THE SEIBERG–WITTEN INVARIANTS OF MANIFOLDS WITH WELLS OF NEGATIVE CURVATURE

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1. Introduction

A 4-manifold with $b_2^+ > 1$ and a nonvanishing Seiberg–Witten invariant cannot admit a metric of positive scalar curvature. This remarkable fact is proved [18] using the Weitzenböck–Lichnerowicz formula for the square of the Spin$^c$ Dirac operator, combined with the ‘curvature’ part of the Seiberg–Witten equations. Thus, in dimension 4, there is a strong generalization of Lichnerowicz’s vanishing theorem [8, 11, 12] for the index of the Dirac operator of a spin manifold with a metric of positive scalar curvature.

In the recent years, the method of semigroup domination [3, 14, 13] has led to a different sort of generalization of Lichnerowicz’s theorem and other theorems in which a positive curvature hypothesis leads to a topological vanishing theorem. Essentially, the hypothesis of positive curvature may be weakened to permit negative curvature on a ‘small’ set. (The precise notion of ‘small’ depends on what kind of curvature is being discussed—see the statement of Theorem 1 for the version we are using.) In this note, we use semigroup domination to show that a 4-manifold with positive scalar curvature away from a set of small volume must have vanishing Seiberg–Witten invariants. Moreover, the same vanishing holds for the Seiberg–Witten invariant of any finite covering space.

We sketch very briefly the definition of the Seiberg–Witten invariants, and refer to [10, 11, 9] for more details. Recall that a Spin$^c$ structure $\sigma$ on a smooth Riemannian 4-manifold $X$ determines a pair of spinor bundles $W^\pm \to X$ which are Hermitian bundles over $X$ of rank 2. A unitary connection $A$ on $L = \det(W^+)$ determines the Dirac operator

$$D_A^+: \Gamma(W^+) \to \Gamma(T^*X \otimes W^+) \to \Gamma(W^-)$$

where $\nabla_A$ is the induced connection on $W^+$ and $\rho$ denotes Clifford multiplication. The Seiberg–Witten equations, for a connection $A$ and spinor $\varphi$ are

$$\begin{cases}
D_A^+ \varphi = 0 \\
\rho(F_A^+) + i\mu^+ = i\tau(\varphi, \varphi)
\end{cases}$$

(1)

Here $\tau(\varphi, \varphi) = (\varphi \otimes \varphi^*)_0$ denotes the traceless part of the endomorphism $\varphi \otimes \varphi^*$ of $W^+$, and $\mu$ is a real 2-form. The solution space to equations (1), modulo gauge...
equivalence, gives the Seiberg–Witten moduli space $\mathcal{M}(X, \sigma)$. It is compact, and for a generic choice of $(g, \mu)$ is an oriented smooth manifold of dimension

$$\dim(\sigma) = \text{ind}_R(D_A^+ - 1/2(c(X) + \text{sign}(X)))$$

If $d = \dim(\sigma)$ is negative or odd, then the Seiberg–Witten invariant is defined to be 0. Otherwise, the Seiberg–Witten invariant is defined as $SW_X(\sigma) = \langle c_1(E) + \text{sign}(X) \rangle$, where $E \to M$ is a naturally defined complex line bundle. If $b_2^+(X) > 1$, then this count is independent of the metric, while if $b_2^+(X) = 1$, there is a mild dependence on the metric. Briefly, in this case, the space of (metrics, 2-forms) is divided by codimension-one ‘walls’ into connected components called chambers, on which the invariant is constant.

To state the main result, let $\mathcal{M}(n, K, D, V)$ be the class of Riemannian $n$-manifolds with Ricci curvature $\geq K$, diameter $\leq D$ and volume $\geq V$.

**Theorem 1.** Let $(X^4, g) \in \mathcal{M}(4, K, D, V)$ have $b_2^+ > 1$, and let $s_0 > 0$. There exists an $\epsilon > 0$, depending only on $K, D, V$, such that if the scalar curvature $s(X, g)$ is bounded below by $s_0$ except on a set of volume less than $\epsilon$, then $SW_X(\sigma) = 0$ for all Spin$^c$ structures on $X$. Moreover, for any finite cover $\pi : \tilde{X} \to X$ and Spin$^c$ structure $\tilde{\sigma} = \pi^* \sigma$ on $\tilde{X}$, we have $SW_{\tilde{X}}(\tilde{\sigma}) = 0$. If $b_2^+(X) = 1$, then these statements hold for the Seiberg–Witten invariant associated to the chamber containing $g$.

One can deduce a statement about Spin$^c$ structures $\tilde{\sigma}$ which do not necessarily pull back from a Spin$^c$ structure on $X$. This requires an additional hypothesis, and will be discussed following the proof of the statement. Note that the same $\epsilon$ works for finite covers of $X$ with arbitrary degree. In principle, a vanishing result for $SW_{\tilde{X}}$ could be deduced directly from the first statement of the theorem. However, in this approach, the constant $\epsilon$ (which depends [3, Proposition 1.2] on the volume and diameter) would decrease rapidly as the degree of the cover increases.

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## 2. A vanishing theorem

The key ingredient in the proof of Theorem 1 is a generalization of Kato’s inequality, referred to as semigroup domination [2, 5, 14]. For a Riemannian metric on $X$, let $\Delta$ denote the Laplacian on functions, and $s$ the scalar curvature function. Likewise, $\tilde{\Delta}$ and $\tilde{s}$ will represent the Laplacian and scalar curvature on a finite covering space $\tilde{X}$. Let $L$ be the self-adjoint operator $\nabla_A^* \nabla_A + s/4$, where $\nabla_A$ is the connection on the bundle of spinors, and $s$ is the scalar curvature of the metric. The operator $L$ is that part of the Weitzenböck decomposition of $D_A^* D_A$ which does not involve the
A VANISHING THEOREM

curvature of the bundle $W^+$. As above, for a Spin connection $\tilde{A}$ on an arbitrary Spin structure on a finite cover of $X$, we will denote by $\tilde{L}$ the operator $\nabla^*_\tilde{A} \nabla_{\tilde{A}} + \tilde{s}/4$.

The principle of semigroup domination, in its strong form [13], yields the following pointwise inequality.

**Lemma 2.** For any unitary connection $A$ on $L$, and any $\psi \in L^2(W^+)$

$$|e^{-tL}(\psi)(x)| \leq 2e^{-t(\Delta + s/4)}|\psi|(x) \tag{2}$$

The smoothing property of the operators $e^{-tL}$ and $e^{-t(\Delta + s/4)}$ implies that the quantities being compared in (2) are continuous functions of $x$, so that the inequality makes sense pointwise. It is also important to note that the right side of this inequality is independent of $A$.

The main step in Theorem 1 is the following vanishing result. From now on, the phrase ‘for sufficiently small $\varepsilon$’ will be used as a shorthand for the hypotheses of Theorem 1.

**Proposition 3.** For sufficiently small $\varepsilon$, any solution to the SW equations

$$D_A^+ \varphi = 0, F_A^+ = \tau(\varphi, \varphi)$$

is reducible (ie has $\varphi \equiv 0$).

**Proof of Proposition 3.** Proposition 2.4 of [14], and the discussion following it shows

Claim A. For sufficiently small $\varepsilon$, the operator $\Delta + s/4$ is positive, as is $\tilde{\Delta} + \tilde{s}/4$ for every covering space of $X$.

The second step is analogous to [14, Theorem 2.2]

Claim B. For sufficiently small $\varepsilon$, the operator $L$ has only positive eigenvalues.

**Proof of Claim B.** Suppose that $L\psi = \lambda \psi$ with $\lambda \leq 0$ and $\psi \neq 0$. By elliptic regularity for $L$, we may assume that $\psi$ is continuous. In particular, there is a point $x_0$ with $\psi(x_0) \neq 0$. According to Claim A, the right hand side of (2), evaluated at $x_0$, goes to 0 as $t \to \infty$. However, the left hand side is $e^{-t\lambda} |\psi(x_0)|$, which goes to infinity if $\lambda < 0$. If $\lambda = 0$, then the left hand side is the non-zero constant $|\psi(x_0)|$, so we still get a contradiction.

Claim C. For $\varepsilon$ sufficiently small, there is no non–zero solution to $L\varphi = -f(x)\varphi$ where $f$ is a continuous function with $f(x) \geq 0$ for all $X$.

**Proof.** Choose an orthonormal basis $\{\psi_i\}$ of eigenspinors for $L$ with eigenvalues $\lambda_i$, which are all positive according to Claim B. Then $\varphi = \sum a_i \psi_i$, where by a standard argument the convergence is in $L^2_\varphi$. It follows that $L\varphi = \sum a_i \lambda_i \psi_i$, so that

$$< L\varphi, \varphi >_{L^2} = \sum \lambda_i a_i^2 > 0$$
But
\[ <L \varphi, \varphi >_{L^2} = - \left( \int f(x) < \varphi, \varphi > \right) \leq 0 \]
which is a contradiction. \[ \square \text{Claim C} \]

To prove Proposition 3, assume that \((A, \varphi)\) is a solution to the Seiberg–Witten equations. By regularity for solutions of the Seiberg–Witten equations, \(\varphi\) is smooth, and so is \(|\varphi|^2\). Using the Weitzenböck formula, and substituting for the curvature term,
\[ 0 = (D_A^+)^* D_A^+ \varphi = (\nabla_A^* \nabla_A + s/4) \varphi + \frac{1}{2} \tau(\varphi, \varphi) \varphi = L \varphi + \frac{1}{4} |\varphi|^2 \varphi \]
So \(L \varphi = -\frac{1}{4} |\varphi|^2 \varphi\) and we conclude that \(\varphi \equiv 0\) by Claim C. \[ \square \]

Proof of Theorem 1. If \(b_+^2 > 1\), then (\cite{1}, \cite{10, §6.3}) the set of ‘generic’ metrics for which the Seiberg–Witten equations admit no reducible solution is open and dense in the space of all metrics (using the \(C^\infty\) topology).

For sufficiently small \(\delta > 0\), Proposition \[3\] provides an \(\epsilon > 0\) such that if \((X, g') \in M(K - \delta, D + \delta, V - \delta)\), and the scalar curvature \(s(X, g') \geq s_0\) except on a set of volume less than \(\epsilon + \delta\), then any solution to the Seiberg–Witten equations on \(X\) is reducible. Now if \((X, g) \in M(4, K, D, V)\), and
\[ \text{vol}_g \{ x \in X | s_g(x) < s_0 \} \leq \epsilon, \]
approximate \(g\) by a generic metric \(g' \in M(K - \delta, D + \delta, V - \delta)\) with
\[ \text{vol}_{g'} \{ x \in X | s_{g'}(x) < s_0 \} \leq \epsilon + \delta. \]
Now any solution to the Seiberg–Witten equations is reducible, but \(g'\) is chosen so that there are no reducible solutions either. Since we may compute it with respect to any metric, the Seiberg–Witten invariant must vanish.

To prove the vanishing statement for the Seiberg–Witten invariant on a covering space \(\tilde{X}\), we use the observation of \cite{12, 3} that the curvature assumption on \(X\) imply that the operator \(\Delta + \tilde{s}/4\) is positive. Thus the proof of Proposition \[3\] applies to show that any solution to the Seiberg–Witten equations on \(\tilde{X}\) must be reducible. This argument does not use the fact that \(\tilde{g}\) is pulled back from \(X\), but merely that the Laplacian on functions is pulled back from \(X\). This, in turn only requires that the metric on \(\tilde{X}\) be the pullback metric.

Now we make use of a simple principle about reducible solutions to the Seiberg–Witten equations.

Claim D. Suppose that \(\tilde{\sigma} = \pi^* \sigma\). Then for a generic metric on \(X\), there are no reducible solutions to the Seiberg–Witten equations on \(\tilde{X}\).

Proof of Claim D. Denote by \(\tilde{\mathcal{L}} = \pi^* \mathcal{L}\) the determinant bundle of the Spin\(^c\) structure \(\tilde{\sigma}\). For a generic metric \(g\) on \(X\), there is a \(g\)-self-dual form \(\alpha \in H^2_\ast (X)\) with \(\alpha \cup c_1(\mathcal{L}) \neq 0\). This is the content of the generic metrics theorem for \(X\). But, pulling back back to \(\tilde{X}\),
this means that $\pi^*\alpha \cup c_1(\tilde{L}) \neq 0$, so $c_1(\tilde{L})$ can’t be represented by a $\pi^*g$–anti-self-dual form. Hence there are no reducible solutions on $\tilde{X}$.

The rest of the proof of the vanishing theorem works exactly as above.

We can still give a vanishing result for the Seiberg–Witten invariants of a covering space, even for $\text{Spin}^c$ structures which don’t pull back from $X$. The result requires an additional mild topological hypothesis; there are perhaps some variations of this method giving similar results.

**Theorem 4.** Let $(X^4, g) \in \mathcal{M}(4, K, D, V)$ have $b_2^+ > 1$, and let $s_0 > 0$. Assume in addition that

$$\frac{1}{2}(e(X) + \text{sign}(X)) = b_2^+(X) - b_1(X) + 1 > 0$$

Then there exists an $\epsilon > 0$, depending only on $K, D, V$, such that if the scalar curvature $s(X, g)$ is bounded below by $s_0$ except on a set of volume less than $\epsilon$, then $SW_{\tilde{X}}(\tilde{\sigma}) = 0$ for all $\text{Spin}^c$ structures $\tilde{\sigma}$ on $\tilde{X}$.

**Proof.** As in the proof given above, for sufficiently small $\epsilon$, and any metric on $X$ satisfying the hypotheses, any solution to the Seiberg–Witten equations on $\tilde{X}$ will be reducible. Let $(\tilde{A}, 0)$ be such a solution, and let $D_{\tilde{A}}^\pm$ be the corresponding Dirac operator. The Weitzenböck formula says that

$$(D_{\tilde{A}}^\pm)^* D_{\tilde{A}}^\pm = \tilde{L} + \rho(F_{\tilde{A}}^\pm)$$

But since $\tilde{A}$ is reducible, $F_{\tilde{A}}^\pm = 0$, and so $(D_{\tilde{A}}^\pm)^* D_{\tilde{A}}^\pm$ is the operator $\tilde{L}$. But the argument above says that $\tilde{L}$ is positive (for sufficiently small $\epsilon$, of course.) It follows that $\ker(D_{\tilde{A}}^\pm) = 0$, and therefore that the index of $D_{\tilde{A}}^\pm$ is less than or equal to 0.

By definition, the Seiberg–Witten invariant $SW_{\tilde{X}}(\tilde{\sigma})$ is non-zero only when the dimension of the Seiberg–Witten moduli space $\mathcal{M}(\tilde{X}, \tilde{\sigma})$ is $\geq 0$. This dimension is given by the formula

$$\dim(\tilde{\sigma}) = \text{ind}(D_{\tilde{A}}^\pm) - \frac{1}{2}(e(\tilde{X}) + \text{sign}(\tilde{X}))$$

$$= \text{ind}(D_{\tilde{A}}^\pm) - n \cdot \frac{1}{2}(e(X) + \text{sign}(X))$$

Since the first term on the right hand side is non-positive, and we have assumed that the second term is negative, this gives a contradiction.

Note that the hypotheses of Theorem 4 are satisfied for $X$ simply connected. So the argument presented above gives a simpler way to prove Theorem 1 in that case.

The reader may wonder if there are examples of manifolds with vanishing Seiberg–Witten invariants to which we can apply the part of Theorem 1 dealing with covering spaces. A class of such manifolds was described by Shuguang Wang [17]; his examples are complex surfaces $\tilde{X}$ admitting a free anti-holomorphic involution $\tau$. He shows that
the quotient $X = \tilde{X}/\tau$ has vanishing Seiberg–Witten invariants; however Theorems 1 and 4 still apply.

3. Higher dimensions

There are more elaborate versions of Lichnerowicz’ index-theoretic obstruction to the existence of a metric of positive scalar curvature on a spin manifold—see [12] for a recent overview. It was noted by Hitchin [6] that the Dirac operator on a spin manifold $M^n$ defines an element $\alpha(M) \in KO_n$, which vanishes if $M$ admits a metric of positive scalar curvature. For $n \geq 5$, it was shown by Stolz [16] that a simply-connected spin manifold $M^n$ for which $\alpha(M) = 0$ admits a metric of positive scalar curvature. The construction of positive scalar curvature metrics uses the surgery method of Schoen–Yau [15] and Gromov–Lawson [4].

Using these results, we show that a metric on a manifold with ‘mostly’ positive curvature can sometimes be traded in for a metric having everywhere positive curvature.

**Theorem 5.** Suppose that $n \geq 5$. Let $(X^n, g) \in \mathcal{M}(n, K, D, V)$ be simply connected, and let $s_0 > 0$. There exists an $\epsilon > 0$, depending only on $n, K, D, V$, such that if the scalar curvature $s(X, g)$ is bounded below by $s_0$ except on a set of volume less than $\epsilon$, then $X$ admits a metric of strictly positive curvature.

**Proof.** Following [14], choose $\epsilon$ sufficiently small so that the operator $\Delta + s/4$ is positive. Then semigroup domination implies that the kernel and cokernel of the Dirac operator must vanish. This implies that the invariant $\alpha(X)$ is 0. By the result of Stolz, it follows that $X$ must in fact admit a metric of positive scalar curvature.

It would be more satisfactory to have a direct method for modifying a metric to eliminate a small region of negative curvature rather than to have to appeal to the theorem of Stolz. Such an argument would have a better chance of applying in dimension 4. Apparently there are results along these lines for Ricci curvature, but none for scalar curvature.

The argument above should extend to non-simply-connected manifolds whose groups satisfy the ‘Gromov–Lawson–Rosenberg’ conjecture. This states that the existence of a positive scalar curvature metric on $M$ is equivalent to the vanishing of an index-theoretic invariant $\alpha(M, f) \in KO_n(C^*\pi)$. Here $f : M \to B\pi$ classifies the universal cover of the fundamental group $\pi$ of $M$ and $KO_n(C^*\pi)$ denotes the K-theory of a $C^*$-algebra associated to $\pi$. The proof of the required vanishing result would require that semigroup domination applies to the Dirac operator with coefficients in the Hilbert space $C^*(\pi)$.

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**A VANISHING THEOREM**

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