On the density of foliations without algebraic solutions on weighted projective planes

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Abstract
We prove that a generic holomorphic foliation on a weighted projective plane has no algebraic solutions when the degree is big enough. We also prove an analogous result for foliations on Hirzebruch surfaces.

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1 INTRODUCTION

One of the most important results of Jouanolou’s celebrated monograph [10] states that the set of holomorphic foliations on the complex projective plane \( \mathbb{P}^2 \) of degree at least 2, which do not have an algebraic solution, is dense in the space of foliations. This result for one-dimensional holomorphic foliations on \( \mathbb{P}^n \) was proved by Lins Neto and Soares in [14]. In [8], the authors prove a generalization of Jouanolou’s result for one-dimensional foliations over any smooth projective variety. On the other hand, in [15], the author gives a different proof of Jouanolou’s theorem by restricting the ideas of [8] to \( \mathbb{P}^2 \). In [12], one can find three versions of Jouanolou’s Theorem for second-order differential equations on \( \mathbb{P}^2 \), for \( k \)-webs (first-order differential equations) on \( \mathbb{P}^2 \), and for webs with sufficiently ample normal bundle on arbitrary projective surfaces.

The main theorem of this work provides an analogous version of Jouanolou’s Theorem for foliations on weighted projective planes.

Theorem A. Let \( l_0, l_1, l_2 \) be pairwise coprimes numbers with \( 1 \leq l_0 \leq l_1 \leq l_2 \). A generic foliation with normal \( \mathbb{Q} \)-bundle of degree \( d \) on the weighted projective plane \( \mathbb{P}(l_0, l_1, l_2) \) does not admit any invariant algebraic curve if \( d \geq l_0l_1l_2 + l_0l_1 + 2l_2 \).

The bound above is not sharp. When \( l_0 = l_1 = 1 \) and \( l_2 = k > 1 \), we denote \( \mathbb{P}(1,1,k) \) by \( \mathbb{P}^2_k \), for which there is a more precise version of Theorem A that is sharp.

Theorem B. A generic foliation with normal \( \mathbb{Q} \)-bundle of degree \( d \) on \( \mathbb{P}^2_k \) with \( k \geq 2 \) does not admit any invariant algebraic curve if \( d \geq 2k + 1 \). Moreover, if \( d < 2k + 1 \) any foliation with normal \( \mathbb{Q} \)-bundle of degree \( d \) on \( \mathbb{P}^2_k \) admits some invariant algebraic curve.

In both statements, by generic we mean the set of foliations having no invariant curves is the complement of a countable union of algebraic closed proper subsets.

It is worth mentioning that there are several works about foliations on weighted projective spaces, see [6, 7] and [13].
It is well known that the minimal resolution of the weighted projective planes $\mathbb{P}_k^2$, $k \geq 2$, are the Hirzebruch surfaces $F_k = \mathbb{P} (\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(k))$ (see [16]). Our next result is a generalized version of Jouanolou’s Theorem for foliations on Hirzebruch surfaces.

**Theorem C.** A generic foliation with normal bundle of bidegree $(a, b)$ on $F_k$ does not admit any invariant algebraic curve if $b \geq 3$ and $a \geq kb + 2$. Moreover, if $a < kb + 2$ or $b < 3$, then any foliation with normal bundle of bidegree $(a, b)$ on $F_k$ admits some invariant algebraic curve.

### 1.1 Organization of the paper

In Section 2, we introduce the notions of weighted projective planes and foliations on weighted projective planes. In Section 3, we study foliations on weighted projective planes with some algebraic solution. We show that the low-degree foliations correspond to rational, logarithmic, and Riccati foliations. In Section 4, we prove our first and second main results about the version of Jouanolou’s theorem for foliations on weighted projective planes. In Section 5, we introduce the notions of foliations on Hirzebruch surfaces and prove our third main result Theorem C.

## 2 Holomorphic Foliations on Weighted Projective Planes

### 2.1 Weighted projective planes

Let $\ell = (l_0, l_1, l_2)$ be a vector of positive integers with $l_0, l_1, l_2$ pairwise coprimes, and call $\ell$ a weighted vector. Consider the $\mathbb{C}^*$-action on $\mathbb{C}^3 \setminus \{0\}$ given by

$$t \cdot (x_0, x_1, x_2) = (t^{l_0}x_0, t^{l_1}x_1, t^{l_2}x_2),$$

where $t \in \mathbb{C}^*$ and $(x_0, x_1, x_2) \in \mathbb{C}^3 \setminus \{0\}$. The weighted projective plane of type $(l_0, l_1, l_2)$ is the quotient space $\mathbb{P}(l_0, l_1, l_2) = (\mathbb{C}^3 \setminus \{0\} / \sim)$, induced by the above action. Consider the decomposition $\mathbb{P}(l_0, l_1, l_2) = U_0 \cup U_1 \cup U_2$, where $U_i$, $0 \leq i \leq 2$, is the open set consisting of all elements $[x_0 : x_1 : x_2]$ with $x_i \neq 0$. For a fixed $i$, let $j < k$ be the elements of $\{0, 1, 2\} \setminus \{i\}$ and denote by $\mu_{l_i} \subset \mathbb{C}^*$ the subgroup of $l_i$-th roots of the unity, which acts on $\mathbb{C}^2$ by $g \cdot (x, y) = (g^{l_j}x, g^{l_k}y)$, for any $g \in \mu_{l_i}$.

We can define an isomorphism $\psi_i : U_i \to \mathbb{C}^2 / \mu_{l_i}$ by

$$\psi_i([x_0 : x_1 : x_2]) = \left(\frac{x_j}{x_j^{l_j/l_i}}, \frac{x_k}{x_k^{l_k/l_i}}\right).$$

Observe that $\psi_i$ is independent of the choice of the $l_i$-th root of $x_i$.

**Remark 2.1.** When $l_0 = l_1 = l_2 = 1$, one obtains the usual complex projective plane, in which case, we write $\mathbb{P}^2$ instead of $\mathbb{P}(1, 1, 1)$.

We can also view the weighted projective plane as follows. Consider the group $G = \mu_{l_0} \times \mu_{l_1} \times \mu_{l_2}$, acting on $\mathbb{P}^2$ by $(g_0, g_1, g_2) \cdot [x_0 : x_1 : x_2] = [g_0x_0 : g_1x_1 : g_2x_2]$. The natural map $\varphi : \mathbb{C}^3 \to \mathbb{C}^3$ given by

$$\varphi(x_0, x_1, x_2) = (x_0^{l_0}, x_1^{l_1}, x_2^{l_2})$$

induces an isomorphism from the set of orbits $\mathbb{P}^2 / G$ to $\mathbb{P}(l_0, l_1, l_2)$.

From now on, we will assume that $1 \leq l_0 \leq l_1 \leq l_2$. For simplicity of notation, we write $\mathbb{P}$ instead of $\mathbb{P}(l_0, l_1, l_2)$ as in the terminology of [9]. The weighted projective plane $\mathbb{P}$ is a singular surface with at worst quotient singularities.
2.2 Twisted differentials defining foliations

If we denote by $\Omega_p^{[1]}$ the sheaf of reflexive differentials on $\mathbb{P}$ ($\Omega_p^{[1]} = \Omega_p^{1}$ in the notation of [9, section 2.1]) and by $\mathcal{O}_p(d)$ the sheaf of $\mathcal{O}_p$-modules associated to the module of quasi-homogeneous polynomials of degree $d$ [9, section 1.4], then we can use the following twisting of Euler’s sequence [9, section 2.1],

$$0 \to \Omega_p^{[1]}(d) \to \bigoplus_{i=0}^{2} \mathcal{O}_p(d - l_i) \to \mathcal{O}_p(d) \to 0,$$

to identify $H^0(\mathbb{P}, \Omega_p^{[1]} \otimes \mathcal{O}_p(d)) = H^0(\mathbb{P}, \Omega_p^{[1]}(d))$ with the $\mathbb{C}$-vector space of quasi-homogenous polynomial 1-forms

$$A_0(x_0, x_1, x_2)dx_0 + A_1(x_0, x_1, x_2)dx_1 + A_2(x_0, x_1, x_2)dx_2$$

with quasi-homogeneous coefficients $A_i$ of degree $d - l_i$, for $i = 0, 1, 2$, which are annihilated by the weighted radial vector field

$$R = l_0x_0 \frac{\partial}{\partial x_0} + l_1x_1 \frac{\partial}{\partial x_1} + l_2x_2 \frac{\partial}{\partial x_2}.$$

A foliation $\mathcal{F}$ on $\mathbb{P}$ is defined by a section $\omega$ of $\Omega_p^{[1]} \otimes \mathcal{O}_p(d)$ with normal sheaf $N_\mathcal{F} = \mathcal{O}_p(d)$.

The singular set of $\mathcal{F}$, denoted by $\text{sing}(\mathcal{F})$ is the subset of $\mathbb{P}$ formed by zeros of $\omega$. A foliation $\mathcal{F}$ is called saturated if $\text{sing}(\mathcal{F})$ is finite.

**Remark 2.2.** In the case $\ell = (1, 1, 1)$, we have $\deg(N_\mathcal{F}) = \deg(\mathcal{F}) + 2$, where $\deg(\mathcal{F})$ is the number of tangencies with general line.

By duality, we also have the following twisting of Euler’s sequence:

$$0 \to \mathcal{O}_p(d - |\ell|) \to \bigoplus_{i=0}^{2} \mathcal{O}_p(d - |\ell| + l_i) \to \mathcal{T}_p(d - |\ell|) \to 0,$$

where $|\ell| = l_0 + l_1 + l_2$. Then, a foliation $\mathcal{F}$ of normal degree $d$ on $\mathbb{P}$ can also be given by a quasi-homogeneous vector field $X \in H^0(\mathbb{P}, T_\mathbb{P} \otimes \mathcal{O}_p(d - |\ell|)) = H^0(\mathbb{P}, T_\mathbb{P}(d - |\ell|))$, that is,

$$X = B_0 \frac{\partial}{\partial x_0} + B_1 \frac{\partial}{\partial x_1} + B_2 \frac{\partial}{\partial x_2},$$

which is not a multiple of $R$, where $B_i$ are quasi-homogeneous polynomials of degree $d - |\ell| + l_i$, respectively. By contracting the volume form $dx_0 \wedge dx_1 \wedge dx_2$ with the vector fields $X$ and $R$, we obtain a 1-form

$$\omega = i_X i_R dx_0 \wedge dx_1 \wedge dx_2 \in H^0(\mathbb{P}, \Omega_p^{[1]}(d)),$$

that defines the foliation $\mathcal{F}$. This vector field $X$ also induces the sheaf $K_\mathcal{F} = \mathcal{O}_p(d - |\ell|)$, called the cotangent or canonical sheaf of $\mathcal{F}$. The dual $K_\mathcal{F}$ is $T_\mathcal{F}$ called the tangent sheaf of $\mathcal{F}$. Note that $N_\mathcal{F}, K_\mathcal{F},$ and $T_\mathcal{F}$ are no longer genuine line bundles but are elements of $\text{Pic}(\mathbb{P}) \otimes \mathbb{Q}$.

2.3 Space of foliations and interpretation

Two 1-forms $\omega$ and $\omega'$ define the same foliation if and only if they differ by multiplication by a nonzero complex constant. Therefore, the space of foliations with normal $\mathbb{Q}$-bundle of degree $d$ on $\mathbb{P}$ is denoted by

$$\text{Fol}(d) := \mathbb{P}H^0(\mathbb{P}, \Omega_p^{[1]}(d)).$$
We denote by $\text{Fol}_d(\mathbb{P}^2)^G$ the space of $G$-invariant foliations of degree $d$ on $\mathbb{P}^2$. Note that the pullback of a quasi-homogeneous 1-form by $\varphi$ (as in 2.3) is a homogeneous 1-form invariant by $G$. Consider the natural map:

$$\varphi^* : \text{Fol}(d + 2) \rightarrow \text{Fol}_d(\mathbb{P}^2)^G$$

$$[\omega] \mapsto [\varphi^* \omega].$$

(2.4)

It is a direct computation to show that the following holds.

Lemma 2.3. The map $\varphi^*$ is an isomorphism.

This means that a foliation on $\mathbb{P}$ is identified with a foliation on $\mathbb{P}^2$ invariant by $G$. Therefore, $\text{Fol}_d(\mathbb{P}^2)^G$ is an irreducible closed set of the space of foliations of degree $d$ on $\mathbb{P}^2$.

Denoting by $S_n \subset \mathbb{C}[x_0, x_1, x_2]$ the space of quasi-homogeneous polynomials of degree $n$, we have the natural application $\phi_n : \mathbb{P}(S_n) \times \text{Fol}(d - n) \rightarrow \text{Fol}(d)$, $([F], [\omega]) \mapsto [F \omega]$, for $1 \leq n \leq d$. The set of the saturated foliations in $\text{Fol}(d)$ is equal to the complement of $\bigcup_{1 \leq n \leq d \dim(\phi_n)}$, which is an open Zariski set in $\text{Fol}(d)$. Note that this open set can be empty, as illustrated by the following example.

Example 2.4. Take $\ell = (1, 1, k)$, $k \geq 2$. Looking at the coefficients of the forms, we can prove that $\text{Fol}(d) = \text{Im}(\phi_{d - 2})$ for all $2 < d \leq k$.

2.4 Invariant algebraic curves

Let $F = [\omega] \in \text{Fol}(d)$ and $C \subset \mathbb{P}$ be an irreducible algebraic curve of degree $n$, that is, defined by a quasi-homogeneous polynomial $F$ of degree $n$. We say that $C$ is $F$-invariant if $i^* \omega \equiv 0$, where $i$ denotes the inclusion of the intersection of smooth loci of both $C$ and $\mathbb{P}$. Equivalently, $C$ is $F$-invariant if there exists a quasi-homogeneous 2-form $\Theta_F$ of degree $n$ such that

$$\omega \wedge dF - F \Theta_F = 0.$$  

(2.5)

Remark 2.5. An important fact about Equation (2.5) is that it still works for reducible curves, that is, if the decomposition of $F$ is $F_1^{n_1} ... F_r^{n_r}$, then Equation (2.5) holds if and only if each irreducible factor $F_j$ defines an $F$-invariant curve.

Let $p_0 := [1 : 0 : 0], p_1 := [0 : 1 : 0], p_2 := [0 : 0 : 1]$, and consider the following sets:

$$C_n(d) := \{ F \in \text{Fol}(d) \mid \text{there is a } F\text{-invariant algebraic curve of degree } n \},$$

$$D_n(d) := \left\{ (x, F) \in \mathbb{P} \times \text{Fol}(d) \mid x \text{ belongs to some } F\text{-invariant algebraic curve of degree } n \right\},$$

and for each $i \in \{0, 1, 2\}$,

$$C_n^{p_i}(d) := \{ F \in \text{Fol}(d) \mid p_i \text{ belongs to some } F\text{-invariant algebraic curve of degree } n \}.$$  

The following lemma will be used in Section 4.

Lemma 2.6. The sets $C_n(d)$, $D_n(d)$, and $C_n^{p_i}(d)$ are closed for every $n$ and $i$.

Proof. The same argument of [15, Proposition 1] shows that $C_n(d)$ and $D_n(d)$ are closed sets. In order to finish, we just note that $\pi_2(\pi_1^{-1}(p_i) \cap D_n(d)) = C_n^{p_i}(d)$, where $\pi_1$ and $\pi_2$ are canonical projections of $\mathbb{P} \times \text{Fol}(d)$.

\end{proof}
2.5 | Saturated foliations

Let $\mathcal{F}$ be a saturated foliation on $\mathcal{P}$ and $p \in \text{sing}(\mathcal{F})$.

**Definition 2.7.** The multiplicity (or Milnor number) of $\mathcal{F}$ at $p$, which is denoted by $m(\mathcal{F}, p)$, is defined as follows:

1. If $p \notin \text{sing}(\mathcal{P})$, and $\mathcal{F}$ is locally given by $\omega = A(x, y) \, dx + B(x, y) \, dy$, then

   $$m(\mathcal{F}, p) := \dim \mathbb{C} \langle A, B \rangle,$$

   where $\mathcal{O}_p$ is the local algebra of $\mathcal{P}$ at $p$.

2. If $p \in \text{Sing}(\mathcal{P})$, then

   $$m(\mathcal{F}, p) := \frac{m(G,(0,0))}{r},$$

   in which $U \simeq \mathbb{C}^2/\mu_r$ is a neighborhood of $p$ and $G$ is the lifting of $\mathcal{F}|_U$ to $\mathbb{C}^2$.

Therefore, we can define:

$$m(\mathcal{F}) := \sum_{p \in \mathcal{P}} m(\mathcal{F}, p).$$

In the index theory of foliations on complex regular surfaces, there are other indices such as tangency, vanishing, Camacho-Sad, and Bam-Bott, for a good reference, see [5]. This theory is also valid on weighted projective planes since they are singular surfaces with at worst quotient singularities, for this, see [4] and [6]. As an adaptation of [5, Proposition 1, p. 21], we have the following formula:

$$m(\mathcal{F}) = c_2(\mathcal{P}) - T_{\mathcal{F}} \cdot N_{\mathcal{F}},$$

where $c_2$ is the 2-Chern class of $\mathcal{P}$.

**Proposition 2.8.** Let $\mathcal{F}$ be a foliation of normal degree $d$ on $\mathcal{P}$. Then,

$$l_0 l_1 l_2 m(\mathcal{F}) = l_0 l_1 + l_0 l_2 + l_1 l_2 + (d - |\ell|)d.$$

(2.6)

**Proof.** It follows from $c_2(\mathcal{P}) = \frac{l_0 l_1 + l_0 l_2 + l_1 l_2}{l_0 l_1 l_2}, T_{\mathcal{F}} = \mathcal{O}_p(|\ell| - d),$ and $N_{\mathcal{F}} = \mathcal{O}_p(d).$ □

It is worth mentioning that other proofs can be found in [6, Proposition 3.2] and [11, Proposition 1.7.6].

Again for a fixed $i$, let $j < k$ be the elements of $\{0, 1, 2\}\setminus\{i\}$. We will work on an affine coordinate system $(x, y) \in \mathbb{C}^2$ where the action of $\mu_i$ on $\mathbb{C}^2$ defines $U_i = \mathbb{C}^2/\mu_i \subset \mathcal{P}$.

In this coordinate system, a foliation $\mathcal{F}$ of normal degree $d$ on $\mathcal{P}$ is defined by a 1-form $\omega$ that can be written as a sum of quasi-homogeneous 1-forms

$$\omega = \omega_r + \omega_{r+l_1} + \omega_{r+2l_1} + \ldots + \omega_{d-l_i} + \omega_d,$$

(2.7)

where, for $s = r, r + l_1, \ldots, d$, $\omega_s$ is a quasi-homogeneous 1-form of degree $s$ (both $x, dx$ and $y, dy$ have degree $l_j$ and $l_k$, respectively) and the contraction of $\omega_d$ with the weighted radial vector field $v = l_j \frac{\partial}{\partial x} + l_k \frac{\partial}{\partial y}$ is equal to zero. If $\omega_d = 0$, then $i_v \omega_{d-l_i} \neq 0$, and the plane at infinity is invariant by the foliation defined by $\omega$. Note that Equation (2.7) is equivalent to $\omega$ is invariant by the action of $\mu_i$. When $\omega_r \neq 0$, we call $r$ the algebraic multiplicity of $\mathcal{F}$ at $p_i$. 

3 | FOLIATIONS WITH ALGEBRAIC LEAVES ON $\mathbb{P}$

3.1 | Logarithmic and rational foliations

The following result has been proved, in a much more general context, by Bogomolov and McQuillan [3]. We will give another proof of the fact that the low-degree foliations on $\mathbb{P}$ have infinitely many algebraic leaves.

**Proposition 3.1.** Let $\mathcal{F}$ be a saturated foliation of normal degree $d$ on $\mathbb{P}$ induced by $\omega$. If $d < |\ell|$, then $\mathcal{F}$ is a rational fibration. Furthermore,

1. If $d = l_0 + l_1$, then $\omega$ is conjugated to $l_1 x_1 dx_0 - l_0 x_0 dx_1$.
2. If $d = l_0 + l_2$, then $\omega$ is conjugated to $l_2 x_2 dx_0 - l_0 x_0 dx_2$.
3. If $l_0 + l_2 < d$ and $d \neq l_1 + l_2$, then $\omega$ is conjugated to
   \[
   \left( d - l_0 \right) \left( x_2 x_0^i + x_1^{i+1} \right) dx_0 - l_0 x_0 d \left( x_2 x_0^i + x_1^{i+1} \right),
   \]
   with the condition $(i + 1)l_0 + l_2 = (j + 1)l_1 + l_0 = d$, $i, j \geq 1$.
4. If $d = l_1 + l_2$, then $\omega$ is conjugated to
   \[
   (a) \ l_2 x_2 dx_1 - l_1 x_1 dx_2, \quad \text{when} \quad 1 = l_0 = l_1 < l_2 \quad \text{or} \quad 1 < l_0 < l_1 < l_2,
   \]
   \[
   (b) \ l_2 x_1 dx_0 - l_0 dx_2 + x_2 x_0^{l_1-1} x_1^{j+1} dx_0 - x_0 dx_2 (x_2 x_0^{l_1-1} x_1^{j+1} + x_1^{j+1}), \quad \text{with the condition} \quad j l_1 + 1 = l_2, \quad j \geq 1,
   \]
   when $1 < l_0 < l_1$.

**Proof.** First notice that $d \geq l_0 + l_1$ and there are no saturated foliations on $\mathbb{P}$ if $l_0 + l_1 < d < l_0 + l_2$. We now proceed to prove Items (1)–(4).

The assumption of Item (1) implies that $\omega = l_1 x_1 dx_0 - l_0 x_0 dx_1$. Item (1) follows.

The assumption of Item (2) implies that $\omega = l_2 x_2 dx_0 - l_0 x_0 dx_2$, if $1 < l_0 < l_1 < l_2$, or $\omega = l_2 x_2 dx_0 (x_0 + x_1) - (l_2 x_0 + b x_1) dx_2$, for some $a, b \in \mathbb{C}$, if $1 = l_0 = l_1 < l_2$. Now, Item (2) follows by performing a suitable change of coordinates.

The assumption of Item (3) implies that $1 \leq l_0 < l_1 < l_2$ and

\[
\omega = x_0^l x_2 (l_2 x_0^i x_2 - l_0 x_0 x_2) + \left( x_0^l B(x_0, x_1) + b x_1^l \right) \left( l_1 x_1 dx_0 - l_0 x_0 dx_1 \right),
\]

\[
= \left( l_2 x_0^l x_2 + l_1 b x_1^{l+1} + l_1 x_1 x_0^l B \right) dx_0 - l_0 x_0 \left( x_0^l x_2 + b x_1^l x_1 x_0^l B \right),
\]

where $(i + 1)l_0 + l_2 = (j + 1)l_1 + l_0 = d$, $i, j \geq 1$, $b \in \mathbb{C}^*$, and $B$ is of degree $d - (i + 1)l_0 - l_1$. To conjugate $\omega$ to $(l_2 x_0^l x_2 + l_1 b x_1^{l+1}) dx_0 - l_0 x_0 (x_0^l x_2 + b x_1^l x_1 x_0^l B)$, we perform the change of coordinates $\phi(x_0, x_1, x_2) = (x_0, x_1, x_2 + H(x_0, x_1))$, where $H$ is chosen from the equation $H x_1 = B$. Also, this last 1-form can be written as

\[
(d - l_0) \left( x_0^l x_2 + \frac{b}{j+1} x_1^{l+1} \right) dx_0 - l_0 x_0 \left( x_0^l x_2 + \frac{b}{j+1} x_1^{l+1} \right),
\]

Up to automorphism of $\mathbb{P}$, we can suppose that $b = j + 1$. Item (3) follows.

The assumption of Item (4) together with Item (2) lets us assume that $1 \leq l_0 < l_1 < l_2$. From Proposition 2.8, we see that $m(\mathcal{F}) = \frac{1}{l_0}$, so we must consider two cases: $l_0 > 1$ and $l_0 = 1$. In the first case, the condition $l_0 > 1$ implies that $\operatorname{sing}(\mathcal{F}) = \{p_0\}$ and that

\[
\omega = \left( l_2 x_2 dx_1 - l_1 x_1 dx_2 \right) + x_1 C(x_0, x_1) \left( l_1 x_1 dx_0 - l_0 x_0 dx_1 \right),
\]

where $C$ is of degree $l_2 - (l_1 + l_0)$. We now conjugate $\omega$ to $l_2 x_2 dx_1 - l_1 x_1 dx_2$ by making the change of coordinates $\phi(x_0, x_1, x_2) = (x_0, x_1, x_2 + H(x_0, x_1))$, where $H$ is chosen from the equation $H x_0 = -x_1 C$. In the second case, $l_0 = 1$, up to an automorphism of $\mathbb{P}$, we can suppose that either $\operatorname{sing}(\mathcal{F}) = \{p_0\}$ or $\operatorname{sing}(\mathcal{F}) = \{p_2\}$. When $\operatorname{sing}(\mathcal{F}) = \{p_0\}$, the conclusion follows from the same argument we employed in the previous case $l_0 > 1$. When $\operatorname{sing}(\mathcal{F}) = \{p_2\}$, we have

\[
\omega = l_0^{-1} \left( l_2 x_2 dx_0 - x_0 dx_2 \right) + \left( x_0^l A(x_0, x_1) + b x_1^l \right) \left( l_1 x_1 dx_0 - x_0 dx_1 \right),
\]

\[
= \left( l_2 x_0^l x_2 + l_1 b x_1^{l+1} + l_1 x_1 x_0^l A \right) dx_0 - x_0 (x_0^l x_2 + b x_1^l x_1 x_0^l A dx_1),
\]

and $\omega$ is conjugated to $(l_2 x_0^l x_2 + l_1 b x_1^{l+1} + l_1 x_1 x_0^l A) dx_0 - x_0 (x_0^l x_2 + b x_1^l x_1 x_0^l A dx_1)$.
where \(j_l + 1 = l_2, j \geq 1, b \in \mathbb{C}^*, \) and \(A\) of degree \(l_2 - (l_1 + 1)\). Now we proceed as in Item (3) to obtain Item (4b). This completes the proof of Item (4), and therefore the proof of the proposition.

For the case of foliations of normal degree \(|\ell|\), we have the following result.

**Proposition 3.2.** If \(d = |\ell|\), then the general element \(\text{Fol}(d)\) is defined by a logarithmic 1-form with poles on three curves of degree \(l_0, l_1,\) and \(l_2\).

Furthermore, a saturated foliation \(\mathcal{F} \in \text{Fol}(|\ell|)\) defined by \(\omega\) is conjugated to one of the following 1-forms:

1. \(x_0x_1x_2 \left( a \frac{dx_0}{x_0} + b \frac{dx_1}{x_1} + c \frac{dx_2}{x_2} \right)\), where \(a, b, c \in \mathbb{C}\) are such that \(al_0 + bl_1 + cl_2 = 0\),
2. \(x_0^{j_1+1}x_1^j \left( l_1 \frac{dx_0}{x_0} - l_0 \frac{dx_1}{x_1} + d \left( \frac{x_2}{x_0^j} \right) \right)\), where \(i \geq 1\) and \(j \geq 0\) are such that \(il_0 + jl_1 = l_2\),
3. \(x_0^{l_2}x_1^{l_1+1} \left( l_2 \frac{dx_0}{x_0} - l_1 \frac{dx_1}{x_1} + d \left( \frac{x_2}{x_0} \right) \right)\), when \(1 = l_0 \leq l_1 < l_2\),
4. \((l_2+1)(x_0x_2 + x_1^{l_2+1})d{\xi}_0 - {\xi}_0d(x_0x_2 + x_1^{l_2+1})\), when \(1 = l_0 = l_1 < l_2\).

**Proof.** We will first prove that a general element \(\mathcal{F} \in \text{Fol}(|\ell|)\) is defined by 1-form conjugated to

\[
\omega = x_0x_1x_2 \left( a \frac{dx_0}{x_0} + b \frac{dx_1}{x_1} + c \frac{dx_2}{x_2} \right),
\]

with the condition \(al_0 + bl_1 + cl_2 = 0\).

In fact, by Proposition 2.8, we have \(m(F) = \frac{1}{l_0} + \frac{1}{l_1} + \frac{1}{l_2}\). Since \(\mathcal{F}\) is a general element, up to an automorphism of \(\mathbb{P}\), we can assume that \(\text{sing}(\mathcal{F}) = \{p_0, p_1, p_2\}\) and therefore \(\mathcal{F}\) is induced by a 1-form \(\omega\) equal to

\[
-(ax_1x_2 + l_0x_0x_1^2A(x_0, x_1))d{\xi}_0 + (bx_0x_2 - l_0x_0^2x_1A(x_0, x_1))d{\xi}_1 + cx_0x_1dx_2,
\]

where \(al_0 + bl_1 + cl_2 = 0\) and \(A\) has degree \(l_2 - l_0 - l_1\).

By proceeding as in the previous proposition, we can conjugate \(\omega\) to the 1-form (3.1). We treat the other cases in a similar way.

\[\square\]

3.2 | Existence of algebraic leaves on \(\mathbb{P}^2_k\)

Recall that \(\mathbb{P}^2_k = \mathbb{P}(1,1,k)\), \(k \geq 2\), and its only singularity is \(p_2 = [0 : 0 : 1] \in U_2 \subset \mathbb{P}^2_k\). The resolution of this singularity can be obtained from the map \(\pi : \hat{U}_2 \to U_2\), where \(\hat{U}_2 = \{(x,y,k), [s : t] \in U_2 \times \mathbb{P}^1 | xt = ys\}\) is a nonsingular surface covered by two open sets \(\hat{U}_2 = V_0 \cup V_1\) such that \(V_i \simeq \mathbb{C}^2\) and

\[
\pi |_{V_0} : V_0 \to U_2,
(x,y) \mapsto (x^{1/k}, y^{1/k})_k,
\]

\[
\pi |_{V_1} : V_1 \to U_2,
(x,y) \mapsto (xy^{1/k}, y^{1/k})_k.
\]

In the same way that \(U_2\) is compactified as \(\mathbb{P}^2_k\), we can compactify \(\hat{U}_2\) as \(\mathbb{F}_k\) and extend \(\pi\) to \(\mathbb{F}_k \to \mathbb{P}^2_k\) with the exceptional divisor \(E = \pi^{-1}(p_2) = \mathbb{P}^1\) satisfying \(E^2 = -k\). See [2, section 2] or [16, section 2.6] for more details.

**Proposition 3.3.** Let \(\mathcal{F}\) be a saturated foliation of normal degree \(d\) on \(\mathbb{P}^2_k\), \(\mathcal{F}\) be the strict transform of a line passing through \(p_2\), \(r\) be the algebraic multiplicity of \(\mathcal{F}\) at \(p_2\), and \(G = \pi^*(\mathcal{F})\) be the pull-back of the foliation \(\mathcal{F}\). Then, the following equality holds:

\[
N_G = dF + \left( \frac{d - e}{k} \right)E,
\]
where

\[
e = \begin{cases} 
  r - k, & \text{if } E \text{ is } G\text{-invariant}, \\
  r, & \text{if } E \text{ is not } G\text{-invariant}.
\end{cases}
\]

Moreover, \( G \) is a Riccati foliation with respect to the natural rational fibration if and only if \( e = d - 2k \).

**Proof.** The first part follows from [2, Proposition 7.3] and the computation of the total transform \( \pi^*(N_{\mathcal{F}}) \). The foliation \( G \) is transversal with respect to the natural fibration if and only if \( N_{\mathcal{G}} \cdot F = 2 \), which is equivalent to \( e = d - 2k \). \( \square \)

**Corollary 3.4.** Under the same hypotheses of Proposition 3.3, \( G \) is a Riccati foliation with respect to the rational fibration and \( E \) is \( G \)-invariant if and only if \( r = d - k \). In particular, \( F \) has an invariant line.

**Proof.** The equivalence follows from Proposition 3.3. We can also see that if \( G \) is a Riccati foliation and \( E \) is \( G \)-invariant, then the fiber through a singularity in \( E \) is invariant by \( G \), thus \( F \) has an invariant line. \( \square \)

In the notation of Proposition 3.3, we will say that \( F \) is a Riccati foliation if \( G \) is a Riccati foliation.

The following proposition characterizes the low-degree foliations on \( \mathbb{P}_2 \) with some algebraic solution.

**Proposition 3.5.** Any foliation of normal degree \( d \) on \( \mathbb{P}_2 \), with \( 2 \leq d \leq 2k \) admits some invariant line. Furthermore, if \( k + 1 \leq d \leq 2k \), then every saturated foliation in \( \text{Fol}(d) \) is a Riccati foliation.

**Proof.** First note that for \( 2 < d \leq k \), there are no saturated foliations. Now, we see that for \( k < d \leq 2k \), a saturated foliation \( F \in \text{Fol}(d) \) admits an invariant line. In fact, \( F \) is given by a 1-form \( \omega \) equal to

\[
A(x_0, x_1)(kx_2dx_0 - x_0dx_2) + B(x_0, x_1)(kx_2dx_1 - x_1dx_2) + C(x_0, x_1)(x_1dx_0 - x_0dx_1),
\]

where \( A, B, C \) are quasi-homogeneous polynomials of degree \( d - k - 1, d - k - 1, \) and \( d - 2 \), respectively. In the open set \( U_2 \cong \mathbb{C}^2/\mu_k \), we lift \( F|_{U_2} \) to \( \mathbb{C}^2 \) and this lifting is given by

\[
\eta = kA(x, y)dx + kB(x, y)dy + C(x, y)(ydx - xdy).
\]

Since \( F \) is saturated, we have \( r = d - k \), where \( r \) is the algebraic multiplicity of \( F \) at \( p_2 \). Therefore, by Corollary 3.4, \( F \) is a Riccati foliation and admits an invariant line. This completes the proof. \( \square \)

It is worth noting that in the general case \( 1 \leq l_0 < l_1 < l_2 \), using the same methods of this subsection, we can show that a foliation of normal degree \( d \leq 2l_2 + l_0 - 1 \) on \( \mathbb{P} \) admits an invariant curve.

## 4 | FOLIATIONS WITHOUT ALGEBRAIC LEAVES ON \( \mathbb{P} \)

### 4.1 | The universal singular set

Set \( X = \mathbb{P} \setminus \text{sing}(\mathbb{P}) \). Recall that \( p_0 = [1 : 0 : 0], p_1 = [0 : 1 : 0], p_2 = [0 : 0 : 1] \). We define the following sets:

\[
S(d) := \{(x, F) \in \mathbb{P} \times \text{Fol}(d) \mid x \in \text{sing}(F)\},
\]

\[
S_X(d) := \{(x, F) \in X \times \text{Fol}(d) \mid x \in \text{sing}(F)\},
\]

\[
S_{p_i}(d) := \{(p_i, F) \in S(d)\}, \ i = 0, 1, 2.
\]

We will see that the set \( S_X(d) \) is irreducible, for this we need the following result, proved by Sylvester in 1894, see [1, page 71], which we will use throughout this section.
Lemma 4.1. For any positive integers $a$ and $b$ with $\gcd(a, b) = 1$, define $g(a, b)$ to be the greatest positive integer $N$ for which the equation

$$a x_1 + b x_2 = N,$$

is not solvable in nonnegative integers. Then, $g(a, b) = ab - a - b$.

Proposition 4.2. For all $d > l_1 l_2$, $S_X(d)$ is an irreducible subvariety and has codimension $2$ in $X \times \text{Fol}(d)$.

Proof. Consider the projection $\pi_1 : S(d) \to \mathbb{P}$. For every $x \in \mathbb{P}$, the fiber $\pi_1^{-1}(x)$ is a subvariety of $\{x\} \times \text{Fol}(d)$, and it is isomorphic to a projective space, since if two 1-forms $\eta$ and $\eta'$ vanish at $x$, then the same is true for a linear combination of them.

The action of the automorphism group of $\mathbb{P}$ on $X$ has four distinct orbits. In order to prove the proposition, we will find for each of these orbits two 1-forms, which are linearly independent at a point of any orbit.

Since $d > l_1 l_2$, we can apply Lemma 4.1 to obtain positive integers $i_1, j_1, i_2$ such that $i_1 l_1 + j_1 l_2 = i_2 l_0 + j_2 l_1 = d$. Moreover, $\alpha = x_1^{i_1 - 1} x_2^{j_1 - 1} (l_2 x_2 d x_1 - l_1 x_1 d x_2)$ and $\beta = x_0^{i_2 - 1} x_1^{j_2 - 1} (l_1 x_1 d x_0 - l_0 x_0 d x_1)$ belong to $H^0(\mathbb{P}, \Omega^{[1]}(d))$.

If $p = [1 : 1 : 1]$, then $V_p = \langle \alpha(p), \beta(p) \rangle$, the vector space generated by the evaluation of $\alpha$ and $\beta$ at $p$, has dimension $2$.

At the point $q_0 = [0 : 1 : 1]$, the analogous vector space has dimension $2$. But we can apply Lemma 4.1 to write $d - l_0 = i_1 l_2$ with $i$ and $j$ positive integers, and see that $\delta_0 = x_1^{i - 1} x_2^j (l_0 x_0 d x_1 - l_1 x_1 d x_0)$ belongs to $H^0(\mathbb{P}, \Omega^{[1]}(d))$. By considering the evaluation of $\delta_0$ and $\alpha$ at $q_0$, we see that they are $\mathbb{C}$-linearly independent. Applying the same argument to the other points $q_1 = [1 : 0 : 1]$ and $q_2 = [1 : 1 : 0]$, we conclude that $\pi_1^{-1}(x) \subset X \times \text{Fol}(d)$ always have codimension $2$. It follows that $S_X(d)$ is a projective bundle over $X$, and therefore is smooth and irreducible.

It is worth mentioning that in general the set $S(d)$ is not irreducible.

Recall that the sets $C_n(d)$, $D_n(d)$, and $C_{p_i}^n(d)$, for each $i \in \{0, 1, 2\}$, are closed (see Lemma 2.6). So, we can state the following proposition.

Proposition 4.3. Assume that $d > l_1 l_2$ and $C_n(d) = \text{Fol}(d)$ for some $n > 0$.

1. If $C_{p_i}^n(d) = C_n(d)$ for some $i$, then $S_{p_i}(d) \subset D_n(d)$.
2. If $C_{p_i}^n(d) \neq C_n(d)$ for every $i$, then $S_X(d) \subset D_n(d)$.

Proof. Item (1) follows from definitions.

We denote by $\pi$ the restriction of the natural projection of $S_X(d)$ to $\text{Fol}(d)$ and define the open set $U_n = \text{Fol}(d) \setminus \bigcup_{i=0}^{n-1} (C_{p_i}^n(d))$. The assumption of Item (2) implies that $U_n$ is a nonempty open set. Let $F \in U_n$ be a saturated foliation. Then, there exists a curve $C$ of degree $n$ invariant by $F$ that does not pass through $p_0$, $p_1$, and $p_2$. Since the map $\overline{\varphi} : \mathbb{P}^2 \to \mathbb{P}$, given by $\overline{\varphi}(x_0 : x_1 : x_2) = [x_0^{l_0} : x_1^{l_1} : x_2^{l_2}]$, is generically finite and any curve on $\mathbb{P}^2$ has positive self-intersection, we see that $C^2 > 0$. Using Camacho–Sad Theorem [4, p. 5] $C^2 = \sum_{p \in X \cap C} CS(F, C, p)$, it follows that there exists $p \in \text{sing}(F) \cap X$. Therefore,

$$\pi^{-1}(U_n) \cap D_n(d) = U_n.$$

By Proposition 4.2, we get $\pi^{-1}(U_n)$ is an irreducible open set and

$$\dim \pi^{-1}(U_n) = \dim \text{Fol}(d).$$

Since $\dim (\pi^{-1}(U_n) \cap D_n(d)) = \dim U_n = \dim \text{Fol}(d)$, we have

$$\pi^{-1}(U_n) \cap D_n(d) = \pi^{-1}(U_n).$$

Taking closure in $\mathbb{P} \times \text{Fol}(d)$, we get $S_X(d) \cap D_n(d) = S_X(d)$. Item (2) follows.
We see that when the conclusion of the above proposition is valid, we have that for each singularity of \( P \) that is also a singularity of each foliation, \( F \in \text{Fol}(d) \) passes an invariant curve by \( F \) of degree \( n \), or for each singularity in \( X \) of each foliation \( F \in \text{Fol}(d) \) passes an invariant curve by \( F \) of degree \( n \). In the next subsection, in order to prove Theorem A, we will construct examples that contradict the conclusion of Items (1) and (2) of the above proposition for \( d \gg 0 \).

### 4.2 Existence of singularities without algebraic separatrix

First, we construct a family of examples that contradict Item (2) of Proposition 4.3 for \( d \gg 0 \). The following example is an adaptation of an example of J. V. Pereira, see [15, p. 5].

Let \( F_0 \) be the foliation on \( \mathbb{P} \) induced by the following 1-form

\[
\beta = x_0 x_1 x_2 G \left( \lambda l_1 l_2 \frac{dx_0}{x_0} + \mu l_0 l_2 \frac{dx_1}{x_1} + + \gamma l_0 l_2 \frac{dx_2}{x_2} - (\lambda + \mu + \gamma) \frac{dG}{G} \right),
\]

where \( G(x_0, x_1, x_2) = x_0^{l_1 l_2} + x_1^{l_0 l_2} + x_2^{l_0 l_1} \). Then, \( \deg(N_{F_0}) = l_0 l_1 l_2 + l_0 + l_1 + l_2 \).

**Proposition 4.4.** If \( \lambda, \mu, \) and \( \gamma \) are \( \mathbb{Z} \)-linearly independent, then there are singularities of \( F_0 \) in \( \mathbb{P} \setminus \{ x_0 x_1 x_2 = 0 \} \) such that no invariant algebraic curves pass through them.

**Proof.** Observe that \( F_0 = \phi^* F \), where

\[
\phi : \mathbb{P} \to \mathbb{P}^2 \\
[ x_0 : x_1 : x_2 ] \mapsto [ x_0^{l_1 l_2} : x_1^{l_0 l_2} : x_2^{l_0 l_1} ],
\]

and \( F \) is the foliation on \( \mathbb{P}^2 \) induced by

\[
\Omega = xyz(x + y + z) \left( \lambda \frac{dx}{x} + \mu \frac{dy}{y} + + \gamma \frac{dz}{z} - (\lambda + \mu + \gamma) \frac{d(x + y + z)}{x + y + z} \right).
\]

Then the proof follows from [15, Lemma 2]. \( \square \)

**Corollary 4.5.** For all \( d > l_0 l_1 l_2 + l_0 l_1 + l_2 \), there exists a foliation \( F \) on \( \mathbb{P} \) with the following properties:

1. its normal \( \mathbb{Q} \)-bundle has degree \( d \) and
2. there are singularities of \( F \) in \( \mathbb{P} \setminus \{ x_0 x_1 x_2 = 0 \} \) such that no \( F \)-invariant algebraic curve passes through them.

**Proof.** Let \( F_0 \) be the foliation on \( \mathbb{P} \) of Proposition 4.4 induced by \( \beta \). Note that, if \( d > l_0 l_1 l_2 + l_0 l_1 + l_2 \), then \( d - \deg(N_{F_0}) > l_0 l_1 - l_0 - l_1 \). By Lemma 4.1, we can find \( i, j > 0 \) such that \( \omega = x_0^i x_1^j \beta \) belongs to \( H^0(\mathbb{P}, \Omega_\beta^{(1)}(d)) \) and has the desired property. \( \square \)

Now, we are going to construct a family of examples that contradict Item (1) of Proposition 4.3 for \( d \gg 0 \).

For every \( j_0 = 1, \ldots, l_2 \), let \( j_1 \) be the unique integer satisfying \( 1 \leq j_1 \leq l_2 \) and \( l_0 j_0 \equiv l_1 j_1 \pmod{l_2} \). Consider the foliation \( F \) in \( \mathbb{C}^2 \) induced by the 1-form

\[
\eta = (x_1^{l_2} - 1)x_0^{j_0 - 1} dx_0 - a(x_0^{l_1} - 1)x_1^{j_1 - 1} dx_1,
\]

in which \( a \in \mathbb{C} \setminus \mathbb{R} \).

**Lemma 4.6.** The foliation \( F \) does not have any \( F \)-invariant algebraic curve passing through the point \((0,0)\).
Proof. We have to consider two cases:

1. **First case**: $j_0 = j_1 = l_2$ (nondicritical case). We extend $F$ to a foliation $\mathcal{G}$ on $\mathbb{P}^2$, which is induced by a 1-form $\omega$ equal to

$$\omega = (x_1^{l_1} - x_2^{l_1}) x_0^{s-1} dx_0 - a(x_0^{l_2} - x_2^{l_2}) x_1^{l_1-1} x_2 dx_1 + (a(x_0^{l_2} - x_2^{l_2}) x_1^{l_1} - (x_0^{l_2} - x_2^{l_2}) x_1^{l_1}) dx_2.$$

Thus, $\deg(\mathcal{G}) = 2 l_2 - 1$, and $\{x_2 = 0\}$, $\{x_1^{l_2} - x_2^{l_2} = 0\}$, $\{x_0^{l_2} - x_2^{l_2} = 0\}$ are $\mathcal{G}$-invariants. Notice that the singularities of $\mathcal{G}$ on $\{x_1^{l_2} - x_2^{l_2} = 0\} \cap \{x_0^{l_2} - x_2^{l_2} = 0\}$ are reduced. Also, over each of these lines, $\mathcal{G}$ has only one extra singularity corresponding to the intersection of the line with $\{x_2 = 0\}$.

Suppose that there exists an algebraic curve $C$ invariant by $\mathcal{G}$ passing through $[0∶0∶1]$. Bézout’s Theorem implies that $C$ must intersect the line $\{x_1 - x_2 = 0\}$. Since the singularities of $\mathcal{G}$ on this line outside of $\{x_2 = 0\}$ are all reduced and possess two separatrices, which do not pass through $[0∶0∶1]$, we conclude that $C$ intersects this line only at the point $[1∶0∶0]$. Let $\pi: M \to \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ at the point $[1∶0∶0]$, $E$ be the exceptional divisor, $\tilde{C}$ be the strict transform of $C$, $L_2$ be the strict transform of $\{x_2 = 0\}$, $F$ be the strict transform of $\{x_1^{l_2} - x_2^{l_2} = 0\}$, and $\tilde{\mathcal{G}} = \pi^*(\mathcal{G})$ be the pullback foliation on $M$. Then, we see that the singularity at the point $[1∶0∶0]$ is nondicritical, the singularities of $\mathcal{G}$ on $E$ are all reduced and are contained in the intersection of $E$ with $L_2 \cup F$. This claim contradicts the fact that $\tilde{C} \cap E$ is contained in $\text{sing}(\tilde{\mathcal{G}})$. This ends the first case.

2. **Second case**: $l_2 > j_0, j_1$ (dicritical case). Again, we extend $F$ to a foliation on $\mathbb{P}^2$ that is denoted by $\mathcal{G}$. Assume, without loss of generality, that $j_0 \geq j_1$. In this case, the foliation $\mathcal{G}$ is induced by

$$\omega = (x_1^{l_1} - x_2^{l_1}) x_0^{s-1} dx_0 - a(x_0^{l_2} - x_2^{l_2}) x_1^{l_1-1} x_2^{l_2-1} x_1^{l_1} dx_1 + f dx_2,$$

where $f(x_0, x_1, x_2) = a(x_0^{l_2} - x_2^{l_2}) x_1^{l_1} x_2^{l_2} - (x_0^{l_2} - x_2^{l_2}) x_1^{l_1}$. Thus, $\deg(\mathcal{G}) = l_2 + j_0 - 1$, and $\{x_2 = 0\}$, $\{x_1^{l_2} - x_2^{l_2} = 0\}$, $\{x_0^{l_2} - x_2^{l_2} = 0\}$ are $\mathcal{G}$-invariants.

Notice that the singularities of $\mathcal{G}$ on $\{x_1^{l_2} - x_2^{l_2} = 0\} \cap \{x_0^{l_2} - x_2^{l_2} = 0\}$ are reduced. Also, over each of these lines, $\mathcal{G}$ has only one extra singularity corresponding to the intersection of the line with $\{x_2 = 0\}$.

Suppose that there exists an algebraic curve $C$ invariant by $\mathcal{G}$ passing through $[0∶0∶1]$. Let $\pi: M \to \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ at the points $[0∶1∶0]$ and $[1∶0∶0]$, $E = E_1 \cup E_2$ be the exceptional divisor, $\tilde{C}$ be the strict transform of $C$, $L_2$ be the strict transform of $\{x_2 = 0\}$, $L_0$ be the strict transform of $\{x_1 - x_2 = 0\}$, $L_1$ be the strict transform of $\{x_0 - x_2 = 0\}$, and $\tilde{\mathcal{G}} = \pi^*(\mathcal{G})$ be the pullback foliation on $M$. Then, we have the singularities at the points $[1∶0∶0]$ and $[0∶1∶0]$ are dicritical and the singularities of $\tilde{\mathcal{G}}$ on $E$ do not belong to the lines $L_1$ and $L_2$.

Now we define the following map:

$$\phi: \text{Pic}(M) \to \mathbb{Z}^2$$

$$D \mapsto (D.L_0, D.L_1).$$

Observe that $\phi$ is a surjective map and $\ker(\phi) = \mathbb{Z}L_2$. By the previous discussion, we see that $\tilde{C} \in \ker(\phi)$, hence we can write $\tilde{C} = bL_2$ in $\text{Pic}(M)$ for some $b \in \mathbb{Z}$. This is a contradiction, since $\tilde{C}$ has positive self-intersection.

**Corollary 4.7.** For all $d \geq l_2 l_1 + l_2 l_0 + l_2$, all $n \in \mathbb{N}$ and every $i = 0, 1, 2$, we have

$$C_n^0(d) \neq \text{Fol}(d).$$

Furthermore, if $l_0 = l_1 = 1$ and $l_2 \geq 2$, then $C_n^l(d) \neq \text{Fol}(d)$, for all $d \geq 2 l_2 + 1$ and all $n \in \mathbb{N}$.

**Proof.** We just show that $C_n^l(d) \neq \text{Fol}(d)$, and similar arguments can be used for the other cases. Take $d \geq l_2 l_1 + l_2 l_0 + l_2$, let $j_0, j_1$ be the unique integers satisfying $1 \leq j_0, j_1 \leq l_2$ and $d \equiv l_0 j_0 \equiv l_1 j_1 \pmod{l_2}$. Then, Lemma 4.6 implies that the foliation $F$ on $\mathbb{C}^2$, given by

$$\eta = (x_1^{l_1} - 1)x_0^{l_1-1} dx_0 - a(x_0^{l_2} - 1)x_1^{l_1-1} dx_1,$$

does not have any $F$-invariant algebraic curve passing through the point $(0,0)$. \hfill \Box
We can extend $\mathcal{F}$ to a foliation $\hat{\mathcal{F}}$ on $\mathbb{P}$, which is induced by $\theta$ and

$$\deg(N\hat{\mathcal{F}}) = \begin{cases} l_0 j_0 + l_1 l_2 + l_2, & \text{if } l_0 j_0 \geq l_1 j_1, \\
 l_1 j_1 + l_0 l_2 + l_2, & \text{if } l_1 j_1 > l_0 j_0. \end{cases}$$

Since $d \equiv \deg(N\hat{\mathcal{F}}) \mod l_2$ and $d \geq l_1 l_2 + l_2 l_0 + l_2$, we can multiply the 1-form $\theta$ by an adequate power of $x_2$ and construct a foliation $\mathcal{H}$ on $\mathbb{P}$ with normal $\mathbb{Q}$-bundle of degree $d$. The foliation $\mathcal{H}$ does not have any $\mathcal{H}$-invariant algebraic curve passing through the point $[0 : 0 : 1]$. Hence $\mathcal{H} \notin C_n^P(d)$.

If $l_0 = l_1 = 1$ and $d \geq 2l_2 + 1$, then $d \equiv j \mod l_2$, for a unique integer $1 \leq j \leq l_2$. Taking $j = j_0 = j_1$, the foliation $\hat{\mathcal{F}}$ constructed above satisfies $\deg(N\hat{\mathcal{F}}) = 2l_2 + j$. By a same procedure, we can construct a foliation $\mathcal{H} \notin C_n^P(d)$ for all $n \in \mathbb{N}$.

### 4.3 Proof of Theorem A

By Lemma 2.6, $C_n(d)$ is an algebraic closed subset of $\text{Fol}(d)$ for all $n \in \mathbb{N}$. We claim that if $d \geq l_0 l_1 l_2 + l_0 l_1 + 2l_2$, then $C_n(d) \neq \text{Fol}(d)$, for all $n \in \mathbb{N}$. In fact, if we had $C_n(d) = \text{Fol}(d)$ for some $n \in \mathbb{N}$, then we could apply Proposition 4.3 and this would contradict Corollary 4.5 and Corollary 4.7. By applying Baire’s Theorem, we finish the proof.

**Corollary 4.8.** Under the assumptions of Theorem A, a generic $G$-invariant foliation of degree $d$ on $\mathbb{P}^2$ does not admit any invariant algebraic curve if $d \geq l_0 l_1 l_2 + l_0 l_1 + 2l_2 - 2$.

**Proof.** It follows from Theorem A and Lemma 2.3. \qed

In general, for $l_0, l_1, l_2$ positive integers not necessarily pairwise coprimes, a reduction method is known to obtain pairwise coprimes as follows:

Let

- $d_0 = \text{mdc}(l_1, l_2)$,
- $d_1 = \text{mdc}(l_0, l_2)$,
- $d_2 = \text{mdc}(l_0, l_1)$,
- $a_0 = \text{lcm}(d_1, d_2)$,
- $a_1 = \text{lcm}(d_0, d_2)$,
- $a_2 = \text{lcm}(d_0, d_1)$,
- $a = \text{lcm}(a_0, a_1, a_2)$,

Then, $l'_i = l_i / a_i$, $i = 0, 1, 2$ are pairwise coprimes. Set $\mathbb{P} := \mathbb{P}(l_0, l_1, l_2)$ and $\mathbb{P}' := \mathbb{P}(l'_{0}, l'_{1}, l'_{2})$. The map

$$\Psi : \mathbb{P} \to \mathbb{P}'$$

$$[x_0 : x_1 : x_2] \mapsto [x_0^{d_0} : x_1^{d_1} : x_2^{d_2}],$$

is an isomorphism, see [9, section 1.3]. This map induces an isomorphism

$$\Psi^* : \text{PH}^0(\mathbb{P}', \mathcal{O}_p^{[1]}(d)) \to \text{PH}^0(\mathbb{P}, \mathcal{O}_p^{[1]}(ad)).$$

We note that in the case $a > 1$, that is, $l_0, l_1, l_2$ are not pairwise coprimes, if the normal degree of a saturated foliation $\mathcal{F}$ on $\mathbb{P}(l_0, l_1, l_2)$ is not a multiple of $a$, then $\mathcal{F}$ has an invariant curve, see [11, Proposition 2.3.11] for more details. From these observations and using Theorem A we can obtain the following.

**Corollary 4.9.** Under the conditions stated above, a generic invariant foliation with normal $\mathbb{Q}$-bundle of degree $ad$ on $\mathbb{P}(l_0, l_1, l_2)$ does not admit any invariant algebraic curve if

$$d \geq \frac{l_0 l_1 l_2}{a_0 a_1 a_2} + \frac{l_0 l_1}{a_0 a_1} + \frac{l_0 l_2}{a_0 a_2} + \frac{l_1 l_2}{a_1 a_2}.$$

In the next subsection, we will construct counterexamples to prove Theorem B.

### 4.4 Existence of singularities without algebraic separatrix on $\mathbb{P}^2_k$

The following family of the examples allows us to obtain the bound of Theorem B.
Let $\mathcal{F}_1$ be the foliation on $\mathbb{P}^2_k$, $k > 1$, induced by the following 1-form:

$$\delta = -kx_2(x_2 - x_0x_1^{k-1})dx_0 + kx_0x_2(x_1^{k-1} - x_0x_1^{k-2})dx_1 + x_0(x_2 - x_1^k)dx_2.$$ 

Notice that $\deg(N_{\mathcal{F}_1}) = 2k + 1$, $\text{sing}(\mathcal{F}_1) = \{[0 : 1 : 0], [1 : 0 : 0], [1 : 1 : 1]\}$ and $\{x_0 = 0\} \cap \text{sing}(\mathcal{F}_1) = \{[0 : 1 : 0]\}$.

**Lemma 4.10.** The foliation $\mathcal{F}_1$ does not have any invariant algebraic curve passing through the point $[1 : 1 : 1]$.

**Proof.** Observe that the lines $\{x_0 = 0\}$ and $\{x_2 = 0\}$ are $\mathcal{F}_1$-invariant. Suppose that there exists an algebraic curve $C$ invariant by $\mathcal{F}_1$ passing through $[1 : 1 : 1]$. Since $\{x_0 = 0\} \cap \text{sing}(\mathcal{F}_1) = \{[0 : 1 : 0]\}$, using Bézout’s Theorem for weighted projective planes [2, Proposition 8.2], we conclude that $\{x_0 = 0\}$ only intersects $C$ at the point $[0 : 1 : 0]$. Note that $[0 : 1 : 0]$ is a saddle-node singularity with only two separatrices $\{x_0 = 0\}$ and $\{x_2 = 0\}$. In particular, $C$ must be one of these lines, contradicting $[1 : 1 : 1] \in C$. □

### 4.5 Proof of Theorem B

It is enough to show that $C_n(d) \neq \text{Fol}(d)$, for all $n \in \mathbb{N}$ and all $d \geq 2k + 1$. Reasoning by contradiction, suppose that this does not hold. Then, Proposition 4.3 and Corollary 4.7 imply $\mathcal{S}(d) \subset \mathcal{D}(d)$, that contradicts Corollary 4.5, for $d > 2k + 1$, and Lemma 4.10, for $d = 2k + 1$.

### 5 Holomorphic Foliations on Hirzebruch Surfaces

Let $\mathbb{F}_k = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(k)^{\oplus 2})$ be the Hirzebruch surface, $\pi : \mathbb{F}_k \to \mathbb{P}^2_k$ be the minimal resolution of $\mathbb{P}^2_k$, $E$ be the exceptional divisor, and $F$ be the strict transform of a line passing through $p_2$, see Section 3.2. Observe that $\text{Pic}(\mathbb{F}_k) = \mathbb{Z}F \oplus \mathbb{Z}E$, in which $E \cdot E = -k, F \cdot F = 0$ and $F \cdot E = 1$.

We denote by $\mathcal{R}(a, b) := \mathbb{P}(H^0(\mathbb{F}_k, \mathcal{O}_{\mathbb{F}_k}(aF + bE)))$ the space of holomorphic foliations with normal bundle of bidegree $(a, b)$ on $\mathbb{F}_k$.

We shall now prove that under suitable conditions on $(a, b)$ each foliation with bidegree $(a, b)$ on $\mathbb{F}_k$ has algebraic solutions.

**Proposition 5.1.** If $a < bk + 2$ or $b < 3$, then any foliation $\mathcal{G} \in \mathcal{R}(a, b)$ admits some invariant algebraic curve.

**Proof.** We first prove that $E$ is $\mathcal{G}$-invariant when $a < bk + 2$. Suppose it is not. Then, by the tangency formula, $0 \leq \text{Tang}(\mathcal{G}, E) = -bk + a - 2$. So, $a \geq bk + 2$, contradicting the hypothesis. We can thus assume that $a - bk \geq 2$. If $b = 0$, then the foliation $\mathcal{G}$ is the rational fibration, so we may also assume $0 < b \leq 2$. Baum–Bott formula implies that $\sum \text{BB}(N_{\mathcal{G}, p}) = N_{\mathcal{G}}^2 = (aF + bE)^2 = b(a + a - bk) > 0$, and therefore there exists a point $p \in \text{sing}(\mathcal{G})$. We claim that the fiber $F$ passing through the point $p$ is $\mathcal{G}$-invariant. If this is not the case, then the tangency formula implies

$$0 < \text{Tang}(\mathcal{G}, F) = N_{\mathcal{G}}F - \chi(F) = b - 2 \leq 0,$$

which is again a contradiction. This completes the proof of the proposition. □

Let $\chi \in \mathbb{Q}[t]$. Define two subsets of $\mathbb{F}_k \times \mathcal{R}(a, b)$ by

$$S(a, b) = \{(x, \mathcal{G}) \in \mathbb{F}_k \times \mathcal{R}(a, b) | x \in \text{sing}(\mathcal{G})\},$$

and

$$D_\chi(a, b) = \{(x, \mathcal{G}) \in \mathbb{F}_k \times \mathcal{R}(a, b) | x \text{ is in some subscheme, invariant by } \mathcal{G}, \text{ of Hilbert polynomial } \chi\}.$$
Proposition 5.2. The following statements hold.

1. If $b \geq 2$ and $a \geq bk + 2$, then $S(a, b)$ is a closed irreducible variety of $\mathbb{F}_k \times R(a, b)$ and $\dim_\mathbb{C} S(a, b) = \dim_\mathbb{C} R(a, b)$.
2. $D_\chi(a, b)$ is a closed subset of $\mathbb{F}_k \times R(a, b)$.

Proof. (1) Let $\Sigma(a, b)$ denote the line bundle $\mathcal{O}_{\mathbb{F}_k}(aF + bE)$ on $\mathbb{F}_k$. Consider the following exact sequence of sheaves:

$$0 \to \ker \psi \to H^0(\mathbb{F}_k, \Omega^1_{\mathbb{F}_k} \otimes \Sigma(a, b)) \to \mathcal{O}_{\mathbb{F}_k} \otimes \Omega^1_{\mathbb{F}_k} \otimes \Sigma(a, b),$$

where $\psi(x, \eta) = \eta(x)$. Notice that if $\ker \psi$ is a vector bundle, then $\mathbb{P}(\ker \psi) = S(a, b)$ is an irreducible variety of codimension 1 on $\mathbb{F}_k \times R(a, b)$. Thus, it suffices to prove $\ker \psi$ is a vector bundle, which we do now. Since

$$\pi|_{\mathbb{F}_k \setminus E} : \mathbb{F}_k \setminus E \to \mathbb{P}^2_k \setminus \{(0 : 0 : 1)\}$$

is an isomorphism and the automorphism group of $\mathbb{P}^2_k$ acts transitively on $\mathbb{P}^2_k \setminus \{(0 : 0 : 1)\}$, then the automorphism group of $\mathbb{P}^2_k$ acts transitively on $\mathbb{F}_k \setminus E$. Therefore, $\ker \psi_x$ has dimension $\dim_\mathbb{C} R(a, b) - 2$ for all $x \in E$. Indeed, we take two foliations $F_1, F_2$ on $\mathbb{P}^2_k$ with normal $\mathbb{Q}$-bundle of degree $a$ induced by the 1-forms $\omega = x_0^b A(x_0, x_1)(x_1 dx_0 - x_0 dx_1)$ and $\eta = x_0^{b-2} B(x_0, x_1)(k x_2 dx_0 - x_0 dx_2)$, respectively, where $A$ and $B$ are homogeneous polynomials of degree $a - bk - 2$ and $a - (b - 1)k - 1$ that can be chosen such that the forms $\omega$ and $\eta$ defining $\pi^*(F_1), \pi^*(F_2) \in R(a, b)$ are linearly independent at $x \in E$. Then, any 1-form $\alpha \in R(a, b)$ with $\alpha(x) \neq 0$ can be written as a $\mathbb{C}$-linear combination of $\omega$ and $\eta$. Thus, $\dim_\mathbb{C} \ker \psi_x$ is constant as a function of $x \in X$, hence $\ker \psi$ is a vector bundle. (2) follows directly from [8, Lemma 5.1, p. 9].

Lemma 5.3. Let $\mathcal{G} \in R(a, b)$ be a foliation on $\mathbb{F}_k$ and $C$ be an algebraic curve invariant by $\mathcal{G}$. If $b \geq 3$ and $a \geq bk + 2$, then

$$C \cap \text{sing}(\mathcal{G}) \neq \emptyset.$$

Proof. We consider two cases:

1. If $C = E$ in Pic($\mathbb{F}_k$), then the Camacho–Sad formula gives $C^2 = -k$, which implies that $C \cap \text{sing}(\mathcal{G}) \neq \emptyset$.
2. Suppose $C = mF + nE$ in Pic($\mathbb{F}_k$), with $m > 0, n \geq 0$, and assume for contradiction that $C \cap \text{sing}(\mathcal{G}) = \emptyset$. Then, $C^2 = 0$ by the Camacho–Sad formula, and from the vanishing formula, we obtain

$$N_{\mathcal{G}} \cdot C = C^2 + Z(\mathcal{G}, C) = 0.$$

But $N_{\mathcal{G}} \cdot C = n(a - bk) + bm > 0$, and this is a contradiction.

5.1 | Proof of Theorem C

Consider the second projection $\pi_2 : S(a, b) \to R(a, b)$ and fix a polynomial $\chi \in \mathbb{Q}[t]$ of degree 1. Suppose that $\pi_2(D_\chi(a, b)) = R(a, b)$; that is, every foliation of bidegree $(a, b)$ has an algebraic invariant curve with Hilbert polynomial $\chi$. By Lemma 5.3, we have

$$\pi_2(D_\chi(a, b) \cap S(a, b)) = \pi_2(S(a, b)).$$

Since $S(a, b)$ is an irreducible variety and $\dim_\mathbb{C} S(a, b) = \dim_\mathbb{C} R(a, b)$, by Proposition 5.2, we get $S(a, b) \cap D_\chi(a, b) = S(a, b)$.

To finish the proof, we take the foliation $F_1$ on $\mathbb{P}^2_k$ of Lemma 4.10, which is induced by $\delta$ and has degree $2k + 1$. Let $F$ be the foliation on $\mathbb{P}^2_k$ with normal $\mathbb{Q}$-bundle of degree $a$ induced by $\omega = x_0^{a-bk+k-1} x_2^{b-1} \delta$. Note that the foliation $\mathcal{G} = \pi^*(F)$ on $\mathbb{F}_k$ has bidegree $(a, b)$, and there is no invariant curve passing through the point $\pi^{-1}(1 : 1 : 1)$, which is a singularity.
of $\mathcal{G}$. This is a contradiction. Because there are only countably many Hilbert polynomials, this completes the proof of the theorem.

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**CONFLICT OF INTEREST STATEMENT**

The authors declare no potential conflict of interests.

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