Lower bounds on the curvature of the Isgur-Wise function

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Abstract

Using the OPE, we obtain new sum rules in the heavy quark limit of QCD, in addition to those previously formulated. Key elements in their derivation are the consideration of the non-forward amplitude, plus the systematic use of boundary conditions that ensure that only a finite number of $j^P$ intermediate states (with their tower of radial excitations) contribute. A study of these sum rules shows that it is possible to bound the curvature $\sigma^2 = \xi''(1)$ of the elastic Isgur-Wise function $\xi(w)$ in terms of its slope $\rho^2 = -\xi'(1)$. Besides the bound $\sigma^2 \geq \frac{5}{4}\rho^2$, previously demonstrated, we find the better bound $\sigma^2 \geq \frac{1}{5}[4\rho^2 + 3(\rho^2)^2]$. We show that the quadratic term $\frac{3}{5}(\rho^2)^2$ has a transparent physical interpretation, as it is leading in a non-relativistic expansion in the mass of the light quark. At the lowest possible value for the slope $\rho^2 = \frac{3}{4}$, both bounds imply the same bound for the curvature, $\sigma^2 \geq \frac{15}{16}$. We point out that these results are consistent with the dispersive bounds, and, furthermore, that they strongly reduce the allowed region by the latter for $\xi(w)$.

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1 Introduction

In a recent paper [1] we have set a systematic method to obtain sum rules (SR) in the heavy quark limit of QCD, that relate the derivatives of the elastic Isgur-Wise (IW) function $\xi(w)$ to sums over IW functions of excited states. The method is based on the Operator Product Expansion (OPE) [2] applied to heavy quark transitions [3] and its key element is the consideration, following Uraltsev [4], of the non-forward amplitude, i.e. $B(v_i) \rightarrow D^{(n)}(v') \rightarrow B(v_f)$ with in general $v_i \neq v_f$. Then, the OPE side of the SR contains the elastic IW function $\xi(w_{if})$ and therefore the SR depend in general on three variables $w_i$, $w_f$ and $w_{if}$ that lie within a certain domain. By differentiation relatively to these variables within the domain and taking the limit to its boundary, one finds a very general class of SR that have interesting consequences on the shape of $\xi(w)$.

Let us be more quantitative. As shown in ref. [1], using the OPE – as formulated for example in [5] and generalized to $v_i \neq v_f$ [4] –, the trace formalism [6] and arbitrary heavy quark currents

$$J_1 = \bar{h}^{(c)}_{v'} \Gamma_1 h^{(b)}_{v_i}, \quad J_2 = \bar{h}^{(b)}_{v_f} \Gamma_2 h^{(c)}_{v'}$$

(1)

the following sum rule can be written in the heavy quark limit [1] :

$$\left\{ \sum_{D=P,V} \sum_n Tr \left[ \bar{B}_f(v_f) \Gamma_2 D^{(n)}(v') \right] Tr \left[ D^{(n)}(v') \Gamma_1 B_i(v_i) \right] \xi^{(n)}(w_i)\xi^{(n)}(w_f) \right. \right.$$

$$+ \text{ Other excited states } \left. \right\} = -2 \xi(w_{if}) Tr \left[ \bar{B}_f(v_f) \Gamma_2 P'_+ \Gamma_1 B_i(v_i) \right].$$

(2)

In this formula $v'$ is the intermediate meson four-velocity, the projector

$$P'_+ = \frac{1}{2}(1 + \gamma')$$

(3)

comes from the residue of the positive energy part of the $c$-quark propagator, and $\xi(w_{if})$ is the elastic Isgur-Wise function that appears because one assumes $v_i \neq v_f$. $B_i$ and $B_f$ are the $4 \times 4$ matrices of the ground state $B$ or $B^*$ meson and $D^{(n)}$ those of all possible ground state or excited state $D$ mesons coupled to $B_i$ and $B_f$ through the currents. In formula (2) we have made explicit the $j = \frac{1}{2}^-$ $D$ and $D^*$ mesons and their radial excitations.
The variables $w_i, w_f$ and $w_{if}$ are defined as

$$w_i = v_i \cdot v', \quad w_f = v_f \cdot v', \quad w_{if} = v_i \cdot v_f.$$  \hfill (4)$$

The domain of $(w_i, w_f, w_{if})$ is

$$w_i, w_f \geq 1, \quad w_i w_f - \sqrt{(w_i^2 - 1)(w_f^2 - 1)} \leq w_{if} \leq w_i w_f + \sqrt{(w_i^2 - 1)(w_f^2 - 1)} \quad (5)$$

There is a subdomain for $w_i = w_f = w$:

$$w \geq 1, \quad 1 \leq w_{if} \leq 2w^2 - 1.$$ \hfill (6)$$

Calling now $L(w_i, w_f, w_{if})$ the l.h.s. and $R(w_i, w_f, w_{if})$ the r.h.s. of (2), this SR writes

$$L(w_i, w_f, w_{if}) = R(w_i, w_f, w_{if})$$ \hfill (7)$$

where $L(w_i, w_f, w_{if})$ is the sum over the intermediate $D$ states and $R(w_i, w_f, w_{if})$ is the OPE side. Within the domain (5) one can derive relatively to any of the variables $w_i, w_f$ and $w_{if}$

$$\frac{\partial^{p+q+r} L}{\partial w_i^p \partial w_f^q \partial w_{if}^r} = \frac{\partial^{p+q+r} R}{\partial w_i^p \partial w_f^q \partial w_{if}^r}$$ \hfill (8)$$

and obtain different SR taking different limits to the frontiers of the domain. One must take care in taking these limits, as we point out below.

Let us parametrize the elastic Isgur-Wise function $\xi(w)$ near zero recoil,

$$\xi(w) = 1 - \rho^2(w - 1) + \frac{\sigma^2}{2}(w - 1)^2 - \cdots$$ \hfill (9)$$

From the SR (2), we gave in ref. \cite{1} a simple and straightforward demonstration of both Bjorken \cite{7} \cite{8} and Uraltsev \cite{4} SR. Both SR imply the lower bound on the elastic slope

$$\rho^2 = -\xi'(1) \geq \frac{3}{4}. \hfill (10)$$

A crucial simplifying feature of the calculation was to consider, for the currents \cite{1}, vector or axial currents aligned along the initial and final velocities $v_i$ and $v_f$. In
we also obtained, modulo a very natural phenomenological hypothesis, a new bound on the curvature.

\[ \sigma^2 = \xi''(1) \geq \frac{5}{4} \rho^2 \geq \frac{15}{16}. \]  

This bound was obtained from the consideration in the SR of the whole tower of \( j^P \) intermediate states [9]. A crucial feature of the calculation was the needed derivation of the projector on the polarization tensors of particles of arbitrary integer spin [10].

Using the SR involving the whole sum over all \( j^P \) intermediate states, we pursued our study in [11] and did demonstrate that the IW function \( \xi(w) \) is an alternate series in powers of \((w - 1)\). Moreover, we did obtain the bound for the \( n \)-th derivative at zero recoil \((-1)^n \xi^{(n)}(1)\)

\[ (-1)^n \xi^{(n)}(1) \geq \frac{2n + 1}{4} (-1)^{n-1} \xi^{(n-1)} \geq \frac{(2n + 1)!!}{2^{2n}} \]  

demonstrating rigorously the bound (11) and generalizing (10) and (11) to any derivative.

The aim of this paper is to investigate whether a systematic use of the sum rules can allow to obtain better bounds on the curvature. As we will see below, the answer is positive. The reason is that only a finite number of \( j^P \) states, with their radial excitations, contribute to the relevant sum rules and one is left with a relatively simple set of algebraic linear equations. As we will see, this is due to the crucial fact that we adopt particular conditions at the boundary of the domain [5].

## 2 Vector and Axial Sum Rules

We choose as initial and final states the \( B \) meson,

\[ B_i(v_i) = P_{i+}(-\gamma_5) \quad B_f(v_f) = P_{f+}(-\gamma_5) \]  

where the projectors \( P_{i+}, P_{f+} \) are defined like in [3]. Moreover, we consider vector or axial currents are projected along the \( v_i \) and \( v_f \) four-velocities. Choosing the vector currents
\[ J_1 = \bar{h}_{\nu v}^{(c)} \gamma_\nu \gamma_\mu h_{\mu v}^{(b)} \] 
\[ J_2 = \bar{h}_{\nu v}^{(b)} \gamma_\mu h_{\mu v}^{(c)} \]  
and gathering the formulas (48) and (89)-(91) of ref. [1] we obtain for the SR [2] with the sum of all excited states \( j^P \), as written down in [11]:

\[ (w_i + 1)(w_f + 1) \sum_{\ell \geq 0} \frac{\ell + 1}{2\ell + 1} S_{\ell}(w_i, w_f, w_{if}) \sum_n \tau_{\ell+1/2}^{(f)(n)}(w_i)\tau_{\ell+1/2}^{(f)(n)}(w_f) \]
\[ + \sum_{\ell \geq 1} S_{\ell}(w_i, w_f, w_{if}) \sum_n \tau_{\ell-1/2}^{(f)(n)}(w_i)\tau_{\ell-1/2}^{(f)(n)}(w_f) = (1 + w_i + w_f + w_{if})\xi(w_{if}) \]  
Choosing instead the axial currents

\[ J_1 = \bar{h}_{\nu v}^{(c)} \gamma_\nu \gamma_5 h_{\mu v}^{(b)} \] 
\[ J_2 = \bar{h}_{\nu v}^{(b)} \gamma_\mu h_{\mu v}^{(c)} \] the SR [2] writes, from the formulas (48) and (92)-(94) of ref. [1], obtained in [11]:

\[ \sum_{\ell \geq 0} S_{\ell+1}(w_i, w_f, w_{if}) \sum_n \tau_{\ell+1/2}^{(f)(n)}(w_i)\tau_{\ell+1/2}^{(f)(n)}(w_f) \]
\[ + (w_i - 1)(w_f - 1) \sum_{\ell \geq 1} \frac{\ell}{2\ell - 1} S_{\ell-1}(w_i, w_f, w_{if}) \sum_n \tau_{\ell-1/2}^{(f)(n)}(w_i)\tau_{\ell-1/2}^{(f)(n)}(w_f) \]
\[ = -(1 - w_i - w_f + w_{if})\xi(w_{if}) \]  
Following the formulation of heavy-light states for arbitrary \( j^P \) given by Falk [9], we have defined in ref. [1] the IW functions \( \tau_{\ell+1/2}^{(f)(n)}(w) \) and \( \tau_{\ell-1/2}^{(f)(n)}(w) \), that correspond to the orbital angular momentum \( \ell \) of the light quark relative to the heavy quark, \( j = \ell \pm \frac{1}{2} \) being the total angular momentum of the light cloud. For the lower values of \( \ell \), one has the identities with the traditional notation of Isgur and Wise [8]:

\[ \tau_{1/2}^{(0)}(w) \equiv \xi(w) \] 
\[ \tau_{1/2}^{(1)}(w) \equiv 2\tau_{1/2}(w) \] 
\[ \tau_{3/2}^{(1)}(w) \equiv \sqrt{3} \tau_{3/2}(w) \]  
where a radial quantum number is implicit. Therefore, the functions \( \tau_{1/2}^{(1)}(w) \) and \( \tau_{3/2}^{(1)}(w) \) correspond, respectively, to the functions \( \zeta(w) \) and \( \tau(w) \) defined by Leibovich, Ligeti, Steward and Wise [12].

In equations (14) and (16) the quantity \( S_n \) is defined by

\[ S_n = v_{f_1} \cdots v_{f_n} T^{\nu_1 \cdots \nu_n \mu_1 \cdots \mu_n} v_{i_1} \cdots v_{i_{\mu_n}} \]
and the polarization projector $T^\nu_1\cdots\nu_n,\mu_1\cdots\mu_n$, given by

$$T^\nu_1\cdots\nu_n,\mu_1\cdots\mu_n = \sum_\lambda \varepsilon^{(\lambda)\nu_1\cdots\nu_n} \varepsilon^{(\lambda)\mu_1\cdots\mu_n} \quad (20)$$

depends only on the four-velocity $v'$. The tensor $\varepsilon^{(\lambda)\mu_1\cdots\mu_n}$ is the polarization tensor of a particle of integer spin $J = n$, symmetric, traceless, i.e. $\varepsilon^{(\lambda)\mu_1\cdots\mu_n} g_{\mu_i\mu_j} = 0 \ (i \neq j \leq n)$, and transverse to $v'$, $v'_\mu \varepsilon^{(\lambda)\mu_1\cdots\mu_n} = 0 \ (i \leq n)$. 

Moreover, as demonstrated in the Appendix A of ref. $[1]$, $S_n$ is given by the following expression:

$$S_n(w_i, w_f, w_{if}) = \sum_{0 \leq k \leq \frac{n}{2}} C_{n,k} (w_i^2 - 1)^k (w_f^2 - 1)^k (w_i w_f - w_{if})^{n-2k} \quad (21)$$

with

$$C_{n,k} = (-1)^k \frac{(n!)^2}{(2n)!} \frac{(2n-2k)!}{k!(n-k)!(n-2k)!} \quad (22)$$

The relation

$$L^V(w_i, w_f, w_{if}) \bigg|_{w_{if}=1, w_i=w_f=w} = R^V(w_i, w_f, w_{if}) \bigg|_{w_{if}=1, w_i=w_f=w} \quad (23)$$

gives, dividing by $2(w+1)$, Bjorken SR $[7] [8]$, including now the whole sum of intermediate states:

$$\frac{w+1}{2} \sum_{\ell \geq 0} \frac{\ell + 1}{2\ell + 1} C_\ell (w^2 - 1)\ell \sum_n \tau_n^{(\ell)(n)}(w) \tau_{\ell+1/2}^{(n)}(w)$$

$$+ \frac{w-1}{2} \sum_{\ell \geq 1} C_\ell (w^2 - 1)^{\ell-1} \sum_n \tau_{\ell-1/2}^{(\ell)(n)}(w) \tau_{\ell+1/2}^{(n)}(w) = 1 \quad (24)$$

where, from $[21]$ and $[22]$

$$S_n(w, w, 1) = C_n (w^2 - 1)^n \quad C_n = \sum_{0 \leq k \leq \frac{n}{2}} C_{n,k} = 2^n \frac{(n!)^2}{(2n)!} \quad (25)$$

Remember that, usually, the first terms in the sum $[24]$ are written in the notation $[18]$ of Isgur and Wise $[8]$: 

$$\frac{w+1}{2} \sum_n \left[ \xi^{(n)}(w) \right]^2 + (w-1) \sum_n \left[ 2 \left[ \tau_{1/2}^{(n)}(w) \right]^2 + (w+1)^2 \left[ \tau_{3/2}^{(n)}(w) \right]^2 \right] + \cdots = 1 \quad (26)$$
Going now to the axial current SR (17), the condition
\[ L^A(w_i, w_f, w_{if}) \big|_{w_{if}=1, w_i=w_f=w} = R^A(w_i, w_f, w_{if}) \big|_{w_{if}=1, w_i=w_f=w} \]
gives again, dividing this time by \(2(w - 1)\), the complete Bjorken SR (24). Notice that, as it must, one obtains the same SR from the vector (14) and the axial current (16) because, from (25), one has
\[ (2n + 1)C_{n+1} = (n + 1)C_n. \]

## 3 Equations from the Vector Sum Rule

In what follows, to look for independent relations, we make use of the fact that the SR (15) and (17) are symmetric in the exchange \(w_i \leftrightarrow w_f\).

Let us first consider the derivatives of the SR for vector currents (15) relatively to \(w_{if}\) with the boundary condition \(w_{if} = 1\). For \(w_{if} = 1\), the domain (5) implies:
\[ w_i = w_f = w. \]

We define therefore
\[ L_V(w_{if}, w) \equiv L_V(w_{if}, w_i, w_f) \big|_{w_i=w_f=w}, \]
\[ R_V(w_{if}, w) \equiv R_V(w_{if}, w_i, w_f) \big|_{w_i=w_f=w}. \]

We then take the \(p + q\) derivatives
\[ \left( \frac{\partial^{p+q} L_V}{\partial w_{if}^p \partial w^q} \right)_{w_{if}=w=1} = \left( \frac{\partial^{p+q} R_V}{\partial w_{if}^p \partial w^q} \right)_{w_{if}=w=1} \]
and exploit systematically the obtained relations. To get information on the curvature \(\sigma^2\) of the elastic IW function (9) we need to go up to the second order derivatives. As we will see, a crucial feature of the adopted boundary condition is that only a finite number of \(j^P\) states, with their tower of radial excitations, contribute to the sum rule. Notice that we could have derived first relatively to \(w_i\) and take the limit
\( w_i = 1 \), and then derive with respect to \( w = w_if = w_f \). We do not obtain however new information from these sum rules that with the former boundary conditions.

Let us proceed with care and begin with the first order derivatives. From (13) and (31), we obtain the following results.

With the notation (31), for \( p = q = 0 \) we obtain the trivial result \( \xi(1) = \xi(1) \), while for the derivatives \( p = 1, q = 0 \) we obtain Bjorken SR for the slope \( \rho^2 \):

\[
\rho^2 = \frac{1}{4} + \frac{2}{3} \sum_n \left[ \tau_{3/2}^{(2)(n)}(1) \right]^2 + \frac{1}{4} \sum_n \left[ \tau_{1/2}^{(1)(n)}(1) \right]^2 .
\] (32)

The relation to the Isgur-Wise notation is given by (18).

Going now to the second order derivatives, we find the following relations. For \( p = 0, q = 1 \) we get \( \xi(1) = \xi(1) \). For a purpose that will appear clear below, we make explicit the IW functions between \( j^P = \frac{1}{2}^- \) states using the notation of Isgur and Wise \( \xi^{(n)}(w) \) (18).

For \( p = 2, q = 0 \):

\[
\rho^2 - 2\sigma^2 + \frac{12}{5} \sum_n \left[ \tau_{5/2}^{(2)(n)}(1) \right]^2 + \sum_n \left[ \tau_{3/2}^{(2)(n)}(1) \right]^2 = 0
\] (33)

\( p = 1, q = 1 \):

\[
\rho^2 - \frac{4}{3} \sum_n \left[ \tau_{3/2}^{(1)(n)}(1) \right]^2 \\
- \frac{8}{3} \sum_n \tau_{3/2}^{(1)(n)}(1) \tau_{3/2}^{(1)(n)'}(1) - \sum_n \tau_{1/2}^{(1)(n)}(1) \tau_{1/2}^{(1)(n)'}(1) \\
- 2 \sum_n \left[ \tau_{3/2}^{(2)(n)}(1) \right]^2 - \frac{24}{5} \sum_n \left[ \tau_{5/2}^{(2)(n)}(1) \right]^2 = 0
\] (34)

\( p = 0, q = 2 \):

\[
1 - 8\rho^2 + 4\sigma^2 + 4 \sum_n \left[ \xi^{(n)'}(1) \right]^2 + 8 \sum_n \left[ \tau_{3/2}^{(1)(n)}(1) \right]^2 + \sum_n \left[ \tau_{1/2}^{(1)(n)}(1) \right]^2 \\
+ \frac{32}{3} \sum_n \tau_{3/2}^{(1)(n)}(1) \tau_{3/2}^{(1)(n)'}(1) + 4 \sum_n \tau_{1/2}^{(1)(n)}(1) \tau_{1/2}^{(1)(n)'}(1) \\
+ \frac{8}{3} \sum_n \left[ \tau_{3/2}^{(2)(n)}(1) \right]^2 + \frac{32}{5} \sum_n \left[ \tau_{5/2}^{(2)(n)}(1) \right]^2 = 0 .
\] (35)
The equations (33)-(35) are a set of linear equations in the elastic slope $\rho^2$ and the curvature $\sigma^2$, and the following quantities, that are series on the radial excitations, indicated by the sums over $n$:

\[
\sum_n \left[ \xi^{(n)'} (1) \right]^2
\]
\[
\sum_n \left[ \tau_{3/2}^{(1)(n)} (1) \right]^2
\]
\[
\sum_n \left[ \tau_{1/2}^{(1)(n)} (1) \right]^2
\]
\[
- \sum_n \tau_{3/2}^{(1)(n)} (1) \tau_{3/2}^{(1)(n)'} (1)
\]
\[
- \sum_n \tau_{1/2}^{(1)(n)} (1) \tau_{1/2}^{(1)(n)'} (1)
\]
\[
\sum_n \left[ \tau_{3/2}^{(2)(n)} (1) \right]^2
\]
\[
\sum_n \left[ \tau_{5/2}^{(2)(n)} (1) \right]^2
\]

We realize first that, due to the fact that we compute the second derivatives in (31) ($p + q = 2$) and use the boundary conditions $w_{if} = w = 1$, the series in $j^P$ states is truncated and includes at most the $\ell = 2$ states $j^P = \frac{3}{2}^-, \frac{5}{2}^-$, corresponding to the unknowns (41) and (42). On the other hand (36) is the square of the derivatives at zero recoil of the lowest $j^P = \frac{1}{2}^-$, and (37) and (38) depend on the IW functions of the transitions to the $P$-wave states $j^P = \frac{1}{2}^+; \frac{3}{2}^+$, that are simply related to the slope $\rho^2$ through Bjorken and Uraltsev SR, as we write down below again. Finally, we have two other unknowns (39) and (40) that involve the derivatives of the $P$-wave IW functions $\tau_{3/2}^{(1)(n)} (w)$, $\tau_{1/2}^{(1)(n)} (w)$ at zero recoil. These quantities were already introduced in ref. [1].

4 Equations from the Axial Sum Rule

Let us now consider likewise the derivatives of the SR for axial currents (17) with the boundary condition $w_{if} = 1$, $w_i = w_f = w \rightarrow 1$:

\[
\left( \frac{\partial^{p+q} L_A}{\partial w_{ij}^p \partial w^q} \right)_{w_{if} = w = 1} = \left( \frac{\partial^{p+q} R_A}{\partial w_{ij}^p \partial w^q} \right)_{w_{if} = w = 1}.
\]
Since all terms in (17) vanish for $w_i = w_f = w_{if} = 1$, to obtain information on the curvature $\sigma^2$ we will need to go up to the third order derivatives.

With the notation (13), for $p = q = 0$ we get $0 = 0$, and for $p = 1$, $q = 0$ and $p = 0$, $q = 1$, $\xi(1) = \xi(1)$.

For the second order derivatives we obtain the following results. For $p = 2$, $q = 0$ and $p = q = 1$ we get the same relation

$$\rho^2 = \sum_n \left[ \tau_{3/2}^{(1)}(1) \right]^2$$  \hspace{1cm} (44)

while for $p = 0$, $q = 2$, we get Bjorken SR:

$$\rho^2 = \frac{1}{4} + \frac{2}{3} \sum_n \left[ \tau_{3/2}^{(1)}(1) \right]^2 + \frac{1}{4} \sum_n \left[ \tau_{1/2}^{(1)}(1) \right]^2 .$$  \hspace{1cm} (45)

Both equations (44) and (45) imply Uraltsev SR

$$\frac{1}{3} \sum_n \left[ \tau_{3/2}^{(1)}(1) \right]^2 - \frac{1}{4} \sum_n \left[ \tau_{1/2}^{(1)}(1) \right]^2 = \frac{1}{4}$$  \hspace{1cm} (46)

using the notation (18).

Going now to the third order derivatives, we obtain the following results.

For $p = 3$, $q = 0$ :

$$\sigma^2 = 2 \sum_n \left[ \tau_{5/2}^{(2)}(1) \right]^2$$  \hspace{1cm} (47)

$p = 2$, $q = 1$ :

$$\sigma^2 = 2 \sum_n \tau_{3/2}^{(1)(n)}(1)\tau_{3/2}^{(1)(n)'}(1) + 6 \sum_n \left[ \tau_{5/2}^{(2)}(1) \right]^2$$  \hspace{1cm} (48)

$p = 1$, $q = 2$ :

$$\sigma^2 + \sum_n \left[ \xi^{(n)'}(1) \right]^2 + 2 \sum_n \left[ \tau_{3/2}^{(1)}(1) \right]^2$$
$$+ 8 \sum_n \tau_{3/2}^{(1)(n)}(1)\tau_{3/2}^{(1)(n)'}(1) + \frac{2}{3} \sum_n \left[ \tau_{5/2}^{(2)}(1) \right]^2 + \frac{48}{5} \sum_n \left[ \tau_{5/2}^{(2)}(1) \right]^2 = 0$$  \hspace{1cm} (49)

$p = 0$, $q = 3$ :

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\[-3\rho^2 + 3\sigma^2 + 3 \sum_n \left[ \xi^{(n)}(1)' \right]^2 + 4 \sum_n \left[ \tau_{3/2}^{(1)}(1) \right]^2 \\
+ 8 \sum_n \tau_{3/2}^{(1)}(1) \tau_{3/2}^{(1)(n)'(1)} + 3 \sum_n \tau_{1/2}^{(1)}(1) \tau_{1/2}^{(1)(n)'(1)} \\
+ 2 \sum_n \left[ \tau_{3/2}^{(2)}(1) \right]^2 + \frac{24}{5} \sum_n \left[ \tau_{5/2}^{(2)}(1) \right]^2 = 0 \ . \quad (50)\]

The equations (47)-(50) depend on \( \rho^2, \sigma^2 \) and the same set of unknowns listed in (36)-(42). Let us now look for the set of linearly independent equations that can be obtained from the sets of equations obtained from the Vector and Axial SR.

## 5 Linearly independent relations

Let us concentrate on the equations (33)-(35) and (47)-(50) obtained respectively from the Vector and Axial Sum Rules.

Using Bjorken SR

\[\rho^2 = \frac{1}{4} + \frac{2}{3} \sum_n \left[ \tau_{3/2}^{(1)}(1) \right]^2 + \frac{1}{4} \sum_n \left[ \tau_{1/2}^{(1)}(1) \right]^2 , \quad (51)\]

the relation

\[\frac{4}{3} \rho^2 - 1 = \sum_n \left[ \tau_{1/2}^{(1)}(1) \right]^2 \quad (52)\]

obtained from (32) and (44), and (44) and (50) we obtain finally the following set of relations that are linearly independent

\[\rho^2 = - \frac{4}{5} \sum_n \tau_{3/2}^{(1)}(1) \tau_{3/2}^{(1)(n)'(1)} + \frac{3}{5} \sum_n \tau_{1/2}^{(1)}(1) \tau_{1/2}^{(1)(n)'(1)} \quad (53)\]

\[\sigma^2 = - \sum_n \tau_{3/2}^{(1)}(1) \tau_{3/2}^{(1)(n)'(1)} \quad (54)\]

\[\sigma^2 = 2 \sum_n \left[ \tau_{5/2}^{(2)}(1) \right]^2 \quad (55)\]

\[\rho^2 - \frac{4}{5} \sigma^2 + \sum_n \left[ \tau_{3/2}^{(2)}(1) \right]^2 = 0 \quad (56)\]
\[ \frac{4}{3} \rho^2 - \frac{5}{3} \sigma^2 + \sum_n \left[ \xi^{(n)'}(1) \right]^2 = 0. \] \quad (57)

Relations (53) and (54) were obtained in ref. [1], and relation (55) and (56) in [11]. The systematic study of the present paper using all possibilities (31) and (43) involving the curvature gives the new equation (57).

6 Bounds on the curvature

The last two equations (56) and (57) involve the curvature with a negative sign and positive definite quantities. Making explicit in the sum \( \sum_n [\xi^{(n)'}(1)]^2 \) the ground state IW function slope \( \xi^{(0)'}(1) = -\rho^2 \), one obtains the two equations

\[ \rho^2 - \frac{4}{5} \sigma^2 + \sum_n |\tau_{3/2}^{(2)(n)}(1)|^2 = 0. \] \quad (58)

\[ \frac{4}{3} \rho^2 + (\rho^2)^2 - \frac{5}{3} \sigma^2 + \sum_{n \neq 0} |\xi^{(n)'}(1)|^2 = 0 \] \quad (59)

that imply respectively the bounds:

\[ \sigma^2 \geq \frac{5}{4} \rho^2 \] \quad (60)

\[ \sigma^2 \geq \frac{1}{5} \left[ 4 \rho^2 + 3 (\rho^2)^2 \right]. \] \quad (61)

The bound (60) was obtained in ref. [1] using the relations (53) and (54) and making the assumption

\[ - \sum_n \tau_{1/2}^{(1)(n)}(1) \tau_{1/2}^{(1)(n)'}(1) \geq 0. \] \quad (62)

Later, (60) was demonstrated rigorously in ref. [11] and generalized to the \( n \)-th derivative. However, in this latter paper, only derivatives relatively to \( w_i \) were taken, while in the present work a systematic use of (31) and (43) is carried out.

The inequality (61) is the best of the bounds that we have obtained for \( \sigma^2 \) for any value of \( \rho^2 \), and is the main result of this paper.

Interestingly enough, both bounds (60) and (61) coincide at the lower bound \( \rho^2 \geq \frac{3}{4} \) implied by Bjorken and Uraltsev SR, (32) and (46). At the value \( \rho^2 = \frac{3}{4} \) one
then gets indeed the same absolute bound (i.e., independent of $\rho^2$) for $\sigma^2$, namely
\[ \sigma^2 \geq \frac{15}{16}. \]

7 Implication on the P-wave IW functions at zero recoil

Let us now express the sums of products of the $P$-wave Isgur-Wise functions $\frac{1}{2}^- \to \frac{1}{2}^+$ and their derivatives $\sum_n \tau^{(1)(n)}_{3/2}(1) \tau^{(1)(n)'}_{3/2}(1)$ and $\sum_n \tau^{(1)(n)}_{1/2}(1) \tau^{(1)(n)'}_{1/2}(1)$ in terms of $\rho^2$ and $\sigma^2$. From (53) and (54) we obtain, using now the notation of Isgur and Wise (18):

\[ -\sum_n \tau^{(n)}_{3/2}(1) \tau^{(n)'}_{3/2}(1) = \frac{1}{3} \sigma^2 \]  
(63)

\[ -\sum_n \tau^{(n)}_{1/2}(1) \tau^{(n)'}_{1/2}(1) = -\frac{5}{12} \rho^2 + \frac{1}{3} \sigma^2. \]  
(64)

Using the bounds (10) and (11) for $\rho^2$ and $\sigma^2$ one finds

\[ -\sum_n \tau^{(n)}_{3/2}(1) \tau^{(n)'}_{3/2}(1) \geq \frac{5}{16} \]  
(65)

\[ -\sum_n \tau^{(n)}_{1/2}(1) \tau^{(n)'}_{1/2}(1) \geq 0. \]  
(66)

Strictly speaking, these relations do not give information on the slope of the lowest $n = 0$ IW functions $\tau^{(0)'}_{3/2}(1)$ and $\tau^{(0)'}_{1/2}(1)$. However, if the $n = 0$ state dominates the sum, the inequalities (65) and (66) imply that the slopes $\tau^{(0)'}_{3/2}(1)$ and $\tau^{(0)'}_{1/2}(1)$ are negative, as it is plausible on physical grounds for form factors that do not involve radially excited states.

This is indeed the case for the Bakamjian-Thomas type of quark models, that satisfy IW scaling [13] and Bjorken and Uraltsev sum rules [14]. We have conjectured in [1] that this class of models presumably satisfy all the SR of the heavy quark limit of QCD.

In the Bakamjian-Thomas model one finds for the phenomenologically successful spectroscopic model of Godfrey and Isgur [15], the numbers

\[ \text{numbers} \]
that by themselves satisfy the preceding bounds, so that the \( n = 0 \) state seems to
give a dominant contribution to the l.h.s. of (65) and (66).

8 Non-relativistic limit of the bounds

There is a simple intuitive argument to understand the term \( \frac{3}{5}(\rho^2)^2 \) in the best
bound (61). Let us consider the non-relativistic quark model, i.e. a non-relativistic
light quark \( q \) interacting with a heavy quark \( Q \) through a potential. The form factor
– to be identified with the IW function – has then the simple form:

\[
F(k^2) = \int d\mathbf{r} \; \phi_0^+(r) \exp \left( i \frac{m_q}{m_q + m_Q} \mathbf{k} \cdot \mathbf{r} \right) \phi_0(r)
\]

where \( \phi_0(r) \) is the ground state radial wave function. In the small momentum
transfer limit, the IW variable \( w \) writes, in the initial heavy hadron rest frame:

\[
w \approx 1 + \frac{v^2}{2} = 1 + \frac{k^2}{2 m_Q^2}.
\]

Identifying the non-relativistic IW function \( \xi_{NR}(w) \) with the form factor \( F(k^2) \) (69),
one finds, because of rotational invariance:

\[
\xi_{NR}(w) \approx 1 - m_q^2 < 0 | z^2 | 0 > (w - 1) + \frac{1}{2} \frac{1}{3} m_q^4 < 0 | z^4 | 0 > (w - 1)^2 + \cdots
\]

where \( | 0 > \) stands for the ground state wave function, and we have neglected in
the \( (w - 1)^2 \) coefficient subleading terms in powers of \( 1/(m_q z) \) (internal velocity).
Therefore, one has the following expressions for the slope and the curvature, in the
non-relativistic limit:

\[
\rho_{NR}^2 = m_q^2 < 0 | z^2 | 0 > \quad \sigma_{NR}^2 \approx \frac{1}{3} m_q^4 < 0 | z^4 | 0 >.
\]

From spherical symmetry one has
\[ <0|z^4|0> = \frac{1}{5} <0|r^4|0> . \] (73)

Using now completeness \( \sum_n |n><n| = 1, \)

\[ <0|r^4|0> = |<0|r^2|0>|^2 + \sum_{n \neq 0} |<n|r^2|0>|^2 \] (74)

we use again spherical symmetry

\[ <0|r^4|0> = 9|<0|z^2|0>|^2 + 9 \sum_{n \neq 0, rad} |<n|z^2|0>|^2 \] (75)

where the latter sum runs only over radial excitations.

Therefore, from (72)-(75) we can rewrite \( \sigma_{NR}^2 \) under the form

\[ \sigma_{NR}^2 = \frac{3}{5} \left\{ \left[ m_q^2 <0|z^2|0> \right]^2 + m_q^4 \sum_{n \neq 0, rad} |<n|z^2|0>|^2 \right\} \] (76)

or

\[ \sigma_{NR}^2 = \frac{3}{5} \left[ \rho_{NR}^2 \right]^2 + \frac{3}{5} m_q^4 \sum_{n \neq 0, rad} |<n|z^2|0>|^2 \] (77)

and therefore

\[ \sigma_{NR}^2 \geq \frac{3}{5} \left[ \rho_{NR}^2 \right]^2 . \] (78)

Notice that, denoting by \( R \) the bound state radius and \( m_q \) the light quark mass, in the non-relativistic limit, just from expressions (72), one can see that \( \rho_{NR}^2 \) scales like \( m_q^2 R^2 \), while \( \sigma_{NR}^2 \) scales like \( m_q^4 R^4 \) and both the l.h.s. and the r.h.s. in (78) scale in the same way.

Going back to the relativistic bounds (60)-(61), we observe that the terms proportional to \( \rho^2 \) are subleading in the non-relativistic expansion and correspond to relativistic corrections specific to QCD in the heavy quark limit. In the non-relativistic limit \( \rho^2 \sim m_q^2 R^2 \gg 1 \), and the power \( (\rho^2)^2 \) is leading. We can understand therefore the appearance of the term \( \frac{2}{5}(\rho^2)^2 \) in the r.h.s. of the inequality (61).
9 An example of fit to the data

An interesting phenomenological remark is that the simple parametrization for the IW function \[\xi(w) = \left(\frac{2}{w+1}\right)^{2\rho^2}\] (79)
gives

\[\sigma^2 = \frac{\rho^2}{2} + (\rho^2)^2\] (80)
that satisfies the inequalities (60)-(61) if \(\rho^2 \geq \frac{3}{4}\), i.e. for all values allowed for \(\rho^2\). Moreover, interestingly, at the lowest bound of the slope \(\rho^2 = \frac{3}{4}\), (80) implies precisely the lower bound of the curvature \(\sigma^2 = \frac{15}{16}\), as pointed out in [11].

Notice that, in ref. [15], within the class of Bakamjian-Thomas quark models, the approximate form (79) was found with \(\rho^2 = 1.02\) in the particular case of the spectroscopic model of Godfrey and Isgur. This gives a curvature (80) \(\sigma^2 = 1.55\), close to the bound (61), that gives \(\sigma^2 \geq 1.44\), stronger than the bound (60), that implies \(\sigma^2 \geq 1.27\).

As a simple example of a fit with the simple function (79), we can use BELLE data on \(\bar{B}^0 \rightarrow D^{*+} e^- \bar{\nu}\) for the product \(|V_{cb}| \mathcal{F}^*(w)|[16]\), as shown in the Figure.

The function \(\mathcal{F}^*(w)\) is equal to the Isgur-Wise function \(\xi(w)\) in the heavy quark limit. Assuming only departures of this limit at \(w = 1\), i.e. fitting \(\xi(w)\) from the data with

\[|V_{cb}| \mathcal{F}^*(w) = |V_{cb}| \mathcal{F}^*(1) \xi(w)\] (81)
we obtain the following results for the normalization and the slope:

\[\mathcal{F}^*(1)|V_{cb}| = 0.036 \pm 0.002\]
\[\rho^2 = 1.15 \pm 0.18\] (82)
with the other derivatives of \(\xi(w)\) fixed by (79) (Fig. 1).

As we can see, the determination of \(\mathcal{F}^*(1)|V_{cb}|\) is rather precise, while \(\rho^2\) has a larger error. However, the values obtained for \(|V_{cb}|\) and \(\rho^2\) are strongly correlated. It is important to point out that the most precise data points are the ones at large \(w\), so that higher derivatives contribute importantly in this region. Due to the alternate
Figure 1: Fit to $\mathcal{F}^*(w)|V_{cb}|$ using the phenomenological formula (79) and the BELLE data for $\bar{B} \to D^* \ell \nu$ [16], assuming only violations to the heavy quark limit at $w = 1$. The fit gives the results [25].
character of $\xi(w)$ as a series of $(w - 1)$, one does not see strongly the curvature of $\xi(w)$ in the Figure, but the curve is definitely not close to a straight line. Linear fits, as are commonly used, should be ruled out at the view of the bounds that we have found.

We must emphasize that the fit that we present is a simple exercise in the heavy quark limit. Radiative corrections and $1/m_Q$ corrections, that enter in the relation between the actual function $F(w)$ and its heavy quark limit $\xi(w)$, should be taken into account, although this does not seem to be an easy task [17].

10 Comparison with the dispersive bounds

A considerable effort has been developed to formulate dispersive constraints on the shape of the form factors in $\bar{B} \to D^{(*)}\ell\nu$ [18]-[22]. The starting point are the analyticity properties of two-point functions and positivity of the corresponding spectral functions. Then, dispersion relations relate the hadronic spectral functions to the QCD two-point functions in the deep Euclidean region, and positivity allows to bounds the contribution of the relevant states, leading to constraints on the semileptonic form factors.

We will now compare our method, that gives information on the derivatives of the Isgur-Wise function, with the dispersive approach.

A first remark to be made is that our approach, based on Bjorken-like SR, holds in the physical region of the semileptonic decays $\bar{B} \to D^{(*)}\ell\nu$ and in the heavy quark limit. However, concerning this last simplifying feature, we should underline that there is no objection of principle to include in the calculation radiative corrections and subleading corrections in powers of $1/m_Q$.

The dispersive approach starts from bounds in the crossed channel by comparison of the OPE and the sum over hadrons coupled to the corresponding current, $\bar{B}\bar{D}$, $\bar{B}\bar{D}^*$, · · · Then, one analytically continues to the physical region of the semileptonic decays. This is done for a single reference form factor, for example the combination

$$V_1(w) = h_+(w) - \frac{m_B - m_D}{m_B + m_D} h_-(w)$$

(83)

that enters in the $\bar{B} \to D\ell\nu$ rate. In the heavy quark limit $h_-(w) = 0$, $V_1(w) =$
$h_+(w) = \xi(w)$. Ratios of the remaining form factors to $V_1(w)$ are computed in the physical region by introducing $1/m_Q$ and $\alpha_s$ corrections to the heavy quark limit. The dispersive approach considers physical quark masses, in contrast with the heavy quark limit of our method.

The two approaches are quite different in spirit and in their results. However, it can be interesting to compare numerically our bounds with the ones of the dispersive approach, as they happen to be complementary. We must however keep in mind precisely the differences between the two methods.

We have demonstrated in [11] that the IW function $\xi(w)$ is an alternating series in powers of $(w - 1)$, with the moduli of the derivatives satisfying the bounds (12) and (61).

10.1 Comparison with the work of Caprini, Lellouch and Neubert

Let us consider the main results of ref. [21], that are summarized by the one-parameter formula

$$\frac{V_1(w)}{V_1(1)} \cong 1 - 8\rho^2 z + (51\rho^2 - 10)z^2 - (252\rho^2 - 84)z^3$$

(84)

with the variable $z(w)$ defined by

$$z = \frac{\sqrt{w+1} - \sqrt{2}}{\sqrt{w+1} + \sqrt{2}}$$

(85)

and the allowed range for $\rho^2$ being

$$-0.17 < \rho^2 < 1.51.$$  

(86)

Of course, the function $\frac{V_1(w)}{V_1(1)}$ contains finite mass corrections that are absent at present in our method. Nevertheless, let us first compare these results with our lower bounds (12), assuming the rough approximation

$$\frac{V_1(w)}{V_1(1)} \cong \xi(w).$$

(87)

Of course, since the expansion (84) stops at third order in $z$, it would only make sense in the comparison to go up to the third derivative of $\xi(w)$. Using our notation,
the results of Section 4 of ref. [21] for the first derivatives write, from the expansion (84), in terms of the slope $\rho^2$:

$$\xi''(1) = \frac{1}{32}(67\rho^2 - 10)$$ (88)

$$\xi'''(1) = -\frac{1}{256}(1487\rho^2 - 372)$$ (89)

with $\rho^2$ in the range (86). From (88) and (89), using the notation $\xi(w) = 1 - \rho^2(w - 1) + c(w - 1)^2 + d(w - 1)^3 + \cdots$ one gets the numerical relations [21]: $c \approx 1.05\rho^2 - 0.15$, $d \approx -0.97\rho^2 + 0.24$.

The lower bound (86) on $\rho^2$ is very weak: there is a region for $\rho^2$ that is below the Bjorken bound $\rho^2 \geq 1/4$, and this reflects on the regions allowed for the higher derivatives. Of course, we must always keep in mind that in these considerations we are neglecting finite mass and $\alpha_s$ corrections. Moreover, one should notice that the series is not alternate in the whole range (86). We have verified analytically that the expression (84) is an alternate series in powers of $(w - 1)$ for values of $\rho^2 \geq \frac{1}{4}$, i.e. for $\rho^2$ satisfying precisely Bjorken bound. It is not clear to us whether this is just a numerical coincidence or it has a deeper significance, i.e. if it extends to the complete series in powers of $z$.

Let us now comment on the implications of our bounds (12). The first important remark is that, within the simplifying hypothesis (87), the range (86) is considerably tightened by the lower bound on $\rho^2 \geq \frac{3}{4}$ implied by Bjorken and Uraltsev sum rules. Therefore, we will consider hereafter, instead of (86), the improved range

$$\frac{3}{4} \leq \rho^2 < 1.51$$ (90)

that shows that our type of lower bounds are complementary to the upper bounds obtained from dispersive methods. Within the hypothesis of the heavy quark limit, the region allowed by the dispersive bounds for $\xi(w)$ with $\rho^2$ within the range (86) is obviously much reduced by the bounds (90) (Fig. 2).

On the other hand, it follows that the second and third derivatives (88), (89) satisfy the bounds (12)

$$\xi''(1) = \frac{1}{32}(67\rho^2 - 10) \geq \frac{5}{4}\rho^2$$ (91)
Figure 2: The upper (lower) curves are the representations of \( \xi(w) \) according to the dispersive approach of Caprini et al [21] (84)-(87). The upper (lower) curve correspond to \( \rho^2 = -0.17 \) (\( \rho^2 = 1.51 \)). The shadowed region is the region forbidden by the Uraltsev bound \( \rho^2 \geq \frac{3}{4} \). The remaining allowed region corresponds to (90). The curve within this allowed region is the best fit to BELLE data [16], normalized to \( w = 1 \), that gives \( F^*(1) |V_{cb}| = 0.036 \pm 0.002 \), \( \rho^2 = 1.16 \pm 0.15 \), in practice the same fit as (82) with the phenomenological formula (79).

\[
-\xi'''(1) = \frac{1}{256} (1487\rho^2 - 372) \geq \frac{7}{4} \xi''(1) = \frac{7}{4} \frac{1}{32} (67\rho^2 - 10) \tag{92}
\]

respectively for \( \rho^2 \gtrsim 0.36 \) and \( \rho^2 \gtrsim 0.42 \). Therefore, for values of \( \rho^2 \) that are within the range (90), these inequalities are a fortiori satisfied, but they are not satisfied in the whole range (86).

Finally, let us look for the implications of our improved bound on the curvature, eq. (61). Combining the linear dependence obtained from dispersive methods (88) with the inequality (61) one obtains the condition

\[
\frac{1}{32} (67\rho^2 - 10) \geq \frac{1}{5} \left[ 4\rho^2 + 3(\rho^2)^2 \right] \tag{93}
\]

that gives the range
Interestingly, the condition (93) gives by itself an upper bound for $\rho^2$ that is of the same order than the upper bound (86). Moreover, the range (94) contains the improved range (90), and things appear to be coherent.

10.2 Comparison with the work of Boyd, Grinstein and Lebed

Let us now compare with the dispersive method results of the work of Boyd, Grinstein and Lebed [22]. In this work, the QCD part of the calculation includes $\alpha_s$ and non-perturbative (condensate) corrections, and new poles below the annihilation threshold ignored in [21]. In a form that allows to make the comparison with our results, the authors of [22] obtain the following expansion for the scalar form factor

\[
\tilde{f}_0(w) = \tilde{f}_0(1) + \left[1.72a_1 - 0.77\tilde{f}_0(1)\right](w-1) \\
+ \left[-1.74a_1 + 0.21a_2 + 0.55\tilde{f}_0(1)\right](w-1)^2 + \cdots
\]  

where $\tilde{f}_0(w)$ has been defined by Caprini and Neubert [20]

\[
\tilde{f}_0(w(q^2)) = \frac{f_0(q^2)}{(M_B - M_D)\sqrt{M_B M_D}(w+1)}
\]  

with

\[
f_0(q^2) = (M_B^2 - M_D^2) f_+(q^2) + q^2 f_-(q^2)
\]  

$f_\pm(q^2)$ being the form factors governing the rate $\bar{B} \to D\ell\nu$ ($q = p - p'$):

\[
< D(p')|V_\mu|B(p) > = f_+(q^2)(p + p')_\mu + f_-(q^2)(p - p')_\mu
\]  

Heavy quark symmetry implies

\[
\tilde{f}_0(w(q^2)) \approx \xi(w).
\]
The coefficients \(a_n\) in (95) are defined by the expression for a generic form factor [21], [22]

\[
F(z) = \frac{1}{P(z)\varphi(z)} \sum_{n=0}^{\infty} a_n z^n
\]  

(100)

where \(z\) is defined by (85). The functions \(P(z)\) and \(\varphi(z)\) – respectively the Blaschke factor and the outer function – contain the subthreshold singularities in the annihilation channel, respectively the \(B_c\) poles and the kinematic singularities. The basic result of the dispersive approach is that the coefficients \(a_n\) of the series obey

\[
\sum_{n=0}^{\infty} a_n^2 \leq 1.
\]  

(101)

To compare with our bounds we proceed like we did above. Since in the heavy quark limit (99) holds, we set \(\tilde{f}_0(1) = 1\) and write the Isgur-Wise function in terms of the coefficients \(a_n\):

\[
\xi(w) \cong 1 + (1.72a_1 - 0.77)(w - 1) + (-1.74a_1 + 0.21a_2 + 0.55)(w - 1)^2 + \cdots
\]  

(102)

Notice that in (95) and (102) it does not make much sense to consider higher powers \((w - 1)^n\ (n \geq 3)\) unless the corresponding \(a_n\ (n \geq 3)\) are introduced. Then, our lower bounds [12] write

\[-1.72a_1 + 0.77 \geq \frac{3}{4} \]

\[2 (-1.74a_1 + 0.21a_2 + 0.55) \geq \frac{5}{4} (-1.72a_1 + 0.77)
\]  

(103)

implying, respectively

\[a_1 \leq 0.01\]

\[a_2 \geq 3.17a_1 - 0.33.
\]  

(104)

Since, from (104) and (101) we have
\[-1 \le a_1 \le 0.01 \quad (105)\]

and the coefficient of $a_1$ in (104) is large, the whole range

\[-1 \le a_2 \le 1 \quad (106)\]

is allowed. This seems to support the statement of ref. [22] that $a_2$ cannot always be neglected.

Moreover, using now the quadratic bound (61), one obtains

\[3(\rho^2)^2 - 6\rho^2 + 2(1 - a_2) \lesssim 0 \quad (107)\]

and therefore

\[-0.5 \lesssim a_2 \le 1 \quad (108)\]

giving the following range for $\rho^2$ in terms of $a_2$:

\[1 - \sqrt{\frac{1 + 2a_2}{3}} \lesssim \rho^2 \lesssim 1 + \sqrt{\frac{1 + 2a_2}{3}} \quad (109)\]

giving the wide range

\[0 \lesssim \rho^2 \lesssim 2 \quad (110)\]

For $a_2 = 0$, implicitly assumed in ref. [21], one finds the range

\[0.42 \lesssim \rho^2 \lesssim 1.58 \quad (111)\]

a domain qualitatively consistent but somewhat narrower than the corresponding one (94) obtained from the linear relation between the curvature and the slope given by ref. [21].

In conclusion, there is no contradiction between the dispersive bounds and the type of bounds that we have obtained using Bjorken-like sum rules in the heavy quark limit. The latter appear rather as lower bounds that are complementary to the upper bounds of the dispersive approach, tightening considerably the allowed range for $\rho^2$ and for the higher derivatives of $\xi(w)$ as well.
11 A phenomenological Ansatz for the Isgur-Wise function and the dispersive constraints

At the light of the preceding discussion, we are now going to address the question of whether the phenomenological Ansatz for the IW function proposed in Section 9,

$$\xi(w) = \left(\frac{2}{w + 1}\right)^{2w^2}$$  \hspace{1cm} (112)

satisfies, assuming the heavy quark limit (87) or (99), the constraints of the dispersive approach.

We will follow here the formulation of Boyd et al. \cite{22} and consider the form factors $f_+(q^2)$ and $f_0(q^2)$ defined by (97) and (98). In the heavy quark limit, one has the relations

$$f_+(q^2(w)) \approx \frac{M_B + M_D}{2\sqrt{M_B M_D}} \xi(w)$$  \hspace{1cm} (113)

$$f_0(q^2(w)) \approx (M_B - M_D)\sqrt{M_B M_D}(w + 1)\xi(w).$$  \hspace{1cm} (114)

We denote now generically any of these form factors by $F(q^2(w))$, or through the transformation (85), $F(q^2(z))$.

We adopt the phenomenological formula (112) for $\xi(w)$ and define the corresponding series (100)

$$\sum_{n=0}^{\infty} a_n z^n = P(z) \varphi(z) F(z)$$  \hspace{1cm} (115)

where $P(z)$ and $\varphi(z)$ are the Blaschke factor and the outer function of the corresponding form factors.

We want now to compare the coefficients $a_n$ obtained from (113)-(114), assuming $F(z) = \xi(w(z))$ given by (112), to the condition (101)

$$\sum_{n=0}^{\infty} a_n^2 \leq 1.$$  \hspace{1cm} (116)

The outer functions $\varphi(z)$ and the Blaschke factors $P(z)$ for $f_+(q^2)$ and $f_0(q^2)$ are given in ref. \cite{22}, respectively by formula (4.23) and Table 1 and by formula (4.25) and Table 3. We have singled out $f_+(q^2)$ and $f_0(q^2)$ as given by (113) and (114) but
we could have taken any other form factor related, up to kinematic factors, to $\xi(w)$.
Of course, the results for the coefficients $a_n$ would differ according to the considered form factor.

We use the numerical parameters of this paper, and two choices for $\rho^2$ in formula (112), namely $\rho^2 = 1.023$, that corresponds to the Isgur-Wise function obtained within the Bakamjian-Thomas scheme from the Godfrey-Isgur spectroscopic model, as found in ref. [15], and $\rho^2 = 1.15$ obtained from the fit of Section 9.

Denoting the Blaschke factor and outer function for each form factor by the corresponding subindices, we find, for $\rho^2 = 1.023$, the series (115) for $f_+(q^2(z))$:

$$P_+(z)\varphi_+(z)f_+(q^2(z)) = 0.0143 - 0.0179z - 0.1164z^2 + 0.3277z^3 - 0.1995z^4 - 0.4497z^5 + 1.2347z^6 + \cdots \quad (117)$$

and for $f_0(q^2(z))$:

$$P_0(z)\varphi_0(z)f_0(q^2(z)) = 0.0834 - 0.1750z - 0.1725z^2 + 0.8673z^3 - 1.1600z^4 - 0.8943z^5 - 0.4346z^6 + \cdots \quad (118)$$

For $\rho^2 = 1.15$ we find, respectively:

$$P_+(z)\varphi_+(z)f_+(q^2(z)) = 0.0143 - 0.0326z - 0.0907z^2 + 0.4294z^3 - 0.5779z^4 - 0.0306z^5 + 1.3868z^6 + \cdots$$

and

$$P_0(z)\varphi_0(z)f_0(q^2(z)) = 0.0834 - 0.2599z + 0.0484z^2 + 0.9094z^3 - 2.0079z^4 + 2.5089z^5 - 2.3596z^6 + \cdots$$

Comparing to the condition (116), we observe two points. First, the first coefficients have squares well below 1, specially for $f_+(q^2)$. That this happens to be the case for this form factor, that has three Blaschke factors, reinforces the idea that one should be closer to the IW function, according to our hypothesis (113)-(114), as
the number of Blaschke factors increases. Second, high powers of $z$ have coefficients that can be of $O(1)$ and they are strongly dependent on the value of $\rho^2$, specially for $f_0(q^2)$, that has only two Blaschke factors. We have actually observed that the behaviour of the coefficients oscillate, as can be seen in (118).

Our conclusion is that, owing to the fact that the coefficients, up to order $z^3$ included, satisfy the condition (116), the “dipole” formula (112) gives, on phenomenological grounds, a good enough representation of the form factors (113), (114).

12 Conclusions

In conclusion, using sum rules in the heavy quark limit of QCD, as formulated in ref. [1], we have found an improved bound for the curvature of the Isgur-Wise function $\sigma^2 = \xi''(1) \geq \frac{1}{5}[4\rho^2 + 3(\rho^2)^2]$ that implies the already demonstrated [11] absolute bound $\sigma^2 \geq \frac{15}{16}$.

Beyond the simple Ansatz (79) introduced above, any phenomenological parametrization of $\xi(w)$ intending to fit the CKM matrix element $|V_{cb}|$ in $B \rightarrow D^{(*)}\ell\nu$ should have, for a given slope $\rho^2$ satisfying the bound (10), a curvature $\sigma^2$ satisfying the new bound (61).

Moreover, we discuss these bounds in comparison with the dispersive approach. We show that there is no contradiction, our bounds restraining the region for $\xi(w)$ allowed by this latter method.

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