ON THE EIGENPROBLEM FOR GAUSSIAN BRIDGES

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Abstract. Spectral decomposition of the covariance operator is one of the main building blocks in the theory and applications of Gaussian processes. Unfortunately it is notoriously hard to derive in a closed form. In this paper we consider the eigenproblem for Gaussian bridges. Given a base process, its bridge is obtained by conditioning the trajectories to start and terminate at the given points. What can be said about the spectrum of a bridge, given the spectrum of its base process? We show how this question can be answered asymptotically for a family of processes, including the fractional Brownian motion.

1. Introduction

The eigenproblem for a centered process $X = (X_t, t \in [0,1])$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ consists of finding all pairs $(\lambda, \varphi)$ satisfying the equation

$$\int_0^1 K(s,t)\varphi(s)ds = \lambda \varphi(t), \quad t \in [0,1],$$

(1.1)

where $K(s,t) = \mathbb{E}X_sX_t$ is the covariance function of $X$. If $K$ is square integrable, this problem is well known to have countably many solutions: the eigenvalues $\lambda_n, n \in \mathbb{N}$ are nonnegative and converge to zero, when put in the decreasing order, and the corresponding eigenfunctions $\varphi_n$ form an orthonormal basis in $L^2([0,1])$.

One of the earliest and most influential implications of this result is the Karhunen–Loève theorem, which asserts that $X$ admits the representation as the $L^2(\Omega)$-convergent series

$$X_t = \sum_{n=1}^{\infty} \langle X, \varphi_n \rangle \varphi_n(t)$$

(1.2)

where the scalar products $\langle X, \varphi_n \rangle = \int_0^1 X_s \varphi_n(s)ds$ are orthogonal zero mean random variables with variance $\mathbb{E}(\mathbb{E}(X, \varphi_n)^2 = \lambda_n$.

Spectral decomposition (1.2) is useful in both theory and applications (see, e.g., [1], [16]). However explicit solutions to eigenproblem (1.1) are notoriously hard to find and they are available only in special cases [12], [9, 8], [21], [22, 23], including the Brownian motion with $K(s,t) = t \wedge s$:

$$\lambda_n = \frac{1}{((n - \frac{1}{2})\pi)^2} \quad \text{and} \quad \varphi_n(t) = \sqrt{2} \sin \left((n - \frac{1}{2})\pi t\right)$$

(1.3)

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and the Brownian bridge with covariance function $\tilde{K}(s, t) = s \wedge t - st$:

$$\tilde{\lambda}_n = \frac{1}{(\pi n)^2} \text{ and } \tilde{\varphi}_n(t) = \sqrt{2} \sin(\pi nt). \quad (1.4)$$

These formulas are obtained by reduction of the eigenproblem for integral operators to explicitly solvable boundary value problems for ordinary differential equations. Similar kind of reduction also works for a number of processes, related to the Brownian motion, putting (1.1) into the framework of Sturm–Liouville type theory [22].

In this paper we consider the eigenproblem for Gaussian bridges. For a base process $X = (X_t, t \in [0, 1])$, the corresponding bridge $\tilde{X} = (\tilde{X}_t, t \in [0, 1])$ is obtained by “restricting” the trajectories to start and terminate at the given points. For Gaussian processes, such restriction amounts to the usual conditioning. Hence if $X$ is a centered Gaussian base process with the starting point $X_0 = 0$ and covariance function $K(s, t)$, the corresponding zero-to-zero bridge is the centered Gaussian process

$$\tilde{X}_t = X_t - \frac{K(t, 1)}{K(1, 1)} X_1, \quad t \in [0, 1]$$

with the covariance function

$$\tilde{K}(s, t) = K(s, t) - \frac{K(s, 1)K(t, 1)}{K(1, 1)}. \quad (1.5)$$

Various aspects of general Gaussian bridges are discussed in [11], [25]. Aside of being interesting mathematical objects, they are important ingredients in applications, such as statistical hypothesis testing [15], exact sampling of diffusions [3], etc.

The covariance operator of the bridge with kernel (1.5) is a rank one perturbation of the covariance operator of its base process. This explains similarity between (1.3) and (1.4) and suggests that the spectra of the two processes must be closely related in general. This is indeed the case and one can find an exact expression for the Fredholm determinant of $\tilde{K}$ in terms of the Fredholm determinant of $K$ even for more general finite rank perturbations (see, e.g., [26], Ch.II, 4.6 in [13]). As mentioned above, the precise formulas for the eigenvalues and eigenfunctions of $K$ are rarely known; however, the exact asymptotic approximations can be more tractable. This raises the following question:

*Can the exact asymptotics of the eigenvalues and the eigenfunctions for the bridge be deduced from those of the base process?*

A rough answer to this question is given by the general perturbation theory [14], which implies that the eigenvalues of $K$ and $\tilde{K}$ agree in the leading asymptotic term, as it happens for (1.3) and (1.4) (see, e.g., the proof of Lemma 2 in [4]). More delicate spectral discrepancies are harder to exhibit and seem to be highly dependent on the perturbation structure. This is vividly demonstrated in the paper [21], where the kernels of the following form are considered, cf. (1.5):

$$\tilde{K}_Q(s, t) = K(s, t) + Q\psi(s)\psi(t). \quad (1.6)$$

Here $Q$ is a scalar real valued parameter and $\psi$ is a function in the range of $K$. It turns out that for any $Q$ greater than a certain critical value $Q^*$, the spectrum of $\tilde{K}_Q$ coincides with
that of \( K \) in the first two asymptotic terms. For \( Q = Q^* \) their spectra depart in the second term. The deviation is quantified in [21], when \( \psi \) is an image of an \( L^2([0,1]) \) function, under the action of \( K \). The bridge process under consideration corresponds precisely to the critical case, but with \( \psi(x) = K(1, x) \) being an image of the distribution \( \delta(t-1) \), rather than of a square integrable function; hence the approach of [21] is not directly applicable here.

In this paper we will take a different route towards answering the above question, using the particular structure of the perturbation inherent to bridges. Observe that the eigenproblem \( \tilde{K}\tilde{\varphi} = \tilde{\lambda}\tilde{\varphi} \) can be written in terms of the covariance operator of the base process

\[
\int_0^1 K(s,t)\tilde{\varphi}(s)ds - K(1,t)\int_0^1 K(1,s)\tilde{\varphi}(s)ds = \tilde{\lambda}\tilde{\varphi}(t), \quad t \in [0,1],
\]

where, without loss of generality, \( X \) is assumed to be normalized so that \( K(1,1) = 1 \).

Taking scalar product with the eigenfunction \( \varphi_n \) of \( K \) gives

\[
\langle \tilde{\varphi}, \varphi_n \rangle = c\frac{\lambda_n}{\lambda_n - \tilde{\lambda}}\varphi_n(1), \quad \tilde{\lambda} \notin \{\lambda_1, \lambda_2, \ldots\}
\]

where \( c := \int_0^1 K(1,s)\tilde{\varphi}(s)ds \). If \( K(s,t) \) is continuous, by Mercer’s theorem

\[
K(s,t) = \sum_{n \geq 1} \lambda_n \varphi_n(t)\varphi_n(s)
\]

where the convergence is absolute and uniform, and hence, in view of (1.8),

\[
c = \int_0^1 K(1,s)\tilde{\varphi}(s)ds = \sum_{n \geq 1} \lambda_n \varphi_n(1)\langle \tilde{\varphi}, \varphi_n \rangle = c\sum_{n \geq 1} \frac{\lambda_n^2}{\lambda_n - \tilde{\lambda}}\varphi_n(1)^2
\]

Since \( \sum_{n \geq 1} \lambda_n \varphi_n(1)^2 = K(1,1) = 1 \) and \( c \neq 0 \) whenever \( \tilde{\lambda} \notin \{\lambda_1, \lambda_2, \ldots\} \), we obtain the following transcendental equation for the eigenvalues of the bridge

\[
\sum_{k=1}^{\infty} \frac{\lambda_k}{\lambda_k - \tilde{\lambda}}\varphi_k(1)^2 = 0,
\]

and the corresponding expression for its eigenfunctions:

\[
\tilde{\varphi}(t) = c\sum_{n \geq 1} \frac{\lambda_n}{\lambda_n - \tilde{\lambda}}\varphi_n(1)\varphi_n(t), \quad t \in [0,1].
\]

Note that the roots of (1.9) are not determined solely by the eigenvalues of the base process, but also require some information on its eigenfunctions.

The objective of this paper is to show how equations (1.9) and (1.10) can be used to construct asymptotic approximation for the solutions to the bridge eigenproblem (1.7), given the exact asymptotics of the eigenvalues and eigenfunctions of the corresponding base process.
2. The main result

For definiteness we will work with a particular process, though the same approach applies whenever similar spectral approximation for the base process is available (as, e.g., for the processes considered in [6, 7]). Our study case will be the fractional Brownian motion (fBm), that is, the centered Gaussian process $B^H = (B^H_t, t \in [0, 1])$ with covariance function

$$K(s, t) = \frac{1}{2} \left( t^{2H} + s^{2H} - |t - s|^{2H} \right), \quad s, t \in [0, 1],$$

where $H \in (0, 1]$ is the Hurst exponent.

This is the only $H$-selfsimilar Gaussian process with stationary increments. For $H = \frac{1}{2}$ it coincides with the standard Brownian motion, but otherwise exhibits quite different properties. In particular, for $H \neq \frac{1}{2}$, it is neither semimartingale nor a Markov process. For $H > \frac{1}{2}$ the covariance sequence of its increments on the grid of integers is not summable. This long range dependence property makes the fBm a powerful tool in modelling, see [24], [2].

The fBm has been extensively studied since its introduction in [19] (see, e.g., [10], [20], [24]) and there seems to be little hope to obtain exact solutions to (1.1), see [18]. Hence efficient approximations are of significant interest. A few largest eigenvalues and the corresponding eigenfunctions can be approximated numerically, as, e.g., in [28], but relative accuracy of such approximations quickly deteriorates as the $\lambda_n$’s get smaller and, in our experience, the problem becomes computationally intractable already for $n \geq 50$.

Smaller eigenvalues and the corresponding eigenfunctions can be approximated using the following asymptotics (see [4, 5], [23], [17], [27] for earlier results):

**Theorem 2.1** (Theorem 2.1 in [6]).

1. For the fractional Brownian motion with $H \in (0, 1)$, the ordered sequence of the eigenvalues satisfies

$$\lambda_n = \frac{\sin(\pi H)\Gamma(2H + 1)}{\nu_n^{2H+1}} n = 1, 2, \ldots$$

where $\nu_n = \pi n + \pi \gamma_H + O(n^{-1})$ as $n \to \infty$ and

$$\gamma_H := -\frac{1}{2} + \frac{(H - \frac{1}{2})^2}{H + \frac{1}{2}}$$

(2.2)

2. The corresponding normalized eigenfunctions admit the approximation

$$\varphi_n(t) = \sqrt{2} \sin \left( \nu_n t + \pi \eta_H \right) + \int_0^\infty f_0(u)e^{-\nu_n u}du + (-1)^n \int_0^\infty f_1(u)e^{-(1-t)\nu_n u}du + n^{-1} r_n(t),$$

where $f_0(\cdot)$ and $f_1(\cdot)$ are explicit functions defined in [6] and

$$\eta_H := \frac{H - \frac{3}{2}}{4} \frac{H - \frac{1}{2}}{H + \frac{1}{2}}.$$  

(2.4)

\[\text{constants } \gamma_H \text{ and } \eta_H \text{ appear in [6] in a slightly more subtle, but equivalent form.}\]
The residual $r_n(t)$ in (2.3) is bounded uniformly over $n \in \mathbb{N}$ and $t \in [0, 1]$. Moreover, the values of the eigenfunctions at $t = 1$ satisfy
\[ \varphi_n(1) = (-1)^n \sqrt{2H + 1} (1 + O(n^{-1})). \] (2.5)

In principle, the spectral approximation technique developed in [6, 7] is applicable to the fractional Brownian bridge directly. However, somewhat surprisingly to the authors, it does not produce results quite as accurate as those of Theorem 2.1. In a nutshell, the approach in those papers hinges on asymptotic analysis of certain auxiliary integral equation with a large parameter (see e.g. (4.9) in [6]). It is relatively easy to find the leading order asymptotic term of its solution as the parameter tends to infinity. For all the processes considered in [6, 7], including the fBm, this turns out to be sufficient for approximating the eigenvalues, exactly up to the second order.

For the fractional Brownian bridge, the solution to the auxiliary equation still admits a reasonably simple first order asymptotics, but it cancels out in further calculations, failing to produce a condition bearing useful information about the second order in the eigenvalues expansion. Consequently, only the rough, first order asymptotics of the eigenvalues becomes available, without further, more delicate analysis of the auxiliary integral equation. This turns out to be a far more complicated task, which remains out of reach at the moment.

In this paper an alternative approach, based on the equations (1.9) and (1.10), is suggested. We will show how the exact spectral asymptotics of the bridge can be derived from that of the base process. Specifically, we will prove the following result:

**Theorem 2.2.**

1. For the fractional Brownian bridge with $H \in (0, 1)$, the ordered sequence of the eigenvalues satisfies
\[ \tilde{\lambda}_n = \frac{\sin(\pi H) \Gamma(2H + 1)}{\tilde{\nu}_n^{2H+1}} \quad n = 1, 2, \ldots \] (2.6)
where $\tilde{\nu}_n = \pi n + \pi \tilde{\gamma}_H + O(n^{-1} \log n)$ as $n \to \infty$ and
\[ \tilde{\gamma}_H := \gamma_H + \frac{H}{H + \frac{1}{2}}. \]

2. The corresponding eigenfunctions admit the approximation
\[ \tilde{\varphi}_n(t) = \sqrt{2} \sin \left( \tilde{\nu}_n t + \pi \tilde{\gamma}_H \right) + \int_0^\infty f_0(u) e^{-\tilde{\nu}_n u} du + \]
\[ (-1)^n \int_0^\infty e^{-\tilde{\nu}_n (1-t) u} \left( \sin(\pi (\tilde{\gamma}_H - \gamma_H)) f_1(u) du + \cos \pi (\tilde{\gamma}_H - \gamma_H) f_1(u) \right) du + \]
\[ (-1)^n \sin \pi (\tilde{\gamma}_H - \gamma_H) \int_0^\infty \tilde{g}_1(\tilde{\nu}_n (1-t) u) f_1(u) du + n^{-1} \log n \tilde{r}_n(t) \] (2.7)
where $f_1$ and $\tilde{g}_1$ are functions, defined in closed forms by (3.20) and (3.21) below, and the residual $\tilde{r}_n(t)$ is bounded, uniformly over $n \in \mathbb{N}$ and $t \in [0, 1]$. 

Remark 2.3.

a. The eigenvalues of the fBm and its bridge differ by a constant shift in the second order asymptotic term of the corresponding “frequencies”:

\[ \pi \tilde{\gamma}_H - \pi \gamma_H = \pi \frac{H}{H + \frac{1}{2}} \]

which reduces to the familiar constant \( \pi/2 \) in the standard Brownian case \( H = \frac{1}{2} \). The residuals in \( \nu_n \) and \( \tilde{\nu}_n \) differ by the \( \log n \) factor, which may well be an artifact of the approach.

b. The eigenfunctions of the bridge inherit the oscillatory term in (2.7) from the corresponding term of the base process (2.3), however, with a frequency shift. A more complicated modification occurs in the integral terms, which are responsible for the boundary layer: their contribution is asymptotically negligible away from the endpoints of the interval, but is persistent near the boundary. For the base process, these terms force the eigenfunctions to vanish at \( t = 0 \) and approach the alternating values (2.5) at \( t = 1 \); for the bridge, they push the eigenfunctions to zero at both endpoints. Consequently the change is more significant near 1 than near the origin.

Tracing back the definitions of all the functions involved, it can be seen that the boundary layer vanishes for \( H = \frac{1}{2} \) and the leading asymptotic term in (2.7) reduces to the familiar formula (1.4) for the standard Brownian motion. Note also that the fractional Brownian bridge is time reversible, that is, its law is invariant under the time change \( t \mapsto 1 - t \). This property, however, does not play a role in our approach, at least, explicitly.

c. Some additional insight into the problem can be gained by considering a slightly more general perturbation of the base covariance operator (cf. (1.5) and (1.6))

\[ \tilde{K}(s,t) = K(s,t) + Q \frac{K(s,1)K(t,1)}{K(1,1)} \]  \hspace{1cm} (2.8)

with \( Q \geq -1 \), which corresponds to the bridge for \( Q = -1 \). It can be seen that for \( Q > -1 \), the method of [6], [7] still applies and, for the fBm as the base process, shows that the eigenvalues of the perturbed operator coincide with those of the base operator, at least up to the second order. Further asymptotic terms are negligible, but not uniformly over \( Q \): at \( Q = -1 \) the residual explodes and the second order asymptotics changes completely. Hence the case of the bridge is “critical” in the our previous approach [6], [7], just as it is in the method of A.Nazarov in [21]. The alternative technique in this paper applies to the perturbed operator in (2.8), without breaking down at \( Q = -1 \).

d. The basic equation (1.9), which relates the eigenvalues of the bridge to those of the base process, involves some information on the eigenfunctions, namely their values at the endpoint of the interval \( \varphi_n(1) \). The analogous equation, relating the eigenvalues of the base process to those of the bridge has the form

\[ \sum_{n \geq 1} \frac{\tilde{\beta}_n^2}{\lambda - \lambda_n} = 1, \quad \lambda \in \mathbb{R}_+ \]
where
\[ \tilde{\beta}_n = \int_0^1 K(1,s)\tilde{\varphi}_n(s)ds. \]

Hence by calculations, similar to those in this paper, the exact asymptotics of \( \lambda_n \) can be derived, if the sufficiently precise asymptotics of both \( \tilde{\lambda}_n \) and \( \tilde{\beta}_n \) is known.

3. Proof of Theorem 2.2

3.1. A preview. Before giving the full proof, it is insightful to consider the special case \( H = \frac{1}{2} \), corresponding to the standard Brownian motion. Let us see how the formulas (1.4) can be derived from (1.3), using the equations (1.9) and (1.10). To this end, it will be convenient to change the variables to
\[ \mu_k := \frac{1}{\pi \sqrt{\lambda_k}} = k - \frac{1}{2} \quad \text{and} \quad \tilde{\mu} = \frac{1}{\pi \sqrt{\lambda}} \]
so that in view of (1.3) the equation (1.9) becomes
\[ g(\tilde{\mu}) := \sum_{k=1}^{\infty} \frac{1}{(k - \frac{1}{2})^2 - \tilde{\mu}^2} = 0. \] (3.1)

The explicit formula for this series is well known:
\[ g(\tilde{\mu}) = \frac{\pi}{2\tilde{\mu}} \tan(\pi \tilde{\mu}) \] (3.2)
and can be obtained by means of the residue calculus. It will be instructive to recall the calculation: define the function
\[ f(z) = \text{ctg} \frac{\pi (z + \frac{1}{2})}{z^2 - \tilde{\mu}^2}, \quad z \in \mathbb{C} \]
which is meromorphic with the simple poles at \( z_{\pm} = \pm \tilde{\mu} \) and \( z_k = k - \frac{1}{2} \). Integrating \( f(z) \) over a circular contour of radius \( R \) and taking the limit \( R \to \infty \) we find that
\[ \text{Res}\{f; z_+\} + \text{Res}\{f; z_-\} + \sum_{k \in \mathbb{Z}} \text{Res}\{f; z_k\} = 0. \]
Here the residues are
\[ \text{Res}\{f; z_+\} = \text{Res}\{f; z_-\} = -\frac{1}{2\tilde{\mu}} \tan(\pi \tilde{\mu}) \]
\[ \text{Res}\{f; z_k\} = \frac{1}{\pi (k - \frac{1}{2})^2 - \tilde{\mu}^2}. \]
Since the sequence \( (k - \frac{1}{2})^2 - \tilde{\mu}^2, \quad k \in \mathbb{Z} \) is symmetric around \( \frac{1}{2} \), the expression (3.2) is obtained and the equation (3.1) produces the roots \( \tilde{\mu}_n = \pi n, \quad n = 1, 2, ..., \) confirming the formula for the eigenvalues in (1.4).

The corresponding eigenfunctions can be found using (1.10):
\[ \tilde{\varphi}_n(t) = 2c_n \tilde{\mu}_n^2 \sum_{k=1}^{\infty} \frac{(-1)^k \sin \mu_k \pi t}{\mu_k^2 - \tilde{\mu}^2} = 2c_n \tilde{\mu}_n^2 \sum_{k=1}^{\infty} \frac{(-1)^k \sin(k - \frac{1}{2}) \pi t}{n^2 - (k - \frac{1}{2})^2}. \]
Using similar residue calculus, the series can be computed exactly:
\[ \tilde{\phi}_n(t) = -c_n n \pi (-1)^n \sin \pi nt, \]
which agrees with the formula in (1.4), after normalizing to the unit norm.

The more general case \( H \in (0, 1) \) is different in two aspects:

1. The function \( g(\tilde{\mu}) \) for \( H \neq \frac{1}{2} \) involves a power function with non-integer exponent (see (3.3) below) and hence, in addition to the poles, has a discontinuity across the branch cut. Consequently the Cauchy theorem cannot be applied as before and a different contour is to be chosen. A natural choice is the boundary of the half disk, which lies in the right half plane, but such integration produces an additional integral term along the imaginary axis. Asymptotic analysis shows that its contribution is non-negligible on the relevant scale for all values of \( H \) but \( \frac{1}{2} \); thus it is “invisible” in the case of standard Brownian motion.

2. The exact formulas for the eigenvalues and eigenfunctions for \( H \neq \frac{1}{2} \) are unavailable beyond their precise asymptotics as in (2.1)-(2.2). It is then reasonable to consider first the perturbed version of the equation (1.9), in which \( \lambda_k \) and \( \varphi_k \) are replaced with the corresponding asymptotic approximations from Theorem 2.1. This gives the main terms in the eigenvalues formula (2.6). It remains then to show that the roots of the perturbed and the exact equations get close asymptotically on the suitable scale. Once the asymptotics of \( \tilde{\lambda}_n \) becomes available, it can be plugged into (1.10), along with the expressions for \( \lambda_k \) and \( \varphi_k(t) \), to construct the approximations for the bridge eigenfunctions.

### 3.2. The eigenvalues.

Let us change the variable to \( \tilde{\mu} \) such that
\[ \tilde{\lambda} = \frac{\sin(\pi H) \Gamma(2H + 1)}{\pi^{2H + 1}}, \]
in which case equation (1.9) becomes
\[ g(\tilde{\mu}) := \sum_{k=1}^{\infty} \frac{\varphi_k(1)^2}{\mu_k^{2H+1} - \tilde{\mu}^{2H+1}} = 0, \]  
where we defined \( \mu_k := \nu_k / \pi \). Observe that \( g(\cdot) \) is continuous, increases on \( \mathbb{R}_+ \setminus \{ \mu_k, k \in \mathbb{N} \} \) and
\[ \lim_{\tilde{\mu} \searrow \mu_k} g(\tilde{\mu}) = -\infty \quad \text{and} \quad \lim_{\tilde{\mu} \nearrow \mu_k} g(\tilde{\mu}) = +\infty, \quad k \in \mathbb{N}. \]
Consequently it has a unique root \( \tilde{\mu}_n \) at each one of the intervals \( (\mu_n, \mu_{n+1}) \).

In view of the asymptotics (2.1)-(2.3), it makes sense to consider first the perturbed equation
\[ g^0(\tilde{\mu}) := \sum_{k=1}^{\infty} \frac{2H + 1}{(k + \gamma H)^{2H+1} - \tilde{\mu}^{2H+1}} = 0, \]  
where the eigenvalues and eigenfunctions of the base process are replaced with their asymptotic approximations. The next step is to argue that the roots of the exact and perturbed equations (3.3) and (3.4) are close on an appropriate scale, asymptotically as \( n \to \infty \).
These steps are implemented in Lemmas 3.1 and 3.2 respectively, which together imply assertion 1 of Theorem 2.2.

**Lemma 3.1.** The unique root $\tilde{\mu}_n^a \in (n + \gamma_H, n + 1 + \gamma_H)$ of equation (3.4) satisfies

$$\tilde{\mu}_n^a = n + \gamma_H + \frac{H}{H + \frac{1}{2}} + O(n^{-1}).$$

**Proof.** A convenient expression can be found for $g^a(\bar{\mu})$ using the residue calculus. To this end note that the principal branch of the function

$$f(z) := \frac{\text{ctg}(\pi(z - \gamma_H))}{z^{2H+1} - \bar{\mu}^{2H+1}}, \quad z \in \mathbb{C} \setminus \mathbb{R}$$

is meromorphic on the right half plane with simple poles at $z_0 := \bar{\mu}$ and $z_k := k + \gamma_H$, $k = 1, 2, \ldots$ Note that $z_k > 0$ since $\gamma_H \in (-\frac{3}{4}, -\frac{1}{2})$ for $H \in (0, 1)$. Integrating this function over the boundary of the half disk of radius $R \in \mathbb{N}$ in the right half plane gives

$$\int_{-\infty}^{-iR} f(z) dz + \int_{iR}^{\infty} f(z) dz = 2\pi i \text{Res} \{f, z_0\} + 2\pi i \sum_{k=1}^{R} \text{Res} \{f, z_k\}$$

where $C_R$ denotes the half circle arc. Since $\text{ctg}(\cdot)$ is bounded on $C_R$, by Jordan’s lemma the integral over $C_R$ vanishes as $R \to \infty$ and we obtain

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} f(it) dt = -\sum_{k=1}^{\infty} \text{Res} \{f, z_k\}. \quad (3.5)$$

Computing the residues

$$\text{Res} \{f, z_0\} = \frac{\text{ctg}(\pi(\bar{\mu} - \gamma_H))}{\lim_{z \to \bar{\mu}} z^{2H+1} - \bar{\mu}^{2H+1}} = \text{ctg}(\pi(\bar{\mu} - \gamma_H)) \frac{\bar{\mu}^{2H} - \gamma_H}{2H + 1}$$

$$\text{Res} \{f, z_k\} = \frac{1}{(k + \gamma_H)^{2H+1} - \bar{\mu}^{2H+1}} \lim_{z \to z_k} \frac{\cos(\pi z)}{\sin(\pi z)} = \frac{1}{\pi (k + \gamma_H)^{2H+1} - \bar{\mu}^{2H+1}}$$

and plugging these expressions into (3.5) gives the more explicit formula

$$g^a(\bar{\mu}) = -\frac{2H + 1}{2} \int_{-\infty}^{\infty} f(it) dt - \frac{\pi}{\mu^{2H}} \text{ctg}(\pi(\bar{\mu} - \gamma_H)) =$$

$$= -\frac{2H + 1}{\mu^{2H}} \text{Re} \left\{ \int_{0}^{\infty} \frac{\text{ctg}(\pi(i\tau \bar{\mu} - \gamma_H))}{(i\tau)^{2H+1} - 1} d\tau \right\} - \frac{\pi}{\mu^{2H}} \text{ctg}(\pi(\bar{\mu} - \gamma_H)).$$

Hence the equation (3.4) becomes

$$\frac{\text{ctg}(\pi(\bar{\mu} - \gamma_H))}{\mu^{2H}} = -\frac{2H + 1}{\pi} \text{Re} \left\{ \int_{0}^{\infty} \frac{\text{ctg}(\pi(i\tau \bar{\mu} - \gamma_H))}{(i\tau)^{2H+1} - 1} d\tau \right\}. \quad (3.6)$$

Let $\tilde{\mu}_n^a$ be the unique root of (3.4) in the interval $(n + \gamma_H, n + 1 + \gamma_H)$, then

$$\int_{0}^{\infty} \frac{\text{ctg}(\pi(i\tau \tilde{\mu}_n^a - \gamma_H))}{(i\tau)^{2H+1} - 1} d\tau = \frac{1}{i} \int_{0}^{\infty} \frac{1 + e^{-2\pi(\tau \tilde{\mu}_n^a + i\gamma_H)}}{1 - e^{-2\pi(\tau \tilde{\mu}_n^a + i\gamma_H)}} \frac{1}{(i\tau)^{2H+1} - 1} d\tau$$

$$= \frac{1}{i} \int_{0}^{\infty} \frac{1}{(i\tau)^{2H+1} - 1} d\tau + R_n \quad (3.7)$$
with the residual satisfying
\[ \tilde{\mu}_n^a |R_n| \leq \int_0^\infty \left| \frac{2e^{-2\pi t}}{1 - e^{-2\pi t} - 2\pi i H} \frac{1}{(it/\tilde{\mu}_n^a)^{2H+1} - 1} \right| dt \leq \int_0^\infty \frac{2e^{-2\pi t}}{(it/\tilde{\mu}_n^a)^{2H+1} - 1} dt \xrightarrow{n \to \infty} \frac{1}{\pi} \]
(3.8)
where the second inequality holds since \( \cos(2\pi \gamma_H) \leq 0 \). The real part of the integral on the right hand side of (3.7) can be computed explicitly:
\[ \frac{1}{i} \int_0^\infty \frac{1}{(it)^{2H+1} - 1} d\tau = \frac{1}{i^{2H+2}} \int_0^\infty \frac{1}{\tau^{2H+1} - 1/i^{2H+1}} d\tau = \frac{1}{i^{2H+2} 2H + 1} \int_0^\infty \frac{u^{2H+1} - 1}{u - 1/i^{2H+1}} du = \frac{\pi}{2H + 1} \sin \frac{\pi}{2H+1} \]
and hence
\[ \text{Re} \left\{ \frac{1}{i} \int_0^\infty \frac{1}{(it)^{2H+1} - 1} d\tau \right\} = \frac{\pi}{2H + 1} \frac{\pi}{\text{ctg} \frac{\pi}{2H+1}}. \]
(3.9)
Plugging (3.8) and (3.9) into (3.7) and (3.6) and recalling that \( \tilde{\mu}_n^a - \gamma_H \in (n, n+1) \) and \( \gamma_H < 0 \), we obtain the claimed asymptotics:
\[ \tilde{\mu}_n^a = n + \gamma_H + \frac{1}{2} + \frac{1}{2H + 1} + O(n^{-1}), \quad n \to \infty. \]

The next step is to show that the roots of (3.4) and (3.3) are close on a suitable scale:

**Lemma 3.2.** The unique root \( \tilde{\mu}_n^a \in (\mu_n, \mu_{n+1}) \) of equation (3.3) satisfies
\[ \tilde{\mu}_n^a - \tilde{\mu}_n^a = O(n^{-1} \log n). \]

**Proof.** Suppose \( f : I \mapsto \mathbb{R} \) is a function on an open interval \( I \) with \( \frac{d}{dx} f(x) \geq r > 0 \) and a root \( x_0 \in I \). Let \( h : I \mapsto \mathbb{R} \) be a continuous strictly increasing function with \( \sup_{x \in I} |f(x) - h(x)| \leq b \) and assume that \( [x_0 - b/r, x_0 + b/r] \subset I \). Then \( h \) has a unique root \( y_0 \) and it satisfies
\[ |y_0 - x_0| \leq b/r. \]
(3.10)
We will apply this elementary bound to \( f := g^a \) and \( h := g \) on the interval \( I_n \) with the endpoints at \( n + \gamma_H \pm \delta \), where \( \delta > 0 \) is fixed small enough so that \( I_n \subset (\mu_n, \mu_{n+1}) \). Recall that by Lemma 3.1, the unique root \( \tilde{\mu}_n^a \in (n + \gamma_H, n + 1 + \gamma_H) \) of \( g^a \) belongs to \( I_n \) for all sufficiently large \( n \). The function \( g^a \) is differentiable on \( \mathbb{R}_+ \setminus \{k + \gamma_H : k \in \mathbb{N}\} \) and
\[ \inf_{\tilde{\mu} \in I_n} \frac{d}{d\tilde{\mu}} g^a(\tilde{\mu}) = \inf_{\tilde{\mu} \in I_n} \sum_{k=1}^{\infty} \frac{(2H+1)^2 \tilde{\mu}^{2H}}{((k + \gamma_H)^{2H+1} - \tilde{\mu}^{2H+1})^2} \geq \frac{\tilde{\mu}^{2H}}{((n + \gamma_H)^{2H+1} - \tilde{\mu}^{2H+1})^2} \geq \frac{(n + \gamma_H - \delta)^{2H}}{((n + \gamma_H)^{2H+1} - (n + \gamma_H - \delta)^{2H+1})^2} \geq c n^{-2H} \]
(3.11)
with a constant $c > 0$.

Next let us estimate the oscillation of $g^a(\tilde{\mu}) - g(\tilde{\mu})$ on $I_n$:

$$g^a(\tilde{\mu}) - g(\tilde{\mu}) = \sum_{k=1}^{\infty} \frac{(2H + 1) - \varphi_k(1)^2}{(k + \gamma_H)^{2H+1} - \tilde{\mu}^{2H+1}} + \sum_{k=1}^{\infty} \varphi_k(1)^2 \frac{\mu_k^{2H+1} - (k + \gamma_H)^{2H+1}}{(k + \gamma_H)^{2H+1} - \tilde{\mu}^{2H+1})} =: D_1(\tilde{\mu}) + D_2(\tilde{\mu}).$$

(3.12)

In view of (2.5),

$$\sup_{\tilde{\mu} \in I_n} |D_1(\tilde{\mu})| \lesssim \sup_{\tilde{\mu} \in I_n} \sum_{k=1}^{n} \frac{1/k}{(k + \gamma_H)^{2H+1} - \tilde{\mu}^{2H+1}} \lesssim \sum_{k=1}^{n} \frac{1/k}{(n + \gamma_H - \delta)^{2H+1} - (k + \gamma_H)^{2H+1}} + \sum_{k=n+1}^{\infty} \frac{1/k}{(k + \gamma_H)^{2H+1} - (n + \gamma_H + \delta)^{2H+1}} =: A_n + B_n,$$

where $x \lesssim y$ means $x \leq C y$ for some constant $C$. The first sum on the right satisfies

$$A_n \leq \int_{1}^{n^{-1}} \frac{1/x}{(n + \gamma_H - \delta)^{2H+1} - (x + \gamma_H)^{2H+1}} dx + O(n^{-2H-1}) =
\int_{1/n}^{-1/n} \frac{1/y}{1 - y^{2H+1}} dy + O(n^{-2H-1}) = O(n^{-2H-1} \log n)$$

Similar bound holds for $B_n$ and therefore

$$\sup_{\tilde{\mu} \in I_n} |D_1(\tilde{\mu})| \lesssim O(n^{-2H-1} \log n).$$

The second term in (3.12) satisfies

$$\sup_{\tilde{\mu} \in I_n} |D_2(\tilde{\mu})| \lesssim \sup_{\tilde{\mu} \in I_n} \sum_{k=1}^{\infty} \left| \frac{k^{2H-1}}{(k + \gamma_H)^{2H+1} - \tilde{\mu}^{2H+1})} \right| =
\frac{1}{n^{2H+2}} \int_{0}^{1-1/n} \frac{y^{2H-1}}{(1 - y^{2H+1})^2} dy + \frac{1}{n^{2H+2}} \int_{1/n}^{\infty} \frac{y^{2H-1}}{(1 - y^{2H+1})^2} dy + O(n^{-2H-1}) = O(n^{-2H-1}).$$

Hence the second sum in (3.12) is asymptotically negligible, that is,

$$\sup_{\tilde{\mu} \in I_n} |g^a(\tilde{\mu}) - g(\tilde{\mu})| \lesssim n^{-2H-1} \log n.$$

Plugging this estimate and (3.11) into (3.10) gives the claimed asymptotics. □
3.3. The eigenfunctions. The approximation (2.7) is obtained by plugging the asymptotics (2.1), (2.3) and (2.6) into the formula (1.10):

\[ \tilde{\varphi}_n(t) = c_n \sum_{k=1}^{\infty} \frac{\lambda_k}{\lambda_k - \lambda_n} \varphi_k(1) \varphi_k(t) = -c_n \mu_n^{2H+1} \sum_{k=1}^{\infty} \frac{1}{\mu_k^{2H+1} - \mu_n^{2H+1}} \varphi_k(1) \varphi_k(t) \]

where we set \( \mu_n := \nu_n / \pi \) and \( \bar{\mu}_n := \tilde{\nu}_n / \pi \) as in Lemma 3.1.

As before, we will first replace the exact values by their leading asymptotic terms and then will argue that the error, thus introduced, is negligible on the suitable scale. To this end, define (c.f. (2.3)):

\[ \tilde{\varphi}_n^{1,a}(t) = -c_n \mu_n^{2H+1} \sqrt{2H + 1} \sqrt{2} \sum_{k=1}^{\infty} (-1)^k \sin \left( \pi (k + \gamma_H) t + \pi \eta_H \right) \frac{1}{(k + \gamma_H)^{2H+1} - \bar{\mu}_n^{2H+1}} \]  
(3.13)

\[ \tilde{\varphi}_n^{2,a}(t) = -c_n \mu_n^{2H+1} \sqrt{2H + 1} \int_0^{\infty} f_0(u) \left( \sum_{k=1}^{\infty} \frac{(-1)^k e^{-(k+\gamma_H)\pi tu}}{(k + \gamma_H)^{2H+1} - \bar{\mu}_n^{2H+1}} \right) du \]  
(3.14)

\[ \tilde{\varphi}_n^{3,a}(t) = -c_n \mu_n^{2H+1} \sqrt{2H + 1} \int_0^{\infty} f_1(u) \left( \sum_{k=1}^{\infty} \frac{e^{-(k+\gamma_H)\pi (1-t)u}}{(k + \gamma_H)^{2H+1} - \bar{\mu}_n^{2H+1}} \right) du \]  
(3.15)

where \( \eta_H \) is the constant defined in (2.4).

Lemma 3.3. The function \( \tilde{\varphi}_n^a(t) = \tilde{\varphi}_n^{1,a}(t) + \tilde{\varphi}_n^{2,a}(t) + \tilde{\varphi}_n^{3,a}(t) \), satisfies

\[ \frac{\tilde{\varphi}_n^a(t)}{\| \tilde{\varphi}_n^a \|} = \sqrt{2} \sin \left( \tilde{\nu}_n t + \pi \eta_H \right) + \int_0^{\infty} f_0(u) e^{-\tilde{\nu}_n tu} du + \]

\[ (-1)^n \int_0^{\infty} e^{-\tilde{\nu}_n (1-t)u} \left( \sin(\pi (\gamma_H - \gamma_H)) f_1(u) du + \cos \pi (\gamma_H - \gamma_H) f_1(u) du \right) du + \]

\[ (-1)^n \sin \pi (\gamma_H - \gamma_H) \int_0^{\infty} \tilde{g}_1(\tilde{\nu}_n (1-t)u) f_1(u) du + n^{-1} \tilde{r}_n(t) \]

where \( \tilde{f}_1 \) and \( \tilde{g}_1 \) are explicit functions, defined in (3.20) and (3.21) below, and the residual \( \tilde{r}_n(t) \) is bounded uniformly over \( n \in \mathbb{N} \) and \( t \in [0,1] \).

Proof. The claimed approximation is obtained by finding the leading term asymptotics of the functions in (3.13)-(3.15) and normalizing their sum by a suitable common factor.

1) Asymptotics of (3.13). For fixed \( t \in [0,1] \) and \( \bar{\mu} > 0 \), consider the series

\[ h(\bar{\mu}) := \sum_{k=1}^{\infty} (-1)^k \sin \left( \pi (k + \gamma_H) t + \pi \eta_H \right) \frac{1}{(k + \gamma_H)^{2H+1} - \bar{\mu}^{2H+1}}. \]

A closed form formula for this expression can be found by means of residue calculus as in Lemma 3.1. To this end, consider the principal branch of the function

\[ f(z) := \frac{\sin \left( \pi (zt + \eta_H) \right)}{z^{2H+1} - \bar{\mu}^{2H+1}} \frac{1}{\sin(\pi (z - \gamma_H))} \quad z \in \mathbb{C} \setminus \mathbb{R}_- \]  
(3.16)
which is meromorphic on the right half plane with the simple poles at \( z_0 := \tilde{\mu} \) and \( z_k = k + \gamma_H, \ k \in \mathbb{N} \). Integrating \( f(z) \) over the half disc boundary in the right half plane we get
\[
\int_{-iR}^{iR} f(z)dz + \int_{C_R} f(z)dz = 2\pi i \text{Res}\{f; z_0\} + 2\pi i \sum_{k=1}^{R} \text{Res}\{f; z_k\}
\]
where \( C_R \) stands for the half circle arc with radius \( R \in \mathbb{N} \). The ratio of sines in (3.16) is bounded for any \( t \in [0, 1] \) and therefore, applying Jordan’s lemma, we get
\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} f(it)dt = -\text{Res}\{f; z_0\} - \sum_{k=1}^{\infty} \text{Res}\{f; z_k\},
\]
where the residues are
\[
\text{Res}\{f; z_0\} = \frac{\sin(\pi(\tilde{\mu} + \eta_H))}{\sin(\pi(\tilde{\mu} - \gamma_H))} \frac{1}{2H + 1} \frac{1}{\tilde{\mu}^{2H}}
\]
\[
\text{Res}\{f; z_k\} = \frac{\sin(\pi(k + \gamma_H)t + \eta_H))}{(k + \gamma_H)^{2H+1} - k^{2H+1}} (-1)^k \pi.
\]
Plugging these expressions into (3.17), we get
\[
h(\tilde{\mu}) = -\frac{\pi}{2} \frac{1}{2H + 1} \frac{1}{\tilde{\mu}^{2H}} \text{Re} \left\{ \int_{0}^{\infty} \frac{1}{(iu)^{2H+1} - 1} \frac{\sin(\pi(iu\tilde{\mu} + \eta_H))}{\sin(\pi(iu\tilde{\mu} - \gamma_H))} du \right\}.
\]
The second term simplifies to
\[
\text{Re} \left\{ \int_{0}^{\infty} \frac{1}{(iu)^{2H+1} - 1} \frac{\sin(\pi(iu\tilde{\mu} + \eta_H))}{\sin(\pi(iu\tilde{\mu} - \gamma_H))} du \right\} =
\int_{0}^{\infty} e^{-\pi u\tilde{\mu}}(1-t) \text{Re} \left\{ \frac{e^{-\pi (\gamma_H + \eta_H)}}{(iu)^{2H+1} - 1} du + R(t, \tilde{\mu}) \right\}
\]
with the residual satisfying
\[
\tilde{\mu} |R(t, \tilde{\mu})| \leq \tilde{\mu} \int_{0}^{\infty} \frac{2e^{-\pi u\tilde{\mu}}}{|(iu)^{2H+1} - 1|} du = \int_{0}^{\infty} \frac{2e^{-\pi s}}{|(is/\tilde{\mu})^{2H+1} - 1|} ds \xrightarrow{\tilde{\mu} \to \infty} \frac{2}{\pi}
\]
Plugging these expressions back gives
\[
\tilde{\varphi}_n^{1,\alpha}(t) \simeq \sqrt{2} \sin \left( \pi(\tilde{\mu}_n t + \eta_H) \right) + (-1)^n \sin \left( \pi(\tilde{\gamma}_H - \gamma_H) \right) \int_{0}^{\infty} e^{-\pi \tilde{\mu} \gamma_n (1-t)} u f_1(u) du + n^{-1}\tilde{\varphi}_n^{(1)}(t)
\]
(3.18)
where \( x \simeq y \) means \( x = C y \) with a constant \( C \) and we normalized by the factor
\[
\tilde{\varphi}_n := \epsilon_n \tilde{\varphi}_n \frac{\pi}{\sqrt{2H + 1}} \sin \pi(\tilde{\gamma}_H - \gamma_H).
\]
(3.19)
It can be seen (as in the calculation, concluding section 5.1.6. in [6]), that the norm of the integral term in (3.18) is of order \( O(n^{-1}) \) and hence the norm of \( \tilde{\varphi}_n^{1,\alpha} \) is asymptotic to
1 as \( n \to \infty \). The residual \( \tilde{r}_n^{(1)}(t) \) is uniformly bounded over \( n \in \mathbb{N} \) and \( t \in [0, 1] \) and the function \( \tilde{f}_1 \) is given by the formula

\[
\tilde{f}_1(u) := \frac{2H + 1}{\pi} \sqrt{2} \text{Re}\left\{ \frac{e^{-i\pi(\eta_H + \gamma_H)}}{(iu)^{2H+1} - 1} \right\}.
\] (3.20)

2) **Asymptotics of (3.14).** A closed form expression for the series

\[
h(\tilde{\mu}) := \sum_{k=1}^{\infty} \frac{(-1)^k}{(k + \gamma_H)^{2H+1} - \tilde{\mu}^{2H+1}} e^{-(k+\gamma_H)\pi tu}
\]

can be found by integrating the principal branch of the function

\[
f(z) := \frac{e^{-z\pi tu}}{z^{2H+1} - \tilde{\mu}^{2H+1} \sin(\pi(z - \gamma_H))}
\]

over the half disk boundary in the right half plane. As before,

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} f(it)dt = - \text{Res}\{f; z_0\} - \sum_{k=1}^{\infty} \text{Res}\{f; z_k\},
\]

with the same poles as defined above. The residues are given by

\[
\text{Res}\{f; z_0\} = \frac{1}{2H + 1} \frac{1}{\tilde{\mu}^{2H} \sin(\pi(\tilde{\mu} - \gamma_H))} e^{-\tilde{\mu}tu}
\]

\[
\text{Res}\{f; z_k\} = \frac{1}{\pi} \frac{(-1)^k}{(k + \gamma_H)^{2H+1} - \tilde{\mu}^{2H+1}} e^{-(k+\gamma_H)\pi tu}
\]

and therefore

\[
h(\tilde{\mu}) = - \frac{\pi}{2H + 1} \frac{1}{\tilde{\mu}^{2H} \sin(\pi(\tilde{\mu} - \gamma_H))} e^{-\tilde{\mu}tu}
\]

\[
- \frac{1}{\tilde{\mu}^{2H}} \text{Re}\left\{ \int_{0}^{\infty} \frac{e^{-is\tilde{\mu}tu}}{(is)^{2H+1} - 1} \sin(\pi(is\tilde{\mu} - \gamma_H)) \frac{1}{\sin(\pi(is\tilde{\mu} - \gamma_H))} ds \right\}.
\]

The integral term satisfies

\[
\left| \int_{0}^{\infty} \frac{e^{-is\tilde{\mu}tu}}{(is)^{2H+1} - 1} \frac{1}{\sin(\pi(is\tilde{\mu} - \gamma_H))} ds \right| \leq \frac{1}{\tilde{\mu}} \int_{0}^{\infty} \frac{2e^{-\pi s}}{(is/\tilde{\mu})^{2H+1} - 1} ds
\]

and hence, normalizing by the constant (3.19), we get

\[
\tilde{\varphi}_{n}^{2a}(t) \simeq \int_{0}^{\infty} f_0(u)e^{-\tilde{\mu}nu} du + n^{-1}\tilde{r}_n^{(2)}(t)
\]

with a uniformly bounded residual \( \tilde{r}_n^{(2)} \).

3) **Asymptotics of (3.15).** An explicit formula for the series

\[
h(\tilde{\mu}) := \sum_{k=1}^{\infty} \frac{e^{-(k+\gamma_H)\pi(1-t)u}}{(k + \gamma_H)^{2H+1} - \tilde{\mu}^{2H+1}}
\]
is obtained by integrating the principal branch of the function
\[
f(z) := \frac{e^{-z\pi(1-t)u}}{z^{2H+1} - \tilde{\mu}^{2H+1} - \tilde{\mu}^{2H+1}} \text{ctg}(\pi(z - \gamma_H)) \quad z \in \mathbb{C} \setminus \mathbb{R}.
\]
over the same contour as above:
\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} f(it) dt = -\text{Res}\{f; z_0\} - \sum_{k=1}^{\infty} \text{Res}\{f; z_k\}.
\]
The residues are
\[
\text{Res}\{f; z_0\} = \frac{1}{2H + 1} \frac{1}{\tilde{\mu}^{2H+1}} e^{-\tilde{\mu}(1-t)u} \text{ctg}(\pi(\tilde{\mu} - \gamma_H))
\]
\[
\text{Res}\{f; z_k\} = \frac{1}{\pi} \frac{e^{-(k+\gamma_H)\pi(1-t)u}}{(k + \gamma_H)^{2H+1} - \tilde{\mu}^{2H+1}}
\]
and therefore
\[
h(\tilde{\mu}) = -\frac{\pi}{2H + 1} \frac{1}{\tilde{\mu}^{2H+1}} e^{-\tilde{\mu}(1-t)u} \text{ctg}(\pi(\tilde{\mu} - \gamma_H))
\]
\[
- \frac{1}{\tilde{\mu}^{2H}} \text{Im} \left\{ \int_{0}^{\infty} \frac{e^{-is\pi(1-t)u}}{(is)^{2H+1} - 1} ds \right\} - \frac{1}{\tilde{\mu}^{2H}} R(\tilde{\mu}),
\]
where the function $R(\tilde{\mu})$ is bounded
\[
|R(\tilde{\mu})| \leq \frac{1}{\tilde{\mu}} \int_{0}^{\infty} \frac{2e^{-2\pi s}}{\left| (is/\tilde{\mu})^{2H+1} - 1 \right|} ds.
\]
Plugging these into (3.15) and normalizing by (3.19) gives
\[
\tilde{\varphi}_{n}^{3,0}(t) \approx (-1)^n \sin(\pi(\tilde{\gamma}_H - \gamma_H)) \int_{0}^{\infty} f_1(u) \tilde{g}_1(\tilde{\mu}_n \pi(1-t)u) du +
\]
\[
(-1)^n \cos \pi(\tilde{\gamma}_H - \gamma_H) \int_{0}^{\infty} f_1(u)e^{-\tilde{\mu}_n \pi(1-t)u} du + n^{-1} \tilde{\gamma}_n^{(3)}(t)
\]
where we defined
\[
\tilde{g}_1(x) = \frac{2H + 1}{\pi} \text{Im} \left\{ \int_{0}^{\infty} \frac{e^{-isx}}{(is)^{2H+1} - 1} ds \right\}. \tag{3.21}
\]

Finally, it is left to check that the eigenfunctions of the bridge are asymptotic to the expressions found in Lemma 3.3:

**Lemma 3.4.** For any $H \in (0, 1)$,
\[
\frac{\tilde{\varphi}_n(t)}{||\tilde{\varphi}_n||} - \frac{\tilde{\varphi}_n^{3,0}(t)}{||\tilde{\varphi}_n^{3,0}||} \leq Cn^{-1} \log n, \quad t \in [0, 1]
\]
for some constant $C$. 
Proof. Denote by \( \varphi_n^a(t) \) the leading asymptotic term in the eigenfunctions approximation (2.3) for the base process, satisfying

\[
|\varphi_n^a(t) - \varphi_n(t)| = |r_n(t)| n^{-1} \leq C n^{-1}
\]

with a constant \( C \). Then after normalizing by the factor (3.19)

\[
|\tilde{\varphi}_n(t) - \tilde{\varphi}_n^a(t)| / c_n \leq \frac{2H}{\mu_n} \sum_{k=1}^{\infty} \frac{\varphi_k(1)\varphi_k(t)}{\mu_k^{2H+1} - \mu_n^{2H+1}} - \frac{\varphi_n^a(1)\varphi_n^a(t)}{(k + \gamma_H)^{2H+1} - \mu_n^{2H+1}} \lesssim
\]

\[
\frac{2H}{\mu_n} \sum_{k=1}^{\infty} \left( \frac{1}{\mu_k^{2H+1} - \mu_n^{2H+1}} \right) + \frac{2H}{\mu_n^2} \sum_{k=1}^{\infty} \frac{1}{\mu_k^{2H+1} - \mu_n^{2H+1}} \lesssim n^{-1} \log n
\]

where all the estimates are obtained as in Lemma 3.2. The claim follows since \( \|\tilde{\varphi}_n / c_n\| = 1 + O(n^{-1}) \). \[\square\]

References

[1] Alain Berlinet and Christine Thomas-Agnan. *Reproducing kernel Hilbert spaces in probability and statistics*. Kluwer Academic Publishers, Boston, MA, 2004. With a preface by Persi Diaconis.

[2] Corinne Berzin, Alain Latour, and José R. León. *Inference on the Hurst parameter and the variance of diffusions driven by fractional Brownian motion*, volume 216 of *Lecture Notes in Statistics*. Springer, Cham, 2014. With a foreword by Aline Bonami.

[3] Alexandros Beskos and Gareth O. Roberts. *Exact simulation of diffusions*. *Ann. Appl. Probab.*, 14(4):2422–2444, 2005.

[4] Jared C. Bronski. Asymptotics of Karhunen-Loève eigenvalues and tight constants for probability distributions of passive scalar transport. *Comm. Math. Phys.*, 238(3):563–582, 2003.

[5] Jared C. Bronski. Small ball constants and tight eigenvalue asymptotics for fractional Brownian motions. *J. Theoret. Probab.*, 16(1):87–100, 2003.

[6] Pavel Chigansky and Marina Kleptsyna. Exact asymptotics in eigenproblems for fractional Brownian covariance operators. *Stochastic Process. Appl.*, 128(6):2007–2059, 2018.

[7] P. Chigansky, M. Kleptsyna, and D. Marushkevych. Exact spectral asymptotics of fractional processes. arXiv:1802.09045, 2018.

[8] Paul Deheuvels and Guennadi V. Martynov. A Karhunen-Loève decomposition of a Gaussian process generated by independent pairs of exponential random variables. *J. Funct. Anal.*, 255(9):2363–2394, 2008.

[9] Paul Deheuvels and Guennady Martynov. Karhunen-Loève expansions for weighted Wiener processes and Brownian bridges via Bessel functions. In *High dimensional probability, III* (Sandberg, 2002), volume 55 of *Progr. Probab.*, pages 57–93. Birkhäuser, Basel, 2003.

[10] Paul Embrechts and Makoto Maejima. *Selfsimilar processes*. Princeton Series in Applied Mathematics. Princeton University Press, Princeton, NJ, 2002.

[11] Dario Gasbarra, Tommi Sottinen, and Esko Valkeila. Gaussian bridges. In *Stochastic analysis and applications*, volume 2 of *Abel Symp.*, pages 361–382. Springer, Berlin, 2007.

[12] Jacques Istas. Karhunen-Loève expansion of spherical fractional Brownian motions. *Statist. Probab. Lett.*, 76(14):1578–1583, 2006.

[13] L. V. Kantorovich and V. I. Krylov. *Approximate methods of higher analysis*. Translated from the 3rd Russian edition by C. D. Benster. Interscience Publishers, Inc., New York; P. Noordhoff Ltd., Groningen, 1958.

[14] Tosio Kato. *Perturbation theory for linear operators*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition.
[15] E. L. Lehmann and Joseph P. Romano. *Testing statistical hypotheses*. Springer Texts in Statistics. Springer, New York, third edition, 2005.

[16] Mikhail Lifshits. *Lectures on Gaussian processes*. SpringerBriefs in Mathematics. Springer, Heidelberg, 2012.

[17] Harald Luschgy and Gilles Pagès. Sharp asymptotics of the functional quantization problem for Gaussian processes. *Ann. Probab.*, 32(2):1574–1599, 2004.

[18] B. B. Mandelbrot. On an eigenfunction expansion and on fractional Brownian motions. *Lett. Nuovo Cimento* (2), 33(17):549–550, 1982.

[19] Benoit B. Mandelbrot and John W. Van Ness. Fractional Brownian motions, fractional noises and applications. *Siam Rev.*, 10:422–437, 1968.

[20] Yuliya S. Mishura. *Stochastic calculus for fractional Brownian motion and related processes*, volume 1929 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2008.

[21] A. I. Nazarov. On a family of transformations of Gaussian random functions. *Teor. Veroyatn. Primen.*, 54(2):209–225, 2009.

[22] A. I. Nazarov and Ya. Yu. Nikitin. Exact $L_2$-small ball behavior of integrated Gaussian processes and spectral asymptotics of boundary value problems. *Probab. Theory Related Fields*, 129(4):469–494, 2004.

[23] A. I. Nazarov and Ya. Yu. Nikitin. Logarithmic asymptotics of small deviations in the $L_2$-norm for some fractional Gaussian processes. *Teor. Veroyatn. Primen.*, 49(4):695–711, 2004.

[24] V. Pipiras and M.S. Taqqu. *Long-Range Dependence and Self-Similarity*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, 2017.

[25] Tommi Sottinen and Adil Yazigi. Generalized Gaussian bridges. *Stochastic Process. Appl.*, 124(9):3084–3105, 2014.

[26] Shashikala Sukhatme. Fredholm determinant of a positive definite kernel of a special type and its application. *Ann. Math. Statist.*, 43:1914–1926, 1972.

[27] Seiji Ukai. Asymptotic distribution of eigenvalues of the kernel in the Kirkwood-Riseman integral equation. *J. Mathematical Phys.*, 12:83–92, 1971.

[28] Mark S. Veillette and Murad S. Taqqu. Properties and numerical evaluation of the Rosenblatt distribution. *Bernoulli*, 19(3):982–1005, 2013.