Detection and correction of errors with quantum tomography

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Abstract
It is shown that quantum tomography can detect and correct unlimited number of errors during the evaluation of quantum algorithms on quantum computer.

1 Introduction
Two main problems of constructing the quantum computer:

1. Quantum decoherence, which is inevitable part of quantum computer. Due to the decoherence [16], the errors start to appear in quantum algorithms, which destroy the information. To remove the errors during the evaluation of quantum algorithms a number of error correcting algorithms [7, 8, 16, 18, 20, 21, 29, 32, 34, 35] have been discovered. But none of them can correct unlimited number of errors. For example, the classical/quantum Hamming error-correcting code can correct only one error [7, 19, 18, 27]. Moreover, classical/quantum algorithms need additional bits/qubits for the correction purposes [7, 27]. For example, if we have codeword of three symbols (n = k = 3), \( u = (u_1, u_2, u_3) \). Where \( u_i \) takes value 0 or 1. After decoherence the codeword changes the value of symbol from 0 to 1 or vice-versa. To know whether the codeword \( u \) has changed its configuration or not after decoherence, we extend the length of codeword from \( n = k = 3 \) to \( n = k + 1 \) or \( n = k + 2 \). The 4th and 5th positions in the codeword, we keep for the parity symbols [1, 9]. Now the codeword or vector \( u = (u_1, u_2, u_3, u_4, \ldots) \) should satisfy the condition \( H.u^T = 0 \) [7, 27]. Where \( H \) is parity matrix, which we choose ourself, so that the condition \( H.u^T \) should satisfy [7, 27].

2. The problems of higher moments. As we know in NMR (nuclear magnetic resonance), the higher order moments (total spin \( s > 1 \)) are uncontrollable or they are not observable with the experimental technique, which we have at the moment. It means that the quantum computer, which consists of
more than 2 spins of 1/2 (i.e., total spin \( s > 1 \)) always has uncontrollable moments. So, the density matrix for more than 2 spins could not reconstructed fully, with the NMR technique or quantum tomography method experimentally. But we can reconstruct the density matrix theoretically with the help of quantum tomography.

We will discuss points 1 and 2 in more rigorous way in sections 2, 3, 4, and 5.

2 Error-correction with quantum tomography

The main purpose of error in quantum computer is to flip the phase of the quantum state. The state vector of one spin 1/2 can be written as

\[
|\phi\rangle = a|0\rangle + b|1\rangle.
\]  

(1)

Where \( |0\rangle \) or \( |1\rangle \) stand for the plus or minus direction of the spin projection on axis \( z \). \( a, b \) are the amplitudes of state \( |0\rangle \) and \( |1\rangle \).

The decoherence or three Pauli matrices \( \sigma_x, \sigma_y, \) and \( \sigma_z \) can change the phase and flip the positions of the vectors \( |0\rangle \) to \( |1\rangle \) and \( |1\rangle \) to \( |0\rangle \) i.e., \( \sigma_x|\phi\rangle \) flips the complex amplitudes of the state vector (1), \( \sigma_z|\phi\rangle \) changes the phase of the state (1), \( \sigma_y|\phi\rangle = i\sigma_x\sigma_z|\phi\rangle \) changes the phase and flipping the state vector (1).

The same procedure will be for the cases of higher spins, entanglement, and for mixed states.

To correct the quantum errors, we usually use ancilla (additonal) qubits. The ancilla qubits contain the error information, which we can extract by projecting state vector to the corresponding state [29,32,34,35]. To correct the error in one qubit, we need 2 ancilla qubits. It means the total qubits will be three or total spin \( s = s_1 + s_2 + s_3 \) and the uncontrollable moments will appear in the density matrix, which are not reconstructable by the experiment. So, the ancilla qubits will complex our problems more than we will have without them, which we will show later.

3 Problems of higher moments during the reconstruction of density matrix.

3.1 Polarization/multipole tensors

Any square matrix can be decomposed into the polarization tensors [39,6]

\[
\hat{A} = \sum_{L=0}^{2\hat{S}} \sum_{M=-L}^{L} A_{L,M} \tilde{T}_{L,M}(\hat{S}).
\]  

(2)
Where $A_{L,M} = Sp\{\hat{T}^+_L, \hat{A}\}$ is decomposition coefficient of matrix $\hat{A}$ and $\hat{T}^+_L, \hat{M}(\hat{S}) = (-1)^M \hat{T}^+_L, -M$ is complex conjugated polarization tensor of $\hat{T}_L, M$ of total spin $\hat{S}$

$$\hat{T}_L, M(\hat{S}) = \sqrt{\frac{2L + 1}{2S + 1}} \sum_{m,m'} C^S_{m', m, L, M} \chi_{S, m'} \chi^+_S, m.$$  \hspace{1cm} (3)

Where $\chi_{S, m}$ - spin vector of spin $\hat{S}$ with projection $m$ on $z$-axis and $C^S_{S, m, L, M}$ - is Clebsch-Gordan coefficient \cite{6,10,39}. $L$ and $M$ take the values between $0 \leq L \leq 2\hat{S}$ and $-L \leq M \leq L$.

3.2 Rotation of polarization tensors

Since the polarization tensors are invariant \cite{39,6} under any rotations. So, we will rotate our reference frame with Wigner function $\hat{D}(\alpha, \beta, \gamma)$ \cite{10,6,39} with Euler angles $\alpha$, $\beta$ and $\gamma$

$$\hat{T}^\prime_{L, M'} = \hat{D}(\alpha, \beta, \gamma) \hat{T}_L, M \hat{D}^{-1}(\alpha, \beta, \gamma) = \sum_{M = -\hat{S}}^{\hat{S}} D^L_{M, M'} \hat{T}_L, M$$  \hspace{1cm} (4)

4 Density matrices

By using (2), we can decompose density matrix $\hat{\rho}(\hat{S})$ of dimensions $n \times n$ with decomposition elements $\rho_{L, M}(\hat{S})$ of spin $\hat{S}$ i.e.,

$$\hat{\rho}(\hat{S}) = \sum_{L = 0}^{2\hat{S}} \sum_{M = -L}^{L} \rho_{L, M}(\hat{S}) \hat{T}_L, M(\hat{S}).$$  \hspace{1cm} (5)

4.1 Kronecker product of density matrices

Let the total spin $\hat{S}_{1...n}$ is consist of $n$ number of spins i.e., $\hat{S}_{1...n} = \hat{S}_1 + \hat{S}_2 + \cdots + \hat{S}_n$. The density matrix of a system of $n$ spins is written

$$\hat{\rho}_{1...n}(\hat{S}_{1...n}) = \otimes^n_{i=1} \hat{\rho}_i(\hat{S}_i).$$  \hspace{1cm} (6)

Where $\otimes^n_{i=1} \hat{\rho}_i(\hat{S}_i) = \hat{\rho}_1(\hat{S}_1) \otimes \hat{\rho}_2(\hat{S}_2) \otimes \hat{\rho}_3(\hat{S}_3) \otimes \cdots \otimes \hat{\rho}_n(\hat{S}_n)$ The $1^{st}$ underbrace

$$\underbrace{\hat{\rho}_1(\hat{S}_1) \otimes \hat{\rho}_2(\hat{S}_2) \otimes \hat{\rho}_3(\hat{S}_3) \otimes \cdots \otimes \hat{\rho}_n(\hat{S}_n)}_{1^{st} \text{ underbrace}}$$

$$\underbrace{\hat{\rho}_1(\hat{S}_1) \otimes \hat{\rho}_2(\hat{S}_2) \otimes \hat{\rho}_3(\hat{S}_3) \otimes \cdots \otimes \hat{\rho}_n(\hat{S}_n)}_{2^{nd} \text{ underbrace}}$$

$$\underbrace{\hat{\rho}_1(\hat{S}_1) \otimes \hat{\rho}_2(\hat{S}_2) \otimes \hat{\rho}_3(\hat{S}_3) \otimes \cdots \otimes \hat{\rho}_n(\hat{S}_n)}_{n^{th} \text{ underbrace}}$$

denotes the kronecker product of $\hat{\rho}_1(\hat{S}_1)$ and $\hat{\rho}_2(\hat{S}_2)$, the $2^{nd}$ denotes the kronecker product of the result of $1^{st}$ underbrace and $\hat{\rho}_3(\hat{S}_3)$ and so on. By using
(5), we can decompose (6) into

$$\hat{\rho}_{1\ldots n}(\hat{S}_{1\ldots n}) = \sum_{L_1,L_2,\ldots,L_n=0,0,\ldots,0}^{L_1+L_2,\ldots,L_n} \sum_{M_1,M_2,\ldots,M_n=-L_1-L_2,\ldots,-L_n}^{|L_1,L_2,\ldots,L_n|} \{[(\hat{T}_{L_1},M_1) (\hat{S}_1)] \
\otimes \rho_{L_2,M_2} (\hat{S}_2)) \otimes \rho_{L_3,M_3} (\hat{S}_3) \otimes \cdots \otimes \rho_{L_n,M_n} (\hat{S}_n) \} \{ \hat{T}_{L_1,M_1}(\hat{S}_1) \
\otimes \hat{T}_{L_2,M_2}(\hat{S}_2) \otimes \hat{T}_{L_3,M_3}(\hat{S}_3) \otimes \cdots \otimes \hat{T}_{L_n,M_n}(\hat{S}_n) \} \}.$$  

The kronecker product [6,10,39] of two polariation tensors $\hat{O}_{L_1,M_1}(\hat{S}_1)$ and $\hat{O}_{L_2,M_2}(\hat{S}_2)$ having spins $\hat{S}_1$ and $\hat{S}_2$ can be written as

$$\hat{O}_{L_1,M_1}(\hat{S}_1) \otimes \hat{O}_{L_2,M_2}(\hat{S}_2) = \sum_{S_{12}=|L_1-L_2|}^{L_1+L_2} \sum_{M=-S_{12}}^{S_{12}} C_{L_1,M_1,L_2,M_2}^{S_{12},M} \hat{O}_{L,M}(\hat{S}_{12}).$$  

By applying the kronecker product property (8), we can get the density matrix (7) of dimensions $n \times n$ in the following way: first we apply (8) to density matrices $\hat{\rho}_1(\hat{S}_1) \otimes \hat{\rho}_2(\hat{S}_2)$ and then the kronecker product of result of $\hat{\rho}_1(\hat{S}_1) \otimes \hat{\rho}_2(\hat{S}_2)$ and $\hat{\rho}_3(\hat{S}_3)$ and so on.

### 4.2 Examples in quantum information theory

First of all, we will consider the density matrices for two spins $\hat{S}_1$, $\hat{S}_2$ and then gradually increase the $n$ number of spins upto 3. Since, we are interested in quantum information theory, so we are considering all the spins with spin 1/2.

#### 4.2.1 Density matrix of two spins

By using (3), (5), (6), (7), and (8), we can write density matrix $\hat{\rho}_{12}$ of two spins $\hat{S}_1$ and $\hat{S}_2$

$$\hat{\rho}_{12}(\hat{S}_{12}) = \hat{\rho}_1(\hat{S}_1) \otimes \hat{\rho}_2(\hat{S}_2)$$

$$= \sum_{L_1,L_2=0,0}^{L_1+L_2} \sum_{M_1,M_2=-L_1-L_2}^{L_1+L_2} \rho_{L_1,M_1}(\hat{S}_1) \otimes \rho_{L_2,M_2}(\hat{S}_2)$$

$$\{ \hat{T}_{L_1,M_1}(\hat{S}_1) \otimes \hat{T}_{L_2,M_2}(\hat{S}_2) \} = \sum_{L_1,L_2=0,0}^{L_1+L_2} \sum_{M_1,M_2=-L_1-L_2}^{L_1+L_2} \sum_{S_{12}=|L_1-L_2|}^{L_1+L_2} \sum_{M=-S_{12}}^{S_{12}} C_{L_1,M_1,L_2,M_2}^{S_{12},M} \hat{O}_{L,M}(\hat{S}_{12}).$$  

$\hat{T}_{S_{12},M_{12}}$ can be defined from (3). The spin vectors $\chi$ for $\hat{S}_{12}$ can be defined by diagonalizing the spin Hamiltonian.
5 Problems in the reconstruction of density matrix

The problems arise when we want to reconstruct some elements of density matrix for spin $\hat{S} > 1$. That is, we could not reconstruct all the elements of density matrix experimentally. Theoretically, there are no problems of reconstruction of density matrix elements. The higher order moments [23] are responsible for the reconstruction of density matrix for $\hat{S} \geq 1$. We will discuss later about the moments, which can be detected with experiment and which not.

5.1 Power expansion of electrostatic and magnetostatic field

If the distance between the two charged particles (nuclei) is large as compare to their dimensions then the electrostatic and magnetostatic field between them can be expanded into power series [23,24,30]

5.1.1 Power expansion of electrostatic field

$$\varphi^{(l)}_{\text{electro}} = \frac{1}{R_0^{(l+1)}} \sum_{m=-l}^{l} \sqrt{\frac{4\pi}{2l+1}} Q_m^{(l)} Y^*_{l,m}(\Theta, \Phi)$$  \hspace{1cm} (10)

$$Q_m^{(l)} = \sum_a e_a r_a^{(l)} \sqrt{\frac{4\pi}{2l+1}} Y_{l,m}(\Theta_a, \Phi_a)$$  \hspace{1cm} (11)

Where $e_a, r_a$ – electric charge and radius of $a^{th}$ nucleus relative to its adjacent nucleus. $R_0$ – radius vector of all nuclei to the point of observation. $Q_m^{(l)}$ – tensor [5,6,10,11,14,31,39] of rang $l$ with $2l + 1$ independent components. $Y^*_{l,m}(\Theta, \Phi)$ – complex conjugated spherical harmonic of angles $(\Theta, \Phi)$ between vectors $\vec{R}, \vec{r}$, and coordinate axes.

By using the spherical harmonic properties [24,14,39], we can express spherical harmonics into eigenvectors of integer spin $\hat{S}$. For example, for $\hat{S} = 1$, we have

$$Y_{1,0} = \sqrt{\frac{3}{4\pi}} \cos(\Theta) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$  \hspace{1cm} (12)

$$Y_{1,1} = -\sqrt{\frac{3}{8\pi}} \sin(\Theta) \exp(i\Phi) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$  \hspace{1cm} (13)

$$Y_{1,-1} = \sqrt{\frac{3}{8\pi}} \sin(\Theta) \exp(-i\Phi) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$  \hspace{1cm} (14)
5.1.2 Power expansion of magnetostatic field

The power expansion of magnetostatic field can be written similarly as in the case of electrostatic field, i.e.,

$$\varphi_{magnetic}^{(l)} = \frac{1}{R_0^{(l+1)}} \sum_{m=-l}^{l} \sqrt{\frac{4\pi}{2l+1}} Q_m^{(l)} Y_{l,m}^* (\Theta, \Phi)$$  \hspace{1cm} (15)

$$Q_m^{(l)} = \sum_a e_{a} t_a^{(l)} \sqrt{\frac{4\pi}{2l+1}} Y_{l,m}(\Theta_a, \Phi_a)$$  \hspace{1cm} (16)

5.2 Electric quadrupole moments

Nuclei with spins $\hat{S} = 1/2$ have symmetric distribution of their charges. So, the electric quadrupole moments have value $Q = 0$. Nuclei with spins $\hat{S} \geq 1$ have asymmetric distribution of charges which create electric quadrupole moment $Q \neq 0$. The $Q$ moment can interact with gradient of electric field at nucleus. To find the electric quadrupole moments of a system of spins, we have two options: 1. By applying gradient of electric field. 2. Through acoustic nuclear magnetic resonance.

5.3 Longitudinal ($T_1$) and transverse ($T_2$) relaxations

Spin-spin interactions, spin-lattice interaction are responsible for relaxation times $T_1$ and $T_2$.

5.4 Quantum tomography of spin systems

Theoretically, we can reconstruct the density matrix by applying the quantum tomography to spin systems. It means, we have to rotate density matrix with Euler angles $(\alpha, \beta, \gamma)$ and take integration over $\alpha, \beta, \gamma$, which is not possible experimentally.

Examples:

5.5 Higher order moments

As the number of spin increases, the higher order moment problems arise. For example:

For the case of two spins $1/2$ ($\hat{S}_1 = 1/2$, $\hat{S}_2 = 1/2$ and $\hat{S} = \hat{S}_1 + \hat{S}_2$), we have magnetic dipole moment and electric quadrupole moment $[5,6,10,11,31,39]$, which can be defined from the experiment.

For the case of three spins $1/2$ ($\hat{S}_1 = 1/2$, $\hat{S}_2 = 1/2$, $\hat{S}_3 = 1/2$ and $\hat{S} = \hat{S}_1 + \hat{S}_2 + \hat{S}_3$), we have magnetic dipole moment, electric quadrupole moment, magnetic quadrupole moment and electric octopole moment. The higher order moments: magnetic quadrupole moments, electric octopole moments are not
extractable from the experiment due to the absence of complicated experiment scheme.
For the case of higher number of spins $1/2$. The higher order moments appear, which are undefined from the experiment except magnetic dipole moment and electric quadrupole moments.

5.5.1 Spin Hamiltonian for 2 spins $1/2$

Hamiltonian of two spins $\hat{\sigma}_1$ and $\hat{\sigma}_2$ defined in linear space $S_1, S_2$. $\hat{\sigma}_{1z}, \hat{\sigma}_{2z}$ coupling with hyperfine interaction $J_{12}$ are placed parallel to applied constant magnetic field $B_0||z-axis$. For simplicity, we are taking $\hbar = 1$.

$$\hat{H}_2 = -\mu B_0 \cdot (\hat{\sigma}_z \otimes \hat{E}_2) - \mu B_0 \cdot (\hat{E}_1 \otimes \hat{\sigma}_{x2}) + J_{12}(\hat{\sigma}_{x1} \otimes \hat{\sigma}_{x2}) + J_{12}(\hat{\sigma}_{y1} \otimes \hat{\sigma}_{y2})$$

5.5.2 Spin Hamiltonian for 3 spins $1/2$

Hamiltonian of three spins $\hat{\sigma}_1$, $\hat{\sigma}_2$ and $\hat{\sigma}_3$ are defined in linear space $S_1, S_2, S_3$ and coupling with hyperfine interaction $J_{12}$ between $\hat{\sigma}_1$ and $\hat{\sigma}_2$, $J_{23}$ between $\hat{\sigma}_2$ and $\hat{\sigma}_3$ and $J_{31}$ between spins $\hat{\sigma}_3$ and $\hat{\sigma}_1$. $\hat{\sigma}_{1z}, \hat{\sigma}_{2z}$ and $\hat{\sigma}_{3z}$ are placed parallel to applied constant magnetic field $B_0||z-axis$:

$$\hat{H}_3 = -\mu B_0 \cdot (\hat{\sigma}_z \otimes \hat{E}_2 \otimes \hat{E}_3) - \mu B_0 \cdot (\hat{E}_1 \otimes \hat{\sigma}_{z2} \otimes \hat{E}_3)$$

$$-\mu B_0 \cdot (\hat{E}_1 \otimes \hat{E}_2 \otimes \hat{\sigma}_{z3}) + J_{12}(\hat{\sigma}_{x1} \otimes \hat{\sigma}_{x2} \otimes \hat{E}_3)$$

$$+ J_{23}(\hat{E}_1 \otimes \hat{\sigma}_2 \otimes \hat{\sigma}_3) + J_{31}(\hat{\sigma}_3 \otimes \hat{\sigma}_1 \otimes \hat{E}_2)$$

Hamiltonians of higher number of spins $1/2$ can be written in the same way as for 2 and 3 spins $1/2$.

5.6 Kronecker product in quantum information theory to get the spin Hamiltonians

To write the spin Hamiltonian, first of all we should write $\hat{S}_z$, $\hat{S}_y$, $\hat{S}_z$, $\hat{S}^2$ and then add them, we will get the spin Hamiltonians (e.g. $\hat{H}_2$ and $\hat{H}_3$) in the following way.

Let we want to write the spin Hamiltonian of $n$ nuclear spins in NMR (Nuclear Magnetic Resonance):

1. Total projection of spins on z-axis is conserved.

$$\hat{S}_z = 1/2(\hat{\sigma}_{1z} \left\{ \sum_{i=2}^{n} \hat{E}_i, \right. \text{ if } n \geq 2$$

$$\frac{1}{1} \text{ if } n = 1 \right\} + \hat{E}_1 \otimes \hat{\sigma}_{2z} \left\{ \sum_{i=3}^{n} \hat{E}_i, \right. \text{ if } n \geq 3$$

$$\frac{1}{0} \text{ if } n = 2$$

$$\frac{0}{0} \text{ if } n < 2 \right\}$$
\[+\hat{E}_1 \otimes \hat{E}_2 \otimes \hat{\sigma}_{3z} \left\{ \begin{array}{ll} \otimes_{i=4}^{n} \hat{E}_i, & \text{if } n \geq 3 \\ 1 & \text{if } n = 3 \\ 0 & \text{if } n < 3. \end{array} \right\} + \cdots \] (17)

\[\hat{S}_x \text{ and } \hat{S}_y \text{ can be written by putting } \hat{\sigma}_x \text{ and } \hat{\sigma}_y \text{ in place of } \hat{\sigma}_z.\]

2. The square of the total spin \(\hat{S}^2 = \hat{S} \cdot (\hat{S} + 1)\) is conserved.

\[
\hat{S} = 1/2(i\hat{\sigma}_x + j\hat{\sigma}_y + k\hat{\sigma}_z)
\]

\[
\hat{S}^2 = 1/4(\hat{\sigma}_x^2 + \hat{\sigma}_y^2 + \hat{\sigma}_z^2)
\]

3. Equations (14) and (3) are constant of motion. That is \([\hat{S}_z, \hat{S}^2] = 0\). It means that the eigenvalues and eigenvectors of (14) and (3) are identicals.

4. The Hamiltonians \(\hat{H}_2\) and \(\hat{H}_3\) are consist of two parts, addition of (14) and (3). So the eigenvalues and eigenvectors of equations (14) and (3) are also the eigenvalues and eigenvectors of Hamiltonians \(\hat{H}_2\) and \(\hat{H}_3\).

6 Conclusion

We propose an alternative technique (quantum tomography), by using that we can correct the errors and can reconstruct the higher moments. So, the ancilla qubits make the error-correction problem more complicated. To remove the errors in quantum computer, we need to develop the experimental techniques how to detect higher moments.

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