RESTRICTION THEOREM FOR THE FOURIER-HERMITE TRANSFORM
ASSOCIATED WITH THE NORMALIZED HERMITE POLYNOMIALS AND THE
ORNSTEIN-UHLENBECK-SCHRÖDINGER EQUATION

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Abstract. In this article, we prove the analogue theorems of Stein-Tomas and Strichartz on the discrete
surface restrictions of Fourier-Hermite transforms associated with the normalized Hermite polynomials
and obtain the Strichartz estimate for the system of orthonormal functions for the Ornstein-Uhlenbeck
operator $L = -\frac{1}{2} \Delta + (x, \nabla)$ on $\mathbb{R}^n$. Further, we show an optimal behavior of the constant in the
Strichartz estimate as limit of a large number of functions.

1. Introduction

A long-standing but persistent classical topic in harmonic analysis is the so-called restriction problem.
Originally emerged by the works of Stein in the late 1960s, the restriction problem is a key problem for
understanding the general oscillatory integral operators. The restriction problem and its applications are
crucial from the point of view of their credible implementation in many areas of mathematical analysis,
geometric measure theory, combinatorics, harmonic analysis, number theory, including the Bochner-
Riesz conjecture, Kakeya conjecture, the estimation of solutions to the wave, Schrödinger, and Helmholtz
equations, and the local smoothing conjecture for PDE’s [21]. For a Schwartz class function $f$ on $\mathbb{R}^n$,
the Fourier transform and the inverse Fourier transform of $f$ is defined as

$$\mathcal{F}(f)(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(x) e^{i\xi \cdot x} dx, \quad \xi \in \mathbb{R}^n,$$

and

$$\mathcal{F}^{-1}(f)(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(x) e^{-i\xi \cdot x} d\xi, \quad x \in \mathbb{R}^n,$$

respectively.

Given a surface $S$ embedded in $\mathbb{R}^n$ with $n \geq 2$, the classical restriction problem is the following:

**Problem A:** For which exponents $1 \leq p \leq 2$, the Fourier transform of a function $f \in L^p(\mathbb{R}^n)$ belongs
to $L^q(S)$, $1 \leq q \leq \infty$, where $S$ is endowed with its $(n-1)$-dimensional Lebesgue measure $d\sigma$?

A model case of the restriction problem which is often considered in the literature is the case $q = 2$
(see [15, 19, 22]). For example, the celebrated Stein-Tomas Theorem (see [15, 22, 23]) gives an affirmative
answer to Fourier restriction problem for compact surfaces with non-zero Gaussian curvature if and only
if $1 \leq p \leq \frac{2(n+1)}{n+3}$. For quadratic surfaces, Strichartz [19] gave a complete solution to Fourier restriction
problem, when $S$ is a quadratic surface given by $S = \{ x \in \mathbb{R}^n : R(x) = r \}$, where $R(x)$ is a polynomial
of degree two with real coefficients and $r$ is a real constant. For a more detailed study on the history
of the restriction problem, we refer to the excellent survey of Tao [21]. The Stein-Tomas Theorem is
further generalized to a system of orthonormal functions with respect to the Fourier transform by Frank-
Lewin-Lieb-Seiringer [9] and Frank-Sabin [8] and the corresponding Strichartz bounds to the Schrödinger
equations up to the end point are obtained.

The main aim of this article is to investigate the validity of Problem A for the Fourier-Hermite transform
and obtain a Strichartz type estimate for a system of orthonormal functions for the Ornstein-Uhlenbeck
operator $L = -\frac{1}{2} \Delta + (x, \nabla)$ on $\mathbb{R}^n$. 

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Schrödinger equation.
Let \( \gamma_n(dx) \) be the Gaussian measure on \( \mathbb{R}^n \) given by \( \pi^{-\frac{n}{2}} e^{-|x|^2} dx \). For \( 1 \leq p \leq \infty \), we denote \( L^p(\mathbb{R}^n, \gamma_n) \) as \( L^p(\gamma_n) \). For \( f \in L^1(\gamma_n) \) the Fourier-Hermite transform of \( f \) is defined by

\[
\hat{f}_n(\mu) = \int_{\mathbb{R}^n} f(x) e^{-i\mu \cdot x} \gamma_n(dx), \quad \mu \in \mathbb{N}_0^n,
\]

where \( \mathbb{N}_0 \) denotes the set of all non-negative integers and \( e^{\mu} \) are the \( n \)-dimensional normalized Hermite polynomials (defined in section 2). If \( f \in L^2(\gamma_n) \) then \( \{\hat{f}_n(\mu)\} \in \ell^2(\mathbb{N}_0^n) \) and the Plancherel formula is of the form

\[
||f||_2^2 = \sum_{\mu \in \mathbb{N}_0^n} |\hat{f}_n(\mu)|^2.
\]

The inverse Fourier-Hermite transform is given by

\[
f(x) = \sum_{\mu \in \mathbb{N}_0^n} \hat{f}_n(\mu) e^{i\mu \cdot x}.
\]

Given a discrete surface \( S \in \mathbb{N}_0^n \times \mathbb{Z} \), we define the restriction operator \( R_S := \{\hat{f}(\mu,\nu)\}_{(\mu,\nu) \in S} \) and the operator dual to \( R_S \) (called the extension operator) as

\[
R_S(\{\hat{f}(\mu,\nu)\}) := \sum_{(\mu,\nu) \in S} \hat{f}(\mu,\nu) \mu \cdot e^{-i\nu}.
\]

where \( \hat{f}(\mu,\nu) = \int_{\mathbb{R}^n} f(t,x) e^{it\nu} \gamma_n(dx) dt \).

For \( 1 \leq p < \infty \), we denote \( L^p((-\pi,\pi) \times \mathbb{R}^n) \) as the class of functions \( f(t,x) \) whose \( p \)th power is absolutely integrable with respect to the Lebesgue measure in \( t \)-variable and Gaussian measure in \( x \)-variable throughout the article.

Now we consider the following problems:

**Problem 1:** For which exponents \( 1 \leq p \leq 2 \), the sequence of Fourier-Hermite transforms of a function \( f \in L^p((-\pi,\pi) \times \mathbb{R}^n) \) belongs to \( \ell^2(S) \)?

This question can be reframed to the boundedness of the operator \( E_S \) from \( \ell^2(S) \) to \( L^p((-\pi,\pi) \times \mathbb{R}^n) \), where \( p' \) is the conjugate exponent of \( p \). Since \( E_S \) is bounded from \( \ell^2(S) \) to \( L^p((-\pi,\pi) \times \mathbb{R}^n) \) if and only if \( T_S := E_S(E_S)^* \) is bounded from \( L^p((-\pi,\pi) \times \mathbb{R}^n) \) to \( L^p((-\pi,\pi) \times \mathbb{R}^n) \), Problem 1 can be re-written as follows:

**Problem 2:** For which exponents \( 1 \leq p \leq 2 \), the operator \( T_S := E_S(E_S)^* \) is bounded from \( L^p((-\pi,\pi) \times \mathbb{R}^n) \) to \( L^p((-\pi,\pi) \times \mathbb{R}^n) \)?

Note that Hölder’s inequality implies that the operator \( T_S := E_S(E_S)^* \) is bounded from \( L^p((-\pi,\pi) \times \mathbb{R}^n) \) to \( L^p((-\pi,\pi) \times \mathbb{R}^n) \) if and only if for any \( W_1, W_2 \in L^p((\mathbb{R}^n), \mathbb{R}^n, \mu) \), the operator \( W_1T_SW_2 \) (composition of the multiplication operator associated with \( W_1, T_S \) and the multiplication operator associated with \( W_2 \)) is bounded on \( L^p((-\pi,\pi) \times \mathbb{R}^n) \) with

\[
||W_1T_SW_2||_{L^p((-\pi,\pi) \times \mathbb{R}^n)} \leq C ||W_1||_{L^p((\mathbb{R}^n), \mathbb{R}^n, \mu)} ||W_2||_{L^p((\mathbb{R}^n), \mathbb{R}^n, \mu)},
\]

for some \( C > 0 \).

Another motivation to consider Problem 1 is due to the connection to Frame theory: The Fourier-Hermite restriction and extension operators seem to be what is called analysis and synthesis operator in Gabor Analysis [5, 12]. More precisely, the extension operator as defined by (1.1) is not only a synthesis operator, it is already the frame operator for the union of modulated ONBs consisting of Hermite functions. The question about the boundedness of the extension operator defined by (1.1) is therefore a question about the boundedness of the frame operator of a degenerated multi-window Gabor system. The degeneracy stems from the fact that no translations are used. The multi-window Gabor system is built from the eigenfunctions of a Daubechies localization operator [6] (see also [7]).

To achieve this goal, we introduce an analytic family of operators (\( T_z \)) defined on the strip \( a \leq \text{Re} z \leq b \) in the complex plane such that \( T_S = T_c \) for some \( c \in (a,b) \) and show that the operator \( W_1T_SW_2 \) belongs to a Schatten class with

\[
||W_1T_SW_2||_{\mathfrak{S}^n(L^2((-\pi,\pi) \times \mathbb{R}^n))} \leq C ||W_1||_{L^p((\mathbb{R}^n), \mathbb{R}^n, \mu)} ||W_2||_{L^p((\mathbb{R}^n), \mathbb{R}^n, \mu)},
\]

for some \( C > 0 \) and some \( \alpha > 0 \), which is more general result \( L^p - L^{p'} \) boundedness of \( T_z \).

The Strichartz inequality for the system of orthonormal functions for the Hermite operator has been proved in [4] using the classical Strichartz estimates for the free Schrödinger propagator for orthonormal
systems [8, 9] and the link between the Schrödinger kernel and the Mehler kernel associated with the Hermite semigroup [14]. Motivated by the study of Gaussian harmonic analysis (see [17]) we prove the Strichartz inequality for the system of orthonormal functions for the Ornstein-Uhlenbeck operator $L = -\frac{1}{2}\Delta + \langle x, \nabla \rangle$ on $\mathbb{R}^n$ with respect to gaussian measure. Obtaining such result for the Ornstein-Uhlenbeck operator using the link between the Schrödinger kernel and the Mehler kernel associated with the Ornstein-Uhlenbeck semigroup is quite challenging. Therefore we prove the restriction theorem for the system of orthonormal functions with respect to the Fourier-Hermite transform associated with the normalized Hermite polynomials and obtain the Strichartz estimate for the system of orthonormal functions for the Ornstein-Uhlenbeck operator as a by product. To the best of our knowledge, the study on restriction theorem with respect to the Fourier-Hermite transform has not been considered in the literature so far. However, we prove the restriction theorem for the Fourier-Hermite transform and obtain the full range Strichartz estimate for the system of orthonormal functions for the Ornstein-Uhlenbeck operator as an application. Further, we show that the constant obtained the Strichartz inequality is optimal in terms of the limit of a large number of functions.

The paper is organized as follows: In Section 2, we discuss the spectral theory of the Ornstein-Uhlenbeck operator and the kernel estimates for the Ornstein-Uhlenbeck semigroup. In Section 3, we obtain the duality principle in terms of Schatten bounds of the operator $W \exp(-itL)(e^{-itL})^{*}W$ and give an affirmative answer to Problem 2 when $p = \frac{n+1}{2m}$ for some $\lambda_0 > 1$. In Section 4, we obtain the Strichartz estimate for $1 \leq q < \frac{n+1}{2m}$, for the system of orthonormal functions associated with the Ornstein-Uhlenbeck operator as the restriction of the Hermite-Fourier transform to the discrete surface $S = \{ (\mu, \nu) \in \mathbb{N}_0^n \times \mathbb{Z} : \nu = |\mu| \}$. Finally we prove the optimality of the Schatten exponent of the Strichartz estimate in Section 5.

2. Preliminary

In this section we discuss some basic definitions and provide necessary background information about the Hermite semigroup.

2.1. Ornstein-Uhlenbeck Operator and the Spectral theory. Let $\mathbb{N}_0$ be the set of all non-negative integers. Let $H_k$ denote the Hermite polynomial on $\mathbb{R}$, defined by

$$H_k(x) = (-1)^k \frac{d^k}{dx^k} (e^{-x^2}) e^{x^2}, \quad k \in \mathbb{N}_0$$

and $h_k$ denote the normalized Hermite polynomial on $\mathbb{R}$ defined by

$$h_k(x) = (2^k k!)^{-\frac{1}{2}} H_k(x), \quad k \in \mathbb{N}_0.$$

The higher dimensional Hermite polynomials denoted by $h_\alpha$ are obtained by taking tensor product of one dimensional Hermite polynomials. Thus for any multi-index $\alpha \in \mathbb{N}_0^n$ and $x \in \mathbb{R}^n$, we define $h_\alpha(x) = \prod_{j=1}^{n} h_{\alpha_j}(x_j)$. The family $\{h_\alpha\}$ forms an orthonormal basis for $L^2(\gamma_\alpha)$. They are eigenfunctions of the Ornstein-Uhlenbeck operator $L = -\frac{1}{2}\Delta + \langle x, \nabla \rangle$ corresponding to eigenvalues $|\alpha|$, where $|\alpha| = \sum_{j=1}^{n} \alpha_j$. Given $f \in L^2(\gamma_\alpha)$, we have the Fourier-Hermite expansion

$$f = \sum_{\alpha \in \mathbb{N}_0^n} (f, h_\alpha)_\gamma h_\alpha = \sum_{k=0}^{\infty} \sum_{|\alpha| = k} (f, h_\alpha)_\gamma h_\alpha = \sum_{k=0}^{\infty} J_k f,$$

where $J_k$ denotes the orthogonal projection of $L^2(\gamma_\alpha)$ onto the eigenspace spanned by $\{h_\alpha : |\alpha| = k\}$. The operator $L$ defines a semigroup called the Ornstein-Uhlenbeck semigroup $\exp(-tL), t \geq 0$, defined by

$$\exp(-tL) f = \sum_{k=0}^{\infty} e^{-kt} J_k f$$

for $f \in L^2(\gamma_\alpha)$. For any $t > 0$ the integral representation for $\exp(-tL)$ is given by

$$\exp(-tL) f(x) = \int_{\mathbb{R}^n} f(y) M_t(x,y) \gamma_\alpha(dy),$$

where the kernel $M_t(x,y)$ is given by the expansion

$$M_t(x,y) = \sum_{\alpha \in \mathbb{N}_0^n} e^{-|\alpha|t} h_\alpha(x) h_\alpha(y).$$
For \( z' = r + it, r > 0, t \in \mathbb{R} \), the kernel of the operator \( e^{-z'L} \) is given by

\[
M_{z'}(x, y) = \sum_{k=0}^{\infty} e^{-z'k} \sum_{|\alpha|=k} \mathbf{h}_\alpha(x) \mathbf{h}_\alpha(y).
\]

Using Mehler’s formula, the kernel of the operator \( e^{-z'L} \) can be obtained as

\[
M_{z'}(x, y) = \frac{e^{-2\pi i n}}{(2\pi \sinh z')^2} e^{\frac{1}{2}((1-\coth z')(|x|^2+|y|^2)+\frac{2\pi y}{\sinh z'})}.
\]

For \( t \in \mathbb{R} \setminus \mathbb{Z} \), letting \( r \to 0 \), the kernel of the operator \( e^{-itL} \) can be written as

\[
M_{it}(x, y) = e^{-\frac{it}{2}} e^{\frac{it}{2}(|x|^2+|y|^2)} e^{\frac{1}{2}(\cot t(|x|^2+|y|^2) - \frac{2\pi y}{t\sinh t})}.
\]

For \( t \in \mathbb{R} \setminus \mathbb{Z} \),

\[
M_{-it}(x, y) = \overline{M_{it}(x, y)} \quad \text{and} \quad M_{it+t}(x, y) = e^{it\gamma_n} M_{it}(-x, y).
\]

For real valued functions \( f \) the \( L^p \) norm of \( e^{-itL}f \) is even and \( \pi \)-periodic as a function of \( t \).

We refer to [17] for a detailed study on the kernel associated with the operator \( e^{-itL} \).

2.2. Schatten class and the duality principle. Let \( \mathcal{H} \) be a complex and separable Hilbert space in which the inner product is denoted by \( \langle \cdot, \cdot \rangle_{\mathcal{H}} \). Let \( T : \mathcal{H} \to \mathcal{H} \) be a compact operator and let \( T^* \) denotes the adjoint of \( T \). For \( 1 \leq r < \infty \), the Schatten space \( \mathcal{G}^r(\mathcal{H}) \) is defined as the space of all compact operators \( T \) on \( \mathcal{H} \) such that

\[
\sum_{n=1}^{\infty} (s_n(T))^r < \infty,
\]

where \( s_n(T) \) denotes the singular values of \( T \), i.e., the eigenvalues of \( |T| = \sqrt{T^*T} \) counted according to multiplicity. For \( T \in \mathcal{G}^r(\mathcal{H}) \), the Schatten \( r \)-norm is defined by

\[
\|T\|_{\mathcal{G}^r} = \left( \sum_{n=1}^{\infty} (s_n(T))^r \right)^{\frac{1}{r}}.
\]

An operator belongs to the class \( \mathcal{G}^1(\mathcal{H}) \) is known as Trace class operator. Also, an operator belongs to \( \mathcal{G}^2(\mathcal{H}) \) is known as Hilbert-Schmidt operator.

3. The Restriction Theorem

In this section, we set a platform to prove the restriction theorem with respect to the Fourier-Hermite transform for a given discrete surface \( S \subset \mathbb{N}_0^n \times \mathbb{Z} \). Recall that, a family of operators \( (T_z) \) on \( \mathbb{R}^n \) defined on a strip \( a \leq \text{Re} z \leq b \) with \( a < b \), in the complex plane is analytic in the sense of Stein [20] if for all simple functions \( f, g \) on \( \mathbb{R}^n \), the map \( z \mapsto \langle g, T_z f \rangle \) is analytic in \( a < \text{Re} z < b \), continuous in \( a \leq \text{Re} z \leq b \), and if \( \sup_{a < x < b} |\langle g, T_{x+i\alpha} f \rangle| \leq C(s) \), where \( C(s) \) with at most a (double) exponential growth in \( s \). The following proposition assures an affirmative answer to Problem 2 under certain assumptions.

In order to obtain the Strichartz inequality for the system of orthonormal functions we need the duality principle lemma in our context. We refer to Proposition 1 and Lemma 3 of [9] with appropriate modifications to obtain the following two results:

**Proposition 3.1.** Let \( (T_z) \) be an analytic family of operators on \( (-\pi, \pi) \times \mathbb{R}^n \) in the sense of Stein defined on the strip \(-\lambda_0 \leq \text{Re} z \leq 0\) for some \( \lambda_0 > 1 \). Assume that we have the following bounds

\[
\|T_{zs}\|_{L^2((-\pi, \pi) \times \mathbb{R}^n)} \leq M_0 e^{a|s|}, \quad \|T_{-\lambda_0+is}\|_{L^1((-\pi, \pi) \times \mathbb{R}^n)} \to L^{\infty}((-\pi, \pi) \times \mathbb{R}^n) \leq M_1 e^{b|s|}
\]

for all \( s \in \mathbb{R} \), for some \( a, b, M_0, M_1 \geq 0 \). Then, for all \( W_1, W_2 \in L^{2\lambda_0}((-\pi, \pi) \times \mathbb{R}^n, \mathbb{C}) \) the operator \( W_1 T_{-\lambda_0} W_2 \) belongs to \( \mathcal{G}^{2\lambda_0} \) \( (L^2((-\pi, \pi) \times \mathbb{R}^n)) \) and we have the estimate

\[
\|W_1 T_{-\lambda_0} W_2\|_{\mathcal{G}^{2\lambda_0}((L^2((-\pi, \pi) \times \mathbb{R}^n))} \leq M_0^{\frac{1}{2}} M_1^{\frac{1}{2}} \|W_1\|_{L^{2\lambda_0}((-\pi, \pi) \times \mathbb{R}^n)} \|W_2\|_{L^{2\lambda_0}((-\pi, \pi) \times \mathbb{R}^n)}.
\]

**Lemma 3.2.** (Duality principle) Let \( p, q \geq 1 \) and \( \alpha \geq 1 \). Let \( A \) be a bounded linear operator from \( L^{q} \) to \( L^{p} \). Then the following statements are equivalent.
(1) There is a constant $C > 0$ such that
\[
\left\| W A^* W \right\|_{L^2((\pi,\pi) \times \mathbb{R}^n)} \leq C \|W\|^2_{L^2((-\pi,\pi) \times \mathbb{R}^n)}
\] (3.3)
for all $W \in L^2_{t,x} L^2_{y,z}((-\pi,\pi) \times \mathbb{R}^n)$, where the function $W$ is interpreted as an operator which acts by multiplication.

(2) For any orthonormal system $(f_j)_{j \in J}$ in $L^2(\gamma)$ and any sequence $(n_j)_{j \in J} \subset \mathbb{C}$, there is a constant $C' > 0$ such that
\[
\left\| \sum_{j \in J} n_j A f_j \right\|_{L^2((-\pi,\pi) \times \mathbb{R}^n)} \leq C' \left( \sum_{j \in J} |n_j|^{1/\alpha'} \right)^{\omega}. \tag{3.4}
\]

Note that Lemma 3.2 and Proposition 3.1 are also valid in the domain $(-\pi,\pi) \times \mathbb{R}^n$.

Let $S$ be the discrete surface $S = \{(\mu,\nu) \in \mathbb{N}_0^2 \times \mathbb{Z} : R(\mu,\nu) = 0\}$, where $R(\mu,\nu)$ is a polynomial of degree one, with respect to the counting measure.

For $-1 < \text{Re} z \leq 0$, consider the analytic family of generalized functions
\[
G_z(\mu,\nu) = \psi(\alpha) R(\mu,\nu)^\lambda, \tag{3.5}
\]
where $\psi(z)$ is an appropriate analytic function with a simple zero at $z = -1$ with exponential growth at infinity when $\text{Re}(z) = 0$ and
\[
R(\mu,\nu)^\lambda = \begin{cases} R(\mu,\nu)^\lambda & \text{for } R(\mu,\nu) > 0, \\ 0 & \text{for } R(\mu,\nu) \leq 0. \end{cases} \tag{3.6}
\]
Restricting the Schwartz class function $\phi$ (defined on $\mathbb{Z}^{n+1}$) to $\mathbb{N}_0^2 \times \mathbb{Z}$, we have
\[
\langle G_z, \phi \rangle := \psi(z) \sum_{\mu,\nu} R(\mu,\nu)^\lambda \phi(\mu,\nu) := \psi(z) \sum_{\mu,\nu} k^\lambda_+ \sum_{\{\mu,\nu), \mu(\nu) = k\}} \phi(\mu,\nu)
\]
and $\lim_{z \to -1} \langle G_z, \phi \rangle = \sum_{\{\mu,\nu), \mu(\nu) = k\}} \phi(\mu,\nu).$ We refer to [10] for the distributional calculus of $R(\mu,\nu)^\lambda$, where $k^\lambda_+$ is defined as in (3.6). Thus $G_{-1} = \delta g$. For $-1 < \text{Re} z \leq 0$, define the analytic family of operators $T_z$ (on Schwartz class functions on $(-\pi,\pi) \times \mathbb{R}^n$) by
\[
T_z g(t,x) := \sum_{\mu,\nu} \hat{g}(\mu,\nu) g_z(\mu,\nu) h_\mu(x) e^{-i\omega t},
\]
where $\hat{g}(\mu,\nu) = \int_{\mathbb{R}^n} \int_{(-\pi,\pi)} g(t,x) h_\mu(x) e^{i\omega t} \gamma_n(dx) dt$. Then $(T_z)$ is an analytic in the sense of Stein defined on the strip $-\lambda_0 \leq \text{Re} z \leq 0$ for some $\lambda_0 > 1$ and
\[
T_z g(t,x) = \int_{\mathbb{R}^n} (K_z(x,y,t) * g,(y))(t) \gamma_n(dy), \tag{3.7}
\]
where $K_z(x,y,t) = \sum_{\mu,\nu} h_\mu(x) h_\nu(y) g_z(\mu,\nu) e^{-i\omega t}$. When $\text{Re}(z) = 0$, we have
\[
\|T_{is}\|_{L^2((-\pi,\pi) \times \mathbb{R}^n) \to L^1((-\pi,\pi) \times \mathbb{R}^n)} = \|G_{is}\|_{L^\infty((-\pi,\pi) \times \mathbb{R}^n)} \leq |\psi(is)|. \tag{3.8}
\]
Again an application of Hölder and Young inequalities in (3.7) gives
\[
|T_z g(t,x)| \leq \sup_{t \in (-\pi,\pi), \ y \in \mathbb{R}^n} |K_z(x,y,t)| \|g\|_{L^1((-\pi,\pi) \times \mathbb{R}^n)} \tag{3.9}
\]
for $g \in L^1((-\pi,\pi) \times \mathbb{R}^n)$. Denoting $T_S = \mathcal{E}_S^* \mathcal{E}_S$, we obtain the following Schatten bound (see (3.10) below) for the operator $W_1 T_S W_2$, where $W_1, W_2 \in L^2_{t,x}((-\pi,\pi) \times \mathbb{R}^n)$.

Theorem 3.3. (Fourier-Hermite restriction theorem) Let $n \geq 1$ and let $S \subset \mathbb{N}_0^2 \times \mathbb{Z}$ be a discrete surface. Suppose that for each $x, y \in \mathbb{R}$, $|K_z(x,y,t)|$ is bounded and has at most exponential growth at infinity when $z = -\lambda_0 + i$ for some $\lambda_0 > 1$. Then $T_S$ is bounded from $L^p((-\pi,\pi) \times \mathbb{R}^n)$ to $L^{p'}((-\pi,\pi) \times \mathbb{R}^n)$ for $p = \frac{2\lambda_0}{1 + \lambda_0}$. 


Proof. It is enough to show
\[ ||W_1 T_S W_2 ||_{L^2_{t,ω}(\mathbb{R}^n)} \leq C ||W_1||_{L^2_{t,ω}(\mathbb{R}^n)} ||W_2||_{L^2_{t,ω}(\mathbb{R}^n)} \]  
(3.10)
for all \( W_1, W_2 \in L^2_{t,ω}(\mathbb{R}^n) \) by Lemma 3.2. By our assumption, together with (3.8), (3.9) and Proposition 3.1, we get (3.10). \( \square \)

4. Strichartz inequality for system of orthonormal functions

Consider the Schrödinger equation associated with the Ornstein-Uhlenbeck operator \( L = -\frac{1}{2} \Delta + (x, \nabla) \):
\[ i\partial_t u(t, x) = Lu(t, x) \quad t \in \mathbb{R}, \ x \in \mathbb{R}^n \]
\[ u(x, 0) = f(x). \]
(4.1)
If \( f \in L^2(\gamma_{\alpha}) \), the solution of the initial value problem (4.1) is given by \( u(t, x) = e^{-itL}f(x) \). The solution to the initial value problem (4.1) can be realized as the extension operator of some function \( f \) on \((-\pi, \pi) \times \mathbb{R}^n\). To estimate the solution to the initial value problem (4.1) is equivalent to obtain the Schatten bound (3.3) with \( A = e^{-itL} \).

Let \( S \) be the discrete surface \( S = \{(\mu, \nu) \in \mathbb{N}_0^n \times \mathbb{Z} : \nu = |\mu|\} \) with respect to the counting measure. Then for all \( \hat{f} \in \ell^1(S) \) and for all \( (t, x) \in [-\pi, \pi] \times \mathbb{R}^n \), the extension operator can be written as
\[ \mathcal{E}_S f(t, x) = \sum_{\mu, \nu \in S} \hat{f}(\mu, \nu) h_{\mu}(x)e^{-it\nu}, \]
(4.2)
where \( \hat{f}(\mu, \nu) = \int_{\mathbb{R}^n} \int_{(-\pi, \pi)} f(t, x) h_{\mu}(x)e^{it\nu} dx dt \).

Choosing
\[ \hat{f}(\mu, \nu) = \left\{ \begin{array}{ll}
\hat{u}(\mu) & \text{if } \nu = |\mu|, \\
0 & \text{otherwise},
\end{array} \right. \]
for some \( u : \mathbb{R}^n \to \mathbb{C} \) in (4.2), we get
\[ \mathcal{E}_S f(t, x) = \sum_{\mu, \nu \in S} \hat{f}(\mu, \nu) h_{\mu}(x)e^{-it\nu} = \int_{\mathbb{R}^n} \left( \sum_{\mu} h_{\mu}(x) |h_{\mu}(y)|^{-i|\mu|} \right) u(y) \, dy \]
\[ = e^{-it\nu} u(x). \]
Restricting the Schwartz class functions \( \phi \) on \( \mathbb{N}_0^n \times \mathbb{Z} \), using the discrete Taylor series expansion (see [18]), we have
\[ \langle k_+^z, \phi \rangle = \sum_{k \in \mathbb{N}_0^n \times \mathbb{Z}} k_+^z \phi(k) \]
(4.3)
\[ = \sum_{k \in \mathbb{N}_0^n \times \mathbb{Z}} k_+^z \left[ \phi(k) - \sum_{|\alpha| < M} \frac{1}{\alpha!} k_+^z \Delta^\alpha \phi(0) \right] + \sum_{|\alpha| < M} \frac{1}{\alpha!} \Delta^\alpha \phi(0) \sum_{k \in \mathbb{N}_0^n \times \mathbb{Z}} k_+^x \]
(4.4)
The above formula is valid for \( z \neq -1, -2, \ldots \), regularizing (4.3). Notice that (4.4) shows that \( \langle k_+^z, \phi \rangle \) is treated as a function of \( z = -1, -2, \ldots \).

Setting \( \psi(z) = \frac{1}{\Gamma(z+1)} \) and \( R(\mu, \nu) = \nu - |\mu| \) in (3.5), we get
\[ G_z(\mu, \nu) = \frac{1}{\Gamma(z+1)}(\nu - |\mu|)_+^z \]
and restricting the Schwartz class function \( \phi \) to \( \mathbb{N}_0^n \times \mathbb{Z} \), we have
\[ \lim_{z \to -1} \langle G_z, \phi \rangle = \psi(z) \sum_{k \in \mathbb{Z}} k_+^z \sum_{\{(\mu, \nu) : R(\mu, \nu) = k\}} \phi(\mu, \nu) = \sum_{(\mu, \nu) \in S} \phi(\mu, \nu). \]
Thus \( G_{-1} = \delta_S \) and
\[ T_z g(t, x) = \int_{\mathbb{R}^n} (K_z(x, y, \cdot) * g(y, \cdot))(t) \, dy, \]
(4.5)
Proof. For \( \tau > 0 \), we calculate the inverse Fourier transform of \( u_+^* e^{-\tau u} \).

\[
\mathcal{F}^{-1}[u_+^* e^{-\tau u}](x) = \frac{1}{(ix)^{z+1}} \int_L \xi^z e^{-\xi} \, d\xi = \frac{\Gamma(z + 1)}{(ix)^{z+1}},
\]

where the contour \( L \) of the integral is a ray from origin to infinity whose angle with respect to the real axis is given by \( \arg \xi = \arg s + \frac{\pi}{2} \). Letting \( \tau \to 0 \) in (4.8), we have

\[
\mathcal{F}^{-1}[u_+^*](x) = \frac{\Gamma(z + 1)}{(ix)^{z+1}}.
\]

By analytic continuation (4.9) is valid for all \( z \neq -1 \).

We use the idea given in Theorem 2.17 of [16] to prove (4.7). To make the paper self contained, we will only indicate the main steps. Let us consider a function \( \eta \in C^\infty(\mathbb{R}) \) such that \( \eta(x) = 1 \) if \( |x| \geq 1 \), and vanishes in a neighborhood of the origin. Let \( F(x) = \eta(x) x_+^* \) for \( x \in \mathbb{R} \). Writing \( F(x) = x_+^* + (\eta(x) - 1)x_+^* \), using (4.9) and denoting \( f \) to be the inverse Fourier transform of \( F \) in the sense of distributions, we have \( f(x) = \Gamma(z + 1)(ix)^{-z-1} + b_1(x) \), where \( b_1 \) is the inverse Fourier transform of the integrable function \( (\eta(x) - 1)x_+^* \), whose support is bounded. Moreover, \( b_1 \in C^\infty(\mathbb{R}) \) and \( f \in L^1(\mathbb{R}) \).

Applying Poisson summation formula (see page 250 of [16]) to the function \( f \) and using the fact \( \hat{f} = F \), we get

\[
\sum_{k=0}^{\infty} k_+^z e^{-itk} = \sum_{k \in \mathbb{Z}} F(k)e^{-itk} = \sum_{k \in \mathbb{Z}} \sum f(2k\pi + t)
\]

\[
= f(t) + \sum_{|k|>0} f(2k\pi + t)
\]

\[
= \Gamma(z + 1)(it)^{-z-1} + b_1(t) + \sum_{|k|>0} f(2k\pi + t)
\]

\[
= \Gamma(z + 1)(it)^{-z-1} + b(t),
\]

where \( b(t) = b_1(t) + \sum_{|k|>0} f(2k\pi + t) \in C^\infty[-\pi, \pi] \).

\[
\square
\]

From the above Proposition, \( \lim_{z \to 1} \frac{1}{\Gamma(z + 1)} \sum_{k=0}^{\infty} k_+^z e^{-itk} = (it)^{-z-1} \) for \( t \in (-\pi, \pi) \setminus \{0\} \). Now we are in a position to prove the following Strichartz inequality for the diagonal case.
Theorem 4.2. \((\text{Diagonal case})\) Let \(n \geq 1\). Then for any (possibly infinite) system \((u_j)\) of orthonormal functions in \(L^2(\gamma_n)\) and any coefficients \((n_j) \subset \mathbb{C}\), we have
\[
\left\| \sum_j n_j |e^{-itL}u_j|^2 \right\|_{L_{t,x}^2((-\pi,\pi)\times \mathbb{R}^n)} \leq C \left( \sum_j |n_j| \right)^{\frac{n+1}{n+2}} \tag{4.10}
\]
where \(C > 0\) is independent of \(n\) and \(q\).

Proof. To prove (4.10), it is enough to show
\[
\|W_1T_3W_2\|_{\mathcal{G}^{n+2}(L^2((-\pi,\pi)\times \mathbb{R}^n))} \leq C \|W_1\|_{L_{t,x}^{n+2}((-\pi,\pi)\times \mathbb{R}^n)} \|W_2\|_{L_{t,x}^{n+2}((-\pi,\pi)\times \mathbb{R}^n)} \tag{4.11}
\]
for all \(W_1, W_2 \in L_{t,x}^{n+2}((-\pi,\pi)\times \mathbb{R}^n)\), where \(S = \{(\mu, \nu) \in N_0 \times \mathbb{Z} : \nu = |\mu|\}\). Since applying Lemma 3.2 to (4.11) gives
\[
\left\| \sum_j n_j |e^{-itL}u_j|^2 \right\|_{L_{t,x}^2((-\pi,\pi)\times \mathbb{R}^n)} \leq C \left( \sum_j |n_j| \right)^{\frac{n+1}{n+2}} \tag{4.12}
\]
Using the kernel properties (2.2) of the semigroup \(e^{-itL}\) the range of \(t\) can be extended to \((-\pi, \pi)\). We show that the family of operators \((T_z)\) (defined in (4.5)) satisfies (3.1). When \(\text{Re} (z) = 0\), we have
\[
\|T_{is}\|_{L^2((-\pi,\pi)\times \mathbb{R}^n) \rightarrow L^2((-\pi,\pi)\times \mathbb{R}^n)} \leq \|G_{is}\|_{L^\infty((-\pi,\pi)\times \mathbb{R}^n)} \leq \frac{1}{|1 + is|} \leq Ce^{|s|}/2. \tag{4.13}
\]
When \(z = -\lambda_0 + is\), (3.9) gives \(T_z\) is bounded from \(L^1((-\pi,\pi)\times \mathbb{R}^n)\) to \(L^\infty((-\pi,\pi)\times \mathbb{R}^n)\) if and only if \(|K_z(x,y,t)|\) is bounded for each \(x, y \in \mathbb{R}^n\). But by (4.6), Proposition 4.1 and (2.1), we get
\[
|K_z(x,y,t)| \sim \frac{Ce^{|s|}}{|1 + is|} \tag{4.14}
\]
So for each \(x, y \in \mathbb{R}^n\), \(|K_z(x,y,t)|\) is bounded if and only if \(\text{Re} (z) = \frac{-\lambda_0}{2}\). The conclusion of the theorem follows by choosing \(\lambda_0 = \frac{-\lambda_0}{2}\) by Proposition 3.1 and the identity \(T_{S} = T_{-1}\).

To obtain the Strichartz inequality for the general case we need to observe the following.

Theorem 4.3. Let \(S\) be the discrete surface \(S = \{(\mu, \nu) \in N_0 \times \mathbb{Z} : \nu = |\mu|\}\) with respect to the counting measure. Then for all exponents \(p, q \geq 1\) satisfying
\[
\frac{2}{p} + \frac{n}{q} = 1, \quad q > n + 1
\]
we have
\[
\|W_1T_3W_2\|_{\mathcal{G}^q(L^2((-\pi,\pi)\times \mathbb{R}^n))} \leq C \|W_1\|_{L_{t,x}^qL_t^3((-\pi,\pi)\times \mathbb{R}^n)} \|W_2\|_{L_{t,x}^qL_t^3((-\pi,\pi)\times \mathbb{R}^n)} \tag{4.15}
\]
with \(C > 0\) independent of \(W_1, W_2\).

Proof. The operator \(T_{-\lambda_0 + is}\) is an integral operator with kernel \(K_{-\lambda_0 + is}(x, x', t - t')\) defined in (4.5). An application of Hardy-Littlewood-Sobolev inequality (see page 39 in [1]) along with (4.13) and (4.14) yields
\[
\|W_1^{\lambda_0 - is}T_{-\lambda_0 + is}W_2^{\lambda_0 - is}\|_{\mathcal{G}^q}^2
\]
\[
= \int_{\mathbb{R}^{2n}} W_1(t,x)^{2\lambda_0} |K_{-\lambda_0 + is}(x, t, x', t')|^2 W_2(t', x')^{2\lambda_0} \gamma_n(dx)\gamma_n(dx')dtd't'
\]
\[
\leq C_1 \int_{\mathbb{R}^{2n}} W_1(t,x)^{2\lambda_0} \|W_2(t', x')\|_{L_{t', x'}^{2\lambda_0}(\gamma_n)} \gamma_n(dx)\gamma_n(dx')dtd't'
\]
\[
\leq C_1 e^{\pi |s|} \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} \|W_1(t)\|_{L_{t,x}^{2\lambda_0}(\gamma_n)} \|W_2(t')\|_{L_{t,x}^{2\lambda_0}(\gamma_n)} dtd't'
\]
\[
\leq C_1 e^{\pi |s|} \|W_1\|_{L_{t,x}^{2\lambda_0}(\gamma_n)} \|W_2\|_{L_{t,x}^{2\lambda_0}(\gamma_n)}
\]
\[
\leq C_1 e^{\pi |s|} \|W_1\|_{L_{t,x}^{2\lambda_0}(\gamma_n)} \|W_2\|_{L_{t,x}^{2\lambda_0}(\gamma_n)}
\]

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Proof. Using the fact that the operator $e^{-itL}$ is unitary, triangle inequality gives (4.16) for the pair $(p, q) = (\infty, 1)$. Equivalently, the operator 

$$W \in L_1^\infty L_2^2((-\pi, \pi) \times \mathbb{R}^n) \mapsto W e^{-itL}(e^{-itL})^* W \in \mathcal{G}^\infty$$

is bounded by Lemma 3.2. Similarly, by Theorem 4.2, the operator 

$$W \in L_1^{n+2} L_2^{n+2}((-\pi, \pi) \times \mathbb{R}^n) \mapsto W e^{-itL}(e^{-itL})^* W \in \mathcal{G}^{n+2}$$

is bounded. Applying the complex interpolation method [2](chapter 4), the operator 

$$W \in L_1^{\frac{2n}{n+1}} L_2^{\frac{2n}{n+1}}((-\pi, \pi) \times \mathbb{R}^n) \mapsto W e^{-itL}(e^{-itL})^* W \in \mathcal{G}^n$$

is bounded for $2 \leq \frac{2n}{n+1} \leq n + 2$ and $n + 2 \leq \frac{2n}{n+1} \leq \infty$. Again applying Lemma 3.2, the inequality (4.16) holds for the range $1 \leq q \leq 1 + \frac{n}{n+1}$. By Theorem 4.3, (4.16) is valid when $1 + \frac{n}{n+1} < q < \frac{n+2}{n+1}$.

5. Optimality of the Schatten exponent

In this section, we show that the power $\frac{2(n+1)}{n+2}$ on the right hand side in (4.16) is optimal. The inequality (4.16) can also be written in terms of the operator 

$$\gamma_0 := \sum_j |n_j| \langle u_j \rangle$$

on $L^2(\gamma_0)$, where the Dirac’s notation $|u\rangle\langle v|$ stands for the rank-one operator $f \mapsto \langle v, f \rangle_{\gamma_0} u$. For such $\gamma_0$, let 

$$\gamma(t) := e^{-itL} \gamma_0 e^{itL} = \sum_j |n_j| e^{-itL} u_j \langle e^{-itL} u_j \rangle.$$

Then the density of the operator $\gamma(t)$ is given by 

$$\rho(t) := \sum_j |n_j| e^{-itL} u_j |e^{-itL} u_j|^2.$$

With these notations (4.16) can be rewritten as 

$$\|\rho(t)\|_{L_t^{\frac{n+2}{n+1}} L_x^\infty((-\pi, \pi) \times \mathbb{R}^n)} \leq C_{n,q} \|\gamma_0\|_{\mathcal{G}^{\frac{2n}{n+1}}}^{\frac{2n}{n+1}},$$

where $\|\gamma_0\|_{\mathcal{G}^{\frac{2n}{n+1}}} = \left( \sum_j |n_j| \frac{2n}{n+1} \right)^{\frac{n+1}{2n}}$. 

Proposition 5.1. (Optimality of the Schatten exponent). Assume that $n, p, q \geq 1$ satisfy $\frac{2n}{n+1} = n$. Then we have 

$$\sup_{\gamma_0 \in \mathcal{G}^{\frac{2n}{n+1}}} \frac{\|\rho e^{-itL} \gamma_0 e^{itL}\|_{L_t^p L_x^q((-\pi, \pi) \times \mathbb{R}^n)}}{\|\gamma_0\|_{\mathcal{G}^{\frac{2n}{n+1}}}} = +\infty$$

for all $r > \frac{2n}{q+1}$. 

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Proof. We introduce a family of operators $\gamma_0$ depending on three positive parameters $\beta, \nu$ and $\mu$, which will be chosen later appropriately. To define $\gamma_0$ we use the coherent states $F_{x, \xi}(z) = (2\beta)^{-\frac{n}{2}} \frac{e^{\frac{1}{2} t}}{2\pi} e^{\frac{(x-z)^2}{4\beta}} e^{i\xi \cdot z}$ depending on parameters $x, \xi \in \mathbb{R}^n$. The functions $F_{x, \xi}$ are normalized and satisfy
\[
\iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{dx \, d\xi}{(2\pi)^n} |F_{x, \xi}|^2 = 1.
\]
Now we construct the family of operators
\[
\gamma_0 = \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{dx \, d\xi}{(2\pi)^n} e^{-\frac{x^2}{2\beta} - \frac{\xi^2}{\mu}} |F_{x, \xi}|^2.
\]
By Mehler’s formula we get
\[
e^{itL}F_{x, \xi}(z) = e^{-\frac{i\pi n}{2} (2\beta)^{-\frac{n}{2}} \frac{e^{\frac{1}{2} t}}{2\pi} \int_{\mathbb{R}^n} \frac{dx}{(2\pi)^n} \frac{1}{2\beta} e^{\frac{(x^2+y^2)}{4\beta}} e^{-\frac{i}{2\beta} \log(\tau^2)} e^{i\xi \cdot y} e^{i\xi \cdot \gamma_n(dy)}.
\]
Therefore
\[
|e^{itL}F_{x, \xi}(z)| = \left(\frac{2\beta}{\pi(4\beta^2 \cos^2 t + \sin^2 t)}\right)^{\frac{n}{2}} \frac{e^{\frac{1}{2} t}}{2\pi} e^{-\frac{(1+2\beta \tau^2)}{2\beta \tau^2} \cos^2 t + (1 + 2\beta \sin^2 t)}
\]
and
\[
\rho_\gamma(t) \cdot \gamma_0 e^{-it\gamma_0}(z) = \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{dx \, d\xi}{(2\pi)^n} e^{-\frac{x^2}{2\beta} - \frac{\xi^2}{\mu}} |e^{itL}F_{x, \xi}(z)|^2
\]
\[
= \left(\frac{\beta \nu \tau^2}{2\pi} \frac{1}{(4\beta^2 + 2\beta \tau^2) \cos^2 t + (1 + 2\beta \sin^2 t)}\right)^{\frac{n}{2}} \frac{e^{\frac{1}{2} t}}{2\pi} e^{-\frac{(1+2\beta \tau^2)}{2\beta \tau^2} \cos^2 t + (1 + 2\beta \sin^2 t)}
\]
So
\[
\|\rho_\gamma(t)\|_{L^q_1(\gamma_0)} \geq \left(\frac{1}{2q} \right)^{\frac{n}{2}} \left(\frac{1}{2\pi} \right)^{\frac{n}{2}} \beta^{\frac{n}{2}} \left(\frac{2\beta + 2\beta \tau^2}{\cos^2 t + (1 + 2\beta \sin^2 t)}\right)^{\frac{n}{2} - 1}.
\]
Using the fact that $n(q-1)p = 2q$, we have
\[
\|\rho_\gamma(t)\|_{L^p_2((-\pi, \pi) \times \mathbb{R}^n)} \geq \left(\frac{1}{2q} \right)^{\frac{n}{2}} \left(\frac{1}{2\pi} \right)^{\frac{n}{2}} \beta^{\frac{n}{2}} \left(\frac{2\beta + 2\beta \tau^2}{\cos^2 t + (1 + 2\beta \sin^2 t)}\right)^{\frac{n}{2} - 1} \frac{1}{\tau^2}.
\]
Thus
\[
\|\rho_\gamma(t)\|_{L^p_2((-\pi, \pi) \times \mathbb{R}^n)} \geq A_{n, p} (\mu^2)^{\frac{n}{2}} \left(\frac{1}{\tau^2} + 1\right)^{\frac{n}{2}} \left(\frac{1}{\mu^2} + 2\right)^{\frac{n}{2}}.
\]
Using the fact that $\frac{n}{2} \left(1 + \frac{1}{q}\right) = \frac{n}{2} - \frac{n}{p}$ and choosing $1/\mu < \beta < \tau^2$, we obtain
\[
\|\rho_\gamma(t)\|_{L^p_2((-\pi, \pi) \times \mathbb{R}^n)} \geq A_{n, p} 2^{-\frac{1}{2p}} N^{\frac{n}{4p}}
\]
where
\[
N = \int_{\mathbb{R}^n} \gamma_0(z, z) \gamma_n(dz) = \int \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{dx \, d\xi}{(2\pi)^n} e^{-\frac{x^2}{2\beta} - \frac{\xi^2}{\mu}} |F_{x, \xi}(z)|^2 \gamma_n(dz)
\]
\[
= \int \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{dx \, d\xi}{(2\pi)^n} e^{-\frac{x^2}{2\beta} - \frac{\xi^2}{\mu}} = A_n \tau^2 \mu^{\frac{n}{2}}.
\]
An application of Berezin-Lieb inequality gives [3, 11] gives that
\[
\text{Tr} \gamma_0 \leq \int \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{dx \, d\xi}{(2\pi)^n} e^{-\frac{x^2}{2\beta} - \frac{\xi^2}{\mu}} = r^{-n} N,
\]
where $r \geq 1$ and $N = \left(\frac{(\mu^2)^{\frac{n}{2}}}{2\pi}\right)^{\frac{1}{n}}$. Therefore
\[
\|\rho_{e^{itL_0}e^{it\gamma_0}}\|_{L^p_2((-\pi, \pi) \times \mathbb{R}^n)} \geq A_{n, p} 2^{-\frac{1}{2p}} N^{\frac{n}{4p}} \frac{(\mu^2)^{\frac{n}{2}}}{r^{-n} N},
\]
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