On the nature of ill-posedness of the forward-backward heat equation.

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Abstract  
We study the Cauchy problem with periodic initial data for the forward-backward heat equation defined by the J-self-adjoint linear operator \( L \) depending on a small parameter. The problem has been originated from the lubrication approximation of a viscous fluid film on the inner surface of the rotating cylinder. For a certain range of the parameter we rigorously prove the conjecture, based on the numerical evidence, that the set of eigenvectors of the operator \( L \) does not form a Riesz basis in \( L^2(-\pi, \pi) \). Our method can be applied to a wide range of the evolutional problems given by \( PT \)-symmetric operators.

1 Introduction  
Analysis of the dynamic of a thin film of liquid which is entrained on the inside of a rotating cylinder is of great importance in lots of applications. For example when liquid thermosetting plastic is placed inside a rotating mould the best quality can be achieved if distribution of the liquid is as uniform as possible. More details about this application can be found in [26]. The same problem arises in the coating of fluorescent light bulbs when suspension consisting of a coating solute and a solvent is placed inside a spinning glass tube. The model for the coating was described for example in [7].

The lubrication approximation is used extensively to study flows in thin films. Under the assumption that the film is thin enough for viscous entrainment to compete with gravity, the time evolution of a thin film of liquid on the inner surface of a rotating in a gravity field cylinder can be described by the forward-backward heat equation:

\[
h_t + Lh = 0, \quad \theta \in (-\pi, \pi), \quad t \in (0, T),
\]

where

\[
Lh = \varepsilon \partial_\theta (\sin \theta h_\theta) + h_\theta, \quad h(-\pi) = h(\pi), \quad \varepsilon > 0.
\]
The effect of the surface tension is neglected in this linearized model derived by Benilov, O’Brien and Sazonov in \[2\].

We prove that the related to this equation Cauchy problem

\[ h|_{t=0} = h_0(\theta), \quad h(-\pi, t) = h(\pi, t). \] \tag{1.3}

does not have a weak in the Sobolev sense solution \( h(\theta, t) \) even locally in time if \( h_0(\theta) \) belongs to the class of finitely smooth functions with \( \text{supp} h_0 \cap (\delta, \pi - \delta) \neq \emptyset \).

The statement above can be roughly understood from the classic theory of the parabolic equations that states that regularity of a generalized solution depends on the regularity of the equation coefficients (in our case all coefficients are in \( C^\infty(-\pi, \pi) \)) and from the time-reversibility of the equation, i.e simultaneous change of the time variable \( t \) to \(-t\) and the space variable \( x \) to \(-x\) leads to the same partial differential equation. Time-reversibility and infinite regularity generally imply ill-posedness.

The physical explanation of this explosive blow up of solutions is related to a drop of fluid that can be detached from the film in the upper part of cylinder, where the effect of the gravity is the strongest \[2\] p. 217].

The eigenvalues of the operator \( L \) were studied asymptotically, with application of the modified WKB approximation and numerically, with application of the analytic continuation method, by Benilov, O’Brien and Sazonov \[2\] and they came to a very interesting set of hypotheses: all eigenvalues of the operator \( L \) are located on the imaginary axis, they are all simple and the set of eigenfunctions is complete in \( L^2(-\pi, \pi) \) that is not typical for the ill-posed time-evolution problem.

The analysis of the spectral properties of this operator was continued by Chugunova, Pelinovsky \[9\] and by Davies \[11\]. Using different approaches they analytically justified that if the parameter \(|\varepsilon| < 2\) then the operator is well defined in the sense that it admits closure in \( L^2(-\pi, \pi) \) with non-empty resolvent set. Analyzing tridiagonal matrix representation of the operator \( L \) with respect to the Fourier basis, Davies \[11\] showed that \( L \) admits an orthogonal decomposition with respect to three invariant subspaces \( \mathcal{H}^{2,0}(\mathbb{D}), \mathcal{H}^{2,0}(\overline{\mathbb{C}} \setminus \mathbb{D}) \), and \( \text{Ker}(L) = \{ c \|, \ c \in \mathbb{C} \} \) (see Section 4 below) and used this fact to prove that the nontrivial part \( \tilde{L} := L \upharpoonright \mathcal{H}^{2,0}(\mathbb{D}) \oplus \mathcal{H}^{2,0}(\overline{\mathbb{C}} \setminus \mathbb{D}) \) of \( L \) has a compact inverse of the Hilbert-Schmidt type. Therefore the spectrum of the original operator \( L \) is discrete with the only possible accumulation point at infinity.

Under the additional condition \( (1/\varepsilon \text{ is not integer}) \) it was proved by Weir \[30\] that if there exists an eigenvalue \( \lambda \) of the operator \( L \), then \( \mu = i \frac{2\pi}{\varepsilon} \) is an eigenvalue of some symmetric operator, hence \( \lambda \) can be only pure imaginary. The elegant proof is based on the continuation of the eigenfunctions into the Hardy space \( \mathcal{H}^2(\mathbb{D}) \) in the unit disk \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}. \) In this paper we sharpen this result showing that the additional condition above can be omitted. Note also that the similar problem was studied in the recent preprint \[5\] for a class operators that includes the operator \( L \).

It was shown numerically \[9\] \[11\] that the angle between the subspace spanned by the \( N \)-first eigenfunctions and the \( (N + 1)\)-th eigenfunction of the operator \( L \) tends to 0 as \( N \) goes to infinity. This gives the numerical evidence that the eigenfunctions do not form a Riesz basis in \( L^2(-\pi, \pi) \) because the related projectors are not uniformly bounded. One of the main goals of this paper is to prove analytically this numerical conjecture justifying that the operator \( L \) is not similar to a self-adjoint.

We also prove that the system of eigenvectors \( L \) is complete in \( L^2(-\pi, \pi) \) (see Theorem 4.3 and Remark 5.2). Hence, this implies that \( L \) has infinite number of pure imaginary eigenvalues that accumulate to \( \pm i\infty \). As a consequence, due to the linearity, the original Cauchy problem has infinitely
many global in time solutions which are linear combinations of harmonics $e^{i\lambda_n t}u_{\lambda_n}(x)$ where $i\lambda_n$ is an eigenvalue of the operator $L$ and $u_{\lambda_n}(x)$ is the related eigenfunction.

The operator $L$ is $J$-self-adjoint in the Krein space with $J(f(\theta)) = f(\pi - \theta)$ and therefore it belongs to the class of $PT$-symmetric operators. Interesting development of the spectral theory of $PT$-symmetric operators which are not similar to self-adjoint ones can be found in [20, 24, 25, 5].

**Notations:** In the sequel, $C_1, C_2, \ldots$ denote constants that may change from line to line but remain independent of the appropriate quantities. We also use $h', \partial_\theta h$, and $h_\theta$ for $\frac{dh}{d\theta}$. The symbol $\|\|$ denotes the function that identically equals 1 for $\theta \in (-\pi, \pi)$. Let $T$ be a linear operator in a Hilbert space $H$. The following classic notations are used: $\text{Dom}(T)$, $\text{Ker}(T)$, $\text{Ran}(T)$ are the domain, the kernel, and the range of $T$, respectively; $\sigma(T)$ and $\rho(T)$ denote the spectrum and the resolvent set of $T$; $\sigma_p(T)$ stands for the set of eigenvalues of $T$. We write $f(x) \asymp g(x)$ ($x \to x_0$) if both $f/g$ and $g/f$ are bounded functions in a certain neighborhood of $x_0$. By $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ we denote the open unit disc in $\mathbb{C}$.

## 2 Analysis of the differential equation.

One of the most general linear second-order differential equations with periodic coefficients that can be solved by a trigonometric series with a three-term recursion relation between the coefficients was studied by Magnus and Winkler [1] and has the form

$$(A + B \cos(2\theta))\frac{d^2y}{d\theta^2} + C \sin(2\theta)\frac{dy}{d\theta} + (D + E \cos(2\theta))y = 0, \quad (2.1)$$

where $A, B, C, D, E$ are constants. Under the additional condition that the coefficient $A + B \cos(2\theta)$ does not have zeros located on the real axis they studied existence of the periodic solutions to (2.1).

In this section we study the basic properties of the differential equation $\ell[h](x) = f(x)$ where the differential expression $h$ is given by

$$\ell[h] := \varepsilon \frac{d}{d\theta} \left( \sin(\theta) \frac{dh}{d\theta} \right) + \frac{dh}{d\theta}, \quad \theta \in (-\pi, \pi). \quad (2.2)$$

and use these properties to define the maximal periodic differential operator associated with $\ell$ and its inverse at the end of the section. This equation can be transformed to the form (2.1) but all singularities are located on the real axis so the additional condition of Magnus and Winkler is not satisfied.

Let $f \in L^2(-\pi, \pi)$ and $\varepsilon > 0$. Denote $\mathcal{I}_+ = (0, \pi)$, $\mathcal{I}_- = (-\pi, 0)$. Consider the differential equation

$$\ell[h](x) = f(x) \quad \text{a.e. on } (-\pi, \pi) \quad (2.3)$$

assuming that

the functions $h$ and $\varepsilon \sin(\theta)h' + h$ are in $AC_{loc}(\mathcal{I}_- \cup \mathcal{I}_+)$, \quad (2.4)

i.e., are absolutely continuous on each closed subinterval of $\mathcal{I}_- \cup \mathcal{I}_+$.

**Lemma 2.1.** Let $h$ satisfy (2.4). Then $h$ is a solution of the equation $\ell[h](x) = f(x)$ if and only if $h$ has the form

$$h(\theta) = |\cot(\theta/2)|^{1/\varepsilon} \left( k_2^\pm - \int_{\pm\pi/2}^\theta f(t) \tan(t/2)|^{1/\varepsilon} dt \right) + \int_0^\theta f(t)dt + k_1^\pm, \quad \theta \in \mathcal{I}_\pm, \quad (2.5)$$

where $k_1^\pm$ and $k_2^\pm$ are arbitrary constants.
The proof is based on direct calculations.

**Proposition 2.1.** Assume that $\varepsilon \in (0,2)$. A function $h \in L^2(\mathbb{R})$ satisfies (2.4) and is a solution of the equation $\ell[h] = f$ with $f \in L^2(\mathbb{R})$ if and only if $h$ has the form

$$h(\theta) = - \left| \cot \frac{\theta}{2} \right|^{1/\varepsilon} \int_0^\theta f(t) \left| \tan \frac{t}{2} \right|^{1/\varepsilon} dt + \int_0^\theta f(t) dt + k_1^\pm, \quad \theta \in \mathbb{I}_\pm,$$

where $k_1^\pm$ are arbitrary constants.

**Proof.** Assume that $h$ is an $L^2(-\pi, \pi)$-solution of $\ell[h] = f$. Then it has the form (2.5). Note that

$$f(t) \left| \tan \frac{t}{2} \right|^{1/\varepsilon} \in L^1(0, \delta) \text{ for any } \delta \in (0, \pi).$$

Therefore there exist the finite limit

$$C_1 := \lim_{\theta \to 0+} \left( k_1^\pm - \int_{\pm \pi/2}^\theta f(t) \left| \tan \frac{t}{2} \right|^{1/\varepsilon} dt \right).$$

If $C_1 \neq 0$, then (2.5) implies that $|h(\theta)| \geq |C_2| \theta^{-1/\varepsilon}$ for $\theta > 0$ small enough, where $C_2 > 0$. Since $\varepsilon < 2$, we see that $h \notin L^2(0, \pi)$. This shows that $C_1 = 0$, and therefore $h$ has the form (2.6) on $(0, \pi)$. Similarly, one can show that $h$ has the form (2.6) on $(-\pi, 0)$.

Let us prove that any function $h$ of the form (2.6) belongs to $L^2(0, \pi)$ (the proof for $L^2(-\pi, 0)$ is the same). It is enough to check that $h \in L^2(0, \delta)$ and $h \in L^2(\pi - \delta, \pi)$ for sufficiently small $\delta > 0$.

For $\theta \in (0, \delta)$, we have

$$\int_0^\theta |f(t)| \left| \tan \frac{t}{2} \right|^{1/\varepsilon} dt \leq C_3 \|f\|_{L^2} \left( \int_0^\theta t^{2/\varepsilon} dt \right)^{1/2} = C_3 \|f\|_{L^2} \theta^{1/2 + 1/\varepsilon}.$$

Hence,

$$\left| \cot \frac{\theta}{2} \right|^{1/\varepsilon} \int_0^\theta |f(t)| \left| \tan \frac{t}{2} \right|^{1/\varepsilon} dt \leq 2^{-1/\varepsilon} C_3 \|f\|_{L^2} \theta^{1/2},$$

and we finally see that $h \in L^2(0, \delta)$.

For $\theta \in (\pi - \delta, \pi)$, we have

$$\int_0^\theta |f(t)| \left| \tan \frac{t}{2} \right|^{1/\varepsilon} dt \leq C_4 + 2^{1/\varepsilon} \|f\|_{L^2} \left( \int_{\pi-\delta}^\theta (\pi - t)^{-2/\varepsilon} dt \right)^{1/2} \leq C_5 + 2^{1/\varepsilon} \|f\|_{L^2} (\pi - \theta)^{1/2 - 1/\varepsilon}.$$

Hence,

$$\left| \cot \frac{\theta}{2} \right|^{1/\varepsilon} \int_0^\theta |f(t)| \left| \tan \frac{t}{2} \right|^{1/\varepsilon} dt \leq C_6 (\pi - \theta)^{1/\varepsilon} + C_7 \|f\|_{L^2} (\pi - \theta)^{1/2}.$$

So $h \in L^2(\pi - \delta, \pi)$. □
In particular, we have proved that
\[
\lim_{\theta \to \pm 0} h(\theta) = k_1^\pm \quad \text{and} \quad \lim_{\theta \to \pm \pi \pm 0} h(\theta) = k_1^\pm + \int_0^{\pm \pi} f(t) dt
\]
hold for any \( L^2 \)-solution \( h \). This implies that the condition
\[
h \quad \text{is continuous on } [-\pi, \pi] \quad \text{and periodic} \quad \text{(2.8)}
\]
is fulfilled exactly when
\[
k_1^+ = k_1^- \quad \text{and} \quad f \perp .
\]

Let the symbol \( W_{2p}^k(-\pi, \pi) \) stand for the subspace of the space \( W_2^k(-\pi, \pi) \) consisting of periodic functions, i.e., functions satisfying the conditions \( u(i)(\pi) = u(i)(-\pi) \) \( (i = 0, 1, \ldots, k - 1) \). The norm in this space coincides with that of the Sobolev space \( W_2^k(-\pi, \pi) \).

\textbf{Proposition 2.2.} Let \( \varepsilon \in (0, 2), f \in L^2(-\pi, \pi), \) and \( \int_{-\pi}^{\pi} f(\theta) d\theta = 0 \). Then an \( L^2 \)-solution of \( \ell[h] = f \) satisfies \( \text{(2.8)} \) if and only if
\[
h(\theta) = -\cot \frac{\theta}{2} \left| \theta \right|^{1/\varepsilon} \left| f\left( \theta \right) \tan \frac{\theta}{2} \right|^{1/\varepsilon} d\theta + \int_0^\theta f(\theta) d\theta + k_1 \quad \text{for a.a. } \theta \in (-\pi, \pi), \quad \text{(2.9)}
\]
where \( k_1 \) is an arbitrary constant. Moreover, any function \( h \) of the form \( \text{(2.9)} \) possesses the following properties:

(i) \( h \in AC[-\pi, \pi], h' \in L^2(-\pi, \pi), \) and
\[
\| h' \|_{L^2} \leq K (|k_1| + \| f \|_{L^2}), \quad \text{(2.10)}
\]
where \( K \) is a constant independent of \( f \).

(ii) \( \sin(\theta) h' \in AC[-\pi, \pi] \) and \( \sin(\theta) h' \)' \( \in L^2(-\pi, \pi) \).

\textit{Proof.} To show that \( h' \in L^2(-\pi, \pi) \), it is enough to prove that \( h' \in L^2(0, \delta) \) for any \( \delta > 0 \) small enough. Since
\[
h'(\theta) = -\frac{1}{\varepsilon \sin \theta} \cot^{1/\varepsilon} \frac{\theta}{2} \int_0^\theta f(t) \tan^{1/\varepsilon} \frac{t}{2} dt, \quad \theta \in (0, \delta),
\]
it is sufficient to show that
\[
\int_0^\delta \theta^{-2/\varepsilon} \left( \int_0^\theta f(t) t^{1/\varepsilon} dt \right)^2 d\theta \leq C_1 \int_0^\delta |f(\theta)|^2 d\theta, \quad \text{(2.11)}
\]
for any \( f \in L^2(0, \delta) \).

Denote \( g(t) := f(t)t^{1/\varepsilon} \). Then \( \text{(2.11)} \) takes the form
\[
\int_0^\delta \theta^{-2/\varepsilon} \left( \int_0^\theta g(t) dt \right)^2 d\theta \leq C_1 \int_0^\delta |g(\theta)|^2 \theta^{-2/\varepsilon} d\theta. \quad \text{(2.12)}
\]
This is a weighted norm inequality for the Hardy operator. Applying [21] (see also [23] and references therein), we see that
\[
\sup_{\theta \in [0, \delta]} \left( \int_{\theta}^{\delta} t^{-2-2/\varepsilon} dt \right)^{1/2} \left( \int_{0}^{\theta} \left( \frac{\theta^{-2/\varepsilon}}{\varepsilon} \right)^{1/2} dt \right)^{1/2} < \infty, \tag{2.13}
\]
and therefore (2.12) holds true. It is easy to see that the latter implies (2.10) and statement (i) of the theorem.

If \( f \perp \| \), then \( \sin(\theta) h' + h \in W^{1}_{2p}(-\pi, \pi) \) and, by statement (i), so is \( \sin(\theta) h' \).

Introduce the space \( X_{2} = \{ h \in W^{1}_{2p}(-\pi, \pi) : (\sin \theta) h_{\theta} \in W^{1}_{2}(-\pi, \pi) \} \) endowed with the norm
\[
\| h \|_{2}^{2} = \| h_{\theta} \|_{L^{2}(-\pi, \pi)}^{2} + \| (\sin \theta) h_{\theta} \|_{W^{1}_{2}(-\pi, \pi)}^{2}.
\]

Denote by \( X_{0}^{2} \) the subspace of \( X_{2} \) comprising the functions \( h \) with the property \( \int_{-\pi}^{\pi} h(\theta) d\theta = 0 \). As a consequence of the definitions, we obtain that if \( h \in X_{2} \), then the function \( (\sin \theta) h_{\theta} \) is absolutely continuous (may be after a change on a set of zero measure) and
\[
(\sin \theta) h_{\theta}|_{\theta=0} = (\sin \theta) h_{\theta}|_{\theta=\pi} = (\sin \theta) h_{\theta}|_{\theta=-\pi} = 0.
\]

Let the symbol \( L^{p}_{2}(-\pi, \pi) \) stand for the subspace of \( L^{2}(-\pi, \pi) \) comprising the functions \( f \) with the property \( \int_{-\pi}^{\pi} f(\theta) d\theta = 0 \).

We write below a set of corollaries of Proposition 2.2. We also give the alternative prove of this result using the Galerkin method in Appendix A.

Denote by \( L \) the operator acting in \( L^{2}(-\pi, \pi) \) and defined by
\[
Lf = \ell[f] \quad \text{for} \quad f \in \text{Dom}(L) := X_{2}. \tag{2.14}
\]

Clearly, \( \text{Ker}(L) = \{ c \|, \ c \in \mathbb{C} \} \).

Let us denote by \( \bar{L} \) the restriction of \( L \) on \( L^{p}_{2}(-\pi, \pi) \) \((= \text{Ker}(L) \perp)\),
\[
\bar{L} := L \upharpoonright L^{2}_{p}(-\pi, \pi), \quad \text{Dom}(\bar{L}) := \text{Dom}(L) \cap L^{2}_{p}(-\pi, \pi). \tag{2.15}
\]

It follows from the remark after (2.8) that \( \text{Ran} \ L \subset L^{2}_{p}(-\pi, \pi) \). So \( \bar{L} \) is an operator in the Hilbert space \( L^{2}_{p}(-\pi, \pi) \).

To find the inverse operator \( \bar{L}^{-1} \), let us symmetrize (2.9) as
\[
h(\theta) = -\cot \frac{\theta}{2} \left( \frac{\theta}{2} \right)^{1/\varepsilon} \int_{0}^{\theta} f(t) \left( \tan \frac{t}{2} \right)^{1/\varepsilon} - \left| \tan \frac{\theta}{2} \right|^{1/\varepsilon} dt + k_{1} \quad \text{for a.a.} \ \theta \in (-\pi, \pi). \tag{2.16}
\]

Solving the equation \((h, 1) = 0\), we get
\[
k_{1} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[ \cot \frac{\theta}{2} \left( \frac{\theta}{2} \right)^{1/\varepsilon} \int_{0}^{\theta} f(t) \left( \tan \frac{t}{2} \right)^{1/\varepsilon} - \left| \tan \frac{\theta}{2} \right|^{1/\varepsilon} dt \right] d\theta \tag{2.17}
\]
So for \( f \in \text{Ran}(L) \), we have \( \bar{L}^{-1} f = h \) with \( h \) defined by (2.16)-(2.17).
Corollary 2.1.

(i) The operator $L$ defined by (2.14) is a closed operator in $L^2(-\pi, \pi)$ (with the dense domain $X_2$).

(ii) Its kernel $\text{Ker}(L)$ is the one-dimensional subspace of constants $\{c \parallel, c \in \mathbb{C}\}$.

(iii) The range $\text{Ran}(L)$ of $L$ is the orthogonal complement to $\text{Ker}(L)$, $\text{Ran}(L) = L^2_p(-\pi, \pi)$.

(iv) The operator $\tilde{L}$ defined by (2.15) (and acting in $L^2_p(-\pi, \pi)$) has a compact inverse $\tilde{L}^{-1}$.

(v) The operator $\tilde{L} : X^0_2 \rightarrow L^2_p(-\pi, \pi)$ is an isomorphism of $X^0_2$ onto $L^2_p(-\pi, \pi)$.

Proof. (i)-(iii) Let $k_1 = k_1(f)$ be the linear functional defined by (2.17). It is easy to see that $k_1$ is bounded on $L^2(-\pi, \pi)$. \hfill (2.18)

This and Proposition [2.2] imply immediately that $\text{Ran}(L) = L^2_p(-\pi, \pi)$. So $\text{Dom}(\tilde{L}) = L^2_p(-\pi, \pi)$. It follows from (2.10) and $\|h\|_{L^2(-\pi, \pi)} \geq \|h\|_{L^2(-\pi, \pi)}$, $h \in L^2_p(-\pi, \pi)$, that $\tilde{L}^{-1}$ is a bounded linear operator on $L^2_p(-\pi, \pi)$. Therefore $\tilde{L}$ is closed and so is $L$.

(iv)-(v) It follows from (2.10) and (2.18) that $\tilde{L}^{-1}$ is bounded as an operator from $L^2_p(-\pi, \pi)$ onto $X^0_2$. This proves statements (iv) and (v).

Remark 2.1. The adjoint operation $\ell^*$ is given by

$$\ell^*[h] := \frac{1}{\pi} \int d\theta \left( \sin(\theta) \frac{dh}{d\theta} \right) - \frac{dh}{d\theta}, \quad \theta \in (-\pi, \pi).$$

It is easy to see that the adjoint operator $L^*$ is unitary equivalent to $L$ and has the form

$$L^*h = \ell^*[h], \quad \text{Dom}(L^*) = X^0_2.$$

Indeed, $L = JL^*J$, where $(Jh)(\theta) = h(\pi - \theta)$. In turn, this implies that the statements analogous to that of Corollary 2.1 are valid for $L^*$. Note that $J$ is a signature operator, i.e., $J = J^* = J^{-1}$, so the operator $L$ is $J$-self-adjoint and belongs to the class of the PT-symmetric operations.

3 The ill-posedness of the Cauchy problem for the forward-backward heat equation.

The linearized model of the thin film dynamic (1.1) was derived without taking into account the smoothing effect of the surface tension. It’s very natural to expect that a drop of fluid will detach itself from the "ceiling" of the rotating cylinder and it will inevitably fall down that perfectly fits into the ill-posed nature of the Cauchy problem for the forward-backward heat equation (1.1). The intuition based on the classic theory of backward heat equation, says that global in time classic solutions can exist only for some class of analytic in vertical strip $(-\pi, \pi)$ functions with exponentially fast decaying Fourier coefficients.
From there on in this section we assume that the parameter \(0 < \varepsilon < 2\). Let us consider the parabolic problem

\[
\begin{align*}
    h_t + Lh &= 0, \quad \theta \in (-\pi, \pi), \quad t \in (0, T), \\
    h|_{t=0} &= h_0(\theta), \quad u(-\pi, t) = u(\pi, t).
\end{align*}
\]

(3.1) (3.2)

Note that, after the change of variables \(\theta \rightarrow -\theta\), equation (3.1) can be replaced with the equation

\[
u_t - Lu = 0.
\]

Let \(Q = (-\pi, \pi) \times (0, T)\). We prove in this section that the problem (3.1), (3.2) is ill-posed in the classes of finite smoothness. In what follows, the symbol \((\cdot, \cdot)\) stands for the inner product in the space \(L^2\) in the corresponding domain \((Q, Q_1, \ldots)\).

**Definition 3.1.** By a generalized solution to the problem (3.1), (3.2) from the space \(W^{1,0}_2(Q)\) we imply a function \(h \in L^2(0, T; W^{1}_2(-\pi, \pi)) \cap C([0, T]; L^2(-\pi, \pi))\) such that

\[-(h, v_t) - (\sin \theta h_\theta, v_\theta) + (h_\theta, v) = \int_{-\pi}^{\pi} h_0(\theta)v(\theta, 0) \, d\theta\]

for all \(v \in L^2(0, T; W^{1}_2(-\pi, \pi)) : v_t \in L^2(Q), v(\theta, T) = 0\).

Note that we can use other definitions of generalized solutions (see [19]). In particular, it is possible to prove that the above definition is equivalent to the following definitions.

**Definition 3.2.** A function \(h \in L^2(0, T; W^{1}_2(-\pi, \pi)) \cap C([0, T]; L^2(-\pi, \pi))\) such that \(h|_{t=0} = u_0(\theta)\) is called a generalized solution to the problem (3.1), (3.2) if

\[-(h, v_t) - (\sin \theta h_\theta, v_\theta) + (h_\theta, v) = 0\]

for all \(v \in L^2(0, T; W^{1}_2(-\pi, \pi)) : v_t \in L^2(Q), v(\theta, T) = 0, v(\theta, 0) = 0\).

**Definition 3.3.** A function \(h \in L^2(0, T; W^{1}_2(-\pi, \pi)) \cap C([0, T]; L^2(-\pi, \pi))\) such that \(h|_{t=0} = h_0(\theta)\) and \(h_t \in L^2(0, T; W^{-1}_2(-\pi, \pi))\) is called a generalized solution to the problem (3.1), (3.2) if

\[h_t + Lh = 0\]

and this equality holds in the space \(L^2(0, T; W^{-1}_2(-\pi, \pi)) (W^{-1}_2(-\pi, \pi)\) is the negative space constructed on the pair \(W^1_2(-\pi, \pi), L^2(-\pi, \pi)\).

**Theorem 3.1 (nonexistence).** Assume that there exists \(k \in \mathbb{N}\) such that \(h_0 \in W^{k}_2(-\pi, \pi), \supp h_0 \in (0, \pi)\), and \(h_0 \not\in W^{k+1}_2(-\pi, \pi)\). Then there is no a generalized solution from the space \(W^{1,0}_2(Q)\) for problem (3.1), (3.2).

**Proof.** Assume the contrary, i.e., that such a solution exists. Let \(\supp h_0 \subset (\delta, \pi - \delta)\). Put \(Q_0 = (\delta, \pi - \delta) \times (0, T/2)\). We have

\[-(h, v_t) - (\sin \theta h_\theta, v_\theta) + (h_\theta, v) = \int_{-\pi}^{\pi} h_0(\theta)v(\theta, 0) \, d\theta\]

(3.3)
for all functions \( v \) from Definition 3.1 such that \( \operatorname{supp} v \subset Q_0 \cup \{ (\theta, 0) : \theta \in (\delta, \pi - \delta) \} \). Make the change of variables \( \tau = T/2 - t \). Then the function \( h \) is a generalized solution of the equation

\[
    h_\tau - Lh = 0
\]

in \( Q_0 \) and \( h \) satisfies the condition

\[
    h|_{\tau=T/2} = h_0(\theta). \tag{3.4}
\]

Here, we understand a generalized solution in the sense that the function \( h \in \mathcal{L}^2(0, T/2; W^1_2(\delta, \pi - \delta)) \cap C([0, T/2]; \mathcal{L}^2(\delta, \pi - \delta)) \) is such that

\[
    - (h, v_\tau) + (\sin \theta h_\theta, v_\theta) - (h_\theta, v) = -\int_\delta^{\pi-\delta} h_0(\theta)v(\theta, T/2) \, d\theta \tag{3.5}
\]

for all \( v \in \mathcal{L}^2(0, T/2; W^1_2(\delta, \pi - \delta)) : v_\tau \in \mathcal{L}^2(Q_0), v(\theta, 0) = 0, v|_{\theta=\delta} = v|_{\theta=\pi-\delta} = 0 \). Now construct a function \( h_1 \) being a solution to the problem

\[
    h_\tau - Lh = 0, \quad h|_{\theta=\delta} = h|_{\theta=\pi-\delta} = 0, \quad h|_{\tau=T/2} = h_0(\theta),
\]

in the domain \( Q_1 = (\delta, \pi - \delta) \times (T/2, T) \). Since \( k \geq 1 \), we see that a solution of this problem exists and belongs at least to the space \( W^{2,1}_2(Q_1) \) (see [19 Theorem 3.6.1]). This solution satisfies the integral identity

\[
    - (h_1, v_\tau) + (\sin \theta h_{1\theta}, v_\theta) - (h_{1\theta}, v) = \int_\delta^{\pi-\delta} h_0(\theta)v(\theta, T/2) \, d\theta \tag{3.6}
\]

for all \( v \in \mathcal{L}^2(0, T/2; W^1_2(\delta, \pi - \delta)) : v_\tau \in \mathcal{L}^2(Q_1), v(\theta, T) = 0, v|_{\theta=\delta} = v|_{\theta=\pi-\delta} = 0 \). Summing (3.5) and (3.6) we conclude that

\[
    - (h_2, v_\tau) + (\sin \theta h_{2\theta}, v_\theta) - (h_{2\theta}, v) = 0 \tag{3.7}
\]

for all \( v \in \mathcal{L}^2(0, T; W^1_2(\delta, \pi - \delta)) : v_\tau \in \mathcal{L}^2(Q_2), v(\theta, T) = 0, v|_{\theta=\delta} = v|_{\theta=\pi-\delta} = 0 \). Here the function \( h_2 \) coincides with \( h \) for \( t \in (0, T/2) \) and with \( h_1 \) for \( t \in (T/2, T) \). So the function \( h_2 \) is a generalized solution of the equation

\[
    h_\tau - Lh = 0
\]

in \( Q_2 \) in the sense of the integral identity (3.7). By [19 Theorem III.12.1], we have \( u \in C^\infty(Q_2) \). Therefore, \( h_2(\theta, T/2) = h_1(\theta, T/2) = h_0(\theta) \in C^\infty(\delta, \pi - \delta) \). This contradicts to the fact that \( h_0 \notin W^{k+1}_{2p}(-\delta, \delta) \).

**Remark 3.1.** It is easily seen from the proof that it does not matter where the support of \( h_0 \) lies. The main condition is that \( h_0 \notin W^{k+1}_{2p}(\delta, \pi - \delta) \) for some \( \delta > 0 \). So the theorem can be strengthened.

**Theorem 3.2** (instability). Assume that the problem (3.1), (3.2) is densely solvable in the following sense: for \( h_0 \) from some set \( K \) of smooth functions that is dense in the space \( W^k_{2p}(-\pi, \pi) \) \((k \geq 2)\), problem (3.1), (3.2) has a generalized solution \( h \) in the sense of Definition 3.1. Then there is no constant \( c > 0 \) such that, for every generalized solution of the problem (3.1), (3.2) with initial value \( h_0 \in K \), the estimate

\[
    \|h\|_{\mathcal{L}^2(Q)} \leq c\|h_0\|_{W^{k+1}_{2p}(-\pi, \pi)} \tag{3.8}
\]

holds.
Proof. We use the arguments of Theorem 3.1. Let us find a function $h_0 \in W^k_{2p}(-\pi, \pi)$ such that $\text{supp } h_0 \subset (0, \pi)$, and $h_0 \notin W^{k+1}_{2p}(-\pi, \pi)$. Find $\delta > 0$ such that $\text{supp } u_0 \subset (\delta, \pi - \delta)$. Put $Q_0 = (\delta/2, \pi - \delta/2) \times (0, T/2)$.

Assume the contrary, i.e., that (3.5) holds. Construct a sequence $h_{0n} \in K : \|h_{0n} - h_0\|_{W^k_{2}(-\pi, \pi)} \to 0$ as $n \to \infty$. In this case,

$$\|h_{0n}\|_{W^{k+1}_{2}(\delta/4, \pi - 3\delta/4)} \to \infty \text{ as } n \to \infty. \quad (3.9)$$

Denote by $h_n$ the corresponding generalized solutions to our parabolic problem. As in the proof of Theorem 3.1, the change of variables $\tau = T/2 - t$, shows that the functions $h_n = h_n(\theta, T/2 - \tau)$ satisfy the integral identity

$$-(\tilde{h}_n, v_\tau) + (\sin \theta \tilde{h}_{n \theta}, v_\theta) - (\tilde{h}_{n \theta}, v) = -\int_{\delta/2}^{\pi - \delta/2} h_{0n}(\theta) v(\theta, T/2) \, d\theta \quad (3.10)$$

for all $v \in \mathcal{L}^2(0, T/2; W^1_{2}(\delta/2, \pi - \delta/2)) : v_t \in \mathcal{L}^2(Q_0), v(\theta, 0) = 0, v|_{\theta = \pm \delta/2} = v|_{\theta = \pi - \delta/2} = 0$.

Thus, the functions $h_n$ are generalized solutions to a parabolic equation in $Q_0$. Using Theorems III.8.1 and III.12.1 and Theorem IV.10.1 in [19], we obtain that the function $h_n(\theta, T/2 - \tau)$ is infinitely differentiable in $Q_0$ and the norm of this function in any Hölder space $H^{2+\alpha, 1+\alpha/2}(Q_1)$ with $Q_1 = (\delta', \pi - \delta') \times (\varepsilon_0, T/2)$ ($\delta' > 0, \varepsilon_0 > 0$) is estimated by some constant depending on $\delta', \varepsilon_0, \alpha$, and the norm $\|h_n\|_{\mathcal{L}^2(Q_0)}$. In view of 3.8 with $h_n, h_{0n}$ substituted for $h, h_0$ and the fact that the norms $\|h_{0n}\|_{W^k_{2}(-\pi, \pi)}$ are bounded, we can assume that this constant is independent of $n$. As a consequence, we have the estimate

$$\|h_n(\theta, 0)\|_{W^{k+1}_{2}(\delta/4, \pi - 3\delta/4)} \leq c\|h_n\|_{\mathcal{L}^2(Q_0)} \quad (3.11)$$

where the constant $c$ is independent of $n$. Comparing 3.11 with 3.9 we arrive at a contradiction. \hfill \Box

4 The completeness property for the operator $L$

As in the previous section we restrict the parameter to the interval $0 < \varepsilon < 2$. In this section, we prove that the system of all eigenvectors and generalized eigenvectors of the operator $L$ is complete in $\mathcal{L}^2(-\pi, \pi)$. In particular, this implies that $L$ has infinite number of eigenvalues.

Denote by $\mathcal{H}^{2,0}(\mathbb{D})$ and $\mathcal{H}^{2,0}(\overline{\mathbb{C}} \setminus \mathbb{D})$ the subspaces of the Hardy spaces $\mathcal{H}^2(\mathbb{D})$ and $\mathcal{H}^2(\overline{\mathbb{C}} \setminus \mathbb{D})$, respectively (see e.g. [12], Section 2.1) that are orthogonal to the function $c$ ($\equiv 1$). In the sequel, we use the standard identification of the function $u(z) \in \mathcal{H}^2(\mathbb{D})$ with the function $u(e^{i\theta}) := \lim_{r \to 1^0} u(re^{i\theta})$, which belongs to $\mathcal{L}^2(-\pi, \pi)$, and also use the similar agreement for $u(z) \in \mathcal{H}^2(\overline{\mathbb{C}} \setminus \mathbb{D})$. Then $\mathcal{H}^2(\mathbb{D})$ and $\mathcal{H}^2(\overline{\mathbb{C}} \setminus \mathbb{D})$ are the subspaces of $\mathcal{L}^2(-\pi, \pi)$ in $\mathcal{L}^2(\overline{\mathbb{C}} \setminus \mathbb{D}) = \{c \|, c \in \mathbb{C}\}$. In these terms, the space $\mathcal{L}^2(-\pi, \pi)$ admits the orthogonal decomposition

$$\mathcal{L}^2(-\pi, \pi) = \mathcal{H}^{2,0}(\mathbb{D}) \oplus \{c \|, c \in \mathbb{C}\} \oplus \mathcal{H}^{2,0}(\overline{\mathbb{C}} \setminus \mathbb{D}). \quad (4.1)$$

Define the operator $L_{\text{fin}}$ in the Hilbert space $\mathcal{L}^2(-\pi, \pi)$ by $L_{\text{fin}}h := \ell[h], \ Dom(L_{\text{fin}}) = P_{\text{fin}}$, where $\ell$ is the differential expression defined in Section 2 and $P_{\text{fin}}$ is the set of finite trigonometric polynomials

$$h(\theta) = (2\pi)^{-1/2} \sum_{n=-N}^{N} v_ne^{in\theta}, \quad N < \infty, \quad v_n \in \mathbb{C}.$$
It is easy to see that $L^*_{\text{fin}}$ is densely defined, and hence the closure

$$L_{\text{min}} := \overline{L_{\text{fin}}}$$

exists as an operator in $L^2(-\pi, \pi)$.

Let $L$ be the indefinite convection-diffusion operator defined by (2.14) and let $\tilde{L}$ be its restriction defined by (2.15). It is easy to see that Remark 2.1 implies that $L^*_{\text{fin}} = L^*$ and therefore $L_{\text{min}} = L$. Below we give another proof of this fact using the results of [11]. This proof allows us to use the orthogonal decomposition of $L$ obtained in [11] (see also [9]).

**Proposition 4.1** (Theorems 11 and 13 in [11]).

(i) The operator $L_{\text{min}}$ admits the orthogonal decomposition $L_{\text{min}} = L_- \oplus 0 \oplus L_+$ with respect to (4.1).

(ii) The operators $L_\pm$ are invertible, and their inverses $L_\pm^{-1}$ are Hilbert-Schmidt operators.

(iii) $L_+^{-1}$ and $(-L_-)^{-1}$ are unitary equivalent.

**Proposition 4.2.** If $\varepsilon \leq 2$, then $L = L_{\text{min}}$ and $\tilde{L} = L_- \oplus L_+$.

**Proof.** By Corollary 2.1 (i), $L$ is a closed extension of $L_{\text{fin}}$. Hence $L_{\text{min}} \subset L$. Let us show that $L = L_{\text{min}}$.

By Proposition 4.1 the operator $L_+^{-1} \oplus L_-^{-1}$ is compact and is acting in $L^2_p(-\pi, \pi)$. So $\sigma(L_- \oplus L_+)$ is at most countable. Then

$$\text{Ran}(L_- \oplus L_+ - \lambda I) = L^2_p(-\pi, \pi) \quad \text{for any} \quad \lambda \in \rho(L_- \oplus L_+).$$  \hfill (4.2)

By Corollary 2.1 (iv), the operator $\tilde{L}^{-1}$ acting in $L^2_p(-\pi, \pi)$ is compact. So $\tilde{L}$ possesses the same properties, that is, $\sigma(\tilde{L})$ is at most countable and (4.2) holds for $\tilde{L}$.

Assume that $L_+ \oplus L_- \subset \tilde{L}$ (which is equivalent to $L_{\text{min}} \subset \tilde{L}$). Then Eqs. (4.2) for $L_+ \oplus L_-$ and $\tilde{L}$ imply that $\lambda \in \sigma_p(\tilde{L})$ whenever $\lambda \in \rho(\tilde{L}) \cap \rho(L_+ \oplus L_-)$, a contradiction. \hfill $\Box$

**Definition 4.1** (e.g. [14]). By $S_p$, $0 < p < \infty$, we denote the class of all bounded linear operators $A$ acting on a Hilbert space $H$ for which

$$|A|_p := \left( \sum_{j=1}^{\infty} s_j^p(A) \right)^{1/p} < \infty$$

where $s_j(A)$ are singular numbers of $A$, i.e., eigenvalues of the self-adjoint operator $(A^*A)^{1/2}$ that are enumerated in decreasing order, counted with multiplicities.

Two $S$-classes were given special names: $S_2$ is the class of Hilbert-Schmidt operators and $S_1$ is the class of nuclear operators. It was proved by Davies [11] that the operators $A_\pm^{-1}$, where

$$A_\pm := -iL_\pm,$$  \hfill (4.3)

belong to the class $S_2$ and so does $\tilde{L}^{-1}$. We prove in this section that, actually, the operator $\tilde{L}^{-1}$ is nuclear.

We need the following result (see [13], its weaker version can be found e.g. in [14, Section III.7.8]):
Theorem 4.1 (Gohberg, Markus). If $0 < p \leq 2$, then the linear operator $A$ acting in a Hilbert space $H$ belongs to $S_p$ if and only if for at least one orthonormal basis $\{e_j\}$ of $H$ the inequality
\[
\sum_{j=1}^{\infty} |Ae_j|^p < \infty
\] (4.4)
holds. In addition,
\[
|A|^p \leq \sum_{j=1}^{\infty} |Ae_j|^p \leq \sum_{j,k=1}^{\infty} |(Ae_j, e_k)|^p.
\]

It follows from [11, Theorem 11 and Eq. (15)] that the operator $iL^{-1} (= A^{-1})$ in the Fourier basis $\{e_n\}_{n=1}^{\infty}, e_n(\theta) = e^{i\theta n}$, is represented by matrix $(\rho_{m,n})$ which has the following properties
\[
|\rho_{m,n}| \leq C_1 m^{-1+1/\epsilon} n^{-1-1/\epsilon}, \quad m \leq n,
\]
\[
|\rho_{m,n}| \leq C_1 m^{-1-1/\epsilon} n^{-1+1/\epsilon}, \quad n < m.
\] (4.5)

Proposition 4.3. The operator $\tilde{L}^{-1}$ belongs to the class $\mathfrak{S}_1$ for any $\epsilon \in (0, 2)$. More precisely:

(i) if $\epsilon \in (0, 1]$, then $\tilde{L}^{-1} \in \mathfrak{S}_p$ for any $p > 2/3$,

(ii) if $\epsilon \in (1, 2)$, then $\tilde{L}^{-1} \in \mathfrak{S}_p$ for any $p > 2\epsilon/(\epsilon + 2)$.

Proof.

(i) $\epsilon \in (0, 1]$.

Using (4.5), one can obtain that
\[
\|L_+^{-1}e_n\|_{L_2}^2 = \sum_{m=1}^{\infty} |\rho_{m,n}|^2 \leq C_1^2 n^{-2+2/\epsilon} \sum_{m=1}^{n} m^{-2+2/\epsilon} + C_2^2 n^{-2+2/\epsilon} \sum_{m=n+1}^{\infty} m^{-2-2/\epsilon}
\]
\[
\leq C_1^2 n^{-3} + C_2^2 n^{-2+2/\epsilon} \int_{n}^{\infty} m^{-2-2/\epsilon} dm \leq C_2 n^{-3}.
\] (4.6)

Hence,
\[
\sum_{n=1}^{\infty} \|L_+^{-1}e_n\|_{L_2}^p \leq C_3 \sum_{n=1}^{\infty} n^{-3p/2}.
\]

So the Gohberg-Markus criterion (Theorem 4.1) shows that $L_+^{-1} \in \mathfrak{S}_p$ for any $2/3 < p \leq 2$. Thus $L_+^{-1}$ belongs to $\mathfrak{S}_p$ for any $p > 2/3$ and so does $\tilde{L}^{-1}$ due to Propositions 4.1 (ii) and 4.2

(ii) $\epsilon \in (1, 2]$.

Using (4.5), one can obtain that
\[
\|L_+^{-1}e_n\|_{L_2}^2 \leq C_1^2 n^{-2-2/\epsilon} \sum_{m=1}^{n} m^{-2+2/\epsilon} + C_2^2 n^{-2+2/\epsilon} \sum_{m=n+1}^{\infty} m^{-2-2/\epsilon} \leq C_1^2 n^{-1-2/\epsilon} + C_2^3 n^{-3} \leq C_3 n^{-1-2/\epsilon}.
\]

Therefore, $\tilde{L}^{-1} \in \mathfrak{S}_p$ whenever $p(\epsilon + 2)/(2\epsilon) > 1$. 

\[ \square \]
Although the weaker result that the operator \( \tilde{L}^{-1} \in \mathcal{S}_p \) for \( p > 1 \) can be obtained directly from the factorization of the operator \( L \) found by Chugunova and Strauss in \cite{10}, the fact that the operator \( \tilde{L}^{-1} \) actually belongs to the class of nuclear operators \( \mathcal{S}_1 \) is crucial, as you see below, for the proof of completeness.

Following \cite{14} Section IV.4, we will call an operator \( T \) acting in a Hilbert space \( H \) dissipative if
\[
\text{Im}(Tf, f) \geq 0 \quad \text{for all } f \in \text{Dom}(T). \tag{4.7}
\]

**Proposition 4.4.** The operators \( L_+, (-L_+)^{-1}, -L_- \) and \( L_1^{-1} - L_1 \) are dissipative.

**Proof.** Using the tridiagonal matrix representations of \( A_+ (= -iL_+) \) with respect to the Fourier basis \( \{e^{i\theta}\}_{\theta=0}^{\infty} \) (see \cite{9, 11}), we get:
\[
A_+ = (a_{n,m})_\infty = \begin{bmatrix}
1 & -\varepsilon & 0 & 0 & \cdots \\
\varepsilon & 2 & -3\varepsilon & 0 & \cdots \\
0 & 3\varepsilon & 3 & -6\varepsilon & \cdots \\
0 & 0 & 6\varepsilon & 4 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}, \tag{4.8}
\]
where \( a_{n,n} = n, \quad a_{n-1,n} = \varepsilon^2 n(n-1), \quad a_{n,n+1} = -\frac{\varepsilon}{2} n(n+1), \quad n = 1, 2, \ldots. \)

This representation implies that
\[
\text{Im}(L_+ h, h) = \text{Re}(A_+ h, h) \geq 0 \tag{4.9}
\]
for all \( h \in \mathcal{H}^2(\mathbb{D}) \cap P_{\text{fin}} \). Since \( L = \overline{L_+} \), one gets (4.9) for all \( h \in \text{Dom}(L_+) \), i.e., \( L_+ \) is dissipative. Substituting \( h = (L_+)^{-1} f \) into (4.9), we see that so is \((-L_+)^{-1}\). Proposition 4.1 (iii) completes the proof.

**Theorem 4.2** (Lidskii, see e.g. Theorem V.2.3 \cite{14}). *If the dissipative operator \( A \) acting in a Hilbert space \( H \) belongs to the class \( \mathcal{S}_1 \), then its system of all eigenvectors and generalized eigenvectors is complete in \( H \).*

Now the main result of this section can be obtained using Propositions 4.1, 4.3, 4.4 and Lidskii’s theorem.

**Theorem 4.3.** The operator \( L \) has infinitely many eigenvalues. The system of its eigenvectors and generalized eigenvectors is complete in \( \mathcal{L}^2(\pi, \pi) \).

**Proof.** By Propositions 4.3, 4.4 and Lidskii’s theorem, the system of all eigenvectors and generalized eigenvectors of the operators \( L_1^{-1}, L_1^{-1}, \) and \( \tilde{L}^{-1} \) are complete in \( \mathcal{H}^2(\mathbb{D}), \mathcal{H}^2(\mathbb{C} \setminus \mathbb{D}) \), and \( \mathcal{L}^2_p(\pi, \pi) \), respectively. Since \( \text{Ker}(\tilde{L}^{-1}) = \{0\} \), all generalized eigenspaces of \( \tilde{L}^{-1} \) corresponding to its eigenvalues \( \alpha_n \) are generalized eigenspaces of \( \tilde{L} \) corresponding to eigenvalues \( \lambda_n = 1/\alpha_n \). Since all eigenvalues of the compact operator \( \tilde{L}^{-1} \) have finite algebraic multiplicities, we see that \( \tilde{L}^{-1} \) and \( \tilde{L} \) have infinitely many eigenvalues. The finite-dimensional spectral mapping theorem implies that the completeness property holds for \( L \) and, consequently, for \( L \).
5 Pure imaginary eigenvalues and the Riesz basis property

In this section, the following result is obtained: all eigenvalues of \( L \) are pure imaginary (this statement was proved by Weir \[30\] under the additional assumption that \( 1/\varepsilon \notin \mathbb{Z} \)). We use this fact to prove that eigenvectors of \( L \) do not form a Riesz basis in \( L^2(-\pi, \pi) \).

Recall that \( L_+ \) is an operator in the Hilbert space \( \mathcal{H}^{2,0}(\mathbb{D}) = H^2(\mathbb{D}) \oplus \{ c \|, c \in \mathbb{C} \} \). We identify the function \( u(z) \in H^2(\mathbb{D}) \) with \( u(e^{i\theta}) \in L^2(-\pi, \pi) \). Note that \( u \perp \| \) is equivalent to \( u(0) = 0 \).

So the last equality holds for all \( u \in \mathcal{H}^{2,0}(\mathbb{D}) \).

Let \( u(z), z \in \mathbb{D} \), be an eigenfunction of the operator \( A_+ (= -iL_+) \). Consider the restriction \( u \) of the function \( u \) on the interval \([0, 1) \subset \mathbb{D}, u(x) = u(x) \) for \( x \in [0, 1) \). (5.1)

The following proposition obtained by Weir \[30\] shows that if \( u \) is an eigenfunction of the operator \( A_+ (= -iL_+) \), then its restriction \( u \) is a solution of a Sturm-Liouville eigenvalue problem with real-values coefficients.

**Proposition 5.1** (\[30\]). Assume that \( \lambda \) is an eigenvalue of the operator \( L_+, u(e^{i\theta}) \) is a corresponding eigenvector, and \( u \) is its restriction defined by (5.1). Then \( b[u](x) = \mu u(x) \) for all \( x \in (0, 1)(\subset \mathbb{D}), \) where \( \mu = -2i\lambda/\varepsilon \) and the differential expression \( b \) is defined by

\[
 b[u] = -\frac{1}{w}(pu')',
 p(x) = (1 - x)^{1+1/\varepsilon}(x + 1)^{1-1/\varepsilon}, \quad w(x) = x^{-1}(1 - x)^{1/\varepsilon}(x + 1)^{-1/\varepsilon}.
\]

Let \( B_{\text{max}} \) be an operator in \( L^2((0, 1); w) \) associated with the differential expression \( b[\cdot] \) and defined on its maximal domain

\[
 B_{\text{max}}u = b[u], \quad \text{Dom}(B_{\text{max}}) = \{ u \in L^2((0, 1); w) : u, u' \in AC_{\text{loc}}(0, 1), \quad b[u] \in L^2((0, 1); w) \}.
\]

Note that all points of the interval \((0, 1)\) are regular for the differential expression \( b \), but the endpoints 0 and 1 are singular (1 is singular since \( p^{-1} \notin L^1(1/2, 1) \)).

**Proposition 5.2.** Let \( \varepsilon > 0 \).

(i) \( b \) is in the limit-point case at 0,

(ii) \( b \) is in the limit-point case at 1 exactly when \( \varepsilon \leq 1 \).

(iii) \( B_{\text{max}} \) is self-adjoint in \( L^2((0, 1); w) \) exactly when \( 0 < \varepsilon \leq 1 \).

**Proof.** (i) Clearly, \( \| \) is a solution of \( b[u] = 0 \) and \( \| \notin L^2((0, 1/2); w) \). Weyl’s alternative (see e.g. \[29\] Theorem 5.6)) completes the proof.
(ii) The general solution of $b[u] = 0$ on $(0, 1)$ is
$$u(x) = k_1 \int_{1/2}^{x} \frac{1}{p(s)} ds + k_2,$$
$k_1, k_2 \in \mathbb{C}$.

Clearly,
$$u(x) \asymp k_1 \int_{1/2}^{x} (1-s)^{-1-1/\epsilon} ds + k_2 \asymp k_1 (1-x)^{-1/\epsilon} + k_2, \quad x \to 1 - 0.$$

Hence all solutions of $b[u] = 0$ belong to $L^2((1/2, 1); w)$ if and only if $\epsilon > 1$.

(iii) follows from (i) and (ii).

\begin{proof}
By (5), we have
$$|u(x)| \leq | \int_{1/2}^{x} ds | w(x) dx | \leq 2 l \leq 1,$$

since $\max_{x \in [1/2, 1]} w(x) \leq 2$. So $M(\cdot)$ is a Carleson measure (see e.g. [12 Sec. 4.3]). Therefore,
$$\int_{1/2}^{1} |u(x)|^2 w(x) dx \leq C_1 \|u\|_{L^2, 0}^2,$$

where $C_1$ is a constant independent of $u$. This completes the proof.
\end{proof}

\begin{proposition}
Let $\epsilon \in (0, 2)$. Then all eigenvalues of the operator $A_+ = -iL_+$ are real and positive.
\end{proposition}

\begin{proof}
Let $u(e^{i\theta}) \in \mathcal{H}^2,0(\mathbb{D})$ and $L_+ u = \lambda u$. By Proposition 5.3, $u \in L^2((0, 1); w)$. Proposition 5.1 implies that $b[u] = \mu u$ with $\mu = -2i\lambda/\epsilon$. Let us split the interval $(0, 2)$ into two parts.

If $\epsilon \leq 1$ then the proof is simple. Clearly, $u \in Dom(B_{\max})$, and therefore $\mu$ is an eigenvalue of the nonnegative self-adjoint operator $B_{\max}$. Thus, $\mu \geq 0$.

If $1 < \epsilon < 2$ then the proof requires additional analysis. By Proposition 2.2, $g(e^{i\theta}) := \frac{du(z)}{dz} e^{i\theta} \in L^2(-\pi, \pi)$. It is easy to see from the representation $u(e^{i\theta}) = (2\pi)^{-1/2} \sum_{n=1}^{\infty} v_n e^{i\theta}$ and that $g(e^{i\theta}) \in \mathcal{H}^2(\mathbb{D})$ (on the other hand, the latter follows from [11 Theorem 16]) and $g(e^{i\theta}) = \lim_{r \to 1-0} g(re^{i\theta})$ where $g(z) = z \frac{du(z)}{dz}$, $z \in \mathbb{D}$. By [12 Problem II.5 (a)], $|g(x)| \leq \|g\|_{L^2} (1 - |x|^2)^{-1/2}$ for $x \in (0, 1)$ and therefore, for $x \in (1/2, 1)$,
$$\left| \frac{du(x)}{dx} \right| \leq \|g\|_{L^2} |x|^{-1} (1 - |x|^2)^{-1/2} \leq C_1 (1 - x)^{-1/2}. \quad (5.2)$$

By Proposition 5.8 (ii), the operator $B_{\|}$ defined by $B_{\|} u := b[u]$ on the domain
$$Dom(B_{\|}) := \{ u \in Dom(B_{\max}) : ||u||_1 = 0 \}, \quad ||u||_1 := \lim_{x \to 1-0} p(x) u'(x),$$

\end{proof}
is self-adjoint. Note that the limit $|[\cdot, u]|_1$ exists for any $u \in \text{Dom}(B_{max})$ due to [29, Theorem 3.10]).

It follows from (5.2) that, for any eigenvector $u(e^{i\theta})$ of $A_+$, its restriction $u$ belongs to $\text{Dom}(B_{||})$. Indeed, it was shown in the step (1) of the proof that $u \in \text{Dom}(B_{max})$. On the other hand, it follows from (5.2) that

$$
[\cdot, u]_1 = \lim_{x \to 1-0} (1 - x)^{1+1/\varepsilon} (x + 1)^{1-1/\varepsilon} u'(x) = 0.
$$

So $\mu$ is an eigenvalue of the operator $B_{||} = B_{||}^*$.

It follows from (5.2) that $u(x) = u(1/2) + \int_{1/2}^x u'(t)dt$ has a finite limit as $x \to 1 - 0$ (this fact also follows from [11, Theorem 16]). Therefore,

$$
(B_{||} u, u)_{L^2((0,1); w(x))} = - \int_0^1 (p(x)u'(x))' \overline{u(x)} dx = \int_0^1 p(x)|u'(x)|^2 dx - \lim_{x \to 1-0} p(x)u'(x)\overline{u(x)}
$$

(5.3)

$$
= \int_0^1 p(x)|u'(x)|^2 dx \geq 0.
$$

(5.4)

Thus, $B_{||} \geq 0$ and therefore $\mu \geq 0$.

Finally, note that $\text{Ker}(L_+) = 0$ and therefore $\mu \neq 0$.

**Remark 5.1.** For $\varepsilon \in (0, 2)$ such that $1/\varepsilon \notin \mathbb{Z}$, a slightly different form of Proposition 5.4 was proved in [20, Theorem 2.3] by means of Proposition 5.1, the Frobenius theory, and the deep analysis of eigenvectors of the corresponding recursion relation given in [11, Sections 2 and 3]. Our proof that removes the condition $1/\varepsilon \notin \mathbb{Z}$, is based on the description of $\text{Dom}(L)$ given in Section 2 and the Hardy spaces theory.

**Theorem 5.1.** If $\varepsilon \in (0, 2)$, then the set of eigenvectors of the operator $L$ does not form a Riesz basis in $L^2(-\pi, \pi)$.

**Proof.** Assume that the set $\{u_n\}_{n=1}^\infty$ of all (linearly independent) eigenvectors of $L$ form a Riesz basis in $L^2(-\pi, \pi)$. Then Proposition 5.4 implies that $iL$ is similar to a certain self-adjoint operator $Q$. That is, there exists a bounded and boundedly invertible operator $S$ such that $S\text{Dom}(Q) = \text{Dom}(L)$ and $iL = SQS^{-1}$.

The spectral theorem for a self-adjoint operator implies that, for arbitrary $u_0 \in \text{Dom}(L) (= X_2)$, the problem

$$
u_t + Lu = 0, \quad u|_{t=0} = u_0, \quad t \in \mathbb{R},
$$

has a unique solution $u(\cdot, t)$ in the sense of [16, Definition I.1.1 and Eq. (I.1.2)] (such solutions are sometimes called strong solutions). Moreover, this solution has the form $u(\cdot, t) = Se^{-itQ}S^{-1}u_0(\cdot)$. Therefore, for any $T > 0$,

$$
u \in C([0, T]; X_2) \subset C([0, T]; \mathcal{W}_{2p}^1(-\pi, \pi)),
$$

$$
u_t(\cdot, t) \in C([0, T]; L^2(-\pi, \pi)), \quad \text{and} \quad Lu(\cdot, t) \in C([0, T]; L^2(-\pi, \pi)).
$$

It is easy to see that $u$ is a generalized solution of (3.1), (3.2) in the sense of Definitions 3.1-3.3. Since $e^{-itQ}$ is a unitary operator,

$$
\|u(\cdot, t)\|_{L^2(-\pi, \pi)} \leq \|S\|\|S^{-1}\|\|u_0\|_{L^2(-\pi, \pi)}, \quad t \in \mathbb{R}.
$$

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Hence, for any $T > 0$, we have
\[
\int_0^T \|u(\cdot, t)\|_{L^2(-\pi, \pi)}^2 dt \leq CT\|u_0\|_{L^2(-\pi, \pi)}^2,
\]
where $C = \|S\|^2 \|S^{-1}\|^2 < \infty$. The latter contradicts Theorem \ref{thm:3.2} since $\text{Dom}(L)$ is dense in $W^2_{2p}(-\pi, \pi)$ and $\|u_0\|_{L^2(-\pi, \pi)} \leq \|u_0\|_{W^2_{2p}(-\pi, \pi)}$.

\begin{remark}
Arguments of Proposition \ref{prop:5.4} show that all eigenvalues of the operators $L_\pm$ are simple. Combining this with Theorems \ref{thm:4.3} and \ref{thm:5.1} one can show that (under the assumption $\varepsilon \in (0, 2)$) the system \{un\}_1^\infty of eigenvectors of $L$ is complete in $L^2(-\pi, \pi)$, but does not form a Riesz basis in $L^2(-\pi, \pi)$.

Indeed, let us show that all eigenvalues of $L_+$ are simple in the case $\varepsilon \in (1, 2)$. Assume that $u_1(e^{i\theta}) \in H^2_0(\mathbb{D}) \setminus \{0\}$ is a generalized first order eigenvector of $L_+$, i.e., $(L_+ - \lambda I)u_1 = u$, where $u(e^{i\theta}) \in H^2_0(\mathbb{D}) \setminus \{0\}$ and $L_+ u = \lambda u$. Consider the restrictions $u, u_1$ of the functions $u$ and $u_1$ on the interval $[0, 1) \subset \mathbb{D}$. It follows from the proof of Proposition \ref{prop:5.4} that $u, u_1 \in \text{Dom}(B_{\|}) \subset L^2((0, 1); w)$. On the other hand, computations analogous to that of \cite[Lemma 2.1 and Theorem 2.3]{30} show that $b[u_1] - \mu u_1 = -\frac{\mu}{2} u$ with $\mu = -\frac{\alpha}{2}$. Therefore $u_1$ is a generalized eigenvector of the self-adjoint operator $B_{\|}$, a contradiction. The proof for the case $\varepsilon \in (0, 1]$ is similar.

We would like to note that the linear partial differential equation \ref{eq:1.1} is an interesting example when the nature of explosive blow-up and instability of solutions has its roots not in location of the eigenvalues but in geometric properties of the eigenfunctions.

\section{Further discussion}

When eigenfunctions related to neutrally stable eigenvalues of some linearized problem form the complete set, representation of a solution of the nonlinear problem as a series of these eigenfunctions is one of general approaches to the nonlinear stability problem. The lack of a basis property of the eigenfunction set is an obstacle for the applicability of this particular method.

Due to the ill-posed nature of the forward-backward heat equation all eigenmodes are linearly unstable \cite{2} and it is common to use the smoothing effect of the surface tension to stabilize them. The lubrication approximation that takes into account the influence of the capillary effects and/or surface tension leads to the initial value problem for the fourth order nonlinear partial differential equation \cite{8}. Some stability properties of its linearization were studied in \cite{3, 4, 6}. They came to the conclusion that almost all but some first modes are getting stable even if the surface tension is relatively weak.

We would also like to mention that the main assumption about the parameter range $|\varepsilon| < 2$ comes naturally from the theory of mixed type equations and for the case when $|\varepsilon| > 2$ all properties of this backward forward heat equation can be changed significantly.

Let us consider the equation
\[
k(x, t) u_{tt} + \alpha(t, x) u_t + \Delta u = 0, \quad x \in \Omega, \quad t > 0
\]
where the coefficient $k(x, t)$ can change sign in the domain where the operator is considered. So equation \ref{eq:6.1} is an equation of the mixed type, i.e. it is of the same type as the well-known Tricomi equation.

\begin{center}
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\end{center}
On the lateral boundary of the cylinder $\Omega \times (0, T)$ we pose the Dirichlet boundary condition and there are two additional boundary condition on the lower and upper base of the cylinder:

$$u|_{t=T} = 0, \quad u|_{S^+} = 0, \quad u|_{S^-} = 0, \quad S^+ = \{(0, x) : k(x, 0) > 0\}, \quad S^- = \{(T, x) : k(x, T) < 0\}.$$  

This boundary value problem and close problems were studied by many authors (see, for instance, [27, 28]). It was demonstrated that the condition

$$\alpha - \frac{k_t}{2} \geq \delta_0 > 0 \quad \forall (x, t),$$

where $\delta_0$ is a positive constant, ensures the existence of generalized solutions to the above-described boundary value problem. Stronger conditions of the type

$$\alpha - \frac{|k_t|(2k - 1)}{2} \geq \delta_0 > 0 \quad \forall (x, t)$$

ensure existence of smooth solutions and uniqueness of generalized solutions. The existence of solutions of non-linear forward-backward heat equations was studied by Hollig [15] and by Pyatkov [22]. Among last results devoted to the nonlinear forward-backward parabolic problems we would like to mention Kuznecov papers [17, 18].

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A Proof of Corollary 2.1 using Galerkin method.

The second proof of Corollary 2.1. Let $\{\omega_j\}_{j=1}^\infty$ be a basis for the Hilbert space $H = \{h \in W^2_{2p}(-\pi, \pi) : \int_{-\pi}^\pi h(\theta) d\theta = 0\}$. Find functions $\varphi_j$ such that $\varphi_j \theta = \omega_j, \int_{-\pi}^\pi \varphi_j(\theta) d\theta = 0$. We look for an approximate solution to equation (2.3) in the form

$$h_n = \sum_{j=1}^n c_{jn} \varphi_j,$$

where the constants $c_{jn}$ are determined from the system of algebraic equations

$$(Lh_n, \omega_j) = (f, \omega_j), \quad j = 1, 2, \ldots, n, \quad (A.1)$$

(the brackets denote the inner product in $L^2(-\pi, \pi)$, i.e., $(h, v) = \int_{-\pi}^\pi h(\theta)v(\theta) d\theta$).

Multiply (A.1) by $c_{jn}$ and summarize the equalities obtained. We arrive at the relation

$$(Lh_n, h_{n\theta}) = (f, h_{n\theta}).$$

Integrating by parts we derive the estimate

$$\|h_{n\theta}\|_{L^2(-\pi,\pi)} \leq c\|f\|_{L^2(-\pi,\pi)}, \quad (A.2)$$

where $c$ is a constant independent of $n$. This estimate implies that the system (A.1) is solvable. Note that there exists a constant $c_1$ independent of $n$ such that

$$\|h_n\|_{L^2(-\pi,\pi)} \leq c\|h_{n\theta}\|_{L^2(-\pi,\pi)} \quad (A.3)$$
We have used the equality $$Re h$$ Multiply the equation by real part, we arrive at the inequality 

$$\exists \lambda$$ show that there exist set of constants

Proof. We can use the same arguments as in the second proof of Corollary 2.1. It is not difficult to show that there exist set of constants $$\lambda_i > 0$$ such that in the equivalent inner product in the space $$W^k_2(-\pi, \pi)$$ 

$$(h, v)_k = \sum_{j=0}^{k} \lambda_j (h^{(j)}, v^{(j)})$$
we have the inequality
\[(Lh, h_0)_k \geq \delta_0 \|h\|^2_{W^{k+1}_2(-\pi, \pi)}, \quad \forall h \in W^{k+2}_2(-\pi, \pi),\]
where the constant \(\delta_0 > 0\) is independent of \(h\). Next we apply the same arguments as those in the theorem 1 but we use the inner product \((h, v)_k\) rather than the inner product \((h, v)\) in \(L^2(-\pi, \pi)\). So the Galerkin method is applicable here.

Remark A.2. It is also possible to use some functional arguments based on the Hahn-Banach theorem.

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