ON A CERTAIN SUM OF THE DERIVATIVES OF
DIRICHLET L-FUNCTIONS

HIROTAKA KOBAYASHI

Abstract. We consider a sum of the derivatives of Dirichlet L-functions over the zeros of Dirichlet L-functions. We give an asymptotic formula for the sum.

1. Introduction

Let $s = \sigma + it$ denote a complex variable. The Dirichlet L-function attached to $\chi$ is defined by

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \quad (\sigma > 1),$$

where $\chi(n)$ is a Dirichlet character modulo $q$. For $\chi \pmod{1}$ we get the Riemann $\zeta$-function $L(s, \chi) = \zeta(s)$. The Generalized Riemann Hypothesis (GRH) states that all zeros of every Dirichlet L-function in the strip $0 < \sigma < 1$ lie on the line $\sigma = 1/2$. We denote the zeros in the strip $0 < \sigma < 1$ by $\rho_{\chi} = \beta_{\chi} + i\gamma_{\chi}$. A Dirichlet character is said to be primitive when it is not induced by any other character of modulus strictly less than $q$. The unique principal character modulo $q$ is denoted by $\chi_0$. When $\chi = \chi_0$, we have $L(s, \chi_0) = \zeta(s) \prod_{p \mid q} (1 - p^{-s})$, where, and in what follows, $p$ denotes a prime number. For a Dirichlet character $\chi \pmod{q}$ the Gauss sum is defined by

$$\tau(\chi) = \sum_{a=1}^{q} \chi(a) \exp \left( \frac{2\pi i a}{q} \right).$$

For a primitive character $\chi \pmod{q}$ we have $|\tau(\chi)| = \sqrt{q}$.

In this paper, $T$ is a positive number which always tends to $+\infty$ and $\varepsilon > 0$. Our main theorem is

2000 Mathematics Subject Classification. 11M06, 11M26.

Key words and phrases. Dirichlet L-function, Derivative, Zeros.
**Theorem 1.1.** Let $c_1$ be a positive constant. Let $\chi \pmod{q}$ be a primitive character. Then, uniformly for $q \leq \exp(c_1 \sqrt{\log T})$, we have

$$
\sum_{0 < \gamma \leq T} L'(\rho, \chi) = \frac{1}{4\pi} T \left( \frac{\log qT}{2\pi} \right)^2 + a_1 \frac{T}{2\pi} \log qT + a_2 \frac{T}{2\pi} + a_3 + O \left( T \exp \left( -c \sqrt{\log T} \right) \right),
$$

where the implicit constant is absolute, $c$ is a positive absolute constant depends on $c_1$ and

$$
a_1 = \sum_{p \mid q} \frac{\log p}{p - 1} + \gamma_0 - 1,
$$

$$
a_2 = \frac{1}{2} \left( \sum_{p \mid q} \frac{\log p}{p - 1} \right)^2 + (\gamma_0 - 1) \sum_{p \mid q} \frac{\log p}{p - 1} - \frac{3}{2} \sum_{p \mid q} p \left( \frac{\log p}{p - 1} \right)^2 + 1 - \gamma_0 - \gamma_0^2 + 3\gamma_1
$$

with the stieltjes constants $\gamma_0, \gamma_1$ and

$$
a_3 = \frac{\omega \chi(-1) \tau(\chi) \tau(\overline{\chi})}{q \varphi(q)} \frac{L'(\beta, \omega)}{\beta} \left( \frac{qT}{2\pi} \right)^\beta
$$

when $L(s, \omega)$ with a quadratic character $\omega \pmod{q}$ has an exceptional zero $\beta$, otherwise $a_3 = 0$.

Assuming the GRH, we can replace the error term by $(qT)^{\frac{1}{4} + \varepsilon}$ uniformly for $q \ll T^{1-\varepsilon}$.

**Remark 1.** Let $q$ be a prime power. If we could obtain the estimate

$$(1) \quad \sum_{\gamma \chi \leq T} |L'(\rho, \chi)|^2 \ll T (\log qT)^4,$$

where the implicit constant is absolute, we could replace the error term by $\sqrt{qT (\log qT)^2}$ under the GRH. We will give the details at the last section. In view of Gonek’s formula \cite{12}, the above estimate \((1)\) may be plausible.

When $q = 1$, the above theorem implies Fujii’s Theorem 1 in \cite{2}. Our proof is a generalization of his method. However, it is not easy to obtain his Theorem 2 in \cite{2} and we give a weaker statement. Kaptan, Karabulut and Yıldırım \cite{6} consider more general cases and give the
asymptotic formula, that is for \( \mu \geq 1 \) and \( q \leq (\log T)^A \) with any fixed \( A > 0 \)

\[
\sum_{0 \leq \gamma \leq T} L^{(\mu)}(\rho, \chi) = \frac{(-1)^\mu}{\mu + 1} T \left( \log \frac{qT}{2\pi} \right)^{\mu+1} + O(T(\log T)^{\mu+\varepsilon})
\]

for any fixed \( \varepsilon > 0 \). Our result is the case \( \mu = 1 \) in their paper and gives a more sophisticated formula. Jakhliouti and Mazhouda [5] consider the sum

\[
\sum_{0 < \gamma \leq T} L'(\rho_{a,x}, \chi)X^{\rho_{a,x}},
\]

where \( \rho_{a,x} = \beta_{a,x} + i\gamma_{a,x} \) are the zeros of \( L(s, \chi) - a \) for any fixed complex number \( a \) and \( X \) is a fixed positive number. They also fix \( \chi \) throughout their paper. Hence our main theorem treats a special case of their sum, but our result gives a more precise form because we do not fix \( \chi \).

2. Preliminaries

The Dirichlet \( L \)-function attached to a primitive character \( \chi \mod q \) satisfies the functional equation

\[
L(s, \chi) = \Delta(s, \chi)L(1 - s, \overline{\chi}),
\]

where

\[
\Delta(s, \chi) = \varepsilon(\chi)2^{s}{\pi}^{s-1}{q}^{\frac{s}{2}}\Gamma(1 - s)\sin \frac{\pi}{2}(s + \kappa)
\]

when we put

\[
\kappa = \frac{1 - \chi(-1)}{2}
\]

and

\[
\varepsilon(\chi) = \frac{\tau(\chi)}{i^\kappa\sqrt{q}}.
\]

We note that \( \Delta(s, \chi) \) is a meromorphic function with only real zeros and poles satisfying the functional equation

\[
\Delta(s, \chi)\Delta(1 - s, \overline{\chi}) = 1.
\]

By Stirling’s formula, we can show that
Lemma 2.1. For $-1 \leq \sigma \leq 2$ and $t \geq 1$, we have

$$\Delta(1 - s, \chi) = \frac{\tau(\chi)}{\sqrt{q}} e^{-\frac{\pi i}{4}} \left( \frac{qt}{2\pi} \right)^{\sigma - \frac{1}{2}} \exp \left( it \log \frac{qt}{2\pi e} \right) \left( 1 + O \left( \frac{1}{t} \right) \right)$$

and

$$\Delta'(s, \chi) = -\log \frac{qt}{2\pi} + O \left( \frac{1}{t} \right).$$

A theorem from [4] and an application of the Phragmen-Lindelöf principle yields the estimate

$$L(s, \chi) \ll (q(|t| + 2)) \frac{1}{\log qT}$$

for $\frac{1}{2} \leq \sigma \leq 1 + \frac{1}{\log qT}$,

$$L(s, \chi) \ll (q(|t| + 2)) \frac{1}{\log qT}$$

for $-1 \leq \sigma \leq 2$, $|t| \geq 1$ uniformly in $|t| \ll T$ for any non-principal Dirichlet character $\chi \pmod{q}$. When we assume the GRH, the bound of (5) can be replaced by $(q(|t| + 2))^\varepsilon$. For the principal character, we need the restriction $|s - 1| \gg 1$ in (5). For the logarithmic derivative it is known that for $q \geq 1$ and $\chi \pmod{q}$

$$\frac{L'}{L}(s, \chi) = \sum_{|t - \rho\chi| \leq 1} \frac{1}{s - \rho\chi} + O(\log q(|t| + 2))$$

for $-1 \leq \sigma \leq 2$, $|t| \geq 1$ (see [9, p. 225]). For $q \geq 1$, $\chi \pmod{q}$ and $t \geq 0$ we have (see [9, p. 220])

$$N(t + 1, \chi) - N(t, \chi) := \# \{ \rho\chi = \beta\chi + i\gamma\chi : t < \gamma\chi \leq t + 1 \} \ll \log q(t + 2).$$

Hence for any $T_0 \geq 0$, there exists a $t = t(\chi)$, $t \in (T_0, T_0 + 1]$, such that

$$\min_{\gamma\chi} |t - \gamma\chi| \gg \frac{1}{\log q(t + 2)}.$$

By the expression (7), it follows that for $q \geq 1$, $\chi \pmod{q}$ and $t$ satisfying (9)

$$\frac{L'}{L}(\sigma + it, \chi) \ll (\log q(|t| + 2))^2$$

for $-1 \leq \sigma \leq 2$. 
uniformly. This estimate is valid for $|s - \rho_\chi| \gg (\log(q(|t| + 2)))^{-1}$ though $t$ is not satisfying (9).

We will apply the following approximate functional equation for $L(s, \chi)$.

**Lemma 2.2** (A. F. Lavrik [7]). We let $0 \leq \sigma \leq 1$, $2\pi xy = t$, $x \geq 1$ and $y \geq 1$. Then for $t > 0$, we get

$$L(s, \chi) = \sum_{n \leq x} \frac{\chi(n)}{n^s} + \Delta(s, \chi) \sum_{n \leq y} \frac{\chi(n)}{n^{1-s}}$$

$$+ O \left( \sqrt{q} \left( y^{-\sigma} + x^{\sigma-1}(qt)^{\frac{1}{2}-\sigma} \right) \log 2t \right).$$

On the other hand, for $t > t_0 > 0$ and $\sigma > 1$, using partial summation, we get

$$L(s, \chi) = \sum_{n \leq qt} \frac{\chi(n)}{n^s} + O \left( \frac{|s|}{\sigma} (qt)^{-\sigma} \right).$$

We will use the following modified Gonek’s lemma ([3, Lemma 5]).

**Lemma 2.3.** Let $\{b_n\}_{n=1}^\infty$ be a sequence of complex numbers such that $b_n \ll n^\epsilon$ for any $\epsilon > 0$. Let $a > 1$ and let $m$ be a non-negative integer. Then for any sufficiently large $T$,

$$\int_1^T \left( \sum_{n=1}^\infty \frac{b_n}{n^{a+it}} \right) \Delta(1-a-it, \chi) \left( \log \frac{qt}{2\pi} \right)^m dt$$

$$= \frac{\tau(\chi)}{q} \sum_{1 \leq n \leq qT/2\pi} b_n e \left( -\frac{n}{q} \right) (\log n)^m + O \left( \sum_{n=1}^\infty \frac{b_n}{n^m} \right) \left( qT^{a-1/2}(\log qT)^m \right).$$

This is provided implicitly by Steuding in [10].

### 3. Proof in the unconditional case

In this section we prove the claim of the unconditional part of Theorem [11]. Let $(\log 2q)^{-1} \ll b \leq 1$ and $T \geq 2$ be such that

$$\min_{\gamma_\chi} |b - \gamma_\chi| \gg \frac{1}{\log 2q} \quad \text{and} \quad \min_{\gamma_\chi} |T - \gamma_\chi| \gg \frac{1}{\log qT}.$$
\[
\sum_{0 < \gamma \chi \leq T} L'(\rho, \chi) = \frac{1}{2\pi i} \int_{C} \frac{L'(s, \chi)L'(s, \chi)}{L(s, \chi)} ds + E,
\]
where \(E\) consists of the terms \(L'(\rho, \chi)\) with \(0 < \gamma \chi < b\).

For zeros \(\rho = \beta + i\gamma \chi\) with \(0 < \gamma \chi < b\) we have
\[
L'(\rho, \chi) \ll q^{\frac{1}{2}} (\log 2q)^2
\]
by (5), (6) and the Cauchy’s integral formula applied to the circle with centre \(\rho\) and radius \((\log 2q)^{-1}\). Therefore, by (8), we have
\[
E = \sum_{0 < \gamma \chi < b} L'(\rho, \chi) \ll q^{\frac{1}{2}} (\log 2q)^2 \sum_{0 < \gamma \chi < b} 1 \ll q^{\frac{1}{2}} (\log 2q)^3.
\]

Next we consider the contour integral
\[
\frac{1}{2\pi i} \int_{C} \frac{L'(s, \chi)L'(s, \chi)}{L(s, \chi)} ds
\]
\[
= \frac{1}{2\pi i} \left\{ \int_{a+ib}^{a+iT} + \int_{1-a+iT}^{1-a+ib} + \int_{a+iT}^{a+ib} + \int_{1-a+ib}^{1-a+iT} \right\} \frac{L'(s, \chi)L'(s, \chi)}{L(s, \chi)} ds
\]
\[
= I_1 + I_2 + I_3 + I_4,
\]
say.

By the Laurent expansion of the Riemann \(\zeta\)-function, it is easily seen that
\[
I_1 = \frac{1}{2\pi} \int_{b}^{T} \frac{L'(a + it, \chi)L'(a + it, \chi)}{L(a + it, \chi)} dt
\]
\[
= \frac{1}{2\pi} \sum_{m=2}^{\infty} \sum_{n=2}^{\infty} \chi(m)\Lambda(m)\chi(n) \log n \int_{b}^{T} \frac{dt}{(mn)^{it}}
\]
\[
\ll \left| \frac{\zeta'(a)}{\zeta(a)} \right| |\zeta'(a)| \ll (\log qT)^3,
\]
where \(\Lambda(m)\) is the von-Mangoldt function. To estimate the integral on the horizontal line, we will show the following lemma.

**Lemma 3.1.** Let \(\chi\) be a primitive character, then
\[
\int_{1-a}^{a} L'(\sigma + iT, \chi) d\sigma \ll \sqrt{qT} \log qT.
\]

**Proof.** Let
\[
\delta = \frac{1}{\log qT}.
\]
A CERTAIN SUM OF THE DERIVATIVES OF DIRICHLET $L$-FUNCTIONS

Then $L(w, \chi)$ is analytic on the disk $|s - w| \leq \delta$, for $s = \sigma + iT$ with $1 - a \leq \sigma \leq a$. Therefore, by Cauchy’s integral formula,

$$L'(s, \chi) = \frac{1}{2\pi i} \int_{|s-w|=\delta} \frac{L(w, \chi)}{(s-w)^2} dw$$

$$\ll \log qT \int_0^{2\pi} |L(s + \delta e^{i\theta}, \chi)| d\theta.$$

Thus it suffices to prove that

$$\int_{1-a}^a \int_0^{2\pi} |L(s + \delta e^{i\theta}, \chi)| d\theta d\sigma = \int_0^{2\pi} \int_{1-a}^a |L(s + \delta e^{i\theta}, \chi)| d\sigma d\theta \ll \sqrt{qT}.$$

From the functional equation and, for $1 - a \leq \sigma \leq 1/2$, we have

$$\int_{1-a}^{1/2} |L(s + \delta e^{i\theta}, \chi)| d\sigma$$

$$= \int_{1-a}^{1/2} |\Delta(s + \delta e^{i\theta}, \chi)L(1 - s - \delta e^{i\theta}, \overline{\chi})| d\sigma$$

$$= \int_{1/2}^{a} |\Delta(1 - \sigma + iT + \delta e^{i\theta}, \chi)L(\sigma - iT - \delta e^{i\theta}, \overline{\chi})| d\sigma.$$

On the second equality, we change the variable $\sigma$ to $1 - \sigma$. Since

$$\Delta(1 - (\sigma - iT - \delta e^{i\theta}), \chi)$$

$$= \Delta(1 - (\sigma + iT - \delta e^{-i\theta}), \overline{\chi})$$

$$= \tau(\chi) \frac{e^{\pi i}}{\sqrt{q}} \left( \frac{qT}{2\pi} \right)^{\sigma - \delta \cos \frac{\theta}{2}} \exp \left( iT \log \frac{qT}{2\pi e} \right) \left( 1 + O \left( \frac{1}{T} \right) \right)$$

by Lemma 2.1, the integral can be bounded by

$$\int_{1/2}^{a} (qT)^{\sigma - \delta \cos \frac{\theta}{2}} |L(\sigma + iT - \delta e^{-i\theta}, \chi)| d\sigma.$$

Therefore we obtain
\[
\int_{1-a}^{a} |L(s + \delta e^{i\theta}, \chi)|d\sigma \\
\ll \int_{\frac{1}{2}}^{a} |L(\sigma + iT + \delta e^{i\theta}, \chi)|d\sigma \\
+ \int_{\frac{1}{2}}^{a} (qT)^{\sigma - \delta \cos \frac{\theta}{2}} |L(\sigma + iT - \delta e^{-i\theta}, \chi)|d\sigma \\
\ll \int_{\frac{1}{2}}^{a} (qT)^{\sigma - \frac{1}{2}} |L(\sigma + iT \pm \delta e^{\pm i\theta}, \chi)|d\sigma.
\]

On the last inequality, we use the facts that
\[
(qT)^{\delta} = e
\]
with \(\delta = (\log qT)^{-1}\). This integral is
\[
= \left\{ \int_{\frac{1}{2}}^{1} + \int_{1}^{a} \right\} (qT)^{\sigma - \frac{1}{2}} |L(\sigma + iT \pm \delta e^{\pm i\theta}, \chi)|d\sigma \\
= S_1 + S_2,
\]
say. Using Lemma 2.2 we have
\[
S_1 \ll (qT)^{-\frac{1}{2}} \sum_{n \ll \sqrt{qT}} n^{\delta} \int_{\frac{1}{2}}^{1} \left( \frac{qT}{n} \right)^{\sigma} d\sigma + \sum_{n \ll \sqrt{qT}} n^{\delta - 1} \int_{\frac{1}{2}}^{1} n^{\sigma} d\sigma \\
+ \sqrt{q \log 2T} \int_{\frac{1}{2}}^{1} (qT)^{\sigma + \frac{\delta - 1}{2}} d\sigma \ll \sqrt{qT}.
\]

On the other hand, by (11), we get
\[
S_2 \ll (qT)^{-\frac{1}{2}} \sum_{n \leq \frac{qT}{2}} n^{\delta} \int_{1}^{a} \left( \frac{qT}{n} \right)^{\sigma} d\sigma + \sqrt{qT} \int_{1}^{a} \frac{d\sigma}{\sigma} \\
\ll \sqrt{qT}.
\]

Hence we complete the proof. \(\square\)

By (10) and the above lemma, we get
\[
I_3 + I_4 \ll (\log qT)^2 \int_{1-a}^{a} |L'(\sigma + iT, \chi)|d\sigma \\
\ll \sqrt{qT}(\log qT)^3.
\]

Now we consider \(I_2\). By the functional equation, we have
\[
\frac{L'}{L}(1 - a + it, \chi)L'(1 - a + it, \chi) \\
= \left(\frac{\Delta'}{\Delta}(1 - a + it, \chi) - \frac{L'}{L}(a - it, \overline{\chi})\right) \\
\times (\Delta'(1 - a + it, \chi)L(a - it, \overline{\chi}) - \Delta(1 - a + it, \chi)L'(a - it, \overline{\chi})) \\
= \frac{\Delta'}{\Delta}(1 - a + it, \chi)\Delta'(1 - a + it, \chi)L(a - it, \overline{\chi}) \\
- 2\Delta'(1 - a + it, \chi)L'(a - it, \overline{\chi}) \\
+ \Delta(1 - a + it, \chi)L'(a - it, \overline{\chi}).
\]

Thus we can divide \( I_2 \) into the following three integrals:

\[
I_2 = \frac{1}{2\pi} \int_{b}^{T} \frac{L'}{L}(1 - a + it, \chi)L'(1 - a + it, \chi)dt \\
= \frac{1}{\pi} \int_{b}^{T} \Delta'(1 - a + it, \chi)L'(a - it, \overline{\chi})dt \\
- \frac{1}{2\pi} \int_{b}^{T} \Delta'(1 - a + it, \chi)\Delta'(1 - a + it, \chi)L(a - it, \overline{\chi})dt \\
- \frac{1}{2\pi} \int_{b}^{T} \Delta(1 - a + it, \chi)L'(a - it, \overline{\chi})dt \\
= J_1 + J_2 + J_3,
\]
say. We take complex conjugates of \( J_i \) \((i = 1, 2, 3)\) to apply Lemma 2.3. Then we have

\[
\mathcal{J}_1 = \frac{1}{\pi} \int_{b}^{T} \Delta'(1 - a + it, \chi)L'(a - it, \overline{\chi})dt \\
= \frac{1}{\pi} \int_{b}^{T} \Delta'(1 - a - it, \overline{\chi})L'(a + it, \chi)dt \\
= -\frac{1}{\pi} \int_{b}^{T} L'(a + it, \chi)\Delta(1 - a - it, \overline{\chi})\log \frac{qt}{2\pi} dt \\
+ O\left(\sum_{n=1}^{\infty} \frac{\log n}{n^a} \int_{b}^{T} \frac{(qt)^{a-\frac{1}{2}}}{t} dt\right) \\
= \frac{1}{\pi} \int_{b}^{T} \sum_{n=1}^{\infty} \frac{\chi(n)\log n}{n^{a+it}} \Delta(1 - a - it, \overline{\chi})\log \frac{qt}{2\pi} dt + O\left((qT)^{a-\frac{1}{2}}(\log qT)^2\right)
\]
On the third equality, we use the approximation (4). For convenience, we put \( x = \frac{qT}{2\pi} \). By partial summation, the above sum can be calculated as

\[
\sum_{1 \leq n \leq x} \chi(n) e \left( -\frac{n}{q} \right) (\log n)^2
\]

\[
= (\log x)^2 \sum_{m=1}^{q} \chi(m) e \left( -\frac{m}{q} \right) \sum_{n \leq x \mod m} 1
\]

\[
- 2 \int_{1}^{x} \left( \sum_{m=1}^{q} \chi(m) e \left( -\frac{m}{q} \right) \sum_{n \leq y \mod m} 1 \right) \log y \frac{dy}{y}
\]

\[
= \left( \frac{x}{q} \chi(-1) \tau(\chi) + O(\sqrt{q}) \right) (\log x)^2 - 2 \int_{1}^{x} \left( \frac{y}{q} \chi(-1) \tau(\chi) + O(\sqrt{q}) \right) \frac{\log y}{y} dy
\]

\[
= \frac{\chi(-1) \tau(\chi)}{q} (x(\log x)^2 - 2 \int_{1}^{x} \log y dy) + O \left( \sqrt{q}(\log x)^2 + \sqrt{q} \int_{1}^{x} \log y dy \right)
\]

\[
= \frac{\chi(-1) \tau(\chi)}{q} (x(\log x)^2 - 2x \log x + 2x) + O \left( \sqrt{q}(\log x)^2 \right),
\]

and we can see that

\[
\frac{\chi(-1) \tau(\chi) \tau(\chi)}{q^2} = \frac{\tau(\chi) \tau(\chi)}{q^2} = \frac{q}{q^2} = \frac{1}{q}.
\]

Therefore we obtain

\[
J_1 = 2 \left( \frac{T}{2\pi} \left( \log \frac{qT}{2\pi} \right)^2 - \frac{T}{2\pi} \log \frac{qT}{2\pi} + \frac{T}{\pi} \right) + O \left( (qT)^{a-\frac{1}{2}} (\log qT)^3 \right).
\]

Next we consider \( J_2 \). We have, by (4) again,
\[ J_2 = -\frac{1}{2\pi} \int_b^T \frac{\Delta'}{\Delta} (1 - a + it, \chi) \Delta'(1 - a + it, \chi) L(a - it, \chi) dt \]
\[ = -\frac{1}{2\pi} \int_b^T L(a + it, \chi) \frac{\Delta'}{\Delta} (1 - a - it, \chi) \Delta'(1 - a - it, \chi) dt \]
\[ = -\frac{1}{2\pi} \int_b^T \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{a+it}} \Delta(1 - a - it, \chi) \left( \log \frac{qt}{2\pi} \right)^2 dt \]
\[ + O \left( \sum_{n=1}^{\infty} \frac{1}{n^a} \int_b^T (qt)^{a-\frac{1}{2}} \log qt \frac{dt}{t} \right) \]
\[ = -\frac{\tau(\chi)}{q} \sum_{1 \leq n \leq qT/2\pi} \chi(n) e \left( -\frac{n}{q} \right) (\log n)^2 + O \left( (qT)^{a-\frac{1}{2}} (\log qT)^3 \right). \]

This sum is the same as the previous one. Hence we get
\[ J_2 = - \left( \frac{T}{2\pi} \left( \log \frac{qT}{2\pi} \right)^2 - \frac{T}{\pi} \log \frac{qT}{2\pi} + \frac{T}{\pi} \right) + O \left( (qT)^{a-\frac{1}{2}} (\log qT)^3 \right). \]

Finally, we calculate \( J_3 \). We have
\[ J_3 = -\frac{1}{2\pi} \int_b^T \Delta(1 - a + it, \chi) \frac{L'}{L}(a - it, \chi) L'(a - it, \chi) dt \]
\[ = -\frac{1}{2\pi} \int_b^T \frac{L'}{L} (a + it, \chi) L'(a + it, \chi) \Delta(1 - a - it, \chi) dt \]
\[ = -\frac{1}{2\pi} \int_b^T \left( \sum_{n=1}^{\infty} \frac{\chi(n) \Lambda(n)}{n^{a+it}} \right) \left( \sum_{n=1}^{\infty} \frac{\chi(n) \log n}{n^{a+it}} \right) \Delta(1 - a - it, \chi) dt \]
\[ = -\frac{\tau(\chi)}{q} \sum_{1 \leq mn \leq qT/2\pi} \chi(mn) e \left( -\frac{mn}{q} \right) \Lambda(m) \log n + O \left( (qT)^{a-\frac{1}{2}} (\log qT)^3 \right). \]

By the orthogonality of Dirichlet characters, we see that
\[ \sum_{mn \leq x} \chi(m) \chi(n) e \left( -\frac{mn}{q} \right) \Lambda(m) \log n \]
\[ = \sum_{a=1}^{q} \chi(a) \sum_{b=1}^{q} \chi(b) e \left( -\frac{ab}{q} \right) \sum_{mn \leq x \atop m \equiv a \mod q \atop n \equiv b \mod q} \Lambda(m) \log n \]
\[
\sum_{mn \leq x} \psi(m)\psi'(n)\Lambda(m) \log n.
\]

We will divide the sum into four parts, according to the following conditions:

(i) \(\psi = \psi_0, \ \psi' = \psi'_0\),

(ii) \(\psi = \psi_0, \ \psi' \neq \psi'_0\),

(iii) \(\psi \neq \psi_0, \ \psi' = \psi'_0\),

(iv) \(\psi \neq \psi_0, \ \psi' \neq \psi'_0\),

where \(\psi_0 = \psi'_0\) is the principal character modulo \(q\). Before discussing further, we will remind some facts on the sum of Dirichlet characters (see \[1, \text{Sec. 8}\]). We define \(G(n, \chi)\) as

\[
G(n, \chi) := \sum_{a=1}^{q} \chi(a)e \left( \frac{an}{q} \right).
\]

If a Dirichlet character \(\chi\) (mod \(q\)) is primitive, then we have

\[
G(a, \chi) = \overline{\chi(a)}\tau(\chi).
\]

Now we consider the above four parts.

(i) In this case, we have

\[
\frac{1}{\varphi(q)^2} \sum_{a=1}^{q} \chi(a) \sum_{b=1}^{q} \chi(b)e \left( -\frac{ab}{q} \right) \sum_{mn \leq x} \psi_0(m)\psi_0(n)\Lambda(m) \log n
\]

\[
= \frac{1}{\varphi(q)^2} \sum_{a=1}^{q} \chi(a)G(-a, \chi) \sum_{mn \leq x} \psi_0(m)\psi_0(n)\Lambda(m) \log n
\]

\[
= \frac{\chi(-1)\tau(\chi)}{\varphi(q)} \sum_{mn \leq x} \psi_0(m)\psi_0(n)\Lambda(m) \log n.
\]

By Perron's formula we get

\[
\sum_{mn \leq x} \psi_0(m)\Lambda(m)\psi_0(n) \log n
\]

\[
= \frac{1}{2\pi i} \int_{a-iU}^{a+iU} \frac{L'(s, \psi_0)L'(s, \psi_0)}{L(s, \psi_0)} \frac{x^s}{s} ds + R,
\]
where \( R \) is the error term appearing in Perron’s formula (see \([8, \text{p.}140]\)) and satisfies that
\[
R \ll \sum_{\frac{x}{2} < mn < x \atop mn \neq x} |\Lambda(m) \log n| \min \left(1, \frac{x}{U|x-mn|} \right)
\]
\[
+ \frac{(4x)^a}{U} \sum_{mn=1}^{\infty} |\Lambda(m) \log n| (mn)^a.
\]
We will choose an appropriate \( U \) later. The first term of the error term \( R \) can be estimated as follows;
\[
\frac{x}{U} \sum_{\frac{x}{2} < mn \leq x-1} \frac{\Lambda(m) \log n}{x-mn} + \sum_{x-1 < mn \leq x+1} \Lambda(m) \log n
\]
\[
\ll \frac{x}{U} \log x \sum_{m < x-1} \frac{\Lambda(m)}{m} \sum_{x-1 < n < x-1-m} \frac{1}{x-m-n}
\]
\[
+ (\log x)^2 \sum_{x-1 \leq l \leq x+1} \sum_{l=mn} 1
\]
\[
+ \frac{x}{U} \log x \sum_{m < 2x} \frac{\Lambda(m)}{m} \sum_{x+1 < n < 2x} \frac{1}{x-m-n}
\]
\[
\ll \frac{x}{U} (\log x)^2 \sum_{m < 2x} \frac{\Lambda(m)}{m} + (\log x)^2 \sum_{x-1 \leq l \leq x+1} d(l)
\]
\[
\ll \frac{x}{U} (\log x)^3 + x^\varepsilon,
\]
where \( d(l) \) is the divisor function. On the last estimates, we use
\[
\sum_{m \leq x} \frac{\Lambda(m)}{m} = \log x + O(1)
\]
and
\[
d(x) \ll x^\varepsilon.
\]
The second is
\[
\ll \frac{(4x)^a}{U} \sum_{mn=1}^{\infty} |\Lambda(m) \log n| (mn)^a \ll \frac{x^a}{U} (\log qT)^3.
\]
Therefore
\[
R \ll \frac{x}{U} (\log x)^3 + x^\varepsilon.
\]
Since $L(s, \psi_0) = \zeta(s) \prod_{p \mid q} (1 - p^{-s})$, there is an absolute constant $C > 0$ such that

$$L(s, \psi_0) \neq 0 \quad \text{for} \quad \sigma \geq 1 - \frac{C}{\log(|t| + 2)}$$

(see [8, p.172]). With regard to this zero-free region for $L(s, \psi_0)$, let $a' = 1 - C/\log U$ and $U = \exp (4c_1 \sqrt{\log qT})$. By the residue theorem, the integral is

$$\frac{1}{2\pi i} \int_{a-iU}^{a+iU} \frac{L'(s, \psi_0)}{L(s, \psi_0)} \frac{x^s}{s} ds \times$$

$$= \text{Res}_{s=1} \frac{L'(s, \psi_0)}{L(s, \psi_0)} \frac{x^s}{s} + \frac{1}{2\pi i} \left\{ \int_{a+iU}^{a'+iU} + \int_{a'+iU}^{a-iU} + \int_{a-iU}^{a+iU} \right\} \frac{L'(s, \psi_0)}{L(s, \psi_0)} \frac{x^s}{s} ds.$$

By an argument similar to the proof of Lemma 3.1 we can see that the integral on the horizontal line can be estimated as

$$\int_{a\pm iU}^{a'\pm iU} \frac{L'(s, \psi_0)}{L(s, \psi_0)} \frac{x^s}{s} ds \ll \frac{(\log qU)^3}{U} a''(qU)^{\frac{3}{16}+\varepsilon} (a - a')$$

$$\ll xU^{-\frac{1}{2}} = x \exp \left( -2c_1 \sqrt{\log x} \right),$$

noting the condition $q \leq \exp (c_1 \sqrt{\log T}) \leq \exp (4c_1 \sqrt{\log qT}) = U$ and (10). Since $L'/L(s, \psi_0) \ll |s - 1|^{-1}$ and $L'(s, \psi_0) \ll |s - 1|^{-2}$ in the neighbourhood around $s = 1$, the integral on the vertical line can be bounded by

$$\ll x''(qU)^{\frac{3}{16}+\varepsilon} (\log qU)^3 \int_U^{\infty} \frac{dt}{1 + |t|} + x''(\log qU)^3 \int_{-1}^{1} \frac{dt}{|a' + it|}$$

$$\ll x''(qU)^{\frac{3}{16}+\varepsilon} (\log U)^4$$

$$\ll x''U^{\frac{1}{2}} = x \exp \left( -2c_1 - \frac{C}{4c_1} \right) \sqrt{\log x}.$$
When we put $c_1 = \sqrt{C}/4$, we obtain that
\[
\frac{1}{2\pi i} \int_{a-iU}^{a+iU} \frac{L'(s, \psi_0) L'(s, \psi_0) x^s}{s} \, ds
= \text{Res}_{s=1} \frac{L'}{L}(s, \psi_0) L'(s, \psi_0) x^s/s + O \left( x \exp \left( -\frac{\sqrt{C}}{2} \sqrt{\log x} \right) \right).
\]

Note that
\[
\text{Res}_{s=1} \frac{L'}{L}(s, \psi_0) L'(s, \psi_0) x^s/s
= \frac{1}{2!} \lim_{s \to 1} \frac{d^2}{ds^2} (s-1)^3 \frac{L'}{L}(s, \psi_0) L'(s, \psi_0) x^s/s.
\]

To calculate this residue, we observe that
\[
L'(s, \psi_0) = \zeta'(s) \prod_{p|q} (1 - p^{-s}) + \sum_{p|q} \log p \frac{p}{p^s - 1}
\]
and
\[
\frac{L'}{L}(s, \psi_0) = \frac{\zeta'(s)}{\zeta(s)} \sum_{p|q} \log p \frac{p}{p^s - 1}
= -\frac{1}{s - 1} + \sum_{k=0}^{\infty} \frac{\eta_k (s - 1)^k}{k!} \prod_{p|q} \log p \frac{p^s - 1}{p^s - 1},
\]
where $\gamma_k$ is the $k$-th Stieltjes constant and can be defined by the limit
\[
\gamma_k = \lim_{n \to \infty} \left\{ \left( \sum_{m=1}^{n} \frac{(\log m)^k}{m} \right) - \frac{(\log n)^{k+1}}{k+1} \right\},
\]
and $\eta_k$ can be represented by the sum
\[
\eta_k = (-1)^k \frac{k+1}{k!} \gamma_k + \sum_{n=0}^{k-1} \frac{(k-n-1)! \gamma_k}{(k-n-1)!} \eta_n \gamma_{k-n-1}.
\]
Hence we get
\[ \text{Res}_{s=1} \frac{L'(s, \psi_0)L'(s, \psi_0)x^s}{s} = \frac{1}{2!} \lim_{s \to 1} \frac{d^2}{ds^2} \prod_{p|q} \left( 1 - \frac{1}{p^s} \right) \frac{x^s}{s} \]

\[ - \frac{2}{2!} \lim_{s \to 1} \frac{d}{ds} \prod_{p|q} (1 - p^{-s}) \left( \sum_{p|q} \frac{\log p}{p^s - 1} + \eta_0 + \sum_{p|q} \frac{\log p}{p^s - 1} \right) \frac{x^s}{s} \]

\[ - \frac{2}{2!} \lim_{s \to 1} \prod_{p|q} \left( 1 - \frac{1}{p^s} \right) \left\{ \gamma_1 + \gamma_0 \sum_{p|q} \frac{\log p}{p^s - 1} + \eta_1 \right\} \frac{x^s}{s} \]

\[ \frac{\varphi(q)}{q} x \left\{ \frac{1}{2} (\log x)^2 - \left( \sum_{p|q} \frac{\log p}{p - 1} + \gamma_0 + 1 \right) \log x - \frac{1}{2} \left( \sum_{p|q} \frac{\log p}{p - 1} \right)^2 \right. \]

\[ \left. + \frac{3}{2} \sum_{p|q} p \left( \frac{\log p}{p - 1} \right)^2 + (1 - \gamma_0) \sum_{p|q} \frac{\log p}{p - 1} + \gamma_0^2 + \gamma_0 - 3 \gamma_1 + 1 \right\} . \]

Therefore we can see that

\[ \frac{\chi(-1)\tau(\chi)}{\varphi(q)} \sum_{mn \leq x} \psi_0(m)\psi_0(n)\Lambda(m) \log n \]

\[ = \frac{\tau(\chi)}{q} x \left\{ \frac{1}{2} (\log x)^2 - \left( \sum_{p|q} \frac{\log p}{p - 1} + \gamma_0 + 1 \right) \log x - \frac{1}{2} \left( \sum_{p|q} \frac{\log p}{p - 1} \right)^2 \right. \]

\[ \left. + \frac{3}{2} \sum_{p|q} p \left( \frac{\log p}{p - 1} \right)^2 + (1 - \gamma_0) \sum_{p|q} \frac{\log p}{p - 1} + \gamma_0^2 + \gamma_0 - 3 \gamma_1 + 1 \right\} \]

\[ + O \left( x \exp \left( -\frac{\sqrt{C}}{2} \sqrt{\log x} \right) \right) . \]

Here we note that \( \tau(\chi)/\varphi(q) \ll 1. \)
(ii) In the same way, we obtain
\[
\frac{1}{\varphi(q)^2} \sum_{\psi' \mod q} \sum_{b=1}^{q} \overline{\psi'}(b) \chi(b) \sum_{a=1}^{q} \chi(a) e\left(-\frac{ab}{q}\right) \\
\times \sum_{mn \leq x} \psi_0(m) \psi'(n) \Lambda(m) \log n
\]
\[
= \frac{\chi(-1)\tau(\chi)}{\varphi(q)^2} \sum_{\psi' \neq \psi'_0} \sum_{b=1}^{q} \overline{\psi'}(b) \sum_{mn \leq x} \psi_0(m) \psi'(n) \Lambda(m) \log n.
\]
The sum of $\overline{\psi'}$ is 0. Hence we see that the sum in this case vanishes.

(iii) This case is the same as the case (ii).

(iv) This case is the same as the case (ii).

To show the last equality, we use the fact that the sum over $a$ does not equal to 0 if and only if $\psi = \psi'$. 

\[
\frac{1}{\varphi(q)^2} \sum_{\psi' \neq \psi'_0} \sum_{a=1}^{q} \overline{\psi}(a) \chi(a) \sum_{b=1}^{q} \overline{\psi'}(b) \chi(b) e\left(-\frac{ab}{q}\right) \\
\times \sum_{mn \leq x} \psi(m) \psi'(n) \Lambda(m) \log n
\]
\[
= \frac{1}{\varphi(q)^2} \sum_{\psi' \neq \psi'_0} \sum_{a=1}^{q} \overline{\psi}(a) \chi(a) \psi'(-a) \chi(-a) \tau(\overline{\psi'} \chi) \\
\times \sum_{mn \leq x} \psi(m) \psi'(n) \Lambda(m) \log n
\]
\[
= \frac{\chi(-1)}{\varphi(q)^2} \sum_{\psi' \neq \psi'_0} \sum_{a=1}^{q} \overline{\psi}(a) \psi'(-a) \chi(-a) \tau(\overline{\psi'} \chi) \sum_{mn \leq x} \psi(m) \psi'(n) \Lambda(m) \log n
\]
\[
= \frac{\chi(-1)}{\varphi(q)^2} \sum_{\psi' \neq \psi'_0} \psi'(-1) \tau(\overline{\psi'} \chi) \sum_{a=1}^{q} \overline{\psi}(a) \psi'(a) \sum_{mn \leq x} \psi(m) \psi'(n) \Lambda(m) \log n
\]
\[
= \frac{\chi(-1)}{\varphi(q)} \sum_{\psi' \neq \psi'_0} \psi(-1) \tau(\overline{\psi} \chi) \sum_{mn \leq x} \psi(m) \psi(n) \Lambda(m) \log n.
\]
In this case, we know the fact that there is an absolute constant $C' > 0$ such that

$$L(s, \chi) \neq 0 \quad \text{for} \quad \sigma > 1 - \frac{C'}{\log q(|t| + 2)}$$

unless $\chi$ is a quadratic character, in which case $L(s, \chi)$ has at most one, necessarily real, zero $\beta < 1$ (see [8, p. 360]). By the same argument as in the case (i), when we put $c_1 = \sqrt{C'/4}$ we have

$$\sum_{mn \leq x} \psi(m)\Lambda(m)\psi(n) \log n = -L'(\beta, \psi)\frac{x^\beta}{\beta} + O \left( x \exp \left( -\frac{\sqrt{C'}}{2} \sqrt{\log x} \right) \right)$$

when $L(s, \psi)$ with a quadratic character $\omega$ has an exceptional zero $\beta$. If there is no exceptional zero, then the first term vanishes. Hence when $L(s, \omega)$ has an exceptional zero $\beta$ we have

$$\frac{\chi(-1)}{\varphi(q)} \sum_{\psi \neq \psi_0} \psi(-1)\tau(\psi\chi) \sum_{mn \leq x} \psi(m)\psi(n)\Lambda(m) \log n$$

$$= -\frac{\chi(-1)}{\varphi(q)} \omega(-1)\tau(\bar{\omega}\chi)L'(\beta, \omega)\frac{x^\beta}{\beta} + O \left( \sqrt{q}x \exp \left( -\frac{\sqrt{C'}}{2} \sqrt{\log x} \right) \right),$$

otherwise the main term does not appear.

From the above, when we put $c_1 = \min\{\sqrt{C}/4, \sqrt{C'/4}\}$ and $c = c_1/2$, we have

$$J_3 = -\frac{T}{2\pi} \left\{ \frac{1}{2} \left( \log \frac{qT}{2\pi} \right)^2 - \left( \sum_{p \mid q} \log p \frac{1}{p-1} + \gamma_0 + 1 \right) \log \frac{qT}{2\pi} - \frac{1}{2} \left( \sum_{p \mid q} \log p \frac{1}{p-1} \right)^2 \right\}$$

$$+ \frac{3}{2} \sum_{p \mid q} p \left( \log p \frac{1}{p-1} \right) + (1 - \gamma_0) \sum_{p \mid q} \log p \frac{1}{p-1} + \gamma_0^2 + \gamma_0 + \gamma_1 + 1$$

$$+ \frac{\omega\chi(-1)\tau(\bar{\chi})L'(\beta, \omega)}{q\varphi(q)} \left( \frac{qT}{2\pi} \right)^{\beta} \frac{1}{\beta} + O \left( T \exp \left( -c\sqrt{\log T} \right) \right).$$

We note that $\tau(\chi)\sqrt{q}/q \ll 1$.

To complete the proof, we take away the condition on $T$. When $T$ increases continuously in $|T - \gamma_\chi| \ll (\log qT)^{-1}$, the number of relevant $L'(\rho_\chi, \chi)$ is at most $O(\log qT)$ and the order of each term is $O((qT)^{\frac{1}{2} + \epsilon})$. Thus the contribution of these terms is smaller than the
A certain sum of the derivatives of Dirichlet $L$-functions

error in our main theorem. Therefore the proof in the unconditional case is completed.

4. The conditional estimate

In this section, we assume the GRH. We choose $a' = 1/2 + (\log qT)^{-1}$ and $U = qT$. In the case (i), by Cauchy’s theorem,

$$\frac{1}{2\pi i} \int_{a-iU}^{a+iU} \frac{L'}{L}(s, \psi_0) L'(s, \psi_0) \frac{x^s}{s} ds = \lim_{s \to 1} \frac{L'(s, \psi_0) L'(s, \psi_0)}{L(s, \psi_0) L'(s, \psi_0)} \frac{x^s}{s} ds,$$

The integral on the horizontal line is

$$\int_{a\pm iU}^{a\pm iU} \frac{L'}{L}(s, \psi_0) L'(s, \psi_0) \frac{x^s}{s} ds \ll \frac{x^a}{U} (qU)^\varepsilon (\log qU)^3 \ll (qT)^\varepsilon.$$

As for the vertical line, we note that

$$\frac{L'}{L}(s, \psi_0) = \frac{\zeta'}{\zeta}(s) + \sum_{\nu|q} \frac{\log p}{p^s - 1} \ll \log 2q$$

for $s = a' + it$ and $0 \leq |t| \leq 1$. Thus we have

$$\int_{s-a'it}^{s+a'it} \frac{L'}{L}(s, \psi_0) L'(s, \psi_0) \frac{x^s}{s} ds \ll x^{a'} (\log qU)^3 \int_1^U \frac{(qt)^\varepsilon}{t} dt + x^{a'} (\log 2q)^2 \int_{-1}^1 \frac{q^\varepsilon}{a'} dt \ll (qT)^{1/2 + \varepsilon}.$$

Concerning the case (iv), we can see that

$$\sum_{nm \leq x} \psi(m) \Lambda(m) \psi(n) \log n \ll (qT)^{1/2 + \varepsilon}$$

by the similar argument. Therefore we can replace the error term in our theorem by $(qT)^{1/2 + \varepsilon}$. 
5. The Details of Remark 1

We consider the case when $q$ is a prime power. Let $q = p^a$, $a' = -(\log qT)^{-1} = 1 - a$ and $U = qT$. In the case (i), by the residue theorem

$$\frac{1}{2\pi i} \int_{\alpha-iU}^{a+iU} \frac{L'(s, \psi_0)L'(s, \psi_0)}{L(s, \psi_0)L(s, \psi_0)} \frac{x^s}{s} ds$$

$$= \text{Res}_{s=1} \frac{L'(s, \psi_0)L'(s, \psi_0)}{L(s, \psi_0)L(s, \psi_0)} \frac{x^s}{s} + \text{Res}_{s=0} \frac{L'(s, \psi_0)L'(s, \psi_0)}{L(s, \psi_0)L(s, \psi_0)} \frac{x^s}{s}$$

$$+ \sum_{\rho \neq 0 \atop |\Im \rho| \leq U} \frac{L'(\rho, \psi_0)x^\rho}{\rho}$$

$$+ \frac{1}{2\pi i} \left\{ \int_{\alpha+iU}^{a'+iU} + \int_{a'+iU}^{a-iU} + \int_{a-iU}^{a+iU} \right\} \frac{L'(s, \psi_0)L'(s, \psi_0)}{L(s, \psi_0)L(s, \psi_0)} \frac{x^s}{s} ds,$$

where $\rho$ runs over the zeros of $L(s, \psi_0)$. With regard to the residue at $s = 0$, we can see that

$$\frac{L'(s, \psi_0)}{L(s, \psi_0)} = \frac{\zeta'(s)}{\zeta(s)} + \frac{\log p}{p^s - 1} \quad (q = p^a)$$

and

$$\frac{\log p}{p^s - 1} = \frac{1}{s} \cdot \frac{s \log p}{e^{s \log p} - 1} = \frac{1}{s} \sum_{n=0}^{\infty} \frac{B_n}{n!} (s \log p)^n,$$

where $B_n$ is the $n$-th Bernoulli number, and hence we have

$$\text{Res}_{s=0} \frac{L'(s, \psi_0)L'(s, \psi_0)}{L(s, \psi_0)L(s, \psi_0)} \frac{x^s}{s}$$

$$= \lim_{s \to 0} \frac{d}{ds} \frac{L'(s, \psi_0)L'(s, \psi_0)}{L(s, \psi_0)L(s, \psi_0)} x^s$$

$$= \lim_{s \to 0} \frac{d}{ds} \left( \frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s} \sum_{n=0}^{\infty} \frac{B_n}{n!} (s \log p)^n \right) L'(s, \psi_0) x^s$$

$$= L''(0, \psi_0) + \left( \frac{\zeta'(0)}{\zeta(0)} + B_1 \log p + \log x \right) L'(0, \psi_0)$$

$$= 3\zeta'(0) \log p - \frac{3}{2} \zeta(0) (\log p)^2 + \zeta(0) \log x \ll (\log qT)^2.$$
The integral on the horizontal line is
\[
\int_{1-a \pm iU}^{a \pm iU} \frac{L'}{L}(s, \psi_0)L'(s, \psi_0) \frac{x^s}{s} ds
\]
\[
\ll \left\{ \int_{\frac{1}{2} \pm iU}^{a \pm iU} + \int_{1-a \pm iU}^{\frac{1}{2} \pm iU} \right\} \frac{L'}{L}(s, \psi_0)L'(s, \psi_0) \frac{x^s}{s} ds
\]
\[
\ll \frac{x^a}{U} (qU)^{\varepsilon} (\log qU)^3 + \frac{x^{\frac{1}{2}}}{U} (qU)^{\frac{1}{2}} (\log qU)^4
\]
\[
\ll (qU)^{\varepsilon} (\log qU)^3 + \sqrt{q}(\log qU)^4.
\]

On the integral along the vertical line, since \(|s - \rho_0| \gg 1\), by (7), we can see that
\[
\frac{L'}{L}(s, \psi_0) \ll \log q(|t| + 2).
\]

Therefore we have
\[
\int_{1-a-iU}^{1-a+iU} \frac{L'}{L}(s, \psi_0)L'(s, \psi_0) \frac{x^s}{s} ds
\]
\[
= i \int_{-U}^{U} \frac{L'}{L}(1-a+it, \psi_0) L'(1-a+it, \psi_0) \frac{x^{1-a+it}}{1-a+it} dt
\]
\[
\ll (\log qU)^2 \left| \int_{-U}^{U} \zeta(1-a+it) \frac{dt}{1-a+it} \right|
\]
\[
\ll (\log qU)^2 \left( \log U \int_{1}^{U} t^{-\frac{1}{2}} dt + \int_{-1}^{1} \frac{dt}{|1-a+it|} \right)
\]
\[
\ll \sqrt{U}(\log qU)^3.
\]

Here we use the well-known estimate
\[
\zeta(s) \ll (|t| + 2)^{\frac{1}{2}} \log(|t| + 2) \quad \text{for} \quad -\frac{1}{\log T} \leq \sigma < \frac{1}{2}.
\]

The sum over \(\rho\) consists of two sums as
\[
\sum_{\rho \neq 0, |\Im \rho| \leq U} \frac{L'(\rho, \psi_0) x^\rho}{\rho} = \sum_{|\gamma| \leq U} L' \left( \frac{1}{2} + i\gamma, \psi_0 \right) x^{\frac{1}{2} + i\gamma}
\]
\[
+ \sum_{|\frac{2\pi k}{\log p} - \gamma| \leq U} L' \left( \frac{2\pi i k}{\log p}, \psi_0 \right) x^{\frac{2\pi k}{\log p} \log p} \frac{2\pi i k}{2\pi i k}
\]
\[
= S_1 + S_2,
\]
say. Since
\[ L'(\frac{1}{2} + i\gamma, \psi_0) = \zeta' \left( \frac{1}{2} + i\gamma \right) (1 - p^{-\frac{1}{2} - i\gamma}), \]
we have
\[
S_1 \ll x^{\frac{1}{2}} \sum_{\gamma \leq U} \frac{|L'(\frac{1}{2} + i\gamma, \psi_0)|}{\gamma}
\ll x^{\frac{1}{2}} \left( \sum_{\gamma \leq U} \frac{|\zeta'(\frac{1}{2} + i\gamma)|^2}{\gamma} \right)^{\frac{1}{2}} \left( \sum_{\gamma \leq U} \frac{1}{\gamma} \right)^{\frac{1}{2}}
\ll x^{\frac{1}{2}} (\log U)^\frac{3}{2}
\]
by partial summation and the fact that
\[
(12) \quad \sum_{0 < \gamma \leq T} \left| \zeta' \left( \frac{1}{2} + i\gamma \right) \right|^2 \asymp T (\log T)^4
\]
proved by Gonek [3].

On the other hand, since
\[ L' \left( \frac{2\pi ik}{\log p}, \psi_0 \right) = \zeta \left( \frac{2\pi ik}{\log p} \right) \log p, \]
we see that
\[
S_2 \ll (\log p)^2 \sum_{\frac{2\pi ik}{\log p} \leq U} \left| \frac{\zeta \left( \frac{2\pi ik}{\log p} \right)}{2\pi k} \right| \ll \sqrt{U} \log U \sqrt{U \log q}^2
\]
by the estimate
\[ \zeta(s) \ll (|t| + 2)^{\frac{1}{4}} \log(|t| + 2) \quad \text{for} \quad -\frac{1}{\log T} \leq \sigma < \frac{1}{2} \]
again. Therefore we can see that

\[
\frac{\chi(-1)\tau(\chi)}{\phi(q)} \sum_{\chi \leq x} \psi_0(m)\psi_0(n) \Lambda(m) \log n
\]

\[
= \frac{\tau(\chi)}{q} x \left\{ \frac{1}{2} (\log x)^2 - \left( \sum_{\nu|q} \frac{\log p}{p - 1} + \gamma_0 + 1 \right) \log x - \frac{1}{2} \left( \sum_{\nu|q} \frac{\log p}{p - 1} \right)^2 \right. \\
+ \frac{3}{2} \sum_{\nu|q} \left( \frac{\log p}{p - 1} \right)^2 + (1 - \gamma_0) \sum_{\nu|q} \frac{\log p}{p - 1} + \gamma^2_0 + \gamma_0 - 3\gamma_1 + 1 \right\} \\
+ O \left( x^\frac{1}{2} (\log U)^2 \right).
\]

As for the case (iv), we need to deal with the Dirichlet \( L \)-functions with primitive and also imprimitive characters. However, it is sufficient to consider these with only primitive characters, for we put \( q = p^\alpha \). For primitive characters, the integral on the vertical line can be estimated as

\[
\int_{1-a-iU}^{1-a+iU} L'(s, \psi)L'(s, \psi) \frac{x^s}{s} ds \\
= \int_{1-a-iU}^{1-a+iU} \Delta(s, \psi) \left\{ \left( \frac{\Delta'}{\Delta}(s, \psi) \right)^2 L(1-s, \psi) \\
- \frac{\Delta'}{\Delta}(s, \psi)L'(1-s, \psi) + \frac{L'}{L}(1-s, \psi)L'(1-s, \psi) \right\} \frac{x^s}{s} ds \\
\ll q^{a-\frac{1}{2}} \left( \int_0^U \left( t^{a-\frac{1}{2}} \exp \left( it \log \frac{2\pi e}{qt} \right) + O(t^{a-\frac{3}{2}}) \right) \\
\times \left( (\log qU)^2 L(a - it, \psi) + \frac{L'}{L}(a - it, \psi)L'(a - it, \psi) \right) \frac{x^{1-a+it}}{1-a+it} dt \right) \\
\ll x^{1-a} q^{a-\frac{1}{2}} \left( (\log U)^2 \sum_{n=1}^\infty \frac{1}{n^a} \left| \int_1^U \left( t^{a-\frac{3}{2}} \exp \left( it \log \frac{2\pi e x n}{qt} \right) + O(t^{a-\frac{3}{2}}) \right) dt \right| \\
+ \sum_{m=2}^{\infty} \Lambda(m) \sum_{n=1}^{\infty} \frac{\log n}{n^a} \left| \int_1^U \left( t^{a-\frac{3}{2}} \exp \left( it \log \frac{2\pi e x mn}{qt} \right) + O(t^{a-\frac{3}{2}}) \right) dt \right| \\
+ O(q^{a-\frac{1}{2}} (\log U)^3) \right) \\
+ O(q^{a-\frac{1}{2}} (\log U)^3) \\
+ O(q^{a-\frac{1}{2}} (\log U)^3) \\
\right\}
\]

Since

\[
\frac{d^2}{dt^2} \left( t \log \frac{2\pi e x n}{qt} \right) = -t^{-1},
\]
by the second derivative test,
\[
\int_1^U t^{a-\frac{3}{2}} \exp \left( it \log \frac{2\pi exn}{qt} \right) dt 
\ll \sum_{l \leq \lceil \log U \rceil + 1} \int_{\frac{U}{2^l}}^{\frac{U}{2^{l+1}}} t^{a-\frac{3}{2}} \exp \left( it \log \frac{2\pi exn}{qt} \right) dt 
\ll \sum_{l \leq \lceil \log U \rceil + 1} 1 \ll \log U.
\]

Therefore we obtain
\[
\int_{1-a+iU}^{1-a-iU} \frac{L'(s, \psi)L'(s, \psi)}{s} ds \ll q^{a-\frac{1}{2}} (\log U)^4.
\]

On the sum \( S_1 \), we assume the estimate \( \square \). By partial summation and this assumption, we have
\[
S_1 \ll x^{\frac{1}{2}} \sum_{0 < \gamma \psi \leq U} \frac{|L'(\frac{1}{2} + i\gamma \psi, \psi)|}{\gamma \psi} 
\ll x^{\frac{1}{2}} \left( \sum_{0 < \gamma \psi \leq U} \frac{|L'(\frac{1}{2} + i\gamma \psi, \psi)|}{\gamma \psi} \right)^{\frac{1}{2}} \left( \sum_{0 < \gamma \psi \leq U} \frac{1}{\gamma \psi} \right)^{\frac{1}{2}} 
\ll x^{\frac{1}{2}} (\log U)^{\frac{7}{2}}.
\]

On the other hand, the counterpart of the sum \( S_2 \) does not appear. When \( \psi \pmod{q} \) is induced by \( \psi^* \pmod{d} \) with \( d \mid q \), we see that
\[
L(s, \psi) = L(s, \psi^*) \prod_{\substack{p \mid q \\ p \nmid d}} \left( 1 - \frac{\psi^*(p)}{p^s} \right).
\]

However we assume that \( q = p^\alpha \). Thus the products on the right-hand side is 1. Hence there is no zeros on the imaginary axis.

Therefore we can replace the estimate of the error term by
\[
\sqrt{qT(\log qT)^\frac{7}{2}}.
\]

**ACKNOWLEDGEMENT**

I would like to thank my supervisor Professor Kohji Matsumoto for useful advice. I am grateful to the seminar members for some helpful remarks and discussions.
REFERENCES

[1] T. M. Apostol, ‘Introduction to Analytic Number Theory’ (Springer-Verlag, New York, 1976).
[2] A. Fujii, ‘On a Conjecture of Shanks’, Proc. Japan Acad. 70 (1994) 109-114.
[3] S. M. Gonek, ‘Mean values of the Riemann zeta-function and its derivatives’, Invent. Math. 75 (1984) 123-141.
[4] D. R. Heath-Brown, ‘Hybrid bounds for Dirichlet L-functions II’, Q. J. Math. 31 (1980) 157-167.
[5] M. T. Jakhlouti and K. Mazhouda, ‘Distribution of the values of the derivative of the Dirichlet L-functions at its α-points’, Bull. Korean Math. Soc. 54 (2017) No. 4, 1141-1158.
[6] D. A. Kaptan, Y. Karabulut and C. Y. Yildirim, ‘Some mean value theorems for the Riemann zeta-function and Dirichlet L-functions’, Comment. Math. Univ. St. Pauli 60 (2011), No. 1-2, 83-87.
[7] A. F. Lavrik, ‘The approximate functional equation for Dirichlet L-functions’, Tr. Mosk. Mat. Obs. 18 (1968) 91-104.
[8] H. L. Montgomery and R. C. Vaughan, ‘Multiplicative Number Theory: I. Classical Theory’, Cambridge Studies in Advanced Mathematics, vol. 97 (Cambridge University Press, Cambridge, 2006).
[9] K. Prachar, ‘Primzahlverteilung’, (Springer-Verlag, 1957).
[10] J. Steuding, ‘Dirichlet series associated to periodic arithmetic functions and the zeros of Dirichlet L-functions’, Anal. Proba. Methods in Number Theory, Proc. 3rd Intern. Conf. in Honour of J. Kubilius, Palanga, Lithuania, A. Dubickas et al. (eds.), TEV, Vilnius, (2002) 282-296.
[11] E. C. Titchmarsh, ‘The theory the Riemann Zeta-Function’, Second edition, Edited and with a preface by D. R. Heath-Brown, (The Clarendon Press, Oxford University Press, New York, 1986).

Graduate School of Mathematics, Nagoya University, Furocho, Chikusaku, Nagoya 464-8602, Japan
Email address: m17011z@math.nagoya-u.ac.jp