STU Black Holes and SgrA*

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ABSTRACT

The equations of null geodesics in the STU family of rotating black hole solutions of supergravity theory, which may be considered as deformations of the vacuum Kerr metric, are completely integrable. We propose that they be used as a foil to test, for example, with what precision the gravitational field external to the black hole at the centre of our galaxy is given by the Kerr metric. By contrast with some metrics proposed in the literature, the STU metrics satisfy by construction the dominant and strong energy conditions. Our considerations may be extended to include the effects of a cosmological term. We show that these metrics permit a straightforward calculation of the properties of black hole shadows.
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1 Introduction

Current and forthcoming observations of the galactic centre, and in particular its shadow [1-3], should ultimately allow a rather precise determination of the metric around the black hole associated with the source SgrA*. Since the black hole is likely to be rotating at a near maximal rate, i.e. with the ratio of angular momentum $|J|$ to mass $M^2$ close to unity, an important goal of these observations is to check how accurately the metric agrees with that given by the Kerr solution [4] of the Einstein equations. If this goal can be achieved one should have a powerful observational test of the Einstein equations at the fully non-linear level (see, e.g., [5-7]). Moreover, according to a well known series of so-called No-Hair theorems following rigorously from Einstein’s equations the possible stationary metric should depend only on the two parameters $J$ and $M$. In making such a test it is thus desirable to have at hand a family of metrics depending on say $n$ additional dimensionless parameters $\lambda_i$, $i = 1, 2, \ldots, n$, such that if $\lambda_i = 0$, we obtain the Kerr solution. The observational goal then becomes to determine upper bounds on the magnitudes of the parameters $Q_i$. Of course this programme is not limited to the case of SgrA* and applies to any plausible astrophysical black hole candidate but its seems that it is SgrA* that offers the greatest promise of progress in the near future [5].

Since the observational signals from black holes depend to a large extent on the behaviour of timelike and null geodesics in the vicinity of the horizon, it is computationally highly desirable that the geodesic equations for the family of metrics should possess the non-trivial property that they be integrable, as they are in the Kerr case [8]. Possible families of metrics for this purpose were introduced in [9, 10], but they suffer from various disadvantages and a more convenient family of metrics was introduced in [11]. These have the merit of being expressible in a “Kerr-Schild-like” form, which facilitates fully relativistic magnetohydrodynamic calculations.

Imposing the requirement that the Hamilton-Jacobi equation

$$-rac{\partial S}{\partial \tau} = \frac{1}{2} g^{\mu \nu} \partial_\mu S \partial_\nu S$$

should have separable solutions of the form

$$S = \frac{1}{2} \mu^2 \tau - Et + L\phi + S_r(r) + S_\theta(\theta)$$

leads to a family of metrics with four arbitrary functions of $r$ and four arbitrary functions of $\theta$. Imposing various regularity and asymptotic flatness conditions reduces this to just four functions of $r$ that have large-distance expansions in inverse powers of $r$.

This provides a powerful general framework but it is perhaps too general, in the sense that with such infinite-dimensional arbitrariness it may be possible to fit almost any observational data. It may therefore be necessary to restrict the range of possibilities. One such restriction would arise from imposing the dominant and the strong energy conditions, so that the family of metrics would at least be generated by physically-allowable matter sources. However this will still not substantially reduce the infinite-dimensional manifold of possible metrics.
In this note we would like to suggest that an interesting family of possibilities, not completely overlapping with those introduced in [11], is contained within the class of much studied and completely explicit asymptotically-flat rotating black hole solutions of STU supergravity theory [12,13]. These include the Kerr and Kerr-Newman metrics as special cases, and in general depend on six parameters, two of which correspond to the total energy $M$ and angular momentum $J$, with the other four associated to generalised charges $Q_i$ carried by gauge fields in the supergravity theory. The most general STU black holes solutions, with the additional (fifth) independent generalised charge parameter, were recently obtained by Chow and Compère [14,15].

Some of these possibilities within the asymptotically-flat STU black hole solutions have been considered in this context already, including the Kerr-Newman metric, in [16], and the Kaluza-Klein rotating dilaton black hole, in [17,19]. Another case which has been extensively investigated is the so-called Kerr-Sen black hole [20,22]. Because in this case the timelike geodesics are also integrable, it has also been the subject of studies of the Banados-Silk-West (BSW) [23] effect [24,25].

The integrability of the timelike geodesics turns out to be a more general phenomenon. In [14,15] it was shown that a wider class of axisymmetric-stationary metrics admits separation for both the null and timelike geodesics. Note, that this result is a consequence of the separability of the minimally coupled massless scalar equations in the STU black hole background [14,15,26].

The Chow-Compère solutions include the most general seed from which to generate, using an $SO(4,4)$ solution generating procedure, all rotating charged solutions of $\mathcal{N} = 8$ un-gauged supergravity theory provided one works in string conformal frame rather than Einstein-conformal frame. The two conformal frames in the case of the Sen Black holes In fact comparing equation (5.15) of [15] with equation (17) of [11] one finds agreement in the case of null geodesics. See also [19] in the Kaluza-Klein case. This observation of Chow and Compère opens the way to a much more extensive comparison of astronomical observations and discussions of such topics as the BSW effect with the STU metrics.

Some other cases not directly related to STU supergravity include the rotating braneworld black hole [27], and the rotating black hole in extended Chern-Simons modified gravity [28] for rotating black holes with exotic matter [29]. For a review of some of these see [30].

The paper is organised as follows.

In section 2 we introduce the 4-charge STU family of charged rotating black holes and describe their separability properties. They depend on 4 real charge parameters $\delta_i$, in addition to the mass and rotation parameters. We present formulae for the ADM mass, total angular momentum, four charges and four magnetic dipole moments. We show that these satisfy some interesting inequalities. By specialising the parameters we obtain various well-known special cases, some of which have already been studied in the astrophysical literature. In the non-rotating case we give the values of the Parametrised Post Newtonian (PPN) parameters $\beta$ and $\gamma$ for spherically-symmetric metrics in terms of the parameters $\delta_i$.

In section 3 we discuss null geodesics and in section 4 equatorial geodesics, both timelike and null.
Section 5 is devoted to the study of shadows. We give a general discussion and then specialise to the case where the four electric charges are set pairwise equal, for which the problem can be reduced to a straightforward numerical procedure. We have collected some details about the pairwise-equal charged black holes of STU supergravity in an appendix.

2 Rotating, Asymptotically Flat STU Black Holes

2.1 The STU metrics

In this section we shall focus on rotating black holes in ungauged STU supergravity, characterised by the mass the $M$, the angular momentum $J$ and four charges $Q_i$. Employing solution-generating techniques, these metrics were first obtained in [12], with all the sources explicitly displayed in [13]. Here we shall present the metric only:

$$\begin{align*}
\text{ds}^2 = & -\Delta^{-1/2} G(dt + \mathcal{A})^2 + \Delta^{1/2} \left( \frac{dr^2}{X} + d\theta^2 + \frac{X}{G} \sin^2 \theta d\phi^2 \right), \\
\text{where} \\
X = & \ r^2 - 2mr + a^2, \\
G = & \ r^2 - 2mr + a^2 \cos^2 \theta, \\
\mathcal{A} = & \ 2ma \sin^2 \theta \left[ (\Pi_c - \Pi_s)r + 2m\Pi_s \right] d\phi,
\end{align*}$$

(2.1)

and

$$\Delta = \prod_{i=1}^{4} (r + 2ms_i^2) + a^4 \cos^4 \theta$$

$$+ 2a^2 \cos^2 \theta |r|^2 + mr \sum_{i=1}^{4} s_i^2 + 4m^2(\Pi_c - \Pi_s)\Pi_s - 2m^2 \sum_{i<j<k} s_i^2 s_j^2 s_k^2. \tag{2.3}$$

We are employing the following abbreviations:

$$\Pi_c \equiv \prod_{i=1}^{4} c_i, \quad \Pi_s \equiv \prod_{i=1}^{4} s_i, \quad s_i = \sinh \delta_i, \quad c_i = \cosh \delta_i. \tag{2.4}$$

The solution is parametrised by the bare mass parameter $m$, the rotational parameter $a$ and four charge parameters $\delta_i$ ($i = 1, 2, 3, 4$). The solution is written as a fibration over a three-dimensional base that is itself independent of the charge parameters, and with a warp factor denoted by $\Delta$.

Defining

$$\mathcal{A}_{\text{red}} = 2m[(\Pi_c - \Pi_s)r + 2m\Pi_s], \quad \nu = \frac{\Delta - \mathcal{A}_{\text{red}}^2}{G}, \tag{2.5}$$

1For the STU Lagrangian of $\mathcal{N} = 2$ supergravity coupled to three vector supermultiplets, and the explicit form of the full solution, see [13]. See also, [31–33]. The black hole solutions of the STU theory are generating solutions of $\mathcal{N} = 4$ and $\mathcal{N} = 8$ supergravity theory, which can be obtained as a toroidal compactification on an effective heterotic string theory and Type IIA superstring theory, respectively. The full set of solutions of these maximally supersymmetric supergravity theories can be obtained by acting with a subset of respective $\{S,T\}$- and $U$-duality transformations. (See e.g., [34, 35].)
and noting that \( G = X - a^2 \sin^2 \theta \), it can be seen that the inverse of the metric (2.1) is given by

\[
\Delta^{1/2} \left( \frac{\partial}{\partial s} \right)^2 = -\frac{1}{X} \left( A_{\text{red}} \frac{\partial}{\partial t} + a \frac{\partial}{\partial \phi} \right)^2 + X \left( \frac{\partial}{\partial r} \right)^2 + \frac{1}{\sin^2 \theta} \left( \frac{\partial}{\partial \phi} \right)^2 - \nu \left( \frac{\partial}{\partial t} \right)^2 , \quad (2.6)
\]

### 2.2 Electric charges, angular momentum and magnetic dipole moments

The electromagnetic properties of rotating electrically charged STU black holes are reviewed in [41].

Employing the notation introduced above the mass \( M \), angular momentum \( J \), the four electric charges \( Q_i \) and the four induced dipole charges \( \mu_i \) are given by\(^2\)

\[
M = \frac{m}{4} \sum_{i=1}^{4} \cosh 2\delta_i , \quad J = ma(\Pi_c - \Pi_s) \quad (2.7)
\]
\[
Q_i = m \sinh 2\delta_i , \quad \mu_i = 2ma(s_i \Pi^i_c - c_i \Pi^i_s) , \quad (2.8)
\]

where \( c_i, s_i, \Pi_s \) and \( \Pi_c \) are defined in (2.4), \( \Pi^i_c = \Pi_c / c_i \) and \( \Pi^i_s = \Pi_s / s_i \). Evidently

\[
4M \geq \sum_i |Q_i| . \quad (2.9)
\]

One also has

\[
\frac{\mu_i}{2J} = \frac{s_i \Pi^i_c - c_i \Pi^i_s}{\Pi_c - \Pi_s} . \quad (2.10)
\]

If we assume that \( s_i > 0 \) for all \( i \), then

\[
\frac{\mu_i}{2J} \leq \frac{s_i (\Pi^i_c - \Pi^i_s)}{c_i (\Pi^i_c - \Pi^i_s)} \leq \tanh \delta_i \leq 1 . \quad (2.11)
\]

There are some special cases which coincide with solutions of an Einstein-Maxwell-Dilaton theory for particular choices of the dilaton coupling to the electromagnetic field:

**Einstein-Maxwell Black Holes:**

Here \( \delta_1 = \delta_2 = \delta_3 = \delta_4 = \delta \). Thus from (2.8)

\[
M = m \cosh 2\delta , \quad J = ma \cosh 2\delta \quad (2.12)
\]
\[
Q_i = m \sinh 2\delta , \quad \mu_i = ma \sinh 2\delta \quad (2.13)
\]

and hence if we set \( Q = Q_i \) and \( \mu = \mu_i \), corresponding to a canonically-normalised electromagnetic field \( F = F_i \) such that the Lagrangian is \( \mathcal{L} = \sqrt{-g}(R - F^2) \), we find that \( g = 2 \) and

\[
\frac{\left| \mu \right|}{\left| J \right|} = \tanh 2\delta = \frac{|Q|}{M} \leq 1 . \quad (2.14)
\]

\(^2\)Here we are using the normalisation of the four \( U(1) \) gauge fields in which, in the absence of scalar fields, the Lagrangian has the form \( \mathcal{L} \sim \sqrt{-g}(R - \frac{1}{4}F^2) \), with the charges being of the form \( Q_i \sim 1/(4\pi) \int \ast F_i \). This is to be contrasted with what we refer to later as the “canonical normalisation” for an electromagnetic field, for which the Lagrangian has the form \( \mathcal{L} \sim \sqrt{-g}(R - F^2) \) and the charge is defined as \( Q = 1/(4\pi) \int \ast F \).
Kerr-Kaluza-Klein Charged Black Holes:

Here \( \delta_1 = \delta, \delta_2 = \delta_3 = \delta_4 = 0 \), and so

\[
M = \frac{m}{4}(3 + \cosh 2\delta), \quad J = ma \cosh \delta \tag{2.15}
\]
\[
Q_1 = m \sinh 2\delta, \quad \mu_1 = 2ma \sinh \delta. \tag{2.16}
\]

If we set \( Q = \frac{1}{2}Q_1 \) and \( \mu = \frac{1}{2}\mu_1 \), corresponding to a canonically-normalised electromagnetic field \( F = \frac{1}{2}F_1 \), we find that

\[
\beta = \frac{|\mu|}{|J|} = \tanh \delta \leq 1. \tag{2.17}
\]

To obtain agreement with [42, 43] we set

\[
v = \tanh \delta. \tag{2.18}
\]

We also have

\[
M \geq \frac{1}{2}|Q| \tag{2.19}
\]

which is consistent with the general Bogomolnyi inequality for Einstein-Maxwell Dilaton black holes [45].

Kerr-Sen Black Holes [44]:

Here \( \delta_1 = \delta_3 = \delta, \delta_2 = \delta_4 = 0 \). Thus

\[
M = \frac{1}{2}m(1 + \cosh 2\delta), \quad J = \frac{1}{2}ma(1 + \cosh 2\delta) \tag{2.20}
\]
\[
Q_1 = Q_2 = m \sinh 2\delta, \quad \mu_1 = \mu_2 = ma \sinh 2\delta. \tag{2.21}
\]

If we set \( Q = (\sqrt{2})^{-1}Q_1 = (\sqrt{2})^{-1}Q_3 \) and \( \mu = (\sqrt{2})^{-1}\mu_1 = (\sqrt{2})^{-1}\mu_3 \), corresponding to a canonically-normalised electromagnetic field \( F = (\sqrt{2})^{-1}F_1 = (\sqrt{2})^{-1}F_3 \), then

\[
\frac{|\mu|}{|J|} = \sqrt{2}\tanh \delta, \tag{2.22}
\]

and

\[
\frac{|Q|}{M} = \sqrt{2}\tanh \delta \leq \sqrt{2}. \tag{2.23}
\]

We find that that the gyromagnetic ratio is \( g = 2 \), and we find consistency with the general Bogomolnyi inequality for Einstein-Maxwell Dilaton black holes [45] and the Kerr-Sen black hole [44], provided one replaces Sen’s \( \alpha \) by \( 2\delta \).

Black Holes with Pairwise-Equal Charges:

A further subclass class of solutions which affords considerable simplifications arises when the four charges are taken to be pairwise equal, with \( \delta_3 = \delta_1 \) and \( \delta_4 = \delta_2 \). We give an explicit expression for the Lagrangian describing this truncation, and we discuss also its duality symmetries and give
the black hole solutions, in appendix A. After making the rescalings so that the two $U(1)$ gauge fields have the canonical normalisations, as given in the appendix, we see that

$$J = aM = \frac{ma}{2} (\cosh 2\delta_1 + \cosh 2\delta_2), \quad \mu_i = aQ_i = \frac{ma}{\sqrt{2}} \sinh 2\delta_i, \quad (2.24)$$

and so

$$\frac{|\mu_i|}{|J|} = \frac{|Q_i|}{M} = \frac{\sqrt{2} \sinh 2\delta_i}{\cosh 2\delta_1 + \cosh 2\delta_2} \leq \sqrt{2} \tanh \delta_i. \quad (2.25)$$

The pairwise-equal charge solutions encompass the Kerr-Sen solution if we set $\delta_2 = 0$ (and hence $\delta_4 = 0$ also). They also specialise to the Kerr-Newman case if we set $\delta_1 = \delta_2$ (and hence equal also to $\delta_3$ and $\delta_4$).

### 2.3 Parametrised Post-Newtonian parameters

We follow the analysis of [46]. The PPN or Eddington parameters $\beta$ and $\gamma$ are defined for asymptotically flat metrics by

$$-g_{00} = 1 - 2 \frac{M}{|x|} + 2 \beta \frac{M^2}{|x|^2} + O(\frac{M^3}{|x|^3}) \quad (2.26)$$

$$g_{ij} = \delta_{ij} \left(1 + 2\gamma \frac{M}{|x|} + O(\frac{M^2}{|x|^2})\right). \quad (2.27)$$

We recall [47] that the static STU metrics may be cast in isotropic coordinates as

$$ds^2 = -(1 - \frac{m^2}{4\rho^2})^2 \Pi^{-\frac{1}{2}} dt^2 + \Pi^{\frac{1}{2}} \left(d\rho^2 + \rho^2(d\theta^2 + \sin^2 \theta d\phi^2)\right) \quad (2.28)$$

where

$$\Pi = \prod_i \left(1 + \frac{m \cosh 2\delta_i}{\rho} + \frac{m^2}{4\rho^2}\right) \quad (2.29)$$

whence

$$\Pi \approx 1 + \frac{4M}{\rho} + \frac{m^2}{\rho^2} \sum_{i<j} \cosh 2\delta_i \cosh 2\delta_j + \frac{m^2}{\rho^2}, \quad (2.30)$$

and so

$$-g_{00} = (1 - \frac{m^2}{4\rho^2})^2 \Pi^{-\frac{1}{2}} = 1 - \frac{2M}{\rho} + \frac{6M^2}{\rho^2} - \frac{m^2}{\rho^2} - \frac{m^2}{2\rho^2} \sum_{i<j} \cosh 2\delta_i \cosh 2\delta_j. \quad (2.31)$$

Thus

$$\gamma = 1, \quad M = \frac{m}{4} \sum_i \cosh 2\delta_i, \quad (2.32)$$

and

$$\beta = 3 - \frac{8}{\left(\sum_i \cosh 2\delta_i\right)^2} - \frac{4 \sum_{i<j} \cosh 2\delta_i \cosh 2\delta_j}{\left(\sum_k \cosh 2\delta_k\right)^2}. \quad (2.33)$$

This can be rewritten as

$$\beta = 1 + \frac{2 \sum_i \sinh^2 2\delta_i}{\left(\sum_j \cosh 2\delta_j\right)^2}. \quad (2.34)$$

In comparing (2.34) with observational/experimental constraints on $\beta$, one should bear in mind that these are typically derived in the case the when the gravitating body is a star or planet rather than a black hole.
3 Hamilton-Jacobi Equation and Null Geodesics

3.1 Hamilton-Jacobi equation in the STU black hole background

Taking $S$ in the Hamilton-Jacobi equation (1.1) to have the form

$$S = \frac{1}{2} \mu^2 \tau - Et + J\phi + W(r, \theta),$$

one sees from the inverse metric (2.6) for the STU black holes that

$$-\nu E^2 - \frac{1}{X} (EA_{\text{red}} - aj)^2 + \frac{j^2}{\sin^2 \theta} + X(\partial_r W)^2 + (\partial_\theta W)^2 = -\Delta^{1/2} \mu^2,$$

(3.1)

where $(\mu, E, j)$ are rest mass, conserved energy and angular momentum of the particle respectively. The quantities $X$ and $A_{\text{red}}$ depend only on $r$, while $\nu$, which is a function of $r$ and $\theta$, is in fact the sum of a function of $r$ and a function of $\theta$:

$$\nu(\theta, r) = \nu_\theta(\theta) + \nu_r(r),$$

(3.2)

with

$$\nu_r = r^2 + 2m(1 + \sum_i s_i^2) r + 8m^2 (\Pi_c - \Pi_s) \Pi_a - 4m^2 \sum_{i<j<k} s_i^2 s_j^2 s_k^2,$$

$$\nu_\theta = a^2 \cos^2 \theta,$$

(3.3)

It therefore follows that, as was shown in [36], the Hamilton-Jacobi equation for massless geodesics in the 4-charge STU black hole background separates. As can be seen from (2.3), for a general 4-charge STU black hole the function $\Delta^{1/2}$ appearing in (3.3) will be an irrational function of $r$ and $\cos \theta$, and thus the Hamilton-Jacobi equation for massive geodesics will not be separable. However, in the case of the pairwise-equal charge black holes, given in appendix A, $\Delta$ becomes a perfect square and in fact one then has

$$\Delta^{1/2} = (r + 2ms_1^2)(r + 2ms_2^2) + a^2 \cos^2 \theta.$$

(3.4)

It is then evident, since $\Delta^{1/2}$ is a sum of a function of $r$ and a function of $\theta$, that the Hamilton-Jacobi equation is now separable also for massive geodesics. Of course this will continue to be the case also for the specialisations of the pairwise-equal charge black holes to Kerr-Sen or to Kerr-Newman black holes.

3.2 Null geodesics

Setting $\mu = 0$ in (3.1), we see that the Hamilton-Jacobi equation for null geodesics separates if we write

$$W(r, \theta) = R(r) + \Theta(\theta),$$

(3.5)

The separation of the Hamilton-Jacobi equation for null geodesics is in fact a corollary of the previously discovered fact [26, 31] that the massless wave equation is separable for all metrics in the family. It is interesting to note that the separability of the wave equation for a massless scalar field also holds for the most general five-dimensional STU black holes [37], as shown in [38], as well as for those with a cosmological constant [39], as shown in [40].
leading to the ordinary differential equations

\[ \frac{dR}{dr}^2 - \nu r E^2 - \frac{1}{X}(E A_{\text{red}} - a j)^2 = K \]  
(3.6)

\[ \left( \frac{d\Theta}{d\theta} \right)^2 + \frac{j^2}{\sin^2 \theta} - \nu \theta E^2 = -K, \]  
(3.7)

where the separation constant \( K \) corresponds to the well-known Carter constant in the case of the uncharged (Kerr) black hole. The geodesic equation \( \dot{x}^\mu = g_{\mu\nu} \partial_\nu S \) then gives

\[ \Delta \left( \frac{dr}{d\lambda} \right)^2 = (E A_{\text{red}} - a j)^2 + (\nu r E^2 + K) X, \]  
\[ \Delta \dot{\Theta}^2 = a^2 E^2 \cos^2 \theta - K - \frac{j^2}{\sin^2 \theta}, \]  
\[ \Delta^{1/2} \dot{\phi} = \frac{a(E A_{\text{red}} - a j)}{X} + \frac{j}{\sin^2 \theta}, \]  
\[ \Delta^{1/2} i = (\nu r + a^2 \cos^2 \theta) E + \frac{A_{\text{red}}}{X} (E A_{\text{red}} - a j). \]  
(3.8)

### 4 Motion in the Equatorial Plane

This can be treated analytically for both massive as well as massless particles even if the general motion out of the plane is not integrable, since it reduces to a one-dimensional problem. We begin by noting from the general metric (2.1) that the horizon, which is a null hypersurface, is located at the larger zero, \( r = r_+ \), of \( X \). Furthermore the ergosurface, which is a timelike hypersurface on which the Killing vector \( \frac{\partial}{\partial \phi} \) becomes null, is given by \( G = 0 \).

One may consistently restrict the motion to the equatorial plane \( \theta = \frac{\pi}{2} \), since this is totally geodesic by virtue of being fixed under the reflection symmetry \( \theta \rightarrow \pi - \theta \).

Taking into account the conservation of energy \( E \) and angular momentum \( j \), it follows from (3.8) that the radial equation reduces to

\[ \Delta \left( \frac{dr}{d\lambda} \right)^2 = X(\nu r E^2 - \Delta \frac{\mu^2}{X} - j^2) + (E A_{\text{red}} - a j)^2 \]  
\[ = \left( A_{\text{red}}^2 + \nu r X \right)(E - V^+(r))(E - V^-(r)). \]  
(4.1)

The “effective potentials” \( V^\pm(r) \) depend upon the mass parameter \( m \), the charge parameters \( \delta_i \), the rotation parameter \( a \) and the impact parameter \( j \).

It is convenient to plot, at least in one’s mind, the two branches \( E = V^\pm(r) \). The motion may be envisaged as a horizontal line in the \( E - r \) plane with turning points when the straight line intersects one of the two branches \( E = V^\pm(r) \). The two branches correspond to prograde or retrograde motion. At infinity \( V^\pm \) limits to \( \pm \mu \), where \( \mu \) is the mass of the particle, and they coalesce at the horizon \( r = r_+ \), at which

\[ E = E_+ = V^+(r_+) = V^-(r_+) = \frac{a J}{A_{\text{red}}(r_+)} \]  
(4.3)

Circular orbits correspond to local maxima or minima of \( V^\pm(r) \). For \( V^+(r) \) the maxima are unstable and the minima stable. The opposite is true for \( V^-(r) \).
5 Shadows and Superficial Orbits

It was recognised early on in the subject [48–50] that among the null geodesics were a family that stayed at constant values of the Boyer-Lindquist coordinate $r$. These are sometimes referred to as spherical geodesics [7,51] or circular geodesics or even closed geodesics [10]. None of these terms is entirely appropriate. Unlike the spherically symmetric non-rotating case, where not only is this a single round 2-sphere of constant curvature on which the null geodesics project to great circles, and so are necessarily closed and of constant curvature, in the rotating case there is no single radius, but rather a range of radii and none of the 2-surfaces $r = \text{constant}$ onto which the null geodesics project are of constant curvature. Moreover the projected curves are not, unless special directions are chosen, closed curves, and even then, not of constant curvature. Finally, while it is obvious in the spherical case that every great circle may be regarded as the projection of a unique null geodesic, this is not obviously true in the rotating case.

In [52] examples were given of static spacetimes admitting spacelike spherical 2-surfaces such that every geodesic lying in the 2-surface may be regarded as the projection of a unique null geodesic. In [53] it was proposed that such surfaces, which are totally geodesic submanifolds of a suitably defined optical metric, be termed photon surfaces or anti-photon surfaces, depending upon whether the null geodesics were unstable or stable respectively. Because of the misgivings we have voiced above, in what follows we shall refer to null geodesics lying in a spacelike 2-surface as superficial orbits, and the surfaces as photon surfaces or anti-photon surfaces.

The first discussion of the shadow phenomenon was by Synge [54]. In effect, he pointed out that a compact spherical body surrounded by or enclosing a photon sphere would cast a shadow on the night sky, subtended by photons entering an observer’s eye that had just grazed the photon sphere. The discussion in the rotating case is similar, although the shadow is no longer circular, nor central with respect to the line of centres even if the observer is located in the equatorial plane of the black hole. We commence by noting that the general four charged black rotating black holes of the STU theory share an important simplifying feature that holds for photons crossing the equator in the Kerr case. Our discussion will follow the lucid papers of Bardeen and Teo [50,51].

5.1 Shadow theory

The basic strategy for finding the shadow was described, for example, in [50]. We shall follow a more recent discussion in [57]. An observer at a large distance $r = r_0$ from the black hole, at co-latitude $\theta_0$, can set up a Cartesian coordinate system $(\alpha, \beta)$ in the plane orthogonal to the line from the black hole to the observer. Thus every photon reaching the observer defines a point in the $(\alpha, \beta)$ plane. The totality of such points fill out a region representing the observer’s unobstructed night sky. Its complement is the the shadow of the black hole, and we seek its boundary. This consists of photon orbits that just fail to escape falling into the black hole. These are the solutions of

$$R(r) = 0, \quad R'(r) = 0,$$

(5.1)
subject to the requirement that the orbits are unstable. For each such superficial orbit, one can
calculate the corresponding point in the \((\alpha, \beta)\) plane, as described below:

In general the values of \(\alpha\) and \(\beta\) for a light ray reaching the observer are related to the asymptotic
values of \(d\phi/dr\) and \(d\theta/dr\) by

\[
\alpha = -r_0^2 \sin \theta_0 \left. \frac{d\phi}{dr} \right|_{r=r_0}, \quad \beta = r_0^2 \left. \frac{d\theta}{dr} \right|_{r=r_0}.
\]  

(5.2)

From (3.8), and from the asymptotic forms of the various metric functions, we therefore find

\[
\alpha = -\frac{\lambda}{\sin \theta_0}, \quad \beta = \left[ \eta + a^2 \cos^2 \theta_0 - \lambda^2 \cot^2 \theta_0 \right]^{1/2},
\]  

(5.3)

where we have defined the \textit{impact parameters}

\[
\lambda = j/E, \quad \eta = Q/E^2.
\]  

(5.4)

Here, \(Q\) is defined such that the additive constant in the \(\dot{\theta}^2\) equation in (3.8) vanishes on the
equatorial plane \(\theta = \pi/2\), i.e.

\[
Q = -K - j^2.
\]  

(5.5)

Imposing the conditions (5.1) for superficial orbits around the black hole allows one to solve for
the impact parameters \((\lambda, \eta)\) as functions of the orbital radius \(r\), where the allowed range of \(r\) runs
between the values \(r_{\text{inner}}\) and \(r_{\text{outer}}\) of the two circular geodesics that lie in the equatorial plane.
Using this information, and the relations (5.3), one can plot a curve in the \((\alpha, \beta)\)-plane. This curve
defines the boundary of the shadow cast by the black hole.

### 5.2 Orbits crossing the equator with \(\dot{\phi} = 0\)

Teo [51] observes that if he considers the spherical orbit at the radius that maximises his Carter
constant \(Q\), it has the property that it has \(\dot{\phi} = 0\) at the moment when it crosses the equator. We
find that this generalises to all the STU black holes (including all Chow-Compere examples). Here,
we show this for the 4-charge black holes discussed above.

We can scale the energy \(E\) of the particle by scaling the affine parameter. We shall assume now
that we scale it so that \(E = 1\), without loss of generality. By analogy with Teo, we define \(Q\) to be
the quantity such that in the \(\theta\) equation we have

\[
\Delta \dot{\theta}^2 = Q + O(\cos^2 \theta),
\]  

(5.6)

i.e. \(Q\) is the value of \(\Delta \dot{\theta}^2\) on the equator. Thus we see from (3.8) that

\[
Q = -K - j^2.
\]  

(5.7)

The radial equation is then \(\Delta r^2 = H(r)\) with

\[
H(r) = (A_{\text{red}} - aj)^2 + (\nu_r - Q - j^2) X.
\]  

(5.8)
The conditions for spherical orbits, $H = 0$ and $H' = 0$, can be viewed as determining $j = j(r_0)$ and $Q = Q(r_0)$ in terms of the orbital radius $r_0$. Thus we shall have

$$H = (A_{\text{red}} - aj)^2 + (\nu_r - Q - j^2) X = 0,$$

$$H' = 2(A_{\text{red}} - aj) A_{\text{red}}' + \nu_r' X + (\nu_r - Q - j^2) X' = 0,$$  \hspace{1cm} (5.9)

where all quantities are evaluated at the orbital radius $r_0$.

We now look for the value of $r_0$ that maximises $Q$. Thus we may now differentiate the $H = 0$ equation in (5.9) with respect to $r_0$, where now we include the differentiation of $j(r_0)$ and $Q(r_0)$. Since we then require $Q'(r_0) = 0$ at its maximum, this means we get an equation like the $H' = 0$ equation in (5.9), except with $j'$ terms also. Thus:

$$2(A_{\text{red}} - aj) A_{\text{red}}' - 2a(A_{\text{red}} - aj) j' + \nu_r' X - 2jj' X + (\nu_r - Q - j^2) X' = 0.$$  \hspace{1cm} (5.10)

Subtracting the $H' = 0$ equation in (5.9), and assuming $j'(r_0) \neq 0$, we therefore find

$$a(A_{\text{red}} - aj) + jX = 0,$$  \hspace{1cm} (5.11)

for the case of the spherical orbit whose radius maximises $Q(r_0)$. It is easy to see from the $\dot{\phi}$ equation in (5.8) that eqn (5.11) implies that $\dot{\phi} = 0$ when $\theta = \pi/2$. In other words, the spherical orbits at the radius that maximises $Q$ always have the property that they cross the equator with zero velocity in the $\phi$ direction.

### 5.3 Pairwise-equal charges

Considerable simplifications occur if the charges in the four-charge rotating black hole solutions are set pairwise equal. These solutions are presented explicitly in (A.9). Generically, the two equations in (5.9) lead to quadratic equations for $Q$ and $j$ that do not factorise over the rational functions, thus giving rise to rather complicated expressions involving square roots. However, in the case of pairwise equal charges the equations do factorise. The two solutions, which we shall denote by $(j_1, Q_1)$ and $(j_2, Q_2)$, are as follows:

\[
\begin{align*}
    j_1 &= \frac{r^2 + a^2 + 2mr(s_1^2 + s_2^2) + 4m^2 s_1^2 s_2^2}{a^2}, \\
    Q_1 &= -\frac{(r + 2ms_1^2)(r + 2ms_2^2)}{a^2}, \\
    j_2 &= \frac{-r^3 - 3mr^2 - 2m^2 r(s_1^2 + s_2^2) + 2s_1^2 s_2^2 + 4m^2 s_1^2 s_2^2 + 4m^3 s_1^2 s_2^2}{a(r - m)}, \\
    Q_2 &= \frac{1}{a^2(r - m)^2} \left\{ - \left[ r^3 - 3mr^2 - 2m^2 r(s_1^2 + s_2^2) + 2s_1^2 s_2^2 + 4m^3 s_1^2 s_2^2 \right] \\
    &\quad + 4m^2 [r^3 (1 + s_1^2 + s_2^2) + m^2 (s_1^4 + s_2^4 + s_1^2 + s_2^2 + 6s_1^2 s_2^2)] \\
    &\quad - 4m^2 r s_1^2 s_2^2 (1 - s_1^2 - s_2^2) - 4m^3 s_1^2 s_2^2 (s_1^2 + s_2^2) \right\}.  \hspace{1cm} (5.12)
\end{align*}
\]

In the first solution $Q$ is negative, and hence a necessary condition for the existence of a solution of the $\theta$ equation is that

$$a^2 - Q - j^2 > 0.$$  \hspace{1cm} (5.13)
However, for this solution we find
\[ a^2 - Q_1 - j^2_1 = -2(r^2 + 2mr(s_1^2 + s_2^2) + 4m^2 s_1^2 s_2^2), \]  
(5.14)
which is always negative. Hence the first solution is unphysical.

6 Conclusion

In this paper we have shown that the STU family of rotating charged black hole solutions of supergravity theory provide a perfectly manageable family of exact metrics which permit separation of variables for null and timelike geodesics and are sufficiently explicit as to allow the calculation of black hole shadows up to some simple numerical procedures. While the most general class of charged STU rotating black holes \[14, 15\] also admits a separation for both null and timelike geodesics, we primarily focused on studies of the much simpler four-charge parameter black holes \[12, 13\]. Furthermore, significant further simplifications occur for pairwise-equal charge parameters. Although we have restricted our attention to photon orbits and the shadow phenomenon, the utility of these metrics is not restricted to the behaviour of photon orbits, and we anticipate further applications in the future.

We should note that classes of rotating black holes in gauged supergravity theories also admit a separation for both the null and timelike geodesics, which is again a consequence of the separability of the minimally-coupled massless scalar equations in these backgrounds, c.f., \[55\]. Thus the extension of the analysis presented in this paper to supergravity black holes with non-zero cosmological constant would provide interesting examples for studying the effects of the cosmological constant in the calculation of the properties of black hole shadows.

Acknowledgements

We are very grateful to the Mitchell Family Foundation for hospitality at the Brinsop Court workshop on strings and cosmology, where some of this work was carried out. The research of M.C. is supported in part by the DOE Grant Award de-sc0013528, the Fay R. and Eugene L. Langberg Endowed Chair and the Slovenian Research Agency (ARRS). C.N.P. is supported in part by DOE grant DE-FG02-13ER42020.

A STU Black Holes with Pairwise-Equal Charges

In this appendix we collect some results from \[13\], where the STU supergravity rotating black holes with pairwise-equal charges were presented. The bosonic Lagrangian describing these black holes is
given by

\[ \mathcal{L} = \sqrt{-g} \left[ R - \frac{1}{2} (\partial \varphi)^2 - \frac{1}{2} e^{2\varphi} (\partial \chi)^2 - \frac{1}{(1 + \chi^2 e^{2\varphi})} [e^{2\varphi} F_{1\mu}^{\nu} F_{1\mu}^{\nu} - \frac{1}{2} \chi e^{2\varphi} \epsilon_{\mu
u\rho\sigma} F_{1\mu}^{\nu} F_{1\rho}^{\sigma}] \right. \\
\left. - e^{-\varphi} F_{2\mu}^{\nu} F_{2\mu}^{\nu} - \frac{1}{2} \chi \epsilon_{\mu
u\rho\sigma} F_{2\mu}^{\nu} F_{2\rho}^{\sigma} \right]. \quad (A.1) \]

If we define \( \tau = \chi + i e^{-\varphi} \) and

\[ F_{1\mu}^{\nu} = \frac{1}{2} \left( F_{1\mu}^{\nu} \pm \frac{i}{2} \epsilon_{\mu
u\rho\sigma} F_{1}^{\rho\sigma} \right), \quad F_{2\mu}^{\nu} = \frac{1}{2} \left( F_{2\mu}^{\nu} \pm \frac{i}{2} \epsilon_{\mu
u\rho\sigma} F_{2}^{\rho\sigma} \right), \quad (A.2) \]

then the Lagrangian \( (A.1) \) can be written as

\[ \mathcal{L} = \sqrt{-g} \left[ R - \frac{|\partial \tau|^2}{2 \tau^2} - 2 \Im \left[ \tau F_{2\mu}^{\nu} F_{2\mu}^{\nu} + \left( - \frac{1}{\tau} \right) F_{1\mu}^{\nu} F_{1\mu}^{\nu} \right] \right], \quad (A.3) \]

where \( \tau_2 = \Im(\tau) = e^{-\varphi} \). The Bianchi identities and equations of motion for the field strengths can be written as

\[ \nabla_\mu \Im(F_{1\mu}^{\nu}) = 0, \quad \nabla_\mu \Im(F_{2\mu}^{\nu}) = 0 \quad (A.4) \]

and

\[ \nabla_\mu \Im(G_{1\mu}^{\nu}) = 0, \quad \nabla_\mu \Im(G_{2\mu}^{\nu}) = 0 \quad (A.5) \]

respectively, where

\[ G_{1\mu}^{\nu} = \left( - \frac{1}{\tau} \right) F_{1\mu}^{\nu}, \quad G_{2\mu}^{\nu} = \tau F_{2\mu}^{\nu}. \quad (A.6) \]

The Bianchi identities and equations of motion transform covariantly under the global \( SL(2, R) \) symmetry

\[ \tau \to \frac{a \tau + b}{c \tau + d}, \quad \begin{pmatrix} F_{1}^{+} \\ G_{1}^{+} \end{pmatrix} \to \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \begin{pmatrix} F_{1}^{+} \\ G_{1}^{+} \end{pmatrix}, \quad \begin{pmatrix} F_{2}^{+} \\ G_{2}^{+} \end{pmatrix} \to \begin{pmatrix} d & c \\ b & a \end{pmatrix} \begin{pmatrix} F_{2}^{+} \\ G_{2}^{+} \end{pmatrix}, \quad (A.7) \]

where \( ad - bc = 1 \). Note that this is a symmetry only at the level of the equations of motion, and not of the Lagrangian. There is, in addition, a discrete \( Z_2 \) symmetry of both the equations of motion and the Lagrangian, under

\[ \tau \to -\frac{1}{\tau}, \quad A_1 \to A_2, \quad A_2 \to A_1. \quad (A.8) \]

The metric, dilatonic scalar, axion and gauge potentials are given by

\[ ds^2 = W \left( \frac{d\varphi^2}{\Sigma} + d\theta^2 \right) - \frac{\sum}{W} \left[ dt - a \sin^2 \theta d\phi \right]^2 + \frac{\sin^2 \theta}{W} \left[ adt - (r_1 r_2 + a^2) d\phi \right]^2, \]

\[ e^\varphi = \frac{r_1^2 + a^2 \cos^2 \theta}{r_1 r_2 + a^2 \cos^2 \theta}, \quad \chi = \frac{2 ma(s_2^2 - s_1^2) \cos \theta}{r_1^2 + a^2 \cos^2 \theta}, \]

\[ A_1 = \frac{\sqrt{2} m_1 c_1 r_1}{W} \left[ dt - a \sin^2 \theta d\phi \right], \quad A_2 = \frac{\sqrt{2} m_2 c_2 r_1}{W} \left[ dt - a \sin^2 \theta d\phi \right], \quad (A.9) \]

---

\(^4\)We have dualised the field strength \( F_1 \), relative to the one in [13], so that both field strengths now carry electric charges in the black hole solutions. We also rescaled the field strengths so that they each have the “canonical normalisation” where \( \mathcal{L} \sim \sqrt{-g} (R - F_{1}^{2} - F_{2}^{2}) \) in the absence of scalars.
\[ \Sigma = r^2 - 2mr + a^2, \quad W = r_1 r_2 + a^2 \cos^2 \theta, \]
\[ r_1 = r + 2ms_1^2, \quad r_2 = r + 2ms_2^2. \quad (A.10) \]

As usual, \( s_i = \sinh \delta_i, \) \( c_i = \cosh \delta_i, \) and the two constants \( \delta_1 \) and \( \delta_2 \) parameterise the electric charges carried by the two field strengths \( F_{\mu \nu}. \) The mass, angular momentum and electric charges (with the field strengths normalised as in (A.1)) are given by
\[ M = \frac{m}{2} (\cosh 2\delta_1 + \cosh 2\delta_2), \quad J = \frac{ma}{2} (\cosh 2\delta_1 + \cosh 2\delta_2), \quad Q_i = \frac{m}{\sqrt{2}} \sinh 2\delta_i. \quad (A.11) \]

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