Modules over a Polynomial Ring Obtained from Representations of a Finite-dimensional Associative Algebra

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Introduction

This paper, concerned with both commutative and noncommutative algebra, is devoted to studying an algebraic generalization of a problem which arose from applications of commutative algebra to linear PDEs with constant coefficients. These applications are rather old: an invariant way to describe a system of linear PDEs

\[
\begin{align*}
D_{11}f_1 + \cdots + D_{1m}f_m &= 0, \\
\cdots & \\
D_{n1}f_1 + \cdots + D_{nm}f_m &= 0,
\end{align*}
\]

where \(f_j\) being functions of several real variables, \(D_{ij}\) differential operators, is to consider a (left) module \(M\) over the ring of differential operators (a \(D\)-module) which is the quotient of a free module of rank \(m\) modulo the submodule generated by the rows of the matrix \((D_{ij})\). Then if we consider the ring of smooth functions (analytic functions, distributions) \(O\) as a module over the ring of differential operators, we easily see that the space of smooth (resp., analytical, generalized) solutions of our system can be identified with the space of homomorphisms of \(D\)-modules \(\text{Hom}(M, O)\): the generators of \(M\) are taken to functions \(f_j\), which satisfy the equations, because there are relations between the generators. But the ring of differential operators with constant coefficients is the ring of (commutative) polynomials in operators \(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\), partial derivations w.r.t. the variables, so in case of constant coefficients we obtain a module over a polynomial ring. Many meaningful properties of the solutions of a system of differential equations have a natural restatement in terms of commutative algebra of this module, see, e.g., [17].

The module corresponding this way to Cauchy–Fueter equations defining quaternionic-differentiable functions has appeared in [1]. This is \(M_n = R^4/\langle A_n \rangle\), where \(A_n\) is the matrix \(U_1| \ldots |U_n, \langle A_n \rangle\) the submodule generated by its columns,

\[
U_i = \begin{pmatrix}
x_i & y_i & z_i & t_i \\
y_i & x_i & t_i & -z_i \\
z_i & -t_i & x_i & y_i \\
t_i & z_i & -y_i & x_i
\end{pmatrix},
\]

and \(R = k[\{x_i, y_i, z_i, t_i\}_{i=1}^n]\) for a field \(k\).

The authors showed that the projective dimension of the module equals \(2n - 1\), whence they derived that the flabby dimension of the sheaf \(O\) of quaternionic-differentiable functions in \(n\) variables also equals \(2n - 1\) [1] Theorem 3.1] and that for any open set \(U \subset \mathbb{H}^n\) \(H^i(U, O) = 0\) for \(i \geq 2n - 1\) [1 Cor. 3.4].

The authors used some concepts and methods of commutative algebra that we are going to recall. See Section 0.1 for further details.
Definition. Let $R$ be a commutative Noetherian ring, $M$ an $R$-module. The sequence $a_1, \ldots, a_n \in R$ is called $M$-regular, if $(a_1, \ldots, a_n)M \neq M$ and for $i$ between $1$ and $n$ the multiplication by $a_i$ is injective on $M/(a_1, \ldots, a_{i-1})M$.

Definition. Let $R$ be a commutative Noetherian ring, $M$ an $R$-module, $I \subset R$ an ideal and $IM \neq M$. The length depth$(I, M)$ of any maximal $M$-regular sequence in $I$ is called depth of $I$ on $M$. When considering the depth of the ideal of the polynomials vanishing at the origin on graded modules over a polynomial ring, we shall omit the ideal and talk of the depth of a module.

Auslander–Buchsbaum formula. For a graded module $M$ over a polynomial ring $R$ one has $\text{depth } M + \text{pd } M = \dim R$.

To compute the projective dimension, the authors used the Auslander–Buchsbaum formula, not the explicit construction of the resolution, as they thought the latter too complicated. The depth of the module, which is needed to apply the Auslander–Buchsbaum formula and equals $2n + 1$, was calculated in [1] by means of an explicit construction of an $\mathcal{M}_n$-sequence, using Gröbner bases. The Krull dimension of $\mathcal{M}_n$ was also determined in the paper, to which end the tangential space to the support of the module in $\mathbb{C}^{4n} = \text{Specm } R$ for $k = \mathbb{C}$ was considered. This way the Cohen-Macaulayness of the module, i.e. the equality of the Krull dimension to the depth (Def. 0.1.6), was proved.

In a subsequent paper [2] the authors continued to investigate this module with Gröbner bases, finding the (graded) Betti numbers (i.e. the ranks and degrees of generators for the components of the graded minimal free resolution of $\mathcal{M}_n$), the Hilbert series (i.e. the dimensions of the homogeneous components of the module) and the multiplicity of $\mathcal{M}_n$ (i.e. the asymptotics of the dimension of a homogeneous component of the module as a function in its degree). We recall precise definitions and basic properties of these concepts in Section 0.3.

Similar studies of other systems of differential operators with constant coefficients were undertaken in [10], [18], [19].

As explained above, in [11] the matrix $A_n$ was obtained from the system of linear partial differential equations for quaternionic-differentiable functions by transposing and replacing the partial derivatives w. r. t. different variables by the variables themselves. Now one can see (cf. [2, Introduction]) that $U_i$ is the matrix of the left multiplication by $x_i - y i + z j - t k$ w. r. t. the basis $1, i, j, k$. This interpretation of $A_n$ allows, as E. S. Golod has observed, to understand completely the structure of the module $\mathcal{M}_n$, in particular that of its projective resolution. Let us complexify the algebra of quaternions. As a change of basis in the algebra leads to a conjugation of the matrix $U_i$ together with a linear change of variables in it, this change does not affect the structure of $\mathcal{M}_n$. Therefore the isomorphism $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong M_2(\mathbb{C})$ ($M_2(\mathbb{C})$ being the matrix algebra) allows, by choosing the basis consisting of matrix elements, to transform $U_i$ into

$$\begin{pmatrix}
    a_i & b_i \\
    c_i & d_i \\
    a_i & b_i \\
    c_i & d_i
\end{pmatrix},$$

i.e. $\mathcal{M}_n = \mathcal{M}_n' \oplus \mathcal{M}_n'$, $\mathcal{M}_n'$ being the quotient module of $R^2$ modulo the columns of a generic $2 \times 2n$-matrix. Now $\mathcal{M}_n'$ is known [8] to be a Cohen-Macaulay module of projective dimension $2n - 2 + 1 = 2n - 1$ and to have an Eagon-Northcott complex (the Buchsbaum-Rim complex of [14], see Section 0.2) for its minimal resolution. This description of the resolution allows to simplify the proofs of the main results of [2], see Chapter 3 below.
From these considerations the following generalization of the problem has emerged. Let \( A \) be a finite-dimensional associative algebra with identity over a field \( k \) with a \( k \)-basis \( f_1, \ldots, f_d \) and \( \rho: A \to M_n(k) \) be its matrix representation corresponding to an \( A \)-module \( M \), \( \dim_k M = n \). Let us fix a positive integer \( l \) and consider the polynomial ring \( R = \mathbb{k}[x_{11}, \ldots, x_{dl}] \) and the module \( F_l(M) \) over it (\( F_l^R(M) \), if the ring should be mentioned explicitly, as we shall sometimes consider a polynomial ring with some additional variables), which is the quotient of a free \( R \)-module \( R^n \) modulo the submodule generated by the columns of the matrices \( \text{Id}_j = \sum_i \rho(f_i)x_{ij}, \ j = 1, \ldots, l \) (\( \text{Id}_j^A \) if we need to mention explicitly the algebra \( A \) in order to avoid confusion). We explore the question of the Cohen-Macaulay property and the dimensions of modules \( F_l(M) \) and of their annihilators.

The answer to this question happens to be connected with the class of maximally central algebras introduced by Azumaya in [3, 4]:

**Definition** [3, §2]. A finite-dimensional associative algebra \( A \) with identity over a field \( k \) is called maximally central, if \( A \) is a direct sum of algebras \( A_i \), whose quotients modulo the radical are simple and

\[
\dim_k A_i \leq t_i^2 \dim_k Z(A_i),
\]

\( t_i^2 \) being the rank of \( A_i/\text{rad}A_i \) over its center and actually an equality takes place.

If \( t_i \) are the same for all the summands, we call a maximally central algebra equidimensional.

Further equivalent definitions of these algebras are collected in Section 0.7. See, e. g., [12] for the results of further development of the works of Azumaya that lead to the concept of Azumaya algebras.

In the present paper we prove

**Theorem 1.** Suppose that either \( l = 1 \) or \( A \) is maximally central. Then

1) \( F_l(\cdot) \) is an exact fully faithful functor from the category of finite-dimensional \( A \)-modules to the category of graded \( R \)-modules and homogeneous homomorphisms of degree 0;

2) if \( l = 1 \) or \( A \) is equidimensional with \( t_i = n \), then \( F_l(\cdot) \) transforms finite-dimensional \( A \)-modules into Cohen-Macaulay \( R \)-modules of projective dimension \((l - 1)n + 1 \) (which equals 1 for \( l = 1 \)), and for any maximally central algebra \( A \) \( F_l(\cdot) \) transforms indecomposable finite-dimensional \( A \)-modules into Cohen-Macaulay \( R \)-modules;

3) for \( l = 1 \) and every \( M \) the annihilator of \( F_l(M) \) is a principal ideal.

There is a converse statement:

**Theorem 2.** If for some \( l > 1 \) either \( F_l \) is exact or for all indecomposable modules \( M \) the modules \( F_l(M) \) are Cohen-Macaulay, then \( A \) is a maximally central algebra, and if the indecomposability condition is omitted, then \( A \) is an equidimensional maximally central algebra. Furthermore, for any (associative unitary) algebra \( A \), some \( A \)-module \( M \) and some \( l > 1 \) \( F_l(M) \) is Cohen-Macaulay, if and only if \( A/\text{ann}M \) is an equidimensional maximally central algebra.

The proof of Theorem 1 (namely, Lemma 1.9) supplies information on the minimal resolution of \( F_l(M) \), which allows to determine various invariants of these modules, in particular, to prove again and generalize the main results of [2]:

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Theorem 3. Suppose that a maximally central algebra $A$ is equidimensional and all $t_i$ equal $n$. Then for every $A$-module $M$ the invariants of $F_i(M)$ are equal to $\dim_k M/n$ times the invariants of $F_i(P)$ for a simple $\overline{A}$-module $P$, where $\overline{A} = A \otimes_k \overline{k}$ is a $\overline{k}$-algebra and $\overline{k}$ is the algebraic closure of $k$. Here the invariants of $F_i(P)$ have the following values:

- the Betti numbers $b_0 = n, b_1 = ln, b_i = \binom{ln}{n-i} \binom{n+i-3}{n-1}$ for $i \geq 2$, concentrated in degree $0, 1, n + i - 1$ respectively (i.e. the ranks of the modules $F_i$ in the minimal graded free resolution equal $b_i$ and each $F_i$ has generators only in one degree, namely, in degree $0$ for $i = 0, 1$ for $i = 1$, $n + i - 1$ for $i \geq 2$);

- the Cohen-Macaulay type $t = b(l-1)n+1 = \binom{ln-2}{n-1}$;

- the Hilbert series ($\sum_k \dim_k F_i(P)_k t^k$, $F_i(P)_k$ being the homogeneous component of $F_i(P)$ of degree $k$)

$$F_i(P)(t) = (1 - t)^{(l-1)n+1-\dim R} \sum_{i=0}^{n-1} \binom{ln}{i} (n - i) t^i (1 - t)^{n-1-i};$$

- the multiplicity

$$e = (\dim R - (l - 1)n - 2)! \lim_{k \to \infty} \dim_k F_i(P)_k / k^{\dim R - (l - 1)n - 2} = \binom{ln}{n - 1}.$$ 

Let us remark that the construction described in the beginning of the introduction makes it possible to associate some systems of PDEs with constant coefficients to the modules over polynomial rings that we consider, so it is possible to restate the results in terms of PDEs. Besides, the results of this work may be applied to quaternionic analysis: it would be interesting to obtain from the Eagon-Northcott complex, which serves as the minimal resolution for the module $M_n$ of [1], an explicit acyclic resolution for the sheaf of quaternionic-differentiable functions (a simile of the Dolbeault complex for holomorphic functions) and to supply by means of this resolution Theorem 3.1 and Corollary 3.4 of [1] with analytic proofs, that the authors lacked. Let us also remark that in [19] the authors have managed to study the corresponding modules by means of Gröbner bases only with machine computations for two and three “variables”, so maybe applying to this problem the methods of commutative algebra similar to those used in the present paper will result in further progress.

Chapter 0 is devoted to preliminaries: Sections 0.1–0.3 deal with commutative algebra and sections 0.5–0.7 with finite-dimensional associative algebras. In Section 0.7 equivalence of several definitions of a maximally central algebra is shown (Proposition 0.7.1), of which Definitions 2) and 7) seem to be new. There is also an example to show that Definition 7) is equivalent to the others only in case of a perfect field, and otherwise it is more restrictive. Three following chapters contain proofs of the respectively numbered theorems.

This is an English translation of the author’s Ph. D. thesis defended in December 2004 at The Department of Mechanics and Mathematics of the Moscow State (Lomonosov) University. The results have been published in [22], [23], [24], [25], [26], but the proof of the second part of Theorem 2 (the one concerning just one module) appears here for the first time. Also the exposition here is more compact and straightforward as in the papers above, unnecessary repetitions have been removed. I would like to thank my mentor Prof. E. S. Golod for his constant attention.
and instructive remarks, the anonymous referee of [23] for simplifying some arguments and also A. A. Gerko for a motivation to finish this work.
Notations

\( \mathbb{R} \) — the field of real numbers,
\( \mathbb{C} \) — the field of complex numbers,
\( \mathbb{H} \) — the skew-field of quaternions,
\( \mathbb{F}_p \) — the finite field with \( p \) elements,
\( M_n(K) \) — the ring of \( n \times n \) matrices with elements in a ring \( K \),
\( \mathfrak{k} \) — a field,
\( A \) — a finite-dimensional associative algebra with identity over a field \( \mathfrak{k} \),
\( k \) — the algebraic closure of \( \mathfrak{k} \),
\( A = A \otimes_k \overline{\mathfrak{k}} \),
\( f_1, \ldots, f_d \) — a vector space basis for \( A \) over \( \mathfrak{k} \),
\( R = \mathfrak{k}[x_{11}, \ldots, x_{d\ell}] \) — the ring of polynomial functions on an affine space \( A^\ell \),
\( \rho_M \) — the matrix representation of an algebra \( A \) corresponding to a finite-dimensional \( A \)-module \( M \),
\( \text{Id}_j \) or \( \text{Id}^A_j \) — a generic element of an algebra \( A \), see Introduction for a coordinate description, the beginning of Chapter 1 for an invariant one,
\( A|B \) — an \((m + n) \times k\) matrix obtained by writing the \( n \times k \) matrix \( B \) to the right of the \( m \times k \) matrix \( A \),
\( F_1(\cdot) \) or \( F_R(\cdot) \) or \( F_K(\cdot) \) — the functor we study, see Introduction for a coordinate description, the beginning of Chapter 1 for an invariant one; here \( R \) (maybe with primes) denotes the polynomial ring used in the construction, \( K \) (some letter that is not \( R \) with primes) the finite dimensional algebra over which the construction is performed,
\( S^i G \) — a symmetric power (of a free module over a commutative ring),
\( S(G) \) — the symmetric algebra of a vector space,
\( \bigwedge^i G \) — an exterior power (of a free module over a commutative ring),
\( G^* \) — the dual module (of a free module over a commutative ring),
\( \dim M \) — the Krull dimension (of a module \( M \) over a polynomial ring),
\( \text{supp} M \) — the support of a module over a commutative ring,
\( \dim_k M \) — the dimension of a vector space \( M \) over \( k \),
\( l(M) \) — the length of a module \( M \) over a finite-dimensional algebra,
\( \text{depth} M \) — the depth of the homogeneous maximal ideal in a polynomial ring on a graded module over this ring (see Section 1.1),
\( Q(K) \) — the field of fractions of a commutative ring \( K \),
\( k(p) = Q(K/p) \) — the residue field of a prime \( p \) in a commutative ring \( K \),
\( \text{pd} M \) — the projective dimension of a module over a polynomial ring,
\( M_i \) — the homogeneous component of degree \( i \) of a graded module \( M \) over a polynomial ring,
\( M[i] \) — a grading shift for a graded module over a polynomial ring \((M[i]_j = M_{i+j})\),
\( M(t) \in \mathbb{Z}[[t]] \) — the Hilbert series of a graded module \( M \) over a polynomial ring,
\( \text{ht } I \) — the height of an ideal in a commutative ring,
\( \text{Ass } M \) — the set of associated primes for a module over a commutative ring,
\( \text{ann } M \) — the annihilator of a module,
\( \mathbb{Z}(K) \) — the center of a ring \( K \),
\( K^0 \) — the opposite ring of a ring \( K \) (the additive group is the same as in \( K \), and the product \( ab \) in \( K^0 \) equals the product \( ba \) in \( K \)),
\( \text{rad } K \) — the radical of a finite-dimensional algebra \( K \),
\( \text{Br}(F) \) — the Brauer group of a field \( F \),
\( \text{Br}(F, L) \) — the subgroup in \( \text{Br}(F) \) comprising the classes of central simple algebras that split over the extension \( L \) of \( F \),
\( H^2(G, K^*) \) — the second group cohomology of a group \( G \) with coefficients in the multiplicative group of a field \( K \).
Chapter 0
Preliminaries

0.1 Depth and Cohen-Macaulayness

We use here the graded versions of these concepts, analogous to the local ones considered in [20], where depth is called “codimension homologique”. An exposition of the graded case can be found in [7, §1.5], but we tried to give references to sources available in Russian wherever possible.

Definition 0.1.1. ([14, Chap. 17, p. 423]; [20, chap. IV, A.4], M-suite.) Let $R$ be a commutative Noetherian ring, $M$ an $R$-module. A sequence $a_1, \ldots, a_n \in R$ is called $M$-regular, if $(a_1, \ldots, a_n)M \neq M$ and for $i$ between 1 and $n$ the multiplication by $a_i$ on the module $M/(a_1,\ldots,a_{i-1})M$ is injective.

Proposition 0.1.2. ([14, Theorem 17.4]) Let $I$ be an ideal in $R$ with $IM \neq M$. Then all maximal $M$-regular sequences in $I$ (i.e. those that cannot be continued) have the same length.

Definition 0.1.3. ([14, 17.2, p. 429], [20, chap. IV, A.4, définition 6].) Under the hypotheses of the previous proposition the length $\text{depth}(I, M)$ of any maximal $M$-regular sequence is called the depth of $I$ on $M$.

When considering the depth of the ideal of the polynomials vanishing at the origin on graded modules over a polynomial ring, we shall omit the ideal and talk of the depth of a module.

Proposition 0.1.4. ([8, Theorem 16.11], [14 Prop. 18.4, Cor. 18.6], [20 chap. IV, A.4, prop. 6 ff.].) For every finitely generated $R$-module $M$ one has:

$$\text{depth}(I, M) \leq \dim M, \text{ dim } M \text{ being the Krull dimension;}$$

$$\text{depth}(I, M) = \min \{i \mid \text{Ext}_R^i(N, M) \neq 0\}$$

for a finitely generated module $N$ with the support equal to the closed subset defined by $I$, so that the depth does not change if one replaces $I$ by its radical;

$$\text{depth}(I, M \oplus N) = \min \{\text{depth}(I, M), \text{depth}(I, N)\};$$

in a short exact sequence $0 \to M \to N \to P \to 0$ one has

$$\text{depth}(I, N) \geq \min \{\text{depth}(I, M), \text{depth}(I, P)\},$$

$$\text{depth}(I, M) \geq \min \{\text{depth}(I, N), \text{depth}(I, P) + 1\}.$$
Proposition 0.1.5 (the Auslander-Buchsbaum formula). ([20, chap. IV, D.1, prop. 21, [14 Exercise 19.8].) For a graded module $M$ over a polynomial ring $R$ one has $\text{depth} M + \text{pd} M = \dim R$.

Definition 0.1.6. ([20 chap. IV, B.1, déf. 1], see also [14, Chap. 18, p. 451].) A module $M$ over $R$ is called Cohen-Macaulay, if for every maximal ideal $m$ in $R$ one has $\text{depth}(m, M) = \dim M$.

Proposition 0.1.7. ([8, Prop. 16.20] Let $M$ be Cohen-Macaulay. Then a sequence $a_1, \ldots, a_s$ is $M$-regular iff $M/(a_1, \ldots, a_s)M \neq 0$ and

$$\dim M/(a_1, \ldots, a_s)M = \dim M - s,$$

(where for a Cohen-Macaulay $M$ and any sequence of length $s$ factorizing $M$ modulo the sequence reduces its dimension for at most $s$), and then the quotient module $M/(a_1, \ldots, a_s)M$ is Cohen-Macaulay.

0.2 The Eagon-Northcott Complex

Definition 0.2.1. ([8 2.C], $D_1(g)$; [14 A2.6.1, p. 600], $C^1$, the Buchsbaum-Rim complex.) Let $\varphi = (\varphi_{ij})$ be a $(g \times f)$-matrix determining a homomorphism of free modules $\varphi: F \to G$, and let the ranks of these modules be denoted by the corresponding lowercase letters. The number 1 Eagon-Northcott complex constructed from $\varphi$ is the complex

$$0 \to F_k \xrightarrow{\varphi_k} \ldots \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0$$

of free modules, where $k = f - g + 1$, $F_0 = G$, $F_1 = F$, $F_i = \bigwedge^{g+i-1}(F) \otimes S^{i-2}(G)^*$ for $i \geq 2$, and the differentials are as follows. Let $f_1, \ldots, f_f$ and $g_1, \ldots, g_g$ be the bases of $F$ and $G$ respectively, then

$$\varphi_1(f_j) = \sum \varphi_{ij}g_i, \quad \varphi_2(f_j \wedge \cdots \wedge f_{g_2}) = \sum_{k=0}^{g} (-1)^k M_{j_0 \ldots j_k \ldots j_g} f_{j_k}$$

(here and in the display below $\wedge$ denotes omitting an item from a list), $M_{j_1 \ldots j_g}$ being the $(g \times g)$-minor of $\varphi$ consisting of the columns $j_1, \ldots, j_g$ in the specified order, and

$$\varphi_1(f_{j_0} \wedge \cdots \wedge f_{j_g+i-2} \otimes (g_{i_1} \ldots g_{i_{g-2}})^*) =$$

$$= \sum_{k=0}^{g+i-2} (-1)^k \sum_{k'=1}^{i-2} \varphi_{ij_{k'}j_k} f_{j_0} \wedge \cdots \wedge \hat{f}_{j_k} \cdots \wedge f_{j_{g+i-2}} \otimes (g_{i_1} \ldots \hat{g}_{i_k} \ldots g_{i_{g-2}})^*.$$

Remark 0.2.2. In [14] similar number $l$ complexes for all integers $l$ are defined, but we need only this particular case.

For such complexes one has the following exactness criterion.
Proposition 0.2.3. ([8] Theorem 16.15], see also [2] Theorem], [14] Chap. 20.3).) Let \( R \) be a commutative Noetherian ring, \( M \neq 0 \) a finitely generated \( R \)-module,

\[
A = (0 \to F_k \xrightarrow{\varphi_k} F_{k-1} \to \ldots \to F_1 \xrightarrow{\varphi_1} F_0)
\]
a complex of finitely generated free \( R \)-modules. Let

\[ r(j) := \sum_{i=j}^{k} (-1)^{i-j} \text{rk} \, F_i \]

be the rank (i.e. the order of the maximal nonzero minor) of \( \varphi_j \) in the case when this complex is exact. Let \( I_r(\varphi) \) denote the ideal generated by all the \((r \times r)\)-minors of \( \varphi \). Then \( A \otimes_R M \) is exact, iff for all \( j \) between 1 and \( k \) \( I_{r(j)}(\varphi_j) \) contains an \( M \)-sequence of length \( j \) or \( I_{r(j)}(\varphi_j)M = M \).

The following fact offers a great simplification of this criterion for the Eagon-Northcott complex:

Proposition 0.2.4. ([14] Theorem A2.10 b] For any matrix \( \varphi \) the rank of \( \varphi_j \) in the corresponding Eagon-Northcott complex does not exceed \( r(j) \), and the ideal \( I_{r(j)}(\varphi_j) \) of the rank \( r(j) \) minors of \( \varphi_j \) lies in the ideal \( I_m(\varphi) \) of the maximal minors of \( \varphi \) and has the latter for its radical.

Therefore, to prove the exactness of a complex of type \( A \otimes_R M \), \( A \) being the Eagon-Northcott complex, it suffices to check that \( I_m(\varphi) \) contains an \( M \)-sequence of length \( f - g + 1 \): then Propositions 0.2.4 and 0.1.4 show this to be the case for all \( I_{r(j)}(\varphi_j) \), so Proposition 0.2.3 applies.

0.3 The Invariants of Modules over Commutative Rings

Definition 0.3.1. (cf. [14] Exercise A3.18]) Let \( R = \bigoplus_{i \geq 0} R_i \) be a graded Noetherian ring, \( R_0 = k \) a field, \( R_+ = \bigoplus_{i > 0} R_i \) the maximal homogeneous ideal, \( R[n] \) the free \( R \)-module with shifted grading (the generator having degree \(-n\)). Let \( M \) be a finitely generated graded \( R \)-module. It is known [14] 19.1; Theorem 20.2], that in this case there exists a unique free resolution

\[
\cdots \to F_i \to \cdots \to F_1 \to F_0 \to M \to 0
\]

over \( R \), such that \( F_i = \bigoplus_{j \geq 0} R[-j]^{b_{ij}} \), the homomorphisms are homogeneous of degree 0 and \( d(F_i) \subset R_+ F_{i-1} \). Graded Betti numbers of a graded module \( M \) are the \( b_{ij} \), Betti numbers \( b_i = \sum_j b_{ij} = \dim_k \text{Tor}^R_i(k, M) \). If for some \( i \) \( b_{ij} = 0 \) for every \( j \) except \( j_i \), we say that the \( i \)th Betti number is concentrated in degree \( j_i \).

Definition 0.3.2. ([14] Exercises 10.12–10.13] The Hilbert series \( M(t) \) of a graded module \( M \) is the series \( \sum \dim_k M_i t^i \).

Then one has \( M(t) = p(t)/(1 - t)^{\dim M} \) for a finitely generated module \( M \) over a polynomial ring and a polynomial \( p(t) \) with \( p(1) \neq 0 \). The Hilbert series is additive in short exact sequences of graded modules and homogeneous homomorphisms of degree 0, which follows from the additivity of dimension for homogeneous components of each degree. In particular, if \( x \) is a homogeneous nonzerodivisor of degree \( d \) on a graded module \( M \), the exact sequence

\[
0 \to M[-d] \xrightarrow{x} M \to M/xM \to 0
\]
shows that \((M/xM)(t) = (1 - t^d)M(t)\). Induction derives from this a formula for the Hilbert series of the quotient module modulo a homogeneous regular sequence, which we shall use.

**Definition 0.3.3.** The multiplicity of a module \(M\) (w. r. t. \(R_+\)) is \(e = p(1)\). The definition shows the mutiplicity to be also additive in short exact sequences. One can easily check that for graded modules this number coincides with the one introduced in \([20\text{ chap. } V, A.2]\) and \([14\text{ 12.1}\) (the last definition was included in the statement of Theorem 3).

**Definition 0.3.4.** (cf. \([14\text{ Exercise 21.14}\]) The Cohen-Macaulay type of a Cohen-Macaulay module \(M\) is the number \(t(M) = \dim_k \text{Ext}_R^d(M, M)\), where \(k = R/R_+\) as an \(R\)-module.

**Proposition 0.3.5.** Over a polynomial ring \(R\) one has \(t(M) = b_{pdR \cdot M}\).

**Proof.** Over a regular ring \(\text{Tor}_i^R(k, M)\) is isomorphic to \(\text{Ext}_R^{d-i}(k, M)\) \([20\text{ chap. IV, D.1, cor. 1 au théorème } 5]\), and the Auslander-Buchsbaum formula yields the required equality. \(\square\)

### 0.4 Gröbner Bases

As most expositions of the theory of Gröbner bases treat the case of ideals, not submodules as used here, we recall the basic statements from \([14\text{ Chap. 15}].\)

Let \(R\) be a polynomial ring over a field \(k\), and \(F\) a free \(R\)-module with a chosen basis \(e_1, \ldots, e_s\). A **monomial** in \(F\) is an element of the form \(m_i e_i\), \(m\) being a monomial in \(R\) (i.e., a product of powers of the variables). A **monomial order** in \(F\) is a total order on the set of all monomials in \(F\) such that if \(m_1 > m_2\) are two monomials in \(F\) and \(n \neq 1\) is a monomial in \(S\), one has \(nm_1 > nm_2 > m_2\). Every such order is Artinian (every nonempty subset has a least element).

Let us describe several ways to construct these orders that we shall use. Take any total order on the set of variables in \(R\). Then one can induce the lexicographic order on monomials in \(R\): compare the degrees w. r. t. the greatest variable, if they are equal, proceed to the next variable, and so on. One can induce the degree-lexicographic order: first compare the total degree of monomials, and in case of equality compare them lexicographically. The same can be done for noncommutative polynomials, where monomials are words, so the lexicographic order compares them letter-by-letter (in the noncommutative case, not all orders are Artinian, but this one is). Given an order on the monomials in the ring and a total order on the basis elements of a module, there are two ways to construct an order on the monomials in the module: “term over position”, comparing first the coefficients w. r. t. the order in the ring and then basis elements in case of equality, and “position over term”, comparing first the basis elements and then the coefficients.

The **initial term** of an element \(f = \sum a_i m_i \in F\), for \(a_i \in k^*\) and \(m_i\) being different monomials in \(F\), is \(a_0 m_0\), \(m_0\) being the greatest of the \(m_i\) involved. If \(M \subset F\) is a submodule, the **initial module** of \(M\) is the submodule in \(F\) generated by the initial terms of all the elements in \(M\). Then the images of the monomials in \(F\) not contained in the initial submodule of \(M\) constitute a vector space basis for \(F/M\). In particular, a homogeneous \(M\) has the same Hilbert series as its initial submodule, so the same is true for the quotient modules \([14\text{ Theorem } 15.26]\).

Suppose a monomial \(n\) involved in an element \(g\) with coefficient \(a\) is divisible by the initial term \(m\) of a element \(f\) (multiplying \(f\) by a scalar one can assume that the coefficient at \(m\) in \(f\) equals 1). Then the **reduction** of \(g\) by \(f\) is replacing \(g\) with \(g - a(n/m) f\). If no monomial in \(g\) is divisible by the initial terms of \(f_1, \ldots, f_k\), \(g\) is said not to be reducible by \(f_1, \ldots, f_k\). The remark
in Section 0.7 uses a version of reduction for two-sided ideals in a noncommutative polynomial ring: divisibility means that \( n = n_1mn_2 \) and reduction replaces \( g \) with \( g - an_1fn_2 \). The Artinian property of the order ensures that there can be no infinite sequence of reductions.

A set of elements \( g_1, \ldots, g_k \) in module \( M \) is called a Gröbner basis for \( M \) w. r. t. a given order, if the initial terms of these elements generate the initial submodule of \( M \) (in the homogeneous case this condition can be verified by means of Hilbert functions). Then these elements generate \( M \).

If \( m_1 \) and \( m_2 \) are two monomials in \( F \) containing the same basis element \( e_i \), the least common multiple \( m \) of these monomials is defined in an obvious fashion. If these monomials are the initial terms of \( f, g \in F \) respectively, the \( S \)-polynomial corresponding to the pair \((f, g)\) is the element \((m/m_1)f - (m/m_2)g \in F\).

Buchberger’s criterion [14, Theorem 15.8] says that the set of elements \( g_1, \ldots, g_k \in F \) is a Gröbner basis of the submodule they generate iff any \( S \)-polynomial corresponding to a pair of elements in this set (with the same basis element of \( F \) in their initial terms) can be reduced to zero by a sequence of reductions by the elements of this set, and then if we apply arbitrary reductions by \( g_i \) to an \( S \)-polynomial and after some steps obtain a element not reducible by \( g_1, \ldots, g_k \), this element is zero.

### 0.5 The Direct Sum Decomposition of Finite-Dimensional Algebras

A finite-dimensional associative algebra \( A \), considered as a left module over itself, can be decomposed into a direct sum of indecomposable left ideals [11, Theorem 14.2], and the decomposition into a direct sum of subalgebras is obtained from this one by grouping the summands [11, §§54–55]. Namely, two such ideals \( a \) and \( b \) are called linked [11, Def. 55.1], if there is such a chain of indecomposable left ideals \( a = a_1, a_2, \ldots, a_n = b \), that any two neighbouring ideals have a composition factor in common. Direct sums of linkage classes — called blocks — are indecomposable two-sided ideals in \( A \), they are uniquely determined and \( A \) is the direct sum of them [11, Theorem 55.2].

We shall apply this result in the case when every indecomposable \( A \)-module has only 1 type of composition factors. Then every block has only 1 type of composition factors, so the algebra is a direct sum of algebras, where each summand has only 1 simple module.

Let us as well recall the description of direct sum decompositions of an algebra in terms of idempotents [11, §25, exercise 2]: there is a one-to-one correspondence between the decompositions of algebra into a direct sum of two-sided ideals and the decompositions of identity into a sum of central orthogonal idempotents, i.e. idempotents from the center of the algebra with all pairwise products equal to 0. The correspondence is natural: idempotents are taken to the ideals they generate and the idempotents corresponding to a direct sum decomposition are the projections of identity into the summands.
0.6 Simple and Separable Algebras

We write $Z(B)$ for the center of a finite-dimensional algebra $B$, $\text{rad} B$ for its radical, $B^0$ for the opposite algebra (i.e. the same vector space with the multiplication $(b_1, b_2) \mapsto b_2 b_1$).

**Proposition 0.6.1.** [11, Theorem 68.1] Let $S$ be a simple algebra, $K$ its center and $L$ a field extension of $K$. Then $S \otimes_K L$ is a simple algebra with center $L$.

**Definition 0.6.2.** [11, Def. 71.1] A semisimple finite-dimensional algebra over a field is called separable, if it remains semisimple after any extension of the base field.

**Remark 0.6.3.** Any semisimple algebra over a perfect field is separable, because Theorem 69.4 of [11] guarantees it to remain semisimple after any finite (separable) extension of the base field, hence after taking the algebraic closure, for otherwise the nilradical will be defined over some finite extension. So over the algebraic closure of the base field this algebra is isomorphic to a direct sum of matrix ones, and Theorem 71.2 of [11] says that an algebra isomorphic over some extension of the base field to a direct sum of matrix algebras is separable.

**Proposition 0.6.4 (Wedderburn–Malcev Theorem).** [11, Theorem 72.19] Let $B$ be a finite-dimensional algebra over a field, such that $B/\text{rad} B$ is a separable algebra. Then $B$ contains a subalgebra $S$, so that one has a semi-direct sum decomposition $B = S \oplus \text{rad} B$.

**Proposition 0.6.5.** [15, Chap. IV, Theorem 4.4.2] If $S \subset A$ is a finite-dimensional simple subalgebra of an $L$-algebra $A$ containing the identity element of $A$ and $Z(S) = L$, then $A = S \otimes_{Z(S)} K$, $K$ being the centralizer of $S$ in $A$.

The following well-known lemmas will be used in the sequel.

**Lemma 0.6.6.** Suppose that all the simple quotients of an algebra $A$ are of dimension $n^2$ over their centers. Then the same is true for the algebra $\overline{A}$ obtained from $A$ by extending the base field to its algebraic closure.

**Proof.** As an extension of the base field takes nilradical into nilradical, one can suppose that $A$ is semisimple, and considering every summand in turn one can suppose that is is simple. Let $\overline{k}$ be the algebraic closure of $k$. Then

$$A \otimes_k \overline{k}/\text{rad} \overline{A} = A \otimes_{Z(A)} ((Z(A) \otimes_k \overline{k})/\text{rad}(Z(A) \otimes_k \overline{k})),$$

as the tensor product of a central simple algebra and a semisimple one is semisimple [11 Theorems 68.1 and 71.10]. But the second factor in this formula is a commutative semisimple algebra over $\overline{k}$, i.e. the sum of several copies of $\overline{k}$, and $A \otimes_{Z(A)} \overline{k}$ is a simple algebra of the same dimension over its center [11 Theorem 68.1].

**Lemma 0.6.7.** Let $A$ be a finite-dimensional associative unitary algebra over an algebraically closed field $k$ with its simple quotients of dimension $n^2$ over their centers. Then $A = M_n(K)$ for some algebra $K$. 

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Proof. As the field is algebraically closed, the semisimple quotient of $A$ is the sum of several copies of $M_n(k)$. Proposition 0.6.4 shows that this quotient can be embedded as a subalgebra of $A$, so that $A$ is a semi-direct sum of this subalgebra and the radical of $A$. As the radical is nilpotent, it is easy to see that the subalgebra contains the identity element of the whole algebra. If we embed $M_n(k)$ as the diagonal of this subalgebra, then the matrix algebra will be also embedded in $A$ as a subalgebra containing the identity of $A$. As the center of $M_n(k)$ equals $k$, one can apply Proposition 0.6.5 and conclude that $A = M_n(k) \otimes_k K = M_n(K)$. 

0.7 Maximally Central Algebras

Definition [3, §2]. A finite-dimensional associative algebra $A$ over $k$ with identity element is called maximally central, if $A$ is a direct sum of algebras $A_i$ the semisimple quotients of which are simple and

$$\dim_k A_i \leq t_i^2 \dim_k Z(A_i),$$

where $t_i^2$ is the rank of $A_i/\text{rad} A_i$ over its center and actually an equality takes place.

If the $t_i$ are the same for all the summands, we call a maximally central algebra equidimensional.

The definition of maximally central algebras is generalized in [4] to algebras over a Henselian ring that are free modules of finite rank over that ring, but we restrict ourselves to the case of finite-dimensional algebras over a field. This definition allows for a lot of equivalent restatements (in particular, the main results of this paper can be interpreted as such a restatement), mostly given in [3, 4]:

Proposition 0.7.1. The following conditions on a finite-dimensional associative algebra $A$ with 1 over a field $k$ are equivalent:

1) $A$ is a maximally central algebra;

2) if $A$ is projected onto any of its quotient algebras $B$, the center of $A$ is mapped surjectively onto the center of $B$;

3) the algebra $\overline{A}$ obtained from $A$ by extending the base field to its algebraic closure is a direct sum of matrix algebras over their centers;

4) the algebra $A \otimes_k L$ is maximally central, $L$ being an arbitrary field extension of $k$;

5) $A$ is a direct sum of algebras $A_i$ with centers $Z_i$ in such a way that $A_i$ are free $Z_i$-modules and $\text{End}_{Z_i} A_i = A_i \otimes_{Z_i} A_i^0$;

6) $A$ is an Azumaya algebra over its center $Z_i$, i.e. ([16] Chap. VI, §41, Definition 41.5) a projective $Z$-module such that for any prime ideal $p \subset Z$ $A \otimes Z_k(p)$ is a central simple algebra over $k(p)$.

If the base field is perfect, these conditions are equivalent to the following:

7) $A = \bigoplus_i S_i \otimes_{Z(S_i)} K_i$, where $S_i$ are simple algebras over $k$ and $K_i$ are commutative algebras over the corresponding $Z(S_i)$ ($A$ is “a linear combination of simple algebras with commutative coefficients”).
[12] Chap. 2, §§2,3] contains some equivalent definitions of Azumaya algebras, allowing to give some more restatements of condition 6).

Proof. The equivalence of 1), 3) and 4) is proved in [3, §2, Theorem 2, Corollary]. The equivalence of 1) and 5) is proved in [4] at the end of Section 4.

The equivalence of 5) and 6) follows from Theorem 15 of [4]. This theorem states that a \( \mathbb{Z}_i \)-algebra \( A_i \) that is a free \( \mathbb{Z}_i \)-module, \( \mathbb{Z}_i \) being a commutative ring, satisfies the property required from the algebra denoted this way in condition 5), iff for every maximal ideal \( \mathfrak{p} \subset \mathbb{Z}_i \), \( A_i/\mathfrak{p}A_i \) is a central simple algebra over \( \mathbb{Z}_i/\mathfrak{p} \). The center of \( A \) is a commutative Artinian ring that can be decomposed into a sum of local Artinian rings, and then the same idempotents give a decomposition of \( A \) into a direct sum of algebras with local centers, so the summands are indecomposable. The condition 6) is local w. r. t. the center, so it can be transferred to every summand and, as a finitely generated projective module over a local ring is free, it means that the summands are free over their centers. The condition 5) can also be transferred to summands, if one decomposes both sides of the equation into a sum of the modules over the summands of the center. Now the equivalence of these conditions for one summand is claimed in Theorem 15 of [4], as quoted above.

3) \( \Rightarrow \) 2): given a quotient algebra of \( A \), it is enough to check the surjectivity of the map of the centers for the corresponding quotient of \( A \) after passing to the algebraic closure of the field: (a faithfully flat) base change commutes with taking the center, as the latter is defined by linear equations. So it suffices to establish this condition for \( A \). This algebra is a direct sum of matrix algebras with commutative coefficients, so any two-sided ideal in it is a direct sum of ideals in the summands, and any two-sided ideal in a matrix algebra over a ring is the set of matrices with elements from some two-sided ideal in the ring (see [5, chap. VIII, §7, exercise 6 b)], where it is presented in generality proper to the author). So any quotient algebra is a direct sum of matrix algebras over some quotients of their centers. As the center of a matrix algebra over a commutative ring coincides with that ring, the map of the centers is surjective.

2) \( \Rightarrow \) 1): let us decompose the semisimple quotient of \( A \) as a direct sum of simple algebras. This decomposition is given by a complete family of central orthogonal idempotents in the semisimple quotient. As the center of \( A \) maps onto the center of the semisimple quotient by hypothesis, Theorem 24 of [4] (or Corollary 7.5 of [14]) allows one to lift this family to a complete family of orthogonal idempotents in the center of \( A \), the latter giving a decomposition of \( A \) into a direct sum of algebras, each of which has only one simple quotient and satisfies the condition 2).

For every summand \( A_i \) we show the formula by induction on Loewy length, i.e. the nilpotence degree of the nilradical. If the nilradical is zero, \( A_i \) is simple and its center is a field of, obviously, required dimension. Otherwise let \( (\text{rad} \ A_i)^n \) be the last non-zero power of the radical. Then by induction assumption the center of \( A_i/(\text{rad} \ A_i)^n \) has the dimension required. The center of \( A_i \) is mapped onto it surjectively with kernel equal to \( Z(A_i) \cap (\text{rad} \ A_i)^n \), and

\[
\dim_k A_i - \dim_k A_i/(\text{rad} \ A_i)^n = \dim_k(\text{rad} \ A_i)^n,
\]

so it suffices to show that

\[
\dim_k Z(A_i) \cap (\text{rad} \ A_i)^n = \dim_k(\text{rad} \ A_i)^n/\dim_{Z(S_i)} S_i
\]

for \( S_i = A_i/\text{rad} \ A_i \). The \( (\text{rad} \ A_i)^n \) under consideration is an \( A_i \)-bimodule, i.e. an \( A_i \otimes_k A_i^0 \)-module, but we can notice that, as the left action of \( Z(A_i) \) is the same as the right one, it is
actually a module over $A_i \otimes_{Z(A_i)} A_i^0$, and, as both actions of the radical are trivial and $Z(A_i)$ is mapped into the center of the (semi)simple quotient $S_i$, it is actually a module over the simple algebra $S_i \otimes_{Z(S_i)} S_i^0$, i.e. a direct sum of bimodules isomorphic to $S_i$. The intersection of the center of $A_i$ with the power of the radical under consideration is the set of the elements on which two actions of $S_i$ in this bimodule structure coincide, i.e. the direct sum of the centers of $S_i$. Hence this intersection has the dimension required.

Now we show that these conditions are equivalent to the last one over a perfect field. It can be easily seen that $\Box \Rightarrow \Box$ over any field. Let us derive condition $\Box$ from the other ones over a perfect field. By $\Box$ the map from $Z(A)$ into the center of the semisimple quotient of $A$ is surjective, so the system of central orthogonal idempotents giving the decomposition of the semisimple quotient into a sum of simple algebras can be lifted to a system of central orthogonal idempotents giving a decomposition of $A$ into a direct sum of algebras $A_i$ the semisimple quotients $S_i$ of which are simple. As the field is perfect, by Remark $0.6.3$ every semisimple algebra over it is separable, so by Wedderburn–Malcev Theorem $Z(S_i)$ is embedded into $Z(A_i)$ as a subalgebra with $1$, so $A_i$ can be regarded as an algebra over the field $Z(S_i)$. A finite extension of a perfect field, $Z(S_i)$ is itself perfect, so we can apply the Wedderburn–Malcev theorem over it and obtain an embedding of $S_i$ into $A_i$ as a $Z(S_i)$-subalgebra with $1$. Now we can apply Proposition $0.6.5$ and obtain that $A_i = S_i \otimes_{Z(S_i)} K_i$, $K_i$ being the centralizer of $S_i$ in $A_i$, so that $K_i \supseteq Z(A_i)$. The tensor product decomposition yields that $\dim_k K_i = \dim_k A_i/t_i^*$, and then the inequality from the definition of a maximally central algebra and the inclusion yield that $Z(A_i) = K_i$, so $K_i$ is commutative.

Remark 0.7.2. Applying condition $\Box$ and Lemma $0.6.6$ to every $A_i$ from the definition of a maximally central algebra, we obtain that equidimensional maximally central algebras are exactly those that become isomorphic to a direct sum of matrix algebras of the same rank over commutative ones after extending the base field to its algebraic closure, i.e. just to a matrix algebra over a commutative one.

Remark 0.7.3. Over a non-perfect field the last condition is not equivalent to the previous ones, as the following example shows.

Set $k = \mathbb{F}_p(x)$ for a variable $x$. Consider a purely inseparable extension $F = \mathbb{F}_p(t) = k[t]/(t^p - x)$ of this field. First we construct a finite-dimensional central skew-field over $F$ that cannot be obtained from such a field over $k$ by extension of scalars.

Let $L = \mathbb{F}_p(u) = F[u]/(u^p - u - t)$ be a cyclic Galois extension of degree $p$ over $F$, $\sigma$ a generator of the Galois group taking $u$ to $u + 1$. Then, according to $\Box$ 114], the elements of the Brauer group of $F$ that are trivial over $L$ are represented by cyclic algebras, i.e. algebras of the form $F\langle \sigma, u \rangle/(\sigma^p - u^p - u - t, \sigma u - (u + 1)\sigma)$ (the variables commuting with $F$ but not with one another), $\alpha$ being an element of $F^*$ $\Box$ 94, 4]. Furthermore, two such algebras are isomorphic iff the corresponding $\alpha$’s differ by a factor that is a norm in $L/F$, in particular, a cyclic algebra is the total ring of matrices iff $\alpha$ is a norm $\Box$ 114, Aufgabe 3]. Let us consider a cyclic algebra with $\alpha = t - 1$: it is a central simple algebra of dimension $p^2$ over $F$, so it is either a skew-field or a matrix algebra over $F$. As $t - 1 = u^p - u - 1$ is an irreducible polynomial in $\mathbb{F}_p[u]$, it is no norm (i.e. no product of $p$ conjugates) in the extension $L/F$, so our cyclic algebra is a non-trivial element of the Brauer group, so it is a skew-field. If this skew-field could be obtained from a skew-field over $k$ by extension of scalars, the skew-field over $k$ would have
rank $p^2$ and would lie in the $p$-torsion of $\text{Br}(k)$ by Theorem 4.4.5 of [15], which claims that the class of a skew-field in a Brauer group is annihilated by the square root of the rank of this skew-field. But the map $\text{Br}(k) \to \text{Br}(F)$ induced by the extension of scalars takes $p$-torsion to zero, as we are going to show. To this end, we use the fact that the Brauer group $\text{Br}(k)$ is the union of its subgroups $\text{Br}(k, K)$ formed by algebras that become isomorphic to a matrix algebra after tensoring with $K$, $K$ varying over all finite Galois extensions of $k$ [21 §113–114]. And $\text{Br}(k, K)$ is isomorphic to $H^2(G, K^*)$, $G$ being the Galois group of $K/k$ [6 §6, n°8, prop. 11] (this is essentially a reformulation of the description of Brauer group in terms of systems of factors). But, according to [6, §5, n°3, prop. 6], i.e. the $p$th power map in $K$ induces multiplication by $p$ in $\text{Br}(k, K)$. Now we notice that the $p$th power homomorphism induces an isomorphism of $F$ onto $k$, so it induces an isomorphism of Brauer groups, and its composition with the embedding $k \subset F$ is the $p$th power map (and can be continued as the $p$th power map to finite Galois extensions of $k$), so the homomorphism $\text{Br}(k) \to \text{Br}(F)$ becomes multiplication by $p$ after composing with an isomorphism, and it takes $p$-torsion to zero.

Thus we have constructed the required skew-field. Now we describe an example of a maximally central algebra over a perfect field that does not satisfy the condition [7] of Proposition 0.7.1. We take the field $k$ as above, set $B = k[\tilde{t}]/(\tilde{t}^p - x^p)$ and $A = B(\sigma, u)/(\sigma^p - (\tilde{t} - 1), u^p - u - \tilde{t}, \sigma u - (u + 1)\sigma)$ (here $\sigma, u$ commute with $B$). This algebra is a cyclic crossed product in the sense of [11, Section 6]. We see also that $a = \tilde{t}^p - x$ is a nilpotent of degree $p$ in $B$ and $B/(a) = F$, so that $A/(a)$ is the skew-field we constructed. Let us note also that $B \subset Z(A)$. If we set $\sigma > u$ and take the degree-lexicographic order, then the initial terms of the defining relations of $A$ are $\sigma^p, u^p, \sigma u$, and the monomials not divisible by them are only $u^i \sigma^j, 0 \leq i, j < p$, so $A$ is generated by $p^2$ monomials as a $B$-module. Hence the condition [11] of Proposition 0.7.1 is satisfied, and from this (from the fact that the inequality cannot be strict) it follows that $Z(A) = B$. If $A$ satisfied [7], then, as the quotient of $A$ modulo the nilradical is a skew-field, the sum would consist of just one summand, i.e. $A = S \otimes_{Z(S)} K$ for a simple $S$ and a commutative local $K$. Then $A/\text{rad}A = S \otimes_{Z(S)} K/\text{rad} K$, i.e. the skew-field we constructed is obtained from $S$ by extension of scalars from $Z(S)$ to $K/\text{rad} K$. Now, the center of our skew-field is $F$, and $[F : k] = p$, so $k \subset Z(S) \subset F$ implies that one of these inclusions is an equality. But the skew-field we constructed cannot be obtained by extension of scalars from a skew-field over $k$, thus $Z(S) = F$. Therefore $F$ should be embedded into $Z(A) = B$ as a $k$-subalgebra with 1, but it cannot be embedded this way: every preimage in $B$ of $t \in F$ has the form $\tilde{t} + ab$ for some $b \in B$, and $(\tilde{t} + ab)^p = \tilde{t}^p + a^p b^p = \tilde{t}^p = x + a \neq x$. 

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Chapter 1

Proof of Theorem 1

First we describe \( M \mapsto \mathcal{F}_l(M) \) as a functor. Let \( A \) be a finite-dimensional associative algebra with 1, \( M \) a finite-dimensional \( A \)-module. Let also \( \{f_i\} \) be a \( k \)-basis of \( A \), \( A^l \) the \( l \)-fold direct product of \( A \) with itself, considered as an algebraic variety (an affine space),

\[
R = k[A^l] = S((A^*)^l) = k[\{x_{ij} \mid i \in [1, \dim_k A], j \in [1, l]\}]
\]

its coordinate ring, where \( S \) stands for the symmetric algebra of a vector space and \( x_{ij} \) is the coefficient before \( f_i \) in the \( j \)th term of \( A^l \). We set

\[
\tilde{A} = R \otimes_k A, \quad \text{Id}_j = \sum_{i=1}^{\dim A} x_{ij} \otimes f_i \in \tilde{A}
\]

generic elements of \( A \), i.e. the images of the identity operator \( \text{Id} \in A^* \otimes_k A \) under the mapping of \( A^* \otimes_k A \) into \( \tilde{A} \), induced by the embedding of \( A^* \) onto the \( j \)th summand of \( (A^*)^l \subset S((A^*)^l) = R \), \( \tilde{M} = R \otimes_k M \) an \( \tilde{A} \)-module. (According to [13] I, §4, the coordinate construction of a generic element of an algebra \( A \) as an element in \( A \otimes_k Q(R) \), \( Q(R) \) being the quotient field of the polynomial ring \( R \) with \( l = 1 \), dates back to Kronecker.) Then

\[
\mathcal{F}_l(M) = \tilde{M}/(\text{Id}_1, \ldots, \text{Id}_l)\tilde{M} = \mathcal{F}_l(A) \otimes_{\tilde{A}} \tilde{M} = \\
= \mathcal{F}_l(A) \otimes_{\tilde{A}} (R \otimes_k A) \otimes_A M = \mathcal{F}_l(A) \otimes_A M,
\]

where \( \mathcal{F}_l(A) = \tilde{A}/(\text{Id}_1, \ldots, \text{Id}_l)\tilde{A} \) has the structure of a right \( \tilde{A} \)- (in particular \( A \)-) module induced from \( \tilde{A} \). This shows that \( \mathcal{F}_l(\cdot) \) is an additive \( k \)-linear right-exact functor from the category of \( A \)-modules into the category of graded \( R \)-modules, where the grading on \( \mathcal{F}_l(M) \) is defined as follows. If we consider \( M \) as a vector space concentrated in degree 0, then the grading of \( R \) and this grading of \( M \) define a grading on \( \tilde{M} \), w. r. t. which \( (\text{Id}_1, \ldots, \text{Id}_l)\tilde{M} \) is a homogeneous \( R \)-submodule, so this grading induces a grading on the quotient \( \mathcal{F}_l(M) \).

We remark that the fiber over \( (a_1, \ldots, a_l) \) of the sheaf on \( A^l \) corresponding to \( \mathcal{F}_l(M) \) is the vector space \( M/(a_1, \ldots, a_l)M \).

1.1 The \( l = 1 \) Case

We write simply \( \text{Id} \) for \( \text{Id}_1 \).
Lemma 1.1. For every $A$-module $M$ the sequence of $R$-modules

$$0 \rightarrow \widetilde{M} \xrightarrow{\text{Id}} \widetilde{M} \rightarrow F_1(M) \rightarrow 0$$

is exact.

Proof. Note that $F_1(M) = \widetilde{M}/\text{Id}{\widetilde{M}}$ and that one has to check only the exactness in the leftmost term, i.e. the vanishing of the kernel of $\text{Id}$ on $\widetilde{M}$. We prove this by contradiction: let $x \in \widetilde{M}$ be a nonzero element in $\text{Ker Id}$. Then $(\text{det Id})x = 0$, and as $\widetilde{M}$ is a free $R$-module and $R$ is a polynomial ring, one has $\text{det Id} = 0$. But if we give the variables $x_i$ such values $c_i \in k$ that $\sum c_if_i = 1_A$, we shall obtain that $\text{det Id}(c_1, \ldots, c_{\dim A}) = \text{det 1}_M = 1$, hence $\text{det Id} \neq 0$ — a contradiction. \qed

Remark 1.2. Actually we have just checked a particular case of the exactness criterion of Prop. 0.2.3 and also shown that $\text{det Id}$ is a regular element in $\text{ann } F_1(M)$. Therefore $\dim F_1(M) \leq \dim R - 1$, but our exact sequence is the minimal resolution of $F_1(M)$ over $R$, hence Prop. 0.1.5 gives that $F_1(M)$ is a Cohen-Macaulay module and $\text{pd}_R M = 1$, so part 2) of the theorem is proved.

Next we prove the exactness of $F_1(\cdot)$. Set $C = (0 \rightarrow \tilde{A} \xrightarrow{\text{Id}} \tilde{A} \rightarrow 0)$ to be a free resolution of $F_1(A)$ over $\tilde{A}$ and $A$ (according to Lemma 1.1), then for any $A$-module $M$ $H_i(C \otimes_A M) = \text{Tor}_i^A(F_1(A), M)$, but

$$C \otimes_A M = \left(0 \rightarrow \tilde{M} \xrightarrow{\text{Id}} \tilde{M} \rightarrow 0\right),$$

whence it follows by Lemma 1.1 that for every $M$ $\text{Tor}_i^A(F_1(A), M) = 0$. Now for every exact sequence of $A$-modules $0 \rightarrow M \rightarrow N \rightarrow P \rightarrow 0$ one has an exact sequence

$$0 = \text{Tor}_1^A(F_1(A), P) \rightarrow F_1(A) \otimes_A M \rightarrow F_1(A) \otimes_A N \rightarrow F_1(A) \otimes_A P \rightarrow 0,$$

as we needed.

Now we prove that $F_1(\cdot)$ is a fully faithful functor into the category of graded modules. The minimal resolution of $F_1(M)$ over $R$ is of the form

$$0 \rightarrow \widetilde{M} \xrightarrow{\text{Id}} \widetilde{M} \rightarrow F_1(M) \rightarrow 0,$$

and any homogeneous homomorphism of $R$-modules $\varphi : F_1(M) \rightarrow F_1(N)$ can be extended to a graded homomorphism of their minimal resolutions:

$$\begin{array}{c@{}c@{}c@{}c@{}c@{}c}
0 & \rightarrow & \tilde{M} & \xrightarrow{\text{Id}} & \tilde{M} & \rightarrow F_1(M) & \rightarrow 0 \\
\downarrow{\tilde{\varphi}} & & \downarrow{\tilde{f}} & & \downarrow{\varphi} & & \\
0 & \rightarrow & \tilde{N} & \xrightarrow{\text{Id}} & \tilde{N} & \rightarrow F_1(N) & \rightarrow 0.
\end{array}$$

The homomorphisms $\tilde{f}$ and $\tilde{\varphi}$ between free modules are given by matrices over $R$, and as these homomorphisms are homogeneous of degree 0, the matrices are actually over $k$, i.e. $\tilde{f}$
and $\tilde{g}$ result from homomorphisms of vector spaces $f, g: M \to N$. The commutativity of the diagram yields $\tilde{f} \circ \text{Id} = \text{Id} \circ \tilde{g}$. Substituting into $\text{Id}$ the values $a_i$, for which $\sum a_i f_i = 1_A$, we obtain $f = g$, and substituting all other values we obtain that $f \in \text{Hom}_A(M, N)$. As $F_i(f) = \varphi$, the map between the spaces of homomorphisms is onto. Since after choosing a basis in $M$ and projecting it to a $k$-basis in $F_i(M)/mF_i(M)$ with $m = (x_1, \ldots, x_{\dim A})$, $\tilde{f}$ and $(R/m) \otimes \varphi$ are given by the same matrix, the matrix of $f$ can be recovered from $\varphi$ and the map between the spaces of homomorphisms is injective.

**Remark 1.3.** For non-isomorphic $M$ and $N$ the modules $F_i(M)$ and $F_i(N)$ are not isomorphic even as ungraded modules, as the homogeneous component of degree $0$ of an isomorphism is itself an isomorphism.

**Remark 1.4.** If we represent $k[A]$ as the quotient of the ring $k[A^t]$ modulo the ideal generated by the variables $x_{ij}$ for $j \geq 2$, we see that

$$F_i(M) \otimes_{k[A^t]} k[A] = F_i(M),$$

since this multiplication does not change $\text{Id}_1$, taking all other $\text{Id}_{ij}$ to zero. As the modules $F_i(M)$ are generated in degree $0$, an element in $\text{Hom}^0_R(F_i(M), F_i(N))$ is given by a matrix in $\text{Hom}_k(M, N)$, i.e. the functors $F_i(\cdot)$ and the tensor product are injective on morphisms. As their composition is bijective on morphisms, we obtain the fact that $F_i(\cdot)$ is fully faithful as a functor into the category of graded modules for any associative algebra $A$ with $1$.

We prove part 3. First we remark that the minimal number of generators of the annihilator of a module is preserved under extension of the base field, so until the end of the proof of part 3 we assume $k$ to be algebraically closed.

**Lemma 1.5.** If $M$ is a simple $A$-module, then $\text{ann} F_i(M)$ is a principal prime ideal.

**Proof.** As $M$ is a simple $A$-module, it is a module over a simple summand of the semisimple quotient algebra of $A$, i.e. the module $k^n$ over $M_n(k)$, according to the theory of finite-dimensional associative algebras ([21, Kap. 13–14], [11]). Then $F_i(M) = R^n/\langle A \rangle$ with $A = (a_{ij})$, $a_{ij}$ being independent variables. Now for such a module one knows that, $\det A$ being irreducible, the ideal $(\det A)$ is prime and equals $\text{ann} F_i(M)$ ((\det A) \subset \text{ann} F_i(M)$, $(\det A)$ is a prime ideal of height $1$, and $\text{pd}_R F_i(M) = 1 \Rightarrow \text{ht} \text{ann} F_i(M) \leq 1$).

Let us prove part 3 by induction on the length of the $A$-module $M$. For $l(M) = 1$ this is Lemma 1.3. If $l(M) > 1$, then there is a simple submodule $N \subset M$. Then part 2 gives us an exact sequence $0 \to F_1(N) \to F_1(M) \to F_1(P) \to 0$. As $l(P) < l(M)$, by the induction assumption $\text{ann} F_1(P)$ is a principal ideal, say $(a)$. Take an $x \in \text{ann} F_1(M)$, then $x$ annihilates $F_1(P)$. Hence $x = ay$, where $y$ annihilates $aF_1(M)$. Now $aF_1(M)$ is a submodule of $F_1(N)$; if $aF_1(M) = 0$, then $x$ annihilates $F_1(M)$ iff $x \in (a)$, so ann $F_1(M) = (a)$; but if $aF_1(M) \neq 0$, then $\text{Ass} aF_1(M) \subset \text{Ass} F_1(N)$. But $\text{Ass} F_1(N) = \{ (p) \}$ for $(p) = \text{ann} F_1(N)$, because by part 2, $F_1(N)$ is Cohen-Macaulay, so $\text{Ass} F_1(N)$ coincides with the set of the minimal primes over $\text{ann} F_1(N)$, and $\text{ann} F_1(N)$ is prime. Then, as all the associated primes of $aF_1(M)$ contain its annihilator, one has $\text{ann} aF_1(M) \subset (p) = \text{ann} F_1(N)$, but $aF_1(M)$ is a submodule of $F_1(N)$, therefore $\text{ann} F_1(N) \subset \text{ann} aF_1(M)$. Hence $\text{ann} aF_1(M) = (p)$. Thus $\text{ann} F_1(M) = a \text{ann} aF_1(M) = (ap)$. 

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Remark 1.6. Let us look at this construction under more general settings. Assume that we consider representations of a vector space \( V = \langle f_1, \ldots, f_d \rangle_k \) with \( 1 = f_1 \), i.e. with an element that acts as identity in any representation. Then we take for \( A \) a free algebra \( k\langle f_2, \ldots, f_d \rangle \) and consider the representations of \( V \) as \( A \)-modules. In this case the description of the functor \( F_1(\cdot) \) remains the same (\( R = k[V] = k[x_1, \ldots, x_d] \)) and the same proofs are valid for two first parts of Theorem 1, only the fact that \( C \) is a resolution for \( F_1(A) \) does not follow from Lemma 1.1 but is obtained by considering the lowest homogeneous component of \( \text{Id} u \) w. r. t. the grading of the free algebra \( A \), as this component equals the lowest component of \( v \) times \( x_1 \).

The presence of the identity is essential for the proofs of parts 1) and 2): for example, if one takes the adjoint representation of a non-Abelian finite-dimensional Lie algebra and constructs the corresponding module, it won't be Cohen-Macaulay, but will be of Krull dimension \( \dim R \) and of projective dimension \( \geq 2 \): let \( L \) be the Lie algebra, then \( \text{Id} \in \tilde{L}, F_1(L) = \tilde{L}/[\text{Id}, \tilde{L}] \), hence

\[
\tilde{L} \xrightarrow{[\text{Id}, \cdot]} \tilde{L} \rightarrow F_1(L) \to 0
\]

is the beginning of the minimal resolution of \( F_1(L) \) over \( R \) and the resolution continues to the left, because \( \text{Id} \in \text{Ker}[\text{Id}, \cdot] \). But \( \dim F_1(L) = \dim R \), for, as observed by the referee of [23], \( \text{supp} F_1(L) \) in \( k^{\dim L \cong L} \) equals

\[
\{ x \in L \mid L/[x, L] \neq 0 \} = L,
\]

because \( \forall x \in L \dim_k L - \dim_k [x, L] = \dim_k \text{Ker}[x, \cdot] > 0 \).

1.2 The \( l > 1 \) Case

First of all, we remark that a maximally central algebra is by definition a direct sum of equidi-

mensal maximally central ones, the modules over it are direct sums of modules over the

summands and the same is true for short exact sequences and homomorphisms (i.e. for cate-

gories). In particular, indecomposable modules are modules over one of the summands. So it

suffices to prove the theorem for equidimensional maximally central algebras: it suffices as well

to check the exactness of the functor over each of the summands.

Then come some general remarks on the extension of scalars.

Lemma 1.7. 1) The functor \( F_1(\cdot) \) commutes with the extension of scalars.

2) The module \( F_1(M) \) is Cohen-Macaulay after the extension of scalars iff it was Cohen-

Macaulay before the extension.

3) Statement 1) of Theorem 4 is satisfied after the extension of scalars iff it was satisfied

before the extension also.

Proof.

1) Evident from the construction.
2) We note that for every graded $R$-module $D$ extension of scalars preserves the minimal graded resolution of $D$ over $R$, hence the projective dimension and the Krull dimension (as the Hilbert function of a module is the Euler characteristic of the resolution w. r. t. the Hilbert function) of $D$, so the presence or absence of Cohen-Macaulayness as well (according to prop. 0.1.5 and 0.1.7) $D$ is Cohen-Macaulay $\Leftrightarrow$ pd $D + \dim D = \dim R$.

3) Hom and homology commute with the extension of scalars.

Thus it suffices to prove the theorem for an equidimensional maximally central algebra and after passing to the algebraic closure of the base field. So by Remark 0.7.2 it suffices to prove the theorem for algebras of the form $M_n(K)$, $K$ being commutative, over an algebraically closed field $k$, and this will occupy the rest of the section.

Note that for $A = M_n(K)$ the tensor multiplication by the $(A, K)$- and $(K, A)$-bimodule $K^n$ defines an equivalence between the categories of $A$-modules and $K$-modules (Morita-equivalence). We now describe the functor which associates to an $A$-module $M$ the minimal resolution of $F_i(M)$ over $R$.

**Notations 1.8.** The ring $R$ can be described as $k[[\{x_{aij} \mid \alpha \in [1, \dim_k K], i \in [1, n], j \in [1, ln]\}]]$. We set $R[T] = R[[T_{ij} \mid i \in [1, n], j \in [1, ln]]]$, $\varphi = (T_{ij})$ an $(n \times ln)$-matrix, $C$ the number 1 Eagon-Northcott complex corresponding to $\varphi$ (Def. 1.2.1), $\widetilde{K} = K \otimes_k R$, $\text{Id}_{ij}^K = \sum_\alpha f_\alpha x_{aij} \in \widetilde{K}$, $f_\alpha$ being a $k$-basis for $K$. Let $\mathcal{K}$ denote the $(\widetilde{K}, \widetilde{A})$-bimodule $\widetilde{K}^n$. The operation of $T_{ij}$ on $\mathcal{K}$ by means of the commuting $(K$ being commutative) endomorphisms $\text{Id}_{ij}^K$ defines an $R[T]$-module structure on $\mathcal{K}$. Set $\mathcal{C} = C \otimes_{R[T]} \mathcal{K}$, a complex of projective (as direct sums of $K$) right $\widetilde{A}$-modules.

Then $F_i(M) = H_0(C \otimes_A M)$ for every $M$: is is $\text{Coker}(\widetilde{M}^{ln} \to \widetilde{M}^n)$ with $\widetilde{M} = \mathcal{K} \otimes_{\widetilde{A}} \widetilde{M}$, and w. r. t. $R$-bases this homomorphism is given by a block matrix $\varphi = (\text{Id}_{ij}^K)$. The complex $\mathcal{C} \otimes_A M = C \otimes_{R[T]} \widetilde{M}$ is a minimal complex of length $(l - 1)n + 1$ and it is the minimal resolution of $F_i(M)$ and $\mathcal{C}$ is a projective resolution of $F_i(A)$ as a right $\widetilde{A}$- (and $A$-) module by virtue of the following lemma:

**Lemma 1.9.** For any $M$ and any $i > 0$ $H_i(\mathcal{C} \otimes_A M) = 0$.

**Proof.** According to the exactness criterion for the Eagon-Northcott complex (Subsection 0.2), it suffices to prove that the ideal $I_\alpha(\varphi)$ of the maximal minors of $\varphi$ contains an $\widetilde{M}$-sequence of legth $(l - 1)n + 1$. This sequence, as we show, consists of the minors corresponding to $n$ columns in succession. $\widetilde{M}$ is a Cohen-Macaulay $R$-module, hence a Cohen-Macaulay $R[T]$-module (as a graded module over a graded ring, where the new variables $T_{ij}$ also have degree 1), because its Krull dimension, determined from the Hilbert function of the grading, is the same over both rings, and a maximal homogeneous $\widetilde{M}$-sequence in $R$ is also a maximal homogeneous $\widetilde{M}$-sequence in $R[T]$. Therefore (Prop. 0.1.8), a regular $\widetilde{M}$-sequence is a sequence, after factoring which out the dimension of the module decreases by the length of this sequence.

$\widetilde{M}$ is a free $R$-module, and the quotient module of $\widetilde{M}$ by our sequence is the quotient by the columns of the matrices which are the sums of products of $\text{Id}_{ij}^K$ corresponding to the minors. Let $f_1$ be the identity of $K$. We order the variables $x_{aij}$ the following way:

$$x_{aij} > x_{bim} \Leftrightarrow (i < l) \lor (i = l & j < m) \lor (i = l & j = m & \alpha < \beta)$$
and take the corresponding degree-lexicographic order in $R$ and the corresponding “term over position” order in $\mathcal{M}$ (for some order of the basis elements of $\mathcal{M}$).

What are the initial terms of our columns of relations w.r.t. this order? Every monomial in every element of the matrix corresponding to $T_{i_1j_1} \cdots T_{i_nj_n}$ is $x_{\alpha_1i_1j_1} \cdots x_{\alpha_ni_nj_n}$ for some $\alpha$. The greatest of these monomials is $x_{1i_1j_1} \cdots x_{1inj_n}$. The fact that the substitution $x_{2ij} = \cdots = x_{(\dim K)ij} = 0$ results in the matrix $x_{1i_1j_1}E \cdots x_{1inj_n}E$, $E$ being the identity matrix, shows that this monomial appears in all diagonal elements, therefore in all columns. So the initial terms of the columns of the matrix corresponding to $T_{i_1j_1} \cdots T_{i_nj_n}$ are the columns of the matrix $x_{1i_1j_1} \cdots x_{1inj_n}E$.

Now we see, that if we add two products corresponding to different sets $(i_1j_1, \ldots, i_nj_n)$, then the initial terms are different and do not cancel out, so the initial terms of the columns of the matrix corresponding to the minor from the columns $j + 1, \ldots, j + n$ are the columns of the matrix

$$\max_{\sigma \in S_n} x_{11(j+\sigma(1))} \cdots x_{1n(j+\sigma(n))}E = x_{11(j+1)} \cdots x_{1n(j+n)}E.$$

Indeed $x_{11(j+1)}$ is the greatest variable occurring in the initial monomials, $x_{12(j+2)}$ is the greatest one occurring in the monomials containing $x_{11(j+1)}$ etc. Thus the initial terms of our relations are the columns of the matrices $y_1E, y_2E, \ldots, y_{(l-1)n+1}E$ with $y_i = x_{1i_1} \cdots x_{1in+i-1}$.

As any two leading terms either have different basis vectors or depend from disjoint sets of variables, there are no critical pairs, so the columns we consider constitute a Gröbner basis of the minors, underwent the required decrease.

Thus our complex is really the minimal resolution of $F_l(M)$, and also $\forall M \ Tor^A_1(F_l(A), M) = 0$. Now one can show the exactness of $F_l(\cdot)$ in the same fashion as for $l = 1$:

$$0 \to M \to N \to P \to 0 \text{ is exact } \Rightarrow \quad 0 = Tor^A_1(F_l(A), P) \to F_l(A) \otimes_A M \to F_l(A) \otimes_A N \to F_l(A) \otimes_A P \to 0$$

is exact. The full faithfulness of $F_l(\cdot)$ follows from Remark 1.4, so part 1) is proved.

Let us prove the Cohen-Macaulayness. For this one might similarly construct a regular sequence in $\text{ann} F_l(M)$, but we pursue another way: induction on the length of $M$. If $k$ is algebraically closed and $M$ is simple, then $F_l(M)$ is the quotient of a free module by the columns of the generic ($n \times ln$)-matrix, and its Cohen-Macaulayness is well-known ([14] Appendix 2.6), [8]). Further, let $l(M)$ be greater than 1 and $0 \to M_l \to M \to M_2 \to 0$ be an exact sequence of $A$-modules with $l(M_i) < l(M)$. Then by the induction assumption $\dim M_l = \text{depth } M_l = \dim R - ((l - 1)n + 1)$ and by the features of depth and of Krull dimension (Prop. 0.1.4 [20 chap. III, B.1, chap. I, C.1, prop. 10]) $\dim M \leq \max_i \dim M_i, \text{depth } M \geq \min_i \text{depth } M_i$, that is, $\dim M = \text{depth } M$ and $M$ is also Cohen-Macaulay.

Remark 1.10. Part 3) emerged from the hope that $R/\text{ann} F_l(M)$ is Cohen-Macaulay and $F_l(M)$ is a maximal Cohen-Macaulay module over it (i.e. of maximal dimension). But this is not the case already for $l = 2$ and for the standard representation of the algebra of diagonal
$2 \times 2$-matrices, when this ring has the form $k[x_1, x_2, y_1, y_2]/(x_i y_j)$ and Krull dimension 2, and after factoring out the regular element $x_1 - y_1$ the element $x_1$ will be annihilated by all the variables.
Chapter 2

Proof of Theorem 2

Throughout this chapter we assume \( l > 1 \).

2.1 The Case of Cohen-Macaulayness

We prove the strongest claim at once, since it is easily reduced to the case of an algebraically closed field.

**Proposition 2.1.** If for some \( l > 1 \) and some \( A \)-module \( M \) \( F_l(M) \) is Cohen-Macaulay, then \( A/\text{ann} \ M \) is an equidimensional maximally central algebra.

**Proof.** Note that we can assume the field to be algebraically closed, as the Cohen-Macaulayness of \( F_l(M) \) does not depend on this (part 2) of Lemma 1.7). So by Remark 1.7.2 we have to show that \( A/\text{ann} \ M \) is a direct sum of matrix algebras of the same rank over commutative ones. Passing to the quotient of \( A \) modulo \( \text{ann} \ M \) we can assume \( \text{ann} \ M = 0 \). So in the sequel of the section we assume that \( k \) is algebraically closed and \( \text{ann} \ M = 0 \) (\( M \) is a faithful module).

**Lemma 2.2.** \( A \) is a direct sum of algebras \( A_i \) the semisimple quotients of which are simple, and these quotients are matrix algebras of the same rank.

**Proof.** Choose an embedding of \( A/\text{rad} \ A = M_{n_1}(k) \oplus \cdots \oplus M_{n_k}(k) \) into \( A \) as a subalgebra with 1 (you can choose one over an algebraically closed field by the Wedderburn–Malcev Theorem, see Prop. 0.6.3), choose a composition series \( 0 = M_0 \subset M_1 \subset \cdots \subset M_m = M \) in \( M \) and choose a \( k \)-basis \( e_1, \ldots, e_r \) in \( M \) that conforms to these choices, i.e. that \( M_i = \langle e_1, \ldots, e_{k_i} \rangle_k \) and that \( \langle e_{k_i+1}, \ldots, e_{k_{i+1}} \rangle_k \) are simple submodules over the subalgebra \( A/\text{rad} \ A \). In the decomposition \( A = M_{n_1}(k) \oplus \cdots \oplus M_{n_k}(k) \oplus \text{rad} \ A \) decompose \( \text{rad} \ A \) further into isotypic components as a bimodule over the semisimple part and choose a \( k \)-basis in \( A \) that conforms to the resulting decomposition (matrix elements being the basis of the semisimple part). Then the matrices \( \text{Id}_j \) become blockwise upper triangular, with the diagonal occupied by square blocks of independent variables corresponding to the simple quotients of \( A \) over which the corresponding composition factors are simple modules, and with linear forms in variables corresponding to the radical of \( A \) above the diagonal. One can say more: the intersection of a row that has a simple quotient of type \( P \) on the diagonal and a column that has a simple quotient of type \( Q \) contains linear forms in the variables corresponding to the \((P,Q)\)-isotypic component of the radical.
We shall illuminate the behavior of the matrices $\text{Id}_j$ under the factorizations we do in the proof by the example of a module of length 4 with successive composition factors of types $P, P, Q, P$:

\[
\text{Id}_j = \begin{pmatrix}
  P & (P, P) & (P, Q) & (P, P) \\
  P & (P, Q) & (P, P) & \\
  0 & Q & (Q, P) & P \\
\end{pmatrix}.
\]

Let $M_{n_1}(\mathbb{k})$ be the simple quotient of the algebra to which corresponds the last composition factor $P$ of our composition series (i.e. the simple quotient of our module). Take the quotient of $F_l(M)$ modulo the sequence of the variables corresponding to the radical of the algebra and then by the sequence of the variables corresponding to the remaining simple quotients of the algebra in elements $\text{Id}_2, \ldots, \text{Id}_i$. Then take the quotient of the resulting module modulo the sequence of the variables that correspond to the remaining simple quotients in $\text{Id}_1$ and do not occupy its principal diagonal and modulo the sequence $y_s - 1$, $y_s$ running over the variables that correspond to the other simple quotients in $\text{Id}_1$ and do occupy its principal diagonal. Then we obtain a module $X'$ over the ring $R'$ of polynomials in the remaining variables (i.e. corresponding to the simple quotient chosen), for which the presentation matrix has only blocks of variables that correspond to this quotient on the diagonal, zeros above it, and the blocks that correspond to other simple quotients of $A$ are turned to identity matrices in $\text{Id}_1$ and to zero matrices in $\text{Id}_2, \ldots, \text{Id}_i$:

\[
\text{Id}_1' = \begin{pmatrix}
  P & 0 & 0 & 0 \\
  P & 0 & 0 & \\
  0 & 1 & 0 & P \\
\end{pmatrix}, \quad \text{Id}_j' = \begin{pmatrix}
  P & 0 & 0 & 0 \\
  P & 0 & 0 & \\
  0 & 0 & 0 & P \\
\end{pmatrix}.
\]

So we can cross out the rows and columns in the presentation matrix containing the identity matrices and realize that we obtain a direct sum of several copies of $F_l^{R'}(P)$ over the polynomial ring in the remaining variables.

The Krull dimension of $F_l^{R'}(P)$ (the quotient modulo the columns of an $(n_1 \times ln_1)$-matrix of independent variables) is known to be $\dim R' - (l - 1)n_1 - 1$ [8]. Since $F_l$ is right-exact (see the beginning of Chap. [1]), $F_l^{R'}(P)$ is a quotient of $F_l(M)$. Thus $\dim F_l(M) \geq \dim F_l^{R'}(P) = \dim R' - (l - 1)n_1 - 1$. As $F_l(M)$ is Cohen-Macaulay, its dimension decreased with the factorization by at most the length of the sequence factored out, namely $\dim R - \dim R'$, and in case of equality this sequence is regular (Prop. [0, 0, 0]). Therefore $\dim F_l(M) = \dim F_l^{R'}(P)$ and the sequence is regular.

Now we consider the quotient of $F_l(M)$ modulo a part of this sequence: we take only those variables corresponding to nilradical in $\text{Id}_2, \ldots, \text{Id}_i$ that correspond to isotypic components not isomorphic to direct sums of $P$ as left modules over the semisimple part of $A$. Then $\text{Id}_1$ and diagonals of other matrices change as above, and above the diagonal of $\text{Id}_2, \ldots, \text{Id}_i$ the rows corresponding to composition factors of type $P$ remain the same, while other rows vanish:

\[
\text{Id}_1'' = \begin{pmatrix}
  P & 0 & 0 & 0 \\
  P & 0 & 0 & \\
  0 & 1 & 0 & P \\
\end{pmatrix}, \quad \text{Id}_j'' = \begin{pmatrix}
  P & (P, P) & (P, Q) & (P, P) \\
  P & (P, Q) & (P, P) & \\
  0 & Q & (Q, P) & P \\
\end{pmatrix}.
\]
If we induce a filtration on this quotient $X''$ by the composition series of $M$, the adjoint factors of this filtration are either quotients of $F_i^{R''}(P)$ over the ring in the remaining variables, as the relations contain at least the diagonal blocks, or zeroes, if the diagonal block is an identity matrix. Moreover, the last factor is exactly $F_i^{R''}(P)$. So the Krull dimension of $X''$ equals $\dim R'' - (l-1) n_1 - 1$, and the same argument as in the previous paragraph shows that the sequence is regular and thus (Prop. 0.1.8) $X''$ is Cohen-Macaulay.

As in the previous factorization, we can remove rows and columns occupied by identity matrices in $\text{Id}_1$ and obtain a presentation of $X''$ by a matrix of linear forms. Thus $X''$ is a graded module and the passage from it to $X'$ is factoring out a homogeneous regular sequence of degree 1. Hence the Hilbert series of $X'$ can be obtained from the one of $X''$ by multiplying by $1 - t$ raised to the power equal to the length of the sequence (see Section 0.3). So $X''$ and $X' \otimes_{R'} R''$ have the same Hilbert function. If we consider the filtration on these modules induced by the composition series of $M$, then, as remarked in the previous paragraph, the adjoint factors for $X''$ are quotients of those for $X' \otimes_{R'} R''$, so there is actually no further factorization. Suppose that after a factor of type $P$ we have a factor of another type $Q$ in the composition series (as in the example). Then the part of the presentation matrix obtained from $\text{Id}_2$ has some forms in the variables that correspond to the $(P, Q)$-isotypic component of the nilradical to the right of the block corresponding to the first factor and above the block corresponding to the second factor, with zeroes in the place of the second block on the diagonal and below:

$$\text{Id}_2'' = \begin{pmatrix} P & (P, P) & (P, Q) & (P, P) \\ P & (P, Q) & (P, P) & 0 \\ 0 & 0 & 0 & P \end{pmatrix}.$$  

Thus if these forms are nonzero, they give an additional factorization of the adjoint factor, which cannot be. Therefore they are zero and if we transpose the corresponding groups of the basis vectors of $M$, we can put the composition factor of $M$ isomorphic to $P$ after the one isomorphic to $Q$:

$$\text{Id}_j'' = \begin{pmatrix} P & (P, Q) & (P, P) & (P, P) \\ Q & 0 & (Q, P) & (P, P) \\ 0 & P & (P, P) & (P, P) \end{pmatrix}.$$  

Thus we can suppose that in the composition series of $M$ all the factors isomorphic to $P$ go after other factors and form a quotient module $M_P$. Then if we factor out a sequence that does the same with the variables corresponding to the semisimple part and all the variables corresponding to the nilradical but for those corresponding to the $(P, P)$-isotypic component, $F_i(M)$ becomes $F_i^{R''}(M_P)$ over the polynomial ring in the remaining variables:

$$\text{Id}_1''' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ P & (P, P) & (P, P) & 0 \\ 0 & P & (P, P) & (P, P) \end{pmatrix}, \quad \text{Id}_j''' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ P & (P, P) & (P, P) & 0 \\ 0 & P & (P, P) & (P, P) \end{pmatrix}.$$  

The same filtration shows that

$$\dim F_i^{R''}(M_P) = \dim F_i^{R''}(P) = \dim R'' - (l-1) n_1 - 1,$$
so we were factoring out a regular sequence and \( F_i^R(M_P) = F_i^{R''} (M_P) \otimes_{R'} R \) is a Cohen-Macaulay module of the same dimension as \( F_i(M) \). Therefore, according to the behavior of the depth (Prop. 0.1.4 and Krull dimension 20 chap. III, D.3, prop. 15], and the same is true for any of its quotient rings), the depth equals \( \text{ht}(\text{ann} F_i(M_P)) \). As our modules have the same dimension, the leftmost module in the exact sequence

\[ 0 \to Y \to F_i(M) \to F_i(M_P) \to 0 \]

is Cohen-Macaulay of the same dimension. We show that this sequence is split. Because of Yoneda’s interpretation of Ext [6 §7 n°3] it will suffice for this to show that \( \text{Ext}^1_R(F_i(M_P), Y) = 0 \).

It is known (Prop. 0.1.4) that

\[ \min\{i \mid \text{Ext}^1_R(M, N) \neq 0\} = \text{depth}(\text{ann} M, N). \]

It is enough to show that in our case this depth is at least 2. As the modules under consideration are Cohen-Macaulay and the ring is a polynomial ring (so that for a (prime, and hence any) ideal \( I \) one has \( \text{ht} I + \dim R/I = \dim R \) 20 chap. III, D.3, prop. 15], and the same is true for any of its quotient rings), the depth equals \( \text{ht}(\text{ann} Y + \text{ann} F_i(M_P)) - \text{ht} \text{ann} Y \) 13 Exercise 18.4], and all the annihilators can be replaced by their radicals. As our modules have the same dimension, \( \text{ht} \text{ann} Y = \text{ht} \text{ann} F_i(M_P) \). Thus we have to show that

\[ \text{ht}(\text{ann} Y + \text{ann} F_i(M_P)) \geq \text{ht}(\text{ann} F_i(M_P)) + 2. \]

The filtration induced by the composition series shows that the radical of the annihilator of \( F_i(M_P) \) equals the radical of the annihilator of \( F_i(P) \), namely, the ideal of the maximal minors of the matrix obtained by writing the blocks corresponding to \( P \) in \( \text{Id}_j \) one after another. It also shows that the radical of the annihilator of \( Y \) contains the product of the ideals of maximal minors of matrices obtained similarly for other simple quotients of \( A \). Such an ideal for an \( n_i \times l n_i \) matrix is of height \( (l - 1)n_i + 1 \geq (2 - 1)1 + 1 = 2 \), thus their product is also of height \( \geq 2 \) and is generated by polynomials in variables not involved in the generators of the radical of \( \text{ann} F_i(M) \), so, if we add this product to the radical of \( \text{ann} F_i(M) \), its height increases by at least 2, as required.

Remark 14 says that \( F_i \), though not always exact, is always fully faithfull as a functor into the graded category. Thus if \( F_i(M) \to F_i(M_P) \) is a split epimorphism (as the homogeneous component of degree 0 of a left inverse to the projection is itself a left inverse, it doesn’t matter whether the epimorphism is split in the graded or in the usual category), \( M \to M_P \) is also a split epimorphism. Hence \( M = M_P \oplus M' \) with \( M' \) having no composition factors of type \( P \). Thus we can choose a composition series in \( M \) in which the last quotient has some other type \( Q \).

Repeating the previous argument for this composition series, we see that \( M = M_P \oplus M_Q \oplus M'' \) with \( M'' \) containing no composition factors of types \( P \) and \( Q \). Induction on the number of composition factors for which the corresponding “isotypic components” are direct summands of \( M \) yields us the conclusion that \( M = \bigoplus_P M_P \) is a direct sum of modules having only one type of composition factors each. The matrices of the representaion of \( A \) in a k-basis of \( M \) conforming to this decomposition and further decompositions as in the beginning of the proof are blockwise diagonal, and, as this representation is faithful, the idempotents that correspond to the identities of the summands \( M_{n_i}(k) \) and are represented by matrices having one block identity and others zero lie in the center of \( A \) and decompose it into a direct sum of algebras \( \rho_{M_P}(A) \), each having only one simple module.
In the beginning of the proof we saw that \( \dim F_l(M) = \dim F_l(P) \). Now we see that it is true for every \( P \), and as the latter dimension equals \( \dim R - (l - 1)n_i - 1 \) and we have \( l > 1 \), all the \( n_i \) are equal.

Now we can apply Lemma 0.6.7 and write \( A = M_n(K) \) for \( K \) a direct sum of algebras with semisimple quotients equal to \( k \) (because the central idempotents that describe the decomposition of \( A \) lie in \( K \)). In Notations 1.8 Morita-equivalence of the categories of modules over \( A \) and \( K \) associates to a faithful \( A \)-module \( M \) a faithful \( K \)-module \( \mathcal{M} \). Let \( l' = (l-1)n+1 \geq (2-1)l+1 = 2 \). A composition series of \( \mathcal{M} \) shows that \( \dim F_{l'}(\mathcal{M}) = \dim k[K^{l'}] - l' \).

**Lemma 2.3.** \( F_{l'}(\mathcal{M}) \) is a Cohen-Macaulay module, \( \mathcal{M} \) being regarded as a \( K \)-module.

**Proof.** In the notations 1.8 \( F_l(M) \) is the quotient modulo the columns of an \( n \times ln \) block matrix \( (\text{Id}_{ij})^K \). Let \( f_1 \) be the identity of \( K \). Consider the sequence

\[
\{x_{ij} - \delta_{ij}\}_{i,j=1}^{n}, \{x_{aij}\}_{i,j=2,a=2}^{\dim K}, \{x_{aij}\}_{i=2,j \notin [n]}, \{x_{aij}\}_{j=2}^{n},
\]

i. e. the one modulo which the matrix above becomes

\[
\begin{pmatrix}
\text{Id}_{11}^K & 0 & \text{Id}_{1,n+1}^K & \cdots & \text{Id}_{1,ln}^K \\
0 & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \cdots & \vdots \\
0 & 0 & 1 & \cdots & 0
\end{pmatrix}.
\]

The quotient modulo the columns of the resulting matrix is easily seen to be \( F_{l'}(\mathcal{M}) \), and one sees also that \( k[(M_n(K))^l]/(\mathbf{x}) = k[K^{l'}] \), i. e. \( F_{l'}(\mathcal{M}) = F_l(M)/\mathbf{x}F_l(M) \). We need to prove that \( F_{l'}(\mathcal{M}) \) is Cohen-Macaulay, and since \( F_l(M) \) is Cohen-Macaulay by hypothesis, it suffices to prove by Prop. 0.1.8 that \( \dim F_{l'}(\mathcal{M}) = \dim F_l(M) - l(\mathbf{x}) \). Now, \( \mathbf{x} \) is a regular sequence of length \( \dim k[(M_n(K))^l] - \dim k[K^{l'}] \) in \( k[(M_n(K))^l] \), and we have remarked above that the dimensions \( \dim F_l(M) = \dim k[(M_n(K))^l] - l' \) and \( \dim F_{l'}(\mathcal{M}) = \dim k[K^{l'}] - l' \) are as required, therefore \( F_{l'}(\mathcal{M}) \) is Cohen-Macaulay.

**Lemma 2.4.** Let \( K \) and \( M \) be as above. Choose an embedding of \( K/\text{rad} \ K = k^k \) into \( K \) as a subalgebra containing \( 1_K \), choose a composition series \( 0 = \mathcal{M}_0 \subset \mathcal{M}_1 \subset \cdots \subset \mathcal{M}_m = \mathcal{M} \) in \( \mathcal{M} \) and choose a \( k \)-basis \( e_1, \ldots, e_m \) for \( \mathcal{M} \) conforming to these choices, i. e. so that \( \mathcal{M}_i = \langle e_1, \ldots, e_i \rangle_k \) and \( k e_i \) are simple submodules over the subalgebra \( K/\text{rad} \ K \). Choose a \( k \)-basis for \( K \) conforming to the decomposition \( K = k^k \oplus \text{rad} K \) and a monomial order in \( R = k[K^{l'}] \). Let \( e_1 < e_2 < \cdots < e_m \) and introduce the “position over term” monomial order in \( R^m \). Then the columns of the matrix Id_{11}^K \ldots \text{Id}_{1,n}^K \) form a Gröbner basis of the submodule they generate.

**Proof.** Note that under these choices \( \text{Id}_{ij}^K \) is upper triangular, the diagonal occupied by the variables corresponding to simple quotients of \( K \) and the space above it by linear forms in the variables corresponding to the radical of \( K \), so that the initial terms of the columns of this matrix are obtained by substituting zero into all the variables corresponding to the radical.

In analogy with the proof of Lemma 2.2 consider the quotient of \( F_{l'}(\mathcal{M}) \) modulo a sequence of the variables corresponding to the basis of \( \text{rad} K \). It is a regular sequence in \( k[K^{l'}] \) and \( k[(K/\text{rad} \ K)^{l'}] \) is the quotient of the ring by it, while the quotient of \( F_{l'}^K(\mathcal{M}) \) modulo this
sequence is \( F^K_{\nu} / \text{rad} K (\mathcal{M}) \) for the restriction of \( \mathcal{M} \) to the semisimple quotient embedded as a subalgebra. The standard argument involving a composition series and the right-exactness of \( F_{\nu} \) shows that the Krull dimensions of both modules equal the dimensions of the corresponding rings less \( l' \). Thus this sequence is \( F^K_{\nu} (\mathcal{M}) \)-regular, so the Hilbert series of the quotient is obtained from the Hilbert series of \( F^K_{\nu} (\mathcal{M}) \) through multiplication by \( 1 - t \) raised to the power equal to the length of the sequence. Therefore if we tensor \( F^K_{\nu} / \text{rad} K (\mathcal{M}) \) over \( \mathbb{k} \) with the polynomial ring in the variables corresponding to the radical, we obtain a module with the same Hilbert function as \( F^K_{\nu} (\mathcal{M}) \). But the module we obtain is the quotient of \( R^n \) modulo the initial terms of the relation columns for \( F^K_{\nu} (\mathcal{M}) \), whence these columns are a Gröbner basis.

**Lemma 2.5.** The algebra \( K \) considered above is commutative.

**Proof.** Consider the commutator of two generic elements. Its columns lie in the submodule of relations and should be reducible to zero, but they depend only on the variables corresponding to the nilradical, as the matrices multiplied by the variables corresponding to \( \mathbb{k}^n \) are central idempotents, whereas the initial terms of the Gröbner basis are divisible by a variable corresponding to the semisimple quotient. So the commutator equals zero.

We have proved the proposition. Now we prove the remainder of the theorem. If \( F_{\nu} \) takes all \( A \)-modules to Cohen-Macaulay ones, then, applying the proposition to the left regular representation of the algebra (which is faithful) we see that the algebra is equidimensional maximally central. It also follows from the proposition that if \( F_{\nu} (M) \) is Cohen-Macaulay for an indecomposable \( A \)-module \( M \), then all the composition factors of \( M \) are isomorphic. Then it follows from the description of the decomposition of finite-dimensional algebras into a direct sum in Section 0.5 that \( A \) is a direct sum of algebras with simple semisimple quotients. So for every summand all the indecomposable modules are taken into Cohen-Macaulay modules of the same dimension (as for the simple module), so all modules are taken into Cohen-Macaulay ones, so the summands, and therefore the whole \( A \), are maximally central.

### 2.2 The Case of an Exact Functor

First we prove that the functor \( F_{\nu} (\cdot) \) remains exact over \( \overline{\mathbb{k}} \), the algebraic closure of \( \mathbb{k} \). We have seen at the beginning of Chapter 1 that \( F_{\nu} (\cdot) \) is the tensor multiplication over \( A \) by the right \( A \)-module \( F_{\nu} (A) \). But the flatness of a module is known to be equivalent to preserving the exactness of sequences of finitely generated modules under tensoring with this module \([6, \S 4, n^6, \text{théorème } 2]\).

So the exactness of \( F_{\nu} (\cdot) \) is equivalent to \( F_{\nu} (A) \) being a flat right \( A \)-module. Now, this condition is preserved under extension of scalars and under passing to induced modules in general, which is immediate from the associativity of the tensor product.

So we can assume \( \mathbb{k} \) to be algebraically closed. Now we show that non-isomorphic simple \( A \)-modules form no nontrivial extensions.

**Lemma 2.6.** Let

\[
0 \rightarrow M \rightarrow N \rightarrow P \rightarrow 0
\]

be a non-split exact sequence of \( A \)-modules, \( M \) and \( P \) be nonisomorphic simple modules and the base field \( \mathbb{k} \) be algebraically closed. Then the sequence \( 0 \rightarrow F_{\nu} (M) \rightarrow F_{\nu} (N) \) is not exact.
Proof. We denote by $\rho_M : A \to \text{End}_k M$ the representation of $A$ corresponding to the module $M$. Choosing a $k$-basis for $N$, conforming to the composition series (2.1), we obtain

$$\rho_N(A) \subset \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} \right\} \text{ with } \alpha \in M_{n_1}(k), \beta \in M_{n_1 \times n_2}(k), \gamma \in M_{n_2}(k)$$

and $\rho_N(a) = \left( \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} \right) \Rightarrow \rho_M(a) = \alpha$, $\rho_P(a) = \gamma$.

As $M \not\cong P$ are simple, we have $\rho_{M \oplus P}(A) = \text{End}_k M \oplus \text{End}_k P$, i.e. $\rho_N(A) \nrightarrow \rho_M(A) \oplus \rho_P(A)$ is an epimorphism; if it were an isomorphism, $\rho_N(A)$ would be semisimple and $N$ would be a direct sum of simple modules and a trivial extension, contradicting the hypothesis of the lemma. Thus

$$\exists \beta \neq 0 : \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix} \in \rho_N(A);$$

the matrix multiplication in $\rho_N(A)$ and the surjectivity of $p$ show that $\beta$’s of this kind form a subrepresentation in the representation of $M_{n_1}(k) \otimes M_{n_2}(k)$ on $M_{n_1 \times n_2}(k)$ by left and right multiplications respectively, and the irreducibility of this representation shows that

$$\rho_N(A) = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} \right\} \text{ for } \alpha \in M_{n_1}(k), \beta \in M_{n_1 \times n_2}(k), \gamma \in M_{n_2}(k).$$

So

$$F_i(N) = \overline{\langle \left( \begin{array}{cc} A & B \\ 0 & C \end{array} \right) \rangle};$$

$A, B, C$ being generic matrices of corresponding sizes, say, $A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$, and $F_i(M) = \overline{\langle A \rangle}$, $F_i(P) = \overline{\langle C \rangle}$. This shows that $\mathcal{M}$, the image of $F_i(M)$ in $F_i(N)$, equals

$$\overline{\langle A \rangle} \cong \overline{\langle \left( \begin{array}{cc} A & B \\ 0 & C \end{array} \right) \rangle}.$$ 

Now we show that

$$\overline{\langle A \rangle} \not\subset \overline{\langle \left( \begin{array}{cc} A & B \\ 0 & C \end{array} \right) \rangle}. \quad (2.2)$$

Let $\sum M_j \varepsilon_j \in \langle \varepsilon_1, \ldots, \varepsilon_{ln} \rangle_R$ be a basis syzygy on the columns of the matrix $C$. Then $\sum M_j b_j$ ($b_j$ being the $j$th column of $B$) belongs to the left-hand side of (2.2) and depends only on $b.$ and $c.$, as in the basis syzygies on the columns of $C$ $M_j$ are $n \times n$-cofactors in some $n \times (n+1)$ submatrix of $C$ (here we use that $l$ is greater than 1 and there are such submatrices).

Thus the left-hand side of (2.2) is strictly larger than the right-hand side and $\mathcal{M} \neq F_i(M)$, i.e. the functor is not exact.

So we see that every indecomposable $A$-module has only one type of composition factors. Since for a simple module $P$ the module $F_i(P)$ is always Cohen-Macaulay and the functor $F_i$ is exact, the behavior of the depth (Prop. 0.1.4) and Krull dimension (20, chap. III, B.1, chap. I, C.1, prop. 10) in short exact sequences together with the induction on the length show that for every indecomposable $M F_i(M)$ is Cohen-Macaulay. So the water is poured out of the kettle and we can apply the remaining part of the theorem.
Chapter 3

Proof of Theorem 3

Reduction to the module $P$. First, we can pass to the algebraic closure of the base field, as all these invariants are determined by the structure of the minimal resolution of $F_i(M)$, so we assume that $k$ is algebraically closed. Further, the Hilbert series and the multiplicity are additive in short exact sequences in the category of graded modules, so these invariants for $F_i(M)$ are $\dim_k M/\dim_k P$ times as big as for $F_i(P)$, and $\dim_k P = n$. For Betti numbers, in particular (Prop. 0.3.5) for the Cohen-Macaulay type, this follows from the fact that with Notations 1.8 the minimal resolution of $F_i(M)$ equals $C \otimes_{R[T]} \tilde{M}$, so the Betti numbers are $\text{rk}_R \tilde{M} = \dim_k M/n$ times as big as the ranks of the components of $C$, and for $P$ this coefficient equals 1. Thus it suffices to calculate all the invariants for $H_0(C)$. We do the calculations for an Eagon-Northcott complex of a $(g \times f)$-matrix and then substitute our values $f = ln, g = n$.

The Betti numbers of $F_i(P)$. The formulas are immediate from the definition of the Eagon-Northcott complex: the matrix $\varphi$ has only entries of degree 1, so, to give the differentials in the Eagon-Northcott complex degree zero when the generators of $F_0$ have degree zero, the generators of $F_1$ should have degree 1, the generators of $F_2$ degree $g + 1$, as the elements of the matrix of the differential are $g \times g$-minors of $\varphi$, and then the degree should advance by 1. As the rank of $\bigwedge^i F \otimes (S^j G)^*$ equals $(\frac{f}{g+j-1})$, we get the required values for Betti numbers.

The Hilbert series and the multiplicity. The Hilbert series is an additive function on the graded modules, so the Euler-Poincare characteristic of the minimal resolution

$$0 \to F_k \to \cdots \to F_0 \to F_i(P) \to 0$$

of the module $F_i(P)$ w. r. t. the Hilbert series equals zero, i. e. $F_i(P)(t) = \sum_i (-1)^i F_i(t)$. If we consider the Poincare series $P(s,t) = \sum_i s^i t^j b_{ij}$, $b_{ij}$ being the graded Betti numbers of $F_i(P)$, then $F_i(P)(t) = P(-1,t)/(1-t)^{\dim R}$. We have:

$$P(s,t) = g + fst + \sum_{k=g+1}^{f} \left( \begin{array}{c} f \\ k \end{array} \right) \left( \begin{array}{c} k-2 \\ g-1 \end{array} \right) s^{k+1-g} t^k =$$

$$= (1 - (-s)^{1-g})(g + fst) + \frac{1}{(g-1)!} s^{2} \left( \frac{\partial}{\partial s} \right)^{g-1} (s^{-2}(1+st)^{f}) =$$

$$= (1 - (-s)^{1-g})(g + fst) + (1 + st)^{f-g+1} \sum_{k=0}^{g-1} \left( \begin{array}{c} f \\ k \end{array} \right) (g-k) t^k (-1/s - t)^{g-1-k}.$$
(if we expand the $f$th power in the middle line by the binomial formula and then differentiate termwise, we obtain the previous expression, and if we differentiate the product $s^{-2} \cdot (1 + st)^f$ according to the Leibniz formula, we obtain the last expression).

Substituting $s = -1$ and cancelling common factors with the denominator we obtain the required expression:

$$F_l(P)(t) = (1 - t)^{f-g+1-\dim R} \sum_{i=0}^{g-1} \left( \binom{f}{i} \right) (g - i) t^i (1 - t)^{g-1-i}$$

(here, we recall, $f = \ln, g = n$), and the sum is exactly the polynomial $p(t)$ in the definition of the multiplicity. Substituting $t = 1$, we find the multiplicity: only the summand with $i = g - 1$ remains.\[\square\]
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