Thermoelastic Problem for a Free Square Plate: Exact Solution

A P Kerzhæv

Institute of Earthquake Prediction Theory and Mathematical Geophysics, Russian Academy of Sciences, Moscow 117997, Russia
E-mail: alex_kerg@mail.ru

Abstract. The paper presents a method for determining thermal stresses in an elastic free square plate (the plane problem). First, we construct the solution to the nonhomogeneous temperature problem for an infinite plane. Then, we add to this solution the solution for a square, with the help of which the required boundary conditions on its sides are satisfied. The temperature factors are obtained in the form of explicit series in Papkovich–Fadle eigenfunctions.

1. Introduction
Knowledge of the values of thermal stresses and strains in engineering structures is of great practical importance. The methods of their determination began to be developed intensively in the middle of the XX century. This was due to problems arising in the design and construction of steam and gas turbines, rocket engines and nuclear reactors. At present, as a rule, numerical methods are used to determine temperature factors (stresses and strains). Approximate analytical ones [1-6] are used less often. In the proposed paper, an example of an exact solution is constructed for a free square (even-symmetric deformation with respect to the central axes). It is based on the exact solution to the boundary value problem for a rectangle with normal and tangential stresses given on its sides [7]. The solution to the problem is represented in the form of series in Papkovich–Fadle eigenfunctions, the coefficients of which are determined in a simple closed form by using the functions biorthogonal to the Papkovich–Fadle eigenfunctions.

2. Formulation of the problem and its solution
Let us consider the thin square plate \( \{ S : |y| \leq 1, |x| \leq 1 \} \) with free sides and the given temperature distribution \( t(x, y) = x^2 + y^2 \).

We introduce the following notation: \( U(x, y) = Gu(x, y), V(x, y) = Gv(x, y) \), where \( u(x, y) \) and \( v(x, y) \) are the displacements in the plate in the direction of the \( x \) and \( y \) axes, respectively; \( G \) is the shear modulus; \( \nu \) is Poisson’s ratio.

Using the method of initial functions [8, 9], we find a particular solution for an unbounded plane:

\[
\begin{align*}
U'(x, y) &= xy^2, \\
V'(x, y) &= x^2y, \\
\sigma'_{xx}(x, y) &= -2x^2, \\
\sigma'_{yy}(x, y) &= -2y^2, \\
\tau'_{xy}(x, y) &= 4xy.
\end{align*}
\]
Adding to the solution (1) the well-known solution [10] for the square $S$ with the constant normal stresses $\sigma_y(\pm 1, y) = \sigma_y(x, \pm 1) = 2$ on its sides, we obtain

$$U'(x, y) = x \left( y^2 + \frac{1 - \nu}{1 + \nu} \right), \quad V'(x, y) = y \left( x^2 + \frac{1 - \nu}{1 + \nu} \right).$$  \hspace{1cm} (2)

$$\sigma_y'(x, y) = -2x^2 + 2, \quad \sigma_y''(x, y) = -2y^2 + 2, \quad \tau_{xy}'(x, y) = 4xy.$$  

On the sides $x = \pm 1$ and $y = \pm 1$, we now have

$$\sigma_y''(\pm 1, y) = 0, \quad \tau_{xy}'(\pm 1, y) = \mp 4y;$$  

$$\sigma_y''(x, \pm 1) = 0, \quad \tau_{xy}'(x, \pm 1) = \pm 4x.$$  \hspace{1cm} (3)

Now we need to satisfy the boundary conditions for the tangential stresses. In order to do this, we add to the solution (2) the solution for the square $S$ loaded with the tangential stresses (3) taken with the opposite sign.

### 2.1. Solution for the square with tangential stresses at the ends

Let us consider the square $S$ with the free horizontal sides $y = \pm 1$ in which, at the ends $x = \pm 1$, the tangential stresses $\tau(y) = -\tau_{xy}'(\pm 1, y) = \mp 4y$ opposite in sign to (3) are given and the normal ones are equal to zero. Given the symmetry of the problem, we will seek the solution in the form of the series

$$U(x, y) = \sum_{k=1}^{\infty} a_k \xi(\lambda_k, y) \sin \lambda_k x + \bar{a}_k \bar{\xi}(\lambda_k, y) \sin \bar{\lambda}_k x,$$

$$V(x, y) = \sum_{k=1}^{\infty} a_k \chi(\lambda_k, y) \cosh \lambda_k x + \bar{a}_k \bar{\chi}(\lambda_k, y) \cosh \bar{\lambda}_k x,$$

$$\sigma_y'(x, y) = \sum_{k=1}^{\infty} a_k s_y(\lambda_k, y) \cosh \lambda_k x + \bar{a}_k \bar{s}_y(\lambda_k, y) \cosh \bar{\lambda}_k x,$$  \hspace{1cm} (4)

$$\sigma_y''(x, y) = \sum_{k=1}^{\infty} a_k \bar{s}_y(\lambda_k, y) \cosh \lambda_k x + \bar{a}_k \bar{\bar{s}}_y(\lambda_k, y) \cosh \bar{\lambda}_k x,$$

$$\tau_{xy}'(x, y) = \sum_{k=1}^{\infty} a_k \tau_{xy}(\lambda_k, y) \sin \lambda_k x + \bar{a}_k \bar{\tau}_{xy}(\lambda_k, y) \sin \bar{\lambda}_k x$$

in the Papkovich–Fadle eigenfunctions

$$\xi(\lambda_k, y) = \left[ \frac{1 - \nu}{2} \sin \lambda_k - \frac{1 + \nu}{2} \lambda_k \cos \lambda_k \right] \cos \lambda_k y - \frac{1 + \nu}{2} \lambda_k \sin \lambda_k \sin \lambda_k y,$$

$$\chi(\lambda_k, y) = \left[ \frac{1 + \nu}{2} \lambda_k \cos \lambda_k + \sin \lambda_k \right] \sin \lambda_k y - \frac{1 + \nu}{2} \lambda_k \sin \lambda_k \cos \lambda_k y,$$

$$s_y(\lambda_k, y) = (1 + \nu) \lambda_k \left[ (\sin \lambda_k - \lambda_k \cos \lambda_k) \cos \lambda_k y - \lambda_k \sin \lambda_k \sin \lambda_k y \right],$$  \hspace{1cm} (5)

$$\bar{s}_y(\lambda_k, y) = (1 + \nu) \lambda_k \left[ (\sin \lambda_k + \lambda_k \cos \lambda_k) \cos \lambda_k y + \lambda_k \sin \lambda_k \sin \lambda_k y \right],$$

$$\bar{\tau}_{xy}(\lambda_k, y) = (1 + \nu) \lambda_k^2 \left[ \cos \lambda_k y - y \sin \lambda_k \cos \lambda_k y \right].$$
where \( \lambda_k \), \( \lambda_k^\ast \) (\( \text{Re} \lambda_k < 0 \)) are all the complex zeros of the entire function \( L(\lambda) = \lambda + \sin \lambda \cos \lambda \) \cite{11, 12}. Since \( s_1(\lambda_k, \pm 1) = r_1(\lambda_k, \pm 1) = 0 \), the boundary conditions at \( x = \pm 1 \) are satisfied exactly.

Satisfying the boundary conditions at the ends of the square (for example, at \( x = -1 \)), we obtain the following two functional equations:

\[
\begin{align*}
\sum_{k=1}^{\infty} a_k s_1(\lambda_k, y) \cosh \lambda_k + a_k^* s_1(\lambda_k, y) \cosh \lambda_k^* &= 0; \\
\sum_{k=1}^{\infty} a_k t_1(\lambda_k, y) \sinh \lambda_k + a_k^* t_1(\lambda_k, y) \sinh \lambda_k^* &= -\tau(y).
\end{align*}
\] (6)

The unknown coefficients \( a_k, a_k^* \) are determined from here by using the functions biorthogonal to the Papkovich–Fadle eigenfunctions \( s_1(\lambda_k, y) \) and \( t_1(\lambda_k, y) \). The equations for their determination are as follows \cite{11, 12}:

\[
\begin{align*}
\int_{-\infty}^{\infty} s_1(\lambda, y) X_k(y) dy = \frac{L(\lambda)}{\lambda^2 - \lambda_k^2}, & \quad \int_{-\infty}^{\infty} t_1(\lambda, y) T_k(y) dy = \frac{\lambda L(\lambda)}{\lambda^2 - \lambda_k^2}.
\end{align*}
\] (7)

As a result, from (6) for each number \( k = 1, 2, \ldots \) we obtain the system of two algebraic equations:

\[
\begin{align*}
 & a_k b_k \cosh \lambda_k + a_k^* b_k \cosh \lambda_k^* = 0; \\
 & a_k^* b_k \sinh \lambda_k + a_k b_k^* \sinh \lambda_k^* = -(\bar{\tau_k} + \tau_k),
\end{align*}
\] (8)

where

\[
\tau_k = \int_{-1}^{1} \tau(y) \eta_k(y) dy, \quad \tau_k(y) = -\frac{\sin \lambda_k y}{2(1 + v) \sin \lambda_k^*}.
\] (9)

The numbers \( \tau_k \) are the Lagrange coefficients of the expanded function \( \tau(y) \), the functions \( t_k(y) \) are the finite parts of the functions biorthogonal to the Papkovich–Fadle eigenfunctions \( t_1(\lambda_k, y) \), and \( M_k = L'(\lambda_k) / 2 \) are normalizing factors.

Solving (8), we find

\[
a_k = \frac{-(\bar{\tau_k} + \tau_k) \cosh \lambda_k}{M_k (\lambda_k \sinh \lambda_k - \lambda_k^* \sinh \lambda_k^* \cosh \lambda_k^*)}.
\] (10)

Substituting (10) into (4) and separating the zero-series in the obtained expressions, as was done in \cite{12, 13}, for example, we obtain the solution to the problem:

\[
U(x, y) = -2 \sum_{k=1}^{\infty} \text{Re} \left( \frac{\tau_k s_1(\lambda_k, y)}{\lambda_k b_k M_k} \right) \frac{\text{Im}(\lambda_k \cosh \lambda_k \sinh \lambda_k^* \cosh \lambda_k^*)}{\text{Im}(\lambda_k \sinh \lambda_k \cosh \lambda_k^*)},
\]

\[
V(x, y) = -2 \sum_{k=1}^{\infty} \text{Re} \left( \frac{\tau_k s_1(\lambda_k, y)}{\lambda_k b_k M_k} \right) \frac{\text{Im}(\lambda_k \cosh \lambda_k \cosh \lambda_k^* \sinh \lambda_k^*)}{\text{Im}(\lambda_k \sinh \lambda_k \cosh \lambda_k^*)},
\]

\[
\sigma_s(x, y) = -2 \sum_{k=1}^{\infty} \text{Re} \left( \frac{\tau_k s_1(\lambda_k, y)}{M_k} \right) \frac{\text{Im}(\cosh \lambda_k \cosh \lambda_k^* \cosh \lambda_k^*)}{\text{Im}(\lambda_k \sinh \lambda_k \cosh \lambda_k^*)}.
\] (11)
\[ \sigma_\nu(x, y) = -\sum_{k=1}^\infty 2 \text{Re} \left( \frac{s_k(\lambda_k, y)}{\lambda_k M_k} \right) \frac{\text{Im}(\lambda_k^2 \cosh \lambda_k \cosh \lambda_k x)}{\text{Im}(\lambda_k \sinh \lambda_k \cosh \lambda_k)} . \]

\[ \tau_\nu(x, y) = -\sum_{k=1}^\infty 2 \text{Re} \left( \frac{t_k(\lambda_k, y)}{\lambda_k M_k} \right) \frac{\text{Im}(\lambda_k \cosh \lambda_k \sinh \lambda_k x)}{\text{Im}(\lambda_k \sinh \lambda_k \cosh \lambda_k)} . \]

2.2. Solution for the square with tangential stresses on the horizontal sides

The solution for the square \( S \) with the free ends \( x = \pm 1 \) in which, on the horizontal sides \( y = \pm 1 \), the tangential stresses \( \tau(x) = -\tau_\nu(x, \pm 1) = \mp 4x \) are given and the normal ones are equal to zero is obtained by replacing \( x \) with \( y \) in the solution (11). One should keep in mind that in this case the displacements \( U(x, y) \) will be described by formulas (11) for \( V(x, y) \) and vice versa. The same applies to the stresses \( \sigma_\nu(x, y) \) and \( \sigma_\nu(x, y) \).

Adding the obtained solution to the solutions (11) and (2), we obtain the complete solution to the problem under consideration. Figures 1–4 show the graphs illustrating the solution. It is assumed that \( \nu = 1/3 \).

**Figure 1.** Distribution of the displacements \( U(-1, y) \) and \( V(x, -1) \).

**Figure 2.** Distribution of the displacements \( V(-1, y) \) and \( U(x, -1) \).

**Figure 3.** Distribution of the stresses \( \sigma_\nu(0, y) \) and \( \sigma_\nu(x, 0) \).

**Figure 4.** Distribution of the stresses \( \sigma_\nu(0, y) \) and \( \sigma_\nu(x, 0) \).
3. Conclusion
In the paper, an example of solving the thermoelastic problem for a thin free square plate has been given. Similarly, solutions for rectangular plates with different boundary conditions on their sides and with different temperature fields can be obtained. The obtained solution is exact because the expansion coefficients of the series in the Papkovich–Fadle eigenfunctions, in the form of which the solution is represented, are determined by simple closed formulas. The solution is very simple and easy to use in engineering.

4. References
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