Biased random walk on random networks in presence of stochastic resetting: exact results

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Abstract

We consider biased random walks on random networks constituted by a random comb comprising a backbone with quenched-disordered random-length branches. The backbone and the branches run in the direction of the bias. For the bare model as also when the model is subject to stochastic resetting, whereby the walkers on the branches reset with a constant rate to the respective backbone sites, we obtain exact stationary-state static and dynamic properties for a given disorder realization of branch lengths sampled following an arbitrary distribution. We derive a criterion to observe in the stationary state a non-zero drift velocity along the backbone. For the bare model, we discuss the occurrence of a drift velocity that is non-monotonic as a function of the bias, becoming zero beyond a threshold bias because of walkers trapped at very long branches. Further, we show that resetting allows the system to escape trapping, resulting in a drift velocity that is finite at any bias.

Keywords: stochastic resetting, random walk, nonequilibrium stationary state

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(Some figures may appear in colour only in the online journal)
Random walk (RW) on random networks such as random comb (RC) lattices, inspired by Pierre de Gennes’ ‘Ant-in-a-Labyrinth’ [1], is a much-studied research topic [2–16]. An RC, comprising a backbone with random-length branches, encodes essential features of physical problems, e.g., finitely-ramified fractals and percolation clusters [17–19]. Biased RW on RCs yields many nontrivial results, e.g., a drift varying non-monotonically with bias [2–4], anomalous diffusion [5, 8, 10, 11]. Dynamics on comb-like structures find wide applications in modeling many natural phenomena, e.g., transport in spiny dendrites [20], rectification in biological ion channels [21], superdiffusion of ultra-cold atoms [22], reaction–diffusion processes [23], crowded-environment diffusion [24], cancer proliferation [25], and even human migration along river networks [26].

In recent years, stochastic resetting has been extensively studied in the area of nonequilibrium statistical mechanics. The setup involves repeated interruptions of a dynamics at random times with a reset to the initial condition [27, 28]. Resetting results in a nonequilibrium stationary state (NESS) with remarkable static and dynamic features. Examples include a wide spectrum of dynamics: diffusion [29–35], RWs [36, 37], Lévy flights [38], Bernoulli trials [39], discrete-time resets [40], active motion [41] and transport in cells [42], search problems [43–49], RNA-polymerase dynamics [50, 51], enzymatic reactions [52], ecology [53, 54], interacting systems [55–61], stochastic thermodynamics [62], quantum dynamics [63], etc.

In this letter, we revisit the classic problem of biased non-interacting RWs in continuous time and on RC, with a twist, namely, with stochastic resets. As regards resetting, we address an unexplored theme: resetting in a system with quenched disorder. The RC-backbone (figure 1(a)) is a one-dimensional (1d) lattice of \( N \) sites, to each of which is attached a branch of a 1d lattice with a random number of sites (all lattice-spacings are unity). Let \( M \) denote the maximum-allowed branch length. Denote the sites by \((n, m)\), wherein \( 0 \leq n \leq N-1 \) labels the backbone sites and \( 0 \leq m \leq L_n \) labels the \((L_n + 1)\) number of sites on the branch attached to the \( n \)th backbone site. The site \((n, m = 0)\) being shared by the backbone and the branch, we will from now on refer to branch sites as those with \( m > 0 \). The \( L_n \)'s are quenched-disordered random variables drawn independently from an arbitrary distribution \( P_L \). The backbone and branches run along a field or a bias with strength \( g \); \( 0 < g < 1 \). Representative \( P_L \)'s are an exponential and a power-law given respectively by

\[
P_L = \begin{cases} 
1 - e^{-1/\xi}, & 0 \leq L \leq M, \\
1 - e^{-(M+1)/\xi} e^{-L/\xi}, & 0 \leq L \leq M, \\
\sum_{L=1}^{M} L^{-k} -1, & k > 1, 1 \leq L \leq M. 
\end{cases}
\]

As \( M \to \infty \), power-law \( P_L \) has finite mean for \( k > 2 \), while that of the exponential is always finite. The dynamics in time \([t, t + dt]\) involves a walker on a site performing either (i) biased hopping with probability \( 1 - rdr \): hop to nearest-neighbor (NN) site(s) along (respectively, against) the bias with rate \( \alpha \equiv W(1 + g) \) (respectively, \( \beta \equiv W(1 - g) \)), or, (ii) resetting with probability \( rdr \). The latter involves (a) reset from a branch to the respective backbone site; (b) reset from a backbone site to itself, with \( r \) the resetting rate. We assume respectively periodic and reflecting boundary conditions for backbone and open end of the branches, and define \( f \equiv \alpha / \beta > 1 \).
Figure 1. (a) A RC comprising a backbone, with random-length branches; broken and continuous arrows denote respectively resetting and biased hopping in presence of bias g. (b) Dramatic consequence of resetting on stationary-state transport shown schematically: no resetting results in walkers trapped towards the end of very-long branches (shown here is one such branch) and consequently, zero drift velocity along backbone. Long-range instantaneous jumps due to resetting allow walkers from the open end to get to the backbone, implying no trapping (thus, vanishing probability to find the walkers towards the open end) and hence, a nonzero drift velocity along backbone. (c) Stationary-state drift velocity versus $g$ from theory (equation (12), continuous line) and numerics (symbols), with $W = 0.5$, number of backbone sites $N = 200$ and for a typical branch-length realization sampled from exponential distribution. Numerics correspond to standard Monte Carlo simulations of the dynamics [64].

The system, in absence ($r = 0$) and presence ($r \neq 0$) of resetting, settles at long times into an NESS. Even with $r = 0$, analytical characterization of the NESS is a long-standing open problem, with approximate analysis pursued until now. For instance, in analyzing transport properties, physical arguments assuming zero current in the branches [2, 3], or, a mean-field approach [8, 10, 11] based on self-consistent scaling and continuous-time RW was invoked. A remarkable revelation is that, for exponential $P_L$, the stationary-state drift velocity along the backbone, $v_{\text{drift}}$, varies non-monotonically with $g$ as $M \to \infty$, becoming zero beyond a threshold $g_c$ because of trapping at long branches.

We motivate our study thus: referring to figure 1(b), consider a random walker aiming to reach a destination lying ahead (which defines the bias direction) on the backbone but is unaware of the path to it. At every branch-backbone junction, it either enters the branch or continues on the backbone. While on a branch, it may at a random time realize that it may not eventually get to the destination, and deterministically walks back to the junction point. The deterministic motion being on a fast time scale compared to the RW-dynamics may be treated as an instantaneous resetting on the scale of the latter. A drift along the backbone at long times implies that the walker eventually reaches the destination.

Here, we report for biased RWs on RC exact NESS static and dynamic properties both in absence and presence of resetting and for any disorder realization $\{L_n\}$ corresponding to arbitrary $P_L$. The NESS-distribution of walkers (equation (10)) and the associated $v_{\text{drift}}$ (equation (12)) hold for general $N$; for the latter, we validate earlier results obtained using approximations as $N \to \infty$ and for exponential $P_L$ [3]. Further, we propose and verify a criterion (equation (14)), valid for arbitrary $P_L$, to observe trapping and hence a vanishing drift velocity. We establish a dramatic consequence of resetting (figure 1(b)): in its absence, a choice of $P_L$ that leads to trapping of walkers towards the open end of long branches and a vanishing drift velocity results, with resetting, in a nonzero drift velocity. Resetting allows walkers to make long-range instantaneous jumps to reach the backbone from the open end, implying no trapping and consequently, nonzero drift velocity. This letter reports a rare example of a system with quenched disorder for which we obtain the exact NESS (i) in absence and presence of resetting, (ii) for any disorder realization, and (iii) in the thermodynamic limit ($N \to \infty$) as well as for finite $N$. 

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Resetting on comb-like structures was invoked in discussing diffusion process in three dimensions [65], RWs on comb graphs with equal-length side-chains [66], and diffusion in a two-dimensional comb with continuously-distributed branches [67]. Our setup involving combs with random branch-lengths and focus on exact NESS deviate markedly from these studies.

To proceed, define \( P_{n,m}(t) \equiv P(n, m, t|0, 0, 0) \) as the conditional probability for a walker to be on site \((n, m)\) at time \( t > 0 \), given that it was on \((0, 0)\) at \( t = 0 \). With normalization \( \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} P_{n,m}(t) = 1 \), \( P_{n,m}(t) \) satisfies the master equation (ME):

\[
\dot{P}_{n,m} = \mathcal{L}P_{n,m}(t) - rP_{n,m}(t) + r\delta_{m,0} \sum_{m'=0}^{L_n} P_{n,m'}(t),
\]

with dot denoting time derivative. With \( \mathcal{W}_{(n',m') \rightarrow (n,m)} \) the transition rate from \((n', m')\) to \((n, m)\) and sum running over all \((n', m')\) that are NN-sites of \((n, m)\), the term \( \mathcal{L}P_{n,m}(t) \equiv \sum_{(n',m')} [\mathcal{W}_{(n',m') \rightarrow (n,m)} P_{n',m'}(t) - \mathcal{W}_{(n,m) \rightarrow (n',m')} P_{n,m}(t)] \) represents ways in which \( P_{n,m}(t) \) changes due to biased-RW dynamics. The second and third terms on the right-hand side (rhs) of equation (2) stand for resetting. The latter represents gain in probability at the backbone site due to resetting, while the former denotes the corresponding loss in probability.

To solve (2) for \( P_{n,m}(t) \)'s for a given realization \( \{L_m\} \), apply Laplace transformation (LT) to equation (2): \( \bar{P}_{n,m}(s) \equiv \int_0^\infty e^{-st}P_{n,m}(t) \) [9]. The ME for branch sites, \( \bar{P}_{n,m}(t) = \alpha \bar{P}_{n,m-1}(t) - (\beta + r)\bar{P}_{n,m}(t) + (1 - \delta_{m,1})[\beta \bar{P}_{n,m+1}(t) - \alpha \bar{P}_{n,m}(t)] \), involves three sites, except for the reflecting-end \((m = L_n)\) that involves the last two branch sites. Applying LT to the ME for \( m = L_n \) gives \( \bar{P}_{n,L_n-1}(s) = (s + \beta + r)/\alpha \bar{P}_{n,L_n}(s) \). This helps to relate the LT-transformed probabilities on two consecutive branch sites by considering successively the LT-transformed branch-ME for \( m = L_n - 1, \ldots, 1 \). We get [64]: \( \bar{P}_{n,m}(s) = \Gamma_{L_n-1}(s) \bar{P}_{n,m-1}(s), \quad m = 1, \ldots, L_n \), with finite continued fraction \( \Gamma_M(s, r) \) being

\[
\Gamma_M(s, r) \equiv \frac{1}{\frac{1}{\alpha + \beta + r} - \frac{s}{\alpha + \beta + r}},
\]

containing \( M \) terms in the denominator. In particular, \( \bar{P}_{n,1}(s) = \Gamma_{L_n}(s, r) \bar{P}_{n,0}(s) \). A remarkable transformation \( \cosh \theta \equiv \sqrt{f}((s + \alpha + \beta + r)/(2\alpha)) = (2W + r)/(2W + r)\sqrt{1 - g^2} \) \((1 + s/(2W + r))\) evaluates \( \Gamma_M \) in closed form, yielding for \( M = L_n \),

\[
\Gamma_{L_n} = \sqrt{f} \frac{\sinh L_n \theta}{\sinh(L_n + 1) \theta - \sqrt{f} \sinh L_n \theta}.
\]

The recursion \( \bar{P}_{n,m}(s) = \Gamma_{L_n-m+1}(s) \bar{P}_{n,m-1}(s) \) and the closed-form \( \Gamma_M \) give

\[
\frac{\bar{P}_{n,m}(s)}{\bar{P}_{n,0}(s)} = \frac{\sinh(L_n - m + 1) \theta - \sqrt{f} \sinh(L_n - m) \theta}{\sinh(L_n + 1) \theta - \sqrt{f} \sinh(L_n \theta)}.
\]

We now apply LT to the ME for the backbone:
\[ \dot{P}_{n,0}(t) = \alpha [(1 - \delta_{n,0})P_{n-1,0}(t) + \delta_{n,0}P_{n-1,0}(t)] \\
+ \beta [(1 - \delta_{n,N-1})P_{n+1,0}(t) + \delta_{n,N-1}P_{n,0}(t)] + \beta P_{n,1}(t) \\
- (2\alpha + \beta)P_{n,0}(t) + r \sum_{m' = 1}^{L_n} P_{n,m'}(t); \quad 0 \leq n \leq N - 1, \]  

(6)

where effects of resetting from backbone sites onto themselves cancel out. We get

\[ s\tilde{P}_{n,0}(s) - \delta_{n,0} = \alpha [(1 - \delta_{n,0})\tilde{P}_{n-1,0}(s) + \delta_{n,0}\tilde{P}_{n-1,0}(s)] \\
+ \beta [(1 - \delta_{n,N-1})\tilde{P}_{n+1,0}(s) + \delta_{n,N-1}\tilde{P}_{n,0}(s)] + \beta \tilde{P}_{n,1}(s) \\
- (2\alpha + \beta)\tilde{P}_{n,0}(s) + r \sum_{m' = 1}^{L_n} \tilde{P}_{n,m'}(s). \]  

(7)

For each \( n \), this ME involves three consecutive backbone sites and all the attached branch sites. Using \( \Delta L_n(s, r) = \Gamma L_n \tilde{P}_{n,0}(s) \) and defining \( \Delta L_n(s, r) \) as \( \Delta L_n \tilde{P}_{n,0}(s) = \sum_{m' = 1}^{L_n} \tilde{P}_{n,m'}(s) \forall n \), replace the LT-transformed branch-site probabilities in the ME with \( \tilde{P}_{n,0}(s) \), giving

\[ s\tilde{P}_{n,0}(s) - \delta_{n,0} = \alpha [(1 - \delta_{n,0})\tilde{P}_{n-1,0}(s) + \delta_{n,0}\tilde{P}_{n-1,0}(s)] + \beta \left[ (1 - \delta_{n,N-1}) \right. \]
\[ \times \tilde{P}_{n+1,0}(s) + \delta_{n,N-1}\tilde{P}_{n,0}(s)] + \beta \Gamma L_n \tilde{P}_{n,0}(s) \\
- (2\alpha + \beta)\tilde{P}_{n,0}(s) + r \Delta L_n \tilde{P}_{n,0}(s). \]  

(8)

These \( N \) coupled linear equations involving only the backbone sites write as a matrix equation:

\[ \mathbf{\tilde{A}}(s) = \mathbf{E}, \]  

(9)

with \( \mathbf{\tilde{A}}(s) \equiv \left(\tilde{P}_{0,0}(s), \tilde{P}_{1,0}(s), \ldots, \tilde{P}_{N-1,0}(s)\right)^T \), \( \mathbf{E} \equiv (1, 0, \ldots, 0)^T \), \( T \) denoting transpose. The matrix \( \mathbf{A} \) has elements \( A_{n,m'} = -\alpha \delta_{n-1,m'} + C_n \delta_{n,m'} - \beta \delta_{n+1,m'} \) for \( 0 \leq n, m' \leq N - 1 \), with \( \delta_{-1,m'} = \delta_{N-1,m'}, \delta_{n,N'} = \delta_{0,m'}, \) \( C_n \equiv s + 2\alpha + \beta (1 - \Gamma L_n) - r \Delta L_n \), where \( \Delta L_n \) on using equation (5) evaluates as \([64]; \Delta L_n = (\beta/(s + r))(f - \Gamma L_n) \). Equation (9) gives \( \mathbf{P}(s) = \mathbf{A}^{-1}\mathbf{E} \), which evaluated numerically yields LT-transformed backbone-site probabilities for a given realization \( \{L_n\} \); the same for branch sites are given by equation (5). Inverse LT of \( P_{n,m}(s) \)'s so obtained yields \( P_{n,m}(t) \forall n, m, t > 0 \).

We are interested in the transport properties in the NESS. The latter is characterized by time-independent probabilities \( P_{n,m}^{\text{stat}} = \lim_{t \to \infty} P_{n,m}(t) \), obtained from equation (9) by using the final value theorem (FVT): \( P_{n,m}^{\text{stat}} = \lim_{s \to 0} s \tilde{P}_{n,m}(s) \). Consider \( s \to 0 \) such that for any \( g \) and \( r > 0 \), \( s/g \ll 1 \) and \( s/r \ll 1 \). One then obtains from equation (4) that \( \Gamma L_n(s, r > 0)_{|s\to 0} = \Lambda_L L_n/\Lambda_0 L_n \), with \( \Lambda_{n,m} ≡ (m^g/2)[\lambda^{m-g}(\lambda - \sqrt{T}) - \lambda^{-g}e^{m}(1/\lambda - \sqrt{T})] \); \( m = 0, 1, \ldots, L_n \), and \( \lambda \equiv (2W + r)/(2W\sqrt{1 - g^2})[1 + \sqrt{1 - (4W^2(1 - g^2))}/((2W + r)^2)] \), while \( \Delta L_n(s, r > 0)_{|s\to 0} = (\beta/r)(f - \Gamma L_n(s, r > 0)_{|s\to 0}) \). We thus get \( C_n = s + 2\alpha + \beta (1 - \Gamma L_n(s, r > 0)_{|s\to 0} - r \Delta L_n(s, r > 0)_{|s\to 0} = \alpha + \beta \). Equation (9), on applying FVT, thus gives stationary-state backbone-ME: \( C_n P_{n,0}^{\text{stat}} = \alpha [(1 - \delta_{n,0})P_{n-1,0}^{\text{stat}} + \delta_{n,0}P_{n-1,0}^{\text{stat}}] + \beta [(1 - \delta_{n,N-1})P_{n+1,0}^{\text{stat}} + \delta_{n,N-1}P_{n,0}^{\text{stat}}] \); the rhs denotes gain in probability, which is balanced by the left denoting the corresponding probability loss. \( C_n P_{n,0}^{\text{stat}} \) then gives stationary-state transition rate out of the \( n \)th backbone site.
The result $C_{n|x=0} = \alpha + \beta$ is non-trivial and interesting: it (i) does not involve $r$, (ii) is independent of $n$, or, equivalently, $L_n$, (iii) has the same value as for $L_n = 0$ (for $L_n = 0$, $\Gamma_{L_n} = f$ and $\Delta_{L_n} = 0$ give $C_{n|x=0} = [s + 2\alpha + \beta(1 - f)]|_{x=0} = \alpha + \beta$). Remarkably, the stationary-state backbone-ME has no branch-effects although the underlying dynamics involves hopping and resetting and includes backbone and branch sites. Indeed, this ME is mathematically equivalent to that for single-site probabilities $p_{n}^{st}$; $n = 0, 1, \ldots, N - 1$ for non-interacting random walkers undergoing only hopping to NN sites with rates $\alpha$ and $\beta$ on a 1d periodic lattice of $N$ sites. This equivalence holds key to our exact results on $v_{\text{drift}}^{\alpha}$.

The aforementioned equivalence is by no means obvious and holds only in the NESS. Then, if the stationary-state backbone-ME in presence of hopping and resetting is the same as the one on a 1d periodic lattice with only hopping, how do branch-effects manifest in the former? The answer lies in the normalization of the stationary-state probabilities. The stationary-state ME yields in both cases a uniform probability: uniform $(p_{n}^{st} = p^{st}(\forall n))$ over the backbone, uniform $(= p^{b})$ over the 1d periodic lattice. The normalization condition however reads differently: $\sum_{m=0}^{N-1} \sum_{n=0}^{N-1} L_{n} p_{n,m}^{st} = 1$ and $\sum_{m=0}^{N-1} P^{st}_{n,m} = 1$. Note that for RC, the branch-site probabilities are not uniform. Applying FVT to the equation defining $\Delta_{L_n}$ gives $\Delta_{L_n}(s, r > 0)|_{x=0} p^{st} = (\beta / r)(f - \Gamma_{L_n}(s, r > 0))|_{x=0} p^{st} = \sum_{m'=1}^{N} p_{n,m'}^{st} p^{st}_{m'}|_{x=0}$, which used in the normalization condition gives $p^{st} = (1/N)\left[1 + (1/N)\sum_{m=0}^{N-1} \Delta_{L_n}(s, r > 0)|_{x=0}\right]^{-1}$, while $p^{st} = 1/N$. The stationary-state branch-site probabilities are obtained by applying FVT to equation (5), yielding $P^{st}_{n,m} = (\lambda_{m, L_n}/\lambda_{0, L_n}) p^{st}$.

To obtain the NESS for non-resetting case, we first set $r = 0$ and consider $s \rightarrow 0$ such that $s / g \ll 1$ for any $g$, to get $\Gamma_{L_n}(s, 0)|_{x=0} = f$ and $\Delta_{L_n}(s, 0)|_{x=0} = (f / (f - 1))(f^{L_n} - 1)$, yielding $C_{n|x=0} = \alpha + \beta$. Using the equivalence of the stationary-state backbone-ME with that for a 1d periodic lattice and the following steps as invoked above for $r \neq 0$ yield the exact expression for the backbone-site probabilities for a given realization $\{L_n\}$ as $p^{st} = (1/N)\left[1 + (1/N)\sum_{m=0}^{N-1} \Delta_{L_n}(s, r > 0)|_{x=0}\right]^{-1}$; the same for the branch sites are given by $p_{n,m}^{st} = f^{m} p^{st}$. We thus obtain exact stationary-state probabilities on all RC-sites both in presence and absence of resetting and for a given realization $\{L_n\}$, one of our key results applicable to any RC as in figure 1(a). The backbone probability has the form

$$p^{st} = \frac{1}{N} \left[1 + \frac{1}{N} \sum_{m=0}^{N-1} \Delta_{L_n}(s, r)|_{x=0}\right],$$

with

$$\Delta_{L_n}(s, r)|_{x=0} = \begin{cases} f - 1 \left(f^{L_n} - 1\right); & r = 0, \\ \frac{f}{r} \left(f - \Gamma_{L_n}(s, r > 0)|_{x=0}\right); & r \neq 0. \end{cases}$$

To compute $v_{\text{drift}}^{st} = v_{\text{drift}}^{\alpha}$, consider the equivalent 1d system of non-interacting walkers. The probability $p_{n}(t)$ to be on site $n$ at time $t$ while starting from $n = 0$ at $t = 0$ satisfies the ME $p_{n}(t) = \alpha[(1 - \delta_{n,0})p_{n-1}(t) + \delta_{n,0}p_{n-1}(t)] + \beta[(1 - \delta_{n,N-1})p_{n+1}(t) + \delta_{n,N-1}p_{n}(t)] - (\alpha + \beta)p_{n}(t)$. Let $p_{n+t, L_n}(t)$ be the probability that a walker starting from $n = 0$ at $t = 0$ and undergoing integer $L_n \in (-\infty, \infty)$ number of turns round the periodic lattice arrives at site $n$ at time $t$. Evidently, $p_{n}(t) = \sum_{n=0}^{L_n} p_{n+t, L_n}(t) \forall n, t$, and $p_{n+t, L_n}(t)$ satisfies the same ME as $p_{n}(t)$. The average displacement in time $t$ is $\langle x(t) \rangle = \sum_{n=0}^{L_n} (n + L_n)p_{n+t, L_n}(t)$, yielding drift velocity $v(t) = \partial \langle x(t) \rangle / \partial t = \sum_{n=0}^{L_n} \sum_{n=0}^{L_n} (n + L_{n})p_{n+t, L_n}(t)$. Using the ME, one obtains $v(t) = (\alpha - \beta)\sum_{n=0}^{L_n} \sum_{n=0}^{L_n} p_{n+t, L_n}(t) = (\alpha - \beta)\sum_{n=0}^{L_n} p_{n}(t)$. As $t \rightarrow \infty$, one obtains $v_{\text{drift}}^{st} = (\alpha - \beta)\sum_{n=0}^{L_n} p_{n}(t)$. As $t \rightarrow \infty$, one obtains $v_{\text{drift}}^{st} = (\alpha - \beta)\sum_{n=0}^{L_n} p_{n}(t)$.
\[ (\alpha - \beta) \sum_{n=0}^{N-1} p^n = (\alpha - \beta) N p^0. \] The equivalence of the NESS dynamics on the RC-backbone with that of 1d periodic system implies \( v_{\text{drift}}^{\text{st}} = (\alpha - \beta) N p^0 \) for RC, obtaining

\[ v_{\text{drift}}^{\text{st}} = \frac{(\alpha - \beta)}{1 + \frac{1}{N} \sum_{n=0}^{N-1} \Delta_{L,s}(s,r)|_{r=0}}. \]  

(12)

The result (12) is verified in figure 1(c) against numerical simulations for \( N = 200, W = 0.5, \) exponential \( P_L(M = 20, \xi = 5) \) [64].

Note that \( v_{\text{drift}}^{\text{st}} \) in equation (12) gives the drift velocity for a given disorder realization. As \( N \to \infty \), the law of large numbers lets the sample average \( (1/N) \sum_{n=0}^{N-1} \Delta_{L,s}(s,r)|_{r=0} \) in equation (12) be replaced with expectation \( \langle \Delta_{L,s}(s,r)|_{r=0} \rangle \equiv \sum_{s} \Delta_{L,s}(s,r)|_{r=0} P_L \), when the latter is finite, as is the case with finite \( M \). Such a replacement makes the resulting expression independent of disorder realizations: \( v_{\text{drift}}^{\text{st}} \to v_{\text{drift}}^{\text{st}} \) with overbar denoting the disorder-realization-independent answer.

For exponential \( P_L \), one easily computes for \( r = 0 \) the quantity \( \langle \Delta_{L,s}(s,r)|_{r=0} \rangle \), obtaining

\[ \bar{v}_{\text{drift}}^{\text{st}} = \frac{(\alpha - \beta)}{e^{1/\xi g} - 1} \left[ e^{1/\xi g} G_M(e^{-1/\xi}) - e^{1/\xi g} \eta \right], \]  

(13)

with \( G_M(y) \equiv (1 - y)/(1 - y^{M+1}) \) and the bias-dependent length scale \( L(g) \equiv 1/\ln f = [\ln(1 + g)/(1 - g)]^{-1} \) [3]. For \( r = 0, \bar{v}_{\text{drift}}^{\text{st, RW}} = f_{\text{RW}}^{\text{st}} \) implies that the net stationary-state probability-current due to biased-RW dynamics, \( \bar{v}_{\text{drift}}^{\text{st, RW}}(\alpha, \beta, m) = \alpha \bar{v}_{\text{drift}}^{\text{st}} - \beta \bar{v}_{\text{drift}}^{\text{st}}, \) is zero in the branches, which was a crucial assumption to derive equation (13) in reference [3] and that we show here to be exact. In contrast, for \( r \neq 0, \bar{v}_{\text{drift}}^{\text{st, RW}}(\alpha, \beta, m, r) > 0, \) and the difference of the net stationary-state probability-current into and out of a site is balanced by an outgoing resetting current [64].

For finite \( N, M \), the sample average in equation (12) is finite, and so is \( v_{\text{drift}}^{\text{st}}; N \to \infty \) at finite \( M \), when \( \langle \Delta_{L,s}(s,r)|_{r=0} \rangle \) is always finite, too yields finite \( v_{\text{drift}}^{\text{st}} \). The opposite limit \( M \to \infty \) at finite \( N \) may render the sample average infinite, yielding \( v_{\text{drift}}^{\text{st}} = 0 \) for specific disorder realizations. A case of interest is considering limit \( N \to \infty \), first, when expectations replace sample averages, followed by \( M \to \infty \), and asking: does the disorder-realization-independent \( v_{\text{drift}}^{\text{st}} \) become zero at any \( g \)? For \( v_{\text{drift}}^{\text{st}} \) to be zero, \( \langle \Delta_{L,s}(s,r)|_{r=0} \rangle \) has to diverge. Now, we have \( \langle \Delta_{L,s}(s,r)|_{r=0} \rangle = \sum_{L} P_L \Delta_{L,s}(s,r)|_{r=0} \), wherein, while \( P_L \) is always finite and is a decreasing function of \( L \), the quantity \( \Delta_{L,s}(s,r)|_{r=0} \) is an increasing function of \( L \) with \( \Delta_{L,s}(s,r)|_{r=0} \) becoming zero at \( L = 0 \). Consequently, the product \( P_L \Delta_{L,s}(s,r)|_{r=0} \) will be either (i) a monotonically increasing function of \( L \) that diverges as \( L \to \infty \), or, (ii) a monotonically decreasing function of \( L \) that does not ever diverge at any \( L \) and goes to zero as \( L \to \infty \), or, (iii) a nonmonotonic function of \( L \) that goes to zero at \( L = 0 \) and as \( L \to \infty \), with a peak at a finite value \( L^* \) of \( L \). Then, as \( M \to \infty \), one has the quantity \( \langle \Delta_{L,s}(s,r)|_{r=0} \rangle \) remaining finite in cases (ii) and (iii); in the case of (i), however, \( \langle \Delta_{L,s}(s,r)|_{r=0} \rangle \) will be diverging, owing to the term \( \Delta_{M,s}(s,r)|_{r=0} P_M \) tending to infinity as \( M \to \infty \). We thus conclude that divergence of \( \langle \Delta_{L,s}(s,r)|_{r=0} \rangle \) requires \( \lim_{M \to \infty} \Delta_{M,s}(s,r)|_{r=0} P_M \to \infty \), where we have for brevity suppressed the dependence of \( \Delta_{L} \) on \( s \) and \( r \). If \( n^\ast \) is a backbone site with attached branch length \( M, \Delta_{M,s}(s,r)|_{r=0} P_M = (1/P_M) \sum_{n^\ast} P_{n^\ast} P_M \) diverges in the limit \( M \to \infty \) if

\[ \lim_{M \to \infty} \left( R \equiv P_{M^n} \frac{P_{n^\ast}}{P_M} \right) \to \infty. \]  

(14)
Physically, $\mathcal{R}$ represents the contribution, from those backbone sites with attached branch length equal to $M$ to the quantity $\langle \Delta L_{\text{drift}} \rangle$, of the relative probability $\frac{P_{n,M}}{P_n}$ of walkers to be on the open end of the branch to that on the backbone. Now, $\frac{P_{n,M}}{P_n}$ being a probability can never diverge. Then, a diverging $\mathcal{R}$ that is associated with a zero drift implies that the walkers are trapped at the open end of such branches, so that one has a vanishing probability of finding them on the backbone: $\frac{P_n}{P_{n,M}} = 0$. Such a trapping results when a walker that happens to be at the open end of a branch at any time has to move against the bias to get to the backbone.\n
Equation (14) thus gives the criterion to observe trapping and hence a vanishing $v_{\text{drift}}^{st}$.\n
With no resetting, using $\frac{P_{n,M}}{P_n} = f^M$ we get for exponential $\mathcal{P}_L$ that $\mathcal{R} \sim \exp \left[ M \left( \frac{1}{L(g)} - \frac{1}{\xi} \right) \right]$, involving two competing length scales $\xi$ and $L(g)$. As $M \to \infty$, trapping requires that $L(g) < \xi$. Trapping causes a vanishing $v_{\text{drift}}^{st}$. Thus, $v_{\text{drift}}^{st}$ crosses over from a finite value to zero at $g = g_c$ satisfying $L(g_c) = \xi$. Our derived condition for trapping for exponential $\mathcal{P}_L$ was obtained in reference [3] by analyzing $v_{\text{drift}}^{st}$ in equation (13) as $M \to \infty$. We here go beyond reference [3] in deriving the condition (14) for trapping that is applicable to any distribution $\mathcal{P}_L$. For instance, for power-law $\mathcal{P}_L$ with $k > 2$ so that $f^{(k+1)}$ and hence, $\frac{P_n}{P_{n,M}}$ is finite, $\mathcal{R} \sim \exp \left[ M \left( \frac{1}{L(g)} - k \ln M \right) \right]$ diverges as $M \to \infty$ for any $0 < g < 1$, implying $v_{\text{drift}}^{st} = 0$ at any bias.\n
In the above backdrop, a pertinent question arises: what happens to trapping as one introduces infinitesimal resetting? Using $\mathcal{R} = (\mathcal{P}_L \Lambda_{M,0,M}) / \Lambda_{0,M}$, exponential $\mathcal{P}_L$, and the limit $r \to 0$ yield for large $M$ the result [64]: $\mathcal{R} \sim \exp \left[ M \left( \frac{1}{L(g)} - \frac{1}{\xi} \right) \right] \exp \left[ -(r/(2Wg))(f/f - 1)f^M \right]$, in which the exponential involving $r$ gives the leading contribution in view of $f > 1$. Consequently, one has $\mathcal{R} \to 0$ as $M \to \infty$, leading to a finite $v_{\text{drift}}^{st}$ at any $g$. The power-law $\mathcal{P}_L$ and $r \to 0$ yield for large $M$ that $\mathcal{R} \sim \exp \left[ M \left( \frac{1}{L(g)} - k \ln M \right) \right] \exp \left[ -(r/(2Wg))(f/f - 1)f^M \right]$. Again, it is because of the exponential involving $r$ that condition (14) is not satisfied, yielding a finite $v_{\text{drift}}^{st}$ at any $g$. We thus see a dramatic consequence of resetting: while in its absence, on varying $g$, $v_{\text{drift}}^{st}$ is zero for power-law $\mathcal{P}_L$ or shows a crossover from a finite value to zero for exponential $\mathcal{P}_L$, it is always finite in presence of resetting. Finally, we study how $v_{\text{drift}}^{st}$ changes on introducing infinitesimal resetting. Equation (12) yields [64]\n
$$\frac{v_{\text{drift}}^{st}(r \to 0) - v_{\text{drift}}^{st}(r = 0)}{v_{\text{drift}}^{st}(r = 0)} = \alpha(\alpha - \beta)(b_2) \left(1 + \alpha(b_1)\right)^{-1},$$\n
with $\langle b_1 \rangle \equiv (2Wg)^{-1} \left( f^{(k+1)} - 1 \right)$, and $\langle b_2 \rangle \equiv (4W^2g^2)^{-1} \left[ (f^{(k+1)} - 1)(f^{(k+1)} - 1 - 2(L_n f^{(k)} - f^{(k)})) \right]$. For any $\mathcal{P}_L$, the rhs is non-zero at any $g$, implying finite $v_{\text{drift}}^{st}$ and no trapping on turning on resetting. This is consistent with our earlier discussion on trapping condition not satisfied with resetting. A finite mean time $1/r$ between successive resets guarantees that a walker that is trapped at the open end of a long branch in absence of resetting can in its presence get to the backbone instantaneously through the now-allowed direct jump, thus avoiding trapping.\n
An interesting follow-up involves extending our analysis to a many-particle setup with exclusion interaction [68] and employing reset-setups using optical tweezers [69, 70] to study RC-dynamics.
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Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

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