On Jensen’s inequality for the power function of norm of bounded linear operators

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Abstract. Some Jensen’s inequalities for the power function of norm of bounded linear operators are proposed. First, a Jensen’s inequality for a power function on $\mathbb{R}$ is investigated. Based on it, Jensen’s inequality for the power function of norm of bounded linear operators is established. Finally, the discussion extends to the polynomial with non-negative coefficients of the power function of the norm, and corresponding Jensen’s inequality is obtained.

1. Introduction

During the long history of mathematics, inequalities have been playing a key role in most of mathematical research. As far as we know, the research results on inequalities have been widely applied to various fields, such as functional analysis, matrix analysis, numerical analysis, statistical analysis, optimization theory, cybernetics, physics and so on [1-15].

Jensen’s inequality, the so-called king of inequalities [7], is actually a fundamental theoretical tool for convex analysis and derives many other important inequalities. The study on Jensen’s inequality is always an interesting topic [6-15].

In this paper, via observing the convexity of a power function, we investigate a Jensen’s inequality for the power function of the norm of bounded linear operators, and then build the Jensen’s inequality for the polynomials of the norm of bounded linear operators, where the coefficients of the polynomials are assumed to be non-negative real numbers.

This article is organized as follows. In section 2 we introduce some concepts which will be applied throughout this paper. In subsection 3.1, we study the convexity of a power function on $\mathbb{R}$. In subsection 3.2, based on some primary results, we investigate a Jensen’s inequality for the power function of norm of bounded linear operators. Furthermore, we obtain an inequality for the polynomial with non-negative coefficients of the norm of bounded linear operators. Some other corollaries are also given.

2. Preliminaries

First of all, we introduce some basic concepts that will be used in the remaining sections.

Let $\mathbb{R}$ denote the set of real numbers, and $\lvert \cdot \rvert$ denote the absolute value function on $\mathbb{R}$.

Let $X$, $Y$ represent two normed linear spaces, and the associated norms are denoted by $\lVert \cdot \rVert_X$ and $\lVert \cdot \rVert_Y$ respectively.
Let $L(X, Y)$ denote the space of all the bounded linear operators from $X$ to $Y$.

**Definition 2.1** [16, 17] The norm of a given operator $A \in L(X, Y)$ is defined as follows:

$$
\|A\|_{L(X, Y)} = \sup_{\|x\|_X = 1} \|Ax\|_Y.
$$

(2.1)

As for the famous Jensen’s inequality, there are some fundamental theoretical results as follows.

**Lemma 2.1** [3] Let $f$ be a function from $S$ to $\mathbb{R}$, where $S$ is a convex set. Then $f$ is convex if and only if

$$
f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y), \quad 0 < \lambda < 1,
$$

for every $x$ and $y$ in $S$.

**Lemma 2.2** (Jensen’s inequality) [3] Let $f$ be a function from $S$ to $\mathbb{R}$. Then $f$ is convex if and only if

$$
f(\sum_{i=1}^{m} \lambda_i x_i) \leq \sum_{i=1}^{m} \lambda_i f(x_i)
$$

whenever $\lambda_1 \geq 0, \ldots, \lambda_m \geq 0$, $\lambda_1 + \ldots + \lambda_m = 1$.

**Lemma 2.3** [3] If $f_1, f_2$ are convex functions on $S$, then $f_1 + f_2$ is convex.

Let $P_n$ represent the vector space of algebraic polynomials with real coefficients and at most $n$ degree. And we denote by $P^+_n$ the set of polynomials in $P_n$ and all coefficients are non-negative [1]. The polynomials with non-negative coefficients have some amazing properties, which were introduced in detail in [1].

3. Main results

3.1. A primary Jensen’s inequality on $\mathbb{R}$

We start the discussion by observing the property of a function from $\mathbb{R}$ to $\mathbb{R}$:

$$
f(x) = \begin{cases} \left\|x\right\| & , \quad 1 \leq t < \infty, \\
0 & , \quad x = 0.
\end{cases}
$$

(3.1)

for which we prepare a lemma as follows.

**Lemma 3.1** Function $f(x)$ defined by (3.1) is convex, i.e.,

$$
\left\|(1-\lambda)x + \lambda y\right\| \leq (1-\lambda)\|x\| + \lambda \|y\|, \quad 0 < \lambda < 1, \quad 1 \leq t < \infty,
$$

(3.2)

for every $x$ and $y$ in $\mathbb{R}$.

**Proof.** For arbitrary real numbers $x$ and $y$, generally we suppose $\|y\| \leq |x|$ (Otherwise, if $|x| \leq \|y\|$, the corresponding result can be obtained as well), then there exists a real number $k$, such that $y = kx$, where $|k| \leq 1$.

(3.3)

Comparing the two sides of (3.2) we have

$$
f((1-\lambda)x + \lambda y) - (1-\lambda)f(x) - \lambda f(y) = \left[\|1 - \lambda + \lambda k\| - (1-\lambda) - \lambda \|k\|^2\right]\|x\|.
$$

As for the right side of above equation, the coefficient of $\|x\|$ is a function of $k$, that is

$$
\|x\| = \|1 - \lambda + \lambda k\| - (1-\lambda) - \lambda \|k\|,
$$

(3.4)

and it satisfies
Next we will analyze the monotonicity of \( S_\lambda(k) \) defined as (3.4) for three cases. We note for the case \( 0 \leq k \leq 1 \) (and \( 0 < \lambda < 1 \) which has been presupposed), it holds that \( 0 \leq \lambda(1-k) < 1 \), and
\[
1 - \lambda + \lambda k = 1 - \lambda(1-k) > 0.
\]

Therefore
\[
S_\lambda(k) = \begin{cases} 
[1-\lambda + \lambda k] - (1-\lambda) - \lambda k', & \text{if } 0 \leq k \leq 1; \\
[1-\lambda + \lambda k] - (1-\lambda) - \lambda(-1)'k', & \text{if } -1 \leq k < 0 \text{ and } 1 - \lambda + \lambda k > 0; \\
(-1)'[1-\lambda + \lambda k] - (1-\lambda) - \lambda(-1)'k', & \text{if } -1 \leq k < 0 \text{ and } 1 - \lambda + \lambda k \leq 0,
\end{cases}
\]

the derivative function is:
\[
S_\lambda'(k) = \begin{cases} 
t\lambda \left\{ [1-\lambda + \lambda k]^{-1} - k'^{-1} \right\}, & \text{if } 0 \leq k \leq 1; \\
t\lambda \left\{ [1-\lambda + \lambda k]^{-1} - (k')^{-1} \right\}, & \text{if } -1 \leq k < 0 \text{ and } 1 - \lambda + \lambda k > 0; \\
t\lambda \left\{ [-(1-\lambda + \lambda k)]^{-1} - (k')^{-1} \right\}, & \text{if } -1 \leq k < 0 \text{ and } 1 - \lambda + \lambda k \leq 0.
\end{cases}
\]

For case \( 0 \leq k \leq 1 \), noticing the fact that \( 1 - \lambda + \lambda k - k = (1 - \lambda)(1 - k) \geq 0 \), which means \( 1 - \lambda + \lambda k \geq k \), one can easily get
\[
S_\lambda'(k) = t\lambda \left\{ [1-\lambda + \lambda k]^{-1} - k'^{-1} \right\} \geq 0.
\]

For the case \( -1 \leq k < 0 \) and \( 1 - \lambda + \lambda k > 0 \), it is obvious that
\[
S_\lambda'(k) = t\lambda \left\{ [1-\lambda + \lambda k]^{-1} + (k')^{-1} \right\} \geq 0.
\]

And for the case \( -1 \leq k < 0 \) and \( 1 - \lambda + \lambda k \leq 0 \), since \( -(1 - \lambda + \lambda k) - (k) = (1 - \lambda)(k - 1) < 0 \), we have
\[
S_\lambda'(k) = -t\lambda \left\{ [-(1-\lambda + \lambda k)]^{-1} - (k')^{-1} \right\} > 0.
\]

The above discussion derives the conclusion
\[
S_\lambda'(k) \geq 0, \quad |k| \leq 1.
\]

Combining (3.6) and (3.5) we prove that \( S_\lambda(k) \leq 0 \) provided \(|k|\leq 1\). Thus (3.2) is obtained and the proof is completed.

3.2. Jensen’s inequality on \( L(X, Y) \)

Now we apply the primary result to the norm on \( L(X, Y) \).

**Theorem 3.1** Let the norm \( \| \cdot \|_{L(X, Y)} \) be defined by (2.1). If \( 1 \leq t < \infty \) and \( 0 < \lambda < 1 \), then
\[
\|(1-\lambda)A + \lambda B\|_{L(X, Y)} \leq (1-\lambda)\|A\|_{L(X, Y)} + \lambda\|B\|_{L(X, Y)}, \quad \forall A, B \in L(X, Y).
\]

**Proof.** Since
\[
(1 - \lambda)A + \lambda B \leq \left\{ \sup_{H \in I} \left\{ (1 - \lambda)A + \lambda B \right\} \right\}
\]

(from (2.1))

\[
= \sup_{H \in I} \left\{ (1 - \lambda)A + \lambda B \right\} \leq \sup_{H \in I} \left[ (1 - \lambda)A + \lambda B \right] \right\}
\]

(from Lemma 3.1)

\[
\leq \sup_{H \in I} \left[ (1 - \lambda)A + \lambda B \right] \right\}
\]


\[
= (1 - \lambda)A + \lambda B \leq \sup_{H \in I} \left[ (1 - \lambda)A + \lambda B \right] \right\}
\]

we complete this proof.

**Remark 3.1** Theorem 3.1 indicates that norm \( \| \|_{L(X,Y)} \), \( 1 \leq t < \infty \) is convex.

**Corollary 3.1** Let the norm \( \| \|_{L(X,Y)} \) be defined by (2.1). If \( 1 \leq t < \infty \), then

\[
\frac{1}{2} \left( \| A \|_{L(X,Y)} + \| B \|_{L(X,Y)} \right), \quad \forall A, B \in L(X,Y)
\]

(3.8)

**Proof.** Let \( \lambda = \frac{1}{2} \) in the assumption of Theorem 3.1, we immediately get (3.8) and complete this proof.

**Corollary 3.2** Let the norm \( \| \|_{L(X,Y)} \) be defined by (2.1). If \( 1 \leq t < \infty \), then

\[
\| U \|_{L(X,Y)} \leq \frac{1}{2} \left( \| U + V \|_{L(X,Y)} + \| U - V \|_{L(X,Y)} \right), \quad \forall U, V \in L(X,Y)
\]

(3.9)

**Proof.** For arbitrary operators \( U \) and \( V \), there must exist operators \( A \) and \( B \), where

\[
A = U + V, \quad B = U - V
\]

such that

\[
U = \frac{1}{2} (A + B), \quad V = \frac{1}{2} (A - B)
\]

(3.10)

Combining (3.10) with Corollary 3.1, we get (3.9) and complete this proof.

**Corollary 3.3** Let \( t \in [1, \infty) \). If \( \lambda_1, \lambda_2, \ldots, \lambda_m \) are all non-negative numbers and satisfy \( \sum_{i=1}^{m} \lambda_i = 1 \), then for \( A_i \in L(X,Y) \), \( i = 1, 2, \ldots, m \), the following inequality holds.

\[
\sum_{i=1}^{m} \lambda_i A_i \leq \sum_{i=1}^{m} \lambda_i \| A_i \|_{L(X,Y)}
\]

The proof of Corollary 3.3 can be directly deduced by Theorem 3.1 and Lemma 2.2. The details of this proof are omitted.

**Corollary 3.4** Let \( t \in [1, \infty) \), and \( \lambda_1, \lambda_2, \ldots, \lambda_m \) are all non-negative numbers which satisfy \( \sum_{i=1}^{m} \lambda_i = 1 \). If \( F(x) \in L^p \), then for \( A_i \in L(X,Y) \), \( i = 1, 2, \ldots, m \), we have
\[ \left\| \sum_{i=1}^{m} \lambda_i F \left( A \right) \right\|_{\mathcal{L}(X,Y)} \leq \sum_{i=1}^{m} \lambda_i \left[ F \left( \left\| A \right\|_{\mathcal{L}(X,Y)} \right) \right]^j. \]  \hspace{1cm} (3.11)

Proof. Suppose \( F(x) = \sum_{k=0}^{n} a_k x^k \), where \( a_k \geq 0 \), \( k = 0, 1, \ldots, n \). Since

\[ \left\| \sum_{i=1}^{m} \lambda_i F \left( A \right) \right\|_{\mathcal{L}(X,Y)} \leq \sum_{i=1}^{m} \lambda_i \left[ F \left( \left\| A \right\|_{\mathcal{L}(X,Y)} \right) \right]^j \]  \hspace{1cm} (from Corollary 3.3)

\[ = \sum_{i=1}^{m} \lambda_i \left[ \sum_{k=0}^{n} a_k A^k \right]_{\mathcal{L}(X,Y)} \leq \sum_{i=1}^{m} \lambda_i \left( \sum_{k=0}^{n} a_k \left\| A^k \right\|_{\mathcal{L}(X,Y)} \right) \]  \hspace{1cm} (by assumption \( a_k \geq 0 \))

\[ \leq \sum_{i=1}^{m} \lambda_i \left( \sum_{k=0}^{n} a_k \left\| A^k \right\|_{\mathcal{L}(X,Y)} \right) = \sum_{i=1}^{m} \lambda_i \left[ F \left( \left\| A \right\|_{\mathcal{L}(X,Y)} \right) \right]^j, \]

we get inequality (3.11) and complete this proof.

Corollary 3.4 gives the Jensen’s inequality for the power function of the norm of polynomials with non-negative coefficients. Next we will give the Jensen’s inequality for the polynomials with non-negative coefficients of the power function of the norm.

**Corollary 3.5** Let \( t \in [1, \infty) \), and \( \lambda_1, \lambda_2, \ldots, \lambda_m \) be nonnegative numbers which satisfy \( \sum_{i=1}^{m} \lambda_i = 1 \). If \( F(x) \in P_n^+ \), then for \( A_i \in \mathcal{L}(X,Y), \ i = 1, 2, \ldots, m \), we have

\[ F \left( \left\| \sum_{i=1}^{m} \lambda_i A_i \right\|_{\mathcal{L}(X,Y)} \right) \leq \sum_{i=1}^{m} \lambda_i F \left( \left\| A_i \right\|_{\mathcal{L}(X,Y)} \right). \]  \hspace{1cm} (3.12)

Proof. Suppose \( F(x) = \sum_{k=0}^{n} a_k x^k \), where \( a_k \geq 0 \), \( k = 0, 1, \ldots, n \). Therefore,

\[ F \left( \left\| \sum_{i=1}^{m} \lambda_i A_i \right\|_{\mathcal{L}(X,Y)} \right) = \sum_{k=0}^{n} a_k \left( \sum_{i=1}^{m} \lambda_i A_i \right)^k = a_0 + \sum_{k=1}^{n} a_k \left( \sum_{i=1}^{m} \lambda_i A_i \right)^k \]

\[ \leq a_0 + \sum_{k=1}^{n} a_k \sum_{i=1}^{m} \lambda_i \left\| A_i \right\|_{\mathcal{L}(X,Y)}^k \]  \hspace{1cm} (from Corollary 3.3 and by assumption \( a_k \geq 0 \))

\[ = \sum_{k=0}^{n} a_k \lambda_k \left\| A_i \right\|_{\mathcal{L}(X,Y)}^k = \sum_{k=0}^{n} \lambda_k \sum_{k=0}^{n} a_k \left\| A_i \right\|_{\mathcal{L}(X,Y)}^k = \sum_{i=1}^{m} \lambda_i F \left( \left\| A_i \right\|_{\mathcal{L}(X,Y)} \right), \]

which leads to the inequality (3.12). The proof is completed.

**Remark 3.2** It seems that inequalities (3.11) and (3.12) are only applicable to the case \( F(x) \in P_n^+ \) and are invalid for the general polynomials in \( P_n \).

4. Conclusions

In this paper we investigate the Jensen’s inequality for the power function of norm of bounded linear operators. The results maybe are helpful to some applications of mathematical analysis such as matrix analysis. In fact, we found the existence of this inequality when we studied some problems of matrix computation, and the prototypes of the bounded linear operators were the matrices. Naturally, all the
results involved can be directly applied to the matrix norms. Further refinement or generalization for the existing work is the next goal in the future.

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References
[1] Bowerin P and Erdélyi T 1995 Polynomials and Polynomial Inequalities (New York: Springer-Verlag New York Inc.) 49-79
[2] Bhatia R 1997 Matrix Analysis (New York: Springer-Verlag New York Inc.) 3-7
[3] Rockafellar R T 1997 Convex Analysis (Princeton: Princeton University Press) 23-25
[4] Wang S, Wu M and Jia Z 2006 Matrix Inequalities (in Chinese) (Beijing: Science Press)
[5] Kuang J 2010 Applied Inequalities (in Chinese) (Jinan: Shandong Science and Technology Press)
[6] Dragomir S S 2010 A new refinement of Jensen's inequality in linear spaces with applications Math. Comput. Modelling 52 1497-1505
[7] Wang W 2011 Approaches to Prove Inequalities (in Chinese) (Harbin: Harbin Institute of Technology Press) 99
[8] Mukhopadhyay N 2011 On sharp Jensen's inequality and some unusual applications. Comm. Statist. Theory Methods 40 1283-1297
[9] Yan Z, Yu R and Xiong X 2012 Matrix Inequalities (in Chinese) (Shanghai: Tongji University Press) 162-166
[10] Cirtoaje V 2013 The best lower bound for Jensen's inequality with three fixed ordered variables Banach J. Math. Anal. 7 116-131
[11] Walker S G 2014 On a lower bound for the Jensen inequality SIAM J. Math. Anal. 46 3151-3157
[12] Nakasuji Y 2014 Refinements of some inequalities related to Jensen’s inequality J. Math. Inequal. 8 685-692
[13] Adil Khan M, Ali Khan G, Ali T and Kiliçman A 2015 On the refinement of Jensen’s inequality Appl. Math. Comput. 262 128-135
[14] Sababheh M 2017 Improved Jensen’s inequality Math. Inequal. Appl. 20 389-403
[15] Lu G 2018 New refinements of Jensen’s inequality and entropy upper bounds J. Math. Inequal. 12 403-421
[16] Saad Y 2009 Iterative Methods for Sparse Linear Systems (Beijing: Science Press) 7
[17] Xia D, Wu Z, Yan S and Shu W 2009 Real Variable Function Theory and Functional Analysis (II) (in Chinese) (Beijing: Higher Education Press) 112