Subgraph densities in signed graphons
and the local Sidorenko conjecture

László Lovász*
Institute of Mathematics, Eötvös Loránd University
Budapest, Hungary

April 20, 2010

Contents

1 Introduction 1
  1.1 Notation ...................................... 3
  1.2 Norms ......................................... 4

2 Density inequalities for signed graphons 4
  2.1 Ordering signed graphons ......................... 4
  2.2 Edge weighting models .......................... 5
  2.3 Inequalities between densities ..................... 7
  2.4 Special graphs and examples ..................... 8
  2.5 The Main Bounds ................................ 11

3 Local Sidorenko Conjecture 16
  3.1 Variations ..................................... 18
  3.2 Graphic form .................................. 19

1 Introduction

Let $F$ be a bipartite graph with $k$ nodes and $l$ edges and let $G$ be any graph
with $n$ nodes and $m = p \binom{n}{2}$ edges. Sidorenko conjectured that the number
of copies of $F$ in $G$ is at least $p' \binom{n}{k} + o(p'n^k)$ (where we consider $k$ and $l$ fixed,
and $n \to \infty$). In a weaker form, this was also conjectured by Simonovits.

*Research supported by OTKA Grant No. 67867 and ERC Grant No. 227701.
One can get a cleaner formulation by counting homomorphisms instead of copies of $F$. Let $\text{hom}(F,G)$ denote the number of homomorphisms from $F$ into $G$. Since we need this notion for the case when $F$ and $G$ are multigraphs, we count here pairs of maps $\phi: V(F) \to V(G)$ and $E(F) \to E(G)$ such that incidence is preserved: if $i \in V(F)$ is incident with $e \in E(F)$, then $\phi(i)$ is incident with $\psi(e)$. We will also consider the normalized version $t(F,G) = \text{hom}(F,G)/n^k$. If $F$ and $G$ are simple, then $t(F,G)$ is the probability that a random map $\phi: V(F) \to V(G)$ preserves adjacency. We call this quantity the density of $F$ in $G$.

In this language, the conjecture says that $t(F,G) \geq t(K_2,G)^{|E(F)|}$ (this is an exact inequality, no error terms.) We can formulate this as an extremal result in two ways: First, for every graph $G$, among all bipartite graphs with $l$ edges, it is the graph consisting of disjoint edges (the matching) that has the least density in $G$. Second, for every bipartite graph $F$, among all graphs on $n$ nodes and edge density $p$, the random graph $\mathbb{G}(n,p)$ has the smallest density of $F$ in it (asymptotically, with large probability).

Sidorenko proved his conjecture in a number special cases: for trees $F$, and also for bigraphs $F$ where one of the color classes has at most 4 nodes. Since then, the only substantial progress was that Hatami [2] proved the conjecture for cubes.

Sidorenko gave an analytic formulation of this conjecture, which is perhaps even cleaner, and which we will also use. Let $F$ be a bipartite graph with bipartition $(A,B)$. For each edge $e \in E(F)$, let $a(e)$ and $b(e)$ be the endpoints of $e$ in $A$ and $B$, respectively. Assign a real variable $x_i$ to each $i \in A$ and a real variable $x_j$ to each $j \in B$. Let $W: [0,1]^2 \to \mathbb{R}_+$ be a bounded measurable function, and define

$$t(F,W) = \int_{[0,1]^{V(F)}} \prod_{e \in E(F)} W(x_{a(e)}, y_{b(e)}) \prod_{i \in A} dx_i \prod_{j \in B} dy_j.$$ 

Every graph $G$ can be represented by a function $W_G$: Let $V(G) = \{1, \ldots, n\}$. Split the interval $[0,1]$ into $n$ equal intervals $J_1, \ldots, J_n$, and for $x \in J_i, y \in J_j$ define $W_G(x,y) = 1_{ij \in E(G)}$. (The function obtained this way is symmetric.) Then we have

$$t(F,G) = t(F,W_G).$$

In this analytic language, the conjecture says that for every bipartite graph $F$ and function $W \in \mathcal{W}_+$,

$$t(F,W) \geq t(K_2,W)^{|E(F)|}. \tag{1}$$

Since both sides are homogeneous in $W$ of the same degree, we can scale $W$ and assume that

$$t(K_2,W) = \int_{[0,1]^2} W(x,y) \, dx \, dy = 1.$$
Then we want to conclude that $t(F, W) \geq 1$. In other words, the function $W \equiv 1$ minimizes $t(F, W)$ among all functions $W \geq 0$ with $\int W = 1$.

The goal of this paper is to prove that this holds locally, i.e., for functions $W$ sufficiently close to 1. Most of the time we will work with the function $U = W - 1$, which can have negative values. Most of our work will concern estimates for the values $t(F', U)$ for various (bipartite) graphs $F'$. This type of question seems to have some interest on its own, because it can be considered as an extension of extremal graph theory to signed graphs.

1.1 Notation

For each bigraph, we fix a bipartition and specify a first and second bipartition class. So the complete bigraphs $K_{a,b}$ and $K_{b,a}$ are different. If $F$ is a bigraph, then we denote by $F^\top$ the bigraph obtained by interchanging the classes of $F$.

We have to consider graphs that are partially labeled. More precisely, a $k$-labeled graph $F$ has a subset $S \subseteq V(F)$ of $k$ elements labeled $1, \ldots, k$ (it can have any number of unlabeled nodes. For some basic graphs, it is good to introduce notation for some of their labeled versions. Let $P_n$ denote the unlabeled path with $n$ nodes (so, with $n - 1$ edges). Let $P_n'$ denote the path $P_n$ with one of its endpoints labeled. Let $P_n''$ denote the $P_n$ with both of its endpoints labeled. Let $C_n$ denote the unlabeled cycle with $n$ nodes, and let $C_n'$ be this cycle with one of its nodes labeled. Let $K_{a,b}$ denote the unlabeled complete bipartite graph; let $K_{a,b}'$ denote the complete bipartite graph with its $a$-element bipartition class labeled. Note that $K_{2,2} \cong C_4$, but $K_{2,2}'$ and $C_4'$ are different as partially labeled graphs.

The most important use of partial labeling is to define a product: if $F$ and $G$ are $k$-labeled graphs, then $FG$ denotes the $k$-labeled graph obtained by taking their disjoint union and identifying nodes with the same label.

We set $I = [0, 1]$. For a $k$-labeled graph $F$, $(F)^{\text{ unl}}$ is the graph obtained by unlabeling. We set $F_1 \ast F_2 = (F_1 F_2)^{\text{ unl}}$, and $F^* = F \ast F$.

Let $W$ denote the set of bounded measurable functions $U : I^2 \to \mathbb{R}$; $W_+$ is the set of bounded measurable functions $U : I^2 \to \mathbb{R}_+$, and $W_1$ is the set of measurable functions $U : I^2 \to [-1, 1]$. Every function $U \in W$ defines a kernel operator $L_1(f) \to L_1(f)$ by

$$f \mapsto \int_I U(., y)f(y) \, dy.$$  

For $U, W \in W$, we denote by $U \circ W$ the function

$$(U \circ W)(x, y) = \int_I U(x, z)W(z, y) \, dz$$

(this corresponds to the product of $U$ and $W$ as kernel operators). For every $W \in W$, we denote by $W^\top$ the function obtained by interchanging the variables in $W$.  

3
1.2 Norms

We consider various norms on the space $W$. We need the standard $L_2$ and $L_\infty$ norms

$$
\|U\|_2 = \left(\int_{I^2} U(x, y)^2 \, dx \, dy\right)^{1/2}, \quad \|U\|_\infty = \sup \text{ess} |U(x, y)|.
$$

For graph theory, the cut norm is very useful:

$$
\|U\|_\square = \sup_{S, T \subseteq I} \left| \int_{S \times T} U(x, y) \, dx \, dy \right|.
$$

This norm is only a factor less than 4 away from the operator norm of $U$ as a kernel operator $L_\infty(I) \to L_1(I)$.

The functional $t(F, U)$ can be used define further useful norms. It is trivial that $t(C_2, U)^{1/2} = \|U\|_2$. The value $t(C_{2r}, U)^{1/(2r)}$ is the $r$-th Schatten norm of the kernel operator defined by $U$. It was proved in [1] that for $U \in W_1$,

$$
\|U\|_\square^4 \leq t(C_4, U) \leq \|U\|_\square.
$$

The other Schatten norms also define the same topology on $W_1$ as the cut norm (cf. Corollary 2.10).

It is a natural question for which graphs does $t(F, W)^{1/|E(F)|}$ or $t(F, |W|)^{1/|E(F)|}$ define a norm on $W$. Besides even cycles and complete bipartite graphs, a remarkable class was found by Hatami: he proved that $t(F, |W|)^{1/|E(F)|}$ is a norm if $F$ is a cube. He in fact proved that Sidorenko’s conjecture is true whenever $F$ is such a “norming” graph. However, a characterization of such graphs is open.

2 Density inequalities for signed graphons

2.1 Ordering signed graphons

For two bipartite multigraphs $F$ and $G$, we say that $F \leq G$ if $t(F, U) \leq t(G, U)$ for all $U \in W_1$. We say that $G \geq 0$ if $t(G, U) \geq 0$ for all $U \in W_1$. Note that if $U$ is nonnegative, then trivially $G \subseteq F$ implies that $t(F, U) \leq t(G, U)$; but since we allow negative values, such an implication does not hold in general. For example, $F \geq 0$ cannot hold for any bigraph $F$ with an odd number of edges, since then $t(F, -U) = -t(F, U)$.

We start with some simple facts about this partial order on graphs.

**Proposition 2.1** If $F$ and $G$ are nonisomorphic bigraphs without isolated nodes such that $F \leq G$, then $|E(F)| \geq |E(G)|$, $|E(G)|$ is even, and $G \geq 0$. Furthermore, $t(F, U) \leq t(G, U)$ for all $U \in W_1$.

The proof of this is based on a technical lemma, which is close to facts that are well known, but not in the exact form needed here.
Lemma 2.2 Let $F$ and $G$ be nonisomorphic graphs without isolated nodes. Then for every $U \in \mathcal{W}_1$ and $\varepsilon > 0$ there exists a function $U' \in \mathcal{W}_1$ such that $\|U - U'\|_\infty < \varepsilon$ and $t(F, U') \neq t(G, U')$.

Proof. First we show that if $F$ and $G$ are two graphs without isolated nodes such that $t(F, W) = t(G, W)$ for every $W \in \mathcal{W}_1$, then $F \cong G$. Consider the function $U = 1_{x,y \leq 1/2}$. Then $t(F, U) = 2^{-|V(F)|}$, so $t(F, U) = t(G, U)$ implies that $|V(F)| = |V(G)|$. Using the function $U = 1/2$, we get similarly that $|E(F)| = |E(G)|$. Using this, we get (by scaling $W$) that $t(F, W) = t(G, W)$ for every $W \in \mathcal{W}$.

For every multigraph $H$ we have
\[
t(F, H) = t(F, W_H) = t(G, W_H) = t(G, H),
\]
and hence it follows that
\[
\text{hom}(F, H) = t(F, H)|V(H)|^{|V(F)|} = t(G, H)|V(G)|^{|V(F)|} = \text{hom}(G, H).
\]
From this it follows by standard arguments that $F \cong G$ (e.g., we can apply Theorem 1(iii) of [3] to the 2-partite structures $(V,E,J)$, where $G = (V,E)$ is a multigraph and $J$ is the incidence relation between nodes and edges).

Since $F$ and $G$ are non-isomorphic, this argument shows that there exists a function $W \in \mathcal{W}_1$ such that $t(F, W) \neq t(G, W)$. The values $t(F, (1-s)U + sW)$ and $t(F, (1-s)U + sW')$ are polynomials in $s$ that differ for $s = 0$. Therefore, there is a value $0 \leq s \leq \varepsilon$ for which they differ. Since $(1-s)U + sW \in \mathcal{W}_1$ and $\|U - ((1-s)U + sW)\|_\infty = s\|U - W\|_\infty \leq \varepsilon$, this proves the lemma. \qed

Proof of Proposition 2.1 Applying the definition of $F \leq G$ with $U = 1/2$, we get that $2^{-|E(F)|} \leq 2^{-|E(G)|}$, and hence $|E(F)| \geq |E(G)|$. The relation $F \leq G$ implies that $t(F, U)^2 = t(F, U \otimes U) \leq t(G, U \otimes U) = t(G, U)^2$ also holds, so $|t(F, U)| \leq |t(G, U)|$ for all $U \in \mathcal{W}_1$. By Lemma 2.2 $U$ can be perturbed by arbitrarily little to get a $U' \in \mathcal{W}_1$ with $t(F, U') \neq t(G, U')$, then $t(F, U') < t(G, U')$ and $|t(F, U')| \leq |t(G, U')|$ imply that $t(G, U') > 0$. Since $U'$ is arbitrarily close to $U$, this implies that $t(G, U) \geq 0$, and so $G \geq 0$. Since this holds for $U$ replaced by $-U$, it follows that $G$ must have an even number of edges. \qed

2.2 Edge weighting models

We need the following generalization of Cauchy–Schwarz:

Lemma 2.3 Let $f_1, \ldots, f_n : I^k \to \mathbb{R}$ be bounded measurable functions, and suppose that for each variable there are at most two functions which depend on that variable. Then
\[
\int_{I^k} f_1 \cdots f_n \leq \|f_1\|_2 \cdots \|f_n\|_2.
\]
This will follow from an inequality concerning a statistical physics type model. Let $G = (V, E)$ be a multigraph (without loops), and let for each $i \in V$ let $f_i \in L_2(I^E)$ such that $f_i$ depends only on the variables $x_j$ where edge $j$ is incident with node $i$. Let $f = (f_i : i \in V)$, and define

$$\text{tr}(G, f) = \int_{I^E} \prod_{i \in V} f_i(x) \, dx$$

(where the variables corresponding to the edges not incident with $i$ are dummies in $f_i$).

**Lemma 2.4** For every multigraph $g$ and assignment of functions $f$,

$$\text{tr}(G, f) \leq \prod_{i \in V} \|f_i\|_2.$$

**Proof.** By induction on the chromatic number of $G$. Let $V_1, \ldots, V_r$ be the color classes of an optimal coloring of $G$. Let $S_1 = V_1 \cup \cdots \cup V_{\lceil r/2 \rceil}$ and $S_2 = V \setminus S_1$. Let $E_0$ be the set of edges between $S_1$ and $S_2$, and let $E_i$ be the set of edges induced by $S_i$. Let $x_i$ be the vector formed by the variables in $E_i$. Then

$$\text{tr}(G, f) = \int_{I^{E_0}} \left( \int_{I^{E_1}} \prod_{i \in S_1} f_i(x) \, dx_1 \right) \left( \int_{I^{E_2}} \prod_{i \in S_2} f_i(x) \, dx_2 \right) \, dx_0.$$

The outer integral can be estimated using Cauchy-Schwarz:

$$\text{tr}(G, f)^2 \leq \int_{I^{E_0}} \left( \int_{I^{E_1}} \prod_{i \in S_1} f_i(x) \, dx_1 \right)^2 \, dx_0 \times \int_{I^{E_0}} \left( \int_{I^{E_2}} \prod_{i \in S_2} f_i(x) \, dx_2 \right)^2 \, dx_0. \quad (2)$$

Let $G_1$ be defined as the graph obtained taking a disjoint copy $(S_1', E_1')$ of the graph $(S_1, E_1)$, and connect each node $i \in S_1$ to the corresponding node $i' \in S_1'$ by as many edges as those joining $i$ to $S_2$ is $G$. Note that these newly added edges correspond to the edges of $E_0$ in a natural way. We assign to each node the same function as before, and also the same function (with differently named variables for the edges in $E_1'$) to $i'$. Then the first factor in (2) can be written as

$$\int_{I^{E_0}} \int_{I^{E_1}} \int_{I^{E_1'}} \prod_{i \in S_1 \cup S_1'} f_i(x) \, dx_1 \, dx_0 = \text{tr}(G_1, f).$$

We define $G_2$ analogously, and get that the second factor in (2) is just $\text{tr}(G_2, f)$. So we have

$$\text{tr}(G, f)^2 \leq \text{tr}(G_1, f)\text{tr}(G_2, f) \quad (3)$$

Next we remark that $G_1$ and $G_2$ have chromatic number at most $\lceil r/2 \rceil$, and so if $r > 2$, then we can apply induction and use that

$$\text{tr}(G_j, f) \leq \prod_{i \in V(G_j)} \|f_i\|_2 = \prod_{i \in S_j} \|f_i\|_2^2.$$
If \( r = 2 \), then \( G_j \) has edges connecting pairs \( i, i' \) only, and so
\[
\text{tr}(G_j, f) = \prod_{i \in S_j} \| f_i \|_2^2.
\]
In both cases, the inequality in the lemma follows by (3).

\[\square\]

2.3 Inequalities between densities

Let \( F_1 \) and \( F_2 \) be two \( k \)-labeled graphs. Then the Cauchy–Schwarz inequality implies that for all \( U \in \mathcal{W} \),
\[
t(F_1 \ast F_2, U)^2 \leq t(F_1^* U, U) t(F_2^* U, U).
\]
(4)

With the notation introduced above, this can be written as
\[
(F_1 \ast F_2)^2 \leq F_1^* F_2^*.
\]
(5)

This also implies that for each \( k \)-labeled graph \( F \),
\[
F^* \geq 0.
\]
(6)

Let \( F_{\text{sub}} \) denote the subdivision of graph \( F \) with one new node on each edge.

**Lemma 2.5** If \( F \leq G \), then \( F_{\text{sub}} \leq G_{\text{sub}} \).

**Proof.** For every \( U \in \mathcal{W} \),
\[
t(F_{\text{sub}}, U) = t(F, U \circ U^\top) \leq t(G, U \circ U^\top) = t(G_{\text{sub}}, U).
\]
\[\square\]

**Lemma 2.6** Let \( F \) be a bigraph, let \( S \subseteq V(F) \), and let \( H_1, \ldots, H_m \) be the connected components of \( F \setminus S \). Assume that each node in \( S \) has neighbors in at most two of the \( H_i \). Let \( F_i \) denote the graph consisting of \( H_i \), its neighbors in \( S \), and the edges between \( H_i \) and \( S \). Let us label the nodes of \( S \) in every \( F_i \). Then
\[
F^2 \leq \prod_{i=1}^m F_i^*.
\]

**Proof.** Let \( F_0 \) denote the subgraph induced by \( S \), and consider the nodes of \( F_0 \) labeled \( 1, \ldots, k \); we may assume that these nodes are labeled the same way in every \( F_i \). Then using that \( |t_{x_1 \ldots x_k}(F_0, U)| \leq 1 \), we get
\[
|t(F, U)| = \left| \int_{I^k} \prod_{i=0}^m t_{x_1 \ldots x_k}(F_i, U) \, dx_1 \ldots dx_k \right| 
\leq \int_{I^k} \prod_{i=1}^m |t_{x_1 \ldots x_k}(F_i, U)| \, dx_1 \ldots dx_k.
\]

Lemma 2.3 implies the assertion.
\[\square\]

We formulate some special cases.
Corollary 2.7 If $F$ contains two nonadjacent nodes of degree at least 2, then $F \leq C_4$.

More generally,

Corollary 2.8 Let $v_1, \ldots, v_k$ be independent nodes in $F$ with degrees $d_1, \ldots, d_k$ such that no node of $F$ is adjacent to more than 2 of them. Then

$$F^2 \leq \prod_{i=1}^{k} K_{2,d_i} \leq C_4^k,$$

A hanging path system in a graph $F$ is a set $\{P_1, \ldots, P_m\}$ of openly disjoint paths such that the internal nodes of each $P_i$ have degree 2, and at most two of them start at any node. The value of a hanging path system is the total number of their internal nodes.

Corollary 2.9 Let $F$ be a bigraph that contains a hanging path system with lengths $r_1, \ldots, r_m$. Then

$$F^2 \leq \prod_{i=1}^{m} C_{2r_i}.$$

Combining with Corollary 2.15 we get the following bound, which we will use:

Corollary 2.10 Let $F$ be a simple graph that contains a hanging path system of lengths between 2 and $r$ and value $2r + a - 2$, $a \geq 0$. Then $F \leq C_{2r}C_4^{a/2}$.

Corollary 2.11 Let $F$ be a graph and $S \subseteq V(F)$. Let $F_0$ be obtained by deleting the edges within $S$, and labeling the nodes in $S$. Then

$$F \leq (F_0^2)^{1/2}.$$

2.4 Special graphs and examples

Lemma 2.12 Let $U \in \mathcal{W}_1$. Then the sequence \( t(C_{2k}, U) : \ k = 1, 2, \ldots \) is nonnegative, logconvex, and monotone decreasing.

With the notation introduced above, we have $C_2 \geq C_4 \geq C_6 \geq \cdots \geq 0$.

Proof. We have

$$t(C_{a+b}, U) = \int_{P} t_{xy}(P'_{a}, U) t_{xy}(P'_{b}, U).$$

Taking $a = b = k$, nonnegativity follows. Applying Cauchy–Schwarz,

$$t(C_{a+b}, U) \leq t(C_{2a}, U)^{1/2} t(C_{2b}, U)^{1/2}.$$

This implies logconvexity. Since the sequence remains bounded by 1, it follows that the sequence is monotone decreasing. □
Lemma 2.13 Let \( r_1, r_2, \ldots, r_k \) be positive integers, and \( r = r_1 + \cdots + r_k \). Then
\[
C_r^2 \leq C_{2r_1} \cdots C_{2r_k}.
\]

Corollary 2.14
\[
C_{2k+2} \leq C_{2k}C^{1/2}_4.
\]

Corollary 2.15 If \( 1 \leq r_1, \ldots, r_n \leq r \) and \( \sum_i (r_i - 1) = k(r-1) \), then
\[
\prod_{i=1}^k C_{2r_i} \leq C_{2r}^k.
\]

Corollary 2.16 For all \( k \geq 2 \),
\[
C_{4k-1}^4 \leq C_{4k} \leq C_{2k}^{k/2}.
\]

We can get similar bounds for paths, of which we only state two, which will be needed. Recall that \( P_n \) denotes the path with \( n \) nodes and \( n-1 \) edges.

Lemma 2.17 For all \( a, b \geq 1 \), we have
\[
\begin{align*}
(a) \quad & P_{a+b+1} \leq P_{2a+1}^{1/2} P_{2b+1}^{1/2}; \\
(b) \quad & P_{2a+b+1} \leq P_{2a+1} C_{4b}^{1/4}.
\end{align*}
\]

Proof. Since \( P_{a+b+1} = P_{a+1}^a \ast P_{b+1} \), the first inequality follows by (5). To get the second, we use the first to get
\[
P_{2a+b+1} \leq P_{2a+1}^{1/2} P_{2a+2b+1}^{1/2}.
\]
Cut \( P_{2a+2b+1} \) into pieces \( P_{a+1}, P_{2b+1} \) and \( P_{a+1} \), and apply Lemma 2.6 we get
\[
P_{2a+2b+1} \leq P_{2a+1} C_{4b}^{1/2},
\]
and hence
\[
P_{2a+b+1} \leq P_{2a+1}^{1/2} (P_{2a+1} C_{4b}^{1/2})^{1/2} = P_{2a+1} C_{4b}^{1/4}.
\]

Lemma 2.18 Let \( U \in W_1 \). Then for every \( h \geq 1 \), the sequence \( (t(K_h,2k, U) : k = 1,2,\ldots) \) is nonnegative, logconcave and monotone decreasing.

Proof. The proof is similar, based on the equation
\[
t(K_{h,a+b}, U) = \int_{I_h} t_{x_1 \ldots x_h}(K'_{h,a}, U) t_{x_1 \ldots x_h}(K'_{h,b}, U) \, dx_1 \ldots dx_h.
\]

For complete bipartite graphs, however, we don’t have a bound similar to Corollary 2.16 at least as long as we restrict ourselves to simple graphs (see Example 1). But we do have the following inequality.
Lemma 2.19 For all \( n \geq 3 \), we have
\[
K_{n,n} \leq K_{2,n}C_2^{1/2}.
\]

Proof. Let \( H \) be the 2-labeled graph obtained from \( K_{n,n} \) by deleting an edge and labeling its endpoints. Then \( K_{n,n} = (K_2''H)^{\text{unl}} \), and hence
\[
K_{n,n}^2 \leq (K_2'')^{*2}H^{*2} = C_2H^{*2}.
\]

Now taking two unlabeled nodes from one color class from one copy of \( H \) and two unlabeled nodes from the other color class from the other copy, we get a set of 4 independent nodes of degree \( n \) such that no three have a neighbor in common. Hence by Corollary 2.8,
\[
H^{*2} \leq K_{2,n}^2,
\]
which proves the Lemma.

Example 1 Let \( U : [0,1]^2 \to [-1,1] \) be defined by
\[
U(x, y) = \begin{cases} 
-1, & \text{if } x, y \geq 1/2, \\
\text{otherwise.} & 
\end{cases}
\]

Then it is easy to calculate that for all \( n, m \geq 1 \),
\[
t(K_{n,m}, U) = \frac{1}{4}.
\]

Lemma 2.20 For all \( U \in W \) and \( x \in I \),
\[
0 \leq t_x(C_r', U) \leq t(C_{4r-4}, U)^{1/2}.
\]

Proof. The first inequality follows from the formula
\[
t_x(C_r', U) = \int_{I} t_{ux}(P_{r+1}'', U)^2 \, du.
\]

For the second, write
\[
t_x(C_r', U) = \int_{I^2} U(x, u)t_{uv}(P_{2r-1}'', U)U(v, x) \, du \, dv,
\]
and apply Cauchy–Schwarz:
\[
(t_x(C_r', U))^2 \leq \int_{I^2} U(x, u)^2U(v, x)^2 \, du \, dv \int_{I^2} t_{uv}(P_{2r-1}'', U)^2 \, du \, dv
\]
\[
= t_x(C_2', U)^2 t(C_{4r-4}, U) \leq t(C_{4r-4}, U).
\]

\( \square \)
Lemma 2.21  For all $U \in W$, $k \geq 4$ and $x, y \in I$,

$$|t_{xy}(P''_k, U)| \leq t(C_{2(k-2)}, U)^{1/4}.$$  

Proof.  We can write

$$t_{xy}(P''_k, U) = \int U(x, u) t_{uy}(P''_{k-1}, U) \, du.$$  

Hence by Cauchy–Schwarz,

$$t_{xy}(P''_k, U)^2 \leq \int U(x, u)^2 \, du \int t_{uy}(P''_{k-1}, U)^2 \, du$$

$$\leq \int t_{uy}(P''_{k-1}, U)^2 \, du = t_y(C'_{2(k-2)}, U) \, du.$$  

Applying Lemma 2.20 the proof follows.  

□

2.5  The Main Bounds

Our main Lemma is the following.

Lemma 2.22  Let $F$ be a bipartite graph with all degrees at least 2, with girth $2r$, which is not a single cycle or a complete bipartite graph. Then $F \leq C_{2r} C_4^{1/4}$.

Before proving this lemma, we need some preparation. Let $T$ be a rooted tree. By its min-depth we mean the minimum distance of any leaf from the root. (As usual, the depth of $T$ is the maximum distance of any leaf from the root.) For a rooted tree $T$, we denote by $T^*2$ the graph obtained by taking two copies of $T$ and identifying leaves corresponding to each other.

Lemma 2.23  Let $T$ be a tree with min-depth $h$ and depth $g$. Then $T^*2$ contains a hanging path system with value at least $g + \max(0, h - 3)$, in which the paths are not longer than $\max(g, 2)$.

Proof.  The proof is by induction on $|V(T)|$. We may assume that the root has degree 1, else we can delete all branches but the deepest from the root. Let $a$ denote the length of the path $P$ in $T$ from the root $r$ to the first branching point or leaf $v$.

If $P$ ends at a leaf, then the whole tree is a path of length $a = g = h$. If $a = 1$, we get a hanging path in $T^*2$ of length 2, and so of value $1 = 1 + \max(0, -1)$. If $a \geq 2$, then we can even cut this into two, and get two hanging paths in $T^*2$ of length $a$, which has value $2a - 2 \geq a + \max(0, a - 3)$.

If $P$ ends at a branching point, then we consider two subtrees $F_1, F_2$ rooted at $v$ (there may be more), where $F_1$ has depth $g - a$. Clearly, $F_1$ has min-depth at least $h - a$ and $F_2$ has min-depth and depth at least $h - a$. By induction, $F_1^*2$ and $F_2^*2$ contain hanging path systems of value $g - a + \max(0, (h - a) - 3)$.
and \( h - a + \max(0, (h - a) - 3) \), respectively. The two systems together have value at least \( g + h - 2a \), and they form a valid system since \( v \) (and its mirror image) are contained in at most one path of each system. If \( a = 1 \), we are done, since clearly \( h \geq 2 \) and so \( g + h - 2 \geq g + \max(0, h - 3) \).

Assume that \( a \geq 2 \). Let \( F_3 \) be obtained from \( F_2 \) by deleting its root. By induction, \( F_2^2 \) contains hanging path systems of value \( g - a + \max(0, h - a - 3) \), and \( F_3^2 \) contains a hanging path system of value \( h - a + \max(0, h - a - 4) \). We can add \( P \) and its mirror image, to get a hanging path system of value

\[
(g - a) + (h - a - 1) + \max(0, h - a - 3) + \max(0, h - a - 4) + 2(a - 1) \\
\geq (g - a) + (h - a - 1) + 2(a - 1) = g + h - 3 = g + \max(0, h - 3),
\]

since \( h \geq a + 1 \geq 3 \). We know that every path constructed lies in the tree or its mirror image, except for the paths in the case \( g = 1 \), which are of length 2. \( \square \)

**Proof of Lemma 2.22.** We distinguish several cases.

**Case 1.** \( r = 2 \). By hypothesis, \( F \) is not a complete bipartite graph, and hence we can choose nonadjacent nodes \( v \) and \( w \) from different bipartition classes. Let \( N \) denote the set of neighbors of \( u \), \( |N| = d \), and let \( F_0 \) denote the graph \( F - u \) with the neighbors of \( u \) labeled. Then \( F \cong F_0 \ast K_{d, 1}^\ast \), and hence by (7),

\[
F^2 \leq F_0^2 (K_{d, 1}^\ast)^2 = F_0^2 K_{d, 2} \leq F_0^2 C_4.
\]

Now let \( v_1 \) and \( v_2 \) be the two copies of \( v \) in \( F_0^2 \), and \( w \), any third node in the same bipartition class. These three nodes have no neighbor in common, so by Corollary 2.8, we get that \( F_0^2 \leq C_4^{3/2} \), and so \( F \leq C_4^{5/4} \).

**Case 2.** \( F \) is disconnected. If one of the components is not a single cycle, we can replace \( F \) by this component. If \( F \) is the disjoint union of single cycles, then \( F \leq C_2 + C_2, C_4 \).

So we may assume that \( F \) is connected. Then it must have at least one node of degree larger than 2.

**Case 3.** \( F \) has at most one node of degree larger than 2 in each color class. Let \( u_1 \) and \( u_2 \) be two nodes, one in each color class, such that all the other nodes have degree 2. Then \( F \) must consist of one or more odd paths connecting \( u_1 \) and \( u_2 \), and even cycles attached at \( u_1 \) and/or \( u_2 \).

If there are even cycles attached at \( u_1 \) and also at \( u_2 \), then \( F \) has a hanging path system consisting of 4 paths of length \( r \), and so \( F \leq C_2^r \leq C_2 C_4^{1/2} \) by Lemma 2.6. If (say) \( u_1 \) but not \( u_2 \) has a cycle attached, then there are at least two paths connecting \( u_1 \) and \( u_2 \), and one of them has length at least \( r \). This implies that \( F \leq C_2^{3/2} \leq C_2, C_4^{1/2} \).

So we may assume that \( F \) consists of openly disjoint paths connecting \( u_1 \) and \( u_2 \). Since \( F \) is not a single cycle, there are at least three paths. Let \( a_1 \leq a_2 \leq a_3 \).
be their lengths. Clearly $a_1 + a_2 \geq 2r$. If $a_2 \geq r + 1$, then we have two hanging paths of length $r + 1$, which implies that $F \leq C_{2r+2} \leq C_{2r}C_4^{1/2}$. So we may assume that $a_1 = a_2 = a_3 = r$. If $r \geq 4$, then we can select two of the paths and path of length 2 disjoint from them, which gives $F \leq C_{2r}C_4^{1/2}$.

So we get to the special case when $F$ consists of 3 or more paths of length 3 connecting $u_1$ and $u_2$. In this case, we use Lemma 2.21.

$$t(F, U) = \int_{I_2} t_{xy}(P'_4, U)^3 \, dx \, dy \leq t(C_4, U)^{1/4} \int_{I_2} t_{xy}(P'_4, U)^2 \, dx \, dy$$

$$= t(C_6, U) t(C_4, U)^{1/4}.$$}

**Case 4.** Suppose that there are two nodes $u_1, u_2$ in the same bipartition class of $F$ of degree at least 3.

Let $S_1$ be the set of nodes in $F$ with $d(x, u_1) \leq \min(r - 2, d(x, u_2) - 2)$, and let $S'_1$ be the set of neighbors of $S_1$ in $F$. We define $S_2$ and $S'_2$ analogously. Let $F_i$ be the subgraph induced by $S_i \cup S'_i$.

**Claim 1** $F_i$ is a tree with endnode set $S'_i$. Every $x \in S'_i$ satisfies $d(x, u_1) = \min(r - 1, d(x, u_2))$.

From the fact that $F$ has girth $2r$ it follows that $F_i$ is a tree. The nodes in $S_i$ are not endnodes of $F_i$, since their degree in $F$ is at least 2 and all their neighbors are nodes of $F_i$. It is also trivial that the nodes in $S'_i$ are endnodes.

Let $x \in S'_i$, then $x \notin S_i$ and hence $d(x, u_1) \geq \min(r - 1, d(x, u_2) - 1)$. But $d(x, u_1)$ and $d(x, u_2)$ have the same parity, and hence it follows that $d(x, u_1) \geq \min(r - 1, d(x, u_2))$. On the other hand, $x$ has a neighbor $y \in S_i$, and hence $d(x, u_1) \leq d(y, u_1) + 1 \leq r - 1$, and $d(x, u_1) \leq d(y, u_1) + 1 \leq d(y, u_2) - 1 \leq d(x, u_2)$. This implies that $d(x, u_1) \leq \min(r - 1, d(x, u_2))$, which proves the Claim.

**Claim 2** There is no edge between $S_1$ and $S_2$.

Indeed, suppose that $x_1 x_2$ is such an edge, $x_i \in S_i$. Then $d(x_1, u_1) < d(x_2, u_2)$, which by parity means that $d(x_1, u_1) \leq d(x_2, u_2) - 2$. But then $d(x_2, u_1) \leq d(x_1, u_1) + 1 \leq d(x_2, u_2) - 1 \leq d(x_2, u_1)$, showing that $x_2 \notin S_2$.

Consider the nodes of $F_i$ in $S'_i$ as labeled. Lemma 2.6 implies that

$$F_i^2 \leq F_1^2 F_2^2.$$ 

Hence to complete the proof, it suffices to show that

$$F_i^2 \leq C_{2r}C_4^{1/4}.$$ 

This will follow by Corollary 2.10 if we construct in $F_i$ a hanging path system of paths of length at most $r$ with value $2r - 1$.  

13
Claim 3 Let \( y \neq x \) be two leaves of \( F_1 \). Then \( d(r, x) + d(r, y) + d(x, y) \geq 2r \).

If \( d(r, x) = r - 1 \) or \( d(r, y) = r - 1 \) then this is trivial, so suppose that \( d(r, x), d(r, y) \leq r - 2 \). Then by Claim 1 we must have \( d(x, u_2) = d(x, u_1) \) and \( d(y, u_2) = d(y, u_1) \). Going from \( x \) to \( u_2 \) to \( y \) and back to \( x \) in \( F \), we get a closed walk of length \( d(r, x) + d(r, y) + d(x, y) \), which contains a cycle of length no more than that, which implies the inequality in the Claim.

Claim 4 Each of \( F_1 \) and \( F_2 \) contains a hanging path system of paths with length at most \( r \), with value \( 2r - 2 \). At least one of them contains such a system with value \( 2r - 1 \).

To prove this, we need to distinguish two cases.

Case 4a. All branches of \( F_1 \) are single paths. Let \( a_1 \leq \cdots \leq a_d \) be their lengths. Claim 3 implies that \( a_1 + a_2 \geq r \), so \( a_2 \geq r/2 \). The graph \( F_1^2 \) consists of paths \( Q_1, \ldots, Q_d \) of length \( 2a_1, \ldots, 2a_d \) connecting \( u_1 \) and its mirror image \( u'_1 \). Select subpaths of length \( r \) from \( Q_2 \) and \( Q_3 \), this gives a hanging path system of value \( 2r - 2 \). If \( a_1 \geq 2 \), then we can add to this a path of length \( 2 \) from \( Q_1 \) not containing its endpoints, and we get a path system of value \( 2r - 1 \). So we may assume that \( a_1 = 1 \). Then \( a_2 \geq r - 1 > r/2 \), and so \( 2a_2, 2a_r > r \). Thus we can select the paths of length \( r \) from \( Q_2 \) and \( Q_3 \) so that one of them misses \( u_1 \) and the other one misses \( u'_1 \). The we can add \( Q_1 \) to the system, and conclude as before.

Case 4b. At least one of the branches of \( F_1 \), say \( A \), is not a single path. Let \( a \) be the length of the path \( Q \) from the root \( u_1 \) to the first branch point \( v \). Let \( T_1, T_2 \) be two subtrees of \( A \) rooted at \( v \), of depth \( d_1 \) and \( d_2 \). Let \( B \) and \( C \) be two further branches, of depth \( b \) and \( c \), respectively, where \( b \geq c \). By Claim 3 we have

\[
d_1 + d_2 + a \geq r \quad \text{and} \quad b + c \geq r.
\]

We start with a simple computation showing that we can get a hanging path system in \( F_1^2 \) of value \( 2r - 2 \). If \( a = 1 \), then we choose a hanging path system from \( T_1^2 \) of value \( d_1 \), from \( T_2^2 \) of value \( d_2 \), from \( B^2 \) of value \( b \) and from \( C^2 \) of value \( c \). This is a total of \( d_1 + d_2 + b + c \geq 2r - 1 \).

If \( a \geq 2 \), then we choose a hanging path system from \( T_1^2 \) of value \( d_1 \), from \( (T_2 - v)^2 \) of value \( d_2 - 1 \), from \( B^2 \) of value \( b \) and from \( (C - u_1)^2 \) of value \( c - 1 \). Leaving out \( v \) from \( T_2 \) and \( u_1 \) from \( C \) allows us to add \( Q \) and its mirror image of value \( 2(a - 1) \). This is a total of

\[
d_1 + d_2 - 1 + 2(a - 1) + b + c - 1 \geq 2r + a - 4 \geq 2r - 2.
\]
If equality holds in all estimates, then \(a + b + c = r\). It also follows that \(b \leq 3\), or else we get a larger system in \(B\). Note that the depth of \(A\) is at least \(a + 1 = 3\), and \(c \leq r/2 \leq b \leq 3\).

If \(B\) is a single path, then we can select a hanging path of length \(r\) from \(B^2\), of value \(r - 1 > b - 1\), and we have gained 1 relative to the previous construction. So we may assume that \(B\) is not a single path. Then applying the same argument as above with \(A\) and \(B\) interchanged, we get that \(b = 3\), and the depth of \(A\) is also 3. Hence \(d_1 = d_2 = b = 1\) and \(r = d_1 + d_2 + a = 4\). It follows that \(c = r - b = 1\), so \(C\) consists of a single edge.

If \(u_1\) has degree larger than 3, then applying the argument to \(A, B, D\), we get that \(D\) must have depth 1, but this contradicts Claim 3. Hence the degree of \(u_1\) is 3.

If \(A\) has at least 3 leaves, then these must be connected to \(u_2\) by disjoint paths of length 3. Since \(u_2\) must be connected to the endpoint of \(C\) as well by Claim 1, we get that \(u_2\) has degree at least 4, and so \(F_2 \geq C_{2r}C_4^{1/2}\).

So \(A\) and similarly \(B\) have two leaves, and \(F_1\) is a 10-node tree consisting of a path with 5 nodes and 2 endnodes hanging from its endnodes and 1 from its middle node. \(F_2\) must be the same, or else we are done. There is only one way to glue two copies of this tree together at their endnodes to get a graph of girth 8, and this yields the subdivision of \(K_{3,3}\) (by one node on each edge). To settle this single graph, we use that

\[
K_{3,3} \leq C_{2r}^{1/2} K_{3,2} \leq C_{2r}^{1/2} C_4
\]

by Lemmas 2.19 and 2.18, and so by Lemma 2.25, we have

\[
F = K_{3,3}^{sub} \leq (C_{2r}^{sub})^{1/2} C_4^{sub} = C_4^{1/2} C_8.
\]

Thus we know that \(F_1^2 F_2^2 \geq C_{2r}\), and for at least one of them \(F_i^2 \geq C_{2r} C_4^{1/2}\), which implies that \(F^2 \geq F_1^2 F_2^2 \geq C_{2r} C_4^{1/2}\). \(\Box\)

Lemma 2.6 implies that if \(F\) is a graph with two nonadjacent nodes \(u, v\) of degree 1, then \(F \leq P_3\). We need a stronger bound:

**Lemma 2.24** Let \(F\) be a graph with two nonadjacent nodes \(u, v\) of degree 1, which is not a star and has at least 3 edges. Then \(F \leq P_3 C_4^{1/4}\).

**Proof.** Case 1. First, suppose that \(F\) has two nonadjacent nodes \(u, v\) of degree 1 whose neighbors \(u'\) and \(v'\) are different. If there is a node \(w \neq u, v, u', v'\) of degree \(d \geq 2\), then we can apply Lemma 2.6 to the stars of \(u, v\) and \(w\), to get

\[
F \leq P_3 K_{2d}^{1/2} \leq P_3 C_4^{1/2}.
\]

If there is a node \(w \neq u, v, u', v'\) of degree 1, then a similar application of Lemma 2.6 gives that

\[
F \leq P_3^{3/2} \leq P_3 C_4^{1/4}.
\]
Finally, if \( V(F) = \{u, v, u', v'\} \), then \( F = P_4 \), and the bound follows from Lemma 2.17(b).

**Case 2.** Suppose that all nodes of \( F \) of degree 1 have a common neighbor \( w \). Let \( F_0 \) denote the subgraph obtained by deleting the nodes of degree 1. Since \( F \) is not a star, \( F_0 \) must have at least two edges. If \( F_0 \) has a node not adjacent to \( w \), then we conclude similarly as above. So suppose that \( w \) is adjacent to all the other nodes of \( F_0 \). Let \( F_1 \) denote the subgraph formed by the edges incident with \( w \) (a star), with the nodes in \( F_0 - w \) labeled. \( \square \)

**Lemma 2.25** Let \( F \) be a bigraph with exactly one node of degree 1 and with girth \( 2r \). Then
\[
F \leq \frac{1}{2}(C_{2r} + P_3)C_4^{1/8}.
\]

**Proof.** Let \( v \) be the unique node of degree 1. We can write \( F \cong F_0 \ast P_2 \), where \( F_0 \) is a 1-labeled graph in which all nodes except possibly the labeled node \( v \) have degrees at least 2. By (7), we get that \( F \leq (P_2) \ast F_0 \cong P_3F_0^2 \). Here \( F_0^2 \) is a graph with girth \( 2r \) and all degrees at least 2. Hence Lemma 2.22 we get
\[
F \leq P_3C_2C_4^{1/4}. \]
Thus
\[
|t(F, U)| \leq \sqrt{t(P_3, U)t(C_{2r}, U)t(C_4, U)^{1/8}} \\
\leq \frac{1}{2}(t(C_{2r}, U) + t(P_3, U))t(C_4, U)^{1/8}.
\]

\( \square \)

3 Local Sidorenko Conjecture

The Sidorenko Conjecture asserts that \( t(F, W) \) is minimized by the function \( W \equiv 1 \) among all functions \( W \geq 0 \) with \( \int W = 1 \). The following theorem asserts that this is true at least locally.

**Theorem 3.1** Let \( F = (V, E) \) be a simple bigraph. Let \( W \in W \) with \( \int W = 1 \), \( 0 \leq W \leq 2 \) and \( \|W - 1\|_{\square} \leq 2^{-8m} \). Then \( t(F, W) \geq 1 \).

**Proof.** We may assume that \( F \) is connected, since otherwise, the argument can be applied to each component. Let \( U = W - 1 \), then we have the expansion
\[
t(F, W) = \sum_{F'} t(F', U), \tag{10}
\]
where \( F' \) ranges over all spanning subgraphs of \( F \). Since isolated nodes can be ignored, we may instead sum over all subgraphs with no isolated nodes (including the term \( F' = K_0 \), the empty graph). One term is \( t(K_0, U) = 1 \), and every term containing a component isomorphic to \( K_2 \) is 0 since \( t(K_2, U) = \int U = 0 \).
Based on (6), we can identify two special kinds of nonnegative terms in (10), corresponding to copies of $P_3$ and to cycles in $F$. We show that the remaining terms do not cancel these, by grouping them appropriately.

(a) For each node $i \in V$, let $\sum_{\mathcal{V}(i)}$ denote summation over all subgraphs $F'$ with at least two edges that consist of edges incident with $i$. Let $d_i$ denote the degree of $i$ in $F$, assume that $d_i \geq 2$, and set $t(x) = t_x(K'_2, U)$. Then using that $t(x) \geq -1$ and Bernoulli’s Inequality,

$$\sum_{\mathcal{V}(i)} t(F', U) = \int \sum_{k=2}^{d_i} \binom{d_i}{k} t(x)^k \, dx = \int (1 + t(x))^{d_i} - 1 - d_i t(x) \, dx$$

$$\geq \int (1 + t(x))(1 + (d_i - 1)t(x)) - 1 - d_i t(x) \, dx$$

$$= \int (d_i - 1)t(x)^2 \, dx = (d_i - 1)t(P_3, U).$$

Hence the terms in (10) that correspond to stars sum to at least

$$\sum_{\text{stars}} t(F', U) \geq (d_i - 1)t(P_3, U) = (2m - n)t(P_3, U).$$

(b) Next, consider those terms $F'$ with at least two endnodes that are not stars. For such a term we have

$$|t(F', U)| \leq t(P_3, U)t(C_4, U)^{1/4} \leq 2^{-2m}t(P_3, U)$$

(if there are two nonadjacent endpoints, then the left hand side is 0; else, this follows from Lemma 2.24). The sum of these terms is, in absolute value, at most

$$2^m 2^{-2m}t(P_3, U) < t(P_3, U).$$

(c) The next special sum we consider consists of complete bipartite graphs that are not stars. Fixing a subset $A$ with $|A| \geq 2$ in the first bipartition class of $F$ with $h \geq 2$ common neighbors, and fixing the variables in $A$, the sum over such complete bigraphs with $A$ as one of the bipartition classes is

$$\sum_{j=2}^{h} \binom{h}{j} \left( \int \prod_{i \in A} U(x_i, y) \, dy \right)^j \geq (h - 1) \left( \int \prod_{i \in A} U(x_i, y) \, dy \right)^2$$

by the same computation as above. This gives that this sum is nonnegative.

(d) If $F'$ has all degrees at least 2 and girth $2r$, and it is not a single cycle or complete bipartite, then $F' \leq C_{2r} C_4^{1/4}$ by Lemma 2.22 and so

$$|t(F', U)| \leq t(C_{2r}, U)t(C_4, U)^{1/4} \leq 2^{-2m}t(C_{2r}, U).$$

So if we fix $r$ and sum over all such subgraphs, we get, in absolute value, at most

$$2^m 2^{-2m}t(C_{2r}, U) < \frac{1}{2}t(C_{2r}, U).$$
(e) Finally, if $F'$ has exactly one node of degree 1 and girth $2r$, then by Lemma 2.25

$$|t(F', U)| \leq \frac{1}{2} (t(P_3, U) + t(C_{2r}, U)) t(C_4, U)^{1/8}$$

$$\leq 2^{-m-1} (t(P_3, U) + t(C_{2r}, U)).$$

If we sum over all such subgraphs $F'$, then we get less than $t(P_3, U) + \frac{1}{2} \sum_{r \geq 2} t(C_{2r}, U)$.

The sum in (a) is sufficient to compensate for the sum in (b) and the first term in (e), while the sum over cycles compensates for the sum in (d) and the second sum in (e). This proves that the total sum in (10) is nonnegative. \qed

3.1 Variations

We could do the computations above more carefully, and use the Neumann-Schatten norm $t(C_4, U)^{1/4}$ instead of the cut norm in the statement. The best one can achieve this way is to replace the bound of $2^{-m}$ by about $2^{1-8m}$:

**Theorem 3.2** Let $F = (V, E)$ be a simple bigraph. Let $W \in \mathcal{W}$ with $\int W = 1$ and $0 \leq W \leq 2$ and $t(C_4, W) \leq 2^{-4m}$. Then $t(F, W) \geq 1$.

One can combine the conditions and assume a bound on $\|W - 1\|_{\infty}$. It follows from the Theorem that $\|W - 1\|_{\infty} \leq 2^{-8m}$ suffices. Going through the same arguments (in fact, in a somewhat simpler form) we get:

**Theorem 3.3** Let $F = (V, E)$ be a simple bigraph. Let $W \in \mathcal{W}$ with $\int W = 1$ and $\|W - 1\|_{\infty} \leq 1/(4m)$. Then $t(F, W) \geq 1$.

The condition that $\|W - 1\|_{\infty} \leq 1/(4m)$ implies trivially that $0 \leq W \leq 2$. It would be interesting to get rid of the condition that $W \leq 2$ under an appropriate bound on $\|W - 1\|_{\Box}$. We can only offer the following result.

**Theorem 3.4** Let $F = (V, E)$ be a simple bigraph with $m$ edges, let $0 < \varepsilon < 2^{-1-8m}$, and let $W \in \mathcal{W}$ such that $\int W = 1$, $\int_{S \times T} W \leq 2\lambda(S)\lambda(T)$ whenever $\lambda(S), \lambda(T) \geq 2^{-4/\varepsilon^2}$, and $\|W - 1\|_{\Box} \leq 2^{-1-8m}$. Then $t(F, W) \geq 1 - \varepsilon$.

**Proof.** For every function $W \in \mathcal{W}$ and partition $\mathcal{P} = \{V_1, \ldots, V_k\}$ of $I$ into a finite number of measurable sets with positive measure, let $W_{\mathcal{P}}$ denote the function obtained by averaging $W$ over the partition classes; more precisely, we define

$$W_{\mathcal{P}}(x, y) = \frac{1}{\lambda(V_i)\lambda(V_j)} \int_{V_i \times V_j} W(u, v) du dv$$

for $x \in V_i$ and $y \in V_j$. 18
The Weak Regularity Lemma of Frieze and Kannan in the form used in [1] implies that there is a partition $\mathcal{P}$ into $K \leq 2^{4l^2/\varepsilon^2}$ equal measurable sets such that the function $W_\mathcal{P}$ satisfies

$$\|W_\mathcal{P} - W\| \leq \frac{\varepsilon}{m},$$

and hence by the Counting Lemma 4.1 in [4],

$$|t(F,W_\mathcal{P}) - t(F,W)| \leq \varepsilon.$$

Clearly $\int W_\mathcal{P} = 1$, $W_\mathcal{P} \geq 0$, and for all $x \in V_i$ and $y \in V_j$,

$$W_\mathcal{P}(x,y) = \frac{1}{\lambda(V_i)\lambda(V_j)} \int_{V_i \times V_j} W(u,v) \, du \, dv \leq 2.$$

Furthermore,

$$\|W_\mathcal{P} - 1\| \leq \|W_\mathcal{P} - W\| + \|W - 1\| \leq 2^{-8m},$$

Thus Theorem 3.1 implies that $t(F,W_\mathcal{P}) \geq 1$, and hence $t(F,W) \geq t(F,W_\mathcal{P}) - \varepsilon \geq 1 - \varepsilon$.

3.2 Graphic form

We end with a graph-theoretic consequence of Theorem 3.1.

**Corollary 3.5** Let $F$ be a bipartite graph with $n$ nodes and $m$ edges, and let $G$ be a graph with $N$ nodes and $M = p\binom{N}{2}$ edges. Let $\varepsilon > 0$. Assume that

$$|e_G(S,T) - p|S| \cdot |T|\| \leq (2^{-8m}p - \varepsilon)N^2$$

for all $S,T \subseteq V(G)$, and

$$e_G(S,T) \leq 2p|S| \cdot |T|$$

for all $S,T \subseteq V(G)$ with $|S|, |T| \geq 2^{-4m^2/\varepsilon^2}N$. Then

$$t(F,G) \geq p^l - \varepsilon.$$

**Proof.** This follows by applying Theorem 3.4 to the function $W_G/p$. □

**References**

[1] C. Borgs, J.T. Chayes, L. Lovász, V.T. Sós, and K. Vesztergombi: Convergent Graph Sequences I: Subgraph frequencies, metric properties, and testing, *Advances in Math.* (2008), 10.1016/j.aim.2008.07.008.
[2] H. Hatami: Graph norms and Sidorenko’s conjecture, *Israel J. of Math.* (to appear),
http://arxiv.org/abs/0806.0047

[3] L. Lovász: On the cancellation law among finite relational structures, *Periodica Math. Hung.* 1 (1971), 145-156.

[4] L. Lovász, B. Szegedy: Limits of dense graph sequences, *J. Comb. Theory B* 96 (2006), 933–957.

[5] A.F. Sidorenko: Inequalities for functionals generated by bipartite graphs (Russian) *Diskret. Mat.* 3 (1991), 50–65; translation in *Discrete Math. Appl.* 2 (1992), 489–504.

[6] A.F. Sidorenko: A correlation inequality for bipartite graphs, *Graphs and Combin.* 9 (1993), 201–204.

[7] A.F. Sidorenko: Randomness friendly graphs, *Random Struc. Alg.* 8, 229–241.

[8] M. Simonovits: Extremal graph problems, degenerate extremal problems, and supersaturated graphs, in: *Progress in Graph Theory*, NY Academy Press (1984), 419–437.