Queueing theory has been recently proposed as a framework to model the heavy tailed statistics of human activity patterns. The main predictions are the existence of a power-law distribution for the interevent time of human actions and two decay exponents $\alpha = 1$ and $\alpha = 3/2$. Current models lack, however, a key aspect of human dynamics, i.e. several tasks require, or are determined by, interactions between individuals. Here we introduce a minimal queueing model of human dynamics that already takes into account human-human interactions. To achieve large scale simulations we obtain a coarse-grained version of the model, allowing us to reach large interevent times and reliable scaling exponents estimations. Using this we show that the interevent distribution of interacting tasks exhibit the scaling exponents $\alpha = 2$, $3/2$ and a series of numerable values between $3/2$ and $1$. This work demonstrates that, within the context of queueing models of human dynamics, interactions change the exponent of the power-law distributed interevent times. Beyond the study of human dynamics, these results are relevant to systems where the event of interest consists of the simultaneous occurrence of two (or more) events.

Understanding the timing of human activities is extremely important to model human related activities such as communication systems \cite{1} and the spreading of computer viruses \cite{2}. In the recent years we have experienced an increased research activity in this area motivated by the increased availability of empirical data. We now count with measurements of human activities covering several individuals and several events per individual \cite{3,4,5,6,7}. Thanks to this data we are in a position to investigate the laws and patterns of human dynamics using a scientific approach.

Barabási has taken an important step in this direction reconsidering queueing theory \cite{8,9} as framework to model human dynamics \cite{5}. Within this framework, the to do list of an individual is modeled as a finite length queue with a task selection protocol, such as highest priority first. The main predictions are the existence of a power law distribution of interevent times $P_\tau \sim \tau^{-\alpha}$ and two universality classes characterized by exponents $\alpha = 1$ \cite{5,10,11} and $\alpha = 3/2$ \cite{6,11}. These universality classes have been corroborated by empirical data for email \cite{5,11} and regular mail communications \cite{6,11}, respectively, motivating further theoretical research \cite{12,13,14}.

The models proposed so far have been limited, however, to single individual dynamics. In practice people are connected in social networks and several of their activities are not performed independently. This reality forces us to model human dynamics in the presence of interactions between individuals. Our past experience with phase transitions has shown us that interactions and their nature are a key factor determining the universality classes and their corresponding scaling exponents \cite{15}. Furthermore, beyond the study of human dynamics, there are several systems where the event of interest consists of the simultaneous occurrence of two (or more) events. For example, collective phenomena in disordered media, such as the interaction of two (or more) particles in cluster formation.

To investigate the impact of human-human interactions on the timing of their activities we consider a minimal model consisting of two agents, A and B (Fig. 1). Each agent is modeled by a priority list containing two tasks, interacting task (I) and aggregate non-interacting task (O). The interacting task models a common activity such as meeting each other, requiring the simultaneous execution of that task by both agents. On the other hand, the non-interacting task represents an aggregate meta-activity accounting for all other tasks the agents execute, which do not require an interaction between them. To each task we assign random priorities $x_{ij}$ ($i = I, O; j = A, B$) extracted from a probability density function (pdf) $f_{ij}(x)$ (see Fig. 1).

The rules governing the dynamics are as follows. **Initial condition:** We start with a random initial condition,
assigning a priority to the I and O tasks from their corresponding pdf. Updating step: At each time step, both agents select the task with higher priority in their list. If (i) both agents select the interacting task then it is executed, (ii) otherwise each agent executes the O task, representing the execution of any of their non-interacting tasks.

Our aim is to determine the impact of the interaction between the agents and the shape of $f_{ij}(x)$ on the scaling exponent $\alpha$ of the interevent time distribution of the interacting task I. For simplicity, we focus on the following priority distribution. Consider the case where each agent has $L_j$ ($j = A, B$) tasks, one I task and $L_j - 1$ non-interacting tasks, their priorities following a uniform distribution in the interval $[0, 1]$. The pdf of the highest priority among $L_j - 1$ tasks is in this case given by $(L_j - 1)x^{L_j - 2}$, resulting in

$$f_{ij}(x) = \begin{cases} 1, & i = I, \\ (L_j - 1)x^{L_j - 2}, & i = O. \end{cases}$$  \hspace{1cm} (1)$$

This example shows that the priorities pdf of task I and O are in general different. All the results shown below were obtained using the pdf in Eq. (1).

To investigate the interevent time distribution we perform extensive numerical simulations. Figure 2 shows the interevent time distribution as obtained from direct simulations of the model introduced above. It becomes clear that for large $L_A$ and/or $L_B$ we do not obtain a good statistics, even after waiting for $10^{11}$ updating steps. This observation is a consequence of the behavior of $f_{O}(x)$ when $L_A$ and/or $L_B$ are large (Fig. 3). Focusing on agent A, as $L_A$ increases $f_{OA}(x)$ gets more concentrated around priority one, while the priority of the I task remains uniformly spread between zero and one. This fact results in increasingly large interevent times between the execution of the I task.

To speed-up the numerical simulations we derive a coarse-grained version of the model, allowing us to analyze the scaling behavior of the interevent time distribution over several orders of magnitude (inset of Fig. 2). We start by noticing that, given $(x_{IA}, x_{IB})$, the joint pdf of $(x_{OA}, x_{OB})$ factorizes and the probability $q(x_{IA}, x_{IB})$ that both agents execute I right after O is given by

$$q(x_{IA}, x_{IB}) = \int_{0}^{x_{IA}} dx_{OA} f_{OA}(x_{OB}) \int_{x_{OB}}^{x_{IB}} dx_{OB} f_{OB}(x). \hspace{1cm} (2)$$

This factorization is possible because the execution of the I task requires its priority to be the largest for both agents. In turn, with probability $1 - q(x_{IA}, x_{IB})$ both agents continue to execute O. Thus, the probability distribution $Q_\tau(x_{IA}, x_{IB})$ that I waits $\tau > 1$ steps before being executed follows the geometric distribution

$$Q_\tau(x_{IA}, x_{IB}) = q(x_{IA}, x_{IB})[1 - q(x_{IA}, x_{IB})]^\tau - 2. \hspace{1cm} (3)$$

Once the I task is executed it can be executed again resulting in interevent times of one step ($\tau = 1$). The overall interevent time distribution of the I task is given by

$$P_\tau = \begin{cases} P_1, & \tau = 1, \\ (1 - P_1)Q_\tau(x_{IA}, x_{IB}), & \tau > 1. \end{cases} \hspace{1cm} (4)$$

where
\[ P_1 = \frac{S_1}{S_1 + 1}, \]  

(5)

\( S_1 \) is the expected number of consecutive executions of the I task and \( \langle \cdots \rangle \) denotes the expectation over different realizations of \( (x_{IA}, x_{IB}) \), just at the step of switching from task I to O. Finally, at the step of switching from O to I, the O task priority of both agents must fall below that of the I task. Therefore, the pdf of \( x_{Oj} \) (\( j = A, B \)) just after the switch from O to I is given by

\[ f^*_O(x|x_{IJ}) = \frac{f_{Oj}(x)}{\int_0^{x_{IJ}} f_{Oj}(x')dx'}, \]  

(6)

where \( 0 \leq x < x_{IJ} \). This later result together with Eq. (3) allow us to condense all steps with consecutive executions of the O task into a single coarse-grained step. More important, this mapping is exact.

Putting all together the coarse-grained model runs as follows. Initial condition: We start with random initial priorities extracted from the pdfs \( f_{ij}(x) \). Updating step: At each step, (i) if for both agents the I task priority is larger than that for the O task we run the model as defined above, both agents executing the I task and updating their I task priorities using the pdfs \( f_{ij} \) (\( j = A, B \)). (ii) Otherwise, we generate a random interevent time \( \tau \) from the probability distribution \( f_{ij} \) and a new O task priority for each agent using the pdf \( f^*_O(x|x_{IJ}) \). This second step avoids going over successive executions of the O task which, for a large number of non-interacting tasks, significantly slow down the simulations.

The second step of the coarse-grained model requires us to extract a random number from a geometric distribution. This can be achieved very efficiently exploiting the fact that the integer part of a real random variable with an exponential distribution follows a geometric distribution. Using this fact, when \( \tau > 1 \), we extract \( \tau \) exactly from the distribution in Eq. (3), which differs from the corresponding branch of Eq. (4). Normalization by the total number of task I executions, including those with \( \tau = 1 \), provides \( \tau > 1 \) distributed according to Eq. (4).

The I task interevent time distribution obtained from simulations of the coarse-grained model is plotted in Fig. 4a. When \( L_A = L_B = L = 2 \) it follows a power-law tail with exponent \( \alpha = 2 \). As \( L \) increases \( \alpha \) approaches one. A guess to this dependence, in good agreement with the measured values, is given by \( \alpha = 1 + 1/\max(L_j - 1) \) (inset of Fig. 4a). The numerical results indicate that there are several numerable universality classes parameterized by \( L_A \) and \( L_B \). Notice that the second largest value of \( \alpha \) (obtained when \( L_A = 2 \) and \( L_B = 3 \), or vice-versa) is close to 3/2 and, therefore, our results do not show universality classes with exponent \( \alpha \) between 3/2 and 2 (unless we assume real valued queue lengths).

The power laws in Fig. 4a exhibit a cutoff at a certain value of \( \tau \). To investigate if this is a natural cutoff or just a finite size effect, we investigate the shape of the interevent time distribution as a function of the observation time window \( T \). The later is defined as the total number of steps considering both the I and O task and satisfy

\[ T = \sum_{i=1}^{N} \tau_i, \]  

(7)

where \( N \) is the number of executions of the I task within the time window \( T \) and \( \tau_i (i = 1, \ldots, N) \) is the sequence.
of interevent times between executions of the I task. We assume that the cutoff is determined by the finite time window and that the interevent time distribution follows the scaling form

\[ P(\tau) = A\tau^{-\alpha}g\left(\frac{\tau}{T}\right) \]  

(8)

where \( A \) is a constant, \( z > 0 \) is a scaling exponent and \( g(x) \) is a scaling function with the asymptotic behaviors \( g(x) \approx 1 \) when \( x \ll 1 \) and \( g(x) \ll 1 \) when \( x \gg 1 \). Under this assumption \( P(\tau) \sim \tau^{-\alpha} \) when \( T \to \infty \), with \( 1 < \alpha < 2 \). Given this power law tail and exponent, the number of interevent times \( N \) necessary to cover the window \( T \) is of the order of magnitude of \( T^{\alpha-1} \) \[10\]. In turn, the mean intervent time is of the order of

\[ \langle \tau \rangle = \frac{1}{N} \sum_{i=1}^{N} \tau_i \sim T^{2-\alpha}. \]  

(9)

From Eqs. (8) and (9) it follows that \( z = 1 \).

To check our scaling assumption we plot \( P_\tau T^\alpha \) as a function of \( \tau/T \) (Fig. 4). The symbols corresponding to different time windows \( T \) clearly overlap into a single curve, demonstrating that the scaling assumption in Eq. (8) is correct with \( z = 1 \). Thus, in the \( T \to \infty \) limit the I task intervent time distribution exhibits a true power law tail \( P_\tau \sim \tau^{-\alpha} \).

Within the context of queuing models of human dynamics, only two universality classes were previously identified, corresponding to the single queue models of Cobham \[6, 17\] (\( \alpha = 3/2 \)) and Barabási \[5\] (\( \alpha = 1 \)). The analysis of the two interacting agents model reveals that the interaction between agents results in a richer set of exponents. Our numerical results provide evidence of a new universality class with exponent \( \alpha = 2 \) and exponents between \( 3/2 \) and \( 1 \). It is worth noticing that the exponents \( 2 \) and \( 1 \) may also result from a Poisson processes with a time dependent rate \[18, 19\].

Because the exponent \( \alpha \) depends on the systems details, here represented by the agent’s queue lengths \( L_A \) and \( L_B \), we conclude that the model with two interacting agents exhibits non-universal behavior. Interestingly, the exponent \( \alpha = 1 \) is asymptotically reached when the number of tasks of one or both agents becomes large. As humans get engaged in several tasks this later asymptotic behavior may explain the ubiquitous observation of the exponent \( \alpha = 1 \) \[11\].

We use the number of non-interacting tasks as a mean to modulate the distribution of the non-interacting aggregate task priority. Yet, it is the distribution shape the primary factor determining the scaling exponent \( \alpha \). The effect of increasing \( L_A \) and/or \( L_B \) is a concentration of the non-interacting aggregate task priority around priority one, resulting in values of \( \alpha \) that approaches one.

This means that the limit \( \alpha = 1 \) is achieved for low priority interacting tasks that remain most of the time in the queue without being executed, at expenses of the execution of tasks which in general have a higher priorities.

Considering the interaction between agents we also solve one of the longstanding problems of the original single queue Barabási model, related to the stationarity of the interevent time distribution \[10, 13\]. In the Barabási single queue model the task with highest priority is executed with a probability \( p \), otherwise a task is selected at random for execution. When \( p \) is close to one the interevent time distribution exhibits a peak at one step and \( P_1 \to 1 \) when \( p \to 1 \). When \( p = 1 \) the distribution is non-stationary and \( P_1 \to 1 \) when time \( t \to \infty \). In contrast, in the model considered here there is no need to introduce the random selection rule and the corresponding model parameter \( p \). The interacting task interevent time distribution is stationary even when the - highest priority first - selection rule is applied. In turn, the exponent \( \alpha \) is not exactly one, but reaches one asymptotically with increasing the number of tasks. Finally, the interevent time distribution of the Barabási model exhibits a natural cutoff determined by the parameter \( p \), while for the model introduced here it is a true power law up to finite size effects. However, it is worth noticing that in a recent work \[20\] it has been found that the original Barabási’s model with variable task execution rate can generate interevent time distributions with exponent \( \alpha = 1.25 \). In principle, in our model also different choice of model parameters could result in other exponents in the range between 1 and 3/2.

This work represents the first step in understanding how interactions among agents affect their activity pattern. Based on recent works using queueing theory we describe the model in the context of human dynamics. It can be generalized to consider a larger number of agents connected by a specific social network. Also, the model can potentially be used more generally to study the time statistics of events requiring synchronization between two physical systems.

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