Quantum Dust Black Holes

A Statistical Mechanics Derivation
of Black Holes Thermodynamics

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ABSTRACT

By analysing the infinite-dimensional midisuperspace of spherically symmetric dust matter universes, and applying it to collapsing dust stars, one finds that the general quantum state is a bound state. This leads to a discrete spectrum. To an outside observer, the geometry is static if the initial radius of the collapsing star is smaller than the Schwarzschild radius. In that case the discrete spectrum implies Bekenstein area quantization: the area of the black hole is an integer multiple of the Planck area. Knowing the microscopic (quantum) states, we suggest a microscopic interpretation of the thermodynamics of black holes: by calculating the degeneracy of the quantum states forming a black hole, one gets the Bekenstein-Hawking entropy (the entropy is proportional to the surface area of the black hole). All other thermodynamical quantities can be derived by using the standard definitions.

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1 Introduction

Recently a simple minisuperspace describing the Oppenheimer - Snyder (OS) collapsing star was found [1]. The semiclassical wave function of that model (e.g. the semiclassical solution of the Wheeler-DeWitt equation) is a bound state. This leads to quantization conditions. The corresponding (Bohr-Sommerfeld) quantization condition can be written in the form

\[ F(M, R_0) = \hbar \left( n + \frac{1}{2} \right) , \quad n = 0, 1, 2, ... \] (1.1)

where \( M \) is the mass of the collapsing star, \( R_0 \) is its initial radius, and \( F(M, R_0) \) is a function of \( M \) and \( R_0 \) to be given later [1]. For fixed \( R_0 \), (1.1) implies mass quantization.

The idea of mass quantization is an old one. Using general arguments from quantum mechanics (of adiabatic variables) and general relativity, Bekenstein [2] got the black hole area quantization condition

\[ M^2_{ir} = \frac{1}{2} \hbar n , \quad n = 1, 2, 3, ... \] (1.2)

where \( M_{ir} \) is the irreducible mass of the black hole, which is related to its surface area, \( A = 16\pi M^2_{ir} \). This discrete spectrum can be related to the thermodynamic properties of black holes [3].

We will show that in the case of black holes, one can get (1.2) from (1.1), which is correct for any OS star. Beside the quantization condition (1.2), one can find in this explicit model also the general quantum states (the solutions of the Wheeler-DeWitt equation) forming this star. By that one can hope to understand better not just the quantization conditions, but also the thermodynamical properties of black holes. This is the purpose of this paper.
While it has been known for two decades, there are still some open questions concerning the entropy of black holes. The classical considerations [4], which use the analogy between some geometrical properties of black holes and thermodynamics, gives the “laws of black hole thermodynamics”, from which one can get the entropy. Semiclassical considerations [5], on the other hand, use the (formal) path integrals (approach) to find the partition function, and then the entropy. But both approaches do not use basic statistical mechanics reasoning, namely, finding the entropy by calculating the number of different “microscopic states” that correspond to the same “macroscopic state” that we call a black hole. This “missing link” is very important, because it requires the understanding of microscopic states forming a (macroscopic) black hole. Those microscopic states are the quantum states, so their importance to the understanding of black holes thermodynamics is obvious. The OS model gives a one dimensional minisuperspace, in which of course one cannot hope to get a degeneracy that will give a nonzero entropy. So one must extend the model. One such an extension is the inclusion of inhomogeneous (spherically symmetric dust) distributions. This was done a long time ago by Lund [6]. Lund used the dust matter as a “clock” and then fixed the gauge completely, reducing the constrained matter-gravity theory to an unconstrained one. We will use Lund’s infinite dimensional “midisuperspace”, and apply it to the collapsing star case. Though infinite dimensional, Lund’s midisuperspace shares some resemblance with the OS model, so one can analyse it in the same manner and find the quantum states that correspond to a classical black hole. Knowing the microscopic (quantum) states, one can calculate their degeneracy, and find the entropy of the black hole. Then by using the standard thermodynamical definitions one can get the other thermodynamical quantities (e.g. the temperature).
We will consider both the static “eternal black hole picture” [7], and the dynamic Hawking evaporation one [8]. They both can be studied in our framework, and the results that we get are in agreement with the known ones.

In this work we use the semiclassical approximation only. This is for two reasons: first, the OS model as well as Lund’s one, are correct only semiclassically. And second, we use Einstein gravity (coupled to matter) which should be (at least) a good approximation semiclassically.

We use geometrical units $G = c = 1$.

The paper is organized as follows: in chapter 2 we describe the OS model, solve (semiclassically) the corresponding Wheller-DeWitt equation, and get the mass and area quantization conditions. In chapter 3 we describe Lund’s midisuperspace and find the (semiclassical) solutions to the Wheeler-DeWitt equation. In chapter 4 we describe the midisuperspace of a collapsing star, and find the quantum states forming this star (black hole). In chapter 5 we study the thermodynamical properties of black holes in this framework, and chapter 6 presents some concluding remarks.

2 The OS Model

2.1 The OS Minisuperspace

In 1939 Oppenheimer and Snyder [9] found a very simple solution (of Einstein gravity couple to dust matter) describing a collapse of a spherically symmetric homogeneous dust star. In their solution the Schwarzschild exterior is smoothly connected to the interior region, which is a slice of a Friedmann universe.
The interior region is described by the Friedmann line element

\[ ds^2 = -N^2(t)dt^2 + a^2(t)[d\chi^2 + \sin^2\chi d\Omega_2^2] \]  

(2.1)

The range of \( \chi \) is \( 0 \leq \chi \leq \chi_0 \), where \( \chi_0 \leq \pi/2 \). At \( \chi = \chi_0 \) the interior is matched to the exterior Schwarzschild solution. If \( M \) and \( R_0 \) are the mass and initial radius of the star, then the matching conditions are

\[ M = \frac{1}{2}a_0 \sin^3\chi_0 \]
\[ R_0 = a_0 \sin\chi_0 \]  

(2.2)

where \( a_0 \) is the initial Friedmann radius.

The gravitational Lagrangian may be split into its interior and exterior parts,

\[ L_G = 4\pi \int_0^{\chi_0} \sin^2\chi d\chi \left[ \frac{3a}{N} \dot{a}^2 - 3Na \right] + \int_{r \geq r_s} \sqrt{-g} R d^3x \]  

(2.3)

where \( r_s \) is the surface radius of the collapsing star.

The matter Lagrangian is

\[ L_M = -8\pi \int \sqrt{-g} \rho U^\mu U_\mu d^3x \]  

(2.4)

where \( \rho \) is the density of the star and \( U_\mu \) is the four-velocity of the matter particles. Energy momentum conservation, \( \nabla_\mu T^{\mu\nu} = 0 \), implies \( \rho = \rho_0/a^3 \), where \( \rho_0 \) is a constant to be determined by the initial conditions. The OS model requires \( \rho_0 = 3a_0/8\pi \). So using (2.3),(2.4) and \( U^\mu U_\mu = -1 \) we get the total Lagrangian

\[ L = L_G + L_M = 12\pi \int_0^{\chi_0} \sin^2\chi \left[ \frac{a\dot{a}^2}{N} - N(a - a_0) \right] + \int_{r > r_s} \sqrt{-g} R d^3x \]  

(2.5)

The Hamiltonian corresponding to (2.5) is

\[ H = N \left[ \frac{1}{4\alpha_0 a} P_a^2 + \alpha_0 (a - a_0) \right] + \int_{r > r_s} \mathcal{H} d^3x \]  

(2.6)
where \( P_a = \partial L / \partial \dot{a} \), and \( \alpha_0 = 12\pi \int_0^{\chi_0} \sin^2 \chi d\chi \). Because the classical solution for \( r > r_s \) is the Schwarzschild space-time, for which \( R_{Sch.} = 0 \), only the first term in (2.5) (or (2.6)) will contribute to the semiclassical dynamics.

The Wheeler-DeWitt equation is the quantum version of the classical Hamiltonian constraint, \( \partial H / \partial N = 0 \), in the coordinate representation: \( |\Psi> = \psi(a) \), \( P_a = -i\hbar \partial / \partial a \). Using (2.6) we get the “Schrödinger equation”

\[
\left( -\hbar^2 \frac{d^2}{da^2} + V(a) \right) \psi(a) = 0 \tag{2.7}
\]

where \( V(a) = 4\alpha_0^2 a(a - a_0) \). If we define \( x \equiv a - a_0 / 2 \), we get

\[
\left( -\hbar^2 \frac{d^2}{dx^2} + \frac{1}{4} \omega^2 x^2 \right) \psi(x) = E\psi(x) \tag{2.8}
\]

where \( \omega = 4\alpha_0 \) and \( E = \alpha_0^2 a_0^2 \). As we can see from (2.8), \( \psi \) describes a harmonic oscillator (with mass \( m = 1/2 \)). So the semiclassical wave function describes a one-dimensional harmonic oscillator, which is of course a bound state.

As in the Hartle-Hawking case [10], the solution of (2.8) describes a superposition of two “universes”, one that collapses to form a black hole, and one that expands, a white hole.

\[\text{Using the path integral approach, the semiclassical wave function is}\]

\[\psi_{WKB} = A\exp[iS_{class.}/\hbar]\]

and we see that only the first term in (2.5) will contribute. \[\text{The solution can be written as}\]

\[\psi(x) = A(e^{ip(x)/\hbar} + e^{-ip(x)/\hbar})\]

where \( p(x) = \int_{x_0}^x \sqrt{|V(x')|} dx' \).
2.2 Mass and Area Quantization

Because the wave function of the OS model describes a bound state, the spectrum is quantized. Semiclassically we should use the Bohr-Sommerfeld quantization condition, but in the case of an harmonic oscillator it is exact,

\[ E(n) = \hbar \omega \left( n + \frac{1}{2} \right), \quad n = 0, 1, 2, \ldots \]  

(2.9)

Using the definition of \( E \) and \( \omega \) (see below (2.8)) we have

\[ \frac{1}{4} \alpha_0 a_0^2 = \hbar \left( n + \frac{1}{2} \right) \]  

(2.10)

In [1] we consider only the case \( R_0 > 2M \) which corresponds to the usual cosmological situation. In that case we have \( \alpha_0 \simeq 4\pi \chi_0^3 \), and using (2.2) and (2.10) we get [1]

\[ MR_0^3(n) = \frac{1}{2\pi^2} \hbar^2 \left( n + \frac{1}{2} \right)^2 \]  

(2.11)

For fixed initial radius, (2.11) gives mass quantization.

Of course all the above describes a dynamical process: the collapse of the star. In this work we try to understand the quantum properties of a “static black hole,” as seen by an outside observer, and especially to find its entropy and temperature (as measured by that observer). For an outside observer, the above picture can be static only if \( R_0 \leq 2M \). In that case the geometry outside the horizon is always Schwarzschild, which is of course static. The OS model requires \( R_0 \geq 2M \) [9], so only if \( R_0 = 2M \) this model can describe a static geometry everywhere outside the horizon.

In the case \( R_0 = 2M \) we have \( \chi_0 = \pi/2 \), and we get from (2.2) and (2.10)

\[ M^2(n) = \frac{1}{3\pi^2} \hbar \left( n + \frac{1}{2} \right), \]  

(2.12)

a surface area quantization: the surface area of the black hole\(^4\) goes linearly with the quantum number \( n \).

\(^4\) For a Schwarzschild black hole \( M_{tr} = M \), so \( A = 16\pi M^2 \).
This is Bekenstein’s result [2], but we got it by using an explicit model, and by solving the corresponding Wheeler-DeWitt equation. The fact that we use a very simple (and even a non-realistic) model (the OS star), and still get his results, suggests that these are quite general.

In Bekenstein’s original paper [2] the prefactor for $M^2$ was $\hbar/2$, but further considerations by Mukhanov [3] suggest that the prefactor should be $\hbar \ln 2/4\pi$. In our case the prefactor is $\hbar/3\pi^2$. This prefactor is a model dependent, and the best that we can hope (using our simple model⁵) is to get the same order of magnitude. This is in fact what we got.

In the OS model, the (microscopic) state of a black hole with a (macroscopic) mass $M$, is $|\Psi_n>$, where $n$ satisfy (2.12). So in this model, for each macroscopic state (labeled by the mass $M$, or the area $A$) there is only one microscopic state⁶ (labeled by the quantum number $n$). So the entropy of the black hole in the OS model is zero⁷. This is because the OS minisuperspace is “too small” (one dimensional). If we want to understand the thermodynamics of black holes, we must extend the model. In the next chapter we describe a much bigger midisuperspace (an infinite dimensional one), which will turn out to be much more appropriate for studying black holes thermodynamics.

3 Lund’s Midisuperspace

⁵ For example, we take only $R_0 = 2M$. A more reasonable model should take some average between $R_0 = 0$ and $R_0 = 2M$.

⁶ We have a one dimensional harmonic oscillator, so there is no degeneracy.

⁷ The entropy goes like

\[ S \sim \ln(\text{number of microscopic states}) = \ln(1) = 0 \]
3.1 The ADM Reduction Process

For a global hyperbolic space-time, $M = \mathbb{R} \times \Sigma^{(3)}$, one can use the ADM splitting [11], and write the Hamiltonian of a dust matter coupled to Einstein gravity in the form [6]

$$ H = \int d^3 x dt \left[ N (\mathcal{H}^0 + \mathcal{E}) + N_i (\mathcal{H}^i + \mathcal{P}^i) \right] $$  \hspace{1cm} (3.1)

where $N$ and $N_i$ are the lapse function and shift vector respectively. $\mathcal{H}^0$ and $\mathcal{H}^i$ are the gravitational super-hamiltonian and super-momentum

$$ \mathcal{H}^0 = h^{1/2} \left[ \pi^{ij} \pi_{ij} - \frac{1}{2} \pi^2 - R^{(2)} \right] $$  \hspace{1cm} (3.2)

$$ \mathcal{H}^i = -2h^{1/2} D_j \pi^{ij} $$  \hspace{1cm} (3.3)

where $h_{ij}$ is the induced 3-metric on $\Sigma^{(3)}$ ($h = \text{det}(h_{ij})$), and $\pi^{ij}$ its conjugate momenta ($\pi = \pi^i_i$). $\mathcal{E}$ and $\mathcal{P}^i$ are the dust matter Hamiltonian and momentum respectively

$$ \mathcal{E} = h^{1/2} n^\mu n^\nu T_{\mu\nu} $$  \hspace{1cm} (3.4)

$$ \mathcal{P}^i = h^{1/2} n^\mu h^{ij} T_{\mu j} $$  \hspace{1cm} (3.5)

where $n^\mu$ is a unit normal vector to the hypersurface $\Sigma^{(3)}$, and $T_{\mu\nu}$ is (using (2.4))

$$ T_{\mu\nu} = 16\pi \rho U_\mu U_\nu $$  \hspace{1cm} (3.6)

If we define the scalar field $\phi$

$$ U_\mu \equiv \nabla_\mu \phi $$  \hspace{1cm} (3.7)

we can treat it as a dynamical variable (and $\rho$, which is not a dynamical variable, will be a function of $h_{ij}$, $\pi^{ij}$, $\phi$, $\nabla_\mu \phi$, to be determined later.).
In the spherically symmetric case one can use the \((R, \theta, \phi)\)-coordinates on \(\Sigma^{(3)}\) in which
\[
d s^2_{\Sigma^{(3)}} = e^{2\mu} dR^2 + e^{2\lambda} d\Omega^2_2
\] (3.8)
where \(\mu\) and \(\lambda\) are functions of \(t\) and \(R\), and \(d\Omega^2_2\) is the volume element in \(S^2\). If we define \(\pi_\mu\) and \(\pi_\lambda\) such that
\[
\pi^{ij} = \text{diag}\left(\frac{1}{2} e^{-2\mu} \pi_\mu, \frac{1}{4} e^{-2\lambda} \pi_\lambda, \frac{1}{4} e^{-2\lambda} \sin^{-2}\theta \pi_\lambda\right)
\] (3.9)
we get
\[
H^0 = e^{-(\mu+2\lambda)} \left(\frac{\pi^2_\mu}{8} - \pi_\mu \pi_\lambda / 4 - 2e^{2(\mu+2\lambda)} \left[e^{-2\lambda} - e^{-2\mu} (2\lambda'' - 2\lambda' \mu' + 3(\lambda')^2)\right]\right)
\] (3.10)
\[
H^R = -e^{-2\mu} (\pi'_\mu - \mu' \pi_\mu - \lambda' \pi_\lambda)
\] (3.11)
\[
\mathcal{E} = \left(16\pi \rho h^{1/2}\right)^{-1} p^2_\phi
\] (3.12)
\[
P^R = p_\phi h^{11} \phi'
\] (3.13)
where prime denote differentiation with respect to \(R\), and \(p_\phi = \partial L / \partial \dot{\phi} = -16\pi N h^{1/2} U^0 \rho\). We see that if we choose the coordinates for which \(N^R = 0\), and using \(U^\mu U_\mu = -1\), we get \(\rho = (16\pi h^{1/2})^{-1} (1 + h^{11}(\phi')^2)^{-1/2} p_\phi\), so from (3.12) we get
\[
\mathcal{E} = \left(1 + h^{11}(\phi')^2\right)^{1/2} p_\phi
\] (3.14)
and we see that \(\mathcal{E}\) goes linearly with \(p_\phi\). This suggests that we can use \(\phi\) as a time variable. And indeed taking \(\phi = -t\), \(N^R = 0\) gives (using (3.10)-(3.13)) the known general solutions [12].

To complete the gauge fixing (the reduction process) one must choose also the \(R\)-coordinate. From the equations of motion (derive from (3.10)-(3.13)) one can get that \(\lambda e^{\lambda - \mu}\) is a function of \(R\) only, so one can choose
\[
R = \lambda e^{\lambda - \mu}
\] (3.15)

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Now the conjugate momenta are
\begin{align*}
\pi_R &= -\left(\lambda' e^{\lambda-\mu}\right)^{-1} \pi_{\mu} \quad (3.16) \\
\bar{\pi}_{\lambda} &= \pi_{\lambda} - e^{\lambda} \left(\lambda'^{-1} e^{-\lambda} \pi_{\mu}\right)' \quad (3.17)
\end{align*}

Using (3.11) and (3.13), the supermomentum constraint, $\mathcal{H}^R + \mathcal{P}^R = 0$, is now
\[\pi_R + \bar{\pi}_{\lambda} \lambda' + p_{\phi} \phi' = 0 \quad (3.18)\]

After solving the constraints, one ends up with the reduced Lagrangian (or Hamiltonian)
\[S_{\text{red}} = 4\pi \int dt dR (\pi_y \dot{y} - \mathcal{H}_{\text{ADM}}) \quad (3.19)\]

where
\begin{align*}
y &= 8 e^{\lambda} \quad (3.20) \\
\pi_y &= \frac{1}{8} e^{-\lambda} \bar{\pi}_{\lambda} \quad (3.21)
\end{align*}

and
\[\mathcal{H}_{\text{ADM}} = R^2 \left(\frac{1}{y^2} \pi_y^2 + (2R)^{-2} (R^{-2} - 1)y\right) \quad (3.22)\]

So we end up with $\infty^1$ unconstraint degrees of freedom, the $y(r)$ field, with the Hamiltonian (3.22). The space of all $y(r)$-field solutions is what we call “Lund’s midisuperspace”.

### 3.2 Quantum States

The reduced Hamiltonian is
\[H = 4\pi \int \mathcal{H}_{\text{ADM}} dR = 4\pi \int R^2 \left(\frac{1}{y} \pi_y^2 + (2R)^{-2} (R^{-2} - 1)y\right) dR \quad (3.23)\]
We use the coordinate representation (in Lund’s midisuperspace)

\[\hat{y} = y\] (3.24)

\[\hat{\pi}_y = \frac{\hbar}{i} \frac{\delta}{\delta y}\] (3.25)

So the corresponding Schrödinger equation (in a different context we could call it the Wheeler-DeWitt equation, see sec. 2) is

\[i\hbar \frac{\partial \Psi[y; t]}{\partial t} = \int \left( -\frac{\hbar^2}{y} \frac{\delta^2}{\delta y^2} + \frac{1}{4} (R^{-4} - R^{-2}) y \right) \Psi[y; t] R^2 dR\] (3.26)

where \(|\Psi\rangle\) is the quantum state, and \(\Psi[y; t] = \langle \Psi | y(R, t) >\) is the wave functional. In this representation it is a functional of the field \(y(r)\) and a function of time \(t\). As one can see, we use the “\(y\pi\)-ordering”\(^{8}\) in (3.26), but different ordering will not change our results, which anyway are correct only semiclassically.

A very important feature of (3.23) is that there are no \(R\)-derivatives in \(H\). This means that the infinite degrees of freedom (d.o.f.) are decoupled. Let \(R_s\) be the surface of the dust ball, then we can divide \(R_s\) to \(N\) equal parts, \(R_k = \frac{R_s}{N} k\) \((k = 1, 2, \ldots, N)\). The “continuum limit” is \(N \to \infty\). The vector space \(\{y(r)\}\) is now \(\{\vec{y} = (y_1, y_2, \ldots, y_N)\}\) where \(y_k = y(R_k)\), and the Schrödinger equation (3.26) becomes

\[i\hbar \frac{\partial \Psi(\vec{y}; t)}{\partial t} = \sum_{k=1}^{N} \left[ -a_k \frac{\hbar^2}{y_k} \frac{\partial^2}{\partial y_k^2} + b_k y_k \right] \Psi(\vec{y}, t)\] (3.27)

where \(a_k = R_k^2\) and \(b_k = (R_k^{-2} - 1)/4\) are positive constants \((b_k\) is positive because \(0 \leq R \leq 1\) \([6]\)). Because of the decoupling we can write \(|\Psi\rangle\) as a direct product

\[|\Psi\rangle = |\Psi_1 \rangle \otimes |\Psi_2 \rangle \otimes \ldots \otimes |\Psi_N \rangle\] (3.28)

\(^{8}\)The field \(y\) is always to the left of its conjugate momenta.
and from (3.27) we have now
\[ i\hbar \frac{\partial \Psi_k(y_k, t)}{\partial t} = \left[ -a_k \frac{\hbar^2}{y_k \partial y_k^2} + b_k y_k \right] \Psi_k(y_k, t) \] (3.29)
where \( \Psi_k(y_k, t) = \langle \Psi_k | y_k(t) \rangle \). The ADM Hamiltonian is time-independent so we have \( H_{ADM}(y_k) = E_k = \text{const.} \) and
\[ \left[ -a_k \frac{\hbar^2}{y_k \partial y_k^2} + b_k y_k \right] \Psi_k(y_k, t) = E_k \Psi_k(y_k) \] (3.30)
If we define
\[ x_k \equiv y_k - \frac{1}{2} E_k \] (3.31)
we get the following harmonic oscillator Schrödinger equation
\[ \left( -\frac{\hbar^2}{2m_k} \frac{\partial^2}{\partial x_k^2} + \frac{1}{2} m_k \omega_k^2 x_k^2 \right) \Psi_k(x_k) = \epsilon_k \Psi_k(x_k) \] (3.32)
where \( m_k = 1/2a_k \), \( \omega_k = \sqrt{8a_k b_k} \) and \( \epsilon_k = m_k (\omega_k E_k)^2/8 \). The solutions of (3.32) are \( |\Psi_k > = |n_k > \), where \( |n_k > \) is a one-dimensional harmonic oscillator with energy \( \epsilon_k = \hbar \omega_k (n_k + 1/2) \), or
\[ \frac{1}{8} a_k^{1/2} E_k^2 = \hbar \left( n_k + \frac{1}{2} \right) \] (3.33)
So the space of quantum states describing this spherically symmetric dust “universe” is spanned by
\[ \{|\Psi_{n_1, n_2, ..., n_N} > = |n_1 > |n_2 > \cdots |n_N > \} \] (3.34)
The total energy is
\[ E = \sum_{k=1}^{N} E_k = \sum_{k=1}^{N} \left( 8\hbar [2(R_k^{-4} - R_k^{-2})]^{-1/2} \right)^{1/2} (n_k + 1/2)^{1/2} \] (3.35)
In the \( N \to \infty \) limit, one can have a finite energy only if one uses the Wick order, so \( n_k + 1/2 \) must be replaced with \( n_k \), but we will come to that later.
4 Midisuperspace for a Collapsing Star

4.1 The Homogeneous Case

In the homogeneous case (the OS case), it is convenient to use (2.1). The 3-metric in the \((R, \theta, \phi)\)-coordinates, is

\[
h_{ij} = \text{diag} \left( a^2 \left( \frac{d\chi}{dR} \right)^2, a^2 \sin^2 \chi, a^2 \sin^2 \chi \sin^2 \phi \right)
\]  

(4.1)

and from (3.8), (3.15) and (4.1) we get

\[
R = \lambda e^{\lambda - \rho} = \cos \chi
\]

(4.2)

We see that \(R\) is not a “usual radial coordinate”. For example the origin \((r = 0)\) is \(R = 1\), and \(R_{\text{min}} = \cos \chi_0 \geq 0\). From (3.15) one can see that \(R = r_c/r_e [6]\), where \(r_c\) is the circumference radius, and \(r_e\) is the extrinsic radius of curvature. Because the horizon is a minimal area, \(R(\text{horizon}) = 0\). One can get this explicitly in the OS model, because a static black hole corresponds to \(\chi_0 = \pi/2\), so \(R(\text{horizon}) = \cos \chi_0 = 0\). For asymptotically flat space \(R\) goes to unity at spatial infinity. It is convenient to use a different radial coordinate\(^9\)

\[
r \equiv \sin \chi = (1 - R^2)^{1/2}
\]

(4.3)

Now one should replace (3.23) with

\[
H = \int_0^{r_s} \mathcal{H}(r) dr = \int_0^{r_s} \left( \frac{32r\sqrt{1 - r^2} P_y^2}{3\pi y} + \frac{3\pi r}{2\sqrt{1 - r^2} y} \right) dr
\]

(4.4)

where \(y = 8e^{\lambda} = 8ar\), and \(P_y = 3\pi y \dot{y}/64r\sqrt{1 - r^2}\). It is very easy to see that (4.4) is the correct Hamiltonian, because from (2.6) and (4.4) we have

\(^9\)The coordinate \(r\) grows with the usual radial coordinate while \(R\) decreases.
$P_y = 3\pi r P_a / 2\alpha_0 \sqrt{1 - r^2}$, so

$$
H = 12\pi \int_0^{\sin \chi_0} dr \frac{r^2}{\sqrt{1 - r^2}} \left( \frac{P^2_a}{4\alpha_0^2 a} + a \right)
= \frac{P^2_a}{4\alpha_0 a} + \alpha_0 a
$$

(4.5)

which is exactly the gravitational part of (2.6).

The Schrödinger equation is

$$
\hat{H} |\Psi\rangle = E |\Psi\rangle
$$

(4.6)

In Lund’s formalism $E$ is an undefined constant, but in the collapsing star case, we must impose the matching conditions: the solution must be smoothly matched to the outside Schwarzschild space-time. This will constrain $E$.

From (2.1) and (2.4) we have

$$
E = 32\pi^2 \int_0^{\chi_0} \sin^2 \chi d\chi a^3 \rho
$$

(4.7)

Because $E$ is time independent, we can easily calculate it at the beginning of the collapse, when the star is at rest. Using $a(t = 0) = a_0$ and $\rho(t = 0) = 3M/4\pi R_0^3$, we get

$$
E = \frac{2\alpha_0 a_0^3 M}{R_0^3}
$$

(4.8)

Now we can use the matching conditions (2.2) to get

$$
E = \alpha_0 a_0
$$

(4.9)

and the Schrödinger equation (4.6) is exactly (2.7), and one can get the results of chapter 2.

Notice that we could get (4.9) from the requirement that the collapse start ($t = 0$) at rest. We will use that in the inhomogeneous case.
In the homogeneous case the difference between Lund’s midisuperspace and a collapsing star midisuperspace is that in the latter the energy $E$ is constraint. In the inhomogeneous case the constraints are more complicated, but one can deal with them in a similar way.

### 4.2 The Inhomogeneous Case

We saw that (in the homogeneous case) one can use Lund’s formalism, impose the energy condition (4.9), and get the OS results. This can be generalized to the inhomogeneous case.

Let us first go back to the homogeneous case. Using (4.4) and (4.9), the Schrodinger equation (4.6) can be written as

$$\int_0^r \left[ \frac{32r \sqrt{1 - r^2}}{3\pi} \frac{P^2}{y} + \frac{3\pi r}{2\sqrt{1 - r^2}} (y - y_0) \right] \Psi[y] dr = 0$$

where $y_0 = y(r, t = 0) = 8a_0 r$.

Now we can generalized to the inhomogeneous case. The result (3.23) (or (4.4)) are correct for any spherically symmetric dust ball (not just for the homogeneous one). In particular, they are correct in the case of a collapsing (spherically symmetric dust) star. If we want the collapse to start ($t = 0$) at rest, which is the generalization of (4.9), then we must have

$$\mathcal{E}(r) = \frac{3\pi r}{2\sqrt{1 - r^2}} y_0$$

where $y_0 = y(r, t = 0) = 8a_0 r$.

Because only in that case $\dot{y}(r, t = 0) = 0$. So the form of (4.10) is quite general: it is correct also in the inhomogeneous case. The only difference is that in the inhomogeneous case, the Friedmann radius, $a$, can be a function of $r$ too. So as a function of $(r, t)$ the field solution is different, but it

\[ E = \int_0^r \mathcal{E}(r) dr, \quad \mathcal{H}(r) = \mathcal{E}(r), \quad \text{and from (4.4) and (4.10) we get (4.11).} \]
has the same form \( y(r, t) = 8a(r, t)r \). The space of all field solutions that describe a collapsing star is of course a subspace of all the field solutions (Lund’s midisuperspace), but at this stage we do not need to know the specific restrictions; what is important is that we can use (4.10).

After making the discretization \( r_k = r_s k/N \) \((k = 1, 2, ..., N)\), we get from (4.10)

\[
\sum_{k=1}^{N} \left( \alpha_k \frac{P^2_{y_k}}{y_k} + \beta_k (y_k - y_k^{(0)}) \right) \Psi(\vec{y}) = 0
\]  

(4.12)

where \( \alpha_k = \frac{(32 r_k \sqrt{1-r_k^2})}{3 \pi} \), \( \beta_k = \frac{3 \pi r_k}{2 \sqrt{1 - r_k^2}} \) and \( y_k^{(0)} = y_0(r_k) \). Using \( \Psi(\vec{y}) = \prod_k \Psi_k(y_k) \) we have a set of \( N \) independent equations

\[
\left( \alpha_k \frac{P^2_{y_k}}{y_k} + \beta_k (y_k - y_k^{(0)}) \right) \Psi_k(y_k) = 0
\]  

(4.13)

Defining \( x_k \equiv y_k - y_k^{(0)}/2 \), we get the (harmonic oscillator) Schrödinger equation

\[
\left( -\frac{\hbar^2}{2m_k} \frac{\partial^2}{\partial x_k^2} + \frac{1}{2} m_k \omega_k^2 x_k^2 \right) \Psi_k(x_k) = \epsilon_k \Psi_k(x_k)
\]  

(4.14)

where \( m_k = 1/2\alpha_k \), \( \omega_k = 8r_k \) and \( \epsilon_k = m_k \omega_k^2 (y_k^{(0)})^2 / 8 \). The quantization conditions are

\[
r_k m_k (y_k^{(0)})^2 = \hbar \left( n_k + \frac{1}{2} \right)
\]  

(4.15)

The total energy is

\[
E = 3\pi \sum_{k=1}^{N} \frac{r_k y_k^{(0)}}{2 \sqrt{1 - r_k^2}}
\]  

(4.16)

and from (4.15) we get

\[
E = \hbar \sum_{k=1}^{N} \Omega_k (n_k + 1/2)
\]  

(4.17)

where \( \Omega_k = 4/a_k^{(0)} \), \( (a_k^{(0)} = a(r_k, t = 0)) \).
So a quantum state describing a collapsing dust star, starting at rest, can be written as

$$|\Psi(\text{star})> = |n_1> |n_2> \cdots |n_N>$$

(4.18)

where $|n_k>$ is a one dimensional harmonic oscillator (excited to the level $n_k$) with frequency $\omega_k = 8r_k$. Each $|n_k>$ and so $|\Psi>$ are bound states, and we end up with quantization conditions.

In the homogeneous case $y_k^{(0)} = 8a_0r_k$, $(a_k^{(0)} = a_0)$ and from (4.17) we get

$$E_{\text{hom}} = \frac{4\hbar}{a_0} \sum_k (n_k + 1/2)$$

(4.19)

and using (4.9) we have

$$\frac{1}{4}a_0a_0^2 = \hbar \sum_k (n_k + 1/2)$$

(4.20)

which is (2.10) (remember that in the homogeneous case there is only one d.o.f. so $N = k = 1$).

In the general inhomogeneous case, $\Omega_k$ is not $k$-independent, but we can use “mean field” reasoning and write (4.17) as

$$E = \hbar \sum_{k=1}^{N} \Omega_k (n_k + 1/2) = \hbar <\Omega> \sum_{k=1}^{N} (n_k + 1/2)$$

(4.21)

One can use (4.21) as a definition of $<\Omega>$, which is the “average” of $\Omega_k$.

The results (4.18) and (4.21) are correct for any collapsing spherically symmetric dust star, which start at rest. But in the case of a static black hole, one can relate $E$ and $<\Omega>$ to the mass of the black hole. In the homogeneous case we have $a_0 = 2M_{\text{bh}}$ so from (4.9) we have $E = 6\pi^2 M_{\text{bh}}$, and from (4.17) $<\Omega> = \Omega_k = (2M_{\text{bh}})^{-1}$. In the inhomogeneous case this can be generalized by dimensional arguments to

$$E \sim \frac{1}{<\Omega>} \sim M_{\text{bh}}$$

(4.22)
where \( \sim \) denote equality up to a constant of order unity. Now using (4.21) and (4.22) we get

\[
M_{bh}^2 \sim \hbar \sum_{k=1}^{N} (n_k + 1/2) \equiv \hbar (n_{tot} + 1/2)
\]  

(4.23)

Which is the generalization of (2.12) to the general inhomogeneous case, in agreement with Bekenstein’s result.

5 Black Hole Thermodynamics

5.1 Entropy

Using the standard statistical mechanics definition, the entropy of a (macroscopic) black hole is

\[
S_{bh} = \ln \left[ \mathcal{N}_{bh}(M) \right]
\]  

(5.1)

where we choose \( k_B \) to be one, and \( \mathcal{N}_{bh}(M) \) is the number of microscopic states that correspond to the same macroscopic state (a black hole) with mass \( M \). The microscopic states are (4.18), and for a given \( M \) (or a given \( E \)), \( \mathcal{N}_{bh}(M) \) is the number of different \( |\Psi(\text{star})\rangle \) -states, that satisfy (4.21) (or (4.23)).

Consider first the limit \( N \to \infty \): according to (4.21) the energy, \( E \), will be finite only if we use the Wick order. Then if only a finite number of d.o.f. are excited, \( E \) will be finite. In that case there are infinitely many other d.o.f. that are in their ground states, and we face two (probably related) problems: first, our semiclassical approximation is very bad when most of the d.o.f. are in their ground state. Second, \( \mathcal{N}_{bh} \) is infinite (there are infinitely many ways to choose a finite number of exited d.o.f. from an infinite number of total d.o.f.) so the entropy will diverge. As a matter of fact, the limit \( N \to \infty \) is
questionable. For example (2.10) is as much a radius quantization condition as it is a mass quantization condition. This means that one cannot simply divide \(a_0\) infinitely many times. This is clearly a quantum gravitational issue. But there is a way to avoid it: If we choose \textit{not} to use the Wick order, then if \(E\) is finite, \(N\) must be finite too. We see that (2.12) or its generalization (4.23) provide us with a natural cutoff\[N_{\text{max}} \sim (M/M_P)^2\], which for a classical black hole is a big, but still finite.

Of course \(1 \leq N \leq N_{\text{max}}\). In a sense, \(N\) describe the amount of inhomogeneity. For \(N = 1\) we have the homogeneous case, for \(N = 2\) the “almost” homogeneous one, and so on until \(N = N_{\text{max}}\) which describe the general inhomogeneous star. We have then

\[
N_{\text{bh}}(M) = \sum_{N=1}^{N_{\text{max}}} N_{\text{bh}}(N, M)
\]  

(5.2)

where \(N_{\text{bh}}(N, M)\) is the corresponding number for a specific \(N\). Our semi-classical approximation is good as long as \(N\) is much smaller then \(N_{\text{max}}\), but we will see that the contribution to (5.2) from \(N > N_{\text{max}}/2\) is the same as from \(N < N_{\text{max}}/2\). So at least we have a good estimate to (5.2).

It is easy to see that the \(N\)’s that will contribute to (5.2) must satisfy \(N = N_{\text{max}} - 2j\), \(j = 0, 1, \ldots, (N_{\text{max}} - 1)/2\). Let us start from \(N = N_{\text{max}}\). In that case we have only one state (4.18), \(|\Psi\rangle = |0 >_1 |0 >_2 \ldots |0 >_{N_{\text{max}}}\rangle\). Next we consider \(N = N_{\text{max}} - 2\), it is easy to see that they are \(N_{\text{max}} - 2\) states\[|\Psi\rangle = |0 >_1 ..|1 >_k ..|0 >_{N_{\text{max}} - 2}\rangle\]. In a similar way, we have for any \(N = N_{\text{max}} - 2j\)

\[
N_{\text{bh}}(N, M) = C^j_{N_{\text{max}} - 1 - j}
\]  

(5.3)

\[\text{11}\text{In our geometrical units } M_P = l_P = h^{1/2}.
\[\text{12}\text{The oscilators are distinguishable because they have different frequencies, } \omega_k = 8r_k.
\text{Or in other words: different } k \text{’s correspond to different shells which are distinguishable because they have different radii.}

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where $C^k_m$ are the binomial coefficients. So

$$N_{bh}(M) = \sum_{j=0}^{(N_{max}-1)/2} C_j^{N_{max}-1-j} \tag{5.4}$$

Though there is no known analytic expression for (5.4) [13], it is elementary to show numerically that

$$\sum_{j=0}^n C_j^{2n-j} \simeq \exp(0.962n - 0.320) \tag{5.5}$$

In our case we have $n = (N_{max} - 1)/2 \sim (M/M_P)^2$, so

$$N_{bh}(M) \sim \exp \left( C \frac{M^2}{M_P^2} \right) \tag{5.6}$$

where $C$ is a constant of order unity. And we find (using (5.1)) that the entropy of the black hole is

$$S_{bh} = C \frac{M^2}{M_P^2} + S_0 \tag{5.7}$$

The entropy is proportional to the surface area of the black hole, or equivalently, it goes linearly with the quantum number $n_{tot}$, see (4.23), in agreement with the Bekenstein-Hawking entropy.

We could use a different approach to calculate the entropy: One can use the Wick order, so (4.22) is replaced with

$$E = \hbar < \Omega > \sum_{k=1}^N n_k \tag{5.8}$$

but still take a finite $N_{max}$. From (2.10) and $R_0 \sim a_0$ we have that $\Delta R_0 \geq \hbar/R_0$, and from $R_0 = a_0 r_s$ we get $(\Delta R_0)_{min} = a_0 r_s/N_{max}$. So in the case of black holes, $r_s = 1$ and $R_0 \sim M$, we have $N_{max} \sim R_0^2/\hbar \sim M^2/M_P^2$. 21
In this case $N$ can take all the integer values between $N_{\text{max}}$ and unity; the degeneracy is

$$N_{\text{bh}}(M) = \sum_{N=1}^{N_{\text{max}}-1} C_N^{N_{\text{max}}-1} = 2^{N_{\text{max}}-1}$$  \hfill (5.9)

Now the entropy is

$$S_{\text{bh}} = \ln 2 \frac{M^2}{M_P^2} + \tilde{S}_0$$  \hfill (5.10)

We see that (5.7) and (5.10) have the same form, the entropy is proportional to the surface area of the black hole, but the prefactors are different. One should get the correct prefactor in the full exact model.

Another thing to notice is that the degeneracy (and so the entropy) of the gravitational d.o.f. is very similar to the degeneracy of other field d.o.f. [14], so it is tempting to think that the gravitational d.o.f. that we use are more appropriate for a unified scheme.

### 5.2 Temperature

Using the standard thermodynamical definition of the temperature we have

$$T_{\text{bh}}^{-1} = \frac{\partial S_{\text{bh}}}{\partial E_{\text{bh}}}$$  \hfill (5.11)

We have $E_{\text{bh}} \sim M$, and using (5.6) we get

$$T_{\text{bh}} \sim \frac{M_P^2}{M}$$  \hfill (5.12)

in agreement with the Hawking temperature [8].

One may argue that the microscopic states (4.18) are microscopic both to a freely falling observer ("Kruskal observer") and to an outside observer ("Schwarzschild observer"). This means that (5.1) should be the same for Kruskal observers as well. Then the entropy (5.7), and temperature (5.12),
are the same for both the Kruskal and Schwarzschild observers. This contradicts the known results that a thermal Schwarzschild state corresponds to a zero Kruskal temperature [7,14]. But remember that though the microscopic states (4.18) are the same for both observers, the macroscopic states are quite different. For a Schwarzschild observer there is an horizon, and from the no-hair theorems, there is only one macroscopic quantity (in the Schwarzschild case) by which one can determine the state. This is of course the mass $M$ of the star. In that case the degeneracy is exactly what we get from (4.21), and indeed we have (5.7) and (5.12). On the other hand, for a Kruskal observer, there is no horizon, and the macroscopic state is determined by an infinite number of macroscopic quantities. For example in our model, we have a global hyperbolic space-time, so a classical solution is determined by the initial data $\left( y(r,t=0), \dot{y}(r,t=0) \right)$. In our case we have $\dot{y}(r,t=0) = 0$, so a classical solution is determine by $y(r,t=0)$, or equivalently by all the moments $P_n = \int_0^r r^n y(r,t=0) dr$, which are macroscopic quantities. This means that all the $\Omega_k$'s in (4.23) are determined, and (at least semiclassically) the state (4.18) is determined completely by the macroscopic state. This means that for a Kruskal observer there is no degeneracy, and the entropy and temperature vanish.

The entropy (5.1) is sometimes called “entangled entropy”, but we think that (at least in the case of black holes) it should be consistent with the thermal entropy. The way to check this is to couple the system to other fields. When we couple the gravitational d.o.f. to other fields, we have the following picture: The fields are in “equilibrium” with the gravitational d.o.f. (the black hole). According to a Schwarzschild observer, it is a thermal equilibrium with the temperature (5.12), and according to a Kruskal observer

\footnote{The question of classical (and of course quantum) observables in gravity is an open one, but in principle one should be able to determine those quantities.}
it is a zero temperature situation. This is a static scenario, in agreement with
the “eternal black hole” picture [7], and with the Thermo-field approach [15].

One can consider also a dynamical situation: a black hole creation and
evaporation. This will be done in the next subsection.

5.3 Hawking Evaporation

So far we have studied only a static picture as seen by a Schwarzschild ob-
server, in which a black hole is in a thermal state in equilibrium with the
outside region. But one can use our formalism to study also the dynamical
process of Hawking evaporation. Using the semiclassical adiabatic argu-
ments, one assumes that at each time the star is in a state (4.18), and the
Hawking radiation is the result of a transition between a level \(|\Psi(n_{tot})\rangle\)
(see (4.23)) to one of the closest levels, \(|\Psi(n_{tot} - 1)\rangle\) [3]. Using energy
conservation and (4.23), the radiation frequency satisfies

\[
\hbar \omega_{rad} = \Delta M(n_{tot}, n_{tot} - 1) \sim \frac{M_P^2}{M}.
\]  

(5.13)

On the other hand the temperature is proportional to the radiating energy
(frequency), so we have

\[
T_H \sim \hbar \omega_{rad} = \Delta M \sim \frac{M_P^2}{M}
\]  

(5.14)

in agreement with (5.12), and with Hawking results.

In this dynamical situation, one can calculate the lifetime of the level
\(|\Psi(n_{tot})\rangle\) [3]. This should be finite, because there is an interaction with the
vacuum state of the radiation fields. Now this is not the Kruskal vacuum
\footnote{Known as the Hartle-Hawking [16], or Israel [15] vacuum.}
(like in the eternal black hole case). The vacuum state is now the Unruh

14Known as the Hartle-Hawking [16], or Israel [15] vacuum.
vacuum [17]. This lifetime can be estimated to be proportional to the inverse of the imaginary part of the effective action (in the Unruh vacuum), and one get the mass rate [3]

\[
\frac{dM}{dt} \sim \frac{T}{\Delta M} \sim \frac{M_P^2}{M^2}
\]  

(5.15)

in agreement with Hawking results, which assume a black body radiation rate.

If we “extrapolate” our results to the quantum region \((n \sim 1)\), we can say that there should be a “quantum remnant” of mass \(M_{rem} \sim M_P\) at the end of the Hawking evaporation [18]. But this is pure speculation because we ignore back-reaction as well as strong quantum effects in our model.

6 Concluding Remarks

In this work we used the canonical quantization approach of spherically symmetric dust matter universes, first given by Lund, and apply it to the case of collapsing stars and black holes. The quantum states describing those universes are bound states and one gets a discrete spectrum.

First let us consider some of the physical consequences of the quantized spectrum. One may question the collapse process itself, because if the collapsing star must satisfy (2.11) for example, then in the space of all masses and initial radii, only a set of measure zero satisfy it, so maybe most of the stars will not collapse at all? This is not the case because though the mass is a constant (by energy conservation), \(R_0\) (or in the general case, all the other geometrical quantities) can fluctuate, and one must calculate \(\Delta R_0/R_0\). If this is a very small number, then the collapse is possible in a general situation.
Using (2.11) we have

$$\frac{\Delta R_0}{R_0} \sim \frac{\hbar^2 n}{M R_0^3} \sim \left(\frac{M_P}{M}\right)^{1/2} \left(\frac{l_P}{R_0}\right)^{3/2} \sim \frac{1}{n}$$  \hspace{1cm} (6.1)$$

which for astronomical objects is a very small number. For example in the case of our sun, we have $\Delta R_\odot/R_\odot \sim 10^{-100}$. This means that for astronomical objects, the mass quantization cannot affect the classical collapse process. On the other hand, if we “extrapolate” our results to Plankc size objects, then the collapse itself may be affected by the quantization conditions. This may be another reason to consider stable Planck size objects (black holes)?

The effect of the mass quantization on the Hawking radiation spectrum, will be mainly on very large wavelengths, $\lambda \geq M_{bh}$. The black hole cannot radiate or absorb radiation with $\lambda > M_{bh}$ because it correspond to $\Delta M$ smaller than the distance between nearest levels. For astronomical objects this effect will be hard to detect.

So we see that for “classical objects” (astronomical stars), for which $n \gg 1$, the correspondence principle works, and the quantum effects are negligible.

It is also quite easy to recover the classical laws of black hole thermodynamics: using (4.23) and $\mathcal{A} = 16\pi M^2$ we get the first law of (Schwarzschild) black hole thermodynamics

$$\delta M = \frac{\partial M}{\partial n} \delta n \sim \frac{M_P^2}{M} \delta \mathcal{A}$$  \hspace{1cm} (6.2)$$

And because $M_{bh} \sim E_{bh}$ we have $S_{bh} \sim \mathcal{A}$ and $T_{bh} \sim M_P^2/M_{bh}$.

The second law is just (5.7) with the fact that $\Delta S_{bh} \geq 0$ for an isolated system, while the generalized second law is $\Delta S_{\text{tot}} \geq 0$, where $S_{\text{tot}} = S_{bh} + S$.

In the case of spherically symmetric dust matter, the infinite number of gravitational degrees of freedom decoupled, and each shell of dust moves independently. It is possible to choose the coordinates and the field variables
such that each shell is an harmonic oscillator. This is a simple generalization
of the homogeneous Oppenheimer- Snyder model.

In the case of black holes, the discrete spectrum gives the Bekenstein
area quantization: the area of the black hole is an integer number times the
Planck area.

It is very easy to calculate the degeneracy of this system (of independent
oscillators), and from it to get the entropy of the black hole. The results
agree with the known Bekenstein-Hawking entropy: the entropy is propor-
tional to the surface area of the black hole. Then one can use the standard
thermodynamic definitions to get all the other thermodynamic quantities
(e.g. Hawking temperature).

It seems surprising that our simple model (of spherically symmetric dust
matter) gives “enough” degeneracy, and the correct Bekenstein-Hawking en-
tropy. One might think that in the general case (no symmetry, and a general
matter field) the degeneracy will be much bigger, and so also the entropy,
which would contradict known results. But this is not necessarily the case.
One should remember that we have been able to quantize the dust system,
because we could fix the gauge completely, which means that we correctly
choose the coordinate system and solved the constraints. A consistent quan-
tization of the general case (if it exists) may be achieved by a “free field
representation”, which will be a generalization of our independent harmonic
oscillators. If this is the case, then the general degeneracy will be quite simi-
lar to what we have in the dust case, as will the black hole thermodynamics.
Maybe this is one thing that we can learn from our simple model

\[15\text{ Though infinite-dimensional.}\]

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is the interaction between the geometrical degrees of freedom that we use and other fields.

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