Dynamics of a novel nonlinear SIR model with double epidemic hypothesis and impulsive effects

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Abstract In this paper, the propagation of a nonlinear delay SIR epidemic using the double epidemic hypothesis is modeled. In the model, a system of impulsive functional differential equations is studied and the sufficient conditions for the global attractivity of the semi-trivial periodic solution are drawn. By use of new computational techniques for impulsive differential equations with delay, we prove that the system is permanent under appropriate conditions. The results show that time delay, pulse vaccination, and nonlinear incidence have significant effects on the dynamics behaviors of the model. The conditions for the control of the infection caused by viruses A and B are given.

Keywords Double epidemic hypothesis · Permanence · Nonlinear incidence · Time delay · SIR epidemic model

1 Introduction

In real world, there are two epidemics, one epidemic caused by virus A and another epidemic caused by virus B. The most likely origin of virus A is a mutation or recombination event from virus B [1]. Also, it has been observed by scientists about possibilities where viruses A and B are of different origins but would cause an overlapping immune response of the host. Both epidemics spread in parallel, and the epidemic caused by virus B, which is rather innocuous, protects against epidemic A. The SIR infections disease model is an important biologic model and has been studied by many authors [2–9]. It is well known that one of the strategies to control infectious diseases is vaccination. Then a number of epidemic models in ecology can be formulated as dynamical systems of differential equations with vaccination [10–17]. Systems with sudden perturbations lead to impulsive differential equations, which have been studied intensively and systematically in [18–23]. It is very important that one investigates under what conditions a given agent can invade partially vaccinated population, i.e., how large a fraction of the population do we have to keep vaccinated in order to prevent the agent from establishing.
Pulse vaccination seems more reasonable than traditional continuous constant vaccination in real world. Pulse vaccination strategy (PVS) [10–14] consists of periodical repetitions of impulsive vaccinations in a population, on all the age cohorts, which is different from the traditional constant vaccination.

A model for the spread of an infectious disease (involving only susceptibles and infective individuals) transmitted by a vector (e.g., mosquitoes) after an incubation time was proposed by Cooke [24]. This is called the phenomena of “time delay.” Many authors have directly incorporated time delays in modeling equations, and, as a result, the models take the form of delay differential equations [2–9, 25–30].

In recent years, the research on delay SIR epidemic models with impulsive perturbations is a relevant, but not totally developed, subject in mathematical biology. See [10, 30] and the references therein. However, this is an interesting problem in mathematical biology. Since an adopted incidence form like the 

\[ \beta e^{-\mu t} S(t) I(t - \omega), \]

which represents an SIR model with epidemics spread via a vector with an incubation time \( \omega \). Here, \( \mu \) is birth and death rate, and \( r \) is a daily recovery rate. Of course, \( \beta, \mu, r \in R_+ \).

Levin et al. have adopted a nonlinear incidence rate form like

\[ \beta e^{-\mu t} S(t) I(t - \omega). \] (2.2)

Let \( I_A \) be the total population of infectives with virus A at time \( t \), and \( I_B \) be the total population of infectives with virus B at time \( t \). Both epidemics spread in parallel, and the epidemic caused by virus B, which is rather innocuous, protects against epidemic caused by virus A. When pulse and the force of infection (2.2) are introduced in (2.1), we have

\[ S'(t) = \mu - \beta_1 e^{-\mu t} S(t) I_A(t - \omega_1) - \beta_2 e^{-\mu t} S(t) I_B(t - \omega_2) - \mu S(t), \]

\[ I'_A(t) = \beta_1 e^{-\mu t} S(t) I_A(t - \omega_1) - \mu I_A(t) - r_1 I_A(t), \]

\[ I'_B(t) = \beta_2 e^{-\mu t} S(t) I_B(t - \omega_2) - \mu I_B(t) - r_2 I_B(t), \]

\[ R'(t) = r_1 I_A(t) + r_2 I_B(t) - \mu R(t), \]

\[ S(t^+) = (1 + \delta) S(t), \]

\[ I_A(t^+) = I_A(t), \]

\[ I_B(t^+) = I_B(t), \]

\[ R(t^+) = R(t) + \delta S(t), \]

where \( \beta_i, r_i, \omega_i \in R_+ \) \( (i = 1, 2) \), \( q_i \in N \) \( (i = 1, 2) \) is positive integer, and \( \delta (0 \leq \delta < 1) \) is the proportion of those vaccinated successfully to all of the susceptible. The \( \beta_i e^{-\mu t} S(t) I(t - \omega_i) \) \( (i = 1, 2) \) term exhibits more clearly the death of the exposed population within finite incubation times (with \( \omega_i \)) than the \( \beta_i S(t) I(t - \omega_i) \) \( (i = 1, 2) \) term. For both systems (2.1) and (2.3), the total population size \( N(t) = S(t) + I_A(t) + I_B(t) + R(t) \) satisfies \( N'(t) = \mu (1 - N(t)), \) and \( N(t) \to 1 \) as \( t \to \infty \). System (2.3)
can be regarded as a model with constant total population. Hence it is sufficient to consider the first three equations in (2.3) with respect to $\Omega = \{ (S, I_A, I_B) \in R^3_+ | S + I_A + I_B \leq 1 \}$.

The initial condition of (2.3) is given as

$$S(\theta) = \phi_I(\theta), \quad I_A(\theta) = \phi_2(\theta), \quad I_B(\theta) = \phi_3(\theta), \quad R(\theta) = \phi_4(\theta) \quad (-\omega \leq \theta \leq 0),$$

where $\omega = \max\{\omega_1, \omega_2\}$ and $\phi = (\phi_1, \phi_2, \phi_3, \phi_4)^T \in C$ are such that $\phi_i(\theta) \geq 0 (-\omega \leq \theta \leq 0, \ i = 1, 2, 3, 4).$ $C$ denotes the Banach space $C([-\omega, 0], R^4)$ of continuous functions mapping the interval $[-\omega, 0]$ into $R^4$. For a biological meaning, we further assume that $\phi_i(0) > 0$ for $i = 1, 2, 3, 4$.

Note that the variable $R$ does not appear in the first three equations of system (2.3); hence we only need to consider the following subsystem of (2.3):

$$S'(t) = \mu - \beta_1 e^{-t(\omega_1)} S(t) I_A(t - \omega_1),$$

$$I_A'(t) = -\beta_2 e^{-t(\omega_2)} S(t) I_B(t - \omega_2) - \mu I_A(t),$$

$$I_B'(t) = \beta_1 e^{-t(\omega_1)} S(t) I_A(t - \omega_1) - \mu I_B(t) - r I_A(t),$$

$$S(t^+) = (1 - \delta) S(t),$$

$$I_A(t^+) = I_A(t),$$

$$I_B(t^+) = I_B(t),$$

where $\omega_1 = \max\{\omega_1, \omega_2\}$ and $\phi = (\phi_1, \phi_2, \phi_3, \phi_4)^T \in C$ are such that $\phi_i(\theta) \geq 0 (-\omega \leq \theta \leq 0, \ i = 1, 2, 3, 4).$ $C$ denotes the Banach space $C([-\omega, 0], R^4)$ of continuous functions mapping the interval $[-\omega, 0]$ into $R^4$. For a biological meaning, we further assume that $\phi_i(0) > 0$ for $i = 1, 2, 3, 4$.

Before starting our theorem, we give the following lemma.

**Lemma 2.1** (See [33]) Consider the following impulse differential inequalities:

$$w'(t) \leq (\geq) p(t) w(t) + q(t), \quad t \neq t_k,$$

$$w(t_k^+) \leq (\geq) d_kw(t_k) + b_k, \quad t = t_k, \ k \in N,$$

where $p(t), q(t) \in C[R_+, R]$, and $d_k \geq 0$ and $b_k$ are constants. Assume that:

(A0) the sequence $\{t_k\}$ satisfies $0 \leq t_0 < t_1 < t_2 < \cdots$ with $\lim_{t \to \infty} t_k = \infty$;

(A1) $w \in PC[R_+, R]$, and $w(t)$ is left continuous at $t_k, \ k \in N$.

Then

$$w(t) \leq (\geq) w(t_0) \prod_{t_0 < t_k < t} \exp \left( \int_{t_k}^{t} p(s) ds \right) + \sum_{t_0 < t_k < t} \left( \prod_{t_k < t_j < t} d_j \exp \left( \int_{t_k}^{t_j} p(s) ds \right) \right) b_k + \int_{t_0}^{t} \prod_{s < t_k < t} d_k \exp \left( \int_{s}^{t} p(\theta) d\theta \right) q(s) ds,$$

$t \geq t_0$.

**Lemma 2.2** (Kuang [34]) Consider the delay differential equation

$$\frac{dx(t)}{dt} = ax(t - \omega) - bx(t),$$

where $a, b, \omega$ are all positive constants, and $x(t) > 0$ for $t \in [-\omega, 0]$:

(i) If $a < b$, then $\lim_{t \to \infty} x(t) = 0$.

(ii) If $a > b$, then $\lim_{t \to \infty} x(t) = +\infty$.

Let $R_+ = [0, +\infty)$, $R^3_+ = \{ X \in R^3 : X \geq 0, X = (S, I_A, I_B) \}, \Omega = \text{int} R^3_+$, and $N$ be the set of nonnegative integers. Denote by $f = (f_1, f_2, f_3)^T$ the map defined by the right-hand side of the anterior three equations of system (2.4). Let $V : R^3_+ \times R^3_+ \to R^3_+$. Then $V$ is said to belong to class $V_0$ if:

(i) $V$ is continuous in $(n\tau, (n + 1)\tau] \times R^3_+$, and for all $X \in R^3_+$ and $n \in N$, $\lim_{(t,y) \to ((n\tau)^+,X)} V(t,y) = V((n\tau)^+,X)$ exists.

(ii) $V$ is locally Lipschitzian in $X$.

**Lemma 2.3** [33] Let $V : R^3_+ \times R^3_+ \to R^3_+$ and $V \in V_0$. Assume that

$$D^+ V(t, z(t)) \leq (\geq) g(t, V(t, z(t))), \quad t \neq n\tau,$$

$$V(t, z(t)^+) \leq (\geq) \Psi_n(V(t, z(t))), \quad t = n\tau,$$

where $g : R^3_+ \times R_+ \to R$ is continuous in $(n\tau, (n + 1)\tau] \times R^3_+$, for all $x \in R^3_+$ and $n \in N$, $\lim_{(t,y) \to ((n\tau)^+,x)} g(t,y) = g((n\tau)^+,x)$ exists; and $\Psi_n : R^3_+ \to R^3_+$ is nondecreasing. Let $r(t) = r(t, 0, u_0)$ be the maximal (minimal) solution of the scalar impulsive differential equation

$$u'(t) = g(t, u), \quad t \neq n\tau,$$

$$u(t^+) = \Psi_n(u(t)), \quad t = n\tau.$$
existing on \([0, \infty)\). Then \(V(0^+, z_0) \leq (\geq) u_0\) implies that \(V(t, z(t)) \leq (\geq) r(t), \ t \geq 0\), where \(z(t) = z(t, 0, z_0)\) is any solution of (2.4) existing on \([0, \infty)\).

3 Main results

**Definition 3.1** System (2.4) is said to be permanent if there exists a compact region \(\Omega \subset \text{int}R_+^3\) such that every solution of system (2.4) with initial conditions \(\phi\) eventually enters and remains in region \(\Omega\).

We begin the analysis of (2.4) by first demonstrating the existence of an infection-free solution in which infectious individuals are entirely absent from the population permanently, i.e.,

\[
I_A(t) = 0, \quad I_B(t) = 0, \quad t \geq 0.
\]

(3.1)

Assuming (3.1), we know that the growth of the susceptible in the time interval \(nt < t \leq (n + 1)t\) and give some basic properties of the subsystem of (2.4)

\[
\begin{aligned}
S'(t) &= -\mu S(t) + \mu, \quad t \neq n\tau, \ n \in N, \\
S(t^+) &= (1 - \delta)S(t), \quad t = n\tau, \ n \in N.
\end{aligned}
\]

(3.2)

Solving (3.2) between pulses and using the discrete dynamical system determined by a fixed-point theory in Poincaré map yields

\[
\begin{aligned}
\hat{S}(t) &= 1 - \frac{\delta}{1 - (1 - \delta)e^{-\mu t}} e^{-\mu (n\tau - t)}, \\
& \quad t \in (n\tau, (n + 1)\tau], \ n \in N, \\
\hat{S}(0^+) &= \hat{S}(n\tau^+) = 1 - \frac{\delta}{1 - (1 - \delta)e^{-\mu}},
\end{aligned}
\]

(3.3)

which is a unique globally asymptotically stable positive periodic solution of system (3.2).

Since the solution of (3.2) is

\[
\begin{aligned}
S(t) &= (S(0^+) - (1 - \frac{\delta}{1 - (1 - \delta)e^{-\mu t}}))e^{-\mu t} \\
& \quad + \hat{S}(t), \ t \in (nT, (n + 1)T], \\
\hat{S}(0^+) &= 1 - \frac{\delta}{1 - (1 - \delta)e^{-\mu}},
\end{aligned}
\]

(3.4)

we have the following Lemma 3.1.

**Lemma 3.1** System (3.2) has a unique positive periodic solution \(\hat{S}(t)\), that is, system (2.4) has an infection-free periodic solution \((\hat{S}(t), 0, 0)\) for \(t \in (n\tau, (n + 1)\tau]\), \(n \in N\), and for any solution \((S(t), I_A(t), I_B(t))\) of (2.4) with positive initial conditions, we have \(S(t) \to \hat{S}(t)\) as \(t \to \infty\). Denote

\[
\begin{aligned}
R_1 &= \frac{\beta_1 e^{-\mu_0}}{r_1 + \mu} \left(\frac{e^{\mu_1} - 1}{e^{\mu_1} - 1 + \delta}\right)^{q_1}, \\
R_2 &= \frac{\beta_2 e^{-\mu_0}}{r_2 + \mu} \left(\frac{e^{\mu_1} - 1}{e^{\mu_1} - 1 + \delta}\right)^{q_2}.
\end{aligned}
\]

(3.5)

**Theorem 3.1** If \(R_1 = \max\{R_1, R_2\} < 1\), then the infection-free periodic solution \((\hat{S}(t), 0, 0)\) of system (2.4) is globally attractive.

**Proof** Let \((S(t), I_A(t), I_B(t))\) be any solution of system (2.4). Since \(R_1 < 1\), we can easily see that

\[
\begin{aligned}
\beta_1 e^{-\mu_0} \left(\frac{e^{\mu_1} - 1}{e^{\mu_1} - 1 + \delta}\right)^{q_1} < r_1 + \mu, \\
\beta_2 e^{-\mu_0} \left(\frac{e^{\mu_1} - 1}{e^{\mu_1} - 1 + \delta}\right)^{q_2} < r_2 + \mu.
\end{aligned}
\]

(3.6)

Then we can choose an \(\epsilon > 0\) small enough such that

\[
\begin{aligned}
\beta_1 e^{-\mu_0} \left(\frac{e^{\mu_1} - 1}{e^{\mu_1} - 1 + \delta} + \epsilon\right)^{q_1} < r_1 + \mu, \\
\beta_2 e^{-\mu_0} \left(\frac{e^{\mu_1} - 1}{e^{\mu_1} - 1 + \delta} + \epsilon\right)^{q_2} < r_2 + \mu.
\end{aligned}
\]

Note that \(S'(t) \leq \mu - \mu S(t)\). Then we consider the impulse differential inequalities

\[
\begin{aligned}
\hat{S}'(t) &\leq \mu - \mu S(t), \quad t \neq n\tau, \ n \in N \\
\hat{S}(t^+) &= (1 - \delta)\hat{S}(t), \quad t = n\tau, \ n \in N.
\end{aligned}
\]

Using Lemma 2.1, we have

\[
\begin{aligned}
S(t) &\leq S(n\tau^+) \prod_{n\tau < t < t} \left(1 - \delta\right) \exp\left(\int_{n\tau}^{t} -\mu ds\right) \\
& \quad + \mu \int_{n\tau}^{t} \prod_{s < t} \left(1 - \delta\right) \exp\left(\int_{s}^{t} -\mu d\theta\right) ds \\
& = \Delta_1 + \Delta_2,
\end{aligned}
\]

where

\[
\begin{aligned}
\Delta_1 &= S(n\tau^+) \left(\prod_{n<T \leq s < t/\tau} \left(1 - \delta\right) \times \exp\left(\int_{n\tau}^{t/\tau} -\mu d\tau d\xi\right)\right) \\
& = S(n\tau^+) (1 - \delta)^{(t/\tau)} e^{-\mu(t-n\tau)},
\end{aligned}
\]
\[
\Delta_2 = e^{-\mu t} \int_{n_1\tau}^{t} \prod_{\xi < n\tau < t} (1 - \delta) e^{\mu_{\xi} t} d\mu_{\xi}
\]
\[
= e^{-\mu t} \int_{n_1}^{t/\tau} \prod_{\xi < n\tau < t/\tau} (1 - \delta) e^{\mu_{\xi} t} d\mu_{\xi}
\]
\[
= e^{-\mu t} \left[ \int_{n_1}^{n_1+1} \prod_{\xi < n\tau < t/\tau} (1 - \delta) e^{\mu_{\xi} t} d\mu_{\xi} + \cdots + \int_{n_1+1}^{n_1+2} \prod_{\xi < n\tau < t/\tau} (1 - \delta) e^{\mu_{\xi} t} d\mu_{\xi} \right] + \int_{n_1}^{t/\tau} \prod_{\xi < n\tau < t/\tau} (1 - \delta) e^{\mu_{\xi} t} d\mu_{\xi} + \int_{n_1}^{t/\tau} \prod_{\xi < n\tau < t/\tau} (1 - \delta) e^{\mu_{\xi} t} d\mu_{\xi}
\]
\[
= e^{-\mu t} \left[ (1 - \delta) [t/\tau - n_1] e^{\mu t} + (1 - \delta) [t/\tau - n_1] e^{\mu t} + \cdots + (1 - \delta) e^{\mu t} - e^{\mu t} [t/\tau] \right]
\]
\[
= e^{-\mu t} \left[ \left( (1 - \delta) [t/\tau - n_1] e^{\mu t} - e^{\mu t} [t/\tau] \right) \right]
\]
\[
\text{Thus,}
\]
\[
S(t) \leq \Delta_1 + \Delta_2 = \gamma(t) + 1 - \frac{\delta e^{\mu t} [t/\tau + 1 - t/\tau]}{e^{\mu t} - 1 + \delta},
\]
where
\[
\gamma(t) = e^{-\mu t} \left[ S(n_1\tau^+) (1 - \delta) [t/\tau] e^{n_1\tau^\mu} - (1 - \delta) [t/\tau - n_1] e^{n_1\tau^\mu} \right] - \frac{e^{-\mu t} S(n_1\tau^+) (1 - \delta) [t/\tau] e^{n_1\tau^\mu}}{e^{\mu t} - 1 + \delta}
\]
\[
\leq e^{-\mu t} S(n_1\tau^+) (1 - \delta) [t/\tau] e^{n_1\tau^\mu} - e^{-\mu t} S(n_1\tau^+) (1 - \delta) [t/\tau + 1] e^{n_1\tau^\mu} < S(n_1\tau) e^{n_1\tau^\mu} e^{-\mu t}.
\]
Then
\[
S(t) \leq S(n_1\tau) e^{n_1\tau^\mu} e^{-\mu t} + 1 - \frac{\delta}{e^{\mu t} - 1 + \delta},
\]
which implies
\[
\limsup_{t \to \infty} S(t) \leq 1 - \frac{\delta}{e^{\mu t} - 1 + \delta},
\]
so there exist a positive integer \( n_1 \) and an arbitrarily small positive constant \( \varepsilon \) such that for all \( t \geq n_1\tau \),
\[
S(t) \leq 1 - \frac{\delta}{e^{\mu t} - 1 + \delta} + \varepsilon \equiv S^\Delta.
\]
From (3.7) and from the second and third equations of (2.4), we get that
\[
\begin{align*}
I'_A(t) & \leq \beta_1 e^{-\mu_1 t} (S^\Delta)^{q_1} I_A(t - \omega_1) \\
& - (r_1 + \mu) I_A(t), \quad t > n_1\tau + \omega_1,
\end{align*}
\]
\[
\begin{align*}
S^\Delta(t) & \leq \beta_2 e^{-\mu_2 t} (S^\Delta)^{q_2} I_B(t - \omega_2) \\
& - (r_2 + \mu) I_B(t), \quad t > n_1\tau + \omega_2.
\end{align*}
\]
Consider the following comparison equation:
\[
\begin{align*}
\zeta'_A(t) & = \beta_1 e^{-\mu_1 t} (S^\Delta)^{q_1} \zeta_A(t - \omega_1) \\
& - (r_1 + \mu) \zeta_A(t), \quad t > n_1\tau + \omega_1,
\end{align*}
\]
\[
\begin{align*}
\zeta'_B(t) & = \beta_2 e^{-\mu_2 t} (S^\Delta)^{q_2} \zeta_B(t - \omega_2) \\
& - (r_2 + \mu) \zeta_B(t), \quad t > n_1\tau + \omega_2.
\end{align*}
\]
From (3.6) we have that \( \beta_i e^{-\mu_{i_0}} (S^\Delta)^{q_i} < r_i + \mu, \ i = 1, 2 \). By Lemma 2.2 we obtain that \( \lim_{t \to \infty} S_A(t) = 0 \) and \( \lim_{t \to \infty} S_B(t) = 0 \). Since \( I_A(s) = z_A(s) = \phi_2(s) > 0 \) and \( I_B(s) = z_B(s) = \phi_3(s) > 0 \) for all \( s \in [-\omega, 0] \), by the comparison theorem for differential equations and the nonnegativity of solution (with \( I_A(t) \geq 0 \) and \( I_B(t) \geq 0 \), we have that \( I_A(t) \to 0 \) and \( I_B(t) \to 0 \) as \( t \to \infty \). Without loss of generality, for any sufficiently small \( 0 < \varepsilon_i < \mu e^{\mu_{i_0}} / \beta_i, \ i = 1, 2 \), we may assume that \( 0 < I_A(t) < \varepsilon_1 \) and \( 0 < I_B(t) < \varepsilon_2 \) for all \( t \geq 0 \). From the first equation of system (2.4) we have
\[
S'(t) \geq \mu - \left( \mu + \beta_1 e^{-\mu_1 t} \varepsilon_1 + \beta_2 e^{-\mu_2 t} \varepsilon_2 \right) S(t).
\]
Then we have \( z_1(t) \leq S(t) \), where \( z_1(t) \) is the solution of the following system (3.10) with initial value \( z_1(0^+) = S(0^+) \), and \( z_1(t) \) is the unique positive periodic solution of
\[
\begin{align*}
\Bar{z}_1'(t) & = \mu - \left( \mu + \beta_1 e^{-\mu_1 t} \varepsilon_1 + \beta_2 e^{-\mu_2 t} \varepsilon_2 \right) \Bar{z}_1(t), \quad t \neq n\tau, \ n \in N, \\
\Bar{z}_1(t^+) & = (1 - \delta) \Bar{z}_1(t), \quad t = n\tau, \ n \in N, \quad \Bar{z}_1(0^+) = S(0^+).
\end{align*}
\]
From (3.10) we have that, for $n\tau < t \leq (n + 1)\tau$,
\[
\widetilde{z}_1(t) = \frac{\mu}{\mu + \beta_1 e^{-\mu \omega_1} e_1 + \beta_2 e^{-\mu \omega_2} e_2} \times \left[ 1 - \frac{\delta e^{-(\mu + \beta_1 e^{-\mu \omega_1} e_1 + \beta_2 e^{-\mu \omega_2} e_2) \tau - nt}}{1 - (1 - \delta) e^{-(\mu + \beta_1 e^{-\mu \omega_1} e_1 + \beta_2 e^{-\mu \omega_2} e_2) \tau}} \right].
\]
Hence, for any $\varepsilon_3 > 0$, there exists an integer $n_2 > n_1$ such that
\[
S(t) > \widetilde{z}_1(t) - \varepsilon_3 \quad \text{for all } t > n_2 \tau. \tag{3.11}
\]
On the other hand, from the first equation of (2.4) it follows that $S'(t) \leq \mu - \mu S(t)$. By Lemma 2.3, we have $S(t) \leq \widetilde{S}(t)$. Then, for any above $\varepsilon_3 > 0$, there exists an integer $n_3 > n_2$ such that
\[
S(t) < \widetilde{S}(t) + \varepsilon_3 \quad \text{for } t > n_3 \tau. \tag{3.12}
\]
Let $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$. Then from (3.11) and (3.12) it follows that
\[
\widetilde{S}(t) - \varepsilon_3 < S(t) < \widetilde{S}(t) + \varepsilon_3
\]
for $t$ large enough, which implies that $S(t) \to \widetilde{S}(t)$ as $t \to \infty$. The proof is completed. □

**Corollary 3.1** (i) If $\beta_i e^{-\mu \omega_i} \leq r_i + \mu$, $i = 1, 2$, then the infection-free periodic solution $(S(t), 0, 0)$ is globally attractive.

(ii) If $\beta_i e^{-\mu \omega_i} > r_i + \mu$ ($i = 1, 2$) and $\delta > \delta^*$, or $\tau < \tau_*$, or $\omega > \omega^*$, or $q > q^*$, then the infection-free periodic solution $(S(t), 0, 0)$ is globally attractive, where the critical values $\delta^*$, $\tau_*$, $\omega^*$, and $q^*$ are listed as follows:

\[
\delta^* = \max \left\{ (e^{\mu \tau} - 1) \left( \frac{q_i^1 B_i e^{-\mu \omega_i}}{r_i + \mu} - 1 \right), \right.
\]
\[
\left. \frac{(e^{\mu \tau} - 1) \left( \frac{q_2^1 B_2 e^{-\mu \omega_2}}{r_2 + \mu} - 1 \right)}{e^{\mu \tau} - 1} \right\},
\]
\[
\tau_* = \min \left\{ \frac{1}{\mu} \ln \left( 1 + \frac{\delta}{q_i^1 B_i e^{-\mu \omega_i} / r_i + \mu} \right), \right.
\]
\[
\left. \frac{1}{\mu} \ln \left( 1 + \frac{\delta}{q_2^1 B_2 e^{-\mu \omega_2} / r_2 + \mu} \right) \right\},
\]
\[
\omega^* = \max \left\{ \frac{1}{\mu} \ln \left[ \frac{\beta_1}{r_1 + \mu} \left( e^{\mu \tau} - 1 - \delta \right) q_1 \right], \right.
\]
\[
\left. \frac{1}{\mu} \ln \left[ \frac{\beta_2}{r_2 + \mu} \left( e^{\mu \tau} - 1 - \delta \right) q_2 \right] \right\},
\]
\[
q^* = \max \left\{ \log \frac{e^{\mu \tau - 1}}{e^{\mu \tau - 1 + \delta}} \frac{r_1 + \mu}{\beta_1 e^{-\mu \omega_1}}, \log \frac{e^{\mu \tau - 1}}{e^{\mu \tau - 1 + \delta}} \frac{r_2 + \mu}{\beta_2 e^{-\mu \omega_2}} \right\}.
\]

From the proof of this section, we can easily obtain the following **Corollary 3.2**.

**Corollary 3.2**

(i) If $R_1 < 1$ and $R_2 > 1$, then the epidemic $I_A$ dies out, and the epidemic $I_B$ is permanent.

(ii) If $R_1 > 1$ and $R_2 < 1$, then the epidemic $I_A$ is permanent, and the epidemic $I_B$ dies out.

**Denote**

\[
R_3 = \frac{\beta_1 e^{-\mu \omega_1}}{r_1 + \mu} \times \left( \frac{\mu - \beta_2 e^{-\mu \omega_2}}{\mu} \cdot \frac{(1 - \delta)(e^{\mu \tau} - 1)}{e^{\mu \tau} - 1 + \delta} \right) q_1,
\]
\[
m_1^* = \frac{(\mu - \beta_2 e^{-\mu \omega_2})(q_i \sqrt{R_i} - 1)}{\beta_1 \mu \sqrt{R_1 e^{-\mu \omega_1}}},
\]
\[
R_4 = \frac{\beta_2 e^{-\mu \omega_2}}{r_2 + \mu} \times \left( \frac{\mu - \beta_1 e^{-\mu \omega_1}}{\mu} \cdot \frac{(1 - \delta)(e^{\mu \tau} - 1)}{e^{\mu \tau} - 1 + \delta} \right) q_2,
\]
\[
m_2^* = \frac{(\mu - \beta_1 e^{-\mu \omega_1})(q_2 \sqrt{R_4} - 1)}{\beta_2 \mu \sqrt{R_2 e^{-\mu \omega_2}}}.
\]
\[
S_1^* = q_1 \frac{r_1 + \mu}{\beta_1 e^{-\mu \omega_1}}, \quad S_2^* = q_2 \frac{r_2 + \mu}{\beta_2 e^{-\mu \omega_2}}.
\]

**Theorem 3.2** If $\mathcal{R}_2 = \min\{R_3, R_4\} > 1$, then there exist constants $\gamma_i : 0 < \gamma_i < 1$, $i = 1, 2$, such that

\[
\lim \inf_{i \to \infty} I_A(t) \geq \min \left\{ \frac{\gamma_i m_1^*}{2}, \gamma_i m_1^* e^{-(r_1 + \mu) \omega_1} \right\} \equiv m_1,
\]
\[
\lim \inf_{i \to \infty} I_B(t) \geq \min \left\{ \frac{\gamma_2 m_2^*}{2}, \gamma_2 m_2^* e^{-(r_2 + \mu) \omega_2} \right\} \equiv m_2.
\]

**Proof** Suppose that $(S(t), I_A(t), I_B(t))$ is any positive solution of system (2.4). Since $\mathcal{R}_2 = \min\{R_3, R_4\} > 1$,
we have $m_1^* > 0$ and $m_2^* > 0$. In the following, we claim that for any $\gamma_i : 0 < \gamma_i < 1$, $i = 1, 2$, we have $I_A(t) > \gamma_1 m_1^*$ and $I_B(t) > \gamma_2 m_2^*$ for $t$ large enough. For convenience, we will show this in two steps.

Step I. We claim that there exist $t_1, t_2 \in (0, \infty)$ such that $I_A(t_1) \geq \gamma_1 m_1^*$ and $I_B(t_2) \geq \gamma_2 m_2^*$ for all $t > 0$. Otherwise, there will be three cases:

(i) There exists $t_2 > 0$ such that $I_B(t_2) \geq \gamma_2 m_2^*$, but $I_A(t) < \gamma_1 m_1^*$ for all $t > t_2$; 
(ii) There exists $t_1 > 0$ such that $I_A(t_1) \geq \gamma_1 m_1^*$, but $I_B(t) < \gamma_2 m_2^*$ for all $t > t_1$; 
(iii) $I_A(t) < \gamma_1 m_1^*$ and $I_B(t) < \gamma_2 m_2^*$ for all $t > 0$.

We first consider case (i). According to the above assumption, we get

\[
\begin{aligned}
S'(t) &\geq \mu - \beta_1 \gamma_1 m_1^* e^{-\mu_1 t} - \beta_2 e^{-\mu_2 t} - \mu S(t), \\
S^+(t) &= (1 - \delta)S(t), \quad t \neq n\tau, \quad n \in \mathbb{N}, \quad \tau = 1.
\end{aligned}
\]

Let $\eta = \frac{\mu - \beta_1 \gamma_1 m_1^* e^{-\mu_1 t} - \beta_2 e^{-\mu_2 t} - \mu S(t)}{\mu} > 0$. By Lemma 2.1, there exists $\varepsilon > 0$ small enough such that $S(t) > \eta - \varepsilon$. Then

\[
I_A'(t) \geq \beta_1 e^{-\mu_1 t} (\eta - \varepsilon)^{q_1} I_A(t - \omega_1) - (\mu + r_1) I_A(t).
\]

Since $R_1 > 1$, it is clear that

\[
\beta_1 e^{-\mu_1 t} (\eta - \varepsilon)^{q_1} > \mu + r_1.
\]

Using Lemma 2.2 along with (3.13) and (3.14), we have $I_A(t) \to \infty$ as $t \to \infty$, which is a contradiction.

Similarly, we can prove that $I_B(t) \to \infty$ as $t \to \infty$ in case (ii), which also is a contradiction.

Last, we consider case (iii). For $t \geq 0$, we define the differentiable function $V(t)$ by

\[
V(t) = I_A(t) + I_B(t) + \beta_1 e^{-\mu_1 t} (S_1^{q_1})^q_1 \int_{t - \omega_1}^t I_A(\theta) d\theta + \beta_2 e^{-\mu_2 t} (S_2^{q_2})^q_2 \int_{t - \omega_2}^t I_B(\theta) d\theta.
\]

Then, the derivative of $V(t)$ satisfies

\[
V'(t) = I_A'(t) + \beta_1 e^{-\mu_1 t} (S_1^{q_1})^q_1 I_A(t) - \beta_1 e^{-\mu_1 t} (S_1^{q_1})^q_1 I_A(t - \omega_1) + I_B'(t) + \beta_2 e^{-\mu_2 t} (S_2^{q_2})^q_2 I_B(t - \omega_2)
\]

\[
\begin{aligned}
&= \beta_1 e^{-\mu_1 t} (S_1^{q_1})^q_1 I_A(t - \omega_1) + \beta_2 e^{-\mu_2 t} (S_2^{q_2})^q_2 I_B(t - \omega_2) \\
&\geq 0.
\end{aligned}
\]

Since $R_2 > 1$, we have $m_1^* > 0$ and $m_2^* > 0$. For any $\gamma_i : 0 < \gamma_i < 1$, $i = 1, 2$, we have that

\[
S_1^* < \frac{\mu - \beta_1 \gamma_1 m_1^* e^{-\mu_1 t} - \beta_2 e^{-\mu_2 t}}{\mu} \times \frac{(1 - \delta)(e^\mu t - 1)}{e^\mu t - 1 + \delta},
\]

\[
S_2^* < \frac{\mu - \beta_2 \gamma_2 m_2^* e^{-\mu_2 t} - \beta_1 e^{-\mu_1 t}}{\mu} \times \frac{(1 - \delta)(e^\mu t - 1)}{e^\mu t - 1 + \delta}.
\]

Then there exist two positive constants $\varepsilon_1$ and $\varepsilon_2$ small enough such that

\[
S_1^* < \frac{\mu - \beta_1 \gamma_1 m_1^* e^{-\mu_1 t} - \beta_2 e^{-\mu_2 t}}{\mu} \times \frac{(1 - \delta)(e^\mu t - 1)}{e^\mu t - 1 + \delta} - \varepsilon_1 \triangleq S_{\Delta 1},
\]

\[
S_2^* < \frac{\mu - \beta_2 \gamma_2 m_2^* e^{-\mu_2 t} - \beta_1 e^{-\mu_1 t}}{\mu} \times \frac{(1 - \delta)(e^\mu t - 1)}{e^\mu t - 1 + \delta} - \varepsilon_2 \triangleq S_{\Delta 2}.
\]
Under the assumption of case (iii), from the first and fourth equations of system (2.4) we have

\[
\begin{align*}
S'(t) &\geq \mu - \beta_1 \gamma_1 m_1^* e^{-\mu_0 t} - \beta_2 e^{-\mu_0 t} - \mu S(t), \\
S(t^+) &= (1 - \delta) S(t), \quad t = n \pi, \quad n \in \mathbb{N}, \\
S(t^+) &= (1 - \delta) S(t), \\
T &\geq n \pi, \quad n \in \mathbb{N},
\end{align*}
\]

and

\[
\begin{align*}
S'(t) &\geq - \beta_2 \gamma_2 m_2^* e^{-\mu_0 t} - \beta_1 e^{-\mu_0 t} - \mu S(t), \\
S(t^+) &= (1 - \delta) S(t), \quad t = n \pi, \quad n \in \mathbb{N}.
\end{align*}
\]

Using Lemma 2.1, we know that there exists \( T \geq t_0 + \omega \) such that, for \( t \geq T \),

\[
\begin{align*}
S(t) > \frac{\mu}{\beta_1 \gamma_1 m_1^* e^{-\mu_0 t} - \beta_2 e^{-\mu_0 t}} \\
\times (1 - \delta) (e^{\mu t} - 1) + \varepsilon_1 = S_{\Delta_1}, \\
S(t) > \frac{\mu}{\beta_2 \gamma_2 m_2^* e^{-\mu_0 t} - \beta_1 e^{-\mu_0 t}} \\
\times (1 - \delta) (e^{\mu t} - 1) + \varepsilon_2 = S_{\Delta_2}.
\end{align*}
\]

Therefore, for \( t \geq T \), inserting (3.17) and (3.18) into (3.15), we have

\[
V(t) > q_1 \beta_1 e^{-\mu_0 t} (S_{\Delta_1}^{q_1 - 1} - S_{\Delta_1}) I_A(t) + q_2 \beta_2 e^{-\mu_0 t} (S_{\Delta_2}^{q_2 - 1} - S_{\Delta_2}) I_B(t),
\]

which is a contradiction to \( \dot{I}_A(T_1 + \omega + T_2) \leq 0 \). Similarly, we can obtain that it also is a contradiction to \( \dot{I}_B(T_1 + \omega + T_2) \leq 0 \). Hence, we get that \( I_A(t) \geq I_A^* > 0 \) and \( I_B(t) \geq I_B^* > 0 \) for all \( t \geq T_1 \). Therefore, for all \( t \geq T_1 + \omega \),

\[
V(t) > q_1 \beta_1 e^{-\mu_0 t} (S_{\Delta_1}^{q_1 - 1} - S_{\Delta_1}) I_A^* + q_2 \beta_2 e^{-\mu_0 t} (S_{\Delta_2}^{q_2 - 1} - S_{\Delta_2}) I_B^* > 0,
\]

which implies \( V(t) \to +\infty \) as \( t \to +\infty \). This is a contradiction to \( V(t) \leq 2 + \beta_1 \omega e^{-\mu_0 t} (S_{\Delta_1}^{q_1} I_A + 2 \beta_2 e^{-\mu_0 t} (S_{\Delta_2}^{q_2} I_B) \) for large enough. Hence, the claim is proved.

From the above three cases we conclude that there exist \( t_1, t_2 \) such that \( I_A(t_1) \geq \gamma_1 m_1^* > 0 \) and \( I_B(t_2) \geq \gamma_2 m_2^* > 0 \).

Step II. In the rest, we are left to consider two cases:

(i) \( I_A(t) \geq \gamma_1 m_1^* \) and \( I_B(t) \geq \gamma_2 m_2^* \) for all \( t \) large enough;

(ii) \( I_A(t) \) oscillates about \( \gamma_1 m_1^* \), or \( I_B(t) \) oscillates about \( \gamma_2 m_2^* \) for all large \( t \).

Case (i). Our aim is obtained. Clearly, we only need to consider Case (ii).

Case (ii). If \( I_A(t) \) oscillates about \( \gamma_1 m_1^* \), then we will show that \( I_A(t) \geq m_1 \) for all large \( t \), where

\[
m_1 = \min \left\{ \frac{\gamma_1 m_1^*}{2}, \gamma_1 m_1^* e^{-(r_1 + \mu) \omega} \right\}.
\]

There exist two positive constants \( \tilde{\iota}, \psi \) such that

\[
I_A(\tilde{\iota}) = I_A(\tilde{\iota} + \psi) = \gamma_1 m_1^*,
\]

and

\[
I_A(t) < \gamma_1 m_1^* \quad \text{for} \quad \tilde{\iota} < t < \tilde{\iota} + \psi.
\]
Since $I_A(t)$ is continuous and ultimately bounded and is not affected by impulses, we conclude that $I_A(t)$ is uniformly continuous. Hence there exists a constant $T_3 \,(0 < T_3 < \omega_1)$ independent of the choice of $\tilde{t}$ such that $I_A(t) > \frac{m_1}{2}$ for all $t \leq \tilde{t} < \tilde{t} + T_3$. If $\psi \leq T_3$, our aim is obtained. If $T_3 < \psi \leq \omega_1$, from the second equation of (2.4) we have that $I_A(t) \geq -(r_1 + \mu)I_A(t)$ for $\bar{t} < t \leq \bar{t} + \psi$. Then we have $I_A(t) \geq \gamma_1 m_1^* e^{-(r_1 + \mu)\psi}$ for $\bar{t} < t \leq \bar{t} + \psi \leq \bar{t} + \omega_1$ since $I_A(\bar{t}) = \gamma_1 m_1^*$. If $\psi > \omega_1$, then we have that $I_A(t) \geq \gamma_1 m_1^* e^{-(r_1 + \mu)\omega_1}$ for $\bar{t} < t \leq \bar{t} + \psi$. Then, proceeding exactly as in the proof of the above claim, we can show that $I_A(t) \geq \gamma_1 m_1^* e^{-(r_1 + \mu)\omega_1}$ for $\bar{t} + \omega_1 \leq t \leq \bar{t} + \omega_1$. In fact, if not, there exists $T_4 > 0$ such that $I(t) \geq \gamma_1 m_1^* e^{-(r_1 + \mu)\omega_1}$ for $\bar{t} \leq t \leq \bar{t} + \omega_1 + T_4$, $I(\bar{t} + \omega_1 + T_4) = \gamma_1 m_1^* e^{-(r_1 + \mu)\omega_1}$, and $I_A(\bar{t} + \omega_1 + T_4) \leq 0$. When $\bar{t}$ is large enough, the inequality $S(t) > S_{\Delta_1}$ holds for $\bar{t} < t < \bar{t} + \psi$. On the other hand, we have from the second equation of (2.4) that

$$I_A(\tilde{t} + \omega_1 + T_4) \geq (\beta_1 e^{-\mu\omega_1} S^1(t) - (r_1 + \mu)) \gamma_1 m_1^* e^{-(r_1 + \mu)\omega_1}$$

$$= (r_1 + \mu) \left( \frac{\beta_1 e^{-\mu\omega_1}}{r_1 + \mu} S^1(t) - 1 \right) \gamma_1 m_1^* e^{-(r_1 + \mu)\omega_1}$$

$$> (r_1 + \mu) \left( \frac{S_{\Delta_1}}{S^1} \right)^{q_1} \gamma_1 m_1^* e^{-(r_1 + \mu)\omega_1}$$

$$> 0.$$

This is a contradiction to $I_A(\bar{t} + \omega_1 + T_4) \leq 0$. Therefore, $I_A(t) \geq m_1$ for $t \in [\bar{t}, \bar{t} + \psi]$. Since this kind of interval $[\bar{t}, \bar{t} + \psi]$ is arbitrarily chosen, we get that $I_A(t) \geq m_1$ for $t$ large enough in Case (ii). In view of our arguments above, the choice of $m_1$ is independent of the positive solution of (2.4), which satisfies that $I_A(t) \geq m_1$ for sufficiently large $t$.

Similarly, if $I_B(t)$ oscillates about $\gamma_2 m_2^*$, then we can prove that $I_B(t) \geq m_2$ for all large $t$, where

$$m_2 = \min \left\{ \frac{\gamma_2 m_2^*}{2}, \gamma_2 m_2^* e^{-(r_2 + \mu)\omega_2} \right\}.$$

This completes the proof of Theorem 3.2. \hfill \Box

**Theorem 3.3** If $R_2 > 1$, then system (2.4) is permanent.

**Proof** Suppose that $(S(t), I_A(t), I_B(t))$ is any positive solution of system (2.4). From the first and fourth equations of system (2.4) it is easy to see that

$$S'(t) \geq \mu - (\mu + \beta_1 e^{-\mu\omega_1} + \beta_2 e^{-\mu\omega_2}) S(t),$$

$$S(t^+) = (1 - \delta) S(t), \quad t = n\tau, n \in \mathbb{N}.$$  \hfill (3.19)

Let

$$m = \frac{\mu e^{\mu\tau}}{\mu + \beta_1 e^{-\mu\omega_1} + \beta_2 e^{-\mu\omega_2}} \times \frac{(1 - \delta)(1 - e^{-(\mu + \beta_1 e^{-\mu\omega_1} + \beta_2 e^{-\mu\omega_2})\tau})}{1 - (1 - \delta)e^{-(\mu + \beta_1 e^{-\mu\omega_1} + \beta_2 e^{-\mu\omega_2})\tau} - \epsilon} > 0,$$

where $\epsilon$ is a sufficiently small positive constant. Similarly, using Lemma 2.1 along with (3.19), we have $S(t) \geq m$ for $t$ large enough.

Set

$$\Omega = \{(S, I_A, I_B) \in R^3_+ \mid m \leq S(t) \leq 1, m_1 \leq I_A(t) \leq 1, m_2 \leq I_B(t) \leq 1\}.$$

Then $\Omega$ is a bounded compact region which has positive distance from coordinate planes. By Theorem 3.2, one obtains that every solution of system (2.4) with initial condition $\phi$ eventually enters and remains in the region $\Omega$. This completes the proof. \hfill \Box

**Corollary 3.3** If $\delta < \delta_\ast$, or $\tau > \tau_\ast$, or $r < r_\ast$, or $q < q_\ast$, then system (2.4) is permanent, that is, the disease can generate an endemic, where the critical values $\delta_\ast$, $\tau_\ast$, $\omega_\ast$, and $q_\ast$ are listed as follows:

$$\delta_\ast = \min_{i,j=1,2,i \neq j} \left\{ 1 - \frac{\mu e^{\mu\tau}}{\mu + (e^{\mu\tau} - 1)(\mu - \beta_j e^{-\mu\omega_j})} \right\},$$

$$\tau_\ast = \max_{i,j=1,2,i \neq j} \left\{ \frac{1}{\mu} \ln \left( 1 - \frac{\delta \mu}{(1 - \delta)(\mu - \beta_j e^{-\mu\omega_j})} \right) \right\}.$$
Note that \( R_1 \to 0 < 1 \) and \( R_2 \to 0 < 1 \) as \( q \to \infty \), which implies that the epidemic disease will die out eventually when nonlinear incidence \( q \) is gradually increasing. This is very interesting since nonlinear incidence has a significant effect on the dynamics of epidemic model.

4 Conclusions

As an example, let \( \delta = 0 \) and \( I_B = 0 \): then (2.3) becomes the following system without pulse:

\[
\begin{aligned}
S'(t) &= -\beta e^{-\mu t} S(t) I(t - \tau) - \mu S(t) + \mu, \\
I'(t) &= \beta e^{-\mu t} S(t) I(t - \tau) - \mu I(t) - r I(t), \\
R'(t) &= r I(t) - \mu R(t).
\end{aligned}
\]  

(4.1)

According to Theorems 3.1 and 3.2, we can deduce the same results for system (4.1) as Ma and Song in [5, 6].

In this paper, we introduce time delay (with \( \omega_i, i = 1, 2 \)), pulse vaccination (with \( \delta \) and \( \tau \)), and nonlinear incidence (with \( q_i, i = 1, 2 \)) into SIR model and obtain that the latent period of disease, pulse vaccination, and nonlinear incidence can bring effects on infection-eradication and the permanence of epidemic disease. The main results show that a short period of pulsing (with \( \tau < \tau^* \)), or a large pulse vaccination rate (with \( \delta > \delta^* \)), or a long latent period of the disease (with \( \omega \geq \omega^* \)), or a large nonlinear incidence (with \( q > q^* \)) is a sufficient condition for the global attractivity of infection-eradication periodic solution \( (\tilde{S}(t), 0, 0) \); if not, the system becomes permanent. Therefore, we can choose the vaccination period (with \( \tau \)) and increase the proportion (with \( \delta \) ) of those vaccinated successfully to all of the susceptible such that \( R_1 < 1 \) in order to prevent the epidemic disease from generating endemic.

We find that infection caused by viruses A and B can be controlled when \( R_1 < 1 \) and \( R_2 < 1 \). Since \( I_B \) competes \( I_A \), the milder infection caused by virus B acts like a vaccine against the virus A. Hence, with the help of this study, there is a possibility in the future to develop a vaccination strategy to fight the epidemic \( I_A \).

Note that \( R_1 > R_2 \), and we obtain the results for \( R_1 < 1 \) or \( R_2 > 1 \). However, for the closed interval \([R_2, R_1]\), the dynamical behavior of model (2.4) has not been studied, and the threshold parameter for the reproducing number (or the pulse vaccination rate) between the extinction of the disease and the permanence of the disease has not been obtained. It is worthwhile for us to study the case for \( R_1 > 1 \) and \( R_2 < 1 \) in the future work. Finding the threshold value \( R = R_1 = R_2 \) is left for our future consideration.

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