ASSOCIATIVE AND LIE DEFORMATIONS OF POISSON ALGEBRAS

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ABSTRACT. Considering a Poisson algebra as a non associative algebra satisfying the Markl-Remm identity, we study deformations of Poisson algebras as deformations of this non associative algebra. This gives a natural interpretation of deformations which preserves the underlying associative structure and to study deformations which preserve the underlying Lie algebra.

1. Generalities on Poisson algebras

Let $\mathbb{K}$ be a field of characteristic 0. A $\mathbb{K}$-Poisson algebra is a $\mathbb{K}$-vector space $\mathcal{P}$ equipped with two bilinear products denoted by $x \cdot y$ and $\{x, y\}$, with the following properties:

1. The couple $(\mathcal{P}, \cdot)$ is an associative commutative $\mathbb{K}$-algebra.
2. The couple $(\mathcal{P}, \{\cdot,\cdot\})$ is a $\mathbb{K}$-Lie algebra.
3. The products $\cdot$ and $\{\cdot,\cdot\}$ satisfy the Leibniz rule:

$$\{x \cdot y, z\} = x \cdot \{y, z\} + \{x, z\} \cdot y$$

for any $x, y, z \in \mathcal{P}$.

The product $\{\cdot,\cdot\}$ is usually called Poisson bracket and the Leibniz identity means that the Poisson bracket acts as a derivation of the associative product.

Classical examples

- Let $M$ be a symplectic manifold. The space $C^\infty(M)$ of real-valued smooth functions over $M$ is an associative algebra for the classical product and a Poisson algebra. When the symplectic manifold is $\mathbb{R}^{2n}$ with the standard symplectic structure, then the Poisson bracket is

$$\{F, G\} = \sum_{i=1}^{n} \frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i}.$$

- Let $M$ be a contact manifold. The associative algebra $C^\infty(M)$ of real-valued smooth functions over $M$ is a Poisson algebra for the Poisson-Libermann bracket.

- Let $\phi$ be a polynomial of the commutative associative algebra $\mathbb{K}[X_1, X_2, X_3]$. Then the bracket

$$\{P, Q\}_\phi = (\frac{\partial \phi}{\partial X_1} \frac{\partial}{\partial X_2} \wedge \frac{\partial}{\partial X_3} + \frac{\partial \phi}{\partial X_2} \frac{\partial}{\partial X_3} \wedge \frac{\partial}{\partial X_1} + \frac{\partial \phi}{\partial X_3} \frac{\partial}{\partial X_1} \wedge \frac{\partial}{\partial X_2})(P, Q)$$

is a Poisson bracket on $\mathbb{K}[X_1, X_2, X_3]$. 

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• Let \( \mathfrak{g} \) be a real finite dimensional Lie algebra and let \( \mathfrak{g}^* \) be the dual vector space of \( \mathfrak{g} \). For any \((f, g)\) in \( C^\infty(\mathfrak{g}^*, \mathbb{R}) \), the product

\[
\{f, g\}(x) = x[ df(x), dg(x) ]
\]

is a Poisson bracket (here we identify \( \mathfrak{g} \) with its bidual). If we identify the symmetric algebra \( S(\mathfrak{g}) \) of \( \mathfrak{g} \) with the algebra of polynomials on \( \mathfrak{g}^* \), then \( S(g) \) is a Poisson algebra.

**Classification of complex Poisson algebras of dimension 2 and 3**

If \( e \) is an idempotent of the associative algebra, then the Leibniz rule implies that such an element is in the center of the Lie algebra corresponding to the Poisson bracket. In fact if \( e \) satisfies \( e \cdot e = e \), thus \( \{ e \cdot e, x \} = 2e \cdot \{ e, x \} = \{ e, x \} \). But if \( y \) is a non zero vector with \( e \cdot y = \lambda y \), thus

\[
(e \cdot e) \cdot y = e \cdot y = \lambda y = e \cdot (e \cdot y) = \lambda^2 y.
\]

This gives \( \lambda^2 = \lambda \), that is, \( \lambda = 0 \) or 1. Since we have \( e \cdot \{ e, x \} = 2^{-1} \{ e, x \} \), the vector \( \{ e, x \} \) is zero for any \( x \) and \( e \) is in the center of the Lie algebra corresponding to the Poisson bracket. This remark simplifies the determination of Poisson brackets when the associative product is given. In the following, we give the associative and Lie products in a fixed basis \( \{ e_i \} \) and the null products or the products which are deduced by commutativity or skew-symmetry are often not written.

**Dimension 2**

1. \( P^2_1 \) \( e_1 \cdot e_i = e_i, i = 1, 2; \ e_2 \cdot e_2 = e_2 \) \( \{ e_i, e_j \} = 0 \).
2. \( P^2_2 \) \( e_1 \cdot e_i = e_i, i = 1, 2 \) \( \{ e_i, e_j \} = 0 \).
3. \( P^2_3 \) \( e_1 \cdot e_1 = e_2 \) \( \{ e_i, e_j \} = 0 \).
4. \( P^2_4 \) \( e_1 \cdot e_1 = e_1 \) \( \{ e_i, e_j \} = 0 \).
5. \( P^2_5 \) \( e_1 \cdot e_j = 0 \) \( \{ e_1, e_2 \} = ae_2, a = 0 \) or \( a = 1 \).

**Dimension 3**

1. \( P^3_1 \) \( e_1 \cdot e_i = e_i, i = 1, 2, 3; \ e_2 \cdot e_2 = e_2; \ e_3 \cdot e_3 = e_3; \) \( \{ e_i, e_j \} = 0 \).
2. \( P^3_2 \) \( e_1 \cdot e_i = e_i, i = 1, 2, 3; \ e_2 \cdot e_2 = e_2; \ e_3 \cdot e_3 = e_2 - e_1; \) \( \{ e_i, e_j \} = 0 \).
3. \( P^3_3 \) \( e_1 \cdot e_i = e_i, i = 1, 2, 3; \ e_2 \cdot e_2 = e_2; \) \( \{ e_i, e_j \} = 0 \).
4. \( P^3_4 \) \( e_1 \cdot e_i = e_i, i = 1, 2, 3; \ e_3 \cdot e_3 = e_2; \) \( \{ e_i, e_j \} = 0 \).
5. \( P^3_5(a) \) \( e_1 \cdot e_i = e_i, i = 1, 2; \ e_2 \cdot e_2 = e_2; \ e_3 \cdot e_3 = e_3 \) or \( 0 \) \( \{ e_i, e_j \} = 0 \).
6. \( P^3_6 \) \( e_1 \cdot e_i = e_i, i = 1, 2, 3 \) \( \{ e_i, e_j \} = 0 \).
7. \( P^3_7(a) \) \( e_1 \cdot e_1 = e_1 \) \( \{ e_2, e_3 \} = e_3 \) or \( 0 \).
8. \( P^3_8 \) \( e_1 \cdot e_i = e_i, i = 1, 2 \) \( \{ e_i, e_j \} = 0 \).
9. \( P^3_9 \) \( e_1 \cdot e_1 = e_1; \ e_2 \cdot e_2 = e_3 \) \( \{ e_i, e_j \} = 0 \).
10. \( P^3_{10}(a, b) \) \( e_1 \cdot e_1 = e_2 \) \( \{ e_1, e_3 \} = ae_2 + be_3 \).
11. \( P^3_{11} \) \( e_1 \cdot e_1 = e_2; \ e_1 \cdot e_2 = e_3 \) \( \{ e_i, e_j \} = 0 \).
12. \( P^3_{12}(a) \) \( e_i \cdot e_j = 0 \) any Lie algebra.
2. Poisson algebra viewed as nonassociative algebra

In [8], one proves that any Poisson structure on a \( \mathbb{K} \)-vector space is also given by a nonassociative product, denoted by \( xy \) and satisfying the non associative identity

\[
3A(x, y, z) = (xz)y + (yz)x - (yx)z - (zx)y.
\]

where \( A(x, y, z) \) is the associator \( A(x, y, z) = (xy)z - x(yz) \). In fact, if \( \mathcal{P} \) is a Poisson algebra with associative product \( xy \) and Poisson bracket \( \{x, y\} \), then \( xy \) is given by \( xy = \{x, y\} + x\cdot y \).

Conversely, the Poisson bracket and the associative product of \( \mathcal{P} \) are the skew-symmetric part and the symmetric part of the product \( xy \). Thus it is equivalent to present a Poisson algebra classically or by this nonassociative product.

If \( \mathcal{P} \) is a Poisson algebra given by the nonassociative product (1), we denote by \( g_\mathcal{P} \) the Lie algebra on the same vector space \( \mathcal{P} \) whose Lie bracket is \( \{x, y\} = \frac{xy - yx}{2} \) and by \( A_\mathcal{P} \) the commutative associative algebra, on the same vector space, whose product is \( x \cdot y = \frac{xy + yx}{2} \).

In [5], we have studied algebraic properties of the non associative algebra \( \mathcal{P} \). In particular we have proved that this algebra is flexible, power-associative, and admits a Pierce decomposition.

3. Deformations of Poisson algebras

In this section, we mean by Poisson algebra a \( \mathbb{K} \)-algebra defined by a non associative product satisfying Identity (1). Let \( R \) be a complete local augmented ring such that the augmentation \( \varepsilon \) is with values in \( \mathbb{K} \). If \( B \) is a \( R \)-Poisson algebra, we consider the \( \mathbb{K} \)-Poisson algebra \( \overline{B} = \mathbb{K} \otimes_R B \) given by \( \alpha(\beta \otimes b) = \alpha \beta \otimes b \), with \( \alpha, \beta \in \mathbb{K} \) and \( b \in B \). It is clear that \( \overline{B} \) satisfies (1). A \( R \)-deformation of a \( \mathbb{K} \)-Poisson algebra \( A \) is a \( R \)-Poisson algebra \( B \) with a \( \mathbb{K} \)-algebra homomorphism

\[
\phi : \overline{B} \to A.
\]

A formal deformation of \( A \) is a \( R \)-deformation with \( R = \mathbb{K}[[t]] \), the local ring of formal series on \( \mathbb{K} \). We assume also that \( B \) is a \( R \)-free module isomorphic to \( R \otimes A \).

Let \( \mathbb{K}[\Sigma_3] \) be the \( \mathbb{K} \)-group algebra of the symmetric group \( \Sigma_3 \). We denote by \( \tau_{ij} \) the transposition exchanging \( i \) and \( j \) and by \( c \) the cycle \( (1, 2, 3) \). Every \( \sigma \in \Sigma_3 \) defines a natural action on any \( \mathbb{K} \)-vector space \( W \) by:

\[
\phi_\sigma : W^3 \to W^3
\]

\[
x_1 \otimes x_2 \otimes x_3 \mapsto x_\sigma(1) \otimes x_\sigma(2) \otimes x_\sigma(3).
\]

We extend this action of \( \Sigma_3 \) to an action of the algebra \( \mathbb{K}[\Sigma_3] \). If \( v = \Sigma_i a_i \sigma_i \in \mathbb{K}[\Sigma_3] \), then

\[
\phi_v = \Sigma_i a_i \phi_{\sigma_i}.
\]

Consider \( v_\mathcal{P} \) the vector of \( \mathbb{K}[\Sigma_3] \)

\[
v_\mathcal{P} = 3Id - \tau_{23} + \tau_{12} - c + c^2.
\]

Let \( \mathcal{P} \) be a Poisson algebra and \( \mu_0 \) its (non-associative) multiplication. Identity (1) writes

\[
(\mu_0 \circ_1 \mu_0) \circ \phi_{v_\mathcal{P}} - 3(\mu_0 \circ_2 \mu_0) = 0
\]

where \( \circ_1 \) and \( \circ_2 \) are the \( \text{comp}_i \) operations given by

\[
(\mu \circ_1 \mu')(x, y, z) = \mu(\mu'(x, y), z),
\]

\[
(\mu \circ_2 \mu')(x, y, z) = \mu(x, \mu'(y, z))
\]
for any bilinear map $\mu$ and $\mu'$.

**Theorem 1.** A formal deformation $B$ of the $\mathbb{K}$-Poisson algebra $A$ is given by a family of linear maps

$$\{\mu_i : A \otimes A \to A, \ i \in \mathbb{N}\}$$

satisfying

1. $\mu_0$ is the multiplication of $A$,
2. $(D_k)$: $\sum_{i+j=k, \ i,j \geq 0} (\mu_i \circ_1 \mu_j) \circ \phi_{vp} = 3 \sum_{i+j=k, \ i,j \geq 0} \mu_i \circ_2 \mu_j$ for each $k \geq 1$.

**Proof.** The multiplication in $B$ is determined by its restriction to $A \otimes A$ ([I]). We expand $\mu(a, b)$ for $a, b \in A$ into the power serie

$$\mu(x, y) = \mu_0(x, y) + t\mu_1(x, y) + t^2\mu_2(x, y) + \cdots + t^n\mu_n(x, y) + \cdots$$

then $\mu$ is a Poisson product if and only if the family $\{\mu_i\}$ satisfies the condition $(D_k)$.

**Remark.** As $R$ is a complete ring, this formal expansion is convergent. It is also the case if $R$ is a valued local ring (see [I]).

Let $\mathbb{K} = \mathbb{C}$ or an algebraically closed field. If $\{e_1, \cdots, e_n\}$ is a fixed basis of $\mathbb{K}^n$, we denote by $\mathcal{P}ois\mathcal{s}(\mathbb{K}^n)$ the set of all Poisson algebra structures on $\mathbb{K}^n$, that is, the set of constant structures $\{\Gamma^k_{ij}\}$ given by $\mu(e_i, e_j) = \sum_{k=1}^n \Gamma^k_{ij} e_k$. Relation (I) is equivalent to

$$\sum_{l=1}^n 3\Gamma^l_{ij} \Gamma^a_{lk} - 3\Gamma^a_{il} \Gamma^l_{jk} - \Gamma^l_{ik} \Gamma^a_{lj} - \Gamma^a_{jk} \Gamma^l_{li} + \Gamma^l_{ji} \Gamma^a_{lk} + \Gamma^a_{ki} \Gamma^l_{lj} = 0.$$

Thus $\mathcal{P}ois\mathcal{s}(\mathbb{K}^n)$ is an affine algebraic variety. If we replace $\mathcal{P}ois\mathcal{s}(\mathbb{K}^n)$ by a differential graded scheme, we call Deformation cohomology, the cohomology of the tangent space of this scheme.

**Remark.** This cohomology of deformation is defined in same manner for any $\mathbb{K}$-algebra and more generally for any $n$-ary algebra. If we denote by $H_{\text{def}}^n(A) = \oplus_{n \geq 0} H_{\text{def}}^n(A)$ the space of the deformations cohomology of the algebra $A$, then $H_{\text{def}}^0(A) = \mathbb{K}$, $H_{\text{def}}^1(A)$ is the space of outer derivations of $A$ and the coboundary operator $\delta_{\text{def}}^1$ corresponds to the operator of derivation, and the space of 2-cocycles is determinate by the linearization of the identities defining $A$. Thus, in any case, the three first spaces of cohomology are easy to compute. But the determination of the spaces $H_{\text{def}}^n(A)$ for $n \geq 3$ is not usually easy, we cannot deduce for example $H_{\text{def}}^3(A)$ directly from the knowledge of $H_{\text{def}}^2(A)$. However we have the following result

**Proposition 2.** Let $\mathcal{P}_A$ the quadratic operad related to $A$. If $\mathcal{P}_A$ is a Koszul operad, thus $H_{\text{def}}(A)$ coincides with the natural operadic cohomology.

For example, if $A$ is a Lie algebra or an associative algebra, the corresponding operads $\mathcal{L}ie$ and $\mathcal{A}ss$ are Koszul and $H_{\text{def}}(A)$ coincide with the operadic cohomology, that is, respectively, the Chevalley cohomology and the Hochschild cohomology. Examples of determination of $H_{\text{def}}^3(A)$ in the non-Koszul cases can be found in ([13], [3], [7]). A theory of deformations on non-Koszul operads is presented in ([9]).
We propose now to determinate the cohomology of deformations of Poisson algebras. In this case the operad is Koszul and we have to determinate this operadic cohomology. The condition \((D_1)\) writes
\[
(\mu_0 \circ_1 \mu_1 + \mu_1 \circ_1 \mu_0) \circ \phi_{\nu P} = 3(\mu_0 \circ_2 \mu_1 + \mu_1 \circ_2 \mu_0)
\]
that is
\[
3\mu_1(\mu_0(X,Y),Z) - 3\mu_1(X,\mu_0(Y,Z)) - \mu_1(\mu_0(X,Z),Y) - \mu_1(\mu_0(Y,Z),X) + \mu_1(\mu_0(Y,X),Z) + 3\mu_0(\mu_1(X,Y),Z) - 3\mu_0(X,\mu_1(Y,Z)) - \mu_0(\mu_1(X,Z),Y) + \mu_0(\mu_1(Y,X),Z) + \mu_0(\mu_1(Z,Y),X) = 0.
\]

**Some identities.** Let \(\varphi\) be a bilinear map on the Poisson algebra \((\mathcal{P}, \mu_0)\) (this means that \(\mu_0\) is the nonassociative product of the Poisson algebra \(\mathcal{P}\)). We consider the following trilinear maps:

1. \(\delta^2_P\varphi = (\mu_0 \circ_1 \varphi + \varphi \circ_1 \mu_0) \circ \phi_{\nu P} - 3\mu_0 \circ_2 \varphi + \varphi \circ_2 \mu_0\).
2. \(2\delta^2_C\varphi = (\varphi \circ_1 \mu_0 + \mu_0 \circ_1 \varphi) \circ \phi_{\nu L}\) with \(v_L = Id - \tau_{12} - \tau_{13} - \tau_{23} + c + c^2\).
3. \(2\delta^2_H\varphi = \varphi \circ_1 \mu_0 \circ \phi_{Id+\tau_{12}} - \varphi \circ_2 \mu_0 \circ \phi_{Id+\tau_{23}} + \mu_0 \circ_1 \varphi \circ \phi_{Id+c^2} - \mu_0 \circ_2 \varphi \circ \phi_{Id+c}\).
4. \(2\mathcal{L}_1(\varphi) = \varphi \circ_1 \mu_0 \circ \phi_{-3Id+3\tau_{12}} - \mu_0 \circ_1 \varphi \circ \phi_{3\tau_{23}+c} - \mu_0 \circ_2 \varphi \circ \phi_{3\tau_{12}+c}\).
5. \(2\mathcal{L}_2(\varphi) = \varphi \circ_1 \mu_0 \circ \phi_{-3Id+3\tau_{23}} + \mu_0 \circ_1 \varphi \circ \phi_{3\tau_{12}-c^2} + \mu_0 \circ_2 \varphi \circ \phi_{3\tau_{12}+c}\).

**Proposition 3.** For every bilinear map \(\varphi\), we have
\[
\delta^2_P\varphi = 4\delta^2_C\varphi + 6\delta^2_H\varphi + 2\mathcal{L}_1 + \mathcal{L}_2 - 2(\mathcal{L}_1 - \mathcal{L}_2)\tilde{\varphi}
\]
where \(\tilde{\varphi}(x,y) = \varphi(y,x)\).

**Proof.** This follows from a long but simple and direct computation.

For every bilinear map \(\varphi\) we denote by \(\varphi_a\) its skew-symmetric part, that is, \(\varphi_a = \frac{\varphi - \tilde{\varphi}}{2}\) and \(\varphi_s\) its symmetric part, that is, \(\varphi_s = \frac{\varphi + \tilde{\varphi}}{2}\). We deduce

**Corollary 4.** For any bilinear map \(\varphi\) we have
\[
(2) \quad \delta^2_P\varphi = 4(\delta^2_C\varphi_a + \delta^2_C\varphi_s + \delta^2_H\varphi_a + 2\delta^2_H\varphi_s + \mathcal{L}_1(\varphi_a) + \mathcal{L}_2(\varphi_s)).
\]

**Proposition 5.** Let \(\varphi\) be a bilinear map and \(\varphi_a\) and \(\varphi_s\) the skew-symmetric and the symmetric parts of \(\varphi\). We have:

\[
12\delta^2_C\varphi_a(X,Y,Z) = \delta^2_P\varphi(X,Y,Z) + 3\delta^2_P\varphi(Y,X,Z) + 3\delta^2_P\varphi(Z,Y,X) - \delta^2_P\varphi(X,Z,Y) + \delta^2_P\varphi(Y,Z,X) + \delta^2_P\varphi(Z,X,Y).
\]

\[
12\delta^2_H\varphi_s(X,Y,Z) = \delta^2_P\varphi(X,Y,Z) - \delta^2_P\varphi(Z,Y,X) + \delta^2_P\varphi(X,Z,Y) - \delta^2_P\varphi(Z,X,Y).
\]

**Proof.** The proof is a straightforward calculation.
4. Lie deformations of Poisson multiplication

Definition 6. We say that the formal deformation $\mu$ of the Poisson multiplication $\mu_0$ is a Lie formal deformation if the corresponding commutative associative multiplication is conserved, that is, if

$$\mu_0(x, y) - \mu_0(y, x) = \mu(x, y) - \mu(y, x)$$

for any $x, y$.

As $\mu(x, y) = \mu_0(x, y) + \sum_{n \geq 1} t^n \mu_n(x, y)$, if $\mu$ is a Lie deformation of $\mu_0$, then

$$\mu(x, y) - \mu(y, x) = \mu_0(x, y) - \mu_0(y, x) + \sum_{n \geq 1} t^n (\mu_n(x, y) - \mu_n(y, x)).$$

Thus

$$\sum_{n \geq 1} t^n (\mu_n(x, y) - \mu_n(y, x)) = 0$$

and

$$\mu_n(x, y) - \mu_n(y, x) = 0$$

for any $n \geq 1$. Each bilinear maps $\mu_n$ is skew-symmetric. In particular $\mu_1$ is skew-symmetric and $(\mu_1)_s = 0$. As $\delta_\mathcal{P}^2 \mu_1 = 0$, Relation (2) writes

$$\delta_\mathcal{C}^2 \mu_1 + \delta_\mathcal{H}^2 \mu_1 + \mathcal{L}_1(\mu_1) = 0.$$ 

But, from Proposition 6 $\delta_\mathcal{P}^2 \mu_1 = 0$ implies $\delta_\mathcal{C}^2 \mu_1 = 0$. Thus we have $\delta_\mathcal{H}^2 \mu_1 + \mathcal{L}_1(\mu_1) = 0$. But $\mathcal{L}_1(\mu_1)(x, y, z) = \mathcal{L}_1(\mu_1)(x, y, z) + \mathcal{L}_1(\mu_1)(y, z, x)$, that is,

$$\delta_\mathcal{H}^2 \mu_1 = \mathcal{L}_1(\mu_1) \circ \phi_{1d+c}.$$ 

Thus

$$\delta_\mathcal{H}^2 \mu_1 + \mathcal{L}_1(\mu_1) = \mathcal{L}_1(\mu_1) \circ \phi_{21d+c}.$$ 

We deduce

$$\mathcal{L}_1(\mu_1) \circ \phi_{21d+c} = 0$$

and this implies that

$$\mathcal{L}_1(\mu_1) = 0.$$

Theorem 7. If $\mu(x, y) = \mu_0(x, y) + \sum_{n \geq 1} t^n \mu_n(x, y)$ is a Lie deformation of the Poisson product $\mu_0$, then $\mu_1$ is a skew-symmetric map satisfying

$$\begin{cases} 
\delta_\mathcal{C}^2 \mu_1 = 0 \\
\mathcal{L}_1(\mu_1) = 0.
\end{cases}$$

Corollary 8. The deformation cohomology corresponding to Lie deformations of a Poisson multiplication is the Poisson-Lichnerowicz cohomology.

A survey of the definition of Poisson-Lichnerowicz cohomology can be read in [11].

Remark. Usually, one considers only Lie deformations of Poisson algebra. This is a consequence of the classical problem of considering Poisson algebras on the associative commutative algebra of differential functions on a manifold. In this context, the associative algebra is preserved when we consider deformations of Poisson structure on this algebra. Moreover, such an associative structure is rigid, then it is not appropriated to consider deformations
Definition 9. We say that the formal deformation $\mu$ of the Poisson multiplication $\mu_0$ is an associative formal deformation if the corresponding Lie multiplication is conserved, that is if

$$\mu_0(x, y) + \mu_0(y, x) = \mu(x, y) + \mu(y, x)$$

for any $x, y$.

As $\mu(x, y) = \mu_0(x, y) + \sum_{n \geq 1} t^n \mu_n(x, y)$, if $\mu$ is an associative deformation of $\mu_0$, then

$$\mu(x, y) + \mu(y, x) = \mu_0(x, y) + \mu_0(y, x) + \sum_{n \geq 1} t^n (\mu_n(x, y) + \mu_n(y, x)).$$

Thus

$$\sum_{n \geq 1} t^n (\mu_n(x, y) + \mu_n(y, x)) = 0$$

and

$$\mu_n(x, y) + \mu_n(y, x) = 0$$

for any $n \geq 1$. Each of the bilinear map $\mu_0$ is symmetric. In particular $\mu_1$ is symmetric and $(\mu_1)_a = 0$. As $\delta^2_P \mu_1 = 0$, Relation (8) writes

$$\delta^2_C \mu_1 + 2 \delta^2_P \mu_1 + L(\mu_1) = 0.$$

But, from Proposition 5 $\delta^2_P \mu_1 = 0$ implies $\delta^2_P \mu_1 = 0$. Thus

$$\delta^2_C \mu_1 + L(\mu_1) = 0.$$

If we denote by $\{x, y\}$ the Lie bracket associated with $\mu_0$, that is $\{x, y\} = \frac{\mu_0(x, y) - \mu_0(y, x)}{2}$, then

$$(\delta^2_C \mu_1 + L(\mu_1))(x_1, x_2, x_3) = 2\mu_1(x_1, x_2, x_3) + \mu_1(x_2, x_3, x_1) + \mu_1(x_1, x_2, x_3) - 2\mu_1(x_2, x_3, x_1) - \mu_1(x_1, x_2, x_3).$$

We deduce that

$$(\delta^2_C \mu_1 + L(\mu_1)) \circ \phi_{Id + c + c^2} = 0,$$

that is

$$\{\mu_1(x_1, x_2), x_3\} + \{\mu_1(x_2, x_3), x_1\} + \{\mu_1(x_3, x_1), x_2\} = 0.$$

Thus, Identity $$(\delta^2_C \mu_1 + L(\mu_1)) \circ \phi_{Id + c + c^2} = 0$$ implies

(3) $$\{\mu_1(x_1, x_2), x_3\} - \mu_1(x_1, x_3, x_2) - \mu_1(x_1, x_2, x_3) = 0.$$ 

Definition 10. Let $\mathcal{P}$ be a Poisson algebra and let $\{x, y\}$ be its Poisson bracket. A bilinear map $\varphi$ on $\mathcal{P}$ is called a Lie-biderivation if

$$\varphi(x_1, x_2, x_3) - \varphi(x_1, x_3, x_2) - \varphi(x_2, x_3, x_1) = 0$$

for any $x_1, x_2, x_3 \in \mathcal{P}$.

We deduce that $\mu_1$, which is a symmetric map, is a Lie biderivation.
Theorem 11. If $\mu(x, y) = \mu_0(x, y) + \sum_{n \geq 1} t^n \mu_n(x, y)$ is an associative deformation of the Poisson product $\mu_0$, then $\mu_1$ is a symmetric map such that

1. $\delta_1^2 \mu_1 = 0$.
2. $\mu_1$ is a Lie biderivation.

In case of Lie deformation of the Poisson product $\mu_0$, we have seen that the relations concerning $\mu_1$ can be interpreted in terms of Lichnerowicz-Poisson cohomology. We propose a same approach for the Lie deformations of $\mu_0$.

The Hochschild-Poisson cohomology. We denote by $x \cdot y$ the associative commutative product associated with the Poisson product $\mu_0$, that is $x \cdot y = \frac{\mu_0(x, y) + \mu_0(y, x)}{2}$.

Lemma 12. Let $\varphi$ be a symmetric bilinear map on $\mathcal{P}$ which is a Lie biderivation. If $\delta_1^3 \varphi$ denotes the trilinear map

$$\delta_1^3 \varphi(x_1, x_2, x_3) = x_1 \cdot \varphi(x_2, x_3) - \varphi(x_1 \cdot x_2, x_3) + \varphi(x_1, x_2 \cdot x_3) - \varphi(x_1, x_2) \cdot x_3,$$

then we have

$$\{\delta_1^3 \varphi(x_1, x_2, x_3), x_4\} = \delta_1^2 \varphi(\{x_1, x_4\}, x_2, x_3) + \delta_1^2 \varphi(x_1, \{x_2, x_4\}, x_3) + \delta_1^2 \varphi(x_1, x_2, \{x_3, x_4\})$$

for any $x_1, x_2, x_3, x_4 \in \mathcal{P}$.

Proof. As $\varphi$ is a Lie bi-derivation, we have

$$\{\varphi(x_1, x_2), x_3\} - \varphi(\{x_2, x_3\}, x_1) - \varphi(\{x_1, x_3\}, x_2) = 0.$$

Thus, using the definition of $\delta_1^2 \varphi$, we obtain

$$\{\delta_1^2 \varphi(x_1, x_2, x_3), x_4\} = \{x_1 \cdot \varphi(x_2, x_3), x_4\} - \{\varphi(x_1 \cdot x_2, x_3), x_4\} + \{\varphi(x_1, x_2 \cdot x_3), x_4\}$$

for any $x_1, x_2, x_3, x_4 \in \mathcal{P}$. As $\varphi$ is a Lie bi-derivation,

$$\{\delta_1^2 \varphi(x_1, x_2, x_3), x_4\} = x_1 \cdot \varphi(\{x_2, x_3\}, x_4) + x_1 \cdot \varphi(x_2 \cdot \{x_3, x_4\}, x_1) - x_3 \cdot \varphi(\{x_1, x_4\}, x_2)$$

for any $x_1, x_2, x_3, x_4 \in \mathcal{P}$.

But

$$\delta_1^2 \varphi(\{x_1, x_4\}, x_2, x_3) = \{x_1, x_4\} \cdot \varphi(x_2, x_3) - \varphi(\{x_2, x_3\}, x_1 \cdot x_2, x_3) + \varphi(\{x_1, x_4\}, x_2 \cdot x_3)$$

for any $x_1, x_2, x_3, x_4 \in \mathcal{P}$.

As the product $\cdot$ is commutative, we deduce

$$\{\delta_1^2 \varphi(x_1, x_2, x_3), x_4\} = \delta_1^2 \varphi(\{x_1, x_4\}, x_2, x_3) + \delta_1^2 \varphi(x_1, \{x_2, x_4\}, x_3) + \delta_1^2 \varphi(x_1, x_2, \{x_3, x_4\}).$$

We note that, this last identity is not a consequence of the symmetry of $\varphi$. It is satified for any bilinear Lie bi-derivation. Now, we can generalize these identities.
Definition 13. Let $\phi$ be a $k$-linear map on $P$. We say that $\phi$ is a Lie $k$-derivation if

$$\{\phi(x_1, \cdots, x_k), x_{k+1}\} = \sum_{i=1}^{k} \phi(x_1, \cdots, \{x_i, x_{k+1}\}, \cdots, x_k)$$

for any $x_1, \cdots, x_{k+1} \in P$, where $\{x, y\}$ denotes the Lie bracket associated with the Poisson product.

For example, from the previous lemma, if $\varphi$ is a Lie 2-derivation (or bi-derivation), then $\delta_H^2 \varphi$ is a Lie 3-derivation.

For any $(k - 1)$-linear map on $p$, let $\delta_H^{k-1} \varphi$ the $k$-linear map given by

$$\delta_H^{k-1} \varphi(x_1, \cdots, x_k) = x_1 \cdot \varphi(x_2, \cdots, x_k) - \varphi(x_1 \cdot x_2, \cdots, x_k) + \varphi(x_1, x_2 \cdot x_3, \cdots, x_k) + \cdots + (-1)^k \varphi(x_1, x_2, \cdots, x_{k-1} \cdot x_k) + (-1)^k \varphi(x_1, x_2, \cdots, x_{k-1} \cdot x_k).$$

This operator is the coboundary operator of the Hochschild complex related to the associative algebra $\text{Ass}_k$.

Theorem 14. If $\varphi$ is a Lie $k$-derivation of $P$, then $\delta_H^k \varphi$ is a Lie $(k + 1)$-derivation of $P$.

Proof. It is analogous to the proof detailed for $k = 3$. It depends only of the symmetry of the associative product $x \cdot y$.

Recall that a $k$-linear map $\varphi$ on a vector space is called symmetric if it satisfies $\varphi \circ \phi_V = 0$ where $V_k$ gives the symmetrizer, that is, $V_k = \sum_{\sigma \in \Sigma_k} \varepsilon(\sigma) \sigma = 0$.

Lemma 15. For any $k$-linear map $\varphi$ on $P$, the $(k + 1)$-linear map $\delta_H^k \varphi$ is symmetric.

Proof. In fact, let us consider the first and last terms of $\delta_H^k \varphi(x_1, \cdots, x_{k+1})$ that is

$$u = x_1 \cdot \varphi(x_2, \cdots, x_{k+1}) + (-1)^{k+1} \varphi(x_1, \cdots, x_k) \cdot x_{k+1}.$$  

The permutation $\sigma \in \Sigma_k$ defined by $\sigma(1) = k + 1, \sigma(i) = i - 1$ for $i = 2, \cdots, k$, satisfies

$$u + \varepsilon(\sigma)u = 0.$$  

As for any $\sigma' \in \Sigma_{k+1}$, we have

$$\varepsilon(\sigma')\sigma'(u) + \varepsilon(\sigma\sigma')\sigma \circ \sigma'(u) = 0,$$

thus

$$\phi_{V_k}(x_1 \cdot \varphi(x_2, \cdots, x_{k+1}) + (-1)^{k+1} \varphi(x_1, \cdots, x_k) \cdot x_{k+1}) = 0.$$  

We have the same process for all the terms of $\delta_H^k \varphi$. We deduce the theorem.

Let $C_{PH}^k(P, P)$ be the vector space constituted by $k$-linear maps on $P$ which are symmetric and which are Lie $k$-derivations. From the previous result, the image of the $C_{PH}^k(P, P)$ by the map $\delta_H^k$ is contained in $C_{PH}^{k+1}(P, P)$. As these maps coincide with the coboundary operators of the Hochschild complex, we obtain a complex $(C_{PH}^k(P, P), \delta_H^k)$ whose associated cohomology is called the Poisson-Hochschild cohomology.

Theorem 16. Let $P$ be a Poisson algebra whose (non-associative) product is denoted $\mu_0$. For any associative deformation $\mu = \sum_{n \geq 0} t^n \mu_i$ of $\mu_0$, the linear term $\mu_1$ is a 2-cocycle for the Poisson-Hochschild cohomology.
Example: Poisson structures on rigid Lie algebras. Such Poisson structures have been studied in [2], [12] and [5]. We will study these structures in terms of Poisson-Hochschild cohomology. Consider, for example, the 3-dimensional complex Poisson algebra given, in a basis \( \{ e_1, e_2, e_3 \} \), by

\[
e_1 e_2 = 2e_2, \quad e_1 e_3 = -2e_3, \quad e_2 e_3 = e_1.
\]

If \( \{ , \} \) and \( \cdot \) denote respectively the Lie bracket and the commutative associative product attached with the Poisson product, we have

\[
\{ e_1, e_2 \} = 2e_2, \quad \{ e_1, e_3 \} = -2e_3, \quad \{ e_2, e_3 \} = e_1
\]

and

\[
e_i \cdot e_j = 0,
\]

for any \( i, j \). If \( \varphi \) is a Lie biderivation, it satisfies

\[
\{ \varphi(e_i, e_j), e_k \} = \varphi(\{ e_i, e_k \}, e_j) + \varphi(\{ e_j, e_k \}, e_i).
\]

This implies \( \varphi = 0 \) and the Poisson algebra is rigid.

6. Poisson cohomology

6.1. The operadic cohomology. Let \( \mathcal{P}_{\text{poiss}} \) be the quadratic binary operad associated with Poisson algebras. Recall briefly its definition. Let \( E = \mathbb{K}[\Sigma_2] \) be the \( \mathbb{K} \)-group algebra of the symmetric group on two elements. The basis of the free \( \mathbb{K} \)-module \( F(E)(n) \) consists of the ”\( n \)-parenthesized products” of \( n \) variables \( \{ x_1, ..., x_n \} \). Let \( R \) be the \( \mathbb{K}[\Sigma_3] \)-submodule of \( F(E)(3) \) generated by the vector

\[
u = 3x_1(x_2x_3) - 3(x_1x_2)x_3 + (x_1x_3)x_2 + (x_2x_3)x_1 - (x_2x_1)x_3 - (x_3x_1)x_2.
\]

Then \( \mathcal{P}_{\text{poiss}} \) is the binary quadratic operad with generators \( E \) and relations \( R \). It is given by

\[
\mathcal{P}_{\text{poiss}}(n) = (F(E)/R)(n) = \frac{F(E)(n)}{R(n)}
\]

where \( R \) is the operadic ideal of \( F(E) \) generated by \( R \) satisfying \( R(1) = R(2) = 0, R(3) = R \). The dual operad \( \mathcal{P}_{\text{poiss}}^! \) is equal to \( \mathcal{P}_{\text{poiss}} \), that is, \( \mathcal{P}_{\text{poiss}} \) is self-dual. In [14] we defined, for a binary quadratic operad \( \mathcal{E} \), an associated quadratic operad \( \hat{\mathcal{E}} \) which gives a functor

\[
\mathcal{E} \otimes \hat{\mathcal{E}} \to \mathcal{E}.
\]

In the case \( \mathcal{E} = \mathcal{P}_{\text{poiss}} \), we have \( \hat{\mathcal{E}} = \mathcal{P}_{\text{poiss}}^! = \mathcal{P}_{\text{poiss}} \). All these properties show that the operad \( \mathcal{P}_{\text{poiss}} \) is a Koszul operad. In this case the cohomology of deformation of \( \mathcal{P}_{\text{poiss}} \)-algebras coincides with the natural operadic cohomology. An explicit presentation of the space of \( k \)-cochains is given in [10]:

\[
C^k(\mathcal{P}, \mathcal{P}) = \text{Lin}(\mathcal{P}_{\text{poiss}}(n)^! \otimes_{\Sigma_n} V^\otimes n, V)
\]

where \( V \) is the underlying vector space (here \( \mathbb{C}^n \)).

Remark. We proved in [6] that for any \( \mathbb{K}[\Sigma_3]^2 \)-associative algebra \( (\mathcal{A}, \mu) \) defined by the relation

\[
A^L_\mu \circ \phi_v - A^R_\mu \circ \phi_w = 0,
\]
with \( v, w \in \mathbb{K}[\Sigma_3] \), the cochains \( \varphi \in \mathcal{C}^i(\mathcal{A}, \mathcal{A}) \) can be chosen invariant under \( F_v^\perp \cap F_w^\perp \) (for the notations see [4]). For a Poisson algebra we have \( v = 1d, w = 3Id - \tau_{23} + \tau_{12} - c_1 + c_1^2 \). Then \( F_v^\perp \cap F_w^\perp = \{0\} \) and if \( \mathcal{C}^k(\mathcal{P}, \mathcal{P}) \) is the space of \( k \)-cochains of \( \mathcal{P} \), we obtain

\[
\mathcal{C}^k(\mathcal{P}, \mathcal{P}) = \text{End}(\mathcal{P}^\otimes k, \mathcal{P}).
\]

We can see that \( \text{End}(\mathcal{P}^\otimes k, \mathcal{P}) \) is isomorphic to \( \text{Lin}(\text{POiss}(n)^\dagger \otimes \Sigma_{\mathbb{K}} V^\otimes n, V) \). If \( \delta^k \mathcal{P} \) denotes the sequence of coboundary operators:

\[
\delta^k \mathcal{P} : \mathcal{C}^k(\mathcal{P}, \mathcal{P}) \to \mathcal{C}^{k+1}(\mathcal{P}, \mathcal{P}),
\]

the cohomology associated with the complex \( (\mathcal{C}^k(\mathcal{P}, \mathcal{P}), \delta^k \mathcal{P})_k \) is denoted by \( H^*_\mathcal{P}(\mathcal{P}, \mathcal{P}) \). To end the description of this complex, it remains to precise the coboundary operators.

**Notations.** Let \( \mathcal{P} \) be a Poisson algebra whose non-associative product \( \mu_0(X, Y) \) is denoted by \( X \cdot Y \). Let \( \mathfrak{g}_\mathcal{P} \) and \( \mathcal{A}_\mathcal{P} \) be its corresponding Lie and associative algebras. We denote by \( H^*_\mathcal{C}(\mathfrak{g}_\mathcal{P}, \mathfrak{g}_\mathcal{P}) \) the Chevalley cohomology of \( \mathfrak{g}_\mathcal{P} \) and by \( H^*_\mathcal{H}(\mathcal{A}_\mathcal{P}, \mathcal{A}_\mathcal{P}) \) the Hochschild cohomology of \( \mathcal{A}_\mathcal{P} \).

We will define coboundary operators \( \delta^k_\mathcal{P} \) on \( \mathcal{C}^k(\mathcal{P}, \mathcal{P}) \).

1) \( k = 0 \). We put

\[
H^0(\mathcal{P}, \mathcal{P}) = \{ X \in \mathcal{P} \text{ such that } \forall Y \in \mathcal{P}, X \cdot Y = 0 \}.
\]

2) \( k = 1 \).

For \( f \in \text{End}(\mathcal{P}, \mathcal{P}) \), we put

\[
\delta^1_\mathcal{P} f(X, Y) = f(X) \cdot Y + X \cdot f(Y) - f(X \cdot Y)
\]

for any \( X, Y \in \mathcal{P} \). Then we have

\[
H^1(\mathcal{P}, \mathcal{P}) = H^1_\mathcal{C}(\mathfrak{g}_\mathcal{P}, \mathfrak{g}_\mathcal{P}) \cap H^1_\mathcal{H}(\mathcal{A}_\mathcal{P}, \mathcal{A}_\mathcal{P}).
\]

3) \( k = 2 \).

For \( \varphi \in \mathcal{C}^2(\mathcal{P}, \mathcal{P}) \) we have, by linearization of \( (1) \)

\[
\delta^2_\mathcal{P} \varphi(X, Y, Z) = 3 \varphi(X \cdot Y, Z) - 3 \varphi(X, Y \cdot Z) - \varphi(X, Z, Y) - \varphi(Y \cdot Z, X)
\]

\[
+ \varphi(Y \cdot X, Z) + \varphi(Z \cdot X, Y) + 3 \varphi(X, Y) \cdot Z - 3 X \cdot \varphi(Y, Z)
\]

\[
- \varphi(X, Z) \cdot Y - \varphi(Y, Z) \cdot X + \varphi(Y, X) \cdot Z + \varphi(Z, X) \cdot Y.
\]

We have seen in Proposition 2 and its corollary that

\[
\delta^2_\mathcal{P} \varphi = 4 \delta^2_\mathcal{C} \varphi + 6 \delta^2_\mathcal{H} \varphi + 2 \delta^2_\mathcal{H} \varphi + 2(\mathcal{L}_1 + \mathcal{L}_2) \varphi - 2(\mathcal{L}_1 - \mathcal{L}_2) \varphi
\]

where \( \varphi(x, y) = \varphi(y, x) \) or if \( \varphi_a \) denotes its skew-symmetric part and \( \varphi_s \) its symmetric part

\[
\delta^2_\mathcal{P} \varphi = 4(\delta^2_\mathcal{C} \varphi_a + \delta^2_\mathcal{C} \varphi_s + \delta^2_\mathcal{H} \varphi_a + 2 \delta^2_\mathcal{H} \varphi_s + \mathcal{L}_1(\varphi_a) + \mathcal{L}_2(\varphi_s)).
\]

Suppose that the Poisson product satisfies \( X \cdot Y = -Y \cdot X \). Then \( \{X, Y\} = X \cdot Y \) and \( X \cdot Y = 0 \). If \( \varphi \in \mathcal{C}^2(\mathcal{P}, \mathcal{P}) \) is also skew-symmetric, then

\[
\delta^2_\mathcal{P} \varphi(X, Y, Z) = 2 \varphi(X \cdot Y, Z) + 2 \varphi(Y \cdot Z, X) - 2 \varphi(X \cdot Z, Y) + 2 \varphi(X, Y) \cdot Z + 2 \varphi(Y, Z) \cdot X - 2 \varphi(X, Z) \cdot Y
\]

\[
= \delta^2_\mathcal{H} \varphi(X, Y, Z).
\]

We recognize the formula of Lichnerowicz-Poisson differential.
A consequence of Proposition (5) is

**Corollary 17.** Let \( \varphi_s \) and \( \varphi_a \) be the symmetric and skew-symmetric parts of \( \varphi \in C^2(\mathcal{P}, \mathcal{P}) \). If \( \varphi \in Z^2(\mathcal{P}, \mathcal{P}) \), then \( \varphi_s \in Z^2_H(\mathcal{A}_\mathcal{P}, \mathcal{A}_\mathcal{P}) \) and \( \varphi_a \in Z^2_C(\mathbb{g}_\mathcal{P}, \mathbb{g}_\mathcal{P}) \).

6.2. **Relation between \( Z^2(\mathcal{P}, \mathcal{P}) \) and \( Z^2_H(\mathcal{A}_\mathcal{P}, \mathcal{A}_\mathcal{P}) \), \( Z^2_C(\mathbb{g}_\mathcal{P}, \mathbb{g}_\mathcal{P}) \).** In the previous section, we have defined the following operators

\[ L_1, L_2 : C^2(\mathcal{P}, \mathcal{P}) \to C^3(\mathcal{P}, \mathcal{P}) \]

given by

\[ L_1(\varphi)(X, Y, Z) = \varphi(X \cdot Y, Z) - \varphi(X, Z) \cdot Y - X \cdot \varphi(Y, Z) \]

and

\[ L_2(\varphi)(X, Y, Z) = -3\varphi(X, \{Y, Z\}) + \{\varphi(X, Y), Z\} - \{\varphi(X, Z), Y\}. \]

**Theorem 18.** Let \( \varphi \) be in \( C^2(\mathcal{P}, \mathcal{P}) \) and let \( \varphi_s \), \( \varphi_a \) be its symmetric and skew-symmetric parts. Then the following propositions are equivalent:

1. \( \delta^2_H \varphi = 0 \).
2. \( \{ i \} \delta^2_C \varphi_a = 0, \delta^2_H \varphi_s = 0. \)
   \( \{ ii \} \delta^3_C \varphi_s + \delta^3_H \varphi_a + L_1(\varphi) + L_2(\varphi_s) = 0 \)

**Proof.** 2 \( \Rightarrow \) 1 is a consequence of Corollary 17. 1 \( \Rightarrow \) 2 is a consequence of Corollary 17 and Relation (2).

**Applications.**

Suppose that \( \varphi \) is skew-symmetric. Then \( \varphi = \varphi_a \) and \( \varphi_s = 0 \). Then \( \delta^2_H \varphi = 0 \) if and only if \( \delta^2_C \varphi = 0 \) and \( \delta^3_H \varphi + L_1(\varphi) = 0 \). Moreless if we suppose that \( \varphi \) is a biderivation on each argument, that is, \( L_1(\varphi) = 0 \), then Theorem 18 implies that \( \delta^2_H \varphi = 0 \) if and only if \( \delta^2_H \varphi = 0 \).

But

\[ \delta^3_H \varphi(X, Y, Z) = \varphi(X, Y) \cdot Z - X \cdot \varphi(Y, Z) + \varphi(X \cdot Y, Z) - \varphi(X, Y \cdot Z) \]

\[ = L_1(\varphi)(X, Y, Z) + L_1(\varphi)(Y, Z, X) \]

Thus \( \delta^3_H \varphi = 0 \) as soon as \( L_1(\varphi) = 0 \).

**Proposition 19.** Let \( \varphi \) be a skew-symmetric map which is a biderivation, that is \( \varphi \) is a Lichnerowicz-Poisson 2-cochain. Then \( \varphi \in Z^2_L(\mathcal{P}, \mathcal{P}) \) if and only if \( \varphi \in Z^2_C(\mathbb{g}_\mathcal{P}, \mathbb{g}_\mathcal{P}) \).

Similarly, if \( \varphi \) is symmetric, then \( \delta^2_H \varphi = 0 \) if and only if \( \delta^2_C \varphi = 0 \) and \( \delta^3_C \varphi + L_2(\varphi) = 0 \).

6.3. **The case \( k=3 \).** Concerning the case \( k = 3 \), let \( \psi \) be a trilinear map. We denote by \( \psi_a \) and \( \psi_s \) the skew-symmetrized and the symmetrized trilinear part associated with \( \psi \). We have

\[ 48 \delta^3_C \psi_a(X_1, X_2, X_3, X_4) = \sum_{\sigma \in \Sigma_4} \varepsilon(\sigma) \delta^3_P \psi(X_\sigma(1), X_\sigma(2), X_\sigma(3), X_\sigma(4)) \]

\[ 48 \delta^3_H \psi_s(X_1, X_2, X_3, X_4) = \delta^3_P \psi(X_1, X_2, X_4, X_3) + \delta^3_P \psi(X_1, X_4, X_2, X_3) + \delta^3_P \psi(X_1, X_3, X_2, X_4) + \delta^3_P \psi(X_1, X_3, X_4, X_2) + \delta^3_P \psi(X_4, X_1, X_2, X_3) + \delta^3_P \psi(X_4, X_1, X_3, X_2) + \delta^3_P \psi(X_4, X_2, X_1, X_3) + \delta^3_P \psi(X_4, X_2, X_3, X_1) \]

\[ + \delta^3_P \psi(X_4, X_3, X_1, X_2) + \delta^3_P \psi(X_4, X_3, X_2, X_1). \]
Moreover the operators $\mathcal{L}_1$ and $\mathcal{L}_2$ can be extended to trilinear map.

$$\mathcal{L}_1(\psi)(X_1, X_2, X_3, X_4) = \psi(X_1X_2, X_3, X_4) + \psi(X_2X_1, X_3, X_4) - X_1\psi(X_2, X_3, X_4)$$

$$- \psi(X_2, X_3, X_4)X_1 - X_2\psi(X_1, X_3, X_4) - \psi(X_1, X_3, X_4)X_2.$$

In particular

$$\delta^3_P \psi = 0 \Rightarrow \delta^3_C \psi_a = \delta^3_H \psi_s = 0.$$

**References**

[1] Doubek, M.; Markl, M.; Zima, P. Deformation theory (lecture notes). Arch. Math. (Brno) 43 (2007), no. 5, 333–371.

[2] Goze Nicolas. Poisson structures associated with rigid Lie algebras. Journal of Generalized Lie theory and Applications. Vol 10. (2010).

[3] Goze Nicolas, Elisabeth Remm. Dimension theorem for Free (3)-ary partially associative algebras and applications. Preprint Mulhouse 2010.

[4] Goze, Michel; Remm, Elisabeth. Valued deformations of algebras. J. Algebra Appl. 3 (2004), no. 4, 345–365.

[5] Goze Michel, Remm Elisabeth. Poisson algebras in terms of non-associative algebras. J. Algebra 320 (2008), no. 1, 294–317.

[6] Goze Michel, Remm Elisabeth. A class of nonassociative algebras. Algebra Colloq. 14 (2007), no. 2, 313–326.

[7] Goze Michel, Elisabeth Remm. A class of nonassociative algebras including flexible and alternative algebras, operads and deformations. arXiv:0910.0700

[8] Markl Martin, Remm Elisabeth. Algebras with one operation including Poisson and other Lie-admissible algebras. J. Algebra 299 (2006), no. 1, 171–189.

[9] Markl Martin, Remm Elisabeth. Non)Koszulity of operads for $n$-ary algebras, cohomology and deformations. arXiv:math.RA 0907.

[10] Markl M., Sniider S., Stasheff J., Operads in algebra, topology and physics. Mathematical Surveys and Monographs, 96. American Mathematical Society, Providence, RI, 2002.

[11] Pichereau A., Poisson (co)homology and isolated singularities, *Journal of Algebra*, 299/2 (2006),747-777

[12] Remm Elisabeth. Non-Associative algebras, operads. Deformations of algebras on Non Koszul operads. Habilitation à diriger les recherches. Mulhouse. (2010)

[13] Remm Elisabeth. On the NonKoszulity of ternary partially associative Operads. Proceedings of the Estonian Academy of Sciences, 59, 4, (2010) 355–363.

[14] Elisabeth Remm, Michel Goze. On algebras obtained by tensor product. Journal of Algebra, Volume 327, Issue 1, 1 February 2011, Pages 13-30

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