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Existence Results for $p_1(x, \cdot)$ and $p_2(x, \cdot)$ Fractional Choquard–Kirchhoff Type Equations with Variable $s(x, \cdot)$-Order

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Abstract: In this article, we study a class of Choquard–Kirchhoff type equations driven by the variable $s(x, \cdot)$-order fractional $p_1(x, \cdot)$ and $p_2(x, \cdot)$-Laplacian. Assuming some reasonable conditions and with the help of variational methods, we reach a positive energy solution and a negative energy solution in an appropriate space of functions. The main difficulties and innovations are the Choquard nonlinearities and Kirchhoff functions with the presence of double Laplace operators involving two variable parameters.

Keywords: Choquard–Kirchhoff type equations; fractional $p_1(x, \cdot)$ and $p_2(x, \cdot)$-Laplacian; variational methods; positive energy solutions; negative energy solutions

1. Introduction

In this paper, we study the existence of solutions for the following Choquard–Kirchhoff type equations

$$\begin{align*}
(P_\lambda): & \quad \sum_{i=1}^{2} M_i \left( \int_{\Omega \times \Omega} \frac{|v(x) - v(y)|^{p_i(x,y)}}{p_i(x,y)|x-y|^{N+\gamma_i(x,y)N_{p_i(x,y)}}} dxdy \right) (-\Delta)^{s(x,\cdot)}_{p_i(x,\cdot)} v \\
& = \lambda f(x) |v|^{\gamma(x)-2} v + \left( \int_{\Omega} \frac{G(y,v(y))}{|x-y|^{\mu(x,y)}} dy \right) g(x, v), \quad x \in \Omega, \\
v = 0, \quad x \in \partial \Omega,
\end{align*}$$

where $M_i(i = 1, 2)$ is a model of Kirchhoff coefficient, $\Omega$ is a bounded smooth domain in $\mathbb{R}^N$, $\lambda$ is a real parameter, and $f, g, \mu, \gamma, N$ are generally continuous functions whose assumptions will be introduced in what follows. The operators $(-\Delta)^{s(x,\cdot)}_{p_i(x,\cdot)}$ are called variable $s(x, \cdot)$-order $p_i(x, \cdot)$-fractional Laplacian, given $p_i(x, \cdot): \overline{\Omega} \times \overline{\Omega} \rightarrow (1, +\infty)$ ($i = 1, 2$) and $s(x, \cdot): \overline{\Omega} \times \overline{\Omega} \rightarrow (0, 1)$ with $N > s(x, y)p_i(x, y)$ for all $(x, y) \in \Omega \times \Omega$, which can be defined as

$$(-\Delta)^{s(x,\cdot)}_{p_i(x,\cdot)} v(x) := P.V. \int_{\Omega} \frac{|v(x) - v(y)|^{p_i(x,y)-2} (v(x) - v(y))}{|x-y|^{N+s(x,y)p_i(x,y)}} dy, \quad x \in \Omega,$$

where $v \in C^0_0(\Omega)$ and $P.V.$ stands for the Cauchy principal value. Especially, when $s(x, \cdot) \equiv$ constant and $p_i(x, \cdot) \equiv$ constant, we observe that the operators $(-\Delta)^{s(x,\cdot)}_{p_i(x,\cdot)}$ in $(P_\lambda)$ reduce to the fractional $p$-Laplace operators $(-\Delta)_p^\gamma$—see [1–6].

Throughout this paper, we assume that $s(x, y)$ is a continuous function and satisfies the following condition:

(S1): $s(x, y)$ is symmetric function, i.e., $s(x, y) = s(y, x)$, and $0 < s^- := \inf_{(x, y) \in \overline{\Omega} \times \overline{\Omega}} s(x, y) \leq s^+ := \sup_{(x, y) \in \Omega \times \Omega} s(x, y) < 1$ for all $(x, y) \in \overline{\Omega} \times \overline{\Omega}$ with $x = s(x, x)$. 

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The variable exponents $p_i(x, y) (i = 1, 2)$, we assume the following assumption:

(P1): $p_i(x, y)$ are symmetric functions, i.e., $p_i(x, y) = p_i(y, x)$, $1 < p_i^- := \inf_{(x, y) \in \Omega} p_i(x, y) \leq p_i^+ := \sup_{(x, y) \in \Omega} p_i(x, y) < +\infty$. We denote $p_{max}(x, y) = \max\{p_1(x, y), p_2(x, y)\}$ and $p_{min}(x, y) = \min\{p_1(x, y), p_2(x, y)\}$ for all $(x, y) \in \Omega \times \Omega$ with $\Omega(x) = p_i(x, y)$.

The nonlinear Choquard equation was studied by Ph. Choquard in [7], in which he established a model as follows

$$-\Delta v(x) + v(x) = (I_2 * |v(x)|^2) v(x), \text{ in } \mathbb{R}^3, \quad (3)$$

this type of model is widely used in the field of physics, such as quantum physics, Newtonian gravity, self-field coupling, and it has been studied in depth by many scholars, we refer interested readers to [8–11].

On the other hand, many researchers gradually devoted themselves to the study of equations with nonlinear convolution terms, such as Hartree type and Choquard type nonlinearities. The nonlinear term in $(P_i)$ is inspired by the following problem

$$-\Delta v + V(x)v = (|x|^{-\mu} * G(x, v))g(x, v), \text{ in } \mathbb{R}^N, \quad (4)$$

which has arisen in various fields of mathematical physics. Such kind of problems was elaborated by Pekar in his description of the quantum theory of stationary polarizons—see [12]. More recently, Penrose settled (4) as model of self-gravitating matter in [13]. More particularly, Moroz et al. in [14], surveyed the existing results and studied the existence and multiplicity of solutions for nonlinear Choquard equations, some of its variants and extensions. In this direction, D’Avenia et al. investigated, in [15], for the first time in the literature, a class of fractional Choquard equation, starting from this paper, a lot of people were interested in solving this class of equations and systems—see [16,17].

Especially, for Choquard–Kirchhoff equations with variable exponent in [18], Bahrouni et al. dealt with Strauss and Lions type theorems and studied the existence and multiplicity of weak solutions. Furthermore, for nonlocal Choquard–Kirchhoff problems in [19], Biswas et al. obtained the existence of ground state solution, and infinitely many weak solutions, which the conditions for nonlinear functions are weaker than the Ambrosetti–Rabinowitz conditions.

The so-called Kirchhoff model was introduced by Kirchhoff in [20], in which he established the following model:

$$\rho \frac{\partial^2 v(x)}{\partial t^2} - \left(\frac{p_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial v(x)}{\partial t} \right|^2 \, dx\right) \frac{\partial^2 v(x)}{\partial x^2} = 0, \quad (5)$$

where $\rho, p_0, h, E, L$ are real constants that represent some specific physical meaning, respectively. From then on, the literature on Kirchhoff type equations and Kirchhoff systems are quite large, here we just list a few—for example, Refs. [21–25] for further details.

The Kirchhoff functions $M_i : R_0^+ \rightarrow R^+(i = 1, 2)$ are continuous, which satisfy the following assumptions:

(M1): There are positive constants $\theta_i \in [1, p_i^{+}(x, y) / p_{max}]$ and $\theta = \max\{\theta_1, \theta_2\}$ such that

$$t M_i(t) \leq \theta \tilde{M}_i(t), \text{ for any } t \in R_0^+, \text{ where } \tilde{M}_i(t) = \int_0^t M_i(\tau) \, d\tau.$$  

(M2): There are $m_i = m_i(\tau) > 0$ for all $\tau > 0$ such that

$$M_i(t) \geq m_i, \text{ for any } t > \tau.$$
The evolution of the Laplace operator has been progressively deepened and has taken many forms so far. Many mathematical scholars have been devoted to the integer Laplace operators, fractional Laplace operators, and variable order fractional Laplace operators. For some important results of variable order fractional Laplace operators, we refer to [26–35]. Note that, in [31], Wang et al. investigated the existence and multiplicity of weak solutions by applying four kinds of different critical point theorems, and the difference with other studies was that Kirchhoff function is zero at zero. In particular, in [32], Xiang et al. studied the multiplicity results for a Schrödinger equation via variational methods. Most importantly, they obtained the embedding theorem for variable-order Sobolev spaces. Moreover, Chen et al. in [33], studied a variable order nonlinear reaction subdiffusion equation, Coimbra et al. in [34], investigated Mechanics with variable-order differential operators, and Birajdar et al. in [35], considered a class of variable-order time-fractional first initial boundary value problems.

Although many materials can be accurately modeled by the classical Lebesgue \( L^p \) and Sobolev spaces \( W^{k,p} \), where \( p \) is a fixed constant and \( s = 1 \), there are some nonhomogeneous materials, for which this is not adequate, for instance, the rheological fluids are characterized by their ability to drastically change their mechanical properties under the influence of an exterior electromagnetic field. Thus, it is necessary for the exponent \( p \) to be variable. The study of various physical and mathematical problems with variable exponent and variable-order has a wide range of applications, concerning elastic mechanics [36], electrorheological fluids [37], image restoration [38], dielectric breakdown and polycrystal plasticity [39], variable-order space-fractional diffusion equations [40].

In recent years, mathematicians began to gradually consider variable exponent Laplace operator \( \Delta_{p(x)} \) and \( \Delta_{p(x),r} \), see the literature [41–46]. It is worth mentioning that Kaufmann et al. in [46] extended the variable exponent Sobolev spaces to the fractional case and established the compact embedding theorem of variable exponent Sobolev spaces. As an application, the existence of weak solutions of a nonlocal problem was studied.

In the framework of variable exponents involving fractional \( p(x,\cdot) \)-Laplace operator with variable \( s(x,\cdot) \)-order, such as Kirchhoff equations, Choquard equations, etc., there have been some papers on this topic—see [19,41,47–51]. We point out that very recently in [47], Biswas et al. firstly proved a embedding theorem for variable exponential Sobolev spaces and Hardy–Littlewood–Sobolev type result, and then they studied the existence of solutions for Choquard equations as follows

\[
\begin{cases}
(-\Delta)^{s(x,\cdot)}_{p(x)} v(x) \\
= \lambda |v(x)|^{\beta(x)-2}v(x) + \left( \int_{\Omega} \frac{G(y,v(y))}{|x-y|^{p(x)}} \, dy \right) g(x,v(x)), \quad x \in \Omega,
\end{cases}
\]

where \((-\Delta)^{s(x,\cdot)}_{p(x,\cdot)}\) is the \( p(x,\cdot) \)-fractional Laplacian with variable \( s(x,\cdot) \)-order. So far, there are already some work [41,47,48] to deal with problems involving variable fractional order \( s(x,\cdot) \) and variable exponent \( p(x,\cdot) \), but without a Kirchhoff coefficient \( M \). While combining this class of operators with Kirchhoff coefficients, Zuo et al. in [50], investigated the critical Kirchhoff type problem in bounded domains,

\[
\begin{cases}
M \left( \int_{\mathbb{R}^N} \frac{|v(x)|^{p(x)} - |v(y)|^{p(x)}}{|x-y|^{N+1}} \, dy \right) (-\Delta)^{s(x,\cdot)}_{p(x,\cdot)} v(x) \\
= |v(x)|^{\sigma(x)-2}v(x) + \lambda f(x,v(x)), \quad x \in \Omega,
\end{cases}
\]

where \( M \) is a model of Kirchhoff coefficient. With the help of variational methods, the authors proved the existence and asymptotic behavior of nontrivial solutions by using the Brézis and Lieb type lemma for fractional Sobolev spaces with variable-order and
variable exponent. In addition, in the whole space $\mathbb{R}^N$, a new variable-order fractional $p(x, \cdot)$-Kirchhoff type problem under two kinds of weaker conditions was studied in [51].

Problem $(P_\lambda)$ comes from the following system:

$$v_i = \text{div}[Dv_i \nabla v_i] + c(x, v),$$

where $Dv = |\nabla v|^{p-2} + |\nabla v|^{q-2}$. Since the system had a wide range of applications in the field of physics and related sciences, this kind of problem has received much attention, we refer to [1,42,49,52–57]. Such as, in the integer order case, the authors in [54] used the constraint minimization to study the subcritical problem with $p$&$q$-Laplacian and proved the existence of this problem without the Ambrosetti–Rabinowitz condition. While concerning a fractional case, Ambrosio et al. in [1] showed the existence and asymptotic behavior of infinitely many solutions for a fractional $p$&$q$ Laplace operator problem with critical Sobolev–Hardy exponents based on the concentration-compactness principle.

There are few papers [42,49] to consider the $p(x, \cdot)$&$q(x, \cdot)$-Laplacian. Problem for example, [42] studied the following problem

$$\left\{ \begin{aligned}
(-\Delta)^{s(x)}_{p_1(x)} v(x) + (-\Delta)^{s(x)}_{p_2(x)} v(x) + |v(x)|^{q_1(x)-2} v(x) \\
\quad = \lambda V_1(x) |v(x)|^{r_1(x)-2} v(x) - \lambda V_2(x) |v(x)|^{r_2(x)-2} v(x), \quad x \in \Omega,
\end{aligned} \right.$$  \hspace{1cm} (9)

where $p_1, p_2, q, r_1$ and $r_2$ are different continuous functions, while $\lambda, \mu$ are real parameters and $V_1, V_2$ are suitable weights. However, in the above problem (9), they considered a local version of the fractional operator, that is with integral set in $\Omega$ and not in the whole space $\mathbb{R}^N$.

Recently, in [49], Zuo et al. analysed a family of the Choquard type problems with $(-\Delta)^{(s(x))}_{p(x, \cdot)}$ and $(-\Delta)^{(s(x))}_{q(x, \cdot)}$ under some appropriate conditions.

$$\left\{ \begin{aligned}
(-\Delta)^{(s(x))}_{p(x, \cdot)} v(x) + (-\Delta)^{(s(x))}_{q(x, \cdot)} v(x) \\
\quad = \lambda |v(x)|^{\beta(x)-2} v(x) + \left( \int_{\Omega} \frac{G(y, \mu(x))}{|x-y|^{p(x)\cdot \cdot(\cdot)}} dy \right) g(x, v(x)) + k(x), \quad x \in \Omega,
\end{aligned} \right.$$  \hspace{1cm} (10)

where the operators $(-\Delta)^{(s(x))}_{p(x, \cdot)}$ and $(-\Delta)^{(s(x))}_{q(x, \cdot)}$ are two fractional Laplace operators with variable order $s(x, \cdot) : \mathbb{R}^{2N} \to (0, 1)$ and different variable exponents $p(x, \cdot), q(x, \cdot) : \mathbb{R}^{2N} \to (1, \infty)$. The results are different with single fractional Laplace operator.

Motivated by the above cited works, we find that there are no results for Choquard–Kirchhoff type equations involving a variable $s(x, \cdot)$-order fractional $p_1(x, \cdot)$&$p_2(x, \cdot)$-Laplacian. Therefore, we will investigate the existence solutions for this kind of equations, which is different from the work of [42,49] and more general than (9) and (10). Our study extends the previous results in some ways.

Throughout this article, $C_j (j = 1, 2, \ldots, N)$ denote distinct positive constants and $i = 1, 2$. For any real-valued function $H$ defined on a domain $\mathcal{D}$ we denote:

$$C_+ (\mathcal{D}) : = \{ H \in C(\mathcal{D}, \mathbb{R}) : 1 < H^- \leq H \leq H^+ < +\infty \},$$

where $H^-$ := $\min_{\mathcal{D}} H \leq H$ and $H^+$ := $\max_{\mathcal{D}} H$ and $p_{\max}(x, y), p_{\min}(x, y) \in C_+ (\overline{\Omega} \times \overline{\Omega})$.

Concerning the continuous function $\mu$, $g$ and $f$, we assume the following hypothesis:

\textbf{(μ1)}: $\mu(x, y) : \overline{\Omega} \times \overline{\Omega} \to (0, N)$ is symmetric function, i.e., $\mu(x, y) = \mu(y, x)$, and $0 < \mu^- := \inf_{(x, y) \in \overline{\Omega} \times \overline{\Omega}} \mu(x, y) \leq \mu^+ := \sup_{(x, y) \in \overline{\Omega} \times \overline{\Omega}} \mu(x, y) < N$ for all $(x, y) \in \overline{\Omega} \times \overline{\Omega}$.

Furthermore, the nonlinearity $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous Carathéodory function, satisfying:
(G1): There exist a positive constant $C_0$ and $q(x) \in C_+ (\overline{\Omega}) \cap \mathcal{N}$ with $p^+_\text{max} < q^- < p^*_\text{s(x_r)}(x)$ such that

$$|q(x,t)| \leq C_0|t|^q(x)^{-1}, \quad \text{for all } (x,t) \in \overline{\Omega} \times \mathbb{R},$$

where $\mathcal{N} = \{q(x) : \mathcal{P}_i(x) \leq q(x)r^- \leq q(x)r^+ < p^*_\text{s(x_r)}(x) \text{ for all } x \in \Omega \}$ with $r \in C_+ (\overline{\Omega} \times \overline{\Omega})$ such that

$$\frac{2}{r(x,y)} + \frac{\mu(x,y)}{N} = 2, \quad \text{for all } (x,y) \in \overline{\Omega} \times \overline{\Omega}.$$

(G2): There exists a positive constant $\theta \in (\theta p^+_\text{max}, +\infty)$ with $\theta$ is given by (M1) such that

$$0 < \theta G(x,t) \leq 2g(x,t)t, \quad \text{for all } t \in \mathbb{R} \setminus \{0\}, \quad \text{and for all } x \in \overline{\Omega}.$$

(F1): $1 < \gamma(x) < p_i(x,y) < \alpha(x)$ for all $(x,y) \in \overline{\Omega} \times \overline{\Omega}$, where $a(x) \in C_+ (\overline{\Omega})$, $f(x) > 0$ and $f(x) \in L^\infty(\Omega)$.

We need to present the corresponding definition and variational framework before stating our main results.

**Definition 1.** We say that $v \in W_0$ is a weak solution of problem $(P_\lambda)$, if

$$\sum_{i=1}^{2} M_i (\delta_p_i (v)) \times (v, \varphi)_{p_i, W_0} = \lambda \int_{\Omega} f(x)|v|^\gamma(x) - 2v \varphi dx + \int_{\Omega \times \Omega} G(x, v(x))g(y, v(y))\varphi(y) \frac{dxdy}{|x-y|^\beta(x,y)}$$

for any $\varphi \in W_0$, where $W_0$ will be introduced in Section 2 and

$$\delta_p_i (v) = \int_{\Omega \times \Omega} \frac{|v(x) - v(y)|^{p_i(x,y)}}{|x-y|^{\beta(x,y)p_i(x,y)}} dxdy,$$

$$(v, \varphi)_{p_i, W_0} = \int_{\Omega \times \Omega} \frac{|v(x) - v(y)|^{p_i(x,y)-2}(v(x) - v(y))(\varphi(x) - \varphi(y))}{|x-y|^{\beta(x,y)p_i(x,y)}} dxdy.$$

The problem $(P_\lambda)$ has a variational form with the Euler function $I : W_0 \to \mathbb{R}$, which is defined as follows:

$$I(v) := \sum_{i=1}^{2} M_i (\delta_p_i (v)) - \lambda \int_{\Omega} \frac{f(x)}{\gamma(x)}|v|^\gamma(x) dx - \frac{1}{2} \int_{\Omega \times \Omega} \frac{G(x, v(x))g(y, v(y))}{|x-y|^\beta(x,y)} dxdy \quad (11)$$

for all $v \in W_0$ and $\delta_p_i$ given in (M1). Moreover, the function $I$ is well-defined on the Sobolev space $W_0$ and belongs to the class $C^1 (W_0, \mathbb{R})$, for which the argument is similar to Lemma 2.15 of [19], and

$$\langle I'(v), \varphi \rangle := \sum_{i=1}^{2} M_i (\delta_p_i (v)) \times (v, \varphi)_{p_i, W_0} - \lambda \int_{\Omega} f(x)|v|^\gamma(x) - 2v \varphi dx$$

$$- \int_{\Omega \times \Omega} \frac{G(x, v(x))g(y, v(y))\varphi(y)}{|x-y|^\beta(x,y)} dxdy \quad (12)$$

for any $\varphi \in W_0$. Thus, under our assumptions, the existence of weak solutions of problem $(P_\lambda)$ is equivalent to the existence of critical points for the Euler function $I$. 
Now, we are ready to state the first result of this paper as follows.

**Theorem 1.** Assume that (M1) and (M2), (µ1), (G1) and (G2) and (F1) are satisfied. Let Ω be a bounded smooth domain of \( \mathbb{R}^N \) with \( N > s(x,y)p(x,y) \) for any \( (x,y) \in \bar{\Omega} \times \bar{\Omega} \), where \( s(x,\cdot) \) and \( p(x,\cdot) \) verify (S1) and (P1). Then, there exists \( \lambda > 0 \) such that for any \( 0 < \lambda \leq \Lambda \), the problem \((P_\lambda)\) admits at least one positive energy solution \( v_1 \) in \( W_0 \).

In order to obtain our other result, we need the following assumption.

(M3): For any \( \tau > 0 \), there are two positive constants \( m_i \) and \( m_i' = m_i(\tau) > 0 \) such that

\[
m_i' \geq M_i(t) \geq m_i, \quad \text{for all } t > \tau,
\]

where \( m_i \) come from (M2) and \( m_i' > m_i \).

**Theorem 2.** Assume that (M1), (M3), (µ1), (G1) and (G2) and (F1) are satisfied. Let Ω be a bounded smooth domain of \( \mathbb{R}^N \) with \( N > s(x,y)p(x,y) \) for any \( (x,y) \in \bar{\Omega} \times \bar{\Omega} \), where \( s(x,\cdot) \) and \( p(x,\cdot) \) verify (S1) and (P1). Then, there exists \( \lambda > 0 \) such that for any \( 0 < \lambda \leq \Lambda \), the problem \((P_\lambda)\) admits at least one negative energy solution \( v_2 \) in \( W_0 \).

**Remark 1.** The main idea to overcome these difficulties lies on the \((-\Delta)^s(x,\cdot)\) and \((-\Delta)^p(x,\cdot)\) Laplace operators developed in [42,49] recently. By using the mountain pass theorem [58], we prove Theorem 1; then, by means of the Ekeland’s variational principle [59], we give the Proof of Theorem 2.

**Remark 2.** Our work is different from the previous papers [1,42,49,54] in the sense because of Kirchhoff terms and the presence of the more complicated operator and Choquard type nonlinearities, which makes our analysis more complicated. The work of this paper is to be of great importance in the development of the \((-\Delta)^s(x,\cdot)\) and \((-\Delta)^\phi(x,\cdot)\)-Laplace operators theory.

The remainder of this paper is organized as follows. Some preliminary results about the fractional Lebesgue spaces and Sobolev spaces are given in Section 2. Theorems 1 and 2 are proved in Section 3. In Section 4, we make a conclusion.

2. Preliminary Results

2.1. Variable Exponents Lebesgue Spaces

In this subsection, we recall some knowledge of generalized variable exponents Lebesgue spaces and give some important lemmas and propositions, which will be used later. For a more detailed information, the reader is invited to consult [43,44,46,60–62].

Let \( \theta(x) \in C_+(\bar{\Omega}) \) and \( v \) be a real-valued function, we introduce the variable exponents Lebesgue spaces as

\[
L^{\theta(x)}(\Omega) := \left\{ v : v \text{ is a measurable and } \int_{\Omega} |v|^{\theta(x)} \, dx < \infty \right\},
\]

with the norm

\[
\|v\|_{\theta(x)} := \inf \left\{ \chi > 0 : \int_{\Omega} \frac{|v|^{\theta(x)}}{\chi} \, dx \leq 1 \right\},
\]

then \((L^{\theta(x)}(\Omega), \| \cdot \|_{\theta(x)})\) is a separable and reflexive Banach space, see [44,62], called generalized Lebesgue spaces.

Let \( \hat{\theta}(x) \) be the conjugate exponent of \( \theta(x) \), that is

\[
\frac{1}{\theta(x)} + \frac{1}{\hat{\theta}(x)} = 1, \text{ for all } x \in \bar{\Omega}.
\]
Lemma 1 (Theorem 2.1 of [62] (Hölder inequality)). Suppose that \( v \in L^{\theta(x)}(\Omega) \) and \( u \in L^{\theta(x)}(\Omega) \), then

\[
\left| \int_{\Omega} v u \, dx \right| \leq \left( \frac{1}{\theta} + \frac{1}{1-\theta} \right) \| v \|_{\theta(x)} \| u \|_{\theta(x)} \leq 2 \| v \|_{\theta(x)} \| u \|_{\theta(x)}.
\]

The modular of \( L^{\theta(x)}(\Omega) \), which is the mapping \( \rho_{\theta(x)} : L^{\theta(x)}(\Omega) \to \mathbb{R} \), is defined by \( \rho_{\theta(x)}(v) := \int_{\Omega} |v|^{\theta(x)} \, dx \). The relation between the modular and Luxemburg norm has the following important properties, given in [44,61].

Proposition 1. Suppose that \( v_n, v \in L^{\theta(x)}(\Omega) \); then, the following properties hold

\( 1 \) \( \| v \|_{\theta(x)} > 1 \Rightarrow \| v \|_{\theta(x)} \leq \rho_{\theta(x)}(v) \leq \| v \|_{\theta(x)}' \),
\( 2 \) \( \| v \|_{\theta(x)} < 1 \Rightarrow \| v \|_{\theta(x)} \leq \rho_{\theta(x)}(v) \leq \| v \|_{\theta(x)}' \),
\( 3 \) \( \| v \|_{\theta(x)} < 1 \) (resp. \( = 1, > 1 \)) \( \Leftrightarrow \rho_{\theta(x)}(v) < 1 \) (resp. \( = 1, > 1 \)),
\( 4 \) \( \| v_n \|_{\theta(x)} \to 0 \) (resp. \( \to +\infty \)) \( \Leftrightarrow \rho_{\theta(x)}(v_n) \to 0 \) (resp. \( \to +\infty \)),
\( 5 \) \( \lim_{n \to \infty} |v_n - v|_{\theta(x)} = 0 \Leftrightarrow \lim_{n \to \infty} \rho_{\theta(x)}(v_n - v) = 0 \).

Remark 3. Note that for any functions \( \theta_1(x), \theta_2(x) \in C_+((\Omega)) \) and \( \theta_1(x) < \theta_2(x) \), there is an embedding \( L^{\theta_1(x)}(\Omega) \hookrightarrow L^{\theta_2(x)}(\Omega) \) for any \( x \in \Omega \). Especially, when \( \theta(x) \equiv \text{constant} \), the results of Proposition 1 still hold.

Lemma 2. Lemma A.1 of [45] Assume that \( \theta_1(x) \in L^\infty(\Omega) \) such that \( \theta_1(x) \geq 0 \) and \( \theta_1(x) \neq 0 \) a.e. in \( \Omega \). Let \( \theta_2(x) : \Omega \to \mathbb{R} \) be a measurable function such that \( \theta_1(x) \theta_2(x) \geq 1 \) a.e. in \( \Omega \). Then, for any \( v \in L^{\theta_1(x)}(\Omega) \),

\[
\left\| v \right\|_{\theta_2(x)} \leq \left\| v \right\|_{\theta_1(x) \theta_2(x)} + \left\| v \right\|_{\theta_1(x) \theta_2(x)'}.
\]

For variable exponents Lebesgue spaces, we state the following propositions given in [43], which is imperative in this paper.

Proposition 2. Let \( \theta_1(x) \) and \( \theta_2(x) \) be measurable functions such that \( \theta_1(x) \in L^\infty(\mathbb{R}) \) and \( 1 \leq \theta_1(x) \theta_2(x) \leq \infty \), for a.e. \( x \in \Omega \). Let \( v \in L^{\theta_2(x)}(\Omega) \), \( v \neq 0 \). Then,

\( 1 \) \( \| v \|_{\theta_1(x) \theta_2(x)} \leq 1 \Rightarrow \| v \|_{\theta_1(x) \theta_2(x)'} \leq \| v \|_{\theta_1(x) \theta_2(x)}, \)
\( 2 \) \( \| v \|_{\theta_1(x) \theta_2(x)'} \geq 1 \Rightarrow \| v \|_{\theta_1(x) \theta_2(x)'} \leq \| v \|_{\theta_1(x) \theta_2(x)} \)

In particular, if \( \theta_1(x) = \theta_1 \) is constant, we have \( \| v \|_{\theta_2(x)} = \| v \|_{\theta_2(x)'} \).

Now, we review a suitable estimate result, given in Proposition 4.1 of [47] and in Proposition 2.4 of [63].

Proposition 3. \( \mu(x,y) \) satisfies (\( \mu 1 \)). Set \( \theta_1'(x,y), \theta_2'(x,y) \in C_+((\Omega) \times \Omega) \) verify

\[
\frac{1}{\theta_1'(x,y)} + \frac{\mu(x,y)}{N} + \frac{1}{\theta_2'(x,y)} = 2, \text{ for any } (x,y) \in \Omega \times \Omega.
\]

Then, for \( u \in L^{\theta_1'(x)}(\Omega) \cap L^{\theta_1'(x)}(\Omega) \) and \( v \in L^{\theta_2'(x)}(\Omega) \cap L^{\theta_2'(x)}(\Omega) \), we have

\[
\left| \int_{\Omega \times \Omega} \frac{u(x)v(x)}{|x-y|^{\mu(x,y)}} \, dx \, dy \right| \leq C \left( \| u \|_{L^{\theta_1'(x)}(\Omega)} \| v \|_{L^{\theta_1'(x)}(\Omega)} + \| u \|_{L^{\theta_2'(x)}(\Omega)} \| v \|_{L^{\theta_2'(x)}(\Omega)} \right)
\]
for a suitable positive constant $C_1$, independent of $u$ and $v$.

2.2. Variable-Order Fractional Sobolev Spaces

From now on, we briefly review some basic properties about fractional Sobolev spaces with variable-order and introduce some important lemmas and propositions, which will be used as tools to prove our main results. We refer to [41,47,48,51] and the references therein for the important knowledge on this subject.

Let $p(x,\cdot) \in C_+ (\overline{\Omega} \times \overline{\Omega})$, $s(x,\cdot) : \overline{\Omega} \times \overline{\Omega} \to (0,1)$ be continuous functions with $N > s(x,y)p(x,y)$ for all $(x,y) \in \overline{\Omega} \times \overline{\Omega}$, and define the Gagliardo seminorm by

$$[v]_{s(x,\cdot),p(x,\cdot)} := \inf \left\{ \chi > 0 : \int_{\Omega \times \Omega} \frac{|v(x) - v(y)|^{p(x,y)}}{\chi^{p(x,y)}} |x-y|^{N+p(x,y)s(x,y)} \, dx \, dy < 1 \right\},$$

where $v : \Omega \to \mathbb{R}$ is continuous. Now, the variable-order fractional Sobolev spaces with variable exponents is denoted as

$$W = W^{s(x,\cdot),p(x,\cdot)}(\Omega) := \left\{ v \in L_{\mathcal{P}(\cdot)}(\Omega) : v \text{ is measurable and } [v]_{s(x,\cdot),p(x,\cdot)} < \infty \right\},$$

endowed with the norm

$$\|v\|_W := \|v\|_{\mathcal{P}(\cdot)} + [v]_{s(x,\cdot),p(x,\cdot)}.$$

Let $W_0$ be the linear space of Lebesgue measurable functions from $\Omega$ to $\mathbb{R}$ such that any function $u = 0$ on $\partial \Omega$ and belongs to $L_{\mathcal{P}(\cdot)}(\Omega)$, and endowed $W_0$ with the norm

$$\|v\|_{W_0} := [v]_{s(x,\cdot),p(x,\cdot)}.$$

then, $(W_0, \| \cdot \|_{W_0})$ is also a reflexive Banach space, see [47]. $W_0'$ denotes the dual spaces of $W_0$.

Define the modular function $\rho_{p(x,\cdot)}^{s(x,\cdot)} : W_0 \to \mathbb{R}$ by

$$\rho_{p(x,\cdot)}^{s(x,\cdot)}(v) = \int_{\Omega \times \Omega} \frac{|v(x) - v(y)|^{p(x,y)}}{|x-y|^{N+p(x,y)s(x,y)}} \, dx \, dy.$$

Proposition 4. Lemmas 2.2 and 2.3 of [41] Let $v \in W_0$ and $\{v_n\} \subset W_0$, then

1. $\|v\|_{W_0} < 1$ (resp. $= 1$, $> 1$) $\Leftrightarrow \rho_{p(x,\cdot)}^{s(x,\cdot)}(v) < 1$ (resp. $= 1$, $> 1$),

2. $\|v\|_{W_0} < 1 \Rightarrow \rho_{p(x,\cdot)}^{s(x,\cdot)}(v) \leq \rho_{p(x,\cdot)}^{s(x,\cdot)}(v) \leq \|v\|_{W_0}$,

3. $\|v\|_{W_0} > 1 \Rightarrow \rho_{p(x,\cdot)}^{s(x,\cdot)}(v) \leq \rho_{p(x,\cdot)}^{s(x,\cdot)}(v) \leq \|v\|_{W_0}$,

4. $\lim_{n \to \infty} \|v_n\|_{W_0} = 0$ (resp. $\to +\infty$) $\Leftrightarrow \lim_{n \to \infty} \rho_{p(x,\cdot)}^{s(x,\cdot)}(v_n) = 0$ (resp. $\to +\infty$),

5. $\lim_{n \to \infty} \|v_n - v\|_{W_0} = 0$ $\Leftrightarrow \lim_{n \to \infty} \rho_{p(x,\cdot)}^{s(x,\cdot)}(v_n - v) = 0$.

We now introduce a compact embedding theorem for $W_0$, whose proof can be inspired by Theorem 3.1 of [47] and adapted in a $p_1(x,\cdot) \& p_2(x,\cdot)$ setting.

Lemma 3. Assume that $s(x,\cdot)$, $p(x,\cdot)$ fulfill (S1), (P1) with $N > p(x,y)s(x,y)$ for any $(x,y) \in \overline{\Omega} \times \overline{\Omega}$. Set $\phi(x) \in C_+ (\overline{\Omega})$ fulfill

$$1 < \phi^- = \min_{x \in \overline{\Omega}} \phi(x) \leq \phi(x) < \phi^+(x), \text{ for any } x \in \overline{\Omega},$$

for a suitable positive constant $C_1$, independent of $u$ and $v$. 
where \( p^s_{\xi} (x) = \frac{Np(x)}{N - p(x) s(x)}, \) \( p(x) = p(x,x) \) and \( s(x) = s(x,x). \) Then, there exists \( C_\phi = C_\phi(N,s,p,\phi,\Omega) > 0 \) such that
\[
\|v\|_{\phi(\cdot)} \leq C_\phi \|v\|_{W_0}
\]
for any \( v \in W_0. \) Moreover, the embedding \( W_0 \hookrightarrow L^p(\Omega) \) is compact.

**Proposition 5** (Theorem 2.1 of \([47]\) (Hardy–Littlewood–Sobolev type inequality)). Let \( s(x, \cdot), \mu(x, y), \) and \( p(x, \cdot) \) satisfy (S1), (\( \mu_1 \)), and (P1) with \( N > p^+ s^+. \) Let \( r \in C_+ (\overline{\Omega} \times \overline{\Omega}) \) be as in (F1). \( q \in C_+ (\overline{\Omega}) \cap \mathcal{N} \) where \( \mathcal{N} \) is defined in (G1). Then, for any \( v \in W_0 \) we have \( |v|^q(\cdot) \in L^{r+}(\Omega) \cap L^{r-}(\Omega) \) with
\[
\left( \int_{\Omega \times \Omega} \frac{|v(x)|^q(x)|v(y)|^q(y)}{|x - y|^{p(x,y)}} \, dx \, dy \right)^{1/q(x)} \leq C_2 \left( \left( \int_{\Omega} |v|^q(\cdot) \right)^{r+} + \left( \int_{\Omega} |v|^q(\cdot) \right)^{r-} \right).
\]
for a suitable positive constant \( C_2, \) independent of \( v. \)

### 3. The Proof of the Main Results

#### 3.1. Palais–Smale Compactness Condition

Let \( W_0 \) be a Banach space, \( I \in C^1(W_0, \mathbb{R}). \) We say that \( I \) satisfies the Palais–Smale condition, if any (PS)_c sequence \( \{v_n\}_{n \in \mathbb{N}} \subset W_0 \) with
\[
I(v_n) \to c, \quad I'(v_n) \to 0 \quad \text{in} \quad W_0^* \quad \text{as} \quad n \to \infty,
\]
possesses a convergent subsequence in \( W_0. \)

**Lemma 4.** Suppose that (P1), (M1) and (M2), (G2) and (F1) are satisfied, moreover, \( I(v_n) \) is bounded and \( I'(v_n) \to 0 \) as \( n \to \infty. \) Then, the sequence \( \{v_n\}_{n \in \mathbb{N}} \) is bounded in \( W_0. \)

**Proof.** We show that the sequence \( \{v_n\}_{n \in \mathbb{N}} \) is bounded in \( W_0. \) There exists a sequence \( \{v_n\}_{n \in \mathbb{N}} \subset W_0, \) such that \( \{I(v_n)\}_{n \in \mathbb{N}} \) is bounded and \( I'(v_n) \to 0 \) as \( n \to \infty. \) Then, there is a positive constant \( c \) such that
\[
|I(v_n)| \leq c \quad \text{and} \quad \langle I'(v_n), v_n \rangle = o_n(1)
\]
for every \( n \in \mathbb{N}. \) We prove this by contrary arguments. Assume that
\[
\|v_n\|_{W_0} \to \infty, \quad \text{as} \quad n \to \infty.
\]
Indeed, from (P1), we can easily derive the following inequality for any \( v \) and \( (x,y) \in \overline{\Omega} \times \overline{\Omega}
\[
\frac{|v(x) - v(y)|^{p_1(x,y)}}{|x - y|^{N + p_1(x,y)s(x,y)}} + \frac{|v(x) - v(y)|^{p_2(x,y)}}{|x - y|^{N + p_2(x,y)s(x,y)}} + \frac{|v(x) - v(y)|^{p_{\max}(x,y)}}{|x - y|^{N + p_{\max}(x,y)s(x,y)}} \geq \frac{|v(x) - v(y)|^{p_{\max}(x,y)}}{|x - y|^{N + p_{\max}(x,y)s(x,y)}}.
\]

According to Proposition 2 and Hölder’s inequality, we obtain that for all $v$

$$\left| \int_{\Omega} \frac{f(x)}{\gamma(x)} \vert v \vert^{\gamma(x)} \, dx \right|$$

$$\leq \frac{1}{\gamma} \int_{\Omega} |f(x)|_{a(x)} \|v\|^\gamma(x) \, dx$$

$$= \begin{cases} \frac{1}{\gamma} \int_{\Omega} |f(x)|_{a(x)} \|v\|^\gamma(x) \, dx & \text{if } |v|_{a(x)} \leq 1, \\ \frac{1}{\gamma} \int_{\Omega} |f(x)|_{a(x)} \|v\|^\gamma(x) \, dx & \text{if } |v|_{a(x)} > 1. \end{cases}$$

(17)

Now, letting $h(x) = \frac{a(x) \gamma(x)}{a(x) - 1}$ and $k(x) = \frac{a(x) \gamma(x)}{a(x) - \gamma(x)}$, from the condition (F1), we have $h(x) < p^*(x, x')$ and $k(x) < p^*(x, x')$ for all $x \in \Omega$. Therefore, under the conditions (P1) and (F1), the embeddings $W_0 \hookrightarrow L^{h(x)}(\Omega)$ and $W_0 \hookrightarrow L^{k(x)}(\Omega)$ are continuous and compact. Thus, there exists $C_{13} > 0$ such that

$$\|v\|_{h(x)} \leq C_{13} \|v\|_{W_0}, \text{ for any } v \in W_0.$$  

(18)

Thus, using (G2), (M1) and (M2), (14), (16), (18), Propositions 1 and 2, Lemma 3, and Hölder’s inequality, there exists $c_{13} > 0$ such that

$$c + c_{13} \|v_n\|_{W_0} + o_\theta(1)$$

$$\geq \theta I(v_n) - \langle I'(v_n), v_n \rangle$$

$$\geq \theta \sum_{i=1}^{2} M_i(\delta_{p_i}(v_n)) - \theta \lambda \int_{\Omega} f(x) |v_n|^{\gamma(x)} \, dx$$

$$- \frac{\theta}{2} \int_{\Omega \times \Omega} \frac{G(x, v_n(x)) G(y, v_n(y))}{|x - y|^{\theta(x,y)}} \, dxdy$$

$$- \sum_{i=1}^{2} M_i(\delta_{p_i}(v_n)) \times (v_n, v_n)_{p_i, W_0} + \lambda \int_{\Omega} f(x) |v_n|^{\gamma(x)} \, dx$$

$$+ \int_{\Omega \times \Omega} \frac{G(x, v_n(x)) G(y, v_n(y)) v_n(y)}{|x - y|^{\theta(x,y)}} \, dxdy$$

$$\geq \left( - \frac{\theta}{\theta p_{max}} - 1 \right) \sum_{i=1}^{2} M_i(\delta_{p_i}(v_n)) \times (v_n, v_n)_{p_i, W_0}$$

$$- \lambda \int_{\Omega} \left( - \frac{\theta}{\theta p_{max}} - 1 \right) f(x) |v_n|^{\gamma(x)} \, dx$$

$$- \int_{\Omega \times \Omega} \frac{G(x, v_n(x)) |\theta G(y, v_n(y)) - G(y, v_n(y))| v_n(y)}{|x - y|^{\theta(x,y)}} \, dxdy$$

$$\geq \left( - \frac{\theta}{\theta p_{max}} - 1 \right) \sum_{i=1}^{2} M_i(\delta_{p_i}(v_n)) \times (v_n, v_n)_{p_i, W_0}$$

$$- \lambda \int_{\Omega} \left( - \frac{\theta}{\theta p_{max}} - 1 \right) f(x) |v_n|^{\gamma(x)} \, dx$$

$$\geq \left( - \frac{\theta}{\theta p_{max}} - 1 \right) \min\{1, m_1, m_2\} \int_{\Omega \times \Omega} \frac{|v(x) - v(y)|^{p_{max}(x,y)} \, dxdy}{|x - y|^{N + p_{max}(x,y)s(x,y)}}$$

$$- \lambda \left( - \frac{\theta}{\theta} - 1 \right) \int_{\Omega} f(x) |v_n|^{\gamma(x)} \, dx$$

$$\geq \left( - \frac{\theta}{\theta p_{max}} - 1 \right) \min\{1, m_1, m_2\} \|v_n\|_{W_0}^{p_{max}}$$

$$- \lambda \left( - \frac{\theta}{\theta} - 1 \right) \|f(x)\|_{a(x)} C_{13}^{p_{max}} \|v_n\|_{W_0}^{p_{max}}$$
Since \( \theta > \theta_{p_{\text{max}}}^+ \) and \( 1 < \gamma^+ < p_{\text{max}}^+ \) we immediately get a contradiction from the above estimate. Hence, the sequence \( \{v_n\}_{n \in \mathbb{N}} \) is bounded in \( W_0 \).  

**Lemma 5.** Assume that (P1), (M1), (M3), (G1) and (F1) hold. Suppose that \( I(v_n) \) is bounded and \( I'(v_n) \rightarrow 0 \) as \( n \rightarrow \infty \). Then, the sequence \( v_n \in W_0 \) is bounded.

**Proof.** We follow the proof of Lemma 4, it is easy to obtain the conclusion.  

**Lemma 6.** Assume that (M2), (G1) and (F1) hold, and \( \lambda \in \mathbb{R} \). If the sequence \( \{v_n\}_{n \in \mathbb{N}} \subset W_0 \) is a (PS) sequence of \( I \), then \( \{v_n\}_{n \in \mathbb{N}} \) has a strong convergent subsequence.

**Proof.** If the sequence \( \{v_n\}_{n \in \mathbb{N}} \subset W_0 \) is a Palais–Smale sequence of \( I \), then \( \{I(v_n)\}_{n \in \mathbb{N}} \) is bounded and \( I'(v_n) \rightarrow 0 \) as \( n \rightarrow \infty \), and we infer from Lemma 4 that \( \{v_n\}_{n \in \mathbb{N}} \) is bounded in \( W_0 \). Thus, there exists \( v \in W_0 \), and we can extract a subsequence, still denoted by \( \{v_n\}_{n \in \mathbb{N}} \), satisfying

\[
v_n \rightharpoonup v \text{ in } W_0, \quad v_n \rightarrow v \text{ in } L^{\theta(x)}(\Omega), \quad v_n \rightarrow v \text{ a.e. in } \Omega.
\]

Furthermore, we have as \( n \rightarrow \infty \)

\[
|\langle I'(v_n), v_n - v \rangle| \leq \|I'(v_n)\| \left(\|v_n\|_{W_0} + \|v\|_{W_0}\right) \rightarrow 0.
\]

Since \( v_n \) is bounded in \( W_0 \) and \( I'(v_n) \rightarrow 0 \), it follows that

\[
\langle I'(v_n), v_n - v \rangle \rightarrow 0, \text{ as } n \rightarrow \infty.
\]

We derive that

\[
\alpha_n(1) = \langle I'(v_n), v_n - v \rangle \\
= M_1(\delta_{p_1}(v_n)) \times \langle v_n, v_n - v \rangle_{L^p_{\Omega}, W_0} \\
+ M_2(\delta_{p_2}(v_n)) \times \langle v_n, v_n - v \rangle_{L^p_{\Omega}, W_0} \tag{20}
- \int_{\Omega} \lambda f(x)|v_n|^{\gamma(x)-2}v_n(v_n - v) \, dx \\
- \int_{\Omega \times \Omega} \frac{G(x, v_n(x))g(y, v_n(y))(v_n - v)(y)}{|x - y|^{\mu(x,y)}} \, dx \, dy.
\]

Therefore, from (G1), Proposition 1 and Lemma 3, we obtain

\[
\|G(\cdot, v_n)\|_{r^+} \leq C_3 \left( \int_{\Omega} |v_n|^{q(x)r^+} \, dx \right)^{\frac{1}{r^+}} \\
\leq C_3 \max \left\{ \|v_n\|_{q(x)r^+}^{q(x)r^+}, \|v_n\|_{q(x)r^+}^{q(x)r^+} \right\} \tag{21}
\]

that is \( G(\cdot, v_n) \in L^{r^+}(\Omega) \). Similarly, we have

\[
\|G(\cdot, v_n)\|_{r^-} \leq C_3 \max \left\{ C_{q(x)r^-}^-, \|v_n\|_{W_0}^{q(x)r^-}, C_{q(x)r^-}^+, \|v_n\|_{W_0}^{q(x)r^-} \right\}.
\]
Thus, combined with (21) and (22) and Proposition 3, we obtain
\[
\left| \int_{\Omega \times \Omega} \frac{G(x, v_n(x))g(y, v_n(y))(v_n - v)(y)}{|x - y|^p(x,y)} \, dx \, dy \right| \\
\leq C_4 \left( \|G(x, v_n(x))\|_{r+} \|g(y, v_n(y))\|_r \right)_{r+} + \|G(x, v_n(x))\|_{r-} \|g(y, v_n(y))\|_r_{r-} \\
\leq C_5 \max \left\{ C_{q(x)r+}^\delta \|v_n\|_{W_0}^q, C_{q(x)r-}^\delta \|v_n\|_{W_0}^q \right\} \\
\times \|g(y, v_n(y))\|_{r+} \|v_n - v\|_r \|v_n - v\|_{r-} \\
+ C_5 \max \left\{ C_{q(x)r+}^\delta \|v_n\|_{W_0}^q, C_{q(x)r-}^\delta \|v_n\|_{W_0}^q \right\} \\
\times \|g(y, v_n(y))\|_{r-} \|v_n - v\|_r \|v_n - v\|_{r+}. 
\]

(23)

Next, using (G1), Lemmas 1 and 2, and (19), we obtain as \( n \to \infty \),
\[
\|g(y, v_n(y))(v_n - v)(y)\|_{r+}^r \leq C_6 \int_{\Omega} |v_n|^{(q(x)-1)r+} |v_n - v| r+ \, dx \\
\leq C_7 \|v_n^{q(x)-1r+} \|_{q(x)}^r \|v_n - v\|_{q(x)}^r \\
\leq C_8 \left( \|v_n^{q(x)-1r+} \|_{q(x)}^r + \|v_n^{q(x)-1r+} \|_{q(x)}^r \right) \|v_n - v\|_{q(x)}^r \\
\leq C_9 \left( \|v_n^{q(x)-1r+} \|_{W_0}^r + \|v_n^{q(x)-1r+} \|_{W_0}^r \right) \|v_n - v\|_{q(x)}^r \\
\leq C_{10} \|v_n - v\|_{q(x)r+}^r = o_n(1). 
\]

(24)

Similarly, we have as \( n \to \infty \),
\[
\|g(y, v_n(y))(v_n - v)(y)\|_{r-}^r = o_n(1). 
\]

(25)

Hence, combining with (23) and (25), we derive
\[
\lim_{n \to \infty} \int_{\Omega \times \Omega} \frac{G(x, v_n(x))g(y, v_n(y))(v_n - v)(y)}{|x - y|^p(x,y)} \, dx \, dy = 0. 
\]

(26)

Since \( \{v_n\} \) is bounded in \( W_0 \). Thus, there exists a subsequence \( \{v_{n_k}\} \) converges weakly to \( v \) in \( W_0 \). As \( k(x) = \frac{a(x)\gamma(x)}{a(x) - \gamma(x)} < p_s^*(x) \) for all \( x \in \Omega \), so we deduce that there exists a compact embedding \( W_0 \to L^{k(x)}(\Omega) \); hence, the sequence \( \{v_n\} \) converges strongly to \( v \) in \( L^{k(x)}(\Omega) \).

According to the hypothesis (F1), using Hölder's inequality, we infer
\[
\int_{\Omega} f(x)|v_n|^{\gamma(x)} - 2v_n(v_n - v) \, dx \\
\leq |f(x)|_{a(x)} \|v_n|^{\gamma(x)} - 2v_n(v_n - v)\|_{h(x)} \\
\leq |f(x)|_{a(x)} \|v_n|^{\gamma(x)} - 2v_n\|_{\frac{\gamma(x)}{\gamma(x)-1}} \|v_n - v\|_{a(x)}. 
\]

(27)

Now, if \( \|v_n\|^{\gamma(x)} - 2v_n\|_{\frac{\gamma(x)}{\gamma(x)-1}} > 1 \) then we obtain \( \|v_n\|^{\gamma(x)} - 2v_n\|_{\frac{\gamma(x)}{\gamma(x)-1}} \leq |v_n|^{\gamma(x)} \). The compact embedding \( W_0 \to L^{\gamma(x)}(\Omega) \) helps us to show that
\[
\lim_{n \to \infty} \int_{\Omega} f(x)|v_n|^{\gamma(x)} - 2v_n(v_n - v) \, dx = 0. 
\]

(28)
Therefore, from (26) and (28), we obtain
\[
\lim_{n \to \infty} \langle I'(v_n), v_n - v \rangle = \lim_{n \to \infty} \left[ M_1(\delta_{p_1}(v_n)) \times \langle v_n, v_n - v \rangle_{p_1, W_0} + M_2(\delta_{p_2}(v_n)) \times \langle v_n, v_n - v \rangle_{p_2, W_0} \right] = 0,
\]
combining this with relations (16) and (M2), it follows that as \( n \to \infty \)
\[
o(1) = M_1(\delta_{p_1}(v_n)) \times \langle v_n, v_n - v \rangle_{p_1, W_0} + M_2(\delta_{p_2}(v_n)) \times \langle v_n, v_n - v \rangle_{p_2, W_0} \geq \min\{1, m_1, m_2\} \langle v_n, v_n - v \rangle_{p_{\max}, W_0}.
\]
Fixed \((x, y) \in (\Omega \times \Omega)\), by the Young inequality and direct calculations, we obtain
\[
|v_n(x) - v_n(y)|^{p_{\max}(x, y)} \leq \frac{1}{p'(x, y)} |v_n(x) - v_n(y)|^{p_{\max}(x, y)} + \frac{1}{p(x, y)} \langle v(x) - v(y) \rangle_{p_{\max}(x, y)}
\]
(30)
such that
\[
\min\{1, m_1, m_2\} \langle v_n, v_n - v \rangle_{p_{\max}, W_0} \geq C_{11} \left( \int_{\Omega \times \Omega} \frac{|v_n(x) - v_n(y)|^{p_{\max}(x, y)}}{|x - y|^{N + p_{\max}(x, y)}} dxdy \right)^{\frac{1}{p_{\max}(x, y)}} - \int_{\Omega \times \Omega} \frac{|v_n(x) - v_n(y)|^{p_{\max}(x, y)} - 1}{|x - y|^{N + p_{\max}(x, y)}} dxdy \geq C_{12} \left( \int_{\Omega \times \Omega} \frac{|v_n(x) - v_n(y)|^{p_{\max}(x, y)}}{|x - y|^{N + p_{\max}(x, y)}} dxdy \right)^{\frac{1}{p_{\max}(x, y)}} - \int_{\Omega \times \Omega} \frac{|v(x) - v(y)|^{p_{\max}(x, y)}}{|x - y|^{N + p_{\max}(x, y)}} dxdy.
\]
From (19) and the Fatou lemma,
\[
\liminf_{n \to \infty} \int_{\Omega \times \Omega} \frac{|v_n(x) - v_n(y)|^{p_{\max}(x, y)}}{|x - y|^{N + p_{\max}(x, y)}} dxdy \geq \int_{\Omega \times \Omega} \frac{|v(x) - v(y)|^{p_{\max}(x, y)}}{|x - y|^{N + p_{\max}(x, y)}} dxdy,
\]
which combined with (29) and (31) yields
\[
\lim_{n \to \infty} \int_{\Omega} \frac{|v_n(x) - v_n(y)|^{p_{\max}(x,y)}}{|x - y|^{N + p_{\max}(x,y)/s(x,y)}} \, dx \, dy
= \int_{\Omega} \frac{|v(x) - v(y)|^{p_{\max}(x,y)}}{|x - y|^{N + p_{\max}(x,y)/s(x,y)}} \, dx \, dy.
\tag{32}
\]

However, using (19) and the Brézis–Lieb type lemma for variable exponent in Lemma 2.4 of [50], we obtain
\[
\int_{\Omega} \frac{|v_n(x) - v_n(y)|^{p_{\max}(x,y)}}{|x - y|^{N + p_{\max}(x,y)/s(x,y)}} \, dx \, dy - \int_{\Omega} \frac{|v(x) - v(y)|^{p_{\max}(x,y)}}{|x - y|^{N + p_{\max}(x,y)/s(x,y)}} \, dx \, dy
= \int_{\Omega} \frac{|v_n(x) - v_n(y) - v(x) + v(y)|^{p_{\max}(x,y)}}{|x - y|^{N + p_{\max}(x,y)/s(x,y)}} \, dx \, dy + o_n(1),
\]
which joint with (32), we obtain
\[
\lim_{n \to \infty} \int_{\Omega} \frac{|v_n(x) - v_n(y) - v(x) + v(y)|^{p_{\max}(x,y)}}{|x - y|^{N + p_{\max}(x,y)/s(x,y)}} \, dx \, dy
= \lim_{n \to \infty} p_{\max}(\cdot)(v_n - v)
= 0,
\]
according to Proposition 4, we finally achieve the strong convergence of \( v_n \to v \) as \( n \to \infty \) in \( W_0 \). \( \square \)

**Lemma 7.** Assume that (M3), (G1) and (F1) hold, and \( \lambda \in \mathbb{R} \). If the sequence \( \{v_n\}_{n \in \mathbb{N}} \subset W_0 \) is a (PS) sequence of \( I \), then \( \{v_n\}_{n \in \mathbb{N}} \) has a strong convergent subsequence.

**Proof.** The proof is a slight modification of Lemma 6 and is omitted. \( \square \)

3.2. Proof of Theorem 1

In what follows, we prove Theorem 1 by applying the mountain pass theorem [58].

**Lemma 8.** Assume that (S1), (P1), (M1) and (M2), (G1), and (F1) satisfy. Then, for all \( \rho_0 \in (0, \min\{1, 1/C_{\rho}\}) \), there exists positive constants \( \lambda \) and \( \alpha_0 = \alpha_0(\rho_0) \) such that for all \( \lambda \in (0, \lambda] \) we have the function \( I \geq \alpha_0(\rho_0) \) for all \( v \in W_0 \) with \( \|v\|_{W_0} = \rho_0 \).

**Proof.** Let \( \rho \in W_0 \) be such that
\[
\|v\|_{W_0} = \min\{1, 1/C_{\rho}\},
\]
where \( C_{\rho} \) is given by Lemma 3. Hence, using the definition of the function \( I \), conditions (M1) and (M2), relations (16), (18), (21), (22), Hölder’s inequality, Lemma 3, and Proposition 1, 2, 4, 5, we deduce that for any \( v \in W_0 \) with \( \|v\|_{W_0} = \rho_0 \in (0, \min\{1, 1/C_{\rho}\}) \),
with (34), we obtain
\[ M(35), \text{there exists a constant} \] 
\[ \lambda \text{for any } t > 0 \in \mathbb{R}. \] 

Therefore, putting \( \lambda = \rho_0^{p_{\max}^- - \gamma} M_{\min} \gamma^- / 4 \rho_0^{p_{\max}^+} \| f(x) \|_{\alpha(x)} C_{13}^{-\gamma} \) \( = 0 \) and combining with (34), we obtain
\[ I(v) \geq \rho_0^{p_{\max}^+} \left( \frac{M_{\min}}{2 \rho_0^{p_{\max}^+}} \right) = \alpha(\rho_0) \] **Lemma 9.** Assume that (S1), (P1), (μ1), (M1), (G2), and (F1) satisfy. Then, for all \( \lambda \in \mathbb{R}, \) there exists \( \varphi_0 \in W_0 \) with \( \| \varphi_0 \|_{W_0} > \rho_0, \) where \( \rho_0 \) is given in Lemma 8, such that \( I(\varphi_0) < 0 \) for all \( t > 0 \) sufficient large.
Proof. Set \( \lambda \in \mathbb{R} \), from (G2), there exist two positive numbers \( L_1, L_2 \) such that
\[
G(x, t) \geq L_1 |t|^{\theta/2}, \text{ for any } x \in \Omega \text{ and } |t| \geq L_2. \tag{36}
\]

Using the condition (M1), we have
\[
\overline{M}_i(t) \leq M_i(1)t^\theta, \text{ for any } t \geq 1. \tag{37}
\]

Take \( \varphi_0 \in C_0^\infty(\Omega) \) with \( \varphi_0 > 0 \) and let \( t \in \mathbb{R} \) such that \( t\varphi_0^- \geq L_2 \). Combining with (36) and (37), we obtain
\[
I(t\varphi_0) = \sum_{i=1}^2 \overline{M}_i(\delta_{p_i}(t\varphi_0)) - \lambda \int_{\Omega} \frac{f(x)|\varphi_0|^\gamma(x)}{|x-y|^{\mu(x,y)}} dx
- \frac{1}{2} \int_{\Omega \times \Omega} G(x, t\varphi_0(x))G(y, t\varphi_0(y))
\]
\[
\leq \sum_{i=1}^2 \overline{M}_i(1)\left(\delta_{p_i}(t\varphi_0)\right)^\theta - \lambda \int_{\Omega} \frac{f(x)|\varphi_0|^\gamma(x)}{|x-y|^{\mu(x,y)}} dx
- \frac{1}{2} \int_{\Omega \times \Omega} L_1|\varphi_0(x)|^{\theta/2}L_1|\varphi_0(y)|^{\theta/2}
\]
\[
\leq \frac{1}{(p_{\text{min}})^\theta} \max \{1, \overline{M}_1(1), \overline{M}_2(1)\}
\times \sum_{i=1}^2 \left( \int_{\Omega \times \Omega} \frac{|t\varphi_0(x) - t\varphi_0(y)|^{p_i(x,y)}}{|x-y|^{N+\tau(x,y)p_i(x,y)}} dx \right)^\theta
\]
\[
- \frac{\lambda}{\gamma} \int_{\Omega} f(x)|\varphi_0|^\gamma(x) dx
- \frac{L_1^2}{2} \int_{\Omega \times \Omega} \frac{|\varphi_0(x)|^{\theta/2}|\varphi_0(y)|^{\theta/2}}{|x-y|^{\mu(x,y)}} dx. \tag{38}
\]

Since \( \theta > \theta_{p_{\text{max}}}^+ \), we deduce that \( I(t\varphi_0) \to -\infty \) as \( t \to \infty \). Then, for all \( \lambda \in \mathbb{R} \), there exists \( \varphi_0 \in W_0 \) with \( \|\varphi_0\|_{W_0} > \rho_0 \), where \( \rho_0 \) is given in Lemma 8, such that \( I(t\varphi_0) < 0 \) for all \( t > 0 \) sufficiently large.

Proof of Theorem 1. According to Lemmas 4, 6, 8 and 9, we know that all conditions of the mountain pass lemma are fulfilled, and therefore there exists a Palais–Smale subsequence \( v_n \), such that \( v_n \to v_1 \) in \( W_0 \) as \( n \to \infty \). So, \( v_1 \) is a nontrivial solution of problem \((P_\lambda)\) with positive energy \( I(v_1) \geq \alpha_0(\rho_0) > 0 \). \( \square \)

3.3. Proof of Theorem 2

In what follows, we prove Theorem 2 by using Ekeland’s variational principle [59].
Lemma 10. Let (S1), (P1), (μ1), (M3), (G2), and (F1) hold. Then, for any \( \lambda \in (0, A] \), there exists \( \nu_0 \in W_0 \) such that
\[
-\infty < I(\nu_0) = \inf \{ I(v) : v \in B_{\rho_0} \} < 0,
\]
where \( A, \rho_0 \) are given by Lemma 8 and \( B_{\rho_0} := \{ v \in W_0 : \| v \|_{W_0} \leq \rho_0 \} \).

Proof. Choose \( \xi > 0 \) in \( W_0 \), for positive number \( t \) sufficiently small such that \( \| t\xi \|_{W_0} < 1 \). Thus, for any \( \lambda \in (0, A] \), from (36) and (M3), we obtain
\[
I(t\xi) = 2 \sum_{i=1}^{2} \frac{M_{i}(\xi)}{\gamma(x)} \int_{\Omega} \lambda f(x) |t\xi|^{\gamma(x)} dx
\]
\[
-\frac{1}{2} \int_{\Omega} \frac{\mathcal{G}(x, t\xi(x)) \mathcal{G}(y, t\xi(y))}{|x-y|^{\mu(x,y)}} dxdy
\]
\[
\leq 2 \sum_{i=1}^{2} m_{i}(\xi) \int_{\Omega} \lambda f(x) |t\xi|^{\gamma(x)} dx
\]
\[
-\frac{1}{2} \int_{\Omega} \frac{L_{1} |t\xi(x)|^{\beta/2} L_{1} |t\xi(y)|^{\beta/2}}{|x-y|^{\beta(x,y)}} dxdy
\]
\[
\leq \frac{M_{\max}}{p_{\min}} \sum_{i=1}^{2} \left( \int_{\Omega} \frac{|t\xi(x) - t\xi(y)|^{\mu(x,y)}}{|x-y|^{\mu(x,y)}} dxdy \right)
\]
\[
- \frac{L_{1}^{2} \lambda}{2} \int_{\Omega} \frac{|t\xi(x)|^{\beta/2} |t\xi(y)|^{\beta/2}}{|x-y|^{\beta(x,y)}} dxdy
\]
\[
\left( 39 \right)
\]
where \( M_{\max} = \max\{1, m_{1}, m_{2}\} \). Since \( \gamma < p_{\min} \), we deduce that \( I(t\xi) < 0 \) as \( t \to 0^{+} \). Then, for \( t > 0 \) small enough, there exists \( \nu_{0} \in W_0 \) such that we obtain
\[
-\infty < I(\nu_{0}) = \inf \{ I(v) : v \in B_{\rho_0} \} < 0,
\]
where \( \rho_0 \) is given by Lemma 8 and \( B_{\rho_0} := \{ v \in W_0 : \| v \|_{W_0} \leq \rho_0 \} \).

Proof of Theorem 2. From Lemma 8, we know that
\[
0 < a_0 < \inf_{B_{\rho_0}} I(\nu),
\]
and from Lemma 10, we have
\[
-\infty < c = \inf_{B_{\rho_0}} I(\nu) < 0.
\]

Thus, we use the Ekeland’s variational principle [59], there exists \( \nu_{j} \in B_{\rho_0} \) such that
\[
c \leq I(\nu_{j}) \leq c + \frac{1}{j}
\]
\[
\text{and } I(\nu_{j}) \leq I(\nu) + \frac{1}{j} \| \nu_{j} - \nu \|_{W_0}
\]
\[
\left( 40 \right)
\]
for all \( v \in \overline{B}_{\rho_0} \) and \( j \in \mathbb{N} \). Fixing \( j \in \mathbb{N} \) and for all \( u \in \partial B_0 \), where \( B_0 = \{ v \in W_0 : \|v\|_{W_0} = 1 \} \), taking \( \zeta > 0 \) small enough so that \( v_j + \zeta u \in \overline{B}_{\rho_0} \). By (40), we obtain

\[
I(v_j + \zeta u) - I(v_j) \geq -\frac{\zeta}{j}.
\]

Since \( I \) is Gâteaux differentiable in \( W_0 \), for all \( u \in B_0 \), we have

\[
\langle I'(v_j), u \rangle = \lim_{\zeta \to 0} \frac{I(v_j + \zeta u) - I(v_j)}{\zeta} \geq -\frac{1}{j}.
\]

Therefore, we obtain

\[
\|I'(v_j)\|_{W_0^*} \leq \frac{1}{j}.
\]

So, we conclude that there exists a sequence \( \{v_j\} \subset \overline{B}_{\rho_0} \) as \( j \to \infty \), such that

\[
I(v_j) \to c < 0 \quad \text{and} \quad \|I'(v_j)\|_{W_0^*} \to 0.
\]

According to Lemma 7 there exists a convergent sequence \( \{v_j\} \) such that \( v_j \to v_2 \) in \( W_0 \) as \( j \to \infty \). Therefore, we have that problem \((P_{1})\) has another nontrivial solution \( v_2 \) which satisfies \( I(v_2) < 0 \) and \( \|v_2\|_{W_0} \leq \rho_0 \). □

**Remark 4.** From the above argument, since \( v_1 \) obtained by mountain pass theorem is a solution of \((P_{1})\) with positive energy \( I(v_1) > 0 \), \( v_2 \) obtained by Ekeland’s variational principle is a solution of \((P_{1})\) with negative energy \( I(v_2) < 0 \); therefore, they are different.

### 4. Conclusions

In this article, we study a class of Choquard–Kirchhoff type problems involving a variable \( s(x, \cdot) \)-order fractional \( p_1(x, \cdot) \) and \( p_2(x, \cdot) \)-Laplacian. Under some reasonable assumptions of \( g \) and \( f \), we obtain the existence of two solutions for this problem by applying some analytical techniques. Several recent results of the literature are extended and improved.

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