SMOOTH GLOBAL LAGRANGIAN FLOW FOR THE 2D EULER AND SECOND-GRADE FLUID EQUATIONS

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Abstract. We present a very simple proof of the global existence of a \( C^\infty \) Lagrangian flow map for the 2D Euler and second-grade fluid equations (on a compact Riemannian manifold with boundary) which has \( C^\infty \) dependence on initial data \( u_0 \) in the class of \( H^s \) divergence-free vector fields for \( s > 2 \).

1. INCOMPRESSIBLE EULER EQUATIONS

Let \((M,g)\) be a \( C^\infty \) compact oriented Riemannian 2-manifold with smooth boundary \( \partial M \), let \( \nabla \) denote the Levi-Civita covariant derivative, and let \( \mu \) denote the Riemannian volume form. The incompressible Euler equations are given by

\[
\begin{align*}
\partial_t u + \nabla u \cdot u &= \text{grad} \, p, \\
\text{div} \, u &= 0, \quad u(0) = u_0, \quad g(u, n) = 0 \quad \text{on} \quad \partial M,
\end{align*}
\]

where \( p(t, x) \) is the pressure function, determined (modulo constants) by solving the Neumann problem \(-\triangle p = \text{div} \, \nabla u \) with boundary condition \( g(\text{grad} \, p, n) = S_n(u) \), \( S_n \) denoting the second-fundamental form of \( \partial M \).

The now standard global existence result for two-dimensional classical solutions states that for initial data \( u_0 \in \chi^s \equiv \{ v \in H^s(TM) \mid \text{div} \, v = 0, \, g(v, n) = 0 \} \), \( s > 2 \), the solution \( u \) is in \( C^0([0, \infty), \chi^s) \) and has \( C^0 \) dependence on \( u_0 \) (see, for example, Taylor’s book \[8\]). Equation (1.1) gives the Eulerian or spatial representation of the dynamics of the fluid. The Lagrangian representation which is in terms of the volume-preserving fluid particle motion or flow map \( \eta(t, x) \) is obtained by solving

\[
\begin{align*}
\partial_t \eta(t, x) &= u(t, \eta(t, x)), \\
\eta(0, x) &= x.
\end{align*}
\]

This is an ordinary differential equation on the infinite dimensional volume-preserving diffeomorphism group \( D^s_\mu \), the set of \( H^s \) class bijective maps of \( M \) into itself with \( H^s \) inverses which leave \( \partial M \) invariant. Ebin & Marsden \[3\] proved that \( D^s_\mu \) is a \( C^\infty \) manifold whenever \( s > 2 \). They also showed that for an interval \( I \), whenever \( u \in C^0(I, \chi^s) \) and \( s > 3 \), there exists a unique solution \( \eta \in C^1(I, D^s_\mu) \) to (1.2). Thus, for \( s > 3 \) the existence of a global \( C^1 \) flow map immediately follows from the fact that \( u \) remains bounded in \( H^s \) for all time. It is often essential, however, for the Euler flow to depend smoothly on the initial data; in the case of vortex methods, for example, Hald in Assumption 3 of \[5\] requires this as a necessary condition to establish convergence.

Theorem 1.1. For \( u_0 \in \chi^s \), \( s > 2 \), there exists a unique global solution to (1.1) which is in \( C^\infty(\mathbb{R}, TD^s_\mu) \) and has \( C^\infty \) dependence on \( u_0 \).
Proof. The smoothness of the flow map follows by considering the Lagrangian version of (1.1) given by

\[ \frac{D}{dt} \partial_t \eta(t, x) = - \text{grad} p(t, \eta(t, x)), \quad \text{det} T \eta(t, x) = 1, \]

\[ \partial_t \eta(0, x) = u_0(x), \quad \eta(0, x) = x, \]

where \( T \eta(t, x) \) denotes the tangent map of \( \eta \) (which in local coordinates is given by the 2x2 matrix of partial derivatives \( \partial \eta / \partial x^j \)), and where \( D/dt \) is the covariant derivative along the curve \( t \mapsto \eta(t, x) \) (which in Euclidean space is the usual partial time derivative). Since

\[ \text{grad} \circ \eta = \text{grad} \Delta^{-1} [\text{Tr}(\nabla u \cdot \nabla u) + \text{Ric}(u, u)] \circ \eta, \]

where \( \text{Ric} \) is the Ricci curvature of \( M \), and since \( S_n \) is \( C^\infty \) and \( H^{s-1}(TM) \) forms a multiplicative algebra whenever \( s > 2 \), we see that the linear operator \( u \mapsto \text{grad} \Delta^{-1} [\text{Tr}(\nabla u \cdot \nabla u) + \text{Ric}(u, u)] \) maps \( H^s \) back into \( H^s \). Denote by \( f : TD^s_\mu \to TTD^s_\mu \) the vector field

\[ (\eta, \partial_t \eta) \mapsto \text{grad} \Delta^{-1} [\text{Tr}(\nabla u \cdot \nabla u) + \text{Ric}(u, u)] \circ \eta. \]

Then,

\[ f(\eta, \partial_t \eta) = \text{grad}_\eta \Delta^{-1}_\eta [\text{Tr}(\nabla_\eta \partial_t \eta \cdot \nabla_\eta \partial_t \eta) + \text{Ric}_\eta(\partial_t \eta, \partial_t \eta)], \]

where \( \text{grad}_\eta g = [\text{grad}(g \circ \eta^{-1})] \circ \eta \) for all \( g \in H^s(M) \), \( \text{div}_\eta X_\eta = [\text{div}(X_\eta \circ \eta^{-1})] \circ \eta \) and \( \nabla_\eta(X_\eta) = [\nabla(X_\eta \circ \eta^{-1})] \circ \eta \) for all \( X_\eta \in T_\eta D^s_\mu \), \( \Delta_\eta = \text{div}_\eta \circ \text{grad}_\eta \), and \( \text{Ric}_\eta = \text{Ric} \circ \eta \). It follows from Lemmas 4, 5, and 6 in [1] and Appendix A in [2] that \( f \) is a \( C^\infty \) vector field. Thus (1.1) is an ordinary differential equation on tangent bundles \( TD^s_\mu \) governed by a \( C^\infty \) vector field on \( TD^s_\mu \); it immediately follows from the fundamental theorem of ordinary differential equations on Hilbert manifolds, that (1.3) has a unique \( C^\infty \) solution on finite time intervals which depends smoothly on the initial velocity field \( u_0 \), i.e., there exists a unique solution \( \partial_t \eta \in C^\infty((-T, T), TD^s_\mu) \) with \( C^\infty \) dependence on initial data \( u_0 \), where \( T \) depends only on \( \|u_0\|_{H^s} \).

When \( s > 3 \), this interval can be extended globally to \( \mathbb{R} \) by virtue of \( \eta \) remaining in \( D^s_\mu \). Unfortunately, the global existence and uniqueness of a \( C^\infty \) flow map \( \eta(t, x) \) does not follow for initial data \( u_0 \in \chi^s \) for \( s \in (2, 3] \), so we provide a simple argument to fill this gap. We must show that \( \eta \) can be continued in \( D^s_\mu \). It suffices to prove that \( T \eta \) and \( T \eta^{-1} \) are both bounded in \( H^{s-1} \). This is easily achieved using energy estimates. We have that

\[ \frac{D}{dt} T \eta = \nabla \partial_t \eta = \nabla u \cdot T \eta \]

and

\[ \frac{D}{dt} T \eta^{-1} = - T \eta^{-1} \cdot \nabla \partial_t \eta \cdot T \eta^{-1} = - T \eta^{-1} \cdot \nabla u. \]

Computing the \( H^{s-1} \) norm of \( T \eta \) and \( T \eta^{-1} \), respectively, we obtain

\[ \frac{1}{2} \frac{d}{dt} \| T \eta \|_{H^{s-1}} = \langle D^{s-1}(\nabla u \cdot T \eta), D^{s-1} T \eta \rangle_{L^2}, \]

and

\[ \frac{1}{2} \frac{d}{dt} \| T \eta^{-1} \|_{H^{s-1}} = \langle D^{s-1}(T \eta^{-1} \cdot \nabla u), D^{s-1} T \eta^{-1} \rangle_{L^2}. \]
It is easy to estimate
\[ \langle D^{s-1}(\nabla u \cdot T\eta), D^{s-1}T\eta \rangle_{L^2} \leq C(\|\nabla u\|_{L^\infty} \|T\eta\|_{H^{s-1}}^2 + \|\nabla u\|_{H^{s-1}} \|T\eta\|_{L^\infty} \|T\eta\|_{H^{s-1}}) \leq C(\|\nabla u\|_{L^\infty} \|T\eta\|_{H^{s-1}}^2 + \|u\|_{H^{s-1}} \|T\eta\|_{H^{s-1}}^2) \]
where the first inequality is due to Cauchy-Schwartz and Moser’s inequalities and the second is the Sobolev embedding theorem. Similarly,
\[ \langle D^{s-1}(-T\eta^{-1} \cdot \nabla u), D^{s-1}T\eta^{-1} \rangle_{L^2} \leq C(\|\nabla u\|_{L^\infty} \|T\eta^{-1}\|_{H^{s-1}}^2 + \|u\|_{H^{s-1}} \|T\eta^{-1}\|_{H^{s-1}}^2) \]
Since the solution \( u \) to (1.3) is in \( \mathcal{X}^s \) for all \( t \), we have that \( \|u\|_{H^s} \) is bounded for all \( t \). Because the vorticity \( \omega = \text{curl } u \) is in \( L^\infty \), we have by Lemma 2.4 in Chapter 17 of [3] that \( \|\nabla u\|_{L^\infty} \leq C(1 + \log \|u\|_{H^s}) \); hence \( \|\nabla u\|_{L^\infty} \) is bounded for \( t \). It then follows that \( \eta \) and \( \eta^{-1} \) are in \( D^s_\mu \) for all time. \( \square \)

2. Second-grade fluid equations

In this section, we establish the global existence of a \( C^\infty \) Lagrangian flow map for the second-grade fluids equations, also known as the isotropic averaged Euler or Euler-\( \alpha \) equations, which has \( C^\infty \) dependence on initial data. These equations are given on \( (M, g) \) by
\[
\begin{align*}
\partial_t (1 - \alpha \Delta_\ell) u - \nu \Delta_\ell u + \nabla_\ell (1 - \alpha \Delta_\ell) u - \alpha (\nabla_\ell u)^t \cdot \Delta_\ell u &= -\text{grad } p, \\
\text{div } u &= 0, \quad u(0) = u_0, \quad u = 0 \text{ on } \partial M
\end{align*}
\]
(2.1)
(see [4]), and were first derived in 1955 by Rivlin & Ericksen [5] in Euclidean space \((\text{Ric}= 0)\) as a first-order correction to the Navier-Stokes equations. In Euclidean space the operator \( \Delta_\ell \) is just the component-wise Laplacian, and the equation may be written as
\[
\partial_t (1 - \alpha \Delta) u - \nu \Delta u + \text{curl} (1 - \alpha \Delta) u \times u = -\text{grad } p.
\]
For convenience, we set \( \alpha = 1 \). We define the unbounded, self-adjoint operator \((1 - \ell)(1 - 2\text{Def}^\ast \text{Def})\) on \( L^2(TM) \) with domain \( H^2(TM) \cap H^0_\partial(TM) \). The operator \( \text{Def}^\ast \) is the formal adjoint of \( \text{Def} \) with respect to \( L^2; \) \( 2\text{Def}^\ast \text{Def} u = -(\Delta + \text{grad div } + 2\text{Ric})u \) so that \( 2\text{Def}^\ast \text{Def} u = -(\Delta + 2\text{Ric})u \) if \( \text{div } u = 0 \). We let \( D^s_\mu, D^s_\mu, D^s_\mu \) denote the subgroup of \( D^s_\mu \) whose elements restrict to the identity on the boundary \( \partial M \). \( D^s_\mu, D^s_\mu, D^s_\mu \) is a \( C^\infty \) manifold (see [3] and [5]). Let \( \chi_D^s = \{ u \in \chi^s | u = 0 \text{ on } \partial M \} \).

The following is Proposition 5 in [3].

Proposition 2.1. For \( s > 2 \), let \( \eta(t) \) be a curve in \( D^s_\mu, D^s_\mu, D^s_\mu \), and set \( u(t) = \partial_t \eta \circ \eta(t)^{-1} \). Then \( u \) is a solution of the initial-boundary value problem (2.1) with Dirichlet boundary conditions \( u = 0 \) on \( \partial M \) if and only if
\[
\begin{align*}
\overline{\nabla} \circ \left[ \frac{\nabla u}{\nabla^2} + [-\nu(1 - \ell)^{-1} \Delta_\ell u + \mathcal{U}(u) + \mathcal{R}(u)] \circ \eta \right] &= 0, \quad \text{Det } T\eta(t, x) = 1, \\
\partial_t \eta(0, x) &= u_0(x), \\
\eta(0, x) &= x
\end{align*}
\]
(2.2)
where
\[
\begin{align*}
\mathcal{U}(u) &= (1 - \ell)^{-1} \{ \text{div } [\nabla u \cdot \nabla^t u + \nabla u \cdot \nabla u - \nabla u^t \cdot \nabla u] + \text{grad } \text{Tr}(\nabla u \cdot \nabla u) \} \\
\mathcal{R}(u) &= (1 - \ell)^{-1} \{ \text{Tr } [\nabla (R(u, \cdot) u) + R(u, \cdot) \nabla u + R(\nabla u, \cdot) u] \\
&\quad + \text{grad } \text{Ric}(u, u) - (\nabla u \text{Ric}) \cdot u + \nabla u^t \cdot \text{Ric}(u) \},
\end{align*}
\]
and $\overline{\mathcal{P}}_\eta : T_\eta D^*_D \rightarrow T_\eta D^*_\mu,D$ is the Stokes projector defined by

$$\overline{\mathcal{P}}_\eta : T_\eta D^*_\mu,D \rightarrow T_\eta D^*_\mu,D,$$

$$\overline{\mathcal{P}}_\eta(X_\eta) = [\mathcal{P}_e(X_\eta \circ \eta^{-1})] \circ \eta,$$

and where $\mathcal{P}_e(F) = v$, $v$ being the unique solution of the Stokes problem

$$(1 - \mathcal{L})v + \text{grad} \, p = (1 - \mathcal{L})F;$$
$$\text{div} \, v = 0,$$
$$v = 0 \text{ on } \partial M.$$

Equation (2.2) is an ordinary differential equation for the Lagrangian flow. Notice again that $H^{s-1}, s > 2$, forms a multiplicative algebra, so that both $\mathcal{U}$ and $\mathcal{R}$ map $H^s$ into $H^s$.

**Theorem 2.1.** For $u_0 \in \chi^s_D, s > 2,$ and $\nu \geq 0$, there exists a unique global solution to (2.2) which is in $C^\infty(\mathbb{R}, T^s_D)$ and has $C^\infty$ dependence on $u_0$.

We note that one cannot prove the statement of this theorem from an analysis of (2.1) alone (see [2] and [4], and references therein).

**Proof.** The ordinary differential equation (2.2) can be written as $\partial_t u = S(\eta, \partial_t \eta)$ (see page 23 in [6]). Remarkably, $S : T^s_D \rightarrow T^s_D$ is a $C^\infty$ vector field, and Theorem 2 provides the existence of a unique short-time solution to (2.2) in $C^\infty((-T, T), T^s_D)$ which depends smoothly on $u_0$, and where $T$ only depends on $\|u_0\|_{H^s}$.

Thus, it suffices to prove that the solution curve $\eta$ does not leave $D^*_\mu,D$. Following the proof of Theorem 1.1, and using the fact that the solution $u(t, x)$ to (2.1) remains in $H^s$ for all time ([2, 4]), it suffices to prove that $\nabla u$ is bounded in $L^\infty$.

Letting $q = \text{curl}(1 - \alpha \triangle_r)u$ denote the potential vorticity, and computing the curl of (2.1), we obtain the 2D vorticity form as

$$\partial_t q + g(\text{grad} \, q, u) = \nu \text{curl} \, u.$$

It follows that for all $\nu \geq 0$, $q(t, x)$ is bounded in $L^2$ (conserved when $\nu = 0$) and therefore by standard elliptic estimates $\nabla u(t, x)$ is bounded in $H^2$, and hence in $L^\infty$.

As a consequence of Theorem 2.1 being independent of viscosity, we immediately obtain the following:

**Corollary 2.1.** Let $\eta^\nu(t, x)$ denote the Lagrangian flow solving (2.3) for $\nu > 0$, so that $u^\nu = \partial_t \eta^\nu \circ \eta^{\nu-1}$ solves (2.4). Then for $u_0 \in \chi^s_D, s > 2$, the viscous solution $\eta^\nu \in C^\infty(\mathbb{R}, T^s_D)$ converges regularly (in $H^s$) to the inviscid solution $\eta^0 \in C^\infty(\mathbb{R}, T^s_D)$. Consequently $u^\nu \rightarrow u^0$ in $H^s$ on infinite-time intervals.

This gives an improvement of Busuioc’s result in [1] in two ways: 1) we are able to prove the regular limit of zero viscosity on manifolds with boundary, and 2) in the Lagrangian framework, we are able to get $C^\infty$ in time solutions.

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