A Generalization of the Hausdorff Dimension Theorem for Deterministic Fractals

Mohsen Soltanifar 1,2

1 Biostatistics Division, Dalla Lana School of Public Health, University of Toronto, 620-155 College Street, Toronto, ON M5T 3M7, Canada; mohsen.soltanifar@alum.utoronto.ca
2 Real World Analytics, Cytel Canada Health Inc., 802-777 West Broadway, Vancouver, BC V5Z 1J5, Canada

Abstract: How many fractals exist in nature or the virtual world? In this paper, we partially answer the second question using Mandelbrot’s fundamental definition of fractals and their quantities of the Hausdorff dimension and Lebesgue measure. We prove the existence of aleph-two of virtual fractals with a Hausdorff dimension of a bi-variate function of them and the given Lebesgue measure. The question remains unanswered for other fractal dimensions.

Keywords: cantor set; fractals; Hausdorff dimension; continuum; aleph-two

MSC: 28A80; 03E10

1. Introduction

Benoit Mandelbrot (1924–2010) coined the term fractal and its dimension in his 1975 essay on the quest to present a mathematical model for self-similar, fractured geometric shapes in nature [1]. Both nature and the virtual world have prominent examples of fractals: some natural examples include snowflakes, clouds and mountain ranges, while some virtual instances are the middle third Cantor set, the Sierpinski triangle and the Vicsek fractal. A key element in the definition of the term fractal is its dimension usually indexed by the Hausdorff dimension. This dimension has played vital roles in developing the concept of fractal dimension and the definition of the term fractal and has had extensive applications [2–5]. For numerical instances, the usual Vicsek has the Hausdorff dimension equal to \( \log_3 (5) \). While each fractal has its Hausdorff dimension as a unique number in \((0, +\infty)\), there has been little evidence on the inverse existence statement. Given that constructive existence theorems play a key role in a wide spectrum of mathematical fields and computer science [6,7], one may ask: Is there any fractal for given Hausdorff dimension? How many? In addition, in case of affirmative response, What are the features of the set of such fractals? To the best of the author’s knowledge, the only available existence theorem of fractals in the literature is limited to the case of the fractals with Lebesgue measure zero (thin fractals) [8] and its very minor version in \( \mathbb{R} \) [9]. Furthermore, among the most well-known applied approaches have been those based on topological diversity [10], the divergence of continuous functions [11], and number theory [12]. However, on one hand, there are fractals in the Euclidean spaces with a positive Lebesgue measure (fat fractals) applied to model real physical systems [13,14]. Examples include the fat Cantor set with a Hausdorff dimension of one, the fat Vicsek fractal with Hausdorff dimension of two, and the fat Menger sponge with Hausdorff dimension of three. On the other hand, the current existential results in the literature [8–12] lacks their provision for the highest potential cardinal of aleph-two and a given Lebesgue measure. This issue becomes more complex given the fact that fractals with a positive Lebesgue measure must have a positive integer Hausdorff dimension.

This work presents a parallel existence result for fractals of a given Hausdorff dimension and a positive Lebesgue measure in \( n \)-dimensional Euclidean spaces. In light of this aim, this work provides a comprehensive deterministic framework, providing a second...
proof for the former results and, in particular, generalizes them in terms of Lebesgue measure and cardinal number.

2. Preliminaries

The reader who has studied fractal geometry is well-equipped with the following definitions and key properties of the Uniform Cantor sets, topological dimension and the Hausdorff dimension. Henceforth, in this paper we consider the n-dimensional Euclidean space \( \mathbb{R}^n \) with its conventional Euclidean metric and the Lebesgue measure of \( \lambda_n (n \in \mathbb{N}) \).

2.1. The Uniform Cantor Set

The Uniform Cantor set, or Smith–Volterra–Cantor set, or the fat Cantor set, was introduced in a series of publications by Henry Smith in 1875, Vito Volterra in 1881, and George Cantor in 1883, respectively. To construct the uniform Cantor set, let \( \{ \beta_n \}_{n \in \mathbb{N}} \)—called the removing sequence—be a sequence of positive numbers with \( \sum_{n \in \mathbb{N}} \beta_n = 1 - l \in (0, 1) \). Take \( s \in \mathbb{N} \), and set \( I_{s,0} = [0,1] \). Define \( I_{s,n} (n \in \mathbb{N}) \) recursively by removing \( s \) symmetrically located open intervals with equal length of \( \frac{\beta_n}{s(s+1)^{n-1}} \) from the middle of each interval in \( I_{s,n} - 1 \). Then, \( I_{s,n} \) is the union of \( (s + 1)^n \) disjoint closed intervals \( I_{s,n}^j \) of equal length \( \delta_n = \frac{1 - \sum_{j=1}^{s} \beta_j}{(s+1)^n} \), furthermore, the sequence \( \{ I_{s,n} \}_{n \in \mathbb{N}_0} \) is a decreasing sequence with a finite intersection property in the compact space \([0,1] \).

Definition 1. The Uniform Cantor set of order \( s \geq 1 \) and Lebesgue measure \( l \geq 0 \) associated with the removing sequence \( \{ \beta_n \}_{n} \) is defined as:

\[
C_{\{\beta_n\}}(s, l) = \bigcap_{n=0}^{\infty} I_{s,n}.
\]

It is well known that the Uniform Cantor set \( C_{\{\beta_n\}}(s, l) \) is nowhere dense, totally disconnected, perfect and uncountable [8]. Next, by definition, \( \lambda_1 ([0,1] - I_{s,n}) = \sum_{k=1}^{s} \beta_k (n \in \mathbb{N}) \). Consequently, the Lebesgue measure of the Uniform Cantor set \( C_{\{\beta_n\}}(s, l) \) is given by:

\[
\lambda_1 (C_{\{\beta_n\}}(s, l)) = 1 - \lambda_1 ([0,1] - C_{\{\beta_n\}}(s, l)) = 1 - \lambda_1 (\bigcup_{n \in \mathbb{N}} ([0,1] - I_{s,n}))
\]

\[
= 1 - \lim_{n \to \infty} \lambda_1 ([0,1] - I_{s,n}) = 1 - \sum_{n \in \mathbb{N}} \beta_n = l.
\]

A prominent family of the Uniform Cantor sets in the literature [8,15–17] is obtained for the case of

\[
\beta_n = \beta_n (s, \beta, l) = ((s + 1)\beta)^{n-1}(1 - (s + 1)\beta)(1 - l)
\]

where \( s \geq 1, 0 < \beta < \frac{1}{s+1} \) and \( 0 \leq l < 1 \). In particular, the ordinary middle third Cantor set is obtained whenever \( s = 1, \beta = \frac{1}{3}, l = 0 \). It is simultaneously the first term of two distinctive sequences of the Uniform Cantor sets: one including \( s \geq 1, \beta = \beta (s) = \frac{1}{s+1} (s \geq 1), l = 0 \) [8] and another including \( s = 1, \beta = \beta (s^*) = \frac{1}{s+1} (s^* \geq 1), l = 0 \), [15].

2.2. Topological Dimension

The topological dimension considered here is the Urysohn–Menger small inductive dimension. It is defined inductively by setting \( \dim_{ind}(\phi) = -1 \). We then say that for a given \( C \subseteq \mathbb{R}^n \), we have \( \dim_{ind}(C) \leq k \) whenever there is a base \( \{ U_i \}_{i \in I} \) of open sets of \( C \) such that \( \dim_{ind}(U_i) \leq k - 1 (i \in I) \). We then say \( \dim_{ind}(C) = k \) whenever \( \dim_{ind}(C) \leq k \) but \( \dim_{ind}(C) \not\leq k - 1 (k \in \mathbb{N}_0) \), [17].

For the case of a zero-topological dimension, we may consider a more informative definition based on the idea of clopen (i.e., simultaneously open and closed) sets as follows [17]:
**Definition 2.** The space $C \subseteq \mathbb{R}^n$ has zero topological dimension (i.e., $\dim_{\text{ind}}(C) = 0$) whenever every finite open cover $\{U_i\}_{i \in I}(|I| < \infty)$ of $C$ has a finite refinement $\{V_i\}_{i \in I}(|I| \leq |I| < \infty)$ that is a clopen partition of $C$.

Using above definition, it has been shown that for the case of middle third Cantor set $C = C_{\{\frac{1}{3}\}}(1, 0)$ we have $\dim_{\text{ind}}(C) = 0$, [17]. A straightforward generalization of the proof shows that for the uniform Cantor set $C = C_{\{\beta_n\}}(s, l)$ we also have the same conclusion. The following theorem summarizes some prominent properties of the small inductive dimension required in this paper [17]:

**Theorem 1.** Let $\{C_i\}_{i \in I}$ be a countable family of subsets of $\mathbb{R}^n \ (n \geq 1)$. Then:

(i) $\dim_{\text{ind}}(C_i) \in \{-1, 0, \ldots, n\}$ where $i \in I$,

(ii) $\dim_{\text{ind}}(C_i) \leq \dim_{\text{ind}}(C_j)$ where $C_i \subseteq C_j (i \neq j)$,

(iii) $\dim_{\text{ind}}(cC_i + d) = \dim_{\text{ind}}(C_i)$ where $c, d \in \mathbb{R}^n$ and $i \in I$,

(iv) $\dim_{\text{ind}}(\cup_{i \in I} C_i) \leq k$ for some fixed $k \in \mathbb{N}$ whenever we have $\dim_{\text{ind}}(C_i) \leq k(i \in I)$.

(v) $\dim_{\text{ind}}(\cap_{i \in I} C_i) \leq \sum_{i \in I} \dim_{\text{ind}}(C_i)$ whenever $I$ is finite.

As a corollary of this theorem, for the case of a finite family of uniform Cantor sets $\{C_i\}_{i \in I}$, it follows that $\dim_{\text{ind}}(\cup_{i \in I} C_i) = 0$ and $\dim_{\text{ind}}(\cap_{i \in I} C_i) = 0$.

2.3. Hausdorff Dimension

The Hausdorff dimension, or Hausdorff–Besicovitch dimension, is considered as the one of the most prominent dimensions for fractals. It was first introduced by Felix Hausdorff in 1918 and was later improved in terms of computational techniques by Abram S. Besicovitch. It is definable for any subset of the real line as follows [16]:

**Definition 3.** Let $C \subseteq \mathbb{R}$ and given $s \geq 0$. Then, given the $s$-dimensional Hausdorff measure of $C$ by:

$$H^s(C) = \lim_{\delta \to 0} \left( \inf \left\{ \sum_{i \in \mathbb{N}} |U_i|^s \mid C \subseteq \bigcup_{i \in \mathbb{N}} U_i \text{ and } 0 < |U_i| \leq \delta \text{ for each } i \in \mathbb{N} \right\} \right),$$

the Hausdorff dimension of $C$ is defined as:

$$\dim_{\text{H}}(C) = \inf \{s \geq 0 \mid H^s(C) = 0\}. 
\quad (5)$$

Using this definition, Hausdorff showed that the middle third Cantor set $C = C_{\{\frac{1}{3}\}}(1, 0)$ has a Hausdorff dimension equal to $\frac{\log(2)}{\log(3)}$. More generally [17]:

**Theorem 2.** The Hausdorff dimension of the Uniform Cantor set $C = C_{\{\beta_n\}}(s, l)$ is given by:

$$\dim_{\text{H}}(C_{\{\beta_n\}}(s, l)) = \begin{cases} \frac{\log(s+1)}{\log(s+1) + \limsup_{n \to \infty} \left( \frac{\log(\beta_n)}{n} \right)} & \text{if } s \geq 1, \quad l = 0, \quad \inf_{n \in \mathbb{N}} \left( \frac{\beta_n}{1 - \sum_{k=1}^{n-1}}} \beta_k \right) > 0, \\ 1 \quad & \text{if } s \geq 1, \quad 0 < l < 1. \end{cases} \quad (6)$$

As a Corollary, the Hausdorff dimension of the above prominent family of Cantor sets in Equation (3) is given by:

$$\dim_{\text{H}}(C_{\{\beta_n(s,\beta)\}}(s, l)) = \begin{cases} \frac{\log(s+1)}{\log(\beta)} & \text{if } s \geq 1, \quad 0 < \beta < \frac{1}{s+1}, \quad l = 0, \\ 1 \quad & \text{if } s \geq 1, \quad 0 < \beta < \frac{1}{s+1}, \quad 0 < l < 1. \end{cases} \quad (7)$$

Next, we consider useful notation for the Uniform Cantor sets with Hausdorff dimension $r > 0$ as follows:
Notation 1. The linear transform of the Uniform Cantor set of order $s \geq 1$, Hausdorff dimension $r > 0$ and Lebesgue measure $I \geq 0$ associated with the removing sequence $\{\beta_n\}_n$ is denoted as:

$$F_{r,s,d} = c \times C_{\{\beta_n\}}(s,I) + d$$

for some $c > 0, d \geq 0$. (8)

The following theorem summarizes some key properties of the Hausdorff dimension required in this paper [16]:

**Theorem 3.** Let $\{C_i\}_{i \in I}$ be a countable family of subsets of $\mathbb{R}^n (n \geq 1)$. Then:

1. $0 \leq \dim_H(C_i) \leq n$ where $i \in I$,
2. $\dim_H(C_i) \leq \dim_H(C_j)$ where $C_i \subseteq C_j (i \neq j)$,
3. $\dim_H(cC_i + d) = \dim_H(C_i)$ where $c,d \in \mathbb{R}^+$ and $i \in I$,
4. $\dim_H(\bigcup_{i \in I} C_i) = \sup_{i \in I}(\dim_H C_i)$,
5. $\dim_H(\prod_{i \in I} C_i) = \sum_{i \in I} \dim_H(C_i)$ whenever $I$ is finite and one of $C_i$s is a uniform Cantor set.
6. $\dim_H(C_i) = n(i \in I)$ whenever $\lambda_\alpha(C_i) > 0(i \in I)$.

Finally, throughout this paper we refer to the fractal in terms of Mandelbrot’s definition [1]:

**Theorem 4.** A subset $C \subseteq \mathbb{R}^n$ with Hausdorff dimension $\dim_H(C)$ and the topological inductive dimension $\dim_{ind}(C)$ is a fractal whenever:

$$\dim_H(C) > \dim_{ind}(C).$$

### 2.4. General Cartesian Product Distribution over Unions

We finish this section with a review of the relationship between union and Cartesian products. As the Cartesian product is distributive over unions [18], we proved the following general result easily by induction on the dimension of the product $n$:

**Theorem 5.** Let $I_j(1 \leq j \leq n)$ be indexing sets and $\{C_{ij}\}_{i \in I_j}(1 \leq j \leq n)$ be families of subsets of $\mathbb{R}$ indexed by them, respectively. Then:

$$\prod_{j=1}^{n} (\bigcup_{i \in I_j} C_{ij}) = \bigcup_{(i_1,\cdots,i_n) \in \prod_{j=1}^{n} I_j} (\prod_{j=1}^{n} C_{ij}),$$

(10)

where $\cup$ denotes the union and $\prod$ denotes the Cartesian product.

### 3. Main Results

We generalize the existence Hausdorff dimension theorem from fractals with a Lebesgue measure zero (thin fractals) to those with a non-negative Lebesgue measure (fat fractals) with the existence of the higher cardinal of aleph-two. The construction process is accomplished in four stages: (i) showing the existence of fractals with a plausible Hausdorff dimension and a Lebesgue measure in $\mathbb{R}$; (ii) expanding the cardinality of fractals in the first stage to the continuum; (iii) extending the result in the second stage to the higher dimensional Euclidean spaces $\mathbb{R}^n (n > 1)$, and, (iv) generalizing the result in the third stage to the cardinal of aleph-two. We begin with the following existence result whose very special case has been stated in [9]:

**Lemma 1.** For any real $0 < r < 1$ and $l \geq 0$, there is a fractal with the Hausdorff dimension $r.1_{(0)}(l) + 1_{(0,\infty)}(l)$ and the Lebesgue measure $l$ in $\mathbb{R}$.

**Proof of Lemma 1.** We consider three scenarios:

1. $0 < r < 1, l = 0$;
Fix $s_0 \in \mathbb{N}$, and consider the family of Uniform Cantor sets $C_{\{\beta_n(s_0,0)\}}(s_0,0)$, $(0 < \beta < \frac{1}{s_0+1})$. Then, their Hausdorff dimensions are given by $f_{s_0}(\beta) = \frac{\log(s_0+1)}{\log(\beta)}$ $(0 < \beta < \frac{1}{s_0+1})$. Since $f_{s_0}$ is a continuous increasing function from $(0, \frac{1}{s_0+1})$ onto $(0,1)$, such that $f(0^+) = 0$ and $f(\frac{1}{s_0+1}^-) = 1$, by an application of the Intermediate Value Theorem, there exists $\beta_0 \in (0, \frac{1}{s_0+1})$ such that $f_{s_0}(\beta_0) = r$. Now, it is enough to consider the following fractal in the interval $[0,1]$:

$$F_{r,0,s_0} = C_{\{\beta_n(s_0,0)\}}(s_0,0)$$  \hspace{1cm} (11)

(ii) $r = 1, l = 0$:

Again, fix $s_0 \in \mathbb{N}$ and consider the sequence $r_n = \frac{n}{n+1} \ (n \in \mathbb{N})$. From part (i), there is a corresponding sequence of fractals $F_{r_n,0,s_0} \ (n \in \mathbb{N})$ such that $\dim_H(F_{r_n,0,s_0}) = r_n$, $\dim_{ind}(F_{r_n,0,s_0}) = 0$, and $\xi_1(F_{r_n,0,s_0}) = 0, \ (n \in \mathbb{N})$. Take:

$$F_{1,0,s_0} = \bigcup_{n \in \mathbb{N}} F_{r_n,0,s_0}$$  \hspace{1cm} (12)

Accordingly, two consecutive applications of Theorems 1 (iv) and 3 (iv) prove the desired result.

(iii) $r = 1, l > 0$:

Finally, for fixed $s_0 \in \mathbb{N}$ consider $F_{1,\frac{1}{l+1},s_0} = ([l] + 1) \times C_{\{\beta_n(s_0,0)\}}(s_0,0) \ \frac{1}{l+1}$ with $\beta_n$ given by Equation (3). Hence, two consecutive applications of Theorems 1 (iii) and 3 (iii) yield the desired result. □

A closer look at the proof of the Lemma 1 and changing values $s_0 \in \mathbb{N}$ shows that indeed for any real $0 < r \leq 1$ and $l \geq 0$ there are countably infinite fractals with the Hausdorff dimension $r.1_{[0]}(l) + 1_{(0,\infty)}(l)$ and the Lebesgue measure $l$ in $\mathbb{R}$. The following Lemma expands this result to the cardinality of the Continuum.

**Lemma 2.** For any real $0 < r \leq 1$ and $l \geq 0$ there is a continuum of distinctive fractals with the Hausdorff dimension $r.1_{[0]}(l) + 1_{(0,\infty)}(l)$ and the Lebesgue measure $l$ in $\mathbb{R}$.

**Proof of Lemma 2.** We consider three scenarios similar to the proof of Lemma 1. In each scenario, we partition a given interval (e.g., $[0,1]$) to an infinite union of its shrinking sub-intervals and, using linear transformations, put into them Cantor fractals.

(i) $0 < r < 1, l = 0$:

By construction, in the part(i) of the proof of Lemma 1, there is a sequence of fractals $\{F_{r,0,s}\} \in \mathbb{N}$ with $F_{r,0,s} = \frac{1}{2}F_{r,0,s+1} + \frac{1}{2}I$ in the interval $[\frac{1}{2} - \frac{1}{2}, \frac{1}{2}]$ such that: $\dim_H(F_{r,0,s}) = r$, $\dim_{ind}(F_{r,0,s}) = 0$, and $\xi_1(F_{r,0,s}) = 0 \ (s \in \mathbb{N})$. Let $I \in \mathcal{P}(\mathbb{N})$ be infinite and define:

$$F^I_{r,0} = \bigcup_{s \in I} F_{r,0,s}$$  \hspace{1cm} (13)

Then, again an application of Theorems 1 (iii,iv) and 3 (iii,iv) and uncountability of the set of such $I$’s proves the assertion.

(ii) $r = 1, l = 0$:

Again, consider the sequence $r_n = \frac{n}{n+1} \ (n \in \mathbb{N})$. From part (i), there is a corresponding sequence of fractals $F^I_{r_n,0}(n \in \mathbb{N})$ and an infinite set $I \in \mathcal{P}(\mathbb{N})$ such that $\dim_H(F^I_{r_n,0}) = r_n$, $\dim_{ind}(F^I_{r_n,0}) = 0$, and $\xi_1(F^I_{r_n,0}) = 0 \ (n \in \mathbb{N})$. Put:

$$F^I_{1,0} = \bigcup_{n \in \mathbb{N}} \left( \frac{1}{2}F^I_{r_n,0} + \frac{1}{2^n} \right)$$  \hspace{1cm} (14)
Now, by two applications of Theorems 1 (iii,iv) and 3 (iii,iv) and uncountability of the set of such \( l \)'s, the assertion follows.

(iii) \( r = 1, l > 0 \):

By construction, in part (iii) of the proof of Lemma 1, there is a sequence of fractals \( \{ F_{1, \frac{j}{\lfloor |r| \rfloor}, \frac{s}{|r|}} \}_{s \in \mathbb{N}} \) in the interval \([0, |l| + 1]\) such that: \( \dim_H(F_{1, \frac{j}{\lfloor |r| \rfloor}, \frac{s}{|r|}}) = 1 \), \( \dim_{\text{ind}}(F_{1, \frac{j}{\lfloor |r| \rfloor}, \frac{s}{|r|}}) = 0 \), and \( \lambda_1(F_{1, \frac{j}{\lfloor |r| \rfloor}, \frac{s}{|r|}}) = l \) (s \( \in \mathbb{N} \)). Let \( F_{1, \frac{j}{\lfloor |r| \rfloor}, \frac{s}{|r|}} = \frac{|l|+1}{2}F_{1, \frac{j}{\lfloor |r| \rfloor}, \frac{s}{|r|}} + \frac{|l|+1}{2} (s \in \mathbb{N}) \), and \( l \in \mathcal{P}(\mathbb{N}) \) be infinite and define:

\[
F_{1,l}^s = \frac{l}{\lambda_1(\bigcup_{s \in I} F^s_{1, \frac{j}{\lfloor |r| \rfloor}, \frac{s}{|r|}})} \times \bigcup_{s \in I} F^s_{1, \frac{j}{\lfloor |r| \rfloor}, \frac{s}{|r|}} 
\]

Finally, from two applications of Theorems 1 (iii,iv) and 3 (iii,iv) and uncountability of the set of such \( l \)'s the assertion follows.  

An investigation into Lemma 2 reveals its limitation to providing fractals with a Hausdorff dimension in the range \( 0 < r \leq 1 \). To obtain fractals with Hausdorff dimension \( r > 1 \) we need to consider the higher dimensional Euclidean spaces. Using the idea of the countable unions of \( n \)-dimensional Cantor fractal dusts (as the Cartesian product of the Uniform Cantor sets defined above), our next result addresses this situation.

**Lemma 3.** For any real \( r > 0 \) and \( l \geq 0 \), there is a continuum of distinctive fractals with the Hausdorff dimension \( r.1_{\{0\}}(I) + n.1_{\{\infty\}}(I) \) and Lebesgue measure \( l \) in \( \mathbb{R}^n \) where \( \lfloor |r| \rfloor \leq n \).

**Proof of Lemma 3.** Let \( r > 0, l \geq 0 \) and for given \( \lfloor |r| \rfloor \leq n \) define \( r' = \frac{r}{n} \in (0,1] \). Then, using Equations (13)–(15), we consider the following scenarios with the Hausdorff dimension \( r' \) and Lebesgue measure \( l \geq 0 \) given different values of \( r', l \):

Now, we consider \( n \) copies from each cell in Table 1 and consider the following Cartesian products:

| \((r', l)\)  | \( l = 0 \) | \( l > 0 \) |
|------------|-------------|-------------|
| \( 0 < r' < 1 \) | \( F_{r', l}^I \) (Equation (13)) | DNE |
| \( r' = 1 \) | \( F_{r', l}^I \) (Equation (14)) | \( F_{1,l}^I \) (Equation (15)) |

Notes: \( I \in \mathcal{P}(\mathbb{N}) \) is infinite; DNE: Does Not Exist.

Finally, considering Table 2, the assertion follows by an application of Theorems 1 (v), 3 (iii,v) and 5, and that there are uncountably infinite sets of \( I_j \in \mathcal{P}(\mathbb{N}) (1 \leq j \leq n) \).

| \((r, l)\)  | \( l = 0 \) | \( l > 0 \) |
|------------|-------------|-------------|
| \( r \notin \mathbb{N} \) | \( \prod_{j=1}^n F_{r_j,l}^i \) | DNE |
| \( r \in \mathbb{N} \) | \( \prod_{j=1}^n F_{r_j,l}^* \) | \( \left( \frac{1}{r} \right)^{n-1} \times \prod_{j=1}^n F_{r_j,l}^* \) |

Notes: \( I_j \in \mathcal{P}(\mathbb{N})(1 \leq j \leq n) \) are infinite; DNE: Does Not Exist.

The constructed fractals in Lemma 3 have two key features as follows: First, they are centrally asymmetric with respect to the point symmetric of \( \frac{r}{n} = \frac{1}{2}, \cdots, \frac{1}{2} \) where \( \lfloor |r| \rfloor \leq n \). This is a direct result of the fact that being centrally asymmetric is invariant under Cartesian products and unions and the building blocks of the fractals in the Lemma are centrally asymmetric fat Cantor sets with point symmetric of \( \frac{r}{n} = \frac{1}{2} \). However, the Lemma can be expanded to the symmetric fractals with consideration of transformed symmetric fat Cantor sets. The details of the proof are minor modifications of our proof for the asymmetric case presented here with
replacement of fractals $F$ in Equations (13)–(15) with $\frac{1}{2}((F \cup (-F)) + 1)$. Second, there is a continuum of them with the same Hausdorff dimension and the same Lebesgue measure in the Euclidean spaces $\mathbb{R}^{n+1} - \mathbb{R}^n$ ($|r| \leq n$) as those in $\mathbb{R}^n$ ($|r| \leq n$). Here, $\mathbb{R}^n$ is considered isomorphic to the subspace of $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$. This result is the direct consequence of the construction process in the proof of the Lemma. Finally, while the Lemma 3 guarantees at least a continuum of fractals in n-dimensional Euclidean space with given properties, it does not guarantee the existence of the aleph-two (or both-two) fractals given the generalized continuum hypothesis [19]. This lack of precision of the cardinal number (between either the continuum or aleph-two) creates ground for further investigation. Our main result provides the answer.

\textbf{Theorem 6 (The Generalized Hausdorff Dimension Theorem).} For any real $r > 0$ and $l \geq 0$, there are aleph-two (symmetric) fractals with the Hausdorff dimension $r.1_{(0)}(l) + n.1_{(0,\infty)}(l)$ and Lebesgue measure $1$ in $\mathbb{R}^n$ where $\{r\} \leq n$.

\textbf{Proof of Theorem 6.} We accomplish the proof in four steps as follows:

First, let $r > 0, l \geq 0$, and fix $n \geq |r|$. Then, for the continuous function $h_n(\beta) = \frac{n.\log(2)}{-\log(2)} (0 < \beta < \frac{1}{2})$, given $h_n(0^+) = 0$ there is $\beta_{00} \in (0, \frac{1}{2})$ such that $h_n(\beta_{00}) < \frac{r}{2}$. Second, take a fractal $F_{00} = (C + [1] + 1)^n$ in $\mathbb{R}^n$ where $C = C_{\beta_0}$ with $\beta_{00} = \beta_n(1, \beta_{00}, 0)$ ($n \geq 1$) as in Equation (3). Then, $\lambda(F_{00}) = 0$, $\dim_{ind}(F_{00}) = 0$, and $\dim_H(F_{00}) < \frac{r}{2}$. By two applications of Theorems 1 (ii) and 3 (iii), for any non-empty $F \subseteq F_{00}$, $\lambda(F) = 0$, $\dim_{ind}(F) = 0$, and $\dim_H(F) < \frac{r}{2}$. Accordingly, define the power set of $F_{00}$ minus empty set (with cardinal of aleph-two) as follows:

$$\mathcal{F}_1(r,l) = \{F \subseteq \mathbb{R}^n | F \subseteq F_{00}, \dim_H(F) < \frac{r}{2}, \lambda(F) = 0, \dim_{ind}(F) = 0\}. \quad (16)$$

Third, consider the family of fractals given by Lemma 3:

$$\mathcal{F}_2(r,l) = \{F \subseteq \mathbb{R}^n | \dim_H(F) = r.1_{(0)}(l) + n.1_{(0,\infty)}(l), \lambda(F) = l, \dim_{ind}(F) = 0\}. \quad (17)$$

Fourth, given two families in Equations (16) and (17), define the following family of fractals with cardinal of aleph-two:

$$\mathcal{F}(r,l) = \{F_1 \cup F_2 | F_1 \in \mathcal{F}_1(r,l), F_2 \in \mathcal{F}_2(r,l)\}. \quad (18)$$

Finally, let $F \in \mathcal{F}(r,l)$. Then, by construction, there are $F_i \in \mathcal{F}_i$ ($i = 1, 2$) such that: $F = F_1 \cup F_2$. Accordingly, by another application of Theorems 1 (iv) and 3 (iv):

$$\dim_H(F) = \max(\dim_H(F_1), \dim_H(F_2)) = \dim_H(F_2) = r.1_{(0)}(l) + n.1_{(0,\infty)}(l), \quad (19)$$

$$\lambda(F) = \lambda(F_2) = l, \quad (20)$$

$$\dim_{ind}(F) = 0. \quad (21)$$

This completes the proof for the case of asymmetric fractals. For the case of symmetric fractals, as before, we replace any $F \in \mathcal{F}(r,l)$, with $\frac{1}{2}((F \cup (-F)) + 1)$. □

The assertion of Theorem 6 and its proof methodology have two immediate consequences: first, in terms of having the cardinality of aleph-two, the Theorem’s assertion and an application of the Cantor–Schröder–Bernstein theorem [20] put the set of virtual fractals of $\mathbb{R}^n$ in the same category of mathematical sets such as the sigma-algebra of Lebesgue measurable sets in $\mathbb{R}^n$, the power set of $\mathbb{R}^n$, the set of functions from $\mathbb{R}^n$ to $\mathbb{R}^n$, and, the Stone–Cech compactification of $\mathbb{R}^n$. Second, the Theorem’s proof can be modified to show that the cardinality of the set of non-fractals in $\mathbb{R}^n$ is aleph-two as well. Simply, consider the set of non-fractals $\mathcal{F}(n) = \{I^n \cup P_{C,n} | P_{C,n} \subseteq (C + 1)^n\}$ where $n \geq 1$, $I^n$ is the unit cube in $\mathbb{R}^n$ and $C$ is the standard middle third Cantor set.

4. Discussion

This work presented an existence theorem for fractals of a given Hausdorff dimension and a Lebesgue measure with the highest possible cardinal number of aleph-two. In addi-
tion, it generalized the former existence theorem in terms of the Lebesgue measure and cardinal number.

This work’s contributions to the fractal geometry literature covers three perspectives: first, it highlights the advantage of the Cantor sets to other well-known classical fractals in showing existence of fractals with any Hausdorff dimension and Lebesgue measure. Other prominent fractals lack this feature given not being defined in the one dimensional Euclidean space $\mathbb{R}$ and having Hausdorff dimension larger than one. Examples include the Sierpinski triangle, Takagi curve, Julia set, Triflake, Koch curve, and Apollonian gasket. Second, the existence theorem is equipped with constructive proof (versus pure existence proof) presenting real fractals for a given Hausdorff dimension and the Lebesgue measure [21]. This key feature helps us to explore other properties of the constructed fractals yielding more comprehensive information on them. Finally, it presents another example of sets with cardinal number of aleph-two providing more knowledge on the cardinal number.

There are some limitations in this work that create four new lines of research for the interested reader. First, we considered only Mandelbrot’s strict mathematical definition to present the existence result. However, there are other agreed-upon key descriptive features in the definition of fractal that need to be considered. Examples of these characteristics are self-similarity type (exact, quasi, statistical), fine structure, irregularity (local, global), and the recursive definition [16]. Second, while we considered only Hausdorff dimension as the index of fractal dimension in the Mandelbrot’s definition, the existence case for the other indices of fractal dimension remains to be investigated. For instance, we can consider the Renyi dimension (with special cases, such as the Minkowski dimension [16], Information dimension [22], Correlation dimension [23]), the Higuchi dimension [24], the Lyapunov dimension [25], Packing dimension [26], the Assouad dimension [27] and the generalization of the Hausdorff dimension [28]. Third, a more rigorous and comprehensive method is to investigate the existence problem of fractals for the generalized fractal space equipped with a fractal structure and the generalized fractal dimension [29]. Finally, the existence result in this work is limited to deterministic fractals constructed by their associated deterministic recursive processes. However, its validity for the more general random fractals remains an open question. We summarize the above points as the following set of open problems:

**Open Problems**

Given the $n$-dimensional Euclidean space $\mathbb{R}^n (n \geq 1)$.

1. Does the precise cardinality of the set of all distinctive fractals vary by the applied fractal dimension?
2. Does the precise cardinality of the set of all distinctive fractals depend on the generalized fractal structure?
3. Does the precise cardinality of the set of all distinctive fractals depend on the deterministic or random nature of the fractal?

**5. Conclusions**

There has been a growing interest in fractal dimension and its calculations from different methods since the early 1900s without retrospective exploration of the existence of any set of fractals for a given fractal dimension and its maximum cardinality. This work addressed the problem in part by presenting a constructive existence result from a deterministic viewpoint for the cardinality of aleph-two, paving the way for working through more unsolved problems in this topic.

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References
1. Mandelbrot, B. The Fractal Geometry of Nature; Lecture Notes in Mathematics; W. H. Freeman: San Francisco, CA, USA, 1982; Volume 1358.
2. Mandelbrot, B. How Long is the Coast of Britain? Statistical Self-Similarity and Fractional Dimension. Science 1967, 156, 636–638. [CrossRef] [PubMed]
3. Mandelbrot, B. Fractals and Chaos; Springer: New York, NY, USA, 2004; p. 38.
4. Gordon, N.L. Introducing Fractal Geometry; Icon Books: Duxford, UK, 2000; p. 71
5. Husain, A.; Reddy, J.; Bisht, D.; Sajid, M. Fractal Dimension of Coastline of Australia. Sci. Rep. 2021, 11, 6304. [CrossRef] [PubMed]
6. Dowek, G. Chapter 8: Constructive Proofs and Algorithms. In Computation, Proof, Machine: Mathematics Enters a New Age; Guillon, P., Roman, M., Eds.; Cambridge University Press: Cambridge, UK, 2015.
7. Klaus, M.; Schuster, P.; Schwichtenberg, H. (Eds.) Proof and Computation: Digitalization in Mathematics, Computer Science, and Philosophy; World Scientific: Hackensack, NJ, USA, 2018; pp. 1–46.
8. Soltanifar, M. On A Sequence of Cantor Fractals. Rose Hulman Undergrad. Math. J. 2006, 7, 9.
9. Squillace, J. Estimating the Fractal Dimension of Sets Determined by Nonergodic Parameters. Discret. Contin. Dyn. Syst. 2017, 37, 5843–5859. [CrossRef]
10. Gryszka, K. Hausdorff dimension is onto. Pr. Kola Mat. Uniw. Ped. W Krak 2019, 5, 13–22.
11. Gryszka, K.; Pasteczka, P. On the L1 and pointwise divergence of continuous functions. arXiv 2020. Available online: https://arxiv.org/abs/2007.12752 (accessed on 13 May 2021).
12. Jarnik, V. Uber die simultanen diophantischen approximationen. Math. Z. 1931, 33, 505–543. [CrossRef]
13. Anitas, E.M. Structural Investigations of Fractals Using Small-angle Scattering. J. Phys. Conf. Ser. 2015, 574, 1–4. [CrossRef]
14. Anitas, E.M. Small-angle Scattering from Fat Fractals. Eur. Phys. J. B 2014, 87, 1–7. [CrossRef]
15. Soltanifar, M. A Different Description of A Family of Middle-α Cantor Sets. Am. J. Undergrad. Res. 2006, 5, 9–12. [CrossRef]
16. Falconer, K. Fractal Geometry: Mathematical Foundations and Applications, 3rd ed.; John Wiley & Sons: Chichester, UK, 2014; pp. 44, 45, 48, 49, 72, 112.
17. Edgar, G. Measure, Topology, and Fractal Geometry, 2nd ed.; Springer: New York, NY, USA, 2008; pp. 85–87, 104, 106, 109.
18. Smullyan, R.M.; Fitting, M. Set Theory and the Continuum Problem; Oxford University Press: Oxford, UK, 1996; p. 23.
19. Moore, G.H. Early history of the generalized continuum hypothesis: 1878–1938. Bull. Symb. Log. 2011, 17, 489–532. [CrossRef]
20. Bajnok, B. An Invitation to Abstract Mathematics, 2nd ed.; Springer: Cham, Switzerland, 2020; p. 325.
21. Bridges, D.; Palmgren, E. Constructive Mathematics; Zalta, E.N., Ed.; The Stanford Encyclopedia of Philosophy, Metaphysics Research Lab, Stanford University: San Francisco, CA, USA, 2018; retrieved 17 June 2020.
22. Rényi, A. On the dimension and entropy of probability distributions. Acta Math. Acad. Sci. Hung. 1959, 10, 193–21. [CrossRef]
23. Peter, G. Generalized Dimensions of Strange Attractors. Phys. Lett. A1983, 97, 227–230. [CrossRef]
24. Higuchi, T. Approach to an irregular time-series on the basis of the fractal theory. Physical D 1988, 31, 277–283. [CrossRef]
25. Frederickson, F.; Kaplan, J.; Yorke, E.; Yorke, J. The Liapunov dimension of strange attractors. J. Differ. Equ. 1983, 49, 185–207. [CrossRef]
26. Tricot, C., Jr. Two definitions of fractional dimension. Math. Proc. Camb. Philos. Soc. 1982, 91, 57–74. [CrossRef]
27. Robinson, J.C. Dimensions, Embeddings, and Attractors; Cambridge University Press: Cambridge, UK, 2010; p. 5.
28. Fernández-Martínez, M.; Sánchez-Granero, M.A. Fractal dimension for fractal structures: A Hausdorff approach revisited. J. Math. Anal. Appl. 2014, 409, 321–330. [CrossRef]
29. Fernández-Martínez, M.; Sánchez-Granero, M.A. Fractal dimension for fractal structures. Topol. Appl. 2014, 163, 93–111. [CrossRef]