Vacuum Non Singular Black Hole Solutions in Tetrad Theory of Gravitation

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Starting from a spherically symmetric tetrad with three unknown functions of the radial coordinate, a general solution of Møller’s field equations in case of spherical symmetry non-singular black hole is derived. The previously obtained solutions are verified as special cases of the general solution. The general solution is characterized by an arbitrary function and two constants of integration. The general solution gives no more than the spherically symmetric nonsingular black hole solution. The energy content of the general solution depends on the asymptotic behavior of the arbitrary function, and is different from the standard one.
1. Introduction

Møller has shown that the problem of the energy-momentum complex has no solution in the framework of gravitational field theories based on Riemannian space \[1\]. In a series of papers, \([1, 2, 3]\) he was able to obtain a general expression for a satisfactory energy-momentum complex in the absolute parallelism space. The Lagrangian formulation of the theory was given by Pellegrini and Plebanski \([4]\). Quite independently Hayashi and Nakano \([5]\) formulated the tetrad theory of gravitation as a gauge theory of the space-time translation group. In these attempts, the admissible Lagrangians were limited by the assumption that the field equations has the Schwarzschild solution. Møller \([6]\) abandoning this assumption and look for a wider class of Lagrangians by constructing a new field theory. His aim was to get a theory free from singularity while retaining the main merits of general relativity as far as possible. Meyer \([7]\) showed that Møller’s theory is a special case of the Poincaré gauge theory \([8, 9]\). Sáez \([10]\) generalized Møller theory into a scalar tetradic theory of gravitation.

In an earlier paper \([11]\) the author used a spherically symmetric tetrad constructed by Robertson \([12]\) to derive two different spherically symmetric vacuum nonsingular black hole solutions of Møller’s field equations. He also calculated the energy content of these solutions \([11]\). It is the purpose of the present work to derive the general solution of Møller’s tetrad theory of gravitation assuming a specific form of the stress-energy momentum tensor as given by Dymnikova \([13]\), then calculated the energy content of this general solution.

In section 2 we briefly review Møller’s tetrad theory of gravitation. In section 3 the structure of the tetrad spaces having spherical symmetry as well as the previously obtained solutions of Møller’s field equations are reviewed. Assuming a specific form of the stress-energy momentum tensor, the general solution of Møller field equations is derived in section 4. The energy content of this general solution is derived in section 5. Section 6 is devoted to the main results and conclusions.

Computer algebra system Maple V Release 4 is used in some calculations.

2. Møller’s tetrad theory of gravitation

Møller’s constructed a gravitational theory based on Weitzenböck space-time. In this theory the field variables are the 16 tetrad components \(\lambda^\mu_i\), from which the metric is derived by

\[
g^{\mu\nu} \overset{\text{def}}{=} \lambda^\mu_i \lambda^\nu_i. \tag{1}
\]

We assume an imaginary values for the vector \(\lambda^\mu_0\) in order to have a Lorentz signature. We note that, associated with any tetrad field \(\lambda^\mu_i\) there is a metric field defined uniquely by (1), while a given metric \(g^{\mu\nu}\) does not determine the tetrad field completely; for any local Lorentz transformation of the tetrads \(\lambda^\mu_i\) leads to a new set of tetrads which also satisfy (1). The
Lagrangian $L$ is an invariant constructed from $\gamma_{\mu\nu\rho}$ and $g^{\mu\nu}$, where $\gamma_{\mu\nu\rho}$ is the contorsion tensor given by

$$\gamma_{\mu\nu\rho} \overset{\text{def}}{=} \lambda_{\mu} \lambda_{\nu\rho}, \quad (2)$$

where the semicolon denotes covariant differentiation with respect to Christoffel symbols. The most general Lagrangian density invariant under the parity operation is given by the form

$$\mathcal{L} \overset{\text{def}}{=} (-g)^{1/2} \left( \alpha_1 \Phi^\mu \Phi_\mu + \alpha_2 \gamma^{\mu\nu\rho} \gamma_{\mu\nu\rho} + \alpha_3 \gamma^{\mu\nu\rho} \gamma_{\rho\nu\mu} \right), \quad (3)$$

where

$$g \overset{\text{def}}{=} \text{det}(g_{\mu\nu}), \quad (4)$$

and $\Phi_\mu$ is the basic vector field defined by

$$\Phi_\mu \overset{\text{def}}{=} \gamma_\rho_{\mu\rho}. \quad (5)$$

Here $\alpha_1$, $\alpha_2$, and $\alpha_3$ are constants determined by Møller such that the theory coincides with general relativity in the weak fields:

$$\alpha_1 = -\frac{1}{\kappa}, \quad \alpha_2 = \frac{\lambda}{\kappa}, \quad \alpha_3 = \frac{1}{\kappa}(1 - 2\lambda), \quad (6)$$

where $\kappa$ is the Einstein constant and $\lambda$ is a free dimensionless parameter. The same choice of the parameters was also obtained by Hayashi and Nakano [5].

Møller applied the action principle to the Lagrangian density (3) and obtained the field equation in the form

$$G_{\mu\nu} + H_{\mu\nu} = -\kappa T_{\mu\nu}, \quad (7)$$

$$F_{\mu\nu} = 0, \quad (8)$$

where the Einstein tensor $G_{\mu\nu}$ is defined by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R. \quad (9)$$

Here $H_{\mu\nu}$ and $F_{\mu\nu}$ are given by

$$H_{\mu\nu} \overset{\text{def}}{=} \lambda \left[ \gamma_{\rho\sigma\mu} \gamma^{\rho\sigma\nu} + \gamma_{\rho\sigma\mu} \gamma^{\nu\rho\sigma} + \gamma_{\rho\sigma\nu} \gamma^{\rho\mu\sigma} + g_{\mu\nu} \left( \gamma^{\rho\sigma\lambda} \gamma_{\rho\sigma\lambda} - \frac{1}{2} \gamma_{\rho\sigma\lambda} \gamma^{\rho\sigma\lambda} \right) \right], \quad (10)$$

and

$$F_{\mu\nu} \overset{\text{def}}{=} \lambda \left[ \Phi_{\mu,\nu} - \Phi_{\nu,\mu} - \Phi_\rho \left( \gamma^\rho_{\mu\nu} - \gamma^\rho_{\nu\mu} \right) + \gamma_{\mu\nu} \gamma^{\rho}_{;\rho} \right], \quad (11)$$

and they are symmetric and skew symmetric tensors, respectively.

Møller assumed that the energy-momentum tensor of matter fields is symmetric. In the Hayashi-Nakano theory, however, the energy-momentum tensor of spin-1/2 fundamental

*Throughout this paper we use the relativistic units, $c = G = 1$ and $\kappa = 8\pi$. 

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particles has nonvanishing antisymmetric part arising from the effects due to intrinsic spin, and the right-hand side of (8) does not vanish when we take into account the possible effects of intrinsic spin.

It can be shown [14] that the tensors, $H_{\mu\nu}$ and $F_{\mu\nu}$, consist of only those terms which are linear or quadratic in the axial-vector part of the torsion tensor, $a_\mu$, defined by

$$a_\mu \equiv \frac{1}{3} \epsilon_{\mu\rho\sigma\tau} \gamma^{\rho\sigma}, \quad (12)$$

where $\epsilon_{\mu\rho\sigma}$ is defined by

$$\epsilon_{\mu\rho\sigma} \equiv (-g)^{1/2} \delta_{\mu\rho\sigma} \quad (13)$$

with $\delta_{\mu\rho\sigma}$ being completely antisymmetric and normalized as $\delta_{0123} = -1$. Therefore, both $H_{\mu\nu}$ and $F_{\mu\nu}$ vanish if the $a_\mu$ is vanishing. In other words, when the $a_\mu$ is found to vanish from the antisymmetric part of the field equations, (8), the symmetric part (7) coincides with the Einstein equation.

3. Spherically symmetric nonsingular black hole solutions

The structure of the Weintzenböck spaces with spherical symmetry has been studied by Robertson [12]. The tetrad space having three unknown functions of radial coordinate with spherical symmetry in spherical polar coordinates, can be written as

$$\begin{pmatrix} iA \\ iDr \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & B \sin \theta \cos \phi & \frac{B}{r} \cos \theta \cos \phi & \frac{B \sin \phi}{r \sin \theta} \\ 0 & B \sin \theta \sin \phi & \frac{B}{r} \cos \theta \sin \phi & \frac{B \cos \phi}{r \sin \theta} \\ 0 & B \cos \theta & -\frac{B}{r} \sin \theta & 0 \end{pmatrix}, \quad (14)$$

where the vector $\lambda^\mu_0$ has taken to be imaginary in order to preserve the Lorentz signature for the metric, i.e., the functions $A$ and $D$ have to be taken as imaginary.

Applying (14) to the field equations (7) and (8) we note that the two tensors $H_{\mu\nu}$ and $F_{\mu\nu}$ are vanishing identically regardless of the values of the functions $A$, $B$ and $D$. Thus Møller field equations reduce for the tetrad (14) to Einstein's equations (9). The corresponding field equations (7) and (8) have given rise to the following equations.
\[ \kappa T_{00} = \frac{1}{r A^2 B^4} \left\{ \left( 3D^2 + B'^2 \right) D - 2 \left( D B'' + B'D' \right) B \right\} r^3 B^2 D - \left\{ 2 \left( D B'' + B'D' \right) B - 5 D B'^2 \right\} r^5 D^3 - \left( 2 BB'' - 3 D'^2 - 3 B'^2 \right) r B^4 + \right. \\
\left. 2 \left( B D' - 4 D B' \right) r^4 BD + 2 \left( B D' - 6 D B' \right) r^2 B^3 D - 4 B^5 B' \right\}, \\
\kappa T_{01} = \frac{D}{A B^4} \left\{ 2 \left( D B'' + B'D' \right) B - 5 D B'^2 \right\} r^3 D + \left( 2 BB'' - 3 D'^2 - 3 B'^2 \right) r B^2 - \right. \\
\left. 2 \left( B D' - 4 D B' \right) r^2 BD + 4 B^3 B' \right\}, \\
\kappa T_{11} = \frac{1}{r A B^4} \left\{ \left( 3D^2 + B'^2 \right) A + 2 B A' B' \right\} r B^2 - \left\{ 2 \left( DB'' + B'D' \right) B - 5 D B'^2 \right\} r^3 A D + \right. \\
\left. 2 \left( B D' - 4 D B' \right) r^2 A B D - 2 A B^3 B' - 2 B^4 A' \right\}, \\
\kappa T_{22} = \frac{r}{A^2 B^4} \left\{ \left( D A'' + 3 A' D' \right) B - 3 D A' B' \right\} A B D + \left( 2 D B'' + 5 B'D' \right) B D - \right. \\
\left. \left( D D'' + D'^2 \right) B^2 - 5 D B'^2 \right\} A^2 - 2 B^2 D^2 A'^2 \right\} r^3 + \right. \\
\left. \left( B'^2 - 3 D^2 \right) A^2 - A B^2 A'' - B'' B A^2 + 2 B^2 A'^2 \right\} r B^2 - \right. \\
\left. 2 \left( 3 B D' - 4 D B' \right) A - 2 B D A' \right\} r^2 A B D + A^2 B^3 B' + A B^4 A' \right\], \\
T_{33} = \sin \theta^2 T_{22}. \\
(15) \\
\text{where } A' = \frac{dA}{dr}, B' = \frac{dB}{dr} \text{ and } D' = \frac{dD}{dr}.
\]

Now we are going to review some exact solutions to the partial differential equations (15). A first trivial flat space-time solution for the field equations (15) is obtained by taking

\[ A = 1, \quad B = 1, \quad D = 0. \]

(16)

A second non-trivial solution can be obtained by taking \( A = 1, B = 1, D \neq 0 \) and solving for \( D \), the result is

\[ A = 1, \quad B = 1 \quad D = \sqrt{\frac{2m}{r^3}} \left( 1 - e^{-r^3/r_1^3} \right). \]

(17)

A third non-trivial solution can be obtained by taking \( D = 0 \) and solve for \( A \) and \( B \). This case is studied by the author [11] where he obtained
\[ A = \frac{1}{\sqrt{1 - \frac{2m}{R} \left(1 - e^{-R^3/r_1^3}\right)}} \]
\[ B = \frac{1}{\sqrt{1 - \frac{2m}{R} \left(1 - e^{-R^3/r_1^3}\right)}}. \]  

where the stress-energy momentum tensor has the form

\[ T^0_0 = T^1_1 = \epsilon_0 e^{-R^3/r_1^3}, \]
\[ T^2_2 = T^3_3 = \epsilon_0 e^{-R^3/r_1^3} \left(1 - \frac{3R^3}{2r_1^3}\right), \]

where \( R \) and \( r_1 \) are defined as

\[ R = \frac{r}{B}, \]
\[ r_1^3 = r_g r_0^2, \]
\[ r_g = 2m, \]
\[ r_0^2 = \frac{3}{8\pi\epsilon_0}. \]

4. General Black Hole solution of Møller’s Field Equations

Mikhail and Wanas [15] constructed a generalized field theory based on the Weintzenböck space. Wanas [16] obtained a spherically symmetric solutions using the tetrad (14) in the case of +ve definite. Mazumber and Ray [17] completely integrated the field equations of Mikhail and Wanas for the tetrad (14) by a suitable change of variables. Mikhail et al. [18] obtained a general solution in Møller’s tetrad theory of gravitation for the tetrad (14) in the vacuum case. It is our purpose to find a general solution for the tetrad (14) when the stress-energy momentum tensor is not vanishing and has the form

\[ T^0_0 = T^1_1, \]
\[ T^2_2 = T^3_3, \]

where all the other mixed spatial components equal to zero [13]. Then the left hand side of the second equation of equations (15) is equal zero and we can find a solution of the unknown function \( D \) in terms of the unknown function \( B \) in the form

\[ D = \frac{1}{\left(1 - \frac{rB'}{B}\right)} \sqrt{\frac{k_1 B^3}{r^3} \left(1 - e^{-r^3/r_1^3}\right) + \frac{BB'}{r} \left(\frac{rB'}{B} - 2\right)}. \]
where \( k_1 \) is a constant of integration. From the first and third equations of (15) using (21) and (22), we get the unknown function \( A \) in the form

\[
A = \frac{k_2}{\left( 1 - \frac{rB'}{B} \right)}, \tag{23}
\]

with \( k_2 \) being another constant of integration. The general solution (22) and (23) satisfy the field equations (15) when the stress-energy momentum tensor has the form

\[
\begin{align*}
T_0^0 &= \frac{\epsilon_0 k_1 B^3 e^{-r^3/r_1^3}}{2m \left( 1 - \frac{rB'}{B} \right)}, \\
T_2^2 &= T_3^3 = \frac{\epsilon_0 k_1 B^3 e^{-r^3/r_1^3}}{2m \left( 1 - \frac{rB'}{B} \right)^3} \left[ \frac{1 - 3r^3}{2r_1^3} + \frac{3r^3}{2r_1^3} \left( \frac{rB'}{B} \right) + \frac{r^2 B''}{2B} - \left( \frac{rB'}{B} \right)^2 \right].
\end{align*} \tag{24}
\]

The line-element squared of (14) takes the form

\[
ds^2 = -\frac{(B^2 - D^2 r^2)}{A^2 B^2} dt^2 - \frac{2Dr}{AB^2} dr dt + \frac{1}{B^2} (dr^2 + r^2 d\Omega^2) \tag{25}\]

with \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 \). We assume \( B(r) \) to be nonvanishing so that the surface area of the sphere of a constant \( r \) be finite. We also assume that \( A(r) \) and \( B(r) \) satisfy the asymptotic condition, \( \lim_{r \to \infty} A(r) = \lim_{r \to \infty} B(r) = 1 \) and \( \lim_{r \to \infty} rB' = 0 \). Then, we can show from (22), (23) and (25) that

(1) \( k_2 = 1 \),

(2) \( B(r) > 0 \),

(3) \( \lim_{r \to \infty} rD(r) = 0 \), and

(4) if \( B - rB' \) vanishes at some point, then \( \left( 1 - \frac{B(r)k_1 \left( 1 - e^{(-r^3/r_1^3)} \right)}{r} \right) < 0 \) at that point.

Using the coordinate transformation

\[
dT = dt + \frac{ADr}{B^2 - D^2 r^2} dr, \tag{26}\]

we can eliminate the cross term of (25) to obtain

\[
ds^2 = -\eta_1 dT^2 + \frac{dr^2}{\eta_1 A^2 B^2} + \frac{r^2 d\Omega^2}{B^2} \tag{27}\]

with \( \eta_1 = (B^2 - D^2 r^2)/A^2 B^2 \). Taking the new radial coordinate \( R = r/B \), we finally get

\[
ds^2 = -\eta_1 dT^2 + \frac{dR^2}{\eta_1} + R^2 d\Omega^2, \tag{28}\]
where

$$\eta_1(R) = \left(1 - k_1 \left(1 - e^{(-R^3/r_1^3)}\right)\right).$$  \hfill (29)

Then, (29) coincides with the nonsingular black hole solution given before by Dymnikova with the mass, \(m = k_1/2\), and hence the general solution in the case of the spherically symmetric tetrad when the stress-energy momentum tensor is nonvanishing gives no more than the nonsingular black hole solution when \(1 - r B'/B\) has no zero and \(R\) is monotonically increasing function of \(r\). If \(1 - r B'/B\) has zeroes, the line-element (25) is singular at these zeroes which lie inside the event horizon as is seen from the property (4) mentioned above. We shall study in the future whether this singularity at zero-points of \(1 - r B'/B\) is physically acceptable or not.

After using the above transformations, the tetrad (14) can be put in the form

$$\left(\lambda^\mu_i\right) = \begin{pmatrix}
\frac{iA}{1 - D^2 R^2} & iD R (1 - R B') & 0 & 0 \\
\frac{AD R \sin \theta \cos \phi}{1 - D^2 R^2} & (1 - R B') \sin \theta \cos \phi & \frac{\cos \theta \cos \phi}{R} & -\frac{\sin \phi}{R \sin \theta} \\
\frac{AD R \sin \theta \sin \phi}{1 - D^2 R^2} & (1 - R B') \sin \theta \sin \phi & \frac{\cos \theta \sin \phi}{R} & \frac{\cos \phi}{R \sin \theta} \\
\frac{AD R \cos \theta}{1 - D^2 R^2} & (1 - R B') \cos \theta & \frac{-\sin \theta}{R} & 0
\end{pmatrix}.$$  \hfill (30)

Here \(A\) and \(D\) are given in terms of the unknown function \(B(R)\) as

$$A(R) = \frac{1}{1 - R B'},$$

$$D(R) = \frac{1}{1 - R B'} \sqrt{\frac{2m}{R^3} \left(1 - e^{(-R^3/r_1^3)}\right) + \frac{B'}{R} (R B' - 2)},$$  \hfill (31)

where \(B' = \frac{dB(R)}{dR}\). It is of interest to note that the general solution (31) satisfies the field equations of Møller’s theory when the stress-energy momentum tensor has the form

$$T_0^0 = T_1^1 = \epsilon_0 e^{-R^3/r_1^3},$$

$$T_2^2 = T_3^3 = \epsilon_0 e^{-R^3/r_1^3} \left(1 - \frac{3R^2}{2r_1^3}\right),$$  \hfill (32)

The previously obtained solutions can be verified as special cases of the general solution (31). The choice

$$B(R) = 1,$$  \hfill (33)

reproduces the solution (17). On the other hand, the choice

$$B(R) = \int \frac{1}{R} \left(1 - \sqrt{1 - \frac{2m(1 - e^{-R^3/r_1^3})}{R}}\right) dR,$$  \hfill (34)
reproduces the solution (18). It is of interest to note that if the exponential term is equal zero then the general solution (31) reduces to that obtained before by Mikhail et al. [18] and the two choices (33) and (34) will give the Schwarzschild solution in its standard form.

5. The Energy Associated with the General Solution

The superpotential of Møller’s theory is given by Mikhail et al. [19] as

\[ U_{\mu}^{\nu\lambda} = \frac{(-g)^{1/2}}{2\kappa} P_{\chi\rho\tau\nu\lambda} \left[ \Phi_{\rho}^{\sigma} g^{\sigma\chi} g_{\mu\tau} - \lambda g_{\tau\mu} \gamma^{\chi\rho\sigma} - (1 - 2\lambda) g_{\tau\mu} \gamma^{\sigma\rho\chi} \right], \tag{35} \]

where \( P_{\chi\rho\tau\nu\lambda} \) is

\[ P_{\chi\rho\tau\nu\lambda} \overset{\text{def.}}{=} \delta_{\chi}^{\tau} g_{\rho\sigma}^{\nu\lambda} + \delta_{\rho}^{\tau} g_{\sigma\chi}^{\nu\lambda} - \delta_{\sigma}^{\tau} g_{\chi\rho}^{\nu\lambda} \tag{36} \]

with \( g_{\rho\sigma}^{\nu\lambda} \) being a tensor defined by

\[ g_{\rho\sigma}^{\nu\lambda} \overset{\text{def.}}{=} \delta_{\rho}^{\nu} \delta_{\sigma}^{\lambda} - \delta_{\rho}^{\lambda} \delta_{\sigma}^{\nu}. \tag{37} \]

The energy is expressed by the surface integral [20]

\[ E = \lim_{r \to \infty} \int_{r = \text{constant}} U_{0}^{0\alpha} n_{\alpha} dS, \tag{38} \]

where \( n_{\alpha} \) is the unit 3-vector normal to the surface element \( dS \).

Now we are in a position to calculate the energy associated with the general solution (31) using the superpotential (35). Thus substituting from (31) into (35) we obtain the following nonvanishing values

\[ U_{0}^{0\alpha} = \frac{2X^{\alpha}}{\kappa R^{3}} \left[ 2m \left( 1 - e^{-R^{3}/r^{3}} \right) - R^{2} B' \right]. \tag{39} \]

Substituting from (39) into (38) we get

\[ E(R) = 2m \left( 1 - e^{-R^{3}/r^{3}} \right) - R^{2} B'. \tag{40} \]

As is clear from (40), the energy content depends on the arbitrary function \( B(R) \). If \( B(R) = 1 \) then the energy content (40) will coincide with that of solution (17) [11], and if \( B(R) \) takes the form (34) then the energy content will coincide with that of solution (18) [11, 24, 25].
6. Concluding Remarks

In this paper we have obtained the general solution of Møller’s tetrad theory of gravitation in case of spherical symmetry and when the stress-energy momentum tensor has a specific form. The previously obtained solutions have been verified as a special case of the general solution. The general solution gave no more than the spherically symmetric nonsingular black hole solution \[11, 13\].

The general solution has been found to contain an arbitrary function and two constants of integration. Hence Møller’s theory does not fix the tetradic geometry in case of spherical symmetry; up to finite number of arbitrary constants. It is of interest to note that in the case of spherical symmetry when the stress-energy momentum tensor is vanishing \[18\] the general solution does not fix the tetradic geometry too. Also in the cases of cosmology and stationary axisymmetric \[23, 24\] although no general solution is obtained, it is proved that the solutions do not fix the field equations of Møller’s theory. Sáez in his scalar tetradic theory of gravitation \[25\] has discussed the point that if the tetradic geometry can be fixed from the field equations of the Weitzenböck theory. He did not find a conclusive answer. Indeed, the present work has the advantage that it gives a reasonable reply for his question.

The energy content of the general solution is calculated. It is found that the energy depends on the asymptotic behavior of the arbitrary function \(B(R)\). If \(B(R) \sim 1/R\) it will contribute to the energy content \((40)\) and if \(B(R) \sim 1\) it will not contribute to the total energy \((40)\).

The general solution of a tetrad having spherical symmetry with three unknown functions is obtained. As for the general tetrad having spherical symmetry which is given by Robertson \[12\], the general solution is obtained in the vacuum case \[26\], but when the stress-energy momentum tensor is not vanishing the general solution is has not yet been obtained. This will be done in a future research.

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