MARGINALLY TRAPPED MERIDIAN SURFACES OF PARABOLIC TYPE IN THE FOUR-DIMENSIONAL MINKOWSKI SPACE

GEORGI GANCHEV AND VELICHKA MILOUSHEVA

ABSTRACT. A marginally trapped surface in the four-dimensional Minkowski space is a spacelike surface whose mean curvature vector is lightlike at each point. We introduce meridian surfaces of parabolic type as one-parameter systems of meridians of a rotational hypersurface with lightlike axis in Minkowski 4-space and find their basic invariants. We find all marginally trapped meridian surfaces of parabolic type and give a geometric construction of these surfaces.

1. Introduction

The concept of trapped surfaces was introduced by Roger Penrose in [15] and is closely related to the theory of cosmic black holes playing an important role in general relativity. These surfaces were defined in order to study global properties of spacetime. In Physics, a surface in the 4-dimensional spacetime is called marginally trapped if it is closed, embedded, spacelike and its mean curvature vector is lightlike at each point of the surface. Recently, marginally trapped surfaces have been studied from a mathematical viewpoint. In the mathematical literature, it is customary to call a surface in a semi-Riemannian manifold marginally trapped if its mean curvature vector \( H \) is lightlike at each point, and removing the other hypotheses, i.e. the surface does not need to be closed or embedded.

Classification results in four-dimensional Lorentz space forms were obtained imposing some extra conditions on the mean curvature vector, the Gauss curvature or the second fundamental form. For example, marginally trapped surfaces with positive relative nullity in Lorenz space forms were classified by B.-Y. Chen and J. Van der Veken in [1]. They also proved the non-existence of marginally trapped surfaces in Robertson-Walker spaces with positive relative nullity [2] and classified marginally trapped surfaces with parallel mean curvature vector in Lorenz space forms [3].

Marginally trapped surfaces in Minkowski 4-space which are invariant under spacelike rotations were classified by S. Haesen and M. Ortega in [9]. In [8] they classified marginally trapped surfaces in Minkowski 4-space which are invariant under boost transformations (hyperbolic rotations). The classification of marginally trapped surfaces in Minkowski 4-space which are invariant under a group of screw rotations (a group of Lorenz rotations with an invariant lightlike direction) was obtained in [10].

Surfaces in the 4-dimensional Minkowski space \( \mathbb{R}^4_1 \) which are invariant under spacelike rotations, hyperbolic rotations or screw rotations are the three types of standard rotational surfaces with two-dimensional axis known also as rotational surfaces of elliptic, hyperbolic or parabolic type, respectively. A rotational surface of elliptic type is an orbit of a regular curve under the action of the orthogonal transformations of \( \mathbb{R}^4_1 \) which leave a timelike plane point-wise fixed. Similarly, a rotational surface of hyperbolic type is an orbit of a regular curve under the action of the orthogonal transformations of \( \mathbb{R}^4_1 \) which leave a spacelike plane point-wise fixed.

2000 Mathematics Subject Classification. Primary 53A35, Secondary 53B25.

Key words and phrases. Marginally trapped surfaces in the four-dimensional Minkowski space, lightlike mean curvature vector, meridian surfaces in Minkowski space.
point-wise fixed. A rotational surface of parabolic type is an an orbit of a regular curve under the action of the orthogonal transformations of $\mathbb{R}^4_1$ which leave a degenerate plane point-wise fixed. Some classification results for rotational surfaces of elliptic, hyperbolic or parabolic type with classical extra conditions have been obtained. A classification of all timelike and spacelike hyperbolic rotational surfaces with non-zero constant mean curvature in the three-dimensional de Sitter space $S^3_1$ is given in [13] and a classification of the spacelike and timelike Weingarten rotational surfaces of the three types in $S^3_1$ is found in [14]. In [5] we described all Chen spacelike rotational surfaces of hyperbolic or elliptic type.

In [7] we studied marginally trapped surfaces in the four-dimensional Minkowski space $\mathbb{R}^4_1$ and developed an invariant theory of these surfaces based on the principal lines generated by the second fundamental form. Using the principal lines, we introduced a geometrically determined moving frame field at each point of such a surface and obtained seven invariant functions which determine the surface up to a motion in $\mathbb{R}^4_1$.

We applied our theory to a special class of spacelike surfaces lying on rotational hypersurfaces with timelike or spacelike axis. We constructed two-dimensional surfaces which are one-parameter systems of meridians of the rotational hypersurface and called these surfaces meridian surfaces. The geometric construction of the meridian surfaces is different from the construction of the standard rotational surfaces with two-dimensional axis. Hence, the class of meridian surfaces is a new source of examples of two-dimensional surfaces in $\mathbb{R}^4_1$. We found all marginally trapped meridian surfaces lying on rotational hypersurfaces with spacelike or timelike axis [7].

In the present paper we continue the study of meridian surfaces considering a rotational hypersurface with lightlike axis in $\mathbb{R}^4_1$ and construct two-dimensional surfaces which are one-parameter systems of meridians of the rotational hypersurface. We call these surfaces meridian surfaces of parabolic type. We calculate their basic invariants and find all marginally trapped meridian surfaces of parabolic type. They are described in Proposition 3.1 and Theorem 3.2. We give a geometric construction of marginally trapped meridian surfaces of parabolic type.

Summarizing, we can say that we have described all marginally trapped meridian surfaces of elliptic, hyperbolic and parabolic type.

2. Preliminaries

Let $\mathbb{R}^4_1$ be the Minkowski space endowed with the metric $\langle \cdot, \cdot \rangle$ of signature $(3,1)$ and $Oe_1e_2e_3e_4$ be a fixed orthonormal coordinate system in $\mathbb{R}^4_1$, i.e. $e_1^2 = e_2^2 = e_3^2 = 1$, $e_4^2 = -1$, giving the orientation of $\mathbb{R}^4_1$. The standard flat metric is given in local coordinates by $dx^2_1 + dx^2_2 + dx^2_3 - dx^2_4$.

A surface $M^2$ in $\mathbb{R}^4_1$ is said to be spacelike if $\langle \cdot, \cdot \rangle$ induces a Riemannian metric $g$ on $M^2$. Thus at each point $p$ of a spacelike surface $M^2$ we have the following decomposition

$$\mathbb{R}^4_1 = T_pM^2 \oplus N_pM^2$$

with the property that the restriction of the metric $\langle \cdot, \cdot \rangle$ onto the tangent space $T_pM^2$ is of signature $(2,0)$, and the restriction of the metric $\langle \cdot, \cdot \rangle$ onto the normal space $N_pM^2$ is of signature $(1,1)$.

A surface $M^2$ in $\mathbb{R}^4_1$ is said to be timelike if the induced metric $g$ on $M^2$ is a metric with index 1, i.e. at each point $p$ of a timelike surface $M^2$ we have the following decomposition

$$\mathbb{R}^4_1 = T_pM^2 \oplus N_pM^2$$
with the property that the restriction of the metric $\langle , \rangle$ onto the tangent space $T_p M^2$ is of signature $(1, 1)$, and the restriction of the metric $\langle , \rangle$ onto the normal space $N_p M^2$ is of signature $(2, 0)$.

Denote by $\nabla'$ and $\nabla$ the Levi Civita connections on $\mathbb{R}^4$ and $M^2$, respectively. Let $x$ and $y$ denote vector fields tangent to $M$ and let $\xi$ be a normal vector field. Then the formulas of Gauss and Weingarten give a decomposition of the vector fields $\nabla_x'y$ and $\nabla_x'\xi$ into a tangent and a normal component:

$$\nabla_x'y = \nabla_x y + \sigma(x, y);$$

$$\nabla_x'\xi = -A_x x + D_x \xi,$$

which define the second fundamental tensor $\sigma$, the normal connection $D$ and the shape operator $A_x$ with respect to $\xi$. The mean curvature vector field $H$ of the surface $M^2$ is defined as $H = \frac{1}{2} \tr \sigma$.

Let $M^2 : z = z(u, v), \ (u, v) \in D \ (D \subset \mathbb{R}^2)$ be a local parametrization on a spacelike surface in $\mathbb{R}^4$. The tangent space at an arbitrary point $p = z(u, v)$ of $M^2$ is $T_p M^2 = \text{span}\{z_u, z_v\}$, where $\langle z_u, z_u \rangle > 0, \langle z_v, z_v \rangle > 0$ since $M^2$ is spacelike. We use the standard denotations $E(u, v) = \langle z_u, z_u \rangle, \ F(u, v) = \langle z_u, z_v \rangle, \ G(u, v) = \langle z_v, z_v \rangle$ for the coefficients of the first fundamental form

$I(\lambda, \mu) = EX^2 + 2F\lambda\mu + G\mu^2, \ \lambda, \mu \in \mathbb{R}$

and we set $W = \sqrt{EG - F^2}$. We choose a normal frame field $\{n_1, n_2\}$ such that $\langle n_1, n_1 \rangle = 1, \langle n_2, n_2 \rangle = -1$, and the quadruple $\{z_u, z_v, n_1, n_2\}$ is positively oriented in $\mathbb{R}^4$. Then we have the following integral formulas:

$$\nabla_{z_u} z_u = z_{uu} = \Gamma^1_{11} z_u + \Gamma^2_{11} z_v + c^1_{11} n_1 - c^2_{11} n_2;$$

$$\nabla_{z_v} z_v = z_{vv} = \Gamma^1_{12} z_u + \Gamma^2_{12} z_v + c^1_{12} n_1 - c^2_{12} n_2;$$

$$\nabla_{z_v} z_v = z_{vv} = \Gamma^1_{22} z_u + \Gamma^2_{22} z_v + c^1_{22} n_1 - c^2_{22} n_2,$$

where $\Gamma^k_{ij}$ are the Christoffel’s symbols and the functions $c^k_{ij}, \ i, j, k = 1, 2$ are given by

$c^1_{11} = \langle z_{uu}, n_1 \rangle; \ c^1_{12} = \langle z_{uv}, n_1 \rangle; \ c^1_{22} = \langle z_{vv}, n_1 \rangle; \ c^2_{11} = \langle z_{uu}, n_2 \rangle; \ c^2_{12} = \langle z_{uv}, n_2 \rangle; \ c^2_{22} = \langle z_{vv}, n_2 \rangle.$

Obviously, the surface $M^2$ lies in a 2-plane if and only if $M^2$ is totally geodesic, i.e. $c^k_{ij} = 0, \ i, j, k = 1, 2$. So, we assume that at least one of the coefficients $c^k_{ij}$ is not zero.

The second fundamental form $II$ of the surface $M^2$ at a point $p \in M^2$ is introduced by the following functions

$L = \frac{2}{W} \begin{vmatrix} c^1_{11} & c^1_{12} \\ c^2_{11} & c^2_{12} \end{vmatrix}; \ M = \frac{1}{W} \begin{vmatrix} c^1_{11} & c^1_{22} \\ c^2_{11} & c^2_{22} \end{vmatrix}; \ N = \frac{2}{W} \begin{vmatrix} c^1_{12} & c^1_{22} \\ c^2_{12} & c^2_{22} \end{vmatrix}.$

Let $X = \lambda z_u + \mu z_v, \ (\lambda, \mu) \neq (0, 0)$ be a tangent vector at a point $p \in M^2$. Then

$$II(\lambda, \mu) = L\lambda^2 + 2M\lambda\mu + N\mu^2, \ \lambda, \mu \in \mathbb{R}.$$

The second fundamental form $II$ is invariant up to the orientation of the tangent space or the normal space of the surface.

The condition $L = M = N = 0$ characterizes points at which the space $\{\sigma(x, y) : x, y \in T_p M^2\}$ is one-dimensional. We call such points flat points of the surface. These points are analogous to flat points in the theory of surfaces in $\mathbb{R}^3$. In [11] and [12] such points are called inflection points. The notion of an inflection point is introduced for 2-dimensional surfaces.
in a 4-dimensional affine space $A^4$. E. Lane [11] has shown that every point of a surface is an inflection point if and only if the surface is developable or lies in a 3-dimensional subspace.

We consider surfaces free of flat points, i.e. $(L, M, N) \neq (0, 0, 0)$.

The second fundamental form $II$ determines conjugate, asymptotic, and principal tangents at a point $p$ of $M^2$ in the standard way. A line $c : u = u(q), v = v(q); q \in J \subset \mathbb{R}$ on $M^2$ is said to be an asymptotic line, respectively a principal line, if its tangent at any point is asymptotic, respectively principal.

The second fundamental form $II$ generates two invariant functions:

$$k = \frac{LN - M^2}{EG - F^2}, \quad \kappa = \frac{EN + GL - 2FM}{2(EG - F^2)}.$$ 

The functions $k$ and $\kappa$ are invariant under changes of the parameters of the surface and changes of the normal frame field [6]. The sign of $k$ is invariant under congruences and the sign of $\kappa$ is invariant under motions in $\mathbb{R}^4$. However, the sign of $\kappa$ changes under symmetries with respect to a hyperplane in $\mathbb{R}^4$. It turns out that the invariant $\kappa$ is the curvature of the normal connection of the surface. The number of asymptotic tangents at a point of $M^2$ is determined by the sign of the invariant $k$.

3. Meridian Surfaces of Elliptic, Hyperbolic, and Parabolic Type in $\mathbb{R}^4$

In [1] we constructed a family of surfaces lying on a standard rotational hypersurface in the four-dimensional Euclidean space $\mathbb{R}^4$. These surfaces are one-parameter systems of meridians of the rotational hypersurface, that is why we called them meridian surfaces. We described the meridian surfaces with constant Gauss curvature, with constant mean curvature, and with constant invariant $k$.

In the four-dimensional Minkowski space there are three types of rotational hypersurfaces: rotational hypersurfaces with timelike axis, with spacelike axis, and with lightlike axis. In [7] we used the idea from the Euclidean case to construct special families of two-dimensional spacelike surfaces lying on rotational hypersurfaces in $\mathbb{R}^4_1$ with timelike or spacelike axis. The construction was the following.

Let $\{e_1, e_2, e_3, e_4\}$ be the standard orthonormal frame in $\mathbb{R}^4_1$, i.e. $e_1^2 = e_2^2 = e_3^2 = 1, e_4^2 = -1$. First we consider the standard rotational hypersurface with timelike axis.

Let $f = f(u), g = g(u)$ be smooth functions, defined in an interval $I \subset \mathbb{R}$, such that $f'^2(u) - g'^2(u) > 0, u \in I$. We assume that $f(u) > 0, u \in I$. The standard rotational hypersurface $\mathcal{M}'$ in $\mathbb{R}^4_1$, obtained by the rotation of the meridian curve $m : u \to (f(u), g(u))$ about the $Oe_4$-axis, is parameterized as follows:

$$\mathcal{M}' : Z(u, w^1, w^2) = f(u) \cos w^1 \cos w^2 e_1 + f(u) \cos w^1 \sin w^2 e_2 + f(u) \sin w^1 e_3 + g(u) e_4.$$ 

The rotational hypersurface $\mathcal{M}'$ is a two-parameter system of meridians. If $w^1 = w^1(v), w^2 = w^2(v), v \in J, J \subset \mathbb{R}$, we construct a surface $\mathcal{M}'_m$ lying on $\mathcal{M}'$ in the following way:

$$\mathcal{M}'_m : z(u, v) = Z(u, w^1(v), w^2(v)), \quad u \in I, v \in J.$$ 

Since $\mathcal{M}'_m$ is a one-parameter system of meridians of $\mathcal{M}'$, we call $\mathcal{M}'_m$ a meridian surface on $\mathcal{M}'$.

In a similar way we consider meridian surfaces lying on the rotational hypersurface in $\mathbb{R}^4_1$ with spacelike axis. Let $f = f(u), g = g(u)$ be smooth functions, defined in an interval $I \subset \mathbb{R}$, such that $f'^2(u) + g'^2(u) > 0, f(u) > 0, u \in I$. The rotational hypersurface $\mathcal{M}''$ in $\mathbb{R}^4_1$, obtained by the rotation of the meridian curve $m : u \to (f(u), g(u))$ about the $Oe_1$-axis is parameterized as follows:

$$\mathcal{M}'' : Z(u, w^1, w^2) = g(u) e_1 + f(u) \cosh w^1 \cos w^2 e_2 + f(u) \cosh w^1 \sin w^2 e_3 + f(u) \sinh w^1 e_4.$$ 

$$\mathcal{M}'_m : z(u, v) = Z(u, w^1(v), w^2(v)), \quad u \in I, v \in J.$$ 

Since $\mathcal{M}'_m$ is a one-parameter system of meridians of $\mathcal{M}'$, we call $\mathcal{M}'_m$ a meridian surface on $\mathcal{M}'$.
If \( w^1 = w^1(v), w^2 = w^2(v), v \in J, J \subset \mathbb{R}, \) we construct a surface \( \mathcal{M}''_m \) in \( \mathbb{R}^4_1 \) in the following way:

\[
\mathcal{M}''_m : z(u, v) = Z(u, w^1(v), w^2(v)), \quad u \in I, v \in J.
\]

We call \( \mathcal{M}''_m \) a meridian surface on \( \mathcal{M}' \), since \( \mathcal{M}''_m \) is a one-parameter system of meridians of \( \mathcal{M}' \).

In \([7]\) we found all marginally trapped meridian surfaces lying on the rotational hypersurfaces \( \mathcal{M}' \) or \( \mathcal{M}'' \). We call the meridian surfaces on \( \mathcal{M}' \) or \( \mathcal{M}'' \) meridian surfaces of elliptic type or meridian surfaces of hyperbolic type, respectively.

Now we shall use the same idea to construct families of two-dimensional spacelike surfaces lying on a rotational hypersurface in \( \mathbb{R}^4_1 \) with lightlike axis.

For convenience we shall use the pseudo-orthonormal base \( \{e_1, e_2, \xi_1, \xi_2\} \) of \( \mathbb{R}^4_1 \), defined by \( \xi_1 = \frac{e_3 + e_4}{\sqrt{2}}, \xi_2 = \frac{-e_3 + e_4}{\sqrt{2}}. \) Let \( \langle \xi_1, \xi_1 \rangle = 0, \langle \xi_2, \xi_2 \rangle = 0, \langle \xi_1, \xi_2 \rangle = -1. \) The rotational hypersurface with lightlike axis can be parameterized by

\[
\mathcal{M}''': Z(u, w^1, w^2) = f(u) w^1 \cos w^2 e_1 + f(u) w^1 \sin w^2 e_2 + \left( f(u) \frac{(w^1)^2}{2} + g(u) \right) \xi_1 + f(u) \xi_2,
\]

where \( f = f(u), g = g(u) \) are smooth functions, defined in an interval \( I \subset \mathbb{R} \), such that \( -f'(u)g'(u) > 0, f(u) > 0, u \in I. \)

Let \( w^1 = w^1(v), w^2 = w^2(v), v \in J, J \subset \mathbb{R} \) and assume that \( (w^1)^2 + (w^2)^2 \neq 0. \) We consider the surface \( \mathcal{M}'''_m \) in \( \mathbb{R}^4_1 \) defined by

\[
(1) \quad \mathcal{M}'''_m : z(u, v) = Z(u, w^1(v), w^2(v)),
\]

where \( u \in I, v \in J. \) The surface \( \mathcal{M}'''_m \), defined by \([11]\), is a one-parameter system of meridians of the rotational hypersurface \( \mathcal{M}''' \) with lightlike axis. We shall call \( \mathcal{M}'''_m \) a meridian surface of parabolic type.

In the present section we shall find all marginally trapped meridian surfaces of parabolic type.

Without loss of generality we assume that \( w^1 = \varphi(v), w^2 = v. \) Then the surface \( \mathcal{M}'''_m \) is parameterized as follows:

\[
(2) \quad \mathcal{M}'''_m : z(u, v) = f(u) \varphi(v) \cos v e_1 + f(u) \varphi(v) \sin v e_2 + \left( f(u) \frac{(\varphi(v))^2}{2} + g(u) \right) \xi_1 + f(u) \xi_2.
\]

First we shall study the parametric \( u \)-lines and \( v \)-lines of the meridian surface of parabolic type.

Let \( v = v_0 = \text{const} \) and denote \( c = \varphi(v_0), \alpha = \cos v_0, \beta = \sin v_0. \) The parametric \( u \)-line \( v = v_0 = \text{const} \) is given by

\[
(3) \quad c_u : z(u) = c \alpha f(u) e_1 + c \beta f(u) e_2 + \left( \frac{c^2}{2} f(u) + g(u) \right) \xi_1 + f(u) \xi_2.
\]

Using \([3]\) we calculate the unit tangent vector field \( t_{c_u} \) of \( c_u \):

\[
t_{c_u} = \frac{1}{\sqrt{-2 f' g'}} \left( c \alpha f' e_1 + c \beta f' e_2 + \left( \frac{c^2}{2} f' + g' \right) \xi_1 + f' \xi_2 \right).
\]

We denote by \( s \) the arc-length of \( c_u \) and calculate the derivative

\[
\frac{dt_{c_u}}{ds} = \frac{t'_{c_u}}{s'} = \frac{f' g'' - g' f''}{(-2 f'') g'} \left( c \alpha f' e_1 + c \beta f' e_2 + \left( \frac{c^2}{2} f' - g' \right) \xi_1 + f' \xi_2 \right).
\]
Hence \( \frac{dt_{c_u}}{ds}, \frac{dt_{c_v}}{ds} = \frac{(f'g'' - g'f'')^2}{(-2f'g')^2} \). We set

\[
(4) \quad n_{c_u} = \frac{1}{\sqrt{-2f'g'}} \left( \alpha \alpha f' e_1 + c \beta f' e_2 + \left( \frac{c^2}{2} f' - g' \right) \xi_1 + f' \xi_2 \right).
\]

Note that \( n_{c_u} \) is a timelike vector field, since \( \langle n_{c_u}, n_{c_u} \rangle = -1 \). Differentiating (4) with respect to \( s \) we get

\[
\frac{dn_{c_u}}{ds} = \frac{n'_{c_u}}{s'} = \frac{f'g'' - g'f''}{(-2f'g')^{3/2}} \left( \alpha \alpha f' e_1 + c \beta f' e_2 + \left( \frac{c^2}{2} f' + g' \right) \xi_1 + f' \xi_2 \right).
\]

Thus we obtain the formulas

\[
\frac{dt_{c_u}}{ds} = \frac{f'g'' - g'f''}{(-2f'g')^{3/2}} n_{c_u};
\]
\[
\frac{dn_{c_u}}{ds} = \frac{f'g'' - g'f''}{(-2f'g')^{3/2}} t_{c_u},
\]

which imply that the curvature of \( c_u \) is \( \frac{f'g'' - g'f''}{(-2f'g')^{3/2}} \). For each \( v = \text{const} \) the parametric lines \( c_u \) are congruent in \( \mathbb{R}^1 \). These curves are the meridians of \( \mathcal{M}_m'' \). We denote \( \kappa_m(u) = \frac{f'g'' - g'f''}{(-2f'g')^{3/2}} \).

Now let us consider the parametric \( v \)-lines of \( \mathcal{M}_m'' \). Let \( u = u_0 = \text{const} \) and denote \( a = f(u_0), b = g(u_0) \). The corresponding parametric \( v \)-line is given by

\[
(5) \quad c_v : z(v) = a \varphi(v) \cos v e_1 + a \varphi(v) \sin v e_2 + \left( \frac{a \varphi^2(v)}{2} + b \right) \xi_1 + a \xi_2.
\]

Using (5) we calculate the unit tangent vector field \( t_{c_v} \) of \( c_v \):

\[
t_{c_v} = \frac{1}{\sqrt{\dot{\varphi}^2 + \varphi^2}} \left( (\dot{\varphi} \cos v - \varphi \sin v) e_1 + (\dot{\varphi} \sin v + \varphi \cos v) e_2 + \varphi \dot{\varphi} \xi_1 \right).
\]

where \( \dot{\varphi} \) denotes the derivative with respect to \( v \). Knowing \( t_{c_v} \) we calculate the curvature \( \kappa_{c_v} \) of \( c_v \) and obtain that \( \kappa_{c_v} = \frac{\dot{\varphi} \dot{\varphi} - 2 \dot{\varphi} \varphi - \varphi^2}{a(\dot{\varphi}^2 + \varphi^2)^{3/2}} \).

Let us denote \( \kappa(v) = \frac{\dot{\varphi} \dot{\varphi} - 2 \dot{\varphi} \varphi - \varphi^2}{(\dot{\varphi}^2 + \varphi^2)^{3/2}} \). Then, for each \( u = u_0 = \text{const} \) the curvature of the corresponding parametric \( v \)-line is expressed as \( \kappa_{c_v} = \frac{1}{a} \kappa(v) \), where \( a = f(u_0) \).

Now we shall find the coefficients of the first and the second fundamental forms of the meridian surface of parabolic type. From (2) we find the tangent vector fields of \( \mathcal{M}_m'' \):

\[
(6) \quad z_u = f' \varphi \cos v e_1 + f' \varphi \sin v e_2 + \left( f' \frac{\varphi^2}{2} + g' \right) \xi_1 + f' \xi_2;
\]
\[
z_v = f(\dot{\varphi} \cos v - \varphi \sin v) e_1 + f(\dot{\varphi} \sin v + \varphi \cos v) e_2 + f \varphi \dot{\varphi} \xi_1.
\]

Hence, the coefficients of the first fundamental form of \( \mathcal{M}_m'' \) are

\[
E = -2f'(u)g'(u); \quad F = 0; \quad G = f^2(u)(\dot{\varphi}^2(v) + \varphi^2(v)).
\]
The first fundamental form is positive definite, since \(-f'g' > 0\). So, \(\mathcal{M}''_m\) is a spacelike surface is \(\mathbb{R}^4_1\).

Let us denote \(x = \frac{z_u}{\sqrt{-2f'g'}}\), \(y = \frac{z_v}{f\sqrt{\dot{\varphi}^2 + \varphi^2}}\). Then \(|x, y|\) is an orthonormal tangent frame field of \(\mathcal{M}''_m\). We consider the orthonormal normal frame field, defined by

\[
n_1 = \frac{1}{\sqrt{\dot{\varphi}^2 + \varphi^2}} \left( (\dot{\varphi} \sin v + \varphi \cos v) e_1 + (-\dot{\varphi} \cos v + \varphi \sin v) e_2 + \varphi^2 \xi_1; \right) \tag{7}
\]

\[
n_2 = \sqrt{-\frac{f'}{2g'}} \left( \varphi \cos v e_1 + \varphi \sin v e_2 + \frac{f' \varphi^2 - 2g'}{2f'} \xi_1 + \xi_2 \right). \tag{8}
\]

Thus we obtain a frame field \(|x, y, n_1, n_2|\) of \(\mathcal{M}_m\), such that \(\langle n_1, n_1 \rangle = 1, \langle n_2, n_2 \rangle = -1, \langle n_1, n_2 \rangle = 0\).

Taking into account (6), we calculate the second partial derivatives of \(z(u, v)\):

\[
\begin{align*}
z_{uu} & = f'' \varphi \cos v e_1 + f'' \varphi \sin v e_2 + \left( \frac{f'' \varphi^2}{2} + g'' \right) \xi_1 + f'' \xi_2; \\
z_{uv} & = f' (\dot{\varphi} \cos v - \varphi v \sin v) e_1 + f' (\dot{\varphi} \sin v + \varphi \cos v) e_2 + f' \varphi \xi_1; \\
z_{vv} & = f ((\ddot{\varphi} - \varphi) \cos v - 2\dot{\varphi} \sin v) e_1 + f ((\ddot{\varphi} - \varphi) \sin v + 2\dot{\varphi} \cos v) e_2 + f (\varphi^2 + \varphi \ddot{\varphi}) \xi_1.
\end{align*}
\]

Then equalities (7) and (8) imply

\[
\begin{align*}
c^1_{11} = \langle z_{uu}, n_1 \rangle &= 0; & c^2_{11} = \langle z_{uu}, n_2 \rangle &= \frac{f'' g' - g'' f'}{\sqrt{-2f' g'}}; \\
c^1_{12} = \langle z_{uv}, n_1 \rangle &= 0; & c^2_{12} = \langle z_{uv}, n_2 \rangle &= 0; \\
c^1_{22} = \langle z_{vv}, n_1 \rangle &= f \frac{\dot{\varphi} \ddot{\varphi} - \varphi^2 - 2\dot{\varphi}^2}{\sqrt{\dot{\varphi}^2 + \varphi^2}}; & c^2_{22} = \langle z_{vv}, n_2 \rangle &= -f \sqrt{-\frac{f'}{2g'}} (\varphi^2 + \varphi^2).
\end{align*}
\]

Hence, the coefficients of the second fundamental form are:

\[
L = 0; \quad M = \frac{f' g'' - g' f''}{2f' g'} \frac{\dot{\varphi} \ddot{\varphi} - \varphi^2 - 2\dot{\varphi}^2}{\dot{\varphi}^2 + \varphi^2}; \quad N = 0.
\]

Then the invariants \(k\) and \(\varkappa\) of the meridian surface of parabolic type are expressed as follows:

\[
k = -\frac{\kappa_m^2(u) \varpi^2(v)}{f^2(u)}; \quad \varkappa = 0.
\]

The equality \(\varkappa = 0\) implies that \(\mathcal{M}_m''\) is a surface with flat normal connection. Using (9) we obtain

\[
\sigma(x, x) = -\varkappa m(u) n_2; \\
\sigma(x, y) = 0; \\
\sigma(y, y) = \frac{\varpi(v)}{f(u)} n_1 - \frac{1}{f(u)} \sqrt{-\frac{f'(u)}{2g'(u)}} n_2.
\]

\[
\sigma(x, y) = \frac{\varpi(v)}{f(u)} n_1 - \frac{1}{f(u)} \sqrt{-\frac{f'(u)}{2g'(u)}} n_2.
\]
Taking into account \((10)\), we find the Gauss curvature \(K\) and the mean curvature vector field \(H\) of \(\mathcal{M}_m''\):

\[
K = -\frac{\kappa_m(u)|f'(u)|}{f(u)\sqrt{-2f'(u)g'(u)}};
\]

\[
(11) \quad H = \frac{\pi(v)}{2f(u)} n_1 - \frac{1}{2} \left( \kappa_m(u) + \frac{|f'(u)|}{f\sqrt{-2f'(u)g'(u)}} \right) n_2.
\]

We can distinguish two special classes of meridian surfaces of parabolic type.

I. \(\pi(v) = 0\). In this case \(\mathcal{M}_m''\) is a surface consisting of flat points, since \(L = M = N = 0\). It follows from \((7)\) that for each meridian surface of parabolic type the next formulas hold:

\[
\nabla'_x n_1 = 0;
\]

\[
\nabla'_y n_1 = \frac{\pi(v)}{f\sqrt{\varphi'^2 + \varphi^2}} \left((\varphi \sin v - \varphi \cos v) e_1 - (\varphi \cos v + \varphi \sin v) e_2 - \varphi \dot{\xi}_1\right).
\]

Having in mind that \(\pi(v) = 0\) we get \(\nabla'_x n_1 = 0; \nabla'_y n_1 = 0\), which imply that the normal vector field \(n_1\) is constant. Hence, \(\mathcal{M}_m''\) lies in the hyperplane \(\mathbb{R}^3_1\) of \(\mathbb{R}^4\) orthogonal to \(n_1\), i.e. \(\mathcal{M}_m''\) lies in \(\mathbb{R}^4 = \text{span}\{x, y, n_2\}\).

In the case \(\pi(v) = 0\) the mean curvature vector field is:

\[
H = -\frac{1}{2} \left( \kappa_m(u) + \frac{|f'(u)|}{f\sqrt{-2f'(u)g'(u)}} \right) n_2.
\]

Hence, \(\langle H, H \rangle = 0\) if and only if \(H = 0\). Consequently, there are no marginally trapped meridian surfaces of parabolic type in the class \(\pi(v) = 0\).

II. \(\kappa_m(u) = 0\). In this case \(\mathcal{M}_m''\) is again a surface consisting of flat points (\(L = M = N = 0\)). Since \(\kappa_m(u) = 0\), without loss of generality we assume that the meridian curve is determined by \(f = u, g = au + b\), where \(a, b\) are constants, \(a < 0\). Hence, \(\mathcal{M}_m''\) is a 1-parameter system of straight-lines, i.e. \(\mathcal{M}_m''\) is a ruled surface, parameterized as follows:

\[
(12) \quad \mathcal{M}_m'' : z(u, v) = u \varphi(v) \cos v e_1 + u \varphi(v) \sin v e_2 + \left(\frac{u(\varphi(v))^2}{2} + au + b\right) \xi_1 + u \xi_2.
\]

Let us consider the curve \(c : z(v) = \varphi(v) \cos v e_1 + \varphi(v) \sin v e_2 + \left(\frac{\varphi^2(v)}{2} + a\right) \xi_1 + \xi_2\). Then the ruled surface \(\mathcal{M}_m''\) is given by

\[
(13) \quad \mathcal{M}_m'' : z(u, v) = P_0 + uz(v),
\]

where \(P_0 = b \xi_1\) is a fixed point in \(\mathbb{R}^4\). The tangent space is spanned by the vector fields \(z(v)\) and \(\dot{z}(v)\), and obviously the tangent space is one and the same at the points of each fixed generator of \(\mathcal{M}_m''\). Hence, \(\mathcal{M}_m''\) is a developable ruled surface in \(\mathbb{R}^4\). The parametrization \((13)\) shows that \(\mathcal{M}_m''\) is a cone in \(\mathbb{R}^4\) determined by the point \(P_0\) and the curve \(c : z = z(v)\).

We shall describe the marginally trapped meridian surfaces of parabolic type in the special class \(\kappa_m(u) = 0\).

**Proposition 3.1.** Let \(\mathcal{M}_m''\) be a developable meridian surface of parabolic type, defined by \((12)\). Then \(\mathcal{M}_m''\) is marginally trapped if and only if \(\pi^2(v) = -\frac{1}{2a}\).
Proof: In the case $\kappa_m(u) = 0$ the mean curvature vector field is:

$$H = \frac{\pi(v)}{2u} n_1 - \frac{1}{2u} \sqrt{-\frac{1}{2a}} n_2.$$

The condition $\langle H, H \rangle = 0$ is equivalent to $\kappa^2(v) = -\frac{1}{2a}$. \qed

Further we shall consider general meridian surfaces of parabolic type, i.e. we assume that $\kappa(v) \neq 0$ and $\kappa_m(u) \neq 0$.

**Theorem 3.2.** The general meridian surface of parabolic type $M_m''$ is marginally trapped if and only if $\kappa(v) = a = \text{const, } a \neq 0$, and the meridian curve is defined by

$$f(u) = u; \quad g(u) = \pm \frac{1}{2a^3} \left( \frac{a^2u^2 \mp 2auc}{c \mp au} - 2c \ln |c \mp au| + b \right),$$

where $b$ and $c$ are constants, $c \neq 0$.

**Proof:** Using that $\kappa_m(u) = \frac{f'g'' - f''g'}{(-2f'g')^{\frac{3}{2}}}$ from (11) we obtain that the mean curvature vector field is

$$H = \frac{\pi}{2f} n_1 + \frac{f(f''g' - f'g'') + 2f'g'|f'|}{2f(-2f'g')^{\frac{3}{2}}} n_2.$$

Hence, $\langle H, H \rangle = 0$ if and only if

$$\kappa^2(v) = \frac{(f(u)(f''(u)g'(u) - f'(u)g''(u)) + 2f'(u)g'(u)|f'(u)|)^2}{(-2f'(u)g'(u))^3}.$$

The last equality implies

$$\kappa(v) = a = \text{const, } a \neq 0; \quad \frac{f(f''g' - f'g'') + 2f'g'|f'|}{(-2f'g')^{\frac{3}{2}}} = \pm a. \tag{14}$$

Assuming that the meridian curve is given by $f = u; \ g = g(u)$, from equation (14) we get

$$-ug'' + 2g' = \pm a(-2g')^{\frac{3}{2}}. \tag{15}$$

After the change $\frac{1}{\sqrt{-2g'(u)}} = h(u)$ the above equation is transformed into

$$h' + \frac{h}{u} \pm \frac{a}{u} = 0. \tag{16}$$

The general solution of equation (16) is given by

$$h(u) = \frac{c \mp au}{u}, \quad c = \text{const, } c \neq 0.$$ 

Hence,

$$g'(u) = \frac{-u^2}{2(c \mp au)^2}. \tag{17}$$
Integrating (17) we obtain that all solutions of differential equation (15) are given by the formula
\[ g(u) = \pm \frac{1}{2a^3} \left( \frac{a^2 u^2 \pm 2auc}{c \mp au} - 2c \ln |c \mp au| + b \right), \]
where \( b \) and \( c \) are constants, \( c \neq 0 \). \( \square \)

Theorem 3.2 and Proposition 3.1 give all marginally trapped meridian surfaces of parabolic type.

4. Geometric description of marginally trapped meridian surfaces of parabolic type

In this section we give a complete geometric description of the marginally trapped meridian surfaces of parabolic type.

Note that for the class of marginally trapped meridian surfaces (in both general and special case) we have \( \kappa(v) = a = \text{const}, \ a \neq 0 \). Now we shall clear up the geometric meaning of the condition \( \kappa(v) = \text{const} \).

Each parametric \( v \)-line \( u = u_0 = \text{const} \) of the meridian surface \( \mathcal{M}_m'' \) is given by (5). Let us consider the curve \( \mathcal{C} : \bar{z} = \bar{z}(v) \), defined by
\[ \bar{z}(v) = \phi(v) \cos v e_1 + \phi(v) \sin v e_2 + \frac{\varphi^2(v)}{2} \xi_1 + \xi_2. \] (18)
Then each parametric \( v \)-line is expressed as
\[ c_v : z(v) = f(u_0) \bar{z}(v) + g(u_0) \xi_1. \]
Hence, all parametric \( v \)-lines of \( \mathcal{M}_m'' \) are generated by the curve \( \mathcal{C} \).

Note that \( \langle z(v), z(v) \rangle = 0 \). The curve \( \mathcal{C} \) lies on the paraboloid \( \mathcal{P}^2 \), defined by
\[ \mathcal{P}^2 : z(w^1, w^2) = w^1 \cos w^2 e_1 + w^1 \sin w^2 e_2 + \frac{(w^1)^2}{2} \xi_1 + \xi_2. \]
We shall prove that in the case \( \kappa(v) = a = \text{const}, \ a \neq 0 \) the curve \( \mathcal{C} \) is a plane curve on \( \mathcal{P}^2 \).

4.1. Curves on \( \mathcal{P}^2 \) with constant curvature. Let \( \mathcal{C} \) be the curve on \( \mathcal{P}^2 \), given by (18). It follows from (18) that the unit tangent vector field \( \tilde{t}(v) \) of \( \mathcal{C} \) is
\[ \tilde{t}(v) = \frac{1}{\sqrt{\varphi^2 + \varphi^2}} \left( (\phi \cos v - \varphi \sin v) e_1 + (\phi \sin v + \varphi \cos v) e_2 + \varphi \phi \xi_1 \right). \] (19)
Obviously \( \mathcal{C} \) is a spacelike curve, since \( \langle \tilde{t}(v), \tilde{t}(v) \rangle = 1 \).
We denote by \( \bar{s} \) the arc-length of \( \mathcal{C} \) and calculate the derivative
\[ \frac{d\tilde{t}}{d\bar{s}} = \frac{\frac{d\tilde{t}}{dt}}{\frac{dt}{d\bar{s}}} = \frac{\varphi \phi - 2\phi^2 - \varphi^2}{(\phi^2 + \varphi^2)^2} \left( (\phi \sin v + \varphi \cos v) e_1 + (-\phi \cos v + \varphi \sin v) e_2 + \frac{\varphi^4 + \varphi^3 \phi}{\varphi \phi - 2\phi^2 - \varphi^2} \xi_1 \right). \]

Hence, the curvature of \( \mathcal{C} \) is \( \pi(v) = \frac{\varphi \phi - 2\phi^2 - \varphi^2}{(\phi^2 + \varphi^2)^3} \).

**Proposition 4.1.** Let \( \mathcal{C} \) be the curve on \( \mathcal{P}^2 \), defined by (18). If \( \pi(v) = a = \text{const}, \ a \neq 0 \), then \( \mathcal{C} \) is a plane curve.
We denote

\[ \pi(v) = \frac{1}{\sqrt{\dot{\phi}^2 + \varphi^2}} \left( (\dot{\phi} \sin v + \varphi \cos v) e_1 + (-\dot{\phi} \cos v + \varphi \sin v) e_2 + \frac{\dot{\phi}^4 + \varphi^3 \dot{\phi}}{\varphi \dot{\phi} - 2\dot{\phi}^2 - \varphi^2} \xi_1 \right). \]

Then, we have the formula \( \frac{d\tau}{ds} = \kappa \pi \). Calculating the derivative \( \dot{\pi}(v) \) we get

\[ \dot{\pi}(v) = \pi(v) \left( (-\dot{\phi} \cos v + \varphi \sin v) e_1 - (\dot{\phi} \sin v + \varphi \cos v) e_2 + \frac{\alpha}{\dot{\phi}^2 - 2\dot{\phi}^2 + \varphi^2} \xi_1 \right), \]

where \( \alpha = (\dot{\phi}^2 + \varphi^2)^\frac{3}{2} \frac{d}{dv} \left( \frac{\dot{\phi}^4 + \varphi^3 \dot{\phi}}{\sqrt{\dot{\phi}^2 + \varphi^2(\dot{\phi}^2 + 2\dot{\phi}^2 - \varphi^2)}} \right). \)

Let \( \kappa(v) = a, a \neq 0 \). Then \( \varphi \dot{\phi} - 2\dot{\phi}^2 - \varphi^2 = a(\dot{\phi}^2 + \varphi^2)^\frac{2}{3} \), which implies

\[ \begin{align*}
\varphi \dot{\phi} - \varphi^2 &= (\dot{\phi}^2 + \varphi^2) \left( 1 + a\sqrt{\dot{\phi}^2 + \varphi^2} \right); \\
\varphi(\dot{\phi} + \varphi) &= (\dot{\phi}^2 + \varphi^2) \left( 2 + a\sqrt{\dot{\phi}^2 + \varphi^2} \right); \\
\varphi \ddot{\phi} - \varphi \dot{\phi} &= \dot{\phi}(\dot{\phi} + \varphi) \left( 2 + 3a\sqrt{\dot{\phi}^2 + \varphi^2} \right). 
\end{align*} \]

Using the last equalities by straightforward computation we get

\[ \frac{d}{dv} \left( \frac{\dot{\phi}^4 + \varphi^3 \dot{\phi}}{(\dot{\phi}^2 + \varphi^2)^2} \right) = -a^2 \varphi \dot{\phi}, \]

which implies

\[ \alpha = (\dot{\phi}^2 + \varphi^2)^\frac{3}{2} \frac{d}{dv} (\frac{\dot{\phi}^4 + \varphi^3 \dot{\phi}}{\sqrt{\dot{\phi}^2 + \varphi^2(\dot{\phi}^2 + 2\dot{\phi}^2 - \varphi^2)}}) = -\varphi \dot{\phi}. \]

Hence, from equalities (19), (20) and (21) we obtain the formulas

\[ \frac{d\tau}{ds} = \kappa \pi; \]

\[ \frac{d\pi}{ds} = -\kappa \tau, \]

which imply that the curve \( \tau \) is a plane curve lying in the plane span\{\( \tau \), \( \pi \)\}. \( \square \)

Let us consider the vector fields \( T_1 = \frac{\dot{\phi} \tau + \varphi \pi}{\sqrt{\dot{\phi}^2 + \varphi^2}} \) and \( T_2 = \frac{\varphi \tau - \dot{\phi} \pi}{\sqrt{\dot{\phi}^2 + \varphi^2}} \). In the case \( \kappa(v) = a, a \neq 0 \) we calculate that

\[ T_1 = \cos v e_1 + \sin v e_2 + \left( \frac{\varphi}{a\sqrt{\dot{\phi}^2 + \varphi^2}} \right) \xi_1; \]

\[ T_2 = -\sin v e_1 + \cos v e_2 - \frac{\dot{\phi}}{a\sqrt{\dot{\phi}^2 + \varphi^2}} \xi_1. \]

\( T_1 \) and \( T_2 \) are unit spacelike vector fields such that \( \langle T_1, T_2 \rangle = 0 \) and span\{\( \tau \), \( \pi \)\} = span\{\( T_1 \), \( T_2 \)\}. Since \( \dot{\phi}^2 + \varphi^2 \neq 0 \), the lightlike vector field \( \xi_1 \) does not lie in the plane span\{\( T_1 \), \( T_2 \)\}.

Each curve lying on \( P^2 \) admits a parametrization of the form \( w^1 = \varphi(v), w^2 = v \) for some smooth function \( \varphi \). From Proposition 4.1 it follows that each curve on \( P^2 \) with constant
curvature is a plane section of $\mathcal{P}^2$ with a plane which does not contain $\xi_1$. Now we shall prove that the converse statement is also true.

**Proposition 4.2.** Let $c$ be a curve on $\mathcal{P}^2$, obtained as a plane section with a plane which does not contain $\xi_1$. Then $c$ has constant curvature.

**Proof:** We shall use the notations $z_1, z_2, \eta_1, \eta_2$ for the coordinate functions of an arbitrary vector field with respect to the base $\{e_1, e_2, \xi_1, \xi_2\}$, respectively. The paraboloid $\mathcal{P}^2$ has the following coordinate parametric equations:

\[
z_1 = w^1 \cos w^2; \\
z_2 = w^1 \sin w^2; \\
\mathcal{P}^2: \\
\eta_1 = \frac{(w^1)^2}{2}; \\
\eta_2 = 1.
\]

Note that the paraboloid $\mathcal{P}^2$ lies in the hyperplane of $\mathbb{E}_4^1$, determined by the equation $\eta_2 = 1$. An arbitrary plane $\pi$ lying in this hyperplane is defined by an equation of the following form:

\[
\pi: A_0 z_1 + B_0 z_2 + C_0 \eta_1 + D_0 = 0,
\]

where $A_0, B_0, C_0, D_0$ are constants. Hence, the plane section of $\mathcal{P}^2$ with $\pi$ is determined by the equation

\[
A_0 w^1 \cos w^2 + B_0 w^1 \sin w^2 + C_0 \frac{(w^1)^2}{2} + D_0 = 0.
\]

Since we consider plane sections of $\mathcal{P}^2$ with planes which does not contain $\xi_1$, we assume that $C_0 \neq 0$ and $A_0^2 + B_0^2 - 2C_0D_0 > 0$. We denote $A = \frac{A_0}{C_0}, B = \frac{B_0}{C_0}, C = \frac{D_0}{C_0}$ and obtain the equation

\[
A w^1 \cos w^2 + B w^1 \sin w^2 + \frac{(w^1)^2}{2} + C = 0,
\]

or equivalently

\[
(22) \quad \frac{(w^1)^2}{2} + (A \cos w^2 + B \sin w^2) w^1 + C = 0,
\]

where $A^2 + B^2 - 2C > 0$. The solution of equation (22) is

\[
w^1 = -(A \cos w^2 + B \sin w^2) \pm \sqrt{(A \cos w^2 + B \sin w^2)^2 - 2C}.
\]

Setting $w^1 = \varphi(v), \ w^2 = v$ we obtain

\[
(23) \quad \varphi(v) = -(A \cos v + B \sin v) \pm \sqrt{(A \cos v + B \sin v)^2 - 2C}.
\]

Now we have to prove that the function $\varphi(v)$, given by formula (23), satisfies the condition

\[
\frac{\varphi \ddot{\varphi} - 2 \dot{\varphi}^2 - \varphi^2}{(\dot{\varphi}^2 + \varphi^2)^{\frac{3}{2}}} = \text{const}.
\]

Let us denote $\theta(v) = A \cos v + B \sin v$. Then $\varphi = -\theta \pm \sqrt{\theta^2 - 2C}$. By long but straightforward computation we get

\[
\dot{\varphi}^2 + \varphi^2 = \frac{(A^2 + B^2 - 2C)(\theta \mp \sqrt{\theta^2 - 2C})^2}{\theta^2 - 2C};
\]

\[
\varphi \ddot{\varphi} - 2 \dot{\varphi}^2 - \varphi^2 = \frac{2(A^2 + B^2 - 2C)}{(\theta^2 - 2C)^{\frac{3}{2}}} \left((2\theta^2 - C)(\pm \theta - \sqrt{\theta^2 - 2C}) \mp 2C \theta\right).
\]
Using the last two equalities we calculate that
\[
\frac{\varphi\ddot{\varphi} - 2\varphi^2 - \varphi^2}{(\varphi^2 + \varphi^2)\ddot{\varphi}} = \pm \frac{1}{\sqrt{A^2 + B^2 - 2C}}.
\]

Consequently, the plane section of \(\mathcal{P}^2\) with \(\pi\) is a curve with constant curvature.

4.2. Geometric construction. Let us consider again the meridian surface of parabolic type \(\mathcal{M}'''_m\), defined by (2). Each parametric line \(c_u (v = v_0 = \text{const})\) lies in the plane span\(\{t_{c_u}, n_{c_u}\}\). Note that
\[
t_{c_u} - n_{c_u} = \frac{2g'}{\sqrt{-2f'g'}} \xi_1;
\]
\[
t_{c_u} + n_{c_u} = \frac{2f'}{\sqrt{-2f'g'}} \left( \varphi(v_0) \cos v_0 e_1 + \varphi(v_0) \sin v_0 e_2 + \frac{\varphi^2(v_0)}{2} \xi_1 + \xi_2 \right).
\]

Hence, for each \(v = v_0 = \text{const}\) the \(u\)-line is a plane curve lying in the plane spanned by the lightlike vector fields \(\xi_1\) and \(\xi(v_0)\). So, the meridians of \(\mathcal{M}'''_m\) are congruent curves lying in the planes span\(\{\xi_1, \xi(v)\}\), where \(\xi(v)\) is the position vector of the curve \(\xi\) on the paraboloid \(\mathcal{P}^2\).

Now, taking into account Proposition 3.1, Theorem 3.2, Proposition 4.1 and Proposition 4.2, we obtain a complete description of all marginally trapped meridian surfaces of parabolic type. They can be constructed as follows.

I. The general case \(\kappa_m(u) \neq 0\).

- Let \(\tau : \xi = \xi(v)\) be a curve on the paraboloid \(\mathcal{P}^2\), obtained by the intersection of \(\mathcal{P}^2\) with an arbitrary plane which does not contain \(\xi_1\). The position vector \(\xi(v)\) of \(\tau\) is given by (18), where \(\varphi\) is determined by (23).
- The one-parameter system of meridian curves lying in the plane span\(\{\xi_1, \xi(v)\}\) and given by
  \[
  f(u) = u;
  \]
  \[
  g(u) = \pm \frac{1}{2a^3} \left( \frac{a^2 u^2 + 2au}{c \mp au} - 2c \ln |c \mp au| + b \right); \quad b = \text{const}, \ c = \text{const} \neq 0
  \]
  determines a marginally trapped meridian surface of parabolic type.

II. The special case \(\kappa_m(u) = 0\).

In this case \(\tau\) is again a plane curve on the paraboloid \(\mathcal{P}^2\), obtained by the intersection of \(\mathcal{P}^2\) with an arbitrary plane which does not contain \(\xi_1\). The marginally trapped meridian surface, which is a 1-parameter system of straight-lines, lies in the three-dimensional space \(\mathbb{R}^3_1\) spanned by the plane of the curve \(\xi\) and the fixed point \(P_0\). Hence, the surface \(\mathcal{M}'''_m\) is a cone in \(\mathbb{R}^3_1\) with lightlike normal vector field.

References

[1] Chen, B.-Y., Van der Veken, J., "Marginally trapped surfaces in Lorenzian space with positive relative nullity," Class. Quantum Grav. 24, 551–563 (2007).
[2] Chen, B.-Y., Van der Veken, J., "Spacial and Lorenzian surfaces in Robertson-Walker space-times," J. Math. Phys. 48, 073509, 12 pp, (2007).
[3] Chen, B.-Y., Van der Veken, J., "Classification of marginally trapped surfaces with parallel mean curvature vector in Lorenzian space forms," Houston J. Math. 36, 421–449 (2010).
[4] Ganchev, G., Milousheva, V., "Invariants and Bonnet-type theorem for surfaces in $\mathbb{R}^4$," Cent. Eur. J. Math. 8 (6), 993–1008 (2010).

[5] Ganchev, G., Milousheva, V., "Chen rotational surfaces of hyperbolic or elliptic type in the four-dimensional Minkowski space", C. R. Acad. Bulg. Sci. 64 (5), 641–652 (2011).

[6] Ganchev, G., Milousheva, V., "An invariant theory of spacelike surfaces in the four-dimensional Minkowski space," Mediterr. J. Math. 9, 267–294 (2012), DOI: 10.1007/s00009-010-0108-2.

[7] Ganchev, G., Milousheva, V., "An invariant theory of marginally trapped surfaces in the four-dimensional Minkowski space," J. Math. Phys. 53, 033705 (2012), DOI: 10.1063/1.3693976.

[8] Haesen, S., Ortega, M., "Boost invariant marginally trapped surfaces in Minkowski 4-space," Class. Quantum Grav. 24, 5441–5452 (2007).

[9] Haesen, S., Ortega, M., "Marginally trapped surfaces in Minkowski 4-space invariant under a rotational subgroup of the Lorenz group," Gen. Relativ. Grav. 41, 1819–1834 (2009).

[10] Haesen, S., Ortega, M., "Screw invariant marginally trapped surfaces in Minkowski 4-space," J. Math. Anal. Appl. 355, 639–648 (2009).

[11] Lane, E., "Projective differential geometry of curves and surfaces," University of Chicago Press, Chicago, 1932.

[12] Little, J., "On singularities of submanifolds of higher dimensional Euclidean spaces," Ann. Mat. Pura Appl., IV Ser 83, 261–335 (1969).

[13] Liu H., Liu G., Hyperbolic rotation surfaces of constant mean curvature in 3-de Sitter space, Bull. Belg. Math. Soc., 2000, 7, 455–466.

[14] Liu H., Liu G., Weingarten rotation surfaces in 3-dimensional de Sitter space, J. Geom., 2004, 79, 156–168.

[15] Penrose, R. "Gravitational collapse and space-time singularities," Phys. Rev. Lett., 14, 57–59 (1965).