WEIGHTED INTEGRABILITY OF POLYHARMONIC FUNCTIONS IN THE HIGHER DIMENSIONAL CASE

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ABSTRACT. This paper is concerned with the $L^p$ integrability of $N$-harmonic functions with respect to the standard weights $(1 - |x|^2)^\alpha$ on the unit ball $B$ of $\mathbb{R}^n$, $n \geq 2$. More precisely, our goal is to determine the real (negative) parameters $\alpha$, for which $(1 - |x|^2)^{\alpha/p}u(x) \in L^p(B)$ implies that $u \equiv 0$, whenever $u$ is a solution of the $N$-Laplace equation on $B$. This question is motivated by the uniqueness considerations of the Dirichlet problem for the $N$-Laplacian $\Delta^N$.

Our study is inspired by a recent work of Borichev and Hedenmalm [4], where a complete answer to the above question in the case $n = 2$ is given for the full scale $0 < p < \infty$. When $n \geq 3$, we obtain an analogous characterization for $2 \geq p \geq 1$, and remark that the remaining case can be genuinely more difficult. Also, we extend the remarkable cellular decomposition theorem of Borichev and Hedenmalm to all dimensions.

1. INTRODUCTION

A complex-valued function $u$ defined on a bounded domain $\Omega$ in the Euclidean space $\mathbb{R}^n$ is polyharmonic of order $N$ (or $N$-harmonic) if $u$ is $2N$ times continuously differentiable and

$$\Delta^N u(x) = 0 \quad \text{for all } x \in \Omega,$$

where $\Delta^N$ is the $N$-th iterate of the Laplacian

$$\Delta := \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}.$$

A polyharmonic function of order 1 is just a harmonic function; for $N = 2$, the term biharmonic function, which is important in elasticity theory, is used. There is a vast literature on polyharmonic functions, see [3] and [6] for basic references.

We denote by $\text{PH}_N(\Omega)$ the linear space of all $N$-harmonic functions on $\Omega$. Also, we let $L^p_{\alpha}(\Omega)$ be the space of measurable functions $f : \Omega \to \mathbb{C}$ with

$$\|f\|_{p,\alpha} := \int_{\Omega} |f(x)|^p \left[ \text{dist}(x, \partial \Omega) \right]^{\alpha} dV(x) < \infty,$$

where $dV$ is the Lebesgue measure on $\mathbb{R}^n$. We put

$$\text{PH}_{N,\alpha}^p(\Omega) := \text{PH}_N(\Omega) \cap L^p_{\alpha}(\Omega),$$

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and endow it with the norm or quasi-norm structure of $L^p_\alpha(\Omega)$. This is obviously the subspace of $L^p_\alpha(\Omega)$ consisting of $N$-harmonic functions.

In their remarkable paper [4], Borichev and Hedenmalm raised the following question.

**Problem 1.1.** For which triples $(N, p, \alpha)$ do we have that $\text{PH}^p_{N,\alpha}(\Omega) = \{0\}$?

The interesting case is when $\alpha$ is negative. Then the integrability asks for the function to decay in mean at some rate along the boundary. This is closely related to the uniqueness issues associated with the Dirichlet problem for the $N$-Laplacian equation

\[
\begin{align*}
\Delta^N u &= 0 \quad \text{in } \Omega, \\
\partial^j_n u &= f_j \quad \text{on } \partial\Omega \text{ for } j = 0, 1, \ldots, N - 1,
\end{align*}
\]

where $\partial_n$ stands for the (interior) normal derivative. See [4, Subsection 1.3] for a detailed background.

There clearly exists a critical number $\beta(N, p)$ such that

$$\text{PH}^p_{N,\alpha}(\Omega) = \{0\} \quad \text{for } \alpha < \beta(N, p)$$

and

$$\text{PH}^p_{N,\alpha}(\Omega) \neq \{0\} \quad \text{for } \alpha > \beta(N, p).$$

In fact, $\beta(N, p)$ can be given explicitly by

$$\beta(N, p) := \inf\{\beta_p(u) : u \in \text{PH}_N(\Omega) \setminus \{0\}\},$$

where for a Borel measurable function $u : \Omega \to \mathbb{C}$,

$$\beta_p(u) := \inf\{\alpha \in \mathbb{R} : u \in L^p_\alpha(\Omega)\}.$$ 

If $u \notin L^p_\alpha(\Omega)$ for every $\alpha \in \mathbb{R}$, we write $\beta_p(u) := +\infty$. Following [4], we call the function $p \mapsto \beta(N, p)$ the critical integrability type curve for the $N$-harmonic functions, and the function $(N, p) \mapsto \beta(N, p)$ the critical integrability type curves for the polyharmonic functions.

When $n = 2$ and $\Omega$ is the unit disk $\mathbb{D}$ in the plane, Borichev and Hedenmalm [4] completely resolved Problem 1.1 by giving an explicit formula for $\beta(N, p)$, the critical integrability type curves for the polyharmonic functions. To avoid repetition, we do not include the detailed results here.

The aim of this paper is to extend the main results of [4] to all dimensions. Let $B$ stand for the open unit ball of $\mathbb{R}^n$. Also, we write $S$ for the unit sphere, the boundary of $B$. By $d\sigma$, we mean the $(n - 1)$-dimensional surface measure on $S$, normalized so that $\sigma(S) = 1$. We investigate the Problem 1.1 when $\Omega = B$, for $n \geq 2$. Our first main result is the following:

**Theorem 1.2.** The critical integrability type curve for the polyharmonic functions on $B$ is given by

$$\beta(N, p) = \min_{j: 0 \leq j \leq N} b_{j,N}(p)$$

for $N \in \mathbb{N}$ and $\frac{n-2}{n-1} \leq p < \infty$, where

$$b_{0,N}(p) := -1 - (N - 1)p,$$

$$b_{j,N}(p) := \max\{-1 - (N + j - 1)p, -n - (N - j - n + 1)p\}$$
for \( j = 1, \ldots, N \). In particular, when \( n \geq 3 \),
\[
\beta(n, p) = \begin{cases} 
-1 - Np, & \text{if } \frac{n-2}{n-1} \leq p < \frac{n-1}{n}, \\
-n - (N - n)p, & \text{if } \frac{n-1}{n} \leq p < 1, \\
-1 - (N - 1)p, & \text{if } p \geq 1.
\end{cases}
\]

Here and throughout this paper, when \( n = 2 \), the expression \( \frac{n-2}{n-1} \leq p < \infty \) should be interpreted as \( 0 < p < \infty \).

The requirement \( p \geq \frac{n-2}{n-1} \) stems from the subharmonicity of the gradient (see \([12]\)), which is a non-issue when \( n = 2 \). It is natural to expect that the formula \([1.2]\) in Theorem \([1.2]\) is true for the full range of \( p \): \( 0 < p < \infty \). Unfortunately, this is not the case if \( n \geq 3 \). See Section 8 for an explanation of why.

Borichev and Hedenmalm \([4]\) also found a novel structure decomposition theorem of polyharmonic functions on the unit disk, referred to as the cellular decomposition theorem, which decomposes the polyharmonic weighted \( L^p \) space in a canonical fashion. The cellular decomposition theorem is closely related to the classical Almansi representation. However, here the terms are mixed in a way that is optimal for the boundary behavior. Our second result is a higher-dimensional generalization of this decomposition.

**Theorem 1.3** (Structure theorem). Let \( 0 < p < \infty \), \( N \in \mathbb{N} \) and \( \alpha \in \mathbb{R} \). Then every \( u \in \text{PH}^p_{N, \alpha}(\mathbb{B}) \) has a unique decomposition
\[
u = w_0 + M[w_1] + \cdots + M^{N-1}[w_{N-1}],
\]
where each term \( M^j[w_j] \) is in \( \text{PH}^p_{N, \alpha}(\mathbb{B}) \), while the functions \( w_j \) are \((N-j)\)-harmonic and solve \( L_{N-j-1}[w_j] = 0 \) on \( \mathbb{B} \), for \( j = 0, \ldots, N - 1 \). Here \( L_{\theta} \) is the second order elliptic partial differential operator given by
\[
L_{\theta}[u] := (1 - |x|^2)\Delta u + 4\theta R[u] + 2\theta(n - 2 - 2\theta)u,
\]
where \( \theta \) is a real parameter and \( R[u](x) := x \cdot \nabla f(x) \) is the radial derivative of \( u \).

The differential operator \( L_{\theta} \) defined by \([1.7]\) is the higher dimensional analogue of the differential operator introduced by Borichev and Hedenmalm in \([4]\) p. 474, (3.5)), which plays a crucial role in our analysis. It has appeared implicitly in \([7, 8, 9]\) and is closely related to the theory of axially symmetric potentials due to Weinstein (see, e.g., \([13]\)). Also, when \( n = 2 \), the operator \( L_{\theta} \) is related in a simple way to the operators \( D_{\alpha} \) introduced by Olofsson in \([10]\). See \([4]\) p. 474).

The result stated in Theorem \([1.3]\) can be improved by specifying which terms in the decomposition \([1.6]\) must necessarily vanish.

Following \([4]\), we denote by \( \mathcal{A}_N \) the open set
\[
\mathcal{A}_N := \{(p, \alpha) \in \mathbb{R}^2 : 0 < p < +\infty \text{ and } \alpha > \beta(N, p)\}
\]
for fixed \( N \geq 2 \), and refer to it as the admissible region. So the definition of \( \beta(N, p) \) is equivalent to the statement
\[
(p, \alpha) \in \mathcal{A}_N \iff \text{PH}^p_{N, \alpha}(\mathbb{B}) \neq \{0\}.
\]
Denote by \( \overline{\mathcal{A}}_N \) the subset of \( \mathcal{A}_N \):
\[
\overline{\mathcal{A}}_N := \{(p, \alpha) \in \mathbb{R}^2 : \frac{n-2}{n-1} \leq p < +\infty \text{ and } \alpha > \min_{j,0 \leq j \leq N} b_j,N(p)\}.
\]
For a point \((p, \alpha) \in \mathcal{A}_N\), we put

\[
J(p, \alpha) := \{ j \in \{0, \ldots, N-1\} : \alpha > a_{N-j,N}(p) \},
\]

where

\[
a_{j,N}(p) := \min \{ b_{j,N}(p), -1 - (N-j)p \}
\]

for \(N \in \mathbb{N}\) and \(j \in \{1, \cdots, N\}\).

**Theorem 1.4.** Suppose \((p, \alpha) \in \bar{\mathcal{A}}_N\). Then every \(u \in \text{PH}^p_{N,\alpha}(\mathbb{B})\) has a unique decomposition

\[
u = \sum_{j \in J(p, \alpha)} M^j[w_j],
\]

where each term \(M^j[w_j]\) is in \(\text{PH}^p_{N,\alpha}(\mathbb{B})\), while the functions \(w_j\) are \((N-j)\)-harmonic and solve \(L_{N-j-1}[w_j] = 0\) on \(\mathbb{B}\), for \(j \in J(p, \alpha)\).

Note that each term \(M^j[w_j]\) with \(j \in J(p, \alpha)\) is allowed to be nontrivial, so the above result is sharp.

We follow the strategy of [4] whenever applicable. There some notable differences, such as the lack of powerful tools from complex analysis that only work in the plane. In addition, instead of defining \(N\)-harmonicity in the sense of distribution theory, we can use the more elementary standard definition (but our results remain valid in the former case). This is the case, because we use the simpler test functions in Lemma 2.10 without resorting to method of Olofsson [10].

The rest of the paper is organized as follows: Section 2 is devoted to the basic properties of the differential operator \(L_\theta\). Our main results, Theorems 1.3 and 1.2 will be proved in Section 3 and Sections 4-6, respectively. Theorem 1.4 is then proved in Section 7. The last Section 8 is devoted to concluding remarks and open problems.

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2. The differential operator \(L_\theta\)

2.1. Some elementary identities. Let \(\lambda\) be a real number. We define the multiplication operator \(M^\lambda\) by

\[
M^\lambda[u](x) := (1 - |x|^2)^\lambda u(x), \quad x \in \mathbb{B},
\]

and in particular, \(M := M^1\). We also write \(M^0 := I\).

The following proposition is called the correspondence principle.

**Proposition 2.1.** For any \(\theta, \lambda \in \mathbb{R}\), we have

\[
L_\theta M^\lambda = M^\lambda L_{\theta-\lambda} + 4\lambda(\lambda - 1 - 2\theta)M^{\lambda-1}.
\]
Proof. We first compute
\[
\Delta \left\{ (1 - |x|^2)\lambda u(x) \right\} = (1 - |x|^2)\lambda \Delta u(x) + 2\nabla \left\{ (1 - |x|^2)\lambda \right\} \cdot \nabla u(x) \\
+ u(x)\Delta \left\{ (1 - |x|^2)\lambda \right\} \\
= (1 - |x|^2)\lambda \Delta u(x) - 4\lambda(1 - |x|^2)^{\lambda-1}R u(x) \\
- 2\lambda(2\lambda + n - 2)(1 - |x|^2)^{\lambda-1} u(x) \\
+ 4\lambda(\lambda - 1)(1 - |x|^2)^{\lambda-2} u(x),
\]
which can be written as
\[
(2.2) \quad \Delta M^\lambda = M^\lambda \Delta - 4\lambda M^{\lambda-1} R - 2\lambda(2\lambda + n - 2)M^{\lambda-1} + 4\lambda(\lambda - 1)M^{\lambda-2}.
\]
Also, it is easy to verify that
\[
RM^\lambda = M^\lambda R + 2\lambda M^{\lambda-1} - 2\lambda M^{\lambda-1}.
\]
Therefore,
\[
\begin{align*}
L_\theta M^\lambda &= M\Delta M^\lambda + 4\theta RM^\lambda + 2\theta(n - 2 - 2\theta)M^\lambda \\
&= M \left\{ M^\lambda \Delta - 4\lambda M^{\lambda-1} R - 2\lambda(2\lambda + n - 2)M^{\lambda-1} + 4\lambda(\lambda - 1)M^{\lambda-2} \right\} \\
&\quad + 4\theta(M^\lambda R + 2\lambda M^{\lambda-1} - 2\lambda M^{\lambda-1}) + 2\theta(n - 2 - 2\theta)M^\lambda \\
&= M^\lambda \left\{ M\Delta + 4(\theta - \lambda)R + 2(\theta - \lambda)(n - 2 - 2\theta + 2\lambda)I \right\} \\
&\quad + 4\lambda(\lambda - 1 - 2\theta)M^{\lambda-1} \\
&= M^\lambda L_{\theta-\lambda} + 4\lambda(\lambda - 1 - 2\theta)M^{\lambda-1},
\end{align*}
\]
as desired.

We single out two special cases of Proposition 2.1 as separate statements.

Corollary 2.2. For any \( \theta \in \mathbb{R} \) we have
\[
L_\theta M = ML_{\theta-1} - 8\theta I.
\]
More generally,
\[
L_\theta M^j = M^j L_{\theta-j} + 4j(j - 1 - 2\theta)M^{j-1}, \quad j = 1, 2, \ldots.
\]

Corollary 2.3. For any \( \theta \in \mathbb{R} \) we have
\[
L_\theta M^{1+2\theta} = M^{1+2\theta} L_{-\theta-1}.
\]

Proposition 2.4. We have that
\[
\Delta L_\theta = L_{\theta-1}\Delta.
\]
More generally,
\[
\Delta^j L_\theta = L_{\theta-j}\Delta^j, \quad j = 1, 2, \ldots.
\]

Proof. It is clear that
\[
\Delta R = R\Delta + 2\Delta.
\]
Also, by applying (2.2) to \( \Delta u \), we get
\[
\Delta M\Delta = M\Delta^2 - 4R\Delta - 2n\Delta.
\]
It follows that
\[
\Delta L_\theta = \Delta M \Delta + 4\theta \Delta R + 2\theta(n - 2 - 2\theta)\Delta
= (M \Delta^2 - 4R \Delta - 2n \Delta) + 4\theta(R \Delta + 2\Delta) + 2\theta(n - 2 - 2\theta)\Delta
= \{M \Delta + 4(\theta - 1)R + 2(\theta - 1)(n - 2\theta)\Delta\} \Delta
= L_\theta \Delta.
\]
The identity (2.7) follows by iteration of (2.6). \qed

The next result for \(n = 2\) is Proposition 6.1 from [4]. It explains the usefulness of the operators \(L_\theta\).

**Proposition 2.5.** We have the following factorization:
\[
L_0 L_1 \cdots L_{N-1} = M^N \Delta^N, \quad N = 1, 2, 3, \ldots
\]

*Proof.* Since, by definition, \(L_0 = M \Delta\), the assertion holds trivially for \(N = 1\). Suppose now that it holds for \(N = k\);
\[
L_0 L_1 \cdots L_{k-1} = M^k \Delta^k.
\]
Then, by (2.7),
\[
L_0 L_1 \cdots L_k = M^k \Delta^k L_k = M^k L_0 \Delta^k = M^k (M \Delta) \Delta^k = M^{k+1} \Delta^{k+1}.
\]
The proof is completed by virtue of the induction principle. \qed

**Corollary 2.6.** If \(u\) solves \(L_{N-1} [u] = 0\) in \(\mathbb{B}\), then \(u\) is \(N\)-harmonic in \(\mathbb{B}\). More generally, if \(u\) solves \(L_{N-j-1} [u] = 0\) with \(j \in \{0, \ldots, N-1\}\), then \(M^j [u]\) is \(N\)-harmonic in \(\mathbb{B}\).

*Proof.* Since \(L_{N-j-1} [u] = 0\), using (2.4), we have
\[
L_{N-1} [M^j [u]] = M^j [L_{N-j-1} [u]] + 4j(j - 2N + 1)M^{j-1} [u]
= 4j(j - 2N + 1)M^{j-1} [u].
\]
We proceed iteratively and discover that for \(k = 1, \ldots, N\),
\[
L_{N-k} \cdots L_{N-1} [M^j [u]] = 4^k (j - k + 1)_k (j - 2N + 1)_k M^{j-k} [u],
\]
where \((a)_0 := 1\) and \((a)_k := a(a + 1) \cdots (a + k - 1)\) for \(k = 1, 2, \ldots\) are the ascending Pochhammer symbols. When \(k > j\), the right hand side of (2.9) vanishes. In particular, when \(k = N\), (2.9) reads
\[
L_0 \cdots L_{N-1} [M^j [u]] = 0.
\]
In view of (2.8), this implies that \(M^j [u]\) is \(N\)-harmonic in \(\mathbb{B}\). \qed

**Corollary 2.7.** If \(u\) is \(N\)-harmonic in \(\mathbb{B}\), then \(L_{N-1} [u]\) is \((N - 1)\)-harmonic. If \(N = 1\), this should be interpreted as \(L_0 [u] = 0\).

2.2. **Special solutions of the equation** \(L_\theta [u] = 0\). For \(\zeta \in \mathbb{S}\), let
\[
P_\theta(x, \zeta) := C_\theta \left(\frac{1 - |x|^2}{|x - \zeta|^{n+2\theta}}\right)^{1/2}, \quad x \in \mathbb{B},
\]
where
\[
C_\theta := \frac{\Gamma(n/2 + \theta)\Gamma(1 + \theta)}{\Gamma(n/2)\Gamma(1 + 2\theta)}.
\]
Lemma 2.8. Let $\theta \in \mathbb{R}$. Then
\begin{equation}
L_\theta [P_0(\cdot, \zeta)] = 0
\end{equation}
holds for any fixed $\zeta \in \mathbb{S}$.

Proof. In view of (2.5), it suffices to show that
\begin{equation}
L_{-1-\theta} \left[ \frac{1}{|x-\zeta|^{n+2\theta}} \right] = 0,
\end{equation}
where the differentiation is with respect to $x$. Simple calculations yield
\begin{equation}
\Delta \left[ \frac{1}{|x-\zeta|^{n+2\theta}} \right] = (2 + 2\theta)(n + 2\theta) \frac{1 - |x|^2}{|x-\zeta|^{n+2\theta+2}}
\end{equation}
\begin{equation}
\text{and}
\end{equation}
\begin{equation}
R \left[ \frac{1}{|x-\zeta|^{n+2\theta}} \right] = (-n - 2\theta) \frac{|x|^2 - x \cdot \zeta}{|x-\zeta|^{n+2\theta+2}}.
\end{equation}
It follows that
\begin{equation}
L_{-1-\theta} \left[ \frac{1}{|x-\zeta|^{n+2\theta}} \right] = (2 + 2\theta)(n + 2\theta) \frac{1 - |x|^2}{|x-\zeta|^{n+2\theta+2}}
\end{equation}
\begin{equation}
+ 4(-1-\theta)(-n - 2\theta) \frac{|x|^2 - x \cdot \zeta}{|x-\zeta|^{n+2\theta+2}}
\end{equation}
\begin{equation}
+ 2(-1-\theta)[n - 2 - 2(-1-\theta)] \frac{1}{|x-\zeta|^{n+2\theta}}
\end{equation}
\begin{equation}
= 0,
\end{equation}
as desired. \qed

For every function $f \in L^1(\mathbb{S}, d\sigma)$ we define a function $P_\theta[f]$ on $\mathbb{B}$ as follows.
\begin{equation}
P_\theta[f](x) := \int_{\mathbb{S}} P_0(x, \zeta) f(\zeta) d\sigma(\zeta), \quad x \in \mathbb{B}.
\end{equation}
The function $P_\theta[f]$ will be called the $\theta$-Poisson integral of $f$.

Lemma 2.9 ([8, Theorem 2.4]). Let $\theta > -1/2$. The Dirichlet problem
\begin{equation}
\begin{cases}
L_\theta[u] = 0, & \text{in } \mathbb{B} \\
u = f, & \text{on } \mathbb{S}
\end{cases}
\end{equation}
has a unique solution, which is given by $u = P_\theta[f]$.

We consider the hypergeometric differential equation
\begin{equation}
z(1-z)f'''(z) + [c - (a + b + 1) z] f'(z) - abf(z) = 0,
\end{equation}
where $a, b, c$ are complex parameters. For $c \neq 0, -1, -2, \ldots$, the hypergeometric function
is defined by the power series
\begin{equation}
\textstyle \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \quad |z| < 1,
\end{equation}
where $(a)_0 := 1 \text{ and } (a)_k := a(a+1) \cdots (a+k-1)$ for $k = 1, 2, \ldots$. It is well-known and
straightforward to check that the function $\textstyle \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}$ in the unit disk $|z| < 1$. See [2] for a complete account on the subject.
Lemma 2.10. The function

\[(2.16) \quad \Phi_\theta(x) := {}_2F_1 \left( -\theta, \frac{n}{2} - 1 - \theta; \frac{n}{2}; |x|^2 \right) \]

solves the equation \( L_\theta[u] = 0 \) in \( \mathbb{B} \).

**Proof.** In the spherical-polar coordinates \( x = r\zeta, r > 0, \zeta \in \mathbb{S} \), the Laplace operator \( \Delta \) can be written as

\[(2.17) \quad \Delta = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_\mathbb{S}, \]

where

\[ \Delta_\mathbb{S} := \sum_{i=1}^{n-1} \frac{\partial^2}{\partial \zeta_i^2} - \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \zeta_i \zeta_j \frac{\partial^2}{\partial \zeta_i \partial \zeta_j} - (n-1) \sum_{i=1}^{n-1} \zeta_i \frac{\partial}{\partial \zeta_i} \]

is the Laplace-Beltrami operator on the unit sphere \( \mathbb{S} \). See for instance [5, Lemma 1.4.1].

Now we consider radial solutions of the equation \( L_\theta[u] = 0 \). Suppose that \( u(x) = f(|x|^2) \), where \( f \) is a \( C^2 \) function on the interval \( (0, 1) \). Then, with \( r = |x| \),

\[ \Delta u(x) = 4r^2 f''(r^2) + 2nf'(r^2), \quad R[u](x) = 2r^2 f'(r^2) \]

and hence

\[ L_\theta[u](x) = 4r^2(1-r^2)f''(r^2) + [2n(1-r^2) + 8\theta r^2] f'(r^2) + 2\theta(n - 2 - 2\theta) f(r^2). \]

Therefore, the differential equation \( L_\theta[u] = 0 \) deduces to

\[(2.18) \quad z(1-z)f''(z) + \left\{ \frac{n}{2} - \left( \frac{n}{2} - 2\theta \right) z \right\} f'(z) + \theta \left( \frac{n}{2} - 1 - \theta \right) f(z) = 0. \]

This is the hypergeometric differential equation, with parameters

\[ a = -\theta, \quad b = \frac{n}{2} - 1 - \theta, \quad c = \frac{n}{2} \]

The hypergeometric function

\[ {}_2F_1 \left( -\theta, \frac{n}{2} - 1 - \theta; \frac{n}{2}; z \right) \]

satisfies the equation (2.18) in the unit disk \( |z| < 1 \), and hence the function \( \Phi_\theta \) solves the equation \( L_\theta[u] = 0 \) in \( \mathbb{B} \).

**Lemma 2.11.** \( \Phi_\theta \) is bounded on \( \mathbb{B} \) if and only if \( \theta > -1/2 \).

**Proof.** By definition,

\[ \Phi_\theta(x) = \sum_{k=0}^{\infty} \frac{(-\theta)_k(n/2 - 1 - \theta)_k}{(1)_k(n/2)_k} |x|^{2k}. \]

It is easy to see that the coefficients in the series are of order \( k^{-2\theta-2} \) as \( k \to \infty \), and the assertion of the lemma follows.

**Corollary 2.12.** Suppose that \( 0 < p < \infty, N \in \mathbb{N} \) and \( j \in \{0, 1, \ldots, N - 1\} \). The function \( M^j[\Phi_{N-j-1}] \) is in \( \text{PH}_{N,\alpha}^p(\mathbb{B}) \) for any \( \alpha > -1 - jp \).

**Proof.** By Lemma 2.10 and Corollary 2.6, the function \( M^j[\Phi_{N-j-1}] \) is \( N \)-harmonic. In view of Lemma 2.11, the function \( \Phi_{N-j-1} \) is bounded in \( \mathbb{B} \), so it is easy to check that \( M^j[\Phi_{N-j-1}] \) is in \( \text{PH}_{N,\alpha}^p(\mathbb{B}) \) for any \( \alpha > -1 - jp \).
2.3. Mapping properties of \( L_\theta \). We will next analyse the image of \( \text{PH}^p_{N,\alpha} (\mathbb{B}) \) under \( L_\theta \).

Lemma 2.13 ([11] Lemma 5). Suppose that \( 0 < p < \infty \) and \( u \) is \( N \)-harmonic in \( \mathbb{B} \). Then

\[
|u(x)|^p \lesssim r^{-n} \int_{B(x,r)} |u(y)|^p dV(y)
\]

for all \( x \in \mathbb{B} \) and \( r \in (0,1) \), where the implicit constant depends only on \( p, N \) and \( n \).

Lemma 2.14. Suppose \( 0 < p < \infty \) and \( \alpha \in \mathbb{R} \). Then

\[
|u(x)| \lesssim (1 - |x|^2)^{-(n+\alpha)/p} |u|_{p,\alpha}
\]

for all \( u \in \text{PH}^p_{N,\alpha} (\mathbb{B}) \) and \( x \in \mathbb{B} \).

Proof. Let \( u \in \text{PH}^p_{N,\alpha} (\mathbb{B}) \) and \( x \in \mathbb{B} \) be fixed. Applying Lemma 2.13 with \( r = \frac{1}{2} (1 - |x|) \), we have

\[
|u(x)|^p \lesssim (1 - |x|)^{-n} \int_{B(x,\frac{1}{2}(1-|x|))} |u(y)|^p dV(y)
\]

Note that if \( y \in B(x, \frac{1}{2}(1 - |x|)) \) then \( 1 - |y|^2 \approx 1 - |x|^2 \). It follows from (2.21) that

\[
|u(x)|^p \lesssim (1 - |x|)^{-n-\alpha} \int_{B(x,\frac{1}{2}(1-|x|))} |u(y)|^p (1 - |y|^2)^\alpha dV(y)
\]

\[
\lesssim (1 - |x|)^{-n-\alpha} |u|_{p,\alpha}
\]

as desired. \( \square \)

Lemma 2.15 ([11] Lemma 6). Suppose that \( 0 < p < +\infty \), \( N \in \mathbb{N} \) and \( \alpha \in \mathbb{R} \). If \( u \in \text{PH}^p_{N,\alpha} (\mathbb{B}) \) then \( \partial_j u \in \text{PH}^p_{N,\alpha+p} (\mathbb{B}) \), \( j = 1, \ldots, n \).

Corollary 2.16. Suppose that \( 0 < p < +\infty \), \( N \in \mathbb{N} \) and \( \alpha \in \mathbb{R} \). If \( u \in \text{PH}^p_{N,\alpha} (\mathbb{B}) \) then

\[
\Delta^k u \in \text{PH}^p_{N-k,\alpha+2k\alpha} (\mathbb{B}) \text{ for each } k \in \{1, \ldots, N-1\}.
\]

Proposition 2.17. Suppose that \( 0 < p < +\infty \), \( N \in \mathbb{N} \) and \( \alpha \in \mathbb{R} \). If \( u \in \text{PH}^p_{N,\alpha} (\mathbb{B}) \) then \( L_\theta [u] \in \text{PH}^p_{N,\alpha+p}(\mathbb{B}) \).

Proof. Suppose \( u \in \text{PH}^p_{N,\alpha} (\mathbb{B}) \). We show that each term on the right hand side of (1.7) belongs to \( \text{PH}^p_{N,\alpha+p} (\mathbb{B}) \). First, by Corollary 2.16 we have \( \Delta u \in \text{PH}^p_{N-1,\alpha+2\alpha} (\mathbb{B}) \) and hence \( M \Delta u \in \text{PH}^p_{N,\alpha+p} (\mathbb{B}) \). Next, it is easy to check that \( \Delta^N R = R \Delta^N + 2N \Delta^N \). So \( R[u] \) is \( N \)-harmonic. It then follows from Lemma 2.15 that \( R[u] \in \text{PH}^p_{N,\alpha+p} (\mathbb{B}) \). We also have \( u \in \text{PH}^p_{N,\alpha+p} (\mathbb{B}) \) because trivially \( \text{PH}^p_{N,\alpha} (\mathbb{B}) \subset \text{PH}^p_{N,\alpha+p} (\mathbb{B}) \). By linearity, we are done. \( \square \)

3. Proof of Theorem 1.3

Having proved the identities (2.4), (2.7) and Proposition 2.17, the proof of Theorem 1.3 follows the same line of reasoning as that of Theorem 3.4 of [4]. For the readers’ convenience, we repeat it here.
Uniqueness. It suffices to show that if

\[ \sum_{j=0}^{N-1} M_j[w_j] = 0 \]

with \( w_j \) satisfying \( L_{N-j-1}[w_j] = 0, j = 0, \ldots, N-1 \), then all the functions \( w_j \) vanish.

To prove this we proceed by induction on \( N \). Clearly, when \( N = 1 \), then (3.1) just states that \( w_0 = 0 \), as needed. For the induction step, assume the above assertion holds for \( N = N_0 \).

Suppose now that

\[ \sum_{j=0}^{N_0-1} M_j[w_j] = 0 \]

with \( w_j \) satisfying \( L_{N_0-j}[w_j] = 0, j = 0, \ldots, N_0 \). Applying the operator \( L_{N_0} \) to both sides of (3.2), and using (2.4), we obtain

\[
\sum_{j=0}^{N_0-1} \left\{ M_jL_{N_0-j}[w_j] + 4j(j-2N_0-1)M^{j-1}_j[w_j] \right\} = 0.
\]

Since \( L_{N_0-j}[w_j] = 0, j = 0, \ldots, N_0 \), setting \( \tilde{w}_j := (j+1)(j-2N_0)w_{j+1} \), the equation becomes

\[
\sum_{j=0}^{N_0-1} M_j[\tilde{w}_j] = 0.
\]

By induction, it is straightforward to deduce uniqueness from this.

Existence. Again, we argue by induction on \( N \). The case \( N = 1 \) is trivial. For the induction step, assume the assertion of the theorem holds for \( N = N_0 > 1 \).

Now, we suppose that \( u \in PH_{N_0,\frac{N_0-1}{4}}(\mathbb{B}) \).

By Proposition 2.17, \( L_{N_0}[u] \in PH_{N_0,\alpha+(j+1)p}(\mathbb{B}) \). Then by the induction hypothesis,

\[ L_{N_0}[u] = \sum_{j=0}^{N_0-1} M_j[v_j], \]

where each term \( M_j[v_j] \) is in \( PH_{N_0,\alpha+p}(\mathbb{B}) \), with \( v_j \) solving \( L_{N_0-j-1}[v_j] = 0 \). Moreover,

\[ v_j \in PH_{N_0-j,\alpha+(j+1)p}(\mathbb{B}). \]

Write

\[ V := \frac{1}{4} \sum_{j=0}^{N_0-1} \frac{1}{(j+1)(2N_0-j)} M^{j+1}_j[v_j]. \]
Note that $V \in \text{PH}_{N,0+1,\alpha}(\mathbb{B})$. So $u + V \in \text{PH}_{N,0+1,\alpha}(\mathbb{B})$, and

$$
L_{N,0}[u + V] = \sum_{j=0}^{N-1} \left\{ M_j[v_j] + \frac{1}{4(j+1)(2N_0 - j)} L_{N,0} M^{j+1}[v_j] \right\} = \sum_{j=0}^{N-1} \left\{ M_j[v_j] + \frac{1}{4(j+1)(2N_0 - j)} (M^{j+1} L_{N,0-j-1}[v_j] - 4(j+1)(2N_0 - j) M_j[v_j]) \right\} = 0,
$$

where we used (2.4) and $L_{N,0-j-1}[v_j] = 0$. So, with $w_0 := u + V$ and

$$
w_j := \frac{1}{4j(2N_0 - j + 1)} v_{j-1}, \quad j = 1, \ldots, N_0,
$$

we see that

$$
u = \sum_{j=0}^{N_0} M_j[w_j],
$$

where $w_j$ is $(N_0 + 1 - j)$-harmonic with $L_{N_0-j}[w_j] = 0$, for $j = 0, \ldots, N_0$. Moreover, (3.4) together with Corollary 2.6 lead to $w_j \in \text{PH}_{N_0+1-j,\alpha+j}(\mathbb{B})$ and $M_j[w_j] \in \text{PH}_{N_0+1,\alpha}(\mathbb{B})$.

### 4. Proof of Theorem 1.2 Part 1

When $n \geq 3$, the formula (1.5) follows immediately from (1.2). So, we only prove (1.2). For convenience, we divide the proof into two separate theorems.

**Theorem 4.1.** Suppose that $0 < p < \infty$, $N \in \mathbb{N}$ and $\alpha$ is real. Then

$$
\text{PH}_{N,\alpha}(\mathbb{B}) = \{0\} \quad \iff \quad \alpha \leq \min_{j,0 \leq j \leq N} b_{j,N}(p).
$$

**Theorem 4.2.** Suppose that $N \in \mathbb{N}$, $\frac{n-2}{n-1} \leq p < \infty$ and $\alpha$ is real. Then

$$
\alpha \leq \min_{j,0 \leq j \leq N} b_{j,N}(p) \quad \implies \quad \text{PH}_{N,\alpha}(\mathbb{B}) = \{0\}.
$$

Note that even for $n \geq 3$ we do not require that $p \geq \frac{n-2}{n-1}$ in Theorem 4.1. This section is devoted to the proof of Theorem 4.1 and Theorem 4.2 will be proved in Section 6.

Given $N \in \mathbb{N}$ and $j \in \{1, \ldots, N\}$, let

$$
U_{j,N}(x) := \frac{(1 - |x|^2)^{N+j-1}}{|x - e_1|^{n+2j-1}}, \quad x \in \mathbb{B},
$$

where $e_1 = (1,0,\ldots,0)$ is the first coordinate vector in $\mathbb{R}^n$, while for $j = 0$ we put

$$
U_{0,N}(x) := (1 - |x|^2)^{N-1}.
$$

**Lemma 4.3.** For $N = 1, 2, 3, \ldots$ and $j = 0, 1, \ldots, N$, the functions $U_{j,N}$ are all $N$-harmonic in $\mathbb{B}$.

**Proof.** The function $U_{0,N}$ is clearly $N$-harmonic in $\mathbb{B}$. For $j \in \{0, \ldots, N\}$, note that $U_{j,N} = M^{N-j}[P_{j-1}(\cdot, e_1)]$, where $P_0$ is defined as in (2.10). By Lemma 2.8 $P_{j-1}(\cdot, e_1)$ solves $L_{j-1}[u] = 0$. Hence, by Corollary 2.6 $U_{j,N}$ is $N$-harmonic in $\mathbb{B}$.

$\square$
Lemma 4.4. Let \( a, b \in \mathbb{R} \). The integral
\[
I(a, b) := \int_{\mathbb{S}} \frac{(1 - |x|^2)^a}{|x - e_1|^n + a + b} dV(x)
\]
is finite if and only if \( a > -1 \) and \( b < 0 \). Moreover, if \( a > -1 \) and \( b < 0 \) then
\[
I(a, b) = \frac{\pi^{n/2} \Gamma(1 + a) \Gamma(-b)}{\Gamma((n + a - b)/2) \Gamma((2 + a - b)/2)}.
\]

Proof. We first recall the following formula (see [8, lemma 2.1]):
\[
\int_{\mathbb{S}} \frac{d\sigma(\zeta)}{|y - \zeta|^{2t}} = \, _2F_1\left( t, t - \frac{n}{2} + 1; \frac{n}{2}; |y|^2 \right), \quad y \in \mathbb{S},
\]
where \( t \) is a real parameter. By integrating in polar coordinates and using the above formula, we find that
\[
I(a, b) = \omega_{n-1} \int_{0}^{1} r^{n-1}(1 - r^2)^a \left\{ \int_{S} \frac{d\sigma(\zeta)}{|re_1 - \zeta|^{n+a+b}} \right\} dr
\]
\[
= \omega_{n-1} \int_{0}^{1} r^{n-1}(1 - r^2)^a \, _2F_1\left( \frac{n + a + b}{2}, \frac{2 + a + b}{2}; \frac{n}{2}; r^2 \right) dr
\]
\[
= \frac{\omega_{n-1}}{2} \sum_{j=0}^{\infty} \frac{(n + a + b)/2)_j}{(n/2)_j} \frac{(2 + a + b)/2)_j}{(1)_j} \int_{0}^{1} r^{j+n/2-1}(1 - r)^a dr,
\]
where \( \omega_{n-1} := 2\pi^{n/2}/\Gamma(n/2) \) stands for the area of the unit sphere \( \mathbb{S} \). For \( a \leq -1 \), we have \( I(a, b) = +\infty \). For \( a > -1 \), we evaluate the (Beta) integral to obtain
\[
I(a, b) = \frac{\pi^{n/2} \Gamma(1 + a)}{\Gamma(n/2 + 1 + a)} \sum_{j=0}^{\infty} \frac{(n + a + b)/2)_j}{(n/2 + 1 + a)_j} \frac{(2 + a + b)/2)_j}{(1)_j}.
\]
Using the well-known Stirling formula
\[
\frac{\Gamma(j + t)}{\Gamma(j + s)} \sim j^{t-s} \quad \text{as} \quad j \to +\infty,
\]
we find that the sum on the right-hand side of (4.2) converges if and only if
\[
\sum_{j=1}^{\infty} j^{b-1} < \infty,
\]
if and only if \( b < 0 \).

Now we assume that \( a > -1 \) and \( b < 0 \). Then the sum on the right-hand side of (4.2) equals
\[
\, _2F_1\left( \frac{n + a + b}{2}, \frac{2 + a + b}{2}; \frac{n}{2} + 1 + a; 1 \right) = \frac{\Gamma(n/2 + 1 + a) \Gamma(-b)}{\Gamma((n + a - b)/2) \Gamma((2 + a - b)/2)},
\]
where we have used the well-known formula of Gauss
\[
\, _2F_1(\alpha, \beta; \gamma; 1) = \frac{\Gamma(\gamma) \Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha) \Gamma(\gamma - \beta)}, \quad \text{Re}(\gamma - \alpha - \beta) > 0.
\]
This completes the proof. \( \square \)
Lemma 4.5. For each fixed \( N \in \mathbb{N} \) and \( j \in \{0, 1, \ldots, N\} \), the function \( U_{j,N} \) is in \( \text{PH}_{N,\alpha}(\mathbb{B}) \) if and only if \( \alpha > b_{j,N}(p) \).

Proof. Clearly, \( U_{0,N} \in \text{PH}_{N,\alpha}(\mathbb{B}) \) if and only if \((N-1)p + \alpha > -1\), which is exactly \( \alpha > b_{0,N}(p) \). For \( j \in \{1, \ldots, N\} \), to decide when \( U_{j,N} \in \text{PH}_{N,\alpha}(\mathbb{B}) \), we note that

\[
\|U_{j,N}\|_{p,\alpha}^{p} = \int_{\mathbb{B}} \frac{(1 - |x|^{2})^{(N+j-1)p + \alpha}}{|x - e_{1}|^{(\alpha + 2j - 2)p}} dV(x),
\]

which is finite if and only if

\[
(N + j - 1)p + \alpha > -1, \quad \text{and} \quad (n + 2j - 2)p - n - (N + j - 1)p - \alpha < 0,
\]

in view of Lemma 4.4. The claim follows, since the condition (5.2) is exactly the same as \( \alpha > b_{j,N}(p) \).

Lemma 4.5 shows that if \( \alpha \) satisfies

\[
\alpha > \min_{j:0 \leq j \leq N} b_{j,N}(p),
\]

then one of the functions \( U_{0,N}, U_{1,N}, \ldots, U_{N,N} \) will be in \( \text{PH}_{N,\alpha}(\mathbb{B}) \), so that in particular, \( \text{PH}_{N,\alpha}(\mathbb{B}) \neq \{0\} \). This completes the proof of Theorem 4.1.

5. Preliminaries for the proof of Theorem 4.2

According to the classical Almansi representation, \( u \) is \( N \)-harmonic if and only if it is of the form

\[
u(x) = u_{0}(x) + |x|^{2}u_{1}(x) + \cdots + |x|^{2N-2}u_{N-1}(x),
\]

where all the functions \( u_{j} \) are harmonic in \( \mathbb{B} \) (see, e.g., Section 32 of [3]). This can be rearranged to obtain

\[
u(x) = v_{0}(x) + (1 - |x|^{2})v_{1}(x) + \cdots + (1 - |x|^{2})^{N-1}v_{N-1}(x),
\]

where the functions \( v_{j} \) are given as

\[
v_{j} := (-1)^{j} \sum_{k=j}^{N-1} \binom{k}{j} u_{k},
\]

which are harmonic functions on \( \mathbb{B} \).

The following result, which generalizes Proposition 4.11 in [4], provides us with a condition that guarantees that an \( N \)-harmonic function \( u(x) \) can be written as \((1 - |x|^{2})\bar{u}(x)\), where \( \bar{u} \) is \( (N - 1) \)-harmonic.

Proposition 5.1. Suppose that \( 0 < p < \infty \), \( \alpha \leq \min\{(n-1)p - n, -1\} \) and \( N \in \mathbb{N} \). Suppose \( u \in \text{PH}_{N,\alpha}(\mathbb{B}) \).

(i) If \( N = 1 \), then \( u = 0 \);

(ii) if \( N \geq 2 \) then \( u \) has the form \( u = M[\bar{u}] \) with \( \bar{u} \in \text{PH}_{N-1,\alpha+p}(\mathbb{B}) \).

Proof. We first show that

\[
\liminf_{r \to 1} \int_{S} |u(r\zeta)| d\sigma(\zeta) = 0.
\]
Case 1: $0 < p < 1$. Since then $\alpha \leq (n-1)p-n$, we have $u \in \text{PH}^p_{N,(n-1)p-n}(\mathbb{B})$, and

$$\|u\|_{p,(n-1)p-n}^p \leq \|u\|_{p,\alpha}^p < +\infty.$$  

By Lemma 2.14 we have

(5.3) $$\sup_{x \in \mathbb{B}} |u(x)|^p (1 - |x|^2)^{(n-1)p} \leq \|u\|_{p,(n-1)p-n}^p < +\infty.$$  

Thus,

$$\|u\|_{1,-1} = \int_\mathbb{B} |u(x)|^p (1 - |x|^2)^{(n-1)p-n} \left\{ |u(x)|^p (1 - |x|^2)^{(n-1)p} \right\}^{(1-p)/p} dV(x) \leq \|u\|_{p,(n-1)p-n}^p \left\{ \sup_{x \in \mathbb{B}} |u(x)|^p (1 - |x|^2)^{(n-1)p} \right\}^{(1-p)/p} < +\infty.$$  

Now we prove (5.2) by contradiction. Assume that

(5.4) $$\liminf_{r \to 1^-} \int_{\mathcal{S}} |u(r\zeta)| d\sigma(\zeta) > 0.$$  

Then there exists a $\delta > 0$ such that

$$\inf_{1 - \delta < r < 1} \int_{\mathcal{S}} |u(r\zeta)| d\sigma(\zeta) > 0.$$  

It follows that

$$\|u\|_{1,-1} = \int_0^1 \frac{r^{n-1}}{1 - r^2} \left\{ \int_{\mathcal{S}} |u(r\zeta)| d\sigma(\zeta) \right\} dr \leq \left\{ \frac{\int_0^1 r^{n-1} dr}{1 - r^2} \right\} \left\{ \inf_{1 - \delta < r < 1} \int_{\mathcal{S}} |u(r\zeta)| d\sigma(\zeta) \right\} = +\infty.$$  

A contradiction.

Case 2: $1 \leq p < +\infty$. Since $\alpha \leq -1$, we have

$$\|u\|_{p,-1}^p = \int_\mathbb{B} |u(x)|^p (1 - |x|^2)^{-1} dV(x) \leq \int_\mathbb{B} |u(x)|^p (1 - |x|^2)^{\alpha} dV(x) < +\infty.$$  

By the same elementary argument as above, we deduce that

(5.5) $$\liminf_{r \to 1^-} \int_{\mathcal{S}} |u(r\zeta)|^p d\sigma(\zeta) = 0.$$  

and (5.2) follows from this and an application of Hölder’s inequality.

Now we proceed to prove the proposition. By the alternative Almansi representation (5.1), we see that

$$u(x) = v_0(x) + (1 - |x|^2)v_1(x) + \cdots + (1 - |x|^2)^{N-1}v_{N-1}(x),$$
where $v_0, v_1, \ldots, v_{N-1}$ are harmonic functions on $\mathbb{B}$. It follows that
\[
\int_{\mathbb{S}} u(r\zeta) \frac{1 - |x|^2}{|x - \zeta|^n} d\sigma(\zeta) = \sum_{j=0}^{N-1} (1 - r^2)^j \int_{\mathbb{S}} v_j(r\zeta) \frac{1 - |x|^2}{|x - \zeta|^n} d\sigma(\zeta)
\]
\[
= \sum_{j=0}^{N-1} (1 - r^2)^j v_j(rx).
\]
Letting $r \to 1^-$, we obtain
\[
(5.6) \quad v_0(x) = \lim_{r \to 1^-} \int_{\mathbb{S}} \frac{1 - |x|^2}{|x - \zeta|^n} u(r\zeta) \, d\sigma(\zeta)
\]
for every $x \in \mathbb{B}$. It follows that
\[
|v_0(x)| = \lim_{r \to 1^-} \left| \int_{\mathbb{S}} \frac{1 - |x|^2}{|x - \zeta|^n} u(r\zeta) \, d\sigma(\zeta) \right|
\]
\[
\leq \frac{1 + |x|}{(1 - |x|)^{n-1}} \liminf_{r \to 1^-} \int_{\mathbb{S}} |u(r\zeta)| d\sigma(\zeta) = 0
\]
for all $x \in \mathbb{B}$. If $N = 1$, we are done. If $N \geq 2$, we obtain instead that $u(x) = (1 - |x|^2)\bar{u}(x)$ where
\[
\bar{u}(x) := \frac{u(x)}{1 - |x|^2} = v_1(x) + (1 - |x|^2)v_2(x) + \cdots + (1 - |x|^2)^{N-2}v_{N-1}(x),
\]
is $(N - 1)$-harmonic. Moreover, this gives $\bar{u} \in \text{PH}_{N-1,\alpha}^{p}(\mathbb{B})$.

The following is a sufficient criterion for the triviality of a polyharmonic function. We note that the restriction $p \geq \frac{n-2}{n-1}$ enters the picture here.

**Proposition 5.2.** Suppose that $\frac{n-2}{n-1} \leq p < +\infty$ and $N \in \mathbb{N}$. Then $\text{PH}_{N,\alpha}^{p}(\mathbb{B}) = \{0\}$ for all $\alpha \leq -1 - (2N - 1)p$.

**Proof.** Since the spaces $\text{PH}_{N,\alpha}^{p}(\mathbb{B})$ grow with $\alpha$, we only need to prove the result when $\alpha = -1 - (2N - 1)p$.

The proof is by induction on $N$. We first prove the claim for $N = 1$:
\[
\text{PH}_{1,-1-p}^{p}(\mathbb{B}^n) = \{0\}.
\]
Let $u \in \text{PH}_{1,-1-p}^{p}(\mathbb{B}^n)$ be arbitrary. Then by Lemma 2.15, $\partial_j u \in \text{PH}_{1,-1}^{p}(\mathbb{B})$, $j = 1, \ldots, n$. Note that
\[
(5.7) \quad \|\nabla u\|_{-1}^p = \omega_n^{-1} \int_{0}^{1} \frac{r^{n-1}}{1 - r^2} \left\{ \int_{\mathbb{S}} |\nabla u(r\zeta)|^p d\sigma(\zeta) \right\} dr,
\]
where, as usual,
\[
|\nabla u| := \left( \sum_{j=1}^{n} \left| \frac{\partial u}{\partial x_j} \right|^2 \right)^{1/2} \quad \text{and} \quad \|\nabla u\|_{\alpha} := \|\nabla u\|_{\alpha}.
\]
Since $u$ is harmonic in $B$, $|\nabla u|^p$ is subharmonic when $p \geq \frac{n-2}{n-1}$, by [12] Theorem A. Hence the function
\[
t \mapsto \int_B |\nabla u(t\zeta)|^p \, d\sigma(\zeta)
\]
is increasing. It then follows from (5.7) that
\[
\|\nabla u\|^p_{p,-1} \geq \left\{ \frac{1}{t} \int_1^t \frac{r^{n-1}}{1-r^2} \, dr \right\} \left\{ \int_B |\nabla u(t\zeta)|^p \, d\sigma(\zeta) \right\}
\]
for every $0 < t < 1$. Thus, $\|\nabla u\|^p_{p,-1} < +\infty$ forces $\nabla u = 0$, and hence $u$ must be constant. As the only constant function in $\text{PH}_{1,-1-p}(B)$ is the zero function, we obtain $u = 0$.

For the induction step, we assume that the above assertion holds for $N = N_0$:
\[
\text{PH}_{N_0,1-(2N_0-1)p}(B) = \{0\}.
\]

Let $u \in \text{PH}_{N_0+1,1-(2N_0+1)p}(B)$ be arbitrary. Put $v := \Delta u$. By Corollary [2.16] $v \in \text{PH}_{N_0-1-(2N_0-1)p}(B)$. Then $v = 0$, by the induction hypothesis. This means that $u$ is harmonic and furthermore $u \in \text{PH}_{1,-1-(2N_0+1)p}(B)$. But $\text{PH}_{1,-1-(2N_0+1)p}(B) \subset \text{PH}_{0,1-p}(B) = \{0\}$, we find that $u = 0$. Consequently,
\[
\text{PH}_{N_0+1,1-(2N_0+1)p}(B) = \{0\}.
\]
The proof is complete. 

\section{6. Proof of Theorem 1.2 Part 2}

In this section, we shall prove Theorem 4.2, which together with Theorem 4.1 will complete the proof of Theorem 1.2.

For fixed $N \in \mathbb{N}$ and $j \in \{1, \cdots, N\}$, we define
\[
a_{j,N}(p) := \min\{b_{j,N}(p), -1 - (N - j)p\},
\]
where
\[
b_{j,N}(p) := \max\{-1 - (N + j - 1)p, -n - (N - j - n + 1)p\},
\]
as defined in (1.4). Note that $a_{j,N}(p) = b_{j,N}(p)$ for $0 < p < 1$ and
\[
\min_{j:1 \leq j \leq N} a_{j,N}(p) = \min_{j:0 \leq j \leq N} b_{j,N}(p).
\]

Thus, we can reformulate Theorem 4.2 as follows.

\section*{Theorem 4.2.} Suppose that $\frac{n-2}{n-1} \leq p < \infty$ and $N \in \mathbb{N}$. Then
\[
\alpha \leq \min_{j:1 \leq j \leq N} a_{j,N}(p) \implies \text{PH}_{N,\alpha}^p(B) = \{0\}.
\]

According to Theorem 1.3, any $u \in \text{PH}_{N,\alpha}^p(B)$ can be uniquely written as
\[
u = w_0 + M[w_1] + \cdots + M^{N-1}[w_{N-1}],
\]
where each term $M^j[w_j]$ remains in the space $\text{PH}_{N,\alpha}^p(B)$, with $w_j$ solving $L_{N-j-1}[w_j] = 0$ on $B$. Therefore, to show $u = 0$, we just need to test each term $M^j[w_j]$ separately. Thus the proof of Theorem 4.2 reduces to proving the following proposition.
Proposition 6.1. Suppose that \( \frac{n-2}{n-1} \leq p < +\infty \), \( N \in \mathbb{N} \), \( j \in \{0, 1, \ldots, N-1\} \). If \( \alpha \leq a_{N-j,N}(p) \) and \( u \in \text{PH}_N^p,\alpha(B) \) is of the form \( u = M^j[w] \), with \( w \) satisfying \( L_{N-j-1}[w] = 0 \), then \( u = 0 \).

**Proof.** It is clear that \( w \in \text{PH}_{N-j,\alpha+jp}(B) \). The assumption \( \alpha \leq a_{N-j,N}(p) \) can be written as
\[
\alpha + jp \leq a_{N-j,N}(p) + jp = a_{N-j,N-j}(p).
\]
Let \( N' := N - j \) and \( \alpha' := \alpha + jp \). We are reduced to proving the following

**Claim.** Assume that \( \alpha' \leq a_{N',N'}(p) \). If \( w \in \text{PH}_{N',\alpha'}(B) \) solves \( L_{N'-1}[w] = 0 \), then \( w = 0 \).

First note that, in the case when \( \frac{n-2}{n-1} \leq p < \frac{n-1}{n+2N'-2} \) (this is only possible if \( N' = 1 \)),
\[
a_{N',N'}(p) = -1 - (2N' - 1)p.
\]
The assertion \( w = 0 \) is then immediate from Proposition \( \ref{prop:unique_decomposition} \).

Now we assume that \( p > \frac{n-1}{n+2N'-2} \). Then
\[
a_{N',N'}(p) = \min \{ (n-1)p - n, -1 \}.
\]
Since \( \alpha' \leq a_{N',N'}(p) \), by Proposition \( \ref{prop:unique_decomposition} \), \( w \) can be written as \( w = M[^{\alpha'}[\tilde{w}] \), with \( \tilde{w} \in \text{PH}_{N',\alpha'}(B) \). If \( N' = 1 \), this should be understood as \( \tilde{w} = 0 \) and we are done. If \( N' \geq 2 \), by Theorem \( \ref{thm:main_result} \), \( \tilde{w} \) has a unique decomposition
\[
\tilde{w} = \sum_{j=0}^{N'-2} M_j[v_j],
\]
where each term \( M_j[v_j] \in \text{PH}_{N'-1,\alpha'+p}(B) \) with \( v_j \) satisfying \( L_{N'-j-2}[v_j] = 0 \). This means that \( w = M[^{\alpha'}[\tilde{w}] \) has the expansion
\[
w = \sum_{j=1}^{N'-1} M_j[v_{j-1}] = \sum_{j=1}^{N'-1} M_j[\tilde{v}_j]
\]
where each term \( M_j[\tilde{v}_j] \) is in \( \text{PH}_{N',\alpha'}(B) \), with \( \tilde{v}_j := v_{j-1} \) satisfying \( L_{N'-j-1}[\tilde{v}_j] = 0 \). Rewrite \( \ref{eq:weight_decomposition} \) as
\[
0 = -w + M[^1[\tilde{v}_1] + \cdots + M[N-1][\tilde{v}_{N-1}].
\]
From the uniqueness of the decomposition in Theorem \( \ref{thm:main_result} \), we see that this is only possible if \( w = 0 \). This proves the claim, and the proof of Theorem \( \ref{thm:main_result} \) is complete. \( \square \)

7. **Proof of Theorem \( \ref{thm:main_result} \)**

Again, we analyze each term in the cellular decomposition separately. We begin with the following proposition.

Proposition 7.1. Suppose that \( 0 < p < \infty \), \( N \in \mathbb{N} \) and \( j \in \{0, 1, \ldots, N-1\} \). If \( \alpha > a_{N-j,N}(p) \) then there exists a nontrivial \( u \in \text{PH}_N^p(\mathbb{N}) \) of the form \( u = M^j[w] \), with \( w \) satisfying \( L_{N-j-1}[w] = 0 \).

**Proof.** When \( 0 < p < 1 \), we consider the function \( u = M^1[P_{N-j-1}(\cdot, e_1)] \), where \( P_0 \) is given by \( \ref{eq:polyharmonic_solution} \). Explicitly,
\[
u(x) = U_{N-j,N}(x) = \frac{(1-|x|^2)^{2N-j-1}}{|x-e_1|^{n+2(N-j-1)}}, \quad x \in \mathbb{B}.
\]
By Lemma 4.5, \( u \) is in \( \text{PH}_{N, \alpha}^\nu(\mathbb{B}) \) if and only if \( \alpha > b_{N-j, N}(p) \). Note that \( a_{N-j, N}(p) = b_{N-j, N}(p) \) for \( 0 < p < 1 \). Hence there exists a nontrivial \( u \in \text{PH}_{N, \alpha}^\nu(\mathbb{B}) \) of the form \( u = M^j[w] \), with \( w \) satisfying \( L_{N-j-1}[w] = 0 \), provided \( \alpha > a_{N-j, N}(p) \).

When \( 1 \leq p < \infty \), we can consider the function \( u = M^j[\Phi_{N-j-1}] \), where \( \Phi_\theta \) is defined by (2.16). By Corollary 2.12, \( M^j[\Phi_{N-j-1}] \) is in \( \text{PH}_{N, \alpha}^\nu(\mathbb{B}) \) for any \( \alpha > -1 - j p \). In view of that \( a_{N-j, N}(p) = -1 - j p \), this completes the proof.

**Proof of Theorem 1.4.** It is a matter of checking which terms actually occur in the decomposition of Theorem 1.3. This is easy to do using Propositions 6.1 and 7.1.

8. CONCLUDING REMARKS

We conclude this paper with several remarks and problems which naturally arise from our results.

**Problem 8.1.** Find an explicit formula for the critical integrability type \( \beta(N, p) \) in the range \( 0 < p < \frac{n-2}{n-1} \).

It is natural to expect that the formula (1.2) in Theorem 1.2 is still valid for the range of \( 0 < p < \frac{n-2}{n-1} \). Nevertheless, it turns out that this is not true even in the simplest case \( N = 1 \). According to Aleksandrov [1, p. 526, Remark], if \( n \geq 3 \) and \( 0 < p < \frac{n-2}{n} \), then there exists an \( \varepsilon_0 = \varepsilon_0(p) > 0 \) and a nonzero harmonic function \( v \) such that

\[
M_p(v, r) = o\left((1-r)^{1+2\varepsilon/p}\right) \quad (r \to 1)
\]

for \( 0 < \varepsilon < \varepsilon_0 \), where

\[
M_p(v, r) := \left\{ \int_S |v(r \zeta)|^p d\sigma(\zeta) \right\}^{1/p}.
\]

It follows that

\[
\|M^{N-1}[v]\|_{p, -1-Np-\varepsilon}^p = \omega_{n-1} \int_0^1 M_p(v, r) (1-r^2)^{-1-p-\varepsilon} r^{n-1} dr 
\]

\[
\lesssim \int_0^1 (1-r^2)^{-1+p} dr < +\infty,
\]

which means that

\[
(8.1) \quad M^{N-1}[v] \in \text{PH}_{N, -1-Np-\varepsilon}^\nu(\mathbb{B}) \quad \text{for all} \quad \varepsilon \in (0, \varepsilon_0).
\]

In particular, when \( N = 1 \), this implies that

\[
\beta(1, p) < -1 - p - \varepsilon
\]

for sufficiently small \( \varepsilon \). On the other hand, it is easy to check that

\[
\min\{b_{0, 1}(p), b_{1, 1}(p)\} = -1 - p \quad \text{for} \quad 0 < p < \frac{n-1}{n}.
\]

We then see that

\[
\beta(1, p) < \min\{b_{0, 1}(p), b_{1, 1}(p)\}
\]

for \( 0 < p < \frac{n-2}{n} \). We have not been able to solve this problem, and it could be very difficult.

Borichev and Hedenmalm [4] also found an interesting entanglement phenomenon in the decomposition (1.6).
3.3], for ease of exposition. It was shown in [4, Proposition 3.6] that, when such that the space PH contains no nontrivial functions of the form M^{N-1}[v] with v harmonic. The complement \( N_N := A_N \setminus \mathcal{E}_N \) is referred as to the unentangled region.

Note that we have reformulated the definition of the entangled region \( \mathcal{E}_N \) in [4, Section 3.3], for ease of exposition. It was shown in [4, Proposition 3.6] that, when \( n = 2 \),

\[
\mathcal{E}_N = \left\{ (p, \alpha) \in A_N : 0 < p < \frac{4}{3} \text{ and } \alpha \leq -1 - Np \right\}.
\]

**Problem 8.2.** Describe the entangled region \( \mathcal{E}_N \) when \( n \geq 3 \).

When \( n \geq 3 \), in view of (8.2), one may conjecture that

\[
\mathcal{E}_N = \left\{ (p, \alpha) \in A_N : 0 < p < \frac{n-1}{n+1} \text{ and } \alpha \leq -1 - Np \right\}.
\]

However, by (8.1), we see that, for each \( 0 < p < \frac{n-2}{n+1} \) there exists an \( \varepsilon_0 = \varepsilon_0(p) > 0 \) such that the space \( \text{PH}_{N,\alpha}(B) \) contains a nontrivial functions of the form \( M^{N-1}[v] \) with \( v \) harmonic, whenever \( \alpha > -1 - Np - \varepsilon(p) \). This means that \( \mathcal{E}_N \) excludes the region

\[
\left\{ (p, \alpha) \in A_N : 0 < p < \frac{n-2}{n+1} \text{ and } -1 - Np - \varepsilon_0(p) < \alpha < -1 - Np \right\}.
\]

It seems to us that the situation in the higher dimensional case \( n \geq 3 \) is rather complicated.

We put

\[
\mathcal{N}_N^{(1)} := \left\{ (p, \alpha) \in \mathcal{N}_N : u \in \text{PH}_{N,\alpha}(B) \implies u = M^{N-1}[v] \text{ for some harmonic } v \right\}
\]
and refer to it as the principal unentangled cell. Next result is a higher-dimensional extension of [4, Proposition 3.7].

**Proposition 8.3.** Let \( p \geq \frac{n-2}{n+1} \) and \( N \in \mathbb{N} \) be fixed. Every \( u \in \text{PH}_{N,\alpha}(B) \) has the form \( u = M^{N-1}[v] \) with \( v \) harmonic on \( B \) if and only if

\[
\alpha \leq \min\{-n - (N - n - 1)p, -1 - (N - 2)p\}.
\]

In other words,

\[
\mathcal{N}_N^{(1)} \cap \mathcal{A}_N = \left\{ (p, \alpha) : p \geq \frac{n-2}{n+1} \text{ and } \alpha \leq \min\{-n - (N - n - 1)p, -1 - (N - 2)p\} \right\}.
\]

**Proof of Proposition 8.3.** In terms of the decomposition in Theorem 1.3 it is a matter of deciding for which \( (p, \alpha) \) the functions \( w_j \), with \( j = 0, \ldots, N - 2 \), must all equal 0. This can be done by using Propositions 6.3 and 7.1. \( \square \)

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