FOUR-PERIODIC INFINITE STAIRCASES FOR
FOUR-DIMENSIONAL POLYDISKS

CADEN FARLEY, TARA S. HOLM, NICKI MAGILL, JEMMA SCHRODER, ZICHEN WANG,
MORGAN WEILER, AND ELIZAVETA ZABELINA

ABSTRACT. The ellipsoid embedding function of a symplectic four-manifold measures
the amount by which its symplectic form must be scaled in order for it to admit an
embedding of an ellipsoid of varying eccentricity. This function generalizes the Gromov
width and ball packing numbers. In the one continuous family of symplectic four-
manifolds that has been analyzed, one-point blowups of the complex projective plane,
there is an open dense set of symplectic forms whose ellipsoid embedding functions are
completely described by finitely many obstructions, while there is simultaneously a
Cantor set of symplectic forms for which an infinite number of obstructions are needed.
In the latter case, we say that the embedding function has an infinite staircase. In
this paper we identify a new infinite staircase when the target is a four-dimensional
polydisk, extending a countable family identified by Usher in 2019. Our work computes
the function on infinitely many intervals and thereby indicates a method of proof for a
conjecture of Usher.

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1. Introduction

A **symplectic form** on a $2n$-dimensional smooth manifold $X$ is a differential 2-form satisfying:

- $d\omega = 0$, i.e., $\omega$ is closed, and
- $\omega^n \neq 0$, i.e., $\omega$ is nondegenerate.

A symplectic form can be thought of as a skew-symmetric version of a Riemannian metric, providing area rather than length measurement. Symplectic geometry forms the mathematical framework for classical mechanics and is a go-between from Riemannian to complex geometry.

The **volume** $\text{vol}(X)$ of a symplectic manifold is the quantity $\int_X \omega^n$. We say a smooth embedding $\varphi : (X, \omega) \to (X', \omega')$ is **symplectic** if $\varphi^*(\omega') = \omega$, and we denote symplectic embedding by $\varphi : (X, \omega) \overset{\varphi}{\hookrightarrow} (X', \omega')$, or $X \overset{\varphi}{\hookrightarrow} X'$ when the symplectic form is clear from context and we are not emphasizing the specific embedding $\varphi$.

Let $(X, \omega)$ be a four-dimensional symplectic manifold. Its **ellipsoid embedding function**\(^1\) is

$$c_X(z) := \inf \left\{ \lambda \mid (E(1, z), \omega_0) \overset{\lambda}{\hookrightarrow} (X, \lambda \omega) \right\}, \quad (1.0.1)$$

where $z \in \mathbb{R}_{>0}$, $\lambda X := (X, \lambda \omega)$ is $X$ with the symplectic form scaled, the ellipsoid $E(c, d) \subset \mathbb{C}^2$ is the set

$$E(c, d) = \left\{ (\zeta_1, \zeta_2) \in \mathbb{C}^2 \mid \pi \left( \frac{|\zeta_1|^2}{c} + \frac{|\zeta_2|^2}{d} \right) < 1 \right\},$$

and $\omega_0$ is the standard symplectic form $dx_1 \wedge dy_1 + dx_2 \wedge dy_2$ on $\mathbb{C}^2$. Note that the associated volume form is twice the standard volume form on $\mathbb{R}^4$, thus $\text{vol}(E(c, d)) = cd$. There is a symmetry that allows us to reduce to $z \geq 1$. Namely, for $0 < z < 1$ we have $c_X(z) = z c_X(1/z)$, because $\omega_0$ restricted to $E(1, z)$ equals $z \omega_0$ restricted to $E(1/z, 1)$ under the diffeomorphism $(\zeta_1, \zeta_2) \mapsto (\zeta_1/\sqrt{z}, \zeta_2/\sqrt{z})$. Therefore, from now on we restrict the domain of $c_X(z)$ to $\mathbb{R}_{\geq 1}$.

The ellipsoid embedding function generalizes the Gromov width\(^2\) via

$$c_{Gr}(X, \omega) = \frac{1}{c_X(1)}$$

and the fraction of the volume of $X$ that can be filled by $n \in \mathbb{Z}_{\geq 1}$ equal balls can, by [Mc1, Thm. 1.1], be computed from $c_X$ via

$$\frac{n}{c_X(n)^2 \text{vol}(X)}.$$

---

\(^1\)It is sometimes also called the **embedding capacity function** or **capacity function**.

\(^2\)The **Gromov width** of a symplectic manifold is $\sup \{ r \mid E(r, r) \overset{\varphi}{\hookrightarrow} (X, \omega) \}$, or the largest ball that embeds into $(X, \omega)$.
For a class of targets \((X, \omega)\) called “finite type convex toric domains” (see §2.1) which includes the polydisks that we study, the ellipsoid embedding function satisfies several key properties.

**Proposition 1.0.1** ([CGHMP, p. 4, Prop. 2.1]). Let \((X, \omega)\) be a finite type convex toric domain. The ellipsoid embedding function \(c_X(z)\) satisfies the following properties.

1. \(c_X(z) \geq \sqrt{\frac{z}{\text{vol}(X)}}\);
2. \(c_X\) is nondecreasing;
3. \(c_X\) is sublinear: for all \(t \geq 1\), we have \(c_X(tz) \leq tc_X(z)\);
4. \(c_X(z)\) is continuous (in \(z\));
5. \(c_X(z)\) is equal to the volume curve for sufficiently large values of \(z\); and
6. \(c_X(z)\) is piecewise linear, when not equal to the volume curve and not at the limit of singular points.

We say \(c_X\) or \(X\) has an **infinite staircase** if it is nonsmooth at infinitely many points. An **outer corner** is a nonsmooth point near which the function is concave while an **inner corner** is one near which the function is convex. By Proposition 1.0.1 (v), the set of nonsmooth points is bounded. By [CGHMP, Thm. 1.13] (see Theorem 2.1.2 for a statement in our case), the nonsmooth points of \(c_X\) have a unique finite limit point called the **accumulation point**, whose \(z\)-coordinate we denote by \(\text{acc}(X)\). (By abuse of notation, we also refer to this \(z\)-coordinate as the “accumulation point.”) We say an infinite staircase is **ascending** if the nonsmooth points accumulate from the left and **descending** if the nonsmooth points accumulate from the right. These concepts are illustrated in Figure 1.0.2. In this paper, we will establish the existence of an ascending staircase.

**Figure 1.0.2.** In blue, the graph of the embedding capacity function for a ball \(X = B^4(1)\) is shown on the domain indicated. The graph in red is the volume lower bound established in Proposition 1.0.1(i). The point marked O is an outer corner and the point marked I is an inner corner. This target has an ascending infinite staircase, first identified by McDuff and Schlenk [McSc] and called the Fibonacci staircase in the literature. The green point is the accumulation point.
1.1. Summary of results. Our target of choice will be the polydisk, defined for \( \beta \in \mathbb{R}_{\geq 1} \) by

\[
P(1, \beta) := \{ (\zeta_1, \zeta_2) \in \mathbb{C}^2 \mid \pi |\zeta_1|^2 \leq 1, \pi |\zeta_2|^2 \leq \beta \}\.
\]

We denote by \( c_\beta \) its ellipsoid embedding function \( c_{P(1, \beta)} \). The polydisk is a finite type convex toric domain, so \( c_\beta \) satisfies Proposition 1.0.1. In this case there are two functions

\[
\text{acc}(\beta) := \text{acc}(P(1, \beta)) : [1, \infty) \to \left[ 3 + 2\sqrt{2}, \infty \right)
\]

\[
\text{vol}(\beta) := \sqrt{\frac{\text{acc}(\beta)}{\text{vol}(P(1, \beta))}} : [1, \infty) \to \left[ 1 + \frac{\sqrt{2}}{2}, 1 \right]
\]

where if \( c_\beta \) has an infinite staircase, its accumulation point has coordinates \((\text{acc}(\beta), \text{vol}(\beta))\) by [CGHMP, Thm. 1.13]; see Lemma 2.1.3.

The first ellipsoid embedding function was computed for \( X = B^4 := E(1, 1) \) by McDuff and Schlenk in [McSc]. They found that its graph contained an infinite staircase whose inner and outer corners were derived from the Fibonacci numbers. Further work by Frenkel and Müller in [FM] exhibited a similar infinite staircase in \( c_1 \) governed by the Pell numbers, while on the other hand work of Cristofaro-Gardiner, Frenkel, and Schlenk showed that the property of having an infinite staircase is not universal: the functions \( c_n \) for \( n \in \mathbb{Z}_{>1} \) do not contain infinite staircases [CGFS]. More generally, a conjecture of Cristofaro-Gardiner, Holm, Mandini, and Pires in [CGHMP] suggests that \( c_\beta \) should not contain an infinite staircase for any rational \( \beta \).

However, work by Usher [U1] suggested that the set of irrational \( \beta \) for which \( c_\beta \) has an infinite staircase might be quite rich: he identified a bi-infinite family \( L_{n,k} \in \mathbb{R}_{\geq 1} \) for which \( c_{L_{n,k}} \) have infinite staircases.\(^3\) Of particular interest to us are his

\[
L_{n,0} := \sqrt{n^2 - 1}, \quad n \geq 2
\]

which generate the \( k > 0 \) values of \( L \) with infinite staircases (see §4.4). See Figure 1.1.1 for a visualization of these results via a plot of the relevant accumulation points.

\(^3\)In this paper as well as in the closely related papers [BHM], [MM], and [MMW] we use \( k \) to denote the staircase step and \( i \) to denote the image of a step, staircase, or \( b \) value under a symmetry analogous to Usher’s Brahmagupta moves ([U1, Def. 2.10]). Our notation differs from Usher’s in that what the \( i \) and \( k \) indices denote are switched. We generally stick to our convention throughout but use Usher’s convention here.
Figure 1.1.1. This figure shows the parameterized curve \( \text{acc}(\beta), \text{vol}(\beta) \) in red. The point on the curve at \( \beta \) represents a point at which an infinite staircase for \( c_\beta \) must accumulate, if it exists. The red dot is the accumulation point of the Pell stairs of Frenkel-Müller; the blue dots are the \( L_{n,0} \) staircases of Usher; and the black \( \times \)s indicate values of \( \beta \) without infinite staircases, proved by Cristofaro-Gardiner–Frenkel–Schlenk. The accumulation points of the new infinite staircases of Theorem 1.1.1 and Conjecture 1.1.2 are indicated by green dots.

Work by Bertozzi, Holm, Maw, McDuff, Mwakyoma, Pires, and Weiler [BHM] and by Magill and McDuff [MM] proved an analogous result for the target

\[
H_b := \{ (\zeta_1, \zeta_2) \in \mathbb{C}^2 \mid \pi|\zeta_1|^2 + \pi|\zeta_2|^2 \leq 1, \pi|\zeta_2|^2 \leq 1 - b \}.
\]

(The region \( H_b \) is equivalent in terms of ellipsoid embeddings, see §2.1.1, to \( \mathbb{C}P^2 \# \mathbb{C}P^2 \), thus in the literature on infinite staircases it is also called the Hirzebruch surface.) They showed that there are two bi-infinite families \( b_{n,i,\delta} \), with \( n, i \in \mathbb{Z}_{\geq 0} \) and \( \delta \in \{0, 1\} \), for which \( c_{H_b} \) has an ascending infinite staircase. Moreover, each ascending infinite staircase comes paired with a descending infinite staircase.

One feature that all infinite staircases described so far appear to have in common is that their outer corners are at \( z \)-values whose continued fractions grow by a predictable pattern of adding pairs of integers. Recall that real numbers can be described by their continued fractions, e.g.

\[
[m, n, \ell] = m + \frac{1}{n + \frac{1}{\ell}},
\]

with repeated parts denoted by

\[
[m, \{n, \ell\}]^k = [m, n, \ell], \quad [m, \{n, \ell\}]^\infty = [m, n, \ell, n, \ell, n, \ell, \ldots].
\]

Every positive real number has a continued fraction with all entries positive integers; rational numbers have finite continued fractions, quadratic irrational numbers (irrational roots of quadratic equations with rational coefficients) have infinite periodic continued
fractions, and non-quadratic irrational numbers have infinite non-periodic continued fractions. We will abuse notation and occasionally allow the last entry in a finite continued fraction to be a real number, e.g. in the proof of Theorem 1.1.1 (ii) in §3.1. Doing so is just a matter of notation, because if $a_i \in \mathbb{Z}_{>0}$ and $z \in \mathbb{R}$ has continued fraction $[b_0, b_1, \ldots]$ then $[a_0, \ldots, a_n, z] = [a_0, \ldots, a_n, b_0, b_1, \ldots]$. Allowing the last number to be real can be helpful when trying to understand the algebraic relationships among the continued fraction's rational approximations, as in Lemma 3.1.5.

Recall that an outer corner of $c_X$ is a nonsmooth point near which $c_X$ is convex; see Figure 3.0.2. The outer corners of the Fibonacci stairs of McDuff-Schlenk have continued fractions $[2], [5], [6, 1, 5, 2], [6, 1, 5, 1, 4], [6, 1, 5, 1, 5, 2], [6, 1, 5, 1, 5, 1, 4], \ldots$

The accumulation points of all infinite staircases discussed so far are quadratic irrationals with two-periodic continued fractions. We say an infinite staircase is $2m$-periodic if the continued fraction of the $k$th outer corner equals that of the $(k - 2)$th outer corner with a fixed length $2m$ sequence of integers added after a fixed sequence of integers at the beginning. For example, in the sequence above, the Fibonacci stairs are 2-periodic with a pair 1,5 inserted recursively after the 6.

In [MMW], Magill, McDuff, and Weiler showed that between each of the pairs of adjacent ascending and descending infinite staircases studied in [BHM, MM] there is a further Cantor set of values of $b$ for which $c_{H_b}$ has an infinite staircase. These include infinite staircases whose outer corners and accumulation points appear to have higher-periodic continued fractions, as well as infinite staircases whose accumulation points may not be quadratic irrational. They were obtained by generalizing the procedure to construct an infinite staircase whose outer corners have four-periodic continued fractions accumulating to $\{7, 5, 3, 1\}^\infty$ from the descending staircase accumulating to $[7, \{5, 1\}^\infty]$ and the ascending staircase accumulating to $\{7, 3\}^\infty$.

We predict a very close correspondence between the cases of the polydisk and $H_b$ in Conjecture 1.2.1. Our main theorem provides evidence for this conjecture.

**Theorem 1.1.1.** Set

$$\beta = \frac{6 + 5\sqrt{30}}{12}.$$

(i) The function $c_{\beta}$ has an infinite staircase.

(ii) It is four-periodic, with $\text{acc}(\beta) = \{8, 6, 4, 2\}^\infty$.

See Figure 1.1.2 for a visualization. Detail on the location of the accumulation point of the infinite staircase of Theorem 1.1.1 is given in Figure 1.1.3.

Of note is the fact that we prove Theorem 1.1.1 in §3 by computing it on infinitely many intervals. In §3 we also outline a procedure for computing $c_{\beta}$ on the entire interval $[1, \text{acc}(\beta)]$ containing the infinite staircase. This would prove an analogue of [U1, Conj. 4.23], with the role of his $A$ classes being played by our $E$ classes and his $\hat{A}$ classes replaced by our $\hat{E}$ classes: see the preamble to §3 for the definitions $E$ and $\hat{E}$, and see §4.2 for further discussion of Usher’s conjecture.
Figure 1.1.2. This figure depicts the infinite staircase $c_\beta$ of Theorem 1.1.1. In both figures, with $\beta$ as in Theorem 1.1.1, the orange curve is $\text{vol}_\beta(z)$ and $c_\beta$ is in blue. The accumulation point curve $(\text{acc}(\beta), \text{vol}(\beta))$ is in red – for this curve, $\beta$ varies. Thus the accumulation point of $c_\beta$ occurs at the intersection of these three curves. In (b), we have zoomed in; the obstructions from $E_0, \hat{E}_1,$ and $E_2$ are visible. See sections §2.2 and §3 for these definitions.

Figure 1.1.3. This figure uses the same color scheme as Figure 1.1.1. More detail near the infinite staircase of Theorem 1.1.1 is shown. The new staircase’s accumulation point is the green dot, while the two blue dots are Usher’s staircases with $\beta = L_{2,0}$ and $L_{3,0}$.

We furthermore expect (from experimental evidence and by combining Conjecture 1.1.2 with [MMW, Thm. 1.1.1]) that our result generalizes to all $n \in \mathbb{Z}_{\geq 2}$:

**Conjecture 1.1.2.** Let $\beta_n$ be of the form

$$\beta_n = \frac{1}{2} + \frac{(2n + 1) \sqrt{n^3 + 2n^2 - 1}}{2n(n + 1)}$$
with $n \in \mathbb{Z}_{\geq 2}$.

(i) The function $c_{\beta_n}$ has an infinite staircase.

(ii) It is four-periodic, with $\text{acc}(\beta_n) = \{2n + 4, 2n + 2, 2n, 2n - 2\}$.

Note that setting $n = 2$ in Theorem 1.1.2 reduces to Theorem 1.1.1.

1.2. Connections to other targets. There is a function

$$\text{acc}_H : [0, 1) \rightarrow [3 + 2\sqrt{2}, \infty),$$

which is analogous to acc in the following way: if $c_{H_b}$ has an infinite staircase then $\text{acc}_H(b)$ is the $z$-coordinate of its accumulation point. It is 2-1 in general but when restricted to $[1/3, 1)$ it is 1-1 with range $[3 + 2\sqrt{2}, \infty)$. These facts have a similar proof to Lemma 2.1.3.

Conjecture 1.2.1. Define a function $f : \mathbb{R} \rightarrow \mathbb{R}$ by sending $z$ to the number whose continued fraction is obtained from the continued fraction of $z$ by subtracting one from each entry.

Let $\beta \geq \sqrt{3}$. The function $c_\beta$ has an infinite staircase if and only if the function $c_{H_b}$ has an infinite staircase, where

$$b = \text{acc}_H^{-1} \circ f \circ \text{acc}(\beta).$$

The infinite staircases $c_{L_n,0}$ of [U1], $c_{\beta_n}$ of Conjecture 1.1.2, and the fact that $c_n, n \in \mathbb{Z}$ do not contain infinite staircases from [CGFS] all support Conjecture 1.2.1. Further evidence is explored in [MPW, §3 and Remark 3.1.7]. It is possible to extend Conjecture 1.2.1 to all $\beta$ using the Brahmagupta moves of Usher and their counterparts for the Hirzebruch surface from [MM]; explaining the extension is beyond the scope of this paper. Because of the similarities between §3.2 and [M1], we expect there is a more direct relationship between $b$ and $\beta$ than via the accumulation point function $\text{acc}$, but this has not yet been discovered.

In Section 3.2, to construct the desired embeddings, we followed the sequences of mutations on almost toric fibrations found by Magill in [M1] and [M2]. In [M1], ATF mutations for ellipsoid embeddings into $H_b$ were considered. The formulas found for the mutations in Section 3.2 mirror the formulas found in [M1]. In fact, the formulas in Lemma 3.2.9 and 3.2.12 could be easily generalized to mirror the formulas of [M1, Def. 3.8]. Therefore, we expect a generalization of Prop 3.2.10 similar to [M1, Thm. 1.1]. This is more evidence for the correspondence between the embedding functions for polydisks and $H_b$.

In [M2], ellipsoid embeddings into a two-fold blow up of $\mathbb{C}P^2$ were considered. In Section 3.2, we follow the same mutation sequences Magill used to compute some of the inner corners of the function. One new addition in Section 3.2 is Conjecture 3.0.7 that adding one extra mutation to Magill’s sequences will compute all the inner corners of the function for $c_\beta$. The work of Casals and Vianna in [CV] and Cristofaro-Gardiner, Holm, Mandini, and Pires in [CGHMP] show that almost toric mutations give all embeddings for particular rational convex toric domains. Conjecture 3.0.7 would imply a similar statement holds for $c_\beta$ where $\beta = \frac{6+5\sqrt{30}}{12}$. It would be interesting to see if similar statements hold for $H_b$ and the two fold blow up of $\mathbb{C}P^2$. 
1.3. Outline of the paper. We introduce the necessary tools to analyze $c_\beta$ in §2, prove Theorem 1.1.1 in §3, and in §4 we outline future work supported by other experimental evidence discovered in summer 2022. The section §2.1.1 requires a graduate-level background in geometry and can be skipped on a first reading.

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2. Tools for obstructing and constructing embeddings

In this section we define the tools we use to prove Theorem 1.1.1.

2.1. Embedding functions of toric domains. A toric domain $X_\Omega$ in $\mathbb{C}^2$ is the preimage of a domain $\Omega \subset \mathbb{R}^2_{\geq 0}$ under the map $\mu : \mathbb{C}^2 \to \mathbb{R}^2$ given by

$$(\zeta_1, \zeta_2) \mapsto (\pi |\zeta_1|^2, \pi |\zeta_2|^2).$$

We call the map $\mu$ the moment map and the domain $\Omega$ the moment polygon of $X_\Omega$, as they are analogous to the moment maps and moment polygons associated to closed toric symplectic manifolds. We say that a toric domain $X_\Omega$ is convex if the domain $\Omega$ is a closed, connected region of $\mathbb{R}^2$ and is convex as a polygon in $\mathbb{R}^2$. As a consequence of the presence of factors of $\pi$ in the expression for the moment map $\mu$, the volume of a toric domain $X_\Omega$ coincides with twice the area of its moment polygon $\Omega$.

When $(X, \omega) = (X_\Omega, \omega_0)$, instead of (1.0.1) we write

$$c_X(z) := \inf \left\{ \lambda \mid E(1, z) \xrightarrow{\delta} X_\lambda \right\},$$

dropping the symplectic forms from the notation.

We say that a convex toric domain $X_\Omega$ is of finite type if $\Omega$ has only finitely many sides and all of these sides have rational slopes. For these finite type toric domains, the accumulation points of potential infinite staircases can be computed as solutions to an explicit quadratic equation. For details of this result and the following definition, see the paper [CGHMP].
Definition 2.1.1. Let $L$ be a line segment in $\mathbb{R}^2$. The affine length of $L$ is the length of the image $AT(L)$ of $L$ under a composition of a translation $T$ with a linear transformation $A \in \text{SL}(2, \mathbb{Z})$, where $A$ and $T$ are chosen so that $AT(L)$ lies along the $x$-axis.

If $\Omega$ is a polygon in $\mathbb{R}^2_0$ with only finitely many sides each of which has a rational slope, define the affine perimeter of $\Omega$ to be the sum of the affine lengths of its sides, and denote this quantity by $\text{per}(\Omega)$.

With these definitions, we can now state the following result about the accumulation points of infinite staircases of finite type convex toric domains:

Theorem 2.1.2 ([CGHMP, Thm. 1.13]). Let $X_\Omega$ be a finite type convex toric domain. If the ellipsoid embedding function $c_{X_\Omega}(z)$ has an infinite staircase, then it accumulates at $\text{acc}(\Omega) \geq 1$, a real solution\footnote{The solutions to this equation have product one and are either positive or complex. For the polydisk, there is always a unique real solution larger than one.} to the quadratic equation

$$z^2 - \left(\frac{\text{per}(\Omega)^2}{2 \cdot \text{area}(\Omega)} - 2\right)z + 1 = 0.$$

In this case, at $\text{acc}(\Omega)$, the ellipsoid embedding function touches the volume curve:

$$c_{X_\Omega}(\text{acc}(\Omega)) = \sqrt{\frac{\text{acc}(\Omega)}{2 \cdot \text{area}(\Omega)}}.$$

In the setting of this paper, $X_\Omega$ will be the polydisk $P(1, \beta)$, which has moment polygon $\Omega_\beta$ a rectangle situated at the origin with sides of length 1 and $\beta$ parallel to the $x$- and $y$-axes. Here, the affine perimeter of $\Omega_\beta$ is the same as its regular perimeter, $\text{per}(\Omega_\beta) = 2(\beta + 1)$, and the area of $\Omega_\beta$ is $\text{area}(\Omega_\beta) = \beta$. In this case, the quadratic equation in Theorem 2.1.2 becomes

$$z^2 - \left(\frac{2(\beta + 1)^2}{\beta} - 2\right)z + 1 = 0. \quad (2.1.1)$$

In addition to providing an explicit way to calculate accumulation points, Theorem 2.1.2 also describes a necessary condition for the existence of an infinite staircase for different values of $\beta$. We call the difference $c_{X_\Omega}(\text{acc}(\Omega)) - \sqrt{\frac{\text{acc}(\Omega)}{2 \cdot \text{area}(\Omega)}} \geq 0$ the staircase obstruction of $X_\Omega$. Theorem 2.1.2 indicates that if the ellipsoid embedding function $c_{X_\Omega}(z)$ has an infinite staircase, then the staircase obstruction of $X_\Omega$ vanishes. For the case where $X_\Omega = P(1, \beta)$, if the staircase obstruction does not vanish for a particular value of $\beta$, we say that this $\beta$-value is blocked, and we conclude that the ellipsoid embedding function $c_{\beta}(z)$ does not have an infinite staircase.

Finally, because the accumulation point of an infinite staircase is on the volume obstruction, the formula on the right hand side of Proposition 1.0.1 (i) specialized to the case of the polydisk will be key throughout; we set the notation

$$\text{vol}_\beta(z) := \sqrt{\frac{z}{2\beta}} = \frac{z}{\sqrt{2 \cdot \text{area}(\Omega_\beta)}}.$$

We compute the ranges of $\text{acc}$ and $\text{vol}$ to motivate Figures 1.1.1 and 1.1.3.
Lemma 2.1.3. Setting \( acc(\beta) = acc(\Omega_\beta) \), we have
\[
acc : [1, \infty) \to \left[ 3 + 2\sqrt{2}, \infty \right)
\]
and \( acc \) is increasing. If we set \( vol(\beta) = vol_\beta(acc(\beta)) \) then
\[
vol : [1, \infty) \to \left[ 1 + \frac{\sqrt{2}}{2}, 1 \right)
\]
and \( vol \) is decreasing.

Proof. Solving (2.1.1) we obtain
\[
acc(\beta) = z = \beta + 1 + \frac{1}{\beta} + \sqrt{\beta^2 + 2\beta + 2 + \frac{2}{\beta} + \frac{1}{\beta^2}},
\]
thus
\[
acc(1) = 3 + \sqrt{8} = 3 + 2\sqrt{2},
\]
and \( \lim_{\beta \to \infty} acc(\beta) = \infty \) because \( acc(\beta) > \beta \). The function \( acc(\beta) \) is increasing because
\[
\frac{\partial}{\partial \beta} \left( \beta + 1 + \frac{1}{\beta} \right) = 1 - \frac{1}{\beta^2}
\]
and
\[
\frac{\partial}{\partial \beta} \left( \beta^2 + 2\beta + 2 + \frac{2}{\beta} + \frac{1}{\beta^2} \right) = 2\beta + 2 - \frac{2}{\beta^2} - \frac{2}{\beta^3},
\]
which are both positive if \( \beta > 1 \).
Because \( vol_\beta(z) \) has \( \beta \) in the denominator, it is decreasing if \( z \) is increasing, so \( vol(\beta) \) is decreasing in \( \beta \). We compute
\[
\left( 1 + \frac{\sqrt{2}}{2} \right)^2 = 1 + \sqrt{2} + \frac{1}{2} = 3 + \frac{2\sqrt{2}}{2} = \sqrt{\frac{acc(1)}{2}} = vol(1).
\]
Finally, by the fact that \( vol \) and \( acc \) are continuous and defined on \([1, \infty)\),
\[
\left( \lim_{\beta \to \infty} vol(\beta) \right)^2 = \lim_{\beta \to \infty} \frac{\beta + 1 + \frac{1}{\beta} + \sqrt{\beta^2 + 2\beta + 2 + \frac{2}{\beta} + \frac{1}{\beta^2}}}{2\beta}
\]
\[
= \lim_{\beta \to \infty} \left[ \frac{1}{2} + \frac{1}{2\beta} + \frac{1}{2\beta^2} + \frac{1}{4} + \frac{1}{2\beta} + \frac{1}{2\beta^2} + \frac{1}{2\beta^3} + \frac{1}{4\beta^4} \right]
\]
\[
= 1.
\]

2.1.1. Closed toric symplectic manifolds. Our methods rely on the fact that ellipsoid embeddings into certain finite type convex toric domain targets are equivalent to ellipsoid embeddings into certain closed symplectic manifolds, specifically toric blowups of \( \mathbb{C}P^2 \).
Topologically, symplectic blowup is a procedure where an open ball is removed from a manifold, and the resulting boundary sphere is collapsed along the Hopf fibration. This can be achieved in a symplectic manner if the ball was symplectically embedded; see [McSal, Thm. 7.1.21]. In the special case when the initial manifold \( M \) is four-dimensional, the symplectic blowup procedure is equivalent to the symplectic connected sum \( M \# \mathbb{C}P^2 \);
see [McSal, Ex. 7.1.4]. Moreover, when $M$ is a four-dimensional toric symplectic manifold and the blowup respects the action, then at the level of moment polygons, the toric blowup has the impact of truncating a vertex [McSal, Ex. 7.1.15].

Toric symplectic manifolds are classified by their moment polytope, up to equivariant symplectomorphism of the manifold and up to affine equivalence of the polytopes. Those polytopes which are the moment polytope of some toric symplectic manifold are called Delzant polytopes. For four-dimensional toric symplectic manifolds, Delzant polygons are those that have edges with rational slope and for each vertex, the two primitive vectors pointing in the directions of the edges form a $\mathbb{Z}$-basis of the integer lattice in $\mathbb{R}^2$. Because we work up to affine equivalence of Delzant polytopes, we may assume that a Delzant polygon has a vertex at the origin, that the edges emanating from the origin point along the positive $x$- and $y$-axes, and the polygon is contained in the positive quadrant. Almost toric fibrations, defined in §2.4, and natural operations on them allow us to modify the Delzant polygon of $M_\Omega$ to indicate new fibrations. We use the modified Delzant polygon to identify new embeddings $E(c,d) \hookrightarrow M_\Omega$ and [CGHMP, Thm. 1.4], stated below, to prove there is thus an embedding into $X_\Omega$.

**Proposition 2.1.4** ([CGHMP, Theorem 1.4]). If $M_\Omega$ is the toric symplectic manifold with Delzant polygon $\Omega$, then

$$E(c,d) \hookrightarrow M_\Omega \iff E(c,d) \hookrightarrow X_\Omega.$$ 

### 2.2. Quasi-perfect Diophantine classes.

Embeddings of rational ellipsoids into finite type convex toric domains are completely characterized by the homology classes of symplectically immersed spheres in blow ups of $\mathbb{C}P^2$, a method due to McDuff and Polterovich (see the proof of [Mc2, Prop. 3.2] and the original reference of [MP]). We will not review this entire story, but refer the reader to the original proof, the in-depth survey [H2] for the case of ellipsoid targets, or the shorter, more general summary in [CGHMP, §2.3]. Here we make the definitions and simplifications used in this paper.

Define the integral weight expansion $W(p,q)$ of a pair of coprime integers $p > q$ recursively by

$$W(q,p) = W(p,q) = (q) \cup W(p-q,q),$$

and the weight expansion $w(z)$ of a rational number $z = p/q$ to be

$$w(z) := W(p,q)/q.$$ 

The weights of $z$ are the entries in its weight expansion. Irrational numbers also have (infinite) weight expansions $w(z) := W(z,1)$.

**Example 2.2.1.** We compute

$$W(41,5) = (5) \cup W(36,5)$$

$$= (5,5) \cup W(31,5)$$

$$= \cdots = (5 \times 8) \cup W(5,1)$$

$$= (5 \times 8, 1 \times 5),$$

thus $w(41/5) = (1 \times 8, 1/5 \times 5)$. 

Remark 2.2.2.  (i) The continued fraction of \( z \) equals the list of multiplicities of its weights, e.g.,
\[
[8, 5] = 8 + \frac{1}{5} = \frac{41}{5}.
\]
(ii) By [McSc, Lem. 1.2.6], if \( w(p/q) = (w_1, \ldots, w_M) \), then
\[
\sum_{i=1}^M w_i^2 = \frac{p}{q} \quad \sum_{i=1}^M w_i = \frac{p}{q} + 1 - \frac{1}{q}.
\]

Definition 2.2.3. We call a 5-tuple of integers \( E = (d, e, p, q, t) \) with \( p \) and \( q \) coprime a quasi-perfect Diophantine class if
\[
2(d + e) = p + q, \quad 2de = pq - 1, \quad t = \sqrt{p^2 + q^2 - 6pq + 8}. \tag{2.2.1}
\]
We say \( p/q \) is the center of \( E \), and call the first two equations in (2.2.1) the Diophantine equations.

Let \( \mu_{E, \beta} \) be the obstruction function defined by
\[
\mu_{E, \beta}(z) := \frac{W(p, q) \cdot w(z)}{d + e\beta}.
\]

Our computations in §3 will rely on the fact that
\[
c_{\beta}(z) \geq \mu_{E, \beta}(z) \tag{2.2.2}
\]
for all quasi-perfect Diophantine classes \( E \). This follows from the fact that \( E \) represents the homology class of a symplectically immersed sphere in a blowup of \( \mathbb{C}P^2 \); the fact that the immersion is symplectic means the sphere has positive area, providing us with an inequality. For the purposes of this paper, (2.2.2) may be taken as a black box following from [Mc2, Prop. 3.2].

Remark 2.2.4.  (i) Computing \( \mu_{E, \beta} \) at the center of \( E \) is particularly simple by Remark 2.2.2 (ii):
\[
\mu_{E, \beta} \left( \frac{p}{q} \right) = \frac{qW(p/q) \cdot w(p/q)}{d + e\beta} = \frac{p}{d + e\beta}. \tag{2.2.3}
\]
Many outer corners of \( c_{\beta} \), including those in the infinite staircase of Theorem 1.1.1, have \( z \)-values equal to centers of quasi-perfect Diophantine classes, and near those centers \( c_{\beta}(z) = \mu_{E, \beta}(z) \).

(ii) The fact that \( t \) is an integer is redundant:
\[
t^2 = 4(d + e)^2 - 16de = 4(d - e)^2.
\]
Finally, we have the following identities relating \( d, e, p, q, \) and \( t \):

Lemma 2.2.5. A integral tuple \((d, e; p, q, t)\) is a quasi-perfect Diophantine class if and only if \( t \) is defined from \( p, q \) as in (2.2.1) and there are integers \((d, e)\) such that
\[
4d = p + q + t \quad \text{and} \quad 4e = p + q - t.
\]
Proof. Using the linear Diophantine equation (2.2.1), we solve for \( e \):
\[
e = \frac{p + q}{2} - d.
\]
We then plug this into the quadratic Diophantine equation, giving us
\[
d(p + q - 2d) = pq - 1 \iff 2d^2 - d(p + q) + (pq - 1) = 0
\]
\[
\iff d = \frac{p + q + \sqrt{(p + q)^2 - 8(pq - 1)}}{4}
\]
\[
\iff 4d = p + q + t,
\]
using the fact that \( d > e \). The formula for \( e \) follows in exactly the same way, using the fact that \( e < d \) to obtain the other solution in the quadratic formula. \( \Box \)

2.3. ECH capacities. Another way to obtain a lower bound on the ellipsoid embedding function of a symplectic manifold is through embedded contact homology (ECH). Computing these lower bounds is algorithmic, and so allows us to explore the space of ellipsoid embedding functions \( c_\beta \) efficiently. In Lemma 2.3.5 we relate ECH obstructions to quasi-perfect Diophantine classes.

Defined in [H1], the **ECH capacities** of a convex toric domain \( X_\Omega \) form a sequence
\[
0 = c_0(X_\Omega) < c_1(X_\Omega) \leq c_2(X_\Omega) \leq \cdots \leq \infty,
\]
which obstruct symplectic embeddings:
\[
X_\Omega \hookrightarrow X_{\Omega'} \Rightarrow c_k(X_\Omega) \leq c_k(X_{\Omega'}) \forall k.
\]
Our computation of ECH capacities for \( P(1, \beta) \) is based on [CG1, App. A].

**Definition 2.3.1.** A **convex lattice path** \( \Lambda : [0, 1] \to \mathbb{R}_{\geq 0} \) is a continuous map satisfying

1. piecewise linearity,
2. all vertices (nonsmooth points) lie in \( \mathbb{Z}^2 \),
3. \( \Lambda(0) \) is on the \( y \)-axis and \( \Lambda(a) \) is on the \( x \)-axis,
4. the region enclosed by \( \Lambda \) and the axes is convex.

*Its edges* are the vector differences between adjacent vertices.

The function \( \mathcal{L}(\Lambda) \) counts the number of lattice points enclosed by \( \Lambda \), which includes points on \( \partial \Lambda \) and those lying on the axes. We further define the **\( \Omega \)-length** \( \ell_\Omega(\Lambda) \) of a given path \( \Lambda \) as
\[
\sum_{\nu \in \text{Edges}(\Lambda)} \det [\nu \, p_{\Omega, \nu}]
\]
where \( p_{\Omega, \nu} \in \partial \Omega \) is the unique point where \( \nu \), shifted to be based at \( p_{\Omega, \nu} \), is tangent to \( \partial \Omega \) and where \( \Omega \) lies entirely to the right-hand side of \( \nu \). See Figure 2.3.1.
Figure 2.3.1. With $\Omega$ the region outlined in red and $\nu$ in blue, the black point is $p_{\Omega,\nu}$.

**Theorem 2.3.2** ([CG1, Cor. A.5]). If $X_\Omega$ is a convex toric domain, then

$$c_k(X_\Omega) = \min \left\{ \ell_\Omega(\Lambda) : \text{convex lattice paths } \Lambda \text{ where } \mathcal{L}(\Lambda) = k + 1 \right\}.$$  

The ECH capacities of an ellipsoid $E(a,b)$ can also be computed via:

**Proposition 2.3.3** ([H1, Prop. 1.2]). Let $N(a,b)$ be the sequence of elements of the array $(am+bn)_{m,n \in \mathbb{N}}$ listed in ascending order with repetitions. The $k$th element indexed from zero of this sequence, $N_k(a,b)$, is exactly equal to $c_k(E(a,b))$.

The use of ECH capacities to obstruct symplectic embeddings of ellipsoids into some target relies on the following result of Frenkel-Müller and Hutchings, which is also a special case of a theorem of Cristofaro-Gardiner.

**Theorem 2.3.4** ([FM, Cor. 1.5], [H2, Cor. 11], [CG1, Thm. 1.2]). There exists a symplectic embedding

$$E(1,z) \xrightarrow{\gamma} P(1,\beta)$$

if and only if

$$c_k(E(1,z)) \leq c_k(P(1,\beta))$$

for all $k \in \mathbb{Z}_{\geq 0}$.

Since our target is $P(1,\beta)$, which is convex, we used the methods of [BHM, §5] as well as Theorems 2.3.2, 2.3.4 and Proposition 2.3.3 to compute a lower bound for $c_{\beta}$. We identify $(P(1,\beta),\lambda \omega_0) = (P(\lambda,\lambda \beta),\omega_0)$ by the diffeomorphism $(\zeta_1,\zeta_2) \mapsto (\sqrt{\lambda}\zeta_1,\sqrt{\lambda}\zeta_2)$.

$$\lambda \geq c_{\beta}(z) \iff E(1,z) \xrightarrow{\gamma} P(\lambda,\lambda \beta)$$

$$\iff c_k(E(1,z)) \leq c_k(P(\lambda,\lambda \beta)) \forall k$$

$$\iff c_k(E(1,z)) \leq \lambda \cdot c_k(P(1,\beta)) \forall k$$

$$\iff \frac{c_k(E(1,z))}{c_k(P(1,\beta))} \leq \lambda \forall k,$$

where the third line follows by the conformality of ECH capacities [H1, (2.5)]. Because $c_{\beta}(z)$ is the infimum over all such $\lambda$, we obtain

$$c_{\beta}(z) = \sup_k \frac{c_k(E(1,z))}{c_k(P(1,\beta))}.$$  \hfill (2.3.2)
It is (2.3.2) which allowed us to explore the space of functions $c_\beta$ for potential infinite staircases and identify our values in Theorem 1.1.1 and Conjecture 1.1.2, by computing
\[
\max_{k \leq K} \frac{c_k(E(1,z))}{c_k(P(1,\beta))} \leq c_\beta(z),
\]
for $K$ large (e.g. $K = 25,000$ or $100,000$). The maximum in (2.3.3) is a good approximation for $c_\beta$ when $K$ is large by [CGHR, Thm. 1.1].

We can use ECH capacities to identify outer corners of $c_\beta$. Complementary to (2.2.3) and (2.2.2), we can use individual convex lattice paths, Theorem 2.3.2, and (2.3.2) to compute precise lower bounds to values of $c_\beta$ at specific values of $z$. That is, to prove
\[
c_\beta(z) \geq \lambda,
\]
it is enough to find a single lattice path $\Lambda$ for which $N_k(1, z)/\ell_{\Omega_\beta}(\Lambda) = \lambda$: see Remark 3.1.3. This is the method used in [CGHMP].

However, in order to make use of [M1], we prove Theorem 1.1.1 using quasi-perfect Diophantine classes rather than ECH capacities. Analogously to [BHM, Lem. 92], we may translate between these perspectives:

**Lemma 2.3.5.** If $E = (d, e, p, q, t)$ is a quasi-perfect Diophantine class, then
\[
\mu_{E,\beta}\left(\frac{p}{q}\right) \leq \frac{c_k(E(1,p/q))}{c_k(P(1,\beta))},
\]
where $k = (d+1)(e+1) - 1 = \frac{(p+1)(q+1)}{2} - 1$.

Lemma 2.3.5 allows us to translate between the obstructions from ECH capacities, which are algorithmic and thus good tools for analyzing $c_\beta$ visually by Theorem 2.3.2 (see Figures 3.2.9 and 4.3.1), and quasi-perfect Diophantine classes, which carry more information. Note that if $c_\beta(p/q) = \mu_{E,\beta}(p/q)$ then the conclusion of Lemma 2.3.5 is an equality.

**Proof.** It suffices to provide a lattice path $\Lambda_E$ with $\mu_{E,\beta}(z) \leq N_k(1, z)/\ell_{\Omega_\beta}(\Lambda_E)$: this is simply the rectangle with corners the origin, $(0, e), (d, e),$ and $(d, 0)$. We check the conclusions of the lemma.

Firstly,
\[
(d+1)(e+1) = \frac{(p+1)(q+1)}{2} \iff 2de + 2(d+e) + 1 = pq + p + q + 1,
\]
which follows from (2.2.1).

Secondly, the edges of $\Lambda_E$ are $(d, 0)$ and $(0, -e)$. For both, we may use $p_{\Omega_\beta,\nu} = (\beta, 1)$. Thus
\[
\ell_{\Omega_\beta}(\Lambda_E) = \det\begin{pmatrix} d & \beta \\ 0 & 1 \end{pmatrix} + \det\begin{pmatrix} 0 & \beta \\ -e & 1 \end{pmatrix} = d + e\beta,
\]
which is the denominator of $\mu_{E,\beta}$.

Finally, it remains to show that $N_k(1,p/q) = W(p,q) \cdot w(p/q) = p$. Identify the nonnegative integer linear combinations of 1 and $z = p/q$ with lattice points in $\mathbb{Z}_{\geq 0}^2$. We will show that if $(x_0, y_0)$ is either $(p, 0)$ or $(0, q)$, there are exactly $k = (p+1)(q+1)/2 - 1$ lattice points in the first quadrant with $x + py/q < x_0 + py_0/q$, and thus $N_k(1,p/q) = x_0 + py_0/q = p$. 
Let $T$ be the triangle below the line $x + py/q < p$ and above the axes. If $I$ denotes the number of interior points of $T$ and $B$ its number of boundary points, the number of lattice points in the first quadrant below the line $x + py/q < p$ is $I + B - 2$. By Pick’s Theorem applied to $T$,

$$I + \frac{B}{2} - 1 = \frac{pq}{2} \iff I + B - 2 = \frac{pq}{2} - 1 + \frac{B}{2} = \frac{pq}{2} - 1 + \frac{p + q + 1}{2} = k,$$

as desired.

Note that it is not too difficult to extend the conclusion of Lemma 2.3.5 to an interval containing $p/q$ as in [BHM, Lem. 92], but we do not need this here. We conclude this subsection with a figure illustrating the constraint a single obstruction at a single $z$-value imposes on the embedding capacity function.

![Figure 2.3.4](image)

Figure 2.3.4. The figure depicts the effect of an obstruction providing a lower bound for $c_X$ at the indicated blue point. The ellipsoid embedding function $c_X$ must lie in the blue shaded region by Proposition 1.0.1 (ii) and (iii).

2.4. Almost toric fibrations. Symplectic embeddings provide a useful counterpoint to the obstructions described in sections 2.2-2.3. We will use combinatorial techniques developed in the theory of almost toric fibrations (ATFs) to establish the existence of embeddings. Introduced by Symington [S] and developed further in [LS, E], an ATF is a completely integrable system on a compact symplectic four-manifold with elliptic and focus-focus singularities. This framework provides a map from the manifold $M$ to $\mathbb{R}^2$ whose image is called the base diagram. There are combinatorial operations called nodal trades, nodal slides, and mutation on the base diagram that correspond to symplectomorphisms of the corresponding manifolds. These allow us to discover embeddings of ellipsoids into the manifold $M$ by identifying appropriate triangles inside the variously manipulated base diagrams. We may then use Proposition 2.1.4 to deduce that the convex toric domain $X$ also has the same ellipsoid embeddings.
In order to study the polydisk \( P(1, \beta) \) with ATFs, we first find a compact manifold that has an ATF with base diagram the \( 1 \times b \) rectangle. The manifold \( M_\beta = \mathbb{C}P^1 \times \mathbb{C}P^1_\beta \), a product of two copies of projective space, the first with size 1 and the second with size \( \beta \), is equipped with a toric \( T^2 \) action by rotation in each factor. This has moment map image the Delzant polygon which is the \( 1 \times \beta \) rectangle. This is our starting point for manipulations using ATF tools. These tools will change the map \( M_\beta \to \mathbb{R}^2 \) and its image, but not the manifold \( M_\beta \) itself.

The first step is to apply a nodal trade at each of the three vertices \( X, Y \) and \( V \) which are not at the origin in \( \mathbb{R}^2 \). Geometrically in \( M_\beta \), this means excising the neighborhood of the fixed point corresponding to a vertex and then gluing in a local model of a focus-focus singularity. At the level of the base diagram, this corresponds to adding a ray with a marked point emanating from the anchor vertex \( P \). Above the marked point on the ray, there is a pinched torus. The pinch point is the new focus-focus singularity for the updated map \( M_\beta \to \mathbb{R}^2 \). If we let \( \overrightarrow{E} \) and \( \overrightarrow{F} \) denote the primitive vectors (in \( \mathbb{Z}^2 \)) pointing along the edges emanating from \( P \), then the smoothness of \( M_\beta \) guarantees that \( \overrightarrow{E} \) and \( \overrightarrow{F} \) form a \( \mathbb{Z} \) basis of \( \mathbb{Z}^2 \). With this notation, then, the nodal ray that we introduce points in the direction \( \overrightarrow{E} + \overrightarrow{F} \). A useful fact, which follows from a straightforward linear algebraic calculation, is that both pairs \( (\overrightarrow{E}, \overrightarrow{E} + \overrightarrow{F}) \) and \( (\overrightarrow{E} + \overrightarrow{F}, \overrightarrow{F}) \) are \( \mathbb{Z} \) bases of \( \mathbb{Z}^2 \).

The second operation we can apply to a base diagram is called a nodal slide. The local model for a focus-focus singularity has one degree of freedom, corresponding to moving the pinched torus further or closer to the level set above the vertex. In the base diagram, this corresponds to moving the marked point along the nodal ray.

The third operation is mutation along a nodal ray of the base diagram. This changes the shape of the diagram. At the level of the function \( M_\beta \to \mathbb{R}^2 \), if the marked point’s location does not move, this corresponds to taking the same function, but choosing a different branch cut to visualize the image of the function.

Combinatorially, the base diagram is divided in two by the line generated by the nodal ray. The mutation operation leaves one piece unchanged (which for us will always be the piece containing the origin) and acts on the other piece by an affine linear transformation that

- fixes the anchor vertex;
- fixes the nodal ray; and
- aligns the two edges emanating from the anchor vertex.

There is a unique transformation in \( ASL_2(\mathbb{Z}) \) that achieves this, as a consequence of the linear algebraic fact about the edge rays and nodal rays noted above. The other changes to the base diagram are the creation of a new (anchor) vertex and nodal ray (the negative of the previous). This is illustrated in Figure 2.4.1 below.

Procedurally, we apply a sequence of mutations with the goal of finding wider and wider triangles inside the mutated base diagram. The impact that one mutation has on triangles that fit inside the base diagram is illustrated in Figure 2.4.2.

**Remark 2.4.1.** When discussing ATF base diagrams and their mutations, we will use the following conventions.
We apply a mutation about the vertex $V$ to the figure in (a) to obtain the figure in (b). The mutation fixes the red region and applies an affine linear transformation encoded in a matrix $M$ to the blue region. The effect on the vertices and is indicated. The nodal rays in (b) that differ from (a) are given by $\vec{n}_V = -\vec{n}_X$ and $\vec{n}_X = M \cdot \vec{n}_V$.

The figures in (a) and (b) are related by a mutation, as in Figure 2.4.1. The fact that the green triangles centered at $O$ have different proportions indicates that we have embeddings of ellipsoids with different eccentricities into the corresponding polydisk.

**Vertices:** We set $O = (0,0)$, use $X$ and $Y$ to denote the vertices on the $x$- and $y$-axes, respectively, and use $V$ to denote the vertex strictly in the positive quadrant.

**Nodal rays:** The nodal ray of vertex $A$ is labeled $\vec{n}_A$.

**Side directions:** The primitive integral vector parallel to the side $AB$ is denoted $\overrightarrow{AB}$.

**Affine lengths:** The affine length of the side $AB$ is denoted $|AB|$.

**Mutations:** The new vertex at its position (relative to the axes) after a mutation at vertex $A$ has a subscript lowercase $a$. For example, the vertex on the $y$-axis after mutation at $A$ is denoted $Y_a$.

**Sequences of mutations:** We denote a sequence of mutations by a word in the lowercase letters $x, y, v$, read from left to right. E.g. $v^2yx$ means “mutate at $V$ twice, then mutate at $Y$, then mutate at $X$.”

We note that after a mutation, the nodal rays are transformed in one of three ways: not at all; by taking the negative; or by applying the mutation matrix $M$. Because our base diagrams are polygons, we will make use of the key identity

$$|OY_a|\overrightarrow{OY_a} + |Y_aV_a|\overrightarrow{Y_aV_a} - |X_aV_a|\overrightarrow{X_aV_a} - |OX_a|\overrightarrow{OX_a} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

(2.4.3)

derived from the fact that the four sides must close up.

The following result makes precise the relationship between triangles in the base diagram and symplectic embeddings of ellipsoids.
Proposition 2.4.2 ([CGHMP, Prop. 2.35]). Suppose that a symplectic manifold $X$ is equipped with an almost toric fibration with base diagram $\Delta_X$ that consists of a closed region in $\mathbb{R}^2_\geq$ that is bounded by the axes and a convex (piecewise-linear) curve from $(a,0)$ to $(0,b)$, for $a,b \in \mathbb{R}^+$. Suppose in addition that there is no nodal ray emanating from $(0,0)$. Then there exists a symplectic embedding of the ellipsoid $(1-\varepsilon) \cdot E(a,b)$ into $X$ for any $0 < \varepsilon < 1$.

While the obstructions in sections 2.2-2.3 give lower bounds on $c_X$, as indicated in Figure 2.3.4, a single embedding forces certain upper bounds on the embedding capacity function. As we will see, the combination of the two can strongly restrict $c_X$.

![Figure 2.4.4.](image)

Figure 2.4.4. By contrast to Figure 2.3.4, an embedding provides an upper bound for $c_X$ at the indicated red point. The function $c_X$ must lie in the red shaded region by Proposition 1.0.1 (ii) and (iii).

2.5. Combining obstructions and embeddings. Combining the effects in Figures 2.3.4 and 2.4.4, we see how to prove that the combination of lower bounds provided by obstructions (quasi-perfect Diophantine classes or ratios of ECH capacities) with upper bounds provided by an embedding allows us to establish the existence of an infinite staircase. A combination of obstructions and embeddings allows us to nail down the ellipsoid embedding function for some ranges of $z$-values (indicated by violet segments in Figure 2.5.1), and provides bounds on $c_X$ for other ranges of $z$-values (indicated by violet regions in Figure 2.5.1). In this way, one can establish the existence of an infinite staircase without computing the entire function. Or if the embeddings and obstructions are lined up just so, one might just compute the entire function. Note, this is usually only a effective strategy before the accumulation point.
Figure 2.5.1. This figure indicates several obstructions at the blue dots and embeddings at the red dots. Combining the bounds forced by these as shown in Figures 2.3.4 and 2.4.4, we deduce that the ellipsoid embedding function must equal the violet segments and must lie in the violet shaded regions. In particular, it must be constant along the horizontal segment between the blue and red points at the same \( \lambda \)-value; and equal the line when a red point and blue point lie on a line through the origin.

3. Proof of the main theorem

In this section we prove that the polydisk \( P(1, \beta) \) has an infinite staircase accumulating to \( \text{acc}(\beta) \), where

\[
\beta = \frac{6 + 5\sqrt{30}}{12} \quad \text{and} \quad \text{acc}(\beta) = \frac{54 + 11\sqrt{30}}{14}.
\]

The fact that \( \text{acc}(\beta) \) satisfies (2.1.1) with \( \text{acc}(\beta) = z \) can be verified by hand. Furthermore, set

\[ E = (17, 6, 41, 5, 22). \]

The utility of \( E \) is that it is a quasi-perfect Diophantine class whose obstruction \( \mu_{E, \beta} \) equals the function \( c_\beta \) for \( z \in (\text{acc}(\beta), 41/5] \). We do not prove this latter claim, but note that on \( (\text{acc}(\beta), 41/5] \) we do know (as shown in Figure 2.3.4 by setting the blue point equal to \( (41/5, \mu_{E, \beta}(41/5)) \)) that

\[
c_\beta(z) \geq z \frac{\mu_{E, \beta}(41/5)}{41/5} = \frac{5z}{17 + 6\beta}.
\]

This is a special case of the analogous [BHM, Prop. 42]. The numerics of \( E \) will be crucial for studying \( c_\beta \).

We next define the obstructions which we will use to prove that \( c_\beta \) has an infinite staircase, following the procedure outlined in §2.5.

**Definition 3.0.1.** We define the **outer class**

\[ E_k := dE_{k-1} - E_{k-2} = (d_k, e_k, p_k, q_k, t_k) \]
where $E_0 = (3, 1, 7, 1, 4)$ and $E_1 = (64, 23, 155, 19, 82)$. The recursion constant is $t = 22$ for all $k$.

**Definition 3.0.2.** We define the inner class

$$
\hat{E}_k := t_{k-1}E_k - E = (\hat{d}_k, \hat{e}_k, \hat{p}_k, \hat{q}_k, \hat{t}_k).
$$

*Note,* $\hat{E}_1 = (239, 86, 579, 71, 250)$.

**Remark 3.0.3.** In the sense of [MMW], the inner class $\hat{E}_k$ is the $x$-mutation of the triple $(E_{k-1}, E_k, E)$. We discovered the $\hat{E}_k$ classes after Mike Usher pointed out the relationship between the $\hat{A}$ classes in [U1] and $x$-mutation, see §4.2.

The outer corners of the $E_k$ and $\hat{E}_k$ classes alternate in the sense that

$$
\cdots < \frac{p_{k-1}}{q_{k-1}} < \frac{\hat{p}_k}{\hat{q}_k} < \frac{p_k}{q_k} < \cdots,
$$

while the values their obstructions take at these $z$-values also alternate. See Lemma 3.1.4, which is illustrated by Figure 3.0.2.

**Figure 3.0.2.** This figure indicates the arrangement of the first several outer and inner corners of $c_\beta$. The black outer corners, labeled $O_k$, arise from the outer $E_k$ classes and their coordinates are given in Proposition 3.0.4 (i). The red outer corners, labeled $\hat{O}_k$, arise from the inner $\hat{E}_k$ classes and their coordinates are given in Proposition 3.0.4 (ii). Because $c_\beta$ does not equal the obstruction $\mu_{E_k,\beta}$ from the outer $E_k$ classes near the intersection of $\mu_{E_k,\beta}$ and $\mu_{E_{k+1},\beta}$, these obstructions are indicated by dashed black lines where the obstructions $\mu_{E_{k+1},\beta}$ are larger.
In §3.1 we will prove the following proposition computing the value of $c_\beta$ at the outer corners of its infinite staircase:

**Proposition 3.0.4.** We establish the following lower bounds on $c_\beta$.

(i) The outer classes $E_k$ determine the lower bounds

$$c_\beta \left( \frac{p_k}{q_k} \right) \geq \frac{p_k}{d_k + e_k \beta}.$$ 

(ii) The inner classes $\hat{E}_k$ determine the lower bounds

$$c_\beta \left( \frac{\hat{p}_k}{\hat{q}_k} \right) \geq \frac{\hat{p}_k}{d_k + \hat{e}_k \beta}.$$ 

We will also prove that the claimed outer corners $z = p_k/q_k$ have four-periodic continued fractions, which, upon proving Theorem 1.1.1 (i), proves Theorem 1.1.1 (ii).

Our final definition in this section provides notation for the intersections between the obstructions from $E_k$ and $\hat{E}_k$.

**Definition 3.0.5.** We set the following notation.

- We denote the points discussed in Proposition 3.0.4 by $O_k = \left( \frac{p_k}{q_k}, \frac{p_k}{d_k + e_k \beta} \right)$, and $\hat{O}_k = \left( \frac{\hat{p}_k}{\hat{q}_k}, \frac{\hat{p}_k}{d_k + \hat{e}_k \beta} \right)$.

- We extend the lower bounds at $O_k$ and $\hat{O}_k$ by horizontal lines and lines through the origin, using Proposition 1.0.1 (ii, iii), as illustrated in Figure 2.3.4.

  - Denote by $I_{k+1} = (z_{in}^{k+1}, \lambda_{in}^{k+1})$ the intersection between the horizontal line through $O_k$ and the line through the origin and $\hat{O}_{k+1}$.
  - Denote by $\hat{I}_{k+1} = (\hat{z}_{in}^{k+1}, \hat{\lambda}_{in}^{k+1})$ the intersection between the horizontal line through $\hat{O}_{k+1}$ and the line through the origin and $O_{k+1}$.

In §3.2 we will use ATFs to construct embeddings computing the value of $c_\beta$ at the points $I_k$, proving that they are inner corners. Specifically, we will show:

**Proposition 3.0.6.** At the intersections of the obstructions from $E_k$ and $\hat{E}_{k+1}$, we have the following upper bound:

$$c_\beta(z_{in}^{k+1}) \leq \lambda_{in}^{k+1}.$$ 

Next we state our conjecture which would, if proven, fully compute $c_\beta$ on $[1, \text{acc}(\beta)]$. See Remark 3.2.15 for a discussion of the complications which arise in its potential proof.

**Conjecture 3.0.7.** At the intersections of the obstructions from $\hat{E}_{k+1}$ and $E_{k+1}$, we have the following upper bound:

$$c_\beta(\hat{z}_{in}^{k+1}) \leq \hat{\lambda}_{in}^{k+1}.$$ 

In the following lemma we compute the coordinates of $I_{k+1}$ and $\hat{I}_{k+1}$.

**Lemma 3.0.8.** (i) We have

$$\left( z_{in}^{k+1}, \lambda_{in}^{k+1} \right) = \left( \frac{p_k(d_{k+1} + \hat{e}_{k+1} \hat{\beta})}{q_{k+1}(d_k + e_k \beta)}, \frac{p_k}{d_k + e_k \beta} \right).$$
(ii) We have

\[
(\hat{z}_{k+1}^{in}, \hat{\lambda}_{k+1}^{in}) = \left( \frac{\hat{p}_{k+1} (d_{k+1} + e_{k+1} \beta)}{q_{k+1} (d_{k+1} + e_{k+1} \beta)} \right). \]

Proof. The values of \(\lambda_{k+1}^{in}\) and \(\hat{\lambda}_{k+1}^{in}\) are immediate because they are the \(\lambda\)-values of the obstructions from \(E_k\) and \(\hat{E}_{k+1}\), respectively.

To compute \(z_{k+1}^{in}\), we solve

\[
\frac{p_k}{d_k + e_k \beta} = \frac{\hat{p}_{k+1}}{q_{k+1}} \hat{z}_{k+1}^{in}
\]

for \(z_{k+1}^{in}\), while to compute \(\hat{z}_{k+1}^{in}\), we solve

\[
\frac{\hat{p}_{k+1}}{d_{k+1} + e_{k+1} \beta} = \frac{p_{k+1}}{q_{k+1}} \hat{z}_{k+1}^{in}
\]

for \(\hat{z}_{k+1}^{in}\). □

Proof. (of Theorem 1.1.1 (i)) The lower bounds in Propositions 3.0.4 (i) and 3.0.4 (ii) combined with the upper bound in Proposition 3.0.6 prove by Lemma 3.0.8 (i) that \(c_\beta\) has infinitely many nonsmooth points at the inner corners between the obstructions from \(E_k\) and \(\hat{E}_{k+1}\), as indicated in Figure 2.5.1. (These inner corners are labeled \(I_{k+1}\) in Figure 3.0.2.) Note that to conclude that \(c_\beta(\frac{p_k}{q_k}) = c_\beta(\hat{z}_{k+1}^{in})\) we use the fact that \(c_\beta\) is increasing, which requires Lemma 3.1.4 to know that

\[
\frac{p_k}{q_k} \leq \hat{z}_{k+1}^{in} \leq \frac{\hat{p}_{k+1}}{q_{k+1}}.
\]

□

Remark 3.0.9. Note that if we could show Conjecture 3.0.7, then by Lemma 3.0.8 and similar reasoning to the proof of Theorem 1.1.1 (i) we would be able to compute the entire function \(c_\beta\) between the center 7 of \(E_0\) and \(\text{acc}(\beta) = \frac{54 + 11\sqrt{30}}{14}\). (It is very little extra work to compute \(c_\beta\) on \([1, 7]\), since it requires identifying only two outer and two inner corners.)

3.1. Outer corners. To prove Propositions 3.0.4, it suffices by (2.2.3) to show that the recursively defined families \(E_k\) and \(\hat{E}_k\) satisfy the Diophantine equations (2.2.1).

For our proof, we use the ideas developed in [MM, Section 2.2] to think of a quasi-perfect class as an integral point \((p, q, t)\) on a quadratic surface \(X\) where \(t = \sqrt{p^2 + q^2 - 6pq + 8}\). In particular, as noted in Lemma 2.2.5, a tuple \((d, e; p, q, t)\) will satisfy the Diophantine equations if given an integral tuple \((p, q, t) \in X\), we define\(^5\) \(d, e\) by

\[
4d = p + q + t, \quad \text{and} \quad 4e = p + q - t.
\]

\(^5\)Note, as defined in this way \((d, e)\) might not be integers for all integral choices of \((p, q, t)\). Thus, not all points on \(X\) correspond to quasi-perfect classes.
We then use the result [MM, Lem. 3.1.2] which allows us to see that we can produce new tuples \((p, q, t) \in X\) via recursion assuming certain compatibility conditions hold. Let
\[
A := \begin{pmatrix} -1 & 3 & 0 \\ 3 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x := \begin{pmatrix} p \\ q \\ t \end{pmatrix}.
\]
Then the surface \(X = \{x^T Ax = 8\}\), as \(x^T Ax = 6pq - p^2 - q^2 + t^2\).

The lemma then states:

**Lemma 3.1.1.** [MM, Lemma 3.1.2] Suppose that \(x_0\) and \(x_1\) are integral vectors that satisfy the following conditions for some integer \(\nu > 0\):
\[
x_i^T A x_i = 8, \quad i = 0, 1, \tag{3.1.1}
\]
\[
x_0^T A x_0 = 4\nu. \tag{3.1.2}
\]

Then, the vectors \(x_2 := \nu x_1 - x_0, x_1\) also satisfy these conditions for the given \(\nu\).

We can then restate [MM, Cor. 3.1.1] for our purposes as

**Corollary 3.1.2.** Any two integral triples \(x_i = (p_i, q_i, t_i), i = 0, 1\) that satisfy (3.1.1) and (3.1.2) for a given \(\nu\) can be extended to a sequence
\[
x_i := \nu x_{i-1} - x_{i-2}, \quad i \geq 0,
\]
and each successive adjacent pair satisfies these conditions. Further, the corresponding quantities
\[
d_i = \frac{1}{4}(p + q + t), \quad e_i = \frac{1}{4}(p + q - t)
\]
also satisfy this recursion and hence are integers, provided that they are integers for \(i = 0, 1\).

We now proceed in proving Prop 3.0.4 giving the bounds for the outer corners at \(z_k = p_k/q_k\) and \(\hat{z}_k = \hat{p}_k/\hat{q}_k\).

**Proof.** (of Proposition 3.0.4) To prove (i) and (ii), we must check that the classes \(E_k = t E_{k-1} - E_{k-2}\) and \(\hat{E}_k = t_{k-1} E_k - E\) are Diophantine classes. By Cor 3.1.2 and Lemma 2.2.5, it is enough to verify:
- \(E_k\) and \(E\) satisfy (3.1.1).
- \(E_0 = (3, 1, 7, 1, 4)\) and \(E_1 = (64, 23, 155, 19, 82)\) satisfy (3.1.2) for \(\nu = t = 22\).
- \(\hat{E}_k\) and \(E = (17, 6, 41, 5, 22)\) satisfy (3.1.2) for \(\nu = t_{k-1}\).

By Cor 3.1.2, (3.1.1) will hold for \(E_k\) if it holds for \(E_0\) and \(E_1\). Thus, we must check this for \(E_0, E_1, \) and \(E\). We have
\[
E_0 : 6(7) - 7^2 - 1^2 + 4^2 = 8,
\]
\[
E_1 : 6(155)(19) - 155^2 - 19^2 + 82^2 = 8,
\]
\[
E : 6(41)(5) - 41^2 - 5^2 + 22^2 = 8.
\]
Now, we check (3.1.2) for \(E_0, E_1\) with \(\nu = 22:\)
\[
1(3 \cdot 155 - 19) + 7(3 \cdot 19 - 155) + 82 \cdot 4 = 4 \cdot 22.
\]
To check (3.1.2) for the pair $E_k$ and $\hat{E}_k$ with $\nu = t_{k-1}$, this involves verifying
\[5(3p_k - q_k) + 41(3q_k - p_k) + 22t_k = 4t_{k-1}.\]
As this is a linear equation, we can verify it holds by induction by checking for $k = 1, 2$. This is an easy computation.

Thus, $E_k$ and $\hat{E}_k$ are quasi-perfect Diophantine classes, and (i) and (ii) follow by (2.2.2) and (2.2.3).

$\square$

Remark 3.1.3. As in the proof of Lemma 2.3.5, if $E = (d, e, p, q, t)$ and $\Lambda_E$ represents the convex lattice path with corners the origin, $(0, e)$, $(d, e)$, and $(d, 0)$, then with
\[k = L(\Lambda_E) = \frac{(p + 1)(q + 1)}{2} - 1 = (d + 1)(e + 1) - 1,
\]
we have
\[c_\beta \left(\frac{p}{q}\right) \geq \frac{N_k(1, p/q)}{\ell_{\Omega_\beta}(\Lambda_E)} = \frac{p}{d + e\beta}.
\]
Thus to prove Propositions 3.0.4 (i) and 3.0.4 (ii) it would also suffice to simply identify the lattice paths $\Lambda_{E_k}$ and $\Lambda_{\hat{E}_k}$.

Next we prove that the centers of the quasi-perfect Diophantine classes $E_k$ and $\hat{E}_k$ are arranged as depicted in Figure 3.0.2.

Lemma 3.1.4. (i) The centers of the classes $E_k$ and $\hat{E}_k$ alternate:
\[\cdots < \frac{p_k}{q_k} < \frac{\hat{p}_{k+1}}{\hat{q}_{k+1}} < \frac{p_{k+1}}{q_{k+1}} < \cdots \]

(ii) The obstructions from the classes $E_k$ and $\hat{E}_k$ alternate:
\[\cdots < \frac{p_k}{d_k + e_k\beta} < \frac{\hat{p}_{k+1}}{d_{k+1} + e_{k+1}\beta} < \frac{p_{k+1}}{d_{k+1} + e_{k+1}\beta} < \cdots \]

Proof. Our goal is to show
\[\frac{p_k}{q_k} < \frac{\hat{p}_{k+1}}{\hat{q}_{k+1}} < \frac{p_{k+1}}{q_{k+1}}. \tag{3.1.3}\]

The first inequality in (3.1.3) is equivalent to
\[pk\hat{q}_{k+1} < q_k\hat{p}_{k+1}, \]
\[p_k(t_kq_{k+1} - 5) < q_k(t_{k+1}p_k - 41), \]
\[t_kp_kq_{k+1} - 5p_k < t_{k+1}p_kq_k - 41q_k,
\]
which follows if we can show that
\[\frac{p_k}{q_k} < \frac{p_{k+1}}{q_{k+1}} \quad \text{and} \quad \frac{p_k}{q_k} < \frac{41}{5}. \tag{3.1.4}\]

Similarly, the second inequality in (3.1.3) is equivalent to
\[\hat{p}_{k+1}\hat{q}_{k+1} < p_{k+1}\hat{q}_{k+1}, \]
\[(t_kp_{k+1} - 41)q_k + 1 < p_{k+1}(t_kq_{k+1} + 5), \]
\[t_kp_kq_{k+1} - 41q_k + 1 < t_{k+1}p_kq_k + 1 - 5p_{k+1}, \]
which follows from the second inequality in (3.1.4).

The first inequality in (3.1.4) is
\[ p_k(22q_k - q_{k-1}) < q_k(22p_k - p_{k-1}) \iff p_{k-1}q_k < p_kq_{k-1}, \]
thus follows by induction and the base case \( k = 1 \):
\[ \frac{p_0}{q_0} = 7, \quad \frac{p_1}{q_1} = \frac{155}{19} \approx 8.158. \]

The second inequality in (3.1.4) is equivalent to a linear inequality in \( p_k, q_k \), which holds because they both satisfy the same recursion and it holds for \( k = 0 \):
\[ 5p_0 < 41q_0 \iff 5 \cdot 7 < 41 \cdot 1. \]

To prove (ii), notice that
\[ \hat{d}_{k+1} + \hat{e}_{k+1} + 1 = t_k(d_{k+1} + e_k + 1) + 17 + 6\beta, \]
thus by the same logic as in the proof of (i), all we need to show is
\[ \frac{p_k}{d_k + e_k + 1} < \frac{p_{k+1}}{d_{k+1} + e_{k+1} + 1}, \quad \frac{p_k}{d_k + e_k + 1} < \frac{41}{17 + 6\beta}. \]

The first inequality in (3.1.5) is
\[ p_k(22d_k - d_{k-1} + 22e_k + e_{k-1}) < (22p_k - p_{k-1})(d_k + e_k + 1) \iff \frac{p_{k-1}}{d_{k-1} + e_{k-1} + 1} < \frac{p_k}{d_k + e_k + 1}, \]
which follows by induction and the base case \( k = 1 \):
\[ \frac{7}{3 + \beta} < \frac{155}{64 + 23\beta} \iff 6\beta < 17, \]
which holds because \( 6\beta \approx 16.693. \)

The second inequality in (3.1.5) is equivalent to a linear inequality in terms satisfying the same recursion, thus we simply need to check it for \( k = 0 \):
\[ \frac{7}{3 + \beta} \approx 1.211 < 1.217 \approx \frac{41}{17 + 6\beta}. \]

\( \square \)

Proof. (of Theorem 1.1.1 (ii), assuming (i))

As above, let \( \{\frac{p_k}{q_k}\} \) be the sequence of rational numbers described by the recursion with seeds \( p_0 = 7, q_0 = 1 \) and \( p_1 = 155, q_1 = 19 \) and relation
\[ p_k = 22p_{k-1} - p_{k-2}, \quad q_k = 22q_{k-1} - q_{k-2} \]
for \( k \geq 2 \). We prove that this sequence coincides with the sequence of continued fractions of the form
\[ \left[ 8, 6, 4, 2, \frac{u_{k-2}}{v_{k-2}} \right] = \frac{u_k}{v_k} \]
for all \( k \geq 2 \). Here, we assume that the seeds of both recursions are equal, so \( u_j = p_j \) and \( v_j = q_j \) for \( j = 0, 1 \).

To prove this equality, we use the following standard result of number theory, which is explained in Chapter 2.1 of [Ha].
Lemma 3.1.5. Let the continued fraction for a real number \( \alpha \) be \( [a_0, a_1, a_2, \ldots] \). If \( \{r_n/s_n\} \) denotes the sequence of convergents of \( \alpha \) obtained by truncating this continued fraction expansion, then for any real number \( z \),

\[
[a_0, a_1, a_2, \ldots, a_n, z] = \frac{zr_n + r_{n-1}}{zs_n + s_{n-1}}.
\]

Furthermore, \( r_{n+1}s_n - r_ns_{n+1} = (-1)^n \) for all \( n \geq 0 \).

The number \( \alpha = \frac{54 + 11\sqrt{30}}{14} = \text{acc}(\beta) \) has the 4-periodic continued fraction

\[
\alpha = \left\{ 8, 6, 4, 2 \right\}^\infty.
\]

The numerators and denominators of the second and third convergents of this continued fraction are \( r_2 = 204, s_2 = 25, r_3 = 457, \) and \( s_3 = 56 \). Lemma 3.1.5 combined with our recurrence relation yields

\[
\begin{pmatrix} u_k \\ v_k \end{pmatrix} = \begin{pmatrix} 8 & 6 & 4 & 2 \\ r_3 & r_2 & u_{k-2} & v_{k-2} \end{pmatrix} = \begin{pmatrix} r_3 & r_2 \\ s_3 & s_2 \end{pmatrix} \begin{pmatrix} u_{k-2} \\ v_{k-2} \end{pmatrix}
\]

for all \( k \geq 2 \). This can also be written using matrix notation:

\[
\begin{pmatrix} u_k \\ v_k \end{pmatrix} = \begin{pmatrix} r_3 & r_2 \\ s_3 & s_2 \end{pmatrix} \begin{pmatrix} u_{k-2} \\ v_{k-2} \end{pmatrix} \quad (3.1.6)
\]

We assume \( x_j = 22x_{j-1} - x_{j-2} \) for \( j < k \) and \( x_j = u_j, v_j \). By (3.1.6), we have

\[
\begin{align*}
 u_k &= 22u_{k-1} - u_{k-2} \\
r_3u_{k-2} + r_2v_{k-2} &= 22(r_3u_{k-3} + r_2v_{k-3}) - (r_3u_{k-4} + r_2v_{k-4}),
\end{align*}
\]

which follows from the inductive hypothesis. Similarly, we may obtain \( v_k = 22v_{k-1} - v_{k-2} \) from \( v_k = s_3u_{k-2} + s_2v_{k-2} \).

Thus, the sequence of rational numbers \( \{u_k/v_k\} \) is determined by the same seeds and the same recurrence relation as the sequence \( \{r_k/s_k\} \), as claimed. \( \square \)

We have also shown:

**Corollary 3.1.6.** The limit of the outer corners is

\[
\lim_{k \to \infty} \frac{p_k}{q_k} = \text{acc}(\beta).
\]

**Proof.** The continued fractions of the ratios \( p_k/q_k \) converge to the continued fraction of \( \alpha = \left\{ 8, 6, 4, 2 \right\}^\infty = \text{acc}(\beta) \). \( \square \)

Corollary 3.1.6 may also be proved by solving the recursion in Definition 3.0.1, see [BHM, Prop. 49], however we do not do this here.

**3.2. Inner corners.** We describe a family of mutations whose existence proves Proposition 3.0.6 and explains the reasoning behind Conjecture 3.0.7. Throughout we will freely use the conventions discussed in Remark 2.4.1.

The following definition is a version of [MMW, Def. 2.1.1]. It describes algebraic relations between various classes, which later will be helpful in showing various identities hold that arise in the ATF proofs.
**Definition 3.2.1.** Two quasi-perfect classes \( E := (d, e, p, q, t) \), \( E' := (d', e', p', q', t') \) are said to be adjacent if after renaming so that \( p/q < p'/q' \) (if necessary), the following relation holds:

\[
(p + q)(p' + q') - tt' = 8pq'.
\]

Further, they are called \( t'' \)-compatible if

\[
\frac{tt' - 4t''}{pp'} = 3(qp' + qq') + qq', \quad \text{i.e.} \quad x^T A x' = 4t''.
\]

The following lemma is from [MMW, Lem. 2.1.2] about \( t \)-compatibility and adjacency. It proves how compatibility and adjacency hold throughout a recursive sequence. Note that this proof did not use the \((d, m)\) coordinates used in [MMW] and just uses the \((p, q, t)\) coordinates, and thus, the lemma holds for our classes here.

**Lemma 3.2.2.** (i) Suppose that the points \( x_0 := (p_0, q_0, t_0), x_1 := (p_1, q_1, t_1) \) are \( t \)-compatible for some \( t \geq 3 \) and have coordinate by coordinate \( x_0 < x_1 \). Then \( x_2 := t x_1 - x_0 \geq 0 \). Also, \( x_1 < x_2 \) and the pair \( x_1, x_2 \) is \( t \)-compatible. Further, if \( x_0, x_1 \) are adjacent, so are \( x_1, x_2 \). Thus, if \( E_0, E_1 \) satisfy \( p_0 < p_1, q_0 < q_1, t_0 < t_1 \) and are adjacent and \( t \)-compatible, then so are the components of all successive pairs in the sequence obtained from \( E_0, E_1 \) by \( t \)-recursion.

(ii) If \( E, E' \) are adjacent, then they are \( t'' \)-compatible exactly if

\[
|p'q - pq'| = t''.
\]

Recall that \( E = (d, e, p, q, t) = (17, 6, 41, 5, 22) \). The following lemma comes from [M1, Lem. 4.6].

**Lemma 3.2.3.** For the classes \( E_\lambda, E_\mu, E_\rho \) where we have \((E_\lambda, E_\mu, E_\rho) := (E_k, E_{k+1}, E)\) or \((E_\lambda, E_\mu, E_\rho) := (E_k, \hat{E}_{k+1}, E_{k+1})\), the following identities hold:

(i) \( p_\lambda + q_\lambda = q_\mu t_\rho - q_\rho t_\mu \) and \( 7p_\lambda - q_\lambda = p_\mu t_\rho - t_\mu p_\rho \)

(ii) \( p_\rho + q_\rho = p_\mu t_\lambda - q_\lambda t_\mu \) and \( p_\rho - 7q_\rho = q_\lambda t_\mu - q_\mu t_\lambda \)

(iii) \( 7q_\mu + q_\mu = q_\mu t_\lambda + p_\mu t_\rho, \quad 7p_\mu - q_\mu = 6p_\mu t_\rho + p_\rho t_\lambda - q_\lambda t_\rho, \) and

\[ 7q_\mu - p_\mu = 6q_\mu t_\lambda + q_\mu t_\rho - p_\rho t_\lambda \]

(iv) \( p_\lambda (p_\rho - 6q_\rho) + q_\lambda q_\rho = t_\mu \)

(v) \( q_\mu t_\lambda + q_\rho t_\mu + q_\mu t_\mu = q_\mu t_\lambda t_\rho \)

(vi) \[ t_\lambda \left( 1 + p_\mu q_\mu - 6q_\mu^2 \right) = q_{xy} \left( p_\mu - 6q_\mu \right) + q_\mu \left( p_\rho - 6q_\rho \right) \]

(vii) \[ -t_\rho \left( \frac{-q_{x\mu}^2}{q_\mu p_\mu - 1} \right) = q_{xy} \left( \frac{q_\mu}{-p_\mu} \right) + q_\mu \left( \frac{q_\lambda}{-p_\lambda} \right) \]

**Proof.** These identities are a reformulation of the recursion compatibility and adjacency equations proven in [M1, Lem. 4.6] using the facts that:

- \( E_\lambda \) and \( E_\mu \) are \( t_\rho \)-compatible and adjacent
- \( E_\rho \) and \( E_\mu \) are \( t_\lambda \)-compatible and adjacent,

which can be proved for the triple \((E_k, E_{k+1}, E)\) by induction using Lemma 3.2.2 and for the triple \((E_k, \hat{E}_{k+1}, E_{k+1})\) using [MMW, Prop. 2.1.9]. Note that in the proof of this proposition, the \((d, m)\) coordinates used in [MMW] are not needed, and only the properties of \((p, q, t)\) were needed which are the same coordinates we are using here. □
The first sequence of mutations we consider is $v^2yxy$. Note, we found this sequence by adapting an analogous case found by Magill in [M1, Prop. 3.9].

We show in Proposition 3.2.10 that this sequence gives embeddings $$(1 - \varepsilon) \cdot E(1, z_k) \hookrightarrow P(\vol(\beta), \vol(\beta))$$ for a sequence $z_k$ such that $\lim_{k \to \infty} z_k = \text{acc}(\beta)$. Each of the points $(z_k, \vol(\beta))$ lie strictly above the embedding function.

Then, for each $k$, we will perform several additional mutations that provide embeddings $$(1 - \varepsilon) \cdot E(1, z) \hookrightarrow P(\lambda, \lambda \beta)$$ where $(z, \lambda)$ does lie on the graph of the embedding function: specifically, at the inner corners between the obstructions from $E_k$ and $\hat{E}_{k+1}$, proving Proposition 3.0.6.

The effects of the successive mutations in the sequence $v^2y^2$ are illustrated in Figure 3.2.1.

We will frequently use the following simplification of $\vol(\beta)$.

**Lemma 3.2.4.** We have the relation

$$\frac{1}{\vol(\beta)} = -1 + \frac{\sqrt{30}}{3} = \frac{4\beta - 7}{5}.$$  

**Proof.** Following the method of proof of [MM, Lem. 2.2.7] and replacing $3 - b$ with the affine perimeter in our case, which is $2 + 2\beta$, for any value of $\beta$ we have

$$\vol(\beta) = \frac{1 + \text{acc}(\beta)}{2 + 2\beta}.$$  

\footnote{In [M1], the mutation sequences are instead written from right to left.}

\footnote{Recall that $\vol(\beta) = \vol_0(\text{acc}(\beta))$.}
With $\beta = (6 + 5\sqrt{30})/12$ and $\text{acc}(\beta) = \frac{54 + 11\sqrt{30}}{14}$, we simplify

$$\frac{1}{\text{vol}(\beta)} = 2 + \frac{2\sqrt{30}}{12} = -1 + \frac{\sqrt{30}}{3}.$$ 

This proves the first equality. The second is a simple computation. \hfill \Box

Remark 3.2.5. The conclusion of Lemma 3.2.4 is similar to [M1, Lem. 5.1 (iii)] where we find that in the case of the Hirzebruch surface, if to the right of the accumulation the function $c_{\beta}(z)$ is given by a class $E = (d, m, p, q, t)$, then

$$\text{vol}(b) = q \left( m - q \right) - \left( d - 3q \right),$$

where $\text{vol}(b) = \text{vol}_H(b)$, noting that the volume obstruction $\text{vol}_b(z)$ has the formula $\sqrt{z/(1 - b^2)}$ when the target is $H_b$. In our case, with $E = (17, 6, 41, 5, 22)$, $\beta = (6 + 5\sqrt{30})/12$, and our definition of $\text{vol}$, we have

$$\text{vol}(\beta) = \frac{5}{4\beta - 7} = \frac{q}{(2q - e)\beta + (2q - d)}. \tag{3.2.2}$$

We compute the result of the first four mutations, illustrated in Figure 3.2.1.

Lemma 3.2.6. After performing the sequence of mutations $v^2yx$ to the diagram $\Omega_\beta$, the nodal rays are

$$\vec{n}_Y = \begin{pmatrix} 1 \\ -7 \end{pmatrix}, \quad \vec{n}_V = \begin{pmatrix} -3 \\ -1 \end{pmatrix}, \quad \vec{n}_X = \begin{pmatrix} 11 \\ 5 \end{pmatrix},$$

the direction vectors are

$$\overrightarrow{OY} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \overrightarrow{OX} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \overrightarrow{YV} = \begin{pmatrix} 1 \\ -6 \end{pmatrix}, \quad \overrightarrow{XV} = \begin{pmatrix} 56 \\ 25 \end{pmatrix},$$

and the affine lengths are

$$|OY| = 3 + \beta, \quad |OX| = \frac{1}{\text{vol}(\beta)}, \quad |YV| = \frac{7 + 4\beta}{19}, \quad |XV| = \frac{3 - \beta}{95}.$$ 

Proof. The diagram $\Omega_\beta$ has nodal rays

$$\vec{n}_Y = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \vec{n}_V = \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \quad \vec{n}_X = \begin{pmatrix} -1 \\ 1 \end{pmatrix},$$

direction vectors

$$\overrightarrow{OY} = \overrightarrow{XV} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \overrightarrow{OX} = \overrightarrow{YV} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and affine lengths

$$|OY| = |XV| = 1, \quad |OX| = |YV| = \beta.$$ 

Step 1: first mutation at $V$. The nodal ray $\vec{n}_V$ hits the side $OX$ at $(\beta - 1, 0)$, giving us the affine lengths

$$|OY_v| = |X_vV_v| = 1, \quad |OX_v| = |Y_vV_v| = \beta - 1, \quad |Y_vV_v| = \beta + 1.$$ 

The matrix $M$ for mutation at $V$ must satisfy

$$M\vec{n}_V = \vec{n}_V, \quad M\overrightarrow{X} = \overrightarrow{YV} \iff M = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}.$$
Thus the result of a \(V\)-mutation has nodal rays
\[
\vec{n}_Y = \vec{n}_Y = \left( \begin{array}{c} 1 \\ -1 \end{array} \right), \quad \vec{n}_V = M \vec{n}_X = \left( \begin{array}{c} -3 \\ -1 \end{array} \right), \quad \vec{n}_X = -\vec{n}_V = \left( \begin{array}{c} 1 \\ 1 \end{array} \right).
\]

The unchanged direction vectors are
\[
\overrightarrow{OY}_v = \left( \begin{array}{c} 0 \\ 1 \end{array} \right), \quad \overrightarrow{OX}_v = \overrightarrow{YV}_v = \left( \begin{array}{c} 1 \\ 0 \end{array} \right),
\]
and
\[
\overrightarrow{XV}_v = M \overrightarrow{OX} = \left( \begin{array}{c} 4 \\ 1 \\ 2 \end{array} \right).
\]

**Step 2: second mutation at \(V\).** We now replace each result \(A_v\) of Step 1 with \(A\) so that we do not have to stack subscripts. The nodal ray \(\vec{n}_V = (-3, -1)\) hits the side \(OX\) at \((b + 1, 1) + (-3, -1) = (b - 2, 0)\), giving us the affine lengths
\[
|OY_v| = |XV_v| = 1, \quad |OX_v| = \beta - 2, \quad |YV_v| = \beta + 2.
\]

The mutation matrix \(M\) must satisfy
\[
M \vec{n}_V = \vec{n}_V, \quad M \vec{YV} = \vec{YV} \iff M = \left( \begin{array}{cc} 4 & -9 \\ 1 & -2 \end{array} \right).
\]

Thus the nodal rays are
\[
\vec{n}_Y = \vec{n}_Y = \left( \begin{array}{c} 1 \\ -1 \end{array} \right), \quad \vec{n}_V = M \vec{n}_X = \left( \begin{array}{c} -5 \\ -1 \end{array} \right), \quad \vec{n}_X = -\vec{n}_V = \left( \begin{array}{c} 3 \\ 1 \end{array} \right).
\]

The unchanged direction vectors are
\[
\overrightarrow{OY}_v = \left( \begin{array}{c} 0 \\ 1 \end{array} \right), \quad \overrightarrow{OX}_v = \overrightarrow{YV}_v = \left( \begin{array}{c} 1 \\ 0 \end{array} \right),
\]
and
\[
\overrightarrow{XV}_v = M \overrightarrow{OX} = \left( \begin{array}{c} 4 \\ 1 \end{array} \right).
\]

**Step 3: mutation at \(Y\).** Again, we replace \(A_v\) with \(A\). The nodal ray \(\vec{n}_Y\) hits the side \(XV\), because its \(x\)-intercept is at \((1, 0)\) and \(\beta - 2 < 1\). The mutation matrix \(M\) must satisfy
\[
M \vec{n}_Y = \vec{n}_Y, \quad M \vec{YV} = \vec{YV} \iff M = \left( \begin{array}{cc} 0 & -1 \\ 1 & 2 \end{array} \right).
\]

Thus the nodal rays are
\[
\vec{n}_Y = M \vec{n}_V = \left( \begin{array}{c} 1 \\ -7 \end{array} \right), \quad \vec{n}_Y = -\vec{n}_Y = \left( \begin{array}{c} -1 \\ 1 \end{array} \right), \quad \vec{n}_X = \vec{n}_X = \left( \begin{array}{c} 3 \\ 1 \end{array} \right).
\]

We know the affine lengths
\[
|OY_y| = |OY| + |YV| = \beta + 3, \quad |OX_y| = |OX| = \beta - 2,
\]
and the unchanged direction vectors
\[
\overrightarrow{OY}_y = \left( \begin{array}{c} 0 \\ 1 \end{array} \right), \quad \overrightarrow{OX}_y = \overrightarrow{OX} = \left( \begin{array}{c} 1 \\ 0 \end{array} \right), \quad \overrightarrow{XV}_y = \overrightarrow{XY}_y = \left( \begin{array}{c} 4 \\ 1 \end{array} \right).
\]
Furthermore,
\[ \overrightarrow{Y_yV_y} = M \overrightarrow{X} = M \begin{pmatrix} -4 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -6 \end{pmatrix}. \]

Finally, we solve (2.4.3) with \( a = y \) to obtain
\[ |Y_yV_y| = \frac{\beta + 2}{5}, \quad |X_yV_y| = \frac{3 - \beta}{5}; \]

note that \( |Y_yV_y| + |X_yV_y| = |XV| = 1 \).

Step 4: mutation at \( X \). We replace \( A_y \) with \( A \). The nodal ray \( \vec{n}_X \) hits the side \( YV \) because it has positive slope. The mutation matrix \( M \) must satisfy
\[ M \vec{n}_X = \vec{n}_X, \quad M \overrightarrow{XV} = \overrightarrow{OX} \iff M = \begin{pmatrix} -2 & 9 \\ -1 & 4 \end{pmatrix}, \]

thus the nodal rays are
\[ \vec{n}_{Y_x} = \vec{n}_Y = \begin{pmatrix} 1 \\ -7 \end{pmatrix}, \quad \vec{n}_{V_x} = -\vec{n}_X = \begin{pmatrix} -3 \\ -1 \end{pmatrix}, \quad \vec{n}_{X_x} = M \vec{n}_V = \begin{pmatrix} 11 \\ 5 \end{pmatrix}. \]

We know the affine lengths
\[ |OY_x| = |OY| = \beta + 3, \quad |OX_x| = |OX| + |XV| = \beta - 2 + \frac{3 - \beta}{5} = \frac{4 \beta - 7}{5}, \]

and the unchanged direction vectors
\[ \overrightarrow{OY_x} = \overrightarrow{OY} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \overrightarrow{Y_xV_x} = \overrightarrow{YY} = \begin{pmatrix} 1 \\ -6 \end{pmatrix}, \quad \overrightarrow{OX_x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \]

Furthermore,
\[ \overrightarrow{X_xV_x} = M \overrightarrow{Y} = M \begin{pmatrix} -1 \\ 6 \end{pmatrix} = \begin{pmatrix} 56 \\ 25 \end{pmatrix}. \]

Finally, we solve (2.4.3) with \( a = x \) to obtain
\[ |Y_xV_x| = \frac{7 + 4 \beta}{19}, \quad |X_xV_x| = \frac{3 - \beta}{95}. \]

The next lemma allows us to compute the effect of \( k \) additional mutations at the corner \( Y \).

**Lemma 3.2.7.** While performing the sequence \( v^2 y x y k \), for each of the final \( y \) mutations, the nodal ray \( \vec{n}_Y \) always intersects the side \( XV \).

**Proof.** We first compute the exact corners of the effect of the sequence \( v^2 y x \) applied to \( \Omega_{\beta} \). From Lemma 3.2.6 we obtain the vertices \( X = (1/\text{vol}(\beta), 0) = (-1 + \sqrt{30}/3, 0) \) and \( Y = (0, \beta + 3) = (0, (42 + 5\sqrt{30})/12) \).

Let \( h_k \) denote the height of the quadrilateral along the \( y \)-axis after the mutation sequence \( v^2 y x y k \), with \( h_0 = (42 + 5\sqrt{30})/12 \) as above. Likewise, let \( (x_k, y_k) \) denote the vector \( \vec{n}_Y \) at the vertex \( (0, h_k) \) along which we are mutating, with \( (x_0, y_0) = (1, -7) \) by Lemma 3.2.6, and let \( t_k \) denote the \( x \)-coordinate of the intersection point of the line through the point \( (0, h_k) \) in the direction of the vector \( (x_k, y_k) \) with the line through the point \( (-1 + \sqrt{30}/3, 0) \) with slope \( \frac{25}{56} \). In terms of \( h_k, x_k, \) and \( y_k, t_k \) is given by the formula
\[ t_k = \frac{x_k(168h_k + 25(\sqrt{30} - 3))}{3(25x_k - 56y_k)}. \]  

(3.2.3)

By definition of mutation, \( h_k > h_{k-1} \) for all \( k \). Assume by induction that \( \vec{n}_Y \) intersects \( \overrightarrow{XY} \) for the first \( k - 1 \) mutations by \( y \). Letting \( V_j \) denote the vertex \( V \) after the mutation sequence \( v^2yx'y' \), our inductive hypothesis implies that \( V_j \) has both \( x \) and \( y \) coordinates less than \( V_{j-1} \) if \( j < k \), so if \( V_j = (x_v, y_v) \),

\[ \frac{y_k}{x_k} < \frac{h_k - y_v}{x_v} < -6, \]  

(3.2.4)

which is the slope of the initial side \( \overrightarrow{YV} \).

Assume by way of contradiction that

\[ t_k \leq -1 + \frac{\sqrt{30}}{3}. \]

Using the formula in (3.2.3) for \( t_k \), we have

\[ \frac{x_k(168h_k + 25(\sqrt{30} - 3))}{3(25x_k - 56y_k)} \leq \frac{-3 + \sqrt{30}}{3} \iff h_k \leq \frac{(3 - \sqrt{30})y_k}{3x_k}. \]

Then, by the inequality (3.2.4), we obtain

\[ h_k \leq \frac{(3 - \sqrt{30})y_k}{3x_k} < \frac{(-6)(3 - \sqrt{30})}{3} = 2(\sqrt{30} - 3), \]

However, \( h_k \geq h_0 = \frac{42 + 5\sqrt{30}}{12} \) for all \( k \in \mathbb{Z}_0 \) and \( h_0 = \frac{42 + 5\sqrt{30}}{12} > 2(\sqrt{30} - 3) \). Thus we have a contradiction. \( \square \)

We now compute the nodal rays and directions of the sides after the mutation sequence \( v^2yx'y' \).

**Lemma 3.2.8.** After performing the sequence \( v^2yx'y' \) the nodal rays are given by:

\[ \vec{n}_Y = \begin{pmatrix} q_k \\ -p_k \end{pmatrix}, \quad \vec{n}_V = \begin{pmatrix} -q_{k-1} \\ p_{k-1} \end{pmatrix}, \quad \vec{n}_X = \begin{pmatrix} 11 \\ 5 \end{pmatrix} \]

and the direction vectors are given by:

\[ \overrightarrow{YV} = \begin{pmatrix} q_k^2 \\ -p_kq_k + 1 \end{pmatrix}, \quad \overrightarrow{XV} = \begin{pmatrix} 56 \\ 25 \end{pmatrix} \]

**Proof.** We first must check the base case when \( k = 0 \), which was computed in Lemma 3.2.6. This is seen as by the defining recursion \( x_k = 22x_{k-1} - x_{k-2} \) we have \( p_{-1} = -1 \) and \( q_{-1} = 3 \). For the inductive step, we explain how this lemma is equivalent to [M1, Lem. 6.6], so follows by the proof there.

In [M1, Lem. 6.6], the lemma is similarly looking at \( y \) mutations to a quadrilateral where the nodal ray intersects the \( |XY|\overrightarrow{XY} \) side of the polygon. The lemma assumes that the quadrilateral is defined via a triple \( \mathcal{T} \) notated as \( Q(\mathcal{T}) \). Looking at the definition of \( Q(\mathcal{T}) \) in [M1, Def. 3.8], we see that if we set \( E_\lambda = E_k, E_\mu = E_{k+1}, \) and \( E_\rho = E \), the definition for the nodal rays and direction vectors are the same as the identities we must prove. Further, after assuming which side the nodal ray will hit, checking formulas for nodal rays and direction vectors after a mutation do not depend on the side lengths.
Thus, this lemma is equivalent to [M1, Lem. 6.6]. The proof uses the identities we already established in Lemma 3.2.3.

□

Next we compute the affine lengths of the sides after the mutation sequence $v^2yxy^k$.

**Lemma 3.2.9.** After performing the sequence $v^2yxy^k$ the affine lengths are given by:

$$
|OY| = \frac{d_k + e_k\beta}{q_k}, \quad |OX| = -1 + \frac{\sqrt{30}}{3} = \frac{4\beta - 7}{5} = \frac{1}{\text{vol}(\beta)}
$$

$$
|YV| = \frac{4\beta + 7}{q_kq_{k+1}}, \quad |XV| = \frac{d_k - e_k\beta}{5q_{k+1}}
$$

**Proof.** We will show this by induction on $k$. The base case $k = 0$ is proved in Lemma 3.2.6. Suppose that the conclusions hold for $k$. Then, for $k + 1$, by Lemma 3.2.7, in performing the consecutive $y$-mutations, the nodal ray will hit the $|XV|$ side of the polygon. Therefore, the $|OX|$ side remains constant.

For $|OY|$, it suffices to show that $|OY| = |OY| + |YV|$. By the induction hypothesis, we have

$$
|OY| + |YV| = \frac{d_k + e_k\beta}{q_k} + \frac{4\beta + 7}{q_kq_{k+1}} = \left(\frac{q_{k+1}e_k + 4}{q_kq_{k+1}}\right)\beta + \left(\frac{q_{k+1}d_k + 7}{q_kq_{k+1}}\right).
$$

It remains to show that $q_{k+1}e_k + 4 = q_k e_{k+1}$ and $q_{k+1}d_k + 7 = q_k d_{k+1}$. Both hold by induction. The base case is easily checked. For the inductive step, we show the details of former and the latter follows similarly. Assuming the equality holds for $k$, by Definition 3.0.1,

$$
q_{k+1}e_k + 4 = (22q_k - q_{k-1})e_k + 4 = 22q_ke_k - (q_{k-1}e_k - 4) = 22q_ke_k - q_ke_{k-1} = q_k(22e_k - e_{k-1}) = q_ke_{k+1}
$$

This completes the proof for $|OY|$.

For $|XV|$, we must show

$$
|XV| = \frac{d_{k+1} - e_{k+1}\beta}{5q_{k+2}}.
$$

By adding the sides of the quadrilateral that is fixed during mutation, we have the following equality

$$
\begin{pmatrix} 0 & |OX| \end{pmatrix} + \begin{pmatrix} |OY| \end{pmatrix} + |XV| \begin{pmatrix} 56 \\ 25 \end{pmatrix} = \begin{pmatrix} s \end{pmatrix} \begin{pmatrix} q_k \\ -p_k \end{pmatrix}
$$

where $s$ is the length of the nodal ray $\vec{n}_Y$ where it intersects the side $XV$. This equality gives us two equations

$$
|OX| + 56|XV| = sq_k
$$

$$
-|OY| + 25|XV| = -sp_k.
$$
We can solve the first one for $s$ and substitute it into the second one to get the equation

$$-|OY| + 25|XV_y| = -\frac{p_k}{q_k}(|OX| + 56|XV_y|).$$

Solving for $|XV_y|$, we get

$$|XV_y| = \frac{q_k|OY| - p_k|OX|}{25q_k + 56p_k}.$$

First, we consider the denominator. We must show that

$$25q_k + 56p_k = q_{k+2}.$$

This follows from (3.1.6). We then consider the numerator where we substitute in the formulas for $|OY|$ and $|OX|$

$$q_k|OY| - p_k|OX| = (e_k\beta + d_k) - \frac{p_k(4\beta - 7)}{5}$$

$$= \frac{1}{5}(b(5e_k - 4p_k) + (5d_k + 7p_k)).$$

It remains to show that

$$-4p_k + 5e_k = -e_{k+1}$$

$$5d_k + 7p_k = d_{k+1}.$$

Again, we prove the first identity by induction and the second follows similarly. The base case is easily checked. Suppose $4p_k - 5e_k = e_{k+1}$ for all $k$. Then, for $k + 1$, by Definition 3.0.1,

$$4p_{k+1} = 4(22p_k - p_{k-1})$$

$$= 22(e_{k+1} + 5e_k) - (e_k + 5e_{k-1})$$

$$= (22e_{k+1} - e_k) + 5(22e_k - e_{k-1})$$

$$= e_{k+2} + 5e_{k+1}. $$

It remains to show that the formula for $|YV|$ holds. We can verify this by checking the first equality in

$$\frac{4\beta + 7}{q_{k+1}q_{k+2}} = \frac{d_k - e_k\beta}{5q_{k+1}} - \frac{d_{k+1} - e_{k+1}\beta}{5q_{k+2}} = |VX| - |V_yX| = |Y_yV_y|. $$

This is equivalent to showing

$$\beta(-e_kq_{k+2} + e_{k+1}q_{k+1}) + (q_{k+2}d_k - d_{k+1}q_{k+1}) = 5(4\beta + 7).$$

Therefore, we must show that

$$-e_kq_{k+2} + e_{k+1}q_{k+1} = 20$$

$$q_{k+2}d_k - d_{k+1}q_{k+1} = 35.$$
We prove the first identity here and the second follows by a similar manner. Suppose that for all $k$, we have $-e_k q_{k+2} + e_{k+1} q_{k+1} = 20$. Then, for $k + 1$,

$$\begin{align*}
- e_{k+1} q_{k+3} + e_{k+2} q_{k+2} &= -e_{k+1} (22 q_{k+2} - q_{k+1}) + e_{k+2} q_{k+2} \\
&= -22 e_{k+1} q_{k+2} + e_{k+1} q_{k+1} + e_{k+2} q_{k+2} \\
&= -22 e_{k+1} q_{k+2} + (20 + e_k q_{k+2}) + e_{k+2} q_{k+2} \\
&= 20 - (22 e_{k+1} - e_{k}) q_{k+2} + e_{k+2} q_{k+2} \\
&= 20 - e_{k+2} q_{k+2} + e_{k+2} q_{k+2} \\
&= 20.
\end{align*}$$

□

**Figure 3.2.5.** This figure illustrates Lemmas 3.2.8 and 3.2.9. The nodal rays are drawn in light blue, with a square indicating their marked point. The fact that $\vec{n}_Y$ intersects $\overrightarrow{XV}$, Lemma 3.2.7, is indicated by the dashed blue line. The affine lengths of $|OX|$ and $|OY|$ are in black, while the vectors $|Y V|$ and $|X V|$ are labeled in red, with their directions indicated by arrowheads.

Together, these lemmas prove:
**Proposition 3.2.10.** There is a full filling at the accumulation point. That is, 
\[ c_\beta(\text{acc}(\beta)) = \text{vol}(\beta). \]

**Proof.** By Lemma 3.2.9 and Lemma 3.2.4, the sequence \( v^2xy^k \) of mutations of the rectangle \( \Omega_\beta \) is a convex quadrilateral containing the right triangle abutting the axes. By multiplying by \( \text{vol}(\beta) \), we invoke Proposition 2.4.2 to obtain an embedding
\[ (1 - \varepsilon) \cdot E\left(1, \frac{\text{vol}(\beta)(d_k + e_k\beta)}{q_k}\right) \rightarrow P(\text{vol}(\beta), \text{vol}(\beta)\beta) \]
for all \( \varepsilon > 0 \). It therefore remains to show
\[ \lim_{k \to \infty} \frac{\text{vol}(\beta)(d_k + e_k\beta)}{q_k} = \text{acc}(\beta). \]

First we note that
\[ \text{vol}(\beta) = \frac{5\text{acc}(\beta)}{17 + 6\beta}, \tag{3.2.6} \]
which can be checked using the formulas for \( \beta \) and \( \text{acc}(\beta) \) in (3.0.1). Thus our goal becomes
\[ \lim_{k \to \infty} \frac{d_k + e_k\beta}{q_k} = \frac{17 + 6\beta}{5}. \]

We find a closed form for the recursion \( x_k = 22x_{k-1} - x_{k-2} \) with \( x_k = d_k, e_k, q_k \). Set
\[ r = 11 + 2\sqrt{30}, \]
\[ d = \frac{3}{2} + \frac{31}{120}\sqrt{30}, \quad \overline{d} = \frac{3}{2} - \frac{31}{120}\sqrt{30}, \]
\[ e = \frac{1}{2} + \frac{1}{10}\sqrt{30}, \quad \overline{e} = \frac{1}{2} - \frac{1}{10}\sqrt{30}, \]
\[ q = \frac{1}{2} + \frac{1}{15}\sqrt{30}, \quad \overline{q} = \frac{1}{2} - \frac{1}{15}\sqrt{30}. \]

Then
\[ d_k = dr^k + \overline{d}r^{-k}, \]
\[ e_k = er^k + \overline{e}r^{-k}, \]
\[ q_k = qr^k + \overline{q}r^{-k}. \]
Then we have
\[
\lim_{k \to \infty} \frac{d_k + e_k \beta}{q_k} = \lim_{k \to \infty} \frac{d \tau^k + d \tau^{-k} + (e \tau^k + e \tau^{-k})}{q \tau^k + q \tau^{-k}} \frac{\tau r^k + \tau r^{-k}}{q r^k + q r^{-k}} \\
= \lim_{k \to \infty} \frac{d + e \beta}{q + q r^{-2k}} = \frac{d + e \beta}{q} = \frac{17 + 6 \beta}{5}.
\]

\[\square\]

Remark 3.2.11. Notice that the slope of $\overrightarrow{Y V}$ has limit
\[
\lim_{k \to \infty} \frac{p_k q_k - 1}{q_k^2} = \lim_{k \to \infty} \frac{p_k}{q_k} = \text{acc}(\beta)
\]
by Corollary 3.1.6. Coupled with the fact that $\overrightarrow{X V} = \frac{d_k - e_k \beta}{5 q_k + 1} \left( \frac{56}{25} \right)$,

to prove Proposition 3.2.10 it would suffice to show that $d_k / e_k \to \beta$, so that the short side $\overrightarrow{X V}$ approaches zero and thus the ratio $|OY|/|OX|$ approaches the slope of $\overrightarrow{Y V}$. However, this would also require solving the recursion.

To get the points on the capacity function, we now consider the sequence $v^2 y x y^k x y^2$. This allows us to prove Proposition 3.0.6 and support Conjecture 3.0.7, because
- the sequence of mutations $v^2 y x y^k x y$ provides an embedding realizing the inner corner between the obstructions from $E_k$ and $\hat{E}_{k+1}$, while
- the sequence of mutations $v^2 y x y^k x y^2$ conjecturally provides an embedding realizing the inner corner between the obstructions from $\hat{E}_{k+1}$ and $E_{k+1}$.

We will use the notation
\[d' := 2q - d, e' := 2q - e.\]

In the following lemma, we are going to use no subscripts to denote the vertices from $v^2 y x y^k$, and then add a subscript of $x$ to get the vertices from $v^2 y x y^k x$.

Lemma 3.2.12. Beginning with the data from Lemma 3.2.9 from performing the sequence $v^2 y x y^k$, one mutation by $x$ gives the nodal rays
\[
\vec{n}_{Y_x} = \left( \frac{q_k}{-p_k} \right), \quad \vec{n}_{V_x} = \left( \frac{-11}{-5} \right), \quad \vec{n}_{X_x} = \left( \frac{121 p_{k-1} + 54 q_{k-1}}{56 p_k + 25 q_k} \right),
\]
the direction vectors
\[
\vec{Y_x V_x} = \left( \frac{q_k^2}{-p_k q_k + 1} \right), \quad \vec{X_x V_x} = \left( \frac{-54 q_k^2 - 121 p_k q_k + 121}{-25 q_k^2 - 56 p_k q_k + 1} \right).
\]
and the affine lengths

\[ |OY_x| = \frac{d_k + e_k \beta}{q_k}, \quad |OX_x| = \frac{d_{k+1} + e_{k+1} \beta}{q_{k+1}}, \]
\[ |Y_x V_x| = \frac{-d_{k+1} + e_{k+1} \beta}{q_k q_{k+1}}, \quad |X_x V_x| = \frac{d_k - e_k \beta}{q_{k+1} q_{k+1}}. \]

Proof. We first give the proof for the direction vectors and nodal rays.

By Lemma 3.2.8, after performing the sequence \( v^2 y x y^k \) we have \( \vec{n}_X = (11, 5) \), \( \vec{n}_V = (-q_{k-1}, p_{k-1}) \), and \( \vec{X} \vec{V} = (56, 25) \). Note that for the next mutation at \( X \), because \( \vec{n}_X \) has positive slope, it will always hit the edge \( \vec{Y} \vec{V} \). Thus, the mutation matrix should satisfy

\[ M \vec{n}_X = \vec{n}_X \quad \text{and} \quad \vec{X} \vec{V} = \vec{O} \vec{X} \iff M = \begin{pmatrix} -54 & 121 \\ -25 & 56 \end{pmatrix}. \]

The polygon after mutation at \( X \) should thus have \( \vec{n}_{Y_x} = -\vec{n}_Y \), \( \vec{n}_{V_x} = \vec{n}_V \), and \( \vec{n}_{X_x} = \vec{M} \vec{n}_V = \begin{pmatrix} 121p_{k-1} + 5q_k \beta \\ 56p_{k-1} + 25q_k \end{pmatrix}, \)

and

\[ \vec{X}_x \vec{V}_x = -M \vec{Y} \vec{V} = \begin{pmatrix} -54q_k^2 - 121pq_k + 121 \\ -25q_k^2 - 56pq_k + 56 \end{pmatrix}, \]

while \( \vec{X}_x \vec{V}_x = \vec{Y} \vec{V} \) because \( \vec{n}_X \) hits \( \vec{Y} \vec{V} \).

We now give the proofs for the affine lengths. Note that \( |OY_x| = |OY| \) because \( Y_x = Y \). Next, to compute \( |OX_x| \), we check that given the formulas for \( |OX| \) and \( |XV| \) from Lemma 3.2.9, we have

\[ |OX_x| = |OX| + |XV| = \frac{4\beta - 7}{5} + \frac{d_k - e_k \beta}{5q_{k+1}} = \frac{d_{k+1} + e_{k+1} \beta}{q_{k+1}}. \]

This follows from the identities

\[ 4q_{k+1} - e_k = 5e_{k+1}, \quad -7q_{k+1} + d_k = 5d_{k+1}, \]

which hold by induction because they are linear identities and the \((d_k, e_k, q_k)\) (and thus \(d_k'\) and \(e_k'\)) satisfy the same linear recursion.

We now look at \( |Y_x V_x| \). Following the proof of [M1, Lem. 6.1 (ii)] (which solves for \( |Y_x V_x| \) using the fact that the sides of the quadrilateral which is fixed under \( x \)-mutation, with sides \( |OX| \vec{OX} \), \( |OY_x| \vec{OY_x} \), \( -|Y_x V_x| \vec{Y} \vec{V} \), \( |OY| \vec{OY} \), and a side parallel to \( \vec{n}_X \), must add to zero), the stated formula for \( |Y_x V_x| \) holds if our analogue of [M1, (6.0.2)] gives us the claimed value for \( |Y_x V_x| \), that is

\[ |Y_x V_x| = \frac{11 |OY| + 5 |OX|}{-11 + q_k (41p_k + 5q_k - 30p_k)} = \frac{-d_{k+1} + e_{k+1} \beta}{q_k q_{k+1}}. \quad (3.2.7) \]

For the denominator of (3.2.7), note that

\[ -11 + q_k (41p_k + 5q_k - 30p_k) \overset{(s)}{=} t_{k+1} q_k - 11 \overset{(**)}{=} t_k q_{k+1} - 5 = q_{k+1}, \]
where (*) uses Lemma 3.2.3 (iv) and (**) uses the second conclusion of Lemma 3.2.3 (ii), both applied to \((E_k, E_{k+1}, E)\).

For the numerator of (3.2.7), we must show
\[
11|OY| + 5|OX| = \frac{11(d_k + e_k\beta)}{q_k} + (4\beta - 7),
\]
so we must check that
\[
11e_k + 4q_k = e_{k+1}',
\]
\[
11d_k - 7q_k = -d_{k+1}'.
\]
As in the proof of the \(|OX_x|\) formula, these hold by induction, using the fact that all terms satisfy the same recursion.

Finally, to verify the formula for \(|X_xV_x|\), we check that
\[
4\hat{q}_{k+1} - e_{k+1}'q_{k+1} = -e_k q_k
\]
\[
7\hat{q}_{k+1} + d_{k+1}'q_{k+1} = d_k q_k.
\]
The first formula can be verified as follows:
\[
4\hat{q}_{k+1} - e_{k+1}'q_{k+1} = -e_k q_k
\]
\[
4(22t_k q_{k+1} - 5) - (2q_{k+1} - e_{k+1})q_{k+1} = -e_k q_k
\]
\[
16t_k q_{k+1} - 80 - 7q_{k+1} + p_{k+1}q_{k+1} - t_{k+1}q_{k+1} = -p_k q_k - q_k^2 + t_k q_k
\]
by Lemma 2.2.5
\[
-6t_k q_{k+1} + 30 - 7q_{k+1} + p_{k+1}q_{k+1} = -p_k q_k - q_k^2
\]
by Lemma 3.2.3 (v)
\[
5t_k q_{k+1} + 30 - 22q_k q_{k+1} = -p_k q_k - q_k^2
\]
by Lemma 3.2.3 (iii)
\[
t_k q_{k+1} + 6 = t_{k+1} q_k
\]
by Lemma 3.2.3 (i),
which holds by Lemma 3.2.3 (ii). All applications of Lemma 3.2.3 use the triple \((E_k, E_{k+1}, E)\). The verification of the second formula uses the exact same sequence of identities.

The next lemma is the key step which allows us to prove Proposition 3.0.6 and thus Theorem 1.1.1. Similar to Lemma 3.2.12, we use no subscripts to denote the vertices after the sequence \(v^2yxy^kx\), and a subscript \(y\) to denote the vertices after the final \(y\) mutation.

**Lemma 3.2.13.** After performing the sequence of mutations \(v^2yxy^kx\), the affine lengths of the axis sides are
\[
|OY_y| = \frac{\hat{d}_{k+1} + \hat{e}_{k+1}\beta}{\hat{q}_{k+1}}
\]
\[
|OX_y| = \frac{d_k + e_k\beta}{p_k}
\]
Proof. First we show that for the last \( y \)-mutation, \( \vec{n}_Y \) extends to hit the side \( \overrightarrow{OX} \) (rather than \( \overrightarrow{XV} \) as for earlier \( y \)-mutations in Lemmas 3.2.6 and 3.2.7). As a consequence, we prove the formula for \(|OX_y|\). By Lemma 3.2.12, we need to show that

(i) the \( x \)-coordinate of the \( x \)-intercept of the line of slope \(-p_k/q_k\) (the slope of \( \vec{n}_Y \)) through the point \((0, |OY|) = (0, (d_k + e_k\beta)/q_k)\) is less than \(|OX| = (d'_{k+1} + e'_{k+1}\beta)/q_{k+1}\), and that

(ii) this \( x \)-coordinate equals the claimed value \((d_k + e_k\beta)/p_k\) of \(|OX_y|\).

Note that this \( x \)-coordinate is at the solution to

\[
- \frac{d_k + e_k\beta}{q_k} = - \frac{p_k x}{q_k} \iff x = \frac{d_k + e_k\beta}{p_k},
\]

so proving (i) suffices to prove (ii). To prove (ii), we need to show

\[
\frac{d_k + e_k\beta}{p_k} < \frac{d'_{k+1} + e'_{k+1}\beta}{q_{k+1}}.
\] (3.2.8)

Recall that in Lemma 3.2.10, we solved the recursion defining the \( E_k \) and found

\[
r = 11 + 2\sqrt{30},
\]
\[
d = \frac{3}{2} + \frac{31}{120} \sqrt{30}, \quad d' = \frac{3}{2} - \frac{31}{120} \sqrt{30},
\]
\[
e = \frac{1}{2} + \frac{1}{10} \sqrt{30}, \quad \varpi = \frac{1}{2} - \frac{1}{10} \sqrt{30},
\]
\[
q = \frac{1}{2} + \frac{1}{15} \sqrt{30}, \quad \bar{q} = \frac{1}{2} - \frac{1}{15} \sqrt{30}.
\]

We can further compute

\[
p = \frac{7}{2} + \frac{13}{20} \sqrt{30}, \quad \bar{p} = \frac{7}{2} - \frac{13}{20} \sqrt{30}.
\]

Expanding (3.2.8) using \( x_k = x r^k + \varpi \bar{r}^{-k} \) for \( x = d, e, p, q \), we want to show

\[
c_{2k+1} r^{2k+1} + c_1 r + c_{-1} r^{-1} + c_{-2k-1} r^{-2k-1} > 0,
\]

where

\[
c_{2k+1} = (2q - d)p + (2q - e)p\beta - dq - eq\beta = 0,
\]
\[
c_1 = (2q - d)\bar{p} + (2q - e)\bar{p}\beta - d\bar{q} - e\bar{q}\beta = -\frac{17}{12} + \frac{31}{120} \sqrt{30} \approx -0.0017,
\]
\[
c_{-1} = (2\bar{q} - d)p + (2\bar{q} - e)p\beta - d\bar{q} - e\bar{q}\beta = \frac{27}{8} + \frac{37}{60} \sqrt{30} \approx 6.7526, \text{ and}
\]
\[
c_{-2k-1} = (2\bar{q} - d)\bar{p} + (2\bar{q} - e)\bar{p}\beta - d\bar{q} - e\bar{q}\beta = -\frac{215}{24} + \frac{39}{24} \sqrt{30} \approx -0.0578.
\]

Because \( c_{2k+1} = 0 \), \( r > 1 \), \( k \geq 0 \), and \( c_{-2k-1} < 0 \), for all \( k \geq 1 \),

\[
c_{2k+1} r^{2k+1} + c_1 r + c_{-1} r^{-1} + c_{-2k-1} r^{-2k-1} > c_1 r + c_{-1} r^{-1} + c_{-2k-1} r^{-2(k-1)-1},
\]
which means
\[ c_{2k+1}r^{2k+1} + c_1r + c_{-1}r^{-1} + c_{-2k-1}r^{-2k-1} > c_1r + (c_{-1} + c_{-2k-1})r^{-1} \]
\[ = -196 + 215\sqrt{\frac{5}{6}} \approx 0.2673 > 0. \]

Finally we prove the statement about \(|OY|\). By Lemma 3.2.12, we must verify that
\[ |OY| = |OY| + |YV| = \frac{d_k + e_k\beta}{q_k} + \frac{-d'_{k+1} + e'_{k+1}\beta}{q_kq_{k+1}} = \frac{\hat{d}_{k+1} + \hat{e}_{k+1}\beta}{q_k}. \]

Therefore, the formula for \(|OY|\) will hold if
\[ e'_{k+1} = q_k\hat{e}_{k+1} - \hat{q}_{k+1}e_k \]
\[ d'_{k+1} = \hat{q}_{k+1}d_k - q_k\hat{d}_{k+1}. \]

Using Remark 3.2.5, and Lemma 2.2.5 to replace all \(d, e\) terms with \(p, q, t\), the first identity becomes
\[ 7q_{k+1} - p_{k+1} + t_{k+1} = q_k\hat{p}_{k+1} + q_k\hat{q}_{k+1} - q_{k+1}\hat{p}_k - \hat{q}_{k+1}q_k + \hat{q}_{k+1}t_k \]
\[ 7q_{k+1} - p_{k+1} = -q_k\hat{t}_{k+1} + \hat{q}_{k+1}t_k \] by Lemma 3.2.3 (vii),
which holds by Lemma 3.2.3 (ii). (Both uses of Lemma 3.2.3 (vii) are applied to the triple \((E_k, \hat{E}_{k+1}, E_{k+1})\).) The proof of the second identity is almost identical. \(\Box\)

**Remark 3.2.14.** Note, that in Lemma 3.2.13 the nodal ray of the additional \(y\) mutation hits the side \(|OX|\) of the quadrilateral rather than the side \(|XV|\) like in Lemma 3.2.9. This implies the formulas in Lemma 3.2.13 are no longer in parallel with the formulas derived by Magill in [M1]. Instead, as found in [M2], when the nodal ray emanating from \(Y\) changes the side it intersects with this corresponds to moving from an embedding strictly above the function to one that lies on the function.

We prove Proposition 3.0.6, proving that there is a full filling at the inner corner between the obstructions from \(E_k\) and \(E_{k+1}\).

**Proof.** (of Proposition 3.0.6) By Lemma 3.2.13, there is an embedding

\[ (1 - \varepsilon) \cdot E\left(\frac{d_k + e_k\beta}{p_k}, \frac{\hat{d}_{k+1} + \hat{e}_{k+1}\beta}{\hat{q}_{k+1}}\right) \overset{s}{\rightarrow} P(1, \beta) \]

for all \(\varepsilon > 0\). Thus there is an embedding

\[ E\left(1, \frac{p_k(\hat{d}_{k+1} + \hat{e}_{k+1}\beta)}{\hat{q}_{k+1}(d_k + e_k\beta)}\right) \overset{s}{\rightarrow} \frac{1}{1 - \varepsilon} \cdot P\left(\frac{p_k}{d_k + e_k\beta}, \frac{p_k\beta}{d_k + e_k\beta}\right), \]

implying that

\[ c_\beta\left(\frac{p_k(\hat{d}_{k+1} + \hat{e}_{k+1}\beta)}{\hat{q}_{k+1}(d_k + e_k\beta)}\right) \leq \frac{p_k}{d_k + e_k\beta} \]

once we take the infimum defining \(c_\beta\). \(\Box\)

To close this section, we explain our reasoning behind Conjecture 3.0.7.
Remark 3.2.15. We initially believed that
\[ c_\beta \|1_{\text{acc}}(\beta)\| = \sup_k \left\{ \mu_{E_k, \beta} \|1_{\text{acc}}(\beta)\| \right\}, \]
and thought that Theorem 1.1.1 would be proved as a consequence of the lower bounds from Proposition 3.0.4 (i) and an embedding providing an upper bound at the intersection between the horizontal line through \( O_{k-1} \) and the line through the origin and \( O_k \) (see Figure 3.0.2 or 3.2.10). Specifically, we thought there might be an infinite sequence of mutations starting with \( v^2 y x y^k x y \) which approached this hypothetical inner corner. We held on to this expectation because the obstructions from the classes \( \hat{E}_k \) are extremely difficult to visualize computationally.\(^8\) The obstruction from \( \hat{E}_2 \) corresponds to the 4,769,607,569th ECH capacity by Lemma 2.3.5, while comparing the ratios between even the first 10,000,000 ECH capacities is very computationally expensive!

\(^8\)We also naively failed to notice that the intersection between the obstructions from \( E_0 \) and \( E_1 \) is below the volume obstruction. This is not true for all \( k \), however.
By comparison with [U1, U2], we eventually discovered the inner classes \( \hat{E}_k \), and found the same sequence considered by Magill in [M2], namely \( v^2 yxy^k xy \), proves Proposition 3.0.6. In order to compute all of \( c_\beta \) on \([1, \text{acc}(\beta)]\), we would need to prove Conjecture 3.0.7 by identifying a sequence of mutations of \( P(1, \beta) \) so that

\[
|OX| = \frac{d_{k+1} + e_{k+1} \beta}{p_{k+1}}, \quad |OY| = \frac{d_{k+1} + e_{k+1} \beta}{q_{k+1}}.
\]

Our hypothesized sequence of mutations is \( v^2 yxy^k xy^2 \), which would provide a full filling and thus upper bound for \( c_\beta \) at the inner corner between the obstructions from \( \hat{E}_{k+1} \) and \( E_{k+1} \): in Figure 3.2.10, this would mean that \( c_\beta \) equals the dashed black line.

**Figure 3.2.10.** This figure provides more detail on the schematic presented in Figure 2.5.1. Using Propositions 3.0.4 (i), 3.0.4 (ii), and 3.0.6, along with Lemma 3.0.8 (i), we have computed \( c_\beta \) along the solid black lines. However, we would need to prove Conjecture 3.0.7 to compute \( c_\beta \) in between \( \hat{O}_k \) and \( O_k \). We know the function must lie in the violet quadrilateral (possibly on its boundary), and we conjecture the function is given by its lower boundary, the dashed black line.

However, computing \( |OX| \) and \( |OY| \) for the sequence \( v^2 yxy^k xy^2 \) – even though it differs from the sequence \( v^2 yxy^k xy \) considered in Lemma 3.2.13 by only one \( y \)-mutation – is considerably more time consuming because for the final two \( y \) mutations, the nodal ray \( \vec{n}_Y \) hits the bottom side of the quadrilateral \( |OX| \) rather than the side \( |XV| \) that the previous \( y^k \) mutations hit. Thus, to determine the combinatorics for the quadrilaterals of the \( y \) mutations hitting the \( |OX| \) would involve many new computations. While this could be done, computing the whole function is not necessary to claim there is an infinite staircase.
4. Other properties of the embedding function

In this section we collect several observations about the structure of the ellipsoid embedding function for polydisks which may be useful for future work.

4.1. Towards Conjecture 1.1.2. Proving Conjecture 1.1.2 would require analogues of Propositions 3.0.4 (i) and 3.0.6. For the obstructions providing the outer corners, this means identifying new Diophantine classes and proving that they are Diophantine. In analogy to [MMW], we expect that the correct classes $E_{k,n}$ and $\hat{E}_{k,n}$ can be obtained from $E_{k,2} := E_k$ and $\hat{E}_{k,2} := \hat{E}_k$ as follows:

- Modify $p_{k,2}/q_{k,2} := p_k/q_k$ by adding $2n - 4$ to each entry in its continued fraction; this is $p_{k,n}/q_{k,n}$.
- Use Lemma 2.2.5 to define $d_{k,n}$ and $e_{k,n}$.

To prove the analogue of Proposition 3.0.6 requires identifying new embeddings. In analogy to [M1] we expect that this amounts to performing the mutations $v^n$ at the start of every sequence of mutations considered in §3.2, rather than just $v^2$.

It would also be possible to prove Conjecture 1.1.2 using [U1, Thm. 4.4]. However, this would require proving that the Diophantine quasi-perfect classes $E_{k,n}$ are perfect,\(^9\) which we do not do in this paper.

Finally we discuss the obstructions analogous to $E$ appearing after the accumulation point of the conjectural staircases $c_{j,n}$.

**Remark 4.1.1.**

(i) We predict that the ECH capacity which gives the step after the accumulation point, generalizing the obstruction for $E$ in the case $n = 2$, for the infinite staircases of Conjecture 1.1.2 will have index

$$k_n = (2n+1)(2n^2+6n+5) = 4n^3+14n^2+16n+5$$

for $n \geq 2$. At these steps, the $z-$coordinate of the associated outer corner is given by the fraction $p_n/q_n$, where

$$p_n = 4n^2 + 10n + 5,$$

and

$$q_n = 2n + 1.$$

The first few of these values are summarized in the table below:

| $n$ | $p_n$ | $q_n$ | $k_n$ |
|-----|-------|-------|-------|
| 2   | 41    | 5     | 125   |
| 3   | 71    | 7     | 287   |
| 4   | 109   | 9     | 549   |
| 5   | 155   | 11    | 935   |
| 6   | 209   | 13    | 1469   |

\(^9\)For a class to be **perfect**, it must be represented by a symplectically embedded sphere, rather than one that is only immersed. That the classes $E_{k,n}$ and $\hat{E}_{k,n}$ are quasi-perfect can be proved following the $n = 2$ case discussed in the proof of Proposition 3.0.4.
One can check using the formulas for \(p_n, q_n,\) and \(k_n\) above that
\[
k_n = \frac{(p_n + 1)(q_n + 1)}{2} - 1
\]
for every \(n \geq 2,\) as predicted by the proof of Lemma 2.3.5.
Also, note that
\[
p_n + q_n = 2(2n^2 + 6n + 3) = t_{n+1},
\]
which is the predicted coefficient of the recursion governing the outer corners of
the next infinite staircase in this family.

(ii) Another way to identify the steps after the accumulation point is to compare to
the case of \(H_b\) and use Conjecture 1.2.1. For \(H_b\), these steps are the obstructions
from the quasi-perfect Diophantine classes centered at
\[
[7,4], [9,6], \ldots, [2n + 3, 2n].
\]
Thus for \(P(1, \beta),\) we expect the centers to be at \([2n + 4, 2n + 1].\) This agrees
with the \(p_n\) and \(q_n\) computed in (i):
\[
[2n + 4, 2n + 1] = 2n + 4 + \frac{1}{2n + 1} = \frac{(2n + 4)(2n + 1) + 1}{2n + 1}.
\]
We can then use
\[
t_n = \sqrt{p^2_n - 6p_nq_n + q^2_n} + 8
\]
and Lemma 2.2.5 to identify the corresponding quasi-perfect Diophantine classes
\(E_n := (2n^2 + 4n + 1, 2n + 2, 4n^2 + 10n + 5, 2n + 1, 2(2n^2 + 2n - 1)).\)
Notice \(E_2\) is what we have been referring to as \(E\).

(iii) The relevance of the \(E_n\)s to the staircases \(c_{\beta_n}\) can also be seen in the fact
that the obstruction \(\mu_{E_n,\beta_n}(z)\) crosses through the volume obstruction \(\text{vol}_{\beta_n}(z)\) at
\(z = \text{acc}(\beta_n).\) That is,
\[
\mu_{E_n,\beta_n}(\text{acc}(\beta_n)) = \frac{q_n\text{acc}(\beta_n)}{d_n + e_n\beta_n} = \sqrt{\frac{\text{acc}(\beta_n)}{2\beta_n}} = \frac{1 + \text{acc}(\beta_n)}{2 + 2\beta_n} = \text{vol}(\beta_n),
\]
implying that (3.2.2) holds with \(q_n, d_n,\) and \(e_n\) replacing \(q, d,\) and \(e.\) We won’t
prove either of these claims here, but we do note that it is a straightforward if
tedious computation using the formulas for \(\beta_n\) and \(\text{acc}(\beta_n)\) in Conjecture 1.1.2
and for \(q_n, d_n,\) and \(e_n\) in (ii) above.

4.2. Usher’s Conjecture. In [U1], Usher considers the family
\(L_{n,0} := \sqrt{n^2 - 1},\)
proving that \(c_{L_{n,0}}\) has an infinite staircase. He does this by proving that a sequence of
classes \(A_{k,n}\) are perfect.
Usher’s classes \(A_{k,n}\) play the role of our classes \(E_k\) (or \(E_{k,n}\) more generally). However, he also identified other obstructions
\(\hat{A}_{k,n} := l_{k-1,n}A_{k,n} - E(n),\)
\[\textit{10}\]
Here we are starting with \(n = 2\) as in [U1], as opposed to \(n = 0\) as in [BHM, MMW].
\[\textit{11}\]
Again note that we use \(k\) to denote a step of the staircase where Usher uses \(i.\)
where $E(n) := E = (n + 1, 1, 2n + 3, 1, 2n)$.

The classes $E(n)$ are similar to our $E = (17, 6, 41, 5, 22)$. Our new staircase $c_\beta$ accumulates to precisely the point where the obstruction from $E$ for $z < 41/5$ crosses the volume curve: see (3.2.6). Meanwhile, Usher’s staircases satisfy

$$\mu_{E(n), L_0}(\mbox{acc}(L_0)) = \frac{\mbox{acc}(L_0)}{n + 1 + L_0} = \mbox{vol}_{L_0}(\mbox{acc}(L_0)).$$

Usher conjectured he could compute the whole function up to the accumulation point. His conjecture says

**Conjecture 4.2.1 ([U1, Conj. 4.23]).** Between the center of $A_{0,n}$ and $\mbox{acc}(L_0)$, $c_{L_0}$ equals the supremum of the obstructions $\mu_{E, L_0}$, where $E$ is one of the $A_{k,n}$ or $\hat{A}_{k,n}$.

Our proof of Theorem 1.1.1 (i) proves that $c_\beta$ is determined on $[p_k/q_k, \hat{p}_{k+1}/\hat{q}_{k+1}]$ by only the classes $E_k$ and $\hat{E}_k$; proving Conjecture 3.0.7 would solve our version of Usher’s conjecture. We expect that Usher’s conjecture for the $c_{L_0}$ staircases could be solved by proving:

- the analogue of Proposition 3.0.6 using the mutation sequence $v^{n-1}y^{k+1}xy$ to compute the inner corner between $A_{k,n}$ and $\hat{A}_{k+1,n}$;

- the analogue of Conjecture 3.0.7 using the mutation sequence $v^{n-1}y^{k+1}xy^2$ to compute the inner corner between $\hat{A}_{k+1,n}$ and $A_{k+1,n}$.

### 4.3. Descending staircases and fractal structure.

If Conjecture 1.2.1 is true, then the set of $\sqrt{3} < \beta \leq \sqrt{5}$ for which $c_\beta$ has an infinite staircase is homeomorphic to the Cantor set. Key to proving this analogy is understanding how $c_\beta$ with $\beta = (6 + 5\sqrt{30})/12$ can be obtained from $c_{\sqrt{3}}$ and a descending infinite staircase $c_\beta$ “mirroring” $c_{\sqrt{3}}$.

In fact, the descending staircase with

$$\beta = \frac{24 + 7\sqrt{3}}{13}$$

was one of the first conjectural infinite staircases we found via computer exploration, and it is precisely this mirror! It is shown in Figure 4.3.1. The reason why we think of $\beta = \frac{24 + 7\sqrt{3}}{13}$ and $\sqrt{3}$ as paired is that both accumulate to the point where the obstruction from $(3, 1, 7, 1, 4)$ intersects the volume curve $\mbox{vol}_\beta(z)$; the ascending staircase $c_{\sqrt{3}}$ from below, where the obstruction has positive slope, and the descending staircase $c_\beta$ from above, where the obstruction is horizontal.

The first two steps of $c_\beta$ are $(5, 1, 9, 1, 6)$ and $E = (17, 6, 41, 5, 22)$. These three classes form what is called in [MMW] a “compatible triple.” Thus we expect the same Cantor set structure to arise for $P(1, \beta)$ infinite staircases with $7 < \mbox{acc}(\beta) < 9$ as does for the target $H_b$ with $6 < \mbox{acc}_H(b) < 8$, see [MMW, Thm. 1.1.1].

Moreover, the language of “blocking classes” developed in [BHM] and relying on the accumulation point formula from [CGHIMP] provides more detail for understanding the results of [CGFS]. In §2.1 we defined the notion of a blocked $\beta$-value. Using the lower bound (2.2.2) on $c_\beta$ by the obstructions $\mu_{E, \beta}$, we say a quasi-perfect Diophantine class...
This figure depicts the conjectural infinite staircase $c_\beta$ with $\beta = (24 + 7\sqrt{3})/13$. In both figures the orange curve is $\text{vol}_\beta(z)$ and $c_\beta$ is in blue. The accumulation point curve $(\text{acc}(\beta), \text{vol}(\beta))$ is in red with $\beta$ varying. Thus the accumulation point of $c_\beta$ ought to occur at the intersection of these three curves. Figure (a) suggests that the accumulation point is precisely where the horizontal obstruction from the class $(3, 1, 7, 1, 4)$ intersects the volume curve. In (b), we have zoomed in.

**E blocks** $c_\beta$ from having an infinite staircase if $\mu_{E_\beta}(\text{acc}(\beta)) > \text{vol}_\beta(\text{acc}(\beta))$. We expect that $c_n, n \in \mathbb{Z}_{\geq 2}$ are blocked by the perfect classes $E(n)$. This illustrates the power of [CGHMP, Thm. 1.13]: it reaffirms why the classes $E(n)$ (which appeared in different notation as the classes $E_n$ in [CGFS, (1-4)]) are natural key players for the computations of Cristofaro-Gardiner, Frenkel, and Schlenk.

### 4.4. Brahmagupta moves.

In [MM], [MMW], and [U1], the authors found a symmetry, referred to as a Brahmagupta move by [U1], that acts on quasi-perfect classes to construct infinitely many different targets that have infinite staircases given one target with an infinite staircase.

It is more natural to define these symmetries via their action on the $z$-variable than the $\beta$-variable in the case of the polydisk (or the $b$-variable in the case of the Hirzebruch surface). Given an infinite sequence of classes $\{E_k\}$ centered at $p_k/q_k$ that form the steps of one infinite staircase, the symmetry sends the class $E_k$ to $S(E_k)$ where $S(E_k)$ is centered at $(6p_k - q_k)/q_k$. On the $z$-coordinate, the symmetry can be expressed as the function $S(z) = 6 - 1/z$. In the work of [MM], [MMW], and [U1], the authors proved that for the staircases being considered the classes $\{S(E_k)\}$ form the steps of a new infinite staircase. Further, by iterating $S$, for each positive integer $i$, $\{S^i(E_k)\}$ is an infinite staircase. Further, by iterating $S$, for each positive integer $i$, $\{S^i(E_k)\}$ is an infinite staircase.

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13In the special cases where $\beta = 1/2$ or $b = 1/3$, when the infinite staircase accumulates to $3 + 2\sqrt{2}$, the symmetry $S$ actually fixes the staircase, but in all other cases studied a new staircase is found.
infinite sequence of classes, which in practice always corresponds to the steps of a new infinite staircase. However, it has not been proven in general that $S$ sends staircases to staircases, and in [MM], [MMW], and [U1] the authors required specific estimates about the starting $\{E_k\}$ staircases to conclude that for each $i$, the $S^i(E_k)$ also form staircases.

For the polydisk, because the function acc is 1-1, we can also consider the effect of the symmetries on the parameter $\beta$ of the polydisk via

$$\beta \mapsto \text{acc}^{-1} \circ S \circ \text{acc}(\beta).$$

The transformation on the $b$-coordinate is exactly the same, but with acc$_H$ replacing acc and its domain restricted to account for the fact that acc$_H$ is 2-to-1 in general.

In [MMW] it was proved that the images of the four-periodic infinite staircase accumulating to $[\{7,5,3,1\}^\infty]$ under the symmetries also have infinite staircases in their ellipsoid embedding functions. Thus we expect that the images of $P(1, (6 + 5\sqrt{30})/12)$ under the Brahmagupta moves likely also have infinite staircases.

**Conjecture 4.4.1.** The functions $c_{\beta_i}$ have infinite staircases, where

$$\beta_i := \text{acc}^{-1} \circ S^i \circ \text{acc} \left( \frac{6 + 5\sqrt{30}}{12} \right).$$

5. Code for exploring ATFs

In §2.4, we gave a detailed introduction to ATFs. In this section, we will continue the topic to discuss the Python realization of mutations.

To start, we use Decimal data type for the calculation. As seen in the mutation sequences providing a full filling at the accumulation point (Proposition 3.2.10), the affine lengths of edges can get extremely small after only a few steps of mutation. This goes beyond the limit of any type of traditional floating data type and could lead to errors and breakdowns of the program. With Decimal, however, one can set however many digits needed with exact precision. This helps greatly when looking for the inner corners near the accumulation point. Further, with enough digits, one can compute the continued fractions of the ratios $|OY|/|OX|$ and, after the periodic pattern is clear, reverse engineer the precise quadratic irrational accumulation points. Here is the code for our setup, with 29 digits:

```python
import decimal
from decimal import Decimal as D

# number of digits calculated:
decimal.getcontext().prec = 10000
# number of digits printed:
N = 29

We construct the node class to integrate the vertex, the nodal ray at the vertex, and the edge departing clockwise from that vertex.

class node (object):
    def __init__ (self, vertex, nodal_ray, edge, affine_length):
```

The next definition, \texttt{init\_polydisk}(b), initializes the polydisk \(P(1, \beta)\)

\begin{verbatim}
def init_polydisk (b):
global n
global nodes
n = 4
nodes = [None] * 4
nodes[0] = node([0,0], [1,1], [0,1], 1.)
nodes[1] = node([0,1], [1,-1], [1,0], b)
nodes[2] = node([b,1], [-1,-1], [0,-1], 1)
nodes[3] = node([b,0], [-1,1], [-1,0], b)
\end{verbatim}

The following two functions, \texttt{dist} and \texttt{dot}, are hand-written helper functions to facilitate the usage of \texttt{Decimal}. We then compute the mutation matrix \(M\).

\begin{verbatim}
def dist (x,y):
    # distance between x and y
    return ( (x[0]-y[0])**2 + (x[1]-y[1])**2 ).sqrt ()
def dot (mat, vec):
    # multiplication of 2*2mat and 2*1vec
    return [ mat[0][0]*vec[0]+mat[0][1]*vec[1] ,
             mat[1][0]*vec[0]+mat[1][1]*vec[1] ]

def solve_matrix (v1, v2, w1, w2):
    # solve the matrix M such that M(v1)=v2, M(w1)=w2
    mat = [ [w1[1], -v1[1]] ,
            [-w1[0], v1[0]] ]
    res = [ dot(mat, [v2[0],w2[0]]),
            dot(mat, [v2[1],w2[1]]) ]
    res[0][0] = res[0][0] / (v1[0]*w1[1]-v1[1]*w1[0])
    res[0][1] = res[0][1] / (v1[0]*w1[1]-v1[1]*w1[0])
    res[1][0] = res[1][0] / (v1[0]*w1[1]-v1[1]*w1[0])
    res[1][1] = res[1][1] / (v1[0]*w1[1]-v1[1]*w1[0])
    return res
\end{verbatim}

In the program we label the vertices clockwise using the numbers 0 – 3, starting from the origin as 0. Then \texttt{intersect\_one}(i,j) solves for the intersection point between the
lines of the $i$-th nodal ray and the $j$-th edge. The function will return the intersection point if it lies on the edge segment and $[-1, -1]$ otherwise.

```python
def intersect_one (i, j):
    # solve the intersection between $i$-th nodal ray
    # and $j$-th edge
    global n
    global nodes

    # copy as local variables
    n1 = nodes[i].vertex
    n2 = nodes[j].vertex
    n3 = nodes[(j+1)%n].vertex
    v1 = nodes[i].nodal_ray
    v2 = nodes[j].edge

    # solve for the intersection point
    vec = [v1[1]*n1[0]-v1[0]*n1[1],
           v2[1]*n2[0]-v2[0]*n2[1]]
    mat = [[-v2[0], v1[0]],
           [-v2[1], v1[1]]]

    itx = dot(mat, vec)
    itx[0] = itx[0] / (v1[0]*v2[1]-v1[1]*v2[0])
    itx[1] = itx[1] / (v1[0]*v2[1]-v1[1]*v2[0])

    # check if the intersection is on the edge
    if abs(n2[0]-n3[0]) == 0:
        lmbda = (itx[1]-n3[1]) / (n2[1]-n3[1])
    else:
        lmbda = (itx[0]-n3[0]) / (n2[0]-n3[0])

    if (lmbda<0 or lmbda>1):
        return [-1, -1]

    return itx
```

The next function, `intersect_all(x)`, solves for the edge that the $x$-th nodal ray intersects with. This is achieved by finding the intersections of the $x$-th nodal ray with all other edges, throwing away invalid intersections, and keeping the one with the shortest distance to the $x$-th vertex.

```python
def intersect_all (x):
    # solve the intersecting edge for the $x$-th nodal ray
    global n
    global nodes
```
```python
# the variables for the intersecting edge
min_edge = x
min_itx = []
min_dis = math.inf

for i in range(n):
    # i is adjacent to x
    if (i==x or i==(x-1)%n):
        continue

    # the intersection of x-th nodal ray
    # and i-th edge is invalid
    itx = intersect_one(x, i)
    if (itx == [-1,-1]):
        continue

    # maintain the closest intersection
    dis = dist(nodes[x].vertex, itx)
    if (dis < min_dis):
        min_edge = i
        min_itx = itx
        min_dis = dis

return (min_edge, min_itx)
```

With the above foundations, the function `mutate(x)` calculates the polygon after mutating the x-th nodal ray. It has two secondary helper functions `mutate_counterclockwise` and `mutate_clockwise`, depending on whether the intersecting edge is to the left or right of the the nodal ray. Here we demonstrate the code for the former as the two are extremely similar.

```python
def mutate_counterclockwise (head, tail, itx):
    # mutate with nodal_ray < intersecting edge
    global n
    global nodes

    mat = solve_matrix( nodes[head].nodal_ray, nodes[head].nodal_ray, nodes[head].edge, nodes[(head-1)%n].edge )

    # construct the new node
    new_length = nodes[ tail ].affine_length
```

* dist (itx, 
  nodes[((tail+1)%n].vertex) 
/ dist (nodes[tail].vertex, 
  nodes[((tail+1)%n].vertex)
new = node(itx, [-nodes[head].nodal_ray[0], 
  -nodes[head].nodal_ray[1]], 
  nodes[tail].edge, new_length)

nodes = np.insert(nodes, tail+1, new)

# adjust the head and tail node
nodes[tail].affine_length -= new_length
nodes[head-1].affine_length += nodes[head].affine_length
nodes = np.delete(nodes, head)

# update remaining nodes
for i in range(head, tail):
  pre = nodes[(i-1)%n]
  nodes[i].vertex[0] = pre.vertex[0] 
    + pre.affine_length*pre.edge[0]
  nodes[i].vertex[1] = pre.vertex[1] 
    + pre.affine_length*pre.edge[1]
  nodes[i].nodal_ray = dot(mat, nodes[i].nodal_ray)
  nodes[i].edge = dot(mat, nodes[i].edge)

def mutate (x):
  # mutate once by x-th nodal_ray
  global n
  global nodes

  # y is the intersecting edge
  # itx is the intersection point
  (y, itx) = intersect_all(x)

  if (x<y):
    mutate_counterclockwise(x,y,itx)
    return y
  else:
    mutate_clockwise(y,x,itx)
    return y+1

Finally, we have two interface functions plot_nodes and print_embd that output direct information for use. The first, plot_nodes, plots the polygon with respect to edge length ratio; print_embd prints the staircase coordinate (z, λ) such that $E(1,z) \leftrightarrow P(\lambda, \lambda b)$. 
The embedding is constructed by fitting the right triangle $\triangle OXY$ into the polygon. Below is an example that gets the $v^2y^2$ example above.

```python
b = (6 + 5 * D(30).sqrt()) / 12
init_polydisk(b)
mutable(2)
mutable(2)
mutable(1)
mutable(1)
plot_nodes()
print_embd()
```

It should be easy to generalize the initialization functions for other types of polygons beyond rectangles, such as triangles and trapezoids. For a complete file of the code, see the Github Repository.

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CF: Rice University
*Email address: ctf4@rice.edu*

TH: Cornell University
*Email address: tara.holm@cornell.edu*

NM: Cornell University
*Email address: nm627@cornell.edu*

JS: Massachusetts Institute of Technology
*Email address: jemma@mit.edu*

ZW: Cornell University
*Email address: zw336@cornell.edu*

MW: Cornell University
*Email address: morgan.weller@cornell.edu*

EZ: Cornell University
*Email address: ez283@cornell.edu*