Solitary waves of nonlinear nonintegrable equations *

Robert CONTE † and Micheline MUSSETTE ‡
† Service de physique de l’état condensé (URA no. 2464)
CEA–Saclay, F–91191 Gif-sur-Yvette Cedex, France
E-mail: Conte@drecam.saclay.cea.fr
‡ Dienst Theoretische Natuurkunde, Vrije Universiteit Brussel
Pleinlaan 2, B–1050 Brussels, Belgium
E-mail: MMusette@vub.ac.be

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Abstract

Our goal is to find closed form analytic expressions for the solitary waves of nonlinear nonintegrable partial differential equations. The suitable methods, which can only be nonperturbative, are classified in two classes.

In the first class, which includes the well known so-called truncation methods, one *a priori* assumes a given class of expressions (polynomials, etc) for the unknown solution; the involved work can easily be done by hand but all solutions outside the given class are surely missed.

In the second class, instead of searching an expression for the solution, one builds an intermediate, equivalent information, namely the first order autonomous ODE satisfied by the solitary wave; in principle, no solution can be missed, but the involved work requires computer algebra.

We present the application to the cubic and quintic complex one-dimensional Ginzburg-Landau equations, and to the Kuramoto-Sivashinsky equation.

Keywords: solitary waves, complex one-dimensional Ginzburg-Landau equation, Kuramoto-Sivashinsky equation, complex Swift-Hohenberg equation, Briot and Bouquet equations, elliptic function, genus, Painlevé property, meromorphy, truncation.

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1 Introduction

Many nonlinear partial differential equations (PDEs) encountered in physics are autonomous, i.e. do not depend explicitly on the independent variables $x$ (space) and $t$ (time). In such a case, they admit a reduction, called traveling wave reduction, to an autonomous nonlinear ordinary differential equation (ODE), defined in the simplest case by $u(x,t) = U(\xi)$, $\xi = x - ct$, with $c$ a constant speed. A solitary wave is then defined as any solution to this nonlinear autonomous ODE. Physically relevant solitary waves must satisfy some decaying condition when $\xi$ goes to $\pm \infty$.

The distinctive feature of this chapter is to explain the methods to find closed form expressions to these solitary waves when the PDE and the reduced ODE are algebraic and nonintegrable. These solitary waves may have the topology of a front (for instance $\tanh$), a pulse (for instance $\sech$), a source, a sink, etc, but we will not discard an apparently physically uninteresting solution, because it might appear interesting to another field.

Why “algebraic”? This only excludes equations impossible to convert to an algebraic form. For instance, the sine-Gordon equation $u_{xt} - \sin u = 0$ is not excluded because it is algebraic in $e^{iu}$.

Why “nonintegrable”? Because the integrable ones (nonlinear Schrödinger (NLS), coupled NLS in the Manakov case, etc) are “easy” to solve using powerful tools like the inverse spectral transform (IST) \cite{1}. The difficulty with the nonintegrable equations is the absence of a general method to achieve the goal.

Why “autonomous”? Irrelevant for the truncation methods (section \ref{sec:truncation}), this restriction is essential for the mathematical method of section \ref{sec:analytic}. Physically, this is not an important restriction, since many interesting PDEs are autonomous, see the examples below.

Why “closed form expressions”? Because a solution represented by a series can be misleading (illusoire, used to say Painlevé). Consider for instance a chaotic deterministic dynamical system for which no analytic solution exists. Around a regular point, it admits a solution represented by a Taylor series, but one can conclude nothing before some analytic continuation has been performed.

On the contrary, the Laurent series around a movable singularity (i.e. one whose location depends on the initial conditions) provides some constructive information (see section \ref{sec:laurent}) about the (global) integrability of the equation.

The methods described here are all based on the a priori singularities \cite{26} of the solutions of the given ODE. In particular, we do not consider the group theoretical methods \cite{40}.

These methods can be applied mainly to dissipative equations of importance in physics, nonlinear optics, mechanics, etc. Our specific examples are the following.

1. The one-dimensional cubic complex Ginzburg-Landau equation (CGL3)

$$iA_t + pA_{xx} + q|A|^2A - i\gamma A = 0, \quad pq\gamma \neq 0, \quad \text{Im}(p/q) \neq 0, \quad (A, p, q, \gamma) \in \mathbb{C}, \quad \gamma \in \mathbb{R},$$

(1)

(and its complex conjugate, i.e. a total differential order four), in which $p, q, \gamma$ are constants, a generic equation which describes many physical phenomena, such as the propagation of a signal in an optical fiber \cite{22}, spatiotemporal intermittency in spatially extended dissipative systems \cite{22,14,34}. We will restrict ourselves to the CGL3 case properly said $\text{Im}(p/q) \neq 0$.

For analytic results on two coupled CGL3 equations, see \cite{10}.

2. The Kuramoto and Sivashinsky (KS) equation,

$$\varphi_t + \nu \varphi_{xxxx} + b \varphi_{xxx} + \mu \varphi_{xx} + \varphi \varphi_x = 0, \quad \varphi \in \mathbb{C}, \quad (\nu, b, \mu) \in \mathbb{R}, \quad \nu \neq 0.$$

(2)

in which $\nu, b, \mu$ are constants. This PDE is obeyed by the variable $\varphi = \text{arg} A$ of the above field $A$ of CGL3 under some limit \cite{32,24}, hence its name of phase turbulence equation.

3. The one-dimensional quintic complex Ginzburg-Landau equation (CGL5),

$$iA_t + pA_{xx} + q|A|^2A + r|A|^4A - i\gamma A = 0, \quad pr \neq 0, \quad \text{Im}(p/r) \neq 0, \quad (A, p, q, r, \gamma) \in \mathbb{C}, \quad \gamma \in \mathbb{R}. \quad (3)$$

4. The Swift-Hohenberg equation \cite{37,21}

$$iA_t + bA_{xxxx} + pA_{xx} + q|A|^2A + r|A|^4A - i\gamma A = 0, \quad br \neq 0, \quad (A, b, p, q, r, \gamma) \in \mathbb{C}, \quad \gamma \in \mathbb{R}, \quad (4)$$

in which $b, p, q, r, \gamma$ are constants.
For the CGL3, KS, and Swift-Hohenberg equations (with one exception, KS with \( b^2 = 16\mu \nu \)),
all the solitary wave solutions \( |A|^2 = f(\xi), \varphi = \Phi(\xi), \xi = x - ct \), which are known hitherto are just polynomials in \( \tanh k\xi \).

So, two natural questions arise:

1. Is it possible or impossible that other solitary waves exist?
2. If such other solutions may exist, can one find not just a few more but all of them?

Let us from now on denote the reduced ODE as
\[
E(u^{(N)}, \ldots, u', u) = 0, \quad ' = \frac{d}{d\xi}, \quad \xi = x - ct.
\]

and let us assume that it is also nonintegrable.

The chapter is organized as follows.

In section 2, one recalls the analytic expressions of the known solitary waves of the examples. This list, to be retrieved or augmented by the singularity based methods, will allow us to rate the efficiency of the various methods.

In section 3, one investigates the amount of integrability of the equation, by applying the so-called Painlevé test. More specifically, one checks the existence of particular solutions which admit a local representation as a Laurent series. This allows us to count the gap, strictly positive because of the assumed nonintegrability, between the differential order of the ODE and the maximal number of available integration constants.

In section 4, we discuss the choice of the suitable dependent variable to be used in the subsequent sections.

In section 5, one tries to obtain a global representation for the local information (Laurent series) previously found. One introduces the distinction between two main classes of methods, according to the following criteria: i) computations easy enough to be carried out by hand, ii) generality or particularity of the expected solution.

The section 6 is devoted to the first class of methods, which are known as “truncation methods”. The input is a class of a priori expressions for \( u \) (usually polynomials), in some intermediate variable \( \chi \) which satisfies a given first order ODE (e.g. Riccati, Weierstrass, Jacobi). Then, by a direct computation, easy to carry out by hand, one checks whether there indeed exist solutions in the given class. The solutions with a simple profile (such as \( \tanh \) for a front, \( \text{sech} \) for a pulse), are easily found by this class of methods.

In the second class of methods \[27\], presented in section 7, rather than directly looking for the unknown solution
\[
u = f(\xi - \xi_0),
\]
in which \( \xi_0 \) is an arbitrary complex constant, one looks as an intermediate information for the first order nonlinear ODE
\[
F(u, u') = 0,
\]

obtained by eliminating \( \xi_0 \) between \(6\) and its derivative, in which \( F \) is as unknown as \( f \). Indeed, provided that \( f \) is singlevalued, by a classical theorem recalled in Appendix, there is equivalence between the knowledge of the solution \( f \) and that of the subequation \( F \) which it satisfies. The way to obtain the subequation \( F \) is to require that it be satisfied by the Laurent series obtained in a previous step.

The difference between the two classes of methods is the following. The solutions found by the first class of methods can only be a subset of those found by the second class. However, the computations involved can easily be performed by hand for the first class, while for the second class a computer algebra package is highly recommended.

### 2 The known solutions of the examples

None of the expressions listed below represents the largest analytic solution which one could find, and their distance to this largest, yet unknown, analytic solution will be computed precisely in section 3.
2.1 CGL3

The traveling wave reduction of (11)

\[ A(x, t) = \sqrt{M(\xi)}e^{i(-\omega t + \varphi(\xi))}, \xi = x - ct, (c, \omega, M, \varphi) \in \mathcal{R}, \]

\[ \frac{M''}{2M} - \frac{M'^2}{4M^2} + i\varphi'' - \varphi'^2 + i\frac{\varphi'}{M} - i\frac{c}{2p} \frac{M'}{M} + \frac{1}{p} (c\varphi' + \omega) + \frac{q}{p} M - \frac{i\gamma}{p} = 0, \tag{9} \]

introduces two additional real constants \((c, \omega)\) and it is convenient to define the six real parameters

\[ d_r, d_i, s_r, s_i, g_r, g_i, \]

\[ d_r + id_i = \frac{q}{p}, s_r - is_i = \frac{1}{p}, g_r + ig_i = \frac{\gamma + i\omega}{p} + \frac{1}{4} c^2 s_i^2. \tag{10} \]

In the CGL3 case properly said \(d_i \neq 0\) to which we restrict here, only three solutions are currently known. Denoting \(A_0^2\) and \(\alpha\) two real constants defined by the complex equation

\[ (-1 + i\alpha)(-2 + i\alpha)p + A_0^2q = 0, \tag{11} \]

these three solutions are the following.

1. A heteroclinic source or propagating hole \([11]\)

\[
\begin{aligned}
A &= A_0 \left[ \frac{k}{2} \tanh \frac{k}{2} \xi - \frac{iqp_i}{2(1 - i\alpha)p|p|^2d_i} \right] e^{i[\alpha \log \cosh k \xi + \frac{q_i}{2|p|^2d_i} c \xi - \omega t]}, \\
\frac{i\gamma - \omega}{p} &= \left( \frac{c}{2p} \right)^2 - (2 - 3i\alpha) \frac{k^2}{4},
\end{aligned}
\tag{12}
\]

in which the velocity \(c\) is arbitrary. Indeed, the real and imaginary parts of the last equation define the value of \(\omega\) and a linear relation between \(c^2\) and \(k^2\), see [11] Eq. (79)].

2. A homoclinic pulse or solitary wave \([30]\)

\[
\begin{aligned}
A &= A_0 (-ik \text{sech} kx) e^{i[\alpha \log \cosh kx - \omega t]}, \\
i\gamma - \omega &= (1 - i\alpha)^2 k^2, c = 0.
\end{aligned}
\tag{13}
\]

3. A heteroclinic front or shock \([28]\)

\[
\begin{aligned}
A &= A_0 \frac{k}{2} \left[ \tanh \frac{k}{2} \xi + \varepsilon \right] e^{i[\alpha \log \cosh \frac{k}{2} \xi + \frac{3p_r + \alpha p_i}{6|p|^2d_i} c \xi - \omega t]}, \\
i\gamma - \omega &= \left( \frac{c}{2p} \right)^2 + \frac{k^2}{4}, \frac{k}{2} = \varepsilon \frac{p_r c}{6|p|^2d_i}.
\end{aligned}
\tag{14}
\]

None of these three solutions requires any constraint on \(p, q, \gamma\), and they depend on an additional sign resulting from the resolution of (11),

\[ A_0^2 = \frac{3(3d_r + \varepsilon_1 \Delta)}{2d_i^2}, \alpha = \frac{3d_r + \varepsilon_1 \Delta}{2d_i}, \Delta = \sqrt{9d_r^2 + 8d_i^2}, \varepsilon_1^2 = 1. \tag{15} \]

In all of them \(M\) is a much simpler expression, namely a second degree polynomial in

\[ \tau = (k/2) \tanh k\xi/2, k^2 \in \mathcal{R}. \tag{16} \]

Therefore, if one wants to extend the three above solutions, it is advisable to eliminate \(\varphi\) between the system of two real equations equivalent to (11),

\[
\begin{aligned}
\frac{M''}{2M} - \frac{M'^2}{4M^2} - (c M' + \gamma) + s_r (c \varphi' + \omega) + d_r M = 0, \\
\varphi'' + \varphi' \frac{c M'}{M} - s_r (c \varphi' + \omega) - s_i (c \varphi' + \omega) + d_i M = 0,
\end{aligned}
\tag{17}
\]

\[ M'' = \frac{M'^2}{4M^2} - \varphi'^2 - s_i (c M' + \gamma) + s_r (c \varphi' + \omega) + d_r M = 0, \]

\[ \varphi'' + \varphi' \frac{c M'}{M} - s_r (c \varphi' + \omega) - s_i (c \varphi' + \omega) + d_i M = 0, \]
which results in
\[ \varphi' = \frac{cs_r}{2} + \frac{G' - 2cs_rG}{2M^2(g_r - d_r M)}, \quad \left(\varphi' - \frac{cs_r}{2}\right)^2 = \frac{G}{M^2}, \quad (18) \]
\[ (G' - 2cs_rG)^2 - 4GM^2(d_r M - g_r)^2 = 0, \quad (19) \]
\[ G = \frac{1}{2} MM'' - \frac{1}{4} M'^2 - \frac{cs_i}{2} MM' + d_r M^3 + g_r M^2, \quad (20) \]
and to concentrate on the single third order equation \[19\] for \( M = |A|^2 \).

2.2 KS

The traveling wave reduction is defined as
\[ \varphi(x, t) = c + u(\xi), \quad \xi = x - ct, \quad \left[\nu u''' + bu'' + \mu u' + \frac{u^2}{2}\right]' = 0, \quad (\nu, \mu) \in \mathbb{R}, \quad \nu \neq 0, \quad (21) \]
which integrates once as
\[ \nu u''' + bu'' + \mu u' + \frac{u^2}{2} + A = 0, \quad (22) \]
in which \( A \) in an integration constant. It has a chaotic behavior \[22\], and it depends on two dimensionless parameters, \( b^2/(\mu \nu) \) and \( \nu A/\mu^3 \).

The known solutions are one elliptic solution, six trigonometric solutions, and one rational solution.

The unique known elliptic solution exists for one constraint between the parameters \( \nu, b, \mu \) of the PDE \[12, 18\],
\[ b^2 - 16\mu \nu = 0 : \quad u = -60\nu \varphi' - 15b \varphi - \frac{b \mu}{4\nu}, \quad g_2 = \frac{\mu^2}{12\nu^2}, \quad g_3 = \frac{13\mu^3 + \nu A}{1080\nu^3}. \quad (23) \]
in which \( \varphi \) is the elliptic function of Weierstrass, defined by the ODE
\[ \varphi'^2 = 4\varphi^3 - g_2 \varphi - g_3. \quad (24) \]

The six trigonometric solutions \[19\] \[16\], all of them rational in \( e^{k \xi} \), exist at the price of one constraint between \( \nu, b, \mu \) and another one on \( A \),
\[ u = 120\nu r^3 - 15br^2 + \left(\frac{60}{19} - 30\nu k^2 - \frac{15b^2}{4 \times 19\nu}\right) r + \frac{5}{2} \frac{bk^2}{32 \times 19\nu^2} - \frac{13b^3}{4 \times 19\nu^2} + \frac{7\mu b}{4 \times 19\nu}, \quad (25) \]
\[ \tau = \frac{k}{2} \tanh \frac{k}{2}(\xi - \xi_0), \]
the allowed values being listed in Table 1.

| \( b^2/(\mu \nu) \) | \( \nu A/\mu^3 \) | \( \nu k^2/\mu \) |
|-----------------|-----------------|-----------------|
| 0 | -4950/19^3, 450/19^3 | 11/19, -1/19 |
| 144/47 |-1800/47^3 | 1/47 |
| 256/73 | -4050/73^3 | 1/73 |
| 16 | -18, -8 | 1, -1 |

Finally, the unique known rational solution
\[ b = 0, \mu = 0, \quad A = 0 \quad u = 120\nu(\xi - \xi_0)^{-3}, \quad (26) \]
This linear operator $D$ and the convenient constants are known.

A heteroclinic front or shock [35],

$$\psi$$ is a limit of all the above solutions.

A nice property common to all those solutions is to admit the representation

$$u = D \log \psi + \text{constant}, \quad D = 6D \frac{d^3}{d\xi^3} + 15b \frac{d^2}{d\xi^2} + \frac{15(16\mu - b^2)}{7b\nu} \frac{d}{d\xi},$$  \hspace{1cm} (27)

in which $\psi$ is an entire function (i.e. one without any singularity at a finite distance) whose ODE is easy to build, respectively,

$$(-\log \psi)'' - 4(-\log \psi)'^3 + g_2(-\log \psi)'' + g_3 = 0,$$  \hspace{1cm} (28)

$$\psi'' - \frac{k^2}{4} \psi = 0,$$  \hspace{1cm} (29)

$$\psi'' = 0.$$  \hspace{1cm} (30)

This linear operator $D$, which captures the singularity structure, is called the singular part operator.

### 2.3 CGL5

The traveling wave reduction is quite similar to that of CGL3, so we do not repeat it. Again, $A_0^2$ and $\alpha$ denote two real constants defined by the complex equation

$$(-1/2 + i\alpha)(-3/2 + i\alpha)p + A_0^2 r = 0,$$  \hspace{1cm} (31)

and the convenient constants are

$$e_r + i e_t = \frac{r}{p}, \quad s_r - is_t = \frac{1}{p}, \quad g_r + ig_t = \frac{\gamma + i\omega}{p} + \frac{1}{4} e^2 s_r^2.$$  \hspace{1cm} (32)

In the CGL5 case properly said $e_t \neq 0$ to which we restrict here, only two solutions are currently known.

1. A heteroclinic front or shock [35],

$$A = A_0 \left( \frac{k \sinh ka}{\cosh k\xi + \cosh ka} + r_0 \right)^{1/2} e^{i[\alpha \log \cosh k\xi/2 + K\xi - \omega t]}, \quad \varepsilon^2 = 1,$n

$$(-1/2 + i\alpha) [i(c - 2pK) + 2\varepsilon(-2 + i\alpha)pr] p - A_0^2 q = 0,$$  \hspace{1cm} (33)

$$\frac{i\gamma - \omega}{p} = \left( \frac{c}{2p} \right)^2 - \left( K - \frac{c}{2p} + \varepsilon(1 - i\alpha)k/2 \right)^2.$$  \hspace{1cm}

The number of constraints among $(p, q, r, \gamma)$ is either two (case $s_t = 0$, $c$ arbitrary), or zero (case $s_t \neq 0$, with a fixed velocity).

2. A homoclinic source or sink [23],

$$A = A_0 \left( \frac{k \sinh ka}{\cosh k\xi + \cosh ka} + r_0 \right)^{1/2} e^{i[\alpha \log \cosh k\xi + \cosh ka] + K\xi - \omega t]},$$  \hspace{1cm}

$$c - 2pK = 0, \quad \text{which implies} \quad c p = 0,$n

$$(-1/2 + i\alpha) [-2k\mu_0(1 - i\alpha) + 2(-2 + i\alpha)r_0] p - A_0^2 q = 0,$$  \hspace{1cm} (34)

$$\frac{i\gamma - \omega}{p} = \left( \frac{c}{2p} \right)^2 + \left( 1/2 - i\alpha \right)^2 k^2 + \left( 3 - 10i\alpha - 4\alpha^2 \right)k\mu_0 r_0 + \left( 3 - 8i\alpha - 2\alpha^2 \right) r_0^2/2, -$$  \hspace{1cm}

$$r_0 \left( r_0^2 + 2k\mu_0 r_0 + k^2 \right) = 0.$$  \hspace{1cm}

in which $K, k^2, r_0, \mu_0 = \coth ka$ are real constants. The number of constraints among $(p, q, r, \gamma)$ is either two (case $s_t = 0$, with $c$ arbitrary), or one (case $s_t \neq 0, c = 0$).

Each of these solutions depends on two additional signs arising from the resolution of (31)

$$A_0^2 = \varepsilon_1 \sqrt{\frac{2e_r + \varepsilon_1 \Delta}{\varepsilon_1^2}}, \quad \alpha = \frac{2e_r + \varepsilon_1 \Delta}{2e_t}, \quad \Delta = \sqrt{4e_t^2 + 3e_r^2}, \quad \varepsilon_1^2 = \varepsilon_2^2 = 1.$$  \hspace{1cm} (35)
2.4 Swift-Hohenberg

Again, $A_0^4$ and $\alpha$ denote two real constants defined by

$$\begin{aligned}
(-1 + i\alpha)(-2 + i\alpha)(-3 + i\alpha)(-4 + i\alpha)b + A_0^4r &= 0. \\
\text{(36)}
\end{aligned}$$

In the case $\text{Im}(r/b) \neq 0$ to which we restrict ourselves, only two solutions seem to be currently known.

1. A stationary front [24, Eq. (127)]

$$\begin{aligned}
\begin{cases}
A &= A_0(k/2) \tanh kx/2 e^{i[{\alpha \text{Log cosh } kx/2 - \omega t}]}, \\
k^2 &= \frac{2}{5(2 - i\alpha)b} \left[ \frac{1 - i\alpha}{2 - i\alpha} + p \right], \\
i\gamma - \omega &= \frac{16 - 30i\alpha - 15\alpha^2}{16}bk^4 + \frac{-2 + 3i\alpha}{4}\nu k^2.
\end{cases}
\end{aligned}$$

2. A stationary pulse [24, Eq. (119)]

$$\begin{aligned}
\begin{cases}
A &= A_0(-ik \text{sech } kx) e^{i[{\alpha \text{Log cosh } kx - \omega t}]}, \\
k^2 &= \frac{-1}{2(5 - 4i\alpha - \alpha^2)b} \left[ \frac{A_0^2\nu}{(1 - i\alpha)(2 - i\alpha)} + p \right], \\
i\gamma - \omega &= (1 - i\alpha)^2 \left[ (1 - i\alpha)^2bk^4 + \nu k^2 \right].
\end{cases}
\end{aligned}$$

In both solutions, the number of constraints on $(b, p, q, r, \gamma)$ is two (defined by the vanishing of the imaginary part of the relations for $k^2$ and $i\gamma - \omega$).

3 Investigation of the amount of integrability

3.1 Counting arguments based on singularity analysis

Because the ODE [13] is assumed nonintegrable, the number of integration constants which can be present in any closed form solution is strictly smaller than the differential order of the ODE. Let us first compute precisely this difference, an indicator of the amount of integrability of the ODE. The technique to do so is just the Painlevé test (see Ref. [7] for the basic vocabulary of this technique). Let us present it on the KS example [29].

Looking for a local algebraic behaviour near a movable singularity $x_0$ (movable means: which depends on the initial conditions),

$$u \sim x \rightarrow x_0 \ u_0 \chi^p, \ u_0 \neq 0, \ \chi = x - x_0,$$

one first obtains the usual balancing conditions (here, between the highest derivative and the nonlinearity)

$$\begin{aligned}
p - 3 &= 2p, \ p(p - 1)(p - 2)\nu u_0 + \frac{u_0^2}{2} = 0, \\
\text{(40)}
\end{aligned}$$

easily solved as

$$\begin{aligned}
p &= -3, \ u_0 = 120\nu, \\
\text{(41)}
\end{aligned}$$

which yields the Laurent series,

$$u^{(0)} = 120\nu \chi^{-3} - 15b\chi^{-2} + \frac{15(16\nu - b^2)}{4 \times 19\nu} \chi^{-1} + \frac{13(4\nu - b^2)b}{32 \times 19\nu^2} + O(\chi^1),$$

from which two out of the three arbitrary constants are missing. These two constants appear in perturbation [14].

$$u = u^{(0)} + \varepsilon u^{(1)} + \varepsilon^2 u^{(2)} + \ldots,$$

$$\text{(43)}$$
in which the small parameter $\varepsilon$ is not in the ODE \( \text{(22)} \). The linearized equation around $u^{(0)}$

$$\left( \nu \frac{d^3}{dx^3} + b \frac{d^2}{dx^2} + \mu \frac{d}{dx} + u^{(0)} \right) u^{(1)} = 0,$$

(44)

is then of the Fuchsian type near $x = x_0$, with an indicial equation ($q = -6$ denotes the singularity degree of the lhs $E$ of \( \text{(22)} \))

$$\lim_{\chi \to 0} \chi^{-q}(\nu \partial_3^3 + u_0 \partial_3^0)\chi^{j+p}$$

$$= \nu(j - 3)(j - 4)(j - 5) + 120 \nu = \nu(j + 1)(j^2 - 13j + 60)$$

(45)

$$= \nu(j + 1) \left( j - \frac{13 + i\sqrt{71}}{2} \right) \left( j - \frac{13 - i\sqrt{71}}{2} \right) = 0.$$ 

(46)

The local representation of the general solution,

$$u(x_0, \varepsilon c_+, \varepsilon c_-) = 120 \nu \chi^{-3} \{\text{Taylor}(\chi) + \varepsilon [c_+ \chi^{(13+i\sqrt{71})/2} \text{Taylor}(\chi) + c_- \chi^{(13-i\sqrt{71})/2} \text{Taylor}(\chi)] + O(\varepsilon^2) \},$$

in which “Taylor” denotes converging series of $\chi$, does depend on three arbitrary constants ($x_0, \varepsilon c_+, \varepsilon c_-$) (the Fuchs index $-1$ only represents a shift of $x_0$). The dense movable branching arising from the two irrational indices characterizes the chaotic behaviour, and the only way to remove it is to require $\varepsilon c_+ = \varepsilon c_- = 0$, i.e. $\varepsilon = 0$, thus restricting to a single arbitrary constant the analytic part of the solution.

To summarize, let us introduce two notions. The first one is trivial. One calls irrelevant any integration constant which, because of some symmetry, is always present in any solution. The KS ODE has one such irrelevant integration constant, the origin $\xi_0$ of $\xi$, and we will systematically omit to write it. The traveling wave reduction of CGL3 has two irrelevant integration constants, the origins of $\xi$ and $\varphi$, etc.

The second notion is quite an important property of the equation. We will call unreachable any constant of integration which cannot participate to any closed form solution. The KS ODE has two unreachable integration constants.

We will also call general analytic solution the closed form solution which depends on the maximal possible number of reachable integration constants, and our goal is precisely to exhibit a closed form expression for this general analytic solution, whose local representation is a Laurent series like \( \text{(42)} \).

The above notions (irrelevant, unreachable) are attached to an equation, not to a solution. Let us similarly introduce two integer numbers, attached to a solution, allowing one to quantify how far this solution is distant from the general analytic solution.

We will call deficiency of a closed form solution the number of reachable integration constants, excluding the irrelevant ones, which are missing in this solution. In KS for instance, the elliptic solution has a zero deficiency ($A$ is arbitrary), and all the trigonometric solutions have deficiency one ($A$ is fixed).

Let us finally define the codimension of a closed form solution of an equation as the number of constraints on the fixed parameters (fixed means: which occur in the definition of the equation). Thus, the elliptic and trigonometric solutions of KS have codimension one, and the rational solution has codimension two.

### 3.2 Evidence for unknown solutions

Computer simulations as well as real experiments (for a recent review, see \[34\]) sometimes display regular patterns in the $(x,t)$ plane, and some of them are indeed described by some analytic solution. For the remaining patterns, the guess is that there should exist analytic expressions, to be found, corresponding to these patterns.

For the KS equation \( \text{(2)} \), one thus observes a homoclinic solitary wave \( \text{[39 Fig. 7]} \) $\varphi = f(\xi)$, $\xi = x - ct$, while all solutions known to date are heteroclinic.
For the CGL3 equation, it has been predicted the existence of a fourth physically interesting solution, which is a codimension-one homoclinic hole solution with an arbitrary velocity $c$.

Table 2 gathers the current state of the known solutions for the various nonintegrable equations considered in this chapter. In Ref. [14], another counting, based on the various possible topological structures, is made for CGL3 and provides the same results.

Table 2: Integer numbers rating the particular solutions of a nonintegrable equation. The vocabulary (irrelevant, unreachable, deficiency, codimension) is defined in section 3.1. The column “Available” indicates the number of relevant, reachable integration constants, this is the algebraic sum “Order” – “Irrelevant” – “Unreachable”. The last two columns indicate the properties of the solutions in their order of appearance section 2. The best solution would be one with deficiency and codimension both equal to zero, with the reduction parameters $c$ and $\omega$ arbitrary.

| Equation | Order | Irrelevant | Unreachable | Available | Deficiency | Codimension |
|----------|-------|------------|-------------|-----------|------------|-------------|
| CGL3     | 4     | 2 = $\xi_0$, $\varphi_0$ | 2           | 0         | 0,0,0      | 1,2,2       |
| CGL5     | 4     | 2 = $\xi_0$, $\varphi_0$ | 2           | 0         | 0,0        | 1,1         |
| KS       | 4     | $1 = \xi_0$ | 2           | 1         | 0,1,1,1,1 | 1,1,1,1,1,1,2 |
| SH       | 8     | 2 = $\xi_0$, $\varphi_0$ | 6           | 0         | 0,0        | 2,2         |

4 Selection of possibly single valued dependent variables

Whatever be the class of methods to be applied, a prerequisite is to determine a variable whose dominant behaviour is single valued and which satisfies some algebraic ODE (or more generally PDE). This is the case for KS since the solution $p$ of (40) is integer, but in CGL3, CGL5, or SH this is the case of neither $(A, \overline{A})$, nor arg $A$, and for CGL5 this is not even the case of $|A|$. Indeed, considering CGL3 for instance, the dominant terms are

$$pA_{xx} + q|A|^2 A,$$

and one easily checks that $|A|$ generically behaves like simple poles. Let us therefore define the dominant behaviour of the two fields $(A, \overline{A})$ as

$$A \sim A_0 \chi^{-1 + i\alpha}, \quad \overline{A} \sim \overline{A_0} \chi^{-1 - i\alpha}, \quad A_0 \in \mathbb{C}, \quad \alpha \in \mathbb{R},$$

in which $(A_0, \alpha)$ are constants to be determined. The resulting complex equation (equivalent to two real equations)

$$(-1 + i\alpha)(-2 + i\alpha)p + |A_0|^2 q = 0,$$

is precisely the one artificially introduced earlier as (11), and solved in (15), the convention that $A_0$ is real being allowed by the phase invariance of CGL3. The same applies to CGL5

$$A \sim A_0 \chi^{-1/2 + i\alpha}, \quad \overline{A} \sim \overline{A_0} \chi^{-1/2 - i\alpha},$$

see (31) and (35), and to SH,

$$A \sim A_0 \chi^{-1 + i\alpha}, \quad \overline{A} \sim \overline{A_0} \chi^{-1 - i\alpha},$$

see (36).

In all three examples CGL3, CGL5, SH, the variable $M = |A|^2$ satisfies an algebraic ODE, which can be built by elimination of arg $A$, and it has a single valued dominant behaviour (respectively movable double poles, simple poles, double poles). Moreover, again for CGL3, CGL5, SH, for all the solitary wave solutions which are known to date (see section 2), this variable $M$ is represented
by quite simple mathematical expressions, namely either polynomials in one elementary variable $\tau$, Eq. (16), which satisfies a Riccati equation
\[
\frac{d}{dz}\tau(z) = 1 - \tau^2, \quad \tau = \tanh z,
\] (53)
or (source solution of CGL5, Eq. (34)) polynomials in two elementary variables $(\sigma, \tau)$ which satisfy a projective Riccati system \[9\]
\[
\frac{d}{dz}\tau = 1 - \tau^2 - \mu_0 \sigma, \quad \frac{d}{dz}\sigma = -\sigma \tau, \quad \sigma^2 - \tau^2 - 2\mu_0 \sigma + 1 = 0,
\] (54)
in which $\mu_0$ is a constant, and whose solution can be expressed as
\[
\tau = \frac{\sinh z}{\cosh z + \cosh ka}, \quad \sigma = \frac{\sinh ka}{\cosh z + \cosh ka}, \quad \mu_0 = \coth ka.
\] (55)

When $\mu_0(\mu_0^2 - 1) = 0$, the class of polynomials in $(\sigma, \tau)$ degenerates to the class of polynomials in $(\text{sech}, \tanh)$.

Therefore $M = |A|^2$ will be our best choice to search for closed form solutions of CGL3, CGL5, SH.

Remark. Despite the multivalued dominant behaviour of the complex amplitude $A$ of CGL3 and SH, one can define two variables with a single valued dominant behaviour. In this complex modulus representation \[9\]
\[
A = A_0 Z(\xi)e^{i[\Phi(\xi) - \omega t]}, \quad \overline{A} = A_0 Z(\xi)e^{-i[\Phi(\xi) - \omega t]},
\] (56)
with $Z$ complex and $\Phi$ real, the dominant behaviour is
\[
Z \sim \chi^{-1}, \quad \overline{Z} \sim \chi^{-1}, \quad \Phi' \sim \alpha \chi^{-1},
\] (57)
and the truncation of $(Z, \overline{Z}, \Phi')$ might prove to be much more economical than that of $M$. All the solutions listed in section \[2\] for CGL3, CGL5, SH have been written in this representation.

5 On the price to obtain closed form expressions

Let us now give some details on the distinction between the two main classes of methods outlined in the introduction.

In the first class of methods, one gives as an input some class of expressions $f(\xi)$ (for instance polynomials in $\text{sech} \, k\xi$ and $\tanh \, k\xi$), and by a direct computation one checks whether there indeed are some solutions in the given class. We will call for shortness these methods sufficient, because they for sure miss any solution outside the given class, e.g. for the ODE
\[
M'^2 + \left(12M^2 - \frac{3}{2}\right)M' + 36M^4 - \frac{17}{2}M^2 + \frac{1}{2} = 0,
\] (58)
its solution since it is rational in $\tanh k\xi$,
\[
M = \frac{\tanh(\xi - \xi_0)}{2 + \tanh^2(\xi - \xi_0)}.
\] (59)

In the second class of methods, the search for first order autonomous subequations \[7\] requires no a priori assumption at all, and, from the classical results recalled in Appendix, the knowledge of the first order subequation is indeed equivalent to the knowledge of the explicit expression \[6\]. As opposed to the previous methods, which are “sufficient” as said above, the proposed method can be qualified as “necessary”.

The difference between the two classes of methods is obvious: the class of expressions $f(\xi)$ is an output of the second method, while it is an input of the first one. This is why the second method can find, if they exist, not only some but all the solutions which are elliptic or trigonometric.

Remark. The cost of the method of first order autonomous subequations is an increasing function of the positive integer $m$ occuring in \[168\], but $m$, which is an input of the method, is not bounded. Indeed, any rational function $u = P_N(\tanh(\xi - \xi_0))/P_D(\tanh(\xi - \xi_0))$ satisfies an ODE \[7\] of order one and degree $\max(N, D)$. By considering only some differential consequence of this ODE, one cannot guess the correct value of $m$ in advance.
6 First class of methods: truncations

After having selected, as indicated in section 4, dependent variables with a single valued leading behaviour, the methods called truncations consist in defining for each such dependent variable some single valued closed form class of expressions, then in checking whether there exist solutions in that class.

The class of expressions to choose as an input depends on the number of families of movable singularities of the considered dependent variable. Thus, the field \( u \) of KS has only one family, i.e. one value of \( u_0 \), while the field \( M = |A|^2 \) of CGL3, CGL5 or SH has respectively two, four, and four families. Let us start with the simplest class of expressions.

6.1 Polynomials in \( \tanh \) (one-family truncation)

The class of polynomials in \( \tanh(k/2)\xi \) is the most frequently encountered class of closed form solutions of autonomous PDEs. This fact is the direct consequence of a quite remarkable property. Indeed, by a result of Painlevé, the variable \( \tau \) in (53) is the unique variable to be at the same time single valued and closed by differentiation: if \( u \) is such a polynomial,

\[
 u = \sum_{j=0}^{-p} u_j \chi_j k^{j+p}, \quad \chi^{-1} = \frac{k}{2} \tanh \frac{k}{2}, \quad \xi = x - ct,
\]

the lhs \( E(u,x,t) \) of the equation of the PDE is also such a polynomial,

\[
 E = \sum_{j=0}^{-q} E_j \chi_j^{-q},
\]

and its identification to the null polynomial

\[
 \forall j : E_j = 0,
\]

generates the smallest possible number of determining equations \( E_j = 0 \). As compared to the Laurent series (42), the series (60) terminates, hence its name of truncation.

The truncation (60) involves only one value of \( u_0 \), it is called for this reason a one-family truncation. Let us give a few examples.

6.1.1 One-family truncation of the KS equation

The symbols \( u_0 \) and \( p \) denoting the leading behaviour of the ODE (22), the truncation (60) defines \(-q+1 = 7\) determining equations, the first four being

\[
 E_0 = -60\nu u_0 + \frac{u_0^2}{2} = 0,
\]
\[
 E_1 = 12bu_0 + (u_0 - 24\nu)u_1 = 0,
\]
\[
 E_2 = -3\mu u_0 + \frac{57}{2}k^2\nu u_0 + 6bu_1 + \frac{1}{2}u_1^2 + (u_0 - 6\nu)u_2 = 0,
\]
\[
 E_3 = -\frac{9}{2}bk^2 u_0 - 2\mu u_1 + 10k^2\nu u_1 + 2bu_2 + u_1 u_2 + u_0 u_3 = 0.
\]

The structure of this kind of algebraic determining equations is always the same: one algebraic equation for \( u_0 \) \((j = 0)\), followed by \(-p\) equations linear in \( u_j, j = 1, \cdots, -p \). Equation \( j = 0 \) has already been solved, see (41), and the next equations \( j = 1, \ldots, -p \) have the same solution \( u_j \) as in the infinite Laurent series (42). The truncated expansion (60) then evaluates to

\[
 u = \mathcal{D} \log \psi + \text{constant},
\]

in which \( \mathcal{D} \) is the singular part operator defined in (23) from the Laurent series, and \( \psi \) is the logarithmic primitive of \( \chi^{-1} \), an entire function defined by

\[
 \psi'' - \frac{k^2}{4} \psi = 0,
\]
whose value can be chosen without loss of generality as
\[ \psi = \cosh \frac{k}{2} \xi. \]  
(69)

After the operator \( \mathcal{D} \) has been computed, the two equations (67), (68) are an equivalent way of defining a one-family truncation, much more elegant than with (60), (53).

The remaining \(-q+p\) equations \( j = -p+1, \ldots, -q \) are algebraic in \( k^2 \) and the parameters appearing in the definition of the equation (22) (one says the fixed parameters).

\[ E_4 \equiv -\frac{5}{2} b^2 k^2 + \frac{44}{19} \mu^2 + \frac{131}{304} b^4 \nu - \frac{87}{38} b^2 \mu \nu^{-1} + 40 k^2 \mu \nu - 76 k^4 \nu^2 = 0, \]  
(70)
\[ E_5 \equiv b(b^2 - 16 \mu \nu) \left( 5 k^2 + \frac{13}{152} b^2 \nu - \frac{7}{19} \mu \nu^{-1} \right) = 0, \]  
(71)
\[ E_6 \equiv 32 A + 3 \nu u_{0} k^6 + 4 (b u_1 - \nu u_2) k^4 + 8k^2 \mu u_2 + 16 u_3^2 = 0, \]  
(72)
and they admit only the six solutions listed in Table 1.

6.1.2 One-family truncation of the real modulus of CGL3

Whatever be the chosen representation (couple \((M, \varphi)\), \((Z, \bar{Z}, \Phi)\), etc), the CGL3 equation has more than one family, see (15), therefore any one-family truncation only captures part of the whole singularity structure and cannot yield the general analytic solution. Nevertheless, as already noticed in section 2.1 Eq. (16), the one-family truncation of \( M = |A|^2 \) must provide at least the three currently known solutions. Let us perform it.

The field \( M \) has two families of movable double pole-like singularities
\[ M = \frac{3(3 d_r \pm \Delta)}{2 d_i^2} \chi^{-2} \left( 1 + \frac{c s_i}{3} \chi + O(\chi^2) \right), \]  
(73)
with singular part operators \( \mathcal{D}_\pm \) equal to
\[ \mathcal{D}_\pm = \frac{3(3 d_r \pm \Delta)}{2 d_i^2} \left( -\partial_x^2 + \frac{c s_i}{3} \partial_x \right). \]  
(74)

In its elegant definition, the one-family truncation,
\[ \begin{cases} 
M = \mathcal{D}_\pm \log \psi + m, \\
\psi'' + \frac{S}{2} \psi = 0, 
\end{cases} \]  
(75)
transforms (19) into the truncated Laurent series
\[ \sum_{j=0}^{14} E_j \chi^{j-14} = 0, \]  
(76)
and one must solve the 15 real determining equations \( E_j = 0 \) for the two constant unknowns \((S, m)\) and the five parameters \( d_r, d_i, g_r, g_i, c_s i \) occurring in (19). By construction of \( \mathcal{D}_\pm \), equations \( E_j = 0, j = 0, 1, \) are identically zero.

To avoid carrying heavy expressions, let us make the following nonrestrictive simplification. Out of the five parameters \( d_r, d_i, g_r, g_i, c_s i \) of the ODE (19), only three are essential \((g_r, g_i, c, \text{ equivalent to } \gamma, \omega, c)\). Indeed, \( p \) and \( q \) (i.e. \( d_r + i d_i \) and \( s_r - is_i \)) can be rescaled to convenient numerical values, such as
\[ p = -1 - 3i, \quad q = 4 - 3i, \quad d_r = \frac{1}{2}, \quad d_i = \frac{3}{2}, \quad s_r = -\frac{1}{10}, \quad s_i = \frac{3}{10}, \quad \Delta = \frac{9}{2}. \]  
(77)
Choosing the + sign in (75), one has $D_+ = 4(-\partial_x^2 - (c/10)\partial_x)$. As seen from the first few determining equations,

$$E_2 \equiv \frac{57}{100} c^2 + 156c_2 + 13k^2 + 4g_i + 16g_r = 0,$$

$$E_3 \equiv \left( -\frac{39}{25} c^2 - 432c_2 - 28k^2 - 16g_i - 48g_r \right) c = 0,$$

the resolution presents no difficulty. In particular, after solving the equations numbered $j = 0, \ldots, 6$, all the remaining equations are identically zero, a fact which indicates a high redundancy in these determining equations, which are therefore not at all optimal. In the CGL3 case properly said $\text{Im}(p/q) \neq 0$, for each sign in (74) one obtains three solutions,

$$
\begin{align*}
M &= -2 \left( \tau - \frac{c}{10} \right)^2 + \left( \frac{c}{10} \right)^2, \quad \varphi' - \frac{cs_r}{2} = -\tau - \frac{c}{20} - \frac{c}{5M} \left( \tau^2 - \frac{k^2}{4} \right), \\
k^2 &= -7 \left( \frac{c}{10} \right)^2 - \frac{4}{3} g_r, \quad 3g_i + 2g_r + \frac{3c^2}{50} = 0,
\end{align*}
$$

(80)

$$
\begin{align*}
M &= -2 \left( \tau - \frac{k^2}{4} \right), \quad \varphi' - \frac{cs_r}{2} = -\tau, \\
k^2 &= 2g_r, \quad c = 0, \quad g_i = 0,
\end{align*}
$$

(81)

$$
\begin{align*}
M &= -2 \left( \tau \pm \frac{k^2}{2} \right)^2, \quad \varphi' - \frac{cs_r}{2} = -\tau + \frac{c}{20}, \\
k^2 &= \left( \frac{c}{10} \right)^2, \quad g_r = 0, \quad g_i - \frac{c^2}{50} = 0,
\end{align*}
$$

(82)

and

$$
\begin{align*}
M &= 4 \left( \tau - \frac{c}{10} \right)^2 + \left( \frac{c}{10} \right)^2, \quad \varphi' - \frac{cs_r}{2} = 2\tau - \frac{c}{20} - \frac{c}{5M} \left( \tau^2 - \frac{k^2}{4} \right), \\
k^2 &= -\left( \frac{c}{10} \right)^2 + \frac{2}{3} g_r, \quad 3g_i - g_r + \frac{3c^2}{80} = 0,
\end{align*}
$$

(83)

$$
\begin{align*}
M &= 4\tau^2, \quad \varphi' - \frac{cs_r}{2} = 2\tau, \\
k^2 &= \frac{2}{3} g_r, \quad c = 0, \quad 3g_i - g_r = 0,
\end{align*}
$$

(84)

$$
\begin{align*}
M &= 4 \left( \tau \pm \frac{k^2}{2} \right)^2, \quad \varphi' - \frac{cs_r}{2} = 2\tau - \frac{c}{10}, \\
k^2 &= \left( \frac{c}{10} \right)^2, \quad g_r = 0, \quad g_i - \frac{c^2}{50} = 0.
\end{align*}
$$

(85)

These solutions are identical to those listed, in the same order, in section 2.31.

### 6.1.3 One-family truncation of CGL3 in the complex modulus representation

As already outlined at the end of section 4, the one-family truncation of $(Z, \overline{Z}, \Phi')$

$$
\begin{align*}
Z &= \chi^{-1} + X + iY, \\
\overline{Z} &= \chi^{-1} + X - iY, \\
\Phi &= \alpha \log \psi + K \xi,
\end{align*}
$$

(86)

with the gradient definitions

$$
\begin{align*}
(\log \psi)' &= \chi^{-1}, \\
\chi' &= 1 - \frac{k^2}{4} \chi^2,
\end{align*}
$$

(87)

puts the lhs of Eq. (1) in the form

$$
\sum_{j=0}^{3} E_j \chi^{j-3} = 0,
$$

(88)
thus generating four complex determining equations $E_j = 0$, (i.e. eight real, to be compared with the fifteen of section 6.1.2). These equations must first be solved as a linear system on $C$, as follows [9, Appendix A]. The first equation $E_0 = 0$, identical to (11), is linear in $p$ and $q$, let us solve it for $q$,

$$q = -(1 - i\alpha)(2 - i\alpha)A_0^{-2}p. \quad (89)$$

The next equation $j = 1$ is then linear in $K, X, Y, c$, let us solve it for instance for $K$,

$$K = (3i + \alpha)X - Y + \frac{c}{2p}. \quad (90)$$

The equation $j = 2$, linear in $\gamma, \omega, k^2$, is solved for $(i\gamma - \omega)/p$

$$\frac{i\gamma - \omega}{p} = \left(\frac{c}{2p}\right)^2 + |X - (1 - i\alpha)iY|^2 - (2 - 3i\alpha) \left[\frac{k^2}{4} - (X + iY)^2\right], \quad (91)$$

and the advantage of this pivoting elimination is that the last equation $j = 3$, which does not depend on $(q, K, \gamma)$ by construction, is also independent of $(p, c, \omega, A_0)$. It only depends on $(X, Y, \alpha, k^2)$, and it factorizes as

$$E_3 \equiv (2X - \alpha Y)(4(X + iY)^2 - k^2) = 0, \quad (92)$$

thus defining two solutions on $C$.

Finally, considering now the system (89)–(92) for the real unknowns or parameters $(A_0^2, \alpha, K, c, X, Y, \gamma, \omega, k^2)$, it is quite easy to obtain the three solutions listed in section 2.1.

### 6.2 Polynomials in tanh and sech (two-family truncation)

The class of polynomials in $\tanh$ and $\sech$

$$u = \left(\sum_{j=0}^{-p} a_j \tanh k\xi\right) + \left(\sum_{j=0}^{-p-1} b_j \tanh k\xi\right) \sech k\xi, \quad (93)$$

can equivalently be represented by the class of powers of tanh ranging from $p$ to $-p$ [31],

$$u = \sum_{j=0}^{-2p} u_j \chi^{j+p}, \quad \chi^{-1} = \frac{k}{2} \tanh \frac{k}{2} \xi, \quad \xi = x - ct, \quad u_0 u_{-2p} \neq 0, \quad (94)$$

because of the elementary identities [9]

$$\tanh z - \frac{1}{\tanh z} = -2i \sech \left[2z + i\frac{\pi}{2}\right], \quad \tanh z + \frac{1}{\tanh z} = 2 \tanh \left[2z + i\frac{\pi}{2}\right]. \quad (95)$$

A solution in this class can only exist for ODEs admitting at least two families with the same $p$. Indeed, if for this $p$ there exists only one value of $u_0$, only the second combination $\tanh +1/ \tanh = 2 \tanh$ can contribute. For instance, the KS equation cannot admit such a solution.

More generally, the class of polynomials in $\tau$ and $\sigma$ defined in [35],

$$u = \left(\sum_{j=0}^{-p} a_j \tau^j\right) + \left(\sum_{j=0}^{-p-1} b_j \tau^j\right) \sigma, \quad (a_{-p}, b_{-p-1}) \neq (0, 0), \quad (96)$$

is equivalently defined as [9, Appendix A]

$$\begin{align*}
\begin{cases}
\psi'''' + S \psi'' + m \psi = 0, & S = \frac{k^2}{2} = \text{constant}, \\
\psi_1 \psi_2 = \frac{k}{2} \mu_0 \left(\frac{\psi_1'}{\psi_1} - \frac{\psi_2'}{\psi_2}\right), & S = -\frac{k}{2} \text{Lag}
\end{cases}
\end{align*} \quad (97)$$
In this writing, which is the natural extension of (75) to two families, the linear operators $D_1$ and $D_2$ are the singular part operators of two different families, the entire functions $\psi_1$ and $\psi_2$ obey the same second order linear equation, but with a different choice of the integration constants,

$$\psi_1 = \cosh \frac{k}{2}(\xi + a), \quad \psi_2 = \cosh \frac{k}{2}(\xi - a), \quad \mu_0 = \coth ka.$$  

(98)

The case $\mu_0(\mu_0^2 - 1) = 0$ reduces to the class of polynomials in tanh and sech. The practical implementation is the following.

1. For the class of polynomials in tanh and sech, one puts the lhs $E(u)$ of the nonlinear ODE under the same form as $u$,

$$u = \sum_{j=0}^{\infty} u_j \chi^{j+q}, \quad u_0 u_{-2q} \neq 0,$$

$$\chi' = 1 + \frac{S}{2} \chi^2, \quad S = -\frac{k^2}{2},$$

$$E = \sum_{j=0}^{\infty} E_j \chi^{j+q},$$

$$\forall j : E_j = 0.$$  

(99)

and one solves the set of $-2q + 1$ determining equations $E_j = 0$.

(100)

2. For the class of polynomials in $\tau$ and $\sigma$ defined in (55), under the assumption (97), the lhs $E(u)$ is first expressed as a polynomial of the two variables $\psi_1'/\psi_1, \psi_2'/\psi_2, j = 1, 2$

$$\sum_{k=0}^{-q} \sum_{l=0}^{-q-k} E_{k,l} \left( \frac{\psi'_1}{\psi_1} \right)^k \left( \frac{\psi'_2}{\psi_2} \right)^l = 0,$$  

(101)

which further reduces, thanks to the third line of (97), to the sum of two polynomials of one variable,

$$E_0 + \left( \sum_{j=1}^{-q} E_j^{(1)} \left( \frac{\psi'_1}{\psi_1} \right)^j \right) + \left( \sum_{j=1}^{-q} E_j^{(2)} \left( \frac{\psi'_2}{\psi_2} \right)^j \right) = 0.$$  

(102)

One then requires the vanishing of the $-2q + 1$ determining equations

$$E_0 = 0, \quad E_j^{(1)} = 0, \quad E_j^{(2)} = 0, \quad j = 1, \ldots, -q.$$  

(103)

As an example, let us apply this to the ODE

$$E(u) \equiv \left( \frac{du}{d\xi} \right)^2 - \alpha^2 (u^2 - b^2)^2 + c = 0.$$  

(104)

It admits two families, with singular part operators $D_1 = \alpha^{-1} \partial_\xi, D_2 = -\alpha^{-1} \partial_\xi$. The relation $\partial_2 = -D_1$ implies $E_j^{(1)} + (-1)^j E_j^{(2)} \equiv 0, \quad j = 1, 2, 3, 4$, and only 5 out of the 9 determining equations are linearly independent. Moreover, by construction of the singular part operators, the two equations $j = 4$ are identically satisfied. The next equation $j = 3$

$$E_3^{(1)} \equiv -2\alpha^{-2} k \mu_0 - 4\alpha^{-1} m = 0,$$  

(105)

is solved for $m$. Then, the equation $j = 2$

$$E_2^{(1)} \equiv 2b^2 + \alpha^{-2} k^2 - \frac{3}{2} \alpha^{-2} (k \mu_0)^2 = 0,$$  

(106)
is solved for \( k^2 \), considering \( k\mu_0 \) as a single variable. The remaining system

\[
E_1^{(1)} = -2b^2 + \frac{1}{2}a^{-2}(k\mu_0)^2, \quad E_2^{(1)} = 0, \quad E_3^{(2)} = 0, \quad E_4^{(2)} = 0,
\]

admits two solutions. The first one \( c = 0, (k\mu_0)^2 = (2ab)^2 \) corresponds to a factorization of the equation \( E(u) = 0 \) into two Riccati equations and therefore must be rejected. The second one

\[
k\mu_0 = 0, \quad \alpha^2b^4 - c = 0,
\]

defines a solution, provided the indicated constraint on the fixed parameters \((\alpha, b, c)\) is satisfied. This solution

\[
\mu_0 = 0, \quad m = 0, \quad k^2 = -2(ab)^2, \quad u = \alpha^{-1}\frac{d}{d\xi} \log \frac{\cosh(k/2)(\xi + a)}{\cosh(k/2)(\xi - a)}
\]

is nothing else than \( u = i(k/\alpha) \tanh k\xi \), using the relation \( \mu_0 = \coth ka \).

Indeed, as opposed to the function \( \tanh \), which satisfies an ODE admitting only one family of movable singularities (the Riccati equation), the function \( \sech \) (or more generally its homographic transform \( \sigma \)) satisfies a first order second degree ODE

\[
\sech^2 + \sech^4 - \sech^2 = 0,
\]

which admits two families of movable simple poles with opposite residues

\[
\sech(\xi - \xi_0) \sim \pm i(\xi - \xi_0)^{-1}.
\]

6.2.1 Two-family truncation of the real modulus of CGL3

\( M \) admits exactly two families, so its two-family truncation is quite appropriate. The two singular part operators are, for each family, defined in [14], with \( D_1 = D_+, D_2 = D_- \). The assumption \( 0 \), with \( p = -2, q = -14 \), transforms [13] into the sum [12] of two polynomials of one variable, and the four equations \( j = 14, 13 \) are identically zero, by definition of \( D_\pm \).

For \( p, q, \gamma \) arbitrary, the resolution of the 25 remaining determining equation is impossible to carry out by hand. But the hand computation becomes possible with the generic numerical values [24]. First, the system \( j = 12 \)

\[
E_{12}^{(1)} = \frac{57}{5}c^2 - 780m - 520k^2 + 78ck\mu_0 + 390(k\mu_0)^2 + 80g_i + 320g_r = 0, \quad E_{12}^{(2)} = \frac{3}{5}c^2 + 120m - 40k^2 - 24ck\mu_0 + 120(k\mu_0)^2 + 20g_i - 40g_r = 0,
\]

is solved as a linear system for \( m \) and \( k^2 \). The next system \( j = 11 \)

\[
E_{11}^{(1)} = \frac{39}{125}c^3 - \frac{254}{25}c^2k\mu_0 + \frac{468}{5}c(k\mu_0)^2 - 312(k\mu_0)^3 - \frac{156}{5}cg_i - 168k\mu_0g_i - \frac{104}{5}cg_r - 48k\mu_0g_r = 0, \quad E_{11}^{(2)} = \frac{177}{500}c^3 + \frac{218}{25}c^2k\mu_0 - 39c(k\mu_0)^2 + 156(k\mu_0)^3 - \frac{87}{5}cg_i + 84k\mu_0g_i - 2cg_r + 24k\mu_0g_r = 0,
\]

is linear in \( (g_r, g_i) \), with a Jacobian \( J = c(3c - 5k\mu_0) \). For the first subcase \( J \neq 0 \), after solving for \( (g_r, g_i) \) as functions of \( (c, k\mu_0) \), the next system \( j = 10 \) only depends on \( k\mu_0/c \) and it admits no solution. The discussion of the second subcase \( J = 0 \) leads to the conclusion, only using the next system \( j = 10 \), that no solution exists. An identical result is achieved for arbitrary values of \( (p, q, \gamma) \) using computer algebra.
This unfortunate situation is exceptional, and only reflects the difficulty of CGL3. Should such a solution exist, it would have the form

\[ M = \left( \frac{\Delta}{2d_1^2} \tanh + c_1 \right) \sech + \frac{9d_2}{2d_1^2} \tanh^2 + c_3 \tanh + c_4, \]  

(117)

and the constraint \( c_3 = 0 \) could define a homoclinic hole solution, just like the (yet analytically unknown) one of van Hecke [13].

6.2.2 Two-family truncation of CGL3 in the complex modulus representation

Let us denote \((A_0, \alpha)\) and \((A_2, \alpha_2)\) two different solutions of (15).

In the complex modulus representation (56), the two-family truncation of \((Z, Z, \Phi)\) is defined as [9, Appendix A],

\[
\begin{align*}
A &= (A_0(\partial_\xi \log \psi_1(\xi) + X + iY) + A_2(\partial_\xi \log \psi_2(\xi)) e^{i[-\omega t + \Phi(\xi)]}, \\
A &= (A_0(\partial_\xi \log \psi_1(\xi) + X - iY) + A_2(\partial_\xi \log \psi_2(\xi)) e^{-i[-\omega t + \Phi(\xi)]},
\end{align*}
\]  

(118)

with the definitions for the derivatives of \((\psi_1, \psi_2)\) given by the last two lines of (97). The lhs of Eq. (1) then takes the form (102) with \( q = -3 \), and one solves the seven complex determining equations as a linear system on \( C \), similarly to what has been done in section 6.1.3.

From the two equations \( j = 3 \),

\[
\begin{align*}
E_3^{(1)} &\equiv A_0 ((1 - i\alpha)(2 - i\alpha)p + A_0^2 q) = 0, \\
E_3^{(2)} &\equiv A_2 ((1 - i\alpha_2)(2 - i\alpha_2)p + A_2^2 q) = 0,
\end{align*}
\]  

(119, 120)

and the two relations implied by (15),

\[
\alpha = \frac{d_4}{3} A_0^2, \quad \alpha_2 = \frac{d_4}{3} A_2^2.
\]  

(121)

one first proves that the only possibility is \( A_2 = -A_0 \), therefore the two complex equations \( j = 3 \) are solved as

\[
A_2 = -A_0, \quad \alpha_2 = \alpha, \quad q = -(1 - i\alpha)(2 - i\alpha)A_0^{-2} p.
\]  

(122)

At the level \( j = 2 \), the symmetric combination

\[
E_2^{(1)} + E_2^{(2)} \equiv p(1 - i\alpha) \left[ (i\alpha - 3)X - iY + \left( i\alpha - \frac{3}{2} \right) k\mu_0 \right] = 0,
\]  

(123)

is solved for the two pieces of information

\[
X = -\frac{1}{2} k\mu_0, \quad Y = \frac{1}{2} \alpha k\mu_0,
\]  

(124)

then the antisymmetric combination

\[
E_2^{(1)} - E_2^{(2)} \equiv p(1 - i\alpha) \left[ \frac{c}{2p} - K \right] = 0,
\]  

(125)

is solved as

\[
K = \frac{c}{2p}.
\]  

(126)

At the level \( j = 1 \), the symmetric combination is identically zero, and the antisymmetric combination is solved for \((i\gamma - \omega)/p\) (we omit the expression). The remaining equation

\[
E_0 \equiv \mu_0 \left[ (2 + (\alpha^2 - 2)\mu_0^2) + i\alpha(-4 + (\alpha^2 + 4)\mu_0^2) \right] = 0
\]  

(127)
admits as only solution \( \mu_0 = 0 \). Therefore, one obtains the unique solution
\[
A_2 = -A_0, \quad \alpha_2 = \alpha, \quad q = -(1 - i\alpha)(2 - i\alpha)A_0^{-2}p,
\]
\[
X = 0, \quad Y = 0, \quad \mu_0 = 0, \quad K = \frac{c}{2p},
\]
\[
\frac{i\gamma - \omega}{p} = \left(\frac{c}{2p}\right)^2 + (1 - i\alpha)^2k^2,
\]
The value of \( K \) implies \( cp = 0 \), and the case \( c = 0 \) represents the homoclinic pulse (13).

6.3 Polynomials in \( \wp \) and \( \wp' \)

A class of polynomial elliptic functions can be defined for instance with the Weierstrass function \( \wp \) and its derivative [17, 33],
\[
u = \left(\sum_{j=0}^{[-p/2]} a_j \wp(\xi)^j\right) + \left(\sum_{j=0}^{[(p-3)/2]} b_j \wp(\xi)^j\right) \wp'(\xi),
\]
Since \( \wp \) admits only one family, such a solution may exist for any ODE. It will be quite useful to take advantage of the value of the singular part operator of \( \wp(\xi) \),
\[
\mathcal{D} = -\frac{d^2}{d\xi^2}.
\]

For KS, the assumption to seek for solutions in the above class
\[
u = c_0 \wp' + c_1 \wp + c_2, \quad c_0 \neq 0.
\]

This together with the knowledge of the singular part operators (27) and (132), first yields the correct values of \( c_0 \) and \( c_1 \),
\[
u = -60\nu \wp' - 15b \wp + c_2,
\]
a truncation which defines the four determining equations
\[
\begin{align*}
b_2 - 16\mu \nu &= 0, \\
b_\mu + 4\nu c_2 &= 0, \\
b c_2 - 120\nu^2 g_2 &= 0, \\
A + \frac{1}{2} c_2^2 + \frac{15}{2} b_2^2 g_2 + 30\mu \nu g_2 - 1080\nu^2 g_3 &= 0.
\end{align*}
\]
Their unique solution is (23).

For CGL3, the assumption that \( M \) be in this class
\[
M = a_2 \wp + c_2, \quad a_2 \neq 0,
\]
generates 10 determining equations, in the parameters and unknowns \( (a_2, c_2, g_2, g_3, d_r, d_i, s_r, c_s; g_r, g_i) \).

The equation with the highest singularity degree
\[
((d_4 a_2)^2 - 9a_2 d_r - 18)(2 + a_2 d_r) = 0
\]
in which the vanishing of the second factor is forbidden, is first solved as a linear equation for \( d_r \)
\[
d_r = \frac{(a_2 d_i)^2 - 18}{9a_2}.
\]
The next equation yields \( c s_i = 0 \). The next equations are successively solved for \( g_r, g_2, g_3 \), and the elliptic discriminant \( g_2^2 - 27g_3^2 \) is then divisible by the unique remaining determining equation. Therefore, one finds as unique solution the pulse (13).

Finally, let us mention the Ansatz made for CGL5 [25, 3]
\[
A = a(x) e^{i[-2\alpha \text{Log} a(x) - \omega t]}, \quad (\omega, \alpha, a) \in \mathcal{R},
\]
which sets an \emph{a priori} constraint between the amplitude and the phase (similar to that made for CGL3 in Ref. [3]), together with the assumption that \( a^2 \) obeys a first order second degree elliptic equation. This allows one to retrieve (34) in the particular case \( r_0 = 0, c = 0 \).
7 Second class of methods: first order subequation

The requirement that the solution (6) be shared by the \(N\)th order ODE (5) and the first order ODE (7) characterizes, as recalled in the Appendix, the singlevalued expressions \(f\) as being elliptic or degenerate elliptic (i.e. trigonometric or rational), i.e. the class

\[ u = R(\varphi', \varphi) \rightarrow R(e^{k\xi}) \rightarrow R(\xi), \quad (140) \]

in which \(R\) denotes rational functions and \(\rightarrow\) denotes the degeneracy. This class contains all the classes considered in previous sections (polynomials in \(\tanh\), in \((\tanh, \text{sech})\), in \((\sigma, \tau)\), in \((\varphi, \varphi')\)), but it also contains in addition expressions like (59).

The algorithm to obtain all the elliptic solutions combines two pieces of information:

1. a local one, a Laurent series representing the largest analytic solution of the \(N\)-th order ODE near a movable pole-like singularity,

2. a global one, the necessary form (108) of (7),

by requiring that the Laurent series satisfies the first order subequation (108).

This provides the explicit form of the first order subequation \(F(u, u') = 0\). Then one computes the solution \(u = f(\xi - \xi_0)\) from this equation \(F = 0\).

The successive steps are [27, Section 5].

1. Choose a positive integer \(m\) and define the Briot and Bouquet first order ODE

\[ F(u, u') \equiv \sum_{k=0}^{m} \sum_{j=0}^{\left\lfloor (m-k)(p-1)/p \right\rfloor} a_{j,k} u^j u'^k = 0, \quad a_{0,m} \neq 0, \quad (141) \]

in which \(\lfloor z \rfloor\) denotes the integer part function. The upper bound on \(j\) implements the condition \(m(p-1) \leq jp + k(p-1)\) that no term can be more singular than \(u^m\), identically satisfied if \(p = -1\). The polynomial \(F\) contains at most \((m+1)^2\) unknown constants \(a_{j,k}\).

2. Compute \(J\) terms of the Laurent series, with \(J\) slightly greater than the number of unknown constants \(a_{j,k}\).

\[ u = \chi^P \left( \sum_{j=0}^{J} M_j \chi^j + \mathcal{O}(\chi^{J+1}) \right), \quad \chi = \xi - \xi_0, \quad (142) \]

where \(p = -3\) for the KS equation [22], \(-2\) for the variable \(|A|^2\) of CGL3, etc.

3. Require the Laurent series to satisfy the Briot and Bouquet ODE, i.e. require the identical vanishing of the Laurent series for the lhs \(F(U, U')\) up to the order \(J\)

\[ F \equiv \chi^D \left( \sum_{j=0}^{J} F_j \chi^j + \mathcal{O}(\chi^{J+1}) \right), \quad D = m(p-1), \quad (143) \]

\[ \forall j : \quad F_j = 0. \quad (144) \]

If it has no solution for \(a_{j,k}\), increase \(m\) and return to first step.

4. For every solution, integrate the first order autonomous ODE (141).

Let us give two examples.

---

20
7.1 First order autonomous subequations of KS

The Laurent series of (22) is (42).

In the second step, the smallest integer m which allows a movable triple pole (p = −3) in (141) is m = 3. With the normalization a0,3 = 1, the subequation contains ten coefficients, which are first determined by the Cramer system of ten equations F_j = 0, j = 0; 6, 8, 9, 12. The first few are

\[ F_0 = -9a_{0,3} + 40\nu a_{4,0} = 0, \]
\[ F_1 = 9ba_{0,3} + 12\nu a_{1,2} - 80\nu a_{4,0} = 0, \]
\[ F_2 = (2120\nu^2 + 2560\nu^3) a_{4,0} - (105\nu^2 + 144\mu\nu)a_{0,3} - 532\nu a_{1,2} - 608\nu^2 a_{2,1} = 0, \]
\[ F_3 = (5\nu^2 + 72\mu\nu) ba_{0,3} + (137\nu^2 + 240\nu^3) a_{1,2} - (442b^2 + 2656\mu\nu) b a_{4,0} + 608\nu^2 a_{2,1} + 608\nu^3 a_{3,0} = 0. \]

The remaining infinitely overdetermined nonlinear system for (ν, b, μ, A) contains as greatest common divisor (gcd) b^2 − 16μ (see Eq. 20), which defines a first solution

\[ \frac{b^2}{\mu^2} = 16, \quad \left(u' + \frac{b}{2\nu} u_s\right)^2 \left(u' - \frac{b}{4\nu} u_s\right) + \frac{9}{40\nu} \left(u_s^2 + \frac{15b^6}{1024\nu^4} + \frac{10A}{3}\right)^2 = 0, \]
\[ u_s = u + \frac{3b^3}{32\nu^2}. \]

After division by this gcd, the remaining system for (ν, b, μ, A) admits exactly four solutions (stopping the series at j = 16 is enough to obtain the result), namely the first three lines of Table 14 each solution defining the same kind of subequation,

\[ b = 0, \]
\[ \left(u' + \frac{180\mu^2}{192\nu^2}\right)^2 \left(u' - \frac{360\mu^2}{192\nu^2}\right) + \frac{9}{40\nu} \left(u^2 + \frac{30\mu^2}{19}\right)^2 = 0, \]
\[ b = 0, \quad u_s^3 + \frac{9}{40\nu} \left(u^2 + \frac{30\mu^2}{19}\right) = 0, \]
\[ b^2 = \frac{144}{47}, \quad u_s = u - \frac{5b^3}{144\nu^2}, \quad \left(u' + \frac{b}{4\nu} u_s\right)^3 + \frac{9}{40\nu} u_s^2 = 0, \]
\[ b^2 = \frac{256}{73}, \quad u_s = u - \frac{45b^3}{2048\nu^2}, \quad \left(u' + \frac{b}{8\nu} u_s\right)^2 \left(u' + \frac{b}{2\nu} u_s\right) + \frac{9}{40\nu} \left(u_s^2 + \frac{5b^3}{1024\nu^2} u_s + \frac{5b^2}{128\nu^2} u_s'\right)^2 = 0, \]

In order to integrate the two sets of subequations (149), (150)–(153), one must first compute their genus 1, which is one for (149), and zero for (150)–(153). Therefore (149) has an elliptic general solution, listed above as (23), and initially found [12, 17] by other methods.

As to the general solution of the four others (150)–(153), this is the third degree polynomial in tanh \( \frac{\chi}{2} (C - \xi_0) \).

These four solutions, obtained for the minimal choice of the subequation degree m, constitute all the analytic results currently known on (22). For m = 4, no additional solution is obtained [11]. The computation for m = 5 is in progress.

7.2 First order autonomous subequations of CGL3

We consider the variable \( M = |A|^2 \), i.e. \( p = -2 \). The smallest value of m is then 2. With the numerical values \( \text{[77]} \), the two Laurent series are

\[ M_+ = \chi^2 \left(-2 + \frac{c}{5} \lambda + \left(\frac{g_r}{3} - \frac{g_i}{6} - \frac{c^2}{200}\right) \chi^2 + \mathcal{O}(\chi^3)\right), \]

\[ M_- = \chi^{-2} \left(2 + \frac{c}{5} \lambda + \left(\frac{g_r}{3} - \frac{g_i}{6} - \frac{c^2}{200}\right) \chi^2 + \mathcal{O}(\chi^3)\right). \]

\[ 1 \text{For instance with the Maple command } \text{genus of the package algcurves [13], which implements an algorithm of Poincaré.} \]
investigation but preliminary results seem to indicate the absence of any new solution, and we are
This situation is quite similar to the absence of solution in the class \(117\), and it just reflects the
since the main step is a linear computation.

\[ M_+ = \chi^{-2} \left( 4 - \frac{2c}{5} \chi + \left( \frac{16g_r}{39} + \frac{4g_i}{39} + \frac{19c^2}{1300} \right) \chi^2 + O(\chi^3) \right). \quad \text{(155)} \]

The existence of two Laurent series, rather than just one, is a feature which the subequation
must also possess, and this has the effect of setting the lower bound to \(m = 4\) instead of 2. Indeed,
the lowest degree subequations

\[
\begin{align*}
F_2 & \equiv M'^2 + M'(a_{1,1}M + a_{0,1}) + a_{3,0}M^3 + a_{2,0}M^2 + a_{1,0}M + a_{0,0} = 0, \\
F_3 & \equiv M'^3 + M'^2(a_{1,2}M + a_{0,2}) + M'(a_{3,1}M^3 + a_{2,1}M^2 + a_{1,1}M + a_{0,1}) \\
& + a_{4,0}M^4 + a_{3,0}M^3 + a_{2,0}M^2 + a_{1,0}M + a_{0,0} = 0,
\end{align*}
\]

have the respective dominant terms \(M'^2 + a_{3,0}M^3\) and \(M'^3 + a_{3,1}M' M^3\), which define only one
family of movable double poles.

Let us nevertheless start with \(m = 2\), for which \(156\) can only be satisfied by one series,
\(\text{e.g. }157\), thus preventing the full desired result to be obtained. The six coefficients \(a_{j,k}\) of \(158\) are
first computed as the unique solution of the linear system of six equations \(F_j = 0\), \(j = 0,1,2,3,4,6\).
Then the \(J + 1 - 6\) remaining equations \(F_j = 0\), \(j = 5,7 : J\), which only depend on the fixed
parameters \((g_r, g_i, c)\), have the greatest common divisor (gcd) \(3g_r + 2g_i + 3c^2/50\), and this factor
defines the first solution \(159\) below. After division par this gcd, the system of equations \(F_j = 0\), \(j = 5,7,8\), provides two and only two two other solutions, see \(160\) and \(161\) below, with the
respective constraints \((c = 0, g_r = 0)\) and \((g_r = 0, 50g_i - c^2 = 0)\), and all the remaining equations
\(F_j = 0, j \geq 9\), are identically satisfied.

Therefore, with this lower bound \(m = 2\), one already recovers all the known first order sub-
equations. These are, with the series \(154\),

\[
\begin{align*}
\left( M' + \frac{c}{5} M + \frac{c^3}{250} \right)^2 + 2 \left( M + \frac{c^2}{50} \right) \left( M - \frac{c^2}{50} - \frac{2}{3} g_r \right)^2 &= 0, 3g_i + 2g_r + \frac{3c^2}{50} = 0, (158) \\
M'^2 + 2(M - g_r)M^2 &= 0, c = 0, g_i = 0, (159) \\
\left( M' + \frac{c}{5} M \right)^2 + 2M^3 &= 0, g_r = 0, g_i - \frac{c^2}{50} = 0. (160)
\end{align*}
\]

Finally, for each of the three subequations, the fourth step finds a zero value for the genus and
returns the general solution as a rational function of \(e^{i(\xi - \xi_0)}\), which basic trigonometric identities
then allow to convert to the second degree polynomials in \((k/2) \tanh k(\xi - \xi_0)/2\) listed in \(80\)–\(82\).

Similarly, with the other series \(155\), one obtains

\[
\begin{align*}
\left( M' + \frac{c}{5} M - \frac{c^3}{500} \right)^2 - \left( M - \frac{c^2}{100} \right) \left( M + \frac{c^2}{100} - \frac{2}{3} g_r \right)^2 &= 0, 3g_i - g_r + \frac{3c^2}{80} = 0, (161) \\
M'^2 - M \left( M - \frac{2}{3} g_r \right)^2 &= 0, c = 0, 3g_i - g_r = 0, (162) \\
\left( M' + \frac{c}{5} M \right)^2 - M^3 &= 0, g_r = 0, g_i - \frac{c^2}{50} = 0. (163)
\end{align*}
\]

With the correct two-family lower bound \(m = 4\), which corresponds to 18 unknowns \(a_{j,k}\) and at
least 24 terms in the series, we have checked that there is no solution other than the above three.
This situation is quite similar to the absence of solution in the class \(117\), and it just reflects the
difficulty of the CGL3 equation.

The case \(m = 8\) (60 unknowns \(a_{j,k}\) and at least 66 terms in the series) is currently under
investigation but preliminary results seem to indicate the absence of any new solution, and we are
now automating the computer algebra program in order to handle much larger values of \(m\).

### 7.3 Domain of applicability of the method

As we have seen, the subequation method contains the truncation methods and its cost is minimal
since the main step is a linear computation.

The two key assumptions behind this “subequation method” are,
1. a Laurent series should exist,

2. a first order autonomous algebraic subequation should exist.

Its best applicability is therefore nonintegrable $N$-th order autonomous nonlinear ODEs admitting a Laurent series which only depends on one movable constant, such as the CGL3 ODE \[19\] or the traveling wave reduction \[22\] of the Kuramoto-Sivashinsky equation \[8, 41\].

Two examples of inapplicability are

1. the Lorenz model, in which the Laurent series generically does not exist and has to be replaced by a psi-series \[36\],

2. the autonomous ODE $u''' - 12uu' - 1 = 0$, which admits the first Painlevé transcendent as its general solution, a case in which no first order subequation exists.

8 Conclusion

How do these two classes of methods (truncations, first order subequations) really compare, independently of the amount of computation involved?

Let us first recall a preliminary, classical result.

The class \[60\] of polynomials of degree $-p$ in tanh obeys a first order equation of degree $m = -p$. For instance, given the polynomial

$$u = \tanh^2 + 2a \tanh + b,$$  \hspace{1cm} (164)

this amounts to eliminate tanh between the two algebraic equations

$$\begin{cases} 
\tanh^2 + 2a \tanh + b - u = 0, \\
2(\tanh + a)(1 - \tanh^2) - u' = 0,
\end{cases}$$ \hspace{1cm} (165)

which results in

$$
\begin{vmatrix}
0 & 0 & 1 & 2a & b - u \\
0 & 1 & 2a & b - u & 0 \\
1 & 2a & b - u & 0 & 0 \\
0 & -2 & -2a & 2 & 2a - u' \\
-2 & -2a & 2 & 2a - u' & 0 \\
\end{vmatrix} = (u' - 4a(u - b + a^2))^2 - 4(u - b + 2a^2 - 1)^2(u - b + a^2) = 0, \hspace{1cm} (166)
$$

an equation with degree $m = 2 = -p$, having genus zero.

Similarly, the class of polynomials of global degree $-p$ in $(\tanh, \text{sech})$ or $(\sigma, \tau)$ obeys a first order ODE with degree $m = -2p$.

Last, the class \[131\] of polynomials of $(\varphi, \varphi')$ of singularity degree at most equal to $p$ obeys a first order ODE with degree $m = -p$.

Therefore, given a value of $p$ (the singularity degree of the ODE) and a truncation considered in section \[2\] there exists a value of $m$ (either $-p$ or $-2p$) at which the result of the truncation can be found by the method of first order subequations.

Conversely, given a value of $m$ (the degree of a first order subequation), the class of solutions of the method of first order subequations is made of the rational functions of $(\varphi, \varphi')$ or of $e^{k\xi}$ i.e. $(k/2)\tanh k\xi/2$, a class richer than the polynomials.

This proves the identity of the two classes of methods, provided the truncations assume rational functions instead of polynomials.

However, from the practical point of view of the amount of computation involved, the increasing order of difficulty seems to be

1. Truncations of polynomials.

\[{2} This formula, due to Sylvester, expresses the resultant of two polynomials of degrees $m$ and $n$ as a determinant of order $m + n$.\[
2. First order subequations.

3. Truncations of rational functions.

How does this compare with the approach of Chow (see e.g. [8]) to find solutions of PDEs in terms of elliptic functions? To be definite, let us consider a PDE in $(x, t)$. The solutions which do depend on both $x$ and $t$ (i.e. which do not satisfy some ODE) are richer than those here described. As to the solutions of the solitary wave type $f(x - ct)$, the method of Chow belongs to the first class of methods, i.e. it may or it may not find the most general elliptic solution which exists.

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9 Appendix. Classical results on first order autonomous equations

The following results were mainly obtained by Briot and Bouquet, Fuchs, Poincaré and put in final form by Painlevé [29, pages 58–59].

Theorem. Given an algebraic first order autonomous ODE (7), whose general solution is therefore (6), the following properties are equivalent.

1. Its general solution is singlevalued.
2. Its general solution is an elliptic function, possibly degenerate.
3. The genus of the algebraic curve (7) is one or zero.
4. There is equivalence between the knowledge of \( f \) and that of \( F \).
5. There exist a positive integer \( m \) and \((m + 1)^2 \) complex constants \( a_{j,k} \), with \( a_{0,m} \neq 0 \), such that the polynomial \( F \) of two variables has the necessary form

\[
F(u, u') \equiv \sum_{k=0}^{m} \sum_{j=0}^{2m-2k} a_{j,k} u^j u'^k = 0, \quad a_{0,m} \neq 0.
\]

6. If the genus is one, there exist two rational functions \( R_1, R_2 \), such that the general solution is

\[
u = R_1(\wp) + \wp' R_2(\wp), \tag{169}
\]

in which \( \wp = \wp(\xi - \xi_0, g_2, g_3) \) is the Weierstrass elliptic function characterized by [24].

If the genus is zero, there exists a (possibly zero) constant \( a \) and a rational function \( R \) such that the general solution is

\[
u = R(e^{a\xi}), \tag{170}
\]

with the degeneracy \( u = R(\xi) \) in case \( a \) is zero.

References

[1] M. J. Ablowitz and P. A. Clarkson, Solitons, nonlinear evolution equations and inverse scattering (Cambridge University Press, Cambridge, 1991).
[2] G. P. Agrawal, Nonlinear fiber optics, 3rd edition (Academic press, Boston, 2001).
[3] N. N. Akhmediev and V. V. Afanasiev, Novel arbitrary-amplitude soliton solutions of the cubic-quintic complex Ginzburg-Landau equation, Phys. Rev. Lett. 75 (1995) 2320–2323.
[4] N. Bekki and K. Nozaki, Formations of spatial patterns and holes in the generalized Ginzburg-Landau equation, Phys. Lett. A 110 (1985) 133–135.
[5] F. Cariello and M. Tabor, Painlevé expansions for nonintegrable evolution equations, Physica D 39 (1989) 77–94.
[6] Kwok W. Chow, A class of doubly periodic waves for nonlinear evolution equations, Wave Motion 35 (2002) 71–90.
[7] R. Conte, The Painlevé approach to nonlinear ordinary differential equations, The Painlevé property, one century later, 77–180, ed. R. Conte, CRM series in mathematical physics (Springer, New York, 1999). [http://arXiv.org/abs/solv-int/9710020]
[8] R. Conte and M. Musette, Painlevé analysis and Bäcklund transformation in the Kuramoto-Sivashinsky equation, J. Phys. A 22 (1989) 169–177.
[9] R. Conte and M. Musette, Linearity inside nonlinearity: exact solutions to the complex Ginzburg-Landau equation, Physica D 69 (1993) 1–17.

[10] R. Conte and M. Musette, Analytic expressions of hydrothermal waves, Reports on mathematical physics 46 (2000) 77–88. [http://arXiv.org/abs/nlin.SI/0009022]

[11] R. Conte, A. P. Fordy, and A. Pickering, A perturbative Painlevé approach to nonlinear differential equations, Physica D 69 (1993) 33–58.

[12] J.-D. Fournier, E. A. Spiegel, and O. Thual, Meromorphic integrals of two nonintegrable systems, *Nonlinear dynamics*, 366–373, ed. G. Turchetti (World Scientific, Singapore, 1989).

[13] M. van Hecke, Building blocks of spatiotemporal intermittency, Phys. Rev. Lett. 80 (1998) 1896–1899.

[14] M. van Hecke, C. Storm, and W. van Saarlos, Sources, sinks and wavenumber selection in coupled CGL equations and experimental implications for counter-propagating wave systems, Physica D 133 (1999) 1–47. [http://arXiv.org/abs/patt-sol/9902005]

[15] Mark van Hoeij, package “algcurves”, Maple V (1997). [http://www.math.fsu.edu/~hoeij/algcurves.html]

[16] N. A. Kudryashov, Exact soliton solutions of the generalized evolution equation of wave dynamics, Prikladaia Matematika i Mekhanika 52 (1988) 465–470 [English : Journal of applied mathematics and mechanics 52 (1988) 361–365]

[17] N. A. Kudryashov, Exact solutions of a generalized equation of Ginzburg-Landau, Matematicheskoye modelirovanie 1 (1989) 151–158.

[18] N. A. Kudryashov, Exact solutions of the generalized Kuramoto-Sivashinsky equation, Phys. Lett. A 147 (1990) 287–291.

[19] Y. Kuramoto and T. Tsuzuki, Persistent propagation of concentration waves in dissipative media far from thermal equilibrium, Prog. Theor. Phys. 55 (1976) 356–369.

[20] J. Lega, Traveling hole solutions of the complex Ginzburg-Landau equation: a review, Physica D 152–153 (2001) 269–287.

[21] J. Lega, J. V. Moloney, and A. C. Newell, Swift-Hohenberg equation for lasers, Phys. Rev. Lett. 73 (1994) 2978–2981.

[22] P. Manneville, *Dissipative structures and weak turbulence* (Academic Press, Boston, 1990). French adaptation: *Structures dissipatives, chaos et turbulence* (Aléa-Saclay, Gif-sur-Yvette, 1991).

[23] P. Marcq, H. Chaté, and R. Conte, Exact solutions of the one-dimensional quintic complex Ginzburg-Landau equation, Physica D 73 (1994) 305–317. [http://arXiv.org/abs/patt-sol/9310004]

[24] K. Maruno, A. Ankiewicz, and N. N. Akhmediev, Exact soliton solutions of the one-dimensional complex Swift-Hohenberg equation, Physica D 176 (2003) 44–66.

[25] J. D. Moores, On the Ginzburg-Landau laser mode-locking model with fifth-order saturable absorber term, Optics Communications 96 (1993) 65–70.

[26] M. Musette, Painlevé analysis for nonlinear partial differential equations, *The Painlevé property, one century later*, 517–572, ed. R. Conte, CRM series in mathematical physics (Springer, New York, 1999).

[27] M. Musette and R. Conte, Analytic solitary waves of nonintegrable equations, Physica D 181 (2003) 70–79. [http://arXiv.org/abs/nlin.PS/0302051]

[28] K. Nozaki and N. Bekki, Exact solutions of the generalized Ginzburg-Landau equation, J. Phys. Soc. Japan 53 (1984) 1581–1582.
[29] P. Painlevé, *Leçons sur la théorie analytique des équations différentielles* (Leçons de Stockholm, 1895) (Hermann, Paris, 1897). Reprinted, *Oeuvres de Paul Painlevé*, vol. I (Éditions du CNRS, Paris, 1973).

[30] N. R. Pereira and L. Stenflo, Nonlinear Schrödinger equation including growth and damping, Phys. Fluids 20 (1977) 1733–1743.

[31] A. Pickering, A new truncation in Painlevé analysis, J. Phys. A 26 (1993) 4395–4405.

[32] Y. Pomeau and P. Manneville, Stability and fluctuations of a spatially periodic flow, J. Physique Lett. 40 (1979) L609–L612.

[33] A. M. Samsonov, Nonlinear strain waves in elastic waveguides, *Nonlinear waves in solids*, 349–382 eds. A. Jeffrey and J. Engelbrecht (Springer-Verlag, Wien, 1994).

[34] W. van Saarloos, Front propagation into unstable states, Physics reports 386 29–222 (2003).

[35] W. van Saarloos and P. C. Hohenberg, Fronts, pulses, sources and sinks in generalized complex Ginzburg-Landau equations, Physica D 56 (1992) 303–367. Erratum 69 (1993) 209.

[36] H. Segur, Solitons and the inverse scattering transform, *Topics in ocean physics*, 235–277, eds. A. R. Osborne and P. Malanotte Rizzoli (North-Holland publishing co., Amsterdam, 1982).

[37] J. Swift and P. C. Hohenberg, Hydrodynamic fluctuations at the convective instability, Phys. Rev. A 15 (1977) 319–328.

[38] O. Thual and U. Frisch, Natural boundary in the Kuramoto model, *Combustion and nonlinear phenomena*, 327–336, eds. P. Clavin, B. Larrouturou, and P. Pelcé (Éditions de physique, Les Ulis, 1986).

[39] S. Toh, Statistical model with localized structures describing the spatio-temporal chaos of Kuramoto-Sivashinsky equation, J. Phys. Soc. Japan 56 (1987) 949–962.

[40] P. Winternitz, Symmetry reduction and exact solutions of nonlinear partial differential equations, *The Painlevé property, one century later*, 591–660, ed. R. Conte, CRM series in mathematical physics (Springer, New York, 1999).

[41] Yee T.-l., R. Conte, and M. Musette, Sur la “solution analytique générale” d’une équation différentielle chaotique du troisième ordre, 195–210, *From combinatorics to dynamical systems*, eds. F. Fauvet and C. Mitschi, IRMA lectures in mathematics and theoretical physics 3 (de Gruyter, Berlin, 2003). [http://arXiv.org/abs/nlin.PS/0302056](http://arXiv.org/abs/nlin.PS/0302056) Journées de calcul formel, Strasbourg, IRMA, 21–22 mars 2002.