CHARACTERIZATION OF THE MONOTONICITY BY THE INEQUALITY

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Abstract. Let \( \varphi \) be a normal state on the algebra \( B(H) \) of all bounded operators on a Hilbert space \( H \), \( f \) a strictly positive, continuous function on \( (0, \infty) \), and let \( g \) be a function on \( (0, \infty) \) defined by \( g(t) = \frac{f}{f(t)} \). We will give characterizations of matrix and operator monotonicity by the following generalized Powers-Størmer inequality:

\[
\varphi(A + B) - \varphi(|A - B|) \leq 2\varphi(f(A)^{\frac{1}{2}}g(B)f(A)^{\frac{1}{2}}),
\]

whenever \( A, B \) are positive invertible operators in \( B(H) \).

1. Introduction

Throughout the paper, \( M_n \) stands for the algebra of all \( n \times n \) matrices, \( M_n^+ \) denote the set of positive semi-definite matrices. We call a function \( f \) matrix convex of order \( n \) or \( n \)-convex in short (resp. matrix concave of order \( n \) or \( n \)-concave) whenever the inequality

\[
f(\lambda A + (1 - \lambda)B) \leq \lambda f(A) + (1 - \lambda)f(B), \; \lambda \in [0, 1]
\]

(resp. \( f(\lambda A + (1 - \lambda)B) \geq \lambda f(A) + (1 - \lambda)f(B) \), \( \lambda \in [0, 1] \)) holds for every pair of selfadjoint matrices \( A, B \in M_n \) such that all eigenvalues of \( A \) and \( B \) are contained in \( I \). Matrix monotone functions on \( I \) are similarly defined as the inequality

\[
A \leq B \implies f(A) \leq f(B)
\]

for any pair of selfadjoint matrices \( A, B \in M_n \) such that \( A \leq B \) and all eigenvalues of \( A \) and \( B \) are contained in \( I \). We call a function \( f \) operator convex (resp. operator concave) if for each \( k \in \mathbb{N} \), \( f \) is \( k \)-convex (resp. \( k \)-concave) and operator monotone if for each \( k \in \mathbb{N} \) \( f \) is \( k \)-monotone.

Let \( n \in \mathbb{N} \) and \( f : [0, \alpha) \to \mathbb{R} \). In [5] the second and the third author discussed about the following 3 assertions at each level \( n \) among them in order to see clear insight of the double piling structure of matrix monotone functions and of matrix convex functions:

(i) \( f(0) \leq 0 \) and \( f \) is \( n \)-convex in \( [0, \alpha) \),

(ii) For each matrix \( a \) with its spectrum in \( [0, \alpha) \) and a contraction \( c \) in the matrix algebra \( M_n \),

\[
f(c^*ac) \leq c^*f(a)c,
\]

(iii) The function \( f(t) \) (\( = g(t) \)) is \( n \)-monotone in \( (0, \alpha) \).

It was shown in [5] that

\[
(i)_{n+1} \prec (ii)_{n} \sim (iii)_{n} \prec (i)_{\frac{n}{n+1}},
\]

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where denotation \((A)_m \prec (B)_n\) means that “if \((A)\) holds for the matrix algebra \(M_m\), then \((B)\) holds for the matrix algebra \(M_n\)”.

In this article, using an idea in [4] we can get the concave version of the above observation. Namely, for \(n \in \mathbb{N}\) and \(f : [0, \alpha) \to \mathbb{R}\) we consider the following assertions:

(iv) \(f(0) \geq 0\) and \(f\) is \(n\)-concave in \([0, \alpha)\),
(v) For each matrix \(a\) with spectrum in \([0, \alpha)\) and a contraction \(c\) in the matrix algebra \(M_n\),
\[f(c^* ac) \geq c^* f(a)c,\]
(vi) The function \(\frac{f(t)}{f(t)}\) is \(n\)-monotone in \((0, \alpha)\).

We will show that \((iv)_{n+1} \prec (v)_{n} \sim (vi)_{n} \prec (iv)_{\frac{n}{2}}\).

As an application we investigate the generalized Powers-Størmer inequality from the point of matrix functions, which was introduced in [4]. Let \(\varphi\) be a normal state on the algebra \(B(H)\) of all bounded operators on a Hilbert space \(H\), \(f\) be a strictly positive, continuous function on \((0, \infty)\), and let \(g\) be a function on \((0, \infty)\) defined by \(g(t) = \frac{t}{f(t)}\).

We will consider the following inequality
\[
\varphi(A + B) - \varphi(|A - B|) \leq 2\varphi(f(A)\frac{1}{2} g(B) f(A)^{\frac{1}{2}}),
\]
where \(A, B\) are positive invertible operators in \(B(H)\).

It will be shown that:

1. If the inequality holds true for any positive invertible \(A, B\), then the function \(g\) is operator monotone.
2. When \(\dim H = n < \infty\), if \(\varphi\) is canonical trace and \(f\) is \((n + 1)\)-concave, then the inequality holds.
3. When \(\dim H = n < \infty\), if the inequality holds, then the state \(\varphi\) has the trace property if and only if the function \(g\) satisfies the condition
\[
\inf_{\lambda > \mu} \frac{\sqrt{g'(\lambda)g'(\mu)}}{\sqrt{g(\lambda) - g(\mu)}} = 0.
\]

2. Hansen-Pedersen’s inequality for matrix functions

For a long time it has been known the following equivalency. When \(f\) is strictly positive, continuous function on \((0, \infty)\), the followings are equivalent ([3, 2.6. Corollary]):

1. \(f\) is operator concave.
2. \(\frac{f}{f(t)}\) is operator monotone

The following result is the matrix function versions of the above observation.

**Theorem 2.1.** Let \(n \in \mathbb{N}\) and \(f : [0, \alpha) \to \mathbb{R}\) be a continuous function for some \(\alpha > 0\) such that \(0 \notin f([0, \alpha))\). Let us consider the following assertions:

4. \(f\) is \(n\)-concave with \(f(0) \geq 0\).
5. For all operators \(A \in M_n\) with its spectrum in \([0, \alpha)\) and all contraction \(C\)
\[f(C^* AC) \geq C^* f(A)C.\]
(6) \( g(t) = \frac{t}{f(t)} \) is \( n \)-monotone on \((0, \alpha)\).

Then we have

\[
(4)_{n+1} \prec (5)_n \sim (6)_n \prec (4)_{\frac{n}{2}}.
\]

**Proof.** The implication \((4)_{n+1} \prec (5)_n\):

Since \( f \) is \((n+1)\)-concave, \(-f\) is \((n+1)\)-convex. From [5] we know that for an operator \( A \in M_n \) with its spectrum in \([0, \alpha)\) and a contraction \( C \)

\[
(-f)(C^*AC) \leq C^*(-f)(A)C,
\]

hence

\[
f(C^*AC) \geq C^*f(A)C.
\]

The implication \((5)_n \sim (6)_n\):

Since \( f(C^*AC) \geq C^*f(A)C \) for an operator \( A \in M_n \) with its spectrum in \([0, \alpha)\) and a contraction \( C \), we know that

\[
(-f)(C^*AC) \leq C^*(-f)(A)C.
\]

Then, from [5] we know that \(-\frac{f(t)}{t}\) is \( n \)-monotone. Since the function \(-\frac{1}{t}\) is operator monotone, \(-\frac{1}{\frac{f(t)}{t}}\) is \( n \)-monotone, that is, \( \frac{t}{f(t)} \) is \( n \)-monotone.

Conversely, if \( \frac{t}{f(t)} \) is \( n \)-monotone, then \(-\frac{1}{\frac{f(t)}{t}} = \frac{(-f)(t)}{t}\) is \( n \)-monotone from the operator monotonicity of \(-\frac{1}{t}\), hence we know in [5] that for an operator \( A \in M_n \) with its spectrum in \([0, \alpha)\) and a contraction \( C \)

\[
(-f)(C^*AC) \leq C^*(-f)(A)C,
\]

that is,

\[
f(C^*AC) \geq C^*f(A)C.
\]

The implication \((6)_n \prec (4)_{\frac{n}{2}}\): Since \( \frac{t}{f(t)} \) is \( n \)-monotone, the function \(-\frac{f(t)}{t}\) is \( n \)-monotone by [4] Lemma 2.2. Hence, \(-f\) is \([\frac{n}{2}]\)-convex by [5], that is, \( f \) is \([\frac{n}{2}]\)-concave. \( \Box \)

### 3. Characterization of matrix monotonicity

The following result was proved in [4] under the condition that the function \( f \) is \( 2n \)-monotone. But using the concavity we will show that the condition of \( f \) is weakened. Note that the \( 2n \)-monotonicity of a function \( f \) on \([0, \infty)\) implies the \( n \)-concavity of \( f \) by [1] Theorem V.2.5.

**Theorem 3.1.** Let \( \text{Tr} \) be the canonical trace on \( M_n \) and \( f \) be a \((n+1)\)-concave function on \([0, \infty)\) such that \( f((0, \infty)) \subset (0, \infty) \). Then for any pair of positive matrices \( A, B \in M_n \)

\[
\text{Tr}(A) + \text{Tr}(B) - \text{Tr}(|A - B|) \leq 2 \text{Tr}(f(A)^{\frac{n}{2}}g(B)f(A)^{\frac{n}{2}}),
\]

where \( g(t) = \left\{ \begin{array}{ll} \frac{t}{f(t)} & (t \in (0, \infty)) \\ 0 & (t = 0) \end{array} \right. \).
Proof. Since \( f \) is \((n + 1)\)-concave, we know that the function \( g \) is \( n \)-monotone from Theorem 2.1. Moreover, by a standard argument (see, for example, [1, Theorem V.2.5]) it is clear that the function \( f \) is \((n + 1)\)-monotone, and hence \( n \)-monotone. Repeating the similar argument as in the proof of [4, Theorem 2.1] with mentioned properties of \( f \) and \( g \), we will get the conclusion. \( \square \)

From the above Theorem 3.1 we consider the following converse problems.

Let \( n \in \mathbb{N} \) and \( \varphi \) be a faithful positive linear functional on \( M_n \), \( f \) be a strictly positive, continuous function on \((0, \infty)\), and let \( g \) be a function on \((0, \infty)\) defined by \( g(t) = f(t)^{\frac{1}{2}} \).

Suppose that for any positive invertible \( A, B \in M_n \)

\[
\varphi(A + B) - \varphi(|A - B|) \leq 2\varphi(f(A^{\frac{1}{2}})g(B)f(A^{\frac{1}{2}})).
\]

Then:
**Problem 1:** Is it true that \( f \) is \( n \)-monotone?
**Problem 2:** Is \( \varphi \) a scalar positive multiple of the canonical trace?

The following examples give a contribution to the attempt to answer problem 1.

**Example 3.2.** Let \( f(t) = t^2 \) on \((0, \infty)\). It is well-known that \( f \) is not 2-monotone. We now show that the function \( f \) does not satisfy the inequality (1). Indeed, let us consider the following matrices

\[
A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.
\]

Then we have

\[
AB^{-1}A = \frac{2}{3}A.
\]

Set \( \tilde{A} = A \oplus \text{diag}(1, \cdots, 1) \) and \( \tilde{B} = B \oplus \text{diag}(1, \cdots, 1) \) in \( M_n \). Then, \( \tilde{A} \leq \tilde{B} \) and for a faithful linear functional \( \varphi \) on \( M_n \)

\[
\varphi(f(\tilde{A}^{\frac{1}{2}})g(\tilde{B})f(\tilde{A}^{\frac{1}{2}})) = \varphi(\tilde{A}B^{-1}\tilde{A})
\]

\[
= \varphi\left(\frac{2}{3}A \oplus \text{diag}(1, \cdots, 1)\right)
\]

\[
< \varphi(A \oplus \text{diag}(1, \cdots, 1))
\]

\[
= \varphi(\tilde{A}).
\]

On the contrary, since \( \tilde{A} \leq \tilde{B} \), from the inequality (1) we have

\[
\varphi(\tilde{A}) + \varphi(\tilde{B}) - \varphi(\tilde{B} - \tilde{A}) \leq 2\varphi(f(\tilde{A}^{\frac{1}{2}})g(\tilde{B})f(\tilde{A}^{\frac{1}{2}})),
\]

or

\[
\varphi(\tilde{A}) \leq \varphi(f(\tilde{A}^{\frac{1}{2}})g(\tilde{B})f(\tilde{A}^{\frac{1}{2}})),
\]

and we have a contradiction. \( \square \)
Now we will show that for $p > 1$ the function $f(t) = t^p$ does not satisfy the special inequality from inequality (1) for a faithful positive linear functional.

**Example 3.3.** It will be shown that for $f(t) = t^p$ ($p > 1$) and a faithful positive linear functional $\varphi$ on $M_n$, the following inequality does not always hold:

\[
\varphi(A) \leq \varphi(f(A)^{1/2}g(B)f(A)^{1/2}) \quad (0 \leq A \leq B).
\]

Note that when $0 < A \leq B$ the inequality (2) can be deduced from the inequality (1) directly.

Since $A \leq B$, we can suppose that $B = A + tC$ for some positive number $t$ and positive matrix $C$. Hence inequality (1) becomes

\[
\varphi(A) \leq \varphi(f(A)^{1/2}g(A + tC)f(A)^{1/2}).
\]

On the other hand, we have

\[
g(A + tC) = g(A) + t \cdot \left. \frac{dg(A + tC)}{dt} \right|_{t=0} + R(A, C, t),
\]

where $\lim_{t \to 0} \frac{||R(A, C, t)||}{t} = 0$.

From (3) and (4) we get

\[
\varphi(A) \leq \varphi(f(A)^{1/2}(g(A) + t \cdot \left. \frac{dg(A + tC)}{dt} \right|_{t=0} + R(A, C, t))f(A)^{1/2})
\]

\[
= \varphi(A) + t\varphi(f(A)^{1/2} \cdot \left. \frac{dg(A + tC)}{dt} \right|_{t=0} \cdot f(A)^{1/2} + o(t).
\]

From that, we get

\[
\varphi(f(A)^{1/2} \cdot \left. \frac{dg(A + tC)}{dt} \right|_{t=0} \cdot f(A)^{1/2}) \geq 0 \quad (\forall A, C \geq 0).
\]

Let us assume that $\varphi(\cdot) = \text{Tr}(S \cdot)$, where $S = \text{diag}(s, 1)$ ($s \in [0, 1]$). For $\beta > 0$ and $\alpha \in [0, 1]$, let us consider the $2 \times 2$ matrices

\[
C = \begin{pmatrix}
\frac{\alpha^2}{\alpha^2 - 1} & \frac{\alpha\sqrt{1 - \alpha^2}}{1 - \alpha^2} \\
\frac{\alpha\sqrt{1 - \alpha^2}}{1 - \alpha^2} & \frac{1}{\alpha^2}
\end{pmatrix} \quad \text{and} \quad A = \beta P_1 + \alpha P_2,
\]

where

\[
P_1 = \begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}, \quad P_2 = \begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix}.
\]

Then by identifying $M_2$ as a $C^*$-subalgebra $M_2 \oplus O_{n-2}$ of $M_n$, we may assume that $\varphi(\cdot) = \text{Tr}(S \cdot)$, where $S = \text{diag}(s, 1)$ ($s \in [0, 1]$).

Then (5) becomes

\[
\alpha^2(1 - \frac{\beta f'(\beta)}{f(\beta)}) + (1 - \alpha^2)(1 - \frac{\alpha f'(\alpha)}{f(\alpha)}) \geq 0.
\]

Since $f(t) = t^p$, for any $\alpha, \beta > 0$ we have

\[
1 - \frac{\beta f'(\beta)}{f(\beta)} = 1 - \frac{\alpha f'(\alpha)}{f(\alpha)} = 1 - p < 0.
\]

It implies that the latter inequality (3) does not hold true for any $\alpha \in [0, 1]$, and the inequality (2) will not hold true.

In particular, even $\varphi$ is the canonical trace on $M_n$, the inequality (2) will not hold true.
From above argument, we can find many counterexamples for the functions not of the form \( f(t) = t^p \) \((p > 1)\). For example, if function \( f \) on some \((0, a)\) satisfies condition \( f(t) < tf'(t) \), then inequality (2) is not true. \(\square\)

Here we will give a positive answer on problem 2 for some class of functions \( g \), namely, it will be shown that inequality (2) characterizes the trace property of \( \varphi \).

**Proposition 3.4.** Let \( n \in \mathbb{N} \) and \( \varphi \) be a faithful positive linear functional on \( M_n \). Let \( f \) be a strictly positive, continuous function on \((0, \infty)\), and let \( g \) be a function on \((0, \infty)\) defined by \( g(t) = t^p f(t) \). Suppose that

\[
\varphi(A) \leq \varphi(f(A)^{1/2}g(B)f(A)^{1/2}),
\]

whenever any pair of positive invertible \( A, B \in M_n \) such that \( 0 < A \leq B \).

Then \( \varphi \) has the trace property if and only if \( g \) satisfies the condition

\[
\inf_{\lambda > \mu} \frac{\sqrt{\lambda} g'(\lambda)}{\sqrt{\lambda} g'(\mu)} = 0.
\]

**Proof.** The conclusion follows from the same steps in the proof of [6, Theorem 2.2], but we put the sketch of the proof for readers.

Let \( S \) be a positive definite matrix such that \( \varphi(X) = \text{Tr}(XS) \) \((X \in M_n)\). Then the trace property of \( \varphi \) is equivalent to the condition that \( S \) is a positive scalar multiple of the identity matrix. Taking into consideration \( \varphi(V^* \cdot V) = \text{Tr}((\cdot VS)V^*) \)

for all unitary \( V \) and that \( VSV^* \) is diagonal for a unitary \( U \), we may assume that that \( \varphi(\cdot) = \text{Tr}(\cdot S^{1/2}) \), where \( S = \text{diag}(s, 1) \) \((s \in [0, 1])\). For \( \beta > 0 \) and \( \alpha \in [0, 1] \), let us consider the matrices

\[
U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} \frac{s}{\alpha} & \frac{s}{\alpha^2} \\ \frac{s}{\alpha \sqrt{1 - \alpha^2}} & \frac{s}{\lambda \sqrt{1 - \alpha^2}} \end{pmatrix}
\]

and

\[
A = \lambda P_1 + \mu P_2,
\]

where

\[
P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
\]

And (7) becomes

\[
\frac{\alpha}{\sqrt{1 - \alpha^2}} \frac{f(\lambda)^{1/2} f'(\lambda)}{f(\lambda)^{1/2}} g'(\lambda) + \frac{\sqrt{1 - \alpha^2}}{\alpha} \frac{f(\mu)^{1/2} f'(\mu)}{f(\lambda)^{1/2}} g'(\mu) \geq 2 \frac{1 - s g(\lambda) - g(\mu)}{1 + s} \frac{\lambda - \mu}{\lambda \mu}.
\]

Put

\[
t = \frac{\alpha}{\sqrt{1 - \alpha^2}} \quad \text{and} \quad \delta = \frac{1 - s}{1 + s}.
\]

Note that \( 0 < \alpha < 1 \iff 0 < t < \infty \).
Then it is clear that \( s = 1 \) iff \( \delta = 0 \). The latter inequality is described as
\[
\frac{1}{2} \left( t \frac{f(\lambda)^{1/2}}{f(\mu)^{1/2}} g'(\lambda) + \frac{1}{t} \frac{f(\mu)^{1/2}}{f(\lambda)^{1/2}} g'(\mu) \right) \geq \delta \frac{g(\lambda) - g(\mu)}{\lambda - \mu}.
\]
Hence, by considering arithmetic-geometric mean inequality in the left-hand side, we have
\[
\sqrt{g'(\lambda)g'(\mu)} \geq \delta \frac{g(\lambda) - g(\mu)}{\lambda - \mu}.
\]
Therefore, the condition that \( \varphi \) has the trace property, that is, the condition \( s = 1 \) or \( \delta = 0 \) is given by
\[
\inf_{\lambda > \mu} \frac{\sqrt{g'(\lambda)g'(\mu)}}{g(\lambda) - g(\mu)} = 0.
\]
\( \square \)

**Example 3.5.** For \( g(x) = t^2 \) (i.e. \( f(t) = 1/t \)) on \((0, \infty)\) which satisfies the condition \( [\delta] \), and for any \( n \in \mathbb{N} \)
\[
\text{Tr}(A) \leq \text{Tr}(f(A)^{1/2}g(B)f(A)^{1/2})
\]
whenever \( 0 < A \leq B \) in \( M_n \).

Indeed, by assumption we have \( B^{-1} \leq A^{-1} \). Consequently,
\[
A \leq B = BB^{-1}B \leq BA^{-1}B.
\]
Therefore,
\[
\text{Tr}(A) \leq \text{Tr}(BA^{-1}B) = \text{Tr}(A^{-1/2}B^2A^{-1/2}).
\]
\( \square \)

We have the following inequality for the exponential functions \( g(t) = e^t \) on \((a, \infty)\) which satisfies the condition \( [\delta] \).

**Example 3.6.** For any natural number \( n \), we have
\[
\text{Tr}(A) \leq \text{Tr}((Ae^{-A})^{1/2}e^B(Ae^{-A})^{1/2})
\]
whenever \( 0 < A \leq B \) in \( M_n \).

Indeed, let \( A, B \) in \( M_n \) such that \( 0 < A \leq B \). Since \( 0 < A \), we have
\[
0 < Ae^{-A} \quad \text{and} \quad 0 < \log(Ae^{-A}).
\]
Besides,
\[
0 < A \leq B \quad \Rightarrow \quad 0 < A \leq \log(Ae^{-A}) + B.
\]
Consequently,
\[
0 < A \leq \log(Ae^{-A}) + B \leq e^{\log(Ae^{-A})+B}.
\]
On account of Golden-Thompson’s Inequality, from the latter inequality it follows
\[
\text{Tr}(A) \leq \text{Tr}(e^{\log(Ae^{-A})+B})
\leq \text{Tr}(e^{\log(Ae^{-A})}e^B)
\leq \text{Tr}(Ae^{-A}e^B)
= \text{Tr}((Ae^{-A})^{1/2}e^B(Ae^{-A})^{1/2}).
\]
On the contrary, when \( g(t) = t^2 \), the inequality \( \mathbf{\square} \) does not hold always.

**Example 3.7.** Let \( g(t) = t^2 \). Suppose that the inequality \( \mathbf{\square} \) holds for \( 0 < A \leq B \) in \( M_2 \). Then we have
\[
\text{Tr}(A) \leq \text{Tr}(A^{-1}BA^{-1})
\]
for \( 0 < A \leq B \). Set \( A = \text{diag}(2, 2) \) and \( B = A \). Since \( A^{-1} = \text{diag} \left( \frac{1}{2}, \frac{1}{2} \right) \), we have
\[
4 = \text{Tr}(A) \leq \text{Tr}(A^{-1}AA^{-1})
= \text{Tr}(A^{-1}) = 1,
\]
and a contradiction. \( \square \)

4. Characterization of Operator Monotonicity

The following lemma is obvious.

**Lemma 4.1.** Let \( A = (a_{ij}) \), \( B = (b_{ij}) \) be positive invertible in \( M_n \) and \( S \) be the density operator on an infinite dimensional, separable Hilbert space \( H \). Suppose that \( a_{11} > b_{11} \). Then there exist an orthogonal system \( \{\xi_i\}_{i=1}^\infty \subset H \) and \( \{\lambda_i\}_{i=1}^\infty \subset [0, 1) \) such that
\[
\sum_{i=1}^\infty \lambda_i = 1, \quad S\xi_i = \lambda_i\xi_i, \quad \text{and} \quad \sum_{i=1}^n a_{ii}\lambda_i > \sum_{i=1}^n b_{ii}\lambda_i.
\]

**Theorem 4.2.** Let \( \varphi \) be a normal state on \( B(H) \), \( f \) be a strictly positive, continuous function on \( (0, \infty) \), and let \( g \) be a function on \( (0, \infty) \) defined by \( g(t) = \frac{1}{t^m} \). Suppose that for any positive invertible \( A, B \in B(H) \)
\[
\varphi(A + B) - \varphi(|A - B|) \leq 2\varphi(f(A)^{\frac{1}{2}}g(B)f(A)^{\frac{1}{2}}).
\]
Then the function \( g \) on \( (0, \infty) \) is operator monotone.

**Proof.** Let \( S_\varphi \) be a density operator on \( H \) such that \( \varphi(X) = \text{Tr}(S_\varphi X) \) for all \( X \in B(H) \).

Suppose that \( g \) is not operator monotone. We have, then, there exist \( n \in \mathbb{N} \) and invertible positive matrices \( A, B \) in \( M_n \) with \( A \leq B \) such that \( g(A) \not\leq g(B) \). Hence, \( A \not\leq f(A)^{\frac{1}{2}}g(B)f(A)^{\frac{1}{2}} \).

Let \( A = [a_{ij}] \) and \( f(A)^{\frac{1}{2}}g(B)f(A)^{\frac{1}{2}} = [b_{ij}] = B' \). Since \( S_\varphi \) is a density operator, from Lemma 4.1 there exist an orthogonal system \( \{\xi_i\}_{i=1}^\infty \subset H \) and \( \{\lambda_i\}_{i=1}^\infty \subset [0, 1) \) such that
\[
\sum_{i=1}^\infty \lambda_i = 1 \quad \text{and} \quad \sum_{i=1}^n a_{ii}\lambda_i > \sum_{i=1}^n b_{ii}\lambda_i.
\]

Let \( \rho: M_n \to (\sum_{i=1}^n |\xi_i\rangle\langle\xi_i|)B(H)(\sum_{i=1}^n |\xi_i\rangle\langle\xi_i|) \) be a canonical inclusion defined by \( \rho([x_{ij}]) = \sum_{i,j=1}^n x_{ij}|\xi_i\rangle\langle\xi_j| \). Let \( C = \rho(A) + \sum_{i=n+1}^\infty |\xi_i\rangle\langle\xi_i| \) and \( D = \rho(B) + \sum_{i=n+1}^\infty |\xi_i\rangle\langle\xi_i| \).

We have, then, both operators \( C \) and \( D \) are invertible on \( H \) with \( C \leq D \). Note that
\[
\rho(f(A)^{\frac{1}{2}})\rho(g(B))\rho(f(A)^{\frac{1}{2}}) = \rho(f(A)^{\frac{1}{2}}g(B)f(A)^{\frac{1}{2}})
= \sum_{i=1}^n b_{ij}|\xi_i\rangle\langle\xi_j|.
\]
We have, then,
\[ f(C) = \rho(f(A)) + f(1) \sum_{i=n+1}^{\infty} |\xi_i\rangle\langle\xi_i| \]
\[ f(C)^{\frac{1}{2}} g(D) f(C)^{\frac{1}{2}} = (\rho(f(A)) + f(1) \sum_{i=n+1}^{\infty} |\xi_i\rangle\langle\xi_i|)^{\frac{1}{2}} \]
\[ \left( \rho(g(B)) + \frac{1}{f(1)} \sum_{i=n+1}^{\infty} |\xi_i\rangle\langle\xi_i| (\rho(f(A)) + f(1) \sum_{i=n+1}^{\infty} |\xi_i\rangle\langle\xi_i|)^{\frac{1}{2}} \right) \]
\[ \rho(f(A))^{\frac{1}{2}} \rho(g(B)) \rho(f(A))^{\frac{1}{2}} + \sum_{i=n+1}^{\infty} |\xi_i\rangle\langle\xi_i| \]
\[ = \rho(f(A))^{\frac{1}{2}} \rho(g(B)) \rho(f(A))^{\frac{1}{2}} + \sum_{i=n+1}^{\infty} |\xi_i\rangle\langle\xi_i| \]
\[ = \sum_{i,j=1}^{n} b_{ij} |\xi_i\rangle\langle\xi_j| + \sum_{i=n+1}^{\infty} |\xi_i\rangle\langle\xi_i| \]

But
\[ \varphi(C) = \text{Tr}(S_{\varphi}(\rho(A)) + \sum_{i=n+1}^{\infty} |\xi_i\rangle\langle\xi_i|) \]
\[ = \sum_{i=1}^{n} (S_{\varphi}(\rho(A))|\xi_i\rangle|\xi_i\rangle) + \sum_{i=n+1}^{\infty} \lambda_i \]
\[ = \sum_{i=1}^{n} a_{ii} \lambda_i + \sum_{i=n+1}^{\infty} \lambda_i \]
\[ > \sum_{i=1}^{n} b_{ii} \lambda_i + \sum_{i=n+1}^{\infty} \lambda_i \]
\[ = \sum_{i=1}^{n} (S_{\varphi}(\rho(f(A))^{\frac{1}{2}} g(D) f(A)^{\frac{1}{2}})|\xi_i\rangle|\xi_i\rangle) + \sum_{i=n+1}^{\infty} \lambda_i \]
\[ = \text{Tr}(S_{\varphi}(\rho(f(A)^{\frac{1}{2}} g(D) f(A)^{\frac{1}{2}}) + \sum_{i=n+1}^{\infty} |\xi_i\rangle\langle\xi_i|) \]
\[ = \varphi(f(C)^{\frac{1}{2}} g(D) f(C)^{\frac{1}{2}}) \]

On the contrary, since \( 0 < C \leq D \) we have from the assumption
\[ \varphi(C + D) - \varphi(|C - D|) \leq 2\varphi(f(C)^{\frac{1}{2}} g(D) f(C)^{\frac{1}{2}}) \]
\[ 2\varphi(C) \leq 2\varphi(f(C)^{\frac{1}{2}} g(D) f(C)^{\frac{1}{2}}) \]
\[ \varphi(C) \leq \varphi(f(C)^{\frac{1}{2}} g(D) f(C)^{\frac{1}{2}}), \]
and this is a contradiction. Therefore, the function \( g \) is operator monotone. \( \square \)
Corollary 4.3. Under the same conditions in Theorem 4.2 $f$ is operator monotone on $(0, \infty)$.

Proof. This follows from [3, Corollary 6].

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