Time-Inconsistent Mean-Utility Portfolio Selection with Moving Target

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Abstract

In this paper, we solve the time inconsistent portfolio selection problem by using different utility functions with a moving target as our constraint. We solve this problem by finding an equilibrium control under the definition given as our optimal control. We derive a sufficient equilibrium condition for $C^2$ utility functions and use power functions of order two, three and four in our problem, and find the respective conditions for obtaining an equilibrium for our different problems. In the last part of the paper, we also consider use another definition of equilibrium to solve our problem when the utility function we use in our problem is $x^{-}$ and also find the conditions for obtaining an equilibrium.

Key words: time inconsistency, stochastic control, equilibrium control, forward-backward stochastic differential equation, utility function, portfolio selection
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1 Introduction

Stochastic control is now a mature and well established subject of study. Here we quote as an introduction a summary of the studies in time inconsistent control problems from Hu, Jin and Zhou[1] as follows. In the study of stochastic control, though not explicitly stated at most of the times, a standard assumption is time consistency, which is a fundamental property of conditional expectation with respect to a progressive filtration. As a result of that, an optimal control viewed from today will remain optimal viewed from tomorrow. Time consistency provides the theoretical foundation of the dynamic programming approach including the resulting HJB equation, which is in turn a pillar of the modern stochastic control theory. However, there are overwhelmingly more time inconsistent problems than their time consistent counterparts. Hyperbolic discounting Ainslie[7] and continuous-time mean–variance portfolio selection model Basak Chabakauri[8], Zhou and Li[9] provide two well-known examples of time inconsistency. Probability distortion, as in behavioral finance models Jin and Zhou[10], is yet another distinctive source of time inconsistency. Motivated by practical applications especially in mathematical finance, time inconsistent control problems have recently attracted considerable research interest and efforts attempting to seek equilibrium controls instead of optimal controls. At a conceptual level, the idea is that a decision the controller makes at every instant of time is considered as a game against all the decisions the future incarnations of the controller are going to make. An “equilibrium” control is therefore the one that any deviation from it at any time instant will be worse off. Taking this game perspective, Ekeland and Lazrak[11] approach the deterministic time inconsistent optimal control, and Bjork and Murgoci[3] extends the idea to the stochastic setting, derive an albeit very complicated HJB equation, and apply the theory to a dynamic Markowitz problem. Yong[12] investigate a time inconsistent deterministic linear quadratic control problem and derive equilibrium controls via some integral equations. However, the study of time inconsistent control is still in its infancy in general.

In this paper, we solve the time inconsistent portfolio selection problem by using different utility functions with a moving target as our constraint. We solve this problem by finding an equilibrium control under the given definition as our optimal control. Firstly, we derive a sufficient equilibrium condition for C^2 utility functions. Then we use power functions of order two, three and four in our problems and find the respective conditions for obtaining an equilibrium for our different problems. In the last part of the paper, we consider use another definition of equilibrium to solve our problem when the utility function that we use in our problem is x^− and we also find the conditions for obtaining an equilibrium for this problem.

The structure of this paper is as follows. In section 2, we set the problems that we want to study and give the definition of equilibrium when we use our power utility functions in our problem. We also prove a sufficient equilibrium condition for C^2 utility functions in this section. In section 3, we use h (x) = X^2 as the utility functions in our problem and find the conditions for obtaining an equilibrium for our problem by setting a deterministic process taking the position as a Lagrangian multiplier. In sections 4 and 5 we use h (x) = −X^3 and X^4 in our problem respectively, and we directly transform our problem into a simpler form and find the conditions for obtaining an equilibrium for our problem. In section 6 we use h (x) = x^− in our problem. we give a different definition of equilibrium that we work on in this section, and as usual we find the conditions for obtaining an equilibrium. Finally we make some remarks in section 7.
2 Problems setting

2.1 The notations

The notations used in this paper are listed as follows

\[
\begin{align*}
L_\infty (t, T; \mathbb{R}^d) : & \quad \text{the set of essentially bounded } \{\mathcal{F}_s\}_{s \in [t, T]}\text{-adapted processes.} \\
L_\mathbb{F}^2 (t, T; \mathbb{R}^d) : & \quad \text{the set of } \{\mathcal{F}_s\}_{s \in [t, T]}\text{-adapted processes} \\
L_G (t, T; \mathbb{R}^d) : & \quad f = \{f_s : t \leq s \leq T\} \text{ with } \mathbb{E} \left[ f_t^T |f_s|^2 ds \right] < \infty \\
L_\mathbb{P}^p (\Omega; C (t, T; \mathbb{R}^d)) : & \quad \text{the set of continuous } \{\mathcal{F}_s\}_{s \in [t, T]}\text{-adapted processes} \\
(W_t)_{t \in [0, T]} : & \quad (W^1_t, \cdots, W^d_t)_{t \in [0, T]} \text{ a d-dimensional Brownian motion on } (\Omega, \mathcal{F}, \mathbb{P}) \\
\mathbb{E} [\cdot] : & \quad \mathbb{E} [\cdot | \mathcal{F}_t]
\end{align*}
\]

For any \( t \in [0, T) \), we consider a pair of portfolio and wealth process \((\pi, X)\) satisfying the following equation

\[
\begin{align*}
\begin{cases}
dX_s = \left[ r_s X_s + \pi_s^T \left( \mu_s^x - r_s \mathbf{1} \right) \right] ds + \pi_s^T \sigma_s dW_s & s \in [t, T] \\
X_t = x_t
\end{cases}
\end{align*}
\]

where \( \mathbf{1} \) is a \( d \)-dimensional vector of ones, \( W \) is a \( d \)-dimensional standard Brownian motion, \( r \) is the risk-free rate process, \( \mu \) is the drift rate vector process of risky assets and \( \sigma \) is the volatility process of risky assets which is a \( d \times d \) matrix and is assumed to be invertible. Throughout this paper we assume that \( r \in L_\mathbb{F}^\infty (0, T; \mathbb{R}), \mu^x \in L_\mathbb{F}^\infty (0, T; \mathbb{R}^d), \sigma \in L_\mathbb{F}^\infty (0, T; \mathbb{R}^{d \times d}) \) and \( \sigma^{-1} \in L_\mathbb{F}^\infty (0, T; \mathbb{R}^{d \times d}) \), which means they are all bounded, in addition, we also assume that \( r \) and \( \sigma \) are always deterministic. Actually except in the case when we use \( \frac{r^2}{2} \) as our utility function, we would also assume \( \mu^x \) to be deterministic. Then the above dynamics of the wealth process can be written as

\[
\begin{align*}
\begin{cases}
dX_s = \left[ r_s X_s + \pi_s^T \sigma_s \theta_s \right] ds + \pi_s^T \sigma_s dW_s & s \in [t, T] \\
X_t = x_t
\end{cases}
\end{align*}
\]

where \( \theta = \sigma^{-1} (\mu^x - r \mathbf{1}) \) is the market price of risk and it is clear that \( \theta \in L_\mathbb{F}^\infty (0, T; \mathbb{R}^d) \) which means \( \theta \) is also bounded.

2.2 Motivation of our problems

A standard static mean variance portfolio selection problem with a fixed mean target \( l \) is

\[
\begin{align*}
\min_\pi \quad & \frac{1}{2} \text{Var}(X_T) \\
\text{s.t.} \quad & \begin{cases}
dX_s = \left( r_s X_s + \pi_s^T \sigma_s \theta_s \right) ds + \pi_s^T \sigma_s dW_s \\
X_0 = x_0 \\
\mathbb{E}[X_T] = l \\
\pi \in U_{\pi}^d = \{ \pi | \pi \in L_\mathbb{F}^2 (0, T; \mathbb{R}^d) \}
\end{cases}
\end{align*}
\]
In this paper we want to extend the above problem in the following two ways. Firstly, we want to have a constraint on our control at any time $t \in [0, T]$ which means we want to solve a family of the following problems for any $t \in [0, T]$

\[
\min_{\pi} \mathbb{E}_t \left[ h \left( X_T - X_t e^{\int_t^T \mu_s ds} \right) \right]
\]

\[
\text{s.t. } \begin{cases} 
X_t = x_t \\
\pi \in \mathcal{U}_{ad}^t = \{ \pi \mid \pi \in L^2_T (0, T ; \mathbb{R}^d) \} 
\end{cases}
\] (2.9)

According to Bjork and Murgoci\[3\], if the terminal evaluation function $h(x)$ in our above kind family of problems depends on $X_t = x_t$, then this $X_t$ will cause time inconsistency if it can not be factorized outside the given utility function $h(x)$. Here our utility function $h(x)$ depends on the term $X_t e^{\int_t^T \mu_s ds}$ and thus our family of problems (2.9) is time inconsistent in most cases.
When we use the utility function \( h(x) = e^{-\alpha x} \) for some \( \alpha > 0 \), the objective of our family of problems (2.9) is

\[
J(t, X_t; \pi) = \mathbb{E}_t \left[ h \left( X_T - X_t e^{\int_t^T \mu_s ds} \right) \right]
\]

\[
= \mathbb{E}_t \left[ e^{-\alpha (X_T - X_t e^{\int_t^T \mu_s ds})} \right]
\]

\[
= e^{\alpha X_t} e^{\int_t^T \mu_s ds} \mathbb{E}_t \left[ e^{-\alpha X_T} \right]
\]

which implies that our family of problems is equivalent to a standard control problem with objective

\[
\hat{J}(t, X_t; \pi) = \mathbb{E}_t \left[ e^{-\alpha X_T} \right]
\]

which becomes a time consistent problem and can be solved by dynamic programming when \( X \) is a Markov process. But for power and \( x^- \) utility functions time inconsistency would be caused and thus this paper focuses on the following utility functions

\[
h(x) = x^2, -x^3, x^4, x^-$

by using which our family of problems (2.9) becomes a time inconsistent problem.

### 2.3 Definition of equilibrium for our power utility functions

In terms of time inconsistency, the notion “optimality” needs to be defined in an appropriate way. Here we adopt the concept of the equilibrium control which is optimal only for spike variation in an infinitesimal way for any \( t \in [0, T) \).

Given a control \( u^* \), for any \( t \in [0, T), \varepsilon > 0 \) and \( v \in L^2_{\mathcal{F}_t} (\Omega; \mathbb{R}^d) \), define

\[ u_s^{t, \varepsilon, v} = u_s^* + v 1_{s \in [t, t + \varepsilon)}, \quad s \in [t, T] \]  

(2.10)

**Definition 2.1.** Let \( u^* \in U_{ad} \) be a given control with \( U_{ad} \) being the set of admissible controls. Let \( X^* \) be the state process corresponding to \( u^* \). The control \( u^* \) is called an equilibrium if

\[
\lim_{\varepsilon \downarrow 0} \frac{J(t, X_t^*; u_s^{t, \varepsilon, v}) - J(t, X_t^*; u^*)}{\varepsilon} \geq 0
\]

(2.11)

for any \( t \in [0, T) \) and \( v \in L^2_{\mathcal{F}_t} (\Omega; \mathbb{R}^d) \) s.t. \( u_s^{t, \varepsilon, v} \in U_{ad} \), where \( u_s^{t, \varepsilon, v} \) is defined by (2.10).

Notice that here we use the definition of an equilibrium control defined in Hu, Jin and Zhou\[1\], which is defined in the class of open-loop controls and is different form the one in Bjork and Murgoci\[3\] where the definition is based on the feedback controls. In this definition, the perturbation of the control in \([t, t + \varepsilon]) \) will not change the control process in \([t + \varepsilon, T] \). We use this definition for our power utility functions \( h(x) = x^2, -x^3, x^4, x^- \) only. For \( h(x) = x^- \) we will use another definition of equilibrium which will be defined in that section.

### 2.4 A sufficient equilibrium condition for \( C^2 \) utility functions

In the following 3 sections when we use our power utility functions in our problem, we will have our problem transformed into a family of problems of the following form for any \( t \in [0, T) \)

\[
\min_u \mathbb{E}_t [h (Y_T - Y_t)]
\]

\[
s.t. \quad \begin{cases} 
  dY_s = (\hat{r}_s Y_s + u_s^T \theta_s) \, ds + u_s^T dW_s \\
  Y_t = y_t \\
  u \in U_{ad} = \{ u | u \in L^2_{\mathcal{F}_t} (0, T; \mathbb{R}^d) \}
\end{cases}
\]

(2.12)
where \( \theta \) is bounded, \( \hat{r} \) is a deterministic process, and \( h(x) \) is the given \( C^2 \) utility function. Here we have \( J(t, Y_t; u) = E_t [h(Y_T - Y_t)] \). For this family of problems, the definition (2.1) for an equilibrium control is translated into the following proposition.

**Proposition 2.2.** Let \( u^* \in L^2_T (0, T; \mathbb{R}^d) \) be a given control and \( Y^* \) be the state process corresponding to \( u^* \). The control \( u^* \) is an equilibrium for the family of problems (2.12) if

\[
\lim_{\epsilon \to 0} \frac{J(t, Y^*_t; u^{t, \epsilon, v}) - J(t, Y^*_t; u^*)}{\epsilon} \geq 0
\]

for any \( t \in [0, T) \) and \( v \in L^2_T (\Omega; \mathbb{R}^d) \), where \( u^{t, \epsilon, v} \) is defined by (2.10).

Here we give a sufficient condition for equilibrium controls when \( Y \) is the process defined in (2.12) by using the second order Taylor expansion at any \( t \in [0, T) \), which is inspired by the idea used in Hu, Jin and Zhou [1]. Let \( u^* \) be a fixed control and \( Y^* \) be the corresponding state process. For any \( t \in [0, T) \), we define in the time interval \([t, T]\) the processes \( p^t \in L^2_T (t, T; \mathbb{R}) \), \( q^t \in L^2_T (t, T; \mathbb{R}^d) \), \( P^t \in L^2_T (t, T; \mathbb{R}) \) and \( Q^t \in L^2_T (t, T; \mathbb{R}^d) \) which satisfy the following system of BSDEs

\[
\begin{cases}
    dp^t_s = -\hat{r}^s p^s ds + (q^s)^T dW_s, & s \in [t, T] \\
    P^t_s = \frac{d(h(Y^*_t - Y^*_s))}{dx} \\
    dP^t_s = -2\hat{r}^s P^t ds + (Q^t)^T dW_s, & s \in [t, T] \\
    P^t_T = \frac{d^2 h(Y^*_t - Y^*_s)}{dx^2}
\end{cases}
\]

**Proposition 2.3.** For any \( \epsilon > 0, t \in [0, T) \), \( v \in L^2_T (\Omega; \mathbb{R}^d) \) and \( u^{t, \epsilon, v} \) defined by (2.10). We have that

\[
J(t, Y^*_t; u^{t, \epsilon, v}) - J(t, Y^*_t; u^*) = E_t \int_t^{t+\epsilon} v^T \Lambda^t_s + \frac{1}{2} P^t_s |v|^2 ds + O(\epsilon)
\]

where \( \Lambda^t_s = p^t_s \theta_s + q^t_s \) for any \( s \in [t, T] \).

**Proof.** Here we use the standard perturbation approach in Yong and Zhou [2]. Let \( Y^{t, \epsilon, v} \) be the state process corresponding to \( u^{t, \epsilon, v} \), we have that

\[
Y^{t, \epsilon, v}_s = Y^*_s + I^{t, \epsilon, v}_s + Z^{t, \epsilon, v}_s, s \in [t, T]
\]

where \( I \equiv I^{t, \epsilon, v} \) and \( Z \equiv Z^{t, \epsilon, v} \) satisfy that

\[
\begin{cases}
    dI_s = \hat{r}^s I_s ds + v^T 1_{s \in [t, t+\epsilon)} dW_s \\
    I_t = 0
\end{cases}
\]

\[
\begin{cases}
    dZ_s = [\hat{r}^s Z_s + v^T \theta_s 1_{s \in [t, t+\epsilon)}] ds \\
    Z_t = 0
\end{cases}
\]

and we have that

\[
E_t \left[ \sup_{s \in [t, T]} |I_s|^2 \right] = O(\epsilon), \quad E_t \left[ \sup_{s \in [t, T]} |Z_s|^2 \right] = O(\epsilon^2)
\]
which implies that
\[
\mathbb{E}_t \left[ \sup_{s \in [t,T]} |I_s + Z_s|^2 \right] \leq \mathbb{E}_t \left[ \sup_{s \in [t,T]} (|I_s| + |Z_s|)^2 \right] \leq 2 \mathbb{E}_t \left[ \sup_{s \in [t,T]} |I_s|^2 + |Z_s|^2 \right] \leq 2 \left( \mathbb{E}_t \left[ \sup_{s \in [t,T]} |I_s|^2 \right] + \mathbb{E}_t \left[ \sup_{s \in [t,T]} |Z_s|^2 \right] \right) = O(\varepsilon)
\]
(2.17)

then we have that
\[
J(t,Y^*_t;u^{t,\varepsilon,v}) - J(t,Y^*_t;u^*) = \mathbb{E}_t \left[ h(Y^t-\varepsilon,Y^*_t) - h(Y^t_0, Y^*_0) \right] = \mathbb{E}_t \left[ \frac{dh}{dx}(Y^t_0 - Y^*_0) \right] (Y^t_0 - Y^*_0) + \frac{1}{2} \mathbb{E}_t \left[ \frac{d^2h}{dx^2}(Y^t_0 - Y^*_0)^2 \right] (Y^t_0 - Y^*_0)^2 + o\left( \mathbb{E}_t \left[ (Y^t_0 - Y^*_0)^2 \right] \right)
\]
(2.18)

because by (2.17) we have
\[
\mathbb{E}_t \left[ (I_T + Z_T)^2 \right] \leq \mathbb{E}_t \left[ \sup_{s \in [t,T]} |I_s + Z_s|^2 \right] \leq O(\varepsilon)
\]

Since we have that
\[
\begin{cases}
\frac{d}{ds} (I_s + Z_s) &= \left[ \tilde{r}_s (I_s + Z_s) + v^T \theta_s 1_{s \in [t,t+\varepsilon]} \right] ds + v^T 1_{s \in [t,t+\varepsilon]} dW_s \\
I_t + Z_t &= 0
\end{cases}
\]

we could calculate that
\[
\begin{align*}
\frac{d}{ds} \left[ p_s^t (I_s + Z_s) \right] &= p_s^t \frac{d}{ds} (I_s + Z_s) + (I_s + Z_s) \frac{dp_s^t}{ds} + d \left( p^t, I + Z \right) \\
&= p_s^t \left[ \tilde{r}_s (I_s + Z_s) + v^T \theta_s 1_{s \in [t,t+\varepsilon]} \right] ds - \tilde{r}_s p_s^t (I_s + Z_s) ds + v^T q_s^t 1_{s \in [t,t+\varepsilon]} ds \\
&\quad + \left[ p_s^t v^T 1_{s \in [t,t+\varepsilon]} + (I_s + Z_s) \left( q_s^t \right)^T \right] dW_s \\
&= \left[ p_s^t v^T \theta_s 1_{s \in [t,t+\varepsilon]} + v^T q_s^t 1_{s \in [t,t+\varepsilon]} \right] ds + [\cdots] dW_s
\end{align*}
\]

thus
\[
\mathbb{E}_t \left[ p_s^t (I_T + Z_T) \right] = \mathbb{E}_t \int_t^T p_s^t v^T \theta_s 1_{s \in [t,t+\varepsilon]} + v^T q_s^t 1_{s \in [t,t+\varepsilon]} ds \\
&= \mathbb{E}_t \int_t^{t+\varepsilon} v^T \left[ p_s^t \theta_s + q_s^t \right] ds \\
&= \mathbb{E}_t \int_t^{t+\varepsilon} v^T \Lambda_s ds
\]
(2.19)
and that
\[
d\left[P_s^t (I_s + Z_s)^2\right] = P_s^t d (I_s + Z_s)^2 + (I_s + Z_s)^2 dP_s^t + d\left<P^t, (I + Z)^2\right>_s,
\]
\[
= P_s^t \left[2 (I_s + Z_s) d (I_s + Z_s) + d \langle I + Z \rangle_s\right] + (I_s + Z_s)^2 \left[-2 \hat{r}_s P_s^t ds + (Q_s^t)^T dW_s\right],
\]
\[
+ 2 (I_s + Z_s) d \langle P^t, I + Z \rangle_s,
\]
\[
= 2 P_s^t (I_s + Z_s) \left[\hat{r}_s (I_s + Z_s) + v^T \theta_s 1_{s \in [t,t+\varepsilon]}\right] ds + P_s^t v^T v 1_{s \in [t,t+\varepsilon]} ds
\]
\[
- 2 \hat{r}_s P_s^t (I_s + Z_s)^2 ds + 2 (I_s + Z_s) v^T Q_s^t 1_{s \in [t,t+\varepsilon]} ds + \cdots dW_s,
\]
\[
= 2 (I_s + Z_s) v^T [P_s^t \theta_s + Q_s^t] 1_{s \in [t,t+\varepsilon]} ds + P_s^t v^T v 1_{s \in [t,t+\varepsilon]} ds + \cdots dW_s.
\]
thus
\[
E_t \left[P^t_T (I_T + Z_T)^2\right] = E_t \int_t^T P^t v^T v 1_{s \in [t,t+\varepsilon]} ds + E_t \int_t^T 2 (I_s + Z_s) v^T [P_s^t \theta_s + Q_s^t] 1_{s \in [t,t+\varepsilon]} ds
\]
\[
= E_t \int_t^{t+\varepsilon} P^t v^T v ds + E_t \int_t^{t+\varepsilon} 2 (I_s + Z_s) v^T [P_s^t \theta_s + Q_s^t] ds
\]
\[
\overset{2.21}{=} E_t \int_t^{t+\varepsilon} P^t |v|^2 ds + o(\varepsilon)
\] (2.20)
because we have
\[
E_t \int_t^{t+\varepsilon} 2 (I_s + Z_s) v^T [P_s^t \theta_s + Q_s^t] ds \leq 2E_t \int_t^{t+\varepsilon} \left(\sup_{u \in [t,T]} |I_u + Z_u|\right) |v^T (P_s^t \theta_s + Q_s^t)| ds
\]
\[
= 2E_t \left[\left(\sup_{u \in [t,T]} |I_u + Z_u|\right) \int_t^{t+\varepsilon} |v^T (P_s^t \theta_s + Q_s^t)| ds\right]
\]
\[
\leq 2 \sqrt{O(\varepsilon)} \sqrt{E_t \left[\left(\int_t^{t+\varepsilon} |v|^2 ds\right)^2\right]} \sqrt{E_t \left[\left(\int_t^{t+\varepsilon} |P_s^t \theta_s + Q_s^t|^2 ds\right)^2\right]}, \text{by Cauchy-Schwarz}
\]
\[
\leq \sqrt{O(\varepsilon)} \sqrt{\varepsilon E_t \left[|v|^2 \int_t^{t+\varepsilon} |P_s^t \theta_s + Q_s^t|^2 ds\right]} \overset{\text{by Cauchy-Schwarz}}{=} O(\varepsilon) \sqrt{\varepsilon E_t \left[|v|^2 \int_t^{t+\varepsilon} |P_s^t \theta_s + Q_s^t|^2 ds\right]}
\]
\[
= O(\varepsilon) \sqrt{\varepsilon E_t \left[|v|^2 \int_t^{t+\varepsilon} |P_s^t \theta_s + Q_s^t|^2 ds\right]}, \text{by Cauchy-Schwarz}
\]
\[
= O(\varepsilon) \left[|v|^2 \int_t^{t+\varepsilon} E_t \left[|P_s^t \theta_s + Q_s^t|^2\right] ds\right]
\]
\[
= o(\varepsilon)
\] (2.21)
as we have
\[
\lim_{\varepsilon \to 0} \sqrt{\varepsilon} \int_t^{t+\varepsilon} E_t \left[ \left| P_s^t \theta_s + Q_s^t \right|^2 \right] ds = 0
\]
which is implied by the above assumptions that \( v \in L^2_T (\Omega; \mathbb{R}^d) \), \( P^t \in L^2_T (t; \mathbb{R}) \), \( Q^t \in L^2_T (t; \mathbb{R}^d) \) and \( \theta \) is bounded. Then plug \( 2.19 \) and \( 2.20 \) into \( 2.18 \) we get \( 2.16 \).

**Theorem 2.4.** If the following system of equations
\[
\begin{align*}
dY_s &= (\hat{r}_s Y + \theta^T \upsilon_s) \, ds + u_s^T \, dW_s, \quad s \in [0, T] \\
Y_0 &= y_0 \\
dp_s &= -\hat{r}_s p_s^T ds + (q_s^T) \, dW_s, \quad s \in [t, T] \\
p^T_T &= \frac{d^2 h(Y^*_T - Y^*_t)}{dx^2}
\end{align*}
\]
admits a solution \((u^*, Y^*, p^*, q^*)\) for any \( t \in [0, T) \), s.t.
\[
\begin{align*}
u^* &\in L^2_T (0, T; \mathbb{R}^d) \\
E_t \int_T^T |\Lambda_s^t| \, ds < \infty \quad \text{and} \quad \lim_{s \to t} E_t[\Lambda_s^t] = 0 \\E_t \left[ \frac{d^2 h(Y^*_s - Y^*_t)}{dx^2} \right] &\geq 0
\end{align*}
\]
then \( u^* \) is an equilibrium control.

**Proof.** Suppose there exits solution \((u^*, Y^*, p^*, q^*)\) which satisfies the above condition \( 2.22 \). Then it is given by \( 2.16 \) that
\[
\begin{align*}
dP_s^t &= -2\hat{r}_s p_s^t ds + (Q_s^t) \, dW_s, \quad s \in [t, T] \\
P^T_T &= \frac{d^2 h(Y^*_T - Y^*_t)}{dx^2}
\end{align*}
\]
which means
\[
\begin{align*}
d \left( e_s^{T-t} 2\hat{r}_u du \, P_s^t \right) &= e_s^{T-t} 2\hat{r}_u du \, (Q_s^t) \, dW_s, \quad s \in [t, T] \\
e_s^{T-t} 2\hat{r}_u du \, P^T_T &= e_s^{T-t} 2\hat{r}_u du \, \frac{d^2 h(Y^*_t - Y^*_t)}{dx^2}
\end{align*}
\]
thus
\[
e_s^{T-t} 2\hat{r}_u du \, P^t_s = E_s \left[ e_s^{T-t} 2\hat{r}_u du \, \frac{d^2 h(Y^*_T - Y^*_t)}{dx^2} \right]
\]
so we get
\[
E_t \left[ P^t_s \right] = E_t \left[ E_s \left[ e_s^{T-t} 2\hat{r}_u du \, \frac{d^2 h(Y^*_T - Y^*_t)}{dx^2} \right] \right]
\]
\[
= E_t \left[ e_s^{T-t} 2\hat{r}_u du \, \frac{d^2 h(Y^*_T - Y^*_t)}{dx^2} \right]
\]
\[
= e_s^{T-t} 2\hat{r}_u du E_t \left[ \frac{d^2 h(Y^*_T - Y^*_t)}{dx^2} \right]
\]
\[
\geq 0
\]
(2.24)
as \( \hat{r} \) is a deterministic and \( \mathbb{E}_t \left[ \frac{d^2 h(Y_t^\varepsilon - Y^\varepsilon)}{d x^2} \right] \geq 0 \) by our assumption. Then for any \( t \in [0, T) \) and \( v \in L^2_{T^t} (\Omega; \mathbb{R}^d) \) we have by proposition 2.3 that

\[
\lim_{\varepsilon \downarrow 0} J (t, Y^\ast_t; u^T, v) - J (t, Y^\ast_t; u^\ast) = \lim_{\varepsilon \downarrow 0} \mathbb{E}_t \left[ \int_t^{t+\varepsilon} v^T \Lambda^t_s + \frac{1}{2} P^t_s \left| v \right|^2 \, ds + o (\varepsilon) \right] = \lim_{\varepsilon \downarrow 0} \mathbb{E}_t \left[ \int_t^{t+\varepsilon} v^T \Lambda^t_s \right] ds \geq v^T \left( \lim_{\varepsilon \downarrow 0} \inf_{s \in [t, t+\varepsilon]} \mathbb{E}_t \left[ \Lambda^t_s \right] \right) \]

\[
= v^T \left( \lim_{\varepsilon \downarrow 0} \mathbb{E}_t \left[ \Lambda^t_{t+\varepsilon} \right] \right) \]

\[
= 0
\]

as \( \lim \mathbb{E}_t [ \Lambda^t_{t+\varepsilon} ] = 0 \) by our assumption, which proves that \( u^\ast \) is an equilibrium control by proposition 2.2. \( \Box \)
3 Utility function $h(x) = \frac{x^2}{2}$ for mean variance portfolio selection

As being given in the section 2, a standard static mean variance portfolio selection problem with a fixed mean target $l$ is

$$\min_{\pi} \quad Var(X_T)$$

$$s.t. \begin{cases} &dX_s = (r_sX_s + \pi_s^T \sigma_s^T \theta_s) \, ds + \pi_s^T \sigma_s \, dW_s, \\ &X_0 = x_0, \\ &\mathbb{E}[X_T] = l, \\ &\pi \in U_{ad} = \{\pi \mid \pi \in L^2_{F}(0, T; \mathbb{R}^d)\} \end{cases} \quad (3.1)$$

for the constraint $\mathbb{E}[X_T] = l$, we could introduce a Lagrangian multiplier $2\lambda$ and consider the following problem

$$\min_{\pi} \quad \mathbb{E} \left[ (X_T - \lambda)^2 \right]$$

$$s.t. \begin{cases} &dX_s = (r_sX_s + \pi_s^T \sigma_s^T \theta_s) \, ds + \pi_s^T \sigma_s \, dW_s, \\ &X_0 = x_0, \\ &\pi \in U_{ad} = \{\pi \mid \pi \in L^2_{F}(0, T; \mathbb{R}^d)\} \end{cases} \quad (3.2)$$

and if there exits $\lambda^*$ s.t. the optimal solution $(u^*, X^*)$ to the above problem (3.2) satisfies $\mathbb{E}[X^*_T] = l$, then $(u^*, X^*)$ is also optimal for (3.1).

Here, as having been described in section 2, we consider a moving target $\mathbb{E}_t[X_T] = X_te^{\int_t^T \mu \, ds}$ for any $t \in [0, T)$, where $\mu$ is our required return process which is assumed to be deterministic and bounded. Then the mean variance portfolio selection with moving target that we want to solve is a family of following problems

$$\min_{\pi} \quad \frac{1}{2} Var_t(X_T)$$

$$s.t. \begin{cases} &dX_s = (r_sX_s + \pi_s^T \sigma_s^T \theta_s) \, ds + \pi_s^T \sigma_s \, dW_s, \\ &X_t = x_t, \\ &\pi \in U_{ad} = \{\pi \mid \pi \in L^2_{F}(0, T; \mathbb{R}^d) \text{ and } \mathbb{E}_t[X_T] = X_te^{\int_t^T \mu \, ds}, \forall t \in [0, T)\} \end{cases} \quad (3.3)$$

which is the family of following problems

$$\min_{\pi} \quad \mathbb{E}_t \left[ \frac{(X_T - X_te^{\int_t^T \mu \, ds})^2}{2} \right]$$

$$s.t. \begin{cases} &dX_s = (r_sX_s + \pi_s^T \sigma_s^T \theta_s) \, ds + \pi_s^T \sigma_s \, dW_s, \\ &X_t = x_t, \\ &\pi \in U_{ad} = \{\pi \mid \pi \in L^2_{F}(0, T; \mathbb{R}^d) \text{ and } \mathbb{E}_t[X_T] = X_te^{\int_t^T \mu \, ds}, \forall t \in [0, T)\} \end{cases} \quad (3.4)$$

or we can write the family of problems as the following form.
min \pi \mathbb{E}_t \left[ h \left( X_T - X_t e^{\int_t^T \mu \cdot ds} \right) \right] \\

s.t. \begin{align*}
  dX_s &= (r_s X_s + \pi_s^T \sigma_s \theta_s) \, ds + \pi_s^T \sigma_s \, dW_s \\
  X_t &= x_t \\
  \pi &\in U_{ad}^{\pi} = \left\{ \pi \mid \pi \in L^2_x (0, T; \mathbb{R}^d) \right\}
\end{align*}

(3.5)

which is one of our problems being set in section 2 with \( h(x) = \frac{x^2}{2} \)

### 3.1 Introducing a Lagrangian multiplier for our problem

Inspired by the static Lagrangian multiplier method we introduce a deterministic process \( \lambda \) with \( \int_0^T |\lambda_s| \, ds < \infty \) and consider a family of problems for any \( t \in [0, T] \) as follows

\[
\min_{\pi} \mathbb{E}_t \left[ \frac{(X_T - X_t e^{\int_t^T \lambda \cdot ds})^2}{2} \right] \\

\begin{align*}
  dX_s &= (r_s X_s + \pi_s^T \sigma_s \theta_s) \, ds + \pi_s^T \sigma_s \, dW_s \\
  X_t &= x_t \\
  \pi &\in U_{ad}^{\pi} = \left\{ \pi \mid \pi \in L^2_x (0, T; \mathbb{R}^d) \right\}
\end{align*}

(3.6)

**Theorem 3.1.** If there exists a deterministic process \( \lambda^* \) with \( \int_0^T |\lambda^*_s| \, ds < \infty \) s.t. an equilibrium solution \((\pi^*, X^*)\) to the above family of problems \((3.6)\) having \( \lambda^* \) as parameter satisfies the condition that \( \mathbb{E}_t [X_T^*] = X_t^* e^{\int_t^T \mu \cdot ds} \) for any \( t \in [0, T] \), then \((\pi^*, X^*)\) is also equilibrium for the family of problems \((3.4)\).

**Proof.** Suppose \((\pi^*, X^*)\) is an equilibrium solution to the family of problems \((3.6)\) having \( \lambda^* \) as parameter with \( \int_0^T |\lambda^*_s| \, ds < \infty \) s.t. \( \mathbb{E}_t [X_T^*] = X_t^* e^{\int_t^T \mu \cdot ds} \), which says \((\pi^*, X^*)\) is an equilibrium solution to the family of following problems for any \( t \in [0, T] \)

\[
\min_{\pi} \mathbb{E}_t \left[ \frac{(X_T - X_t e^{\int_t^T \lambda \cdot ds})^2}{2} \right] \\

\begin{align*}
  dX_s &= (r_s X_s + \pi_s^T \sigma_s \theta_s) \, ds + \pi_s^T \sigma_s \, dW_s \\
  X_t &= x_t \\
  \pi &\in U_{ad}^{\pi} = \left\{ \pi \mid \pi \in L^2_x (0, T; \mathbb{R}^d) \right\}
\end{align*}

(3.7)

Firstly, \( \mathbb{E}_t [X_T^*] = X_t^* e^{\int_t^T \mu \cdot ds} \) implies that \( \pi^* \in U_{ad}^{\pi} \) which can be used with definition \((3.4)\) to prove that \((\pi^*, X^*)\) is also equilibrium for the family of following problems for any \( t \in [0, T] \)

\[
\min_{\pi} \mathbb{E}_t \left[ \frac{(X_T - X_t e^{\int_t^T \lambda \cdot ds})^2}{2} \right] \\

\begin{align*}
  dX_s &= (r_s X_s + \pi_s^T \sigma_s \theta_s) \, ds + \pi_s^T \sigma_s \, dW_s \\
  X_t &= x_t \\
  \pi &\in U_{ad}^{\pi} = \left\{ \pi \mid \pi \in L^2_x (0, T; \mathbb{R}^d) \right\}
\end{align*}

(3.8)
Here we define $J(t, X_t; \pi)$ for all problems in the same way as before. Since the $J(t, X_t^*; \pi^*)$ term used in definition 3.3 for problems (3.8) above and the one used for problems (3.9) below are the same after calculation, we deduce that the family of problems is equivalent to

$$\min_{\pi} \frac{1}{2} \left\{ \mathbb{E}_t [X_T^2] - 2X_t e^{\int_t^T \lambda_s ds} \mathbb{E}_t [X_T] \right\} + \frac{1}{2} X_t^2 \left[ 2e^{\int_t^T \lambda_s ds} e^{\int_t^T \mu_s ds} - e^{2 \int_t^T \mu_s ds} \right]$$

s.t. \begin{align*}
& \frac{dX_t}{ds} = (r_s X_t + \pi_s^T \sigma_s \theta_s) ds + \pi_s^T \sigma_s dW_s \\
& \pi \in U_{ad}^t \left\{ \pi | \pi \in L^2_T (0, T; \mathbb{R}^d) \text{ and } \mathbb{E}_t [X_T] = X_t e^{\int_t^T \mu_s ds}, \forall t \in [0, T) \right\}
\end{align*}

(3.9)

which after calculation is

$$\min_{\pi} \frac{1}{2} \left\{ \mathbb{E}_t [X_T^2] - \left( X_t e^{\int_t^T \mu_s ds} \right)^2 \right\}$$

s.t. \begin{align*}
& \frac{dX_t}{ds} = (r_s X_t + \pi_s^T \sigma_s \theta_s) ds + \pi_s^T \sigma_s dW_s \\
& \pi \in U_{ad}^t \left\{ \pi | \pi \in L^2_T (0, T; \mathbb{R}^d) \text{ and } \mathbb{E}_t [X_T] = X_t e^{\int_t^T \mu_s ds}, \forall t \in [0, T) \right\}
\end{align*}

(3.10)

which again by using the constraint $\mathbb{E}_t [X_T] = X_t e^{\int_t^T \mu_s ds}$ can be written as

$$\min_{\pi} \mathbb{E}_t \left[ \left( X_T - X_t e^{\int_t^T \mu_s ds} \right)^2 \right]$$

s.t. \begin{align*}
& \frac{dX_t}{ds} = (r_s X_t + \pi_s^T \sigma_s \theta_s) ds + \pi_s^T \sigma_s dW_s \\
& \pi \in U_{ad}^t \left\{ \pi | \pi \in L^2_T (0, T; \mathbb{R}^d) \text{ and } \mathbb{E}_t [X_T] = X_t e^{\int_t^T \mu_s ds}, \forall t \in [0, T) \right\}
\end{align*}

(3.11)

thus the family of problems (3.9) is equivalent to the family of following problems

$$\min_{\pi} \mathbb{E}_t \left[ \frac{\left( X_T - X_t e^{\int_t^T \mu_s ds} \right)^2}{2} \right]$$

s.t. \begin{align*}
& \frac{dX_t}{ds} = (r_s X_t + \pi_s^T \sigma_s \theta_s) ds + \pi_s^T \sigma_s dW_s \\
& \pi \in U_{ad}^t \left\{ \pi | \pi \in L^2_T (0, T; \mathbb{R}^d) \text{ and } \mathbb{E}_t [X_T] = X_t e^{\int_t^T \mu_s ds}, \forall t \in [0, T) \right\}
\end{align*}

(3.12)

which is exactly the family of problems (3.4), thus we deduce that $(u^*, X^*)$ is also equilibrium for (3.4). \hfill \Box

### 3.2 Transformation of our problem

By the above theorem, we try to solve the family of problems (3.4) instead, that is

$$\min_{\pi} \mathbb{E}_t \left[ \frac{\left( X_T - X_t e^{\int_t^T \lambda_s ds} \right)^2}{2} \right]$$

s.t. \begin{align*}
& \frac{dX_t}{ds} = (r_s X_t + \pi_s^T \sigma_s \theta_s) ds + \pi_s^T \sigma_s dW_s \\
& \pi \in U_{ad}^t \left\{ \pi | \pi \in L^2_T (0, T; \mathbb{R}^d) \right\}
\end{align*}

(3.13)
by letting for any $s \in [0, T]$

$$Y_s = X_s e^{\int_t^T \lambda_s du}$$

the above family of problems is equivalent to

$$\min_u \mathbb{E}_t \left[ \frac{(Y_T - Y_t)^2}{2} \right]$$

s.t. \hspace{1em}

\begin{align*}
\begin{cases}
  dY_s = (\tilde{r}_s Y_s + u_s^T \theta_s) \, ds + u_s^T \, dW_s \\
  Y_t = y_t \\
  u \in U_{ad} = \{ u \mid u \in L^2_F(0, T; \mathbb{R}^d) \}
\end{cases}
\end{align*}

(3.15)

where $\tilde{r}_s = r_s - \lambda_s$, $u_s = e^{\int_t^T \lambda_u du} \sigma_s^T \pi_s$, $y_t = x_t e^{\int_t^T \lambda_u du}$ and $\pi \in L^2_F(0, T; \mathbb{R}^d)$ if and only if $u \in L^2_F(0, T; \mathbb{R}^d)$ which is implied by our previous assumption $\sigma$ is bounded and that $e^{\int_t^T \lambda_u du}$ is bounded due to $\int_0^T |\lambda_s| \, ds < \infty$.

The above family of problems can be written as

$$\min_u \mathbb{E}_t [h(Y_T - Y_t)]$$

s.t. \hspace{1em}

\begin{align*}
\begin{cases}
  dY_s = (\tilde{r}_s Y_s + u_s^T \theta_s) \, ds + u_s^T \, dW_s \\
  Y_t = y_t \\
  u \in U_{ad} = \{ u \mid u \in L^2_F(0, T; \mathbb{R}^d) \}
\end{cases}
\end{align*}

(3.16)

where $h(x) = \frac{x^2}{2}$, and this is exactly a family of problems of the form in (2.12), so we have the following system of BSDEs by (2.14) and (2.15)

\begin{align*}
\begin{cases}
  dp_s = -\tilde{r}_s p_s + (q_t^*)^T dW_s, \hspace{1em} s \in [t, T] \\
  p_T = \frac{dh(Y_T - Y_t^*)}{dx} = Y_T^* - Y_t^* \\
  dP_s = -2 \tilde{r}_s P_s + (Q_s^*)^T dW_s, \hspace{1em} s \in [t, T] \\
  P_T = \frac{d^2h(Y_T - Y_T^*)}{dx^2} = 1
\end{cases}
\end{align*}

(3.17)

(3.18)

**Proposition 3.2.** If (3.16) and (3.14) admit a solution $(u^*, Y^*, p^*, q^*)$ for any $t \in [0, T)$ s.t.

\begin{align*}
\begin{cases}
  u^* \in L^2_F(0, T; \mathbb{R}^d) \\
  \mathbb{E}_t \left[ \int_t^T |\Lambda_s| \, ds < \infty \right] \text{ and } \lim_{s \uparrow t} \mathbb{E}_t [\Lambda_s] = 0 \hspace{1em} \text{where } \Lambda_s = p_s^* \theta + q_s^*
\end{cases}
\end{align*}

(3.19)

then $u^*$ is an equilibrium control for the family of problems in (3.14).

**Proof.** Since one of the sufficient conditions in (2.23) $\mathbb{E}_t \left[ \frac{d^2h(Y_T - Y_T^*)}{dx^2} \right] = 1 \geq 0$ has been met, thus we have the required result by theorem (2.24). \hfill \square

### 3.3 Details of finding a potential equilibrium

By the assumptions we have made so far we have that $\theta$ is bounded and $\tilde{r}$ is deterministic with $\int_0^T |\tilde{r}_s| \, ds < \infty$. Firstly, we allow $\mu^x$ which is our drift rate vector process of risky assets to be random, i.e. $\theta$ could be random.

Then for any $t \in [0, T)$ we make the following Ansatz

$$p_s^* = M_s Y_s^* - \Gamma_s Y_t^*$$

(3.20)
where \((M, K), (\Gamma, \phi)\) are the solutions of the following BSDEs

\[
\begin{align*}
\begin{cases}
    dM_s &= -f_{M, K}(s, M_s, K_s) \, ds + K_s^T \, dW_s \\
    M_T &= 1
    
    d\Gamma_s &= -f_{\Gamma, \phi}(s, \Gamma_s, \phi_s) \, ds + \phi_s^T \, dW_s \\
    \Gamma_T &= 1
\end{cases}
\end{align*}
\] (3.21)

by Itô formula we get

\[
\begin{align*}
d(M_s Y_s^*) &= M_s dY_s^* + Y_s^* dM_s + d(Y^*, M)_s \\
&= M_s \left[ \left( \alpha^T \right) \theta_s + \left( u^*_s \right)^T \, ds + \left( u^*_s \right)^T \, dW_s \right] \\
&+ \left[ M_s \left( \alpha^T \right) \theta_s + K_s^T \, Y_s^* - f_{M, K}(s, M_s, K_s) Y_s^* + Y_s^* f_{\Gamma, \phi}(s, \Gamma_s, \phi_s) \right] \, ds \\
&+ \left[ M_s \left( u^*_s \right)^T + K_s^T \, Y_s^* \right] \, dW_s
\end{align*}
\] (3.23)

then

\[
\begin{align*}
dp_s &= dM_s Y_s^* - Y_s^* d\Gamma_s \\
&= \left[ M_s \left( \alpha^T \right) \theta_s + K_s^T \, Y_s^* - f_{M, K}(s, M_s, K_s) Y_s^* \right] \, ds \\
&+ \left[ M_s \left( u^*_s \right)^T + K_s^T \, Y_s^* \right] \, dW_s
\end{align*}
\] (3.24)

by comparing the \(dW\) term of (3.24) and (3.17), we get

\[
q^*_s = M_s u^*_s + K_s Y^*_s - Y_s^* \phi_s
\] (3.25)

Suppose \(\Lambda^t\) is continuous and bounded, then \(\lim\mathbb{E}_s \left[ \Lambda^t_s \right] = 0, \forall t \in [0, T)\) is ensured by \(\Lambda^t_s = 0, \forall t \in [0, T)\). To get a possible linear feedback \(u^*\), we try by setting

\[
0 = \Lambda^s_s = p^s \theta_s + q^s_s, \quad s \in [0, T]
\]

that is

\[
0 = (M_s - \Gamma_s) Y^*_s \theta_s + M_s u^*_s + K_s Y^*_s - Y^*_s \phi_s
\]

from which we get

\[
u^*_s = \left[ \left( \frac{\Gamma_s}{M_s} - 1 \right) \theta_s + \left( \frac{K_s}{M_s} + \frac{\phi_s}{M_s} \right) \right] Y^*_s
\] (3.26)

where \(\alpha_s = \left( \frac{\Gamma_s}{M_s} - 1 \right) \theta_s - \frac{K_s}{M_s} + \frac{\phi_s}{M_s}\) based on the assumption that \(M_s \neq 0, \forall s \in [0, T)\)

also by comparing the \(ds\) term of (3.24) and (3.17), we get

\[
-\tilde{r}_s (M_s Y^*_s - \Gamma_s Y_s^*) = M_s \left( \alpha^T \right) \theta_s + M_s \left( u^*_s \right)^T \, ds + K_s^T \, u^*_s
\]

\[
-\tilde{r}_s M_s Y^*_s + \tilde{r}_s \Gamma_s Y^*_s = \left[ M_s \left( \alpha^T \right) \theta_s + K_s^T \, Y_s^* \right] \, ds + M_s \alpha^T \, \theta_s \, Y_s^*
\]

\[
\Leftrightarrow [\tilde{r}_s \Gamma_s - f_{\Gamma, \phi}(s, \Gamma_s, \phi_s)] Y_s^* = \left[ M_s \left( \alpha^T \right) \theta_s + K_s^T \, Y_s^* \right] \, ds + M_s \alpha^T \, \theta_s \, Y_s^* + K_s^T \alpha \, \phi_s + \tilde{r}_s \alpha_s
\]

\[
\Leftrightarrow [\tilde{r}_s \Gamma_s - f_{\Gamma, \phi}(s, \Gamma_s, \phi_s)] Y_s^* = \left[ M_s \left( \alpha^T \right) \theta_s + K_s^T \, Y_s^* \right] \, ds + M_s \alpha^T \, \theta_s \, Y_s^*
\]
which leads to the following equations

\[ f_{M,K}(s, M_s, K_s) = 2\hat{r}_s M_s + (M_s \theta_s^T + K_s^T) \alpha_s \quad (3.27) \]

\[ f_{\Gamma,\phi}(s, \Gamma_s, \phi_s) = \hat{r}_s \Gamma_s \quad (3.28) \]

plug (3.28) back into (3.22) we get

\[
\begin{aligned}
    \left\{ 
    d\Gamma_s &= -\hat{r}_s \Gamma_s ds + \phi_s^T dW_s \\
    \Gamma_T &= 1 
    \right. 
\end{aligned}
\]

that is

\[
\begin{aligned}
    \left\{ 
    d\Gamma_s e^{\int_0^s \hat{r}_u du} &= e^{\int_0^s \hat{r}_u du} \phi_s^T dW_s \\
    \Gamma_T e^{\int_0^T \hat{r}_u du} &= e^{\int_0^T \hat{r}_u du} 
    \right. 
\end{aligned}
\]

which could be solved and we get

\[ \Gamma_s = E_s \left[ e^{\int_s^T \hat{r}_u du} \right] = e^{\int_s^T \hat{r}_u du} \quad (3.31) \]

which implies

\[ \phi = 0 \quad (3.32) \]

and plug (3.32) back into (3.25) and (3.26) we get

\[ q_s = M_s u_s^* + K_s Y_s^* \quad (3.33) \]

and

\[ u_s^* = \left[ \left( \frac{\Gamma_s}{M_s} - 1 \right) \theta_s - \frac{K_s}{M_s} \right] Y_s^* \]

\[ = \alpha_s Y_s^* \quad (3.34) \]

where \( \alpha_s = \left( \frac{\Gamma_s}{M_s} - 1 \right) \theta_s - \frac{K_s}{M_s} \)

plug (3.27) into (3.21) we get that

\[
\begin{aligned}
    \left\{ 
    dM_s &= -\left[ 2\hat{r}_s M_s + (M_s \theta_s^T + K_s^T) \alpha_s \right] ds + K_s^T dW_s \\
    M_T &= 1 
    \right. 
\end{aligned}
\]

and plug \( \alpha \) into (3.35) we have

\[
\begin{aligned}
    \left\{ 
    dM_s &= -\left[ 2\hat{r}_s M_s + (M_s \theta_s^T + K_s^T) \left( \left( \frac{\Gamma_s}{M_s} - 1 \right) \theta_s - \frac{K_s}{M_s} \right) \right] ds + K_s^T dW_s \\
    M_T &= 1 
    \right. 
\end{aligned}
\]

that is

\[
\begin{aligned}
    \left\{ 
    dM_s &= -\left[ \left( 2\hat{r}_s - |\theta_s|^2 \right) M_s + \Gamma_s |\theta_s|^2 - 2K_s^T \theta_s + \left( \frac{\Gamma_s}{M_s} - 1 \right) \theta_s - \frac{K_s}{M_s} \right] ds + K_s^T dW_s \\
    M_T &= 1 
    \right. 
\end{aligned}
\]

Here we list the fact about the BMO martingale in Kazamaki[4] which will be used to prove the existence of a solution to the above BSDE (3.37).
Fact 3.3. A process $\int_0^T Z_t^s dW_s$ is a BMO martingale if and only if there exists a constant $C > 0$ s.t.
\[ \mathbb{E} \left[ \int_\tau^T |Z_t|^2 \, ds \bigg| \mathcal{F}_\tau \right] \leq C \] (3.38)
for any stopping time $\tau \leq T$. A BMO martingale $\int_0^T Z_t^s dW_s$ has the property that the stochastic exponential $\mathbb{E} \left( \int_0^T Z_t^s dW_s \right)$ is a martingale. Thus by defining $\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t} = \mathbb{E} \left( \int_0^T Z_t^s dW_s \right)$, we have that $W_t^\mathbb{Q} = W_t - \int_0^t Z_s ds$ is a $\mathbb{Q}$-Brownian motion.

Proposition 3.4. The BSDE (3.37) admits a solution $(M, K) \in L^\infty (0, T; \mathbb{R}) \times L^2 (0, T; \mathbb{R}^d)$ s.t. $M \geq \eta$ for some constant $\eta > 0$.

Proof. Here we use the truncation method. Choose any constant $c > 0$ and bound the $\frac{1}{M}$ in the BSDE (3.37) by the chosen $c$, which leads to the following BSDE
\[
\begin{align*}
&\left\{ \begin{array}{l}
dM_s = - \left( 2r_s - |\theta_s|^2 \right) M_s + \Gamma_s |\theta_s|^2 - 2K_s^T \theta_s + \frac{\Gamma_s}{M^\vee c} K_s^T \theta_s - \frac{|K_s|^2}{M^\vee c} \right) ds + K_s^T dW_s \\
M_T = 1
\end{array} \right.
\end{align*}
(3.39)

which can be written as
\[
\begin{align*}
&\left\{ \begin{array}{l}
dM_s = - \left( 2r_s - |\theta_s|^2 \right) M_s + \Gamma_s |\theta_s|^2 \right) ds + K_s^T \left[ dW_s - \left( -2\theta_s + \frac{\Gamma_s}{M^\vee c} \theta_s - \frac{K_s}{M^\vee c} \right) ds \right] \\
M_T = 1
\end{array} \right.
\end{align*}
(3.40)

(3.39) is a standard quadratic BSDE. Hence there exists a solution $(M^c, K^c) \in L^\infty (0, T; \mathbb{R}) \times L^2 (0, T; \mathbb{R}^d)$ depending on the chosen constant $c$ and $\int_0^T (K_s^c)^T dW_s$ is a BMO martingale according to the results in Kobylanski[5] and Morlais[6].

since $\int_0^T (K_s^c)^T dW_s$ is a BMO martingale and $\frac{1}{M^\vee c}, \theta, \Gamma$ are all bounded, we have that
\[
\int_0^t \left( -2\theta_s + \frac{\Gamma_s}{M^\vee c} \theta_s - \frac{K_s}{M^\vee c} \right)^T dW_s
\] (3.41)
is also a BMO martingale by definition according to fact 3.3. Thus by defining
\[
\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t} = \mathbb{E} \left[ \int_0^t \left( -2\theta_s + \frac{\Gamma_s}{M^\vee c} \theta_s - \frac{K_s}{M^\vee c} \right)^T dW_s \right]
\]
we have that
\[
W_t^\mathbb{Q} = W_t - \int_0^t \left( -2\theta_s + \frac{\Gamma_s}{M^\vee c} \theta_s - \frac{K_s}{M^\vee c} \right) ds
\] is a $\mathbb{Q}$-Brownian motion, and thusly using (3.30) we have
\[
\begin{align*}
&\left\{ \begin{array}{l}
dM_s^c e^{\int_0^s 2r_u - |\theta_u|^2 \, du} = -e^{\int_0^s 2r_u - |\theta_u|^2 \, du} \Gamma_s^\vee c \theta_s^2 \, ds + e^{\int_0^s 2r_u - |\theta_u|^2 \, du} (K_s^c)^T dW_s^\mathbb{Q} \\
M_T^c e^{\int_0^T 2r_u - |\theta_u|^2 \, du} = e^{\int_0^T 2r_u - |\theta_u|^2 \, du}
\end{array} \right.
\end{align*}
(3.42)

which is
\[
\begin{align*}
&\left\{ \begin{array}{l}
d \left( M_s^c e^{\int_0^s 2r_u - |\theta_u|^2 \, du} + \int_0^s e^{\int_0^u 2r_v - |\theta_v|^2 \, dv} \Gamma_u \theta_u^2 \, du \right) = -e^{\int_0^s 2r_u - |\theta_u|^2 \, du} (K_s^c)^T dW_s^\mathbb{Q} \\
M_T^c e^{\int_0^T 2r_u - |\theta_u|^2 \, du} + \int_0^T e^{\int_0^u 2r_v - |\theta_v|^2 \, dv} \Gamma_u \theta_u^2 \, du = e^{\int_0^T 2r_u - |\theta_u|^2 \, du} + \int_0^T e^{\int_0^u 2r_v - |\theta_v|^2 \, dv} \Gamma_u \theta_u^2 \, du
\end{array} \right.
\end{align*}
(3.43)
thus

\[ M_\alpha = \mathbb{E}^Q \left[ e^{\int_0^T 2r_u \left| \theta_u \right|^2 du} + \int_0^T e^{\int_0^u 2r_v \left| \theta_v \right|^2 dv} \Gamma_u \left| \theta_u \right|^2 du \right] \]  

(3.44)

it implies that there exists a constant \( \eta > 0 \) s.t. we have \( M_\alpha \geq \eta \) for any chosen constant \( c > 0 \), and we could choose \( c = \eta \) in the BSDE (3.39) and deduce that the BSDE (3.37) admits a solution \((M, K) \in L^\infty_T (0, T; \mathbb{R}) \times L^2_T (0, T; \mathbb{R}^d)\) s.t. \( M \geq \eta \).

Since we have proved proposition (3.44), we could have our linear feedback as

\[
u_* = \left[ \left( \frac{\Gamma_s}{M_s} - 1 \right) \theta_s - \frac{K_s}{M_s} \right] Y^*_s = \alpha_s Y^*_s
\]

(3.45)

where \( \alpha_s = \left( \frac{\Gamma_s}{M_s} - 1 \right) \theta_s - \frac{K_s}{M_s} \) has the process \( \lambda \) as its parameter, i.e. \( \alpha_s = f_s(\lambda) \) with \( f_s = \left( \frac{\Gamma_s}{M_s} - 1 \right) \theta_s - \frac{K_s}{M_s} \) for any \( s \in [0, T] \).

**Theorem 3.5.** The control \( \nu_* \) in (3.44) is an equilibrium control for the family of problems in (3.25).

**Proof.** Let \( Y^* \) be the corresponding state process with respect to the above control \( \nu_* \) in (3.44). We have for \( s \in [0, T] \)

\[
u_* = \alpha_s Y^*_s
\]

Plug the \( \nu_* \) into the dynamics of \( Y \) in (3.10) we get

\[
dY^*_s = \left( \Gamma_s Y^*_s + (\nu^*_s)^T \theta_s \right) ds + (\nu^*_s)^T dW_s
\]

\[ = Y^*_s \left[ (\Gamma_s + \alpha_s^2 \theta_s) ds + \alpha_s^2 dW_s \right]
\]

(3.46)

thus we have that

\[
Y^*_t = y_0 e^{\int_0^t \hat{\gamma}_s ds} + \alpha_s^2 \theta_s \int_0^t e^{\int_0^s \hat{\gamma}_u ds} \theta_u^2 ds + \alpha_s^2 \hat{\gamma}^T_e T dW_s,
\]

(3.47)
by applying Itô formula to \( \log(M) \) we get

\[
d \log (M_t) = \frac{1}{M_t} dM_t - \frac{1}{2M_t^2} d \langle M \rangle_t
\]

from which we get

\[
\log \left( \frac{M_t}{M_0} \right) = \int_0^t -2\hat{r}_s \, ds + \int_0^t \left\{ \frac{|\alpha_s|^2}{2} - \left( \frac{\Gamma_s}{M_s} - 1 \right) |\theta_s|^2 - \frac{1}{2} \left( \frac{\Gamma_s}{M_s} - 1 \right)^2 \frac{K_s^T \theta_s}{M_s^2} \right\} \, ds + \frac{K_s^T}{M_s} \left( dW_s + \theta_s \, ds \right)
\]

\[= -2\hat{r}_s + \left\{ \frac{|\alpha_s|^2}{2} - \left( \frac{\Gamma_s}{M_s} - 1 \right) |\theta_s|^2 - \frac{1}{2} \left( \frac{\Gamma_s}{M_s} - 1 \right)^2 \frac{K_s^T \theta_s}{M_s^2} \right\} \, ds + \frac{K_s^T}{M_s} \left( dW_s + \theta_s \, ds \right)\]

\[
(3.48)
\]

by combining (3.47) and (3.49) we get

\[
Y_t^* = y_0 e^{\hat{r}_0 t - \hat{r}_s \, ds} \frac{M_0}{M_t} \mathcal{E} \left( \int_0^t \left( \frac{\Gamma_s}{M_s} - 1 \right) \theta_s^T \, dW_s \right)
\]

\[\text{(3.50)}\]

So we have as \( \int_0^T |\hat{r}_s| \, ds < \infty \) and \( \Gamma, M, \frac{1}{M_t}, \theta \) are all bounded that

\[
\mathbb{E} \left[ \sup_{t \in [0, T]} (Y_t^*)^2 \right] = \mathbb{E} \left\{ \sup_{t \in [0, T]} \left( y_0 e^{\hat{r}_0 t - \hat{r}_s \, ds} \frac{M_0}{M_t} \right)^2 \mathcal{E} \left( \int_0^t \left( \frac{\Gamma_s}{M_s} - 1 \right) \theta_s^T \, dW_s \right)^2 \right\}
\]

\[
\leq \mathbb{E} \left\{ \sup_{t \in [0, T]} \left( y_0 e^{\hat{r}_0 t - \hat{r}_s \, ds} \frac{M_0}{M_t} \right)^2 \sup_{\tau \in [0, T]} \mathcal{E} \left( \int_0^\tau \left( \frac{\Gamma_s}{M_s} - 1 \right) \theta_s^T \, dW_s \right)^2 \right\}
\]

\[
\leq y_0^2 \sup_{t \in [0, T]} e^{\hat{r}_0 t - 2\hat{r}_s t} \sup_{\tau \in [0, T]} \left( \frac{M_0}{M_t} \right)^2 \sup_{\tau \in [0, T]} \mathcal{E} \left( \int_0^\tau \left( \frac{\Gamma_s}{M_s} - 1 \right) \theta_s^T \, dW_s \right)^2
\]

\[
\leq CE \left[ \sup_{t \in [0, T]} \mathcal{E} \left( \int_0^t \left( \frac{\Gamma_s}{M_s} - 1 \right) \theta_s^T \, dW_s \right)^2 \right], \text{ for some constant } C
\]

and \( \mathcal{E} \left( \int_0^t \left( \frac{\Gamma_s}{M_s} - 1 \right) \theta_s^T \, dW_s \right) \) is a martingale by Novikov

\[
\leq 4CE \left[ \mathcal{E} \left( \int_0^T \left( \frac{\Gamma_s}{M_s} - 1 \right) \theta_s^T \, dW_s \right)^2 \right], \text{ by } L^p \text{Maximal Inequality}
\]

\[
= 4CE \left[ e^{\hat{r}_0 (\frac{\Gamma_s}{M_s} - 1)^2 |\theta_s|^2 \, ds} \mathcal{E} \left( \int_0^T 2 \left( \frac{\Gamma_s}{M_s} - 1 \right) \theta_s^T \, dW_s \right) \right]
\]

\[< \infty\]

\[\text{(3.51)}\]
as we also have $E \left[ \mathcal{E} \left( \int_0^T 2 \left( \frac{\theta_s^T}{M_s} - 1 \right) \theta_s^T dW_s \right) \right] = 1$, thus we deduce that

$$Y^* \in L^2_T (\Omega; C (0, T; \mathbb{R}))$$  \hspace{1cm} (3.52)

Since $(u^*, Y^*)$ can be regraded as the solution to the following BSDE

$$
\begin{align*}
    dY_s &= \left( \bar{r}_s Y_s + (u_s)^T \theta_s \right) ds + (u_s)^T dW_s \\
    Y_T &= Y^*_T
\end{align*}
$$

(3.53)

and we have

$$
\begin{align*}
    d \left( e^{-\int_0^t \bar{r}_s u_s} Y^*_s \right) &= \left[ \left( e^{-\int_0^t \bar{r}_s u_s} \right)^T \theta_s \right] ds + \left[ \left( e^{-\int_0^t \bar{r}_s u_s} \right)^T \right] dW_s \\
    e^{-\int_0^T \bar{r}_s u_s} Y^*_T &= e^{-\int_0^T \bar{r}_s u_s} Y^*_T
\end{align*}
$$

(3.54)

which implies $e^{-\int_0^T \bar{r}_s u_s} Y^*_s, e^{-\int_0^T \bar{r}_s u_s} Y^*_s)$ can be regraded as the solution to the following BSDE

$$
\begin{align*}
    d\tilde{Y}_s &= (\tilde{u}_s)^T \theta_s ds + (\tilde{u}_s)^T dW_s \\
    \tilde{Y}_T &= \left( e^{-\int_0^T \bar{r}_s u_s} Y^*_s \right)
\end{align*}
$$

(3.55)

since the above BSDE is Lipschitz as $\theta$ is bounded and $e^{-\int_0^T \bar{r}_s u_s} Y^*_s \in L^2_T (\Omega; \mathbb{R})$, thus we deduce that $e^{-\int_0^T \bar{r}_s u_s} Y^*_s \in L^2_T (0, T; \mathbb{R}^d)$ which implies that

$$u^* \in L^2_T (0, T; \mathbb{R}^d)$$  \hspace{1cm} (3.56)

Also we have that for any $t \in [0, T)$

$$A'_t = p_t \theta_s + q'_s$$

$$= \left[ M_s Y^*_s - \Gamma_s Y^*_s \right] \theta_s + M_s u^*_s + K_s Y^*_s$$

$$= \left\{ M_s \theta_s + M_s \left[ \left( \frac{\Gamma_s}{M_s} - 1 \right) \theta_s - \frac{K_s}{M_s} \right] + K_s \right\} Y^*_s - \Gamma_s Y^*_s \theta_s$$

$$= \Gamma_s \theta_s Y^*_s - \Gamma_s Y^*_s \theta_s$$

$$= \Gamma_s \theta_s (Y^*_s - Y^*_t)$$  \hspace{1cm} (3.57)

Since $\Gamma, \theta$ are bounded, it is clearly by \hspace{1cm} (3.52) that

$$E_t \int_t^T |A'_s| ds < \infty$$  \hspace{1cm} (3.58)

and we have

$$\lim_{s \uparrow t} E_t \left[ A'_s \right] = \lim_{s \uparrow t} E_t \left[ \Gamma_s \theta_s (Y^*_s - Y^*_t) \right]$$

$$\hspace{1cm} (3.59)$$

Then \hspace{1cm} (3.56), \hspace{1cm} (3.58), \hspace{1cm} (3.59) are exactly the required conditions in \hspace{1cm} (3.15) and we deduce that $u^*$ is an equilibrium control for the family of problems in \hspace{1cm} (3.15) by proposition 3.2.

\[ \square \]
Since \( u^*_s = e^{\int_0^s \lambda_s \, du} \sigma_s^* \pi^*_s \) by (3.15) and \( u^*_s = \alpha_s Y^*_s = \alpha_s e^{\int_0^s \lambda_s \, du} X^*_s \) by (3.54) and (3.14). Thus we have that
\[
\pi^*_s = (\alpha_s)^{-1} \alpha_s X^*_s
\]
which says \( \pi^*_s \) is equilibrium to the family of problems in (3.60).

### 3.4 Conditions for obtaining an equilibrium for our problem

**Theorem 3.6.** If there exists a deterministic process \( \lambda^* \) with \( \int_0^T |\lambda^*_s| \, ds < \infty \) s.t. \( \mathbb{E}^Q_t \left[ e^{\int_0^T \alpha_s^* \theta_s \, ds} \right] = e^{\int_0^t \mu_s - r_s \, ds} \) for any \( t \in [0, T) \) with \( \frac{d\mathbb{Q}}{d\mathbb{P}} \bigg|_{\mathcal{F}_t} = \mathcal{E} \left( \int_0^t \alpha_s^* dW_s \right) \), then the above \( \pi^* \) in (3.60) is an equilibrium control for our original family of problems (3.4).

**Proof.** Suppose there exists a deterministic process \( \lambda^* \) with \( \int_0^T |\lambda^*_s| \, ds < \infty \) s.t. \( \mathbb{E}^Q_t \left[ e^{\int_0^T \alpha_s^* \theta_s - \mu_s \, ds} \right] = 1 \) for any \( t \in [0, T) \) with \( \frac{d\mathbb{Q}}{d\mathbb{P}} \bigg|_{\mathcal{F}_t} = \mathcal{E} \left( \int_0^t \alpha_s^* dW_s \right) \). Firstly, we verify that \( \frac{d\mathbb{Q}}{d\mathbb{P}} \bigg|_{\mathcal{F}_t} = \mathcal{E} \left( \int_0^t \alpha_s^* dW_s \right) \) is well defined. Since \( \int_0^T (K_s) dW_s \) is a BMO martingale and \( \mu_s, \theta_s \) are all bounded, we deduce that \( \int_0^T (\alpha_s)^T dW_s \) is also a BMO martingale by definition according to fact (3.3). Thus we could define
\[
\frac{d\mathbb{Q}}{d\mathbb{P}} \bigg|_{\mathcal{F}_t} = \mathcal{E} \left( \int_0^t \alpha_s^* dW_s \right)
\]

By combining (3.14) and (3.40) we could get
\[
X^*_t e^{\int_0^T \lambda^*_s \, ds} = X^*_0 e^{\int_0^T \lambda^*_s \, ds} e^{\int_0^t \mu_s - r_s - \lambda^*_s \, ds + \frac{1}{2} |\lambda^*_s|^2 \, ds} dW_s
\]
\[
\Rightarrow X^*_t = x_0 e^{\int_0^t \mu_s - r_s + \lambda^*_s \, ds + \frac{1}{2} |\lambda^*_s|^2 \, ds + \int_0^t \alpha_s^* \, ds}
\]
where \( \alpha_s = \left( \frac{\Gamma_s}{M_s} - 1 \right) \theta_s - \frac{K_s}{M_s} \), and we have that
\[
\mathbb{E}_t [X^*_t] = X^*_t e^{\int_0^t \mu_s \, ds} \mathbb{E}_t \left[ e^{\int_0^T \mu_s - r_s - \lambda^*_s \, ds + \frac{1}{2} |\lambda^*_s|^2 \, ds + \int_0^T \alpha_s^* \, ds} \right] = X^*_t e^{\int_0^t \mu_s \, ds} = X^*_t e^{\int_0^t \mu_s \, ds}
\]
for any \( t \in [0, T) \). Since \( \pi^*_s \) is equilibrium to the family of problems in (3.6), thus we could deduce that having \( \lambda^* \) as parameter \( \pi^*_s \) is an equilibrium control for our original family of problems (3.4) by theorem 3.1.

If we also assume that \( \mu^x \) is deterministic, i.e. \( \theta \) is deterministic, then we have as \( K = 0 \) that
\[
u^*_s = \left( \frac{\Gamma_s}{M_s} - 1 \right) \theta_s Y^*_s = \alpha_s Y^*_s
\]
where \( \alpha_s = \left( \frac{\Gamma_s}{M_s} - 1 \right) \theta_s \) for any \( s \in [0, T] \) and we could solve (3.37) and get
\[
\begin{align*}
\Gamma_s &= e^{\int_0^s T \theta_s \, du} \\
M_s &= e^{\int_0^s \theta_s \, du} + \int_0^T \Gamma_u |\theta_u|^2 e^{\int_s^T 2 \theta_r \, dr} \, du
\end{align*}
\]
In this case, theorem 3.6 becomes that if there exists a deterministic process \( \lambda^* \) with \( \int_0^T |\lambda^*_s| \, ds < \infty \) s.t. \( \int_0^T r_s + \alpha^*_s \theta_s - \mu_s \, ds = 0 \) for any \( t \in [0, T) \), then the above \( \pi^* \) in (3.60) is an equilibrium control for our original family of problems in (3.4).
4 Utility function $h(x) = -\frac{x^3}{3}$ for mean-cubic portfolio selection

Some investors are risk seekers and they may choose a utility function that looks quite risky as the one we will use here. In this section, we want to solve our moving target portfolio selection problem when we choose to use $h(x) = -\frac{x^3}{3}$ as the utility function. That means we want to solve the family of following problems for any $t \in [0, T)$

$$\min_{\pi} \mathbb{E}_t \left[ -\frac{(X_T - X_t e^{\int_t^T \mu_sds})^3}{3} \right]$$

s.t. $\begin{cases} 
    dX_s = (r_sX_s + \pi_s^T \sigma_s \theta_s) ds + \pi_s^T \sigma_s dW_s \\
    X_t = x_t \\
    \pi \in \mathcal{U}_{ad} = \{ \pi \mid \pi \in L^2_\mathbb{F} (0, T; \mathbb{R}^d) \text{ and } \mathbb{E}_t [X_T] = X_t e^{\int_t^T \mu_sds}, \forall t \in [0, T) \} 
\end{cases}$

(4.1)

where $\mu$ is our required return process which is bounded and deterministic.

4.1 Transformation of our problem

Here we use another approach to solve our problem rather than the Lagrangian multiplier method used in the previous section for $h(x) = x^2$. By letting for any $s \in [0, T]$

$$Y_s = X_s e^{\int_s^T \mu_u du}$$

(4.2)

we have the family of following problems for any $t \in [0, T)$, which is equivalent to the above family of problems (4.1)

$$\min_u \mathbb{E}_t \left[ -\frac{(Y_T - Y_t)^3}{3} \right]$$

s.t. $\begin{cases} 
    dY_s = (\hat{r}_s Y_s + u_s^T \theta_s) ds + u_s^T dW_s \\
    Y_t = y_t \\
    u \in \mathcal{U}_{ad} = \{ u \mid u \in L^2_\mathbb{F} (0, T; \mathbb{R}^d) \text{ and } \mathbb{E}_t [Y_T] = Y_t, \forall t \in [0, T) \} 
\end{cases}$

(4.3)

where $\hat{r}_s = r_s - \mu_s$, $u_s = e^{\int_s^T \mu_u du} \sigma_s^T \pi_s$, $y_t = x_t e^{\int_t^T \mu_u du}$.

$\mathbb{E}_t [Y_T] = Y_t, \forall t \in [0, T)$ implies that $Y$ must be a martingale as

$$\forall \hat{t} \in (t, T), \mathbb{E}_t [Y_{\hat{t}}] = \mathbb{E}_t [\mathbb{E}_t (Y_T)] = \mathbb{E}_t [Y_T] = Y_t$$

(4.4)

which is an admissible constraint on $u$ and thus we could firstly consider the family of following problems

$$\min_u \mathbb{E}_t \left[ -\frac{(Y_T - Y_t)^3}{3} \right]$$

s.t. $\begin{cases} 
    dY_s = (\hat{r}_s Y_s + u_s^T \theta_s) ds + u_s^T dW_s \\
    Y_t = y_t \\
    u \in \mathcal{U}_{ad} = \{ u \mid u \in L^2_\mathbb{F} (0, T; \mathbb{R}^d) \} 
\end{cases}$

(4.5)
Proposition 4.1. If an equilibrium solution \((u^*, Y^*)\) to the above family of problems \((4.5)\) satisfies 
\[r_s Y^*_s + (u^*_s)^T \theta_s = 0 \text{ for any } s \in [0, T),\]
then \((u^*, Y^*)\) is also equilibrium for the family of problems in \((4.3)\).

Proof. Suppose \((u^*, Y^*)\) is an equilibrium solution to \((4.5)\) which satisfies 
\[r_s Y^*_s + (u^*_s)^T \theta_s = 0 \text{ for any } s \in [0, T),\]
then we have that 
\[dY^*_s = u^*_s dW_s, \forall s \in (0, T)\]
which implies that \(Y^*\) is a martingale. So we have 
\[\mathbb{E}_t [Y^*_T] = Y^*_t, \forall t \in [0, T)\]
and thus \(u^* \in \mathcal{U}_{ad}\), which can be used together with definition \((2.1)\) to deduce that \((u^*, Y^*)\) is also equilibrium for the family of problems \((4.3)\).

By the above proposition, we try to solve the family of problems \((4.5)\) instead, that is
\[
\min_u \mathbb{E}_t [h (Y_T - Y_t)] \\
\text{s.t.} \left\{ 
\begin{aligned}
dY_s &= (\hat{r}_s Y_s + u^*_s \theta_s) \, ds + u^*_s \, dW_s \\
Y_t &= y_t \\
u &\in U^u_{ad} = \{u \mid u \in L^2_T (0, T; \mathbb{R}^d)\}
\end{aligned} \right.
\]
where \(h(x) = -\frac{x^2}{2}\), and this is again exactly a family of problems of the form in \((2.12)\), so we have the following system of BSDEs by \((2.14)\) and \((2.15)\)
\[
\begin{aligned}
&\left\{ 
\begin{aligned}
dp^t_s &= -\hat{r}_s p^t_s \, ds + (q^t_s)^T \, dW_s, \ s \in [t, T] \\
p^t_T &= \frac{d(h(Y^*_{T} - Y^*_t))}{dx} = -(Y^*_T - Y^*_t)^2 \\
\end{aligned} \right.
\end{aligned}
\]
and
\[
\begin{aligned}
&\left\{ 
\begin{aligned}
dP^t_s &= -2\hat{r}_s P^t_s \, ds + (Q^t_s)^T \, dW_s, \ s \in [t, T] \\
P^t_T &= \frac{d(h(Y^*_{T} - Y^*_t))}{dx} = -2(Y^*_T - Y^*_t)
\end{aligned} \right.
\end{aligned}
\]

Proposition 4.2. If \((4.6)\) and \((4.7)\) admit a solution \((u^*, Y^*, p^t, q^t)\) for any \(t \in [0, T)\) s.t.
\[
\begin{aligned}
&u^* \in L^2_T (0, T; \mathbb{R}^d) \\
&\mathbb{E}_t \int_t^T |\Lambda^t_s| \, ds < \infty \text{ and } \lim_{s \to t} \mathbb{E}_t [\Lambda^t_s] = 0 \text{ where } \Lambda^t_s = p^t_s \theta_s + q^t_s; \\
&\hat{r}_t Y^*_t + (u^*_t)^T \theta_t = 0
\end{aligned}
\]
then \(u^*\) is an equilibrium control for the family of problems \((4.3)\).

Proof. Since the additional admissible condition \(\hat{r}_s Y^*_s + (u^*_s)^T \theta_s = 0, \forall s \in [0, T)\) implies that 
\[\mathbb{E}_t [Y^*_T] = Y^*_t, \forall t \in [0, T),\]
so we have that 
\[\mathbb{E}_t \left[ \frac{d^2 (h(Y^*_T - Y^*_t))}{dx^2} \right] = -2 \left( \mathbb{E}_t [Y^*_T] - Y^*_t \right) = 0, \forall t \in [0, T).\]
Thus one of the sufficient condition for equilibrium, i.e. 
\[\mathbb{E}_t \left[ \frac{d^2 (h(Y^*_T - Y^*_t))}{dx^2} \right] \geq 0, \text{ in } (2.23)\]
derived theorem \((2.3)\) is covered by the condition \(\hat{r}_s Y^*_s + (u^*_s)^T \theta_s = 0, \forall s \in [0, T)\), and here we could make a replacement. Then by combining theorem \((2.3)\) and proposition \((4.1)\) we deduce that \(u^*\) is an equilibrium for the family of problems \((4.3)\).

4.2 Details of finding a potential equilibrium

By the assumptions we have made, \(\hat{r}\) and \(\sigma\) are deterministic and bounded, \(\mu^x\) is bounded. Here we also assume that \(\mu^x\) is deterministic, i.e. \(\theta\) is deterministic and bounded. Then for any \(t \in [0, T)\), we make the following Ansatz
\[
p^t_s = -M_s (Y^*_s)^2 + N_s Y^*_s Y^*_s - \Gamma_s (Y^*_s)^2, \ s \in [t, T]
\]
where \(M, N, \Gamma\) are deterministic functions which are differentiable with \(M_T = 1, N_T = 2, \Gamma_T = 1\) by Itô formula we have

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\[
\begin{align*}
    d (Y_s^*)^2 &= 2Y_s^* dY_s^* + d \langle Y_s^* \rangle \\
    &= \left[2Y_s^* \left( \tilde{r}_s Y_s^* + (u_s^*)^T \theta_s \right) + (u_s^*)^T u_s^* \right] ds + 2Y_s^* (u_s^*)^T dW_s \quad (4.11)
\end{align*}
\]

\[
\begin{align*}
    d \left[ M_s (Y_s^*)^2 \right] &= M_s d (Y_s^*)^2 + (Y_s^*)^2 dM_s \\
    &= \left\{ M_s \left[ 2Y_s^* \left( \tilde{r}_s Y_s^* + (u_s^*)^T \theta_s \right) + (u_s^*)^T u_s^* \right] \right\} ds \\
    &\quad + 2M_s Y_s^* (u_s^*)^T dW_s \quad (4.12)
\end{align*}
\]

\[
\begin{align*}
    d (N_s Y_t^* Y_s^*) &= Y_t^* [N_s dY_s^* + Y_s^* dN_s] \\
    &= Y_t^* \left[ N_s \left( \tilde{r}_s Y_s^* + (u_s^*)^T \theta_s \right) + Y_s^* N_s \right] ds + Y_s^* N_s (u_s^*)^T dW_s \quad (4.13)
\end{align*}
\]

So by applying Itô formula to (4.10) with respect to \( M_s \) and (4.14), we get that

\[
\begin{align*}
    dp_t^s &= -d \left[ M_s (Y_s^*)^2 \right] + d (N_s Y_t^* Y_s^*) - d \left[ \Gamma_s (Y_t^*)^2 \right] \\
    &= \left\{ - (Y_s^*)^2 M_s (u_s^*)^T u_s^* - 2M_s Y_s^* \left( \tilde{r}_s Y_s^* + (u_s^*)^T \theta_s \right) \right\} ds \\
    &\quad + \left( Y_t^* Y_s^* N_s + Y_t^* N_s \left( \tilde{r}_s Y_s^* + (u_s^*)^T \theta_s \right) \right) ds - \Gamma_s (Y_t^*)^2 ds \\
    &\quad + \left[ -2M_s Y_s^* + Y_t^* N_s \right] (u_s^*)^T dW_s \quad (4.14)
\end{align*}
\]

by comparing the \( dW \) terms of \( dp_t^s \) in (4.7) and (4.14), we get that

\[
q_t^s = [-2M_s Y_s^* + Y_t^* N_s] u_s^*, \quad s \in [t, T] \quad (4.15)
\]

we again hope to find a possible linear feedback \( u^* \) and as before try by setting

\[
0 = \Lambda_s^* = p_s^* \theta_s + q_s^*, \quad s \in [0, T] \quad (4.16)
\]

which leads to the equation

\[
\begin{align*}
    [(-M_s + N_s - \Gamma_s) Y_s^* \theta_s + (N_s - 2M_s) u_s^*] Y_s^* &= 0 \quad (4.17)
\end{align*}
\]

from which we get

\[
u_s^* = \alpha_s \theta_s Y_s^* \quad (4.18)
\]

where

\[
\alpha_s = \begin{cases} 
-M_s + N_s - \Gamma_s, & \forall s \in [0, T) \\
\lim_{s \uparrow T} -M_s + N_s - \Gamma_s, & s = T
\end{cases}
\]

(4.19)

based on the assumption that \( 2M_s - N_s \neq 0, \forall s \in [0, T) \) and \( \lim_{s \uparrow T} -M_s + N_s - \Gamma_s \) exists.
By comparing the $ds$ terms of $dp'$ in (1.17) and (1.14), we get that

$$-\dot{r}_s \left[ -M_s (Y'_s)^2 + N_s Y'_s Y''_s - \Gamma_s (Y'_s)^2 \right]$$

$$= \left[ - (Y'_s)^2 M'_s - M_s (u'_s)^T u'_s - 2M_s Y'_s \left( \dot{r}_s Y'_s + (u'_s)^T \theta_s \right) \right] + \left[ Y'_t Y'_s N'_s + Y'_s N'_s \left( \dot{r}_s Y'_s + (u'_s)^T \theta_s \right) \right] - \Gamma'_s (Y'_s)^2$$

$$\Leftrightarrow -\dot{r}_s \left[ -M_s (Y'_s)^2 + N_s Y'_s Y''_s - \Gamma_s (Y'_s)^2 \right]$$

$$= \left[ - (Y'_s)^2 M'_s - M_s \alpha_s^2 |\theta_s|^2 (Y'_s)^2 - 2M_s Y'_s \left( \dot{r}_s Y'_s + \alpha_s |\theta_s|^2 Y'_s \right) \right] + \left[ Y'_t Y'_s N'_s + Y'_s N'_s \left( \dot{r}_s Y'_s + \alpha_s |\theta_s|^2 Y'_s \right) \right] - \Gamma'_s (Y'_s)^2$$

$$\Leftrightarrow \ddot{r}_s M_s (Y'_s)^2 - \dot{r}_s N_s Y'_s Y''_s + \ddot{r}_s \Gamma_s (Y'_s)^2$$

$$= \left[ - M'_s - M_s \alpha_s^2 |\theta_s|^2 - 2M_s \left( \dot{r}_s + \alpha_s |\theta_s|^2 \right) \right] (Y'_s)^2 + \left[ N'_s + N_s \left( \dot{r}_s + \alpha_s |\theta_s|^2 \right) \right] Y'_s Y''_s - \Gamma'_s (Y'_s)^2$$

after rearrangement we get

$$0 = (\Gamma'_s + \ddot{r}_s \Gamma_s) (Y'_s)^2 - \left[ N'_s + 2\ddot{r}_s N_s + \alpha_s |\theta_s|^2 N_s \right] Y'_s Y''_s$$

$$+ \left[ M'_s + 3\ddot{r}_s M_s + 2\alpha_s |\theta_s|^2 M_s + \alpha_s^2 |\theta_s|^2 M_s \right] (Y'_s)^2$$

(4.20)

which leads to the following system of ODEs

$$\begin{cases}
M'_s + \left( 3\ddot{r}_s + 2\alpha_s |\theta_s|^2 + \alpha_s^2 |\theta_s|^2 \right) M_s = 0, & s \in [0, T] \\
M_T = 1
\end{cases}$$

(4.21)

$$\begin{cases}
N'_s + \left( 2\ddot{r}_s + \alpha_s |\theta_s|^2 \right) N_s = 0, & s \in [0, T] \\
N_T = 2
\end{cases}$$

(4.22)

$$\begin{cases}
\Gamma'_s + \ddot{r}_s \Gamma_s = 0, & s \in [0, T] \\
\Gamma_T = 1
\end{cases}$$

(4.23)

the solution to equation (4.23) is $\Gamma_s = e^{T \dot{r}_s du}$, which makes the unsettled system contains only (4.21) and (4.22) as follows

$$\begin{cases}
M'_s + \left( 3\ddot{r}_s + 2\alpha_s |\theta_s|^2 + \alpha_s^2 |\theta_s|^2 \right) M_s = 0, & s \in [0, T] \\
M_T = 1
\end{cases}$$

(4.24)

$$\begin{cases}
N'_s + \left( 2\ddot{r}_s + \alpha_s |\theta_s|^2 \right) N_s = 0, & s \in [0, T] \\
N_T = 2
\end{cases}$$

Remark 4.3. If we need $\lim_{s \to T} \frac{M'_s + N'_s - \Gamma_s}{2M_s - N_s} \neq 0$, we also need to assume that $\lim_{s \to T} \frac{\ddot{r}_s}{\theta''_s} \leq \frac{\alpha}{16}$ which is a necessary condition for the existence of none zero $\lim_{s \to T} \frac{M'_s + N'_s - \Gamma_s}{2M_s - N_s}$ where $\alpha, M, N, \Gamma$ are those defined in (4.19), (4.21), (4.22) and (4.23).
Proof. Suppose \( \lim_{s \to T} \frac{-M_s + N_s - \Gamma_s}{2M_s - N_s} \) exists and does not equal to zero. Let \( \hat{\theta} = \lim_{s \to T} \theta_s \), \( \bar{r} = \lim_{s \to T} r_s \) and \( \hat{\alpha} = \lim_{s \to T} -\frac{M_s + N_s - \Gamma_s}{2M_s - N_s} \) which also implies \( \hat{\alpha} = \lim_{s \to T} \alpha_s \) by the definition of \( \alpha \) in (4.19). Since we have

\[
\lim_{s \to T} \frac{-M_s' + N_s' - \Gamma_s'}{2M_s' - N_s'} = \lim_{s \to T} \left( \frac{3\bar{r}_s + 2\alpha_s |\theta_s|^2 + \alpha^2_s |\theta_s|^2}{2\bar{r}_s + \alpha_s |\theta_s|^2} \right) M_s - \left( 2\bar{r}_s + \alpha_s |\theta_s|^2 \right) N_s + \bar{r}_s \Gamma_s
\]

\[
= \lim_{s \to T} \left( \frac{3\bar{r}_s + 2\alpha_s |\theta_s|^2 + \alpha^2_s |\theta_s|^2}{2\bar{r}_s + \alpha_s |\theta_s|^2} \right) M_s - \left( 2\bar{r}_s + \alpha_s |\theta_s|^2 \right) N_s + \bar{r}_s \Gamma_s
\]

\[
= \lim_{s \to T} \frac{2\alpha_s^2 |\theta_s|^2}{2\bar{r}_s + \alpha_s |\theta_s|^2 - 2\bar{r}}
\]

\[
= \frac{\hat{\alpha}^2}{-2\hat{\alpha}^2 - 2\hat{\alpha} - 2\eta}
\]

(4.25)

where \( \eta = \frac{\bar{r}}{|\theta|^2} \), thus we must have

\[
\hat{\alpha} = \lim_{s \to T} \frac{-M_s + N_s - \Gamma_s}{2M_s - N_s} = \lim_{s \to T} \frac{-M_s' + N_s' - \Gamma_s'}{2M_s' - N_s'} = \frac{\hat{\alpha}^2}{-2\hat{\alpha}^2 - 2\hat{\alpha} - 2\eta}
\]

which implies the rearrangement is

\[
(2\hat{\alpha}^2 + 3\hat{\alpha} + 2\eta) \hat{\alpha} = 0
\]

(4.26)

which implies that \( \eta \leq \frac{9}{10} \) is a necessary condition for the existence of none zero \( \hat{\alpha} \)

### 4.3 Conditions for obtaining an equilibrium for our problem

**Theorem 4.4.** If the system of ODEs (4.24) admits a solution \((M, N)\) s.t. the corresponding \( \alpha \) defined in (4.19) satisfies that \( \bar{r}_t + \alpha_t \theta_t = 0 \) for every \( t \in [0, T] \), then \( u^* \) is an equilibrium control for the family of problems in (4.5).

**Proof.** Suppose \((M, N)\) is a solution to (4.24) s.t. \( \bar{r}_t + \alpha_t \theta_t = 0 \) for every \( t \in [0, T] \). Since the deterministic \( M, N, \Gamma \) are continuous and thus are bounded on \([0, T]\), we have that \( \alpha \) is also deterministic and bounded on \([0, T]\) according to (4.19). We have in (4.18) for \( s \in [0, T] \) that

\[
u^*_s = \alpha_s \theta_s Y^*_s
\]

(4.27)

and thus we have

\[
dY^*_s = \left( \bar{r}_s Y^*_s + (u^*_s)^T \theta_s \right) ds + (u^*_s)^T dW_s
\]

\[
= Y^*_s \left( \left( \bar{r}_s + \alpha_s |\theta_s|^2 \right) ds + \alpha_s (\theta_s)^T dW_s \right)
\]

(4.28)

which leads to

\[
Y^*_t = y_0 e^{\int_0^t \bar{r}_s + \alpha_s |\theta_s|^2 - \frac{3}{2} \alpha^2_s |\theta_s|^2 ds + \int_0^s \alpha_s (\theta_s)^T dW_s}
\]

(4.29)

thus
\[
\begin{align*}
\mathbb{E} \left[ \sup_{t \in [0,T]} (Y^*_t)^2 \right] &= \mathbb{E} \left[ \sup_{t \in [0,T]} \left( y_0 e^{\int_0^t \hat{r}_s + \alpha_s \theta_s^2 ds} \right)^2 \left( e^{\int_0^t - \frac{1}{2} \alpha_s^2 \theta_s^2 ds + \int_0^t \alpha_s (\theta_s)^T dW_s \right)^2 \right] \\
&\leq \sup_{t \in [0,T]} \left( y_0 e^{\int_0^t \hat{r}_s + \alpha_s \theta_s^2 ds} \right)^2 \mathbb{E} \left[ \sup_{t \in [0,T]} \left( e^{\int_0^t - \frac{1}{2} \alpha_s^2 \theta_s^2 ds + \int_0^t \alpha_s (\theta_s)^T dW_s \right)^2 \right] \\
&\leq \sup_{t \in [0,T]} \left( y_0 e^{\int_0^t \hat{r}_s + \alpha_s \theta_s^2 ds} \right)^2 4 \mathbb{E} \left[ e^{\int_0^T - \frac{1}{2} \alpha_s^2 \theta_s^2 ds + \int_0^T \alpha_s (\theta_s)^T dW_s \right] \\
&= 4 e^{\int_0^T \alpha_s^2 \theta_s^2 ds} \sup_{t \in [0,T]} \left( y_0 e^{\int_0^t \hat{r}_s + \alpha_s \theta_s^2 ds} \right)^2 \\
&< \infty
\end{align*}
\]

as \( \alpha, \theta, \hat{r} \) are bounded, which implies that
\[
Y^* \in L^2_F (\Omega; C (0, T; \mathbb{R}))
\]
and thus we have
\[
\mathbb{E} \left[ \int_0^T |u^*_s|^2 ds \right] = \mathbb{E} \left[ \int_0^T \alpha_s^2 |\theta_s|^2 (Y^*_s)^2 ds \right] \leq \sup_{s \in [0,T]} \left( \alpha_s^2 |\theta_s|^2 \right) \int_0^T \mathbb{E} \left[ \sup_{u \in [0,T]} (Y^*_u)^2 \right] ds
\]
\[
= \sup_{s \in [0,T]} \left( \alpha_s^2 |\theta_s|^2 \right) \mathbb{E} \left[ \sup_{u \in [0,T]} (Y^*_u)^2 \right] T < \infty
\]
which implies that
\[
u^* \in L^2_F (0, T; \mathbb{R}^d)
\]
also we have that for any \( t \in [0, T) \)
\[
\Lambda^t_s = p^t_s \theta_s + q^t_s
\]
\[
= \left[ -M_s (Y^*_s)^2 + N_s Y^*_s Y^*_s - \Gamma_s (Y^*_s)^2 \right] \theta_s + (Y^*_s N_s - M_s Y^*_s) u^*_s
\]
\[
= \left[ -M_s (Y^*_s)^2 + N_s Y^*_s Y^*_s - \Gamma_s (Y^*_s)^2 + (Y^*_s N_s - M_s Y^*_s) \alpha_s Y^*_s \right] \theta_s
\]
\[
= \left[ -(1 + 2 \alpha_s) M_s (Y^*_s)^2 + (1 + \alpha_s) N_s Y^*_s Y^*_s - \Gamma_s (Y^*_s)^2 \right] \theta_s
\]

Since \( \alpha, M, N, \Gamma, \theta \) are bounded, it is clearly by (4.31) that
\[
\mathbb{E}_t \int_t^T |\Lambda^t_s| ds < \infty
\]
and
\[
\lim_{s \downarrow t} \mathbb{E}_t \left[ \Lambda^t_s \right] = \lim_{s \downarrow t} \mathbb{E}_t \left[ -(1 + 2 \alpha_s) M_s (Y^*_s)^2 + (1 + \alpha_s) N_s Y^*_s Y^*_s - \Gamma_s (Y^*_s)^2 \right]
\]

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since we have

\[
\lim_{s \to t} E_t \left[ - \left( 1 + 2\alpha_s \right) M_s (Y_s^*)^2 + (1 + \alpha_s) N_s Y_s^* - \Gamma_s (Y_s^*)^2 \right] = - \left( 1 + 2\alpha_t \right) M_t \lim_{s \to t} \left( Y_s^* \right)^2 + (1 + \alpha_t) N_t Y_t^* - \Gamma_t (Y_t^*)^2
\]

by Dominated Convergence

\[
= - (1 + 2\alpha_t) M_t (Y_t^*)^2 + (1 + \alpha_t) N_t (Y_t^*)^2 - \Gamma_t (Y_t^*)^2
\]

\[
\left[ -M_t + N_t - \Gamma_t - (2M_t - N_t) \alpha_t \right] (Y_t^*)^2
\]

\[
\lim_{s \to t} \left( \alpha_s \right) M_s = 0 \quad (4.37)
\]

and also \( \theta \) is bounded, thus we deduce that

\[
\lim_{s \to t} E_t \left[ \Lambda_s^T \right] = 0 \quad (4.38)
\]

we also have \( \hat{r}_t + \alpha_t |\theta_t|^2 = 0, \forall t \in [0, T) \) which implies that

\[
\left( \hat{r}_t + \alpha_t |\theta_t|^2 \right) Y_t^* = 0, \forall t \in [0, T)
\]

\[
\Rightarrow \hat{r}_t Y_t^* + \alpha_t \theta_t^T \theta_t Y_t^* = 0, \forall t \in [0, T)
\]

\[
\Rightarrow \hat{r}_t Y_t^* + (u_t^*)^T \theta_t = 0, \forall t \in [0, T)
\]

which means \((u^*, Y^*)\) satisfies

\[
\hat{r}_t Y_t^* + (u_t^*)^T \theta_t = 0, \forall t \in [0, T) \quad (4.39)
\]

Then \((4.33), (4.35), (4.38)\) and \((4.40)\) are exactly the required conditions in \((4.9)\) and we deduce that \(u^*\) is an equilibrium control for the family of problems in \((4.3)\) by proposition \((4.2)\). \(\square\)

### 4.4 A particular solution to our problem

Although we had not managed to proved the general conditions for the existence of the solutions for \((4.24)\), we found two particular solutions to \((4.24)\) as follows.

We have got by \((4.23)\) that \(\Gamma_s = e^T \int_s^t \hat{r}_u du\). Here we rearrange \((4.19)\) to get

\[
(2M_s - N_s) \alpha_s = - M_s + N_s - \Gamma_s
\]

which is equivalent to

\[
- (2\alpha_s + 1) M_s + (\alpha_s + 1) N_s - \Gamma_s = 0
\]

\[
(4.42)
\]

We could set two constant solutions for \(\alpha\), i.e. \(\alpha_s = -\frac{1}{2}\) or \(-1\) for \(s \in [0, T]\), to solve the system of ODEs \((4.24)\).

Firstly we set \(\alpha_s = -\frac{1}{2}\) for \(s \in [0, T]\), then by \((4.12)\) we get

\[
N_s = 2\Gamma_s = 2e^T \int_s^t \hat{r}_u du \quad (4.43)
\]
if we plug (4.43) and \( \alpha_s = -\frac{1}{2} \) back into (4.22), we get
\[
\begin{cases}
(\hat{r}_s - \frac{1}{2}|\theta_s|^2) N_s = 0, & s \in [0, T] \\
N_T = 2
\end{cases}
\]
(4.44)
then if we set \( \hat{r} = \frac{1}{2}|\theta|^2 \), i.e. we set our required return \( \mu = r - \frac{1}{2}|\theta|^2 \), we have that \( N_s = 2e^{\int_s^T \hat{r}_u du} \) is a solution to (4.22). Now we plug \( \hat{r} = \frac{1}{2}|\theta|^2 \) and \( \alpha_s = -\frac{1}{2} \) into (4.21) and get
\[
\begin{cases}
M_s' + (\hat{r}_s + \frac{1}{4}|\theta_s|^2) M_s = 0, & s \in [0, T] \\
M_T = 1
\end{cases}
\]
(4.45)
by solving which we get \( M_s = e^{\int_s^T \hat{r}_u du} \), thus we get a solution for the system of ODEs (4.24) as follows
\[
\begin{cases}
M_s' + (\hat{r}_s + \frac{1}{4}|\theta_s|^2) M_s = 0, & s \in [0, T] \\
M_T = 1
\end{cases}
\]
(4.46)
and we verify that
\[
\hat{r}_t Y_t^* + (u_t^*)^T \theta_t = (\hat{r}_t - \frac{1}{2}|\theta_t|^2) Y_t^* = 0, \forall t \in [0, T)
\]
(4.48)
(4.46) and (4.48) deduce that \( u^* \) in (4.47) is an equilibrium control for the family of problems in (4.3) by theorem 4.3.

If we plug \( u_s^* = e^{\int_s^T \hat{r}_u du} \) into (4.47), then we get
\[
e^{\int_s^T \mu_{\theta, \sigma} du} \pi_s^* \]
(4.49)
and thus get
\[
\pi_s^* = -\frac{1}{2} \left( \sigma^{-1} \right)^T \theta_s X_s^*
\]
(4.50)
This says that when we have our required return \( \mu = r - \frac{1}{2}|\theta|^2 \), we could find an equilibrium control \( \pi^* = -\frac{1}{2} \left( \sigma^{-1} \right)^T \theta X^* \) for the family of problems (4.1), although it sounds a bit unusual as our required return \( \mu \) is below the risk free rate \( r \).

If we set \( \alpha_s = -1 \) then we could not deduce that the resulting \( u^* \) is equilibrium by our theorem, which is explained as follows. We set \( \alpha_s = -1 \) for \( s \in [0, T] \), then by (4.42) we get
\[
\begin{cases}
M_s' = \Gamma_s \\
M_T = e^{\int_s^T \hat{r}_u du}
\end{cases}
\]
(4.51)
if we plug (4.51) and \( \alpha_s = -1 \) back into (4.44), we get
\[
\begin{cases}
(\hat{r}_s - |\theta_s|^2) M_s = 0, & s \in [0, T] \\
M_T = 1
\end{cases}
\]
(4.52)
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Again we set $\hat{r} = \frac{1}{2} |\theta|^2$, i.e. we set our required return $\mu = r - \frac{1}{2} |\theta|^2$ which happens to be the same as that in the case $\alpha_s = -\frac{1}{2}$, we have that $M_s = e^{T_s \hat{r}_s du}$ is a solution to (4.21). Now we plug $\hat{r} = \frac{1}{2} |\theta|^2$ and $\alpha_s = -1$ into (4.22) and get

$$\begin{align*}
N_s' &= 0, \quad s \in [0, T] \\
N_T &= 2
\end{align*}$$

(4.53)

by solving which we get $N_s = 2$, thus we get a solution for the system of ODEs (4.24) as follows

$$\begin{align*}
M &= e^{T_s \hat{r}_s du}, \quad s \in [0, T] \\
N &= 2, \quad s \in [0, T]
\end{align*}$$

(4.54)

in this case, by (4.18) we get

$$u_s^* = -\theta_s Y_s^*$$

(4.55)

and we verify that

$$\begin{align*}
\hat{r}_t Y_t^* + (u_t^*)^T \theta_t &= \left(\hat{r}_t - |\theta_t|^2\right) Y_t^* \\
&= -\frac{1}{2} |\theta_t|^2 Y_t^* \\
&\neq 0, \exists t \in [0, T)
\end{align*}$$

(4.56)

(4.56) implies that we cannot deduce $u^*$ in (4.55) is an equilibrium control for the family of problems in (4.1) by our theorem 4.4. Since for our required return $\mu = r - \frac{1}{2} |\theta|^2$ we have already found an equilibrium control $\pi^* = -\frac{1}{2} (\sigma^{-1})^T \theta X^*$ for the family of problems (4.1), thus the result here does not matter.
5 Utility function $h(x) = \frac{x^4}{4}$ for strong risk aversion

Some investors have strong risk aversion and they would like to use a kind of utility function that we use here. In this section, we want to solve the portfolio selection problem when we choose to use $h(x) = \frac{x^4}{4}$ as the utility function. That means we want to solve the family of following problems for any $t \in [0, T)$

$$\min_{\pi} \mathbb{E}_t \left[ \frac{(X_T - X_t e^{\int_t^T \mu_s ds})^4}{4} \right]$$

s.t.

$$dX_t = (r_s X_t + \pi_s^T \sigma_s \theta_s) ds + \pi_s^T \sigma_s dW_s$$

$$x_t = x_t$$

$$\pi \in \mathbb{U}_{ad}^s = \left\{ \pi \mid \pi \in L_2^T (0, T; \mathbb{R}^d) \text{ and } \mathbb{E}_t [X_T] = X_t e^{\int_t^T \mu_s ds}, \forall t \in [0, T) \right\}$$

where $\mu$ as usual is our required return process which is bounded and deterministic.

5.1 Transformation of our problem

Here we again use same approach which is used above for $h(x) = -\frac{x^3}{3}$ to solve our problem. Again by letting for any $s \in [0, T]$

$$Y_s = X_se^{\int_t^T \mu_s ds}$$

we have the family of following problems for any $t \in [0, T)$, which is equivalent to the above family of problems \(5.1\)

$$\min_{u} \mathbb{E}_t \left[ \frac{(Y_T - Y_t)^4}{4} \right]$$

s.t.

$$dY_s = (\tilde{r}_s Y_s + u_s^T \theta_s) ds + u_s^T dW_s$$

$$Y_t = y_t$$

$$u \in \mathbb{U}_{ad}^s = \{ u \mid u \in L_2^T (0, T; \mathbb{R}^d) \text{ and } \mathbb{E}_t [Y_T] = Y_t, \forall t \in [0, T) \}$$

where $\tilde{r}_s = r_s - \mu_s$, $u_s = e_s^T \mu_u d\sigma_s T \pi_s$, $y_t = x_t e^{\int_t^T \mu_s ds}$.

$\mathbb{E}_t [Y_T] = Y_t, \forall t \in [0, T)$ again implies that $Y$ must be a martingale which is an admissible constraint on $u$ and thus as usual we could firstly consider the family of following problems

$$\min_{u} \mathbb{E}_t \left[ \frac{(Y_T - Y_t)^4}{4} \right]$$

s.t.

$$dY_s = (\tilde{r}_s Y_s + u_s^T \theta_s) ds + u_s^T dW_s$$

$$Y_t = y_t$$

$$u \in \mathbb{U}_{ad}^s = \{ u \mid u \in L_2^T (0, T; \mathbb{R}^d) \}$$

Proposition 5.1. If an equilibrium solution $(u^*, Y^*)$ to the above family of problems \(5.3\) satisfies $\tilde{r}_s Y_s^* + (u_s^*)^T \theta_s = 0$ for any $s \in [0, T)$, then $(u^*, Y^*)$ is also equilibrium for the family of problems in \(5.3\)

Proof. Suppose $(u^*, Y^*)$ is an equilibrium solution to \(5.4\) which satisfies $\tilde{r}_s Y_s^* + (u_s^*)^T \theta_s = 0$ for any $s \in [0, T)$, then we have that $dY_s^* = u_s^* dW_s, \forall s \in [0, T)$ which implies that $Y^*$ is a martingale. So we have $\mathbb{E}_t [Y_s^*] = Y_t^*, \forall t \in [0, T)$, which as before is used together with definition \(5.3\) to deduce that $(u^*, Y^*)$ is also equilibrium for the family of problems \(5.3\) \(\square\)

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By the above proposition, we try to solve the family of problems (5.4) instead, that is

\[
\min_u \mathbb{E}_t [h (Y_T - Y_t)]
\]

\[
dY_s = (\dot{r}_s Y_s + u_s^T \theta_s) \, ds + u_s^T \, dW_s \\
Y_t = y_t \\
u \in U_t = \{ u \mid u \in L^2_T (0, T; \mathbb{R}^d) \}
\]

(5.5)

where \( h (x) = \frac{x^2}{2} \), and this is once again a family of problems of the form in (2.12), so we have the following system of BSDEs by (2.14) and (2.15)

\[
M, N, \quad \text{where} \quad \mu^x \text{ is deterministic}
\]

Following the assumptions we have made, \( \hat{r} \) and \( \sigma \) are deterministic and bounded, \( \mu^x \) is bounded. Here we could get

\[
\begin{aligned}
\min_u \mathbb{E}_t [h (Y_T - Y_t)] \\
\mathbb{E}_t \int_t^T |\Lambda_t^s| \, ds < \infty \quad \text{and } \lim_{s \uparrow t} \mathbb{E}_t [\Lambda_t^s] = 0 \quad \text{where } \Lambda_t^s = p_t^s \theta_s + q_t^s \\
\hat{r}_t Y_t^s + (u_t^*)^T \theta_t = 0
\end{aligned}
\]

(5.8)

then \( u^* \) is an equilibrium control for the family of problems (5.5).

**Proof.** Since one of the sufficient condition for equilibrium \( \mathbb{E}_t \left[ \frac{d^2 h (Y_t^*, Y_s^*)}{d s^2} \right] = 3 \mathbb{E}_t \left[ (Y_t^* - Y_s^*)^2 \right] \geq 0 \) in (2.23) under theorem 2.4 has already been satisfied. Then by combining theorem 2.4 and proposition 5.1 we deduce that \( u^* \) is an equilibrium for the family of problems (5.3).

\[\square\]

### 5.2 Details of finding a potential equilibrium

By the assumptions we have made, \( \hat{r} \) and \( \sigma \) are deterministic and bounded, \( \mu^x \) is bounded. Here we also assume that \( \mu^x \) is deterministic, i.e. \( \theta \) is deterministic and bounded. Then for any \( t \in [0, T) \), we make the following Ansatz

\[
p_t^s = M_s (Y_t^*)^3 - N_s (Y_t^*)^2 Y_t^* + \Gamma_s Y_t^* (Y_t^*)^2 - \Phi_s (Y_t^*)^3 \quad s \in [t, T]
\]

(5.9)

where \( M, N, \Gamma, \Phi \) are deterministic functions which are differentiable with \( M_T = 1, N_T = 3, \Gamma_T = 3, \Phi_T = 1 \).

By applying Itô formula to (5.9) with respect to \( s \) we could get

\[
dp_t^s = d \left[ M_s (Y_t^*)^3 \right] - d \left[ N_s (Y_t^*)^2 Y_t^* \right] + d \left[ \Gamma_s Y_t^* (Y_t^*)^2 \right] - d \left[ \Phi_s (Y_t^*)^3 \right]
\]

\[
= \left[ 3M_s (Y_t^*)^2 - 2N_s Y_t^* Y_t^* + \Gamma_s (Y_t^*)^2 \right] \left( \hat{r}_s Y_t^* + (u_t^*)^T \theta_s \right) \, ds
\]

\[
+ \left[ M_s (Y_t^*)^3 - N_s (Y_t^*)^2 Y_t^* + Y_t^* (Y_t^*)^2 \Gamma_s - \Phi_s (Y_t^*)^3 \right] \, ds
\]

\[
+ \left[ 3M_s Y_t^* (u_t^*)^T u_t^* - N_s (u_t^*)^T \right] \, ds
\]

(5.10)

\[
\text{by comparing the } dW \text{ terms of } dp^t \text{ in (5.6) and (5.10), we get that}
\]

\[
q_t^s = \left[ 3M_s (Y_t^*)^2 - 2N_s Y_t^* Y_t^* + (Y_t^*)^2 \Gamma_s \right] u_t^* \quad s \in [t, T]
\]

(5.12)
we again hope to find a possible linear feedback $u^*$ and as before try by setting

$$0 = \Lambda_s^* = p_s^* \theta_s + q_s^*, \ s \in [0, T]$$

(5.13)

which leads to the equation

$$[(M_s - N_s + \Gamma_s - \Phi_s) Y^*_s \theta_s + (3M_s - 2N_s + \Gamma_s) u^*_s] (Y^*_s)^2 = 0$$

(5.14)

from which we get

$$u^*_s = \alpha_s \theta_s Y^*_s$$

(5.15)

where

$$\alpha_s = \begin{cases} \frac{M_s-N_s+\Gamma_s-\Phi_s}{-3M_s+2N_s-\Gamma_s}, & \forall s \in [0, T) \\ \lim_{s \to T} \frac{M_s-N_s+\Gamma_s-\Phi_s}{-3M_s+2N_s-\Gamma_s}, & s = T \end{cases}$$

(5.16)

based on the assumption that $-3M_s + 2N_s - \Gamma_s \neq 0, \forall s \in [0, T)$ and $\lim_{s \to T} \frac{M_s-N_s+\Gamma_s-\Phi_s}{-3M_s+2N_s-\Gamma_s}$ exists.

By comparing the $ds$ terms of $dp^t$ in (5.6) and (5.10), we get that

$$-\hat{r}_s \left[ M_s (Y^*_s)^3 - N_s (Y^*_s)^2 Y^*_s + \Gamma_s Y^*_s (Y^*_s)^2 - \Phi_s (Y^*_s)^3 \right]$$

$$= \left[ 3M_s (Y^*_s)^2 - 2N_s Y^*_s + \Gamma_s (Y^*_s)^2 \right] \left( \hat{r}_s Y^*_s + (u^*_s)^T \theta_s \right)$$

$$+ \left[ M'_s (Y^*_s)^3 - N'_s Y^*_s (Y^*_s)^2 + Y^*_s (Y^*_s)^2 \Gamma_s' - \Phi'_s (Y^*_s)^3 \right]$$

$$+ \left[ 3M_s Y^*_s (u^*_s)^T u^*_s - N_s Y^*_s (u^*_s)^T u^*_s \right]$$

$$\iff -\hat{r}_s \left[ M_s (Y^*_s)^3 - N_s (Y^*_s)^2 Y^*_s + \Gamma_s Y^*_s (Y^*_s)^2 - \Phi_s (Y^*_s)^3 \right]$$

$$= \left[ 3M_s (Y^*_s)^2 - 2N_s Y^*_s + \Gamma_s (Y^*_s)^2 \right] \left( \hat{r}_s Y^*_s + \alpha_s |\theta_s|^2 Y^*_s \right)$$

$$+ \left[ M'_s (Y^*_s)^3 - N'_s Y^*_s (Y^*_s)^2 + Y^*_s (Y^*_s)^2 \Gamma_s' - \Phi'_s (Y^*_s)^3 \right]$$

$$+ \left[ 3M_s Y^*_s - N_s Y^*_s \alpha_s |\theta_s|^2 (Y^*_s)^2 \right]$$

$$\iff -\hat{r}_s M_s (Y^*_s)^3 + \hat{r}_s N_s (Y^*_s)^2 Y^*_s - \hat{r}_s \Gamma_s Y^*_s (Y^*_s)^2 + \hat{r}_s \Phi_s (Y^*_s)^3$$

$$= \left[ M'_s + 3M_s \hat{r}_s + 3\alpha_s |\theta_s|^2 M_s + 3\alpha_s^2 |\theta_s|^2 M_s \right] (Y^*_s)^3$$

$$- \left[ N'_s + 2N_s \hat{r}_s + 2\alpha_s |\theta_s|^2 N_s + \alpha_s^2 |\theta_s|^2 N_s \right] Y^*_s (Y^*_s)^2$$

$$+ \left[ \Gamma'_s + \Gamma_s \hat{r}_s + \alpha_s |\theta_s|^2 \Gamma_s \right] Y^*_s (Y^*_s)^2 - \Phi'_s (Y^*_s)^3$$

$$= \left[ M'_s + 3M_s \hat{r}_s + 3\alpha_s |\theta_s|^2 M_s + 3\alpha_s^2 |\theta_s|^2 M_s \right] (Y^*_s)^3$$

$$- \left[ N'_s + 2N_s \hat{r}_s + 2\alpha_s |\theta_s|^2 N_s + \alpha_s^2 |\theta_s|^2 N_s \right] Y^*_s (Y^*_s)^2$$

$$+ \left[ \Gamma'_s + \Gamma_s \hat{r}_s + \alpha_s |\theta_s|^2 \Gamma_s \right] Y^*_s (Y^*_s)^2 - \Phi'_s (Y^*_s)^3$$
after rearrangement we get

\[
0 = (\Phi_s' + \hat{r}_s \Phi_s)(Y_s^*)^3 - \left[\Gamma_s' + 2\hat{r}_s \Gamma_s + \alpha_s |\theta_s|^2 \Gamma_s \right] Y_s^* (Y_s^*)^2 \\
+ \left[N_s' + 3\hat{r}_s N_s + 2\alpha_s |\theta_s|^2 N_s + \alpha_s^2 |\theta_s|^2 N_s \right] (Y_s^*)^2 Y_t^* \\
- \left[M_s' + 4\hat{r}_s M_s + 3\alpha_s |\theta_s|^2 M_s + 3\alpha_s^2 |\theta_s|^2 M_s \right] (Y_s^*)^3
\] (5.17)

which leads to the following system of ODEs

\[
\begin{cases}
M_s' + \left(4\hat{r}_s + 3\alpha_s |\theta_s|^2 + 3\alpha_s^2 |\theta_s|^2 \right) M_s = 0, & s \in [0, T] \\
M_T = 1
\end{cases}
\] (5.18)

\[
\begin{cases}
N_s' + \left(3\hat{r}_s + 2\alpha_s |\theta_s|^2 + \alpha_s^2 |\theta_s|^2 \right) N_s = 0, & s \in [0, T] \\
N_T = 3
\end{cases}
\] (5.19)

\[
\begin{cases}
\Gamma_s' + \left(2\hat{r}_s + \alpha_s |\theta_s|^2 \right) \Gamma_s = 0, & s \in [0, T] \\
\Gamma_T = 3
\end{cases}
\] (5.20)

\[
\begin{cases}
\Phi_s' + \hat{r}_s \Phi_s = 0, & s \in [0, T] \\
\Phi_T = 1
\end{cases}
\] (5.21)

the solution to equation (5.21) is \( \Phi_s = e^{\int_s^T \hat{r}_s \, du} \), which makes the unsettled system contains (5.18), (5.19) and (5.20) as follows

\[
\begin{cases}
M_s' + \left(4\hat{r}_s + 3\alpha_s |\theta_s|^2 + 3\alpha_s^2 |\theta_s|^2 \right) M_s = 0, & s \in [0, T] \\
M_T = 1 \\
N_s' + \left(3\hat{r}_s + 2\alpha_s |\theta_s|^2 + \alpha_s^2 |\theta_s|^2 \right) N_s = 0, & s \in [0, T] \\
N_T = 3 \\
\Gamma_s' + \left(2\hat{r}_s + \alpha_s |\theta_s|^2 \right) \Gamma_s = 0, & s \in [0, T] \\
\Gamma_T = 3
\end{cases}
\] (5.22)

5.3 Conditions for obtaining an equilibrium for our problem

**Theorem 5.3.** If the system of ODEs (5.22) admits a solution \((M, N)\) s.t. the corresponding \(\alpha\) defined in (5.16) satisfies that \(\hat{r}_t + \alpha_t |\theta_t|^2 = 0, \forall t \in [0, T]\), then \(u^*\) is an equilibrium control for the family of problems in (7.3).

**Proof.** Suppose \((M, N)\) is a solution to (5.22) s.t. \(\hat{r}_t + \alpha_t |\theta_t|^2 = 0, \forall t \in [0, T]\). Since the deterministic \(M, N, \Gamma\) are continuous and thus are bounded on \([0, T]\), we have that \(\alpha\) is also deterministic and bounded on \([0, T]\) according to (5.10). We have in (5.10) for \(s \in [0, T]\) that

\[
u_s^* = \alpha_s \theta_s Y_s^*
\] (5.23)

and thus we have

\[
dY_s^* = \left(\hat{r}_s Y_s^* + (u_s^*)^T \theta_s \right) ds + (u_s^*)^T dW_s
\] (5.24)

which leads to

\[
Y_t^* = y_0 e^{\int_0^t \hat{r}_s ds + \frac{1}{2} \alpha_s^2 |\theta_s|^2 ds + \int_0^t \alpha_s(s) \, dW_s}
\] (5.25)
thus

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} |Y^*_t|^3 \right] = \mathbb{E} \left[ \sup_{t \in [0,T]} \left| y_0 e^{\int_0^t \left( \hat{r}_r + \alpha_s(\theta) \right) ds} \right|^3 \left| e^{\int_0^t - \frac{3}{2} \alpha_s^2(\theta_s) ds + \int_0^t \alpha_s(\theta) dW_s} \right|^3 \right]
\]

\[
\leq \left\{ \sup_{t \in [0,T]} \left| y_0 e^{\int_0^t \left( \hat{r}_r + \alpha_s(\theta) \right) ds} \right|^3 \right\} \mathbb{E} \left[ \sup_{t \in [0,T]} \left( e^{\int_0^t - \frac{3}{2} \alpha_s^2(\theta_s) ds + \int_0^t \alpha_s(\theta) dW_s} \right)^3 \right]
\]

\[
\leq \left\{ \sup_{t \in [0,T]} \left| y_0 e^{\int_0^t \left( \hat{r}_r + \alpha_s(\theta) \right) ds} \right|^3 \right\} \frac{27}{8} \mathbb{E} \left[ \left( e^{\int_0^T - \frac{3}{2} \alpha_s^2(\theta_s) ds + \int_0^T \alpha_s(\theta) dW_s} \right)^3 \right]
\]

by $L^p$-Maximal Inequality

\[
= \frac{27}{8} e^{\int_0^T 3 \alpha_s^2(\theta_s) ds} \left\{ \sup_{t \in [0,T]} \left| y_0 e^{\int_0^t \left( \hat{r}_r + \alpha_s(\theta) \right) ds} \right|^3 \right\} \mathbb{E} \left[ e^{\int_0^T - \frac{3}{2} \alpha_s^2(\theta_s) ds + \int_0^T 3 \alpha_s(\theta) dW_s} \right]
\]

\[
< \infty
\]

(5.26)

as $\alpha, \theta, \hat{r}$ are bounded, which implies that

\[
Y^* \in L^3_{\mathbb{F}} \left( \Omega; C \left( 0, T; \mathbb{R} \right) \right)
\]

(5.27)

and thus we have

\[
\mathbb{E} \left[ \int_0^T |u^*_s|^2 ds \right] = \mathbb{E} \left[ \int_0^T \alpha_s^2(\theta_s) (Y^*_s)^2 ds \right]
\]

\[
\leq \sup_{s \in [0,T]} \left( \alpha_s^2(\theta_s)^2 \right) \int_0^T \mathbb{E} \left[ (Y^*_s)^2 \right] ds
\]

(5.28)

\[
\leq \sup_{s \in [0,T]} \left( \alpha_s^2(\theta_s)^2 \right) \int_0^T \left( \mathbb{E} \left[ |Y^*_s|^3 \right] \right)^{\frac{2}{3}} ds, \text{ by Holder's}
\]

(5.29)

\[
\leq \sup_{s \in [0,T]} \left( \alpha_s^2(\theta_s)^2 \right) \int_0^T \left( \mathbb{E} \left[ \sup_{u \in [0,T]} |Y^*_u|^3 \right] \right)^{\frac{2}{3}} ds
\]

\[
= \sup_{s \in [0,T]} \left( \alpha_s^2(\theta_s)^2 \right) \left( \mathbb{E} \left[ \sup_{u \in [0,T]} |Y^*_u|^3 \right] \right)^{\frac{2}{3}} T
\]

(5.30)

(5.31)

which implies that

\[
u^* \in L^3_{\mathbb{F}} \left( 0, T; \mathbb{R}^d \right)
\]

(5.32)

also we have that for any $t \in [0,T)$

\[
A^t_s = p_s^t \theta_s + q_s^t
\]

\[
= \left[ M_s (Y^*_s)^3 - N_s (Y^*_s)^2 Y^*_s + \Gamma_s (Y^*_s)^2 - \Phi_s (Y^*_s)^3 \right] \theta_s
\]

\[
+ \left[ 3M_s (Y^*_s)^2 - 2N_s Y^*_s + (Y^*_s)^2 \Gamma_s \right] u^*_s
\]

\[
= \left[ M_s (Y^*_s)^3 - N_s (Y^*_s)^2 Y^*_s + \Gamma_s (Y^*_s)^2 - \Phi_s (Y^*_s)^3 \right] \theta_s
\]

\[
+ \left[ 3\alpha_s M_s (Y^*_s)^3 - 2\alpha_s N_s (Y^*_s)^2 + \alpha_s (Y^*_s)^2 \Gamma_s \right] \theta_s
\]

\[
= \left[ 1 + 3\alpha_s \right] M_s (Y^*_s)^3 - (1 + 2\alpha_s) N_s Y^*_s (Y^*_s)^2 + (1 + \alpha_s) \Gamma_s (Y^*_s)^2 Y^*_s - \Phi_s (Y^*_s)^3 \right] \theta_s
\]

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Since $\alpha, M, N, \Gamma, \theta$ are bounded, it is clearly by (5.34) that
\[
E_t \int_t^T |A^*_t| \, ds < \infty
\] (5.33)
and
\[
\lim_{s \downarrow t} E_t \left[ A^*_t \right] = \lim_{s \downarrow t} \theta_t E_t \left[ (1 + 3\alpha_s) M_s (Y^*_s)^3 - (1 + 2\alpha_s) N_s Y^*_s (Y^*_s)^2 + (1 + \alpha_s) \Gamma_s (Y^*_s)^2 Y^*_s - \Phi_s (Y^*_s)^3 \right]
\] (5.34)

since we have
\[
\lim_{s \downarrow t} E_t \left[ (1 + 3\alpha_s) M_s (Y^*_s)^3 - (1 + 2\alpha_s) N_s Y^*_s (Y^*_s)^2 + (1 + \alpha_s) \Gamma_s (Y^*_s)^2 Y^*_s - \Phi_s (Y^*_s)^3 \right] = \lim_{s \downarrow t} \left\{ (1 + 3\alpha_s) M_s E_t \left[ (Y^*_s)^3 \right] - (1 + 2\alpha_s) N_s Y^*_s E_t \left[ (Y^*_s)^2 \right] + (1 + \alpha_s) \Gamma_s (Y^*_s)^2 E_t \left[ Y^*_s \right] - \Phi_s (Y^*_s)^3 \right\}
\]
(5.35)

by Dominated Convergence
\[
\left[ (1 + 3\alpha_t) M_t - (1 + 2\alpha_t) N_t + (1 + \alpha_t) \Gamma_t - \Phi_t \right] (Y^*_t)^3
\]
\[
= \frac{M_t - N_t + \Gamma_t - \Phi_t}{-3M_t + 2N_t - \Gamma_t - \alpha_t} \left( -3M_t + 2N_t - \Gamma_t \right) (Y^*_t)^3
\]
(5.36)
and also $\theta$ is bounded, thus we deduce that
\[
\lim_{s \downarrow t} E_t \left[ A^*_t \right] = 0
\]
(5.37)
we also have $\tilde{r}_t + \alpha_t |\theta_t|^2 = 0, \forall t \in [0, T)$ which implies that
\[
\left( \tilde{r}_t + \alpha_t |\theta_t|^2 \right) Y^*_t = 0, \forall t \in [0, T)
\]
\[
\Rightarrow \tilde{r}_t Y^*_t + \alpha_t \theta_t^T \theta_t Y^*_t = 0, \forall t \in [0, T)
\]
\[
\Rightarrow \tilde{r}_t Y^*_t + (u^*_t)^T \theta_t = 0, \forall t \in [0, T)
\]
(5.38)
which means $(u^*, Y^*)$ satisfies
\[
\tilde{r}_t Y^*_t + (u^*_t)^T \theta_t = 0, \forall t \in [0, T)
\]
(5.39)
Then (5.24), (5.34), (5.36) and (5.38) are exactly the required conditions in (5.3) and we deduce that $u^*$ is an equilibrium control for the family of problems in (5.3) by proposition 5.2.

From the studies on $h(x) = -\frac{x^3}{3} - \frac{x^4}{4}$, we could see that in solving the problem for these two power utility functions there is a sort of regular pattern in the systems of ODEs (5.24) and (5.24) obtained above which can be extended to the same problem for higher order power functions.
6 Utility function $h(x) = x^-$ for stronger risk aversion

Some investors are risk averse to the extent that they hope to make the under-performs down to the lowest level. Thus they would like to use the utility function we use here. In this section, we change our view from the previous sections to a different one which makes this section look like that it is not related to the previous ones.

6.1 Notations and definition of equilibrium for this section

Firstly we define $\mathbb{P}$ as our physical measure and $\mathbb{Q}$ as the risk neutral measure with

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E} \left( -\int_0^t \theta_s dW_s \right)$$

(6.1)

In this section, when we define sets $L^2_{\mathbb{P}}(t,T;\mathbb{R}^d)$ and $L^2_{\mathbb{Q}}(\Omega;\mathbb{R}^d)$ which contain the elements that satisfy the corresponding conditions under both of the probability measure $\mathbb{P}$ and $\mathbb{Q}$. That is

$$\begin{cases}
L^2_{\mathbb{P}}(t,T;\mathbb{R}^d) : & \text{the set of } \{F_s\}_{s \in [t,T]} \text{-adapted processes } f = \{f_s : t \leq s \leq T\} \\
& \text{with } \mathbb{E}^\mathbb{P} \left[ \int_t^T |f_s|^2 ds \right] < \infty \text{ and } \mathbb{E}^\mathbb{Q} \left[ \int_t^T |f_s|^2 ds \right] < \infty \\
L^2_{\mathbb{Q}}(\Omega;\mathbb{R}^d) : & \text{the set of random variables } \xi : (\Omega,\mathcal{G}) \to (\mathbb{R}^d,\mathcal{B} (\mathbb{R}^d)) \\
& \text{with } \mathbb{E}^\mathbb{P} \left[ |\xi|^2 \right] < \infty \text{ and } \mathbb{E}^\mathbb{Q} \left[ |\xi|^2 \right] < \infty
\end{cases}$$

For the reason of simplicity, we just write $\mathbb{E} [\cdot]$ to represent $\mathbb{E}^\mathbb{P} [\cdot]$ in the remaining part of the section.

In this section we want to solve the family of following problems

$$\min_u \mathbb{E}_t \left[ X_T - X_t e^{\int_t^T \mu_s ds} \right]$$

s.t.

$$\begin{cases}
\begin{align}
\ dX_s &= (r_s X_s + \pi_s^T \sigma_s \theta_s) \, ds + \pi_s^T \sigma_s dW_s \\
\ X_t &= x_t \\
\ \pi &\in U^\pi_{ad} = \left\{ \pi \mid \pi \in L^2_{\mathbb{P}}(0,T;\mathbb{R}^d) \text{ and } \mathbb{E}_t [X_T] = X_t e^{\int_t^T \mu_s ds}, \forall t \in [0,T) \right\}
\end{align}
\end{cases}$$

(6.2)

where we assume our drift rate of risky assets $\mu^\pi$ is deterministic and thus based on our assumptions made above we have that $r$, $\sigma$ and $\theta$ are bounded and deterministic. Here $\mu$ as usual is our required return process which is assumed to be bounded and deterministic.

Then by letting $u = \sigma^T \pi$ the above family of problems is equivalent to

$$\min_u \mathbb{E}_t \left[ X_T - X_t e^{\int_t^T u_s ds} \right]$$

s.t.

$$\begin{cases}
\ dX_s &= (r_s X_s + u_s^T \theta_s) \, ds + u_s^T dW_s \\
\ X_t &= x_t \\
\ u &\in U^u_{ad} = \left\{ u \mid u \in L^2_{\mathbb{P}}(0,T;\mathbb{R}^d) \text{ and } \mathbb{E}_t [X_T] = X_t e^{\int_t^T \mu_s ds}, \forall t \in [0,T) \right\}
\end{cases}$$

(6.3)

Also in this section we use the following definition of equilibrium which is different from the one used by previous sections.
Given a control \( u^* \), for any \( t \in [0, T) \), \( \varepsilon > 0 \) and \( v \in L^2_{\mathcal{F}} (t, t + \varepsilon; \mathbb{R}^d) \), we define
\[
u^{t,\varepsilon,v}_s = u^*_s + v_s 1_{s \in [t, t + \varepsilon)}, \quad s \in [t, T] \tag{6.4}\]

\[\text{Definition 6.1.} \quad \text{Let } u^* \in U_{ad} \text{ be a given control with } U_{ad} \text{ being the set of admissible controls. Let } X^* \text{ be the state process corresponding to } u^*. \text{ The control } u^* \text{ is called an equilibrium if for any } t \in [0, T), \exists \delta > 0, \text{ s.t. for any } \varepsilon \in (0, \delta) \text{ and } v \in L^2_{\mathcal{F}} (t, t + \varepsilon; \mathbb{R}^d) \text{ s.t. } u^{t,\varepsilon,v}_s \in U_{ad}, \text{ we have that}\]
\[
J(t, X^*_t; u^{t,\varepsilon,v}_t) - J(t, X^*_t; u^*_t) \geq 0 \tag{6.5}
\]

where \( u^{t,\varepsilon,v}_t \) is defined by (6.4).

### 6.2 Details of finding an equilibrium

Firstly we consider the family of following problems
\[
\min_u \mathbb{E}_t \left[ \left( X_t e^{\int_t^T \mu_s ds} - X_T \right)^2 \right]
\]
\[
s.t. \quad \begin{align*}
dX_s &= (r_s X_s + u^*_s \theta_s) ds + u^*_s dW_s \\
X_t &= x_t \\
u_s &\in U^u_{ad} = \{ u | u \in L^2_{\mathcal{F}} (0, T; \mathbb{R}^d) \} \tag{6.6}
\end{align*}
\]

Let \( X^{t,\varepsilon,v}_t \) be the state process corresponding to \( u^{t,\varepsilon,v}_t \). We set
\[
\Delta^{s,\varepsilon,v}_s = X^{t,\varepsilon,v}_s - X^*_s \tag{6.7}
\]

then by letting
\[
\begin{align*}
\tilde{\Delta}^{s,\varepsilon,v}_s &= \Delta^{s,\varepsilon,v}_s e^{-\int_t^s r_u du} \\
\tilde{X}^s &= X^{t,\varepsilon,v}_s e^{-\int_t^s r_u du} \\
\tilde{u}^*_s &= u^*_s e^{-\int_t^s r_u du} \\
\bar{v}^*_s &= v^*_s e^{-\int_t^s r_u du}
\end{align*}
\]

we have
\[
d\Delta^{s,\varepsilon,v}_s = (\bar{v}^{t,\varepsilon,v}_s - \bar{u}^*_s)^T dW^Q_s \tag{6.8}
\]

and
\[
\tilde{\Delta}^{t,\varepsilon,v}_T = \int_t^T \tilde{v}_s^T dW^Q_s = \tilde{\Delta}^{t,\varepsilon,v}_{t+\varepsilon} \tag{6.9}
\]

and we have that
\[
\mathbb{E}_t^Q [\tilde{\Delta}^{t,\varepsilon,v}_{t+\varepsilon}] = 0
\]

and thus
\[
\mathbb{E}_t \left[ \tilde{\Delta}^{t,\varepsilon,v}_{t+\varepsilon} e^{-\int_t^{t+\varepsilon} \theta_s dW_s} \right] = 0 \tag{6.10}
\]
Then for any \( t \in [0, T) \), \( v \in L^2_{\mathcal{F}}(t, t + \varepsilon; \mathbb{R}^d) \) and \( \varepsilon \in (0, \delta) \) for some \( \delta > 0 \) we have

\[
J(t, X_t^*; u^{t, \varepsilon}; v) - J(t, X_t^*; u^*) = \mathbb{E}_t \left[ \left( X_t^* e^{\int_t^T \mu_s ds} - X_T^* \right)^+ - \left( X_t^* e^{\int_t^T \mu_s ds} - X_T^* \right)^+ \right]
\]

By definition \( 6.1 \) and \( 6.11 \), we could deduce that \( u^* \) is an equilibrium for our problem \( 6.6 \) if and only if for any \( t \in [0, T) \), \( \exists \delta > 0 \), s.t. for any \( \varepsilon \in (0, \delta) \) we have \( v = 0 \) is optimal to the following problem

\[
\begin{align*}
\min_v \quad & \mathbb{E}_t \left[ \left( X_t^* e^{\int_t^T \mu_s ds} - X_T^* - \Delta_{t, \varepsilon, v}^T \right)^+ - \left( X_t^* e^{\int_t^T \mu_s ds} - X_T^* \right)^+ \right] \\
\text{s.t.} \quad & \begin{cases}
\int d\Delta_{s}^{t, \varepsilon, v} = (\bar{u}_s^{t, \varepsilon, v} - \bar{u}^*_s)^T dW_s^Q \\
v \in L^2_{\mathcal{F}}(t, t + \varepsilon; \mathbb{R}^d)
\end{cases}
\end{align*}
\]

which is equivalent to say that \( v = 0 \) is optimal to the following problem for the given \( t \) and \( \varepsilon \)

\[
\begin{align*}
\min_v \quad & e^{-\int_t^T r_s ds} \mathbb{E}_t \left[ \left( X_t^* e^{\int_t^T \mu_s ds} - X_T^* - \Delta_{t, \varepsilon, v}^T \right)^+ \right] \\
\text{s.t.} \quad & \begin{cases}
\int d\Delta_{s}^{t, \varepsilon, v} = (\bar{u}_s^{t, \varepsilon, v} - \bar{u}^*_s)^T dW_s^Q \\
v \in L^2_{\mathcal{F}}(t, t + \varepsilon; \mathbb{R}^d)
\end{cases}
\end{align*}
\]

which can be written as

\[
\begin{align*}
\min_v \quad & \mathbb{E}_t \left[ \left( X_t^* e^{\int_t^T \mu_s ds} - X_T^* \right)^+ \right] \\
\text{s.t.} \quad & \begin{cases}
\int d\Delta_{s}^{t, \varepsilon, v} = (\bar{u}_s^{t, \varepsilon, v} - \bar{u}^*_s)^T dW_s^Q \\
v \in L^2_{\mathcal{F}}(t, t + \varepsilon; \mathbb{R}^d)
\end{cases}
\end{align*}
\]

by \( 6.9 \) which can also be written as

\[
\begin{align*}
\min_v \quad & \mathbb{E}_t \left[ \left( X_t^* e^{\int_t^T \mu_s ds} - X_T^* - \Delta_{t, \varepsilon}^T \right)^+ \right] \\
\text{s.t.} \quad & \begin{cases}
\bar{\Delta}_{t}^{t, \varepsilon} = f_t^{t+\varepsilon} \bar{v}_s^{t+\varepsilon} dW_s^Q \\
v \in L^2_{\mathcal{F}}(t, t + \varepsilon; \mathbb{R}^d)
\end{cases}
\end{align*}
\]

then the statement \( v = 0 \) is optimal to the problem \( 6.13 \) is equivalent to the statement that \( \bar{\Delta}_{t+\varepsilon} = 0 \) is optimal to the following problem

\[
\begin{align*}
\min_{\bar{\Delta}_{t+\varepsilon}} \quad & \mathbb{E}_t \left[ \left( X_t^* e^{\int_t^T \mu_s ds} - X_T^* \right)^+ \right] \\
\text{s.t.} \quad & \begin{cases}
\mathbb{E}_Q \left[ \bar{\Delta}_{t+\varepsilon} \right] = 0 \\
\bar{\Delta}_{t+\varepsilon} \in L^2_{\mathcal{F}_{t+\varepsilon}}(\Omega; \mathbb{R})
\end{cases}
\end{align*}
\]

which means that given any \( v \in L^2_{\mathcal{F}}(t, t + \varepsilon; \mathbb{R}^d) \) the corresponding \( \bar{\Delta}_{t+\varepsilon}^{t, \varepsilon, v} \) is an admissible \( \bar{\Delta}_{t+\varepsilon} \) in problem \( 6.10 \), and given any admissible \( \bar{\Delta}_{t+\varepsilon} \) in problem \( 6.10 \) we could find a \( v \in L^2_{\mathcal{F}}(t, t + \varepsilon; \mathbb{R}^d) \) s.t. \( \bar{\Delta}_{t+\varepsilon} = \bar{\Delta}_{t+\varepsilon}^{t, \varepsilon, v} = f_t^{t+\varepsilon} \bar{v}_s^{t+\varepsilon} dW_s^Q \). This is shown as follows:
Proof. On the one hand, we have as $\theta$ is bounded that

$$
\begin{align*}
\mathbb{E}^Q_t \left[ |\tilde{\Delta}_{t+\varepsilon}^{t,\varepsilon,v} |^2 \right] &= \mathbb{E}^Q_t \left[ \int_t^{t+\varepsilon} \tilde{v}^T_s dW_s^Q \right]^2 \\
\mathbb{E} \left[ |\tilde{\Delta}_{t+\varepsilon}^{t,\varepsilon,v} |^2 \right] &\leq 2 \mathbb{E} \left[ \int_t^{t+\varepsilon} \tilde{v}^T_s dW_s^2 \right] \leq 2 \mathbb{E} \left[ \int_t^{t+\varepsilon} |\tilde{v}^T_s |^2 d\theta_s^2 ds \right] (6.17) \\
\forall \varepsilon \in L_2^2 (t, t + \varepsilon ; \mathbb{R}^d) \Rightarrow \left\{ \begin{array}{l}
\mathbb{E}^Q_t \left[ |\tilde{\Delta}_{t+\varepsilon}^{t,\varepsilon,v} |^2 \right] = \mathbb{E}^Q_t \left[ \int_t^{t+\varepsilon} \tilde{v}^T_s dW_s^Q \right]^2 \\
\mathbb{E} \left[ |\tilde{\Delta}_{t+\varepsilon}^{t,\varepsilon,v} |^2 \right] \leq 2 \mathbb{E} \left[ \int_t^{t+\varepsilon} |\tilde{v}^T_s |^2 d\theta_s^2 ds \right]
\end{array} \right.
\end{align*}
$$

which means the corresponding $\tilde{\Delta}_{t+\varepsilon}^{t,\varepsilon,v} \in L_2^2 (\Omega ; \mathbb{R})$ and $\mathbb{E}^Q_t \left[ \tilde{\Delta}_{t+\varepsilon}^{t,\varepsilon,v} \right] = 0$, thus $\tilde{\Delta}_{t+\varepsilon}^{t,\varepsilon,v}$ is an admissible $\tilde{\Delta}_{t+\varepsilon}$ in problem (6.10).

On the other hand, $\forall \tilde{\Delta}_{t+\varepsilon} \in L_2^2 (\Omega ; \mathbb{R})$ with $\mathbb{E}^Q_t \left[ \tilde{\Delta}_{t+\varepsilon} \right] = 0$ we have $\tilde{\Delta}_{t+\varepsilon}$ is $\mathcal{F}_{t+\varepsilon}$ measurable and that

Firstly, $\mathbb{E} \left[ |\tilde{\Delta}_{t+\varepsilon} |^2 \right] < \infty$ implies the following Lipschitz BSDE

$$
\begin{align*}
\left\{ \begin{array}{l}
d\tilde{\Delta}_{t+\varepsilon}^{t,\varepsilon,v} = \tilde{v}_s^T 1_{s \in [t, t+\varepsilon)} (dW_s + \theta_s ds) \\
\tilde{\Delta}_{t+\varepsilon}^{t,\varepsilon,v} (\tilde{v}_1) = \tilde{\Delta}_{t+\varepsilon}
\end{array} \right.
\end{align*}
$$

admits a unique solution $(\tilde{\Delta}_{t+\varepsilon}^{t,\varepsilon,v}, \tilde{v}_1)$ with $\mathbb{E} \left[ \int_t^{t+\varepsilon} |(\tilde{v}_1)_s |^2 ds \right] < \infty$ under $\mathbb{P}$ s.t.

$$
\tilde{\Delta}_{t+\varepsilon} = \tilde{\Delta}_{t+\varepsilon}^{t,\varepsilon,v} = \int_t^{t+\varepsilon} (\tilde{v}_1)_s^T (dW_s + \theta_s ds) = \int_t^{t+\varepsilon} (\tilde{v}_1)_s^T dW_s^Q
$$

Secondly, $\mathbb{E}^Q \left[ |\tilde{\Delta}_{t+\varepsilon} |^2 \right] < \infty$ implies by using martingale representation theorem under $\mathbb{Q}$ that there exists a unique $\tilde{v}_2$ with $\mathbb{E}^Q \left[ \int_t^{t+\varepsilon} |(\tilde{v}_2)_s |^2 ds \right] < \infty$ s.t.

$$
\tilde{\Delta}_{t+\varepsilon} = \mathbb{E}^Q \left[ \tilde{\Delta}_{t+\varepsilon} \right] + \int_t^{t+\varepsilon} (\tilde{v}_2)_s^T dW_s^Q
$$

Then the above two equations for $\tilde{\Delta}_{t+\varepsilon}$ implies that $\tilde{v}_1 = \tilde{v}_2$, which means there exists a $v \in L_2^2 (t, t + \varepsilon ; \mathbb{R}^d)$ with $\tilde{v} = \tilde{v}_1 = \tilde{v}_2$ s.t. $\tilde{\Delta}_{t+\varepsilon} = \int_t^{t+\varepsilon} \tilde{v}^T_s dW_s^Q$. \hfill \Box
Then by (6.11), the problem (6.10) for the given \( t \) and \( \varepsilon \) can be written as

\[
\min_{\Delta_{t+\varepsilon}} \mathbb{E}_t \left[ \left( X^*_t e^{\int_t^T \mu_s - r_s ds} - \bar{X}_T^* - \Delta_{t+\varepsilon} \right)^+ \right]
\]

\[
\text{s.t.} \quad \left\{ \begin{array}{l}
\Delta_{t+\varepsilon} \in L^2_{\mathcal{F}_{t+\varepsilon}} (\Omega; \mathbb{R}) \\
\mathbb{E}_t \left[ \Delta_{t+\varepsilon} \mathcal{E} \left( - \int_t^{t+\varepsilon} \theta_s dW_s \right) \right] = 0
\end{array} \right.
\]

(6.18)

which could be transformed to the following problem using Lagrangian multiplier method

\[
\min_{\Delta_{t+\varepsilon}} \mathbb{E}_t \left[ \left( X^*_t e^{\int_t^T \mu_s - r_s ds} - \bar{X}_T^* - \Delta_{t+\varepsilon} \right)^+ - \lambda \Delta_{t+\varepsilon} \mathcal{E} \left( - \int_t^{t+\varepsilon} \theta_s dW_s \right) \right]
\]

\[
\text{s.t.} \quad \left\{ \Delta_{t+\varepsilon} \in L^2_{\mathcal{F}_{t+\varepsilon}} (\Omega; \mathbb{R}) \right. \]

(6.19)

so we have deduced that \( u^* \) is an equilibrium for our problem (6.6) if and only if for any \( t \in [0, T) \), \( \exists \delta > 0 \), s.t. for any \( \varepsilon \in (0, \delta) \) we have that \( \Delta_{t+\varepsilon} = 0 \) is optimal to the above problem (6.19).

Then problem (6.19) can be written as

\[
\min_{\Delta_{t+\varepsilon}} \mathbb{E}_t \left[ f(\Delta_{t+\varepsilon}) - \lambda \Delta_{t+\varepsilon} \mathcal{E} \left( - \int_t^{t+\varepsilon} \theta_s dW_s \right) \right]
\]

\[
\text{s.t.} \quad \left\{ \Delta_{t+\varepsilon} \in L^2_{\mathcal{F}_{t+\varepsilon}} (\Omega; \mathbb{R}) \right. \]

(6.20)

where \( f(y) = \left( X^*_t e^{\int_t^T \mu_s - r_s ds} - \bar{X}_T^* - y \right)^+ \) and let \( f(y, w) = \left( X^*_t e^{\int_t^T \mu_s - r_s ds} - \bar{X}_T^* - y \right)^+ (w), \forall w \in \Omega \)

since \( f(\Delta_{t+\varepsilon}(w), w) - \lambda \Delta_{t+\varepsilon}(w) \mathcal{E} \left( - \int_t^{t+\varepsilon} \theta_s dW_s \right) (w) \) is a convex and differentiable function with respect to \( \Delta_{t+\varepsilon}(w) \) for any \( w \in \Omega \), then we deduce that \( \Delta_{t+\varepsilon} = 0 \) is optimal if and only if

\[
f'(0, w) - \lambda \mathcal{E} \left( - \int_t^{t+\varepsilon} \theta_s dW_s \right) (w) = 0, \forall w \in \Omega \]

(6.21)

and we have

\[
f'(0, w) = \left[ -1 \begin{pmatrix} 0 & X^*_t e^{\int_0^T \mu_s - r_s ds} - \bar{X}_T^* \end{pmatrix} \right] (w)
\]

(6.22)

since \( f'(0, w) \in [-1, 0] \), while \( \mathcal{E} \left( - \int_t^{t+\varepsilon} \theta_s dW_s \right) (w) \) could blow up towards \( \infty \), we must have \( \lambda = 0 \) to make (6.21) achievable and thus we have

\[
f'(0, w) = 0, \forall w \in \Omega
\]

which is again achievable if and only if

\[
X^*_t e^{\int_0^T \mu_s - r_s ds} \leq \bar{X}_T^* = X^*_t + \int_t^T (\bar{u}^*_s)^T dW_s^2
\]

which means

\[
\int_t^T (\bar{u}^*_s)^T dW_s^2 \geq X^*_t \left( e^{\int_0^T \mu_s - r_s ds} - 1 \right)
\]

(6.23)
thus we have showed that \( \Delta_{t+\varepsilon} = 0 \) is optimal to problem (6.19) if and only if (6.23) is satisfied. Since we have also showed above that \( u^* \) is an equilibrium for our problem (6.6) if and only if for any \( t \in [0, T) \), \( \exists \delta > 0 \), s.t. for any \( \varepsilon \in (0, \delta) \) we have that \( \Delta_{t+\varepsilon} = 0 \) is optimal to problem (6.19), so we conclude by definition (6.1) that

- \( u^* \) is an equilibrium for the family of problems (6.6) if and only if for any \( t \in [0, T) \), \( \exists \delta > 0 \), s.t. for any \( \varepsilon \in (0, \delta) \)

\[
\int_t^T (\bar{u}^*_s)^T dW_s^Q \geq \int_t^T \left( e^{-\int_t^s \mu_s - r_s ds} - 1 \right) X_t^* \left( e^{-\int_t^s \mu_s - r_s ds} - 1 \right)
\]

since \( \mathbb{E}_t [X_t^*] = X_t^* e^{\int_t^T \mu_s ds} \) is equivalent to \( X_t^* \left( e^{\int_t^T \mu_s - r_s ds} - 1 \right) = \mathbb{E}_t \left[ (\bar{u}_s)^T \theta_s \right] \), then \( u^* \in \mathcal{U}_{ad} \) and thus is an equilibrium for the family of problems (6.3) if for any \( t \in [0, T) \), \( \exists \delta > 0 \), s.t. for any \( \varepsilon \in (0, \delta) \) we have

\[
\begin{aligned}
\int_t^T (\bar{u}^*_s)^T dW_s^Q &\geq \int_t^T \left( e^{\int_t^s \mu_s - r_s ds} - 1 \right) X_t^* \left( e^{\int_t^s \mu_s - r_s ds} - 1 \right) \\
\int_t^T \mathbb{E}_t \left[ (\bar{u}_s)^T \theta_s \right] ds &\geq \int_t^T \left( e^{\int_t^s \mu_s - r_s ds} - 1 \right) X_t^* \left( e^{\int_t^s \mu_s - r_s ds} - 1 \right)
\end{aligned}
\]

It is clear that \( u^* = 0 \) is an equilibrium for (6.6) when \( \mu \leq r \) and \( x_0 \geq 0 \) with the corresponding state process \( X_t^* = x_0 e^{\int_0^T \mu_s ds} \). And this \( u^* = 0 \) is also an equilibrium for (6.3) when \( r = \mu \).

### 7 Conclusion

In this paper, we have studied the time inconsistent stochastic control problems of portfolio selection by using different utility functions with a moving target that need to be met. And we solve our problem by finding equilibrium controls under our definition as the optimal controls. This paper has also posed some open questions during the procedure of solving our problems such as how to prove the existence of solutions for our derived system of ODEs when we solve our family of problems using utility function \( h(x) = -x^3 \) and \( h(x) = x^4 \), which could be good further research topics in this area.

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