ON THE COMPACTNESS THRESHOLD IN THE CRITICAL KIRCHHOFF EQUATION

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Abstract. We study a class of critical Kirchhoff problems with a general nonlocal term. The main difficulty here is the absence of a closed-form formula for the compactness threshold. First we obtain a variational characterization of this threshold level. Then we prove a series of existence and multiplicity results based on this variational characterization.

1. Introduction. The purpose of this paper is to study the critical Kirchhoff problem

\[
\begin{aligned}
-\h\left(\int_{\Omega} |\nabla u|^2 \, dx\right) \Delta u &= f(x, u) + |u|^{2^*-2} u \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \), \( N \geq 3 \), \( \h : [0, \infty) \to [0, \infty) \) is a continuous and nondecreasing function, \( 2^* = 2N/(N-2) \) is the critical Sobolev exponent, and \( f \) is a Carathéodory function on \( \Omega \times \mathbb{R} \) satisfying the subcritical growth condition

\[
|f(x, t)| \leq c_1 |t|^{p-1} + c_2 \quad \text{for a.a. } x \in \Omega \text{ and all } t \in \mathbb{R}
\]

for some constants \( c_1, c_2 > 0 \) and \( 1 < p < 2^* \). The class of nonlocal terms considered here includes sums of powers

\[
\h(t) = \sum_{i=1}^{n} a_i t^{\gamma_i-1}, \quad t \geq 0,
\]

where \( a_1, \ldots, a_n > 0 \) and \( 1 \leq \gamma_1 < \cdots < \gamma_n < +\infty \). A model case is

\[
\h(t) = a + bt^{\gamma-1},
\]

where \( a, b \geq 0 \) with \( a + b > 0 \) and \( 1 < \gamma < +\infty \). The classical case \( \h(t) = a + bt \) corresponds to \( \gamma = 2 \).

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As is usually the case with problems of critical growth, problem (1) lacks compactness. The standard approach to such problems is to determine a threshold level below which there is compactness and construct minimax critical levels below this threshold. This approach has been used in the classical case

\[ h(t) = a + bt, \quad a, b \geq 0, \quad a + b > 0 \]

in dimensions \( N = 3 \) and 4 to obtain nontrivial solutions in the recent literature (see, e.g., Huang et al. [7], Liao et al. [9], Naimen [10, 11], Xie et al. [14], Yao and Mu [15], Zhang and Liu [16], and the references therein). However, in the general case considered in the present paper such a threshold level cannot be found in closed form. Our first contribution here is a variational characterization of this threshold level (see Theorem 2.2). Then we give a series of existence and multiplicity results based on this variational characterization (see Section 3). This requires novel arguments due to the absence of a closed-form compactness threshold.

We will state and prove our compactness, existence, and multiplicity results for problem (1) with a general nonlocal term \( h \) in the next two sections. To illustrate our results while keeping the presentation simple, we state them here for the model problem

\[
\begin{cases}
-\left(a + b \left( \frac{1}{\gamma} \int_{\Omega} |\nabla u|^2 \, dx \right)^{\gamma-1} \right) \Delta u = \lambda u + |u|^{2^* - 2} u & \text{in } \Omega \\
 u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \( 1 < \gamma < +\infty, a \geq 0, b > 0, \) and \( \lambda > 0. \)

Weak solutions of this problem coincide with critical points of the functional

\[ J(u) = \frac{a}{2} \int_{\Omega} |\nabla u|^2 \, dx + \frac{b}{2\gamma} \left( \int_{\Omega} |\nabla u|^2 \, dx \right)^{\gamma} - \frac{\lambda}{2} \int_{\Omega} u^2 \, dx - \frac{1}{2} \int_{\Omega} |u|^{2^*} \, dx, \quad u \in H_0^1(\Omega). \]

Recall that \( J \) satisfies the Palais-Smale compactness condition at the level \( c \in \mathbb{R}, \) or the \( (PS)_c \) condition for short, if every sequence \( (u_j) \) in \( H_0^1(\Omega) \) such that \( J(u_j) \to c \) and \( J'(u_j) \to 0 \) has a strongly convergent subsequence. Let \( S \) be the best Sobolev constant (see (6)) and let \( \lambda_1 > 0 \) be the first Dirichlet eigenvalue of the Laplacian in \( \Omega. \) We have the following compactness results for the cases \( \gamma < 2^*/2, \ \gamma = 2^*/2, \) and \( \gamma > 2^*/2 \) (see Corollary 4, Corollary 5, and Corollary 6).

**Theorem 1.1.** Let \( 1 < \gamma < 2^*/2, \ a, b > 0, \) and \( 0 < \lambda \leq a\lambda_1. \) Let \( t_0 \) be the unique positive solution of the equation \( a + bt^{\gamma-1} = S^{-2^*/2} t^{2^*/2-1} \) and set

\[ c^* = \frac{1}{N} at_0 + \left( \frac{1}{2\gamma} - \frac{1}{2^*} \right) bt_0^2. \]

Then \( J \) satisfies the \( (PS)_c \) condition for all \( c < c^*. \)

**Theorem 1.2.** Let \( \gamma = 2^*/2. \)

(i) Let \( a > 0, \ 0 < b < S^{-2^*/2}, \) and \( 0 < \lambda < a\lambda_1. \) Set

\[ c^* = \frac{1}{N} \left( \frac{a^2/2}{S^{-2^*/2} - b} \right)^{2/(2^*-2)}. \]

Then \( J \) satisfies the \( (PS)_c \) condition for all \( c < c^*. \)

(ii) If \( a \geq 0 \) and \( b > S^{-2^*/2}, \) then \( J \) satisfies the \( (PS)_c \) condition for all \( c \in \mathbb{R} \) for any \( \lambda > 0. \)
Theorem 1.3. If \( \gamma > 2^* / 2 \) and
\[
a^{\gamma - 2^*/2} b^{2^*/2 - 1} > \frac{(\gamma - 2^*/2) \gamma - 2^*/2 (2^*/2 - 1)^{2^*/2 - 1}}{(\gamma - 1)^{\gamma - 1}} S^{-(2^*/2)(\gamma - 1)},
\]
then \( J \) satisfies the (PS)\(_c\) condition for all \( c \in \mathbb{R} \) for any \( \lambda > 0 \).

Theorems 1.1–1.3 have the following corollaries for the classical case \( \gamma = 2 \), where \( c^* = +\infty \) means that \( J \) satisfies the (PS)\(_c\) condition for all \( c \in \mathbb{R} \).

Corollary 1. Let \( \gamma = 2 \).

(i) If \( N = 3, a, b > 0, \) and \( 0 < \lambda \leq a \lambda_1 \), then \( c^* = \frac{1}{4} a b S^3 + \frac{1}{24} b^3 S^6 + \frac{1}{24} (4 a S + b^2 S^4)^{3/2} \).

(ii) If \( N = 4, a > 0, 0 < b < S^{-2}, \) and \( 0 < \lambda < a \lambda_1 \), then \( c^* = \frac{a^2}{4 (S^{-2} - b)} \).

(iii) If \( N = 4, a \geq 0, \) and \( b > S^{-2} \), then \( c^* = +\infty \) for any \( \lambda > 0 \).

(iv) If \( N \geq 5 \) and \( a^{N-4} b^2 > \frac{4 (N - 4)^{N-4} (N - 2)^{N-2}}{4} S^{-N} \), then \( c^* = +\infty \) for any \( \lambda > 0 \).

Remark 1. The threshold levels in Corollary 1 (i)–(iii) were also obtained using different arguments in Naïmen [11], Naïmen [10], and Liao et al. [9], respectively.

We have the following existence and multiplicity results for problem (3) (see Corollary 8, Theorem 3.5, and Theorem 3.6).

Theorem 1.4. If \( 1 < \gamma < 2^*/2 \), \( a, b > 0, \) and \( N \geq 4, \) then problem (3) has a nontrivial solution for \( 0 < \lambda < a \lambda_1 \).

Theorem 1.5. Let \( \gamma = 2^*/2 \).

(i) If \( a > 0, 0 < b < S^{-2^*/2}, \) and \( N \geq 4, \) then problem (3) has a nontrivial solution for \( 0 < \lambda < a \lambda_1 \).

(ii) If \( a = 0 \) and \( b > S^{-2^*/2}, \) then problem (3) has a nontrivial solution for all \( \lambda > 0 \).

(iii) If \( a > 0 \) and \( b > S^{-2^*/2}, \) then problem (3) has two nontrivial solutions for \( \lambda > a \lambda_1 \).

Theorem 1.6. If \( \gamma > 2^*/2 \) and
\[
a^{\gamma - 2^*/2} b^{2^*/2 - 1} > \frac{(\gamma - 2^*/2) \gamma - 2^*/2 (2^*/2 - 1)^{2^*/2 - 1}}{(\gamma - 1)^{\gamma - 1}} S^{-(2^*/2)(\gamma - 1)},
\]
then problem (3) has two nontrivial solutions for \( \lambda \geq a \lambda_1 \).

Theorems 1.4–1.6 have the following corollaries for the classical case \( \gamma = 2 \).

Corollary 2. Let \( \gamma = 2 \) and \( N = 4 \).

(i) If \( a > 0 \) and \( 0 < b < S^{-2} \), then problem (3) has a nontrivial solution for \( 0 < \lambda < a \lambda_1 \).

(ii) If \( a = 0 \) and \( b > S^{-2} \), then problem (3) has a nontrivial solution for all \( \lambda > 0 \).

(iii) If \( a > 0 \) and \( b > S^{-2} \), then problem (3) has two nontrivial solutions for \( \lambda > a \lambda_1 \).

Corollary 3. If \( \gamma = 2, N \geq 5, \) and
\[
a^{N-4} b^2 > \frac{4 (N - 4)^{N-4} (N - 2)^{N-2}}{4} S^{-N},
\]
then problem (3) has two nontrivial solutions for \( \lambda \geq a\lambda_1 \).

**Remark 2.** The result in Corollary 2 (i) was also obtained in Naimen [10] using a different method. Liao et al. [9] obtained one nontrivial solution when \( a \geq 0 \), \( b > S^{-2} \), and \( \lambda > a\lambda_1 \). See also Perera and Zhang [13] for a related result in the subcritical case in dimensions \( N \leq 3 \).

**Remark 3.** Corollary 3 complements the results in Naimen and Shibata [12], where two positive solutions were obtained when \( a = 1 \), \( b > 0 \) is sufficiently small, and \( 0 < \lambda < \lambda_1 \). The case where equality holds in (4) was also considered in [12]. See also Hebey [6] where related results have been obtained on a closed Riemannian manifold.

In the borderline case where \( \gamma = 2^* / 2 \) and \( b = S^{-2 / 2} \), lower-order terms come into play. Consider the problem

\[
\begin{cases}
- \left[ a + S^{-2 / 2} \left( \int_\Omega |\nabla u|^2 \, dx \right)^{2 / 2 - 1} + \eta \left( \int_\Omega |\nabla u|^2 \, dx \right)^{\sigma - 1} \right] = \lambda u + |u|^{2^* - 2} u & \text{in } \Omega \\
\quad \quad u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \( a \geq 0 \), \( \eta > 0 \), \( 1 < \sigma < 2^* / 2 \), and \( \lambda > 0 \). We have the following existence and multiplicity result (see Theorem 3.7).

**Theorem 1.7.** Let \( \eta > 0 \) and \( 1 < \sigma < 2^* / 2 \).

(i) If \( a = 0 \), then problem (5) has a nontrivial solution for all \( \lambda > 0 \).

(ii) If \( a > 0 \), then problem (5) has two nontrivial solutions for \( \lambda > a\lambda_1 \).

2. **Compactness threshold.** A weak solution of problem (1) is a function \( u \) that belongs to the Sobolev space \( H^1_0(\Omega) \) and satisfies

\[
h \left( \int_\Omega |\nabla u|^2 \, dx \right) \int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f(x, u) v \, dx + \int_\Omega |u|^{2^* - 2} uv \, dx \quad \forall v \in H^1_0(\Omega).
\]

Weak solutions coincide with critical points of the \( C^1 \)-functional

\[
J(u) = \frac{1}{2} H \left( \int_\Omega |\nabla u|^2 \, dx \right) - \int_\Omega F(x, u) \, dx - \frac{1}{2^*} \int_\Omega |u|^{2^*} \, dx, \quad u \in H^1_0(\Omega),
\]

where \( F(x, t) = \int_0^t f(x, s) \, ds \) is the primitive of \( f \).

**Definition 2.1.** The functional \( J \) satisfies the Palais-Smale compactness condition at the level \( c \in \mathbb{R} \), or the \((PS)_c\) condition for short, if every sequence \( (u_j) \) in \( H^1_0(\Omega) \) such that

\[
J(u_j) \to c \quad J'(u_j) \to 0,
\]

called a \((PS)_c\) sequence, has a strongly convergent subsequence.

Let

\[
S = \inf_{u \in H^1_0(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2 \, dx}{\left( \int_\Omega |u|^{2^*} \, dx \right)^{2 / 2^*}}
\]

be the best Sobolev constant. The set

\[
I = \{ t > 0 : h(t) \leq S^{-2 / 2} t^{2^* - 2 - 1} \}
\]
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will play an important role in our compactness results. We begin with a simple but useful proposition.

**Proposition 1.** If \((u_j)\) is a sequence in \(H^1_0(\Omega)\) such that
\[
J'(u_j) \to 0, \quad u_j \rightharpoonup u, \quad \|u_j - u\|^2 \to t,
\]
then either \(t = 0\) or \(t \in I\). In particular, if \(I = \emptyset\), then every bounded sequence \((u_j)\) in \(H^1_0(\Omega)\) such that \(J'(u_j) \to 0\) has a strongly convergent subsequence.

**Proof.** Since \(J'(u_j) \to 0\),
\[
h \left( \int_\Omega |\nabla u_j|^2 \, dx \right) \int_\Omega \nabla u_j \cdot \nabla v \, dx - \int_\Omega f(x, u_j) v \, dx - \int_\Omega |u_j|^{2^*-2} u_j v \, dx = o(\|v\|),
\]
for all \(v \in H^1_0(\Omega)\), and since \(u_j \rightharpoonup u\) and \(\|u_j - u\|^2 \to t\),
\[
\int_\Omega |\nabla u_j|^2 \, dx \to \int_\Omega |\nabla u|^2 \, dx + t =: s.
\]
Passing to a renamed subsequence, we may assume that \(u_j \to u\) strongly in \(L^p(\Omega)\) and a.e. in \(\Omega\). So taking \(v = u_j\) in (7) gives
\[
h(s) \left( \int_\Omega |\nabla u_j|^2 \, dx + t \right) - \int_\Omega u f(x, u) \, dx - \int_\Omega |u_j|^2 \, dx = o(1),
\]
while taking \(v = u\) and passing to the limit gives
\[
h(s) \int_\Omega |\nabla u|^2 \, dx - \int_\Omega u f(x, u) \, dx - \int_\Omega |u|^2 \, dx = 0.
\]
Since
\[
\int_\Omega |u_j|^{2^*} \, dx - \int_\Omega |u|^{2^*} \, dx = \int_\Omega |u_j - u|^{2^*} \, dx + o(1)
\]
by the Brézis-Lieb lemma (see [2]), subtracting (9) from (8) and using (6) gives
\[
sth(s) = \int_\Omega |u_j - u|^{2^*} \, dx + o(1) \leq S^{-2^*/2} \left( \int_\Omega |\nabla (u_j - u)|^2 \, dx \right)^{2^*/2} + o(1).
\]
If \(t > 0\), then passing to the limit and noting that \(h(s) \geq h(t)\) since \(s \geq t\) and \(h\) is nondecreasing gives \(h(t) \leq S^{-2^*/2} t^{2^*/2-1}\), so \(t \in I\).

First we consider the case where \(I\) is nonempty. Let
\[
H(t) = \int_0^t h(s) \, ds, \quad t \geq 0
\]
be the primitive of \(h\), and set
\[
K(t) = \frac{1}{2} H(t) - \frac{1}{2^*} th(t), \quad t \geq 0.
\]
For \(1 \leq \gamma \leq 2^*/2\), let
\[
\lambda_1(\gamma) = \inf_{u \in H^1_0(\Omega) \setminus \{0\}} \frac{\left( \int_\Omega |\nabla u|^2 \, dx \right)^{\gamma}}{\int_\Omega |u|^{2\gamma} \, dx}
\]
(10)
be the first eigenvalue of the nonlinear eigenvalue problem
\[
\begin{cases}
- \left( \int_{\Omega} |\nabla u|^2 \, dx \right)^{\gamma - 1} \Delta u = \lambda |u|^{2\gamma - 2} u & \text{in } \Omega \\
\quad u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
which is positive by the Sobolev embedding theorem. We note that \( \lambda_1(1) = \lambda_1 \), the first Dirichlet eigenvalue of the Laplacian in \( \Omega \), and \( \lambda_1(2^*/2) = S^{2^*/2} \). We assume that
\[
(A_1) \quad \text{for some constants } \alpha_1, \ldots, \alpha_n > 0, 1 \leq \gamma_1 < \cdots < \gamma_n < 2^*/2, \text{ and } \mu_1 \leq \lambda_1(\gamma_1), \ldots, \\
\mu_n \leq \lambda_1(\gamma_n) \text{ with at least one of the inequalities strict,}
\]
\[
K(t) \geq \sum_{i=1}^{n} \alpha_i t^{\gamma_i} \quad \forall t \geq 0
\]
and
\[
F(x, t) - \frac{1}{2^*} tf(x, t) \leq \sum_{i=1}^{n} \mu_i \alpha_i |t|^{2\gamma_i} \quad \text{for a.a. } x \in \Omega \text{ and all } t \in \mathbb{R};
\]
\[
(A_2) \quad K \text{ is superadditive, i.e.,}
\]
\[
K(t_1 + t_2) \geq K(t_1) + K(t_2) \quad \forall t_1, t_2 \geq 0;
\]
\[
(A_3) \quad h(t)/t^{2^*/2-1} \text{ is strictly decreasing for } t > 0 \text{ and }
\]
\[
0 \leq b := \lim_{t \to +\infty} \frac{h(t)}{t^{2^*/2-1}} < S^{-2^*/2} < \lim_{t \to 0} \frac{h(t)}{t^{2^*/2-1}} \leq +\infty.
\]
We note that \( K \) is nonnegative by \((A_1)\) and hence nondecreasing by \((A_2)\). We have the following theorem.

**Theorem 2.2.** Assume that \( I \neq \emptyset \) and \((2), (A_1), \text{ and } (A_2) \) hold. Set
\[
ce^* = \inf_{t \in I} K(t).
\]
Then \( J \) satisfies the \((PS)_c\) condition for all \( c < c^* \). If, in addition, \((A_3)\) holds, then the equation
\[
h(t) = S^{-2^*/2} t^{2^*/2-1}
\]
has a unique positive solution \( t_0 \) and
\[
c^* = K(t_0),
\]
in particular, \( c^* > 0 \).

**Proof.** First we note that for all \( u \in H_0^1(\Omega), \)
\[
K \left( \int_{\Omega} |\nabla u|^2 \, dx \right) - \int_{\Omega} \left[ F(x, u) - \frac{1}{2^*} uf(x, u) \right] \, dx \geq \sum_{i=1}^{n} \alpha_i \left( 1 - \frac{\mu_i}{\lambda_1(\gamma_i)} \right) \left( \int_{\Omega} |\nabla u|^2 \, dx \right)^{\gamma_i}
\]
by \((A_1)\) and \((10), \)
\[
\frac{1}{2} H \left( \int_{\Omega} |\nabla u_j|^2 \, dx \right) - \int_{\Omega} F(x, u_j) \, dx - \frac{1}{2^*} \int_{\Omega} |u_j|^2 \, dx = c + o(1)
\]
for some sequence \((u_j)\) in \( (PS)_c \).

Let \( c < c^* \) and let \((u_j)\) be a \((PS)_c\) sequence. Then
\[
\frac{1}{2} H \left( \int_{\Omega} |\nabla u_j|^2 \, dx \right) - \int_{\Omega} F(x, u_j) \, dx - \frac{1}{2^*} \int_{\Omega} |u_j|^2 \, dx = c + o(1)
\]
and
\[ h \left( \int_{\Omega} |\nabla u_j|^2 \, dx \right) \int_{\Omega} \nabla u_j \cdot \nabla v \, dx - \int_{\Omega} f(x, u_j) v \, dx - \int_{\Omega} |u_j|^{2^* - 2} u_j v \, dx = o(\|v\|) \]
for all \( v \in H^1_0(\Omega) \). Taking \( v = u_j \) in (16), dividing by \( 2^* \), and subtracting from (15) gives
\[ K \left( \int_{\Omega} |\nabla u_j|^2 \, dx \right) - \int_{\Omega} \left[ F(x, u_j) - \frac{1}{2^*} u_j f(x, u_j) \right] \, dx = c + o(1 + \|u_j\|), \tag{17} \]
which together with (14) implies that \( (u_j) \) is bounded in \( H^1_0(\Omega) \). So a renamed subsequence of \( (u_j) \) converges to some \( u \) weakly in \( H^1_0(\Omega) \), strongly in \( L^p(\Omega) \), and a.e. in \( \Omega \). For a further subsequence, \( \|u_j - u\|^2 \) converges to some \( t \geq 0 \). We will show that \( t = 0 \).

Suppose \( t \neq 0 \). Then \( t \in I \) by Proposition 1 and hence
\[ K(t) \geq c^* \tag{17} \]
by (11). As in the proof of Proposition 1,
\[ \int_{\Omega} |\nabla u_j|^2 \, dx \to \int_{\Omega} |\nabla u|^2 \, dx + t =: s \]
and
\[ sh(s) - \int_{\Omega} u f(x, u) \, dx - \int_{\Omega} |u_j|^{2^*} \, dx = o(1). \tag{18} \]
Moreover, passing to the limit in (15) gives
\[ \frac{1}{2} H(s) - \int_{\Omega} F(x, u) \, dx - \frac{1}{2^*} \int_{\Omega} |u_j|^{2^*} \, dx = c + o(1), \]
and combining this with (18) gives
\[ c = K(s) - \int_{\Omega} \left[ F(x, u) - \frac{1}{2^*} u f(x, u) \right] \, dx. \]
Since
\[ K(s) \geq K(t) + K \left( \int_{\Omega} |\nabla u|^2 \, dx \right) \]
by \((A_2)\) and
\[ K \left( \int_{\Omega} |\nabla u|^2 \, dx \right) - \int_{\Omega} \left[ F(x, u) - \frac{1}{2^*} u f(x, u) \right] \, dx \geq 0 \]
by (14), then
\[ c \geq K(t). \]
This together with (17) gives \( c \geq c^* \), contrary to assumption. So \( t = 0 \).

If \((A_3)\) holds, then the equation \( h(t)/t^{2^*/2 - 1} = S^{-2^*/2} \) has a unique positive solution \( t_0 \) and \( I = [t_0, \infty) \). Since \( K \) is nondecreasing, then \( c^* = K(t_0) \), in particular, \((A_3)\) implies that \( c^* > 0 \). \( \square \)

Next we consider the case where \( I \) is empty. We assume that
\((A_4)\) \( h \) satisfies one of the following conditions:
\((i)\) for some constants \( \eta > 0 \) and \( p/2 < \gamma < 2^*/2 \),
\[ h(t) \geq S^{-2^*/2} t^{2^*/2 - 1} + \eta t^{\gamma - 1} \quad \forall t \geq 0; \]
(ii) for some constant \( b > S^{-2/2} \),
\[
h(t) \geq bt^{2/2 - 1} \quad \forall t \geq 0;
\]
(iii) for some constants \( b > 0 \) and \( \gamma > 2^*/2 \),
\[
h(t) > \max \left\{ S^{-2/2} t^{2^*/2 - 1}, bt^{\gamma - 1} \right\} \quad \forall t > 0.
\]
We have the following theorem.

**Theorem 2.3.** Assume that (2) and \((A_4)\) hold. Then \( J \) is bounded from below, coercive, and satisfies the (PS)\(_c\) condition for all \( c \in \mathbb{R} \). In particular, \( J \) has a global minimizer.

**Proof.** By (2) and (6),
\[
J(u) \geq \frac{1}{2} H\left( \int_\Omega |\nabla u|^2 \, dx \right) - c_3 \int_\Omega |u|^p \, dx - c_4 - \frac{1}{2\gamma} S^{-2/2} \left( \int_\Omega |\nabla u|^2 \, dx \right)^{2/\gamma}
\]
for some constants \( c_3, c_4 > 0 \). By \((A_4)\), \( H \) satisfies one of the following:

(i) \( H(t) \geq \frac{2}{2^*} S^{-2^*/2} t^{2^*/2} + \frac{\eta}{\gamma} t^\gamma \) for all \( t \geq 0 \), where \( \eta > 0 \) and \( p/2 < \gamma < 2^*/2 \);

(ii) \( H(t) \geq \frac{2b}{2^*} t^{2^*/2} \) for all \( t \geq 0 \), where \( b > S^{-2/2} \);

(iii) \( H(t) \geq \frac{b}{2^*} t^\gamma \) for all \( t \geq 0 \), where \( b > 0 \) and \( \gamma > 2^*/2 \).

It follows that \( J \) is bounded from below and coercive.

Let \( c \in \mathbb{R} \) and let \((u_j)\) be a (PS)\(_c\) sequence. By coercivity, \((u_j)\) is bounded. By \((A_4)\), \( h(t) > S^{-2/2} t^{2^*/2 - 1} \) for all \( t > 0 \), so \( I = \emptyset \). So \((u_j)\) has a strongly convergent subsequence by Proposition 1. \(\Box\)

Finally we apply our results to the model problem
\[
\begin{cases}
-\left[ a + b \left( \int_\Omega |\nabla u|^2 \, dx \right)^{\gamma - 1} \right] \Delta u = f(x, u) + |u|^{2^* - 2} u \quad \text{in } \Omega \\
u = 0 \quad \text{on } \partial \Omega,
\end{cases}
\]
where \( a, b \geq 0 \) with \( a + b > 0 \) and \( 1 < \gamma < +\infty \). Here
\[
h(t) = a + bt^{\gamma - 1}, \quad H(t) = at + \frac{b}{\gamma} t^\gamma, \quad K(t) = \frac{1}{N} at + \left( \frac{1}{2\gamma} - \frac{1}{2^*} \right) bt^\gamma
\]
and
\[
J(u) = \frac{a}{2} \int_\Omega |\nabla u|^2 \, dx + \frac{b}{2\gamma} \left( \int_\Omega |\nabla u|^2 \, dx \right)^\gamma - \int_\Omega F(x, u) \, dx - \frac{1}{2^*} \int_\Omega |u|^{2^*} \, dx, \quad u \in H^1_0(\Omega).
\]

Theorem 2.2 has the following corollary for the case \( \gamma < 2^*/2 \).

**Corollary 4.** Let \( 1 < \gamma < 2^*/2 \) and \( a, b \geq 0 \). Assume that \( f \) satisfies (2) and
\[
F(x, t) - \frac{1}{2^*} tf(x, t) \leq \frac{1}{N} \lambda at^2 + \left( \frac{1}{2\gamma} - \frac{1}{2^*} \right) \mu b |t|^{2^*} \quad \text{for a.a. } x \in \Omega \text{ and all } t \in \mathbb{R}
\]
for some constants \( \lambda \leq \lambda_1 \) and \( \mu \leq \lambda_1(\gamma) \) with either \( a > 0 \) and \( \lambda < \lambda_1 \), or \( b > 0 \) and \( \mu < \lambda_1(\gamma) \). Let \( t_0 \) be the unique positive solution of the equation
\[
a + bt^{\gamma - 1} = S^{-2/2} t^{2^*/2 - 1}
\]
and set
\[ c^* = \frac{1}{N} a t_0 + \left( \frac{1}{2}\gamma - \frac{1}{2^*} \right) b t_0^\gamma. \]
Then \( J \) satisfies the (PS)\(_c\) condition for all \( c < c^* \).

**Remark 4.** For \( a = 1 \) and \( b = 0 \), Corollary 4 gives the well-known compactness threshold
\[ c^* = \frac{1}{N} S^{N/2} \]
in the Brézis-Nirenberg problem (see [3]).

**Remark 5.** An interesting special case of problem (19) is
\[
\begin{aligned}
-\left( \int_\Omega |\nabla u|^2 \, dx \right)^{\gamma - 1} \Delta u &= \mu |u|^{2^* - 2} u + |u|^{2^* - 2} \quad \text{in } \Omega \\
&= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]
where \( \gamma > 1 \) and \( \mu > 0 \). For \( \gamma < \frac{2^*}{2} \) and \( \mu < \lambda_1(\gamma) \), Corollary 4 gives the compactness threshold
\[ c^* = \left( \frac{1}{2}\gamma - \frac{1}{2^*} \right) S^{2^*/(2^* - 2\gamma)} \]
for the associated variational functional
\[
J(u) = \frac{1}{2^*} \left( \int_\Omega |\nabla u|^2 \, dx \right)^\gamma - \frac{\mu}{2\gamma} \int_\Omega |u|^{2\gamma} \, dx - \frac{1}{2^*} \int_\Omega |u|^{2^*} \, dx, \quad u \in H^1_0(\Omega).
\]

**Remark 6.** The classical case \( h(t) = a + bt \) when \( N = 3 \) is one of the few cases with both \( a \) and \( b \) positive and \( \gamma < \frac{2^*}{2} \) where \( c^* \) can be found in closed form. Here Corollary 4 gives
\[ c^* = \frac{1}{4} \lambda b S^3 + \frac{1}{24} b^3 S^6 + \frac{1}{24} \left( 4aS + b^2 S^4 \right)^{3/2}. \]
This threshold level was also obtained in Naimen [11, Lemma 2.5] using concentration compactness arguments. Our approach here is simpler.

Theorem 2.2 also has the following corollary for the case \( \gamma = \frac{2^*}{2} \).

**Corollary 5.** Let \( \gamma = \frac{2^*}{2} \), \( a > 0 \), and \( 0 < b < S^{-2}/2 \). Assume that \( f \) satisfies (2) and
\[ F(x, t) - \frac{1}{2^*} tf(x, t) \leq \frac{1}{N} \lambda at^2 \quad \text{for a.a. } x \in \Omega \text{ and all } t \in \mathbb{R} \]
for some constant \( \lambda < \lambda_1 \). Set
\[ c^* = \frac{1}{N} \left( \frac{a^{2^*/2}}{S^{-2}/2 - b} \right)^{2/(2^* - 2)}. \]
Then \( J \) satisfies the (PS)\(_c\) condition for all \( c < c^* \).

**Remark 7.** For the classical case \( h(t) = a + bt \) with \( N = 4 \), \( a > 0 \), and \( 0 < b < S^{-2} \), Corollary 5 gives
\[ c^* = \frac{a^2}{4 (S^{-2} - b)}. \]
This threshold level was also obtained in Naimen [10, Lemma 2.1] by analyzing the behavior of Palais-Smale sequences. Our approach here is simpler again.
Theorem 2.3 has the following corollary for $\gamma \geq 2^*/2$.

**Corollary 6.** Assume that $f$ satisfies (2). Then $J$ is bounded from below, satisfies the (PS)$_c$ condition for all $c \in \mathbb{R}$, and has a global minimizer in each of the following cases:

(i) $\gamma = 2^*/2$, $a \geq 0$, and $b > S^{-2^*/2}$;
(ii) $\gamma > 2^*/2$ and

$$a^{-2^*/2} b^{2^*/2-1} > \frac{(\gamma - 2^*/2)\gamma^{-2^*/2} (2^*/2 - 1)^{2^*/2-1}}{(\gamma - 1)^{\gamma-1}} S^{-(2^*/2)(\gamma-1)}.$$

**Proof.** The minimum of $a + bt^{-1} - S^{-2^*/2} t^{2^*/2-1}$, $t > 0$ is positive if and only if the last inequality holds. \hfill $\square$

**Remark 8.** For the classical case $h(t) = a + bt$ with $N = 4$, $a \geq 0$, and $b > S^{-2}$, Corollary 6 (i) implies that $J$ satisfies the (PS)$_c$ condition for all $c \in \mathbb{R}$. This was also observed in Liao et al. [9, Proposition 2.1].

**Remark 9.** For the classical case $h(t) = a + bt$ with $N \geq 5$, Corollary 6 (ii) implies that $J$ satisfies the (PS)$_c$ condition for all $c \in \mathbb{R}$ if

$$a^{N-4} b^2 > \frac{4(N-4)^{N-4}}{(N-2)^{N-2}} S^{-N}.$$

3. Existence and multiplicity results. In the case where $I$ is nonempty, our main existence result for problem (1) is the following theorem.

**Theorem 3.1.** Assume (2) and (A$_1$)–(A$_3$). Assume further that

$$H(t) \geq b_0 t^{\gamma_0} \quad \text{for } 0 \leq t \leq \delta \quad (20)$$

for some constants $\delta, b_0 > 0$ and $1 \leq \gamma_0 < 2^*/2$,

$$F(x, t) \leq \frac{1}{2} \mu_0 b_0 |t|^{\gamma_0} \quad \text{for a.a. } x \in \Omega \text{ and } |t| \leq \delta \quad (21)$$

for some $\mu_0 < \lambda_1(\gamma_0)$, and

$$F(x, t) \geq \frac{1}{q} \nu t^q \quad \text{for a.a. } x \in B_r(x_0) \text{ and all } t \geq 0 \quad (22)$$

for some ball $B_r(x_0) \subset \Omega$, $\nu > 0$, and $2\gamma_0 \leq q \leq 2\gamma_n$. Then problem (1) has a nontrivial solution in each of the following cases:

(i) $N = 3$ and $q > 4$,
(ii) $N \geq 4$ and $q \geq N/(N-2)$.

We will show that the functional $J$ has the mountain pass geometry and the mountain pass level is below the compactness threshold $c^*$ in (13).

**Lemma 3.2.** If (2), (20), and (21) hold, then $\exists \rho > 0$ such that

$$\inf_{\|u\| = \rho} J(u) > 0. \quad (23)$$

**Proof.** By (2) and (21),

$$F(x, t) \leq \frac{1}{2} \mu_0 b_0 |t|^{\gamma_0} + c_3 |t|^{2^*} \quad \text{for a.a. } x \in \Omega \text{ and all } t \in \mathbb{R}$$
for some constant $c_5 > 0$. This together with (20) implies that for $\|u\| \leq \sqrt{3}$,
\[
J(u) \geq \frac{1}{2} b_0 \left( \left( \int_{\Omega} |\nabla u|^2 \, dx \right)^{\gamma_0} - \mu_0 \int_{\Omega} |u|^{2\gamma_0} \, dx \right) - \left( c_5 + \frac{1}{2r} \right) \int_{\Omega} |u|^{2^*} \, dx
\]
\[
\geq \frac{1}{2} b_0 \left( 1 - \frac{\mu_0}{\lambda_1(\gamma_0)} + o(1) \right) \|u\|^{2\gamma_0} \quad \text{as } \|u\| \to 0
\]
since $2^* > 2\gamma_0$. Since $\mu_0 < \lambda_1(\gamma_0)$, the desired conclusion follows from this. \hfill \square

Next we show that for a suitably chosen $v \in H_0^1(\Omega) \setminus \{0\}$, $J(sv) \to -\infty$ as $s \to +\infty$ and the maximum of $J$ on the ray $sv$, $s \geq 0$ is strictly less than $c^*$. Take a function $\psi \in C_0^{\infty}(B_r(x_0))$ such that $0 \leq \psi \leq 1$ on $B_r(x_0)$ and $\psi = 1$ on $B_{r/2}(x_0)$, and set
\[
u_\varepsilon(x) = \frac{\psi(x)}{(\varepsilon + |x - x_0|^2)^{(N-2)/2}}
\]
and
\[
u_\varepsilon = \frac{\nu_\varepsilon}{\|\nu_\varepsilon\|_{2^*}}
\]
for $\varepsilon > 0$. Then
\[
\int_{\Omega} |\nabla \nu_\varepsilon|^2 \, dx = S + O(\varepsilon^{(N-2)/2})
\]
and
\[
\int_{\Omega} \nu_\varepsilon^q \, dx = \begin{cases} \kappa \varepsilon^{(2N-(N-2)q)/4} + O(\varepsilon^{(N-2)q/4}) & \text{if } q > N/(N-2) \\ \kappa \varepsilon^{N/4} |\log \varepsilon| + O(\varepsilon^{N/4}) & \text{if } q = N/(N-2) \end{cases}
\]
for some constant $\kappa > 0$ (see, e.g., Drábek and Huang [5]).

**Lemma 3.3.** For all sufficiently small $\varepsilon > 0$,
\[
J(sv_\varepsilon) \to -\infty \quad \text{as } s \to +\infty.
\]

**Proof.** Since $|v_\varepsilon|_{2^*} = 1$, $|v_\varepsilon|_p$ is bounded and (2) gives
\[
J(sv_\varepsilon) \leq \frac{1}{2} H(s^2 \int_{\Omega} |\nabla v_\varepsilon|^2 \, dx) + c_6 s^p + c_7 - \frac{s^{2^*}}{2^*}, \quad s \geq 0
\]
for some constants $c_6, c_7 > 0$. Set
\[
t = s^2 \int_{\Omega} |\nabla v_\varepsilon|^2 \, dx.
\]
Then $t \to +\infty$ as $s \to +\infty$ and (27) gives
\[
J(sv_\varepsilon) \leq \frac{1}{2} H(t) + c_6 t^{p/2} \left( \int_{\Omega} |\nabla v_\varepsilon|^2 \, dx \right)^{-p/2} + c_7 - \frac{t^{2^*/2}}{2^{2^*/2}} \left( \int_{\Omega} |\nabla v_\varepsilon|^2 \, dx \right)^{-2^*/2}.
\]
By $(A_3)$,
\[
\lim_{t \to +\infty} \frac{H(t)}{t^{2^*/2}} = \frac{2b}{2^*},
\]
so (28) gives
\[
J(sv_\varepsilon) \leq c_6 t^{p/2} \left( \int_{\Omega} |\nabla v_\varepsilon|^2 \, dx \right)^{-p/2} + c_7 - \frac{t^{2^*/2}}{2^{2^*/2}} \left[ \left( \int_{\Omega} |\nabla v_\varepsilon|^2 \, dx \right)^{-2^*/2} - b + o(1) \right]
\]
as \( t \to +\infty \). Since \( \int_{\Omega} |\nabla v_\epsilon|^2 \, dx \to S \) as \( \epsilon \to 0 \) by (24), \( b < S^{-2}/2 \), and \( p < 2^* \), the desired conclusion follows.

**Lemma 3.4.** In each of the two cases in Theorem 3.1,

\[
\max_{s \geq 0} J(sv_\epsilon) < c^*
\]

for all sufficiently small \( \epsilon > 0 \).

**Proof.** Since \( v_\epsilon = 0 \) outside \( B_r(x_0) \), (22) gives

\[
J(sv_\epsilon) \leq \frac{1}{2} H \left( s^2 \int_{\Omega} |\nabla v_\epsilon|^2 \, dx \right) - \frac{1}{q} \nu s^q \int_{\Omega} v_\epsilon^q \, dx - \frac{s^{2^*}}{2^*} =: z_\epsilon(s),
\]

so it suffices to show that

\[
\max_{s \geq 0} z_\epsilon(s) < c^*
\]

for sufficiently small \( \epsilon > 0 \). Suppose this is false. Then there are sequences \((\epsilon_j)\) and \((s_j)\), with \( \epsilon_j, s_j > 0 \) and \( \epsilon_j \to 0 \), such that

\[
z_{\epsilon_j}(s_j) = \frac{1}{2} H \left( s_j^2 \int_{\Omega} |\nabla v_{\epsilon_j}|^2 \, dx \right) - \frac{1}{q} \nu s_j^q \int_{\Omega} v_{\epsilon_j}^q \, dx - \frac{s_j^{2^*}}{2^*} \geq c^*
\]

and

\[
s_j z_{\epsilon_j}'(s_j) = H \left( s_j^2 \int_{\Omega} |\nabla v_{\epsilon_j}|^2 \, dx \right) s_j^2 \int_{\Omega} |\nabla v_{\epsilon_j}|^2 \, dx - \nu s_j^q \int_{\Omega} v_{\epsilon_j}^q \, dx - s_j^{2^*} = 0. \tag{31}
\]

Set

\[
t_j = s_j^2 \int_{\Omega} |\nabla v_{\epsilon_j}|^2 \, dx.
\]

Then (31) gives

\[
\frac{h(t_j)}{t_j^{2^*/2-1}} = \frac{1}{\left( \int_{\Omega} |\nabla v_{\epsilon_j}|^2 \, dx \right)^{2^*/2} + \nu t_j^{-(2^*-q)/2}} \int_{\Omega} v_{\epsilon_j}^q \, dx - \left( \int_{\Omega} |\nabla v_{\epsilon_j}|^2 \, dx \right)^{q/2}. \tag{32}
\]

If \( t_j \to +\infty \) for a renamed subsequence, then the left-hand side goes to \( b \) by (A3), while the right-hand side goes to \( S^{-2}/2 \) since \( \int_{\Omega} |\nabla v_{\epsilon_j}|^2 \, dx \to S \) by (24) and \( \int_{\Omega} v_{\epsilon_j}^q \, dx \to 0 \) by (25), contradicting our assumption that \( b < S^{-2}/2 \). So \( (t_j) \) is bounded, and hence converges to some \( t \geq 0 \) for a renamed subsequence. Then \( s_j^2 \to S^{-1} t \) and hence passing to the limit in (30) gives

\[
\frac{1}{2} H(t) - \frac{1}{2^*} S^{-2}/2 \, t^{2^*/2} > 0
\]

since \( c^* > 0 \), so \( t > 0 \). On the other hand, passing to the limit in (31) shows that \( t \) satisfies (12). Since \( t_0 \) is the unique positive solution of this equation, it follows that \( t = t_0 \).

Now combining (32) with (24) and (25) gives

\[
\frac{h(t_j)}{t_j^{2^*/2-1}} = S^{-2}/2 + \begin{cases} 
\sigma_j \frac{\epsilon_{j}^{(2N-(N-2)q)/4} + O(\epsilon_{j}^{(N-2)/2})}{\epsilon_{j}^{N/4} \log \epsilon_j + O(\epsilon_{j}^{\min(2/(N-2),2)} \sqrt{\epsilon_j})} & \text{if } q > N/(N-2), \\
\sigma_j \frac{\epsilon_{j}^{N/4} \log \epsilon_j + O(\epsilon_{j}^{\min(2/(N-2),2)} \sqrt{\epsilon_j})} & \text{if } q = N/(N-2),
\end{cases}
\]

for all sufficiently small \( \epsilon > 0 \).
where \( \sigma_j \rightarrow \kappa \nu S^{-\eta/2} t_0^{-(2^*-\eta)/2} > 0 \). It follows from this that in each of the two cases in the lemma,

\[
\frac{h(t_j)}{t_j^{2^*/2-1}} \geq S^{-2^*/2} = \frac{h(t_0)}{t_0^{2^*/2-1}}
\]

for all sufficiently large \( j \). Then \( t_j \leq t_0 \) by \((A_3)\). Since \( K \) is nondecreasing, then

\[
K(t_j) \leq K(t_0) = c^*.
\]

However, dividing (31) by \( 2^* \) and subtracting from (30) gives

\[
K(t_j) - \left( \frac{1}{q} - \frac{1}{2^*} \right) \nu s_j^q \int_{\Omega} v_{t_j}^q \, dx \geq c^*,
\]

so \( K(t_j) > c^* \). This contradiction completes the proof.

We are now ready to prove Theorem 3.1.

**Proof of Theorem 3.1.** Let \( \rho \) be as in Lemma 3.2 and fix \( \varepsilon > 0 \) such that (26) and (29) hold. Then \( \exists R > \rho \) such that \( J(Rv_\varepsilon) \leq 0 \). Let\[
\Gamma = \{ \varphi \in C([0,1], H^1_0(\Omega)) : \varphi(0) = 0, \varphi(1) = Rv_\varepsilon \}
\]
be the class of paths in \( H^1_0(\Omega) \) joining the origin to \( Rv_\varepsilon \), and set

\[
c := \inf_{\varphi \in \Gamma} \max_{u \in \varphi([0,1])} J(u).
\]

By (23), \( c > 0 \). Since the path \( \varphi_0(s) = sRv_\varepsilon, s \in [0, 1] \) is in \( \Gamma \),

\[
c \leq \max_{u \in \varphi_0([0,1])} J(u) \leq \max_{s \geq 0} J(sv_\varepsilon) < c^*,
\]

so \( J \) satisfies the (PS)\(_c\) condition by Theorem 2.2. Hence \( J \) has a critical point \( u \) with \( J(u) = c \) by the mountain pass theorem (see Ambrosetti and Rabinowitz [1]). Then \( u \) is a weak solution of problem (1) and \( u \) is nontrivial since \( c > 0 \).

Theorem 3.1 has many interesting consequences, some of which we now present. First we consider the problem

\[
\begin{cases}
-\nu \int_{\Omega} |\nabla u|^2 \, dx \Delta u = \lambda u + |u|^{2^*-2} u & \text{in } \Omega \\
u s_j^q \int_{\Omega} v_{t_j}^q \, dx & \text{on } \partial \Omega,
\end{cases}
\]

where \( \lambda > 0 \). Assume that

\[
K(t) \geq \alpha t \quad \forall t \geq 0
\]

for some constant \( \alpha > 0 \) and

\[
H(t) \geq a_0 t \quad \text{for } 0 \leq t \leq \delta
\]

for some constants \( \delta, a_0 > 0 \). We have \( f(x, t) = \lambda t \) and

\[
F(x, t) = \frac{1}{2} \lambda t^2, \quad F(x, t) - \frac{1}{2^*} t f(x, t) = \frac{1}{N} \lambda t^2,
\]

so \((A_1)\) holds with \( \mu_1 = \lambda/N\alpha \) if \( \lambda < N\alpha \lambda_1 \), (21) holds with \( \gamma_0 = 1 \) and \( \mu_0 = \lambda/a_0 \) if \( \lambda < a_0 \lambda_1 \), and (22) holds with \( q = 2 \) if \( \lambda > 0 \). So Theorem 3.1 has the following corollary for problem (33).

**Corollary 7.** Assume (34), (A2), (A3), and (35). If

\[
0 < \lambda < \min \{a_0, N\alpha\} \lambda_1
\]

and \( N \geq 4 \), then problem (33) has a nontrivial solution.
In particular, we have the following corollary in the model case \( h(t) = a + bt^{\gamma - 1} \).

**Corollary 8.** The problem

\[
\begin{cases}
- \left[ a + b \left( \int_{\Omega} |\nabla u|^2 \, dx \right)^{\gamma - 1} \right] \Delta u = \lambda u + |u|^{2^*-2} u & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where either \( 1 < \gamma < 2^*/2 \) and \( b \geq 0 \), or \( \gamma = 2^*/2 \) and \( 0 \leq b < S^{-2^*/2} \), has a nontrivial solution if \( 0 < \lambda < a \lambda_1 \) and \( N \geq 4 \).

Next we consider the problem

\[
\begin{cases}
- h \left( \int_{\Omega} |\nabla u|^2 \, dx \right) \Delta u = \mu |u|^{2\gamma - 2} u + |u|^{2^*-2} u & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

(36)

where \( \mu > 0 \) and \( 1 < \gamma < 2^*/2 \). Assume that

\[
K(t) \geq \beta t^\gamma \quad \forall t \geq 0
\]

(37)

for some constant \( \beta > 0 \) and

\[
H(t) \geq b_0 t^\gamma \quad \text{for } 0 \leq t \leq \delta
\]

(38)

for some constants \( \delta, b_0 > 0 \). We have \( f(x, t) = \mu |t|^{2\gamma - 2} t \) and

\[
F(x, t) = \frac{1}{2\gamma} \mu |t|^{2\gamma}, \quad F(x, t) - \frac{1}{2} tf(x, t) = \left( \frac{1}{2\gamma} - \frac{1}{2^*} \right) \mu |t|^{2\gamma},
\]

so \((A_1)\) holds with \( \mu_1 = (1/2\gamma - 1/2^*) \mu / \beta \) if

\[
\mu < \left( \frac{1}{2\gamma} - \frac{1}{2^*} \right)^{-1} \beta \lambda_1(\gamma),
\]

(21) holds with \( \gamma_0 = \gamma \) and \( \mu_0 = \mu / \gamma b_0 \) if \( \mu < \gamma b_0 \lambda_1(\gamma) \), and (22) holds with \( q = 2\gamma \) if \( \mu > 0 \). So Theorem 3.1 has the following corollary for problem (36).

**Corollary 9.** Assume (37), \((A_2)\), \((A_3)\), and (38). If

\[
0 < \mu < \min \left\{ \gamma b_0, \left( \frac{1}{2\gamma} - \frac{1}{2^*} \right)^{-1} \beta \right\} \lambda_1(\gamma)
\]

and \( N \geq 4 \), or \( N = 3 \) and \( \gamma > 2 \), then problem (36) has a nontrivial solution.

In particular, we have the following corollary in the model case \( h(t) = a + bt^{\gamma - 1} \).

**Corollary 10.** The problem

\[
\begin{cases}
- \left[ a + b \left( \int_{\Omega} |\nabla u|^2 \, dx \right)^{\gamma - 1} \right] \Delta u = \mu |u|^{2\gamma - 2} u + |u|^{2^*-2} u & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \( a \geq 0 \) and \( 1 < \gamma < 2^*/2 \), has a nontrivial solution if \( 0 < \mu < b \lambda_1(\gamma) \) and \( N \geq 4 \), or \( N = 3 \) and \( \gamma > 2 \).
Finally we consider the problem
\[
\begin{aligned}
-h \left( \int_\Omega |\nabla u|^2 \, dx \right) \Delta u &= \nu |u|^{q-2} u + |u|^{2^* - 2} u & \text{in } \Omega \\
u u &= 0 & \text{on } \partial \Omega,
\end{aligned}
\]
where \( \nu > 0 \) and \( 2 < q < 2^* \). Assume that for some constants \( \alpha, \beta > 0 \) and \( q/2 < \gamma < 2^*/2 \),
\[K(t) \geq \frac{1}{2} \alpha t^2 + \frac{1}{2} \beta |t|^{2\gamma}.\]
Since \( h \) is nonnegative, \( H(t) \geq 2K(t) \geq 2\alpha t \), so (20) holds with \( b_0 = 2\alpha \) and \( \gamma_0 = 1 \). We have \( f(x, t) = \nu |t|^{q-2} t \) and
\[F(x, t) = \nu |t|^q, \quad F(x, t) - \frac{1}{2} tf(x, t) = \left( \frac{1}{q} - \frac{1}{2^*} \right) \nu |t|^q.
\]
Since \( q > 2 \), (21) holds for any \( \mu_0 > 0 \) if \( \delta > 0 \) is sufficiently small. Theorem 3.1 has the following corollary for problem (39).

**Corollary 11.** Assume (40), (A2), and (A3). If
\[0 < \nu < (2\gamma - 2) \left( \frac{1}{q} - \frac{1}{2^*} \right) \left( \frac{\alpha \lambda_1}{2\gamma - q} \right)^{(2\gamma - q)/(2\gamma - 2)} \left( \frac{\beta \lambda_1(\gamma)}{q - 2} \right)^{(q-2)/(2\gamma - 2)} \]
and \( N \geq 4 \), or \( N = 3 \) and \( q > 4 \), then problem (39) has a nontrivial solution.

**Proof.** To see that (A1) holds, note that the minimum of
\[\lambda \alpha t^2 + \mu |t|^{2\gamma} - \left( \frac{1}{q} - \frac{1}{2^*} \right) \nu |t|^q, \quad t \in \mathbb{R}\]
is nonnegative if and only if
\[\nu \leq (2\gamma - 2) \left( \frac{1}{q} - \frac{1}{2^*} \right) \left( \frac{\alpha \lambda_1}{2\gamma - q} \right)^{(2\gamma - q)/(2\gamma - 2)} \left( \frac{\beta \mu}{q - 2} \right)^{(q-2)/(2\gamma - 2)}.\]

In the case where \( I \) is empty, first we consider the model problem
\[
\begin{aligned}
- \left( a + b \left( \int_\Omega |\nabla u|^2 \, dx \right)^{\gamma - 1} \right) \Delta u &= \lambda u + g(x, u) + |u|^{2^* - 2} u & \text{in } \Omega \\
u u &= 0 & \text{on } \partial \Omega,
\end{aligned}
\]
where \( a, b \geq 0 \) and \( 2^*/2 \leq \gamma \leq +\infty \) satisfy one of the two conditions in Corollary 6, \( \lambda > 0 \), and \( g \) is a Carathéodory function on \( \Omega \times \mathbb{R} \) satisfying
\[g(x, t) = o(t) \quad \text{as } t \to 0, \text{ uniformly a.e. in } \Omega\]
and
\[|g(x, t)| \leq c_8 |t|^{p-1} + c_9 \quad \text{for a.a. } x \in \Omega \text{ and all } t \in \mathbb{R}\]
for some constants \( c_8, c_9 > 0 \) and \( 2 < p < 2^* \). The associated variational functional is
\[J(u) = \frac{a}{2} \int_\Omega |\nabla u|^2 \, dx + \frac{b}{2^*} \left( \int_\Omega |\nabla u|^2 \, dx \right)^{\gamma} - \frac{\lambda}{2} \int_\Omega u^2 \, dx - \int_\Omega G(x, u) \, dx - \frac{1}{2^*} \int_\Omega |u|^{2^*} \, dx, \quad u \in H_0^1(\Omega),\]
where \( G(x, t) = \int_0^t g(x, s) \, ds \) is the primitive of \( g \). We note that
\[
\int_\Omega G(x, u) \, dx = o(\|u\|^2) \quad \text{as} \quad \|u\| \to 0
\]
by (42) and (43).

When \( a = 0 \), we have the following existence result.

**Theorem 3.5.** Assume that \( g \) satisfies (42) and (43). If \( \gamma = 2^*/2 \), \( a = 0 \), and \( b > S^{-2^*/2} \), then problem (41) has a nontrivial solution for all \( \lambda > 0 \).

**Proof.** By Corollary 6, \( J \) has a global minimizer \( u_0 \). For any \( u \in H^1_0(\Omega) \setminus \{0\} \),
\[
J(su) = -\frac{\lambda s^2}{2} \int_\Omega u^2 \, dx + o(s^2) \quad \text{as} \quad s \to 0
\]
by (44), so \( J(su) < 0 \) if \( s > 0 \) is sufficiently small. So \( J(u_0) = \inf_{H^1_0(\Omega)} J \) < 0 and hence \( u_0 \) is nontrivial. \( \square \)

When \( a > 0 \), we prove a multiplicity result. Let \( 0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \) be the Dirichlet eigenvalues of \(-\Delta\) on \( \Omega \), repeated according to multiplicity.

**Theorem 3.6.** Assume that \( g \) satisfies (42) and (43).

(i) If \( \gamma = 2^*/2 \), \( a > 0 \), and \( b > S^{-2^*/2} \), then problem (41) has at least two nontrivial solutions in each of the following cases:
- \( a\lambda_k < \lambda < a\lambda_{k+1} \) for some \( k \geq 1 \);
- \( a\lambda_k < \lambda = a\lambda_{k+1} \) for some \( k \geq 1 \) and \( G(x, t) \leq 0 \) for a.a. \( x \in \Omega \) and \( |t| \leq \delta \) for some \( \delta > 0 \).

(ii) If \( \gamma > 2^*/2 \) and
\[
a^{-2^*/2} b^{2^*/2-1} > \frac{(\gamma - 2^*/2)^{\gamma - 2^*/2} (2^*/2 - 1)^{2^*/2-1}}{(\gamma - 1)^{\gamma - 1}} S^{-(2^*/2)(\gamma - 1)},
\]
then problem (41) has at least two nontrivial solutions in each of the following cases:
- \( a\lambda_k < \lambda < a\lambda_{k+1} \) for some \( k \geq 1 \);
- \( a\lambda_k = \lambda < a\lambda_{k+1} \) for some \( k \geq 1 \) and \( G(x, t) \geq 0 \) for a.a. \( x \in \Omega \) and \( |t| \leq \delta \) for some \( \delta > 0 \).

We will prove this theorem using the following result of Brezis and Nirenberg [4, Theorem 4].

**Proposition 2.** Let \( J \) be a \( C^1 \)-functional on a Banach space \( X \). Assume that \( J \) is bounded from below, \( \inf_X J < 0 \), and \( J \) satisfies the \((PS)_c\) condition for all \( c \in \mathbb{R} \).

Assume further that \( X \) has a direct sum decomposition \( X = V \oplus W \), \( u = v + w \) with \( \dim V < +\infty \) and
\[
\begin{cases}
J(v) \leq 0 & \text{for} \ v \in V \cap B_r(0) \\
J(w) \geq 0 & \text{for} \ w \in W \cap B_r(0)
\end{cases}
\]
for some \( r > 0 \). Then \( J \) has at least two nontrivial critical points.

**Proof of Theorem 3.6.** By Corollary 6, \( J \) is bounded from below and satisfies the \((PS)_c\) condition for all \( c \in \mathbb{R} \). We have the direct sum decomposition \( H^1_0(\Omega) = \)}
V ⊕ W, u = v + w, where V is the span of the eigenfunctions associated with λ₁, ..., λₖ and W is the orthogonal complement of V. For v ∈ V,

\[ J(v) \leq -\frac{1}{2} \left( \frac{\lambda}{\lambda_k} - a \right) \int_{\Omega} |\nabla v|^2 \, dx + \frac{b}{2\gamma} \left( \int_{\Omega} |\nabla v|^2 \, dx \right)^\gamma - \int_{\Omega} G(x, v) \, dx - \frac{1}{2\gamma} \int_{\Omega} |v|^2 \, dx \]

\[ = -\frac{1}{2} \left( \frac{\lambda}{\lambda_k} - a \right) \int_{\Omega} |\nabla v|^2 \, dx + o(\|v\|^2) \quad \text{as } \|v\| \to 0 \quad (45) \]

by (44), so J(v) < 0 if \( \lambda > a\lambda_k \) and \( \|v\| > 0 \) is sufficiently small. For w ∈ W,

\[ J(w) \geq \frac{1}{2} \left( a - \frac{\lambda}{\lambda_{k+1}} \right) \int_{\Omega} |\nabla w|^2 \, dx + \frac{b}{2\gamma} \left( \int_{\Omega} |\nabla w|^2 \, dx \right)^\gamma - \int_{\Omega} G(x, w) \, dx - \frac{1}{2\gamma} \int_{\Omega} |w|^2 \, dx \]

\[ = \frac{1}{2} \left( a - \frac{\lambda}{\lambda_{k+1}} \right) \int_{\Omega} |\nabla w|^2 \, dx + o(\|w\|^2) \quad \text{as } \|w\| \to 0 \]

by (44), so J(w) ≥ 0 if \( \lambda < a\lambda_{k+1} \) and \( \|w\| \) is sufficiently small. So J has at least two nontrivial critical points by Proposition 2 in the cases (i)(a) and (ii)(a).

In the case (i)(b), (6) gives

\[ J(w) \geq \int_{\Omega} \frac{1}{2} \left( |\nabla w|^2 - \lambda_{k+1} w^2 \right) - G(x, w) \, dx + \frac{b - S^{-2^*/2}}{2^*} \left( \int_{\Omega} |\nabla w|^2 \, dx \right)^{2^*/2} \quad \forall w \in W. \]

The local sign condition on G in this case implies that the first integral on the right-hand side is nonnegative if \( \|w\| \) is sufficiently small (see Li and Willem [8]). Since \( b > S^{-2^*/2} \), then J(w) ≥ 0 when \( \|w\| \) is small. In the case (ii)(b), (45) gives

\[ J(v) \leq \frac{b}{2\gamma} \left( \int_{\Omega} |\nabla v|^2 \, dx \right)^\gamma - \int_{\{|v| > \delta\}} G(x, v) \, dx - \frac{1}{2^*} \int_{\Omega} |v|^2 \, dx \quad \forall v \in V. \]

Since V is a finite dimensional subspace of \( H^1_0(\Omega) \) consisting of \( L^\infty \)-functions and \( \gamma > 2^*/2 \), it follows from this that J(v) < 0 if \( \|v\| > 0 \) is sufficiently small. So J has two nontrivial critical points in these cases also. \( \square \)

In the borderline case where \( \gamma = 2^*/2 \) and \( b = S^{-2^*/2} \), lower-order terms come into play. We consider the problem

\[
\begin{align*}
-\Delta u &= \lambda u + g(x, u) + |u|^{2^* - 2} u \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

where

\[ h(t) = a + S^{-2^*/2} t^{2^*/2-1} + \eta t^{\sigma-1}, \quad t \geq 0, \]

\[ a \geq 0, \eta > 0, p/2 < \sigma < 2^*/2, \lambda > 0, \text{ and } g \text{ is a Carathéodory function on } \Omega \times \mathbb{R}, \]

satisfying (42) and (43). The associated functional is

\[ J(u) = \frac{a}{2} \int_{\Omega} |\nabla u|^2 \, dx + \frac{S^{-2^*/2}}{2^*} \left( \int_{\Omega} |\nabla u|^2 \, dx \right)^{2^*/2} + \frac{\eta}{2\sigma} \left( \int_{\Omega} |\nabla u|^2 \, dx \right)^\sigma \]

\[ - \frac{\lambda}{2} \int_{\Omega} u^2 \, dx - \int_{\Omega} G(x, u) \, dx - \frac{1}{2} \int_{\Omega} |u|^2 \, dx, \quad u \in H^1_0(\Omega), \]

where G(x, t) = \( \int_0^t g(x, s) \, ds \) satisfies (44) as before. We have the following existence and multiplicity result.
Theorem 3.7. Let \( \eta > 0 \) and \( p/2 < \sigma < 2^*/2 \), and assume that \( g \) satisfies (42) and (43).

(i) If \( a = 0 \), then problem (46) has a nontrivial solution for all \( \lambda > 0 \).

(ii) If \( a > 0 \), then problem (46) has at least two nontrivial solutions in each of the following cases:

(a) \( a\lambda_k < \lambda < a\lambda_{k+1} \) for some \( k \geq 1 \);

(b) \( a\lambda_k < \lambda = a\lambda_{k+1} \) for some \( k \geq 1 \) and \( G(x, t) \leq 0 \) for a.a. \( x \in \Omega \) and 
|t| \( \leq \delta \) for some \( \delta > 0 \).

Proof. (i) By Theorem 2.3, \( J \) has a global minimizer \( u_0 \). For any \( u \in H^1_0(\Omega) \setminus \{0\} \),

\[ J(su) = -\frac{\lambda s^2}{2} \int_{\Omega} u^2 dx + o(s^2) \quad \text{as } s \to 0 \]

by (44), so \( J(su) < 0 \) if \( s > 0 \) is sufficiently small. So \( J(u_0) = \inf_{H^1_0(\Omega)} J < 0 \) and hence \( u_0 \) is nontrivial.

(ii) By Theorem 2.3, \( J \) is bounded from below and satisfies the (PS)\(_c\) condition for all \( c \in \mathbb{R} \). We have the direct sum decomposition \( H^1_0(\Omega) = V \oplus W \), \( u = v + w \), where \( V \) is the span of the eigenfunctions associated with \( \lambda_1, \ldots, \lambda_k \) and \( W \) is the orthogonal complement of \( V \). For \( v \in V \),

\[
J(v) \leq -\frac{1}{2} \left( \frac{\lambda}{\lambda_k} - a \right) \int_{\Omega} |\nabla v|^2 dx + \frac{S^{-2/2}}{2} \left( \int_{\Omega} |\nabla v|^2 dx \right)^{2/2} + \frac{\eta}{\sigma} \left( \int_{\Omega} |\nabla v|^2 dx \right)^{\sigma} - \int_{\Omega} G(x, v) dx - \frac{1}{2} \int_{\Omega} |v|^2 dx = -\frac{1}{2} \left( \frac{\lambda}{\lambda_k} - a \right) \int_{\Omega} |\nabla v|^2 dx + o(||v||^2) \quad \text{as } ||v|| \to 0
\]

by (44), so \( J(v) < 0 \) if \( \lambda > a\lambda_k \) and \( ||v|| > 0 \) is sufficiently small. For \( w \in W \),

\[
J(w) \geq \frac{1}{2} \left( a - \frac{\lambda}{\lambda_{k+1}} \right) \int_{\Omega} |\nabla w|^2 dx + \frac{S^{-2/2}}{2} \left( \int_{\Omega} |\nabla w|^2 dx \right)^{2/2} + \frac{\eta}{\sigma} \left( \int_{\Omega} |\nabla w|^2 dx \right)^{\sigma} - \int_{\Omega} G(x, w) dx - \frac{1}{2} \int_{\Omega} |w|^2 dx = \frac{1}{2} \left( a - \frac{\lambda}{\lambda_{k+1}} \right) \int_{\Omega} |\nabla w|^2 dx + o(||w||^2) \quad \text{as } ||w|| \to 0
\]

by (44), so \( J(w) \geq 0 \) if \( \lambda < a\lambda_{k+1} \) and \( ||w|| \) is sufficiently small. So \( J \) has at least two nontrivial critical points by Proposition 2 in the case (a). In the case (b), (6) gives

\[
J(w) \geq \int_{\Omega} \left[ \frac{a}{2} \left( |\nabla w|^2 - \lambda_{k+1} w^2 \right) - G(x, w) \right] dx \quad \forall w \in W.
\]

The local sign condition on \( G \) implies that the right-hand side is nonnegative when \( ||w|| \) is small (see Li and Willem [8]). So \( J \) has two nontrivial critical points in this case also.

\[
\Box
\]

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