Generalized Kitaev Models and Slave Genons

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We present a wide class of partially integrable lattice models with two-spin interactions, which generalize the Kitaev honeycomb model. These models have an infinite number of conserved quantities associated with each plaquette of the lattice, conserved large loop operators on the torus, and protected topological degeneracy. We introduce a ‘slave-genon’ approach, which generalizes the Majorana fermion representation of the original Kitaev honeycomb model. The Hilbert space of our spin model can be embedded into an enlarged Hilbert space of non-Abelian twist defects, referred to as genons.

In the enlarged Hilbert space, the spin model is exactly reformulated as a model of non-Abelian genons coupled to a discrete gauge field. We discuss in detail a particular $Z_3$ generalization, and show that in a certain limit the model is analytically tractable and may produce a non-Abelian topological phase with chiral parafermion edge states.

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**Introduction**—The Kitaev honeycomb model $[1]$ is an exactly solvable spin model on the two-dimensional hexagonal lattice, which can realize different exotic topologically ordered phases of matter, along with non-Abelian quasiparticle excitations. Over the past decade, this model has generated remarkable excitement$[2]$: its solvability has provided a theoretical framework to study the emergence of topological order and non-Abelian anyons from microscopic models, while its simplicity supports the hope for experimental realization, either in Mott insulators with strong spin orbit coupling, such as various Iridate compounds $[3, 4]$, or directly engineered Mott insulators with strong spin orbit coupling, such as the Kitaev model for most of the paper, and discuss more general models in the end of the draft. We introduce a constraint on their overall fusion channel. This generalizes the Majorana fermion representation of the original Kitaev honeycomb model $[1]$. While the transformed problem is itself a non-trivial interacting problem, certain results in 1+1 dimensional critical phenomena can then be utilized to solve the model in certain limits.

We will focus on a particular $Z_n$ rotor generalization of the Kitaev model for most of the paper, and discuss more general models in the end of the draft. We introduce a graphical method to perform the slave genon technique, making use of genons in bilayer FQH states $[7, 8]$, with a $1/n$ Laughlin state in each layer. In the case $n = 2$, the genons localize Majorana fermion zero modes, thus reproducing Kitaev’s construction. More generally they localize parafermion zero modes $[7, 15]$. For the case $n = 3$, we present some preliminary numerical results, and discuss the possible realization of a non-Abelian $Z_3$ parafermion phase, which contains the non-Abelian Fibonacci anyon $[6]$ in its excitation spectrum.

$Z_n$ Kitaev model—We consider the following Hamiltonian on the honeycomb lattice with $n$ states per site:

$$H = - \sum_{\langle ij \rangle} J_{s_{ij}} \langle T_{ij}^{s_{ij}} \rangle + H.c.,$$

where $s_{ij} = x, y, z$ depends on the direction of the link $ij$ (Fig. 1). $T_{ij}^x$ and $T_{ij}^y$ are $n \times n$ matrices satisfying the relations: $T_{ij}^x T_{jk}^y = T_{ik}^y T_{jk}^x \omega$, $(T_{ij}^x)^n = (T_{ij}^y)^n = 1$, where $\omega = e^{2\pi i/n}$. We further define: $T_{ij}^z \equiv (T_{ij}^x T_{ij}^y)^{1/2}$, which implies $T_{ij}^z T_{jk}^z = T_{ik}^z \omega$, $T_{ij}^z T_{jk}^z = T_{ik}^z T_{jk}^z \omega$. $T_{ij}^z$ from different

**FIG. 1:** The links of a honeycomb lattice are labelled $x$, $y$, or $z$, depending on their orientation. Sites on a plaquette are labelled 1,...,6, as shown. Red and blue circles illustrate the path $L_1$, purple and orange squares illustrate the path $L_2$, which are used to define the string operators $\Phi_1$ and $\Phi_2$ in a system with periodic boundary conditions.
sites commute with each other. The case $n = 2$ corresponds to the original Kitaev model.

The key fact about this model is that there is a conserved operator associated with each plaquette. Define:

$$W_p = \prod_{\langle ij \rangle \in \mathcal{P}} K_{ij} = (\omega T^x_1 T^y_2 T^z_3 T^x_4 T^y_5 T^z_6)^\dagger$$

(2)

where the site labels are shown in Fig. 1. Following Kitaev, we define $K_{jk} = T^a_j T^{a^*}_k$. It can be verified directly that $[W_p, H] = 0$, so that the spectrum can be decomposed into eigenstates of $W_p$. Note that $W_p^p = 1$.

In addition to the above conserved plaquette operators, the model (for $n \geq 3$) with periodic boundary conditions also admits conserved, non-commuting, loop operators:

$$\Phi_1 \equiv \prod_{2i-1,2i \in L_1} T^z_{2i-1} T^z_{2i}, \quad \Phi_2 \equiv \prod_{2i-1,2i \in L_2} T^z_{2i-1} T^y_{2i}$$

(3)

where $[\Phi_1, H] = [\Phi_2, H] = 0$, and $\Phi_3 \Phi_1 = \Phi_1 \Phi_3 \omega^2$. The loops $L_1$ and $L_2$ are shown in Fig. 1 and describe non-contractible paths around the hexagonal lattice in the two directions. Since these operators are conserved, eigenstates must form a representation of their algebra. This rigorously implies a ground state degeneracy on the torus that is a multiple of $n$ for $n$ odd, and $n/2$ for $n$ even.

Just as in the original Kitaev model, the generalized model can be defined on any planar trivalent graph. A key difference between the $n \geq 3$ and $n = 2$ cases is that for $n \geq 3$, the three operators $T_i^{x,y,z}$ on each site must be ordered with the same chirality. In other words, the direction $x \rightarrow y \rightarrow z \rightarrow x$ must be either all counter-clockwise or all clockwise on all sites. This requirement also means that the model can only be defined on planar graphs. Physically, this is because the large loops $\Phi_1, \Phi_2$ defined above can be considered as Wilson loops of a particle with statistical angle $\frac{2\pi}{n}$. For $n > 2$ this particle is an Abelian anyon, which can only be defined in two-dimensions, while for $n = 2$ it is a fermion. Multi-site terms can be added to the Hamiltonian without affecting the conservation laws, as long as they are products of bond terms $K_{ij}$ and/or $K^\dagger_{ij}$. In the supplementary materials[10], we present more details of the computation of commutation relations and conserved quantities by setting up convenient diagrammatic rules.

Anisotropic limit and the Abelian phase – Similar to the original model[1], the anisotropic limit $J_z \gg J_x, J_y$ can be easily solved. In this limit, we first diagonalize the $J_z$ terms in the Hamiltonian. To do this, let us pick a basis of $n$ states on each site, $|a\rangle$, which diagonalize $T_i^z$: $T_i^z |a\rangle = \omega^a_i |a\rangle$, for $a = 0, ..., n - 1$. Pairs of sites $i, j$ coupled by $J_y$ have their $n^2$ states split into $n$ degenerate lowest energy states, $|a\rangle |n - a\rangle_j$, for $a = 0, ..., n - 1$. These states are separated by a gap of order $J_z$ relative to the remaining $n^2 - n$ states. For large $J_z$, we can treat pairs of sites separated by vertical links effectively as a single site, thus obtaining at low energies a square lattice with $n$ states per site. Within the degenerate $n$-dimensional space on each site, we can define a new set of $Z_n$ rotor operators $L^x_i, L^y_i$, such that $L^x_i |a\rangle |n - a\rangle_j = \omega^a |a\rangle |n - a\rangle_j$, and $L^y_i |a\rangle |n - a\rangle_j = |a - 1\rangle |n - a + 1\rangle_j$.

Within this low-energy subspace, the remaining $J_x$ and $J_y$ terms can be treated within perturbation theory. The lowest order term that does not change the $J_z$ bond energy is $J_z^2 \sum_{ij} K_{ij} K_{ij}^\dagger$, and its contribution to the Hamiltonian is $H_{eff} = J_z^2 \sum_{ij} K_{ij} K_{ij}^\dagger$, which is the $Z_n$ toric code Hamiltonian [17, 19].

Slave Genons – In order to further analyze the model beyond this strongly anistropic limit, we introduce a ‘slave genon’ approach, which maps the spin model to a model of coupled non-Abelian twist defects [7, 8] [10, 13, 20, 27], referred to as genons [7, 8], in a topologically ordered state. This generalizes the Majorana fermion representation introduced in the original Kitaev honeycomb model [1], along with well-known slave fermion/boson techniques [28]. A key difference in the $n \geq 3$ $Z_n$ models is that the slave particles must be non-local topological defects instead of fermions or bosons.

Consider a Laughlin $1/n$ fractional quantum Hall (FQH) state on the surface shown in Fig. 2 (a). The surface is obtained by introducing a branch cut line in a bilayer system, such that the two layers are exchanged across the branch cut line. A genon is defined as the
the Hamiltonian can be rewritten as:

\[ H = \sum_{\langle ij \rangle} J_{s_{ij}} u_{ij} W_{ij} + H.c. \]

where \( W_{ij} \) and \( u_{ij} \) are the loop operators corresponding to the operation of moving charge \( 1/n \) Laughlin quasi-particles around the loops shown in Fig. 3. Note that \( u_{ij} \) only appears in the Hamiltonian in the term \( T_i^{s_i} T_j^{s_j} \). From Fig. 3 we deduce that \( [u_{ij}, W_{ij}] = 0 \), \( [u_{ij}, W_{kl}] = 0 \) and therefore \( [u_{ij}, H] = 0 \). We can hence replace the \( u_{ij} \) by \( c \)-numbers, associated with different superselection sectors. \( W_{ij} \) can be considered as a two-dimensional “parafermion hopping” term, while the eigenvalues of \( u_{jk} \) can be considered as a \( Z_3 \) gauge field coupled to the parafermions \( \alpha \). The precise meaning of the parafermion coupling will be discussed in next paragraph.

From the pictorial representation, we readily infer that the Hamiltonian can be rewritten as:

\[ H = -\sum_{\langle ij \rangle} J_{s_{ij}} u_{ij} W_{ij} + H.c. \quad (4) \]

FIG. 3: (a) The interaction terms in the Hamiltonian correspond to the three types of loops. The blue loop around each site represents the local constraint which commute with the Hamiltonian terms. (b) A loop corresponding to the interaction \( T_i^{s_i} T_j^{s_j} \) can be decomposed into two non-overlapping loops, \( W_{ij} \) and \( u_{ij} \).

endpoint of the branch cut line\[7, 8, 20\]. Now consider 4 genons with the constraint that they fuse to vacuum. As is shown in Fig. 2 (c), this constraint means a Laughlin quasiparticle going around the 4 genon cluster obtains no Berry’s phase. With this constraint, the disk region with 4 genons is topologically equivalent to a torus with a single layer of \( 1/n \) state\[20\], which thus has \( n \) topological ground states. The slave genon approach is defined by introducing an \( \alpha \)-genon with the constraint that they fuse to vacuum. As is shown in Fig. 2 (d), during topological deformations of the Wilson loops, we also require that a double loop around a genon is contractible, as is illustrated in Fig. 2 (c). Physically this removes the ambiguity that a genon may trap a Laughlin quasiparticle. We emphasize that the genons are entirely auxiliary degrees of freedom – the spin model is not required to have a FQH state physically.

In this representation, the spin model is mapped to a two-dimensional array of genons, with couplings given by Wilson loop operators. The two-site terms \( K_{ij} \) in the Hamiltonian simply correspond to Wilson loops surrounding 4 genons, as is shown in Fig. 3. Importantly, the Hamiltonian commutes with the local constraint at each site, since the Wilson loop corresponding to the local constraint commutes with that of \( K_{ij} \), as is illustrated in Fig. 3 (a). On each site, the constraint can be expressed in the spin operators \( T_i^{s_i} \) as \( D_i \equiv T_i^{s_i} T_{\alpha}^{s_{\alpha}} = 1 \), which projects the \( n^2 \) states of 4 genons\[20\] to \( n \) states of the physical spin.

From the pictorial representation, we readily infer that the Hamiltonian can be rewritten as:

\[ H = \sum_{\langle ij \rangle} J_{s_{ij}} u_{ij} W_{ij} + H.c. \quad (4) \]

FIG. 4: Hamiltonian (4) describes a hexagonal array of coupled genons, or \( Z_3 \) parafermion zero modes. For \( J_x = J_y \), and in the absence of interchain interactions, each chain is at criticality, which in the \( n = 3 \) case is described by a \( Z_3 \) parafermion CFT. Interchain coupling terms can be added to gap out counterpropagating parafermion modes from each chain, leading to a gapped topologically ordered state with a chiral \( Z_3 \) parafermion edge mode. Red bonds correspond to the next neighbor interactions (see \[5\]).
(CFT) with central charge $c = 4/5$\cite{34}. At small but
finite $J_z$, the system can be viewed as coupled parafermion
chains, as is illustrated in Fig. 4. It is known that a
“chiral” coupling between 1D gapless chains can realize a
dirac 2D topologically ordered state\cite{15, 33, 35, 36}, if
the right-moving (left-moving) states of a chain are only
coupled to the left-moving (right-moving) states of the
chain below (above) by a relevant coupling. In our $n = 3$
system, such a coupling, if realized, will result in a non-
Abelian topological state with chiral $Z_3$ parafermion edge
states. This is similar to the proposal of \cite{15}, although
the latter is not a local spin model and therefore real-
izes a different topological order. In $n > 2$ models, the
$J_z$ coupling breaks time-reversal symmetry, so that it is
possible for the system with some proper $J_z$ to be in the
same non-Abelian phase as the ideal system with only
chiral coupling.

Numerical Results – To gain further understanding of the
$n = 3$ system, we have performed preliminary nu-
merical analyses. For the single chain with $J_x = J_y$, our
DMRG results \cite{37, 38} for the entanglement entropy
shows that the chain is indeed described by a conformal
field theory with central charge $c = 4/5$, as shown in
Fig. 5(a). In the opposite limit $J_z \gg J_x, J_y$, we have
verified through exact diagonalization that the system is
gapped, with a 9-fold ground state degeneracy. As $J_z$
is lowered relative to $J_x, J_y$, we expect a phase transi-
tion from the Abelian phase to the isotropic phase. Fig.
5(b) shows DMRG results for the second derivative of the
ground state energy density, $-d^2E_0/dJ_z^2$, which in-
deeds shows evidence of a sharp phase transition. These
numerical results confirm non-trivial features of the $Z_3$
Kitaev model, while they do not fully establish the nature
of the isotropic phase. More complete numerical study of
the non-Abelian phase will be left for future works.

Multi-site terms and the controlled limit– In the original
Kitaev model\cite{11}, a three site term drives the model into the non-Abelian Ising phase. Similarly,
for $n = 3$ it is possible to consider a modification of the Hamiltonian \cite{15} that makes the non-Abelian state
more tractable. As is pointed out in Ref. \cite{15, 39},
there is a known correspondence between the lattice
parafermion operators and continuous fields in the
$Z_3$ Potts model CFT. Using this correspondence,
one can see that the parafermion coupling of the form

$$\lambda \sum_{j,m} \left( \alpha_{R,j+1}^r \left( \alpha_{R,j}^l + \alpha_{R,j+1}^r \right) \alpha_{L,j+m+1} + \alpha_{L,j+1}^r \right) + h.c.$$

between two neighboring chains labelled by $m$ and
$m + 1$ induces the chiral coupling between the right
movers of the $m$-th chain and the left movers of the
$m + 1$-th chain. Since this is a direct application of Ref.
\cite{15, 39}’s result, we will leave more detailed derivation
of this term for the supplementary materials\cite{15}.

Using the Wilson loop representation, the chiral
coupling between parafermions reviewed above can be
achieved in a local spin Hamiltonian:

$$H' = H - J_z \sum_\Omega \mathcal{O}_\Omega,$$

with $\mathcal{O}_\Omega = (T_\uparrow \bar{T}_\uparrow T_\uparrow \bar{T}_\uparrow + \bar{T}_\uparrow T_\uparrow \bar{T}_\uparrow T_\uparrow +
T_\uparrow \bar{T}_\uparrow T_\downarrow \bar{T}_\downarrow + H.c.)$, and $H$ given by Eq. (1). Therefore, the above Hamiltonian, with $J_x = J_y \gg J_z > 0$, could
realize a gapped, 2D topologically ordered state, with a
robust chiral $Z_3$ parafermion CFT propagating along
its boundary. The topological order can then be read
off from the primary field content of the $Z_3$ parafermion
CFT \cite{34, 40}.

We emphasize that the possibility of realizing a coupled
array of parafermion chains, with couplings that involve
only single parafermion operators from different chains,
is highly non-trivial. This is not possible with the usual
transverse field Potts model, but is possible with the ap-
proach described here. The slave genon transformation
thus provides a way to design general interactions in 2D
lattices of parafermions, in terms of local interactions of a
2D spin model. We expect a similar method can be
employed for much more general models, which may en-
able a spin model realization of the anyon lattice models
studied in the literature \cite{41, 42}.

Further generalizations – The model described here ad-
mits much more generalization. For example, one can

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig5.png}
\caption{(a) Entanglement entropy of an open $N$ site chain.
The fit to $S(x) = \frac{2}{5} \ln(x) + \text{const}$ extrapolates to a central
charge $c = 4/5$, where $x = \frac{\pi}{2} \sin\left(\frac{\pi}{N}\right)$ and $l$ is subsystem
length. (b) Second derivative of the ground state energy
density as a function of the distortion $J_z$, computed from DMRG
with 3 chains. A phase transition is resolved when $J_z$ is tuned
between the isotropic and anisotropic limits.}
\end{figure}
consider genons in a generic Abelian FQH state. Quasi-particles in each layer are labeled by integer vectors \( \vec{l} \), with the fractional mutual statistics \( \theta_{\vec{l},\vec{m}} = 2\pi i \vec{K} \cdot \vec{\rho} - \vec{l} \cdot \vec{\rho} \) and self statistics \( \theta_{\vec{l}} = \pi \vec{I} \cdot \vec{K} - \vec{l} \cdot \vec{I} \) determined by an integer valued \( K \) matrix [28]. 4 genons with the local constraint in Fig. 2 (d). These operators satisfy the algebra

\[
T_{\vec{l}}^x T_{\vec{m}}^y = T_{\vec{m}}^y T_{\vec{l}}^x e^{2\pi i \vec{I} \cdot \vec{K} - \vec{l} \cdot \vec{I}},
\]

\[
T_{\vec{l}}^z T_{\vec{m}}^z = T_{\vec{m}}^z T_{\vec{l}}^z + 1 \quad \text{for all} \quad \vec{m}, \vec{l} \in \mathbb{Z}^N.
\]

Therefore, we can consider the more general Kitaev-type Hamiltonian:

\[
H = \sum_{\vec{r} \in \mathbb{Z}^N} \sum_{\{ij\}} J_{ij} T_{\vec{r}}^{x_{ij}} T_{\vec{r}}^{x_{ij}} + H.c.
\]

This model can be analyzed similarly to the \( Z_n \) generalization presented earlier. In particular, there are conserved quantities associated with each plaquette and conserved large loop operators on torus geometry, and one can consider an exact transformation to a lattice model of interacting genons or, alternatively, generalized parafermion zero modes [14].

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In this section, we will explain the general rules we use to define the $\mathbb{Z}_n$ Kitaev model and obtain the conserved quantities. These rules can then be generalized to define these models on generic trivalent lattices. We start with the $\mathbb{Z}_n$ algebra

$$
T^x_i T^y_i = T^y_i T^x_i \omega, \quad T^y_i T^z_i = T^z_i T^y_i \omega, \quad T^z_i T^x_i = T^x_i T^z_i \omega
$$

(6)

with $\omega = e^{i 2 \pi / n}$. This algebra can be summarized by drawing a triangle on each site of the honeycomb lattice, as is shown in Fig. 6 (a). The three vertices of the triangle represent the three operators $T^x,y,z_i$, with the arrow an indicator of their commutation relation. For any two of the three operators $T^{\alpha}_i$, $T^{\beta}_i$, $T^{\gamma}_i$, $T^{\alpha}_i T^{\beta}_i = T^{\beta}_i T^{\gamma}_i \omega$ if $\alpha \to \beta$ is along the arrow direction, while the phase factor is $\omega^{-1}$ if $\alpha \to \beta$ is the reverse of the arrow direction. This arrow rule will be helpful when we check the commutation relation between terms in the Hamiltonian and define the conserved quantities.

Consider the Hamiltonian given by Eq. (1) of the manuscript. We denote the term on each bond as $K_{ij} = T^{s_{ij}}_i T^{s_{ij}}_j$, with $s_{ij} = x, y, z$ depending on the bonds. Consider the plaquette formed by sites 1,2,...,6 shown in Fig. 6 (b). Using the arrow rule, and remembering that the spins on different sites commute with each other, we can easily check that

$$K_{12} K_{61} = K_{61} K_{12}, \quad K_{12} K_{23} = K_{23} K_{12} \omega^{-1}
$$

(7)

This is simply because the arrow goes out of bond 12 at site 1, while it goes into the bond at site 2. In the plaquette operator

$$W_p = K_{12} K_{23} K_{34} K_{45} K_{56} K_{61}
$$

(8)

$K_{16}$ and $K_{23}$ are the only two terms which do not commute with $K_{12}$. Therefore we see that the factor given by the two terms cancel, and we obtain

$$[K_{12}, W_p] = 0
$$

(9)

This proves that the Hamiltonian commutes with $W_p$, and also proves that the plaquette operators $W_p$ commute with each other, since they are products of $K_{ij}$.

Compared with the $\mathbb{Z}_2$ Kitaev model, in the $\mathbb{Z}_n$ model with $n > 2$ a new requirement needs to be satisfied in order for the plaquette conserved quantities to be defined: the arrows at each site must have the same chirality. In the choice we make, the arrows go around the triangle in a counter-clockwise order. If the arrow is reversed on some site...
FIG. 7: (a) The naive construction of large loop operator $\tilde{\Phi}_1$ which does not commute with bond terms $K_{ij}$. (b) The correct large loop operator $\Phi_1$ which commutes with every $K_{ij}$. (c) Another large loop operator along a different direction. (d) A generic loop operator, which is a product of $T^{\alpha}_i$ at all right-turn corners, and $T^{\alpha\dagger}_i$ at all left-turn corners.

(which can be done by replacing $T^{\alpha}_i \rightarrow T^{\alpha\dagger}_i$ only on that site), the two arrows connected to a given bond will be both out or both in, which makes the bond operator $K_{ij}$ non-commuting with $W_p$.

Now we define the large loop operators in a system defined on the torus. In the $Z_2$ Kitaev model, the large loop operator can be obtained by multiplying bond terms $K_{ij}$. In the $Z_n$ case, if we follow this definition and define, for example,

$$\tilde{\Phi}_1 = \prod_{i \in L_1} K_{i,i+1} = \omega^N \prod_{i \in L_1} T^{z\dagger}_i \quad (10)$$

as the product of $K_{ij}$ along the zigzag line $L_1$ shown in Fig. 7 (a) (where $L_1$ contains $2N$ sites), it does not commute with $K_{16}$, since

$$K_{16} K_{12} = K_{12} K_{16}^{-1}, \quad K_{16} K_{67} = K_{67} K_{16}^{-1}. \quad (11)$$

To define the correct large loop operator, we can change the arrow direction every other site by replacing $T^z_{2i}$ by $T^z_{2i}\dagger$ on all even sites. This results in the loop operator shown in Fig. 7 (b), defined as

$$\Phi_1 = \prod_{2i-1,2i \in L_1} T^{z\dagger}_{2i-1} T^{z\dagger}_{2i} \quad (12)$$

which can be verified as commuting with the Hamiltonian. Following this rule, another large loop operator can be defined on the other large loop $L_2$ of the torus, as is shown in Fig. 7 (c), and written in Eq. (3) of the main text. Interestingly, the two large loops do not commute with each other, as has been discussed in the main text.

More generally, a loop operator can be defined for any given loop $L$ drawn on the honeycomb lattice, as is illustrated in Fig. 7 (d). The general rule is the following. i) For each site $i \in L$, pick $\alpha_i \ (x, y, z)$ to be the bond type of the bond that is not included in $L$. ii) We define an orientation of the loop. Following this orientation, define a number
FIG. 8: Definition of the $Z_n$ Kitaev model on the square-octahedron lattice. The terms in the Hamiltonian are $J_z T_z^i T_z^j + h.c.$ on the red bonds, and $J_x T_x^i T_y^j + h.c.$ on the blue bonds.

$s_i = \pm 1$ for each vertex $i$, such that $s_i = 1$ or $-1$ if the loop turns right or left at the vertex, respectively. Then the loop operator is defined as

$$\Phi_L = \prod_{i \in L} [T_\alpha^{s_i}]$$

(13)

(It should be remembered that $[T_\alpha^s]^{-1} = T_\alpha^{-s}$. )

Using the arrow rules discussed above, the $Z_n$ Kitaev model can be generalized to all planar trivalent lattices. At each vertex, the operators $T_{x,y,z}^i$ can be assigned to the three bonds connecting this site, with the order of $x,y,z$ bonds following the same chirality at each site. Denote the spin operator assigned to a bond $ij$ at site $i$ as $T_\alpha^{i,j}$, we can write the Hamiltonian as

$$H = \sum_{\langle ij \rangle} [J_{ij} T_\alpha^{i,j} T_\alpha^{j,i} + h.c.]$$

(14)

In general, $\alpha_{ij} \neq \alpha_{ji}$. An example of a different trivalent lattice and the corresponding operator assignment is shown in Fig. 8. The plaquette conserved quantities and the generic loop operators can all be defined in the same way as on honeycomb lattice.

MAPPING TO PARAFERMIIONS AND MODIFIED HAMILTONIAN

Mapping to parafermion array

Let us begin with the model, eq. (1) of the main text, defined along a one-dimensional chain. As explained in the main text, after applying the slave genon transformation and considering the case where the $Z_n$ gauge fields are uniformly equal to one, we obtain:

$$H_{1D} = - \sum_i (J_x W_{2i-1,2i} + J_y W_{2i,2i+1} + H.c.),$$

(15)

with

$$W_{2i-1,2i} W_{2i,2i+1} = W_{2i,2i+1} W_{2i-1,2i} \omega.$$ 

(16)

It will be useful to introduce a second representation of (15) in terms of genons, as shown in Fig. 9. To understand this, suppose that (15) contains $2N$ sites of the honeycomb lattice. In the second representation, we introduce $N + 1$ genons, as shown in Fig. 9 with the loops $W_{i,i+1}$ as shown.
FIG. 9: (a) The slave genon representation for a single chain of the honeycomb model. The loops corresponding to the quasiparticle loop operators $W_{ij}$ are shown. (b) The Hamiltonian [15], which describes a 1D Potts model, can be reformulated in a slightly different genon representation. Each pair of genons, which gives rise to $n$ states, is effectively a single site of the Potts model. The Wilson loop operators $W_{2i-1,2i}$ and $W_{2i,2i+1}$ are shown, and acquire non-trivial commutation relations due to the single crossing. A set of reference defects, labelled $R_0$ and $R_1$ and in green color, are used as well, although they are not directly associated with any site of the Potts chain. They are useful for regulating the strings of the parafermion operators used later. Their necessity can be understood by recalling that if we start with $N$ pairs of genons on a sphere, the resulting topological degeneracy is $n^{N-1}$ [8]. Therefore, if the Potts chain has $N$ sites, we need $N + 1$ pairs of genons. $R_0$ and $R_1$ can be thought of as this extra pair.

$H_{1D}$ is equivalent to the transverse field $Z_n$ Potts model. To see this, we group the genons $2i - 1, 2i$ into a single site with $n$ states, and define

$$\tau_i \equiv W_{2i-1,2i}, \quad \sigma_i^\dagger \sigma_{i+1} \equiv W_{2i,2i+1},$$

such that $\sigma_j \tau_i = \tau_i \sigma_j \omega^\delta_{ij}$, and $\tau_i^n = \sigma_i^n = 1$, where $\delta_{ij}$ is the Kronecker delta function. In these variables,

$$H_{1D} = -\sum_i (J_x \tau_i + J_y \sigma_i^\dagger \sigma_{i+1} + H.c.),$$

which is the familiar form for the $Z_n$ Potts model.

We can define “parafermion” operators:

$$\alpha_{R,2j-1} = \sigma_j \mu_j^{-1}, \quad \alpha_{R,2j} = \omega \sigma_j \mu_j,$$

$$\alpha_{L,2j-1} = \sigma_j^\dagger \mu_j^{-1}, \quad \alpha_{L,2j} = \omega^{-1} \sigma_j \mu_j^\dagger.$$
where \( \mu_j = \prod_{k \leq j} \tau_k \). The parafermion operators satisfy the algebra:

\[
\alpha_{Rj}\alpha_{Rj} = e^{i \frac{2\pi \text{sgn}(j-i)}{n}} \alpha_{Rj}\alpha_{Ri},
\]

\[
\alpha_{Lj}\alpha_{Lj} = e^{-i \frac{2\pi \text{sgn}(j-i)}{n}} \alpha_{Lj}\alpha_{Li},
\]

(20)

with \((\alpha_{Li})^n = (\alpha_{Ri})^n = 1\). Note that \( \alpha_L \) and \( \alpha_R \) are not independent degrees of freedom. In terms of these lattice parafermion operators,

\[
\tau_j = \omega^* \alpha_R^{\dagger} \alpha_{R,2j-1} \alpha_{R,2j} = \omega^* \alpha_L^{\dagger} \alpha_{L,2j} \alpha_{L,2j-1} \\
\sigma_j \sigma_{j+1} = \omega^2 \alpha_R^{\dagger} \alpha_{R,2j+1} \alpha_{R,2j} = (\omega^*)^2 \alpha_L^{\dagger} \alpha_{L,2j+1} \alpha_{L,2j}.
\]

(21)

Therefore, in terms of the lattice parafermions, the Hamiltonian is

\[
H_{1D} = - \sum_i (J_x \omega \alpha_R^{\dagger} \alpha_{R,2j-1} \alpha_{R,2j} + J_y \omega^2 \alpha_L^{\dagger} \alpha_{L,2j} + H.c.)
\]

\[
= - \sum_i (J_x \omega^* \alpha_R^{\dagger} \alpha_{R,2j} \alpha_{R,2j-1} + J_y \omega^2 \alpha_L^{\dagger} \alpha_{L,2j} \alpha_{L,2j+1} + H.c.).
\]

(22)

In Fig. 10 we show how the parafermion operators can be understood in terms of the genon representation as Wilson loop operators of Abelian quasiparticles.

Now let us turn to the 2D version of the model, eq. (4) in the main text. Again, for simplicity we will consider the ground state sector where \( u_{ij} \) are uniform in space. The 2D model can be understand as an array of 1D chains, together with an appropriate interchain coupling:

\[
H = \sum_m H_{1D}[m] + H_{\text{inter}},
\]

(23)

where now

\[
H_{1D}[m] = - \sum_i (J_x \omega \alpha_R^{\dagger} \alpha_{R,2j,m} \alpha_{R,2j-1,m} + J_y \omega^2 \alpha_L^{\dagger} \alpha_{L,2j,m} \alpha_{R,2j+1,m} + H.c.),
\]

(24)

and \( m \) is the chain index, and

\[
H_{\text{inter}} = - \sum_{\langle ij \rangle = \text{z-link}} J_z W_{ij} + H.c.,
\]

(25)

where \( \langle ij \rangle \) is a vertical z-link of the honeycomb lattice.

From Fig. 11a, it is straightforward to see that the Wilson loop operators which couple different parafermion chains can be written as

\[
H_{\text{inter}} = - \sum_m J_z \left( u_{0,m} \alpha_R^{\dagger} \alpha_{R,2j,m} \alpha_{L,2j,m+1} + H.c. \right),
\]

(26)

where \( u_{0,m} \) is the loop operator shown in Fig. 11a, which encloses the reference defects. Since this loop encloses the reference defects and commutes with all terms in the Hamiltonian of the Potts chain, it can be treated as a c-number.

The parafermion operators \( \alpha_{R/L,i} \) can also be written in terms of the original spins of the generalized Kitaev model. To understand this, let us first observe that for a single chain of the generalized Kitaev model (see Fig. 4a), we have the relation:

\[
T^1_1 W_{12} = W_{12} T^1_1 \omega^*.
\]

(27)

In the alternate representation of Fig. 4b, we can therefore associate

\[
\alpha_{R1} \propto T^1_1, \quad \alpha_{L1} \propto T^1_1^\dagger,
\]

(28)
FIG. 11: Depiction of interchain couplings, between chains $m$ and $m+1$. (a) The loops corresponding to the operators $\alpha_{R,1,m}^\dagger \alpha_{L,2,m+1}$ and $\alpha_{R,4,m}^\dagger \alpha_{L,4,m+1}$ are shown. These loops also contain a loop that encloses the reference genons from the two chains, which we have labelled $u_{0,m}$. $u_{0,m}$ acts completely trivially in the Hilbert space of the Potts chain. (b) The loops associated with the interchain parafermion interactions in (32). (c) The equivalent loops of (b), but shown in the original slave genon representation of the honeycomb model. We label the associated loop operators as $W_{21}, W_{36}, W_{26},$ and $W_{31}$.

in order to reproduce (27). Furthermore, from (19), we see that in the Potts model $\alpha_{R,1} = \alpha_{L,1} = \sigma_1$, and so

$$\alpha_{R,2j-1} = \alpha_{R1} \prod_{k=1}^{j-1} \sigma_{k+1}^z \prod_{k=1}^{j-1} \tau_k \propto T^\dagger_1 W_{12} W_{23} \ldots W_{2j-2}, 2j-1$$

$$\alpha_{R,2j} = \omega \alpha_{R1} \prod_{k=1}^{j-1} \sigma_{k+1}^z \prod_{k=1}^{j} \tau_k \propto T^\dagger_1 W_{12} W_{23} \ldots W_{2j-1}$$

$$\alpha_{L,2j-1} = \alpha_{L1} \prod_{k=1}^{j-1} \sigma_{k+1}^z \prod_{k=1}^{j-1} \tau_k^\dagger \propto T^\dagger_1 W_{12} W_{23} W_{34} \ldots W_{2j-2j-1}$$

$$\alpha_{L,2j} = \omega^* \alpha_{L1} \prod_{k=1}^{j-1} \sigma_{k+1}^z \prod_{k=1}^{j} \tau_k^\dagger \propto T^\dagger_1 W_{12} W_{23} W_{34} \ldots W_{2j-1}$$

(29)

By including the $Z_3$ gauge fields $u_{ij}$ along the chain, these operators can be made to be gauge-invariant and therefore expressible in terms of the local spin operators of the generalized Kitaev chain:

$$\alpha_{R,j} \propto T^\dagger_1 K_{12} K_{23} \ldots K_{j-1,j}$$

$$\alpha_{L,j} \propto T^\dagger_1 K_{12} K_{23} K_{34} \ldots K_{j-1,j}^s$$

(30)

where $s_j = -(\omega)^j$. 

Therefore, (34) reduces to (32), in the sector with spatially uniform $Z$ can be rewritten as

$$W_{26} = W_{21} W_{16},$$

allowing us to write down generic interchain couplings in terms of local spin interactions.

**Controlled limit**

Let us now return to the 1D Hamiltonian, and specialize to the case $n = 3$. When $J_x = J_y$, the model is self-dual and lies at a critical point between the ordered and disordered phase of the $Z_3$ Potts model. This critical point is described by a $Z_3$ parafermion conformal field theory.

As was pointed out recently [15, 39], at the critical point the lattice parafermion operators can be expanded in terms of the continuum fields of the $Z_3$ parafermion CFT as:

$$\alpha_R, j \sim a \psi_R + (-1)^j b \epsilon_R \sigma_L + ..., \quad \alpha_L, j \sim a \psi_L + (-1)^j b \epsilon_L \sigma_R + ..., \quad (31)$$

where $\psi_R/L$ are the right/left moving $Z_3$ parafermion fields, $\sigma_R/L$ are the $Z_3$ order parameter fields, and $\epsilon_R/L$ are the energy operators for the right/left moving sectors of the theory. $a$ and $b$ are constants. The ... include less relevant terms with higher scaling dimensions.

Using the above expansion, let us consider the following interchain coupling between the uncoupled 1D chains:

$$H_{\text{inter}} = -\lambda \sum_{j,m} u_{0,m}^\dagger (\alpha_{R,2j,m} + \alpha_{R,2j+1,m}) (\alpha_{L,2j,m+1} + \alpha_{L,2j+1,m+1}) + \text{H.c.}$$

$$\sim -4\sigma^2 \lambda \sum_m u_{0,m}^\dagger (\bar{\psi}_{R,m} \psi_{L,m+1} + \text{H.c.}) + ... \quad (32)$$

where recall $u_{0,m}^\dagger$ is a c-number here, which we can set to 1. When $\lambda > 0$, the above perturbation gaps out counterpropagating $Z_3$ parafermion modes from different chains [15]. This leaves a gapped two-dimensional bulk, with a chiral $Z_3$ parafermion mode propagating along the boundary.

$H_{\text{inter}}$ above is written in terms of the lattice parafermion operators. Now we would like to find the appropriate interchain coupling in terms of the original spin Hamiltonian, which reduces to $H_{\text{inter}}$ after the slave genon transformation for the uniform choice of $Z_3$ gauge fields. In order to do this, we use the graphical loop representation developed in this paper. $H_{\text{inter}}$ can be written as

$$H_{\text{inter}} = -\lambda \sum (W_{21} + W_{31} + W_{26} + W_{36} + \text{H.c.}), \quad (33)$$

where the loop operators $W_{21}, W_{31}, W_{61}$, and $W_{36}$ are shown in Fig. 11.

It is then straightforward to show (for example, see Fig. 12) that the following interactions in the spin model:

$$H_{\text{inter}} = -J_z (T_1 T_2^y T_1^x T_2^y T_1^y T_2^y + T_3 T_2^y T_1^x T_1^y T_2^y T_1^y + T_3 T_2^y T_1^x T_1^y T_2^y T_1^y + \text{H.c.}), \quad (34)$$

can be rewritten as

$$H_{\text{inter}} = -J_z (u_{21} W_{21} + u_{21} u_{16} W_{26} + u_{32} u_{21} u_{16} W_{36} + \text{H.c.}). \quad (35)$$

Therefore, (34) reduces to (32), in the sector with spatially uniform $Z_3$ gauge fields $u_{ij}$.
It follows that the modified Hamiltonian,

\[ H' = -\sum_{\langle ij \rangle} J_{s_{ij}} T^{s_{ij}} T^*)^{s_{ij}} + H.c. - J_z \sum_{O} O_O, \tag{36} \]

with

\[ O_O = T^x_1 T^y_1 T^z_2 T^x_6 + T^x_3 T^y_2 T^z_2 T^x_1 T^y_6 + T^x_3 T^y_2 T^z_2 T^y_1 + H.c., \tag{37} \]

is expected to realize a non-Abelian topologically ordered phase when \( J_z \ll J_x = J_y \), and \( J_z, J_x, J_y > 0 \). This non-Abelian phase has a chiral \( Z_3 \) parafermion CFT propagating along its boundary. Therefore, it contains a non-Abelian “Fibonacci” anyon.