Transformation of the Couette vortex along the length of the tubular structure

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Abstract. The article is a further research of a circular-longitudinal flow created in a cylindrical pipe by a continuous swirler called Couette vortex, which the author started to study in his previous works. The key question is how Couette modified vortex is transformed along the channel (pipe). The author regards variation of azimuthal velocities (u) and the Heeger – Baer swirl number (Sn) in turbulent irregular circular-longitudinal flow, which is described by the model of modified Couette vortex along the cylindrical channel. It is confirmed that the model of the modified Couette vortex and free-forced Burgers – Batchelor vortex show almost similar results in calculations and both vortex models can be equally used in engineering practice in calculations and the analysis of circulating and longitudinal flow operating modes (swirl flows).

1. Introduction
Quite often tubular structures: small tubular culverts in road construction, corrugated metal road pipes, pipelines for water supply and drainage systems, pipelines for hydroelectric power plants operate with swirling flows. The underestimation of the peculiarities of the movement of swirling flows leads to unjustified construction costs, to an underestimation of the volume of water passed through, and often to accidents associated with an incorrect determination of the throughput of the designed pipelines. This article is devoted to accounting for the peculiarities of swirling flows hydraulics.

In one of my previous articles [1] I regarded steady regular circular-longitudinal flow created in a cylindrical pipe by a continuous swirler (strap, tape or other). The flow is called Couette vortex. I also showed that it is possible to modify Couette vortex and find azimuthal velocities radial distribution formula, corresponding to irregular motion profiles of non-uniform motion of circular-longitudinal flows in a pipe, formed by local swirlers (bladed or tangential). The classical example of such profile is given in [2] (Figure 1).
2. Results and Discussion
The modified Couette profile is described by the formula

\[ u = u_R \frac{r(R^2 + r_m^2)}{R(r_m^2 + r^2)}. \]  

(1)

The decoding of the designations in the formula (1) is given in the figure caption to Figure 1.

![Figure 1. Azimuthal (tangential) velocity profile in turbulent circular-longitudinal flow: where \( u \) and \( u_R \) – azimuthal velocity on the reference radius \( r \) and velocity on the border of turbulent flow core with a boundary layer thick \( \delta \); \( r, r_m \) and \( R \) are reference radius, radius of peak azimuthal (tangential) velocities \( (u_m) \) and cylindrical pipe radius; line \( b \) – free (potential) rotation; line \( a \) – forced vortex (rigid-body rotation); point \( m \) – maximum tangential velocity point.]

However, in [1] the key question remained open. How is the modified Couette vortex transformed along the length of the channel (pipe)? To answer it, we turn to articles [3, 4], in which, along with the Couette vortex, the Burgers – Batchelor vortex is considered

\[ u = u_{R0} \frac{R}{r} \left[ 1 - \exp \left( -\eta \frac{r^2}{r_m^2} \right) \right], \]  

(2)

where \( u_{R0} \) – azimuthal velocity on the border of a boundary layer at the channel inlet, i.e. directly behind the local swirler forming irregular (fading along the cylindrical channel) circular-longitudinal flow; \( \eta \) – constant, equal to \( \eta = 1.256 \).

The free-forced Burgers – Batchelor vortex (2) in a turbulent regime of fluid motion, as is known, is obtained as a result of the transformation of a free vortex along the length of a cylindrical tube with the distribution of azimuthal velocities according to the law of dynamic rotation

\[ uR = u_{R0}R = \text{const}. \]

It is also known that the modified Couette vortex (1) and the free-forced Burgers – Batchelor vortex (2) give practically the same radial profile of azimuthal velocities in the calculations. In this case, the following relations are characteristic of the Burgers – Batchelor vortex [5]

\[ \frac{r_m^2}{\eta R^2} = \chi \sqrt{2 \lambda} \frac{z}{R}, \]  

(3)

\[ u_m r_m = u_{R0} R \left[ 1 - \exp(-\eta) \right] = 0.7152 u_{R0} R = \text{const}, \]  

(4)

where \( \chi \) is a universal constant equal to \( \chi = 0.2 \) for water; \( \lambda \) is flow frictional resistance.

But, according to (1), for the modified Couette profile, we have

\[ u_m r_m = u_R \frac{R^2 + r_m^2}{2R}. \]
and

\[ u = u_m \frac{2r_m r}{r_m^2 + r^2}. \]  

(5)

Then, taking equalities (3) and (4) valid not only for the Burgers – Batchelor vortex, but also for the modified Couette vortex, we obtain

\[ u = u_R(2rR[1 - \exp(-\eta)]) \eta \sqrt{2zR} + r^2. \]  

(6)

Formula (6) describes the radial distribution of azimuthal velocities in a turbulent circulation-longitudinal flow in an arbitrary pipe section. For the calculation, you only need to know the value of the hydraulic resistance coefficient along the length of \( \lambda \). As practice has shown, for a turbulent circulation-longitudinal flow, the coefficient \( \lambda \) can be assigned in accordance with the equivalent equal-grained absolute roughness of the pipe walls \( (k_e) \) according to the well-known formulas Prandtl – Nikuradze

\[ \frac{1}{\sqrt{\lambda}} = 2\log \left( \frac{D}{k_e} \right) + 1.14 \]

or Shifrinson

\[ \lambda = 0.11 \left( \frac{k_e}{D} \right)^{0.25}, \]

where \( D \) is a pipe diameter, \( D = 2R \).

In this case, hydraulic friction will occur not only in the axial, but also in the azimuthal direction.

Figure 2,a shows the calculated radial profiles of normalized azimuthal velocities \((u/u_R)\) in a turbulent circulation-longitudinal flow in seven sections along the pipe length at a distance of \( z = 25, 50, 100, 200, 400, 800, \) and 1600 mm from the local swirler. The pipe has a diameter \( D = 50 \) mm, the material of the pipe walls is polished organic glass with an equivalent equal-grained absolute roughness equal to \( k_e = 0.02 \) mm.

![Figure 2](image_url)

**Figure 2.** Radial profiles of azimuthal velocities (a) and reduction of Heeger – Baer swirl number along the pipe length (b).
Let us consider the change along the length of the cylindrical channel of the integral characteristic of the circulation-longitudinal flow - the Heeger – Baer swirl number [6]

\[ S_n = \frac{M}{RI} \]  

(7)

here M and I are the angular momentum and the momentum of the circulation-longitudinal flow, respectively.

\[ M = R \int_0^R \rho rv^2 2\pi r \, dr \]  

(8)

\[ I = R \int_0^R \rho v^2 2\pi r \, dr = \rho \alpha_0 Q V \]  

(9)

where \( \rho \) is the density of the liquid; \( v \) is the axial flow velocity at the current radius \( r \); \( Q \) is flow rate; \( V \) is average flow velocity, \( V = Q / \pi R^2 \); \( \alpha_0 \) is the Boussinesq coefficient.

According to (7) and (9), the change in the swirl number \( S_n \) to within the Boussinesq correction \( \alpha_0 \) is associated exclusively with the drop in the angular momentum of the circulation-longitudinal flow along the channel length caused by hydraulic friction. In this case, the ratio of the angular momentum of the circulation-longitudinal flow in an arbitrary section of the pipe to its initial value at the entrance to the pipe behind the local swirler is

\[ \frac{S_n}{S_n_0} = \frac{M}{M_0} \]  

(10)

Let us find the angular momentum of the circulation-longitudinal flow in an arbitrary section along the length of the pipe, taking integral (8) with the radial distribution of azimuthal velocities according to formula (5) and taking the radial distribution of axial velocities is uniform \( v = V \)

\[ M = R \int_0^R \rho rv^2 2\pi r \, dr = 2\rho u_m r_m Q \left[ 1 - \frac{r_m^2}{R^2} \ln \left( 1 + \frac{R^2}{r_m^2} \right) \right] . \]  

(11)

But, the density \( \rho \), the flow rate \( Q \) and the product \( u_m r_m \) are constants that are the same in any section of the conduit, including in its initial section, where \( r_m \) tends to zero, \( u_m \) tends to infinity, and their product is a finite number equal to \( u_m r_m = u_m R \left[ 1 - \exp(-\eta) \right] \). Then the initial moment of momentum of the circulation-longitudinal flow in the initial section of the pipe behind the local swirler according to (11) will be equal

\[ M = 2\rho u_m r_m Q = 2\rho u_m R Q \left[ 1 - \exp(-\eta) \right] . \]

From this, using (10), we find

\[ \frac{S_n}{S_n_0} = 1 - \frac{r_m^2}{R^2} \ln \left( 1 + \frac{R^2}{r_m^2} \right) \]  

(12)

or taking into account (3)

\[ \frac{S_n}{S_n_0} = 1 - \eta \chi \sqrt{2\lambda z} \frac{R}{\eta \chi \sqrt{2\lambda z}} \]  

(13)

The graph of the change the Heeger – Baer swirl number along the length of the cylindrical channel for the design case is shown in Figure 2.b.
Analysis of the results obtained shows that as the flow follows the pipe along the axial coordinate \( z \), the azimuthal velocities \( u \) and the swirl number \( Sn \) decrease. Moreover, at \( z = R/\eta \chi \sqrt{2\lambda} \), the maximum tangential velocities reach the pipe walls, then, according to (12) and (13), we have \( r_m = R \) and

\[
\frac{Sn}{Sn_0} = 1 - \ln 2 = 0.3069.
\]

Consequently, the ratio \( Sn/Sn_0 < 0.3069 \), which corresponds to the value of the parameter \( z = R/\eta \chi \sqrt{2\lambda} \), characterizes the rotation of the liquid according to the law of a rigid body, that is, the stage of degeneration of the circulation flow.

In the design example, quasi-rigid rotation occurs at \( z > 564 \text{ mm} \), which is more than 11 pipe diameters.

3. Conclusion

Complete damping of the flow swirl occurs at \( Sn = 0 \), which is achieved only at \( z \to \infty \). Thus, the damping of the azimuthal velocities and the Heeger – Beer swirl number along the length of the channel in the flow that had free (dynamic) rotation at the inlet is impossible except by gradual transition to quasi-rigid rotation.

Note that the results obtained on the transformation of the modified Couette vortex along the length of the cylindrical channel are close to the analogous characteristics of the change in the parameters of the free-forced Burgers – Batchelor vortex. Both the first and the second vortex model can be equally used in engineering practice when calculating and analyzing the modes of motion of circulation-longitudinal flows (swirling flows).

References
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