Smooth dynamical (de)-phantomization of a scalar field in simple cosmological models

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Simple scalar field cosmological models are considered describing gravity assisted crossing of the phantom divide line. This crossing or (de)-phantomization characterized by the change of the sign of the kinetic term of the scalar field is smooth and driven dynamically by the Einstein equations. Different cosmological scenarios, including the phantom phase of matter are sketched.

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I. INTRODUCTION

The discovery of cosmic acceleration \([1]\) has become one of the main challenges for modern theoretical physics, because it has put forward the problem of the so called dark energy, responsible for this acceleration. This dark energy should possess a negative pressure such that the relation between pressure and energy density is less than \(-1/3\) \([2]\). A considerable amount of dark energy models was proposed (see, e.g. \([3, 4]\)), which now have to confront with growing quantity of observational data.

Being enigmatic by itself, the phenomenon of the cosmic acceleration - dark energy - negative pressure hides inside another subphenomenon, which is not yet so well confirmed experimentally, but if confirmed, it would become, perhaps, more intriguing than all the preceding cosmological puzzles. We mean here, the phenomenon of the so called phantom dark energy, i.e. the matter with the relation between the pressure and dark energy \(k = p/\rho < -1\) \([5]\). Some observations indicate that the present day value of the parameter \(k < -1\) provides the best fit. According to some authors, the analysis of observations, permits to specify the existence of the moment when the universe changes the value of the parameter \(k\) from that the region \(k > -1\) to \(k < -1\) \([6]\). This transition is called “the crossing of the phantom divide line”. Other authors speak about double crossing of the phantom divide line \([7]\). In any case the existence of the phantom phase of cosmological evolution put forward some important theoretical problems. For example, considering a perfect fluid with barotropic equation of state, where \(k = const\) and \(k < -1\) one can easily see that a universe during its evolution after a finite amount of time encounters a special kind of cosmological singularity, characterized by an infinite value of the velocity of expansion, which has been dubbed “Big Rip” \([8, 9]\). Being a little bit unusual in comparison with well accepted Big Bang and Big Crunch singularities, the Big Rip singularity is quite similar to them from the formal (geometrical) point of view. Notice, that in the modern cosmological literature also other “exotic” types of singularities are under consideration \([10, 11, 12, 13, 14]\).

Another difficulty connected with the problem of phantom energy has a more fundamental character. It is well-known that the models with minimally coupled scalar field are very efficient both for the description of the inflationary stage of the development of the early universe and for the explanation of the late-time cosmic acceleration phenomenon \([8]\). However, it is easy to see, that the standard minimally coupled scalar field cannot give rise to the phantom dark energy, because in this model the absolute value of energy density is always greater than that of pressure, i.e. \(|k| < 1\). A possible way out of this situation is the consideration of the scalar field models with the negative kinetic term. Again, being a little unusual from the point of view of common wisdom, these models have become quite popular nowadays.

Perhaps, the most thrilling problem arising in connection with the phantom energy is the crossing of the phantom divide line. The general belief is that this crossing is not admissible in simple minimally coupled models and explanation of this phenomenon requires more complicated models such a multifield ones or models with non-minimal coupling between scalar field and gravity (see e.g. \([15, 16, 17, 18]\)).

In the present paper, we would like to explore the consequences of the possibility of existence of simple minimally coupled scalar field models allowing the crossing of the phantom divide line. As a matter of fact, we would like to present some toy examples, suggesting that such a crossing could be unavoidable and dictated by the continuity of the solution of Einstein equations. In the next section we discuss in some detail a simple toy model, giving an illustration
of the change of sign of the kinetic term in the scalar field cosmology. In the third section we sketch some other models and in the last section we attempt to discuss the above consideration in a more general context.

II. TOY MODEL WITH A CROSSING OF THE PHANTOM DIVIDE LINE

Let us consider a simple cosmological model, representing a flat Friedmann universe with a metric

$$ds^2 = dt^2 - a^2(t)dl^2$$  \(1\)

filled with a barotropic fluid with an equation of state

$$p = k\rho,$$  \(2\)

where \(\rho\) and \(p\) are energy density and pressure of the fluid respectively and \(-1 < k \leq 1\). Choosing conveniently the normalisation of the constants in the theory one can write the Friedmann equation in a simple form

$$h^2 = \rho,$$  \(3\)

where the Hubble variable is defined as \(h = \frac{\dot{a}}{a}\). The energy conservation law is

$$\dot{\rho} = -3h(\rho + p)$$  \(4\)

which together with Eq. \(2\) gives immediately

$$\rho = \frac{\rho_0}{a^{3(1+k)}}.$$  \(5\)

From Eqs. \(5\) and \(3\) one gets

$$a = a_0 t^{2/3(1+k)},$$  \(6\)

or, in other terms,

$$h = \frac{2}{3(1+k)t}.$$  \(7\)

Another useful equation obtained by differentiation of the Eq. \(3\) and substitution into it Eq. \(4\) reads:

$$\dot{h} = -\frac{3}{2}(\rho + p).$$  \(8\)

The pressure can be written down as

$$p = -\frac{2}{3}\dot{h} - h^2.$$  \(9\)

It is well known that for a given cosmological evolution \(h(t)\) satisfying some simple conditions one can find a minimally coupled scalar field cosmological model with Lagrangian

$$L = \frac{\dot{\phi}^2}{2} - V(\phi),$$  \(10\)

which contains this evolution as a particular solution, provided suitable initial conditions are chosen (see, e.g. [10], [19] and references therein).

The energy density and pressure of the scalar field \([10]\) are

$$\rho = \frac{\dot{\phi}^2}{2} + V(\phi),$$  \(11\)

$$p = \frac{\dot{\phi}^2}{2} - V(\phi).$$  \(12\)
From Eqs. (11), (12), (3) and (9) one has
\[ \dot{\phi}^2 = (\rho + p) = -\frac{2}{3} \dot{h}, \quad (13) \]
\[ V(\phi) = \frac{1}{2} (\rho - p) = \frac{\dot{h}}{3} + h^2. \quad (14) \]
Equation (14) gives the potential \( V(t) \) as a function of \( t \). Finding \( \phi(t) \) by integrating Eq. (13), inverting this function and substituting a function \( t(\phi) \) into Eq. (14), one finds the potential \( V \) as a function of the scalar field \( \phi \). Sometimes this can be done explicitly. For example, the evolution (7) could be reproduced in the extensively studied [10, 20] cosmological model with an exponential potential
\[ V(\phi) \sim \exp \left( \frac{3\sqrt{1 + k\phi}}{2} \right). \quad (15) \]
Apparently, the super-accelerated evolution (7) with \( k < -1 \) cannot be reproduced by means of the minimally coupled scalar field model (10), but can be easily obtained in the framework of the model with phantom scalar field
\[ L = -\frac{\dot{\phi}^2}{2} - V(\phi). \quad (16) \]
Our starting point is the remark made in [21, 22, 23], attracting the attention to some interesting features of the potential (14) represented as a function of time \( t \). The point is that the volume function
\[ \psi(t) = a^3(t) \quad (17) \]
satisfies a simple second-order differential equation
\[ \ddot{\psi} = 9V(t)\psi, \quad (18) \]
which can be easily obtained from Eq. (14). As has been already noticed above, the potential \( V(t) \) is chosen in such a way to provide a given cosmological evolution. For example, for the evolution (7) the form of the potential is
\[ V(t) = \frac{2(1 - k)}{9(1 + k)^2 t^2}. \quad (19) \]
For simple forms of the potential one can find the general solution of Eq. (18), which contains together with the solution used for the construction of this potential also another independent solution. For the potential (19) the general solution looks like
\[ \psi(t) = \psi_1 t^{\alpha_1} + \psi_2 t^{\alpha_2}, \]
\[ \alpha_1 = \frac{2}{1 + k}, \quad \alpha_2 = \frac{k - 1}{1 + k}, \quad (20) \]
where \( \psi_1 \) and \( \psi_2 \) are nonnegative constants. (In papers [22, 23] the solution (20) was written down for the case of dust \( k = 0 \).) Now, knowing \( \psi(t) \) one can find \( h(t) = \frac{\dot{\psi}}{3\psi} \) and \( \dot{h} \). Substituting the obtained \( \dot{h}(t) \) into Eq. (14) one can try to find the dependence \( \dot{\phi}(t) \). Inversion of this dependence \( t(\phi) \) and its substitution into the expression for the potential as a function of time (19) would have given the potential \( V(\phi) \), whose form is different from the exponential one (15) when \( \psi_2 \neq 0 \).

One can derive the equation for the functions \( t(\phi) \) compatible with the potential \( V(t) \) from Eq. (15). First of all, notice that the field \( \phi \) should satisfy the Klein-Gordon equation
\[ \ddot{\phi} + 3h\dot{\phi} + \frac{dV}{d\phi} = 0. \quad (21) \]
Dividing Eq. (21) by \( \dot{\phi} \) one has
\[ \frac{\ddot{\phi}}{\dot{\phi}} + 3h + \frac{1}{\dot{\phi}} \frac{dV}{d\phi} = 0. \quad (22) \]
It is convenient to rewrite the second time derivative of the scalar field as
\[ \ddot{\phi} = \dot{\phi} \frac{d\dot{\phi}}{d\phi} = \frac{1}{t^2} \left( \frac{1}{t^3} \right) ', \]  \hspace{1cm} (23)
where we have used the relation
\[ t' = \frac{1}{\phi}, \] \hspace{1cm} (24)
with the “prime” denoting the differentiation with respect to the scalar field \( \phi \). Now, we would like to take the time derivative of Eq. (22). The first term gives
\[ \frac{d}{dt} \left( \frac{1}{t^2} \left( \frac{1}{t^3} \right)' \right). \] \hspace{1cm} (25)
Then, using the relations (14) and (24) one gets
\[ 3 \dot{h} = - \frac{9}{2t^2}. \] \hspace{1cm} (26)
Then we make the following transformation:
\[ \frac{1}{\phi} \frac{dV}{d\phi} = t^2 \dot{\psi}. \] \hspace{1cm} (27)
The time derivative of the expression in the left hand side of Eq. (27) is
\[ \frac{d}{dt} \left( \frac{1}{\phi} \frac{dV}{d\phi} \right) = \frac{1}{t^2} (t^2 \dot{\psi})'. \] \hspace{1cm} (28)
Combining expressions (26), (28), and multiplying the resulting equation by \( t' \) one arrives to the following equation:
\[ \left( \frac{1}{t^2} \right)' - \frac{9}{2t^2} - \frac{4(1-k)}{9(1+k)^2} \left( \frac{1}{t^3} \right)' = 0. \] \hspace{1cm} (29)
Particular solutions of this equation being substituted into Eq. (19) generate different potentials \( V(\phi) \) corresponding to different cosmological evolutions (20). It is easy to check that the solution
\[ t = \exp \frac{3\sqrt{1+k\phi}}{2}, \] \hspace{1cm} (30)
corresponding to the potential (16) and to the evolution with \( \psi_2 = 0 \), satisfies Eq. (29). Instead of looking for other exact solutions (which is not an easy task) we concentrate now on the qualitative study of Eq. (29) or its equivalents.

First of all, let us notice that the solution (20) tacitly assumes that the time parameter runs from 0 to \( +\infty \). When \( t \rightarrow +\infty \) one can neglect the influence of the second term in the solution (20). Thus, the potential \( V(\phi) \) in this limit has the asymptotic form (15). It is possible to calculate also the subleading term. The form of the Hubble variable corresponding to the evolution (20) is
\[ h = \frac{\alpha_1 \psi_1 t^{\alpha_1 - 1} + \alpha_2 \psi_2 t^{\alpha_2 - 1}}{3(\psi_1 t^{\alpha_1} + \psi_2 t^{\alpha_2})}, \] \hspace{1cm} (31)
while its time derivative is
\[ \dot{h} = - \frac{\alpha_1 \psi_1^2 t^{2\alpha_1} + \alpha_2 \psi_2^2 t^{2\alpha_2} + (1 - (\alpha_1 - \alpha_2)^2) \psi_1 \psi_2 t^{\alpha_1 + \alpha_2}}{3t^2(\psi_1 t^{\alpha_1} + \psi_2 t^{\alpha_2})^2}. \] \hspace{1cm} (32)
At large values of \( t \), \( \dot{h} \) written down up to the first non-leading term reads:
\[ \dot{h} \approx - \frac{2}{3(1+k)t^2} \left( 1 + \frac{2(k-3)\psi_2}{(1+k)\psi_1} \right). \] \hspace{1cm} (33)
Substituting the expression \[33\] into Eq. \[13\] one finds the subleading correction to the scalar field \(\phi\) as a function of \(t\). Inverting this relation and inserting the result into Eq. \[14\] we obtain the potential \(V(\phi)\) including the first nonleading term:

\[
V(\phi) \sim \exp \left( -\frac{3\sqrt{1+k}\phi}{2} \right) \left( 1 + \frac{3\psi_2}{\psi_1} \frac{k - 3}{\sqrt{1+k}} \exp \frac{3(k-3)\phi}{2\sqrt{1+k}} \right). \tag{34}
\]

The above formula can be elucidated as follows: if one consider the cosmological evolution described by Eq. \[20\] with non-zero value of the parameter \(\psi_2\), one can state that there exists a potential describing this evolution. In spite of the absence of an explicit expression for this potential we know that its asymptotic behaviour is described by the formula \[34\].

Now, let us consider the evolution of a cosmological model, which is characterized by Eq. \[31\] and the potential whose asymptotic form is given by Eq. \[34\]. It is easy to show that at some moment \(t_0\) such as \(0 < t_0 < \infty\), the time derivative of the Hubble variable vanished. Indeed, a simple analysis of the formula \[32\] gives this value

\[
t_0 = \left( \frac{(3-k)\sqrt{2(1-k)} + 4(1-k)\psi_2}{2(1+k)} \right)^{\frac{1}{1+k}}. \tag{35}
\]

Obviously, \(t_0\) exists for any value of \(\psi_2 \neq 0\). The evolution of the model given by Eq. \[31\] is consistently defined at \(t > t_0\) in the standard scalar model \[10\] with a potential whose asymptotic form at large values of \(t\) was presented above \[34\]. (The consistency is based on the fact that at \(t > t_0\) the time derivative of the Hubble constant is always negative). It is possible also to find the approximate expression for the potential \(V(\phi)\) for \(t \to t_0^+\). it is well known that the Klein-Gordon and Friedmann equations allow a shift of the scalar field by some constant. It is convenient to fix the scalar field as \(\phi \to 0\) when \(t \to t_0^+\). The time derivative of the Hubble variable at \(t \to t_0^+\) behaves as

\[
h = -H(t - t_0),
\]

where

\[
H = \sqrt{8(1-k)(3-k)^2\psi_1\psi_2t_0^{\frac{4\psi_2}{1+k}}} \left( \frac{1}{1+k} \right)^{\frac{2\psi_1}{1+k} + \frac{\psi_2}{2\psi_1}}. \tag{37}
\]

Substituting the expression \[36\] into Eq. \[14\] one can integrate the latter, getting

\[
\phi \approx \sqrt{\frac{8H}{27}}(t - t_0)^{3/2}, \tag{38}
\]

whose inversion gives

\[
t = t_0 + \left( \sqrt{\frac{27}{8H\phi}} \right)^{2/3}. \tag{39}
\]

Thus, the potential results

\[
V(\phi) = \frac{2(1-k)}{9(1+k)^2} \left( t_0 + \left( \sqrt{\frac{27}{8H\phi}} \right)^{2/3} \right)^2. \tag{40}
\]

It is rather straightforward to verify that the expressions for \(\phi, \dot{\phi}\) and the potential \(V(\phi)\) are regular when \(t \to t_0^+\) while the derivative \(\frac{dV(\phi)}{d\phi}\) and \(\ddot{\phi}\) are singular.

Now, before discussing the subtle question of the behavior of the cosmological model under consideration at \(t = t_0\) and of the possibility of crossing this point, we shall study briefly another branch of the evolution \[31\]: namely, that where \(t < t_0\). For small values of \(t\) to reproduce the evolution \[20\] one should use the phantom Lagrangian \[16\]. Proceeding in the similar way one can find the corresponding potential of the phantom field in the subleading approximation:

\[
V(\phi) = \frac{2(1-k)}{9(1+k)^2} \exp \left( -3\sqrt{\frac{2(1+k)}{1-k}} \phi \right) \left( 1 + \frac{4\psi_1(1+k)}{\psi_2(3-k)} \exp \left( 3\phi(3-k)\sqrt{\frac{1}{2(1-k^2)}} \right) \right). \tag{41}
\]
One can easily describe what is going on in the model characterized by a negative kinetic term and the potential whose asymptotic form is given by (41) in the vicinity of the point \( t \to t_0^- \).

Now, we are in a position to scrutinize the possibility of matching these two branches of the evolution at \( t < t_0 \) and \( t > t_0 \). Are they really incompatible? Let us consider a scalar field model with a negative kinetic term and a potential whose asymptotic form is that of Eq. (41). Further, let us suppose that at some moment \( t < t_0 \) one has initial conditions on the values of \( h, \phi \) and \( \dot{\phi} \) which provide the evolution (20) where the values of the parameter \( s \) are consistent with those of the potential \( V(\phi) \). Then approaching the time moment \( t \to t_0^- \) one arrives to the regime when \( \dot{h}, \dot{\phi} \) and \( \ddot{\phi} \) tend to vanish. Nevertheless, all the geometric characteristics of the spacetime remain well defined (due to homogeneity and isotropy of the Friedmann cosmology all the curvature invariants are expressible through the cosmological factor \( a \) and its time derivatives, which are obviously finite). In contradistinction, the second time derivative of the scalar field at \( t = t_0 \) and \( \frac{dV(\phi)}{d\phi} \) diverge, but this divergence is an integrable one. This offers us an opportunity (and, perhaps, necessity) of a continuation of the spacetime geometry and field configurations beyond this “divide line”. Clearly, such a smooth dynamical continuation which respects Einstein equations, entails the change of the sign of the kinetic term and the transition to the regime for \( t > t_0 \) described above.

III. DIFFERENT REGIMES OF CROSSING OF PHANTOM DIVIDE LINE

The cosmological evolution presented in the preceding section describes the following scenario: the universe begins its evolution from the cosmological singularity of the “anti-Big Rip” type and its squeezing is driven by a scalar field with a negative kinetic term. Then at the moment (easily calculated from Eq. (31))

\[
t_1 = \left(\frac{(1-k)\psi_1}{2\psi_2}\right)^{\frac{3+k}{2}}
\]

the contraction of the universe is replaced by an expansion. At the moment \( t = t_0 > t_1 \) (see Eq. (35) the kinetic term of the scalar field changes sign. Then, with the time growing the universe undergoes an infinite power-law expansion. The graphic of the time dependence of \( h(t) \) is presented in Fig. 1. Changing the sign of time parameter in all the equations the range \(-\infty < t < 0\) one can consider the cosmological evolution running from \( t = -\infty \) to \( t = 0 \). It begins from the non-singular contraction and ends in a “standard” Big Rip singularity. In this case at the moment \( t = t_0 \) the scalar field undergoes the “phantomization”, while the transition at the moment \( t = t_0 \), considered in the preceding section can be called “dephantomization”.

The toy model discussed above provides an illustration of these phenomena of phantomization and dephantomization, but from observational point of view, it would be more interesting to get an evolution beginning from the Big Bang and ending in the Big Rip singularity, after undergoing a phantomization transition. Instead of trying to construct some potential and cosmological evolution describing such a process we shall limit ourselves to giving a graphical presentation of the \( h(t) \)-dependence which could be responsible for such a scenario (see Fig. 2). This picture represents the function \( h(t) = \frac{A_0}{t(t_R-t)} \). Here the moment \( t = t_R \) is the moment of Big Rip, while the the moment of phantomization is \( t_0 = t_R/2 \). In principle, the function \( h(t) \) determines the model completely, including the form of the potential, even if when it is not written explicitly.

As has been already mentioned in Introduction, some observations favor double crossing of the phantom divide line \[7\]. The corresponding graphic \( h(t) \) is presented in Fig. 3. Also to this time dependence of \( h(t) \) one can associate some potential and the evolution, in course of which the scalar field undergoes two smooth transitions: phantomization at the point of minimum of the curve \( h(t) \) and the subsequent dephantomization at the point of its maximum.
FIG. 2: $h(t)$ dependence in the model, describing the Big Rip.

FIG. 3: $h(t)$ dependence in the model, describing a double crossing of the phantom divide line.

It is suggestive to remark that the diagram Fig.3 reminds topologically the well-known van der Waals isotherm curve. The part of the van der Waals curve situated between the maximum and minimum, describing a metastable state (of supercool vapor or superheated liquid), is analogous to the phantom phase in our Fig. 3. Models with double crossing of the phantom divide line were considered also in [22]. The idea in [22] consisted in cutting off the phantom piece of the evolution with the subsequent sewing of remaining branches of the trajectory removes the problems connected with the crossing of the phantom divide line. In other terms, the trajectories, considered in [22] correspond graphically to $h(t)$ which instead of one maximum and one minimum (see Fig. 3) have one inflection point. Continuing the analogy with the van der Waals isoterms, one sees that this curves corresponds to the critical van der Waals curve. Surely, such models can be studied, but their existence does not exclude the existence of models described by Fig. 3 and in a frame of a concrete model with fixed Lagrangian and initial conditions these two types of trajectories cannot be transformed one into other.

IV. DISCUSSION

Above we have discussed the properties of some simple cosmological models based on minimally coupled scalar fields, which could, in our opinion, describe the crossing of the phantom divide line, changing the sign of the kinetic term in the corresponding Lagrangian. As a starting point of our consideration we have used the observation [22, 23] that having a simple form of the scalar field potential, written as a function of time $V(t)$, one can find the general solution of the Einstein (Friedmann) equations for the Hubble variable $h(t)$ (see Eq. (20)). Different particular solutions describing the evolution of the Hubble variable correspond to different potentials as functions of the scalar field $V(\phi)$. All these solutions (excluding the obvious, when $\psi_2 = 0$) contain the time moment when the time derivative of the Hubble variable vanishes (see Eq. (35)) and the only consistent way of passing of this point respecting the continuity of the Einstein equations is the change of the sign of the kinetic term for the scalar field. This change can be called (de)-phantomization of the field.

All above are of technical nature and are confirmed by a direct analysis of the corresponding differential equations. As far as physical interpretation is concerned one has two alternatives. One can say that the sign of the kinetic term should be fixed a priori (see, e.g. [17]) and, hence the models, which imply the change of this sign should be discarded as non-physical ones. Here, we attempted to treat another alternative: i.e. to take this theory seriously because from mathematical point of view it looks consistent. In choosing this alternative we were guided by a belief that Einstein equations are more fundamental then the concrete form of the action for other fields. The idea about the dominant role of Einstein equations goes back to the classical works by Einstein, Infeld and Hoffmann [24, 25], where it was shown that the motions of mass points (geodesics law) are determined by the Einstein equations for the gravitational
field. Later this approach was confirmed in works by Lanczos \[26\], Fock \[27\] and others. In this respect the general relativity is quite different from electrodynamics, where the Lorentz’s force law is quite independent of the Maxwell field equations (for a discussion see e.g. \[28\]). The derivability of the equation of motion for particles is based on the fact that the Einstein equations are non-linear and also they are subject to additional identities, namely the Bianchi identities which reduce the number of independent equations. These Bianchi identities imply conservation laws for matter, present in the model under consideration.

In fact, the argumentation, based on the Bianchi identities could be used also for getting information about the possible field configurations for non-gravitational fields. Indeed, the Einstein equations for fields, minimally coupled to gravity, have the form

\[
R^\mu_\nu - \frac{1}{2} \delta^\mu_\nu R = \kappa T^\mu_\nu.
\] (43)

The Bianchi identities

\[
\nabla_\mu \left( R^\mu_\nu - \frac{1}{2} \delta^\mu_\nu R \right) = 0
\] (44)

imply some kind of the energy-momentum conservation law:

\[
\nabla_\mu T^\mu_\nu = 0.
\] (45)

For the case of the minimally coupled spatially homogeneous time-dependent scalar field with the standard sign of the kinetic term Eq. \[45\] is reduced to

\[
\left( \ddot{\phi} + 3h \dot{\phi} + \frac{dV}{d\phi} \right) \dot{\phi} = 0,
\] (46)

which is equivalent to the Klein-Gordon equation when \( \dot{\phi} \neq 0 \). For the phantom scalar field one has similarly

\[
\left( -\ddot{\phi} - 3h \dot{\phi} + \frac{dV}{d\phi} \right) \dot{\phi} = 0.
\] (47)

The comparison between Eqs. \[46\] and \[47\] points to the opportunity of the change of type of the Klein-Gordon equation at the moment when \( \dot{\phi} = 0 \). In other words, while the Einstein equations require the change of the type of the scalar field Lagrangian provided some form of the scalar field potential and the cosmological evolution are chosen, the Bianchi identities show why this transformation is possible.

Before concluding, let us notice that we have shown that simple scalar field models driving the universe through the phantom divide line do exist. However, the natural question arises: given the potential as a function of the scalar field (not of the time) how general are the initial conditions providing the reaching of the point \( \dot{\phi} = 0 \). In other words, do we need the fine tuning (cf. \[18\]) ?. The answer to this question is not simple technically and, perhaps, is worth further investigations. Another interesting question is concerned with an opportunity of using some kind of modified local Lagrangian formalism for the description of smooth dynamical transition from one form kinetic term to another one.

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[1] A. Riess et al., Astron. J. \textbf{116}, 1009 (1998); S.J. Perlmutter et al., Astroph. J. \textbf{517}, 565 (1999).
[2] V. Sahni and A.A. Starobinsky, Int. J. Mod. Phys. D \textbf{9}, 373 (2000); T. Padmananbhan, Phys. Rep. \textbf{380}, 235 (2003); P.J.E. Peebles and B. Ratra, Rev. Mod. Phys. \textbf{75}, 559 (2003); V. Sahni, Class. Quantum Grav. \textbf{19}, 3435 (2002).
[3] R.R. Caldwell, R. Dave and P.J. Steinhardt, Phys. Rev. Lett. \textbf{80}, 1582 (1998); J.A. Frieman, I. Waga, Phys. Rev. D \textbf{57}, 4642 (1998).
F. Lucchin and S. Matarrese, Phys. Rev. D 32, 1316 (1985); J.J. Halliwell, Phys. Lett. B 185, 341 (1987); J.D. Barrow, Phys. Lett. B 187, 12 (1987); A.B. Burd and J.D. Barrow, Nucl. Phys. B 308, 929 (1988); B. Ratra, Phys. Rev. D 45, 1913 (1992); J.D. Barrow, Phys. Lett. B 235, 40 (1990); J.E. Lidsey, Class. Quantum Grav. 9, 1239 (1992); S. Capozziello, R. de Ritis, C. Rubano and P. Scudellaro, Riv. Nuovo Cimento 19, 1 (1996); A.A. Coley, J. Ibanez and R.J. van den Hoogen, J. Math. Phys. 38, 5256 (1997); C. Rubano and P. Scudellaro, Gen. rel. Grav. 34, 307 (2002).

S.V. Chervon and V.M. Zhuravlev, gr-qc/9907051.

A.V. Yurov, astro-ph/0305019.

A.V. Yurov and S.D. Vereshchagin, Theor. Math. Phys. 139, 787 (2004).

A. Einstein, L. Infeld and B. Hoffmann, Ann. Math. 39, 65 (1938).

A. Einstein and L. Infeld, Ann. Math. 41, 455 (1940).

C. Lanczos, Phys. Rev. 59, 813 (1941).

V. Fock, JETP 9 (4), 375 (1939).

P.G. Bergmann, Introduction to the theory of relativity, (Prentice-Hall, New York, 1942).