Semiclassical time evolution and quantum ergodicity for Dirac-Hamiltonians

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Abstract

Within the framework of Weyl calculus we establish a quantum-classical correspondence for the time evolution of observables generated by a Dirac-Hamiltonian. This includes a semiclassical separation of particles and antiparticles. We then prove quantum ergodicity for Dirac-Hamiltonians under the condition that a skew product of the classical relativistic translational motion and relativistic spin precession is ergodic.
1 Introduction

Dynamical properties of quantum systems have recently attracted considerable attention, in particular in connection with the question of quantum chaos. In this context a central problem is to characterise eigenstates of a quantum Hamiltonian semiclassically in terms of the dynamical behaviour of the associated classical system. Phase space lifts of eigenfunctions have proved to be especially suited for such investigations since in the semiclassical limit they converge to classically invariant measures on phase space. This property follows from a dynamical version of the correspondence principle, which is established in a mathematically rigorous form through Egorov’s theorem \[8\]. Prominent examples of invariant measures on phase space are ergodic ones. On the quantum side a classically ergodic Liouville (i.e. microcanonical) measure corresponds to the property of quantum ergodicity. This concept goes back to Shnirelman \[14\] and denotes the semiclassical convergence of the phase space lifts of almost all quantum eigenfunctions to Liouville measure; it has been proven in several situations \[16, 5, 10\].

Quantum systems that possess spin degrees of freedom in addition to their translational ones require an extension of Egorov’s theorem and of the concept of quantum ergodicity. In previous studies of non-relativistic quantum systems with spin \[1, 3\], whose dynamics are generated by a Pauli-type operator, we noticed that in general classical ergodicity of the translational motion is not sufficient to guarantee quantum ergodicity; the spin dynamics have to be considered as well. Here we present an extension of these investigations to Dirac-Hamiltonians. The main difference to the non-relativistic situation lies in the coexistence of particle and antiparticle states. This requires a separation of the Hilbert space into two subspaces that are almost invariant under the quantum time evolution. Only in these subspaces the usual semiclassical methods can be applied. As a consequence, quantum ergodicity for Dirac-Hamiltonians is concerned with projections of eigenspinors to the almost invariant subspaces, which in general are only approximate eigenspinors of the Hamiltonian.

Due to lack of space we cannot present proofs here. Together with further references these can be found in \[2\], where we have investigated quantum ergodicity for more general matrix valued Hamiltonians.

2 Background

We consider the Dirac equation

\[
i\hbar \frac{\partial}{\partial t} \psi(t, x) = \hat{H}_D \psi(t, x)
\]  

in a fixed inertial frame in which the potentials \(A\) and \(\phi\) are time-independent,

\[
\hat{H}_D = c\alpha \cdot \left( \frac{\hbar}{i} \nabla - \frac{e}{c} A(x) \right) + mc^2 \beta + e\phi(x) ,
\]
and the Clifford algebra, realised in the usual Dirac representation, is generated by $\alpha$ and $\beta$. The propagation of Dirac spinors prescribed by (2.1) therefore takes place in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3) \otimes \mathbb{C}^4$.

For the purpose of semiclassical approximations it is useful to represent quantum observables as Weyl operators. In the case of the Dirac-Hamiltonian (2.2) this means

$$(\hat{H}_D \psi)(\mathbf{x}) = \left( \text{op}^W[H_D] \psi \right)(\mathbf{x}) = \frac{1}{(2\pi \hbar)^3} \int \int e^{i \mathbf{p} \cdot (\mathbf{x} - \mathbf{y})} H_D(\mathbf{p}, \frac{\mathbf{x} + \mathbf{y}}{2}) \psi(\mathbf{y}) \, d\mathbf{y} \, d\mathbf{p} \quad (2.3)$$

Here the Weyl symbol $H_D(\mathbf{p}, \mathbf{x}) \in C^\infty(\mathbb{R}^3 \times \mathbb{R}^3) \otimes \mathbb{C}^{4 \times 4}$ is the smooth function on the classical phase space $\mathbb{R}^3 \times \mathbb{R}^3$ that arises from (2.2) by replacing $\frac{\hbar}{i} \nabla$ with $\mathbf{p}$. It hence takes values in the hermitian $4 \times 4$ matrices, and for each $(\mathbf{p}, \mathbf{x})$ possesses the two doubly degenerate eigenvalues

$$h_\pm(\mathbf{p}, \mathbf{x}) = e\phi(\mathbf{x}) \pm \sqrt{(c\mathbf{p} - eA(\mathbf{x}))^2 + m^2c^4} \quad (2.4)$$

with associated projection matrices $\pi_0^\pm(\mathbf{p}, \mathbf{x})$ to the $h_\pm(\mathbf{p}, \mathbf{x})$-eigenspaces in $\mathbb{C}^4$. The functions (2.4) can be identified as the classical Hamiltonians of relativistic systems with positive or negative energy, corresponding to particles or antiparticles, respectively. They generate the Hamiltonian flows $\Phi^t_\pm$ on the phase space $\mathbb{R}^3 \times \mathbb{R}^3$, representing the translational part of the classical analogue to the quantum dynamics generated by (2.1).

It will turn out that the spin part of the relevant classical dynamics is determined by the equations

$$\dot{s}_\pm(t) = C_\pm(\Phi^t_\pm(\mathbf{p}, \mathbf{x})) \times s_\pm(t) \quad (2.5)$$

that describe the Thomas precession of a classical normalised spin $\mathbf{s} \in S^2$ along the trajectories $\Phi^t_\pm(\mathbf{p}, \mathbf{x})$. Here $C_\pm(\mathbf{p}, \mathbf{x})$ contains the electromagnetic fields and potentials as known from the relativistic Thomas precession [17]. For each classical Hamiltonian $h_\pm$ the dynamics of the translational and the spin degrees of freedom can be combined into a single (non-Hamiltonian, skew product) flow

$$Y^t_\pm(\mathbf{p}, \mathbf{x}, \mathbf{s}) = (\Phi^t_\pm(\mathbf{p}, \mathbf{x}), s_\pm(t)) \quad \text{with} \quad Y^0_\pm(\mathbf{p}, \mathbf{x}, \mathbf{s}) = (\mathbf{p}, \mathbf{x}, \mathbf{s}) \quad (2.6)$$

on the combined phase space $\mathbb{R}^3 \times \mathbb{R}^3 \times S^2$. On $\Omega^\pm_E \times S^2$, where $\Omega^\pm_E = h_\pm^{-1}(E) \subset \mathbb{R}^3 \times \mathbb{R}^3$ is an energy shell of $h_\pm$, the flow $Y^t_\pm$ leaves the product measure $d\mu^\pm_E$ that consists of Liouville measure on $\Omega^\pm_E$ and the normalised area measure on $S^2$ invariant. If the energy shells are compact we will always assume that their Liouville measures are normalised. Then we can consider situations in which the flows $Y^t_\pm$ are ergodic with respect to the probability measures $d\mu^\pm_E$.

There exists an extensive calculus for Weyl operators of the form (2.3) when their symbols obey certain restrictions, see e.g. [13, 7]. If, in the present case, $|A(\mathbf{x})|$ is bounded by some power $|\mathbf{x}|^N$, one can introduce the order function

$$M(\mathbf{p}, \mathbf{x}) := \sqrt{(c\mathbf{p} - eA(\mathbf{x}))^2 + m^2c^4} \quad (2.7)$$
It defines the symbol class $S(M)$, which consists of all smooth, matrix valued functions $B(p, x)$ that satisfy the estimates
\[
\|\partial_p^\alpha \partial_x^\beta B(p, x)\|_{4 \times 4} \leq C_{\alpha\beta} M(p, x),
\]
where $\| \cdot \|_{4 \times 4}$ is some matrix norm. The symbol $H_D(p, x)$ of the Dirac-Hamiltonian then is in $S(M)$, if the potentials $\phi(x)$ and $A_k(x)$ ($k = 1, 2, 3$) are smooth and $\phi(x)$ as well as all derivatives of $\phi(x)$ and $A_k(x)$ are bounded. In this case the estimate
\[
\|(H_D(p, x) + i)^{-1}\|_{4 \times 4} \leq C M(p, x)^{-1}
\]
holds and implies that for small enough $\hbar$ the Dirac-Hamiltonian $\hat{H}_D$ is essentially self-adjoint on the domain $C^\infty_0(\mathbb{R}^3) \otimes \mathbb{C}^4$. It therefore generates a unitary time evolution.

### 3 Semiclassical projections

On the level of Weyl symbols there exist the two projection matrices $\pi_0^\pm$ onto the two-dimensional eigenspaces of $H_D(p, x)$ in $\mathbb{C}^4$. These lead to the two classical flows introduced in the preceding section that correspond to positive and negative energies, respectively. In the quantum system a related division of states into particles and antiparticles would also be desirable. Due to the effects of pair creation and annihilation such a separation, however, can only possibly be achieved in the semiclassical limit. Weyl quantisation suggests $\text{op}^W[\pi_0^\pm]$ as first candidates for the desired quantum projectors, but these (bounded) operators in fact are no projectors on the Hilbert space $\mathcal{H}$. One therefore proceeds to an inductive construction by correcting the symbols order by order in $\hbar$, starting with $\pi_{0}^\pm$ as the lowest order term,
\[
\hat{\Pi}^\pm := \text{op}^W[\pi^\pm] \quad \text{with} \quad \pi^\pm(p, x) \sim \sum_{k=0}^{\infty} \hbar^k \pi_k^\pm(p, x),
\]
see [3]. The expansion in powers of $\hbar$ has to be understood in the sense of an asymptotic series in the symbol class $S(1)$ that consists of all $B \in C^\infty(\mathbb{R}^3 \times \mathbb{R}^3) \otimes \mathbb{C}^{4 \times 4}$ with bounded derivatives, $\|\partial_x^\alpha \partial_p^\beta B(p, x)\|_{4 \times 4} \leq C_{\alpha\beta}$. The Weyl quantisation of such a symbol then is a bounded operator on $\mathcal{H}$ [4]. The terms in the asymptotic expansion (3.1) are uniquely fixed by the requirement that (in operator norm)
\[
(\hat{\Pi}^\pm)^* = \hat{\Pi}^\pm, \quad \| (\hat{\Pi}^\pm)^2 - \hat{\Pi}^\pm \| = O(\hbar^\infty), \quad \|[\hat{H}_D, \hat{\Pi}^\pm]\| = O(\hbar^\infty).
\]
That such almost projection operators exist is guaranteed by:

**Proposition 3.1.** Let $A_k$ ($k = 1, 2, 3$) and $\phi$ be smooth potentials such that $\phi$ and all derivatives of $A_k$ and $\phi$ are bounded. Then there exist two bounded almost projection operators $\hat{\Pi}^\pm$ that fulfill (3.3) and that are Weyl quantisations of symbols $\pi^\pm \in S(1)$ with asymptotic expansions (3.1). Moreover, the almost projectors provide a semiclassical resolution of unity on $\mathcal{H}$, in the sense that $\|\hat{\Pi}^+ + \hat{\Pi}^- - 1_H\| = O(\hbar^\infty)$.
A similar result can be found in [12]. Following [11] one can now construct genuine projection operators \( \hat{P}^\pm \) from \( \hat{\Pi}^\pm \) through the Riesz formula,

\[
\hat{P}^\pm := \frac{1}{2\pi i} \int_{|z-1|=\frac{1}{2}} (\hat{\Pi}^\pm - z)^{-1} \, dz .
\] (3.3)

These projectors are themselves Weyl operators with symbols in \( S(1) \) and are semiclassically close to the almost projectors, \( \| \hat{P}^\pm - \hat{\Pi}^\pm \| = O(\hbar^\infty) \). However, the operators \( \hat{P}^\pm \) in general do not commute with the Dirac-Hamiltonian. They only almost commute with \( \hat{H}_D \) in the same way as the operators \( \hat{\Pi}^\pm \). Nevertheless, one can relate the semiclassical projectors \( \hat{P}^\pm \) to almost commute with the Dirac-Hamiltonian is that the projections of eigenvectors \( \psi_n \in \mathcal{H} \) of \( \hat{H}_D \), \( \hat{H}_D \psi_n = E_n \psi_n \), are quasimodes with discrepancies of \( O(\hbar^\infty) \). This means that \( \hat{P}_n^\pm \psi_n \in \mathcal{H}_\pm \) are almost eigenvectors of \( \hat{H}_D \) with errors whose \( \mathcal{H} \)-norms are of \( O(\hbar^\infty) \): \( \| (\hat{H}_D - E_n) \hat{P}_n^\pm \psi_n \| = O(\hbar^\infty) \). Appropriate phase space lifts of the normalised quasimodes \( \phi_n^\pm := \hat{P}_n^\pm \psi_n / \| \hat{P}_n^\pm \psi_n \| \) can now be studied in the semiclassical limit and can be related to the dynamical behaviour of the respective classical flows \( Y^\pm_\pm \).

**Proposition 3.2.** If \( E \in (E_-, E_+) \) is neither in the spectrum of \( \hat{H}_D \), nor an accumulation point thereof, and if the same conditions as in Proposition 3.1 hold, the semiclassical projectors are close to the spectral projectors, \( \| \hat{P}^\pm - \hat{P}_n^\pm \| = O(\hbar^\infty) \).

The projectors that almost commute with the Dirac-Hamiltonian allow to introduce the subspaces \( \mathcal{H}^\pm := \mathcal{P}^\pm \mathcal{H} \) that in the above semiclassical sense can be viewed as particle and antiparticle spaces. The time evolution generated by \( \hat{H}_D \) leaves these subspaces invariant up to the semiclassically long time scale \( T(\hbar) = O(\hbar^{-\infty}) \).

A further consequence of the projectors \( \hat{P}^\pm \) to almost commute with the Dirac-Hamiltonian is that the projections of eigenvectors \( \psi_n \in \mathcal{H} \) of \( \hat{H}_D \), \( \hat{H}_D \psi_n = E_n \psi_n \), are quasimodes with discrepancies of \( O(\hbar^\infty) \). This means that \( \hat{P}_n^\pm \psi_n \in \mathcal{H}_\pm \) are almost eigenvectors of \( \hat{H}_D \) with errors whose \( \mathcal{H} \)-norms are of \( O(\hbar^\infty) \): \( \| (\hat{H}_D - E_n) \hat{P}_n^\pm \psi_n \| = O(\hbar^\infty) \). Appropriate phase space lifts of the normalised quasimodes \( \phi_n^\pm := \hat{P}_n^\pm \psi_n / \| \hat{P}_n^\pm \psi_n \| \) can now be studied in the semiclassical limit and can be related to the dynamical behaviour of the respective classical flows \( Y^\pm_\pm \).

### 4 Semiclassical time evolution

We now consider the time evolution of observables generated by \( \hat{H}_D \). For convenience we restrict attention to bounded Weyl operators \( \hat{B} = \text{op}^W[B] \), with symbols from the class \( S(1) \) that possess an asymptotic expansion in integer powers of \( \hbar \), compare (3.1). We call such operators semiclassical observables. The semiclassical projectors \( \hat{P}^\pm \) are of this type and provide a natural separation of an observable \( \hat{B} \) into diagonal and off-diagonal blocks, \( \hat{B} = \hat{B}_d + \hat{B}_{od} + O(\hbar^\infty) \), with

\[
\hat{B}_d := \hat{P}^+ \hat{B} \hat{P}^+ + \hat{P}^- \hat{B} \hat{P}^- \quad \text{and} \quad \hat{B}_{od} := \hat{P}^+ \hat{B} \hat{P}^- + \hat{P}^- \hat{B} \hat{P}^+ .
\] (4.1)
One expects that the diagonal blocks will be propagated semiclassically by the classical dynamics associated with the eigenvalue functions $h_+$ and $h_-$, respectively. For the off-diagonal blocks it is not so obvious how a semiclassical propagation works. In fact, under the quantum time evolution the off-diagonal blocks will cease to be semiclassical observables. A related discussion can be found in [6].

For a precise statement we require in addition to the assumptions on the potentials made previously that their first and all higher derivatives are bounded:

**Proposition 4.1.** Let $\hat{B} = \text{op}^W[B]$ be a semiclassical observable. Then for $t > 0$ its time evolution $\hat{B}(t)$ generated by the Dirac-Hamiltonian is a semiclassical observable, $\hat{B}(t) = \text{op}^W[B(t)]$ with symbol $B(t) \in S(1)$ and asymptotic expansion of the type (3.1), if and only if $\hat{B}_{\text{od}} = O(\hbar^\infty)$.

Hence the propagation of the diagonal blocks can be analysed semiclassically. To leading order this will happen in terms of the Hamiltonian flows $\Phi^t_{\pm}$ for the translational degrees of freedom and the spin-transport matrices $D_{\pm} \in \text{SU}(2)$ that follow from the equation

$$
\dot{D}_{\pm}(p, x, t) + \frac{1}{2} C_{\pm}(\Phi^t_{\pm}(p, x)) \cdot \sigma D_{\pm}(p, x, t) = 0 , \quad D_{\pm}|_{t=0} = 1_2 .
$$

With suitable isometries $V_{\pm}(p, x) : C^2 \to \pi_0^\pm(p, x)C^4$ these spin transport matrices can be converted to $d_{\pm}(p, x, t) \in U(4)$ via $D_{\pm}(p, x, t) = V_{\pm}^*(\Phi^t_{\pm}(p, x))d_{\pm}(p, x, t)V_{\pm}(p, x)$. By further introducing the notation $\hat{B}^\pm := \hat{P}^\pm \hat{B} \hat{P}^\pm = \text{op}^W[B^\pm]$ we can now state the relevant Egorov theorem:

**Theorem 4.2.** Under the conditions specified above the quantum time evolution of the diagonal part $\hat{B}_{\text{d}}$ of a semiclassical observable $\hat{B} = \text{op}^W[B]$ is again a semiclassical observable, $\hat{B}_{\text{d}}(t) = \text{op}^W[B_{\text{d}}(t)]$, with symbol $B_{\text{d}}(t) \in S(1)$ and asymptotic expansion

$$
B_{\text{d}}(t)(p, x) \sim \sum_{k=0}^{\infty} \hbar^k B_{\text{d}}(t)_k(p, x) .
$$

The $\hbar$-independent, leading term is completely determined by the Hamiltonian flows $\Phi^t_{\pm}$ generated by the eigenvalue functions $h_{\pm}$, and by the unitary spin-transport matrices $d_{\pm}$,

$$
B_{\text{d}}(t)_0(p, x) = \sum_{\nu \in \{+,-\}} d_{\nu}^{-1}(p, x, t) B^\nu_0(\Phi^t_{\nu}(p, x)) d_{\nu}(p, x, t) .
$$

The two types of dynamics that enter on the right-hand side of (1.4) can be combined into skew product flows on $\mathbb{R}^3 \times \mathbb{R}^3 \times \text{SU}(2)$ given by

$$
\hat{Y}^t_{\pm}(p, x, g) := (\Phi^t_{\pm}(p, x), D_{\pm}(p, x, t)g) .
$$

The double covering map $R : \text{SU}(2) \to \text{SO}(3)$ now allows to relate these flows to the genuinely classical skew product flows (2.0), if one sets $s_{\pm}(t) = R(D_{\pm}(p, x, t))s$. Moreover, the dynamical properties of the flows $Y^t_{\pm}$ and $\hat{Y}^t_{\pm}$ are closely related. E.g., $Y^t_{\pm}$ is ergodic.
with respect to the product measure $d\mu_\pm$ on $\Omega_E^\pm \times S^2$, if and only if $\tilde{Y}^t_\pm$ is ergodic on $\Omega_E^\pm \times SU(2)$ with respect to the product of Liouville and (normalised) Haar measure. Thus, to leading semiclassical order the time evolution of block-diagonal observables, generated by the Dirac-Hamiltonian $\hat{H}_D$, can be completely described by the two classical flows $Y^t_\pm$ that combine the translational and the spin degrees of freedom of particles and antiparticles, respectively.

5 Quantum ergodicity

In quantum systems without spin quantum ergodicity means that phase space lifts of almost all eigenfunctions of the quantum Hamiltonian semiclassically converge to Liouville measure on an appropriate energy shell, if the classical Hamiltonian flow on this energy shell is ergodic. Apart from an Egorov theorem, a proof of this statement requires a (Szegö-type) limit theorem for averaged phase space lifts of eigenfunctions.

In the case of a Dirac-Hamiltonian one first has to ensure the very existence of eigen-spinors $\psi_n \in \mathcal{H}$. To this end we require that there exists an energy $E$ such that all energy shells $\Omega_{E'}^\pm$ are compact when $E'$ varies in $[E - \varepsilon, E + \varepsilon]$ with some $\varepsilon > 0$. These $E'$ shall moreover be no critical values of the eigenvalue functions (classical Hamiltonians) $h_\pm(p, x)$.

In addition, at least one of the energy shells $\Omega_{E}^\pm$ shall be non-empty. Then, for sufficiently small $\hbar$, the spectrum of $\hat{H}_D$ is discrete in the interval $I(E, \hbar) := [E - \hbar \omega, E + \hbar \omega]$, comprising of $N_I > 0$ eigenvalues.

The desired limit theorem is concerned with averages of the expectation values of a semiclassical observable $\hat{B} = \text{op}^W[B]$ in normalised eigenstates $\psi_n$ of $\hat{H}_D$ with eigenvalues $E_n \in I(E, \hbar)$. For this to hold we require that the periodic orbits of the Hamiltonian flows $\Phi_t^\pm$ on $\Omega_E^\pm$ are of Liouville measure zero; e.g., this condition is fulfilled if the flows are ergodic.

**Proposition 5.1.** Under the conditions imposed above on the Dirac-Hamiltonian the number $N_I$ of eigenvalues in the interval $I(E, \hbar)$ grows semiclassically according to

$$N_I = \frac{2\omega \text{vol } \Omega_E^+ + \text{vol } \Omega_E^-}{(2\pi \hbar)^2} \left(1 + O(\hbar)\right).$$

Moreover, the Szegö-type limit formula,

$$\lim_{\hbar \to 0} \frac{1}{N_I} \sum_{E_n \in I(E, \hbar)} \langle \psi_n, \hat{B} \psi_n \rangle = \frac{1}{2} \sum_{\nu \in \{+, -\}} \frac{\text{vol } \Omega_E^\nu \text{ tr } (\pi_{m_0} B_0 \pi_{m_0}^\nu)^{E, \nu}}{\text{vol } \Omega_E^+ + \text{vol } \Omega_E^-},$$

holds for any semiclassical observable $\hat{B} = \text{op}^W[B]$.

Here $(\pi_{m_0} B_0 \pi_{m_0}^\nu)^{E, \nu}$ denotes an average over $\Omega_E^\nu$ with respect to Liouville measure. On average, therefore, expectation values of observables in eigenstates of the Hamiltonian semiclassically converge to a weighted average of the ‘classical’ projections $\pi_{m_0} B_0 \pi_{m_0}^\nu$ of the
‘classical’ observable $B_0(p, x)$; the weights being determined by the relative volumes of the respective energy shells. The factor $\frac{1}{2}$ accounts for the two dimensions of the spin space (i.e. the range of $\pi_0^\nu$ in $C^4$). An important consequence of the limit formula (5.2) is that only the diagonal part $\hat{B}_d$ of an observable gives a non-vanishing contribution to the semiclassical average.

To prove quantum ergodicity now requires to combine Szegő-type limits with the Egorov-Theorem 4.2. The latter, however, is only concerned with block-diagonal observables. We therefore consider only observables of the type $\hat{P}_\nu \hat{B} \hat{P}_\nu$. For these Proposition 5.1 relates expectation values of $\hat{B}$ in the projected eigenspinors $\hat{P}_\nu \psi_n$ to the classical weighted averages of $\pi^\nu_0 B_0 \pi^\nu_0$. At least some of these projected eigenspinors can possibly vanish as $\hbar \to 0$. But Proposition 5.1 allows to conclude that a positive fraction of them retains a positive norm in the semiclassical limit; hence these can safely be normalised. As discussed in section 3 the projected (and normalised) eigenspinors in general only yield quasimodes for the Dirac-Hamiltonian with discrepancies $O(\hbar^\infty)$. Thus, quantum ergodicity in the present context is concerned with quasimodes rather than with actual eigenstates. The reason for this lies in the fact that only after the projection can one associate a definite classical dynamics to eigenspinors.

**Theorem 5.2.** Suppose that all the conditions hold that have previously been imposed on the Dirac-Hamiltonian $\hat{H}_D$, as well as that the skew product flow $Y^t_\nu (\nu \in \{+, -\} \text{ fixed})$ defined in eq. (2.6) is ergodic on $\Omega^\nu_E \times S^2$. Then in every sequence $\{\phi^\nu_n\}_{n \in \mathbb{N}}$ of normalised projected eigenspinors of $\hat{H}_D$, with $\|\hat{P}_\nu \psi_n\| \geq \delta$ ($\delta > 0$ small enough and fixed), there exists a subsequence $\{\phi^\nu_{n_j}\}_{j \in \mathbb{N}}$ of density one, i.e.,

$$\lim_{\hbar \to 0} \frac{\#\{j; \|\hat{P}_\nu \psi_{n_j}\| \geq \delta, E_{n_j} \in I(E, \hbar)\}}{\#\{n; \|\hat{P}_\nu \psi_n\| \geq \delta, E_n \in I(E, \hbar)\}} = 1,$$

(5.3)

such that for every semiclassical observable $\hat{B} = \text{op}^W[B]$,

$$\lim_{\hbar \to 0} \langle \phi^\nu_{n_j}, \hat{B} \phi^\nu_{n_j} \rangle = \frac{1}{2} \text{tr} (\pi^\nu_0 B_0 \pi^\nu_0) E^\nu_{\nu, E}.$$

(5.4)

The subsequence $\{\phi^\nu_{n_j}\}_{j \in \mathbb{N}}$ can be chosen to be independent of the observable $\hat{B}$.

The statement of this theorem can be made more transparent, if one chooses an explicit phase space representation of the quasimodes $\phi^\nu_n$. E.g., if one introduces the matrix valued Wigner transform

$$W[\psi](p, x) := \int e^{-\frac{i}{\hbar}p \cdot y} \psi(x - \frac{1}{2}y) \otimes \psi(x + \frac{1}{2}y) \, dy$$

(5.5)

of $\psi \in \mathcal{H}$, the expectation value of $\hat{B}$ in a state $\phi^\nu_n$ reads

$$\langle \phi^\nu_n, \hat{B} \phi^\nu_n \rangle = \frac{1}{(2\pi \hbar)^d} \int \int \text{tr}(W[\phi^\nu_n](p, x) (\pi^\nu_0 B_0 \pi^\nu_0)(p, x)) \, dp \, dx \left(1 + O(\hbar)\right).$$

(5.6)
The conclusion (5.4) of Theorem 5.2 can therefore be rephrased in terms of the Wigner transform (5.5) as

$$
\lim_{\hbar \to 0} \frac{1}{(2\pi\hbar)^d} W[\phi_{\nu n_j}](p, x) = \frac{1}{2\text{vol } \Omega_E^\nu} \delta(h_{\nu}(p, x) - E) \mathbb{1}_2.
$$

(5.7)

Here the limit along the density-one subsequence \(\{\phi_{\nu n_j}\}_{j \in \mathbb{N}}\) has to be understood in a weak sense, namely after integration with \(\pi_0^\nu B_0 \pi_0^\nu\). Thus, the Wigner transforms of the quasimodes converge to a uniform distribution on the energy shell \(\Omega_E^\nu\). In the matrix aspect, which represents spin, the result is an equidistribution of ‘spin up’ and ‘spin down’. With an appropriate Weyl calculus for spin the latter can also be rephrased in terms of an equidistribution on the unit sphere \(S^2\), see [3, 2].

In Theorem 5.2 it was supposed that one of the two skew product flows \(Y_{\nu}^\nu\) is ergodic; no assumption was made about the complementary classical system. If, however, the other energy shell at \(E\) is empty, the norms \(\|\hat{P}^\nu \psi_n\|\) must converge to one. Then the statement of the theorem applies to a density-one subsequence \(\{\psi_{n_j}\}_{j \in \mathbb{N}}\) of all eigenspinors \(\psi_n\) with \(E_n \in I(E, \hbar)\) in the form

$$
\lim_{\hbar \to 0} \langle \psi_{n_j}, \hat{P}^\nu \hat{B} \hat{P}^\nu \psi_{n_j} \rangle = \frac{1}{2} \text{tr} \left( \pi_0^\nu B_0 \pi_0^\nu \right)^{E,\nu}.
$$

(5.8)

In this case quantum ergodicity is hence concerned with the eigenspinors themselves. For a Dirac-Hamiltonian this situation is not untypical, since both the particle and the antiparticle energy shell to be non-empty at the same energy requires very strong potentials, with magnitudes comparable to the rest energy \(mc^2\). But then also the description of a relativistic quantum system in a single-particle framework begins to become questionable.

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