Applications of Riemann–Liouville Fractional Integral of $q$-Hypergeometric Function for Obtaining Fuzzy Differential Sandwich Results

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Abstract: Studies regarding the two dual notions are conducted in this paper using Riemann–Liouville fractional integral of $q$-hypergeometric function for obtaining certain fuzzy differential subordinations and superordinations. Fuzzy best dominants and fuzzy best subordinants are given in the theorems investigating fuzzy differential subordinations and superordinations, respectively. Moreover, corollaries are stated by considering particular functions with known geometric properties as fuzzy best dominant and fuzzy best subordinant in the proved results. The study is completed by stating fuzzy differential sandwich theorems followed by related corollaries combining the results previously established concerning fuzzy differential subordinations and superordinations.

Keywords: Riemann–Liouville fractional integral; $q$-hypergeometric function; sandwich-type theorem; fuzzy differential subordination; fuzzy differential superordination

MSC: 30C45; 30A10; 33C15

1. Introduction

Fractional calculus and $q$-calculus have been widely used in studies regarding geometric function theory. The famous notion of subordination was interpreted in the fuzzy concept in 2011 [10] and fuzzy differential subordination was introduced in 2012 [11] as an extension of the classical notion of differential subordination due to Miller and Mocanu [12,13]. The

The notion of fuzzy set introduced by Lotfi A. Zadeh in 1965 [2] has tremendous applications in various fields of science and technique. Extensions of many branches of mathematics have developed in the fuzzy context created by introducing the notion of fuzzy set into the studies. Applications of the notion related to integro-differential equations can be seen in [3,4]. Aspects of fuzzy linear fractional programming embedding the concept are presented in [5]. Proportional Integral Derivative (PID) and Fuzzy-PID control are used in [6] and an evolution of the concept of fuzzy normed linear spaces is presented in [7]. Some highlights of the applications of the fuzzy set concept can be read in [8,9]. The

The famous notion of subordination was interpreted in the fuzzy concept in 2011 [10] and fuzzy differential subordination was introduced in 2012 [11] as an extension of the classical notion of differential subordination due to Miller and Mocanu [12,13]. The
theory started to develop in the next years [14–16] and the dual notion of fuzzy differential superordination was introduced in 2017 [17]. Later, studies continued to emerge using both theories and different types of operators and various notions familiar to the classical theory of differential subordination [18–20]. Some highlights on the development of the theories of fuzzy differential subordination and superordination can be read in [21].

In recent studies, fractional operators have been defined and used in correlation to fuzzy differential subordinations and superordinations. Fractional integral of confluent hypergeometric function is applied for obtaining new results in [22–24], fractional integral of Gaussian hypergeometric function is used in [25], fractional derivative is associated to the study of fuzzy differential subordinations in [26], Riemann–Liouville fractional integral of Ruscheweyh and Sălăgean operators helps in obtaining new results in [27] and Wanans operator is associated to fractional calculus aspects for obtaining fuzzy differential subordinations in [28].

Another step in the development of the theories of fuzzy differential subordination and superordination is taken in the present paper by adding quantum calculus aspects alongside fractional calculus.

Quantum calculus has a long history related to mathematical fields. Being known as the study of calculus without limits, it has many applications related to combinatorics [29], orthogonal polynomials [30–32], number theory [33] or basic hypergeometric functions [34]. Certain aspects regarding fundamentals of \(q\)-calculus and how it was adapted to different theories can be seen in [35–37]. In 1911, Jackson introduced the notions of \(q\)-derivative [38] and \(q\)-integral [39]. The first applications of \(q\)-calculus in geometric function theory are seen in [40] where the authors define the class of \(q\)-starlike functions. However, the favorable context for the development of studies involving \(q\)-calculus in geometric function theory is created by the publication of the book chapter authored by Srivastava [41]. It is the same book chapter which mentions \(q\)-hypergeometric function as potentially important for future studies connecting \(q\)-calculus to univalent functions.

Geometric function theory uses many \(q\)-operators for obtaining geometrical interpretations of certain special classes of functions. For instance, the famous the \(q\)-analogue of the Ruscheweyh and Sălăgean differential operators were defined [42,43], and many applications occurred [44–46]. A comprehensive indexing of \(q\)-operators is listed by Srivastava in a recent review paper [1]. Many applications of \(q\)-hypergeometric function in geometric function theory can be listed. A good description of the use of \(q\)-hypergeometric function in early studies from geometric function theory is given in [47] and new aspects are shown in [48]. Some results on applications of \(q\)-hypergeometric functions could be mentioned as [49–52]. The research presented in this paper connects an operator defined as Riemann–Liouville fractional integral of \(q\)-hypergeometric function with fuzzy differential subordination and superordination theory. Before presenting the operator and the fuzzy context in which the study is conducted, the basic notations and notions familiar to geometric function theory are recalled.

With \(U = \{z \in \mathbb{C} : |z| < 1\}\) denoting the unit disc of the complex plane, the class of holomorphic functions in \(U\) is denoted by \(\mathcal{H}(U)\). Certain important subclasses of \(\mathcal{H}(U)\) are defined as:

\[
\mathcal{A}_n = \left\{ f \in \mathcal{H}(U) : f(z) = z + a_n z^{n+1} + \ldots, z \in U \right\},
\]

with \(\mathcal{A}_1 = \mathcal{A}\), and

\[
\mathcal{H}(a,n) = \left\{ f \in \mathcal{H}(U) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots, z \in U \right\},
\]

with \(a \in \mathbb{C}, \ n \in \mathbb{N}^*\).

The important tool represented by Riemann–Liouville fractional integral is defined as presented in [53,54]:
We note that (5) differential subordination is satisfied such that \( h \prec f \) function \( f \) is fuzzy subordinate to \( g \), written \( f \prec g \), following form, after a simple calculation:

\[
D_z^{-\lambda}(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(t)}{(z-t)^{1-\lambda}} dt.
\]

The \( q \)-hypergeometric function is given by:

**Definition 2** ([55]). The \( q \)-hypergeometric function \( \phi(m, n; q, z) \) is defined by

\[
\phi(m, n; q, z) = \sum_{k=0}^{\infty} \frac{(m, q)_k}{(n, q)_k (q, q)_k} z^k,
\]

where

\[
(m, q)_k = \begin{cases} 1, & k = 0, \\ (1-m)(1-mq)(1-mq^2)\ldots(1-mq^{k-1}), & k \in \mathbb{N}, \end{cases}
\]

and \( 0 < q < 1 \).

Using Definitions 1 and 2, Riemann–Liouville fractional integral of \( q \)-hypergeometric function in introduced as:

**Definition 3** ([56]). Let \( m, n \) be complex numbers with \( m \neq 0, -1, -2, \ldots \) and \( \lambda > 0 \), \( 0 < q < 1 \). We define the Riemann–Liouville fractional integral of \( q \)-hypergeometric function

\[
D_z^{-\lambda} \phi(m, n; q, z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{\phi(m, n; q, t)}{(z-t)^{1-\lambda}} dt = \frac{1}{\Gamma(\lambda)} \sum_{k=0}^{\infty} \frac{(m, q)_k}{(n, q)_k (q, q)_k} t^k \int_0^z \frac{t^k}{(z-t)^{1-\lambda}} dt.
\]

The Riemann–Liouville fractional integral of \( q \)-hypergeometric function has the following form, after a simple calculation:

\[
D_z^{-\lambda} \phi(m, n; q, z) = \sum_{k=0}^{\infty} \frac{(m, q)_k}{(n, q)_k (q, q)_k} (k+1)_\lambda z^{k+1}.
\]

We note that \( D_z^{-\lambda} \phi(m, n; q, z) \in H[0, \lambda] \).

Next, the fuzzy context where the research was conducted is shown.

**Definition 4** ([10]). Fuzzy subset of \( X \) is a pair \((A, F_A)\), with \( F_A : X \to [0, 1] \) and \( A = \{ x \in X : 0 < F_A(x) \leq 1 \} \). The support of the fuzzy set \((A, F_A)\) is the set \( A \) and the membership function of \((A, F_A)\) is \( F_A \). It is denoted \( A = \text{supp}(A, F_A) \).

**Definition 5** ([10]). Consider \( D \subset \mathbb{C} \), the functions \( f, g \in H(D) \) and \( z_0 \in D \) a fixed point. The function \( f \) is fuzzy subordinate to \( g \), written \( f \prec_g f \), if the following conditions are satisfied:

1. \( f(z_0) = g(z_0) \),
2. \( F_{f(D)}(z) \leq F_{g(D)}(z), \quad z \in D. \)

**Definition 6** ([11], Definition 2.2). Consider \( h \) a univalent function in \( U \) and \( \psi : \mathbb{C}^3 \times U \to \mathbb{C} \), such that \( h(0) = \psi(a, 0, 0) = a \). When \( p \) is analytic in \( U \), such that \( p(0) = a \) and the fuzzy differential subordination is satisfied

\[
F_{\psi(C^3 \times U)}(\psi(p(z), zp'(z), z^2p''(z), z) \leq F_{h(U)}(h(z), z \in U,
\]

\[
(3)
\]
then $p$ is a fuzzy solution of the fuzzy differential subordination. The univalent function $q$ is a fuzzy dominant of the fuzzy solutions of the fuzzy differential subordination, if $F_{p(U)}(z) \leq F_{q(U)}(z)$, $z \in U$, for all $p$ satisfying (3). A fuzzy dominant $\tilde{q}$ that satisfies $F_{\tilde{q}(U)}(z) \leq F_{q(U)}(z)$, $z \in U$, for all fuzzy dominants $q$ of (3) is the fuzzy best dominant of (3).

**Definition 7** ([17]). Consider $h$ an analytic function in $U$ and $\varphi : \mathbb{C}^3 \times U \rightarrow \mathbb{C}$. When $p$ and $q(p(z), zp'(z), z^2p''(z); z)$ are univalent functions in $U$ and the fuzzy differential superordination is satisfied

$$F_h(U)\varphi(z) \leq F_{\varphi(U)}(z) \varphi(p(z), zp'(z), z^2p''(z); z), \quad z \in U,$$

then $p$ is a fuzzy solution of the fuzzy differential superordination. An analytic function $q$ is fuzzy subordinant of the fuzzy differential superordination if

$$F_{\tilde{q}(U)}(z) \leq F_{p(U)}(z), \quad z \in U,$$

for all $p$ satisfying (4). A univalent fuzzy subordinant $\tilde{q}$ that satisfies $F_{\tilde{q}(U)}(z) \leq F_{p(U)}(z)$ for all fuzzy subordinants $q$ of (4) is the fuzzy best subordinant of (4).

**Definition 8** ([11]). $Q$ represents the set of all analytic and injective functions $f$ on $\overline{U} \setminus E(f)$, with $f'(_c) \neq 0$ for $\xi \in \partial U \setminus E(f)$, where $E(f) = \{\xi \in \partial U : \lim_{z \to \xi} f(z) = \infty\}$.

The following lemmas are needed as tools in the proof of the theorems stated in the next section.

**Lemma 1** ([57]). Consider $g$ a univalent function in the unit disc $U$ and $\varphi, \gamma$ analytic functions in a domain $D \supseteq g(U)$, such that $\varphi(u) \neq 0$ for $u \in g(U)$. Let $G(z) = zg'(z)\gamma(g(z))$ and $h(z) = G(z) + \varphi(g(z))$, supposing $G$ is starlike univalent in $U$ and $\text{Re}\left(\frac{\varphi'(z)}{G(z)}\right) > 0$ for $z \in U$.

When $p$ is an analytic function such that $p(0) = g(0)$, $p(U) \subseteq D$ and

$$F_{p(U)}(z) + zp'(z)\gamma(p(z)) \leq F_{h(U)}(z) + zg'(z)\gamma(g(z)),$$

then

$$F_{p(U)}(z) \leq F_{g(U)}(z),$$

and the fuzzy best dominant is $g$.

**Lemma 2** ([58]). Consider $g$ a convex univalent function in $U$ and $\varphi, \gamma$ analytic functions in a domain $D \supseteq g(U)$. Assume that $\text{Re}\left(\frac{\varphi'(g(z))}{G(g(z))}\right) > 0$ for $z \in U$ and $G(z) = zg'(z)\gamma(g(z))$ is a starlike univalent function in $U$. When $p(z) \in \mathcal{H}[g(0), 1] \cap Q$, with $p(U) \subseteq D$ and $\varphi(p(z)) + zp'(z)\gamma(p(z))$ is a univalent function in $U$ and

$$F_{g(U)}(z) + zg'(z)\gamma(g(z)) \leq F_{p(U)}(z) + zp'(z)\gamma(p(z)),$$

then

$$F_{g(U)}(z) \leq F_{p(U)}(z),$$

and the fuzzy best subordinant is $g$.

In the next section, fuzzy differential subordinations and superordinations will be investigated using the operator in (1) with its particular form as seen in (2). For the fuzzy differential subordinations and superordinations under investigation, fuzzy best dominant and fuzzy best subordinant are obtained, respectively. As applications of the results contained in the theorems, corollaries emerge when fuzzy best dominant and fuzzy best subordinant are considered particular functions known in geometric function theory to have certain geometric properties very useful in studies related to differential subordinations and superordinations. Two sandwich-type theorems and several corol-
Theorem 1. Consider \( g \) an analytic and univalent function in \( U \) such that \( g(z) \neq 0 \), for all \( z \in U \), and \( \frac{z(D_{2}^{-}\psi(m,n,q,z))}{(D_{2}^{-}\phi(m,n,q,z))'} \in \mathcal{H}(U) \), with \( m, n \) be complex numbers with \( n \neq 0, -1, -2, \ldots \), and \( \lambda > 0 \), \( 0 < q < 1 \). Assume that \( \frac{\dot{z}g'(z)}{g(z)} \) is starlike univalent in \( U \),

\[
\Re\left(\frac{\dot{g}}{g} + \frac{2e}{\dot{g}}g'^2(z) + 1 + z \frac{g''(z)}{\dot{g}(z)} - z \frac{g'z^2}{g(z)}\right) > 0,
\]

for \( \beta, \psi, \epsilon, \lambda, \delta \in \mathbb{C}, \delta \neq 0, z \in U \) and

\[
\Psi_{\lambda}^{m,n,q}(\beta, \psi, \epsilon, \delta; z) := \beta + \delta + (\psi - \dot{\delta})\frac{z(D_{2}^{-}\psi(m,n,q,z))}{(D_{2}^{-}\phi(m,n,q,z))'} + \epsilon \frac{z}{(D_{2}^{-}\phi(m,n,q,z))'} + \dot{\delta} \frac{z}{(D_{2}^{-}\phi(m,n,q,z))''}.
\]

If the following fuzzy differential subordination is satisfied by \( g \)

\[
F_{\Psi_{\lambda}^{m,n,q}(U)} \Psi_{\lambda}^{m,n,q}(\beta, \psi, \epsilon, \delta; z) \leq F_{g(U)} \left( \beta + \psi g(z) + \epsilon (g(z))^2 + \delta \frac{\dot{g}^2(z)}{g(z)} \right),
\]

for \( \beta, \psi, \epsilon, \delta \in \mathbb{C}, \delta \neq 0 \), then

\[
F_{D_{2}^{-}\phi(U)} \left( \frac{z(D_{2}^{-}\phi(m,n,q,z))}{(D_{2}^{-}\phi(m,n,q,z))'} \right) \leq F_{g(U)}g(z),
\]

and the fuzzy best dominant is \( g \).

Proof. Consider \( p(z) := \frac{z(D_{2}^{-}\phi(m,n,q,z))}{(D_{2}^{-}\phi(m,n,q,z))'} \), \( z \in U \), \( z \neq 0 \). We have \( p'(z) = \frac{(D_{2}^{-}\phi(m,n,q,z))'}{(D_{2}^{-}\phi(m,n,q,z))'} \) and

\[
\frac{zp'(z)}{p(z)} = 1 + z \frac{\dot{\phi}(m,n,q,z)}{(D_{2}^{-}\phi(m,n,q,z))'}.
\]

Let \( \phi(u) := \beta + \psi u + eu^2 \) and \( \gamma(u) := \frac{\dot{\delta}}{u} \), it is easy to show that \( \phi \) is analytic in \( \mathbb{C} \), \( \gamma \) is analytic in \( \mathbb{C} \setminus \{0\} \) and that \( \gamma(u) \neq 0, u \in \mathbb{C} \setminus \{0\} \).

Also, setting \( G(z) = zg'(z)/(g(z)) = \frac{\dot{g}^2(z)}{g(z)} \), which is starlike univalent in \( U \), and \( h(z) = G(z) + \phi(g(z)) = \beta + \psi g(z) + \epsilon (g(z))^2 + \delta \frac{\dot{g}^2(z)}{g(z)} \), we obtain by differentiating it

\[
h'(z) = \psi g'(z) + 2\epsilon g(z)g''(z) + \delta \frac{\dot{g}^2(z)}{g(z)} + \delta \frac{\dot{g}^2(z)}{g(z)} = \frac{\psi g'(z)}{g(z)} + 2\epsilon g(z)g''(z) + \delta \frac{\dot{g}^2(z)}{g(z)} = \frac{\psi g'(z)}{g(z)} + 2\epsilon g(z)g''(z) + \delta \frac{\dot{g}^2(z)}{g(z)}.
\]

We get that \( \Re \left( \frac{\dot{h}'(z)}{C_{\lambda}(z)} \right) = \Re \left( \frac{\psi g'(z)}{g(z)} + 2\epsilon g(z)g''(z) + 1 + z \frac{g''(z)}{g(z)} - z \frac{g'(z)}{g(z)} \right) > 0 \).
Using relation (9), we get

\[
\beta + \psi p(z) + \epsilon(p(z)) + \delta z p'(z) = \beta + \delta + (\psi - \delta)
\]

By using fuzzy differential subordination (7), we obtain

\[
F_p(U) \left( \beta + \psi p(z) + \epsilon(p(z)) + \delta z p'(z) \right) \leq F_g(U) \left( \beta + \psi g(z) + \epsilon(g(z)) + \delta z g'(z) \right).
\]

Applying Lemma 1, we have

\[
F_p(U) p(z) \leq F_g(U) g(z), \quad z \in U, \text{i.e., } F_{D_z^{-\lambda}} p(U) \leq F_{D_z^{-\lambda}} g(U), \quad z \in U \text{ and the fuzzy best dominant is } g. \quad \Box
\]

**Example 1.** Let \( \phi(3, 2, q, z) = 1 + \frac{2}{1-q} z + \frac{2(1-3q)}{1-q^2} z^2 \) for \( m = 3, n = 2, z \in U \).

Then \( D_z^{-\lambda} \phi(3, 2, q, z) = \frac{\phi(3, 2, q, z)}{z} = \frac{1 + z}{1-q} \frac{1}{1+q} \int_0^z \left( 1 + \frac{2}{1-q} z + \frac{2(1-3q)}{1-q^2} z^2 \right) (z-t)^{\lambda-1} dt \) and after changing the variable \( z-t = u \) and making a simple calculus, we get \( D_z^{-\lambda} \phi(3, 2, q, z) = \frac{z^{\lambda-1}}{1+q}, z \in U \).

Consider \( g(z) = 1 - \frac{1}{1+q}, z \in U \) and differentiating it we obtain \( g'(z) = -\frac{2}{(1+q)^2} \) and \( g''(z) = \frac{4}{(1+q)^3}, z \in U \).

Choosing \( \beta = 0, \delta = 1, \psi = 1, \epsilon = 1 \), we obtain after a long calculus that \( \Psi_{3,2,q}^3(0,1,1;1) = \lambda^2 - \lambda \).

Using Theorem 1, when \( Re z - 6z + 2 + 4 (1+1) > 0 \) and \( 2g(z) = \frac{-2z}{1-z} \) is starlike univalent in \( U \), we obtain \( F_U \left( \lambda^2 - \lambda \right) \leq F_U \left( \frac{-6z + 2}{(1+1)(1+1)} \right), z \in U, \text{ induce } F_U(\lambda - 1) \leq F_U \left( \frac{z}{1+1} \right) \).

**Corollary 1.** Consider \( m, n \) complex numbers with \( n \neq 0, -1, -2, \ldots; \lambda > 0, 0 < q < 1 \) and suppose that relation (5) holds. Then

\[
F_{\Psi_{\lambda}}^m n \ (U) \leq F_{\Psi_{\lambda}}^m n \ (U) \leq F_{\Psi_{\lambda}}^m n \ (U) \leq F_{\Psi_{\lambda}}^m n \ (U)
\]

and the fuzzy best dominant is \( \frac{Mz+1}{Nz+1} \).

**Proof.** Put \( g(z) = \frac{Mz+1}{Nz+1}, -1 \leq N < M \leq 1 \) in Theorem 1 and we obtain the corollary. \( \Box \)

**Corollary 2.** Consider \( m, n \) complex numbers with \( n \neq 0, -1, -2, \ldots; \lambda > 0, 0 < q < 1 \) and suppose that relation (5) holds. Then

\[
F_{\Psi_{\lambda}}^m n \ (U) \leq F_{\Psi_{\lambda}}^m n \ (U) \leq F_{\Psi_{\lambda}}^m n \ (U) \leq F_{\Psi_{\lambda}}^m n \ (U)
\]

and the fuzzy best dominant is \( \frac{Mz+1}{Nz+1} \).
Consider $g$ an analytic and univalent function in $U$ with the properties $g(z) \neq 0$ and $\frac{z g'(z)}{g(z)}$ is starlike univalent in $U$. Suppose that

$$\text{Re}\left(\frac{1}{\eta} g(z) g'(z) + \frac{2\varepsilon}{\eta} g^2(z) g'(z)\right) > 0, \text{ for } \eta, \delta, \varepsilon \in \mathbb{C}, \delta \neq 0. \quad (10)$$

The next theorem and the corollaries associated involve fuzzy differential superordination aspects.

**Theorem 2.** Consider $g$ an analytic and univalent function in $U$ with the properties $g(z) \neq 0$ and $\frac{z g'(z)}{g(z)}$ is starlike univalent in $U$. Suppose that

$$F_{g(U)}\left(\beta + \psi g(z) + \varepsilon (g(z))^2 + \delta \frac{z g'(z)}{g(z)}\right) \leq F_{\Psi^{m,n,q}_{\Lambda}}\left(\beta, \psi, \varepsilon, \delta; z\right) \quad (11)$$

implies

$$F_{g(U)}\left(\beta + \psi g(z) + \varepsilon (g(z))^2 + \delta \frac{z g'(z)}{g(z)}\right) \leq F_{D_{\phi_{\eta}}}(g)\left(\frac{z (D_{\phi_{\eta}}{\Lambda})(m,n,q)z}{D_{\phi_{\eta}}{\Lambda}(m,n,q)}\right), \quad z \in U, \quad (12)$$

and the fuzzy best subordinant is $g$.

**Proof.** Define $p(z) := \frac{z (D_{\phi_{\eta}}{\Lambda})(m,n,q)z}{D_{\phi_{\eta}}{\Lambda}(m,n,q)}$, $z \in U, z \neq 0$.

Let $\phi(u) := \beta + \psi u + \varepsilon u^2$ and $\gamma(u) := \frac{\delta}{\eta}$ and it is easy to show that $\phi$ is analytic in $\mathbb{C}$, $\gamma$ is analytic in $\mathbb{C}\setminus\{0\}$ with $\gamma(u) \neq 0, u \in \mathbb{C}\setminus\{0\}$.

We can write $\frac{\phi'(g(z))}{\phi(g(z))} = \frac{g'(z)g(z)\psi + 2\varepsilon g(z)}{\eta}$, and

$$\text{Re}\left(\frac{\phi'(g(z))}{\phi(g(z))}\right) = \text{Re}\left(\frac{g'(z)g(z)\psi + 2\varepsilon g(z)}{\eta}\right) > 0, \text{ for } \psi, \delta, \varepsilon \in \mathbb{C}, \delta \neq 0.$$

By using relations (9) and (11) we get

$$F_{g(U)}\left(\beta + \psi g(z) + \varepsilon g(z)^2 + \delta \frac{z g'(z)}{g(z)}\right) \leq$$

$$F_{p(U)}\left(\beta + \psi p(z) + \varepsilon (p(z))^2 + \delta \frac{z p'(z)}{p(z)}\right).$$

Applying Lemma 2, we obtain

$$F_{g(U)}\left(\beta + \psi g(z) + \varepsilon g(z)^2 + \delta \frac{z g'(z)}{g(z)}\right) \leq F_{D_{\phi_{\eta}}}(g)\left(\frac{z (D_{\phi_{\eta}}{\Lambda})(m,n,q)z}{D_{\phi_{\eta}}{\Lambda}(m,n,q)}\right), \quad z \in U,$$

and the fuzzy best subordinant is $g$. \hfill \Box

**Example 2.** Taking the same functions as in Example 1, $\phi(3,2,q,z) = 1 + \frac{2}{1-q} z + \frac{2(1-4q)}{1-2q} z^2$ for $m = 3, n = 2, z \in U$, with $D_{\phi_{\eta}}{\Lambda}(3,2,q/z) = \frac{\eta^{\lambda-1}}{\eta^{\lambda}}, z \in U$, and $g(z) = \frac{1}{1+z}, z \in U$ with $g'(z) = -\frac{2}{(1+z)^2}$ and $g''(z) = \frac{4}{(1+z)^3}, z \in U.$

Choosing $\beta = 0, \delta = 1, \psi = 1, \varepsilon = 1, we get $\Psi^{3,2,\eta}_{\Lambda}(0,1,1;z) = \lambda^2 - \lambda.$
Using Theorem 2, when \( \text{Re}\left(-\frac{6z^2+8z-6}{(1+z)^3}\right) > 0 \) and \( \frac{z_g'(z)}{g(z)} = \frac{-2z}{1-z} \) is starlike univalent in \( U \), we obtain \( F_U\left(\frac{-6z^2+8z-6}{(1+z)^3}\right) \leq F_U(\lambda^2 - \lambda), z \in U \), induce \( F_U\left(\frac{1-z}{1+z}\right) \leq F_U(\lambda - 1) \).

**Corollary 3.** For \( m, n \) complex numbers with \( n \neq 0, -1, -2, \ldots; \lambda > 0, 0 < q < 1 \), suppose that relation (10) holds. When \( \frac{z(D_z^{-\lambda} \psi(m, n, q, z))'}{D_z^{-\lambda} \phi(m, n, q, z)} \in \mathcal{H}[g(0), 1] \cap Q \) and

\[
F_{g(U)}\left(\beta + \psi\left(\frac{Mz+1}{Nz+1} + \epsilon\left(\frac{Mz+1}{Nz+1}\right)^2 + \frac{\delta(M-N)z}{(Mz+1)(Nz+1)}\right)\right) 
\leq F_{\psi_{\lambda}^{m,n,q}(U)}^{\psi_{\lambda}^{m,n,q}}(\beta, \psi, \epsilon, \delta; z),
\]

where \( \beta, \psi, \delta, \epsilon \in \mathbb{C}, \delta \neq 0, -1 \leq N < M \leq 1, \) and \( \psi_{\lambda}^{m,n,q} \) defined by relation (6), then

\[
F_{g(U)}\left(\frac{Mz+1}{Nz+1}\right) \leq F_{D_z^{-\lambda} \phi(U)}\left(\frac{z(D_z^{-\lambda} \phi(m, n, q, z))'}{D_z^{-\lambda} \phi(m, n, q, z)}\right),
\]

and the fuzzy best subordinant is \( \frac{Mz+1}{Nz+1} \).

**Proof.** Considering \( g(z) = \frac{Mz+1}{Nz+1} \), \( -1 \leq N < M \leq 1 \) in Theorem 2 we have the corollary. \( \Box \)

**Corollary 4.** For \( m, n \) complex numbers with \( n \neq 0, -1, -2, \ldots; \lambda > 0, 0 < q < 1 \), suppose that relation (10) holds. When \( \frac{z(D_z^{-\lambda} \psi(m, n, q, z))'}{D_z^{-\lambda} \phi(m, n, q, z)} \in \mathcal{H}[g(0), 1] \cap Q \) and

\[
F_{g(U)}\left(\beta + \psi\left(\frac{z+1}{1-z} + \epsilon\left(\frac{z+1}{1-z}\right)\right)\right) 
\leq F_{\psi_{\lambda}^{m,n,q}(U)}^{\psi_{\lambda}^{m,n,q}}(\beta, \psi, \epsilon, \delta; z),
\]

where \( \beta, \psi, \delta, \epsilon \in \mathbb{C}, \delta \neq 0, 0 < \alpha \leq 1, \) and \( \psi_{\lambda}^{m,n,q} \) defined by relation (6), then

\[
F_{g(U)}\left(\frac{z+1}{1-z}\right) \leq F_{D_z^{-\lambda} \phi(U)}\left(\frac{z(D_z^{-\lambda} \phi(m, n, q, z))'}{D_z^{-\lambda} \phi(m, n, q, z)}\right),
\]

and the fuzzy best subordinant is \( \left(\frac{z+1}{1-z}\right)^\alpha \).

**Proof.** Letting \( g(z) = \left(\frac{z+1}{1-z}\right)^\alpha, 0 < \alpha \leq 1 \) in Theorem 2 we obtain the corollary. \( \Box \)

Combining the results obtained in Theorems 1 and 2, the following sandwich-type result can be established.

**Theorem 3.** Consider \( g_1, g_2 \) analytic and univalent functions in \( U \) with the properties \( g_1(z) \neq 0, g_2(z) \neq 0 \), for all \( z \in U \), and \( \frac{z_g'(z)}{g(z)} = \frac{z_{g_1}'(z)}{g_1(z)} = \frac{z_{g_2}'(z)}{g_2(z)} \) are starlike univalent functions. Assume that \( g_1 \) satisfies relation (5) and \( g_2 \) satisfies relation (10). For \( m, n \) complex numbers with \( n \neq 0, -1, -2, \ldots; \lambda > 0, 0 < q < 1 \), when \( \frac{z(D_z^{-\lambda} \phi(m, n, q, z))'}{D_z^{-\lambda} \phi(m, n, q, z)} \in \mathcal{H}[g(0), 1] \cap Q \) and \( \psi_{\lambda}^{m,n,q} \) defined by relation (6) is univalent in \( U \), then

\[
F_{g_1(U)}\left(\beta + \psi g_1(z) + \epsilon(g_1(z))^2 + \frac{\delta(z g_1'(z))}{g_1(z)}\right) \leq F_{\psi_{\lambda}^{m,n,q}(U)}^{\psi_{\lambda}^{m,n,q}}(\beta, \psi, \epsilon, \delta; z)
\]

\[
\leq F_{g_2(U)}\left(\beta + \psi g_2(z) + \epsilon(g_2(z))^2 + \frac{\delta(z g_2'(z))}{g_2(z)}\right),
\]
for \(\beta, \psi, \delta, \epsilon \in \mathbb{C}, \delta \neq 0\), implies
\[
F_{g_1}(u)g_1(z) \leq F_{D_z^{-\lambda}\phi(u)}\left(\frac{z(D_z^{-\lambda}\phi(m, n; q, z))'}{D_z^{-\lambda}\phi(m, n; q, z)}\right) \leq F_{g_2}(u)g_2(z),
\]
and the fuzzy best subordinant is \(g_1\) and the fuzzy best dominant is \(g_2\).

Considering \(g_1(z) = \frac{M_1z+1}{N_1z+1}\) and \(g_2(z) = \frac{M_2z+1}{N_2z+1}\), with \(-1 \leq N_2 < N_1 < M_1 < M_2 \leq 1\), we get the following corollary.

**Corollary 5.** For \(m, n\) complex numbers with \(n \neq 0, -1, -2, \ldots; \lambda > 0, 0 < q < 1\), suppose that relations (5) and (10) hold. When \(\frac{z(D_z^{-\lambda}\phi(m, n; q, z))'}{D_z^{-\lambda}\phi(m, n; q, z)} \in H[g(0), 1] \cap Q\) and
\[
F_{g_1}(u)\left(\beta + \psi \frac{M_1z+1}{N_1z+1} + \epsilon \left(M_1z+1\right)^2 + \frac{\delta}{(M_1-1)(N_1z+1)}\right) 
\leq F_{\Psi^{\lambda, n, q}}(u)\left(\Psi^{\lambda, n, q}(\beta, \psi, \epsilon, \delta; z)\right)
\]
\[
F_{g_2}(u)\left(\beta + \psi \frac{M_2z+1}{N_2z+1} + \epsilon \left(M_2z+1\right)^2 + \frac{\delta}{(M_2-1)(N_2z+1)}\right) 
\leq F_{g_1}(u)\left(\frac{M_2z+1}{N_2z+1}\right),
\]
with \(\beta, \psi, \delta, \epsilon \in \mathbb{C}, \tau \delta \neq 0, -1 \leq N_2 < N_1 < M_1 < M_2 \leq 1\), and \(\Psi^{\lambda, n, q}\) defined by relation (6), then
\[
F_{g_1}(u)\left(\frac{M_1z+1}{N_1z+1}\right) \leq F_{D_z^{-\lambda}\phi(u)}\left(\frac{z(D_z^{-\lambda}\phi(m, n; q, z))'}{D_z^{-\lambda}\phi(m, n; q, z)}\right) \leq F_{g_2}(u)\left(\frac{M_2z+1}{N_2z+1}\right),
\]
the fuzzy best subordinant is \(\frac{M_1z+1}{N_1z+1}\) and the fuzzy best dominant is \(\frac{M_2z+1}{N_2z+1}\), respectively.

Let \(g_1(z) = \left(\frac{z+1}{1-z}\right)^{a_1}\) and \(g_2(z) = \left(\frac{z+1}{1-z}\right)^{a_2}\), with \(0 < a_1 < a_2 \leq 1\), we get the following corollary.

**Corollary 6.** For \(m, n\) complex numbers with \(n \neq 0, -1, -2, \ldots; \lambda > 0, 0 < q < 1\), suppose that relations (5) and (10) hold. When \(\frac{z(D_z^{-\lambda}\phi(m, n; q, z))'}{D_z^{-\lambda}\phi(m, n; q, z)} \in H[g(0), 1] \cap Q\) and
\[
F_{g_1}(u)\left(\beta + \psi \left(\frac{z+1}{1-z}\right)^{a_1} + \epsilon \left(\frac{z+1}{1-z}\right)^{2a_1} + \frac{2\alpha_1 \delta z}{1-z^2}\right) 
\leq F_{\Psi^{\lambda, n, q}}(u)\left(\Psi^{\lambda, n, q}(\beta, \psi, \epsilon, \delta; z)\right)
\]
\[
F_{g_2}(u)\left(\beta + \psi \left(\frac{z+1}{1-z}\right)^{a_2} + \epsilon \left(\frac{z+1}{1-z}\right)^{2a_2} + \frac{2\alpha_2 \delta z}{1-z^2}\right),
\]
with \(\beta, \psi, \delta, \epsilon \in \mathbb{C}, \delta \neq 0, 0 < \alpha_1 < \alpha_2 \leq 1\), and \(\Psi^{\lambda, n, q}\) defined by relation (6), then
\[
F_{g_1}(u)\left(\frac{z+1}{1-z}\right)^{a_1} \leq F_{D_z^{-\lambda}\phi(u)}\left(\frac{z(D_z^{-\lambda}\phi(m, n; q, z))'}{D_z^{-\lambda}\phi(m, n; q, z)}\right) \leq F_{g_2}(u)\left(\frac{z+1}{1-z}\right)^{a_2},
\]
the fuzzy best subordinant is \(\left(\frac{z+1}{1-z}\right)^{a_1}\) and the fuzzy best dominant is \(\left(\frac{z+1}{1-z}\right)^{a_2}\), respectively.

Considering \(\varphi(u) := \beta u\) and \(\gamma(u) := \delta\) in Lemmas 1 and 2, we obtain the following results regarding fuzzy differential subordination:
Theorem 4. Consider \( g \) a convex and univalent function in \( U \) such that \( g(0) = \lambda, z \in U \), and \( z(D_z^{-}\lambda \phi(m,n;q,z))' \) \( \in \mathcal{H}(U), z \in U \), where \( \lambda > 0, 0 < q < 1, m, n \) are complex numbers with \( n \neq 0, -1, -2, \ldots \). Suppose that
\[
\Re\left(\frac{\beta + \delta}{\delta} + z\phi''(z)\frac{\phi''(z)}{\phi'(z)}\right) > 0, \tag{13}
\]
for \( \beta, \delta \in \mathbb{C}, \delta \neq 0, z \in U \), and

\[
\Psi_{\lambda}^{m,n,q}(\beta, \delta; q, z) := (\beta + \delta)\frac{z(D_z^{-}\lambda \phi(m,n;q,z))'}{D_z^{-}\lambda \phi(m,n;q,z)} + \delta\frac{z^2(D_z^{-}\lambda \phi(m,n;q,z))''}{D_z^{-}\lambda \phi(m,n;q,z)} \tag{14}
\]
and

\[
-\delta \left( z(D_z^{-}\lambda \phi(m,n;q,z))' \right)^2.
\]

If \( g \) verifies the fuzzy differential subordination
\[
F_{\Psi_{\lambda}^{m,n,q}(U)}(\beta, \delta; q, z) \leq F_{g(U)}(\beta g(z) + \delta z g'(z)), \tag{15}
\]
for \( \beta, \delta \in \mathbb{C}, \delta \neq 0, z \in U \), then
\[
F_{D_z^{-}\lambda \phi(U)} \left( \frac{z(D_z^{-}\lambda \phi(m,n;q,z))'}{D_z^{-}\lambda \phi(m,n;q,z)} \right) \leq F_{g(U)g}(z), z \in U, \tag{16}
\]
and the fuzzy best dominant is \( g \).

Proof. Define again \( p(z) := \frac{z(D_z^{-}\lambda \phi(m,n;q,z))'}{D_z^{-}\lambda \phi(m,n;q,z)}, z \in U, z \neq 0 \), which is analytic in \( U \) and \( p(0) = \lambda \).

Differentiating it we can write \( p'(z) = \frac{z(D_z^{-}\lambda \phi(m,n;q,z))'}{D_z^{-}\lambda \phi(m,n;q,z)} + z\phi(z) - z\phi''(z) \).

Then
\[
zp'(z) = \frac{z(D_z^{-}\lambda \phi(m,n;q,z))'}{D_z^{-}\lambda \phi(m,n;q,z)} + \frac{z^2(D_z^{-}\lambda \phi(m,n;q,z))''}{D_z^{-}\lambda \phi(m,n;q,z)} - \left( \frac{z(D_z^{-}\lambda \phi(m,n;q,z))'}{D_z^{-}\lambda \phi(m,n;q,z)} \right)^2. \tag{17}
\]

Set \( \phi(u) := \beta u \) and \( \gamma(u) := \delta \), we can show easily that \( \phi \) is analytic in \( \mathbb{C} \), \( \gamma \) is analytic in \( \mathbb{C} \setminus \{0\} \) and \( \gamma(u) \neq 0, u \in \mathbb{C} \setminus \{0\} \).

Also, we set \( G(z) := z g'(z), g(z) := \delta z g'(z) \), which is starlike univalent in \( U \).

Let \( h(z) := G(z) + \phi(g(z)) = \beta g(z) + \delta z g'(z) \).

We get \( \Re\left(\frac{\phi''(z)}{G(z)}\right) = \Re\left(\frac{\beta + \delta}{\delta} + z\phi''(z)\frac{\phi''(z)}{\phi'(z)}\right) > 0.\)

Taking account the relation (17), we can write \( \beta p(z) + \delta z p'(z) = (\beta + \delta)\frac{z(D_z^{-}\lambda \phi(m,n;q,z))'}{D_z^{-}\lambda \phi(m,n;q,z)} + \delta\frac{z^2(D_z^{-}\lambda \phi(m,n;q,z))''}{D_z^{-}\lambda \phi(m,n;q,z)} - \left( \frac{z(D_z^{-}\lambda \phi(m,n;q,z))'}{D_z^{-}\lambda \phi(m,n;q,z)} \right)^2. \)

Using fuzzy differential subordination (15), we get \( F_{p(U)}(\beta p(z) + \delta z p'(z)) \leq F_{g(U)}(\beta g(z) + \delta z g'(z)) \).

Applying Lemma 1, we obtain \( F_{p(U)p}(z) \leq F_{g(U)g}(z), z \in U \), i.e., \( F_{D_z^{-}\lambda \phi(U)} \left( \frac{z(D_z^{-}\lambda \phi(m,n;q,z))'}{D_z^{-}\lambda \phi(m,n;q,z)} \right) \leq F_{g(U)g}(z), z \in U \), and the fuzzy best dominant is \( g \). \( \square \)
Corollary 7. Consider $\lambda > 0$, $0 < q < 1$, $m$, $n$ complex numbers with $n \neq 0, -1, -2, ...$ and $g(z) = \frac{Mz + 1}{Nz + 1}$, $z \in U$, $-1 \leq N < M \leq 1$. Suppose that relation (13) holds. Then

\[ F_{\Psi_{\lambda}^{m,n,q}(U)}(\Psi_{\lambda}^{m,n,q}(\beta, \delta, q, z)) \leq F_{\tilde{g}(U)} \left( \frac{Mz + 1}{Nz + 1} + \frac{\delta(M - N)z}{(Nz + 1)^2} \right), \]

with $\beta, \delta \in \mathbb{C}$, $\delta \neq 0$, $-1 \leq N < M \leq 1$, and $\Psi_{\lambda}^{m,n,q}$ defined by relation (14), then

\[ F_{\tilde{D}_{z}^{-\lambda}\phi(U)} \left( \frac{z(D_{z}^{-\lambda}\phi(m,n,q,z))'}{D_{z}^{-\lambda}\phi(m,n,q,z)} \right) \leq F_{\tilde{g}(U)} \left( \frac{Mz + 1}{Nz + 1} \right), \]

and the fuzzy best dominant is $\frac{Mz + 1}{Nz + 1}$.

Proof. Consider $\tilde{g}(z) = \frac{Mz + 1}{Nz + 1}$, $-1 \leq N < M \leq 1$, in Theorem 4 and we obtain the corollary. □

Corollary 8. Consider $\lambda > 0$, $0 < q < 1$, $m$, $n$ complex numbers with $n \neq 0, -1, -2, ...$ and $g(z) = \left( \frac{z + 1}{z - 1} \right)^{a}$. Suppose that relation (13) holds. Then

\[ F_{\Psi_{\lambda}^{m,n,q}(U)}(\Psi_{\lambda}^{m,n,q}(\beta, \delta, q, z)) \leq F_{\tilde{g}(U)} \left( \beta \left( \frac{z + 1}{1 - z} \right)^{a} + \frac{2\alpha\delta z}{1 - z} \left( \frac{z + 1}{1 - z} \right)^{a} \right), \]

with $\beta, \delta \in \mathbb{C}$, $0 < \alpha \leq 1$, $\delta \neq 0$, and $\Psi_{\lambda}^{m,n,q}$ defined by relation (14), then

\[ F_{\tilde{D}_{z}^{-\lambda}\phi(U)} \left( \frac{z(D_{z}^{-\lambda}\phi(m,n,q,z))'}{D_{z}^{-\lambda}\phi(m,n,q,z)} \right) \leq F_{\tilde{g}(U)} \left( \frac{z + 1}{1 - z} \right)^{a}, \quad z \in U, \]

and the fuzzy best dominant is $\left( \frac{z + 1}{z - 1} \right)^{a}$.

Proof. Corollary follows considering in Theorem 4 $\tilde{g}(z) = \left( \frac{z + 1}{z - 1} \right)^{a}$, $0 < \alpha \leq 1$. □

The next theorem and the corollaries associated involve fuzzy differential superordination aspects, too.

Theorem 5. Consider $g$ a convex and univalent function in $U$ such that $g(0) = \lambda$, with $0 < q < 1$, $\lambda > 0$, $m$, $n$ complex numbers with $n \neq 0, -1, -2, ...$. Suppose that

\[ \text{Re} \left( \frac{\beta}{\delta} g'(z) \right) > 0, \quad \text{for } \beta, \delta \in \mathbb{C}, \delta \neq 0. \] (18)

When $\frac{z(D_{z}^{-\lambda}\phi(m,n,q,z))'}{D_{z}^{-\lambda}\phi(m,n,q,z)} \in H[g(0), 1] \cap Q$ and $\Psi_{\lambda}^{m,n,q}(\beta, \delta, q, z)$ defined by relation (14) is univalent in $U$, then

\[ F_{\tilde{g}(U)}(\beta g(z) + \delta z g'(z)) \leq F_{\Psi_{\lambda}^{m,n,q}(U)}(\Psi_{\lambda}^{m,n,q}(\beta, \delta, q, z)) \] (19)

implies

\[ F_{\tilde{g}(U)} g(z) \leq F_{\tilde{D}_{z}^{-\lambda}\phi(U)} \frac{z(D_{z}^{-\lambda}\phi(m,n,q,z))'}{D_{z}^{-\lambda}\phi(m,n,q,z)}, \quad z \in U, \] (20)

and the fuzzy best subordinant is $g$.

Proof. Define also $p(z) := \frac{z(D_{z}^{-\lambda}\phi(m,n,q,z))'}{D_{z}^{-\lambda}\phi(m,n,q,z)}$, $z \in U$, $z \neq 0$, which is analytic in $U$ and $p(0) = \lambda$. □
Let \( \varphi(u) := \beta u \) and \( \gamma(u) := \delta \), it is easy to show that \( \varphi \) is analytic in \( \mathbb{C} \), \( \gamma \) is analytic in \( \mathbb{C} \setminus \{0\} \) with \( \gamma(0) \neq 0 \), \( u \in \mathbb{C} \setminus \{0\} \).

Since \( \frac{\varphi'(g(z))}{\varphi'(z)} = \frac{\beta}{\delta} g'(z) \), it yields that \( \text{Re} \left( \frac{\varphi'(g(z))}{\varphi'(z)} \right) = \text{Re} \left( \frac{\beta}{\delta} g'(z) \right) > 0 \), for \( \beta, \delta \in \mathbb{C} \), \( \delta \neq 0 \).

The fuzzy differential superordination (19) can be written as

\[
F_{\varphi}(u) (\beta g(z) + \delta z g'(z)) \leq F_{\psi}(u) (\beta p(z) + \delta z p'(z)), \quad z \in U,
\]

and applying Lemma 2, we obtain

\[
F_{\varphi}(u) g(z) \leq F_{D_{\lambda}^{-1} \phi(u)} \left( z \left( D_{\lambda}^{-\lambda} \phi(m, n; q, z) \right) \phi(m, n; q, z) \right), \quad z \in U,
\]

and the fuzzy best subordinant is \( g \).  □

**Corollary 9.** For \( \lambda > 0, 0 < q < 1, m, n \) complex numbers with \( n \neq 0, -1, -2, \ldots \) and \( g(z) = \frac{Mz + 1}{Nz + 1}, -1 \leq N < M \leq 1, z \in U \), suppose that relation (18) holds. When \( \frac{z(D_{\lambda}^{-\lambda} \phi(m, n; q, z))}{D_{\lambda}^{-\lambda} \phi(m, n; q, z)} \in \mathcal{H}[g(0), 1] \cap \mathbb{Q} \), and

\[
F_{\varphi}(u) \left( \beta \frac{Mz + 1}{Nz + 1} + \frac{\delta(M - N)z}{(Nz + 1)^2} \right) \leq F_{\psi^{\lambda, n,q}}(u) \Psi^{\lambda, n,q}_{m,n,q}(\beta, \delta; q, z),
\]

with \( \beta, \delta \in \mathbb{C} \), \( \delta \neq 0 \), \( -1 \leq N < M \leq 1 \), and \( \Psi^{\lambda, n,q}_{m,n,q} \) defined by relation (14), then

\[
F_{\varphi}(u) \left( \frac{Mz + 1}{Nz + 1} \right) \leq F_{D_{\lambda}^{-1} \phi(u)} \left( z \left( D_{\lambda}^{-\lambda} \phi(m, n; q, z) \right) \phi(m, n; q, z) \right),
\]

and the fuzzy best subordinant is \( \frac{Mz + 1}{Nz + 1} \).

**Proof.** Considering \( g(z) = \frac{Mz + 1}{Nz + 1}, -1 \leq N < M \leq 1 \), in Theorem 5 we obtain the corollary.  □

**Corollary 10.** For \( \lambda > 0, 0 < q < 1, m, n \) complex numbers with \( n \neq 0, -1, -2, \ldots \) and \( g(z) = \left( \frac{z + 1}{1 - z} \right)^{\alpha} \), suppose that relation (18) holds. When \( \frac{z(D_{\lambda}^{-\lambda} \phi(m, n; q, z))}{D_{\lambda}^{-\lambda} \phi(m, n; q, z)} \in \mathcal{H}[g(0), 1] \cap \mathbb{Q} \) and

\[
F_{\varphi}(u) \left( \beta \frac{z + 1}{1 - z} \right)^{\alpha} + \frac{2\alpha \delta z}{2 - z^{2}} \left( \frac{z + 1}{1 - z} \right)^{\alpha} \leq F_{\psi^{\lambda, n,q}}(u) \Psi^{\lambda, n,q}_{m,n,q}(\beta, \delta; q, z),
\]

with \( \beta, \delta \in \mathbb{C} \), \( 0 < \alpha \leq 1 \), \( \delta \neq 0 \), and \( \Psi^{\lambda, n,q}_{m,n,q} \) defined by relation (14), then

\[
F_{\varphi}(u) \left( \frac{z + 1}{1 - z} \right)^{\alpha} \leq F_{D_{\lambda}^{-1} \phi(u)} \left( z \left( D_{\lambda}^{-\lambda} \phi(m, n; q, z) \right) \phi(m, n; q, z) \right),
\]

and the fuzzy best subordinant is \( \left( \frac{z + 1}{1 - z} \right)^{\alpha} \).

**Proof.** Corollary follows considering in Theorem 5 \( g(z) = \left( \frac{z + 1}{1 - z} \right)^{\alpha}, 0 < \alpha \leq 1 \).  □

Combining the results obtained in Theorems 4 and 5, the following sandwich-type result can be established.

**Theorem 6.** Consider \( g_1, g_2 \) convex and univalent functions in \( U \) with the properties \( g_1(z) \neq 0, g_2(z) \neq 0 \), for all \( z \in U \). Assume that \( g_1 \) satisfies relation (13) and \( g_2 \) satisfies relation (18).
For $\lambda > 0$, $0 < q < 1$, $m$, $n$ complex numbers with $n \neq 0$, $-1$, $-2$, ..., when $\frac{z(D_{2}^{-}\lambda \phi(m, n; q, z))'}{D_{2}^{-}\lambda \phi(m, n; q, z)} \in \mathcal{H}[g(0), 1] \cap \mathbb{Q}$, and $\Psi_{\lambda}^{m,n,\alpha}(\beta, \delta; q, z)$ defined by relation (14), then

$$F_{g_{1}}(z) \leq \frac{z(D_{2}^{-}\lambda \phi(m, n; q, z))'}{D_{2}^{-}\lambda \phi(m, n; q, z)} \leq F_{g_{2}}(z), \quad z \in U,$$

for $\beta, \delta \in \mathbb{C}$, $\delta \neq 0$, implies

$$F_{g_{1}}(z) \leq F_{D_{2}^{-}\lambda \phi(U)} \left( \frac{z(D_{2}^{-}\lambda \phi(m, n; q, z))'}{D_{2}^{-}\lambda \phi(m, n; q, z)} \right) \leq F_{g_{2}}(z), \quad z \in U,$$

and the fuzzy best subdominant is $g_{1}$ and the fuzzy best dominant is $g_{2}$.

Considering $g_{1}(z) = \frac{M_{1}z+1}{N_{1}z+1}$ and $g_{2}(z) = \frac{M_{2}z+1}{N_{2}z+1}$, with $-1 \leq N_{2} < N_{1} < M_{1} < M_{2} \leq 1$, we get the following corollary.

**Corollary 11.** For $m$, $n$ complex numbers with $n \neq 0$, $-1$, $-2$, ...; $\lambda > 0$, $0 < q < 1$, suppose that relations (13) and (18) hold for $g_{1}(z) = \frac{M_{1}z+1}{N_{1}z+1}$ and $g_{2}(z) = \frac{M_{2}z+1}{N_{2}z+1}$, respectively. When

$$\frac{z(D_{2}^{-}\lambda \phi(m, n; q, z))'}{D_{2}^{-}\lambda \phi(m, n; q, z)} \in \mathcal{H}[g(0), 1] \cap \mathbb{Q}$$

and

$$F_{g_{1}(U)} \left( \frac{z(D_{2}^{-}\lambda \phi(m, n; q, z))'}{D_{2}^{-}\lambda \phi(m, n; q, z)} \right) \leq F_{g_{2}(U)} \left( \frac{z(D_{2}^{-}\lambda \phi(m, n; q, z))'}{D_{2}^{-}\lambda \phi(m, n; q, z)} \right), \quad z \in U,$$

with $\beta, \delta \in \mathbb{C}$, $\delta \neq 0$, $-1 \leq N_{2} < N_{1} < M_{1} < M_{2} \leq 1$, and $\Psi_{\lambda}^{m,n,\alpha}$ defined by relation (14), then

$$F_{g_{1}(U)} \left( \frac{z(D_{2}^{-}\lambda \phi(m, n; q, z))'}{D_{2}^{-}\lambda \phi(m, n; q, z)} \right) \leq F_{g_{2}(U)} \left( \frac{z(D_{2}^{-}\lambda \phi(m, n; q, z))'}{D_{2}^{-}\lambda \phi(m, n; q, z)} \right), \quad z \in U,$$

the fuzzy best subdominant is $\frac{M_{1}z+1}{N_{1}z+1}$ and the fuzzy best dominant is $\frac{M_{2}z+1}{N_{2}z+1}$, respectively.

Considering $g_{1}(z) = \left( \frac{z+1}{1-z} \right)^{a_{1}}$ and $g_{2}(z) = \left( \frac{z+1}{1-z} \right)^{a_{2}}$, with $0 < a_{1} < a_{2} \leq 1$, we obtain the following corollary.

**Corollary 12.** For $m$, $n$ complex numbers with $n \neq 0$, $-1$, $-2$, ...; $\lambda > 0$, $0 < q < 1$, suppose that relations (13) and (18) hold for $g_{1}(z) = \left( \frac{z+1}{1-z} \right)^{a_{1}}$ and $g_{2}(z) = \left( \frac{z+1}{1-z} \right)^{a_{2}}$, respectively.

When

$$\frac{z(D_{2}^{-}\lambda \phi(m, n; q, z))'}{D_{2}^{-}\lambda \phi(m, n; q, z)} \in \mathcal{H}[g(0), 1] \cap \mathbb{Q}$$

and

$$F_{g_{1}(U)} \left( \left( \frac{z+1}{1-z} \right)^{a_{1}} + \frac{2a_{1}\delta z}{1-z^{2}} \left( \frac{z+1}{1-z} \right)^{a_{1}} \right) \leq F_{g_{2}(U)} \left( \left( \frac{z+1}{1-z} \right)^{a_{2}} + \frac{2a_{2}\delta z}{1-z^{2}} \left( \frac{z+1}{1-z} \right)^{a_{2}} \right), \quad z \in U,$$

with $\beta, \delta \in \mathbb{C}$, $\delta \neq 0$, $0 < a_{1} < a_{2} \leq 1$, and $\Psi_{\lambda}^{m,n,\alpha}$ defined by relation (14), then

$$F_{g_{1}(U)} \left( \left( \frac{z+1}{1-z} \right)^{a_{1}} \right) \leq F_{g_{2}(U)} \left( \left( \frac{z+1}{1-z} \right)^{a_{2}} \right), \quad z \in U,$$
the fuzzy best subordinant is \( (\frac{z+1}{1-z})^{α_1} \) and the fuzzy best dominant is \( (\frac{z+1}{1-z})^{α_2} \), respectively.

3. Conclusions

This paper presents new fuzzy differential subordinations and superordinations obtained by involving in the studies the operator presented in (1). The first theorem refers to a new fuzzy differential subordination for which the fuzzy best dominant is provided. Two corollaries follow as applications of the new result by considering two particular functions with nice geometric properties as fuzzy best subordinant of the first fuzzy differential subordination investigated in this study. Next, similar results are presented regarding a fuzzy differential superordination for which the fuzzy best dominant is obtained and two corollaries follow as applications. The sandwich-type theorem connecting Theorems 1 and 2 is stated as Theorem 3 and the corollaries formulated as a consequence of this theorem appear naturally by combining the results presented in the corollaries related to Theorems 1 and 2. Finally, a special fuzzy differential subordination is investigated in Theorem 4 and the dual result concerning a fuzzy differential superordination is contained in Theorem 5. Naturally, specific corollaries also accompany these theorems.

As future applications of the results presented in this paper, they could serve as inspiration for Riemann–Liouville fractional integral to be applied on other q-calculus functions and operators. Symmetry properties might be investigated regarding the operator shown in (1) considering the fuzzy subordinations and superordinations obtained here. In addition, classes of analytic functions could be introduced due to the geometric properties which can be interpreted from the corollaries. Those classes could also be investigated related to symmetry properties given by the use of a fractional operator in their definition.

Keeping in mind the notable applications of fuzzy sets theory and fractional calculus in real life contexts as it can be seen for example in [59–61], hopefully, this new fractional q-operator will find applications in future studies.

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