Non-Fermi liquid behavior in Bose-Fermi mixtures at two dimensions

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In this paper we study the low temperature behaviors of a system of Bose-Fermi mixtures at two dimensions. Within a self-consistent ladder diagram approximation, we show that at nonzero temperatures \(T \to 0\) the fermions exhibit non-fermi liquid behavior. We propose that this is a general feature of Bose-Fermi mixtures at two dimensions. An experimental signature of this new state is proposed.

The issue of Bose-Fermi mixture (BFm) can be traced back to the study of mixture of Helium-4 and Helium-3. \(^4\)He atoms carry integer spin and obey Bose-Einstein statistics whereas \(^3\)He atoms carry half-integer spin and obey Fermi-Dirac statistics. A natural question is what physics a system of mixture of these two types of particles will display. It was found that the critical temperature of the \(^4\)He condensate is suppressed with increasing the concentration of \(^3\)He before phase separation. Similar phenomena also occur in atomic BFm with richer physics. Through exchanging Bogoliubov excitons in Bose condensate, fermions gain effective attraction between themselves, leading to a rich possibility of different orders. In strong coupling limit, composite fermions comprising of one fermion and several bosons or bosonic holes may form, the residual interactions between these composite fermions, according to the system parameters, will result in different states such as a density wave, a superfluid liquid and an insulator with fermionic boson pairing, i.e.

\[
\int d^d x n^b(x) n^f(x) = \sum_{k,q} f^\dagger_{k-q} f_{k-q} \bar{b}^\dagger_q b_q \to \sum_{k,q} f^\dagger_{k-q} b^+_k c_k + \sum_{k,q} \bar{c}^+_k c_k - \sum_k c^+_k c_k, \quad (2)
\]

leading to an effective BFm action

\[
S = S_b + \sum_k (-i\omega + \xi^b_k) f^+_k f_k + U_b \sum_k c^+_k c_k - \frac{U_0}{\sqrt{N_b}} \sum_{k,q} (f^+_k b^+_q c_k + c.c.)(3a)
\]

where

\[
S_b = \sum_q (-i\Omega + \varepsilon^b_q) b^+_q b_q + U_b \int d^d x (n^b(x))^2 \quad (3b)
\]

is the (pure) boson action. \(q = (\vec{q}, i\Omega)\) and \(k = (\vec{k}, i\omega)\) with \(\Omega = 2n\pi/\beta\) and \(\omega = (2n+1)\pi/\beta\) in which \(n\) = integers and \(\beta = (k_B T)^{-1}\). \(c_k\)'s describe pairing of fermions and bosons and are the analogue of pairing order parameters \(\Delta_k\) in BCS theory for superconductors. The major difference between the two situations is that \(c_k\)'s are Grassman numbers here. As a result, we cannot have \(\langle c_k\rangle \neq 0\) in boson-fermion mixture and a more elaborated technique beyond BCS mean-field theory is needed to treat the \(c_k\) fields.

1. non-interacting bosons

To proceed further we first consider non-interacting fermions, i.e. \(U_b = 0\). In this case we can integrate out the bosons in \(S\) straightforwardly since it is quadratic in

\[
H = \sum_q \varepsilon^b_q b^+_q b_q + \sum_k \xi^b_k f^\dagger_k f_k
\]

where \(b\) and \(f\) represent boson and fermion, respectively, \(\varepsilon^b = \vec{q}^2/2m_B - \mu_B\) and \(\xi^b_k = k_f^2/2m_f - \mu_f\). \(\mu_B(f), m_B(f)\) are the chemical potentials and masses for bosons (fermions), respectively and \(n^b_f(\vec{x})\) are the corresponding density operators. \(U_b\) is the direct interaction between bosons and \(U_0\) is the interaction strength between bosons and fermions. We set \(\hbar = 1\) in this paper. We shall consider weak interactions \(U_b\) and \(U_0\) in the following.

To treat boson-fermion interaction we go to momentum space and introduce a fermion (Grassman) Hubbard-Strotonovich field \(c_k\) which describes fermion-boson pairing, i.e.

\[
\int d^d x n^b(x) n^f(x) = \sum_{k,q} f^\dagger_{k-q} f_{k-q} \bar{b}^\dagger_q b_q \to \sum_{k,q} f^\dagger_{k-q} b^+_k c_k + \sum_{k,q} \bar{c}^+_k c_k - \sum_k c^+_k c_k, \quad (2)
\]
function. In this approximation, \( \Pi_{c}^{q} \) where \( f^{*}c^*e \sim (f^{*}f)e^*c + f^{*}f(e^c) - \langle f^{*}f \rangle \langle e^c \rangle \), where \((f^{*}f)(e^c) = G_{f}(k)(G_{c}(k))\) are \( f - \) and \( c - \) fermion-Green’s functions but not \( c \)-numbers. The mean-field theory leads to the self-consistent equations

\[
G_{c}(k) = \frac{-1}{U_{0} + U_{0}^{2}\Pi_{c}(k)} G_{f}(k) = \frac{1}{i\omega - \xi_{k}^{c} - U_{0}^{2}\Pi_{f}(k)}
\]

where

\[
\Pi_{c}(k) = \frac{-1}{\beta N} \sum_{q} G_{f}(k - q)g_{0}(q),
\]

\[
\Pi_{f}(k) = \frac{-1}{\beta N} \sum_{q} G_{c}(k + q)g_{0}(q).
\]

\( G_{c} \) and \( G_{f} \) are determined self-consistently from Eq. \((5)\) and \((6)\) and can be viewed as the generalization of BCS theory to the case of fermion-boson binding.

To gain insight to the equations we first examine the fermion-boson bound states in the lowest order ladder diagram approximation where we approximate \( G_{f}(k - q) \sim g_{f}(k - q) \) in Eq. \((5)\), where \( g_{f}(k) \) is the free fermion Green’s function. In this approximation, \( \Pi_{c}(k) \rightarrow \Pi_{c}^{(0)}(k) \), where

\[
\Pi_{c}^{(0)}(k) = \frac{-1}{\beta N} \sum_{q} g_{f}(k - q)g_{0}(q)
\]

is the (bare) fermion-boson bubble diagram. The feedback effects of fermion-boson bound state on the fermion green’s function \( g_{f} \) is missing in this approximation.

Performing the sum over frequencies, we obtain \( \Pi_{c}^{(0)}(k, i\omega) = \Pi_{0B}(k, i\omega) + \Pi_{0F}(k, i\omega) \), where

\[
\Pi_{0B}(k, i\omega) = \int \frac{d^{d}q}{(2\pi)^{d}} \frac{n_{B}(\varepsilon_{q})}{i\omega - \varepsilon_{q} - \xi_{k-q}}
\]

\[
\Pi_{0F}(k, i\omega) = \int \frac{d^{d}q}{(2\pi)^{d}} \frac{1 - n_{F}(\varepsilon_{k-q})}{i\omega - \varepsilon_{q} - \xi_{k-q}}.
\]

It is straightforward to show that in two dimensions,

\[
\Pi_{0F}(k \rightarrow k_{F}, \omega \rightarrow \xi_{k}) \rightarrow C - \frac{im_{f}}{4\pi\varepsilon_{k}} \sqrt{4\varepsilon_{k}\omega - (\omega - \xi_{k})^{2}}
\]

in the low temperature limit \( \beta^{-1} \rightarrow 0 \), where \( C \sim m_{f} \) is a constant, i.e. \( \Pi_{F}(k \rightarrow k_{F}, \omega \rightarrow 0) \) is a non-singular function at the Fermi surface. The situation is quite different for \( \Pi_{B} \). Approximating \( n_{B}(\varepsilon) \sim 0 \) for \( \varepsilon \gg \beta^{-1} \) and \( n_{B}(\varepsilon) \sim 1/\beta \varepsilon \) for \( \varepsilon \ll \beta^{-1} \), we obtain

\[
\Pi_{0B}(k, i\omega) \sim \frac{1}{4\pi^{2}\beta} \int \frac{d^{2}q}{\varepsilon_{q}(i\omega - \varepsilon_{q} - \xi_{k-q})^{2}},
\]

the integral is diverging logarithmically at small \( q \) at any finite temperature \( k_{B}T \gg \Delta_{c} \), where \( \Delta_{c} \) is the (boson) energy level spacing. This singular behavior is a particular feature of (free) boson propagators in two dimensions and is missing in three-dimension systems. The leading effect of the \( q \rightarrow 0 \) singularity can be extracted by expanding the function \( (i\omega - \varepsilon_{q} - \xi_{k-q})^{-2} \) in a power series in \( q \). Performing the expansion, we obtain

\[
\Pi_{0B}(k, i\omega) \sim g_{f}(k) \left( x + O \left[ \frac{m_{b}k_{B}T \ln \left( \frac{(i\omega - \xi_{k})^{2}m_{f}\beta}{\varepsilon_{k}^{2}m_{b}} \right) }{\varepsilon_{k}^{2}m_{b}} \right] \right)
\]

where \( x \) is the density of bosons. We have used the result

\[
x = \int \frac{d^{d}q}{(2\pi)^{d}} n_{B}(\varepsilon_{q}) \sim \frac{1}{4\pi^{2}\beta} \int \frac{d^{2}q}{\varepsilon_{q}^{2}},
\]

in deriving \((7d)\), which is valid for non-interacting bosons at two dimensions where Bose-Einstein condensation is absent at any finite temperature. The singular behavior of the integral permits us to extract the leading contribution to \( \Pi_{0B}(k, i\omega) \) analytically at \( T \rightarrow 0 \).

We now return to Eq. \((5)\). The self-consistent equations are difficult to solve in general. However, the leading singular contribution at \( q \rightarrow 0 \) at two dimensions to the propagators \( \Pi_{c} \) and \( \Pi_{f} \) at \( k \rightarrow k_{F}, \omega \rightarrow 0 \) can be extracted in the limit \( \beta^{-1} \rightarrow 0 \) rather straightforwardly. Using the spectral representation

\[
G_{c}(\vec{k}, i\omega) = \frac{1}{\pi} \int d\varepsilon A_{c}(\vec{k}, \varepsilon) \frac{1}{i\omega - \varepsilon},
\]

we can perform the sum over frequencies in \( \Pi_{c} \) and \( \Pi_{f} \) and \( \Pi_{c}(f) \) can be separated into \( \Pi_{c}(f) = \Pi_{B}(f) + \Pi_{F}(f) \) as in Eq. \((7)\), where \( \Pi_{c}(f) \) is regular in the limit \( k \rightarrow k_{F} \) and \( \omega \rightarrow 0 \) and

\[
\Pi_{c}(f)(k, i\omega) = \int \frac{d^{2}q}{(2\pi)^{d}} n_{B}(\varepsilon_{q})G_{c}(\vec{k} - \vec{q}, i\omega - \varepsilon_{q})G_{f}(k - \vec{q}, i\omega - \varepsilon_{q})
\]

\[
\sim xG_{c}(\vec{k}, i\omega) + O(\beta^{-1})
\]

where we have followed the same analysis for Eq. \((10)\) and \((11)\) to reach the last result.

Therefore in the limit \( \beta^{-1} \rightarrow 0 \), as far as the leading contributions from \( q \rightarrow 0 \) is concerned, Eq. \((5)\) can be approximated by

\[
G_{c}(k) = \frac{1}{U(1 + xu_{eff}G_{f}(k))}
\]

\[
G_{f}(k) = \frac{1}{i\omega - \xi_{k}^{c} - xU^{2}G_{c}(k)}
\]
where \( U_{\text{eff}} = U/(1 + U\Pi_F^P(k_F,0)) \) is the approximate T-matrix for bosons scattering with fermions on the Fermi surface evaluated in the absence of other bosons. \( z = U_{\text{eff}}/U \) and \( \xi_k = \xi_k + U^2\Pi_F^P(k_F,0) - \mu_f \).

Eq. (9) can be solved easily since it represents quadratic equations for \( G_c \) and \( G_f \) when the two equations are combined. There are two branches of solutions where \( z \neq 0 \) can be computed rather straightforwardly with the approximated Green’s function (10). After some algebra (see supplementary materials) we obtain \( f(T \to 0) = F(T \to 0)/V = F_M(T \to 0) + f_B(T \to 0) + f_F(T \to 0) \), where \( f_B \) and \( f_F \) are the free energy densities for the corresponding non-interacting Bose and fermi gases, respectively,

\[
f_M(T \to 0) = -\frac{\rho}{2} \left( \frac{\Gamma^2}{2} + \Gamma\mu_f \right) \tag{12a}\]

is the mean-field free energy coming from boson-fermion interaction, where \( \rho = m_f/2\pi \) is the (2D) fermion density of states. Using the thermodynamics equality \( n_f = -\partial f_M/\partial \mu_f \), we find that the density of fermions is given by \( n_f = \rho(\mu_f + \xi_k/2) \). The corresponding ground state energy density is

\[
\varepsilon_g = \varepsilon_h + \varepsilon_f - U_{\text{eff}} n_{fx} - \frac{\rho}{2}(U_{\text{eff}}x)^2, \tag{12b}
\]

where \( \varepsilon_h \) and \( \varepsilon_f \) are the ground state energies for the corresponding non-interacting boson and fermion gases, respectively. \( U_{\text{eff}}n_{fx} \) is the usual Hartree energy. Our self-consistent theory introduces an additional energy correction \( -\frac{\rho}{2}(U_{\text{eff}}x)^2 \) corresponding to an effective attractive interaction \( -\rho U_{\text{eff}}^2 \) between bosons mediated through fermions.

The \( T \to 0 \) fermion occupation number in momentum space \( n_k \) can be calculated by \( n_k = \partial f/\partial \xi_k \). \( n_k \) is shown in figure (2) for two different values of \( U_{\text{eff}}x = \pm0.05 \) with fix fermion density \( n_f \). There is no discontinuity in \( n_k \) across \( \xi_k = 0 \) and the non-Fermi liquid nature of the system is obvious. Notice that for fixed fermion number \( n_f \), \( \mu_f \) is reduced (increased) from its non-interacting value when \( U_{\text{eff}} \) is attractive (repulsive).

![Figure 1: The spectral function \( A_f(\vec{k},\omega) \) of the fermions. (a) For \( U_{\text{eff}} = -1 \); (b) for \( U_{\text{eff}} = 1 \) for two different values of \( x = 0.02 \) and 0.05.](image)

Notice that the corresponding composite spectral function \( A_c(\vec{k},\omega) = -\frac{1}{2} ImG_c(\vec{k},\omega) \sim |\omega - \xi_k|A_f(\vec{k},\omega) \) is nonzero at the same range of frequencies where \( A_f(\vec{k},\omega) \) is nonzero. Moreover, \( A_c(\vec{k},\omega) \) has no peak and is quite structureless, indicating that the non-Fermi liquid behavior is not coming from formation of stable, sharp fermion-boson bound states, but is a particular feature of fermion-boson mixture in two dimensions.

The mean-field free energy density of the Bose-Fermi mixture at \( T \to 0 \) can be computed rather straightforwardly with the approximated Green’s function (10). After some algebra (see supplementary materials) we obtain \( f(T \to 0) = F(T \to 0)/V = F_M(T \to 0) + f_B(T \to 0) + f_F(T \to 0) \), where \( f_B \) and \( f_F \) are the free energy densities for the corresponding non-interacting Bose and fermi gases, respectively,

\[
f_M(T \to 0) = -\frac{\rho}{2} \left( \frac{\Gamma^2}{2} + \Gamma\mu_f \right) \tag{12a}\]

is the mean-field free energy coming from boson-fermion interaction, where \( \rho = m_f/2\pi \) is the (2D) fermion density of states. Using the thermodynamics equality \( n_f = -\partial f_M/\partial \mu_f \), we find that the density of fermions is given by \( n_f = \rho(\mu_f + \xi_k/2) \). The corresponding ground state energy density is

\[
\varepsilon_g = \varepsilon_h + \varepsilon_f - U_{\text{eff}} n_{fx} - \frac{\rho}{2}(U_{\text{eff}}x)^2, \tag{12b}
\]

where \( \varepsilon_h \) and \( \varepsilon_f \) are the ground state energies for the corresponding non-interacting boson and fermion gases, respectively. \( U_{\text{eff}}n_{fx} \) is the usual Hartree energy. Our self-consistent theory introduces an additional energy correction \( -\frac{\rho}{2}(U_{\text{eff}}x)^2 \) corresponding to an effective attractive interaction \( -\rho U_{\text{eff}}^2 \) between bosons mediated through fermions.

The \( T \to 0 \) fermion occupation number in momentum space \( n_k \) can be calculated by \( n_k = \partial f/\partial \xi_k \). \( n_k \) is shown in figure (2) for two different values of \( U_{\text{eff}}x = \pm0.05 \) with fix fermion density \( n_f \). There is no discontinuity in \( n_k \) across \( \xi_k = 0 \) and the non-Fermi liquid nature of the system is obvious. Notice that for fixed fermion number \( n_f \), \( \mu_f \) is reduced (increased) from its non-interacting value when \( U_{\text{eff}} \) is attractive (repulsive).

![Figure 2: The occupation number of the fermions in momentum space \( n_k \): (a) For \( U_{\text{eff}} = -1 \); (b) for \( U_{\text{eff}} = 1 \). \( x=0.05 \). The same density of fermions are assumed in both cases.](image)
We note that $\alpha S_{\Pi}^{\epsilon}(q)$ has the same logarithmic divergence at $\vec{q}_0$ as in Eq. (12b) except that the non-interacting boson Green’s function $G_b(q)$ is replaced by the corresponding Boson Green’s function $G_b^\epsilon(q)$ in the presence of interaction. The ground state energy is weak and $S_{\Pi}$ assume a self-consistent Ladder diagram approximation in evaluating the effect of fermion-boson interaction.

Assuming that the interaction between bosons are weak and $S_b$ can be approximated by the usual phase action $S_b$, where the boson operators are approximated by

$$b(\vec{x}) \sim \sqrt{\rho_s e^{i\theta(\vec{x})}},$$

where $\rho_s$ is the superfluid density, we obtain at low temperature $T \ll T_{KT}$, where $T_{KT}$ is the Kosterlitz-Thouless transition temperature [13],

$$G_b(\vec{q},\Omega) \sim \rho_s \frac{\delta(\Omega)}{|\vec{q}|^{2-\alpha}} + G_{reg}(\vec{q},\Omega),$$

(13)

where $\alpha = m_b/(2\pi\rho_s\beta)$ and $G_{reg}(\vec{q},\Omega)$ is a regular function in the limit $\vec{q},\Omega \to 0$.

Putting Eq. (13) into Eq. (6), we obtain $\Pi_{\epsilon(f)}(k) = \Pi_{\epsilon(f)}^S(k) + \Pi_{\epsilon(f)}^R(k)$, where

$$\Pi_{\epsilon(f)}^R(k) = -\frac{1}{\beta N} \sum_q G_{f(c)}(k - (+)q)G_{reg}(q)$$

is regular on the Fermi surface and

$$\Pi_{\epsilon(f)}(\vec{k},\omega) = \rho_s \int \frac{d^2q}{(2\pi)^2} \frac{1}{|\vec{q}|^{2-\alpha}} G_{f(c)}(\vec{k} - (+)\vec{q},\omega).$$

(14)

We note that $\alpha \to 0$ in the limit $T \to 0$, and the integral has the same logarithmic divergence at $\vec{q} \to 0$ as in Eq. (12) or (3), indicating that we can extract the leading divergence at $\vec{q} \to 0$ in the same way as before, i.e.

$$\Pi_{\epsilon(f)}(\vec{k},\omega) \to \rho_s G_{f(c)}(\vec{k},\omega) + O(\beta^{-1}).$$

suggesting that the same non-Fermi liquid behavior will be obtained as before except $x \to \rho_s$ in the presence of boson-boson interaction. The ground state energy is of the same form as in equation (12) except $x \to \rho_s$ and $\varepsilon_b \to \varepsilon_b = \text{ground state energy of the corresponding interacting bosons.}$

3. discussions

By summing the leading infra-red singularities in a self-consistent ladder diagram approximation, we show in this paper the existence of a $T \to 0(k_B T >> \Delta \varepsilon)$ non-Fermi liquid state in Bose-Fermi mixtures in two dimensions. The non-Fermi liquid state is a consequence of the absence of Bose-condensation (for non-interacting bosons) in two dimension at temperatures $k_B T >> \Delta \varepsilon$ and may persist even when bosons are interacting. The true Fermi liquid ground state is recovered only at $k_B T \leq \Delta \varepsilon$, where we may set $n_B(\varepsilon) \sim 0$. The non-Fermi liquid state we obtained is characterized by a fermion spectral function with no poles, with absence of sharp Fermi surface in the corresponding fermion occupation number $n_k$.

The change in shape of the fermion occupation number $n_k$ as the sign of fermion-boson interaction changes may be used as an experimental indicator for this non-Fermi liquid state in cold atom systems. The non-Fermi liquid state we obtained has lower energy than usual Hartree/Bogoliubov mean-field states and gives rise to effective attractive interaction $\sim \rho U^2_{F}f$ between bosons. This is in qualitative agreement with the result obtained in Ref.[11] that a sufficiently strong boson-boson repulsion is needed to stabilize the Bose-Fermi mixture for both repulsive and attractive Bose-Fermi interaction.

It should be cautioned that our result is obtained from a particular form of mean-field theory. The range of validity of the mean-field theory is unclear and can be addressed only through a careful Renormalization Group (RG) analysis which is out of the scope of the present paper. Nevertheless, our analysis demonstrates that the absence of Bose-condensation in two dimension may have a profound effect on Bose-Fermi systems and may lead to unconventional states not covered by conventional mean-field theories. A more rigorous theoretical analysis of the problem will be carried out in a future paper.

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Supplementary Material

In this Supplementary Material, we provide the derivation of the free energy and the occupation number of the fermions \( n_f(\vec{k}) \) for the bose-fermi mixtures in our mean-field theory.

A. The free energy

We start from the expression for the mean-field free energy term,

\[
F = -\frac{1}{\beta V} \sum_{k=\vec{k},i\omega} \left[ \ln G_f^{-1}(k) + \ln G_c^{-1}(k) \right] + \frac{U_0^2}{\beta V} \sum_{k=\vec{k},i\omega} \Pi_f(k)G_f(k).
\]  

(15)

Since \( \Pi_f(k) \sim xG_c(k) (k = (\vec{k}, i\omega)) \) and \( G_c(k) = -\frac{z}{\epsilon}g_0(k)G_f(k) \) where \( g_0(k) = (i\omega - \xi(k))^{-1} \) (Eq.(10) in main text), we have

\[
F \sim -\frac{1}{\beta V} \sum_{k=\vec{k},i\omega} \left[ 2 \ln G_f^{-1}(k) - \ln g_0^{-1}(k) \right] - \frac{zU_0x}{\beta V} \sum_{k=\vec{k},i\omega} \frac{G_f(k)^2}{g_0(k)}
\]

(16)

where we have dropped a \( \beta \) independent constant which has no thermodynamical effect.

We first consider \( F_f = -\frac{1}{\beta V} \sum_{k=\vec{k},i\omega} \ln G_f^{-1}(k) \). Writing \( \ln G_f^{-1}(k) = -\frac{1}{\pi} \int_{-\infty}^\infty \frac{A(\vec{k},\varepsilon)}{i\omega - \varepsilon} d\varepsilon \) where

\[
A(\vec{k}, \varepsilon) = \text{Im} \ln \left[ G_f^{-1}(\vec{k}, \varepsilon + i\delta) \right] = -\tan^{-1} \left( \frac{\text{Im}G_f(\vec{k}, \varepsilon + i\delta)}{\text{Re}G_f(\vec{k}, \varepsilon + i\delta)} \right)
\]

(17)

We have \( F_f = -\frac{1}{\beta V} \sum_{k=\vec{k},i\omega} \int_{-\infty}^\infty \frac{d\varepsilon}{\pi} n_f(\varepsilon) \tan^{-1} \left( \frac{\text{Im}G_f(\vec{k}, \varepsilon + i\delta)}{\text{Re}G_f(\vec{k}, \varepsilon + i\delta)} \right) \) where \( n_f(\varepsilon) = \left( e^{\beta\varepsilon} + 1 \right)^{-1} \) is the Fermi distribution function. Using \( G_f(k) = \frac{1}{\pi} (1 + 2i\delta g_0(k) - 1) \), we obtain also

\[
\frac{zU_0x}{\beta V} \sum_{k=\vec{k},i\omega} \frac{G_f(k)^2}{g_0(k)} \sim -\frac{1}{\beta V} \sum_{k=\vec{k},i\omega} \frac{G_f(k)}{g_0(k)} = \frac{1}{V} \sum_{\vec{k}} \int_{-\infty}^\infty \frac{d\varepsilon}{\pi} n_f(\varepsilon) \text{Im} \left( \frac{G_f(\vec{k}, \varepsilon + i\delta)}{g_0(\vec{k}, \varepsilon + i\delta)} \right)
\]

(18)

Therefore

\[
F = -\frac{zU_0x}{\beta V} \sum_{k=\vec{k},i\omega} \frac{G_f(k)^2}{g_0(k)} - F_f
\]

(19)

where \( F_f = \frac{1}{\beta V} \sum_{k=\vec{k},i\omega} \ln g_0^{-1}(k) = -\frac{m_f}{2\pi} \frac{\mu^2}{2} \) at zero temperature is the free energy for free fermions.

Notice that \( \xi(k) = \frac{k^2}{2m_f} - \mu \) where \( \mu = \mu_f - U_0^2 \Pi_f^c(kf, 0) \) and \( d\xi = kd\kappa/\mu_f \), we may change the summation over \( \vec{k} \) from \( d^2k \) to \( d\xi \), and the free energy of the system becomes,

\[
F = -\frac{m_f}{2\pi} \int_{-\infty}^\infty \frac{d\xi}{\pi} n_f(\varepsilon) \int_{-\mu}^\infty d\varepsilon \left[ \tan^{-1} \left( \frac{\text{Im}G_f(\vec{k}, \varepsilon + i\delta)}{\text{Re}G_f(\vec{k}, \varepsilon + i\delta)} \right) \right] - F_f.
\]

(20)

Denoting \( x = \varepsilon - \xi + \Gamma \), It is straightforward to show that \( \text{Im} \left( \frac{G_f(\vec{k}, \varepsilon + i\delta)}{g_0(\vec{k}, \varepsilon + i\delta)} \right) = -\sqrt{1 - \frac{x^2}{\Gamma^2}} \) when \( |x| < |\Gamma| \) and is equal to zero otherwise. We have also

\[
\text{Im} G_f(\vec{k}, \varepsilon + i\delta) = \frac{\delta}{|\varepsilon - \Gamma|^{1/2} - \Gamma}, \quad \text{Re} G_f(\vec{k}, \varepsilon + i\delta) = \frac{1}{\Gamma} \left( \frac{\varepsilon + \Gamma + 1}{\varepsilon - \Gamma} - 1 \right), \quad (|x| \leq |\Gamma|)
\]

\[
\text{Im} G_f(\vec{k}, \varepsilon + i\delta) = -\frac{\delta}{|\varepsilon - \Gamma|^{1/2} + \Gamma}, \quad \text{Re} G_f(\vec{k}, \varepsilon + i\delta) = \frac{1}{\Gamma} \left( \frac{\varepsilon + \Gamma + 1}{\varepsilon - \Gamma} + 1 \right), \quad (|x| > |\Gamma|)
\]

(21)

and

\[
\tan^{-1} \left( \frac{\text{Im} G_f(\vec{k}, \varepsilon + i\delta)}{\text{Re} G_f(\vec{k}, \varepsilon + i\delta)} \right) = \begin{cases} 
-\tan^{-1} \left( \frac{\pi}{\Gamma} \right) \sqrt{\frac{\varepsilon + \Gamma + 1}{\varepsilon - \Gamma}}, & (|x| \leq |\Gamma|) \\
\pi, & (x > |\Gamma|) \\
0, & (x < -|\Gamma|)
\end{cases}
\]

(22)
Changing the integration variable from $\xi$ to $x$ and considering the case of zero temperature, we obtain

$$F = -\frac{m_f}{2\pi} \int_{-\infty}^{0} \frac{d\xi}{\pi} \int_{-\Gamma}^{\xi+\Gamma+\mu} dx \left[ 2\tan^{-1} \left( \frac{\text{Im} G_f(\vec{k}, x + i\delta)}{\text{Re} G_f(\vec{k}, x + i\delta)} \right) + i\text{Im} \left( \frac{G_f(\vec{k}, x + i\delta)}{g_0(\vec{k}, x + i\delta)} \right) \right] - F_{0f}$$

$$= -\frac{m_f}{2\pi} \int_{-\infty}^{0} \frac{d\xi}{\pi} \left\{ \theta(\varepsilon + \mu + \Gamma + |\Gamma|) \int_{-|\Gamma|}^{\min(\varepsilon + \Gamma + \mu, |\Gamma|)} dx \left[ 2\tan^{-1} \left( \frac{\Gamma + x}{|\Gamma|\sqrt{\Gamma + x - \frac{x^2}{\Gamma^2}}} \right) - \sqrt{1 - \frac{x^2}{\Gamma^2}} \right] + \theta(\varepsilon + \mu + \Gamma - |\Gamma|) \int_{|\Gamma|}^{\varepsilon + \Gamma + \mu} dx 2\pi \right\} - F_{0f}. \quad (23)$$

It is straightforward to obtain after performing the integrals,

$$F = F_{0f} - \frac{m_f}{2\pi} \left( \frac{\Gamma^2}{4} + \frac{\Gamma\mu}{2} \right) = -\frac{\rho}{2} \left( \frac{\Gamma^2}{2} + \Gamma\mu + \mu^2 \right) \quad (24)$$

### B. The Fermion occupation number

Using Equation (6) and considering the case of zero temperature, we have

$$F = -\frac{m_f}{2\pi} \int_{-\infty}^{0} \frac{d\xi}{\pi} \int_{-\Gamma}^{\xi+\Gamma+\mu} \frac{dx}{\pi} \left[ 2\tan^{-1} \left( \frac{\text{Im} G_f(\vec{k}, \xi + i\delta)}{\text{Re} G_f(\vec{k}, \xi + i\delta)} \right) + i\text{Im} \left( \frac{G_f(\vec{k}, \xi + i\delta)}{g_0(\vec{k}, \xi + i\delta)} \right) \right] - F_{0f} \quad (25)$$

In the case of $\Gamma > 0$, carrying out the integral in $\varepsilon$ first, the free energy becomes

$$F = -\frac{m_f}{2\pi} \int_{-\Gamma}^{0} d\xi \int_{-\Gamma}^{\xi+\Gamma+\mu} dx \frac{1}{\pi} \left[ 2\tan^{-1} \left( \sqrt{1 + \frac{x}{\Gamma - x}} - \sqrt{1 - \frac{x^2}{\Gamma^2}} \right) \right] - F_{0f}$$

$$= \frac{m_f}{2\pi} \int_{-\Gamma}^{0} d\xi \int_{-\Gamma}^{\Gamma - \xi} dx \frac{1}{\pi} \left[ 2\tan^{-1} \left( \sqrt{1 + \frac{x}{\Gamma - x}} - \sqrt{1 - \frac{x^2}{\Gamma^2}} \right) \right]$$

$$= \frac{m_f}{2\pi} \int_{-\Gamma}^{0} d\xi \left( \frac{\Gamma - \xi}{2} \right) + \frac{m_f}{2\pi} \int_{0}^{2\Gamma} \frac{d\xi}{2\pi} \left[ \left( \sin^{-1} \frac{\alpha + \frac{\pi}{2}}{\frac{\pi}{2}} \right) \left( 1 - 2\alpha \right) + \left( \alpha - 2 \right) \sqrt{1 - \alpha^2} \right]$$

$$= \frac{1}{V} \sum_{\vec{k}(\xi_{-\mu < \xi_{\vec{k}} < 0})} \left( \frac{\xi_{\vec{k}} - \Gamma}{2} \right) + \frac{1}{V} \sum_{\vec{k}(0 < \xi_{\vec{k}} < 2\Gamma)} \frac{\Gamma}{2\pi} \left( \sin^{-1} \frac{\alpha + \frac{\pi}{2}}{\frac{\pi}{2}} \right) \left( 1 - 2\alpha \right) + \left( \alpha - 2 \right) \sqrt{1 - \alpha^2} \quad (26)$$

in which $\alpha = \frac{\Gamma - \xi_{\vec{k}}}{\Gamma - \xi_{\vec{k}}}$.

The Fermion occupation number can be determined by the thermodynamics equality $n_{\vec{k}} = V \frac{\partial F}{\partial \xi_{\vec{k}}}$. We obtain,

$$n_{\vec{k}} = \begin{cases} 1, & \text{if } \xi_{\vec{k}} < 0, \\ \frac{1}{\alpha} (\sin^{-1} \frac{\alpha + \frac{\pi}{2}}{\frac{\pi}{2}} - \sqrt{1 - \alpha^2}), & 0 < \xi_{\vec{k}} < 2\Gamma \\ 0, & \xi_{\vec{k}} \geq 2\Gamma \end{cases} \quad (27)$$

For $\Gamma < 0$, it is also easy to show that

$$G_f^R(x'; -\Gamma) = -G_f^A(-x'; \Gamma) \quad (28)$$

in which $x'$ denotes $\omega - \xi_{\vec{k}}$ and therefore

$$n(\xi_{\vec{k}}; \Gamma) = 1 - n(-\xi_{\vec{k}}; \Gamma) \quad (29)$$
We therefore obtain the expression for $\Gamma < 0$,

$$n_k = \begin{cases} 
\frac{1}{\pi} \left( -\sin^{-1} \alpha + \frac{\pi}{2} + \sqrt{1 - \alpha^2} \right), & \bar{\xi}_k > 0 \\
0, & 2\Gamma < \bar{\xi}_k < 0 \\
1, & \bar{\xi}_k < 2\Gamma
\end{cases}$$

(30)