A cluster-separable Born approximation for the
3D reduction of the three-fermion Bethe-Salpeter
equation.

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Abstract
We perform a 3D reduction of the two-fermion Bethe-Salpeter equation
based on Sazdjian’s explicitly covariant propagator, combined with a co-
variant substitute of the projector on the positive-energy free states. We
use this combination in the two fermions in an external potential and in the
three-fermion problems. The covariance of the two-fermion propagators
insures the covariance of the two-body equations obtained by switching
off the external potential, or by switching off all interactions between any
pair of two fermions and the third one, even if the series giving the 3D
potential is limited to the Born term or more generally truncated. The co-
variant substitute of the positive energy projector preserves the equations
against continuum dissolution without breaking the covariance.

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1 Introduction

The Bethe-Salpeter equation \[1,2\] is the usual tool for the study of relativistic bound states. The principal difficulty in the treatment of this equation comes from the existence of unphysical relative time variables. The most usual practice in the two-fermion problem consists in eliminating the relative time variable (3D reduction). This 3D reduction is most often based on the replacement of the free Green function by an expression combining a delta fixing the relative energy and a 3D propagator. The exact equivalence (in what concerns the physically measurable quantities of the pure two-fermion problem) with the original Bethe-Salpeter equation can be obtained by recuperating the difference with the original free Green function in a series of correction terms to the 3D potential.

The 3D reduction of the two-fermion Bethe-Salpeter equation has been performed by many authors \[3-19\]. All methods are theoretically equivalent at
the limit of all correction terms included. Beyond the two-fermion problem, we have the cases of two fermions in an external potential (2\frac{1}{2}-body problem) and of three fermions (3-body problem). In these cases we meet new difficulties which do not appear or can be easily solved in the two-fermion case: the Lorentz invariance / cluster separability problem and the continuum dissolution problem.

— Lorentz invariance and cluster separability. It is always possible to render an equation Lorentz invariant by working in the general rest frame (center of mass reference frame) and by building invariants with the total 4-momentum vector. The real difficulty appears when we start with a three-fermion equation and "switch off" all interactions between fermion 3 (for example) and the other ones. We should get a free equation for fermion 3 and an acceptable equation for the (12) system (cluster separability). In particular, this implies that the equation for the (12) cluster must not refer to the global center of mass frame anymore, as the momentum of fermion 3 enters in the definition of this frame. This equation must therefore be invariant, explicitly or implicitly (i.e. after rearrangements).

— Solution of the continuum dissolution problem. In the relativistic equations for several relativistic particles, the physical bound states are degenerate with a continuum of states combining asymptotically free particles with opposite energy signs. This often neglected fact forbids the building of normalizable solutions in the $N > 2$-body problem (including the two-body plus potential problem). The usual solution consists in including positive-energy projection operators into the zero-order propagator [20-26]. The modified equations must of course continue to satisfy the Lorentz invariance / cluster separability requirement.

In our preceding works on the two-particles in an external potential problem [23] and, more recently, on the three-fermion problem [24], our approach was based on the cluster separability requirement. In the three-fermion problem the 3D potential was obtained by adding the three 3D potentials obtained by the 3D reduction of the three two-fermion equations in the three-fermion rest frame. Each two-fermion potential depending in general of the total energy of the subsystem, this two-fermion energy was taken as the three-fermion total energy, minus the free Dirac hamiltonian of the spectator fermion. The continuum dissolution was avoided in both cases by choosing a two-fermion reduction based on a 3D propagator containing a projector on the positive eigenvalues eigenstates of the Dirac’s free hamiltonian. With these choices the three two-cluster limits are exact: switching off the two interaction terms with a spectator fermion gives the exact 3D equation which would be obtained by the reduction of the two-body Bethe-Salpeter equation. From this exact 3D equation for two fermions it is possible to go back to the original two-fermion Bethe-Salpeter equation and to perform the 3D reduction again in another reference frame, so that we can consider this two-fermion equation as implicitly covariant. This covariance is however quite implicit and global: neither the approached propa-
gator nor the positive-energy projector being covariant, the terms of the series giving the 3D propagator are not individually covariant, so that the implicit covariance becomes an approximated covariance when the series giving the 3D potential is truncated, for example by keeping only the Born term.

In the present work we combine Sazdjian’s explicitly covariant propagator, based on the second-order equations, with a covariant form of the positive-energy projectors. When used in the 3D reduction of a two-fermion Bethe-Salpeter equation, this combination leads to a 3D potential given by a series in which each term is individually covariant. This potential can then be used in the two-fermion in an external potential problem and in the three-fermion problem, leading to continuum dissolution-free equations which can be truncated (for example by keeping only the Born terms) without losing the covariance of their cluster-separated limits.

Although our 3D equations have exact two-cluster limits, they are themselves approximations. In the $2\frac{1}{2}$-body problem, one should take into account the modifications brought by the external potential to the fermion propagators also inside the Bethe-Salpeter irreducible kernel. In the three-body problem one has to take into account the irreducible three-body terms at the Bethe-Salpeter equation level and also the three-body terms generated at the 3D level by the reduction. We tried to do that recently [26].

In section 2 we build a covariant propagator by combining Sazdjian’s covariant propagator with a covariant form of positive-energy projector. We use it to perform a 3D reduction of the two-fermion Bethe-Salpeter equation and study the resulting 3D equation (one-body limit, transition matrix elements, symmetrization of the potential). We present also a "covariant Salpeter propagator", slightly more complicated but preserving the particle-antiparticle symmetry. In the two next sections (3 and 4) we exploit these results by using our 3D potentials in the two fermions in an external potential problem (with examples) and in the three-fermion problem (with examples and with a calculation of the two-body limits). Section 5 is devoted to conclusions.

2 Two fermions.

2.1 Notations.

We shall write the Bethe-Salpeter equation for the bound states of two fermions [1] as

\[ \Phi = G^0 K \Phi, \]  

(1)

where \( \Phi \) is the Bethe-Salpeter amplitude, function of the positions \( x_1, x_2 \) or of the momenta \( p_1, p_2 \) of the fermions, according to the representation chosen. The operator \( K \) is the Bethe-Salpeter kernel, given by the sum of the irreducible two-fermion Feynman graphs. The operator \( G^0 \) is the free propagator, given by
the product $G^0_i G^0_2$ of the two individual fermion propagators

$$G^0_i = \frac{1}{p_{0i} + h_i + i\epsilon h_i} \beta_i = \frac{p_{0i} + h_i}{p_{i}^2 - m_{i}^2 + i\epsilon \beta_i}$$

(2)

where the $h_i$ are the Dirac free hamiltonians

$$h_i = \vec{\alpha}_i \cdot \vec{p}_i + \beta_i m_i \quad (i = 1, 2).$$

(3)

We shall define the total (or external, CM, global) and relative (or internal) variables:

$$X = \frac{1}{2}(x_1 + x_2) , \quad P = p_1 + p_2 ,$$

(4)

$$x = x_1 - x_2 , \quad p = \frac{1}{2}(p_1 - p_2).$$

(5)

and give a name to the corresponding combinations of the free hamiltonians:

$$S = h_1 + h_2 , \quad s = \frac{1}{2}(h_1 - h_2).$$

(6)

We know that, at the no-interaction limit, we shall have to get a pair of free Dirac equations:

$$(p_{10} - h_1)\Psi = 0 , \quad (p_{20} - h_2)\Psi = 0 ,$$

(7)

where $\Psi$ depends on $x_1, x_2$. Let us also write their iterated versions

$$(p_{10}^2 - E_{1}^2)^2\Psi = 0 , \quad (p_{20}^2 - E_{2}^2)^2\Psi = 0$$

(8)

with

$$E_i = \sqrt{h_i^2} = (\vec{p}_i^2 + m_i^2)^{1/2}. $$

(9)

Interesting combinations can be obtained from the sum and differences of the equations (7) or of the iterated equations (8):

$$(P_0 - S)\Psi = 0 , \quad (p_0 - s)\Psi = 0 ,$$

(10)

$$H^0\Psi = 0 , \quad (p_0 - \mu)\Psi = 0$$

(11)

with

$$H^0 = 2[(p_1^2 - m_1^2) + (p_2^2 - m_2^2)]_{p_0 = \mu} = P_0^2 - 2(E_1^2 + E_2^2) + 4\mu^2 ,$$

(12)

$$\mu = \frac{1}{2P_0}(E_1^2 - E_2^2) = \frac{1}{2P_0}(h_1^2 - h_2^2) = \frac{sS}{P_0} .$$

(13)
2.2 Sazdjian’s explicitly covariant propagator.

The free propagator $G^0$ can be written

$$G^0 = G^{02} (p_{10} + h_1)(p_{20} + h_2)\beta_1\beta_2,$$

where $G^{02}$ is the second-order (spinless) free propagator:

$$G^{02} = \frac{1}{(p_{10}^2 - E_1^2 + i\epsilon)(p_{20}^2 - E_2^2 + i\epsilon)}.$$  

(15)

In a two-boson Bethe-Salpeter equation the free propagator $G^{02}$ could be approached with the product $G^{S2}$ of a constraint $\delta(p_0 - \mu)$ fixing the relative energy and forcing a solution of the free difference equation (second equation (11)) by a 3D propagator which could be the integral of $G^{02}$ with respect to the relative energy $p_0$:

$$G^{S2} = \delta(p_0 - \mu) \int dp_0 \ G^{02}(p_0) = -4i\pi \delta(p_0 - \mu) \frac{\sigma}{P_0 H^0}$$

(16)

with

$$\sigma = \frac{1}{2}(\sigma_1 + \sigma_2), \quad \sigma_1 = \frac{P_0 + 2\mu}{2E_1}, \quad \sigma_2 = \frac{P_0 - 2\mu}{2E_2}.$$  

(17)

On the mass shell $P_0 = h_1 + h_2$, the operator $\sigma_i$ is the sign $h_i/E_i$ of the energy of the fermion i. The presence of $\sigma$ annihilates thus the residues of the poles at $P_0 = \pm(E_1 - E_2)$, so that $G^{S2}$ can be written

$$G^{S2} = -2i\pi \delta(p_0 - \mu) \frac{E_1 + E_2}{2E_1 E_2} \frac{1}{P_0^2 - (E_1 + E_2)^2 + i\epsilon}.$$  

(18)

The propagator (16) could easily be written in covariant form if it did not contain the operator $\sigma$. If we replace $\sigma$ by $\epsilon(P_0)$ and add the spinor part of (14), we obtain Sazdjian’s propagator [12] :

$$G^{SZ} = -4i\pi \frac{\delta(p_0 - \mu)}{|P_0| H^0} (p_{10} + h_1)(p_{20} + h_2)\beta_1\beta_2$$

(19)

which can also be written in covariant form

$$G^{SZ} = -2i\pi \delta(P \cdot p - \frac{m_1^2 - m_2^2}{2}) \frac{(p_1 \cdot \gamma_1 + m_1)(p_2 \cdot \gamma_2 + m_2)}{p_1^2 + p_2^2 - (m_1^2 + m_2^2) + i\epsilon}$$

(20)

or in a form which exhibits the pole in $P_0 - S$ :

$$G^{SZ} = \frac{-2i\pi}{P_0 - S + i\epsilon P_0} \frac{P_0 + S}{2|P_0|} \delta(p_0 - \mu) \beta_1\beta_2.$$  

(21)
2.3 Covariant positive-energy projectors.

The combination of two fermions with opposite energy signs leads to continuum dissolution when an external potential or a third particle is added. The operator $\sigma$ would have kill the unwanted poles of Sazdjian’s propagator, but we could not re-introduce it without losing the covariance. We have thus to search for an invariant substitute of $\sigma$. A Lorentz invariant operator forcing a positive energy for fermion $i$ could be

$$\theta_i = \theta(p_{i0})\theta(p_i^2) = \theta(p_{i0} - |\vec{p}_i|) \quad (22)$$

as the sign of the energy is invariant for positive squared 4-momenta. Taking the product of two such operators (one for each fermion) and replacing $p_0$ by $\mu$ gives

$$\hat{\theta} = \theta \left( \frac{P_0}{2} + \frac{\vec{p}_1^2 - \vec{p}_2^2}{2P_0} - |\vec{p}_1| + \frac{m_1^2 - m_2^2}{2P_0} \right) \theta \left( \frac{P_0}{2} + \frac{\vec{p}_2^2 - \vec{p}_1^2}{2P_0} - |\vec{p}_2| + \frac{m_2^2 - m_1^2}{2P_0} \right). \quad (23)$$

Let us assume that the heaviest fermion is fermion 2. We can then write (23) as

$$\hat{\theta} = \theta \left( \frac{P_0}{2} + \frac{\vec{p}_1^2 - q_2^2}{2P_0} - |\vec{p}_1| \right) \theta \left( \frac{P_0}{2} + \frac{q_2^2 - \vec{p}_1^2}{2P_0} - |\vec{p}_2| \right) \quad (24)$$

with

$$q_2 = \sqrt{\vec{p}_2^2 + m_2^2 - m_1^2}. \quad (25)$$

The sum of the arguments of the two $\theta$ is $P_0 - |\vec{p}_1| - |\vec{p}_2|$. The argument of the first $\theta$ is $(P_0 - |\vec{p}_1| + q_2)(P_0 - |\vec{p}_1| - q_2)/2P_0$, which implies that $P_0$ must be outside the interval $(|\vec{p}_1| - q_2, |\vec{p}_1| + q_2)$. The combination of these two results implies $P_0 > |\vec{p}_1| + q_2$. The second $\theta$ brings no supplementary restriction, its argument being always positive when $P_0 > |\vec{p}_1| + q_2$. We can thus write

$$\hat{\theta} = \theta \left( P_0 - |\vec{p}_1| - \sqrt{\vec{p}_2^2 + m_2^2 - m_1^2} \right). \quad (26)$$

We see that our projection operator introduces a cutoff on the high $|\vec{p}_i|$ for a given $P_0$.

It is interesting to write $\hat{\theta}$ also in terms of the $h_i$, for comparison with the $\Lambda^{++} = \theta(h_1)\theta(h_2)$ projector widely used by us [23, 20] and by others, and also for future use in the two-fermion plus potential problem. We get

$$\theta_i = \theta(p_{i0})\theta(p_i^2) = \theta(p_{i0}) \theta(p_{i0}^2 - h_i^2 + m_i^2) \quad (27)$$

so that

$$\hat{\theta} = \left[ \theta(p_{10}) \theta(p_{10}^2 - h_1^2 + m_1^2) \theta(p_{20}) \theta(p_{20}^2 - h_2^2 + m_2^2) \right]_{p_0=\mu}. \quad (28)$$
The \( p_0 = \mu \) constraint gives

\[
p_{10} = \frac{P_0}{2} + \frac{h_1^2 - h_2^2}{2P_0}, \quad p_{20} = \frac{P_0}{2} + \frac{h_2^2 - h_1^2}{2P_0}
\]

and also \( p_{10}^2 - h_1^2 = p_{20}^2 - h_2^2 \) so that the last \( \theta \) which concerns the heaviest fermion can be omitted. For a given bound state energy \( P_0 \), \( \hat{\theta} \) eliminates the mixed energy-sign continuum which would share this energy. When \( P_0 = S \), (28) reduces indeed to \( \theta(h_1)\theta(h_2) \). In fig. 1, we draw a map of the \( \hat{\theta} = 1 \) region in the \((h_1, h_2)\) plane, for \( P_0 \) fixed to 2 (in arbitrary units) and different values of the lowest mass \( m_1 : 0, 0.5, 1 \) (solid line), 1.5, 2, 2.5 and 3 (while \( m_2 \) can take any value larger or equal to \( m_1 \)). We draw only the first quadrant, the other ones being the mirror images of the first one. The \( \hat{\theta} = 1 \) region is bounded by the two \( p_{i0} = 0 \) curves, independent of the masses, and by the \( p_i^2 = 0 \) curve corresponding to the lowest mass \( m_1 \). In each quadrant, this region is divided into four parts, corresponding to the signs of \( h_i^2 - m_i^2 \) as an example, we indicate this partition in the \( m_1 = 1, m_2 = 1.1 \) case, for which \( P_0 \) is thus 0.1 below the threshold 2.1). When no external potential is present, both expressions are positive definite, the negative values being for the bound states of the corresponding fermion in an external potential.

While the widely used \( \Lambda^{++} = \theta(h_1)\theta(h_2) \) noncovariant projector simply selects the first quadrant, the \( \hat{\theta} = 1 \) region is divided into four symmetric parts, one for each quadrant, but there is a cutoff on the high-\(|h_i|\) values. For moderately relativistic systems, the important regions are near the \((m_1, m_2)\) point.

We shall thus finally approach the free propagator by the product of the covariant Sazdjian propagator with our covariant projector:

\[
G^\delta = \hat{\theta} G^{SZ}.
\]

### 2.4 3D reduction of the two-fermion Bethe-Salpeter equation.

We shall write the free propagator as the sum of the zero-order propagator, plus a remainder:

\[
G^0 = G^\delta + G^R.
\]

The Bethe-Salpeter equation becomes then the inhomogeneous equation

\[
\Phi = G^0 K \Phi = (G^\delta + G^R) K \Phi = \Psi + G^R K \Phi,
\]

with

\[
\Psi = G^\delta K \Phi \quad (= G^\delta (G^0)^{-1} \Phi).
\]

Solving (formally) the inhomogeneous equation \( \Phi \) with respect to \( \Phi \) and putting the result into \( \Psi \), we get

\[
\Psi = G^\delta K (1 - G^R K)^{-1} \Phi = G^\delta K^T \Psi
\]
where
\[ K^T = K(1 - G^R K)^{-1} = K + KG^R K + ... = (1 - KG^R)^{-1} K \quad (35) \]
obeys
\[ K^T = K + KG^R K^T = K + K^T G^R K. \quad (36) \]
The reduction series (35) re-introduces in fact the reducible graphs into the Bethe-Salpeter kernel, but with \( G^0 \) replaced by \( G^R \). Equation (34) is a 3D equivalent of the Bethe-Salpeter equation. It depends on the choice of \( G^\delta \).

The relative energy dependence of eq. (34) can be easily eliminated:
\[ \Psi = \delta(p_0 - \mu) \psi \quad (37) \]
and \( \psi \) satisfies the equation
\[ \psi = -\frac{2i\pi}{P_0 - S + i\epsilon P_0} \frac{P_0 + S}{2|P_0|} \hat{\theta} \beta_1 \beta_2 K^T(\mu, \mu) \psi \quad (38) \]
with
\[ \beta_1 \beta_2 K^T(\mu, \mu) \equiv \int dp'_0 dp_0 \delta(p'_0 - \mu) \beta_1 \beta_2 K^T(p'_0, p_0) \delta(p_0 - \mu). \quad (39) \]
Note that we write \((p'_0, p_0)\) but \((\mu, \mu)\), as we keep \( \mu \) in operator form. This operator is diagonal in the spatial momentum space. The eigenvalue will depend on the position of \( \mu \) in the formula: the eigenvalue of the first \( \mu \) in (39) will be built with the final momenta and that of the last \( \mu \) will be built with the initial momenta. Defining
\[ V = -2i\pi \hat{\theta} \beta_1 \beta_2 K^T(\mu, \mu) \hat{\theta} \quad (40) \]
we get
\[ \psi = \frac{1}{P_0 - S + i\epsilon P_0} \frac{P_0 + S}{2 P_0} V \psi. \quad (41) \]
We included a final \( \hat{\theta} \) in the definition of \( V \), as the wave function at right is already projected, and we replaced \(|P_0|\) by \( P_0 \), which is always positive in the \( \hat{\theta} = 1 \) region.

The inversion of the reduction is given by
\[ \Phi = (1 - G^R K)^{-1} \Psi = (1 + G^R K^T) \Psi = (1 + G^0 K^T - G^\delta K^T) \Psi = G^0 K^T \Psi \quad (42) \]
or, explicitating the relative energy variable
\[ \Phi(p'_0) = G^0(p'_0) K^T(p'_0, \mu) \psi. \quad (43) \]
2.5 Covariant Salpeter equation.

A Bethe-Salpeter equation with an instantaneous kernel (i.e. a kernel independent of the relative energy) leads directly to Salpeter’s 3D equation [3]:

$$\psi = \frac{1}{P_0 - S + i\epsilon P_0} V^S \psi$$  \hspace{1cm} (44)

with

$$V^S = -2i\pi \tau \beta_1 \beta_2 K \tau^2$$  \hspace{1cm} (45)

$$\tau = \theta(h_1)\theta(h_2) - \theta(-h_1)\theta(-h_2).$$  \hspace{1cm} (46)

When the kernel is not instantaneous, we can make an expansion based on the approached propagator

$$G^{\delta S} = \delta(p_0-\mu) \int dp_0 \ G^0(p_0) = -2i\pi \delta(p_0-\mu) \ \frac{\tau}{P_0 - S + i\epsilon P_0} \beta_1 \beta_2$$  \hspace{1cm} (47)

in the same way as above. This operator $\tau$, which projects on the $\tau^2 = 1$ subspace with a change of sign for the negative-energy part, is also a continuum dissolution-preserving operator.

A “covariant Salpeter equation” can easily be obtained by combining Sazdjian’s covariant operator with a covariant substitute of $\tau$ instead of a covariant substitute of $\theta(h_1)\theta(h_2)$. We could indeed use

$$\theta^S = \left[ \theta(p_{10}) \theta(p_{20}) - \theta(-p_{10}) \theta(-p_{20}) \right] \theta(p_{12}) \theta(p_{21}) \bigg|_{p_0=\mu}.$$  \hspace{1cm} (48)

Our 3D equation would then again be (41), but with the potential

$$V = -2i\pi \theta^S \beta_1 \beta_2 K^T(\mu, \mu) (\theta^S)^2$$  \hspace{1cm} (49)

where $K^T$ must now be built using $\theta^S$ instead of $\hat{\theta}$.

As the second term of (46) does not contribute much in practical calculations, it is often omitted. For the same reason, we shall omit the second term of (48) and use simply $\hat{\theta}$ in the present work. From a more fundamental point of view, however, the use of $\tau$ or of $\theta^S$ preserves a particle-antiparticle symmetry which is an important feature of relativistic theories.

2.6 One-body limit.

If we go to the center of mass reference frame, write $P_0 = m_2 + W_1$ (for $P_0 > 0$) and make $m_2 \to \infty$ in the 3D equation (41), we get the equation

$$\psi = \frac{1}{W_1 - h_1 + i\epsilon} V_\infty \psi, \quad V_\infty = \lim_{m_2 \to \infty} V.$$  \hspace{1cm} (50)
In [17], we showed that a 3D reduction performed with a propagator \( \Lambda G^{SZ} \), \( \Lambda \) being a projector, leads to a 3D potential given by

\[
V^\Lambda = \Lambda \left[ V^{SZ} + V^{SZ} \frac{1 - \Lambda}{P_0 - S} V^{SZ} + \cdots \right] \Lambda. \tag{51}
\]

We also showed that the limit of \( V^{SZ} \) (let us call it \( V^C \)) is given by the limit of the Born term (a Coulomb potential in QED). Since the limit of \( \hat{\theta} \) is \( \theta(W_1 - |\vec{p}_1|) \), we have thus

\[
V^\infty = \theta(W_1 - |\vec{p}_1|) V^C \left( 1 - \frac{\theta(|\vec{p}_1| - W_1)}{W_1 - h_1} V^C \right)^{-1} \theta(W_1 - |\vec{p}_1|). \tag{52}
\]

Equation (50) is related to the equation

\[
\psi^C = \frac{1}{W_1 - h_1 + i\epsilon} V^C \psi^C \tag{53}
\]

which describes the light fermion 1 in the potential generated by the heavy fermion 2. We have

\[
\psi = \theta(W_1 - |\vec{p}_1|) \psi^C. \tag{54}
\]

Equation (50) is thus the exact equation for the projection (54) provided the expansion of (52) is not truncated. If, however, the two-body potential is truncated to the Born term, its one-body limit will be the projection of \( V^C \) instead of \( V^\infty \). The introduction of the projector \( \hat{\theta} \) has destroyed the exact one-body limit of the Born approximation, as it would also be the case with other continuum dissolution-preserving operators like \( \Lambda^{++} \).

2.7 Transition matrix elements.

The 3D off mass shell transition matrix element is

\[
T^{3D} = V + V \frac{1}{P_0 - S + i\epsilon P_0} \frac{P_0 + S}{2P_0} V + \cdots = -2i\pi \hat{\theta} \beta_1 \beta_2 \left[ K^T + K^T G^\delta K^T + \cdots \right] (\mu, \mu) \hat{\theta} \tag{55}
\]

\[
K^T + K^T G^\delta K^T + \cdots = K(1 - G^R K)^{-1}(1 - G^\delta K(1 - G^R K)^{-1})^{-1}
\]

so that

\[
T^{3D} = -2i\pi \hat{\theta} \beta_1 \beta_2 T(\mu, \mu) \hat{\theta}. \tag{57}
\]

On the positive-energy mass shell \( P_0 = E_1 + E_2, \hat{\theta} = 1 \) and \( T(\mu, \mu) \) is the physical on mass shell scattering matrix element of field theory. Although the Bethe-Salpeter equation (1) was written only for bound states, our final 3D equation can still be used for describing the scattering.
2.8 Symmetric form.

In our 3D equation \((41)\), written in the form
\[
(P_0 - S) \psi = \frac{P_0 + S}{2P_0} V \psi
\]
(58)
the interaction term (the operator before \(\psi\) at right) is not symmetric and depends on \(P_0\) via the operator before \(V\) and also via \(V\) itself. This does not excludes a real energy spectrum, but this can not be seen directly on (58). In the two-fermion framework, this is not a problem as there exist many ways of transforming (58) into a symmetric equation. We shall however use these two-fermion interaction terms in a three-fermion equation below (in section 4) and, in this new framework, their symmetrization will not be possible anymore (it would demand a different transformation for each two-fermion pair). In order to be sure to get a real energy spectrum from our three-fermion equation, we should thus symmetrize the interaction terms at the two-fermion level. This can be done by simply multiplying (58) by \(1 + V/2P_0\), which gives
\[
(P_0 - S) \psi = U \psi,
\]
(59)
\[
U = V - \{P_0 - S, \frac{V}{2P_0}\} + \frac{V}{2P_0} (P_0 + S) \frac{V}{2P_0}.
\]
(60)
In a 3D reduction based on the noncovariant projector \([47]\), we would have simply \(V^S\). The first-order energy shift \(- < V^2/2P_0 >\) corresponding to the difference \(U - V\) would then be included in the contribution of the first ladder term \(KG^{RS}K\), which enters in the definition of \(V^S\).

3 Two fermions in an external potential.

3.1 3D equation.

The two-fermion in an external potential problem is already much more complicated than the pure two-fermion problem, although it exhibits some simplifying features when compared with the three-fermion problem. The principal new difficulty is the non-conservation of the total spatial momentum \(\vec{P}\). In the pure two-fermion case, this conservation law forbids the mixing of the physical bound states with the mixed-energy states (continuum dissolution). When an external potential is present, it becomes possible to get any given energy in an infinity of ways by combining fermions with opposite free energy signs. The positive-energy bound states will then not be normalizable, being mixed with a continuum.

The two-fermion plus external potential equations can easily be obtained with the simple generalizations \([24]\)
\[
h_i \rightarrow h_{ie} = \vec{\alpha}_i \vec{\beta}_i + \beta_i m_i + V_i(\vec{x}_i, \gamma_i)
\]
(61)
where \( V_i \) is the external potential acting on fermion \( i \). These substitutions must be done in the free Bethe-Salpeter propagator \( G^0 \) and in the manipulations leading to the 3D equation (this is an approximation, as we should also modify the fermion propagators inside the irreducible graphs, but this gives the exact cluster-separated limits). We get

\[
P_0 \psi = \left[ S_e + \frac{P_0 + S_e}{2P_0} V_e \right] \psi, \tag{62}
\]

with

\[
S_e = h_{1e} + h_{2e} = h_1 + h_2 + V_1(\vec{x}_1, \gamma_1) + V_2(\vec{x}_2, \gamma_2), \tag{63}
\]

\[
V_e = -2i\pi \hat{\theta}_e \int dp_0 dp_0' \delta(p_0' - \mu_e) \beta_1 \beta_2 K_e^T(p_0', p_0) \delta(p_0 - \mu_e) \hat{\theta}_e, \tag{64}
\]

\[
\mu_e = \frac{1}{2P_0} (h_{1e}^2 - h_{2e}^2) \tag{65}
\]

\[
\hat{\theta}_e = \theta \left( \frac{P_0}{2} + \mu_e \right) \theta \left( \frac{P_0}{2} - \mu_e \right) \theta \left( \left[ \frac{P_0}{2} + \mu_e \right]^2 - h_{1e}^2 \right), \tag{66}
\]

\[
K_e^T = K + Kg eK + \cdots \tag{67}
\]

\[
G^R_e(p_0) = \left[ \frac{1}{\frac{1}{2}P_0 + p_0 - h_{1e} + i\epsilon h_{1e}} \frac{1}{\frac{1}{2}P_0 - p_0 - h_{2e} + i\epsilon h_{2e}} \right] \beta_1 \beta_2 \tag{68}
\]

Besides the free-free continuum, the \((h_{1e}, h_{2e})\) spectrum (for which we can use figure 1) contains now bound-free or free-bound combinations (lines) and bound-bound combinations (points). The equations are written in the laboratory reference frame, defined as the reference frame in which the external potential is static. When the mutual interaction is switched off, we get a pair of independent Dirac equations in the external potential. When the external potential is switched off, we get the equation of a system of two mutually interacting fermions. This last equation can be written in covariant form, the contribution of each term of (67) being separately invariant. We can keep only the Born term (only \( K \) in \( K_e \)) and only the ladder term in \( K \) without losing this covariance.

### 3.2 Examples.

#### 3.2.1 Two electrons in the field of a nucleus.

In this case, the external potentials are given by (in configuration space):

\[
V_1 = \frac{-Ze^2}{|\vec{x}_1|}, \quad V_2 = \frac{-Ze^2}{|\vec{x}_2|}. \tag{69}
\]
For the mutual interaction kernel we have to choose a gauge. In Feynman’s gauge, the Born term is

$$K(p', p) = -\frac{2ie^2}{(2\pi)^3} \frac{1}{k^2 + i\epsilon} (\gamma_1 \cdot \gamma_2), \quad k = p - p'. \quad (70)$$

In Coulomb’s gauge, it is

$$K(p', p) = -\frac{2ie^2}{(2\pi)^3} \frac{\beta_1 \beta_2}{k^2} - \frac{1}{k^2 + i\epsilon} \left(\gamma_1 \cdot \gamma_2 - \frac{\gamma_1 \cdot \vec{k} \cdot \vec{\gamma}_2}{k^2}\right). \quad (71)$$

This last gauge being not covariant, the space-time separation refers to the laboratory frame. This gauge is not a priori a good choice for our covariant calculation. In the two-fermion problem, however, it is well known that the best choice for the calculation of the energy spectrum is Coulomb’s gauge in the center of mass reference frame. We could take this fact into account by writing (71) for a pure two-fermion system in the two-fermion center of mass reference frame $\vec{P} = 0$, making everything covariant by using $P$:

\[
\begin{align*}
\beta_1 \beta_2 &\rightarrow \frac{\gamma_1 \cdot (P)}{P^2}, \\
\vec{a} \cdot \vec{b} &\rightarrow - \frac{(a \cdot P)(b \cdot P)}{P^2} - (a \cdot b),
\end{align*}
\]

and importing the result into the two-fermion in an external potential problem, where $\vec{P}$ is no more conserved. We could call this the “covariant Coulomb gauge”.

We have now to compute $V_e$ by using (64), with $K_T^e = K$ given by (70). The principal difficulty comes from the fact that the modified Dirac operators $h_ie$ are diagonal neither in momentum space, nor in configuration space. If we expand the potential $V_e$ into the eigenfunctions of the $h_ie$, it will be given by (in Feynman’s gauge):

$$V_e(p', p) = \sum_{\omega_i, \omega_i} \psi_{\omega_i}^+(\vec{p}_1) \psi_{\omega_i}^-(\vec{p}_2) < \omega_i' | V_e | \omega_i > \psi_{\omega_i}^+(\vec{p}_1) \psi_{\omega_i}^-(\vec{p}_2) \quad (73)$$

with

$$< \omega_i' | V_e | \omega_i > = -\frac{\sqrt{e}}{2\pi^2} \hat{\theta}_e(\omega'_i) \int d^3p_1 d^3p'_2 d^3p_2 d^3p_2 \psi_{\omega_i}^+(\vec{p}_1) \psi_{\omega_i}^+(\vec{p}_2) \hat{\theta}_e(\omega_i)$$

where $\psi_{\omega_i}(\vec{p}_1)$ is an eigenstate of $h_{ie}$ with eigenvalue $\omega_i$. The sum in (73) includes also an integral on the continuum. This expression is well adapted to a perturbation calculation starting with an eigenstate of $S_e$. The first-order energy shift will indeed be $< \omega_i' | V_e | \omega_i >$, in which $\hat{\theta}_e(\omega_i)$ will be equal to 1, as $P_0 = \omega_i + \omega_2$. 


3.2.2 Positronium in an external potential.

For the positronium in an external potential, the mutual interaction kernel and potential in the Born approximation (neglecting the annihilation graph) are the same as for the two-electron pair above, with a change of sign. The external potentials could again be the potentials generated by a nucleus, and we would write again (69), with a change of sign for the positron. We could also consider non-central potentials (as $V_1 = e x_{13}$, $V_2 = -e x_{23}$ for a constant electric field along the third axis). The representation of $V_e$ in the space of the eigenfunctions of the modified Dirac operators gives a meaning to our rather formal expression (64), but it is not well adapted to this new framework, as the principal interaction is now the binding mutual interaction. We shall therefore isolate a zero-order Coulomb contribution from $V_e$:

$$V_e(p_i', p_i) = V_C(p_i') + \sum_{\omega'_{i,\omega_i}} \psi_{\omega'_{i,\omega_i}}(p_i') \psi_{\omega_{i,\omega_i}}(p_i) <\omega'_{i,\omega_i} | (V_e - V_C) | \omega_{i,\omega_i} > \psi_{\omega'_{i,\omega_i}}(p_i') \psi_{\omega_{i,\omega_i}}(p_i)$$

(75)

with

$$<\omega'_{i,\omega_i} | (V_e - V_C) | \omega_{i,\omega_i} > = \frac{e^2}{2\pi^2} \hat{\theta}_e(\omega_i) \int d^3p_1' d^3p_2' d^3p_1 d^3p_2 \psi_{\omega'_{i,\omega_i}}^+(p_i') \psi_{\omega_{i,\omega_i}}^-(p_i)$$

$$\left[ \frac{1 - \hat{\alpha}_1 \cdot \hat{\alpha}_2}{[\mu_e(\omega_i') - \mu_e(\omega_i)]^2 - (\vec{p}' - \vec{p})^2} + \frac{1}{(\vec{p}' - \vec{p})^2} \right] \psi_{\omega'_{i,\omega_i}}(p_i') \psi_{\omega_{i,\omega_i}}(p_i)$$

(76)

4 Three fermions.

4.1 3D equation.

It is easy to build a three-fermion 3D equation with three two-fermion potentials $U_{ij}(P_{ij})$ (we explicitate for a while the dependence in the total energy of the two fermions):

$$(P_0 - S) \psi = [U_{12}(P_0 - h_3) + U_{23}(P_0 - h_1) + U_{31}(P_0 - h_2)] \psi$$

(77)

with $S = h_1 + h_2 + h_3$. We have replaced the arguments $P_{ij}$, by the operators $P_0 - h_k$ in order to get the exact cluster-separated limits. At the $V_{23} = V_{31} = 0$ limit, for example, we get indeed

$$\left[ P_0 - h_1 - h_2 - h_3 \right] \psi = U_{12}(P_0 - h_3) \psi.$$ 

(78)

Writing

$$\psi = \psi_{12} \psi_3, \quad P_0 = P_{120} + p_{30}$$

(79)

we get two independent equations:

$$\left[ P_{120} - h_1 - h_2 \right] \psi_{12} = U_{12}(P_{120}) \psi_{12}, \quad p_{30} \psi_3 = h_3 \psi_3$$

(80)
as \( P_0 - h_3 \) can be replaced by \( P_0 - p_{30} = P_{120} \). Our 3D equation (77) satisfies thus clearly the cluster separability requirement. Furthermore, the three cluster-separated limits are exactly the 3D equations we would get by 3D-reducing the corresponding two-fermion Bethe-Salpeter equations. This cluster separability is a property of the equation, or, if we prefer, of the full Green function. For a given scattering solution it is also possible to take the cluster-separated limit at fixed \( P \). This is not possible for the bound state solutions.

Our use of a covariant approached propagator results in an explicit covariance of the cluster-separated limit equations like (80): this equation can directly be written in covariant form and each term of the series giving the 3D potential is separately covariant. We could thus neglect all terms of \( K^T \) but the Born term without losing the covariance of the cluster-separated limits.

Let us recapitulate the elements of our 3D equation (77) (we give only \( U_{12} \)):

\[
U_{12} = V_{12} - \left\{ P_0 - S, \frac{V_{12}}{2(P_0 - h_3)} \right\} + \frac{V_{12}(P_0 + S - 2h_3) V_{12}}{4(P_0 - h_3)^2} \tag{81}
\]

\[
V_{12} = -2i\pi \theta_{12} \beta_1 \beta_2 K^T_{12}(\mu_{12}, \mu_{12}) \hat{\theta}_{12} \tag{82}
\]

\[
\hat{\theta}_{12} = \theta(P_0 - h_3 - |\vec{p}_1| - \sqrt{\vec{p}_2^2 + m_2^2 - m_1^2}) \tag{83}
\]

\[
\mu_{12} = \frac{1}{2(P_0 - h_3)} (E_1^2 - E_2^2) \tag{84}
\]

\[
K^T_{12} = K_{12} + K_{12} G^{R}_{12} K_{12} + \cdots \tag{85}
\]

\[
G^{R}_{12} = \left( \frac{1}{(p_{10} - h_1 + i\epsilon h_1)(p_{20} - h_2 + i\epsilon h_2)} \right) \left( \frac{P_0 + S - 2h_3}{P_0 - S + i\epsilon} \delta(p_{120} - \mu_{12}) \hat{\theta} \right) \beta_1 \beta_2 \tag{86}
\]

We could keep only the Born terms in the \( K^T_{ij} \) without losing the covariance of the cluster-separated limit equations.

### 4.2 Two-body limits.

It is interesting to compare the two-body limits (say \( m_3 \to \infty \)) of our three-fermion equations with our two-fermion in an external potential equations. Writing \( P_0 = m_3 + P_{120} \) and making \( m_3 \to \infty \), we get

\[
P_{120} \psi = (h_1 + h_2 + V_1 + V_2 + U_{12}) \psi \tag{87}
\]

where \( U_{12} \) is still given by (81), with \( P_0 - h_3 \) replaced by \( P_{120} \), while

\[
V_1 = \theta(P_{120} - h_2 - |\vec{p}_1|) V_1^C \left( 1 - \frac{\theta(-P_{120} + h_2 + |\vec{p}_1|)}{P_{120} - h_1 - h_2} V_1^C \right)^{-1} \theta(P_{120} - h_2 - |\vec{p}_1|) \tag{88}
\]
\[ V_2 = \theta (P_{120} - h_1 - |\vec{p}_2|) V_2^C \left( 1 - \frac{\theta (\frac{P_{120} - h_1 + |\vec{p}_2|}{P_{120} - h_1 - h_2})}{V_2^C} \right)^{-1} \theta (P_{120} - h_1 - |\vec{p}_2|) \] (89)

We note the following differences with the two-fermion in an external potential equation:

- In the mutual interaction term (which takes its symmetrized form), the free Dirac operators \( h_i \) are not replaced by the operators \( h_{ie} \).

- The external potentials are projected, as in the one-body limit (5 2). The energy argument \( W_1 \) is here \( P_{120} - h_2 \) (and vice-versa). If \( V_1^C \) is energy-dependent (to consider more general cases than a pure Coulomb potential) its energy argument will also be replaced by \( P_{120} - h_2 \).

The difference between both approaches consists clearly in three-body terms. The cluster-separated limits are still exact, as they were before taking the \( m_3 \to \infty \) limit. These cluster-separated limits appear when two of the three two-body interactions are "switched off". When only the mutual interaction \( V_{12} \) is "switched off", we do not completely get two independent equations as in section 3. Writing \( P_{120} = p_{10} + p_{20} \), we see that \( V_1 \) will depend on \( p_{20} - h_2 \) and \( V_2 \) on \( p_{10} - h_1 \). It must however be noted that the external potentials obtained by the \( m_3 \to \infty \) limit are less general than the external potential we could consider in the formalism of section 3: here we expect central potentials decreasing with the distance. With such potentials, it is not possible to let fermion 1 (for example) go to infinity without having both \( V_{12} \) and \( V_{31} \) going to zero.

### 4.3 Examples.

#### 4.3.1 Two electrons and a nucleus.

For two electrons and a nucleus of spin \( \frac{1}{2} \), we have the Born terms

\[ K_{12}(p'_{12}, p_{12}) = \frac{-2ie^2}{(2\pi)^3} \frac{1}{k_{12}^2 + i\epsilon} (\gamma_1 \cdot \gamma_2) \] (90)

\[ K_{23}(p'_{23}, p_{23}) = \frac{2iZe^2}{(2\pi)^3} \frac{1}{k_{23}^2 + i\epsilon} (\gamma_2 \cdot \gamma_3) \] (91)

\[ K_{31}(p'_{31}, p_{31}) = \frac{2iZe^2}{(2\pi)^3} \frac{1}{k_{31}^2 + i\epsilon} (\gamma_3 \cdot \gamma_1) \] (92)

The 3D potentials are, in the momentum representation (we omit the momentum arguments of the \( V_{ij} \) here and in the next subsection):

\[ V_{12} = \frac{-e^2}{2\pi^2} \theta (P_0 - h_3 - |\vec{p}'_1| - |\vec{p}'_2|) \frac{1 - \vec{a}_1 \cdot \vec{a}_2}{k_{12}^2 - k_{12}^2} \theta (P_0 - h_3 - |\vec{p}_1| - |\vec{p}_2|) \] (93)

\[ V_{23} = \frac{Ze^2}{2\pi^2} \theta (P_0 - h_1 - |\vec{p}'_2| - \sqrt{p_3'^2 + m_3^2 - m_2^2}) \]
\[
\frac{1 - \vec{\alpha}_2 \cdot \vec{\alpha}_3}{k_{120}^2 - k_{23}^2} \theta \left( P_0 - h_1 - |\vec{p}_2| - \sqrt{\vec{p}_3^2 + m_3^2 - m_2^2} \right)
\]

where \( k_{120}^2 \) and \( k_{23}^2 \) must be taken at

\[
k_{120}^2 = \left( \frac{\vec{P}_{12} \cdot \vec{k}_{12}}{P_0 - h_3} \right)^2, \quad k_{23}^2 = \left( \frac{\vec{P}_{23} \cdot \vec{k}_{23}}{P_0 - h_1} \right)^2,
\]

while \( V_{31} = V_{23} (1 \leftrightarrow 2) \). We could also use the "invariant Coulomb gauge" (in fact, one gauge for each two-fermion subsystem). If we had started with a true three-fermion Bethe-Salpeter equation we could not use three different gauges simultaneously, but here our aim is only to get continuum dissolution-free equations with the correct cluster-separated limits.

4.3.2 Three protons exchanging mesons.

For three protons exchanging a vector, a scalar and a pseudoscalar meson, the contribution of the Born terms to \( V_{12} \) is

\[
V_{12} = \frac{-1}{2\pi^2} \theta \left( P_0 - h_3 - |\vec{p}_1'| - |\vec{p}_2'| \right) \left\{ \frac{g_\rho^2}{k_{120}^2 - k_{12}^2 - m_\rho^2} (1 - \vec{\alpha}_1 \cdot \vec{\alpha}_2 - \frac{k_{120}^2 - \vec{\alpha}_1 \cdot \vec{k} \cdot \vec{\alpha}_2 \cdot \vec{k}}{m_\rho^2}) + \frac{g_\sigma^2 \beta_1 \beta_2}{k_{120}^2 - k_{12}^2 - m_\sigma^2} + \frac{g_\pi^2 \beta_1 \beta_2 \gamma_5 \gamma_{25}}{k_{120}^2 - k_{12}^2 - m_\pi^2} \theta \left( P_0 - h_3 - |\vec{p}_1| - |\vec{p}_2| \right), \]

where \( k_{120}^2 \) must again be replaced by (95), and similarly for \( V_{23} \) and \( V_{31} \).

5 Conclusions.

Using the explicitly covariant Sazdjian propagator combined with our covariant substitute of the positive-energy projector, we built a 3D equation for two fermions in an external potential and for three fermions. The two-fermion equations obtained at the cluster separated limits are these which would be obtained by a 3D reduction of the two-fermion Bethe-Salpeter equation. In our preceding works this implied that these cluster-separated limits were implicitly covariant if not truncated, even if they remained written in a specific reference frame (the laboratory frame or the three-fermion rest frame). In the present work, however, they are explicitly covariant, which implies that the covariance will survive the possible truncations of the 3D potentials. The invariant substitute of the positive energy projector we use brings a cutoff on the spatial momenta, which could be welcome when insuring the existence of some integrals. This cutoff appear for rather low, but nevertheless truly relativistic values. It is not
by itself an approximation: it reflects the choice to work with a given projection of the two-fermion equation.

The 3D equations we write in this work are more complicated than the corresponding equations in our preceding works. Keeping in mind that in both cases the 3D potentials are given by series to be truncated anyway, we hope that the Lorentz invariance / cluster separability properties of the truncations of the present equations will lead to a better approach of the real physics in the first terms.
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Figure 1. The \( \hat{\theta} = 1 \) region.