Observable currents and a covariant Poisson algebra of physical observables

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Abstract

Observable currents are conserved gauge invariant currents; physical observables may be calculated integrating them on appropriate hypersurfaces. Due to the conservation law the hypersurfaces become irrelevant up to homology, and the main objects of interest become the observable currents themselves. Gauge inequivalent solutions can be distinguished by means of observable currents. With the aim of modelling spacetime local physics, we work on spacetime domains \( U \subset M \) which may have boundaries and corners. Hamiltonian observable currents are those satisfying

\[
d_v F = -\iota_V \Omega_L + d_h \sigma_F
\]

and a certain boundary condition. The family of Hamiltonian observable currents is endowed with a bracket that gives it a structure which generalizes a Lie algebra in which the Jacobi relation is modified by the presence of a boundary term; however, boundary terms vanish when integrating over a hypersurface with \( \partial \Sigma \subset \partial U \) making the resulting algebra of observables a Lie algebra. In the resulting framework algebras of observable currents are associated to bounded domains, and the local algebras obey interesting glueing properties. These results are due to a revision of the concept of gauge invariance in bounded spacetime domains. A perturbation of the field on a bounded spacetime domain is regarded as gauge if: (i) the “first order holographic imprint” that it leaves in any hypersurface locally splitting a spacetime domain into two subdomains is negligible according to the linearized glueing field equation, and (ii) the perturbation vanishes at the boundary of the domain. A current is gauge invariant if the variation induced by any gauge perturbation vanishes up to boundary terms.

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1 Motivation

The multisymplectic approach to classical field theory (see for example [18, 9]) encodes the symplectic structure present in the space of gauge equivalence classes of solutions of a classical field theory by means of a local object in the jet bundle: the pre multisymplectic form, which may also be called the pre symplectic current, associated to a Lagrangian density $L$. In the first order Lagrangian framework that we will use in this article the premultisymplectic form will be denoted by $\Omega_L$ and when integrated on a hypersurface $\Sigma$ it yields a closed two form $\omega_{L\Sigma}$ on the space of first order data on $\Sigma$. The multisymplectic formula says that, when the history under consideration $\phi$ is a solution, given any two perturbations of the field $v, w$ (which are parametrized by vector fields in the space of first order data $V, W \in X_v(J^1Y|_{U}))$ there is a conservation law

$$\omega_{L\Sigma}(v, w) = \int_\Sigma j^1\phi^*\iota_W\iota_V\Omega_L = \omega_{L\Sigma'}(v, w)$$

for any $\Sigma' = \Sigma + \partial U'$ for some region inside of the domain of interest $U' \subset U$. The multisymplectic approach to field theory recognizes the spacetime local object

$$\Omega_L$$

as the carrier of geometric structure and brings it to the forefront.

In a similar way, it is natural to be interested in functions, $f_\Sigma$, of first order data on hypersurfaces that arise from a spacetime local object $F$ that is subject to a conservation law stating that when the history under consideration $\phi$ is a solution then

$$f_\Sigma(\phi) = \int_\Sigma j^1\phi^*F = f_{\Sigma'}(\phi).$$

The main objective of this article is the study of conserved currents of this type, that furthermore are gauge invariant. In Section 3 we introduce them and call them observable currents. The explicit knowledge of a rich enough family of physical observables in a nonlinear field theory is as hard a problem as the explicit knowledge of all the solutions of that theory. Our goal is not to explicitly construct observables, but to study the family of covariant objects that precede observables of a particular type. A part from the family of Noether currents there is a large family of observable currents corresponding to observable currents that generalize the notion of the symplectic product function of classical mechanics $\omega(v, w)$. We prove that observable currents are capable of separating solutions modulo gauge. This result is in sharp contrast with previous reports stating that in nonlinear field theories, besides Noether currents, there are no interesting families of conserved currents; see for example [14, 19, 21, 23, 17]. For a compact review of the notion of observables within multisymplectic approaches to field theory see [31]. Another related approach is Vitagliano’s version of the covariant phase space approach to classical field theory [32], which is based on the so called secondary calculus. His results are closely related to part of our results.

With the intention of modelling spacetime local physics, we work on spacetime domains $U \subset M$ which are allowed to have boundaries and corners.
Working in this spacetime local context forced us to review the notion of gauge equivalence in first order Lagrangian field theory over bounded spacetime domains. Subsection 2.2 is dedicated to a detailed presentation of a definition of gauge vector fields. The definition is motivated from a novel point of view; an expanded presentation with relatively few formulas is presented in an essay entitled “Gauge from holography” [34].

In Hamiltonian mechanics the symplectic structure dictates an association of vector fields to functions by the formula $df = -\iota_v \omega$. In the multisymplectic approach to classical field theory it is natural to look for a version of this relation in the jet bundle that when integrated in a hypersurface $\Sigma$ induces the mentioned relation between a Hamiltonian vector field and a physical observable. We propose

$$d_v F = -\iota_V \Omega_L + dh \sigma^F,$$

where the boundary term $\sigma^F$ does not have any effect after integration on a hypersurface with $\partial \Sigma \subset \partial U$. Our proposal is derived from the study of the geometrical structure participating in this version of classical field theory. The equation above is introduced in Section 4 after the appropriate notion of generators of multisymplectomorphisms is identified. An observable current participating in the equation given above together with an associated Hamiltonian vector field will be referred to as a Hamiltonian observable current.

The presence of the boundary term makes the formula less rigid than $d_v F = -\iota_V \Omega_L$ which could be guessed as a natural generalization of the formula that appears in mechanics. In the absence of boundaries Hamiltonian observable currents are enough to distinguish gauge inequivalent solutions, and it is natural to conjecture that all observable currents are Hamiltonian.

We define a bracket for Hamiltonian observable currents in Section 5. When integrating over a hypersurface with $\partial \Sigma \subset \partial U$, the bracket induced in the corresponding algebra of observables makes it a Poisson algebra. However, the bracket among the currents turns out not to be a Lie bracket because the Jacobi relation is modified by a boundary term. The resulting structure is a Lie $n$-algebra [28, 5].

In the resulting framework algebras of observable currents are associated to local domains; in Section 7 we study the properties of local algebras corresponding to nested and glued domains.

The framework used in this article uses tools and notations from the variational bicomplex. Since we restrict to first order Lagrangian densities, all the core ingredients of the framework live in the first and second jet bundles; thus excluding certain boundary terms whose specific form does not figure in the formalism, our work takes place in those finite dimensional spaces. For the convenience of the reader we include an appendix with the minimal set of definitions needed to read the article. A very good brief introduction can be found in [2].

The example of the Maxwell field is presented in a minimalistic style in Section 8. We provide all the necessary elements for the reader to go through the calculations by herself or himself with the intention of providing a familiar example that the interested reader can use to work out each aspect of the formalism without significant effort.
2 General framework

We work in a local Lagrangian first order formulation of field theory in which we allow domains with boundaries and corners. In this section we start with a brief review of standard material to fix notation, and spell out some (possibly unusual) assumptions that are essential in our framework. Then we carefully review the definition of what perturbations of the field are considered to be gauge.

2.1 First order Lagrangian field theory

Histories of the field are local sections \( \phi : U \subset M \to Y \) of the bundle \( Y \to M \) where \( U \subset M \) is a compact domain with piecewise smooth boundary. Physical histories are selected by Hamilton’s principle according to the action \( S_U(\phi) = \int_U L(j^1\phi) \), defined by a Lagrangian density \( L(x, \phi(x), D\phi(x)) \) whose domain is the first jet bundle, \( J^1Y \).

The derivative of the action in the direction prescribed by a variation of the field may be calculated by integration of a local object acting on vector field in the jet associated to the given variation. The mentioned local object is given by the variational formula

\[
d_v L = E(L) + d_h \Theta_L,
\]

where the differential in the jet has been written as \( d = d_h + d_v \). We stress that \( d_v L \) is a differential form in the first jet bundle \( J^1Y \) and not in the space of fields. The left hand side of the field equation \( E(L) = Id_v L \) is obtained from an integration by parts operator acting on \( d_v L \); the reminding term is horizontally exact (leading to the boundary term in the variation of the action) and becomes the corner stone for the geometric structure of this formulation of field theory. A reader who knows a different derivation of the field equations and the geometric structure will still be able to read the paper without problems. For the convenience of the reader, a minimal set of definitions of the variational bicomplex is given in the appendix. In addition, the case of the Maxwell field is presented in Section 8. The intention is helping the interested reader become familiarized with this framework working on a familiar example. Thus, the last section should not necessarily be read at the end; when the reader feels the need of a more concrete explanation she or he can work it out in the example. A very good brief introduction to the variational bicomplex can be found in [2], and for detailed references see for example [24, 3, 29].

Our notation for the space of solutions to the field equation as contained in the space of histories is \( \text{Sols}_U \subset \text{Hists}_U \). However, we will rarely talk about the space of solutions; instead, we will often refer to the subspace \( \mathcal{E}_L \subset J^2Y \) in which \( E(L) \) vanishes and \( C_L \subset J^1Y \) which is the projection of \( \mathcal{E}_L \) to the first jet. Variations of histories are parametrized by vector fields in \( J^1Y \) which are of the form \( j^1V \in \mathfrak{X}(J^1Y|_U) \) for a vertical

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\( ^1 \) Differential calculus simplifies in the infinite jet bundle \( J^\infty Y \), the space that contains all the jets of any finite order. Each differential form in \( J^\infty Y \) fits in a given jet of finite order. In first order field theory most relevant objects live in the first or second jets, which are finite dimensional manifolds. We will use the simplicity of calculation native to \( J^\infty Y \), but we will often say that our objects live in the first or second jet. It is an abuse of notation that we will commit through the paper.
vector field $V \in \mathfrak{X}_v(Y|_U)$. The space of variations of histories which moreover satisfy the linearized field equation, $\mathcal{L}_{j^2V}E(L)|_{\mathcal{E}_L} = 0$, will be referred to as perturbations and denoted by $\mathcal{F}_U \subset \mathfrak{X}_v(J^1Y|_U)$.

The premultisymplectic form $\Omega_L = -d\Theta_L$, a form of vertical degree two and horizontal degree $n - 1$, is the local precursor of the presymplectic form $\omega_{L\Sigma}$ on space of first order data on any hypersurface $\Sigma \subset M$. The conservation law for $\Omega_L$ following from $d^2L = 0$ and the variational formula (3) is the multisymplectic formula written in the introduction (1), and it is the geometric structure behind the conservation of the presymplectic form appearing in first order formulations of field theory in terms of initial data.

**Remark 1** (Cohomology classes vs a local description). Notice that since $\Theta_L$ arises as the boundary term in the variation of the action it is to be integrated at boundaries or connected components of boundaries which are cycles. We can also see from formula (4) that $\Theta_L$ is ambiguous up to horizontally exact terms. Thus, even when we write $\Theta_L$, it may seem more appropriate to think about its horizontal cohomology class, and this remark extends to the premultisymplectic form $\Omega_L$. However, the resulting framework would not be appropriate to model local physics—like describing what happens in a laboratory during the course of an experiment—because the local objects in the framework would be integrable only on extended hypersurfaces that could not be split into smaller pieces; we would not be able to compute local observables to compare them with the measurements performed in the laboratory. Thus, we will force the framework to let us work in compact spacetime domains $U \subset M$ in such a way that integration on hypersurfaces with $\partial \Sigma \subset \partial U$ can be done. Two stages are needed to accomplish this goal: (i) out of the mentioned cohomology classes we have to choose a representative (which is something that we are used to), and (ii) the choice has to be consistent with gauge equivalence (which is the subject of the next subsection).

A separate issue is that Lagrangian densities leading to the same variational problem should be considered equivalent, and adding boundary terms $L \rightarrow L + d_b b$ does not modify the problem stated by Hamilton’s principle of least action. Thus, the field equation $E(L)$ remains invariant under the addition of boundary terms, while the horizontally exact term changes as $\Theta_L \rightarrow \Theta_L - d_b b$ and the premultisymplectic form $\Omega_L$ also remains invariant.

Hence, $\Omega_L$ is the carrier of invariant geometrical structure in this framework associating presymplectic forms $\omega_{L\Sigma}$ to spacetime hypersurfaces with $\partial \Sigma \subset \partial U$ in a compatible way as phrased by the multisymplectic formula (1).

### 2.2 Gauge freedom

In physics, a description includes gauge freedom if physically distinct configurations do not correspond to points in the space that hosts it, but to equivalence classes. Often the equivalence classes are the orbits of certain vector fields declared to be gauge vector

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2 A vector field is called vertical if at every point it is tangential to the fibers where the field takes values.

3 Notice that the linearized field equation is written in terms of the prolongation of the vector field to the second jet, $j^2V$, because the Euler-Lagrange equations involve second derivatives.
fields. In the Lagrangian first order formalism gauge freedom can be understood considering the propagation of perturbations through hypersurfaces: A perturbation whose first order holographic imprint on any hypersurface is negligible is declared to be gauge. Gauge equivalence and locality have a delicate relation; our framework is phrased within a compact spacetime domain which may be glued to other spacetime regions through its boundary and our goal is to compute physically meaningful objects as appropriate compositions of objects defined on compact spacetime domains. Below we give a precise definition of the notion of gauge freedom and explain its motivation. For different arguments leading to a closely related related but inequivalent definition of gauge freedom see Wald and Lee [25].

Definition 1 (Gauge vector fields). A solution of the linearized field equation $X \in \mathfrak{X}(J^1Y|_U)$ is declared to be a gauge vector field, $X \in \mathcal{G}_U$, if and only if

1. $\iota_X \Omega_L$ is horizontally exact when evaluated in $C^L$, and
2. The restriction of $X$ to the subbundle over $\partial U$ vanishes, $X|_{\partial U} = 0$.

Below we will prove that $\mathcal{G}_U$ is a Lie algebra. Thus, this definition induces a notion of gauge equivalence classes.

If we are working on a domain of the type $U = \Sigma \times [0,1]$ endowed with a foliation $\Sigma_t$, we may replace Condition 2 by $X|_{\partial \Sigma \times [0,1]} = 0$; this will be addressed below.

Condition 1 in the definition says that the first order requirement for glueing perturbations considers $X$ as equivalent to the null perturbation. Now we give a more detailed explanation about glueing perturbations supporting this statement. Consider a spacetime domain $U \subset M$ and an arbitrary partition of it into two pieces separated by a hypersurface, $U = U_1 \cup U_2$ with $\Sigma = U_1 \cap U_2$. Either $\Sigma$ is a cycle (i.e. $\partial \Sigma = \emptyset$) or $\partial \Sigma \subset \partial U$. Let us write the field as the glueing of its restriction to the pieces of the domain $\phi = \phi_1 \#_{\Sigma} \phi_2$, where the use of the glueing symbol assumes that the field is continuous at $\Sigma$. The action and its variation are additive under such a subdivision of the domain, $dS_U = dS_{U_1} + dS_{U_2}$. However, when we split the domain in two pieces the degree of differentiability of the field over $\Sigma$ is relaxed and the usual variation of the action $dS_U|\{\phi_0\} = \int_U j^1 \phi^* t_j^1 V E(L) + \int_{\partial U} j^1 \phi^* t_j^1 V \Theta_L$ following from (3) acquires an extra term associated to $\Sigma$

$$\int_{\Sigma} (j^1 \phi_1^* - j^1 \phi_2^*) t_j^1 V \Theta_L.$$ (4)

If we look for extrema of $S_U$, apart from field equations at $U_1$ and $U_2$ there is a glueing field equation at $\Sigma$ requiring that the above integral vanishes for any variation that vanishes at $\partial U$. The local incarnation of this condition is $(j^1 \phi_1^* - j^1 \phi_2^*) I(\Theta_L|_{\Sigma}) = 0$, where we have written the field as $\phi = \phi_1 \#_{\Sigma} \phi_2$ and $I$ is the integration by parts operator. Now consider a one parameter family of fields $\phi_t$ (with $\phi_{t=0} = \phi$ and with the variation at $t = 0$ given by $V = V_1 \#_{\Sigma} V_2$) solving the field equation in $U_1$ and $U_2$ and solving the

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4 Notice that even when $X$ is an object in the first jet, we wrote its restriction to the restriction of the jet over $\partial U$ in an abbreviated fashion. We will continue to use this simplified notation throughout the paper.

5 In the appendix we recall the definition of the integration by parts operator.
glueing problem over \( \Sigma \). Since for each value of the parameter the field \( \phi_t \) is an extremum of (4), at first order in \( t \) we have \( I(\Theta_L(j^1(\phi + t(V_1 - V_2)))|\Sigma) = 0 \). Thus, the linearized glueing equation is

\[
I(\mathcal{L}_{j^1V_1 - j^1V_2}\Theta_L|\Sigma) = -I(\iota_{j^1V_1 - j^1V_2}\Omega_L|\Sigma) = 0,
\]

where we have simplified the expression using Cartan’s identity for the Lie derivative of vertical vector fields \( \mathcal{L}_X = \iota_X d_v + d_v \iota_X \) and the fact that \( (V_1 - V_2)|\Sigma = 0 \) implies that the vector field in the first jet \( j^1V_1 - j^1V_2 \) is in the kernel of \( \Theta_L \), as can be readily verified from its expression within a coordinate system. Notice that this is a condition on the perturbation that does not explicitly involve the field, but due to its origin, we should demand that the condition holds in \( C_{\mathcal{L}} \subset J^1Y \). Since the integration by parts operator decomposes any \( n-1 \) horizontal form in \( \Sigma \) as \( \mu = I(\mu) + d_h \sigma \), and it satisfies \( Id_h = 0 \), we see that the linearized glueing field equation is equivalent to requiring that \( j^1V_1 - j^1V_2 \Omega_L|\Sigma \) be horizontally exact. Thus, the requirements for glueing perturbations across hypersurface \( \Sigma \) are: (C) Continuity of the perturbation at the dividing hypersurface; that is, \( (V_1 - V_2)|\Sigma = 0 \). (LG) The linearized glueing field equation, that \( (\iota_{j^1V_1 - j^1V_2}\Omega_L|\Sigma)|_{C_{\mathcal{L}}} \) be horizontally exact. This equation contains a germ of information regarding the bulk; more precisely, it contains partial derivatives of the perturbations in directions transversal to the dividing hypersurface. We call this information the \textit{first order holographic imprint} of the perturbation. The linear operator which appears in the linearized glueing equation may have a nontrivial null space. Such a linearized glueing equation would find the imprint left by some non zero perturbations as negligible. For those perturbations propagation through a dividing hypersurface proceeds without any trace of bulk information. \textit{Vector fields satisfying Condition 1 may have a complicated form in the bulk, but as far as propagation through \( \star \) any \( \star \) dividing hypersurface all this information is lost; the definition of gauge vector fields declares those degrees of freedom as physically unimportant. This is the motivation for Condition 1 in the definition of gauge vector fields.} Further support for Condition 1 is given in Remark 10 of section 4 where we consider the notion of multisymplectomorphisms and related locally Hamiltonian vector fields. In Section 8 we show how in the case of Maxwell’s field the familiar notion of gauge arises from Condition 1.

Due to a technical lemma [30] reviewed in the appendix, an equivalent condition to Condition 1 is the equation \( (d_h(\iota_{j^1V_1 - j^1V_2}\Omega_L)|\Sigma) = 0 \) which may be simpler to verify.

We mentioned that since we work with a first order Lagrangian density, most objects in our formalism fit in the first or second jet bundles. We must warn the reader that Condition 1 demanding that an object in the first jet be horizontally exact means that there is a form \( \sigma \) such that its horizontal differential gives \( \iota_X \Omega \), but \( \sigma \) is not restricted to live in the first jet; if one insists in working in the first jet one also must allow \( \sigma \) to be a differential operator of any finite order.

Condition 2 of the definition of gauge vector fields is essential for the integration of currents on hypersurfaces with \( \partial \Sigma \subset \partial U \) producing gauge invariant quantities. Below, in Remark (3) we will spell out the condition on a current to be gauge invariant. From the definition it is clear that without Condition 2 demanding gauge invariance would force us to work only with cohomology classes that we would be able to integrate only on cycles.
rendering most allowed calculations at a compact domain \( U \subset M \) trivial. Additionally, Remark 5 shows that measuring properties of the system in a domain \( U \) may need a reference at \( \partial U \) and preserving that reference frame may be essential for talking about those properties. Yet another reason for including Condition 2 in our definition comes from the standard definition of gauge vector fields as generators of Lagrange symmetries depending on arbitrary local parameters. Wald and Lee [25] start with a precise version of that definition and arrive to our Condition 1, but along their argument they assume that if there is a boundary it is located at infinity which (together with appropriate fall-off conditions on the field) lets them conclude that the Noether charge associated to a gauge vector field \( X \) according to their definition vanishes identically \( Q^X_\Sigma = 0 \). The interested reader is invited to try to reproduce the mentioned argument by Wald and Lee in the context of a domain with boundary using the result shown in Remark 12; Condition 2 will emerge naturally. Recently, Freidel and Donnelly [12] emphasized that in domains with boundary a condition in the spirit of Condition 2 is necessary and gives rise to “would be gauge degrees of freedom” living at the boundary; see Remark 6. Their motivation came from entanglement entropy in gauge theories [11] and general relativity in spacetime domains with corners [15].

The definition of gauge vector fields given above and an expanded version of this supporting argument are presented in a essay entitled Gauge from holography [34].

Remark 2 (Other definitions of gauge). A definition of gauge vector fields very closely related to Condition 1 of our definition appears in the work of Vitagliano in the context of the covariant phase space in the language of variational bicomplex and secondary calculus [32]. Other references in the context of classical field theory and the variational bicomplex also give definitions closely related to Condition 1 [26, 29]. The work of Wald and Lee is the reference for the subject in the context of the covariant phase space formulation of field theory [25]. The rough idea behind those other definitions of gauge, clearly stated in [25] is that families of symmetries depending on locally independent parameters become an obstacle for predictability of the theory and should be regarded as gauge. A complementary feature of that notion of gauge symmetries is that they are linked with relations among the field equations (and the linearized field equations) that show up in the form of constraints or as the statement that the Noether currents associated to the gauge symmetries vanish on-shell (up to boundary terms); this phenomenon is the content of Noether’s second theorem. Another important property is that the evaluation of a (pre)symplectic product of variations is independent of changes of the variations in gauge directions.

Every gauge vector field according to Wald and Lee satisfies Condition 1 of our definition [32, 25].

Remark 3 (Gauge invariance). A function of the first jet is gauge invariant if it remains constant along orbits of the gauge vector fields (the existence of the mentioned orbits is justified below). Since in our work currents play a central role, we need to spell out the meaning of gauge invariance for them. The natural gauge invariance requirement for a current is to ask that its integration on cycles produces gauge invariant functions (when evaluated on solutions). The corresponding local requirement in the jet is to call a current...
\( F \) (a \( n - 1 \) horizontal form in \( J^1Y \)) gauge invariant if for every \( X \in \mathcal{G}_U \)

\[ \mathcal{L}_X F|_{\mathcal{C}_L} = d_h \sigma \text{ for some } \sigma. \tag{5} \]

Notice that \( \sigma \) must be linear in \( X \) and that Condition 2 in Definition \( \{1\} \) implies that the restriction of \( \sigma \) to the subbundle over \( \partial U \) vanishes, \( \sigma|_{\partial U} = 0 \). Any spacetime cycle \( \Sigma \subset M \) may be decomposed as a sum of hypersurfaces contained in compact domains \( U_i \) with \( \partial \Sigma_i \subset \partial U_i \) and we may write \( f_\Sigma(\phi) = f_{\Sigma_i} \mathcal{J}^1 \phi^* F \). Due to Condition 2 each \( f_{\Sigma_i}(\phi) \) is gauge invariant. However, if \( \Sigma \) is a hypersurface with boundary and \( \partial \Sigma_i \) is not contained in \( \partial U \) then \( f_{\Sigma}(\phi) \) is not gauge invariant; if we choose a representative in the cohomology class of \( F \) to calculate \( f_{\Sigma}(\phi) \) a gauge transformation would not preserve our choice and the resulting boundary term in the integral would not vanish.

Gauge vector fields \( X \in \mathcal{G}_U \) preserve the premultisymplectic form in the sense that \( \mathcal{L}_X \Omega_L|_{\mathcal{C}_L} \) is horizontally exact. Thus, the presymplectic form obtained by integration on any cycle as in formula \( \{7\} \) will be gauge invariant, \( \mathcal{L}_X \omega|_{\Sigma \cap \text{Sols}} = 0 \). Additionally, if a hypersurface is not a cycle but \( \partial \Sigma \subset \partial U \) then \( \omega|_{\Sigma} \) would also be preserved by gauge transformations.

**Remark 4** (Gauge equivalence classes). We need to talk about equivalence classes in \( \mathcal{C}_L \subset J^1Y|_U \) arising from the orbits of gauge vector fields. The local existence of such orbits follows from \( \mathcal{G}_U \) being a Lie subalgebra of \( \mathfrak{F}_U \). Given any \( X, Y \in \mathcal{G}_U \) (with \( \iota_X \Omega_L = d_h \rho^X, \iota_Y \Omega_L = d_h \rho^Y \)) a short calculation yields

\[ \iota_{[X,Y]} \Omega_L = d_h (\mathcal{L}_X \rho^Y - \mathcal{L}_Y \rho^X). \]

Thus, the flows of these vector fields define the local action of a group, the gauge group \( G \), in a neighborhood of \( \mathcal{C}_L \subset J^1Y \) preserving \( \mathcal{C}_L \). A more precise geometric picture is obtained looking at the field equation in the second jet \( \mathcal{E}_L \subset J^2Y \) where again the gauge group acts on a neighborhood of \( \mathcal{E}_L \) preserving it. For heuristic arguments it will be relevant to have in mind the local product structure induced by gauge equivalence. Each point of \( \mathcal{E}_L \) is contained in a neighborhood \( \Delta \) that is decomposed as a product of gauge orbits over a space of gauge classes \( (\mathcal{E}_L/G)_{\Delta} \) that is a bundle over \( U \). However, we will continue to work in \( J^1Y \) (and \( J^2Y \)) looking for objects that are appropriately gauge invariant.

Vector fields \( V \in \mathfrak{X}_v(J^1Y|_U) \) that are gauge orbit preserving are also invariant under the flow of gauge vector fields (modulo gauge vector fields); then, we will refer to these vector fields frequently as gauge invariant. The Lie subalgebra of gauge invariant solutions of the linearized field equations will be denoted by

\[ \mathfrak{F}_U^G := \{ V \in \mathfrak{F}_U : \mathcal{L}_V X \in \mathcal{G}_U, \forall X \in \mathcal{G}_U \}. \]

Since \( \mathcal{G}_U \subset \mathfrak{F}_U^G \) is a Lie ideal, the quotient makes sense and inherits a Lie algebra structure leading to a reduced space \( \mathfrak{F}_U^G/\mathcal{G}_U := \mathfrak{F}_U^G/\mathcal{G}_U \) in which the premultisymplectic form \( \Omega_L \) becomes non degenerate in the appropriate sense.
Remark 5 (Isolated systems and measuring with respect to the boundary). We can apply our formalism in the context of asymptotically flat General Relativity formulated la Palatini \cite{4}. The spacetime domain considered in this case is of the type $U = \Sigma \times [0, 1]$ with the boundary $\partial \Sigma \times [0, 1]$ being a world tube at spatial infinity (and possibly an inner boundary modelling a horizon); it is known that diffeomorphisms induce variations such that $\iota_X \Omega_L$ is horizontally exact, which implies that $X$ satisfies Condition 1 of the definition of gauge vector fields. However, regarding variations that do not vanish at infinity as gauge is inappropriate because they modify the reference frame needed to define energy, momentum and angular momentum. Thus, preserving a reference frame at the boundary that may be used as a reference for measurements is another motivation for Condition 2 of the definition of gauge.

In domains of the type $U = \Sigma \times [0, 1]$ endowed with a foliation $\Sigma_t$ it may be desirable that Condition 2 is replaced by $X|_{\partial \Sigma \times [0, 1]} = 0$. If we use this condition all leaves $\Sigma_t$ in a foliation would be analogous to the leaves of initial and final conditions at $t = 0, 1$. This way of working introduces an asymmetry regarding gluing domains in the “time” direction and in the other direction of the product. The alternative is to work on this type of domains using Condition 2 considering leaves $\Sigma_t$ in a foliation with $t \in (1, 0)$.

Remark 6 (“Would be gauge” degrees of freedom at the boundary). Condition 2 in the definition of gauge vector fields had the main purpose of allowing for a local description of physics. For the sake of this discussion consider the Lie algebra of vector fields $\hat{G}_U$ satisfying the linearized field equation and Condition 1 of the definition of gauge without imposing Condition 2. In the following heuristic argument based on the local product structure described in Remark 2 we will think of an action of the group of gauge transformations $G$ on a neighborhood of $E_L$ leading to the bundle $(E_L/G)_{\Delta}$ over $U$, and we will also think of a larger group of transformations $\hat{G}$ induced by the vector fields that would be gauge if we ignore Condition 2. Notice that $\hat{G}_U \subset \hat{G}_U$ is a Lie ideal. The quotient $(\hat{G}/G)_U$ is characterized by vector fields at $\partial U$ which are extendible to gauge vector fields on the bundle over $U$. The resulting classes of transformations would further reduce the bundle $(E_L/G)_{\Delta}$ to $(E_L/\hat{G})_{\Delta}$, and the mentioned reduction is due to a group action which is not trivial only over $\partial U$. In this sense our restriction to gauge fields that vanish over $\partial U$ has the effect of adding boundary degrees of freedom in $(E_L/G)_{\Delta}$ as compared to $(E_L/\hat{G})_{\Delta}$. A formalism to study gauge theories in the presence of boundaries was recently put forward by Donnelly and Freidel in which boundary degrees of freedom are added to the system \cite{12}.

From the perspective of our formalism the “dynamics” of these degrees of freedom “added” at the boundary is not dictated by new independent field equations. The field is bounded to be the restriction to $\partial U$ of a solution to the bulk field equation; additionally, there is a symmetry acting non trivially on those degrees of freedom generated by $(\hat{G}/G)_U$. We will mention in Remark 12 that a class of vector fields in $(\hat{G}/G)_U$ which comes from local Lagrangian symmetries may have an associated Noether current which does not vanish. All these properties seem to be in agreement with \cite{12}. It would be interesting to have a detailed understanding of the relation between the formalism that we describe here and theirs.
Remark 7 (Glueing spacetime domains). Consider a domain that is constructed by glueing two subdomains \( U = U_1 \cup U_2 \) over a codimension one cycle \( \Sigma = U_1 \cap U_2 \). Some gauge vector fields over \( U \) are composed by a pair of a gauge vector fields over \( U_1 \) and a gauge vector fields over \( U_2 \). Notice that due to Condition 2 the given pair trivially satisfies the continuity condition at \( \Sigma \), and it also trivially satisfies the linearized glueing field equation due to Condition 1. However, there are some gauge vector fields at \( U \) that do not vanish over \( \Sigma \). As mentioned in the previous remark, these gauge vector fields when considered over \( U_i \) were symmetry generators and after the domains are glued they become gauge vector fields. In terms of the bundles used in the heuristic argument of the previous remark \( (E_L/G)_{\Delta U} \) is constructed in two steps from the bundles \( (E_L/G)_{\Delta U_i} \): First we consider “the diagonal” of the cartesian product obtained by imposing glueing field equations over \( \Sigma \). Second we take a quotient by the group generated by the shared “would be gauge” vector fields \( (\hat{\mathcal{G}}/\mathcal{G})_{U_i} \) over \( \Sigma \). This glueing procedure also seems to be in agreement with the construction of Donnelly and Freidel [12]. We will return to the subject of glueing subdomains further ahead in the paper when we consider the algebras of observable currents associated to spacetime domains.

3 Observable currents

Physical observables, functions of the space of solutions modulo gauge, may be constructed by integration of currents on hypersurfaces as in formula (2). Gauge invariant conserved currents are the central object of this work; in order to emphasize the use that we will give them, we will call them observable currents.

**Definition 2** (Observable currents). (i) A current \( F \in \Omega^{n-1,0}(J^1 Y) \) is conserved if \( \partial_h F|_{E_L} = 0 \). (ii) It is gauge invariant if \( \mathcal{L}_X F|_{E_L} \) is horizontally exact for every \( X \in \mathcal{G}_U \).

An observable current is a gauge invariant conserved current. We will write \( F \in OC_U \).

The objective of any current \( F \in \Omega^{n-1,0}(J^1 Y) \), in its whole existence, is to be paired with an oriented hypersurface \( \Sigma \) so they together beget a function \( f_\Sigma : Hist_U \rightarrow \mathbb{R} \) through integration

\[
f_\Sigma(\phi) = \int_\Sigma j^1 \phi^* F.
\]

The function \( f_\Sigma \) is defined for any oriented hypersurface, and the conservation law obeyed by observable currents when evaluating on \( Sols_U \subset Hist_U \) disregards \( \Sigma \) as unimportant (except for its homology class) and most of the features of \( f_\Sigma \) have origin in

\[
F \in OC_U.
\]

Notice that if \( \Sigma_1 \sim \Sigma_2 \) and the hypersurfaces are not cycles then \( \partial \Sigma_1 = \partial \Sigma_2 \).

Functions induced by observable currents defined by equation (6) are gauge invariant if \( \partial \Sigma \subset \partial U \). When we restrict these functions to act on solutions we will call them physical observables.

---

\* Equivalently, one may demand that \( j^1 \phi^* \mathcal{L}_X F \) be exact for any solution \( \phi \in Sols_U \).
Definition 3 (Observables from observable currents). The space of physical observables

\[ f_\Sigma : \text{Sols}_U \to \mathbb{R} \]

associated to a hypersurface such that \( \partial \Sigma \subset \partial U \) will be denoted by \( \text{Obs}_\Sigma \).

Remark 8 (Domain with a foliation). In a domain of the type \( U = \Sigma \times [0, 1] \) endowed with a foliation \( \Sigma_t \) we may be interested in studying evolution of functions \( f_{\Sigma_t} \) as functions of the “time” parameter. In this situation the conservation law tells us that \( f_{\Sigma_t} - f_{\Sigma_0} = \int_{\partial \Sigma \times [0, t]} \cdot \). Thus, if the boundary conditions and \( F \in \text{OC}_U \) are such that \( \int_{\partial \Sigma \times [0, t]} = 0 \) the conservation law will simply state that the value of \( f_{\Sigma_t} \) is time independent. In particular, this is expected to be the case for a class of observable currents of physical interest when the physical boundary \( \partial \Sigma \times [0, t] \) is located “at infinity”. However, in general the term \( \int_{\partial \Sigma \times [0, t]} \) will be relevant. This remark also applies to the conservation of the presymplectic form \( \omega_{\Sigma_t} \).

Gauge invariance is delicate because \( \Sigma_t \) has a boundary. The requirement of gauge invariance is that \( \mathcal{L}_X F |_{\mathcal{L}_L} \) be horizontally exact, but since gauge vector fields satisfy \( X |_{\partial \Sigma \times [0, 1]} = 0 \) the differential form \( \sigma \) satisfying \( \mathcal{L}_X F |_{\mathcal{L}_L} = \delta_h \sigma \) vanishes at the boundary, \( \sigma |_{\partial \Sigma \times [0, 1]} = 0 \). Then, in the general case \( f_{\Sigma_t} \) is not gauge invariant when \( \Sigma_t \) is not a cycle due to boundary terms, but if \( \partial \Sigma \subset \partial \Sigma \times [0, 1] \) the boundary term breaking gauge invariance vanishes and the function is gauge invariant. Thus due to Condition 2, functions \( f_{\Sigma_t} \) associated to the leaves of the foliation \( \Sigma_t \) are gauge invariant.

In Remark 8 we mentioned that in domains endowed with a foliation one may opt to replace Condition 2 in the definition of gauge vector fields by \( X |_{\partial \Sigma \times [0, 1]} = 0 \). This modification has the effect of making all leaves with \( t \in [0, 1] \) equivalent.

Noether’s theorem stating that symmetries lead to conserved quantities is crystal clear in this framework.

Theorem 1 (Noether). A Lagrange symmetry is a vector field \( V \in \mathfrak{x}_x(\mathcal{J}^1 Y |_U) \) satisfying \( \mathcal{L}_V L = \delta_h \sigma_L^V \). Every Lagrange symmetry has a corresponding Noether current \( N^V \in \text{OC}_U \) given by

\[ N^V = -t_V \Theta - \sigma_L^V. \]

We include a proof of this classical theorem in the appendix; a more detailed presentation can be found in [29]. Proving conservation of the Noether current is trivial, but on the other hand gauge invariance requires the use of a technical lemma of Takens [30].

A large family of observable currents is given below. Given our previous definitions the proof of this result is simple, we state it as a theorem because in the context of multisymplectic formulations of classical field theory the existence of a rich family of gauge invariant conserved currents has been a long standing problem (see for example [14, 19, 21, 23, 17]).

Definition 4 (Symplectic product current). Given a pair of solutions of the linearized field equations \( V, W \in \mathfrak{s}_U \) their symplectic product is the current

\[ F^{VW} = t_W t_V \Omega_L. \]
Theorem 2. The symplectic product current of two gauge invariant solutions to the linearized field equation \( V, W \in \mathcal{F}_U \subset \mathfrak{h}_U \) is an observable current

\[ F^{VW} \in OC_U. \]

Proof. Conservation of \( F^{VW} \) is the statement that the multisymplectic formula, described in Section 2, holds. Gauge invariance follows from the gauge invariance of \( V, W \) and \( \Omega_L \).

In Section 8 we give the elements to evaluate symplectic product observable currents in the case of the Maxwell field. However, the resulting observable currents are trivial due to the linearity of the field. An explicit example (with spacetime being one dimensional) showing that symplectic product observable currents are generically non-trivial is rigid body motion [1], where the configuration of the system at time \( t \) is given by \( q \in SO(3) \). Let us denote left invariant vector fields in \( SO(3) \) by \( \xi \in \mathcal{X}(SO(3)) \). In the first order Lagrangian framework, the state of the system at time \( t \) is given by \((q, \xi)\) \( \in TSO(3) \). Perturbations corresponding to generators of rotations may be parametrized by left invariant vector fields in \( SO(3) \); let us denote such perturbations by \( V^\xi \in \mathcal{X}(TSO(3)) \). Consider the system at time \( t = 0 \) at state \((q, \xi)\) \( \in TSO(3) \) and two perturbations of the system at that time \( V^\xi_1 \) and \( V^\xi_2 \). Evolution according to the Euler-Lagrange equation will yield \((q(t), \xi(t)) \in TSO(3) \); the perturbations will also evolve according to the linearized equation, and yield \( V^\xi_1(t) \) and \( V^\xi_2(t) \). The evaluation of the symplectic product \( f^{V^\xi_1(t)V^\xi_2(t)} \) using the symplectic form in \( TSO(3) \) induced by the Lagrangian (or equivalently by Legendre transformation of the symplectic form of \( T^*SO(3) \)) is 

\[ \omega_L(V^\xi_1(t), V^\xi_2(t))_{(q(t), \xi(t)_{q(t)})} = -d\theta_L(V^\xi_1(t), V^\xi_2(t))_{(q(t), \xi_1(t)_{q(t)})}, \]

where the symplectic potential is basically the angular momentum calculated in the body reference frame. The body angular momentum is not constant in time and the perturbations also evolve in time, but their combination in \( f^{V^\xi_1(t)V^\xi_2(t)} \) is a conserved quantity. This is a family of conserved quantities parametrized by the choice of two elements of the Lie algebra \( \xi_1, \xi_2 \) which encode information regarding the state of the system. The Hamiltonian vector field associated to the observable shown above is the commutator of the vector fields, \([V^\xi_1(t), V^\xi_2(t)]\). The same logic can be used in the case of the Yang-Mills field [20] to obtain nontrivial explicit observable currents of the symplectic product type.

Remark 9 (Observable currents in linear field theories). If we have a theory in which \( \text{Sols}_U \) is a linear subspace of \( \text{Hist}_U \) then the spaces \( \text{Sols}_U \) and \( \mathfrak{h}_U \) may be identified. This trick, extensively used by Wald in the quantization of linear fields [33], leads to the following special type of observable currents \( F^V \in OC_{SU} \) parametrized by an element \( V \in \mathfrak{h}_U \),

\[ F^V(j^1\phi) = i_W(j^1\phi) i_V \Omega_L(j^1\phi), \]

where \( W(\phi) \) is an element of \( \mathfrak{h}_U \) that is compatible with the solution \( \phi \in \text{Sols}_U \). By construction \( d_v F^V = -i_V \Omega_L \). We give an explicit example in Section 8.
4 Locally Hamiltonian vector fields and Hamiltonian observable currents

In the multisymplectic framework for field theory described in Section 2, the core geometrical structure associated to a field theory is given by the structure of $J^1Y$ (and $J^2Y$), the field equations $\mathcal{E}_L \subset J^2Y$ and the premultisymplectic form $\Omega_L$. Thus, it is natural to look for the structure preserving automorphisms of $J^1Y|_U$. Automorphisms $\Phi : J^1Y|_U \to J^1Y|_U$ are diffeomorphisms that preserve the fibers of the fibration over $Y$ and of the fibration over $U \subset M$ and that moreover send sections of the type $j^1\phi : U \to J^1Y|_U$ to other sections of the same type. In this work we will be particularly interested on vertical automorphisms; that is, those inducing the identity map on $U$. Then, we are interested on vertical automorphisms such that $j^2\Phi(\mathcal{E}_L) = \mathcal{E}_L$ and such that $\Phi^*\Omega_L = \Omega_L$ up to a horizontally exact term vanishing on $\pi|_{\partial U}$ (when we evaluate on $\mathcal{C}_L$); we may call these maps premultisymplectomorphisms. The generators of such vertical automorphisms are solutions to the linearized field equation $V \in \mathfrak{F}_U \subset \mathcal{X}_v(J^1Y|_U)$ that furthermore satisfy

$$\mathcal{L}_V\Omega_L = \text{dh}\sigma_V$$

when restricted to $\mathcal{C}_L$ for some boundary term such that $\sigma_V|_{\partial U} = 0$. These generators of premultisymplectomorphisms will be referred to as locally Hamiltonian vector fields, and the space of such vector fields will be denoted by $\mathfrak{F}^\text{LH}_U \subset \mathfrak{F}_U$. For a locally Hamiltonian vector field and a hypersurface with $\partial \Sigma \subset \partial U$ we get $\mathcal{L}_V\omega_{LS} = 0$.

There are two remarks relating locally Hamiltonian vector fields and gauge vector fields. First, if $X$ is a gauge vector field then it is locally Hamiltonian, $\mathcal{G}_U \subset \mathfrak{F}^\text{LH}_U$. Second, locally Hamiltonian vector fields are gauge invariant, $\mathfrak{F}^\text{LH}_U \subset \mathfrak{F}^\text{G}_U$; this is because preserving $\Omega_L$ implies preserving Condition 1 of Definition 1 for $\mathcal{G}_U$, and Condition 2 of the definition is also preserved.

The equation above says that, when restricted to $\mathcal{C}_L$, the form $\iota_V\Omega_L$ is vertically closed (up to horizontally exact terms vanishing at the boundary). Thus, it is natural to study if it can be promoted to be vertically exact (up to horizontally exact terms vanishing at the boundary). More concretely, we look for an observable current $F$ such that

$$\text{d}_vF = -\iota_V\Omega_L + \text{dh}\sigma^F$$

when restricted to $\mathcal{C}_L$ for some boundary term such that $\sigma^F|_{\partial U} = 0$.

**Definition 5** (Hamiltonian observable currents). An observable current $F \in OC_U$ and a gauge invariant solution of the linearized field equations $V \in \mathfrak{G}_U$ participating in equation (7) are called Hamiltonian observable current, $F \in \mathcal{H}OC_U$, and Hamiltonian vector field, $V \in \mathfrak{F}^\text{H}_U \subset \mathfrak{F}^\text{LH}_U$.

A unique gauge equivalence class of Hamiltonian vector fields is assigned to the vertical differential of a Hamiltonian observable current. If two Hamiltonian vector fields $V_1, V_2$ are associated to the same vertical differential of a Hamiltonian observable current, $\text{d}_vF$, then, when restricted to $\mathcal{C}_L$, $\iota_{V_1-V_2}\Omega_L = \text{dh}(\sigma^1F - \sigma^2F)$ and $\rho^{V_1-V_2}|_{\partial U} \equiv \sigma^1F - \sigma^2F|_{\partial U} = 0$ proving that the vector fields differ by a gauge vector field. The same
argument shows that gauge vector fields are Hamiltonian vector fields associated to any vertically constant observable current, \( \mathcal{G}_U \subset \mathfrak{g}^H_U \).

A Hamiltonian observable current associated to a given locally Hamiltonian vector field is not uniquely determined by its Hamiltonian vector field. In classical mechanics the association is unique up to an integration constant. Here we have the ambiguity due to integration constants: if \( H \) is such that \( d_H H = 0 \) then \( F + H \) has the same associated Hamiltonian vector field and boundary term. However, in field theory there is a further source of ambiguity arising from the fact that observable currents consisting of a pure boundary term \( H = d_h \bar{\sigma} \) (for any \( n - 2 \) horizontal form such that \( d_v \bar{\sigma} |_{\partial U} = 0 \)) are Hamiltonian observable currents with associated Hamiltonian vector field \( V = 0 \).

The obstruction for the existence of a Hamiltonian observable current associated to a given locally Hamiltonian vector field is the non triviality of the cohomology group \( H^{n-1,1}_d(J^\infty Y|_U) \), where furthermore we are identifying forms that differ by horizontally exact terms vanishing over \( \partial U \). Later in the text we will allow observable currents that are defined only on neighborhoods of \( j_1 \phi(U) \); in that context there is a Hamiltonian observable current for any given locally Hamiltonian vector field.

**Remark 10** (Further support for the definition of gauge vector fields). *Notice that every conserved current satisfying equation (7) is gauge invariant. In addition, notice that Condition 2 is essential for the gauge invariance of a current implying the gauge invariance of its associated observable after integration on a hypersurface with \( \partial \Sigma \subset \partial U \). Thus, regarding observable currents as generators of multisymplectomorphisms gives further support for our definition of gauge vector fields.*

**Remark 11** (Hamiltonian observables). *Equation (7) induces on \( \text{Obs}_\Sigma \) the all important equation of symplectic geometry

\[
d_f \Sigma = -\iota_V \omega_{L\Sigma}.
\]

The resulting space of Hamiltonian observables is denoted by \( \text{HObs}_\Sigma \subset \text{Obs}_\Sigma \).

*Notice that the absence of a boundary term is due to our initial consideration of multisymplectomorphisms as those preserving the multisymplectic form up to a horizontally exact term that vanishes over \( \partial U \).*

Of course the first examples of Hamiltonian observable currents are Noether currents.

**Theorem 3.** *A Noether current \( N^V = -\iota_V \Theta_L - \sigma_L^V \) satisfies the equation

\[
d_v N^V = -\iota_V \Theta_L + d_h \sigma_N^V.
\]

If \( \sigma_N^V |_{\partial U} = 0 \) then it is is a Hamiltonian observable current according to our definition \( N^V \in \text{HOC}_U \) with \( V \) as its Hamiltonian vector field.*

A proof for the existence of such a horizontally exact term requires the use of Takens’ lemma [30], and it is given in the appendix. Notice that it is not a priori clear if \( d_h \sigma_N^V |_{\partial U} = 0 \); in general Noether currents are not Hamiltonian Observable currents.
Remark 12 (Conserved charges associated to “would be gauge” symmetries). An element $X$ of a family of Lagrange symmetries depending on parameters with possible arbitrary local variation satisfies Condition 1 of the definition of gauge vector fields, and it has a corresponding conserved Noether current $N^X_X$; if $X$ also satisfies the locality condition in the definition of gauge vector fields requiring that $X|_{\partial U} = 0$, then $\sigma^X_X|_{\partial U} = 0$ which implies that $N^X_X \in \text{HOC}_U$ and that $n^X_X(\phi) = \int_{\Sigma} j^1 \phi^* N^X_X = 0$ for any hypersurface with $\partial \Sigma \subset \partial U$. In this case the differential of $n^X_X$ must also vanish, which means that $\sigma^X_X|_{\partial U} = 0$.

Now consider one of this generators of local Lagrangian symmetries $X$ which does not vanishing over $\partial U$, a “would be gauge” vector field. The Noether charge $n^X_X(\phi) = \int_{\Sigma} j^1 \phi^* N^X_X$ would vanish if $\Sigma$ is a cycle; thus, the current must be horizontally exact $N^X_X = d_h \nu_X$. In our case we have

$$n^X_X(\phi) = \int_{\Sigma} j^1 \phi^* \nu_X,$$

which would not vanish in general. Moreover, since any hypersurface $\Sigma'$ homologous with $\Sigma$ has the same boundary our ability to move the hypersurface to a region where the vector field vanishes (as used in the argument in the absence of boundaries) is crucially diminished, and the boundary integral in general does not vanish. This is consistent with the results of [12].

In general the boundary term $\sigma^X_X$ participating in equation (7) for a Noether current $N^X_X$ of this type will not vanish over $\partial U$ making the current not Hamiltonian according to our definition. One reason that we excluded this type of currents is that they have associated vector fields $X, X'$ both satisfying equation (7) even when $X - X' \notin G$; another reason is that the flow of these vector fields does not preserve the presymplectic forms $\omega_{L\Sigma}$ associated to hypersurfaces with $\partial \Sigma \subset \partial U$.

A result of Wald and Lee [25, 32] says that gauge and “would be gauge vector fields” satisfy $\iota_X \Omega_L = d_h \rho^X$. Thus, the vertical differential of the associated Noether current is a pure boundary term. The differential of the corresponding charge is

$$dn^X_X(\phi) = - \int_{\partial \Sigma} j^1 \phi^* d_v \nu_X = \int_{\partial \Sigma} j^1 \phi^* (\sigma^X_X - \rho^X).$$

In abelian Chern-Simons theory over a bounded domain Noether charges associated to would be gauge vector fields do not vanish. The interested reader can perform the calculation following the notation presented in the example of Section 8 and arrive to the results recently presented in [16] expressed in our language.

Theorem 4. Let $F^{VW}$ be a symplectic product observable current associated to two locally Hamiltonian vector fields $V, W \in \mathfrak{sl}_U^{LH}$. Then $F^{VW} \in \text{HOC}_U$ with Hamiltonian vector field $[V, W] \in \mathfrak{sl}_U^H$.

$$d_v F^{VW} = - \iota_{[V, W]} \Omega_L + d_h \sigma^{VW}$$

where

$$\sigma^{VW} = \iota_W \sigma^V - \iota_V \sigma^W.$$
Proof. A short calculation yields $d_v F^{VW} = -\iota_{[V,W]} \Omega_L + \iota_V \mathcal{L}_W \Omega_L - \iota_W \mathcal{L}_V \Omega_L$. The proof is completed noticing that $\mathcal{L}_V \Omega_L = d_h \sigma^V$ and $\mathcal{L}_W \Omega_L = d_h \sigma^W$. 

Apart from describing a property of an important family of observable currents, the previous result has the following corollary.

**Corollary 1.** $\mathfrak{H}^U \subset \mathfrak{L}^U \subset \mathfrak{H}^U$ are Lie subalgebras and $[\mathfrak{L}^U, \mathfrak{L}^U] \subset \mathfrak{H}^U$ is a Lie ideal. Additionally, this structure is compatible with reduction by gauge vector fields producing the natural inclusions

$[\mathfrak{L}^U, \mathfrak{L}^U] / \mathcal{G}_U \rightarrow \mathfrak{H}^U / \mathcal{G}_U \rightarrow \mathfrak{H}^U / \mathcal{G}_U \rightarrow \mathfrak{H}^U / / \mathcal{G}_U := \mathfrak{H}^U / \mathcal{G}_U$.

In symplectic geometry every function of phase space has an associated Hamiltonian vector field. However, from the families of examples given above (Noether and symplectic product observable currents) we see that not all observable currents are Hamiltonian observable currents.

Section 6 proves that, in the presence of a Cauchy surface with no boundary, Hamiltonian observable currents are capable to distinguishing gauge inequivalent solutions. Thus, assuming completeness in the space of locally Hamiltonian vector fields and boundary terms, this result implies that, in the presence of a Cauchy surface with no boundary, every observable current is Hamiltonian. The corresponding statement in a context closely related to ours was proven by Vitagliano in [32].

5 A bracket for observable currents and the Poisson algebra of local observables

Given two Hamiltonian vector fields $V, W$, with associated Hamiltonian observable currents $F, G \in \text{HOC}_U$, their commutator is another Hamiltonian vector field. We would like to find a Hamiltonian observable current associated to $[V, W]$. It would be even nicer if the resulting observable current could be calculated only from $F$ and $G$ and the assignment made the vector space of Hamiltonian observable currents $\text{HOC}_U$ into a Lie algebra extending the Lie algebra of Hamiltonian vector fields $\mathfrak{H}^U$. Below we will show several different Hamiltonian observable currents which have $[V, W]$ as their Hamiltonian vector field; they differ by horizontally exact terms vanishing over $\partial U$. Thus, when these different candidates are integrated over hypersurfaces with $\partial \Sigma \subset \partial U$ they all coincide. In this way $\text{HObs}_\Sigma$ acquires the structure of a Lie algebra. Furthermore, if there is a Cauchy surface $\Sigma$ without boundary we will see below that $\text{HObs}_\Sigma$ becomes a Poisson algebra.

Consider any two Hamiltonian observable currents $F, G \in \text{HOC}_U$ with choices of Hamiltonian vector fields $V, W \in \mathfrak{H}^U$, respectively. We have already shown (see Proposition 4) that $F^{VW} \in \text{HOC}_U$ with Hamiltonian vector field $[V, W]$. This gives us a natural definition of a bracket among Hamiltonian observable currents; however, different choices of Hamiltonian vector fields for $F$ and $G$ lead to results that differ by a boundary term. Here is the formal statement.
Definition 6 (Bracket for observable currents). Let $F, G \in \text{HOC}_U$ with choices of Hamiltonian vector fields $V, W \in \mathfrak{h}_U^H$ respectively. The bracket
\[
\{F^V, G^W\} = \iota_W \iota_V \Omega_L
\] (8)
defines a Hamiltonian observable current $\{F^V, G^W\} \in \text{HOC}_U$.

Remark 13 (Dependence on the choice of Hamiltonian vector fields). Consider $V_1, V_2$ Hamiltonian vector fields for $F \in \text{HOC}_U$. Thus,
\[
\{F^{V_1}, G^W\} - \{F^{V_2}, G^W\} = \iota_W \iota_{V_1} - \iota_{V_2} \Omega_L = -d_h \iota_{V_1} \rho^{V_1 - V_2}.
\]
Since, as shown just after Definition 6, $V_1 - V_2 \in \mathfrak{g}_U$. Thus, different choices of Hamiltonian vector fields lead to observable currents that differ by a boundary term vanishing over $\partial U$.

Notice, however, that the induced bracket on $\text{Hobs}_\Sigma$ is independent of any choice.

The following result is proven by a short calculation.

Lemma 1. Let $F, G \in \text{HOC}_U$ be observable currents with Hamiltonian vector fields $V, W \in \mathfrak{h}_U^H$ respectively. Then
\[
d_v\{F^V, G^W\} = -\iota_{\{V, W\}} \Omega_L + d_h \sigma^{VW},
\]
with $\sigma^{VW} = \iota_W \sigma^V - \iota_V \sigma^W$.

Other Hamiltonian observable currents with $[V, W]$ as Hamiltonian vector field are $\mathcal{L}_V G$ and $-\mathcal{L}_W F$, which have are geometrically interesting since they associate observable currents to Lie derivatives in the jet. However, they have the disadvantage of not being skew symmetric, but it is also possible to skew symmetrize them. Here is the relation between the mentioned Hamiltonian observable currents.

\[
\begin{align*}
\{F^V, G^W\} &= \mathcal{L}_V G + d_h \iota_V \sigma^G = -\mathcal{L}_W F - d_h \iota_V \sigma^F \\
&= \frac{1}{2} (\mathcal{L}_V G - \mathcal{L}_W F) + \frac{1}{2} d_h \left( \iota_V \sigma^G - \iota_V \sigma^F \right).
\end{align*}
\]

It is clear that our bracket is bilinear and skew symmetric. However, it does not satisfy a Jacobi relation. On the other hand, it is a straight forward calculation to verify that the Lie derivative bracket $\{F^V, G\}_l = \mathcal{L}_V G$, which is not skew symmetric, satisfies a Jacobi identity
\[
\begin{align*}
\{F_{1, 2}^V, \{F_3^V, F_3^W\}\}_l &= \mathcal{L}_{V_1, V_2} F_3 = \mathcal{L}_{[V_1, V_2]} F_3 + \mathcal{L}_{V_2} \mathcal{L}_{V_1} F_3 \\
&= \left\{ \{F_{1, 2}^V, F_3^V\}, F_3^W \right\}_l + \left\{ F_{1, 2}^V, \{F_3^V, F_3^w\}\right\}_l.
\end{align*}
\]

From this result and repeated use of the identity $\{F^V, G\}_l = \{F^V, G^W\} - d_h \iota_V \sigma^G$ we can see that our bracket is subject to a Jacobi relation that is modified by a horizontally exact term
\[
\begin{align*}
\{F_{1, 2}^V, \{F_3^W, F_3^V\}\}_l + d_h J = \left\{ \{F_{1, 2}^V, F_3^W\}, F_3^V \right\}_l + \left\{ F_{1, 2}^V, \{F_3^W, F_3^V\}\right\}_l,
\end{align*}
\]
with $J = -\iota_{V_1} \sigma^{V_2 V_3} + \iota_{V_2} \sigma^{V_1 V_3} + (\iota_{V_1} \mathcal{L}_{V_2} - \iota_{V_2} \mathcal{L}_{V_1} - \iota_{\{V_1, V_2\}}) \sigma^{F_3}$. Notice that $J$ vanishes over $\partial U$. 

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Remark 14 (Lie n-algebra of observable currents). The structure in $\text{HOC}_U$ given by the brackets defined above fits into the general structure described by Rogers [28] for multisymplectic field theory. Baez and Rogers studied the case of the classical bosonic string with particular detail [5]. Our bracket $\{F^V, G^W\}$ corresponds to the hemibracket, and $\{F^V, G\}$ corresponds to semibracket in their notation.

In [6] Barnich et al use the variational bicomplex to develop a framework appropriate for spacetime-localized observables. For us it would be of great interest to understand the relation between their work and ours.

There is further work [13] with the motivation of studying algebraic properties of Noether currents in multisymplectic field theory.

The class of physical observables $\text{HObs}_\Sigma$ inherits a bracket satisfying the Jacobi identity. Before stating the result formally, we recall that as any space of functions $\text{HObs}_\Sigma$ is endowed with the spacetime non-local product of pointwise evaluation $(f \cdot g)_{\Sigma}(\phi) = f_{\Sigma}(\phi)g_{\Sigma}(\phi)$. In Remark 15 we comment on the nontrivial issue of whether any product observable is realizable as the integral of a current or approximated by observables of this class. There we see that if $\Sigma$ has no boundary it is reasonable to conjecture that $\text{Obs}_\Sigma = \text{HObs}_\Sigma$. Furthermore, if $\Sigma$ is a Cauchy surface then it is reasonable to assume that all observables are realized as integrals of currents over $\Sigma$.

Proposition 1. If $\partial \Sigma \subset \partial U$ the bracket induced by the equation

$$[f, g]_\Sigma = \int_{\Sigma} j^1\phi^*\{F^V, G^W\}$$

(9)

gives $\text{HObs}_\Sigma$ the structure of a Lie algebra.

If $\Sigma$ is a Cauchy surface without boundary, it is reasonable to conjecture that product observables are realized as Hamiltonian observable currents integrated over $\Sigma$. Provided that this is true, $\text{HObs}_\Sigma$ is a Poisson algebra.

Proof. The bracket $\{\cdot, \cdot\}_\Sigma$ in $\text{HObs}_\Sigma$ inherits bilinearity and skew symmetry from the bracket $\{\cdot, \cdot\}$ in $\text{HOC}_U$. Jacobi’s identity holds because $J$ vanishes over $\partial U$. If $\Sigma$ is a Cauchy surface without boundary, the assumption guarantees that bracket observables are again in $\text{HObs}_\Sigma$ and Leibnitz’s rule is satisfied because for any $f_{\Sigma} \in \text{HObs}_\Sigma$ the bracket induces a derivative operator $[f, g]_\Sigma = \int_{\Sigma} j^1\phi^*\mathcal{L}_V G$.

These results allow us to refine Theorem 3 about Noether currents; the following result is a corollary of that theorem and Lemma 1.

Corollary 2 (Algebra of Noether currents). A Lie algebra of Lagrange symmetries $\mathcal{S}_L$ induces a vector space of observable currents $\mathcal{O}_\mathcal{S}_L \subset \text{HOC}_U$ which is compatible with the brackets in the sense that given $V, W \in \mathcal{S}_L$ we have

$$\{N^V, N^W\} = N^{[V,W]} + d_h\sigma^V_N.$$

with boundary term $\sigma^V_N = \nu_N \sigma^V_N - \nu_V \sigma^N_W$. Moreover, the boundary term satisfies $\sigma^W_N|_{\partial U} = 0$ if the symmetry algebra obeys the locality condition $V \in \mathcal{S}_L \implies \mathcal{L}_V \Omega_L = d_h\sigma^V$ with $\sigma^V|_{\partial U} = 0$.
At the level of Noether charges if the locality condition written above is satisfied by the symmetry algebra then for any \( V, W \in \mathcal{J}_L \) we have
\[
\mathcal{J}_L \rightarrow \text{Hobs}_\Sigma.
\]

A general framework to study the algebraic properties of Noether currents in multisymplectic field theory extending the work of Rogers [28] is developed in [13]. With the aim of understanding the association of currents to symmetries in a finer way a framework for homotopy moment maps was introduced in [8].

Symplectic product observable currents are a large family of observable currents. In the special case when the vector fields \( V, W \) are locally Hamiltonian we gave explicit formula for the Hamiltonian vector field associated to \( F_{VW} \). The following result about the algebra of symplectic product currents is a trivial consequence of the definitions, and complements Corollary [11].

**Proposition 2** (Algebra of symplectic product currents). Let \( F_{V_1W_1} \) and \( F_{V_2W_2} \) be symplectic product observable currents associated to the locally Hamiltonian vector fields \( V_1, W_1; V_2, W_2 \in \mathfrak{H}_U \) respectively. Then
\[
\{F_{V_1W_1}, F_{V_2W_2}\} = F[V_1, W_1][V_2, W_2].
\]

We have just proven that symplectic product observable currents form a Lie algebra isomorphic to the commutator subalgebra of the Lie algebra of Hamiltonian vector fields. This result together with other results of this section and Section 4 may be summarized in the following diagram. In the diagram \( \text{SPOC}_U \subset \text{HOC}_U \) denotes the Lie algebra of symplectic product observable currents as a sub Lie \( n \)-algebra of the algebra of Hamiltonian observable currents which is an extension of the Lie algebra of Hamiltonian vector fields.

\[
\begin{array}{ccc}
[\mathfrak{S}_U^{1H}, \mathfrak{S}_U^{1H}] / \mathcal{G}_U & \sim & \mathfrak{S}_U^H / \mathcal{G}_U \\
\uparrow & & \uparrow \\
\text{SPOC}_U & \rightarrow & \text{HOC}_U \\
\downarrow & & \downarrow \\
\{H = d_\theta \sigma : d_\nu \sigma |_{\partial U} = 0\} & \rightarrow & \{H = d_\theta \tilde{\sigma} : d_\nu \tilde{\sigma} |_{\partial U} = 0\}
\end{array}
\]

6 Observable currents separate solutions modulo gauge

In previous work the multisymplectic approach to classical field theory it is argued that the set of physical observables that can be obtained from currents is very limited including almost nothing besides Noether currents (see for example [14, 19, 21, 23, 17]). Here we defined the notion of observable currents and exhibited the large family of symplectic product observable currents. In order to be conclusive showing that observable currents are an interesting source of physical observables we (i) prove that if our domain

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is foliated by Cauchy surfaces with no boundary the algebra of Hamiltonian observable currents is capable of distinguishing between gauge inequivalent solutions, and (ii) we give supporting evidence to conjecture that for any spacetime domain gauge inequivalent solutions may be distinguished by means of observable currents. With the aim of making the task more transparent we will prove a local version of statement (i), and then discuss an extension of our argument to address more general cases.

**Theorem 5.** Assume that the spacetime domain $U$ is endowed with a foliation by Cauchy surfaces with no boundary. Consider any curve of solutions $\phi_t \in \text{Sols}_U$ starting at $\phi_0 = \phi$. Thus, if the tangent of the curve at $t = 0$ is compatible with a vector field $W \in \mathfrak{F}_G^U$ that is not a gauge vector field, there is a Hamiltonian observable current $F \in \text{HOC}_U$ defined at least in a neighborhood of $j^1\phi_0(U)$ such that for any Cauchy surface $f_\Sigma \in \text{HObs}_\Sigma$ satisfies

$$\frac{d}{dt}|_{t=0} f_\Sigma(\phi_t) \neq 0.$$  

**Proof.** Consider any Cauchy surface $\Sigma$; if $W \in \mathfrak{F}_G^U$ is not a gauge vector field there is a variation $v$ such that $\iota_W \iota_v \omega_{L\Sigma} \neq 0$, where $w$ is the variation corresponding to $W \in \mathfrak{F}_G^U$, and the presymplectic form was calculated using $\phi$. Moreover, since $\Sigma$ is a Cauchy surface we know that it is possible to choose the variation $v$ to be a locally Hamiltonian vector field for $\omega_{L\Sigma}$. Then, at least in a neighborhood of $j^1\phi(\Sigma)$, there is a vector field which is locally Hamiltonian $V \in \mathfrak{F}_{LH}^U$ corresponding to the variation $v$. Given our strong assumptions, and the conservation of $\iota_W \iota_V \Omega_L$ we know that $V$ is extendible to a neighborhood of $j^1\phi(U)$ and that $\iota_W \iota_V \Omega_L$ is not horizontally exact. Thus, at least in a neighborhood of $j^1\phi(U)$, there is a Hamiltonian observable current $F \in \text{HOC}_U$ such that

$$\mathcal{L}_WF = \iota_W d_v F = -\iota_W \iota_V \Omega_L - d_h \iota_W \sigma F.$$  

It follows that $f_\Sigma$ distinguishes between $\phi$ and neighboring solutions in the curve $\phi_t$.  

**Remark 15** (General observables approximated by observable currents). Not all observables $f : \text{Sols}_U \to \mathbb{R}$ are of the type $f_\Sigma \in \text{Obs}_\Sigma$ for some hypersurface. However, under the conditions of Theorem 5 any observable can be approximated by means of observables induced by Hamiltonian observable currents.

Observables that are of special importance for Proposition 7 are the products of observables in $\text{HObs}_\Sigma$. The observable $(fg)_\Sigma$ defined as the product of the evaluations pointwise in $\text{Sols}_U$ may be approximated by means of observables induced by Hamiltonian observable currents. Then it is reasonable to conjecture that $(fg)_\Sigma$ belongs to $\text{HObs}_\Sigma$ or to an appropriate completion of it. Thus, under the conditions of Proposition 7 $\text{HObs}_\Sigma$ has a Poisson algebra structure. Moreover, in this case it is known that the complete algebra of observables is encoded in $\text{HObs}_\Sigma$. For a rigorous treatment focussing on gauge invariant conserved currents see Vitagliano’s work on the covariant phase space [32].

In the next section we will talk about glueing domains; a given hypersurface $\Sigma \subset U$ may be a portion of a Cauchy surface and $\text{Obs}_\Sigma$ may be completed to be able to separate solutions modulo gauge after glueing it with the space of observables associated with the rest of the Cauchy surface.
The previous theorem relies on the strong simplifying assumption of \( U \) being foliated by Cauchy surfaces with no boundary. This assumption helped us avoid dealing with the condition on locally Hamiltonian vector fields \( V \in \mathcal{F}^{LH}_U \) to yield a boundary term (in \( \mathcal{L}_V \Omega_L = d_\hbar \sigma^V \)) such that \( \sigma^V|_{\partial U} = 0 \) when restricted to \( \mathcal{C}_L \). Before addressing the extension of the result to regions without a foliation by Cauchy surfaces with no boundary there is an important cautionary remark to consider: We know that in the presence of a boundary there are non trivial observable currents that are not Hamiltonian according to our definition, a family of examples being the Noether currents associated to would be gauge symmetries. The origin of this peculiarity is that would be gauge symmetries are associated to vector fields that do not preserve the presymplectic forms \( \omega_{L_\Sigma} \) associated to hypersurfaces with \( \partial \Sigma \subset \partial U \). Thus, there are observable currents which can not be approximated by Hamiltonian observable currents. We do not claim that in the presence of boundaries Hamiltonian observable currents do not distinguish gauge inequivalent solutions, but we know that there are some aspects of the field that are naturally measured by observable currents which are not Hamiltonian.

Remark 16 (In the absence of a Cauchy surface). Now we work on a spacetime domain \( U \) that may not have a Cauchy surface, and consider the issue of distinguishing gauge inequivalent solutions by means of observable currents (even if they are not Hamiltonian).

In this case a similar argument to the one used in the proof of Theorem 3 may work if it is true that for any \( W \in \mathcal{F}^G_U \) which is not a gauge vector field there is a vector field \( V \in \mathcal{F}^G_U \) satisfying
\[
\mathcal{L}_V \Omega_L = d_\hbar \sigma^V
\]
without demanding that \( \sigma^V|_{\partial U} = 0 \) such that such that \( \iota_W \iota_V \Omega_L \) is not horizontally exact. Then we could assure that there is an observable current \( F \) (defined at least in a neighborhood of \( j^1 \phi(U) \)) such that \( d_\hbar F = -\iota_V \Omega_L + d_\hbar \sigma^F \), but which would not be Hamiltonian in general because the \( \sigma^F \) may not fulfill the boundary condition. That observable current would be such that \( \mathcal{L}_W F \) is not horizontally exact. The condition expressed in equation (10) seems to be reasonable and it would not be difficult to check in a given theory.

If we have succeeded finding an observable current such that \( \mathcal{L}_W F \) is not horizontally exact, we need to face the second challenge of converting it into a nontrivial observable in the absence of a Cauchy surface. We know that there may not be a single hypersurface such that first order data over it fully characterizes a solution over \( U \). However, if \( \mathcal{L}_W F \) is not horizontally exact it is reasonable to conjecture that there is some hypersurface \( \Sigma \) inside \( U \) such that \( \mathcal{L}_\omega f_\Sigma \neq 0 \).

We consider that the argument given above provides enough evidence to conjecture that observable currents are capable of distinguishing gauge inequivalent solutions over any type of spacetime domain. Without a general proof when a case of interest is presented, the natural way to proceed is to check the validity of the assumptions spelled out above.

Remark 17 (Localized measurement and observable currents). Spacetime localized measurement is the source of observables with direct physical interest. Peierls constructed a bracket for this type of observables [27] whose core ingredient is a locally Hamiltonian vector field \( V_A \) constructed from a the deformation of extrema of the action \( \phi_\lambda \) induced
by a density modelling the localized measurement $A$ modifying the Lagrangian density up to first order in a deformation parameter $L \rightarrow L_{\lambda A} = L + \lambda A$. Any two observable currents $F_A, F'_A$ in the class of with the same associated Hamiltonian vector field differ only by covariant counterpart of a constant function and a boundary term

$$d_v (F_A - F'_A) = d_h (\sigma^F - \sigma^{F'}),$$

where moreover the boundary term is constrained by $d_v d_h \sigma^F = d_v d_h \sigma^{F'} = d_h \sigma^{V_A}$. That observable currents with a given vertical differential exist is guaranteed at least in a neighborhood of $j^1 \phi(U)$ for a given regular solution $\phi \in \text{Sols}_U$, and as shown in [22] local existence of observables should be our only objective. Thus, Peierls’ procedure yields a narrow class of observables which are candidates to model a given localized measurement. A more thorough description of this argument and its consequences will be treated elsewhere [7].

A direct treatment of localized observables in a covariant field theory formalism based on the variational bicomplex is given by Barnich et al [6]. It would be interesting to explore the relation between their formalism and ours.

### 7 Observable algebras of nested and glued domains

Let us start with the case of a domain contained in another one $U' \subset U$. The elements of $OC_U$ are differential forms that may be restricted to $U'$; moreover, the conservation law that they obey will continue to hold after their restriction to $U'$. Let us now study the issue of gauge invariance. Since gauge vector fields in $G_{U'}$ vanish over $\partial U'$ they may be extended by zero to gauge vector fields in $G_U$. Thus, any $F \in OC_U$ is invariant with respect to the gauge vector fields that can be imported from $U'$. This means that $F|_{U'}$ is $G_{U'}$ invariant and we have a map

$$OC_U \rightarrow OC_{U'}$$

which does not have to be injective nor surjective. In the case of spacetime localized observables there is a natural map from the space of observables corresponding to the smaller domain to the space of observables corresponding to the bigger domain; the map for observable currents goes in the opposite direction. In Remark [7] we briefly commented on observable currents induced by localized measurements and its relation to Peierls’ bracket. In the context of Peierls’ procedure, observable currents on the bigger domain can be induced by localized measurements in the smaller domain as expected for observables associated to localized measurement [10].

Now consider a domain that is composed by two subdomains that intersect along a hypersurface $U = U_1 \# \Sigma U_2$. Our previous argument shows that there are maps $OC_U \rightarrow OC_{U_1}$; additionally, in the following definition we show how to glue compatible observable currents of the subdomains to produce any observable current in $OC_U$.

---

7 At the level of the associated Hamiltonian vector fields it is clear that a Hamiltonian vector field for $F|_{U'}$ is the restriction of a Hamiltonian vector field associated to $F$.  

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Definition 7 (Glueing algebras of adjacent domains). The construction needs the following definitions:

- \( OC_{U_1 \#_\Sigma U_2} = \{(F_1, F_2) : F_i \in OC_{U_i} \text{ with } F_1|\Sigma = F_2|\Sigma\} \).
- \( \hat{G}_{U_1} \) is the subalgebra of \( \mathcal{F}_{U_1} \) whose elements satisfy Condition 1 for gauge vector fields: \( \iota_X \Omega_L \) is horizontally exact when evaluated in \( \hat{G}_L \).
- \( \hat{G}_{U_1 \#_\Sigma \hat{G}_{U_2}} = \{(V_1, V_2) : V_i \in \hat{G}_{U_i} \text{ with } V_1|\Sigma = V_2|\Sigma\} \) and \( \hat{G}_{U_1 \#_\Sigma \hat{G}_{U_2}} \) denotes simply pairs of elements of \( \hat{G}_{U_i} \).
- \( G_{\Sigma} = \hat{G}_{U_1 \#_\Sigma \hat{G}_{U_2}}|_{\Sigma} \).
- \( \text{Inv}_{G_{\Sigma}}(OC_{U_1 \#_\Sigma U_2}) \) denotes the subspace of \( OC_{U_1 \#_\Sigma U_2} \) that is invariant under \( G_{\Sigma} \).

The following proposition follows trivially from the definitions.

Proposition 3.

\[ OC_U = \text{Inv}_{G_{\Sigma}}(OC_{U_1 \#_\Sigma U_2}). \]

Now consider the situation in which a domain \( U = U_1 \#_\Sigma U_2 \) with a Cauchy surface \( \Sigma \) is divided into two subdomains in such a way that the Cauchy surface is also subdivided as \( \Sigma = \Sigma_1 \#_\Delta \Sigma_2 \) by a codimension two surface \( \Delta \). Due to Condition 2 in the definition of gauge vector fields we have a subalgebra of observables associated to each of the portions of Cauchy surface \( \text{Obs}_{\Sigma_i} \); moreover it is clear from the definitions of the observables that any element of \( \text{Obs}_{\Sigma} \) is a sum of two terms \( f_{\Sigma} = f_{\Sigma_1} + f_{\Sigma_2} \) belonging to \( \text{Obs}_{\Sigma_i} \). Thus, \( \text{Obs}_{\Sigma} \) is recoverable from \( \text{Obs}_{\Sigma_1} \) and \( \text{Obs}_{\Sigma_2} \).

8 Example: Maxwell field

In this section we give the notation and initial setup to treat the Maxwell field in this formalism. The presentation is not pedagogical; the aim of this section is only to be used as a reference for the reader to be able to work on this familiar example by him self or her self. We also mention particularly illustrative results that are easily obtainable in this prime example of a linear gauge field theory.

The notation for the general case is given in the appendix; in this section we follow that notation only in its essence. In the general case a field is denoted by \( \phi^a \), and partial derivatives in a coordinate chart are written as \( \partial_i \phi = \frac{\partial \phi^a}{\partial x^i} \). In this example the field is taken to be the potential one form \( A \).

Let \( M = \mathbb{R}^4 \) with the Minkowski metric \( \eta \). Histories, i.e. local sections, are one forms; then \( Y = T^*\mathbb{R}^4 \). The notation for elements in the first jet will be \( j^1A(x) = (x^\mu; A_\nu(x); v_{\nu\mu} = \partial_\mu A_\nu(x)) \in J^1Y \). A general point in the infinite jet will be denoted by \( (x^\mu; A_\nu; v_{\nu\mu}; v_{\nu\mu\rho}; \ldots) \).
In the Lagrangian density only the skew symmetric combination $F_{\mu\nu} = v_{\nu\mu} - v_{\mu\nu}$ appears,

$$L = \frac{-1}{4} F_{\mu\nu} F^{\mu\nu} d^4x.$$ 

Basic vector fields in the infinite jet are denoted by \{\partial_{\mu} = \frac{\partial}{\partial x^\mu}; \partial_A^\nu = \frac{\partial}{\partial A^\nu}; \partial^\mu = \frac{\partial}{\partial \nu^\mu}; \ldots\}. The generators of the exterior algebra of differential forms in the infinite jet are \{dx^\mu; \theta_\nu = dA_\nu - v_{\nu\mu} dx^\mu; \theta_{\nu\mu} = dv_{\nu\mu} - v_{\mu\nu} dx^\rho; \ldots\}.

The non zero horizontal differentials of the coordinates and basic forms are: $d_0 x^\mu = dx^\mu; d_0 A_\nu = v_{\nu\mu} dx^\mu; d_\nu v_{\nu\mu} = v_{\nu\mu\rho} dx^\rho; \ldots$ The non zero vertical differentials are: $d_0 A_\nu = \theta_\nu; d_0 v_{\nu\mu} = \theta_{\nu\mu}; \ldots d_0 F_{\mu\nu} = \theta_{\nu\mu} - \theta_{\mu\nu}; d_0 L = \frac{-1}{2} (\theta_{\nu\mu} - \theta_{\mu\nu}) F^{\mu\nu} d^4x$.

The left hand side of the field equation is

$$I(d_0 L) = \frac{1}{2} \theta_{\sigma} \wedge D_{\rho} [\theta_{\rho \sigma} (\theta_{\tau \mu} - \theta_{\mu \tau})] F^{\mu\nu} d^4x = (v_{\nu\mu} - v_{\mu\nu}) \theta_\mu \wedge d^4x.$$ 

From the equation $d_0 L = I(d_0 L) + d_0 \Theta L$ it is easy to verify that a premultisymplectic potential that works is $\Theta_L = F^{\mu\nu} \theta_\mu \wedge d^3 x_\nu$; which yields

$$\Omega_L = -d_0 \Theta_L = -(\theta^{\nu\mu} - \theta^{\mu\nu}) \wedge \theta_\mu \wedge d^3 x_\nu.$$ 

In our framework field perturbations play a central role. In linear field theories generic perturbations correspond to one parameter families of solutions of the type $A_\nu(t) = A_\nu + t A'_\nu$, where both $A_\nu$ and $A'_\nu$ are solutions. The corresponding vector field in the bundle $Y$ may be written as $X^{A'} = A'_\nu \partial_A^\nu$. Since the perturbation is independent of the field, its prolongation to the infinite jet is

$$X^{A'} = A'_\nu \partial_A^\nu + \partial_\nu (A'_\nu) \partial^{\nu\mu} + \ldots.$$ 

The field perturbation $X^f$ corresponding to $A_\nu(t) = A_\nu + t \partial_\nu f$ is $X^f = \partial_\nu f \partial_A^\nu$, and its prolongation to the infinite jet is

$$X^f = \partial_\nu f \partial_A^\nu + \partial_\nu (\partial_\nu f) \partial^{\nu\mu} + \ldots.$$ 

Notice that since $X^f_{\mu\nu} = \partial_\mu (\partial_\nu f)$ is symmetric it does not modify $F_{\mu\nu}$; this will be relevant below. A field perturbation $V^k$ corresponding to superposing a plane wave $A_\nu(t) = A_\nu + t V^k_{\nu} \partial_A^\nu$ may be written in the bundle $Y$ as: $V^k = \text{Re}(V_\nu e^{i k x} \partial_A^\nu)$. In the infinite jet its prolongation is

$$V^k = V_\nu \partial_A^\nu + V_{\nu\mu} \partial^{\nu\mu} + \ldots = V_\nu \text{Re} \left( e^{i k x} \partial_A^\nu + i k_\mu e^{i k x} \partial^{\nu\mu} + \ldots \right).$$ 

A perturbation is considered gauge if its associated vector field $X$ is a solution of the linearized field equations that satisfies Definition 1 Condition 1 of that definition is satisfied if $d_0 t X \Theta_L|_{C_L} = 0$. For a generic perturbation, after using the field equation, we find

$$d_0 t X \Theta_L|_{C_L} = [(D_\nu X_\mu - D_\mu X_\nu) \theta^{\nu\mu} - \eta^{\rho\mu} D_\rho (D_\nu X_\mu - D_\mu X_\nu) \theta^{\rho\mu}] \wedge d^4 x.$$ 

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The solutions of this condition are vector fields in which $X_\mu$ are the components of a closed one form: locally, solutions are of the form $X^I$ given above. Thus, we recover the usual notion of gauge freedom of the Maxwell field from Condition 1. A new element is Condition 2; for a perturbation $X^I$ to be gauge we require that $\partial_\nu f|_{\partial U} = 0$.

On the other hand, $V^k$ corresponds to a nontrivial perturbation. Moreover, since $d_\nu V^k_\nu = 0$ we see that it is a locally Hamiltonian vector field

\[ \mathcal{L}_{V^k} \Omega_L = d_\nu (d_\nu V^k_\nu - D_\nu d_\nu V^k_\nu) \wedge \theta^\mu \wedge d^3 x^\nu + d_\nu V^k_\nu (\theta^\mu - \theta^\nu) \wedge d^3 x^\nu = 0. \]

Thus, one may try to find a Hamiltonian observable current with $V^k$ as associated vector field. Since the Maxwell field is linear the observable current we are looking for is $F^{V^k} \in \mathcal{O}_{\mathcal{S}_U}$ as defined in Remark 9.

If we use two perturbations $V^k$ and $V^l$ we can write their symplectic product observable current $F^{V^k V^l}$. The result is the simplest observable current—a constant current—and integrated on a hypersurface it yields a constant function. We could do the calculation directly, but one can also notice that the Hamiltonian vector field associated to $F^{V^k V^l}$ is $[V^k, V^l] = 0$ which the reader may verify from the definition of these vector fields.

For an example that exhibits abelian gauge freedom and nonlinearities it may be a good idea to explore the Born-Infield model.

Appendix: Minimal set of definitions about the variational bicomplex

This minimalistic revision of the variational bicomplex may serve the purpose of letting someone that knows another presentation of classical field theory, like the covariant phase space formalism, read this article. For an introductory presentation of the ideas of the subject the reader is referred to Anderson’s brief introduction [2].

Let $M$ be an $n$–dimensional manifold and $\pi : Y \to M$ be a fiber bundle with $m$–dimensional fiber $\mathcal{F}$.

Points in the $k$–jet bundle $\pi_{k,0} : J^k Y \to Y$, $k = 1, 2, \ldots$ correspond to equivalence classes of local sections of $\pi$ that agree up to $k$-th order partial derivatives when evaluated at a given point $x \in M$. If in the restriction of $Y$ over a coordinate chart of the base $U \subset M$ we use coordinates such that the evaluation of a local section is $\phi(x) = (x^1, \ldots, x^i, \ldots, x^n; u^1, \ldots, u^a, \ldots, u^m) \in Y|_U$, then we get the following coordinates for the $k$-jet

\[ (x; u^{(k)}_I) := (x^1, \ldots, x^i, \ldots, x^n; u^1, \ldots, u^a, \ldots, u^m; u^{\alpha}_I, \ldots), \in J^k Y|_U \]

where $i = 1, \ldots, n; a = 1, \ldots, m;$ and $I = (i_1, \ldots, i_n)$ denotes a multiindex of degree $|I| := i_1 + \cdots + i_n = 0, 1, \ldots, k$, $i_j \geq 0, i_j \in \mathbb{Z}$. For $I = 0$, we define $u^{\alpha}_0 = u^\alpha$.

The projection $\pi_{k+r,k} : J^{k+r} Y \to J^{k} Y$ is defined by forgetting the coordinates corresponding to partial derivatives of higher order. The infinite jet $J^{\infty} Y$ may be defined as the inverse limit of this system of projections, and it is the space where the formalism
of the variational bicomplex takes place. The jets of finite order can be thought of as truncations of it corresponding to neglecting all the partial derivatives of order higher than a certain cut-off specifies.

For a local section \( \phi : U \subset M \to Y|_U \), its prolongation to the \( k \)-jet \( j^k \phi : U \subset M \to J^k Y|_U \) is the section

\[
j^k \phi (x) = \left( x^1, \ldots, x^i, \ldots, x^n; \phi^1(x), \ldots, \phi^m(x); \ldots, \frac{\partial^{|I|} \phi^a}{\partial x^{i_1} \cdots \partial x^{i_n}}, \ldots \right),
\]

where \( k \) may be taken finite or \( k = \infty \).

The exterior algebra of differential forms in \( J^\infty Y \) is generated by the set of one forms \( \{ dx^i, \vartheta^a_I \} \), where

\[
\vartheta^a_I := du^a_I - \sum_{j=1}^n u^a_{(I,j)} dx^j.
\]

A general \( p \)-form is written as a sum of terms with products of \( p \) of such generators; factors of the type \( dx^i \) are called “horizontal”, factors of the type \( \vartheta^a_I \) are called “vertical”. Thus, the space of \( p \)-forms becomes a direct sum of spaces \( \Omega^r,s(J^\infty Y) \) of forms which are products of exactly \( r \) horizontal one forms and \( s \) vertical one forms. The differential brings up the degree of forms by one and the direct sum structure mentioned makes the differential split as a sum of operators

\[
d = d_h + d_v,
\]

where \( d_h : \Omega^{r,s}(J^\infty Y) \to \Omega^{r+1,s}(J^\infty Y) \) and \( d_v : \Omega^{r,s}(J^\infty Y) \to \Omega^{r,s+1}(J^\infty Y) \) are characterized by their action on functions

\[
d_h f = \left( \frac{\partial f}{\partial x_i} + u^a_{(I,i)} \frac{\partial f}{\partial u^a_J} \right) dx^i = (D_i f) dx^i, \quad d_v f = \frac{\partial f}{\partial u^a_i} \vartheta^a_I.
\]

For the generating one forms we get

\[
d_h dx^i = 0, \quad d_v dx^i = 0, \quad d_h \vartheta^a_I = dx^i \wedge \vartheta^a_{(I,i)}, \quad d_v \vartheta^a_I = 0.
\]

The following identities hold

\[
d_h^2 = 0, \quad d_v d_h = -d_h d_v, \quad d_v^2 = 0.
\]

Other identities that we use repeatedly are

\[
\iota_X d_h F = -d_h \iota_X F, \quad j^k \phi^* d_h F = d j^k \phi^* F,
\]

where the differential form \( d_h F \) of the infinite jet fits in the \( k \)-jet. (All differential forms in the infinite jet are required to fit in a finite jet for some \( k \) and be lifted to \( J^\infty Y \) with the pull back of the projection map.)
Vector fields in the jet are called vertical if they are annihilated by all horizontal one forms. A vertical vector field \( X \) that arises from the prolongation of one parameter family of local sections \( \phi_t \) starting at \( \phi \) may be viewed as an infinitesimal field variation,

\[
\frac{d}{dt}|_{t=0} F(j^1 \phi_t) = \mathcal{L}_X F(j^1 \phi) = (\iota_{j^k \phi} d\nu + d\nu \iota_{j^k \phi}) F(j^1 \phi),
\]

where the form \( F \) needs to fit in \( J^k Y \).

The left hand side of the field equation of field theory \((E(L) = 0)\) appears in the variation of the Lagrangian density (which in the terminology just given means a form of horizontal degree \( n \) and vertical degree zero), \( d\nu L = E(L) + dh \Theta_L \). From our first encounter with the Euler-Lagrange equation we know that integration by parts is an essential step in its derivation. In the language of the variational bicomplex the definition is

\[
E(L) = Id_\nu L,
\]

where the integration by parts operator \( I : \Omega^{n,s}(J^\infty Y) \to \Omega^{n,s}(J^\infty Y) \) is defined by

\[
I = \frac{1}{s} \delta^a F_a, \quad F_a \mu = \sum_{|J|} \text{sgn}(|J|) D_J \partial_a \mu,
\]

where \( \partial_a^J = \frac{\partial}{\partial u^J}, \text{sgn}(|J|) \) is positive for \( |J| \) even, and the sum stops at \( k \) if \( \mu \) fits in \( J^k Y \) (i.e. if \( \mu \) is a differential form of order \( k \)). The integration by parts operator \( I \) and \( F_a \) have the following properties

\[
F_a \circ d_h = 0, \quad \mu = I(\mu) + d_h \eta, \quad I^2 = I.
\]

The differential operators \( d_h, d_\nu \) among the spaces \( \Omega^{r,s} \) are complemented by the map \( I \) and the spaces \( F^s = I(\Omega^{n,s}) \) and the maps \( E = Id_\nu : \Omega^{n,0} \to F^1, \quad \delta = Id_h : F^s \to F^{s+1} \) to form an augmented variational bicomplex. The Euler-Lagrange complex resides at the corner of the augmented variational bicomplex starting at the spaces \( \Omega^{r,0} \) moving with \( d_h \) and then turning with \( E \) to the spaces \( F^s \) and moving with the differential \( \delta \).

In our definition of gauge vector fields the multisymplectic form \( \Omega_L \) plays an important role. In the context in which it appears, the glueing field equation it is natural to consider it as restricted to a hypersurface and integration by parts becomes necessary to obtain the glueing field equation. Thus, in a slight abuse of notation we give the name \( I \) to the operator \( \frac{1}{s} \delta^a F_a : \Omega^{n-1,1}(J^\infty Y) \to \Omega^{n-1,1}(J^\infty Y) \).

In a few instances during the article we alluded to “Takens’ Lemma” [30]. Here we state the part of the mentioned lemma that we need.

**Lemma 2.** For every \( d_h - \) closed form \( \tau \in \Omega^{n-1,1}(J^1 Y) \) with \( d_h \tau = 0 \) there exists \( \sigma \in \Omega^{n-2,1}(J^1 Y) \) such that \( \tau = d_h \sigma \).

For essential geometrical arguments that we did not give and for important features of the variational bicomplex that we did not cover (because they are not essential in this article) see Anderson’s introduction [2].

We finish the appendix stating and proving a version of Noether’s theorem.
Theorem 6 (Noether). A Lagrange symmetry is a vector field \( V \in \mathfrak{X}(J^1Y|U) \) satisfying \( \mathcal{L}_V L = \delta h \sigma^V_L \). Every Lagrange symmetry has a corresponding Noether current \( N^V = -\iota_V \Omega_L - \sigma^V_N \) which satisfies the equation
\[
d_N^V = -\iota_V \Omega_L + \delta h \sigma^V_N
\]
for some differential form \( \sigma^V_N \).

If \( \sigma^V_N|_{\partial U} = 0 \) then \( N^V \in \text{HOC}_U \) with \( V \) as its Hamiltonian vector field.

Proof. First it is clear that, being a symmetry generator, \( V \) preserves the variational principle. Thus, it is a solution of the linearized field equation.

The proof that the current is conserved is trivial from the definition of \( N^V \). Below we prove gauge invariance, but first we will prove that \( N^V \) is Hamiltonian; that is, \( d_N^V = -\iota_V \Omega_L + \delta h \sigma^V_N \) for some differential form \( \sigma^V_N \).

Direct calculation leads to \( d_N^V = -\iota_V \Omega_L - \mathcal{L}_V \Theta_L - d_N^V \). Since \( \delta h d_N^V|_{\mathcal{C}_L} = 0 \) we get \( d_N^V (\mathcal{L}_V \Theta_L - d_N^V)|_{\mathcal{C}_L} = 0 \). Now we use Takens’ Lemma to conclude that there is a form \( \sigma^V_N \) such that \( -\mathcal{L}_V \Theta_L - d_N^V = \delta h \sigma^V_N \) when restricted to \( \mathcal{C}_L \). Thus according to our definition, if the boundary term satisfies the condition \( \sigma^V_N|_{\partial U} = 0 \) the current \( N^V \) is a Hamiltonian observable current.

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