MEDIAN LOCATION PROBLEM WITH TWO PROBABILISTIC LINE BARRIERS: EXTENDING THE HOOK AND JEEVES ALGORITHM

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(Communicated by Harish Garg)

Abstract. We consider a median location problem in the presence of two probabilistic line barriers on the plane under rectilinear distance. It is assumed that the two line barriers move on their corresponding horizontal routes uniformly. We first investigate different scenarios for the position of the line barriers on the plane and their corresponding routes, and then define the visibility and invisibility conditions along with their corresponding expected barrier distance functions. The proposed problem is formulated as a mixed-integer nonlinear programming model. Our aim is to locate a new facility on the plane so that the total weighted expected rectilinear barrier distance is minimized. We present efficient lower and upper bounds using the forbidden location problem for the proposed problem. To solve the proposed model, the Hooke and Jeeves algorithm (HJA) is extended. We investigate various sample problems to test the performance of the proposed algorithm and appropriateness of the bounds. Also, an empirical study in Kingston-upon-Thames, England, is conducted to illustrate the behavior and applicability of the proposed model.

1. Introduction. Facility location analysis is a branch of operations research and computational geometry with numerous applications in many scientific fields, such as decision making in management, economy and production planning. Considering a finite set of existing facilities, a well-known example of single-facility location problems (SFLP) is the classical Weber problem, first introduced by [50], where a facility is located on the plane by minimizing the distance between the new facility and the existing ones. The classical formulation of this problem neglects to model some real-life restrictions and constraints of real-life situations. In location problems, due to the presence of some forbidden regions, where facility placement is forbidden but there is no restriction on traveling through them, congested regions, where placement is prohibited but travelling through is possible with a penalty, barrier regions, where both traveling and facility placement are forbidden, it is not possible to move between a new facility and existing facilities. Examples of such
regions may include protected lands, lakes that can be crossed only using a jolly boat and military areas. So, optimal locations are to be found with respect to these restrictions. The distance measurement criterion is a main component in modeling an analytical facility location problem. Among the various distance functions, the rectilinear distance functions, due to its applicability, is widely used in the literature, in which the distance between two facilities is measured along the nearest orthogonal path between them. It is obvious that presence of the aforementioned regions can affect the distance between any two facilities. An overview of the restricted facility location problems investigated in the literature shows that the shape of these regions may be linear, circular, polyhedral or arbitrary. Decisions on location should be made considering the geometric shape of the restrictions. Also, locations of these regions can be fixed or variable based on a probabilistic distribution function. A review of the literature shows that considering a probabilistic barrier in the facility location problems was first studied by [9], providing motivation for further developments. But, all these studies consider only a single probabilistic barrier. Here, we extend the available studies to consider a single facility location problem having two horizontal routes on the plane with two line barriers moving uniformly on their corresponding routes. A special contribution of our study is the presence of two probabilistic line barriers on the plane. The objective is to locate a new facility on the plane so that the total weighted expected rectilinear barrier distance is minimized. The rest of our work is organized as follows: in the next section, an overview of some studies related to our work is discussed. In Section 3, preliminaries and the problem description are provided. The visibility conditions and computation of the expected barrier distance between the new facility and the existing ones are presented in Section 4. In Section 5, a mathematical mixed-integer nonlinear programming model, the lower and upper bounds and a solution approach are provided. In section 6 we investigate the applicability of the proposed model and the effectiveness of solution approaches using several sample problems and an empirical study in Kingston-upon-Thames, England. Finally, in Section 7, concluding remarks and some directions for future research are provided.

2. Literature review. Considering the Euclidean metric and a circular barrier, for the first time [21] formulated a facility location problem in the presence of barrier regions. They showed that the objective function of the problem was non-convex and suggested a heuristic algorithm for solving the proposed problem that was based on a sequential unconstrained minimization technique (SUMT). Later, the study was extended by [26], in which the feasible set was subdivided into a polynomial number of sub-regions with convex objective functions. The results showed that as the number of existing facilities increased, making the convex subsets turned to be more cumbersome. Later, [8] presented a genetic algorithm along with the Weiszfeld technique to ease the difficulty. [7] formulated a multi facility location problem with polyhedral barriers and the Euclidean metric where they introduced two heuristics using two methods for finding a starting point. [28] presented an efficient algorithm for determining the shortest feasible path under the rectangular distance function when the model recognizes polygonal barrier regions on the plane. [29] obtained the discretization results for a similar problem in the presence of arbitrarily shaped barriers. Considering a grid construction procedure, the proposed problem was converted into an equivalent p-median problem to select an optimal set of facility locations. [6] extended the obtained results for both
convex forbidden regions and arbitrarily shaped barriers to provide a stochastic queue problem. Also, [36] extended the study of [29] for a center objective function. A new concept to categorize cells according to their cell corners was introduced and for each group of cells, a solution approach with an overall polynomially bounded complexity was presented. Then, [41] extended the work by considering the finite size center location problem under rectangular distances with the presence of barriers considering variable locations. [10] investigated a center location problem with the convex polygonal barriers being located on the plane using the rectangular distance function. To solve the proposed problem, a polynomial time algorithm based on dominating sets was presented. Later, [11] generalized this problem to the block normal distances. In the case of polyhedral barrier sets, [23] decomposed a non-convex barrier problem into a finite number of related unconstrained sub-problems to obtain an optimal solution of the original problem. [16] studied a center location problem with the Euclidean distances at the presence of polyhedral barriers and designed a solution approach based on propagation of circular wave fronts. For the first time, [42] considered a finite size location problem with rectangular distances in which both facilities and barriers had arbitrarily shapes. They considered interaction between demand points in addition to interaction between a facility and demand points and presented a heuristic approach to solve the problem. [22] extended this study so that both facility and barriers were rectangular shaped considering the barrier locations as variables, and used a contour line method to solve the proposed problem. [33] considered the rectilinear deviation distance to visit a facility from preplanned routes in which both points and lines were used to estimate the facilities. [34] presented an analytical study on facility location problems in the presence of a square barrier with probabilistic rectilinear distance. The author used two patterns of square and diamond lattices of facilities for obtaining the distribution of the barrier distance. Later, [35] considered the shape of barrier be rectangular and provided a bi-objective model to locate a new facility on a continuous plane using the rectilinear distance function with the barrier distances being minimized. [37] provided a modeling structure for solving facility location problems with barriers or forbidden regions on the plane. To model the regions, they used the phi-objects tool. [37] formulated a new model for the continuous facility location problem under restricted regions and multicommodity flows with unknown destinations and presented a Benders’ decomposition algorithm to solve the proposed model. [39] studied a facility location problem in the presence of polygonal and elliptical forbidden regions simultaneously and presented a Further Reduced Search Area (FRSA) technique to solve the problem.

For the first time, [24] presented a facility location problem in the presence of a line barrier on which there were a certain number of passages making it possible to travel from one side of the barrier to the other. It was shown that the complexity of the problem grew exponentially by increasing the number of passages but remained polynomial for a fixed number of passages. To solve the non-convex original problem, the author presented a decomposition approach. This study was extended to multi-criteria location problems by [27], introducing an efficient algorithm for solving the problem. Recently, researchers have been interested in constrained facility location problems with the barrier not having a fixed position. For the first time, [9] considered a horizontal route on the plane where a line barrier with certain length and negligible width is distributed uniformly. By dividing the feasible region into two separate half-planes and introducing the visibility conditions, they presented
a solution algorithm to find the optimal location of a new facility minimizing the total of the weighted expected rectilinear distances. Later, [2] used this problem with a center objective function and presented an efficient heuristic algorithm to solve the problem. [43] also extended the study of [9] to a multi-facility location problem considering the interaction between new facilities in addition to the interaction between the new and the existing facilities. They formulated a mixed-integer quadratic programming model. For solving large problems, a genetic algorithm and an imperialist competitive algorithm were presented. [19] studied the same problem under a multi-period planning horizon. They provided a lower bound technique based on the forbidden region and two solution algorithms to solve the proposed problem. [4] focused on the single facility location-dependent relocation problem with a center objective function under the same assumptions. They formulated the proposed problem as a mixed-integer nonlinear programming model affected by two elements: the rate of demands and the location of demand points during the planning horizon. [3] considered a Weber location problem in the presence of a probabilistic polyhedral barrier with a left triangular distribution function on the plane. After modifying the polyhedral barrier, they formulated a mixed integer nonlinear programming model with a nonconvex solution space. They presented an exact heuristic solution approach based on decomposing the feasible space. [5] extended the study of [4] considering a probabilistic polyhedral barrier on a horizontal route. Under the same assumptions and by considering a threshold for the barrier size, the proposed problem was first formulated as a mixed integer quadratic-restricted mathematical program and then simplified as a mixed integer linear programming model. To solve the problem, a lower bound technique and a decomposition method were presented. [1] formulated a mixed integer quadratic-constrained programming model for a multi-facility minimax location problem in the presence of a probabilistic line barrier under the rectangular metric. To solve the proposed model, a heuristic approach by combining a bounding method and a split-divide-and-conquer technique was proposed. They conducted a case study in Kingston-Upon-Thames and reported the obtained results. [9], [43] and [1] summarized different features of the existing studies on restricted facility location problems in tabular forms. Since our study is focused on probabilistic barriers, in Table 1 below, we provide more details about the cited studies on restricted facility location problems with probabilistic barriers. In the first column, the related studies are pointed out. Also, according to the number of new facilities being found, the studies are classified as single facility or multi facilities. The objective functions are grouped as Median or Center problems. The shapes of the barriers are categorized into two groups of line and polyhedral. We note that all the available studies considered only a single barrier on the plane, and our present work here extends the available studies by considering two probabilistic line barriers. In the last column, the available solution approaches are specified.

The available study of facility location problems consider only one probabilistic barrier, while presence of two probabilistic line barriers on the plane is very likely in reality. For example, consider a factory having an inside rail transportation system with two certain rail routes on each of which, a wagon with a certain length and a negligible width moves repeatedly from one end to the other. Thus, due to the presence of wagons anywhere on the routes, they can interfere with the flow of material. At the first glance, this may not seem to be different from the single probabilistic line barrier case, but a more precise attention will show that consideration of two
probabilistic line barriers are quite different from the situation where there is only a single probabilistic line barrier on the plane. Existence of two line barriers affects the visibility conditions between the points significantly, and consequently, making decisions about location of the new facility will be more complicated. Therefore, identifying the visibility conditions of two mutual points relative to two line barriers on the plane is the most important issue of our study here. Depending on the number of existing line barriers between the points, calculation of the expected barrier distance will be different. The most significant innovation of our work is the presentation of a heuristic approach to specify the possible visibility conditions of two mutual points using some conditional relations. Also, provision of a mathematical model to effectively handle complex problem situations is very useful. Here, we formulate a mixed-integer nonlinear programming model for a single facility location problem on the plane in the presence of two probabilistic line barriers with uniform distribution functions. For modeling, we first consider the different positions of two arbitrary points on the plane relative to each other and then define the visibility conditions of the problem using some conditional relations. Our aim is to find an optimal location of the new facility so that the total weighted expected rectilinear barrier distances is minimized. The main contributions of our study are: (1) this is the first effort to model a rectilinear median location problem considering two probabilistic line barriers on the plane. Due to non-linearity and non-convexity of the formulated model, solving the model appears to be harder than the relevant problems presented in the literature, (2) a heuristic approach is presented to specify the possible visibility conditions, (3) effective upper bound and lower bound for the proposed problem are proposed using the forbidden location problem and the location problem when the model ignores the line barriers, (4) a new solution approach is presented based on a local search. Two main features of this solution approach are high solution quality and short computing time, and (5) an empirical study is conducted in Kingston-upon-Thames, England, to show the behavior and applicability of the proposed model and the sensitivity of the problem parameters.

3. Problem description. Consider a plane (e.g., \( \mathbb{R}^2 \)) with \( m \) existing facilities having the coordinates \( X_i = (x_i, y_i), \) \( i = 1, \ldots, m \), and two horizontal line barriers. Let \( \mathcal{E} = \{ X_i \in \mathbb{R}^2 : i = 1, \ldots, m \} \) be the set of locations for the \( i \) existing facilities and \( X = (x, y) \) denote the location of a new facility. Let \( R_1 \) and \( R_2 \) be two horizontal routes on the plane, where two line barriers called \( B_1 \) and \( B_2 \), with
lengths $L_1$ and $L_2$ and negligible widths, are present. To avoid the complexity of the analysis, it is assumed that $L_1 = L_2 = L$. These line barriers respectively move on $R_1$ and $R_2$ routes and their $y$-coordinates are fixed at $b_k$, $k = 1, 2$, with $b_1 < b_2$, and their $x$-coordinates are distributed randomly. The starting points of the line barriers, being continuous random variables, are shown by $X_{sk}$, $k = 1, 2$. It is obvious that depending on the kind of the distribution function of $X_{sk}$, the shortest barrier distance functions will be different. Here, it is assumed that these points are uniformly distributed random variables with parameters $U(u_1, u_2)$, where $u_1 \leq \min_{i \in M} \{x_i\}$ and $u_2 \geq \max_{i \in M} \{x_i\}$. The probability density function of $X_{sk}$ can be written as

$$f(X_{sk}) = \begin{cases} \frac{1}{u_2 - u_1}, & u_1 \leq X_{sk} \leq u_2, \quad k = 1, 2 \\ 0, & \text{otherwise} \end{cases} \quad (1)$$

Also, $X_{ek} = X_{sk} + L$, $k = 1, 2$, denote the end points of the $k$th barrier (see Figure 1). Now, the barrier routes are defined as follows:

$$R_k = \{(x, y) \in \mathbb{R}^2 | u_1 \leq x \leq u_2 + L, y = b_k\}, \quad k = 1, 2 \quad (2)$$

where $u_1 \leq u_2$, $L > 0$ and $b_1 < b_2$. The two line barriers move independently, that is, the random variables $X_{s1}$ and $X_{s2}$ are independent. So, the combined density function of the starting points of the line barriers can be defined as follows:

$$f(X_{s1}, X_{s2}) = \begin{cases} \frac{1}{(u_2 - u_1)^2}, & u_1 \leq x_{s1}, x_{s2} \leq u_2 \\ 0, & \text{otherwise} \end{cases} \quad (3)$$

Here, $R = R_1 \cup R_2$ and $B = B_1 \cup B_2$ are forbidden and barrier regions, respectively, and $\mathcal{F} = \mathbb{R}^2 \setminus R$ is the feasible region of the problem. As shown in Figure 1, these feasible regions are divided into three regions with respect to the line barriers: lower region, middle region and upper region. It is desired to locate a single new facility on the plane to serve the existing facilities. According to the probabilistic occurrence of barriers, making it impossible to compute the exact barrier distances, the objective is to find a new facility location $X(x, y) \in \mathbb{R}^2$ such that the expected
weighted sum of the $p$-norm barrier distances between the new facility and the expected existing ones is minimized. Generally, the objective function of the single facility location problem with probabilistic line barrier can be written as:

$$f(X) = \sum_{i=1}^{m} w_i E[d_p^B(X, X_i)]$$  \(\text{(4)}\)

where $w_i$ is a positive weight associated with the existing facility $X_i$, which can be interpreted as the demand of facility $X_i$, and $E[d_p^B(X, X_i)]$ stands for the expected $p$-norm barrier distance between the points $X$ and $X_i$, which is the infimum of the lengths of all the permitted paths between $X$ and $X_i$ (i.e., shortest path between $X$ and $X_i$).

In general, in the presence of a barrier, the two arbitrary points $X$ and $X_i$ are called $p$-visible from each other, if $d_p^B(X, X_i) = d_p(X, X_i)$ and if $d_p^B(X, X_i) > d_p(X, X_i)$, then they are called $p$-shadow from each other. In the condition of $p$-shadow, the distance between the points will be a barrier distance, generally having a non-convex function. Also, a barrier $B_k, k = 1, 2$, is effective, if $d_p^B(X, X_i) > d_p \setminus B_k(X, X_i)$. For more discussion on the concept of visibility in the location problems, see [25]. In our study, the rectilinear distance function is considered (i.e., $p = 1$). In Figure 2, some visibility cases for the rectilinear distance are shown. Figure 2 shows the possible visibility conditions of two mutual points $X$ and $X_i$ relative to two line barriers on the plane. Case (a) shows that the points are 1-visible from each other and the barriers do not affect the distance between the two points, whereas cases (b) and (c) display that the points $X$ and $X_i$ are 1-shadow from each other. In case (b), the line barrier $B_1$ and in case (c), both line barriers $B_1$ and $B_2$ affect the distance between the points. In cases (b) and (c), the additional distances which should be traveled are indicated by red dashed lines. For rectilinear distance, [9] showed that the barrier affects only the $x$-distance between the new facility and the existing facilities and the $y$-distance will remain the same. Using this property, in the objective function (4), $d_p^B(X, X_i)$ is separable as $d_p^B(X, X_i) = d_1^B(x, x_i) + d_1^B(y, y_i)$, where $d_1^B(y, y_i) = d_1(y, y_i) = |y - y_i|$. So, the objective function (4) can be rewritten as:

$$f(X) = \sum_{i=1}^{m} w_i E[d_1^B(x, x_i) + d_1^B(y, y_i)]$$

$$= \sum_{i=1}^{m} w_i E[d_1^B(x, x_i) + |y - y_i|]$$  \(\text{(5)}\)

$$= \sum_{i=1}^{m} w_i (E[d_1^B(x, x_i)] + |y - y_i|)$$

$$= \sum_{i=1}^{m} w_i E[d_1^B(x, x_i)] + \sum_{i=1}^{m} w_i |y - y_i|.$$

The above equation shows that for the expected barrier distances we only need to calculate the $x$-coordinate of the distance function. In the next section, we discuss the expected barrier $x$-distance between two points in rectilinear plane.

4. **Expected barrier $x$-distance.** The possible visibility conditions of two mutual points $X$ and $X_i$ relative to line barriers is the most important issue in calculating the expected barrier $x$-distance between the points. As shown in Figure 1, the
barrier routes $R_k, k = 1, 2$, divide the feasible region into three regions. So, for two arbitrary points $X$ and $X_i$, depending on the number of existing line barriers between them, the set $I_z$ can be defined as follows:

$$I_z = \{ i : \text{there is } z \text{ barrier route between } X \text{ and } X_i \}, z \in \{0, 1, 2\}. \quad (6)$$

Generally, for each two points $X$ and $X_i$, one of the three following cases may occur. **Case 1.** There are no barrier routes between $X$ and $X_i$. In this case, both points are located in the same region. For this case, the expected barrier $x$-distance is equal to the $x$-distance:

$$E[d^B_1(x, x_i)|i \in I_0] = d_1(x, x_i) = |x - x_i|. \quad (7)$$

**Case 2.** There is one barrier route between $X$ and $X_i$. For this case, to compute the expected barrier distance between $x$ and $x_i$, we use Theorem 1 below, established by [9].

**Theorem 1.** If there is one barrier route between $X$ and $X_i$, then we have

$$E[d^B_1(x, x_i)|i \in I_1] = \begin{cases} 
\frac{(L - |x - x_i|)^2}{2(u_2 - u_1)} + |x - x_i|, & |x - x_i| < L \\
|x - x_i|, & |x - x_i| \geq L,
\end{cases} \quad (8)$$

According to Theorem 1, if $|x - x_i| < L$, then one of the line barriers affects the distance between the two points; otherwise, the barrier does not affect the distance
between the two points and the expected barrier distance is equal to the rectilinear distance between them.

Case 3. Routes of both barriers $B_1$ and $B_2$ are located between $X$ and $X_i$. In this condition, $E[d_1^B(x, x_i)]$ depends on the distance of $x$ and $x_i$ so that if $|x - x_i| \geq 2L$, $L \leq |x - x_i| < 2L$ or $|x - x_i| < L$, then the corresponding function for $E[d_1^B(x, x_i)]$ will be different. In total, there are 5 regions for $(x_{s1}, x_{s2})$ in which $X$ and $X_i$ are invisible. We present these regions by $\psi_n, 1 \leq n \leq 5$. In each of these regions, the barrier rectilinear distance of $x$ and $x_i$, which is affected by the locations of the barriers, is displayed by $D_n(x_{s1}, x_{s2})$.

**Theorem 2.** If there are two barrier routes between $X$ and $X_i$, then we have

$$E[d_1^B(x, x_i)|i \in I_2] = \left\{ \begin{array}{ll}
|x - x_i|, & |x - x_i| \geq 2L \\
\pi_1 e_1 + (1 - \pi_1)|x - x_i|, & L \leq |x - x_i| < 2L \\
\sum_{n=2}^{5} \pi_n e_n + (1 - \sum_{n=2}^{5} \pi_n)|x - x_i|, & |x - x_i| > 2L 
\end{array} \right. $$

(9)

where

$$\pi_n = P(X_s \in \psi_n) = \int_{\psi_n} \frac{1}{(u_2 - u_1)^2} dx_{s1}.dx_{s2},$$

(10)

$$e_n = E[d_1^B(x, x_i)|X_s \in \psi_n] = \int_{\psi_n} \frac{D_n(x_{s1}, x_{s2})}{\pi_n} \frac{1}{(u_2 - u_1)^2} dx_{s1}.dx_{s2},$$

(11)

and

$$D_n : \psi_n \rightarrow \mathbb{R}$$

$$1 \leq n \leq 5$$

$$X_s \rightarrow d_1^B(x, x_i)|X_{s} \in \psi_n$$

(12)

denotes the barrier distance function on the $\psi_n$ region.

Proof. In the next calculations, for two mutual points $X$ and $X_i$, it is assumed that $y > y_i$ (since the computations for $y < y_i$ is similar, we disregard it).

Case $|x - x_i| \geq 2L$. It is obvious that if $|x - x_i| \geq 2L$, then $X$ and $X_i$ will be visible and the locations of the line barriers have no effect on the visibility and we have

$$E[d_1^B(x, x_i)] = |x - x_i|. $$

(13)

Case $L \leq |x - x_i| < 2L$. In this case, $\psi_1$ shows a region in which if $(x_{s1}, x_{s2})$ is located, then $X$ and $X_i$ will be invisible. As displayed in Figure 3(a), considering $x \leq x_i$, region $\psi_1$ is stated as follows:

$$\psi_1 : \left\{ \begin{array}{l}
x_i - L \leq x_{s1} \leq x_{s2} + L \\
x_i - 2L \leq x_{s2} \leq x.
\end{array} \right. $$

(14)

For $(x_{s1}, x_{s2}) \in \psi_1$, the barrier rectilinear distance between $x$ and $x_i$ is computed as follows:

$$d_1^B(x, x_i) = D_1(x_{s1}, x_{s2}) = \min \left( \frac{(x_i - x_{s2}) + (x - x_{s2})}{(x_{s1} - x_i) + (x_{s2} + L - x_{s1}) + (x_{s2} + L - x)} \right)$$

$$= \min \left( \frac{-2x_{s2} + x + x_i}{2x_{s1} + 2x_{s2} + 2L - x + x_i} \right) $$

(15)
Also, for \( x > x_i \), we have
\[
\psi_1 : \left\{ \begin{array}{l}
x - 2L < x_{s1} < x_i \\
x - L \leq x_{s2} \leq x_{s1} + L.
\end{array} \right.
\] (16)

In this case, the barrier rectilinear distance between \( x \) and \( x_i \) is calculated as follows:
\[
d^B_1(x, x_i) = D_1(x_{s1}, x_{s2}) = \min \left( \begin{array}{l}
(x_{s1} + L - x_i) + (x_{s1} + L - x_{s2}) + (x - x_{s2}) \\
(x_{s2} + L - x_i) + (x_{s2} + L - x)
\end{array} \right)
\] 
\[
= \min \left( \begin{array}{l}
-2x_{s1} + x + x_i \\
2x_{s2} - 2x_{s2} + 2L + x - x_i
\end{array} \right)
\] (17)

In Figure 3(b), an example of this case is displayed. Due to the existing symmetries, the value of \( E[d^B_1(x, x_i)] = \pi_1 e_1 + (1 - \pi_1)|x - x_i| \) for this case is equal to the one for the case \( x \leq x_i \).

**Case** \( L > |x - x_i| \). In this case, for the invisible \( X \) and \( X_i \), one of the following two situations occurs.

**Situation1**: Only one of the barriers \( B_1 \) or \( B_2 \) affects the distance between \( X \) and \( X_i \) and the presence of the other barrier is ineffective. For this situation, it is necessary that \((x_{s1}, x_{s2})\) be located in one of the regions \( \psi_2 \) or \( \psi_3 \). Next, we show these regions and the corresponding values of \( d^B_1(x, x_i) \).

- **Region** \( \psi_2 \): Line barrier \( B_1 \) blocks visibility of \( X \) and \( X_i \) and the presence of \( B_2 \) is ineffective. Examples for placement of \((x_{s1}, x_{s2})\) in this region are shown in Figure 4. In this region, we have
\[
\psi_2 : \left\{ \begin{array}{l}
\max\{x - L, x_i - L\} \leq x_{s1} \leq \min\{x, x_i\} \\
u_1 \leq x_{s2} \leq x - L \text{ or } x \leq x_{s2} \leq u_2
\end{array} \right.
\] (18)
and also,

\[(x_{s1}, x_{s2}) \in \psi_2 \implies D_2(x_{s1}, x_{s2}) = \begin{cases} 
2x_{s1} + 2L - x_i - x_{s1} & \text{if } x_{s1} \leq \frac{(x + x_i - L)}{2} \\
2x_i - 2x_{s1} & \text{if } x_{s1} > \frac{(x + x_i - L)}{2}
\end{cases},
\]

\[\implies d_B^1(x, x_i) = D_2(x_{s1}, x_{s2}) = \min \left( \frac{2x_{s1} + 2L - x_i}{x + x_i - 2x_{s1}} \right) \quad (19)\]

Calculating the expected barrier distance when a line barrier is located between two points has already been stated in Theorem 1.

**Figure 4.** Examples for \((x_{s1}, x_{s2}) \in \psi_2\).

- **Region \(\psi_3\)**: Line barrier \(B_2\) blocks visibility of \(X\) and \(X_i\) and the presence of \(B_1\) is ineffective. Examples of placement of \((x_{s1}, x_{s2})\) in this region are shown in Figure 5. In this region, we have

\[\psi_3 : \begin{cases} 
2x_{s1} + 2L - x_i - x_{s1} & \text{if } x_{s1} \leq \frac{(x + x_i - L)}{2} \\
2x_i - 2x_{s1} & \text{if } x_{s1} > \frac{(x + x_i - L)}{2}
\end{cases},
\]

\[\implies d_B^1(x, x_i) = D_3(x_{s1}, x_{s2}) = \min \left( \frac{2x_{s1} + 2L - x_i}{x + x_i - 2x_{s1}} \right) \quad (20)\]

In this situation, Theorem 1 is also used to calculate the expected barrier distance.

**Situation 2** : Both barriers \(B_1\) and \(B_2\) block visibility of \(X\) and \(X_i\). This situation occurs in regions \(\psi_4\) and \(\psi_5\). In the following, these regions and the corresponding values of \(d_B^1(x, x_i)\) are shown.

- **Region \(\psi_4\)**: Generally, in this region, we have

\[\psi_4 : \begin{cases} 
x_i - L \leq x_{s1} \leq \min\{x, x_i\} \
\max\{x_{s1}, x - L\} \leq x_{s2} \leq \min\{x_{s1} + L, x\}
\end{cases},
\]

In the above statement, if \(x \leq x_i\), then \(x - L \leq x_i - L\), and since in region \(\psi_4\) we have \(x_i - L \leq x_{s1}\), we get

\[\psi_4 : \begin{cases} 
x_i - L \leq x_{s1} \leq x \\
x_{s1} \leq x_{s2} \leq x
\end{cases} \quad (22)\]
Also, if \( x > x_i \), then, using \(|x - x_i| < L\), we have \( x_i - L < x - L < x_i \). Therefore, the range of \( x_s1 \) is broken down as follows:

\[
\begin{align*}
(i) & \quad x_i - L \leq x_s1 \leq x - L \implies x - L \leq x_s2 \leq x_s1 + L \\
(ii) & \quad x - L \leq x_s1 \leq x_i \implies x_s1 \leq x_s2 \leq x.
\end{align*}
\]

So, we have

\[
\psi_4 : \\
x_i - L \leq x_s1 \leq x - L \implies x - L \leq x_s2 \leq x_s1 + L \\
x - L \leq x_s2 \leq x_s1 + L \cup \{ x - L \leq x_s1 \leq x_i \\
x_s1 \leq x_s2 \leq x \}.
\]

Finally, for \((x_s1, x_s2) \in \psi_4\), the barrier rectilinear distance between \( x \) and \( x_i \) is calculated as follows:

\[
d_B^B(x, x_i) = D_4(x_s1, x_s2) = \min \left( \begin{array}{c} (x_i - x_s1) + (x - x_s1) \\
(x_s1 + L - x_i) + (x_i + L - x_s2) + (x - x_s2) \\
(x_s2 + L - x_i) + (x_s2 + L - x) \end{array} \right) \\
= \min \left( \begin{array}{c} -2x_s1 + x + x_i \\
2x_s1 - 2x_s2 + 2L + x - x_i \\
2x_s2 + 2L - x - x_i \end{array} \right)
\]

An example of placement of \((x_s1, x_s2)\) in region \(\psi_4\) is shown in Figure 6.

**Region** \(\psi_5\) : Generally, in this region, we have

\[
\psi_5 : \begin{align*}
\max\{x_s2, x_iL\} & \leq x_s1 \leq \min\{x_s2 + L, x_i\} \\
x - L & \leq x_s2 \leq \min\{x, x_i\}.
\end{align*}
\]

The calculations are similar to the previous region, and we omit the details.

\[
\begin{align*}
\text{If } x \leq x_i & \quad \frac{x - L \leq x_s1 \leq x_s2}{x \leq x_i} & \frac{x_s2 \leq x_s1 \leq x_i}{x - L \leq x_s2 \leq x_i} \\
\text{If } x > x_i & \quad \frac{x_i - L \leq x - L \leq x_s2}{x_i \leq x_s1 \leq x_i} & \frac{x_s2 \leq x_s1 \leq x_i}{x - L \leq x_s2 \leq x_i}.
\end{align*}
\]

![Figure 5. Examples for \((x_s1, x_s2) \in \psi_3\).](image-url)
Finally, for \((x_1, x_2) \in \psi_5\), the barrier rectilinear distance between \(x\) and \(x_i\) is calculated as follows:

\[
d^B_1(x, x_i) = D_5(x_1, x_2) = \min \left( \begin{array}{c} (x_i - x_2) + (x - x_2) \\ \frac{1}{2}(x_i - x_1) + (x_2 + L - x_1) + (x_2 + L - x) \\ (x_1 + L - x_i) + (x_1 + L - x) \end{array} \right) \\
= \min \left( \begin{array}{c} -2x_2 + x + x_i \\ 2x_1 + 2x_2 + 2L - x + x_i \\ 2x_1 + 2L - x - x_i \end{array} \right)
\]

An example of the placement of \((x_1, x_2)\) in region \(\psi_5\) is displayed in Figure 7.

Now, as a summary, invisible regions \(\psi_n\) and their corresponding distance functions \(D_n\), \(1 \leq n \leq 5\), for \(i \in \mathcal{I}_2\), are shown in Table 2.

Also, we have designed a heuristic approach to specify the possible visibility conditions of two mutual points \(X_i\) and \(X\) relative to the two line barriers. In Algorithm 1, a pseudo code for the proposed heuristic is provided.
Table 2. Invisible regions and their corresponding distance functions for \( i \in \mathcal{I}_2 \).

| \( n \times \) | \( x \) | \( \Psi_n \) | \( D_n(x_{s1},x_{s2}) \) |
|---|---|---|---|
| 1 | \( x \leq x_i \) \( x \leq x_i \) | \( x_i - L \leq x_{s1} \leq x_{s2} + L \) | \( -2x_{s1} \leq x + x_i \) \( 2x_{s1} + 2L \leq x - x_i \) | \( \min \left(-2x_{s1} + 2x_{s2} + 2L - x + x_i \right) \) |
| 2 | \( x > x_i \) \( x \in \mathcal{F} \) | \( x < 2L \leq x_{s1} \leq x_{s2} \leq x_{s1} + L \) | \( -2x_{s2} + x + x_i \) \( 2x_{s2} + 2L + x - x_i \) | \( \min \left(2x_{s1} - 2x_{s2} + 2L + x - x_i \right) \) |
| 3 | \( x \in \mathcal{F} \) \( x \leq x_i \) \( x \) \( x \leq x_{s1} \leq x_{s2} \leq x \) | \( \max(x - L, x_i - L) \leq x_{s1} \leq \min[x, x_i] \) \( u_1 \leq x_{s2} \leq x - L \) \( u_2 \) | \( 2x_{s1} + 2L - x - x_i \) \( x_{s1} \leq \frac{(x + x_i - L)}{2} \) \( x_{s1} - 2x_{s1} \geq \frac{(x + x_i - L)}{2} \) |
| 4 | \( x > x_i \) \( x \in \mathcal{F} \) \( x \) \( x_{s1} \leq x \leq x_{s2} \leq x_{s1} + L \) \( x_{s1} \leq x_{s2} \leq x_{s1} + L \) \( u_1 \) \( u_2 \) | \( \max(x - L, x_i - L) \leq x_{s1} \leq \min[x, x_i] \) \( z \leq x_{s2} \leq x_{s1} \) \( x_{s1} \leq x_{s2} \leq x \) | \( 2x_{s2} + 2L - x - x_i \) \( x_{s2} \leq \frac{(x + x_i - L)}{2} \) \( x_{s2} - 2x_{s2} \geq \frac{(x + x_i - L)}{2} \) |
| 5 | \( x < x_i \) \( x \) \( x_{s1} \leq x_{s2} \leq x_{s1} \) \( x_{s1} \leq x_{s2} \leq x_{s1} + L \) \( x_{s1} \leq x_{s2} \leq x_{s1} + L \) \( u_1 \) \( u_2 \) | \( \max(x - L, x_i - L) \leq x_{s1} \leq x_{s2} \leq x_{s1} + L \) \( x_{s1} \leq x_{s2} \leq x_{s1} + L \) \( x_{s1} \leq x_{s2} \leq x_{s1} + L \) \( u_1 \) \( u_2 \) | \( -2x_{s2} + x + x_i \) \( 2x_{s2} + 2L - x - x_i \) | \( \min \left(-2x_{s1} + 2x_{s2} + 2L + x - x_i \right) \) |

5. Mathematical model and solution approach.

5.1. Mathematical model. Our proposed model is to locate one new facility among \( m \) existing facilities in presence of two probabilistic line barriers minimizing the total weighted expected rectilinear barrier distances. It should be pointed out that the new facility is not allowed to be located on the routes of the line barriers. Now, according to the discussion in Section 4, the objective function (5) can be rewritten as follows:

\[
\sum_{i=1}^{m} w_i E[d^B_1 (X, X_i)] = \sum_{i=1}^{m} w_i E[d^B_1 (X, x_i)] + \sum_{i=1}^{m} w_i |y - y_i| \\
= \left( \sum_{i \in \mathcal{I}_0} w_i |x - x_i| + \sum_{i \in \mathcal{I}_1} w_i E[d^B_1 (X, x_i)] \right) + \sum_{i \in \mathcal{I}_2} w_i |y - y_i| \quad \text{(32)}
\]

To specify the constraints, we first define the binary parameters \( q_{ki} \), the binary decision variables \( g_k \) and \( \Gamma_{ki} \) and the variables \( \lambda_i \) as follows:

\[
g_k = \begin{cases} 
1, & \text{if } y > b_k \quad k = 1, 2 \\
0, & \text{otherwise.} 
\end{cases} \quad \text{(33)}
\]

\[
q_{ki} = \begin{cases} 
1, & \text{if } y_i > b_k \quad i = 1, \ldots, m \text{ and } k = 1, 2 \\
0, & \text{otherwise.} 
\end{cases} \quad \text{(34)}
\]
Algorithm 1: Pseudo code for specifying the possible visibility conditions.

START
For $i = 1, ..., m$ do
    if $i \in I_0$ then
        $X$ and $X_i$ are visible
    elseif $i \in I_1$ then
        if $|x - x_i| \geq L$ then
            $X$ and $X_i$ are visible
        else
            $X$ and $X_i$ are invisible
        endif
    elseif $i \in I_2$ then
        if $|x - x_i| \geq 2L$ then
            $X$ and $X_i$ are visible
        else
            if $L \leq |x - x_i| < 2L$ then
                if $(x_1, x_2) \in \psi_1$ then
                    $X$ and $X_i$ are visible
                else
                    $X$ and $X_i$ are invisible
                endif
            elseif $(x_1, x_2) \in \psi_2$ then
                $X$ and $X_i$ are invisible
            elseif $(x_1, x_2) \in \psi_3$ then
                $X$ and $X_i$ are invisible
            elseif $(x_1, x_2) \in \psi_4$ then
                $X$ and $X_i$ are invisible
            endif
        endif
    endif
endfor
END

$$
\lambda_i = \begin{cases} 
0, & \text{if } i \in I_0 \\ 
1, & \text{if } i \in I_1 \quad i = 1, \ldots, m. \\ 
2, & \text{if } i \in I_2.
\end{cases} \quad (35)
$$

$$
\Gamma_{ki} = \begin{cases} 
1, & \text{if } |x - x_i| < kL, \quad i = 1, \ldots, m \text{ and } k = 1, 2 \\ 
0, & \text{otherwise}.
\end{cases} \quad (36)
$$

where the decision variables $g_k$ determine the position of new facility relative to the line barriers $B_1$ and $B_2$. The variables specify the new facilities being located in the lower, middle or upper regions. Similarly, the parameters $q_{ki}$ show the position of existing facilities relative to the line barriers. The number of barrier routes between $X$ and $X_i$ are defined by the variable $\lambda_i$. Finally, the binary variable $\Gamma_{ki}$ specifies whether the difference of $x$-coordinates of the existing facility $X_i$ and the new facility $X$ is less than $kL$ or not. Now, using the defined relations in the previous section, the objective function (32), the parameters and variables (33)-(36), the proposed mathematical model for the single facility location problem in
presence of two probabilistic line barriers is formulated as follows:

$$\min_{X \in \mathcal{R}} \sum_{i=1}^{n} w_i \left( \frac{1}{2} (\lambda_i - 1)(\lambda_i - 2)|x - x_i| - \lambda_i(\lambda_i - 2) \left( \Gamma_{1i} \frac{(L - |x - x_i|)^2}{2(u_2 - u_1)} + |x - x_i| \right) 
+ \frac{1}{2} \lambda_i(\lambda_i - 1) \left( (1 - \Gamma_{2i})|x - x_i| + \Gamma_{2i} (1 - \Gamma_{1i}) (\pi_1 e_1 + (1 - \pi_1)|x - x_i|) \right) 
+ \Gamma_{1i} \left( \sum_{\pi_2=n=2}^{5} \pi_2 e_2 + (1 - \sum_{\pi_2=n=2}^{5} \pi_2)|x - x_i| \right) + |y - y_i| \right) \right)$$

s.t. relations (10) and (11) hold,

$$2g_k(y - b_k) \geq (y - b_k), \quad k = 1, 2, \tag{38}$$

$$2g_k(y_i - b_k) \geq (y_i - b_k), \quad i = 1, \ldots, m \text{ and } k = 1, 2, \tag{39}$$

$$|q_1(1 + g_2) - q_2(1 + g_2i)| = \lambda_i, \quad i = 1, \ldots, m, \tag{40}$$

$$2\Gamma_{ki}(|x - x_i| - k.L) \leq |x - x_i| - k.L, \quad i = 1, \ldots, m \text{ and } k = 1, 2, \tag{41}$$

$$g_k, q_{ki}, \Gamma_{ki} \in \{0, 1\}, \lambda_i \in \{0, 1, 2, 3\}, \quad x, y \geq 0, \quad i = 1, \ldots, m \text{ and } k = 1, 2. \tag{42}$$

The objective function (37) gives the total weighted expected rectilinear barrier distance from new facility to the existing facilities, where depending on the number of barrier routes between X and $X_i$ only one of the coefficients $\frac{1}{2} (\lambda_i - 1)(\lambda_i - 2), -\lambda_i(\lambda_i - 2)$ or $\frac{1}{2} \lambda_i(\lambda_i - 1)$ is equal to one, and the others are equal to zero. Then, depending on i belonging to the sets $I_0, I_1$ or $I_2$, using the $\Gamma_{ki}$ coefficient, $E[d_B^i(x, x_i)]$ is calculated. The statements (33) and (34) are now expressed by the constraints (38) and (39), respectively, which determine the position of existing and new facilities relative to line barriers $B_1$ and $B_2$. Expression (35) has been replaced by the equations (40) showing the number of barrier routes between the ith existing facility and the new facility. Expression (36) for specifying that if the difference of x-coordinates of the existing facility $X_i$ and the new facility X is less than k.L, then $\Gamma_{ki}$ is set to 1, otherwise, zero, is now transformed into constraints (41). Constraints (42) define the binary and nonnegative variables.

5.2. Upper and lower bounds of the proposed problem. Since location problems with barrier are generally global optimization problems, problem relaxations provide possibility to determine the appropriate bounds that are necessary to solve these problems. Here, to find good bounds for the proposed problem, we extend the work of [25].

5.2.1. Lower bound for the proposed problem. To find lower bounds for location problems with barriers, instead of solving a problem with barriers, we solve the corresponding unconstrained problem; that is, based on the notation of [17], instead of solving a problem of type $1/\mathbb{R}^2/B = 2ProbL/d_1/\sum$, a problem of type $1/\mathbb{R}^2/\cdot/d_1/\sum$ is solved where the black solid dot signifies that there are no line barriers on the plane $\mathbb{R}^2$. In this case, considering $z_B^*$ as the optimal objective value of a location problem of type $1/\mathbb{R}^2/B = 2ProbL/d_1/\sum$ and $X^*$ as an optimal solution of a location problem of type $1/\mathbb{R}^2/\cdot/d_1/\sum$, we have $f(X^*) \leq z_B^*$. It is possible that $X^*$ in problem of type $1/\mathbb{R}^2/\cdot/d_1/\sum$ is infeasible for the location problem of type $1/\mathbb{R}^2/B = 2ProbL/d_1/\sum$. Then, a different relaxation of
1/\mathbb{R}^2/\mathcal{B} = 2\text{ProbL}/d_1/\sum \text{ for a restricted location problem with forbidden regions with the notation } 1/\mathbb{R}^2/\mathcal{R} : \mathcal{B} = 2\text{ProbL}/d_1/\sum \text{ should be considered (in forbidden regions, placement of a facility is not allowed but travelling through the regions is permitted without any penalty). In this case, considering } X^*_R \text{ as an optimal solution of a location problem of type } 1/\mathbb{R}^2/\mathcal{R} : \mathcal{B} = 2\text{ProbL}/d_1/\sum \text{, we have } f(X^*_R) \leq z^*_B. \text{ In Figure 8, the location problems of type } 1/\mathbb{R}^2/\mathcal{B} = 2\text{ProbL}/d_1/\sum \text{ and } 1/\mathbb{R}^2/\mathcal{R} : \mathcal{B} = 2\text{ProbL}/d_1/\sum \text{ are depicted.}

5.2.2. Upper bound for the proposed problem. Since $X^*_R$ obtained from a constrained location problem of type $1/\mathbb{R}^2/\mathcal{R} : \mathcal{B} = 2\text{ProbL}/d_1/\sum$ is always a feasible solution for a location problem of type $1/\mathbb{R}^2/\mathcal{B} = 2\text{ProbL}/d_1/\sum$, the objective value corresponding to $X^*_R$ evaluated by the barrier objective function $f_B$ can be an upper bound for location problem of type $1/\mathbb{R}^2/\mathcal{B} = 2\text{ProbL}/d_1/\sum$; that is, considering $z^*_B$ as the optimal objective value of a location problem of type $1/\mathbb{R}^2/\mathcal{B} = 2\text{ProbL}/d_1/\sum$ and $X^*_R$ as an optimal solution of a location problem of type $1/\mathbb{R}^2/\mathcal{R} : \mathcal{B} = 2\text{ProbL}/d_1/\sum$, we have $z^*_B \leq f_B(X^*_R)$, where $f_B(X^*_R)$ gives the objective value of a barrier location problem for solution of the forbidden location problem.

5.3. The HJA. To solve the median location problem in the presence of two probabilistic horizontal barriers under the stated assumptions, first the $y$-coordinates of the existing facilities are sorted in a nondecreasing order:

$$y_1 \leq \cdots \leq y_i \leq b_1 \leq y_{i+1} \leq \cdots \leq y_{i_2} \leq b_2 \leq y_{i+1} \leq y_m.$$  

So, a vector $i = (i_1, i_2)$ divides the existing facilities based on the route of the line barriers. In the following, a solution approach is presented for solving the proposed problem.

In this study here we deal with a location problem in a continuous space where the objective function (37) is not indeterminate. So, to find an optimal location of the facility, it is necessary to search the solution space of the problem exactly. The HJA is considered to be appropriate because of the following advantages: (1) It is a search direction algorithm, (2) It is an optimization method searching for the optimal values at the problem variables ([20]), (3) it generates a set of search directions iteratively and in each direction, the search space is explored completely, and (4) It
combines the exploratory move (EM) and the heuristic pattern move repeatedly. An exploratory move tests the local behavior of the objective function systematically to reach the best point around the current point. A pattern move utilizes two such points to generate a new point by considering a single step from the current base point along a direction connecting the previous point to the present base point. The general steps of the proposed algorithm are stated in Figure 9. As shown, in the first

Figure 9. General steps of the proposed algorithm.

step we need to choose an appropriate starting point. An appropriate starting point helps to decrease the number of exploratory and pattern moves and turns to speed up the algorithm in reaching a favorable solution. Considering section 5.2.2, and performing several tests, we found out that $X^*_R$ as an optimal solution of a forbidden location problem of type $1/R^2/R : B = 2ProbL/d_1/\sum$ corresponding to an upper bound of the barrier location problem of type $1/R^2/B = 2ProbL/d_1/\sum$ can be considered as an appropriate starting point for the proposed algorithm ($x^{(0)} = X^*_R$). Having a starting point $x^{(0)}$ as a base point, the exploration searches take place by certain step sizes which may be different for each coordinate direction. These step sizes may be changed during the various searches. For each coordinate direction, the step will be successful if the objective value is not increased and otherwise, it will be designated as failed and replaced by a step in the opposite direction. The exploration move will continue until all the coordinates are investigated. The obtained point of the exploratory move is considered as a base point. Here, we consider the horizontal
and vertical coordinates. So, we have \( N = 2 \) and \( \Delta = (\Delta x_1, \Delta x_2) \) and using a step reduction factor \( \alpha > 1 \), the step size will be reduced by \( \Delta x_j / \alpha \) for \( j = 1 \) and 2.

In the above flowchart (Figure 9), by using a pattern move a single step from the current base point along a direction connecting the previous point to the present base point is considered. The point \( x_p^{(K+1)} \) presents a new pattern point considered as a temporary base point for a new exploratory search. As shown, in the proposed algorithm, the stopping criterion is \( \Delta < T \) or equivalently \( (\Delta x_1, \Delta x_2) < (t_1, t_2) \). In our studied problem, this stopping criterion is considered to be \( \Delta x_1 < t_1 \). This is due to the fact that based on the problem assumptions, the vertical distance of the facility is not affected by the barriers, and thus the stopping criterion is conditioned only on the horizontal distance. More details about this algorithm can be found in [40].

6. Numerical results. Here, the performance of the proposed model and the computational efficiency of the proposed algorithm are tested. Two categories of sample problems and a case study are considered. In all the samples, our proposed model was implemented in the LINGO software environment applying global solver and our proposed solution method was coded in MATLAB 7.10. The programs were executed on a Dell 1564 laptop with 4 GB RAM and a 2.27 GHz processor running Windows 7. Finding some optimal solutions needed excessive amount of time depending on the size of the instance, and a time limit of 18000 seconds was considered for the LINGO software package. If after this period, an optimal solution is not obtained, the LINGO solver reports the best feasible solution obtained and the instance is reported unsolvable. In our proposed algorithm, the values of the step length (\( \Delta \)), the step reduction factor (\( \alpha \)) and the threshold value in the stopping criterion (\( T \)) are \([0.5, 0.5], 2, 0.001\), respectively. These values were determined to be appropriate after conducting numerous runs for each set of parameters.

6.1. Sample problems. To evaluate the performance of the proposed algorithm, we tested several sample problems with different sizes. The number of existing facilities on the plane specifies the size of each sample problem. The sample problems are generated under the following assumptions: the \( x \) and \( y \) coordinates of the existing facilities are selected randomly in the ranges \([4, 30]\) and \([0, 40]\) excluding \( b_1 \) and \( b_2 \), respectively, the barrier lengths are \( L_1 = L_2 = 4 \), the starting point of both line barriers have a uniform distribution, \( U(0, 35) \), and the \( y \)-coordinates are fixed respectively at \( b_1 = 10 \) and \( b_2 = 25 \). Also, the weight corresponding to every facility is assumed to be 1.

As shown in tables 3 and 4, the results are reported in two categories: (1) small and medium problems and, (2) large problems. In total, 45 sample problems were generated. In all the sample problems, two performance measures were explored for the algorithm: (1) solution quality, and (2) computing time. To evaluate solution quality of the algorithm (\( f_{Alg} \)), it is benchmarked using three values: (1) the objective value obtained using the LINGO software package (\( f_{LINGO} \)), (2) the lower bound obtained using the proposed technique in Section 5.2.1 (\( f_{LB} \)), and (3) the upper bound obtained using the proposed technique in Section 5.2.2 (\( f_{UB} \)). That is, besides the optimal solution of the original problem, optimal solution of a location problem without line barriers (\( 1/R^2/\sum d_i \)) and optimal solution of a forbidden location problem (\( 1/R^2/\sum \cdot B = 2ProbL/d_i \)) were considered as benchmarks. In tables 3 and 4, using the defined benchmarks, we computed the relative gaps in determining the solution quality of the algorithm for each sample problem. At the
bottom of Table 3, computational formulas for the relative gaps are given. These gaps stand for the relative difference between the benchmark objective value and the obtained objective value by the proposed algorithm. Also, in Figure 10, values of these gaps for all the sample problems are illustrated graphically. The obtained results in the column labeled as $Gap_{exact}^\%$ in tables 3 and 4 and Figure 10 show that in all the samples (samples 1 to 20), after obtaining the optimal solutions, the relative gaps between $f_{Lingo}$ and $f_{Alg}$ are almost equal to zero. For samples 21 to 31, by increasing the number of existing facilities, the relative gaps between the best feasible objective values were obtained after 18000 seconds and $f_{Alg}$ increased. This is due to the fact that by increasing the size of the sample, the problem becomes harder and the quality of the obtained solution within the time limitation deteriorates; for sample problems of larger than 31, no feasible solutions are found. In Figure 10, to show a more proper comparison, only positive gaps are shown. Also, the obtained results in columns labeled as $Gap_{LB}^\%$ and $Gap_{UB}^\%$ in tables 3 and 4 and their corresponding curves in Figure 10 show that the values of $Gap_{LB}^\%$ are more than the ones for $Gap_{UB}^\%$ in all the samples. It can be seen that the values of $Gap_{LB}^\%$ vary from 2.2% to 3.5% and the values of $Gap_{UB}^\%$ vary from 0.8% to 2%. It is clear from the gaps that: (1) the obtained solutions of the studied problem are closer to the obtained optimal solutions of a forbidden location problem, and, (2) the presented algorithm has a relatively robust performance with respect to the size of the problem.

For the computing times of the proposed algorithm, in tables 3 and 4, the time needed to solve the studied problem by the LINGO software package ($t$) and the proposed algorithm ($t_{Alg}$) are reported in seconds. Also, in Figure 11, a graphical comparison of the times of the two methods with respect to the size of the sample problems is provided. The results show a significant difference of the required time to solve the original problem using LINGO and the proposed algorithm. For the proposed algorithm, only two samples with time more than 2000 seconds are reported while for LINGO, for all the samples having 35 or more existing facilities this time is exceeded, and for samples having more than 220 existing facilities, the full 18000 seconds is consumed. Unlike LINGO for which the computing time grows quickly with increase in the size of the sample problem, for our proposed algorithm a specific increasing trend is not observed. This is due to the search-based structure of the algorithm, where regardless of the number of existing facilities, in exploratory moves, search around a current point continues until a best point is reached. So, the time of this local search cannot be prespecified.

6.2. An empirical study. Here, we show the impact of presence of two probabilistic line barriers in locating a single facility on the plane using the case studied by [1]. Also, the behavior of the proposed model and its sensitivity relative to the number and length of the line barriers will be analyzed. This empirical study relates to Kingston-Upon-Thames, England, having 16 local areas, which in terms of our model, are considered as existing facilities. The Cartesian coordinates of the locations of the existing facilities were given by [1]. The Cartesian coordinates of the starting and end points of the barrier routes are given in Table 5. In the Cartesian coordinates, Easting and Northing coordinates represent the horizontal and vertical-measured distances, respectively. Here, we extended the case study on the basis of our proposed problem. That is, in addition to an overground rail line between Surbiton and Beverley, we considered another overground rail line between Tolworth & Hook Rise and St. James (see Figure 12a) with a train having negligible
width passing through each rail line. The movements of the trains across the roads, which block the traffic in the roads, are considered as two probabilistic line barriers. These trains are illustrated as the bold lines in Figure 12a. Also, the routes of movement of these trains across the roads are depicted as the dashed lines \( R_1 \) and \( R_2 \). It is assumed that the trains (line barriers) move with an almost constant speed.
and their starting points are uniformly distributed over the length of the routes. Here, the objective is to locate a police department as a new facility to respond to reports received on criminal activities in Kingston in the presence of two trains moving through predetermined rail lines, so that a minimal distance, in terms of weighted summation of the weighted expected rectilinear barrier distances to the existing facilities, is achieved. Other assumptions are the same as stated earlier.

**Table 5.** The Cartesian coordinates of the barrier routes.

| Cartesian coordinates | Starting points of the barrier routes | End points of the barrier routes |
|-----------------------|--------------------------------------|----------------------------------|
|                       | $R_1$      | $R_2$      | $R_1$      | $R_2$      |
| Easting               | 517500     | 517500     | 521700     | 517700     |
| Northing              | 167000     | 166300     | 169000     | 168300     |

To have a better understanding of the behavior of the proposed model and to show
the impacts of two probabilistic trains passing through the rail lines on the objective function value, two different scenarios are considered: (1) solving a location problem by ignoring the presence of the train lines (Figure 12b), and (2) solving the original problem (Figure 12a). In the first scenario, the optimal solution can easily be determined using the technique of [32]. In Figure 12, the optimal location of a police department as a new facility along with the objective function values for both scenarios are illustrated. The results show that in both scenarios, the models found the same location in Berrylands. Although the location of the police department in both scenarios are the same (this may not be the case, in general), the summation of the weighted expected barrier distances when the model considers the train lines is almost 1.35% more than the case that the model ignores the train lines, as being expected. Also, to show the impact of number of probabilistic line barriers and the coordinates of their routes, we considered two other scenarios in Figure 12a: (1) solving a location problem by considering a train line in route $R_1$, and, (2) solving a location problem by considering a train line in route $R_2$. In these scenarios, we used the technique proposed by [9] to find the optimal solutions. The optimal location of the police department and the objective function values for both scenarios are shown in Figure 12a. The results show that by changing the movement route of the train from $R_1$ to $R_2$, the minimum weighted expected barrier distance decreases almost 0.6%, which is due to changing the visibility conditions. Also, by comparing the obtained results in these scenarios and the one in the original problem, it can be seen that when the model considers both train lines, the minimum weighted expected barrier distance increases almost 0.5% and 1.06% as compared to the case that the model considers a train line in routes $R_1$ and $R_2$, respectively.
Next, impact of the lengths of the trains on the decisions made for the proposed problem is also investigated. For this, we considered trains having lengths 2, 6, 8 and 10. These numbers are chosen based on the scale of the map. In Figure 13, the obtained results for the scenarios are illustrated. For the trains having lengths 2 and 6, the optimal locations of the police department are similar to what was found in the original problem and for the trains with lengths 8 and 10, the model found two different locations in Berrylands. It can be seen that when the length of both trains is 2, the minimum weighted expected barrier distances is almost 1% less than the objective value of the original problem. This is due to the fact that when the length of the train becomes smaller, the regions will be more visible and thus, the value of the weighted expected barrier distance decreases. Also, when the length of both trains is 6, 8 or 10, the minimum weighted expected barrier distance increases almost 1.8%, 4.26% and 7.5%, respectively. This is due to the more difficult visibility conditions of the regions, which turn to increase the value of the weighted expected barrier distance.

![Figure 13. Impact of lengths of the trains.](image)

7. **Conclusion.** We presented a single facility location problem on the plane in the presence of two horizontal line barriers being distributed uniformly on their corresponding routes. After introducing different visibility conditions, we formulated the proposed problem as a mixed-integer nonlinear programming model. To solve the problem, an effective extended Hooke and Jeeves algorithm (HJA) was presented. Also, we introduced efficient upper and lower bounds for the proposed problem using the forbidden location problem. Different sample problems and various graphic curves were provided to evaluate the performance of the proposed algorithm. The results showed that the proposed algorithm produced solutions having better qualities using less computing time. Also, an empirical study in Kingston-Upon-Thames, England, was conducted to show the performance of the proposed
model in real world conditions. The behavior of the proposed model and its sensitivity relative to the number and lengths of the line barriers were also analyzed. The results showed that the presence of two probabilistic line barriers on the plane as opposed to considering a single probabilistic line barrier or ignoring the presence of line barrier, has a significant impact on the summation of the weighted expected barrier distance, a fact which cannot be ignored by decision makers in their location decisions. Also, by increasing the length of the barriers, the visibility conditions become more complicated and consequently, the value of the total weighted expected rectilinear barrier distance increases.

Modeling the proposed problem when more than two line barriers are moving on the plane may be considered in a future work. In this case, visibility between two points will depend on the position of each line barrier on the plane. So, an efficient algorithm to identify the visibility and invisibility conditions between two points is the main concern. In our study here, the distribution functions for both line barriers were assumed to be the same. Different distribution functions for the line barriers may be considered in a future work. In this situation, calculation of the expected barrier distance needs to be revised. Investigation of other objective functions such as minimax can be considered as a further development of the proposed model. Also, consideration of other distance functions such as Euclidean or block norm metrics may turn to be useful. A comparative study of our proposed method with some of the most representative computational intelligence algorithms like monarch butterfly optimization (MBO) ([47], [49], [14], [15] and [12]), earthworm optimization algorithm (EWA) ([46]), elephant herding optimization (EHO) ([45], [48] and [30]), moth search (MS) algorithm ([44] and [13]), slime mould algorithm (SMA) ([31]), and Harris hawks optimization (HHO) ([18]) may turn to be useful in identifying preferred solution approaches for solving the proposed problem in various situations.

Acknowledgments. The first author acknowledges University of Garmsar and the second author thanks Sharif University of Technology for supporting this work.

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Received October 2020; 1st revision April 2021; 2nd revision April 2021; early access August 2021.

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