VISIBILITY FOR SELF-SIMILAR SETS OF DIMENSION ONE IN THE PLANE

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Abstract. We prove that a purely unrectifiable self-similar set of finite 1-dimensional Hausdorff measure in the plane, satisfying the Open Set Condition, has radial projection of zero length from every point.

1. Introduction

For \( a \in \mathbb{R}^2 \), let \( P_a \) be the radial projection from \( a \):

\[
P_a : \mathbb{R}^2 \setminus \{a\} \to S^1, \quad P_a(x) = \frac{x - a}{|x - a|}.
\]

A special case of our theorem asserts that the “four corner Cantor set” of contraction ratio \( 1/4 \) has radial projection of zero length from all points \( a \in \mathbb{R}^2 \).

Figure 1. The radial projection of the four corner set
See Figure 1 where we show the second-level approximation of the four corner Cantor set and the radial projection of some of its points.

Denote by $\mathcal{H}^1$ the one-dimensional Hausdorff measure. A Borel set $\Lambda$ is a 1-set if $0 < \mathcal{H}^1(\Lambda) < \infty$. It is said to be invisible from $a$ if $P_a(\Lambda \setminus \{a\})$ has zero length.

**Theorem 1.1.** Let $\Lambda$ be a self-similar 1-set in $\mathbb{R}^2$ satisfying the Open Set Condition, which is not on a line. Then $\Lambda$ is invisible from every $a \in \mathbb{R}^2$.

Recall that a nonempty compact $\Lambda$ is self-similar if $\Lambda = \bigcup_{i=1}^m S_i(\Lambda)$ for some contracting similitudes $S_i$. This means that

$$S_i(x) = \lambda_i O_i x + b_i,$$

where $0 < \lambda_i < 1$, $O_i$ is an orthogonal transformation of the plane, and $b_i \in \mathbb{R}^2$. The Open Set Condition holds if there exists an open set $V \neq \emptyset$ such that $S_i(V) \subset V$ for all $i$ and $S_i(V) \cap S_j(V) = \emptyset$ for all $i \neq j$. For a self-similar set satisfying the Open Set Condition, being a 1-set is equivalent to $\sum_{i=1}^m \lambda_i = 1$.

A Borel set $\Lambda$ is purely unrectifiable (or irregular), if $\mathcal{H}^1(\Lambda \cap \Gamma) = 0$ for every rectifiable curve $\Gamma$. A set $\Lambda$ satisfying the assumptions of Theorem 1.1 is purely unrectifiable by Hutchinson [5] (see also [8]). A classical theorem of Besicovitch [2] (see also [3, Theorem 6.13]) says that a purely unrectifiable 1-set has orthogonal projections of zero length on almost every line through the origin. We use it in our proof.

In [10, Problem 12] (see also [9, 10.12]) Mattila raised the following question: Let $\Lambda$ be a Borel set in $\mathbb{R}^2$ with $\mathcal{H}^1(\Lambda) < \infty$. Is it true that for $\mathcal{H}^1$ almost all $a \in A$, the intersection $\Lambda \cap L$ is a finite set for almost all lines $L$ through $a$? If $\Lambda$ is purely unrectifiable, is it true that $\Lambda \cap L = \{a\}$ for almost all lines through $a$? Our theorem implies a positive answer for a purely unrectifiable self-similar 1-set $\Lambda$ satisfying the Open Set Condition. The general case of a purely unrectifiable set remains open. On the other hand, M. Csörnyei and D. Preiss proved recently that the answer to the first part of the question is negative [personal communication].

Note that we prove a stronger property for our class of sets, namely, that the set is invisible from every point $a \in \mathbb{R}^2$. It is easy to construct examples of non-self-similar purely unrectifiable 1-sets for which this property fails. Marstrand [6] has an example of a purely unrectifiable 1-set which is visible from a set of dimension one.
We do not discuss here other results and problems related to visibility; see [4] Section 6] for a recent survey. We only mention a result of Mattila [7, Th.5.1]: if a set \( \Lambda \) has projections of zero length on almost every line (which could have \( \mathcal{H}^1(\Lambda) = \infty \)), then the set of points \( \Xi \) from which \( \Lambda \) is visible is a purely unrectifiable set of zero 1-capacity. A different proof of this and a characterization of such sets \( \Xi \) is due to Csörnyei [3].

2. Preliminaries

We have \( S_i(x) := \lambda_i \mathcal{O}_i x + b_i \), where \( 0 < \lambda_i < 1 \),

\[
\mathcal{O}_i = \begin{bmatrix}
\cos(\varphi_i) & -\varepsilon_i \sin(\varphi_i) \\
\sin(\varphi_i) & \varepsilon_i \cos(\varphi_i)
\end{bmatrix},
\]

\( \varphi_i \in [0, 2\pi) \), and \( \varepsilon_i \in \{-1, 1\} \) shows whether \( \mathcal{O}_i \) is a rotation through the angle \( \varphi_i \) or a reflection about the line through the origin making the angle \( \varphi_i/2 \) with the \( x \)-axis.

Let \( \Sigma := \{1, \ldots, m\}^\mathbb{N} \) be the symbolic space. The natural projection \( \Pi : \Sigma \to \Lambda \) is defined by

\[
\Pi(i) = \lim_{n \to \infty} S_{i_1 \ldots i_n}(x_0), \quad \text{where } i = (i_1 i_2 i_3 \ldots) \in \Sigma,
\]

and \( S_{i_1 \ldots i_n} = S_{i_1} \circ \cdots \circ S_{i_n} \). The limit in (1) exists and does not depend on \( x_0 \).

Denote \( \lambda_{i_1 \ldots i_n} = \lambda_{i_1} \cdots \lambda_{i_n} \) and \( \varepsilon_{i_1 \ldots i_k} = \varepsilon_{i_1} \cdots \varepsilon_{i_k} \). We can write

\[
S_{i_1 \ldots i_n}(x) = \lambda_{i_1 \ldots i_n} \mathcal{O}_{i_1 \ldots i_n} x + b_{i_1 \ldots i_n},
\]

where

\[
\mathcal{O}_{i_1 \ldots i_n} := \mathcal{O}_{i_1} \circ \cdots \circ \mathcal{O}_{i_n} = \begin{bmatrix}
\cos(\varphi_{i_1 \ldots i_n}) & -\varepsilon_{i_1 \ldots i_n} \sin(\varphi_{i_1 \ldots i_n}) \\
\sin(\varphi_{i_1 \ldots i_n}) & \varepsilon_{i_1 \ldots i_n} \cos(\varphi_{i_1 \ldots i_n})
\end{bmatrix},
\]

\( \varphi_{i_1 \ldots i_n} := \varphi_{i_1} + \varepsilon_{i_1} \varphi_{i_2} + \varepsilon_{i_1 i_2} \varphi_{i_3} + \cdots + \varepsilon_{i_1 \ldots i_{n-1}} \varphi_{i_n} \),

and

\[
b_{i_1 \ldots i_n} = b_{i_1} + \lambda_{i_1} \mathcal{O}_{i_1} b_{i_2} + \cdots + \lambda_{i_1 \ldots i_{n-1}} \mathcal{O}_{i_1 \ldots i_{n-1}} b_{i_n}.
\]

Since \( \sum_{i=1}^m \lambda_i = 1 \), we can consider the probability product measure \( \mu = (\lambda_1, \ldots, \lambda_m)^\mathbb{N} \) on the symbolic space \( \Sigma \) and define the natural measure on \( \Lambda \):

\[
\nu = \mu \circ \Pi^{-1}.
\]
By a result of Hutchinson [3, Theorem 5.3.1(iii)], as a consequence of the Open
Set Condition we have

\[(2) \quad \nu = c\mathcal{H}^1|_{\Lambda}, \quad \text{where} \quad c = (\mathcal{H}^1(\Lambda))^{-1}.\]

To $\theta \in [0, \pi)$ we associate the unit vector $e_\theta = (\cos \theta, \sin \theta)$, the line $L_\theta = \{te_\theta : t \in \mathbb{R}\}$, and the orthogonal projection onto $L_\theta$ given by $x \mapsto (e_\theta \cdot x)e_\theta$. It is more
convenient to work with the signed distance of the projection to the origin, which
we denote by $p_\theta$:

\[p_\theta : \mathbb{R}^2 \to \mathbb{R}, \quad p_\theta x = e_\theta \cdot x.\]

Denote $A := \{1, \ldots, m\}$ and let $A^* = \bigcup_{i=1}^\infty A^i$ be the set of all finite words over
the alphabet $A$. For $u = u_1 \ldots u_k \in A^k$ we define the corresponding “symbolic”
cylinder set by

\[[u] = [u_1 \ldots u_k] := \{i \in \Sigma : i_\ell = u_\ell, 1 \leq \ell \leq k\}.\]

We also let

\[\Lambda_u = S_u(\Lambda) = \lambda_u O_u \Lambda + b_u\]

and call $\Lambda_u$ the cylinder set of $\Lambda$ corresponding to the word $u$. Let $d_\Lambda$ be the
diameter of $\Lambda$; then $\text{diam}(\Lambda_u) = \lambda_u d_\Lambda$. For $\rho > 0$ consider the “cut-set”

\[W(\rho) = \{u \in A^* : \lambda_u \leq \rho, \lambda_{u'} > \rho\}\]

where $u'$ is obtained from $u$ by deleting the last symbol. Observe that for every
$\rho > 0$,

\[(3) \quad \Lambda = \bigcup_{u \in W(\rho)} \Lambda_u.\]

In view of (2), we have $\nu(\Lambda_u \cap \Lambda_v) = 0$ for distinct $u, v \in W(\rho)$, hence

\[\nu(\Lambda_u) = \lambda_u \quad \text{for all} \quad u \in A^*.\]

Denote $\lambda_{\text{min}} := \min\{\lambda_i : i \leq m\}$; then $\mu(\Lambda_u) = \lambda_u \in (\rho\lambda_{\text{min}}, \rho)$ for $u \in W(\rho)$.

We identify the unit circle $S^1$ with $[0, 2\pi)$ and use additive notation $\theta_1 + \theta_2$ understood mod $2\pi$ for points on the circle. For a Radon measure $\eta$ on the line
or on $S^1$, the upper density of $\eta$ with respect to $\mathcal{H}^1$ is defined by

\[
\overline{D}(\eta, t) = \limsup_{r \to 0} \frac{\eta([t-r, t+r])}{2r}.
\]

The open ball of radius $r$ centered at $x$ is denoted by $B(x, r)$. 

3. Proof of the main theorem

In the proof of Theorem 1.1 we can assume, without loss of generality, that \( a \not\in \Lambda \), and

\[ P_a(\Lambda) \text{ is contained in an arc of length less than } \pi. \]

Indeed, \( \Lambda \setminus \{a\} \) can be written as a countable union of self-similar sets \( \Lambda_u \) for \( u \in A^* \), of arbitrarily small diameter. If each of them is invisible from \( a \), then \( \Lambda \) is invisible from \( a \).

Let

\[ \Omega := \{ i \in \Sigma : \forall u \in A^* \exists n \text{ such that } \sigma^n i \in [u] \}, \]

that is, \( \Omega \) is the set of sequences which contain each finite word over the alphabet \( A = \{1, \ldots, m\} \). It is clear that every \( i \in \Omega \) contains each finite word infinitely many times and \( \mu(\Sigma \setminus \Omega) = 0 \).

**Lemma 3.1** (Recurrence Lemma). For every \( i \in \Omega \), \( \delta > 0 \), and \( j_1, \ldots, j_k \in \{1, \ldots, m\} \), there are infinitely many \( n \in \mathbb{N} \) such that

\[ (5) \quad \phi_{i_1 \ldots i_n} \in [0, \delta], \quad \varepsilon_{i_1 \ldots i_n} = 1, \quad \text{and } \sigma^n i \in [j_1 \ldots j_k]. \]

If the similitudes have no rotations or reflections, that is, \( \phi_i = 0 \) and \( \varepsilon_i = 1 \) for all \( i \leq m \) (as in the case of the four corner Cantor set), then the conditions on \( \phi \) and \( \varepsilon \) in (3) hold automatically and the lemma is true by the definition of \( \Omega \). The proof in the general case is not difficult, but requires a detailed case analysis, so we postpone it to the next section.

Let

\[ \Theta := \{ \theta \in [0, \pi) : \mathcal{H}^1(p_{\theta}(\Lambda)) = 0 \} \quad \text{and} \quad \Theta' := (\Theta + \pi/2) \cup (\Theta + 3\pi/2) \]

(recall that addition is considered mod \( 2\pi \)). Since \( \Lambda \) is purely unrectifiable, \( \mathcal{H}^1([0, \pi) \setminus \Theta') = 0 \) by Besicovitch’s Theorem [2]. The following proposition is the key step of the proof. We need the following measures:

\[ \nu_a := \nu \circ P_a^{-1} \quad \text{and} \quad \nu_\theta := \nu \circ p_\theta^{-1}, \; \theta \in [0, \pi). \]

We also denote \( \Lambda' = \Pi(\Omega) \).

**Proposition 3.2.** If \( \theta' \in P_a(\Lambda') \cap \Theta' \), then \( \mathcal{D}(\nu_a, \theta') = \infty \).
Proof of Theorem 1.1 assuming Proposition 3.2. By Proposition 3.2 and [9, Lemma 2.13] (a corollary of the Vitali covering theorem), we obtain that $H^1(P_a(\Lambda') \cap \Theta') = 0$. As noted above, $\Theta'$ has full $H^1$ measure in $S^1$. On the other hand,

$$\mu(\Sigma \setminus \Omega) = 0 \Rightarrow \nu(\Lambda \setminus \Lambda') = 0 \Rightarrow H^1(\Lambda \setminus \Lambda') = 0 \Rightarrow H^1(P_a(\Lambda \setminus \Lambda')) = 0,$$

and we conclude that $H^1(P_a(\Lambda)) = 0$, as desired. □

Proof of Proposition 3.2. Let $x \in \Lambda'$ and $\theta' = P_a(x) \in \Theta'$. Let $\theta := \theta' - \pi/2 \mod [0, \pi)$. By the definition of $\Theta'$ we have $H^1(p_\theta(\Lambda)) = 0$.

First we sketch the idea of the proof. Since $H^1(p_\theta(\Lambda)) = 0$, we have $\nu_\theta \perp H^1$, and this implies that for every $N \in \mathbb{N}$ there exist $N$ cylinders of $\Lambda$ approximately the same diameter (say, $\sim r$), such that their projections to $L_\theta$ are $r$-close to each other. Then there is a line parallel to the segment $[a, x]$, whose $Cr$-neighborhood contains all $\Lambda_u$, $j = 1, \ldots, N$. By the definition of $\Lambda' = \Pi(\Omega)$, we can find similar copies of this picture near $x \in \Lambda'$ at arbitrarily small scales. The Recurrence Lemma 3.1 guarantees that these copies can be chosen with a small relative rotation. This will give $N$ cylinders of $\Lambda$ of diameter $\sim r_0r$ contained in a $C'r_0r$-neighborhood of the ray obtained by extending $[a, x]$. Since $a$ is assumed to be separated from $\Lambda$, we will conclude that $D(\nu_\theta, \theta') \geq C''N$, and the proposition will follow. Now we make this precise. The proof is illustrated in Figure 2.

Claim. For each $N \in \mathbb{N}$ there exists $r > 0$ and distinct $u^{(1)}, \ldots, u^{(N)} \in \mathcal{W}(r)$ such that

$$|p_\theta(b_{u^{(j)}} - b_{u^{(i)}})| \leq r, \quad \forall i, j \leq N. \tag{6}$$

Indeed, for every $u \in \mathcal{A}^*$,

$$\Lambda_u = \lambda_u \mathcal{O}_u \Lambda + b_u \Rightarrow \Lambda_u \subset B(b_u, d_\Lambda \lambda_u),$$

hence for every interval $I \subset \mathbb{R}$ and $r > 0$,

$$\nu_\theta(I) \leq \sum_{u \in \mathcal{W}(r)} \{ \lambda_u : \text{dist}(p_\theta(b_u), I) \leq d_\Lambda r \}.$$ 

If the claim does not hold, then there exists $N \in \mathbb{N}$ such that for every $t \in \mathbb{R}$ and $r > 0$,

$$\nu_\theta([t - r, t + r]) \leq N(2(1 + d_\Lambda) + 1)r.$$
Then $\nu_\theta$ is absolutely continuous with respect to $\mathcal{H}^1$, which is a contradiction. The claim is verified. \hfill \Box

We are given that $x \in \Lambda' = \Pi(\Omega)$, which means that $x = \pi(i)$ for an infinite sequence $i$ containing all finite words. We fix $N \in \mathbb{N}$ and find $r > 0$, $u^{(1)}, \ldots, u^{(N)} \in \mathcal{W}(r)$ from the Claim. Then we apply Recurrence Lemma with $j_1 \ldots j_k := u^{(i)}$ and $\delta = r$ to obtain infinitely many $n \in \mathbb{N}$ satisfying (3). Fix such an $n$. Denote

$$w := i_1 \ldots i_n \quad \text{and} \quad v^{(j)} = wu^{(j)}, \ j = 1, \ldots, N.$$
Observe that \( i \) starts with \( v^{(1)} \), so \( x = \Pi(i) \in \Lambda_{v^{(1)}} \), hence

(7) \[ |p_\theta(x - b_{v^{(1)}})| \leq |x - b_{v^{(1)}}| \leq d_\Lambda \lambda_{v^{(1)}} \leq d_\Lambda \lambda_w r. \]

Here we used that \( u^{(1)} \in \mathcal{W}(r) \), so \( \lambda_{v^{(1)}} = \lambda_w \lambda_{u^{(1)}} \leq \lambda_w r \). We have for \( z \in \mathbb{R}^2 \),

\[ \lambda_{v^{(j)}} \mathcal{O}_{v^{(j)}} z + b_{v^{(j)}} = S_{v^{(j)}}(z) = S_w \circ S_{u^{(j)}}(z) = \lambda_w \mathcal{O}_w(\lambda_{u^{(j)}} \mathcal{O}_{u^{(j)}} z + b_{u^{(j)}}) + b_w, \]

hence

\[ b_{v^{(j)}} = \lambda_w \mathcal{O}_w b_{u^{(j)}} + b_w. \]

It follows that

\[ p_\theta(b_{v^{(i)}} - b_{v^{(j)}}) = \lambda_w p_\theta(\mathcal{O}_w(b_{v^{(i)}} - b_{u^{(j)}})). \]

By (7), we have \( \varepsilon_w = 1 \) and \( \phi := \phi_w \in [0, r) \); therefore, \( \mathcal{O}_w = R_\theta \) is the rotation through the angle \( \phi \). One can check that \( p_\theta R_\phi = p_{\theta - \phi} \), which yields

(8) \[ |p_\theta(b_{v^{(i)}} - b_{v^{(j)}})| = \lambda_w |p_{\theta - \phi}(b_{u^{(i)}} - b_{u^{(j)}})|. \]

Clearly, \( ||p_\theta - p_{\theta - \phi}|| \leq |\phi| \leq r \), where \( || \cdot || \) is the operator norm, so we obtain from (7) and (8) that

\[ |p_\theta(b_{u^{(i)}} - b_{u^{(j)}})| \leq \lambda_w (|b_{u^{(i)}} - b_{u^{(j)}}| r + r) \leq \lambda_w (d_\Lambda + 1) r. \]

Recall that \( i \) starts with \( v_1 \), so \( x = \Pi(i) \in \Lambda_{v^{(1)}} \), hence for each \( j \leq N \), for every \( y \in \Lambda_{v^{(j)}} \),

\[ |p_\theta(x - y)| \leq |x - b_{v^{(1)}}| + |p_\theta(b_{v^{(1)}} - b_{v^{(j)}})| + |b_{v^{(j)}} - y| \]

(9) \[ \leq d_\Lambda (\lambda_{v^{(1)}} + \lambda_{v^{(j)}}) + \lambda_w (d_\Lambda + 1) r \leq \lambda_w (3d_\Lambda + 1) r. \]

Now we need a simple geometric fact: given that

\[ P_a(x) = \theta', \quad \theta = \theta' + \pi/2 \mod [0, \pi), \quad |p_\theta(x - y)| \leq \rho, \quad |y - a| \geq c_1, \quad \text{and (1)} \]

holds, we have

\[ |P_a(y) - \theta'| = |P_a(y) - P_a(x)| = \arcsin \frac{|p_\theta(y - x)|}{|y - a|} \leq \frac{\pi}{2c_1} \rho. \]

This implies, in view of (1), that for \( c_2 = \pi (3d_\Lambda + 1)/(2c_1) \),

\[ \nu_a([\theta' - c_2 \lambda_w r, \theta' + c_2 \lambda_w r]) \geq \sum_{j=1}^{N} \nu(\Lambda_{v^{(j)}}) = \sum_{j=1}^{N} \lambda_{v^{(j)}} = \lambda_w \sum_{j=1}^{N} \lambda_{u^{(j)}} \geq \lambda_w N \lambda_{\text{min}} r, \]

where \( \lambda_{\text{min}} = \min\{\lambda_1, \ldots, \lambda_n\} \), by the definition of \( \mathcal{W}(r) \). Recall that \( n \) can be chosen arbitrarily large, so \( \lambda_w \) can be arbitrarily small, and we obtain that

\[ \overline{\nu}(\nu_3, \theta') \geq c_2^{-1} \lambda_{\text{min}} N. \]
Since \( N \in \mathbb{N} \) is arbitrary, the proposition follows. \( \square \)

4. Proof of the recurrence lemma 4.1

Let \( K \in \{0, \ldots, m\} \) be the number of \( i \) for which \( \varphi_i \notin \pi \mathbb{Q} \). Without loss of generality we may assume the following: if \( K \geq 1 \) then \( \varphi_1, \ldots, \varphi_K \notin \pi \mathbb{Q} \).

We distinguish the following cases:

**A:** \( \varphi_i \in \pi \mathbb{Q} \) for all \( i \leq m \).

**B:** there exists \( i \) such that \( \varphi_i \notin \pi \mathbb{Q} \) and \( \varepsilon_i = 1 \).

**C:** \( K \geq 1 \) and \( \varepsilon_i = -1 \) for all \( i \leq K \).

**C1:** there exist \( i, j \leq K \) such that \( \varphi_i - \varphi_j \notin \pi \mathbb{Q} \).

**C2:** there exists \( r_i \in \mathbb{Q} \) such that \( \varphi_i = \varphi_1 + r_i \pi \) for \( 1 \leq i \leq K \).

**C2a:** \( K < m \) and there exists \( j \geq K + 1 \) such that \( \varepsilon_j = -1 \).

**C2b:** \( K < m \) and for all \( j \geq K + 1 \) we have \( \varepsilon_j = 1 \).

**C2c:** \( K = m \).

Denote by \( R_\phi \) the rotation through the angle \( \phi \). We call it an irrational rotation if \( \phi \notin \pi \mathbb{Q} \). Consider the semigroup generated by \( O_i, i \leq m \), which we denote by \( S \). We begin with the following observation.

**Claim.** Either \( S \) is finite, or \( S \) contains an irrational rotation.

The semigroup \( S \) is clearly finite in Case A and contains an irrational rotation in Case B. In Case C1 we have \( O_iO_j = R_{\varphi_i - \varphi_j} \), which is an irrational rotation. In Case C2a we also have that \( O_iO_j = R_{\varphi_i - \varphi_j} \) is an irrational rotation, since \( \phi \notin \pi \mathbb{Q} \) and \( \varphi_j \in \pi \mathbb{Q} \). We claim that in remaining Cases C2b and C2c the semigroup is finite. This follows easily; then \( S \) is generated by one irrational reflection and finitely many rational rotations.

**Proof of Lemma 4.1 when \( S \) is finite.** A finite semigroup of invertible transformations is necessarily a group. Let \( S = \{s_1, \ldots, s_t\} \). By the definition of the semigroup \( S \) we have \( s_i = O_{w(i)} \) for some \( w(i) \in A^* \), \( i = 1, \ldots, t \). For every \( v \in A^* \) we can find \( \tilde{v} \in A^* \) such that \( O_{\tilde{v}} = O_v^{-1} \). Fix \( u = j_1 \ldots j_k \) from the statement of the lemma. Consider the following finite word over the alphabet \( A \):

\[
\omega := \tau_1 \ldots \tau_t, \quad \text{where} \quad \tau_j = (w^{(j)}u)(\overline{w^{(j)}u}), \quad j = 1, \ldots, t.
\]

Note that \( O_{\tau_j} = I \) (the identity). By the definition of \( \Omega \), the sequence \( \textbf{i} \in \Omega \) contains \( \omega \) infinitely many times. Suppose that \( \sigma^j \textbf{i} \in [\omega] \). Since \( O_{\textbf{i}^\ell} \in S \), there exists \( w^{(j)} \) such that \( O_{w^{(j)}} = O_{\textbf{i}^\ell}^{-1} \). Then the occurrence of \( u \) in \( \tau_j \), the \( j \)th factor
of \( \omega \), will be at the position \( n \) such that \( O_{\ell|n} = I \), so we will have \( \phi_{\ell|n} = 0 \in [0, \delta] \) and \( \varepsilon_{\ell|n} = 1 \), as desired.

**Proof of Lemma 2.7 when \( S \) is infinite.** By the claim above, there exists \( w \in A^* \) such that \( \phi_w \not\in \pi Q \) and \( \varepsilon_w = 1 \). Fix \( u = j_1 \ldots j_k \) from the statement of the lemma. Let

\[
v := \begin{cases} uu, & \text{if } \phi_u \not\in \pi Q; \\ uuw, & \text{if } \phi_u \in \pi Q.\end{cases}
\]

Observe that \( \phi_v \not\in \pi Q \) and \( \varepsilon_v = 1 \). Let \( v^k = v \ldots v \) (the word \( v \) repeated \( k \) times).

Since \( \phi_v/\pi \) is irrational, there exists an \( N \) such that every orbit of \( R_{\phi_v} \) of length \( N \) contains a point in every subinterval of \( [0, 2\pi) \) of length \( \delta \). Put

\[
\omega := \begin{cases} v^N, & \text{if } \varepsilon_i = 1, \forall i \leq m; \\ v^N j^* w^N, & \text{if } \exists j^* \text{ such that } \varepsilon_{j^*} = -1.\end{cases}
\]

By the definition of \( \Omega \), the sequence \( i \in \Omega \) contains \( \omega \) infinitely many times. Let \( \ell \in \mathbb{N} \) be such that \( \sigma^\ell \in [\omega] \). Suppose first that \( \varepsilon_{\ell|\ell} = 1 \). Then we have, denoting the length of \( v \) by \( |v| \),

\[
\sigma^{\ell+k|v|} i \in [u], \quad \phi_{\ell|\ell+k|v|} = \phi_{\ell|\ell} + k\phi_v \, (\text{mod } 2\pi), \quad \varepsilon_{\ell|\ell+k|v|} = 1,
\]

for \( k = 0, \ldots, N - 1 \). By the choice of \( N \), we can find \( k \in \{0, \ldots, N - 1\} \) such that \( \phi_{\ell|\ell+k|v|} \in [0, \delta] \), then \( n = \ell + k|v| \) will be as desired. If \( \varepsilon_{\ell|\ell} = -1 \), then we replace \( \ell \) by \( \ell^* := \ell + N|v| + 1 \) in (10), that is, we consider the occurrences of \( u \) in the second factor \( v^N \). The orientation will be switched by \( O_{j^*} \) and we can find the desired \( n \) analogously. \( \square \)

5. CONCLUDING REMARKS

Consider the special case when the self-similar set \( \Lambda \) is of the form

\[
\Lambda = \bigcup_{i=1}^m (\lambda_i \Lambda + b_i), \quad b_i \in \mathbb{R}^2.
\]

In other words, the contracting similitudes have no rotations or reflections, as for the four corner Cantor set. Then the projection \( \Lambda^\theta := p_\theta(\Lambda) \) is itself a self-similar set on the line:

\[
\Lambda^\theta = \bigcup_{i=1}^m (\lambda_i \Lambda^\theta + p_\theta(b_i)), \quad \text{for } \theta \in [0, \pi).
\]

Let \( \Lambda_i^\theta = \lambda_i \Lambda^\theta + p_\theta(b_i) \). As above, \( \nu \) is the natural measure on \( \Lambda \). Let \( \nu_\theta \) be the natural measure on \( \Lambda^\theta \), so that \( \nu_\theta = \nu \circ p_\theta^{-1} \).
Corollary 5.1. Let $\Lambda$ be a self-similar set of the form (11) that is not on a line, such that $\sum_{i=1}^{m} \lambda_i \leq 1$. If $\Lambda$ satisfies the Open Set Condition condition, then

$$\nu_\theta(\Lambda^\theta_i \cap \Lambda^\theta_j) = 0, \quad i \neq j,$$

for a.e. $\theta \in [0, \pi)$.

Proof. Let $s > 0$ be such that $\sum_{i=1}^{m} \lambda_i^s = 1$. By assumption, we have $s \leq 1$. This number is known as the similarity dimension of $\Lambda$ (and also of $\Lambda^\theta$ for all $\theta$).

Suppose first that $s = 1$. Then we are in the situation covered by Theorem 1.1 and $\nu$ is just the normalized restriction of $H^1$ to $\Lambda$. Consider the product measure $\nu \times L$, where $L$ is the Lebesgue measure on $[0, \pi)$. Theorem 1.1 implies that

$$\nu \times L \{ (x, \theta) \in \Lambda \times [0, \pi) : \exists y \in \Lambda, \ y \neq x, \ p_\theta(x) = p_\theta(y) \} = 0.$$

By Fubini’s Theorem, it follows that for $L$ a.e. $\theta$, for $\nu_\theta$ a.e. $z \in L^\theta$, we have that $p_\theta^{-1}(z)$ is a single point. This proves the desired statement, in view of the fact that $\nu(\Lambda_i \cap \Lambda_j) = 0$ for $\Lambda$ satisfying the Open Set Condition.

In the case when $s < 1$ we can use [11, Proposition 1.3], which implies that the packing measure $P^s(\Lambda^\theta)$ is positive and finite for $L$ a.e. $\theta$. By self-similarity and the properties of $P^s$ (translation invariance and scaling), we have $P^s(\Lambda^\theta_i \cap \Lambda^\theta_j) = 0$ for $i \neq j$. Then we use [11, Corollary 2.2], which implies that $\nu_\theta$ is the normalized restriction of $P^s$ to $\Lambda^\theta$, to complete the proof. \qed

Remark. In [11, Proposition 2] it is claimed that if a self-similar set $K = \bigcup_{i=1}^{m} K_i$ in $\mathbb{R}^d$ has the Hausdorff dimension equal to the similarity dimension, then the natural measure of the “overlap set” $\bigcup_{i \neq j} (K_i \cap K_j)$ is zero. This would imply Corollary 5.1 since the Hausdorff dimension of $\Lambda^\theta$ equals $s$ for $L$ a.e. $\theta$ by Marstrand’s Projection Theorem. Unfortunately, the proof in [11] contains an error, and it is still unknown whether the result holds [C. Bandt, personal communication]. (It should be noted that [11, Proposition 2] was not used anywhere in [11].)

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