INVERSION OF ADJUNCTION FOR $F$-SIGNATURE

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Abstract. Let $(R, \Delta + D)$ be a log $\mathbb{Q}$-Gorenstein pair of index prime to $p$ where $R$ is an $F$-finite, Noetherian, normal local domain of characteristic $p > 0$, $\Delta \geq 0$ is a $\mathbb{Q}$-divisor and $D$ is an integral $\mathbb{Q}$-Cartier divisor. We show that the left derivative of the $F$-signature function $s(R, \Delta + tD)$ at $t = 1$ is equal to $-s(\mathcal{O}_D, \text{Diff}_D(\Delta))$. This equality can be interpreted as a quantitative form of inversion of adjunction for strong $F$-regularity. Indeed, we obtain the inequality $s(R, \Delta) \geq s(\mathcal{O}_D, \text{Diff}_D(\Delta))$ as an immediate corollary of our main theorem.

1. Introduction

Inversion of adjunction refers to any phenomenon in which a property of a divisor $D$ on a scheme $X$ implies that $X$ has that property near $D$. This notion plays a crucial role in the study of singularities in the minimal program due to its role in arguments via induction on dimension [Kol13, Chapter 4]. Given the deep connections between singularities of the minimal model program and $F$-singularities, it is natural to expect that inversion of adjunction results hold for $F$-singularities. Indeed, such results have been discovered, especially for strongly $F$-regular singularities and the test ideal (see e.g. [Sch09] [TW04] [Tak04] [Das15] [DS17]). In this paper, we prove an inversion of adjunction statement for $F$-signature, an important numerical invariant of singularities of pairs in positive characteristic. In particular, our result can be viewed as a quantitative inversion of adjunction for strong $F$-regularity which refines the above results. For a function $f : \mathbb{R} \to \mathbb{R}$, we denote by $D_f(c)$ the left-derivative of $f$ at $c$.

Theorem 1.1. Let $(R, \Delta + D)$ be a log $\mathbb{Q}$-Gorenstein pair with index prime to $p$ where $R$ is a Noetherian, $F$-finite, normal local ring of dimension $d$, $\Delta$ is an effective $\mathbb{Q}$-divisor, and $D$ is an irreducible, reduced $\mathbb{Q}$-Cartier divisor such that $D \cap \Delta = 0$. Let $s_R^{\Delta, D}(t) = s(R, \Delta + tD)$. Then

$$D_{-s_R^{\Delta, D}(1)} = -s(\mathcal{O}_D, \text{Diff}_D(\Delta))$$

where $\text{Diff}_D(\Delta)$ is the different of $\Delta$ on $D$.

The different $\text{Diff}_D(\Delta)$ is a $\mathbb{Q}$-divisor that acts as a correction factor in adjunction statements; see Section 3.1 for its construction. In the case where $R$ is regular and $\Delta = 0$, we have $\text{Diff}_D(\Delta) = 0$, and Theorem 1.1 reduces to [BST13, Theorem 4.6]. One may interpret Theorem 1.1 as a quantitative version of inversion of adjunction for $F$-regularity. Indeed, one of its immediate consequences is the following inequality for $F$-signature.

Corollary 1.2. With notation as in Theorem 1.1, we have

$$s(R, \Delta) \geq s(\mathcal{O}_D, \text{Diff}_D(\Delta))$$

with equality if and only if the function $s(R, \Delta + tD)$ is linear on the interval $[0, 1]$ with slope $s(R, \Delta)$. 

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An immediate consequence of this inequality is the (previously known) fact that if \((D, \text{Diff}_D(\Delta))\) is strongly \(F\)-regular, then \((R, \Delta)\) is strongly \(F\)-regular (see e.g. [Das15]). However, Corollary 1.2 yields even more information. Indeed, it states that, with respect to \(F\)-signature, the singularity of \((R, \Delta)\) is no worse than that of \((\mathcal{O}_D, \text{Diff}_D(\Delta))\). This inequality introduces the possibility of using induction on dimension as a strategy for placing lower bounds on \(F\)-signature.

The behavior of normalized volume, defined by Li in [Li18], provides another reason to expect that a statement like Theorem 1.1 holds. The normalized volume is a numerical invariant of klt singularities in characteristic 0 which provides a local notion of stability and is believed to have deep connections with the \(F\)-signature. Indeed, these invariants exhibit similar behavior in a number of situations [LLX19] [MPST19]. In pursuit of a theory of adjunction for normalized volume, Li, Liu, and Xu proved the following.

**Theorem 1.3.** ([LLX19, Proposition 6.8]) Let \(x \in (X, \Delta)\) be an \(n\)-dimensional klt singularity. Let \(D\) be a normal \(\mathbb{Q}\)-Cartier divisor containing \(x\) such that \((X, D + \Delta)\) is plt. Denote by \(\text{Diff}_D(\Delta)\) the different of \(\Delta\) on \(D\). Then

\[
\lim_{\varepsilon \to 0^+} \frac{\hat{\text{vol}}(x, X, \Delta + (1 - \varepsilon)D)}{n^n \varepsilon} = \frac{\hat{\text{vol}}(x, D, \text{Diff}_D(\Delta))}{(n-1)^{n-1}}.
\]

Up to sign conventions and normalization of the invariant, this theorem is precisely the formula in our main theorem. So \(F\)-signature and normalized volume behave identically with respect to adjunction along codimension one centers, which supports the belief that these invariants are connected in some way.

The paper is structured as follows. In Section 2, we gather the necessary background on Cartier subalgebras, \(F\)-signature, and Hilbert-Kunz multiplicity. In Section 3, we recall Schwede’s theory of \(F\)-adjunction [Sch09] and derive some consequences for the \(F\)-signature. In particular, we show that when \(D\) is Cartier, the \(F\)-signature of the pair \((\mathcal{O}_D, \text{Diff}_D(\Delta))\) may be computed as a limit involving data from the pair \((R, \Delta)\) and the defining equation of \(D\) (see Lemma 3.6). In Section 4, we review the cyclic cover construction and transformation rules for \(F\)-signature under finite morphisms. This construction allows us to reduce to the case where \(D\) is Cartier. The proof of the main theorem occupies Section 5.

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2. **Cartier Subalgebras, \(F\)-Signature, and Hilbert-Kunz Multiplicity**

2.1. **Conventions.** All rings are assumed to be a commutative with 1, Noetherian, and of characteristic \(p > 0\) unless otherwise specified. If \(R\) is such a ring, we have the **Frobenius maps** \(F^e : R \to R\) given by \(r \mapsto r^p^e\) for each \(e \geq 1\). If \(I \subseteq R\), we denote by \(I[p^e] = \langle a^p^e : a \in I \rangle\), the \(p^e\)th **Frobenius power** of \(I\). If \(M\) is an \(R\)-module, we denote by \(F^e_* M\) the restriction of scalars of \(M\) along \(F^e\). That is, \(R\) acts on \(F^e_* M\) via \(r \cdot F^e_* m = F^e_* (r^p^e m)\). A ring \(R\) is **\(F\)-finite** if \(F^e_* R\) is a finite \(R\)-module.

_In what follows, we assume that all rings of characteristic \(p > 0\) are \(F\)-finite._
Note that any ring essentially of finite type over an $F$-finite field is $F$-finite.

The index of a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $\Delta$ is the smallest integer $m$ such that $m\Delta$ is Cartier. A pair $(R, \Delta)$ is said to be $\mathbb{Q}$-Gorenstein if $K_X + \Delta$ is $\mathbb{Q}$-Cartier on $X = \text{Spec } R$, and the index of the pair $(X, \Delta)$ is the index of $K_X + \Delta$. For divisors $D = \sum a_iD_i$ and $D' = \sum a'_iD_i$, we define $D \wedge D' = \sum \min\{a_i, a'_i\}D_i$.

2.2. Cartier Subalgebras. The primary reference for this subsection is [BST12]. Other useful sources include [Bli13, BS13, ST12].

Let $M$ and $N$ be $R$-modules. A $p^{-k}$-linear map from $M$ to $N$ is an $R$-linear map $\phi : F^e_*M \to N$. The abelian group $\text{Hom}_R(F^e_*M, N)$ has an $R$-module structure via pre-multiplication and an $F^e_*R$-module structure via pre-multiplication. The direct sum

$$\mathcal{C}^R = \bigoplus_{e \geq 0} \mathcal{C}_e^R = \bigoplus_{e \geq 0} \text{Hom}_R(F^e_*R, R)$$

has a natural (non-commutative) graded $R$-algebra structure where multiplication of homogeneous elements in $\mathcal{C}^R$ is given by composition. That is, if $\phi_e \in \text{Hom}_R(F^e_*R, R)$ and $\phi_f \in \text{Hom}_R(F^f_*R, R)$, then $\phi_e \cdot \phi_f$ is defined to be the composition

$$F^e_*R \xrightarrow{F^f_*\phi_f} F^f_*R \xrightarrow{\phi_e} R.$$ 

Definition 2.1. [BST12, Definition 2.4] The ring $\mathcal{C}^R$ constructed above is the (total) Cartier algebra of $R$. A Cartier subalgebra on $R$ is a graded $\mathbb{F}_p$-subalgebra $\mathcal{D} = \bigoplus_{e \geq 0} \mathcal{D}_e$ of $\mathcal{C}^R$ such that $\mathcal{D}_0 = \text{Hom}_R(R, R) = R$ and $\mathcal{D}_e \neq 0$ for some $e > 0$. For a Cartier subalgebra $\mathcal{D}$, the set $\Gamma_{\mathcal{D}} := \{e \in \mathbb{Z}_{\geq 0} : \mathcal{D}_e \neq 0\}$ is a semigroup.

Our main interest is in Cartier subalgebras arising from triples $(R, \Delta, a^t)$ where $R$ is a normal ring, $\Delta$ is an effective $\mathbb{Q}$-divisor on $\text{Spec } R$, $a \subseteq R$ is an ideal not contained in a minimal prime of $R$, and $t$ is a positive real number. Given such a triple $(R, \Delta, a^t)$, we define

$$\mathcal{C}^{\Delta, a^t} = \bigoplus_{e \geq 0} \mathcal{C}^\Delta_{e} a^t = \bigoplus_{e \geq 0} \text{Hom}_R(F_*R([\lceil p^e - 1 \rceil \Delta]), R) \cdot F^e_*a^{t(p^e - 1)}.$$ 

For an ideal $J \subseteq R$, we denote by $\text{Hom}_R(F^e_*M, N) \cdot F^e_*J$ the submodule consisting of elements of the form $F_*a \cdot \phi$ where $a \in J$ and $\phi \in \text{Hom}_R(F^e_*M, N)$.

The classical definitions of $F$-regularity and $F$-purity extend to general Cartier subalgebras.

Definition 2.2. [HR76, HH89, Sch11, BST12] Let $\mathcal{D}$ be a Cartier subalgebra on a local ring $R$. We say $(R, \mathcal{D})$ is (sharply) $F$-pure if there is a surjective homomorphism in $\mathcal{D}_e$ for some $e$. We say $(R, \mathcal{D})$ is (strongly) $F$-regular if for all $e \in R$ not contained in a minimal prime, there exists $\phi \in \mathcal{D}_e$ for some $e$ such that $\phi(F^e_*c) = 1$. A triple $(R, \Delta, a^t)$ is called $F$-pure (resp. $F$-regular) if the Cartier subalgebra $\mathcal{C}^{\Delta, a^t}$ is $F$-pure (resp. $F$-regular).

Proposition 2.3. [AE03, Sch10, BST12] Let $\mathcal{D}$ be a Cartier subalgebra on a local ring $(R, \mathfrak{m})$. For every $e \geq 1$, let $I_e^{\mathcal{D}} = \{r \in R : \phi(F^e_*r) \in \mathfrak{m} \text{ for all } \phi \in \mathcal{D}_e\}$. The ideal

$$P_{\mathcal{D}} := \bigcap_{e \in \Gamma_{\mathcal{D}}} I_e^{\mathcal{D}}.$$ 

is called the splitting prime of $(R, \mathcal{D})$. The ideal $P_{\mathcal{D}}$ is proper if and only if $(R, \mathcal{D})$ is $F$-pure, and in this case, $P_{\mathcal{D}}$ is prime. Furthermore, $P_{\mathcal{D}} = 0$ if and only if $(R, \mathcal{D})$ is $F$-regular.
**Definition 2.4.** [BST12, Sch10] Let $\mathcal{D}$ be a Cartier subalgebra on $R$. An ideal $J \subseteq R$ is $\mathcal{D}$-compatible if for every $e \in \Gamma_\mathcal{D}$ and every $\phi \in \mathcal{D}_e$, we have $\phi(F^e J) \subseteq J$. In this case, there is an induced map $\phi_J : F^e_*(R/J) \to R/J$ which makes the following diagram commute

$$
\begin{array}{ccc}
F^e_* R & \xrightarrow{\phi} & R \\
\downarrow & & \downarrow \\
F^e_*(R/J) & \xrightarrow{\phi_J} & R/J
\end{array}
$$

where the vertical arrows are the natural quotient maps. The induced Cartier subalgebra $\mathcal{D}_J$ on $R/J$ is the Cartier subalgebra whose $e$th graded piece is $(\mathcal{D}_J)_e := \{ \phi_J : \phi \in \mathcal{D}_e \}$.

**Remark 2.5.** For a pair $(R, \mathcal{D})$, the splitting prime $P_\mathcal{D}$ is the largest $\mathcal{D}$-compatible ideal. The subscheme defined by $P_\mathcal{D}$ is called the minimal center of $F$-purity [Sch10].

**Example 2.6.** (cf. [Sch09, Section 7]) Suppose $(R, \Delta)$ is a pair where $R$ is a normal local domain and $\Delta$ is an effective $\mathbb{Q}$-divisor. Suppose $D$ is a normal, reduced, irreducible $\mathbb{Q}$-Cartier divisor on Spec $R$ which is not a component of Supp $\Delta$. Let $p$ be the ideal defining $D$. Suppose $K_X + \Delta + D$ is $\mathbb{Q}$-Cartier with index prime to $p$. The pair $(X, \Delta + D)$ is $F$-pure at the generic point of $D$ since $X$ is normal (and hence regular in codimension 1). Also $p$ is $\mathcal{O}^{\Delta + D}$-compatible, so $p \subseteq P_{\mathcal{O}^{\Delta + D}}$.

Before concluding this subsection, we record an inclusion of ideals which will be used in the proof of the main theorem.

**Lemma 2.7.** (cf. [Tuc12, Lemma 4.4]) Suppose $R$ is a reduced local ring and $\Delta$ is an effective $\mathbb{Q}$-divisor on Spec $R$. For any element $f \in R$ not in a minimal prime of $R$ and positive integers $r \geq e$, we have

$$(I^\Delta_e : f)^{[p^{r-e}]} \subseteq (I^\Delta_r : f^{p^{r-e}}).$$

**Proof.** If $a \in (I^\Delta_e : f)$ and $\phi \in \text{Hom}_R(F^e_* R([p^{r-1} \Delta]), R)$, then

$$\phi(F^e_* (f^{p^{r-e}} a^{p^{r-e}})) = \phi|_{F^e_* R}(F^e_* (fa)) \in \mathfrak{m}.$$
exists and is called the $F$-signature of $(R, \mathcal{D})$. When $\mathcal{D} = \mathcal{D}^{\Delta, a^t}$ is the Cartier subalgebra associated to a triple $(R, \Delta, a^t)$, we write $s(R, \Delta, a^t)$.

The next proposition is useful for computations. One would not lose much in what follows by taking it as the definition of $F$-signature.

**Proposition 2.9.** [BST13, Proposition 2.2] Let $(R, \Delta, a^t)$ be a triple. Then

$$s(R, \Delta, a^t) = \lim_{e \to \infty} \frac{1}{p^e d} \ell_R \left( \frac{R}{(I_e^{\Delta} : a^{(t/p^e-1)})} \right).$$

We also have

$$s(R, \Delta, a^t) = \lim_{e \to \infty} \frac{1}{p^e d} \ell_R \left( \frac{R}{(I_e^{\Delta} : a^{(t/p^e)})} \right).$$

As stated previously, the $F$-signature measures singularities, with smaller $F$-signature corresponding to worse singularities.

**Theorem 2.10.** Let $R$ be a local ring.

1. [HL02, Corollary 16] The ring $R$ is regular if and only if $s(R) = 1$.
2. [AL03, BST12] If $\mathcal{D}$ is a Cartier subalgebra on $R$, then $(R, \mathcal{D})$ is $F$-regular if and only if $s(R, \mathcal{D}) > 0$.

In the case $(R, \mathcal{D})$ is $F$-pure but not $F$-regular (so $s(R, \mathcal{D}) = 0$), there is still a numerical invariant to consider.

**Theorem 2.11.** [BST12, Theorem 4.2] Let $\mathcal{D}$ be a Cartier subalgebra on a local ring $R$. Suppose $(R, \mathcal{D})$ is $F$-pure with $F$-splitting prime $P_{\mathcal{D}}$. Then the limit

$$r_F(R, \mathcal{D}) = \lim_{e \in \Gamma_{\mathcal{D}} \to \infty} \frac{1}{p^e d \dim R/P_{\mathcal{D}}} \ell_R \left( R/I_e^\mathcal{D} \right)$$

exists and is called the $F$-splitting ratio. The integer $\dim R/P_{\mathcal{D}}$ is called the splitting dimension of $(R, \mathcal{D})$. Furthermore, if $\mathcal{D}_{P_{\mathcal{D}}}$ denotes the induced Cartier subalgebra on $R/P_{\mathcal{D}}$, then

$$r_F(R, \mathcal{D}) = s(R/P_{\mathcal{D}}, \mathcal{D}_{P_{\mathcal{D}}})$$

so that, in particular, $0 < r_F(R, \mathcal{D}) \leq 1$.

Our main theorem concerns the behavior of $s(R, \Delta + tD)$ as a function of the real parameter $t$. Blickle, Schwede, and Tucker established some formal properties of the function $s(R, \Delta, a^t)$ in [BST13].

**Proposition 2.12.** [BST13, Section 3] Suppose $R$ is a normal local domain, $\Delta$ is an effective $\mathbb{R}$-divisor on Spec $R$, and $D$ is a $\mathbb{Q}$-Cartier divisor. Then the function $s_{\Delta, D}(t) := s(R, \Delta + tD)$ is continuous and convex in $t$. In particular, $D \cdot s_{\Delta, D}^{\Delta, D}(1)$ exists.

**Proof.** Suppose $nD$ is Cartier with defining equation $f \in R$. Then we have

$$s(R, \Delta + tD) = s \left( R, \Delta + \frac{t}{n} \text{div } f \right) = s(R, \Delta, f_{t/n}).$$

From [BST13, Theorem 3.9], we have that $s(R, \Delta, a^t)$ is continuous. Furthermore, it is convex if $a$ is principal. So $s(R, \Delta + tD)$ is continuous and convex, as it is equal to $f_{t/n}$ for a continuous and convex function $f$. □
2.4. Hilbert-Kunz Multiplicity.

**Definition 2.13.** Let $I \subseteq R$ be an ideal of finite colength. We define the Hilbert-Kunz multiplicity of $I$ along $R$ by

$$e_{HK}(I; R) := \lim_{e \to \infty} \frac{1}{p^e} \ell_R \left( R/I^{[p^e]} \right).$$

Monsky proved that this limit exists in [Mon83]. The first proof of the existence of $F$-signature in general used the existence of Hilbert-Kunz multiplicity [Tuc12], establishing the following precise relationship between the two invariants.

**Theorem 2.14.** [Tuc12, BST12] Suppose $\mathcal{D}$ is a Cartier algebra on a local ring $R$ of dimension $d$. Then

$$s(R, \mathcal{D}) = \lim_{e \in \Gamma_{\mathcal{D}} \to \infty} \frac{1}{p^e} e_{HK}(I^g_e, R).$$

3. $F$-Adjunction

3.1. The Different. Let $(R, \Delta)$ be a pair where $R$ is a normal, local domain with an effective $\mathbb{Q}$-divisor $\Delta$ on $\text{Spec } R$. Let $D$ be a normal, prime $\mathbb{Q}$-Cartier divisor on $\text{Spec } R$ such that $\Delta \wedge D = 0$. Furthermore, assume that the pair $(R, \Delta + D)$ is log $\mathbb{Q}$-Gorenstein with index prime to $p$. When $X$ and $D$ are regular, the classical adjunction formula states that $(K_X + D)|_D = K_D$. In what follows, we construct this divisor via $p^{-e}$-linear maps.

As laid out explicitly in [Sch09], for any normal $X = \text{Spec } R$, there is a bijection

$$\{ \text{Effective } \mathbb{Q}\text{-divisors } \Delta \text{ such that } (p^e - 1)(K_X + \Delta) \text{ is Cartier} \} \leftrightarrow \{ \text{Non-zero elements of } \text{Hom}_R(F_e^e R, R) \} / \sim$$

where two maps are $\sim$-equivalent if they differ by pre-multiplication by a unit of $R$. Indeed, Grothendieck duality for a finite map gives an isomorphism $\text{Hom}_R(F_e^e R, R) \cong R((1 - p^e)K_X)$ [Har66]. Given a map $\phi : F_e^e R \to R$, we associate an effective divisor $D_\phi \sim (1 - p^e)K_X$.

We normalize $D_\phi$ by defining $\Delta_\phi := \left(\frac{1}{p^e - 1}\right) D_\phi$. See [BS13, Section 4] for the details of this correspondence and its generalizations.

**Definition 3.1.** [Sch09] With notation as above, $(p^e - 1)(K_X + \Delta + D)$ is Cartier for some $e$, so $\Delta + D$ corresponds to an $R$-linear map $\phi_{\Delta + D} : F_e^e R \to R$. Since $D$ is a $\mathcal{C}^{\Delta + D}$-compatible subscheme, there is an induced map $\overline{\phi_{\Delta + D}} : F_e^e \mathcal{O}_D \to \mathcal{O}_D$ making the following diagram commute

$$
\begin{array}{ccc}
F_e^e R & \xrightarrow{\phi_{\Delta + D}} & R \\
\downarrow & & \downarrow \\
F_e^e \mathcal{O}_D & \xrightarrow{\overline{\phi_{\Delta + D}}} & \mathcal{O}_D.
\end{array}
$$

Since $D$ is normal, the correspondence (1) holds for $D$, and the divisor associated to $\overline{\phi_{\Delta + D}}$, denoted $\text{Diff}_D(\Delta)$, is called the different of $\Delta$ on $D$.

The proposition below lists the relevant properties of the different. Both follow quickly from construction and are proven carefully in [Sch09].
Proposition 3.2. Let \((X, \Delta + D)\) be a log \(\mathbb{Q}\)-Gorenstein pair with index prime to \(p\) where \(D\) is a normal, irreducible \(\mathbb{Q}\)-Cartier divisor. Suppose \((p^e - 1)(K_X + \Delta + D)\) is Cartier.

1. \((p^e - 1)(K_D + \text{Diff}_D(\Delta))\) is Cartier.
2. The projection map
   \[\text{Hom}_R(F^e_* R((p^e - 1)(\Delta + D)), R) \rightarrow \text{Hom}_{O_D}(F^e_* O_D((p^e - 1) \text{Diff}_D(\Delta)), O_D)\]
   is surjective for all \(e\) such that \(e_0|e\).

Remark 3.3. This construction of the different was introduced in [Sch09], where it was called the \(F\)-different, in order to formulate adjunction statements in the context of \(F\)-singularities. In birational geometry, there is a divisor called the different which appears in the theory of adjunction [Kol13, Chapter 4]. In our case (i.e. where \(D\) is a divisor), the different and the \(F\)-different are equal [Das15, Theorem 5.3], so for the rest of the paper, we drop the “\(F\)” prefix. It should be noted however that for higher codimension centers of \(F\)-purity, the \(F\)-different and the different do not coincide in general [DS17].

The proof of our main theorem only uses the construction of Definition 3.1, but for completeness, we briefly review the geometric construction of the different closely following [Kol13, Chapter 4]. For the most efficient path to the proof of the main theorem, the reader can skip directly to Section 3.2.

Let \(X\) be a normal variety, \(D\) a reduced, effective Weil divisor and \(\Delta\) an effective \(\mathbb{Q}\)-divisor that shares no components with \(D\). Suppose \(m(K_X + D + \Delta)\) is Cartier for some \(m \geq 1\), and let \(\nu : D' \rightarrow D\) be the normalization map\(^1\). Let \(Z \subseteq D\) be the non-regular locus of \(D\) and \(D \cap \text{Supp}(\Delta)\). Note that \(Z\) has codimension at least 1 in \(D\) by assumption. Since \(D \setminus Z\) is regular, the Poincaré residue map provides an isomorphism

\[\mathcal{R}_D : \omega_X(D)|_{D \setminus Z} \sim \omega_D|_{D \setminus Z}.\]

Moreover, \(\nu|_{D' \setminus \nu^{-1}(Z)} : D' \setminus \nu^{-1}(Z) \rightarrow D \setminus Z\) is an isomorphism and \(\text{Supp}(\Delta) \cap S \subseteq Z\). So taking reflexive powers\(^2\) of \(\mathcal{R}_D\) yields an isomorphism

\[\mathcal{R}^m_D : \nu^*(\omega_X^m(mD + m\Delta))|_{D' \setminus \nu^{-1}(Z)} \sim \omega_{D'}^m|_{D' \setminus \nu^{-1}(Z)}.\]

Let \(D'_{\text{reg}} \subseteq D'\) be the regular locus of \(D'\). So \(\nu^* \omega_X^m(mD + m\Delta)\) and \(\omega_{D'}^m\) are invertible sheaves on \(D'_{\text{reg}}\) which implies

\[\mathcal{H}om_{O_{D'\setminus \nu^{-1}(Z)}}(\nu^*(\omega_X^m(mD + m\Delta)), \omega_{D'}^m)\]

is invertible on \(D'_{\text{reg}}\). Furthermore, \(\mathcal{R}^m_D\) is a rational section of this invertible sheaf. As such, there is a unique divisor \(\Delta_{D'_{\text{reg}}}\) on \(D'_{\text{reg}}\) such that \(\mathcal{R}^m_D\) extends to an isomorphism

\[(2) \quad \mathcal{R}^m_{D'_{\text{red}}} : \nu^*(\omega_X^m(mD + m\Delta)) \sim \omega_{D'}^m(\Delta_{D'_{\text{reg}}})\]

\(^1\)The different exists in greater generality. See [Kol13, Chapter 4] for a discussion under a set of minimal assumptions.

\(^2\)For a line bundle \(L\), the symbol \(L^m\) denotes the reflexification (or S2-ification) of the sheaf \(L^{\otimes m}\). That is \(L^m := (L^{\otimes m})^{**}\). This is the only time we use the reflexive powers notation, so it should not be confused with the notation for the Frobenius power of an ideal.
on all of $D'_{reg}$. As $D'$ is normal, $D' \setminus D'_{reg}$ has codimension at least 2 in $D'$, and $\Delta_{D'_{reg}}$ extends to a uniquely to a divisor $\Delta_{D'}$ on $D'$. The different of $\Delta$ on $D$ is defined to be $\text{Diff}_{D'}(\Delta) = \frac{1}{m} \Delta_{D'}$. Note that $(K_X + D + \Delta)|_{D'} \sim_{Q} K_{D'} + \text{Diff}_{D'}(\Delta)$.

Example 3.4. (An singularities, [Kol13, Example 4.3]) Let $R = S/f = \mathbb{F}_p[[x, y, z]]/(xy - z^{n+1})$ with $n \geq 1$ and $p \geq 3$, and let $X = \text{Spec } R$. Let $D = V(x, z)$, so $D$ is $\mathbb{Q}$-Cartier with index $n+1$. Indeed, $(n+1)D = \text{div } x$. We compute the different $\text{Diff}_D(0)$ in two ways. First, we use the geometric construction. The canonical module $\omega_X$ is generated by $\frac{1}{y} dy \wedge dz$. So $\omega_X^{[n+1]}((n+1)D)$ is locally free with generator

$$\frac{(dy \wedge dz)^{\otimes(n+1)}}{xy^{n+1}} = \frac{(dx \wedge dz)^{\otimes n}}{z^{n+1}y^n} = \frac{(dz)^{\otimes n+1}}{z^{n+1} - \frac{dx}{x^n}}.$$ 

The residue of the generator is $\left(\frac{n}{n+1}\right) [0]$ where $[0]$ is the class of the point $0 \in D \cong A^1$ with coordinate $y$. So the different is $\text{Diff}_D(0) = (1-1/(n+1))[0]$. Note that $\text{Diff}_D(0)$ is non-zero despite the fact that $D$ is regular.

Now, let us compute the different via Frobenius. With this approach, we require that the index of $D$ be prime to $p$, so there is some $e$ such that $n|p^e - 1$. Let $\Phi^e : F_*^e S \to S$ be the map defined by

$$\Phi^e(x^i y^j z^k) = \begin{cases} 1 & i = j = k = p^e - 1 \\ 0 & \text{otherwise} \end{cases}$$

for $0 \leq i, j, k \leq p^e - 1$. This map generates $\text{Hom}_S(F_*^e S, S)$ as an $F_*^e S$-module. Note that $\Phi^e(f^{p^e-1} \cdot -)$ induces a map $\phi^e : F_*^e R \to R$ that generates $\text{Hom}_R(F_*^e R, R)$ (see [BST12, Example 4.3.2]). The map associated to the divisor $D$ is $\phi^e(x^{(p^e-1)/(n+1)} \cdot -)$. Since $D$ is $\phi^e(x^{(p^e-1)/(n+1)} \cdot -)$-compatible, there is an induced map $\phi_{\text{Diff}_D(0)}^e : F_*^e \mathcal{O}_D \to \mathcal{O}_D$. Since $\mathcal{O}_D = \mathbb{F}_p[[y]]$, the map $\phi_{\text{Diff}_D(0)}^e$ is determined by its values on $F_*^e y^i$ for $0 \leq i \leq p^e - 1$. We have a diagram

$$\begin{array}{ccc} F_*^e S & \longrightarrow & F_*^e R \\
\Phi^e(f^{p^e-1}x^{\frac{p^e-1}{n+1}} \cdot -) & | & \phi^e(x^{\frac{p^e-1}{n+1}} \cdot -) \\
S & \longrightarrow & R \\
\phi_{\text{Diff}_D(0)}^e & | & \phi_{\text{Diff}_D(0)}^e \\
& | & \mathbb{F}_p[[y]] \end{array}$$

That is, $\phi_{\text{Diff}_D(0)}^e(F_*^e y^i) = \Phi^e(F_*^e(f^{p^e-1}x^{\frac{p^e-1}{n+1}} y^i)) \pmod{(x, z)}$. So we compute

$$\Phi^e \left( F_*^e(f^{p^e-1}x^{\frac{p^e-1}{n+1}} y^i) \right) = \Phi^e \left( F_*^e \left( \sum_{k=0}^{p^e-1} \binom{p^e-1}{k} x^{k + \frac{p^e-1}{n+1}} y^{k+i} z^{(n+1)(p^e-1-k)} \right) \right)$$

$$\equiv \Phi^e \left( F_*^e \left( \frac{p^e-1}{n(p^e-1)} \right) x^{p^e-1} y^{\frac{n(p^e-1)}{n+1} + i} z^{p^e-1} \right) \pmod{(x, z)}$$

$$\equiv \begin{cases} (n(p^e-1)) & \pmod{(x, z)} & i = \frac{p^e-1}{n+1} \\ 0 & \pmod{(x, z)} & \text{otherwise} \end{cases}$$
Since this binomial coefficient is a unit in $\mathbb{F}_p[[y]]$, we see that the divisor associated to this map is

$$\text{Diff}_D(0) = \frac{1}{p^e - 1} \left( \frac{n(p^e - 1)}{n + 1} \right) \text{div } y = \left( 1 - \frac{1}{n + 1} \right) [0].$$

3.2. \textit{F-Adjunction for Cartier Subalgebras and F-signature.} In this subsection, we specify the relationship between $\mathcal{C}^{\Delta + D}$ on $R$ and $\mathcal{C}^{\text{Diff}_D(\Delta)}$ on $\mathcal{O}_D$.

Proposition 3.5. Let $(R, \Delta + D)$ be a log $\mathbb{Q}$-Gorenstein pair with index prime to $p$. Assume $R$ is a normal local ring, $\Delta$ is an effective $\mathbb{Q}$-divisor, and $D$ is an irreducible, reduced, normal, $\mathbb{Q}$-Cartier divisor with $D \land \Delta = 0$.

1. The $th$ graded piece of $\mathcal{C}^{\text{Diff}_D(\Delta)}$ is equal to the $th$ graded piece of the Cartier subalgebra on $\mathcal{O}_D$ induced by $\mathcal{C}^{\Delta + D}$ (as in Definition 2.4) for sufficiently divisible $e$.

2. For sufficiently divisible $e$, the ideal $I_{e}^{\text{Diff}_D(\Delta)}$ is the extension of $I_{e}^{\Delta + D}$ in $\mathcal{O}_D$. In particular, $\ell_R(R/I_{e}^{\Delta + D}) = \ell_{\mathcal{O}_D}(\mathcal{O}_D/I_{e}^{\text{Diff}_D(\Delta)})$ for sufficiently divisible $e$.

Proof. Assume $(p^e - 1)(K_X + D + \Delta)$ is Cartier. The $th$ graded piece of the Cartier subalgebra induced by $\mathcal{C}^{\Delta + D}$ is the collection of maps $\overline{\psi} : F^e_*\mathcal{O}_D \to \mathcal{O}_D$ which fit into a diagram

$$
\begin{array}{ccc}
F^e_*R & \xrightarrow{\psi} & R \\
\downarrow & & \downarrow \\
F^e_*\mathcal{O}_D & \xrightarrow{\overline{\psi}} & \mathcal{O}_D
\end{array}
$$

for some $\psi : F^e_*R \to R$, i.e. it is the image of the projection map

$$\text{Hom}(F^e_*R((p^e - 1)(\Delta + D)), R) \to \text{Hom}_{\mathcal{O}_D}(F^e_*\mathcal{O}_D((p^e - 1)\text{Diff}_D(\Delta)), \mathcal{O}_D).$$

By Proposition 3.2, this map is surjective for all $e$ such that $e_0 \mid e$. So the first statement follows.

Part (2) follows from (1) given the definition of the ideals $I_{e}^{\text{Diff}_D(\Delta)}$ and $I_{e}^{\Delta + D}$. The length statement uses the fact since $D$ is $\mathcal{C}^{\Delta + D}$-compatible, so its ideal is contained in each $I_{e}^{\Delta + D}$.

The following consequence of Proposition 3.5 says that we may compute the $F$-signature of the pair $(\mathcal{O}_D, \text{Diff}_D(\Delta))$ from the pair $(R, \Delta + D)$. Our proof of Theorem 5.3 only requires the case where $D$ is Cartier, but we include the general case since its proof is no harder.

Lemma 3.6. Let $(R, \Delta + D)$ be a log $\mathbb{Q}$-Gorenstein pair with index prime to $p$. Assume $R$ is a normal local ring of dimension $d$, $\Delta$ is an effective $\mathbb{Q}$-divisor, and $D$ is a normal $\mathbb{Q}$-Cartier divisor of index $m$ such that $D \land \Delta = 0$ and $(m, p) = 1$. If $mD = \text{div}(f)$, and $m|(p^e - 1)$, then

$$s(\mathcal{O}_D, \text{Diff}_D(\Delta)) = \lim_{e \to \infty} \frac{1}{p^e(d-1)} \ell_R \left( \frac{R}{I_{e}^{\Delta + D} : f^{(p^e - 1)/m}} \right).$$

If $D$ is the minimal center of $F$-purity of the pair $(R, \Delta + D)$ then $s(\mathcal{O}_D, \text{Diff}_D(\Delta)) = r(R, \Delta + D)$. Otherwise $s(\mathcal{O}_D, \text{Diff}_D(\Delta)) = 0$. 

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Proposition 4.1. Let $(R, m)$, $D$, and $C(D)$ be defined as above.

(1) The ring $C(D)$ is local with maximal ideal $n = m \oplus \left( \bigoplus_{i=1}^{n-1} R(-iD) \right)$, and $\pi$ is a finite map of degree $n$.
(2) If $R$ is strongly $F$-regular, then $C(D)$ is strongly $F$-regular.
(3) The divisor $D' := \pi^* D$ is a Cartier divisor on $\text{Spec } C(D)$.
(4) The restriction $\pi|_{D'} : D' \to D$ is a finite cover of degree $n$.

Proof. For (1), the maximal ideal of $C(D)$ is computed in [CR18a, Proposition 5.3.1]. Since each $R(-iD)$ is a rank one reflexive module, we see that $\text{Frac} C(D) = C(D) \otimes_R \text{Frac } R \cong (\text{Frac } R)^{\oplus n}$; so the map is finite of degree $n$ (see [TW92] for more details).

Part (2) is [CR18a, Corollary 6.3.1].
For (3), let $nD = \text{div}(f)$. Then $(ft^{n-1})^n = f^n/f^{n-1} = f$ in $C(D)$. So

$$n \cdot \pi^* D = \pi^* \text{div}_R(f) = \text{div}_{C(D)}(f) = \text{div}_{C(D)}((ft^{n-1})^n) = n \cdot \text{div}_{C(D)}(ft^{n-1}).$$

Dividing both sides by $n$ gives $\pi^* D = \text{div}_{C(D)}(ft^{n-1})$ as required.

For (4), we compute the ring $\mathcal{O}_{D'}$ explicitly. Let $p$ be the ideal of $D$ in $R$. By (3), the ideal defining $D'$ is $(ft^{n-1})$. We claim that $\mathcal{O}_{D'} = R/p \oplus (p/p^{(2)}) t \oplus \cdots \oplus (p^{(n-1)}/p^{(n)}) t^{n-1}$. Clearly $(ft^{n-1}) \cap R = p$. For the higher degree terms, any element of $(ft^{n-1}) \cap p^{(j)}$ is of the form $(ft^{n-1})(at^{j+1})$ for $a \in p^{j+1}$. Then $(ft^{n-1})(at^{j+1}) = aft^{n+j} = at^j \in p^{j+1}$. For the reverse inclusion, if $at^j \in p^{j+1}t^j$, then $at^j = at^{j+1} \cdot ft^{n-1} \in (ft^{n-1})$. The map $D' \to D$ is induced by the inclusion of $R/p$ into the degree 0 piece of $\mathcal{O}_{D'}$, and the result follows.

The behavior of $F$-signature under finite maps was studied in [CRST18] [CR18b] and [CR18a]. We will use the following transformation rule to lift our calculation to the cyclic cover.

**Theorem 4.2.** [CR18a, Theorem 3.0.1], [CR18b, Theorem 4.9] Let $(R, \mathfrak{m}) \subseteq (S, \mathfrak{n})$ be a local extension of normal domains with corresponding morphism of schemes $f : Y \to X$. Let $\Delta$ be an effective $\mathbb{Q}$-divisor on $X$. Suppose there is a nonzero morphism of $S$-modules $\tau : S \to \text{Hom}_R(f_*, S, R) = \omega_{S/R}$ such that $T := \tau(1)$ is surjective, $T(\mathfrak{n}) \subseteq \mathfrak{m}$ and $\Delta^* = f^* \Delta - \text{Diff}_{\mathfrak{n}}$ is effective on $Y$. Then

$$[\kappa(\mathfrak{n}) : \kappa(\mathfrak{m})] \cdot s(S, \Delta^*) = [\text{Frac} S : \text{Frac} R] \cdot s(R, \Delta).$$

Furthermore, if $(R, \mathfrak{m}) \subseteq (S, \mathfrak{n})$ is merely a local extension where $R$ is a domain and $S$ is a reflexive $R$-module and $\tau$ is an isomorphism, then

$$[\kappa(\mathfrak{n}), \kappa(\mathfrak{m})] \cdot s(S, \Delta^*) = \dim_K S_K \cdot s(R)$$

where $K = S_K$ is the generic fiber of $R \subseteq S$.

**Remark 4.3.** This theorem is more general than the one proven in [CRST18] which assumes that the extension is separable. In the separable case, the field trace $\text{Tr}_{L/K} : L \to K$ restricts to an $R$-linear map $\text{Tr}_{S/R} : S \to R$. Theorem 4.2 generalizes [CRST18] by replacing $\text{Tr}_{S/R}$ with the map $T$ which satisfies similar formal properties. This extra generality allows us to extend our theorem to the case where $p$ divides the index of $D$. Indeed for cyclic covers, we have the map $T : C(D) \to R$ given by projection onto the degree 0 summand. This divisor generates $\text{Hom}_R(C(D), R)$ as a $C(D)$-module, so $D_T = 0$. For further details, see [CR18a].

**Remark 4.4.** Other invariants of Cartier subalgebras seem to behave well with respect to finite morphisms. In [CRS19], Carvajal-Rojas and Stäbler systematically treat the behavior of $F$-signature, $F$-splitting ratio, splitting primes, and the test ideal of arbitrary Cartier subalgebras under finite maps.

**Corollary 4.5.** Let $(R, \Delta)$ be a strongly $F$-regular pair, where $R$ is an $F$-finite local ring and $\Delta \geq 0$. Let $D$ be a reduced, irreducible $\mathbb{Q}$-Cartier divisor such that $\mathcal{O}_D$ is strongly $F$-regular. For the extension $R \subseteq C(D)$ described above, with associated map of schemes $\pi : \text{Spec } C(D) \to \text{Spec } R$, we have the following.

1. $m \cdot s(R, \Delta) = s(C(D), \pi^* \Delta)$.
2. If $D' = \pi^* D$, then $\mathcal{O}_{D'}$ is strongly $F$-regular.
3. If the index of $(R, \Delta + D)$ is prime to $p$, then $m \cdot s(D, \text{Diff}_D(\Delta)) = s(D', \text{Diff}_{D'}(\pi^* \Delta))$.  

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Proof. By Proposition 4.1, \(C(D)\) is a normal local ring with maximal ideal \(n = m \oplus \left( \bigoplus_{i=1}^{m-1} R(-iD)u^i \right)\). In particular, \(\kappa(m) = \kappa(n)\). Moreover, \([\text{Frac} S : \text{Frac} R] = m\) since each \(R(iD)\) is a rank one reflexive sheaf. Hence the transformation rule (1) holds by Theorem 4.2 and the fact that the projection to the degree 0 piece \(T : \text{Frac} C(D) \to R\) generates \(\text{Hom}_R(C(D), R)\) as a \(C(D)\)-module. Note that this statement does not require the strong \(F\)-regularity of \(D\).

For (2), \(\mathcal{O}_{D'}\) is a \(\mathbb{Z}/n\mathbb{Z}\)-graded ring over \(\mathcal{O}_D\) so is \(S_2\) by [TW92]. So the final statement of Theorem 4.2 implies that \(s(\mathcal{O}_{D'}) > 0\) as long as \(s(\mathcal{O}_D) > 0\). So \(F\)-regularity of \(\mathcal{O}_{D'}\) follows from the \(F\)-regularity of \(D\).

For (3), since strongly \(F\)-regular rings are normal, we have the divisor-maps correspondence discussed in Section 3.1. Let \(\pi|_{\mathcal{O}_D'} : D' \to D\) be the restriction of \(\pi\) to \(D'\). Let \(\mathcal{T} : \mathcal{O}_{D'} \to \mathcal{O}_D\) be the restriction of the projection map \(T : \text{Frac} C(D) \to R\). Note that \(\mathcal{T}\) is surjective and \(\mathcal{T}(m_{\mathcal{O}_D}) = m_{\mathcal{O}_D}\), so it satisfies the hypotheses on the map in Theorem 4.2. Furthermore, the residue fields of \(\mathcal{O}_D\) and \(\mathcal{O}_{D'}\) agree by the explicit calculation of \(\mathcal{O}_{D'}\) done in Proposition 4.1. So Theorem 4.2 allows us to conclude once we show that \(\text{Diff}_{D'}(\pi^*\Delta) = \pi|^{|D'}_{\Delta'}(\text{Diff}_D(\Delta)) - D\mathcal{T}\). To that end, consider the following diagrams defining \(\text{Diff}_D(\Delta)\) and \(\text{Diff}_{D'}(\pi^*\Delta)\). The vertical arrows are the natural quotient maps.

We may connect these two diagrams with \(T\) and \(\mathcal{T}\) to form the following commutative cube

The top of the cube commutes by the Schwede-Tucker transposition criterion [ST14, Theorem 5.7] and the fact that \(D_T = 0\). In particular, \(\phi_{\text{Diff}_{D'}(\pi^*\Delta)}\) lifts the map \(\phi_{\text{Diff}_D(\Delta)}\), and the statement follows from the transposition criterion [ST14, Theorem 5.7].

Lemma 4.6. Let \(R\) be a local ring, \(D\) an integral \(\mathbb{Q}\)-Cartier divisor on \(X = \text{Spec} R\), and \(\pi : Y = \text{Spec} C(D) \to X\) be the associated cyclic cover. If \((X, \Delta)\) is a pair of index prime to \(p\), then \((Y, \pi^*\Delta)\) has index prime to \(p\).

Proof. As before, let \(T : C(D) \to R\) be the map which projects onto the degree zero piece of \(C(D)\). We saw in the proof of Corollary 4.5 that \(D_T = 0\), so we have the diagram
$F^e_*C(D) \xrightarrow{\phi_{\pi^e} \Delta} C(D) \xrightarrow{F^e_*T} T \xrightarrow{\phi\Delta} R^e_*$

By the divisor-maps correspondence discussed in Section 3.1, we see that $(p^e - 1)(K_Y + \pi^e \Delta)$ is Cartier, as required.

Example 4.7. Let us carry out the cyclic cover construction for $R = \mathbb{F}_p[x, y, z]/(xy - z^{n+1})$ and $D = V(x, z)$. Then $C = R \oplus (xt, zt) \oplus (xt^2, z^2t^2) \oplus \cdots \oplus (xt^n, z^n t^n)$ with $t^{n+1} = 1/x$. It is easy to see that $C/(xt^n) \cong \mathbb{F}_p[y, z]/(y - (zt)^{n+1})$, and the inclusion $R \subseteq C$ is given by $y \mapsto y$. We can also explicitly verify the equality of Proposition 4.1(4). We have $\text{Ram}_{\pi|D'} = n[0]$ where $[0]$ is the divisor corresponding to the point $(0, 0)$ on $D'$ in the $(y, zt)$-plane. Also,

$$\pi|_{D'} \text{ Diff}_D(0) = \pi|_{D'} \left( \frac{n}{n+1} \right) [0] = n[0].$$

So $\pi|_{D'} \text{ Diff}_D(0) - \text{Ram}_{\pi|_{D'}} = 0$. Since $D'$ is a smooth Cartier divisor, $C$ is Gorenstein. Hence $\text{Diff}_{D'}(0)$ is easily seen to be 0 from the definitions. From these computations and Example 3.4, one sees that

$$(n + 1) \cdot s(D, \text{Diff}_{D}(\Delta)) = (n + 1) \cdot s\left( \mathbb{F}_p[y], \left( \frac{n}{n+1} \right) [0] \right) = 1$$

and

$$s(D', \pi|_{D'} \text{ Diff}_D(\Delta) - \text{Ram}_{\pi|_{D'}}) = s(\mathbb{F}_p[y]) = 1$$

which verifies part (2) of Corollary 4.5 in this case.

5. Inversion of Adjunction for F-Signature

This section contains the proof of the main theorem. In the proof, we employ two standard exact sequence computations whose proofs we include for completeness.

Lemma 5.1. Let $R$ be a ring. If $I \subseteq R$ is an ideal of finite colength, $g \in R$ is a non-zero element, then

$$\ell_R \left( \frac{R}{(I : g)} \right) = \ell_R \left( \frac{R}{I} \right) - \ell_R \left( \frac{R}{(I, g)} \right).$$

Proof. Take the kernel and cokernel of the map $R/I \xrightarrow{\times g} R/I$ to obtain the exact sequence

$$0 \rightarrow \frac{(I : g)}{I} \rightarrow \frac{R}{I} \xrightarrow{\times g} \frac{R}{I} \rightarrow \frac{R}{(I, g)} \rightarrow 0.$$ 

So $\ell_R((I : g)/I) = \ell(R/(I, g))$. We also have $\ell((I : g)/I) = \ell(R/I) - \ell(R/(I : g))$. Equating the two expressions for $\ell((I : g)/I)$ gives the result.

Lemma 5.2. Let $R$ be a ring, and let $M$ be an $R$-module of finite length. If $f \in R$ is a non-zero element, then

$$\ell_R \left( \frac{M}{f^n M} \right) = n \cdot \ell_R \left( \frac{M}{f M} \right) - \sum_{i=1}^{n-1} \ell_R \left( \frac{(f^{i+1}M : f^i)}{f M} \right).$$
Proof. For \( n \geq 1 \), we have the isomorphism \( \frac{f^n M}{f^{n+1} M, f^n} \cong \frac{f^n M}{f^{n+1} M} \) and exact sequences
\[
0 \to \frac{f^n M}{f^{n+1} M} \to \frac{M}{f^{n+1} M} \to \frac{M}{f^n M} \to 0
\]
\[
0 \to \frac{(f^{n+1} M : f^n)}{f M} \to \frac{M}{f M} \to \frac{(f^{n+1} M : f^n)}{f M} \to 0.
\]
The base case of \( n = 1 \) being trivial, the result follows by induction on \( n \).

Now, we have all that we need to prove our main theorem. Recall that for a function \( f(t) \), we denote by \( D_f(t_0) \) the left derivative of \( f \) at \( t_0 \).

**Theorem 5.3.** Let \((R, \Delta + D)\) be a log \( \mathbb{Q} \)-Gorenstein pair with index prime to \( p \). Assume \( R \) is a normal local ring of dimension \( d \), \( \Delta \) is an effective \( \mathbb{Q} \)-divisor, and \( D \) is a reduced, irreducible \( \mathbb{Q} \)-Cartier divisor such that \( D \wedge \Delta = 0 \). Then
\[
D_{-s_R^{\Delta, D}}(1) = -s(O_D, \text{Diff}_D(\Delta))
\]
where \( s_R^{\Delta, D}(t) = s(R, \Delta + tD) \).

**Proof.** First, we may assume that \((R, \Delta + D)\) has splitting dimension \( d - 1 \). Otherwise, the \( F \)-pure threshold of the pair is strictly less than one, and the result is trivial. Thus, we may assume that \( D \) is the minimal center of \( F \)-purity for \((R, \Delta + D)\), so is \( F \)-regular and hence normal [Sch09].

We begin with the case that \( D = \text{div}(f) \) is Cartier. The limit defining \( D_{-s_R^{\Delta, D}}(1) \) exists by Proposition 2.12, so we may compute \( D_{-s_R^{\Delta, D}}(1) \) via any sequence converging to 1 from below. We have
\[
D_{-s_R^{\Delta, D}}(1) = \lim_{t \to 1} \frac{s(R, \Delta + tD) - s(R, \Delta + D)}{t - 1}
= \lim_{e \to \infty} \frac{s(R, \Delta + (1 - 1/p^e)D) - s(R, \Delta + D)}{-1/p^e}
= - \lim_{e \to \infty} \lim_{r \to \infty} \frac{1}{p^{rd-e}} \left( \ell_R \left( \frac{R}{(I_r^\Delta : f^{p^e-p^{e-r}})} \right) - \ell_R \left( \frac{R}{(I_r : f^{p^e})} \right) \right)
\]
Applying Lemmas 5.1 and 5.2 gives
\[
D_{-s_R^{\Delta, D}}(1) = - \lim_{e \to \infty} \lim_{r \to \infty} \frac{1}{p^{rd-e}} \left( \ell_R \left( \frac{R}{(I_r^\Delta, f^{p^e})} \right) - \ell_R \left( \frac{R}{(I_r : f^{p^e})} \right) \right)
= - \lim_{e \to \infty} \lim_{r \to \infty} \frac{1}{p^{rd-e}} \left( p^{r-e} \ell_R \left( \frac{R}{(f, I_r^\Delta)} \right) - \sum_{i=p^e-p^{e-r}}^{p^e-1} \ell_R \left( \frac{f^{i+1}}{f} \right) \right)
\]
where \( f \) denotes the image of \( f \) in \( R/I_r^\Delta \). Note that the ideals \((f^{i+1} : f_i)\) form an increasing chain
\[
(f^2 : f) \subseteq (f^3 : f^2) \subseteq \cdots \subseteq (f^{p^e} : f^{p^e-1}) = (0 : f^{p^e-1})
\]
In particular, $\ell_R \left( \frac{f^{i+1}}{f} \right) < \ell_R \left( \frac{f^i}{f} \right)$ for $i \leq p^s - 1$. So we have the inequality

$$D_- s_R^{\Delta, D}(1) \leq - \lim_{e \to \infty} \lim_{r \to \infty} \frac{1}{p^{r(d-1)}} \left( \ell \left( \frac{R}{(I^\Delta, f)} \right) - \ell \left( \frac{0 : f^{p^e-1}}{f} \right) \right)$$

$$= - \lim_{r \to \infty} \frac{1}{p^{r(d-1)}} \left( \ell \left( \frac{R}{(I^\Delta, f)} \right) - \ell \left( \frac{R}{(I^\Delta, f)} \right) + \ell \left( \frac{R}{(I^\Delta, f^{p^e-1})} \right) \right)$$

$$= - \lim_{r \to \infty} \frac{1}{p^{r(d-1)}} \ell \left( \frac{R}{(I^\Delta, f^{p^e-1})} \right)$$

where the final equality follows from Lemma 3.6.

For the reverse inequality, the inclusion $(I^\Delta : f^{p^e-1})[p^{r-e}] \subseteq (I^\Delta, f^{p^e-p^e-e})$ of Lemma 2.7 implies

$$D_- s_R^{\Delta, D}(1) = \lim_{e \to \infty} \frac{s(R, f^{1-1/p^e})}{-1/p^e}$$

$$= - \lim_{e \to \infty} \lim_{r \to \infty} \frac{1}{p^{r-d-e}} \ell_R \left( \frac{R}{(I^\Delta, f^{p^e-p^e-e})} \right)$$

$$\geq - \lim_{e \to \infty} \lim_{r \to \infty} \frac{1}{p^{r-d-e}} \ell_R \left( \frac{R}{(I^\Delta, f^{p^e-1}[p^{r-e}])} \right)$$

$$= - \lim_{e \to \infty} \frac{\ell_R ((I^\Delta : f^{p^e-1})[p^{r-e}] ; R)}{p^{r(d-1)}}.$$

For any ideal $J \subseteq R$ containing $f$, $\ell_R (R/J) = \ell_{R/f} (R/(J, f))$. By Proposition 3.5, the extension of $(I^\Delta : f^{p^e-1})$ in $R/f$ is $I^\Delta_{Diff_D(\Delta)}$ for large and divisible $e$. Thus

$$D_- s_R^{\Delta, D}(1) \geq - \lim_{e \to \infty} \frac{\ell_R ((I^\Delta : f^{p^e-1})[p^{r-e}] ; O_D)}{p^{r(d-1)}} = - s(O_D, Diff_D(\Delta))$$

where the last equality follows from Theorem 2.14 and Proposition 3.5.

Now, we use the results of Section 4 to reduce to the case where $D$ is Cartier. Let $p$ be the ideal cutting out the $\mathbb{Q}$-Cartier divisor $D$. Let $n$ be the index of $D$, and let $nD = \text{div}(x)$. Form the finite ring map $R \to C(D) := R \oplus pt \oplus p(2) \oplus \cdots \oplus p(n-1) \cdot t^{n-1}$ with $\pi : \text{Spec } C(D) \to \text{Spec } R$ the corresponding map of schemes. By Proposition 4.1, $D' := \pi^* D$ is Cartier. By Corollary 4.5, $D'$ is also $F$-regular, so in particular, $D'$ is normal. Moreover, the pair $(C(D), \pi^* \Delta + D')$ has index prime to $p$ by Lemma 4.6. We compute

$$D_- s_R^{\Delta, D}(1) = \lim_{e \to \infty} \frac{s(R, \Delta + (1 - 1/p^e)D)}{-1/p^e} = \lim_{e \to \infty} \frac{s(C(D), \pi^* \Delta + (1 - 1/p^e)D')}{-m/p^e}$$

$$= - \frac{s(D', \text{Diff}_{D'}(\pi^* \Delta))}{m} = - s(D, \text{Diff}_D(\Delta)).$$

The first equality is part (1) of Corollary 4.5, the second follows from the Cartier case, and the third follows from part (2) of Corollary 4.5. □
**Remark 5.4.** One might wonder whether the cyclic cover construction is necessary for the proof of Theorem 5.3. The proof in the Cartier case can be carried out in the \( \mathbb{Q} \)-Cartier case, using Lemma 3.6 in its most general form. However, this approach only yields the bound

\[-s(D, \text{Diff}_D(\Delta)) \leq Ds_-(1) \leq -s(D, \text{Diff}_D(\Delta))/m\]

where \( m \) is the index of \( D \).

**Remark 5.5.** Due to the extra generality of the transformation rule in [CR18a], we need not assume that the index of \( D \) is prime to \( p \). However, this increase in generality is mitigated by the fact that the index of \((R, \Delta + D)\) must be prime to \( p \). One would hope that Theorem 5.3 holds with no index assumptions entirely. To prove such a result, one must work with the geometric definition of the different since the construction of the different via Frobenius requires the existence of a map \( \phi_{\Delta+D} : F^e_*R \to R \) representing \( \Delta + D \).

As noted previously, one can think of Theorem 5.3 as a quantitative version of inversion of adjunction. Indeed, the theorem immediately implies the following inequality.

**Corollary 5.6 (Inversion of Adjunction for \( F \)-Signature).** In the setup of Theorem 5.3, we have

\[ s(R, \Delta) \geq s(D, \text{Diff}_D(\Delta)) \]

with equality if and only if the function \( s(R, \Delta+tD) \) is linear for \( t \in [0,1] \) with slope \( s(R, \Delta) \).

**Proof.** By Proposition 2.12, the \( F \)-signature function \( s(R, \Delta+tD) \) is convex. Thus \( -D_+s(1) \leq s(R, \Delta) \), and the statement follows from the inequality \( s(D, \text{Diff}_D(\Delta)) \leq -Ds_-(1) \) of Theorem 5.3. The characterization of equality is an immediate consequence of the convexity of the function \( s(R, \Delta+tD) \). \( \square \)

In particular, if \((D, \text{Diff}_D(\Delta))\) is strongly \( F \)-regular, so is \( s(R, \Delta) \). This statement is known as strongly \( F \)-regular inversion of adjunction and was originally shown in [Das15, Corollary 5.4]. As stated in the introduction, inversion of adjunction is a key tool in the MMP. For example, in [Hac17], Das’s inversion of adjunction is used to establish the existence of generalized PLT blowups in dimension 3.

**Example 5.7.** Let \( R = \mathbb{F}_p[[x,y, z]]/(xy - z^{n+1}) \) and \( D = V(x, z) \). Then Corollary 5.4 shows that \( s(R, tD) = \frac{1}{n+1} \) for all \( t \in [0,1] \). Indeed, \( s(R) = 1/(n+1) \) by [HL02, Example 18] and as computed in Example 3.4, \( s(D, \text{Diff}_D(0)) = s(\mathbb{F}_p[[y]], (\frac{n}{n+1})[0]) = \frac{1}{n+1} \).

One should note that the \( A_n \) singularities, used as a running example in this paper, are toric. So the \( F \)-signature function can also be computed with toric methods [VK11].

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