QUANTUM NOISE AND ITS IMPORTANCE TO THE QUANTUM CLASSICAL TRANSITION PROBLEM: ENERGY MEASUREMENT ASPECTS

Adélcio C. Oliveira- e-mail adelcio@ufs.jedu.br
Departamento de Física e Matemática, Universidade Federal de São João Del Rei,
Ouro Branco, 36420 000, Minas Gerais, Brazil
Zolacir T. Oliveira Junior, Nestor Santos Correia
Departamento de Ciências Exatas e Tecnológicas, Universidade Estadual de Santa Cruz,
Ilhéus, 45662 000, Bahia, Brazil

Abstract. In the present contribution we discuss the role of experimental limitations in the classical limit problem. We studied some simple models and found that Quantum Mechanics does not re-produce classical mechanical predictions, unless we consider the experimental limitations ruled by uncertainty principle. We have shown that the discrete nature of energy levels of integrable systems can be accessed by classical measurements. We have defined a precise limit for this procedure. It may be used as a tool to define the classical limit as far as the discrete spectra of integrable systems are concerned.

Keywords: quantum noise, complementarity, classical limit, energy spectrum, correspondence principle

1. Introduction

There were no conclusive explanation for the disappearance of quantum effects in the macroscopic regime. Which of the quantum features we don’t see in our day life experience? Does quantum mechanics reproduce the observed results of macroscopic experiments? The first try to answer this question is attributed to Bohr, the correspondence principle. Bohr’s correspondence principle (Bohr, 1998). In this contribution, the main goal is to investigate the role of quantum noise in the quantum-classical transition problem. We focus on the spectra discreteness. We propose a procedure to use spectroscopic information from model Hamiltonians and time energy Heisenberg relations in order to decide whether a quantum system can be described by CM. The quantum behavior is characterized by the discreteness of energy spectra. We are considering a gedankenexperiment, where the experimentalist does not know Quantum Mechanics but tries obtain the energy spectrum. Our results confirm previous one (Angelo, 2007; Oliveira et al., 2012) that asserts the importance of the quantum noise to understand quantum classical transition problem.
2. The quantum noise in position and momentum measurements

In this section we show that quantum uncertainty are inherent to any system. After a position measurement we assume that the quantum state is $|x\rangle$. We prepare an ensemble of $N$ identical particles prepared in the same initial state, then the result of the measurement of position for $i$-th particle is $x_i$ and its mean is $\overline{x_i} = r$, thus $x_i = r + \xi_i$, where $\xi_i$ is $N(0, \delta x^2)$, i.e. normal random variable with variance $s^2$, thus the state after the measurement is

$$|\varphi\rangle = |\overline{x}\rangle,$$

(1)

the overbar represents ensemble mean. Using the position displacement operator, $\hat{S}(\lambda) = \exp\left(-\frac{i}{\hbar}\hat{p}\lambda\right)$, and the resolution of unity in $|p\rangle$ bases we have

$$|\varphi\rangle = \int dp \exp\left(-\frac{i}{\hbar}p\xi_i\right) |p\rangle \langle p | r \rangle.$$

(2)

This ensemble mean can be easily computed (Oksendal, 2000; Gardiner & Zoller, 2010; Oliveira, 2013), it is $\exp\left(-\frac{i}{\hbar}p\xi_i\right) = \exp\left(-\frac{p^2 \Delta x^2}{2\hbar^2}\right)$, also we have $\langle p | r \rangle = (2\pi\hbar)^{-1/2} \exp\left(-\frac{ipr}{\hbar}\right)$, then

$$|\varphi\rangle = (2\pi\hbar)^{-1/2} \int dp \exp\left(-\frac{p^2 \Delta x^2}{2\hbar^2}\right) \exp\left(-\frac{ipr}{\hbar}\right) |p\rangle.$$

(3)

and

$$\varphi(x) = \langle x | \varphi\rangle = (2\pi\hbar)^{-1} \int dp \exp\left(-\frac{p^2 \Delta x^2}{2\hbar^2}\right) \exp\left(-\frac{ip(r-x)}{\hbar}\right)$$

(4)

integrating (4) we obtain

$$\varphi(x) = \frac{1}{\sqrt{2\pi}s^2} \exp\left(-\frac{(x-r)^2}{2\delta x^2}\right).$$

(5)

For the momentum measurement we observe that $\hat{T}(\lambda) = \exp\left(\frac{i}{\hbar}\hat{q}\lambda\right)$ is the momentum displacement operator and $p_i = d + \zeta_i$, again $\zeta_i$ is $N(0, \delta p^2)$ then

$$\varphi(p) = \frac{1}{\sqrt{2\pi}\alpha^2} \exp\left(-\frac{(p-d)^2}{2\delta p^2}\right),$$

(6)

where $d$ is the ensemble mean of $p_i$. The maximum precision is achieved when $\delta x\delta p = \hbar/2$, this is the standard quantum limit (SQL) (Lynch, 1985; Appleby, 1998). Even if one measures the state with greater precision than SQL (Rozena et al., 2012), one can not use it as an initial state, as pointed by Ballentine (Ballentine, 1970) “the uncertain principle restricts the degree of statistical homogeneity which is possible to achieve in an ensemble of similarly prepared systems and thus it limits the precision for any system can made”.
3. The large quantum numbers limit

It is well known that time and frequency are conjugated variables in a pair of Fourier transforms in classical physics, the duration of a signal and the respective frequency are subjected to an unsharpness relation \( \Delta t \Delta \omega \geq \frac{1}{2} \) that is classical indeed. Frequency, in quantum theory, is another way of speaking about energy. An uncertainty relation between time and energy must be seriously considered, studied and interpreted, although there are objections due to the fact that time is not associated to a dynamical operator canonically conjugated to the Hamiltonian. An exhaustive examination of this matter is made by Peres in his book [Peres, 2002]. On the other hand we have to assume that QM is not an objective description of physical reality. It only predicts the probability of occurrence of stochastic macroscopic events, following specified preparation procedures.

The main stream of our proposal is not face the discussion of the time-energy uncertainty relation, we face the problem of to decide under what conditions a system may be considered as Classical or Quantum. It is in this sense that we tackle the problem of dealing with the physical reality: how can one measure and what is in fact measured. Thus, we use the product of energy differences between neighbor levels by the corresponding classical period differences to compare with the time-energy uncertainty relation to classify a system as classical or quantum. If this product fulfills the time-energy uncertainty relation it is quantum otherwise it is classical.

To do this we add an element connected to this subject that concerns integrable systems: the decision wether a system is classical or quantum depends on the experimental apparatus which are essentially classical. One undoubtedly quantum feature is the discreteness of at least part of the energy spectrum. The essential idea here is that if one tries to measure the energy of the system in question using classical canonical pairs \( q, p \), the information about the quantum nature of the particle will be lost. In spite of this fact, as we show in what follows, the function

\[
y(n) = |\Delta E_n \Delta \tau_n|
\]

where \( \Delta E_n = (E_n - E_{n-1})/2 \) is merely the energy difference between two neighbor levels, (it is the maximum uncertainty in energy for the state \( a |E_n\rangle + b |E_{n-1}\rangle \)) and \( \Delta \tau_n = (\tau_n - \tau_{n-1})/2 \), with \( \tau_n \) being the classical period associated to the energy \( E_n \). If the experiment has an accuracy \( \delta t \) it limits the period measurement precision, in real systems it is desirable that \( \delta t << \Delta \tau \), we assume that they are of same order, \( \Delta \tau \approx \delta t \).

The \( y(n) \) function can be heuristically justified if we consider the Bohr-Sommerfeld quantization rule for periodic systems, that states

\[
I = \oint pdq = 2\pi \hbar n
\]

thus we have

\[
\langle K \rangle = \frac{\pi n \hbar}{\tau},
\]

where \( \langle K \rangle \) is the the mean kinetic energy, \( \tau \) being the classical period associated with \( K \) and \( n \) is a quantum number.
Observing that
\[ \delta I = \oint \delta H \, dt \]  
(10)
Consider two neighboring periodic motions of the same system. Then H is a constant, and we have
\[ \delta I = \tau \delta H. \]  
(11)
Note that (11) determines that the energy of the system depends only on I, then Bohr’s quantization rule determines the energy of the system. The Bohr-Sommerfeld quantization rule makes a direct connection between kinetics energy and classical period, it was also demonstrated (Oliveira, 2012) that, for integrable systems, is possible to reconstruct semiclassically the quantum state using classical dynamics. Since you know the Wigner function of the system it is possible to infer the period related to the state (Oliveira, 2012; Oliveira & Nemes, 2001). Alternatively, we can use the semiclassical quantization rule (Berry & Balazs, 1978; Stockmann, 2000). This function (7) is stated for a more clear definition of a classical limit. As we will show, the fact that one will only be able to see a continuum of energies does not mean that the classical limit has been reached. It only reveals that we have become “myope” to see the nature of the spectrum.

In what follows, we will consider a simple one dimensional systems with discrete spectra. Let us assume that a state has been prepared in the following way
\[ |\Psi(0)\rangle = a |E_n\rangle + b |E_{n-1}\rangle \]  
(12)
where \{ |E_n\rangle \} are eigenstates of the hamiltonian \( H \)
\[ \hat{H} |E_n\rangle = E_n |E_n\rangle \]  
(13)
The variance in energy difference, for any time, of this state is given by
\[ \Delta E = |a| |b| [E_n - E_{n-1}] \]  
(14)
Using the normalization condition \( |a|^2 + |b|^2 = 1 \), it is easy to check that the maximum for this energy difference is achieved for \( |a| = |b| = 1/\sqrt{2} \), and is given by
\[ \Delta E(n) = 1/2 |E_n - E_{n-1}|. \]  
(15)
How do we define the classical limit in this simple case? Let us assume the following \( \lim_{n \to \infty} \frac{\Delta E_n}{E_n} = 0 \). The above expression means that as the energy increases \( (E_n) \), the uncertainty in energy of the state \( |\Psi(t)\rangle \) becomes negligible as compared to \( E_n \). Let us discuss a few examples.
Harmonic Oscillator

For the harmonic oscillator we have

$$\Delta E = \frac{1}{2} \hbar \omega \Rightarrow \frac{\Delta E}{E_n} = \frac{1/2}{(n + 1/2)}$$

and we clearly have

$$\lim_{n \to \infty} \frac{\Delta E_n}{E_n} = 0.$$  \hspace{1cm} (16)

Particle in a box

In this case we have

$$E_n = \hbar^2 \frac{n^2 \pi^2}{2ma^2},$$

where $n = 1, 2, 3, \ldots$ and $m$ is the mass of the particle and $a$ is the box width. Then we have

$$\Delta E_n = \frac{\hbar^2}{4} \left| \frac{(2n - 1)\pi^2}{ma^2} \right|.$$  \hspace{1cm} (17)

Once again we have

$$\lim_{n \to \infty} \frac{\Delta E_n}{E_n} \to 0.$$  \hspace{1cm} (18)

The fact that this limit is zero is the argument usually found in textbooks in order to define the classical limit (Home, 1997; Cohen et al., 1977; Eisberg & Resnick, 1994). Of course it is true that when one considers the high energy limit, it becomes increasingly difficult to obtain good experimental resolution. However, as we show below, this limit does not necessarily imply that it is impossible to obtain the needed resolution.

As we know, quantization has been observed in some macroscopic systems like superconducting Josephson junctions (Richard et al., 1981). Besides, one can use something analogous to the Heisenberg microscope (Braginsky & Khalili, 1992) in order to obtain the velocity of the particle. In the case of a particle in a box, discussed above, once we know the velocity, the energy is easily obtained. The measurement of the velocity implies a perturbation of the position which depends on the characteristics of the apparatus and in principle, is independent of the quantum number $n$. Thus, if one uses the adequate measurement the discrete character of a spectrum can be verified.

4. Classical measurement of the Energy

When one deals with realistic systems, say, atoms, the spectrum is obtained from the emitted or absorbed electromagnetic radiation. A completely different approach is used for macroscopic systems. Within CM, for closed systems, the energy is a function of position and momentum $E = f(p, q)$. So, from now on, we will call Classical Measurement of energy every process which makes use of the relation $E = f(p, q)$ in order to obtain the energy of a given system, classical or quantum. As an example, we may look at the particle in a box again. In this case, the energy is a direct function of the velocity, so that once we determine the velocity, the energy will be defined. In practice, one may measure the time it takes for $2s$ inversions in the momentum and so determine the period of the motion.
Classically, the period is given by
\[
\tau = \frac{2a}{v} = a\sqrt{\frac{2m}{E}}.
\] (19)

In the above expression, we are considering the correspondent classical period for a specific energy eigenvalue. Since the energy levels are determined by QM, regardless of the energy scale one is talking about, the allowed values of the period are
\[
\tau_n = \frac{2a^2m}{\hbar n\pi}.
\] (20)

So, the difference in period for two quantum neighboring levels is
\[
\Delta\tau = \frac{\tau_n - \tau_{n-1}}{2} = \frac{a^2m}{\hbar} \frac{1}{n(n-1)}.
\] (21)

From the above expression one can see that it becomes increasingly difficult to distinguish two higher adjacent energy levels by this method. Let us take a look at the product \(|\Delta E_n \Delta \tau_n|\)

where \(\Delta E_n = (E_n - E_{n-1})/2\) and \(\Delta \tau_n = (\tau_n - \tau_{n-1})/2\). \(\tau_n\) is the classical period associated with the energy \(E_n\). It is interesting to observe the behavior of the function
\[
y(n) = |\Delta E_n \Delta \tau_n|.
\] (22)

There is no mathematical restriction for \(y(n)\), but the set \(\{n \in |y(n) < \hbar/2\}\) is beyond the SQL since Quantum Mechanics forbids such precision.\(^5\) For the case in question,
\[
y(n) = \frac{\hbar\pi}{4} \frac{(2n - 1)}{(n - 1)n}.
\] (23)

If we take \(n \geq 4\), we find \(y(4) < \hbar/2\). Here, we are just using the fact that the time-energy uncertainty relation has to be respected since we are considering that Quantum Mechanics must prevent Classical Mechanics. Also it is easy to see that \(\lim_{n\to\infty} y(n) = 0\).

In the more realistic case of a hydrogenoid atom we have
\[
E_n = -\frac{\mu Z^2 e^4}{2\hbar^2 n^2}
\] (24)

where \(Ze\) is the total charge interacting with the electron of charge \(e\), and \(\mu\) is the reduced mass of the system. As opposed to the previous case, the energy levels become closer as \(n\) grows. Thus, the question is: from which \(n\) can we say that the spectrum is continuous from the point of view of Classical Mechanics?

---

\(^5\)The time-energy uncertainty relation \(\Delta E \Delta t \geq \hbar/2\) is a first principle limitation and has nothing to do with experimental errors. A deeper discussion on the subject can be found in Refs. (Braginsky & Khalili, 1992; Appleby, 1998; Rajeev, 2003)
In this case
\[ \Delta E = \frac{\mu Z^2 e^4}{4 \hbar^2} \left| \frac{2n - 1}{n^2(n - 1)^2} \right| , \]  
(25)
\[ \Delta T = \frac{\pi \hbar^3}{Z^2 e^4 \mu} \left[ 3n^2 - 3n + 1 \right] , \]  
(26)
and
\[ y(n) = \frac{\pi \hbar (2n - 1) [3n^2 - 3n + 1]}{4n^2(n - 1)^2} . \]  
(27)

Now, for the $1/r$ potential, we have $y(n) < \hbar/2$ for $n \geq 9$, so one won’t be able to observe the quantization for $n$ bigger than 9 while using classical measurement of the energy.

Other interesting case is the Morse Potential, frequently used to describe the spectra of molecules (Oliveira & Nemes, 2001). The Morse potential is defined as
\[ U(x) = D \left( e^{-2\alpha x} - 2 e^{-\alpha x} \right) , \]  
(28)
where $D$ and $\alpha$ are constants experimentally determined. For $s$ waves, i.e., the orbital angular momentum is zero, we have
\[ E(n) = -D + \hbar \omega \left[ \frac{1}{2} (n + 1/2) - \frac{1}{\zeta} (n + 1/2)^2 \right] . \]  
(29)
$\zeta$ is also experimentally determined. The classical period corresponding to each $E(n)$ is given by
\[ \tau(n) = 2\pi \sqrt{\frac{MRo^2}{2 |E(n)| \alpha^2}} \]  
(30)
where $Ro$ is a function of $\alpha$ and $\zeta$. It is easy to verify that, for the hydrogen molecule parameters, $y(n) < \hbar/2$ for all possible $n$. Since the Morse potential is quasi-harmonic for low energies we observe that it is not possible to distinguish neighboring discrete states with a classical measurement. From this example, we may conclude that any potential that is approximately harmonic have no assessable discrete spectrum through a classical measurement of the energy.

4.1 Harmonic Oscillator

For a harmonic oscillator, $E = \frac{p^2}{2m} + \frac{1}{2}kq^2$, its period is $\tau = \frac{2\pi}{\omega}$ where $\omega = \sqrt{\frac{k}{m}}$ which is energy independent, thus we conclude that it can not be used to characterize the spectrum. Another way of classically determining the energy can be obtained by measuring $q$ and $p$ at same time, therefore classical energy uncertain is given by
\[ \delta E = \frac{p}{m} \delta p + kq \delta q \]  
(31)
Without lost of generality, we choose $\delta p = \sqrt{\frac{m\omega}{2}}a$ and $\delta q = \sqrt{\frac{\hbar}{2m\omega}}a$ then we obtain the relation

$$\delta p \delta q = a^2\hbar/2.$$  

(32)

In case of $a=1$ we have the minimum uncertain defined under Hisenberg relation, in general we have

$$\delta E = \left|p\sqrt{\frac{\hbar\omega}{2m}} + k|q|\sqrt{\frac{\hbar}{2m\omega}} \right| a$$  

(33)

then

$$\delta E^2 = \frac{2}{\hbar} \left(p\sqrt{\frac{\hbar\omega}{2m}} + kq\sqrt{\frac{\hbar}{2m\omega}} \right)^2 \delta p \delta q$$  

(34)

From (16) we have

$$\delta E = \frac{1}{2}\hbar\omega$$  

(35)

taking into account that its energy is $E = \hbar\omega(n + 1/2)$, then after some straightforward algebra we find that $\delta p \delta q < \hbar/2$ for all $n$, then we need an experimental resolution that is beyond SQL. Thus we can say that his spectrum can not be resolved by Classical Energy Measurement.

5. Conclusion

In this work we have shown that the discrete nature of the energy levels can be accessed by classical measurements in some cases. We also defined a precise limit for this procedure using the function $y(n) = |\Delta E_n \Delta \tau_n|$ and comparing it with the time-energy uncertainty principle. This maneuver gives us a complementarity principle and a well defined mathematical limit dictated by the experiment. Of course, the fact that we are not able to recognize the discrete nature of a spectrum does not necessarily mean it is not discrete. It only means how “myope” we are, suggesting that Classical Mechanics can be viewed as a blurring of essential aspects of Quantum Mechanics and also explains why it took so long to find quantum effects.
REFERENCES

Angelo, R. M., 2007. Phys. Rev. A., 76, 052111.

Appleby, D. M., 1998. International Journal of Theoretical Physics, 37, 1491.

Ballentine, L. E., 1970. Rev. Mod. Phys., 42, 358.

Berry, M. V., Balazs, N. L., 1978. J. Phys. A, 12 625.

Bohr, N., 1998. Causality and Complementarity, supplementary papers edited by Jan Faye and Henry Folse as The Philosophical Writings of Niels Bohr, Vol. IV, Woodbridge: Ox Bow Press.

Braginsky, V. B., Khalili, F. Y., 1992. Quantum Measurement (Cambridge University Press, Cambridge).

Cohen-Tannouji, C., Diu, B., Laloë, F., 1977. Quantum Mechanics, vol. 1 (Wiley, New York).

Eisberg, R. M., Resnick, R., 1994. Física Quântica: Átomos, Moléculas, Sólidos, Núcleos e Partículas (Campus, São Paulo).

Gardiner, C. W., Zoller, P. 2010. Quantum Noise (Springer-Verlag, Berlin, Heidelberg).

Home, D., 1997. Conceptual Foundations of Quantum Physics. An Overview from Modern Perspectives (Plenum Press, New York and London).

Lynch, R., 1985. Phys. Rev. Lett., 54, 1599.

Novaes, M., Aguiar, M. A. M., 2005. Phys. Rev. A, 71 012104.

Oksendal, B., 2000. Stochastic Differential Equations: An introduction with applications, 5 ed., Springer-Verlag Heidelberg New York.

Oliveira, A. C., Nemes, M. C., 2001. Physica Scripta, 64, 279.

Oliveira, A. C., Bosco de Magalhães, A. R., Peixoto Faria, J. G., 2012. Physica A, 391, 5082.

Oliveira, A. C., 2012. Jour. Mod. Phys., 3, 694.

Oliveira, A. C., 2013. Classical Limit of Quantum Mechanics Induced by Continuous Measurements, Submited Physica A.

Peres, A., 2002. Quantum Theory: Concepts and Methods (Kluver Academic Publishers, New York, Boston, Dordrecht, London, Moscow), chapter 12, item 12-8 and references therein cited.
Rajeev, S. G., 2003. *A theory of errors in quantum measurement*, arXiv: [quant-ph/0306037](http://arxiv.org/abs/quant-ph/0306037).

Richard, F. V., Web, R. A. 1981. Phys. Rev. Lett., **47**, 265.

Rozema, L. A., Darabi, A., Mahler., D. H., Hayat, A., Soudagar, Y., Steinberg, A. M., 2012. Phys. Rev. Lett., **109**, 100404.

Stöckmann, H. J., 2000. *Quantum Chaos: an introduction*, Cambridge University Press, New York.

Wisniacki, D. A., Vergini, E., Benito, R. M., Borondo, F., 2006. Phys. Rev. Lett., **97** 094101.