Model-dependent radiative corrections to $\tau^- \rightarrow \pi^- \pi^0 \nu$ revisited

A. Flores-Tlalpa$^a$, F. Flores-Baéz$^a$, G. López Castro$^a$ and G. Toledo Sánchez$^b$†

$^a$Departamento de Física, Cinvestav, Apdo. Postal 14-740, México, D.F., México

$^b$Instituto de Física, UNAM, A. P. 20-364, 01000 México, D.F., México

The long-distance electromagnetic radiative corrections to $\tau^- \rightarrow \pi^- \pi^0 \nu$ are re-evaluated. A meson dominance model is used to describe the emission of real photons in this decay. Results obtained for the hadronic spectrum and the decay rate in photon inclusive reactions are compared with previous calculations based on the chiral resonance theory. Independent tests in $\tau \rightarrow \pi\pi\nu\gamma$ that can help to validate the predictions of one of the two models are briefly discussed.

1. INTRODUCTION

Radiative corrections to $\tau^\pm \rightarrow \pi^\pm \pi^0 \nu_i$ ($\tau_{2\pi}$) decays are important for several reasons:

- The current precision in the world average of $\tau_{2\pi}$ measured branching ratios is reaching the 0.4% level [1]. A correct comparison of theory and experiment requires the inclusion of $O(\alpha)$ radiative corrections.
- The conserved vector current (CVC) hypothesis, valid in the isospin symmetry limit, predicts the equality of the weak (measured in $\tau_{2\pi}$ decays) and electromagnetic form factors. Measurements exhibit departures [2] beyond expected isospin symmetry breaking effects (which include radiative corrections).
- Predictions of the two-pion vacuum polarization contribution to the $\mu$ anomalous magnetic moment based on $\tau$ and $e^+e^-$ data, should be equal on the basis of CVC. However, they differ by more than 3$\sigma$'s [2].

The CVC hypothesis has been verified with high accuracy (at the level of $10^{-4}$) in decay rates of superallowed Fermi transitions [3]. The discrepancies pointed out above suggest that unaccounted effects, either in experimental data of pion form factors or in isospin breaking corrections, may have escaped consideration.

In this contribution we revisit the long-distance (LD) model-dependent radiative corrections to $\tau_{2\pi}$ decays. LD corrections provide an energy-dependent source of isospin breaking correction to be applied to the two-pion spectral functions in $\tau$ lepton decays. Previous calculations of the corrections to the hadronic invariant mass spectrum in this decay were studied in refs. [4,5] within the framework of chiral resonance theory [6]. A different approach to compute LD corrections, based on a meson dominance model, was considered in refs. [7,8]. Here we discuss the different results obtained from the two approaches and suggest how independent tests can be carried out in the corresponding radiative $\tau^- \rightarrow \pi^- \pi^0 \nu\gamma$ to distinguish between the two models.

2. LONG-DISTANCE CORRECTIONS TO THE HADRONIC SPECTRUM

The radiative corrected hadronic invariant mass distribution in $\tau_{2\pi}$ decays, is obtained by adding the virtual corrections of $O(\alpha)$ and the real photon corrections, shown in Figure 1, to the zero order expression ($t = (p_{\pi^-} + p_{\pi^0})^2$ is the square of the momentum transfer):

$$\frac{d\Gamma(\tau_{2\pi}(\gamma))}{dt} = \frac{d\Gamma^0}{dt} + \frac{d\Gamma^1}{dt} + \frac{d\Gamma^1}{dt}. \quad (1)$$
Figure 1. Virtual (a) and real (b) photon corrections of $O(\alpha)$ to $\tau_{2\pi}$ decays.

If we also add to the above expression the short-distance corrections arising from the emission and reabsorption of gauge and the Higgs bosons, we get the fully radiative corrected expression:

$$\frac{d\Gamma(\tau_{2\pi}^{\gamma}))}{dt} = \frac{d\Gamma_0}{dt} S_{EW} G_{EM}(t).$$

The factor $S_{EW} = 1.026 \pm 0.0003$ in eq. (2) summarizes the short-distance corrections and includes the effects of resummation of dominant logarithms to all orders \([9]\) and the remaining electromagnetic corrections of order $\alpha$ \([10]\). $S_{EW}$ includes also the resummation of sub-leading strong interaction effects which were recently discussed in ref. \([11]\).

Given that high energy virtual corrections probes the quark level structure in semileptonic decays ($\tau^- \rightarrow \bar{ud}\nu\tau$), $S_{EW}$ is believed to be independent of the specific $\Delta S = 0$ lepton decay.

The long-distance radiative corrections are included in the factor $G_{EM}(t)$ and are model-dependent. The couplings of photons to hadrons are calculated on the basis of scalar QED and also include the effects of model-dependent couplings of the photon to hadrons in all possible ways. It is defined from eq. (1) as follows:

$$G_{EM}(t) = 1 + \frac{d\Gamma_1}{dt} + \frac{d\Gamma_{1,m.i.}}{dt} + \frac{d\Gamma_{1,m.d.}}{dt}$$

$$= G_0^{EM}(t) + G_{rest}^{EM}(t),$$

where we have separated the real photon corrections into its model-independent (m.i.) parts (see \([18]\) for details). The model-independent correction $G_0^{EM}(t)$ includes the sum of virtual corrections and m.i. piece of real photon emission necessary to cancel infrared divergences; the remaining piece $G_{rest}^{EM}(t)$ is regular and model-dependent.

Let us comment that to get the rates for real photon emission in eq. (3) we have integrated over all the photon energies; thus, the $G_{EM}(t)$ correction can be applied to photon inclusive $\tau_{2\pi}$ measurements only.

The correction factor $G_{EM}(t)$, was first calculated in refs. \([4,5]\). Both, virtual and real corrections, were computed in the framework of the chiral resonance theory \([6]\) by considering the exchange of $\rho$ \([12]\) and $a_1$ \([13]\) resonances. The axial couplings to the weak current in real photon corrections were assumed in that model to include the axial anomalous terms \([14]\). Figure 2 displays the results obtained in ref. \([5]\) for $G_0^{EM}(t)$ (short-dashed line) and $G_{EM}(t)$ (long-dashed line). Notice that the contribution of model-dependent corrections $G_{rest}^{EM}(t)$ are small and negative for $t \geq 0.5$ GeV$^2$ and positive and rapidly increasing as the threshold is approached.

Figure 2. Energy dependence of the long-distance correction $G_{EM}(t)$: model-independent corrections $G_0^{EM}(t)$ (short-dashed), full corrections of ref. \([5]\) (long-dashed) and ref. \([8]\) (solid) are shown.
Model-dependent radiative corrections to $\tau^− \rightarrow \pi^− \pi^0 \nu$ revisited

The region of $t$ very close to threshold must be handled with care. The apparent divergent behavior in that region arises from the kinematical suppression of the tree-level spectrum that appears in the denominator of eq. (3). The definition given in eq. (1) must be directly used in that case.

In refs. [7,8] the emission of real photons was considered in the framework of a meson dominance model. The idea behind our approach is that given the large momentum transfer released in $\tau$ decays, all intermediate states involving the production and decay of light resonances ($\rho(770), \omega(782), a_1(1620)$) (see Figure 3) that are allowed by their quantum numbers, must contribute. The different couplings entering in the model-dependent contributions were determined from independent low energy processes (see [7] for details). As we will see below, the diagram (g) in Figure 3, which has not been considered in the calculation of ref. [5], will play an important role from a numerical point of view. In Figure 2 we plot our long-distance correction factor $G_{EM}(t)$ (solid line) as a function of the squared momentum transfer $t$. In the region above $t = 0.7$ GeV$^2$, our radiative corrections are smaller than the ones calculated in ref [5]. Below that value, however, our results are larger. Despite the fact that the predictions of both models give small long-distance corrections, the difference in the integrated observables, $\tau$-decay rate and $a_{\mu}^{\pi,\text{had.}}$ turns out to be interesting (see below).

The difference between our calculation of $G_{EM}(t)$ and that of ref. [5] stems almost completely from the anomalous $\rho\omega\pi$ vertex (Figure 3g). As it was discussed in ref. [8], the predictions of both models coincide when such diagram is excluded from our calculations. This also confirms that the axial-vector contributions, despite their very different origin in both models, are almost negligible. Since the $\rho\omega\pi$ coupling becomes an important contribution within our model, the question arises whether the energy dependence of this coupling would affect our predictions in a sizable way. To have an idea of the answer, we have allowed a variation of $\pm 30\%$ around the central value ($g_{\rho\omega\pi} = 0.012$ MeV$^{-1}$) used for this coupling in our calculations. We have found that a similar variation is obtained, for example, in our estimate of the shift in $a_{\mu}^{\pi,\text{LO}}$ (eq. 5 below).

A simple and useful analytical expression can be obtained for the long-distance correction factor. In almost all the interval of $t$, $G_{EM}(t)$ can be approximated very well by the polynomial function ($x = t/m_{\tau}^2$) [8]:

$$G_{EM}(x) = 1.107 - 1.326x + 5.667x^2 - 10.95x^3 + 9.735x^4 - 3.2776x^5.$$  

(4)

As is well known, experimental data on the $\tau\pi$ spectral function can be used to predict the dominant part of the hadronic vacuum polarization contribution to the muon anomalous magnetic moment ($a_{\mu}^{\pi,\text{LO}}$). The model-dependent long-distance corrections affect the prediction of ($a_{\mu}^{\pi,\text{LO}}$) based on tau data (additional sources of isospin breaking corrections were discussed in ref. [15]), particularly when photon inclusive mea-

Figure 3. Feynman diagram contributions to radiative $\tau \rightarrow \pi\pi\nu\gamma$ decays. The purely model-dependent contributions are shown in diagrams (e-k).
measurements are used. The correction due to LD radiative effects to be applied to the prediction of $a_{\mu,LO}^{\pi\pi}$ based on $\tau_{2\pi}$ data can be estimated from the following formula [4]:

$$\Delta a_{\mu}^{\pi\pi,LO} = \frac{1}{4\pi^3} \int dt K(t) \left[ \frac{K_{\sigma}(t)}{K_{1}(t)} \frac{d\Gamma_{\pi\pi(\gamma)}}{dt} \right] \times \left( \frac{1}{G_{EM}(t)} - 1 \right) = -3.7 \times 10^{-10}. \quad (5)$$

The expression $K_{\sigma}(t)$ ($K_{1}(t)$) in eq. (5) contains the kinematical factors and fundamental constants for $2\pi$ production in $e^+e^-$ annihilation ($\tau$ decays) [5] and $K(t)$ is the kernel function associated to radiative corrections [16]. The function $d\Gamma_{\pi\pi(\gamma)}/dt$ must be the measured inclusive photon invariant mass distribution (for the purposes of our estimate and of comparison with previous calculations we have used eqs. (4.1) and (5.6) from ref. [4]).

The result shown in eq. (5) is almost 4 times larger than the one reported in ref. [5] ($\Delta a_{\mu}^{\pi\pi,LO} = -1.0 \times 10^{-10}$). It is of a size similar to the shift in $a_{\mu}^{\pi\pi,LO}(\tau$, based) produced by the effects of $\rho - \omega$ mixing or the pion mass difference in pion form factors [15].

3. CORRECTIONS TO THE DECAY RATE

The corrections to the decay rate can be obtained from direct integration of eq. (1). If the emission of hard photons ($E_{\gamma} \geq \omega_0$) is discriminated by experiments, it becomes useful to define the corrected rate that includes the emission of soft photons:

$$\Gamma(\pi\pi(\gamma), E_{\gamma} \leq \omega_0) = \Gamma(\pi\pi) \cdot (1 + \delta_{LD}) , \quad (6)$$

where $\Gamma(\pi\pi)$ is the decay rate without long-distance corrections.

In the second column of Table 1 we show our long-distance corrections $\delta_{LD}$ for different values of the cutoff $\omega_0$ for hard photons. The correction to the photon inclusive rate corresponds to $\omega_0 = \omega_{max}$. In this case, our result turns out to be less than twice smaller that the correction obtained using the model of ref. [4] shown in the third column (this result was not given in that reference; we have estimated its value by setting $g_{\rho\omega\pi} = 0$ in our model).

On the other hand, we observe from Table 1 that the correction to the decay rate due to hard photons (let say $E_{\gamma} \geq 300$ MeV) is around +0.17%. This result is much smaller than the estimate (+0.8%) given recently in ref. [17] based on the infrared logarithmic term of radiative events.

4. AN INDEPENDENT TEST FOR OUR MODEL

One may wonder if there is an independent way to discriminate between the predictions of meson dominance and chiral resonance models. The answer is yes, and radiative $\tau \rightarrow \pi\pi\nu\gamma$ ($\tau_{2\pi\gamma}$) decays can be useful for that purpose.

In figure 4 we plot the branching ratio for this decay as a function of the minimum photon energy cutoff (photons of energy larger than $E_{\gamma}^{min}$ are detectable in a given experiment). The solid line denotes the result of our calculation including all the diagrams of Figure 3, while the model-independent (diagrams $a - d$ of figure 3) result is represented by the dashed line. Just for comparison, we also show (dotted line) the result obtained when the diagram involving the $\omega(782)$ meson (Figure 3g) is excluded. The branching ratios obtained in ref. [5] are displayed as three squares at $E_{\gamma}^{min} = 100, 300$ and 500 MeV. Clearly, our

| $\omega_0$ (MeV) | $\delta_{LD}$ (%) | $\delta_{LD}$ (%) |
|-----------------|-----------------|-----------------|
| 300             | -0.31           | -               |
| 400             | -0.27           | -               |
| 500             | -0.23           | -               |
| 600             | -0.19           | -               |
| 700             | -0.16           | -               |
| 800             | -0.15           | -               |
| $\omega_{max}$  | -0.15           | -0.38           |

Table 1: Long-distance corrections $\delta_{LD}$ to the integrated rate of $\tau_{2\pi}$ decays.
branching ratios differs significantly from the results of ref. [5] for $E_{\gamma}^{\text{min}} \geq 150$ MeV and can be a useful discriminator between the two models. Other observables associated to $\tau_2\pi$ decays that can help to distinguish the predictions of the two models have been discussed in ref. [7].

In summary, we have compared the long-distance radiative corrections obtained in the context of the meson dominance model proposed in refs. [7,8] with those obtained in the chiral resonance model discussed in refs. [4,5]. Despite the very different assumptions involved in such models, we have found that the only (numerically) important difference obtained in the calculation of long-distance corrections arise from the real photon emission diagram involving the $\rho\omega\pi$ vertex (Figure 3).

This difference is noticeable in the calculation of the radiative correction to the di-pion spectrum and in other observables associated to the radiative $\tau$ lepton decay. In particular, long-distance corrections shift the two-pion hadronic vacuum polarization contribution to $a_{\mu}$ extracted from tau data by $-3.7 \times 10^{-10}$, which is four times larger than the prediction of ref. [5].

Finally, we have pointed out that some observables associated to radiative $\tau_2\pi$ decays can help to distinguish between the predictions of both models.

REFERENCES

1. W. M. Yao et al. [Particle Data Group], J. Phys. G 33 (2006) 1.
2. M. Davier, these proceedings.
3. J. C. Hardy and I. S. Towner, Eur. Phys. J. A 25 (2005) 695; W. Marciano and A. Sirlin, Phys. Rev. Lett. 96 (2006) 032002.
4. V. Cirigliano, G. Ecker and H. Neufeld, Phys. Lett. B 513 (2001) 361. arXiv:hep-ph/0104267.
5. V. Cirigliano, G. Ecker and H. Neufeld, JHEP 0208 (2002) 002.
6. G. Ecker, J. Gasser, A. Pich and E. de Rafael, Nucl. Phys. B 321 (1989) 311.
7. A. Flores-Tlalpa, G. López Castro and G. Toledo Sánchez, Phys. Rev. D 72 (2005) 113003.
8. F. Flores-Baez, A. Flores-Tlalpa, G. López Castro and G. Toledo Sánchez, Phys. Rev. D 64 (2006) 071301(R).
9. A. Sirlin, Rev. Mod. Phys. 50 (1978) 573 [Erratum-ibid. 50 (1978) 905]; W. J. Marciano and A. Sirlin, Phys. Rev. Lett. 61 (1988) 1815; A. Sirlin, Nucl. Phys. B 196 (1982) 83.
10. E. Braaten and C. S. Li, Phys. Rev. D 42 (1990) 3888.
11. J. Erler, Rev. Mex. Fis. 50 (2004) 200.
12. F. Guerrero and A. Pich, Phys. Lett. B 412 (1997) 382.
13. J. H. Kuhn and A. Santamaria, Z. Phys. C 48 (1990) 445.
14. J. Wess and B. Zumino, Phys. Lett. B 37 (1971) 95; E. Witten, Nucl. Phys. B 223 (1983) 422.
15. M. Davier, S. Eidelman, A. Hocker, and Z. Zhang Eur. Phys. J. C 27 (2003) 497.
16. S. Brodsky and E. de Rafael, Phys. Rev. 168 (1968) 1620.
17. J. F. de Troconiz and F. J. Yndurain, Phys. Rev. D 71 (2005) 073008.