ON THE O-MINIMAL HILBERT’S FIFTH PROBLEM

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Abstract. Let $M$ be an arbitrary o-minimal structure. Let $G$ be a definably compact definably connected abelian definable group of dimension $n$. Here we compute the new the intrinsic o-minimal fundamental group of $G$; for each $k > 0$, the $k$-torsion subgroups of $G$; the o-minimal cohomology algebra over $\mathbb{Q}$ of $G$. As a corollary we obtain a new uniform proof of Pillay’s conjecture, an o-minimal analogue of Hilbert’s fifth problem, relating definably compact groups to compact real Lie groups, extending the proof already known in o-minimal expansions of ordered fields.

1. Introduction

In this paper we work in an arbitrary o-minimal structure $M = (M, <, (c)_{c \in C}, (f)_{f \in F}, (R)_{R \in R})$ and are interested in the geometry of definable groups in $M$. We refer the reader to [7] for basic o-minimality. O-minimality is the analytic part of model theory and deals with theories of ordered, hence topological, structures satisfying certain tameness properties. It generalizes PL-geometry ([7]), semi-algebraic geometry ([4]) and globally sub-analytic geometry ([27], also called finitely sub-analytic in [6]) and it is claimed to be the formalization of Grothendieck’s notion of tame topology (topologie modérée). See [7] and [8].

A definable group in an o-minimal structure $M$ is a group whose underlying set is a definable set and the graph of the group operation is a definable set. The notion of definably compact is the analogue of the notion of semi-algebraically complete and was introduced by Peterzil and Steinhorn in [35]. The theory of definable groups, which includes real algebraic groups and semi-algebraic groups, began with Pillay’s paper [36] and has since then grown into a well developed branch of mathematics. The literature contains many interesting results about definable groups which have an analogue in the theory of Lie groups - see [36], [31], [32], [33], [10], [17] and [11]. All these fundamental results hinted at a deeper connection between definably compact definable groups and compact real Lie groups, which were finally formulated in the paper [37] by Pillay. Pillay’s conjecture is a non-standard analogue of Hilbert’s fifth problem for locally compact topological groups. Roughly it says that after taking the quotient by a “small subgroup” (a smallest type-definable subgroup of bounded index) the quotient when equipped with the so
called logic topology is a compact real Lie group of the same dimension. For more on definable groups and on Pillay’s conjecture see [28] and [29].

Pillay’s conjecture was solved in the following cases: (i) o-minimal expansions of fields [26] using new model-theoretic tools and the computation of m-torsion subgroups of definably compact abelian groups [17] (based on o-minimal singular (co)homological arguments); (ii) linear o-minimal expansions of ordered groups using direct methods [25]; (iii) semi-bounded non-linear o-minimal expansions of ordered groups [30] by reduction to the field case using a refinement of the dichotomy bounded/unbounded for semi-bounded sets studied in [9], namely the dichotomy short/long.

Here we extend the computation of m-torsion subgroups of definably compact abelian groups from [17] in o-minimal expansions of real closed fields to arbitrary o-minimal structures, using o-minimal sheaf cohomology instead of o-minimal singular cohomology and after defining a new o-minimal fundamental group in arbitrary o-minimal structures extending the one from o-minimal expansions of ordered groups:

**Theorem 1.1** (Structure Theorem). Let $G$ be a definably compact definably connected abelian definable group of dimension $n$. Then,

(a) the intrinsic o-minimal fundamental group of $G$ is isomorphic to $\mathbb{Z}^n$;

(b) for each $k > 0$, the $k$-torsion subgroup of $G$ is isomorphic to $(\mathbb{Z}/k\mathbb{Z})^n$, and

(c) the o-minimal cohomology algebra over $\mathbb{Q}$ of $G$ is isomorphic to the exterior algebra over $\mathbb{Q}$ with $n$ generators of degree one.

As pointed out in [26] (see Remark 4 and the end of Section 8), the proof of the Pillay’s conjecture given in that paper requires the presence of an ambient real closed field only in two places, namely, in the computation of $m$-torsion subgroups of definably compact abelian groups [17] and in the following fact on the theory of generic definable subsets first proved in o-minimal expansions of real closed fields in [34, Theorem 2.1].

**Fact 1.2.** Let $G$ be a definably compact group defined over a small model $\mathcal{M}_0$. If $X \subseteq G$ is a closed definable subset, then the set of $\mathcal{M}_0$-conjugates of $X$ is finitely consistent if and only if $X$ has a point in $\mathcal{M}_0$.

Since Fact 1.2 was established in [21, Theorem 3.2] in arbitrary o-minimal structures after it was generalized to o-minimal expansions of ordered groups in [30] (see point 1 at the beginning of Section 8), we also now have Pillay’s conjecture proved in arbitrary o-minimal structures:

**Theorem 1.3** (Pillay’s conjecture). Let $G$ be a definable group in a $\kappa$-saturated o-minimal structure $\mathcal{M}$ ($\kappa$ large). Then:

1. $G$ has a smallest type-definable normal subgroup of bounded index $G^{00}$.
2. $G/G^{00}$, equipped with the logic topology, is isomorphic, as a topological group, to a compact real Lie group.
3. If $G$ is definably compact, then $\dim_{\text{Lie}}(G/G^{00}) = \dim_{\mathcal{M}}(G)$.

We now explain the details of the proof of our main result, Theorem 1.1, pointing out to the reader the important points and techniques.

The strategy is the same as that of the proof of its analogue in o-minimal expansions of real closed fields ([17]), but we have to use o-minimal sheaf cohomology
(14) instead of the o-minimal singular homology ([38]) and cohomology ([22]) as well as a new o-minimal fundamental group in arbitrary o-minimal structures generalizing the o-minimal fundamental group from o-minimal expansions of fields ([2]) or ordered groups ([25], [12]).

Let \( G \) be a definably compact, definably connected, abelian, definable group of dimension \( n \).

From the o-minimal (co)homology side we need: (i) the Künneth formula to show that the cohomology of \( G \) with coefficients in \( \mathbb{Q} \) is a graded Hopf algebra of finite type; (ii) the theory of o-minimal \((\mathbb{Z}-)\)orientability to show that \( G \) is orientable and so the (co)homology of \( G \), with coefficients in \( \mathbb{Z} \), in degree \( n \) is \( \mathbb{Z} \); (iii) degree theory for continuous definable maps between orientable definably compact manifolds. These three parts in combination with the fact that the definable homomorphism \( p_k : G \rightarrow G : x \mapsto kx \) is a definable covering map, gives a lower bound on the size \( \#G[k] \) of the subgroup of \( k \)-torsion points of \( G \) of the form \( k^r \leq \#G[k] \) where \( r \) is the number of generators of the Hopf algebra of \( G \).

From the o-minimal fundamental group side we need: (iv) the new o-minimal fundamental group is well-connected with the theory of definable covering maps, giving us that \( G[k] \simeq (\mathbb{Z}/k\mathbb{Z})^s \) where \( s \) is number of generators of the new o-minimal fundamental group of \( G \); (v) the Hurewicz theorem relating the o-minimal (co)homology in degree one.

The Hurewicz and the universal coefficients theorem (from the cohomology side) shows that \( s \leq r \) and so, since we have \( k^r \leq k^s \), we obtain \( r = s \). Since also the sum of the degrees of the \( r \) generators of the Hopf algebra of \( G \) must be \( n \), because the (co)homology of \( G \) in degree \( n \) is \( \mathbb{Q} \), we obtain that \( r = s = n \) as required.

Given the above strategy let us now point out exactly which difficulties we had to face in order to implement it.

(i) The Künneth formula for the o-minimal singular homology is rather easy from the definitions as in the classical topological case (see [17] for details). The Künneth formula for o-minimal sheaf cohomology (even in coefficients in constant sheaves) turned out to be rather complicated and is obtained only after the formalism of the Grothendieck six operations on o-minimal sheaves is developed. This formalism was developed in the recent paper [20], but for definable spaces in full subcategories \( A \) of the category of definable spaces such that:

(A0) cartesian products of objects of \( A \) are objects of \( A \) and locally closed subsets of objects of \( A \) are objects of \( A \);

(A1) in every object of \( A \) every open definable subset is a finite union of open and definably normal definable subsets;

(A2) every object of \( A \) has a definably normal definable completion in \( A \).

Moreover, Künneth formula holds for objects \( X \) of such a subcategory \( A \) if furthermore:

(A3) for every elementary extension \( S \) of \( M \) and every sheaf \( F \) on the o-minimal site on \( X \) we have an isomorphism

\[
H^*_c(X ; F) \simeq H^*_c(X(S); F(S))
\]

where \( H^*_c \) is the o-minimal cohomology with definably compact supports ([18, Example 2-10 and Definition 2.12]).

Therefore, in order to use the Künneth formula here, we had to show that:
(*) the full subcategory of locally closed definable subsets of definably compact
definable groups satisfies conditions (A0), (A1) and (A2) and definably
compact groups satisfy condition (A3).

(ii) O-minimal Z-orientability theory is rather technical both with o-minimal
singular homology ([2], [3]) and with o-minimal sheaf cohomology, the difficult
part being the proof of the existence of relative fundamental classes associated to
orientations. Here this is obtained using a consequence of the o-minimal Alexander
duality theorem proved in [20] (as another consequence of the formalism of the
Grothendieck six operations on o-minimal sheaves).

(iii) Having a good orientation theory available, degree theory is rather classical.
The novelty here is that we work with the o-minimal Borel-Moore homology, but
since we only need to work in homology groups of top degree, we actually don’t
introduce formally the o-minimal Borel-Moore homology and use instead the de-
scription of the o-minimal Borel-Moore homology groups in top degree, given by
the o-minimal Alexander duality theorem, as the Z-dual of the relative o-minimal
cohomology group in top degree.

In both cases, in [17] and here, the existence of relative fundamental classes
associated to orientations and in fact also even the existence of local orientations
(resp. o-minimal orientation sheaf) depends crucially on the existence of finite
covers by open definable subsets of definably compact definable manifolds for which
we can compute some relative o-minimal singular cohomology groups (resp. the o-
minimal cohomology with definably compact supports).

Therefore, we had to show that:

(**) definably compact definable groups have such finite covers by open definable
subsets and, have o-minimal orientation sheaves and are orientable.

(iv) and (v) The existence of an o-minimal fundamental group in arbitrary o-
minimal structures extending the o-minimal fundamental group from o-minimal
expansions of fields ([2]) or ordered groups ([25], [12]) is one of the main novelties
of this paper. As observed in the concluding remarks of the paper [13], this new o-
minimal fundamental group, when relativized to a full subcategory P of the category
of locally definable spaces, will have all the properties proved in [13] (including
the good connection to definable covering maps and a Hurewicz theorem) if the
following hold:

(P1) (a) every object of P which is definably connected is uniformly definably
path connected;
(b) definable paths and definable homotopies in objects of P can be lifted
uniquely to locally definable coverings of such objects;

(P2) Every object of P has admissible covers by definably simply connected, open
definable subsets refining any admissible cover by open definable subsets.

Therefore, we also had to show that:

(***) definably compact definable groups leave in such subcategories P on which
the relativization of the new o-minimal fundamental group has the proper-
ties (P1) and (P2).

The main tool we use to obtain (*), (**) and (***) is a consequence of the
following result ([24, Theorem 3]):
Fact 1.4. If $G$ is a definable group, then there is a definable injection $G \to \prod_{i=1}^m J_i$, where each $J_i \subseteq M$ is a definable group-interval.

Note that Fact 1.4 does not imply that $G$ is somehow definably built from definable subgroups $G_1, \ldots, G_m$ with each $G_i$ definable in the definable group-interval $J_i$ with its induced structure, which would reduce all questions about $G$ to questions about definable groups in o-minimal expansions of (partial) ordered groups. Nevertheless we are able to use quite extensively Fact 1.4 to prove (*) , (** ) and (***) directly or in combination with an extension to the context of cartesian products of definable group-intervals of some techniques used by Berarducci and Fornasiero ([1]) in o-minimal expansions of ordered groups. These techniques from [1] were already used in [19] to prove (A3) for $G$.

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2. Preliminaries

In this section we prove a covers by open cells result and define a new o-minimal fundamental group.

2.1. A general covers by open cells result. Here we show that in o-minimal structures with definable choice functions every open definable subset is a finite union of open definable subsets each definably homeomorphic, by reordering of coordinates, to an open cell.

The following is obtained from the definition of cells ([7, Chapter 3, §2]):

Remark 2.1. Let $C \subseteq M^n$ be a $d$-dimensional cell. Then by definition of cells, $C$ is a $(i_1, \ldots, i_n)$-cell for some unique sequence $(i_1, \ldots, i_n)$ of 0’s and 1’s and there are $\lambda(1) < \cdots < \lambda(d)$ indices $\lambda \in \{1, \ldots, n\}$ for which $i_\lambda = 1$. Moreover, if $p_{(i_1, \ldots, i_n)} : M^n \to M^d : (x_1, \ldots, x_n) \mapsto (x_{\lambda(1)}, \ldots, x_{\lambda(d)})$ is the projection, then $C' := p_{(i_1, \ldots, i_n)}(C)$ is an open $d$-dimension cell in $M^d$ and the restriction $p_C := p_{(i_1, \ldots, i_n)}(C) : C \to C'$ is a definable homeomorphism.

Let $\tau(1) < \cdots < \tau(n - d)$ be the indices $\tau \in \{1, \ldots, n\}$ for which $i_\tau = 0$. For each such $i_\tau$, by definition of cells, there is a definable continuous function $f_\tau : \pi_{(i_1, \ldots, i_{\tau - 1})}(C) \subseteq M^{n-1} \to M$ where, for each $k = 1, \ldots, n$, $\pi_{(i_1, \ldots, i_k)} : M^n \to M^k$
is the projection onto the first \( k \)-coordinates. Moreover we have \( \pi_{(i_1, \ldots, i_r)}(C) = \{(x, f_i(x)) : x \in \pi_{(i_1, \ldots, i_{r-1}}(C)\}. \)

Let \( f = (f_1, \ldots, f_{n-d}) : C' \to M^{n-d} \) be the definable continuous map where for each \( l = 1, \ldots, n-d \) we set \( f_l = f_{\tau(l)} \circ \pi_{(i_1, \ldots, i_{\tau(l)-1})} \circ \pi_{C}^{-1}. \) Let \( \sigma : M^n \to M^n : (x_1, \ldots, x_n) \mapsto (x_{\lambda(1)}, \ldots, x_{\lambda(d)}, x_{\tau(1)}, \ldots, x_{\tau(n-d)}). \) Then we clearly have
\[
\sigma(C) = \{(x, f(x)) : x \in C'\}.
\]

**Theorem 2.2.** Suppose that \( M \) has definable choice functions. Let \( U \) be an open definable subset of \( M^n. \) Then \( U \) is a finite union of open definable sets definably homeomorphic, by reordering of coordinates, to open cells.

**Proof.** It suffices to prove the following. If \( C \subset U \) is a cell, then there are finitely many open subsets of \( U \) definably homeomorphic, by reordering of coordinates, to open cells such that \( C \) is contained in the union of them.

We proceed by induction on the dimension of \( C. \) The zero-dimensional case is immediate. Let \( C \) be \( d \)-dimensional and assume the statement for cells of lower dimension.

Modulo reordering of the coordinates (Remark 2.1) we may assume
\[
C = \{(x, f(x)) : x \in C'\}
\]
where \( C' \subset M^d \) is a \( d \)-dimensional open cell and \( f : C' \subseteq M^d \to M^{n-d} \) is a continuous definable function. Since \( U \) is open, for every \( x \in C' \) there are \( u, v \in M^{n-d} \) such that \( u_i < f_i(x) < v_i \) for every \( i = 1, \ldots, n-d, \) and
\[
\{x\} \times [u_1, v_1] \times \ldots \times [u_{n-d}, v_{n-d}] \subset U
\]
By definable choice there are definable functions \( g = (g_1, \ldots, g_{n-d}) : C' \to M^{n-d} \) and \( h = (h_1, \ldots, h_{n-d}) : C' \to M^{n-d} \) such that for every \( x \in C' \) we have \( g_i(x) < f_i(x) < h_i(x) \) for every \( i = 1 \ldots n-d, \) and
\[
\{x\} \times [g_1(x), h_1(x)] \times \ldots \times [g_{n-d}(x), h_{n-d}(x)] \subset U.
\]
Let \( O \subset C' \) be the definable set of the continuity points of \( g \) and \( h. \) Let \( C_1, \ldots, C_m \) be the \( d \)-dimensional cells of a cell decomposition of \( C' \) compatible with \( O. \) Then for each \( i = 1, \ldots, m \) define
\[
V_i = \{(x, z_1, \ldots, z_{n-d}) : x \in C_i \text{ and } g_l(x) < z_l < h_l(x) \text{ for } l = 1, \ldots, n-d\}
\]
These sets are clearly definably homeomorphic to open cells (in fact they are open cells), and \( \bigcup_i V_i \) covers \( \{(x, f(x)) : x \in C' \setminus \bigcup_i C_i \} \subset C. \) Hence we conclude by the induction hypothesis observing that \( \dim(\{(x, f(x)) : x \in C' \setminus \bigcup_i C_i \}) = \dim(C' \setminus \bigcup_i C_i) < d. \)

2.2. **A general o-minimal fundamental group functor.** Here we introduce an o-minimal fundamental group functor in arbitrary o-minimal structures. We also prove some basic properties of this new general o-minimal fundamental group.

First we recall the definition of the category of locally definable manifolds with continuous locally definable maps.

A locally definable manifold (of dimension \( n \)) is a triple \((S, (U_i, \theta_i)_{i \leq \kappa})\) where:

- \( S = \bigcup_{i \leq \kappa} U_i; \)
- \( \theta_i : U_i \to M^n \) are definable continuous functions.

...
each \( \theta_i : U_i \to \mathbb{R}^n \) is an injection such that \( \theta_i(U_i) \) is an open definable subset of \( M^n \);

for all \( i, j \), \( \theta_i(U_i \cap U_j) \) is an open definable subset of \( \theta_i(U_i) \) and the transition maps \( \theta_{ij} : \theta_i(U_i \cap U_j) \to \theta_j(U_i \cap U_j) : x \mapsto \theta_j(\theta_i^{-1}(x)) \) are definable homeomorphisms.

We call the \((U_i, \theta_i)\)'s the \textit{definable charts} of \( S \). If \( \kappa < \aleph_0 \) then \( S \) is a \textit{definable manifold}.

A locally definable manifold \( S \) is equipped with the topology such that a subset \( U \) of \( S \) is open if and only if for each \( i \), \( \theta_i(U \cap U_i) \) is an open definable subset of \( \theta_i(U_i) \).

We say that a subset \( A \) of \( S \) is \textit{definable} if and only if there is a finite \( I_0 \subseteq \kappa \) such that \( A \subseteq \bigcup_{i \in I_0} U_i \) and for each \( i \in I_0 \), \( \theta_i(A \cap U_i) \) is a definable subset of \( \theta_i(U_i) \). A subset \( B \) of \( S \) is \textit{locally definable} if and only if for each \( i \), \( B \cap U_i \) is a definable subset of \( S \). We say that a locally definable manifold \( S \) is \textit{definably connected} if it is not the disjoint union of two open and closed locally definable subsets.

If \( U = \{ U_\alpha \}_{\alpha \in I} \) is a cover of \( S \) by open locally definable subsets, we say that \( U \) is \textit{admissible} if for each \( i \leq \kappa \), the cover \( \{ U_\alpha \cap U_i \}_{\alpha \in I} \) of \( U_i \) admits a finite subcover.

If \( V = \{ V_\beta \}_{\beta \in J} \) is another cover of \( S \) by open locally definable subsets, we say that \( V \) \textit{refines} \( U \), denoted by \( V \leq U \), if there is a map \( \epsilon : J \to I \) such that \( V_\beta \subseteq U_{\epsilon(\beta)} \) for all \( \beta \in J \).

A map \( f : X \to Y \) between locally definable manifolds with definable charts \((U_i, \theta_i)_{i \leq \kappa_X}\) and \((V_j, \delta_j)_{j \leq \kappa_Y}\) respectively is a \textit{locally definable map} if for every finite \( I \subseteq \kappa_X \) there is a finite \( J \subseteq \kappa_Y \) such that:

- \( f(\bigcup_{i \in I} U_i) \subseteq \bigcup_{j \in J} V_j \);
- the restriction \( f_i : \bigcup_{i \in I} U_i \to \bigcup_{j \in J} V_j \) is a definable map between definable manifolds, i.e., for each \( i \in I \) and every \( j \in J \), \( \delta_j \circ f \circ \theta_i^{-1} : \theta_i(U_i) \to \delta_j(V_j) \) is a definable map between definable sets.

Thus we have the category of locally definable manifolds with locally definable continuous maps.

By a \textit{basic d-interval}, short for \textit{basic directed interval}, we mean a tuple

\[
\mathcal{I} = \langle [a, b], (0_\mathcal{I}, 1_\mathcal{I}) \rangle
\]

where \( a, b \in \mathbb{M} \) with \( a < b \) and \( (0_\mathcal{I}, 1_\mathcal{I}) \in \{ (a, b), (b, a) \} \). The \textit{domain} of \( \mathcal{I} \) is \([a, b]\) and the \textit{direction} of \( \mathcal{I} \) is \((0_\mathcal{I}, 1_\mathcal{I})\). The \textit{opposite} of \( \mathcal{I} \) is the \textit{basic d-interval}

\[
\mathcal{I}^{\text{op}} = \langle [a, b], (0_\mathcal{I}^{\text{op}}, 1_\mathcal{I}^{\text{op}}) \rangle
\]

with the same domain and opposite direction \((0_\mathcal{I}^{\text{op}}, 1_\mathcal{I}^{\text{op}}) = \langle 1_\mathcal{I}, 0_\mathcal{I} \rangle \).

If \( \mathcal{I}_i = \langle [a_i, b_i], (0_{\mathcal{I}_i}, 1_{\mathcal{I}_i}) \rangle \) are basic d-intervals, for \( i = 1, \ldots, n \), we define the \textit{d-interval}, short for \textit{directed interval},

\[
\mathcal{I}_1 \wedge \cdots \wedge \mathcal{I}_n := \left\{ \bigcup_i [c_i] \times [a_i, b_i] \right\}_\sim, \quad (0_{\mathcal{I}_1 \wedge \cdots \wedge \mathcal{I}_n}, 1_{\mathcal{I}_1 \wedge \cdots \wedge \mathcal{I}_n})
\]

where \( c_1, \ldots, c_n \) are \( n \) distinct points of \( M \), \( (c_i, 1_{\mathcal{I}_i}) \sim (c_{i+1}, 0_{\mathcal{I}_{i+1}}) \) for each \( i = 1, \ldots, n - 1 \) and \( 0_{\mathcal{I}_1 \wedge \cdots \wedge \mathcal{I}_n} = \langle c_1, 0_{\mathcal{I}_1} \rangle \) and \( 1_{\mathcal{I}_1 \wedge \cdots \wedge \mathcal{I}_n} = \langle c_n, 1_{\mathcal{I}_n} \rangle \). The \textit{domain} of \( \mathcal{I}_1 \wedge \cdots \wedge \mathcal{I}_n \) is

\[
[a_1, b_1] \wedge \cdots \wedge [a_n, b_n] := \bigcup_i [c_i] \times [a_i, b_i]_\sim
\]
and the \textit{direction} of $I_1 \wedge \cdots \wedge I_n$ is $(0_{I_1 \wedge \cdots \wedge I_n}, 1_{I_1 \wedge \cdots \wedge I_n})$. The \textit{opposite} of $I_1 \wedge \cdots \wedge I_n$ is the $d$-interval $(I_1 \wedge \cdots \wedge I_n)^{op} = ([a_1, b_1] \wedge \cdots \wedge [a_n, b_n], \langle 0_{(I_1 \wedge \cdots \wedge I_n)^{op}}, 1_{(I_1 \wedge \cdots \wedge I_n)^{op}} \rangle)$ with the same domain and opposite direction $(0_{(I_1 \wedge \cdots \wedge I_n)^{op}}, 1_{(I_1 \wedge \cdots \wedge I_n)^{op}}) = (1_{I_1 \wedge \cdots \wedge I_n}, 0_{I_1 \wedge \cdots \wedge I_n})$.

\textbf{Fact 2.3.} If $I_i = ([a_i, b_i], \langle 0_{I_i}, 1_{I_i} \rangle)$ are basic $d$-intervals, for $i = 1, \ldots, n$, then $(I_1 \wedge \cdots \wedge I_n)^{op} = I_1^{op} \wedge \cdots \wedge I_n^{op}$.

\textbf{Lemma 2.4.} Let $I = \langle I, \langle 0_I, 1_I \rangle \rangle$ be a $d$-interval. Then the domain $I$ of $I$ is a definable space of dimension one which is equipped with a definable total order $<_I$.

\textbf{Proof.} Let $I_i = ([a_i, b_i], \langle 0_{I_i}, 1_{I_i} \rangle)$ be basic $d$-intervals, for $i = 1, \ldots, n$, and suppose that $I = I_1 \wedge \cdots \wedge I_n$. Then $I = [a_1, b_1] \wedge \cdots \wedge [a_n, b_n]$ is clearly a definable space of dimension one.

For each $i$ let $<_I$ be the total order on $[a_i, b_i]$ which is $<$ if $\langle 0_{I_i}, 1_{I_i} \rangle = \langle a_i, b_i \rangle$ or $>$ if $\langle 0_{I_i}, 1_{I_i} \rangle = \langle b_i, a_i \rangle$. Then total ordering on $I$ is given by $x <_I y$ if $x \sim y$ and either $x, y \in [a_i, b_i]$ for some $i$, or $x <_I y$, or $y \in I_i$ and $y \in I_j$ with $i < j$.

Due to Lemma 2.4 below we will identify a $d$-interval $I = \langle I, \langle 0_I, 1_I \rangle \rangle$ with its domain equipped with the definable total order $<_I$. In particular, since the domain $I$ of $I^{op}$ is a definable space of dimension one which is equipped with the definable total order $>_I$, we have an order reversing definable homeomorphism $o_I : I \to I^{op}$ given by the identity on the domain.

Given two $d$-intervals $I = I_1 \wedge \cdots \wedge I_n$ and $J = J_1 \wedge \cdots \wedge J_m$, we define the $d$-interval $I \wedge J = I_1 \wedge \cdots \wedge I_n \wedge J_1 \wedge \cdots \wedge J_m$ and we will regard $I$ and $J$ as definable subsets of $I \wedge J$.

We say that $I$ and $J$ are equal, denoted $I = J$, if $n = m$ and $I_i = J_i$ for all $i = 1, \ldots, n$.

Let $X$ be a locally definable manifold. A \textit{definable path} $\alpha : I \to X$ is a continuous definable map from some $d$-interval $I$ to $X$. We define $\alpha_0 := \alpha(0_I)$ and $\alpha_1 := \alpha(1_I)$ and call the them the end points of the definable path $\alpha$.

A definable path $\alpha : I \to X$ is \textit{constant} if $\alpha_0 = \alpha(t)$ for all $t \in I$. Below, given a $d$-interval $I$ and a point $x \in X$, we denote by $c^I x$ the constant definable path in $X$ with endpoints $x$.

A definable path $\alpha : I \to X$ is a \textit{definable loop} if $\alpha_0 = \alpha_1$. The \textit{inverse} $\alpha^{-1}$ of a definable path $\alpha : I \to X$ is the definable path $\alpha^{-1} := \alpha \circ o_{I}^{-1} : I^{op} \to X$.

A concatenation of two definable paths $\gamma : I \to X$ and $\delta : J \to X$ with $\gamma(1_I) = \delta(0_J)$ is a definable path $\gamma \cdot \delta : I \wedge J \to X$ with:

$$(\gamma \cdot \delta)(t) = \begin{cases} 
\gamma(t) & \text{if } t \in I \\
\delta(t) & \text{if } t \in J.
\end{cases}$$
We say that $X$ is **definably path connected** if for every $u, v$ in $X$ there is a definable path $\alpha : I \rightarrow X$ such that $\alpha_0 = u$ and $\alpha_1 = v$.

Let $X$ be a locally definable manifold. Given two definable continuous maps $f, g : Y \rightarrow X$, we say that a definable continuous map $F(t, s) : Y \times J \rightarrow X$ is a **definable homotopy** between $f$ and $g$ if $f = F(0, \cdot) := F_{0, J}$ and $g = F(1, \cdot) := F_{1, J}$, where $\forall s \in J, F_s := F(\cdot, s)$. In this situation we say that $f$ and $g$ are **definably homotopic**, denoted $f \sim g$.

Two definable paths $\gamma : I \rightarrow X$, $\delta : J \rightarrow X$, with $\gamma_0 = \delta_0$ and $\gamma_1 = \delta_1$, are called **definably homotopic**, denoted $\gamma \approx \delta$, if there are $d$-intervals $I'$ and $J'$ such that $J' \cap I = J \cap I'$, and there is a definable homotopy

$$c_{\gamma}^{\delta}, \gamma \sim \delta \cdot c_{\gamma}^{\delta}$$

fixing the end points (i.e., they are definably homotopic by a definable homotopy $F : K \times J \rightarrow X$ such that $F(0, \beta, s) = \gamma_0 = \delta_0$ and $F(1, \beta, s) = \gamma_1 = \delta_1$ for all $s \in J$.)

The goal now is to show that definable homotopy of definable paths $\approx$ is an equivalence relation compatible with concatenation. The next two observations show that definable homotopy $\sim$ is an equivalence relation compatible with concatenation, however we have to do more since the relation $\approx$ does not assume that the domains of the definable paths are the same.

**Remark 2.5.** Let $X$ be a locally definable manifold. Then definable homotopy of definable continuous maps $Y \rightarrow X$ is an equivalence relation.

Indeed, $F : Y \times J \rightarrow X : (t, s) \mapsto f(t)$ is a definable homotopy between $f$ and $f'$; if $F : Y \times J \rightarrow X$ is a definable homotopy between $f$ and $g$, then $H := F \circ (id_Y \times \sigma_J)$ : $Y \times J^\text{op} \rightarrow X$ is a definable homotopy between $g$ and $f$; if $F : Y \times J \rightarrow X$ is a definable homotopy between $f$ and $g$ and if $G : Y \times K \rightarrow X$ is a definable homotopy between $g$ and $h$, then $H : Y \times (J \land K) \rightarrow X$ with

$$H(t, s) = \begin{cases} F(t, s) & \text{if } s \in J \\ G(t, s) & \text{if } s \in K. \end{cases}$$

is a definable homotopy between $f$ and $h$.

**Remark 2.6.** Let $X$ be a locally definable manifold. If $\gamma_i : I \rightarrow X$ ($i = 1, 2$) and $\delta : J \rightarrow X$ are definable paths with $\gamma_1 \sim \gamma_2$ and $(\gamma_i)_1 = \delta_0$ for $i = 1, 2$, then $\gamma_1 \cdot \delta \sim \gamma_2 \cdot \delta$.

Let $F : I \times A \rightarrow X$ be a definable homotopy between $\gamma_1$ and $\gamma_2$. Let $i : I \rightarrow I \land J$ and $j : J \rightarrow I \land J$ be the obvious definable immersions. Then $H : (I \land J) \times A \rightarrow X$ with

$$H(t, s) = \begin{cases} F(t', s) & \text{for } t = i(t') \text{ and } s \in A \\ \delta(t') & \text{for } t = j(t') \end{cases}$$

is a definable homotopy between $\gamma_1 \cdot \delta$ and $\gamma_2 \cdot \delta$.

Similarly, if $\lambda : J \rightarrow X$ is a definable path with $\lambda_1 = (\gamma_i)_0$ for $i = 1, 2$, then $\lambda \cdot \gamma_1 \sim \lambda \cdot \gamma_2$. 
Therefore, by transitivity of $\sim$ (Remark 2.5), if $\delta_i : J \to X$ ($i = 1, 2$) are definable paths with $\delta_1 \sim \delta_2$ and $(\gamma_i)_0 = (\delta_i)_0$ for $i = 1, 2$, then $\gamma_1 \cdot \delta_1 \sim \gamma_2 \cdot \delta_2$.

**Remark 2.7.** Let $X$ be a locally definable manifold. If $\gamma : I \to X$ is a definable path and $J$ is any $d$-interval, then

$$c_{I \cup J}^{\gamma} \cdot \gamma \sim \gamma \cdot c_{I \cup J}^{\gamma}.$$ 

Indeed let

$$i_1 : I \to I \land (J \land I)$$
$$i_2 : I \to (I \land J) \land I$$

be the two immersions of $I$ in $I \land J \land I$. Then $H : (I \land J \land I) \times I \to X$ with

$$H(t, s) = \begin{cases} 
\gamma(t') & \text{for } t = i_1(t') \text{ with } t' < s \\
\gamma(t') & \text{for } t = i_2(t') \text{ with } s < t' \\
\gamma(s) & \text{otherwise}
\end{cases}$$

is the required definable homotopy.

It follows from Remarks 2.7 and 2.6 that:

**Remark 2.8.** Let $X$ be a locally definable manifold. If $\delta_i : J \to X$ ($i = 1, 2$) are definable paths with $\delta_1 \sim \delta_2$, then $\delta_1 \approx \delta_2$.

**Lemma 2.9.** Let $X$ be a locally definable manifold. Let $\gamma : I \to X$ and $\delta : J \to X$ be definable paths with $\gamma_0 = \delta_0$ and $\gamma_1 = \delta_1$. Then the following are equivalent:

1. $\gamma \approx \delta$.
2. There are four $d$-intervals $A$, $B$, $C$, and $D$, such that $A \land I \land B = C \land J \land D$ and $c_A^{\gamma} \cdot \gamma \cdot c_B^{\gamma} \sim c_C^{\delta} \cdot \delta \cdot c_D^{\delta}$.

**Proof.** Assume (1). Consider $d$-intervals $I'$ and $J'$ such that $J' \land I = J \land I'$ and there is a definable homotopy $c_{I'}^{\gamma} \cdot \gamma \sim \delta \cdot c_{I'}^{\delta}$. Let $A = J'$, $B = J \land I'$, $C = J' \land I$ and $D = I'$. Then $A \land I \land B = C \land J \land D$ and we have by Remark 2.6

$$c_A^{\gamma} \cdot \gamma \cdot c_B^{\gamma} = (c_{I'}^{\gamma} \cdot \gamma) \cdot c_B^{\gamma} \sim (\delta \cdot c_{I'}^{\delta}) \cdot c_B^{\delta} = (\delta \cdot c_{I'}^{\delta}) \cdot c_{I' \land J}^{\delta} = (\delta \cdot c_{I' \land J}^{\delta}) \cdot c_{I' \land J}^{\delta}.$$ 

Since

$$c_{I'}^{\delta} \cdot \delta \cdot c_D^{\delta} = (c_{I'}^{\delta} \cdot \delta) \cdot c_{I'}^{\delta} = (c_{I' \land J}^{\delta} \cdot \delta) \cdot (c_{I' \land J}^{\delta} \cdot \delta),$$

we conclude by Remarks 2.7 and again 2.6 and transitivity of $\sim$ (Remark 2.5).

Assume (2). Consider four $d$-intervals $A$, $B$, $C$, and $D$, such that $A \land I \land B = C \land J \land D$ and $c_A^{\gamma} \cdot \gamma \cdot c_B^{\gamma} \sim c_C^{\delta} \cdot \delta \cdot c_D^{\delta}$. Let

$$J' = J \land A \land I \land B$$
$$I' = C \land J \land D \land I.$$ 

Then $J' \land I = J \land I'$ and by Remark 2.7 we also have

$$c_{J'}^{\gamma} \cdot \gamma = c_{J'}^{\gamma} \cdot c_A^{\gamma} \cdot c_I^{\gamma} \cdot \gamma \sim c_{J'}^{\gamma} \cdot (c_A^{\gamma} \cdot \gamma \cdot c_B^{\gamma}) \cdot c_I^{\gamma}.$$ 

$$\delta \cdot c_{I'}^{\delta} = \delta \cdot c_{I'}^{\delta} \cdot c_D^{\delta} \cdot c_{I'}^{\delta} \sim c_{I'}^{\delta} \cdot (c_D^{\delta} \cdot \delta \cdot c_D^{\delta}) \cdot c_{I'}^{\delta}.$$
We conclude by Remark 2.6 and transitivity of \( \sim \) (Remark 2.5).

Proposition 2.10. Let \( X \) be a locally definable manifold and \( x_0, x_1 \in X \). Let \( \mathbb{P}(X, x_0, x_1) \) denote the set of all definable paths in \( X \) that start at \( x_0 \) and end at \( x_1 \). Then the restriction of \( \approx \), the relation of being definably homotopic, to \( \mathbb{P}(X, x_0, x_1) \times \mathbb{P}(X, x_0, x_1) \) is an equivalence relation on \( \mathbb{P}(X, x_0, x_1) \).

Proof. For reflexivity, let \( \gamma : I \to X \) be a definable path in \( \mathbb{P}(X, x_0, x_1) \), and take \( I' = I \). Then \( I' \land I = I \land I' \) and \( c_{\gamma}^{00} \cdot \gamma \approx \gamma \cdot c_{\gamma}^{11} \) by Remark 2.7.

Symmetry follows at once from Lemma 2.9.

For transitivity consider definable paths \( \gamma : I \to X, \lambda : J \to X \) and \( \delta : J \to X \) in \( \mathbb{P}(X, x_0, x_1) \) and assume that \( \gamma \approx \lambda \) and \( \lambda \approx \delta \). Then there are \( d \)-intervals \( J', J'' \), and \( Y'' \), \( Y' \), such that \( Y' \land I = Y' \land I' \), \( J' \land Y' = J' \land Y'' \) and \( c_{\gamma}^{00} \cdot \gamma \approx \lambda \cdot c_{\gamma}^{11} \) \( c_{\lambda}^{00} \cdot \lambda \approx \delta \cdot c_{\lambda}^{11} \). By Remark 2.6 we have

\[
\begin{align*}
\delta \cdot c_{\gamma}^{00} \cdot \gamma &\approx c_{\gamma}^{00} \cdot \gamma \sim \lambda \cdot c_{\gamma}^{11} \\
\delta \cdot c_{\gamma}^{00} \cdot \gamma &\approx \lambda \cdot c_{\gamma}^{11} \sim \delta \cdot c_{\lambda}^{11} \cdot \lambda \sim \delta \cdot c_{\lambda}^{11}.
\end{align*}
\]

Since \( \gamma \approx \delta \), \( \lambda \approx \delta \), and \( \gamma \approx \delta \), \( \lambda \approx \delta \), then \( \gamma \cdot \delta \approx \gamma' \cdot \delta' \).

Lemma 2.11. Let \( X \) be a locally definable manifold. Let \( \gamma, \gamma', \delta \) and \( \delta' \) be definable paths in \( X \) such that \( \gamma_1 = \delta_0 \) and \( \gamma'_1 = \delta'_0 \). If \( \gamma \approx \gamma' \) and \( \delta \approx \delta' \), then \( \gamma \cdot \delta \approx \gamma' \cdot \delta' \).

Proof. By transitivity of \( \approx \) (Proposition 2.10) it suffices to prove the case \( \delta = \delta' \). Suppose that \( \gamma : I \to X, \gamma' : J \to X \) and \( \delta : J \to X \). By hypothesis there are \( d \)-intervals \( J' \) and \( J'' \) such that \( J' \land I = J' \land I' \) and \( c_{\gamma}^{00} \cdot \gamma \approx \gamma' \cdot c_{\gamma'}^{11} \). By Remarks 2.6 and 2.7 we obtain

\[
\begin{align*}
c_{\gamma}^{00} \cdot \gamma \cdot \delta \cdot c_{\gamma}^{11} &\approx \gamma' \cdot c_{\gamma'}^{11} \cdot \delta \cdot c_{\gamma'}^{11} \\
\gamma' \cdot c_{\gamma'}^{11} \cdot \delta &\approx \gamma' \cdot c_{\gamma'}^{11} \cdot \delta \\
\gamma' \cdot c_{\gamma'}^{11} \cdot \delta &\approx \gamma' \cdot c_{\gamma'}^{11} \cdot \delta.
\end{align*}
\]

and we conclude by Lemma 2.9 (using also Remark 2.6).

Lemma 2.12. Let \( X \) be a locally definable manifold and let \( \gamma : I \to X \) be a definable path in \( X \). Then \( \gamma \cdot \gamma^{-1} \approx c_{\gamma \land I}^{00} \) and so \( \gamma \cdot \gamma^{-1} \approx c_{\gamma \land I}^{00} \).

Proof. We have that \( H : (I \land I^0) \times I^0 \to X \) with

\[
H(t, s) = \begin{cases} 
\gamma(t) & \text{if } t \in I \text{ and } s < o_I(t) \\
\gamma^{-1}(t) & \text{if } t \in I^0 \text{ and } s < t \\
\gamma^{-1}(s) & \text{otherwise}
\end{cases}
\]

is the definable homotopy \( \gamma \cdot \gamma^{-1} \approx c_{\gamma \land I}^{00} \) and the rest follows from Remark 2.8.

Let \( X \) be a locally definable manifold and \( e_X \in X \). If \( L(X, e_X) \) denotes the set of all definable loops that start and end at a fixed element \( e_X \) of \( X \) (i.e. \( L(X, e_X) = \), then
\( \mathbb{P}(X, e_X, e_X) \), the restriction of \( \approx \) to \( \mathbb{L}(X, e_X) \times \mathbb{L}(X, e_X) \) is an equivalence relation on \( \mathbb{L}(X, e_X) \). We define the *o-minimal fundamental group* \( \pi_1(X, e_X) \) of \( X \) by

\[
\pi_1(X, e_X) := [\mathbb{L}(X, e_X)] / \approx
\]

and we set \([\gamma] := \text{the class of } \gamma \in \mathbb{L}(X, e_X)\). By Lemmas 2.11 and 2.12, \( \pi_1(X, e_X) \) is indeed a group with group operation given by \([\gamma][\delta] = [\gamma \cdot \delta]\) and identify the class \( a \) of constant loop at \( e_X \). Also this group depends on the topology on \( X \).

If \( f : X \to Y \) is a locally definable continuous map between two locally definable manifolds with \( e_X \in X \) and \( e_Y \in Y \) such that \( f(e_X) = e_Y \), then we have an induced homomorphism \( f_* : \pi_1(X, e_X) \to \pi_1(Y, e_Y) : [\sigma] \mapsto [f \circ \sigma] \) with the usual functorial properties.

We define the *o-minimal fundamental groupoid* \( \Pi_1(X) \) of \( X \) to be the small category \( \Pi_1(X) \) given by

\[
\text{Ob}(\Pi_1(X)) = X,
\]

\[
\text{Hom}_{\Pi_1(X)}(x_0, x_1) = \mathbb{P}(X, x_0, x_1) / \approx
\]

We set \([\gamma] := \text{the class of } \gamma \in \mathbb{P}(X, x_0, x_1)\). By Lemma 2.11, the small category \( \Pi_1(X) \) is indeed a groupoid with operations

\[
\text{Hom}_{\Pi_1(X)}(x_0, x_1) \times \text{Hom}_{\Pi_1(X)}(x_1, x_2) \to \text{Hom}_{\Pi_1(X)}(x_0, x_2)
\]

given by \([\delta] \circ [\gamma] = [\gamma \cdot \delta]\).

Note that if \( x \in X \), then \( \mathbb{P}(X, x, x) = \mathbb{L}(X, x) \) and so

\[
\pi_1(X, x) = \text{Hom}_{\Pi_1(X)}(X, x, x).
\]

If \( X \) is a locally definable manifold and \( x \in X \), we define \( \Pi_1(X, x) \) to be the category given by

\[
\text{Ob}(\Pi_1(X, x)) = \{x\},
\]

\[
\text{Hom}_{\Pi_1(X, x)}(x, x) = \pi_1(X, x).
\]

If \( f : X \to Y \) is a locally definable continuous map between locally definable manifolds, then we have an induced functor \( f_* : \Pi_1(X) \to \Pi_1(Y) \) which is a morphism of groupoids sending the object \( x \in X \) to the object \( f(x) \in Y \) and a morphism \([\gamma]\) of \( \Pi_1(X) \) to the morphism \([f \circ \gamma]\) of \( \Pi_1(Y) \).

**Lemma 2.13.** Let \( X \) and \( Y \) be locally definable manifolds. Then

1. If \( X \) is definably path connected then the natural functor \( \Pi_1(X, x) \to \Pi_1(X) \) is an equivalence for every \( x \in X \).
2. The natural functor \( \Pi_1(X \times Y) \to \Pi_1(X) \times \Pi_1(Y) \) given by projection is an equivalence.

**Proof.**
1. The functor \( \Pi_1(X, x) \to \Pi_1(X) \) sends the object \( x \) of \( \Pi_1(X, x) \) to the object \( x \) of \( \Pi_1(X) \) and sends a morphism of \( \Pi_1(X, x) \) represented by a definable loop at \( x \) to the morphism of \( \Pi_1(X) \) represented by the same definable loop at \( x \). By definition this morphism is fully faithfull. Since \( X \) is definably path connected, every object of \( \Pi_1(X) \) is isomorphic to the object \( x \). So the functor is also essentially surjective. Therefore, it is an equivalence.
2. The functor \( \Pi_1(X \times Y) \to \Pi_1(X) \times \Pi_1(Y) \) sends a morphism of \( \Pi_1(X \times Y) \) represented by a definable path \( \rho \) in \( X \times Y \) to the morphism of \( \Pi_1(X) \times \Pi_1(Y) \)
represented in each coordinate by the definable paths \( q_1 \circ \rho \) in \( X \) and \( q_2 \circ \rho \) in \( Y \) where \( q_1 \) and \( q_2 \) are the projections onto \( X \) and \( Y \), respectively. This functor is an isomorphism with inverse given by the functor \( \Pi_1(X) \times \Pi_1(Y) \to \Pi_1(X \times Y) \) that sends the object \((x, y)\) of \( \Pi_1(X) \times \Pi_1(Y) \) to the object \((x, y)\) of \( \Pi_1(X \times Y) \) and sends a morphism of \( \Pi_1(X) \times \Pi_1(Y) \) represented by a pair of definable paths \( \gamma \) in \( X \) and \( \delta \) in \( Y \) to the morphism of \( \Pi_1(X \times Y) \) represented by the definable path in \( X \times Y \) with coordinates \( \gamma \) and \( \delta \). \( \square \)

**Corollary 2.14.** Let \( X \) and \( Y \) be locally definable manifolds with \( e_X \in X \) and \( e_Y \in Y \). Then

1. If \( X \) is definably path connected then \( \pi_1(X, e_X) \simeq \pi_1(X, x) \) for every \( x \in X \).
2. \( \pi_1(X, e_X) \times \pi_1(Y, e_Y) \simeq \pi_1(X \times Y, (e_X, e_Y)) \).

**Notation:** As usual for a definably path connected locally definable manifold \( X \) if there is no need to mention a base point \( e_X \in X \), then by Corollary 2.14 (1), we may denote \( \pi_1(X, e_X) \) by \( \pi_1(X) \).

3. **Topology on Products of Definable Group-Intervals**

In this section we study some topology on products of definable group-intervals including: definable normality, locally definable covering maps and the relativized new o-minimal fundamental group, cohomology with definably compact supports of cells, the orientability and degree theory for definable manifolds in products of definable group-intervals.

### 3.1. Products of Definable Group-Intervals

Here we recall a few notions about products of definable group-intervals. The results we will need came from [24] or are built from what is done in that paper.

Recall the following ([24, Definition 3.1]):

**Definition 3.1.** A **definable group-interval** \( J = ((-b, b), 0, +, <) \) is an open interval \((-b, b) \subseteq M\), with \(-b < b \) in \( M \cup \{-\infty, +\infty\}\), together with a binary partial continuous definable operation \(+ : J^2 \to J\) and an element \( 0 \in J\), such that:

- \( x + y = y + x \) (when defined), \( (x + y) + z = x + (y + z) \) (when defined) and \( x < y \Rightarrow x + z < y + z \) (when defined);
- for every \( x \in J \) with \( 0 < x \), the set \( \{y \in J : 0 < y \text{ and } x + y \text{ is defined}\} \) is an interval of the form \((0, r(x))\);
- for every \( x \in J \) with \( 0 < x \), then \( \lim_{z \to 0}(x + z) = x \) and \( \lim_{z \to r(x)-(x + z)} = b; \)
- for every \( x \in J \) there exists \( z \in J \) such that \( x + z = 0 \).

The definable group-interval \( J \) is **unbounded** (resp. **bounded**) if the operation \(+\) in \( J \) is total (resp. not total). The notion of a **definable homomorphism** between definable group-intervals is defined in the obvious way.

By the properties above, it follows that: (i) for each \( x \in J \) there is a unique \( z \in J \) such that \( x + z = 0 \), called the inverse of \( x \) and denoted by \(-x\); (ii) for each \( x \in J \) we have \(-0 = 0, -(x) = x \) and \( 0 < x \) if and only if \(-x < 0\); (iii) the maps \( J \to J : x \mapsto -x \) and \((-b, 0) \to (0, b) : x \mapsto -x \) are continuous definable bijections;
(iv) for every $x \in J$ with $x < 0$, the set $\{y \in J : y < 0 \text{ and } x + y \text{ is defined}\}$ is an interval of the form $(-r(x), 0)$; (v) for every $x \in J$ with $x < 0$, then $\lim_{x \to 0} (x + z) = x$ and $\lim_{x \to -r(x)} (x + z) = -b$; (vi) for every $x \in J$ we have $x + 0 = x$ (both sides are defined and they are equal).

By the proof of [24, Lemma 3.5] we have:

**Fact 3.2.** Let $J = ((-b, b), 0, +, −, <)$ is a definable group-interval. Then there exists an injective, continuous definable homomorphism $\tau : J \to J$ given by $\tau(x) = \frac{x}{2}$ such that if $x, y \in \tau(J) = (-\frac{b}{2}, \frac{b}{2})$, then $x + y, x - y$ and $\frac{x}{2}$ are defined in $J$.

Fix a cartesian product $J = \Pi_{i=1}^{m} \mathcal{J}$ of definable group-intervals $\mathcal{J}_i = ((-b_i, b_i), \emptyset, +_i, -, _i, <)$. We say that $X$ is a $J$-bounded subset if $X \subseteq \Pi_{i=1}^{m} [-c_i, c_i]$ for some $c_i > 0$, in $\mathcal{J}_i$.

Let $l \in \{1, \ldots, m - 1\}$. For a definable subset $X \subseteq \Pi_{i=1}^{l} \mathcal{J}_i$, we set $L^l(X) = \{f : X \to J_{l+1} : f \text{ is definable and continuous}\}$ and $L_{\infty}^l(X) = L^l(X) \cup \{-i_{l+1} b_{l+1}, b_{l+1}\}$, where we regard $-i_{l+1} b_{l+1}$ and $b_{l+1}$ as constant functions on $X$. If $f \in L^l(X)$, we denote by $\Gamma(f)$ the graph of $f$. If $f, g \in L^l_{\infty}(X)$ with $f(x) < g(x)$ for all $x \in X$, we write $f < g$ and set $(f, g)_X = \{(x, y) \in X \times J_{l+1} : f(x) < y < g(x)\}$.

Then,

- a $J$-cell in $J_1$ is either a singleton subset of $J_1$, or an open interval with endpoints in $J_1 \cup \{-b_1, b_1\}$,
- a $J$-cell in $\Pi_{i=1}^{l+1} \mathcal{J}_i$ is a set of the form $\Gamma(f)$, for some $f \in L^l(X)$, or $(f, g)_X$, for some $f, g \in L_{\infty}^l(X)$, $f < g$, where $X$ is a $J$-cell in $\Pi_{i=1}^{l} \mathcal{J}_i$.

In either case, $X$ is called the domain of the defined cell. The dimension of a $J$-cell in $\Pi_{i=1}^{m} \mathcal{J}_i$ is defined as usual ([7, Chapter 3 (2.3) and Chapter 4 (1.1)])

We refer the reader to [7, Chapter 3 (2.10)] for the definition of a decomposition of $J$. A $J$-decomposition is then a decomposition $\mathcal{C}$ of $J$ such that each $B \in \mathcal{C}$ is a $J$-cell. The following can be proved similarly to [7, Chapter 3 (2.11)].

**Theorem 3.3 (J-CDT).**

1. If $A_1, \ldots, A_k \subseteq \Pi_{i=1}^{m} \mathcal{J}_i$ are definable sets, then there is a $J$-decomposition $\mathcal{C}$ that partitions each $A_i$.
2. If $A \subseteq \Pi_{i=1}^{m} \mathcal{J}_i$ is a definable set and $f \in L^{m+1}(A)$, then there is a $J$-decomposition $\mathcal{C}$ that partitions $A$ such that the restriction $f|_B$ to each $B \in \mathcal{C}$ with $B \subseteq A$ is continuous.

To $J$ there is an associated definable o-minimal structure $\mathbb{J}$ such that: (i) the domain of $\mathbb{J}$ is the definable set $\text{dom}(\mathbb{J}) = (-b_1, b_1) \cup \{-b_2, b_2\} \cup \{c_m\} \cup \{-m b_m, b_m\}$ where the $c_i$’s are new elements (each definable in $\mathbb{M}$), with the obvious induced definable total order; (ii) the $\mathbb{J}$-definable subsets are the subsets $X \subseteq \text{dom}(\mathbb{J})^k$ such that $X$ is a definable set.

By [24, Fact 4.4] we have:

**Fact 3.4.** The o-minimal structure $\mathbb{J}$ has $\mathbb{J}$-definable choice.

The following remark will allow us to work in $\mathbb{J}$ instead of in $\mathbb{M}$ when convenient.
Remark 3.5. Let \( X \subseteq \Pi_{i=1}^m J_i \) be a definable subset. Then \( X \) is a \( \mathbb{J} \)-definable set and a definable subset of \( X \) is relatively open if and only if it is a relatively open \( \mathbb{J} \)-definable subset of \( X \).

We call a (locally) definable manifold (resp. space) a (locally) definable \( \mathbf{J} \)-bounded manifold (resp. space) whenever it has definable charts \((U_i, \phi_i)\) with \( \phi_i(U_i) \) a definable \( \mathbf{J} \)-bounded subset.

3.2. Definable normality in products of definable group-intervals. Here we study the notion of definably normal in products of definable group-intervals extending what was known in o-minimal expansions of ordered groups ([7, Chapter 6, §3]).

Recall that a definable space \( X \) is definably normal if one of the following equivalent conditions holds:

1. for every disjoint closed definable subsets \( Z_1 \) and \( Z_2 \) of \( X \) there are disjoint open definable subsets \( U_1 \) and \( U_2 \) of \( X \) such that \( Z_i \subseteq U_i \) for \( i = 1, 2 \).
2. for every \( S \subseteq X \) closed definable and \( W \subseteq X \) open definable such that \( S \subseteq W \), there is an open definable subsets \( U \) of \( X \) such that \( S \subseteq U \) and \( U \subseteq W \).

Definable normality is quite useful since it gives the shrinking lemma (compare with [7, Chapter 6, (3.6)]):

**Fact 3.6** (The shrinking lemma). Suppose that \( X \) is a definably normal definable space. If \( \{U_i : i = 1, \ldots, n\} \) is a covering of \( X \) by open definable subsets, then there are open definable subsets \( V_i \) and definable closed subsets \( C_i \) of \( X \) (\( 1 \leq i \leq n \)) with \( V_i \subseteq C_i \subseteq U_i \) and \( X = \cup \{V_i : i = 1, \ldots, n\} \).

Recall that two definable intervals \((b, b') \subseteq M \) and \((a, a') \subseteq M \) are non-orthogonal if there are sub-intervals \((c, c') \subseteq (b, b') \) and \((d, d') \subseteq (a, a') \) together with a definable bijection \( \sigma : (c, c') \rightarrow (d, d') \). The intervals are orthogonal if they are not non-orthogonal.

**Lemma 3.7.** Let \( \mathbf{J} = \Pi_{i=1}^m J_i \) be a cartesian product of non-orthogonal definable group-intervals \( J_i = \langle (-b_i, b_i), 0_i, +_i, -_i, \cdot_i \rangle \). Then every definable subset of \( \Pi_{i=1}^m J_i \) is \( \mathbb{J} \)-definably normal.

**Proof.** By Fact 3.2 we only have to show that every definable subset of \( \Pi_{i=1}^m (-\frac{1}{m}, \frac{1}{m}) \) is \( \mathbb{J} \)-definably normal.

By non-orthogonality, for each \( l, k \in \{1, \ldots, m\} \), consider sub-intervals \((c_l, d_l) \subseteq (-b_l, b_l) \) and \((c_k, d_k) \subseteq (-b_k, b_k) \) together with a definable bijection \( \sigma_{lk} : (c_l, d_l) \rightarrow (c_k, d_k) \).

By o-minimality and Fact 3.2, we may assume that \( \sigma_{lk} \) is continuous and \((c_l, d_l) \subseteq (-\frac{1}{m}, \frac{1}{m}) \) (resp. \((c_k, d_k) \subseteq (-\frac{1}{m}, \frac{1}{m}) \)). By Fact 3.2, composing \( \sigma_{lk} \) with the translation \( x \mapsto x + \frac{d_l + d_k}{2} \) on the right and with the translation \( x \mapsto x - \frac{d_l + d_k}{2} \)
on the left, we may assume that \( c_i = -d_i \) and \( c_k = -d_k \). In this situation we have definable subgroup-intervals \( I_i = \langle -d_i, d_i \rangle, 0_i, +, i, < \rangle \) of \( J_i \) and \( I_k = \langle -d_k, d_k \rangle, 0_k, +, k, < \rangle \) of \( J_k \) together with a continuous definable bijection \( \sigma_{ik} : I_i \rightarrow I_k \).

Let \( I = \langle -d, d \rangle, 0, +, -, \langle \rangle \) be the definable group-interval \( I_1 \). For each \( i \), set \( \sigma_i = \sigma_{1i} \) and consider \( \delta_i : (-\frac{b_i}{4}, \frac{b_i}{4}] \times (-\frac{b_i}{4}, \frac{b_i}{4}) \rightarrow I \) given by

\[
\delta_i(x, y) = \begin{cases} 
\sigma_i^{-1}(|x - y|) & \text{if } |x - y| \in \sigma_i(I) \\
\text{d} & \text{otherwise.}
\end{cases}
\]

Now let \( \delta : \Pi_{i=1}^m (-\frac{b_i}{4}, \frac{b_i}{4}] \times \Pi_{i=1}^m (-\frac{b_i}{4}, \frac{b_i}{4}) \rightarrow I \)
be given by \( \delta(u, v) = \max\{\delta_i(u_i, v_i) : i = 1, \ldots, m\} \) whenever \( u = \langle u_1, \ldots, u_m \rangle \) and \( v = \langle v_1, \ldots, v_m \rangle \). Clearly, \( \delta \) is a continuous \( \mathbb{J} \)-definable function such that if \( A \subseteq B \subseteq \Pi_{i=1}^m (-\frac{b_i}{4}, \frac{b_i}{4}] \) are definable subsets with \( A \) closed in \( B \), then the definable function \( \delta_{A,B} : B \rightarrow I \) given by \( \delta_{A,B}(u) = \inf\{\delta(u, v) : v \in A\} \) is continuous with \( A = \{u \in B : \delta_{A,B}(u) = 0\} \). Therefore, if \( C \) and \( D \) are disjoint definable subset of a definable subset \( B \subseteq \Pi_{i=1}^m (-\frac{b_i}{4}, \frac{b_i}{4}] \) which are closed in \( B \), then the definable subsets \( U = \{v \in B : \delta_{C,B}(v) < \delta_{D,B}(v)\} \) and \( W = \{v \in B : \delta_{D,B}(v) < \delta_{C,B}(v)\} \) of \( B \) are open in \( B \), disjoint and such that \( C \subseteq U \) and \( D \subseteq W \).

Let \( J = \Pi_{i=1}^m J_i \) and \( I = \Pi_{j=1}^n I_j \) be cartesian products of definable intervals. We say that \( J \) and \( I \) are orthogonal if for any \( l \in \{1, \ldots, m\} \) and any \( k \in \{1, \ldots, n\} \) we have that \( J_l \) is orthogonal to \( I_k \).

**Lemma 3.8.** Let \( I \) and \( J \) be orthogonal cartesian products of definable intervals. Let \( A \subseteq I \times J \) be a definable set and consider the uniformly definable family \( \{A_x : x \in I\} \) of definable subsets \( A_x = \{y \in J : (x, y) \in A\} \) of \( J \). Then there are \( x_0, \ldots, x_s \in I \) such that \( \{A_x : x \in I\} = \{A_{x_0}, \ldots, A_{x_s}\} \). Moreover we have

\[
A = \bigcup_{i=0}^s (\{v \in I : A_v \supseteq A_{x_i}\} \times A_{x_i})
\]

and if \( A \) is open, then \( \{v \in I : A_v \supseteq A_{x_i}\} \) and \( A_{x_i} \) are open definable sets for each \( i = 0, \ldots, s \).

In particular, for every closed definable subset \( B \subseteq I \times J \) there are finitely many closed definable subsets \( C_1, \ldots, C_k \subseteq I \) and \( D_1, \ldots, D_l \subseteq J \) such that

\[
B = \bigcup \{C_i \times D_j : i = 1, \ldots, k \text{ and } j = 1, \ldots, l\}.
\]

**Proof.** Take a cell decomposition \( C \) of \( A \). Then \( \{C \cap A_x : C \in C\} \) is an induced uniform cell decomposition on each fiber \( A_x \). Replacing \( A \) by each \( C \in C \), we may suppose that each \( A_x \) is a cell of a fixed dimension \( k \).

We proceed by induction on \( k \). When \( k = 0 \), let \( a_x, b_x \) be the endpoints in the definition of \( A_x \) (possibly \( a_x = b_x \)). Then at least one of \( a_x, b_x \) must vary infinitely and definably with \( x \) if \( A_x \) does, contradicting orthogonality.
For the case $k > 1$, assume the result fails. Then at least one coordinate of $A_x$ must vary infinitely and definably as $x$ does. By induction, we know there are only finitely many sets of the form $\pi(A_x)$, where $\pi$ is projection to the first $k - 1$ coordinates. Fix a $\pi(A_x)$ such that there are infinitely many distinct $A_x$ with this projection, and restrict to this family. We suppose that all cells $A_x$ are “open” in the $k$-th coordinate – the other case is much the same – and that $f_x : \pi(A_x) \to J_k$, the function giving the lower boundary of $A_x$ in the $k$-th coordinate, varies infinitely and definably in $x$. If there exists a point in $\pi(A_x)$ at which $f_x$ takes on infinitely many values as $x$ varies, then we have a contradiction to orthogonality. So given any $y$ in the domain of $f_x$, there are finitely many values $f_x$ can assume at $y$, say at most $m$ independent of $y$. Let $D^i_x$ be the set of all $y$ such that $f_x(y)$ is the $i$-th possible value of $f_x$ at $y$. Note that the set $\{D^1_x, \ldots, D^m_x\}$ partitions $\pi(A_x)$ and determines the function $f_x$. Thus, for at least one $i \leq m$, the set $D^i_x$ varies infinitely as $x$ varies. Since $D^i_x$ lies in an ambient space of dimension $k - 1$, we are done by induction.

Clearly we have $A = \bigcup_i \{\{v \in I : A_v \supseteq A_{x_i}\} \times A_{x_i}\}$ and if $A$ is open then each fiber $A_{x_i}$ is also open.

Suppose that $A$ is open but some $\{v \in I : A_v \supseteq A_{x_i}\}$ is not open. Fix $v \in \{v \in I : A_v \supseteq A_{x_i}\}$. Thus, we can find points $z$ as close as we like to $v$ such that $A_{x_i} \not\subseteq A_z$ for any such $z$. Consider the family of definable sets $\{A_{x_i} \setminus A_z : z \in I\}$ and $A_{x_i} \not\subseteq A_z$. By the first part of the lemma, there are only finitely many sets in this family, so there is one that occurs for $z$ arbitrarily close to $v$, say $A_{x_i} \setminus A_{z_0}$. Fix any point $y \in A_{x_i} \setminus A_{z_0}$. Then for any open box $B$ containing $v$, we can find $z \in B$ with $y \not\in A_z$. But then any box in $I \times J$ around the point $(v, y) \in A$ must contain a point not in $A$, namely $(z, y)$ for such a $z$, contradicting that $A$ is open.

\begin{lemma}
Let $I_1$ and $I_2$ be orthogonal cartesian products of definable group-intervals and set $I = I_1 \times I_2$. Let $A \subseteq I_1$ be an $I_1$-definably normal definable subset and let $B \subseteq I_2$ be an $I_2$-definably normal definable subset. Then $A \times B \subseteq I$ is an $I$-definably normal definable subset.

\begin{proof}
Let $S, T \subseteq A \times B$ be closed, disjoint definable subsets. Then by Lemma 3.8, $S = \cup\{S_{i_1} \times S_{i_2} : i = 1, \ldots, s\}$ with each $S_{i_1} \subseteq A$ a closed (in $A$) definable subset and each $S_{i_2} \subseteq B$ a closed (in $B$) definable subset. Similarly, $T = \cup\{T_{i_2} \times T_{i_2} : j = 1, \ldots, t\}$.

First suppose $s = 1$. Since $T$ is disjoint from $S$ we have that each $T_{i_1} \times T_{i_2}$ has empty intersection with $S$ and therefore, either $T_{i_1}$ has empty intersection with $S_{i_1}$ or $T_{i_2}$ has empty intersection with $S_{i_2}$. Suppose the first case holds. Since $A \subseteq I_1$ is $I_1$-definably normal and $B \subseteq I_2$ is $I_2$-definably normal, there exist $U_{i_1} \subseteq A$ an definable subset open in $A$ and $V_{i_2} \subseteq B$ an definable subset open in $B$, containing $S_{i_1}$ and $T_{i_1}$ respectively, with empty intersection. Let $U_{i_1} \subseteq A$ be an arbitrary open (in $A$) definable subset and let $V_{i_2} \subseteq B$ be an arbitrary open (in $B$) definable subset. Then the products $U_{i_1} \times U_{i_2} \subseteq A \times B$ and $V_{i_2} \times V_{i_2} \subseteq A \times B$ are definable subsets open in $A \times B$, with empty intersection and containing $S$ and $T_{i_1} \times T_{i_2}$ respectively. Now take $U = \cap\{U_{i_1} : j = 1, \ldots, t\}$ and $V = \cup\{V_{i_2} : j = 1, \ldots, t\}$. Then $U$ and $V$ are definable subsets open in $A \times B$, with empty intersection and containing $S$ and $T$ respectively.
\end{proof}
\end{lemma}
If $s > 1$, by the above, one can take definable subsets $U_i$ and $V_i$ open in $A \times B$, with empty intersection and containing $S_{1i} \times S_{2i}$ and $T$ respectively. Let $U = \cup\{U_i : i = 1, \ldots, s\}$ and $V = \cap\{V_i : i = 1, \ldots, s\}$. Then $U$ and $V$ are definable subsets open in $A \times B$, with empty intersection and containing $S$ and $T$ respectively.

We are ready to prove the main observation of this subsection:

**Proposition 3.10.** Let $J$ be a cartesian product of definable groups intervals. Then every open definable subset of $J$ is a finite union of open $J$-definably normal definable subsets.

**Proof.** Let $J_1, \ldots, J_k$ be cartesian products of non-orthogonal definable group-intervals, with $J_i$ and $J_j$ orthogonal for $i \neq j$, and $J = \Pi_{i \leq k} J_i$. We prove the result by induction on $k$.

If $k = 1$ then every open definable subset of $J$ is $J$-definably normal by Lemma 3.7. On the other hand, the inductive step follows from Lemmas 3.8 and 3.9.

The following consequence of Proposition 3.10 will be useful later:

**Corollary 3.11.** Let $X$ be a locally definable $J$-bounded manifold. If $Z$ is a locally closed definable subset of $X$, then every open definable subset of $Z$ if a finite union of open definably normal definable subsets.

**Proof.** Intersecting $Z$ with the definable charts of $X$ we may identify $Z$ with a definable subset of $J$ and the result follows from Proposition 3.10.

We this subsection with some observations. Recall that a definable space $X$ is completely definably normal if one of the following equivalent conditions holds:

1. every definable subset $Z$ of $X$ is a definably normal definable subspace.
2. every open definable subset $U$ of $X$ is a definably normal definable subspace.
3. for every closed definable subsets $Z_1$ and $Z_2$ of $X$, if $Z_0 = Z_1 \cap Z_2$, then there are open definable subsets $V_1$ and $V_2$ of $X$ such that:
   - (i) $Z_i \setminus V_i = Z_0$, $i = 1, 2$.
   - (ii) $V_1 \cap V_2 = \emptyset$.
   - (iii) $V_1 \cap V_2 \subseteq Z_0$.
4. for every definable subsets $S_1$ and $S_2$ of $X$, if $S_1 \cap S_2 = \overline{S_1} \cap S_2 = \emptyset$, then there are disjoint open definable subsets $U_1$ and $U_2$ of $X$ such that $S_i \subseteq U_i$ for $i = 1, 2$.

In topology a Hausdorff compact space is normal and moreover completely definably normal. In the paper [16] it was showed that if $M$ has definable choice, then every Hausdorff definably compact definable space is definably normal, however the following shows that definable choice functions is not enough to guarantee complete definable normality:

**Example 3.12.** Let $I_1 = ((-d_1, d_1), 0_1, +, -1, <)$ and $I_2 = ((-d_2, d_2), 0_2, +, -2, <)$ be two orthogonal definable group-intervals. Let $(b_1, b'_1) \subseteq I_1$ and $(b_2, b'_2) \subseteq I_2$ be definable sub-intervals bounded in $I_1$ and $I_2$ respectively and consider also
a point \((a, b) \in (b_1, b'_1) \times (b_2, b'_2)\). Take \(X = [b_1, b'_1] \times [b_2, b'_2]\) and \(U = ((b_1, b'_1) \times (b_2, b'_2)) \setminus \{(a, b)\}\), and take \(A = \{a\} \times (b_2, b'_2)\) and \(B = (b_1, b'_1) \times \{b\}\).

Let \(I = I_1 \times I_2\). Then \(I\) has definable choice (Fact 3.4). \(X\) is \(I\)-definably compact and \(I\)-definably normal (Lemma 3.9) but the open \(I\)-definable subset \(U\) of \(X\) is not \(I\)-definably normal. Indeed, \(A\) and \(B\) are closed disjoint in \(U\) which, by the description of open definable subsets of \(I\) (Lemma 3.8), cannot be separated by disjoint open \(I\)-definable subsets.

### 3.3. Covering maps in products of definable group-intervals

Here we study the locally definable covering maps between locally definable \(J\)-bounded manifolds extending the results proved in o-minimal expansions of ordered group in [13].

Below we let \(J = \Pi_{i=1}^m J_i\) be a cartesian product of definable group-intervals \(J_i = (\langle -b_i, b_i \rangle, 0, +i, -i, \langle \rangle)\).

In this subsection we will need to relativize the notions and results of Subsection 2.2 to \(J\) in the following way:

- A basic \(d\)-\(J\)-interval is a basic \(d\)-interval \(I = \langle [a, b], \langle 0, 1 \rangle \rangle\) with \([a, b] \subseteq J_i\)
- If \(X\) is a locally definable \(J\)-bounded manifold, then a definable \(J\)-path (resp. constant definable \(J\)-path, or definable \(J\)-loop) is a definable path (resp. constant definable path or definable loop) \(\alpha : I \rightarrow X\) with \(I\) a \(d\)-\(J\)-interval; \(X\) is definably \(J\)-path connected if for every \(u, v\) in \(X\) there is a definable \(J\)-path \(\alpha : I \rightarrow X\) such that \(\alpha_0 = u\) and \(\alpha_1 = v\).
- If \(X\) and \(Y\) are locally definable \(J\)-bounded manifolds, then two definable continuous maps \(f, g : Y \rightarrow X\) are definable \(J\)-homotopy, denoted \(f \sim_J g\), if there is a definable homotopy \(F(t, s) : Y \times J \rightarrow X\) between \(f\) and \(g\) with \(J\) a \(d\)-\(J\)-interval; two definable \(J\)-paths \(\gamma : I \rightarrow X\), \(\delta : J \rightarrow X\), with \(\gamma_0 = \delta_0\) and \(\gamma_1 = \delta_1\), are definably \(J\)-homotopic, denoted \(\gamma \approx_J \delta\), if there are \(d\)-\(J\)-intervals \(I'\) and \(J'\) such that \(J' \wedge I = J \wedge I'\), and there is a definable \(J\)-homotopy

\[c_{I', J'} \cdot \gamma \sim_J \delta \cdot c_{I', J'}^{-1}\]

fixing the end points.

The results proved in Subsection 2.2 for the relations \(\sim\) and \(\approx\) hold also for \(\sim_J\) and \(\approx_J\) respectively.

Let \(X\) be a locally definable \(J\)-bounded manifold, \(e_X \in X\) and \(x_0, x_1 \in X\). Let \(\mathbb{P}^J(X, x_0, x_1)\) denote the set of all definable \(J\)-paths in \(X\) that start at \(x_0\) and end at \(x_1\) and let \(\mathbb{L}^J(X, e_X)\) denotes the set of all definable \(J\)-loops that start and end at a fixed element \(e_X\) of \(X\) (i.e. \(\mathbb{L}^J(X, e_X) = \mathbb{P}^J(X, e_X, e_X)\)). Then the restriction of \(\approx_J\) to \(\mathbb{P}^J(X, x_0, x_1) \times \mathbb{P}^J(X, x_0, x_1)\) is an equivalence relation on \(\mathbb{P}^J(X, x_0, x_1)\) and

\[\pi^J_1(X, e_X) := \mathbb{L}^J(X, e_X) / \approx_J\]

is a group, the \(o\)-\textit{minimal} \(J\)-\textit{fundamental group} \(\pi^J_1(X, e_X)\) of \(X\), with group operation given by \([\gamma][\delta] = [\gamma \cdot \delta]\) and identity the class \([\alpha]\) of constant \(J\)-loop at \(e_X\).

Moreover, if \(f : X \rightarrow Y\) is a locally definable continuous map between two locally
definable \( J \)-bounded manifolds with \( e_X \in X \) and \( e_Y \in Y \) such that \( f(e_X) = e_Y \), then we have an induced homomorphism \( f_* : \pi_1^J(X, e_X) \to \pi_1^J(Y, e_Y) : [\sigma] \mapsto [f \circ \sigma] \) with the usual functorial properties.

**Notation:** As usual for a definably \( J \)-path connected locally definable \( J \)-bounded manifold \( X \) if there is no need to mention a base point \( e_X \in X \), then by Corollary 2.14 (1), we may denote \( \pi_1^J(X, e_X) \) by \( \pi_1^J(X) \).

We start with the following in which the proof of (1) is similar to that of [7, Chapter 6, Proposition (3.2)] and in the proof of (2) we use the observation that, as in o-minimal expansions of ordered groups ([1, Lemmas 3.1 and 3.2]), there is a definable \( J \)-deformation retract from a \( J \)-cell which is \( J \)-bounded to a \( J \)-cell of lower dimension.

**Lemma 3.13.** Let \( C \subseteq \Pi_{i=1}^m (-\frac{b_i}{4}, \frac{b_i}{4}) \) be a \( J \)-cell. Then:

1. \( C \) is definably \( J \)-path connected. In fact there is a uniformly definable family of definable \( J \)-paths connecting a given fixed point in \( C \) to any other point in \( C \).
2. If \( C \) is a \( J \)-bounded subset, then \( C \) is definably \( J \)-simply connected, i.e. \( \pi_1^J(C) = 1 \).

**Proof.** (1) We prove the result of the inductive definition of \( J \)-cell. If \( C \) is a \( J \)-cell in \( J_1 \), then \( C \) is either a point or an open interval with endpoints in \( J_1 \cup \{ -\frac{b_i}{4}, \frac{b_i}{4} \} \) and (1) is clear in both cases. Otherwise, there is \( l \in \{ 1, \ldots, m-1 \} \) such that \( C \) is a \( J \)-cell in \( \Pi_{i=1}^{l+1} (-\frac{b_i}{4}, \frac{b_i}{4}) \) and \( C \) is either of the form \( \Gamma(f) \), for some \( f \in L^J(B) \), or \( (f, g)_B \) for some \( f, g \in L^J(B) \), \( f < g \), where \( B \) is a \( J \)-cell in \( \Pi_{i=1}^l (-\frac{b_i}{4}, \frac{b_i}{4}) \).

If \( C = \Gamma(f) \), then the projection of \( C \) onto \( B \) is a definable homeomorphism and (1) follows by induction hypothesis. Suppose that \( C = (f, g)_B \), and moreover assume also that \( -\frac{b_{l+1}}{4} < f < g < \frac{b_{l+1}}{4} \), if not the argument is easier. Let \( \langle x, u \rangle, \langle x', u' \rangle \in C \) with \( x, x' \in B \).

By Fact 3.2 we can do the operations \( \frac{f(x)+g(x)}{2} \) in the component \( J_{l+1} \) and so \( \langle x, \frac{f(x)+g(x)}{2} \rangle, \langle x', \frac{f(x')+g(x')}{2} \rangle \in C \). Let \( \alpha : J \to C \) be the vertical definable \( J \)-path with \( \alpha_0 = \langle x, u \rangle \) and \( \alpha_1 = \langle x, \frac{f(x)+g(x)}{2} \rangle \) and let \( \alpha' : J' \to C \) be the vertical definable \( J \)-path with \( \alpha_{0}' = \langle x', \frac{f(x')+g(x')}{2} \rangle \) and \( \alpha_{1}' = \langle x', u' \rangle \). By the induction hypothesis, let \( \beta : J \to B \) be a definable \( J \)-path with \( \beta_0 = x \) and \( \beta_1 = x' \). Then \( \alpha \cdot \gamma \cdot \alpha' : J \land J \land J' \to C \) where \( \gamma(t) = \langle \beta(t), \frac{f(\beta(t)) + g(\beta(t))}{2} \rangle \) is a definable \( J \)-path in \( C \) connecting \( \langle x, u \rangle \) to \( \langle x', u' \rangle \). Since the definable \( J \)-paths \( \alpha, \alpha' \) and \( \beta \) can be chosen uniformly, the definable \( J \)-path \( \gamma \) can also be defined uniformly.

(2) By Fact 3.2 we can apply the operations \( x-iy, x+iy \) and \( \frac{1}{4} \) in each coordinate of \( \Pi_{i=1}^m J_i \) just like in the proof of [1, Lemmas 3.1 and 3.2], to show that:

**Claim 3.14.** If \( C \) is a \( J \)-cell in \( J_1 \) which is \( J \)-bounded subset, then \( C \) is definably \( J \)-contractible to a point in \( J_1 \).

**Claim 3.15.** If \( C \) is a \( J \)-cell in \( \Pi_{i=1}^{l+1} J_i \) which is a \( J \)-bounded subset, then there is a definable \( J \)-deformation retract of \( C \) to a \( J \)-cell \( B \subseteq C \) of strictly lower dimension.
A definable \( J \)-deformation retract from \( C \) to \( B \) is a definable \( J \)-homotopy \( F : C \times I \to C \) between \( \text{id}_C \) and \( i_B \circ r \), where \( r : C \to B \) is given by \( r(x) = F(x, 1) \) and \( i_B : B \to C \) is the inclusion, such that \( F(b, t) = b \) for all \( b \in B \) and all \( t \in I \). Since \( r \circ i_B = \text{id}_B \) and \( i_B \circ r \), it follows that, if \( b \in B \subseteq C \), then \( i_{B*} : \pi_1^J(B, b) \to \pi_1^J(C, b) \) is an isomorphism with inverse \( r_* \).

By induction on the dimension of \( C \) we conclude that \( \pi_1^J(C) = 1 \) as required. \( \square \)

By Theorem 2.2 and Lemma 3.13 we have the following. Compare with the corresponding results [13, Lemma 2.9 and Proposition 3.1] in \( o \)-minimal expansions of ordered groups.

**Corollary 3.16.** Let \( X \) be a definable \( J \)-bounded manifold of dimension \( n \). Then the following hold:

1. \( X \) is definably connected if and only if \( X \) is definably \( J \)-path connected.
   In fact, for any definably connected definable subset \( D \) of \( X \) there is a uniformly definable family of definable \( J \)-paths in \( D \) connecting a given fixed point in \( D \) to any other point in \( D \).
2. \( X \) has an admissible cover \( \{ O_s \}_{s \in S} \) by open definably connected definable subsets such that:
   - \( \{ O_s \}_{s \in S} \) refines the definable charts of \( X \);
   - for each \( s \in S \), \( O_s \) is definably homeomorphic to a \( J \)-cell of dimension \( n \), in particular, the \( o \)-minimal \( J \)-fundamental group \( \pi_1^J(O_s) \) is trivial.

We will need one further crucial result. Compare with [17, Section 2] in \( o \)-minimal expansions of fields or [13, Lemma 2.13] in \( o \)-minimal expansions of ordered groups. But first we need to recall a few definitions. See for example [13].

Given a definably connected locally definable manifold \( S \), a locally definable manifold \( X \) and an admissible cover \( \mathcal{U} = \{ U_\alpha \}_{\alpha \in I} \) of \( S \) by open definable subsets, we say that a continuous surjective locally definable map \( p_X : X \to S \) is a locally definable covering map trivial over \( \mathcal{U} = \{ U_\alpha \}_{\alpha \in I} \) if the following hold:

- \( p_X^{-1}(U_\alpha) = \bigcup_{1 \leq s} U_\alpha^s \) a disjoint union of open definable subsets of \( X \);
- each \( p_X|_{U_\alpha} : U_\alpha \to U_\alpha \) is a definable homeomorphism.

A locally definable covering map \( p_X : X \to S \) is a locally definable covering map trivial over some admissible cover \( \mathcal{U} = \{ U_\alpha \}_{\alpha \in I} \) of \( S \) by open definable subsets.

We say that two locally definable covering maps \( p_X : X \to S \) and \( p_Y : Y \to S \) are locally definably homeomorphic if there is a locally definable homeomorphism \( F : X \to Y \) such that:

- \( p_X = p_Y \circ F \).

Such \( F : X \to Y \) is called a locally definable covering homeomorphism.

A locally definable covering map \( p_X : X \to S \) is trivial if it is locally definably homeomorphic to a locally definable covering map \( S \times M \to S : (s, m) \mapsto s \) for some set \( M \).

Let \( p_Y : Y \to T \) be a locally definable covering map, \( X \) be a locally definable manifold and let \( f : X \to T \) be a locally definable map. A lifting of \( f \) is a continuous map \( \tilde{f} : X \to Y \) such that \( p_Y \circ \tilde{f} = f \). Note that a lifting of a continuous
locally definable map need not be a locally definable map. However, if $X$ is definably connected, then any two continuous locally definable liftings which coincide in a point must be equal [13, Lemma 2.8].

**Lemma 3.17.** Let $X$ and $S$ be locally definable $J$-bounded manifolds. Suppose that $p_X : X \to S$ is a locally definable covering map. Then the following hold.

1. Let $\gamma : I \to X$ be a definable $J$-path in $S$ and $x \in X$. If $p_X(x) = \gamma_0$, then there is a unique definable $J$-path $\overline{\gamma} : I \to X$ in $X$, lifting $\gamma$, such that $\overline{\gamma}_0 = x$.

2. Suppose that $F : I \times J \to X$ is a definable $J$-homotopy between the definable $J$-paths $\gamma$ and $\sigma$ in $S$. Let $\overline{\gamma}$ be a definable $J$-path in $X$ lifting $\gamma$. Then there is a unique definable lifting $\overline{F} : I \times J \to X$ of $F$, which is a definable $J$-homotopy between $\overline{\gamma}$ and $\overline{\sigma}$, where $\overline{\sigma}$ is a definable $J$-path in $X$ lifting $\sigma$.

**Proof.** Let $U = \{U_\alpha\}_{\alpha \in I}$ be an admissible cover of $S$ by open definable subsets over which $p_X : X \to S$ is trivial. We may assume that $U = \{U_\alpha\}_{\alpha \in I}$ refines the definable chart $s$ of $S$ witnessing the fact that $S$ is a locally definable $J$-bounded manifold.

1. First we assume that $I$ is a basic $d$-$J$-interval $([a, b], \langle 0_I, 1_I \rangle)$. We may also assume that the definable total order $<_I$ on the domain $[a, b]$ of $I$ is $<$. If not, the argument is similar, one just has to construct the lifting from right to left instead of from left to right.

   Let $L \subseteq I$ be a finite subset such that $\gamma([a, b]) \subseteq \bigcup_{i \in L} U_i$. Then $[a, b] \subseteq \bigcup_{i \in L} \gamma^{-1}(U_i)$, with the $\gamma^{-1}(U_i)$’s open in $[a, b]$. By Lemma 3.7, $[a, b]$ is definably normal and so by the shrinking lemma (Fact 3.6), for each $l \in L$ there is $W_l \subseteq [a, b]$, open in $[a, b]$ such that $W_l \subseteq \gamma^{-1}(U_l)$ and $[a, b] \subseteq \bigcup_{l \in L} W_l$. Therefore, there are $a = s_0 < s_1 < \cdots < s_r = b$ such that for each $i = 0, \ldots, r - 1$ we have $\gamma([s_i, s_{i+1}]) \subseteq U_{t(i)}$ (and $\gamma(s_{i+1}) \in U_{t(i) \cap U_{t(i+1)}}$).

   Lift $\gamma_1 = \gamma|_{[s_0, s_1]}$ to $\overline{\gamma}_1 = (p_{U_{t(0)}}|_{U_{t(0)}})^{-1} \circ \gamma|_{[a, s_1]}$, with $\overline{\gamma}_1(0) = x$, using the definable homeomorphism $p_{U_{t(0)}} : U_{t(0)} \to U_{t(0)}$, where $U_{t(0)}$ is the definable connected component of $p^{-1}(U_{t(0)})$ in which $x$ lays. Repeat the process for each $\gamma_{i+1} = \gamma|_{[s_i, s_{i+1}]}$ with $\overline{\gamma}_i(s_i)$ instead of $x$. Patch the liftings together to obtain $\overline{\gamma}$.

   Now if $I = I_1 \wedge \ldots \wedge I_k$ with each $I_i$ a basic $d$-$J$-interval apply the previous process to lift $\gamma_1 = \gamma|_{I_1}$ to $\overline{\gamma}_1$, with $\overline{\gamma}_1(0) = x$ and repeat the process for each $\gamma_{i+1} = \gamma|_{I_{i+1}}$ with $\overline{\gamma}_i(1_I)$ instead of $x$. Patch the liftings together to obtain $\overline{\gamma}$.

   Uniqueness follows (in each step) from [13, Lemma 2.8].

2. First assume that $J$ is a basic $d$-$J$-interval $([c, d], \langle 0_J, 1_J \rangle)$. We may also assume that the definable total order $<_J$ on the domain $[c, d]$ of $J$ is $<$. If not, the argument is similar, one just has to construct the lifting from top to bottom instead of from bottom to top.

   To proceed we also assume that $I$ is a basic $d$-$J$-interval $([a, b], \langle 0_I, 1_I \rangle)$. We may furthermore assume that the definable total order $<_I$ on the domain $[a, b]$ of $I$ is $<$. If not the argument is similar, one just has to construct the lifting from right to left instead of from left to right.

   Let $L \subseteq I$ be a finite subset such that $F([a, b] \times [c, d]) \subseteq \bigcup_{l \in L} U_l$. Then $[a, b] \times [c, d] \subseteq \bigcup_{l \in L} F^{-1}(U_l)$, with the $F^{-1}(U_l)$’s open in $[a, b] \times [c, d]$. By Lemma 3.7, both $[a, b]$ and $[c, d]$ are definably normal and so by Lemma 3.7 or Lemma 3.9...
whose objects are the locally definable
coverings maps and o-minimal fundamental groups in o-minimal expan-
sions of or-
Remark 3.18. Let $J$-decomposition of $[a, b] \times [c, d]$ compatible with the $W_l$'s. This $J$-decomposition is given by a decom-
position $a = t_0 < t_1 < \cdots < t_r = b$ of $[a, b]$ together with definable continuous
functions $f_{i,j} : [t_i, t_{i+1}] \to [c, d]$ for $i = 0, \ldots, r - 1$ and $j = 0, \ldots, k_i$ such that: (i) $f_{i,0} < f_{i,1} < \cdots < f_{i,k_i}$ for $i = 0, \ldots, r - 1$; (ii) $\Gamma(f_{i,0}) = [t_i, t_{i+1}] \times \{c\}$ and $\Gamma(f_{i,k_i}) = [t_i, t_{i+1}] \times \{d\}$ for $i = 0, \ldots, r - 1$; (iii) the two-dimensional $J$-cells are of form $C_{i,j} = (f_{i,j}, f_{i,j+1})(t_i, t_{i+1})$. For each two-dimensional $J$-cell $C_{i,j}$ and each $l(i,j)$ such that $C_{i,j} \subset W_{l(i,j)}$, we have $F(C_{i,j}) \subset U_{l(i,j)}$ and for any two-dimensional $J$-cells $C_{i,j}$ and $C_{i',j'}$ in $[a, b] \times [c, d]$, and for each $l(i,j), l(i', j')$, such that $C_{i,j} \subset W_{l(i,j)}$ and $C_{i',j'} \subset W_{l(i',j')}$, we also have $F(C_{i,j}) \cap F(C_{i',j'}) \subset U_{l(i,j)} \cap U_{l(i',j')}$. Lift $F_0,1 = F_{|U_0,1}$ to $F_0,1 = (p^{-1}_{|U_0,1})^{-1} \circ F_{|U_0,1}$, using the definable homeomor-
phism $p^{-1}_{|U_0,1} : U_0,1 \to U_0,1$, where $U_{|U_0,1}$ is the definable connected com-
ponent of $p^{-1}(U_{|U_0,1})$ in which $\bar{\gamma}(l_0, t_1)$ lays. Repeat the process for each $F_{0,j+1} = F_{|U_0,j+1}$ with $F_{0,j+1}(\Gamma(f_{0,j}))$ instead of $\bar{\gamma}(l_0, t_1)$. Patch the liftings together to obtain $F_0 : [l_0, t_1] \times [c, d] \to X$ a definable lifting of $F_{|[l_0, t_1] \times [c, d]}$ which is a definable
$J$-homotopy between $\bar{\gamma}_{l_0, t_1}$ and $\bar{\sigma}_{l_0, t_1}$. Repeat the above process again but now for each $i = 1, \ldots, r - 1$, starting in each case with $\bar{\gamma}(l_i, t_{i+1})$ and obtain the liftings $\bar{F}_i : [t_i, t_{i+1}] \times [c, d] \to X$ a definable lifting of $F_{|[t_i, t_{i+1}] \times [c, d]}$ which is a definable
$J$-homotopy between $\bar{\gamma}_{t_i, t_{i+1}}$ and $\bar{\sigma}_{t_i, t_{i+1}}$. These liftings patch together to give a definable lifting $\bar{F} : [a, b] \times [c, d] \to X$ of $F$ which is a definable $J$-homotopy between $\bar{\gamma}$ and $\bar{\sigma}$. Now if $I = I_1 \land \cdots \land I_k$ with each $I_i$ a basic $d$-$J$-interval apply the previous process to lift $F_1 = F_{|I_1 \times [c, d]}$ to $\bar{F}_1$, with $\bar{F}_1(I_1, c) = \bar{\gamma}(I_1)$ and repeat the process for each $F_{i+1} = F_{|I_i \times [c, d]}$ with $\bar{\gamma}(I_{i+1})$ instead of $\bar{\gamma}(I_i)$. Then patch these liftings together to obtain a definable lifting $\bar{F} : I \times J \to X$ of $F$ which is a definable
$J$-homotopy between $\bar{\gamma}$ and $\bar{\sigma}$. Now if $J = J_1 \land \cdots \land J_k$ with each $J_i$ a basic $d$-$J$-interval apply the previous process to lift $F_1 = F_{|I \times J_i}$ to $\bar{F}_1$, with $\bar{F}_1(I, 0, J_i) = \bar{\gamma}(I)$ and repeat the process for each $F_{i+1} = F_{|I \times J_{i+1}}$ with $\bar{F}_1(I, 1, J_i)$ instead of $\bar{F}_1(I, 0, J_i)$. To finish patch these liftings together to obtain a definable lifting $\bar{F} : I \times [c, d] \to X$ of $F$ which is a definable $J$-homotopy between $\bar{\gamma}$ and $\bar{\sigma}$. As above, uniqueness follows from [13, Lemma 2.8]. We end by observing that all the main results from [13] about locally definable covering maps and o-minimal fundamental groups in o-minimal expansions of ordered
discrete structures also hold for locally definable covering maps between locally definable
$J$-bounded manifolds and o-minimal $J$-fundamental groups.

Remark 3.18. Let $P$ be at the full subcategory of locally definable spaces in $M$
whose objects are the locally definable $J$-bounded manifolds. Then in the category
$P$ the following hold:

(P1) (a) every object of $P$ which is definably connected is uniformly definably
$J$-path connected;
(b) given a locally definable covering map \( p_X : X \to S \) in \( \mathbf{P} \) then: (i) every definable \( J \)-path \( \gamma \) in \( S \) has a unique lifting \( \tilde{\gamma} \) which is a definable \( J \)-path in \( X \) with a given base point; (ii) every definable \( J \)-homotopy \( F \) between definable \( J \)-paths \( \gamma \) and \( \sigma \) in \( S \) has a unique lifting \( \tilde{F} \) which is a definable \( J \)-homotopy between the definable \( J \)-paths \( \tilde{\gamma} \) and \( \tilde{\sigma} \) in \( X \).

(P2) Every object of \( \mathbf{P} \) has admissible covers by definably \( J \)-simply connected, open definable subsets refining any admissible cover by open definable subsets.

As observed in the Concluding remarks of the paper [13], with (P1) and (P2) one proves in exactly the same way all the main result of the paper [13]. In fact, besides (P1) and (P2) (and their consequences) everything else that is required is: [15, Lemma 2.1 (1)] (in the proofs of [13, Theorem 3.4 and Proposition 4.3], [15, Corollary 2.2] (in [13, Remark 3.12]), [15, Corollary 2.3] (in [13, Remark 3.13]) and [7, Chapter 6, (3.6)] (in [13, Theorem 3.16 and 4.7]). Now the quoted results from [15] hold in arbitrary o-minimal structures (and for locally definable spaces as well). On the other hand, [7, Chapter 6, (3.6)] is used to notice that the domains of the “good” definable paths are definably normal. In our case here the good definable paths are the definable \( J \)-paths and their domains are definably normal by Proposition 3.10.

The fact that (P1) and (P2) are the only requirements needed to develop this kind of theory is somewhat not surprising. Indeed in topology, where we have good notions of paths and homotopies with the lifting of paths and homotopies property, all one needs is existence of such nice open covers as in (P2). In the o-minimal context (here and in [13]), the role that (P1) (b) and (P2) play is similar to the role the analogue properties play in topology. However, (P2) is often used in combination with the results from [15] mentioned above to get local definability. Also (P1) (a) is required essentially only once and to get local definability (see [13, Proposition 2.18]), the other places where it is used, it is used to replace definably connected by definably path connected.

Due to Remark 3.18, in the rest of the paper, when needed, we will freely use the results of [13] in our context locally definable coverings maps between locally definable \( J \)-bounded manifolds and o-minimal \( J \)-fundamental groups.

**Question:** Let \( X \) be definably connected, locally definable \( J \)-bounded manifold. It is not difficult to prove that \( \text{id}_X : X \to X \) induces a well defined surjective homomorphism \( \iota : \pi^J_1(X) \to \pi_1(X) \). Indeed, a definable \( J \)-homotopy is a definably homotopy and by Theorem 2.2 and Lemma 3.13 every definable path in \( X \) is definably homotopic to a definable \( J \)-path. Is \( \iota : \pi^J_1(X) \to \pi_1(X) \) always an isomorphism?

3.4. **Cohomology with definably compact supports of \( J \)-cells.** Here we compute the o-minimal cohomology with definably compact supports of \( J \)-cells. This will be necessary later for the theory of orientability.

We start with the following easy observation:
Fact 3.19. Let $h : X \rightarrow Y$ a continuous definable map between definable spaces. If $K \subseteq X$ is a definably compact subset with definable choice for uniformly definable families of closed definable subsets, then $h(K)$ is a definably compact definable subset of $Y$.

Below we let $J = \Pi_{i=1}^{m} J_{i}$ be a cartesian product of definable group-intervals $J_{i} = ((-b_{i}, b_{i}), 0, +, -, <)$.

Let $C$ be a $J$-cell which is a $J$-bounded subset and of dimension $r$. By Fact 3.2, we assume that $C \subseteq \Pi_{i=1}^{m} [-c_{i}, c_{i}]$ for some $0 < c_{i} < \frac{b_{i}}{2}$ in $J_{i}$ and the group-interval operations $x - y$, $x - y$ and $\frac{b_{i}}{2}$ are all defined in each coordinate of $\Pi_{i=1}^{m} J_{i}$.

Following a similar construction in o-minimal expansions of ordered groups from [1, Lemma 7.1], it was defined in [19, Lemma 4.11] the definable family $\{C_{t_{1}, \ldots, t_{m}} : 0 < t_{i} < \frac{b_{i}}{2}, i = 1, \ldots, m\}$ of closed and $J$-bounded subsets by induction on $l \in \{1, \ldots, m - 1\}$ in the following way.

(1) If $l = 1$ and $C$ is a singleton in $J_{1}$, we define $C_{t_{1}} = C$.

(2) If $l = 1$ and $C = (d, e) \subseteq J_{1}$, then $C_{t_{1}} = [d + \gamma_{t_{1}}, e - 1 - \gamma_{t_{1}}]$ where $\gamma_{t_{1}} = \min\{\frac{b_{1}}{2e}, 1, t_{1}\}$, (in this way $C_{t_{1}}$ is non empty).

(3) If $l > 1$ and $C = \Gamma(f)$, where $f \in L^{l}(B)$ is a continuous definable map and $B$ is $\Pi_{i \leq l} J_{i}$-cell which is a $\Pi_{i \leq l} J_{i}$-bounded subset. By induction $B_{t_{1}, \ldots, t_{l}}$ is defined. We put $C_{t_{1}, \ldots, t_{l-1}, t_{l+1}} = \Gamma(f_{B_{t_{1}, \ldots, t_{l+1}}})$.

(4) If $l > 1$ and $C = (f, g)_{B}$, where $f, g \in L^{l}(B)$ are continuous definable maps, $B$ is $\Pi_{i \leq l} J_{i}$-cell which is a $\Pi_{i \leq l} J_{i}$-bounded subset and $f < g$. By induction $B_{t_{1}, \ldots, t_{l}}$ is defined. We put $C_{t_{1}, \ldots, t_{l-1}, t_{l+1}} = [f + t_{l+1} \gamma_{t_{l+1}}^{l+1}, g - t_{l+1} \gamma_{t_{l+1}}^{l+1}]_{B_{t_{1}, \ldots, t_{l}}}$, where $\gamma_{t_{l+1}}^{l+1} := \min\{\frac{b_{l+1}}{2}, |t_{l+1}|, t_{l+1}\}$.

By construction we have:

Remark 3.20. Let $C$ be a $J$-cell which is a $J$-bounded subset. Assume that $C \subseteq \Pi_{i=1}^{m} [-c_{i}, c_{i}]$ for some $0 < c_{i} < \frac{b_{i}}{2}$ in $J_{i}$. Then the following hold:

(1) $C = \bigcup_{t_{1}, \ldots, t_{m}} C_{t_{1}, \ldots, t_{m}}$ where the union is over all $m$-tuples $t_{1}, \ldots, t_{m}$.

(2) If $0 < t_{i} < t_{i}$ for all $i = 1, \ldots, m$, then $C_{t_{1}, \ldots, t_{m}} \subset C_{t_{1}, \ldots, t_{m}}$.

(3) There is a point $p_{C} \in C$ such that for all $t_{1}, \ldots, t_{m}$, if $c_{i} < t_{i}$ for all $i = 1, \ldots, m$, then $C_{t_{1}, \ldots, t_{m}} = \{p_{C}\}$.

Lemma 3.21. Let $C$ be a $J$-cell which is a $J$-bounded subset. Then $C_{t_{1}, \ldots, t_{m}}$ is a closed (hence definably compact) definable subset of $C$ for any $t_{1}, \ldots, t_{m}$.

Proof. The proof is by induction on $l \in \{1, \ldots, m - 1\}$ and the construction. If $l = 1$ and $C$ is a singleton in $J_{1}$, then the claim is clear. If $l = 1$ and $C = (d, e) \subseteq J_{1}$, then the claim is also clear.

Let $l > 1$ and $C = \Gamma(f)$, where $f \in L^{l}(B)$ is a continuous definable map and $B$ is $\Pi_{i \leq l} J_{i}$-cell which is a $\Pi_{i \leq l} J_{i}$-bounded subset. Consider the restriction $\pi_{C} : C \rightarrow B$ of the projection. This is a definable homeomorphism. By induction hypothesis $B_{t_{1}, \ldots, t_{l}}$ is closed for any $t_{1}, \ldots, t_{l}$. On the other hand, since $C_{t_{1}, \ldots, t_{l}, t_{l+1}} = \Gamma(f_{B_{t_{1}, \ldots, t_{l+1}}})$ for any $t_{l+1}$, the claim follows.

Let $l > 1$ and $C = (f, g)_{B}$, where $f, g \in L^{l}(B)$ are continuous definable maps, $B$ is $\Pi_{i \leq l} J_{i}$-cell which is a $\Pi_{i \leq l} J_{i}$-bounded subset and $f < g$. Let $\pi_{C} : C \rightarrow B$ be
the restriction of the projection. By induction $B_{t_1, \ldots, t_l}$ is closed for any $t_1, \ldots, t_l$.

Also, since

$$C_{t_1, \ldots, t_l, t_{l+1}} = [f + t_{l+1} \gamma_{t_{l+1}}^{l+1}, g - t_{l+1} \gamma_{t_{l+1}}^{l+1}]_{B_{t_1, \ldots, t_l}}$$

(recall that $\gamma_{t_{l+1}}^{l+1} := \min(|f - t_{l+1} g|_{l+1}, t_{l+1})$) then, for each $x \in B_{t_1, \ldots, t_l}$, the fiber $(\pi_C)^{-1}(x) \cap C_{t_1, \ldots, t_l, t_{l+1}}$ is closed. Let $(x, y) \in C$ be an element in the closure of $C_{t_1, \ldots, t_l, t_{l+1}}$. Then $x \in B_{t_1, \ldots, t_l}$ and $(x, y) \in (\pi_C)^{-1}(x) \cap C_{t_1, \ldots, t_l, t_{l+1}} \subseteq C_{t_1, \ldots, t_l, t_{l+1}}$. Since $C_{t_1, \ldots, t_l, t_{l+1}}$ is also bounded in $M^m$, it follows by [35, Theorem 2.1] that $C_{t_1, \ldots, t_l, t_{l+1}}$ is definably compact.

**Lemma 3.22.** Let $C$ be a $J$-cell which is a $J$-bounded subset. Let $K \subseteq C$ be a definably compact definable subset. Then there are $t_1, \ldots, t_m$ such that $K \subseteq C_{t_1, \ldots, t_m}$.

**Proof.** The proof is by induction on $l \in \{1, \ldots, m - 1\}$ and the construction. If $l = 1$ and $C$ is a singleton in $J_1$, then the claim is clear. If $l = 1$ and $C = (d, e) \subseteq J_1$, then the claim is also clear.

Let $l > 1$ and $C = \Gamma(f)$, where $f \in L^1(B)$ is a continuous definable map and $B$ is $\Pi_{\leq j} J_l$-cell which is a $\Pi_{\leq j} J_l$-bounded subset. Consider the restriction $\pi_C : C \rightarrow B$ of the projection. This is a definable homeomorphism. By induction hypothesis there are $t_1, \ldots, t_l$ such that $\pi_C(K) \subseteq B_{t_1, \ldots, t_l}$. Since $C_{t_1, \ldots, t_l, t_{l+1}} = \Gamma(f|_{B_{t_1, \ldots, t_l}})$ for any $t_{l+1}$, the claim follows.

Let $l > 1$ and $C = (f, g)_B$, where $f, g \in L^1(B)$ are continuous definable maps, $B$ is $\Pi_{\leq j} J_l$-cell which is a $\Pi_{\leq j} J_l$-bounded subset and $f < g$. Let $\pi_C : C \rightarrow B$ be the restriction of the projection and let $\pi_C' : C \rightarrow J_{l+1}$ be the other projection. By Facts 3.4 and fact def skolem def compl, $\pi_C(K)$ is a definably compact definable subset of $B$. By induction there are $t_1, \ldots, t_l$ such that $\pi_C(K) \subseteq B_{t_1, \ldots, t_l}$. Similarly, for each $x \in B_{t_1, \ldots, t_l}$, if $K_x = \{(x, y) \in C : (x, y) \in K\}$, then $\pi_C(K_x)$ is a definably compact definable subset of $J_{l+1}$. So define maps $s : \pi_C(K) \rightarrow J_{l+1} : s(x) = \min(\pi_C'(K_x))$ and $s' : \pi_C(K) \rightarrow J_{l+1} : s'(x) = \max(\pi_C'(K_x))$. Of course we have $f(x) < s(x) \leq s'(x) < g(x)$ for all $x \in \pi_C(K)$. We claim that there is $0_{l+1} < t_{l+1} < c_{l+1}$ (in $J_{l+1}$) such that $f(x) + t_{l+1} s_{l+1} < s(x) < g(x) - t_{l+1} s_{l+1}$ for all $x \in \pi_C(K)$. If not, then by definable choice, there are $0_{l+1} < c_{l+1} < c_{l+1}$ (in $J_{l+1}$) and a definable map $\gamma : (0_{l+1}, c_{l+1}) \subseteq J_{l+1} \rightarrow \pi_C(K)$ such that $g(\gamma(t)) - t_{l+1} f \leq s'(\gamma(t))$ (resp. $s(\gamma(t)) \leq f(\gamma(t)) + t_{l+1} t$). By o-minimality, after shrinking $(0_{l+1}, c_{l+1})$ if necessary, we may assume that $\gamma$ and $s' \circ \gamma$ (resp. $s \circ \gamma$) are continuous. Since $\pi_C(K)$ is definably compact, the limit $\lim_{t \rightarrow 0_{l+1}} \gamma(t)$ exists in $\pi_C(K)$, say it is equal to $k \in \pi_C(K)$. But then we obtain, $g(k) \leq s'(k)$ (resp. $s(k) \leq f(k)$) which is absurd.

Now if we take

$$C_{t_1, \ldots, t_l, t_{l+1}} = [f + t_{l+1} \gamma_{t_{l+1}}^{l+1}, g - t_{l+1} \gamma_{t_{l+1}}^{l+1}]_{B_{t_1, \ldots, t_l}}$$

(recall that $\gamma_{t_{l+1}}^{l+1} := \min(|f - t_{l+1} g|_{l+1}, t_{l+1})$), then the result follows in this case also.

Below we denote by $Z_Z$ the constant sheaf with value $Z$ on a definable space $Z$ equipped with the o-minimal site $Z_{\text{def}}$. 
By [19, Lemma 4.8], if $C$ be a $J$-cell which is a $J$-bounded subset, then $C$ is acyclic, i.e. $H^p(C;\mathbb{Z}_C) = 0$ for $p > 0$ and $H^0(C;\mathbb{Z}_C) = \mathbb{Z}$.

Regarding the o-minimal cohomology with definably compact supports we have:

**Proposition 3.23.** Let $C$ be a $J$-cell of positive dimension $r$ which is a $J$-bounded subset. Then

$$H^l_c(C;\mathbb{Z}_C) = \begin{cases} \mathbb{Z} & \text{if } l = r \\ 0 & \text{if } l \neq r. \end{cases}$$

Moreover, the inclusion induces an isomorphism $H^l_c(C;\mathbb{Z}_C) \simeq H^l_p(C;\mathbb{Z}_C)$ for every $t_1, \ldots, t_m$.

**Proof.** For short, write $C_{\mathcal{T}}$ (resp. $C_{\mathcal{T}'}$) instead of $C_{t_1, \ldots, t_m}$ (resp. $C_{t'_1, \ldots, t'_m}$). If $0 < t'_i < t_i$ for all $i = 1, \ldots, m$ then we have a commutative diagram

\[
\begin{array}{cccc}
H^l_{C_{\mathcal{T}}}(C) & \rightarrow & H^l(C) & \rightarrow & H^l(C \setminus C_{\mathcal{T}}) & \rightarrow & H^{l+1}_{C_{\mathcal{T}}}(C) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^l_{C_{\mathcal{T}'}'}(C) & \rightarrow & H^l(C) & \rightarrow & H^l(C \setminus C_{\mathcal{T}'}) & \rightarrow & H^{l+1}_{C_{\mathcal{T}'}'}(C)
\end{array}
\]

(where we omitted the coefficients) given by the pairs $(C, C \setminus C_{\mathcal{T}})$ and $(C, C \setminus C_{\mathcal{T}'})$.

By the five lemma and [19, Lemmas 4.8 and 4.11 (2)], we conclude that

$$H^l_{C_{t_1, \ldots, t_m}}(C;\mathbb{Z}_C) \simeq H^l_{C_{t'_1, \ldots, t'_m}}(C;\mathbb{Z}_C).$$

In particular, we have $H^0_c(C;\mathbb{Z}_C) \simeq H^0_{pc}(C;\mathbb{Z}_C) \simeq 0$ (using the notation of Remark 3.20, the fact that $r > 0$ and [19, Lemma 4.8]), and so from the top long exact sequence in the previous commutative diagram we obtain

$$H^l_{C_{\mathcal{T}}}(C;\mathbb{Z}_C) \simeq H^{l-1}(C \setminus C_{\mathcal{T}};\mathbb{Z}_C).$$

On the other hand, by Lemma 3.22, for every definably compact definable subset $A$ of $C$ there is $\mathcal{T}$ such that $A \subseteq C_{\mathcal{T}}$. Therefore, by definition of o-minimal cohomology with definably compact supports, we have:

$$H^*_c(C;\mathbb{Z}_C) = \lim_{A \subseteq C} H^*_A(C;\mathbb{Z}_C) = \lim_{\mathcal{T}} H^*_c(C;\mathbb{Z}_C),$$

where $c$ denotes the family of definably compact definable subsets of $C$. Therefore the result follows from [19, Lemma 4.11 (3)].

By Theorem 2.2 and Proposition 3.23 we have:

**Corollary 3.24.** Let $X$ be a definable $J$-bounded manifold of dimension $n$. Then every open definable subset $U \subseteq X$ is a finite union of open definable subsets
$U_1, \ldots, U_i \subseteq U$ each of which is definably homeomorphic to a $J$-cell of dimension $n$ and such that, for each $i$, we have:

$$H^p_c(U_i; \mathbb{Z}_X) = \begin{cases} 
\mathbb{Z} & \text{if } p = n \\
0 & \text{if } p \neq n. 
\end{cases}$$

### 3.5. On the orientability of definable $J$-bounded manifolds

Here we introduce the notion of orientability for definable $J$-bounded manifolds and prove a criteria using locally definable $\mathbb{Z}$-covering maps.

As before, below we let $J = \Pi_{i=1}^m J_i$ be a cartesian product of definable group-intervals $J_i = \langle (-i b_i, b_i), 0_i, +i, -i, < \rangle$.

Since a definable $J$-bounded manifold can be assume to be $J$-definable and $J$ has definable choice functions and on the other hand, by [16], Hausdorff $J$-definably compact spaces are $J$-definably normal, it follows that, if $X$ is definably locally compact, then the family $c$ of definably compact supports on $X$ is a definably normal family of supports.

By Corollary 3.24 and [20] we then have:

**Fact 3.25.** If $X$ is a definably locally compact definable $J$-bounded manifold of dimension $n$, then $X$ has an orientation sheaf $\mathcal{O}_r X$ (relative to the $\alpha$-minimal site $X_{\text{def}}$ on $X$) whose sections are given by

$$\Gamma(U; \mathcal{O}_r X) \simeq \text{Hom}(H^p_c(U; \mathbb{Z}_X), \mathbb{Z})$$

for each open definable subset $U \subseteq X$. Moreover, the sheaf $\mathcal{O}_r X$ is locally constant.

Following [20] we say that $X$ is orientable if there exists an isomorphism $\mathbb{Z}_X \simeq \mathcal{O}_r X$ of sheaves on $X_{\text{def}}$. If $X$ is orientable, then the orientation class $\mu_X \in \Gamma(X; \mathcal{O}_r X)$ is the section image of the section $1_X \in \Gamma(X; \mathbb{Z}_X)$ by the orientation.

By [13, Proposition 4.3] (see also [13, Example 4.2]) we have an equivalence between the category of locally constant $\mathbb{Z}_X$-sheaves on $X_{\text{def}}$ and the category of locally definable $\mathbb{Z}$-covering maps of $X$. Note that [13, Proposition 4.3] is proved in o-minimal expansions of ordered groups but it only uses [15, Lemma 2.1 (1)] which holds in arbitrary o-minimal structures. By this equivalence of categories (and its proof) we have:

**Fact 3.26.** Let $X$ be a definably locally compact definable $J$-bounded manifold of dimension $n$. Then there is a canonically associated locally definable $\mathbb{Z}$-covering map

$$w_\mathcal{O} : W_\mathcal{O} \to X$$

where $W_\mathcal{O} = \bigsqcup_{x \in X} \mathcal{O}_r x$ and $w_\mathcal{O}(x) = x$, such that $X$ is orientable (i.e. there is an isomorphism $\mathcal{O}_r \simeq \mathbb{Z}_X$ of sheaves on $X_{\text{def}}$) if and only if the locally definable $\mathbb{Z}$-covering map $w_\mathcal{O} : W_\mathcal{O} \to X$ is trivial. Moreover, if $\{U_j\}_{j \in J}$ is an admissible cover of $X$ by open definable subsets such that for each $j \in J$ the restriction $\mathcal{O}_r | U_j \simeq \mathbb{Z}_X | U_j$, then $w_\mathcal{O} : W_\mathcal{O} \to X$ is a locally definable $\mathbb{Z}$-covering map trivial over $U = \{U_j\}_{j \in J}$ with, for each $j \in J$
\[
\begin{align*}
\bullet & \ w^{-1}(U_j) = \bigsqcup_{m \in \mathbb{Z}} U_j^m; \\
\bullet & \ \text{each } w_{O(U_j)} : U_j^m \to U_j \text{ is a definable homeomorphism}
\end{align*}
\]

where for \( m \in \mathbb{Z} \), \( U_j^m = \{ s_x : \Gamma(U_j; O_{rX}) \simeq \mathbb{Z} : s \mapsto m, \ x \in U_j \} \).

Above \( O_{rX,x} \) are the stalks
\[
O_{rX,x} = \lim_{\xrightarrow{\longrightarrow}} \Gamma(U; O_{rX})
\]
of the orientation sheaf and the limit is over open definable subsets \( U \subseteq X \) of \( X \) such that \( x \in U \).

By Fact 3.25 and Corollary 3.24 we have:

**Lemma 3.27.** Let \( X \) be a definably locally compact definable \( J \)-bounded manifold of dimension \( n \). Then for each \( x \in X \) we have
\[
O_{rX,x} \simeq \text{Hom}(H^n_{\{x\}}(X; \mathbb{Z}_X), \mathbb{Z}).
\]

**Proof.** Let \( x \in X \) and \( V \subseteq X \) an open definable subset of \( X \) such that \( x \in V \). Then by excision isomorphism we have \( H^n_{\{x\}}(V; \mathbb{Z}_X) \simeq H^n_{\{x\}}(X; \mathbb{Z}_X) \). On the other hand, if \( U \) is definably homeomorphic to an open \( J \)-cell \( C \) and \( x \) corresponds to \( p_C \in C \) under the definable homeomorphism, where \( p_C \) is the point of \( C \) given by Remark 3.20 (3), then by Proposition 3.23 (including also the moreover part) and Remark 3.20 (3), we have also
\[
H^n_{\{x\}}(X; \mathbb{Z}_X) \simeq H^n_{\{x\}}(U; \mathbb{Z}_X) \simeq H^n_{\{x\}}(U; \mathbb{Z}_X) \simeq \mathbb{Z}.
\]

Since, using first Corollary 3.24, we have for every open definable subset \( V \) of \( X \) such that \( x \in V \) there is an open definable subset \( U \) of \( V \) such that \( x \in U \) and \( U \) is definably homeomorphic to an open \( J \)-cell \( C \) and \( x \) corresponds to \( p_C \in C \) under the definable homeomorphism, by Fact 3.25 and the above isomorphisms, we have that
\[
O_{rX,x} = \lim_{\xrightarrow{\longrightarrow}} \Gamma(V; O_{rX})
\]
\[
\simeq \lim_{\xrightarrow{\longrightarrow}} \text{Hom}(H^n_c(V; \mathbb{Z}_X), \mathbb{Z})
\]
\[
\simeq \text{Hom}(H^n_{\{x\}}(X; \mathbb{Z}_X), \mathbb{Z}).
\]

\( \square \)

### 3.6. Degree theory for definable \( J \)-bounded manifolds.

Here we introduce degree theory for continuous definable maps between definably locally compact, definable \( J \)-bounded manifold of positive dimension \( n \).

We recall the following consequence of Alexander duality proved in [20]:

**Fact 3.28.** Let \( X \) be a definably locally compact, definable \( J \)-bounded manifold of positive dimension \( n \) which is orientable. If \( Z \) a definably compact definable subset with \( l \) definably connected components, then there exists an isomorphism
\[
H^n_Z(X; \mathbb{Z}_X) \simeq \text{Hom}(H^0(Z; \mathbb{Z}_X), \mathbb{Z}) \simeq \mathbb{Z}^l
\]
induced by the given orientation. In particular, by excision, if \( U \) is an open definable subset of \( X \) such that \( Z \subseteq U \), then we have an isomorphism

\[
H^n_Z(U; \mathbb{Z}_X) \simeq \mathbb{Z}^l
\]

compatible with the inclusions of open definable neighborhoods of \( Z \) in \( X \).

Since the functor \( \text{Hom}(\bullet, \mathbb{Z}) \) on the category of abelian groups when restricted to the subcategory of torsion free abelian groups is exact, below we will denote it by \( (\bullet)^\vee \). In particular, we will use quite often the fact that if \( f : A \to B \) is an isomorphism of torsion free abelian groups, then \( f^\vee : B^\vee \to A^\vee \) is also an isomorphism of abelian groups.

**Definition 3.29.** Let \( X \) be a definably locally compact, definable \( J \)-bounded manifold of positive dimension \( n \) which is orientable. Let \( Z \) be a definably compact definable subset with \( l \) definably connected components and \( U \) an open definable subset of \( X \) such that \( Z \subseteq U \).

We call the element \( \zeta_Z \in H^n_Z(U; \mathbb{Z}_X)^\vee \) corresponding to \((1, \ldots, 1) \in \mathbb{Z}^l\) the fundamental class around \( Z \). If \( X \) is definably connected and definably compact, then we call \( \zeta_X \) the fundamental class of \( X \).

**Remark 3.30.** Let \( X \) be as above. Let \( Z, Z_1 \subseteq Z_2 \) be definably compact, definable subset of \( X \). Then:

1. If \( Z = \emptyset \), then \( \zeta_Z = 0 \).
2. \( \zeta_{Z_1} \) is the image of \( \zeta_{Z_2} \) under the homomorphism

\[
H^n_{Z_2}(U; \mathbb{Z}_X)^\vee \to H^n_{Z_1}(U; \mathbb{Z}_X)^\vee
\]

induced by inclusion.
3. If \( Z \) is definably connected, then \( H^n_Z(U; \mathbb{Z}_X)^\vee \simeq \mathbb{Z} \) and \( \zeta_Z \in H^n_Z(U; \mathbb{Z}_X)^\vee \) is a generator.
4. If \( X \) is definably connected and definably compact, then the fundamental class \( \zeta_X \in H^n(X; \mathbb{Z}_X)^\vee \) of \( X \) corresponds to the orientation class \( \mu_X \in \Gamma(X; \text{Or}_X) \).

**Definition 3.31.** Let \( X \) and \( Y \) be definably locally compact, \( J \)-bounded manifolds of positive dimension \( n \) which are orientable. Let \( f : Y \to X \) a definable continuous map. Let \( Z \) be a definable connected, definably compact, nonempty definable subset of \( X \) such that \( f^{-1}(Z) \) is a definably compact definable subset of \( Y \).

We call degree of \( f \) over \( Z \) the unique element \( \deg_Z f \in \mathbb{Z} \) such that the image of the fundamental class around \( f^{-1}(Z) \) under the map

\[
(f^*)^\vee : H^n_{f^{-1}(Z)}(Y; \mathbb{Z}_Y)^\vee \to H^n_Z(X; \mathbb{Z}_X)^\vee.
\]

is given by

\[
(f^*)^\vee(\zeta_{f^{-1}(Z)}) = \deg_Z f \zeta_Z.
\]

If \( Y \) is definably compact and \( X \) definably connected, then \( \deg f := \deg_X f \) is called the degree of \( f \). Note that \( \deg_Z f = 0 \) if \( f^{-1}(Z) = \emptyset \).

The next lemmas establish some basic properties of the degree. Their proofs are classical but we include them for completeness.
Lemma 3.32. Let $X$ and $Y$ be definably locally compact, $J$-bounded manifolds of positive dimension $n$ which are orientable. Suppose that $X$ is definably compact. Let $V$ be a definable open subset of $Y$. Then the following holds.

(1) Let $f : V \to X$ be the inclusion map. Let $Z$ be a nonempty definably connected definably compact definable subset of $V$. Then, $\deg_Z f = 1$.

(2) Let $f : Y \to X$ be a definable homeomorphism onto an open definable subset of $X$. Suppose that $Z$ is a nonempty definably connected definably compact definable subset of $X$ such that $f^{-1}(Z)$ is definably compact. Then, $\deg_Z f = \pm 1$.

Proof. (1) The dual $H^n_Z(X; \mathbb{Z})^\vee \to H^n_Z(V; \mathbb{Z})^\vee$ of the excision isomorphism is the inverse of $(f^*)^\vee : H^n_Z(V; \mathbb{Z})^\vee \to H^n_Z(X; \mathbb{Z})^\vee$ and hence $(f^*)^\vee(\zeta_Z) = \zeta_Z$ (we have identified $\zeta_Z$ with its image through the dual of the excision isomorphism).

(2) By $H^n_Z(X; \mathbb{Z})^\vee \simeq \mathbb{Z}$ and the dual $H^n_Z(f(Y); \mathbb{Z})^\vee \cong H^n_Z(X; \mathbb{Z})^\vee$ of the excision, the composition

$$H^n_{f^{-1}(Z)}(Y; \mathbb{Z})^\vee \xrightarrow{(f^*)^\vee} H^n_Z(f(Y); \mathbb{Z})^\vee \cong H^n_Z(X; \mathbb{Z})^\vee$$

must take $\zeta_{f^{-1}(Z)}$ to $\pm \zeta_Z$. \qed

Lemma 3.33. Let $X$ and $Y$ be definably locally compact, $J$-bounded manifolds of positive dimension $n$ which are orientable. Suppose that $X$ is definably compact. Let $f : Y \to X$ be a definable continuous map. Let $Z \subset Z_1$ be definably compact nonempty subsets of $X$ such that $Z$ is definably connected and $f^{-1}(Z)$ and $f^{-1}(Z_1)$ are definably compact. Then,

$$(f^*)^\vee : H^n_{f^{-1}(Z_1)}(Y; \mathbb{Z})^\vee \to H^n_{Z_1}(X; \mathbb{Z})^\vee$$

takes $\zeta_{f^{-1}(Z_1)}$ into $(\deg_Z f) \zeta_{Z_1}$. Moreover, if $Z_1$ is also definably connected then we have $\deg_Z f = \deg_{Z_1} f$.

Proof. Consider the following commutative diagram

$$
\begin{array}{ccc}
H^n_{f^{-1}(Z)}(Y; \mathbb{Z})^\vee & \xrightarrow{(f^*)^\vee} & H^n_Z(X; \mathbb{Z})^\vee \\
\downarrow (i^*)^\vee & & \downarrow (j^*)^\vee \\
H^n_{f^{-1}(Z_1)}(Y; \mathbb{Z})^\vee & \xrightarrow{(f^*)^\vee} & H^n_{Z_1}(X; \mathbb{Z})^\vee,
\end{array}
$$

where $(i^*)^\vee$ and $(j^*)^\vee$ are induced by the respective inclusion maps. Chasing $\zeta_{f^{-1}(Z)}$ through the diagram gives $\zeta_{f^{-1}(Z)} \mapsto \deg_Z f \zeta_Z \mapsto \deg_Z f \zeta_{Z_1}$, respectively $\zeta_{f^{-1}(Z_1)} \mapsto \zeta_{f^{-1}(Z_1)} \mapsto (f^*)^\vee(\zeta_{f^{-1}(Z_1)})$. \qed

We have the following useful particular case:

Corollary 3.34. Let $X$ and $Y$ be definably compact, $J$-bounded manifolds of positive dimension $n$ which are orientable. Suppose that $X$ is definably connected. Let $f : Y \to X$ be a definable continuous map. Then $\deg_Z f := \deg_{\{x\}} f = \deg f$, for any $x \in X$.

Finally we will also need:
Lemma 3.35. Let $X$ and $Y$ be definably locally compact, J-bounded manifolds of positive dimension $n$ which are orientable. Suppose that $X$ is definably compact. Let $f : Y \to X$ be a definable continuous map. Let $Z \subset X$ be a definably compact set such that $f^{-1}(Z)$ is definably compact. Suppose $Y = \bigcup_{\lambda=1}^{m} Y_{\lambda}$ such that each $Y_{\lambda}$ is an open definable subset of $Y$ and $f^{-1}(Z) = \bigcup_{\lambda=1}^{m} f^{-1}(Z) \cap Y_{\lambda}$. Then $\deg_{Z} f = \sum_{\lambda=1}^{m} \deg_{Z} f_{\lambda}$, where $f_{\lambda} = f|_{Y_{\lambda}} : Y_{\lambda} \to X$.

Proof. Firstly, note that the $Z_{\lambda}' = f^{-1}(Z) \cap Y_{\lambda}$ are clopen in $f^{-1}(Z)$ and hence definably compact, so it makes sense to speak about the fundamental class around $Z_{\lambda}'$. Then observe that if $i_{\lambda} : (Y_{\lambda}, Y_{\lambda} \setminus Z_{\lambda}') \to (Y, Y \setminus f^{-1}(Z))$ is the inclusion map, then the following diagram commutes

$$
\begin{array}{ccc}
\oplus_{\lambda=1}^{m} H_{Z_{\lambda}}^{n}(Y_{\lambda}; Z_{Y}) & \xrightarrow{\left(\sum_{\lambda=1}^{m} f_{\lambda}^*\right)^{\vee}} & H_{f^{-1}(Z)}^{n}(Y; Z_{Y}) \\
\sum_{\lambda=1}^{m} (f_{\lambda}^*) & \downarrow & f^* \\
H_{Z}^{n}(X; Z_{X}) & \xrightarrow{\left(\sum_{\lambda=1}^{m} f_{\lambda}^*\right)^{\vee}} & H_{f^{-1}(Z)}^{n}(Y; Z_{Y}) \\
\end{array}
$$

Now, for each $p \in f^{-1}(Z)$, consider the maps

$$
\oplus_{\lambda=1}^{m} H_{Z_{\lambda}}^{n}(Y_{\lambda}; Z_{Y}) \xrightarrow{(i_{\lambda}^*)} H_{f^{-1}(Z)}^{n}(Y; Z_{Y}) \xrightarrow{(p^*)} H_{p}^{n}(Y; Z_{Y})
$$

and observe that $(p^* \circ (i_{\lambda}^*))((\zeta_{Z_{\lambda}'})) = \zeta_{p}$ (all the components of $\zeta_{Z_{\lambda}'}$ go to zero except the component $\zeta_{Z_{\lambda}'}$ containing $p$ which goes to $\zeta_{p}$). Hence, by unicity of the fundamental class $(i_{\lambda}^*)(\zeta_{Z_{\lambda}'}) = \zeta_{f^{-1}(Z)}$. Therefore, by the above diagram, we have

$$(\deg_{Z} f)(\zeta_{Z}) = (f^*)(\zeta_{f^{-1}(Z)}) = (f^*)(\sum_{\lambda=1}^{m} (i_{\lambda}^*)(\zeta_{Z_{\lambda}'}) = (f^*)(\sum_{\lambda=1}^{m} f_{\lambda}^*\zeta_{Z_{\lambda}'}) = (\sum_{\lambda=1}^{m} \deg_{Z} f_{\lambda})\zeta_{Z}.
$$

4. Applications to definably compact abelian groups

In this section we prove our results about definably compact abelian groups as explained in the Introduction.

4.1. Definably compact groups and products of definable group-intervals.

The following result ([24, Theorem 3]) establishes the connection between definable groups and cartesian products of definable group-intervals:

Fact 4.1. If $G$ is a definable group, then there is a definable injection $G \to \Pi_{i=1}^{m} J_{i}$, where each $J_{i} \subseteq M$ is a definable group-interval.

Using Fact 4.1, we have the following result from [19, Lemma 4.18]:

Fact 4.2. If $G$ is a definably compact definable group, then there is a cartesian product $J = \Pi_{i=1}^{m} J_{i}$ of definable group-intervals such that $G$ has definable charts $\{(U_{i}, \phi_{i})\}_{i=1}^{k}$ such that $\phi_{i}(U_{i}) \subseteq \Pi_{i=1}^{m} J_{i}$ for each $i$. 

Due to Fact 4.2, for the rest of the paper, we will always assume that definably compact definable groups are definable $J$-bounded manifolds. Also since $J$ is constructed from $G$ below we will call $\pi_1^J(G)$ the intrinsic o-minimal fundamental group of $G$.

It follows from Remark 3.18 that we can extend to intrinsic o-minimal fundamental groups of definably compact definable groups in $\text{M}$ (an arbitrary o-minimal structure), the results about o-minimal fundamental groups already proved in o-minimal expansions of ordered fields ([17, Section 2]) or in o-minimal expansions of ordered groups ([12]).

**Theorem 4.3.** Let $G$ a definably compact, definably connected definable group. Then there exists a universal locally definable covering homomorphism $\tilde{\rho}: \tilde{G} \to G$ where $\tilde{G}$ is a locally definable $J$-bounded manifold. Moreover, the intrinsic o-minimal fundamental group $\pi_1^J(G)$ of $G$ is abelian and finitely generated.

**Proof.** By Remark 3.18 and [13, Theorem 1.2] there is a universal locally definable covering map $\tilde{\rho}: \tilde{G} \to G$ in the category of locally definable $J$-bounded manifolds. So $\tilde{G}$ is a locally definable $J$-bounded manifold. By [13, Proposition 2.28] the definable group operations of $G$ can be lifted to locally definable group operations on $\tilde{G}$ making $\tilde{G}$ a locally definable group and $\tilde{\rho}: \tilde{G} \to G$ a locally definable homomorphism (compare with the proof of [12, Claims 3.9 and 3.10]).

By Remark 3.18 and [13, Theorem 1.1] the o-minimal $J$-fundamental group $\pi_1^J(G)$ of $G$ is finitely generated. As in [17, Lemma 2.3], $\pi_1^J(G)$ is abelian. □

The following is proved in exactly the same as the proof of its analogue in o-minimal expansions of ordered fields ([17, Theorem 2.1]):

**Theorem 4.4.** Let $G$ a definably compact, definably connected definable abelian group. Then there is $s \in \mathbb{N}$ such that:

(a) the intrinsic o-minimal fundamental group $\pi_1^J(G)$ of $G$ is isomorphic to $\mathbb{Z}^s$, and

(b) the subgroup $G[k]$ of $k$-torsion points of $G$ is isomorphic to $(\mathbb{Z}/k\mathbb{Z})^s$, for each $k \in \mathbb{N}$.

**Proof.** Since [17, Proposition 2.10] holds in arbitrary o-minimal structures, $p_k: G \to G: x \mapsto kx$ is a definable covering homomorphism with $G[k] \cong \text{Aut}(p_k: G \to G)$ (the group of definable covering homeomorphisms). By [13, Theorem 3.9] (and Remark 3.18) we have also $G[k] \cong \pi_1^J(G)/p_{ks}(\pi_1^J(G))$. Since [17, Lemma 2.3 and 2.4] hold here as well, $p_{ks}(\pi_1^J(G)) = k\pi_1^J(G)$. Finally, by [13, Corollary 2.17] (and Remark 3.18), $p_{ks}: \pi_1^J(G) \to \pi_1^J(G)$ is injective and so the finitely generated abelian group $\pi_1^J(G)$ is free. □

4.2. **Orientability of definably compact groups.** Here we show that definably compact definable groups are orientable.

Let $G$ a definably compact, definably connected definable group. By Fact 4.2, fix a cartesian product $J = \prod_{i=1}^m J_i$ of definable group-intervals such that $G$ is definable $J$-bounded manifold. Since $G$ is definably normal ([21, Corollary 2.3]), $G$
is definably locally compact. By Fact 3.25, $G$ has an orientation sheaf $\mathcal{O}_R$ (relative to the o-minimal site on $G$) whose sections are given by

$$\Gamma(U; \mathcal{O}_R) \simeq \text{Hom}(H^n_c(U; \mathcal{O}_G), \mathbb{Z})$$

for each open definable subset $U \subseteq G$.

**Theorem 4.5.** Let $G$ be a definably connected, definably compact, definable group of positive dimension $n$. Then $G$ is orientable. In particular, $H^n(G; \mathcal{O}_G) \simeq \mathbb{Z}$.

**Proof.** By Fact 3.26, there is a canonically associated locally definable $\mathbb{Z}$-covering map

$$w_{\mathcal{O}} : W_{\mathcal{O}} \to G$$

where $W_{\mathcal{O}} = \bigsqcup_{x \in G} \mathcal{O}_{R,x}$ and $w_{\mathcal{O}}(s_x) = x$, such that $G$ is orientable if and only if the locally definable $\mathcal{O}$-covering map $w_{\mathcal{O}} : W_{\mathcal{O}} \to G$ is trivial. Moreover, if $U_1, \ldots, U_l$ is an admissible cover of $G$ by open definable subsets such that for each $j = 1, \ldots, l$, the restriction $\mathcal{O}_{R_i \mid U_j} \simeq \mathcal{O}_{G \mid U_j}$, then $w_{\mathcal{O}} : W_{\mathcal{O}} \to G$ is a locally definable $\mathcal{O}$-covering map trivial over $U = \{U_j\}_{j=1}^l$ with, for each $j = 1, \ldots, l$:

- $w_{\mathcal{O}}(U_j) = \bigsqcup_{m \in \mathbb{Z}} U^m_{\mathcal{O}_{U,j}}$;
- each $w_{\mathcal{O}_{U,j}} : U^m_{\mathcal{O}_{U,j}} \to U_j$ is a definable homeomorphism

where for $m \in \mathbb{Z}$, $U^m_{\mathcal{O}_{U,j}} = \{s_x : \Gamma(U_j; \mathcal{O}_R) \simeq \mathbb{Z} : s \mapsto m, \ x \in U_j\}$.

By Lemma 3.27, for each $x \in G$ we have

$$\mathcal{O}_{R,x} \simeq \text{Hom}(H^n_{\{1\}}(G; \mathcal{O}_G), \mathbb{Z}).$$

Let $e_G \in G$ be the identity element and suppose that $e_G \in U_1$. Let $\omega_{e_G} \in \mathcal{O}_{R,x}$ corresponding to $1 \in \mathbb{Z}$ under the isomorphism $\mathcal{O}_{R,x} \simeq \mathbb{Z}$ induced by the isomorphism $\mathcal{O}_{R,G[U_1]} \simeq \mathcal{O}_{G[U_1]}$.

For $z \in G$ let $L_z : G \to G : u \mapsto zu$ be the left translation by $z$. Then $L_z^* : H^n_{\{z\}}(G; \mathcal{O}_G) \to H^n_{\{e_G\}}(G; \mathcal{O}_G)$ is an isomorphism. So we can define $\omega_z \in \mathcal{O}_{R,G,z}$ to be the image $(L_z^*)^!(\omega_{e_G})$ of $\omega_{e_G}$ under the dual of the isomorphism $L_z^*$.

**Claim 4.6.** For each $j$, there is a unique generator

$$\omega_j \in \Gamma(U_j; \mathcal{O}_R) \simeq \text{Hom}(H^n_{\{1\}}(U_j; \mathcal{O}_G), \mathbb{Z}) \simeq \mathbb{Z}$$

such that $\omega_z = (\omega_j)_{z \in U_j}$ for all $z \in U_j$.

Since $U_j$ is definably homeomorphic to an open $J$-cell $C_j$, if $u_j \in U_j$ corresponds to $p_{C_j} \in C_j$ under the definable homeomorphism, where $p_{C_j}$ is the point of $C_j$ given by Remark 3.20 (3), then by excision and Proposition 3.23 (including also the moreover part) and Remark 3.20 (3), we have

$$H^n_{\{u_j\}}(G; \mathcal{O}_G) \simeq H^n_{\{u_j\}}(U_j; \mathcal{O}_G) \simeq H^n_c(U_j; \mathcal{O}_G) \simeq \mathbb{Z}.$$
Since \( L_u = L_{u^{-1}} \circ L_y \) we have \((L_u^*)^\vee = (L_{u^{-1}})^\vee \circ (L_y^*)^\vee \) and so \( \omega_u = (L_{u^{-1}})^\vee (\omega_y) \). This together with the (dual) of the commutative diagram above shows that there is a unique generator \( \omega_j \in \Gamma(U_j; O_{\text{Or}_G}) \simeq \text{Hom}(H^*_c(U_j; \mathbb{Z}_G), \mathbb{Z}) \simeq \mathbb{Z} \) such that \( \omega_u = (\omega_j)_{u_j} \) and \( \omega_z = (\omega_j)_{z_j} \).

**Claim 4.7.** Let \( s : G \to W_O \) be the map given by \( s(z) = \omega_z \). Then \( s : G \to W_O \) is a continuous locally definable section to \( W_O : W_O \to G \) (i.e. such that \( w_O \circ s = \text{id}_G \)).

We have \( w_O \circ s = \text{id}_G \) and moreover, since \( G \) is definably connected, by Claim 4.6, for each \( j \), we have \( s(U_j) = U_j^1 = \{ s_x : \Gamma(U_j; O_{\text{Or}_G}) \simeq \mathbb{Z} : s \mapsto 1, \ x \in U_j \} \). Hence, \( s|_{U_j : U_j \to U_j^1} \) is the inverse to the definable homeomorphism \( w_O|_{U_j : U_j} \to U_j \). Therefore, \( s \) is continuous and locally definable as required.

Finally, if \( \alpha : W_O \times \mathbb{Z} \to W_O \) is the locally definable \( \mathbb{Z} \)-action making \( w_O : W_O \to G \) into a locally definable \( \mathbb{Z} \)-covering map, then \( W_O \to G \times \mathbb{Z} : v \mapsto (w_O(v), l_v) \) where \( l_v \in \mathbb{Z} \) is the unique element such that \( v = \alpha(s(w_O(v)), l_v) \) is a locally definable \( \mathbb{Z} \)-covering homeomorphism showing that \( w_O : W_O \to G \) is trivial.

**4.3. The Hopf algebra a definably compact group.** Here we show that the o-minimal cohomology \( H^*(G; k_G) \) of a definably connected, definably compact definable group \( G \) with coefficients in a field \( k \) is a connected, bounded, Hopf algebra over \( k \) of finite type.

First we make a general observation:

By going to \( \overset{\sim}{\text{Def}} \), using the isomorphism \( \text{Mod}(k_{X_{\text{def}}}) \simeq \text{Mod}(k_X) \) ([14, Proposition 3.2]) and [5, Chapter II, Section 7 and (8.2)] we have:

**Fact 4.8.** Let \( X \) be a definable space. Let \( k \) be a field. Then there is a cup product operation

\[
\cup : H^p(X; k_X) \otimes H^q(X; k_X) \to H^{p+q}(X; k_X)
\]

making \( H^*(X; k_X) \) into a graded, associative, skew-commutative \( k \)-algebra with unit in \( H^0(X; k_X) \). This product is functorial and the algebra is connected if \( X \) is definably connected.

In order to prove the main result of the subsection we need to use the K"unneth formula relating the cohomology of \( G \times G \) with the cohomology of \( G \). Since cohomology in \( \text{Def} \) is the same as cohomology in \( \overset{\sim}{\text{Def}} \) and the tilde functor \( \text{Def} \to \overset{\sim}{\text{Def}} \) does not commute with products we cannot use the K"unneth formula for cohomology in topology. However, as explained in the Introduction, after the work developed in [20], we do have K"unneth formula for \( G \times G \) after we show that the full subcategory of locally closed definable subsets of definably compact definable groups satisfies conditions (A0), (A1) and (A2) and definably compact groups satisfy condition (A3).

But (A0) follows from that fact that a product of locally closed definable subsets of a cartesian product of definably compact definable groups is also a locally
closed definable subset of a definably compact definable group; (A1) follows from Fact 4.2 and Corollary 3.11; (A2) follows since: (i) a definably compact group is definably normal ([21, Corollary 2.3]) and (ii) a locally closed definable subset of a cartesian product of a given definably compact definable group has a definably normal completion, namely its closure; (A3) was proved in [19, Theorem 1.1].

So by [20] we have:

**Fact 4.9** (Künneth formula). Let $G$ be a definably compact definable group and $k$ a field. Then there is a natural isomorphism

$$H^*(G \times G; k_{G \times G}) \simeq \bigoplus_{p+q=*} (H^p(G; k_G) \otimes H^q(G; k_G)).$$

**Remark 4.10.** Since the Künneth isomorphism (Fact 4.9) is natural. We have:

- If $f : G \to G$ and $g : H \to H$ are definable continuous maps between definably compact definable groups, then we can make the identification $(f \times g)^* = f^* \otimes g^*$.
- If $q_i : G \times G \to G$ ($i = 1, 2$) is the projection onto the $i$th coordinate, then using $\{0\} \times G = G = G \times \{0\}$, it follows that for $z \in H^p(G; k_G)$ we can make the identifications $q_1^* (z) = z \otimes 1$ and $q_2^* (z) = 1 \otimes z$.

We can now prove the main result of this subsection:

**Theorem 4.11.** Let $G$ be a definably connected, definably compact definable group. Let $k$ be a field. Then the $o$-minimal sheaf cohomology $H^*(G; k_G)$ of $G$ with coefficients in $k$ is a connected, bounded, Hopf algebra over $k$ of finite type. Moreover, if $\text{char}(k) = 0$, then we have a Hopf algebra isomorphism

$$H^*(G; k_G) \simeq \bigwedge \{y_1, \ldots, y_r\}_k$$

with the exterior algebra with the $y_i$'s of odd degree and primitive.

**Proof.** By Fact 4.8, $H^*(G; k_G)$ is a connected, graded, associative, skew-commutative $k$-algebra with unit in $H^0(G; k_G)$.

Let $m : G \times G \to G$ the multiplication map. Let

$$\mu : H^*(G; k_G) \to \bigoplus_{p+q=*} (H^p(G; k_G) \otimes H^q(G; k_G))$$

be the composition of $m^* : H^*(G; k_G) \to H^*(G \times G; k_{G \times G})$ with the isomorphism $H^*(G \times G; k_{G \times G}) \simeq \bigoplus_{p+q=*} (H^p(G; k_G) \otimes H^q(G; k_G))$ given by the Künneth formula (Fact 4.9). From the properties of $m$, it is standard to show that $\mu$ is a co-multiplication making $H^*(G; k_G)$ into a Hopf algebra over $k$. Compare with [17, Corollary 3.5].

Since $G$ is definably normal ([21, Corollary 2.3]), by [14, Proposition 4.2] we have $H^p(G; k_G) = 0$ for all $p > \dim G$ and therefore $H^*(G; k_G)$ is a bounded Hopf algebra. By [19, Theorem 1.2], it is of finite type.

In the case $\text{char}(k) = 0$, the description of the Hopf algebra $H^*(G; k_G)$ follows from the Hopf-Leray theorem. See [17, Corollary 3.6] for details.
4.4. Computing the torsion subgroups. Here we compute the torsion subgroups of a definably compact abelian definable group.

Below we will omit the subscript on the field $\mathbb{Q}$ when we consider the constant sheaf it determines on a definably compact definable group $G$. We also consider $G$ with a fixed orientation (Theorem 4.5).

Lemma 4.12. Let $G$ be a definably compact, definably connected, abelian definable group. For each $k > 0$, consider the map $p_k: G \rightarrow G: x \mapsto kx$. Then we have $\deg p_k \leq |p_k^{-1}(0)|$, where $0$ is the neutral element of $G$.

Proof. Fix a $k > 0$. By Corollary 3.34, $\deg p_k = \deg_0 p_k$. Also, by [17, Corollary 2.12] (which holds in arbitrary o-minimal structures), we know that the homomorphism $p_k: G \rightarrow G$ is a definable covering map. Let $G = \bigcup_{l \in L} U_l$ be as in the definition of definable covering map. Fix $l_0 \in L$ such that $0 \in U_{l_0}$ and let $Y = p_{k}^{-1}(U_{l_0})$. Now consider the map $f = (p_k)_{|Y}: Y \rightarrow G$. Note that, by the dual of the excision, $\deg_0 p_k = \deg_0 f$. On the other hand, if we let $Y = \bigcup_{\lambda=1}^{m} Y_{\lambda}$ with the $Y_{\lambda}$'s being the definably connected components of $Y$ so that $f^{\lambda} = f|_{Y_{\lambda}}: Y_{\lambda} \rightarrow G$ is a homeomorphism onto an open subset of $G$, namely $U_{l_\lambda}$. Now the data $f: Y \rightarrow G$, $\{0\} \subset G$ and $Y = \bigcup_{\lambda} Y_{\lambda}$ satisfy the hypothesis of Lemma 3.35, therefore we can conclude that $\deg_0 f = \sum_{\lambda} \deg_0 f^{\lambda}$. Finally, by Lemma 3.32, $\deg_0 f^{\lambda} = \pm 1$, for each $\lambda$, and hence $\deg_0 f \leq m = |f^{-1}(0)|$. By the above, this last relation means $\deg p_k \leq |p_k^{-1}(0)|$. \qed

By Theorem 4.11 we have a Hopf algebra isomorphism

$$H^*(G; \mathbb{Q}) \cong \bigwedge[y_1, \ldots, y_r]_{\mathbb{Q}}$$

with the exterior algebra with the $y_i$'s of odd degree and primitive. This means that $\mu(y_i) = y_i \otimes 1 + 1 \otimes y_i$ for each $i = 1, \ldots, r$ where the co-multiplication

$$\mu = m^* : H^*(G; \mathbb{Q}) \rightarrow H^*(G \times G; \mathbb{Q}) \cong \bigoplus_{p+q=*} (H^p(G; \mathbb{Q}) \otimes H^q(G; \mathbb{Q}))$$

is given by the composition of homomorphism $m^* : H^*(G; \mathbb{Q}) \rightarrow H^*(G \times G; \mathbb{Q})$ induce by the multiplication map $m : G \times G \rightarrow G$ on $G$ with the isomorphism $H^*(G \times G; \mathbb{Q}) \cong \bigoplus_{p+q=*} (H^p(G; \mathbb{Q}) \otimes H^q(G; kG))$ given by the Künneth formula (Fact 4.9).

We call an element $x \in H^*(G; \mathbb{Q})$ a monomial of length $l$ if $x = y_{i_1} \cup \cdots \cup y_{i_l}$ where $1 \leq i_1 < \cdots < i_l \leq r$.

Lemma 4.13. Let $G$ be a definably connected, definably compact, definable group. For each $k > 0$, consider the definable continuous map $p_k : G \rightarrow G: a \mapsto a^k$, for each $a \in G$. Then, the map $p_k^* : H^*(G; \mathbb{Q}) \rightarrow H^*(G; \mathbb{Q})$ sends each monomial $x$ of length $l$ to $k^lx$.

Proof. First we prove by induction on $k$ that, for $y \in \{y_1, \ldots, y_r\}$, we have $p_k^*(y) = ky$.

For $k = 1$, we have $p_k = \text{id}$ and so this case is trivial. For the induction step, using $p_{k+1} = m \circ (p_k \times \text{id}) \circ \Delta$ where $\Delta : G \rightarrow G \times G$ is the diagonal map in $G$, we
have

\[ p_{k+1}^r(y) = (m \circ (p_k \times \text{id}) \circ \Delta^*)^r(y) \]

\[ = (\Delta^* \circ (p_k \times \text{id})^* \circ m^*)(y) \]

\[ = \Delta^* \circ (p_k^r \times \text{id})(y \otimes 1 + 1 \otimes y) \]

\[ = \Delta^*(k^r y \otimes 1 + 1 \otimes y) \]

\[ = \Delta^*(q_1^r(k^r y) + q_2^r(y)) \]

\[ = k^r y + y \]

\[ = (k + 1)^r y. \]

In these equalities we used Remark 4.10 and \( q_i \circ \Delta = \text{id} \) where \( q_i : G \times G \to G \) \((i = 1, 2)\) is the projection onto the \( i \)th coordinate.

Finally, we get \( p_{k+1}^r(x) = (k + 1)^r x \), for each \( k > 0 \), since \( p_{k+1}^r \) is an algebra morphism. \( \square \)

**Proof of Theorem 1.1:** Let \( G \) be a definably connected definably compact abelian group of dimension \( n \). Consider also the Hopf algebra \( H^*(G; \mathbb{Q}) \approx \bigwedge[y_1, \ldots, y_r] \mathbb{Q} \) of \( G \).

As we saw in Remark 3.30 the orientation class \( \mu_G^\gamma \in \Gamma(G; \mathcal{O}_G) \) determines the fundamental class \( \zeta_G^\gamma \in H^n(G; \mathbb{Z}_G) \). Let \( \zeta_G^\gamma \in H^n(G; \mathbb{Z}_G) \) be the dual of \( \zeta_G^\gamma \) and let \( \omega_G \in H^n(G; \mathbb{Q}) \) be the image of \( \zeta_G^\gamma \) under the isomorphism

\[ H^n(G; \mathbb{Z}_G) \otimes \mathbb{Q} \approx H^n(G; \mathbb{Q}) \]

given by the universal coefficients formula ([20]).

Now fix a \( k > 0 \), and consider the definable continuous map \( p_k : G \to G : a \mapsto ka \).

By definition of degree of a map we obtain \( p_k^r(\omega_G) = (\deg p_k) \omega_G \). Since \( \omega_G \) generates \( H^n(G; \mathbb{Q}) \), and \( 0 \neq y_1 \cup \cdots \cup y_r \in H^n(G; \mathbb{Q}) \) we can suppose \( \omega_G = y_1 \cup \cdots \cup y_r \). By Lemma 4.13, \( p_k^r(\omega_G) = k^r \omega_G \), and so \( \deg p_k = k^r \).

On the other hand, by Theorem 4.4, there is an \( s \geq 0 \) such that \( \pi^1_1(G) \cong \mathbb{Z}^s \) and \( p_k^{-1}(0) = G[k] \cong (\mathbb{Z}/k\mathbb{Z})^s \). By Corollary 4.12, \( \deg p_k \leq \deg(p_k)^{-1}(0) = k^s \), and hence \( r \leq s \). By the Hurewicz theorem ([13, Theorem 4.16] (and Remark 3.18)),

\[ \text{Hom}(\pi^1_1(G)^{op}, \mathbb{Z}) \cong \tilde{H}^1(G; \mathbb{Z}), \]

and on the other hand, since \( G \) is definably normal ([21, Corollary 2.3]), \( \tilde{H}^1(G; \mathbb{Z}) \cong H^1(G; \mathbb{Z}_G) \) ([14, Proposition 4.1]) and hence

\[ \text{Hom}(\pi^1_1(G)^{op}, \mathbb{Z}) \otimes \mathbb{Q} \cong H^1(G; \mathbb{Q}). \]

Therefore, since \( H^1(G; \mathbb{Q}) \) is a subspace of \( H^*(G; \mathbb{Q}) \) (and the elements of \( H^1(G; \mathbb{Q}) \) cannot be decomposable), among \( \{y_1, \ldots, y_r\} \) there must be exactly \( s \) elements of degree one. Hence \( s = r \) and all \( y_i \)'s are of degree one. Finally, since \( \omega_G = y_1 \cup \cdots \cup y_r \in H^n(G; \mathbb{Q}) \) we met have \( s = r = n \). \( \square \)

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