A GENERALIZATION OF THE GINZBURG–LANDAU THEORY TO p-WAVE SUPERCONDUCTORS

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We succeed in building up a straightforward theoretical model for spin-triplet p-wave superconductors, by introducing a second-order parameter and a nonlinear interaction between the two mean fields in the Ginzburg–Landau theory. Such interaction breaks the isotropy of the original medium and allows pairs of electrons to arrange into $S = 1$ Cooper pairs. The present model predicts a thermodynamical and magnetic behavior analogous to that observed in conventional s-wave superconductors.

Keywords: Spin-triplet superconductors; p-wave superconductivity; Ginsburg–Landau theory.

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1. “Doubling” the Ginzburg–Landau Theory

In early works based on the mean-field “macroscopic” approach,\textsuperscript{1} superfluidity was described in terms of a single-order parameter, $\phi_1 = \sqrt{\rho} e^{i\theta}$. Subsequently, besides the phonon propagation, Ziman\textsuperscript{2} also considered the roton motion by introducing, in addition to the usual couple of fields $\rho$ and $\theta$, two other real fields, $\chi$ and $\psi$,
entering the Clebsh term of the most general expansion of a generic velocity field: 
\[ \mathbf{v} = \nabla \theta + \chi \nabla \psi. \]
Correspondingly, in addition to the usual complex quantum field \( \phi_1 \), Ziman introduced a second complex quantum field accounting for rotational motions, 
\[ \phi_2 \equiv (\psi_a + i \psi_b)/2h, \]
where the real and imaginary parts are related to \( \chi \) and \( \psi \) as follows: 
\[ \psi_a \equiv \sqrt{2 \rho} \sqrt{\chi}, \quad \psi_b \equiv \sqrt{2 \rho}. \]
Hence the current density can be re-written as a sum of an irrotational (superfluid) part and a non-potential (non-superfluid) part:
\[ j = \rho \mathbf{v} = \rho (\nabla \theta + \chi \nabla \psi) = \frac{1}{2} \hbar (\phi_1^* \nabla \phi_1 - \phi_1 \nabla \phi_1^*) + \frac{1}{2} \hbar (\phi_2^* \nabla \phi_2 - \phi_2 \nabla \phi_2^*). \]
Finally the complete Hamiltonian, entailing also roton kinetic energy and phonon-roton interaction terms, can be written straightforwardly as the sum of the total kinetic energy and of a suitable pressure potential:
\[ H = \frac{1}{2} \rho \mathbf{v}^2 + \rho \int \frac{p^2}{2m} \, dp. \]
Therefore two independent complex order parameters are needed for a complete picture of the superfluid dynamics and the original Landau irrotational mean-field theory should be “doubled” in order to describe the rotational degrees of freedom.
Ginzburg and Landau\(^3\) started from the analogy with superfluid \(^3\)He to derive their theory for superconductivity based on the existence of an underlying mean field \( \phi \) in the bulk of a superconducting medium, whose field can be interpreted as the wavefunction of the Cooper pair in its center-of-mass frame. At the same time, within a quantum field framework, the order parameter can be conceived as a self-interacting Higgs field which undergoes condensation for U(1) symmetry breaking when the temperature approaches a critical value. The Ginzburg–Landau (GL) theory was built up simply by taking into account the electric and magnetic properties of the Cooper pairs already discovered in the theoretical framework of the fermionic superfluidity. Of course, if we introduce a second-order parameter in the GL theory of superconductivity by analogy with the Ziman approach, we will not describe rotons but, actually, two different superconducting systems: \textit{spin-singlet two-phase superconductors} or, as we shall show in the next section, \textit{spin-triplet one-phase superconductors}.

With regards to the former system, in recent works\(^4\)–\(^6\) we have considered two scalar charged fields \( \phi_w \) and \( \phi_s \) corresponding to Cooper pairs formed by electrons bound by a weaker or stronger attractive force, respectively. In so doing we have obtained a theoretical model for superconductors endowed with two distinct superconducting phases, since the two-order parameters may condensate at different critical temperatures.\(^a\) In Ref. 5, we found some deviations in basic thermodynamical quantities with respect to the Ginzburg–Landau one-phase superconductors. In particular, by contrast with the usual case where only one jump in the specific heat \( C_V \) takes place at the normal-superconductor transition temperature, we have actually predicted an additional discontinuity for \( C_V \) when passing from one superconducting phase to the other. Furthermore, on analyzing the magnetic behavior of such systems,\(^6\) we found some observable differences with respect to the case of

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\(^a\)Let us recall that, as proved in the above-quoted papers, the introduction of additional degrees of freedom (two complex fields rather than one) \textit{does not involve new unknown physical constants} since both scalar fields are endowed with equal \textit{bare} masses and self-interaction coupling constants.
conventional Ginzburg–Landau superconductors. In particular, at low temperature the London penetration length is strongly reduced and the Ginzburg–Landau parameter $\kappa$ becomes a function of the temperature. In contrast, in the temperature region between the two phase transitions, $\kappa$ is constant and the system is a type-I or type-II superconductor depending on the ratio between the critical temperatures. Recently, a thermodynamical and magnetic behavior qualitatively similar to the one predicted by our model has been seemingly observed in the MgB$_2$ system $^b$ (see, for example, Ref. 7 and references therein).

By introducing two mutually interacting order parameters, in the present letter we will not describe two-phases superconductors but, actually, spinning Cooper pairs and rotational degrees of freedom in superconductivity. In the Bardeen, Cooper, and Schrieffer (BCS) theory for conventional superconductors, the electrons are paired into a zero total angular momentum state, with zero spin and zero orbital angular momentum: $J = L = S = 0$. As a matter of fact, in BCS superconductors, the s-wave is shown to correspond to the minimum energy state with maximum attraction between the electrons in a Cooper pair. Indeed, soon after the BCS theory was advanced, Kohn and Luttinger$^9$ predicted that, if the mutual interaction is repulsive in all partial wave channels, the Cooper pairs will be bonded by a weak residual attraction (out of the Coulomb repulsion) in higher angular momentum channels: this is the so-called Kohn–Luttinger effect. On the other hand, there exists also a p-wave Cooper-pairing in superfluid $^3$He (which is, as said above, the liquid counterpart of GL superconductors). Actually, we can meet p-wave superconductivity in certain “heavy-electron” compounds (heavy fermion systems as, e.g., UPt$_3$) and in special materials recently discovered as, e.g., Sr$_2$RuO$_4$ (Ref. 10) which is the only known metal oxide displaying p-wave superconductivity. Let us recall that the p-wave Cooper pairs are always spin-triplets ($S = 1$) because of Pauli’s exclusion principle applied to systems composed of a pair of particles endowed with odd ($L = 1$) total orbital quantum number. Taking into account this property, in the next section we shall put forward a simple GL-like model just for spin-triplet superconductors.

2. The Model

Let us consider a physical system described by one doublet of complex scalar fields $\phi_1, \phi_2$ through the following Lagrangian density:

$$\mathcal{L} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \phi_1)^*(D^\mu \phi_1) + (D_\mu \phi_2)^*(D^\mu \phi_2) - \lambda \left( |\phi_1|^2 - \frac{1}{2} \phi_0^2 \right)^2$$

$^b$Several two-band theories$^d$ try to explain the observed behavior of the specific heat and the peculiar temperature dependence of the upper and lower critical fields of MgB$_2$. Probably there exists a correspondence between the two “classical-macroscopic” (since they represent “collective” wave-functions for the condensate) order parameters in GL-like approaches, as the present one, and the two “quantum-microscopic” gaps in quasi-particle energy spectra predicted for MgB$_2$ by some BCS-like theories.
Here, the covariant derivative $D_\mu \equiv \partial_\mu + ieA_\mu$ describes the minimal electromagnetic interaction of the two scalar fields, while the first term in the Lagrangian (with $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$) accounts for the kinetic energy of the free electromagnetic field $A_\mu$. The complete potential term for the interaction of the two scalar fields is composed of three different terms, $V = V_{A} + V_{\text{self}} + V(\phi_1, \phi_2)$, the first two of them describing the usual interaction between the electromagnetic field and the charged scalar field (coming from the covariant derivative), and $V_{\text{self}}$ ruling the self interaction of the scalar fields: $V_{\text{self}} \equiv \lambda|\phi_1|^4 + \mu|\phi_2|^4$. For the interaction between the two scalar fields we instead adopt the following nonlinear term:

$$V(\phi_1, \phi_2) = -\frac{\lambda\phi_0^4}{8}\ln 2 \frac{\phi_1 \phi_2^*}{\phi_1^* \phi_2}. \quad (2)$$

Let us study the small fluctuations of the two scalar fields around the minimum of the energy corresponding to $\phi_1 = \phi_2 = \phi_0/\sqrt{2}$ by expanding both scalar fields as follows:

$$\phi_1 \equiv \frac{1}{\sqrt{2}}(\phi_0 + \eta_1)e^{i\theta_1}/\phi_0, \quad (3)$$
$$\phi_2 \equiv \frac{1}{\sqrt{2}}(\phi_0 + \eta_2)e^{i\theta_2}/\phi_0, \quad (4)$$

where $\eta_1$, $\eta_2$, $\theta_1$, $\theta_2$ are real fields. From these definitions, the above interaction term can be written more simply as follows:

$$V(\phi_1, \phi_2) = \frac{\lambda\phi_0^4}{2} (\theta_1 - \theta_2)^2. \quad (5)$$

Notice that $V(\phi_1, \phi_2)$ is positive-definite, describing a repulsion between the two fields with strength $\lambda\phi_0^4$ equal to the mass squared $m^2_B$ (see below). Note also that $V(\phi_1, \phi_2)$ corresponds to the main term of the expansion for small phase differences\(^\text{11}\) of the Legget interaction $\gamma(\phi_1^* \phi_2 + \phi_1 \phi_2^*)$.

By inserting Eqs. (3) and (4) into the Lagrangian density (1) and performing the gauge transformation: $A_\mu \rightarrow A_\mu + \partial_\mu \Lambda$ with

$$\Lambda \equiv -\frac{1}{2e\phi_0}(\theta_1 + \theta_2), \quad (6)$$

we obtain the following Lagrangian, up to quadratic terms in the fields:

$$\mathcal{L} \approx -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + e^2\phi_0^2A_\mu A^{\mu} + \frac{1}{2}\partial_\mu\eta_1 \partial^\mu\eta_1 + \frac{1}{2}\partial_\mu\eta_2 \partial^\mu\eta_2$$

$$+ \frac{1}{2}\partial_\mu(\theta_1 - \theta_2)\partial^\mu(\theta_1 - \theta_2) + \lambda\phi_0^2\eta_1^2 + \lambda\phi_0^2\eta_2^2 + \frac{\lambda\phi_0^4}{2}(\theta_1 - \theta_2)^2. \quad (7)$$

Let us set

$$\eta_3 \equiv \frac{1}{\sqrt{2}}(\theta_1 - \theta_2), \quad (8)$$

Where

$$-\lambda\left(|\phi_2|^2 - \frac{1}{2}\phi_0^2\right)^2 + V(\phi_1, \phi_2). \quad (1)$$
and define the triplet field $W_a \equiv (\eta_1, \eta_2, \eta_3)$. The Lagrangian describing our physical system now becomes
\begin{equation}
\mathcal{L} \approx -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + m_A^2 A_\mu A^\mu + \frac{1}{2} (\partial_\mu W_a) (\partial^\mu W_a) + m_W^2 W_a W_a \tag{9}
\end{equation}
with
\begin{equation}
m_A^2 = e^2 \phi_0^2, \quad m_W^2 = \lambda \phi_0^2. \tag{10}
\end{equation}
As a result, only one of the original four degrees of freedom embedded into two-charged (complex) scalar fields disappears, giving rise to a massive photon as in the standard GL model. By virtue of the interaction potential in Eq. (2), the remaining three degrees of freedom all have the same mass, and can thus be combined to form a triplet field $W$ (i.e. a triplet spinor representation of SU(2)), suitable to describe a so-called $p$-wave superconductor. We stress that, notwithstanding the simultaneous condensation of two real degrees of freedom, the key point in our model is the particular interaction term we have introduced, which prevents a gauge transformation to re-absorb one more degree of freedom (only the sum of the phases of the complex fields turns out to be “eaten”, but not even the difference). Such a very peculiar interaction breaks the isotropy of the original medium and allows pairs of electrons to arrange into possible $S = 1$ (instead of $S = 0$) Cooper pairs. As a matter of fact, the emergence of a triplet field is a signal of the occurred “anisotropization” of the system, which can no more be described by a singlet scalar field.

3. Application to Triplet Superconductors

The order parameter describing $p$-wave superconductors may be associated in our model to the above triplet Higgs field $W_a$ which is responsible of the U(1) spontaneous symmetry breaking occurring during the normal state-superconducting-phase transition. Therefore, from the Lagrangian (1), the effective free energy density at finite temperature $T$, resulting from the quantum fields calculation, including one-loop radiative corrections,\cite{12,13} is given by
\begin{equation}
F(T) = F_n(T) + a(T)|\phi_1|^2 + a(T)|\phi_2|^2 + \lambda |\phi_1|^4 + \lambda |\phi_2|^4 + a(T) \frac{\phi_2^2}{8} \ln \left| \frac{\phi_1}{\phi_1^* \phi_2} \right|^2,
\end{equation}
where
\begin{equation}
a(T) = -m_W^2 + \frac{\lambda + e^2}{4} T^2, \tag{12}
\end{equation}
with label $n$ referring to the normal (non superconducting) phase. The coefficient $a$ vanishes when the temperature approaches a critical value given by
\begin{equation}
T_c = \sqrt{\frac{4m_W^2}{\lambda + e^2}}. \tag{13}
\end{equation}
Below $T_c$, the expectation values of the scalar fields $\phi_1$ and $\phi_2$ which minimize the free-energy function become

$$|\phi_1(T)| = |\phi_2(T)| = \sqrt{-\frac{a(T)}{2\lambda}},$$

while the third-degree of freedom defined in Eq. (8) fluctuates around the zero expectation value, corresponding to $\theta_1 = \theta_2$. This last occurrence directly comes from the fact that the nonlinear characteristic potential term in Eq. (2) is non-negative definite, so that the minimum of the free energy is reached when it vanishes. In this case, our model practically reduces to a “simple” doubling of the standard GL theory making recourse to two scalar order parameters. As a consequence, it is very easy to re-obtain the usual main properties for $p$-wave superconductors considered here.

The London penetration length of the magnetic field inside the superconductor arises due to the presence of a massive photon, that is

$$\delta = \frac{1}{m_A} = \frac{1}{e\phi_0},$$

while the coherence length of the Cooper pairs described by the triplet scalar field is given by

$$\xi = \frac{1}{m_W} = \frac{1}{\phi_0\sqrt{\lambda}} = \frac{\xi_0}{\sqrt{1 - \frac{T^2}{T_c^2}}}.$$  

The critical magnetic field $H_c$, measuring the condensation energy $F(T) - F_n(T) = -\mu_0 H_c^2/2$ of the superconductor system can be obtained as follows:

$$H_c^2 = \frac{1}{\mu_0} \frac{a^2(T)}{\lambda} = H_c^0 \left(1 - \frac{T^2}{T_c^2}\right)^2.$$  

By taking the derivative of the free energy function with respect to temperature, we easily get the entropy gain with respect to the normal phase:

$$S - S_n = \frac{\partial}{\partial T} \left(-\frac{a^2(T)}{2\lambda}\right) = S_0 \left(1 - \frac{T^2}{T_c^2}\right) \frac{T}{T_c}. $$

Finally, we can write down the expected discontinuity of the specific heat at the critical point as

$$\Delta C_V = T \frac{\partial}{\partial T} (S - S_n) = S_0 \left(1 - 3 \frac{T^2}{T_c^2}\right) \frac{T}{T_c}. $$

4. Conclusion

We have extended the standard GL theory in order to describe $p$-wave superconductors by means of two mutually interacting order parameters which condensate simultaneously at a same critical temperature (since the $\lambda\phi^4$ self-interaction is the
same for both fields). After the condensation via the Higgs mechanism, we remain with three massive degrees of freedom (in addition to a massive photon, related to the Meissner-effect) which can be put in correspondence to the three components of a $S = 1$ triplet mean-field describing spinning $p$-wave Cooper pairs. In our model, the main magnetic and thermodynamical properties (including the discontinuity in the specific heat) of $p$-wave superconductors turn out to be essentially the same as for conventional $s$-wave superconductors.

The non-chiral spin-triplet superconductors described in this paper can be compared with chiral $p$-wave superconductors; a generalization of the GL theory for the description of such media has been considered in Ref. 14. Similarly to what presented here, in Ref. 14 two complex-order parameters form a two-component vector which describes a chiral system endowed with cylindrical symmetry (an effective bi-dimensional system), and the GL free-energy function is written just in terms of these components (see Eq. (3.2) of Ref. 14). In Ref. 14, the authors focused on particular properties of chiral superconductors, i.e. a spontaneous Hall effect, so that several peculiar terms describing anomalous electromagnetic interactions of the medium were introduced in the expression for the free-energy (for simplicity, here we have not considered such terms). The \textit{ab initio} introduction of a vector field (with two or three components for bi- or three-dimensional systems, respectively) is well-justified in the case considered there, since that particular chiral system does not respect, in a sense, the Lorentz symmetry (it is not an isotropic medium). Hence, no general principle prevents the free-energy to have a minimum for a non-vanishing vacuum expectation value of that vector field along a given spatial direction. Actually the present case is quite different since we have considered a (three-dimensional) \textit{isotropic} medium. Nevertheless, it is very intriguing that completely different systems may exhibit similar physical properties, thus encouraging further theoretical and experimental studies in this direction.

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