Continuous limits of linear and nonlinear quantum walks

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Abstract

In this paper, we consider the continuous limit of a nonlinear quantum walk (NLQW) that incorporates a linear quantum walk as a special case. In particular, we rigorously prove that the walker (solution) of the NLQW on a lattice $\delta\mathbb{Z}$ uniformly converges (in Sobolev space $H^s$) to the solution to a nonlinear Dirac equation (NLD) on a fixed time interval as $\delta \to 0$. Here, to compare the walker defined on $\delta\mathbb{Z}$ and the solution to the NLD defined on $\mathbb{R}$, we use Shannon interpolation.

1 Introduction

Quantum walks (QWs), or more precisely discrete time QWs, are quantum counterparts of classical random walks [1, 2, 19]. We can find the early prototypes in the context of Feynman path integral [14, 38] and Quantum Lattice Boltzmann methods [6, 28, 43]. QWs are now attracting diverse interest because of its connection to various fields of mathematics and physics such as orthogonal polynomials on the unit circle [10], quantum search algorithms [3], topological insulators [21, 22]. Further, since the early works studying QWs [6, 14, 19, 38, 43] are all more or less motivated by the discretization of Dirac equation, the relation between QWs and Dirac equation and other wave equation have repeatedly discussed by many authors from various viewpoints [4, 7, 9, 23, 30, 31, 32, 41, 42].

Nonlinear QWs (NLQWs), which are nonlinear versions of the usual (linear) QWs with the nonlinearity coming into the dynamics through the state-dependence of the quantum coin, was first proposed in [35] as an “optical Galton board [8]” with Kerr effect. From then, several models of NLQWs have been proposed motivated by simulating nonlinear Dirac equations (NLD) [23, 33] and studying the nonlinear effect to the topological insulators [15]. See also [24, 25, 26] for the study of scattering phenomena, weak limit theorem and soliton-like behavior for NLQWs. We note that for continuous time QWs, which are substantially described by discrete Schrödinger equations, nonlinear models are also attracting interest because it can speed up the quantum search [29].

In this paper, motivated by the above works, we study the connection between NLQWs and nonlinear Dirac equations (NLD). In particular, we show that the walker (or the solution) of NLQWs converges to the solution to the NLD. Very roughly, we show that for fixed $T > 0$,

$$
\|u(m\delta) - v_\delta(m\delta)\|_{L^2} \to 0 \text{ as } \delta \to 0, \quad \text{uniformly for } m \in \mathbb{Z}, \ 0 \leq m \leq T/\delta,
$$

(1.1)

where $u$ is the solution to the NLD, $v_\delta$ is the walker of NLQW on $\delta\mathbb{Z}$(for the precise statement, see Theorem 1.14 below). Thus, we see that the walker converges to the solution to the NLD uniformly in a fixed time interval. We emphasize that our model incorporates a linear quantum walk as a particular case and hence Theorem 1.14 says that the walker of the linear QW also converges to the solution to a Dirac equation.
This paper is organized as follows. In Subsections 1.1 and 1.2 we introduce NLQW and NLD respectively. In Subsection 1.3, we state our main result Theorem 1.14. In Section 2 we recall some facts of the Shannon interpolation. In Section 3, we prove Theorem 1.14.

1.1 Nonlinear quantum walks

We now introduce NLQWs, which are space-time discretized dynamics conserving $l^2$ norm (in the linear case, it is a unitary dynamics). Let $\delta > 0$ be a constant and set

$$\delta \mathbb{Z} := \{\delta n \mid n \in \mathbb{Z}\}.\quad (1.2)$$

We set

$$\mathcal{H}_{\delta} := l^2(\delta \mathbb{Z}, \mathbb{C}^2), \quad \langle u, v \rangle_{\mathcal{H}_{\delta}} := \delta \sum_{x \in \delta \mathbb{Z}} \langle u(x), v(x) \rangle_{\mathbb{C}^2} \text{ and } \|u\|_{\mathcal{H}_{\delta}}^2 := \langle u, u \rangle_{\mathcal{H}_{\delta}},\quad (1.3)$$

where $\langle \cdot, \cdot \rangle_{\mathbb{C}^2}$ is the inner-product of $\mathbb{C}^2$, i.e. for $u = (u_1, u_2) \in \mathbb{C}^2$ and $v = (v_1, v_2) \in \mathbb{C}^2$, $\langle u, v \rangle_{\mathbb{C}^2} = \sum_{j=1,2} u_j \overline{v}_j$. We also use the standard Pauli matrices

$$\sigma_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.\quad (1.4)$$

We define a shift operator $S_{\delta} : \mathcal{H}_{\delta} \to \mathcal{H}_{\delta}$ by

$$S_{\delta} := \begin{pmatrix} T_{+,\delta} & 0 \\ 0 & T_{-,\delta} \end{pmatrix}, \quad T_{\pm,\delta} f := f(\cdot \mp \delta).\quad (1.5)$$

It is clear that $S_{\delta}$ is a unitary operator on $\mathcal{H}_{\delta}$.

**Remark 1.1.** Recall (formally) $(e^{i0x} f) (x) = f(x + t)$. Thus, we can express $S_{\delta}$ as

$$S_{\delta} = \begin{pmatrix} e^{-i\partial_x} & 0 \\ 0 & e^{i\partial_x} \end{pmatrix} = e^{-i\delta \sigma_3 \partial_x} = e^{iA}, \quad A := i\sigma_3 \partial_x.\quad (1.6)$$

In the following, we will use the expression of (1.6) for $S_{\delta}$.

We next fix a smooth function $s = (s_0, s_1, s_2, s_3) : \mathbb{R} \to \mathbb{R}^4$ and set a linear coin operator $C_{\delta,s} : \mathcal{H}_{\delta} \to \mathcal{H}_{\delta}$ by

$$\langle C_{\delta,s} u \rangle (x) := e^{-i\delta s(x)A} u(x), \quad x \in \delta \mathbb{Z},$$

where $s(x) \cdot A = \sum_{i=0}^{3} s_i(x) \sigma_i$. For simplicity, we often suppress the explicit dependence on $s$ and write $C_{\delta}$ for $C_{\delta,s}$. Since for each $x \in \delta \mathbb{Z}$, $e^{-i\delta s(x)A}$ is a $2 \times 2$ unitary matrix, it is clear that $C_{\delta}$ is unitary operator on $\mathcal{H}_{\delta}$.

To define a nonlinear (or state-dependent) coin operator, we fix $\gamma$ to be a $2 \times 2$ Hermitian matrix and smooth function $g \in C^\infty(\mathbb{R}, \mathbb{R})$. We now define the nonlinear coin $N_{\delta,\gamma,g} : \mathcal{H}_{\delta} \to \mathcal{H}_{\delta}$ by

$$\langle N_{\delta,\gamma,g} u \rangle (x) := e^{-i\delta g(u(x), \gamma u(x))_{\mathbb{C}^2}} u(x), \quad x \in \delta \mathbb{Z}.\quad (1.7)$$

When there is no ambiguity we drop the dependence on $\gamma$ and $g$ and write just $N_{\delta}$ for $N_{\delta,\gamma,g}$. Notice that since $\langle N_{\delta} u(x), N_{\delta} u(x) \rangle_{\mathbb{C}^2} = \langle u(x), u(x) \rangle_{\mathbb{C}^2}$, we have

$$\|N_{\delta} u\|_{\mathcal{H}_{\delta}} = \|u\|_{\mathcal{H}_{\delta}}.\quad (1.8)$$
**Definition 1.2.** For \( u_0 \in \mathcal{H}_\delta \) and \( m \in \mathbb{Z}, m \geq 0 \), we define \( \mathcal{U}_\delta(m)u_0 \in \mathcal{H}_\delta \) by the recurrence relation

\[
\mathcal{U}_\delta(0)u_0 = u_0, \quad \mathcal{U}_\delta(m + 1)u_0 = S_\delta C_\delta N_\delta (\mathcal{U}_\delta(m)u_0). 
\] (1.9)

**Remark 1.3.** If \( g = 0 \), then \( \mathcal{U}_\delta \) is a linear unitary operator. However, if \( g \neq 0 \), \( \mathcal{U}_\delta \) becomes a nonlinear operator. This is the reason why we need to define \( \mathcal{U}_\delta(t)u_0 \) be the recurrence relation (1.9).

We give several examples of our model, which cover various QWs appeared in the literature.

**Example 1.4 (Free QWs).** When \( g = 0 \) and \( s \) do not depend on \( x \in \mathbb{Z} \), we will call the corresponding QW a free QW. This quantum walk is also called homogeneous since the coin operator is spatially homogeneous. A typical example is the case \( s = (0, -1, 0, 0) \), which appeared in the Feynman checkerboard model [14]. In particular, the coin operator in this case have the form

\[
C_{\delta,(0,-1,0,0)} = e^{i\delta s_1} = \begin{pmatrix} \cos \delta & i \sin \delta \\ i \sin \delta & \cos \delta \end{pmatrix}. 
\]

Another important example is the Hadamard walk [2], which is usually considered for the case \( \delta = 1 \), and the coin operator is given by the Hadamard matrix:

\[
C_{1,(2,0,1,-2)} = e^{-i\frac{\pi}{4}(2,0,1,-2)}\sigma = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \] (1.10)

**Example 1.5 (Linear QWs).** When \( g = 0 \), we will call the corresponding QW a linear QW. A typical example will be the case of a free QW. This quantum walk is also called homogeneous since the coin operator is spatially homogeneous. A typical example is the case \( s = (0, -1, 0, 0) \), which appeared in the Feynman checkerboard model [14]. In particular, the coin operator in this case have the form

\[
C_{1} = R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} 
\]

is spatially inhomogeneous and called a position-dependent coin. Such a model appears in the context of topological insulators [21] and the scattering theory for linear QWs are studied in [27, 34, 39, 40, 44].

**Example 1.6 (NLQWs, I).** When \( g \neq 0 \), we will call the corresponding QW a NLQW. NLQW first proposed in [35] is of the form

\[
\mathcal{U}_{\text{NPR}} := S_1 C_{1,(2,0,1,-2)} N_{1,\gamma_1,\gamma_2} N_{1,\gamma_2,\gamma_3}, 
\]

where the linear coin is given by the Hadamard matrix (1.10) and the two nonlinear coins are defined by \( g(s) = \lambda s \) \((\lambda \in \mathbb{R})\), \( \gamma_1 = \frac{1}{2}(\sigma_0 + \sigma_3) \) and \( \gamma_2 = \frac{1}{2}(\sigma_0 - \sigma_3) \). In particular, for \( u = (u_1, u_2) \), we have

\[
N_{1,\gamma_1,\gamma_2} u(x) = \begin{pmatrix} e^{i\lambda_1 |u_1(x)|^2} & 0 \\ 0 & 1 \end{pmatrix} u(x), \quad N_{1,\gamma_2,\gamma_3} u(x) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\lambda_2 |u_2(x)|^2} \end{pmatrix} u(x). 
\]

**Remark 1.7.** Even though our result in this paper is proved for the case of a single nonlinear coin, it can be extended to the case of two nonlinear coins as stated above without difficulty.

**Example 1.8 (NLQWs, II).** Another example of NLQW, which was proposed in [23] as a simulator of a nonlinear Dirac equation, is

\[
\mathcal{U}_{\text{LKN}} := S_\delta C_\delta N_{\delta,\sigma_1,\gamma_2}, 
\]
where $g(s) = s$, $C_1 = R(\theta)$ ($\theta \in \mathbb{R}$) and $j = 0$ or $3$. The case $j = 3$ is for simulating the Gross-Neveu model (scaler type interaction) and the nonlinear coin is of the form

$$N_{3, \sigma_3, g} u(x) = \begin{pmatrix} e^{-i\sigma_3 (|u_1|^2 - |u_2|^2)} & 0 \\ 0 & e^{i\sigma_3 (|u_1|^2 - |u_2|^2)} \end{pmatrix}.$$

The case $j = 0$ is for simulating the Thirring model (vector type interaction) and the nonlinear coin is of the form

$$N_{3, \sigma_3, g} u(x) = e^{-i\sigma_3 (|u_1|^2 + |u_2|^2)} u(x).$$

### 1.2 Nonlinear Dirac equations in 1+1 space-time

The Dirac equation on $\mathbb{R}$ is given by

$$i\partial_t u = -i\sigma_3 \partial_x u + s \cdot \sigma u + g(u, \gamma u) \gamma u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}, \quad u : \mathbb{R} \to \mathbb{C}^2.$$  (1.11)

Here, $s : \mathbb{R} \to \mathbb{R}^4$, $\gamma$ is a $2 \times 2$ Hermitian matrix and $g : \mathbb{R} \to \mathbb{R}$ corresponds to the ones given in the definition of the NLQW. Indeed, we will show that a solution to the NLQW converges to a solution to the NLD with the same $s$, $\gamma$ and $g$. We will denote the solution $u = u(t)$ to the Dirac equation (1.11) with the initial condition $u(0) = u_0$ by

$$u(t) = U_{\text{Dirac}}(t) u_0.$$

Note that if the Dirac equation (1.11) is nonlinear (i.e. if $g$ is not a constant), then so is $U_{\text{Dirac}}$. For a comprehensive introduction for the linear Dirac equation, see [45].

As the NLQW, we introduce several examples of the NLD.

**Example 1.9** (NLD: Gross-Neveu model and Thirring model). For the case $s = (0, 0, m, 0)$ ($m$ is a constant), $\gamma = \sigma_3$ (resp. $\gamma = \sigma_0$), $g(s) = s$, NLD (1.11) is called the Gross-Neveu model [18] (resp. Thirring model [46]). Further, a generalized Gross-Neveu model, which is the case of general $g \in C^\infty(\mathbb{R}, \mathbb{R})$, has been studied in [12].

**Example 1.10** (Nonlinear coupled mode equations). Let $s = (V, \kappa, 0, 0)$ and suppose that $V, \kappa \in C^\infty(\mathbb{R}, \mathbb{R})$ are bounded functions and $\gamma_1 = \frac{1}{2}(\sigma_0 + \sigma_3)$, $\gamma_2 = \frac{1}{2}(\sigma_0 - \sigma_3)$. Then the NLD becomes

$$i\partial_t u = -i\sigma_3 u + s \cdot \sigma u + \gamma_1 u - \gamma_2 u.$$

Such a model appears in the study of nonlinear propagation of light in an optical fiber waveguide [16, 17]. A similar model also appears in the study of Bose-Einstein condensation. In particular, in [37], the following model is studied:

$$i\partial_t u = -i\sigma_3 u + s \cdot \sigma u + \gamma_1 u - \gamma_2 u.$$

As we remarked in Example 1.6, our result in this paper can be generalized to the case of several nonlinear coins without difficulty.

We introduce some mathematical results about NLD. To do so, we prepare several notations.

We set $L^2 = L^2(\mathbb{R}, \mathbb{C}^2)$ and $H^s := H^s(\mathbb{R}, \mathbb{C}^2)$ ($s \in \mathbb{N}$), the $\mathbb{C}^2$-valued Sobolev spaces. The inner product of $L^2$ will be denoted by

$$\langle u, v \rangle := \int_{\mathbb{R}} \langle u(x), v(x) \rangle_{\mathbb{C}^2} \, dx.$$
We set \( \|u\|_{L^2} := \langle u, u \rangle^{1/2} \). The norm of \( H^s \) is defined by
\[
\|u\|^2_{H^s} := \sum_{j=0}^s \|\partial_x^j u\|_{L^2}^2.
\]
(1.12)
We further, define the inner product of \( H^s \) by
\[
\langle u, v \rangle_{H^s} := \sum_{j=0}^s \langle \partial_x^j u, \partial_x^j v \rangle.
\]
Since for \( s \geq 1 \), \( H^s \) becomes an algebra, one can show the following result by standard fixed point argument.

**Proposition 1.11.** Let \( s \geq 1 \) and suppose that \( \|s\|_{L^\infty} + \|s'\|_{H^{-1}} < \infty \). Let \( L > 0 \). Then there exists \( T > 0 \) such that a unique solution \( u(t) = U_{\text{Dirac}}(t)u_0 \in C([0,T], H^s) \) of NLD (1.11) exists for any \( u_0 \in H^s \) with \( \|u_0\|_{H^s} \leq L \). Further, for \( u_j \in H^s \) with \( \|u_j\|_{H^s} \leq L \) \( (j = 1, 2) \), \( U_{\text{Dirac}}(t)u_j \) satisfies
\[
\sup_{0 \leq t \leq T} \|U_{\text{Dirac}}(t)u_1 - U_{\text{Dirac}}(t)u_2\|_{H^s} \leq C\|u_1 - u_2\|_{H^s},
\]
where \( C \) is a constant depends only on \( L \).

Inspired by the above result for the solutions of nonlinear Dirac equations, we define the condition (Lip)_s as follows.

**Definition 1.12.** Let \( s \geq 1 \) and \( T, L > 0 \). We say that the pair \((T, L)\) satisfies condition (Lip)_s if there exists a constant \( C_{T,L} > 0 \) such that for \( u_j \in H^s \) with \( \|u_j\|_{H^s} \leq L \) \( (j = 1, 2) \), \( U_{\text{Dirac}}(\cdot)u_j \in C([0,T], H^s) \) and
\[
\sup_{0 \leq t \leq T} \|U_{\text{Dirac}}(t)u_1 - U_{\text{Dirac}}(t)u_2\|_{H^s} \leq C_{T,L}\|u_1 - u_2\|_{H^s}.
\]
(1.13)

**Remark 1.13.** By proposition 1.11, for any \( L > 0 \), we can always find \( T > 0 \) such that \((T, L)\) satisfies (Lip)_s. Moreover, if NLD (1.11) is globally wellposed, we can take \( T = \infty \) in proposition 1.11 for arbitrary \( L > 0 \) and hence (Lip)_s holds for arbitrary \((T, L) \in \mathbb{R}_+ \times \mathbb{R}_+ \). It is known that if the nonlinearity comes only from \( \sigma_0, \gamma_1 \) and \( \gamma_2 \) (as Example 1.10), then the NLD is globally wellposed (see [36] for more information).

### 1.3 Main results

To state our result precisely, we introduce some notation. When there exists a constant \( C > 0 \) such that \( a \leq Cb \), we write \( a \lesssim b \) or \( b \gtrsim a \). If the implicit constant \( C \) depends on some parameter \( \alpha \) \( (C = C_\alpha) \), then we write \( a \lesssim_\alpha b \). If \( a \lesssim b \) and \( b \gtrsim a \), we write \( a \sim b \).

For Banach spaces \( X, Y \), we set \( \mathcal{L}(X, Y) \) to be the Banach space of all bounded linear operators from \( X \) to \( Y \). We set \( \mathcal{L}(X) := \mathcal{L}(X, X) \).

We set \( \mathcal{H}_\delta := L^2(\mathbb{R}/2\pi\delta, \mathbb{C}^2) \) and define the inner product and norm by
\[
\langle u, v \rangle_{\mathcal{H}_\delta} := \int_{-\pi/\delta}^{\pi/\delta} \langle u(\xi), v(\xi) \rangle_{\mathbb{C}^2} d\xi, \quad \|u\|^2_{\mathcal{H}_\delta} := \langle u, u \rangle_{\mathcal{H}_\delta}.
\]
We define the discrete Fourier transform \( \mathcal{F}_\delta \in \mathcal{L}(\mathcal{H}_\delta, \hat{\mathcal{H}}_\delta) \) and its inverse \( \mathcal{F}_\delta^{-1} \in \mathcal{L}(\hat{\mathcal{H}}_\delta, \mathcal{H}_\delta) \) as

\[
\mathcal{F}_\delta u(\xi) := \frac{\delta}{\sqrt{2\pi}} \sum_{x \in \delta \mathbb{Z}} e^{-i\xi x} u(x), \quad \mathcal{F}_\delta^{-1} v(x) = \frac{1}{\sqrt{2\pi}} \int_{-\pi/\delta}^{\pi/\delta} e^{i\xi x} v(\xi) d\xi,
\]

where \( \mathcal{H}_\delta \) is defined in (1.3). Then, we have

\[
\langle \mathcal{F}_\delta u, \mathcal{F}_\delta v \rangle_{\hat{\mathcal{H}}_\delta} = \langle u, v \rangle_{\mathcal{H}_\delta}, \quad \langle \mathcal{F}_\delta^{-1} u, \mathcal{F}_\delta^{-1} v \rangle_{\hat{\mathcal{H}}_\delta} = \langle u, v \rangle_{\hat{\mathcal{H}}_\delta}.
\]

We denote the Fourier transform on \( L^2 \) by \( F \). In particular, we set

\[
F u(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi x} u(x) dx, \quad F^{-1} u(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\xi x} u(x) dx.
\]

We define \( H_\delta \subset L^2 \) by

\[
H_\delta := \{ u \in L^2 \mid \text{supp} Fu \subset [-\pi/\delta, \pi/\delta] \}.
\]

Since the Fourier transform has compact support, we have \( H_\delta \subset \cap_{r \geq 0} H^r \). We define the projection to \( H_\delta \) by

\[
j_\delta = F^{-1} \chi_{[-\pi/\delta, \pi/\delta]} F \in \mathcal{L}(L^2), \quad (1.14)
\]

where \( \chi_A \) is the characteristic function of \( A \). Obviously, we have

\[
j_\delta^2 = j_\delta, \quad \text{Ran} j_\delta = H_\delta, \quad \langle j_\delta u, v \rangle = \langle u, j_\delta v \rangle. \quad (1.15)
\]

We next define the Shannon interpolation (see, e.g. [5]), which is an isometry from \( H_\delta \) to \( H_\delta \subset L^2 \), by

\[
\mathcal{J}_\delta := F^{-1} \circ \hat{\mathcal{J}}_\delta \circ \mathcal{F}_\delta \in \mathcal{L}(H_\delta, L^2), \quad (1.16)
\]

where \( \hat{\mathcal{J}}_\delta \) is the natural identification between \( \hat{\mathcal{H}}_\delta \) and \( H_\delta \) given by

\[
\hat{\mathcal{J}}_\delta : \hat{\mathcal{H}}_\delta \to H_\delta \subset L^2, \quad \hat{\mathcal{J}}_\delta u(\xi) := \begin{cases} u(\xi), & \xi \in [-\pi/\delta, \pi/\delta] \\ 0, & |\xi| > \pi/\delta. \end{cases} \quad (1.17)
\]

We are now in a position to state our main result precisely.

**Theorem 1.14.** Let \( s \geq 1 \), \( T > 0 \) and \( L > 0 \). Assume \( \| s \|_{L^\infty} + \| s' \|_{H^s(\mathbb{R}, \mathbb{R}^4)} < \infty \) and \( (T, L) \) satisfies condition (Lip). Assume \( \| u_0 \|_{H^{s+1}} \leq L \). Then, there exists \( \delta_0 > 0 \) such that for \( \delta \in (0, \delta_0] \),

\[
\sup_{m \in \mathbb{Z}, 0 \leq \delta m \leq T} \| \mathcal{J}_\delta \circ \mathcal{U}_\delta(m) \circ \mathcal{J}_\delta^{-1} \circ j_\delta u_0 - U_{\text{Dirac}}(m\delta)u_0 \|_{H^r} \lesssim_{T, L, \delta}, \quad (1.18)
\]

where the implicit constant is independent of \( \delta \).

Since \( \| \cdot \|_{L^2} \leq \| \cdot \|_{H^s} \) for \( s \geq 0 \), we have the following continuous limit.

**Corollary 1.15.** Under the same assumptions as in Theorem 1.14, the walker of the NLQW converges to the solution of the NLD in the following sense:

\[
\lim_{\delta \to 0} \sup_{m \in \mathbb{Z}, 0 \leq \delta m \leq T} \| \mathcal{J}_\delta \circ \mathcal{U}_\delta(m) \circ \mathcal{J}_\delta^{-1} \circ j_\delta u_0 - U_{\text{Dirac}}(m\delta)u_0 \|_{L^2} = 0.
\]
Remark 1.16. Theorem 1.14 calls for some explanation. For the solution $u(t) = U_{\text{Dirac}}(t)u_0$ to the NLD in $\mathbb{R}$ with the initial condition $u(0) = u_0 \in H^{s+1}$ (on $\mathbb{R}$), we discretize the initial condition $u_0$ by $\mathcal{J}_\delta^{-1} \circ j_\delta$, evolve it by the NLQW by (1.9). After $m$ steps of the NLQW evolution, we can put it back to a function by the Shannon interpolation $\mathcal{I}_\delta^{-1}$ since it is unitary from $\mathcal{H}_\delta$ to $\mathcal{H}_\delta$ (see Lemma 2.1). Theorem 1.14 ensures that the resulting function $\mathcal{J}_\delta \circ U_\delta(m) \circ \mathcal{J}_\delta^{-1} \circ j_\delta u_0$ successively approximates the solution to the NLD. In this sense, we can say that the continuous limit of the NLQW is the NLD.

Although many works discuss the continuous limit of (linear and nonlinear) QWs, it seems that they just informally compare the equation of QWs and the Dirac equation by, for instance, expanding the equation or referring the Trotter-Kato formula. What is really needed is the estimate of the difference of the walker of the QW and solution to the Dirac equation as given in (1.18). The only result of such kind we are aware is [4], where the authors show (1.18) for $m = 1$. In this sense, our result is new even in the linear QWs.

From the viewpoint of numerical analysis, the NLQW gives a splitting method of the NLD. Splitting methods are now popular numerical schemes for approximating semilinear Hamiltonian partial differential equations such as nonlinear Schrödinger equations [13]. In this point of view, it may be interesting to investigate higher-order methods such as Strang splitting for the NLD. However, to make our paper simple, we will not investigate them.

For the proof of Theorem 1.14, we employ the energy method of Holden-Karlsen-Risebro-Tao [20], which was originally applied to the KdV equation.

2 Preliminary

In this section, we collect technical tools which we use in the proof of Theorem 1.14. Recall the definitions of $\langle \cdot, \cdot \rangle_{\mathcal{H}_\delta}$ and $\mathcal{J}_\delta$ given in (1.3) and (1.16) respectively.

**Lemma 2.1** ([5]). $\mathcal{J}_\delta : \mathcal{H}_\delta \rightarrow \mathcal{H}_\delta$ is unitary and

$$\mathcal{J}_\delta u(x) = u(x), \quad x \in \delta \mathbb{Z}. \quad (2.1)$$

**Proof.** By definition, $\mathcal{J}_\delta$ is an isometry and the image of $\mathcal{J}_\delta$ is $H_\delta$. Hence, $\mathcal{J}_\delta$ is unitary. (2.1) is shown by

$$\mathcal{J}_\delta u(x) = \frac{1}{\sqrt{2\pi}} \int_\mathbb{R} e^{i \xi \mathcal{J}_\delta} \mathcal{F}_\delta u(\xi) \, d\xi = \frac{1}{\sqrt{2\pi}} \int_{-\pi/\delta}^{\pi/\delta} e^{i \xi \mathcal{F}_\delta u(\xi)} \, d\xi = \mathcal{F}_\delta^{-1} \circ \mathcal{F}_\delta u(x) = u(x).$$

Recall $j_\delta$ given in (1.14).

**Lemma 2.2.** For $\sigma \geq 1$,

$$\| (1 - j_\delta) u \|_{H^\sigma} \lesssim \delta^\sigma \| u \|_{H^{\sigma+\sigma}}. \quad (2.2)$$

**Proof.** By the definition of $j_\delta$ and the norm of $H^\sigma$ given in (1.12), we have

$$\| (1 - j_\delta) u \|_{H^\sigma} = \| \langle \cdot \rangle^\sigma \chi_{\{|\cdot| > \pi/\delta\}} \mathcal{F} u \|_{L^2} \lesssim \delta^\sigma \| \langle \cdot \rangle^{\sigma+\sigma} \mathcal{F} u \|_{H^\sigma} = \delta^\sigma \| u \|_{H^{\sigma+\sigma}},$$

where we have used the fact that $(\delta \langle \xi \rangle)^\sigma \geq 1$ if $|\xi| > \pi/\delta$. Therefore, we have (2.2).
Recall $T_{-\delta}u(x) = u(x + \delta)$. We set

$$D_\delta := \delta^{-1}(T_{-\delta} - 1) \in \mathcal{L}(\mathcal{H}_\delta).$$

(2.3)

Formally, we can write $D_\delta = \frac{e^{\delta \partial_x} - 1}{\delta}$. Further, we set

$$D_\delta := \frac{e^{\delta \partial_x} - 1}{\delta} \in \mathcal{L}(L^2).$$

Remark 2.3. The operators $D_\delta$ and $D_\delta$ formally have the same definition. However, $D_\delta$ is defined on $\mathcal{H}_\delta = l^2(\delta \mathbb{Z}, \mathbb{C}^2)$ and $D_\delta$ is defined on $L^2(\mathbb{R}, \mathbb{C}^2)$.

Lemma 2.4. Let $s \geq 0$. For $u \in H^1$, we have

$$\|D_\delta u\|_{L^2} \leq \|\partial_x u\|_{L^2}.$$

Proof. By $e^{\delta \partial_x} = 1 + \delta \int_0^1 e^{\delta \partial_x} dt \partial_x$, we have

$$\|D_\delta u\|_{L^2} \leq \|\int_0^1 e^{\delta \partial_x} dt\|_{L^2} \|\partial_x u\|_{L^2}.$$

Since

$$\|\int_0^1 e^{\delta \partial_x} v dt\|_{L^2} = \|\int_0^1 v(\cdot + \delta t) dt\|_{L^2} \leq \|v\|_{L^2},$$

we have $\|\int_0^1 e^{\delta \partial_x} dt\|_{L^2} \leq 1$. Therefore, we have the conclusion.

Lemma 2.5. Let $s \geq 0$ and $u \in H^{s+1}$. Then, for $0 \leq j \leq s$, we have

$$\delta \sum_{x \in \delta \mathbb{Z}} \|D^j_x u(x)\|_{L^2}^2 \leq \|\partial_x^j u\|_{L^2}^2 + 2\|\partial_x^j u\|_{L^2} \|\partial_x^{j+1} u\|_{L^2}.$$

(2.4)

Remark 2.6. By Sobolev embedding, we have $u \in H^1(\mathbb{R}, \mathbb{C}^2) \hookrightarrow C^1(\mathbb{R}, \mathbb{C}^2)$. Therefore $u$ is defined pointwise and $D^j_x u(x)$ has a meaning.

Proof. We first prove the case $j = 0$. Fix $x \in \delta \mathbb{Z}$. Set $F_x(t) = \|u(x + t)\|_{L^2}^2$. Then, since $\partial_t F_x = 2\text{Re} \langle u(x + t), \partial_x u(x + t) \rangle_{\mathbb{C}^2} \in L^1(\mathbb{R})$, we have

$$F_x(t) = F_x(0) + \int_0^t \partial_t F_x(s) ds.$$

By the Fubini Theorem,

$$\delta \sum_{x \in \delta \mathbb{Z}} \|u(x)\|_{L^2}^2 = \sum_{x \in \delta \mathbb{Z}} \int_0^\delta F_x(0) dt = \sum_{x \in \delta \mathbb{Z}} \int_0^\delta F_x(t) dt - \sum_{x \in \delta \mathbb{Z}} \int_0^\delta \int_0^t \partial_t F_x(s) ds dt$$

$$= \|u\|_{L^2}^2 - 2\text{Re} \sum_{x \in \delta \mathbb{Z}} \int_0^\delta (\delta - s) \langle u(x + s), \partial_x u(x + s) \rangle_{\mathbb{C}^2} ds$$

$$\leq \|u\|_{L^2}^2 + 2\delta \int_R \|u(x)\|_{L^2} \|\partial_x u(x)\|_{L^2} dx.$$
Therefore, by Schwartz, we have the conclusion.

Next, for \( j \geq 1 \), assume that we have (2.4) for \( j - 1 \). Then, by Lemma 2.4, since \( \partial_x \) and \( D_\delta \) commute, we have

\[
\delta \sum_{x \in \mathbb{Z}^d} \|D_\delta^j u(x)\|_C^2 \leq \|\partial_x^{j-1} D_\delta u\|_C^2 + 2\delta \|\partial_x^{j-1} D_\delta u\|_L^2 \|\partial_x^j D_\delta u\|_L^2
\]

\[
\leq \|\partial_x^j u\|_C^2 + 2\delta \|\partial_x^j u\|_L^2 \|\partial_x^{j+1} u\|_L^2.
\]

Therefore, we have the conclusion. \( \square \)

**Lemma 2.7.** Let \( u \in H_\delta \). Then, we have

\[
\|\partial_x^j u\|_L^2 \sim \|D_\delta^j u\|_{L^2}.
\]

Here, the implicit constant is independent of \( \delta \).

**Proof.** First, for \( |\eta| \leq \pi \), we have

\[
| \int_0^1 e^{i\eta t} \, dt | \sim 1.
\]

We have

\[
\|D_\delta^j u\|_{L^2}^2 = \int_{-\pi/\delta}^{\pi/\delta} \left| \frac{e^{i\eta \xi} - 1}{\delta} \right|^{2j} |\hat{u}(\xi)|^2 \, d\xi = \int_{-\pi/\delta}^{\pi/\delta} |\xi|^{2j} |\hat{u}(\xi)|^2 \, d\xi
\]

\[
\sim \int_{-\pi/\delta}^{\pi/\delta} |\xi|^{2j} |\hat{u}(\xi)|^2 \, d\xi = \|\partial_x^j u\|_{L^2}^2.
\]

Therefore, we have the conclusion. \( \square \)

**Proposition 2.8.** Let \( s \geq 0, \sigma \geq 1 \). Let \( u \in H_\delta \) and \( v \in H^{s+\sigma} \) with \( u(x) = v(x) \) for all \( x \in \delta \mathbb{Z} \subset \mathbb{R} \). Then, we have

\[
\|u - v\|_{H^s} \lesssim \delta^\sigma \|v\|_{H^{s+\sigma}}. \tag{2.5}
\]

Here, the implicit constant is independent of \( u, v \) and \( \delta \).

**Remark 2.9.** Notice that the right hand side of (2.5) does not depend on \( u \).

**Proof.** First, by Lemma 2.2, we have

\[
\|u - v\|_{H^s} \lesssim \|u - j \delta v\|_{H^s} + \delta^\sigma \|v\|_{H^{s+\sigma}}. \tag{2.6}
\]

For \( 0 \leq j \leq s \), since \( u - j \delta v \in H_\delta \), by Lemma 2.7 we have

\[
\|\partial_x^j (u - j \delta v)\|_{L^2} \sim \|D_\delta^j (u - j \delta v)\|_{L^2}.
\]

By Lemmas 2.1, 2.2 and 2.5, we have

\[
\|D_\delta^j (u - j \delta v)\|_{L^2}^2 = \|\mathcal{F}^{-1} D_\delta^j (u - j \delta v)\|_{L^2}^2 = \delta \sum_{x \in \mathbb{Z}^d} \|D_\delta^j (u - j \delta v) (x)\|_{C^2}^2
\]

\[
= \delta \sum_{x \in \mathbb{Z}^d} \|D_\delta^j (j \delta - 1) v(x)\|_{C^2}^2 \lesssim \| (j \delta - 1) \partial_x^j v\|_{L^2}^2 + \delta \| (j \delta - 1) \partial_x^j v\|_{L^2} \| (j \delta - 1) \partial_x^{j+1} v\|_{L^2}
\]

\[
\lesssim \delta^{2s} \|v\|_{H^{j+\sigma}}^2.
\]

Therefore, we have the conclusion. \( \square \)
3 Proof of Theorem 1.14

In this section, we prove Theorem 1.14. In the following, as claimed in Theorem 1.14, we fix \( T, L > 0 \) and \( s \geq 1 \) and assume

\[
\|u_0\|_{H^{s+1}} \leq L, \quad \|s\|_{L^\infty(\mathbb{R}, \mathbb{R}^4)} + \|s'\|_{H^s(\mathbb{R}, \mathbb{R}^4)} < \infty
\]  

(3.1)

and

\[
(T, L) \text{ satisfies condition (Lip)\(_s\).} 
\]  

(3.2)

Since \( s \) is fixed, we will not denote the dependence of \( \|s\|_{L^\infty(\mathbb{R}, \mathbb{R}^4)} + \|s'\|_{H^s(\mathbb{R}, \mathbb{R}^4)} \) in the implicit constant in the inequalities below.

We start with decomposing \( \hat{\mathcal{H}}_\delta \circ \mathcal{U}_\delta(m) \circ \hat{\mathcal{H}}_\delta^{-1} \circ j_\delta u_0 - \mathcal{U}_{\text{Dirac}}(m\delta)u_0 \) \( \|_{H^s} \) as

\[
\|\hat{\mathcal{H}}_\delta \circ \mathcal{U}_\delta(m) \circ \hat{\mathcal{H}}_\delta^{-1} \circ j_\delta u_0 - \mathcal{U}_{\text{Dirac}}(m\delta)u_0\|_{H^s} \leq \|\hat{\mathcal{H}}_\delta \circ \mathcal{U}_\delta(m) \circ \hat{\mathcal{H}}_\delta^{-1} \circ j_\delta u_0 - \mathcal{U}_{\text{Dirac}}(m\delta)j_\delta u_0\|_{H^s} + \|\mathcal{U}_{\text{Dirac}}(m\delta)j_\delta u_0 - \mathcal{U}_{\text{Dirac}}(m\delta)u_0\|_{H^s}. 
\]  

(3.3)

For \( m\delta \leq T \), one can estimate the second term of the right hand side of (3.3) by the assumptions (3.1) and (3.2). Indeed, by Lemma 2.2,

\[
\|\mathcal{U}_{\text{Dirac}}(m\delta)j_\delta u_0 - \mathcal{U}_{\text{Dirac}}(m\delta)u_0\|_{H^s} \lesssim_{T, L} \|(j_\delta - 1)u_0\|_{H^s} \lesssim_{T, L, \delta} .
\]  

(3.4)

We further decompose the first term of (3.3) as

\[
\|\hat{\mathcal{H}}_\delta \circ \mathcal{U}_\delta(m) \circ \hat{\mathcal{H}}_\delta^{-1} \circ j_\delta u_0 - \mathcal{U}_{\text{Dirac}}(m\delta)j_\delta u_0\|_{H^s} \leq \|\mathcal{U}_\delta(0)u_0\|_{H^s} + \|\mathcal{U}_\delta(m+1)u_0\|_{H^s}.
\]  

(3.5)

where

\[
\mathcal{U}_\delta(0)u_0 = u_0, \quad \mathcal{U}_\delta(m+1)u_0 = \mathcal{S}_\delta \mathcal{C}_\delta \mathcal{N}_\delta (\mathcal{U}_\delta(m)u_0).
\]  

(3.6)

and

\[
\mathcal{S}_\delta := \begin{pmatrix} e^{-\delta \partial_x} & 0 \\ 0 & e^{\delta \partial_x} \end{pmatrix}, \quad \mathcal{C}_\delta := e^{-i\delta \langle \cdot \rangle \sigma} \quad \text{and} \quad \mathcal{N}_\delta := e^{-i\delta \langle (\cdot, \gamma) \rangle \gamma}. 
\]  

(3.7)

Remark 3.1. \( \mathcal{U}_\delta, \mathcal{S}_\delta, \mathcal{C}_\delta \) and \( \mathcal{N}_\delta \) are the continuous counterparts of \( \mathcal{U}_\delta, \mathcal{S}_\delta, \mathcal{C}_\delta \) and \( \mathcal{N}_\delta \) respectively. That is, \( \mathcal{U}_\delta, \mathcal{S}_\delta, \mathcal{C}_\delta \) and \( \mathcal{N}_\delta \) is defined on \( L^2(\mathbb{R}, \mathbb{C}^2) \) instead of \( \mathcal{H}_\delta \) with formally the same definition as \( \mathcal{U}_\delta, \mathcal{S}_\delta, \mathcal{C}_\delta \) and \( \mathcal{N}_\delta \).

We next bound the second term in (3.5) following Holden-Karlsen-Risebro-Tao [20]. To this end, we introduce \( v_\delta(t_1, t_2, t_3) \) as follows. Let

\[
\Omega_\delta = \cup_{m \in \mathbb{Z} \geq 0} \Omega_\delta^m, 
\]  

(3.8)

where \( \Omega_\delta^m := [m\delta, (m+1)\delta]^3 \). We define self-adjoint operators \( A \) and \( B \) as

\[
A = -i\sigma_3 \partial_x, \quad B = s \cdot \sigma.
\]  

We define a nonlinear operator \( G \) as

\[
G(v) = g((v, \gamma v)_{\mathbb{C}^2}) \gamma v, \quad v \in L^2(\mathbb{R}, \mathbb{C}^2).
\]
Let \( v_\delta(0, 0, 0) = j_\delta u_0 \) and define \( v_\delta(t_1, t_2, t_3) \) for \( (t_1, t_2, t_3) \in \Omega_\delta^n \) by
\[
\begin{align*}
\partial_t v_\delta &= G(v_\delta), & t_2 = t_3 = \delta m, \\
\partial_{t_1} v_\delta &= B v_\delta, & t_3 = \delta m, \\
\partial_{t_3} v_\delta &= A v_\delta.
\end{align*}
\] (3.9)

**Lemma 3.2.** Let \( v_\delta \) be the solution to (3.9) with \( v_\delta(0, 0, 0) = j_\delta u_0 \). Then, \( v_\delta \) correspond to \( U_\delta(\cdot)j_\delta u_0 \) at the diagonal lattice point. That is, we have
\[
v_\delta(\delta m, \delta m, \delta m) = U_\delta(m)j_\delta u_0.
\] (3.10)

**Proof.** Recall (3.6) and (3.7). We prove (3.10) by induction. Thus, we can assume (3.10). Our goal will be to show (3.10) with \( m \) replaced by \( m + 1 \). We first show
\[
v_\delta((m + 1)\delta, m\delta, m\delta) = N_\delta(U_\delta(m)j_\delta u_0).
\] (3.11)

By the first equation of (3.9),
\[
\frac{d}{dt_1} \langle v_\delta(t_1, \delta m, \delta m), \gamma v_\delta(t_1, \delta m, \delta m) \rangle = \langle -iG(v_\delta), \gamma v_\delta \rangle - \langle v_\delta, i\gamma G(v_\delta) \rangle
\]
\[
= \langle -i(\iota \gamma v_\delta)_{\mathbb{C}^2}, \gamma v_\delta \rangle - \langle v_\delta, i\gamma (\iota v_\delta)_{\mathbb{C}^2} \rangle \gamma^2 v_\delta = 0.
\]
Hence, \( \langle v_\delta(t_1, \delta m, \delta m), \gamma v_\delta(t_1, \delta m, \delta m) \rangle \) conserves. By (3.7) and the first equation of (3.9) again, we obtain (3.11). Similarly, from the second and third equations of (3.9), we can prove
\[
v_\delta((m + 1)\delta, (m + 1)\delta, (m + 1)\delta) = C_\delta N_\delta(U_\delta(m)j_\delta u_0),
\]
and
\[
v_\delta((m + 1)\delta, (m + 1)\delta, (m + 1)\delta) = S_\delta C_\delta N_\delta(U_\delta(m)j_\delta u_0).
\]
Therefore, we have the conclusion. \( \square \)

Setting \( v_\delta(t) := v_\delta(t, t, t) \), we show the following proposition.

**Proposition 3.3.** For sufficiently small \( \delta > 0 \), we have
\[
\sup_{t \in [0, T]} \| v_\delta(t) - u_\delta(t) \|_{H^s} \lesssim_{T, \delta} \delta,
\] (3.12)
where \( u_\delta(t) := U_{\text{Dirac}}(t)j_\delta u_0 \).

**Remark 3.4.** By Lemma 3.2 and Proposition 3.3, we obviously have
\[
\| U_\delta(m)u_0 - U_{\text{Dirac}}(m\delta)j_\delta u_0 \|_{H^s} \lesssim_{T, \delta} \delta, \quad \text{for } m \in \mathbb{N}, \ m\delta \leq T,
\] (3.13)
where the implicit constant are independent of \( m, \delta \). Thus, we obtain the bound for the second term of (3.5). It remains to obtain the bound for the first term of (3.5).

Before proving Proposition 3.3, we prepare several notations and lemmas. First, we set
\[
G'(v) = 2g(\langle v, \gamma v \rangle_{\mathbb{C}^2}) \text{Re} \langle \cdot, \gamma v \rangle_{\mathbb{C}^2} \gamma v + g(\langle v, \gamma v \rangle_{\mathbb{C}^2})\gamma v
\] (3.14)
and
\[
[X, G](v) := XG(v) - G'(v)Xv \text{ for } X = A, B.
\] (3.15)
Lemma 3.5. We have
\[ \| [A, B] v \|_{H^s} \lesssim \| v \|_{H^s}, \quad \| [A, G] v \|_{H^s} \lesssim \| v \|_{H^{s+1}}, \quad \| [B, G] v \|_{H^s} \lesssim \| v \|_{H^s}. \]

Proof. First, recall (3.1). Thus, by
\[ [A, B] v = -i(s' \cdot \sigma_3) v, \]
the bound for \( \| [A, B] v \|_{H^s} \) is obvious since for \( s \geq 1 \), \( H^s \) becomes an algebra. Next, we have
\[ [A, G] (v) = i g(\langle v, \gamma v \rangle C_2) [\gamma, \sigma_3] \partial_x v + 2g'(\langle v, \gamma v \rangle C_2) (\text{Re} \langle i \sigma_3 \partial_x v, \gamma v \rangle C_2 \gamma v - \langle v, \gamma v \rangle C_2 i \sigma_3 \gamma v). \]
Again, since \( H^s \) is an algebra, we can bound each term by using the elementary inequality
\[ \| hf \|_{H^s} \lesssim (\| h \|_{L^\infty} + \| \partial_x h \|_{H^{s-1}}) \| f \|_{H^s}. \]
By a similar manner, we have the estimate for \( \| [B, G] v \|_{H^s} \).

Lemma 3.6. Let \( T' > 0 \). Suppose there exists \( \delta_1 > 0 \) such that for \( \delta \in (0, \delta_1) \),
\[ \sup_{0 \leq t \leq T'} \| v_\delta(t) \|_{H^s} \leq M. \]
Then, there exists \( \delta_0 > 0 \) such that for \( \delta \in (0, \delta_0) \),
\[ \sup_{0 \leq t \leq T'} \| v_\delta(t) \|_{H^{s+1}} \lesssim T', L, M 1. \]
In particular, the implicit constant is independent of \( \delta \).

Proof. For \( 0 \leq \tau \leq \delta \), we set
\[ v_{\delta, 1}(\delta m + \tau) := v_\delta(\delta m + \tau, \delta m, \delta m). \quad (3.16) \]
Since we have
\[ v_\delta(\delta m + \tau_1, \delta m + \tau_2, \delta m + \delta m) = e^{-i \tau_3 A} v_\delta(\delta m + \tau_1, \delta m + \tau_2, \delta m), \]
by the 1st line of (3.9), we see
\[ \| v_\delta(\delta m + \tau_1, \delta m + \tau_2, \delta m) \|_{H^{s+1}} = \| v_\delta(\delta m + \tau_1, \delta m + \tau_2, \delta m) \|_{H^{s+1}}. \]
Similarly, by the 2nd line of (3.9), we have
\[ v_\delta(\delta m + \tau_1, \delta m + \tau_2, \delta m) = v_{\delta, 1}(\delta m + \tau_1) - i \int_0^{\tau_2} B v_\delta(\delta m + \tau_1, \delta m + \sigma, \delta m) \, d\sigma. \]
Thus,
\[ \sup_{0 \leq \tau_2 \leq \tau} \| v_\delta(\delta m + \tau, \delta m + \tau_2, \delta m) \|_{H^{s+1}} \leq \| v_{\delta, 1}(\delta m + \tau) \|_{H^{s+1}} + \bar{C} \tau \sup_{0 \leq \tau_2 \leq \tau} \| v_\delta(\delta m + \tau, \delta m + \tau_2, \delta m) \|_{H^{s+1}}. \]
Further, since we can express

$$v_{\delta,1}(\delta m + \tau)$$

and by (3.18), we have

Next, by (3.9), we have

Then, we have

where we have used assumption (3.1). Therefore, we conclude

$$\|v_{\delta}(\delta m + \tau, \delta m + \tau, \delta m)\|_{H^{s+1}} \leq (1 + C\tau)\|v_{\delta,1}(\delta m + \tau)\|_{H^{s+1}} \leq e^{C\tau}\|v_{\delta,1}(\delta m + \tau)\|_{H^{s+1}}. \quad (3.17)$$

Now, since $v_{\delta,1}(\delta m + \tau)$ is the solution to $i\partial_{\tau}v_{\delta,1} = G(v_{\delta,1})$ with $v_{\delta,1}(\delta m) = v_\delta(\delta m)$. Therefore, since we can express $v_{1,\delta} = e^{i\theta(v_{\delta}(\delta m), \gamma v_{\delta}(\delta m))} v_\delta(\delta)$, we have

$$\|v_{1,\delta}(\delta m + \tau)\|_{H^{s}} \lesssim_M 1. \quad (3.18)$$

Further, since

$$\left|\frac{d}{d\tau}\|v_{\delta,1}\|_{H^{s+1}}\right| \leq \sum_{j=0}^{s+1} \sum_{k=0}^{j} 2j C_k \left|\left\langle \partial_{\tau}^{j} g\left(\langle v_{\delta,1}, \gamma v_{\delta,1}\rangle \zeta_{2}\right)\right\rangle \right|,$$

and by (3.18), we have

$$\left|\frac{d}{d\tau}\|v_{\delta,1}\|_{H^{s+1}}\right| \lesssim_M \|v_{\delta,1}\|_{H^{s+1}}. \quad (3.19)$$

Therefore, by comparison theorem of ordinary differential equation (or Gronwall’s inequality), we have

$$\|v_{\delta,1}(\delta m + \tau)\|_{H^{s+1}} \leq e^{C_M\tau}\|v_{\delta}(\delta m)\|_{H^{s+1}},$$

where $C_M > 0$ is the implicit constant in (3.19). Combining (3.17) and (3.19), we have

$$\|v_{\delta}(\delta m + \tau)\|_{H^{s+1}} \leq e^{C_0\tau}\|v_{\delta}(\delta m)\|_{H^{s+1}},$$

with $C_0 = C + C_M$. Thus for $0 \leq t \leq T'$, we have

$$\|v_{\delta}(t)\|_{H^{s+1}} \leq e^{C_0T'}\|u_0\|_{H^{s+1}}.$$

This gives us the conclusion. \qed

**Lemma 3.7.** Let $T' > 0$ and suppose

$$\sup_{0 \leq t \leq T'} \|v_{\delta}(t)\|_{H^{s}} \leq M. \quad (3.20)$$

Then, we have

$$\sup_{0 \leq t \leq T'} \|v_{\delta}(t) - u_{\delta}(t)\|_{H^{s}} \lesssim_{T',L,M} \delta. \quad (3.21)$$

**Proof of Proposition 3.3.** Set $w_{\delta}(t) := v_{\delta}(t) - u_{\delta}(t)$. By (3.20) and Lemma 3.6, we have

$$\sup_{0 \leq t \leq T'} \|v_{\delta}(t)\|_{H^{s+1}} \lesssim_{T',L,M} 1.$$

Next, by (3.9), we have

$$i\partial_{\tau}w_{\delta} = i\partial_{t_1}v_{\delta} + i\partial_{t_2}v_{\delta} + i\partial_{t_3}v_{\delta} - i\partial_{t}u_{\delta} = i\partial_{t_1}v_{\delta} - G(v_{\delta}) + i\partial_{t_2}v_{\delta} - Bv_{\delta} + (Aw_{\delta} + Bw_{\delta} + G(v_{\delta}) - G(u_{\delta})).$$

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Then, setting $F_{12}(t_1, t_2, t_3) := i \partial_t v_3 - G(v_3) + i \partial_x v_3 - Bv_3$, we have

$$\frac{d}{dt} \|w_3(t)\|_{H^s}^2 = 2 \text{Re} \left( (iw_3, F_{12})_{H^s} + (iw_3, G(v_3) - G(u))_{H^s} + (iw_3, Bw_3)_{H^s} \right).$$

and thus

$$\frac{d}{dt} \|w_3(t)\|_{H^s} \leq \|F_{12}\|_{H^s} + \|G(v_3) - G(u)\|_{H^s} + \|Bw_3\|_{H^s}. \tag{3.22}$$

Recall we have $\|Bw_3\|_{H^s} \lesssim \|w_3\|_{H^s}$.

By (3.9) we have $F_{12}(t, \delta m, \delta m) = 0$. Further,

$$i \partial_t F_{12} = i \partial_t (Av_3) - G'(v_3)(Av_3) + i \partial_x (Av_3) - BA v_3 = AF_{12} + [A, G](v_3) + [A, B]v_3.$$

By lemma 3.5, we have

$$\|[A, G]v_3\|_{H^s} + \|[A, B](v_3)\|_{H^s} \lesssim_{T', L, M} \|v_3\|_{H^{s+1}} \lesssim_{T', L, M} 1.$$

Therefore, we have

$$\|F_{12}(t, t, t)\|_{H^s} \leq \|F_{12}(t, t, \delta m)\|_{H^s} + C \delta, \tag{3.23}$$

where $C = C_{T', L, M} > 0$ is a constant. Now, we set

$$F_1(t_1, t_2) := F_{12}(t_1, t_2, \delta m) = i \partial_t v_3 - G(v_3).$$

By (3.9), we have $F_1(t, \delta m) = 0$ and

$$i \partial_x F_1 = \partial_t (Bv_3) + i G'(v_3)(Bv_3) = BF_1 + [B, G]v_3.$$

The estimate of $\|F_1(t, t)\|_{H^s}$ need a little care since $B$ do not commute with the derivatives. Since, by Lemma 3.5, we have $\|[B, G]v_3\|_{H^s} \lesssim_{T', L, M} 1$, we first get the estimate

$$\|F_1(t, t)\|_{L^2} \lesssim_{T', L, M} \delta. \tag{3.24}$$

Suppose that for $s' \leq s$, we have the estimate

$$\|F_1(t, t)\|_{H^{s'-1}} \lesssim_{T', L, M} \delta. \tag{3.25}$$

Since

$$\text{Re} \left( \partial_x^j F_1, -i \partial_x^j (BF_1) \right) = \sum_{k=0}^{j-1} j C_k \text{Re} \left( \partial_x^j F_1, -i \partial_x^{-k} s \cdot \sigma \partial_x^k F_1 \right),$$

we have

$$\partial_t \|F_1(t, t)\|_{H^{s'}} \lesssim_{T', L, M} \delta + 1.$$

Therefore, we obtain (3.25) with $s' - 1$ replaced by $s'$ and thus by induction we have (3.25) with $s' - 1$ replaced by $s$. Therefore, substituting (3.24) into (3.23), we have

$$\|F_{12}\|_{H^s} \lesssim_{T', L, M} \delta. \tag{3.26}$$
Next, since
\[ G(v_\delta) - G(u_\delta) = \int_0^1 G'(u_\delta + \tau w_\delta) w_\delta \, d\tau. \]
we have
\[ \|G(v_\delta) - G(u_\delta)\|_{H^s} \lesssim_{T', L, M} \|w_\delta\|_{H^s}. \tag{3.27} \]
Therefore, by (3.22), (3.26) and (3.27), we have
\[ \frac{d}{dt}\|w_\delta\|_{H^s} \lesssim_{T', L, M} \delta + \|w_\delta\|_{H^s}. \tag{3.28} \]
This gives us the conclusion. Indeed, if we have such inequality, setting \( A(t) \) by
\[ A(0) = \|w_\delta(0)\|_{H^s} = 0, \quad A'(t) = \tilde{C}(A + \delta), \]
we have \( \|w(t)\|_{H^s} \leq A(t) \), where \( \tilde{C} = \tilde{C}_{T', L, M} > 0 \) is the implicit constant in (3.28). Moreover, since
\[ A(t) = e^{\tilde{C}t}(A(0) + \delta) - \delta, \]
we can conclude
\[ \sup_{0 \leq t \leq T'} \|w(t)\|_{H^s} \leq e^{\tilde{C}T'} \delta, \]
which is the desired estimate. \( \square \)

**Proof of Proposition 3.3.** By Lemma 3.7, it suffices to prove (3.20) for \( T' = T \). Let \( \tilde{C}_{T, L} \) be the constant given by the assumption that \((T, L)\) satisfies (Lip). Without loss of generality, we can assume \( \tilde{C}_{T, L} L > \max(1, L) \). Set \( M = M_{T, L} := 4\tilde{C}_{T, L} L \). Let \( C_{T, L, M} \) the implicit constant given in (3.21) in Lemma 3.7. Set \( \delta_{T, L} := \frac{1}{C_{T, L, M}} \). For \( \delta \in (0, \delta_{T, L}) \), we set
\[ T_\delta := \{ T' \in [0, T] \mid \sup_{0 \leq t \leq T'} \|v_\delta(t)\|_{H^s} < M \}. \]
Then, it suffices to show \( T \in T_\delta \).

First, \( 0 \in T_\delta \) so \( T_\delta \) is not empty. Further, since \( v_\delta \) is continuous in \( H^s \), we see that \( T_\delta \) is an open interval in \([0, T]\) (i.e. there exists an open interval \( \mathcal{O} \subset \mathbb{R} \) s.t. \( T_\delta = [0, T] \cap \mathcal{O} \)). Now, suppose \( T^* := \sup T_\delta < T \). Then, for any \( T' < T^* \), by Lemma 3.7, we have
\[
\sup_{0 \leq t \leq T'} \|v_\delta(t)\|_{H^s} \leq \sup_{0 \leq t \leq T'} \|u_\delta(t)\|_{H^s} + \sup_{0 \leq t \leq T'} \|v_\delta(t) - u_\delta(t)\|_{H^s} \\
\leq \tilde{C}_{T, L} L + C_{T, L, M} \delta \leq C_{T, L, M} + 1 < \frac{1}{2} M.
\]
Therefore, by continuity, we have
\[ \sup_{0 \leq t \leq T'} \|v_\delta(t)\|_{H^s} \leq \frac{1}{2} M, \]
and thus for sufficiently small \( \epsilon > 0 \), we have \( T^* + \epsilon \in T_\delta \), which contradicts the definition of \( T^* \). Therefore, we have \( T \in T_\delta \). \( \square \)
Proof of Theorem 1.14. As mentioned in Remark 3.4, we need only bound the first term of (3.5). We note that from Proposition 3.3, Lemma 3.6 and the assumption (3.2) we have
\[
\sup_{0 \leq m \leq \lfloor T/\delta \rfloor} \| U_\delta(m) j_\delta u_0 \|_{H^{s+1}} \lesssim T, L.
\] (3.29)

Further, notice that from the definition of $U_\delta$ and $U_\delta$ and (2.1), we have
\[
\hat{\mathcal{I}}_\delta \circ U_\delta(m) \circ \hat{\mathcal{I}}_\delta^{-1} \circ j_\delta u_0(x) = U_\delta(m) j_\delta u_0(x) \text{ for each } x \in \delta \mathbb{Z}.
\] (3.30)

Thus we can apply Proposition 2.8 and obtain
\[
\| \mathcal{I}_\delta \circ U_\delta(m) \circ \mathcal{I}_\delta^{-1} \circ j_\delta u_0 - U_\delta(m) j_\delta u_0 \|_{H^s} \lesssim \delta \| U_\delta(m) j_\delta u_0 \|_{H^{s+1}} \lesssim T, L, \delta.
\] (3.31)

This completes the proof of Theorem 1.14.

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