (40) and using

$$H + \tilde{H} = -b\alpha_s/(8\pi)G^2, \hspace{1cm} \langle H \rangle = (3b/2)\langle m\bar{\Psi}_L\Psi_R \rangle,$$

we obtain

$$i \int dx\langle 0\mid T\{\frac{\alpha_s}{12\pi}G^2(x) m\bar{\Psi}_L\Psi_R(0)\}\mid 0\rangle = \frac{8}{3b} \langle m\bar{\Psi}_L\Psi_R \rangle, \hspace{1cm} (42)$$

and analogously from Eqs. (39),(41)

$$i \int dx\langle 0\mid T\{\frac{\alpha_s}{12\pi}G^2(x) m\bar{\Psi}_R\Psi_L(0)\}\mid 0\rangle = \frac{8}{3b} \langle m\bar{\Psi}_R\Psi_L \rangle. \hspace{1cm} (43)$$

On the other hand, taking the differences for the same pairs of Eqs. (38-41), we obtain

$$i \int dx\langle 0\mid T\{\frac{\alpha_s}{8\pi}G\tilde{G}(x) m\bar{\Psi}_L\Psi_R(0)\}\mid 0\rangle = i\frac{8}{3b} \langle m\bar{\Psi}_L\Psi_R \rangle, \hspace{1cm} (44)$$
\[ i \int dx \langle 0 \mid T \left\{ \frac{\alpha_s}{8\pi} G \tilde{G}(x) m \bar{\Psi}_R \Psi_L(0) \right\} \mid 0 \rangle = -i \frac{8}{3b} \langle m \bar{\Psi}_R \Psi_L \rangle. \quad (45) \]

Eqs. (42-45) are precisely the relations suggested earlier in Ref. [11] with a different motivation to find the topological susceptibility in terms of the gluon condensate. Taking the difference of Eqs. (44) and (45) and using Eqs. (27), we obtain

\[ i \int dx \langle 0 \mid T \left\{ \frac{\alpha_s}{8\pi} G \tilde{G}(x) \frac{\alpha_s}{8\pi} G \tilde{G}(0) \right\} \mid 0 \rangle = \left( \frac{4}{3b} \right)^2 \langle -b \alpha_s \rangle, \quad (46) \]

which is exactly the result obtained in [11]. (As in [11], this scheme dependent result implies a particular regularization prescription where correlation functions are defined via the path integral with perturbative contributions subtracted, see the discussion above.) We have thus closed the circle: starting with Eq. (3) with an unspecified parameter \( \xi \), we have found the value of \( \xi \) using the effective Lagrangian, obtained from the anomalous Ward identities (1-3), along with the integrating in/out technique, and eventually fixed the initial Eq. (3). The introduction of a heavy quark was crucial to find the parameter \( \xi \). One should note that the main assumptions of the present paper and that of Ref. [11] are quite different, and yet lead to the same result (46). We consider this as an evidence in favor of correctness and self-consistency of our approach.

## 5 Discussion and comparison with related works

The purpose of this section is a qualitative discussion of the results obtained in the present paper. We will first address the counting of vacua following with our methods and compare its \( N_c \) dependence with the results obtained for softly broken SUSY models [17] and the behavior found for the lattice \( Z_p \) gauge models [18]. Another purpose of this section is a qualitative comparison of our picture of the vacuum structure and \( \theta \) dependence in YM theory with a recent work by Witten [12] who studies the same issues in the limit \( N_c \to \infty \) on the basis of the AdS/CFT correspondence [19].

As was discussed in [8], the value of \( \xi \) determines the number of different nondegenerate \( \theta \) vacuum sectors in pure YM theory. When the rational number \( p/q = 1/(2\xi) \) is fixed with \( p \) and \( q \) being relatively prime, the number of different non-degenerate \( \theta \) vacua is \( p \). More precisely:

\[
\text{number of different } \theta \text{ sectors} = p = 11 \cdot N_c \text{ for } N_c \text{ odd}
\]

\[
p = \text{min.integer}\left[ \frac{11N_c}{2}; \frac{11N_c}{4}; \frac{11N_c}{8} \right] \text{ for } N_c = \text{ even},
\quad (47)
\]

where we have used \( \xi = 4/(3b) \). In particular, it follows from Eq. (47) that the number of vacua is 11 for \( N_c = 2 \), and 33 for \( N_c = 3 \). Although these numbers may look strange, they seem to be the only ones compatible with the effective Lagrangian and integrating in procedure considered in this paper. (They would be a wrong answer if one of the assumptions of the present approach were incorrect.) As was pointed out in [8], this

\[10\]Here we would like to recall that, for any generic value of \( \theta \), there is only one true physical vacuum which is a lowest energy state among all \( \theta \) vacua. For a fixed value of \( \theta \), additional vacua show up in the thermodynamics limit only indirectly via the value of parameter \( \xi \) in the correlation function (3).
counting of vacua disagrees with what could be expected starting from SUSY YM theory broken softly by a small gluino mass $m_g \ll \Lambda$ [17]. The latter theory predicts $N_c$ vacua for small $m_g$. Here is how it comes about (see a discussion by Shifman in [2] for more detail). In the limit of small $m_g$ the VEV of the holomorphic combination $G^2 + ig\bar{G}G$ is proportional to the VEV $m_g \langle \lambda \lambda \rangle$ where the gluino condensate $\langle \lambda \lambda \rangle$ is to be calculated in the supersymmetric limit $m_g = 0$. The $\theta$ dependence of the latter is known [10]:

$$\langle \lambda \lambda \rangle \sim \exp(i\theta/N_c + 2\pi k/N_c), \quad k = 0, 1, \ldots, N_c - 1,$$

which corresponds to $N_c$ degenerate vacua. When $m_g \neq 0$, the vacuum degeneracy is lifted. For $N_c = 3$ and $\theta = 0$, we have one state with negative energy $E = -m_g \Lambda_{SYM}^3$, and two degenerate states with positive energy $E = (1/2)m_g \Lambda_{SYM}^3$. The former is the true vacuum state of softly broken SUSY gluodynamics, while the latter are metastable states with broken CP. The lifetime of the metastable states is very large for small $m_g$, and decreases as $m_g$ approaches $\Lambda_{SYM}$. When $\theta$ is varied, the three states intertwine, thus restoring the physical $2\pi$ periodicity in $\theta$. This picture implies the values $p = N_c$, $q = 1$, different from those suggested by the present approach.

The problem with the above SUSY-motivated scenario is that the genuine case of pure YM theory corresponds to the limit $m_g \gg \Lambda_{SYM}$ which is not controlled in this approach. Thus, although naively one could expect that increasing of $m_g$ to higher values $m_g \geq \Lambda$ does not change the number of vacua of the theory, this expectation is unwarranted. It is conceivable that an additional level splitting occurs with passing the region $m_g \sim \Lambda_{SYM}$ where the SUSY methods become inapplicable. Our results imply that this is indeed what happens, i.e. that there exists a sort of phase transition that separates the softly broken SUSY YM theory with $m_g \ll \Lambda$ from pure non-supersymmetric gluodynamics. This expectation is consistent with numerous evidences within the soft SUSY breaking theories indicating that the naive decoupling limit $m_g \gg \Lambda_{SYM}$ produces results incompatible with the known infrared features of QCD. In particular, they include the wrong $N_c$ dependence of the effective chiral dynamics [20] and the run-away behavior for $N_f < N_c$ in the presence of supersymmetry breaking [21]. On the other hand, it is curious to mention that the same value $p = 11N_c$ follows within a non-standard non-soft SUSY breaking suggested recently [22] as a toy model to match the conformal anomaly of non-supersymmetric YM theory at the effective Lagrangian level.

Our next remark concerns with another feature of the $N_c$ dependence in Eq.(47). Naively, one could expect (and supersymmetric models support this expectation) that small variations of $N_c$ lead to small variations in the number of vacuum states. Our results suggest quite a different picture for certain values of $N_c$: when the number of colors changes from $N_c = 8k - 1$ to $N_c = 8k$, the number of vacuum states abruptly changes from $11(8k - 1)$ to $11k$. At the same time, for generic values of $N_c$, the variation of a number of vacua is quite smooth. Have we ever met with such kind of behavior in physics? The answer is yes. When a theory possesses two (or more) relevant integer parameters, the vacuum structure of the theory may undergo very dramatic changes with variations of these parameters. As an example one could consider $Z_p$ lattice gauge models in 4 dimensions in the presence of a $\theta$ term which takes the values $\theta/(2\pi) = l/q$ where $l$ and $q$ are integers [13]. In this case the physics is very sensitive to the numerical value of

11 We thank the Referee for asking this question.
q, namely whether it is proportional to l or not. The physics changes drastically exactly at the points when \( q = l \cdot n \) where \( n \) is another integer, much like in our case described above. (The concrete examples when \( \theta/(2\pi) = 1/N \) and \( \theta/(2\pi) = 2/N \) with \( N \) odd, were considered in detail in Ref. [18].) The physical explanation for such a behavior in the \( Z_p \) model is based on the existence of a dual description where the relevant degrees of freedom are quite different from the original ones. In particular, exactly at these points one can construct a unique composite field which may condense according to different patterns depending on the value of \( \theta/(2\pi) \).

In a sense, the supersymmetric models are similar to the \( Z_p \) lattice gauge models with \( \theta/(2\pi) = 1/N \) where a very smooth behavior is expected. We believe that the non-supersymmetric models are closer to the case when \( \theta/(2\pi) = \frac{l}{q} \) in the \( Z_p \) lattice gauge model with \( l, q \) being relatively prime. We do not know whether the explanation using the dual description for the \( Z_p \) models can be extended to the case of gluodynamics, but an analogy with the examples discussed in Ref. [18] suggest that this might be the case.

Finally, we would like to comment on another related development. Very recently, Witten [12] has shown how the qualitative features of the \( \theta \) dependence in non-supersymmetric YM theory - such as a multiplicity of vacua \( \sim N_c \), existence of domain walls and exact vacuum doubling at some special values of \( \theta \) - can be understood using the AdS/CFT duality. The latter [19] provides a continuum version of the strong coupling limit, with a fixed ultraviolet cutoff, for YM theory with \( N_c \to \infty, g_{YM}^2 N_c \to \infty \). As was shown in [12], in this regime the \( \theta \) dependence of the vacuum energy in YM theory takes the form

\[
E_{\text{vac}}(\theta) = C \min_k (\theta + 2\pi k)^2 + O(1/N_c),
\]

where \( C \) is some constant. We would like to make two comments on a comparison of our results with the picture advocated by Witten in the large \( N_c \) limit. First, we note that the structure of Eq.(48) agrees with our modified definition of the path integral including summation over all branches of a multi-valued (effective) action. Indeed, Eq.(48) suggests the correspondence

\[
C \min_k (\theta + 2\pi k)^2 \leftrightarrow \lim_{V \to \infty} \left( -\frac{1}{V} \right) \log \left[ \sum_k e^{-V C(\theta + 2\pi k)^2} \right]
\]

using the definition of the vacuum energy through the thermodynamic limit of the path integral. With this definition which prescribes the way the volume \( V \) appears in the formula for the vacuum energy, the correspondence (49) appears to be the only possible one. On the other hand, the latter expression is exactly what arises (in the large \( N_c \) limit) with our definition of the improved effective potential (10). Therefore, our prescription of summation over all branches of a multi-valued (effective) action seems to be consistent with the picture developed by Witten using an approach based on the AdS/CFT correspondence. In particular, our picture of bubbles of metastable vacua bounded by domain walls considered in the context of QCD within an effective Lagrangian approach in [27] is in qualitative agreement with that suggested by Witten [14] for the pure YM case.

Second, one may wonder whether the approach of Ref. [12] can provide an alternative way to fix the parameters \( p, q \) of interest. We note that Eq.(48) indicates a non-analyticity at the values \( \theta_c = \pi \ (mod \ 2\pi) \) only, where \( CP \) is broken spontaneously. If the technique

\[17\]
based on the AdS/CFT duality could be smoothly continued to the weak coupling regime of non-supersymmetric YM theory, this would result in the values $q = 1, p \sim N_c$. However, the possibility of such extrapolation is unclear, as for small $\lambda = g_{YM}^2 N_c$ the background geometry develops a singular behavior and the supergravity approach breaks down. There might well be a phase transition \cite{23} when the effective YM coupling $g_{YM}^2 N_c$ is reduced. That such a phase transition should occur in the supergravity approach to QCD$_3$ was argued in \cite{24}. Other reservations about the use of the supergravity approach to the non-supersymmetric YM theory in D=4 have been expressed in \cite{25} where no perturbative indication was found for decoupling of unwanted massive Kaluza-Klein states of string theory. On the other hand, there exist some evidences from lattice simulations that a critical value of $\theta$ moves from $\theta_c = \pi$ in the strong coupling regime to $\theta_c < \pi$ in the weak coupling regime \cite{26}. In terms of parameters $p, q$, such a case corresponds to $q \neq 1$.

Therefore, we conclude that if no phase transition existed in the supergravity approach, our results would be in conflict with the latter which would imply $p = O(N_c), q = 1$. In this case, the assumptions made in the present work would have to be reconsidered. Alternatively, there might be no conflict between the two approaches if such a phase transition does occur.

6 Conclusions

In this paper we suggested using the integrating in/out procedure to study the properties of a low energy Lagrangian for gluodynamics obtained in \cite{8}. We have shown that a particular holomorphic structure of this effective Lagrangian naturally corresponds to holomorphy in the fermion mass in an extended theory with a heavy fermion, provided the standard form of the fermion mass term is used. This observation supports the proposals of Refs. \cite{16} and \cite{11} where the idea of holomorphy in the regulator or heavy fermion mass was applied, respectively, to the study of matrix elements and correlation functions of the topological density operator $G\tilde{G}$. We have argued that the integrating in/out method provides not only a perturbative matching of the dimensional scale parameters in different theories, but also has a global, “topological”, content. This holomorphic and “topological” matching of the theories with and without a heavy fermion was shown to fix, under certain assumptions, the number of different non-degenerate $\theta$ vacua for YM theory. The result obtained implies that the number of vacua for softly broken SUSY YM gluodynamics changes discontinuously when the gluino mass becomes large. A check of this conclusion by different methods and analysis of its possible consequences would be an interesting problem for a further study. In particular, we have seen that the analysis of the $\theta$ dependence is relevant for the question of presence or absence of a phase transition in the supergravity approach to the non-supersymmetric YM theory. In a more general context, it is perhaps worthwhile to point out that the modified path integral prescription of summation over all branches of a multi-valued (effective) action suggested for the case of pure YM theory in \cite{8} (and previously proposed in different settings in Refs. \cite{28}, \cite{2}) is consistent with the picture developed by Witten using an approach based on the AdS/CFT correspondence. Analogous modifications of the partition function for the case of lattice regularized sigma models and Abelian lattice models are found to be the
only self-consistent method [23] for the analysis of dualities on the lattice. This may provide further indications that such modifications of the partition function is the correct way to work with multi-valued actions.
References

[1] N. Seiberg, Phys. Lett. B318 (1993) 469; Phys. Rev. D49 (1994) 6857.

[2] K. Intriligator and N. Seiberg, hep-th/9509096.
M. Peskin, hep-th/9702094.
M. Shifman, hep-th/9704114.

[3] G. Veneziano and S. Yankielowicz, Phys. Lett. 113B (1982) 231.
T. Taylor, G. Veneziano and S. Yankielowicz, Nucl. Phys. B218 (1983) 439.

[4] K. Intriligator, R.G. Leigh, and N. Seiberg, Phys. Rev. D50 (1994) 1092.
K. Intriligator, Phys. Lett. B336 (1994) 409.

[5] I. Affleck, M.Dine, and N. Seiberg, Nucl. Phys. B241 (1984) 493; Nucl. Phys. B256 (1985) 557.

[6] L. Intriligator and N. Seiberg, Nucl. Phys. B431 (1994) 484.

[7] D. Finnell and P. Pouliot, Nucl. Phys. B453 (1995) 225.

[8] I. Halperin and A. Zhitnitsky, Phys. Rev. D58 (1998) in press; hep-ph/9711398

[9] A. Kovner and M. Shifman, Phys. Rev. D56 (1997) 2396; hep-th/9702174.

[10] M.A. Shifman and A.I. Vainshtein, Nucl. Phys. B296 (1988) 445.

[11] I. Halperin and A. Zhitnitsky, Mod. Phys. Lett. A13 (1998) 1955; hep-ph/9707286.

[12] E. Witten, hep-th/9807109.

[13] V.A. Novikov, M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, Nucl. Phys. B191 (1981) 301.

[14] E. Witten, Ann. Phys. 128 (1980) 363.
P. Di Vecchia and G. Veneziano, Nucl. Phys. B171 (1980) 253.

[15] I. Halperin and A. Zhitnitsky, hep-ph/9803301.

[16] J.H. Kühn and V.I. Zakharov, Phys. Lett. B252 (1990) 615.

[17] A. Masiero and G. Veneziano, Nucl. Phys. B249 (1985) 593.

[18] J. Cardy and E. Rabinovici, Nucl. Phys. B205 (1982) 1.
J. Cardy, Nucl. Phys. B205 (1982) 17.

[19] J. Maldacena, hep-th/9711200.
S. Gubser, I.R. Klebanov and A.M. Polyakov, hep-th/9802109.
E. Witten, hep-th/9802150, hep-th/9803131.

[20] S. Martin and J. Wells, hep-th/9801157.
[21] J.L.F. Barbon and A. Pasquinucci, hep-th/9804029.
[22] F. Sannino and J. Schechter, hep-th/9708113.
[23] D. Gross and H. Ooguri, hep-th/9805129.
[24] J. Greensite and P. Olesen, hep-th/9806235.
[25] H. Ooguri, H. Robins and J. Tannenhauser, hep-th/9806171.
[26] G. Schierholz, Nucl. Phys. Proc. Suppl. 42 (1995) 270; hep-lat/9412083.
   A.S. Hassan, M. Imachi, N. Tsuzuki and H. Yoneyama, Prog. Theor. Phys. 95 (1996) 175.
   H. Yoneyama, talk at Lattice-98.
[27] T. Fugleberg, I. Halperin and A. Zhitnitsky, to appear.
[28] A.V. Smilga, Phys. Rev. D49 (1994) 6836.
[29] S. Jaimungal, hep-th/9805211, hep-th/9808018.