Stochastic Conditional Gradient++

Hamed Hassani
Department of Electrical and Systems Engineering
University of Pennsylvania

Amin Karbasi
Department of Electrical Engineering and Computer Science
Yale University

Aryan Mokhtari
Laboratory for Information and Decision Systems
Massachusetts Institute of Technology

Zebang Shen
College of Computer Science and Technology
Zhejiang University

Abstract
In this paper, we develop Stochastic Continuous Greedy++ (SCG++), the first efficient variant of a conditional gradient method for maximizing a continuous submodular function subject to a convex constraint. Concretely, for a monotone and continuous DR-submodular function, SCG++ achieves a tight $(1 - 1/e)OPT - \epsilon$ solution while using $O(1/\epsilon^2)$ stochastic oracle queries and $O(1/\epsilon)$ calls to the linear optimization oracle. The best previously known algorithms either achieve a suboptimal $(1/2)OPT - \epsilon$ solution with $O(1/\epsilon^2)$ stochastic gradients or the tight $(1 - 1/e)OPT - \epsilon$ solution with suboptimal $O(1/\epsilon^3)$ stochastic gradients. SCG++ enjoys optimality in terms of both approximation guarantee and stochastic stochastic oracle queries. Our novel variance reduction method naturally extends to stochastic convex minimization. More precisely, we develop Stochastic Frank-Wolfe++ (SFW++) that achieves an $\epsilon$-approximate optimum with only $O(1/\epsilon)$ calls to the linear optimization oracle while using $O(1/\epsilon^2)$ stochastic oracle queries in total. Therefore, SFW++ is the first efficient projection-free algorithm that achieves the optimum complexity $O(1/\epsilon^2)$ in terms of stochastic oracle queries.

Keywords: submodular maximization, stochastic optimization, conditional gradient methods

1. Introduction
In this paper, we consider the following non-oblivious stochastic optimization problem:

$$\max_{x \in C} F(x) := \max_{x \in C} \mathbb{E}_{z \sim p(z; x)} [\tilde{F}(x; z)],$$

(1)

where $x \in \mathbb{R}^d$ is the decision variable, $C \subseteq \mathbb{R}^d$ is a convex feasible set, $z \in Z$ is a random variable with distribution $p(z; x)$, and the objective function $F : \mathbb{R}^d \to \mathbb{R}$ is defined as the expectation of a set of stochastic functions $\tilde{F} : \mathbb{R}^d \times Z \to \mathbb{R}$. Problem (1) is called non-oblivious as the underlying distribution depends on the variable $x$ and may change during the optimization procedure.

The authors are listed in alphabetical order.
One should note that the usual oblivious stochastic (convex/non-convex) optimization, in which the probability distribution $p$ is independent of $x$, is a special case of problem (1). We focus on providing efficient solvers for (1) in terms of the sample complexity of $z$ (a.k.a calls to the stochastic oracle), for the settings that the objective function is either (non-convex) continuous submodular or concave. A canonical example of problem (1) is the multi-linear extension of a discrete submodular function where the stochasticity crucially depends on the decision variable $x$ at which we evaluate (the definition is deferred to Section 5).

When the objective function $F$ is monotone and continuous DR-submodular, Hassani et al. (2017) showed that the projected Stochastic Gradient Ascent (SGA) method finds a solution to problem (1) with a function value no less than $[(1/2)\text{OPT} - \epsilon]$ after computing at most $O(1/e^2)$ stochastic gradients. Here, and throughout the paper, OPT denotes the optimal value of problem (1). Hassani et al. (2017) also provided examples for which SGA cannot achieve better than $1/2$ approximation ratio, in general. Later, Mokhtari et al. (2018a) proposed Stochastic Continuous Greedy (SCG), a conditional gradient method that achieves the tight $[(1 - 1/e)^2\text{OPT} - \epsilon]$ solution by $O(1/e^3)$ calls to the linear optimization oracle while using $O(1/e^3)$ stochastic gradients. While both SCG and SGA are first-order methods, meaning that they rely on stochastic gradients, SCG provably achieves a better result at the price of being slower. The first contribution of this paper is to answer the following question:

"Can we achieve the best of both worlds? That is, can we find a $[(1 - 1/e)^2\text{OPT} - \epsilon]$ solution after at most $O(1/e^2)$ calls to the stochastic oracle?"

A very similar question arises in the case of stochastic convex minimization (concave maximization) using conditional gradient methods (a.k.a. Frank Wolfe). It is well known that such methods, due to solving a linear optimization program, are highly sensitive to stochasticity and may easily diverge (unlike projected gradient methods). To overcome this issue, Hazan and Luo (2016) proposed a variance reduced method, by using an increasing mini-batch sizes, that achieves $\epsilon$-approximate optimum by using $O(1/e^3)$ stochastic gradients. Recently, Mokhtari et al. (2018b) proposed a momentum method that achieves the same rate while fixing the size of mini-batches to 1. Our next contribution is to answer the following question:

"Can we achieve an $\epsilon$-approximate optimum with $O(1/e^2)$ stochastic oracle calls?"

Our contributions. We answer the above questions affirmatively. More precisely, we develop Stochastic Continuous Greedy++ (SCG++), the first algorithm that achieves the tight $[(1 - 1/e)\text{OPT} - \epsilon]$ solution for problem (1) with $O(1/e)$ calls to the linear optimization program while using $O(1/e^2)$ stochastic gradients in total. Our technique relies on a novel variance reduction method that estimates the difference of gradients in the non-oblivious stochastic setting without introducing extra bias. This is crucial in our analysis, as all the existing variance reduction methods fail to correct for this bias and can only operate in the oblivious/classical stochastic setting. We further show that our result is optimal in all aspects. In particular, in Theorem 5.3, we provide an information-theoretic lower bound to showcase the necessity of $O(1/e^2)$ stochastic oracle queries in order to achieve $[(1 - 1/e)\text{OPT} - \epsilon]$. Note that under natural complexity assumptions, one cannot achieve an approximation ratio better than $(1 - 1/e)$ for monotone submodular functions (Feige, 1998). Finally, we extend our result to the stochastic concave maximization (i.e., $F$ is concave) and propose Stochastic Frank-Wolfe++ (SFW++), that finds an $\epsilon$-suboptimal solution using $O(1/e^3)$ stochastic gradients with $O(1/e)$ calls to the linear optimization oracle. To the best
of our knowledge, SFW++ is the first efficient conditional gradient method that achieves the optimal stochastic oracle complexity $O(1/\epsilon^2)$ for smooth concave maximization (resp., smooth convex minimization).

2. Related Work

Submodular Maximization. Submodular set functions (Nemhauser et al., 1978) capture the intuitive notion of diminishing returns and have become increasingly important in various machine learning applications. Examples include data summarization (Lin and Bilmes, 2011b,a), crowd teaching (Singla et al., 2014), neural network interpretation (Elenberg et al., 2017), dictionary learning (Das and Kempe, 2011), and variational inference (Djolonga and Krause, 2014), to name a few. The celebrated result of Nemhauser et al. (1978) shows that for a monotone submodular function and subject to a cardinality constraint, a simple greedy algorithm achieves the tight $(1 - 1/\epsilon)$ approximation guarantee. However, the vanilla greedy method does not provide the tightest guarantees for many classes of feasibility constraints. To circumvent this issue, the continuous relaxation of submodular functions, through the multilinear extension, have been extensively studied (Vondrák, 2008; Calinescu et al., 2011; Chekuri et al., 2014; Feldman et al., 2011; Gharan and Vondrák, 2011; Sviridenko et al., 2015). In particular, it is known that the Continuous Greedy algorithm achieves the tight $(1 - 1/\epsilon)$ approximation guarantee for monotone submodular functions under a general matroid constraint (Calinescu et al., 2011).

Continuous DR-submodular functions, an important subclass of non-convex functions, generalize the notion of diminishing returns to the continuous domains (Bian et al., 2017). Such functions naturally arise in machine learning applications such as optimum experimental design (Chen et al., 2018), Map inference for Determinantal Point Processes (Kulesza and Taskar, 2012), and revenue maximization (Niazadeh et al., 2018). It has been recently shown that monotone continuous DR-submodular functions can be (approximately) maximized over convex bodies using first-order methods (Bian et al., 2017; Hassani et al., 2017; Mokhtari et al., 2018a). When exact gradient information is available, Bian et al. (2017) showed that the Continuous Greedy algorithm, a variant of the conditional gradient method, achieves $[(1 - 1/\epsilon)OPT - \epsilon]$ with $O(1/\epsilon)$ gradient evaluations. However, the problem becomes considerably more challenging when we only have access to a stochastic first-order oracle. In particular, Hassani et al. (2017) showed that the stochastic gradient ascent achieves $[(1/2)OPT - \epsilon]$ by using $O(1/\epsilon^2)$ stochastic gradients. In contrast, Mokhtari et al. (2018a) proposed the stochastic variant of the continuous greedy algorithm that achieves $[(1 - 1/\epsilon)OPT - \epsilon]$ by using $O(1/\epsilon^3)$ stochastic gradients. This paper shows how achieving $[(1 - 1/\epsilon)OPT - \epsilon]$ is possible by $O(1/\epsilon^2)$ stochastic gradient evaluations.

Convex minimization. The problem of minimizing a stochastic convex function subject to a convex constraint using stochastic projected gradient descent-type methods has been studied extensively in the past (Robbins and Monro, 1951; Nemirovski and Yudin, 1978; Nemirovskii et al., 1983). Although stochastic gradient computation is inexpensive, the cost of projection step can be prohibitive (Fujishige and Isotani, 2011) or intractable (Collins et al., 2008). In such cases, the projection-free algorithms, a.k.a., Frank-Wolfe or conditional gradient, are the method of choice (Frank and Wolfe, 1956; Jaggi, 2013). In the stochastic setting, the online Frank-Wolfe algorithm proposed by Hazan and Kale (2012) requires $O(1/\epsilon^4)$ stochastic gradient evaluations to reach an $\epsilon$-approximate optimum, i.e., $F(x) \leq OPT + \epsilon$, under the assumption that the objective function is convex and has bounded gradients. The stochastic variant of Frank-Wolfe studied by Hazan and Luo (2016), uses an increas-
ing batch size of \( b = O(t^2) \) (at iteration \( t \)) to obtain an improved stochastic oracle complexity of \( O(1/e^3) \) under the assumptions that the expected objective function is smooth and Lipschitz continuous. Recently, Mokhtari et al. (2018b) proposed a momentum gradient estimator thorough which they achieve a similar \( O(1/e^3) \) stochastic gradient evaluations while fixing the batch-size to 1. Our work improves these results by introducing the first stochastic conditional gradient method that obtains an \( \epsilon \)-suboptimal solution after computing at most \( O(1/e^2) \) stochastic gradients.

3. Preliminaries

We first recap some standard definitions for both discrete and continuous submodular maximization, and then review variance reduced methods for solving stochastic optimization problems.

Submodularity. A set function \( f : 2^V \rightarrow \mathbb{R}_+ \), defined on the ground set \( V \), is submodular if

\[
f(A) + f(B) \geq f(A \cap B) + f(A \cup B),
\]

for all subsets \( A, B \subseteq V \). Even though submodularity is mostly considered in the discrete domain, the notion can be naturally extended to arbitrary lattices (Fujishige, 2005). To this aim, let us consider a subset of \( \mathbb{R}_+ \) of the form \( \mathcal{X} = \prod_{i=1}^n X_i \) where each \( X_i \) is a compact subset of \( \mathbb{R}_+ \). A function \( F : \mathcal{X} \rightarrow \mathbb{R}_+ \) is continuous submodular (Wolsey, 1982) if for all \( (x, y) \in \mathcal{X} \times \mathcal{X} \), we have

\[
F(x) + F(y) \geq F(x \vee y) + F(x \wedge y),
\]

where \( x \vee y = \max(x, y) \) (component-wise) and \( x \wedge y = \min(x, y) \) (component-wise). A submodular function is monotone if for any \( x, y \in \mathcal{X} \) such that \( x \leq y \), we have \( F(x) \leq F(y) \) (here, by \( x \leq y \) we mean that every element of \( x \) is less than that of \( y \)). When twice differentiable, \( F \) is submodular if and only if all cross-second-derivatives are non-positive (Bach, 2015), i.e.,

\[
\forall i \neq j, \forall x \in \mathcal{X}, \quad \partial^2 F(x) / \partial x_i \partial x_j \leq 0.
\]

The above expression makes it clear that continuous submodular functions are not convex nor concave in general, as concavity (convexity) implies that \( \nabla^2 F \leq 0 \) (resp. \( \nabla^2 F \geq 0 \)). A proper subclass of submodular functions are called DR-submodular (Bian et al., 2017; Soma and Yoshida, 2015) if for all \( x, y \in \mathcal{X} \) such that \( x \leq y \) and any standard basis vector \( e_i \in \mathbb{R}^n \) and a non-negative number \( z \in \mathbb{R}_+ \) such that \( ze_i + x \in \mathcal{X} \) and \( ze_i + y \in \mathcal{X} \), then, \( F(ze_i + x) + F(x) \geq F(ze_i + y) - F(y) \). One can easily verify that for a differentiable DR-submodular function the gradient is an antitone mapping, i.e., for all \( x, y \in \mathcal{X} \) such that \( x \leq y \) we have \( \nabla F(x) \succeq \nabla F(y) \) (Bian et al., 2017).

An important example of a DR-submodular function is the multilinear extension (Calinescu et al., 2011) which will be studied in Section 5.

Variance Reduction. Beyond the vanilla stochastic gradient, variance reduced algorithms (Schmidt et al., 2017; Johnson and Zhang, 2013; Defazio et al., 2014; Nguyen et al., 2017; Reddi et al., 2016; Allen-Zhu, 2018) have been successful in reducing stochastic first-order oracle complexity in oblivious stochastic optimization

\[
\max_{x \in C} F(x) := \max_{x \in C} \mathbb{E}_{x \sim p(z)} f(x; z),
\]

where each component function \( f(\cdot; z) \) is \( L \)-smooth. In contrast to (1), the underlying distribution \( p \) of (4) is invariant to the variable \( x \) and is hence called oblivious. We will now explain a recent
Stochastic Conditional Gradient++

variance reduction technique for solving (4) using stochastic gradient information. Consider the following *unbiased* estimate of the gradient at the current iterate $x^t$:

$$g^t \equiv g^{t-1} + \nabla f(x^t; \mathcal{M}) - \nabla f(x^{t-1}; \mathcal{M}), \quad (5)$$

where $f(y; \mathcal{M}) \equiv \frac{1}{|\mathcal{M}|} \sum_{z \in \mathcal{M}} \nabla f(y; z)$ for some $y \in \mathbb{R}^d$, $g^{t-1}$ is an unbiased gradient estimator at $x^{t-1}$, and $\mathcal{M}$ is a mini-batch of random samples drawn from $p(z)$. Fang et al. (2018) showed that, with the gradient estimator (5), $O(1/\epsilon^3)$ stochastic gradient evaluations are sufficient to find an $\epsilon$-first-order stationary point of problem (4), improving upon the $O(1/\epsilon^4)$ complexity of SGD. A crucial property leading to the success of the variance reduction method given in (5) is that $\nabla f(x^t; \mathcal{M})$ and $\nabla f(x^{t-1}; \mathcal{M})$ use the same minibatch sample $\mathcal{M}$ in order to exploit the $L$-smoothness of component functions $f(\cdot; z)$. Such a construction is only possible in the oblivious setting where $p(z)$ is invariant to the choice of $x$. In fact, (5) would introduce bias in the more general non-oblivious case (1): To see this, let $\mathcal{M}$ be the minibatch of random variable $z$ sampled according to distribution $p(z; x^t)$. We have $\mathbb{E}[\nabla f(x^t; \mathcal{M})] = \nabla F(x^t)$ but $\mathbb{E}[\nabla f(x^{t-1}; \mathcal{M})] \neq \nabla F(x^{t-1})$ since the distribution $p(z; x^{t-1})$ is not the same as $p(z; x^t)$. The same argument renders all the existing variance reduction techniques inapplicable for the non-oblivious setting of problem (1).

4. Stochastic Continuous Greedy++

In this section, we present the Stochastic Continuous Greedy++ (SCG++) algorithm, the first method that obtains a $[(1-1/e)\text{OPT}-\epsilon]$ solution with $O(1/\epsilon^2)$ stochastic oracle complexity. The SCG++ algorithm is a variant of a conditional gradient method. To be more precise, at each iteration $t$, given a gradient estimator $g^t$, SCG++ solves the subproblem

$$v^t = \arg\max_{v \in C} \langle v, g^t \rangle \quad (6)$$

to obtain an element $v^t$ in $C$ as an ascent direction, which is then added to the current iterate $x^t$ with a scaling factor $1/T$, i.e., the new iterate $x^{t+1}$ is computed by following the update

$$x^{t+1} = x^t + \frac{1}{T}v^t, \quad (7)$$

Here, and throughout the paper, $T$ is the total number of iterations of the algorithm. The iterates are assumed to be initialized at the origin which may not belong to the feasible set $C$. Even though each iterate $x^t$ may not necessarily be in $C$, the feasibility of the final iterate $x^T$ is guaranteed by the convexity of $C$. Note that the sequence of iterates $\{x^s\}_{s=0}^T$ can be regarded as a path from the origin (as we manually force $x^0 = 0$) to some feasible point in $C$. The key idea in SCG++ is to exploit the high correlation between the consecutive iterates to maintain a highly accurate estimate $g^t$, which is the focus of the rest of this section. Note that by replacing the gradient approximation vector $g^t$ in the update of SCG++ by the exact gradient of the objective function, we recover the update of the continuous greedy method (Calinescu et al., 2011; Bian et al., 2017).

We now proceed to describe our approach for evaluating the gradient approximation $g^t$ when we face a non-oblivious problem (1). Given a sequence of iterates $\{x^s\}_{s=0}^t$, the gradient of the objective function $F$ at iterate $x^t$ can be written in a path-integral form as follows

$$\nabla F(x^t) = \nabla F(x^0) + \sum_{s=1}^{t} \left\{ \Delta^s \equiv \nabla F(x^s) - \nabla F(x^{s-1}) \right\}. \quad (8)$$
Then, estimates for \( s < t \) directly estimating the difference is an approach for the issue in the non-oblivious case (see the discussion at the end of section 3). By obtaining an unbiased estimate of \( \Delta \), we obtain recursively an unbiased estimator of the Hessian-vector product:

\[
\nabla \tilde{\Delta} = \int_0^1 \nabla^2 F(x(a))(x^t - x^{t-1}) \, da = \left[ \int_0^1 \nabla^2 F(x(a)) \, da \right] (x^t - x^{t-1}),
\]

(9)

where \( x(a) \) is defined as \( a \cdot x^t + (1-a) \cdot x^{t-1} \) for \( a \in [0, 1] \). Therefore, if we sample the parameter \( a \) uniformly at random from the interval \([0, 1]\), it can be easily verified that \( \tilde{\Delta} := \nabla^2 F(x(a))(x^t - x^{t-1}) \) is an unbiased estimator of the gradient difference \( \Delta \) since

\[
\mathbb{E}_a[\nabla^2 F(x(a))(x^t - x^{t-1})] = \nabla F(x^t) - \nabla F(x^{t-1}).
\]

Hence, all we need is an unbiased estimator of the Hessian-vector product \( \nabla^2 F(y)(x^t - x^{t-1}) \) for the non-oblivious objective \( F \) at an arbitrary \( y \in C \). In the following lemma, we present such an unbiased estimator of the Hessian \( \nabla^2 F(y) \) that can be evaluated efficiently.

**Lemma 4.1** For any \( y \in C \), let \( z \) be a random variable with distribution \( p(z; y) \) and define

\[
\nabla \tilde{\nabla}^2 F(y; z) \overset{\text{def}}{=} \tilde{F}(y; z)[\nabla \log p(z; y)][\nabla \log p(z; y)]^\top + [\nabla \tilde{F}(x; z)][\nabla \log p(z; y)]^\top
\]

\[
+ [\nabla \log p(z; y)][\nabla \tilde{F}(y; z)]^\top + \nabla^2 \tilde{F}(y; z) + \tilde{F}(y; z) \nabla^2 \log p(z; y).
\]

(11)

Then, \( \nabla \tilde{\nabla}^2 F(y; z) \) is an unbiased estimator of \( \nabla^2 F(y) \), i.e., \( \mathbb{E}_{z \sim p(z; y)}[\nabla \tilde{\nabla}^2 F(y; z)] = \nabla^2 F(y) \).

---

**Algorithm 1 Stochastic Continuous Greedy++ (SCG++)**

**Input:** Minibatch size \(|\mathcal{M}_0|\) and \(|\mathcal{M}|\), and total number of rounds \( T \)

1. Initialize \( x^0 = 0 \);
2. for \( t = 1 \) to \( T \) do
3.  if \( t = 1 \) then
4.  Sample a minibatch \( \mathcal{M}_0 \) of \( z \) according to \( p(z; x^0) \) and compute \( g^0 \overset{\text{def}}{=} \nabla \tilde{F}(x^0; \mathcal{M}_0) \);
5.  else
6.  Sample a minibatch \( \mathcal{M} \) of \( z \) according to \( p(z; a) \) where \( a \) is a chosen uniformly at random from \([0, 1]\) and \( x(a) := a \cdot x^t + (1-a) \cdot x^{t-1} \);
7.  Compute the Hessian approximation \( \nabla^2 \tilde{F} \) corresponding to \( \mathcal{M} \) according to (12);
8.  Construct \( \tilde{\Delta} \) based on (13) (Option I) or (20) (Option II);
9.  Update the stochastic gradient approximation \( g^t := g^{t-1} + \tilde{\Delta} \);
10. end if
11. Compute the ascent direction \( v^t := \arg\max_{v \in C} \{ v^\top g^t \} \);
12. Update the variable \( x^{t+1} := x^t + 1/T \cdot v^t \);
13. end for

---

By obtaining an unbiased estimate of \( \Delta \) and reusing the previous unbiased estimates for \( s < t \), we obtain recursively an unbiased estimator of \( \nabla F(x^t) \) which has a reduced variance. Estimating \( \nabla F(x^s) \) and \( \nabla F(x^{s-1}) \) separately as suggested in (5) would cause the bias issue in the non-oblivious case (see the discussion at the end of section 3). Therefore, we propose an approach for directly estimating the difference \( \Delta \) in an unbiased manner.

We construct an unbiased estimator \( g^t \) of the gradient vector \( \nabla F(x^t) \) by adding an unbiased estimate \( \tilde{\Delta} \) of the gradient difference \( \Delta := \nabla F(x^t) - \nabla F(x^{t-1}) \) to \( g^{t-1} \), where \( g^{t-1} \) is an unbiased estimator of the gradient \( \nabla F(x^{t-1}) \). Note that the vector \( \tilde{\Delta} \) can be written as

\[
\tilde{\Delta} = \nabla^2 \tilde{F}(x(a))(x^t - x^{t-1}),
\]

where \( x(a) \) is defined as \( a \cdot x^t + (1-a) \cdot x^{t-1} \) for \( a \in [0, 1] \). Therefore, if we sample the parameter \( a \) uniformly at random from the interval \([0, 1]\), it can be easily verified that \( \tilde{\Delta} \) is an unbiased estimator of the gradient difference \( \Delta \) since

\[
\mathbb{E}_a[\nabla^2 \tilde{F}(x(a))(x^t - x^{t-1})] = \nabla F(x^t) - \nabla F(x^{t-1}).
\]

Hence, all we need is an unbiased estimator of the Hessian-vector product \( \nabla^2 F(y)(x^t - x^{t-1}) \) for the non-oblivious objective \( F \) at an arbitrary \( y \in C \). In the following lemma, we present such an unbiased estimator of the Hessian \( \nabla^2 F(y) \) that can be evaluated efficiently.
The result in Lemma 4.1 shows how to evaluate an unbiased estimator of the Hessian $\nabla^2 F(y)$. If we consider $a$ to be a random variable with a uniform distribution over the interval $[0, 1]$, then we can define a random variable $z(a)$ with the probability distribution $p(z(a); x(a))$ where $x(a) := a \cdot x^t + (1 - a) \cdot x^{t-1}$. Considering these two random variables, combined with Lemma 4.1, we can construct an unbiased estimator of the integral $\int_0^1 \nabla^2 F(x(a)) \, da$ in (9) by

$$\hat{\nabla}_t^2 \overset{\text{def}}{=} \frac{1}{|M|} \sum_{(a, z(a)) \in M} \nabla^2 F(x(a); z(a)),$$

where $M$ is a minibatch containing $|M|$ samples of the random pair $(a, z(a))$.

Once we have access to $\hat{\nabla}_t^2$, which is an unbiased estimator of $\int_0^1 \nabla^2 F(x(a)) \, da$, we can approximate the gradient difference $\Delta^t$ by its unbiased estimator defined as

$$\tilde{\Delta}^t := \hat{\nabla}_t^2 (x^t - x^{t-1}).$$

Note that for the general objective $F(\cdot)$, the matrix-vector product $\hat{\nabla}_t^2 (x^t - x^{t-1})$ requires $O(d^2)$ computation and memory. To resolve this issue, in Section 4.1 we provide an implementation of (13) using only first-order information which has a computational and memory complexity of $O(d)$.

Using $\tilde{\Delta}^t$ as an unbiased estimator for the gradient difference $\Delta^t$, we can define our objective function gradient estimator as

$$g^t = \nabla \tilde{F}(x^0; M_0) + \sum_{i=1}^t \tilde{\Delta}^t.$$  \hfill (14)

Indeed, this update can also be written in a recursive way as

$$g^t = g^{t-1} + \tilde{\Delta}^t,$$  \hfill (15)

if we set $g^0 = \nabla \tilde{F}(x^0; M_0)$. Note that the proposed approach for gradient approximation in (14) has an inherent variance reduction mechanism which leads to the optimal complexity of SCG++ in terms of the number of calls to the stochastic oracle. We further highlight this point in Section 4.2.

4.1. Implementation of the Hessian-Vector Product

In this section, we focus on the computation of the gradient difference approximation $\tilde{\Delta}^t$ introduced in (13). We aim to come up with a scheme that avoids explicitly computing the matrix estimator $\tilde{\nabla}_t^2$, which has a complexity of $O(d^2)$, and present an approach directly approximating $\tilde{\Delta}^t$ that only uses the finite differences of gradients with a complexity of $O(d)$. Recall the definition of the Hessian approximation $\hat{\nabla}_t^2$ in (12). Computing $\hat{\nabla}_t^2 (x^t - x^{t-1})$ is equivalent to computing $|M|$ instances of $\hat{\nabla}_t^2 F(y; z)(x^t - x^{t-1})$ for some $y \in \mathcal{C}$ and $z \in \mathcal{Z}$. Denote $d = x^t - x^{t-1}$ and use the expression in (11) to write

$$\hat{\nabla}_t^2 F(y; z) \cdot d = \tilde{F}(y; z)[\nabla \log p(z; y) \top d] \nabla \log p(z; y) + [\nabla \log p(z; y) \top d] \nabla \tilde{F}(x; z)$$

$$+ [\nabla \tilde{F}(y; z) \top d] \nabla \log p(z; y)] + \hat{\nabla}_t^2 \tilde{F}(y; z) \cdot d + \tilde{F}(y; z) \nabla^2 \log p(z; y) \cdot d.$$  \hfill (16)

Note that the first three terms can be computed in time $O(d)$ and only the last two terms on the right hand side of (16) involve $O(d^2)$ operations, which can be approximated by the following finite
gradient difference scheme. For any twice differentiable function $\psi : \mathbb{R}^d \to \mathbb{R}$ and arbitrary $d \in \mathbb{R}^d$ with bounded Euclidean norm $\|d\| \leq D$, we compute, for some small $\delta > 0$,

$$\phi(\delta; \psi) \overset{\text{def}}{=} \frac{\nabla \psi(y + \delta \cdot d) - \nabla \psi(y - \delta \cdot d)}{2\delta} \simeq \nabla^2 \psi(y) \cdot d.$$

(17)

By considering the second-order smoothness of the function $\psi(\cdot)$ with constant $L_2$ we can show that for arbitrary $x, y \in \mathbb{R}^d$ it holds $\|\nabla^2 \psi(x) - \nabla^2 \psi(y)\| \leq L_2\|x - y\|$. Therefore, the error of the above approximation can be bounded by

$$\|\nabla^2 \psi(y) \cdot d - \phi(\delta; \psi)\| = \|\nabla^2 \psi(y) \cdot d - \nabla^2 \psi(\tilde{x}) \cdot d\| \leq D^2 L_2 \delta,$$

(18)

where $\tilde{x}$ is obtained from the mean value theorem. This quantity can be made arbitrary small by decreasing $\delta$. Later, we show that setting $\delta = \mathcal{O}(\epsilon^2)$ is sufficient, where $\epsilon$ is the target accuracy. By applying the technique of (17) to the two functions $\psi(y) = F(y; z)$ and $\psi(y) = \log p(z; y)$, we can approximate (16) in time $\mathcal{O}(d)$:

$$\xi_\delta(y; z) \overset{\text{def}}{=} F(y; z) [\nabla \log p(z; y) \cdot d] \nabla \log p(z; y) + [\nabla \log p(z; y) \cdot d] ^\top \nabla F(x; z)$$

$$+ [\nabla F(y; z) \cdot d] [\nabla \log p(z; y)] + \phi(\delta; F(y; z)) + \phi(\delta; \log p(z; y)).$$

(19)

We further can define a minibatch version of such implementation as

$$\xi_\delta(x; M) \overset{\text{def}}{=} \frac{1}{|M|} \sum_{(a, z(a)) \in M} \xi_\delta(x(a); z(a)),$$

(20)

which is used in Option II of Step 8 in Algorithm 1.

4.2. Convergence Analysis

In this section, we analyze the convergence property of Algorithm 1 using (20) as the gradient-difference estimation. The result for (13) can be obtained accordingly. We note that (13) is a special case of (20) by taking $\delta \to 0$ (e.g., by letting $\delta = \mathcal{O}(\epsilon^2)$). We first specify the assumptions required for the analysis of the SCG++.

Assumption 4.1 (monotonicity and DR-submodularity) The function $F : \mathcal{X} \to \mathbb{R}_+$ is non-negative, monotone, and continuous DR-submodular.

Assumption 4.2 (function value at the origin) The function $F$ at the origin is $F(0) = 0$.

Assumption 4.3 (bounded stochastic function value) The stochastic function $\tilde{F}(x; z)$ has bounded function value for all $z \in \mathcal{Z}$ and $x \in \mathcal{C}$: $\max_{z \in \mathcal{Z}, x \in \mathcal{C}} \tilde{F}(x; z) \leq B$.

Assumption 4.4 (compactness of feasible domain) The set $\mathcal{C}$ is compact with diameter $D$.

Assumption 4.5 (bounded gradient norm) For all $x \in \mathcal{C}$, the stochastic gradient $\nabla \tilde{F}$ has bounded norm: $\forall z \in \mathcal{Z}, \|\nabla \tilde{F}(x; z)\| \leq G_{\tilde{F}}$, and the norm of the gradient of $\log p$ has bounded fourth-order moment, i.e., $\mathbb{E}_{z \sim p(x; z)} \|\nabla \log p(z; x)\|^4 \leq G_p^4$. Furthermore, we define $G = \max \{G_{\tilde{F}}, G_p\}$.
Assumption 4.6 (bounded second-order differentials) For all $x \in \mathcal{C}$, the stochastic Hessian $\nabla^2 \tilde{F}$ has bounded spectral norm $\forall z \in \mathcal{Z}, \|\nabla^2 \tilde{F}(x; z)\| \leq L_F$, and the spectral norm of the Hessian of the log-probability function has bounded second order moment: $E_{z \sim p(x; z)}\|\nabla^2 \log p(z; x)\|^2 \leq L_{p}^2$. Furthermore, we define $L = \max\{L_F, L_p\}$.

Assumption 4.7 (continuity of the Hessian) The stochastic Hessian $\nabla^2 \tilde{F}$ is $L_{2, \tilde{F}}$-Lipschitz continuous, i.e., for all $x, y \in \mathcal{C}$ and all $z \in \mathcal{Z}$, i.e., $\|\nabla^2 \tilde{F}(x; z) - \nabla^2 \tilde{F}(y; z)\| \leq L_{2, \tilde{F}}\|x - y\|$. The Hessian of the log probability $\nabla^2 \log p(x; z)$ is $L_{2, p}$-Lipschitz continuous, i.e., for all $x, y \in \mathcal{C}$ and all $z \in \mathcal{Z}$ we have $\|\nabla^2 \log p(x; z) - \nabla^2 \log p(y; z)\| \leq L_{2, p}\|x - y\|$. Furthermore, we define $L_2 = \max\{L_{2, \tilde{F}}, L_{2, p}\}$.

Remark 4.2 Assumption 4.7 is only used to show the finite difference scheme (16) has bounded variance, and the oracle complexity of our method does not depend on $L_{2, \tilde{F}}$ and $L_{2, p}$.

As we mentioned in the previous section, the update for the stochastic gradient vector $g^t$ in SCG++ is designed properly to reduce the noise of gradient approximation. In the following lemma, we formally characterize the variance of the gradient approximation for SCG++. To this end, we also need to properly choose the minibatch sizes $|M_0|$ and $|M|$.

Lemma 4.3 Consider the SCG++ method outlined in Algorithm 1 and assume that in Step 8 we follow the update in (20) to construct the gradient difference approximation $\tilde{\Delta}^t$ (Option II). If Assumptions (4.3), (4.4), (4.5), (4.6), and (4.7) hold, then

$$E\left[\|g^t - \nabla F(x^t)\|^2\right] \leq (1 + \epsilon t)L^2D^2\epsilon^2, \quad \forall t \in \{0, \ldots, T - 1\},$$

where we set $T = 1/\epsilon$, the minibatch sizes to $|M_0| = G^2/(L^2D^2\epsilon^2)$ and $|M| = 2/\epsilon$, and the error of Hessian-vector product approximation $\delta$, which is defined in (18), is sufficiently small (in fact $\delta = \mathcal{O}(\epsilon^2)$ is sufficient). Here $\bar{L}$ is a constant defined by $\bar{L}^2 \overset{\text{def}}{=} 4B^2G^4 + 4G^4 + 4L^2 + 4B^2L^2$.

The result in Lemma 4.3 shows that by $|M| = \mathcal{O}(1/\epsilon)$ calls to the stochastic oracle at each iteration, the variance of gradient approximation in SCG++ after $t$ iterations is of $\mathcal{O}((1 + \epsilon t)\epsilon)$. In the following theorem, we incorporate this bound on the noise of gradient approximation to characterize the convergence guarantee of SCG++.

Theorem 4.4 Consider the SCG++ method outlined in Algorithm 1 and assume that in Step 8 we follow the update in (20) to construct the gradient difference approximation $\tilde{\Delta}^t$ (Option II). If Assumptions 4.1-4.7 hold, then the output of SCG++, denoted by $x^T$, satisfies

$$E\left[F(x^T)\right] \geq (1 - 1/\epsilon)F(x^*) - 2LeD^2,$$

by setting $|M_0| = \frac{G^2}{2LD^2\epsilon^2}$, $|M| = \frac{1}{2\epsilon}$, $T = \frac{1}{\epsilon}$, and $\delta$ is sufficiently small as discussed in Lemma 4.3. Here $\bar{L}$ is a constant defined by $\bar{L}^2 \overset{\text{def}}{=} 4B^2G^4 + 16G^4 + 4L^2 + 4B^2L^2$.

The result in Theorem 4.4 shows that after at most $T = 1/\epsilon$ iterations the objective function value for the output of SCG++ is at least $(1 - 1/\epsilon)\text{OPT} - \mathcal{O}(\epsilon)$. As the number of calls to the stochastic oracle per iteration is of $\mathcal{O}(1/\epsilon)$, to reach a $(1 - 1/\epsilon)\text{OPT} - \mathcal{O}(\epsilon)$ approximation guarantee, SCG++ has an overall stochastic first-order oracle complexity of $\mathcal{O}(1/\epsilon^2)$. We formally characterize this result in the following corollary.
Corollary 4.5 (oracle complexities) To find a $[(1-1/e)OPT - \epsilon]$ solution to problem (I) using Algorithm I with Option II, the overall stochastic first-order oracle complexity is $(2G^2D^2 + 4\bar{L}^2D^4)/\epsilon^2$ and the overall linear optimization oracle complexity is $2\bar{L}D^2/\epsilon$.

5. Discrete Stochastic Submodular Maximization

In this section, we focus on extending our result in the previous section to the case where $F$ is the multilinear extension of a (stochastic) discrete submodular set function $f$. This is an important instance of non-oblivious stochastic optimization, introduced in (1). Indeed, once such a result is achieved, with a proper loss-less rounding scheme, including pipage rounding (Calinescu et al., 2007) or contention resolution method (Vondrák et al., 2011), we can extend our results to the discrete setting.

Let $V$ denote a finite set of $d$ elements, i.e., $V = \{1,\ldots,d\}$. Consider a discrete and monotone submodular function $f : 2^V \to \mathbb{R}_+$, which is defined as an expectation over a set of functions $f_\gamma : 2^V \to \mathbb{R}_+$. Our goal is to maximize $f$ subject to some constraint $I$, where $I$ encodes the collection of feasible solutions. In other words, we aim to solve the following discrete and stochastic submodular function maximization problem

$$\max_{S \subseteq I} f(S) := \max_{S \subseteq I} \mathbb{E}_{\gamma \sim p(\gamma)} [f_\gamma(S)],$$

(22)

where $p(\gamma)$ is an arbitrary distribution. In particular, we assume that the pair $M = \{V, I\}$ forms a matroid with rank $r$. The simplest example is maximization under the cardinality constraint, i.e., for a given integer $r$, find a set $S \subseteq V$, of size $|S| \leq r$, that maximizes $f$. The challenge here is to find a near-optimal solution for the problem (22) without explicitly computing the expectation. That is, we assume access to an oracle that, given a set $S$, outputs an independently chosen sample $f_\gamma(S)$ where $\gamma \sim p(\gamma)$. The focus of this section is to show how SCG++ can solve the discrete optimization problem (22). To do so, we rely on the multilinear extension $F : [0,1]^d \to \mathbb{R}_+$ of a discrete submodular function $f$ defined as

$$F(x) := \sum_{S \subseteq V} f(S) \prod_{i \in S} x_i \prod_{j \not\in S} (1-x_j) = \sum_{S \subseteq V} \mathbb{E}_{\gamma \sim p(\gamma)} [f_\gamma(S)] \prod_{i \in S} x_i \prod_{j \not\in S} (1-x_j),$$

(23)

where $C = \text{conv}\{1_I : I \in I\}$ is the matroid polytope (Calinescu et al., 2007). Note that here $x_i$ denotes the $i$-th component of the vector $x$. In other words, $F(x)$ is the expected value of $f$ over sets wherein each element $i$ is included independently with probability $x_i$. Specifically, in lieu of solving (22) we can maximize its multilinear extension, i.e.,

$$\max_{x \in C} F(x).$$

(24)

To this end, we need access to unbiased estimators of the gradient and the Hessian. In the following lemma, we recall the structure of the Hessian of the objective function (23).

Lemma 5.1 (Calinescu et al., 2007) Recall the definition of $F$ in (23) as the multilinear extension of the set function $f$ defined in (22). Then, for $i = j$ we have $[\nabla^2 F(y)]_{i,j} = 0$, and for $i \neq j$ we have

$$[\nabla^2 F(y)]_{i,j} = F(y; y_i \leftarrow 1, y_j \leftarrow 1) - F(y; y_i \leftarrow 1, y_j \leftarrow 0) - F(y; y_i \leftarrow 0, y_j \leftarrow 1) + F(y; y_i \leftarrow 0, y_j \leftarrow 0),$$

(25)
where for the vector \( \mathbf{y}; y_i \leftarrow c_i, y_j \leftarrow c_j \) the \( i^{th} \) and \( j^{th} \) entries are set to \( c_i \) and \( c_j \), respectively.

Note that every entry in (25) can be estimated direct sampling without introducing any bias. We will now construct the Hessian approximation \( \tilde{\nabla}_k^2 \) using Lemma 5.1. Let \( a \) be a uniform random variable between \([0, 1]\) and let \( e = (e_1, \cdots, e_d) \) be a random vector in which \( e_i \)'s are generated i.i.d. according to the uniform distribution over the unit interval \([0, 1]\). In each iteration, a minibatch \( M \) of \(|M|\) samples of \( \{a, e, \gamma\} \) (recall that \( \gamma \) is the random variable that parameterizes the component function \( f_\gamma \), i.e., \( M = \{a_k, e_k, \gamma_k\}_{k=1}^{\lfloor M/|M| \rfloor} \) is generated. Then for all \( k \in [M] \), we let \( x_{a_k} = a_kx^* + (1 - a_k)x^{k-1} \) and construct the random set \( S(x_{a_k}, e_k) \) using \( x_{a_k} \) and \( e_k \) in the following way: \( s \in S(x_{a_k}, e_k) \) if and only if \( [e_k]_s \leq [x_{a_k}]_s \) for \( s \in [d] \). Having \( S(x_{a_k}, e_k) \) and \( \gamma_k \), each entry of the Hessian estimator \( \tilde{\nabla}_k^2 \in \mathbb{R}^{d \times d} \) is

\[
[\tilde{\nabla}_k^2]_{i,j} = f_{\gamma_k}(S(x_{a_k}, e_k) \cup \{i, j\}) - f_{\gamma_k}(S(x_{a_k}, e_k) \cup \{i\} \setminus \{j\}) - f_{\gamma_k}(S(x_{a_k}, e_k) \cup \{j\} \setminus \{i\}) + f_{\gamma_k}(S(x_{a_k}, e_k) \setminus \{i, j\}),
\]

(26)

where \( i \neq j \), and if \( i = j \) then \( [\tilde{\nabla}_k^2]_{i,j} = 0 \). As linear optimization over the rank-\( r \) matroid polytope always return \( v^j \) with at most \( r \) nonzero entries, the complexity of computing \( \tilde{\nabla}_k^2 \cdot v^j \) is \( O(rd) \).

Now we use the above approximation of the Hessian to solve the multilinear extension as a special case of problem (1) using SCG++. To do so, we first introduce the following assumption.

**Assumption 5.1** Let \( D_\gamma \) denote maximum marginal value of \( f_\gamma \), i.e.,

\[ D_\gamma \overset{\text{def}}{=} \max_{i \in V, S \subseteq V} f_\gamma(S \cup \{i\}) - f_\gamma(S), \]

and further define \( D_f \overset{\text{def}}{=} \sqrt{\mathbb{E}_{\gamma}[D_\gamma^2]} \).

Under Assumption 5.1, the Hessian estimator \( \tilde{\nabla}_k^2 \) has bounded \( \| \cdot \|_{2, \infty} \) norm:

\[
\mathbb{E}\|\tilde{\nabla}_k^2\|_{2, \infty} = \mathbb{E}\left( \max_{i \in [d]} \|\tilde{\nabla}_k^2(:, i)\|^2 \right) \leq 4d \cdot \mathbb{E}_{\gamma}D_\gamma^2 = 4d \cdot D_f^2.
\]

We now analyze the convergence of SCG++ for solving the problem in (24). Compared to Theorem 4.4, Theorem 5.2 has an explicit dependence on the problem dimension \( d \) and exploits the sparsity of \( v^j \) to tighten the bounding terms of dimension-dependency.

**Theorem 5.2** Consider the multilinear extension problem (24) under Assumption 5.1. By using the minibatch size \( |M| = O(\sqrt{r^3d}D_f/\epsilon) \) and \( |M_0| = O(\sqrt{d}D_f/\sqrt{r}\epsilon^2) \), Algorithm 1 finds a \([1 - (1 - 1/\epsilon)OPT - 6\epsilon]\) solution to problem (24) with at most \((\sqrt{r^3d}D_f/\epsilon)\) iterations. Moreover, the overall stochastic oracle cost is \( O(r^3dD_f^2/\epsilon^2) \).

Note that, in multilinear extension case, the smoothness property required for the results in Section 4 is absent, and that is why we need to develop a more sophisticated gradient-difference estimator to achieve a similar theoretical guarantee (more details is available in section 8.5 of the appendix). Theorem 5.2 shows that by using a proper rounding scheme SCG++ finds a \([1 - (1 - 1/\epsilon)OPT - \epsilon]\) approximate solution of the discrete submodular maximization problem (22) after at most \( O(1/\epsilon^2) \) calls to the stochastic oracle. In the following corollary, we show that this complexity bound is optimal.
Algorithm 2 Stochastic Frank-Wolfe++ (SFW++)

**Input:** Number of iteration $T$, minibatch size $|\mathcal{M}_0^t|$ and $|\mathcal{M}_h^t|$, step size $\eta_t := \frac{2}{t+2}$

1: $t = 0$ to $T$
2: if $\log_2 t \in \mathbb{Z}$ then
3: Sample a minibatch $\mathcal{M}_0^t$ of $z$ with distribution $p(z; x^t)$ to compute $g^t \equiv \nabla f(x^t; \mathcal{M}_0^t)$;
4: else
5: Sample a minibatch $\mathcal{M}_h^t$ of $\Theta$ to compute $\tilde{\nabla}_2^t$ using (12)
6: Compute $g^t := g^{t-1} + \tilde{\nabla}_2^t(x^t - x^{t-1})$; $\diamond$ Or use (20) instead.
7: end if
8: $v^t := \text{argmax}_{v \in C}\{v^\top g^t\}$;
9: $x^{t+1} := x^t + \eta_t \cdot (v^t - x^t)$;
10: end for

**Theorem 5.3** There exists a distribution $p(\gamma)$ and a monotone submodular function $f : 2^V \rightarrow \mathbb{R}_+$ given as $f(S) = \mathbb{E}_{\gamma \sim p(\gamma)}[f_\gamma(S)]$, such that the following holds. In order to find a $[(1 - 1/e)\text{OPT} - \epsilon]$ solution for (22) with a $k$-cardinality constraint, any algorithm requires at least $\min\{\exp(\alpha k), \beta/\epsilon^2\}$ stochastic samples $f_\gamma(\cdot)$ where $\alpha, \beta$ are positive constants.

The result in Theorem 5.3 shows that finding a $[(1 - 1/e)\text{OPT} - \epsilon]$ solution for the problem in (22) could require at least $O(1/\epsilon^2)$ calls to the stochastic function $f_\gamma(\cdot)$. Therefore, SCG++ is optimal in terms of number of calls to the stochastic oracle, highlighted in the following remark.

**Remark 5.4 (optimality of stochastic oracle complexity)** To achieve the tight $[(1 - 1/e)\text{OPT} - \epsilon]$ approximate solution, the $O(1/\epsilon^2)$ stochastic oracle complexity in Theorem 5.2 is optimal in terms of its dependency on $\epsilon$.

6. Stochastic Concave Maximization

In this section, we focus on another case of problem (1) when the objective function $F$ is concave. By following the variance reduction technique developed in the gradient approximation of SCG++, we introduce the Stochastic Frank-Wolfe++ method (SFW++) for concave maximization which achieves an $\epsilon$ accurate solution in expectation after at most $O(1/\epsilon^2)$ stochastic gradient computations. We study the general case of non-oblivious stochastic optimization, but, indeed, the results also hold for the oblivious stochastic problem as a special case.

The steps of the SFW++ are summarized in Algorithm 2. Unlike the update of SCG++, in SFW++ we restart the gradient estimation $g$ for the iterates that are powers of 2. This modification is necessary to ensure that the noise of gradient approximation stays bounded by a proper constant when the diminishing step-size $\eta_t$ is used. The rest of gradient approximation scheme is similar to that of SCG++. Once the gradient approximation $g^t$ is evaluated we find the ascent direction $v^t$ by solving the linear optimization program $v^t := \text{argmax}_{v \in C}\{v^\top g^t\}$. Then, we compute the updated variable $x^{t+1}$ by performing the update $x^{t+1} := x^t + \eta_t \cdot (v^t - x^t)$, where $\eta_t$ is a chosen stepsize.

In the following theorem, we characterize the convergence properties of the proposed SFW++ method for solving stochastic concave maximization problems. For simplicity, we analyze the convergence using gradient-difference estimator (13). Similar results can be obtained when using (20).
Theorem 6.1  Consider problem (1) when $F$ is concave. Further, recall the SFW++ method outlined in Algorithm 2. Let $\bar{L}^2 \triangleq 4B^2G^4 + 16G^4 + 4L^2 + 4B^2L^2$. If the conditions in Assumptions 4.2-4.7 hold and we set SFW++ parameters to $\eta_t = 2/(t+2)$, $|\mathcal{M}_h^t| = 16(t+2)$ and $|\mathcal{M}_0^t| = (G^2(t+1)^2)/(\bar{L}^2D^2)$, then the iterates generated by SFW++ satisfy

$$F(x^*) - \mathbb{E}[F(x^t)] \leq \frac{28\bar{L}D^2 + (F(x^*) - F(x^0))}{t+2}.$$  

Theorem 6.1 shows that after at most $O(1/\epsilon)$ iterations, SFW++ reaches an $\epsilon$-approximate solution. To characterize the overall complexity, we need to take into account the number of stochastic gradient evaluations per iteration, as we do in the following corollary.

Corollary 6.2 (oracle complexity for convex case)  Assume that the target accuracy $\epsilon$ satisfies $t = (28LD^2 + (F(x^*) - F(x^0)))/\epsilon = 2^K$ for some $K \in \mathbb{N}$. The overall stochastic complexity is

$$\sum_{i=0}^{t} |\mathcal{M}_i^h| + \sum_{k=0}^{K} |\mathcal{M}_0^{2k}| = O \left( \frac{\bar{L}^2D^2}{\epsilon^2} + \frac{G^2D^2}{\epsilon^2} + \frac{G^2}{L^2D^2} \cdot \frac{(F(x^*) - F(x^0))^2}{\epsilon^2} \right).$$  

According to Corollary 6.2, SFW++ finds an $\epsilon$-approximate solution for stochastic concave maximization after at most computing $O(1/\epsilon^2)$ stochastic gradient evaluations.

In appendix 8.9 we show a similar oracle complexity result for the oblivious case with a significantly reduced dependence on the regularity parameters.

7. Conclusion

In this paper, we developed SCG++, the first efficient variant of a conditional gradient method for maximizing a stochastic continuous DR-submodular function subject to a convex constraint. We showed that SCG++ achieves a tight $[(1 - 1/e)OPT - \epsilon]$ solution while using $O(1/\epsilon^2)$ stochastic oracle queries. We also extended our results to stochastic concave maximization by developing SFW++, the first efficient projection-free algorithm that achieves an $\epsilon$-approximate solution with the optimum complexity $O(1/\epsilon^2)$ in terms of stochastic gradient evaluations.

References

Alekh Agarwal, Martin J Wainwright, Peter L Bartlett, and Pradeep K Ravikumar. Information-theoretic lower bounds on the oracle complexity of convex optimization. In Advances in Neural Information Processing Systems, pages 1–9, 2009.

Zeyuan Allen-Zhu. Natasha 2: Faster non-convex optimization than sgd. In Advances in Neural Information Processing Systems, pages 2680–2691, 2018.

F. Bach. Submodular functions: from discrete to continuous domains. arXiv preprint arXiv:1511.00394, 2015.

Andrew An Bian, Baharan Mirzasoleiman, Joachim Buhmann, and Andreas Krause. Guaranteed non-convex optimization: Submodular maximization over continuous domains. In Artificial Intelligence and Statistics, pages 111–120, 2017.
Gruia Calinescu, Chandra Chekuri, Martin Pál, and Jan Vondrák. Maximizing a submodular set function subject to a matroid constraint. In IPCO, volume 7, pages 182–196. Springer, 2007.

Gruia Calinescu, Chandra Chekuri, Martin Pál, and Jan Vondrák. Maximizing a monotone submodular function subject to a matroid constraint. SIAM Journal on Computing, 40(6):1740–1766, 2011.

Chandra Chekuri, Jan Vondrák, and Rico Zenklusen. Submodular function maximization via the multilinear relaxation and contention resolution schemes. SIAM Journal on Computing, 43(6):1831–1879, 2014.

Lin Chen, Hamed Hassani, and Amin Karbasi. Online continuous submodular maximization. arXiv preprint arXiv:1802.06052, 2018.

Michael Collins, Amir Globerson, Terry Koo, Xavier Carreras, and Peter L Bartlett. Exponentiated gradient algorithms for conditional random fields and max-margin markov networks. Journal of Machine Learning Research, 9(Aug):1775–1822, 2008.

A. Das and D. Kempe. Submodular meets spectral: Greedy algorithms for subset selection, sparse approximation and dictionary selection. ICML, 2011.

Aaron Defazio, Francis Bach, and Simon Lacoste-Julien. SAGA: A fast incremental gradient method with support for non-strongly convex composite objectives. In Advances in neural information processing systems, pages 1646–1654, 2014.

J. Djolonga and A. Krause. From map to marginals: Variational inference in bayesian submodular models. In NIPS, 2014.

Ethan Elenberg, Alexandros G Dimakis, Moran Feldman, and Amin Karbasi. Streaming weak submodularity: Interpreting neural networks on the fly. In Advances in Neural Information Processing Systems, pages 4044–4054, 2017.

Cong Fang, Chris Junchi Li, Zhouchen Lin, and Tong Zhang. Spider: Near-optimal non-convex optimization via stochastic path-integrated differential estimator. In Advances in Neural Information Processing Systems, pages 687–697, 2018.

Uriel Feige. A threshold of ln n for approximating set cover. Journal of the ACM (JACM), 45(4):634–652, 1998.

Moran Feldman, Joseph Naor, and Roy Schwartz. A unified continuous greedy algorithm for submodular maximization. In IEEE 52nd Annual Symposium on Foundations of Computer Science, pages 570–579, 2011.

Marguerite Frank and Philip Wolfe. An algorithm for quadratic programming. Naval Research Logistics (NRL), 3(1-2):95–110, 1956.

S. Fujishige. Submodular functions and optimization, volume 58. Annals of Discrete Mathematics, North Holland, Amsterdam, 2nd edition, 2005. ISBN 0-444-52086-4.

Satoru Fujishige and Shigueo Isotani. A submodular function minimization algorithm based on the minimum-norm base. Pacific Journal of Optimization, 7(1):3–17, 2011.
Shayan Oveis Gharan and Jan Vondrák. Submodular maximization by simulated annealing. In Proceedings of the Twenty-Second Annual ACM-SIAM Symposium on Discrete Algorithms, pages 1098–1116, 2011.

Hamed Hassani, Mahdi Soltanolkotabi, and Amin Karbasi. Gradient methods for submodular maximization. In Advances in Neural Information Processing Systems, pages 5841–5851, 2017.

Elad Hazan and Satyen Kale. Projection-free online learning. In Proceedings of the 29th International Conference on Machine Learning, ICML 2012, Edinburgh, Scotland, UK, June 26 - July 1, 2012, pages 1843–1850, 2012.

Elad Hazan and Haipeng Luo. Variance-reduced and projection-free stochastic optimization. In Proceedings of the 33rd International Conference on Machine Learning, ICML 2016, New York City, NY, USA, June 19-24, 2016, pages 1263–1271, 2016.

Martin Jaggi. Revisiting Frank-Wolfe: Projection-free sparse convex optimization. In Proceedings of the 30th International Conference on Machine Learning, ICML 2013, Atlanta, GA, USA, 16-21 June 2013, pages 427–435, 2013.

Rie Johnson and Tong Zhang. Accelerating stochastic gradient descent using predictive variance reduction. In Advances in neural information processing systems, pages 315–323, 2013.

Alex Kulesza and Ben Taskar. Determinantal point processes for machine learning. arXiv preprint arXiv:1207.6083, 2012.

H. Lin and J. Bilmes. A class of submodular functions for document summarization. In Proceedings of Annual Meeting of the Association for Computational Linguistics: Human Language Technologies, 2011a.

Hui Lin and Jeff Bilmes. Word alignment via submodular maximization over matroids. In Proceedings of the 49th Annual Meeting of the Association for Computational Linguistics: Human Language Technologies: short papers-Volume 2, pages 170–175. Association for Computational Linguistics, 2011b.

Aryan Mokhtari, Hamed Hassani, and Amin Karbasi. Conditional gradient method for stochastic submodular maximization: Closing the gap. In International Conference on Artificial Intelligence and Statistics, pages 1886–1895, 2018a.

Aryan Mokhtari, Hamed Hassani, and Amin Karbasi. Stochastic conditional gradient methods: From convex minimization to submodular maximization. arXiv preprint arXiv:1804.09554, 2018b.

George L Nemhauser and Laurence A Wolsey. Best algorithms for approximating the maximum of a submodular set function. Mathematics of operations research, 3(3):177–188, 1978.

George L Nemhauser, Laurence A Wolsey, and Marshall L Fisher. An analysis of approximations for maximizing submodular set functions—I. Mathematical Programming, 14(1):265–294, 1978.

Arkadi Nemirovskii and D Yudin. On Cezari’s convergence of the steepest descent method for approximating saddle point of convex-concave functions. In Soviet Math. Dokl, volume 19, 1978.
Arkadii Nemirovskii, David Borisovich Yudin, and Edgar Ronald Dawson. Problem complexity and method efficiency in optimization. 1983.

Lam M Nguyen, Jie Liu, Katya Scheinberg, and Martin Takáč. SARAH: A novel method for machine learning problems using stochastic recursive gradient. In International Conference on Machine Learning, pages 2613–2621, 2017.

Rad Niazadeh, Tim Roughgarden, and Joshua R Wang. Optimal algorithms for continuous non-monotone submodular and dr-submodular maximization. 2018.

Sashank J Reddi, Ahmed Hefny, Suvrit Sra, Barnabas Poczos, and Alex Smola. Stochastic variance reduction for nonconvex optimization. In International conference on machine learning, pages 314–323, 2016.

Herbert Robbins and Sutton Monro. A stochastic approximation method. The annals of mathematical statistics, pages 400–407, 1951.

Mark Schmidt, Nicolas Le Roux, and Francis Bach. Minimizing finite sums with the stochastic average gradient. Mathematical Programming, 162(1-2):83–112, 2017.

A. Singla, I. Bogunovic, G. Bartok, A. Karbasi, and A. Krause. Near-optimally teaching the crowd to classify. In ICML, 2014.

T. Soma and Y. Yoshida. A generalization of submodular cover via the diminishing return property on the integer lattice. In NIPS, 2015.

Maxim Sviridenko, Jan Vondrák, and Justin Ward. Optimal approximation for submodular and supermodular optimization with bounded curvature. In Proceedings of the Twenty-Sixth Annual ACM-SIAM Symposium on Discrete Algorithms, pages 1134–1148, 2015.

Jan Vondrák. Optimal approximation for the submodular welfare problem in the value oracle model. In Proceedings of the fortieth annual ACM symposium on Theory of computing, pages 67–74. ACM, 2008.

Jan Vondrák, Chandra Chekuri, and Rico Zenklusen. Submodular function maximization via the multilinear relaxation and contention resolution schemes. In Proceedings of the forty-third annual ACM symposium on Theory of computing, pages 783–792. ACM, 2011.

Laurence A. Wolsey. An analysis of the greedy algorithm for the submodular set covering problem. Combinatorica, 1982.
8. Appendix

8.1. Proof of Lemma 4.1

Proof Recall the definition of $F(y) = \int_{z \in Z} \tilde{F}(y; z)p(z; y) dz$. The first order differential of $F(\cdot)$ can be computed by

$$
\nabla F(y) = \int_{z \in Z} p(z; y) \nabla \tilde{F}(y; z) + \tilde{F}(y; z) \nabla p(z; y) dz
$$

where we use $\nabla \log p(z; y) = \frac{\nabla p(z; y)}{p(z; y)}$ in the second equality. We now compute the second order differential of $F(\cdot)$ by

$$
\nabla^2 F(y) = \int_{z \in Z} \left[ \nabla \tilde{F}(y; z) + \tilde{F}(y; z) \nabla \log p(z; y) \right] \left[ \nabla p(z; y) \right]^T dz
$$

where again we use $\nabla \log p(z; y) = \frac{\nabla p(z; y)}{p(z; y)}$ in the second equality. From such derivation, we have the result.

8.2. Proof of Lemma 4.3

Before we give the proof of Lemma 4.3, we first present a lemma which bounds the second moment of the spectral norm of the Hessian estimator $\nabla^2 \tilde{F}(y; z)$ for any $y \in C$.

Lemma 8.1 Recall the definition of the Hessian estimator $\nabla^2 \tilde{F}(y; z)$ in (11). Under Assumptions 4.3, 4.5, 4.6, for any $y \in C$ we can show that

$$
\mathbb{E}_{z \sim p(x, y)} \| \nabla^2 \tilde{F}(y; z) \|^2 \leq 4B^2G^4 + 16G^4 + 4L^2 + 4B^2L^2 \overset{\text{def}}{=} \bar{L}^2. \quad (28)
$$

Proof [Lemma 8.1] From the definition of the Hessian estimator $\nabla^2 \tilde{F}(y; z)$ (see (11)), we have

$$
\| \nabla^2 F(y; z) \| \leq B \| \nabla \log p(z; y) \|^2 + 2G \| \nabla \log p(z; y) \| + L + B \| \nabla^2 \log p(z; y) \|
$$

where we use Assumptions 4.3 and 4.5 and the triangle inequality. Further, take expectation on both sides and use Assumption 4.6 to bound

$$
\mathbb{E} \| \nabla^2 F(y; z) \|^2 \leq 4B^2G^4 + 16G^4 + 4L^2 + 4B^2L^2. \quad (29)
$$

17
\textbf{Proof} [Lemma 4.3] We prove via induction. When \( t = 0 \), by using the unbiasedness of \( \nabla \tilde{F}(x^0; z) \) and Assumption 4.5, we bound
\[
\mathbb{E}_{\mathcal{M}_0} \| F(x^0) - g^0 \|^2 = \frac{1}{|\mathcal{M}_0|} \mathbb{E} \| F(x^0) - \nabla \tilde{F}(x^0; z) \|^2 \leq \frac{1}{|\mathcal{M}_0|} \mathbb{E} \| \nabla \tilde{F}(x^0; z) \|^2 \leq \frac{G^2}{|\mathcal{M}_0|} \leq \bar{L}^2 D^2 \epsilon^2.
\]
Now assume that we have the result for \( t = \bar{t} \). When \( t = \bar{t} + 1 \), we have from the definition of \( g^t \)
\[
g^t - \nabla F(x^t) = [g^{t-1} - \nabla F(x^{t-1})] + \left[ \xi_d(x; \mathcal{M}) - \tilde{V}^t_\mathcal{M}(x^t - x^{t-1}) \right] + \left[ \tilde{V}^t_\mathcal{M}(x^t - x^{t-1}) - (\nabla F(x^t) - \nabla F(x^{t-1})) \right].
\]
Expand \( \| \nabla F(x^t) - g^t \|^2 \) to obtain
\[
\| \nabla F(x^t) - g^t \|^2 = \| \nabla F(x^t) - \nabla F(x^{t-1}) - \nabla \tilde{V}^t_\mathcal{M}(x^t - x^{t-1}) \|^2 + \| g^{t-1} - \nabla F(x^{t-1}) \|^2 + 2\langle \nabla F(x^t) - \nabla F(x^{t-1}) - \nabla \tilde{V}^t_\mathcal{M}(x^t - x^{t-1}), g^{t-1} - \nabla F(x^{t-1}) \rangle + 2\langle \nabla \tilde{V}^t_\mathcal{M}(x^t - x^{t-1}) - \nabla \tilde{V}^t_\mathcal{M}(x^t - x^{t-1}), g^{t-1} - \nabla F(x^{t-1}) \rangle + \| \nabla \tilde{V}^t_\mathcal{M}(x^t - x^{t-1}) - \nabla \tilde{V}^t_\mathcal{M}(x^t - x^{t-1}) \|^2.
\]
Using the unbiasedness of \( \nabla \tilde{V}^t_\mathcal{M}(x^t - x^{t-1}) \), we have
\[
\mathbb{E} \langle \nabla F(x^t) - \nabla F(x^{t-1}) - \nabla \tilde{V}^t_\mathcal{M}(x^t - x^{t-1}), g^{t-1} - \nabla F(x^{t-1}) \rangle = 0.
\]
Additionally, from the unbiasedness of \( \nabla \tilde{V}^t_\mathcal{M}(x^t - x^{t-1}) \) and \( \| x^t - x^{t-1} \| \leq \epsilon D \), we have
\[
\mathbb{E} \| \nabla \tilde{V}^t_\mathcal{M}(x^t - x^{t-1}) - (\nabla F(x^t) - \nabla F(x^{t-1})) \|^2 \leq \frac{\epsilon^2 D^2}{|\mathcal{M}|} \mathbb{E} \| \nabla F(x^t; z(a)) \|^2 \leq \frac{\epsilon^2 \bar{L}^2 D^2}{|\mathcal{M}|},
\]
where we use Lemma 8.1 in the last inequality. Taking expectation on both sides of (32), we have
\[
\mathbb{E} \| \nabla F(x^t) - g^t \|^2 \leq \mathbb{E} \| \nabla F(x^t) - \nabla F(x^{t-1}) - \nabla \tilde{V}^t_\mathcal{M}(x^t - x^{t-1}) \|^2 + \mathbb{E} \| g^{t-1} - \nabla F(x^{t-1}) \|^2 + 2\mathbb{E} \| \nabla \tilde{V}^t_\mathcal{M}(x^t - x^{t-1}) - \nabla \tilde{V}^t_\mathcal{M}(x^t - x^{t-1}) \|^2 + 2\mathbb{E} \| \nabla \tilde{V}^t_\mathcal{M}(x^t - x^{t-1}) - \nabla \tilde{V}^t_\mathcal{M}(x^t - x^{t-1}) \|^2 + 4D^2 L^2 \delta^2 + 4D^2 L^2 \delta \| g^{t-1} - \nabla F(x^{t-1}) \| + 4D^2 L^2 \delta \| g^{t-1} - \nabla F(x^{t-1}) \| \leq \frac{\bar{L}^2 D^2 \epsilon^2}{|\mathcal{M}|} + (1 + \epsilon(t-1))\bar{L}^2 D^2 \epsilon^2 + 4\delta \left( \frac{D^2 L^2 \bar{L} \delta}{\sqrt{|\mathcal{M}|}} + D^2 L^2 \sqrt{(1 + \epsilon(t-1))} \bar{L} \delta + D^4 L^2 \delta^2 \right).
\]
By taking \( \delta \) sufficiently small such that for all \( t \leq 1/\epsilon \)
\[
4\delta \left( \frac{D^2 L^2 \bar{L} \delta}{\sqrt{|\mathcal{M}|}} + D^2 L^2 \sqrt{(1 + \epsilon(t-1))} \bar{L} \delta + D^4 L^2 \delta^2 \right) \leq \bar{L}^2 D^2 \epsilon^3/2,
\]
we have shown that the induction holds for \( t = \bar{t} + 1 \). Note that the inequality in (35) holds if we set \( \delta = \mathcal{O}(\epsilon^2) \). \[\blacksquare\]
8.3. Proof of Theorem 4.4

Proof From Lemma 8.1, we have the $\bar{L}$-smoothness of $F$:

$$||\nabla^2 F(x)||^2 \leq ||E_{z \sim p(z|x)}\nabla^2 \tilde{F}(x; z)||^2 \leq E_{z \sim p(z|x)}||\nabla^2 \tilde{F}(x; z)||^2 \leq \bar{L}^2. \quad (36)$$

Let $x^*$ be a global maximizer within the constraint set $C$. From the smoothness of $F$, we have

$$F(x^{t+1}) \geq F(x^t) + \langle \nabla F(x^t), x^{t+1} - x^t \rangle - \frac{\bar{L}}{2} ||x^{t+1} - x^t||^2$$

$$= F(x^t) + \frac{1}{T} \langle \nabla^2 F(x^t), v^t \rangle - \frac{\bar{L}}{2T^2} ||v^t||^2$$

$$= F(x^t) + \frac{1}{T} \langle g^t, v^t \rangle + \frac{1}{T} \langle \nabla F(x^t) - g^t, v^t \rangle - \frac{\bar{L}D^2}{2T^2}$$

$$\geq F(x^t) + \frac{1}{T} \langle g^t, x^* \rangle + \frac{1}{T} \langle \nabla F(x^t) - g^t, v^t \rangle - \frac{\bar{L}D^2}{2T^2},$$

where we use the optimality and boundedness of $v^t$ in the last inequality. Take expectation on both sides and use the unbiasedness of $g^t$ to yield

$$\mathbb{E}F(x^{t+1}) \geq \mathbb{E}F(x^t) + \frac{1}{T} \mathbb{E}[F(x^*) - F(x^t)] + \frac{1}{T} \mathbb{E}[\nabla F(x^t) - g^t, v^t] - \frac{\bar{L}D^2}{2T^2}. \quad (38)$$

From the monotonicity of $F$ and the concavity of $F$ along positive directions, we have $\langle \nabla F(x^t), x^* \rangle \geq F(x^*) - F(x^t)$ (Mokhtari et al., 2018a). Additionally, using the Young’s inequality, we write

$$\mathbb{E}F(x^{t+1}) \geq \mathbb{E}F(x^t) + \frac{1}{T} \mathbb{E}[F(x^*) - F(x^t)] - \frac{1}{2L} \mathbb{E}||\nabla F(x^t) - g^t||^2 - \frac{\bar{L}D^2}{T^2}.$$ 

Using Lemma 4.3, we have for all $t \in \{0, \ldots, T - 1\}$

$$\mathbb{E}||\nabla F(x^t) - g^t||^2 \leq 2\bar{L}^2D^2\epsilon^2. \quad (39)$$

Consequently, we have with $T = \frac{1}{\epsilon}$

$$\mathbb{E}F(x^{t+1}) \geq \mathbb{E}F(x^t) + \epsilon \mathbb{E}[F(x^*) - F(x^t)] - 2\bar{L}\epsilon^2D^2,$$

which is equivalent to

$$\mathbb{E}[F(x^*) - F(x^{t+1})] \leq (1 - \epsilon)^T \mathbb{E}[F(x^*) - F(x^t)] - 2\bar{L}\epsilon D^2.$$

In conclusion, we have

$$\mathbb{E}F(x^T) \geq (1 - 1/\epsilon)\mathbb{E}[F(x^*)] - 2\bar{L}\epsilon D^2.$$
8.4. Multilinear Extension as Non-oblivious Stochastic Optimization

We proceed to show that the multilinear extension problem in (23) is captured by (1). To do so, use $\text{Ber}(b; m)$ with $b \in \{0, 1\}$ and $m \in [0, 1]$ to denote the Bernoulli distribution with parameter $m$, i.e.

$$\text{Ber}(b; m) = m^b(1 - m)^{1-b},$$

and denote the $i^{th}$ entries of $\mathbf{z}$ and $\mathbf{x}$ by $z_i$ and $x_i$. The distribution $p(\mathbf{z}, \gamma; \mathbf{x})$ is defined as

$$p(\mathbf{z}, \gamma; \mathbf{x}) = p(\gamma) \times \prod_{i=1}^{d} \text{Ber}(z_i; x_i),$$  \hspace{1cm} (40)

where $p(\gamma)$ is defined in (22). Let $N(\mathbf{z})$ be a subset of $N$ such that $i \in N(\mathbf{z})$ if and only if $z_i = 1$. We then define the stochastic function $\tilde{F}(\mathbf{x}; \mathbf{z}, \gamma)$ as

$$\tilde{F}(\mathbf{x}; \mathbf{z}, \gamma) = f_\gamma(N(\mathbf{z})), \hspace{1cm} (41)$$

where $f_\gamma$ is defined in (22). We emphasize that for a fixed $\mathbf{z}$ the stochastic function $\tilde{F}$ does not depend on $\mathbf{x}$ and hence $\nabla \tilde{F}(\mathbf{x}; \mathbf{z}) = 0$. Considering the definition of the stochastic function $\tilde{F}(\mathbf{x}; \mathbf{z}, \gamma)$ in (41), the multilinear extension function $\tilde{F}$ in (23), and the probability distribution $p(\mathbf{z}, \gamma; \mathbf{x})$ in (40), it can be verified that $\tilde{F}$ is the expectation of the random function $\tilde{F}(\mathbf{x}; \mathbf{z}, \gamma)$, and, therefore, the problem in (23) can be written as (1).

At the first glance, it seems that we can apply the SCG++ method in Algorithm 1 to maximize the multilinear extension function $\tilde{F}$. However, the smoothness conditions required for the result in Theorem 4.4 do not hold in the multilinear setting. To be more specific, following the result in Lemma 4.1, we can derive an unbiased estimator for the second-order differential of (23) using

$$\tilde{\nabla}^2 \tilde{F}(\mathbf{y}; \mathbf{z}) = \tilde{F}(\mathbf{y}; \mathbf{z}) \left[ (\nabla \log p(\mathbf{z}, \gamma; \mathbf{y}) [\nabla \log p(\mathbf{z}, \gamma; \mathbf{y})]^\top + \nabla^2 \log p(\mathbf{z}, \gamma; \mathbf{y}) \right],$$

$$= f_\gamma(N(\mathbf{z})) \left[ \sum_{i=1}^{d} \nabla \log \text{Ber}(z_i; x_i) [\nabla \log \text{Ber}(z_i; x_i)]^\top + \sum_{i=1}^{d} \nabla^2 \log \text{Ber}(z_i; x_i) \right], \hspace{1cm} (42)$$

where we use $\nabla \tilde{F}(\mathbf{x}; \mathbf{z}) = 0$ in the first equality and use (40) and (41) in the second one. Further, note that $[\nabla \log \text{Ber}(z_i; x_i)]^2 + \nabla^2 \log \text{Ber}(z_i; x_i) = 0$ for all $i \in [d]$ and hence, the above estimator can be further simplified to

$$\tilde{\nabla}^2 \tilde{F}(\mathbf{y}; \mathbf{z}, \gamma) = f_\gamma(N(\mathbf{z})) \sum_{i,j=1}^{d} \mathbbm{1}_{i \neq j} [\nabla \log \text{Ber}(z_i; x_i)] [\nabla \log \text{Ber}(z_j; x_j)]^\top. \hspace{1cm} (43)$$

Despite the simple form of (43), the smoothness property in Assumption 4.6 is absent since every entry in $\tilde{\nabla}^2 \tilde{F}(\mathbf{y}; \mathbf{z}, \gamma)$ can have unbounded second-order moment when $x_i \to 0$ or $x_i \to 1$.

8.5. Detailed Implementation of SCG++ for Multilinear Extension

While we have briefly mentioned the Hessian estimator $\tilde{\nabla}_h^2$ in (26), in this section, we describe SCG++ for the Multilinear Extension problem (24) in Algorithm 3. In particular, we specify the gradient construction for $x^0$ using the fact that

$$[\nabla F(x)]_i = F(x; x_i \leftarrow 1) - F(x; x_i \leftarrow 0), \hspace{1cm} (44)$$
for the multilinear extension $F$. Since both terms in (44) are expectation, we can directly sample a mini-batch $\mathcal{M}_0$ of $(\gamma, z)$ pair from (40) to obtain an unbiased estimator of $\nabla F(x)$ by

$$[g^0]_i \overset{def}{=} \frac{1}{|\mathcal{M}_0|} \sum_{(\gamma, z) \in \mathcal{M}_0} f_\gamma(N(z) \cup \{i\}) - f_\gamma(N(z) \setminus \{i\}).$$ (45)

8.6. Proof of Lemma 5.1

Proof First note that

$$\nabla x_i \log \text{Ber}(z_i; x_i) = \frac{z_i}{x_i} - \frac{1 - z_i}{1 - x_i}. \quad (46)$$

We use $z_{i,j}$ to denote the random vector $z$ excluding the $i^{th}$ and $j^{th}$ entries, and denote $z; z_i \leftarrow c_i, z_j \leftarrow c_j$ as the random vector obtained by setting the $i^{th}$ and $j^{th}$ entries of $z$ to corresponding constants $c_i$ and $c_j$. Compute $\mathbb{E}_{z \sim p(z; x)}[\nabla^2 F(y; z, \gamma)]_{i,j}$ using (43)

$$\mathbb{E}_{z \sim p(z; x)}[\nabla^2 F(y; z, \gamma)]_{i,j} = \mathbb{E}_{z \sim p(z; x)}[f(N(z))[\nabla x_i \log \text{Ber}(z_i; x_i)][\nabla x_j \log \text{Ber}(z_j; x_j)]]
= \sum_{c_i, c_j \in \{0, 1\}^2} \mathbb{E}_{z_{i,j}} f(N(z; z_i \leftarrow c_i, z_j \leftarrow c_j))(-1)^{c_i}(-1)^{c_j}$$

where in the first equality we use $\mathbb{E}_x f_\gamma = f$ and the second one uses

$$x_i^{c_i} \cdot (1 - x_i)^{1 - c_i} \cdot \left[\frac{c_i}{x_i} - \frac{1 - c_i}{1 - x_i}\right] = -(-1)^{c_i}. \quad (47)$$

We discuss in detail the case of $c_i = c_j = 1$. The other three cases can be obtained similarly.

$$\mathbb{E}_{z_{i,j}} f(N(z; z_i \leftarrow 1, z_j \leftarrow 1)) = F(y; y_i \leftarrow 1, y_j \leftarrow 1), \quad (48)$$

which recovers the first term in (25). \[\]
8.7. Proof of Theorem 5.2

To prove the claim in Theorem 5.2 we first prove the following lemma. The following lemma exploits the sparsity of \( v^t \) and the upper bound on the \( \| \cdot \|_{2,\infty} \) norm of \( \nabla g_k \) to give a tighter variance bound on \( g^t \) with an explicit dependence on the problem dimension \( d \).

**Lemma 8.2** Recall the constructions of the gradient estimator (15) and the Hessian estimator (26). In the multilinear extension problem (24), under Assumption 5.1, we have the variance bound

\[
\mathbb{E}\|g^t - \nabla F(x^t)\|^2 \leq \frac{4r^2d \cdot \epsilon}{|M|} D_f^2 + \frac{dD_j^2}{|M_0|}.
\]  

**Proof** For \( k = 0 \), we bound

\[
\mathbb{E}\|g^0 - \nabla F(x^0)\|^2 \leq \frac{1}{|M_0|} \sum_{i=1}^d D_j^2 = \frac{dD_j^2}{|M_0|},
\]

\[
\mathbb{E}\|g^t - \nabla F(x^t)\|^2 = \mathbb{E}\|\tilde{g}^t + g^{t-1} - \nabla F(x^t)\|^2
\]

\[
= \mathbb{E}\|\tilde{g}^t - (\nabla F(x^t) - \nabla F(x^{t-1}))\|^2 + \mathbb{E}\|g^{t-1} - \nabla F(x^{t-1})\|^2
\]

\[
= \frac{1}{|M|} \mathbb{E}\|\nabla_1 (x^t - x^{t-1}) - (\nabla F(x^t) - \nabla F(x^{t-1}))\|^2 + \mathbb{E}\|g^{t-1} - \nabla F(x^{t-1})\|^2
\]

\[
\leq \frac{1}{|M|} \mathbb{E}\|\nabla_1 (x^t - x^{t-1})\|^2 + \mathbb{E}\|g^{t-1} - \nabla F(x^{t-1})\|^2
\]

Note that \( x^{t+1} - x^t = \epsilon v^t \), which has \( r \) entries with value \( \epsilon \) and \( d-r \) entries with value 0. Therefore

\[
\mathbb{E}\|g^t - \nabla F(x^t)\|^2 \leq \frac{r^2 \epsilon^2}{|M|} \mathbb{E}\|\nabla_1\|^2_{2,\infty} + \mathbb{E}\|g^{t-1} - \nabla F(x^{t-1})\|^2
\]

\[
\leq \frac{4r^2d \cdot \epsilon^2}{|M|} D_f^2 + \mathbb{E}\|g^{t-1} - \nabla F(x^{t-1})\|^2.
\]

Repeat the above recursion \( t \leq \frac{1}{\epsilon} \) times, we obtain

\[
\mathbb{E}\|g^t - \nabla F(x^t)\|^2 \leq \frac{4r^2d \cdot t \cdot \epsilon^2}{|M|} D_f^2 + \frac{dD_j^2}{|M|} \leq \frac{4r^2d \cdot \epsilon}{|M|} D_f^2 + \frac{dD_j^2}{|M_0|}.
\]  

**Proof** (Proof of Theorem 5.2) From calculus, we know that

\[
|F(x^{t+1}) - F(x^t) - \langle \nabla F(x^t), x^{t+1} - x^t \rangle|
\]

\[
= \frac{1}{2} \int_0^1 \|\nabla^2 F(x(a))(x^{t+1} - x^t), (x^{t+1} - x^t)\| \, da
\]

\[
\leq \frac{1}{2} \int_0^1 \|\nabla^2 F(x(a))(x^{t+1} - x^t)\| \cdot \|x^{t+1} - x^t\| \, da
\]

\[
\leq \frac{1}{2} \int_0^1 \sqrt{r \cdot \|\nabla^2 F(x(a))\|_{2,\infty}^2 \cdot \|x^{t+1} - x^t\|^2} \, da
\]

\[
\leq \frac{1}{2} \sqrt{rD_f^2} \cdot \|x^{t+1} - x^t\|^2,
\]
where \( x(a) = ax^t + (1 - a)x^{t-1} \) with \( 0 \leq a \leq 1 \) and we use \( x^{t+1} - x^t = 1/T \cdot v^t \) which has at most \( r \) non-zero entries in inequality (i). We thus have the following bound on \( F(x^{k+1}) \):

\[
F(x^{t+1}) \geq F(x^t) + \langle \nabla F(x^t), x^{t+1} - x^t \rangle - \sqrt{f_d D_f^2} \| x^{t+1} - x^t \|^2 \\
= F(x^t) + \frac{1}{T} \langle \nabla F(x^t) - g^t, v^t \rangle + \frac{1}{T} \langle g^t, v^t \rangle - \frac{\sqrt{r d D_f^2}}{T^2} \| v^t \|^2 \\
\geq F(x^t) + \frac{1}{T} \langle \nabla F(x^t) - g^t, v^t \rangle + \frac{1}{T} \langle g^t, x^* \rangle - \frac{\sqrt{r d D_f^2}}{T^2} \| v^t \|^2.
\]

Take expectation on both sides and use the unbiasedness of \( g^t \) to yield

\[
\mathbb{E} F(x^{t+1}) \geq \mathbb{E} F(x^t) + \frac{1}{T} \mathbb{E} \langle \nabla F(x^t), x^* \rangle + \frac{1}{T} \mathbb{E} \langle \nabla F(x^t) - g^t, v^t \rangle - \frac{\sqrt{r d D_f^2}}{T^2}. \tag{53}
\]

From the monotonicity of \( F \) and the concavity of \( F \) along positive directions, we have \( \langle \nabla F(x^t), x^* \rangle \geq F(x^*) - F(x^t) \). Additionally, using the Young’s inequality, we write

\[
\mathbb{E} F(x^{t+1}) \geq \mathbb{E} F(x^t) + \frac{1}{T} \mathbb{E} [F(x^*) - F(x^t)] - \frac{1}{2} \mathbb{E} \| \nabla F(x^t) - g^t \|^2 - \frac{2}{T^2} \sqrt{r d D_f^2}.
\]

Recall the variance bound (51)

\[
\mathbb{E} \| g^t - \nabla F(x^t) \|^2 \leq \frac{4r^2 d \cdot \epsilon}{|\mathcal{M}|} D_f^2 + \frac{d D_f^2}{|\mathcal{M}_0|}.
\]

By choosing \( |\mathcal{M}| = \frac{2}{\epsilon} \) and \( |\mathcal{M}_0| = \frac{1}{2r^2} \), we have

\[
\mathbb{E} \| \nabla F(x^t) - g^t \|^2 \leq 4r^2 d \cdot \epsilon^2 D_f^2 \tag{54}
\]

and consequently by setting \( T = \frac{1}{\epsilon} \) we have

\[
\mathbb{E} F(x^{t+1}) \geq \mathbb{E} F(x^t) + \epsilon \mathbb{E} [F(x^*) - F(x^t)] - 6 \sqrt{r^3 d D_f^2} \cdot \epsilon^2,
\]

which can be translated to

\[
\mathbb{E} [F(x^*) - F(x^t)] \leq (1 - \epsilon)^{\frac{1}{2}} \mathbb{E} [F(x^*) - F(x^0)] - 6 \sqrt{r^3 d} \cdot D_f \cdot \epsilon.
\]

In conclusion, we have

\[
\mathbb{E} F(x^t) \geq (1 - 1/e) \mathbb{E} [F(x^*)] - 6 \sqrt{r^3 d} \cdot D_f \cdot \epsilon.
\]
8.8. Proof of Theorem 6.1

To prove the theorem we first prove the following lemma.

**Lemma 8.3 (variance bound)** Recall the definition of the Hessian estimator \( \nabla^2 \tilde{F}(y; z) \) in (11). Under Assumptions 4.3, 4.5, 4.6, by taking \( |M_h| = 16(t + 2) \) and \( |M_0| = \frac{G^2(t+1)^2}{L^2D^2} \), we bound

\[
    \mathbb{E}\|g^t - \nabla F(x^t)\|^2 \leq 2\tilde{L}^2D^2\eta_t^2,
\]

where \( \tilde{L} \) is defined in Lemma 8.1.

**Proof** In the following, we say iteration \( t \) is in the \( k^{th} \) epoch if \( 2^k \leq t < 2^{k+1} \). Now we prove the result in the \( k^{th} \) epoch. For mod\((t, 2^k) \neq 0 \), we have

\[
    \mathbb{E}\|g^t - \nabla F(x^t)\|^2 = \mathbb{E}\|\nabla^2_t [x^t - x^{t-1}] + g^{t-1} - \nabla F(x^t)\|^2
\]

\[
    = \mathbb{E}\|\nabla^2_t [x^t - x^{t-1}] - (\nabla F(x^t) - \nabla F(x^{t-1}))\|^2 + \mathbb{E}\|g^{t-1} - \nabla F(x^{t-1})\|^2
\]

\[
    \leq \frac{1}{|M_h|} \mathbb{E}\|\nabla^2_t [x^t - x^{t-1}] - (\nabla F(x^t) - \nabla F(x^{t-1}))\|^2 + \mathbb{E}\|g^{t-1} - \nabla F(x^{t-1})\|^2
\]

Observe that \( x^{t+1} - x^t = \eta_t(v^t - x^t) \) and therefore

\[
    \mathbb{E}\|g^t - \nabla F(x^t)\|^2 \leq \frac{4\eta_t^2 D^2}{|M_h|} \mathbb{E}\|\nabla^2_t\|^2 + \mathbb{E}\|g^{t-1} - \nabla F(x^{t-1})\|^2
\]

\[
    \leq \frac{4\eta_t^2 D^2 \tilde{L}^2}{|M_h|} + \mathbb{E}\|g^{t-1} - \nabla F(x^{t-1})\|^2,
\]

where we use Lemma 8.1 in the second inequality. By repeat the above recursion \( t - 2^k < 2^{k+1} \) times (since \( t < 2^{k+1} \)), we obtain

\[
    \mathbb{E}\|g^t - \nabla F(x^t)\|^2 \leq \mathbb{E}\|g^{2^k} - \nabla F(x^{2^k})\|^2 + \sum_{i=2^k}^{t} 4D^2 \tilde{L}^2 \cdot \eta_t^2
\]

\[
    \leq \frac{G^2}{|M_0^{2^k}|} + D^2 \tilde{L}^2 \sum_{i=2^k}^{t} \frac{1}{(i + 2)^3}
\]

\[
    \leq \tilde{L}^2D^2\eta_t^2 + \sum_{i=2^k}^{t} \left[ \frac{1}{(i + 1)(i + 2)} - \frac{1}{(i + 2)(i + 3)} \right] \leq \tilde{L}^2D^2\eta_t^2 + \sum_{i=2^k}^{t} \frac{1}{(2k+1)(2k-1+1)} \leq 2\tilde{L}^2D^2\eta_t^2 \leq 2\tilde{L}^2D^2\eta_t^2
\]

Now we are ready to prove the claim in Theorem 6.1. From Lemma 8.1, we have the \( \tilde{L} \)-smoothness of the function \( F' \):

\[
    \|\nabla^2 F(x)\|^2 \leq \|E_{z \sim p(z;x)} \nabla^2 \tilde{F}(x; z)\|^2 \leq \mathbb{E}_{z \sim p(z;x)}\|\nabla^2 \tilde{F}(x; z)\|^2 \leq \tilde{L}^2.
\]

(56)
Let $x^*$ be a global maximizer within the constraint set $C$. From the smoothness of $F$, we have

$$ F(x^{t+1}) \geq F(x^t) + \langle \nabla F(x^t), x^{t+1} - x^t \rangle - \frac{L}{2} \| x^{t+1} - x^t \|^2 $$

$$ = F(x^t) + \eta_t \langle \nabla F(x^t), \nabla^t - x^t \rangle - \frac{\bar{L} \eta_t^2}{2} \| \nabla^t - x^t \|^2 $$

$$ = F(x^t) + \eta_t \langle g^t, \nabla^t - x^t \rangle + \eta_t \langle \nabla F(x^t) - g^t, \nabla^t - x^t \rangle - 2 \bar{L} \eta_t^2 D^2 $$

$$ \geq F(x^t) + \eta_t \langle g^t, x^* - x^t \rangle + \eta_t \langle \nabla F(x^t) - g^t, \nabla^t - x^t \rangle - 2 \bar{L} \eta_t^2 D^2, $$

where we use the optimality and boundedness of $v^t$ in the last inequality. Take expectation on both sides and use the unbiasedness of $g^t$, and the Young’s inequality to yield

$$ \mathbb{E} F(x^{t+1}) \geq \mathbb{E} F(x^t) + \eta_t \langle \nabla F(x^t), x^* - x^t \rangle - \frac{1}{2L} \mathbb{E} \| \nabla F(x^t) - g^t \|^2 - 6 \bar{L} \eta_t^2 D^2. $$

From the convexity of $F$, we have $\langle \nabla F(x^t), x^* - x^t \rangle \geq F(x^*) - F(x^t)$ and thus

$$ \mathbb{E} F(x^{t+1}) \geq \mathbb{E} F(x^t) + \eta_t \mathbb{E} [F(x^*) - F(x^t)] - \frac{1}{2L} \mathbb{E} \| \nabla F(x^t) - g^t \|^2 - 6 \bar{L} \eta_t^2 D^2. $$

Using Lemma 4.3 with $|\mathcal{M}_0| = \frac{G^2}{L^2D^2\eta_t}$ and $|\mathcal{M}_h| = \frac{1}{\eta_t}$, we have

$$ \mathbb{E} \| \nabla F(x^t) - g^t \|^2 \leq 2 \bar{L}^2 D^2 \eta_t^2. $$

Let $\delta_t \overset{\text{def}}{=} F(x^*) - F(x^t)$ and $c \overset{\text{def}}{=} \max\{14\bar{L}D^2, \delta_0\}$. Combining (59) and (60) gives

$$ \mathbb{E} \delta_{t+1} \leq (1 - \eta_t) \mathbb{E} \delta_t + c \eta_t^2 / 2. $$

Taking $\eta_t = \frac{2}{t + 2}$, from induction we obtain $\mathbb{E} \delta_t \leq \frac{2c}{t + 2}$. For $t = 0$, it trivially holds. Assume $\mathbb{E} \delta_{t_0} \leq \frac{2c}{t_0 + 2}$ with $t_0 \geq 1$. For $t = t_0 + 1$,

$$ \mathbb{E} \delta_{t_0 + 1} \leq \frac{t_0}{t_0 + 2} \cdot \frac{2c}{t_0 + 2} + \frac{2c}{(t_0 + 2)^2} \leq \frac{2c}{t_0 + 3}. $$

In conclusion, we have

$$ F(x^*) - \mathbb{E} F(x^t) \leq \frac{28 \bar{L} D^2 + (F(x^*) - F(x^0))}{t + 2} $$

8.9. Stochastic Frank-Wolfe++ for Oblivious Stochastic Optimization

While our Stochastic Frank-Wolfe++ method finds an $\epsilon$-approximate solution of the general non-oblivious stochastic convex optimization problem (1) with $O\left(\frac{1}{\epsilon^2}\right)$ stochastic oracle queries, in this section we restrict ourselves to the easier oblivious case, i.e. when the underlying probability $p$ is independent of the variable $x$, and show similar oracle complexity result holds but with a significantly reduced dependence on the regularity parameters.

Note that in the oblivious case, the gradient-difference estimate can be simply constructed by

$$ \tilde{\Delta}^t \overset{\text{def}}{=} \nabla \tilde{F}(x^t; M_t^h) - \tilde{F}(x^{t-1}; M_t^h), $$
where $\mathcal{M}_h^t$ is a minibatch of samples drawn from $p(z)$ and $\nabla \hat{F}(x^t; \mathcal{M}_h^t)$ is defined by:

$$
\nabla \hat{F}(x^t; \mathcal{M}_h^t) \overset{\text{def}}{=} \frac{1}{|\mathcal{M}_h^t|} \sum_{z \in \mathcal{M}_h^t} \nabla \hat{F}(x^t; z).
$$

(63)

To analyze the convergence property of Stochastic Frank-Wolfe++ using (62) as gradient-difference estimator, we make the following assumptions:

**Assumption 8.1 (concavity)** $F$ is concave.

**Assumption 8.2 (compactness of feasible domain)** The set $\mathcal{C}$ is compact with diameter $D$.

**Assumption 8.3 (bounded gradient norm)** For all $x \in \mathcal{C}$, the stochastic gradient $\nabla \hat{F}$ has bounded variance:

$$
\mathbb{E}_{z \sim p(z)} \| \nabla \hat{F}(x; z) - F(x) \|^2 \leq G^2.
$$

**Assumption 8.4 (Lipschitz continuous gradient)** The stochastic gradient $\nabla \hat{F}$ is Lipschitz continuous in the following sense: for all $x, y \in \mathcal{C}$,

$$
\mathbb{E}_{z \sim p(z)} \| \nabla \hat{F}(x; z) - \nabla \hat{F}(y; z) \|^2 \leq L^2 \| x - y \|^2.
$$

It is easy to check that Assumption 8.4 is weaker than Assumption 4.6 on the stochastic function $\hat{F}$.

**Theorem 8.4** Consider Problem (1) when $F$ is concave and oblivious. Further, recall the SFW++ method outlined in Algorithm 2 using (62) as gradient-difference estimator. If the conditions in Assumptions 8.1-8.4 hold and we set SFW++ parameters to $\eta_t = 2/(t + 2)$, $|\mathcal{M}_h^t| = 16(t + 2)$ and $|\mathcal{M}_0| = (G^2(t + 1)^2)/(L^2D^2)$, then the iterates generated by SFW++ satisfy

$$
F(x^*) - \mathbb{E}[F(x^t)] \leq \frac{28LD^2 + (F(x^*) - F(x^0))}{t + 2}.
$$

The proof of Theorem 8.4 is identical to Theorem 6.1 except we use the following variance bound.

**Lemma 8.5 (variance bound)** Recall the definition of the gradient-difference estimator $\tilde{\Delta}^t$ in (62). Under Assumption 8.2, 8.3, and 8.4, by taking $|\mathcal{M}_h^t| = 16(t + 2)$ and $|\mathcal{M}_0^0| = \frac{G^2(t + 1)^2}{L^2D^2}$, we bound

$$
\mathbb{E} \| g^t - \nabla F(x^t) \|^2 \leq 2L^2D^2\eta_t^2.
$$

(64)

Again, the proof of Lemma 8.5 resembles the one of Lemma 8.3.
8.10. Proof of Theorem 5.3
Consider the classical problem of maximizing a monotone submodular function subject to a cardinality constraint, \( \max \{ f(S) : |S| \leq k \} \). It is known that there exists a monotone submodular set function, denoted by \( f_0 \), for which obtaining a \((1 - 1/e + \epsilon)\)OPT solution requires exponentially many, namely \( \exp(\alpha(\epsilon)k) \) for some constant \( \alpha(\epsilon) > 0 \), function value queries no matter what algorithm is used (Nemhauser and Wolsey, 1978). To fix the notation, we assume that \( f_0 \) is defined over the ground set \([n] = \{1, \ldots, n\}\) and let

\[
\text{OPT}(f_0, k) = \max_{S : |S| \leq k} f_0(S).
\]  

(65)

Without loss of generality, let us assume that an optimal solution of (65) contains the element \( n \in [n] \). For \( \delta \in [0, 1/2] \) consider the submodular set function \( f_\delta : 2^{[n+1]} \to \mathbb{R} \) defined as follows:

\[
f_\delta(S) = \begin{cases} 
  f_0((S \setminus \{n + 1\}) \cup \{n\}) + \delta \text{OPT}(f_0, k) & \text{if } n + 1 \in S, \\
  f_0(S) + \frac{\delta}{2} \text{OPT}(f_0, k) & \text{if } n + 1 \notin S \land n \in S, \\
  f_0(S) & \text{o.w.}
\end{cases}
\]

Note that the function \( f_\delta \) can be defined as an expectation in the following way:

\[
f_\delta(S) = \mathbb{E}_{A \sim \text{Ber}(2\delta), B \sim \text{Ber}(\delta)} [f_{A,B}(S)],
\]  

(66)

where

\[
f_{A,B}(S) = \begin{cases} 
  f_0((S \setminus \{n + 1\}) \cup \{n\}) + A \text{OPT}(f_0, k) & \text{if } n + 1 \in S, \\
  f_0(S) + B \text{OPT}(f_0, k) & \text{if } n + 1 \notin S \land n \in S, \\
  f_0(S) & \text{o.w.}
\end{cases}
\]  

(67)

In this way, it is easy to check that \( f_\delta \) is monotone and submodular, and

\[
\text{OPT}(f_\delta, k) = (1 + \delta)\text{OPT}(f_0, k).
\]  

(68)

Finally, we consider a stochastic oracle that, when queried for the function value \( f_\delta(S) \), returns the following unbiased estimate: The oracle first draws independent samples of random variables \( A, B \) and plugs-in the resulting value into (67) to obtain an unbiased sample for \( f_\delta(S) \).

Now, consider an algorithm which aims to maximize \( f_\delta \) subject to the \( k \)-cardinality constraint by assuming only access to the stochastic oracle mentioned above. Note here that the algorithm does not have any prior information about the structure of the function \( f_\delta \), and the only information that it obtains is through the stochastic oracle. In particular, the algorithm does not know a priori that \( f_\delta(\{n + 1\}) \) is larger than \( f_\delta(\{n\}) \).

In order to obtain a \((1 - 1/e - \frac{\delta}{2})\)-optimal solution for this problem, the algorithm has to either find a \((1 - 1/e + \frac{\delta}{2})\)-optimal solution to Problem (65), or it has to know that \( f_\delta(\{n + 1\}) \) is larger than \( f_\delta(\{n\}) \). The former case needs at least \( \exp \{ \alpha(\delta/4)k \} \) queries from the oracle, and the latter case needs at least \( O(1/\delta^2) \) oracle queries since it is equivalent to the problem of distinguishing between two Bernoulli random variables \( A = \text{Bernoulli}(\delta) \), and \( B = \text{Bernoulli}(\delta/2) \)--see Lemma 3 in Agarwal et al. (2009). We thus obtain the desired result by noting that we can assume without loss of generality that \( \text{OPT} \geq 1 \), and hence \((1 - 1/e - \epsilon)\text{OPT} \leq (1 - 1/e)\text{OPT} - \epsilon \).