Probabilistic programs for inferring the goals of autonomous agents

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Abstract

Intelligent systems sometimes need to infer the probable goals of people, cars, and robots, based on partial observations of their motion. This paper introduces a class of probabilistic programs for formulating and solving these problems. The formulation uses randomized path planning algorithms as the basis for probabilistic models of the process by which autonomous agents plan to achieve their goals. Because these path planning algorithms do not have tractable likelihood functions, new inference algorithms are needed. This paper proposes two Monte Carlo techniques for these “likelihood-free” models, one of which can use likelihood estimates from neural networks to accelerate inference. The paper demonstrates efficacy on three simple examples, each using under 50 lines of probabilistic code.

1 INTRODUCTION

Intelligent systems sometimes need to infer the probable goals of people, cars, and robots, based on partial observations of their motion. These problems are central to autonomous driving and driver assistance [Franke et al., 1998; Urmson et al., 2008; Aufrère et al., 2003], but also arise in aerial robotics, reconnaissance, and security applications [Kumar and Michael, 2012; Liao et al., 2006; Tran and Davis, 2008]. In these settings, knowledge of the beliefs and goals of an agent makes it possible to infer their probable future actions.

Because the mental state of another agent is inherently unobservable and uncertain, it is natural to take a Bayesian approach to inferring it. Probabilistic models can be used to describe how an agent’s latent high-level goals and beliefs about the environment interact to yield its probable actions. Most existing work along these lines has focused on modeling goal-directed behavior using Markov decision processes and related approaches from stochastic control [Baker et al., 2007; Ziebart et al., 2009]. While promising, these approaches involve significant task-specific engineering. They also calculate policies that prescribe actions for every possible state of the world, sometimes in the inner loop of an inference algorithm. This leads to fundamental scaling challenges, even for simple environments and goal priors.

This paper introduces a class of probabilistic programs that formulate goal inference problems as approximate inference in generative models of goal-directed behavior. The proposed approach reflects three contributions: First, agents are assumed to follow paths generated by fast randomized path planning code that can incorporate heuristics drawn from video game engines and robotics. This can scale to larger environments than approaches based on optimal control. Second, hierarchical models for goals and paths are represented as probabilistic programs. This allows one to formulate a broad class of single- and multi-agent problems with common modeling and inference machinery. Ordinary probabilistic programming constructs can handle complex maps, hierarchical goal priors, and partially observed environments. Third, this paper proposes an approach to real-time approximate inference, using neural networks to learn proposals for the internal choices made by any path planners. Together, these contributions lead to a practical proposal for goal inference that has the potential to scale to a broad class of real-world problems and real-time applications. We demonstrate the efficacy of prototype implementations of these algorithms on three simple examples, each written in under 50 lines of probabilistic code.

Note that this proposal does not require planning algorithms to be rewritten as probabilistic programs, but instead allows optimized, low-level, or legacy planning codes to be treated as black boxes. This avoids the implementation and performance cost of rewriting an existing path planner in a high-level probabilistic programming...
language, and exposing the thousands of random choices it might make to generic inference algorithms. One difficulty is that such optimized black-box planners may well make too many internal random choices to have tractable input-output likelihoods. This paper proposes two novel Monte Carlo techniques for these “likelihood-free” models, each extending Metropolis-Hastings: (i) a cascading resimulation algorithm that makes joint proposals to ensure cancellation of the unknown likelihoods, and (ii) a nested inference algorithm that uses estimated likelihoods derived from inference over the internal random choices of the planner. Cascading resimulation is simple to implement, but nested inference enables use of a broad class of Monte Carlo, variational, and neural network mechanisms to handle the intractable likelihoods.

2 MODELING GOAL-DIRECTED BEHAVIOR USING RANDOMIZED PATH PLANNERS

This paper defines probabilistic models of goal-directed behavior using randomized path-planning algorithms. Algorithm 1 describes one such planner, called AGENT-PATH. This planner can be applied to a broad class of environments with complex obstacles. The planner assumes a bounded two-dimensional space (e.g., the square $[0,1]^2$) and a world map $M$ that is a set of polygonal obstacles. The planner takes as input a start location $s \in [0,1]^2$, a goal location $g \in [0,1]^2$, the map $M$, and a sequence of $T$ time points $t = (t_1, \ldots, t_T)$, and returns either a sequence of locations $z \in [0,1]^{2T}$ on a path from $s$ to $g$ at each time $t_i$, or ‘no-path-found’. The planner operates by growing a rapidly-exploring random tree (RRT) [LaValle 1998] from the start location $s$ to fill the space, searching for a clear line of sight between the tree and the goal. If a path is found, it is then refined to minimize its length using local optimization. Finally, the agent walks the path at a constant speed, producing the output locations $z$. See Appendix A for more details.

Many variations of this planner are possible, including versions that take into account costs other than path length, and spaces encoding configurations other than geographic position (e.g., configuration spaces of an articulated robot). The planner parameters $N$ and $R$ trade off the cost of planning with the (probable) optimality of the paths (see Figure 1). Figure 2 and Figure 4 show this planner being used as a modeling primitive in the Venture probabilistic programming platform [Mansinghka et al. 2014]. The planner was implemented in C and imported as a foreign modeling primitive into Venture. Venture supports likelihood-free primitives and design of custom inference strategies, including those of Section 3.

Algorithm 1 Model of an agent’s path given destination

```
Require: 
{World map $M$; Start, goal $s, g \in [0,1]^2$} 
Time points $t \in \mathbb{R}^T_+$ 
Refinement amount $N$; Restarts $R$ 
Max. # tree nodes $J$; Min. # tree nodes $S$
1: procedure RRT($M, s, g$) 
2: \hspace{1em} $V \leftarrow \{s\}$ \hspace{1em} \text{Initialize tree with start $s$}
3: \hspace{1em} for $j \leftarrow 1$ to $J$ do \hspace{1em} \text{Grow $j$ tree growth iterations}
4: \hspace{2em} $a \sim \text{Uniform}([0,1] \times [0,1])$ \hspace{1em} \text{Random point}
5: \hspace{2em} if $M.\text{valid-state}(a)$ then
6: \hspace{3em} $b \leftarrow \text{nearest-vertex}(V, a)$
7: \hspace{3em} $c \sim \text{Uniform}([0,1])$
8: \hspace{3em} $c \leftarrow ca + (1-c)b$ \hspace{1em} \text{Propose new vertex}
9: \hspace{2em} if $M.\text{clear-line}(b, c)$ then
10: \hspace{3em} $V.\text{add-edge}(b \rightarrow c)$ \hspace{1em} \text{Extend tree}
11: \hspace{3em} if $M.\text{clear-line}(c, g) \land j > S$ then
12: \hspace{4em} $V.\text{add-edge}(c \rightarrow g)$
13: \hspace{3em} return \text{path-in-tree}($V, s, g$)
14: \hspace{1em} return \text{‘no-path-found’}
15: procedure PLAN-PATH($M, s, g; R, N$) 
16: \hspace{1em} for $r \leftarrow 1$ to $R$ do \hspace{1em} \text{Generate $R$ paths}
17: \hspace{2em} $p^{(r)} \sim \text{RRT}(M, s, g)$
18: \hspace{2em} $p^{(r)} \leftarrow \text{simplify-path}(p^{(r)})$
19: \hspace{2em} $p^{(r)} \sim \text{refine-path}(M, s, g, p^{(r)})$
20: \hspace{2em} $d^{(r)} \leftarrow \text{path-length}(p^{(r)}, s, g)$
21: \hspace{2em} $r^* \leftarrow \text{argmin}(d)$ \hspace{1em} \text{Select best of $R$ paths}
22: \hspace{2em} return $p^{(r^*)}$
23: procedure AGENT-PATH($M, s, g, t; R, N$) 
24: \hspace{1em} $p \sim \text{plan-path}(M, s, g; R, N)$ \hspace{1em} \text{Abstract path}
25: \hspace{1em} $z \leftarrow \text{walk-path}(p, t)$ \hspace{1em} \text{Locations at times $t$}
26: return $z$
```
3 INFERENCE IN PROBABILISTIC PROGRAMS WITH LIKELIHOOD-FREE PRIMITIVES

The path planner \textsc{agent-path} of Algorithm 1 can be used in a probabilistic program either by employing the planner in a probabilistic programming language, or by treating the planner as a primitive random choice. We treat the planner as a random choice, as this allows use of an optimized \( C \) implementation of the planner. However, probabilistic programming languages such as Church, Stan, BLOG, and Figaro all require random choices to have tractable marginal likelihoods [Goodman et al., 2012; Carpenter et al., 2016; Milch et al., 2007; Pfeffer, 2009]. Computing the marginal likelihood of \textsc{agent-path} for outputs \( z \) and inputs \( M, s, g, \) and \( t \) would involve an intractable integral over the (thousands of) internal random choices made in \textsc{agent-path}.

This section introduces two Monte Carlo strategies for inference in probabilistic programs that include random choices with intractable marginal likelihoods, referred to as “likelihood-free” primitives. The first strategy, shown in Algorithm 2, is called Cascading Resimulation Metropolis-Hastings; it makes block proposals to likelihood-free random choices, exploiting cancellation of the unknown likelihoods. The second, shown in Algorithm 3, is called Nested Inference Metropolis-Hastings; it uses Monte Carlo estimates of the unknown likelihoods in place of the likelihoods themselves. Although simple techniques like likelihood-weighting can also be used in the presence of likelihood-free primitives, they tend to work well only when a global proposal that is well-matched to the posterior is available. The algorithms we introduce do not have this limitation.

We first introduce notation. Let \( \mathcal{T} \) be the set of primitive random choices available to a probabilistic program (e.g. \{FLIP, UNIFORM\_CONTINUOUS, AGENT\_PATH\}). For \( t \in \mathcal{T} \), let \( X_t \) denote the set of valid arguments for the primitive, let \( Z_t \) denote the set of possible outputs, and let \( p_t(z|x) \) denote the marginal likelihood of output \( z \in Z_t \) given arguments \( x \in X_t \), where \( \int p_t(z|x)dz = 1 \) for all \( x \in X_t \). We do not require evaluation of \( p_t(z|x) \) to be computationally tractable.

Following Wingate et al. [2011], for a probabilistic program \( P \), we assume there is a name \( i \in I \) assigned to every possible random choice, for some countable \( I \). We assume that distinct random choices are assigned unique names within every execution of \( P \). The set of names used in an execution is some finite set \( I \subseteq I \). We require that all random choices with name \( i \) are of the same type \( t_i \in T \). Each unique completed execution of \( P \) can therefore be represented as the finite set of names \( I \subseteq I \) of those random choices made in the execution, together with the result values. We denote these results \( z \in X_{\cup_{i \in I} Z_{t_i}} \), and denote this complete package \( \rho = (I, z) \). The tuple \( \rho \) is called an execution trace of the probabilistic program \( P \).
This paper focuses on probabilistic programs where $I$ is the same for all executions — that is, the set of random choices made is not affected by any of those choices. Relaxations of this are left for future work; more general formalizations of probabilistic programs can be found in [Wingate et al. 2011, Mansinghka et al. 2014].

We consider random choice $j$ to depend on random choice $i$ if changing the result $z_i$ of $i$ can lead to a change in the inputs $x_j$ of $j$, even if all other results $z_{I \setminus \{i\}}$ are held fixed. We assume that it is possible to construct a directed acyclic dependency graph $G = (I, E)$ among random choices $I$, where an edge $(i, j) \in E \subset I \times I$ exists if and only if random choice $j$ depends on random choice $i$ in the above sense. The parents of a random choice $j$ are denoted by $\pi_G(j) := \{i \in I : (i, j) \in E\}$.

The arguments $x_j$ of each random choice $j$ are then (a deterministic) function $f_j$ of the results of random choices in $\pi_G(j)$, which are denoted $z_{\pi_G(j)}$; we write $x_j = f_j(z_{\pi_G(j)})$. Let $G(i) := \{j \in I : (i, j) \in E\}$ denote the ‘children’ of choice $i$. Also, let $F \subset I$ denote the random choices with intractable likelihoods (the “likelihood-free” choices). Let $C \subseteq (I \setminus F)$ denote the random choices that are constrained based on data, which must have tractable likelihoods. Let $z_C \in \mathbb{X}_{i \in C} Z_i$ denote the values we are constraining those random choices to.

The joint probability density of an execution trace $\rho = (I, z)$ is:

$$p(z) := \prod_{i \in I} p_i(z_i ; f_i(z_{\pi_G(i)}))$$

where we have omitted the dependence on $I$ because it is the same for all executions.

### 3.1 Cascading Resimulation

How can a probabilistic program cope with complex, likelihood-free primitives? Our core insight is that if the proposal distribution $m(z'_i ; \cdot)$ for a random choice $z_i$ is equal to the prior $p_i(z'_i ; f_i(z_{\pi_G(i)}))$, then the likelihoods will cancel in a Metropolis-Hastings (MH) acceptance ratio and therefore do not need to be explicitly computed. Sampling from the prior is achieved simply by simulating the random choice. A (prototypical) acceptance ratio looks like this:

$$\alpha = \frac{p_i(z'_i ; f_i(z_{\pi_G(i)})) \cdot m(z_{\pi_G(i)} ; z_{\pi_G(i)})}{p_i(z_i ; f_i(z_{\pi_G(i)})) \cdot m(z_{\pi_G(i)} ; z_{\pi_G(i)})} = 1$$

We use blocked proposals in which a change to a likelihood-free choice is proposed from the prior whenever a proposal is made to any of its parents. A likelihood-free choice that is proposed may itself have likelihood-free choices as children, in which case these children are also proposed, generating a cascade of proposals. 

Algorithm 2 shows the Cascading Resimulation MH transition operator, which extends an initial custom proposal $m(z'_i ; z)$ to random choice $i$ (which must not be likelihood-free) to also include any likelihood-free random choices $H$ in the cascade, such that the intractable likelihoods cancel.

#### Algorithm 2 Single-site Cascading Resimulation

**Metropolis-Hastings transition**

Require:
- Prob. program with dep. graph $G = (I, E)$
- Likelihood-free random choices $F \subseteq I$
- Proposed-to random-choice $i \in (I \setminus F)$
- Custom proposal density $m(z'_i ; z)$ for choice $i$
- Previous values $z$ for all random choices

1: $z'_i \sim m_i(\cdot ; z)$ \(\triangleright\) Propose a new value for choice $i$
2: $z_{I \setminus \{i\}}' \sim z_{I \setminus \{i\}}$ \(\triangleright\) Initially, no change to other choices
3: $\ell \leftarrow 1$ \(\triangleright\) Unnormalized target density for previous values
4: $\ell' \leftarrow 1$ \(\triangleright\) Unnormalized target density for proposed values
5: $B \leftarrow I \cup \pi_G(i)$ \(\triangleright\) Ask for likelihoods from $j \in B$
6: $H \leftarrow \{\}$ \(\triangleright\) Likelihood-free cascade participants
7: $A \leftarrow \{i\}$ \(\triangleright\) Visited choices with tractable likelihoods
8: while $|B| > 0$ do
9: $j \leftarrow \text{POP}(B)$ \(\triangleright\) Pop in topological order
10: if $j \in F$ then \(\triangleright\) Choice $j$ is likelihood-free
11: $z'_j \sim p_j(\cdot ; f_j(z_{\pi_G(j)}))$ \(\triangleright\) Propose from prior
12: $\text{INSERT}(B, \pi_G(j))$ \(\triangleright\) Ask for child likelihoods
13: $H \leftarrow H \cup \{j\}$
14: else \(\triangleright\) Choice $j$ has tractable likelihood
15: $\ell \leftarrow \ell \cdot p_j(z_j ; f_j(z_{\pi_G(j)}))$
16: $\ell' \leftarrow \ell' \cdot (m(z_i ; z'_i)/m(z_i ; z))$ \(\triangleright\) MH ratio
17: $s \sim \text{Uniform}(0, 1)$
18: if $s \leq \alpha$ then \(\triangleright\) Accept
20: $z \leftarrow z'$

Algorithm 2 is a Metropolis-Hastings transition over the random choices $\{i\} \cup H$ with target density equal to the local posterior $p(z_{I \cup H} | z_{I \setminus \{i\} \cup H})$, and with proposal density:

$$m(z'_i ; z) \prod_{j \in H} p_j(z'_j ; f_j(z'_{\pi_G(j)}))$$

The Metropolis-Hastings acceptance ratio is:

$$\alpha = \left(\prod_{j \in H \cup A} p_j(z'_j ; f_j(z'_{\pi_G(j)})) \prod_{j \in H \cup A} m(z_{\pi_G(j)} ; z_{\pi_G(j)}) \right) \left(\prod_{j \in H} p_j(z_j ; f_j(z_{\pi_G(j)})) \prod_{j \in H} m(z_j ; z'_j) \right)$$

$$= \left(\prod_{j \in H \cup A} p_j(z'_j ; f_j(z'_{\pi_G(j)})) \prod_{j \in H} m(z'_j ; z) \prod_{j \in A} p_j(z'_j ; f_j(z'_{\pi_G(j)})) \right)$$

We illustrate Cascading Resimulation MH in Figure 2 on the task of inferring the goal of a simulated drone in an observed environment.
3.2 NESTED INFERENCE METROPOLIS-HASTINGS

In some problems, Cascading Resimulation MH will generate many expensive simulations of likelihood-free choices, most of which will be rejected. For these problems, and for real-time applications, we propose an alternative Metropolis-Hastings algorithm, called Nested Inference MH, that uses Monte Carlo estimates of the intractable likelihoods in the acceptance ratio. The likelihood estimates are obtained using auxiliary “nested inference” algorithms, which sample probable values for the internal random choices made by a likelihood-free choice (e.g., a randomized planning algorithm) given its inputs and outputs, and calculate a weight that can be used to form an importance sampling estimate of the unknown likelihood.

Nested Inference MH is based on an interpretation of likelihood-free random choices like AGENT-PATH as probabilistic programs in their own right. Let \( u \in U_t \) be an execution trace of a likelihood-free random choice of type \( t \). We denote the joint density on execution traces \( u \) and return values \( z \) of the random choice, given input arguments \( x \), by \( p_t(u; z; x) \). The marginal likelihood of the random choice is given by (the intractable) integral

\[
p_t(z; x) = \int p_t(u; z; x)du.
\]

We denote the conditional trace density for arguments \( x \) and output \( z \) by

\[
p_t(u; z; x) := p_t(u; z; x)/p_t(z; x).
\]

Nested inference assumes the existence of a nested inference algorithm that samples execution traces \( u \) according to some density \( q_t(u; z; x) \) that approximates the conditional density on traces of the likelihood-free choice, i.e., \( q_t(u; z; x) \approx p_t(u; z; x) \). We require that \( q_t(u; z; x) > 0 \) for all \( u \) where \( p_t(u; z; x) > 0 \). Using the nested inference algorithm as an importance sampler, we produce an unbiased importance sampling estimate \( \hat{p}_t(z; x) \) of the random choice’s intractable likelihood for arguments \( x \) and output \( z \) by sampling \( K \) times \( u_k \sim q_t(\cdot; x, z) \) from the inference algorithm, as follows:

\[
\hat{p}_t(z; x) := \frac{1}{K} \sum_{k=1}^{K} \frac{p_t(u_k; z; x)}{q_t(u_k; x, z)} \text{ for } u_k \sim q_t(\cdot; x, z).
\]

(4)

Nested inference also assumes that the ratio \( p_t(u; z; x)/q_t(u; z; x) \) can be evaluated. While in principle the nested inference algorithm can be produced by recoding the likelihood-free primitive in a high-level probabilistic programming language, this is by no means required, nor do we expect it to be the common case. In this paper, we focus on nested inference algorithms that use learned neural networks.

The accuracy of the likelihood estimate is determined by the accuracy of the nested inference algorithm. Specifically, for \( K = 1 \) the variance of the estimate is:

\[
\text{Var}_{u \sim q_t(\cdot; z)} \left[ \frac{p_t(u; z; x)}{q_t(u; x; z)} \right] \propto D_{\chi^2} \left( p_t(u; z; x) | q_t(u; x; z) \right),
\]

where \( D_{\chi^2} \) denotes the chi-square divergence [Nielsen and Nock, 2014], and where \( p_t(u; z; x) \) and \( q_t(u; x; z) \) on the right-hand side represent density functions over \( u \), not specific density values. Similarly, we can view \( \log(p_t(u; z; x)/q_t(u; x; z)) \) for \( u \sim q_t(\cdot; x, z) \) as a (biased) estimator of \( \log p_t(z; x) \), where the bias is:

\[
E_{u \sim q_t(\cdot; z, x)} \left[ \log \frac{p_t(u; z; x)}{q_t(u; x; z)} \right] - \log p_t(z; x) = -D_{KL}(q_t(u; x, z) || p_t(u; z; x)),
\]

where \( D_{KL} \) denotes the Kullback-Leibler (KL) divergence [Kullback and Leibler, 1951].

3.2.1 Nested Inference Metropolis-Hastings

Algorithm 3 describes a Nested Inference MH transition in which a custom proposal is made to a likelihood-free random choice \( i \) that uses estimated likelihoods produced using a nested inference algorithm. It assumes that all children of \( i \) also have nested inference algorithms themselves. Heterogeneous configurations are also possible.

**Algorithm 3** Single-site Nested Inference Metropolis-Hastings transition

| Require: |
| --- |
| Prob. program with dep. graph \( G = (I, E) \), Proposed-to-random choice \( i \), Custom proposal density \( m(z_i' ; z) \), Previous values \( z \) for all random choices, Previous likelihood estimates \( \ell \) for all choices |

1. \( z_i' \sim m(z_i' ; z) \) \( \triangleright \) Propose a new value for choice \( i \)
2. \( z_i' \leftarrow z_i \setminus \{i\} \) \( \triangleright \) No change to other choices
3. for \( k \leftarrow 1 \) to \( K \) do
   4. \( u_k, z_i' \sim q_t(\cdot; x_i, z_i') \) \( \triangleright \) Choice \( i \) nested inference
   5. \( \ell \leftarrow \frac{1}{K} \sum_{k=1}^{K} \frac{p_t(u_k; z_i' ; x_i)}{q_t(u_k; x_i, z_i')} \) \( \triangleright \) Estimate \( p_t(z_i' ; x_i) \)
6. for \( j \in c_G(i) \) do
5. for \( k \leftarrow 1 \) to \( K \) do
   8. \( u_k, z_j' \sim q_t(\cdot; x_j', z_j) \) \( \triangleright \) Choice \( j \) nested inference
   9. \( \ell \leftarrow \frac{1}{K} \sum_{k=1}^{K} \frac{p_t(u_k; z_j' ; x_j)}{q_t(u_k; x_j, z_j')} \) \( \triangleright \) Estimate \( p_t(z_j' ; x_j) \)
10. \( \alpha \leftarrow \prod_{j \in c_G(i) \cup c_G(i)} \left( \frac{\ell_j}{\ell_j'} \right)^{ \frac{m_t(z_i' ; z)}{m_t(z_i ; z)} } \)
11. \( s \sim \text{Uniform}(0, 1) \)
12. if \( s \leq \alpha \) then
   13. \( z_i \leftarrow z_i' \) \( \triangleright \) Accept
   14. for \( j \in \{i\} \cup c_G(i) \) do
   15. \( \ell_j \leftarrow \ell_j' \) \( \triangleright \) Update density estimates

Although this transition uses Monte Carlo estimates of likelihoods in the acceptance ratio, it is a standard Metropolis-Hastings transition on an extended state
It is possible to learn a nested inference algorithm \( q_t(u; x, z) \) that approximates \( p_t(u|z; x) \). The idea of learned inference for probabilistic generative models goes back at least to \cite{Morris2001} and has also been used in \cite{Stuhlmüller2013} and \cite{Kingma2013}. We apply this idea to nested inference as follows. Let \( q_t, \theta(u; x, z) \) denote a nested inference algorithm that is parameterized by \( \theta \) — for example, \( \theta \) might be the weights of a neural network used as part of the inference algorithm. We establish a training distribution \( d_t(x) \) over the arguments to the primitive \( t \), and approximately solve the following optimization problem:

\[
\min_\theta \left\{ \mathbb{E}_{x \sim d_t(x)} \left[ D_{KL}(p_t(u|z; x)||q_t, \theta(u; x, z)) \right] \right\}
\]

\[
= \min_\theta \left\{ \mathbb{E}_{x \sim d_t(x)} \frac{p_t(u|z; x)}{q_t, \theta(u; x, z)} \left[ \log \frac{p_t(u|z; x)}{q_t, \theta(u; x, z)} \right] \right\}
\]

The goal is for \( q_t, \theta(u; x, z) \) to approximate \( p_t(u|z; x) \) well (i.e., have small KL divergence) for typical input arguments \( x \sim d_t(\cdot) \). We approximate this objective function by drawing \( M \) independent sets of input arguments \( x^{(i)} \) from the training distribution, and running a traced execution of the likelihood-free random choice (e.g. planner) on each set of arguments, recording the trace \( u^{(i)} \) and output \( z^{(i)} \):

\[
x^{(i)} \sim d_t(\cdot) \quad \text{Sample planner arguments}
\]

\[
u^{(i)}, z^{(i)} \sim p_t(\cdot|; x^{(i)}) \quad \text{Run likelihood-free planner, record trace } u^{(i)}, 	ext{ output } z^{(i)}
\]

We use the resulting dataset \( D = \left\{ (x^{(i)}, z^{(i)}, u^{(i)}) : i = 1 \ldots M \right\} \) to define an approximate objective function \( J_D(\theta) \) that is an unbiased estimate of the original objective function:

\[
J_D(\theta) := \frac{1}{M} \sum_{i=1}^{M} \log \frac{p_t(u^{(i)}|z^{(i)}, x^{(i)})}{q_t, \theta(u^{(i)}; x^{(i)}, z^{(i)})}
\]

\[
= C - \frac{1}{M} \sum_{i=1}^{M} \log q_t, \theta(u^{(i)}; x^{(i)}, z^{(i)})
\]

where \( C \) does not depend on \( \theta \). Note that minimizing \( J_D(\theta) \) over \( \theta \) is equivalent to maximizing the log-likelihood of the data \( D \). Because we use forward simulations to produce \( u^{(i)}, z^{(i)} \) jointly from \( p_t(\cdot|; x^{(i)}) \), we have one exact conditional sample \( u^{(i)}|z^{(i)} \sim p_t(\cdot|z^{(i)}; x^{(i)}) \) for each training example.\footnote{This training regime cannot be applied to a true black-box path planner, since a recording of its internal randomness is now necessary. However, such recordings can be produced from a straightforwardly instrumented version of the algorithm. The likelihood estimator for the planner can still be treated as a black-box by the Nested Inference MH transition.}

3.2.2 Learning a nested inference algorithm

It is possible to learn a nested inference algorithm \( q_t(u; x, z) \) of the proposed-to-random choice \( i \), \( K \) traces \( u_{i,k} \) of the proposed-to-random choice, and \( K \) traces \( u_{i,k} \) of each child \( j \) of the proposed-to-random choice. The target density on the extended space is:

\[
p(z_i|z_{\setminus\{i\}}) \prod_{j \in \{i\} \cup C_G(i)} \frac{1}{K} \sum_{k=1}^{K} p_j(u_{i,k}|z_i; x_j) \prod_{r \in \{i\} \cup C_G(i)} q_{r,j}(u_{j,r}; x_j, z_j)
\]

The proposal density on the extended space is:

\[
m(z_i^2; z) \prod_{j \in \{i\} \cup C_G(i)} q_{r,j}(u_{j,r}; x_j, z_j)
\]

The values \( z_j \) of other random choices \( j \not\in \{i\} \cup C_G(i) \) are constant. See Appendix C for derivation. The marginal density of \( z_i \) in the extended target density is the local posterior \( p(z_i|z_{\setminus\{i\}}) \) for the result of random choice \( i \) given the values of all other random choices. Single-site Nested Inference MH transitions that propose to different random choices \( i \) but use the same database of nested-inference likelihood estimates \( \ell \) are compared to form Markov chains that converge to the posterior \( p(z_{\setminus\{i\}}|z_C) \).

Our use of unbiased likelihood estimates in place of the true likelihoods when computing the Metropolis-Hastings acceptance ratio in Algorithm 3 is closely related to pseudo-marginal MCMC \cite{Andrieu2009} and particle MCMC \cite{Andrieu2010}. Indeed, each single-site Nested Inference MH transition can be seen as a compositional variant of a ‘grouped-independence MH’ transition \cite{Beaumont2005} in which several pseudo-marginal likelihoods (one for each random choice \( j \in \{i\} \cup C_G(i) \)) are used in the same update. The database of nested-inference likelihood estimates \( \ell \) stores the ‘recycled’ pseudo-marginal likelihood estimates from previous transitions.

The convergence rate of a Markov chain based on Nested Inference MH transition operators depends on the accuracy of the nested inference algorithm and \( K \). In the limit of exact nested inference algorithm \( q_t(u; x, z) = p_t(u|z; x) \) the likelihood estimates are exact, and the algorithm is identical to standard Metropolis-Hastings. If the nested inference algorithm is very inaccurate, it may routinely propose traces \( u \) that are incompatible with the output \( z \) of the random choice, resulting in low acceptance rates. Better characterizing how the convergence rate depends on the accuracy of the nested inference algorithms and on \( K \) is an important area for future work.
4 EXAMPLE APPLICATIONS

We have implemented four example applications, designed to illustrate the flexibility of our framework:

1. Inferring the probable goal of a simulated drone. This example shows that small changes to the environment, such as including an additional doorway, can yield large changes in the inferred goals.

2. Inferring the probable goal of a simulated drone with a more complex planner. Specifically, we model the drone as following a multi-part path produced by a planner that first chooses a waypoint uniformly at random and then recursively solves the two path planning problems induced by the choice of waypoint. This example shows (a) applicability of the framework to more complex models of goal-directed behavior, and (b) that Nested Inference MH with a learned neural network can outperform Cascading Resimulation MH.

3. Inferring whether or not two people walking around tables in a room are headed for the same goal or different goals. This example demonstrates applicability to simple hierarchical models for goals and also demonstrates applicability to real-world (as opposed to synthetic) data.

4. Jointly inferring a simulated agent’s goals and its beliefs about an obstacle in the map whose location, size, and orientation is unknown to the probabilistic network’s stochastic outputs.

4.1 EXAMPLE 1: SENSITIVITY OF GOAL INFERENCE TO SMALL MAP CHANGES

Figure 2 shows a comparison of goal inference in two different maps given the same observations. The map for the scenario on the left has an enclosure with two openings, one on the top and one on the bottom, while the map for the scenario on the right has a single opening. In the map on the left, the inferred goal samples fall outside the enclosure, because if the drone intended to go inside the enclosure, it could have taken a much shorter path. In the map on the right, a significant fraction of goal samples fall inside the enclosure, as relatively efficient paths into the enclosure go through the partial trajectory that has been observed so far. Samples shown are the final states of 480 independent replicates of a Markov chain initialized from the prior, with 1000 Cascading Resimulation MH transitions (Algorithm 2) using the prior as the proposal. Planner parameters are $R = 10$, $N = 1000$, $\epsilon = 0.01$, $v = 0.5$, $J = 10000$, $S = 2000$.

4.2 EXAMPLE 2: HANDLING PATH PLANNERS WITH WAYPOINTS VIA NESTED INFECTION

Next, we used a model where the agent may choose a waypoint and separately plan a path to the waypoint and a path from the waypoint to the goal (AGENT-WAYPOINT-PATH, Algorithm 4). Unlike the simpler AGENT-PATH model, which typically samples from a small number of modes concentrated at efficient routes from the start to the goal, AGENT-WAYPOINT-PATH yields paths that are unpredictable without knowledge of the waypoint. Parameters $R$ and $N$ of PLAN-PATH are omitted for simplicity. We consider the same goal inference task as in Example 1 but with the alternative planner. Cascading Resimulation MH performs poorly on this task, because the prior is a poor proposal for the internal random choices of AGENT-WAYPOINT-PATH.

Algorithm 4 Pseudo-code for a likelihood-free primitive that models observed motion of an agent with known goal but optional unknown waypoint

Require:

\{ World map $M$; Start, goal $s, g \in [0,1]^2$
\{ Time points $t \in \mathbb{R}^+$

1: procedure AGENT-WAYPOINT-PATH($M, s, g, t$)
2: $g' \sim \text{Uniform}([0,1] \times [0,1])$ \hfill \triangleright \text{Pick waypoint}$
3: $w \sim \text{Bernoulli}(0.5)$ \hfill \triangleright \text{Use waypoint?}$
4: if $w$ then
5: $p_1 \sim \text{PLAN-PATH}(M, s, g')$ \hfill \triangleright \text{Start-waypoint}$
6: $p_2 \sim \text{PLAN-PATH}(M, g', g)$ \hfill \triangleright \text{Waypoint-goal}$
7: $p = (p_1, p_2)$ \hfill \triangleright \text{Concatenate paths}$
8: else
9: $p \sim \text{PLAN-PATH}(M, s, g)$ \hfill \triangleright \text{Start to goal}$
10: $\tilde{z} \leftarrow \text{WALK-PATH}(p, t)$ \hfill \triangleright \text{Locations at times $t$}$
11: $z \sim \text{ADD-NOISE}(\tilde{z})$ \hfill \triangleright \text{Add noise to locations}$
12: return $z$ \hfill \triangleright \text{Return noisy agent locations}$

Algorithm 5 shows a nested inference algorithm for AGENT-WAYPOINT-PATH that uses a neural network to propose the waypoint ($g'$) and whether the waypoint is used ($w$), given the goal and observations, and then executes the rest of the planner, conditioned on $w$ and $g'$. The network was trained on 10,000 runs of AGENT-WAYPOINT-PATH with random goal input $g \sim \text{Uniform}([0,1]^2)$ and fixed world map $M$ and start $s$. The nested inference algorithm splits the trace $u$ of AGENT-WAYPOINT-PATH into $u_1 = (w, g')$ and $u_2$ (the random choices made within executions of PLAN-PATH), so that $u = (u_1, u_2)$. The density of the nested inference algorithm is then $q_t(u; x, z) = q_0(u_1; x, z)p_t(u_2; x)$, and the density ratio $p_t(u, x, z)/q_t(u_1; x, z)$, which is used by Nested Inference MH when estimating the planner likelihoods, simplifies to $p_t(z; u; x)/q_0(u_1; x, z)$. To evaluate this ratio, we separately evaluate the density $p_t(z; u; x)$ of ADD-NOISE and the density $q_0(u_1; x, z)$ of the neural network’s stochastic outputs.
We compared three strategies for goal inference: Nested Inference MH using Algorithm 5 and $K = 1$, Cascading Resimulation MH, and Nested Inference MH using a “resimulation” nested inference algorithm ($q_h(u;x,z) = p_h(u;x,z)$) and $K = 2, 10$. Figure 3 shows that neural Nested Inference MH converges faster than the other strategies. Planner parameters were the same as in Example 1. All inference strategies were implemented using a custom Python inference library. Integration of Nested Inference MH with Venture is left for future work.

4.3 EXAMPLE 3: MODELING REAL-WORLD HUMAN MOTION

The Venture program of Figure 4(c) defines a model with two agents whose destinations may or may not be the same. The environment (world) and the start locations of the agents are known. The $is\_common\_goal$ flag determines whether the agents share the same goal destination. The paths of both agents are modeled using AGENT-PATH. The corresponding Bayesian network is shown in Figure 4(e). We collected video of two collaborators walking in a scene containing tables, for two conditions—one in which the they meet at a common location, and one where they diverge. For the common-goal condition we constructed short and extended sequences of observed locations (Figure 4a and (b)). We used Cascading Resimulation MH for inference, initialized from the prior, with a joint prior proposal over all latent variables. We ran 60 chains of 200 transitions each, and rendered the final states in Figures 4(a-b). The speed for each individual was set to their average speed along the observed path. The estimated probabilities of $is\_common\_goal=True$ for the short and extended sequences are 0.63 and 0.82 respectively. This trend qualitatively matches human judgments, shown in Figure 4(d) (the model was not calibrated to match human judgments). See Appendix B for additional results.

5 DISCUSSION

This paper introduced a class of probabilistic programs for formulating goal inference as approximate inference in probabilistic generative models of goal-directed behavior. The technical contributions are (i) a probabilistic programming formulation that makes complex goal and map priors easy to specify; (ii) the use of randomized path planning algorithms as the backbone of generative models; and (iii) the introduction of Monte Carlo techniques that can handle the intractable likelihoods of these path planners. The experiments showed that it is possible for short probabilistic programs to make meaningful inferences about goal-directed behavior.

From the standpoint of robotics, autonomous driving, or reconnaissance, the examples in this paper are quite pre-
Assume \( \text{world} = \ldots \);
Assume \( \text{start}_a = [0.92, 0.31] \);
Assume \( \text{start}_b = [0.45, 0.14] \);
Assume \( \text{random}_\text{goal} = () \rightarrow \{ \text{uniform}_\text{continuous}(0,1), \ \text{uniform}_\text{continuous}(0,1) \} \);
Assume \( \text{common}_\text{goal} = \text{random}_\text{goal}() \);
Assume \( \text{individual}_\text{goal}_a = \text{random}_\text{goal}() \);
Assume \( \text{individual}_\text{goal}_b = \text{random}_\text{goal}() \);
Assume \( \text{is}_\text{common}_\text{goal} = \text{flip}(0.5) \);
Assume \( (\text{goal}_a, \text{goal}_b) = \text{if} \ (\text{is}_\text{common}_\text{goal}) \{
(\text{common}_\text{goal}, \text{common}_\text{goal})
\} \ \text{else} \{
(\text{individual}_\text{goal}_a, \text{individual}_\text{goal}_b)
\} \);
Assume \( \text{times} = [1, 2, 3, 4, 5, 6, 7, 8, 9, 10] \);
Assume \( \text{path}_a = \text{agent}_\text{path}(\text{world}, \text{start}_a, \text{goal}_a, \text{times}, \text{speed}_a) \);
Assume \( \text{path}_b = \text{agent}_\text{path}(\text{world}, \text{start}_b, \text{goal}_b, \text{times}, \text{speed}_b) \);
Observe \( \text{add}_\text{noise}(\text{path}_a[0], \text{sigma}) = [0.88, 0.36] \);
Observe \( \text{add}_\text{noise}(\text{path}_a[1], \text{sigma}) = [0.82, 0.41] \);
Observe \( \text{add}_\text{noise}(\text{path}_b[0], \text{sigma}) = [0.42, 0.16] \);
Observe \( \text{add}_\text{noise}(\text{path}_b[1], \text{sigma}) = [0.34, 0.18] \);

Figure 4: Inferring whether or not two people are headed to the same destination. (c) shows a Venture model of two people. (e) shows a Bayesian network representation of the model with observed variables shaded. (a) and (b) show the map, two pairs of observed trajectories, and approximate posterior samples obtained from Cascading Resimulation MH. Each sample with \( \text{is}_\text{common}_\text{goal}=\text{True} \) is rendered as a single yellow circle. Each sample with \( \text{is}_\text{common}_\text{goal}=\text{False} \) is rendered as two separate magenta and blue circles. In (a) inference is uncertain about goal locations, and the estimated probability of \( \text{is}_\text{common}_\text{goal}=\text{True} \) is 0.63. In (b) inference gives a concentrated common goal region with estimated common-goal probability 0.82. Some probability mass is reserved for the people walking past one another to uncertain destinations. (d) shows judgments from 30 human responders of the likelihood over time that the individuals have different destinations, for the video sequence spanned by the frames in (a,b). The human judgments qualitatively agree with the automated inferences.

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A PLANNER DETAILS

We now describe details of the planner omitted from the main text, including the procedures SIMPLIFY-PATH, REFINE-PATH, and WALK-PATH, which are defined in Algorithm 6. Paths p are represented as sequences of points, with lines connecting the points. The path p begins with start s and ends with goal g. To be a valid path with respect to map M, no point in the path may lie within an obstacle (polygon) of M (i.e. M.IS-VALID(p_i)), and no line between two adjacent path points may intersect an obstacle of M (i.e. M.CLEAR-LINE(p_i, p_{i+1})).

Algorithm 6 Additional details of the AGENT-PATH model of goal-directed behavior.

Require: \{ World map M: Start, goal s, g \in [0,1]^2 \}
\{ Time points t \in \mathbb{R}^2 \}
\{ Refinement amount N; Restarts R \}
\{ Agent speed v \in \mathbb{R}^+ \}
1: procedure SIMPLIFY-PATH(M, p, s, g)
2: \hspace{1em} p'_i \leftarrow s \quad \triangleright \text{Initialize simplified path}
3: \hspace{1em} j \leftarrow 2
4: \hspace{1em} \text{for } i \leftarrow 2 \text{ to NUM-POINTS}(p) - 1 \text{ do}
5: \hspace{2em} \text{if not M.CLEAR-LINE}(p_{i-1}, p_{i+1}) \text{ then}
6: \hspace{3em} p'_i \leftarrow p_i \quad \triangleright \text{Point } p_i \text{ is needed, keep it}
7: \hspace{3em} j \leftarrow j + 1
8: \hspace{2em} \text{else}
9: \hspace{3em} \text{pass} \quad \triangleright \text{Point } p_i \text{ is not needed, skip it}
10: \hspace{1em} \text{return } p'_i
11: procedure REFINE-PATH(M, s, g, p)
12: \hspace{1em} \text{for } i \leftarrow 1 \text{ to } N \text{ do}
13: \hspace{2em} d \leftarrow \text{PATH-LENGTH}(p, s, g)
14: \hspace{2em} l \leftarrow 1 \text{ to } L \text{ do} \quad \triangleright \text{Iterate over } L \text{ path dims.}
15: \hspace{3em} \epsilon \sim \mathcal{N}(0, \sigma^2)
16: \hspace{3em} p' \leftarrow p + \epsilon \cdot e_l \quad \triangleright \text{Change path dim. } l
17: \hspace{3em} d' \leftarrow \text{PATH-LENGTH}(p', s, g)
18: \hspace{2em} \text{if } d' < d \wedge M.CLEAR-PATH(p', s, g) \text{ then}
19: \hspace{3em} \text{return } (d, p) \leftarrow (p', d') \quad \triangleright \text{Accept}
20: \hspace{1em} \text{return } p'
21: procedure WALK-TO(p, t, v)
22: \hspace{1em} d \leftarrow 0.0 \quad \triangleright \text{Path dist. from } s \text{ traveled so far}
23: \hspace{1em} d* \leftarrow tv \quad \triangleright \text{Desired path distance from } s
24: \hspace{1em} \delta \leftarrow 1 \text{ to NUM-POINTS}(p) - 1 \text{ do}
25: \hspace{2em} \text{Dist. to next point}
26: \hspace{3em} \delta \leftarrow ||p_j - p_{j+1}||_2
27: \hspace{2em} \text{if } d + d' > d* \text{ then}
28: \hspace{3em} e \leftarrow d' - d
29: \hspace{3em} \text{return } d_j \leftarrow (1 - \frac{e}{d*}) p_j + (1 - \frac{e}{d*}) p_{j+1}
30: \hspace{1em} \text{return } g \quad \triangleright \text{Once reached goal, stay forever}
31: procedure WALK-PATH(p, t, v)
32: \hspace{1em} \text{for } i \leftarrow 1 \text{ to } T \text{ do}
33: \hspace{2em} \text{Update position based on } t_i, v_i
34: \hspace{1em} \text{return } z

B ADDITIONAL EXPERIMENTS

B.1 JOINTLY INFERRING THE BELIEF AND GOAL OF AN AGENT

The Venture program of Figure 3(a) defines a model in which the belief of an agent about its environment, upon which the agent’s motion plan depends, is uncertain. The environment contains two, static objects (known_objects): a tree and a central divider wall that divides the [0,1] x [0,1] square into a left and right side. There are passageways between the left and right side that go above and below the divider. However, the agent has knowledge of (or belief in) an additional obstacle wall (obstacle), and the agent plans their path to the destination (goal) taking this additional obstacle into account. Figure 4(a) also shows a Bayesian network representation of this model. We seek to infer both the agent’s goal and the agent’s beliefs about the location, orientation, and size of the obstacle.

We used Cascading Resimulation Metropolis-Hastings (Algorithm 2) with a single repeated transition operator based on an independent joint proposal to goal (Uniform([0,1]^2)) and to the unknown parameters of obstacle (start post location, orientation, and length), proposed from the prior. We initialized from the prior. Parameters of the planner AGENT-PATH were R = 10, N = 1000, \epsilon = 0.01, v = 0.5, J = 10000, S = 2000. We ran several independent Markov chains of 1000 iterations each, on a synthetic dataset in which the agent takes a path from the right to the left of the map by going below the divider. The final state of four such chains are visualized in Figure 6(b). For this dataset, the goal destination of the agent is revealed with certainty because the agent reaches and stops in the upper left corner. The obstacle inferences indicate that agent believes the upper route to its goal is blocked, because otherwise the agent would have taken the shorter, upper route, to its goal. However, the specific details of how the obstacle blocks the upper passageway remain uncertain.

B.2 GOAL INFERENCE IN A DRIVING SCENARIO

Figure 7 shows an application of the multi-agent common-goal model of Figure 4 to a driving scenario. We show 60 independent replicates of 3000 iterations of Cascading Resimulation Metropolis-Hastings each. The results illustrate that this model can be used with varied environments.
B.3 REAL-WORLD HUMAN MOTION, ALTERNATE SEQUENCE

We extended the experiment described in Section 4.3 and shown in Figure 4 by running Cascading Resimulation Metropolis-Hastings on an alternate sequence of observed person locations in which the individuals diverge to separate individual goal destination. The inferences, shown in Figure 5, confirm the expectations, with all samples indicating \( \text{is_common_goal} = \text{False} \). Samples were obtained from the final state of 120 independent Markov chains, with initialization from the prior, followed by 1200 iterations of Cascading Resimulation Metropolis-Hastings.

Figure 5: Inferring whether or not two people are headed to the same destination, as in Figure 4, but for a different sequence of observed locations. The final frame shows approximate posterior inference samples obtained from cascading resimulation Metropolis-Hastings. Inference gives low probability of a common goal for this sequence (there were no \( \text{is_common_goal} = \text{True} \) samples).

B.4 INFERENCE WITH WAYPOINT PLANNER

Figure 8 compares waypoints and paths proposed by Nested Inference MH with a neural nested inference algorithm, and Cascading Resimulation MH, when evaluating the MH acceptance ratio for a proposed goal \( g' \) in the center of the enclosure that is the ground truth goal. Because the neural nested inference algorithm generates reasonable proposed waypoints, the proposed paths have a high probability of being consistent with the observed data. Because Cascading Resimulation proposes the waypoint and path from the prior for each proposed goal \( g' \), the paths proposed are unlikely to be consistent with the observations, resulting in a high MH rejection rate, even when the proposed goal \( g' \) is the ground truth goal, as is the case here.

Figure 8: Proposed waypoints and paths produced by Nested Inference MH with a neural nested inference algorithm, and Cascading Resimulation MH, when evaluating the MH acceptance ratio for a proposed goal \( g' \) in the center of the enclosure that is the ground truth goal. Because the neural nested inference algorithm generates reasonable proposed waypoints, the proposed paths have a high probability of being consistent with the observed data. Because Cascading Resimulation proposes the waypoint and path from the prior for each proposed goal \( g' \), the paths proposed are unlikely to be consistent with the observations, resulting in a high MH rejection rate, even when the proposed goal \( g' \) is the ground truth goal, as is the case here.
 Venture code and Bayesian network

```
assume start = [0.9, 0.6];
assume goal ~ [
  uniform_continuous(0, 1),
  uniform_continuous(0, 1),
];
assume obstacle ~ [
  "wall",
  // wall start post location
  uniform_continuous(0, 1),
  uniform_continuous(0, 1),
  // wall orientation
  if (flip(0.5)) { "x" } else { "y" },
  // wall length
  uniform_continuous(0, 1)
];
assume known_objects = ..;
assume world = dict(
  ["xlim", [0, 1]],
  ["ylim", [0, 1]],
  ["objects", append(known_objects, obstacle)],
); assume times = linspace(0.0, 4.0, 20);
assume speed = 0.5;
assume path ~ agent_path(world, start, goal,
  times, speed);
assume sigma = 0.01;
observe add_noise(path[0], sigma) = [0.83, 0.52];
observe add_noise(path[1], sigma) = [0.76, 0.42]; ...
```

Figure 6: Inferring the belief of an agent about the location and shape of an obstacle in its environment, from observations of the agent’s motion. (a) shows a Venture model of the agent’s belief, goal, and resulting motion and a Bayesian network representation of the model with observed variables shaded. (b) shows approximate posterior samples of goal and obstacle obtained with Cascading Resimulation Metropolis-Hastings, for a data set in which the goal is disambiguated to lie in the upper left corner. The obstacle samples in (b) indicate that inference in the model concluded that the agent believes that there is an obstacle blocking the upper route to its goal. Otherwise, the agent would have taken the shorter, upper route.

Figure 7: A synthetic application of the common-goal inference problem from Figure 4 to a different scenario inspired by autonomous driving. Above shows approximate posterior inference samples obtained with independent runs of Cascading Resimulation MH. Each sample with is_common_goal = True is rendered as a single yellow sphere. Each sample with is_common_goal = False is rendered as two blue and red spheres. Left: Inference indicates an approximate 0.5 probability that the cars are both headed for the center of the map. Right: The red car has stopped, revealing its goal, and the blue car continued, indicating that the two cars do not share the same destination.
Figure 9: Additional comparisons of approximate posterior goal inferences using neural Nested Inference Metropolis-Hastings (NNI, top row) and Cascading Resimulation Metropolis-Hastings (CR, bottom row). The drone starts in the lower left corner (orange). Observations of the drone’s location are shown in white. Red dots are independent approximate posterior samples of the drone’s goal. The neural Nested Inference MH strategy converges to a qualitatively correct distribution within 100-300ms (indicating real-time performance), whereas Cascading Resimulation MH requires 10-30 seconds to produce similarly accurate inferences (also see Figure 3(d)).
The variance of the likelihood estimate with $K = 1$ is:

$$\begin{align*}
\text{Var}_{u \sim q_t(x; z)} & \left[ \frac{p_t(u; z; x)}{q_t(u; x, z)} \right] \\
& = p_t(x; z)^2 \cdot \int \left( \frac{p_t(u; z; x)}{q_t(u; x, z)} - 1 \right)^2 q(u; x, z)du \\
& \propto \int \left( \frac{p_t(u; z; x)}{q_t(u; x, z)} - 1 \right)^2 q(u; x, z)du \\
& = D_{KL}(p_t(u; z; x) || q_t(u; x, z))
\end{align*}$$

The bias of the log likelihood estimate with $K = 1$ is:

$$\begin{align*}
E_{u \sim q_t(x; z)} \left[ \log \frac{p_t(u; z; x)}{q_t(u; x, z)} \right] - \log p_t(x; z) \\
& = E_{u \sim q_t(x; z)} \left[ \log \frac{p_t(u; z; x)}{q_t(u; x, z)} \right] \\
& = -D_{KL}(q_t(u; x, z) || p_t(u; z; x))
\end{align*}$$

MH acceptance ratio. However, Algorithm 3 is theoretically justified by recognizing that it is a standard joint MH transition on an extended state space that consists of $z_i$ (the value of the proposed-to random choice), $u_{i,k}$ for $k = 1 \ldots K$ (a set of $K$ traces for the proposed-to random choice), and $u_{j,k}$ for $j \in c_G(i)$ and $k = 1 \ldots K$ (a set of $K$ traces for each of the children of the proposed-to random choice). The extended target density is:

$$\begin{align*}
p(z_i|z_{\setminus\{i\}}) \prod_{j \in \{i\} \cup c_G(i)} \frac{1}{K} \sum_{k=1}^{K} p_{t_j}(u_{j,k}; z_j; x_j) \prod_{r \neq k} q_{t_j}(u_{r,k}; x_j, z_j)
\end{align*}$$

Note that the marginal target density of $z_i$ is the original target density $p(z_i|z_{\setminus\{i\}})$, which is proportional to $p_{t_i}(z_i; x_i) \prod_{j \in c_G(i)} p_{t_j}(z_j; x_j)$. Substituting $p_{t_i}(z_i; x_i) \prod_{j \in c_G(i)} p_{t_j}(z_j; x_j)$ for $p(z_i|z_{\setminus\{i\}})$ in the extended target density expression and simplifying gives the following unnormalized extended target density:

$$\begin{align*}
\prod_{j \in \{i\} \cup c_G(i)} \frac{1}{K} \sum_{k=1}^{K} p_{t_j}(u_{j,k}; z_j; x_j) \prod_{r \neq k} q_{t_j}(u_{r,k}; x_j, z_j)
\end{align*}$$

The extended proposal density is:

$$\begin{align*}
m(z_i'; z) \prod_{j \in \{i\} \cup c_G(i)} \prod_{k=1}^{K} q_{t_j}(u_{j,k}; x_j, z_j)
\end{align*}$$

The ratio of the unnormalized extended target density over the extended proposal density, for proposed values $z_i'$ and $u_{j,k}'$ for all $j \in \{i\} \cup c_G(i)$ and $k = 1 \ldots K$, with all other $z_{\setminus\{i\} \cup c_G(i)} = z_i \setminus \{i\} \cup c_G(i)$ is:

$$\begin{align*}
\frac{1}{m(z_i'; z)} \prod_{j \in \{i\} \cup c_G(i)} \frac{1}{K} \sum_{k=1}^{K} \frac{p_{t_j}(u_{j,k}'; z_j'; x_j')}{q_{t_j}(u_{j,k}; x_j, z_j)}
\end{align*}$$

Note that each factor $\frac{1}{K} \sum_{k=1}^{K} \frac{p_{t_j}(u_{j,k}'; z_j'; x_j')}{q_{t_j}(u_{j,k}; x_j, z_j)}$ within this ratio takes the form of a nested inference likelihood estimate. Composing the full MH acceptance ratio for the extended target and proposal densities gives the acceptance ratio used in Algorithm 3. Note that Algorithm 3 samples the joint state $z_i, u_{i,k}$ for $k = 1 \ldots K$, and $u_{j,k}$ for $j \in c_G(i)$ and $k = 1 \ldots K$ precisely according to the extended proposal density. Finally, note that the nested-inference likelihood estimates $\ell$ for accepted proposals are retained between updates in Algorithm 3. These estimate values serve as summaries of the previous iterates for the traces $u_{i,k}$ and $u_{j,k}$. Although the transition is an MH transition on the extended space including the traces, the previous estimates $\ell$ are sufficient for evaluating the extended MH acceptance ratio and retaining the previous trace iterates themselves is not necessary.