General Flattened Jaffe Models for Galaxies

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Abstract

In this paper we extend oblate and prolate Jaffe models into more general flattened Jaffe models. Since dynamical properties of oblate and prolate Jaffe Models have been studied by Jiang & Moss, they are not repeated here.

Key words: stellar dynamics – celestial mechanics – galaxies: elliptical and lenticular, cD.

1 Introduction

There is a long history of studying the structure of ellipsoidal galaxies by construction of self-consistent density-potential pairs. Some early profiles of modelling ellipsoidal galaxies are spherical and purely empirical to fit the surface brightness of galaxies observed, for example, the de Vaucouleurs (1948) and Hubble (1930) profiles. The de Vaucouleurs profile can fit the brightness profiles of many giant ellipticals such as the giant E1 galaxy NGC 3379. In contrast, the data for NGC 4472 (a giant elliptical galaxy) cannot be fitted by the de Vaucouleurs profile. The Hubble profile fits the observed brightness profiles of some galaxies as well as the de Vaucouleurs laws. However, the Hubble profile predicts that the total mass of a galaxy is infinite. This must involve making some reasonable assumption, such as that the mass/light ratio is constant with position. Also, based on gravitational potential theory or Jeans’s theorems, other spherical models have been constructed by using potential-density pairs or described by using distribution functions, for example, the Plummer (1911) and King profiles. The Plummer profiles fit observations of globular clusters. Although the Plummer profiles can fit the brightness of some galaxies at finite $R$, they cannot fit the brightness profiles of galaxies at positions with large $R$, since the density distributions of galaxies typically decrease according to $O(\frac{1}{r^\alpha})$ in regions of large $r$, where $2 < \alpha \leq 4$ (Binney & Tremaine 1987, Mihalas & Binney 1981). Applying Jeans’s theorems to spherical stellar systems, it follows that the potential-density pairs of Plummer’s profiles can be reproduced by constructing a very simple distribution function. The King profile is constructed from a distribution function (Michie 1963, Michie & Bodenheimer 1963, King 1966, 1981) and fits star counts in globular clusters and dwarf ellipticals (Mihalas & Binney 1981). The brightness profiles of some giant ellipticals such as NGC 4472 can be also well fitted by the King profiles but none of the

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King profiles can fit the observations for the giant E1 galaxy NGC 3379 as well as does the de Vaucouleurs law (Mihalas & Binney 1981), since the King profiles predict that the luminosity of a galaxy vanishes at some finite radius $R_t$. It is thus clear that neither empirical nor theoretical laws can describe all elliptical galaxies.

Of course, there are also many other spherical models for the surface brightness profiles of galaxies. Although spherical models can mimic the surface brightness of some galaxies observed, galaxies are not all spherical. Thus many axisymmetric non-spherical models have been constructed for our overall understanding of galaxy formation and evolution. The basic idea is to extend some spherical models to more general axisymmetric cases. The methods of this extension can roughly be classified into three groups. The first is to use the axisymmetric radius of $\sqrt{R^2 + (a + |z|)^2}$ in place of the spherical radius $r = \sqrt{R^2 + z^2}$, where $a > 0$. This was originally introduced by Kuzmin (1956). It is necessary to discuss this device here although it is not directly relevant to ellipticals and it is used to model discs. The physical significance of this device is that at the point $(R, -|z|)$ below Kuzmin’s disc, the potential of Kuzmin’s (1956) models is identical with that of a point mass located at distance $a$ above the disc’s centre. The second is to use the axisymmetric radius of $\sqrt{R^2 + (z/q)^2}$ in place of the spherical radius, where $q$ is the axial ratio, as in Binney’s logarithmic models (Binney & Tremaine 1987). The third is to use the axisymmetric radius of $\sqrt{R^2 + (\sqrt{z^2 + c^2 + d})^2}$ in place of the spherical radius, where $c$ and $d$ are positive constants. This device was first introduced by Miyamoto and Nagai (1975) for the Plummer potential. There are also other methods of constructing axisymmetric potential-density pairs. For example, a new potential-density pair can be obtained if a known potential is differentiated. Satoh’s (1980) models is obtained by differentiating the potential of Plummer-Kuzimin’s model $n$ times with respect to $c^2$ if it is assumed that the axisymmetric radius of Plummer-Kuzimin’s model is $\sqrt{R^2 + (\sqrt{z^2 + c^2 + d})^2}$ as given above.

Jaffe’s (1983) spherical models can be flattened into the oblate models (Jiang 2000) and they can be also elongated into the prolate models (Jiang & Moss 2002) using Miyamoto and Nagai’s device mentioned above. It have been known that the density of Jaffe’s (1983) spherical model decays radially like $r^{-4}$ at large distances and that, at large distances, the density of the oblate model given by Jiang (2000) decays radially like $r^{-4}$, except on the $R$-axis, and like $r^{-3}$ on the $R$-axis. In contrast with the oblate model, the density of the prolate model (Jiang & Moss 2002) at large distances decays radially like $r^{-4}$, except on the $z$-axis, and like $r^{-3}$ on the $z$-axis. The question addressed here is, can Jaffe’s spherical model be extended into a model whose density at large distances decays radially like $r^{-4}$, except on the two principal axes, and like $r^{-3}$ on the two axes? Yes. In this paper, we construct a class of more general flattened Jaffe models using a similar Miyamoto and Nagai’s device, that is, replacing the spherical radius of the potentials of Jaffe’s models with a more general axisymmetric radius of $\sqrt{(\sqrt{R^2 + a^2 + b})^2 + (\sqrt{z^2 + c^2 + d})^2}$, where $a, b, c$ and $d$ are positive constants. Then we study the properties of the potential-density pairs of the general flattened Jaffe models in order to advance our overall understanding of galaxy formation and evolution.
2 Potential-density Pairs

In order to answer the question mentioned in the last section, we recall Jaffe’s spherical models with potential-density pairs as follows:

\[ \Phi(r) = \frac{GM}{r_J} \ln\left(\frac{r}{r + r_J}\right), \]  
\[ \rho(r) = \left(\frac{M}{4\pi r_J^3}\right) \frac{r_J^4}{r^2(r + r_J)^2}, \]

where and everywhere below, \( r = \sqrt{x^2 + y^2 + z^2} \) is one of three spherical coordinates \((r, \theta, \phi)\) which can be expressed by three Cartesian coordinates \((x, y, z)\), \( M \) and \( r_J \) are positive constants and \( G \) is the gravitational constant. Then we replace \( r \) in (2.1) by

\[ \sqrt{\left(\sqrt{R^2 + a^2 + b^2} + (\sqrt{z^2 + c^2 + d^2})\right)^2}, \]

where and everywhere below, \( a, b, c \) and \( d \) are positive constants, thus giving a family of general axisymmetric elliptical models with potentials

\[ \Phi(R^2, z) = \frac{GM}{r_J} \ln\left(\frac{\sqrt{(\sqrt{R^2 + a^2 + b^2} + (\sqrt{z^2 + c^2 + d^2})^2})}{\sqrt{(\sqrt{R^2 + a^2 + b^2} + (\sqrt{z^2 + c^2 + d^2})^2 + r_J)}}\right). \]  
\[ \rho(R^2, z) = \frac{M}{4\pi r_J^3} \frac{r_J^4}{r^2(R + r_J)^2(\sqrt{R^2 + a^2 + b^2} + (\sqrt{z^2 + c^2 + d^2})^2)} \frac{A\tau^3 + B\tau^2 + (3\tau + 2r_J)r_JC}{\tau^4(\tau + r_J)^2X^3Y^3} \]

Of course, from (2.3), the densities

\[ \rho(R^2, z) = \frac{M}{4\pi r_J^3} \frac{r_J^4}{r^2(R + r_J)^2(\sqrt{R^2 + a^2 + b^2} + (\sqrt{z^2 + c^2 + d^2})^2)} \frac{A\tau^3 + B\tau^2 + (3\tau + 2r_J)r_JC}{\tau^4(\tau + r_J)^2X^3Y^3} \]

\[ A = bX^2Y^3 + a^2bY^3 + c^2dX^3, \]
\[ B = X^3Y^3 + bX^2Y^3 + a^2bY^3 + c^2dX^3, \]
\[ C = a^2X(X + b)Y^3 + c^2X^3Y(Y + d)^2, \]
\[ \tau = \sqrt{(X + b)^2 + (Y + d)^2}, \quad X = \sqrt{R^2 + a^2}, \quad Y = \sqrt{z^2 + c^2}. \]

Obviously, \( \rho(R^2, z) \geq 0 \) for all positive constants \( a, b, c, d \) and \( r_J \). It is also found from (2.3) and (2.4) that the model (2.3)–(2.4) degenerates into the oblate model as \( a \) and \( b \) limit to zero and the prolate model as \( c \) and \( d \) go to zero, respectively. It will be shown in the next section that the density of the model (2.3)–(2.4) at large distances decays radially like \( r^{-4} \), except on the two principal axes, and like \( r^{-3} \) on the two axes.
\section{Properties of Densities}

For the model (2.3)–(2.4), the density \( \rho(R^2, z) \) at large distances has a radial dependence \( \sim r^{-4} \), except on the two principle axes, and like \( r^{-3} \) on the two axes, i.e.,

\[ \rho(R^2, 0) \approx \frac{Md}{4\pi cR^3} + O\left(\frac{1}{R}\right), \tag{3.1} \]

\[ \rho(0, z) \approx \frac{Mb}{2\pi a z^3} + O\left(\frac{1}{z}\right). \tag{3.2} \]

The ratio, \( \alpha \), of \( R \)-axis to \( z \)-axis extent of the contours of finite \( \rho(R^2, z) \) near the origin is of the form

\[ \alpha^2 = \frac{a^2[3ad\tau_0^2(\tau_0 + r_J) + c(c + d)(3ad + b)bc + 5ac)(3\tau_0 + 2r_J)]}{c^2[4bc\tau_0^2(\tau_0 + r_J) + a(a + b)(ad + 4bc + 5ac)(3\tau_0 + 2r_J)]}, \tag{3.3} \]

where \( \tau_0 = \sqrt{(a + b)^2 + (c + d)^2} \). It is worth mentioning that the ratios in the oblate and prolate Jaffe models are degenerate forms of (3.3). In fact, by substituting \( b = 0 \) into (3.3) and letting \( a \) go to zero, (3.3) becomes the ratio in the oblate Jaffe models; similarly, (3.3) with \( d = 0 \) can be changed into the ratio in the prolate Jaffe models as \( c \) tends to 0. It is also below found by element integration that the total mass of the general flattened Jaffe models is \( M \).

In order to consider the total mass of the more general models (2.3)–(2.4) with positive constants \( a, b, c \) and \( d \) briefly, it is necessary to introduce the following notations:

\[ I_1 \equiv \frac{r_J^3}{\tau(\tau + r_J)^2} + \frac{r_Jc^2d}{\tau(\tau + r_J)} \frac{r_J(3\tau + 2r_J)}{\tau^3(\tau + r_J)^2}(a^2 + \frac{c^2(Y + d)^2}{Y^2}), \tag{3.4} \]

\[ \begin{split} I_2 & \equiv \frac{r_Jb}{(\tau + r_J)^2(X + b)} + \frac{r_Ja^2b}{(\tau + r_J)(X + b)} - \frac{r_Jbc^2d}{\tau(\tau + r_J)(X + b)} + \frac{2}{\tau^3} - \frac{1}{r_J\tau^2} + \frac{1}{r_J(\tau + r_J)^2} \left( \frac{a^2b}{X} - \frac{bc^2(Y + d)^2}{(X + b)Y^2} \right), \tag{3.5} \end{split} \]

where \( X, Y \) and \( \tau \) are the same as in (2.4), thus, by (2.4), giving an identity

\[ \frac{4\pi r_J \rho(R^2, z)X\tau}{M(X + b)} = I_1 + I_2. \tag{3.6} \]

Integrating (3.4) about \( \tau \) from \( \sqrt{(a + b)^2 + (Y + d)^2} \) to \( +\infty \), it follows that

\[ \int_{\sqrt{(a + b)^2 + (Y + d)^2}}^{+\infty} I_1 d\tau = \frac{Y^3 + c^2d}{Y^3} \ln \left( \frac{\sqrt{(a + b)^2 + (Y + d)^2 + r_J}}{\sqrt{(a + b)^2 + (Y + d)^2}} \right) \]

\[ - \frac{r_J[(Y + d)^2(Y^2 - c^2) + (b^2 + 2ab)Y^2]}{Y^2[(a + b)^2 + (Y + d)^2][\sqrt{(a + b)^2 + (Y + d)^2} + r_J]}. \tag{3.7} \]
Integrating (3.5) about \( \tau \) gives

\[
\int I_2 d\tau = \left[ 1 - \frac{c^2(Y + d)^2}{r_j^2 Y^2} \right] \frac{r_j b (X + b)}{(Y + d)^2 - r_j^2} \frac{r_j b (X + b)}{(Y + d)^2 - r_j^2} d\tau \]

\[
- \left\{ 1 - \frac{c^2(Y + d)^2}{r_j^2 Y^2} \right\} \frac{r_j^2 b}{(Y + d)^2 - r_j^2} - \frac{bc^2 d}{Y^3} \right\} g(\tau, Y + d) = \frac{bc^2}{Y^3} \arccos \left( \frac{Y + d}{\tau} \right) \]

\[
- \frac{r_j b a^2}{\tau^2(\tau + r_j)X} - \frac{bc^2(X + b)}{\tau^2 Y^2} + \text{constant} \quad (3.8)
\]

for \( Y + d \neq r_j \), where and everywhere below,

\[
g(\tau, Y + d) = \begin{cases} 
\frac{2}{\sqrt{(Y + d)^2 - r_j^2}} \arctan \left( \frac{\sqrt{Y + d - r_j}}{Y + d + r_j} \right) & \text{as } Y + d > r_j, \\
\frac{1}{\sqrt{r_j^2 - (Y + d)^2}} \ln \left( \frac{\sqrt{r_j^2 + Y + d}}{\sqrt{r_j^2 - Y + d}} \right) & \text{as } Y + d < r_j.
\end{cases} \quad (3.9)
\]

With the help of (3.10),

\[
\frac{4 \pi r_j}{M} \int_0^{+\infty} \rho(R^2, z) RdR = \int_0^{+\infty} \frac{4 \pi r_j \rho(R^2, z) X \tau}{\sqrt{(a + b)^2 + (Y + d)^2}} d\tau = Y^3 + \frac{c^2 d}{Y^3} \ln \left( \frac{\sqrt{(a + b)^2 + (Y + d)^2 + r_j}}{\sqrt{(a + b)^2 + (Y + d)^2}} \right) - \frac{r_j [(Y + d)^2(Y^2 - c^2) + (b^2 + ab)Y^2]}{Y^2[(a + b)^2 + (Y + d)^2][\sqrt{(a + b)^2 + (Y + d)^2} + r_j]} + \left[ 1 - \frac{c^2(Y + d)^2}{r_j^2 Y^2} \right] \frac{r_j b}{(Y + d)^2 - r_j^2}
\]

\[
- \left\{ 1 - \frac{c^2(Y + d)^2}{r_j^2 Y^2} \right\} \frac{r_j^2 b}{(Y + d)^2 - r_j^2} - \frac{bc^2 d}{Y^3} \right\} I_0(Y + d, r_j) + \frac{bc^2}{r_j Y^2}
\]

\[
- \left[ 1 - \frac{c^2(Y + d)^2}{r_j^2 Y^2} \right] \frac{r_j b (a + b)}{(Y + d)^2 - r_j^2} \]

\[
+ \left\{ 1 - \frac{c^2(Y + d)^2}{r_j^2 Y^2} \right\} \frac{r_j^2 b}{(Y + d)^2 - r_j^2} - \frac{bc^2 d}{Y^3} \right\} g(\sqrt{(a + b)^2 + (Y + d)^2}, Y + d)
\]

\[
- \frac{bc^2(a + b)}{r_j Y^2 \sqrt{(a + b)^2 + (Y + d)^2}} = \frac{bc^2}{Y^3} \arcsin \left( \frac{Y + d}{\sqrt{(a + b)^2 + (Y + d)^2}} \right) + \frac{bc^2(a + b)}{((a + b)^2 + (Y + d)^2) Y^2} \quad (3.10)
\]
implies that the total mass is finite for the model given by (2.1), as follows. It can be deduced from (3.10) that

\[
\frac{4\pi r_J}{M} \int \left\{ \int_0^{\infty} \rho(R^2, z) RdR \right\} dz = \frac{z(Y + d)}{Y} \ln \left( \frac{\sqrt{(a + b)^2 + (Y + d)^2} + r_J}{\sqrt{(a + b)^2 + (Y + d)^2}} \right) \\
+ \frac{b z(Y + d)}{Y} \left[ I_0(Y + d, r_J) - g((a + b)^2 + (Y + d)^2, Y + d) \right] \\
- \frac{b z}{Y} \arcsin \left( \frac{Y + d}{\sqrt{(a + b)^2 + (Y + d)^2}} \right) + \text{constant}
\]  

(3.11)

for any positive \(c\), since the following equalities hold:

\[
\frac{\partial}{\partial z} \left\{ z(1 + \frac{d}{Y}) \ln \left( \frac{\sqrt{(Y + d)^2 + (a + b)^2} + r_J}{\sqrt{(Y + d)^2} + (a + b)^2} \right) \right\} \\
= \frac{Y^3 + c^2 d}{Y^3} \ln \left( \frac{\sqrt{(a + b)^2 + (Y + d)^2} + r_J}{\sqrt{(a + b)^2} + (Y + d)^2} \right) \\
- \frac{r_J(Y + d)^2(Y^2 - c^2)}{Y^2[(a + b)^2 + (Y + d)^2][\sqrt{(a + b)^2} + (Y + d)^2] + r_J},
\]

(3.12)

\[
\frac{\partial}{\partial z} \left\{ b z(1 + \frac{d}{Y}) I_0(Y + d, r_J) \right\} \\
= \frac{b c^2(Y + d)^3 - r_J^2 b(Y^3 + c^2 d)}{Y^3[(Y + d)^2 - r_J^2]} I_0(Y + d, r_J) + \frac{r_J b(Y^2 - c^2)}{Y^2[(Y + d)^2 - r_J^2]},
\]

(3.13)

\[
\frac{\partial}{\partial z} \left\{ b z(1 + \frac{d}{Y}) g((a + b)^2 + (Y + d)^2, Y + d) \right\} \\
= \frac{b c^2(Y + d)^3 - r_J^2 b(Y^3 + c^2 d)}{Y^3[(Y + d)^2 - r_J^2]} g((a + b)^2 + (Y + d)^2, Y + d) \\
+ \frac{r_J b(a + b)(Y^2 - c^2)}{Y^2[(a + b)^2 + (Y + d)^2 + r_J][Y^2[(Y + d)^2 - r_J^2] - Y^2(\sqrt{(a + b)^2 + (Y + d)^2} + (Y + d)^2 + r_J]}
\]

(3.14)

and

\[
\frac{\partial}{\partial z} \left\{ \frac{b z}{Y} \arcsin \left( \frac{Y + d}{\sqrt{(a + b)^2} + (Y + d)^2} \right) \right\} \\
= \frac{b c^2}{Y^3} \arcsin \left( \frac{Y + d}{\sqrt{(a + b)^2}} + (Y + d)^2 \right) + \frac{b(a + b)(Y^2 - c^2)}{Y^2[(a + b)^2 + (Y + d)^2]},
\]

(3.15)
Furthermore, it can also be shown from (3.11) that
\[ 4\pi \int_0^\infty \int_0^\infty \rho(R^2, z) RdRdz = M, \]
i.e., the total mass is still \( M \) for the more general models (2.3)–(2.4).

**Remark:** The potential \( \Phi(R^2, z) \sim -GM/r + O(1/r^2) \) at large distances and the density is obtained from the potential by Poisson’s equation \( \Delta \Phi(R^2, z) = 4\pi G \rho(R^2, z) \).
The divergence theorem then guarantees that the volume integral of the analytic \( \rho(R^2, z) \) gives a total mass of \( M \), i.e., the Cauchy value of the volume integral can be easily obtained by using the divergence theorem and it limits to \( M \) as \( r \to +\infty \). This idea is very simple. In spite of this, it is necessary to do a set of integrations here since the method used to do these integrations is similar to that of deriving other formulae of the edge-on projected surface densities for the flattened Jaffe models and it is very useful to the readers for their understanding of how to obtain those results.

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