# Cluster Algebras II: Finite Type Classification

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1. INTRODUCTION AND MAIN RESULTS

1.1. Introduction. The origins of cluster algebras, first introduced in [9], lie in the desire to understand, in concrete algebraic and combinatorial terms, the structure of “dual canonical bases” in (homogeneous) coordinate rings of various algebraic varieties related to semisimple groups. Several classes of such varieties—among them Grassmann and Schubert varieties, base affine spaces, and double Bruhat cells—are expected (and in many cases proved) to carry a cluster algebra structure. This structure includes the description of the ring in question as a commutative ring generated inside its ambient field by a distinguished family of generators called cluster variables. Even though most of the rings of interest to us are finitely generated, their set of cluster variables may well be infinite. A cluster algebra has finite type if it has a finite number of cluster variables.

The main result of this paper (Theorem 1.4) provides a complete classification of the cluster algebras of finite type. This classification turns out to be identical to the Cartan-Killing classification of semisimple Lie algebras and finite root systems. This result is particularly intriguing since in most cases, the symmetry exhibited by the Cartan-Killing type of a cluster algebra is not apparent at all from its geometric realization. For instance, the coordinate ring of the base affine space of the group $SL_5$ turns out to be a cluster algebra of the Cartan-Killing type $D_6$. Other examples of similar nature can be found in Section 12, in which we show how cluster algebras of types $ABCD$ arise as coordinate rings of some classical algebraic varieties.

In order to understand a cluster algebra of finite type, one needs to study the combinatorial structure behind it, which is captured by its cluster complex. Roughly speaking, it is defined as follows. The cluster variables for a given cluster algebra are not given from the outset but are obtained from some initial “seed” by an explicit process of “mutations”; each mutation exchanges a cluster variable in the current seed by a new cluster variable according to a particular set of rules. In a cluster algebra of finite type, this process “folds” to produce a finite set of seeds, each containing the same number $n$ of cluster variables (along with some extra information needed to perform mutations). These $n$-element collections of cluster variables, called clusters, are the maximal faces of the (simplicial) cluster complex.

In Theorem 1.13, we identify this complex as the dual simplicial complex of a generalized associahedron associated with the corresponding root system. These complexes (indeed, convex polytopes [27]) were introduced in [11] in relation to our proof of Zamolodchikov’s periodicity conjecture for algebraic $Y$-systems. A generalized associahedron of type $A$ is the usual associahedron, or the Stasheff polytope [24]; in types $B$ and $C$, it is the cyclohedron, or the Bott-Taubes polytope [5, 24].

One of the crucial steps in our proof of the classification theorem is a new combinatorial characterization of Dynkin diagrams. In Section 8 we introduce an equivalence relation, called mutation equivalence, on finite directed graphs with weighted edges. We then prove that a connected graph $\Gamma$ is mutation equivalent to an orientation of a Dynkin diagram if and only if every graph that is mutation equivalent to $\Gamma$ has all edge weights $\leq 3$. We do not see a direct way to relate this description to any previously known characterization of the Dynkin diagrams.

We already mentioned that the initial motivation for the study of cluster algebras came from representation theory; see [27] for a more detailed discussion of the representation-theoretic context. Another source of inspiration was Lusztig’s
theory of total positivity in semisimple Lie groups, which was further developed in a series of papers of the present authors and their collaborators (see, e.g., [19, 8] and references therein). The mutation mechanism used for creating new cluster variables from the initial “seed” was designed to ensure that in concrete geometric realizations, these variables become regular functions taking positive values on all totally positive elements of a variety in question, a property that the elements of the dual canonical basis are known to possess [18].

Following the foundational paper [9], several unexpected connections and appearances of cluster algebras have been discovered and explored. They included: Laurent phenomena in number theory and combinatorics [10], Y-systems and thermodynamic Bethe ansatz [11], quiver representations [21], and Poisson geometry [12].

While this text belongs to an ongoing series of papers devoted to cluster algebras and related topics, it is designed to be read independently of any other publications on the subject. Thus, all definitions and results from [9, 11, 7] that we need are presented “from scratch”, in the form most suitable for our current purposes. In particular, the core concept of a normalized cluster algebra [9] is defined anew in Section 1.2, while Section 3 provides the relevant background on generalized associahedra [11, 7].

The main new results (Theorems 1.4-1.13) are stated in Sections 1.3–1.5. The organization of the rest of the paper is outlined in Section 1.6.

1.2. Basic definitions. We start with the definition of a (normalized) cluster algebra \( A \) (cf. [9, Sections 2 and 5]). This is a commutative ring embedded in an ambient field \( \mathcal{F} \) defined as follows. Let \((P, \otimes, \cdot)\) be a semifield, i.e., an abelian multiplicative group supplied with an auxiliary addition \( \otimes \) which is commutative, associative, and distributive with respect to the multiplication in \( P \). The following example (see [9, Example 5.6]) will be of particular importance to us: let \( P \) be a free abelian group, written multiplicatively, with a finite set of generators \( p_j (j \in J) \), and with auxiliary addition \( \otimes \) given by

\[
\prod_j p_j^{a_j} \otimes \prod_j p_j^{b_j} = \prod_j p_j^{\min(a_j, b_j)}.
\]

We denote this semifield by \( \text{Trop}(p_j : j \in J) \). The multiplicative group of any semifield \( \mathbb{P} \) is torsion-free [9, Section 5], hence its group ring \( \mathbb{ZP} \) is a domain. As an ambient field for \( A \), we take a field \( \mathcal{F} \) isomorphic to the field of rational functions in \( n \) independent variables (here \( n \) is the rank of \( A \)), with coefficients in \( \mathbb{ZP} \).

A seed in \( \mathcal{F} \) is a triple \( \Sigma = (x, p, B) \), where

- \( x \) is an \( n \)-element subset of \( \mathcal{F} \) which is a transcendence basis over the field of fractions of \( \mathbb{ZP} \).
- \( p = (p_j^x)_{x \in x} \) is a \( 2n \)-tuple of elements of \( \mathbb{P} \) satisfying the normalization condition \( p_x^x + p_y^x = 1 \) for all \( x \in x \).
- \( B = (b_{xy})_{x,y \in x} \) is an \( n \times n \) integer matrix with rows and columns indexed by \( x \), which is sign-skew-symmetric [9, Definition 4.1]; that is,

\[
b_{xy} = 0, \quad \text{or} \quad b_{xy}b_{yx} < 0.
\]

We will need to recall the notion of matrix mutation [9, Definition 4.2]. Let \( B = (b_{ij}) \) and \( B' = (b'_{ij}) \) be real square matrices of the same size. We say that \( B' \)
is obtained from $B$ by a matrix mutation in direction $k$ if

\[
(1.3) \quad b'_{ij} = \begin{cases} 
-b_{ij} & \text{if } i = k \text{ or } j = k; \\
-b_{ij} + \frac{|b_{ik}b_{kj} + b_{ik}b_{kj}|}{2} & \text{otherwise}.
\end{cases}
\]

**Definition 1.1.** (Seed mutations) Let $\Sigma = (x, p, B)$ be a seed in $F$, as above, and let $z \in x$. Define the triple $\overline{\Sigma} = (x, p, B)$ as follows:

- $\overline{x} = x - \{z\} \cup \{\overline{z}\}$, where $\overline{z} \in F$ is determined by the exchange relation

\[
(1.4) \quad z\overline{x} = p_z^+ \prod_{x \in x, b_{xz} > 0} x^{b_{xz}} + p_z^- \prod_{x \in x, b_{xz} < 0} x^{-b_{xz}}
\]

- the $2n$-tuple $\overline{p} = (\overline{p}_z^+)_z \in \overline{x}$ is uniquely determined by the normalization conditions $\overline{p}_x^+ \oplus \overline{p}_x^- = 1$ together with

\[
(1.5) \quad \overline{p}_x^+ / \overline{p}_x^- = \begin{cases} 
\overline{p}_x^+ / p_x^+ & \text{if } x = \overline{z}; \\
(p_x^+)^{b_{xz}} p_x^+ / p_x^- & \text{if } b_{xz} \geq 0; \\
(p_x^-)^{b_{xz}} p_x^- / p_x^- & \text{if } b_{xz} \leq 0.
\end{cases}
\]

- the matrix $\overline{B}$ is obtained from $B$ by applying the matrix mutation in direction $z$ and then relabeling one row and one column by replacing $z$ by $\overline{z}$.

If the triple $\overline{\Sigma}$ is again a seed in $F$ (i.e., if the matrix $\overline{B}$ is sign-skew-symmetric), then we say that $\Sigma$ admits a mutation in the direction $z$ that results in $\overline{\Sigma}$.

We note that the exchange relation (1.4) is a reformulation of [9] (2.2), (4.2), while the rule (1.5) is a rewrite of [9] (5.4), (5.5). The elements $\overline{p}_x^+$ are determined by (1.5) uniquely since $p \oplus q = 1$ and $p / q = u$ imply $p = u / (1 + u)$ and $q = 1 / (1 + u)$.

In particular, the first case in (1.5) yields $\overline{p}_x^+ = p_x^+$. It is easy to check that the mutation of $\Sigma$ in direction $\overline{z}$ recovers $\Sigma$.

**Definition 1.2.** (Normalized cluster algebra) Let $S$ be a set of seeds in $F$ with the following properties:

- every seed $\Sigma \in S$ admits mutations in all $n$ conceivable directions, and the results of all these mutations belong to $S$;
- any two seeds in $S$ are obtained from each other by a sequence of mutations.

The sets $x$, for $\Sigma = (x, p, B) \in S$, are called clusters; their elements are the cluster variables; the set of all cluster variables is denoted by $X$. The set of all elements $p_x^\pm \in p$, for all seeds $\Sigma = (x, p, B) \in S$, is denoted by $P$. The ground ring $\mathbb{Z}[P]$ is the subring of $F$ generated by $P$. The (normalized) cluster algebra $A = A(S)$ is the $\mathbb{Z}[P]$-subalgebra of $F$ generated by $X$. The exchange graph of $A(S)$ is the $n$-regular graph whose vertices are labeled by the seeds in $S$, and whose edges correspond to mutations. (This is easily seen to be equivalent to [9] Definition 7.4.)

Definition 1.2 is a bit more restrictive than the one given in [9], where we allowed to use any subring with unit in $\mathbb{Z}[P]$ containing $P$ as a ground ring. Some concrete examples of cluster algebras will be given in Section 22 below.

**Remark 1.3.** There is an involution $\Sigma \mapsto \Sigma^\vee$ on the set of seeds in $F$ acting by $(x, p, B) \mapsto (x, p^\vee, -B)$, where $(p^\vee)^\pm = p_x^\pm$. An easy check show that this involution commutes with seed mutations. Therefore, if a collection of seeds $S$ satisfies
the conditions in Definition 1.2, then so does the collection \( S' \). The corresponding cluster algebras \( \mathcal{A}(S) \) and \( \mathcal{A}(S') \) are canonically identified with each other. (This is a reformulation of [9, (2.8)].)

Two cluster algebras \( \mathcal{A}(S) \subset \mathcal{F} \) and \( \mathcal{A}(S') \subset \mathcal{F}' \) over the same semifield \( \mathbb{P} \) are called strongly isomorphic if there exists a \( \mathbb{ZP} \)-algebra isomorphism \( \mathcal{F} \to \mathcal{F}' \) that transports some (equivalently, any) seed in \( S \) into a seed in \( S' \), thus inducing a bijection \( S \to S' \) and an algebra isomorphism \( \mathcal{A}(S) \to \mathcal{A}(S') \).

The set of seeds \( S \) for a cluster algebra \( \mathcal{A} = \mathcal{A}(S) \) (hence the algebra itself) is uniquely determined by any single seed \( \Sigma = (x, p, B) \in S \). Thus, \( \mathcal{A} \) is determined by \( B \) and \( p \) up to a strong isomorphism, justifying the notation \( \mathcal{A} = \mathcal{A}(B, p) \). In general, an \( n \times n \) matrix \( B \) and a \( 2n \)-tuple \( p \) satisfying the normalization conditions define a cluster algebra \( \mathcal{A}(B, p) \) if and only if any matrix obtained from \( B \) by a sequence of mutations is sign-skew-symmetric. This condition is in particular satisfied whenever \( B \) is skew-symmetrizable [9, Definition 4.4], i.e., there exists a diagonal matrix \( D \) with positive diagonal entries such that \( DB \) is skew-symmetric. Indeed, matrix mutations preserve skew-symmetrizability [9, Proposition 4.5], and any skew-symmetrizable matrix is sign-skew-symmetric.

Every cluster algebra over a fixed semifield \( \mathbb{P} \) belongs to a series \( \mathcal{A}(B, -) \) consisting of all cluster algebras of the form \( \mathcal{A}(B, p) \), where \( B \) is fixed, and \( p \) is allowed to vary. Two series \( \mathcal{A}(B, -) \) and \( \mathcal{A}(B', -) \) are strongly isomorphic if there is a bijection sending each cluster algebra \( \mathcal{A}(B, p) \) to a strongly isomorphic cluster algebra \( \mathcal{A}(B', p') \). (This amounts to requiring that \( B \) and \( B' \) can be obtained from each other by a sequence of matrix mutations, modulo simultaneous relabeling of rows and columns.)

1.3. Finite type classification. A cluster algebra \( \mathcal{A}(S) \) is said to be of finite type if the set of seeds \( S \) is finite.

Let \( B = (b_{ij}) \) be an integer square matrix. Its Cartan counterpart is a generalized Cartan matrix \( A = A(B) = (a_{ij}) \) of the same size defined by

\[
\begin{align*}
    a_{ij} &= \begin{cases} 
        2 & \text{if } i = j; \\
        -|b_{ij}| & \text{if } i \neq j.
    \end{cases}
\end{align*}
\]

The following classification theorem is our main result.

**Theorem 1.4.** All cluster algebras in any series \( \mathcal{A}(B, -) \) are simultaneously of finite or infinite type. There is a canonical bijection between the Cartan matrices of finite type and the strong isomorphism classes of series of cluster algebras of finite type. Under this bijection, a Cartan matrix \( A \) of finite type corresponds to the series \( \mathcal{A}(B, -) \), where \( B \) is an arbitrary sign-skew-symmetric matrix with \( A(B) = A \).

We note that in the last claim of Theorem 1.4, the series \( \mathcal{A}(B, -) \) is well defined since \( A \) is symmetrizable and therefore \( B \) must be skew-symmetrizable.

By Theorem 1.4, each cluster algebra of finite type has a well-defined type (e.g., \( A_n, B_n, \ldots \)), mirroring the Cartan-Killing classification.

We prove Theorem 1.4 by splitting it into the following three statements (Theorems 1.5–1.7 below).
Theorem 1.5. Suppose that
1.7. \( A \) is a Cartan matrix of finite type;
1.8. \( B_0 = (b_{ij}) \) is a sign-skew-symmetric matrix such that \( A = A(B_0) \) and 
\( b_{ij}b_{ik} \geq 0 \) for all \( i, j, k \);
1.9. \( p_0 \) is a 2n-tuple of elements in \( \mathbb{P} \) satisfying the normalization conditions.

Then \( A(B_0, p_0) \) is a cluster algebra of finite type.

It is easy to see that for any Cartan matrix \( A \) of finite type, there is a matrix \( B_0 \)
satisfying (1.8). Indeed, the sign-skew-symmetric matrices \( B \) with \( A(B) = A \) are
in a bijection with orientations of the Coxeter graph of \( A \) (recall that this graph has \( I \)
as the set of vertices, with \( i \) and \( j \) joined by an edge whenever \( a_{ij} \neq 0 \):
under this bijection, \( b_{ij} > 0 \) if and only if the edge \( \{i, j\} \) is oriented from \( i \) to \( j \).
Condition (1.8) means that \( B_0 \) corresponds to an orientation such that every vertex
is a source or a sink; since the Coxeter graph is a tree, hence a bipartite graph,
such an orientation exists.

Theorem 1.6. Any cluster algebra \( A \) of finite type is strongly isomorphic to a
cluster algebra \( A(B_0, p_0) \) for some data of the form (1.7)–(1.9).

Theorem 1.7. Let \( B \) and \( B' \) be sign-skew-symmetric matrices such that \( A(B) \) and 
\( A(B') \) are Cartan matrices of finite type. Then the series \( A(B, -) \) and \( A(B', -) \) are
strongly isomorphic if and only if \( A(B) \) and \( A(B') \) are of the same Cartan-Killing

In the process of proving these theorems, we obtain the following characterizations
of the cluster algebras of finite type.

Theorem 1.8. For a cluster algebra \( A \), the following are equivalent:
1. For (i) \( A \) is of finite type;
2. the set \( X \) of all cluster variables is finite;
3. for every seed \((x, p, B)\) in \( A \), the entries of the matrix \( B = (b_{xy}) \)
satisfy the inequalities \( |b_{xy}b_{yx}| \leq 3 \), for all \( x, y \in X \).
4. \( A = A(B_0, p_0) \) for some data of the form (1.7)–(1.9).

The equivalence (i) \( \iff \) (iv) in Theorem 1.8 is tantamount to Theorems 1.5–1.6.

1.4. Cluster variables in the finite type. The techniques in our proof of Theorem 1.6
allow us to enunciate the basic properties of cluster algebras of finite type.

We begin by providing an explicit description of the set of cluster variables in terms
of the corresponding root system.

For the remainder of Section 1, \( A = (a_{ij})_{i,j \in I} \) is a Cartan matrix of finite type
and \( A = A(B_0, p_0) \) a cluster algebra (of finite type) related to \( A \) as in Theorem 1.5.

Let \( \Phi \) be the root system associated with \( A \), with the set of simple roots \( \Pi = \{\alpha_i: i \in I\} \)
and the set of positive roots \( \Phi_+ \). (Our convention on the correspondence between \( A \) and \( \Phi \)
is that the simple reflections \( s_i \) act on simple roots by \( s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i \).)
Let \( x_0 = \{x_i : i \in I\} \) be the cluster for the initial seed \((x_0, p_0, B_0)\).
(By an abuse of notation, we label the rows and columns of \( B_0 \) by the elements of \( I \)
rather than by the variables \( x_i \), for \( i \in I \).) We will use the shorthand \( x_0 = \prod_{i \in I} x_i^{x_i} \)
for any vector \( \alpha = \sum_{i \in I} a_i \alpha_i \) in the root lattice.

The following result shows that the cluster variables of \( A \) are naturally parameterized by the set \( \Phi_{\geq 1} = \Phi_{>0} \cup (-\Pi) \) of almost positive roots.
Theorem 1.9. There is a unique bijection $\alpha \mapsto x[\alpha]$ between the almost positive roots in $\Phi$ and the cluster variables in $A$ such that, for any $\alpha \in \Phi_{\geq -1}$, the cluster variable $x[\alpha]$ is expressed in terms of the initial cluster $x_0 = \{x_i : i \in I\}$ as

$$x[\alpha] = \frac{P_\alpha(x_0)}{x^\alpha},$$

where $P_\alpha$ is a polynomial over $\mathbb{Z}P$ with nonzero constant term. Under this bijection, $x[-\alpha_i] = x_i$.

Formula (1.10) is an example of the Laurent phenomenon established in [9] for arbitrary cluster algebras: every cluster variable can be written as a Laurent polynomial in the variables of an arbitrary fixed cluster and the elements of $P$. In [9], we conjectured that the coefficients of these Laurent polynomials are always non-negative. Our next result establishes this conjecture (indeed, strengthens it) in the special case of the distinguished cluster $x_0$ in a cluster algebra of finite type.

Theorem 1.10. Every coefficient of each polynomial $P_\alpha$ (see (1.10)) can be written as a polynomial in the elements of $P$ (see Definition 1.2) with positive integer coefficients.

1.5. Cluster complexes. We next focus on the combinatorics of clusters. As before, $A$ is a cluster algebra of finite type associated with a root system $\Phi$.

Theorem 1.11. The exact form of each exchange relation (1.4) in $A$ (that is, the cluster variables, exponents, and coefficients appearing in the right-hand side) depends only on the ordered pair $(z, z')$ of cluster variables, and not on the particular choice of clusters (or seeds) containing them.

In fact, we do more: we describe in concrete root-theoretic terms all pairs $(\beta, \beta')$ of almost positive roots such that the product $x[\beta]x[\beta']$ appears as a left-hand side of an exchange relation, and for every such pair, we describe the exponents appearing on the right. See Definition 4.2 and formula (5.1).

Theorem 1.12. Every seed $(x, p, B)$ in $A$ is uniquely determined by its cluster $x$. For any cluster $x$ and any $x \in x$, there is a unique cluster $x'$ with $x \cap x' = x - \{x\}$.

We conjecture that the requirement of finite type in Theorems 1.11 and 1.12 can be dropped; that is, any cluster algebra conjecturally has these properties.

We define the cluster complex $\Delta(A)$ as the simplicial complex whose ground set is $X$ (the set of all cluster variables) and whose maximal simplices are the clusters. By Theorem 1.12, the cluster complex encodes the combinatorics of seed mutations. Thus, the dual graph of $\Delta(A)$ is precisely the exchange graph of $A$.

Our next result identifies the cluster complex $\Delta(A)$ with the dual complex $\Delta(\Phi)$ of the generalized associahedron of the corresponding type. The simplicial complexes $\Delta(\Phi)$ were introduced and studied in [11]; see also [7] and Section 3 below.

Theorem 1.13. Under the bijection $\Phi_{\geq -1} \rightarrow X$ of Theorem 1.9, the cluster complex $\Delta(A)$ is identified with the simplicial complex $\Delta(\Phi)$. In particular, the cluster complex does not depend on the coefficient semifield $\mathbb{P}$, or on the choice of the coefficients $p_\circ$ in the initial seed.
1.6. **Organization of the paper.** The bulk of the paper is devoted to the proofs of Theorems 1.4–1.8. We already noted that Theorem 1.4 follows from Theorems 1.5–1.7, and that the implications (iv) $\implies$ (i) and (i) $\implies$ (iv) in Theorem 1.8 are essentially Theorems 1.5 and 1.6, respectively. Furthermore, (i) $\implies$ (ii) is trivial, while (ii) $\implies$ (iii) follows from [9, Theorem 6.1]. Thus, we need to prove the following:

- **Theorem 1.5**
- **Theorem 1.6** via the implication (iii) $\implies$ (iv) in Theorem 1.8
- **Theorem 1.7**

Figure 1 shows the logical dependences between these proofs, and the sections containing them. Theorems 1.9–1.13, which only rely on Theorem 1.5, are proved in Sections 5–6 following the completion of the proof of Theorem 1.5.

![Logical dependences among the proofs of Theorems 1.5–1.13](image)

The concluding Section 12 provides explicit geometric realizations for some special cluster algebras of the classical types $ABCD$. These examples are based on a general criterion given in Section 11 for a cluster algebra to be isomorphic to a $\mathbb{Z}$-form of the coordinate ring of some algebraic variety. In particular, we show that a $\mathbb{Z}$-form of the homogeneous coordinate ring of the Grassmannian $\text{Gr}_{2,m}$ ($m \geq 5$) in its Plücker embedding carries two different cluster algebra structures of types $A_{m-3}$ and $B_{m-2}$, respectively.

2. **Cluster algebras via pseudomanifolds**

2.1. **Pseudomanifolds and geodesic loops.** This section begins our proof of Theorem 1.5. Its main result (Proposition 2.3) provides sufficient conditions ensuring that a cluster algebra that arises from a particular combinatorial construction is of finite type.

The first ingredient of this construction is an $(n-1)$-dimensional pure simplicial complex $\Delta$ (finite or infinite) on the ground set $\Psi$. Thus, every maximal simplex in $\Delta$ is an $n$-element subset of $\Psi$ (a “cluster”). A simplex of codimension 1 (i.e., an $(n-1)$-element subset of $\Psi$) is called a wall. The vertices of the dual graph $\Gamma$ are the clusters in $\Delta$; two clusters are connected by an edge in $\Gamma$ if they share a wall.

We assume that $\Delta$ is a pseudomanifold, i.e.,

1. every wall is contained in precisely two maximal simplices (clusters);
2. the dual graph $\Gamma$ is connected.

In view of (2.1), the graph $\Gamma$ is $n$-regular, i.e., there are precisely $n$ edges incident to every vertex $C \in \Gamma$. 

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Example 2.1. Let $n = 1$. Then (2.1) is saying that the empty simplex is contained in precisely two 0-dimensional simplices (points in $\Psi$). Thus a 0-dimensional pseudomanifold must be a disjoint union of two points (a 0-dimensional sphere). The dual graph $\Gamma$ has these points as vertices, with an edge connecting them.

For $n = 2$, a 1-dimensional pseudomanifold is nothing but a 2-regular connected graph—thus, either an infinite chain or a cycle. Ditto for its dual graph.

For a non-maximal simplex $D \in \Delta$, we denote by $\Delta_D$ the link of $D$. This is the simplicial complex on the ground set $\Psi_D = \{ \alpha \in \Psi : D \cup \{ \alpha \} \in \Delta \}$ such that $D'$ is a simplex in $\Delta_D$ if and only if $D \cup D'$ is a simplex in $\Delta$. The link $\Delta_D$ is a pure simplicial complex of dimension $n - |D| - 1$ satisfying property (2.1) of a pseudomanifold.

We will assume that $\Delta$ satisfies the following additional condition:

(2.3) the link of every non-maximal simplex $D$ in $\Delta$ is a pseudomanifold.

Equivalently, the dual graph $\Gamma_D$ of $\Delta_D$ is connected.

Conditions (2.1)–(2.3) can be restated as saying that $\Delta$ is a (possibly infinite, simplicial) abstract polytope in the sense of [1] or [23]; another terminology is that $\Delta$ is a thin, residually connected complex (see, e.g., [2]).

We identify the graph $\Gamma_D$ with an induced subgraph in $\Gamma$ whose vertices are the maximal simplices in $\Delta$ that contain $D$. In particular, for $|D| = n - 2$, the pseudomanifold $\Delta_D$ is 1-dimensional, so $\Gamma_D$ is either an infinite chain or a finite cycle in $\Gamma$. In the latter case, we call $\Gamma_D$ a geodesic loop. (This is a geodesic in $\Gamma$ with respect to the canonical connection on $\Gamma$, in the sense of [4, 14].)

We assume that the fundamental group of $\Gamma$ is generated by the geodesic loops pinned down to a fixed basepoint.

More precisely, by (2.4) we mean that the fundamental group of $\Gamma$ is generated by all the loops of the form $PL\bar{P}$, where $L$ is a geodesic loop, $P$ is a path originating at the basepoint, and $\bar{P}$ is the inverse path to $P$.

Lemma 2.2. Let $\Delta$ be the boundary complex of an $n$-dimensional simplicial convex polytope. Then conditions (2.1)–(2.4) are satisfied.

Equivalently, conditions (2.1)–(2.4) hold if $\Delta$ is (the simplicial complex of) the normal fan of a simple $n$-dimensional convex polytope $\Delta^*$.

Proof. Statements (2.1)–(2.3) are trivial. The link $\Delta_D$ of each non-maximal face $D$ in $\Delta$ is again a simplicial polytope, implying (2.3). Specifically, $\Delta_D$ is canonically identified (see, e.g., [3 Problem VI.1.4.4]) with the dual polytope for the dual face $D^*$ in the dual simple polytope $\Delta^*$. Thus, the graph $\Gamma_D$ is the 1-skeleton of $D^*$.

It remains to check (2.4). The case $n = 2$ is trivial, so let us assume that $n \geq 3$. Each geodesic loop $\Gamma_D$ (for an $(n-2)$-dimensional face $D$) is identified with the 1-skeleton (i.e., the boundary) of the dual 2-dimensional face $D^*$ in $\Delta^*$. The boundary cell complex of $\Delta^*$ is spherical, hence simply connected. On the other hand, the fundamental group of this complex can be obtained as a quotient of the fundamental group of its 1-skeleton $\Gamma$ by the normal subgroup generated by the boundaries of its 2-dimensional cells pinned down to a basepoint (see, e.g.,
Theorem VII.2.1, VII.4.1), or, equivalently, by the subgroup generated by all pinned-down geodesic loops. This proves Proposition 2.3. \qed

2.2. Sufficient conditions for finite type. We next describe the second ingredient of our construction. Suppose that we have a family of integer matrices \( B(C) = (b_{\alpha\beta}(C))_{\alpha, \beta \in C} \), for all vertices \( C \) in \( \Gamma \), satisfying the following conditions:

(2.5) all the matrices \( B(C) \) are sign-skew-symmetric.

(2.6) for every edge \((C, \overline{C})\) in \( \Gamma \), with \( \overline{C} = C - \{\gamma\} \cup \{\overline{\gamma}\} \), the matrix \( B(\overline{C}) \)

is obtained from \( B(C) \) by a matrix mutation in direction \( \gamma \) followed by relabeling one row and one column by replacing \( \gamma \) by \( \overline{\gamma} \).

We need one more assumption concerning the matrices \( B(C) \), which will require a little preparation. Fix a geodesic \( \Gamma_D \) and associate to its every vertex \( C = D \cup \{\alpha, \beta\} \) the integer \( b_{\alpha\beta}(C)b_{\beta\alpha}(C) \). It is trivial to check, using (2.6), that this integer depends only on \( D \), not on the particular choice of \( \alpha \) and \( \beta \). We say that \( \Gamma_D \) is of finite type if \( b_{\alpha\beta}(C)b_{\beta\alpha}(C) \in \{0, -1, -2, -3\} \) for some (equivalently, any) vertex \( C = D \cup \{\alpha, \beta\} \) on \( \Gamma_D \). If this is the case, then we associate to \( \Gamma_D \) the Coxeter number \( h \in \mathbb{Z}_{>0} \) defined by

\[
2 \cos(\pi/h) = \sqrt{|b_{\alpha\beta}(C)b_{\beta\alpha}(C)|},
\]
or, equivalently, by the table

| \( b_{\alpha\beta}(C)b_{\beta\alpha}(C) \) | 0 | 1 | 2 | 3 |
|-----|---|---|---|---|
| \( h(C, x, y) \) | 2 | 3 | 4 | 6 |

Our last condition is:

(2.7) every geodesic loop in \( \Gamma \) is of finite type, and has length \( h + 2 \), where \( h \) is the corresponding Coxeter number.

Proposition 2.3. Assume that a simplicial complex \( \Delta \) and a family of matrices \( (B(C)) \) satisfy the assumptions 2.1, 2.7 above. Let \( B = B(C) \) for some vertex \( C \), and let \( \mathcal{A} = \mathcal{A}(B, p) \) be the cluster algebra associated with \( B \) and some coefficient tuple \( p \). There exists a surjection from the set of vertices of \( \Gamma \) onto the set of all seeds for \( \mathcal{A} \). In particular, if \( \Delta \) is finite, then \( \mathcal{A} \) is of finite type.

We will prove this proposition by showing that, whether \( \Delta \) is finite or infinite, its dual graph \( \Gamma \) is always a covering graph for the exchange graph of \( \mathcal{A}(B, p) \).

To formulate this more precisely, we need some preparation.

Let \( C \) be a vertex of \( \Gamma \). A seed attachment at \( C \) consists of a choice of a seed \( \Sigma = (x, p, B) \) and a bijection \( \alpha \mapsto x[C, \alpha] \) between \( C \) and \( x \) identifying the matrices \( B(C) \) and \( B \), so that \( b_{\alpha\beta}(C) = b_{x[C,\alpha], x[C,\beta]} \). The transport of a seed attachment along an edge \((C, \overline{C})\) with \( \overline{C} = C - \{\gamma\} \cup \{\overline{\gamma}\} \) is defined as follows: the seed \( \Sigma = (x, p, B) \) attached to \( \overline{C} \) is obtained from \( \Sigma \) by the mutation in direction \( x[C, \gamma] \), and the corresponding bijection \( \overline{\Sigma} \) is uniquely determined by \( x[C, \alpha] = x[C, \alpha] \) for all \( \alpha \in C \cap \overline{C} \). (The remaining cluster variable \( x[C, \overline{\gamma}] \) is obtained from \( x[C, \gamma] \) by the corresponding exchange relation.) We note that transporting the resulting seed attachment backwards from \( \overline{C} \) to \( C \) recovers the original seed attachment.

Proposition 2.3 is an immediate consequence of the following lemma.
Lemma 2.4. Let $\Delta$ and $(B(C))$ satisfy (2.1)–(2.7). Suppose we are given a vertex $C_0$ in $\Gamma$ together with a seed $\Sigma_0$ for a cluster algebra $A$.

1. The given seed attachment at $C_0$ extends uniquely to a family of seed attachments at all vertices in $\Gamma$ such that, for every edge $(C, C)$, the seed attachment at $C$ is obtained from that at $C_0$ by transport along $(C, C)$.

2. Let $\Sigma(C)$ denote the seed attached to a vertex $C$. The map $C \mapsto \Sigma(C)$ is a surjection onto the set of all seeds for $A$.

3. For every vertex $C$ and every $\alpha \in \Psi$, the cluster variable $x[C, \alpha]$ attached to $\alpha$ at $C$ depends only on $\alpha$ (so can be denoted by $x[\alpha]$).

4. The map $\alpha \mapsto x[\alpha]$ is a surjection from the ground set $\Psi$ onto the set of all cluster variables for $A$.

Proof. 1. Since $\Gamma$ is connected, we can transport the initial seed attachment at $C_0$ to an arbitrary vertex $C$ along a path from $C_0$ to $C$. We need to show that the resulting seed attachment at $C$ is independent of the choice of a path. For that, it suffices to prove that transporting a seed attachment along a loop in $\Gamma$ brings it back unchanged. By (2.4), it is enough to show this for the geodesic loops. Then the claim follows from (2.7) and [9, Theorem 7.7].

2. Take an arbitrary seed $\Sigma$ for $A$. By Definition 1.2, $\Sigma$ can be obtained from the initial seed $\Sigma_0$ by a sequence of mutations. This sequence is uniquely lifted to a path $(C_0, \ldots, C)$ in $\Gamma$ such that transporting the initial seed attachment at $C_0$ along the edges of this path produces the chosen sequence of mutations. Hence $\Sigma(C) = \Sigma$, as desired.

3. Let $\alpha \in \Psi$, and let $C$ and $C'$ be two vertices of $\Gamma$ such that $\alpha \in C \cap C'$. By (2.3), $C$ and $C'$ can be joined by a path $(C_1 = C, C_2, \ldots, C_\ell = C')$ such that $\alpha \in C_i$ for all $i$. Hence $x[C_1, \alpha] = x[C_2, \alpha] = \cdots = x[C_\ell, \alpha]$, as needed.

4. Follows from Part 2. □

Remark 2.5. Parts 1 and 2 in Lemma 2.4 imply that the map $C \mapsto \Sigma(C)$ induces a covering of the exchange graph of $A$ by the graph $\Gamma$. If, in addition, the map $\alpha \mapsto x[\alpha]$ in Lemma 2.4 is a bijection, then the map $C \mapsto \Sigma(C)$ is also a bijection. Thus, the latter map establishes an isomorphism between $\Gamma$ and the exchange graph of $A$, and between $\Delta$ and the cluster complex of $A$.

3. Generalized associahedra

This section contains an exposition of the results in [11] and [7] that will be used later in our proof of Theorem 1.5.

Let $A = (a_{ij})_{i,j \in I}$ be an indecomposable Cartan matrix of finite type, and $\Phi$ the corresponding irreducible root system of rank $n = |I|$. We retain the notation introduced in Section 1.4. In particular, $\Phi_{\geq -1} = \Phi_{\geq 0} \cup (-\Pi)$ denotes the set of almost positive roots.

The Coxeter graph associated to $\Phi$ is a tree; recall that this graph has the index set $I$ as the set of vertices, with $i$ and $j$ joined by an edge whenever $a_{ij} \neq 0$. In particular, the Coxeter graph is bipartite; the two parts $I_+, I_- \subset I$ are determined uniquely up to renaming. The sign function $\varepsilon : I \to \{+, -\}$ is defined by

$$
\varepsilon(i) = \begin{cases} 
+ & \text{if } i \in I_+; \\
- & \text{if } i \in I_-.
\end{cases}
$$
Let $Q = \mathbb{Z}\Pi$ denote the root lattice, and $Q_\mathbb{R}$ the ambient real vector space. For $\gamma \in Q_\mathbb{R}$, we denote by $[\gamma : \alpha_i]$ the coefficient of $\alpha_i$ in the expansion of $\gamma$ in the basis $\Pi$. Let $\tau_+$ and $\tau_-$ denote the piecewise-linear automorphisms of $Q_\mathbb{R}$ given by

$$[\tau_+ \gamma : \alpha_i] = \begin{cases} -[\gamma : \alpha_i] - \sum_{j \neq i} a_{ij} \max([\gamma : \alpha_j], 0) & \text{if } i \in I_+; \\ [\gamma : \alpha_i] & \text{otherwise}. \end{cases}$$

(3.2)

It is easy to see that each of $\tau_+$ and $\tau_-$ is an involution that preserves the set $\Phi_{\geq -1}$. More specifically, the action of $\tau_+$ and $\tau_-$ on $\Phi_{\geq -1}$ can be described as follows:

$$\tau_\varepsilon(\alpha) = \begin{cases} \alpha & \text{if } \alpha = -\alpha_i, i \in I_{-\varepsilon}; \\ (\prod_{i \in I_\varepsilon} s_i)(\alpha) & \text{otherwise}. \end{cases}$$

(3.3)

(The product of simple reflections $\prod_{i \in I_\varepsilon} s_i$ is well-defined since its factors commute.) To illustrate, consider the type $A_2$, with $I_+ = \{1\}$ and $I_- = \{2\}$. Then

$$\begin{array}{ccccccc}
-\alpha_1 & \tau_+ & 0 & \tau_- & \alpha_1 + \alpha_2 & \tau_+ & 0 & \tau_- & -\alpha_2 \\
\tau_+ & \tau_+ & \tau_+ & \tau_+ & \tau_+ & \tau_+ & \tau_+ & \tau_+ & \tau_+
\end{array}$$

(3.4)

We denote by $\langle \tau_-, \tau_+ \rangle$ the group generated by $\tau_-$ and $\tau_+$.

The Weyl group of $\Phi$ is denoted by $W$, its longest element by $w_0$, and its Coxeter number by $h$.

**Theorem 3.1.** [11] Theorems 1.2, 2.6

1. The order of $\tau_- \tau_+$ is equal to $(h + 2)/2$ if $w_0 = -1$, and to $h + 2$ otherwise. Accordingly, $\langle \tau_-, \tau_+ \rangle$ is a dihedral group of order $(h + 2)$ or $2(h + 2)$.

2. The correspondence $\Omega \mapsto \Omega \cap (-\Pi)$ is a bijection between the $\langle \tau_-, \tau_+ \rangle$-orbits in $\Phi_{\geq -1}$ and the $\langle w_0 \rangle$-orbits in $(-\Pi)$.

We note that Theorem 3.1 is stronger than [11] Theorem 2.6, since in the latter, $\tau_-$ and $\tau_+$ are treated as permutations of the set $\Phi_{\geq -1}$, rather than as transformations of the entire space $Q_\mathbb{R}$. This stronger version follows from [11] Theorem 1.2 by “tropical specialization” (see [11] (1.8)).

According to [11] Section 3.1, there exists a unique function $\langle \alpha, \beta \rangle \mapsto [\alpha \parallel \beta]$ on $\Phi_{\geq -1} \times \Phi_{\geq -1}$ with nonnegative integer values, called the compatibility degree, such that

$$(-\alpha_i \parallel \alpha) = \max([\alpha : \alpha_i], 0)$$

for any $i \in I$ and $\alpha \in \Phi_{\geq -1}$, and

$$[\tau_\varepsilon \alpha \parallel \tau_\varepsilon \beta] = [\alpha \parallel \beta]$$

for any $\alpha, \beta \in \Phi_{\geq -1}$ and any sign $\varepsilon$. We say that $\alpha$ and $\beta$ are compatible if $[\alpha \parallel \beta] = 0$. (This is equivalent to $(\beta \parallel \alpha) = 0$ by [11] Proposition 3.3, Part 2.)

Let $\Delta(\Phi)$ be the simplicial complex on the ground set $\Phi_{\geq -1}$ whose simplices are the subsets of mutually compatible roots. As in Section 2 above, the maximal simplices of $\Delta(\Phi)$ are called clusters.

**Theorem 3.2.** [11] Theorems 1.8, 1.10 | [7] Theorem 1.4

1. Each cluster in $\Delta(\Phi)$ is a $\mathbb{Z}$-basis of the root lattice $Q$; in particular, all clusters are of the same size $n$. 


The cones spanned by the simplices in \( \Delta(\Phi) \) form a complete simplicial fan in \( Q \).

This simplicial fan is the normal fan of a simple \( n \)-dimensional convex polytope, the generalized associahedron of the corresponding type.

\[ \Delta(\Phi) \]

Figure 2. The complex \( \Delta(\Phi) \) and the corresponding polytope in type \( A_2 \)

Generalized associahedra of types \( ABC \) are: in type \( A \), the Stasheff polytope, or ordinary associahedron (see, e.g., \( \text{[29, 17]} \) or \( \text{[13, Chapter 7]} \)); in types \( B \) and \( C \), the Bott-Taubes polytope, or cyclohedron (see \( \text{[5, 20, 24]} \)). Explicit combinatorial descriptions of generalized associahedra of types \( ABCD \) in relation to the root system framework are discussed in \( \text{[11, 7]} \); see also Section 12 below.

**Proposition 3.3.** \( \text{[11, Theorem 3.11]} \) Every vector \( \gamma \in Q \) has a unique cluster expansion, that is, \( \gamma \) can be expressed uniquely as a nonnegative linear combination of mutually compatible roots from \( \Phi_{\geq -1} \) (the cluster components of \( \gamma \)).

**Proposition 3.4.** \( \text{[7, Proposition 1.13]} \) Let \( [\gamma : \alpha]_{\text{clus}} \) denote the coefficient of an almost positive root \( \alpha \) in the cluster expansion of a vector \( \gamma \in Q \). Then we have \( [\sigma(\gamma) : \sigma(\alpha)]_{\text{clus}} = [\gamma : \alpha]_{\text{clus}} \) for \( \sigma \in \langle \tau_+ , \tau_- \rangle \).

We call two roots \( \beta, \beta' \in \Phi_{\geq -1} \) exchangeable if \( (\beta \mid \beta') = (\beta' \mid \beta) = 1 \). The choice of terminology is motivated by the following proposition.

**Proposition 3.5.** \( \text{[7, Lemma 2.2]} \) Let \( C \) and \( C' = C - \{ \beta \} \cup \{ \beta' \} \) be two adjacent clusters. Then the roots \( \beta \) and \( \beta' \) are exchangeable.

The converse of Proposition 3.5 is also true: see Corollary \( \text{[13]} \) below.

**Proposition 3.6.** \( \text{[7, Theorem 1.14]} \) If \( n > 1 \) and \( \beta, \beta' \in \Phi_{\geq -1} \) are exchangeable, then the set
\[ \{ \sigma^{-1}( \sigma(\beta) + \sigma(\beta') ) : \sigma \in \langle \tau_+ , \tau_- \rangle \} \]
consists of two elements of \( Q \), one of which is \( \beta + \beta' \), and the other will be denoted by \( \beta \psi \beta' \). In the special case where \( \beta' = -\alpha_j, j \in I \), we have
\[
(-\alpha_j) \psi \beta = \tau_{-\varepsilon(j)}(-\alpha_j + \tau_{-\varepsilon(j)}(\beta))
\]
\[
= \beta - \alpha_j + \sum_{i \neq j} a_{ij} \alpha_i.
\]

A precise rule for deciding whether an element \( \sigma^{-1}( \sigma(\beta) + \sigma(\beta') ) \) is equal to \( \beta + \beta' \) or \( \beta \psi \beta' \) is given in Lemma \( \text{[14]} \) below.
Remark 3.7. If \( n = 1 \), i.e., \( \Phi \) is of type \( A_1 \) with a unique simple root \( \alpha_1 \), then \( \{ \beta, \beta' \} = \{-\alpha_1, \alpha_1 \} \), and the group \( \langle \tau_+ , \tau_- \rangle \) is just the Weyl group \( W = \langle s_1 \rangle \). Thus, in this case, the set in Proposition 3.6 consists of a single element \( \beta + \beta' = 0 \). It is then natural to set \( \beta \oplus \beta' = 0 \) as well.

Remark 3.8. All results in this section extend in an obvious way to the case of an arbitrary Cartan matrix of finite type (not necessarily indecomposable). In that generality, \( \Phi \) is a disjoint union of irreducible root systems \( \Phi^{(1)}, \ldots, \Phi^{(m)} \), and the clusters for \( \Phi \) are the unions \( C_1 \cup \cdots \cup C_m \), where each \( C_k \) is a cluster for \( \Phi^{(k)} \).

4. Proof of Theorem 1.5

In this section, we complete the proof of Theorem 1.5. The plan is as follows. Without loss of generality, we can assume that the Cartan matrix \( A \) is indecomposable, so the corresponding root system \( \Phi \) is irreducible. By Theorem 3.2 and Lemma 2.2, conditions (2.1)–(2.4) are satisfied. By Proposition 2.3, to prove Theorem 1.5 it suffices to define a family of matrices \( B(C) \), for each cluster \( C \) in \( \Delta(\Phi) \), such that (2.5)–(2.7) hold, together with

\[
\text{(4.1)} \quad \text{for some cluster } C_0, \text{ the matrix } B_0 = B(C_0) \text{ is as in (1.8)}.
\]

Defining the matrices \( B(C) \) requires a little preparation. Throughout this section, all roots are presumed to belong to the set \( \Phi_{\geq -1} \).

Lemma 4.1. There exists a sign function \( (\beta, \beta') \mapsto \varepsilon(\beta, \beta') \in \{\pm 1\} \) on pairs of exchangeable roots, uniquely determined by the following properties:

\[
\text{(4.2)} \quad \varepsilon(-\alpha_j, \beta') = -\varepsilon(j);
\]

\[
\text{(4.3)} \quad \varepsilon(\tau \beta, \tau \beta') = -\varepsilon(\beta, \beta') \text{ for } \tau \in \{\tau_+, \tau_-\} \text{ and } \beta, \beta' \notin \{-\alpha_j : \tau(-\alpha_j) = -\alpha_j\}.
\]

Moreover, this function is skew-symmetric:

\[
\text{(4.4)} \quad \varepsilon(\beta', \beta) = -\varepsilon(\beta, \beta').
\]

Proof. The uniqueness of \( \varepsilon(\beta, \beta') \) is an easy consequence of Theorem 3.1 Part 2. Let us prove the existence. For a root \( \beta \in \Phi_{\geq -1} \) and a sign \( \varepsilon \), let \( k_\varepsilon(\beta) \) denote the smallest nonnegative integer \( k \) such that \( \tau_{\varepsilon}^{(k+1)}(\beta) = \tau_{\varepsilon}^{(k)}(\beta) \in \Pi \), where we abbreviate

\[
\tau_{\varepsilon}^{(k)} = \tau_{\varepsilon} \cdots \tau_{\varepsilon} \tau_{\varepsilon} \tau_{\varepsilon} \cdots \underset{k \text{ factors}}{\overbrace{\cdots}}
\]

(cf. Theorem 3.1). In view of \( \text{Theorem 3.1} \), we always have

\[
\text{(4.5)} \quad k_+(\beta) + k_-(\beta) = h + 1;
\]

in particular, \( k_{\varepsilon(j)}(-\alpha_j) = h + 1 \) and \( k_{\varepsilon(j)}(-\alpha_j) = 0 \). It follows from (4.5) that if \( \beta \) and \( \beta' \) are incompatible (in particular, exchangeable), then \( k_\varepsilon(\beta) < k_\varepsilon(\beta') \) for precisely one choice of a sign \( \varepsilon \). Let us define \( \varepsilon(\beta, \beta') \) by the condition

\[
\text{(4.6)} \quad k_{\varepsilon(\beta, \beta')}(\beta) < k_{\varepsilon(\beta, \beta')}(\beta').
\]

The properties (4.2)–(4.4) are immediately checked from this definition. \( \Box \)

We are now prepared to define the matrices \( B(C) = (b_{\alpha \beta}(C)) \).
Definition 4.2. Let $C$ be a cluster in $\Delta(\Phi)$, that is, a $\mathbb{Z}$-basis of the root lattice $Q$ consisting of a mutually compatible roots. Let $C' = C - \{\beta\} \cup \{\beta'\}$ be an adjacent cluster obtained from $C$ by exchanging a root $\beta \in C$ with some other root $\beta'$. The entries $b_{\alpha,\beta}(C)$, $\alpha \in C$, of the matrix $B(C)$ are defined by
\begin{equation}
 b_{\alpha,\beta}(C) = \varepsilon(\beta, \beta') \cdot [\beta + \beta' - (\beta \cup \beta')] : \alpha \rangle_C,
\end{equation}
where $[\gamma : \alpha]_C$ denotes the coefficient of $\alpha$ in the expansion of a vector $\gamma \in Q$ in the basis $C$.

To complete the proof of Theorem 1.3, all we need to show is that the matrices $B(C)$ described in Definition 4.2 satisfy (4.1) and (2.5)–(2.7).

Proof of (4.1). Let $C = -\Pi$ be the cluster consisting of all the negative simple roots. Applying (4.7), and using (4.2) and (3.7), we obtain
\begin{equation}
 b_{-\alpha,\beta}(C) = -\varepsilon(j) \cdot [- \sum_{k \neq j} a_{kj}\alpha_k : -\alpha_i]_C = \begin{cases} 0 & \text{if } i = j; \\ \varepsilon(j)a_{ij} & \text{if } i \neq j, \end{cases}
\end{equation}
establishing (4.1). \Box

Proof of (2.5). We start by summarizing the basic properties of cluster expansions of $\beta + \beta'$ and $\beta \cup \beta'$ for an exchangeable pair of roots.

Lemma 4.3. Let $\beta, \beta' \in \Phi_{\geq -1}$ be exchangeable.

1. No negative simple root can be a cluster component of $\beta + \beta'$.
2. The vectors $\beta + \beta'$ and $\beta \cup \beta'$ have no common cluster components. That is, no root in $\Phi_{\geq -1}$ can contribute, with nonzero coefficient, to the cluster expansions of both $\beta + \beta'$ and $\beta \cup \beta'$.
3. All cluster components of $\beta + \beta'$ and $\beta \cup \beta'$ are compatible with both $\beta$ and $\beta'$.
4. A root $\alpha \in \Phi_{\geq -1}$, $\alpha \notin \{\beta, \beta'\}$, is compatible with both $\beta$ and $\beta'$ if and only if it is compatible with all cluster components of $\beta + \beta'$ and $\beta \cup \beta'$.
5. If $\alpha \in -\Pi$ is compatible with all cluster components of $\beta + \beta'$, then it is compatible with all cluster components of $\beta \cup \beta'$.

Proof.

1. Suppose $[\beta + \beta' : -\alpha_i]_{\text{clus}} > 0$ for some $i \in I$. (Here we use the notation from Proposition 3.3) Since all roots compatible with $-\alpha_i$ and different from $-\alpha_i$ do not contain $\alpha_i$ in their simple root expansion, it follows that $[\beta + \beta' : \alpha_i] < 0$. This can only happen if one of the roots $\beta$ and $\beta'$, say $\beta$, is equal to $-\alpha_i$; but since $\beta'$ is incompatible with $\beta$, we will still have $[\beta + \beta' : \alpha_i] \geq 0$, a contradiction.

2. Suppose $\alpha$ is a common cluster component of $\beta + \beta'$ and $\beta \cup \beta'$. Applying if necessary a transformation from $\langle \tau_+ , \tau_- \rangle$, we can assume that $\alpha$ is negative simple root (see Proposition 3.3 and Theorem 3.1 Part 2). But this is impossible by Part 1.

3. The claim for $\beta + \beta'$ is proved in [7, Lemma 2.3]. Since $\beta \cup \beta' = \sigma^{-1}(\beta + \sigma(\beta'))$ for some $\sigma \in \langle \tau_+ , \tau_- \rangle$, the claim for $\beta \cup \beta'$ follows from Proposition 3.3.

4. First suppose that $\alpha$ is compatible with both $\beta$ and $\beta'$. The fact that $\alpha$ is compatible with all cluster components of $\beta + \beta'$ is proved in [7, Lemma 2.3]. The fact that $\alpha$ is compatible with all cluster components of $\beta \cup \beta'$ now follows in the same way as in Part 3.

To prove the converse, suppose that $\alpha$ is incompatible with $\beta$. As in Part 2 above, we can assume that $\alpha = -\alpha_i$ for some $i$. Thus, we have $[\beta : \alpha_i] > 0$. Since
In this notation, (4.9) takes the form
\begin{equation}
\text{Lemma 4.7.}
\end{equation}
For a pair of exchangeable roots \(S\) the functions \(\tau S\) (4.10)

Proof. Let us introduce some notation. For every pair of exchangeable roots \((\beta, \beta')\), we denote by \(S_+(\beta, \beta')\) and \(S_-(\beta, \beta')\) the two elements of \(Q\) given by
\begin{align*}
S_+(\beta, \beta') &= \beta + \beta', \\
S_-(\beta, \beta') &= \beta \uplus \beta'.
\end{align*}
In this notation, (4.10) takes the form
\begin{equation}
(\tau_{\pm})^{-1}(\tau_{\epsilon}(\beta) + \tau_{\epsilon}(\beta')) = \begin{cases} 
\beta + \beta' & \text{if } 0 \leq k \leq \min(k_{\epsilon}(\beta), k_{\epsilon}(\beta')); \\
\beta \uplus \beta' & \text{if } \min(k_{\epsilon}(\beta), k_{\epsilon}(\beta')) < k \leq \max(k_{\epsilon}(\beta), k_{\epsilon}(\beta')).
\end{cases}
\end{equation}

Proof. This is a consequence of [17, Lemma 3.2].

We next establish the following symmetry property.

\begin{equation}
\text{Lemma 4.8.}
\end{equation}
For \(\tau \in \{\tau_+, \tau_-, \tau_0\}\), we have \(b_{\tau\alpha, \tau\beta}(\tau C) = -b_{\alpha\beta}(C)\).

Proof. Let us introduce some notation. For every pair of exchangeable roots \((\beta, \beta')\), we denote by \(S_+\) and \(S_-\) the two elements of \(Q\) given by
\begin{align*}
S_+(\beta, \beta') &= \beta + \beta', \\
S_-\epsilon(\beta, \beta') &= \beta \uplus \beta'.
\end{align*}
In this notation, (4.10) takes the form
\begin{equation}
\text{Lemma 4.9.}
\end{equation}
\begin{equation}
S_\epsilon(\beta, \beta') = S_-\epsilon(\beta, \beta') \quad \text{for } \tau \in \{\tau_+, \tau_-, \tau_0\};
\end{equation}
By Lemma 4.3, Part 1, and Proposition 3.4, the element of the set that does not have \( \beta \) where \( k \) proving (4.15), we show that \( k \). As a main step towards proving (4.15), we show that \( k \). Suppose on the contrary that \( k \). Applying Lemma 4.7 with \( k = k \) \( \leq \min (k, k') \), we see that

\[
\tau^{(k)}(\beta + \beta') = \tau^{(k)}(\beta) + \tau^{(k)}(\beta').
\]
To arrive at a contradiction, notice that $\alpha'$ is compatible with all cluster components of $\beta + \beta'$ but is incompatible with $\beta'$ (see Lemma 1.13). It follows that the root $\tau^{(k)}_e(\alpha') \in -\Pi$ is compatible with all cluster components of $\tau^{(k)}_e(\beta) + \tau^{(k)}_e(\beta')$ but incompatible with $\tau^{(k)}_e(\beta')$. But this contradicts Lemma 1.13 Parts 4-5.

Having established the inequality $k_o < k_e_0(\alpha')$, we see that (4.15) becomes a special case of Lemma 4.7. Indeed, if $\varepsilon_0 = \varepsilon(j)$ then $k_{e_0}(\alpha) = k_{e_0}(-\alpha_j) = h + 1$, so $k_o < \min(k_{e_0}(\alpha), k_{e_0}(\alpha'))$; and if $\varepsilon_0 = -\varepsilon(j)$ then $k_{e_0}(\alpha) = k_{e_0}(-\alpha_j) = 0$, so $\min(k_{e_0}(\alpha), k_{e_0}(\alpha')) \leq k_o < \max(k_{e_0}(\alpha), k_{e_0}(\alpha'))$. This concludes our proof of 2.6.

**Proof of 2.6.** Let $C$ and $\overline{C} = C - \{\gamma\} \cup \{\overline{\gamma}\}$ be adjacent clusters. Our task is to show that every matrix entry $b_{\alpha\beta}(\overline{C})$ is obtained from the entries of $B(C)$ by the procedure described in 2.6. We already know that all diagonal entries in $B(C)$ and $B(\overline{C})$ are equal to 0 which is consistent with the matrix mutation rule (1.3). We assume that $\alpha \neq \beta$. If $\beta = \overline{\gamma}$ then the desired equality $b_{\alpha\gamma}(\overline{C}) = -b_{\alpha\gamma}(C)$ is immediate from (1.3) and (1.4). Thus, it remains to treat the case where $\alpha \neq \beta$ and $\beta \neq \overline{\gamma}$. By Lemma 4.8, it is enough to consider the special case $\beta = -\alpha_j \in -\Pi$.

In view of (4.17), (4.18) and (1.24), we have

\begin{align}
(4.16) & \quad b_{\alpha\beta}(C) = -\varepsilon(j)[\delta : \alpha]_C, \\
(4.17) & \quad b_{\alpha\beta}(\overline{C}) = -\varepsilon(j)[\delta : \alpha]_{\overline{C}},
\end{align}

where we use the notation

\[ \delta = \delta_j = -\sum_{i \neq j} a_{ij} \alpha_i. \]

Since the basis $\overline{C}$ is obtained from $C$ by replacing $\gamma$ by $\overline{\gamma}$, we can use the expansion

\[ \gamma + \overline{\gamma} = \sum_{\alpha \in C \cap \overline{C}} [\gamma + \overline{\gamma} : \alpha]_{\text{clus}} \cdot \alpha \]

(cf. Corollary 4.6) to obtain:

\begin{align}
(4.18) & \quad b_{\alpha\beta}(\overline{C}) = -b_{\gamma\beta}(C) \\
(4.19) & \quad b_{\alpha\beta}(\overline{C}) = b_{\alpha\beta}(C) + [\gamma + \overline{\gamma} : \alpha]_{\text{clus}} \cdot b_{\gamma\beta}(C) \quad \text{for } \alpha \in C \cap \overline{C}.
\end{align}

Combining (4.18) and (4.17), we get

\begin{align}
(4.20) & \quad [\gamma + \overline{\gamma} : \alpha]_{\text{clus}} = \max(\varepsilon(\gamma, \overline{\gamma}) b_{\alpha\gamma}(C), 0), \\
(4.21) & \quad [\gamma + \overline{\gamma} : \beta]_{\text{clus}} = \max(\varepsilon(\gamma, \overline{\gamma}) b_{\beta\gamma}(C), 0).
\end{align}

By Lemma 1.3 Part 1, the left-hand side of (4.21) equals 0. Hence

\begin{align}
(4.22) & \quad \varepsilon(\gamma, \overline{\gamma}) = -\text{sgn}(b_{\beta\gamma}(C)) = \text{sgn}(b_{\gamma\beta}(C))
\end{align}

(assuming that $B(C)$ is sign-skew-symmetric). Now (4.19), (4.20), and (4.22) give

\[ b_{\alpha\beta}(\overline{C}) = b_{\alpha\beta}(C) + \max(\text{sgn}(b_{\gamma\beta}(C)) \cdot b_{\alpha\gamma}(C), 0) \cdot b_{\gamma\beta}(C), \]

which is easily seen to be equivalent to the second case in (1.3). This completes the verification that the matrices $B(C)$ satisfy (2.6). □

Proof of (2.7). We proceed by induction on the rank $n$ of a root system $\Phi$. For induction purposes, we need to allow $\Phi$ to be reducible; this is possible in view of Remark 3.8. For $n = 2$, the generalized associahedra of types $A_1 \times A_1$, $A_2$, $B_2$ and $G_2$ are convex polygons with 4, 5, 6, and 8 sides, respectively, matching the claim.

For the induction step, consider a geodesic loop $L$ in the dual graph $\Gamma = \Gamma_{\Delta(\Phi)}$ for a root system $\Phi$ of rank $n \geq 3$. According to the definition of a geodesic, the clusters lying on $L$ are obtained from some initial cluster $C$ by fixing $n - 2$ of the $n$ roots and alternately exchanging the remaining two roots. By (3.6), every transformation from $\langle \tau_+, \tau_- \rangle$ sends geodesics to geodesics. Furthermore, the Coxeter number associated to a geodesic does not change, by Lemma 4.8. Using Theorem 3.1, Part 2, we may therefore assume, without loss of generality, that one of the $n - 2$ fixed roots in the initial cluster $C$ is $-\alpha_j \in -\Pi$.

Lemma 4.10. Let $\Phi'$ be the rank $n - 1$ root subsystem of $\Phi$ spanned by the simple roots $\alpha_i$ for $i \neq j$.

1. The correspondence $C' \mapsto \{-\alpha_j\} \cup C'$ is a bijection between the clusters in $\Delta(\Phi')$ and the clusters in $\Delta(\Phi)$ that contain $-\alpha_j$. Thus, it identifies the cluster complex $\Delta(\Phi')$ with the link $\Delta_{\{-\alpha_j\}}$ of $\{-\alpha_j\}$ in $\Delta(\Phi)$ (see Section 2.1).

2. Assume that the sign function (3.1) for $\Phi'$ is a restriction of the sign function for $\Phi$. The matrix $B(C')$ associated to a cluster $C' \subset \Phi_{\geq -1}$ can be obtained from the matrix $B(\{-\alpha_j\} \cup C')$ by crossing out the row and column corresponding to the root $-\alpha_j$.

Proof. Part 1 follows from Proposition 3.5 (3)]. The assertion in Part 2 is immediately checked in the special case $C' = \{-\alpha_i : i \neq j\}$ (see (4.8)). It is then extended to an arbitrary cluster $C'$ because the graph $\Gamma' = \Gamma_{\Delta(\Phi')}$ is connected, and the propagation rules for the matrices $B(C')$ and $B(\{-\alpha_j\} \cup C')$ are easily seen to be exactly the same. (Here we use the fact that conditions (2.1), (2.5), and (2.6) have already been checked for $\Gamma$ and $\Gamma'$ alike.) □

By Lemma 4.10, Part 1, $L$ can be viewed as a geodesic loop in $\Gamma' = \Gamma_{\Delta(\Phi')}$. To complete the induction step, it remains to notice that, in view of Lemma 4.10 and Part 2, the Coxeter number associated with $L$ in $\Gamma'$ coincides with the original value in $\Gamma$. This completes the proof of (2.7). □

Theorem 1.5 is proved.

5. Proofs of Theorems 1.9 and 1.11–1.13

Proof of Theorems 1.11–1.13 modulo Theorem 1.9. To take Theorems 1.11–1.13 out of the way, we begin by deducing them from Theorem 1.9. We adopt all the conventions and notation of Section 3. In particular, we assume, without loss of generality, that the Cartan matrix $A$ is indecomposable, so the corresponding (finite) root system $\Phi$ is irreducible. We have proved that the complex $\Delta(\Phi)$ on the ground set $\Phi_{\geq -1}$ of almost positive roots, together with the family of matrices $B(C)$ introduced in Definition 4.2 satisfy conditions (2.1)–(2.7).

Let $A = A(B_o, p_o)$ be the cluster algebra of finite type appearing in Theorem 1.5. Here we choose an initial seed $\Sigma_o = (x_o, p_o, B_o)$ for $A$ by identifying the matrix $B_o$ with
with the matrix $B(C_0)$ at the cluster $C_0 = -\Pi$ in $\Delta(\Phi)$ (see \ref{ex}). This gives us a seed attachment at $C_0$. Applying Lemma \ref{seed} we obtain a surjection $\alpha \mapsto x[\alpha]$ from $\Phi_{\geq -1}$ onto the set of all cluster variables in $A$. Note that at this point, we have not yet proved that the variables $x[\alpha]$ are all distinct.

Assume for a moment that Theorem \ref{exchange} has been established. Then the map $\alpha \mapsto x[\alpha]$ is a bijection, and Theorems \ref{exchange} and \ref{exchange'} follow by Remark \ref{fact}.

As for Theorem \ref{w}, it becomes a consequence of Lemma \ref{lemma}. To be more precise, let us associate to every lattice vector $\gamma \in Q$ a monomial in the cluster variables by setting

$$x[\gamma] = \prod_{\alpha} x[\alpha]^{m_{\alpha}}, \quad m_{\alpha} = [\gamma : \alpha]_{\text{clus}}.$$ 

In view of \ref{ex}, every exchange relation \ref{exchange} corresponding to adjacent clusters $C$ and $C - \{\beta\} \cup \{\beta'\}$ can be written in the form

$$(5.1) \quad x[\beta] x[\beta'] = p_{\beta,\beta'}^x(C) x[\beta + \beta'] + p_{\beta,\beta'}^{-x}(C) x[\beta \cup \beta'],$$

for some coefficients $p_{\beta,\beta'}^\pm(C) \in \mathbb{P}$. Thus, the set of cluster variables and the respective nonzero exponents that appear in the right-hand side of \ref{exchange} are uniquely determined by $\beta$ and $\beta'$. The same holds for the coefficients $p_{\beta}^+(C)$, since the cluster variables appearing in the right-hand side are algebraically independent.

We denote $p_{\beta,\beta'}^\pm = p_{\beta}^\pm(C)$. This notation is justified in view of Theorem \ref{exchange}.

**Remark 5.1.** In view of Corollary \ref{c4}, the exchange relation \ref{exchange} holds for every pair $(\beta, \beta')$ of exchangeable roots. Also note that, in view of \ref{four} and \ref{five}, the exchange relation \ref{exchange} takes the following more explicit form if $\beta'$ is negative simple:

$$(5.2) \quad x[\beta] x[-\alpha_j] = p_{\beta,-\alpha_j}^x x[\beta - \alpha_j] + p_{\beta,-\alpha_j}^{-x} x[\beta \cup (-\alpha_j)]$$

$$= p_{\beta,-\alpha_j}^x x[\beta - \alpha_j] + p_{\beta,-\alpha_j}^{-x} x[\beta - \alpha_j + \sum_{i \neq j} a_{ij} \alpha_i].$$

For the classical types, the list of all exchangeable pairs $(\beta, -\alpha_j)$, together with the explicitly given cluster expansions for $\beta - \alpha_j$ and $\beta \cup (-\alpha_j) = \beta - \alpha_j + \sum_{i \neq j} a_{ij} \alpha_i$, was given in \cite[Section 4]{7}.

**Proof of Theorem \ref{exchange}** We prove \ref{exchange} by induction on

$$k(\alpha) = \min(k_+ (\alpha), k_- (\alpha)) \geq 0$$

(see the proof of Lemma \ref{lemma}). If $k(\alpha) = 0$, then $\alpha$ is a negative simple root, and there is nothing to prove. So we assume that $k(\alpha) = k \geq 1$, and that \ref{exchange} holds for all roots $\alpha'$ with $k(\alpha') < k$.

By the definition of $k(\alpha)$, we have

$$\alpha = \tau_{\varepsilon(j)}(-\alpha_j) = \tau_{-\varepsilon(j)}(\alpha_j)$$

for some $j \in I$. Since $\alpha_j$ and $-\alpha_j$ are exchangeable, so are $\alpha$ and $\tau(-\alpha_j)$, where we abbreviate $\tau = \tau_{-\varepsilon(j)}(\alpha_j)$. Let us write the corresponding exchange relation. Using the $(\tau_\pm)$-invariance of the exponents appearing in exchange relations (Lemma \ref{ex}), together with \ref{six} and \ref{seven}, we obtain:

$$(5.3) \quad x[\alpha] x[\tau(-\alpha_j)] = q \prod_{i \neq j} x[\tau(-\alpha_i)]^{-a_{ij}} + r,$$
where $q, r \in \mathbb{P}$. For $k = 1$, we have $\alpha = \alpha_j$, and (5.3) yields
\[
x[\alpha_j] = \frac{q \prod_{i \neq j} x_i^{-a_{ij}} + r}{x_j},
\]
establishing (1.10). Thus, we may assume that $k \geq 2$. In this case, all the roots $\alpha' \neq \alpha$ that appear in (5.3) are positive with $k(\alpha') < k$. Abbreviating
\[
\gamma = \sum_{i \neq j} (-a_{ij}) \cdot \tau(-\alpha_i)
\]
and applying the induction assumption, we can rewrite (5.3) as
\[
(5.4) \quad x[\alpha] = x[\tau(-\alpha_j) - \gamma, \frac{q \prod_{i \neq j} P_{\tau(-\alpha_i)}^{-a_{ij}} + r x[\gamma]}{P_{\tau(-\alpha_j)}},
\]
where all $P_{\alpha'}$ are polynomials over $\mathbb{Z}[\mathbb{P}]$ in the variables from the initial cluster $x_0$ with nonzero constant terms. The next step of the proof relies on the following trivial lemma.

Lemma 5.2. Let $P$ and $Q$ be two polynomials (in any number of variables) with coefficients in a domain $S$, and with nonzero constant terms $a$ and $b$, respectively. If the ratio $P/Q$ is a Laurent polynomial over $S$, then it is in fact a polynomial over $S$ with the constant term $a/b$.

By [9, Theorem 3.1], $x[\alpha]$ is a Laurent polynomial. Hence, by Lemma 5.2, the second factor in (5.4) is a polynomial over $\mathbb{Z}[\mathbb{P}]$ with nonzero constant term. To complete the proof of Theorem 1.9, it remains to compare (5.4) with (1.10), and to observe that
\[
\gamma = \tau\left(\sum_{i \neq j} a_{ij} \alpha_i\right) \quad \text{(by Proposition 3.4)}
\]
\[
= \tau(\alpha_j \cup (-\alpha_j)) \quad \text{(by (3.7))}
\]
\[
= \tau(\alpha_j) + \tau(-\alpha_j) \quad \text{(by Lemma 4.7)}
\]
\[
= \alpha + \tau(-\alpha_j). \quad \square
\]

Remark 5.3. Unfortunately, the argument above does not establish Theorem 1.10 because there is no guarantee that the second factor in (5.4) is a polynomial with coefficients in $\mathbb{Z}_{\geq 0}[\mathbb{P}]$ even if we assume that all the polynomials $P_{\alpha'}$ appearing there have this property. (Recall from Definition 1.2 that $\mathbb{P}$ denotes the set of all coefficients $p_{\beta, \beta'}$ appearing in various exchange relations (5.1); we denote by $\mathbb{Z}_{\geq 0}[\mathbb{P}]$ the set of polynomials with nonnegative integer coefficients in the elements of $\mathbb{P}$.)

The proof of Theorem 1.10 given in Section 6 below does not rely on Theorem 1.9, thus providing an alternative proof of the latter.

6. Proof of Theorem 1.10

We use the nomenclature of root systems given in Bourbaki [6], including the labeling of the simple roots in $\Phi$ by the indices $1, \ldots, n$. On the other hand, our convention on associating a Cartan matrix $A$ to a root system $\Phi$, as described in Section 1.4, is transposed to that in [6]—and the same as that in Kac [15].

We abbreviate $x_i = x[-\alpha_i]$ for $i = 1, \ldots, n$. Our goal is to prove that, for every almost positive root $\alpha$, we can write $x[\alpha]$ as a Laurent polynomial in $x_1, \ldots, x_n$ with coefficients in $\mathbb{Z}_{\geq 0}[\mathbb{P}]$. This time we will proceed by induction on the height of $\alpha$. 
(recall that $ht(\alpha) = \sum [\alpha : \alpha_i]$). The base case $\alpha \in -\Pi$ is trivial. The induction step will follow from the lemma below.

**Lemma 6.1.** For every positive root $\alpha$, there exists an index $j \in I$ such that

$$x_j x[\alpha] = F(x[\beta_1], \ldots, x[\beta_m]),$$

where $F$ is a polynomial with coefficients in $\mathbb{Z}_{\geq 0}[P]$ in some cluster variables $x[\beta_1], \ldots, x[\beta_m]$ such that $ht(\beta_i) < ht(\alpha)$ for all $i$.

The rest of this section is devoted to the proof of Lemma 6.1.

We call a positive root $\alpha$ non-exceptional if there exists a negative simple root $-\alpha_j$ exchangeable with $\alpha$; otherwise, $\alpha$ will be called exceptional. If the root $\alpha$ in Lemma 6.1 is non-exceptional, and $-\alpha_j$ is a negative simple root exchangeable with $\alpha$, then one easily sees that all cluster components of the vectors $\alpha - \alpha_j$ and $\alpha \oplus (-\alpha_j)$ appearing in the right-hand side of (6.2) are of smaller height than $\alpha$, and we are done. Thus, it remains to prove Lemma 6.1 for the exceptional roots. First, we identify them explicitly.

**Lemma 6.2.** The complete list of all exceptional positive roots is as follows:

1. $\Phi$ is of type $E_8$, and $\alpha = \alpha_{\text{max}}$ is the highest root in $\Phi$;
2. $\Phi$ is of type $F_4$, and $\alpha = \alpha_{\text{max}} = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$;
3. $\Phi$ is of type $F_4$, and $\alpha = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_2$;
4. $\Phi$ is of type $G_2$, and $\alpha = \alpha_{\text{max}} = 3\alpha_1 + 2\alpha_2$;
5. $\Phi$ is of type $G_2$, and $\alpha = 2\alpha_1 + \alpha_2$.

**Proof.** As noted in [7, Remark 1.16], $\alpha$ and $-\alpha_j$ are exchangeable if and only if

$$[\alpha : \alpha_j] = [\alpha^\vee : \alpha_j^\vee] = 1,$$

where $\alpha^\vee$ is the coroot corresponding to $\alpha$ under the natural bijection between $\Phi$ and the dual system $\Phi^\vee$. Let $(\alpha, \beta)$ denote a $W$-invariant scalar product on the root lattice $Q$. Then $[\alpha^\vee : \alpha_j^\vee] = \frac{[\alpha : \alpha^\vee]}{[\alpha, \alpha]} [\alpha : \alpha_j]$, so (6.7) is equivalent to

$$[\alpha : \alpha_j] = 1, \quad (\alpha, \alpha) = (\alpha_j, \alpha_j).$$

Thus, we need to verify that for every positive root $\alpha$, there exists a simple root $\alpha_j$ satisfying (6.8), unless $\alpha$ appears on the list (6.2)–(6.6), in which case there is no such simple root. This is checked by direct inspection using, e.g., the tables in [6].

In all classical types, the list of all pairs $(\alpha, -\alpha_j)$ satisfying (6.7) was given in [7].

**Proof of Lemma 6.1 for the type $E_8$ and $\alpha = \alpha_{\text{max}}$.** This case is by far the hardest among (6.2)–(6.6), so we will treat it in detail. We will prove that in this special case, Lemma 6.1 holds with $j = 8$, in the standard numeration of simple roots (see Figure 9).

We will need the following construction. In view of Lemma 4.8 any transformation $\sigma \in (\tau_+, \tau_-)$ gives rise to a “twisted” cluster algebra $\sigma(A)$ whose seeds are the transfers by $\sigma$ of the seeds of $A$; if $\sigma$ is written in terms of $\tau_+$ and $\tau_-$ as a product of an odd number of factors, this transfer involves the change of signs for the matrices $B$ and the corresponding interchange of $p^+$ and $p^-$ for the coefficients, as in Remark 1.3. This twist preserves the Cartan-Killing type.
Direct computation shows that for $\sigma = (\tau_+ \tau_-)^8 = (\tau_- \tau_+)^8$ (cf. Theorem 5.1), we have $\sigma(\alpha_{\text{max}}) = -\alpha_4$ and $\sigma(-\alpha_8) = \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5$. To prove Lemma 3.1 for the type $E_8$ and $\alpha = \alpha_{\text{max}}$, it is therefore sufficient to show that, in the twisted cluster algebra $\sigma(A)$, we have

\[(6.9) \quad x_4 x[\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5] = \tilde{F}(x[\beta_1], \ldots, x[\beta_m]),\]

where $\tilde{F}$ is a polynomial with coefficients in $\mathbb{Z}_{\geq 0}[P]$, and each $\beta_i$ is different from $\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5$.

Figure 3. Dynkin diagrams of types $E_8$ and $D_4$

Let $J = \{2, 3, 4, 5\} \subset I$, and let $\Phi(J)$ denote the type $D_4$ root subsystem of $\Phi$ spanned by the simple roots $\alpha_j$ with $j \in J$. Applying Lemma 4.10 four times, we conclude that the correspondence $C' \mapsto (-\Pi(I - J)) \cup C'$ identifies the cluster complex $\Delta(\Phi(J))$ with the link of $-\Pi(I - J)$ in the cluster complex $\Delta(\Phi)$; here we use the notation

$$-\Pi(I - J) = \{-\alpha_i : i \in I - J\}.$$ 

The exchange graph $\Gamma(J) = \Gamma_{\Delta(\Phi(J))}$ is therefore identified with the induced subgraph in the exchange graph of $A$ whose vertices are all the clusters containing $-\Pi(I - J)$. Let $A'$ denote the subring in $A$ generated by the cluster variables $x[\alpha]$ for $\alpha \in \Phi(J)_{\geq -1}$, together with the “coefficients” in all exchange relations corresponding to the edges in $\Gamma(J)$, where by a “coefficient” we mean the part of a monomial that does not involve the variables $x[\alpha]$ for $\alpha \in \Phi(J)_{\geq -1}$. (Thus, each “coefficient” is a product of an element of $P$ and a monomial in the variables $x_i$ for $i \in I - J$.) By Lemma 4.11 Part 2, the ring $A'$ is a normalized cluster algebra of type $D_4$ (cf. [9, Proposition 2.6]). The claim (6.9) now becomes a consequence of the following lemma.

Lemma 6.3. In the case of type $D_4$, with the notation as in Figure 3, we have

$$x_2 x[\alpha_{\text{max}}] = G(x[\gamma_1], \ldots, x[\gamma_k]),$$

where $\alpha_{\text{max}} = \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$, the almost positive roots $\gamma_1, \ldots, \gamma_k$ are different from $\alpha_{\text{max}}$, and $G$ is a polynomial with coefficients in $\mathbb{Z}_{\geq 0}[P]$.

We note that the roots $\beta_1, \ldots, \beta_m$ appearing in (6.9) are of two kinds: first, the images of $\gamma_1, \ldots, \gamma_k$ under the embedding $D_4 \to E_8$, and second, (some of) the “frozen” roots $-\alpha_1, -\alpha_6, -\alpha_7, -\alpha_8$.

Proof. Figure 4 shows a fragment of the exchange graph in type $D_4$, with each vertex $C$ representing a cluster containing the 4 roots written into the regions adjacent to $C$. The mutual compatibility of the roots in each of these quadruples is easily checked from the definitions.
We next write the exchange relations for some pairs of adjacent clusters shown in Figure 4. In doing so, we use:

- (implicitly) the combinatorial interpretation of almost positive roots of type $D_n$ given in [11, Section 3.5] and reproduced in Section 12.3 below; see specifically [11, Figure 7] for the type $D_4$;
- the resulting explicit expressions for the exchange relations which are consequences of [7, Lemma 4.6] (see Proposition 12.14 below);
- the monomial relations among the coefficients of exchange relations along a geodesic of type $A_2$, as given in [9, (6.11)]; our notation is patterned after [9, Figure 3].

The exchange relations for the left pentagonal geodesic in Figure 4 can be written in the following form, with $p_1, \ldots, p_5 \in \mathcal{P}$:

$$x_4[x_\alpha + x_\beta + x_\gamma] = p_1 [x_\alpha + x_\beta] + p_3 p_4 x_\gamma,$$

$$x_2 x_\alpha = p_2 x_1 + p_4 p_5 x_\beta,$$

$$x_2 x_\beta = p_3 x_2 + p_5 p_1,$$

$$x_2 x_\gamma = p_4 x_\alpha + p_1 p_2 x_1,$$

$$x_\alpha x_\beta = p_5 x_\alpha + p_2 p_3 x_1.$$

(Among these relations, only (6.10), (6.11), and (6.12) are needed in the proof; we wrote all five relations for the sake of clarity.) Similarly, the exchange relations for the right pentagonal geodesic can be written as follows, with $q_1, \ldots, q_5 \in \mathcal{P}$:

$$x_1 x_\alpha = q_1 [x_\alpha + x_\beta] x_\alpha + q_3 q_4 x_\beta,$$

$$x_\alpha x_\beta = q_2 x_\alpha + q_4 q_5,$$

$$x_\beta x_\gamma = q_3 x_\alpha + q_5 q_1 x_\alpha,$$

$$x_\gamma x_\gamma = q_4 x_\alpha + q_1 q_2 x_\alpha,$$

$$x_\alpha x_\gamma = q_5 x_1 + q_2 q_3 x_\alpha.$$
Comparing (6.15) to (6.12), we conclude that

\[ p_5 = q_2 q_3. \]

Successively applying (6.13), (6.10)–(6.11), (6.16), and (6.14), we obtain:

\[ x_{1,2} x'[\alpha_{\text{max}}] = q_1 p_2 x[\alpha_2 + \alpha_3] x[\alpha_2 + \alpha_4] + q_3 q_4 p_2 x[\alpha_2 + \alpha_3 + \alpha_4], \]

\[ = q_1 p_2 x[\alpha_2 + \alpha_4] + q_1 p_4 p_5 x[\alpha_3] x[\alpha_2 + \alpha_4] + q_3 q_4 p_4 x[\alpha_3] x[\alpha_4] + q_3 q_4 p_1 p_2 x_1, \]

\[ = q_1 p_2 x[\alpha_2 + \alpha_4] + q_2 q_3 p_4 x[\alpha_3] x[\alpha_2 + \alpha_4] + q_3 q_4 p_4 x[\alpha_3] x[\alpha_4] + q_3 q_4 p_1 p_2 x_1, \]

which implies

\[ x_{2} x'[\alpha_{\text{max}}] = q_1 p_2 x[\alpha_2 + \alpha_4] + q_3 q_4 x[\alpha_3] x[\alpha_1 + \alpha_2 + \alpha_4] + q_3 q_4 p_1 p_2. \]

and we are done.

Proof of Lemma 6.1 in the types \( F_4 \) and \( G_2 \). One way of handling the non-simply-laced cases is to deduce them from the simply-laced ones by means of the “folding” technique (see, e.g., [11] Section 2.4). Alternatively, one can perform direct computations, which show that, in the type \( F_4 \), we have

\[ x_{1,2} x[\alpha_1 + 2 \alpha_2 + 3 \alpha_3 + 2 \alpha_4] = P_1(x[\alpha_3 + \alpha_4], x[\alpha_2 + \alpha_3], x[\alpha_2 + 2 \alpha_3 + 2 \alpha_4], x[\alpha_2 + 2 \alpha_3 + \alpha_4]), \]

and, in the type \( G_2 \), we have

\[ x_{2} x[2 \alpha_1 + \alpha_2] = P_3(x[\alpha_1], x[\alpha_1 + \alpha_2]), \]

\[ x_{1,2} x[3 \alpha_1 + 2 \alpha_2] = P_4(x[\alpha_2], x[3 \alpha_1 + \alpha_2]), \]

where \( P_1, P_2, P_3, P_4 \) are polynomials with coefficients in \( \mathbb{Z}_{\geq 0}[P] \). Details are left to the reader.

This completes our proofs of Lemma 6.1 and Theorem 1.10.

7. 2-finite matrices

In accordance with the plan outlined in Section 1.6, our next task is to prove the implication (iii) \( \implies \) (iv) in Theorem 1.10, which will in turn imply Theorem 1.6. As a first step, we restate the claim at hand as a purely combinatorial result (see Theorem 7.1 below) on matrix mutations (1.3).

We shall write \( B' = \mu_k(B) \) to denote that a matrix \( B' \) is obtained from \( B \) by a matrix mutation in direction \( k \). Note that \( \mu_k \) preserves integrality of entries, and is an involution: \( \mu_k(\mu_k(B)) = B \). If two matrices can be obtained from each other by a sequence of matrix mutations followed by a simultaneous permutation of rows and columns, we will say that they are mutation equivalent.

A real square matrix \( B = (b_{ij}) \) is sign-skew-symmetric (cf. (1.2)) if, for any \( i \) and \( j \), either \( b_{ij} = b_{ji} = 0 \), or else \( b_{ij} b_{ji} < 0 \); in particular, \( b_{ii} = 0 \) for all \( i \). Furthermore, we say that \( B \) is 2-finite if it has integer entries, and any matrix
$B' = (b'_{ij})$ mutation equivalent to $B$ is sign-skew-symmetric and satisfies $|b'_{ij}b'_{ji}| \leq 3$ for all $i$ and $j$.

In the language just introduced, the implication (iii) $\Rightarrow$ (iv) in Theorem 1.8 can be formulated as follows.

**Theorem 7.1.** Every 2-finite matrix $B$ is mutation equivalent to a matrix $B_0$ from Theorem 1.5.

The converse of Theorem 7.1 also holds: by Theorem 1.5 (which has already been proved), $B_0$ is 2-finite.

Our proof of Theorem 7.1 occupies the rest of Sections 7–9 below. The main result of Section 7 is the following proposition.

**Proposition 7.2.** Every 2-finite matrix is skew-symmetrizable.

(Recall that a square matrix $B$ is skew-symmetrizable if there exists a diagonal matrix $D$ with positive diagonal entries such that $DB$ is skew-symmetric.)

The rest of this section is devoted to the proof of Proposition 7.2.

The crucial role in the sequel will be played by a combinatorial construction that associates with a sign-skew-symmetric matrix its *diagram*, whose role is parallel to that of the Dynkin diagram for a generalized Cartan matrix.

**Definition 7.3.** The *diagram* of a sign-skew-symmetric matrix $B = (b_{ij})_{i,j \in I}$ is the weighted directed graph $\Gamma(B)$ with the vertex set $I$ such that there is a directed edge from $i$ to $j$ if and only if $b_{ij} > 0$, and this edge is assigned the weight $|b_{ij}b_{ji}|$.

More generally, we will use the term *diagram* to denote a finite directed graph without loops and multiple edges, whose edges are assigned positive real weights. By some abuse of notation, we denote by the same symbol $\Gamma$ the underlying directed graph of a diagram. If two vertices of $\Gamma$ are not joined by an edge, we may also say that they are joined by an edge of weight 0.

The following lemma is an analogue of the well-known symmetrizability criterion [15, Exercise 2.1].

**Lemma 7.4.** A matrix $B = (b_{ij})_{i,j \in I}$ is skew-symmetrizable if and only if, first, it is sign-skew-symmetric and, second, for all $k \geq 3$ and all $i_1, \ldots, i_k$, it satisfies

$$(7.1) \quad b_{i_1i_2}b_{i_2i_3} \cdots b_{i_{k-1}i_k} = (-1)^kB_{i_2i_1}b_{i_3i_2} \cdots b_{i_1i_k}.$$ 

**Proof.** The “only if” part is trivial. Thus, let us assume that $B$ is sign-skew-symmetric and satisfies (7.1). Without loss of generality, we also assume that $B$ is indecomposable, i.e., cannot be represented as a direct (block-diagonal) sum of two proper submatrices. It follows that the graph $\Gamma(B)$ is connected. Let $T$ be one of its spanning trees. There exists a diagonal matrix $D = (d_{ij})$ with positive diagonal entries such that $d_{ii}b_{ij} = -d_{jj}b_{ji}$ for every edge $(i, j)$ in $T$. (Such a matrix can be constructed inductively by setting $d_{ii}$ equal to an arbitrary positive number for some vertex $i$, and moving within the tree $T$ away from this vertex.) Then $DB$ is skew-symmetric, for the following reason: by definition of a spanning tree, any edge $(i, j)$ of $\Gamma(B)$ which is not in $T$ belongs to a cycle in which the rest of the edges belong to $T$; then use (7.1).}

**Lemma 7.5.** Let $B$ be a 2-finite matrix. Then the edges of every triangle in $\Gamma(B)$ are oriented in a cyclic way.
Proof. Suppose on the contrary that \( b_{ij}, b_{ik}, b_{kj} > 0 \) for some distinct \( i, j, k \). Then in the matrix \( B' = \mu_k(B) \), we have \( b'_{ij} = b_{ij} + b_{ik}b_{kj} \geq 2 \) and \( b'_{ji} = b_{ji} - b_{jk}b_{ki} \leq -2 \), violating 2-finiteness. \( \square \)

Lemma 7.6. Let \( B \) be a 2-finite matrix. Then

\[
(7.2) \quad b_{ij}b_{jk}b_{ki} = -b_{ji}b_{kj}b_{ik}
\]

for any distinct \( i, j, k \). Also, in every triangle in \( \Gamma(B) \), the edge weights are either \{1, 1, 1\} or \{2, 2, 1\}.

Proof. In view of Lemma 7.5, we may assume without loss of generality that \( B \) is a 3 \( \times \) 3 matrix

\[
(7.3) \quad \begin{pmatrix}
0 & a_1 & -c_2 \\
-a_2 & 0 & b_1 \\
c_1 & -b_2 & 0
\end{pmatrix},
\]

where \( a_1, b_1, c_1, a_2, b_2, c_2 \) are positive integers. (If one of these entries is 0, then \( 7.3 \) is automatic.) Again without loss of generality, we may assume that the entry of maximal absolute value in \( B \) is \(-c_2\). We claim that, under this assumption,

\[
(7.4) \quad c_1 = a_2b_2, \quad c_2 = a_1b_1,
\]

implying \( a_1b_1c_1 = a_2b_2c_2 \), and hence proving \( 7.2 \).

Indeed, we have

\[
(7.5) \quad \mu_2(B) = \begin{pmatrix}
0 & a_1 & -c_2 \\
a_2 & 0 & b_1 \\
-a_2b_2 + c_1 & b_2 & 0
\end{pmatrix}.
\]

Applying Lemma 7.5 to \( \mu_2(B) \), we conclude that

\[
a_1b_1 - c_2 \geq 0, \quad a_2b_2 - c_1 \geq 0,
\]

where either both inequalities are strict, or both are equalities. We need to show that the former case is impossible. Indeed, otherwise we would have had \( a_2b_2 > c_1 \geq 1 \), implying \( \max(a_2, b_2) \geq 2 \); also, \( a_1b_1 > c_2 \geq \max(a_1, b_1) \), implying \( a_1 \geq 2 \) and \( b_1 \geq 2 \). But then \( \max(a_1a_2, b_1b_2) \geq 4 \), contradicting the 2-finiteness of \( B \).

It remains to show that the set of edge weights \{\( a_1a_2, b_1b_2, c_1c_2 \)\} is either \{1, 1, 1\}, or \{2, 2, 1\}. The only other option consistent with both \( 7.4 \) and the inequalities

\[
a_1a_2 \leq 3, \quad b_1b_2 \leq 3, \quad c_1c_2 \leq 3
\]

is \( c_1 = 3, \{a_1, b_1\} = \{3, 1\}, \quad c_1 = a_2 = b_2 = 1 \). Say \( a_1 = 3 \) and \( b_1 = 1 \) (the other case is analogous). Then \( B' = \mu_1(B) \) has \( |b_{23}b'_{32}| = 4 \), violating 2-finiteness. \( \square \)

Now everything is ready for the proof of Proposition 7.2. It suffices to check that every 2-finite matrix satisfies the criterion \( 7.1 \). Suppose this is not the case. Among all instances where \( 7.4 \) is violated for some 2-finite matrix \( B \), pick one with the smallest value of \( k \). Then \( b_{i_j,i_m} = 0 \) for any pair of subscripts \( (i_j, i_m) \) not appearing in \( 7.4 \). (Otherwise we could obtain \( 7.4 \) as a corollary of its counterparts for two smaller cycles.) In other words, the diagram \( \Gamma(B) \) restricted to the vertices \( i_1, \ldots, i_k \) must be a cycle. Pick any two consecutive edges on this cycle that form an oriented 2-path (that is, \( b_{i_j,i_{j+1}}b_{i_{j+1},i_{j+2}} > 0 \)). (If there is no such pair, we will need to first apply a mutation at an arbitrary vertex \( i_j \).) By Lemma 7.6, we have \( k \geq 4 \), hence \( b_{i_{j+1},i_{j-1}} = 0 \). Now apply the mutation \( \mu_{i_j} \). In the resulting matrix, condition \( 7.1 \) for the sequence of indices \( i_1, \ldots, i_{j-1}, i_{j+1}, \ldots, i_k \)
will be equivalent to \((7.1)\) in the original matrix; hence it must fail, contradicting our choice of \(k\).

\[
\square
\]

8. Diagram mutations

Let \(B = (b_{ij})_{i,j \in I}\) be a skew-symmetrizable matrix. Notice that the diagram \(\Gamma(B)\) does not determine \(B\): for instance, the matrix \((-B^T)\) has the same diagram as \(B\). However, the following important property holds.

**Proposition 8.1.** For a skew-symmetrizable matrix \(B\), the diagram \(\Gamma' = \Gamma(\mu_k(B))\) is uniquely determined by the diagram \(\Gamma = \Gamma(B)\) and an index \(k \in I\). Specifically, \(\Gamma'\) is obtained from \(\Gamma\) as follows:

- The orientations of all edges incident to \(k\) are reversed, their weights intact.
- For any vertices \(i\) and \(j\) which are connected in \(\Gamma\) via a two-edge oriented path going through \(k\) (refer to Figure 5 for the rest of notation), the direction of the edge \((i, j)\) in \(\Gamma'\) and its weight \(c'\) are uniquely determined by the rule

\[
\pm \sqrt{c} \pm \sqrt{c'} = \sqrt{ab},
\]

where the sign before \(\sqrt{c}\) (resp., before \(\sqrt{c'}\)) is “+” if \(i, j, k\) form an oriented cycle in \(\Gamma\) (resp., in \(\Gamma'\)), and is “−” otherwise. Here either \(c\) or \(c'\) can be equal to 0.

- The rest of the edges and their weights in \(\Gamma\) remain unchanged.

\[
\begin{array}{ccc}
\includegraphics[width=0.2\textwidth]{figure5.png}
\end{array}
\]

**Figure 5. Diagram mutation**

**Remark 8.2.** If \(B\) has integer entries, then all edge weights in \(\Gamma\) are positive integers. The rule (8.1) ensures that the same is true for \(\Gamma'\): indeed, the fact that \(c' = (\sqrt{ab} \pm \sqrt{c})^2 = ab + c \mp 2\sqrt{abc}\) is an integer (that is, \(abc\) is a perfect square) is an easy consequence of the skew-symmetrizability of \(B\) (more specifically, of the identity \((7.1)\) with \(k = 3\)).

Our proof of Proposition 8.1 is based on the following construction.

**Lemma 8.3.** Let \(B\) be a skew-symmetrizable matrix. Then there exists a diagonal matrix \(H\) with positive diagonal entries such that \(HBH^{-1}\) is skew-symmetric. Furthermore, the matrix \(S(B) = (s_{ij}) = HBH^{-1}\) is uniquely determined by \(B\).

Specifically, the matrix entries of \(S(B)\) are given by

\[
s_{ij} = \text{sgn}(b_{ij}) \sqrt{|b_{ij}b_{ji}|},
\]

(8.2)

**Proof.** Let \(D\) be a skew-symmetrizing matrix for \(B\), i.e., a diagonal matrix with positive diagonal entries such that \(DB\) is skew-symmetric. Setting \(H = D^{1/2}\), we see that \(HBH^{-1} = H^{-1}(DB)H^{-1}\) is skew-symmetric. To prove (8.2), note that

\[
\begin{align*}
\text{sgn}(s_{ij}) &= \text{sgn}(h_i b_{ij} h_j^{-1}) = \text{sgn}(b_{ij}), \\
s_{ij}^2 &= |s_{ij} s_{ji}| = |(h_i b_{ij} h_j^{-1}) \cdot (h_j b_{ji} h_i^{-1})| = |b_{ij} b_{ji}|,
\end{align*}
\]

where the \(h_i\) are the diagonal entries of \(H\). 

\[
\square
\]
Lemma 8.4. Let $B$ be a skew-symmetrizable matrix. Then, for any $k \in \mathbb{I}$, we have $S(\mu_k(B)) = \mu_k(S(B))$.

Proof. Follows from Lemma 8.3 together with the directly checked fact that the mutation rules are invariant under conjugation by a diagonal matrix with positive entries. \hfill \Box

Proof of Proposition 8.1 Formula (8.2) shows that the diagram $\Gamma(B)$ and the skew-symmetric matrix $S(B)$ encode the same information about a skew-symmetrizable matrix $B$: having an edge in $\Gamma(B)$ directed from $i$ to $j$ and supplied with weight $c$ is the same as saying that $s_{ij} = \sqrt{c}$ and $s_{ji} = -\sqrt{c}$. Lemma 8.4 asserts that, as $B$ undergoes a mutation $\mu_k$, so does the matrix $S(B)$. Translating this statement into the language of diagrams, we obtain Proposition 8.1. \hfill \Box

In the situation of Proposition 8.1 we write $\Gamma' = \mu_k(\Gamma)$, and call the transformation $\mu_k$ a diagram mutation in the direction $k$. Two diagrams $\Gamma$ and $\Gamma'$ related by a sequence of diagram mutations are called mutation equivalent, and we write $\Gamma \sim \Gamma'$. A diagram $\Gamma$ is called 2-finite if any diagram $\Gamma' \sim \Gamma$ has all edge weights equal to 1, 2, or 3. Thus a matrix $B$ is 2-finite if and only if its diagram $\Gamma(B)$ is 2-finite. (Here we rely on Proposition 7.2.) Note that a diagram is 2-finite if and only if so are all its connected components.

In the case of 2-finite diagrams, Lemmas 7.5 and 7.6 ensure that every triangle is oriented in a cyclic way, and has edge weights $(1, 1, 1)$ or $(2, 2, 1)$. As a result, the rules of diagram mutations (as given in Proposition 8.1) simplify as follows.

Lemma 8.5. Let $\Gamma$ be a 2-finite diagram, and $k$ a vertex of $\Gamma$. Then the diagram $\mu_k(\Gamma)$ is obtained from $\Gamma$ as follows:

- The orientations of all edges incident to $k$ are reversed, their weights intact.
- For any vertices $i$ and $j$ which are connected in $\Gamma$ via a two-edge oriented path going through $k$, the diagram mutation $\mu_k$ affects the edge connecting $i$ and $j$ in the way shown in Figure 6, where the weights $c$ and $c'$ are related by

\[
\sqrt{c} + \sqrt{c'} = \sqrt{ab};
\]

where either $c$ or $c'$ can be equal to 0.
- The rest of the edges and their weights in $\Gamma$ remain unchanged.

Figure 6. Mutation of 2-finite diagrams

Taking into account Propositions 7.2 and 8.1 we see that Theorem 7.1 becomes a consequence of the following classification of 2-finite diagrams.

Theorem 8.6. Any connected 2-finite diagram is mutation equivalent to an orientation of a Dynkin diagram. (Cf. Figure 4 where all unspecified weights are equal to 1.) Furthermore, all orientations of the same Dynkin diagram are mutation equivalent to each other.
As already noted following Theorem 7.1, the converse is true as well: any diagram mutation equivalent to an orientation of a Dynkin diagram is 2-finite.

![Dynkin diagrams](image_url)

**Figure 7.** Dynkin diagrams

9. **Proof of Theorem 8.6**

Throughout this section, all diagrams are presumed connected, and all edge weights are positive integers. With some abuse of notation, we use the same symbol \( \Gamma \) to denote a diagram and the set of its vertices. A diagram that is not 2-finite will be called 2-infinite.

**Definition 9.1.** A subdiagram of a diagram \( \Gamma \) is a diagram \( \Gamma' \) obtained from \( \Gamma \) by taking an induced directed subgraph on a subset of vertices and keeping all its edge weights the same as in \( \Gamma \). We will denote this by \( \Gamma \supset \Gamma' \).

We will repeatedly use the following obvious fact: any subdiagram of a 2-finite diagram is 2-finite. Equivalently, any diagram that has a 2-infinite subdiagram is 2-infinite.

The proof of Theorem 8.6 will proceed in several steps.
9.1. **Shape-preserving diagram mutations.** Let $k$ be a sink (resp., source) of a diagram $\Gamma$, that is, a vertex such that all edges incident to $k$ are directed towards $k$ (resp., away from $k$). Then a diagram mutation at $k$ reverses the orientations of all edges incident to $k$, leaving the rest of the graph and all the edge weights unchanged. We shall refer to such mutations as *shape-preserving*.

**Proposition 9.2.** Let $T$ be a subdiagram of a diagram $\Gamma$ such that:

(i) $T$ is a tree.

(ii) $T$ is attached to the rest of $\Gamma$ by a single vertex $v \in T$, i.e., no vertex in $T - \{v\}$ is joined by an edge with a vertex in $\Gamma - T$.

Then any diagram obtained from $\Gamma$ by arbitrarily re-orienting the edges of $T$ (while keeping the rest of $\Gamma$ intact) is mutation equivalent to $\Gamma$.

In particular, any two orientations of a tree diagram are mutation equivalent.

(A tree diagram is a diagram whose underlying graph is an orientation of a tree.)

**Proof.** Using induction on the size of $T$, we will show that one can arbitrarily re-orient the edges of $T$ by applying a sequence of shape-preserving mutations at the vertices of $T - \{v\}$. If $T$ consists of a single vertex $v$, there is nothing to prove. Otherwise, pick a leaf $l \in T$ different from $v$, and apply the inductive assumption to the diagram $\Gamma' = \Gamma - \{l\}$ and its subdiagram $T' = T - \{l\}$. So we are able to arbitrarily re-orient the edges of $T'$ by a sequence of shape-preserving mutations of $\Gamma'$ at the vertices of $T' - \{v\}$. To do the same for $T$, we lift this sequence from $\Gamma'$ to $\Gamma$ as follows: each time right before we need to perform a mutation at the unique vertex $k \in T'$ adjacent to $l$, we first mutate at $l$ if necessary to make $k$ a source or sink in $T$, rather than just in $T'$. This way, we can achieve an arbitrary re-orientation of the edges of $T'$ by a sequence of shape-preserving mutations of $\Gamma$ at the vertices of $T' - \{v\}$. The remaining edge $(k, l)$ can then be given an arbitrary orientation by a (shape-preserving) mutation at $l$. □

As a practical consequence of Proposition 9.2, in drawing a diagram $\Gamma$, we do not have to specify orientations of edges in any subdiagram $T \subset \Gamma$ satisfying the conditions of the proposition. Figure 7 provides an example of this (see also Figure 10 below).

Proposition 9.2 also justifies notation of the form $\Gamma \sim A_m$, $\Gamma \supset A_m$, etc.

9.2. **Taking care of the trees.**

**Proposition 9.3.** Any 2-finite tree diagram is an orientation of a Dynkin diagram.

**Proof.** A diagram $\Gamma$ is called an *extended Dynkin tree diagram* if

- $\Gamma$ is a tree diagram with edge weights $\leq 3$;
- $\Gamma$ is not on the Dynkin diagram list;
- every proper subdiagram of $\Gamma$ is a disjoint union of Dynkin diagrams.

(In this definition, we ignore the orientations of the edges.) To prove the proposition, it is enough to show that any extended Dynkin tree diagram is 2-infinite. Direct inspection shows that Figure 5 provides a complete list of such diagrams. Here each tree $X_n^{(1)}$ has $n + 1$ vertices. As before, all unspecified edge weights are equal to 1; in the diagram $G_2^{(1)}$, we have $a \in \{1, 2, 3\}$. We note that all these diagrams are associated with untwisted affine Lie algebras and can be found in the tables in [6] or in [15, Chapter 4, Table Aff 1]. The only diagram from those tables
that is missing in Figure 8 is $A_n^{(1)}$, which is an $(n+1)$-cycle; it will be treated in Section 9.3.

In showing that an extended Dynkin tree diagram is 2-infinite, we can choose its orientation arbitrarily, by Proposition 9.2. Let us start with the three infinite series $B_n^{(1)}$, $C_n^{(1)}$, and $D_n^{(1)}$, and in each case let us orient all the edges left to right. Let us denote the diagram in question by $X_n^{(1)}$; thus, if $X = D$ (resp., $B, C$) then the minimal value of $n$ is equal to 4 (resp., 3, 2). If $n$ is greater than this minimal value, then performing a mutation at the second vertex from the left, and subsequently removing this vertex (together with all incident edges) leaves us with a subdiagram of type $X_{n-1}^{(1)}$. Using induction on $n$, it remains to check the basic cases $D_4^{(1)}$, $B_3^{(1)}$ and $C_2^{(1)}$. For $C_2^{(1)}$, the mutation at the middle vertex produces a triangle with edge weights $(2, 2, 2)$, which is 2-infinite by Lemma 7.6. For $B_3^{(1)}$, mutating at the branching vertex and then removing it leaves us with the subdiagram $C_2^{(1)}$ which

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{extendedDynkinDiagrams.png}
\caption{Extended Dynkin tree diagrams}
\end{figure}
was just shown to be 2-infinite. Finally, for \( D_4^{(1)} \), let the branching point be labeled by 1, and let it be joined with vertices 2 and 4 by incoming edges, and with 3 and 5 by outgoing edges. Then the composition of mutations \( \mu_3 \circ \mu_2 \circ \mu_1 \) makes the subdiagram on the vertices 2, 4 and 5 a 2-infinite triangle.

To see that \( G_2^{(1)} \) is 2-infinite, orient the two edges left to right and mutate at the middle vertex to obtain a 2-infinite triangle. To see that \( F_4^{(1)} \) is 2-infinite, again orient all the edges left to right, label the vertices also left to right, and apply \( \mu_1 \circ \mu_2 \circ \mu_3 \circ \mu_4 \) to obtain a subdiagram \( C_2^{(1)} \) on the vertices 1, 3 and 5.

The remaining three cases \( E_6^{(1)} \), \( E_7^{(1)} \) and \( E_8^{(1)} \) can be treated in a similar manner but we prefer another approach. To describe it, we will need to introduce some notation.

**Definition 9.4.** For \( p, q, r \in \mathbb{Z}_{\geq 0} \), we denote by \( T_{p,q,r} \) the tree diagram (with all edge weights equal to 1) on \( p + q + r + 1 \) vertices obtained by connecting an endpoint of each of the three chains \( A_p \), \( A_q \) and \( A_r \) to a single extra vertex (see Figure 9).

**Figure 9.** The tree diagram \( T_{5,4,2} \).

**Definition 9.5.** For \( p, q, r \in \mathbb{Z}_{>0} \) and \( s \in \mathbb{Z}_{\geq 0} \), let \( S_{p,q,r}^s \) denote the diagram (with all edge weights equal to 1) on \( p + q + r + s \) vertices obtained by attaching three branches \( A_{p-1} \), \( A_{q-1} \), and \( A_{r-1} \) to three consecutive vertices on a cyclically oriented \((s + 3)\)-cycle (see Figure 10).

**Figure 10.** The diagram \( S_{4,3,2}^5 \).

**Lemma 9.6.** The diagram \( S_{p,q,r}^s \) is mutation equivalent to \( T_{p+r-1,q,s} \).

**Proof.** Let us consider the subdiagram of \( S_{p,q,r}^s \) obtained by removing the middle branch \( A_q \). This subdiagram is a copy of \( A_{p+s+r-1} \). We label its vertices consecutively by \( 1, \ldots, p + s + r \), starting with the endpoint of \( A_r \); and we orient the edges
of $A_p$ and $A_r$ so that all the edges of $A_{p+s+r}$ point at the same direction. Now a direct check shows that $\mu_1 \circ \mu_2 \circ \cdots \circ \mu_{s+r}$ transforms $S_{p,q,r}^s$ into $T_{p+r-1,q,s}$. □

The proof of Proposition 9.3 can now be completed as follows:

\[
E_6^{(1)} = T_{2,2,2} \sim S_{2,2,1}^2 \supset D_6^{(1)}; \\
E_7^{(1)} = T_{3,1,3} \sim S_{3,1,1}^3 \supset E_6^{(1)}; \\
E_8^{(1)} = T_{2,1,5} \sim S_{2,1,1}^5 \supset E_7^{(1)}. \]

□

9.3. Taking care of the cycles.

**Proposition 9.7.** Let $\Gamma$ be a 2-finite diagram whose underlying graph is an $n$-cycle for some $n \geq 3$ (with some orientation of edges). Then $\Gamma$ must be one of the diagrams shown in Figure 11. More precisely, one of the following holds:

(a) $\Gamma$ is an oriented cycle with all weights equal to 1.

In this case, $\Gamma \sim D_n$ (with the understanding that $D_3 = A_3$).

(b) $\Gamma$ is an oriented triangle with edge weights 2, 2, 1 shown in Figure 11(b).

In this case, $\Gamma \sim B_3$.

(c) $\Gamma$ is an oriented 4-cycle with edge weights 2, 1, 2, 1 shown in Figure 11(c).

In this case, $\Gamma \sim F_4$.

In particular, the edges in $\Gamma$ must be cyclically oriented.

**Proof.** The case $n = 3$ of Proposition 9.7 follows from Lemmas 7.5 and 7.6 so for the rest of the proof we assume that $n \geq 4$.

We begin by proving the last claim of Proposition 9.7 by induction on $n$. Invoking if necessary a shape-preserving mutation, we may assume that there is a vertex $v \in \Gamma$ that has one incoming and one outgoing edge. Let $\Gamma' = \mu_v(\Gamma)$. Then the subdiagram $\Gamma'' = \Gamma' - \{v\}$ is an $(n-1)$-cycle, which must be cyclically oriented by the induction assumption. Backtracking to $\Gamma$, we obtain the desired claim.

Furthermore, observe that the product of edge weights of $\Gamma''$ is the same as in $\Gamma$. Again using induction together with Lemma 7.6 we conclude that this product is either 1 or 4. In the former case, $\Gamma$ is an oriented $n$-cycle, and we apply Lemma 9.6 to obtain $\Gamma = S_{1,1,1}^{n-3} \sim T_{1,1,n-3} = D_n$, as needed. In the latter case, $\Gamma$ has two edge weights equal to 2, and the rest of them are equal to 1. Then either $\Gamma$ is one of the two diagrams (b) and (c) in Figure 11 or else it contains a 2-infinite subdiagram $C_m^{(1)}$ for some $m \geq 2$. It remains to show that the diagrams in Figure 11(b)-(c) are mutation equivalent to $B_3$ and $F_4$, respectively. This is straightforward. □

![Figure 11. 2-finite cycles](image)
9.4. Completing the proof of Theorem 8.6. The second claim in Theorem 8.6 follows from Proposition 9.2 so we only need to show that a connected 2-finite diagram $\Gamma$ is mutation equivalent to some Dynkin diagram. We proceed by induction on $n$, the number of vertices in $\Gamma$. If $n \leq 3$, then $\Gamma$ is either a tree or a cycle, and the theorem follows by Propositions 9.3 and 9.7. So let us assume that the statement is already known for some $n \geq 3$; we need to show that it holds for a diagram $\Gamma'$ on $n+1$ vertices. Pick a vertex $v \in \Gamma$ such that the subdiagram $\Gamma' = \Gamma - \{v\}$ is connected. Since $\Gamma'$ is 2-finite, it is mutation equivalent to some Dynkin diagram $X_n$. Furthermore, we may assume that $\Gamma'$ is (isomorphic to) our favorite representative of the mutation equivalence class of $X_n$. For each $X_n$, we will choose a representative that is most convenient for the purposes of this proof.

**Case 1.** $\Gamma'$ is a Dynkin diagram with no branching point, i.e., is of one of the types $A_n$, $B_n$, $F_4$, or $G_2$. Let us orient the edges of $\Gamma'$ so that they all point in the same direction. If $v$ is adjacent to exactly one vertex of $\Gamma'$, then $\Gamma$ is a tree, and we are done by Proposition 9.3. If $v$ is adjacent to more than 2 vertices of $\Gamma'$, then $\Gamma$ has a cycle subdiagram whose edges are not cyclically oriented, contradicting Proposition 9.7. Therefore we may assume that $v$ is adjacent to precisely two vertices $v_1$ and $v_2$ of $\Gamma'$. Thus, $\Gamma$ has precisely one cycle $C$, which furthermore must be of one of the types (a)–(c) described in Proposition 9.7 and Figure 11.

**Subcase 1.1.** $C$ is an oriented cycle with unit edge weights. If $\Gamma$ has an edge of weight $\geq 2$, then it contains a subdiagram of type $B_{m}^{(1)}$ or $G_{2}^{(1)}$, unless $C$ is a 3-cycle, in which case $\mu_v(\Gamma) \sim B_{n+1}$. On the other hand, if all edges in $\Gamma$ are of weight 1, then it is one of the diagrams $S_{p,q,r}$ in Lemma 9.6 (with $q = 0$). Hence $\Gamma$ is mutation equivalent to a tree, and we are done.

**Subcase 1.2.** $C$ is as in Figure 11(b). If one of the edges $(v, v_1)$ and $(v, v_2)$ has weight 1, then by Lemma 9.6, $\Gamma$ is mutation equivalent to a tree, and we are done again. So assume that both $(v, v_1)$ and $(v, v_2)$ have weight 2. If at least one edge outside $C$ has weight $\geq 2$, then $\Gamma \supset B_{m}^{(1)}$ or $\Gamma \supset G_{2}^{(1)}$. It remains to consider the case shown in Figure 12 (as before, unspecified edge weights are equal to 1). Direct check shows that $\mu_1 \circ \cdots \circ \mu_2 \circ \mu_1 \circ \mu_2 \circ \mu_v(\Gamma) = B_{n+1}$, and we are done.

**Subcase 1.3.** $C$ is as in Figure 11(c). It suffices to show that any diagram obtained from $C$ by adjoining a single vertex adjacent to one of its vertices is 2-infinite. If this extra edge has weight 1 (resp., 2, 3), then the resulting 5-vertex diagram has a 2-infinite subdiagram of type $B_{m}^{(1)}$ (resp., $C_{2}^{(1)}$, $G_{2}^{(1)}$), proving the claim.

**Case 2.** $\Gamma' \sim D_n$ ($n \geq 4$). By Proposition 9.7(a), we may assume that $\Gamma'$ is an oriented $n$-cycle with unit edge weights.

**Subcase 2.1.** $v$ is adjacent to a single vertex $v_1 \in \Gamma'$. If the edge $(v, v_1)$ has weight $\geq 2$, then $\Gamma$ has a subdiagram $B_{m}^{(1)}$ or $G_{2}^{(1)}$. If the edge $(v, v_1)$ has weight 1, then by Lemma 9.6 $\Gamma$ is mutation equivalent to a tree, and we are done again.
Subcase 2.2. $v$ is adjacent to exactly two vertices $v_1$ and $v_2$ of $\Gamma'$, which are adjacent to each other. Then the triangle $(v, v_1, v_2)$ is either an oriented $3$-cycle with unit edge weights or the diagram in Figure 12b. In the former case, $\mu_v(\Gamma)$ is an oriented $(n+1)$-cycle, so $\Gamma \sim D_{n+1}$. In the latter case, $\mu_v$ reverses the orientation of the edge $(v_1, v_2)$, transforming $\Gamma'$ into an improperly oriented (hence $2$-infinite) cycle (cf. Proposition 9.7).

Subcase 2.3. $v$ is adjacent to two non-adjacent vertices of $\Gamma'$ (and maybe to some other vertices). In this case, $\Gamma$ contains a subdiagram which is a non-cyclically-oriented cycle, contradicting Proposition 9.7.

Case 3. $\Gamma' \sim E_n = T_{1,2,n-4}$, for $n \in \{6, 7, 8\}$. By Lemma 9.6, we may assume that $\Gamma' = S_{1,2}^{n-4}$. In other words, $\Gamma'$ is a cyclically oriented $(n-1)$-cycle $C$ with unit edge weights, and an extra edge of weight $1$ connecting a vertex in $C$ to a vertex $v \notin C$.

Subcase 3.1. $v$ is adjacent to $v_1$, and to no other vertices in $\Gamma'$. If the edge $(v, v_1)$ has weight $\geq 2$, then $\Gamma$ has a $2$-infinite subdiagram $B_3^{(1)}$ or $G_2^{(1)}$. If $(v, v_1)$ has weight $1$, then by Lemma 9.6, $\Gamma$ is mutation equivalent to a tree.

Subcase 3.2. $v$ is adjacent to a vertex $v_2 \in C$, and to no other vertices in $\Gamma'$. Then $\Gamma$ has a subdiagram of type $D_m^{(1)}$ or $B_3^{(1)}$ or $G_2^{(1)}$.

Subcase 3.3. $v$ is adjacent to at least two vertices in $C$. By the analysis in Subcases 2.2–2.3 (with $\Gamma'$ replaced by $C$), we must have $\Gamma - \{v_1\} \sim D_n$, and the problem reduces to Case 2 already treated above (the role of $v$ now played by $v_1$).

Subcase 3.4. $v$ is adjacent to $v_1$ and a single vertex $v_2 \in C$. Let $v_0$ be the only vertex on $C$ adjacent to $v_1$ in $\Gamma'$. If $v_2$ is neither $v_0$ nor a vertex adjacent to $v_0$, then the three cycles in $\Gamma$ cannot be simultaneously oriented. If $v_2 = v_0$, then $\mu_{v_1}$ removes the edge $(v, v_2)$, transforming $\Gamma$ into a diagram mutation equivalent to a tree by Lemma 9.6. If $v_2$ is adjacent to $v_0$, then $\Gamma - \{v_0\}$ has no branching point, and the problem reduces to Case 1 already treated above, with the role of $v$ now played by $v_0$.

This concludes the proof of Theorem 8.6. As a consequence, we obtain Theorems 1.1 and 1.6. □

10. Proof of Theorem 1.7

Let $B$ and $B'$ be sign-skew-symmetric matrices such that both $A = A(B)$ and $A' = A(B')$ are Cartan matrices of finite type. We already proved that both $B$ and $B'$ are $2$-finite. We need to show that $B$ and $B'$ are mutation equivalent if and only if $A$ and $A'$ are of the same type. Without loss of generality, we may assume that $A$ and $A'$ are indecomposable, i.e., the corresponding root systems $\Phi$ and $\Phi'$ are irreducible.

We first prove the “only if” part. If $B$ and $B'$ are mutation equivalent, then the simplicial complexes $\Delta(\Phi)$ and $\Delta(\Phi')$ are isomorphic to each other, by Theorem 11.13. In particular, $\Phi$ and $\Phi'$ have the same rank and the same cardinality. A direct check using the tables in [2] shows that the only different Cartan-Killing types with this property are $B_n$ and $C_n$ for all $n \geq 3$, and also $E_6$, which has the same data as $B_6$ and $C_6$. To distinguish between these types, note that mutation-equivalent skew-symmetrizable matrices share the same skew-symmetrizing matrix $D$. Furthermore, $D$ is skew-symmetrizing for $B$ if and only if it is symmetrizing for $A$; thus, the diagonal entries of $D$ are given by $d_i = (\alpha_i, \alpha_i)$, where $(\alpha, \beta)$ is a $W$-invariant
scalar product on the root lattice. Since the root system of type $B_n$ (resp., $C_n$) has one short simple root and $n - 1$ long ones (resp., one long and $n - 1$ short), the corresponding matrices $B$ and $B'$ cannot be mutation equivalent. The same is true for $E_6$ and $B_6$ (or $C_6$) since all simple roots for $E_6$ are of the same length.

To prove the “if” part, suppose that $A$ and $A'$ are of the same Cartan-Killing type. By Proposition 9.2, we may assume without loss of generality that $B$ and $B'$ have the same diagram. By Lemma 8.3, we have $S(B) = S(B')$. Since $B$ and $B'$ share a skew-symmetrizing matrix $D$, the proof of Lemma 8.3 shows that $S(B) = HBH^{-1}$ and $S(B') = HB'H^{-1}$ for $H = D^{1/2}$. Hence $B = B'$, and we are done. □

11. ON CLUSTER ALGEBRAS OF GEOMETRIC TYPE

In this section we present two general results on cluster algebras of geometric type in the sense of [9, Definition 5.7]. These algebras are not assumed to be of finite type, so all the necessary background is contained in Section 1.2.

Recall that a cluster algebra is of geometric type if it satisfies the following two conditions:

(11.1) The coefficient semifield $P$ is of the form $\text{Trop}(p_j : j \in J)$. That is, the multiplicative group of $P$ is a free abelian group with a finite set of generators $p_j (j \in J)$, and the auxiliary addition $\oplus$ is given by (1.1).

(11.2) Every element $p \in P$, i.e., every coefficient in one of the exchange relations (1.4), is a monomial in the $p_j$ with all exponents nonnegative.

We note a little discrepancy between our choice of the ground ring $\mathbb{Z}[P]$ in Definition 1.2 and the choice described in [9, Section 5], where the ground ring was taken to be the polynomial ring $\mathbb{Z}[p_j : j \in J]$. The following additional assumption guarantees that these two choices coincide:

(11.3) Every generator $p_j$ of $P$ belongs to $P$.

11.1. Geometric realization criterion. Our first result gives sufficient conditions under which a cluster algebra of geometric type can be realized as a Z-form of the coordinate ring $\mathbb{C}[X]$ of some algebraic variety $X$.

We make the following assumptions on $X$:

(11.4) $X$ is a rational quasi-affine irreducible algebraic variety over $\mathbb{C}$.

Irreducibility implies that the ring of regular functions $\mathbb{C}[X]$ is a domain, so its fraction field is well defined. Quasi-affine means Zariski open in some affine variety; this condition is imposed to ensure that the fraction field of $\mathbb{C}[X]$ coincides with the usual field $\mathbb{C}(X)$ of rational functions on $X$. Rationality means that $X$ is birationally isomorphic to an affine space, i.e., $\mathbb{C}(X)$ is isomorphic to the field of rational functions over $\mathbb{C}$ in $\dim(X)$ independent variables.

Let $\mathcal{A}$ be a cluster algebra of rank $n$ whose coefficient system satisfies conditions (11.1) – (11.3), and let $\mathcal{X}$ be the set of cluster variables in $\mathcal{A}$. Suppose the variety $X$ satisfies

(11.5) $\dim(X) = n + |J|$

also suppose we are given a family of functions

$$\{\varphi_y : y \in \mathcal{X}\} \cup \{\varphi_j : j \in J\}$$
in $\mathbb{C}[X]$ satisfying the following conditions:

1. (11.6) the functions $\varphi_y$ and $\varphi_j$ generate $\mathbb{C}[X]$;
2. (11.7) every exchange relation becomes an identity in $\mathbb{C}[X]$ if we replace each cluster variable $y$ by $\varphi_y$, and each coefficient $p_\pm = \prod_{j \in J} p_{j}^{a_j}$ by $\prod_{j \in J} \varphi_j^{a_j}$.

**Proposition 11.1.** Under conditions (11.1)–(11.7), the correspondence

$$y \mapsto \varphi_y \quad (y \in \mathcal{X}), \quad p_j \mapsto \varphi_j \quad (j \in J)$$

extends uniquely to an algebra isomorphism between the cluster algebra $\mathcal{A}$ and the $\mathbb{Z}$-form of $\mathbb{C}[X]$ generated by all $\varphi_y$ and $\varphi_j$.

**Proof.** Pick an arbitrary cluster $x$ of $\mathcal{A}$, and let $\tilde{x} = x \cup \{p_j : j \in J\}$. Since $x$ is a transcendence basis of the ambient field $\mathcal{F}$ over $\mathbb{Z}P$, the set $\tilde{x}$ is a transcendence basis of $\mathcal{F}$ over $\mathbb{Q}$. Furthermore, every cluster variable is uniquely expressed as a rational function in $\tilde{x}$ by iterating the exchange relations away from a seed containing $x$ in the exchange graph of $\mathcal{A}$. In view of (11.7), we can apply the same procedure to express all functions $\varphi_y$ and $\varphi_j$ inside the field $\mathbb{C}(\mathcal{X})$ as rational functions in the set

$$\varphi(\tilde{x}) = \{\varphi_x : x \in x\} \cup \{\varphi_j : j \in J\}.$$ 

Furthermore, we have $|\varphi(\tilde{x})| = \dim(X)$ by (11.5). Since $X$ is rational, we conclude from (11.6) that $\varphi(\tilde{x})$ is a transcendence basis of the field of rational functions $\mathbb{C}(X)$, and that the correspondence (11.8) extends to an embedding of fields $\mathcal{F} \to \mathbb{C}(X)$, and hence to an embedding of algebras $\mathcal{A} \to \mathbb{C}[X]$. This proves Proposition 11.1. □

11.2. Sharpening the Laurent phenomenon. As mentioned in Section 1.1, the *Laurent phenomenon*, established in [9] for arbitrary cluster algebras, says that every cluster variable can be written as a Laurent polynomial in the variables of an arbitrary fixed cluster, with coefficients in $\mathbb{Z}P$. For the cluster algebras of geometric type, this result can be sharpened as follows.

**Proposition 11.2.** In any cluster algebra with the coefficient system satisfying conditions (11.1)–(11.3), every cluster variable is expressed in terms of an arbitrary cluster $x$ as a Laurent polynomial with coefficients in $\mathbb{Z}[P]$.

**Proof.** Fix some generator $p = p_{j_0}$ of the coefficient semifield $P = \text{Trop}(p_j : j \in J)$. We will think of any cluster variable $z$ as a Laurent polynomial $z(p)$ whose coefficients are integral Laurent polynomials in the set $x \cup \{p_j : j \in J, j \neq j_0\}$. Our goal is to show that $z(p)$ is in fact a polynomial in $p;\text{ Proposition 11.2 will then follow by varying a distinguished index } j_0 \text{ over the index set } J$.

Define the *distance* $d(z, x)$ between $z$ and $x$ as the shortest distance in the exchange graph between a seed containing $z$ and a seed whose cluster is $x$. We will use induction on $d(z, x)$ to show the following strengthening of the desired statement:

- $z(p)$ is a polynomial in $p$ whose constant term $z(0)$ is a subtraction-free rational expression in $x \cup \{p_j : j \in J, j \neq j_0\}$ (in particular, $z(0) \neq 0$).
If \( d(z, x) = 0 \), then \( z \in x \), and there is nothing to prove. If \( d(z, x) > 0 \), then, by the definition of the distance, \( z \) participates in an exchange relation \((1.4)\) such that all the other participating cluster variables are at a smaller distance from \( x \) than \( z \). Applying the inductive assumption to all these cluster variables and using Lemma 5.2 together with the fact that, by the normalization condition, \( p \) appears in at most one of the monomials on the right hand side of \((1.4)\), we obtain our claim for \( z \). \( \square \)

12. Examples of geometric realizations of cluster algebras

In this section, we present some examples of concrete geometric realizations of cluster algebras \( \mathcal{A} = \mathcal{A}(B, p) \) of finite type. In all these examples, the Cartan counterpart of \( B \) is a Cartan matrix of one of the classical types \( A_n, B_n, C_n, D_n \), and the coefficient system of \( \mathcal{A} \) satisfies conditions \((11.1)\)–\((11.3)\).

12.1. Type \( A_1 \). We start by presenting four natural geometric realizations of cluster algebras of type \( A_1 \). Such an algebra \( \mathcal{A} \) has only two one-element clusters \( \{x\} \) and \( \{\mathcal{T}\} \), and a single exchange relation

\[
(12.1) \quad x\mathcal{T} = p^+ + p^-,
\]

where \( p^+ \) and \( p^- \) belong to the coefficient semifield \( \mathbb{P} \). By Definition 1.2, \( \mathcal{A} \) is a subalgebra of the ambient field \( \mathcal{F} \) generated by \( x, \mathcal{T}, p^+ \), and \( p^- \).

Example 12.1. Let \( \mathcal{A} \) have the coefficient semifield \( \mathbb{P} = \text{Trop}(p) \) (the free abelian group with one generator), and let the coefficients in \((12.1)\) be given by \( p^+ = p \) and \( p^- = 1 \). Let \( G = SL_2(\mathbb{C}) \) be the group of complex matrices

\[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}
\]

with \( ad - bc = 1 \). The correspondence

\[
x \mapsto a, \quad \mathcal{T} \mapsto d, \quad p \mapsto bc
\]

identifies \( \mathcal{A} \) with the subring of the coordinate ring \( \mathbb{C}[G] \) generated by \( a, d \), and \( bc \). It is easy to see that this ring is a \( \mathbb{Z} \)-form of the ring of invariants \( \mathbb{C}[G]^H \), where \( H \) is the maximal torus of diagonal matrices in \( G \) acting on \( G \) by conjugation.

The next three examples give three different realizations of the same cluster algebra \( \mathcal{A} \) for which the coefficients in \((12.1)\) are the generators of \( \mathbb{P} = \text{Trop}(p^+, p^-) \).

Example 12.2. Let \( N \) be the group of complex matrices of the form

\[
\begin{bmatrix}
1 & a & c \\
0 & 1 & b \\
0 & 0 & 1
\end{bmatrix}
\]

The correspondence

\[
x \mapsto a, \quad \mathcal{T} \mapsto b, \quad p^+ \mapsto c, \quad p^- \mapsto ab - c
\]

identifies \( \mathcal{A} \) with a \( \mathbb{Z} \)-form \( \mathbb{Z}[N] = \mathbb{Z}[a, b, c] \) of the ring \( \mathbb{C}[N] \).
Example 12.3. Let $G = SL_3(\mathbb{C})$, and let $N \subset G$ be the same as in Example [12.2]. Let $X = G/N$ be the base affine space of $G$ taken in the standard embedding into $\mathbb{C}^3 \times \wedge^2 \mathbb{C}^3$. Let $(\Delta_1, \Delta_2, \Delta_3)$ and $(\Delta_{12}, \Delta_{13}, \Delta_{23})$ (here $\Delta_{ij} = \Delta_i \wedge \Delta_j$) be the standard (Plücker) coordinates in $\mathbb{C}^3$ and $\wedge^2 \mathbb{C}^3$, respectively. In these coordinates, the coordinate ring of $X$ is given by

$$\mathbb{C}[X] = \mathbb{C}[\Delta_1, \Delta_2, \Delta_3, \Delta_{12}, \Delta_{13}, \Delta_{23}]/(\Delta_1 \Delta_{23} - \Delta_2 \Delta_{13} + \Delta_3 \Delta_{12}).$$

The correspondence

$$x \mapsto \Delta_2, \quad \pi \mapsto \Delta_{13}, \quad p^+ \mapsto \Delta_{12} \Delta_{23}, \quad p^- \mapsto \Delta_3 \Delta_{12}$$

identifies $\mathcal{A}$ with the subring of $\mathbb{C}[X]$ generated by $\Delta_2, \Delta_{13}, \Delta_1 \Delta_{23}$, and $\Delta_3 \Delta_{12}$. It is easy to see that this ring is a $\mathbb{Z}$-form of the ring of invariants $\mathbb{C}[X]^T$, where $T \subset G$ is the torus of all diagonal matrices of the form

$$\begin{bmatrix}
t & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & t^{-1}
\end{bmatrix},$$

acting on $X$ by left translations.

Example 12.4. Let $X \subset \wedge^2 \mathbb{C}^4$ be the affine cone over the Grassmannian $\text{Gr}_{2,4}$ taken in its Plücker embedding. In the standard coordinates $(\Delta_{ij} : 1 \leq i < j \leq 4)$ on $\wedge^2 \mathbb{C}^4$, the coordinate ring of $X$ is given by

$$\mathbb{C}[X] = \mathbb{C}[(\Delta_{ij})]/(\Delta_{12} \Delta_{44} - \Delta_{13} \Delta_{24} + \Delta_{14} \Delta_{23}).$$

The correspondence

$$x \mapsto \Delta_{13}, \quad \pi \mapsto \Delta_{24}, \quad p^+ \mapsto \Delta_{12} \Delta_{34}, \quad p^- \mapsto \Delta_{14} \Delta_{23}$$

identifies $\mathcal{A}$ with the subring of $\mathbb{C}[X]$ generated by $\Delta_{13}, \Delta_{24}, \Delta_{12} \Delta_{34}$, and $\Delta_{14} \Delta_{23}$. This ring is a $\mathbb{Z}$-form of the ring of invariants $\mathbb{C}[X]^T$, where $T \subset SL_4$ is the torus of all diagonal matrices of the form

$$\begin{bmatrix}
t_1 & 0 & 0 & 0 \\
0 & t_2 & 0 & 0 \\
0 & 0 & t_1^{-1} & 0 \\
0 & 0 & 0 & t_2^{-1}
\end{bmatrix},$$

naturally acting on $X$.

12.2. Type $A_n$ ($n \geq 2$). Here we present a geometric realization of a cluster algebra of type $A_n$ for all $n \geq 2$, for a special choice of a coefficient system, to be specified below.

First, we reproduce the concrete description of the cluster complex of type $A_n$ given in [11] Section 3.5. We identify $\Phi_{\geq -1}$ with the set of all diagonals of a regular $(n + 3)$-gon $\mathbf{P}_{n+3}$. Under this identification, the roots in $-\Pi$ correspond to the diagonals on the “snake” shown in Figure 13. Non-crossing diagonals represent compatible roots, while crossing diagonals correspond to roots whose compatibility degree is 1. (Here and in the sequel, two diagonals are called crossing if they are distinct and have a common interior point.) Thus, each positive root $\alpha_{i,j} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j$ corresponds to the unique diagonal that crosses precisely the diagonals $-\alpha_i, -\alpha_{i+1}, \ldots, -\alpha_j$ from the snake (see Figure 13).

The clusters are in bijection with the triangulations of $\mathbf{P}_{n+3}$ by non-crossing diagonals. The cluster complex is the dual complex of the ordinary associahedron.
Two triangulations are joined by an edge in the exchange graph if and only if they are obtained from each other by a “flip” that replaces a diagonal in a quadrilateral formed by two triangles of the triangulation by another diagonal of the same quadrilateral. See [11, Section 3.5] and [7, Section 4.1] for further details.

![Figure 13. The “snake” in type $A_5$](image)

![Figure 14. Labeling of the diagonals in type $A_2$](image)

We next describe the cluster variables and the exchange relations in concrete combinatorial terms. For a diagonal $[a, b]$, we denote by $x_{ab}$ the cluster variable $x[\alpha]$ associated to the corresponding root. We adopt the convention that $x_{ab} = 1$ if $a$ and $b$ are two consecutive vertices of $P_{n+3}$. Comparing [11] with [7] Lemma 4.2], we conclude that the matrices $B(C)$ can be described as follows.
Proposition 12.5. Let $C$ be the cluster corresponding to a triangulation $T$ of $\mathbb{P}_{n+3}$, and let $B(C) = B(T)$ be the corresponding matrix with rows and columns indexed by the diagonals in $T$. Then each matrix entry $b_{\alpha\beta}$ is equal to 0 unless $\alpha$ and $\beta$ are two sides of some triangle $(a, b, c)$ in $T$; in the latter case, $b_{\alpha\beta} = 1$ (resp., $-1$) if $\alpha = [a, b], \beta = [a, c]$, and the order of points $a, b, c$ is counter-clockwise (resp., clockwise).

In view of Proposition 12.5, the exchange relations (5.1) in a cluster algebra of type $A_n$ have the form

$$x_{ac}x_{bd} = p_{ac,bd}^+ x_{ab} x_{cd} + p_{ac,bd}^- x_{ad} x_{bc},$$

where $a, b, c, d$ are any four vertices of $\mathbb{P}_{n+3}$ taken in counter-clockwise order, and $p_{ac,bd}^\pm$ are elements of the coefficient semifield $\mathbb{P}$. See Figure 15.

![Figure 15. Exchanges in type $A_n$](image)

Let $A_\circ$ denote the cluster algebra of type $A_n$ associated with the following coefficient system. We take

$$p = \operatorname{Trop}(p_{ab} : [a, b] \text{ is a side of } \mathbb{P}_{n+3}),$$

and define the coefficients in (12.4) by

$$p_{ac,bd}^+ = q_{ab} q_{cd}, \quad p_{ac,bd}^- = q_{ad} q_{bc},$$

where

$$q_{ab} = \begin{cases} 1 & \text{if } [a, b] \text{ is a diagonal;} \\ p_{ab} & \text{if } [a, b] \text{ is a side.} \end{cases}$$

Since $p_{ac,bd}^+$ and $p_{ac,bd}^-$ have no common factors, they satisfy the normalization condition $p_{ac,bd}^+ \oplus p_{ac,bd}^- = 1$; a direct check using Proposition 12.5 shows that this choice of coefficients also satisfies the mutation rule (1.5), making the cluster algebra $A_\circ$ well-defined.

Example 12.6 (Geometric realization for $A_\circ$ in type $A_n$). Let $X = X_{n+3}$ be the affine cone over the Grassmannian $\operatorname{Gr}_{2,n+3}$ of 2-dimensional subspaces in $\mathbb{C}^{n+3}$ taken in its Plücker embedding (cf. Example 12.4); simply put, $X$ is the variety of all nonzero decomposable bivectors in $\wedge^2 \mathbb{C}^{n+3}$. Let $(\Delta_{ab} : 1 \leq a < b \leq n+3)$ be the standard Plücker coordinates on $X$. We identify the indices $1, \ldots, n+3$ with

![Diagram](image)
the vertices of $\mathbb{P}_{n+3}$ by numbering these vertices, say, counterclockwise. Thus, we associate the Plücker coordinates with all the sides and diagonals of $\mathbb{P}_{n+3}$. Note that we have previously used the same set $\{(ab) : 1 \leq a < b \leq n + 3\}$ to label the cluster variables $x_{ab}$ and the coefficients $p_{ab}$.

**Proposition 12.7.** The correspondence sending each cluster variable $x_{ab}$ and each coefficient $p_{ab}$ to the corresponding element $\Delta_{ab}$ extends uniquely to an algebra isomorphism between the cluster algebra $A_n$ and the $\mathbb{Z}$-form $\mathbb{Z}[X]$ of $\mathbb{C}[X]$ generated by all Plücker coordinates.

**Proof.** This is a special case of Proposition 11.4. To see this, we need to verify the conditions (11.4), (11.7). The fact that $X$ satisfies (11.4) is well known (for the rationality property, it is enough to note that $X$ has a Zariski open subset isomorphic to an affine space). For the dimension count (11.5), we have

$$\dim(X) = \dim(Gr_{2,n+3}) + 1 = 2n + 3 = n + |J|,$$

as required. The property (11.6) means that $\mathbb{C}[X]$ is generated by all Plücker coordinates, which is trivial. Finally, (11.7) follows from the standard fact that the Plücker coordinates satisfy the Grassmann-Plücker relations

$$\Delta_{ac}\Delta_{bd} = \Delta_{ab}\Delta_{cd} + \Delta_{ad}\Delta_{bc}$$

for all $1 \leq a < b < c < d \leq n + 3$. \hfill $\square$

We note that the ring $\mathbb{Z}[X]$ is naturally identified with the ring of $SL_2$-invariant polynomial functions with coefficients in $\mathbb{Z}$ on the space of $(n+3)$-tuples of vectors in $\mathbb{C}^2$. Representing these vectors as columns of a $2 \times (n + 3)$ matrix $Z = (z_{ij})$, we identify the Plücker coordinates with the $2 \times 2$ minors of $Z$:

$$\Delta_{ab} = z_{1a}z_{2b} - z_{1b}z_{2a} \quad (1 \leq a < b \leq n + 3).$$

**Remark 12.8.** It is classically known that the monomials in the Plücker coordinates that are not divisible by $\Delta_{ac}\Delta_{bd}$ for any $a < b < c < d$, form a $\mathbb{Z}$-basis in $\mathbb{Z}[X]$ (see [10] or [20] for a proof). Let us translate this fact into the setting of cluster algebras. We shall call a monomial $\prod_\alpha x_\alpha^{m_\alpha}$ in the cluster variables compatible if $m_\alpha m_\beta = 0$ whenever the roots $\alpha$ and $\beta$ are incompatible, i.e., whenever the corresponding diagonals cross each other. (Equivalently, all variables contributing to a compatible monomial belong to a single cluster.) In this terminology, the cluster algebra $A_n$ is a free $\mathbb{Z}[^P\mathbb{Z}]$-module with the basis formed by all compatible monomials. We believe that this property remains true for an arbitrary cluster algebra of finite type (we have checked it for all classical types); we plan to investigate it in a separate publication. We note that linear independence of compatible monomials is an immediate consequence of Theorem 11.9 and the uniqueness of cluster expansions (Proposition 5.3).

### 12.3. Types $B_n$ and $C_n$.

Let $\Phi$ be a root system of type $B_n$ or $C_n$. We identify the set $I$ in a standard way with $[1, n]$. As in [7] Section 4.2], in order to treat both cases at the same time, we set $r = 1$ for $\Phi$ of type $B_n$, and $r = 2$ for $\Phi$ of type $C_n$.

Once again, our convention for the Cartan matrices is different from the one in [6] but agrees with that in [10]; thus, we have $a_{n-1,n} = -r$ and $a_{n,n-1} = -2/r$.

We recall the combinatorial description of the cluster complex of type $B_n/C_n$ from [11] Section 3.5. Let $\Theta$ denote the $180^\circ$ rotation of a regular $(2n + 2)$-gon $\mathbb{P}_{2n+2}$. There is a natural action of $\Theta$ on the diagonals of $\mathbb{P}_{2n+2}$. Each orbit of
this action is either a diameter (i.e., a diagonal connecting antipodal vertices) or an unordered pair of centrally symmetric non-diameter diagonals of $P_{2n+2}$. Following [11, Section 3.5], we identify almost positive roots in $\Phi$ with these orbits. Under this identification, each of the roots $-\alpha_i$ for $i = 1, \ldots, n-1$ is represented by a pair of diagonals on the “snake” shown in Figure 16, whereas $-\alpha_n$ is identified with the only diameter on the snake. Two $\Theta$-orbits represent compatible roots if and only if the diagonals they involve do not cross each other. More generally, in type $B_n$ (resp., $C_n$), for $\alpha, \beta \in \Phi_{\geq -1}$, the compatibility degree $(\alpha \parallel \beta)$ is equal to the number of crossings of one of the diagonals representing $\alpha$ (resp., $\beta$) by the diagonals representing $\beta$ (resp., $\alpha$). Thus, each positive root $\beta = \sum b_i \alpha_i$ is represented by the unique $\Theta$-orbit such that every diagonal representing $-\alpha_i$ (resp., $\beta$) crosses the diagonals representing $\beta$ (resp., $-\alpha_i$) at $b_i$ points.

**Figure 16.** The “snake” for the types $B_3$ and $C_3$

The clusters are in bijection with the centrally-symmetric (that is, $\Theta$-invariant) triangulations of $P_{2n+2}$ by non-crossing diagonals. The cluster complex is the dual complex for the Bott-Taubes cyclohedron [5]. Two centrally symmetric triangulations are joined by an edge in the exchange graph if and only if they are obtained from each other either by a flip involving two diameters, or by a pair of centrally symmetric flips. See [11, Section 3.5] and [7, Section 4.2] for details.

For a vertex $a$ of $P_{2n+2}$, let $\overline{a}$ denote the antipodal vertex $\Theta(a)$. For a diagonal $[a, b]$, we denote by $x_{ab}$ the cluster variable $x[\alpha]$ associated to the root corresponding to the $\Theta$-orbit of $[a, b]$. Thus, we have $x_{ab} = x_{ba} = x_{a\overline{b}}$. Similarly to the type $A_n$, we adopt the convention that $x_{ab} = 1$ if $a$ and $b$ are consecutive vertices in $P_{2n+2}$.

Comparing (5.1) with [7, Lemma 4.4], we obtain the following concrete description of the exchange relations in types $B_n$ and $C_n$.

**Proposition 12.9.** The exchange relations in a cluster algebra of type $B_n$ or $C_n$ have the following form:

\[
\begin{align*}
(12.6) & \quad x_{ac} x_{bd} = p^{+}_{ac,bd} x_{ab} x_{cd} + p^{-}_{ac,bd} x_{ad} x_{bc}, \\
(12.7) & \quad x_{ac} x_{a\overline{b}} = p^{+}_{ac,\overline{a}b} x_{ab} x_{\overline{a}c} + p^{-}_{ac,\overline{a}b} x_{a\overline{c}} x_{bc},
\end{align*}
\]

whenever $a, b, c, d, \overline{a}$ are in counter-clockwise order;
whenever \(a, b, c, \overline{a}\) are in counter-clockwise order;

\[
x_{\alpha \pi} x_{\delta_5} = p^+_{\alpha \pi, \delta_5} x^{r}_{ab} + p^-_{\alpha \pi, \delta_5} x^{r}_{ab},
\]
whenever \(a, b, \overline{a}\) are in counter-clockwise order. See Figure 17.

![Figure 17. Exchanges in types \(B_n\) and \(C_n\)](image)

We will provide an explicit realization of a special cluster algebra \(A_\circ\) of type \(B_n\) or \(C_n\) similar to its namesake for \(A_n\). The algebra \(A_\circ\) corresponds to the following special choice of coefficients. We set \(P = \text{Trop}(\{p_\delta\})\), where \(\delta\) runs over all centrally-symmetric pairs of sides of the polygon \(P_{2n+2}\). For such a pair \(\delta = \{[a, b], [\overline{a}, \overline{b}]\}\), we write the corresponding generator of \(P\) as \(p_\delta = p_{ab} = p_{\overline{a}\overline{b}}\). The coefficients in \((12.6)-(12.8)\) are specified in a similar way to \((12.4)-(12.5)\). More precisely, to obtain a coefficient of some monomial in \((12.6)-(12.8)\), take this monomial and replace each of its cluster variables \(x_{ab}\) by 1 (resp., \(p_{ab}\)) if \([a, b]\) is a diagonal (resp., a side) of \(P_{2n+2}\). The fact that these coefficients satisfy the normalization condition is again obvious; we leave to the reader a direct check that they also satisfy the mutation rule \((1.5)\).

So far the material for \(B_n\) has been completely parallel to that for \(C_n\). However, our geometric realizations for these two types are quite different from each other.

**Example 12.10** (Geometric realization for \(A_\circ\) in type \(B_n\)). Somewhat surprisingly, it turns out that the algebra \(A_\circ\) for the type \(B_n\) (for \(n \geq 3\)) is isomorphic,
as a ring, to the cluster algebra $\mathcal{A}_c$ for the type $A_{n-1}$. Recall that the latter is naturally identified with the ring $\mathbb{Z}[X_{n+2}]$ generated by the Plücker coordinates $\Delta_{ab}$ (for $1 \leq a < b \leq n + 2$) on the Grassmannian $\text{Gr}_{2,n+2}$ (see Proposition 12.7).

Let us label the vertices of $P_{2n+2}$ in the counterclockwise order by the indices $1, \ldots, n + 1, \overline{1}, \ldots, \overline{n + 1}$. We associate a function from $\mathbb{Z}[X_{n+2}]$ to every $\Theta$-orbit on the set of all diagonals and sides of $P_{2n+2}$, as follows:

$$[a, \overline{a}] \mapsto \Delta_{a\overline{a}} = \Delta_{a,n+2} \quad (1 \leq a \leq n + 1),$$

(12.9)  $$\{[a, b], [\overline{a}, \overline{b}]\} \mapsto \Delta_{ab} \quad (1 \leq a < b \leq n + 1),$$

$$\{[a, \overline{b}], [\overline{a}, b]\} \mapsto \Delta_{ab} = \Delta_{a,n+2}\Delta_{b,n+2} - \Delta_{ab} \quad (1 \leq a < b \leq n + 1).$$

Proposition 12.11. The correspondence sending each cluster variable and each coefficient for the cluster algebra $\mathcal{A}_c$ of type $B_n$ to the element in (12.9) with the same label extends uniquely to an algebra isomorphism of $\mathcal{A}_c$ with $\mathbb{Z}[X_{n+2}]$.

Proof. The proof is similar to that of Proposition 12.7. It is enough to check that our data satisfy the conditions (11.4)–(11.7). Condition (11.4) was already checked in the proof of Proposition 12.7. The dimension count (11.5) now takes the form:

$$\dim(X_{n+2}) = 2n + 1 = n + |J|,$$

as required. The property (11.6) is clear since all the Plücker coordinates $\Delta_{ab}$ for $1 \leq a < b \leq n$ are among the functions (12.9). Finally, (11.7) amounts to checking the following six identities (with $r = 1$) obtained from the exchange relations (12.6)–(12.8) (we have to take into account possible positions of vertices in Figure 17 among the vertices $1, \ldots, n + 1, \overline{1}, \ldots, \overline{n + 1}$):

(12.10)  $$\Delta_{ac} \Delta_{bd} = \Delta_{ab} \Delta_{cd} + \Delta_{ad} \Delta_{bc} \quad (1 \leq a < b < c < d \leq n + 1),$$

(12.11)  $$\Delta_{ac} \Delta_{bd} = \Delta_{ab} \Delta_{cd} + \Delta_{ad} \Delta_{bc} \quad (1 \leq a < b < c < d \leq n + 1),$$

(12.12)  $$\Delta_{ac} \Delta_{bd} = \Delta_{ab} \Delta_{cd} + \Delta_{ad} \Delta_{bc} \quad (1 \leq a < b < c < d \leq n + 1),$$

(12.13)  $$\Delta_{ac} \Delta_{bd} = \Delta_{ab} \Delta_{cd} + \Delta_{ad} \Delta_{bc} \quad (1 \leq a < b < c \leq n + 1),$$

(12.14)  $$\Delta_{ac} \Delta_{bd} = \Delta_{ab} \Delta_{cd} + \Delta_{ad} \Delta_{bc} \quad (1 \leq a < b < c \leq n + 1),$$

(12.15)  $$\Delta_{ac} \Delta_{bd} = \Delta_{ab} \Delta_{cd} + \Delta_{ad} \Delta_{bc} \quad (1 \leq a < b \leq n + 1).$$

Of these identities, (12.10) is a Grassmann-Plücker relation, and the rest are reduced to this relation by simple algebraic manipulations.

Example 12.12 (Geometric realization for $\mathcal{A}_c$ in type $C_n$). Let $SO_2$ be the group of complex matrices

$$\begin{bmatrix} u & -v \\ v & u \end{bmatrix}$$

with $u^2 + v^2 = 1$. Consider the algebra $\mathcal{R} = \mathbb{C}[\text{Mat}_{2,n+1}]^{SO_2}$ of $SO_2$-invariant polynomial functions on the space of $2 \times (n + 1)$ complex matrices, or, equivalently, on the space of $(n + 1)$-tuples of vectors in $\mathbb{C}^2$. Alternatively, $\mathcal{R}$ can be identified with the ring of invariants $\mathbb{C}[\text{Mat}_{2,n+1}]^T$, where $T \subset SL_2$ is the torus of all diagonal matrices of the form

$$\begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix}.$$
Indeed, we have \( g(SO_2)g^{-1} = T \), where

\[
g = \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix},
\]

so the map \( f \mapsto f^g \) defined by \( f^g(z) = f(gz) \) is an isomorphism

\[
(12.16) \quad \mathbb{C}[\text{Mat}_{2,n+1}]^T \to \mathbb{C}[\text{Mat}_{2,n+1}]^{SO_2}.
\]

The ring \( \mathcal{R} = \mathbb{C}[\text{Mat}_{2,n+1}]^T \) can also be viewed as the coordinate ring \( \mathbb{C}[X] \) of the variety \( X \) of complex \( (n+1) \times (n+1) \) matrices of rank \( \leq 1 \) (even more geometrically, \( X - \{0\} \) is the affine cone over the product of two copies of the projective space \( \mathbb{CP}^n \) taken in the Segre embedding). Specifically, the map

\[
y = \begin{bmatrix} y_{11} & \cdots & y_{1,n+1} \\ y_{21} & \cdots & y_{2,n+1} \end{bmatrix} \mapsto (y_{1a}y_{2b})_{a,b=1,\ldots,n+1} \in X
\]
induces an algebra isomorphism \( \mathbb{C}[X] \to \mathbb{C}[\text{Mat}_{2,n+1}]^T \). Combining this with \( (12.16) \), we obtain an isomorphism \( \mathbb{C}[X] \to \mathbb{C}[\text{Mat}_{2,n+1}]^{SO_2} \) induced by the map

\[
\begin{bmatrix} z_{11} & \cdots & z_{1,n+1} \\ z_{21} & \cdots & z_{2,n+1} \end{bmatrix} \mapsto ((z_{1a} - iz_{2a})(z_{1b} + iz_{2b}))_{a,b=1,\ldots,n+1} \in X.
\]

By analogy with \( (12.14) \), we associate an element from \( \mathcal{R} = \mathbb{C}[X] \) to every Θ-orbit on the set of all diagonals and sides of \( \mathbb{P}_{2n+2} \), as follows:

\[
(12.17) \quad \{[a,b], [\overline{a}, \overline{b}]\} \mapsto \Delta_{ab} = z_{1a}z_{2b} - z_{1b}z_{2a} = \frac{y_{1a}y_{2b} - y_{1b}y_{2a}}{2i} \quad (1 \leq a < b \leq n+1),
\]

\[
\{[a, \overline{b}], [\overline{a}, b]\} \mapsto \Delta_{\alpha \beta} = z_{1a}z_{1b} + z_{2a}z_{2b} = \frac{y_{1a}y_{2b} + y_{1b}y_{2a}}{2} \quad (1 \leq a \leq b \leq n+1).
\]

The following result is a type \( C_n \) counterpart of Proposition \( (12.14) \).

**Proposition 12.13.** The correspondence sending each cluster variable and each coefficient for the cluster algebra \( A_\Phi \) of type \( C_n \) to the element in \( \mathbb{C}[X] \) with the same label extends uniquely to an algebra isomorphism of \( A_\Phi \) with a \( \mathbb{Z} \)-form of \( \mathcal{R} \).

**Proof.** The proof is analogous to that of Proposition \( (12.14) \). The only work involved is to check that the functions in \( (12.17) \) satisfy the identities \( (12.10) \) – \( (12.15) \), this time with \( r = 2 \). This is completely straightforward; for example, \( (12.15) \) becomes

\[
(z_{1a}^2 + z_{2a}^2)(z_{1b}^2 + z_{2b}^2) = (z_{1a}z_{2b} - z_{1b}z_{2a})^2 + (z_{1a}z_{1b} + z_{2a}z_{2b})^2. \quad \square
\]

12.4. **Type \( D_n \).** Let \( \Phi \) be a root system of type \( D_n \) (allowing for \( n = 3 \), in which case \( \Phi \) is of type \( A_3 \)). According to [11] Section 3.5, the almost positive roots (hence the cluster variables) for the type \( D_n \) have a natural surjection onto those for \( B_{n-1} \). This surjection is one-to-one over the roots corresponding to pairs of diagonals of \( \mathbb{P}_{2n} \), and two-to-one over those corresponding to diameters. Thus, the roots in \( \Phi_{\geq -1} \) are represented by Θ-orbits on the set of diagonals in a regular \( 2n \)-gon, in which each diameter can be of one of two different “colors”; we denote the two different kinds of diameters by \([a, \overline{a}]\) and \([\overline{a}, \overline{b}]\). The negative simple roots form a “type \( D \) snake” shown in Figure 18. Two Θ-orbits represent compatible roots if and only if the diagonals they involve do not cross each other; here we use the following convention:

\[
(12.18) \quad \text{diameters of the same color do not cross each other.}
\]
More generally, for \( \alpha, \beta \in \Phi_{\geq -1} \), the compatibility degree \( (\alpha \parallel \beta) \) is equal to the number of \( \Theta \)-orbits in the set of crossing points between the diagonals representing \( \alpha \) and \( \beta \) (again, with the convention (12.18)). Each positive root \( \beta = \sum b_i \alpha_i \) is then represented by the unique \( \Theta \)-orbit such that the diagonals representing \( \beta \) cross the diagonals representing \( -\alpha_i \) at \( b_i \) pairs of centrally symmetric points (counting an intersection of two diameters of different color and location as one such pair).

![Figure 18. Representing the roots in \(-\Pi\) for the type \( D_4 \)](image)

Accordingly, the cluster variables for \( D_n \) can be denoted as \( x_{\alpha} \), for all diagonals \( \alpha \) in \( P_{2n} \) (with the convention \( x_\alpha = x_{\Theta(\alpha)} \)), plus \( n \) extra variables \( \tilde{x}_\beta \) for all diameters \( \tilde{\beta} \).

With the help of [7, Lemma 4.6], we obtain the following analogue of Proposition 12.9.

**Proposition 12.14.** The exchange relations in a cluster algebra of type \( D_n \) have the following form:

(12.19) \[ x_{ac} x_{bd} = p_+^{ac,bd} x_{ab} x_{cd} + p_-^{ac,bd} x_{ad} x_{bc} \]

whenever \( a, b, c, d \) are in counter-clockwise order;

(12.20) \[ x_{ac} x_{\bar{a}b} = p_+^{ac,\bar{a}b} x_{ab} x_{a\bar{c}} + p_-^{ac,\bar{a}b} x_{a\bar{c}} x_{bc} \]

whenever \( a, b, c, \bar{a} \) are in counter-clockwise order;

(12.21) \[ x_{a\bar{c}} x_{\bar{a}b} = p_+^{a\bar{c},\bar{a}b} x_{ab} + p_-^{a\bar{c},\bar{a}b} x_{a\bar{b}} \]

whenever \( a, b, \bar{a} \) are in counter-clockwise order;

(12.22) \[ x_{a\bar{c}} x_{b\bar{c}} = p_+^{a\bar{c},b\bar{c}} x_{ab} x_{c\bar{c}} + p_-^{a\bar{c},b\bar{c}} x_{c\bar{c}} x_{b\bar{c}} \]

and

(12.23) \[ \tilde{x}_{a\bar{c}} x_{b\bar{c}} = \tilde{p}_+^{a\bar{c},b\bar{c}} x_{ab} \tilde{x}_{c\bar{c}} + \tilde{p}_-^{a\bar{c},b\bar{c}} x_{c\bar{c}} \tilde{x}_{b\bar{c}} \]

whenever \( a, b, c, \bar{a} \) are in counter-clockwise order.
We define a special coefficient system and the corresponding cluster algebra $A$ of type $D_n$ in precisely the same way as for the types $B_n$ and $C_n$ above. We conclude this paper with a geometric realization of this algebra similar to the one given in Example 12.10.

Example 12.15 (Geometric realization for $A$ in type $D_n$). Consider the same variety $X_{n+2}$ as in Example 12.10. Let $X$ be the divisor in $X_{n+2}$ given by the equation $\Delta_{n+1,n+2} = 0$; thus, we have

$$\mathbb{C}[X] = \mathbb{C}[X_{n+2}]/(\Delta_{n+1,n+2}).$$

(Geometrically, $X$ is the affine cone over the Schubert divisor in the Grassmannian $\text{Gr}_{2,n+2}$.)

Let $\mathbb{Z}[X]$ denote the $\mathbb{Z}$-form of $\mathbb{C}[X]$ generated by all Plücker coordinates. By analogy with (12.24), we introduce the following family of functions from $\mathbb{Z}[X]$:

$$[a, \overline{a}] \mapsto \Delta_{a,a} = \Delta_{a,n+1}, \quad (1 \leq a \leq n),$$
$$\{a, \overline{a}, b, \overline{b}\} \mapsto \Delta_{ab} = \Delta_{a,n+1}\Delta_{b,n+2} - \Delta_{ab}, \quad (1 \leq a < b \leq n).$$

(Note that in $\mathbb{Z}[X]$, there is a relation $\Delta_{a,n+1}\Delta_{b,n+2} = \Delta_{a,n+2}\Delta_{b,n+1}$ since we have $\Delta_{n+1,n+2} = 0$.)

Proposition 12.16. The correspondence sending each cluster variable and each coefficient for the cluster algebra $A$ of type $D_n$ to the element in (12.24) with the same label extends uniquely to an algebra isomorphism between $A$ and $\mathbb{Z}[X]$.

Proof. The proof is completely analogous to that of Proposition 12.11. Details are left to the reader. □

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