BLOW-UP FOR TWO-COMPONENT CAMASSA-HOLM EQUATION WITH GENERALIZED WEAK DISSIPATION

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Abstract. This paper is concerned with blow-up solution for the Cauchy problem of two-component Camassa-Holm equation with generalized weak dissipation. By Kato’s theorem and monotonicity, we investigate the local well-posedness of Cauchy problem and establish the blow-up criteria and the blow-up rate. Moreover, the property of blow-up points set is characterized.

1. Introduction. It is an interesting phenomenon that the length of wave is much greater than the depth of wave in the water. The various equations have been proposed to investigate the shallow water wave problems. The well-known model of shallow water wave problem is the Camassa-Holm (CH) equation. In 1993, Camassa and Holm proposed the following equation (CH):

\[ u_t - u_{txx} + 3uu_x = 2u_xu_{xx} + uu_{xxx}, \quad t > 0, \; x \in \mathbb{R}. \]  

(1.1)

to model the propagation of shallow water waves. In (1.1), \( u(t, x) \) represents the free surface of fluid above a flat bottom. After this equation was put forward, it has attracted the attention of a large number of researchers. They found two remarkable properties of this equation. The first one is the existence of solutions in the form of peaked solitary waves or peakons, i.e., \( u(t, x) = ce^{-|x-ct|}, c \neq 0 \), which is smooth everywhere except at its crest and the tallest among all waves of fixed energy [2]. It is a feature observed for the traveling waves of largest amplitude which solve the governing equations for water waves [4, 20, 6, 24]. The other remarkable feature is the existence of breaking waves of CH equation, which means that the solution is bounded while the slope is unbounded in finite time [10]. After blow up, the solutions can be continued uniquely as either global conservative or global dissipative solutions [1]. The Cauchy problem of Camassa-Holm equation has global strong solutions modelling permanent waves [5, 22, 23], also has blow-up solutions modelling wave breaking [5, 6, 7, 8]. Whitham said that it is intriguing to know
which mathematical models for shallow water waves exhibit both phenomenon of solution interaction and wave breaking [20]. It is worth to notice that the CH equation was the first one having above characteristics, modelling the soliton interaction of peaked travelling waves and wave breaking [2, 6].

The Camassa-Holm equation possesses many integrable multi-component generalizations. The most popular system is the following one:

\[
\begin{aligned}
  u_t - u_{xxt} + 3uu_x - Au_x - uu_{xxx} - 2u_xu_{xx} + \rho \rho_x = 0, \\
  \rho_t + (\rho u)_x = 0.
\end{aligned}
\] (1.2)

One can get the Camassa-Holm equation by choosing \( \rho = 0 \) and \( A = 0 \) in (1.2), where \( \rho(t, x) \) is related to the free surface elevation from equilibrium (or scalar density), i.e. amplitude, and \( A \geq 0 \) characterizes a linear underlying shearflow.

The Camassa-Holm equation was first obtained by Fokas and Fuchssteiner as a bi-Hamiltonian generalization of KdV [2], then Camassa and Holm derived it by physical methods [2]. Ivanov [15] and Constantin [9] established a rigorous justification of the derivation of system (1.2). Some mathematical properties of the system have been also studied further in many works, refer to [21, 24, 18, 11, 3, 13]. The local well-posedness is investigated by Gui and Liu [14] on the Besov spaces (especially on the Sobolev spaces \( H^s(R) \times H^{s-1}(R), s \geq \frac{5}{2} \)). All of the works showed that the blow-up is determined by either the slope of the first component \( u \) or the slope of the second component \( \rho \) in finite time.

Dissipation is an inevitable phenomenon, so it is necessary to study two-component Camassa-Holm equation with generalized weak dissipation. In 2006, Wu and Yin have investigated the blow-up of the strong solutions of the weakly dissipative Camassa-Holm equation [21]. Constantin and Escher have confirmed for a large class of initial data the exact blow-up set for the corresponding blow-up solutions to the initial value problem [8]. Based on these backgrounds, we studied the following two-component Camassa-Holm equation with generalized weak dissipation:

\[
\begin{aligned}
  u_t - u_{xxt} + ku_x + 3uu_x - 2u_xu_{xx} - uu_{xxx} + \lambda (u - u_{xx}) + \sigma \rho \rho_x = 0, \\
  \rho_t + (\rho u)_x = 0,
\end{aligned}
\] (1.3)

where \( k, \lambda \) are positive constants, \( \sigma \) is a parameter.

The rest of this paper is organized as follows. In Section 2, we give the local well-posedness of the interval value problem associated with equation (1.3) by applying Kato’s theorem. Section 3 presents a blow-up criterion for strong solutions. Section 4 is devoted to the blow-up rate of strong solutions. The blow-up points set is discussed in Section 5.

2. Local well-posedness. In this section we will apply Kato’s theorem to investigate the local well-posedness of Cauchy problem of (1.3).

Consider the following two-component Camassa-Holm equation with generalized weak dissipation:

\[
\begin{aligned}
  u_t - u_{xxt} + ku_x + 3uu_x - 2u_xu_{xx} - uu_{xxx} + \lambda (u - u_{xx}) + \sigma \rho \rho_x = 0, \\
  \rho_t + (\rho u)_x = 0,
\end{aligned}
\] (2.1)

where \( t > 0, x \in R \). Rewrite Eq (2.1) as a quasi-linear nonlocal evolution system of hyperbolic type:
\[
\begin{aligned}
&u_t + uu_x = -\partial_x g \ast (u^2 + \frac{1}{2}u_x^2 + ku + \frac{\sigma}{2} \rho^2) - \lambda u, \\
&\rho_t + (\rho u)_x = 0,
\end{aligned}
\] (2.2)

where \( t > 0, x \in R, \ g(x) := \frac{1}{2}e^{-|x|}, x \in \mathbb{R} \) and \( \ast \) indicates the convolution operator. 

Then \( (1 - \partial_x^2)^{-1}f = g \ast f \) for all \( f \in L^2(R) \) and \( g \ast (u - u_x) = u \). Now we give the theorem of local well-posedness of Cauchy problem of (2.1).

**Theorem 2.1.** Let \( z_0 = (u_0, \rho_0) \in H^s(R) \times H^{s-1}(R) \) with \( s \geq 2 \), there exist a maximal time \( T = T(\|u_0, \rho_0 - 1\|_{H^s \times H^{s-1}}) > 0 \) and a unique solution \( u \) of (2.1) in the interval \([0, T)\) with initial data \((u_0, \rho_0)\), such that the solution depends continuously on the initial data.

The remainder of this section is devoted to the proof of Theorem 2.1.

Define \( Y = H^s(R) \times H^{s-1}(R) \), \( X = H^{s-1}(R) \times H^{s-2}(R) \), \( \land = (1 - \partial_x^2) \frac{1}{2} \),

\[
Q = \begin{pmatrix} \land & 0 \\ 0 & \land \end{pmatrix}, \quad z = (u\rho),
\]

\[
A(z) = \begin{pmatrix} u\partial_x & 0 \\ 0 & u\partial_x \end{pmatrix},
\]

and

\[
f(z) = -\left( \partial_x (1 - \partial_x^2)^{-1}(u^2 + \frac{1}{2}u_x^2 + ku + \frac{\sigma}{2} \rho^2) + \lambda u \right),
\]

for any given \( z \in H^s(R) \times H^{s-1}(R) \). Define,

\[
B(z) = QA(z)Q^{-1} - A(z).
\]

From [12] we know \( \land \) is a topological isomorphism map from \( Y \) to \( X \). \( Q \) is an isomorphism from \( H^s(R) \times H^{s-1}(R) \) to \( H^{s-1}(R) \times H^{s-2}(R) \), we rewrite the Cauchy problem corresponding to (2.1) to the abstract quasi-linear evolution equation with initial data:

\[
\begin{aligned}
&\frac{dz}{dt} + A(z)z = f(z), \quad t \geq 0, \\
&z(0) = z_0.
\end{aligned}
\] (2.6)

Let \( X \) and \( Y \) are Hilbert spaces, \( Y \) is continuously and densely embedded in \( X \), and let \( Q : Y \to X \) is a topological isomorphism, \( L(Y, X) \) denotes the space of all bounded linear operator from \( Y \) to \( X \), hypothesis that:

\((H_1)\) \( A(y) \in L(X) \), for \( \forall y \in X \) with

\[
\| (A(y) - A(z))w \|_X \leq \mu_1 \| y - z \|_X \| w \|_Y,
\]

where \( z, y, w \in Y, A(y) \in G(X, 1, \beta) \), i.e. \( A(y) \) is quasi-m-accretive, uniformly on bounded sets in \( Y \) (we write \( G(X, 1, \beta) \) for the set of all linear operators \( A \) in \( X \) such that \( -A \) generates a \( C_0 \)-semigroup \( T(t) \) on \( X \) and that \( \| T(t) \|_{L(X)} \leq e^{t\beta} \) for all \( t \geq 0 \) [19].

\((H_2)\) \( QA(y)Q^{-1} = A(y) + B(y) \), where \( B(y) \in L(X) \) is uniformly bounded on a bounded set in \( Y \)

\[
\| (B(y) - B(z))w \|_X \leq \mu_2 \| y - z \|_Y \| w \|_X,
\]

where \( z, y \in Y, w \in X \).

\((H_3)\) \( f : Y \to Y \), it can be a mapping from \( X \) to \( Y \), \( f \) is bounded on a bounded set in \( Y \).
where $z, y \in Y, \mu_1, \mu_2, \mu_3, \mu_4$ are constants which only depending $\{\|y\|_Y, \|z\|_Y\}$.

**Lemma 2.2** ([16]). Assume that $H_1, H_2, H_3$ hold, and give $v_0 \in Y$, there exist a $T > 0$ depending only on $\|v_0\|_Y$ and a unique solution $v$ of (2.6) such that

$$v = v(\cdot, v_0) \in C([0, T); Y) \cap C^1([0, T); X).$$

Furthermore, the mapping $v_0 \mapsto v(\cdot, v_0)$ is continuous from $Y$ to $C([0, T); Y) \cap C^1([0, T); X)$.

It is sufficient to verify $A(z), B(z), f(z)$ satisfying $H_1, H_2, H_3$. For this aim, the following lemmas are needed.

**Lemma 2.3** ([16]). Let $r, l$ be real numbers such that $-r < l \leq r$, then

$$\|fg\|_{H^l} \leq c \|f\|_{H^l} \|g\|_{H^l}, \quad r > \frac{1}{2},$$

$$\|fg\|_{H^{l-r}} \leq c \|f\|_{H^r} \|g\|_{H^r}, \quad r < \frac{1}{2},$$

where $c$ is a positive constant which only depending on $r, l$.

**Lemma 2.4** ([16]). Let $f \in H^s$, with $s > \frac{3}{2}$, then

$$\|\Lambda^{-r}[\Lambda^{s+l+1}, M_f] \Lambda^{-1} \|_{L^2} \leq c \|f\|_{H^s}, \quad \|r\| \leq s - 1,$$

where $[\Lambda^{s-1}, u] \partial_z y_1 = \Lambda^{s-1}(u \partial_z y_1) - u \Lambda^{s-1}(\partial_z y_1), M_f$ is the operator of multiplication by $f$ and $c$ is a constant depending only on $r, l$.

**Lemma 2.5** ([16]). Operator $A(z)$ is given in (2.3), where $z \in H^s(R) \times H^{s-1}(R), s \geq 2$, then $A(z) \in G(L^2 \times L^2, 1, \beta)$.

**Lemma 2.6** ([16]). Operator $A(z)$ is given in (2.3), where $z \in H^s(R) \times H^{s-1}(R), s \geq 2$, then $A(z) \in G(H^{s-1}(R) \times H^{s-2}(R), 1, \beta)$.

**Lemma 2.7.** Operator $A(z)$ is given in (2.3), where $z \in H^s(R) \times H^{s-1}(R), s \geq 2$, then $A(z) \in L(H^s(R) \times H^{s-1}(R), H^{s-1}(R) \times H^{s-2}(R))$, and for all $z, y, w \in H^s(R) \times H^{s-1}(R)$, we have

$$\|(A(z) - A(y))w\|_{H^{s-1} \times H^{s-2}} \leq \mu_1 \|z - y\|_{H^s \times H^{s-1}} \|w\|_{H^s \times H^{s-1}}. \quad (2.7)$$

**Proof.** Let $z, y, w \in H^s(R) \times H^{s-1}(R), s \geq 2$, note that $H^{s-1}$ is a Banach algebra [12], we have

$$(A(z) - A(y))w = \begin{pmatrix} (u - y_1) \partial_z & 0 \\ (u - y_1) \partial_z & (u - y_1) \partial_z \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} (u - y_1) \partial_z w_1 \\ (u - y_1) \partial_z w_2 \end{pmatrix}.$$

Then it follows that

$$\|(A(z) - A(y))w\|_{H^{s-1} \times H^{s-2}} \leq \|u - y_1\|_{H^s} \|\partial_z w_1\|_{H^{s-1}} + \|u - y_1\|_{H^s} \|\partial_z w_2\|_{H^{s-1}}$$

$$\leq c \|u - y_1\|_{H^s} \|\partial_z w_1\|_{H^{s-1}} + \|u - y_1\|_{H^s} \|\partial_z w_2\|_{H^{s-1}}$$

$$\leq c \|w_1\|_{H^{s-1}} + \|w_2\|_{H^{s-1}}$$

$$\leq c \|z - y\|_{H^s \times H^{s-1}} \|w\|_{H^s \times H^{s-1}}.$$
Define \( \| z - y \|_{H^s \times H^{s-1}} = \| u - y_1 \|_{H^s} + \| \rho - y_2 \|_{H^{s-1}} \), let \( \mu_1 = c \), then
\[
\| (A(z) - A(y))w \|_{H^s \times H^{s-2}} \leq \mu_1 \| z - y \|_{H^s \times H^{s-1}} \| w \|_{H^s \times H^{s-1}},
\]
where we apply the Lemma 2.3 with \( r = s - 1, l = s - 2 \) and \( y = 0 \) in the above inequality. Then we can obtain \( A(z) \in L(H^s(R) \times H^{s-1}(R), H^{s-1}(R) \times H^{s-2}(R)) \).

The proof of Lemma 2.7 is completed.

**Lemma 2.8.** \( B(z) \) is given in (2.5), where \( z \in H^s(R) \times H^{s-1}(R), s \geq 2 \), then \( B(z) \in L(H^{s-1}(R) \times H^{s-2}(R)) \), and for all \( z, y \in H^s(R) \times H^{s-1}(R) \), \( w \in H^{s-1}(R) \times H^{s-2}(R) \), we have
\[
\| (B(z) - B(y))w \|_{H^{s-1} \times H^{s-2}} \leq \mu_2 \| z - y \|_{H^s \times H^{s-1}} \| w \|_{H^s \times H^{s-1}}. \tag{2.8}
\]

**Proof.** Let \( z, y \in H^s \times H^{s-1}, w \in H^{s-1} \times H^{s-2}, s \geq 2 \), we have
\[
(B(z) - B(y))w = (Q(A(z) - A(y))Q^{-1}) - (A(z) - A(y))w
\]
\[
= \left[ \left( \Lambda(u - y_1)\partial_x \Lambda^{-1} \ 0 \\ 0 \ \Lambda(u - y_1)\partial_x \Lambda^{-1} \right) - \left( \left( (u - y_1)\partial_x \\ 0 \right) \Lambda(u - y_1)\partial_x \Lambda^{-1} \right) \right] (w_1, w_2)
\]
\[
= \Lambda(u - y_1)\partial_x \Lambda^{-1} w_1 - (u - y_1)\partial_x (u - y_1)\partial_x w_1 + \Lambda(u - y_1)\partial_x \Lambda^{-1} w_2 - (u - y_1)\partial_x w_2.
\]

One can abbreviate that
\[
(B(z) - B(y))w = [\Lambda^1, (u - y_1)\partial_x]\Lambda^{-1} w_1 + [\Lambda^1, (u - y_1)\partial_x]\Lambda^{-1} w_2.
\]

Then
\[
\| (B(z) - B(y))w \|_{H^{s-1} \times H^{s-2}} \leq \| \Lambda^1[u - y_1] \partial_x \Lambda^{-1} w_1 \|_{L^2} + \| \Lambda^2[u - y_1] \partial_x \Lambda^{-1} w_2 \|_{L^2}
\]
\[
\leq \| \Lambda^1[u - y_1] \partial_x \Lambda^{-1} w_1 \|_{L^2} \leq \| \Lambda^1[u - y_1] \Lambda^{-1} w_1 \|_{L^2} \leq \mu_2 \| z - y \|_{H^s \times H^{s-1}} \| w \|_{H^s \times H^{s-1}}.
\]

Let \( \mu_2 = c \), then
\[
\| (B(z) - B(y))w \|_{H^{s-1} \times H^{s-2}} \leq \mu_2 \| z - y \|_{H^s \times H^{s-1}} \| w \|_{H^s \times H^{s-1}},
\]
where we apply the Lemma 2.3 with \( r = 1 - s, l = s - 1 \) and \( r = 2 - s, l = s - 2 \) and \( y = 0 \) in the above inequality. One can obtain \( B(z) \in L(H^{s-1}(R) \times H^{s-2}(R)) \). The proof of the lemma 2.8 is finished.

**Lemma 2.9.** \( f(z) \) is defined as (2.4), for any given \( z \in H^s \times H^{s-1}, s \geq 2 \), then \( f \) is uniformly bounded on a bounded set in \( H^s \times H^{s-1} \) and satisfies the following two conditions:
\begin{align*}
(a) \quad & \| f(y) - f(z) \|_{H^s \times H^{s-1}} \leq \mu_3 \| y - z \|_{H^s \times H^{s-1}}, \ z, y \in H^s \times H^{s-1} \tag{2.9} \\
(b) \quad & \| f(y) - f(z) \|_{H^{s-1} \times H^{s-2}} \leq \mu_4 \| y - z \|_{H^{s-1} \times H^{s-2}}, \ z, y \in H^s \times H^{s-1} \tag{2.10}
\end{align*}
Proof. Let \( z, y \in H^s \times H^{s-1} \), then
\[
\| f(y) - f(z) \|_{H^s \times H^{s-1}} \\
\leq \| -\partial_x (1 - \partial_x^2)^{-1}[(y_1^2 - u^2) + \frac{1}{2}(y_1^2 - u_2^2)] + k(y_1 - u) + \frac{\sigma}{2}(y_2^2 - \rho^2) \|_{H^s} \\
+ \| \lambda(y_1 - u) \|_{H^s} + \| u_x \rho - y_1 x y_2 \|_{H^{s-1}} \\
\leq \| (y_1^2 - u^2) + \frac{1}{2}(y_1^2 - u_2^2) + k(y_1 - u) \|_{H^{s-1}} + \frac{\sigma}{2} \| (y_2^2 - \rho^2) \|_{H^{s-1}} \\
+ \| \lambda \| y_1 - u \|_{H^s} + \| (u_x - y_1 x) \rho \|_{H^{s-1}} + \| y_1 x (\rho - y_2) \|_{H^{s-1}} \\
\leq \| (y_1 - u)(y_1 + u) \|_{H^{s-1}} + \frac{1}{2} \| (y_1 - u_x)(y_1 + u_x) \|_{H^{s-1}} \\
+ \| k \| y_1 - u \|_{H^{s-1}} + \| \lambda \| y_1 - u \|_{H^s} + \frac{\sigma}{2} \| y_2 - \rho \|_{H^{s-1}} \| y_2 + \rho \|_{H^{s-1}} \\
+ \| y_1 - u \|_{H^s} \| \rho \|_{H^{s-1}} + \| y_1 \|_{H^s} \| \rho - y_2 \|_{H^{s-1}} \\
\leq \| y_1 - u \|_{H^s} \| y_1 - u \|_{H^s} + \| u \|_{H^{s-1}} + \frac{1}{2} \| y_1 - u \|_{H^s} \| y_1 - u \|_{H^s} + \| u \|_{H^{s-1}} \\
+ \| k \| y_1 - u \|_{H^s} + \| \lambda \| y_1 - u \|_{H^s} + \| y_1 - u \|_{H^s} \| \rho \|_{H^{s-1}} \\
+ \frac{\sigma}{2} \| y_2 - \rho \|_{H^{s-1}} \| y_2 \|_{H^{s-1}} + \| \rho \|_{H^{s-1}} + \| y_1 \|_{H^s} \| \rho - y_2 \|_{H^{s-1}} \\
\leq \| y_1 - u \|_{H^s} \| y_1 - u \|_{H^s} + \frac{3}{2} \| u \|_{H^s} + \| \lambda \| + \| k \| + \| \rho \|_{H^{s-1}} \\
+ \| y_2 - \rho \|_{H^{s-1}} \left( \frac{\sigma}{2} \| y_2 \|_{H^{s-1}} + \frac{\sigma}{2} \| \rho \|_{H^{s-1}} + \| y_1 \|_{H^s} \right).
\]

Let \( \mu_3 = \| \lambda \| + \| k \| + \frac{\lambda + |\sigma|}{2} \| y \|_{H^s \times H^{s-1}} + \frac{3 + |\sigma|}{2} \| z \|_{H^s \times H^{s-1}}, \) then
\[
\| f(y) - f(z) \|_{H^s \times H^{s-1}} \leq \mu_3 \| y - z \|_{H^s \times H^{s-1}}, y, z \in H^s \times H^{s-1}.
\]

Taking \( y = 0 \), we can obtain \( f \) is bounded on bounded set in \( H^s \times H^{s-1} \).

Next we turn to prove (b). Making \( z, y \in H^s \times H^{s-1} \), we derive that
\[
\| f(y) - f(z) \|_{H^s \times H^{s-1}} \\
\leq \| -\partial_x (1 - \partial_x^2)^{-1}[(y_1^2 - u^2) + \frac{1}{2}(y_1^2 - u_2^2)] + k(y_1 - u) + \frac{\sigma}{2}(y_2^2 - \rho^2) \|_{H^{s-2}} \\
+ \| \lambda(y_1 - u) \|_{H^{s-1}} + \| u_x \rho - y_1 x y_2 \|_{H^{s-2}} \\
\leq \| (y_1^2 - u^2) + \frac{1}{2}(y_1^2 - u_2^2) + k(y_1 - u) \|_{H^{s-2}} + \frac{\sigma}{2} \| (y_2^2 - \rho^2) \|_{H^{s-2}} \\
+ \| \lambda \| y_1 - u \|_{H^{s-1}} + \| y_1 - u \|_{H^{s-1}} \| \rho \|_{H^{s-2}} + \| y_1 \|_{H^{s-1}} \| \rho - y_2 \|_{H^{s-2}} \\
\leq \| y_1 - u \|_{H^{s-1}} \| y_1 \|_{H^{s-1}} + \| u \|_{H^{s-1}} + \| k \| y_1 - u \|_{H^{s-1}} \\
+ \frac{1}{2} \| y_1 - u \|_{H^{s-1}} + \| \lambda \| y_1 - u \|_{H^{s-1}} + \| y_1 \|_{H^{s-1}} \| \rho \|_{H^{s-2}} - \| y_2 \|_{H^{s-2}} \\
+ \frac{\sigma}{2} \| y_2 - \rho \|_{H^{s-2}} \| y_2 \|_{H^{s-2}} + \| \rho \|_{H^{s-2}} + \| y_1 \|_{H^{s-1}} \| \rho \|_{H^{s-2}} \\
= \| y_1 - u \|_{H^{s-1}} \left( \frac{3}{2} \| y_1 \|_{H^{s-1}} + \frac{3}{2} \| u \|_{H^{s-1}} + \| \lambda \| + \| k \| + \| \rho \|_{H^{s-2}} \right) \\
+ \| y_2 - \rho \|_{H^{s-2}} \left( \frac{\sigma}{2} \| y_2 \|_{H^{s-2}} + \frac{\sigma}{2} \| \rho \|_{H^{s-2}} + \| y_1 \|_{H^{s-1}} \right).
Let $\mu_4 = |\lambda| + |k| + \frac{5+|\sigma|}{2} \| y \|_{H^{-1} \times H^{s-2}} + \frac{3+|\sigma|}{2} \| z \|_{H^{-1} \times H^{s-2}}$, then
\[
\| f(y) - f(z) \|_{H^{-1} \times H^{s-2}} \leq \mu_4 \| y - z \|_{H^{-1} \times H^{s-2}}, \quad y, z \in H^s \times H^{s-1}.
\]
\[\square\]

**Proof.** From the inequalities (2.7)-(2.10), it is obvious that $A(z), B(z), f(z)$ satisfy $H_1, H_2, H_3$. Then the proof of Theorem 2.1 is completed. \[\square\]

3. **Blow-up.** In this section we will establish the blow-up criterion for solution of (2.1). According to Theorem 2.1, there exists a maximal time $T > 0$ and the unique solution $z = (u, \rho)$, such that
\[
z_0 = z(\cdot, z_0) \in C([0, T]; H^s \times H^{s-1}) \cap C^1([0, T]; H^{s-1} \times H^{s-2}).
\]
Consider the following equation of trajectory:
\[
\begin{align*}
\frac{d}{dt} q(t,x) &= u(t, q(t,x)), \quad t \in [0, T), \\
q(0,x) &= x, \quad x \in R,
\end{align*}
\]
where $u$ is the first component of the solution $z$. By Sobolev embedding theorem we know that $u(t, \cdot) \in H^2 \subset C^m, 0 \leq m \leq \frac{3}{2}, u \in C^1([0, T] \times R, R)$. By the existence theorem of solutions of ODE, we obtain that the (3.1) has a unique solution $q$. The following two critical lemmas are needed.

**Lemma 3.1** ([5, 22]). Let $z_0 = (u_0, \rho_0) \in H^s(R) \times H^{s-1}(R)$ with $s \geq 2$, then (3.1) admits a unique solution $q \in C^1([0, T] \times R, R)$. The mapping $q(t, \cdot)$ is monotonically increasing with respect to $x$, and it is the derivative homeomorphism on $R$, i.e.,
\[
q_x(t,x) = e^{\int_0^t u_x(\tau, q(\tau, x)) d\tau} > 0, \quad \forall (t,x) \in [0, T] \times R.
\]

**Lemma 3.2** ([22]). Let $z_0 = (u_0, \rho_0) \in H^s(R) \times H^{s-1}(R)$ with $s \geq 2$, there exist a maximal time $T > 0$ and a unique solution $u$ of Eq (2.1) with initial data $(u_0, \rho_0)$, then we have
\[
\rho(t, q(t,x))q_x(t,x) = \rho_0(x), \quad \forall (t,x) \in [0, T] \times R.
\]
Furthermore, if there exists $M_1 > 0$ such that $u_x(t,x) \geq -M_1$ for any given $(t,x) \in [0, T] \times R$, then
\[
\parallel \rho(t, \cdot) \parallel_{L^\infty} = \parallel \rho(t, q(t, \cdot)) \parallel_{L^\infty} \leq e^{M_1 T} \parallel \rho_0(\cdot) \parallel_{L^\infty}, \quad \forall t \in [0, T).
\]

If $\rho_0 \in L^1$, then
\[
\int_R |\rho(t,x)| \, dx = \int_R |\rho_0(x)| \, dx, \quad \forall t \in [0, T).
\]

Now, we give the blow-up criterion for sufficiently regular solutions of (2.1).

**Theorem 3.3.** Let $z_0 = (u_0, \rho_0) \in H^s(R) \times H^{s-1}(R)$ with $s \geq \frac{3}{2}$, there exists a maximal time $T > 0$ and a unique solution $u$ of (2.1) with initial data $(u_0, \rho_0)$, then the corresponding solution blows up in finite time, if and only if
\[
\lim_{t \to T^-} \inf_{x \in R} \{u_x(t,x)\} = -\infty \quad \text{or} \quad \lim_{t \to T^-} \sup \{\parallel \rho_x(t, \cdot) \parallel\} = +\infty.
\]

**Proof.** Take the local well-posedness and density argument into consideration, it suffices to prove the desired estimates for $s = 3$. Multiplying the first equation in (2.1) by $u$ and integrating, yield that
\[
\frac{d}{dt} \int_R (u^2 + u_x^2) dx + 2\lambda \int_R (u^2 + u_x^2) dx - \sigma \int_R u^2 u_x dx = 0.
\]
Since $2\lambda \int_R (u^2 + u_x^2)dx \geq 0$, we have
\[
\frac{d}{dt} \int_R (u^2 + u_x^2)dx \leq \sigma \int_R \rho^2 u_x dx.
\] (3.5)

In a similar way, one can get that
\[
\frac{d}{dt} \int_R (u^2 + u_x^2)dx \leq -\sigma \int_R \rho^2 u_{xxx} dx - 3 \int_R u_x^3 dx - 3 \int_R u_x u_{xx}^2 dx.
\] (3.6)

Similarly,
\[
\frac{d}{dt} \int_R (u^2 + u_{xxx}^2)dx \leq 2\sigma \int_R \rho^2 u_{xxx} dx - 5 \int_R u_x u_{xx}^2 dx.
\] (3.7)

Multiplying the second equation of (2.1) by $\rho$ and integrating, give that
\[
\frac{d}{dt} \int_R \rho^2 dx = - \int_R \rho^2 u_x dx.
\] (3.8)

In a similar manner, we deduce that
\[
\frac{d}{dt} \int_R \rho_x^2 dx = -3 \int_R \rho_x^2 u_x dx + \int_R u_{xxx} \rho^2 dx.
\] (3.9)

Analogously,
\[
\frac{d}{dt} \int_R \rho_{xx}^2 dx = -5 \int_R \rho_{xx}^2 u_x dx + 3 \int_R u_{xxx} \rho_x^2 dx - 2 \int_R \rho \rho_{xxx} u_{xxx} dx.
\] (3.10)

Now we prove Theorem 3.1 by contradiction argue. Assume that there exist constants $M_1, M_2 > 0$, such that $u_x(t, x) \geq -M_1$ and $\| \rho_x(t, \cdot) \|_{L^\infty} \leq M_2$. From Lemma 3.2, we get
\[
\| \rho(t, \cdot) \|_{L^\infty} = \| \rho(t, q(t, \cdot)) \|_{L^\infty} \leq e^{M_1 T} \| \rho_0(\cdot) \|_{L^\infty}.
\]

Then by the inequality (3.5) and (3.8), we obtain
\[
\frac{d}{dt} \int_R (u^2 + u_x^2 + \rho^2)dx \leq (\sigma - 1) \int_R u_x \rho^2 dx.
\] (3.11)

The inequality (3.11) can be discussed as follows.

(1) When $\sigma > 1$,
\[
\frac{d}{dt} \int_R (u^2 + u_x^2 + \rho^2)dx \leq (\sigma - 1)e^{M_1 T} \| \rho_0 \|_{L^\infty} \int_R (u_x^2 + \rho^2)dx
\]
\[
\leq (\sigma - 1)e^{M_1 T} \| \rho_0 \|_{L^\infty} \int_R (u^2 + u_x^2 + \rho^2)dx
\]
\[
= c_1 \int_R (u^2 + u_x^2 + \rho^2)dx.
\]

(2) When $\sigma < 1$,
\[
\frac{d}{dt} \int_R (u^2 + u_x^2 + \rho^2)dx
\]
\[
\leq (\sigma - 1) \int_R (u_x \rho^2)dx \leq (1 - \sigma)M_1 \int_R \rho^2 dx
\]
\[
\leq (1 - \sigma)M_1 \int_R (u^2 + u_x^2 + \rho^2)dx = c_2 \int_R (u^2 + u_x^2 + \rho^2)dx.
\]
Let $C_1 = \max\{c_1, c_2\}$, then
\[
\frac{d}{dt} \int_R (u^2 + u_x^2 + \rho^2) dx \leq C_1 \int_R (u^2 + u_x^2 + \rho^2) dx.
\]
Using the Gronwall inequality, we have
\[
\|u\|_{H^1}^2 + \|\rho\|_{L^2}^2 \leq e^{C_1T\|u_0\|_{H^1}^2 + \|\rho_0\|_{L^2}^2}.
\]
Applying Sobolev embedding theorem, yields
\[
\|u\|_{L^\infty}^2 \leq \frac{1}{2} \|u\|_{H^1}^2 \leq \frac{1}{2} e^{C_1T\|u_0\|_{H^1}^2 + \|\rho_0\|_{L^2}^2}.
\] (3.12)
Then it follows from (3.5),(3.6),(3.8) and (3.9) that
\[
\frac{d}{dt} \int_R (u^2 + 2u_x^2 + u_{xx}^2 + \rho^2 + \rho_x^2) dx
\leq (\sigma - 1) \int_R u_x \rho^2 dx + 2(\sigma - 1) \int_R \rho u_x u_{xx} dx
- 3 \int_R u_x^2 dx - 3 \int_R u_{xx} u_x^2 dx - 3 \int_R (\rho_0^2 u_x^2) dx.
\] (3.13)
The inequality (3.13) can be discussed as follows.

(i) When $\sigma > 1$,
\[
\frac{d}{dt} \int_R (u^2 + 2u_x^2 + u_{xx}^2 + \rho^2 + \rho_x^2) dx
\leq (\sigma - 1)e^{M_1T\|\rho_0\|_{L^\infty}} \int_R (\rho^2 + u_x^2) dx + (\sigma - 1)M_2 \int_R (\rho^2 + u_{xx}^2) dx
+ 3M_1 \int_R u_x^2 dx + 3M_1 \int_R u_{xx}^2 dx + 3M_1 \int_R \rho_x^2 dx
\leq (3M_1 + (\sigma - 1)M_2 + (\sigma - 1)e^{M_1T\|\rho_0\|_{L^\infty}}) \int_R (u^2 + u_x^2 + u_{xx}^2 + \rho^2 + \rho_x^2) dx
= c_3 \int_R (u^2 + u_x^2 + u_{xx}^2 + \rho^2 + \rho_x^2) dx.
\]

(ii) When $\sigma < 1$,
\[
\frac{d}{dt} \int_R (u^2 + 2u_x^2 + u_{xx}^2 + \rho^2 + \rho_x^2) dx
\leq (1 - \sigma)M_1 \int_R \rho^2 dx + (1 - \sigma)M_2 \int_R (\rho^2 + u_x^2) dx + 3M_1 \int_R u_x^2 dx
+ 3M_1 \int_R u_{xx}^2 dx + 3M_1 \int_R \rho_x^2 dx
\leq ((3 - \sigma)M_1 + (1 - \sigma)M_2) \int_R (u^2 + u_x^2 + u_{xx}^2 + \rho^2 + \rho_x^2) dx
= c_4 \int_R (u^2 + u_x^2 + u_{xx}^2 + \rho^2 + \rho_x^2) dx.
\]
Let $C_2 = \max\{c_3, c_4\}$, then
\[
\frac{d}{dt} \int_R (u^2 + 2u_x^2 + u_{xx}^2 + \rho^2 + \rho_x^2) dx \leq C_2 \int_R (u^2 + u_x^2 + u_{xx}^2 + \rho^2 + \rho_x^2) dx
By the Gronwall inequality and Sobolev embedding theorem, we obtain
\[
\| u \|_{L^\infty}^2 \leq \frac{1}{2} \| u \|_{H^2}^2 \leq \frac{1}{2} e^{C_2 t} (\| u_0 \|_{H^2}^2 + \| \rho_0 \|_{L^1}^2)
\]  
(3.14)

From (3.5)-(3.10), we derive
\[
\frac{d}{dt} \int_R (u^2 + 2u_x^2 + 2u_{xx}^2 + u_{xxx}^2 + \rho^2 + \rho_x^2 + \rho_{xx}^2) dx \\
\leq (\sigma - 1) \int_R u_x u_x^2 dx + 2(\sigma - 1) \int_R \rho_x u_{xx} dx - 18 \int_R u_x u_{xx}^2 dx - 3 \int_R u_x^3 dx \\
- 3 \int_R \rho_x^2 u_x dx - 5 \int_R u_x u_{xxx} dx - 4\sigma \int_R \rho_x \rho_{xx} u_{xx} dx + 2\sigma \int_R \rho_x \rho_{xx} u_{xxx} dx \\
- 6 \int_R \rho_x \rho_{xx} u_{xx} dx - 5 \int_R \rho_{xx}^2 u_x dx - 2 \int_R \rho_x \rho_{xx} u_{xxx} dx.
\]  
(3.15)

The inequality (3.15) can be discussed as follows.

(i) When \( \sigma > 1 \),
\[
\frac{d}{dt} \int_R (u^2 + 2u_x^2 + 2u_{xx}^2 + u_{xxx}^2 + \rho^2 + \rho_x^2 + \rho_{xx}^2) dx \\
\leq (\sigma - 1) e^{M_1 t} \| \rho_0 \|_{L^\infty} \int_R (u_x^2 + \rho^2) dx + (\sigma - 1) M_2 \int_R (\rho^2 + u_{xx}^2) dx \\
+ 2\sigma M_2 \int_R (\rho_{xx}^2 + u_{xxx}^2) dx + \sigma M_2 \int_R (\rho_{xx}^2 + u_{xxx}^2) dx + 3M_1 \int_R u_x^3 dx \\
+ 18M_1 \int_R u_{xx}^2 dx + 3M_1 \int_R \rho_x^2 dx + 5M_1 \int_R u_{xxx} dx + 3M_2 \int_R (u_{xx}^2 + \rho_{xx}^2) dx \\
+ 5M_1 \int_R \rho_{xx}^2 dx + e^{M_1 t} \| \rho_0 \|_{L^\infty} \int_R (u_{xxx}^2 + \rho_{xx}^2) dx \\
\leq (18M_1 + 3(\sigma + 1) M_2 + (\sigma - 1) e^{M_1 t} \| \rho_0 \|_{L^\infty}) \\
\times \int_R (u^2 + u_x^2 + u_{xx}^2 + u_{xxx}^2 + \rho^2 + \rho_x^2 + \rho_{xx}^2) dx \\
= c_0 \int_R (u^2 + u_x^2 + u_{xx}^2 + u_{xxx}^2 + \rho^2 + \rho_x^2 + \rho_{xx}^2) dx.
\]

(ii) When \( 0 < \sigma < 1 \),
\[
\frac{d}{dt} \int_R (u^2 + 2u_x^2 + 2u_{xx}^2 + u_{xxx}^2 + \rho^2 + \rho_x^2 + \rho_{xx}^2) dx \\
\leq (1 - \sigma) M_1 \int_R \rho_x^2 dx + (1 - \sigma) M_2 \int_R (\rho^2 + u_{xx}^2) dx + 2\sigma M_2 \int_R (\rho_{xx}^2 + u_{xxx}^2) dx \\
+ \sigma M_2 \int_R (\rho_{xx}^2 + u_{xxx}^2) dx + 3M_1 \int_R u_x^3 dx + 18M_1 \int_R u_{xx}^2 dx + 3M_1 \int_R \rho_x^2 dx \\
+ 5M_1 \int_R u_{xxx}^2 dx + 3M_2 \int_R (\rho_{xx}^2 + u_{xx}^2) dx + 5M_1 \int_R \rho_x^2 dx \\
+ e^{M_1 t} \| \rho_0 \|_{L^\infty} \int_R (u_{xxx}^2 + \rho_{xx}^2) dx \\
\leq (18M_1 + 3(\sigma + 4) M_2 + e^{M_1 t} \| \rho_0 \|_{L^\infty}) \\
\times \int_R (u^2 + u_x^2 + u_{xx}^2 + u_{xxx}^2 + \rho^2 + \rho_x^2 + \rho_{xx}^2) dx.
\]
Lemma 3.4. The inequalities (3.12), (3.14) and (3.16) and Sobolev embedding theorem show

\[ C \leq c_{\delta} \int_{\mathbb{R}} (u^2 + u_{x}^2 + u_{xx}^2 + u_{xxx}^2 + \rho^2 + \rho_{x}^2 + \rho_{xx}^2) dx. \]

(iii) When \( \sigma < 0 \),

\[
\frac{d}{dt} \int_{\mathbb{R}} (u^2 + 2u_{x}^2 + 2u_{xx}^2 + u_{xxx}^2 + \rho^2 + \rho_{x}^2 + \rho_{xx}^2) dx \\
\leq (1 - \sigma)M_1 \int_{\mathbb{R}} \rho^2 dx + (1 - \sigma)M_2 \int_{\mathbb{R}} (\rho^2 + u_{xxx}^2) dx - 2\sigma M_2 \int_{\mathbb{R}} (\rho_{xx}^2 + u_{xxx}^2) dx \\
- \sigma M_2 \int_{\mathbb{R}} (\rho_{xx}^2 + u_{xxx}^2) dx + 3M_1 \int_{\mathbb{R}} u_{x}^2 dx + 18M_1 \int_{\mathbb{R}} u_{xx}^2 dx + 3M_1 \int_{\mathbb{R}} \rho_{x}^2 dx \\
+ 5M_1 \int_{\mathbb{R}} u_{xxx}^2 dx + 3M_2 \int_{\mathbb{R}} (\rho_{xx}^2 + u_{xxx}^2) dx + 5M_1 \int_{\mathbb{R}} \rho_{xx}^2 dx \\
+ e^{M_1T} \| \rho_0 \|_{L^\infty} \int_{\mathbb{R}} (u_{xx}^2 + \rho_{xx}^2) dx \\
\leq \left( (18 - \sigma)M_1 + (4 - 3\sigma)M_2 + e^{M_1T} \| \rho_0 \|_{L^\infty} \right) \\
\times \int_{\mathbb{R}} (u^2 + u_{x}^2 + u_{xx}^2 + u_{xxx}^2 + \rho^2 + \rho_{x}^2 + \rho_{xx}^2) dx \\
= c_7 \int_{\mathbb{R}} (u^2 + u_{x}^2 + u_{xx}^2 + u_{xxx}^2 + \rho^2 + \rho_{x}^2 + \rho_{xx}^2) dx.
\]

Let \( C_3 = \max\{c_5, c_6, c_7\} \), then

\[
\frac{d}{dt} \int_{\mathbb{R}} (u^2 + 2u_{x}^2 + 2u_{xx}^2 + u_{xxx}^2 + \rho^2 + \rho_{x}^2 + \rho_{xx}^2) dx \\
\leq C_3 \int_{\mathbb{R}} (u^2 + u_{x}^2 + u_{xx}^2 + u_{xxx}^2 + \rho^2 + \rho_{x}^2 + \rho_{xx}^2) dx.
\]

Similarly,

\[
\| u \|_{L^\infty}^2 \leq \frac{1}{2} \| u \|_{H^3}^2 \leq \frac{1}{2} e^{C_3T} (\| u_0 \|_{H^3}^2 + \| \rho_0 \|_{H^3}^2).
\]

The inequalities (3.12), (3.14) and (3.16) and Sobolev embedding theorem show that the solution does not blow up in finite time. On the other hand, we know that if

\[
\liminf_{t \to T^-} u(x, t) = -\infty, \limsup_{t \to T^-} \| \rho_x(t, \cdot) \| = +\infty,
\]

then solution \( z \) blows up in finite time. This completes the proof of Theorem 3.3.

In order to further study the problem of blow-up of strong solutions, we need the following lemmas to establish the blow-up criterion.

Lemma 3.4. Let \( z_0 = (u_0, \rho_0) \in H^s(R) \times H^{s-1}(R) \) with \( s \geq 2 \), there exist a maximal time \( T > 0 \) and a unique solution \( u \) of Eq (2.1) with initial data \((u_0, \rho_0)\), then we have

\[
\int_{\mathbb{R}} (u^2(t, x) + u_{x}^2(t, x) + \sigma \rho^2(t, x)) dx \leq \int_{\mathbb{R}} (u_0^2(t, x) + u_{0x}^2(t, x) + \sigma \rho_{0}^2(t, x)) dx.
\]

In addition, when \( \sigma \leq 0 \), if there exists \( M > 0 \), such that \( \| \rho(t, \cdot) \|_{L^\infty} \leq M \). For any given \( (t, x) \in [0, T) \times R \) and \( \rho_0 \in L^1 \), then

\[
\| u(t, \cdot) \|_{L^\infty} \leq \frac{\sqrt{2}}{2} \| u(t, \cdot) \|_{H^1} \leq \frac{\sqrt{2}}{2} (\| u_0 \|_{H^1}^2 + \sigma \| \rho_0 \|_{L^2}^2 - \sigma M \| \rho_0 \|_{L^1})^{\frac{1}{2}}.
\]

(3.17)
Proof. From (3.5) and (3.8), we have \( \frac{d}{dt} \int_R (u^2 + u_x^2 + \sigma \rho^2) \, dx \leq 0 \). It yields
\[
\int_R (u^2 + u_x^2 + \sigma \rho^2) \, dx \leq \int_R (u_0^2 + u_{0x}^2 + \sigma \rho_0^2) \, dx. \tag{3.18}
\]
From the inequality (3.18), Lemma 3.2 and \( \sigma \leq 0 \), we obtain
\[
\begin{align*}
&u^2(t,x) = \int_{-\infty}^{x} uu_x \, dx - \int_{x}^{+\infty} uu_x \, dx \leq \int_{-\infty}^{+\infty} |uu_x| \, dx \leq \frac{1}{2} \int_R (u^2 + u_x^2) \, dx \\
&\leq \frac{1}{2} \int_R (u_0^2 + u_{0x}^2) \, dx + \frac{\sigma}{2} \int_R \rho_0^2 \, dx - \frac{\sigma}{2} \int_R \rho^2 \, dx \\
&\leq \frac{1}{2} \int_R (u_0^2 + u_{0x}^2) \, dx + \frac{\sigma}{2} \int_R \rho_0^2 \, dx - \frac{\sigma}{2} M \int_R |\rho_0| \, dx.
\end{align*}
\]
Then
\[
\|u(t,\cdot)\|_{L^\infty} \leq \frac{\sqrt{2}}{2} \|u(t,\cdot)\|_{H^1} \leq \frac{\sqrt{2}}{2} (\|u_0\|_{H^1}^2 + \sigma \|\rho_0\|_{L^2}^2 - \sigma M \|\rho_0\|_{L^1})^{\frac{1}{2}}.
\]
The proof of Lemma 3.4 is completed. \( \square \)

**Lemma 3.5 (0).** Let \( T > 0 \) and \( v \in C^1([0,T); H^1(R)) \), then for every \( t \in [0,T) \), there exists at least one point \( \xi(t) \in R \) with
\[
m(t) := \inf_{x \in R} \{v_x(t,x)\} = v_x(t,\xi(t)). \tag{3.19}
\]
The function \( m(t) \) is absolutely continuous on \((0,T)\) with
\[
\frac{dm(t)}{dt} = v_{tx}(t,\xi(t)) \quad a.e. on \quad (0,T). \tag{3.20}
\]

**Theorem 3.6.** Let \( z_0 = (u_0,\rho_0) \in H^s(R) \times H^{s-1}(R) \) with \( s \geq 2 \), there exist a maximal time \( T > 0 \) and a unique solution \( u \) of (2.1) with initial data \((u_0,\rho_0)\). Assume that there exists \( M > 0 \) such that \( \|\rho(t,\cdot)\|_{L^\infty} \leq M \) for any given \( t \in [0,T) \) and \( \rho_0 \in L^1 \). If there exists \( x_0 \in R \) such that \( u_{0x}(x_0) < -\lambda - \sqrt{\lambda^2 + 2K} \), then the solution of (2.1) blows up in finite time, in addition, if there exists
\[
0 < T < \frac{1}{\sqrt{\lambda^2 + 2K}} \ln \left( \frac{m(0) + \lambda - \sqrt{\lambda^2 + 2K}}{m(0) + \lambda + \sqrt{\lambda^2 + 2K}} \right),
\]
then
\[
\lim_{t \to T, T \in R} \inf_{x \in R} \{u_x(t,x)\} = -\infty,
\]
where \( K = \max\{C_4,C_5\} \),
\[
\begin{align*}
C_4 &= \sqrt{2k}(\|u_0\|_{H^1}^2 + \sigma \|\rho_0\|_{L^2}^2 - \sigma M \|\rho_0\|_{L^1})^{\frac{1}{2}} \\
&+ \frac{1}{4} (\|u_0\|_{H^1}^2 + \sigma \|\rho_0\|_{L^2}^2 - 4\sigma M \|\rho_0\|_{L^1}), \\
C_5 &= \frac{3}{4} + \frac{3k^2}{4} (\|u_0\|_{L^2}^2 + \|u_{0x}\|_{L^2}^2 + \sigma \|\rho_0\|_{L^2}^2) + \frac{3}{4}.
\end{align*}
\]

**Proof.** Let \( z = (u,\rho) \) is the solution of Eq(2.1) with initial data \( z_0 \in H^s(R) \times H^{s-1}(R) \) with \( s \geq 2 \) and \( T \) is the maximal existence time of the solution. Differentiating the first equation of (2.2) with respect to \( x \) and noting that \( \partial_x^2 g \ast f = g \ast f - f \), we obtain
\[
u_{tx} = u^2 - \frac{1}{2} u_x^2 - \lambda u_x + ku + \frac{\sigma}{2} \rho^2 - g \ast (u^2 + \frac{1}{2} u_x^2 + ku + \frac{\sigma}{2} \rho^2) - uu_{xx}.
\]
Denote \( m(t) = u_x(t, \xi(t)) = \inf_{x \to \rho} (u_x(t, x)) \). Then for any given \( t \in [0, T) \), we obtain \( u_{xx}(t, \xi(t)) = 0 \). Notice that the relation \( g * (u^2 + \frac{1}{2} u_x^2) \geq \frac{1}{2} u^2 \), then

\[
\frac{dm(t)}{dt} \leq u^2 - \frac{1}{2} m^2(t) - \lambda m(t) + ku + \frac{\sigma}{2} \rho^2 - g * (u^2 + \frac{1}{2} u_x^2) - g * (ku + \frac{\sigma}{2} \rho^2)
\leq -\frac{1}{2} m^2(t) - \lambda m(t) + \frac{1}{2} u^2 + ku + \frac{\sigma}{2} \rho^2 - g * (ku + \frac{\sigma}{2} \rho^2).
\]

(3.21)

Let \( f = \frac{1}{2} u^2 + ku + \frac{\sigma}{2} \rho^2 - g * (ku + \frac{\sigma}{2} \rho^2) \),

(i) In the case of \( \sigma < 0 \),

\[
f \leq \frac{1}{2} u^2 + ku - \frac{\sigma}{2} \rho^2 + |k g * u| - \frac{\sigma}{2} |g \ast \rho^2|.
\]

From the Young’s inequality and the Lemma 3.4, we get

\[
k \| g * u \|_{L^\infty} \leq k \| g \|_{L^1} \| u \|_{L^\infty} \leq \frac{\sqrt{k}}{2} (\| u_0 \|_{H^1} + \sigma \| \rho_0 \|_{L^2} - \sigma M \| \rho_0 \|_{L^1}) \frac{1}{2}
\]

\[
-\frac{\sigma}{2} \| g \ast \rho^2 \|_{L^\infty} \leq -\frac{\sigma}{2} \| g \|_{L^\infty} \| \rho^2 \|_{L^1} \leq -\frac{\sigma}{4} M \| \rho_0 \|_{L^1},
\]

\[
\frac{1}{2} u^2 \leq \frac{1}{4} (\| u_0 \|_{H^1}^2 + \sigma \| \rho_0 \|_{L^2}^2 - \sigma M \| \rho_0 \|_{L^1}),
\]

\[
k u \leq k \| u \|_{L^\infty} \leq \sqrt{\frac{k}{2}} (\| u_0 \|_{H^1} + \sigma \| \rho_0 \|_{L^2} - \sigma M \| \rho_0 \|_{L^1}) \frac{1}{2}
\]

\[
-\frac{\sigma}{2} \| \rho^2 \| \leq -\frac{\sigma}{2} M \| \rho_0 \|_{L^1}.
\]

Thus

\[
f \leq \sqrt{2} k (\| u_0 \|_{H^1} + \sigma \| \rho_0 \|_{L^2} - \sigma M \| \rho_0 \|_{L^1}) \frac{1}{2}
\]

\[
+ \frac{1}{4} (\| u_0 \|_{H^1} + \sigma \| \rho_0 \|_{L^2} - 4 \sigma M \| \rho_0 \|_{L^1}) = C_4.
\]

(ii) In the case of \( \sigma > 0 \),

\[
f \leq \frac{1}{2} u^2 + ku + \frac{\sigma}{2} \rho^2 + |k g * u| + \frac{\sigma}{2} |g \ast \rho^2|.
\]

We redefine the following estimates

\[
\frac{1}{2} u^2 \leq \frac{1}{4} \int_R (u^2 + u_x^2) dx \leq \frac{1}{4} \| u \|^2_{L^2}.
\]

\[
|ku| \leq k \| u \|_{L^2} \leq \frac{1}{2} + \frac{k^2}{2} \| u \|^2_{L^2}, \quad \frac{\sigma}{2} \| \rho^2 \| \leq \frac{\sigma}{2} \| \rho \|^2_{L^2},
\]

\[
k \| g * u \| \leq k \| g \|_{L^\infty} \| u \|_{L^2} = \frac{k}{2} \| u \|_{L^2} \leq \frac{1}{4} + \frac{k^2}{4} \| u \|_{L^2}^2.
\]

\[
\frac{\sigma}{2} \| g \|_{L^\infty} \| \rho \|_{L^2} = \frac{\sigma}{4} \| \rho \|_{L^2}.
\]

Then

\[
f \leq \frac{3 + 3k^2}{4} (\| u_0 \|_{L^2}^2 + \| u_0x \|_{L^2}^2 + \sigma \| \rho_0 \|_{L^2}^2) + \frac{3}{4} = C_5.
\]

Let \( K = \max\{C_4, C_5\} \), it becomes

\[
\frac{dm(t)}{dt} \leq -\frac{1}{2} m^2(t) - \lambda m(t) + K.
\]

(3.22)

From the inequality (3.22), we have

\[
\frac{dm(t)}{dt} \leq -\frac{1}{2} (m(t) + \lambda - \sqrt{\lambda^2 + 2K}) (m(t) + \lambda + \sqrt{\lambda^2 + 2K}).
\]

(3.23)
The estimate $m(0) = u_{0x}(x_0) < -\lambda - \sqrt{\lambda^2 + 2K}$ means that $m^2(0) > (\lambda + \sqrt{\lambda^2 + 2K})^2$. We will claim $m^2(t) > (\lambda + \sqrt{\lambda^2 + 2K})^2$ is true for any given $t \in [0, T)$. By contradiction, there exists $t_0 \in [0, T)$ such that $m^2(t) > (\lambda + \sqrt{\lambda^2 + 2K})^2$, but $m^2(t_0) = (\lambda + \sqrt{\lambda^2 + 2K})^2$. Consider the inequality (3.23), we get $\frac{dm(t)}{dt} < 0$ for $t \in [0, t_0)$. Because the function $m(t)$ is absolutely continuous, integrating $\frac{dm(t)}{dt} < 0$ we get $m(t_0) < m(0) = u_{0x}(x_0) < -\lambda - \sqrt{\lambda^2 + 2K}$, so this proves the claim $m^2(t) > (\lambda + \sqrt{\lambda^2 + 2K})^2$ for $\forall t \in [0, T)$.

Solving the inequality (3.23), we have

$$\frac{m(0) + \lambda + \sqrt{\lambda^2 + 2K}}{m(0) + \lambda - \sqrt{\lambda^2 + 2K}} e^{\sqrt{\lambda^2 + 2K} t} - 1 \leq \frac{2\sqrt{\lambda^2 + 2K}}{m(t) + \lambda - \sqrt{\lambda^2 + 2K}} \leq 0.$$ 

Since

$$0 < \frac{m(0) + \lambda + \sqrt{\lambda^2 + 2K}}{m(0) + \lambda - \sqrt{\lambda^2 + 2K}} < 1,$$

then there exists

$$0 < T < \frac{1}{\sqrt{\lambda^2 + 2K}} \ln \left( \frac{m(0) + \lambda - \sqrt{\lambda^2 + 2K}}{m(0) + \lambda + \sqrt{\lambda^2 + 2K}} \right),$$

such that

$$\lim_{t \to T^{-}} \inf_{x \in R} \{u_x(t, x)\} = -\infty.$$ 

The proof of Theorem 3.6 is completed.

4. The Blow-up rate. In this section, we will estimate the blow-up rate of strong solutions of (2.1). The result shows that the blow-up rate of strong solutions of (2.1) is not affected by the weakly dissipation.

**Theorem 4.1.** Let $z_0 = (u_0, \rho_0) \in H^s(R) \times H^{s-1}(R)$ with $s \geq 2$, there exist a maximal time $T > 0$ and a unique solution $u$ of (2.1) with initial data $(u_0, \rho_0)$ satisfies the assumption of the Theorem 3.6, then

$$\lim_{t \to T^{-}} \inf_{x \in R} (u_x(t, x)(T - t)) = -2. \tag{4.1}$$

**Proof.** From the proof of Theorem 3.6, we have $\frac{dm(t)}{dt} \leq -\frac{1}{2} m(t) - \lambda m(t) + K$, where $K$ is given in Theorem 3.6. Then

$$-K \leq \frac{dm(t)}{dt} + \frac{1}{2} m^2(t) + \lambda m(t) \leq K, \quad \forall t \in (0, t).$$

That is,

$$-K - \frac{1}{2} \lambda^2 \leq \frac{dm(t)}{dt} + \frac{1}{2} (m(t) + \lambda)^2 \leq K + \frac{1}{2} \lambda^2, \quad \forall t \in (0, t). \tag{4.2}$$

Choose $\varepsilon \in (0, \frac{1}{2})$, due to $\lim_{t \to T^{-}} (m(t) + \lambda) = -\infty$, then there exists $t_0 \in (0, t)$, such that $m(t_0) + \lambda < 0$ and $(m(t_0) + \lambda)^2 > \frac{1}{\varepsilon} (K + \frac{1}{2} \lambda^2)$. $m$ is absolutely continuous due to $m$ is locally Lipschitz, we claim that

$$(m(t) + \lambda)^2 > \frac{1}{\varepsilon} (K + \frac{1}{2} \lambda^2), \quad t \in [t_0, T). \tag{4.3}$$

From (4.1) and (4.2), we obtain

$$-\frac{1}{2} (m(t) + \lambda)^2 - K - \frac{1}{2} \lambda^2 \leq \frac{dm(t)}{dt} \leq K + \frac{1}{2} \lambda^2 - \frac{1}{2} (m(t) + \lambda)^2, \quad \forall t \in (0, t). \tag{4.4}$$
Due to $m$ is locally Lipschitz in $(t_0, T)$, we know $\frac{1}{m}$ is also locally Lipschitz in $(t_0, T)$. From the inequality (4.3), we have

$$\frac{1}{2} - \varepsilon \leq \frac{d}{dt} \left( \frac{1}{m(t) + \lambda} \right) \leq \frac{1}{2} + \varepsilon, \quad t \in (t_0, T). \quad (4.5)$$

Integrating (4.5) over $(t, T)$ with respect to $t \in [t_0, T)$ and noting that $\lim_{t \to T^-} m(t) = -\infty$, then

$$\left( \frac{1}{2} - \varepsilon \right) (T - t) \leq - \left( \frac{1}{m(t) + \lambda} \right) \leq \left( \frac{1}{2} + \varepsilon \right) (T - t), \quad t \in (t_0, T).$$

The arbitrariness of $\varepsilon \in (0, \frac{1}{2})$ yields that

$$\lim_{t \to T^-} \{ (m(t)(T - t) + \lambda(T - t)) \} = -2.$$

That is $\lim_{t \to T^-} \{ m(t)(T - t) \} = -2$. The blow-up rate of strong solutions of Eq(2.1) is not effected by the weakly dissipation. \hfill \Box

5. The Blow-up points set. In this section we will give the properties of blow-up points set.

**Theorem 5.1.** Let $A = \{ x \in R \mid u_{0x}(x) < -\lambda - \sqrt{\lambda^2 + 2K} \}$ is nonempty, where $\lambda$ is the coefficient of weakly dissipative term, $K$ is given in Theorem 3.6, then $A$ is a closed set with finite measure.

**Proof.** Let $\mid A \mid$ is the measure of the set $A$. From the proof of Theorem 3.6, we know $u_{0x}^2(x) > (\lambda + \sqrt{\lambda^2 + 2K})^2$, $x \in A$. Then we have

$$\| u_{0x} \|^2 = \int_R | u_{0x} |^2 dx > \int_A | u_{0x} |^2 dx > \int_A (\lambda + \sqrt{\lambda^2 + 2K})^2 dx = | A | (\lambda + \sqrt{\lambda^2 + 2K})^2.$$

Hence,\n
$$| A | \leq \frac{\| u_{0x} \|^2}{(\lambda + \sqrt{\lambda^2 + 2K})^2}. \quad (5.1)$$

The measure of $A$ is finite. Let the range of points $\{ x_n \}_{n=1}^\infty \subseteq A$ and $\lim_{n \to \infty} x_n = y_0$. Since $u_0 \in H^2$, $u_{0x}$ is continuous function and we have

$$u_{0x}(y_0) = \lim_{n \to \infty} u_{0x}(x_n) = \lim_{n \to \infty} u_{0x}(x_n) \leq -\lambda - \sqrt{\lambda^2 + 2K}. \quad (5.2)$$

Thus $y_0 \in A$ and $A$ is closed. If theorem 5.1 holds, then $A$ is closed with a finite measure. When $\lambda = 0$, $A' = \{ x \in R \mid u_{0x}(x) < -\sqrt{2K} \}$. Using the similar method, we have

$$| A' | \leq \frac{\| u_{0x} \|^2}{(2K)^2}.$$ \hfill \Box

The measure of $A'$ is finite and making $\{ x'_n \}_{n=1}^\infty \subseteq A'$ such that $\lim_{n \to \infty} x'_n = y'_0$, we have

$$u_{0x}(y'_0) = \lim_{n \to \infty} u_{0x}(x'_n) = \lim_{n \to \infty} u_{0x}(x'_n) \leq -\sqrt{2K}.$$

Hence $y_0 \in A'$ and $A'$ is closed with a finite measure. It is obviously that weakly dissipation do not effect the existence of blow-up set.

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