A quantum phase transition in a quantum external field: Superposing two magnetic phases

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We study an Ising chain undergoing a quantum phase transition in a quantum magnetic field. Such a field can be emulated by coupling the chain to a central spin initially in a superposition state. We show that – by adiabatically driving such a system – one can prepare a quantum superposition of any two ground states of the Ising chain. In particular, one can end up with the Ising chain in a superposition of ferromagnetic and paramagnetic phases – a scenario with no analogue in prior studies of quantum phase transitions. Remarkably, the resulting magnetization of the chain encodes the position of the critical point and universal critical exponents, as well as the ground state fidelity.

Quantum phase transitions (QPTs) occur when dramatic changes in the ground state properties of a quantum system are induced by a tiny variation of an external parameter, such as the magnetic field in spin systems or the intensity of a laser beam in cold atom or ion simulators of Hubbard-like models. In all current studies of QPTs, the external parameter is assumed to be classical, i.e., it has a well-defined instantaneous value. However, the field inducing a QPT can be quantum as well, taking on different values by virtue of being in a superposition of states. In fact, tremendous progress with the preparation and manipulation of cold atom/ion systems will allow for creation of scenarios where the quantum nature of the “external” parameter will play a central role.

For instance, cavity-QED systems offer intriguing possibilities to study quantum control parameters. In these systems, photons bouncing off two parallel mirrors interact with ultracold atoms. If the number of photons in the cavity does not fluctuate, atoms experience an “external” periodic potential $\cos(2kx)$, whose amplitude is proportional to the number of intra-cavity photons ($k$ is the photon wave-vector). Atoms in such a system would be either in the superfluid phase or in the Mott insulator phase. It may be possible, however, to create a coherent superposition of the intra-cavity photonic states, giving rise to quantum fluctuations in the number of photons between the mirrors. The atoms would then be exposed to a coherent superposition of periodic potentials with the same period but differing amplitudes. In this case, one can have atoms in a superposition of two quantum phases, i.e., simultaneously in superfluid and Mott insulator ground states. Such a situation has no counterpart in traditional studies of QPTs where the system is either in one phase or another.

An analogous phenomenon can be envisioned in central spin models. These models are used to describe qubit–environment interactions in nitrogen-vacancy centers in diamond, quantum dots in semiconductors, NMR experiments, etc. The focus is typically on the loss of coherence of the qubit while ignoring the environmental degrees of freedom. We will take the opposite perspective and explore the quantum state of the environment subjected to an effective quantum potential originating from the central spin. For an experimental study of such a scenario, one needs a well-controlled system, which we expect will be delivered in the foreseeable future by ion simulators of spin chains.

Results

The model. We will discuss the most striking consequence of a QPT in a quantum potential: The possibility of having the system in a superposition of ground states belonging to different phases, as shown in Fig. 1. We consider a quantum Ising chain uniformly coupled to a (central) spin-1/2 (Fig. 2):
The couplings between the effective spins-1/2 are optically engineered to be spin model in a linear ion chain. The ions emulate the effective spins-1/2. To all the spins-1/2 arranged on a ring. (b) Possible realization of the central spin model in a classical magnetic field: The central spin is equally coupled to the central spin is turned off, which provides freedom to engineer the state of the central spin. The composite wave function is \( |\psi(g(t))\rangle = |S\rangle \langle g(t)| \), where \( |g\rangle \) is a ground state of \( H_I(g) \) and the central spin state is

\[
|S\rangle = c_1 e^{i\phi_1(t)} |\uparrow\rangle + c_2 e^{i\phi_2(t)} |\downarrow\rangle, \quad c_1^2 + c_2^2 = 1,
\]

where \( c_{1,2} > 0 \). By changing both the bias field \( g \) and the coupling \( \delta \), the wave-function evolves according to

\[
|\psi(g(t))\rangle = \mathcal{T} \exp \left( -i \int dt \left[ H_I(g(t), \delta(t)) \right] \right) |\psi(t_0)\rangle,
\]

where \( \mathcal{T} \) is the time-ordering operator.

As was shown in Ref. [13], \( |\psi(g(t))\rangle \) can be simplified. Considering adiabatic evolution, we obtain

\[
|\psi(g(t))\rangle = e^{i\phi_1(t)} c_1 |\uparrow\rangle |g + \delta\rangle + e^{i\phi_2(t)} c_2 |\downarrow\rangle |g - \delta\rangle.
\]

We thus study finite, i.e., gapped, systems so that adiabatic evolution is possible by changing \( g(t) \) and \( \delta(t) \) slow enough. In the state (3), the chain experiences an average magnetic field

\[
\langle \hat{g} \rangle = g + \delta \left( c_1^* c_2^2 - c_1^2 c_2 \right)
\]

with fluctuations

\[
\sqrt{\langle \hat{g}^2 \rangle - \langle \hat{g} \rangle^2} = 2\delta c_1 c_2.
\]

In particular, this shows that once the desired coupling \( \delta \) is adiabatically reached, fluctuations of the quantum potential are fixed.

The state (3) is already a Schrödinger’s cat state, where the two “macroscopically” distinct possibilities are the ferromagnetic and paramagnetic phases, both of which are coupled to the auxiliary two-level system. Since we are interested in the QPT of the Ising chain, we will “trace out” the central spin by measuring its state. If we will do the measurement in the \([|\uparrow\rangle, |\downarrow\rangle] \) basis, the superposition will be destroyed and the state of the Ising chain will be one of the ground states \(|g \pm \delta\rangle\). Measurement in any other basis will result in a superposition of Ising ground states at different magnetic fields.

We assume that the measurement will be done in the eigenbasis of the \( \sigma_3^c \) operator:

\[
|+\rangle = \frac{|\uparrow\rangle + |\downarrow\rangle}{\sqrt{2}}, \quad |-\rangle = \frac{|\uparrow\rangle - |\downarrow\rangle}{\sqrt{2}}.
\]

In this basis,

\[
|\psi(g(t))\rangle = |+\rangle \frac{c_1 e^{i\phi_1} |g + \delta\rangle + c_2 e^{i\phi_2} |g - \delta\rangle}{\sqrt{2}} + |-\rangle \frac{c_1 e^{i\phi_1} |g + \delta\rangle - c_2 e^{i\phi_2} |g - \delta\rangle}{\sqrt{2}},
\]

where we write \( \phi_{1,2} \) as shorthand for \( \phi_{1,2}(t) \). Therefore, the measurement of the central spin in the state \(|\pm\rangle\) leaves the chain in the state
\[ \frac{c_1 e^{i\delta} |g + \delta\rangle + c_1 e^{i\delta} |g - \delta\rangle}{\sqrt{1 + 2c_1 c_1 \cos(\phi_1 - \phi_1) F(g, \delta)}} \]  

where \( F(g, \delta) = (g - \delta |g + \delta\rangle \) is a ground state fidelity, or simply fidelity, whose crucial role in this problem will be carefully discussed below. Without loss of generality, we define it in such a way that \( F(g, \delta) > 0 \).

Now we comment on the measurement of the central spin. The state \(|\pm\rangle\) will occur with probability

\[ P_{\pm} (\phi_1 - \phi_1) = \frac{1}{2} \pm c_1 c_1 \cos(\phi_1 - \phi_1) F, \]

which depends on the relative phase between the ground states in the superposition (3). Since the point of the measurement is to prepare a well-defined superposition state of the Ising chain, we will describe properties of the Ising chain after finding the central spin in, e.g., the \(|+\rangle\) state. Then, the Ising chain will be in the state

\[ |\text{Ising}\rangle = \frac{c_1 e^{i\delta} |g + \delta\rangle + c_1 e^{i\delta} |g - \delta\rangle}{\sqrt{1 + 2c_1 c_1 \cos(\phi_1 - \phi_1) F(g, \delta)}}. \]  

This state is the desired superposition of ferromagnetic and paramagnetic ground states when

\[ g - \delta < g < g + \delta, \]

which is depicted in Fig. 1. We propose to call such a state a Schrödinger magnet. The possibility to create such a novel state of matter is offered by the quantum magnetic field in Eq. (2). Indeed, if there would be no quantum component in the magnetic field, the wave function of the Ising chain after the adiabatic evolution would correspond to either a ferromagnetic or a paramagnetic phase ground state, but never to a superposition of both.

**Ising chain in the superposition state.** For simplicity, we assume that the measurements on the Ising chain are performed immediately after measuring the central spin. The expectation value of an operator \( \hat{O} \) in the state (5) is

\[ \langle \text{Ising} | \hat{O} | \text{Ising} \rangle = \frac{O' + 2c_1 c_1 \cos(\Delta) O^+ -}{1 + 2c_1 c_1 \cos(\Delta) F}, \]

where \( \Delta = \phi_1 - \phi_1 \) and

\[ O' = c_1^2 O^+ + c_1^2 O^-, \quad O^\pm = \langle g \pm \delta | \hat{O} | g \pm \delta \rangle, \]

is the “standard” average, and

\[ O^+ = \langle g + \delta | \hat{O} | g - \delta \rangle \]

designate the cross term that arises. For clarity of presentation, we restrict ourselves to real \( O^- \) in Eq. (6), because \( O^- \) is always real for the operators \( \hat{O} = \sigma^x_n, \sigma^y_n, \sigma^z_n \) that we study. It is not real, however, for all operators (e.g., for \( \hat{O} = \sigma^x_n \)), and it is a straightforward exercise to extend our calculations to these cases. While the standard average does not yield any new information, the cross term provides a non-trivial correction absent in a quantum phase transition in a classical field.

To further simplify the discussion, we average Eq. (6) over several realizations where the appearance of the relative phase \( \Delta \) of the superposition (3) is given by some probability distribution \( p(\Delta) \). For example, such averaging may appear due to preparation of the central spin with random initial phases \( \phi_{1,1} (t) \). We assume below for simplicity that \( p(\Delta) = 1/2\pi \) for \( \Delta \in [0, 2\pi] \). We denote the result of such averaging as \( \overline{O} \), and define its variance through \( \text{var}(O) = \overline{O^2} - \overline{O}^2 \). Finally, we introduce the notation

\[ O^\pm = O^+ - F, \]

because for the operators \( \hat{O} \) that we consider, \( O^\pm \) are well-defined non-zero quantities in the thermodynamic limit in which fidelity typically tends to zero (see the Discussion section).

The phase-averaged observable and its variance are

\[ \overline{O} = \frac{\int_0^{2\pi} d\Delta \hat{O}(\Delta) p(\Delta)}{\int_0^{2\pi} d\Delta p(\Delta)} = O', \]

\[ \text{var}(O) = \left( \frac{\int_0^{2\pi} d\Delta \hat{O}(\Delta) p(\Delta) \overline{O^2} - (\overline{O})^2}{\int_0^{2\pi} d\Delta p(\Delta)} \right) = \left( O' - \overline{O}^2 \right)^2 \frac{1}{1 - \chi^2} - 1 \approx \chi^2, \]

with \( x = 2c_1 c_1 F \). By expanding

\[ \frac{1}{1 - \chi^2} \approx \frac{1}{\chi^2}, \]

we see that the square root of variance is proportional to fidelity when \( x \ll 1 \). The role of fidelity in our problem is discussed in the Discussion section. In the following, we use the exact solution of the Ising model to study expectation values of different observables in the superposition state (5), see the Methods section for details.

We start by looking at \( \hat{O} = M_z = \sigma^z_n, M^\pm_z \) terms have been calculated in Ref. [15]

\[ M^\pm_z = \frac{1 + g + \delta}{\pi(g + \delta)} E(\chi^\pm) + \frac{-1 + g + \delta}{\pi(g + \delta)} K(\chi^\pm), \]

where \( \chi^\pm = 4(g + \delta)/(1 + g + \delta)^2 \), and \( K \) and \( E \) are elliptic functions of the first and the second kind, respectively. Above a large \( N \) limit is assumed to simplify the expressions (see the Methods section for the exact finite \( N \) expressions).

The cross terms can be obtained from the eigen-equation

\[ \hat{H}_g |g \pm \delta\rangle = \hat{N} \hat{e} |g \pm \delta\rangle |g \pm \delta\rangle, \]

where \( \hat{e} \) is the ground state energy per spin. Indeed, one gets from it

\[ M^\pm_z = \frac{1 + g + \delta}{\pi \delta} E(\chi^\pm) - \frac{-1 + g + \delta}{\pi \delta} K(\chi^\pm), \]

In the limit of \( N \to \infty \), \( \hat{e} (g + \delta) = -2/\pi (1 + g + \delta) E(\chi^\pm) \). Consequently,

\[ M^\pm_z = \frac{1 + g + \delta}{\pi \delta} E(\chi^\pm) + \frac{-1 + g + \delta}{\pi \delta} K(\chi^\pm), \]

The dependence of magnetization on the relative phase of the superposition and the variance of magnetization at \( g = 1 \) are depicted in Fig. 3.

We mention in passing that similar expressions can be obtained for \( \hat{O} = \hat{C}_n = \sigma^x_n \sigma^\pm_n \). Indeed, it is known from Ref. [15] that

\[ C^\pm_z = \frac{1 + g + \delta}{\pi} E(\chi^\pm) + \frac{-1 + g + \delta}{\pi} K(\chi^\pm), \]

and one can use again Eq. (7) to derive

\[ C^\pm_z = \frac{\overline{\hat{F}(g, \delta)} - \overline{\hat{F}(g + \delta)} - \overline{\hat{F}(g - \delta)}}{2 \overline{\hat{F}(g, \delta)} - \overline{\hat{F}(g + \delta)} - \overline{\hat{F}(g - \delta)}} \]

Since these results are analogous in structure to the ones already discussed, we will not analyze them.

Next, we study spontaneous magnetization in the \( x \)-direction. The system will acquire such a magnetization when a tiny field breaking the \( \sigma^x_n \to -\sigma^x_n \) symmetry of the Hamiltonian is present. When
necessary, we thus add a $-h \sum_{n=1}^{N} \sigma_n^z$ term to $H_I(g)$ and denote a ground state of the resulting Hamiltonian as $|g, h\rangle$. Without the quantum magnetic field, $\delta = 0$, the Ising chain acquires macroscopic magnetization (along the direction of the symmetry breaking field $h$) only in the ferromagnetic phase. This magnetization can also be calculated by studying the correlation function\textsuperscript{15}

$$\lim_{R \to \infty} \frac{1}{R} \left( \langle \sigma_1^z \sigma_R^z \rangle - \langle \sigma_1^z \rangle \langle \sigma_R^z \rangle \right) = \frac{1}{2} \left( 1 - g^2 \right)^{1/8}.$$  

Importantly, it encodes the critical exponent $\beta = 1/8$ (see Ref. [16]).

To study spontaneous magnetization in the presence of the superposition of ground states, we find numerically the states $|g, h\rangle$ using a periodic version\textsuperscript{17} of the TEBD algorithm\textsuperscript{18,19}. Then, we calculate $M_+^z = \langle g, h^+ | \sigma_n^x | g, h \rangle |_{h=0}$ and $M_-^z = \langle g, h^- | \sigma_n^x | g, h \rangle |_{h=0}$. Naturally, for large enough systems, the standard result is reproduced by numerics:

$$M_{ \pm }^z \equiv \left[ 1 - (g \pm \delta)^2 \right]^{1/8}$$

for $|g \pm \delta| < 1$ and zero otherwise. The results of TEBD calculations are plotted in Fig. 4. The presence of the cross term magnetization, resulting from the superposition of two ground states in Eq. (5), leads to sizable deviations from the “standard” average.

To analyze this deviation more efficiently in the thermodynamic limit, we study the asymptotic behavior of the two-point correlation functions:

$$M_{ \pm }^x = \lim_{R \to \infty} \sqrt{\langle \sigma_1^x \sigma_R^x \rangle_{F}^{\pm}},$$

where $\langle \cdots \rangle_{F}^{\pm} = \langle g + \delta \cdots | g - \delta \rangle / F$. It can be done using the exact solution of the Ising model through fermionization, where we express the correlator as a determinant of a $2R \times 2R$ block Toeplitz matrix, which is then numerically evaluated (see the Methods section for details).

As shown in Fig. 5, we find that the scaling of $M_+^z$ around the critical point is consistent with the ansatz

$$M_+^z = \mathcal{F} \delta g B(c), \quad c = \frac{g - g_c}{\delta},$$

where $\beta = 1/8$ and $g_c = 1$ for the Ising chain that we study, and $B(c)$ is the scaling function. It is nonzero when at least one of the superposed ground states is in the ferromagnetic phase, i.e., $B(c) \neq 0$ for $c < 1$. Far away from the critical point, we observe that $M_+^z \sim M_+^0 \approx M_{\pm}^x$ and so $B(c \ll 1) \approx (-2c)^{1/8}$.

**Discussion**

We have seen that the presence of the quantum external field allows for creation of the superposition state of two distinct ground states in general and two distinct phases in particular. If this happens,
expectation values are altered by the cross terms. The magnitude of this effect can be sizable as depicted in Figs. 3 and 4.

A fundamentally important question can now be answered: What is the role of the system size in a quantum phase transition in a quantum field and what critical information is imprinted onto the cross terms.

To answer this, we note that all the cross terms that we studied are a product of the two terms: The ground state fidelity \( F(g, \delta) \) and a term that has a well-defined non-zero value in the thermodynamic limit. Ground state fidelity, however, typically disappears in the thermodynamic limit of \( N \to \infty \) often invoked in the context of quantum phase transitions.

This is known as Anderson orthogonality catastrophe after the seminal work reported in Ref. [20]. Therefore, we are interested in the studies of systems for which \( N \gg 1 \) (to see quantum criticality), but still \( N < \infty \) (to avoid the catastrophe). There are three options here, which we will discuss below. Instead of providing specific results for fidelity of the Ising chain, we provide general scaling results to highlight the role of critical exponents in our problem and to keep the discussion concise.

First, one can consider the limit of \( \delta \to 0 \) taken at fixed \( N \gg 1 \). Then, fidelity reads\(^{21-23}\)

\[
\ln F(g, \delta) \sim -\delta^2 N^{2/dv}\tag{10}
\]

near the critical point. Here, \( d \) is system dimensionality and \( v \) is the critical exponent (correlation length diverges as \( |g - g_c|^{-\nu} \) near the critical point; \( d = v = 1 \) in the Ising chain that we consider). Since fidelity is close to unity in this limit, the cross terms do not get small. One must remember, however, that they will be dominated by finite system size corrections requiring a separate study, which is beyond the scope of this work.

Second, in the limit of \( N \to \infty \) at fixed \( \delta \) – the one that we assumed in our calculations – one can focus on the “moderately” large size systems. To explain this term we note that near the critical point (in the above-mentioned limit)\(^{24}\)

\[
\ln F(g, \delta) \sim -N\delta^{dv}\tag{11}
\]

The crossover from Eq. (10) to Eq. (11) happens near the critical point when\(^{24}\)

\[
N\delta^{dv} \sim 1.
\]

We define the “moderately” large system to be just large enough to exhibit the scaling of fidelity with \( \delta \) and \( N \) given by Eq. (11) rather than Eq. (10). In the Ising case, Eq. (11) predicts in \( N^{-\delta^{dv}} \) while Eq. (10) predicts in \( N^{-\delta^{dv}} \). For a “moderately” large system fidelity shall not be too small to erase the contribution of the cross term (see, e.g., Fig. 4).

Third, one can study superpositions of two ground states from the same phase far away from the critical point. There in \( F \sim -N\delta^{2/|g - g_c|^{dv/2}} \), and for \( N \gg 1 \) fidelity can be kept close to unity by a proper choice of \( \delta \). The downside of this scenario is that we lose the possibility to superimpose two phases.

From the above discussion, we see that the critical exponent \( v \) is imprinted onto the cross term via fidelity. Also the critical exponent \( \beta \) is seen in the cross term \( M^\pm_v \) contributing to spontaneous magnetization. The location of the critical point is most directly seen in the cross terms \( M^\pm_v \) and \( C^\pm_v \) through “divergence” of their second derivative over \( g \) taken at \( g_c = \pm \delta \). This is caused by the singularity of the second derivative of the ground state energy per spin across the critical point. This singularity will be rounded off in finite systems (\( N < \infty \)), but nevertheless there shall be pronounced peaks visible. We also note that while the “average” averages \( M^\pm_v \) and \( C^\pm_v \) also encode the position of the critical point, they do not encode the critical exponent \( v \).

To observe the superposed phases, experiments will have to keep decoherence to a minimum. The decoherence rate of the state in Eq. (5) will depend on how well the environment distinguishes the two components, which will depend on the system size (see, e.g., Ref. [25]) and the overlap between the two states (fidelity). Thus, \( N \) cannot be too large and \( F \) cannot be too small. This is a similar issue to being able to observe the effect of the cross term, which we discussed above. We thus do not expect decoherence to be overwhelming in a properly prepared setup. Further, the system size can be used as a parameter controlling decoherence, and its manipulation should be sufficient to bring decoherence down to an acceptable level. Looking from a different perspective, studies of decoherence of such a novel macroscopic quantum superposition are fundamentally interesting on its own, e.g., to boost understanding of the quantum-to-classical transition.

To conclude, we considered a quantum phase transition of an Ising chain exposed to a quantum external field. This scenario can be used to create a new state of matter where the system is simultaneously in two distinct quantum phases. Observables on the chain then take on forms that encode the ground state fidelity, the location of the critical point, and the universal critical exponents of the system. These findings set the foundations for developing a scaling theory of quantum phase transitions in quantum fields. Recent advances in cold atom cavity-QED and ion traps may lead to experimental realization of superposed phases.

**Methods**

We provide here some technical details regarding our calculations. The Ising Hamiltonian \( H_I(g) \) is diagonalized using the standard approach (see e.g., Ref. [26]). The Jordan-Wigner transformation,

\[
\sigma^x = 1 - 2d^\dagger d, \quad \sigma^z = (d^\dagger + d^\dagger - d^\dagger d) \prod_{n=0}^N (1 - 2d^\dagger d_n),
\]

where \( d \) are fermionic annihilation operators, transforms the Ising chain to a free-fermion model. After applying the Fourier transform

\[
\tilde{c}_n = -\frac{e^{-i2\pi n k}}{\sqrt{N}} \sum_{k} \tilde{c}_k e^{i2\pi nk},
\]

the Hamiltonian takes the form:

\[
H_I(g) = \sum_k \left( \frac{\tilde{c}_k^\dagger \tilde{c}_k}{2} - g \cos k \right) + \left( \tilde{c}_k^\dagger \tilde{c}_{-k} + \tilde{c}_{-k}^\dagger \tilde{c}_k \right) \sin k,
\]

\[
k = \pm (2s+1) \frac{\pi}{N}, \quad s = 0, \ldots, N/2 - 1.
\]

Diagonalization of the Hamiltonian with the help of the Bogolubov transformation leads to the following ground state wave function

\[
|g \pm \delta\rangle = \prod_{k=0}^{N/2} \left[ \cos \left( \theta^+_k/2 \right) |00\ldots 0\rangle - \sin \left( \theta^+_k/2 \right) |11\ldots 1\rangle \right],
\]

where \( |\psi_0, m, \rangle \) describes the state with \( m = 0 \), 1 pairs of \( c \) quasiparticles with momentum \( k \) and

\[
\tan \theta^+_k = \frac{g + \delta - \cos k}{\delta},
\]

To prepare Fig. 3, we fix the system size \( N \) and use the following exact expressions for magnetization and fidelity

\[
M^+_v \pm = (g \pm \delta |\tilde{c}_v^\dagger \tilde{c}_v| g \mp \delta) = \frac{1}{N} \sum_k \cos \theta^+_k \pm \theta^-_k,
\]

\[
M^-_v \pm = (g \pm \delta |\tilde{c}_v^\dagger \tilde{c}_v| g \mp \delta) = \frac{F}{N} \sum_k \cos \left( \theta^+_k \pm \theta^-_k \right)\cos \left( \theta^+_k \mp \theta^-_k \right),
\]

\[
F = (g + \delta - \delta) = \prod_{k=0}^{N/2} \cos \left( \frac{\theta^+_k - \theta^-_k}{2} \right) > 0.
\]

To prepare Fig. 4, we select \( g, \delta \), and \( h \), and calculate the ground states \( |g \mp \delta, h\rangle \) of the Ising chain exposed to transverse and longitudinal magnetic fields. This is done through imaginary time evolution performed with the periodic TEBD algorithm. A global phase of the wave-functions is then chosen to make \( F = (g - \delta, h |g + \delta, h\rangle \) positive. Then we directly calculate \( M^+_v \pm \) and \( M^-_v \pm \) (both are positive). Putting these results into Eq. (6), one can calculate the spontaneous magnetization in the x-direction in the superposition state (5). The result is still dependent on the relative phase \( \phi \). When this phase is either 0 or \( \pi \), spontaneous magnetization at any fixed \( g, \delta \), and \( h \) reaches an extremum. These extremal values are depicted by solid green lines in Fig. 4.
To prepare Fig. 5, we calculate correlation function

$$C(x) = \frac{\langle \sigma_1 \sigma_{1+x} \rangle}{\langle \sigma_1 \sigma_1 \rangle} = \frac{1}{\langle \sigma_1 \sigma_1 \rangle} \langle \sigma_{1} \sigma_{1+x} \rangle$$

where we introduce \(\sigma_1 = \sigma^z_1 + \sigma^z_2\) and \(\sigma_{1+x} = \sigma^z_{1+x} + \sigma^z_{1+x+2}\). We study it, because \(M_{xx}^y = \lim_{x \to -\infty} C(x)\).

The next step is to use Wick’s theorem extended to such a cross-correlation\(^2\). It can be used as long as the overlap \(Z \neq 0\), which is the case in our calculations. Then extending the results of Ref. [28], we find that \(C(x)\) can be expressed as a Pfaffian of a 2R \(\times\) 2R antisymmetric matrix, which can be converted into a determinant:

$$C(x) = \det[A]$$

where \(A\) is a block Toeplitz matrix. Apart from a few special cases, it is not known how to calculate such a determinant analytically\(^2\). Thus, we use numerics with a large enough \(R\) to obtain a well-converged result. We employ a continuous (i.e. \(N \to \infty\)) approximation for the elements of the Toeplitz matrix.

$$\langle \sigma_1 \sigma_{1+x} \rangle = \frac{1}{2\pi} \int_0^{2\pi} \frac{dk \tan \frac{\delta_1}{2}}{\cos \frac{\delta_1}{2}} \left[ e^{ik} \langle \sigma_1 \sigma_{1+x} \rangle - e^{-ik} \langle \sigma_{1+x} \sigma_1 \rangle \right]$$

and

$$\langle \sigma_{1+x} \sigma_1 \rangle = \frac{1}{2\pi} \int_0^{2\pi} \frac{dk \tan \frac{\delta_1}{2}}{\cos \frac{\delta_1}{2}} \left[ e^{-ik} \langle \sigma_1 \sigma_{1+x} \rangle - e^{ik} \langle \sigma_{1+x} \sigma_1 \rangle \right]$$

Regarding the parameter \(g\), we mention that it has to be of the order of 500 (2000) for \(g = 0.995\) and \(\delta = 0.01\) (\(g = 1.005\) and \(\delta = 0.01\)) in order for the results to be converged to \(R \to \infty\). For every \(g\) and \(\delta\) sufficiently large \(R\) is chosen to calculate data for Fig. 5.

Finally, we provide definition of the elliptic functions that we use in the Results section:

$$K(x) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-x \sin^2 \phi}}$$

$$E(x) = \int_0^{\pi/2} d\phi \sqrt{1-x \sin^2 \phi}$$

1. Sachdev, S. Quantum Phase Transitions (Cambridge University Press, Cambridge, U.K., 2011).
2. Lewenstein, M., Sanpera, A., Ahufinger, V., Damski, B., Sen(De), A. & Sen, U. Ultracold atomic gases in optical lattices: mimicking condensed matter physics and beyond. Adv. Phys. 56, 243 (2007).
3. Maschler, C. & Ritsch, H. Cold atom dynamics in a quantum optical lattice potential. Phys. Rev. Lett. 95, 260401 (2005).
4. Lin, J., Fernández-Vidal, S., Morigi, G. & Lewenstein, M. Quantum stability of Mott-insulator states of ultracold atoms in optical resonators. New J. Phys. 10, 045002 (2008).
5. Brennecke, F., Donner, T., Ritter, S., Bourdel, T., Köhl, M. & Esslinger, T. Cavity QED with a Bose-Einstein condensate. Nature 450, 268 (2007).
6. Hansch, R., Dobrovitski, V. V., Feiguin, A. E., Gwyat, O. & Awschalom, D. D. Coherent dynamics of a single spin interacting with an adjustable spin bath. Science 320, 352 (2008).
7. Bluhm, H. et al. Dephasing time of GaAs electron-spin qubits coupled to a nuclear bath exceeding 200 µs. Nature Phys. 7, 109 (2011).
8. Cywiński, Ł. Dephasing of electron spin qubits due to their interaction with nuclei in quantum dots. Acta Phys. Pol. A 119, 576 (2011).
9. Zhang, J., Peng, X., Rajendran, N. & Suter, D. Detection of quantum critical points by a probe qubit. Phys. Rev. Lett. 100, 100501 (2008).
10. Porras, D. & Cirac, J. I. Effective quantum spin systems with trapped ions. Phys. Rev. Lett. 92, 207901 (2004).
11. Korenblit, S. et al. Effective quantum spin systems with trapped ions. e-print arXiv:1201.0776 (2012).
12. Islam, R. et al. Quantum simulation of spin models on a lattice with trapped ions. e-print arXiv:1201.0776 (2012).
13. Damski, B., Quan, H. T. & Zurek, W. H. Critical dynamics of decoherence. Phys. Rev. A 83, 062104 (2011).
14. Gu, S.-J. Fidelity approach to quantum phase transitions. Int. J. Mod. Phys. B 24, 4371 (2010).
15. Pfeuty, P. The one-dimensional Ising model with a transverse field. Ann. Phys. 37, 79 (1970).
16. Continentino, M. A. Quantum Scaling in Many-Body Systems (World Scientific Publishing, Singapore, 2001).
17. Daneshita, I. & Naidon, P. Bose-Hubbard ground state: Extended Bogoliubov and variational methods compared with time-evolving block decimation. Phys. Rev. A 79, 043601 (2009).
18. Vidal, G. Efficient classical simulation of slightly entangled quantum computations. Phys. Rev. Lett. 91, 147902 (2003).
19. Vidal, G. Efficient simulation of one-dimensional quantum many-body systems. Phys. Rev. Lett. 93, 040502 (2004).
20. Anderson, P. W. Infrared catastrophe in Fermi gases with local scattering potentials. Phys. Rev. Lett. 18, 1049 (1967).
21. Albuquerque, A. F., Alet, F., Sire, C. & Capponi, S. Quantum critical scaling of fidelity susceptibility. Phys. Rev. B 81, 064418 (2010).
22. Baranov, R. A. Quench dynamics as a probe of quantum criticality. e-print arxiv:0910.0255 (2009).
23. Gritsev, V. & Polkovnikov, A. Universal dynamics near quantum critical points. e-print arxiv:0910.3692 (2009).
24. Rams, M. M. & Damski, B. Quantum fidelity in the thermodynamic limit. Phys. Rev. Lett. 106, 055701 (2011).
25. Dziarmaga, J., Zurek, W. H. & Zwolak, M. Non-local quantum superpositions of topological defects. Nature Physics 8, 49 (2012).
26. Dziarmaga, J. Dynamics of a quantum phase transition: Exact solution of the quantum Ising model. Phys. Rev. Lett. 95, 245701 (2005).
27. Balian, R. & Brezin, E. Nonunitary Bogoliubov transformations and extension of Wick’s theorem. Nuovo Cimento 64, 37 (1969).
28. Barouch, E. & McCoy, B. M. Statistical mechanics of the XY model. II. Spin-correlation functions. Phys. Rev. A 3, 786 (1971).
29. Ito, A. R. & Korepin, V. E. The Fisher-Hartwig formula and entanglement entropy. J. Stat. Phys. 137, 1014 (2009).

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Author contributions
All authors contributed to the research described in this manuscript and to its preparation.

Additional information
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