BIDYYNAMICAL POISSON GROUPOIDS

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ABSTRACT. We give relations between dynamical Poisson groupoids, classical dynamical Yang–Baxter equations and Lie quasi-bialgebras. We show that there is a correspondence between the class of bidynamical Lie quasi-bialgebras and the class of bidynamical Poisson groupoids. We give an explicit, analytical and canonical equivariant solution of the classical dynamical Yang–Baxter equation (classical dynamical ℓ-matrices) when there exists a reductive decomposition \( g = l \oplus m \), and show that any other equivariant solution is formally gauge equivalent to the canonical one. We also describe the dual of the associated Poisson groupoid, and obtain the characterization that a dynamical Poisson groupoid has a dynamical dual if and only if there exists a reductive decomposition \( g = l \oplus m \).

INTRODUCTION

The Classical Dynamical Yang–Baxter equation (CDYBE) is an important differential equation in mathematical physics. It was first introduced by Felder [7] in the context of conformal field theory, appearing as a dynamical analogue of the Classical Yang–Baxter equation (CYBE), which plays a central role in the theory of integrable systems; the geometric meaning of (CYBE) was given by Drinfel’d, and gives rise to the theory of Poisson–Lie groups. The geometric meaning of (equivariant solutions of) the (CDYBE) was given by Etingof and Varchenko [6], and is a groupoidal analogue of that of (CYBE): dynamical Poisson groupoids. In the present paper, we explicitly describe dynamical Poisson groupoids with base space containing \( 0 \) which have a dynamical dual — bidynamical Poisson groupoids.

Let \( G \) be a connected, simply connected Lie group, \( L \subset G \) a Lie subgroup, and let \( l = \text{Lie}(L) \) and \( g = \text{Lie}(G) \) be their respective Lie algebras; we denote by \( i : l \to g \) the corresponding inclusion. For an \( \text{Ad}^*_L \)-equivariant subset \( U \subset l^* \) we consider the trivial Lie groupoid \( G = U \times G \times U \) with base \( U \), with the product given by \( (p,x,q)(q,y,r) = (p,xy,r) \). Let \( r : U \to \bigwedge^2 g \) be a differentiable map (we identify \( \bigwedge^2 g \) with the skew-symmetric maps from \( g^* \) to \( g \)). In [5], extending Drinfel’d’s classical work, P. Etingof and A. Varchenko, showed that the following bracket on \( C^\infty(G) \):

\[
\{f,g\}_{(p,x,q)} = \langle p, [\delta f, \delta g] \rangle - \langle q, [\delta' f, \delta' g] \rangle - \langle Dg, i\delta f \rangle - \langle D'g, i\delta' f \rangle + \langle Df, i\delta g \rangle + \langle D'f, i\delta' g \rangle - \langle Df, r_p Dg \rangle + \langle D'f, r_q D'g \rangle
\]

is a Poisson bracket if and only if \( r \) is a classical dynamical \( r \)-matrix, in which case \( G \) turns out to be a Poisson groupoid — which they call dynamical (see Section 1 for the notations). We recall that a classical dynamical \( r \)-matrix is an \( l \)-equivariant solution of the classical dynamical Yang–Baxter equation:

\[
\bigotimes_{\xi,\eta,\zeta} \left( \langle \xi, dp r(i^* \xi) \eta \rangle - \langle \xi, [r_p \xi, r_p \eta] \rangle \right) = \langle \xi \otimes \eta \otimes \zeta, \varphi \rangle,
\]

where \( \varphi \) is any element in \( \left( \bigwedge^3 g \right)^g \). It also appeared that the smallest Poisson submanifold of \( G \) containing the unit of \( G \) is not the unit itself, but the image of the hamiltonian unit \( L = U \times L \)
by the groupoid morphism $I(p,h) = (\text{Ad}_{h^{-1}}^* p, h, p)$. Explicit dynamical $r$-matrices were given and classified when $\mathfrak{g}$ is a complex semi-simple Lie algebra and $I$ a Cartan subalgebra. In [2], A. Alekseev and E. Meinrenken exhibited an analytic classical dynamical $r$-matrix in the case where $\mathfrak{g} = \mathfrak{l}$ is a quadratic Lie algebra. In [3], P. Etingof and O. Schiffmann proved the existence of (formal) classical dynamical $r$-matrices and gave a complete description of the moduli space of classical dynamical $r$-matrices in the case where there exists a reductive decomposition $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{m}$ and for an element $\varphi = \langle \Omega, \Omega \rangle$ with $\Omega \in \langle S^2 \mathfrak{l} \rangle^0$ such that $\Omega \in \mathfrak{l} \oplus \mathfrak{l} \oplus \mathfrak{m} \oplus \mathfrak{m}$.

In [5], L.C. Li and S. Parmentier gave the form of all Poisson groupoid structures on the trivial Lie groupoid $\mathcal{G} = U \times G \times U$ which admit an inclusion of the hamiltonian unit $I$. But in the present paper, we only consider the following special form of Poisson brackets on $\mathcal{G}$ which correspond to the inclusion $I(p,h) = (\text{Ad}_{h^{-1}}^* p, h, p)$:

$$\{f,g\}_{(p,x,g)} = \langle p, [\delta f, \delta g] \rangle - \langle q, [\delta' f, \delta' g] \rangle - \langle Dg, i\delta f \rangle - \langle D'g, i\delta' f \rangle + \langle Df, i\delta g \rangle + \langle D'f, i\delta' g \rangle - \langle Df, l_p Dg \rangle + \langle Df, \pi_x Dg \rangle + \langle D'f, l_q D'g \rangle,$$

where $\pi: G \to \bigwedge^2 \mathfrak{g}$ is a Lie group 1-cocycle, and $l: U \to \bigwedge^2 \mathfrak{g}$. It turns out that Jacobi’s identity is equivalent to the following two conditions:

- There exists an element $\varphi \in \bigwedge^3 \mathfrak{g}$ such that for all $\xi, \eta, \zeta \in \mathfrak{g}^*$ the following identity holds:

$$\langle \xi \otimes \eta \otimes \zeta, \text{ad}_x^{(3)} \varphi \rangle = \bigwedge_{\langle \xi, \eta, \zeta \rangle} \langle \xi, \varphi \varpi_{\eta, \eta} \zeta \rangle,$$

and for all $p \in U$ and $\xi, \eta, \zeta \in \mathfrak{g}^*$ the following identity holds:

$$\bigwedge_{\langle \xi, \eta, \zeta \rangle} \left( \langle \zeta, d_p (i^* \xi) \eta \rangle - \langle \zeta, [l_p \xi, l_p \eta] \rangle - \langle \zeta, \varphi \varpi_{l_p \eta, l_p \eta} \rangle \right) = \langle \xi \otimes \eta \otimes \zeta, \varphi \rangle;$$

- for all $p \in U$, $z \in \mathfrak{l}$, and $\xi \in \mathfrak{g}^*$ the following identity holds:

$$d_p i^* \text{ad}_{l_p^* \xi} \xi + \varphi \varpi_{l_p \xi, l_p \xi} + \text{ad}_{l_p^* i^* \xi} l_p \xi + l_p \text{ad}_{l_p^* i^* \xi} \xi = 0.$$

Such a groupoid $\mathcal{G}$ is called dynamical, and a map $l$ satisfying the two previous conditions will be referred to as a classical dynamical $\ell$-matrix.

It is shown in [11] that the previous conditions have solutions $l: U \to \bigwedge^2 \mathfrak{g}$ only if the quadruple $\mathcal{G} = (\mathfrak{g}, [\cdot, \cdot], \varpi, \varphi)$ is a Lie quasi-bialgebra. From the form of the dual of the Lie algebroid, it is observed that a necessary condition for the groupoid $\mathcal{G}$ to have a dynamical dual is that the Lie algebra $\mathfrak{g}$ admits a reductive decomposition $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{m}$. Under this assumption and additional natural (but restrictive) compatibility conditions between $\mathcal{G}$ and the reductive decomposition, all formal solutions $l$ of the above conditions are given, via an explicit and analytic representative, and the action of a (formal) gauge group. The duality for the groupoid $\mathcal{G}$ is also explicitly described.

The goal of the present paper is to solve the problem when no compatibility condition between the reductive decomposition of $\mathcal{G}$ and the Lie quasi-bialgebra is assumed. In short Section 1 we recall some notations and set the problem. Section 2 is divided into three subsections: first we show that every classical dynamical $\ell$-matrix is gauge equivalent to one satisfying $l_p sp = 0$, and that there is at most one formal classical dynamical $\ell$-matrix satisfying this condition, which we call canonical. Second we find an explicit formula for the canonical $\ell$-matrix, which shows that it is analytic (Theorem 2.11). Then, we define bidynamical objects and morphisms on both the Lie quasi-bialgebra level and groupoid level, and show that there is a functorial correspondence between these bidynamical objects. In Section 3 we give an explicit trivialization isomorphism which enables us to describe explicitly the Poisson groupoid dual to $\mathcal{G}$. The duality for groupoids
induces a duality for the class of bidynamical Lie quasi-bialgebras, which is also described. We obtain the following characterization which was announced in [11]: a dynamical Poisson groupoid (with $0 \in U$) is bidynamical if and only if $\mathfrak{g}$ admits a reductive decomposition $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{m}$. In Section 4 we give a link between the two canonical dynamical $\ell$-matrices associated to dynamical Poisson groupoids in duality, which shows that both $\ell$-matrices have the same domain of analyticity.

The problem will be adapted to the case where $0$ does not belong to $U$ in a forthcoming publication [13].

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1. Dynamical Poisson groupoids at 0

Let $i: \mathfrak{l} \to \mathfrak{g}$ be an inclusion of the Lie algebra $\mathfrak{l}$ into the Lie algebra $\mathfrak{g}$. Let $U$ be an $\text{Ad}^*_L$-invariant, contractible open subset in $\mathfrak{l}^*$ containing $0$ and $\mathcal{G} = (\mathfrak{g}, [\cdot, \cdot], \varpi, \varphi)$ a Lie quasi-bialgebra (see [11] for information on Lie quasi-bialgebras where we use the same conventions and notations as in the present paper).

By definition, a classical dynamical $\ell$-matrix on $U$ associated with $\mathcal{G}$ is a map $\ell: U \to \bigwedge^2 \mathfrak{g}$ which is a solution of the following two equations (we identify $\bigwedge^2 \mathfrak{g}$ with the skew-symmetric maps from $\mathfrak{g}^* \to \mathfrak{g}$):

\[
\bigwedge^2_{(\xi, \eta, \zeta)} \left( \langle \xi, d_p l(i^* \xi) \eta \rangle - \langle \zeta, l_p [\xi, l_p \eta] \rangle - \langle \xi, \varpi l_p \xi \eta \rangle \right) = \langle \xi \otimes \eta \otimes \zeta, \varphi \rangle \quad (1.1)
\]

\[
d_p l(\text{ad}^*_z p) + \varpi_z + \text{ad}_z l_p + l_p \text{ad}_z^* = 0. \quad (1.2)
\]

The set of classical dynamical $\ell$-matrices on $U$ associated with $\mathcal{G}$ is denoted by $\text{Dynl}(U, \mathcal{G})$. Classical dynamical $r$-matrices are just classical dynamical $\ell$-matrices with $\varphi \equiv 0 \bmod \mathfrak{l}$. Let $\mathbb{D}$ be the formal neighborhood of $0$ in $\mathfrak{l}^*$. We also consider classical dynamical $\ell$-matrices which are formal around $0 \in \mathfrak{l}^*$, the set of which is denoted by $\text{Dynl}(\mathbb{D}, \mathcal{G})$.

For all $t \in \bigwedge^2 \mathfrak{g}$, we denote by $\mathcal{G}^t$ the twist of the Lie quasi-bialgebra $\mathcal{G}$ via $t$. The following result is proved in [11]:

**Proposition 1.1.** For all $t \in \bigwedge^2 \mathfrak{g}$,

$$\text{Dynl}(U, \mathcal{G}^t) = \text{Dynl}(U, \mathcal{G}) - t.$$  

This proposition allows us to be only concerned with classical dynamical $\ell$-matrices which vanish at $0$, which form a set denoted by $\text{Dynl}_0(U, \mathcal{G})$.

If we want $\text{Dynl}_0(U, \mathcal{G})$ (or $\text{Dynl}_0(\mathbb{D}, \mathcal{G})$) to be non empty, then we must have:

$$\varpi_0 = 0 \quad \text{and} \quad \varphi \equiv 0 \bmod \mathfrak{l} \quad (1.3)$$

(evaluate equations (1.1) and (1.2) at $0$).

Classical dynamical $\ell$-matrices are related to Poisson groupoids in the following way: Let $\mathcal{G}$ be a connected Lie group with Lie algebra $\mathfrak{g}$. For any point $x \in \mathcal{G}$ and any function $f \in C^\infty(\mathcal{G})$, we denote by $D_x f \in \mathfrak{g}^*$ and $D'_x f \in \mathfrak{g}^*$ the right and left derivatives at $x$:

\[
\langle D_x f, u \rangle = \left. \frac{d}{dt} \right|_{t=0} f(e^{t u} x) \quad (1.4)
\]

\[
\langle D'_x f, u \rangle = \left. \frac{d}{dt} \right|_{t=0} f(x e^{t u}) \quad (1.5)
\]
for all \( u \in \mathfrak{g} \). Let \( L \) be a connected Lie subgroup of \( G \) with Lie algebra \( \mathfrak{l} \), and \( U \) an \( \text{Ad}_p^* \)-invariant open subset in \( \mathfrak{l}^* \) containing 0. We will denote the inclusion by \( : \mathfrak{l} \rightarrow \mathfrak{g} \). Consider the trivial Lie groupoid \( \mathcal{G} = U \times G \times U \) with multiplication:

\[
(p, x, q)(q, y, r) = (p, xy, r)
\]

We say that a multiplicative Poisson bracket on \( \mathcal{G} \) is dynamical if it is of the form:

\[
\{f, g\}(p, x, q) = \langle p, [\delta f, \delta g]_l \rangle - \langle q, [\delta f, \delta g]_l \rangle - \langle Dg, i\delta f \rangle - \langle D'g, i\delta f \rangle + \langle Df, i\delta g \rangle + \langle D'f, i\delta g \rangle - \langle Df, l_p Dg \rangle - \langle Df, \pi_x Dg \rangle + \langle D'f, l_q D'g \rangle
\]

where \( l: U \rightarrow \bigwedge^2 \mathfrak{g} \) is a smooth map, and \( \pi: \mathfrak{g} \rightarrow \bigwedge^2 \mathfrak{g} \) is a group 1-cocycle. In this equation, \( \delta f \in \mathfrak{l} \) and \( \delta' f \in \mathfrak{l} \) denote the derivatives of \( f \) with respect to the first and second \( U \) factors, \( Df \) and \( D'f \) denote the right and left derivatives of \( f \) with respect to the \( G \) factor, and all derivatives are evaluated at \((p, x, q)\). Denote by \( \varpi = T_1 \pi: \mathfrak{g} \rightarrow \bigwedge^3 \mathfrak{g} \) the Lie algebra 1-cocycle associated with \( \pi \). It can be shown (see [8]) that the bracket [17] is Poisson (i.e., satisfies Jacobi’s identity) if and only if \( l \in \text{Dyn}(U, \mathcal{G}) \) with \( \mathcal{G} = (\mathfrak{g}, \mathcal{C}, \varpi, \varphi) \) for some \( \varphi \in \bigwedge^3 \mathfrak{g} \) such that \( \mathcal{G} \) is a Lie quasi-bialgebra.

There is a notion of duality for Poisson groupoids which extends that of Poisson–Lie groups (see [7], [14], and also [8] [11] for our more concrete case). It was already observed in [11] (see [11]) that the Lie algebra of the vertex group \( G^0_\mathfrak{p} \) is (isomorphic to) the lagrangian Lie subalgebra \( \mathfrak{g}_0^0 = \mathfrak{l} \oplus \mathfrak{l}^\perp \) of the double \( \mathfrak{d} \) of the Lie quasi-bialgebra \( \mathcal{G} \), which is a reductive decomposition over \( \mathfrak{l} \) (we recall that a reductive decomposition over \( \mathfrak{b} \) of a Lie algebra \( \mathfrak{a} \) is a vector space decomposition \( \mathfrak{a} = \mathfrak{b} \oplus \mathfrak{c} \) such that \( \mathfrak{b} \) is a Lie subalgebra of \( \mathfrak{a} \) and \( [\mathfrak{b}, \mathfrak{c}] \subset \mathfrak{c} \)). Thus, a necessary condition for the dual to be (a covering of) a dynamical Poisson groupoid, is that \( \mathfrak{g} \) admits a reductive decomposition \( \mathfrak{g} = \mathfrak{l} \oplus \mathfrak{m} \). In this case, it was shown in [11], under additional natural but restrictive compatibility conditions on \( \varphi \) and \( \varpi \), that the dual of \( \mathcal{G} \) is still (a covering of) a dynamical Poisson groupoid.

This result is proved in this paper, without the additional compatibility assumptions on \( \varphi \) and \( \varpi \).

It is shown in [8], using a theorem of Mackenzie [2] that the vertex algebras \( \mathfrak{g}_p^* \) defined as:

\[
\mathfrak{g}_p^* = \{i(z) + \xi \in i(\mathfrak{l}) \oplus \mathfrak{g}^* \mid i^* \xi = \text{ad}_{p^*}^* \xi \} \subset \mathfrak{g} \oplus \mathfrak{g}^*
\]

with the Lie bracket:

\[
[i(z) + \xi, i(z') + \xi'] = (i([z, z'])) + \varpi_{i(z)} \xi' + \text{ad}_{i(z)} l_p \xi' + l_p \text{ad}_{i(z)}^* \xi' - \varpi_{i(z')} \xi - \text{ad}_{i(z')} l_p \xi - l_p \text{ad}_{i(z')}^* \xi

\]

\[
+ [l_p \xi, l_p \xi'] + l_p \text{ad}_{i(z')}^* \xi - l_p \text{ad}_{i(z)}^* \xi

\]

are all isomorphic when \( p \) ranges over \( U \). Isomorphisms between the \( \mathfrak{g}_p^* \)'s are not provided by Mackenzie’s theorem, and are not canonical. In the present paper, we construct such an isomorphism as part of the trivialization. It was observed in [11] that the Lie algebra \( \mathfrak{g}_p^* \) is a lagrangian Lie subalgebra in \( \mathfrak{d}_p^0 \), the canonical double of the twisted Lie quasi-bialgebra \( G^{lp} \).

It is natural to consider the following definition:

**Definition 1.2.** A dynamical Poisson groupoid \( \mathcal{G} \) over \( U \) is said to be bidynamical if its dual is (a covering of) a dynamical Poisson groupoid.
Remark 1.3. It is shown in [11] that if the Lie quasi-bialgebra $G$ is compatible with the reductive decomposition $g = l \oplus m$ in the sense of Example 2.12, the associated dynamical Poisson groupoid is bidynamical.

This paper is devoted to the study of bidynamical Poisson groupoids (at 0). Thus, from now on, we fix a reductive decomposition $g = l \oplus m$ of a Lie algebra $g$, and a Lie quasi-bialgebra $G = (g, [\cdot, \cdot], \varpi, \varphi)$ such that $\varpi_l = 0$ and $\varphi \equiv 0 \mod l$. We also denote by $p_l$ (resp. $p_m$) the projection on $l$ (resp. $m$) along $m$ (resp. $l$), and by $s: l^* \rightarrow g^*$ the adjoint of $p_l$. Notice that $s$ is $l$-equivariant and that its image is $m^!$.

2. Canonical dynamical $\ell$-matrices

For a vector space $E$ and a formal map $f: \mathbb{D} \rightarrow E$, we denote either by $[f]_k$ or $f^k$ its homogeneous term of degree $k$ (the notation $f^k$ is not to be confused with the $k$-th power of $f$).

For a dynamical $\ell$-matrix $l \in \text{Dyn}(\mathbb{D}, G)$ and an $l$-equivariant map $\sigma \in \text{Map}(\mathbb{D}, G)^l$, we denote by $l^\sigma$ the gauge transformation of the map $l$ by $\sigma$ (see [11]):

$$l^\sigma_p = \text{Ad}_{\sigma_p} l_p \text{Ad}^*_p + \theta^\sigma_p + \pi_p,$$

where $\theta^\sigma$ is defined as:

$$\theta^\sigma_p = r_{\sigma_p}^{-1} (T_p \sigma)^* i^* \text{Ad}^*_p - (T_p \sigma)^* r_{\sigma_p}^{-1},$$

and where $\pi: G \rightarrow \bigwedge^2 g$ is the Lie group cocycle integrating $\varpi$. This action is a left action:

$$(l^\sigma)^\sigma' = l^{\sigma'}.$$  

2.1. Canonical dynamical $\ell$-matrices. The following proposition shows that there always exists some distinguished representative in each (formal) gauge orbit of dynamical $\ell$-matrices.

We denote by $\text{Map}(\mathbb{D}, G)$ the group of formal maps from $\mathbb{D}$ to $G$ with pointwise multiplication, by $\text{Map}_0(\mathbb{D}, G)$ the group of formal maps $\sigma$ such that $\sigma_0 = 1$ and by $\text{Map}_0^{(2)}(\mathbb{D}, G)$ the group of formal maps $\sigma$ such that $\sigma_0 = 1$ and $T_0 \sigma = 0$.

Proposition 2.1. For all $l \in \text{Dyn}_0(\mathbb{D}, G)$, there exists a gauge transformation associated with some $\sigma \in \text{Map}_0^{(2)}(\mathbb{D}, G)^l$ such that $l^\sigma_p sp = 0$.

Proof. By induction: Let $k \geq 1$ and assume that $[l_p]_{k-1} sp = 0$. Now, if $\Sigma: \mathbb{D} \rightarrow g$ is an $l$-equivariant homogeneous map of degree $k+1$, then setting $\sigma = e^{\Sigma}$ yields $[l^\sigma]_k = [l]_k + d \Sigma i^* - (d \Sigma)^*$, so that $[l^\sigma]_k sp = [l_p]_k sp + d_p \Sigma(p) - (d_p \Sigma)^*(sp)$. Now, define $\Sigma$ as:

$$\Sigma_p = \frac{1}{k+1} p_m [l_p]_k sp - \frac{1}{k+2} p_l [l_p]_k sp.$$  

Notice that $\Sigma$ is an $l$-equivariant homogeneous map of degree $k + 1$. We have

$$d_p \Sigma(\alpha) = -\left( \frac{1}{k+1} p_m + \frac{1}{k+2} p_l \right) ([l_p]_k sp + d_p [l]_k(\alpha)sp),$$

thus

$$d_p \Sigma(p) = -p_m [l_p]_k sp - \frac{k + 1}{k + 2} [l_p]_k sp.$$
and
\[(d_p \Sigma)^* sp = \frac{1}{k + 2} p_l[l_p] k sp.\]
Hence, \(d_p \Sigma(p) - (d_p \Sigma)^* sp = -[l_p] k sp\) and \([l^p]_{\leq k} sp = 0\). The proof follows by induction. \(\square\)

We will make use of the following notations: for all \(\xi \in g^*\), we define:
\[\xi'_p = p_\mathfrak{g} \cdot \text{ad}_{sp} l_p \xi, \quad \xi''_p = p_\mathfrak{g} \cdot \text{ad}_{sp} \xi, \quad \tilde{\xi}_p = \xi'_p + \xi''_p.\]
The following proposition shows that such a representative is necessarily unique.

**Proposition 2.2.** There exists at most one (formal) dynamical \(\ell\)-matrix \(l\) satisfying \(l_p sp = 0\) and \(l_0 = 0\). It is the unique (formal) solution \(l\) satisfying \(l_0 = 0\) of the following differential equation:
\[d_p l(p) \xi = \text{ad}_{sp}(l_p \xi + \xi) - (l_p \xi'_p + \xi'_p) - p t_p \xi - l_p s t^* \xi\]
for all \(\xi \in g^*\).

**Proof.** Assume that \(l : \mathbb{D} \to \bigwedge^2 g\) is a formal map satisfying \(l_p sp = 0\) and the generalized Yang–Baxter equation (1.1).

For all \(\xi, \eta \in \mathfrak{l}^\perp\), we obtain from equation (1.1):
\[\langle \eta, d_p l(p) \xi \rangle = (\eta, [l_p \xi, l_p \eta + \eta])_\mathfrak{g} + (\eta, [l_p \eta, sp])_\mathfrak{g} + (\eta, \text{ad}_{sp} \xi)_\mathfrak{g}\]
\[= (\eta, -l_p \xi'_p + \text{ad}_{sp} l_p \xi + \text{ad}_{sp} \xi)_\mathfrak{g}.\]
If we define \(L : \mathbb{D} \to \mathcal{L}(g^*, g)\) as:
\[L_p \xi = p_\mathfrak{g} \text{ad}_{sp} \xi + \text{ad}_{sp} l_p \xi - l_p \xi'_p\]
for all \(\xi \in g^*\), then equation (2.7) reads:
\[p_m[l_p] k \xi = \frac{1}{k} p_m[L_p] k \xi\]
for all \(\xi \in \mathfrak{l}^\perp\) and \(k \geq 1\).

For all \(\xi \in \mathfrak{l}^\perp\) and \(\alpha \in \mathfrak{l}^*\), we obtain from equation (1.1):
\[\langle \xi, d_p l(\alpha) sp - d_p l(p) s \alpha \rangle = \langle \xi, L_p s \alpha \rangle.\]
Since \(d_p l(\alpha) sp = -l_p s \alpha\) for all \(\alpha \in \mathfrak{l}^*\), equation (2.10) reads:
\[p_m[l_p] k s \alpha = \frac{1}{k + 1} p_m[L_p] k s \alpha\]
for all \(\alpha \in \mathfrak{l}^*\) and \(k \geq 1\). Similarly, one obtains:
\[p_1[l_p] k \xi = \frac{1}{k + 1} p_1[L_p] k \xi\]
\[p_1[l_p] k s \alpha = \frac{1}{k + 2} p_1[L_p] k s \alpha\]
for all \(\xi \in \mathfrak{l}^\perp, \alpha \in \mathfrak{l}^*\) and \(k \geq 1\). Thus, \([l_p] k\) is uniquely determined by the \([l_p] j\)'s, for \(j \leq k - 1\). It is easily shown that \(l\) satisfies equation (2.6). \(\square\)

In particular, one obtains the following corollary:

**Corollary 2.3.** The reduced moduli space \(\text{Dynl}_0(\mathbb{D}, G)/\text{Map}_0^{(2)}(\mathbb{D}, G)^l\) consists of at most one point.
Proof. Let \( l \) and \( l' \) in \( \text{Dyn}_0(\mathbb{D}, G) \). Then, from Proposition 2.1 there are two maps \( \sigma \) and \( \sigma' \) in \( \text{Map}^{(2)}_0(\mathbb{D}, G) \) such that \( l_{sp} \sigma = (l')_{sp} \sigma' = 0 \). Thus, \( l_{sp} \sigma \) and \( (l')_{sp} \sigma' \) are two solutions of equation (2.6) which vanish at zero. From the uniqueness of such solutions, we must have \( l_{sp} \sigma = (l')_{sp} \sigma' \). \( \square \)

Conversely, we show that a solution of (2.6) is a (formal) dynamical \( \ell \)-matrix:

**Theorem 2.4.** The (formal) solution \( l \) of (2.6) with initial condition \( l_0 = 0 \) is the unique (formal) dynamical \( \ell \)-matrix satisfying \( l_0 = 0 \) and \( l_{sp} = 0 \).

Before proving Theorem 2.4, we introduce some notations and state three lemmas:

**Lemma 2.5.** The (formal) solution \( l \) of equation (2.6) such that \( l_0 = 0 \) takes its values in \( \bigwedge^2 g \), and satisfies equation (1.2).

**Proof.** It is a direct consequence of the skew-symmetry of the map \( \text{ad}_{sp} \), and of the \( l \)-equivariance of the maps \( s \) and \( p \).

**Lemma 2.6.** For all \( \xi \in g^* \), the element \( \tilde{\xi}_{sp} + p s_{sp} \xi \) lies in \( g^* \).

**Proof.** For all \( z \in l, \langle \tilde{\xi}_{sp} + p s_{sp} \xi, z \rangle = (\text{ad}_{sp}(l_{sp} \xi + \xi), z) = (\text{ad}_{sp} l_{sp} \xi, z) = (\text{ad}_{sp} p s_{sp} \xi, z) \), since \( \varpi_l = 0 \). Thus, \( i^* \tilde{\xi}_{sp} = \text{ad}^*_{s_{sp}} l_{sp} \xi \), and Lemma 2.6 is proved. \( \square \)

We denote by \( \text{CDYB} \) the (generalized) classical dynamical Yang–Baxter operator: For any map \( l \in \text{Map}(\mathbb{D}, \bigwedge^2 g) \), \( \text{CDYB}(l) : \mathbb{D} \rightarrow \bigwedge^3 g \) is given by

\[
\langle \xi \otimes \eta \otimes \zeta, \text{CDYB}_p(l) \rangle = \bigwedge^3 (\xi, d_p l(i^* \xi) \eta - [l_{sp}\xi, l_{sp} \eta + \eta]) \quad (2.14)
\]

for all \( \xi, \eta, \zeta \in g^* \), where, as usual, \( \langle \cdot, \cdot \rangle \) is the canonical bilinear form on the double \( \mathfrak{d} \) of \( \mathcal{G} \). Notice that equation (1.4) is equivalent to \( \text{CDYB}(l) = \varphi \).

**Lemma 2.7.** For all \( \xi, \eta, \zeta \in g^* \), we have

\[
d_p \langle \xi \otimes \eta \otimes \zeta, \text{CDYB}_p(l) \rangle(p) = - \bigwedge^3 (\xi \otimes (s i^* \eta + \tilde{\eta} p) \otimes \zeta, \text{CDYB}_p(l) - \varphi). \quad (2.15)
\]

**Proof.** For all \( \xi, \eta, \zeta \in g^* \), using equations (2.6) and (1.2), as well as:

\[
\bigwedge^3 \langle \xi_{sp}' + [l_{sp} \eta, l_{sp} \zeta] \rangle = 0 \quad (2.16)
\]

\[
\bigwedge^3 \langle \xi_{sp}'' + [\eta, \zeta] \rangle = 0 \quad (2.17)
\]
which are consequences of Jacobi's identity on $\mathfrak{d}$, one obtains:

\begin{align*}
\mathcal{D}_p \circ (\xi, [l_p \eta, l_p \zeta + \xi])_\mathfrak{d}(p) &= \mathcal{D}_p \circ (\xi, [l_p \eta, l_p \zeta + \xi] + [l_p \eta, \mathcal{D}_p (l(p)) \zeta] + [p \eta, \mathcal{D}_p (l(p)) \zeta])_\mathfrak{d} \\
&= \mathcal{D}_p \circ (\xi, [\text{ad}_{sp} l_p \eta, l_p \zeta + \xi] + [\text{ad}_{sp} \eta, l_p \zeta] - [l_p \tilde{\eta}_p + \tilde{\eta}_p, l_p \zeta + \xi] - [p \mathcal{D}_p \eta, \zeta])_\mathfrak{d} \\
&\quad - [p \eta \mathcal{D}_p \eta, l_p \zeta] - [l_p s^i \eta, l_p \zeta + \xi] + [l_p \eta, \text{ad}_{sp} (l_p \zeta + \xi) - (l_p \tilde{\zeta}_p + \tilde{\zeta}_p) - p \mathcal{D}_p \zeta - l_p s^i \zeta)_\mathfrak{d} \\
&= \mathcal{D}_p \circ (\xi, [\text{ad}_{sp} [l_p \eta, l_p \zeta + \xi] + [\text{ad}_{sp} \eta, l_p \zeta] - [l_p \tilde{\eta}_p, l_p \zeta + \xi] - [\tilde{\eta}_p, l_p \zeta + \xi])_\mathfrak{d} \\
&\quad + d_p l(i^* \tilde{\eta}_p) \zeta - [l_p s^i \eta, l_p \zeta + \xi] - [l_p \eta, l_p \tilde{\zeta}_p + \tilde{\zeta}_p] - [l_p \eta, l_p s^i \zeta])_\mathfrak{d} \\
&= \mathcal{D}_p \circ (\xi, d_p l(i^* \eta) \zeta - (\xi, d_p l(i^* \eta))_\mathfrak{d} - (\xi, d_p l(i^* \eta) \tilde{\eta}_p) \\
&\quad - (\xi, [l_p \xi, l_p s^i \eta] - (\xi, [l_p s^i \eta, l_p \zeta + \xi])_\mathfrak{d})_\mathfrak{d}.
\end{align*}

Now,

\begin{align*}
(\xi, l_p \zeta) &= \sum_{k \geq 0} ([\xi, l_p \zeta])_k = (\xi, l_0 \zeta) + \sum_{k \geq 1} \frac{1}{k} ([\xi, d_p (l(p)) \zeta])_k \\
&= (\xi, l_0 \zeta) + \sum_{k \geq 1} \frac{1}{k} \left( (\xi, \text{ad}_{sp} (l_p \zeta + \xi) - l_p \tilde{\zeta}_p - p \mathcal{D}_p \zeta - l_p s^i \zeta) \right)_k
\end{align*}

thus,

\begin{align*}
\mathcal{D}_p \circ (\xi, d_p l(i^* \eta) \zeta) &= \mathcal{D}_p \circ \sum_{k \geq 1} \frac{1}{k} \left( [\xi, \text{ad}_{sp} (l_p \zeta + \xi) + \text{ad}_{sp} d_p l(i^* \eta) \zeta - d_p l(i^* \eta) \tilde{\zeta}_p - l_p g^* \text{ad}_{sp} \eta \zeta]ight)_k \\
&\quad - l_p g^* \text{ad}_{sp} d_p l(i^* \eta) \zeta - l_p g^* \text{ad}_{sp} \eta \zeta - p l d_p l(i^* \eta) \zeta - d_p l(i^* \eta) s^i \zeta) \right]_k \\
&= \mathcal{D}_p \circ \sum_{k \geq 1} \frac{1}{k} \left( [s^i \eta, l_p \zeta, l_p \zeta + \xi] + [\xi, \xi] \right)_\mathfrak{d} - (\tilde{\zeta}_p, d_p l(i^* \eta) \zeta) - (\xi, d_p l(i^* \eta) \tilde{\eta}_p) \\
&\quad + (\tilde{\zeta}_p, d_p l(i^* \eta) \zeta) - (\xi, d_p l(i^* \eta) \tilde{\eta}_p) - (\xi, d_p l(i^* \eta) s^i \zeta) \right]_k
\end{align*}

and we obtain

\begin{align*}
\mathcal{D}_p \circ (\xi, d_p l(i^* \eta) \zeta)(p) &= \mathcal{D}_p \circ (\xi, d_p l(i^* \eta) \zeta)(p) \\
&= \mathcal{D}_p \circ (\xi, d_p l(i^* \eta) \zeta) + (\xi, d_p l(i^* \eta) \zeta) - (\xi, d_p l(i^* \eta) \zeta)
\end{align*}

Assembling equations (2.18) and (2.19) proves Lemma 2.7.

We can now prove Theorem 2.4.
Proof of Theorem 2.4. The map $l$ is $\ell$-equivariant and takes its values in $\wedge^2 g$, by Lemma 2.8. We show that $l$ satisfies the generalized Yang–Baxter equation (1.1) by induction:

A computation shows that it is satified at order 0, i.e., $CDYB_0(l) = \varphi$. Indeed, if $\xi, \eta, \zeta \in l^\perp$, then

$$\bigwedge_{(\xi, \eta, \zeta)} (\xi, [\eta_0, l_0 \zeta + \zeta])_0 = 0 = \langle \xi \otimes \eta \otimes \zeta, \varphi \rangle$$

(2.20)
since $\varphi \equiv 0 \mod \ell$. Thus, $p_m^{(3)} CDYB_0(l) = p_m^{(3)} \varphi$. Now, from Lemma 2.7, since $\alpha, \beta, \gamma \in l^\perp$, we obtain:

$$0 = \langle s\alpha \otimes \xi \otimes \eta, CDYB_0(l) - \varphi \rangle$$

(2.21)

$$0 = 2\langle s\alpha \otimes s\beta \otimes \xi, CDYB_0(l) - \varphi \rangle$$

(2.22)

$$0 = 3\langle s\alpha \otimes s\beta \otimes s\gamma, CDYB_0(l) - \varphi \rangle.$$  

(2.23)

Thus, $CDYB_0(l) = \varphi$.

So let $k \geq 0$ and assume that $[CDYB_p(l) - \varphi]_{\leq k} = 0$. From Lemma 2.7, since $l$ satisfies equation (2.6), we have:

$$(k + 1)[[\xi \otimes \eta \otimes \zeta, CDYB_p(l)] + 1] - [\xi \otimes (s\alpha \otimes \eta + s\beta \otimes \varphi)]_{\zeta} + 1)_{k+1} = 0.$$  

(2.24)

for all $\xi, \eta, \zeta \in g^\ast$. Now, since $\bar{\eta}_l = 0$, equation (2.24) reads:

$$(k + 1)[[\xi \otimes \eta \otimes \zeta, CDYB_p(l)] + 1] = -\bigotimes_{(\xi, \eta, \zeta)} [\xi \otimes (s\alpha \otimes \eta + s\beta \otimes \varphi)]_{\zeta} + 1)_{k+1}$$

(2.25)

for all $\xi, \eta, \zeta \in g^\ast$.

1. Let $\xi, \eta, \zeta \in l^\perp$. Then equation (2.25) reads $[[\xi \otimes \eta \otimes \zeta, CDYB_p(l)] + 1] = 0$;
2. Let $\xi, \eta \in l^\perp$ and $\zeta \in m^\perp$. Then equation (2.25) reads:

$$(k + 1)[[\xi \otimes \eta \otimes \zeta, CDYB_p(l)] + 1] = -[\xi \otimes \eta \otimes \zeta, CDYB_p(l)]_{k+1}$$

thus $[[\xi \otimes \eta \otimes \zeta, CDYB_p(l)] + 1] = 0$;
3. Let $\xi \in l^\perp$ and $\eta, \zeta \in m^\perp$. Then equation (2.25) reads:

$$(k + 1)[[\xi \otimes \eta \otimes \zeta, CDYB_p(l)] + 1] = -2[\xi \otimes \eta \otimes \zeta, CDYB_p(l)]_{k+1}$$

thus $[[\xi \otimes \eta \otimes \zeta, CDYB_p(l)] + 1] = 0$;
4. Let $\xi, \eta, \zeta \in m^\perp$. Then equation (2.25) reads:

$$(k + 1)[[\xi \otimes \eta \otimes \zeta, CDYB_p(l)] + 1] = -3[\xi \otimes \eta \otimes \zeta, CDYB_p(l)]_{k+1}$$

thus $[[\xi \otimes \eta \otimes \zeta, CDYB_p(l)] + 1] = 0$.

Hence, the map $l$ satisfies equation (1.1). Thus $l \in Dynl(D, g)$. We have already seen that it is the unique dynamical $\ell$-matrix satisfying $l_0 = 0$ and $l_p sp = 0$. $\square$

**Definition 2.8.** The (formal) $\ell$-matrix defined in Theorem 2.4 is called the canonical dynamical $\ell$-matrix associated with the Lie quasi-bialgebra $g$ and the reductive decomposition $g = l \oplus m$, and is denoted by $l^{can}(g, 1, m)$, or simply $l^{can}$ when no confusion is possible.

**Remark 2.9.** Clearly, when the Lie quasi-bialgebra $g$ is canonically compatible with the reductive decomposition $g = l \oplus m$ of $g$ (see Example 2.12 or 11), then the canonical $\ell$-matrices of Definition 2.8 and of 11 coincide in view of uniqueness of dynamical $\ell$-matrices satisfying $l_p sp = 0$ (see also Example 2.12).
2.2. Canonical dynamical $\ell$-matrices are analytic. We now show that a canonical dynamical $\ell$-matrix is in fact analytic, and find an explicit formula.

For any $t \in \wedge^2 g$ we set

$$\tau_t: g \oplus g^* \rightarrow g \oplus g^*$$

$$x + \xi \mapsto (x + t \xi) + \xi.$$ 

We start with the following proposition:

**Proposition 2.10.** Let $l = l^{can}(G, \text{Im}) \in \text{Dynl}_0(\mathbb{D}, \mathcal{G})$. Then, for all $X_p \in g_p^*$ and for all $\alpha \in \iota^*$, one has:

$$p_m \text{Ad}_{e^{-sp}} \tau_p X_p = 0 \quad (2.26)$$

$$p_l \frac{\text{Ad}_{e^{-sp}} - 1}{\text{ad}_{sp}} \tau_p X_p = -p_l X_p \quad (2.27)$$

$$p_m \text{Ad}_{e^{-sp}} \tau_p s \alpha = -p_m \frac{\text{Ad}_{e^{-sp}} - 1}{\text{ad}_{sp}} s \alpha \quad (2.28)$$

$$p_l \frac{\text{Ad}_{e^{-sp}} - 1}{\text{ad}_{sp}} \tau_p s \alpha = -p_l \frac{\text{Ad}_{e^{-sp}} - 1 + \text{ad}_{sp} s \alpha}{\text{ad}_{sp}^2} \quad (2.29)$$

We recall that $X_p$ belongs to $g_p^*$ if and only if $X_p = z_p + \xi_p \in \iota \oplus g^*$ and $i^* \xi_p = \text{ad}_{z_p}^* p$.

**Proof.** Since $l_0 = 0$, equations (2.26) and (2.27) are clearly satisfied at order 0. Now, let $X_p \in g_p^*$. Then, $X_p$ can be uniquely written $X_p = z + \text{ad}_{sp} z + \xi$, where $z \in \iota$ and $\xi \in \iota^\perp$ (notice that $\text{ad}_{sp} z = \text{ad}_{z_p}^* sp \in m^{\perp}$ for all $z \in \iota$, $p \in \iota^*$, since $\pi_l = 0$).

Since $l$ satisfies equation (2.6), one has:

$$d_{p}(\text{Ad}_{e^{-sp}} \tau_p X_p)(p) = \text{Ad}_{e^{-sp}} \left(-\text{ad}_{sp} \tau_p X_p + \text{ad}_{sp} \tau_p (X_p - z) - \tau_p (\tilde{\xi} + (\tilde{\text{ad}_{sp}} z)_p)\right)$$

$$- p_l l_p (X_p - z) - l_p \text{ad}_{sp} z + l_p \text{ad}_{sp} z$$

$$= -\text{Ad}_{e^{-sp}} \tau_p (\tilde{\xi} + (\tilde{\text{ad}_{sp}} z)_p + p_l l_p (\xi + \text{ad}_{sp} z)).$$

Now, the element $\tilde{\xi} + (\tilde{\text{ad}_{sp}} z)_p + p_l l_p (\xi + \text{ad}_{sp} z)$ lies in $g_p^*$, by Lemma 2.6 and is of degree $\geq 1$. Thus, if equation (2.26) is satisfied modulo terms of degree $\geq k$, then it will also be satisfied modulo terms of degree $\geq k + 1$. Equation (2.26) is thus proved.

Since $l$ satisfies equation (2.6), one has:

$$d_{p} \left(\frac{\text{Ad}_{e^{-sp}} - 1}{\text{ad}_{sp}} \tau_p X_p\right)(p) = -\text{Ad}_{e^{-sp}} \tau_p X_p - \frac{\text{Ad}_{e^{-sp}} - 1}{\text{ad}_{sp}} \tau_p X_p$$

$$+ \frac{\text{Ad}_{e^{-sp}} - 1}{\text{ad}_{sp}} \left(\text{ad}_{sp} \tau_p X_p - \tau_p (\tilde{\xi} + (\tilde{\text{ad}_{sp}} z)_p) - p_l l_p (X_p - z)\right)$$

$$= -\tau_p X_p - \frac{\text{Ad}_{e^{-sp}} - 1}{\text{ad}_{sp}} \tau_p (\tilde{\xi} + (\tilde{\text{ad}_{sp}} z)_p + p_l l_p (\xi + \text{ad}_{sp} z) + X_p).$$

Thus, for $k \geq 1,$

$$\left[\frac{\text{Ad}_{e^{-sp}} - 1}{\text{ad}_{sp}} \tau_p X_p\right]_k = -[p_l l_p (\xi + \text{ad}_{sp} z)]_k$$

$$- \left[\frac{\text{Ad}_{e^{-sp}} - 1}{\text{ad}_{sp}} \tau_p (\tilde{\xi} + (\tilde{\text{ad}_{sp}} z)_p + p_l (\xi + \text{ad}_{sp} z))\right]_k.$$
Now, assume that equation (2.27) is satisfied modulo terms of degree $\geq k$ (for some $k \geq 1$). Then,

$$(k + 1) \left[ p_t \frac{\text{Ad}_{e^{-sp}} - 1}{\text{ad}_{sp}} \tau_p X_p \right]_k = 0$$

and equation (2.27) is satisfied modulo terms of degree $\geq k + 1$, and equation (2.27) is proved.

Let $\alpha \in \ell^*$. Similarly, one obtains:

$$(k + 1) \left[ \text{Ad}_{e^{-sp}}(l_p s\alpha + s\alpha) + \frac{\text{Ad}_{e^{-sp}} - 1}{\text{ad}_{sp}} s\alpha \right] = - \left[ \text{Ad}_{e^{-sp}} \tau_p (\widetilde{s\alpha} + p_t l_p s\alpha) \right]_k.$$ 

Thus, by equation (2.26):

$$p_m \left( \text{Ad}_{e^{-sp}}(l_p s\alpha + s\alpha) + \frac{\text{Ad}_{e^{-sp}} - 1}{\text{ad}_{sp}} s\alpha \right) = 0$$

and equation (2.28) is proved.

Let $\alpha \in \ell^*$. Similarly, one obtains:

$$(k + 2) \left[ \frac{\text{Ad}_{e^{-sp}} - 1}{\text{ad}_{sp}} \tau_p s\alpha + \frac{\text{Ad}_{e^{-sp}} - 1 + \text{ad}_{sp}}{\text{ad}_{sp}^2} s\alpha \right] = - [\tau_p s\alpha]_k - \left[ \frac{\text{Ad}_{e^{-sp}} - 1}{\text{ad}_{sp}} \tau_p (\widetilde{s\alpha} + p_t l_p s\alpha) \right]_k.$$ 

Thus, by equation (2.27),

$$p_t \left( \frac{\text{Ad}_{e^{-sp}} - 1}{\text{ad}_{sp}} \tau_p s\alpha + \frac{\text{Ad}_{e^{-sp}} - 1 + \text{ad}_{sp}}{\text{ad}_{sp}^2} s\alpha \right) = 0$$

and equation (2.29) is proved.

The following theorem gives an explicit analytic formula for $l^\text{can}$:

**Theorem 2.11.** The canonical $\ell$-matrix of Theorem 2.4 is analytic around 0, and is explicitly given by:

$$p_m l_p \xi = (\text{Id}_m - R_p S_p i_m)^{-1} R_p (S_p \xi - \xi)$$

(2.30)

$$p_l l_p \xi = (\text{Id}_l - S_p R_p i_l)^{-1} S_p (R_p \xi - \xi)$$

(2.31)

$$p_m l_p s\alpha = (\text{Id}_m - R_p S_p i_m)^{-1} R_p \left( S_p + K_p + \frac{\text{Ad}_{e^{-sp}} - 1 - \text{ad}_{sp}}{\text{ad}_{sp}} \right) s\alpha$$

(2.32)

$$p_l l_p s\alpha = - (\text{Id}_l - S_p R_p i_l)^{-1} \left( S_p + K_p + S_p R_p \frac{\text{Ad}_{e^{-sp}} - 1 - \text{ad}_{sp}}{\text{ad}_{sp}} \right) s\alpha$$

(2.33)

for all $\xi \in \ell^1$ and $\alpha \in \ell^*$, where $R$ and $S$ are given by:

$$K_p = \left( p_t \frac{\text{Ad}_{e^{-sp}} - 1}{\text{ad}_{sp}} \right)^{-1} \frac{\text{Ad}_{e^{-sp}} - 1 + \text{ad}_{sp}}{\text{ad}_{sp}^2}$$

(2.34)

$$R_p = (p_m \text{Ad}_{e^{-sp}} i_m)^{-1} p_m \text{Ad}_{e^{-sp}}$$

(2.35)

$$S_p = \left( p_t \frac{\text{Ad}_{e^{-sp}} - 1}{\text{ad}_{sp}} \right)^{-1} \frac{\text{Ad}_{e^{-sp}} - 1}{\text{ad}_{sp}}$$

(2.36)

**Proof.** Let $\xi \in \ell^1 \subset \mathfrak{g}^*_p$. We get from equation (2.26): $p_m l_p \xi = - R_p (p_l l_p \xi + \xi)$, and from equation (2.27): $p_l l_p \xi = - S_p (p_m l_p \xi + \xi)$. Combining these two equations, and the fact that both $(\text{Id}_m - R_p S_p i_m)$ and $(\text{Id}_l - S_p R_p i_l)$ are analytically invertible around 0 yields equations (2.30) and (2.31).
Let $\alpha \in \mathfrak{l}^*$. We get from equation (2.28): $p_m l_p s\alpha = -R_p \left(p_l p s\alpha + s\alpha + \frac{1-\text{Ad}_{e^{sp}}}{\text{ad}_{sp}} s\alpha\right)$, and from equation (2.29): $p_l p s\alpha = -K_p s\alpha - S_p s\alpha - S_p p_m l_p s\alpha$. Combining these two equations yields equations (2.32) and (2.33). \hfill $\Box$

Example 2.12 ($\ell$-matrices for a Lie quasi-bialgebra canonically compatible with a reductive decomposition). We say (see [11]) that the Lie quasi-bialgebra $G = (g, [\cdot, \cdot], \varpi, \varphi)$ is canonically compatible with the reductive decomposition $g = \mathfrak{l} \oplus \mathfrak{m}$ if the following conditions hold:

\begin{align*}
\varpi_{\ell} &= 0 \quad (2.37) \\
\langle m^\perp, \varpi m^\perp \rangle &= 0 \quad (2.38) \\
\varphi &\in \text{Alt}(\ell \otimes \ell \otimes \ell \otimes \ell \otimes m \otimes m). \quad (2.39)
\end{align*}

In this case, the expression $S_p$ of equation (2.36) vanishes, and the expression $R_p$ of equation (2.35) reads as:

$$R_p = p_m (p_g \text{Ad}_{e^{-sp}} i_g)^{-1} i_g \text{Ad}_{e^{-sp}}.$$  
(2.40)

Thus, $\ell^{\text{can}}$ has the simpler expression of [11]:

$$\ell^{\text{can}}_p s\alpha = \left(\cotanh \text{ad}_{sp} - \frac{1}{\text{sh ad}_{sp}}\right) s\alpha, \quad \alpha \in \mathfrak{l}^* \quad (2.41)$$

$$\ell^{\text{can}}_p \xi = -(p_g \text{Ad}_{e^{-sp}} i_g)^{-1} i_g \text{Ad}_{e^{-sp}} \xi, \quad \xi \in \mathfrak{l}^\perp. \quad (2.42)$$

Moreover, if $\varpi = 0$, then equation (2.42) reads as:

$$\ell^{\text{can}}_p \xi = \tanh \text{ad}_{sp} \xi. \quad (2.43)$$

This example can be specified to the Etingof–Varchenko case (see Example 3.9).

Example 2.13 (Alekseev–Meinrenken $r$-matrices [2]). Alekseev–Meinrenken $r$-matrices are obtained when $\mathfrak{l} = \mathfrak{g}$, for a cocommutative Lie quasi-bialgebra $G = (g, [\cdot, \cdot], \varpi, \varphi)$. In this case (as $\mathfrak{m} = \{0\}$), the only line of interest in Theorem 2.11 is equation (2.33), which reads as

$$l_p \alpha = (S_p + K_p) \alpha, \quad (2.44)$$

for $\alpha \in \mathfrak{g}^*$. Also, since $\text{ad}_p \mathfrak{g} \subset \mathfrak{g}^*$, $\text{ad}_p \mathfrak{g}^* \subset \mathfrak{g}$,

\begin{align*}
S_p \alpha &= \left(\frac{1}{\text{sh ad}_p} - \cotanh \text{ad}_p\right) \alpha, \quad K_p \alpha = \left(\frac{1}{\text{ad}_p} - \frac{1}{\text{sh ad}_p}\right) \alpha, \quad (2.45) \\
S_p \alpha &= \left(\frac{1}{\text{sh ad}_p} - \cotanh \text{ad}_p\right) \alpha, \quad K_p \alpha = \left(\frac{1}{\text{ad}_p} - \frac{1}{\text{sh ad}_p}\right) \alpha, \quad (2.46)
\end{align*}

for all $\alpha \in \mathfrak{g}^*$, thus we obtain:

$$\ell^{\text{can}}_p \alpha = r_p^{AM} \alpha = \left(\cotanh \text{ad}_p - \frac{1}{\text{ad}_p}\right) \alpha, \quad (2.47)$$

for all $\alpha \in \mathfrak{g}^*$. The Alekseev-Meinrenken $r$-matrix associated with a cocommutative Lie quasi-bialgebra was already constructed in [2].

Example 2.14 ($r$-matrices for the non-compatible case). In this example, we assume that $\varpi = 0$ and that $\varphi \in \text{Alt}(\ell \otimes \ell \otimes m)$. We set:

$$c^\pm x = \frac{1}{2} (\text{ch} x \pm \cos x), \quad s^\pm x = \frac{1}{2} (\text{sh} x \pm \sin x). \quad (2.48)$$
the vector space decomposition \( g \) for all \( \alpha \) reads as:

\[
\begin{align*}
\kappa & \in m^\perp, \\
\kappa & \in l^\perp,
\end{align*}
\]

Thus, for all \( k \in \mathbb{N} \) such that \( k \not\equiv 0 \mod 4 \) one has \( p_l \text{ad}_{sp}^k i_l = 0 \) and \( p_m \text{ad}_{sp}^k i_m = 0 \), thus the map \( K \) of Theorem 2.11 reads as:

\[
K_p s\alpha = \left( \frac{1}{\text{ad}_{sp}} - \frac{1}{s^+ \text{ad}_{sp}}s\alpha \right),
\]

for all \( \alpha \in \mathfrak{l}^* \), and the maps \( R \) and \( S \) are represented by the following block-matrices (relatively to the vector space decomposition \( g = l \oplus m^+ \oplus m \oplus l^\perp \), in this order):

\[
R_p = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
c^- \text{ad}_{sp} & -s^+ \text{ad}_{sp} & 1 & -s^- \text{ad}_{sp} \\
c^+ \text{ad}_{sp} & 0 & 0 & 0
\end{pmatrix}, \quad
S_p = \begin{pmatrix}
-1 & c^+ \text{ad}_{sp} & s^+ \text{ad}_{sp} & c^- \text{ad}_{sp} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

and for \( \alpha \in \mathfrak{l}^* \), \( R_p \frac{\text{Ad}_{sp} - 1 - \text{ad}_{sp}}{\text{ad}_{sp}} s\alpha \) reads as:

\[
R_p \frac{\text{Ad}_{sp} - 1 - \text{ad}_{sp}}{\text{ad}_{sp}} s\alpha = \frac{\text{ad}_{sp} s^+(\text{ad}_{sp}) - c^- (\text{ad}_{sp})}{\text{ad}_{sp} c^+(\text{ad}_{sp})} s\alpha.
\]

Also, the mappings \( (\text{Id}_m - R_p S_p i_m) \) and \( (\text{Id}_l - S_p R_p i_l) \) read as:

\[
\begin{align*}
\text{Id}_m - R_p S_p i_m &= \frac{c^+ \text{ad}_{sp} s^+ \text{ad}_{sp} + c^- \text{ad}_{sp} s^- \text{ad}_{sp} i_m}{c^+ \text{ad}_{sp} s^+ \text{ad}_{sp}}, \\
\text{Id}_l - S_p R_p i_l &= \frac{c^+ \text{ad}_{sp} s^+ \text{ad}_{sp} + c^- \text{ad}_{sp} s^- \text{ad}_{sp} i_l}{c^+ \text{ad}_{sp} s^+ \text{ad}_{sp}}.
\end{align*}
\]

We set

\[
\begin{align*}
F(z) &= \frac{\text{ch}(z) \cos(z) - 1}{\text{ch}(z) \sin(z) + \cos(z) \text{sh}(z)} = \sum_{k \geq 0} F_k z^{4k+3}, \\
G(z) &= \frac{\text{sh}(z) \sin(z)}{\text{ch}(z) \sin(z) + \cos(z) \text{sh}(z)} = \sum_{k \geq 0} G_k z^{4k+1}, \\
H(z) &= \frac{\cos(z)(z \text{ch}(z) - \text{sh}(z)) - \sin(z) \text{ch}(z) + z}{z(\cos(z) \text{sh}(z) + \text{ch}(z) \sin(z))} = \sum_{k \geq 0} H_k z^{4k+3}.
\end{align*}
\]

After computation and simplification, one obtains:

\[
\begin{align*}
p_m^{\text{can}} \xi &= F(\text{ad}_{sp})\xi, \\
p_l^{\text{can}} \xi &= G(\text{ad}_{sp})\xi, \\
p_m^{\text{can}} s\alpha &= G(\text{ad}_{sp})s\alpha, \\
p_l^{\text{can}} s\alpha &= H(\text{ad}_{sp})s\alpha,
\end{align*}
\]

for all \( \xi \in \mathfrak{l}^\perp, \alpha \in \mathfrak{l}^* \).
Equation (2.46) for $l^{\text{can}}$ is equivalent to the following differential system:

\[
\begin{align*}
  zF'(z) &= -z(F(z)^2 + G(z)^2) \\
  zG'(z) &= z - zG(z)(H(z) + F(z)) - G(z) \\
  zH'(z) &= -zG(z)^2 - zH(z)^2 - 2H(z).
\end{align*}
\] (2.63)

2.3. Functoriality.

**Definition 2.15.** A bidynamical Lie quasi-bialgebra (over $l$) is a Lie quasi-bialgebra $G = (g, [\ , \ ], \varpi, \varphi)$ such that $\varpi_1 = 0$, $\varphi \equiv 0 \mod l$ and such that there exists a reductive decomposition $g = l \oplus m$.

A morphism $\phi$ between two bidynamical Lie quasi-bialgebras over $l$, $G = (g = l \oplus m, [\ , \ ], \varpi, \varphi)$ and $G' = (g' = l \oplus m', [\ , \ ], \varpi', \varphi')$, is a Lie algebra morphism $\phi: g \to g'$ such that $\phi(z) = z$ for all $z \in l$, $\phi(m) \subset m'$ and

\[
\begin{align*}
  \phi \varpi \phi^* &= \varpi' \phi(x), \quad \forall x \in g, \\
  \phi^3 \varphi' &= \varphi'.
\end{align*}
\] (2.64) (2.65)

**Proposition 2.16.** Let $G = (g = l \oplus m, [\ , \ ], \varpi, \varphi)$ and $G' = (g' = l \oplus m', [\ , \ ], \varpi', \varphi')$ be two bidynamical Lie quasi-bialgebras, and let $\psi: G \to G'$ be a bidynamical Lie quasi-bialgebra morphism. Then the following equality holds

\[
\psi l^{\text{can}}(G,l,m) \phi^* = l^{\text{can}}(G',l,m').
\] (2.66)

In particular, the Lie groupoid morphism

\[
\Psi: G = U \times G \times U \longrightarrow G' = U \times G' \times U
\]

\[ (p, x, q) \longmapsto (p, \psi(x), q) \]

is a Poisson groupoid morphism, when $G$ and $G'$ are equipped with the Poisson bracket induced by $l^{\text{can}}(G,l,m)$ and $l^{\text{can}}(G',l,m')$ respectively, where $\psi: G \to G'$ is the Lie group morphism integrating the Lie algebra morphism $\psi$.

**Proof.** We set $\lambda_p = \psi l^{\text{can}}(G,l,m) \phi^*$. Then, it is easy to check, from equation (2.46), that $\lambda$ satisfies the same equation as $l^{\text{can}}(G',l,m')$, namely:

\[
d_p \lambda(p) = \tau_{-\lambda_p} \text{ad}_{x'}\tau_{\lambda_p} - p'_s \lambda_p - \lambda_p s'_p t^*_p,
\]

thus $\psi l^{\text{can}}(G,l,m) \phi^* = l^{\text{can}}(G',l,m')$. \qed

3. Trivialization and duality

From now on, we denote by $U$ the domain of analyticity of the canonical $\ell$-matrix $l^{\text{can}}$. Notice that $U$ is $l$-equivariant, but not simply-connected in general.

3.1. **Trivial Lie algebroids.** Let $(g, [\ , \ ]_g)$ be a Lie algebra and $M$ a manifold. Recall (see [9]) that the trivial Lie algebroid on $M$ with vertex algebra $g$ is the vector bundle $A = TM \oplus (M \times g) = TM \times g$ over $M$ (Whitney sum), where the anchor is the projection on $TM$, and the bracket is defined as follows: let $\sigma$ and $\sigma'$ be two sections of the vector bundle $A$, say $\sigma = (X, x)$ and $\sigma' = (X', x')$ where $X$ and $X'$ are two vector fields on $M$ and $x, x': M \to g$, and set

\[
[\sigma, \sigma']_A = [X, X'] \oplus (X \cdot x' - X' \cdot x + [x, x']_g)
\] (3.1)

The bracket in the first component of the right hand side of equation (3.1) is the bracket of vector fields on $M$, and $X \cdot x'$ denotes the derivative of $x'$ in the direction of $X$. 

The Lie algebroid of the trivial groupoid $G = U \times G \times U$ is the trivial Lie algebroid over $U$: $\mathcal{A}(G) = U \times (\mathfrak{g} \oplus \mathfrak{g})$, with the following bracket on its sections:

$$[\sigma, \sigma']_p = (d_p \alpha'(\sigma) - d_p \alpha(\sigma'), d_p x'(\sigma) - d_p x(\sigma') + [x_p, x'_p|_p]),$$

for $p \in U$ and $\sigma_p = \alpha_p + x_p, \sigma'_p = \alpha'_p + x'_p \in \mathfrak{g} \oplus \mathfrak{g}$, and the anchor is:

$$a^{\mathcal{A}(G)}(\sigma) = \alpha.$$

### 3.2. Trivialization

We recall (see [8]) that the Lie algebroid of the dual of the Poisson groupoid $G$ is the vector bundle $N(U) = U \times \mathfrak{g}^*$ over $U$, together with the following bracket on its sections:

$$[(z, \xi), (z', \xi')]^N(U)_p = \left( d_p z'(a_p^N(U)(z_p, \xi_p)) - d_p z(a_p^N(U)(z'_p, \xi'_p)) - [z_p, z'_p] + (\xi, d_p l(\cdot)\xi'), \right.$$

$$\left. d_p \xi'(a_p^N(U)(z_p, \xi_p)) - d_p \xi(a_p^N(U)(z'_p, \xi'_p)) + \text{ad}^*_z p \xi_p - \text{ad}^*_z p \xi_p \right)$$

and the anchor:

$$a_p^{N(U)}(z, \xi) = i^*_z \xi - \text{ad}^*_z p.$$

We want a trivialization of the Lie algebroid $N(U)$, that is a Lie algebroid isomorphism $T: \mathcal{A}(N(U)) \to N(U)$, where $G^* = U \times \mathfrak{g}^* \times U$ is the trivial groupoid over $U$, with vertex group $G^*_0$ such that $\text{Lie}(G^*_0) = \mathfrak{g}^*_0$. Such an isomorphism can be split into two parts (see [8]), a Lie algebra bundle isomorphism $\psi: U \times \mathfrak{g}^*_0 \to \ker a^{N(U)}$ and a flat connection $\nabla: U \times \mathfrak{g}^* \to N(U)$ such that $[\nabla(\alpha), \psi X]^{N(U)} = \psi(d X(\alpha))$ for any smooth section $\alpha \in \Gamma(U \times \mathfrak{g}^*)$ and $X \in \Gamma(U \times \mathfrak{g}^*_0)$ — we recall that by definition, $\nabla$ is a flat connection if $[\nabla \alpha, \nabla \beta] = 0$ for all $\alpha, \beta \in \Gamma(U \times \mathfrak{g}^*)$. Then, setting $T(\alpha + X) = \nabla \alpha + \psi X$ for $\alpha \in \mathfrak{g}^*$, $X \in \mathfrak{g}^*_0$ provides a trivialization $T$.

We denote by $U$ the domain of analyticity of $l^{an}$ (which is $\text{Ad}_{\mathfrak{l}}^*-\text{equivariant}$ since $l^{an}$ is $\mathfrak{l}$-equivariant and $L$ is connected). As a corollary of Theorem 2.11 we have the following:

**Corollary 3.1.** Let $l = l^{an}([G, l^m]) \in \text{Dynl}_0(U, \mathcal{G})$. Then, for all $X_p \in \mathfrak{g}^*_p$, the expression

$$\text{Ad}_{e^{-sp}} \tau_p X_p$$

lies in $\mathfrak{g}^*_0$. In particular, the map

$$\phi_p: \mathfrak{g}^*_p \longrightarrow \mathfrak{g}^*_0,$$

$$X_p \longmapsto \text{Ad}_{e^{-sp}} \tau_p X_p,$$

is a Lie algebra isomorphism, and its inverse is given by

$$\phi_p^{-1}: \mathfrak{g}^*_0 \longrightarrow \mathfrak{g}^*_p,$$

$$z + \xi \longmapsto \left( p_1 \frac{\text{Ad}\circ e^{-sp}}{\text{ad}_{sp}} + p_{0^*} \text{Ad} e^{-sp}\right) (z + \xi).$$

**Proof.** By equation (2.26), we know that the expression (3.6) lies in $\mathfrak{l} \oplus \mathfrak{g}^* \subset \mathfrak{d}$. Also, since $p_{m^\perp} \text{ad}_{sp} = \text{ad}_{sp} p_{l^m}$, applying $\text{ad}_{sp}$ to both sides of equation (2.27) yields:

$$p_{m^\perp} \text{Ad}_{e^{-sp}} \tau_p X_p - p_{m^\perp} X_p = - \text{ad}_{sp} p_{l^m} X_p.$$

Thus, since $p_{m^\perp} X_p = - \text{ad}_{sp} p_{l^m}$, the expression $p_{m^\perp} \text{Ad}_{e^{-sp}} \tau_p X_p$ vanishes, and expression (3.6) lies in $\mathfrak{l} \oplus l^m = \mathfrak{g}^*_0$. Clearly, the map $\phi_p$ is a Lie algebra isomorphism. A simple computation using the relations of Proposition 2.10 shows that

$$\left( p_1 \frac{\text{Ad}\circ e^{-sp}}{\text{ad}_{sp}} + p_{0^*} \text{Ad} e^{-sp}\right) \text{Ad}_{e^{-sp}} \tau_p (z + \text{ad}_{sp} z + \xi) = z + \text{ad}_{sp} z + \xi$$

for all $z \in \mathfrak{l}$ and $\xi \in l^m$. □
Corollary 3.1 implies that the bundle map

\[ \psi: U \times g^* \rightarrow \text{Ker } a^{N(U)} \subset U \times (l \oplus g^*) \]

\[ (p, X) \mapsto (p, -\phi_p^{-1}X) \tag{3.7} \]

is a Lie algebra bundle isomorphism. So, in order to complete the trivialization, we need a flat connection \( \nabla: U \times l^* \rightarrow N(U) \), satisfying \( [\nabla(\alpha), \psi X]^N(U) = \psi(d X(\alpha)) \) for any smooth section \( \alpha \in \Gamma(U \times l^*) \) and \( X \in \Gamma(U \times g^*)_0 \).

For all \( \alpha \in l^* \), we set

\[ \nabla_p(\alpha) = (u_p\alpha, s\alpha + \text{ad}_{sp} u_p\alpha + v_p\alpha), \tag{3.8} \]

where \( u_p: l^* \rightarrow l \) and \( v_p: l^* \rightarrow l^\perp \) are given by:

\[ \begin{cases} u_p\alpha = p_l^\tau \frac{\text{Ad}_{sp} - 1}{\text{ad}_{sp}} s\alpha \\ v_p\alpha = p_l^\tau \frac{\text{Ad}_{sp} - 1}{\text{ad}_{sp}} s\alpha \end{cases} \tag{3.9} \]

for all \( \alpha \in l^* \). We show in Proposition 3.3 below that \( \nabla \) is a flat connection.

We introduce the following notations:

\[ \widetilde{\alpha}_p = p_g \frac{\text{Ad}_{sp} - 1}{\text{ad}_{sp}} s\alpha \tag{3.10} \]

\[ \check{\alpha}_p = p_g \frac{\text{Ad}_{sp} - 1}{\text{ad}_{sp}} s\alpha \tag{3.11} \]

for \( \alpha \in l^* \). Then, \( \nabla \) is given by:

\[ \nabla_p\alpha = (u_p\alpha, \check{\alpha}_p) \tag{3.12} \]

for all \( \alpha \in l^* \). The following lemma, the proof of which is straightforward, will help in our computations:

**Lemma 3.2.** For all \( \alpha \in l^* \),

\[ u_p\alpha = \widetilde{\alpha}_p - l_p \check{\alpha}_p \tag{3.13} \]

\[ = p_g^\tau \frac{\text{Ad}_{sp} - 1}{\text{ad}_{sp}} s\alpha. \tag{3.14} \]

**Proposition 3.3.** The map \( \nabla \) is a flat connection.

**Proof.** To show that \( \nabla \) is a flat connection, we have to show the two following equalities for all \( \alpha, \beta \in l^* \) seen as constant sections:

\[ d_p u(\alpha) \beta - d_p u(\beta) \alpha - [u_p\alpha, u_p\beta] + \langle \alpha_p, d_p l(\cdot) \beta_p \rangle = 0 \tag{3.15} \]

\[ d_p \beta_p(\alpha) - d_p \alpha_p(\beta) - p_g^* [u_p\alpha, \beta_p] - p_g^* [\alpha_p, u_p\beta] - p_g^* [\alpha_p, \beta_p] - p_g^* [l_p \check{\alpha}_p, \beta_p] - p_g^* [\alpha_p, l_p \beta_p] = 0. \tag{3.16} \]

The left hand side of (3.16) vanishes, since \( d_p \beta_p(\alpha) - d_p \alpha_p(\beta) = \left[ \widetilde{\alpha}_p + \check{\alpha}_p, \widetilde{\beta}_p + \check{\beta}_p \right] \), and by Lemma 3.2 Equation (3.15) vanishes too, by (1.1) and (1.2). □
The trivialization is given in the following theorem:

**Theorem 3.4.** Let $l$ be a canonical dynamical $\ell$-matrix. Then, the bundle map:

$$T: U \times l^* \times \mathfrak{g}_0^* \longrightarrow N(U) = U \times l \times \mathfrak{g}^*$$

(3.17)

given by

$$T_p(\alpha, z + \xi) = \left(u_p\alpha - p_{\ell}(\alpha) Ad_{e\ell}(z + \xi), \alpha_p - p_{\ell} Ad_{e\ell}(z + \xi)\right)$$

(3.18)

is a Lie algebroid isomorphism.

**Proof.** It only remains to show that $[\nabla(\alpha), \psi X]^{N(U)} = 0$, for all $\alpha \in l^*$, and all $X \in \mathfrak{g}_0^*$ seen as constant sections.

First, notice that

$$\psi_p z = -\nabla_p Ad_{z}^* p - (z, 0) \quad \text{for all } z \in l.$$ Thus, $[\nabla(\alpha), \psi z]^{N(U)} = -\nabla_p (\alpha, Ad_{z}^*) - [\nabla(\alpha, z), 0]^{N(U)} = \psi_p (d_p z(\alpha)).$

Now, if $\xi \in l^\perp$ (seen as a constant section), the first component of $[\nabla(\alpha), \psi \xi]^{N(U)}$ is:

$$p_{\ell} d_p l(\alpha) Ad_{e\ell}(\xi) - p_{\ell}\tau_{-\ell} p [Ad_{e\ell} - 1, Ad_{e\ell}\xi] + [u_p\alpha, p_{\ell}(\alpha) Ad_{e\ell}(\xi)] - (\alpha_p, d_p l(\cdot)p^\mathfrak{g} Ad_{e\ell}(\xi))$$

$$= -p_{\ell} d_p l(\alpha^\mathfrak{g} Ad_{e\ell}(\xi) + p_{\ell}\tau_{-\ell} p [Ad_{e\ell} - 1, Ad_{e\ell}\xi]) + [u_p\alpha, p_{\ell} Ad_{e\ell}(\xi)]$$

$$= 0$$

and the second component vanishes too. \hfill \square

The trivialization may also be written:

$$T_p(\alpha, z + \xi) = \tau_{-\ell} p \frac{Ad_{e\ell} - 1}{ad_{sp}} s\alpha - \tau_{-\ell} p Ad_{e\ell}(z + \xi)$$

(3.19)

where $\alpha \in l^*$, $z \in l$ and $\xi \in l^\perp$, and also in a way where $l$ does not appear:

$$T_p(\alpha, z + \xi) = p_{\ell} \frac{Ad_{e\ell} - 1}{ad_{sp}} s\alpha + p_{\ell} \frac{Ad_{e\ell} - 1}{ad_{sp}} s\alpha - p_{\ell} \frac{Ad_{e\ell} - 1}{ad_{sp}} (z + \xi) - p_{\ell} Ad_{e\ell}(z + \xi).$$

(3.20)

To show this last equality, compute the adjoint $T_p^*$ of $T_p$, and use Proposition 2.10.

3.3. **Duality.** For a Manin quasi-triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ we denote by $\mathcal{G}_{(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})}$ the corresponding Lie quasi-bialgebra; for a Lie quasi-bialgebra $\mathcal{G} = (\mathfrak{g}, [\cdot, \cdot], \varpi, \varphi)$, we set $\mathcal{G}^- = (\mathfrak{g}, [\cdot, \cdot], -\varpi, \varphi)$.

We start with the definition of a duality for bidynamical Lie quasi-bialgebras, which was introduced in [11].

**Definition 3.5.** Let $\mathcal{G} = (\mathfrak{g} = l \oplus \mathfrak{m}, [\cdot, \cdot], \varpi, \varphi)$ be a bidynamical Lie quasi-bialgebra with canonical double $\mathfrak{d}$. The Lie quasi-bialgebra

$$\mathcal{G}^* = \left(\mathcal{G}_{(\mathfrak{d}, \mathfrak{g} \oplus l^\perp, \mathfrak{m} \oplus \mathfrak{m})}^-\right)$$

is called the **dual over $l$ of the bidynamical Lie quasi-bialgebra** $\mathcal{G}$.

Observe that $\mathfrak{g}^* = l \oplus l^\perp$ is indeed a lagrangian subalgebra of $\mathfrak{d}$, so that the dual over $l$ is well-defined. Also observe that the dual of a bidynamical Lie quasi-bialgebra is again a bidynamical Lie quasi-bialgebra.
Let $\mathbf{op}: g \to g$ be the standard involution associated with the reductive decomposition $g = l \oplus m$:

$$\mathbf{op}(z) = z \quad \mathbf{op}(u) = -u$$

for all $z \in l$ and $u \in m$. We define the Lie bracket $[,]^\mathbf{op}$ on the vector space $g$ as follows:

$$[z, z']^\mathbf{op} = [z, z'] \quad [z, u]^\mathbf{op} = -[z, u] \quad [u, u']^\mathbf{op} = [u, u']$$

for all $z, z', u, u' \in l$ and $u, u' \in m$, and we denote by $g^\mathbf{op}$ the resulting Lie algebra. We also set $G^{\mathbf{op}} = (g^\mathbf{op}, [, ]^\mathbf{op}, \varphi^\mathbf{op})$ where

$$\varphi^\mathbf{op} = \mathbf{op} \varphi \mathbf{op}^* \quad \varphi^\mathbf{op} = (\mathbf{op} \otimes \mathbf{op} \otimes \mathbf{op}) \varphi$$

We denote by $d^*$ the canonical double of $G^*$. First, observe that under the canonical vector space identification $d^* \simeq d$, the Lie algebra $g$ is not a Lie subalgebra of $d^*$ (but the Lie algebra $g$ is isomorphic to the Lie subalgebra $g^{\mathbf{op}} = l \oplus m$ of $d^*$). Second, observe that under the canonical identification $d^* \simeq d$, then $(G^*)^* \neq G$, but rather $(G^*)^* = G^{\mathbf{op}}$, which is isomorphic to $G$.

We now turn to our main duality statement which provides the dual Poisson groupoid of a Poisson groupoid associated with a canonical $\ell$-matrix:

**Theorem 3.6.** The dual Poisson groupoid of the dynamical Poisson groupoid associated with a canonical $\ell$-matrix $l^\text{can}(G^*, t^*)$ is isomorphic to (a covering of) the dynamical Poisson groupoid $U \times G^* \times U$ with the Poisson structure associated with the canonical $\ell$-matrix on $U \times G^*, \ell^* \times G^*$, where $G^*$ is the connected, simply connected Lie group with Lie algebra $g^*$.

**Proof.** We compute $-T^*: U \times l^* \times g \to U \times l \times (g^*_0)^*$ from equation (5.20).

$$-T^*_p(\alpha, x) = \left( -p_1 \frac{\text{Ad}_{\text{osp}} - 1 + \text{ad}_{\text{sp}}}{\text{ad}_{\text{sp}}} \alpha + p_1 \frac{\text{Ad}_{\text{osp}} - 1}{\text{ad}_{\text{sp}}} x, \right.$$

$$\left. - (p_{m^\perp} + p_m) \frac{\text{Ad}_{\text{osp}} - 1}{\text{ad}_{\text{sp}}} \alpha + (p_{m^\perp} + p_m) \text{Ad}_{\text{osp}} x \right)$$

for $\alpha \in l^*$ and $x \in g$.

Now let $T^*$ be the algebroid isomorphism

$$T^*: U \times l^* \times g^{\mathbf{op}} \to U \times l \times (g^*_0)^*$$

given by Theorem 3.4 associated with the datum $G^*$. An easy computation shows that $T^* \circ \tilde{\mathbf{op}} = -T^*$, i.e., that $-T^*$ is indeed a Lie algebroid isomorphism, where $\tilde{\mathbf{op}}: U \times l^* \times g \to U \times l^* \times g^{\mathbf{op}}$ is the Lie algebroid isomorphism given by $\tilde{\mathbf{op}}_p(\alpha, x) = (\alpha, \mathbf{op}(x))$ for all $p \in U$, $\alpha \in l^*$ and $x \in g$. \qed

In this part, we also showed the following:

**Theorem 3.7.** A dynamical Poisson groupoid on $U \subset l^*$ with $0 \in U$ is bidynamical if and only if there is a reductive decomposition $g = l \oplus m$ of the Lie algebra $g$ of the vertex group.

**Proof.** It is shown in [11] that it is a necessary condition. Theorem 3.6 shows that it is indeed a sufficient condition. \qed

**Example 3.8** (Dual of the Alekseev-Meinrenken $r$-matrix). Let $G = (g, [\ , \ ], 0, \varphi)$ be a cocommutative Lie quasi-bialgebra. Clearly, the dual over $g$ of $G$ is $G$ itself, so that $r^{AM}$ is self-dual.
and we obtain the form of the canonical $\ell$ for $\alpha$, $\beta$, $\gamma$ (with $\pm$ simple roots. We denote by $\Delta$ the Killing form). We recall (see [6]) the form of $r_p$ given by

$\ell$-matrix associated with $\Gamma$ is given as follows:

$$t_\alpha(p) = \begin{cases} \cotanh((\alpha, p + \mu)_B) & \text{if } \alpha \in \langle \Gamma \rangle, \\ \pm 1 & \text{if } \alpha \in \bar{\Gamma} \pm \end{cases}$$

($\mu$ is assumed to be chosen such that $t_\alpha$ is defined for $p = 0$). The Etingof–Varchenko dynamical $r$-matrix associated with $\Gamma$ is given as follows:

$$r^EV \alpha = t_\alpha(p)e_{-\alpha}, \quad r^EV \alpha = 0, \quad \alpha \in \Delta. \tag{3.25}$$

We set $t = r^EV_0 \in \bigwedge^2 \mathfrak{g}$, and $t_\alpha = t_\alpha(0)$. Using Proposition [11] we see that $r^EV - t$ is a dynamical $\ell$-matrix for the Lie quasi-bialgebra $G^\ell = (\mathfrak{g}, [\cdot, \cdot], t, \varphi^\ell)$. The dual over $\mathfrak{l}$ of $G^\ell$ is $G^* = (\mathfrak{g}^*, [\cdot, \cdot]^{G^*}, \varpi^*, \varphi^*)$ with

$$\varpi^* e_{\alpha} \beta = -t_\alpha(\alpha, \beta)e_{\alpha}, \quad \langle \alpha \otimes \beta \otimes 1, \varpi \rangle = 0,$n

$$\varpi^* e_{\alpha} \gamma = -N_{\gamma, \alpha} - \gamma e_{\alpha - \gamma}, \quad \langle \alpha \otimes e_\beta \otimes 1, \varpi \rangle = (\alpha, \beta)e^{-\beta},$$

$$\varpi^* e_{\alpha} e_{\alpha} = t_\alpha h_{\alpha}, \quad \langle e_\alpha \otimes e_\beta \otimes 1, \varpi \rangle = \delta_{\alpha, -\beta} h_{\alpha},$$

for $\alpha, \beta, \gamma \in \Delta, \alpha \neq \gamma$. We compute $ad_{sp}^*$ and its powers:

$$ad_{sp}^{2k} e_\alpha = (p, \alpha)^{2k} e_\alpha, \quad ad_{sp}^{2k+1} e_\alpha = (p, \alpha)^{2k+1} (-t_\alpha e_\alpha + e^{-\alpha}),$$

$$ad_{sp}^{2k+1} e_\alpha = (p, \alpha)^{2k+1} ((t_{\alpha}^2 - 1)e_\alpha + t_\alpha e_\alpha),$$

and we obtain the form of the canonical $\ell$-matrix associated with $G^*$:

$$\ell_p^EV e_\alpha = \frac{1}{\cotanh((\alpha, p)_B)} - t_\alpha e^{-\alpha}, \tag{3.26}$$

$$\ell_p^EV \alpha = 0. \tag{3.27}$$

We end this example with the following remark: the $r$-matrix $r^EV$ is linked to the function $x \mapsto \cotanh(x - a)$ which satisfies the following differential equation:

$$f'(x) - f(x)^2 = 1,$$

and the $\ell$-matrix $\ell^EV$ is related to the function $x \mapsto \frac{1}{\cotanh(x - a)}$ which satisfies the following differential equation:

$$f'(x) + (a^2 - 1)f(x)^2 + 2af(x) = -1.$$

**Example 3.10 (Dual of the non-compatible $r$-matrix of Example 2.14).** Let $G = (\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{m}, [\cdot, \cdot], 0, \varphi)$ be the bidynamical Lie quasi-bialgebra of Example 2.14 (with $\varphi \in \text{Alt}(\mathfrak{l} \otimes \mathfrak{l} \otimes \mathfrak{m})$). The dual over $\mathfrak{l}$ of the $G$ is $G^* = (\mathfrak{g}^*, [\cdot, \cdot]^G, \varpi^*, \varphi^*)$ where the Lie bracket $[\cdot, \cdot]^G$ on $\mathfrak{g}^*$ is given by:

$$[z, z']^G = [z, z']_B, \quad [z, \xi]^* = -ad^*_\xi, \quad [\xi, \xi]^* = 0. \tag{3.28}$$
for all \( z, z' \in l, \xi, \xi' \in l^\perp \), the cobracket \( \varpi^* \) is given by:
\[
\varpi^*_{+\xi}(s\alpha + u) = -(\xi \otimes s\alpha \otimes 1, \varphi) - p_{l^\perp} \operatorname{ad}^*_u \xi,
\]
(3.29)
for all \( z \in l, \alpha \in l^\ast, u \in m \) and \( \xi \in l^\perp, \) and \( \varphi^* \) is given by:
\[
\langle (s\alpha + u) \otimes (s\beta + v) \otimes 1, \varphi^* \rangle = p_l[u,v]_g + \operatorname{ad}^*_v s\alpha - \operatorname{ad}^*_u s\beta,
\]
(3.30)
for all \( \alpha, \beta \in l^\ast \) and \( u, v \in m \). We set \( \mathfrak{e} = l \subset g^* \) and \( u = l^\perp \subset g^* \), so that \( g^* = \mathfrak{e} \oplus u \). We denote by \( \operatorname{ad}^* \) the adjoint action of the double \( \mathfrak{d}^* \) of the Lie quasi-bialgebra \( G^* \). One has:
\[
\operatorname{ad}^*_{sp} \mathfrak{e} \subset u \subset \mathfrak{g}^{**},
\]
(3.31)
\[
\operatorname{ad}^*_{sp} \mathfrak{e}^\perp \subset u \subset \mathfrak{g}^*,
\]
(3.32)
We define the following functions:
\[
F^*(z) = \frac{2 \sin(z) \operatorname{sh}(z)}{\operatorname{ch}(z) \sin(z) + \cos(z) \operatorname{sh}(z)} = \sum_{k \geq 0} F^*_k z^{4k+1},
\]
(3.33)
\[
G^*(z) = -2 \frac{-\cos(z) \operatorname{sh}(z) + \sin(z) \operatorname{ch}(z))^2}{\operatorname{ch}(2z) \cos(2z) - 1} = \sum_{k \geq 0} G^*_k z^{4k+2},
\]
(3.34)
\[
H^*(z) = \frac{2z \cos(z) \operatorname{ch}(z) - \sin(z) \operatorname{sh}(z) - \cos(z) \operatorname{sh}(z) \operatorname{ch}(z)}{z(\cos(z) \operatorname{sh}(z) + \sin(z) \operatorname{ch}(z))} = \sum_{k \geq 0} H^*_k z^{4k+3}.
\]
(3.35)
A computation shows that \( l^{\text{can}}(G^*) \) reads as:
\[
\begin{align*}
p_{p^{l^{\text{can}}(G^*)}} u &= F^* (\operatorname{ad}^*_{sp}) u, \quad (3.36) \\
p_{p^{l^{\text{can}}(G^*)}} u &= G^* (\operatorname{ad}^*_{sp}) u, \quad (3.37) \\
p_{\mathfrak{u}^{l^{\text{can}}(G^*)}} s\alpha &= -G^* (\operatorname{ad}^*_{sp}) s\alpha, \quad (3.38) \\
p_{\mathfrak{p}^{l^{\text{can}}(G^*)}} s\alpha &= H^* (\operatorname{ad}^*_{sp}) s\alpha, \quad (3.39)
\end{align*}
\]
where \( u \in \mathfrak{e}^\perp, \alpha \in \mathfrak{e}^* \). Equation (2.8) for \( l^{\text{can}}(G^*) \) is equivalent to the following the differential system:
\[
\begin{align*}
zF^{**}(z) &= z(1 + G^*(z))^2, \\
zG^{**}(z) &= zF^*(z) - zG^*(z)H^*(z) - G^*(z), \\
zH^{**}(z) &= -2zG^*(z) - zH^*(z)^2 - 2H^*(z).
\end{align*}
\]
(4.40)
4. LINK BETWEEN \( l^{\text{can}}(G^,l,m) \) AND \( l^{\text{can}}(G^*,l^\perp) \)

Let \( G = (g, [\cdot, \cdot], \varpi, \varphi) \) be a Lie quasi-bialgebra. It is well-known (see e.g. [1]) that the canonical double \( \mathfrak{d} \) of \( G \) carries a Lie quasi-bialgebra structure:
\[
G^{(2)} = (\mathfrak{d}, [\cdot, \cdot], \partial \mathfrak{r}, j^{(3)} \varphi)
\]
(4.1)
where \( j : g \to \mathfrak{d} \) is the canonical inclusion, and \( \partial \mathfrak{r} \) is the coboundary of the “r-matrix” \( \mathfrak{r} = \frac{1}{2}(p_{\mathfrak{e}^*} - p_g) \in \wedge^2 \mathfrak{d} \):
\[
\partial X \mathfrak{r} = \operatorname{ad}_X \mathfrak{r} + \mathfrak{r} \operatorname{ad}^*_X, \quad X \in \mathfrak{d}.
\]
(4.2)
It is also well-known that twisting \( G^{(2)} \) via \( -\mathfrak{r} \) yields the cocommutative Lie quasi-bialgebra \( (G^{(2)})^{-\mathfrak{r}} = (\mathfrak{d}, [\cdot, \cdot], 0, \frac{1}{2}(\Omega^2, \Omega^0)) \) where \( \Omega^0 \in S^2 \mathfrak{d} \) is the symmetric 2-tensor associated with the canonical bilinear form on \( \mathfrak{d} \).
Now, if $\mathcal{G}$ is a canonical dynamical Lie quasi-bialgebra, then so is $\mathcal{G}^{(2)}$ with reductive decomposition $\mathfrak{d} = \mathfrak{l} \oplus (\mathfrak{m} \oplus \mathfrak{g}^*)$ over $\mathfrak{l}$, and $j$ is thus a canonical dynamical Lie quasi-bialgebra morphism. Thus, it follows from Proposition 2.16 that

$$j^* \text{can}(\mathcal{G}, \mathfrak{l}, \mathfrak{m}) j^* = \mathcal{G}^{(2)}(\mathfrak{l}\mathfrak{m} \oplus \mathfrak{g}^*).$$

(4.3)

We denote by $K$ the Lie algebra isomorphism from the double $\mathfrak{d}$ of $\mathcal{G}$ to the double $\mathfrak{d}^*$ of the dual $\mathcal{G}^*$ given by $K = p_E \mathfrak{l} - p_m \mathfrak{m} \oplus \mathfrak{m}$ when the vector spaces $\mathfrak{d}$ and $\mathfrak{d}^*$ are canonically identified. Let $r^* = \frac{1}{2}(p_{\mathfrak{g}^*} - p_{\mathfrak{g}^*})$ be the $r$-matrix associated to the Manin triple $(\mathfrak{d}, \mathfrak{g}^*, \mathfrak{g}^*)$. Clearly, $\Omega^r = (K \otimes K) \Omega^r$, and $K: (\mathcal{G}^{(2)})^{-r} \rightarrow (\mathcal{G}^{*^{(2)})}^{-r}$ is thus a canonical dynamical Lie quasi-algebra morphism. Hence, using Proposition 2.14 we have:

$$K \left( j^* \text{can}(\mathcal{G}, \mathfrak{l}, \mathfrak{m}) j^* + r^* \right) K^* \in \text{Dynl}(U, (\mathcal{G}^{(2)})^{-r})$$

(4.4)

and thus

$$Kj^* \text{can}(\mathcal{G}, \mathfrak{l}, \mathfrak{m}) j^* K^* + Kr^* K^* + r \in \text{Dynl}(U, \mathcal{G}^{(2)}).$$

(4.5)

A simple computation shows that

$$Kr^* K^* + r = p_l - p_{m^\perp}.$$  

(4.6)

Thus, the transformation of the $\ell$-matrix of equation (4.5) by the element $\zeta$: $p \mapsto e^{sp}$ of $\text{Map}_D(U, D)^l$ (where $D$ is the connected, simply-connected Lie group with Lie algebra $\mathfrak{d}$), satisfies:

$$\left( Kj^* \text{can}(\mathcal{G}, \mathfrak{l}, \mathfrak{m}) j^* K^* + Kr^* K^* + r \right)^\zeta_0 = 0$$

(4.7)

as well as

$$\left( Kj^* \text{can}(\mathcal{G}, \mathfrak{l}, \mathfrak{m}) j^* K^* + Kr^* K^* + r \right)^\zeta sp = 0.$$  

(4.8)

Thus, from the uniqueness result of Proposition 2.2 we must have:

$$Kj^* \text{can}(\mathcal{G}, \mathfrak{l}, \mathfrak{m}) j^* K^* = \left( j^* \text{can}(\mathcal{G}, \mathfrak{l}, \mathfrak{m}) j^* \right)^{-1} - p_l + p_{m^\perp}.$$  

(4.9)

References

[1] A. Alekseev, Y. Kosmann-Schwarzbach, Manin pairs and moment maps, J. Diff. Geom. 56 (2000), 133–165.
[2] A. Alekseev, E. Meinrenken, The non-commutative Weil algebra, Invent. Math. 139 (2000), 135–172.
[3] V. G. Drinfel’d, Quasi-Hopf algebras, Leningrad Math. J. 1 (1990), 1419–1457.
[4] B. Enriquez, P. Etingof, Quantization of Alekseev–Meinrenken dynamical $r$-matrices, in Lie groups and symmet-
ic spaces, 81–98, Amer. Math. Soc. Transl. Ser. 2, 210, Amer. Math. Soc. (2003).
[5] P. Etingof, O. Schiffmann, On the moduli space of classical dynamical $r$-matrices, Math. Res. Lett. 8 (2001), 157–170.
[6] P. Etingof, A. Varchenko, Geometry and classification of solutions of the classical dynamical Yang–Baxter equation, Comm. Math. Phys. 192 (1998), 77–120.
[7] G. Felder, Conformal field theory and integrable systems associated to elliptic curves, Proceedings of the Interna-
tional Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994), Birkhäuser, Basel (1995), 1247–1255.
[8] L. C. Li, S. Parmentier, On dynamical Poisson groupoids I, Mem. Amer. Math. Soc. 174 (2005), no. 824.
[9] K. C. H. Mackenzie, Lie groupoids and Lie algebroids in differential geometry, London Mathematical Society
Lecture Note Series, 124, Cambridge University Press, Cambridge, 1987.
[10] K. C. H. Mackenzie, On symplectic double groupoids and the duality of Poisson groupoids, Internat. J. Math. 10
(1999), no. 4, 435–456.
[11] S. Parmentier, R. Pujol, Quasi-bialgebras and dynamical $r$-matrices, Adv. Math 197 (2005), 41–85.
[12] R. Pujol, Équations de Yang–Baxter dynamiques classiques, groupoïdes de Poisson, quasi-bigèbres de Lie et dualité, Ph.D. thesis (in french), Université Claude Bernard Lyon 1 (2005).
[13] R. Pujol, $A$-dynamical Poisson groupoids, in final preparation.
[14] A. Weinstein, Coisotropic calculus and Poisson groupoids, J. Math. Soc. Japan 40 (1988), 705–727.
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