Higher Siegel theta lifts on Lorentzian lattices, harmonic Maass forms, and Eichler–Selberg type relations

Joshua Males

Received: 6 July 2021 / Accepted: 16 February 2022 / Published online: 21 April 2022
© The Author(s), under exclusive licence to Springer-Verlag GmbH Germany, part of Springer Nature 2022

Abstract
We investigate so-called “higher” Siegel theta lifts on Lorentzian lattices in the spirit of Bruinier–Ehlen–Yang and Bruinier–Schwagenscheidt. We give a series representation of the lift in terms of Gauss hypergeometric functions, and evaluate the lift as the constant term of a Fourier series involving the Rankin–Cohen bracket of harmonic Maass forms and theta functions. Using the higher Siegel lifts, we obtain a vector-valued analogue of Mertens’ result stating that the Rankin–Cohen bracket of the holomorphic part of a harmonic Maass form of weight $\frac{3}{2}$ and a unary theta function, plus a certain form, is a holomorphic modular form. As an application of these results, we offer a novel proof of a conjecture of Cohen which was originally proved by Mertens, as well as a novel proof of a theorem of Ahlgren and Kim, each in the scalar-valued case.

1 Introduction

In recent years, there have been many investigations into certain theta lifts and their relationship to modular objects. Perhaps one of the most striking applications is in realising rationality and algebraicity results. In a recent breakthrough paper, Bruinier et al. [8] made a major advance towards the Gross–Zagier conjecture by proving that a certain two-variable Green function evaluated at CM points in one variable and an average over CM points in the other variable takes algebraic values. A pivotal result that the authors there used was the connection between the Green function and a “higher” Siegel theta lift.1 A further example lies in [3], where Alfes-Neumann, Bringmann, Schwagenscheidt, and the author used the higher Siegel theta lift2 on lattices of signature $(1, 2)$ to investigate traces of cycle integrals of a certain cusp form. By relating the lift to the Fourier coefficients of certain modular objects called harmonic Maass forms, the results also gave an alternative proof of the rationality of such traces of cycle integrals, which had previously been shown in [4].

1 Note that the authors there use a different signature convention than the current paper.
2 The paper [3] also used the higher Millson theta lift.
More recently, Bruinier and Schwagenscheidt [11] investigated the Siegel theta lift on a Lorentzian lattice $L$ (that is, of signature $(1, n)$). For $\tau = u + iv \in \mathbb{H}$ and $z \in \text{Gr}(L)$, the Grassmanian of $L$ i.e. positive lines in $L \otimes \mathbb{R}$, this is given by

$$\int_{\mathcal{F}} \left( f(\tau), \Theta_L(\tau, z) \right) v^k d\mu(\tau),$$

where the regularised integral is defined by $\int_{\mathcal{F}}^\text{reg} := \lim_{T \to \infty} \int_{\mathcal{F}_T}$, where $\mathcal{F}_T$ denotes the standard fundamental domain for $\Gamma$ truncated at height $T$. Here, $\mathcal{F}$ is the standard fundamental domain for $\text{SL}_2(\mathbb{Z})$, $f$ is a vector-valued harmonic Maass form of appropriate weight, $\langle \cdot, \cdot \rangle$ is the natural bilinear pairing on the group ring $\mathbb{C}[L'/L]$ with $L'$ the dual lattice, and $d\mu = \frac{dudv}{v^2}$ is the usual invariant measure on $\mathbb{H}$. Moreover, $\Theta_L(\tau, z)$ is a vector-valued Siegel theta function defined on $\mathbb{H} \times \text{Gr}(L)$. As a function of $\tau \in \mathbb{H}$, it transforms as a vector-valued modular form of weight $(1 - n)/2$ with respect to the Weil representation $\rho_L$ associated to $L$, and is invariant in $z$ under the subgroup of $O(L)$ that fixes $L'/L$ pointwise. Explicitly, this means that for $(g, \phi) \in \text{Mp}_2(\mathbb{Z})$, the metaplectic extension of $\text{SL}_2(\mathbb{Z})$ (see Sect. 2.1) and $h$ in the subgroup of $O(L)$ that fixes $L'/L$ pointwise we have

$$\Theta_L(g\tau, z) = \phi(\tau)\overline{\phi(\tau)}\rho_L(g, \phi)\Theta_L(\tau, z), \quad \Theta_L(\tau, hz) = \Theta_L(\tau, z).$$

In the present paper we consider an extension of this lift, first considered by Bruinier et al. in [8] for the $n = 2$ case. Borrowing their terminology, we call this a higher Siegel theta lift. Let $k := \frac{1-n}{2}$. The case of $j = 0$ was studied centrally in the influential work of Bruinier, Imamoğlu, and Funke [10]. Since the case $j = 0$ is well-understood, let $j \in \mathbb{N}$ throughout unless otherwise stated. For a harmonic Maass form $f \in H_{k-2j,L}$ (see Sect. 2.2 for a precise definition) we consider

$$\Lambda^\text{reg}_j(f, z) := \int_{\mathcal{F}}^\text{reg} \left( R_{k-2j}(f)(\tau), \Theta_L(\tau, z) \right) v^k d\mu(\tau),$$

where $R^\circ_k := R_{k+2n-2} \circ \cdots \circ R_x$ with $R^0_x := \text{id}$ is an iterated version of the Maass raising operator $R_x := 2i \frac{\partial}{\partial \tau} + \frac{x}{2}$. By the results of [7,Section 2.3] for the Siegel theta function the integral converges for every $z \in \mathbb{H}$.

This situation lies at the interface of [3, 8] which included iterated raising operators but restricted to the case of $n = 2$, and [11] where no iterated raising operators were included, but $n$ was unrestricted. We borrow techniques from each of these papers, and in Sect. 3 obtain a description of the lift $\Lambda^\text{reg}_j$ as a series involving Gauss hypergeometric functions $2F_1$, as well as a constant term of a Fourier expansion involving coefficients of $f$, a theta function, and a harmonic Maass form. We also note how one could obtain the Fourier expansion of the lift, but omit the calculation as it is not required here. Note that for $j = 0$ we recover the lift studied by Bruinier [7] and Bruinier–Schwagenscheidt [11], and for $n = 2$ the lift used centrally in [3].

Moreover, in [16,Theorem 1.2] Mertens used holomorphic projection to show that the Fourier coefficients of all scalar-valued mock modular forms of weight $\frac{1}{2}$ satisfy Eichler-Selberg type relations. We provide a vector-valued analogue of these relations, relying on the framework of the higher Siegel theta lifts for $j > 0$. To state the result, we need some notation. When evaluated at a special point $w$ (see Sect. 2.6) the Siegel theta function on $L$ splits as the tensor product $\Theta_P \otimes \Theta_N$ where $P$ is one-dimensional positive-definite, and $N$ is $n$-dimensional negative-definite. For $X \in L$ by $X_*$ we mean the projection of $X$ onto $z$, and furthermore we denote vector-valued Rankin–Cohen brackets by $[\cdot, \cdot]_L$. Finally, let $\Theta^{+}_P$ be the holomorphic part of a preimage of $\Theta_P$ under $\xi_\kappa := 2i v^k \frac{\partial}{\partial w}$. By computing the lift in
two different ways, taking the difference and invoking Serre duality, we prove the following theorem (for undefined notation see Sect. 2).

**Theorem 1.1** Let $L$ be an even lattice of signature $(1, 1)$ and $w$ be a special point. Then for every $j > 0$ there is an explicit constant $C_j \in \mathbb{R}$ such that

$$\left[ G_p^+(\tau), \Theta_{N}^-(\tau) \right]_j + C_j \sum_{m=0}^{\infty} \sum_{q(X)=m} \left( \sqrt{|q(X+w)|} - \sqrt{|q(X)|} \right)^{1+2j} q^m$$

is a holomorphic vector-valued modular form of weight $2j + 2$ for $\rho_L$.

**Remarks** (1) One could obtain a similar vector-valued analogue of [16, Theorem 1.3], i.e. for $G_p^+$ replaced by the holomorphic part of harmonic Maass forms of weight $\frac{1}{2}$ (so including the classical mock theta functions of Ramanujan) by similar methods, replacing $\Theta_{L}$ by a modified theta function $\Theta_{L}^*$ as in [11, Section 4.3]. In doing so, one could also obtain new recurrence relations for the classical mock theta functions. We leave the details for the interested reader.

(2) For $n \geq 2$ it is possible to obtain similar theorems. However, the form on the right-hand side becomes much more complicated. Using the contiguous Gauss hypergeometric functions to rewrite the output of Theorem 3.1, the forms should be linear combinations of those stated in Theorem 1.1 and similarly-shaped sums with factors of $\arcsin \left( \sqrt{\frac{q(X)}{q(X+w)}} \right)$ (compare [11, equation (3.3)]).

As an application, we obtain novel proofs of a conjecture of Cohen [13] which was originally proved by Mertens [17, Theorem 1], as well as a theorem of Ahlgren and Kim [2, Proposition 4], each of which is detailed below. For $\ell \geq 1$ and $n \geq 1$ we let

$$\lambda_{\ell}(n) = \frac{1}{2} \sum_{d|n} \min(d, n/d) \ell,$$

and

$$\mathcal{H}(\tau) = \sum_{n \geq 0} H(n) q^n,$$

where for $n \in \mathbb{N}$, we let $H(n)$ be the Hurwitz class numbers, i.e. the class number of binary quadratic forms of discriminant $-n$, where the class containing $a(x^2 + y^2)$ (resp. $a(x^2 + xy + y^2)$) is counted with multiplicity $\frac{1}{2}$ (resp. $\frac{1}{3}$), and we set $H(0):=-\frac{1}{12}$. Then $\mathcal{H}$ is a mock modular form of weight $\frac{3}{2}$, i.e. the holomorphic part of a scalar-valued harmonic Maass form of weight $\frac{3}{2}$. Let $\theta(\tau):=\sum_{n \in \mathbb{Z}} q^n$ be the classical Jacobi theta function. For $f = \sum_{n} c_f(n) q^n$ we let $f^{\text{odd}} = \sum_{n \text{ odd}} c_f(n) q^n$. Then we have the following theorem of Mertens [17, Theorem 1]. We offer a novel proof in Sect. 4.

**Theorem 1.2** For $j \geq 0$ the function

$$c_j[\mathcal{H}, \theta]^{\text{odd}} + \sum_{n \geq 0} \lambda_{2j+1}(2n + 1) q^{2n+1}$$

with $c_j := \frac{j! \sqrt{\pi}}{\Gamma(j+1/2)}$ is a holomorphic modular form of weight $2j + 2$ for $\Gamma_0(4)$. For $j > 0$ it is a cusp form.
Furthermore, let spt be the classical smallest parts partition function of Andrews [5] and \( p(n) \) the integer partition function. Define

\[
g(\tau) = q^{-\frac{1}{24}} \sum_{n \geq 0} \left( \text{spt}(n) + \frac{1}{12}(24n - 1)p(n) \right) q^n.
\]

Let \( \eta \) be the usual Dedekind \( \eta \)-function and define

\[
G_k(\tau) := - \sum_{r > s > 0} \left( \frac{12}{r^2 - s^2} \right) s^{k-1} q^{\frac{r^2}{12}}.
\]

Then Zagier stated [19, Section 6] and Ahlgren and Kim proved [2, Proposition 1.4] the following theorem.\(^3\) Again, we offer a novel proof in Sect. 4.

**Theorem 1.3** For every \( j > 0 \) we have that

\[
G_{2j+2}(\tau) + 24^j \binom{2j}{j}^{-1} \left[ g\left(\frac{\tau}{24}\right), \eta(\tau) \right]_j
\]

is a modular form of weight \( 2j + 2 \) on \( \text{SL}_2(\mathbb{Z}) \).

To end the introduction, we remark as a cute application that by choosing \( n = j = 1 \) in Theorem 1.1 one obtains a re-proof in terms of higher Siegel lifts of the classical relation

\[
\sum_{n \in \mathbb{Z}} (t - n^2) H(4t - n^2) = \sum_{a,b \in \mathbb{N}} \min(a,b)^3
\]

that may also be concluded by Theorem 1.2. One may obtain proofs of other similar relations of Hurwitz class numbers.

**Outline**

We begin in Sect. 2 by recalling preliminary results needed for the rest of the paper. In Sect. 3 we prove the main results of the paper regarding the higher Siegel lift. Finally, Sect. 4 is devoted to proving Theorem 1.1 and giving novel proofs of Theorems 1.2 and 1.3.

**2 Preliminaries**

We collect some preliminary results needed for the sequel.

**2.1 The Weil representation**

The metaplectic extension of \( \text{SL}_2(\mathbb{Z}) \) is defined as

\[
\tilde{\Gamma} := \text{Mp}_2(\mathbb{Z}) := \left\{ (\gamma, \phi) : \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}), \phi : \mathbb{H} \to \mathbb{C} \text{ holomorphic}, \phi^2(\tau) = c\tau + d \right\},
\]

with generators \( \tilde{T} := \left( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \right) \) and \( \tilde{S} := \left( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau} \right) \). Moreover, we let \( \tilde{\Gamma}_\infty \) be the subgroup generated by \( \tilde{T} \).

\(^3\) For the \( j = 0 \) case Ahlgren and Kim explicitly showed equality with the quasi-modular Eisenstein series \( \frac{1}{12} E_2 \).

Springer
Let $L$ be an even lattice of signature $(r, s)$ with quadratic form $q$ and associated bilinear form $(\cdot, \cdot)$. We let $L'$ denote its dual lattice, which has a group ring of $L'/L$ denoted by $\mathbb{C}[L'/L]$ with standard basis elements $e_\mu$ for $\mu \in L'/L$. Furthermore, there is a natural bilinear form $\langle \cdot, \cdot \rangle$ on the group ring given by $\langle e_\mu, e_\nu \rangle = \delta_{\mu, \nu}$. Then the Weil representation $\rho_L$ associated with $L$ is the representation of $\tilde{G}$ on $\mathbb{C}[L'/L]$ defined by

$$
\rho_L (T) (e_\mu) := e(q(\mu))e_\mu, \quad \rho_L (\tilde{S}) (e_\mu) := \frac{e \left( \frac{1}{8} (s-r) \right)}{\sqrt{|L'|/L|}} \sum_{\nu \in L'/L} e(\langle \nu, \mu \rangle) e_\nu.
$$

Here and throughout $e(x) := e^{2\pi i x}$. The Weil representation $\rho_{L^-}$ associated to the lattice $L^- = (L, -q)$ is called the dual Weil representation of $L$.

### 2.2 Harmonic Maass forms

Let $\kappa \in \frac{1}{2} \mathbb{Z}$ and define the usual slash-operator by

$$
f \mid_{k, \rho_{L}} (\gamma, \phi)(\tau) := \phi(\tau)^{-2\kappa} \rho_{L}^{-1} (\gamma, \phi) f(\gamma \tau),
$$

for a function $f : \mathbb{H} \to \mathbb{C}[L'/L]$ and $(\gamma, \phi) \in \tilde{G}$. A harmonic Maass form of weight $\kappa$ with respect to $\rho_L$ is any smooth function $f : \mathbb{H} \to \mathbb{C}[L'/L]$ that satisfies the following three conditions [9].

1. It vanishes under the action of the weight $\kappa$ Laplace operator

$$
\Delta_\kappa := -v^2 \left( \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) + i \kappa v \left( \frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right),
$$

2. It is invariant under the slash-operator $|_{k, \rho_{L}},$

3. There exists a $\mathbb{C}[L'/L]$-valued Fourier polynomial (the principal part of $f$)

$$
P_f(\tau) := \sum_{\mu \in L'/L} \sum_{n \leq 0} c_+^\tau (\mu, n) e(n\tau) e_\mu
$$

such that $f(\tau) - P_f(\tau) = O(e^{-\varepsilon v})$ as $v \to \infty$ for some $\varepsilon > 0$.

The vector space of harmonic Maass forms of weight $\kappa$ with respect to $\rho_L$ is denoted by $H_{k, L}$, and $M^!_{k, L}$ is the subspace of weakly holomorphic modular forms. It is a classical fact that any $f \in H_{k, L}$ can be decomposed as $f = f^+ + f^-$ where $f^+$ is holomorphic and $f^-$ is non-holomorphic. These have Fourier expansions of the form

$$
f^+(\tau) = \sum_{\mu \in L'/L} \sum_{n \gg -\infty} c_+^\tau (\mu, n) e(n\tau) e_\mu,
$$

$$
f^-(\tau) = \sum_{\mu \in L'/L} \sum_{n < 0} c_-^\tau (\mu, n) \Gamma(1 - \kappa, 4\pi |n| v) e(n\tau) e_\mu,
$$

where $\Gamma(s, x) := \int_x^\infty t^{s-1} e^{-t} dt$ denotes the incomplete Gamma function.

The antilinear differential operator $\xi_{\kappa} = 2i v^2 \frac{\partial}{\partial v}$ from the introduction maps a harmonic Maass form $f \in H_{k, L}$ to a cusp form of weight $2 - \kappa$ for $\rho_{L^-}$. In addition to the Maass raising operator in the introduction, we also require the Maass lowering operator $\text{L}_\kappa := -2i v^2 \frac{\partial}{\partial v}$.
2.3 Maass–Poincaré series

Let $\kappa \in \frac{1}{2}\mathbb{Z}$ with $\kappa < 0$. We let $M_{\mu, \nu}$ be the usual $M$-Whittaker function—see [1, equation 13.1.32], for example—and for $s \in \mathbb{C}$ and $\gamma \in \mathbb{R}\setminus\{0\}$, define

$$M_{\kappa, s}(y) := |y|^{-\frac{\kappa}{2}}M_{\text{sgn}(y)\frac{\kappa}{2}, -\frac{s}{2}}(|y|).$$  \hspace{1cm} (2.1)

For $\mu \in L'/L$ and $m \in \mathbb{Z} - q(\mu)$ with $m > 0$ we define the vector-valued Maass–Poincaré series [7]

$$F_{\mu, -m, \kappa, s}(\tau) := \frac{1}{2\Gamma(2s)} \sum_{(\gamma, \phi) \in \Gamma_{\infty}\setminus\Gamma} (M_{\kappa, s}(-4\pi mv)e(-mu)e_{\mu})|_{\kappa, \rho_L}(\gamma, \phi)(\tau).$$

The series converges absolutely for $\text{Re}(s) > 1$, and at the point $s = 1 - \frac{\kappa}{2}$, the function

$$F_{\mu, -m, \kappa, 1-\frac{\kappa}{2}}(\tau)$$

defines a harmonic Maass form in $H_{\kappa, L}$ with principal part $e(-m\tau)(e_{\mu} + e_{-\mu}) + c$ for some constant $c \in \mathbb{C}[L'/L]$. Let $H_{\kappa, L}^1$ be the space of harmonic Maass forms with at most linear exponential growth at $\infty$, and $H_{\kappa, L}^{\text{cusp}}$ be the subspace of $H_{\kappa, L}^1$ that maps to a cusp form under $\xi_\kappa$. Then it is a classical fact that for $\kappa < 0$ any $f \in H_{\kappa, L}^{\text{cusp}}$ may be written as

$$f(\tau) = \frac{1}{2} \sum_{h \in L'/L} \sum_{m \geq 0} c^+_f(h, -m)F_{\kappa, m, \kappa}(\tau) + c,$$

where $c$ is a constant in $\mathbb{C}[L'/L]$ which may be non-zero only if $\kappa = 0$.

The following lemma is [3, Lemma 2.1], and follows inductively from [8, Proposition 3.4].

**Lemma 2.1** For $n \in \mathbb{N}_0$ we have that

$$R^\mu_n\left(F_{\mu, -m, \kappa, s}\right)(\tau) = (4\pi m)^n \frac{\Gamma(s + n + \frac{\kappa}{2})}{\Gamma(s + \frac{\kappa}{2})} F_{\mu, -m, \kappa + 2n, s}(\tau).$$

2.4 Operators on vector-valued modular forms

Let $L$ be an even lattice. Then the space of $\mathbb{C}[L'/L]$-valued smooth modular forms of weight $\kappa$ with respect to the representation $\rho_L$ is denoted by $A_{\kappa, L}$.

Let $K \subset L$ be a sublattice of finite index. Since $K \subset L \subset L' \subset K'$ we have $L'/K \subset L'/K \subset K'/K$, and therefore obtain a map $L'/K \rightarrow L'/L, \mu \mapsto \tilde{\mu}$. For $\mu \in K'/K$ and $f \in A_{\kappa, L}$, and $g \in A_{\kappa, K}$, we define

$$(f_K)_\mu := \begin{cases} f_{\tilde{\mu}} & \text{if } \mu \in L'/K, \\ 0 & \text{if } \mu \notin L'/K, \end{cases} \quad (g^L)_\mu = \sum_{\alpha \in L/K} g_{\alpha + \mu},$$

where $\mu$ is a fixed preimage of $\tilde{\mu}$ in $L'/K$. The following lemma may be found in [12, Section 3].

**Lemma 2.2** There are two natural maps

$$\text{res}_{L/K} : A_{\kappa, L} \rightarrow A_{\kappa, K}, \quad f \mapsto f_K, \quad \text{tr}_{L/K} : A_{\kappa, K} \rightarrow A_{\kappa, L}, \quad g \mapsto g^L,$$

such that for any $f \in A_{\kappa, L}$ and $g \in A_{\kappa, K}$, we have $(f, \overline{g^L}) = (f_K, \overline{g})$.  

\(\text{Springer}\)
2.5 Rankin–Cohen brackets

Let $K$ and $L$ be even lattices. For $n \in \mathbb{N}_0$ and functions $f \in A_{K,K}$ and $g \in A_{\ell,L}$ with $\kappa, \ell \in \frac{1}{2}\mathbb{Z}$ the $n$-th Rankin–Cohen bracket is defined by

$$[f, g]_n := \frac{1}{(2\pi i)^n} \sum_{r+s \geq 0 \atop r+s = n} (-1)^r \frac{\Gamma(n)}{\Gamma(n+1)} \frac{\Gamma(\ell+n)}{\Gamma(\ell+1)} f^{(r)} \otimes g^{(s)},$$

where $f^{(r)} = \frac{\partial^r}{\partial \tau^r} f$. For two vector-valued functions $f = \sum_{\mu} f_{\mu} \epsilon_{\mu} \in A_{K,K}$ and $g = \sum_{\nu} g_{\nu} \epsilon_{\nu} \in A_{\ell,L}$ their tensor product is defined as

$$f \otimes g := \sum_{\mu,\nu} f_{\mu} g_{\nu} \epsilon_{\mu+\nu} \in A_{K+\ell,K \oplus L}.$$

The following proposition is the vector-valued version of [8, Proposition 3.6].

**Proposition 2.3** Let $f \in H_{K,K}$ and $g \in H_{\ell,L}$ be harmonic Maass forms. For $n \in \mathbb{N}_0$ we have

$$(-4\pi)^n L_{K+\ell+2n} ([f, g]_n) = \frac{\Gamma(n+1)}{\Gamma(n)} L_{K}(f) \otimes R^n_{\ell}(g)$$

$$+ (-1)^n \frac{\Gamma(n+1)}{\Gamma(n+1)} R^n_{\ell}(f) \otimes L_{\ell}(g).$$

2.6 Theta functions

Let $(K, q)$ be a positive-definite lattice of rank $n$. Then one may define the vector-valued theta function

$$\Theta_{K}(\tau) := \sum_{\mu \in K^0} \sum_{X \in K+\mu} e(q(X)\tau) \epsilon_{\mu},$$

which is a holomorphic modular form of weight $\frac{n}{2}$ for the Weil representation $\rho_{K}$.

For an even lattice $L$ of signature $(1, n)$ the Siegel theta function is defined by

$$\Theta_{L}(\tau, z) := v^\frac{n}{2} \sum_{\mu \in L^0/\ell} \sum_{X \in L+\mu} e(q(X) + q(X_{\perp})) \epsilon_{\mu},$$

where $X_{\perp}$ denotes the orthogonal projection of $X$ onto $z$. The Siegel theta function transforms in $\tau$ like a modular form of weight $\frac{1-n}{2}$ for $\rho_{L}$ and is invariant in $z$ under the subgroup $\Gamma_{L}$ of the orthogonal group $O(L)$ which fixes the classes of $L'//L$. If $K \subset L$ is a sublattice of finite index, Lemma 2.2 implies that

$$\Theta_{L} = (\Theta_{K})^L.$$  

(2.3)

As in [11], we call $w \in \text{Gr}(L)$ a special point if it is defined over $\mathbb{Q}$ (i.e. $w \in L \otimes \mathbb{Q}$). Then the orthogonal complement of $w$ in the non-degenerate space $V = L \otimes \mathbb{R}$ is defined over $\mathbb{Q}$ as well, and we denote it by $w^{\perp}$. We have the splitting $L \otimes \mathbb{Q} = w \oplus w^{\perp}$. Let $P = L \cap w$ and $N = L \cap w^{\perp}$ be the corresponding positive definite one-dimensional and negative definite $n$-dimensional sublattices of $L$, and note that $P \oplus N$ has finite index in $L$.

A direct computation shows that the evaluation of the Siegel theta function $\Theta_{L}$ at $w$ splits as

$$\Theta_{L}(\tau, w) = \Theta_{P \oplus N}(\tau, w) = \Theta_{P}(\tau) \oplus v^{\frac{n}{2}} \Theta_{N}(\tau).$$  

(2.4)
2.7 Unary theta functions

Let \( d \in \mathbb{N} \). Then the lattice \( \mathbb{Z}(d) = (\mathbb{Z}, dx^2) \) is one-dimensional positive-definite, has level \( 4d \), and its discriminant group is isomorphic to \( \mathbb{Z}/2d\mathbb{Z} \) with the quadratic form \( x \mapsto x^2/4d \).

We define the unary theta series by

\[
\theta_{1/2,d}(\tau) := \sum_{r \pmod{2d}} \sum_{n \in \mathbb{Z}} q^{n^2/4d} \varepsilon_r.
\]

It is a holomorphic modular form of weight \( \frac{1}{2} \) for the Weil representation of \( \mathbb{Z}(d) \).

Given an exact divisor \( c \mid d \), we can associate an Atkin-Lehner involution \( \sigma_c \) on \( \mathbb{Z}/2d\mathbb{Z} \), defined by the equations \( \sigma_c(x) \equiv -x \pmod{2c} \) and \( \sigma_c(x) \equiv x \pmod{2d/c} \). These involutions act on vector-valued modular forms for the Weil representation associated to \( \mathbb{Z}(d) \) via the action

\[
\left( \sum_{r \pmod{2d}} f_r(\tau) \varepsilon_r \right)^{\sigma_c} \equiv \sum_{r \pmod{2d}} f_{\sigma_c(r)}(\tau) \varepsilon_r.
\]

2.8 Serre duality

Let \( L \) be an even lattice, and let \( k \in \frac{1}{2}\mathbb{Z} \). Then we have the following proposition (compare e.g. [15, Proposition 2.5]).

**Proposition 2.4** (Serre duality) Let

\[
g(\tau) = \sum_{h \in L'/L} \sum_{n \geq 0} c_g(h, n) q^n \varepsilon_h
\]

be a holomorphic \( q \)-series which is bounded at \( i\infty \). Then \( g(\tau) \) is a holomorphic modular form of weight \( k \) for the Weil representation \( \rho_L \) if and only if

\[
\{g, f\} := \sum_{h \in L'/L} \sum_{n \geq 0} c_g(h, n) c_f(h, -n) = 0
\]

for every weakly holomorphic modular form \( f \) of weight \( 2 - k \) for \( \overline{\rho}_L \).

3 The higher Siegel theta lift

In this section we compute the lift \( \Lambda^\text{reg}_j(f, z) \) for certain input functions \( f \) in two different ways (as well as offering a pathway as to how one could compute its Fourier expansion).

3.1 A series representation

We first obtain an expression for \( \Lambda^\text{reg}_j(f, z) \) as a series whenever \( f \in H^\text{cusp}_{k-2j, L} \). Recall that \( k = \frac{1-n}{2} \).
**Theorem 3.1** Assume that $q(X_2) \neq 0$ for every $X \in L + \mu$ with $q(X) = -m$. Let $f \in H_{k-2,j,s}^\text{cap}$. For every $j \in \mathbb{N}$, we have

$$
\Lambda_j^\text{reg}(f, z) = (4\pi)^{\frac{j}{2} + j} \frac{\Gamma(1 + j) \Gamma(\frac{s}{2} + j)}{\Gamma\left(\frac{n+3}{2} + 2j\right)} \sum_{X \in L, \ q(X) < 0} c_j^+(X, q(X)) q(X) \frac{n+1}{2} + 2j \left(-q(X_\perp)\right)^{-\frac{n+1}{2} - j} \times 2 F_1 \left(\frac{n}{2} + j, 1 + j; \frac{n+3}{2} + 2j; \frac{q(X)}{q(X_\perp)}\right).
$$

where $2 F_1$ denotes the classical Gauss hypergeometric function. The series on the right-hand side converges absolutely.

**Proof** We consider the regularized theta lift of the Maass–Poincaré series $F_{\mu,-m,k-2,j,s}$. Applying Lemma 2.1 we obtain

$$
\Lambda_j^\text{reg}(F_{\mu,-m,k-2,j,s}, z) = (4\pi m)^{j} \frac{\Gamma\left(s + \frac{k}{2}\right)}{\Gamma\left(s + \frac{k}{2} - j\right)} \int_{\mathcal{F}}^{\text{reg}} (F_{\mu,-m,k,s}(\tau), \Theta_L(\tau, z)) v^k d\mu(\tau).
$$

The usual unfolding argument yields the above expression as

$$
2(4\pi m)^{j} \frac{\Gamma\left(s + \frac{k}{2}\right)}{\Gamma(2s) \Gamma\left(s + \frac{k}{2} - j\right)} \int_0^\infty \int_0^1 \mathcal{M}_k,s(-4\pi mv e(-mu) \Theta_{L,\mu}(\tau, z)) v^{k-2} du dv,
$$

where $\Theta_{L,\mu}$ denotes the $\mu$-th component of $\Theta_L$. Inserting (2.2) and (2.1), after evaluating the integral over $u$ we are left with

$$
2(4\pi m)^{j} \frac{\Gamma\left(s + \frac{k}{2}\right)}{\Gamma(2s) \Gamma\left(s + \frac{k}{2} - j\right)} \times \sum_{X \in L + \mu \ q(X) = -m} \int_0^\infty M_{-\frac{k}{2},s-\frac{1}{2}}(4\pi m v) v^{-\frac{3}{2} + \frac{k}{2}} e^{-2\pi v \left(q(X) - q(X_\perp)\right)} dv.
$$

The integral is a Laplace transform. Using equation (11) on page 215 of [14] we obtain

$$
2(4\pi m)^{j} \times \frac{\Gamma\left(s + \frac{k}{2}\right)}{\Gamma(2s) \Gamma\left(s + \frac{k}{2} - j\right)} \times \sum_{X \in L + \mu \ q(X) = -m} (-4\pi q(X_\perp))^\frac{k}{2} + \frac{s}{2} - 2 F_1 \left(s - \frac{1}{2} - \frac{k}{2}, s + \frac{k}{2}, 2s; \frac{-m}{q(X_\perp)}\right).
$$

To obtain the statement of the theorem, we plug in the point $s = 1 - \frac{k}{2} + j$ and use that $f$ may be written as a linear combination of the Maass–Poincaré series as in Sect. 2.3. Convergence follows similarly to that of [11, Theorem 3.1].

**Remark 3.2** For small values of $n$ and $j$ the Gauss hypergeometric function on the right-hand side can be evaluated explicitly. For example, choose $n = 1$. Then by [18, Eq. 15.4.17] and using that $2 F_1$ is symmetric in the first two arguments, we have that

$$
2 F_1 \left(\frac{1}{2} + j, 1 + j; 2 + 2j; \frac{q(X)}{q(X_\perp)}\right) = \left(\frac{1}{2} + \frac{1}{2} \left(1 - \frac{q(X)}{q(X_\perp)}\right)^{\frac{1}{2}}\right)^{-1 - 2j}.
$$
For other values of $n$ and $j$, one can use transformations of $2F_1$ along with the contiguous hypergeometric functions to obtain a (lengthy) description in terms of linear combinations of more elementary functions. The coefficients in the linear combination are rational.

### 3.2 Evaluating the theta lift at special points

We now evaluate the theta integral at special points. Recall that at a special point $w$ we have the positive-definite one-dimensional lattice $P = \mathcal{L} \cap w$ and negative-definite $n$ dimensional lattice $N = \mathcal{L} \cap w^\perp$. Recall that $\mathcal{G}_P$ denotes a harmonic Maass form of weight $\frac{3}{2}$ for $\rho_P$ that maps to $\Theta_1$ under $\xi_{\frac{3}{2}}$. We denote its holomorphic part by $\mathcal{G}_P^+$. For simplicity, we now assume that the input $f$ for the regularized theta lift is weakly holomorphic (if this is not the case, then one obtains another integral on the right-hand side).

**Theorem 3.3** Let $f \in M_{k-2j,L}^!$. We have

$$
\Lambda_j^{\text{reg}} (f, w) = \frac{\pi^\frac{j}{2} \Gamma(1 + j)}{2\Gamma(\frac{3}{2} + j)} (4\pi)^j \text{CT} \left( \left[ f_{P \oplus N}(\tau), \left[ \mathcal{G}_P^+(\tau), \Theta_{N^-}(\tau) \right]_j \right] \right).
$$

**Remark 3.4** For fixed $j$, one can evaluate the constant term as sums over lattice vectors as in [11, Remark 3.6]. However, the terms arising from the Rankin–Cohen bracket quickly become unwieldy for large values of $j$.

**Proof of Theorem 3.3** The proof is similar to those of [8, Theorem 5.4] and [3, Theorem 4.1] and so we only sketch the details, for the convenience of the reader. We infer from Lemma 2.2 and (2.3) that

$$
\langle f, \Theta_L \rangle = \left( f, (\Theta_{P \oplus N})^L \right) = \langle f_{P \oplus N}, \Theta_{P \oplus N} \rangle.
$$

This means that one may assume that $L = P \oplus N$ if $f$ is replaced by $f_{P \oplus N}$. For succinctness, we write $f$ instead of $f_{P \oplus N}$ throughout.

The first step is to use the self-adjointness of the raising operator given in [7, Lemma 4.2] to obtain

$$
\int_{\mathcal{F}}^{\text{reg}} \left( R^j_{k-2j} (f)(\tau), \Theta(\tau, w) \right) v^k d\mu(\tau)
$$

$$
= (-1)^j \int_{\mathcal{F}}^{\text{reg}} \left( f(\tau), R^j_{-k} \left( v^k \Theta_L(\tau, w) \right) \right) d\mu(\tau),
$$

after noting that the apparent boundary term appearing disappears in the same way as in the proof of [7, Lemma 4.4]. We next require the formula

$$
R_{\ell-k} \left( v^k g(\tau) \otimes h(\tau) \right) = v^k g(\tau) \otimes R_{\ell} (h)(\tau)
$$

(3.1) which holds for every holomorphic function $g$, every smooth function $h$, and $\kappa, \ell \in \mathbb{R}$. Then using (2.4) and (3.1) yields

$$
R^j_{-k} \left( v^{-k} \Theta_{P \oplus N}(\tau, w) \right) = L^\frac{1}{2} (\mathcal{G}_P)(\tau) \otimes R^j_{1-k} (\Theta_{N^-})(\tau).
$$

Next we use the fact that $L^\frac{1}{2} (\Theta_{N^-}) = 0$ along with Proposition 2.3 together to infer that

$$
L^\frac{1}{2} (\mathcal{G}_P)(\tau) \otimes R^j_{1-k} (\Theta_{N^-})(\tau) = \frac{\pi^\frac{j}{2} \Gamma(j + 1)}{2\Gamma(\frac{3}{2} + j)} (-4\pi)^j L_{\frac{3}{2} - k + 2j}^\frac{5}{2} \left( \left[ \mathcal{G}_P(\tau), \Theta_{N^-}(\tau) \right]_j \right).
$$
Therefore, overall we have
\[
\int_{\mathcal{F}}^\text{reg} \left( R_{-2j, k}^j (f), \Theta_L (\tau, w) \right) v^k d\mu (\tau) = \frac{\pi^{\frac{1}{2}} \Gamma (j + 1)}{2 \Gamma (\frac{j}{2} + j)} (4\pi)^j \int_{\mathcal{F}}^\text{reg} \left( f (\tau), L_{\frac{j}{2} + 2} (\Theta_N (- \tau)) \right) d\mu (\tau).
\]

Applying Stokes' Theorem as in the proof of [9, Proposition 3.5] yields the result, noting that apparent boundary terms arising vanish because \( f \) is weakly holomorphic.

\[
\square
\]

3.3 The Fourier expansion

To end this section, we remark how one could obtain the Fourier expansion of the lift \( \Lambda_j^\text{reg} (f, z) \) for any \( f \in H_{k-2j, L}^1 \). This, however, would be a lengthy but completely clear calculation that we do not require in the present paper, and so we omit the details.

Firstly, note that Lemma 2.1 implies that one can rewrite the lift as a linear combination of the lift
\[
\int_{\mathcal{F}}^\text{reg} \left( F_{\mu, -m, k, s} (\tau), \Theta_L (\tau, z) \right) v^k d\mu (\tau).
\]

Recall that we evaluate at \( s = 1 - \frac{k}{2} + j \). In the case of \( j = 0 \) the Fourier expansion of this lift was calculated by Bruinier in his celebrated habilitation [7]. One could then proceed as in the proof of [7, Theorem 2.15 and Proposition 3.1] to obtain a Fourier expansion for \( \Lambda_j^\text{reg} (f, z) \) which is of a similar shape to the \( j = 0 \) case. Finally, one could also translate this to the hyperbolic model of the Grassmanian in the same fashion as [11, Theorem 3.3].

4 Eichler–Selberg type relations

We are now in a position to use our techniques to offer proofs of Theorem 1.1 along with Theorems 1.2 and 1.3. Let \( n = 1 \) throughout this section.

**Proof of Theorem 1.1** For any \( f \in H_{k-2j, L}^\text{cusp} \) by Theorem 3.1 and the remark after it we have that
\[
\Lambda_j^\text{reg} (f, z) = 4^{1+2j} \pi^{\frac{1}{2}+j} \frac{\Gamma (1 + j) \Gamma \left( \frac{1}{2} + j \right)}{\Gamma (2 + 2j)} \times \sum_{X \in L', \quad q(X) < 0} c_f^j (X, q(X)) \left( \sqrt{|q(X_z)|} - \sqrt{|q(X)|} \right)^{1+2j}.
\]

Simply combine this with Theorem 3.3, take the difference and apply Proposition 2.4 to yield the statement of the theorem with \( C_j = 2^{2j+3} \frac{\Gamma \left( \frac{1}{2} + j \right) \Gamma \left( \frac{3}{2} + j \right)}{\Gamma (2 + 2j)} \).

Finally, we apply these results to see how one can obtain known results for scalar-valued modular forms from the higher Siegel theta lifts. Take the rational quadratic space \( \mathbb{Q}^2 \) with quadratic form \( Q(a, b) = ab \). Then a lattice \( L \) in \( \mathbb{Q}^2 \) has signature \((1, 1)\) and is isotropic. For a special point \( w \) we follow [11] and let \( y = (y_1, y_2) \in L \) be its primitive generator with \( y_1, y_2 > 0 \). Let \( y^\perp \in L \) be the primitive generator of \( w^\perp \) with positive second coordinate.
Then $y_1$ is a positive multiple of $(-y_1, y_2)$. Defining $d_P = y_1 y_2$ and $d_N = y_1^1 y_2^1$ we have

$$P \equiv (\mathbb{Z}, d_P x^2) \text{ and } N \equiv (\mathbb{Z}, -d_N x^2).$$

As in [11], one may identify

$$\Theta_P = \theta_{\frac{1}{2}, d_P}, \quad \Theta_N = \theta_{\frac{1}{2}, d_N}.$$

**Proof of Theorem 1.2** We choose the lattice $L = (\mathbb{Z}, x^2) \oplus (\mathbb{Z}, y^2)$ which we view as a sublattice of $(\mathbb{R}^2, xy)$. The lattice $L$ has a group ring isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$ which we label by $e_{(t,r)}$, and we choose the point $(y_1, y_2) = (1,1)$.

Note the following explicit formulae (recalled from [11] for the convenience of the reader).

We have

$$(X, y) = X_2 y_1 + X_1 y_2, \quad -2X_1 X_2 |y|^2 + (X, y)^2 = (X_1 y_2 - X_2 y_1)^2,$$

along with

$$q(X_w) = \frac{1}{2} \frac{(X, y)^2}{|y|^2}, \quad |q(X_w)| = -X_1 X_2 + \frac{1}{2} \frac{(X, y)^2}{|y|^2}.$$

Combining these yields

$$\sqrt{|q(X_w)|} - \sqrt{|q(X_w)|} = \frac{1}{\sqrt{y_1 y_2}} \min (|X_1 y_2|, |X_2 y_1|).$$

Let $f$ be a weakly holomorphic modular form of appropriate weight. Then Theorems 3.1 and 3.3 along with the duplication formula for the $\Gamma$-function $\pi^\frac{1}{2} \Gamma(2x) = 2^{2x-1} \Gamma(x) \Gamma(x + \frac{1}{2})$ imply that

$$\frac{j^1}{4 \sqrt{\pi} \Gamma \left( \frac{1}{2} + j \right)} \text{CT} \left( (f, [\Theta_P^+, \Theta_N^-])_{j} \right) = \sum_{X=(X_1, X_2) \in L'} \sum_{X_1 X_2 \leq 0} c_j^+(X, X_1 X_2) \min (|X_1|, |X_2|)^{2j+1}.$$

By results of Zagier [20], the generating function

$$G_P^+(\tau) = -8 \pi \left( \sum_{n \geq 0 \, \mod 4} H(n) e \left( \frac{n \tau}{4} \right) \right) \epsilon_0 \left( \sum_{n \geq 0 \, \mod 3} H(n) e \left( \frac{n \tau}{4} \right) \right) \epsilon_1$$

is the holomorphic part of a harmonic Maass form $G_P$ of weight $\frac{3}{2}$ for the dual Weil representation $\rho_L^-$ which maps to $\Theta_P$ under $\epsilon_{\frac{3}{2}}$. We plug this in for the choice of $G_P^+$. Note that $\Theta_N^- = \theta_{\frac{1}{2}, -1}$ and that we pick up a minus sign from (4.1).

Concentrating on the Rankin–Cohen bracket, we have four components, given by

$$\left[ \sum_{n \equiv -r \, \mod 4} H(n) e \left( \frac{n \tau}{4} \right), \sum_{n \equiv r \, \mod 2} e \left( \frac{n^2 \tau}{4} \right) \right]_{j} \epsilon_{(t,r)},$$

for $t, r \pmod{2}$. 

© Springer
Overall, after dividing the result by 2 and letting \([\cdot, \cdot]_j^{[n]}\) denote the \(n\)-th coefficient in the Fourier expansion of the Rankin–Cohen bracket, a short simplification yields that

\[
\frac{j! \sqrt{\pi}}{\Gamma \left( \frac{1}{2} + j \right)} \sum_{t, r \pmod{2}} \sum_{m \geq 0} c_j^+ ((0, r), -m/4) \\
\times \left[ \sum_{n \equiv -t^2 \pmod{4}} \sum_{n \in \mathbb{Z}} H(n)e \left( \frac{n\tau}{4} \right), \sum_{n \equiv r \pmod{2}} e \left( \frac{n^2\tau}{4} \right) \right] \epsilon_{(t, r)}^{[m/4]} + \frac{1}{2} \sum_{t, r \pmod{2}} \sum_{m \geq 0} \sum_{a, b \in \mathbb{N}, ab = m} c_j^+ ((t, r), -m/4) \min (a, b)^{2j+1} \epsilon_{(t, r)}
\]

vanishes. We then use Proposition 2.4 to conclude that

\[
g(\tau) := \sum_{t, r \pmod{2}} \sum_{m \geq 0} c_g ((t, r), m) q^{m/4} \epsilon_{t, r}
\]

where

\[
c_g ((t, r), m) := c_j \left[ \sum_{n \equiv -t^2 \pmod{4}} H(n)e \left( \frac{n\tau}{4} \right), \sum_{n \equiv r \pmod{2}} e \left( \frac{n^2\tau}{4} \right) \right]^{[m/4]} + \frac{1}{2} \sum_{a, b \equiv t+r \pmod{2}} \min (a, b)^{2j+1}
\]

is a holomorphic modular form of weight \(2j + 2\) for \(\rho_L\). Summing the components \(\epsilon_{(1,0)}\) and \(\epsilon_{(0,1)}\) and shifting \(\tau \mapsto 4\tau\) yields precisely the expression in the statement of the theorem. Moreover, a direct computation using the Weil representation shows that this is a modular form of weight \(2j + 2\) on \(\Gamma_0(4)\). That it is a cusp form is also clear.

**Remark 4.1** The above calculations show that \(g(\tau)\) has an even part given by the other two components labeled by \((0, 0)\) and \((1, 1)\). Tracing this through to the scalar-valued case, we obtain that

\[
c_j \left[ H, \vartheta \right]^{\text{even}}_j + \sum_{m \geq 0} \sum_{a, b \in \mathbb{N}, ab = 4m} \min (a, b)^{2j+1} q^{4m}
\]

is a holomorphic modular form of weight \(2j + 2\) on \(\Gamma_0(4)\).

Further examples could also be computed by combining the techniques in [11] with those in the current paper. As remarked previously, for small values of \(j\), these can be written explicitly, but for larger values of \(j\) the terms arising from the Rankin–Cohen brackets quickly become unwieldy. With \(x = (x_1, \ldots, x_j) \in \mathbb{Z}^j\), \(p\) some polynomial and \(Q\) a quadratic form, one could obtain formulae for sums of the shape

\[
\sum_{x \in \mathbb{Z}^j} p(x) H(t - Q(x))
\]
in terms of linear combinations (with rational coefficients) of divisor power sums. This complements the large literature on recurrences for Hurwitz class numbers. It is also clear that one could derive similar examples for other generating functions of weight $\frac{3}{2}$.

**Proof of Theorem 1.3** The proof is very similar to that of Theorem 1.2, and so we only offer a sketch and leave the details to the interested reader. We choose the lattice $L = (\mathbb{Z}, 6x^2) \oplus (\mathbb{Z}, -6y^2)$ viewed as a sublattice of $((\mathbb{R}^2, xy)$ as before. Then $L$ has a group ring isomorphic to $\mathbb{Z}/12\mathbb{Z}$. Recall that

$$g(\tau) = q^{-\frac{1}{24}} \sum_{n \geq 0} \left( \text{spt}(n) + \frac{1}{12}(24n - 1)p(n) \right) q^n.$$  

It is known that $g$ is the holomorphic part of weight $\frac{3}{2}$ harmonic Maass form $F$ on $\Gamma_0(576)$ that maps to $(-\sqrt{6}/4)\eta$ under $\xi^{\frac{3}{2}}$, see [6]. As in [11], we see that

$$\sum_{r \pmod{12}} \left( \frac{12}{r} \right) F(\tau)e_r = F(\tau)(\epsilon_1 - \epsilon_5 - \epsilon_7 + \epsilon_{11})$$

is a harmonic Maass form of weight $\frac{3}{2}$ for the Weil representation of the lattice $(\mathbb{Z}, 6x^2)$. Moreover, under the action of $\xi^{\frac{3}{2}}$ it maps to

$$-\frac{\sqrt{6}}{4} \left( \theta_{\frac{1}{2},6} - \theta_{\frac{5}{2},6}^{\sigma_2} \right).$$

One evaluates the lift at the special points $y = (1, 6)$ and $y = (2, 3)$ and takes the difference. As before, this results in two different evaluations, one in terms of a series of minima, and one as a constant term in a Fourier expansion. After shifting the constants to the CT term, one proceeds as in the proof of Theorem 1.2 to use Proposition 2.4 to obtain a modular form of weight $2j + 2$ whose coefficients are essentially a rescaling of those stated in the theorem. Making use of the fact that $\eta$ may be written as the difference of two unary theta functions yields precisely the function in the statement of the theorem (up to rescaling by a constant factor, and after rewriting the sum over minima by case distinctions and using the duplication formula for the $\Gamma$-function twice). This time, one does not need to shift $\tau$, and so obtains a modular form on $\text{SL}_2(\mathbb{Z})$.  

\[\square\]

**Acknowledgements** The author thanks Markus Schwagenscheidt for many insightful conversations on the contents of the paper, in particular suggesting the connection to Theorems 1.2 and 1.3, as well as useful comments on previous versions of the paper. The author would also like to thank Jan Bruinier and Andreas Mono for helpful comments on an earlier draft of the paper. The research conducted for this paper is supported by the Pacific Institute for the Mathematical Sciences (PIMS). The research and findings may not reflect those of the Institute.
References

1. Abramowitz, M., Stegun, I.: Handbook of mathematical functions with formulas, graphs, and mathematical tables. In: National Bureau of Standards Applied Mathematics Series, vol. 55 (1964)
2. Ahlgren, S., Kim, B.: Congruences for a mock modular form on $SL_2(\mathbb{Z})$ and the smallest parts function. J. Number Theory 189, 81–89 (2018)
3. Allès-Neumann, C., Bringmann, K., Males, J., Schwagenscheidt, M.: Cycle integrals of meromorphic modular forms and coefficients of harmonic Maass forms. J. Math. Anal. Appl. (to appear)
4. Allès-Neumann, C., Bringmann, K., Schwagenscheidt, M.: On the rationality of cycle integrals of meromorphic modular forms. Math. Ann. 376(1–2), 243–266 (2020)
5. Andrews, G.: The number of smallest parts in the partitions of n. J. Reine Angew. Math. 624, 133–142 (2008)
6. Bringmann, K.: On the explicit construction of higher deformations of partition statistics. Duke Math. J. 144(2), 195–233 (2008)
7. Bruinier, J.H.: Borcherds products on $O(2,1)$ and Chern classes of Heegner divisors. In: Lecture Notes in Mathematics, vol. 1780. Springer, Berlin (2002)
8. Bruinier, J.H., Ehlen, S., Yang, T.: CM values of higher automorphic Green functions on orthogonal groups. Invent. Math. 225(3), 693–785 (2021)
9. Bruinier, J.H., Funke, J.: On two geometric theta lifts. Duke Math. J. 125, 45–90 (2004)
10. Bruinier, J.H., Funke, J., Imamoglu, Ö.: Regularized theta liftings and periods of modular functions. J. Reine Angew. Math. 703, 43–93 (2015)
11. Bruinier, J., Schwagenscheidt, M.: Theta lifts for Lorentzian lattices and coefficients of mock theta functions. Math. Z. 297(3–4), 1633–1657 (2020)
12. Bruinier, J., Yang, T.: Faltings heights of CM cycles and derivatives of L-functions. Invent. Math. 177(3), 631–681 (2009)
13. Cohen, H.: Sums involving the values at negative integers of L-functions of quadratic characters. Math. Ann. 217(3), 271–285 (1975)
14. Erdélyi, A., Magnus, W., Oberhettinger, F., Tricomi, F.: Tables of integral transforms. Vol. I. McGraw-Hill Book Company, Inc., New York–Toronto–London (1954). Based, in part, on notes left by Harry Bateman
15. Li, Y., Schwagenscheidt, M.: Mock modular forms with integral fourier coefficients. Adv. Math. (to appear)
16. Mertens, M.: Eichler–Selberg type identities for mixed mock modular forms. Adv. Math. 301, 359–382 (2016)
17. Mertens, M.: Mock modular forms and class number relations. Res. Math. Sci. 1, 16 (2014)
18. NIST digital library of mathematical functions. http://dlmf.nist.gov/, Release 1.1.5 of 2022-03-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain (eds.)
19. Zagier, D.: Modular forms and differential operators. Proc. Indian Acad. Sci. Math. Sci. 104(1), 57–75 (1994)
20. Zagier, D.: Nombres de classes et formes modulaires de poids 3/2. C. R. Acad. Sci. Paris Sér. A-B 281(21), A883–A886 (1975). (French, with English summary)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.