Electron wave functions on $T^2$ in a static magnetic field of arbitrary direction

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Abstract

A basis set expansion is performed to find the eigenvalues and wave functions for an electron on a toroidal surface $T^2$ subject to a constant magnetic field in an arbitrary direction. The evolution of several low-lying states as a function of field strength and field orientation is reported, and a procedure to extend the results to include two-body Coulomb matrix elements on $T^2$ is presented.

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1. INTRODUCTION

Quantum dots with novel geometries have spurred considerable experimental and theoretical interest because of their potential applications to nanoscience. Ring and toroidal structures in particular have been the focus of substantial effort because their topology makes it possible to explore Aharonov-Bohm and interesting transport phenomena [1, 2, 3, 4]. Toroidal InGaAs devices have been fabricated [5, 6, 7, 8] and modelled, [9] and toroidal carbon nanotube structures studied by several groups [4, 10, 11].

This work is concerned with the evolution of one-electron wave functions on $T^2$ in response to a static magnetic field in an arbitrary direction. The problem of toroidal states in a magnetic field has been studied with various levels of mathematical sophistication. Onofri [12] has employed the holomorphic gauge to study Landau levels on a torus defined by a strip with appropriate boundary conditions and Narnhofer has analyzed the same in the context of Weyl algebras [13]. Here, the aim is to do the problem with standard methodology: develop a Schrodinger equation inclusive of surface curvature, evaluate the vector potential on that surface, and proceed to diagonalize the resulting Hamiltonian matrix.

As noted in [14], ideally one would like to solve the $N$-electron case, but the single particle problem is generally an important first step, and while the $N$ electron system on flat and spherical surfaces has been studied [15, 16, 17, 18, 19, 20], the torus presents its own difficulties. In an effort to partially address this issue, the evaluation of Coulombic matrix elements on $T^2$ is also discussed here.

This paper is organized as follows: in section 2 the Schrodinger equation for an electron on a toroidal surface in the presence of a static magnetic field is derived. In section 3 a brief exposition on the basis set employed to generate observables is presented. Section 4 gives results. Section 5 develops the scheme by which this work can be extended to the two electron problem on $T^2$, and section 6 is reserved for conclusions.

2. FORMALISM

The geometry of a toroidal surface of major radius $R$ and minor radius $a$ may be parameterized by

$$r(\theta, \phi) = W(\theta)\rho + a \sin \theta k$$

(1)

with

$$W = R + a \cos \theta,$$  

(2)

$$\rho = \cos \phi i + \sin \phi j.$$  

(3)

The differential of Eq. (1)

$$dr = a d\theta \theta + W d\phi \phi$$

(4)

with $\theta = -\sin \theta \rho + \cos \theta k$ yields for the metric elements $g_{ij}$ on $T^2$

$$g_{\theta\theta} = a^2$$

(5)
The integration measure and surface gradient that follow from Eqs. (5) and (6) become
\[ \sqrt{g}dq^1dq^2 \rightarrow aWd\theta d\phi \] (7)
and
\[ \nabla = \theta \frac{1}{a} \frac{\partial}{\partial \theta} + \phi \frac{1}{W} \frac{\partial}{\partial \phi}. \] (8)

The Schrödinger equation with the minimal prescription for inclusion of a vector potential \( A \) is
\[ H = \frac{1}{2m} (\frac{\hbar}{i} \nabla + qA)^2 \Psi = E \Psi. \] (9)

The magnetic field under consideration will take the form
\[ B = B_1i + B_0k, \] (10)
which by symmetry comprises the general case. In the Coulomb gauge the vector potential \( A(\theta, \phi) = \frac{1}{2}B \times r \) expressed in surface variables reduces to
\[ A(\theta, \phi) = \frac{1}{2} [B_1(W \sin \phi \cos \theta + a \sin^2 \theta \sin \phi) \theta + (B_0W - B_1 a \sin \theta \cos \phi)] \phi + B_1(F \sin \phi \sin \theta - a \cos \theta \sin \theta \sin \phi) \mathbf{n}. \] (11)

with \( \mathbf{n} = \phi \times \theta \). The normal component of \( A \) contributes a quadratic term to the Hamiltonian but leads to no differentiations in the coordinate normal to the surface as per Eq. (8).

There is a wealth of literature concerning curvature effects when a particle is constrained to a two-dimensional surface in three-space \([21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38]\), including some dealing with the torus specifically \([39]\), but the scope of this work will remain restricted to study of the Hamiltonian given by Eq. (9).

The Schrödinger equation (spin splitting will be neglected throughout this work) is more simply expressed by first defining

\[ \alpha = a/R \]
\[ F = 1 + \alpha \cos \theta \]
\[ \gamma_0 = B_0\pi R^2 \]
\[ \gamma_1 = B_1\pi R^2 \]
\[ \gamma_N = \frac{\pi \hbar}{q} \]
\[ \tau_0 = \frac{\gamma_0}{\gamma_N} \]
\[ \tau_1 = \frac{\gamma_1}{\gamma_N} \]
\[ \varepsilon = \frac{2mEa^2}{\hbar^2}, \]

after which Eq. (9) may be written

\[
\left[ \frac{\partial^2}{\partial^2 \theta} - \frac{\alpha \sin \theta}{F} \frac{\partial}{\partial \theta} + \frac{\alpha^2}{F^2} \frac{\partial^2}{\partial^2 \phi} + i \left( \tau_0 \alpha^2 - \frac{\tau_1 \alpha^2}{F} \sin \theta \cos \phi \right) \frac{\partial}{\partial \phi} \right. \\
\left. + i \alpha \tau_1 \sin \phi (\alpha + \cos \theta) \frac{\partial}{\partial \theta} \right] \Psi = \varepsilon \Psi \tag{12} \]

\[ \Rightarrow H_{\tau} \Psi = \varepsilon \Psi. \tag{13} \]

3. CALCULATIONAL SCHEME

To proceed with a basis set expansion, Gram-Schmidt (GS) functions orthogonal over the integration measure \( F = 1 + \alpha \cos \theta \) must be generated. Fortunately, it is possible to construct such functions almost trivially. The method for doing so has been described elsewhere \[40\], so only the salient results will be presented below.

The \( \tau_1 = 0, \theta \rightarrow -\theta \) invariance of \( H_{\tau} \) suggests that the solutions of the Schrödinger equation be split into even and odd functions, and the primitive basis set can be taken to possess this property;

\[ u_n(\theta) = \frac{1}{\sqrt{\pi}} \cos[n\theta], \quad v_n(\theta) = \frac{1}{\sqrt{\pi}} \sin[n\theta]. \tag{14} \]

The GS functions will take the form

\[ \psi^\pm_K(\theta) = \sum_m c_{Km}^\pm \begin{pmatrix} u_m(\theta) \\ v_m(\theta) \end{pmatrix} \tag{15} \]

with the \( c_{Km} \) given by (momentarily suppressing the parity superscripts \[41\])

\[ c_{Km} = (-)^{K+m} N_K(N_{K-1}\beta N_{K-1})(N_{K-2}\beta N_{K-2})\ldots(N_m\beta N_m) \tag{16} \]

and the normalization factors \( N_K \) determined from

\[ N_{k+1}^2 = \frac{1}{1 - \beta^2 N_k^2} \tag{17} \]

starting from \( N_0 = \sqrt{1/2} \) for positive parity states and \( N_1 = 1 \) for negative parity states. The \( K^{th} \) basis state is attained by appending azimuthal eigenfunctions onto the GS functions described above,

\[ \Psi^\pm_{K\nu}(\theta, \phi) = \frac{1}{\sqrt{2\pi}} \sum_m c_{Km}^\pm \begin{pmatrix} u_m(\theta) \\ v_m(\theta) \end{pmatrix} e^{i\nu \phi}. \tag{18} \]
The matrix

$$H_{\tau \nu}^{pq} = \langle \bar{K}^\pm \nu | H_{\tau} | K^\pm \nu \rangle$$

is then easily constructed since the matrix elements can all be written in closed form (see the Appendix), and the eigenvalues and eigenvectors determined here with a 30 state expansion for each $\theta$-parity. The ordering convention adopted for the states was taken as

$$\Psi_{0,-2}^+, \Psi_{0,-1}^+, \ldots \Psi_{5,2}^+, \Psi_{1,-2}^- \ldots \Psi_{6,2}^-$$

yielding a Hamiltonian matrix blocked schematically into

$$
\begin{pmatrix}
H^{++} & H^{+-} \\
H^{-+} & H^{--}
\end{pmatrix}
$$

4. RESULTS

Rather than present a large number of tables conveying little useful information per unit page length, the focus will be on indicating how some low-lying states evolve as a function of magnetic field strength for two distinct orientations. Some remarks will also be made regarding the general trend seen for higher excited states. Here the ratio $\alpha = a/R$ was set to $1/2$ as a compromise between smaller $\alpha$ where the states tend towards decoupled ring functions and larger $\alpha$ which are less likely to be physically realistic.

Fig. 1 illustrates the evolution of the energy eigenvalue for five low-lying states as a function of $\tau_0$ with $\tau_1 = 0$. The states are all distinct and are labelled in the caption. Not shown are values trivially obtained from the $\pm \nu B_0$ splitting arising from $B_0 = 0, \nu \neq 0$ degeneracy. It is interesting that level crossings with attendant movement towards a ground state with different $K\nu$ occurs near integer values of $\tau_0$, though it is not immediately clear if this is of real significance. It is also of interest to show the sensitivity of the dependence of $\Psi^* \Psi F$ on field strength. Fig. 2 shows that even for moderate field values ($\tau_0 = 5$ corresponds to a field of $1.3 \ T$ for a torus with $R = 50 \ nm$) the large effective flux as compared to atomic or molecular dimensions causes substantial modification to $\Psi^* \Psi F$ in the ground state.

The results given in Figs. 1-2 were for a field configuration that did not mix azimuthal basis states. To investigate an asymmetric case, let $\tau_0 = 0$ and vary $\tau_1$ wherein no field threads the torus. Fig. 3 is analogous to Fig. 1 as described above with the notable exception that the $\nu$ splitting is non-trivial (hence fewer distinct states are shown), and level crossings occur at larger values. There is no plot analogous to Fig. 2 for larger values of $\tau_1$ with $\tau_0 = 0$ because $\phi$ dependence increases rapidly in the eigenstates even for small $\tau_1$; a contour plot is preferable. Figs. 4 and 5 show contour plot results for two states at three field strengths. Note that there is slightly more dependence in $\theta$ when $\tau_1 = 0$ for the state displayed in Fig. 6 than in Fig. 5; the integration measure acts to cancel the angular variation of the state displayed in Fig. 5.
5. EXTENSION TO COULOMB INTEGRALS

The two-electron problem on $T^2$ is complicated by the inability (at least by the author) to find a transformation that decouples the relative electron motion from their center of mass motion as is easily done on $R^2 \text{[42]}$. The obvious transformations do not lead to any advantage over the method adopted by workers long used in atomic and molecular physics, which is to evaluate the two-body matrix elements (supressing spin indices and physical constants)

$$\int \int \Phi_i^*(r_1)\Phi_j^*(r_2)V(r_1, r_2)(1 - P_{12})\Phi_k(r_1)\Phi_l(r_2)d^3r_1d^3r_2 \quad (20)$$

with

$$V(r_1, r_2) = 4\pi \sum_{LM} \frac{1}{2L + 1} r^{L} Y_{LM}^*(\theta_1, \phi_1)Y_{LM}^*(\theta_2, \phi_2). \quad (21)$$

Eq. (20) can be adopted on $T^2$ subject to some peculiarities which are due to the restriction of $r_1, r_2$ to a surface. Eq. (20) on $T^2$ with the notation employed in section 3 becomes

$$\int_0^{2\pi} \Psi_{P\nu_1}(\theta, \phi_1)\Psi_{Q\nu_2}(\theta, \phi_2)V(r_1, r_2)(1 - P_{12})$$

$$\Psi_{R\nu_3}(\theta, \phi_1)\Psi_{S\nu_4}(\theta, \phi_2)F(\theta_1)F(\theta_2)d\theta_1d\theta_2d\phi_1d\phi_2. \quad (22)$$

Consider the direct term; in terms of a spherical coordinate system centered at the middle of the torus ($r \sin \theta \cos \phi, r \sin \theta \sin \phi, r \cos \theta$)

$$r \sin \theta_\alpha = R + a \cos \theta \quad (23)$$

$$r \cos \theta_\alpha = a \sin \theta \quad (24)$$

$$r = \sqrt{R^2 + a^2 + 2aR \cos \theta} \quad (25)$$

and defining

$$\rho_{IJ} \equiv \psi^*_I(\theta)\psi_J(\theta) \quad (26)$$

gives for Eq. (22) after some manipulation

$$\sum_{LM} (\delta_{M-N_1-N_3}\delta_{M-N_2-N_4}) \frac{(L - M)!}{(L + M)!} \int_0^{2\pi} \int_0^{2\pi} \rho_{PQ}(\theta_1)\rho_{RS}(\theta_2)$$

$$P_{LM}(\theta_1, \phi_1)P_{LM}(\theta_2, \phi_2)F(\theta_1)F(\theta_2) \frac{(R^2 + a^2 + 2aR\cos \theta_\alpha)^{L/2}}{(R^2 + a^2 + 2aR\cos \theta_\alpha)^{(L_1+1)/2}}d\theta_1d\theta_2. \quad (27)$$

The arguments of the $P_{LM}$ are evaluated with

$$\theta_{\alpha_i} = \arctan \left( \frac{R + a \cos \theta_i}{a \sin \theta_i} \right). \quad (28)$$
To evaluate the integral care must be taken with the $>,<$ character of the radial factor in the integrand. One way to proceed is as follows:

1. Fix $\theta_1 = 0$. At this point $r_1$ is at its maximum; integrate the integrand of Eq.(27) numerically over $d\theta_2$ from $[0, 2\pi]$ by some suitable method to attain a value labelled by, say, $G_0(\theta_1)$.

2. Set $\theta_1 = \delta$. Integrate $d\theta_2$ with $r_\geq = r_1$, $r_\leq = r_2$ over the interval $[\delta, 2\pi - \delta]$, then set $r_\geq = r_2$ and $r_\leq = r_1$ from $[2\pi - \delta, \delta]$. This is $G_\delta(\theta_1)$.

3. Repeat the second step until the entire interval around the toroidal cross section is covered. A table $[G_0(\theta_1), G_\delta(\theta_1), G_{2\delta}(\theta_1), ...]$ results that can then be integrated numerically. The exchange term proceeds similarly; only the densities need modification.

6. CONCLUSIONS

This work presents a method to calculate the spectrum and wave functions for an electron on $T^2$ in an arbitrary static magnetic field. Aside from the character of the solutions and numerical data, perhaps the main result of this paper has to do with the ease with which an arbitrarily large number of GS states can be trivially generated. Because every physical interaction can eventually be expressed as a periodic function on $T^2$, matrix elements for the interaction may then be evaluated in closed form; hence the only restriction to doing any problem on $T^2$ is matrix inversion.

The procedure employed here to generate observables lends itself to easy incorporation of an arbitrary number of surface delta functions or other type of potential. This is important because on a closed nanotube, in contrast to a macroscopic crystal, there are a relatively small number charge carriers, so the continuum approximation may break down for smaller torii. Clearly, the magnetic field treated here will not be sufficient to comprise the general case as soon as any sort of lattice structure breaking azimuthal symmetry is imposed on the torus. However, the extension is simple to implement in Eq. (12), requiring only a few more terms. It would be also be interesting to see the extension of the static case discussed here to a time dependent laser control problem of the type in [43].

Some remarks should be made regarding the curvature potential $V_C$ well known to workers in the field of quantum mechanics on curved surfaces. It was shown in [44] that a full three dimensional treatment of the problem of a particle near, but not necessarily restricted to $T^2$, yields a spectrum consistent with inclusion of $V_C$ added to the two dimensional surface Hamiltonian. Here the potential could not be included without substantially increasing the scope and complexity of the problem undertaken. It was shown in [45] that the inclusion of a vector potential precludes a separation of variables into surface and normal degrees of freedom; $A$ added to the Schrodinger equation requires solving coupled differential equations in the surface and normal variables, or if a basis set expansion is employed, a much more complicated procedure to generate three-dimensional GS states.

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Appendix

This appendix gives closed form expressions for the matrix elements needed to construct the matrix $H_{\tau_{KK\nu\nu}}^{pq} = \langle \bar{K}^+\bar{\nu}|H_{\tau}|K^+\nu \rangle$. First let

$$P_1 = 1 + \frac{3}{2}\alpha^2$$
$$P_2 = 3\alpha + \frac{3}{4}\alpha^3$$
$$P_3 = \frac{3}{2}\alpha^2$$
$$P_4 = \frac{\alpha^3}{4}$$
$$f(\alpha) = \frac{\sqrt{1 - \alpha^2} - 1}{\alpha}$$

and define

$$\delta_{J,K} \equiv \Delta_{J-K}$$

Each operator in Eq. (9) will connect either only like parity states or opposite parity states; no single operator will do both. The matrix elements that connect like positive parities are

$$\langle \bar{K}^+\bar{\nu}\bigg|\frac{\partial^2}{\partial^2\theta}\bigg|K^+\nu \rangle = \frac{\pi}{2} \sum_{m=0}^{\tilde{K}} \sum_{n=0}^{\tilde{K}} c_{Km}c_{Kn} (-n^2) \Delta_\nu^{-\nu} \left[ (\Delta_{m+n} - \Delta_{m-n}) + \frac{\alpha}{2}(\Delta_{m+n+1} + \Delta_{m-n+1} + \Delta_{m+n-1} + \Delta_{m-n-1}) \right]$$

$$\langle \bar{K}^+\bar{\nu}\bigg|\frac{\alpha^2}{F^2} \frac{\partial^2}{\partial^2\phi}\bigg|K^+\nu \rangle = \frac{\alpha^2\pi}{\sqrt{1 - \alpha^2}} \sum_{m=0}^{\tilde{K}} \sum_{n=0}^{\tilde{K}} c_{Km}c_{Kn} (-\nu^2) \Delta_\nu^{-\nu} [f^{n+m}(\alpha) + f^{m-n}(\alpha)]$$

$$\langle \bar{K}^+\bar{\nu}\bigg|i\tau_0\alpha^2 \frac{\partial}{\partial\phi}\bigg|K^+\nu \rangle = \pi\tau_0\alpha^2 \sum_{m=0}^{\tilde{K}} \sum_{n=0}^{\tilde{K}} c_{Km}c_{Kn} (-\nu) \Delta_\nu^{-\nu} \left[ (\Delta_{m+n} + \Delta_{m-n}) + \frac{\alpha}{2}(\Delta_{m+n+1} + \Delta_{m-n+1} + \Delta_{m+n-1} + \Delta_{m-n-1}) \right]$$

$$\langle \bar{K}^+\bar{\nu}\bigg|\frac{-\tau_0^2\alpha^2}{4}F^2\bigg|K^+\nu \rangle = -\frac{\pi\tau_0^2\alpha^2}{4} \sum_{m=0}^{\tilde{K}} \sum_{n=0}^{\tilde{K}} c_{Km}c_{Kn} \Delta_\nu^{-\nu}$$
\[ P_1\Delta_{m-n} + \frac{1}{2}(P_2(\Delta_{m+n-1} + \Delta_{m-n+1} + \Delta_{m-n-1}) + \\
+ P_3(\Delta_{m+n-2} + \Delta_{m-n+2} + \Delta_{m-n-2}) + \\
+ P_4(\Delta_{m+n-3} + \Delta_{m-n+3} + \Delta_{m-n-3})) \] (A5)

\[ \left\langle \tilde{K}^+\tilde{\nu} \right| - \frac{\gamma^2\alpha^2}{4} F^2 \sin^2 \phi \left| K^+\nu \right\rangle = -\frac{\pi \gamma^2\alpha^2}{4} \sum_{m=0}^{\tilde{K}} \sum_{n=0}^{K} c_{\tilde{K}m} c_{Kn} (\frac{1}{2}\Delta_{\nu-\nu} - \frac{1}{4}\Delta_{\nu-\nu}^2 - \frac{1}{4}\Delta_{\nu-\nu}^2) \] (A6)

\[ \left\langle \tilde{K}^+\tilde{\nu} \right| - \frac{\gamma^2\alpha^4}{4} \sin^2 \phi \left| K^+\nu \right\rangle = -\frac{\pi \gamma^2\alpha^2}{8} \sum_{m=0}^{\tilde{K}} \sum_{n=0}^{K} c_{\tilde{K}m} c_{Kn} \Delta_{\nu-\nu} [\Delta_{m+n} + \Delta_{m-n} + \\
\frac{\alpha}{4}(\Delta_{m+n+1} + \Delta_{m-n+1} + \Delta_{m+n-2} + \Delta_{m-n-2}) + \\
\frac{1}{2}(\Delta_{m+n+2} + \Delta_{m-n+2} + \Delta_{m+n-2} + \Delta_{m-n-2}) + \\
\frac{\alpha}{4}(\Delta_{m+n-3} + \Delta_{m-n-3} + \Delta_{m+n-2} + \Delta_{m-n-2})]\right. (A7)

The negative to negative terms are

\[ \left\langle \tilde{K}^-\tilde{\nu} \right| \frac{\partial^2}{\partial^2 \theta} \left| K^-\nu \right\rangle = \pi \sum_{m=1}^{\tilde{K}} \sum_{n=1}^{K} d_{\tilde{K}m} d_{Kn} (\nu^2) \Delta_{\nu-\nu} [\Delta_{m-n} + \frac{\alpha}{2}(\Delta_{m-n+1} + \Delta_{m-n-1})] \] (A8)

\[ \left\langle \tilde{K}^-\tilde{\nu} \right| - \frac{\alpha}{F} \sin \theta \frac{\partial}{\partial \theta} \left| K^-\nu \right\rangle = \alpha \frac{\pi}{2} \sum_{m=1}^{\tilde{K}} \sum_{n=1}^{K} d_{\tilde{K}m} d_{Kn} \Delta_{\nu-\nu} (\Delta_{m-n+1} - \Delta_{m-n-1}) \] (A9)

\[ \left\langle \tilde{K}^-\tilde{\nu} \right| \frac{\alpha^2}{F^2} \frac{\partial^2}{\partial^2 \phi} \left| K^-\nu \right\rangle = \frac{\alpha^2\pi}{\sqrt{1 - \alpha^2}} \sum_{m=1}^{\tilde{K}} \sum_{n=1}^{K} d_{\tilde{K}m} d_{Kn} (\nu^2) \Delta_{\nu-\nu} [\langle \nu^{n+m}(\alpha) - f^{n-m}(\alpha) \rangle] \] (A10)

\[ \left\langle \tilde{K}^-\tilde{\nu} \right| i\tau_0 \alpha^2 \frac{\partial}{\partial \phi} \left| K^-\nu \right\rangle = \pi \tau_0 \alpha^2 \sum_{m=1}^{\tilde{K}} \sum_{n=1}^{K} d_{\tilde{K}m} d_{Kn} (-\nu) \Delta_{\nu-\nu} [\Delta_{m-n} + \\
\frac{\alpha}{2}(\Delta_{m+n+1} + \Delta_{m-n+1} + \Delta_{m+n-1} + \Delta_{m-n-1})] \] (A11)
\[
\left\langle K^{-\nu} \right| - \frac{\tau_0^2 \alpha^2}{4} F^2 \right| K^{-\nu} \right\rangle = -\frac{\pi \tau_0^2 \alpha^2}{4} \sum_{m=1}^{K} \sum_{n=1}^{K} d_{Kn} \Delta_{\nu-\nu} \]

\[
[P_1 \Delta_{m-n} + \frac{1}{2} (P_2 (\Delta_{m-n-1} + \Delta_{n-m-1} - \Delta_{1-m-n}) + 
+ P_3 (\Delta_{m-n-2} + \Delta_{n-m-2} - \Delta_{2-m-n}) 
+ P_4 (\Delta_{m-n-3} + \Delta_{n-m-3} - \Delta_{3-m-n}))]
\]

(A12)

\[
\left\langle K^{-\nu} \right| - \frac{\tau_1^2 \alpha^2}{4} F^2 \sin^2 \phi \right| K^{-\nu} \right\rangle = -\frac{\pi \tau_1^2 \alpha^2}{8} \sum_{m=1}^{K} \sum_{n=1}^{K} d_{Kn} \Delta_{\nu-\nu} \left( \frac{1}{2} \Delta_{\nu-\nu} - \frac{1}{4} \Delta_{\nu+2} - \frac{1}{4} \Delta_{\nu-2} \right)
\]

\[
[P_1 \Delta_{m-n} + \frac{1}{2} (P_2 (\Delta_{m-n-1} + \Delta_{n-m-1} - \Delta_{1-m-n}) + 
+ P_3 (\Delta_{m-n-2} + \Delta_{n-m-2} - \Delta_{2-m-n}) 
+ P_4 (\Delta_{m-n-3} - \Delta_{n-m-3} + \Delta_{3-m-n}))]
\]

(A13)

\[
\left\langle K^{-\nu} \right| - \frac{\tau_1^2 \alpha^4}{4} \sin^2 \theta \cos^2 \phi \right| K^{-\nu} \right\rangle = -\frac{\pi \tau_1^2 \alpha^4}{8} \sum_{m=1}^{K} \sum_{n=1}^{K} d_{Kn} \Delta_{\nu-\nu} \left[ \Delta_{m-n} + \frac{\alpha}{4} (\Delta_{m-n+1} + \Delta_{n-m+1} - \Delta_{1-m-n}) + \frac{1}{2} (-\Delta_{m-n+2} - \Delta_{n-m-2} + \Delta_{2-m-n}) 
+ \frac{\alpha}{4} (-\Delta_{m-n+3} - \Delta_{n-m-3} + \Delta_{3-m-n}) \right].
\]

(A14)

The matrix elements connecting different parities are

\[
\left\langle K^{+\nu} \right| - i \frac{\tau_1 \alpha^3}{F} \sin \phi \cos \phi \frac{\partial}{\partial \phi} \right| K^{-\nu} \right\rangle = \frac{\pi \tau_1 \alpha^3}{4} \sum_{m=0}^{K} \sum_{n=1}^{K} c_{Km} d_{Kn} (-\nu) (\Delta_{\nu-\nu+1} + \Delta_{\nu-\nu-1})
\]

\[
\left[ \Delta_{m+n+1} + \Delta_{n+m+1} - \Delta_{m-n+1} - \Delta_{m+n-1} \right]
\]

(A15)

\[
\left\langle K^{+\nu} \right| i \alpha \tau_1 \sin \phi (\alpha + \cos \theta) \frac{\partial}{\partial \theta} \right| K^{-\nu} \right\rangle = \tau_1 \alpha^3 \pi \sum_{m=0}^{K} \sum_{n=1}^{K} c_{Km} d_{Kn} (\nu) (\Delta_{\nu-\nu+1} - \Delta_{\nu-\nu-1})
\]

\[
\left[ \frac{3 \alpha}{4} (\Delta_{m+n} + \Delta_{n-m}) + \frac{1 + \alpha^2}{4} (\Delta_{m+n+1} + \Delta_{n-m+1} + \Delta_{m+n-1} + \Delta_{m-n-1}) 
+ \frac{\alpha}{4} (\Delta_{m+n+2} + \Delta_{n-m-2} + \Delta_{m+n-2} + \Delta_{m-n-2}) \right]
\]

(A16)
\begin{align*}
\left\langle \bar{K}^+ \bar{\nu} \left| \frac{\tau_0 \tau_1 \alpha^3}{2} \sin \theta \cos \phi \right| F \right| K^- \nu \right\rangle &= \frac{\tau_0 \tau_1 \alpha^3 \pi}{4} \sum_{m=0}^{K} \sum_{n=1}^{K} c_{\tilde{K}m} d_{Kn}(\Delta_{\nu-\rho+1} - \Delta_{\nu-\rho-1}) \\
&= \left\{ \frac{1}{2}(1 + \frac{\alpha^2}{2})(\Delta_{m+n-1} - \Delta_{m+n+1} + \Delta_{m-n-1} - \Delta_{m-n+1}) + \frac{\alpha}{2}(\Delta_{m+n-2} - \Delta_{m+n+2} + \Delta_{m-n-2} - \Delta_{m-n+2}) \\
&\quad + \frac{\alpha^2}{8}(\Delta_{n+m+1} - \Delta_{m+n-1} + \Delta_{m+n+1} - \Delta_{m-n+1}) \\
&\quad + \Delta_{n+m-3} - \Delta_{m+n+3} + \Delta_{n-m-3} - \Delta_{n-m+3}) \right\}. \tag{A17}
\end{align*}

The negative to positive elements are obtained by interchanging all indices, or equivalently, by taking their transpose.
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Figure Captions

Fig. 1: $\varepsilon$ as a function of $\tau_0$ for five low-lying states. Diamonds correspond to the $|\nu K^\pm > = |00^+ >$ state, stars to $|-10^+ >$, squares to $|-20^+ >$, triangles to $|01^- >$ and circles to $|01^+ >$.

Fig. 2: Evolution of $\Psi^*\Psi F$ for the $|00^+ >$ state given for $\tau_0 = 0$ (thin line), $\tau_0 = 2.5$ (medium line) and $\tau_0 = 5.0$ (thickest line).

Fig. 3: $\varepsilon$ as a function of $\tau_1$ for five low-lying states. Diamonds correspond to the $|\nu K^\pm > = |00^+ >$ state, stars/squares to $|-10^+ >$, and triangles/circles to $|-20^+ >$.

Fig. 4: Sequential evolution of the $\Psi^*\Psi F = F|00^+ >$ state on $T^2$ for $\tau_0 = 0$, $\tau_0 = 2.5$ and $\tau_0 = 5.0$.

Fig. 5: Sequential evolution of the $\Psi^*\Psi F = F|-10^+ >$ state on $T^2$ for $\tau_1 = 0$, $\tau_1 = 2.5$ and $\tau_1 = 5.0$. The ground state variation in $\theta$ is partially cancelled by the integration measure.
Fig. 4
Fig. 5