Using the Generalize Laguerre Operational Matrix for The Caputo Fractional Derivatives to Solve Some Classes of Fractional Optimal Control Problems

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Abstract. This paper aims to develop the new numerical solution method of the fractional optimal control problems (FOCPs) based on the generalized Laguerre polynomials. The generalize Laguerre operational matrix for the Caputo fractional derivatives has been derived. The operational matrix was used together with the properties of the generalized Laguerre orthonormal polynomials to reduce FOCPs to the system of algebraic equations by the direct method. Five examples are included to demonstrate efficient and accurate numerical algorithms for some classes of FOCPs is considered.

Keyword: Caputo Fractional Derivatives, Fractional Optimal Control Problem, Laguerre Series, Operational Matrix.

1. Introduction

Recently, fractional calculus is a growing field, it has been successfully applied to problems in many fields of life and engineering [25, 15, 21]. The fractional optimal control problems (FOCPs) are generalized forms of the integer-order ones, which are obtained by replacing the integer-order derivatives by fractional ones. Newly, FOCPs has shown many advantages over the simulation of natural physical processes and dynamic systems. Most fractional optimal control problems don’t have exact solutions, so the numerical techniques and approximate methods must be used. Accordingly, seeking numerical methods to solve FOCPs, with high accuracy, rapid convergence, and less storage is becoming more and more important. In this respect, in this work, we are developed efficient and accurate numerical algorithms for some classes of FOCPs. There were many researchers have been interested in studying the FOCPs and obtain numerical solutions for them, like [2, 7, 16].
The Laguerre polynomial has many useful properties and plays a prominent role in various areas of mathematics. It has frequently been used in the solution of variational problems, the differential equations, integral equations, and approximation theory [22]. The Laguerre polynomial has been used to get approximate method for solving linear fractional Klein-Gordon and Fractional Logistic differential equations, [17, 18, 19].

The main aim of this work is to use the advantage of the generalized Laguerre polynomials to approximate solutions for some classes of FOCPs. Two efficient numerical methods depend upon the Laguerre polynomials have been presented where fractional derivatives are introduced in the Caputo sense.

The structure of this paper was arranged in the following way: In Section two, preliminaries, notations and some properties of the generalized Laguerre polynomials and the FOCP model were given. In Section Three, the basic formulation of the proposed approximate formulas of the fractional derivatives was obtained and the numerical scheme for solving FOCP is presented. In Section Four, the illustrative examples were included to demonstrate the validity and applicability of the proposed technique. In Section Five, a brief conclusion and some remarks.

2. Preliminaries and notations

We introduce some basic definitions and properties of Laguerre polynomials, Caputo fractional derivatives and the basic formulation of the proposed FOCPs.

2.1. Generalized Laguerre Polynomials

Laguerre polynomials, named later than Edmond Laguerre (1834-1886) [20]. Where the weight function \( w(t) = e^{-\lambda t} \) then the generalized Laguerre polynomial is defined as

\[
L_m(t) = \frac{e^{\lambda t}}{m!} \frac{d^m}{dt^m} \left[ t^n e^{-\lambda t} \right]
\]

The sequence of generalized Laguerre polynomials \( \{L_m(t)\}_{m=0}^{\infty} \) is defined in \([0, \infty)\) and \( \lambda > -1 \). The orthogonality relation is:

\[
\int_0^\infty t^\lambda \frac{e^{-t}}{t^{\gamma}} L_i(t) L_j(t) dt = \frac{(m+\lambda)!}{m!} \delta_{ij}, \quad \delta_{ij} \text{ is the Kronecker delta function.}
\]

The Laguerre polynomials can be generated by using the following useful recurrence relation:

\[
L_{m+1}(t) = \frac{1}{m+1} \left( (t - 2m - \lambda - 1)L_m(t) + (m + \lambda)L_{m-1}(t) \right), \quad m = 0, 1, 2, \ldots.
\]

Let \( x(t) \) be any function and \( x(t) \in C^m[a, b], \ t \in [0, \infty) \) then we can write the approximation formula of it as follows:

\[
x(t) \approx x_m(t) = \sum_{i=0}^{m} c_i L_i(t),
\]

\( m \) is sufficient large integer number and the coefficients \( c_i \) are given by:

\[
c_i = \frac{\Gamma(i+1)}{\Gamma(i+\lambda+1)} \int_0^\infty \frac{e^{-t} L_i(t)x(t) dt}{t^{\lambda+i}}, \ i = 0, 1, 2, 3 \ldots m.
\]

For more information see [20].

There are many properties of the generalized Laguerre polynomials. Below, we recall some important properties needed in this present work:

1. \( \frac{d^n}{dt^n} [L_m(t)] = (-1)^n L_{m-n, \lambda+n}(t) \), and \( \int L_i(t) L_j(t) dt = L_{i+j, \lambda-1}(t) \).

2. The inner products \(< t^n, L_m(t) > = \int_0^\infty \frac{e^{-t} L_m(t)}{t^n} dt = 0, \forall \ n < m \)

3. \( \int_0^t L_i(t) dt = L_i(t) - L_{i+1}(t), \ \forall \ i = 0, 1, 2, \ldots m \).

4. Addition formulas:

\[
\int_0^t L_i(t) r^m dr = \frac{1}{m+1} \left[ (m+1) L_i(t) - L_{i+1}(t) \right], \ \forall \ i = 0, 1, 2, \ldots m.
\]
\[ L_{m,\lambda}(t + w) = \sum_{i=0}^{m} L_{i,\mu}(t) L_{m-i,\lambda-\mu-1}(w), \]

and

\[ \sum_{i=0}^{m} L_{i,\lambda}(t) L_{n-i,\lambda}(w) = L_{m,\lambda+m+\mu+1}(t + w) \]

5- Multiple arguments:

\[ L_{m,\lambda}(t, w) = \sum_{i=0}^{m} (m+i)_{\lambda-i} t^i (1-t)^{m-i}, L_{i,\lambda}(w). \]

6- Involving power function:

\[ \int t^{\alpha-1} L_{m,\lambda}(t) dt = \frac{\Gamma(n+\lambda+1)}{\Gamma(\alpha+n+\lambda+1)} \, _2F_1(-m, \alpha; \alpha+1; \lambda; t), \]

\(_2F_1\) is Hypergeometric Function.

7- Monomials are represented as:

\[ t^n = n! \sum_{i=0}^{n} (-1)^i \binom{n+\lambda}{n-i} L_{i,\lambda}(t). \]

For there more details see [20, 23].

2.2 Caputo Fractional Derivative

Let \( 0 \leq \alpha \leq 1 \) is a real number and \( x: [a, b] \rightarrow R \) is continuous function then:

\[ {}_0^C D_t^\alpha x(t) = \int_a^t \frac{1}{\Gamma(1-\alpha)} (t-s)^{-\alpha} \dot{x}(s) ds, \quad (2.5) \]

is the left Caputo fractional derivative (LCFD) of order \( \alpha \). Where \( 0 \leq \alpha \) and \( n+1 \leq \alpha \leq n \), \( n \) is a positive integer number the (LCFD) defined as:

\[ {}_0^C D_t^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} x^{(n)}(s) ds. \quad (2.6) \]

Where \( x(t) = t^m \) then \( {}_0^C D_t^\alpha (t-a)^m = \frac{\Gamma(m+1)}{\Gamma(m+1-\alpha)} (t-a)^{m-\alpha} \), also

\[ {}_0^C D_t^\alpha (t^m) = \begin{cases} 0, & m < \text{nand} \ 1 < \alpha < n \\ \frac{\Gamma(m+1)}{\Gamma(m+1-\alpha)} t^{m-\alpha}, & m > \text{nand} \ 1 < \alpha < n \end{cases} \quad (2.7) \]

for more information see [3, 14, 13, 7, 10, 1, 8, 9, 12, 15].

2.3 Problem statement

In this work, we focus on a classes of FOCPs with Caputo fractional derivative in dynamical system. The problem formulation is as follows:

\[ \text{minimize} \quad J[u] = \int_a^b f(t, x(t), u(t)) dt \quad (2.8) \]

subject to

\[ {}_0^C D_t^\alpha x(t) = g(t, x(t)) + b(t) u(t), \quad n-1 \leq \alpha \leq n \quad n = 2, 3, \ldots \quad (2.9) \]

with initial condition \( x(a) = a_0, x'(a) = a_1, x''(a) = a_2, \ldots, x^{(n-1)}(a) = a_{n-1} \). Here \( f \) and \( g \) are polynomial functions of their arguments and \( b \neq 0 \) is a smooth function. Where \( x(t) \) is the state variable function, \( u(t) \) is the control function, \( t \in [a, b] \) is the time, and \( b(t) \) any real function, see [26].

The problem formulation of the especial case of FOCP (2.8) – (2.9) with the quadratic performance index, is as follows:

\[ J[u] = 1/2 \int_a^b \left[ p(t)x^2(t) + q(t)u^2(t) \right] dt \quad (2.10) \]

\[ {}_0^C D_t^\alpha x(t) = a(t)x(t) + b(t) u(t), \quad x(\xi_a) = \xi \quad (2.11) \]

Where \( q(t) \geq 0 \), \( p(t) \geq 0 \), \( b(t) \neq 0 \) and the fractional derivative is defined in the Caputo sense, see [25, 26].
3. Numerical experiments

In this section, has been established to new theorems to approximate LCFD by using generalized Laguerre polynomials, and has been derived an operational matrix of generalized Laguerre polynomial for LCFD. The first theorem gives the new result to the LCFD for generalized Laguerre polynomials, the second, third and fourth theorems give us more expiration to represent the LCFD to function \( x(t) \) by generalized Laguerre polynomials. The last theorem to estimate the error of approximating LCFD by the Laguerre polynomials and bound.

Theorem 3.1 The LCFD of order \( \alpha \) for generalized Laguerre polynomials of order \( m \) can be expressed in terms of the generalized Laguerre polynomials themselves in the following form:

\[
\frac{\partial}{\partial t^{\alpha}} L_{\lambda}(t) = t^{-\alpha} \Gamma(\alpha + 1) \sum_{j=n}^{\infty} \sum_{k=0}^{j} R_{j,k} L_{k,\lambda}(t),
\]

and

\[
R_{j,k} = \binom{m+j}{j} \binom{j+k}{j-k} (\lambda)^{j} (-1)^{j+k}.
\]

Proof: To prove this theorem we can use the closed form for generalized Laguerre polynomials of degree \( m \), \( L_{\lambda}(u), -1 < \lambda \), and the first property (Linearity) of LCFD where \( n - 1 < \alpha < n \), then we get:

\[
\frac{\partial}{\partial t^{\alpha}} L_{\lambda}(t) = \sum_{j=n}^{\infty} \sum_{k=0}^{j} \binom{m+j}{j} \binom{j+k}{j-k} \lambda^{j} (-1)^{j+k} t^{j+k}.
\]

From the property of Monomials the above equation will be:

\[
\frac{\partial}{\partial t^{\alpha}} L_{\lambda}(t) = t^{-\alpha} \Gamma(\alpha + 1) \sum_{j=n}^{\infty} \sum_{k=0}^{j} \binom{m+j}{j} \binom{j+k}{j-k} \lambda^{j} (-1)^{j+k} t^{j+k}.
\]

Theorem 3.2 Let \( x_{m}(t) = \sum_{j=0}^{\infty} c_{j} L_{j,\lambda}(t) \) is generalized Laguerre approximation for the function \( x(t) \) then:

\[
\frac{\partial}{\partial t^{\alpha}} x_{m}(t) = \sum_{j=0}^{\infty} \sum_{k=0}^{j} \binom{m+j}{j} \binom{j+k}{j-k} \lambda^{j} (-1)^{j+k} c_{j} t^{j+k}.
\]
\[ \frac{c_0}{\alpha} D^\alpha_t x(t) = \sum_{i=n}^m \sum_{k=n}^i c_i \frac{(-1)^k}{\Gamma(k-\alpha + 1)} \binom{k+\frac{\lambda}{\alpha}}{i-k} t^{k-\alpha}, \] (3.2)

and

\[ t^{k-\alpha} = \sum_{j=0}^{k-n} c_{kj} L_j(t), \] (3.3)

from equation (2.3) and using closed form for generalized Laguerre polynomials (2.4) with definition of Gamma function we get

\[ c_{kj} = \frac{\Gamma(j+1)}{\Gamma(j+1+k-\alpha)} \int_0^\infty t^{k+\lambda-\alpha} e^{-t} L_j(t) dt, \quad j = 0, 1, 2, \ldots, m, \] (3.4)

Substituting above equation into (3.3) we obtain

\[ t^{k-\alpha} = \sum_{j=0}^{k-n} \sum_{r=0}^{j-k-1} (-1)^j \frac{\Gamma(j+1+k)}{\Gamma(j+1+k+r)} \frac{(r)!r!}{(j-r)!r!} L_j(t), \] (3.5)

substituting this equation into (3.2) then (3.1) holds. \( \square \)

Theorem 3.3 Let \( x_m(t) = \sum_{j=0}^m c_j L_j(t) \) is generalized Laguerre approximation for the function \( x(t), \) and \( n-1 < \alpha < n, \) then:

1. \[ \frac{c_0}{\alpha} D^\alpha_t x(t) = \frac{(-1)(a-t)^{n-\alpha}}{(n-\alpha)^2 t^{n-\alpha - 1}} x^{(n)}(t) v, \quad \text{put} \quad t - v = -r, \]
2. \[ \frac{c_0}{\alpha} D^\alpha_t x(t) = \frac{1}{\Gamma(n-\alpha)} \sum_{j=0}^m \sum_{k=0}^j (\frac{c_j}{n-\alpha}) \frac{(a-t)^{i+n-\alpha}}{i!} L_j(t+r), \]

Proof: To prove (1)

\[ \frac{c_0}{\alpha} D^\alpha_t x(t) = \frac{(1)}{\Gamma(n-\alpha)} \int_0^t (t-v)^{n-\alpha-1} x^{(n)}(v) dv, \quad \text{put} \quad t - v = -r, \]

\[ \frac{c_0}{\alpha} D^\alpha_t x(t) = \frac{(1)}{\Gamma(n-\alpha)} \int_{-r}^0 (-r)^{n-\alpha-1} x^{(n)}(t+r) dr, \]

\[ \frac{c_0}{\alpha} D^\alpha_t x(t) = \frac{(-1)}{\Gamma(n-\alpha)} \int_0^a (-r)^{n-\alpha-1} x^{(n)}(t+r) dr, \quad \text{put} \quad x^{(n)}(t+r) = \sum_{j=0}^m c_j L_j(t+r), \]

\[ \frac{c_0}{\alpha} D^\alpha_t x(t) = \frac{(-1)}{\Gamma(n-\alpha)} \int_0^a (-r)^{n-\alpha-1} \sum_{j=0}^m c_j L_j(t+r) dr, \] (3.7)

substitute addition formula from properties of generalized Laguerre polynomials in (3.7), put \( \mu = 0 \)

\[ \frac{c_0}{\alpha} D^\alpha_t x(t) = \frac{(1)}{\Gamma(n-\alpha)} \int_0^a (-r)^{n-\alpha-1} \sum_{j=0}^m c_j L_j(t) dr, \]

\[ \frac{c_0}{\alpha} D^\alpha_t x(t) = \frac{(1)}{\Gamma(n-\alpha)} \sum_{j=0}^m \sum_{k=0}^j c_j L_j(t) \int_0^a (-r)^{n-\alpha-1} L_k(t) dr, \]

(3.8)

By using Involving Function from properties of generalized Laguerre polynomials (3.8) will be

\[ \frac{c_0}{\alpha} D^\alpha_t x(t) = \frac{(-1)(n-a)}{(n-\alpha)^2} \sum_{j=0}^m \sum_{k=0}^j c_j L_j(t) \int_0^a (-r)^{n-\alpha-1} L_k(t) dr, \]

\[ \frac{c_0}{\alpha} D^\alpha_t x(t) = \frac{(-1)(a-t)^{n-\alpha}}{(n-\alpha)^2} \sum_{j=0}^m \sum_{k=0}^j c_j L_j(t) \int_0^a (-r)^{n-\alpha-1} L_k(t) dr, \]
this relation between Caputo fractional derivative and Hypergeometric function with generalized Laguerre polynomials.

To prove (2), from the equation (3.8), we can obtain a new expiration between Caputo fractional derivative and generalized Laguerre polynomials.

\[
\begin{align*}
\frac{\partial^n}{\partial t^n} x(t) &= \frac{(-1)^n}{\Gamma(n-\alpha)} \sum_{j=0}^{m} c_j \int_0^a (-r)^{n-\alpha-1} \sum_{l=0}^{k} \frac{(-1)^n}{l!} r^l \frac{\partial^l}{\partial r^l} x(t) dr,
\frac{\partial^n}{\partial t^n} x(t) &= \frac{1}{\Gamma(n-\alpha)} \sum_{j=0}^{m} \sum_{k=0}^{n-\alpha-1} \frac{(-1)^n}{l!} \frac{\partial^l}{\partial r^l} x(t) \frac{\partial^k}{\partial r^k} x(t). 
\end{align*}
\]

Theorem 3.4 Let \( x_m(t) = \sum_{j=0}^{m} c_j L_{j,\lambda}(t) \) is generalized Laguerre approximation for the function \( x(t) \), and \( n - 1 < \alpha < n \) then:

\[
\frac{\partial^n}{\partial t^n} x_m(t) = t^{-\alpha} \Gamma(\alpha + 1) \sum_{m=0}^{p} \sum_{j=0}^{m} \sum_{k=0}^{n-\alpha-1} c_m \beta_{m,j,k,\lambda,\alpha} L_{j,\lambda}(t),
\beta_{m,j,k,\lambda,\alpha} = \frac{(m\lambda)^j}{(m-j)} \frac{(j+\lambda)^k}{(j-k)} \frac{(-1)^{i+k}}{i!}.
\]

Proof: To prove this theorem we can use the first theorem (3.1) with generalized Laguerre polynomials of degree \( p \) obtain:

\[
\frac{\partial^n}{\partial t^n} x_m(t) = \sum_{m=0}^{p} c_m \frac{\partial^n}{\partial t^n} L_{m,\lambda}(t),
\frac{\partial^n}{\partial t^n} x_m(t) = t^{-\alpha} \Gamma(\alpha + 1) \sum_{m=0}^{p} \sum_{j=0}^{m} \sum_{k=0}^{n-\alpha-1} c_m \beta_{m,j,k,\lambda,\alpha} L_{j,\lambda}(t). \]

Theorem 3.5 For the Laguerre polynomials \( L_{j,\lambda}(t) \) we have the following global uniform bounds estimates

\[
|L_{j,\lambda}(t)| \leq \begin{cases} 
\frac{(\lambda+1)^j}{j!} e^{t/2} & \text{for } \lambda \geq 0, t \geq 0 \\
(2 - \frac{(1+\lambda)^j}{j!}) e^{t/2} & \text{for } -1 < \lambda \leq 0, t \geq 0 
\end{cases} \quad (3.9)
\]

where \( j = 0, 1, 2, \ldots, n \)

Proof see [5] and [6].

4 Laguerre operation matrix of LCFD

Let \( x(t) \) be any function and \( x(t) \in C^m[a, b] \), \( t \in [0, \infty) \) then we can write the approximation formula of it as follows:

\[
x(t) = x_m(t) = \sum_{i=0}^{m} c_i L_{i,\lambda}(t) = C^T L, \quad (4.1)
\]

\( c_i \) are known parameters and \( C^T = [c_0, c_1, c_2, c_3, \ldots, c_m] \), also \( L \) is a matrix of Laguerre polynomials elements \( (m + 1) \times 1 \). And let

\[
\frac{\partial^n}{\partial t^n} x(t) = \sum_{j=0}^{m} f_j L_{j,\lambda}(t) = F^T L, \quad (4.2)
\]

where
\[ aCD^\alpha_{Dtn} = aCl_{Dtn}, aCD^\alpha_{Dtn} L, \ldots, aCD^\alpha_{Dtn}L_n, \ldots = 0 \] on \( t_0 \) \( -\Gamma(2)\Gamma(2-\alpha) t_1 - \alpha + \Gamma(3)\Gamma(3-\alpha) u_2 - \alpha / 2 \ldots \sum_{n=0}^{m} u_n = 0 \) \( m = jm + \lambda, \Gamma(j + 1) \Gamma(j + 1 - \alpha) t_j - \alpha \).

The element functions \( \frac{-\Gamma(2)}{\Gamma(2-\alpha)} t^{1-\alpha}, \frac{-4\Gamma(2)}{\Gamma(2-\alpha)} t^{1-\alpha} + \Gamma(3)\Gamma(3-\alpha) t^{2-\alpha} / 2 \ldots \) and \( \sum_{j=0}^{m} \left( -\frac{1}{j!} \right)^{(m+\lambda)} \frac{\Gamma(j+1)}{\Gamma(j+1-\alpha)} t^{j-\alpha} \) also can be represented by generalized Laguerre polynomials of order \( \lambda \), then we get:

\[
\mathcal{C}_a D^\alpha_t L = \begin{pmatrix}
P_{0,0} & P_{0,1} & P_{0,2} & \ldots & P_{0,m} \\
P_{0,0} & P_{0,1} & P_{0,2} & \ldots & P_{0,m} \\
P_{0,0} & P_{0,1} & P_{0,2} & \ldots & P_{0,m} \\
\vdots & \vdots & \vdots & \ldots & \vdots \\
P_{m,0} & P_{m,1} & P_{m,2} & \ldots & P_{m,m}
\end{pmatrix} \left( L_{0,\lambda}, L_{1,\lambda}, L_{2,\lambda}, \ldots, L_{m,\lambda} \right) = \mathcal{C}_a L,
\]

such that

\[
p_{i,j} = \left( -\frac{1}{j!} \right)^{(m+\lambda)} \frac{\Gamma(j+1)}{\Gamma(j+1-\alpha)} \frac{\Gamma(j+1)}{\Gamma(j+1+\lambda+1)} \int_0^\infty t^{\lambda + i - \alpha} e^{-t} L_j(t) dt.
\] (4.3)

By using equation (3.4), we can write equation (4.3) as:

\[
p_{i,j} = \left( -\frac{1}{j!} \right)^{(m+\lambda)} \frac{\Gamma(j+1)}{\Gamma(j+1-\alpha)} \frac{\Gamma(j+1)}{\Gamma(j+1+\lambda+1)} \sum_{r=0}^{\infty} \frac{(-1)^r \Gamma(j+r+\lambda+1)}{r! \Gamma(j+r+1) \Gamma(\lambda+r+1)} t^r L_j(t)
\] (4.4)

where \( r = 0, 1, 2, \ldots, j \) and \( j = 0, 1, 2, \ldots, m \). \( P \) is a known matrix will named the Laguerre operation matrix of LCFD. Therefore the relation between coefficient of Caputo fractional derivative \( F^T \) with coefficient of \( C^T \) can be writ it as:

\[
F^T \mathcal{C}_a L = C^T \mathcal{C}_a D^\alpha_t L,\]

\[
F^T \mathcal{C}_a L = F^T C^T P \mathcal{C}_a L,\]

\[
F^T = C^T P.
\] (4.6)

### 4.1 Solving FOCPs by Laguerre operation matrix

Consider the following fractional optimal control problem (2.9) – (2.8). Let us approximate the state function \( x(t) \) as:

\[
x(t) \approx x_m(t) = \sum_{i=0}^{m} c_i L(t) = C^T \mathcal{C}_a L,
\] (4.7)

and by using equation (4.5) will obtain

\[
\mathcal{C}_a D^\alpha_t x(t) = C^T \mathcal{C}_a D^\alpha_t L = C^T \mathcal{C}_a L.
\] (4.8)

Then the dynamic system (2.9) will be

\[
C^T \mathcal{C}_a L = g(t, C^T L) + b(t) \cdot u(x)
\] (4.9)

\[
u(x) = \frac{1}{b(t)} [C^T \mathcal{C}_a L - g(t, C^T L)]
\] (4.10)

Also the initial condition \( x(a) = a_0, x'(a) = a_1, x''(a) = a_2, \ldots, x^{(n-1)}(a) = a_{n-1}, \) will be

\[
C^T \mathcal{C}_a L_{m,\alpha}(a) = a_i = 0, \quad i = 0, 1, 2, \ldots, n - 1,
\]
\[ t^i [C^T \overline{L}_{m, \lambda}(\alpha) - \alpha_i] = 0, \quad i = 0, 1, 2, \ldots, n - 1. \]

We can merge the previous equation
\[ u(x) = \frac{1}{b(t)} [C^T P L - g(t, C^T \overline{L}) + \sum_{i=0}^{n-1} t^i [C^T \overline{L}_{m, \lambda}(\alpha) - \alpha_i]]. \tag{4.11} \]

By using the equations (4.11) and (4.7) to obtain performance index (2.8)
\[ J_m[C^T] = \int_a^b f(t, C^T \overline{L}_{m, \lambda}, \frac{1}{b(t)} [C^T P L - g(t, C^T \overline{L}) + \sum_{i=0}^{n-1} t^i [C^T \overline{L}_{m, \lambda}(\alpha) - \alpha_i])] dt \tag{4.12} \]

Computing the previous integration and using the Rayleigh–Ritz method to find the necessary conditions for the optimality of the performance index are
\[ \frac{\partial J}{\partial c_a} = 0, \frac{\partial J}{\partial c_s} = 0, \ldots, \frac{\partial J}{\partial c_m} = 0. \tag{4.13} \]

Then the system of algebraic equation introduced above can be solved by using any standard iteration method to find the unknown coefficients \( c_i \). See examples 1, 2, 3, 5, 6. In the next subsection discuss a special case of last optimal control problem with the quadratic performance index and the dynamic system with Caputo fractional derivative.

### 4.2 Solving FOCPs with the quadratic performance index by Laguerre operation matrix

Consider the following FOCP
\[ J = 1/2 \int_a^b [p(t) x^2(t) + q(t) u^2(t)] dt, \]

\[ \zeta D^\alpha g x(t) = a(t)x(t) + b(t)u(t), \quad g(\xi_0) = \xi, \]

to approximation of the state control variables \( x(t) \) and \( u(t) \), we must calculate the product of \( \overline{L} \) and \( \overline{L}^T \), which is we can name the product matrix of the Laguerre polynomials basis. The form of operational matrices \( P \) is
\[ C^T \overline{L} \overline{L}^T = \overline{L}^T P, \tag{4.14} \]

where \( P \) is an \((m + 1) \times (m + 1)\) matrix.

From the closed form for generalized Laguerre polynomials of degree \( m \),
\[ L_{m, \lambda}(t) = \sum_{i=0}^{m} (-1)^i \binom{m+\lambda}{m-i} t^i, \]

we can write the vector \( \overline{L} \) where \( \lambda = 0 \) as:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & \cdots & 0 \\
1 & -1 & 0 & 0 & 0 & \cdots & 0 \\
1 & -2 & 1/2 & 0 & 0 & \cdots & 0 \\
1 & -3 & 3/2 & -1/6 & 0 & \cdots & 0 \\
1 & -4 & 3 & -2/3 & 1/24 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\end{pmatrix}
= \begin{pmatrix}
1 \\
t \\
t^2 \\
t^3 \\
\vdots \\
t^m 
\end{pmatrix},
\]

\[ \overline{L} = B_{(m+1) \times (m+1)} \cdot T_{(m+1) \times 1}, \]

\[ C^T \overline{L} \overline{L}^T = C^T \overline{L}, (T^T, B^T) = [C^T \overline{L}, t(C^T \overline{L}), t^2(C^T \overline{L}), \ldots, t^m(C^T \overline{L})], B^T, \]

\[ C^T \overline{L} \overline{L}^T = \sum_{i=0}^{m} c_i L_{i,0}, \sum_{i=0}^{m} c_i t L_{i,0}, \ldots, \sum_{i=0}^{m} c_i t^m L_{i,0}, \tag{4.15} \]

\]
to approximate all functions $t^nL_{i,0}$ in terms of the sequence $\{L_{i,0}\}_{i=0}^m$ for all $i$ and $n = 0,1,\ldots m$,

$$t^nL_{i,0} = \sum_{j=0}^m \xi_j^{(n,i)}L_{j,0}(t) = \mathbf{c}^T_{n,i}\mathbf{L},$$

such that

$$\mathbf{c}^T_{n,i} = [\xi_0^{(n,i)}, \xi_1^{(n,i)}, \xi_2^{(n,i)}, \ldots, \xi_m^{(n,i)}].$$

To find all coefficient $\xi_j^{(n,i)}$, we can using equation (2.3). Thus we get

$$\xi_j^{(n,i)} = \int_0^\infty e^{-t}t^nL_{i,0}(t)L_{j,0}(t)dt,$$

then we can write:

$$\sum_{i=0}^m c_i t^kL_{i,0}(t) = \sum_{i=0}^m c_i (\sum_{j=0}^m \xi_j^{(k,i)}L_{j,0}(t)) = \sum_{j=0}^m L_{j,0}(t). (\sum_{i=0}^m \xi_j^{(k,i)} c_i),$$

$$\sum_{i=0}^m c_i t^kL_{i,0}(t) = \mathbf{L}^T [\xi_0^{(k,0)}, \xi_1^{(k,0)}, \ldots, \xi_m^{(k,m)}] = \mathbf{L}^T \mathbf{V}_{k+1} \mathbf{C},$$

where $\mathbf{V}_{k+1}$ is matrix $(m+1) \times (m+1)$ and has vectors $\xi_j^{(k,i)}$ for each columns. Now put result of equation (4.16) in (4.15) we get

$$C^T \mathbf{L}^T \mathbf{L} = \mathbf{P}^T [\mathbf{V}_0, \mathbf{V}_1, \ldots, \mathbf{V}_m] \mathbf{C} = \mathbf{P}^T \mathbf{P} \mathbf{C}.$$

Now we solving FOCP (2.10) – (2.11): Suppose that $x(t) \approx C^T \mathbf{L}$ then by using equation (4.6)

$$\mathbf{C}^T \mathbf{D}_t^\alpha x(t) \approx \mathbf{F}^T \mathbf{L} = \mathbf{C}^T \mathbf{P} \mathbf{L},$$

also we approximate functions

$$q(t) = Q^T \mathbf{L}, \quad p(t) = S^T \mathbf{L}, \quad a(t) = A^T \mathbf{L}, \quad b(t) = B^T \mathbf{L},$$

where $C^T = [c_0, c_1, c_2,\ldots, c_m], S^T = [s_0, s_1, s_2,\ldots, s_m], A^T = [a_0, a_1, a_2,\ldots, a_m], B^T = [b_0, b_1, b_2,\ldots, b_m]$, the coefficients $a_i, b_i, s_i, q_i$ we can find it by using (2.3). So the optimal control system

$$J = 1/2 \int_a^b [p(t)x^2(t) + q(t)u^2(t)]dt,$$

and

$$J = J[C,W] = 1/2 \int_a^b (S^T \mathbf{L})(C^T \mathbf{LL}^T C) + (Q^T \mathbf{L})(W^T \mathbf{LL}^T W)dt,$$

where

$$C^T \mathbf{LL}^T = \mathbf{L}^T \mathbf{P},$$

and

$$W^T \mathbf{LL}^T = \mathbf{L}^T \mathbf{W}.$$
Also the dynamical system we can approximate as:

\[ C^T P \bar{L} = A^T \bar{L}^T C + B^T \bar{L}^T W \]  (4.22)

\[ P \] is Caputo fractional operational matrix

\[ C^T P \bar{L} = \bar{L}^T AC + \bar{L}^T B W, \]  (4.23)

We observe that the dynamical system change to the linear system of algebraic equations

\[ C^T P - AC - B W = 0, \]  (4.24)

where \( \mu = [\mu_0, \mu_1, \ldots, \mu_m]^T \) is the unknown Lagrange multiplier. To find necessary conditions for optimality we compute

\[ \frac{\partial J}{\partial C} = 0, \frac{\partial J}{\partial W} = 0, \frac{\partial J}{\partial \mu} = 0. \]

Finally the system of algebraic equation can be solved by using any standard iteration method. Where we finding \( W, \mu, C \) we can determine the approximate values of \( \alpha(t) \) and \( u(t) \). See examples 4,5,6.

5. Numerical results and examples

This section provided some examples to illustrate our main results and to show the efficiency and accuracy of the numerical method has been introduced. We applied it to solved various examples and compared all results that have been obtained by using our method with the exact solution of the problem. In the first example, has been checked the validity of the approximation formula (4.6) of the operation matrix to approximate the functions \( \alpha(t) \). The second example shows solving FOCP by using operation matrix (4.6). The third example illustrate solving FOCP by theorem 3.4. Examples four and five illustrate the solving system of FOCP with the quadratic performance index (2.10) – (2.11). It is seen that in figures of exam examples we can get better results by using large values of the number \( m \) and the approximations are reasonably accurate. All computations to solve examples are done by using Matlab program.

![Figure (1)](image)

**Figure (1)** Analytic various approximation solution to example one, where \( m = 15, \lambda = 0 \).
Figure (2) Analytic various approximation solution to example two, the exact solution is the same approximation solution where $\lambda = 0, m = 5$.

Example 5.1 Consider the function $x(t) = 16t^5 - 20t^3 + 5t$, to find $D^\alpha_0 x(t)$ by using the operation matrix (4.6) to approximate Caputo fractional derivative by generalized Laguerre polynomials. The exact solution is $x(t) = 16t^5 - 20t^3 + 5t$. See Figure one denoted to this example. We observe that where $m = 15$, gets a good approximation with a maximum absolute errors $10^{-5}$.

Example 5.2 Consider the problem of minimizing

$$J = \int_0^1 \left( \frac{16\Gamma(6)}{\Gamma(5.5)} t^{4.5} - \frac{20\Gamma(4)}{\Gamma(3.5)} t^{2.5} + \frac{5}{1.5} t^{0.5} \right) dt,$$

subject to the initial condition $g(0) = 0$. The exact minimizer solution is $x(t) = 16t^5 - 20t^3 + 5t$, see[24]. Since the Lagrangian is always positive, the problem attains minimum when

$$\left( \frac{16\Gamma(6)}{\Gamma(5.5)} t^{4.5} - \frac{20\Gamma(4)}{\Gamma(3.5)} t^{2.5} - \frac{5}{\Gamma(1.5)} t^{0.5} \right) = 0$$

by using equation (4.6) of operation matrix to approximation Caputo fractional derivative by generalized Laguerre polynomials $P$.

$$x(t) = C^T L$$

$$C^T F - F^T L = 0$$

The system of equations (5.1) is the algebraic equation introduced above can be solved by using any standard iteration method. Where we finding $C^T P = -F^T$.

$$x_5(t) = (1099699788248015t^4)/2199023255552 - (45858568410234869/g1\gamma(0))/659706976656 + (41709286404915185/g1\gamma(0))/2199023255552 + (30135552133038085)/1099511627776 - 201348066836479987/7696581394432 + c$$

$$x_5(t) = 16t^5 - 20t^3 + 5t - 9245 + c$$
By using the initial condition gets \( c = 9245 \), we observe that the exact solution of \( x(t) \) is same \( x_5(t) \). We compared the results obtained in [24] with those achieved our technique. The minimum value of \( J \) achieved by our numerical technique is equal to zero that is achieved at \( m = 5 \). The Figure two denoted to this example. We observe where \( m = 5 \) get an exact solution.

Example 5.3 Consider the following fractional problem of the calculus of variations in the sense of Caputo [11].

\[
\begin{align*}
\min J[x] &= \int_0^1 \left[ \frac{5}{2} D_t^{0.5} x(t) - 1.1284 t^{0.5} (2t + 1) \right] dt & \text{and } x(0) = 3, \ x(1) = 5 \ .
\end{align*}
\]

Exact solution is \( x(t) = t^2 + t + 3 \), we can get it by using Euler–Lagrange equation. By using theorem (3.4) to approximate \( \frac{5}{2} D_t^{0.5} x(t) \). To get good approximation let \( p = 11, \lambda = 0, n = 1, \alpha = 0.5 \), and

\[
x_{11}(t) = \sum_{m=0}^{p=11} c_m L_{m,0}(t)
\]

\[
J = \int_0^1 \left[ t^{-0.5} \Gamma(\alpha + 1) \sum_{m=n}^p \sum_{j=n}^{m} \sum_{k=n}^{j} c_m \beta_{m,j,k,\lambda,\alpha} L_{k,j}(t) - 1.1284 t^{0.5} (2t + 1) \right] dt
\]

By using Rayleigh–Ritz method and apply it to evaluate \( c_0, c_1, \ldots, c_{11} \) unknown coefficients. The coefficients \( \beta_{m,j,k,\lambda,\alpha} \) are known. Figures three and four show that approximation for various index \( p \) and the maximum absolute error.

![Figure (3)](image1.png)

**Figure (3)** Analytic and various approximation solution to example three and **Figure (4)** The error to example four where, \( p = 11, \lambda = 0 \).

![Figure (5)](image2.png)

**Figures (5) and (6)** to analytic and various approximation for \( x(t) \) to example four where \( \alpha = 0.5 \) and \( \alpha = 0.7 \) respectively.
Figures (7) and (8) to showing absolute error where approximation \( x(t) \) to example four where \( \alpha = 0.5 \) and \( \alpha = 0.7 \) respectively.

Figures (9) and (10) to Analytic and various approximation for \( u(t) \) to example four where \( \alpha = 0.5 \) and \( \alpha = 0.7 \) respectively.

Figures (11) and (12) to showing absolute errors where approximation \( u(t) \) to example four where \( \alpha = 0.5 \) and \( \alpha = 0.7 \) respectively.

Example 5.4 Consider the following fractional optimal control problem with free final time

\[
J[u] = \frac{1}{2} \int_0^T \left[ t^\beta - 3t^{\beta + \frac{\alpha}{2}} + \frac{9}{4} t^\alpha + 1 - x(t) \right]^2 dt
\]  

(5.2)

with the dynamic system

\[
\mathcal{C} D^\alpha_0 x(t) = -(y(t) - 1)^3 + \frac{40320}{\Gamma(9-\alpha)} t^{\beta - \alpha} - \frac{3}{\Gamma(5-\alpha)} t^{4-\frac{\alpha}{2}} + u(t)
\]  

(5.3)
such that \( x(t) \geq 1 \), \( x(0) = 1 \) and \( x(t_f) = 1 \).

Exact solution to this problem is \( x(t) = e^{\frac{t^\alpha}{\alpha-\alpha}} \), \( x(t) = t^\alpha - 3t^4 + \alpha + 1 \) and \( u(t) = \left( \frac{3t^\alpha}{\alpha} - t^4 \right)^3 + \frac{9}{4} \Gamma(\alpha + 1) \). By using the method has been presented in subsection (4.2) to solved FOCP based on Rietz’s direct method to approximated \( u(t) \) and \( x(t) \). Suppose that \( x(t) = \sum_{i=0}^{n-1} c_i L_i(t) = C^T L \), then we can solve this problem and from solving the tranversility condition we can get \( t_f \) such that for different \( t_f \), like \( t_f = 1.11418 \) if \( \alpha = 0.5 \) and \( t_f = 1.1175 \) if \( \alpha = 0.7 \). The approximate solution of control, state and error functions have been calculated. The figures (5)–(12) are illustrative approximation of control and state functions with a lower error for different values of \( \alpha \).

Example 5.5 To solve an example that was introduced in [26]. Consider the following FOCP:

\[
\min J[u] = \int_0^1 [(x(t) - t^2)^2 + (u(t) + t^4 - 2.0795t^{0.9})^2] dt
\]

The dynamic constraints

\[
D^{1.1}x(t) = t^2x(t) + u(t), \ x(0) = x'(0) = 0
\]

The exact solution \( x(t) = t^2 \) and \( u(t) = -t^4 + 2.0795t^{0.9} \). By using the method has been presented in subsection (4.2) to solved FOCP based on Rietz direct method to approximate \( u(t) \) and \( x(t) \). Where \( x(t) = \sum_{i=0}^{n-1} c_i L_i(t) = C^T L \) and \( u(t) = \sum_{i=0}^{n-1} \eta_i L_{i,0}(t) = R^T L \) The approximate solution of control, state and error functions have been calculated. The figures (13)–(16) are illustrative approximation of control and state functions with a lower error.

Figure (13) and (14) to analytic and various approximation for \( u(t) \) and \( x(t) \) respectively to example five where \( \alpha = 1.1 \).
Figure (15) and (16) to showing absolute error function where approximation \( u(t) \) and \( x(t) \) respectively to example five where \( n = 14 \).

6. Conclusion

In this work, have been proposed a new numerical technique based on the generalized Laguerre polynomials to approximate the solution of FOCPs. The operational matrix of fractional derivatives and the properties of Laguerre polynomials are used to reduce FOCP into the solution of the system of algebraic equations. The fractional derivatives are described in the Caputo sense. Many theorems have been used to approximate the Caputo fractional derivative by using the operation matrix. Special attention has been given to study the error analysis.

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