GENERATORS FOR RINGS OF COMPACTLY SUPPORTED DISTRIBUTIONS

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Abstract. Let $C$ denote a closed convex cone $C$ in $\mathbb{R}^d$ with apex at 0. We denote by $E'(C)$ the set of distributions having compact support which is contained in $C$. Then $E'(C)$ is a ring with the usual addition and with convolution. We give a necessary and sufficient analytic condition on $\hat{f}_1, \ldots, \hat{f}_n$ for $f_1, \ldots, f_n \in E'(C)$ to generate the ring $E'(C)$. (Here $\hat{\cdot}$ denotes Fourier-Laplace transformation.) This result is an application of a general result on rings of analytic functions of several variables by Hörmander. En route we answer an open question posed by Yutaka Yamamoto.

1. Introduction

Let $R$ be a commutative ring with identity. Elements $a_1, \ldots, a_n$ of $R$ are said to generate $R$ if the ideal generated by $a_1, \ldots, a_n$ is equal to $R$, or equivalently, if there exist $b_1, \ldots, b_n$ such that $a_1b_1 + \cdots + a_nb_n = 1$.

For instance, if $R = H^\infty(\mathbb{D})$, the set of all bounded and holomorphic functions on the open unit disc $\mathbb{D}$ centered at 0 in $\mathbb{C}$, then the corona theorem says that $f_1, \ldots, f_n \in H^\infty(\mathbb{D})$ generate $H^\infty(\mathbb{D})$ iff there exists a $C > 0$ such that $|f_1(z)| + \cdots + |f_n(z)| > C$ for all $z \in \mathbb{D}$; see [2].

In this note we address this question when the ring $R$ consists of compactly supported distributions.

Let $C$ denote a closed convex cone $C$ in $\mathbb{R}^d$ with apex at 0. Recall that a convex cone is a subset of $\mathbb{R}^d$ with the following properties:

1. If $x, y \in C$, then $x + y \in C$.
2. If $x \in C$ and $t > 0$, then $tx \in C$.

Let $E'(C)$ be the set consisting of all distributions having a compact support which is contained in $C$. Then $E'(C)$ is a commutative ring with the usual addition of distributions and the operation of convolution. The Dirac delta distribution $\delta$ supported at 0 serves as an identity in the ring $E'(C)$.

Recall that a distribution $f$ with compact support has a finite order, and its Fourier-Laplace transform is an entire function given by

$$\hat{f}(z) = \langle f, e^{-iz} \rangle, \quad z \in \mathbb{C}^d.$$
We use the notation $\| \cdot \|$ for the usual Euclidean 2-norm in $\mathbb{C}^d$. The same notation is also used for the Euclidean norm in $\mathbb{R}^d$.

The supporting function of a convex, compact set $K (\subset \mathbb{R}^d)$ is defined by

$$H_K(\xi) = \sup_{x \in K} \langle x, \xi \rangle, \quad \xi \in \mathbb{R}^d.$$ 

Our main result is the following:

**Theorem 1.1.** Let $C$ denote a closed convex cone in $\mathbb{R}^d$ with apex at 0, and $H$ denote the supporting function of the compact convex set

$$B := C \cap \{ x \in \mathbb{R}^d : \|x\| \leq 1 \},$$

that is, $H(\xi) = \sup_{x \in B} \langle \xi, x \rangle$.

Let $f_1, \ldots, f_n \in \mathcal{E}'(C)$. There exist $g_1, \ldots, g_n \in \mathcal{E}'(C)$ such that

$$f_1 * g_1 + \cdots + f_n * g_n = \delta$$

iff there are positive constants $C, N, M$ such that

$$|\hat{f}_1(z)| + \cdots + |\hat{f}_n(z)| \geq C(1 + \|z\|^2)^{-N} e^{-MH(\text{Im}(z))}. \quad (1.1)$$

**Theorem 1.1** in the case when $d = 1$ and $C = \mathbb{R}$ was known; see [5].

2. Proof of the main result

We will show that our main result follows from the main result given in [3]. We will use the Payley-Wiener-Schwartz theorem, which is recalled below.

**Proposition 2.1** (Payley-Wiener-Schwartz). Let $K$ be a convex compact subset of $\mathbb{R}^d$ with supporting function $H$. If $u$ is a distribution with support contained in $K$, then there exists a positive $N$ such that

$$|\hat{u}(z)| \leq C(1 + \|z\|^2)^N e^{H(\text{Im}(z))}. \quad (2.1)$$

Conversely, every entire analytic function in $\mathbb{C}^d$ satisfying an estimate of the form (2.1) is the Fourier-Laplace transform of a distribution with support contained in $K$.

**Proof.** See for instance [4, Theorem 7.3.1]. The only difference is that we have the term $(1 + \|z\|^2)^N$ instead of $(1 + \|z\|)^N$ in the estimate (2.1), which follows from the observation that $1 + \|z\|^2 \leq (1 + \|z\|)^2 \leq 2(1 + \|z\|^2)$ for every $z \in \mathbb{C}^d$ (and by replacing $N/2$ by $N$).

We also recall the main result from Hörmander [3, Theorem 1, p. 943], which we will use.

Let $p$ be a nonnegative function defined in $\mathbb{C}^d$. Let $A_p$ denote the set of all entire functions $F : \mathbb{C}^d \to \mathbb{C}$ such that there exist positive constants $C_1$ and $C_2$ (which in general depend on $F$) such that

$$|F(z)| \leq C_1 e^{C_2 p(z)}.$$ 

It is clear that $A_p$ is a ring with the usual pointwise operations.
Proposition 2.2 (Hörmander). Let \( p \) be a nonnegative plurisubharmonic function in \( \mathbb{C}^d \) such that

1. all polynomials belong to \( A_p \)
2. there exist nonnegative \( K_1, K_2, K_3, K_4 \) such that whenever \( z, \zeta \in \mathbb{C}^d \) satisfy \( \| z - \zeta \| \leq e^{-K_1 p(z) - K_2} \), there holds that \( p(\zeta) \leq K_3 p(z) + K_4 \).

If there exist positive constants \( C_1, C_2 \) such that

\[
\text{for all } z \in \mathbb{C}^d, \quad |F_1(z)| + \cdots + |F_n(z)| \geq C_1 e^{-C_2 p(z)},
\]

then \( F_1, \ldots, F_n \in A_p \) generate \( A_p \).

Lemma 2.3. Let \( C \) denote a closed convex cone in \( \mathbb{R}^d \) with apex at 0, and \( H \) denote the supporting function of the compact convex set \( B := C \cap \{ x \in \mathbb{R}^d : \| x \| \leq 1 \} \), that is, \( H(\xi) = \sup_{x \in B} \langle \xi, x \rangle \).

Let \( p(z) := \log(1 + \| z \|^2) + H(\text{Im}(z)) \). Then we have the following:

1. \( p \) is nonnegative and subharmonic.
2. \( A_p = \mathcal{E}'(C) \).
3. \( A_p \) contains the polynomials.
4. There exist nonnegative \( K_1, K_2, K_3, K_4 \) such that whenever \( z, \zeta \in \mathbb{C}^d \) satisfy \( \| z - \zeta \| \leq e^{-K_1 p(z) - K_2} \), there holds that \( p(\zeta) \leq K_3 p(z) + K_4 \).

(That is, condition (2) of Proposition 2.2 is satisfied.)

Proof. (1) Clearly \( p \) is nonnegative. Also, the complex Hessian at \( z \) of the map \( z \mapsto \log(1 + |z|^2) \) is easily seen to be

\[
F(z) := \frac{1}{1 + \| z \|^2} I - \frac{1}{(1 + \| z \|^2)^2} zz^*.
\]

So for \( w \in \mathbb{C}^d \), we have that

\[
w^* F(z) w = \frac{1}{1 + \| z \|^2} \| w \|^2 - \frac{1}{(1 + \| z \|^2)^2} |w^* z|^2
\]

\[
= \frac{\| w \|^2 + \| w \|^2 \| z \|^2 - |w^* z|^2}{(1 + \| z \|^2)^2} \geq 0
\]

by the Cauchy-Schwarz inequality. So the map \( z \mapsto \log(1 + |z|^2) \) is plurisubharmonic; see [6, Proposition 4.9, p.88].

We will use the fact that a map \( \varphi : \mathbb{C}^d \to \mathbb{R} \) that depends only on the imaginary part of the variable is plurisubharmonic iff the map is convex; see [6, E.4.8, p.92]. The supporting function \( H_K \) of any convex compact set \( K \) satisfies the properties that

\[
H_K(\xi + \eta) \leq H_K(\xi) + H_K(\eta), \quad H_K(t \xi) = t H_K(\xi)
\]

for all \( \xi, \eta \in \mathbb{R}^d \) and \( t \geq 0 \). It is then clear that \( H_K \) is a convex function. In particular our \( H \) (the supporting function of \( B \)) is convex too. Thus
Let $z \mapsto H(\text{Im}(z))$ is plurisubharmonic. Consequently, $p$, which is the sum of the plurisubharmonic maps $z \mapsto \log(1 + \|z\|^2)$ and $z \mapsto H(\text{Im}(z))$, is plurisubharmonic as well; [3], p.88.

(2) Suppose that $f \in \mathcal{E}'(C)$ has support contained in the compact set $K$ contained in $C$. Then by the Payley-Wiener-Schwartz Theorem, there exist positive $C, N, M$, such that

$$|\hat{f}(z)| \leq C(1 + \|z\|^2)^N e^{MH(\text{Im}(z))}.$$ 

Let $\epsilon > 0$ be such that $\epsilon K \subset B$. Then we have for $\xi \in \mathbb{R}^d$ that

$$H_K(\xi) = \sup_{x \in K} \langle x, \xi \rangle \leq \sup_{x \in \epsilon^{-1}B} \langle x, \xi \rangle = \epsilon^{-1} \sup_{y \in B} \langle y, \xi \rangle = \epsilon^{-1} H(\xi).$$

Thus with $M := \epsilon^{-1}$, we have

$$|\hat{f}(z)| \leq C(1 + \|z\|^2)^N e^{MH(\text{Im}(z))} = C e^{N \log(1 + \|z\|^2) + MH(\text{Im}(z))} \leq C e^{\max\{N,M\} p(z)}.$$ 

So $\hat{f} \in A_p$. Conversely, if $F \in A_p$, then

$$|F(z)| \leq C_1 e^{C_2 p(z)} = C_1 (1 + \|z\|^2)^{C_2} e^{C_2 H(\text{Im}(z))}.$$ 

But for $\xi \in \mathbb{R}^d$ we have

$$C_2 H(\xi) = C_2 \sup_{x \in B} \langle x, \xi \rangle = \sup_{y \in C_2 B} \langle y, \xi \rangle = H_{C_2 B}(\xi).$$

So by the Payley-Wiener-Schwartz theorem, there exists an $f \in \mathcal{E}'(\mathbb{R}^d)$ such that $\hat{f} = F$ and the support of $f$ is contained in $C_2 B \subset C$. Thus $F \in \mathcal{E}'(C)$.

(3) Let $\mathbb{Z}_+ = \{0, 1, 2, 3, \ldots \}$. Let

$$Q(z) = \sum_{k \in \mathbb{Z}_+^n, |k| \leq N} a_k z^k,$$

where for a multi-index $k = (k_1, \ldots, k_n) \in \mathbb{Z}_+^n$,

$$|k| := k_1 + \ldots + k_n, \quad z^k = z_1^{k_1} \ldots z_n^{k_n}, \quad a_k \in \mathbb{C}.$$ 

Consider

$$q = \sum_{k \in \mathbb{Z}_+^n, |k| \leq N} a_k \frac{1}{i^{|k|}} \frac{\partial^{|k|}}{\partial x_1^{k_1} \ldots \partial x_n^{k_n}} \delta \in \mathcal{E}'(C),$$

Then $Q = \hat{q} \in \mathcal{E}'(C) = A_p$, and so $Q \in A_p$.

(4) Let $K_1$ and $K_2$ be nonnegative, and let $z, \zeta$ satisfy $\|z - \zeta\| \leq e^{-K_1 p(z) - K_2}$. Then

$$\|z - \zeta\| \leq e^{-K_1 p(z) - K_2} = e^{-K_1 p(z)} e^{-K_2} \leq 1 \cdot 1 = 1.$$
In particular, \( \|z\| \leq \|z\| + 1 \). Also,
\[
H(\text{Im}(\zeta - z)) = \sup_{x \in B} \langle x, \text{Im}(\zeta - z) \rangle \leq \sup_{x \in B} \|x\| \|\text{Im}(\zeta - z)\| \\
\leq \sup_{x \in B} \|x\| \|\zeta - z\| \leq 1 \cdot 1 = 1.
\]
Thus
\[
p(\zeta) = \log(1 + \|\zeta\|^2) + H(\text{Im}(\zeta)) \leq 2 \log(1 + \|\zeta\|) + H(\text{Im}(z + \zeta - z)) \\
\leq 2 \log(2 + \|z\|) + H(\text{Im}(z)) + H(\text{Im}(\zeta - z)) \\
\leq \log(8(1 + |z|^2)) + H(\text{Im}(z)) + 1 = p(z) + \log 8 + 1.
\]
This completes the proof. \( \square \)

**Proof of Theorem 1.1.** Necessity of the condition (1.1) is not hard to check. Indeed, if there are \( g_1, \ldots, g_n \in \mathcal{E}'(C) \) such that
\[
f_1 \ast g_1 + \cdots + f_n \ast g_n = \delta,
\]
then upon taking Fourier-Laplace transforms, we obtain
\[
\hat{f}_1(z)\hat{g}_1(z) + \cdots + \hat{f}_n(z)\hat{g}_n(z) = 1 \quad (z \in \mathbb{C}^d).
\]
By the triangle inequality,
\[
1 = |\hat{f}_1(z)\hat{g}_1(z) + \cdots + \hat{f}_n(z)\hat{g}_n(z)| \leq |\hat{f}_1(z)||\hat{g}_1(z)| + \cdots + |\hat{f}_n(z)||\hat{g}_n(z)|.
\]
Suppose that \( g_k \) has support contained in the compact convex set \( L_k (\subset C) \), where \( k = 1, \ldots, n \). Then by the Payley-Wiener-Schwartz theorem, we have an estimate
\[
|\hat{g}_k(z)| \leq C_k(1 + \|z\|^2)^{N_k} e^{H_{L_k}(\text{Im}(z))}
\]
for each \( k \). Let \( \epsilon > 0 \) be small enough so that \( \epsilon L_k \subset B \) for all the \( k \). Then we have for \( \xi \in \mathbb{R}^d \) that
\[
H_{L_k}(\xi) = \sup_{x \in L_k} \langle x, \xi \rangle \leq \sup_{x \in \epsilon^{-1} B} \langle x, \xi \rangle = \epsilon^{-1} \sup_{y \in B} \langle y, \xi \rangle = \epsilon^{-1} H(\xi).
\]
Thus we have that for all \( k \),
\[
|\hat{g}_k(z)| \leq C(1 + \|z\|^2)^N e^{M H(\text{Im}(z))},
\]
where \( M := \epsilon^{-1}, C := \max_k C_k \) and \( N := \max_k N_k \). Consequently,
\[
1 \leq (|\hat{f}_1(z)| + \cdots + |\hat{f}_n(z)|)C(1 + \|z\|^2)^N e^{M H(\text{Im}(z))},
\]
and this yields (1.1), completing the proof of the necessity part.

We now show the sufficiency of (1.1). Let \( f_1, \ldots, f_n \in \mathcal{E}'(C) \) be such that their Fourier-Laplace transforms satisfy (1.1). Then by Lemma 2.3, \( \hat{f}_1, \ldots, \hat{f}_n \in A_p \) with \( p(z) = \log(1 + \|z\|^2) + H(\text{Im}(z)) \) \( (z \in \mathbb{C}^d) \). Moreover,
this $p$ satisfies the conditions (1) and (2) of Proposition 2.2. The condition (1.1) gives
\[
|\hat{f}_1(z)| + \cdots + |\hat{f}_n(z)| \geq C(1 + \|z\|^2)^{-N} e^{-MH(\text{Im}(z))} \\
\geq C e^{-N \log(1 + \|z\|^2) - MH(\text{Im}(z))} \\
\geq C e^{-\max\{N,M\} \rho(z)}.
\]

It then follows from Proposition 2.2 that there are some $G_1, \ldots, G_n$ in $A_p$ such that $\hat{f}_1 G_1 + \cdots + \hat{f}_n G_n = 1$ on $\mathbb{C}$. But $A_p = \mathcal{E}'(C)$. Hence there exist $g_1, \ldots, g_n \in \mathcal{E}'(C)$ such that $f_1 * g_1 + \cdots + f_n * g_n = \delta$. \hfill $\square$

3. Special cases of the main result

3.1. The full space $\mathbb{R}^d$. The supporting function $H$ of the unit ball $B$ in $\mathbb{R}^d$ is given by $H(\xi) = \|\xi\|$. So we obtain the following consequence of Theorem 1.1.

**Corollary 3.1.** Let $f_1, \ldots, f_n \in \mathcal{E}'(\mathbb{R}^d)$. There exist $g_1, \ldots, g_n \in \mathcal{E}'(\mathbb{R}^d)$ such that
\[
f_1 * g_1 + \cdots + f_n * g_n = \delta
\]
iff there are positive constants $C, N, M$ such that
\[
\text{for all } z \in \mathbb{C}^d, \quad |\hat{f}_1(z)| + \cdots + |\hat{f}_n(z)| \geq C(1 + \|z\|^2)^{-N} e^{-M \|\text{Im}(z)\|}. \quad (3.1)
\]

3.2. The nonnegative orthant in $\mathbb{R}^d$. Let
\[
\mathbb{R}^d_+ = \{x = (x_1, \ldots , x_d) \in \mathbb{R}^d : x_k \geq 0, \text{ for all } k = 1, \ldots , d\}.
\]
The supporting function $H$ of $B = \{x \in \mathbb{R}^d_+ : \|x\| \leq 1\}$ in $\mathbb{R}^d$ is given by
\[
H(\xi) = \|\xi^+\|,
\]
where
\[
\xi^+ := (\max\{\xi_1, 0\}, \ldots , \max\{\xi_d, 0\})
\]
for $\xi = (\xi_1, \ldots , \xi_d) \in \mathbb{R}^d$. Theorem 1.1 gives the following.

**Corollary 3.2.** Let $f_1, \ldots , f_n \in \mathcal{E}'(\mathbb{R}^d_+)$. There exist $g_1, \ldots , g_n \in \mathcal{E}'(\mathbb{R}^d_+)$ such that
\[
f_1 * g_1 + \cdots + f_n * g_n = \delta
\]
iff there are positive constants $C, N, M$ such that
\[
\text{for all } z \in \mathbb{C}^d, \quad |\hat{f}_1(z)| + \cdots + |\hat{f}_n(z)| \geq C(1 + \|z\|^2)^{-N} e^{-M \|\text{Im}(z)\|^+}. \quad (3.2)
\]

In particular, in the case when $d = 1$, we obtain:

**Corollary 3.3.** Let $f_1, \ldots , f_n \in \mathcal{E}'(\mathbb{R}_+)$. There exist $g_1, \ldots , g_n \in \mathcal{E}'(\mathbb{R}_+)$ such that
\[
f_1 * g_1 + \cdots + f_n * g_n = \delta
\]
iff there are positive constants $C, N, M$ such that
\[
\text{for all } z \in \mathbb{C}, \quad |\hat{f}_1(z)| + \cdots + |\hat{f}_n(z)| \geq C(1 + |z|^2)^{-N} e^{-M \max\{\text{Im}(z), 0\}}. \quad (3.3)
\]
3.3. **The future light cone in** $\mathbb{R}^{d+1}$. Let $C$ be the future light cone, namely,

$$
\Gamma := \{(x, t) \in \mathbb{R}^d \times \mathbb{R} : \|x\| \leq ct, \ t \geq 0\},
$$

where $c$ denotes the speed of light. Then the supporting function of the intersection of $\Gamma$ and the unit ball in $\mathbb{R}^{d+1}$ is given by

$$
\Phi(\xi, \tau) = \begin{cases} 
\sqrt{\|\xi\|^2 + \tau^2} & \text{if } c^{-1}\|\xi\| \leq \tau, \\
\frac{\tau + c\|\xi\|}{\sqrt{c^2 + 1}} & \text{if } -c\|\xi\| \leq \tau \leq c^{-1}\|\xi\|, \\
0 & \text{if } \tau \leq -c\|\xi\|,
\end{cases}
$$

for $(\xi, \tau) \in \mathbb{R}^d \times \mathbb{R}$. Then we have:

**Corollary 3.4.** Let $f_1, \ldots, f_n \in \mathcal{E}'(\Gamma)$. There exist $g_1, \ldots, g_n \in \mathcal{E}'(\Gamma)$ such that

$$
f_1 * g_1 + \cdots + f_n * g_n = \delta
$$

iff there are positive constants $C, N, M$ such that

$$
\text{for all } z \in \mathbb{C}^d, |\hat{f}_1(z)| + \cdots + |\hat{f}_n(z)| \geq C(1 + \|z\|^2)^{-N}e^{-M\Phi(\text{Im}(z))}. \quad (3.4)
$$

4. **Answer to Yamamoto’s question**

We remark that Theorem 1.1 answers an open question of Y. Yamamoto; see question number 2 [7, p.282]. There it was asked if for $f_1, f_2 \in \mathcal{E}'(\mathbb{R})$, the condition that $\hat{f}_1, \hat{f}_2$ have no common zeros in $\mathbb{C}$ is enough to guarantee that there are $g_1, g_2 \in \mathcal{E}'(\mathbb{R})$ such that $f_1 * g_1 + f_2 * g_2 = \delta$.

In light of Theorem 1.1 above, the answer is no, since our analytic condition (3.1) (in the case when $d = 1$) is not equivalent to (and is stronger) than the condition that there is no common zero, as seen in the following example. (The idea behind this example is taken from [5].)

**Example 4.1.** Let $c \in \mathbb{R}_+$ be the Liouville constant, that is,

$$
c = \sum_{n=1}^{\infty} \frac{1}{10^{n!}}.
$$

(See for example, [1].) Then it can be seen that $c$ is irrational. Also, for $K \in \mathbb{N}$, with $p_K, q_K$ defined by

$$
p_K = 10^{K!} \sum_{k=1}^{K} \frac{1}{10^{k!}}, \quad q_K = 10^{K!},
$$

...
we have that
\[
0 < \left| c - \frac{pK}{qK} \right| = \sum_{k=K+1}^{\infty} \frac{1}{10^k!} = \frac{1}{10^{(K+1)!}} + \frac{1}{10^{(K+2)!}} + \frac{1}{10^{(K+3)!}} + \cdots
\leq \frac{1}{10^{(K+1)!}} \cdot \sum_{m=0}^{\infty} \frac{1}{10^m} = \frac{10^/9}{(10^{(K+1)!})10^{K!}}
\leq \frac{1}{(10^{K+1})10^{K!}} = \frac{1}{qK^K}.
\]
(4.1)

Take \( f_1 = \delta - \delta_c \) and \( f_2 = 1_{[0,1]} \), where \( 1_{[0,1]} \) denotes the indicator function of the interval \([0,1]\). Then \( f_1, f_2 \) belong to \( E'(\mathbb{R}) \) and we have that
\[
\hat{f}_1(z) = 1 - e^{-icz}, \quad \hat{f}_2(z) = \begin{cases} 
eq 0 & \text{if } z = 0 \\
 ie^{-iz} & \text{if } z = 0.
\end{cases}
\]

Then \( \hat{f}_1 \) and \( \hat{f}_2 \) have no common zeros (otherwise \( c \) would be rational!). We now show that (3.1) does not hold. Suppose, on the contrary that there exist \( C, N, M \) positive such that
\[
|\hat{f}_1(z) + \hat{f}_2(z)| \geq C(1 + |z|^2)^{-N} e^{-M|\text{Im}(z)|}
\]
for all \( z \in \mathbb{C} \). If \( z = 2\pi qK \), then we have \( \hat{f}_2(2\pi qK) = 0 \). On the other hand,
\[
|\hat{f}_1(2\pi qK)| = |1 - e^{-ic(2\pi qK)}| = \sin(\pi qK) = |\sin(\pi qK - \pi pK)|.
\]
The inequality (4.2) now yields that
\[
|\sin(\pi (cqK - pK))| \geq C(1 + 4\pi^2 qK^2)^{-N}.
\]
But \( |\sin \Theta| \leq |\Theta| \) for all real \( \Theta \), and so we obtain
\[
\pi qK \left| c - \frac{pK}{qK} \right| \geq C(1 + 4\pi^2 qK^2)^{-N}.
\]
In light of (4.1), we now obtain
\[
\pi qK \frac{1}{qK} \geq C(1 + 4\pi^2 qK^2)^{-N},
\]
and rearranging, we have
\[
\pi qK (1 + 4\pi^2 qK^2)^N \geq C.
\]
Passing the limit \( K \to \infty \), we arrive at the contradiction that \( 0 \geq C \).

We remark that in this example \( f_1, f_2 \) actually belong to \( E'(\mathbb{R}_+) \), and with the same argument given above, it can be seen that \( \hat{f}_1, \hat{f}_2 \) don’t satisfy (3.3) either. This also gives another example answering question number 1 in [7], namely, for \( f_1, f_2 \) in \( E'(\mathbb{R}_+) \), whether the condition that \( \hat{f}_1, \hat{f}_2 \) have no common zeros is enough to guarantee that there are \( g_1, g_2 \in E'(\mathbb{R}_+) \) such that \( f_1 \ast g_1 + f_2 \ast g_2 = \delta \).
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