Semiclassical dynamics of Dirac and Weyl particles in rotating coordinates

Ömer F. Dayi, Eda Kilinçarslan and Elif Yunt

Physics Engineering Department, Faculty of Science and Letters, Istanbul Technical University, TR-34469, Maslak–Istanbul, Turkey

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Abstract

The semiclassical kinetic theory of Dirac particles in the presence of external electromagnetic fields and global rotation is established. To provide the Hamiltonian formulation of Dirac particles a symplectic two-form which is a matrix in spin indices is proposed. The particle number and current densities for the Dirac particles are acquired in the helicity basis. Following a similar procedure, semiclassical kinetic theory of the Weyl particles is accomplished. It is shown that the phase-space dynamics of the Weyl and Dirac particles is directly linked. The anomalous chiral effects due to the external electromagnetic fields and angular velocity of the frame are calculated.

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I. INTRODUCTION

Quantum mechanical calculations revealed that one cannot conserve both the vector and axial currents which are originated from spin-1/2 particles, even in the vanishing mass limit due to axial anomaly. In heavy ion collisions this anomaly can yield observable effects like the chiral magnetic effect [1–3] and the chiral separation effect [4, 5]. There are also similar anomalous effects due to global rotation of Fermi liquid which are known as the chiral vortical effect [6] and local (spin) polarization effect [7–9]. Experimental evidences of these anomalous chiral effects in relativistic heavy ion collisions were recently discussed in [10], where a complete list of references can be found.

The massless Dirac equation is invariant under chiral transformations. However, when the external electromagnetic fields are coupled to the massless spin-1/2 particles, chiral invariance is lost due to quantum mechanical effects. On the other hand when the Lorentz force on a particle of charge \( q \), and mass \( m \), is expressed in a rotating coordinate frame, the Coriolis term disappears if the angular velocity of the rotation \( \Omega \) and the magnetic field \( B \) are related as \( 2mc\Omega = -qB \). Hence, global rotations and magnetic fields generate similar effects as far as the centrifugal force is ignored. This resemblance can be extended to the massless fermion i.e. the Weyl particle. These one particle properties can be generalized to many particles within the kinetic theory. It is possible to incorporate chiral anomaly into the classical kinetic theory of chiral particles by adding the first order quantum corrections [11, 12]. This semiclassical approach yields an intuitive understanding of the chiral magnetic and chiral vortical effects. Semiclassical chiral kinetic theory was also studied within the Hamiltonian formalism using symplectic two-forms [13] by associating some classical variables to spin.

Noninteracting, massive spin-1/2 particles obey the Dirac equation which yields two Weyl equations in the massless case. These are relativistic systems where the energy can be positive or negative. Although a complete quantum mechanical wave packet should be formed by positive as well as negative energy solutions, to get an intuitive picture of some quantum phenomena, it is possible to consider wave packets which are composed of positive energy solutions only [14–17]. This is the semiclassical approach which we deal with. This formalism unlike the others include the Berry connection obtained from the free particle solutions. Within the semiclassical wave packet formalism the Berry gauge fields cease to be pure gauge fields, so that they yield a nonvanishing curvature. A combination of the differential form method of [13] with the semiclassical wave packet formalism was presented in [18]. In this approach the spin degrees of freedom are kept explicit, so that the symplectic two-form which is a matrix in spin indices is introduced. This method was
proved to be efficient in deriving the chiral magnetic effect and the chiral anomaly. Within this formalism some aspects of the semiclassical kinetic theory of the Dirac particles in the presence of electromagnetic fields was analyzed in [19] (for another approach see [20]). Moreover, it was demonstrated that the differential form formalism in terms of matrices in spin indices is crucial to express the spin Hall conductivity in terms of the topological spin Chern number of the systems obeying Dirac-like equations [21].

We would like to study the semiclassical kinetic theory of the Dirac and Weyl particles in the presence of the external electromagnetic fields in a uniformly rotating coordinate frame by keeping the usually ignored centrifugal force terms. There are several reasons of dealing with rotating coordinates. First of all, as it has been emphasized in [22] where the Dirac particle was studied in a rotating coordinate frame, the laboratories on Earth are affected by the rotation of Earth. By observing a fluid element in a frame rotating with respect to laboratory frame, one can study vortical effects [12]. From the high energy point of view it is important to analyze chiral particle currents which can produce anomalous chiral transport effects. However, some aspects of the semiclassical kinetic theory of chiral particles like the dispersion relation [23], can be established systematically by studying the massive case [24]. Because of considering both the external electromagnetic fields and global rotation we can keep track of the similarities and differences between the effects generated by them. Moreover, we can handle systematically how they affect each other and if they furnish some joint phenomena.

Spin-dependent interactions of Dirac particles may yield chiral imbalance of the chiral particles. Our method provides a direct relation between the spin-dependent interactions of Dirac fermions with the chiral particles. It constitutes the first step in the systematic study of interactions between the right- and left-handed fermions. On the other hand the semiclassical kinetic theory the Dirac particle in rotating coordinate frames may find applications in condensed matter systems where mechanical rotations of Dirac particles can generate spin currents [25, 26]. Obviously rotation which we consider is not an intrinsic property of the system, so that the rotating coordinate system which we consider is not necessarily Lorentz invariant.

The Dirac equation in the presence of external electromagnetic fields and mechanical rotation was also studied in [25, 26]. By considering the low-energy limit of the Dirac Hamiltonian they derived the related Pauli-Schrödinger equation which has only positive energy solutions given by the unit spinors $(1 0)^T$, $(0 1)^T$. In the absence of rotation, the semiclassical transport of fermions based on the Pauli-Schrödinger equation has been investigated in [16]. They showed that the formalism based on positive energy solutions of the Dirac equation and the one derived from the Pauli-
Schrödinger equation are different. In [26], wave packet formalism was not discussed. Nevertheless, they proposed semiclassical equations of motion in terms of the force they had calculated. To have an idea about the differences between their proposal and our approach, in Appendix we present the force arising in our approach.

Wigner functions are quantum mechanical analogs of classical distribution functions. The Wigner function constructed by Dirac spinor fields satisfies a quantum kinetic equation [27]. In [9], a solution of the quantum kinetic equation was obtained for weak external fields. Within this approach a Lorentz covariant Boltzmann (kinetic) equation was presented in [28] for chiral particles. By integrating it over the energy (zeroth component of the 4-momentum) they obtained a solution for the first time derivatives of three-dimensional phase-space variables. Although, first time derivatives of spatial coordinates which they propose are similar to the ones which we acquired, solutions for the measure and first time derivatives of momentum variables differ considerably as we will discuss in Sec. VII.

The free Dirac Hamiltonian does not commute with spin operator but it commutes with the helicity operator. Thus helicity is a conserved quantity under the time evolution generated by the free Dirac Hamiltonian. Hence it is natural to work in helicity basis to obtain the particle number and current densities. On the other hand chirality is equal to helicity for the massless Dirac equation. Thus, in the helicity basis the effects of imbalance between the right- and left-handed particles can be studied explicitly.

The semiclassical kinetic equations of Dirac and Weyl particles which we established by including the rotation of coordinates beginning from the nonrelativistic particles are novel. In fact, symplectic two forms which we introduce to construct the semiclassical equations by making use of differential form approach are new. We show that our results for the Weyl particles are consistent with the hydrodynamic approach when we deal with Boltzmann equation without collisions. However the main power of our kinetic equations will be clear when we deal with collisions which can directly be introduced within our approach for instance by adopting the relaxation time approximation. Only in the presence of collisions particle currents will acquire some new contributions. These are currently under consideration.

The paper is organized as follows. In the next section we discuss the main ingredients of our semiclassical approach. We present how the Berry curvature arises naturally in the semiclassical wave packet formalism. In Sec. III the symplectic matrix two-form which is suitable to establish semiclassical formulation of the Dirac particles coupled to external electromagnetic fields in rotating coordinates is presented. Sec. IV is devoted to obtain the related Hamiltonian including
terms at the order of Planck constant. In Sec. VA we derived continuity equation for the Dirac particles. Semiclassical kinetic theory of the Weyl particles coupled to electromagnetic fields in rotating coordinates is developed in Secs. VI and VII. In Sec. VIII we present anomalous chiral effects arising in our formalism. They are in accord with the chiral effects obtained within other formalisms. The results acquired and their possible applications are discussed in the last section.

II. SEMICLASSICAL APPROXIMATION AND THE BERRY CURVATURE

Our semiclassical approach is based on the two linearly independent positive energy solutions of the Dirac equation furnished by the Dirac Hamiltonian

$$H_D^{(4)}(p) = \beta m + \alpha \cdot p.$$  \hspace{1cm} (1)

We set the speed of light $c = 1$ and choose the following representation of $\beta$, $\alpha_i; \, i = 1, 2, 3$, matrices,

$$\alpha = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where $\sigma$ are the Pauli spin matrices. The semiclassical wave packet is defined by means of the positive energy solutions $u^{\alpha}(p)\); $\alpha = 1, 2$, as

$$\psi_x(p_c) = \sum_{\alpha} \xi_\alpha u^{\alpha}(p_c) e^{-i p_c \cdot x/\hbar}.$$  

The coefficients $\xi_\alpha$, are chosen to be constant. $x_c$, and $p_c$, denote the phase-space coordinates of wave packet center coinciding with the center of mass. We define the one-form $\eta_0$ through

$$\int [dx] \delta(x_c - x) \Psi^\dagger_x ( -i \hbar d - H_D^{(4)} dt ) \Psi_x = \sum_{\alpha, \beta} \xi^*_\alpha \eta_0^{\alpha \beta} \xi_\beta.$$  

$\eta_0$, which is a matrix in the “spin indices” $\alpha, \beta$, can be written as

$$\eta_0^{\alpha \beta} = -\delta^{\alpha \beta} x_c \cdot dp_c - A^{\alpha \beta} \cdot dp_c - H_D^{\alpha \beta} dt.$$  \hspace{1cm} (2)

Here $H_D^{\alpha \beta}$ is the projection of the Dirac Hamiltonian (1) on the positive energy solutions. Moreover, we introduced the matrix valued Berry gauge field

$$A^{\alpha \beta} = -i \hbar u^{(\alpha)}(p_c) \frac{\partial}{\partial p_c} u^{(\beta)}(p_c).$$  \hspace{1cm} (3)

By relabeling $(x_c, p_c) \rightarrow (x, p)$ and adding an exact differential term, the one-form (2) can be rewritten as

$$\eta_0 = p \cdot dx - A \cdot dp - H_D dt.$$

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Unless necessary the spin indices and the related unit matrix are suppressed. Before proceeding with the Hamiltonian formulation let there be external vector fields, like electromagnetic fields, which are provided by the gauge field \( a(x, p, t) \) and the scalar field \( \phi(x, p, t) \). The presence of external fields will also alter the starting Hamiltonian. Because of dealing with the semiclassical formulation up to first order in the Planck constant, the Hamiltonian would be decomposed as:

\[
H^{(4)} = H_0^{(4)} + \hbar H_1^{(4)}.
\]

Let us consider the first-order Hamiltonian formalism designated by the one-form

\[
\eta = p \cdot dx - A(p) \cdot dp + a(x, p, t) \cdot dx + \phi(x, p, t)dt - H(x, p, t)dt,
\]

where \( H^{(4)} \) denotes the projection of \( H^{(4)} \) on the positive energy solutions of the free Dirac equation. We need to introduce the related symplectic two-form to establish the Hamiltonian formulation. The Berry gauge field \( A(p) \), can be non-Abelian. Moreover, we also consider \( a(x, p, t) \), \( \phi(x, p, t) \), which can be noncommuting with \( A(p) \). Therefore we define the symplectic two-form matrix by

\[
\tilde{\omega}_t = d\eta = dt \frac{\partial \eta}{\partial t} + dx \cdot \frac{\partial \eta}{\partial x} + dp \cdot D\eta
\]

where we introduced the covariant derivative

\[
D \equiv \frac{\partial}{\partial p} + i \frac{\hbar}{2} [A,].
\]

By employing the one-form (4), we acquire

\[
\tilde{\omega}_t = dp_i \wedge dx_i + D_ia_j \ dp_i \wedge dx_j - G + F + \left( \frac{\partial \phi}{\partial x_i} - \frac{\partial a_i}{\partial t} \right) \ dx_i \wedge dt - \frac{\partial H}{\partial x_i} \ dx_i \wedge dt
\]

\[
+ D_i \phi \ dp_i \wedge dt - D_i H \ dp_i \wedge dt.
\]

As usual the repeated indices are summed over. \( F \) is the curvature two-form of the gauge field \( a \),

\[
F = \frac{1}{2} \left( \frac{\partial a_j}{\partial x_i} - \frac{\partial a_i}{\partial x_j} \right) dx_i \wedge dx_j,
\]

and the Berry curvature two-form \( G = \frac{1}{2} G_{ij} dp_i \wedge dp_j \) is defined through the covariant derivative,

\[
G_{ij} = -i\hbar [D_i, D_j] = \left( \frac{\partial A_j}{\partial p_i} - \frac{\partial A_i}{\partial p_j} + i \frac{\hbar}{2} [A_i, A_j] \right) = \epsilon_{ijk} G_k.
\]

Let us calculate the Berry gauge field \( A(p) \), and the Berry curvature \( \epsilon_{ijk} G_k \). The positive energy solutions of the Dirac equation can be written as

\[
u^\alpha(p) = U_0(p)u_\alpha^0,
\]

6
where \( u_0^1 = (1 \ 0 \ 0)^T \), \( u_0^2 = (0 \ 1 \ 0)^T \), are the rest frame solutions and \( U_0(p) \) is the Foldy-Wouthuysen transformation,

\[
U_0(p) = \frac{\beta H_D^{(1)}(p) + E}{\sqrt{2E(E + m)}}.
\]

\( E = \sqrt{p^2 + m^2} \) is the free relativistic energy. Now, the Berry gauge field (3) is expressed as

\[
A = -i\hbar I_+ U_0(p) \frac{\partial U_0^1(p)}{\partial p} I_+.
\]

\( I_+ \) projects onto the positive energy subspace. Hence, we acquire

\[
A = \frac{\hbar \sigma \times p}{2E(E + m)}.
\]

and by plugging it into (6) one establishes

\[
G = \frac{\hbar m^2}{2E^3} \left( \sigma + \frac{p(\sigma \cdot p)}{m(m + E)} \right).
\]

It furnishes the Berry curvature via \( G_{ij} = \epsilon_{ijk} G_k \).

### III. SYMPLECTIC TWO-FORM MATRIX IN ROTATING COORDINATES

We aim to study the Dirac particle in the presence of external electromagnetic fields, in the coordinate frame rotating with the constant angular velocity \( \Omega \). Nonrelativistic global rotations, i.e. which fulfill \( |\Omega \times \mathbf{x}| \ll c \), can be associated with the vector gauge field \( m(\Omega \times \mathbf{x}) \), and with the scalar gauge field \( \frac{m}{2} (\Omega \times \mathbf{x})^2 \), for a nonrelativistic particle of mass \( m \). We can extend them to our relativistic particle formulation by \( a^{\Omega} = \mathcal{E}(\Omega \times \mathbf{x}) \), and \( \phi^{\Omega} = \frac{\mathcal{E}}{2}(\Omega \times \mathbf{x})^2 \), where \( \mathcal{E} \equiv H \) is the dispersion relation which yields \( m \), in the nonrelativistic limit. We then set \( a = a^{EM} + a^{\Omega} \) and \( \phi = \phi^{EM} + \phi^{\Omega} \) in the one-form (4), so that we deal with the one-form matrix,

\[
\eta = p \cdot d\mathbf{x} - A \cdot dp + a^{EM} \cdot d\mathbf{x} + \mathcal{E}(\Omega \times \mathbf{x}) \cdot d\mathbf{x} + \phi^{EM} dt + \frac{\mathcal{E}}{2}(\Omega \times \mathbf{x})^2 dt - H dt.
\]

Equation (10) furnishes the symplectic two-form needed for the Hamiltonian formalism in rotating coordinates as

\[
\tilde{\omega}_t = dp_i \wedge dx_i + \frac{1}{2} \epsilon_{ijk}(qB_k + 2\mathcal{E}\Omega_k) \ dx_i \wedge dx_j - \frac{1}{2} \epsilon_{ijk}G_k \ dp_i \wedge dp_j
+ \epsilon_{ijk}x_j \Omega_k \nu_m dx_i \wedge dp_m - \nu_i \ dp_i \wedge dt + \frac{1}{2} \nu_i (\Omega \times \mathbf{x})^2 dp_i \wedge dt
+ [q\mathcal{E} + (\Omega \times \mathbf{x}) \times (qB + \mathcal{E}\Omega)]_i \ dx_i \wedge dt.
\]
The electromagnetic vector and scalar potentials, \(a^{EM}, \phi^{EM}\), are chosen appropriately to get the terms depending on the external electric and magnetic fields, \(E, B\), as in (11). Moreover, we introduced

\[ \nu = \frac{\partial H}{\partial p} + \frac{i}{\hbar} [A, H] \equiv DH. \]  

(12)

The Hamiltonian \(H \equiv \mathcal{E}\) will be presented in the next section. In (10) we generalized the nonrelativistic fields, \(m(\Omega \times x), \frac{q}{\hbar}(\Omega \times x)^2\), which depend explicitly on \(x\), by substituting \(m\), with \(\mathcal{E}\), so that the fourth and sixth terms appear in the generalized two-form (11). These terms are essential to construct the semiclassical kinetic equation correctly.

The equations of motion can be derived by imposing the condition

\[ i\tilde{v}\omega_t = 0, \]  

(13)

where \(i\tilde{v}\) denotes the interior product of the vector field

\[ \tilde{v} = \frac{\partial}{\partial t} + \hat{x}\frac{\partial}{\partial x} + \hat{p}\frac{\partial}{\partial p}. \]  

(14)

\((\hat{x}, \hat{p})\) are the matrix-valued time evolutions of the phase-space variables \((x, p)\). The equations of motion are deduced by making use of (11) in (13):

\[ \dot{x} + (\Omega \times x) \cdot \dot{x} \nu = \nu(1 - \frac{1}{2}(\Omega \times x)^2) + \hat{p} \times G, \]  

(15)

\[ \dot{p} + (\hat{p} \cdot \nu)\Omega \times x = qE + (\Omega \times x) \times (qB + \mathcal{E}\Omega) + \hat{x} \times (qB + 2\mathcal{E}\Omega). \]  

(16)

The second and third terms in the right-hand side of (16), respectively, encompass the centrifugal and Coriolis forces correctly. The left-hand sides of (15) and (16) resemble the Lorentz transformations of velocity \(\dot{x}\), and force \(\dot{p}\), to a reference frame moving with the velocity \(\nu = \Omega \times x\). To derive the explicit expression of \(\nu\), we need to be acquainted with the underlying Hamiltonian.

**IV. SEMICLASSICAL DIRAC HAMILTONIAN IN ROTATING COORDINATES**

To accomplish the semiclassical energy, we need to clarify the starting Hamiltonian in the classical phase-space variables \((x, p)\). The simplest choice is to deal with the free Dirac Hamiltonian. However, we are interested in the semiclassical approximation where the terms at the first order in \(\hbar\) have been retained. When the Dirac particle is subject to the external magnetic field \(B\), in [30] Bliokh suggested to add the \((-\frac{\hbar q}{2\hbar}\Sigma \cdot B)\) term to the Hamiltonian (11), where

\[ \Sigma = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}, \]
is the spin matrix. Observe that this additional term can be acquired from the magnetic moment-magnetic field interaction term of an electron by the substitution of mass \( m \), with the relativistic free energy \( E \). By making use of the analogy between magnetic field and angular velocity one can write the following Hamiltonian

\[
H^{(4)} = \beta m + \alpha \cdot p - \frac{g}{2} \Sigma \cdot \Omega - \frac{\hbar q}{2E} \Sigma \cdot B.
\] (17)

We introduced the constant \( g \), which should be set as \( g = 2 \), to keep the analogy between magnetic field and angular velocity which shows up in the Coriolis term in 16. However, the Dirac Hamiltonian in a rotating coordinate frame was established in 22, where the spin-angular velocity coupling term appears with \( g = 1 \). For the sake of generality, we retain \( g \).

The Hamiltonian (17) can be decomposed into two parts: \( H^{(4)} = H^{(4)}_D + hH^{(4)}_1 \). The semiclassical Hamiltonian is defined to be the projection of the Hamiltonian (17) onto the positive energy solutions. Instead of doing this calculation straightforwardly we would like to first block diagonalize \( H^{(4)} \) and then project it onto the positive energy subspace. In order to block diagonalize (17) up to \( h \) order, let us introduce the transformation \( U_0 + hU_1 \), yielding

\[
(U_0 + hU_1)(H^{(4)}_D + hH^{(4)}_1)(U_0 + hU_1)^\dagger = U_0 H^{(4)}_D U_0^\dagger + h(U_0 H^{(4)}_D U_1^\dagger + U_0 H^{(4)}_1 U_0^\dagger + U_1 H^{(4)}_1 U_0^\dagger) + O(h^2). \] (18)

\( U_0(p) \) denotes the Foldy-Wouthuysen transformation given in (8). It diagonalizes the Dirac Hamiltonian (11) and acts on the spin dependent part as

\[
U_0 H^{(4)}_1 U_0^\dagger = -\frac{m}{2E^2} (qB + gE\Omega) \cdot \Sigma - \frac{i}{2E^2} [\Sigma \cdot (qB + gE\Omega) \times \beta \alpha] - \frac{1}{2E^2(E + m)} \Sigma \cdot p (qB + gE\Omega) \cdot p.
\] (19)

To get rid of the off-diagonal blocks we choose

\[
U_1 = \frac{1}{2E^3 \sqrt{2E(E + m)}} \begin{pmatrix}
    i \sigma \cdot [p \times (qB + gE\Omega)] (\sigma \cdot p) & -i(E + m) \sigma \cdot [p \times (qB + gE\Omega)] \\
    -i(E + m) \sigma \cdot [p \times (qB + gE\Omega)] & -i \sigma \cdot [p \times (qB + gE\Omega)] (\sigma \cdot p)
\end{pmatrix}.
\]

One can indeed show that the off-diagonal blocks in (19) are canceled by \( (U_1 H^{(4)}_D U_0^\dagger + U_0 H^{(4)}_D U_1^\dagger) \). Therefore (18) provides the block-diagonal Hamiltonian in the rotating frame as follows

\[
H^{(4)} = E\beta - \frac{\hbar m}{2E^2} \Sigma \cdot (qB + gE\Omega) - \frac{\hbar}{2E^2(E + m)} (\Sigma \cdot p)(qB + gE\Omega) \cdot p.
\] (20)

For \( \Omega = 0 \), this block-diagonal Hamiltonian was established in 30, which had been found also in 31 within a systematic but somewhat complicated approach.

Projection of (20) onto the positive energy subspace furnishes the desired Hamiltonian:

\[
H = E - \frac{\hbar m}{2E^2} \sigma \cdot (qB + gE\Omega) - \frac{\hbar}{2E^2(E + m)} (\sigma \cdot p) p \cdot (qB + gE\Omega).
\] (21)
Observe that it is indeed the projection of the initial Hamiltonian, (17), onto the positive energy solutions of the Dirac equation. Equation (21) can be expressed as

$$H = E[1 - \mathbf{G} \cdot (q\mathbf{B} + gE\mathbf{\Omega})]$$

by employing the Berry curvature (9).

V. SEMICLASSICAL TRANSPORT OF DIRAC PARTICLES

To analyze the particle number conservation law we start with the volume form

$$\tilde{\Omega} = \frac{1}{3!} \tilde{\omega}_t \wedge \tilde{\omega}_t \wedge \tilde{\omega}_t \wedge dt$$

$$= \frac{1}{3!} \tilde{\omega} \wedge \tilde{\omega} \wedge \tilde{\omega} \wedge dt. \quad (23)$$

$$\tilde{\omega} \equiv \tilde{\omega}_t|_{dt=0}$$ is the matrix valued symplectic two-form in the ordinary phase-space whose coordinates are $$(x, p)$$. The volume form (23) can be expressed as

$$\tilde{\Omega} = \tilde{\omega}_{1/2} \ dV \wedge dt,$$  \quad (24)

where $$\tilde{\omega}_{1/2}$$ is the Pfaffian of the $$(6 \times 6)$$ matrix,

$$\begin{bmatrix}
\epsilon_{ijk}(q\mathbf{B}_k + 2E\mathbf{\Omega}_k) & -\delta_{ij} + \nu_j(x \times \mathbf{\Omega})_i \\
\delta_{ij} - \nu_i(x \times \mathbf{\Omega})_j & -\epsilon_{ijk}G_k
\end{bmatrix}. \quad (25)$$

We will accomplish the explicit form of the Pfaffian $$\tilde{\omega}_{1/2}$$ in the sequel. To attain the Liouville equation, we need to calculate the Lie derivative of the volume form which can be carried out in two different ways. One of them is to utilize the definition of the volume form in terms of the Pfaffian, (24):

$$\mathcal{L}_p \tilde{\Omega} = (i_p d + di_p)(\tilde{\omega}_{1/2} dV \wedge dt)$$

$$= \left( \frac{\partial \tilde{\omega}_{1/2}}{\partial t} + \frac{\partial}{\partial x} \cdot (\dot{x} \tilde{\omega}_{1/2}) + D \cdot (\tilde{\omega}_{1/2} \dot{p}) \right) dV \wedge dt. \quad (26)$$

The other way is to employ the definition of volume form (23) and directly compute its Lie derivative:

$$\mathcal{L}_p \tilde{\Omega} = (i_p d + di_p)(\frac{1}{3!} \tilde{\omega}^3 \wedge dt)$$

$$= \frac{1}{3!} d\tilde{\omega}^3. \quad (27)$$
Explicit calculation of $\tilde{\omega}_3$ and the comparison of (27) with (26), provide us the explicit form of Pfaffian and $\dot{\tilde{x}}$, which are the solutions of the equations of motion (15)-(16), in terms of the phase-space variables $(x, p)$, as

$$
\tilde{\omega}_{1/2} = 1 + G \cdot (qB + 2E\Omega) - \nu \cdot (x \times \Omega) - (\nu \cdot G)(qB \cdot (x \times \Omega)),
$$

$$
\dot{x}\tilde{\omega}_{1/2} = \nu(1 - \frac{1}{2}(\Omega \times x)^2) + \nu \cdot G
+ (\nu \cdot G)(qB + 2E\Omega)(1 - \frac{1}{2}(\Omega \times x)^2) + (\nu \cdot G)([x \times \Omega] \times e),
$$

$$
\tilde{\omega}_{1/2}\hat{p} = e + \nu \times (qB + 2E\Omega)(1 - \frac{1}{2}(\Omega \times x)^2)
+ G(e \cdot (qB + 2E\Omega)) - [(x \times \Omega) \times e] \times \nu.
$$

$e$ denotes the effective electric field:

$$
e = qE + (\Omega \times x) \times (qB + E\Omega).
$$

The second term in (30), reflects the fact that $2E\Omega$ behaves as an effective magnetic field in the classical limit. By plugging the semiclassical Hamiltonian (21), into the definition (12), one calculates $\nu$ as

$$
\nu = \frac{p}{E} \left[ 1 + 2G \cdot \left( qB + \frac{E}{2}\Omega \right) \right] - \frac{\hbar}{2E^3} (qB + gE\Omega)\sigma \cdot p.
$$

Direct computation of the Lie derivative, (27), leads to

$$
\left( \frac{1}{2}d\tilde{\omega}_t \wedge \tilde{\omega}_t^2 \right)_V = (1 - \frac{1}{2}(\Omega \times x)^2)(qB + 2E\Omega) \cdot (D \times \nu)
= (1 - \frac{1}{2}(\Omega \times x)^2)(qB + 2E\Omega) \cdot (i[G, H]/\hbar)
= (2 - g)\frac{mq\hbar}{2E^3}(1 - \frac{1}{2}(\Omega \times x)^2) \left( \sigma - \frac{\sigma \cdot p}{E(E + m)} p \right) \cdot (\Omega \times B)
$$

Details of this cumbersome calculation are given in Appendix A. The subscript $V$ indicates that the canonical volume form is factored out: $(\frac{1}{2}d\tilde{\omega}_t \wedge \tilde{\omega}_t^2)_V \equiv (\frac{1}{2}d\tilde{\omega}_t \wedge \tilde{\omega}_t^2)/d^3V \wedge dt$. The other subscript, $M$, denotes that the Maxwell equations in rotating coordinates, $\partial B/\partial t = -\nabla \times [E + (\Omega \times x) \times B]$, $\nabla \cdot B = 0$, are employed. The former equation was obtained e.g. in Appendix C of [26], under the condition $|\Omega \times x| \ll c$.

To reduce the Pfaffian and the dynamical equations, (28)-(30), to spin independent scalars we can take their trace. Then, the Liouville equation is satisfied since the trace of (33) vanishes: $\text{Tr}[d(\tilde{\omega}_t)^3] = 0$. Observe that if we choose $g = 2$, Liouville equation is satisfied identically, even before taking the trace of (33). Moreover, for most of the calculations involving distribution
functions one approximates $\nu \approx p/E$. In this case, $D \times \nu = 0$. Hence, we conclude that, for Dirac particles the Liouville equation is satisfied:

$$\mathcal{L}_\nu \tilde{\Omega} = 0. \quad (34)$$

To discuss particle number and current densities within our formalism, one has to introduce matrix valued distribution functions. Distribution functions which are matrices carrying spin indices show up naturally in the framework of spin dependent Fermi liquids as it was mentioned in Chapter 9 of [32]. The semiclassical formulation has been elaborated in the spin basis dictated by the positive energy solutions given in (7). In this basis the distribution function will be a matrix with nonvanishing off-diagonal elements which can hardly have a physical interpretation. However, we can get rid of them by working in the helicity basis rather than the spin basis of (7). Helicity operator commutes with the free Dirac Hamiltonian, so that it is a conserved quantity for the free Dirac particles. In the helicity basis, the distribution function can be expressed through the right-handed and left-handed distribution functions $f_R$ and $f_L$ as

$$f = \begin{pmatrix} f_R & 0 \\ 0 & f_L \end{pmatrix}.$$ 

This separation is also essential to establish the connection between the massive and massless cases.

The semiclassical helicity matrix is defined as

$$\lambda^{\alpha \beta} = u^{\alpha \dagger} \left( \frac{\Sigma \cdot p}{p} \right) u^\beta = \frac{\sigma \cdot p}{p}.$$ 

Employing the spherical coordinates in momentum space we can diagonalize $\lambda$ by the matrix

$$R = \begin{pmatrix} \cos \left( \frac{\theta}{2} \right) & -\sin \left( \frac{\theta}{2} \right) e^{-i\phi} \\ \sin \left( \frac{\theta}{2} \right) e^{i\phi} & \cos \left( \frac{\theta}{2} \right) \end{pmatrix}.$$ 

One can easily observe that $R^\dagger \lambda R = \text{diag} \ (1, -1)$.

To accomplish the continuity equation let us deal with the distribution function satisfying the collisionless Boltzmann equation in the helicity basis:

$$\left( \tilde{\omega}_{1/2} \right)_H \frac{\partial f}{\partial t} + (\tilde{x} \tilde{\omega}_{1/2} )_H \cdot \frac{\partial f}{\partial x} + (\tilde{\omega}_{1/2} \tilde{p})_H \cdot D_H f = 0. \quad (35)$$

The subscript $H$ denotes the matrices written in the helicity basis, like $(\tilde{\omega}_{1/2})_H \equiv R^\dagger \tilde{\omega}_{1/2} R$. By making use of (26) and (34), we obtain

$$\int \frac{d^3p}{(2\pi \hbar)^3} \left( \frac{\partial}{\partial t} ( \left( \tilde{\omega}_{1/2} \right)_H f ) + \frac{\partial}{\partial x} \cdot ( (\tilde{x} \tilde{\omega}_{1/2} )_H f ) + D^H \cdot ( (\tilde{\omega}_{1/2} \tilde{p})_H f ) \right) = 0.$$
The measure in phase-space integrals is proportional to the Pfaffian $\tilde{\omega}_{1/2}$, so that the probability density is $\rho(x,p,t) = (\tilde{\omega}_{1/2})_{Hf}$. Therefore, the particle number density and the particle current density are given by

$$n(x,t) = \int \frac{d^3p}{(2\pi\hbar)^3} \text{Tr}\left[(\tilde{\omega}_{1/2})_{Hf}\right], \quad (36)$$

$$j(x,t) = \int \frac{d^3p}{(2\pi\hbar)^3} \text{Tr}\left[(\dot{\tilde{\omega}}_{1/2})_{Hf}\right]. \quad (37)$$

By setting $\int \frac{d^3p}{(2\pi\hbar)^3} \mathbf{D} \cdot ((\tilde{\omega}_{1/2})_{Hf}) = 0$, the continuity equation follows,

$$\frac{\partial}{\partial t}n(x,t) + \nabla \cdot j(x,t) = 0.$$

Hence, the particle number is conserved. Obviously, to reach this conclusion, one can also work in the spin basis. However, the helicity basis is suitable to inspect the vanishing mass limit as well as to take into account chiral imbalance.

VI. SEMICLASSICAL FORMULATION OF WEYL PARTICLES

The massless limit of the Dirac equation leads to two equations describing the Weyl particles. After solving one of these two component Weyl equations, one can compute the related Berry gauge field. Then, to establish its semiclassical kinetic theory one proceeds as in the massive case. However, our formulation of the Dirac particles in the helicity basis directly provides the semiclassical kinetic theory of the Weyl particles either left- or right-handed. In fact by expressing (9) in the helicity basis and setting $m = 0$, we readily acquire the Berry curvature

$$G_R \equiv b = \hbar \frac{p}{2p^3}. \quad (38)$$

We focus on the right-handed Weyl particle. The symplectic two-form which is the main ingredient of the Hamiltonian approach is deduced from (11), as

$$\omega_t = dp_i \wedge dx_i + \frac{1}{2} \epsilon_{ijk}(qB_k + 2\mathcal{E}_0\Omega_k)dx_i \wedge dx_j + e_{ijk}\Omega_k x_j \nu_0 m dx_i \wedge dp_m$$

$$-\frac{1}{2} \epsilon_{ijk} b_k dp_i \wedge dp_j + \nu_0 (\Omega \times x)^2 dp_i \wedge dt$$

$$-\nu_0 dp_i \wedge dt + [qE + (\Omega \times x) \times (qB + \mathcal{E}_0\Omega)] dx_i \wedge dt. \quad (39)$$

$\nu_0$ is delivered by (32) after expressing it in the helicity basis and setting $m = 0$:

$$\nu_0 = \frac{p}{p} + gh \frac{p}{2p^3} (\Omega \cdot p) - gh \frac{\Omega}{2p} + hq \frac{p}{p^4} (B \cdot p) - hq \frac{B}{2p^2}. \quad (40)$$
On the other hand, υ₀ could also be defined by the semiclassical Weyl Hamiltonian $H₀$, which is established from (21) in the helicity basis as

$$H₀ ≡ \mathcal{E}_0 = p - gp^2(b \cdot \Omega) - qp(B \cdot b).$$

Indeed, $\nu_0 = \partial H_0/\partial p$ produces (40).

To derive the equations of motion let us introduce the vector field

$$v = \frac{\partial}{\partial t} + \dot{x} \frac{\partial}{\partial x} + \dot{p} \frac{\partial}{\partial p}.$$ (42)

The interior product of the vector field (42), with the symplectic two-form (39),

$$i_v \omega_t = 0,$$ (43)

yields the following coupled equations of motion

$$\dot{x} + (\Omega \times x) \cdot \dot{x} \nu_0 = \nu_0(1 - \frac{1}{2}(\Omega \times x)^2) + \dot{p} \times b,$$ (44)

$$\dot{p} + (\dot{p} \cdot \nu) \Omega \times x = qE + (\Omega \times x) \times (qB + \mathcal{E}_0 \Omega) + \dot{x} \times (qB + 2\mathcal{E}_0 \Omega).$$ (45)

These could also be obtained from the massive equations of motion (15), (16). It is worth noting that the second term in (45) is the centrifugal-like force for Weyl particles.

VII. SEMICLASSICAL TRANSPORT OF WEYL PARTICLES

To discuss particle number conservation for the Weyl particles, we need to attain the related Liouville equation which can be accomplished by calculating the Lie derivative of the volume form

$$\Omega = \frac{1}{3!} \omega_t^3 \wedge dt = \frac{1}{3!} \omega^3 \wedge dt,$$ (46)

where $\omega \equiv \omega_t|_{dt=0}$ is the symplectic form in the six-dimensional ordinary phase-space. As we have already seen in the massive case, by expressing the Lie derivative of volume form in two different ways and comparing them, one can solve the coupled equations of motion (44)–(45) for the velocities $(\dot{x}, \dot{p})$, in terms of the phase-space variables $(x, p)$. By making use of (43), the Lie derivative can be calculated as

$$\mathcal{L}_v \Omega = (i_v, d + di_v)(\frac{1}{3!} \omega_t^3 \wedge dt) = \frac{1}{3!} d \omega_t^3.$$ (47)
On the other hand by means of $\sqrt{\omega}$, which is the Pfaffian of the symplectic matrix (25), where $G$ is given by (38), the volume form (46) can be expressed as

$$\Omega = \sqrt{\omega} dV \wedge dt.$$  

(48)

Then the Lie derivative of (48) yields

$$L_v \Omega = \left( \frac{\partial \sqrt{\omega}}{\partial t} + \frac{\partial (\sqrt{\omega} \dot{x})}{\partial x} + \frac{\partial (\sqrt{\omega} \dot{p})}{\partial p} \right) dV \wedge dt.$$  

(49)

Now, by calculating $\omega_t^3$ and comparing (47) with (49), Pfaffian and the velocities of phase-space variables are revealed to be

$$\sqrt{\omega} = 1 + b \cdot (qB + 2p\Omega) - \nu_0 \cdot (x \times \Omega) - (\dot{p} \cdot b)(qB \cdot (x \times \Omega)),$$  

(50)

$$\sqrt{\omega} \dot{x} = \nu_0 (1 - \frac{1}{2}(\Omega \times x)^2) + e \times b$$

$$+ (\dot{p} \cdot b)(qB + 2p\Omega)(1 - \frac{1}{2}(\Omega \times x)^2) + (\dot{p} \cdot b)[(x \times \Omega) \times e],$$  

(51)

$$\sqrt{\omega} \dot{p} = e + \nu_0 \times (qB + 2p\Omega)(1 - \frac{1}{2}(\Omega \times x)^2)$$

$$+ b(e \cdot (qB + 2p\Omega)) - [(x \times \Omega) \times e] \times \nu_0.$$  

(52)

$e$ is given as in (31) for $m = 0$ : $e = qE + (\Omega \times x) \times (qB + \varepsilon_0 \Omega)$. These solutions can also be detected from the massive ones (28)-(30) by setting $m = 0$ and retaining the right-handed part in the helicity basis.

Some other solutions for velocities of phase-space variables and the Pfaffian were proposed in [28]. They start from a Lorentz invariant quantum Boltzmann equation and reduce it to three-dimensions. If we ignore centrifugal terms and the last term, (51) coincides with $\sqrt{\omega} \dot{x}$, presented in [28]. However, the others differ. In our formalism the symmetry between $2p\Omega$ and $B$ is respected up to centrifugal terms; however, this is not the case in [28]: When we switch off the electric field $E$, and ignore centrifugal terms, (52) leads to $\sqrt{\omega} \dot{p} = \dot{p} \times (qB + 2p\Omega)$. This reflects the fact that there is a Coriolis force-like term for Weyl particles which vanishes for $2p\Omega = -qB$, similar to the Dirac particles. But $\sqrt{\omega} \dot{p}$, acquired in [28] yields only the magnetic force for vanishing electric field. For $\Omega = 0$, either our solutions (50)-(52) or the ones given in [28] reproduce the solutions obtained in [12].

The Lie derivative of the volume form can be calculated either by inserting the solutions (50)-(52) into (49) or in terms of the symplectic form (39) as

$$\left( \frac{1}{2} dw_t \wedge w_t \right)_{YM} = 2\pi \hbar q^2 \delta(p) E \cdot B + 4\pi \hbar q \delta(p) E \cdot (p \Omega) - 2\pi \hbar q \delta(p) B \cdot ((\Omega \times x) \times p \Omega)$$

$$= 2\pi \hbar q^2 \delta(p) E \cdot B.$$  

(53)
In Appendix A we presented the details of this calculation. The subscript $VM$ denotes that the canonical volume form $dV \wedge dt$ is factored out and Maxwell equations $\nabla_x \cdot B = 0$ and $\nabla_x \times e = -q\partial B/\partial t$, have been employed. To exhibit the role of monopole located at the origin we presented the second and third terms which are actually vanishing. We conclude that the Liouville equation is anomalous.

To inspect the kinetic theory let us introduce the distribution function for the right-handed fermions, $f_R$, satisfying the Boltzmann equation without collisions:

$$\frac{df_R}{dt} = \frac{\partial f_R}{\partial t} + \frac{\partial f_R}{\partial x} \cdot \dot{x} + \frac{\partial f_R}{\partial p} \cdot \dot{p} = 0.$$ 

Therefore we get

$$\int \frac{d^3p}{(2\pi\hbar)^3} \left( \frac{\partial}{\partial t}(\sqrt{\omega}f_R) + \frac{\partial}{\partial x}(\sqrt{\omega}\dot{x}f_R) + \frac{\partial}{\partial p}(\sqrt{\omega}\dot{p}f_R) \right) = \int \frac{d^3p}{(2\pi\hbar)^3} \left( \frac{1}{2} d\omega_l \wedge \omega^2_{VM} f_R. \right)$$

Measure of the phase-space integrals differs from the canonical value up to the Pfaffian $\sqrt{\omega}$. Hence, particle number and current densities are defined by

$$n_R(x,t) = \int \frac{d^3p}{(2\pi\hbar)^3} \sqrt{\omega}f_R, \quad (54)$$

$$j_R(x,t) = \int \frac{d^3p}{(2\pi\hbar)^3} \sqrt{\omega}\dot{x}f_R. \quad (55)$$

The 4-divergence of the particle 4-current $(n_R, j_R)$ can be written as

$$\frac{\partial n_R}{\partial t} + \nabla \cdot j_R = \int \frac{d^3p}{(2\pi\hbar)^3} \left( \frac{1}{2} d\omega_l \wedge \omega^2_{VM} f_R. \right) \quad (56)$$

We then conclude that Weyl particles satisfy the following continuity equation with source,

$$\frac{\partial n_R(x,t)}{\partial t} + \nabla \cdot j_R(x,t) = \frac{q^2}{4\pi^2\hbar^2} E \cdot B. \quad (57)$$

Following the same procedure for the left-handed fermions we accomplish

$$\frac{\partial n_L(x,t)}{\partial t} + \nabla \cdot j_L(x,t) = -\frac{q^2}{4\pi^2\hbar^2} E \cdot B. \quad (58)$$

Observe that (36)-(37) yield (54)-(55) in the vanishing mass limit. The continuity equations (57) and (58) are consistent with the results of generic chiral hydrodynamics accomplished in [6].

VIII. THE ANOMALOUS CHIRAL EFFECTS

The anomalous chiral transport effects and the related experimental results were recently reviewed in [10]. In the light of recent experiments it is asserted that most probably the anomalous
chiral effects show up in heavy ion collisions. Let us analyze how our semiclassical approach produces the anomalous chiral effects due to external electromagnetic fields and global rotation.

We can readily consider the $m = 0$ limit of the current generated by Dirac particles in the helicity basis (37), which can be written as

$$j = \begin{pmatrix} j_R & 0 \\ 0 & j_L \end{pmatrix}.$$ 

One can observe that $j_R$, coincides with the current (55), which is given by

$$j_R = \int \frac{d^3p}{(2\pi\hbar)^3} \left( \nu_0 (1 - \frac{1}{2} (\Omega \times x)^2) + e \times b \right. \\
+ \left. (\nu_0 \cdot b) (qB + 2p\Omega) (1 - \frac{1}{2} (\Omega \times x)^2) + (\nu_0 \cdot b) [(x \times \Omega) \times e] \right) f_R.$$ 

Similarly we can derive the current for the particles of positive helicity as

$$j_L = \int \frac{d^3p}{(2\pi\hbar)^3} \left( \nu_0^L (1 - \frac{1}{2} (\Omega \times x)^2) - e \times b \right. \\
- \left. (\nu_0^L \cdot b) (qB + 2p\Omega) (1 - \frac{1}{2} (\Omega \times x)^2) - (\nu_0^L \cdot b) [(x \times \Omega) \times e] \right) f_R.$$ 

For the left-handed fermions we introduced

$$\nu_0^L = \frac{p}{\hbar} - g \hbar \frac{p}{2p^2} (\Omega \cdot p) + gh \frac{\Omega}{2p} - hq \frac{p}{p^3} (B \cdot p) + hq \frac{B}{2p^2}.$$ 

In terms of the right- and left-handed particle number current densities $j_R$ and $j_L$, one defines the vector and axial currents:

$$\dot{j}_V = j_R + j_L, \quad \dot{j}_A = j_R - j_L.$$ 

Let us deal with the right- and left-handed fermions obeying Fermi-Dirac distribution whose respective chemical potentials are denoted by $\mu_R$, and $\mu_L : f_{R(L)} = f_{FD}(E, \mu_{R(L)})$. We ignore quantum corrections to the energy (11), so that we can perform the integrals in (59) and (60) by setting $\nu_0 = \nu_0^L = \hat{p}$. In the integrals we set the surface terms equal to zero and make use of the approximation [33] valid for a well-behaved function $F(E)$:

$$\int F(E) \frac{\partial f_{FD}(E, \mu_{R(L)})}{\partial E} dE \approx -F(\mu_{R(L)}) - \frac{1}{6} \pi^2 T^2 \frac{\partial^2 F(E)}{\partial E^2} |_{E = \mu_{R(L)}}.$$ 

The chiral magnetic effect and the chiral separation effect are generated by the terms which are proportional to the magnetic field $B$, respectively, in the vector and axial currents. They are calculated to be

$$\dot{j}_V^{CME} = \frac{q}{2\pi^2 \hbar^2} \mu_5 B,$$

$$\dot{j}_A^{CSE} = \frac{q}{2\pi^2 \hbar^2} \mu B.$$
We introduced the total chemical potential $\mu = \frac{1}{2}(\mu_R + \mu_L)$ and the chiral chemical potential $\mu_5 = \frac{1}{2}(\mu_R - \mu_L)$. On the other hand, the terms which are proportional to the angular velocity $\Omega$, in the vector and axial currents, respectively, generate the chiral vortical effect and the local (spin) polarization effect. They are obtained as

$$J_{CV}^{VE} = \frac{\mu \mu_5}{\pi^2 \hbar^2} \Omega$$
$$J_{LP}^{PE} = \left\{ \frac{1}{2\pi^2 \hbar^2} (\mu^2 + \mu_5^2) + \frac{T^2}{6\hbar^2} \right\} \Omega.$$ 

These are in accord with the results obtained within other approaches (see [10] and the references therein). Temperature dependence has been obtained e.g. in [34] which was then shown to be related to the mixed gauge-gravity anomaly [35].

IX. DISCUSSIONS

The semiclassical formalism which we presented provides an intuitive understanding of the transport phenomena of the Dirac and Weyl particles in the presence of the external electromagnetic fields as well as rotation of the coordinate frame. It delivers the anomalous chiral effects straightforwardly. We considered noninteracting particles, though the kinetic theory is powerful in studying transport equations in the presence of interactions [36], [37]. The results obtained here should be considered as the first step in that direction. Our formulation of the Dirac particles has the advantage of exhibiting spin degrees of freedom explicitly, thus it is adequate to deal with spin dependent interactions. Therefore the exposed straightforward connection between the massive and massless cases can give clues about the systematic study of chirality imbalance.

It is known that rotations of coordinates generate spin currents due to spin-rotation coupling [26], [38], [39]. For condensed matter systems spin currents are mostly studied by considering the third component of spin, although generally spin is not a conserved quantity. However the helicity basis is suitable for analyzing spin currents and calculating the related spin Hall conductivities [21]. Therefore the semiclassical kinetic theory of the Dirac particles developed here will yield a better understanding of spin currents generated by rotations of coordinate frames.

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Appendix A: The Lie Derivatives of Volume Forms for Dirac and Weyl Particles

The Lie derivative of volume form suitable to Dirac particles can be computed directly as

$$\mathcal{L}_\xi \tilde{\Omega} = \frac{1}{2} d\tilde{\omega}_t \wedge \tilde{\omega}_t \wedge \tilde{\omega}_t. \quad (A.1)$$

When the exterior derivative of the symplectic two-form \( \Omega \) is taken there are some terms which cancel each other: (i) the term that includes the covariant derivative of the effective electric field, \( D_j e_i \; dp_j \wedge dx_i \wedge dt \) and \( \nu_i (\Omega \times \mathbf{x})_j dx_j \wedge dp_i \wedge dt \), which is the spatial derivative of \( \frac{1}{2} (\Omega \times \mathbf{x})^2 dp_i \wedge dt \), term in \( \tilde{\omega}_t \), cancel each other, and (ii) \( D_t \mathcal{E} \epsilon_{ijk} \Omega_k dp_l \wedge dx_i \wedge dx_j \) cancels out \( \epsilon_{ijk} \Omega_j \nu_j dx_k \wedge dp_j \wedge dx_i \), which arises from the spatial derivative of \( \nu_m (\Omega \times \mathbf{x})_i dp_m \wedge dx_i \) term in \( \tilde{\omega}_t \). Thus, the only nonvanishing terms are

$$d\tilde{\omega}_t = -\frac{1}{2} \epsilon_{ijk} D_i G_k \; dp_l \wedge dp_i \wedge dp_j - \left( 1 - \frac{1}{2} (\Omega \times \mathbf{x})^2 \right) D_i \nu_j dp_k \wedge dp_i \wedge dt$$

$$+ \frac{q}{2} \epsilon_{ijk} \frac{\partial B_k}{\partial x_l} \; dx_i \wedge dx_l \wedge dx_j + D_i \nu_j \epsilon_{kml} \Omega_l x_m dp_i \wedge dp_j \wedge dx_k$$

$$+ \frac{\partial \nu_i}{\partial x_k} dx_k \wedge dx_l \wedge dt + \frac{q}{2} \epsilon_{ijk} \frac{\partial B_k}{\partial t} \; dx_i \wedge dx_j \wedge dt.$$

In the wedge product of \( d\tilde{\omega}_t \) with \( \omega_t \wedge \tilde{\omega}_t \), most of the terms vanish due to antisymmetry of the wedge product. Thus the nonvanishing terms of \( (A.1) \) are as follows,

$$\frac{1}{2} d\tilde{\omega}_t \wedge \tilde{\omega}_t^2 = \left\{ (D \cdot G)(B + 2 \mathcal{E} \Omega) \cdot e + \left[ 1 - \frac{1}{2} (\Omega \times \mathbf{x})^2 \right] (D \times \nu) \cdot (B + 2 \mathcal{E} \Omega) \right.$$  
$$+ (\nabla \cdot B)(\nu \cdot G) + (q \frac{\partial B}{\partial t} + \nabla \times e) \cdot [G + (\nu \cdot G) (\Omega \times \mathbf{x})] \right\} d^3 V \wedge dt. \quad (A.2)$$

The first term vanishes due to \( D \cdot G = 0 \). Using the Maxwell equations in rotating coordinates, the last line also vanishes. Then, we conclude that \( (A.2) \) leads to \( (33) \):

$$\frac{1}{2} d\tilde{\omega}_t \wedge \tilde{\omega}_t^2 = \left( 1 - \frac{1}{2} (\Omega \times \mathbf{x})^2 \right) (qB + 2E \Omega) \cdot (D \times \nu) \right\} d^3 V \wedge dt.$$

Similar calculations can be done for the massless case by using the symplectic two-form \( \Omega \). While taking the exterior derivative of \( \Omega \), cancellations mentioned before \( \Omega \) occur in the massless case, too. Thus the exterior derivative of \( \Omega \) leads to

$$d\tilde{\omega}_t = -\frac{1}{2} \frac{\partial b_k}{\partial p_i} \epsilon_{ijk} \; dp_l \wedge dp_i \wedge dp_j - \frac{\partial \nu_i q}{\partial p_k} \left( 1 - \frac{1}{2} (\Omega \times \mathbf{x})^2 \right) \; dp_k \wedge dp_i \wedge dt$$

$$+ \frac{q}{2} \epsilon_{ijk} \frac{\partial B_k}{\partial x_n} \; dx_n \wedge dx_i \wedge dx_j + \frac{\partial \nu_i q}{\partial p_k} \epsilon_{lmn} \Omega_m x_n dp_k \wedge dp_i \wedge dx_l$$

$$+ \frac{\partial \nu_i}{\partial x_k} dx_k \wedge dx_i \wedge dt + \frac{q}{2} \epsilon_{ijk} \frac{\partial B_k}{\partial t} \; dx_i \wedge dx_j \wedge dt. \quad (A.3)$$
Both of the \( \partial \nu_{ai}/\partial p_k \) terms in (A.3), multiplied with the terms containing the four-form \( dx^3 dp \) in \( \omega_t \wedge \omega_t \), vanish due to antisymmetry of wedge product. Hence, the Lie derivative of the volume form for Weyl particles is obtained as

\[
L_v \Omega = e \cdot (qB + 2p\Omega)(\nabla \cdot b) + (\nabla \cdot B)(\nu_0 \cdot b) + (\nabla \times e + \frac{\partial B}{\partial t}) \cdot \{b - (x \times \Omega)(\nu_0 \cdot b)\} d^3V \wedge dt. \tag{A.4}
\]

Obviously, the last two terms in (A.4) vanish when the Maxwell equations in rotating coordinates are satisfied. Finally, we obtain

\[
\frac{1}{2} d\omega_t \wedge \omega_t^2 = 2\pi \hbar q^2 \delta(p) E \cdot B d^3V \wedge dt,
\]

which is the result given in (53).

**Appendix B: Comparison With the Pauli-Schroedinger Hamiltonian Approach**

We would like to compare the force which we acquired in terms of the wave packet composed of the positive energy solutions of the Dirac equation with the one presented in [26], where the Pauli-Schrödinger Hamiltonian was employed. By ignoring the terms at \( \hbar^2 \) order, they accomplished the following force,

\[
F = qE' + qv \times B + 2mv \times \Omega - m\Omega \times (\Omega \times x) - \frac{q\hbar}{4m^2}(\sigma \times E') \times (qB + m\Omega) - ((qB + m\Omega) \times \sigma) \times E' + \frac{q\hbar}{4m}(\sigma \cdot (\Omega \times v)B - 2(\Omega \cdot B)\sigma \times v - (B \cdot v)\sigma \times \Omega + \Omega \cdot (x \times B)\sigma \times \Omega - (B \cdot \Omega)\sigma \times (\Omega \times x)), \tag{B.1}
\]

where \( E' = E + (\Omega \times x) \times B \) is the effective electric field in a rotating frame when angular speed is much smaller than the speed of light.

Within our approach the force acting on the Dirac particle in nonrelativistic limit can be established from (30) ignoring the last term which gives rise to higher-order terms in momentum. We also ignore the term proportional to \( (\Omega \times x)^2 \), which is irrelevant for the comparison with (B.1). We invert the Pfaffian at the first order in \( \hbar \) as \( \tilde{\omega}^{-1} \approx 1 - G \cdot (qB + 2E\Omega) \), neglecting the last two terms in (28) since they are higher order terms in momentum. Thus, we acquire the force as

\[
\dot{p} = e + \nu \times (qB + 2E\Omega) - (qB + 2E\Omega) \times (G \times e) + G \cdot (qB + 2E\Omega)[\nu \times (qB + 2E\Omega)].
\]
In the limit $E \to m$, and setting the velocity $v = p/m$, it leads to

\[
\dot{p} = qE' + qv \times B + 2mv \times \Omega - m\Omega \times (\Omega \times x) - \frac{qh}{2m^2}(\sigma \times E') \times (qB + 2m\Omega)
\]

\[
-\frac{qh}{2m} \left[ 2(\sigma \cdot (\Omega \times v))B + g(\Omega \cdot B)\sigma \times v - g(B \cdot v)\sigma \times \Omega - (2 - g)(\sigma \cdot (B \times v))\Omega \right.
\]

\[
+ (2 - g)v(\sigma \cdot (B \times \Omega)) - \Omega \cdot (x \times B)\sigma \times \Omega + (B \cdot \Omega)\sigma \times (\Omega \times x) \bigg] + \frac{qh}{2m^2}(\sigma \cdot B)v \times B + h(g - 2)(\sigma \cdot \Omega)v \times \Omega + \frac{qh}{2m} \left[ (\Omega \times (\Omega \times x)) \times (\sigma \times B) \right]
\]

\[
+h\Omega \times \left[ \sigma \times ((\Omega \times x) \times \Omega) \right] + \frac{h}{4m^2} \sigma \cdot v(qv \cdot B + m(g - 2)v \cdot \Omega)v \times (qB + 2m\Omega)
\]

Neglecting higher-order terms in the velocity to compare with (B.1), we obtain

\[
\dot{p} = qE' + qv \times B + 2mv \times \Omega - m\Omega \times (\Omega \times x) - \frac{qh}{2m^2}(\sigma \times E') \times (qB + 2m\Omega)
\]

\[
-\frac{qh}{2m} \left[ 2(\sigma \cdot (\Omega \times v))B + g(\Omega \cdot B)\sigma \times v - g(B \cdot v)\sigma \times \Omega - (2 - g)(\sigma \cdot (B \times v))\Omega \right.
\]

\[
+ (2 - g)v(\sigma \cdot (B \times \Omega)) - \Omega \cdot (x \times B)\sigma \times \Omega + (B \cdot \Omega)\sigma \times (\Omega \times x) \bigg] + \frac{qh}{2m^2}(\sigma \cdot B)v \times B + h(g - 2)(\sigma \cdot \Omega)v \times \Omega + \frac{qh}{2m} \left[ (\Omega \times (\Omega \times x)) \times (\sigma \times B) \right]
\]

The terms at the order $h^0$ coincide. However, the terms dependent on the centrifugal force dependent terms differ. Although neither $g = 1$ nor $g = 2$ yield an exact match between (B.1) and (B.2), choosing $g = 2$, yields a better match.

In [26], the effects of impurity scatterings were discussed by considering the Dirac particles moving in the plane perpendicular to $\Omega$, which is in the same direction with $B$, by neglecting higher-order terms in $\Omega$ and taking $qB \gg m\Omega$. Under these conditions our formulation for $E = 0$ leads to

\[
\dot{p} = q(B \cdot \Omega)x - \frac{qh}{2m^2}(\sigma \cdot B)(B \cdot \Omega)x + qv \times B,
\]

which has the same form with the one obtained from (B.1), except the second term is smaller by a factor of two. This discrepancy between the semiclassical formulations established by making use of the Dirac and Pauli wave packets, has already been mentioned in [16].

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