The QCD phase transition at high temperature and low density

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We study the thermal properties of QCD in the presence of a small quark chemical potential $\mu$. Derivatives of the phase transition point with respect to $\mu$ are computed at $\mu = 0$ for 2 and 3 flavors of p-4 improved staggered fermions on a $16^3 \times 4$ lattice. Moreover we contrast the case of isoscalar and isovector chemical potentials, quantify the effect of $\mu \neq 0$ on the equation of state, and comment on the screening effect by dynamical quarks and the complex phase of the fermion determinant in QCD with $\mu \neq 0$.

To understand recent heavy-ion collision experiments, theoretical study of the QCD phase transition at high temperature and low density is important. For instance, the interesting regime for RHIC is $\mu_q/T_c \sim 0.1$, where $\mu_q = \mu/\alpha$ is a quark chemical potential. However, the Monte-Carlo method is not directly applicable for simulations at $\mu \neq 0$, which makes the study of finite-density QCD difficult; hence we usually use the reweighting method. Using the identity

$$\langle \mathcal{O} \rangle_{(\beta, \mu)} = \frac{\langle \mathcal{O} \mathcal{W} \rangle_{(\beta_0, 0)}}{\langle \mathcal{W} \rangle_{(\beta_0, 0)}},$$

(1)

the expectation value $\langle \mathcal{O} \rangle$ at $\mu \neq 0$ is computed by a simulation at $\mu = 0$. Here $M$ is the fermion matrix, $S_g$ the gauge action, and $N_f$ the number of flavors. Then, there exists a famous “sign problem”. Because $\det M$ is complex at $\mu \neq 0$, if the complex phase fluctuates rapidly, both numerator and denominator in RHS of eqn.(1) become vanishingly small. For the case of small $\mu$, the complex phase can be written by the odd terms of the Taylor expansion of $\ln \det M$ [1]. Denoting

$$\det M = |\det M|e^{i\theta},$$

the first term is $N_f \text{Im tr}[M^{-1}(\partial M/\partial \mu)\mu]$. From these equations, we find explicitly that the magnitude of $\theta$ is proportional to $\mu$, the volume and $N_f$. Moreover the first term of $\theta$ can be computed by the noise method and has a tendency that the phase fluctuation becomes larger as the quark mass or $T$ decreases. Roughly speaking, the sign problem happens when the fluctuation of $\theta$ becomes larger than $O(\pi/2)$. Thus the sign problem is not serious for the range of small $\mu$, but that region becomes narrower as the volume increases, which suggests that in the thermodynamic limit, since the region where the sign problem is manageable decreases to zero size, the only way of accessing the $\mu \neq 0$ region is via a Taylor expansion since this involves calculating quantities (i.e. derivatives) only at $\mu = 0$.

Last year, we proposed a general formulation to compute the derivatives of physical quantities [2]. We perform a Taylor expansion for $\ln \det M$ and fermionic operators in eqn.(1) and neglect higher order terms of $\mu$. Then the resulting expectation value contains an error of higher order in $\mu$ but which does not affect the calculation of derivatives of lower order than the neglected terms. In this report, we summarize the results obtained by this method. We perform simulations on a $16^3 \times 4$ lattice using a combination of the Symanzik improved gauge and 2 flavors of the p4-improved staggered fermion actions [3]. Details are written in Ref. [4]. We also comment on the $N_f$ de-
Phase transition line

First of all, we discuss the phase transition line in the \((T, \mu)\) plane. Because the first derivative is expected to be zero from the symmetry under exchange of \(\mu\) and \(-\mu\), we calculate second derivative of \(T_c\) with respect to \(\mu\). In Fig. 1, we plot Polyakov loop susceptibility as a function of \(\beta\) for \(\mu = 0\), \(\pm 0.05\) and \(\pm 0.1\) at \(m = 0.2\). This calculation contains errors at \(O(\mu^3)\). From this figure, we find that the peak position of the susceptibility moves left as \(\mu\) increases, which means that \(\beta_c\) or \(T_c\) decreases as \(\mu\) increases. Assuming the peak position is at \(\beta_c\), we determine the second derivative of \(\beta_c\). Combining with results for the chiral susceptibility at \(m = 0.1\) and 0.2, we obtain \(d^2\beta_c/d\mu^2 \approx -1.1\) with 30 - 50% error and any quark mass dependence of \(d^2\beta_c/d\mu^2\) is not visible within the accuracy of our calculation.

The second derivative of \(T_c\) is given by

\[
\frac{d^2 T_c}{d\mu^2} = -\frac{1}{N^2 T_c} \frac{d^2 \beta_c}{d\mu^2} \left( a \frac{d\beta}{da} \right),
\]

with \(a(d\beta/da)\) obtained from the string tension data in Fig. 1. We then find \(T_c(d^2 T_c/d\mu^2) \approx -0.14\). We sketch the phase transition line from the curvature with 50% error in Fig. 2. At the relevant point for RHIC, this shift of \(T_c\) is very small from that at \(\mu = 0\) and the result is roughly consistent with those obtained by other groups.

Screening effect by dynamical quarks

Under the presence of chemical potential, the time reversal symmetry is broken. By the asymmetry, an interesting property is shown in the measurement of the Polyakov loop, an external quark current running in the positive time direction. Negative chemical potential induces the dynamical generation of anti-quarks, which in contrast to quarks can completely screen an external color triplet current. Thus the free energy of a single quark is reduced, especially in the confinement phase, and the singularity at the phase transition point is weakened, since the long range fluctuation is screened. This effect can be seen in Fig. 3, where we denote the Polyakov loop susceptibility \(\chi_L\) at \(\mu < 0\) by dot-dot-dash and dot-dash-dash lines. We see that the peak height of \(\chi_L\) becomes smaller for \(\mu < 0\) corresponding to a weaker singularity, while the peak position is almost the same between positive and negative \(\mu\). Simultaneously, we observe that the Polyakov loop at \(\mu < 0\) is larger than that at \(\mu > 0\), suggesting the free energy of a single quark is reduced. Note that this property cannot be seen in 2-color QCD where systems at \(\mu\) and \(-\mu\) are identical.

Figure 1. Polyakov loop susceptibility at \(m = 0.2\).

Figure 2. Sketch of the phase diagram. The diamond symbol is the critical point obtained by Fodor and Katz. Dotted line is upper bound of the fit range to determine the curvature.
Equation of state

Next we discuss the \(\mu\)-dependence of the equation of state which describes the energy density \(\epsilon\) and pressure \(p\). If we employ the integral method based on the homogeneity of the system, we obtain \(p = -f\), where \(f = -(T/V)\ln Z\). Then derivatives of \(p\) with respect to \(\mu\) are related to the quark number density \(n_\mu\) and the singlet quark number susceptibility \(\chi_s = \partial n_\mu / \partial \mu_\mu\)

\[
\frac{\partial (p/T^4)}{\partial \mu_\mu} = \frac{n_\mu}{T^4} \quad \frac{\partial^2 (p/T^4)}{\partial \mu_\mu^2} = \frac{\chi_s}{T^4}.
\]

(4)

The quark number density is zero at \(\mu = 0\) so the leading correction is \(O(\mu^2)\). Moreover the second derivative of \(\epsilon\) can also be estimated by

\[
\frac{\partial^2 (\epsilon - 3p)/T^4}{\partial \mu_\mu^2} \approx -\frac{1}{T^4} \frac{\partial \chi_s}{\partial \epsilon} \left(\frac{1}{a} \frac{\partial a}{\partial \chi_s}\right)^{-1}.
\]

(5)

Here we neglect \(a(\partial m/\partial a)\), an approximation which is valid in the chiral limit. We obtain \(T^2 \partial^2 (p/T^4)/\partial \mu_\mu^2 \approx 0.69\) and \(T^2 \partial^2 (\epsilon/T^4)/\partial \mu_\mu^2 \approx 10.6\) at \(\beta_c\) for \(m = 0.1\). The discrepancy of \(p/T^4\) \((\epsilon/T)^4\) in the RHIC regime \(\mu_\mu/T_c \approx 0.1\) from its value at \(\mu = 0\) is about 0.0035 \((0.05)\). This is a 1% effect, and hence quite small.

From the second derivatives of \(p\) and \(\epsilon\) with respect to \(\mu\) together with the derivatives with respect to \(T\), we calculate the line of constant pressure and energy density. We find that the slope of the constant pressure (energy density) line is \(T(dT/d(\mu_\mu^2)) \approx -0.11(-0.09)\). As the slope of \(T_c\) in \(\mu_\mu^2\) is \(T_c(dT_c/d(\mu_\mu^2)) \approx -0.07\), this result suggests that the line of constant pressure or energy density is parallel with the phase transition line.

Iso-vector chemical potential

If instead we were to impose an isovector chemical potential \(\mu_I\) having opposite sign for \(u\) and \(d\) quarks, then the quark determinant would become real and positive, enabling simulations using standard Monte-Carlo methods \(\text{[8]}\). In the framework of the Taylor expansion, terms even in \(\mu\) are identical for both \(u\) and \(d\) quarks, but odd terms cancel for the case \(\mu_I \neq 0\). We analyzed the transition point \(\beta_c(\mu_I)\) and do not observe significant difference for \(\beta_c\) between \(\mu_I \neq 0\) in the region of small \(\mu\), which is different from a naive expectation that the phase transition line for \(\mu_I\) runs toward the \(T = 0\) onset threshold of a pion condensate at a critical \(\mu_{I_0} \approx m_{PS}/2 < m_N/3\), and hence the curvature for \(\mu_I\) larger than for isoscalar \(\mu\). However the quark mass we used is still large and \(m_\pi\) is not so small. Simulations at small \(m\) are necessary to check the naive picture.

\(N_f\) dependence

Finally, we comment on the difference between \(N_f = 2\) and 3. We performed an additional simulation of \(N_f = 3\). The preliminary results of the curvature at \(m = 0.1\) are \(T_c(d^2T_c/d\mu_\mu^2) \approx -0.15\) and \(-0.11\) for the cases \(\mu_{ud} = \mu_s = \mu_q\) and \(\mu_{ud} = \mu_q, \mu_s = 0\) respectively. There is no significant difference from \(N_f = 2\). Therefore we expect that our result for \(N_f = 2\) is not so different from real QCD lying between \(N_f = 2\) and 3.

The most interesting point for \(N_f = 3\) is the existence of a critical quark mass \(m_c\) on the \(\mu = 0\) axis between a first order phase transition at small \(m\) and a crossover at large \(m\). We also expect such a critical point at \(\mu \neq 0\) even in the region of large \(m\). Hence to investigate the relation between these critical points is quite important. The first preliminary result is reported by \(\text{[2]}\).

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