CHIRAL ANOMALY ON A LATTICE

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ABSTRACT A calculation of the chiral anomaly on a finite lattice without fermion doubling is presented. The lattice gauge field is defined in the spirit of noncommutative geometry. Standard formulas for the continuum anomaly are obtained as a limit.

1. THE FINITE LATTICE EFFECTIVE ACTION

Chiral fermions have been the Achilles’ heel of lattice approximations in QCD. In standard lattice method one uses a nearest neighbor approximation to the derivative of a fermion field, the gauge field is represented by the Wilson link variables for nearest neighbor lattice points. In this approach one cannot avoid an unphysical doubling of fermion degrees of freedom. In the present paper I want to propose an alternative lattice approximation which uses not only the nearest neighbors but all points on the finite lattice in the construction of the (chiral) Dirac operator. This may sound a bit clumsy, but actually we shall see that it is quite natural, and what is most important, it leads to the correct continuum limit without a fermion doubling.

We study chiral fermions in euclidean space in $2n$ dimensions. In the continuum the fermions are functions in $\mathbb{R}^{2n}$ taking values in $\mathbb{C}^{2n} \otimes \mathbb{C}^N$. The euclidean $\gamma$-matrices operate on the first factor and the compact gauge group $G$ on the second factor. The $\gamma$ matrices satisfy

\begin{equation}
\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_{\mu\nu}.
\end{equation}
The matrix $\gamma_{2n+1} = (-i)^n \gamma_1 \gamma_2 \ldots \gamma_{2n}$ anticommutes with the other $\gamma$ matrices. We define the chiral projections $P_\pm = \frac{1}{2} (1 \pm \gamma_{2n+1})$ and consider the chiral Dirac operator

$$(2) \quad D_A = \gamma_\mu (-i\partial_\mu + A_\mu P_+) = \not{p} + \not{A} P_+$$

coupled to a vector potential $A = \gamma_\mu A_\mu$ with $A_\mu$ functions on $\mathbb{R}^{2n}$ taking values in the Lie algebra $\mathfrak{g}$ of the gauge group $G$.

On a finite lattice $L \subset \mathbb{R}^{2n}$ consisting of points $x = m \cdot \Delta$, where the lattice spacing $\Delta$ is some fixed positive real number and $m = (m_1, m_2, \ldots, m_{2n})$ is a vector with integer components $n_\mu = -q, -q + 1, \ldots, q$, any function is a linear combination of the Fourier modes $f_p(x) = \frac{1}{(2q)^{2n}} e^{ip \cdot x}$ with $p_\mu = \frac{2\pi}{2q \Delta} k$ and $k = -q, -q + 1, \ldots, q$. Thus the vector space of massless fermions on the lattice $L$ has dimension $K = 2^n \cdot N \cdot (2q)^{2n}$. Note that at the end points, corresponding to $k = \pm q$, in the momentum space lattice the Fourier modes are identical; we have periodicity with period $2q$ in the momentum index $k$. We shall later break this periodicity.

When studying the finite volume continuum limit one lets $q \to \infty$ and $\Delta \to 0$ such that $\ell = 2q \Delta$ is fixed. However, we shall not insist on this but we keep the option open for an infinite volume limit.

Now our approach departs from the usual setting. Normally, the Dirac operator is written using the principle that partial derivatives on a lattice are certain finite difference operators involving nearest neighbor points. Here we define the free Dirac operator through the Fourier transform $D_0 \mapsto \gamma_\mu p_\mu$. The vector potential $A = A_\mu \gamma_\mu$, thought of as an operator in the one-particle Hilbert space, is defined as a $K \times K$ matrix with matrix elements

$$(3) \quad (\gamma_\mu A_\mu)_{sp, s'p'} = (\hat{A}_\mu (p - p') \gamma_\mu)_{ss'}$$

where $ss'$ denote both the spin and gauge indices and the ambiguity in the Fourier transform $\hat{A}$ (when restricting the continuum potential to the finite lattice) is resolved by requiring that both $k, k' = -q, -q + 1, \ldots, q$ for $p_\mu = k \cdot \frac{2\pi}{2q \Delta}$. Note that in general the Fourier transform is not periodic and therefore the matrix elements $\hat{A}_{sp, s'p'}$ for $p, p' = \pm \frac{2\pi}{2q \Delta} q$ are all independent. The number of lattice sites is $(2q + 1)^{2n}$ instead of $(2q)^{2n}$. What we done above is simply that we have defined the
lattice operator $A$ (in momentum space) as a $K \times K$ submatrix of the corresponding continuum operator by restricting the momentum indices to the given range.

We shall generalize the above setting in the spirit of noncommutative geometry, [C]. The reason for doing it here is purely technical: We do not need to think all the time whether the matrix representing a vector potential really represents a multiplication by a Lie algebra valued function. A vector potential in the generalized sense is any hermitean $K \times K$ matrix of the form $A_\mu \gamma_\mu$ where the $A_\mu$’s are hermitean matrices acting in $\mathbb{C}^N \times (2q+1)^2$. A gauge transformation is then an unitary $K \times K$ matrix $g$ which commutes with the $\gamma_\mu$’s. The action of $g$ on a potential $A$ is given as

$$A' = g^{-1}Ag + g^{-1}[\hat{p},g].$$

The curvature form is

$$F_{\mu\nu} = [p_\mu, A_\nu] - [p_\nu, A_\mu] + [A_\mu, A_\nu].$$

It transforms as usual, $F_{\mu\nu} \mapsto g^{-1}F_{\mu\nu}g$.

We define the lattice effective action

$$S_{eff}(A) = -\log Z(A), \quad Z(A) = \det_{ren}(\hat{p} + A P_+),$$

where $\det_{ren}$ is a renormalized determinant. Even on a finite lattice we want to use certain renormalized determinants in order that in the limit $q, 1/\Delta \to \infty$ the effective action remains finite and leads to the continuum effective action.

In continuum the first renormalization (‘vacuum subtraction’) is to replace the determinant $\det(\hat{p} + A P_+)$ by $\det(\frac{1}{\hat{p} + i\epsilon} (\hat{p} + A P_+)) \equiv \det (1 + T)$, where $T$ is an operator of order $-1$ in momenta. The term $i\epsilon$, for a real nonzero parameter $\epsilon$, is introduced as an infrared regularization. From now on any $\hat{p}$ in the denominator stands for $\hat{p} + i\epsilon$. Using the formula $\log \det = \text{tr} \log$ and the expansion of the logarithm

$$\text{tr} \log(1 + T) = \text{tr}(T - \frac{1}{2} T^2 + \frac{1}{3} T^3 \ldots)$$

one localizes the potentially diverging terms as the traces of $T^k$ for $k = 1, 2, \ldots, 2n$. The higher powers behave like $|\vec{p}|^{2n-a}$ as $|\vec{p}| \to \infty$ for some positive $a$. Assuming
that the potential $A$ vanishes more rapidly than $|x|^{-2n}$ at infinity, the trace of these terms is finite. In terms of a momentum space cut-off $\Lambda$ one can write an asymptotic expansion

$$\text{tr}_\Lambda(T - \frac{1}{2}T^2 + \cdots - \frac{1}{2n}T^{2n}) = \sum_{i \leq i_0} \Lambda^i a_i + \log(\Lambda/\Lambda_0)a_{\log}.$$  

The scale fixing constant $\Lambda_0$ is determined by the physical requirement that the strength and location of the pole at $q = 0$ of the boson propagator is not affected by the loop diagrams. The renormalized trace is defined as the coefficient $a_0$ in the asymptotic expansion.

It follows that we can define the effective action as proposed by Seiler [Se],

$$-\log \det_k(1 + T) + \text{ a finite number of Feynman diagrams}$$

where

$$\log \det_k(1 + T) = \text{tr} \left( \frac{(-1)^{k-1}}{k} T^k + \frac{(-1)^k}{k+1} T^{k+1} + \cdots \right)$$

with $k = 2n + 1$. The finite set of (renormalized) Feynman diagrams comes from the renormalized traces of $T^k$ for $k \leq 2n$. For example, when $2n = 4$ the only diverging diagrams are the vacuum polarization terms with at most four external gauge boson lines.

In the lattice the modified Fredholm determinant $\det_k(1 + T)$ is defined exactly as in the continuum case. The renormalized determinant is

$$\det_{ren}(1 + \frac{1}{P}AP_+) = \det_{2n+1}(1 + \frac{1}{P}AP_+) \cdot e^{\text{TR}(T + \cdots - \frac{1}{2n}T^{2n})}$$

where the renormalized trace TR is defined as follows.

**Case of $2n = 2$.** The continuum limit of $\text{tr}_\Lambda(\frac{1}{P}AP_+)$ vanishes by a simple parity argument. Thus no renormalization is needed for this term. The next term $\text{tr}_\Lambda(T^2)$ is potentially logarithmically diverging. However, by the trace properties of products of $\gamma$ matrices and parity this term is actually of order $-3$ and gives a finite trace. Thus the effective action is completely determined by $\det_3(1 + T)$ and the finite 1-loop diagram, $\text{TR}(T - \frac{1}{2}T^2) = -\frac{1}{2}\text{tr}_C T^2$, where $\text{tr}_C$ stands for the conditionally convergent trace: one computes first the trace over spin and color.
indices, then integrates over momentum variables with $|p| \leq \Lambda$, followed by the limit $\Lambda \to \infty$ and the integration over $x$.

**Case of $2n = 4$.** The first term $\text{tr} \ T$ vanishes as in the previous case. The next is the 1-loop diagram $\text{tr} \ T^2$ with two external boson lines. In the continuum we must compute traces of operators which are composed of products of Green’s functions (the operators $1/(p + i\epsilon)$) and and of smooth functions $A$. They are examples of pseudodifferential operators. The algebraic manipulations involving PSDO’s are most conveniently performed using the symbol calculus. First let us recall the basic rule of symbol calculus: A pseudodifferential operator is represented by a smooth function of coordinates and momenta, its symbol. The symbol $a \ast b$ for a product of operators is computed from the symbols $a, b$ of the factors as

$$(a \ast b)(x, p) = \sum \frac{(-i)^{n_1 + \ldots + n_d}}{n_1! n_2! \ldots n_d!} \frac{\partial^{n_1} \ldots \partial^{n_d} a}{(\partial p_1)^{n_1} \ldots (\partial p_d)^{n_d}} \frac{\partial^{n_1} \ldots \partial^{n_d} b}{(\partial x_1)^{n_1} \ldots (\partial x_d)^{n_d}}.$$

Now the first term in the expansion of the PSDO $(\frac{1}{i} AP^+)^2$ leads to both quadratic and logarithmic divergencies which are given by the integral

$$-\frac{1}{(2\pi)^4} \int d^4 x \int d^4 p \frac{p^2}{(p^2 + \epsilon^2)^2} \text{tr} A^2.$$  

The lattice version of this is simply

$$-\frac{1}{(2\pi)^4} \text{tr} \left( \frac{p^2}{(p^2 + \epsilon^2)^2} A^2 \right)$$  

In addition, there is are logarithmic divergencies arising from the next terms in the expansion of $T^2$ as well as contributions from the higher order terms which involve the traces $\text{tr}(T^k)$ for $k = 3, 4$. In the continuum case, by a standard Feynman integral calculation carried out in [SABJ], one obtains ($\beta$ is a numerical constant) as the total logarithmic divergence

$$\beta \log(\Lambda/\Lambda_0) \int \text{tr}(F^2 + F \ast F)$$

where $\Lambda_0$ is a renormalization constant and $\ast F$ is the dual of the field tensor. Actually, in our case the second term involving $F \ast F$ vanishes because we are not considering instanton backgrounds, the vector potential is globally defined and vanishes at $|x| \to \infty$. A lattice version of this is

$$\beta \text{tr} \left[ \frac{1}{2 \Lambda_0^2} (F^2 + F \ast F) \right]$$
modulo finite terms in the continuum limit. Thus for \(2n = 4\) the effective action is given by \(Z(A) = \det_5(1 + T) \times e^{\beta(A)}\), where \(\beta(A)\) is the ordinary trace \(\sum_{k=1}^{4} \frac{(-1)^k}{k^k} \text{tr}(\frac{1}{\mathcal{P}} A_+)^k\) minus the diverging terms discussed above.

**General case.** In \(2n\) dimensions there is a finite number of both polynomially and logarithmically diverging terms. A derivative in momentum space makes the diagram better converging. A derivative in momentum space is associated with a differentiation in \(x\) space of one of the \(A\)'s in the expansion of the trace. Since for diverging diagrams we can have only a finite number of differentiations, the coefficient of \(\Lambda^k\) or of \(\log \Lambda\) will be a finite differential polynomial in \(A\). The lattice renormalization is obtained by subtracting these diverging terms from the naive effective action in such a way that the partial derivatives \(\frac{\partial}{\partial x_\mu}\) are replaced by the multiplication operators \(i p_\mu\).

## 2. THE ANOMALY

Next we shall compute the gauge variation of the (lattice) effective action. For that we need some properties of the generalized Fredholm determinants. According to our definition, \([S]\),

\[
\text{(11)} \quad \det_k(1 + X) = \det[(1 + X)e^{\beta_k(X)}],
\]

where

\[
\text{(12)} \quad \beta_k(X) = \sum_{i=1}^{k-1} \frac{(-1)^i}{i} X^i.
\]

If all the traces of powers of \(X\) are finite we can write

\[
\text{(13)} \quad \det_k(1 + X) = \det(1 + X) \cdot e^{\text{tr} \beta_k(X)}.
\]

The generalized determinants have a multiplicative anomaly,

\[
\text{(14)} \quad \det_k[(1 + X)(1 + Y)] = \det_k(1 + X) \cdot \det_k(1 + Y) \cdot e^{\gamma_k(X,Y)},
\]

where

\[
\text{(15)} \quad \gamma_k(X,Y) = \text{tr}[\beta_k(X + X + X X) - \beta_k(X) - \beta_k(Y)].
\]
The first nonzero multiplicative anomaly is \( \gamma_2(X,Y) = -\text{tr}(XY) \), the next is \( \gamma_3(X,Y) = \text{tr}(X^2Y + Y^2X + \frac{1}{2}(XY)^2) \). We shall also need the derivative \( H_k = \gamma_k^{(1)} \) with respect to the first argument at \((X,Y) = (0,Y)\). Its value for a variation \( \delta X \) is easily computed to be

\[
H_k(Y)(\delta X) = \text{tr}(Y^{-1}\delta X).
\]

The gauge variation of \( S_{\text{eff}}(A) \) can be computed as follows. Modulo a variation of a finite polynomial in \( A \), the anomaly is given by the gauge variation of the functional \( Z_k(A) = \det_k(1 + \frac{i}{p}\mathcal{A}_+), \mathcal{A}_+ = \mathcal{A}P_+ \). Denote \( g = gP_+ + P_- \) and \( g = gP_+ + P_- \). We obtain

\[
Z_k(A^g) = Z_k(g^{-1}Ag + g^{-1}dg) = \det_k\left(\frac{1}{p}g_-(\mathcal{P} + \mathcal{A}_+)g_+^{-1}\right) = \det_k\left(g_+^{-1}\frac{1}{p}g_-(1 + \frac{1}{p}\mathcal{A}_+)\right) = \det_k\left(g_+^{-1}\frac{1}{p}g_-(\mathcal{P} + 1 + \frac{1}{p}\mathcal{A}_+)\right) \cdot e^{\gamma_k(g^{-1}\frac{1}{p}[\mathcal{P}, g]P_+, \frac{1}{p}\mathcal{A}_+)}.
\]

For infinitesimal gauge variations we have

\[
\omega(X; A) = \delta X S_k(A) = \left. \frac{d}{dt} S_k(A(t)) \right|_{t=0} = (D_1 \gamma_k(0, \frac{1}{p}\mathcal{A}_+))(\frac{1}{p}[\mathcal{P}, X])P_+),
\]

where \( D_1 \) is the derivative with respect to the first argument and \( A(t) = A^g(t) \) for \( g(t) = e^{tX} \). By (16),

\[
\omega(X; A) = \text{tr}\left(\frac{1}{p}[\mathcal{P}, X](\frac{1}{p}\mathcal{A}_+)^{k-1}\right).
\]

**Example 2n = 2** Because of the accidental property of the two dimensional effective action, \( (\frac{1}{p}\mathcal{A}P_+)^2 \) is conditionally convergent, we may use the determinant \( \det_2 \) (instead of the more complicated \( \det_3 \)). The anomaly is then computed from the multiplicative anomaly \( \gamma_2 \). Both on the lattice and the continuum the anomaly is

\[
\omega(A; X) = \text{tr}\left(\frac{1}{p+i\epsilon}[\mathcal{P}, X]P_+ + \frac{1}{p+i\epsilon}\mathcal{A}P_+\right).
\]

In the continuum a simple computation leads to

\[
\omega(A; X) = \frac{i}{2\pi} \int d^2x \text{tr}A_\mu \partial_\nu X_\mu \nu + \frac{1}{2\pi} \int d^2x \text{tr}A_\mu \partial_\nu X.
\]
The second term is a trivial cocycle, it is the gauge variation of the local functional
\[ \frac{1}{4\pi} \int d^2x \text{tr} A^2. \]

The first term is the nontrivial part of the anomaly. On the lattice the local formula does not make sense, but we still have the trace formula (20), which in the continuum limit leads to an integration in momentum and configuration space, giving the formula (21).

**The general case.** The nontrivial part of the anomaly comes from the difference

\[ \theta_k(A) = \log \det(1 + T) - \log \det_k(1 + T). \]

This follows from the fact that the unrenormalized determinant is perfectly gauge symmetry; the gauge symmetry is spoiled by the renormalization. In the continuum the difference above is infinite, but we may use this formula on the finite lattice, giving

\[ \omega(A; X) = \delta X \theta_k(A) = \delta X \sum_{j=1}^{k-1} \text{tr} \left( \frac{(-1)^{j-1}}{j} T^j \right). \]

Inserting \( T = \frac{1}{p^2} \mathcal{A}_+ \) and \( \delta X \mathcal{A} = [\mathcal{A}, X] + [\mathcal{P}, X] \) in (23) we obtain once more the form (19) of the anomaly.

In the case of periodic boundary conditions in the continuum version (or more generally, with an enough rapid decrease of the gauge fields at \( |x| \to \infty \)) we can prove that (23) has a continuum limit as the momentum space lattice size is increased (the cut-off \( |p| \leq \Lambda \) is removed). This follows from a simple Hölder inequality argument: If \( W \) is any operator such that \( |W|^n \) is a trace-class operator, that is \( W \in L_n \), and \( W_i \) is a sequence of operators in \( L_n \) converging to \( W \) with respect to the \( L_n \) norm then \( \lim \text{tr}(W_i)^n = \text{tr}W^n \). Now \( W = T \) and the \( W_i \)'s are \( i \times i \) lattice approximations to the continuum operator \( T \), \( \lim |W - W_i|_n = 0 \).

The operators under the trace map in (23) become conditionally trace-class for \( k \geq 2n \), [LM1], and the trace of the infinite dimensional matrices (continuum limit) is given by a local formula, the eq. (21) in the case \( 2n = 2 \).

**3. HAMILTONIAN FORMULATION: ANOMALY OF THE CURRENT ALGEBRA**
The same finite lattice approximation can be used also in the real time hamiltonian formulation for fermions in background gauge fields. The hamilton operator $D_A$ in $2n-1$ dimensions is defined exactly in the same way as the $2n$ Dirac operator earlier,

\begin{equation}
D_A = \gamma_0 \gamma_k (p_k + A_k) = \alpha_k (p_k + A_k),
\end{equation}

sum over $k = 1, 2, \ldots, 2n - 1$. The dimension of the space lattice is now $K = 2^n \cdot N \cdot (2q + 1)^{2n-1}$, in case of Dirac fermions; for Weyl fermions the hamiltonian above must be multiplied by the chiral projection $P_+$ and the dimension of the projected subspace is $K = 2^n - 1 \cdot N \cdot (2q + 1)^{2n-1}$.

We shall consider the gauge currents for massless fermions. The left and right components of fermions decouple and we may restrict to (left-handed) Weyl fermions. The current algebra for Dirac fermions becomes just the direct sum of left and right current algebras.

In continuum the current algebra is anomalous. There are Schwinger terms which in the case $2n - 1 = 1$ can be written as

\begin{equation}
[\rho(X), \rho(Y)] = \rho([X, Y]) + \frac{i}{2\pi} \int_M \text{tr} X(x)Y'(x)dx,
\end{equation}

where the integral is over the one dimensional space $M$. When $M$ is a unit circle (25) gives an affine Kac-Moody algebra. Here $\rho(X)$ denotes the charge density integrated with a smooth Lie algebra valued test function $X$,

\[ \rho(X) = \int_M \rho_k(x)X_k(x)dx, \]

where $k$ is a Lie algebra index, $k = 1, 2, \ldots, \dim \mathfrak{g}$. The trace under the integral sign in (25) refers to the representation of the gauge group acting on fermion components. In the case $2n - 1 = 3$ one has, [M1], [F-Sh],

\begin{equation}
[\rho(X), \rho(Y)] = \rho([X, Y]) + \frac{1}{24\pi^2} \int_M \text{tr} A(dX, dY),
\end{equation}

where the 3-form under the integral is defined as an exterior product of the 1-forms $A, dX, dY$.

There are alternative, but equivalent, formulas for the Schwinger terms. Equivalent means again that the difference between the Schwinger terms is a gauge variation, of the type $f, \theta(A; X) = f, \theta(A; X) - \theta(A; [X, Y])$ for some function $\theta$ of $A$. 
and \(X\), linear in the latter argument. In the case \(2n - 1 = 1\) one has in fact an exact formula, [L],

\[
(27) \quad c(X, Y) = \frac{1}{4} \text{tr}\langle \epsilon, X]\epsilon, Y\rangle,
\]

where \(\epsilon\) is the sign of the free 1-particle hamilton operator. In three space dimensions we have, [MR],[LM1],

\[
(28) \quad c(A; X, Y) = \frac{1}{8} \text{tr}_C(\epsilon - F)[[\epsilon, X], [\epsilon, Y]],
\]

where now \(F\) is the sign of the 1-particle hamiltonian \(D_A\) in the external gauge field \(A\). The conditional trace is defined as \(\text{tr}_C K = \frac{1}{2} \text{tr}(\epsilon K + K\epsilon)\).

These formulas, and the corresponding formulas in higher dimensions, can be directly translated to the lattice. When defining the sign operators \(F, \epsilon\) one should be careful in order to have the right continuum limit. We have to split (somewhat artificially) the energy spectrum even in the finite case to positive and negative parts. As before we define the momentum components as \(p_i = \frac{2\pi}{2q\Delta} k\) with \(k = 0, \pm 1, \pm 2, \ldots, \pm q\) for some positive integer \(q\). The energy eigenvalues for the free hamiltonian become then \(E = \pm (p_ip_i)^{1/2}\) and so the spectrum is symmetric around zero.

In the case of periodic boundary conditions (in \(x\) space) in the continuum theory it is simple to prove that one obtains the right continuum limit as the size of the momentum space lattice is increased, that is, when the cut-off \(|E| \leq \Lambda\) is removed. This follows again from a standard Hölder inequality argument. The trace in (28) is convergent, it is known that the operators \([\epsilon, X], [\epsilon, Y]\) belong to the Schatten ideal \(L_4\) of operators \(T\) such that \(|T|^4\) is trace-class; furthermore, the diagonal blocks \(\epsilon(\epsilon - F) + (\epsilon - F)\epsilon\) of \(\epsilon - F\) (the only part of the operator which contributes to (28)) are Hilbert-Schmidt. All these operators can be approximated by finite-dimensional matrices (in the appropriate \(L_k\) norms) and therefore the trace of the product is a limit of traces of finite-dimensional (cut-off) matrices.

We have discussed above the abstract commutation relations of the current algebra but we have said nothing about an operator realization of the currents. In \(1 + 1\) dimensions the theory is well understood; a physically acceptable realization is obtained using highest weight representations of affine algebras. An important example is the basic representation which is a representation in a fermionic Fock
space. This construction has been generalized to the $3 + 1$ dimensional case (and the method works in higher dimensions), [M2]. That representation is also suitable for a lattice approximation, as will be briefly explained below.

The basic idea is to define a continuous family $T_A$ of unitary conjugations in the one-particle fermionic Hilbert space such that the off-diagonal blocks (with respect to the energy polarization $\epsilon$) of currents are reduced such that the resulting (unitarily equivalent) Gauss law generators can be quantized by canonical methods. More precisely, we have to require that

$$[\epsilon, \theta(X; A)]$$ is Hilbert-Schmidt,

where $X$ is any infinitesimal gauge transformation and

$$\theta(X; A) = T_A^{-1}(\delta_X + X)T_A - \delta_X.$$ 

Note that the modified Gauss law generators $\tilde{G}_X = T_A^{-1}G_XT_A$, with $G_X = \delta_X + X$, automatically satisfy the same commutation relations as the generators $X$. The construction of the operators $T_A$ is best understood through an (asymptotic) expansion in powers of the inverse momenta $1/p$. In three space dimensions the first terms of the symbol are, [M2],

$$T_A = 1 - \frac{1}{4p} [\hat{p}, A] \frac{1}{p} + \ldots$$

and the resulting gauge currents

$$\theta(X; A) = X + \frac{i}{4p} [\hat{p}, \sigma_{\mu}] \partial_{\mu}X \frac{1}{p} + \ldots,$$

where we have used the hermitean Pauli matrices $\sigma_{\mu}$ as the 3-space Dirac matrices, $\hat{p} = \sigma_{\mu}p_{\mu}$. In all formulas it is implicitly assumed that an infrared regularization $1/|p| \mapsto 1/(|p| + \lambda)$ is performed. The terms which are of order strictly lower than $-1$ in momenta are Hilbert-Schmidt and therefore not critical for the current renormalization. With the renormalized current operators in hand, one can compute the quantum commutation relations in a straight-forward way. The resulting Schwinger term was calculated in [M2] and found to be equivalent with (26).

These formulas can be translated immediately to the lattice. One replaces the derivatives $-\partial_{\mu}X$ by $[p_{\mu}, X]$ and the momentum symbols in the PSDO’s become multiplication operators by the discrete momentum variables.
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