Algebraic (geometric) $n$-stacks

Carlos Simpson

In the introduction of Laumon-Moret-Bailly ([LMB] p. 2) they refer to a possible theory of algebraic $n$-stacks:

Signalons au passage que Grothendieck propose d’élargir à son tour le cadre précédent en remplaçant les 1-champs par des $n$-champs (grosso modo, des faisceaux en $n$-catégories sur $(\text{Aff})$ ou sur un site arbitraire) et il ne fait guère de doute qu’il existe une notion utile de $n$-champs algébriques . . . .

The purpose of this paper is to propose such a theory. I guess that the main reason why Laumon and Moret-Bailly didn’t want to get into this theory was for fear of getting caught up in a horribly technical discussion of $n$-stacks of groupoids over a general site. In this paper we simply assume that a theory of $n$-stacks of groupoids exists. This is not an unreasonable assumption, first of all because there is a relatively good substitute—the theory of simplicial presheaves or presheaves of spaces ([Bro] [B-G] [Jo] [Ja] [Si3] [Si2])—which should be equivalent, in an appropriate sense, to any eventual theory of $n$-stacks; and second of all because it seems likely that a real theory of $n$-stacks of $n$-groupoids could be developed in the near future ([Br2], [Ta]).

Once we decide to ignore the technical complications involved in theories of $n$-stacks, it is a relatively straightforward matter to generalize Artin’s definition of algebraic 1-stack. The main observation is that there is an inductive structure to the definition whereby the ingredients for the definition of algebraic $n$-stack involve only $n-1$-stacks and so are already previously defined.

This definition came out of discussions with C. Walter in preparation for the Trento school on algebraic stacks (September 1996). He made the remark that the definition of algebraic stack made sense in any category where one has a reasonable notion of smooth morphism, and suggested a general terminology of “geometric stack” for this notion. One immediately realizes that the notion of smooth morphism makes sense—notably—in the “category” of algebraic stacks and therefore according to Walter’s remark, one could define the notion of geometric stack in the category of algebraic stacks. This is the notion of algebraic 2-stack. It is an easy step to go from there to the general inductive definition of algebraic $n$-stack. Walter informs me that he had also come upon the notion of algebraic 2-stack at the same time (just before the Trento school).

Now a note about terminology: I have chosen to write the paper using Walter’s terminology “geometric $n$-stack” because this seems most closely to reflect what is going on: the definition is made so that we can “do geometry” on the $n$-stack, since in a rather strong sense it looks locally like a scheme. For the purposes of the introduction, the terminology “algebraic $n$-stack” would be better because this fits with Artin’s terminology
for $n = 1$. There is another place where the terminology “algebraic” would seem to be useful, this is when we start to look at geometric stacks on the analytic site, which we call “analytic $n$-stacks”. In fact one could interchange the terminologies and in case of confusion one could even say “algebraic-geometric $n$-stack”.

In [Si5] I proposed a notion of presentable $n$-stack stable under homotopy fiber products and truncation. One key part of the notion of algebraic stack is the smoothness of the morphism $X \to T$ from a scheme. This is lost under truncation (e.g. the sheaf $\pi_0$ of an algebraic stack may no longer be an algebraic stack); this indicates that the notion of “geometric stack” is something which combines together the various homotopy groups in a fairly intricate way. In particular, the notion of presentable $n$-stack is not the same as the notion of geometric $n$-stack (however a geometric $n$-stack will be presentable). This is a little bit analogous to the difference between constructible and locally closed subsets in the theory of schemes.

We will work over the site $\mathcal{X}$ of schemes of finite type over $\text{Spec}(k)$ with the etale topology, and with the notion of smooth morphism. The definitions and basic properties should also work for any site in which fiber products exist, provided with a certain class of morphisms analogous to the smooth morphisms. Rather than carrying this generalization through in the discussion, we leave it to the reader. Note that there are several examples which readily come to mind:

— the site of schemes of finite type with the etale topology and the class of etale morphisms: this gives a notion of what might be called a “Deligne-Mumford $n$-stack”;

— the site of schemes of finite type with the fppf topology and the class of flat morphisms: this gives a notion of what might be called a “flat-geometric $n$-stack”;

— the site of schemes of finite type with the qff topology and the class of quasi-finite flat morphisms: this gives a notion of what might be called a “qff-geometric $n$-stack”.

Whereas Artin proves [Ar1] that flat-geometric 1-stacks are also smooth-geometric stacks (i.e. those defined as we do here using smooth morphisms)—his proof is recounted in [LMB]—it seems unlikely that the same would be true for $n$-stacks. Artin’s method also shows that qff-geometric 1-stacks are Deligne-Mumford stacks. However it looks like Deligne-Mumford $n$-stacks are essentially just gerbs over Deligne-Mumford 1-stacks, while on the other hand in characteristic $p$ one could apply Dold-Puppe (see below) to a complex of finite flat group schemes to get a fairly non-trivial qff-algebraic $n$-stack. This seems to show that the implication “qff-geometric $\Rightarrow$ Deligne-Mumford” no longer holds for $n$-stacks. This is why it seems unlikely that Artin’s reasoning for the implication “flat-geometric $\Rightarrow$ smooth-geometric” will work for $n$-stacks.

Here is the plan of the paper. In §1 we give the basic definitions of geometric $n$-stack and smooth morphism of geometric $n$-stacks. In §2 we give some basic properties which amount to having a good notion of geometric morphism between $n$-stacks (which are themselves not necessarily geometric). In §3 we briefly discuss some ways one could obtain
geometric $n$-stacks by glueing. In §4 we show that geometric $n$-stacks are presentable in the sense of [Si5]. This is probably an important tool if one wants to do any sort of Postnikov induction, since presentable $n$-stacks are closed under the truncation processes which make up the Postnikov tower (whereas the notion of geometric $n$-stack is not closed under truncation). In §5 we do a preliminary version of what should be a more general Quillen theory. We treat only the 1-connected case, and then go on in the subsection “Dold-Puppe” to treat the relative (i.e. over a base scheme or base $n$-stack) stable (in the sense of homotopy theory) case in a different way. It would be nice to have a unified version including a reasonable notion of differential graded Lie algebra over an $n$-stack giving an algebraic approach to relatively 1-connected $n$-stacks over $R$, but this seems a bit far off in a technical sense.

In §6 we look at maps from a projective variety (or a smooth formal category) into a geometric $n$-stack. Here again it would be nice to have a fairly general theory covering maps into any geometric $n$-stack but we can only say something interesting in the easiest case, that of maps into connected very presentable $T$, i.e. $n$-stacks with $\pi_0(T) = *$, $\pi_1(T)$ an affine algebraic group scheme and $\pi_i(T)$ a vector space for $i \geq 2$. (The terminology “very presentable” comes from [Si5]). At the end we speculate on how one might generalize to various other classes of $T$.

In §7 we briefly present an approach to defining the tangent stack to a geometric $n$-stack. This is a generalization of certain results in the last chapter of [LMB] although we don’t refer to the cotangent complex.

In §8 we explain how to use geometric $n$-stacks as a framework for looking at de Rham theory for higher nonabelian cohomology. This is sort of a synthesis of things that are in [Si4] and [Si3].

We assume known an adequate theory of $n$-stacks of groupoids over a site $\mathcal{X}$. The main thing we will need is the notion of fiber product (which of course means—as it always shall below—what one would often call the “homotopy fiber product”).

We work over an algebraically closed field $k$ of characteristic zero, and sometimes directly over field $k = \mathbb{C}$ of complex numbers. Note however that the definition makes sense over arbitrary base scheme and the “Basic properties” hold true there.

The term “connected” when applied to an $n$-stack means that the sheaf $\pi_0$ is the final object $*$ (represented by $Spec(k)$). In the case of a 0-stack represented by $Y$ this should not be confused with connectedness of the scheme $Y$ which is a different question.

1. Definitions

Let $\mathcal{X}$ be the site of schemes of finite type over $Spec(k)$ with the etale topology. We will define the following notions: that an $n$-stack $T$ be geometric; and that a morphism
from a geometric $n$-stack to a scheme be smooth. We define these notions together by induction on $n$. Start by saying that a 0-stack (sheaf of sets) is geometric if it is represented by an algebraic space. Say that a morphism $T \to Z$ from a geometric 0-stack to a scheme is smooth if it is smooth as a morphism of algebraic spaces.

Now we give the inductive definitions: say that an $n$-stack $T$ is geometric if:

GS1—for any schemes $X$ and $Y$ and morphisms $X \to T$, $Y \to T$ the fiber product $X \times_T Y$ (which is an $n-1$-stack) is geometric using the inductive definition; and

GS2—there is a scheme $X$ and a morphism of $n$-stacks $f : X \to T$ which is surjective on $\pi_0$ with the property that for any scheme $Y$ and morphism $Y \to T$, the morphism $X \times_T Y \to Y$ from a geometric $n-1$ stack to the scheme $Y$ is smooth (using the inductive definition).

If $T$ is a geometric $n$-stack we say that a morphism $T \to Y$ to a scheme is smooth if for at least one morphism $X \to T$ as in GS2, the composed morphism $X \to Y$ is a smooth morphism of schemes.

This completes our inductive pair of definitions.

For $n = 1$ we recover the notion of algebraic stack, and in fact our definition is a straightforward generalization to $n$-stacks of Artin’s definition of algebraic stack.

The following lemma shows that the phrase “for at least one” in the definition of smoothness can be replaced by the phrase “for any”.

Lemma 1.1 Suppose $T \to Y$ is a morphism from an $n$-stack to a scheme which is smooth according to the previous definition, and suppose that $U \to T$ is a morphism from a scheme such that for any scheme $Z \to T$, $U \times_T Z \to Z$ is smooth (again according to the previous definition, as a morphism from an $n-1$-stack to a scheme). Then $U \to Y$ is a smooth morphism of schemes.

Proof: We prove this for $n$-stacks by induction on $n$. Let $X \to T$ be the morphism as in GS2 such that $X \to Y$ is smooth. Let $R = X \times_T U$. This is an $n-1$-stack and the morphisms $R \to X$ and $R \to U$ are both smooth as morphisms from $n-1$-stacks to schemes according to the above definition. Let $W \to R$ be a surjection from a scheme as in property GM2. By the present lemma applied inductively for $n-1$-stacks, the morphisms $W \to X$ and $W \to U$ are smooth morphisms of schemes. But the condition that $X \to Y$ is smooth implies that $W \to Y$ is smooth, and then since $W \to U$ is smooth and surjective we get that $U \to Y$ is smooth as desired. This argument doesn’t work when $n = 0$ but then $R$ is itself an algebraic space and the maps $R \to X$ (hence $R \to Y$) and $R \to U$ are smooth maps of algebraic spaces; this implies directly that $U \to Y$ is smooth. □

The following lemma shows that these definitions don’t change if we think of an $n$-stack as an $n+1$-stack etc.
Lemma 1.2 Suppose $T$ is an $n$-stack which, when considered as an $m$-stack for some $m \geq n$, is a geometric $m$-stack. Then $T$ is a geometric $n$-stack. Similarly smoothness of a morphism $T \to Y$ to a scheme when $T$ is considered as an $m$-stack implies smoothness when $T$ is considered as an $n$-stack.

Proof: We prove this by induction on $n$ and then $m$. The case $n = 0$ and $m = 0$ is clear. First treat the case $n = 0$ and any $m$: suppose $T$ is a sheaf of sets which is a geometric $m$-stack. There is a morphism $X \to T$ with $X$ a scheme, such that if we set $R = X \times_T X$ then $R$ is an $m-1$-stack smooth over $X$. However $R$ is again a sheaf of sets so by the inductive statement for $n = 0$ and $m-1$ we have that $R$ is an algebraic space. Furthermore the smoothness of the morphism $R \to X$ with $R$ considered as an $m-1$-stack implies smoothness with $R$ considered as a 0-stack. In particular $R$ is an algebraic space with smooth maps to the projections. Since the quotient of an algebraic space by a smooth equivalence relation is again an algebraic space, we get that $T$ is an algebraic space i.e. a geometric 0-stack (and note by the way that $X \to T$ is a smooth surjective morphism of algebraic spaces). This proves the first statement for $(0, m)$. For the second statement, suppose $T \to Y$ is a morphism to a scheme $Y$ which is smooth as a morphism from an $m$-stack. Then choose the smooth surjective morphism $X \to T$; as we have seen above this is a smooth morphism of algebraic spaces. The definition of smoothness now is that $X \to Y$ is smooth. But this implies that $T \to Y$ is smooth. This completes the inductive step for $(0, m)$.

Now suppose we want to show the lemma for $(n, m)$ with $n \geq 1$ and suppose we know it for all $(n', m')$ with $n' < n$ or $n' = n$ and $m' < m$. Let $T$ be an $n$-stack which is geometric considered as an $m$-stack. If $X, Y \to T$ are maps from schemes then $X \times_T Y$ is an $n-1$-stack which is geometric when considered as an $m-1$-stack; by the induction hypothesis it is geometric when considered as an $n-1$-stack, which verifies GS1. Choose a smooth surjection $X \to T$ from a scheme as in property GS2 for $m$-stacks. Suppose $Y \to T$ is any morphism from a scheme. Then $X \times_T Y$ is an $n-1$-stack with a map to $Y$ which is smooth considered as a map from $m-1$-stacks. Again by the induction hypothesis it is smooth considered as a map from an $n-1$-stack to a scheme, so we get GS2 for $n$-stacks. This completes the proof that $T$ is geometric when considered as an $n$-stack.

Finally suppose $T \to Y$ is a morphism from an $n$-stack to a scheme which is smooth considered as a morphism from an $m$-stack. Choose a surjection $X \to T$ as in property GS2 for $m$-stacks; we have seen above that it also satisfies the same property for $n$-stacks. By definition of smoothness of our original morphism from an $m$-stack, the morphism $X \to Y$ is smooth as a morphism of schemes; this gives smoothness of $T \to Y$ considered as a morphism from an $n$-stack to a scheme. This finishes the inductive proof of the lemma.

\[\Box\]
Remarks:

1. We can equally well make a definition of Deligne-Mumford \(n\)-stack by replacing “smooth” in the previous definition with “etale”. This gives an \(n\)-stack whose homotopy group sheaves are finite...

2. We could also make definitions of flat-geometric or qff-geometric \(n\)-stack, by replacing the smoothness condition by flatness or quasifinite flatness. If all of these notions are in question then we will denote the standard one by “smooth-geometric \(n\)-stack”. Not to be confused with “smooth geometric \(n\)-stack” which means a smooth-geometric \(n\)-stack which is smooth!

We now complete our collection of basic definitions in some obvious ways. We say that a morphism of \(n\)-stacks \(R \to T\) is geometric if for any scheme \(Y\) and map \(Y \to T\) the fiber product \(R \times_T Y\) is a geometric \(n\)-stack.

We say that a geometric morphism of \(n\)-stacks \(R \to T\) is smooth if for any scheme \(Y\) and map \(Y \to T\) the morphism \(R \times_T Y \to Y\) is a smooth morphism in the sense of our inductive definition.

**Lemma 1.3** If \(T \to Z\) is a morphism from an \(n\)-stack to a scheme then it is smooth and geometric in the sense of the previous paragraph, if and only if \(T\) is geometric and the morphism is smooth in the sense of our inductive definition.

**Proof:** Suppose that \(T\) is geometric and the morphism is smooth in the sense of the previous paragraph. Then applying that to the scheme \(Z\) itself we obtain that the morphism is smooth in the sense of the inductive definition. On the other hand, suppose the morphism is smooth in the sense of the inductive definition. Let \(X \to T\) be a surjection as in GS2. Thus \(X \to Z\) is smooth. For any scheme \(Y \to Z\) we have that \(X \times_Z Y \to T \times_Z Y\) is surjective and smooth in the sense of the previous paragraph; but in this case (and using the direction we have proved above) this is exactly the statement that it satisfies the conditions of GS2 for the stack \(T \times_Z Y\). On the other hand \(X \times_Z Y \to Y\) is smooth. This implies (via the independence of the choice in the original definition of smoothness which comes from \([1]\)) that that \(T \times_Z Y \to Y\) is smooth in the original sense. As this works for all \(Y\), we get that \(T \to Z\) is smooth in the new sense.

2. Basic properties

We assume that the propositions, lemmas and corollaries in this section are known for \(n-1\)-stacks and we are proving them all in a gigantic induction for \(n\)-stacks. On the other hand, in proving any statement we can use the previous statements for the same \(n\), too.
Proposition 2.1 If $R$, $S$ and $T$ are geometric $n$-stacks with morphisms $R, T \to S$ then the fiber product $R \times_S T$ is a geometric $n$-stack.

Proof: Suppose $R$, $S$ and $T$ are geometric $n$-stacks with morphisms $R \to S$ and $T \to S$. Let $X \to R$, $Y \to S$ and $Z \to T$ be smooth surjective morphisms from schemes. Choose a smooth surjective morphism $W \to X \times_S Z$ from a scheme (by axiom GS2 for $S$).

By base change of the morphism $Z \to T$, the morphism $X \times_S Z \to X \times_S T$ is a geometric smooth surjective morphism. We first claim that the morphism $W \to X \times_S T$ is smooth. To prove this, suppose $A \to X \times_S T$ is a morphism from a scheme. Then $W \times_{X \times_S T} A \to A$ is the composition of

$$W \times_{X \times_S T} A \to (X \times_S Z) \times_{X \times_S T} A \to A.$$ 

Both morphisms are geometric and smooth, and all three terms are $n-1$-stacks (note that in the middle $(X \times_S Z) \times_{X \times_S T} A = A \times_T Z$). By the composition result for $n-1$-stacks (Corollary 2.7 below with our global induction hypothesis) the composed morphism $W \times_{X \times_S T} A \to A$ is smooth, and this for any $A$. Thus $W \to X \times_S T$ is smooth.

Next we claim that the morphism $W \to R \times_S T$ is smooth. Again suppose that $A \to R \times_S T$ is a morphism from a scheme. The two morphisms

$$W \times_{R \times_S T} A \to (X \times S T) \times_{R \times S T} A = X \times_{R} A \to A$$

are smooth and geometric by base change. Again this is a composition of morphisms of $n-1$-stacks so by Corollary 2.8 and our global induction hypothesis the composition is smooth and geometric. Finally the morphism $W \to R \times_S T$ is the composition of three surjective morphisms so it is surjective. We obtain a morphism as in GS2 for $R \times_S T$.

We turn to GS1. Suppose $X \to R \times_S T$ and $Y \to R \times_S T$ are morphisms from schemes. We would like to check that $X \times_{R \times S T} Y$ is a geometric $n-1$-stack. Note that calculating $X \times_{R \times S T} Y$ is basically the same thing as calculating in usual homotopy theory the path space between two points $x$ and $y$ in a product of fibrations $r \times s t$. From this point of view we see that

$$X \times_{R \times S T} Y = (X \times_R Y) \times_{X \times S T} Y.$$ 

Note that the three components in the big fiber product on the right are geometric $n-1$-stacks, so by our inductive hypothesis (i.e. assuming the present proposition for $n-1$-stacks) we get that the right hand side is a geometric $n-1$-stack, this gives the desired statement for GS1.

Corollary 2.2 If $R$ and $T$ are geometric $n$-stacks then any morphism between them is geometric. In particular an $n$-stack $T$ is geometric if and only if the structural morphism $T \to *$ is geometric.
Lemma 2.3 If \( R \to S \to T \) are morphisms of geometric \( n \)-stacks and if each morphism is smooth then the composition is smooth.

Proof: We have already proved this for morphisms \( X \to T \to Y \) where \( X \) and \( Y \) are schemes (see Lemma 1.1). Suppose \( U \to T \to Y \) are smooth morphisms of geometric \( n \)-stacks with \( Y \) a scheme. We prove that the composition is smooth, by induction on \( n \) (we already know it for \( n = 0 \)). If \( Z \to T \) is a smooth surjective morphism from a scheme then the morphism
\[
U \times_T Z \to Z
\]
is smooth by the definition of smoothness of \( U \to T \). Also the map \( Z \to Y \) is smooth by definition of smoothness of \( T \to Y \). Choose a smooth surjection \( V \to U \times T Z \) from a scheme \( V \) and note that the map \( V \to Z \) is smooth by definition, so (since these are morphisms of schemes) the composition \( V \to Y \) is smooth. On the other hand
\[
U \times_T Z \to U
\]
is smooth and surjective, by base change from \( Z \to T \). We claim that the morphism \( V \to U \) is smooth and surjective—actually surjectivity is obvious. To prove that it is smooth, let \( W \to U \) be a morphism from a scheme; then
\[
W \times_U V \to W \times_U (U \times_T Z) = W \times_T Z \to W
\]
is a composable pair of morphisms of \( n-1 \)-stacks each of which is smooth by base change. By our induction hypothesis the composition is smooth. This shows by definition that \( V \to U \) is smooth. In particular the map \( V \to U \) is admissible as in GS2, and then we can conclude that the map \( U \to Y \) is smooth by the original definition using \( V \). This completes the proof in the current case.

Suppose finally that \( U \to T \to R \) are smooth morphisms of geometric \( n \)-stacks. Then for any scheme \( Y \) the morphisms \( U \times_R Y \to T \times_R Y \to Y \) are smooth by base change; thus from the case treated above their composition is smooth, and this is the definition of smoothness of \( U \to R \). □

Lemma 2.4 Suppose \( S \to T \) is a geometric smooth surjective morphism of \( n \)-stacks, and suppose that \( S \) is geometric. Then \( T \) is geometric.

Proof: We first show GS2. Let \( W \to S \) be a smooth geometric surjection from a scheme. We claim that the morphism \( W \to T \) is surjective (easy), geometric and smooth. To show
that it is geometric, suppose \( Y \to T \) is a morphism from a scheme. Then since \( S \to T \) is geometric we have that \( Y \times_T S \) is a geometric \( n \)-stack. On the other hand,

\[
Y \times_T W = (Y \times_T S) \times_S W,
\]

so by Proposition \( \ref{prop:geometric-morphisms} \) \( Y \times_T W \) is geometric. Finally to show that \( W \to T \) is smooth, note that

\[
Y \times_T W \to Y \times_T S \to Y
\]

is a composable pair of smooth (by base change) morphisms of geometric \( n \)-stacks, so by the previous lemma the composition is smooth. The morphism \( W \to T \) thus works for condition GS2.

To show GS1, suppose \( X, Y \to T \) are morphisms from schemes. Then

\[
(X \times_T Y) \times_T W = (X \times_T W) \times_W (Y \times_T W).
\]

The geometricity of the morphism \( W \to T \) implies that \( X \times_T W \) and \( Y \times_T W \) are geometric, whereas of course \( W \) is geometric. Thus by Proposition \( \ref{prop:geometric-morphisms} \) we get that \( (X \times_T Y) \times_T W \) is geometric. Now note that the morphism

\[
(X \times_T Y) \times_T W \to X \times_T Y
\]

of \( n-1 \)-stacks is geometric, smooth and surjective (by base change of the same properties for \( W \to T \)). By the inductive version of the present lemma for \( n-1 \) (noting that the lemma is automatically true for \( n=0 \)) we obtain that \( X \times_T Y \) is geometric. This is GS1.

\( \Box \)

**Corollary 2.5** Suppose \( Y \) is a scheme and \( T \to Y \) is a morphism from an \( n \)-stack. If there is a smooth surjection \( Y' \to Y \) such that \( T' := Y' \times_Y T \to Y' \) is geometric then the original morphism is geometric.

**Proof:** The morphism \( T' \to T \) is geometric, smooth and surjective (all by base-change from the morphism \( Y' \to Y \)). By \( \ref{prop:geometric-morphisms} \) the fact that \( T' \) is geometric implies that \( T \) is geometric. \( \Box \)

This corollary is particularly useful to do etale localization. It implies that the property of a morphism of \( n \)-stacks being geometric, is etale-local over the base.

**Corollary 2.6** Given a geometric morphism \( R \to T \) of \( n \)-stacks such that \( T \) is geometric, then \( R \) is geometric.
Proof: Let \( X \rightarrow T \) be the geometric smooth surjective morphism from a scheme given by GS2 for \( T \). By base change, \( X \times_T R \rightarrow R \) is a geometric smooth surjective morphism. However, by the geometricity of the morphism \( R \rightarrow T \) the fiber product \( X \times_T R \) is geometric; thus by the previous lemma, \( R \) is geometric.

Corollary 2.7 The composition of two geometric morphisms is again geometric.

Proof: Suppose \( U \rightarrow T \rightarrow R \) are geometric morphisms, and suppose \( Y \rightarrow R \) is a morphism from a scheme. Then

\[ U \times_R Y = U \times_T (T \times_R Y). \]

By hypothesis \( T \times_R Y \) is geometric. On the other hand \( U \times_R Y \rightarrow T \times_R Y \) is geometric (since the property of being geometric is obviously stable under base change). By the previous Proposition 2.1 we get that \( U \times_R Y \) is geometric. Thus the morphism \( U \rightarrow R \) is geometric.

Corollary 2.8 The composition of two geometric smooth morphisms is geometric and smooth.

Proof: Suppose \( R \rightarrow S \rightarrow T \) is a pair of geometric smooth morphisms. Suppose \( Y \rightarrow T \) is a morphism from a scheme. Then (noting by the previous corollary that \( R \rightarrow T \) is geometric) \( R \times_T Y \) and \( S \times_T Y \) are geometric. The composable pair

\[ R \times_T Y \rightarrow S \times_T Y \rightarrow Y \]

of smooth morphisms now falls into the hypotheses of Lemma 2.3 so the composition is smooth. This implies that our original composition was smooth.

In a relative setting we get:

Corollary 2.9 Suppose \( a \rightarrow T \rightarrow b \rightarrow R \) is a composable pair of morphisms of \( n \)-stacks. If \( a \) is geometric, smooth and surjective and \( ba \) is geometric (resp. geometric and smooth) then \( b \) is geometric (resp. geometric and smooth).

Proof: Suppose \( Y \rightarrow R \) is a morphism from a scheme. Then

\[ Y \times_R U = (Y \times_R T) \times_T U. \]

The map \( Y \times_R U \rightarrow Y \times_R T \) is geometric, smooth and surjective (since those properties are obviously—from the form of their definitions—invariant under base change). The fact that \( ba \) is geometric implies that \( Y \times_R U \) is geometric, which by the previous lemma
implies that $Y \times_R T$ is geometric. Suppose furthermore that $ba$ is smooth. Choose a
smooth surjection $W \to Y \times_R T$ from a scheme. Then the morphism

$$W \times_{Y \times_R T} (Y \times_R U) \to Y \times_R U$$

is smooth by basechange and the morphism $Y \times_R U \to Y$ is smooth by hypothesis. Thus
$W \times_{Y \times_R T} (Y \times_R U) \to Y$ is smooth. Choosing a smooth surjection from a scheme

$$V \to W \times_{Y \times_R T} (Y \times_R U)$$

we get that $V \to Y$ is a smooth morphism of schemes. On the other hand, the morphism

$$W \times_{Y \times_R T} (Y \times_R U) \to W$$

is smooth and surjective, so $V \to W$ is smooth and surjective. Therefore $W \to Y$ is
smooth. This proves that if $ba$ is smooth then $b$ is smooth. \qed

Examples: Proposition 2.6 allows us to construct many examples. The main examples
we shall look at below are the connected presentable $n$-stacks. These are connected
$n$-stacks $T$ with (choosing a basepoint $t \in T(\text{Spec}(\mathbb{C}))$ $\pi_i(T,t)$ represented by group schemes
of finite type. We apply 2.6 inductively to show that such a $T$ is geometric. Let $T \to \tau_{\leq n-1}T$ be the truncation morphism. The fiber over a morphism $Y \to \tau_{\leq n-1}T$ is (locally
in the etale topology of $Y$ where there exists a section—this is good enough by 2.5)
isomorphic to $K(G/Y, n)$ for a smooth group scheme of finite type $G$ over $Y$. Using the
following lemma, by induction $T$ is geometric.

Lemma 2.10 Fix $n$, suppose $Y$ is a scheme and suppose $G$ is a smooth group scheme
over $Y$. If $n \geq 2$ require $G$ to be abelian. Then $K(G/Y, n)$ is a geometric $n$-stack and the
morphism $K(G/Y, n) \to Y$ is smooth.

Proof: We prove this by induction on $n$. For $n = 0$ we simply have $K(G/Y, 0) = G$ which
is a scheme and hence geometric—also note that by hypothesis it is smooth over $Y$. Now
for any $n$, consider the basepoint section $Y \to K(G/Y, n)$. We claim that this is a smooth
degenerate map. If $Z \to K(G/Y, n)$ is any morphism then it corresponds to a map $Z \to Y$
and a class in $H^n(Z, G|_Z)$. Since we are working with the etale topology, by definition
this class vanishes on an etale surjection $Z' \to Z$ and for our claim it suffices to show
that $Y \times_{K(G/Y, n)} Z'$ is smooth and geometric over $Z'$. Thus we may assume that our map
$Z' \to K(G/Y, n)$ factors through the basepoint section $Y \to K(G/Y, n)$. In particular it
suffices to prove that $Y \times_{K(G/Y, n)} Y \to Y$ is smooth and geometric. But

$$Y \times_{K(G/Y, n)} Y = K(G/Y, n - 1)$$
so by our induction hypothesis this is geometric and smooth over $Y$. This shows that $K(G/Y, n)$ is geometric and furthermore the basepoint section is a choice of map as in GS2. Now the composed map $Y \to K(G/Y, n) \to Y$ is the identity, in particular smooth, so by definition $K(G/Y, n) \to Y$ is smooth.\hfill $\square$

Note that stability under fiber products (Proposition 2.3) implies that if $T$ is a geometric $n$-stack then $\text{Hom}(K, T)$ is geometric for any finite CW complex $K$. See ([Si3] Corollary 5.6) for the details of the argument—which was in the context of presentable $n$-stacks but the argument given there only depends on stability of our class of $n$-stacks under fiber product. We can apply this in particular to the geometric $n$-stacks constructed just above, to obtain some non-connected examples.

If $T = BG$ for an algebraic group $G$ and $K$ is connected with basepoint $k$ then $\text{Hom}(K, T)$ is the moduli stack of representations $\pi_1(K, k) \to G$ up to conjugacy.

3. Locally geometric $n$-stacks

The theory we have described up till now concerns objects of finite type since we have assumed that the scheme $X$ surjecting to our $n$-stack $T$ is of finite type. We can obtain a definition of “locally geometric” by relaxing this to the condition that $X$ be locally of finite type (or equivalently that $X$ be a disjoint union of schemes of finite type). To be precise we say that an $n$-stack $T$ is locally geometric if there exists a sheaf which is a disjoint union of schemes of finite type, with a morphism

$$\varphi : X = \coprod X_i \to T$$

such that $\varphi$ is smooth and geometric.

Note that if $X$ and $Y$ are schemes of finite type mapping to $T$ we still have that $X \times_T Y$ is geometric (GS1).

All of the previous results about fibrations, fiber products, and so on still hold for locally geometric $n$-stacks.

One might want also to relax the definition even further by only requiring that $X \times_T Y$ be itself locally geometric (and so on) even for schemes of finite type. We can obtain a notion that we call slightly geometric by replacing “scheme of finite type” by “scheme locally of finite type” everywhere in the preceding definitions. This notion may be useful in the sense that a lot more $n$-stacks will be “slightly geometric”. However it seems to remove us somewhat from the realm where geometric reasoning will work very well.

4. Glueing

12
We say that a morphism \( U \to T \) of geometric stacks is a Zariski open subset (resp. etale open subset) if for every scheme \( Z \) and \( Z \to T \) the fiber product \( Z \times_T U \) is a Zariski open subset of \( Z \) (resp. an algebraic space with etale map to \( Z \)).

If we have two geometric \( n \)-stacks \( U \) and \( V \) and a geometric \( n \)-stack \( W \) with morphisms \( W \to U \) and \( W \to V \) each of which is a Zariski open subset, then we can glue \( U \) and \( V \) together along \( W \) to get a geometric \( n \)-stack \( T \) with Zariski open subsets \( U \to T \) and \( V \to T \) whose intersection is \( W \). If one wants to glue several open sets it has to be done one at a time (this way we avoid having to talk about higher cocycles).

As a more general result we have the following. Suppose \( \Phi \) is a functor from the simplicial category \( \Delta \) to the category of \( n \)-stacks (say a strict functor to the category of simplicial presheaves, for example). Suppose that each \( \Phi_k \) is a geometric \( n \)-stack, and suppose that the two morphisms \( \Phi_1 \to \Phi_0 \) are smooth. Suppose furthermore that \( \Phi \) satisfies the Segal condition that

\[
\Phi_k \to \Phi_1 \times_{\Phi_0} \ldots \times_{\Phi_0} \Phi_1
\]

is an equivalence (i.e. Illusie weak equivalence of simplicial presheaves). Finally suppose that for any element of \( \Phi_1(X) \) there is, up to localization over \( X \), an “inverse” (for the multiplication on \( \Phi_1 \) that comes from Segal’s condition as in [Se]) up to homotopy. Let \( T \) be the realization over the simplicial variable, into a presheaf of spaces (i.e. we obtain a bisimplicial presheaf, take the diagonal).

**Proposition 4.1** In the above situation, \( T \) is a geometric \( n+1 \)-stack.

**Proof:** There is a surjective map \( \Phi_0 \to T \) and we have by definition that \( \Phi_0 \times_T \Phi_0 = \Phi_1 \). From this one can see that \( T \) is geometric. \( \square \)

As an example of how to apply the above result, suppose \( U \) is a geometric \( n \)-stack and suppose we have a geometric \( n \)-stack \( R \) with \( R \to U \times U \). Suppose furthermore that we have a multiplication \( R \times_{p_2,U,p_1} R \to R \) which is associative and such that inverses exist up to homotopy. Then we can set \( \Phi_k = R \times_U \ldots \times_U R \) with \( \Phi_0 = U \). We are in the above situation, so we obtain the geometric \( n \)-stack \( T \). We call this the \( n \)-stack related to the descent data \((U,R)\).

The original result about glueing over Zariski open subsets can be interpreted in this way.

The simplicial version of this descent with any \( \Phi \) satisfying Segal’s condition is a way to avoid having to talk about strict associativity of the composition on \( R \).

**5. Presentability**

Recall first of all that the category of vector sheaves over a scheme \( Y \) is the smallest abelian category of abelian sheaves on the big site of schemes over \( Y \) containing the...
structure sheaf (these were called “U-coherent sheaves” by Hirschowitz in [H1], who was the first to define them). A vector sheaf may be presented as the kernel of a sequence of 3 coherent sheaves which is otherwise exact on the big site; or dually as the cokernel of an otherwise-exact sequence of 3 vector schemes (i.e. duals of coherent sheaves). The nicest thing about the category of vector sheaves is that duality is involutive.

Recall that we have defined in [Si5] a notion of presentable group sheaf over any base scheme Y. We will not repeat the definition here, but just remark (so as to give a rough idea of what is going on) that if G is a presentable group sheaf over Y then it admits a Lie algebra object Lie(G) which is a vector sheaf with bilinear Lie bracket operation (satisfying Jacobi).

In [Si5] a definition was then made of presentable n-stack; this involves a certain condition on \( \pi_0 \) (for which we refer to [Si5]) and the condition that the higher homotopy group sheaves (over any base scheme) be presentable group sheaves.

For our purposes we shall often be interested in the slightly more restrictive notion of very presentable n-stack. An n-stack T is defined (in [Si5]) to be very presentable if it is presentable, and if furthermore:

1. for \( i \geq 2 \) and for any scheme \( Y \) and \( t \in T(Y) \) we have that \( \pi_i(T|_{X/Y}, t) \) is a vector sheaf over \( Y \); and
2. for any artinian scheme \( Y \) and \( t \in T(Y) \) the group of sections \( \pi_1(T|_{X/Y}, t)(Y) \) (which is naturally an algebraic group scheme over \( \text{Spec}(k) \)) is affine.

For our purposes here we will mostly stick to the case of connected n-stacks in the coefficients. Thus we review what the above definitions mean for \( T \) connected (i.e. \( \pi_0(T) = * \)). Assume that \( k \) is algebraically closed (otherwise one has to take what is said below possibly with some Galois twisting). In the connected case there is essentially a unique basepoint \( t \in T(\text{Spec}(k)) \), a group sheaf over \( \text{Spec}(k) \) is presentable if and only if it is an algebraic group scheme (in [Si5]), so \( T \) is presentable if and only if \( \pi_i(T, t) \) are represented by algebraic group schemes. Note that a vector sheaf over \( \text{Spec}(k) \) is just a vector space, so \( T \) is very presentable if and only if the \( \pi_i(T, t) \) are vector spaces for \( i \geq 2 \) and \( \pi_1(T, t) \) is an affine algebraic group scheme (which can of course act on the \( \pi_i \) by a representation which—because we work over the big site—is automatically algebraic).

**Proposition 5.1** If \( T \) is a geometric n-stack on \( X \) then \( T \) is presentable in the sense of [Si5].

**Proof:** Suppose \( X \to R \) is a smooth morphism from a scheme \( X \) to a geometric n-stack \( R \). Note that the morphism \( R \to \pi_0(R) \) satisfies the lifting properties \( \text{Lift}_n(Y, Y_i) \), since by localizing in the etale topology we get rid of any cohomological obstructions to lifting coming from the higher homotopy groups. On the other hand the morphism \( X \to R \) being smooth, it satisfies the lifting properties (for example one can say that the map \( X \times_R Y \to Y \) is smooth and admits a smooth surjection from a scheme smooth over
Y; with this one gets the lifting properties, recalling of course that a smooth morphism between schemes is vertical. Thus we get that $X \rightarrow \pi_0(R)$ is vertical.

Now suppose $T$ is geometric and choose a smooth surjection $u : X \rightarrow T$. We get from above that $X \rightarrow \pi_0(T)$ is vertical. Note that

$$X \times_{\pi_0(T)} X = im(X \times_T X \rightarrow X \times X).$$

Let $G$ denote the group sheaf $\pi_1(T|_{X/Y}, u)$ over $X$. We have that $G$ acts freely on $\pi_0(X \times_T X)$ (relative to the first projection $X \times_T X \rightarrow X$) and the quotient is the image $X \times_{\pi_0(T)} X$. Thus, locally over schemes mapping into the target, the morphism

$$\pi_0(X \times_T X) \rightarrow X \times_{\pi_0(T)} X$$

is the same as the morphism

$$G \times_X (X \times_{\pi_0(T)} X) \rightarrow X \times_{\pi_0(T)} X$$

obtained by base-change. Since $G \rightarrow X$ is a group sheaf it is an $X$-vertical morphism ([Si5] Theorem 2.2 (7)), therefore its base change is again an $X$-vertical morphism. Since verticality is local over schemes mapping into the target, we get that

$$\pi_0(X \times_T X) \rightarrow X \times_{\pi_0(T)} X$$

is an $X$-vertical morphism. On the other hand by the definition that $T$ is geometric we obtain a smooth surjection $R \rightarrow X \times_T X$ from a scheme $R$, and by the previous discussion this gives a $Spec(\mathbb{C})$-vertical surjection

$$R \rightarrow \pi_0(X \times_T X).$$

Composing we get the $X$-vertical surjection $R \rightarrow X \times_{\pi_0(T)} X$. We have now proven that $\pi_0(T)$ is $P3\frac{1}{2}$ in the terminology of [Si5].

Suppose now that $v : Y \rightarrow T$ is a point. Let $T' := Y \times_T Y$. Then $\pi_0(T') = \pi_1(T|_{X/Y}, v)$ is the group sheaf we are interested in looking at over $Y$. We will show that it is presentable. Note that $T'$ is geometric; we apply the same argument as above, choosing a smooth surjection $X \rightarrow T'$. Recall that this gives a $Spec(\mathbb{C})$-vertical (and hence $Y$-vertical) surjection $X \rightarrow \pi_0(T')$. Choose a smooth surjection $R \rightarrow X \times_{T'} X$. In the previous proof the group sheaf denoted $G$ on $X$ is actually pulled back from a group sheaf $\pi_2(T|_{X/Y}, v)$ on $Y$. Therefore the morphism

$$\pi_0(X \times_{T'} X) \rightarrow X \times_{\pi_0(T')} X$$

15
is a quotient by a group sheaf over \( Y \), in particular it is \( Y \)-vertical. As usual the morphism \( R \to \pi_0(X \times_{T'} X) \) is \( \text{Spec}(C) \)-vertical so in particular \( Y \)-vertical. We obtain a \( Y \)-vertical surjection
\[
R \to X \times_{\pi_0(T')} X.
\]
This finishes the proof that \( \pi_1(T|_{X/Y}, v) \) satisfies property \( P4 \) (and since it is a group sheaf, \( P5 \) i.e. presentable) with respect to \( Y \).

Now note that \( \pi_i(T|_{X/Y}, v) = \pi_{i-1}(T'|_{X/Y}, d) \) where \( d : Y \to T' := Y \times_T Y \) is the diagonal morphism. Hence (as \( T' \) is itself geometric) we obtain by induction that all of the \( \pi_i(T|_{X/Y}, v) \) are presentable group sheaves over \( Y \). This shows that \( T \) is presentable in the terminology of [Si5].

Note that presentability in [Si5] is a slightly stronger condition than the condition of presentability as it is referred to in [Si3] so all of the results stated in [Si3] hold here; and of course all of the results of [Si5] concerning presentable \( n \)-stacks hold for geometric \( n \)-stacks. The example given below which shows that the class of geometric \( n \)-stacks is not closed under truncation, implies that the class of presentable \( n \)-stacks is strictly bigger than the class of geometric ones, since the class of presentable \( n \)-stacks is closed under truncation [Si5].

The results announced (some with sketches of proofs) in [Si3] for presentable \( n \)-stacks hold for geometric \( n \)-stacks. Similarly the basic results of [Si3] hold for geometric \( n \)-stacks. For example, if \( T \) is a geometric \( n \)-stack and \( f : Y \to T \) is a morphism from a scheme then \( \pi_i(T|_{X/Y}, f) \) is a presentable group sheaf, so it has a Lie algebra object \( \text{Lie} \pi_i(T|_{X/Y}, f) \) which is a vector sheaf (or “\( U \)-coherent sheaf” in the terminology of [Hi]) with Lie bracket operation.

Remark: By Proposition 5.1, the condition of being geometric is stronger than the condition of being presentable given in [Si3]. Note from the example given below showing that geometricity is not compatible with truncation (whereas by definition presentability is compatible with truncation), the condition of being geometric is strictly stronger than the condition of being presentable.

Of course in the connected case, presentability and geometricity are the same thing.

**Corollary 5.2** A connected \( n \)-stack \( T \) is geometric if and only if the \( \pi_i(T, t) \) are group schemes of finite type for all \( i \).

**Proof:** We show in [Si5] that presentable groups over \( \text{Spec}(k) \) are just group schemes of finite type. Together with the previous result this shows that if \( T \) is connected and geometric then the \( \pi_i(T, t) \) are group schemes of finite type for all \( i \). On the other hand, if \( \pi_0(T) = \ast \) and the \( \pi_i(T, t) \) are group schemes of finite type for all \( i \) then by the Postnikov decomposition of \( T \) and using 2.6, we conclude that \( T \) is geometric (note that for a group scheme of finite type \( G, K(G, n) \) is geometric).  

\[ \square \]
6. Quillen theory

Quillen in [1] associates to every 1-connected rational space $U$ a differential graded Lie algebra (DGL) $L = \lambda(U)$: a DGL is a graded Lie algebra (over $\mathbb{Q}$ for our purposes) $L = \bigoplus_{p \geq 1} L_p$ (with all elements of strictly positive degree) with differential $\partial : L_p \rightarrow L_{p-1}$ compatible in the usual (graded) way with the Lie bracket. Note our conventions that the indexing is downstairs, by positive numbers and the differential has degree $-1$. The homology groups of $\lambda(U)$ are the homotopy groups of $U$ (shifted by one degree).

This construction gives an equivalence between the homotopy theory of DGL’s and that of rational spaces. Let $L \mapsto |L|$ denote the construction going in the other direction. We shall assume for our purposes that there exists such a realization functor from the category of DGL’s to the category of 1-connected spaces, compatible with finite direct products.

Let $\mathit{DGL}_{C,n}$ denote the category of $n$-truncated $C$-DGL’s of finite type (i.e. with homology groups which are finite dimensional vector spaces, vanishing in degree $\geq n$) and free as graded Lie algebras.

We define a realization functor $\rho^{\text{pre}}$ from $\mathit{DGL}_{C,n}$ to the category of presheaves of spaces over $X$. If $L \in \mathit{DGL}_{C,n}$ then for any $Y \in X$ let

$$\rho^{\text{pre}}(L)(Y) := |L \otimes_C \mathcal{O}(Y)|.$$  

Then let $\rho(L)$ be the $n$-stack associated to the presheaf of spaces $\rho^{\text{pre}}(L)$. This construction is functorial and compatible with direct products (because we have assumed the same thing about the realization functor $|L|$).

Note that $\pi^\text{pre}_0(\rho^{\text{pre}}(L)) = \ast$ and in fact we can choose a basepoint $x$ in $\pi^{\text{pre}}(L)(\text{Spec}(C))$. We have

$$\pi^\text{pre}_i(\rho^{\text{pre}}(L), x) = H_{i-1}(L),$$

(in other words the presheaf on the left is represented by the vector space on the right). This gives the same result on the level of associated stacks and sheaves:

$$\pi_i(\rho(L), x) = H_{i-1}(L).$$

In particular note that a morphism of DGL’s induces an equivalence of $n$-stacks if and only if it is a quasismorphism. Note also that $\rho(L)$ is a 1-connected $n$-stack whose higher homotopy groups are complex vector spaces, thus it is a very presentable geometric $n$-stack.

**Theorem 6.1** The above construction gives an equivalence between the homotopy category $\mathit{ho \mathit{DGL}}_{C,n}$ and the homotopy category of 1-connected very presentable $n$-stacks.
Proof: Let \((L, M)\) denote the set of homotopy classes of maps from \(L\) to \(M\) (either in the world of DGL’s or in the world of \(n\)-stacks on \(X\)). Note that if \(L\) and \(M\) are DGL’s then \(L\) should be free as a Lie algebra (otherwise we have to replace it by a quasiisomorphic free one). We prove that the map

\[(L, M) \to (\rho(L), \rho(M))\]

is an isomorphism. First we show this for the case where \(L = V[n - 1]\) and \(M = U[m - 1]\) are finite dimensional vector spaces in degrees \(n - 1\) and \(m - 1\). In this case (where unfortunately \(L\) isn’t free so has to be replaced by a free DGL) we have

\[(V[n - 1], U[m - 1]) = \text{Hom}(\text{Sym}^{m/n}(V), U)\]

where the symmetric product is in the graded sense (i.e. alternating or symmetric according to parity) and defined as zero when the exponent is not integral. Note that \(\rho(V[n - 1]) = K(V, n)\) and \(\rho(U[m - 1]) = K(U, m)\). The Breen calculations in characteristic zero (easier than the case treated in [Br1]) show that

\[(K(V, n), K(U, m)) = \text{Hom}(\text{Sym}^{m/n}(V), U)\]

so our claim holds in this case.

Next we treat the case of arbitrary \(L\) but \(M = U[m - 1]\) is again a vector space in degree \(m - 1\). In this case we are calculating the cohomology of \(L\) or \(\rho(L)\) in degree \(m\) with coefficients in \(U\). Using a Postnikov decomposition of \(L\) and the appropriate analogues of the Leray spectral sequence on both sides we see that our functor induces an isomorphism on these cohomology groups.

Finally we get to the case of arbitrary \(L\) and arbitrary \(M\). We proceed by induction on the truncation level \(m\) of \(M\). Let \(M' = \tau_{\leq m - 1}M\) be the truncation (coskeleton) with the natural morphism \(M \to M'\). The fiber is of the form \(U[m - 1]\) (we index our truncation by the usual homotopy groups rather than the homology groups of the DGL’s which are shifted by 1). Note that \(\rho(M') = \tau_{\leq m - 1}\rho(M)\) (since the construction \(\rho\) is compatible with homotopy groups so it is compatible with the truncation operations). The fibration \(M \to M'\) is classified by a map \(f : M' \to U[m]\) and the fibration \(\rho(M) \to \rho(M')\) by the corresponding map \(\rho(f) : \rho(M') \to K(U, m + 1)\). The image of

\[(L, M) \to (L, M')\]

consists of the morphisms whose composition into \(U[m]\) is homotopic to the trivial morphism \(L \to U[m]\). Similarly the image of

\[(\rho(L), \rho(M)) \to (\rho(L), \rho(M'))\]
is the morphisms whose composition into $K(U, m + 1)$ is homotopic to the trivial morphism. By our inductive hypothesis

$$\rho : (L, M') \to (\rho(L), \rho(M'))$$

is an isomorphism. The functor $\rho$ is an isomorphism on the images, because we know the statement for targets $U[m]$. Suppose we are given a map $a : L \to M'$ which is in the image. The inverse image of this homotopy class in $(L, M)$ is the quotient of the set of liftings of $a$ by the action of the group of self-homotopies of the map $a$. The set of liftings is a principal homogeneous space under $(L, U[m - 1])$.

Similarly the inverse image of the homotopy class of $\rho(a)$ in $(\rho(L), \rho(M))$ is the quotient of the set of liftings of $\rho(a)$ by the group of self-homotopies of $\rho(a)$. Again the set of liftings is a principal homogeneous space under $(\rho(L), K(U, m))$.

The actions in the principal homogeneous spaces come from maps

$$U[m - 1] \times M \to M$$

over $M'$ and

$$K(U, m) \times \rho(M) \to \rho(M)$$

over $\rho(M')$, the second of which is the image under $\rho$ of the first.

Since $\rho : (L, U[m - 1]) \cong (\rho(L), K(U, m))$, we will get that $\rho$ gives an isomorphism of the fibers if we can show that the images of the actions of the groups of self-homotopy equivalences are the same. Notice that since these actions are on principal homogeneous spaces they factor through the abelianizations of the groups of self-homotopy equivalences.

In general if $A$ and $B$ are spaces then $(A \times S^1, B)$ is the disjoint union over $(A, B)$ of the sets of conjugacy classes of the groups of self-homotopies of the maps from $A$ to $B$. On the other hand a map of groups $G \to G'$ which induces an isomorphism on sets of conjugacy classes is surjective on the level of abelianizations. Thus if we know that a certain functor gives an isomorphism on $(A, B)$ and on $(A \times S^1, B)$ then it is a surjection on the abelianizations of the groups of self-homotopies of the maps.

Applying this principle in the above situation, and noting that we know by our induction hypothesis that $\rho$ induces isomorphisms on $(L, M')$ and $(L \times \lambda(S^1) \otimes_{\mathbb{Z}} k, M')$, we find that $\rho$ induces a surjection from the abelianization of the group of self-homotopies of the map $a : L \to M'$ to the abelianization of the group of self-homotopies of $\rho(a)$. This finally allows us to conclude that $\rho$ induces an isomorphism from the inverse image of the class of $a$ in $(L, M)$ to the inverse image of the class of $\rho(a)$ in $(\rho(L), \rho(M))$. We have completed our proof that

$$\rho : (L, M) \cong (\rho(L), \rho(M)).$$

In order to obtain that $\rho$ induces an isomorphism on homotopy categories we just have to see that any 1-connected very presentable $n$-stack $T$ is of the form $\rho(L)$. We show this
by induction on the truncation level. Put $T' = \tau_{\leq n-1}T$. By the induction hypothesis there is a DGL $L'$ with $\rho(L') \cong T'$ (and we may actually write $\rho(L') = T'$). Now the fibration $T \to T'$ is classified by a map $f : T' \to K(V, n+1)$. From the above proof this map comes from a map $b : L' \to V[n]$, that is $f = \rho(b)$. In turn this map classifies a new DGL $L$ over $L'$. The fibration $\rho(L) \to \rho(L') = T'$ is classified by the map $\rho(b) = f$ so $\rho(L) \cong T$. \hfill \Box

Dold-Puppe

Eventually it would be nice to have a relative version of the previous theory, over any $n$-stack $R$. The main problem in trying to do this is to have the right notion of complex of sheaves over an $n$-stack $R$. Instead of trying to do this, we will simply use the notion of spectrum over $R$ (to be precise I will use the word “spectrum” for what is usually called an “$\Omega$-spectrum”. For our purposes we are only interested in spectra with homotopy groups which are rational and vanish outside of a bounded interval. In absolute terms such a spectrum is equivalent to a complex of rational vector spaces, so in the relative case over a presheaf of spaces $R$ this gives a generalized notion of complex over $R$.

For our spectra with homotopy groups nonvanishing only in a bounded region, we can replace the complicated general theory by the simple consideration of supposing that we are in the stable range. Thus we fix numbers $N, M$ with $M$ bigger than the length of any complex we want to consider and $N$ bigger than $2M+2$. For example if we are only interested in dealing with $n$-stacks then we cannot be interested in complexes of length bigger than $n$ so we could take $M > n$.

An spectrum (in our setting) is then simply an $N$-truncated rational space with a basepoint, and which is $N - M - 1$-connected. More generally if $R$ is an $n$-stack with $n \leq N$ then a spectrum over $R$ is just an $N$-stack $S$ with morphism $p : S \to R$ and section denoted $\xi : R \to S$ such that $S$ is rational and $N - M - 1$-connected relative to $R$. A morphism of spectra is a morphism of spaces (preserving the basepoint).

Suppose $S$ is a spectrum; we define the complex associated to $S$ by setting $\gamma(S)^i$ to be the singular $N - i$-chains on $S$. The differential $d : \gamma(S)^i \to \gamma(S)^{i+1}$ is the same as the boundary map on chains (which switches direction because of the change in indexing). Note that we have normalize things so that the complex starts in degree 0. The homotopy theory of spectra is the same as that of complexes of rational vector spaces indexed in degrees $\geq 0$, with cohomology nonvanishing only in degrees $\leq M$.

If $C^\cdot$ is a complex as above then let $\sigma(C^\cdot)$ denote the corresponding spectrum.

This can be generalized to the case where the base is a 0-stack. If $Y$ is a 0-stack (notably for example a scheme) and if $S$ is a spectrum over $Y$ then we obtain a complex of presheaves of rational vector spaces $\gamma(S/Y)$ over $Y$. Conversely if $C^\cdot$ is a complex of presheaves of rational vector spaces over $Y$ then we obtain a spectrum denoted $\sigma(C^\cdot/Y)$. 20
These constructions are an equivalence in homotopy theories, where the weak equivalence between complexes means quasiisomorphism (i.e. morphisms inducing isomorphisms on associated cohomology sheaves).

If $S$ is a spectrum and $n \leq N$ then we can define the realization $\kappa(S, n)$ to be the $N - n$-th loop space $\Omega^{N-n}S$ (the loops are taken based at the given basepoint). Similarly if $S$ is a spectrum over an $n'$-stack $R$ then we obtain the realization $\kappa(S/R, n) \to R$ as the $N - n$-th relative loop space based at the given section $\xi$.

Taken together we obtain the following construction: if $C$ is a complex of vector spaces then $\kappa(\sigma(C), n)$ is an $n$-stack. If $C$ is a complex of presheaves of rational vector spaces over a 0-stack (presheaf of sets) $Y$ then $\kappa(\sigma(C/Y)/Y, n)$ is an $n$-stack over $Y$. These constructions are what is known as Dold-Puppe. They are compatible with the usual Eilenberg-MacLane constructions: if $V$ is a presheaf of rational vector spaces over $Y$ considered as a complex in degree 0 then

$$\kappa(\sigma(V/Y)/Y, n) = K(V/Y, n).$$

The basic idea behind our notational system is that we think of spectra over $R$ as being complexes of rational presheaves over $R$ starting in degree 0. The operation $\kappa(S/R, n)$ is the Dold-Puppe realization from a “complex” to a space relative to $R$.

We can do higher direct images in this context. If $f : R \to T$ is a morphism of $n$-stacks and if $S$ is a spectrum over $R$ then define $f_*(S)$ to be the $N$-stack $\Gamma(R/T, S)$ of sections relative to $T$. This is compatible with realizations: we have

$$\Gamma(R/T, \kappa(S, n)) = \kappa(f_*(S), n).$$

Suppose that $f : X \to Y$ is a morphism of 0-stacks. Then for a complex of rational presheaves $C$ on $X$ the direct image construction in terms of spectra is the same as the usual higher direct image of complexes of sheaves (applied to the sheafification of the complex):

$$f_*(\sigma(C/X)) = \sigma((Rf_*C)/Y).$$

We extend this just a little bit, in a special case in which it still makes sense to talk about complexes. Suppose $X$ is a 1-stack and $Y$ is a 0-stack, with $f : X \to Y$ a morphism. Suppose $V$ is a local system of presheaves on $X$ (i.e. for each $Z \in X$, $V(Z)$ is a local system of rational vector spaces on $X(Z)$). Another way to put this is that $V$ is an abelian group object over $Z$. We can think of $V$ as being a complex of presheaves over $X$ (even though we have not defined this notion in general) and we obtain the spectrum which we denote by $\sigma(V/X)$ over $X$ (even though this doesn’t quite fit in with the general definition of $\sigma$ above), and its realization $\kappa(\sigma(V/X)/X, n) \to X$ which is what we would otherwise denote as $K(V/X, n)$. The higher direct image $Rf_*(V)$ makes sense as a complex of presheaves on $Y$, and we have the compatibilities

$$f_*\sigma(V/X) = \sigma(Rf_*(V))$$
and
\[ \Gamma(X/Y, \kappa(\sigma(V/X)/X, n)) = \kappa(\sigma(Rf_*(V)), n). \]

**Proposition 6.2** Suppose \( R \) is an \( n \)-stack and \( S \) is a spectrum over \( R \) such that for every map \( Y \to R \) from a scheme, there is (locally over \( Y \) in the etale topology) a complex of vector bundles \( E_Y \) over \( Y \) with \( S \times_R Y \cong \sigma(E_Y/Y) \). Then the realization \( \kappa(S/R, n) \) is geometric over \( R \). In particular if \( R \) is geometric then so is \( \kappa(S/R, n) \).

**Proof:** In order to prove that the morphism \( \kappa(S/R, n) \to R \) is geometric, it suffices to prove that for every base change to a scheme \( Y \to R \), the fiber product \( \kappa(S/R, n) \times_R Y \) is geometric. But
\[ \kappa(S/R, n) \times_R Y = \kappa(\sigma(E_Y/Y)/Y, n), \]
so it suffices to prove that for a scheme \( Y \) and a complex of vector bundles \( E \) on \( Y \), we have \( \kappa(\sigma(E/Y)/Y, n) \) geometric.

Note that \( \kappa(\sigma(E'), n) \) only depends on the part of the complex
\[ E^0 \to E^1 \to \ldots \to E^n \to E^{n+1} \]
so we assume that it stops there or earlier. Now we proceed by induction on the length of the complex. Define a complex \( F^i = E^{i-1} \) for \( i \geq 1 \), which has length strictly smaller than that of \( E \). Let \( E^0 \) denote the first vector bundle of \( E \) considered as a complex in degree 0 only. We have a morphism of complexes \( E^0 \to F^i \) and \( E^i \) is the mapping cone. Thus
\[ \sigma(E'/Y) = \sigma(E^0/Y) \times_{\sigma(F'/Y)} Y \]
with \( Y \to \sigma(F'/Y) \) the basepoint section. We get
\[ \kappa(\sigma(E'/Y)/Y, n) = K(E^0/Y, n) \times_{\kappa(\sigma(F'/Y)/Y, n)} Y. \]
By our induction hypothesis, \( \kappa(\sigma(F'/Y)/Y, n) \) is geometric. Note that \( E_0 \) is a smooth group scheme over \( Y \) so by Lemma 2.10, \( K(E^0/Y, n) \) is geometric \( Y \). By 2.1, \( \kappa(\sigma(E'/Y)/Y, n) \) is geometric. \( \square \)

**Remark:** This proposition is a generalisation to \( n \)-stacks of ([LMB] Construction 9.19, Proposition 9.20). Note that if \( E \) is a complex where \( E^i \) are vector bundles for \( i < n \) and \( E^n \) is a vector scheme (i.e. something of the form \( V(M) \) for a coherent sheaf \( M \) in the notation of [LMB]) then we can express \( E^n \) as the kernel of a morphism \( U^n \to U^{n+1} \) of vector bundles (this would be dual to the presentation of \( M \) if we write \( E^n = V(M) \)). Setting \( U^i = E^i \) for \( i < n \) we get \( \kappa(\sigma(E'), n) = \kappa(\sigma(U'), n) \). In this way we recover Laumon’s and Moret-Bailly’s construction in the case \( n = 1 \).
Corollary 6.3 Suppose \( f : X \to Y \) is a projective flat morphism of schemes, and suppose that \( V \) is a vector bundle on \( X \). Then \( \Gamma(X/Y, K(V/X, n)) \) is a geometric \( n \)-stack lying over \( Y \).

Proof: By the discussion at the start of this subsection,

\[
\Gamma(X/Y, K(V/X, n)) = \kappa(\sigma(\mathbf{R}f_*(V)/Y)/Y, n).
\]

But by Mumford’s method [Mu], \( \mathbf{R}f_*(V) \) is quasiisomorphic (locally over \( Y \)) to a complex of vector bundles. By Proposition 6.2 we get that \( \Gamma(X/Y, K(V, n)) \) is geometric over \( Y \).

Recall that a formal groupoid is a stack \( X_\Lambda \) associated to a groupoid of formal schemes where the object object is a scheme \( X \) and the morphism object is a formal scheme \( \Lambda \to X \times X \) with support along the diagonal. We say it is smooth if the projections \( \Lambda \to X \) are formally smooth. In this case the cohomology of the stack \( X_\Lambda \) with coefficients in vector bundles over \( X_\Lambda \) (i.e. vector bundles on \( X \) with \( \Lambda \)-structure meaning isomorphisms between the two pullbacks to \( \Lambda \) satisfying the cocycle condition on \( \Lambda \times_X \Lambda \)) is calculated by the de Rham complex \( \Omega_\Lambda \otimes_O V \) of locally free sheaves associated to the formal scheme [Il] [Ber].

We say that \( X_\Lambda \to Y \) is a smooth formal groupoid over \( Y \) if \( X_\Lambda \) is a smooth formal groupoid mapping to \( Y \) and if \( X \) is flat over \( Y \).

Corollary 6.4 Suppose \( f : X_\Lambda \to Y \) is a projective smooth formal groupoid over a scheme \( Y \). Suppose that \( V \) is a vector bundle on \( X_\Lambda \) (i.e. a vector bundle on \( X \) with \( \Lambda \)-structure). Then \( \Gamma(X_\Lambda/Y, K(V/X_\Lambda, n)) \) is a geometric \( n \)-stack lying over \( Y \).

Proof: By the “slight extension” in the discussion at the start of this subsection,

\[
\Gamma(X_\Lambda/Y, K(V/X_\Lambda, n)) = \kappa(\sigma(\mathbf{R}f_*(V)/Y)/Y, n).
\]

But

\[
\mathbf{R}f_*(V) = \mathbf{R}f'_*(\Omega_\Lambda \otimes_O V)
\]

where \( f' : X \to Y \) is the morphism on underlying schemes. Again by Mumford’s method [Mu], \( \mathbf{R}f'_*(\Omega_\Lambda \otimes_O V) \) is quasiisomorphic (locally over \( Y \)) to a complex of vector bundles. By the Proposition 6.2 we get that \( \Gamma(X_\Lambda/Y, K(V/X_\Lambda, n)) \) is geometric over \( Y \). □

7. Maps into geometric \( n \)-stacks
Theorem 7.1 Suppose $X \to S$ is a projective flat morphism. Suppose $T$ is a connected $n$-stack which is very presentable (i.e. the fundamental group is represented by an affine group scheme of finite type denoted $G$ and the higher homotopy groups are represented by finite dimensional vector spaces). Then the morphism $\text{Hom}(X/S,T) \to \text{Bun}_G(X/S) = \text{Hom}(X/S, BG)$ is a geometric morphism. In particular $\text{Hom}(X/S,T)$ is a locally geometric $n$-stack.

Proof: Suppose $V$ is a finite dimensional vector space. Let $\mathcal{B}(V,n) = B\text{Aut}(K(V,n))$ be the classifying $n+1$-stack for fibrations with fiber $K(V,n)$. It is connected with fundamental group $\text{GL}(V)$ and homotopy group $V$ in dimension $n+1$ and zero elsewhere. The truncation morphism $\mathcal{B}(V,n) \to B\text{GL}(V)$ has fiber $K(V,n+1)$ and admits a canonical section $o : B\text{GL}(V) \to \mathcal{B}(V,n)$ (which corresponds to the trivial fibration with given action of $\text{GL}(V)$ on $V$—this fibration may itself be constructed as $\mathcal{B}(V,n-1)$ or in case $n = 2$ as $B(\text{GL}(V) \ltimes V)$). The fiber of the morphism $o$ is $K(V,n)$, and $B\text{GL}(V)$ is the universal object over $\mathcal{B}(V,n)$.

Note that $B\text{GL}(V)$ is an geometric 1-stack (i.e. algebraic stack) and by Proposition 2.6 applied to the truncation fibration, $\mathcal{B}(V,n)$ is a geometric $n+1$-stack. If $X \to S$ is a projective flat morphism then $\text{Hom}(X/S, B\text{GL}(V))$ is a locally geometric 1-stack (via the theory of Hilbert schemes). We show that $p : \text{Hom}(X/S, \mathcal{B}(V,n)) \to \text{Hom}(X/S, B\text{GL}(V))$ is a geometric morphism. For this it suffices to consider a morphism $\zeta : Y \to \text{Hom}(X/S, B\text{GL}(V))$ from a scheme $Y/S$ which in turn corresponds to a vector bundle $V_\zeta$ on $X \times_S Y$. The fiber of the map $p$ over $\eta$ is $\Gamma(X \times_S Y/Y; K(V_\zeta,n+1))$ which as we have seen above is geometric over $Y$. This shows that $p$ is geometric. In particular $\text{Hom}(X/S, \mathcal{B}(V,n))$ is locally geometric.

We now turn to the situation of a general connected geometric and very presentable $n$-stack $T$. Consider the truncation morphism $a : T \to T' := \tau_{\leq n-1}T$. We may assume that the theorem is known for the $n-1$-stack $T'$. The morphism $a$ is a fibration with fiber $K(V,n)$ so it comes from a map $b : T' \to \mathcal{B}(V,n)$ and more precisely we have

$$T = T' \times_{\mathcal{B}(V,n)} B\text{GL}(V).$$

Thus

$$\text{Hom}(X/S,T) = \text{Hom}(X/S,T') \times_{\text{Hom}(X/S, \mathcal{B}(V,n))} \text{Hom}(X/S, B\text{GL}(V)).$$

But we have just checked that $\text{Hom}(X/S, B\text{GL}(V))$ and $\text{Hom}(X/S, \mathcal{B}(V,n))$ are locally geometric, and by hypothesis $\text{Hom}(X/S,T')$ is locally geometric. Therefore by the version
of [2,1] for locally geometric n-stacks, the fiber product is locally geometric. This completes the proof.

Theorem 7.2 Suppose $(X, \Lambda) \to S$ is a smooth projective morphism with smooth formal category structure relative to $S$. Let $X_{\Lambda} \to S$ be the resulting family of stacks. Suppose $T$ is a connected very presentable n-stack which is very presentable (with fundamental group scheme denoted $G$). Then the morphism $\text{Hom}(X_{\Lambda}/S, T) \to \text{Hom}(X_{\Lambda}/S, BG)$ is a geometric morphism. In particular $\text{Hom}(X_{\Lambda}/S, T)$ is a locally geometric n-stack.

Proof: The same as before. Note here also that $\text{Hom}(X_{\Lambda}/S, BG)$ is an algebraic stack locally of finite type.

Remark: In the above theorems the base $S$ can be assumed to be any n-stack, one looks at morphisms with the required properties when base changed to any scheme $Y \to S$.

Semistability

Suppose $X \to S$ is a projective flat morphism, with fixed ample class, and suppose $G$ is an affine algebraic group. We get a notion of semistability for $G$-bundles (for example, fix the convention that we speak of Gieseker semistability). Fix also a collection of Chern classes which we denote $c$. We get a Zariski open substack $\text{Hom}_{c}^{se}(X/S, BG) \subset \text{Hom}(X/S, BG)$ (just the moduli 1-stack of semistable $G$-bundles with Chern classes $c$). The boundedness property for semistable $G$-bundles with fixed Chern classes shows that $\text{Hom}_{c}^{se}(X/S, BG)$ is a geometric 1-stack.

Now if $T$ is a connected very presentable n-stack, let $G$ be the fundamental group scheme and let $c$ be a choice of Chern classes for $G$-bundles. Define

$$\text{Hom}_{c}^{se}(X/S, T) := \text{Hom}(X/S, T) \times_{\text{Hom}(X/S, BG)} \text{Hom}_{c}^{se}(X/S, BG).$$

Again it is a Zariski open substack of $\text{Hom}(X/S, T)$ and it is a geometric n-stack rather than just locally geometric.

We can do the same in the case of a smooth formal category $X_{\Lambda} \to S$. Make the convention in this case that we ask the Chern classes to be zero (there is no mathematical need to do this, it is just to conserve indices, since practically speaking this is the only case we are interested in below). We obtain a Zariski open substack

$$\text{Hom}_{c}^{se}(X_{\Lambda}/S, BG) \subset \text{Hom}(X_{\Lambda}/S, BG),$$

25
the moduli stack for semistable $G$-bundles on $X_A$ with vanishing Chern classes. See [St] for the construction (again the methods given there suffice for the construction, although stacks are not explicitly mentionned). Again for any connected very presentable $T$ with fundamental group scheme $G$ we put

$$\text{Hom}^{se}(X_A/S, T) := \text{Hom}(X_A/S, T) \times_{\text{Hom}(X_A/S, BG)} \text{Hom}^{se}(X_A/S, BG).$$

It is a geometric $n$-stack.

Finally we note that in the case of the relative de Rham formal category $X_{DR/S}$ semistability of principal $G$-bundles is automatic (as is the vanishing of the Chern classes). Thus

$$\text{Hom}^{se}(X_{DR/S}/S, T) = \text{Hom}(X_{DR/S}/S, T)$$

and $\text{Hom}(X_{DR/S}, T)$ is already a geometric $n$-stack.

**The Brill-Noether locus**

Suppose $G$ is an algebraic group and $V$ is a representation. Define the $n$-stack $\kappa(G, V, n)$ as the fibration over $K(G, 1)$ with fiber $K(V, n)$ where $G$ acts on $V$ by the given representation and such that there is a section. Let $X$ be a projective variety. We have a morphism

$$\text{Hom}(X, \kappa(G, V, n)) \to \text{Hom}(X, K(G, 1)) = \text{Bun}_G(X).$$

The fiber over a point $S \to \text{Bun}_G(X)$ corresponding to a principal $G$-bundle $P$ on $X \times S$ is the relative section space

$$\Gamma(X \times S/S, K(P \times^G V/X \times S, n)).$$

By the compatibilities given at the start of the section on Dold-Puppe, this relative section space is the $n$-stack corresponding to the direct image $R\pi_*(P \times^G V)$ which is a complex over $S$. Note that this complex is quasiisomorphic to a complex of vector bundles. Thus we have:

**Corollary 7.3** The morphism

$$\text{Hom}(X, \kappa(G, V, n)) \to \text{Bun}_G(X)$$

is a morphism of geometric $n$-stacks.

\[\square\]

*Remark:* The $\text{Spec}(C)$-valued points of $\text{Hom}(X, \kappa(G, V, n))$ are the pairs $(P, \eta)$ where $P$ is a principal $G$-bundle on $X$ and $\eta \in H^n(X, P \times^G V)$.

Thus $\text{Hom}(X, \kappa(G, V, n))$ is a geometric $n$-stack whose $\text{Spec}(C)$-points are the Brill-Noether set of vector bundles with cohomology classes on $X$. 

26
Some conjectures

We give here some conjectures about the possible extension of the above results to any (not necessarily connected) geometric $n$-stacks $T$.

**Conjecture 1** If $T$ is a geometric $n$-stack which is very presentable in the sense of $[SÉ]$ (i.e. the fundamental groups over artinian base are affine, and the higher homotopy groups are vector sheaves) then for any smooth (or just flat?) projective morphism $X \to S$ we have that $\text{Hom}(X/S, T)$ is locally geometric.

**Conjecture 2** If $T = K(G_m, 2)$ then for a flat projective morphism $X \to S$, $\text{Hom}(X/S, T)$ is locally geometric. Similarly if $G$ is any group scheme of finite type (e.g. an abelian variety) then $\text{Hom}(X/S, BG)$ is locally geometric.

Putting together with the previous conjecture we can make:

**Conjecture 3** If $T$ is a geometric $n$-stack whose $\pi_i$ are vector sheaves for $i \geq 3$ then $\text{Hom}(X/S, T)$ is locally geometric.

Note that Conjecture 2 cannot be true if $K(G_m, 2)$ is replaced by $K(G_m, i)$ for $i \geq 3$, for in that case the morphism stacks will themselves be only locally of finite type. Instead we will get a “slightly geometric” $n$-stack as discussed in §3. One could make the following conjecture:

**Conjecture 4** If $T$ is any geometric (or even locally or slightly geometric) $n$-stack and $X \to S$ is a flat projective morphism then $\text{Hom}(X/S, T)$ is slightly geometric.

After these somewhat improbable-sounding conjectures, let finish by making a more reasonable statement:

**Conjecture 5** If $T$ is a very presentable geometric $n$-stack and $X$ is a smooth projective variety then $\text{Hom}(X_{DR}, T)$ is again geometric.

Here, we have already announced the finite-type result in the statement that $\text{Hom}(X_{DR}, T)$ is very presentable $[SÉ]$ (I have not yet circulated the proof, still checking the details...).

GAGA

27
Let $\mathcal{X}^{\text{an}}$ be the site of complex analytic spaces with the etale (or usual—its the same) topology. We can make similar definitions of geometric $n$-stack on $\mathcal{X}^{\text{an}}$ which we will now denote by analytic $n$-stack (in case of confusion...). There are similar definitions of smoothness and so on.

There is a morphism of sites from the analytic to the algebraic sites.

If $T$ is a geometric $n$-stack on $\mathcal{X}$ then its pullback by this morphism (cf \cite{Si6}) is an analytic $n$-stack which we denote by $T^{\text{an}}$.

We have:

**Theorem 7.4** Suppose $T$ is a connected very presentable geometric $n$-stack. Suppose $X \to S$ is a flat projective morphism (resp. suppose $X_{\Lambda} \to S$ is the morphism associated to a smooth formal category over $S$). Then the natural morphism

$$\text{Hom}(X/S, T)^{\text{an}} \to \text{Hom}(X^{\text{an}}/S^{\text{an}}, T^{\text{an}})$$

(resp. $\text{Hom}(X_{\Lambda}/S, T)^{\text{an}} \to \text{Hom}(X_{\Lambda}^{\text{an}}/S^{\text{an}}, T^{\text{an}})$)

is an isomorphism of analytic $n$-stacks.

**Proof:** Just following through the proof of the facts that $\text{Hom}(X/S, T)$ or $\text{Hom}(X_{\Lambda}/S, T)$ are geometric, we can keep track of the analytic case too and see that the morphisms are isomorphisms along with the main induction. \hfill $\Box$

**Remarks:**

(1) This GAGA theorem holds for $X_{\text{DR}}$ with coefficients in any very presentable $T$ (not necessarily connected) \cite{Si3}.

(2) In \cite{Si3} we also give a “GFGA” theorem for $X_{\text{DR}}$ with coefficients in a very presentable $n$-stack.

(3) The GAGA theorem does not hold with coefficients in $T = K(\mathbb{G}_m, 2)$. Thus the condition that the higher homotopy group sheaves of $T$ be vector sheaves is essential. Maybe it could be weakened by requiring just that the fibers over artinian base schemes be unipotent (but this might also be equivalent to the vector sheaf condition).

(4) Similarly the GAGA theorem does not hold with coefficients in $T = BA$ for an abelian variety $A$; thus again the hypothesis that the fibers of the fundamental group sheaf over artinian base be affine group schemes, is essential.

8. The tangent spectrum

We can treat a fairly simple case of the conjectures outlined above: maps from the spectrum of an Artin local algebra of finite type.
**Theorem 8.1** Let \( X = \text{Spec}(A) \) where \( A \) is artinian, local, and of finite type over \( k \). Suppose \( T \) is a geometric \( n \)-stack. Then \( \text{Hom}(X,T) \) is a geometric \( n \)-stack. If \( T \to T' \) is a geometric smooth morphism of \( n \)-stacks then \( \text{Hom}(X,T) \to \text{Hom}(X,T') \) is a smooth geometric morphism of \( n \)-stacks.

**Proof:** We prove the following statement: if \( Y \) is a scheme and \( A \) as in the theorem, and if \( T \to Y \times \text{Spec}(A) \) is a geometric (resp. smooth geometric) morphism of \( n \)-stacks then \( \Gamma(Y \times \text{Spec}(A)/Y,T) \) is geometric (resp. smooth geometric) over \( Y \). The proof is by induction on \( n \); note that it works for \( n = 0 \). Now in general choose a smooth surjection \( X \to T \) from a scheme. Then \( \Gamma(Y \times \text{Spec}(A)/Y,X) \) is a scheme over \( Y \), and if \( X \) is smooth over \( Y \) then the section scheme is smooth over \( Y \). We have a surjection

\[
a : \Gamma(Y \times \text{Spec}(A)/Y,X) \to \Gamma(Y \times \text{Spec}(A)/Y,T),
\]

and for \( Z \to \Gamma(Y \times \text{Spec}(A)/Y,X) \) (which amounts to a section morphism \( Z \times \text{Spec}(A) \to T \)) the fiber product

\[
\Gamma(Y \times \text{Spec}(A)/Y,X) \times_{\Gamma(Y \times \text{Spec}(A)/Y,T)} Z
\]

is equal to

\[
\Gamma(Z \times \text{Spec}(A)/Z,X \times_T (Z \times \text{Spec}(A))).
\]

But \( X \times_T (Z \times \text{Spec}(A)) \) is a smooth \( n-1 \)-stack over \( Z \times \text{Spec}(A) \) so by induction this section stack is geometric and smooth over \( Z \). Thus our surjection \( a \) is a smooth geometric morphism so \( \Gamma(Y \times \text{Spec}(A)/Y,T) \) is geometric. The smoothness statement follows immediately. \( \square \)

We apply this to define the tangent spectrum of a geometric \( n \)-stack. This is a generalization of the discussion at the end of [LMF] §9, although we use a different approach because I don’t have the courage to talk about cotangent complexes!

Recall from [Ad] Segal’s infinite loop space machine: let \( \Gamma \) be the category whose objects are finite sets and where the morphisms from \( \sigma \) to \( \tau \) are maps \( P(\sigma) \to P(\tau) \) preserving disjoint unions (here \( P(\sigma) \) is the set of subsets of \( \sigma \)). A morphism is determined, in fact, by the map \( \sigma \to P(\tau) \) taking different elements of \( \sigma \) to disjoint subsets of \( \tau \) (note that the empty set must go to the empty set). Let \( [n] \) denote the set with \( n \) elements. There is a functor \( s : \Delta \to \Gamma \) sending the the ordered set \( \{0, \ldots , n\} \) to the finite set \( \{1, \ldots , n\} \)—see [Ad] p. 64 for the formulas for the morphisms.

Segal’s version of an \( E_\infty \)-space (i.e. infinite loop space) is a contravariant functor \( \Psi : \Gamma \to \text{Top} \) such that the associated simplicial space (the composition \( \Psi \circ s \)) satisfies Segal’s condition [Ad]. In order to really get an infinite loop space it is also required that \( \Psi(\emptyset) \) be a point (although this condition seems to have been lost in Adams’ very brief treatment).
Segal’s machine is then a classifying space functor $B$ from special $\Gamma$-spaces to special $\Gamma$-spaces. This actually works even without the condition that $\Phi(\emptyset)$ be a point, however the classifying space construction is the inverse to the relative loop space construction over $\Phi(\emptyset)$. Note that since $\emptyset$ is a final object in $\Gamma$ the components of a $\Gamma$-space are provided with a section from $\Phi(\emptyset)$. If $\Phi$ is a special $\Gamma$-space then $B^n\Phi$ is again a special $\Gamma$ space with

$$B^n\Phi(\emptyset) = \Phi(\emptyset)$$

and

$$\Omega^n(B^n\Phi([1])/\Phi(\emptyset)) = \Phi([1]).$$

The notion of $\Gamma$-space (say with $\Phi[1]$ rational over $\Phi(\emptyset) = R$) is another replacement for our notion of spectrum over $R$; we get to our notion as defined above by looking at $B^N\Phi([1])$.

The above discussion makes sense in the context of presheaves of spaces over $\mathcal{X}$ hence in the context of $n$-stacks.

We now try to apply this in our situation to construct the tangent spectrum. For any object $\sigma \in \Gamma$ let $A^\sigma$ be the affine space over $k$ with basis the set $\sigma$. An element of $A^\sigma$ can be written as $\sum_{i \in \sigma} a_i e_i$ where $e_i$ are the basis elements and $a_i \in k$. Given a map $f : \sigma \to P(\tau)$ we define a map

$$A^f : A^\sigma \to A^\tau$$

$$\sum a_i e_i \mapsto \sum_{i \in \sigma} \sum_{j \in f(i) \subset \tau} a_i e_j.$$

For example there are four morphisms from $[1]$ to $[2]$, sending 1 to $\emptyset$, $\{1\}$, $\{2\}$ and $\{1,2\}$ respectively. These correspond to the constant morphism, the two coordinate axes, and the diagonal from $A^1$ to $A^2$. We get a covariant functor from $\Gamma$ to the category of affine schemes.

For a finite set $\sigma$ let $D^\sigma$ denote the subscheme of $A^\sigma$ defined by the square of the maximal ideal defining the origin. These fit together into a covariant functor from $\Gamma$ to the category of artinian local schemes of finite type over $k$.

If $T$ is a geometric $n$-stack thought of as a strict presheaf of spaces, then the functor

$$\Theta : \sigma \mapsto \text{Hom}(D^\sigma, T)$$

is a contravariant functor from $\Gamma$ to the category of geometric $n$-stacks, with $\Theta(\emptyset) = T$.

To see that it satisfies Segal’s condition we have to check that the map

$$\text{Hom}(D^n, T) \to \text{Hom}(D^1, R) \times_R \ldots \times_T \text{Hom}(D^1, T)$$

is an equivalence. Once this is checked we obtain a spectrum over $T$ whose interpretation in our terms is as the $N$-stack $B^N\Phi([1])$. 

30
In the statement of the following theorem we will normalize our relationship between complexes and spectra in a different way from before—the most natural way for our present purposes.

**Theorem 8.2** Suppose $T$ is a geometric $n$-stack. The above construction gives a spectrum $\Theta(T) \to T$ which we call the tangent spectrum of $T$. If $Y \to T$ is a morphism from a scheme then $\Theta(T) \times_T Y$ is equivalent to $\sigma(E/Y)$ for a complex

$$E^{-n} \to \ldots \to E^0$$

with $E^i$ vector bundles ($i < 0$) and $E^0$ a vector scheme over $Y$. Furthermore if $T$ is smooth then $E^0$ can be assumed to be a vector bundle. In particular, $\kappa(\Theta(T)/T, n)$ is geometric, and if $T$ is smooth then $\Theta(T)$ is geometric.

**Proof of 8.2:** The first task is to check the above condition for $\Theta$ to be a special $\Gamma$-space. Suppose in general that $A, B \subset C$ are closed artinian subschemes of an artinian scheme with the extension property that for any scheme $Y$ the morphisms from $C$ to $Y$ are the same as the pairs of morphisms $A, B \to Y$ agreeing on $A \cap B$. We would like to show that for any geometric stack $T$,

$$\text{Hom}(C, T) \to \text{Hom}(A, T) \times_{\text{Hom}(A \cap B, T)} \text{Hom}(B, T)$$

is an equivalence. We have a similar relative statement for sections of a geometric morphism $T \to Y \times C$ for a scheme $Y$. We prove the relative statement by induction on the truncation level $n$, but for simplicity use the notation of the absolute statement. Let $X \to T$ be a smooth geometric morphism from a scheme. Then consider the diagram

$$\begin{array}{ccc}
\text{Hom}(C, X) & \xrightarrow{\cong} & \text{Hom}(A, X) \times_{\text{Hom}(A \cap B, X)} \text{Hom}(B, X) \\
\downarrow & & \downarrow \\
\text{Hom}(C, T) & \to & \text{Hom}(A, T) \times_{\text{Hom}(A \cap B, T)} \text{Hom}(B, T).
\end{array}$$

It suffices to prove that for a map from a scheme $Y \to \text{Hom}(C, T)$ the morphism on fibers is an equivalence. The fiber on the left is

$$\text{Hom}(C, X) \times_{\text{Hom}(C, T)} Y = \Gamma(Y \times C, X \times_T (Y \times C)),$$

whereas the fiber on the right is

$$\Gamma(Y \times A, X \times_T (Y \times A)) \times_{\Gamma(Y \times (A \cap B), X \times_T (Y \times (A \cap B)))} \Gamma(Y \times B, X \times_T (Y \times B)).$$

By the relative version of the statement for the $n-1$-stack $X \times_T (Y \times C)$ over $Y \times C$, the map of fibers is an equivalence, so the map

$$\text{Hom}(C, T) \to \text{Hom}(A, T) \times_{\text{Hom}(A \cap B, T)} \text{Hom}(B, T)$$
is an equivalence.

Apply this inductively with \( C = D^n, A = D^1 \) and \( B = D^{n-1} \) (so \( A \cap B = D^0 \)). We obtain the required statement, showing that \( \Theta \) is a special \( \Gamma \)-space relative to \( T \). It integrates to a spectrum which we denote \( \Theta(T) \to T \).

Note that if \( T = X \) is a scheme considered as an \( n \)-stack then \( \Theta(X) \) is just the spectrum associated to the complex consisting of the tangent vector scheme of \( X \) in degree 0. We obtain the desired statement in this case.

If \( R \to T \) is a morphism of geometric \( n \)-stacks then we obtain a morphism of spectra

\[
\Theta(R) \to \Theta(T) \times_T R.
\]

The cofiber (i.e. \( B \) of the fiber) we denote by \( \Theta(R/T) \). We prove more generally—by induction on \( n \)—that if \( T \to Y \) is a geometric morphism from an \( n \)-stack to a scheme, and if \( Y \to T \) is a section then \( \Theta(T/Y) \times_T Y \) is associated (locally on \( Y \)) to a complex of vector bundles and a vector scheme at the end; with the last vector scheme being a bundle if the morphism is smooth. Note that it is true for \( n = 0 \). For any \( n \) choose a smooth geometric morphism \( X \to T \) and we may assume (by etale localization) that there is a lifting of the section to \( Y \to X \). Now there is a triangle of spectra (i.e. associated to a triangle of complexes in the derived category)

\[
\Theta(X) \times_X Y \to \Theta(T) \times_T Y \to B\Theta(X/T) \times_X Y.
\]

On the other hand,

\[
B\Theta(X/T) \times_X Y = B\Theta(X \times_T Y/Y) \times_{X \times_T Y} Y.
\]

By induction this is associated to a complex as desired, and we know already that \( \Theta(X) \times_X Y \) is associated to a complex as desired. Therefore \( \Theta(T) \times_T Y \) is an extension of complexes of the desired form, so it has the desired form. Note that since \( X \times_T Y \to Y \) is smooth, by the induction hypothesis we get that \( B\Theta(X/T) \times_X Y \) is associated to a complex of bundles.

If the morphism \( T \to Y \) is smooth then the last term in the complex will be a bundle (again following through the same induction).

If \( T \) is a smooth geometric \( n \)-stack and \( P : \text{Spec}(k) \to T \) is a point then we say that the dimension of \( T \) at \( P \) is the alternating sum of the dimensions of the vector spaces in the complex making up the complex associated to \( P^*(\Theta(T)) \). This could, of course, be negative.

For example if \( G \) is an algebraic group then the dimension of \( BG \) at any point is \(-\text{dim}(G)\). More generally if \( A \) is an abelian group scheme smooth over a base \( Y \) then

\[
\text{dim}(K(A/Y, n)) = \text{dim}(Y) + (-1)^n \text{dim}(A).
\]
9. De Rham theory

We will use certain geometric $n$-stacks as coefficients to look at the de Rham theory of a smooth projective variety. The answers come out to be geometric $n$-stacks. (One could also try to look at de Rham theory for geometric $n$-stacks, a very interesting problem but not what is meant by the title of the present section).

If $X$ is a smooth projective variety let $X_{DR}$ be the stack (which is actually a sheaf of sets) associated to the formal category whose object object is $X$ and whose morphism object is the formal completion of the diagonal in $X \times X$. Another cheaper definition is just to say

$$X_{DR}(Y) := X(Y^{\text{red}}).$$

If $f : X \to S$ is a smooth morphism, let

$$X_{DR/S} := X_{DR} \times_{S_{DR}} S.$$ 

It is the stack associated to a smooth formal groupoid over $S$ (take the formal completion of the diagonal in $X \times_S X$).

The cohomology of $X_{DR}$ with coefficients in an algebraic group scheme is the same as the de Rham cohomology of $X$ with those coefficients. We treat this in the case of coefficients in a vector space, or in case of $H^1$ with coefficients in an affine group scheme. Actually the statement is a more general one about formal categories. Suppose $(X, \Lambda) \to S$ is a smooth formal groupoid over $S$ which we can think of as a smooth scheme $X/S$ with a formal scheme $\Lambda$ mapping to $X \times_S X$ and provided with an associative product structure. There is an associated de Rham complex $\Omega_{\Lambda}$ on $X$ (cf [Ber] [I])—whose components are locally free sheaves on $X$ and where the differentials are first order differential operators. Let $X_{\Lambda}$ denote the stack associated to the formal groupoid. It is the stack associated to the presheaf of groupoids which to $Y \in X$ associates the groupoid whose objects are $X(Y)$ and whose morphisms are $\Lambda(Y)$.

Suppose $V$ is a vector bundle over $X_{\Lambda}$, that is a vector bundle on $X$ together with isomorphisms $p_1^*V \cong p_2^*V$ on $\Lambda$ satisfying the cocycle condition on $\Lambda \times_X \Lambda$. We can define the cohomology sheaves on $S$, $H^i(X_{\Lambda}/S, V)$ which will be equal to $\pi_0(\Gamma(X_{\Lambda}/S; K(V, i)))$ in our notations. These cohomology sheaves can be calculated using the de Rham complex: there is a twisted de Rham complex $\Omega_{\Lambda} \otimes_{\mathcal{O}} V$ whose hypercohomology is $H^i(X_{\Lambda}/S, V)$.

When applied to the de Rham formal category (the trivial example introduced in [Ber] in characteristic zero) whose associated stack is the sheaf of sets $X_{DR/S}$, we obtain the usual de Rham complex $\Omega_{X/S}$ relative to $S$. A vector bundle $V$ over $X_{DR/S}$ is the same thing as a vector bundle on $X$ with integrable connection, and the twisted de Rham complex is the usual one. Thus in this case we have

$$\pi_0(\Gamma(X_{DR/S}/S, K(V, i))) = H^i(X/S, \Omega_{X/S} \otimes V).$$
We can describe more precisely $\Gamma(X_{DR/S}, K(V, i))$ as being the $i$-stack obtained by applying Dold-Puppe to the right derived direct image complex $Rf_*(\Omega_{X/S} \otimes V)[i]$ (appropriately shifted).

For the first cohomology with coefficients in an affine algebraic group $G$, note that a principal $G$-bundle on $X_{DR}$ is the same thing as a principal $G$-bundle with integrable connection. We have that the 1-stack $\Gamma(X_{DR/S}, BG)$ on $S$ is the moduli stack of framed principal bundles; the moduli stack is immediately obtained as an algebraic stack, the quotient stack by the action of $G$ on the scheme of framed bundles).

Of course we have seen in [7.2] that for any smooth formal category $(X, \Lambda)$ over $S$ and any connected very presentable $n$-stack $T$, the morphism $n$-stack $Hom(X_\Lambda/S, T)$ is a locally geometric $n$-stack. Recall that we have defined the semistable morphism stack $Hom^{se}(X_\Lambda/S, T)$ which is geometric; but in our case all morphisms $X_{DR/S} \rightarrow BG$ (i.e. all principal $G$-bundles with integrable connection) are semistable, so in this case we find that $Hom(X_{DR/S}, T)$ is a geometric $n$-stack. In fact it is just a successive combination of the above discussions applied according to the Postnikov decomposition of $T$.

De Rham theory on the analytic site

The same construction works for smooth objects in the analytic site. Suppose $f : X \rightarrow S$ is a smooth analytic morphism. Here we would like to consider any connected analytic $n$-stack $R$ whose homotopy groups are represented by analytic Lie groups. Such an $R$ is automatically an analytic $n$-stack (by the analytic analogue of [2.3]). We call these the “good connected analytic $n$-stacks” since we haven’t yet proven that every connected analytic $n$-stack must be of this form (I suspect that to be true but don’t have an argument).

If $G$ is an analytic Lie group, a map $X_{DR/S} \rightarrow BG$ is a principal $G$-bundle $P$ on $X$ together with an integrable connection relative to $S$.

Suppose $A$ is an analytic abelian Lie group with action of $G$. Then we can form the analytic $n$-stack $\kappa(G, A, n)$ with fundamental group $G$ and $\pi_n = A$. Given a map $X_{DR/S} \rightarrow BG$ corresponding to a principal bundle $P$, we would like to study the liftings into $\kappa(G, A, n)$. We obtain the twisted analytic Lie group $A_P := P \times^G A$ over $X$ with integrable connection relative to $S$. Let $V$ denote the universal covering group of $A$ (isomorphic to $\text{Lie}(A)$, thus $G$ acts here) and let $L \subset V$ denote the kernel of the map to $A$. Note that $V$ is a complex vector space and $L$ is a lattice isomorphic to $\pi_1(A)$. Again $G$ acts on $L$. We obtain after twisting $V_p$ and $L_p$. Note that $V_p$ is provided with an integrable connection relative to $S$. The following Deligne-type complex calculates the cohomology of $A_P$:

$$C_D(A_P) := \{ L_P \rightarrow V_P \rightarrow \Omega^1_{X/S} \otimes_\mathcal{O} V_P \rightarrow \ldots \}.$$
The $n$-stack $\Gamma(X_{DR/S}/S, K(A_P, n))$ is again obtained by applying Dold-Puppe to the shifted right derived direct image complex $Rf_*(C_D(A_P))[n]$. We can write $C_D(A_P)$ as the mapping cone of a map of complexes $L_P \to U_P$.

If $f$ is a projective morphism then applying GAGA and the argument of Mumford (actually I think there is an argument of Grauert which treats this for any proper map), we get that

$$Rf_*(C_D(U_P))$$

is quasiisomorphic to a complex of analytic Lie groups (vector bundles in this case). On the other hand, locally on the base the direct image $Rf_*(C_D(L_P))$ is a trivial complex so quasiisomorphic to a complex of (discrete) analytic Lie groups. The direct image $Rf_*(C_D(A_P))$ is the mapping cone of a map of these complexes, so the associated spectrum fits into a fibration sequence. The base and the fiber are analytic $N$-stack so the total space is also an analytic $N$-stack. Thus the spectrum associated to $Rf_*(C_D(A_P))$ is analytic over $S$. In particular its realization $\Gamma(X_{DR/S}/S, K(A_P, n))$ is a geometric $n$-stack over $S$.

For $Hom(X_{DR/S}/S, B\mathbb{G})$ we can use the Riemann-Hilbert correspondence (see below) to see that it is an analytic 1-stack. The same argument as in Theorem 7.1 now shows that for any good connected analytic $n$-stack $T$, the $n$-stack of morphisms $Hom(X_{DR/S}/S, T)$ is an analytic $n$-stack over $S$.

If the base $S$ is a point we don’t need to make use of Mumford’s argument, so the same holds true for any proper smooth analytic space $X$.

Caution: There is (at least) one gaping hole in the above argument, because we are applying Dold-Puppe for complexes of $\mathbb{Z}$-modules such as $L_P$ or its higher direct image, which are not complexes of rational vector spaces. Thus this doesn’t fit into the previous discussion of Dold-Puppe, spectra etc. as we have set it up. In particular there may be problems with torsion, finite groups or subgroups of finite index in the above discussion. The reader is invited to try to figure out how to fill this in (and to let me know if he does).

The Riemann-Hilbert correspondence

We can extend to our cohomology stacks the classical Riemann-Hilbert correspondence. We start with a statement purely in the analytic case. In order to avoid confusion between the analytic situation and the algebraic one, we will append the superscript ‘an’ to objects in the analytic site, even if they don’t come from objects in the algebraic site. We will make clear in the hypothesis whenever our analytic objects actually come from algebraic ones.

**Theorem 9.1** Suppose $T^{an}$ is a good connected analytic $n$-stack, and suppose $X^{an}$ is a smooth proper complex analytic space. Define $X_{DR}^{an}$ as above. Let $X_{B}^{an}$ denote the $n$-stack

$$\text{Hom}(X_{DR}^{an}/S, T^{an})$$

is quasiisomorphic to a complex of analytic Lie groups (vector bundles in this case). On the other hand, locally on the base the direct image $Rf_*(C_D(L_P))$ is a trivial complex so quasiisomorphic to a complex of (discrete) analytic Lie groups. The direct image $Rf_*(C_D(A_P))$ is the mapping cone of a map of these complexes, so the associated spectrum fits into a fibration sequence. The base and the fiber are analytic $N$-stack so the total space is also an analytic $N$-stack. Thus the spectrum associated to $Rf_*(C_D(A_P))$ is analytic over $S$. In particular its realization $\Gamma(X_{DR/S}/S, K(A_P, n))$ is a geometric $n$-stack over $S$.

For $Hom(X_{DR/S}/S, B\mathbb{G})$ we can use the Riemann-Hilbert correspondence (see below) to see that it is an analytic 1-stack. The same argument as in Theorem 7.1 now shows that for any good connected analytic $n$-stack $T$, the $n$-stack of morphisms $Hom(X_{DR/S}/S, T)$ is an analytic $n$-stack over $S$.

If the base $S$ is a point we don’t need to make use of Mumford’s argument, so the same holds true for any proper smooth analytic space $X$.

Caution: There is (at least) one gaping hole in the above argument, because we are applying Dold-Puppe for complexes of $\mathbb{Z}$-modules such as $L_P$ or its higher direct image, which are not complexes of rational vector spaces. Thus this doesn’t fit into the previous discussion of Dold-Puppe, spectra etc. as we have set it up. In particular there may be problems with torsion, finite groups or subgroups of finite index in the above discussion. The reader is invited to try to figure out how to fill this in (and to let me know if he does).

The Riemann-Hilbert correspondence

We can extend to our cohomology stacks the classical Riemann-Hilbert correspondence. We start with a statement purely in the analytic case. In order to avoid confusion between the analytic situation and the algebraic one, we will append the superscript ‘an’ to objects in the analytic site, even if they don’t come from objects in the algebraic site. We will make clear in the hypothesis whenever our analytic objects actually come from algebraic ones.

**Theorem 9.1** Suppose $T^{an}$ is a good connected analytic $n$-stack, and suppose $X^{an}$ is a smooth proper complex analytic space. Define $X_{DR}^{an}$ as above. Let $X_{B}^{an}$ denote the $n$-stack

$$\text{Hom}(X_{DR}^{an}/S, T^{an})$$

is quasiisomorphic to a complex of analytic Lie groups (vector bundles in this case). On the other hand, locally on the base the direct image $Rf_*(C_D(L_P))$ is a trivial complex so quasiisomorphic to a complex of (discrete) analytic Lie groups. The direct image $Rf_*(C_D(A_P))$ is the mapping cone of a map of these complexes, so the associated spectrum fits into a fibration sequence. The base and the fiber are analytic $N$-stack so the total space is also an analytic $N$-stack. Thus the spectrum associated to $Rf_*(C_D(A_P))$ is analytic over $S$. In particular its realization $\Gamma(X_{DR/S}/S, K(A_P, n))$ is a geometric $n$-stack over $S$.

For $Hom(X_{DR/S}/S, B\mathbb{G})$ we can use the Riemann-Hilbert correspondence (see below) to see that it is an analytic 1-stack. The same argument as in Theorem 7.1 now shows that for any good connected analytic $n$-stack $T$, the $n$-stack of morphisms $Hom(X_{DR/S}/S, T)$ is an analytic $n$-stack over $S$.

If the base $S$ is a point we don’t need to make use of Mumford’s argument, so the same holds true for any proper smooth analytic space $X$.

Caution: There is (at least) one gaping hole in the above argument, because we are applying Dold-Puppe for complexes of $\mathbb{Z}$-modules such as $L_P$ or its higher direct image, which are not complexes of rational vector spaces. Thus this doesn’t fit into the previous discussion of Dold-Puppe, spectra etc. as we have set it up. In particular there may be problems with torsion, finite groups or subgroups of finite index in the above discussion. The reader is invited to try to figure out how to fill this in (and to let me know if he does).
associated to the constant presheaf of spaces which to each $Y^{an}$ associates the topological space $X^{top}$. Then there is a natural equivalence of analytic $n$-stacks $\text{Hom}(X^{an}_{DR}, T^{an}) \cong \text{Hom}(X^{an}_{B}, T^{an})$.

Proof: By following the same outline as the argument given in [7.1], it suffices to see this for the cases $T^{an} = B G^{an}$ for an analytic Lie group $G^{an}$, and $T^{an} = B(A^{an}, n)$ for an abelian analytic Lie group $A^{an}$. In the second case we reduce to the case of cohomology with coefficients in a twisted version of $A^{an}$. We now leave it to the reader to verify these cases (which are standard examples of using analytic de Rham cohomology to calculate singular cohomology). \hfill \Box

Remark: For convenience we have stated only the absolute version. We leave it to the reader to obtain a relative version for a smooth projective morphism $f : X \rightarrow S$.

Now we turn to the algebraic situation. We can combine the above result with GAGA to obtain:

**Theorem 9.2** Suppose $T$ is a connected very presentable algebraic $n$-stack, and suppose $X$ is a smooth projective variety. Define $X^{DR}$ as above. Let $X^{an}_{B}$ denote the $n$-stack associated to the constant presheaf of spaces which to each $Y$ associates the topological space $X^{top}$. Then there is a natural equivalence of analytic $n$-stacks $\text{Hom}(X^{an}_{DR}, T^{an}) \cong \text{Hom}(X^{an}_{B}, T^{an})$.

Proof: By GAGA we have

$$\text{Hom}(X^{an}_{DR}, T^{an}) \cong \text{Hom}(X^{an}_{DR}, T^{an}).$$

Similarly the calculation of $\text{Hom}(X^{an}_{B}, T)$ using a cell decomposition of $X^{an}_{B}$ and fiber products yields the equivalence

$$\text{Hom}(X^{an}_{B}, T) \cong \text{Hom}(X^{an}_{B}, T^{an}).$$

Putting these together we obtain the desired equivalence. \hfill \Box

**The Hodge filtration**

Let $H : = \mathbb{A}^{1}/\mathbb{G}_{m}$ be the quotient stack of the affine line modulo the action of the multiplicative group. This has a Zariski open substack which we denote $1 \subset H$; note that $1 \cong \text{Spec}(\mathbb{C})$. There is a closed substack $0 \subset H$ with $0 \cong B \mathbb{G}_{m}$.

As in [8.14] we can define a smooth formal category $X_{Hod} \rightarrow H$ whose fiber over 1 is $X_{DR}$ and whose fiber over 0 is $X_{Dol}$.
Suppose $T$ is a connected very presentable $n$-stack. Then we obtain the relative semistable morphism stack
\[ \text{Hom}^\text{se}(X_{\text{Hod}}/H, T) \to H. \]
In the case $T = BG$ this was interpreted as the Hodge filtration on $\mathcal{M}_{DR} = \text{Hom}(X_{DR}, BG)$. Following this interpretation, for any connected very presentable $T$ we call this relative morphism stack the Hodge filtration on the higher nonabelian cohomology stack $\text{Hom}(X_{DR}, T)$.

Note that when $T = K(\mathcal{O}, n)$ we recover the usual Hodge filtration on the algebraic de Rham cohomology, i.e. the cohomology of $X_{DR}$ with coefficients in $\mathcal{O}$.

The above general definition is essentially just a mixture of the case $BG$ and the cases $K(\mathcal{O}, n)$ but possibly with various twistings.

The analytic case: The above discussion works equally well for a smooth proper analytic variety $X$. For any good connected analytic $n$-stack $T$ we obtain the relative morphism stack
\[ \text{Hom}(X_{\text{Hod}}/H^{\text{an}}, T) \to H^{\text{an}}. \]
Note that there is no question of semistability here. The moduli stack of flat principal $G$-bundles $\text{Hom}(X_{\text{Hod}}/H^{\text{an}}, BG)$ is still an analytic $n$-stack because in the analytic category there is no distinction between finite type and locally finite type.

In case $X$ is projective and $G = \pi_1(T)$ affine algebraic we can put in the semistability condition and get
\[ \text{Hom}^\text{se}(X_{\text{Hod}}/H^{\text{an}}, T) \to H^{\text{an}}. \]
If $T$ is the analytic stack associated to an algebraic geometric $n$-stack then this analytic morphism stack is the analytic stack associated to the algebraic morphism stack.

The Gauss-Manin connection

Suppose $X \to S$ is a smooth projective morphism and $T$ a connected very presentable $n$-stack. The formal category $X_{DR/S} \to S$ is pulled back from the morphism $X_{DR} \to S_{DR}$ via the map $S \to S_{DR}$. Thus
\[ \text{Hom}(X_{DR/S}/S, T) = \text{Hom}(X_{DR/S_{DR}}, T) \times_{S_{DR}} S. \]
Thus the morphism stack $\text{Hom}(X_{DR/S}/S, T)$ descends down to an $n$-stack over $S_{DR}$. If $Y \to S_{DR}$ is a morphism from a scheme then locally in the etale topology it lifts to $Y \to S$. We have
\[ \text{Hom}(X_{DR/S}, T) \times_{S_{DR}} Y = \text{Hom}(X_{DR} \times_{S_{DR}} Y/Y, T) = \text{Hom}(X_{DR/S} \times_{S} Y/Y, T) = \text{Hom}(X_{DR/S}/S, T) \times_{S} Y. \]
The right hand side is a geometric $n$-stack, so this shows that the morphism

$$\text{Hom}(X_{DR}/S_{DR}, T) \to S_{DR}$$

is geometric. This descended structure is the Gauss-Manin connection on $\text{Hom}(X_{DR}/S, T)$.

In the case $T = BG$ this gives the Gauss-Manin connection on the moduli stack of $G$-bundles with flat connection (cf [Si1], [Si4]). In the case $T = K(V, n)$ this gives the Gauss-Manin connection on algebraic de Rham cohomology.

In [Si4] we have indicated, for the case $T = BG$, how to obtain the analogues of Griffiths transversality and regularity for the Hodge filtration and Gauss-Manin connection. Exactly the same constructions work here. We briefly review how this works. Suppose $X \to S$ is a smooth projective family over a quasiprojective base (smooth, let’s say) which extends to a family $\mathcal{X} \to \mathcal{S}$ over a normal crossings compactification of the base. Let $D = \mathcal{X} - X$ and $E = \mathcal{S} - S$. Recall that $\mathcal{X}_{\text{Hod}}(\log D) \to H$ is the smooth formal category whose underlying space (stack, really, since we have replaced $A^1$ by its quotient $H$) is $X \times H$ and whose associated de Rham complex is $(\Omega^{\cdot}X(\log D), \lambda d)$ where $\lambda$ is the coordinate on $H$ (actually to be correct we have to twist everything by line bundles on $H$ to reflect the quotient by $G_m$ but I won’t put this into the notation). Similarly we obtain the formal category $\mathcal{S}_{\text{Hod}}(\log E) \to H$, with a morphism

$$\mathcal{X}_{\text{Hod}}(\log D) \to \mathcal{S}_{\text{Hod}}(\log E).$$

If we pull back by $\mathcal{S} \to \mathcal{S}_{\text{Hod}}(\log E)$ then we get a smooth formal category over $\mathcal{S}$. Thus by 7.2 for any connected very presentable $n$-stack $T$ the morphism

$$\text{Hom}(\mathcal{X}_{\text{Hod}}(\log D)/\mathcal{S}_{\text{Hod}}(\log E), T) \to \mathcal{S}_{\text{Hod}}(\log E), T)$$

is a geometric morphism. The existence of this extension (which over the open subset $S_{DR} \subset \mathcal{S}_{\text{Hod}}(\log E), T)$ is just the Gauss-Manin family $\text{Hom}(X_{DR}/S_{DR}, T)$) combines the Griffiths transversality of the Hodge filtration and regularity of the Gauss-Manin connection. This is discussed in more detail in [Si4] in the case $T = BG$ or particularly $BGL(r)$—I just wanted to make the point here that the same thing goes through for any connected very presentable $n$-stack $T$.

The same thing will work in an analytic setting, but in this case we can use any good connected analytic $n$-stack $T$ as coefficients.

References

[Ad] J. Adams. Infinite Loop Spaces, Princeton University Press Annals of Math. Studies 90 (1978).
[Ar1] M. Artin. Versal deformations and algebraic stacks, *Inventiones Math.* 27 (1974), 165-189.

[Ar2] M. Artin. Approximation of algebraic structures over local rings, *Publ. Math. I.H.E.S.* 36 (1969), 23-58.

[B-D] J. Baez, J. Dolan. Higher dimensional algebra and topological quantum field theory. Preprint q-alg (95-03).

[Ben] J. Bénabou. *Introduction to Bicategories*, Lect. Notes in Math. 47, Springer-Verlag (1967).

[Ber] P. Berthelot. Cohomologie cristalline des schémas de caractéristique p > 0. Springer *Lecture Notes in Math.* 407 (1974).

[Br1] L. Breen. Extensions du groupe additif. *Publ. Math. I.H.E.S.* 48 (1978), 39-126.

[Br2] L. Breen. On the classification of 2-gerbs and 2-stacks. *Astérisque* 225, Soc. Math. de France (1994).

[Bro] K. Brown. Abstract homotopy theory and generalized sheaf cohomology. *Trans. A.M.S.* 186 (1973), 419-458.

[B-G] K. Brown, S. Gersten. *Algebraic K-theory as generalize sheaf cohomology* Springer *Lecture Notes in Math.* 341 (1973), 266-292.

[C-F] C.-L. Chai, G. Faltings. *Degenerations of Abelian Varieties*, Springer-Verlag (1990).

[D-M] P. Deligne, D. Mumford. The irreducibility of the space of curves of a given genus, *Publ. Math. I.H.E.S.* 36, (1969), 75-110.

[GPS] R. Gordon, A.J. Power, R. Street. Coherence for tricategories *Memoirs A.M.S.* 117 (1995), 558 ff.

[Gr] A. Grothendieck. *Pursuing Stacks*, unpublished manuscript.

[Hi] A. Hirschowitz. Cohérence et dualité sur le gros site de Zariski, *Algebraic curves and projective geometry* (Ballico, Ciliberto eds.), *Lecture Notes in Math.* 1389, (1989), 91-102.

[Ill] L. Illusie. *Le complexe cotangent I et II*, *Lecture Notes in Math.* 239, 283, Springer-Verlag, Berlin (1971,1972).
[Ja]  J.F. Jardine. Simplicial presheaves, *J. Pure and Appl. Algebra* 47 (1987), 35-87.

[Jo]  A. Joyal, Letter to A. Grothendieck (referred to in Jardine’s paper).

[LMB] G. Laumon, L. Moret-Bailly. Champs algébriques. Preprint, Orsay 42 (1992).

[Le]  O. Leroy. Sur une notion de 3-catégorie adaptée à l’homotopie. Preprint Univ. de Montpellier 2 (1994).

[Mu]  D. Mumford. *Abelian Varieties*.

[Mo]  S. Mochizuki. The geometry of the compactification of the Hurwitz scheme. *Publ. R.I.M.S. Kyoto University* 31 (1995), 355-441.

[1]  D. Quillen. Rational homotopy theory. *Ann. of Math.* 90 (1969), 205-295.

[Se]  G. Segal. Homotopy everything $H$-spaces. Preprint.

[Si1] C. Simpson. Moduli of representations of the fundamental group of a smooth projective variety, I and II. *Publ. Math. I.H.E.S.* 79 and 80 (1994), 47-129 and 5-79.

[Si2] C. Simpson. Flexible sheaves. Preprint available on q-alg (96-08).

[Si3] C. Simpson. Homotopy over the complex numbers and generalized de Rham cohomology. *Moduli of Vector Bundles*, M. Maruyama (Ed.) *Lecture Notes in Pure and Applied Math.* 179, Marcel Dekker (1996), 229-263.

[Si4] C. Simpson. The Hodge filtration on nonabelian cohomology. Preprint, available on alg-geom (96-04).

[Si5] C. Simpson. A relative notion of algebraic Lie group and application to $n$-stacks. Preprint, available on alg-geom 96-07.

[Si6] C. Simpson. The topological realization of a simplicial presheaf. Preprint, available on q-alg 96-09.

[Ta] Z. Tamsamani. Sur des notions de $n$-catégorie et $n$-groupoïde non-stricte via des ensembles multi-simpliciaux. Thesis, Université Paul Sabatier, Toulouse (1996) available on alg-geom (95-12 and 96-07).

[Th] Thomason. Algebraic K-theory and etale cohomology, *Ann. Sci. E.N.S.* 18 (1985), 437-552.