On the Random $1/2$-Disk Routing Scheme in Wireless Ad Hoc Networks

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Abstract

Random $1/2$-disk routing in wireless ad-hoc networks is a localized geometric routing scheme in which each node chooses the next relay randomly among the nodes within its transmission range and in the general direction of the destination. We introduce a notion of convergence for geometric routing schemes that not only considers the feasibility of packet delivery through possibly multi-hop relaying, but also requires the packet delivery to occur in a finite number of hops. We derive sufficient conditions that ensure the asymptotic convergence of the random $1/2$-disk routing scheme based on this convergence notion, and by modeling the packet distance evolution to the destination as a Markov process, we derive bounds on the expected number of hops that each packet traverses to reach its destination.

Index Terms

Random Geometric Routing Schemes, Guaranteed Delivery, Convergence, Markov Process.

I. INTRODUCTION

A wireless ad-hoc network consists of autonomous wireless nodes that collaborate on communicating information in the absence of a fixed infrastructure. Each of the nodes might act as a source/destination node or as a relay, where communication occurs between a source-destination (S-D) pair through a single-hop transmission if they are close enough, or through multi-hop transmissions over intermediate relaying nodes if they are far apart from each other. The selection of the relaying nodes along the multi-hop path is governed by the routing scheme.
Takagi and Kleinrock [1] introduced the first position-based routing scheme, coined as Most Forward within Radius (MFR), based on the notion of progress: Given a transmitting node $S$ and a destination node $Dst$, the progress at relay node $V$ is defined as the projection of the line segment $SV$ onto the line connecting $S$ and $Dst$. In MFR, each node forwards the packet to the neighbor with the largest progress, or discards the packet if none of its neighbors are closer to the destination than itself. Takagi and Kleinrock [1] also discussed a method in which one of the nodes with forward (i.e., positive) progress is chosen at random, arguing that there is a tradeoff between progress and transmission success. This scheme bears the closest resemblance to the routing scheme that we consider in this paper, where each node picks the next relay uniformly at random among the nodes in its transmission range over the $1/2$-disk oriented towards the destination. There are some other variants of MFR in the literature. In [2], the authors introduced the Nearest Forward Progress (NFP) method that selects the nearest neighbor of the transmitter with forward progress; in [3], the Compass Routing (also referred to as DIR method) was proposed, where the neighbor closest to the line connecting the sender and the destination is chosen.

All the aforementioned work lacks rigorous analysis of the conditions for guaranteed packet delivery and the expected hop-count between the S-D pairs. In particular, most of the existing convergence and hop-count results, based on the notion of progress, ignore the vertical distance of the next hop to the line connecting the current node to the destination, which has a considerable effect on the packet delivery as the packet gets closer to the destination. For other relaying schemes, there have been some results on routing convergence analysis. In particular, Xing et al. [4] showed that the delivery can be guaranteed between any S-D pair using the greedy forwarding scheme (GFR), if the transmission radius is larger than twice the sensing radius in a fully covered homogeneous wireless sensor network. In [5], the authors derived the critical transmission radius that ensures the network connectivity asymptotically based on the GFR scheme.

In this paper, we elaborate on the sufficient conditions for asymptotically guaranteed packet delivery of the random $1/2$-disk routing scheme and derive bounds on the expected number of hops that a packet traverses to reach its destination. The remainder of this paper is organized as follows. In Section [1] we introduce the system model and define the notion of convergence for geometric routing schemes, based on connectivity and finite hop-count properties. Then, we explain the localized properties that ensure the convergence of the random $1/2$-disk routing
scheme with probability approaching one as the network density goes to infinity. In Section \text{III}, we establish conditions on the node transmission range that ensure connectivity for the $1/2$-disk routing scheme in a dense network. In Section \text{IV}, we derive bounds on the expected number of hops between an arbitrary source-destination pair and show that all packets will almost surely reach their destinations in finitely many hops if the network is connected. Section \text{V} concludes the paper.

II. System Model

Let $|A|$ be the area of a convex region $A \subset \mathbb{R}^2$ over which the network resides. Network nodes are distributed according to a homogeneous Poisson point process with density $\lambda$. Each node picks a destination node uniformly at random among all other nodes in the network, and operates with a transmission range $R$.

For a general geometric routing scheme, when the destination is out of the one-hop transmission range of a given source/relay node, the next relay is selected (based on some rule) among the nodes contained in the relay selection region (RSR) of the transmitting node, where the RSR in general can be any subset of a full disk of radius $R$ centered at the transmitting node. We define the rule that governs the selection of the next relay in each node’s RSR as the relay selection rule (RSL). Note that a node can potentially act as a relay only if it is contained in the RSR of the current source/relay. We define the convergence of a routing scheme as follows.

\textbf{Definition 1.} A routing scheme is convergent if the following two conditions are satisfied:

1) Connectivity: Between each pair of nodes in the network that are not already in the transmission range of one another, there exists a sequence of relays complying with the RSL.

2) Finite hop-count: Given such sequences determined in the connectivity condition above, their cardinalities\footnote{Measured by the number of hops/relays.} are finite.

Specifically, in this paper we study a special case of the localized geometric routing schemes, namely the Random $1/2$-disk routing scheme, where for each transmitting node $S$ in the network, as illustrated in Fig. \text{I} the next relay $V$ is chosen randomly among the nodes contained in the
Fig. 1. The random 1/2-disk routing scheme.

1/2-disk of radius $R$ centered at node $S$ and oriented towards the packet destination $Dst$. Hence, the name random 1/2-disk routing scheme is coined. We denote the relay selection region and the relay selection rule of the random 1/2-disk routing scheme by $\frac{1}{2}$RSR and rRSL, respectively. Observe that according to our routing scheme, the next chosen relay might be farther away from the destination than the current node. This fact differentiates our routing scheme from GFR [4] and its variants.

Note that for the network to be connected based on any geometric routing scheme, the connectivity condition in Definition 1 requires the existence of RSL-compliant routing paths between all source-destination pairs in the network. In particular, the network is connected under the random 1/2-disk routing scheme if the $\frac{1}{2}$RSRs of every node $S$ in the network contain at least one node with high probability (note that the $\frac{1}{2}$RSR of node $S$ can be oriented in any direction that $Dst$ may exist). This ensures the existence of a relaying node in every direction of a particular transmitting node and ascertains the possibility of packet delivery to a particular destination from any direction. In fact, a node has a possible relay/destination in any direction if and only if it can serve as relay/destination from any direction. Now, given that the network is connected, the packets are guaranteed to be delivered to their destinations, according to the random 1/2-disk routing scheme, if the cardinality of the rRSL-compliant relay sequence (i.e., the number of hops) starting from the source and ending at the destination is finite almost surely. In the following, Theorem 1 presents the main result of this paper regarding the convergence.

\[^2\]From here on we claim that the following two statements are equivalent: “the packet delivery is guaranteed” and “the routing scheme is convergent”.
of the 1/2-disk routing scheme. For notational convenience, we let \( N = \lambda |A| \) designate the expected number of nodes in the network region of area \(|A|\) and \( d = \frac{\pi R^2}{|A|} \) denote the normalized area of a full disk of radius \( R \) relative to the area of the whole region, such that \( dN \) is the expected number of nodes in such a disk. Also, \( f(n) = o\left(g(n)\right) \) means that there exist positive constants \( c_1 \) and \( k \) such that \( f(n)/g(n) \leq c_1 \) whenever \( n \geq k \), and \( f(n) = \Theta\left(g(n)\right) \) means that there exist positive constants \( c_1 \), \( c_2 \) and \( k \) such that \( c_1 \leq f(n)/g(n) \leq c_2 \) whenever \( n \geq k \).

**Theorem 1.** If \( d = o\left(N^{-2/3}\right) \) and \( dN \to \infty \) as \( N \to \infty \), then the random 1/2-disk routing scheme is convergent, according to Definition 1, with probability approaching one.

**Proof:** Here we only sketch the outline of the proof and present the respective details in the following sections. According to Definition 1, we need to show that asymptotically (i.e., as \( N \to \infty \)) a) the network is connected based on the random 1/2-disk routing scheme; and b) given a connected network, the random 1/2-disk routing scheme delivers each packet to its destination in a finite number of hops.

In Section III, regarding condition a) described above, we derive the upper bound \( \sigma_N \) on the probability that some nodes in the network are not connected to the rest of the network. Then we determine conditions that ensure the convergence of \( \sigma_N \) to zero as \( N \to \infty \). This establishes the sufficient conditions for the network to be connected with probability approaching one as \( N \to \infty \).

In Section IV, regarding condition b) described above, we show that given a connected network, the number of hops that each packet traverses to reach its destination with the proposed routing scheme is finite almost surely, by modeling the distance evolution of the packet to its destination as a Markov process. This, together with the results in Section III, concludes the proof for the asymptotic convergence of the random 1/2-disk routing scheme.

### III. Sufficient Conditions for Network Connectivity

As mentioned earlier, the network is connected if there exists a sequence of rRSL-compliant relays between any two nodes of the network, which are not already in the transmission range of each other. Hence, the problem of proving connectivity amounts to bounding the probability \( \sigma_N \) that there exist some network nodes which are not connected to the rest of the network. A node is connected to the rest of the network if it has a possible relay/destination in any direction,
which is equivalent to being able to serve as relay/destination from any direction. Thus, \( \sigma_N \) equals the probability that, for some network nodes, there are certain directions at which their \( \frac{1}{2} \)RSRs are empty. We can identify two types of network nodes based on their distances to the edge of the network: Nodes that are farther than \( R \) away from the edge of the network, which we call \textit{interior nodes}, and nodes that are closer than \( R \) to the edge of the network, which we call \textit{edge nodes}.

For interior nodes, it is clear that the node distribution in their \( \frac{1}{2} \)RSRs, looking in any direction, is similar. Therefore, the existence probability of an empty \( \frac{1}{2} \)RSR for an interior node is independent of its direction. However, due to the proximity of edge nodes to the boundary of the network, the existence probability of an empty \( \frac{1}{2} \)RSR for an edge node highly depends on its orientation. For example, the \( \frac{1}{2} \)RSRs that fall partly outside the network region are more likely to be empty than the ones that are fully contained in the network region. Hence, we derive the probabilities of a node being disconnected from the rest of the network separately for the interior nodes and the edge nodes, denoted by \( \sigma'_N \) and \( \sigma''_N \), respectively.

We now slightly generalize the problem as follows. Suppose that the RSR of a node is instead a wedge of angle \( 2\pi \eta \) and radius \( R \), with \( 0 < \eta \leq 1 \) (hereafter called \( \eta \)-disk or \( \eta \)RSR, interchangeably). Hence, the \( \frac{1}{2} \)RSR is a special case of \( \eta \)-disk with \( \eta = 1/2 \). Each \( \eta \)-disk has an expected number of nodes \( \eta dN \). As shown in Section III-C, the probability of network disconnectivity increases as \( \eta \) decreases. However, we conjecture that the packet reaches its destination with fewer hops as \( \eta \) decreases. Hence, there may exist a tradeoff between the probability of network disconnectivity and the expected number of hops between a S-D pair parameterized by \( \eta \). We leave the study of this tradeoff to a future work and only derive (in Section IV) the expected number of hops between a source and its destination for the case of \( \eta = 1/2 \).

A. Calculation of \( \sigma'_N \)

Consider an interior node \( x \), fixed for now. Given \( i \geq 1 \) nodes in the transmission range of \( x \), their directions (in reference to \( x \)) are independent and uniformly distributed on \([0, 2\pi]\). The probability that \( x \) is not connected to the rest of the network equals the probability \( U_i(\eta) \) that the angle of the widest wedge containing none of these \( i \) nodes is at least \( 2\eta \pi \). It is not difficult to give a simple upper bound on \( U_i(\eta) \): Of the \( i \) nodes, without loss of generality (W.L.O.G.),
we can assume that (at least) one is at one end of an empty wedge with angle of $2\eta\pi$, while the other $i-1$ are distributed independently and uniformly in the remainder of the full transmission disk, as shown in Fig. 2. Note that, by assuming uniform (and independent) distribution for the remaining $i-1$ nodes over the remainder of the transmission disk, we are also taking into account the possibility of an effective empty wedge with an angle $2\eta'\pi$ greater than $2\eta\pi$, as suggested in Fig. 2. Hence, we obtain $U_i(\eta) \leq i(1-\eta)^{i-1}$. Of course, if $i = 0$ the probability is $U_0(\eta) = 1$.

One can obtain the a more precise expression for $U_i(\eta)$ using results in [6], page 188 as:

$$U_i(\eta) = \sum_{k=1}^{\min\{\lfloor 1/\eta \rfloor, i\}} (-1)^{k-1} \binom{i}{k} (1-k\eta)^{i-1} \leq i(1-\eta)^{i-1},$$

where $\lfloor x \rfloor$ is the largest integer smaller than $x$. This expression is based on the inclusion-exclusion principle for the probability of the union of events, for which the first summand provides an upper bound and the first two summands provide a lower bound.

Therefore, averaging over $i$ (number of the nodes in the transmission range of $x$) and over the number of interior nodes, we have:
\[ \sigma_N' \leq \left(1 - (2 - \sqrt{d})\sqrt{d}\right) N \sum_{i=0}^{\infty} U_i(\eta) \frac{(dN)^i}{i!} e^{-dN} \]
\[ \leq \left(1 - (2 - \sqrt{d})\sqrt{d}\right) N \left[e^{-dN} + \sum_{i=1}^{\infty} U_i(\eta) \frac{(dN)^i}{i!} e^{-dN}\right] \]
\[ \leq (1 - 2\sqrt{d}) N (dN + 1) e^{-\eta dN} , \quad (1) \]

where \((1 - (2 - \sqrt{d})\sqrt{d})N\) is the expected number of interior nodes if, for simplicity, we assume that the network region is a disk of radius \(L = \sqrt{|A|/\pi} = R/\sqrt{d}\).

**B. Calculation of \(\sigma''_N\)**

So far we have considered the interior nodes which are well away from the boundary of the network region. Now, we consider edge nodes which are closer than \(R\) to the edge of the network and do not have any nodes in the portion of their transmission ranges that falls outside of the network region. Therefore, some \(\eta\)-disks (of an edge node) may fall partially (up to half) outside the region, which increases the chance that they are empty. We refer to this phenomenon as *edge effect*. Assuming the network region is circular, the number of such edge nodes equals \((2 - \sqrt{d})\sqrt{d}N\), which is of order \(\Theta(\sqrt{d}N)\). We need to determine how their contribution to the bound on probability of connectivity differs from the previous computations.

Consider an edge node \(e\), \((\delta'R)\)-distance away from the network edge, with \(0 < \delta' < 1\). Define angle \(0\) to be perpendicular to the network edge and pointing towards the edge with node \(e\) as the origin; in other words, as shown in Fig. 3 we take node \(e\) as the pole and the ray \(ev\) (perpendicular to the network edge) as the polar axis of the *local* (polar) coordinates at node \(e\). We observe that, due to the curvature of the network edge, the overlap of node \(e\)’s transmission range with the network region is larger than what it would be if the network’s edge were straight (i.e., the line passing through the intersection points \(A\) and \(B\) in Fig. 3). This area difference (the shaded area in Fig. 3) is no larger than \(\sqrt{d}R^2\) containing an expected number of nodes on the order of \(\Theta(d^{3/2}N)\), where the maximum area difference is obtained when node \(e\) is located on the straight network edge approximation (i.e., in the middle of \(AB\) in Fig. 3). Accordingly, we make a further simplifying assumption that \(d = o \left(N^{-2/3}\right)\); this is equivalent to a practical assumption that the ratio between the transmission range \(R\) and the radius \(L\) of the network.
region goes to zero fast enough such that the expected number of nodes in the shaded area of Fig. 3 goes to zero as \( N \to \infty \). Then the error in calculating the probability of any event in the following will be a factor of no more than \( e^{kd^{3/2}N} \to 1 \), where \( k \) is a finite constant. Henceforth, we proceed as if the network region is straight where it intersects with the node’s transmission disk for large \( N \).

We argued in the beginning of this section that, for edge node \( e \), the probability of an \( \eta \)-disk being empty, depends highly on its orientation. Let us consider this claim more closely. Let \( \varphi = \cos^{-1}(\delta) \in (0, \pi/2) \), as shown in Fig. 4, where \( \delta R \) is the distance between node \( e \) and the straight approximation of the network edge as defined before with \( 0 < \delta < 1 \). Note that all the \( \eta \)-disks are oriented towards the destination of the packets. Hence, for all \( \eta \)-disks that are oriented toward an angle in the range \((-\varphi, \varphi)\), we must have that the destination is within node \( e \)’s transmission range. Therefore, we only need to be concerned with empty \( \eta \)-disks oriented towards an angle in the range \((\varphi, 2\pi - \varphi)\). The \( \eta \)-disks oriented to an angle in either the range \((-\varphi - \eta\pi, -\varphi)\) or the range \((\varphi, \varphi + \eta\pi)\) are partially outside the network region, as illustrated in Fig. 4 and those oriented to any angle in \((\varphi + \eta\pi, 2\pi - \varphi - \eta\pi)\) are fully contained inside the

\(^3\)This rate will apply as long as the region is convex and smooth.
network region; this defines the effective $\eta$RSR range for node $e$. Note that here, all the angles are measured relative to the polar axis $ev$.

We now compute $\sigma''_N$ for node $e$. First, suppose that there are no nodes within the transmission range of node $e$; this event occurs with probability no greater than $e^{-dN/2}$, which corresponds to the case where node $e$’s transmission disk is half outside the network region. Therefore, the probability that some edge nodes having no other nodes within their transmission ranges is upper-bounded by $2\sqrt{dNe^{-dN/2}}$.

Second, suppose that there are $i \geq 1$ nodes in the intersection of node $e$’s transmission range with the network region. If an empty $\eta$-disk exists and it is completely contained within the network region, then, W.L.O.G., there should be a node on its counter-clockwise edge at some angle $\theta \in (\varphi, 2\pi - \varphi - 2\eta\pi)$. However, for an empty $\eta$-disk that is partially contained within the network region there should be, again W.L.O.G., a node at an angle $\theta \in (-\varphi - 2\eta\pi, -\varphi - \eta\pi)$ or $\theta \in (\varphi - \eta\pi, \varphi)$ on the counter-clockwise edge of the $\eta$-disk. Clearly, the existence probability of such empty $\eta$-disks (that is partially contained in the region) increases as either $\delta$ or $|\theta|$ decreases. The area of the intersection between the $\eta$-disk (that is partially contained in the

\[\footnote{The straight edge (of the $\eta$-disk) that over its counter-clockwise direction, the entire $\eta$-disk is located, as shown in Fig. 4.}\]
region) and the network region is that of a wedge with angle $|\theta| - \varphi$ (wedge $eAB$ in Fig. 4) plus a triangle abutting the counter-clockwise edge of the sector (triangle $eBC$ in Fig. 4) with area no more than $\frac{R^2}{2} \cot(\varphi)$. In fact if, for an arbitrary small $\epsilon$, either $\delta \geq \sin(3\epsilon\pi)$ or $\theta \geq \varphi + \eta\pi + 2\epsilon\pi$, then the area of the intersection between $\eta$-disk and the network region is at least $(\eta/2 + \epsilon)\pi R^2$. Otherwise, it is at least $\eta\pi R^2/2$. Thus, after averaging over $\delta$ and $\theta$ (note that the number of nodes approximately $\delta R$ from the edge is uniformly distributed with an expected value proportional to $1 - \sqrt{d\delta}$ when $d$ is very small), the probability that node $e$ is disconnected from the rest of the network is derived to be no more than

$$(1 - 12\pi\epsilon^2)[(1 - 2\eta)e^{-\eta dN} + 2\eta e^{-(\eta/2 + \epsilon)dN}] + 12\pi\epsilon^2 e^{-\eta dN/2} \leq e^{-(\eta/2 + \epsilon)dN} + 12\pi\epsilon^2 e^{-\eta dN/2},$$

(2)

where $12\pi\epsilon^2$ is an upper bound on the probability that $\delta < \sin(3\epsilon\pi)$ and $\theta < \varphi + \eta\pi + 2\epsilon\pi$, and $(1 - 2\eta)$ is the probability that the whole empty $\eta$-disk falls within the network region. Choosing $\epsilon = \frac{2 \log(dN)}{dN}$ maximizes the right-hand side of (2), which leads to the upper bound $\frac{48\pi \log^2(dN) + 1}{(dN)^2} e^{-\eta dN/2}$ for the probability that node $e$ is disconnected from the rest of the network.

Subsequently, accounting for the expected $dN$ nodes in node $e$’s transmission range and the expected $2\sqrt{dN}$ edge nodes, we obtain the upper bound on the probability that some edge nodes are not connected to the rest of the network as:

$$\sigma''_N \leq \frac{96\pi \log^2(dN) + 1}{\sqrt{d}} e^{-\eta dN/2} + 2\sqrt{d} Ne^{-dN/2},$$

(3)

where the second summand on the right-hand side is the probability that some edge nodes have no other nodes within their transmission ranges.

C. Calculation of $\sigma_N$

Finally, summing (1) and (3), we obtain the bound $\sigma_N$ on the probability that the network is not connected:

$$\sigma_N \leq N(dN + 1)e^{-\eta dN} + \frac{96\pi \log^2(dN) + 1}{\sqrt{d}} e^{-\eta dN/2} + 2\sqrt{d} Ne^{-dN/2}.$$

(4)

$^5$This upper bound is looser but, independent of $\epsilon$. 
This bound on $\sigma_N$ is valid for all $N$ and $d = o \left( N^{-2/3} \right)$, and is asymptotic to $\frac{96\pi \log^2(dN)}{\sqrt{d}} e^{-\eta dN/2}$ which goes to zero if $dN \to \infty$ as $N \to \infty$. Particularly, assuming $d = \frac{c \log(N)}{N}$ and $\eta c > 1$ results in the network being connected with probability approaching one as $N \to \infty$, which shows the consistency between our result and the ones derived in [7] and [8] for $\eta = 1$.

**IV. Hop-Count Between S-D Pairs**

Assuming that the network is connected as stated in Definition 1, we now investigate the question of how many hops it takes for a packet to traverse from the source to its destination with the random $1/2$-disk routing scheme. To answer this question we need to have a mathematical description of the process that models the distance evolution from a typical packet to its destination.

Specifically, we set the intended destination node at the origin and assume that the packet starts at the source node $X_0 = (-h, 0)$, where $X_t$ is the (Cartesian) coordinate of the packet and $r_t := \|X_t\|$ is the (Euclidean) distance from the packet to its destination at time $t \in \mathbb{N}$.

The packet starts at the source node $X_0$ with $\frac{1}{2}$RSR $D_0$ that is a $1/2$-disk centered at $X_0$ and oriented towards the destination at $(0, 0)$. The next relay $X_1$ is selected at random from those contained in $D_0$ (the rRSL rule). This induces a new $\frac{1}{2}$RSR $D_1$, also a $1/2$-disk but centered at $X_1$ and oriented towards the destination, such that $D_1$ is both a translated and a rotated version of $D_0$. Relay $X_2$ is selected randomly among the nodes in $D_1$, and the process continues in the same manner until the destination is within transmission range. We claim that the packet reaches its destination whenever it enters the transmission/reception range of the final destination, i.e., $r_{\tau} \leq R$, for some $\tau \in \mathbb{N}$.

Let $|C|$ denote the area of a region $C$, and $C^c := A - C$ denote the complement of $C$ with respect to network region $A$. Each $\frac{1}{2}$RSR $D_t$ has a radius $R$, with $|D_t| = \frac{\pi}{2} R^2$. Define $S_t := h - r_t$ and $Y_t := S_{t+1} - S_t$, and let $N(D_t)$ be the number of nodes in $D_t$. We would like to know how similar $Y_t$ (and consequently $S_t$) is to a Markov process. Note that even though new relays are chosen uniformly within each $\frac{1}{2}$RSR, the increments $Y_0, Y_1, \ldots$ are not independent due to the fact that the orientations of all $\frac{1}{2}$RSRs are pointing to a common node (destination). This results in the dependence of the node distribution in $D_t$, not only on $X_t$, but also possibly on $X_i$, $0 \leq i < t$, where this dependence increases as the packet gets closer to the destination. Note that throughout this section we assume that the network is connected, i.e., $N(D_t) > 0$ for all $t$.
unless otherwise stated.

Since tracking the dependence of $Y_t$ on all $X_i$, $i \leq t$ is tedious, in this work we only investigate the dependence of $Y_t$ on $X_t$ and $X_{t-1}$ and in particular, determine how close is the distribution of the nodes in $D_t$ to a uniform distribution when just knowing the location of the previous node $X_{t-1}$. 

Note that conditional on $X_t$ (or equivalently on $D_t$) and network being connected $(N(D_t) > 0)$, $N(D_t)$ has a Poisson distribution with intensity $\lambda|D_t|$, given it is positive (non-zero). What is less clear, however, is the nature of $N(D_t)$, given $X_{t-1}$.

Observe that $D_t$ only depends on $X_t$. Given $X_t$, $X_{t-1}$, $N(D_{t-1})$, and $N(D_t) > 0$, the number of nodes $N_t'$ in $D_t \cap D_{t-1}$ is Binomial $\left( N(D_{t-1}) - 1, \frac{|D_t \cap D_{t-1}|}{|D_{t-1}|} \right)$ + $I_{(X_{t-1} \in D_t)}$ and independent of the number of nodes $N_t''$ in $D_t \cap D_{t-1}$, which is Poisson$(\lambda|D_t \cap D_{t-1}|)$; where $I_{\{\cdot\}}$ represents the indicator function, i.e., $I_{\{E\}} = 1$ if event $E$ happens and $I_{\{E\}} = 0$ otherwise. Moreover, conditioned additionally on the two random variables $N_t'$ and $N_t''$, each collection of nodes (located in $D_t \cap D_{t-1}$ and $D_t \cap D_{t-1}'$) is uniformly distributed on the respective areas. This does not, however, imply that the combined collection of nodes is uniformly distributed on $D_t$.

Nonetheless, according to the following proposition, the error that results from proceeding as if $X_{t+1}$ is chosen uniformly on $D_t$ is negligible for large $\lambda$. Essentially, knowing $X_t$, the distribution of nodes in $D_t$ is independent of the location of the previous nodes ($X_j$, $j < t$) for large $\lambda$.

**Proposition 1.** Given the location of previous relaying node ($X_{t-1}$), the distribution of the nodes located in the $\frac{1}{2}$-RSR of the current node ($D_t$) approaches the uniform distribution as $\lambda \to \infty$. In particular, the conditional probability of selecting the next node from $D_t \cap D_{t-1}$ satisfies:

$$
\left( 1 - \frac{1}{\lambda|D_{t-1}|} \right) \frac{|D_t \cap D_{t-1}|}{|D_t|} \leq E \left( \frac{N_t'}{N(D_t)} \bigg| N(D_{t-1}) > 0, N(D_t) > 0, X_{t-1}, X_t \right) \leq \frac{|D_t \cap D_{t-1}|}{|D_t|}. 
$$

\[ \tag{5} \]

**Proof:** Refer to Appendix A.

Accordingly, we can model the distance from a packet to its intended destination, $\{r_t\}_{t \in \mathbb{N}}$, as a Markov process solely characterized by its step size $\xi_t := -Y_t$:

\[ \tag{6} \]

The analysis gets more complicated as we consider longer history of the previous relaying nodes, i.e., $X_{t-2}, X_{t-3}, \cdots$, but we conjecture that the error resulted from neglecting $X_{t-2}, X_{t-3}, \cdots$ should remain of order $\Theta(1/\lambda)$, where $1/\lambda$ is the error resulting from neglecting $X_{t-1}$, as shown in Proposition 1. This is because there are only finitely many previous relays whose $\frac{1}{2}$-RSR might intersect with $D_t$, and we can assume, as a worst case scenario, that all of them have roughly similar and independent effects on the node distribution in $D_t$. 

Fig. 5. Distance evolution.

\[ r_{t+1} = r_t + \xi_t I_{\{r_t > R\}}, \]  

(6)

with \( r_0 = h \) being the initial distance between the source and the destination. Here we assumed that the step size at each time is merely a function of the current position of the packet and the effect of the knowledge on the previous locations is negligible, i.e., \( \Pr(\xi_t \leq x \mid r_u, u \leq t) \approx \Pr(\xi_t \leq x \mid r_t) \), which is justified by Proposition 1 for large \( \lambda \). Fig. 5 depicts (a realization of) the evolution of distance of a packet to its destination at time \( t \) based on (6).

The conditional distribution of the step size \( F(x, r_t) := \Pr(\xi_t \leq x \mid r_t) \) for \(-R \leq x \leq \sqrt{r_t^2 + R^2} - r_t\) can be obtained as follows. Given \( r_t \), the probability that the next hop is at least \( x \) distance farther/closer to the destination than the current node is equivalent to the probability that the relay falls in the intersection of \( D_t \) and a circle centered at the destination with radius \( r_t + x \), i.e., the dashed area depicted in Figs. 6(a) and 6(b) for \( x > 0 \) and \( x \leq 0 \), respectively. Any node in the interior of a circle centered at the destination with radius \( r_t + x \) is at least \( x \) farther/closer to the destination than the current transmitting node (which is located at \( r_t \)). Consequently, we can determine the conditional probability distribution of the packet’s next
displacement at time $t$ as:

$$F(x, r_t) := \Pr(\xi_t \leq x|r_t) = \frac{2}{\pi R^2} \left[ R^2 \cos^{-1}\left( \frac{r_t^2 + R^2 - (r_t + x)^2}{2r_t R} \right) 
+ (r_t + x)^2 \cos^{-1}\left( \frac{2r_t(r_t + x) - (R^2 - x^2)}{2r_t(r_t + x)} \right) 
- \frac{1}{2} \sqrt{(R^2 - x^2)(4r_t(r_t + x) - (R^2 - x^2))} \right] 
- \textbf{I}_{\{x>0\}} \left( (r_t + x)^2 \cos^{-1}\left( \frac{r_t}{r_t + x} \right) + r_t \sqrt{(r_t + x)^2 - r_t^2} \right), \quad (7)$$

for $-R \leq x \leq \sqrt{r_t^2 + R^2} - r_t$. Observe that this distribution directly depends on the distance between the packet and the destination in the current time slot, which makes direct analysis of the random process extremely tedious. Hence, as shown in Fig. 7, we rewrite (6) as

$$r_{t+1} = \sqrt{(r_t - x'_t)^2 + y'_t^2}, \quad (8)$$

where $(x'_t, y'_t)$ is the projection of $X_{t+1} - X_t$ onto the local Cartesian coordinates with the current node as the origin and the $x$ axis pointing from the current node to the destination. Based on Proposition 1, $(x'_t, y'_t)$ is distributed uniformly on $D_t$ which makes (8) the Markov approximation
Fig. 7. Distance between the next relay and the current node projected onto the local coordinates at the current node.

Define $\nu_r := \inf\{ t : r_t \leq r \}$, for $r \geq R$ as the first time that the packet gets closer than $r$ to the destination. Hence, $\nu_R$ represents the first time the packet enters the reception range of the destination and $\nu_R + 1$ indicates the number of hops that a packet traverses to reach its destination. It is easy to show that $\nu_r$ is a stopping time \cite{10}. It is clear that

$$r - R \leq r_{\nu_r} \leq r.$$  

Furthermore, let $g(r, x', y') = \sqrt{(r - x')^2 + y'^2} - r$. We observe that $g$ is a decreasing function over $r$, for fixed $(x', y')$. Thus, for $t + 1 \leq \nu_r$, we have $r_t > r$ and

$$-x'_t \leq r_{t+1} - r_t \leq g(r, x'_t, y'_t).$$

Hence, we have

$$r - R \leq r_{\nu_r} \leq h + \sum_{t=1}^{\nu_r} g(r, x'_t, y'_t), \quad (9a)$$

$$h + \sum_{t=1}^{\nu_r} (-x_t) \leq r_{\nu_r} \leq r. \quad (9b)$$

Note, as well, that (refer to Appendix \[B\])
\[-\frac{4R}{3\pi} = E(-x_t) \leq E(g(r, x'_t, y'_t)) \leq E(g(R, x'_t, y'_t)) < -\frac{R}{4} < 0. \tag{10}\]

Now, applying Wald’s equality to (9a) and (9b) and rearranging, we obtain a bound on the expected value of the stopping time \(\nu_r\):

\[
\frac{3\pi(h-r)}{4R} \leq E(\nu_r | h) \leq \frac{h-r+R}{-E(g(r, x'_t, y'_t))} \leq \frac{h}{-E(g(r, x'_t, y'_t))} \leq \frac{4h}{R}. \tag{11}\]

Substituting \(r\) with \(R\) we obtain a general bound for the expected number of hops that a packet traverses to reach the destination (minus one) as:

\[
\frac{3\pi}{4} \left( \frac{h}{R} - 1 \right) \leq E(\nu_R | h) \leq \frac{4h}{R}. \tag{12}\]

This implies that the packet reaches its destination almost surely (a.s.) in a finite number of steps when the network is connected, which happens with probability no less than \(1 - \sigma_N\) as obtained in (4). Specifically, with \(dN \to \infty\) as \(N \to \infty\), we have \(\Pr(\nu_R < \infty) \to 1\).

Moreover, when the ratio \(h/R\) (i.e., the ratio between the source-destination distance and the transmission range) is large, we can obtain a tighter bound on the expected number of hops between a source-destination pair that are \(h\) apart. Since \(r_{\nu_r} < r\), we must have \(E(\nu_R | h) \leq E(\nu_r | h) + E(\nu_R | r)\). Thus, by (11) and proper substitutions, we have

\[
\frac{E(\nu_R | h)}{h/R} \leq \frac{R}{-E(g(r, x'_t, y'_t))} + \frac{4r}{h}, \tag{13}\]

for all \(r \in (R, h)\). From this, it is easy to deduce that

\[
\lim_{h/R \to \infty} \frac{E(\nu_R | h)}{h/R} \leq \min_{r>R} \frac{R}{-E(g(r, x'_t, y'_t))} = \frac{3\pi}{4}. \tag{14}\]

This implies that as \(h/R \to \infty\), the expected number of hops a packet traverses between a source and its destination is asymptotic to \(\frac{3\pi h}{4R}\).

Furthermore, we can determine the expected initial distance \(h\) between an arbitrary source-destination pair in the network by deriving the expected distance between any two points that are randomly located inside the network area. The problem of quantifying \(h\) is well studied in the literature [9], with the following known results for two special convex network regions: If
the region is a planar disc with diameter \( a \), we have \( h = \frac{64a}{(45\pi)} \approx 0.4527a \); and if it is a square with side length \( a \), we have \( h = \left(2 + \sqrt{2} + 5\log(\sqrt{2} + 1)\right) \frac{a}{15} \approx 0.5214a \).

V. Conclusion

In this paper, we determined the sufficient conditions for a localized routing scheme, namely the random 1/2-disk routing scheme, to be convergent based on a convergence notion that not only considers the feasibility of packet delivery to the destination via possibly multi-hop relaying, but also requires the packet delivery to occur in finitely many hops. To this end, we showed that, given certain conditions, there exists a sequence of relays complying with the 1/2-disk routing scheme between any two nodes in the network with probability approaching one as the (network) node density goes to infinity. Furthermore, we showed that given a network satisfying those conditions, the random 1/2-disk routing scheme will deliver the packets to their destinations in finitely many hops almost surely, by modeling the distance evolution from a packet to its destination as a Markov process.

APPENDIX A

Proof of Proposition 1

First, let us consider the distribution of a Poisson point process conditioned on deleting one point. Let \( N \) be a homogeneous Poisson point process with intensity \( \lambda \) and fix a region \( D \). If \( N(D) > 0 \), one point in \( D \) is selected at random and removed. Let \( X \) be the location of that point. The distribution of \( N \) on \( D^c \) remains Poisson and independent of \( N \) on \( D \), and thus independent of both \( N(D) \) and \( X \). Let \( N' \) be the (point) process with the point at \( X \) deleted.

Let \( A_1, A_2, \ldots, A_k \) be a partition of \( D \). Given \( N(D) > 0 \), the points in \( D \) are distributed uniformly. If one point is removed at random, the remaining points are still distributed uniformly on \( D \). Hence,

\[
\Pr\left( \bigcap_{j=1}^{k}\{N'(A_j) = n_j\} \mid N(D) > 0, X \right) = (1 - e^{-\lambda|D|})^{-1} \sum_{i=1}^{k} \prod_{j=1}^{k} \frac{(\lambda|A_j|)^{n_j+I_{j=i}}}{(n_j+I_{j=i})!} e^{-\lambda|A_j|} \frac{n_j+I_{j=i}}{n_1 + \cdots + n_k + 1} \\
= \frac{\lambda|D|}{(1 - e^{-\lambda|D|})(n_1 + \cdots + n_k + 1)} \prod_{j=1}^{k} \frac{(\lambda|A_j|)^{n_j}}{(n_j)!} e^{-\lambda|A_j|},
\]

(15)
since $|A_1| + \cdots + |A_k| = |D|$. Therefore, conditional on $N(D) > 0$, $N'$ is independent of $X$. In particular,

$$
\Pr \left( N'(D) = n \mid N(D) > 0, X \right) = \frac{(\lambda|D|)^{n+1}}{(n+1)! (1-e^{-\lambda|D|})^k} e^{-\lambda|D|}
$$

$$
= \Pr \left( N(D) = n + 1 \mid N(D) > 0 \right).
$$

Furthermore, given $n_1 + \cdots + n_k = n > 0$, we have

$$
\Pr \left( \bigcap_{j=1}^k \{ N'(A_j) = n_j \} \mid N(D) > 0, N'(D) = n, X \right) = \binom{n}{n_1 \cdots n_k} \prod_{j=1}^k \left( \frac{|A_j|}{|D|} \right)^{n_j}.
$$

Thus, for $A \subseteq D$ and given $N'(D) = n > 0$, $N'(A)$ is conditionally binomial $(n, |A|/|D|)$. Without knowing $N'(D) > 0$, however, we obtain from (15) that

$$
\Pr \left( N'(A) = k \mid N(D) > 0, X \right) = \frac{\lambda|D| e^{-\lambda|D|}}{(1-e^{-\lambda|D|})} \sum_{j=0}^\infty \frac{1}{k+j+1} \frac{(\lambda|A|)^k (\lambda|A^cD|)^j}{k! j!}
$$

$$
= \frac{\lambda|D| e^{-\lambda|D|}}{(1-e^{-\lambda|D|})} \frac{(\lambda|A|)^k}{k!} \int_0^1 y^k e^{\lambda|A^cD|y} dy,
$$

(16)

where the second equality is due to

$$
\sum_{j=0}^\infty \frac{1}{k+j+1} \frac{a^j}{j!} = \sum_{j=0}^\infty \frac{1}{a^{k+1}} \int_0^a x^{k+j} \frac{dx}{j!}
$$

$$
= \int_0^a \frac{a^k x^j}{a^{k+1}} e^x dx = \int_0^1 y^k e^{ay} dy.
$$

After the aforementioned preliminaries, we now proceed with the proof of Proposition 1. Suppose $C$ is a random set that depends only on $X$. The points of $N'$, if any, that are in $CD := C \cap D$ are uniformly distributed and independent of the points in $CD^c$, which also are uniformly distributed (if any). The combined points are uniformly distributed on $C$ only if the expected proportion of points in $CD$ is $|CD|/|D|$. However, it is not in our case as we now compute. Given $N'(C) > 0$, the probability that
a randomly selected point in $C$ also is in $D$ is $E \left( \frac{N'(CD)}{N'(C)} \mid N'(C) > 0, N(D) > 0, X \right)$. Let $\frac{N'(CD)}{N'(C)} = 0$ when $N'(C) = 0$. We have

\[
\Pr \left( N'(C) > 0 \mid N(D) > 0, X \right) = 1 - \Pr \left( N'(CD) = 0, N'(CD^c) = 0 \mid N(D) > 0, X \right)
\]

\[
= 1 - \frac{\lambda|D|e^{-\lambda|D|}}{1 - e^{-\lambda|D|}} \frac{e^{\lambda|C^cD|} - 1}{e^{-\lambda|C^cD|}}
\]

\[
= 1 - \frac{|D|}{|C^cD|} \frac{1 - e^{-\lambda|C^cD|}}{1 - e^{-\lambda|D|}} e^{-\lambda|C|},
\]

and so

\[
1 - \frac{|D|}{|C^cD|} e^{-\lambda|C|} \leq \Pr \left( N'(C) > 0 \mid N(D) > 0, X \right) \leq 1 - e^{-\lambda|C|}.
\]

(17)

Using the observation above and (16) we have,

\[
E \left( \frac{N'(CD)}{N'(C)} I_{\{N'(C) > 0\}} \mid N(D) > 0, X \right)
\]

\[
= \frac{\lambda|D|e^{-\lambda|D|}}{1 - e^{-\lambda|D|}} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\lambda|CD|)^n}{n!} \int_0^1 y^n e^{\lambda|C^cD|y} dy \frac{(\lambda|CD^c|)^m}{m!} e^{-\lambda|CD^c|}
\]

\[
= \frac{\lambda|D|e^{-\lambda|C^cD|}}{1 - e^{-\lambda|D|}} \sum_{n=1}^{\infty} \frac{(\lambda|CD|)^n}{(n-1)!} \int_0^1 y^{n-1} e^{\lambda|C^cD|y} \int_0^1 w^{n-1} e^{\lambda|CD^c|w} dw dy
\]

\[
= \frac{\lambda|D|e^{-\lambda|C^cD|}}{1 - e^{-\lambda|D|}} \int_0^1 \int_0^1 |CD| y e^{\lambda|CD|y + |C^cD||y + |CD^c||w|} dy dw
\]

\[
= \frac{|CD| \lambda|D|e^{-\lambda|C^cD|}}{1 - e^{-\lambda|D|}} \int_0^1 |CD| y + |CD^c| \left( e^{\lambda|CD|y + |CD^c||w|} - 1 \right) e^{\lambda|C^cD|y} dy
\]

\[
\leq \frac{|CD| \lambda|D|e^{-\lambda|C^cD|}}{|C|} \int_0^1 \left( e^{\lambda|CD|y + |CD^c||w|} - 1 \right) e^{\lambda|C^cD|y} dy
\]

\[
= \frac{|CD|}{|C|} \Pr \left( N'(C) > 0 \mid N(D) > 0, X \right).
\]

(18)

Therefore,

\[
E \left( \frac{N'(CD)}{N'(C)} \mid N'(C) > 0, N(D) > 0, X \right) \leq \frac{|CD|}{|C|}.
\]

Noting that
\[ ae^{-a} \int_0^1 (1 - y)e^{ay} \, dy = \frac{1 - e^{-ay}}{a} \leq \frac{1}{a}. \]

we also may ascertain that

\[
\mathbb{E} \left( \frac{N'(CD)}{N'(C)} \mid N'(C) > 0, N(D) > 0, X \right) \geq \left( 1 - \frac{1}{\lambda |D|} \right) \frac{|CD|}{|C|}.
\]

So, if \( \lambda \) is very large, the selected point is slightly less likely to be in \( D \) than would the case if we could assume \( N' \) is Poisson on \( C \) (or that the points in \( C \) truly were uniformly distributed).

**APPENDIX B**

**DERIVATION OF INEQUALITY (10)**

We have \( (x'_i, y'_i) \overset{D}{=} (Rv \cos(\theta), Rv \sin(\theta)) \), where \( \theta \sim \text{uniform}(-\pi/2, \pi/2) \) and \( v \sim \beta(2, 1) \) are independent. Thus,

\[
E(x'_i) = \frac{R}{\pi} \int_0^{\pi/2} \cos(\theta) \, d\theta \int_0^1 2v^2 \, dv = \frac{4R}{3\pi}.
\]

Also, by first changing \( x \) to \( 1 - x \) and then using polar coordinates,

\[
\frac{1}{R} E(g(R, x'_i, y'_i)) + 1 = \frac{4}{\pi} \int_0^1 \int_0^1 1_{x^2 + y^2 \leq 1} \sqrt{(1 - x)^2 + y^2} \, dx \, dy
\]

\[
= \frac{4}{\pi} \int_0^1 \int_0^1 1_{(1-x)^2 + y^2 \leq 1} \sqrt{x^2 + y^2} \, dx \, dy
\]

\[
= \frac{2}{\pi} \int_0^{\pi/4} \int_0^{\sec \theta} 2v^2 \, dv \, d\theta + \frac{2}{\pi} \int_0^{\pi/2} \int_{\sec \theta}^{\cos \theta} 2v^2 \, dv \, d\theta
\]

\[
= \frac{4}{3\pi} \int_0^{\pi/4} \left( (\sec \theta)^3 + (2 \sin \theta)^3 \right) \, d\theta
\]

\[
= \frac{3(2^{3/2}) + 6 \log(1 + \sqrt{2}) + 64 - 5(2^{7/2})}{9\pi} \approx 0.7499728.
\]

Hence \( E(g(R, x'_i, y'_i)) < -\frac{R}{4} \).

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