Abstract. We present a novel technique for proving program termination which introduces a new dimension of modularity. Existing techniques use the program to incrementally construct a termination proof. While the proof keeps changing, the program remains the same. Our technique goes a step further. We show how to use the current partial proof to partition the transition relation into those behaviors known to be terminating from the current proof, and those whose status (terminating or not) is not known yet. This partition enables a new and unexplored dimension of incremental reasoning on the program side. In addition, we show that our approach naturally applies to conditional termination which searches for a precondition ensuring termination. We further report on a prototype implementation that advances the state-of-the-art on the grounds of termination and conditional termination.

1 Introduction

The question of whether or not a given program has an infinite execution is a fundamental theoretical question in computer science but also a highly interesting question for software practitioners. The first major result is that of Alan Turing, showing that the termination problem is undecidable. Mathematically, the termination problem for a given program \( \text{Prog} \) is equivalent to deciding whether the transition relation \( R \) induced by \( \text{Prog} \) is well-founded.

The starting point of our paper, is a result showing that the well-foundedness problem of a given relation \( R \) is equivalent to the problem of asking whether the transitive closure of \( R \), noted \( R^+ \), is disjunctively well-founded [22]. That is whether \( R^+ \) is included in some \( W \) (in which case \( W \) is called a transition invariant) such that \( W = W_1 \cup \cdots \cup W_n, n \in \mathbb{N} \) and each \( W_i \) is well-founded (in which case \( W \) is said to be disjunctively well-founded). This result has important practical consequences because it triggered the emergence of effective techniques, based on transition invariants, to solve the termination problem for real-world programs [11, 27, 18].

By replacing the well-foundedness problem of \( R \) with the equivalent disjunctive well-foundedness problem of \( R^+ \), one allows for the incremental construction of \( W \): when the inclusion of \( R^+ \) into \( W \) fails then use the information from the failure to update \( W \) with a further well-founded relation [10]. Although the proof is incremental for \( W \), it is

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important to note that a similar result does not hold for \( R \). That is, it is in general not true that given \( R = R_1 \cup R_2 \), if \( R_1^+ \subseteq W \) and \( R_2^+ \subseteq W \) then \( R^+ \subseteq W \).

We introduce a new technique that, besides being incremental for \( W \), further partitions the transition relation \( R \) separating those behaviors known to be terminating from the current \( W \), from those whose status (terminating or not) is not known yet. Formally, given \( R \) and a candidate \( W \), we shall see how to compute a partition \( \{ R_G, R_B \} \) of \( R \) such that (a) \( R_G^+ \subseteq W \); and (b) every infinite sequence \( s_1 R s_2 R \cdots s_i R s_{i+1} \cdots \) (or trace) has a suffix that exclusively consists of transitions from \( R_B \), namely we have \( s_z R_B s_{z+1} R_B \cdots \) for some \( z \geq 1 \).

It follows that well-foundedness of \( R_B \) implies that of \( R \). Consequently, we can focus our effort exclusively on proving well-foundedness of \( R_B \). In the affirmative, then so is \( R \) and hence termination is proven. In the negative, then we have found an infinite trace in \( R_B \), hence in \( R \). We observed that working with \( R_B \) typically provides further hints on which well-founded relations to add to \( W \). The partition of \( R \) into \( \{ R_G, R_B \} \) enables a new and unexplored dimension of modularity for termination proofs.

Let us mention that the partitioning of \( R \) is the result of adopting a fixpoint centric view on the disjunctive well-foundedness problem and leverage equivalent formulation of the inclusion check. More precisely, we introduce the dual of the check \( R^+ \subseteq W \) by defining the adjoint to the function \( \lambda X. X \circ R \) used to define \( R^+ \). Without defining it now, we write the dual check as follows: \( R \subseteq W^- \). We shall see that while the failure of \( R^+ \subseteq W \) provides information to update \( W \); the failure of \( R \subseteq W^- \) provides information on all pairs in \( R \) responsible for the failure of \( W \) as a transition invariant. This is exactly that information, of semantical rather than syntactical nature, that we use to partition \( R \).

We show that the partitioning of \( R \) can be used not only for termination, but it also serves for conditional termination. The goal here is to compute a precondition, that is a set \( P \) of states, such that no infinite trace starts from a state of \( P \). We show how to compute a (non-trivial) precondition from the relation \( R_B \).

Our contributions are summarized as follows: (i) we present \texttt{Acabar}, a new algorithm which allows for enhanced modular reasoning about infinite behaviors of programs; (ii) we show that, besides termination, \texttt{Acabar} can be used in the context of conditional termination; and (iii) finally, we report on a prototype implementation of our techniques and compare it with the state-of-the-art on two grounds: the termination problem, and the problem of inferring a precondition that guarantees termination.

2 Example

In this section, we informally overview our proposed techniques on an example taken from the literature \cite{9}. Consider the following loop:

\[
\text{while ( } x>0 \text{ )}{ \ x:=x+y; \ y:=y+z; \ }
\]

represented by the transition relation \( R = \{ x \geq 0, x' = x + y, y' = y + z, z' = z \} \), where the primed variables represent the values of the program variables after executing the loop body. Note that, depending on the input values, the program may not terminate (e.g. for \( x = 1, y = 1 \) and \( z = 1 \)). Below we apply \texttt{Acabar} to prove termination. As we will see, this attempt ends with a failure which provide information on which subset
of the transition relation to blame. Then, we will explain how to compute a termination
precondition from this subset.

In order to prove termination of this loop, we seek a disjunctive well-founded relation
W such that R⁺ ⊆ W. To find such a W, Acabar is supported by incrementally (and
automatically) inferring (potential) linear ranking functions for R or R⁺ [9,10]. When
running on R, Acabar first adds the candidate well-founded relation W₁ = {x′ < x, x > 0} to W which is initially empty. Relation W₁ stems from the observation that, in R, x
is bounded from below (as shown by the guard) but not necessarily decreasing. Hence,
using W = W₁, Acabar partitions R into \{R₁, R⁺\} where:

\[ R₁ = \{ x > 0, x' = x + y, y' = y + z, z' = z, y < 0, z \leq 0 \} \]
\[ R⁺ = \{ x > 0, x' = x + y, y' = y + z, z' = z, y < 0, z > 0 \} \]

The partition comes with the further guarantee that every infinite trace in R must have
a suffix that exclusively consists of transitions from R⁺, which means that if R⁺ is
well-founded then so is R. In addition, one can easily see that (R⁺)⁺ ⊆ W.

Next, Acabar calls itself recursively on R⁺ to show its well-foundedness. As before,
it first adds W₂ = {y′ < y, y ≥ 0} to W. Similarly to the construction of W₁, W₂ stems from the observation that, in some parts of R⁺, y is bounded from below but
not necessarily decreasing. Then, using W = W₁ ∨ W₂, Acabar partitions R⁺ into:

\[ R₂ = \{ x > 0, x' = x + y, y' = y + z, z' = z, z < 0 \} \]
\[ R⁺₂ = \{ x > 0, x' = x + y, y' = y + z, z' = z, y ≥ 0, z ≥ 0 \} \]

Again the partition \{R₂, R⁺₂\} of R⁺ comes with a similar guarantee. This time it holds
that that every infinite trace in R must have a suffix that exclusively consists of transi-
tions from R⁺₂. Recursively applying Acabar on R⁺₂ does not yield any further par-
titioning, that is R⁺₂ = R⁺₂. The reason being that no potential ranking function is
automatically inferred. Thus, Acabar fails to prove well-foundedness of R, which is
indeed not well-founded. However, due to the above guarantee, we can use R⁺₂ to infer
a sufficient precondition for the termination of R. We explain this next.

Inferring a sufficient precondition is done in two steps: (i) we infer (an overapproxi-
mation of) the set of all states \( \mathcal{Z} \) visited by some infinite sequence of steps in R⁺₂, and
(ii) we infer (an overapproximation of) the set of all states \( \mathcal{V} \) each of which can reach
\( \mathcal{Z} \) through some steps in R. Turning to the example, we infer \( \mathcal{Z} = \{ x > 0, y ≥ 0, z ≥ 0 \} \) and the following overapproximation \( \mathcal{V} \) of \( \mathcal{V} \):

\[ \mathcal{V} = \{ x ≥ 1, z = 0, y ≥ 0 \} \lor \{ x ≥ 1, z ≥ 1, x + y ≥ 1, x + 2y + z ≥ 1, x + 3y + 3z ≥ 1 \} \]

It can be seen that every infinite trace visits only states in \( \mathcal{V} \), hence the complement of
\( \mathcal{V} \) is a precondition for termination.

Let us conclude this section by commenting on an example for which Acabar proves
termination. Assume that we append \( z := z - 1 \) to the loop body above and call R’
the induced transition relation. Following our previous explanations, running Acabar on
R’ updates W from \( \emptyset \) to W₁, and then to W₁ ∨ W₂. Then, and contrary to the previous
explanations, Acabar will further update W to W₁ ∨ W₂ ∨ W₃ where W₃ is the well-
formed relation \( \{ z' < z, z ≥ 0 \} \). From there, Acabar returns with value R⁺₂ = \( \emptyset \), hence
we have that R’ is well-founded.
3 Preliminaries

A transition system is a pair \((Q,R)\) where \(Q\) is the set of states and \(R \subseteq Q \times Q\) is the transition relation. An initialized transition system includes a further component \(I \subseteq Q\), the set of initial states. For simplicity, we defer the treatment of initial states to Sec.8.

An \(R\)-trace is a sequence \(s_1, s_2, \ldots, s_n\) of states such that for every \(i, 1 \leq i < n\) we have \((s_i, s_{i+1}) \in R\). When \(R\) is clear from the context we simply say trace. An infinite \(R\)-trace is a sequence \(s_1, s_2, \ldots\) of states such that for every \(i \geq 1\) we have \((s_i, s_{i+1}) \in R\). Given \(R' \subseteq R\) and an infinite \(R\)-trace \(\pi\) we say that \(\pi\) has infinitely many steps in \(R'\) if \((s_i, s_{i+1}) \in R'\) for infinitely many \(i \geq 1\).

Given a relation \(R' \subseteq R\) and a set \(Q' \subseteq Q\), define \(\text{post}[R'](Q') \overset{\text{def}}{=} \{s' \in Q \mid \exists s \in Q': (s, s') \in R'\}\). We say that this operator computes the \(R'\)-successors of \(Q'\). Dually, define \(\text{pre}[R'](Q') \overset{\text{def}}{=} \text{post}[R'^{-1}](Q') = \{s \in Q \mid \exists s' \in Q': (s, s') \in R'\}\). We say that this operator computes the \(R'\)-predecessors of \(Q'\).

A relation \(W \subseteq Q \times Q\) is called disjunctively well-founded iff \(W\) coincides with the union of finitely many relations (viz. \(W = W_1 \cup \ldots \cup W_n\)) each of which is well-founded (viz. there is no infinite sequence \(s_1, s_2, \ldots\) such that \((s_i, s_{i+1}) \in W_i\) for all \(i \geq 1\)).

In this paper, we adhere to the following conventions: calligraphic letters \(X, Y, \ldots\) refer to subsets of \(Q\) and capital letters \(X, Y, \ldots\) refer to relations over \(Q\), that is subsets of \(Q \times Q\). Further, throughout the paper the letter \(W\) is used to denote a relation over \(Q\) that is disjunctively well-founded.

A linear expression is of the form \(a_0 + a_1x_1 + \cdots + a_nx_n\) where \(a_i \in \mathbb{Z}\) and \(\bar{x} = (x_1, \ldots, x_n)\) are variables ranging over \(\mathbb{Z}\). An atomic linear constraint \(c\) is of the form \(e_1 \text{ op } e_2\) where \(e_i\) is a linear expression and \(\text{op} \in \{=, \leq, \geq, >, <\}\). A formula \(\psi\) is a Boolean combination of atomic linear constraints. Note that \(\neg\psi\) is also a formula. For the sake of simplicity, a conjunction \(c_1 \land \cdots \land c_n\) of atomic linear constraints is sometimes written as the set \(\{c_1, \ldots, c_n\}\). A solution of a formula \(\psi\) is a mapping from its variables into the integers such that the formula evaluates to true. Sets and relations over, respectively, \(\mathbb{Z}^n\) and \(\mathbb{Z}^n \times \mathbb{Z}^n\) are sometimes specified using formulas, with the customary convention, for relations, of variables and primed variables. For instance, the formula \(\{x \geq 0, x' = x - y, y' = y\}\) defines the relation \(R \subseteq \mathbb{Z}^2 \times \mathbb{Z}^2\) such that \(R = \{(x, y), (x', y')\} \mid x \geq 0 \land x' = x - y \land y' = y\).

Finally, we briefly recall classical results of lattice theory and refer to the classical book of Davey and Priestley [15] for further information. Let \(f\) be a function over a partially ordered set \((L, \sqsubseteq)\). A fixpoint of \(f\) is an element \(l \in L\) such that \(f(l) = l\). We denote by \(\text{lfp } f\) and \(\text{gfp } f\), respectively, the least and the greatest fixpoint, when they exist, of \(f\). The well-known Knaster-Tarski’s theorem states that each order-preserving function \(f \in L \to L\) over a complete lattice \((L, \sqsubseteq, \sqcup, \sqcap, \top, \bot)\) admits a least (greatest) fixpoint and the following characterization holds:

\[
\text{lfp } f = \bigsqcap \{x \in L \mid f(x) \sqsubseteq x\} \quad \text{gfp } f = \bigsqcup \{x \in L \mid x \sqsubseteq f(x)\}.
\]

\[1\] We define \(R'^{-1}, R^*\) and \(R^+\) to be \(R^{-1} = \{(s', s) \mid (s, s') \in R\}\), \(R^* = \bigcup_{i \geq 0} R^i\) and \(R^+ = R \ast R^*\) where \(R^0\) is the identity, \(R^{i+1} = R^i \ast R\) and \(R_1 \circ R_2 = \{(s, s'') \mid \exists s': (s, s') \in R_1 \land (s', s'') \in R_2\}\).
4 Modular Reasoning for Termination

A termination proof based on transition invariants consists in establishing the existence of a disjunctively well-founded transition invariant. That is, the goal is to prove the inclusion of $R^+$, into some $W$. For short, we write $R^+ \subseteq W$. Proving termination is thus reduced to finding some $W$ and prove that the inclusion hold.

In the above inclusion check, $R^+$ coincides with the least fixpoint of the function $\lambda Y. R \cup g(Y)$ where $g \overset{\text{def.}}{=} \lambda Y. Y \circ R$. It is known [13] that if we can find an adjoint function $\tilde{g}$ to $g$ such that $g(X) \subseteq Y$ iff $X \subseteq \tilde{g}(Y)$ for all $X, Y$ then there exists an equivalent inclusion check to $R^+ \subseteq W$. This equivalent check, denoted $R \subseteq W^-$ in the introduction, is such that $W^-$ is defined as a greatest fixpoint of the function $\lambda Y. W \cap \tilde{g}(Y)$. Next, we define $\tilde{g} \overset{\text{def.}}{=} \lambda Y. \neg(\neg Y \circ R^{-1})$.

**Lemma 1.** Let $X, Y$ be subsets of $Q \times Q$ we have: $X \circ R \subseteq Y$ $\iff$ $X \subseteq \neg(\neg Y \circ R^{-1})$.

**Proof.** First we need an easily proved logical equivalence:

$$(\varphi_1 \land \varphi_2) \iff \varphi_3 \iff (\neg \varphi_3 \land \varphi_2) \iff \neg \varphi_1.$$ 

Then we have:

- $X \circ R \subseteq Y$
- $\forall s, s', s_1 : (s, s_1) \in X \land (s_1, s') \in R \Rightarrow (s, s') \in Y$
- $\forall s, s', s_1 : (s, s') \not\in Y \land (s_1, s') \in R \Rightarrow (s, s_1) \not\in X$ by above equivalence
- $\forall s, s', s_1 : (s, s') \not\in Y \land (s', s_1) \in R^{-1} \Rightarrow (s, s_1) \not\in X$ def. of $R^{-1}$
- $\forall s, s', s_1 : (s, s') \in \neg Y \land (s', s_1) \in R^{-1} \Rightarrow (s, s_1) \subseteq \neg X$
- $\forall X \subseteq \neg(\neg Y \circ R^{-1})$

Intuitively, $g$ corresponds to forward reasoning for proving termination while $\tilde{g}$ corresponds to backward reasoning because of the composition with $R^{-1}$. The least fixpoint $\text{lfp } \lambda Y. R \cup g(Y)$ is the least relation $Z$ containing $R$ and closed by composition with $R$, viz. $R \subseteq Z$ and $Z \circ R \subseteq Z$. On the other hand, the greatest fixpoint $\text{gfp } \lambda Y. W \cap \tilde{g}(Y)$ is best understood as the result of removing from $W$ all those pairs $(s, s')$ of states such that $(s, s') \circ R^+ \not\subseteq W$. This process returns the largest subset $Z'$ of $W$ which is closed by composition with $R$, viz. $Z' \subseteq W$ and $Z' \circ R \subseteq Z'$. Using the results of Cousot [13] we find next that termination can be shown by proving either inclusion of Lem. [2]

**Lemma 2 (from [13]).** $\text{lfp } \lambda Y. R \cup g(Y) \subseteq W$ $\iff$ $R \subseteq \text{gfp } \lambda Y. W \cap \tilde{g}(Y)$.

**Proof.**

$$\text{lfp } \lambda Y. R \cup g(Y) \subseteq W \iff \text{exists } A : R \subseteq A \land g(A) \subseteq A \land A \subseteq W \text{ by (1)}$$
$$\iff \exists A : R \subseteq A \land A \subseteq \tilde{g}(A) \land A \subseteq W \text{ Lem. (1)}$$
$$\iff R \subseteq \text{gfp } \lambda Y. W \cap \tilde{g}(Y) \text{ by (1)} \square$$

\[2\] Recall that $W$ is always assumed to be disjunctively well-founded.
As we shall see, the inclusion check based on the greatest fixpoint has interesting consequences when trying to prove termination.

An important feature when proving termination using transition invariants is to define actions to take when the inclusion check \( \text{lfp } \lambda Y. R \cup g(Y) \subseteq W \) fails. In this case, some information is extracted from the failure (e.g., a counter example), and is used to enrich \( W \) with more well-founded relations [10].

We shall see that, for the backward approach, failure of \( R \subseteq \text{gfp } \lambda Y. W \cap \tilde{g}(Y) \) induces a partition of the transition relation \( R \) into \( \{ R_G, R_B \} \) such that (a) \( (R_G)^{\ast} \subseteq W \); together with the following termination guarantee (b) every infinite \( R \)-trace contains a suffix that is an infinite \( R_B \)-trace (Lem. 4). An important consequence of this is that we can focus our effort exclusively on proving termination of \( R_B \). It is important to note that the guarantee that no infinite \( R \)-trace contains infinitely many steps from \( R_G \) is not true for any partition \( \{ R_G, R_B \} \) of \( R \) but it is true for our partition which we define next.

**Definition 1.** Let \( G = \text{gfp } \lambda Y. W \cap \tilde{g}(Y) \), we define \( \{ R_G, R_B \} \) to be the partition of \( R \) given by \( R_G = R \cap G \) and \( R_B = R \setminus R_G \).

**Example 1.** Let \( R = \{ x \geq 1, x' = x + y, y' = y - 1 \} \) and assume \( W = \{ x' < x, x \geq 1 \} \) which is well-founded, hence disjunctively well-founded as well. Evaluating the greatest fixpoint (we omit calculations) yields

\[
R_G = \{ x \geq 1, x' = x + y, y' = y - 1, y < 0 \}
\]
\[
R_B = \{ x \geq 1, x' = x + y, y' = y - 1, y \geq 0 \}
\]

which is clearly a partition of \( R \). The relation \( R_G \) consists of those pairs of states where \( y \) is negative, hence \( x \) is decreasing as captured by \( W \). On the other hand, \( R_B \) consists of those pairs where \( y \) is positive or null. It follows that, when taking a step from \( R_B \), \( x \) does not decrease. This is precisely for those pairs that \( W \) fails to show termination. ■

Next, we state and prove the termination guarantees of the partition \( \{ R_G, R_B \} \).

**Lemma 3.** Given \( R_G \) as in Def. 7, we have \( \text{lfp } \lambda Y. R_G \cup Y \circ R \subseteq W \).

**Proof.**

\[
G \subseteq \tilde{g}(G) \land G \subseteq W \quad \text{def. of } G \text{ and (1)}
\]

only if \( g(G) \subseteq G \land G \subseteq W \quad \text{Lem. (1)}
\]

only if \( R \cap G \subseteq G \land g(G) \subseteq G \land G \subseteq W \)
\[
\text{only if } R_G \subseteq G \land g(G) \subseteq G \land G \subseteq W \quad \text{def. of } R_G
\]

only if \( \text{lfp } \lambda Y. R_G \cup g(Y) \subseteq W \quad \text{by (1)}
\]

\[
\square
\]

An equivalent formulation of the previous result is \( R_G \circ R^{\ast} \subseteq W \), which in turn implies, since \( R_G \subseteq R \), that \( (R_G \circ R^{\ast})^{\ast} \subseteq W \), and also \( (R_G)^{\ast} \subseteq W \).

**Lemma 4.** Every infinite \( R \)-trace has a suffix that is an infinite \( R_B \)-trace.
Proof. Assume the contrary, i.e., there exists an infinite \( R \)-trace \( s_1, s_2, \ldots \) that contains infinitely many steps from \( R_G \). Let \( S = s_i, s_{i+1}, \ldots \) be the infinite subsequence of states such that \((s_j, s_{j+1}) \in R_G \) for all \( j \geq 1 \). Recall also that \( W = W_1 \cup \cdots \cup W_n \) where each \( W_\ell \) is well-founded. For any \( s_i, s_j \in S \) with \( i < j \) it holds that \((s_i, s_j) \in R_G \circ R^* \), and thus, according to Lem. 3 we also have that \((s_i, s_j) \in W_\ell \) for some \( 1 \leq \ell \leq n \). Ramsey’s theorem \([24]\) guarantees the existence of an infinite subsequence \( S' = s_{j_1}, s_{j_2}, \ldots \) of \( S \), and a single \( W_\ell \), such that for all \( s_i, s_j \in S' \) with \( i < j \) we have \((s_i, s_j) \in W_\ell \). This contradicts that \( W_\ell \) is well-founded and we are done. \( \square \)

Remark 1. When fixpoints are not computable, they can be approximated from above or from below \([14]\). It is routine to check that the results of Lemmas 3 and 4 remain valid when replacing \( G = gfp \ \lambda Y. W \cap \tilde{g}(Y) \) in Def. 1 with \( G' \subseteq gfp \ \lambda Y. W \cap \tilde{g}(Y) \). Therefore we have that, even when approximating \( gfp \ \lambda Y. W \cap \tilde{g}(Y) \) from below, the termination guarantees of \( \{R_G, R_B\} \) still hold. In Sec. 6 we shall see how to exploit this result in practice.

Example 2 (cont’d from Ex. 1). We left Ex. 1 with \( W = \{x' < x, x \geq 1\} \) and \( R_B = \{x \geq 1, x' = x + y, \ y' = y - 1, y \geq 0\} \). As argued previously, to prove the well-foundedness of \( R \) it is enough to show that \( R_B \) is well-founded. For clarity, we rename \( R_B \) into \( R_B^{(1)} \).

Next we partition \( R_B^{(1)} \) as we did it for \( R \) in Ex. 1. As a result, we update \( W \) by adding the well-founded relation \( \{y' < y, y \geq 0\} \). Then we evaluate again \( G \) (we omit calculations) which yields \( R_B^{(2)} = \emptyset \). Hence we conclude from Lem. 4 that \( R \) is well-founded. \( \blacksquare \)

Building upon all the previous results, we introduce \texttt{A acabar} that is given at Alg. 1. \texttt{A acabar} is a recursive procedure that takes as input two parameters: a transition relation \( R \) and a disjunctively well-founded relation \( W \). The second parameter is intended for recursive calls, hence the user should invoke \texttt{A acabar} as follows: \texttt{A acabar}(\( R, \emptyset \)). We call it the root call. Upon termination, \texttt{A acabar} returns a subset \( R_B \) of the transition relation \( R \). If it returns the empty set, then the relation \( R \) is well-founded, hence termination is proven. Otherwise (\( R_B \neq \emptyset \)), we can not know for sure if \( R \) is well-founded: there might be an infinite \( R \)-trace. However, Lem. 4 tells us that every infinite \( R \)-trace must have a suffix that is an infinite \( R_B \)-trace. It may also be the case that \( R_B \) is well-founded (and so is \( R \)) in which case it was not discovered by \texttt{A acabar}. Another case is that \( R = R_B \). In this case we have made no progress and therefore we stop. Whenever \( R_B \neq \emptyset \), we call this returned value the problematic subset of \( R \).

Next we study progress properties of \texttt{A acabar}. We start by defining the sequence \( \{R_B^{(i)}\}_{i \geq 0} \) where each \( R_B^{(i)} \) is the argument passed to the \( i \)-th recursive call to \texttt{A acabar}. In particular, \( R_B^{(0)} \) is the argument of the root call. Furthermore, we define the sequences \( \{R_B^{(i)}\}_{i \geq 1} \) and \( \{R_G^{(i)}\}_{i \geq 1} \) where \( \{R_B^{(i)}, R_G^{(i)}\} \) is a partition of \( R^{(i-1)} \) and \( R_B^{(i)} = R^{(i)} \) for all \( i \geq 1 \).

Lemma 5. Given a run of \texttt{A acabar} with at least \( i \geq 1 \) recursive calls, then we have

\[
R^{(0)} \supseteq R^{(1)} \supseteq \cdots \supseteq R^{(i)}.
\]

Proof. The proof is by induction on \( i \), for \( i = 1 \) it follows from the definitions that \( R^{(1)} = R_B^{(1)} \) and \( \{R_B^{(1)}, R_G^{(1)}\} \) is a partition of \( R^{(0)} \). Moreover, since at least \( i = 1 \) recursive calls take place we find that the condition of line 5 fails, meaning neither \( R_B^{(1)} \) nor \( R_G^{(1)} \) is empty, hence \( R^{(1)} \) is a strict subset of \( R^{(0)} \). The inductive case is similar. \( \square \)
Algorithm 1. Enhanced modular reasoning

\begin{verbatim}
Acabar(R, W)
  Input: a relation \( R \subseteq Q \times Q \)
  Input: a relation \( W \subseteq Q \times Q \) such that \( W \) is disjunctively well-founded
  Output: \( R_B \subseteq R \)
  begin
    \( W := W \cup \text{find\_dwf\_candidate}(R) \)
    let \( G \) be such that \( G \subseteq gfp \lambda Y. W \cap \tilde{g}(Y) \)
    \( R_B := R \setminus G \)
    if \( R_B = \emptyset \) or \( R_B = R \) then
      return \( R_B \)
    else
      return \( \text{A acabar}(R_B, W) \)
  end
\end{verbatim}

By Lemmas 4 and 5, we have that every infinite \( R^{(0)} \)-trace has a suffix that is an infinite \( R^{(i)} \)-trace for every \( i \geq 1 \). As a consequence, forcing \texttt{A acabar} to execute line 6 after predefined number of recursive calls, it returns a relation \( R^{(i)}_B \) such that the previous property holds. Incidentally, we find that \texttt{A acabar} proves program termination when it returns the empty set as stated next.

**Theorem 1.** Upon termination of the call \( \text{A acabar}(R, \emptyset) \), if it returns the empty set, then the relation \( R \) is well-founded.

Let us turn to line 2. There, \texttt{A acabar} calls a subroutine \texttt{find\_dwf\_candidate}(\( R \)) implementing a heuristic search which returns a disjunctively well-founded relation using hints from the representation and the domain of \( R \). Details about its implementation, that is inspired from previous work [9,10], will be given at Sec. 6 — we will consider the case of \( R \) being a relation over the integers of the form \( R = \rho_1 \lor \cdots \lor \rho_n \) where each \( \rho_i \) is a conjunction of linear constraints over the variables \( \bar{x} \) and \( \bar{x}' \). Let us intuitively explain this procedure on an example.

**Example 3 (cont’d from Ex. 2).** \texttt{A acabar}(\( R, \emptyset \)) updates \( W \) as follows: (1) \( \emptyset \); (2) \( \{x' < x, x \geq 1\} \); (3) \( \{x' < x, x \geq 1\}, \{y' < y, y \geq 0\} \). The first update from \( \emptyset \) to \( \{x' < x, x \geq 1\} \) is the result of calling \texttt{find\_dwf\_candidate}(\( R \)). The hint used by \texttt{find\_dwf\_candidate} is that \( x \) is bounded from below in \( R \). The second update to \( W \) results from calling \texttt{find\_dwf\_candidate}(\( R_B = \{x \geq 1, x' = x + y, y' = y - 1, y \geq 0\} \)). Since \( R_B \) has the linear ranking function \( f(x, y) = y \), \texttt{find\_dwf\_candidate} returns \( \{y' < y, y \geq 0\} \). ■

## 5 Acabar for Conditional Termination

As mentioned previously, upon termination, \texttt{A acabar} returns a subset \( R_B \) of the transition relation \( R \). If this set is empty then \( R \) is well-founded and we are done. Otherwise, \( R_B \) is a non-empty subset and called the problematic set. In this section, we shall see how to compute, given the problematic set, a precondition \( P \) for termination. More precisely, \( P \) is a set of states such that no infinite \( R \)-trace starts with a state of \( P \). We illustrate our definitions using the simple but challenging example of Sec. 2.
Lemma 4 tells us that every infinite $R$-trace $\pi$ is such that $\pi = \pi_f \pi_\infty$ where $\pi_f$ is a finite $R$-trace and $\pi_\infty$ is an infinite $R_B$-trace. Our computation of a precondition for termination is divided into the following parts: (i) compute those states $Z$ visited by infinite $R_B$-trace; (ii) compute the set $V$ of $R^*$-predecessors of $Z$, that is the set of states visited by some $R$-trace ending in $Z$; and (iii) compute $P$ as the complement of $V$. Formally, (i) is given by a greatest fixpoint expression $\text{gfp} \, \lambda X. \text{pre}[R_B](X)$. This expression is directly inspired by the work of Bozga et al. [6] on deciding conditional termination. This greatest fixpoint is the largest set $V$ of states each of which has an $R_B$-successor in $Z$. Because of this property, every infinite $R_B$-trace visits only states in $Z$. In $\pi = \pi_f \pi_\infty$, this corresponds to the suffix $\pi_\infty$ that is an infinite $R_B$-trace.

Example 4. Consider again the relation $R = \{x > 0, x' = x + y, y' = y + z, z' = z\}$. Upon termination $\text{Accion}$ returns the following relation:

$$R_B = \{x' = x + y, y' = y + z, z' = z, x > 0, y \geq 0, z \geq 0\}$$

which corresponds to all the cases where $x$ is stable or increasing over time.

Example 5. For $R_B$ as given in Ex. 4 we have that $Z = \{z \geq 0, y \geq 0, x > 0\}$ which contains the following infinite $R_B$-trace:

$$(x = 1, y = 0, z = 0) R_B (x = 1, y = 0, z = 0) R_B (x = 1, y = 0, z = 0) R_B \ldots$$

Let us now turn to (ii), that is computing the set $V$ of $R^*$-predecessors of $Z$. It is known that $V$ coincides with $\text{lfp} \, \lambda X. Z \cup \text{pre}[R](X)$. Intuitively, we prepend to those infinite $R_B$-traces a finite $R$-trace. That is, prefixing $\pi_f$ to $\pi_\infty$ results in $\pi = \pi_f \pi_\infty$. Finally, step (iii) results into a precondition for termination $P$ obtained by complementing $V$.

Example 6. Computing $\text{lfp} \, \lambda X. Z \cup \text{pre}[R](X)$ for $Z$ as given in Ex. 5 and $R = \{x' = x + y, y' = y + z, z' = z, x > 0, y \geq 0, z \geq 0\}$ (Ex. 4) gives $V = V_1 \cup V_2$ where

$${V_1} = \{x \geq 1, z = 0, y \geq 0\}$$

$${V_2} = \{x \geq 1, z \geq 1\} \cup \{x + i \ast y + j \ast z \geq 1 \mid i \geq 1, j = \sum_{k=0}^{i-1} k\}.$$ 

Intuitively, the set $V_1$ of states corresponds to entering the loop with $z = 0$ and $y$ non-negative, in which case the loop clearly does not terminate. The set $V_2$ of states corresponds to entering the loop with $z$ positive, and the loop does not terminate after $i$-th iterations for all $i$. Note that $V_2$ consists of infinitely many atomic formulas. Complementing $V$ gives $P$.

Theorem 2. There exists an infinite $R$-trace starting from $s$ iff $s \notin P$.

Approximations. As argued previously, it is often the case that only approximations of fixpoints are available. In our case, any overapproximation of either $Z$ or $V$ can be exploited to infer $P$. Because of approximations, we lose the if direction of the theorem, that is, we can only say that there is no infinite $R$-trace starting from some $s \in P$.

Example 7. Using finite disjunctions of linear constraints, we can approximate $V$ by

$${\{x \geq 1, z = 0, y \geq 0\} \lor \{x \geq 1, z \geq 1, x + y \geq 1, x + 2y + z \geq 1, x + 3y + 3z \geq 1\}}$$
and then the complement $P$ is

$$x \leq 0 \lor x + y \geq 1 \lor x + 2y + z \leq 0 \lor x + 3y + 3z \leq 0 \lor z \leq -1 \lor (y \leq -1 \land z \leq 0)$$

which is a sufficient precondition for termination. Note that the first 4 disjuncts correspond to the executions which terminates after 0, 1, 2 and 3 iterations.

6 Implementation

We have implemented the techniques described in Sec. 4 and 5 for the case of multipath integer linear-constraint loops. These loops correspond to relations of the form $R = \rho_1 \lor \cdots \lor \rho_d$ where each $\rho_i$ is a conjunction of linear constraints over the variables $\vec{x}$ and $\vec{x}'$. In this context, the set $Q$ of states is equal to $\mathbb{Z}^n$ where $n$ is the number of variables in $\vec{x}$. This is a classical setting for termination [5,7,22]. Internally, we represent sets of states and relations over them as DNF formulas where the atoms are linear constraints. In what follows, we explain sufficient implementation details so that our experiments can be independently reproduced if desired. Our implementation is available [1].

We start with line 2 of Alg. 1. Recall that the purpose of this line is to add more well-founded relations to $W$ based on the current relation $R$. In our implementation, $W$ consists of well-founded relations of the form $\{f(\vec{x}) \geq 0, f(\vec{x}') < f(\vec{x})\}$ where $f$ is a linear function [10,9]. Thus, our implementation looks for such well-founded relations. In particular, for each $\rho_i$ of $R$ we add new well-founded relations to $W$ as follows: if $\rho_i$ has a linear ranking function $f(\vec{x})$ that is synthesized automatically [2,15] then $\{f(\vec{x}') < f(\vec{x}), f(\vec{x}) \geq 0\}$ is added to $W$; otherwise, let $\{f_1(\vec{x}) \geq 0, \ldots, f_d(\vec{x}) \geq 0\}$ be the result of projecting each $\rho_i$ on $\vec{x}$ (i.e., eliminating variables $\vec{x}'$ from $\rho_i$), then $\{\{f_i(\vec{x}') < f_i(\vec{x}), f_i(\vec{x}) \geq 0\} | 1 \leq i \leq d\}$ is added to $W$. Because $f_i$ is bounded but not necessarily decreasing, it is called a potential linear ranking function [9].

As for line 3, recall that $G$ is a subset of $\lambda Y. W \cap g(Y)$. Furthermore, the sole purpose of $G$ is to compute $R_B = R \setminus G$. We now observe that $\neg G$, the complement of $G$, is as good as $G$. In fact, $R_B = R \cap (\neg G)$. So by considering $\neg G$ instead, we are looking for is an overapproximation of $\neg(\lambda Y. W \cap \tilde{g}(Y))$. Next we recall Park’s theorem replacing the above expression by a least fixpoint expression.

Theorem 3 (From [20]). Let $(L, \subseteq, [\bigcup], \top, \bot, \neg)$ be a complete Boolean algebra and let $f \in L \rightarrow L$ be an order-preserving function then $f' = \lambda X. \neg(f(\neg X))$ is an order-preserving function on $L$ and $\neg(\lambda f. f') = \lambda f. f'.

Park’s theorem applies in our setting because computations are carried over the Boolean algebra $(\mathbb{Z}^{\geq 0}, \subseteq, [\bigcup], 0, \neg)$. Applying it to $\lambda Y. W \cap \tilde{g}(Y)$ where $\tilde{g}(Y) = \neg(\neg Y \circ R^{-1})$, we find that

$$\neg(\lambda f. \lambda Y. W \cap \neg(\neg Y \circ R^{-1})) = \lambda f. \lambda Y. (\neg Y) \cup Y \circ R^{-1}.$$

Therefore, to implement line 3 we rely on abstract interpretation to compute an over-approximation of $\lambda f. \lambda Y. (\neg Y) \cup Y \circ R^{-1}$, hence, by negation, an underapproximation of $\lambda f. \lambda Y. W \cap \tilde{g}(Y)$ therefore complying with the requirement on $G$.

As far as abstract interpretation is concerned, our implementation uses a combination of predicate abstraction [17] and case splitting. The set of predicates is given by a finite
set of atomic linear constraints and is also closed under negation, e.g., if \( x + y \geq 0 \) is a predicate then \( x + y \leq -1 \) is also a predicate. Abstract values are positive Boolean combination of atoms taken from the set of predicates. Observe that although negation is forbidden in the definition of abstract values, the abstract domain is closed under complement.

The set of predicates is chosen so as the following invariant to hold: each time the control hits line 3 the set contains enough predicates to represent precisely each well-founded relation in \( W \). Our implementation provides enhanced precision by enforcing a stronger invariant: besides the above predicates for \( W \), it includes all atomic linear constraints occurring in the formulas representing \( X_0, \ldots, X_\ell \) where \( \ell \geq 0 \), \( X_0 = (\neg W) \) and \( X_{i+1} = (\neg W) \cup X_i \circ R^{-1} \). The value of \( \ell \) is user-defined and, in our experiments, it did not exceed 1.

To further enhance precision at line 3 we apply case splitting. The set of \( R \)-traces is partitioned using the linear atomic constraints of the form \( f(\bar{x}') < f(\bar{x}) \) that appear in \( W \). More precisely, partitioning \( R \) on \( f(\bar{x}') < f(\bar{x}) \) is done by replacing each \( \rho_i \) by \( (\rho_i \land f(\bar{x}') < f(\bar{x})) \lor (\rho_i \land f(\bar{x}') \geq f(\bar{x})) \).

As for conditional termination, overapproximating \( Z = gfp \lambda X. \text{pre}[R_B](X) \) is done by computing the last element \( X_\ell \) from the finite sequence \( X_0, \ldots, X_\ell \) given by \( X_0 = Q \) and \( X_{i+1} = X_i \land \text{pre}[R_B](X_i) \) where \( \ell \) is predefined. The result is always representable as DNF formula where the atoms can be any atomic linear constraints. As for \( V = lfp \lambda X. \text{pre}[R](X) \cup \text{pre}[R](X) \), an overapproximation is computed in a similar way to that of line 3 i.e., using a combination of predicate abstraction and case splitting.

7 Experiments

We have evaluated our prototype implementation against a set of benchmarks collected from publications in the area [9,8]. In what follows, we present the results of our implementation for those loops, and compare them to existing tools for proving termination [25,8,7] as well as tools for inferring preconditions for termination [9]. We compare the different techniques according to what the corresponding implementations report. We ignore performance because, for the selected benchmarks, little insight can be gained from performance measurements when an implementation was available (which was not always the case [26]).

The benchmarks accompanied with our results are depicted in Table 1. Translating each loop to a relation of the form \( R = \rho_1 \lor \cdots \lor \rho_n \) is straightforward. Every line in the table includes a loop and its inferred termination precondition (\textit{true} means it terminates for any input). In addition, preconditions (different from \textit{true}) marked with \( \bullet \) are optimal, i.e., the corresponding loop is non-terminating for any state in the complement.

We have divided the benchmarks into 3 groups: (1–5), (6–15) and (16–41). With the exception of loop 1, each loop in group (1–5) includes non-terminating executions and thus those loops are suitable for inferring preconditions. Our implementation reports the same preconditions as the tool of Cook et al. [9] save for loop 1 for which their tool is reported to infer the precondition \( x > 5 \lor x < 0 \), while we prove termination for all input. Note that every other tool used in the comparison [8,7,25] fail to prove termination of this loop. Further, the precondition we infer for loop 5 is optimal.
Table 1. Benchmarks used in experiments. Loops (1–5) are taken from [9] and (6–41) from [8]

| # | Loop                              | Termination Precondition                  |
|---|-----------------------------------|------------------------------------------|
| 1 | while (x≥0) x'=-2x+10;            | true                                     |
| 2 | while (x>0) x'=x+y; y'=y+z;       | x≤0 ∨ z≤0 ∨ (z=0 ∧ y<0) ∧ x+y≤0 ∨ x+2y+z≤0 ∨ x+3y+3z≤0 |
| 3 | while (x≤N)                       |                                          |
|    | if (*) { x'=2x+y; y'=y+1; } else x'=x+1; | x > n ∨ x + y ≥ 0                             |
| 4 | @requires n>200 and y<9 while (1) |                                          |
|    | if (x<n) {                         |                                          |
|    | x'=x+y;                           |                                          |
|    | if (x'≥200) break;               |                                          |
| 5 | while (x<>y) if (x>y) x'=x-y; else y'=y-x; | (x ≥ 0 ∨ x + y ≥ 0) ∨ x + 2y ≥ 1 ∨ x + 3y ≥ 3 |
| 6 | while (x<0) x'=x+y; y'=y-1;       |                                          |
| 7 | while (x>0) x'=x+y; y'=2y;        | x ≥ 0 ∨ y ≠ 0                             |
| 8 | while (x<y) x'=x+y; y'=y-2y;      | x ≥ 0 ∨ y ≠ 0                             |
| 9 | while (x<y) x'=x+y; y'=y;         | x ≥ 0 ∨ y ≠ 0                             |
| 10| while (4x-5y>0) x'=2x+4y; y'=4x;  |                                          |
| 11| while (x<5) x'=x-y; y'=x+y;       | x ≠ 0 ∨ y ≠ 0                             |
| 12| while (x>0 and y>0) x'=2x+10y;    | x ≤ 3 ∨ 10y − 3x ≠ 0                      |
| 13| while (x>0) x'=x+y; y'=y;         | x ≤ 0 ∨ y < 0 ∨ x + y ≤ 0                 |
| 14| while (x<0) x'=-y; y'=y+1;        | y ≤ 0 ∨ y < 10                            |
| 15| while (x<0) x'=x+y; y'=y+1; z'=-2y | x ≥ 0 ∨ x + z ≥ 0                          |
| 16| while (x<0 and x<100) x'≥2x+10;   | true                                     |
| 17| while (x<1) -2x'=x;              | true                                     |
| 18| while (x>1) 2x'≤x;               | true                                     |
| 19| while (x>0) 2x'≤x;               | true                                     |
| 20| while (x>0) x'=x+y; y'=y-1;       | true                                     |
| 21| while (4x+y>0) x'=-2x+4y; y'=4x;  | 4x + y ≤ 0 ∨ (x − 4y ≥ 0 ∧ 8x − 15y ≥ 1)  |
| 22| while (x<0 and x<100)            | true                                     |
| 23| while (x>0) x'=x-y; y'=y+1;       | true                                     |
| 24| while (x>0 and x<n) x'=x+y-5;     | true                                     |
| 25| while (x>0 and y≥0) x'≥x+y; y'=y-1; | true                                     |
| 26| while (x<0 and x>100)            | true                                     |
| 27| while (x>0) x'=x+y; y'=y-1;       | true                                     |
| 28| while (x>0) x'=x+y-5; y'=y+1;     | true                                     |
| 29| while (x>0 and y>0) x'=x-1; y'=y-2y; true |
| 30| while (x<0) x'=x+y; y'=y+1;       | true                                     |
| 31| while (x>0) x'=x+y; y'=y-1;       | true                                     |
| 32| while (x>0) x'=x+y; y'=y-1;       | true                                     |
| 33| while (x<0) x'=x+1; y'=z; z'=z;   | true                                     |
| 34| while (x>0) x'=x+y; y'=y+z; z'=z-1; true |
| 35| while (x≥0 and x≤z) x'=2x+y; y'=y+1; z'=z | true |
| 36| while (x>0 and x<z) x'=2x+y; y'=y+1; z'=z | true |
| 37| while (x>0) x'=x+y; z'=z; z'=-z-1; true |
| 38| while (x>0 and x<z) x'=x+y; y'=y+1; z'=z | true |
| 39| while (x>0 and x<y) x'>2x; y'=y; z'=z | true |
| 40| while (x>0 and x<y) x'=x+y+z; y'=z-1; z'=z | true |
| 41| while (x+y≥0 and x≤n) x'=2x+y; y'=z; z'=z+1; n'=n; | true |
All the loops (6–15) are non-terminating. Chen et al. [8] report that their tool cannot handle them since it aims at proving termination and not inferring preconditions for termination. We infer preconditions for all of them, and in addition, most of them are optimal (those marked with ⋄). Unfortunately for those loops we could not compare with the tool of Cook et al. [9], since there is no implementation available [26].

Loops in the group (16–41) are all terminating. Those marked with ⋄ actually have linear ranking functions, those unmarked require disjunctive well-founded transition invariants with more than one disjunct. We prove termination of all of them except loop 21. We point that the tool of Chen et al. [8] also fails to prove termination of loop 21, but also of loop 34. On the other benchmarks, they prove termination. They also report that the tool of PolyRank [7] failed to prove termination of any of the loops that do not have a linear ranking function. In addition, we applied ARMC [25] on the loops of the group (16–41). ARMC, a transition invariants based prover, succeeded to prove termination for all those loops with a linear ranking function (marked with ⋄) and also loop 39.

Next we discuss in details the analysis of two selected examples from Table 1.

**Example 8.** Let us explain the analysis of loop 1 in details starting with the root call Acabar(R, 0) where \( R = \{ x \geq 0, x' = -2x + 10 \} \). At line 2 since \( R \) includes the bound \( x \geq 0 \), i.e., \( f(x) = x \) is a potential linear ranking function, we add \( \{ x' < x, x \geq 0 \} \) to \( W \). Computing \( G \) at line 3 hence \( R_B \) at the following line, results in \( R_B = \rho_1 \lor \rho_2 \) where \( \rho_1 = \{ x' = -2x + 10, x \geq 0, x \leq 3 \} \) and \( \rho_2 = \{ x' = -2x + 10, x \geq 4, x \leq 5 \} \).

Note that \( \rho_1 \) is enabled for \( 0 \leq x \leq 3 \) and in this case \( x' > x \). Also \( \rho_2 \) is enabled for \( x = 4 \) or \( x = 5 \) for which \( x' < x \) and thus \( \rho_2 \subseteq W \), however, after one more iteration, the value of \( x \) increases (this is why \( \rho_2 \) is included in \( R_B \)). Transitions for which \( x > 5 \) are not included in \( R_B \), hence they belong to \( R_G \) itself included in \( W \) (Lem. 3). Hence when \( x > 5 \) termination is guaranteed, this is also easily seen since those transitions terminate after one iteration.

Since \( R_B \) is neither empty nor equal to \( R \), a recursive call to \( \text{Aobar}(R_B, W) \) takes place. At line 2 we add \( \{ -x' < -x, 10 - x \geq 0 \} \) to \( W \) since \( f(x) = 10 - x \) is a linear ranking function for \( \rho_1 \). Note that \( \rho_2 \) has the linear ranking function \( f(x) = x \) already included in \( W \). Computing \( G \) at line 3 hence \( R_B \), yields \( R_B = 0 \) and therefore we conclude that the loop terminates for any input.

**Example 9.** Let us explain the analysis of loop 9 in details starting with the root call Acabar(R, 0) where \( R = \{ x < y, x' = x + y, 2y' = y \} \). At line 2 since \( R \) includes the bound \( y - x > 0 \), i.e., \( f(x, y) = y - x - 1 \) is a potential linear ranking function, we add \( \{ y' - x' < y - x, y - x - 1 \geq 0 \} \) to \( W \). Computing \( G \) at line 3 hence \( R_B \) yields \( R_B = \{ x < y, x' = x + y, 2y' = y, y \leq 0 \} \). Note that \( R_B \) exclusively consists of transitions where \( y \) is not positive, in which case \( x' - y' \geq x - y \) and thus not included in \( W \). Transitions where \( y \) is positive are not included in \( R_B \) (hence they belong to \( R_G \)) since they always decrease \( x - y \), and thus are transitively included in \( W \) (Lem. 3).

Since \( R_B \) is neither empty nor equal to \( R \), we call recursively Acabar(R_B, W). At line 2 since \( R \) includes the bound \( y \leq 0 \) (or equivalently \( -y \geq 0 \), i.e., \( f(x, y) = -y \) is a potential linear ranking function, we add \( \{ -y' < -y, -y \geq 0 \} \) to \( W \). Computing \( G \) at line 3 hence \( R_B \) yields \( R_B = \{ x < y, x' = x + y, 2y' = y, y = 0 \} \). Note that \( R_B \) exclusively consists of transitions where \( y = 0 \), which keeps both values of \( x \) and \( y \) unchanged.
Transitions in which \( y \) is negative belong to \( R_G \), hence they are transitively covered by \( W \) (Lem.3), in particular by the last update (viz. \( \{-y' < -y, -y \geq 0\} \)) to \( W \).

Since \( R_B \) is neither empty nor equal to \( R \), we call recursively \( \text{Acabar}(R_B, W) \). This time our implementation does not further enrich \( W \) with a well-founded relation, and as a consequence, after computing \( G \) at line5, we get that \( R_B = R \). Hence, \( \text{Acabar} \) returns with \( R_B = \{x < y, x' = x + y, 2y' = y, y = 0\} \).

Now, given \( R_B \), we infer a precondition for termination as described in Sec.5. We first compute \( \text{gfp} \lambda X. \pre_{R_B}(X) \), which in this case, converges in two steps with \( Z \equiv y = 0 \land x < 0 \). Then we compute \( \text{lfp} \lambda X. Z \cup \pre_R(X) \), which results in \( V \equiv y = 0 \land x < 0 \). The complement, \( P \equiv y < 0 \lor y > 0 \lor x < 0 \), is a precondition for termination. Note that the result is optimal, i.e., \( V \) is a precondition for non-termination. Optimality is achieved because \( Z \) and \( V \) coincide with the \( \text{gfp} \) and the \( \text{lfp} \) of the corresponding operators, and are not overapproximations.

8 Conclusion

This work started with the invited talk of A. Podelski at ETAPS ’11 who remarked that the inclusion check \( R^+ \subseteq W \) is equivalently formulated as a safety verification problem where states are made of pairs. Back to late 2007, a PhD thesis [16] proposed a new approach to the safety verification problem in which the author shows how to leverage the equivalent backward and forward formulations of the inclusion check. Those two events planted the seeds for the backward inclusion check \( R \subseteq W^- \), and later \( \text{Acabar} \).

Initial States. For the sake of simplicity, we deliberately excluded the initial states \( I \) from the previous developments. Next, we introduce two possible options to incorporate knowledge about the initial states in our framework. The first option consists in replacing \( R \) by \( R' \) that is given by \( R \cap (\text{Acc} \times \text{Acc}) \) where \( \text{Acc} \) denotes (an overapproximation of) the reachable states in the system. Formally, \( \text{Acc} \) is given by the least fixpoint \( \text{lfp} \lambda X. I \cup \text{post}(R)(X) \).

The second option is inspired by the work of Cousot [12] where he mixes backward and forward reasoning. We give here some intuitions and preliminary development. Recall that the greatest fixpoint \( \text{gfp} \lambda Y. W \cap \tilde{g}(Y) \) of line3 is best understood as the result of removing all those pairs \((s, s') \in W\) such that \((s, s') \circ R^+ \not\in W\). We observe that the knowledge about initial states is not used in the greatest fixpoint. A way to incorporate that knowledge is to replace the greatest fixpoint expression by the following one \( \text{gfp} \lambda Y. (B \cap W) \cap \tilde{g}(Y) \) where \( B \) takes the reachable states into account. In a future work, we will formally develop those two options and evaluate their benefit.

Related Works. As for termination, our work is mostly related to the work of Cook et al. [10][11] where the inclusion check \( R^+ \subseteq W \) is put to work by incrementally constructing \( W \). Our approach, being based on the dual check \( R \subseteq W^- \), adds a new dimension of modularity/incrementality in which \( R \) is also modified to safely exclude those transitions for which the current proof is sufficient. The advantage of the dual check was shown experimentally in Sec.7. However, let us note that in our implementation we use potential ranking functions and case splitting, which are not used in ARMC [10]. Moreover, it smoothly applies to conditional termination.
Kroening et al. [18] introduced the notion of compositional transition invariants, and used it to develop techniques that avoid the performance bottleneck of previous approaches [11]. Recently, Chen et al. [8] proposed a technique for proving termination of single-path linear-constraint loops. Contrary to their techniques, we handle general transition relations and our approach applies also to conditional termination. Alias et al. [3] developed a termination analysis for flowcharts by incrementally synthesizing a lexicographical (linear) ranking function. As we do, they discard transitions covered by the current ranking function. They differ from us in the granularity by considering all the transitions corresponding an edge in the flowchart. On the contrary, our reasoning is independent from the system description. As for conditional termination, the work of Cook et al. [9] is the closest to ours. However, we differ in the following points: (a) we do not use universal quantifier elimination, whose complexity is usually very high, depending on the underlying theory used to specify R. Instead, we adapt a fixpoint centric view that allows using abstract interpretation, and thus to control precision and performance; (b) we do not need special treatment for loop with phase transitions (as the one of Sec. 2), they are handled transparently in our framework. Podelski et al. [23] studied the problem of conditional termination for heap manipulating programs. In this context, the inferred conditions are assumptions on the heap (reachability, aliasing, etc.). Bozga et al. [6] studied the problem of deciding conditional termination. Their main interest is to identify family of systems for which $\text{gfp} \lambda X. \text{pre}[R](X)$, the set of non-terminating states, is computable.

It is worth “terminating” by mentioning an alternate formulation of the termination check $R^+ \subseteq W$ [19]. Works based on this alternate formulation, in particular those that construct global ranking functions for $R$ [4], might serve as a starting point to understand some (completeness) properties of our approach. This is left for future work.

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