Surface Charges Toolkit for Gravity

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Abstract: These notes provide a detailed catalog of surface charge formulas for different classes of gravity theories. The present catalog reviews and extends the existing literature on the topic. The main method to define and compute quasi-local surface charges for gauge theories is reviewed in order to clarify conceptual issues and applicability. The computation of the surface charge formulas is carried out in metric, tetrads, Chern-Simons connection, and BF variables. The language of differential forms is exploited. The coupling with matter fields as scalar, Maxwell, Skyrme, Yang-Mills and spinors is considered.


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1 Introduction

In physicist’s eyes one of the most beautiful results of mathematical-physics is the deep connection between global symmetries and conserved quantities elegantly established by Emmy Noether one century ago [1, 2]. She taught us, as a particular consequence, that the celebrated principle of energy conservation in a given theory is a consequence of the symmetry under time translations of that theory. Her general result is known as the First Noether Theorem (for a recent review see [3]).

In the same paper, written in 1918, Noether showed also that applying the same strategy for gauge theories we can not obtain conserved quantities, but instead, she showed another interesting result: For each independent gauge symmetry, a relation among field equations holds (off-shell relations), the so-called Noether identities (e.g. Bianchi identity for gravity). This is the content of the Second Noether Theorem.

As you may appreciate after the great work done by Noether, an important question remained unanswered: Is there still a connection between charges and global symmetries in the case of gauge theories?. As physics developed through the last century it became clear that all fundamental theories in physics are in fact gauge theories. Therefore, it was crucial to have an answer.

First Noether Theorem is still partially\(^1\) useful for gauge theories if the theory, such as Yang-Mills theory, is defined on a fix background spacetime, such as Minkowski spacetime. The global symmetries of spacetime are translated into Noether charges, usually packed into a stress-energy tensor. However, by unfreezing spacetime and considering a more realistic theory coupled to General Relativity, itself a gauge theory of spacetime, hope is lost. Most attempts of directly using First Noether Theorem to define charges out of global symmetries run into troubles or are condemned to ambiguity.

To deal with gauge theories and specially with General Relativity several successful methods where developed during the last fifty years. The most popular are, by name

\(^1\)Because of the gauge symmetry, think of a connection field, there is already an ambiguity on the field symmetry transformation that is inherited in the computation of the stress-energy tensor through this method. See the discussion around Belinfante tensor.
of authors, Abbott-Deser-Tekin [4, 5], Regge-Teitelboim [6], Iyer-Wald [7, 8], Barnich-Brandt [9, 10], and Torre-Anderson [11, 12] method. Their developments were sometimes independent and sometimes cross inseminated. Nevertheless, all of them have common features. One is their relying, directly or indirectly, on the structure of the phase space, more specifically on the symplectic structure. Another common feature is that in all of them the obtained charges are expressed as closed surface integrals (or \((D - 2)\)-surface integrals for \(D\)-dimensional spacetimes). The equivalence and connection among some of them have been established in the literature. However, a systematic study of their connection is still missing.

In these notes we revisit and explore the method to compute surface charges for several gravity theories. We follow in general terms the Iyer-Wald (IW) symplectic method but being well aware about the close Barnich-Brandt symplectic method and their equivalence. One main difference of the approach presented here is the emphasis on the quasi-local nature of the formulas for charges, but of course, they might be used at spacetime asymptotic regions too.\(^2\) To achieve clarity our terminology and notation slightly differ from the original IW treatment but the core logic is the same. For a complementary approach see the lecture [13].

Our purpose with these notes is to make a self-contained systematic review and to widen the applicability of this symplectic method to compute charges. We treat several gravity theories with metric variable but a special emphasis is on gravity theories written in the differential form language. Due to its geometric nature and elegance the use of forms deserves a closer attention in the light of recent developments in physics. Some conceptual issues regarding gauge symmetries are transparent in this language besides that equivalent formulas for charges usually get more compact expressions and become more tractable. An indication of this are the several new results that we encounter when analyzing the surface charges of the theories.

The procedure to compute charges for a given gauge theory can be ordered in five simple steps 1) Identify the fields infinitesimal gauge transformations, 2) Obtain the general surface charge density formula for the theory, 3) Identify the parameters solving the exact symmetry condition (\(e.g.\) generalized Killing equation), 4) Compute the surface charge integral with those parameters (to get a differential charge), and 5) Integrate the differential charge on phase space. In the present notes we extensively explore the first and second steps for different gravity theories. In order to further pursue the rest of the steps a particular family solution of field equations have to be chosen.

There are multiple ways to use this toolkit. Mainly it can be useful for readers interested in any of the exhibited theories. Check the index. In Table the resulting formulas of the surface charges density for each theory are compactly presented. For purposes of comparison we have grouped theories written with metric variable in Section 3 and with a differential form language in Section 4. The Table summarizes the main result of both sections. In each

\(^2\)It is worth to note here that the quasi-local treatment for the charge conservation ensures, given a spacetime with exact symmetries, the independence of the radius. This contrast with some asymptotic approaches where \(r \rightarrow \infty\) is required to get rid of terms appearing on the specific computation of charge formula even if the solution is everywhere specified, \(e.g.\) a black hole solution.
respective subsection the main formula is also highlighted in a square box and accompanied with explanation and comments. It is worth to stress, that for most theories considered, there is an associated appendix with extra comments and a step-by-step calculations that leads to the corresponding surface charge formula.

In this toolkit there is also scope for readers unaware but interested in being introduced to the surface charge method itself. Simple examples are presented in great detail with pedagogical purposes. This is the case of the pure electromagnetic theory treated already in the next introductory Subsection 1.1 or the Subsection 4.5 devoted to Chern-Simons theory in $2 + 1$. Those readers can also follow the general method to compute surface charges for an arbitrary theory presented in full detail in Section 2.

Experts on the surface charge method can also find interesting new results in Section 4. We bring to their attention 1) The connection with the Barnich-Brandt method to compute surface charge density and the use of new expressions of the contracting homotopy operator adapted to Einstein-Cartan or Chern-Simons theories in Eqs. (H.4) and (J.13) of the appendices, respectively, 2) The unexpected results on simple theories with torsion, see Subsection 4.3, where it is shown that contorsion disappears from the formulas, or 3) The compact expression for surface charges one gets by assuming an asymptotic constant curvature in the Einstein-Cartan theory with cosmological constant, see Subsection 4.1.1. Also the general overview and the easy comparison of formulas in the metric and differential form languages for the different or equivalent theories could be useful for a deeper analysis of the surface charge method itself.
1.1 Electromagnetic warm up

To appreciate the relevance of surface charge method we start by naïvely applying the First Noether Theorem for the gauge symmetry of pure Maxwell theory. Then, we stress the problem due to the gauge symmetry of the theory and immediately use the surface charge method to save the day. A reader familiar with the method may skip this subsection.

The Lagrangian is \( L[A] = F \star F \), where the two-form field strength is the exterior derivative of the one-form connection, \( F = dA \), and \( \star \) is the Hodge product.\(^3\) The theory has a \( U(1) \) gauge symmetry, \( A \rightarrow A' = A + d\lambda \), with the infinitesimal version \( \delta_\lambda A = -d\lambda \).

The infinitesimal parameter \( \lambda \) is a spacetime function. The variation of the Lagrangian is

\[
\delta L = E_A \delta A + d\Theta(\delta A) = -2(d \star F) \delta A + 2d(\delta A \star F). \tag{1.1}
\]

If we restrict the variation to an infinitesimal gauge symmetry, off-shell, we get

\[
0 = 2(d \star F)d\lambda - 2d(d\lambda \star F) = -2d((d \star F)\lambda + d\lambda \star F), \tag{1.2}
\]

because trivially \(-2d(d \star F)\lambda = 0\) due to \(d^2 = 0\). We may be tempted to interpret the previous equation as a conservation law, \(dJ_\lambda = 0\), for the current

\[
J_\lambda = -2(d \star F)\lambda - 2d\lambda \star F = -2d(\lambda \star F), \tag{1.3}
\]

however, the second equality tells us that the current is trivially conserved, that is, even without imposing the equation of motion. In the differential form parlance a current built from a gauge symmetry is always an exact form and therefore a closed form, this is another form of the old Second Noether Theorem. Therefore, the would-be Noether current \(J_\lambda\) is trivial and thus is physically meaningless to define a charge with it. In fact (1.2) is an off-shell identity by virtue of the Noether identity \(N_\lambda = -2d(d \star F)\lambda = 0\) which here, in the case of electromagnetism, is a mere consequence of \(d^2 = 0\). This analysis is in agreement with the expected well-known result that gauge symmetries do not produce charges.

Now, a subset of gauge symmetries in special cases may become global symmetries (also called exact symmetries). In the case of pure electromagnetism this stands for solving \(\delta_\lambda A = -d\lambda = 0\) which has the solution \(\lambda = \lambda_0 = \text{cte}\).\(^4\) Now, still the previous analysis will not change its triviality because it was completely general and this is just a particular choice of \(\lambda\). This is important to stress because a naive use of \(J_{\lambda_0}\) as defined before will actually produce here the right formula for the electric charge, however, the logic is misleading and that mistake will hit back in other gauge theories.

To get a sensitive charge from the exact symmetry in gauge theories, \(\delta_{\lambda_0} A = 0\), we should not follow Noether’s approach but use another strategy. Consider the two-form on phase space known as the symplectic structure density

\[
\Omega(\delta_1, \delta_2) \equiv \delta_1 \Theta(\delta_2) - \delta_2 \Theta(\delta_1) = 2\delta_1 A \star d\delta_2 A - 2\delta_2 A \star d\delta_1 A, \tag{1.4}
\]

\(^3\)In components this corresponds to the usual Lagrangian density \(\sqrt{-g} F_{\mu\nu} F^{\mu\nu}\). If you are unfamiliar with this notation check (A.8) in convention’s Appendix A and translate each equation.

\(^4\) Just for electromagnetism this solution does not depend on the fields and thus the rest of the analysis is quite general. In the case of general relativity the analogous equation is the Killing equation which is a property of certain symmetric spacetime, or for the case of Yang-Mills theory \(\delta_\lambda A^i = -d_A \lambda^i = 0\) also depends on the fields an there is no a general solution.
evaluated on an infinitesimal gauge direction $\delta_2 A \rightarrow \delta \lambda A = -d\lambda$ (and $\delta_1 A \rightarrow \delta A$)

$$\Omega(\delta, \delta \lambda) = 2d\lambda * d(\delta A) = 2d(\lambda * d\delta A) - 2\lambda d * d(\delta A) = 2d(\lambda * \delta F), \quad (1.5)$$

where we used that $d^2 = 0$. Further, we used that the field is on-shell and that $\delta A$ satisfies the linearized equation of motion $\delta(d * F) = d * \delta F = d * d(\delta A) = 0$. At this level to consider $\lambda = \lambda_0$ truly makes a sensitive difference because the l.h.s. of (1.5) simply vanishes, because of $\delta\lambda_0 A = 0$, and we get a conservation law

$$2d(\lambda_0 * \delta F) = 0, \quad (1.6)$$

this suggest us to define a (varied) charge, named surface charge, as

$$\delta Q_{\lambda_0} = 2\lambda_0 \int_S * \delta F, \quad (1.7)$$

this is meaningful information we can extract from the exact symmetry assumption. The key point was to establish a conservation law for the surface charge density $k_{\lambda_0} \equiv 2\lambda_0 * \delta F$.

Note that the expression is a variation on phase space, however, because electromagnetism is simple enough the integration is trivial and, taking the integration constant zero, we can just drop the $\delta$ symbols

$$Q_{\lambda_0} = 2\lambda_0 \int_S * F, \quad (1.8)$$

this is the well-known formula for the electric charge obtained as a consequence of an exact symmetry of the theory.

Some remarks are in order:

• Note that (1.6) is a quasi-local expression, the integration is over any close two-surface $S$. The same holds in the general case.

• To get a non-vanishing charge $Q$ the source for the charge should be enclosed by the surface $S$, fields sourcing the charge could be in principle included in the Lagrangian but because the analysis leading to (1.6) is fully local, we do not need to know the field description of the source as far as they are non-zero just in a compact region.

• As stated in footnote 4, electromagnetism is a very special case because we have a conserved charge without knowing the details of the field. In general cases charges are obtained only when the exact symmetry condition holds and that usually depends on particular solutions of the field equations, e.g. symmetric spacetimes.

## 2 Derivation of Surface Charges

In this section, we aim at a pedagogical step-by-step derivation of the surface charges following in general terms the Iyer-Wald [7] approach.

Let $L[\phi]$ be a diffeomorphism-invariant Lagrangian and $\phi$ be a collection of dynamical fields. The first variation of the Lagrangian can always be written as

$$\delta L = E(\phi)\delta\phi + d\Theta(\phi, \delta\phi), \quad (2.1)$$
where \( E(\phi) \) are the equations of motion of the Lagrangian theory and they locally depend on the dynamical fields and their derivatives, while \( \Theta(\phi, \delta \phi) \) locally depends on the dynamical fields \( \phi \), their variations \( \delta \phi \) and derivatives. The letter \( d \) stands for exterior derivative. The boundary term \( \Theta(\phi, \delta \phi) \) is linear in the variations \( \delta \phi \); it is a \((D-1)\)-form and is called the symplectic potential form. It suffers from two types of ambiguities: The first ambiguity arises from adding an exact \( D\)-form to the Lagrangian top-form; the second one arises from adding an exact \((D-1)\)-form to \( \Theta \).

In the following we write the formulas in differential form language using bold letters for them. In parallel we also express the formulas in the more conventional language, without using differential form, with ordinary letters.

Let \( \xi = \xi^\mu \partial_\mu \) be an arbitrary vector field and let us compute the first variation of the Lagrangian with respect to \( \xi \). Since \( \delta \xi L = \mathcal{L}_\xi L \) and \( L \) is a top-form, by using the Cartan’s magic formula,\(^5\) one has

\[
\partial_\mu (\xi^\mu L) = E(\phi)\delta \xi \phi + \partial_\mu \Theta^\mu(\phi, \delta \xi \phi), \quad [d(i_\xi L) = E(\phi)\delta \xi \phi + d\Theta(\phi, \delta \xi \phi)]. \tag{2.2}
\]

Here \( L \) is the Lagrangian density (e.g., in General Relativity \( L[g_{\mu\nu}] = \sqrt{-g} R \)).

We now assume that the field transformations, \( \delta \xi \phi \), are linear in \( \xi \). If so, one can always write the first term in the variation of the Lagrangian density along \( \xi \) as

\[
E(\phi)\delta \xi \phi = \partial_\mu S^\mu_\xi + N_\xi, \quad [E(\phi)\delta \xi \phi = dS_\xi + N_\xi], \tag{2.3}
\]

where \( N_\xi \) collects all terms linear in the vector field. Now by replacing Eq. (2.3) into Eq. (2.2) one can arrange terms in a total derivative as

\[
\partial_\mu \left( \Theta^\mu(\phi, \delta \xi \phi) - \xi^\mu L + S^\mu_\xi \right) = N_\xi, \quad [d(\Theta(\phi, \delta \xi \phi) - i_\xi L + S_\xi) = N_\xi]. \tag{2.4}
\]

We obtained an equation of the form \( \partial_\mu J^\mu_\xi = N_\xi \) (in forms \( dJ_\xi = N_\xi \)). Because \( \xi \) is an arbitrary spacetime function, the very structure of this equation implies that \( N_\xi = 0 \). This result is the Second Noether Theorem and \( N_\xi = 0 \) are known as the Noether identities. There is one of them for each independent arbitrary function \( \xi \) (e.g. in four dimensional GR the four independent diffeomorphisms imply the four component Bianchi identity).

Therefore, one can defines a current vector, \( J^\mu_\xi = \Theta^\mu - \xi^\mu L + S^\mu_\xi \), which is identically conserved, \( \partial_\mu J^\mu_\xi = 0 \), even without the use of the equations of motion. In this sense, one says that \( J^\mu_\xi \) is trivially conserved. When the equations of motion are used, \( J^\mu_\xi \) becomes the sometimes called Noether current \( J^\mu_\xi \approx \Theta^\mu - \xi^\mu L \), which due to the previous analysis is also trivially conserved.

At this stage, we can evoke the Poincaré’s lemma to the identically conserved current \( J^\mu_\xi \). Therefore, there must locally exist a codimension-two-form sometimes called Noether potential, \( \tilde{Q}^\mu_\xi \), such that \( J^\mu_\xi = \partial_\nu \tilde{Q}^\mu_\xi \). Notice that \( \tilde{Q}^\mu_\xi \) is ambiguous up to a closed codimension-two form.

\(^5\)For an arbitrary form \( \omega \) and a vector field \( \xi \) the Lie derivative is \( \mathcal{L}_\xi \omega = d(i_\xi \omega) + i_\xi (d\omega) \). For the inner operation we use either the \( i_\xi \) notation (here) or the \( \xi \cdot \) notation (later).
Because of this trivial conservation, charges should not be defined by using $J^\mu_\xi$. In order to define charges for gauge theories one has to rely on the symplectic structures of the theory. A way to define the symplectic structure current is

$$\Omega^\mu(\phi, \delta_1, \delta_2) \equiv \delta_1 \Theta^\mu(\phi, \delta_2 \phi) - \delta_2 \Theta^\mu(\phi, \delta_1 \phi) - \Theta^\mu(\phi, [\delta_1, \delta_2] \phi),$$

(2.5)

where the last term ensures linearity on the variations.

Now, consider the arbitrary variation of the current $J^\mu_\xi$, we keep the vector field fix, i.e. $\delta \xi^\mu = 0$, then

$$\delta J^\mu_\xi = \delta \Theta^\mu(\phi, \delta_\xi \phi) - \delta \xi \Theta^\mu(\delta \phi) = \xi^\mu (E \delta \phi + \partial_\nu \Theta^\nu(\delta \phi)) - \delta \xi \Theta^\mu(\delta \phi) = \xi^\mu E \delta \phi - \delta \xi \Theta^\mu(\delta \phi)$$

Evaluating the symplectic current on the gauge symmetry, $\delta_2 = \delta_\xi$ and $\delta_1 = \delta$, one obtains

$$\Omega^\mu(\delta \phi, \delta \xi \phi) \equiv \delta \Theta^\mu(\delta \xi \phi) - \delta \xi \Theta^\mu(\delta \phi) = \xi^\mu \left( E \delta \phi + \partial_\nu \Theta^\nu(\delta \phi) \right) - \delta \xi \Theta^\mu(\delta \phi),$$

(2.6)

in second line we replaced $\delta \Theta^\mu(\phi, \delta \phi)$ from Eq. (2.6). In Eq. (2.6), the Lagrangian $\delta L$ and we used the Lie derivative of the tensor density $\Theta^\mu$ as $\delta \xi \Theta^\mu = \mathcal{L}_\xi \Theta^\mu = 2 \partial_\nu (\xi^{[\nu} \Theta^{\rho]} + \xi^{\rho} \partial_\rho \Theta^{\nu}$).

To be consistent, with differential forms we have\footnote{In the following, we are assuming that variations commute, i.e., $[\delta, \delta] \phi = 0$ and, therefore, $\Theta(\phi, [\delta, \delta] \phi) = 0$. This is true if $\xi$ is assumed to be fix on the phase space as we do here. But if besides $\xi$ the theory has more gauge symmetry parameters involved, e.g. a collection $\epsilon = (\xi, \lambda^a, \lambda^i, \ldots)$, then the term $\Theta(\phi, [\delta, \delta] \phi)$ can always be decomposed as (in analogy to Eq. (2.3))

$$\Theta(\phi, [\delta, \delta] \phi) = d B_\delta + C_\delta,$$

(2.8)

where $C_\delta$ collects all terms linear in the variation of the symmetry parameter and it vanishes on-shell, $C_\delta \approx 0$. The term $B_\delta$ will affect the surface charge density as far as some of the symmetry parameter could be field dependent, $\delta \epsilon = (\delta \xi, \delta \lambda^a, \delta \lambda^i, \ldots) \neq 0$, still with $\delta \xi = 0$.}

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forms, $dk_\xi \approx 0$). The exactness of the surface charge density guarantees that the value of its integration on a closed surface is independent of the choice of the surface. By integrating it over any closed surface $S$ (e.g., a $(D - 2)$-dimensional sphere), one obtains the surface charge

$$\delta Q_\xi(\phi, \delta \phi) = \frac{1}{2(D - 2)!} \oint_S k_\xi^{\mu \nu \alpha_3 \ldots \alpha_D} dx^{\alpha_3} \wedge \cdots \wedge dx^{\alpha_D} = \oint_S k_\xi(\phi, \delta \phi), \quad (2.12)$$

naturally on both languages one obtains the same value for the surface charge. Note that this quantity is a differential or one-form on the phase space. The symbol $\delta$ emphasizes that the surface charge is not necessarily an exact differential on the phase space of the solutions. In other words the function $Q_\xi$ may not exists. A sufficient condition for its existence is $\delta \left( \oint_S k_\xi \right) = 0$. If the condition holds, then, after an integration on the phase space it is possible to obtain the finite surface charge $Q_\xi$.

3 Gravity Theories with Metric Variable

In this section, we aim at those gravity theories, in the metric formalism, whose action reads

$$S[g_{\mu \nu}, \phi] = S^{(EH)}[g_{\mu \nu}] + S^{(m)}[g_{\mu \nu}, \phi], \quad (3.1)$$

where $S^{(EH)}[g_{\mu \nu}]$ is the Einstein-Hilbert (EH) action describing General Relativity (GR) and $S^{(m)}[g_{\mu \nu}, \phi]$ is the action describing the dynamics of the matter sector.

To achieve this program, we first study in details the Einstein-Hilbert action and how surface charges are derived in General Relativity.

3.1 Einstein-Hilbert-\(\Lambda\) action

We start by considering the Einstein-Hilbert (EH) action in $D$-dimensional spacetime with an additional cosmological constant term $\Lambda$

$$S^{(EH)}[g_{\mu \nu}] = \frac{\kappa}{2} \int_M dx^D \sqrt{-g} (R - 2\Lambda), \quad (3.2)$$

where $\kappa = c^4/(8\pi G)$, $c$ the speed of light and $G$ the Newton’s constant.

The variation of the metric field under an infinitesimal diffeomorphism amounts for its Lie derivative generated by the vector field $\xi$: $\delta_\xi g_{\mu \nu} = \mathcal{L}_\xi g_{\mu \nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$. This corresponds to the gauge symmetry of GR.\(^7\) Therefore, as we know, to compute charges we should use the surface charge method.

The surface charge density for GR is derived in full detail in the Appendix B, the results is the following expression\(^8\)

$$k_\xi^{\mu \nu} = \sqrt{-g} \kappa \left( \xi^{[\mu} \nabla_\sigma \delta g^{\nu]\sigma} - \xi^{[\mu} \nabla^{\nu]} g + \xi^{[\mu} \nabla_\xi^{\nu]} - \frac{1}{2} \delta g^{[\nu} \nabla^{\mu]} \xi_\sigma - \frac{1}{2} \delta g^{[\mu} \nabla^{\nu]} \xi_\sigma \right) \quad (3.3)$$

\(^7\)We take the perspective of GR as a gauge theory in the sense that for arbitrary vector field the Lie derivative of the metric is both 1) a local transformation and 2) a symmetry of the action (i.e. the varied action after replacement of the symmetry becomes a boundary term).

\(^8\)See also [13]. Note the overall sign difference. It is due to our conventions on the variation symbol $\delta$. We assume it respects $\delta g^{\mu \nu} = -g^{\mu \alpha} g^{\nu \beta} \delta g_{\alpha \beta}$.
where $\delta g = g_{\alpha\beta} \delta g^{\alpha\beta}$. The surface charge density does not depend on the cosmological constant term. Moreover it is linear in the symmetry vector field $\xi$ and in the variation of the metric field $\delta g^{\mu\nu}$. Note also that $k^{\mu\nu}_\xi$ is anti-symmetric: An indication that the differential form language may be appropriated here.

If we assume the condition that $\xi$ is an exact symmetry for the metric field, i.e., $\xi$ is a Killing vector

$$\mathcal{L}_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0,$$

the surface charge density satisfies a conservation, $\partial_\mu k^{\mu\nu}_\xi = 0$, and therefore it is suitable to define a surface charge through the integral expression (2.12).

### 3.2 Einstein-Hilbert-Maxwell action

Let us consider the Einstein-Hilbert-Maxwell action. The theory is described by the following action

$$S[g_{\mu\nu}, A_\mu] = S^{(EH)}[g_{\mu\nu}] - \frac{1}{4} \int_M d^D x \sqrt{-g} F_{\mu\nu} F^{\mu\nu},$$

where $A_\mu$ is the electromagnetic gauge potential and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength.

The infinitesimal gauge transformations for this theory are

$$\delta g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu,$$

$$\delta A_\mu = \delta_{(\xi,\lambda)} A_\mu = \mathcal{L}_\xi A_\mu - \nabla_\mu \lambda' = \xi^\nu F_{\nu\mu} - \nabla_\mu \lambda,$$

where in addition to the diffeomorphisms on the $A_\mu$ field we must consider the $U(1)$ gauge symmetry of the electromagnetism. Note that we use $\lambda = \lambda' - \xi^\mu A_\mu$, which is the prescription for the so-called improved gauge transformations. We use $\epsilon = (\xi, \lambda)$ to pack all gauge symmetry parameters. For convenience we use this prescription all along this toolkit each time there is a gauge transformation acting on connections. The advantage is that the transformations become explicitly covariant (here invariant) as far as the gauge parameter $\lambda$ transforms in a covariant way (here invariant). To undo the prescription, a simple replacement of $\lambda$ in the final formulas is enough.

The surface charge density for Einstein-Hilbert-Maxwell theory is (see Appendix C)

$$k^{\mu\nu}_\xi = \tilde{k}^{\mu\nu}_\xi + \sqrt{-g} \left[ \lambda \left( \frac{1}{2} g F^{\mu\nu} - \delta F^{\mu\nu} \right) + \delta A_\alpha \left( \xi^\alpha F^{\mu\nu} - 2 \xi^{[\mu} F^{\nu]\alpha} \right) \right]$$

where again $\delta g \equiv g_{\alpha\beta} \delta g^{\alpha\beta}$.

In order to define a conserved surface charge the exact symmetry conditions must be satisfied. The conditions stand for equating the infinitesimal gauge transformations (3.6) and (3.7) to zero and solve for the parameters $\epsilon = (\xi, \lambda)$. As we know for pure gravity this is the Killing condition on $\xi$, but here we also need to solve $\lambda$ in terms of $\xi$. For a given Killing field $\lambda = \lambda(\xi)$ is the one needed for computing the charge. Note, that even if there is no Killing field at all, still $\xi = 0$ and $\lambda = \lambda_0$ is a general solution. This is the origin of the electric charge in curved spacetimes.
3.3 Einstein-Hilbert-Λ action with a conformally coupled scalar field

Consider gravity field interacting with a scalar field. A particularly simple model is to add, a so-called, conformally coupled scalar field, $\Phi$. The theory is described by the Lagrangian action (see for instance [14])

$$ S[g_{\mu\nu}, \Phi] = \frac{\kappa}{2} \int_M d^D x \sqrt{-g} \left( R - 2\Lambda - \frac{1}{\kappa} (\partial^\mu \Phi \partial_\mu \Phi - \zeta_D R \Phi^2) \right). \tag{3.9} $$

Although the theory does not possess a conformal symmetry, in this model the equation of motion for the scalar field (a generalization of the Klein-Gordon equation given by $(\Box + \zeta_D R)\Phi = 0$) is conformally invariant\(^9\) for the specific values $\zeta_D = \frac{D-2}{4(D-1)}$.

The infinitesimal diffeomorphism on the metric field is the usual (3.6). There is no a new gauge symmetry in this theory, then, for the scalar field the transformation is simply

$$ \delta_\xi \Phi = \mathcal{L}_\xi \Phi = \xi^\mu \partial_\mu \Phi. \tag{3.10} $$

The surface charge density is (see the details in the Appendix D )

$$ k^{\mu\nu}_\xi = \xi^\nu \left( 1 + \frac{\zeta_D}{\kappa} \Phi^2 \right) + \zeta_D \sqrt{-g} \xi^{\mu} \left( \delta g^{|a|}_\mu \nabla_a \Phi^2 - \nabla^{|a|} \delta \Phi^2 - \frac{2}{\zeta_D} \delta \Phi \nabla^{|a|} \Phi \right) \tag{3.11} $$

which is expressed in terms of pure GR surface charge density (3.3). Again to have a conserved charge one must use a Killing vector field, (3.4), but also the exact symmetry equation on the scalar field must hold: $\xi^\mu \partial_\mu \Phi = 0$. In words, there must be spacetime directions along which the scalar field is constant.

3.4 Einstein-Hilbert-Skyrme action

Here we consider gravity coupled to a Skyrme field, denoted $U(x^\mu)$, which is a $SU(2)$ group valued field on spacetime. The Skyrme model, in flat spacetime, is an effective field theory describing either nuclear and particle physics in the low energy regime of QCD (see the review [15]). Currently, the research on the gravitational Skyrme model is in development. Notable results come form the side of black hole physics (see for instance [16, 18]) and particular self-gravitating Skyrme solutions (see [17]).

The Skyrme-Einstein action is

$$ S[g_{\mu\nu}, U] = \int_M d^D x \sqrt{-g} \left( \frac{\kappa}{2} (R - \Lambda) + \frac{K}{4} \left( R^\mu R_\mu + \lambda \left( F^\mu_\nu F_{\mu\nu} \right) \right) \right), \tag{3.12} $$

with $R_\mu = U^{-1} \partial_\mu U$, therefore $R_\mu$ is valued in the $SU(2)$ algebra ($R_\mu = R_\mu^j \tau_j$, we use the generators $\tau_j = -i \sigma_j$ with $\sigma_j$ the Pauli matrices), and $F_{\mu\nu} = [R_\mu, R_\nu]$ with $[\cdot, \cdot]$ the algebra commutator. The $K$ and $\lambda$ are positive coupling constants; and the $\langle \cdot \rangle$ denotes the trace on the algebra elements.

\(^9\)The conformal transformations are $\Phi \rightarrow \Omega^{1-\frac{D}{2}}(x)\Phi$ and $g_{\mu\nu} \rightarrow \Omega^2(x)g_{\mu\nu}$ for $\Omega(x)$ an arbitrary spacetime function.
Analogously with the previous theory there is no new gauge symmetry. Then, we must consider just the infinitesimal diffeomorphism transformations on the fields

\[ \delta \xi g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \]  

(3.13)

\[ \delta \xi U = \mathcal{L}_\xi U = \xi^\mu \partial_\mu U. \]  

(3.14)

In the Appendix E it is worked out the derivation of the surface charge density. The result is

\[ k^\mu_{\xi\nu} = \frac{K}{\sqrt{-g}} \xi^\mu \left( R^\nu + \frac{\lambda}{4} [R_{\sigma}, F^{\nu\sigma}] \right) U^{-1} \delta U \]  

(3.15)

Again, to have the conservation law, \( \partial_\mu k^\mu_{\xi\nu} \approx 0 \), the exactness symmetry condition must hold. That is, equating (3.13) and (3.14) to zero and solving for \( \xi \) (have a Killing vector).

As an application, this is the formula one should use, within this formalism, to compute the mass/energy of a spherically symmetric black holes in the presence of a Skyrme field.

3.5 Lanczos-Lovelock action

Gravity theories can be extended to higher and lower spacetime dimensions. With the criteria of having second order field equations for the metric Lanczos in 1938 [19] and Lovelock in 1971 [20] wrote the most general expression for the Lagrangian in an arbitrary dimension. In addition to the cosmological constant and the Einstein-Hilbert term, the Lanczos-Lovelock (LL) Lagrangian includes a series of higher power curvature terms that depend on the dimension. For a recent review we suggest [21]. The action is

\[ S[g_{\mu\nu}] = \int_{\mathcal{M}} \sqrt{-g} \sum_{m=0}^{[(D-1)/2]} c_m L_m, \]  

(3.16)

with \( c_m \) arbitrary constants, here \([\cdot]\) is the integer part function, and each term in the series is

\[ L_m = \frac{1}{2m} \delta^{\alpha_1 \beta_1} \ldots \delta^{\alpha_m \beta_m} R_{\alpha_1 \beta_1} \ldots R_{\alpha_m \beta_m} \gamma_1 \gamma_2 \ldots \gamma_m \sigma_m, \]  

(3.17)

where the symbol \( \delta^{\alpha_1 \beta_1} \ldots \) are the totally anti-symmetric Kronecker’s deltas (for instance \( \delta^{\alpha \beta} = \delta^\alpha_\mu \delta^\beta_\nu - \delta^\alpha_\nu \delta^\beta_\mu \)). The \( L_{m=0} = 1 \) is defined to have the cosmological constant term (Einstein-Hilbert-\( \Lambda \), (3.2), is recovered with \( c_{m=0} = -\kappa \Lambda \) and \( c_1 = \kappa / 2 \)).

The only field in this theory is the metric, thus the infinitesimal gauge transformation is \( \delta g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu} \). With the gauge transformation identified we can compute the surface charge density from the general formula where the only gauge symmetry is diffeomorphism, remember, it reads

\[ k^\mu_{\xi\nu} = \frac{\delta \tilde{Q}^\mu_{\xi\nu}}{\delta \delta \xi} + 2 \xi^\rho [\mu \Theta^\nu](\delta). \]  

(3.18)

It is straightforward to compute the specific quantities for LL Lagrangian, they can also be read from [21], the result

\[ \tilde{Q}^\mu_{\xi\nu} = -2 \sqrt{-g} P^{\mu\alpha\beta} \nabla_\alpha \xi_\beta, \]  

(3.19)

\[ \Theta^\nu(\delta) = 2 \sqrt{-g} P^{\mu\alpha\beta} \nabla_\gamma \delta g_{\alpha\beta}, \]  

(3.20)
with
\[ P^{\beta \gamma \delta}_{\alpha \gamma \delta} \equiv \frac{\partial}{\partial R_{\alpha \beta \gamma \delta}} \left( \sum_{m=0}^{[(D-1)/2]} c_m L_m \right) = \sum_{m=1}^{[(D-1)/2]} \frac{c_m}{2m} P^{\alpha \beta \gamma \delta}_{(m)}, \] (3.21)
and each term in the sum is
\[ P^{\alpha \beta \gamma \delta}_{(m)} \equiv \delta^{\alpha \beta_1}_{\gamma \sigma_1} \cdots \delta^{\alpha_{m-1} \beta_{m-1} \alpha_1}_{\gamma_{m-1} \sigma_{m-1} \mu \lambda} g^{\omega \nu} g^{\lambda \kappa} R_{\alpha_1 \beta_1} \gamma_1 \cdots R_{\alpha_{m-1} \beta_{m-1}} \gamma_{m-1} \sigma_{m-1}. \] (3.22)

Note that by its definition \( P^{\alpha \beta \gamma \delta} \) inherits the symmetries of the Riemann tensor \( P^{\alpha \beta \gamma \delta} = -P^{\beta \alpha \gamma \delta} = P^{\gamma \delta \alpha \beta} \).

Now we manipulate each term in (3.18) separately
\[
\delta \tilde{Q}^{\mu \nu} = \sqrt{-g} \left[ \delta g P^{\mu \nu \alpha \beta} \nabla_\alpha \epsilon_\beta - 2 \delta P^{\mu \nu \alpha \beta} \nabla_\alpha \epsilon_\beta - 2 P^{\mu \nu \alpha \beta} \delta (g_{\beta \gamma} \nabla_\alpha \epsilon_\gamma) \right],
\]
\[
= \sqrt{-g} \left[ \delta g P^{\mu \nu \alpha \beta} \nabla_\alpha \epsilon_\beta - 2 \delta P^{\mu \nu \alpha \beta} \nabla_\alpha \epsilon_\beta + 2 P^{\mu \nu \alpha \beta} \delta g_{\beta \gamma} \nabla_\alpha \epsilon_\gamma + 2 P^{\mu \nu \alpha \beta} \nabla_\alpha \delta g_{\beta \gamma} \epsilon_\lambda \right].
\] (3.23)

where, to compute \( \delta (\nabla_\alpha \epsilon_\lambda) = \delta \Gamma^\lambda_{\alpha \beta} \epsilon_\beta \), we used the formula \( \delta \Gamma^\lambda_{\alpha \beta} = \frac{1}{2} g^{\lambda \sigma} (\nabla_\beta g_{\alpha \sigma} + \nabla_\alpha g_{\beta \sigma} - \nabla_\sigma g_{\alpha \beta}) \). We also used \( \delta g_{\alpha \beta} = -g_{\alpha \sigma} \delta g^{\alpha \sigma} g_{\beta \gamma} \). On the other hand, the second term in the expression (3.18), reads
\[
2 \xi^{[\mu} P^{\nu]} g^{\alpha \beta} \nabla_\nu \delta g_{\alpha \beta} = 4 \sqrt{-g} \xi^{[\mu} P^{\nu]} \gamma_\beta \nabla_\gamma \delta g^{\alpha \beta},
\] (3.24)
where we used \( P^{\mu \nu \alpha \beta} = -P^{\mu \nu \beta \alpha} \) and \( \delta g_{\alpha \beta} = -g_{\alpha \sigma} \delta g^{\alpha \sigma} g_{\beta \gamma} \). Plugging back (3.23) and (3.24) into (3.18), the final result is
\[
k^{\mu \nu}_\xi = \sqrt{-g} \left[ \delta g P^{\mu \nu \alpha \beta} \nabla_\alpha \epsilon_\beta - 2 \delta P^{\mu \nu \alpha \beta} \nabla_\alpha \epsilon_\beta + 2 P^{\mu \nu \alpha \beta} \delta g_{\beta \gamma} \nabla_\alpha \epsilon_\gamma + 2 P^{\mu \nu \alpha \beta} \nabla_\alpha \delta g_{\beta \gamma} \epsilon_\lambda \right]
\]
(3.25a)

Analogously to General Relativity, the conservation law \( \partial_\mu k^{\mu \nu} \approx 0 \) is fulfilled when the Killing equation holds, \( \mathcal{L}_\xi g_{\mu \nu} = 0 \) (exact symmetry condition). A short calculation shows that, as expected, in four dimensions the replacement of \( P^{\mu \nu \alpha \beta} = \frac{1}{2} g^{[\mu \alpha} g^{\nu \beta]} \) produces exactly the formula for \( k^{\mu \nu}_\xi \) given by (3.3).

We finish the section with an aside comment. Let us establish the connection of the formula just found and the black hole entropy for LL theories. The observation made in [7], is that eternal black holes possess a bifurcated horizon surface where the horizon generator, which is proportional to a Killing vector, vanishes there, \( \tilde{\xi} \big|_{bf} = 0 \). Over that surface, the formula (3.18) is trivially integrable and therefore the so-called Noether potential, \( \tilde{Q}^{\mu \nu}_\xi \), is enough to compute the full associated charge. This analysis can be generalized to LL theories, see [22] and [23]. A further analysis allows for the obtained quantity to be interpreted as the entropy of the black hole. Here we stress that although the formula simplifies on the bifurcated horizon, one is also able to compute the same value for the entropy outside the horizon, on an arbitrary closed surface containing the black
The reason is that the full formula is protected by the conservation while the Noether potential is not.

4 Gravity Theories in Differential Form Language

Theories of gravity are mainly studied in the metric formalism, where the metric tensor is the dynamical variable describing the gravitational field. However, they can also be recast using the language of differential forms. In the the most common alternative, known as the Cartan formulation of gravity, the metric tensor is traded for tetrads and spin connections as dynamical variables in the action principle.

In this section we also consider in the form language general Chern-Simons (CS) theories in arbitrary dimension and the BF theories. To recover gravity in the CS formulation the tetrad must be encoded in the algebra valued connection field. In the BF formulation, instead, the metric is directly traded by a two-form field and an one-form connection.

Coming back to the Cartan formulation of gravity, though less preferred in the literature, it has undoubtedly advantages to provide an explicit coordinate-invariant description, to describe the coupling with fermionic matter fields, and to study dynamical and non-dynamical torsion.

In the following we start by presenting theories of gravity in tetrad formalism. The reader might appreciate the comparison between the two above-mentioned formalism, and the benefits and disadvantages in either formulations. We aim at presenting the surface charge formulas for the Einstein-Cartan theory coupled to electromagnetic and matter fields. Many of the formulas in this section are equivalent to those derived in the previous section with the metric formalism. We comment about some features, highlighted by the use of differential forms, by deepening on direct consequences of the formulas themselves.

For example, in the case of a pure gravity theory and for asymptotically (anti)-de Sitter spacetimes, we show that the corresponding surface charge evaluated in the asymptotic region gets a very compact form, (4.8), as recently noted in [24, 25]. A new result, as far as the authors’ knowledge concerns, is about torsional gravity theories: We show two relevant examples where the charges are unaffected by the presence of non-dynamical torsion fields.

Preliminaries on Einstein-Cartan formalism

Here we introduce the basics of the Einstein-Cartan formulation of gravity; for a complete review we suggest [26]. In order to write Einstein-Cartan with forms the metric field should be replaced by a vielbein field, $e^{a}_{\mu}$ satisfying $g_{\mu\nu} = e^{a}_{\mu} e^{b}_{\nu} \eta_{ab}$. This variable exhibits a local Lorentz transformation as a gauge symmetry, which is nothing but the symmetries of the tangent space of the manifold at each spacetime point. In four dimensions, with internal indices $a, b, c, \cdots = 0, 1, 2, 3$, the vielbein are in fact the coordinate components of four
one-form fields, the coframe field, \( e^a = e^a_\mu dx^\mu \), which are at the same time an orthogonal basis at each point of the cotangent space.

The elegance of the differential form language for gravity is clear when one adopts the first-order formulation. To have first-order equations of motion, besides \( e^a \), we introduce another one-form field called the spin connection field \( \omega^{ab} = -\omega^{ba} = \omega^{ab}_\mu dx^\mu \). Then, we can define the covariant exterior derivative, \( d_\omega(\cdot) = d(\cdot) + \omega(\cdot) \), which acting on a vielbein defines the torsion field \( T^a \equiv d_\omega e^a \equiv de^a + \omega^a_b \wedge e^b \), and with it we can also define the spacetime curvature two-form \( R^{ab} \equiv d\omega^{ab} + \omega^a_c \wedge \omega^{cb} \).

### 4.1 Einstein-Cartan-Λ

The Einstein-Hilbert action with cosmological constant (3.2) is equivalent to the so-called Einstein-Cartan action with cosmological constant

\[
S[e^a, \omega^{ab}] = \kappa' \int_\mathcal{M} \varepsilon_{abcd} \left( R^{ab} e^c e^d + \frac{1}{2\ell^2} e^a e^b e^d e^d \right),
\]

where the wedge product among forms is left implicit, for instance \( e^a e^b = e^a \wedge e^b = -e^b \wedge e^a \).

The constants are related to the old ones by \( \kappa' = \kappa/4 = c^4/(32\pi G) \) and \( \ell^2 = 3/|\Lambda| \). The ± signs correspond to negative and positive cosmological constants, respectively. As expected from the metric analysis, the surface charges will not depend on the cosmological constant in the Einstein-Cartan formalism either.

In the metric formalism, spacetime symmetries or isometries are encoded in the Killing equation. The general wording used to refer to it on arbitrary fields is the exact symmetry condition. As we showed before, it is not always just a Lie derivative because when a local gauge symmetry is present and a gauge transformation easily spoils the symmetry condition. For the vielbein and connection variables the exact symmetry condition that is gauge invariant is

\[
\begin{align*}
\delta_\epsilon e^a &= d_\omega(\xi \cdot e^a) + \xi(d_\omega e^a) + \lambda^a_b e^b = 0 \quad (4.2) \\
\delta_\epsilon \omega^{ab} &= \xi \cdot R^{ab} - d_\omega \lambda^{ab} = 0 \quad (4.3)
\end{align*}
\]

which in fact can be understood as a Lie derivative on forms (Cartan magic formula: \( \mathcal{L}_\epsilon e^a = d(\xi \cdot e^a) + \xi(d_\omega e^a) \)) plus a specific infinitesimal Lorentz transformation, \( \delta_\lambda e^a = \lambda^a_b e^b \) generated by the field dependent parameter \( \lambda^{ab} = \lambda^{ab}_\mu dx^\mu \). We group both parameters in \( \epsilon = (\xi, \lambda^{ab}) \), such that \( \delta_\epsilon \equiv \mathcal{L}_\xi + \delta_\lambda + \xi \cdot \omega \). This combination of infinitesimal transformation is just a convenient prescription, sometimes called improved transformation, and it has the advantage of being homogeneous under local Lorentz transformations, \( \delta_\epsilon(\Lambda^{a}_b e^b) = \Lambda^{a}_b \delta_\epsilon e^b \), which is crucial to keep the local Lorentz gauge symmetry explicitly free while imposing the exact symmetry. If we do not do this a local Lorentz transformation would change the Killing equation and one has to keep track of the extra piece in all formulas. A simple analysis shows that the exact symmetry condition \( \delta_\epsilon e^a = 0 \) implies the usual Killing equation on the metric field

\[
\mathcal{L}_\xi g = \mathcal{L}_\xi e^a \otimes e_a + e^a \otimes \mathcal{L}_\xi e_a = \delta_\epsilon e^a \otimes e_a + e^a \otimes \delta_\epsilon e_a = 0, \quad (4.4)
\]

\[\text{We use the notation } \xi \cdot \eta = i_\xi \eta \text{ for the interior product, for instance } \xi \cdot e^a = \xi^a e^a_\mu.\]
where we used that $g = g_{\mu\nu}(dx^\mu \otimes dx^\nu) = \eta_{ab}(e^a \otimes e^b)$ with $\otimes$ the symmetric tensor product, and the anti-symmetry of $\lambda^{ab}$ such that $\delta \epsilon^a \otimes e_a = \lambda^{ab} e_b \otimes e_a = 0$. Note also that if the connection can be expressed in terms of the vielbein, $\omega^{ab}(e)$, the condition $\delta \omega^{ab} = 0$ is a trivial consequence of $\delta \epsilon^a = 0$. And note also that the equation is linear and thus it is straightforward to solve for the parameter $\lambda^{ab}$; in fact $\lambda^{ab} = e^a \bullet^\sharp (d\omega^{ab}(\xi \otimes e))$ which is equivalent to $\lambda^{ab} = e^a \gamma^\mu e^b\nabla^\mu \xi^a$, where the anti-symmetry is an explicit consequence of the Killing equation $\nabla_{(\mu} \xi_{\nu)} = 0$.

Now, we consider the surface charge density. For the reader interested in the details we refer to the Appendix F, where the surface charge density for this theory is worked out step by step. The final result is simply (see [27, 28])

$$\hat{k}_e = -\kappa' \epsilon_{abcd} \left( \lambda^{ab} \delta(e^c e^d) - \delta \omega^{ab} \xi_{c} \otimes (e^c e^d) \right)$$

or equivalently

$$\hat{k}_e = -2\kappa' \epsilon_{abcd} \left( \lambda^{ab} \delta e^c - \delta \omega^{ab} \xi_{c} \otimes e^c \right) e^d.$$

Thus, as mentioned earlier, it does not depend explicitly on the cosmological constant. In particular, if there is no a cosmological constant term in the action, the formula for the surface charge density is the same one.

The previous formula can be used to define charges in wherever spacetime region the exact symmetry condition holds. If the spacetime is assumed to have an exact symmetry with parameters $\epsilon$ defined on a limited region, in that region the surface charge density will satisfy $dk_e = 0$ and therefore a surface charge may be defined. In particular for exact solutions with exact symmetries, like simple black holes solutions, the surface charge defines charges quasi-locally: an asymptotic analysis is not needed, and the charge is defined at any two-surface enclosing the black hole. This makes surface charge a useful tool to compute the charges in spacetimes with complicated asymptotic structures.

As a final comment, the surface charges are insensitive to the addition of topological terms (Euler/Gauss-Bonnet, Nieh-Yan, Pontryagin, etc) or any boundary term in the Lagrangian. This may not be completely clear at first sight, because in some cases the surface charge density formula gets modified. Yet, it can be shown explicitly that those modifications disappear after the spacetime integration is done. The proof of this statement can be found in [27].

### 4.1.1 Asymptotically (Anti)-de Sitter Spacetimes

Now, that we have the simple formula at hand, we may consider go to particular situations. Depending of the physical problem we are interested in, further conditions on the fields or even restriction on the gauge symmetries, might be imposed. With them, there is a chance for the formula to become even simpler.

Let us assume that at the spacetime boundary, $\partial M$, the curvature is constant, namely

$$R^{ab}\Big|_{\partial M} = \frac{1}{\ell^2} e^a e^b\Big|_{\partial M}$$

(4.7)
where the plus sign is for de Sitter (positive curvature) and the minus sign is for anti-de Sitter (negative curvature). This is nothing but the equivalent of the constant curvature condition $R_{\mu\nu} = \pm |\Lambda| g_{\mu\nu}$ in form language. To simplify the notation, we define the (anti)-de Sitter curvature as $ar{R}^{ab} = R^{ab} \pm \frac{1}{\ell^2} \epsilon^{a} e^{b}$. With the asymptotic condition at hand, the surface charge density for pure gravity with cosmological constant (4.5), may be evaluated at the asymptotic region. There, it gets the simpler expression

$$\hat{k}_{\epsilon} = \pm \ell^2 \kappa \epsilon_{abcd} \delta \bar{R}^{ab} \lambda^{cd}$$

(4.8)

This beautiful formula is remarkable because in that compact expression is encoded the same information that one would compute in the asymptotic region using all the terms appearing in (3.3) for the metric formalism. This expression is the equivalent one in differential forms language of the result recently obtained by Altas and Tekin [24, 25], and of course it strongly uses a non-vanishing cosmological constant.

4.2 Einstein-Cartan-Maxwell

The electromagnetic field is described by the one-form potential $A$ and the field strength is simply the exterior derivative of the potential, $F = dA$. The Einstein-Cartan action coupled to the electromagnetic field is

$$S[e^a, \omega^{ab}, A] = \int_{M} \left( \kappa' \epsilon_{abcd} R^{ab} e^{c} e^{d} + \alpha F \star F \right),$$

(4.9)

with $\alpha = -1/2$. The coupling with the vielbein field in the second term is through the Hodge star $\star$. Explicit components on the frame field are $F = \frac{1}{2} F_{abc} e^{a} e^{b}$, and the Hodge dual is $\star F = \frac{1}{4} \epsilon_{abcd} F^{ab} e^{c} e^{d}$ (see Appendix A for conventions).

To impose the exact symmetry conditions we still should impose the equations $\delta_{\epsilon} e^a = 0$ and $\delta_{\epsilon} \omega^{ab} = 0$ as in (4.2) and (4.3), but now we have the field $A$ with its own extra gauge symmetry. The corresponding infinitesimal gauge transformation is $\delta_{\lambda} A = -d\lambda'$. Therefore, besides the two previous conditions, we should add a third exact symmetry condition directly on the potential, namely

$$\delta_{\epsilon} A = \xi \lambda' F - d\lambda = 0.$$  

(4.10)

Once more for the symmetry condition, we use the improved transformation that is the composition of a Lie derivative and a particular $U(1)$ gauge transformation with parameter $\lambda' = \lambda + \xi \lambda A$; explicitly $\delta_{\epsilon} A = \mathcal{L}_{\xi} A + \delta_{\epsilon} (\lambda + \xi \lambda A) A = d(\xi \lambda A) + \xi \lambda dA - d(\lambda + \xi \lambda A)$. In this set up we group together all the parameters as $\epsilon = (\xi, \lambda^{ab}, \lambda)$ although $A$ does not transform with local Lorentz and $e^a$ or $\omega^{ab}$ do not change with $U(1)$ gauge transformation.

$^{12}$To explicitly prove (4.8) note that in the asymptotic region we have

$\epsilon_{abcd} \delta R^{ab} \lambda^{cd} = \epsilon_{abcd} (d_\omega \omega^{ab}) \lambda^{cd} = \epsilon_{abcd} \omega^{ab} d_\lambda \lambda^{cd} + d(\cdot) = \epsilon_{abcd} \omega^{ab} \xi \lambda R^{cd} + d(\cdot) = \pm \frac{1}{\ell^2} \epsilon_{abcd} \delta \omega^{ab} \xi \lambda (\epsilon^{cd}) + d(\cdot),$ 

where in the last two equalities we used the exact symmetry condition and the asymptotic condition. Then, we should remember that surface charge densities are defined as equivalence classes that are unaffected by adding exact forms, see (2.12), then we can disregard any $d(\cdot)$ term in the formula. Previous equation allows us to trade the second term in (4.5) and we obtain the desired result.
Then, the surface charge density is the sum of $k_e$, from (4.5), and an extra electromagnetic piece

$$k_e^{ECM} = k_e - 2\alpha(\lambda \delta * F - \delta A \xi * F)$$  \hspace{1cm} (4.11)

The derivation is presented in Appendix G for a general Yang-Mills theory. The surface charge for gravity coupled with extra fields is then given by the surface charge of the pure gravity plus the contributions from the additional fields. The reason for this is that these extra fields enter the boundary term $\Theta(\delta)$ in a linear way. An exception to this structure is the conformally coupled scalar field that we studied in the metric formalism.

### 4.2.1 Einstein-Cartan-Yang-Mills

The previous case of Einstein-Cartan-Maxwell theory is a particular case of the more general derivation of surface charges for Yang-Mills theories (YM). For completeness, we include the result here and work out the details of the derivation in the Appendix G.

This class of gauge theories involves a gauge symmetry described by a semi-simple Lie group $G$, usually the $SU(N)$ group. They are also described by a fundamental one-form gauge connection now valued on the algebra generators of the group, $A = A^i \tau_i$, the indices $i, j, k, \ldots = 1, 2, \ldots, \dim(G)$. The field strength is an algebra valued two-form, $F = F^i \tau_i$, defined as $F = dA + A \wedge A$, and the covariant derivative is $d_A(\cdot) = d(\cdot) + [A, (\cdot)]$ with $d$ the exterior derivative in spacetime and $[\cdot, \cdot]$ the algebra commutator. The Einstein-Cartan-Yang-Mills action is

$$S[e^a, \omega^{ab}, A] = \int_M \left( \kappa \varepsilon_{abcd} R^{ac} \omega^{bd} + \alpha_{YM} \langle F * F \rangle \right),$$  \hspace{1cm} (4.12)

where $*$ is again the Hodge operator, the bracket $\langle \cdot \rangle$ denotes the trace acting on the group generators $\langle F * F \rangle = \text{Tr}(\tau_i \tau_j) F^i * F^j$, and $\alpha_{YM}$ the coupling constant.

The action (4.12) is invariant under diffeomorphisms, local Lorentz transformation, and now also under the non-abelian gauge transformations with the infinitesimal realization $\delta \lambda A^i = -d_A \lambda^i = -d \lambda^i - [A, \lambda]^i$. We group all the associated infinitesimal parameters in $\varepsilon = (\xi, \lambda^{ab}, \lambda^i)$. Again we should impose an extra exact symmetry condition

$$\delta \varepsilon A^i = \xi_i F^i - dA \lambda^i = 0,$$  \hspace{1cm} (4.13)

which is the trivial generalization of (4.10) also with the improved prescription. Then, the surface charge density for gravity coupled to a YM theory is (Appendix G)

$$k_e^{YM} = k_e - 2\alpha_{YM} \langle \lambda \delta * F - \delta A \xi * F \rangle$$  \hspace{1cm} (4.14)

There is an interesting pattern on the structure of the surface charge densities: The contribution from the YM field is structurally similar to the pure gravity one. In fact, even for pure gravity, when considering a Macdowell-Mansouri action [29], that is, a Lagrangian of the form $\varepsilon_{abcd} \tilde{R}^{ab} \tilde{R}^{cd}$ (with $\tilde{R}^{ab}$ the (anti)-de Sitter field strength as defined after (4.7)), the surface charge density resembles even more this structure too (check it in Eq. (3.22) of [27]). As we show later, the same is true for BF theories. This constant pattern is explained by the similarity of the form of the Lagrangians.
It would be also possible to couple Einstein-Cartan theory with a scalar fields or Skyrme fields. However, we will not proceed along this direction. We prefer taking advantage of the differential form approach to discuss gravity theories with non-dynamical torsion.

4.3 Einstein-Cartan with Torsion: Two Examples

In the Einstein-Cartan theory, as a first order formulation, the connection $\omega^{ab}$ is an independent variable. Therefore, in this frame the torsion does not vanish in general. It is enough to have a source in the corresponding field equation to turn on the torsion field. While being a natural option to consider for the geometry of real spacetimes, so far, the torsion seems an elusive feature of physical spacetime and there is no experimental indication for it at the moment. Still, it is not ruled out and thus for the sake of generality it is worth to study.

Here, in particular, we wonder how is that surface charges get modified with the presence of torsion in spacetime? How the torsion affects the spacetime charges?. We do not have a general answer. But by studying two particular and quite different theories, the conclusion for both of them is that torsion field does not affect the general formula for the charges.

The role of torsion in the following charge formulas is analogous with the role played by the cosmological constant term: Although present in gravity theories, explicitly modifying the field equations and its respective solutions, it does not appear as a direct contributing term into surface charges.

The first theory we consider is a simple pure gravity example in $(2+1)$-spacetime where torsion is sourced without adding any extra fields. In fact the term, first introduced by Mielke and Baekler in [30], is built out of the same gravity fields. In a second example, now in $(3+1)$-dimensional spacetime, we consider the well-known Einstein-Cartan-Dirac theory where torsion is sourced by Dirac spinors.

For both theories we find the remarkable fact that torsion is explicitly absent from the charge formula (this result may be contrasted with the recent work [31]). In fact we go further and conjecture that torsion do not enter the charge formulas for theories with non-propagating torsion. The reason for this will be clear as we go into the details.

4.3.1 Einstein-Cartan in 2+1 plus a Torsion Term

Consider the gravitational action in three dimensions

$$S[e^a, \omega^{ab}] = \int_M \left( \varepsilon^{abc} e^a R_{bc}^{bc} + \beta e_a T^a \right),$$

(4.15)

for simplicity we set the overall parameter to one and introduce $\beta$ as the coupling constant for the new term, where $T^a = d_\omega e^a$. As matter of fact $e_a T^a$ can not be written as a boundary term. Furthermore, it produces a source for torsion, as can be checked in the equations of motion (H.2). This action can be seen as a sector of the more general Mielke-Baekler model [30] which contains two more terms: A cosmological constant term and a Chern-Simons term built with $\omega^{ab}$. A general analysis of the surface charges for the Mielke-Baekler model is straightforward but for our purposes this torsional extra term is enough. In fact, an even more elegant perspective can be done by writing the full Mielke-Baekler model as a Chern-Simons theory and use the results of the Chern-Simons Section 4.5 below.
Because the theory depends only on vielbein and spin connection fields, the corresponding exact symmetry conditions are just the same as before, (4.2) and (4.3), which are valid for any dimensions. The computation of the surface charge density is straightforward and is done in full detail in Appendix H. It reads

\[ k_\epsilon = -\epsilon_{abc}(\lambda^{ab}\delta e^c - \delta \omega^{ab}\xi,xe^c) + 2\beta\xi,xe^a\delta e_a, \]  

(4.16)

note that the first two terms have a similar structure than the four dimensional case (4.5), this is not casual: In Section 4.4 we exhibit the general formula in arbitrary dimensions, for Lovelock-Cartan theories.

As explained in the appendix we can go further and split \( \omega^{ab} = \tilde{\omega}^{ab} + \bar{\omega}^{ab} \), such that the torsionless part of the connection satisfies

\[ d\tilde{\omega}_e^a = de^a + \tilde{\omega}_a^bd_e^b = 0, \]

and we can use the equations of motion to solve algebraically the contorsion \( \bar{\omega}^{ab} \). With this, the surface charge formula simplifies to

\[ k_\epsilon = -\epsilon_{abc}(\tilde{\lambda}^{ab}\delta e^c - \delta \bar{\omega}^{ab}\xi,xe^c) \]

(4.17)

where \( \tilde{\lambda}^{ab} = e^a,b(d_\omega(\xi,xe^b)) \) is the parameter that solves the exact symmetry condition with just the torsionless part of the connection, (H.11). Then, (4.17) shows that the contorsion is absent from the seed formula for the charges, that is, even before computing it for any specific solution or symmetry. As we see in a moment this is also true in a completely different theory.

4.3.2 Einstein-Cartan-Dirac

The action for the massless Einstein-Cartan-Dirac theory in four dimensions is

\[ S[e^a,\omega^{ab},\psi] = \int_M \epsilon_{abcd}e^ae^b \left[ \kappa'R_{ed} - \frac{i}{3}\alpha_\psi e^c\left( \bar{\psi}\gamma^d\gamma_5d_\omega\psi + \overline{d_\omega\psi}\gamma^d\gamma_5\psi \right) \right], \]

(4.18)

with \( \alpha_\psi \) the coupling parameter and the \( \gamma \)-matrices satisfying \( \{\gamma_a,\gamma_b\} = \gamma_a\gamma_b + \gamma_b\gamma_a = 2\eta_{ab} \). The covariant derivative acting on spinors is \( d_\omega \psi = d\psi + \frac{1}{2}\omega^{ab}\gamma^{ab}\psi \), with \( \gamma_{ab} = \frac{1}{2}[\gamma_a,\gamma_b] \).

The special matrix \( \gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3 \) satisfies \( \gamma_5\gamma_a = -\gamma_a\gamma_5 \), and we use the bar to denote the complex conjugate.

Besides the exact symmetry condition on \( e^a \) and \( \omega^{ab} \) given by (4.2) and (4.3), we need to impose the exact symmetry condition directly on the spinor field

\[ \delta_\xi \psi = \mathcal{L}_\xi \psi + \lambda' \psi = \xi, d_\omega \psi + \lambda\psi = 0, \]

(4.19)

where again we used and improved version with the algebra valued parameter \( \lambda' = \frac{1}{2}\lambda^{ab}\gamma_{ab} = \frac{1}{2}(\lambda^{ab} + \xi,\omega^{ab})\gamma_{ab} \). Remember that the gauge symmetry for spinors is the local Lorentz symmetry, then \( \epsilon = (\xi, \lambda^{ab}) \). Spinors change under an infinitesimal Lorentz transformation as \( \delta_\lambda' \psi = \lambda' \psi \), with the algebra valued parameter \( \lambda' = \frac{1}{2}\lambda^{ab}\gamma_{ab} \). Hence, in this section \( \lambda' \) without indices is a matrix.

For the surface charge density the calculations are long, details are in Appendix I. The result is

\[ k_\epsilon = \hat{k}_\epsilon - i\alpha_\psi \epsilon_{abcd}\xi,xe^ae^b\delta^c\left( \bar{\psi}\gamma^d\gamma_5\psi \right), \]

(4.20)
which is again a simple modification of the Einstein-Cartan surface charge density produced by the spinor field. The new term comes directly from the spinor contribution to the boundary term in the varied action.

As we saw in the previous section we can go further if we consider the splitting \( \omega^{ab} = \widetilde{\omega}^{ab} + \bar{\omega}^{ab} \) such that \( \widetilde{\omega}^{ab} \) is the torsionless part of the connection and the contorsion field \( \bar{\omega}^{ab} \) is solved from the equations of motion. Replacing this back we find a cancellation to simply get (Appendix I)

\[
\kappa e = -\kappa' \varepsilon_{abcd} \left( \tilde{\lambda}^{ab} \delta(e^c e^d) - \delta \tilde{\omega}^{ab} \xi \langle e^c e^d \rangle \right)
\]

where again \( \tilde{\lambda}^{ab} = e^a \langle d \omega \rangle \langle e^b \rangle \). This surface charge density is exactly \( \tilde{k}_e \) but using on it the Levi-Civita connection, \( \tilde{\omega}^{ab} \), instead of the general connection \( \omega^{ab} \). Therefore, the conclusion for the Einstein-Cartan-Dirac theory is the same, contorsion leaves no trace on charges.

4.4 Lovelock-Cartan action

For completeness we also consider in the differential forms language the generalization of the Einstein-Cartan action in arbitrary dimensions: The Lovelock-Cartan action (see for instance [32]). Similarly than in the case of the metric formalism, as we increase the number of dimensions more and more terms are allowed into the action. A nice difference here is that in the differential form language the allowed terms are precisely those that one is able to write down with two restrictions: First, that each of them is a \( D \)-form, and second, that indices are contracted using the \( D \)-dimensional Levi-Civita tensor. This automatically guarantees that the obtained equations of motion are first order, as expected. This is to be contrasted with the case of the metric language where to have the expected second order equations of motion we need to precisely select the coefficients of the specific contracted Riemann curvature terms (e.g. for \( D = 5 \) the Gauss-Bonnet term is \( R^a_{\alpha \beta \gamma \delta} R_{\alpha \beta \gamma \delta} - 4 R^a_{\alpha \beta} R_{\alpha \beta} + R^2 \)). Thus, in this sense the form language, with vielbein and connection variables, imposes the natural restriction to specify the theory in arbitrary dimensions.

The Lovelock-Cartan action reads

\[
S[e^a, \omega^{ab}] = \int_M \sum_{p=0}^{[D/2]} L_p^D,
\]

with \([\cdot]\) denoting the integer part function, and where \( L_p^D \) is a \( D \)-form given by

\[
L_p^D = \kappa_p^D \varepsilon_{a_1\cdots a_D} R^{a_{p+1}a_2} \cdots R^{a_{2p-1}a_{2p}} e^{a_{2p+1}} \cdots e^{a_D}.
\]

Here \( a_1, a_2, \cdots = 0, \ldots, D - 1 \). Notice that for \( D = 4 \) to have (4.1) we just need to set \( \kappa_0^D = \pm \frac{1}{2\pi} \kappa' \) to recover the cosmological constant term, \( \kappa_1^D = \kappa' \) for the Einstein-Cartan term, and set \( \kappa_2^D = 0 \) to avoid the Euler topological density (also called Gauss-Bonnet). This is a pure gravitational action and the only fields present are the vielbein and the connection, then, the exact symmetry conditions are just \( \delta e^a = 0 \) and \( \delta \omega^{ab} = 0 \) as given by (4.2) and (4.3).
With the theory and the expressions for the exact symmetries at hand we can compute the surface charge density. For the derivation the steps are similar to those shown in Appendix F for the Einstein-Cartan-Λ theory. The final formula is

\[ k^{LC}_\varepsilon = - \sum_{p=1}^{[D/2]} p k^D_p \varepsilon_{a_1 \ldots a_D} (\lambda^{a_1 a_2} \delta - \delta \omega^{a_1 a_2} \xi) R^{a_3 a_4} \ldots R^{a_{2p-1} a_{2p}} e^{a_{2p+1}} \ldots e^{a_D}, \quad (4.24) \]

where both the \( \delta \) and \( \xi \) operations have been factorized and should be understood as acting on all terms on the right. Notice that, as expected, there is no \( p = 0 \) term which means that the cosmological constant term does not affect the formula for the charges in any dimension.

As noted in [27], the previous formula can be rewritten by noticing that each contributing term in the sum of (4.24) can be expressed as the sum of three terms, we suppress indices and write it schematically

\[ \varepsilon(\lambda \delta - \delta \omega \xi) R \cdots Re \cdots e = (D - 2p)\varepsilon(\lambda \delta e - \delta \omega \xi e) R \cdots Re \cdots e + (p - 1)\varepsilon(\lambda \delta e R \cdots Re \cdots e + \delta \omega \delta \omega R \cdots Re \cdots e). \]

The terms inside the curly brackets do not contribute to the surface charges: The first one is an exact form, and the second one is proportional to the exact symmetry condition. Therefore, an equivalent surface charge density is

\[
\begin{align*}
\varepsilon_{a_1 \ldots a_D} (\lambda^{a_1 a_2} \delta - \delta \omega^{a_1 a_2} \xi) R^{a_3 a_4} \ldots R^{a_{2p-1} a_{2p}} e^{a_{2p+1}} \ldots e^{a_D} = \\
(D - 2p)\varepsilon_{a_1 \ldots a_D} (\lambda^{a_1 a_2} \delta e^{a_3} - \delta \omega^{a_1 a_2} \xi e^{a_3}) R^{a_4 a_5} \ldots R^{a_{2p} a_{2p+1}} e^{a_{2p+2}} \ldots e^{a_D}
\end{align*}
\]

For even dimensions this formula has one less term than \( k_\varepsilon \) because in fact Lovelock-Cartan action contains topological terms: The generalization of the Euler/Gauss-Bonnet density. Their contributions to the surface charge density are cleaned out by the previous observation. Surprisingly, this second version of the formula is also obtained with the contracting homotopy operator method, which happens to be a direct and efficient method to get rid of spurious boundary terms appearing in the Lagrangian.

### 4.5 Chern-Simons action

In this section we discuss the surface charge formula for the Chern-Simons (CS) theory in \( 2 + 1 \) dimensions. Because the calculations are simple this example is pedagogical, therefore, we put all the details here. However, in the context of Chern-Simons theories it is also interesting to consider general CS theories for an arbitrary odd dimension, the calculations for those theories are also worked out in full detail but relegated to the Appendix J.

\[ \varepsilon_{a_1 \ldots a_D} (\lambda^{a_1 a_2} \delta - \delta \omega^{a_1 a_2} \xi) R^{a_3 a_4} \ldots R^{a_{2p-1} a_{2p}} e^{a_{2p+1}} \ldots e^{a_D} = \\
(D - 2p)\varepsilon_{a_1 \ldots a_D} (\lambda^{a_1 a_2} \delta e^{a_3} - \delta \omega^{a_1 a_2} \xi e^{a_3}) R^{a_4 a_5} \ldots R^{a_{2p} a_{2p+1}} e^{a_{2p+2}} \ldots e^{a_D}
\]

\[ + (p - 1)\varepsilon_{a_1 \ldots a_D} (\lambda^{a_1 a_2} \delta e^{a_3} R^{a_4 a_5} \ldots R^{a_{2p-1} a_{2p}} e^{a_{2p+1}} \ldots e^{a_D})
\]

\[ + \delta \omega^{a_1 a_2} \delta \omega^{a_3 a_4} R^{a_5 a_6} \ldots R^{a_{2p-1} a_{2p}} e^{a_{2p+1}} \ldots e^{a_D}. \quad (4.25) \]
Consider the Chern-Simons (CS) action in $D = 3$ dimensions

$$S[A] = \kappa_{CS} \int_M \langle A \wedge dA + \frac{2}{3} A \wedge A \wedge A \rangle,$$  \hspace{1cm} (4.26)

the one-form gauge connection $A$ is valued on the Lie algebra defining the theory, $\langle \cdot \rangle$ denotes a group invariant symmetric polynomial here of rank $r = 2$, in this dimension is also named the bilinear form; and $\kappa_{CS}$ the level of the theory which is not relevant for the classical analysis. Under a gauge transformation the CS action (4.26) is not invariant but quasi-invariant because it produces a boundary term, this is of course still a gauge symmetry of the theory as far as the equations of motion are concerned. In the following we consider diffeomorphisms and gauge symmetries and group them in $\epsilon = (\xi, \lambda)$, with $\xi$ a vector field and $\lambda$ a Lie algebra valued gauge parameter. The general infinitesimal symmetry transformation reads

$$\delta_\epsilon A = L_\xi A - d_A \lambda' = \xi \lrcorner F - d_A \lambda,$$  \hspace{1cm} (4.27)

notice that we use the exterior covariant derivative $d_A (\cdot) \equiv d (\cdot) + [A, (\cdot)]$ and define a displaced parameter as $\lambda = \lambda' - \xi \lrcorner A$ to work directly with the improved general transformation, as is our custom through these notes.

Now, the variation of the action produces the equation of motion $F = dA + A \wedge A = 0$ which holds only if one gets rid of the boundary term given by the potential $\Theta(\delta A) = \kappa_{CS} \langle A \wedge \delta A \rangle$. The symplectic structure density is simply $\Omega(\delta_1, \delta_2) = 2 \kappa_{CS} \langle \delta_1 A \wedge \delta_2 A \rangle$, which we evaluate with one of its entries on the gauge symmetry transformation (4.27)

$$\Omega(\delta, \delta_\epsilon) = 2 \kappa_{CS} \langle \delta A \wedge (\xi \lrcorner F - d_A \lambda) \rangle$$  \hspace{1cm} (4.28)

$$= 2 \kappa_{CS} d(\delta A \lambda),$$  \hspace{1cm} (4.29)

where to get second line we used the equation of motion, $F = 0$ and the linearized equation of motion $\delta F = d_A \delta A = 0$. Hence, for the CS theory in $D = 3$ we have the surface charge density

$$k_{CS}^{(3)} = 2 \kappa_{CS} \langle \lambda \delta A \rangle$$  \hspace{1cm} (4.30)

This simple formula covers all CS theories in 3-dimensions in the sense that the group algebra the theory is defined with is not specified yet. In particular we can choose the Poincaré or (anti-)de Sitter group to obtain the surface charge formula for general relativity in $(2 + 1)$-dimensions (as we do in the first torsion example). We leave it as an exercise but, of course, the result is exactly what we got in the Lovelock-Cartan theory for $D = 3$.

The previous derivation is a particular case of the more general derivation for CS theory in $D = 2n + 1$ dimensions. The details of the general calculation are explained in Appendix J. The general result for the surface charge density is

$$k_{CS}^{(2n+1)} = n(n + 1) \kappa_{CS} \langle \lambda \delta A F^{n-1} \rangle$$  \hspace{1cm} (4.31)

which could probably had been guessed, in fact we note there is also a very direct computation to get this result by using the contracting homotopy operator, see Appendix J.1. The infinitesimal symmetry transformation for the connection are the same (4.27) and the actions for these theories are compactly written in equation (J.1).
4.6 BF action

The BF theories are a set of first order alternative formulation of General Relativity where the one-form frame field (vielbein) is traded by a two-form field usually named B. A possible explicit relation among them is $B^{ab} \sim \varepsilon^{ab}_{\ cde} e^c e^d$ but as the $B^{ab}$ field is an independent variable this relation should be obtained as a consequence of the equations of motion. The $F$ letter in the name of the BF theory comes from the curvature two-form that originally was denoted by $F^{ab}$, although here we keep our notation $R^{ab}(\omega) = d\omega^{ab} + \omega^a_c \omega^{cb}$. All BF theories contain a kinetic term in the action of the form $B^{ab} R_{ab}$, originally the BF term. If this is the only term in the action the theory is pure topological: There are non-propagating degrees of freedom present. To recover true General Relativity, with two propagating degrees of freedom, very specific additional constraints and their corresponding Lagrange multipliers should be incorporated into the action, see (4.32) below.

There are at least two good motivations to consider BF theories. The first one is due to its close relation with pure topological theories, this property makes BF actions a very interesting starting point to perform direct quantizations within the path integral formalism. Usually those approaches are based on manifold discretizations, and require several prescriptions to define the fields over the simplicial objects. The models for gravity quantization based on BF theories are called Spin Foams [33]. A second motivation arises because of the very different geometric nature of BF theories. In particular it suggests new ways to generalize General Relativity action, and consider it as a special case of a completely different family of theories that would be impossible to grasp within the metric or frame field formalism [34] (we suggest also the talks [35]). In this sense, within the perspective of GR as an effective field theory, it is worth exploring new windows that allows us to frame the theory in wider contexts.

In addition, another interesting technical thing of BF theories is that they can be seen as intermediate formulations of GR, that further allow for the integration of the Lagrange multipliers and even the $B$ field, to finally get a pure connection formulation of GR [34] (and references therein). These are very special actions for GR whose applications have not been fully explored yet (see recent advances in [36]). In contrast with the BF, that are first order formulations, this pure connection is a second order formulation, as the metric one, and that spoils the simplicity of the formula for surface charge. Hence, here we focus on a BF example and we leave the longer but straightforward surface charge analysis of pure connection formulations for a future work.

To have taste of the BF theories here we pick a particular BF formulation known as Chiral BF theory which have the nice feature of being one of the simplest first order formulations of GR. We use the fact that the Lorentz group algebra can be decomposed as $\mathfrak{so}(3,1,\mathbb{C}) = \mathfrak{sl}(2,\mathbb{C}) \oplus \overline{\mathfrak{sl}(2,\mathbb{C})}$. Then, instead of the $B^{ab}$ and $\omega^{ab}$ variables we use a projector to define their dual and anti-selfdual components. The interesting observation is that GR is redundant in this splitting and to pick only one of the components is enough to describe the full theory. By choosing one of them we obtain a chiral BF theory, which is more economic in components that the full BF. All the details of the equivalence of chiral BF theory and the Einstein-Cartan-\Lambda formulation are in the Appendix K.
The action for a chiral BF formulation of GR is
\[ S[B^i, A^i, \chi^{ij}] = \int_M \left( B^i F^i + \frac{1}{2\ell^2} B^i B^i + \frac{1}{2} \chi^{ij} B^i B^j \right), \tag{4.32} \]
where repeated indices implies summation and the internal metric is just $\delta_{ij}$ so we do not make any difference among upstairs or downstairs indices. The $B^i$ field is a self-dual two-form and with the self-dual spin connection one-form $A^i$ we define the curvature $F^i = dA^i + \frac{1}{2} \varepsilon^{ijk} A^j A^k$.

As before the $\pm$ sign stands for both possible signs of the cosmological constant. The field $\chi^{ij}$ is a zero-form Lagrange multiplier demanded to be traceless $\chi^{ij} \delta_{ij} = 0$. This theory is equivalent to gravity.

To obtain the surface charge density for chiral BF we need to establish the exact symmetry condition for all fields involved, however, as the fields $\chi^{ij}$ is non-dynamical it does not appear in the symplectic structure. Thus, it is enough to impose the improved exact symmetry conditions
\[ \delta_\epsilon A = \xi^i F - d_{\lambda} \lambda = 0 \]
\[ \delta_\epsilon B = \xi^i d_A B + d_A (\xi^i B) + [\lambda, B] = 0. \tag{4.34} \]
Then, the surface charge density, see Appendix K for details, is simply
\[ k_{\epsilon}^{\text{ChBF}} = -\delta B^i \lambda^i + \delta A^i \xi^i B^i \tag{4.35} \]
This expression is the most compact surface charge density formula for pure GR we have found so far, and it is certainly like this because of the easiness in components of the chiral BF formulation of GR.

As a final remark, we note there is a curious connection between the previous surface charge density formula for the chiral BF theory in $(3 + 1)$-dimensions and the formula one gets in the case of the BF-like formulation of the $(1 + 1)$-gravitational Jackiw-Teitelboim model. First note that because $B$, $\lambda$ and $A$ are valued in the algebra we can also write the previous surface charge density as
\[ k_{\epsilon}^{\text{ChBF}} = -\langle \delta B, \lambda \rangle + \langle \delta A, \xi^i B^i \rangle, \tag{4.36} \]
where $\langle \cdot, \cdot \rangle$ is the trace on the algebra generators, therefore if we call $\tilde{X}_i$ the generators we have that $\langle \tilde{X}_i, \tilde{X}_j \rangle$ reduces to a Kronecker delta, $\delta_{ij}$, and previous formula is exactly (4.35). As explained in the Appendix K.3, the BF-like formulation of the Jackiw-Teitelboim model has a surface charge density given by
\[ k_{\epsilon}^{\text{JT}} = -\langle \delta B, \lambda \rangle = -\delta B^i \lambda^i \langle X_i, X_j \rangle = -\delta \tilde{B} \lambda - \delta B^a \lambda_a, \tag{4.37} \]
which is the formula one could deduce from the $3 + 1$ case remembering that in the $1 + 1$ theory the $B$ is a zero-form and therefore the second term in (4.36) must vanish. All the details of this statement are explicit in Appendix K.\textsuperscript{14} In the same spirit of Lovelock-Cartan actions, the previous link certainly suggest a general formula of surface charge density for BF theories in even dimensions, we leave it for a future work.

\textsuperscript{14}A particular difference is that for the $1+1$ BF-like gravity the $X_i$ are $\mathfrak{so}(2,1)$ algebra generators and thus the meaning of $\langle \cdot, \cdot \rangle$ changes accordingly.
5 Discussion

In this toolkit we have presented the method, known as the surface charge method, to compute charges for gauge theories. As explained at the beginning this method extends the old Noether program to theories with gauge redundancy: (Exact) symmetries implies charges also for gauge theories. Here we have reviewed precisely how this happens and extensively applied the method to different gravity theories.

The main difference with the First Noether Theorem is that in the surface charge method the construction is over a one-dimension lower submanifold. That is, for theories without gauge symmetries, the usual Noether current $J_\mu$ corresponds to a $(D-1)$-form and defines a charge over a $(D-1)$-dimensional spacetime slice $\Sigma$ as $Q_\epsilon = \int_\Sigma J_\mu d\Sigma_\mu$. In contrast, for gauge theories, we have the surface charge density $k_{\mu\nu}$, playing the role of a current, that corresponds to a $(D-2)$-form and defines a (differential) charge over a closed $(D-2)$-dimensional surface $S$ as $\delta Q_\epsilon = \oint_S k_{\mu\nu} dS_{\mu\nu}$. Thus, a second difference is that in the surface charge method, instead of finite charges, one defines differential charges on phase space. They require further phase space integration.

The integrability of $\delta Q$, to obtain a $Q$, is usually assumed in the literature, but as we saw in detail is not ensured by the method. Still for exact solutions the integration on phase space of charges is achieved easily in most cases.

In the context of asymptotic charges, which are a consequence of asymptotic symmetries, a lot of work is focused on providing the conditions to have integrable asymptotic charges. In contrast insufficient effort is put in providing an explanation of what those non-conserved asymptotic charges mean. Through these notes it is clear that to have a conservation law an exact symmetry equation is needed; an equation that by definition asymptotic symmetries do not satisfy.

It is on the purposes of these notes to bring clarity on the surface charge method in order to focus its applicability on physical systems in the context of current research. But also this toolkit is written in the hope to easy the work of gravity physicist, specially to those who decide to use alternative variables, and need to analyze the field equation solutions in the light of gauge invariant quantities. It is in this spirit that we systematically wrote the surface charge density formulas for many gravity theories, presented all of them in a Table, and furthermore we also performed all the step-by-step calculation in the appendices.

A key field where the results here are of direct use is in analyzing black holes, more specifically, their thermodynamic properties. The surface charge method allows us to have good control of the space of exact symmetries and the space of differential charges at once. In the case of black holes this analysis can lead to a well-posed first law of black hole mechanics.\textsuperscript{15}

\textsuperscript{15}See [52] for a general treatment or [53] for a recent successful application on rotating magnetic black hole that has a non-usual asymptotic description.
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A Notation and conventions

For a $D$-dimensional spacetime, we use the signature convention for the flat metric $\eta_{ab} = \text{diag}(-1, 1, \ldots, 1)$ and the Levi-Civita totally anti-symmetric symbol $\varepsilon_{\mu_1 \cdots \mu_D}$ such that $\varepsilon_{01 \cdots (D-1)} = 1$, we also have

$$\varepsilon_{\mu_1 \cdots \mu_D} = \bar{\varepsilon}^{\mu_1 \cdots \mu_D} \quad (A.1)$$

$$g^{\mu_1 \nu_1} \cdots g^{\mu_D \nu_D} \varepsilon_{\mu_1 \cdots \mu_D} = \frac{1}{g} \bar{\varepsilon}^{\nu_1 \cdots \nu_D} = \varepsilon_{\nu_1 \cdots \nu_D} \quad (A.2)$$

$$g_{\mu_1 \nu_1} \cdots g_{\mu_D \nu_D} \bar{\varepsilon}^{\mu_1 \cdots \mu_D} = g_{\nu_1 \cdots \nu_D} \varepsilon^{\nu_1 \cdots \nu_D} \quad (A.3)$$

with the spacetime metric determinant $g \equiv \det(g_{\mu \nu})$. We also introduced $\bar{\varepsilon}^{\mu_1 \cdots \mu_D}$ such that $\bar{\varepsilon}^{01 \cdots (D-1)} = 1$, this twiddle symbol is exactly the Levi-Civita symbol but with indices written upstairs. In contrast, $\varepsilon^{\mu_1 \cdots \mu_D}$ is a spacetime function, not the Levi-Civita symbol, its indices are raised with the spacetime metric. Note that we use Greek letters for spacetime indices and Latin letters for internal indices. Thus, similarly to (A.2), to raise and lower indices with the internal flat metric, yields

$$\eta^{a_1 b_1} \cdots \eta^{a_D b_D} \varepsilon_{a_1 \cdots a_D} = \det(\eta)^{-1} \bar{\varepsilon}^{b_1 \cdots b_D} = -\bar{\varepsilon}^{b_1 \cdots b_D} = \varepsilon^{b_1 \cdots b_D}. \quad (A.4)$$

The introduction of the object $\bar{\varepsilon}$ is highly recommended as a way to keep consistent the Einstein notation of index contraction and thus to avoid some usual confusions on the computations.

A.1 Three differential forms operations

Let us consider a $p$-form expressed in a coordinate basis

$$\alpha = \frac{1}{p!} \alpha_{\mu_1 \cdots \mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}. \quad (A.5)$$

The operator exterior derivative on differential forms, $d : \Omega^p \rightarrow \Omega^{p+1}$, has the explicit action

$$d\alpha = \frac{1}{p!} \partial_{\mu_0} \alpha_{\mu_1 \cdots \mu_p} dx^{\mu_0} \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}. \quad (A.6)$$

This operator satisfies the following Leibniz rule $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$, and it is nilpotent, $d^2 = 0$. 

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The Hodge dual operator $\star : \Omega^p \rightarrow \Omega^{D-p}$, acts on a $p$-form as

$$\star \alpha = \frac{1}{(D-p)! p!} \epsilon^{a_1 \cdots a_p} g_{a_1 \cdots a_p b_1 \cdots b_{D-p}} e^{b_1} \wedge \cdots \wedge e^{b_{D-p}}, \quad (A.7)$$

or with the differential form expressed in a coordinate basis

$$\star \alpha = \frac{1}{(D-p)! p!} \sqrt{|g|} \epsilon^{\mu_1 \cdots \mu_p} g_{\mu_1 \cdots \mu_p \nu_1 \cdots \nu_{D-p}} dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_{D-p}}. \quad (A.8)$$

For a vector field $\xi = \xi^a \partial_a$, the interior product on forms is either denoted by $i_\xi$ or also with the alternative notation $\xi \cdot$. This operation lowers by one the form degree $\iota_\xi \equiv \xi : \Omega^p \rightarrow \Omega^{p-1}$. Explicitly, on a $p$-form it has the action

$$\iota_\xi \alpha = \frac{1}{(p-1)!} \epsilon^{\mu_2 \cdots \mu_p} g_{\xi \mu_2 \cdots \mu_p} dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_p}. \quad (A.9)$$

### B Einstein-Hilbert-$\Lambda$ action

Consider the Einstein-Hilbert-$\Lambda$ action

$$S[g_{\mu\nu}] = \frac{\kappa}{2} \int_M dx^D \sqrt{-g} (R - 2\Lambda). \quad (B.1)$$

We first need to compute the variation of the Lagrangian generated by an arbitrary vector field $\xi = \xi^\mu \partial_\mu$. One has, from Eq. (2.2), that

$$\partial_\mu (\xi^\mu L) = E_{\mu\nu} \delta g^{\mu\nu} + \nabla_\mu \Theta^\mu (g, \delta g),$$

$$= -2E_{\mu\nu} \nabla^{(\mu} \xi^{\nu)} + \nabla_\mu \Theta^\mu (g, \delta g),$$

$$= -\nabla^{(\mu} (2E_{\mu\nu} \xi^{\nu)}) + 2\xi^{(\nu} \nabla_\mu E_{\mu\nu} + \nabla_\mu \Theta^\mu (g, \delta g),$$

$$= -\nabla^{(\mu} (2E_{\mu\nu} \xi^{\nu}) + \nabla_\mu \Theta^\mu (g, \delta g),$$

$$= \nabla^\mu [2E_{\mu\nu} \xi^{\nu} + \Theta_\mu (g, \delta g)]. \quad (B.2)$$

In the second line we replaced $\delta g^{\mu\nu} = -\nabla^\mu \xi^\nu - \nabla^\nu \xi^\mu = -2\nabla^{(\mu} \xi^{\nu)}$, in the third line we made use of the Leibniz rule, and in the fourth line we used the symmetry of $E_{\mu\nu}$ and the Noether (Bianchi) identity $\nabla^\mu E_{\mu\nu} = 0$.

The explicit expressions of the quantities in Eq. (B.2) are

$$L = \frac{\kappa}{2} \sqrt{-g} (R - \Lambda), \quad (B.3)$$

$$E_{\mu\nu} = \frac{\kappa}{2} \sqrt{-g} \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} \right), \quad (B.4)$$

$$\Theta^\mu (g, \delta g) = \kappa \sqrt{-g} \nabla^\alpha (g^{[\mu} \delta g_{\alpha]}) = \kappa \sqrt{-g} \left( \nabla_\alpha \nabla^\alpha \xi^\mu - \nabla^\mu \nabla_\alpha \xi_\alpha \right), \quad (B.5)$$

---

$^{16}$Notice that from the variation of the metric, $\delta g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$, and from the identity $0 = \delta (\delta^\nu) = \delta g^{\mu\nu} g_{\mu\nu} + g^{\mu\nu} \delta g_{\mu\nu}$, the variation of the inverse metric gets a minus sign.
where we used the identity \([\nabla_\alpha, \nabla^\mu]\xi^\alpha = R^\mu\nu\xi^\nu\). As seen in the general case, Eq. (2.4), and using Eqs. (B.3)-(B.4)-(B.5), one has the trivially conserved current

\[
J^\mu_\xi = \Theta^\mu(g, \delta g) - \xi^\mu L - 2\xi_\mu E^{\mu\nu},
\]

\[
= \kappa\sqrt{-g}\nabla_\nu \nabla^{[\nu} \xi_{\alpha]},
\]

\[
= \kappa \partial_\nu \left( \sqrt{-g} \nabla^{[\nu} \xi_{\alpha]} \right) = \kappa \partial_\nu \left( -\sqrt{-g} \nabla^{[\nu} \xi_{\alpha]} \right).
\]

(B.6)

The Noether potential

\[
\tilde{Q}^{\mu\nu}_\xi = -\kappa\sqrt{-g}\nabla^{[\mu} \xi_{\nu]},
\]

(B.7)

because its anti-symmetry the current is trivially conserved, \(\partial_\mu \partial_\nu \tilde{Q}^{\mu\nu}_\xi = 0\), without using the equations of motion.

Finally, the surface charge density is given by

\[
k^{\mu\nu}_\xi = \delta \tilde{Q}^{\mu\nu}_\xi + 2\xi^{[\mu} \Theta^{\nu]}(g, \delta g)
\]

\[
= -\kappa \delta \left( \sqrt{-g} \nabla^{[\mu} \xi_{\nu]} \right) + 2\kappa \sqrt{-g} \xi^{[\mu} \nabla^{(\alpha} \left( g^{\nu]\beta} \delta g_{\alpha\beta} \right).
\]

(B.8)

To compute the variation of the first term we use

\[
\delta (\nabla_\alpha \xi^\nu) = \delta \Gamma^\nu_{\alpha\gamma} = \frac{1}{2} g^{\nu\lambda} (\nabla_\gamma \delta g_{\alpha\lambda} + \nabla_\alpha \delta g_{\gamma\lambda} - \nabla_\lambda \delta g_{\alpha\gamma}) \xi^\gamma,
\]

(B.9)

we insert this in the first term of (B.8) as

\[
\delta \left( \sqrt{-g} \nabla^{[\mu} \xi_{\nu]} \right) = \delta \left( \sqrt{-g} g^{[\mu} \nabla_{\alpha} \xi_{\nu]} \right),
\]

(B.10)

\[
= \sqrt{-g} \left( -\frac{1}{2} \delta g \nabla^{[\mu} \xi_{\nu]} + \delta g^{[\mu} \nabla_{\sigma} \xi_{\nu]} - \xi_{\sigma} \nabla^{[\mu} \delta g_{\sigma]} \right),
\]

(B.11)

with \(\delta g \equiv g_{\mu\nu} \delta g^{\alpha\beta}\), then we replace the result in (B.8) to get the final expression of the surface charge in Eq. (3.3).

C Einstein-Hilbert-Maxwell action

The variation of the Lagrangian generated by an arbitrary vector field \(\xi = \xi^\mu A_\mu\) and gauge transformation \(\lambda' = \lambda + \xi^\mu A_\mu\) (with \(\delta_\epsilon = \delta_\xi + \delta_\lambda\))

\[
\delta_\xi L = E_{\mu\nu} \delta g^{\mu\nu} + E_\mu \delta_\epsilon A^\mu + \partial_\mu \Theta^\mu(\delta_\epsilon),
\]

(C.1)

\[
\partial_\mu (\xi^\mu L) = -E_{\rho\nu} (\nabla^\mu \xi^\nu + \nabla^\nu \xi^\mu) + E_\mu (\xi^\nu F^{\mu\nu} - \nabla^\mu \lambda) + \partial_\mu \Theta^\mu(\delta_\epsilon),
\]

(C.2)

\[
\partial_\mu (\xi^\mu L) = \partial_\mu[-2(\xi^\nu E^{\mu\nu}) - E^{\mu\nu} \lambda + \Theta^\mu(\delta_\epsilon)],
\]

(C.3)

where we used the corresponding Noether identities \(F^{\mu\nu} E_{\nu} = 2 \nabla_\mu E^{\mu\nu} = 0\) and \(\nabla_\mu E^{\mu} = 0\). The explicit functions in (C.3) are
\[ L = \sqrt{-g} \left( \frac{\kappa}{2} R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right), \quad (C.4) \]

\[ E^{\mu\nu} = \sqrt{-g} \left[ \frac{\kappa}{2} \left( R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) - \frac{1}{2} \left( g_{\alpha\beta} F^{\alpha\mu} F^{\beta\nu} - \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right) \right], \quad (C.5) \]

\[ E^\mu = \sqrt{-g} \nabla_\nu F^{\mu\nu}, \quad (C.6) \]

\[ \Theta^\mu(\delta_\epsilon) = \sqrt{-g} \left[ \kappa \nabla^\alpha (g^{\mu\nu} \delta g_{\alpha\beta}) - \delta \epsilon A_\epsilon F^{\mu\nu} \right], \quad (C.7) \]

Replacing them we note that the three terms inside the total derivative in (C.3) can be rewritten as a total derivative too

\[ J_\epsilon^\mu \equiv \Theta^\mu(\delta_\epsilon) - \xi^\mu L - 2 \xi_\nu E^{\mu\nu} - \lambda E^\mu = \partial_\nu \left[ -\sqrt{-g} \left( \kappa \nabla^\alpha \xi^\alpha + \lambda F^{\mu\nu} \right) \right], \]

where we used \([\nabla^\mu, \nabla^\nu] \xi_\mu = R^{\mu\nu} \xi_\mu\). The term inside the total derivative is \( \tilde{Q}_\xi^{\mu\nu} \). Then we obtain the surface charge density

\[ k_\xi^{\mu\nu} = \delta \tilde{Q}_\xi^{\mu\nu} + 2 \xi_\nu \Theta^\mu(\delta_\epsilon), \quad (C.8) \]

\[ = -\delta \left[ \sqrt{-g} \left( \kappa \nabla^\mu \xi^\mu + \lambda F^{\mu\nu} \right) \right] + 2 \sqrt{-g} \xi^\mu \left( \kappa \nabla^\alpha (g^{\epsilon\nu}) \delta g_{\alpha\beta} - \delta A_\alpha F^{\mu\alpha} \right). \]

Note that when using improved transformations we have \( \delta \lambda = \delta (\lambda' - \xi^\mu A_\mu) = -\xi^\mu \delta A_\mu \).\(^{17}\)

Expanding all we obtain the surface charge density for this theory (3.8).

\[ \text{D Einstein-Hilbert action with a conformally coupled scalar field} \]

The variation of the Lagrangian (3.9) generated by an arbitrary vector field \( \xi = \xi^\mu \partial_\mu \) is

\[ \delta_\xi L = E_{\mu\nu} \delta_\xi g^{\mu\nu} + E_\Phi \delta_\xi \Phi + \partial_\mu \Theta^\mu(\delta_\xi), \]

\[ \partial_\mu(\xi_\mu L) = -E_{\mu\nu}(\nabla^\mu \xi^\nu + \nabla^\nu \xi^\mu) + E_\Phi \xi_\mu \nabla^\mu \Phi + \partial_\mu \Theta^\mu(\delta_\xi), \]

\[ \partial_\mu(\xi^\mu L) = \partial_\mu \left[ -2(\xi_\nu E^{\mu\nu}) + \Theta^\mu(\delta_\epsilon) \right], \quad (D.1) \]

where we used the corresponding Noether identity \( \frac{1}{2} \nabla^\mu \Phi E_\Phi - \nabla_\nu E^{\mu\nu} = 0 \).

The explicit expressions in (D.1) are

\(^{17}\) Equivalently we can forget this and note that in the symplectic structure density the variations do not commute, thus we should include a term \( \Theta([\delta_\epsilon, \delta_\epsilon]) \) which will contribute to the surface charge. The specific non-commutation in this case is \([\delta_\epsilon, \delta_\epsilon] A_\mu = \delta_\epsilon A_\mu = -\partial_\mu(\xi^\nu A_\nu) \).
\[ L = \frac{\kappa}{2} \sqrt{-g} \left( R - 2\Lambda \right) + \frac{1}{\kappa} \nabla^\mu \Phi \nabla_\mu \Phi + \frac{\zeta_\alpha}{\kappa} R \Phi^2 \]  
(D.2)

\[ E_{\mu\nu} = \frac{\kappa}{2} \sqrt{-g} \left[ \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) \left( 1 + \frac{\zeta_\alpha}{\kappa} \Phi^2 \right) + \Lambda g^{\mu\nu} \right. \]

\[ + \frac{1}{\kappa} \left( \nabla^\mu \Phi \nabla_\nu \Phi + \frac{1}{2} g^{\mu\nu} \nabla^\alpha \Phi \nabla_\alpha \Phi + \zeta_\alpha \left( g^{\mu\nu} \Phi^2 - \nabla^\alpha \nabla^{(\mu} \Phi) \Phi^{(\nu} \right) \right] \]  
(D.3)

\[ \Theta^\mu(\delta \xi) = \kappa \sqrt{-g} \left( \nabla^{[\alpha\beta]}(g^{\mu\beta} \partial_\xi g_{\alpha\beta}) \right) \left( 1 + \frac{\zeta_\alpha}{\kappa} \Phi^2 \right) - \frac{1}{\kappa} \delta_\xi \nabla^\mu \Phi + \frac{\zeta_\alpha}{\kappa} g^{\beta[\alpha \partial_\xi g_{\alpha\beta} \nabla^\beta]} \Phi^2 \]

\[ = \kappa \sqrt{-g} \left( \nabla^\alpha \nabla^{\alpha \xi} - \nabla^\alpha \nabla^{\alpha \xi} \right) \left( 1 + \frac{\zeta_\alpha}{\kappa} \Phi^2 \right) - \frac{1}{\kappa} \nabla_\alpha \Phi \nabla^\mu \Phi \]

\[ + \frac{\zeta_\alpha}{\kappa} \left( \nabla_\alpha \nabla^{\mu} \Phi^2 - \nabla^{(\alpha \Phi) \nabla_{\alpha} \Phi^2} \right) \]  
(D.4)

Replacing them we note that the three terms inside the total derivative in (D.1) can be rewritten as a total derivative too

\[ J^\mu_\xi = \Theta^\mu(\delta \xi) - \xi^\mu L - \xi_\mu E_{\mu\nu} = \partial_\nu \left[ -\kappa \sqrt{-g} \left( \nabla^{[\mu} \xi^{\nu]} \right) \left( 1 + \frac{\zeta_\alpha}{\kappa} \Phi^2 \right) + \frac{2\zeta_\alpha}{\kappa} g^{[\mu} \xi^{\nu]} \Phi^2 \right], \]

where we used \([\nabla^\mu, \nabla^\nu] \xi_\mu = R^{\mu\nu} \xi_\mu\). The term inside the total derivative is \( \tilde{Q}_{\xi}^{\mu\nu} \). Then we obtain the surface charge density

\[ k_\xi^{\mu\nu} = \delta \tilde{Q}_{\xi}^{\mu\nu} + 2 \xi^{[\mu} \Theta^\nu(\delta) \]  
(D.5)

\[ = \delta \left[ -\kappa \sqrt{-g} \left( \nabla^{[\mu} \xi^{\nu]} \right) \left( 1 + \frac{\zeta_\alpha}{\kappa} \Phi^2 \right) + \frac{2\zeta_\alpha}{\kappa} g^{[\mu} \xi^{\nu]} \Phi^2 \right] \]  
(D.6)

the previous expression reduces to (3.11).

E Einstein-Hilbert-Skyrme action

The variation of the Lagrangian (3.12) generated by an arbitrary vector field \( \xi = \xi^\mu \partial_\mu \)

\[ \delta_\xi L = E_{\mu\nu} \delta_\xi g^{\mu\nu} + \{E_U \xi_\mu U\} + \partial_\mu \Theta^\mu(\delta \xi) \]  
(E.1)

\[ \partial_\mu (\xi^\mu L) = -E_{\mu\nu} (\nabla^\mu \xi^\nu + \nabla^\nu \xi^\mu) + \{E_U \xi_\mu \nabla^\mu U\} + \partial_\mu \Theta^\mu(\delta \xi) \]  
(E.2)

\[ \partial_\mu (\xi^\mu L) = \partial_\mu [-2 \xi_\nu E^{\mu\nu} + \Theta^\mu(\delta \xi)] \]  
(E.3)

where we used the Noether identity \( 2 \nabla^\mu E_{\mu\nu} + \{E_U \nabla_\nu U\} = 0 \). The explicit functions in (E.3) are
\[ L = \sqrt{-g} \left[ \frac{\kappa}{2} (R - 2\Lambda) + \frac{K}{4} \left( R^{\mu} R_{\mu} + \frac{\lambda}{8} F_{\mu\nu} F^{\mu\nu} \right) \right] \]  
\[ E^{\mu\nu} = \sqrt{-g} \left[ \frac{\kappa}{2} \left( R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R - \Lambda g^{\mu\nu} \right) + \frac{K}{4} \left( R^{\mu\nu} R_{\mu\nu} - \frac{1}{2} R g^{\mu\nu} \right) \right] \]  
\[ E_U = -\frac{K}{2} \sqrt{-g} \left\langle \nabla_{\mu} \left( R^{\mu\nu} + \frac{\lambda}{4} [R_{\nu}, F^{\mu\nu}] \right) U^{-1} \right\rangle \]  
\[ \Theta^\mu(\delta \xi) = \sqrt{-g} \left[ \kappa \nabla_{[\alpha}(g^{\mu\beta]} \delta g_{\alpha\beta}) + \frac{K}{2} \left\langle \left( R^{\mu\nu} + \frac{\lambda}{4} [R_{\nu}, F^{\mu\nu}] \right) U^{-1} \delta \xi U \right\rangle \right]. \]

and again, after cancellations, we simply have
\[ J^\mu \equiv \Theta^\mu(\delta \xi) - \xi^\mu L - 2\xi_{\nu} E^{\mu\nu} = \partial_{\nu} \left[ -\kappa \sqrt{-g} \nabla^\mu [\xi^\nu] \right], \]  
where the cyclic property of the trace was used to show that \( \langle [R_{\nu}, F^{\mu\nu}] \xi^\alpha R_{\alpha} - \xi_{\nu} F^{\mu\alpha} F^{\nu}_{\alpha} \rangle = 0. \) Therefore \( \tilde{Q}_{\xi}^{\mu\nu} = -\kappa \sqrt{-g} \nabla^\mu [\xi^\nu], \) and we use the surface charge formula \( k_{\xi}^{\mu\nu} = \delta \tilde{Q}_{\xi}^{\mu\nu} + 2\xi^\mu \Theta^\mu(\delta) \) to get (3.15).

**F Einstein-Cartan-\( \Lambda \)**

Consider four-dimensional General Relativity with a cosmological term in the differential form language
\[ S[e^a, \omega^{ab}] = \kappa' \int_M \varepsilon_{abcd} \left( R^{ab} e^c e^d \pm \frac{1}{2\ell^2} e^a e^b e^c e^d \right), \quad \ell^2 = \frac{3}{|\Lambda|}. \]  

The general variation of the Lagrangian density is
\[ \delta L = E_a \delta e^a + E_{ab} \delta \omega^{ab} + d\Theta(\delta \omega). \]

The equations of motion and boundary term are given by
\[ E_a = 2\kappa' \varepsilon_{abcd} \left( R^{bc} e^d \pm \frac{1}{\ell^2} e^a e^b e^c \right) e^d = 0, \]  
\[ E_{ab} = 2\kappa' \varepsilon_{abcd} T^c e^d = 0, \]  
\[ \Theta(\delta \omega) = \kappa' \varepsilon_{abcd} \delta \omega^{ab} e^c e^d. \]

The action (F.1) is invariant under diffeomorphisms and local Lorentz transformations. Infinitesimal generators of these symmetries are a vector field \( \xi \) and the set of parameters \( \lambda^{ab} \), we group them in \( \epsilon = (\xi, \lambda^{ab}) \). Both can be combined such that the dynamical fields transform infinitesimally as
\[ \delta e^a = d_\omega (\xi \omega^a) + \xi_{\cdot\cdot} (d_\omega e^a) + \lambda^a b e^b, \]  
\[ \delta \xi_{\cdot\cdot} = \xi_{\cdot\cdot} R^{ab} - d_\omega \lambda^{ab}. \]
The symplectic structure density computed with these local symmetries as one of its entries is

$$\Omega(\delta, \delta_t) = \delta \Theta(\delta_t \omega) - \delta_t \Theta(\delta \omega) - \Theta([\delta, \delta_t] \omega),$$  \hspace{1cm} (F.8)

$$= \kappa' \varepsilon_{abcd} \left( \delta_t \omega^{ab} \delta(e^c e^d) - \delta \omega^{ab} \delta_t(e^c e^d) \right),$$  \hspace{1cm} (F.9)

$$= 2 \kappa' \varepsilon_{abcd} \left( [\xi, \omega]^{ab} - d_{\omega} \lambda^{ab} \right) \delta(e^c e^d) - \delta \omega^{ab} \delta_t \left( [d_{\omega} \xi, e^c] + [\xi, d_{\omega} e^c] + [\lambda^f, e^f] e^d \right), \hspace{1cm} (F.10)$$

$$= d k_c,$$  \hspace{1cm} (F.11)

note a subtle point, the term \(\Theta([\delta, \delta_t] \omega)\) should be formally included to guarantee the bilinearity on \(\delta\) and \(\delta_t\) because the variations do not commute in general. The surface charge density is

$$k_c = -\kappa' \varepsilon_{abcd} \left( \lambda^{ab} \delta(e^c e^d) - \delta \omega^{ab} \xi, e^c e^d \right).$$  \hspace{1cm} (F.12)

In the last step of (F.11) we used both, the equations of motion and the linearized equations of motion too. Knowing the result of this kind of calculation the strategy is always to rearrange the exterior derivatives on the parameters \(\xi\) and \(\lambda^{ab}\), second and third terms in (F.10), to complete exact differential forms and then check that all the remaining terms vanish due to the equations of motion and the linearized equations of motion, for instance \(T^a = de^a + \omega^{a} e^b = 0\) and \(\delta T^a = d \delta e^a + \delta \omega^{a} e^b + \omega_{ab} \delta e^b = 0\), as a general rule all of them have to be explicitly used.

G  Einstein-Cartan-Yang-Carrills

The four-dimensional Einstein-Cartan gravity coupled to a non-Abelian field reads

$$S[e^a, \omega^{ab}, A] = \int_M \left( \kappa' \varepsilon_{abcd} R^{ab} e^c e^d + \alpha_{YM} \langle F \star F \rangle \right),$$  \hspace{1cm} (G.1)

where \(\langle \cdot \rangle\) denotes an invariant bilinear form on the Lie algebra of the non-Abelian Lie group \(SU(N)\), \(\alpha_{YM}\) is the coupling constant, the two-form \(F\) is defined as \(F = dA + A \wedge A = \frac{1}{2} F_{ab} e^a e^b \tau_i\), the Hodge operator acts as \(\star F = \frac{1}{2} \varepsilon_{abcd} F^{iab} e^c e^d \tau_i\), with \(\tau_i\) the \(SU(N)\) generators.

The variation of the Lagrangian is

$$\delta L = E_a \delta e^a + E_{ab} \delta \omega^{ab} + \langle E_A \delta A \rangle + d \Theta(\delta \omega, \delta A),$$  \hspace{1cm} (G.2)

with the equations of motion and boundary term given by

$$E_a = \kappa' \varepsilon_{abcd} R^{bc} e^d - \alpha_{YM} \langle e_a \star F - F e_a \star F \rangle = 0,$$  \hspace{1cm} (G.3)

$$E_{ab} = 2 \kappa' \varepsilon_{abcd} T^{cd} = 0,$$  \hspace{1cm} (G.4)

$$E_A = -2 \alpha_{YM} d_A \star F = 0,$$  \hspace{1cm} (G.5)

$$\Theta(\delta \omega, \delta A) = \kappa' \varepsilon_{abcd} \delta \omega^{ab} e^c e^d + 2 \alpha_{YM} \langle \delta A \star F \rangle,$$  \hspace{1cm} (G.6)

with the covariant derivative defined by \(d_A(\cdot) = d(\cdot) + [A, \cdot]\). Remember the operation of the interior product \(e_a \star F = \frac{1}{2} F_{bc} e_a \cdot (e^b e^c) = \frac{1}{2} F_{bc} (\delta^b_a e^c - e^b e^c) = F_{ac} e^c\).
The gauge symmetries are diffeomorphisms, local Lorentz transformations, and $SU(N)$ acting on $A$. The parameters of the infinitesimal symmetries are grouped in $\epsilon = (\xi, \lambda^{ab}, \lambda^i)$, with $\lambda^i$ the components of the algebra valued gauge parameter $\lambda = \lambda^i \tau_i$. The improved exact symmetry conditions are

$$\delta \epsilon^a = d \omega (\xi \epsilon^a) + \xi_{,b} (d \omega^a) + \lambda^a b^b = 0,$$

$$\delta \epsilon^{ab} = \xi_{,b} F^{ab} - d \omega \lambda^{ab} = 0,$$

$$\delta \epsilon A^i = \xi_{,i} F^i - d A \lambda^i = 0.$$  

As showed in the general case for the differential form language, the surface charge density is the sum of three terms

$$k_\epsilon = \delta \tilde{Q}_\epsilon - \xi_{,b} \Theta (\delta) - B_{\delta \epsilon}.$$  

We already have the boundary term, (G.6). Evaluating on an infinitesimal gauge symmetry $E_\epsilon \delta_t \epsilon^a + E_{ab} \delta \omega^{ab} + (E_A \delta A) = d S_\epsilon + N_\epsilon$, with the Noether identities $N_\epsilon = 0$, we obtain $S_\epsilon$. Then, as usual, the would-be Noether charge $J_\epsilon = \Theta (\delta_t) - \xi_{,b} L + S_\epsilon = d \tilde{Q}_\epsilon$ is an exact form with

$$\tilde{Q}_\epsilon = - \kappa' \epsilon_{abcd} \delta \lambda^{ab} e^d - 2 \alpha_{YM} \langle \lambda \star F \rangle.$$  

Now, we use that $[\delta, \delta_t] = [\delta, \delta_{\xi} + \delta_{\lambda^{ab}} + \delta_{\lambda^i}] = \delta_{\lambda^{ab} + \delta_{\lambda^i}}$ because the vector field $\xi$ is assumed fixed on the phase space. Then, we have $\theta ([\delta, \delta_t]) = dB_{\delta \epsilon} + C_\epsilon$ such that on-shell $C_\epsilon \approx 0$. Thus, we obtain

$$B_{\delta \epsilon} = - \kappa' \epsilon_{abcd} (\delta \lambda^{ab} + \xi_{,b} \omega^{ab}) e^d - 2 \alpha_{YM} \langle \delta \lambda^i + \xi_{,i} \delta A^i \star F \rangle.$$  

Replacing all back in the general expression (G.10) we get

$$k_\epsilon = - \kappa' \epsilon_{abcd} \lambda^{ab} \delta e^d - \delta \omega^{ab} \xi_{,b} (e^d) - 2 \alpha_{YM} \langle \delta \lambda^i + \xi_{,i} \delta A^i \star F \rangle,$$

with the first two terms just the surface charge density of pure gravity (F.12). Thus, roughly, to consider the extension to a general YM theory from a pure electromagnetic field one should include the $\langle \cdot \rangle$ brackets to deal with the algebra valued fields.

**H Einstein-Cartan in 2 + 1 plus a Torsional Term**

In this appendix we compute the surface charge density explicitly. Because is faster and equivalent, we use the contracting homotopy operator method to do it. Consider the Lagrangian for (2 + 1)-spacetime dimensions

$$L = \epsilon_{abc} e^a R^{bc} + \beta e_a T^a,$$

where $T^a = de^a + \omega^a_\epsilon e^b$, and the term proportional to $\beta$ is the one that would produce torsion. The variation of the Lagrangian is

$$\delta L = \delta e^a (\epsilon_{abc} R^{bc} + 2 \beta T_a) + \delta \omega^{ab} (\epsilon_{abc} T^c - \beta e_a e_b) - d (\epsilon_{abc} \delta \omega^{bc} + \beta e^a \delta e_a),$$
the second equation of motion tells us that torsion does not vanish, and because of the first one, we can advance that in fact it plays the role of a cosmological constant term.

With the improved infinitesimal gauge transformation for the fields, $\delta_e e^a$ and $\delta_\omega \omega^{ab}$ from (4.2) and (4.3) respectively, we rearrange the combination $E_a e^a + E_{ab} \delta_\omega \omega^{ab} = dS_e + N_\epsilon$, such that

$$S_e = \xi_a e^a (\varepsilon_{abc} R^{bc} + 2\beta T_a) - \lambda^{ab} (\varepsilon_{abc} T^c - \beta e_a e_b),$$

and $N_\epsilon = 0$ are the Noether identities. The $S_e$ is what we need to compute the surface charge density using the contracting homotopy operator. For the Einstein-Cartan theory the operator is (see Eq. (3.29) in [27])

$$I_{\delta_e, \delta_\omega} \equiv \delta e^a \frac{\partial}{\partial T^a} + \delta \omega^{ab} \frac{\partial}{\partial R^{ab}},$$

because apart from $e^a$ and $\omega^{ab}$ there are no extra fields in the phase space, this is the full operator for this theory. Then, the surface charge density, $k_\epsilon \equiv I_{\delta_e, \delta_\omega} S_e$, becomes

$$k_\epsilon = -\varepsilon_{abc} (\lambda^{ab} \delta e^c - \delta \omega^{ab} \xi_a e^c) + 2\beta \xi_a e^c \delta e_a$$

By following the prescription that uses the symplectic structure density, $\Omega(\delta, \delta_\epsilon) = dk_\epsilon$, we arrive at exactly the same expression.

We remember that in general, boundary terms (exact forms) at the level of the Lagrangian do not contribute to the surface charges, or at most they contribute as exact forms in the $k_\epsilon$ formula and therefore can be neglected. However, notice that the term used here can not be written as an exact form, the tentative term one would try vanishes identically $d(e^a e_a) = T^a e_a - e^a T_a = 0$. Then, at this level $\beta e_a T^a$ is a genuine term that produces torsion. In particular, it contributes to the surface charge density as is explicit in (H.5) with the last term, and again, this contribution is not an exact form at this level either.

Now, we make a step further to analyse the surface charge density by performing a split of the connection in torsionless and contorsion parts. That is,

$$\omega^{ab} = \tilde{\omega}^{ab} + \omega^{ab},$$

with $\tilde{\omega}^{ab}(e)$ solving $d e^a + \tilde{\omega}^b_a e^b = 0$. Then, the torsion is simply $T^a = \tilde{\omega}^a_b e^b$. From the equation of motion, $\varepsilon_{abc} T^a = \beta e_b e_c$, we solve

$$\omega^{ab} = \frac{\beta}{2} \varepsilon^{abc} e_c,$$

where we used $\varepsilon^{abc} \varepsilon^{d} = -\varepsilon_{abc} \varepsilon^{d} = -2\delta_a^d$.\(^{18}\) The split of the connection induces also a split of the parameter, $\lambda^{ab}$, that solves the exact symmetry condition. In general the condition $\delta_\epsilon e^a = 0$ is solved by

$$\lambda^{ab} = e^a_j (d_j (\varepsilon_a e^b) + d_a e^b)$$

$$= e^a_j (d_j (\varepsilon_a e^b) + \tilde{\omega}^b_j e^c + \tilde{\omega}^c_j e^b)$$

$$= \tilde{\lambda}^{ab} - \xi_a \tilde{\omega}^{ab},$$

\(^{18}\)Remember $\varepsilon^{abc} = \eta^{a'd'} \eta^{b'd'} \eta^{c'd'} = -\varepsilon_{abc} = -\varepsilon^{abc}$ we put a twiddle to the Levi-Civita symbol that do not carry information about the flat metric even if it has upstairs indices. Check this prescription in (A.4).
where we used $d\omega^b = 0$, $e^a \epsilon^c = \eta^{ac}$, and introduced $\lambda^{ab} \equiv e^a \omega(d_\omega(\xi \epsilon^b))$, the torsionless part of the parameter. An equivalent way to define this parameter is to use the exact symmetry condition but improving the transformation only with the torsionless connection

$$\delta \epsilon^a = \xi \omega(d_\omega(\epsilon^a)) + d_\omega(\xi \epsilon^a) + \lambda^a \epsilon^b = 0,$$  \hfill (H.12)

note that the first term vanishes by construction. Collecting all, we can now use the split of the connection and the $\lambda^{ab}$ parameter on the surface charge density formula to show that

$$k_r = -\varepsilon_{abc}(\lambda^{ab} \delta \epsilon^c - \delta \tilde{\omega}^{ab} \xi \epsilon^c) + \varepsilon_{abc}(\xi \tilde{\omega}^{ab} \delta \epsilon^c + \delta \tilde{\omega}^{ab} \xi \epsilon^c) + 2\beta \xi \epsilon a \delta \epsilon^a$$  \hfill (H.13)

$$\tilde{k}_r = \tilde{k}_r + \frac{\beta}{2} \varepsilon_{abc} \varepsilon^{abd}(\xi \epsilon_d \delta \epsilon^c + \delta \epsilon_d \xi \epsilon^c) + 2\beta \xi \epsilon a \delta \epsilon^a$$  \hfill (H.14)

in the second line we defined $\tilde{k}_r$ and used the explicit expression for $\tilde{\omega}^{ab}$ computed at (H.7). To reach the third line remember that $\varepsilon^{abd} = -\varepsilon^{adb}$, as in (A.4), then $\varepsilon_{abc} \varepsilon^{abd} = -2\delta_{d}$. The remarkable fact is the third line. At the level of surface charge density, the torsion contribution to the connection cancels exactly with the extra source term $2\beta \xi \epsilon a \delta \epsilon^a$. In other words, the surface charge can be computed with the usual expression if one uses the Levi-Civita, or torsionless, connection $\tilde{\omega}^{ab}$. 

The equation $k_r = \tilde{k}_r$ tells us that contorsion will never contribute to the charges in this theory. For this simple theory this result could have been expected as from the beginning we knew that the torsional term in the action is equivalent to a cosmological constant term. And we already know, at least in four dimensions but for any dimensions is the same, that a cosmological term do not enter in the surface charge formula.\(^{19}\)

### H.1 From a Chern-Simons perspective

The previous result can be understood from a Chern-Simons (CS) perspective too. In fact, it is well-known that three-dimensional general relativity with negative cosmological constant can be written as a topological Chern-Simons theory of a one-form gauge connection $\tilde{A} = e^b \tilde{P}_a + \omega^a \tilde{J}_a$ valued on the anti-de Sitter algebra in three dimensions ($AdS_3$, $\mathfrak{so}(2,1)$). For the spin connection we use $\omega_a = \frac{1}{2} \varepsilon_{abc} \omega^{ck}$. The algebra reads

$$[\tilde{P}_a, \tilde{P}_b] = \Lambda \varepsilon_{abc} \tilde{J}^c, \quad [\tilde{J}_a, \tilde{P}_b] = \varepsilon_{abc} \tilde{P}^c, \quad [\tilde{J}_a, \tilde{J}_b] = \varepsilon_{abc} \tilde{J}^c,$$  \hfill (H.16)

with $\Lambda$ the cosmological constant. The algebra (H.16) admits a non-degenerate and invariant bilinear form

$$\langle \tilde{J}_a, \tilde{P}_b \rangle = \eta_{ab}.$$  \hfill (H.17)

\(^{19}\)As an extra comment, we contrast our results with the charges computed in [42]. To compare, all contributions coming from the action term, $\omega_d a + \frac{\beta}{2} \omega^3$, in [42], shall be set to zero. Still, due to the $e_a T^a$ term in the action, it is found that the theory admits a so-called BTZ solution with torsion. The formulas for the mass and angular momentum presented in [42] have a direct torsion contribution, and not only through the effective cosmological constant parameter, as can be appreciated in Eqs. (20) and (21) there. This is in tension with our results because, as we just checked, torsion disappears from our charge formulas. Another curiosity is that in [42] the proposed quasilocal charge expressions depend on the $r$ coordinate. This dependence is avoided there, in their final formula, by taking the usual $r \to \infty$. From the surface charges density perspective we adopt here this can not happen simply because of the conservation law, $dk_c = 0$, guarantee independence of the radius.
These two ingredients provide a CS construction for three-dimensional general relativity as follows

\[ L_{CS} = \left\langle \tilde{A} \wedge d\tilde{A} + \frac{1}{3} \tilde{A} \wedge [\tilde{A}, \tilde{A}] \right\rangle \]

\[ = 2e^a R_a(\omega) + \frac{\Lambda}{3} \varepsilon_{abc} e^a e^b e^c + d(e^a \omega_a), \quad (H.18) \]

with the curvature \( R_a(\omega) = d\omega_a + \frac{1}{2} \varepsilon_{abc} \omega^b \omega^c \). Note that the equivalence is up to a boundary term. Now, the physics does not depend on the chosen algebra basis. Let us introduce a different basis for the algebra generators

\[ P_a = \tilde{P}_a + (\beta/2) \tilde{J}_a \]
\[ J_a = \tilde{J}_a, \quad (H.19) \]

with \( \beta \) a constant. Then, the AdS\(_3\) algebra commutators (H.16) in this basis are

\[ [P_a, P_b] = (\Lambda - (\beta^2/4)) \varepsilon_{abc} J^c + \beta \varepsilon_{abc} P^c \]
\[ [J_a, P_b] = \varepsilon_{abc} P^c \]
\[ [J_a, J_b] = \varepsilon_{abc} J^c. \quad (H.21) \]

The invariant and non-degenerate bilinear form associated to (H.21) is now

\[ \left\langle J_a, P_b \right\rangle = \eta_{ab}, \quad \left\langle P_a, P_b \right\rangle = \beta \eta_{ab}. \quad (H.22) \]

Thus, the equivalent Chern-Simons Lagrangian for the one-form gauge connection \( A = e^a P_a + \omega^a J_a \) valued on the algebra (H.16) and associated to the bilinear form (H.22) is

\[ L_{CS} = 2e^a R_a(\omega) + \beta e^a T_a + \frac{\Lambda_{eff}}{3} \varepsilon_{abc} e^a e^b e^c + d(e^a \omega_a), \quad (H.23) \]

with the torsion \( T_a = de_a + \varepsilon_{abc} \omega^b e^c \), and the effective cosmological constant \( \Lambda_{eff} = \Lambda - \frac{3}{4} \beta^2 \).

We can choose the parameter \( \beta^2 = \frac{3}{4} \Lambda \) and the last Lagrangian becomes exactly the Lagrangian (H.1) considered previously. Therefore, we conclude that the torsional Lagrangian (H.1) is just equivalent to the Einstein-Cartan Lagrangian (H.18) with a specific value for the cosmological constant \( \Lambda = \frac{3}{4} \beta^2 \).

### I Einstein-Cartan-Dirac

Consider the gravity contribution to surface charge density in four spacetime dimensions

\[ \hat{k}_e = -\kappa' \varepsilon_{abcd} \left[ \lambda^{ab} \delta(e^c e^d) - \delta e^a \omega^b e^c \right]. \quad (I.1) \]

As a preliminary we will rearrange this formula. First note that the spin connection can have a torsion part, named the contorsion, we want to isolate its contribution into the formula. Exactly as in the previous appendix, (H.11), we perform a split of the spin connection,
\( \omega^{ab} = \tilde{\omega}^{ab}(e) + \dot{\omega}^{ab} \), such that \( d_\omega e^a = 0 \). The exact symmetry condition, \( \delta_ee^a = 0 \), is solved by the parameter
\[
\lambda^{ab} = e^a_{\ j}[(\xi \cdot d_\omega e^b)] + e^a_{\ j}(d_\omega \xi^b e^b) = e^a_{\ j}(d_\omega \xi^b e^b) - \xi \tilde{\omega}^{ab},
\]
thus, the split of the connection is translated in a split of the parameter \( \lambda^{ab} = \tilde{\lambda}^{ab} + \dot{\lambda}^{ab} \).

Then, the gravity contribution to the surface charge density has two parts
\[
\tilde{k}_e = \tilde{k}_e + \tilde{k}_e,
\]
the torsionless part
\[
\tilde{k}_e = -\kappa'\varepsilon_{abcd}\left[\tilde{\lambda}^{ab}\delta(e^c e^d) - \delta\tilde{\omega}^{ab}\xi^c e^d\right],
\]
where \( \tilde{\lambda}^{ab} = e^a_{\ j}(d_\omega (\xi^c e^b)) \), and the contorsion part that we wanted to isolate
\[
\bar{k}_e = -\kappa'\varepsilon_{abcd}\left[\bar{\lambda}^{ab}\delta(e^c e^d) - \delta\bar{\omega}^{ab}\xi^c e^d\right],
\]
where \( \bar{\lambda}^{ab} = -\xi \tilde{\omega}^{ab} \). If we further express the contorsion one-form in frame components, \( \tilde{\omega}^{ab} = \omega^{ab} f e^f \), we can write
\[
\bar{k}_e = 2\kappa'\varepsilon_{abcd}\left[\tilde{\omega}^{ab} f e^c (\xi^c \omega^d f) \delta e^d + \xi^c \omega^d f \delta e^f - \delta\tilde{\omega}^{ab} f e^c e^d\right].
\]

Now, we compute the whole surface charge density. Consider the Einstein-Cartan-Dirac action
\[
S[e^a, \omega^{ab}, \psi] = \int_M \varepsilon_{abcd} e^a e^b \left[\kappa' R^{cd} - \frac{i}{3} \alpha_\psi e^c \left(\bar{\psi} \gamma^d \gamma_5 d_\omega \psi + d_\omega \bar{\psi} \gamma^d \gamma_5 \psi\right)\right],
\]
with \( d_\omega \psi = d\psi + \frac{1}{2} \omega_\gamma \gamma^\mu \psi \) and \( \gamma_{ab} \equiv \frac{1}{4}[\gamma_a, \gamma_b] \) satisfying the Lorentz algebra. The special matrix \( \gamma_5 \equiv \gamma_0 \gamma_1 \gamma_2 \gamma_3 \) satisfies \( \gamma_5^a \gamma_a = -\gamma_5 \gamma_a \). The following computation of surface charge is very sensitive to the coefficients, therefore we make a \textit{scriptsize detour} to be self-contained and to check the consistence of our conventions.

The \( \gamma \)-matrices satisfy the Clifford algebra \( \{\gamma_a, \gamma_b\} = \gamma_a \gamma_b + \gamma_b \gamma_a = 2\eta_{ab} \). Then, we have also \( \gamma_{ab} (\gamma_{ae} \gamma_c - \gamma_{ec} \gamma_a) = 4(\eta_{ab} \gamma_c - \eta_{ac} \gamma_b) \). If we define \( \gamma_{ab} \equiv \frac{1}{4}[\gamma_a, \gamma_b] \), we can check that \( \gamma_{ab} \gamma_{ac} = \eta_{ab} \gamma_c - \eta_{ac} \gamma_b \), and the matrices \( \gamma_{ab} \) satisfy the Lorentz algebra, namely
\[
[\gamma_{ab}, \gamma_{cd}] = \eta_{bd} \gamma_{ac} - \eta_{ad} \gamma_{bc} + \eta_{ac} \gamma_{bd} - \eta_{bc} \gamma_{ad}.
\]
this fix the 1/4 normalization of the \( \gamma_{ab} \) definition. The coefficient multiplying the connection in the covariant derivative acting on a spinor is fixed by the defining equation of the covariant derivatives on spinors
\[
d_\omega (\Lambda \psi) = \Lambda d_\omega \psi.
\]
The Lorentz transformation is \( \Lambda = \exp(\frac{1}{2} \lambda^{ab} \gamma_{ab}) \), thus we define the algebra valued coefficient \( \lambda = \frac{1}{2} \lambda^{ab} \gamma_{ab} \). We check that the 1/2 is consistent with (1.9). If we expand to first order in the Lorentz transformation both sides of (1.9), we get
\[
d\lambda \psi + \lambda d_\omega \psi - \frac{1}{2} (d_\omega \lambda) \gamma^{ab} \psi + \frac{1}{2} \omega_{ab} \gamma^{ab} \lambda \psi = \lambda d_\omega \psi + \frac{1}{2} \lambda \omega_{ab} \gamma^{ab} \psi,
\]
where we used \( \omega^{ab} = \omega_{ab} + \delta \omega_{ab} \) with \( \delta \omega_{ab} = -d_\omega \lambda_{ab} = -d_\omega \lambda_{ab} + \lambda_{ac} \omega^{cb} - \lambda^{cb} \omega_{ac} \). The last expression brings into the formula the convention for the infinitesimal transformation \( \lambda_{ab} \) used in the other variables \( \delta_ee^a = \lambda^{ab} e^b \) or
equivalently $\delta_{\lambda} \omega_{ab} = -d_{\lambda} \gamma_{ab}$. Then we check that (I.10) is in fact and identity because our conventions are correct such that $d\lambda = \frac{1}{2}d\lambda_{ab}\gamma^{ab}$; and

$$
\omega_{ab}[\lambda, \gamma^{ab}] = \frac{1}{2} \omega_{ab} \lambda_{cd} [\gamma^{cd}, \gamma^{ab}] = -\frac{1}{2} \omega_{ab} \lambda_{cd} [\gamma_{ab}, \gamma^{cd}]
$$

with $(\lambda_{ab} \omega^c_e - \lambda^c_e \omega_{ab}) \gamma^{ab}$.

(I.11)

Those checks set correctly the three coefficients in $\gamma_{ab} = \frac{1}{2}[\gamma_{ab}, \gamma_{cd}]$, $\lambda = \frac{1}{2} \lambda_{ab} \gamma^{ab}$, and $d_{\lambda} \psi = d\psi + \frac{1}{2} \omega_{ab} \gamma^{ab} \psi$.

Now, besides the usual exact symmetry conditions on the gravity fields, (4.2) and (4.3), we should impose the exact symmetry condition on the spinor field. Spinor field transform under an infinitesimal local Lorentz transformation as $\delta\chi, \psi = \lambda \psi$. Therefore, the correct exact symmetry condition is

$$
\delta_{\epsilon} \psi = \mathcal{L}_\xi \psi + \lambda \psi = \xi_{\lambda} d_{\lambda} \psi + \lambda \psi = 0,
$$

(I.13)

with, again, the improved prescription given by $\lambda = \frac{1}{2} \gamma_{ab}(\lambda^{ab} - \xi_{\omega} \omega^{ab})$ (remember $\lambda_{ab} = \lambda_{ab} - \xi_{\omega} \omega_{ab}$).

A general formula for the surface charge density in differential form language is (see Eq. (2.19) in [27])

$$
k_{\epsilon} = \delta \tilde{Q}_{\epsilon} - \xi_{\omega} \Theta(\delta) - B_{6\epsilon}.
$$

(I.14)

The variation of the Lagrangian is

$$
\delta L = E_\alpha \delta e^a + E_{ab} \delta \omega^{ab} + E_{\psi} \delta \psi + + E_{\bar{\psi}} \delta \bar{\psi} + d\Theta(\delta),
$$

(I.15)

with the boundary term

$$
\Theta(\delta) = \varepsilon_{abcd} e^a e^b \left( \kappa' \delta \omega^{cd} + i \frac{3}{2} \alpha_{\psi} e^\delta \left( \bar{\psi} \gamma^{d} \gamma_{5} \psi \right) \right).
$$

(I.16)

This is the middle term we need in (I.14). For the first term we compute the trivial current $J_{\epsilon} = \Theta(\delta)_{\xi} - \xi \omega + S_{\epsilon} = d\tilde{Q}_{\epsilon}$, and after cancellations we get the usual $\tilde{Q}_{\epsilon} = -\kappa' \varepsilon_{abcd} e^a e^b \lambda^{cd}$ we find for pure Einstein-Cartan theory. For the third term in (I.14), the prescription tells us that the symplectic potential term $\Theta(\delta_{\epsilon}, \delta_{\epsilon}) = dB_{6\epsilon} + C_{6\epsilon}$ with $C_{6\epsilon} \approx 0$, thus we use the commutation of variations $[\delta, \delta_{\epsilon}] = [\delta, \mathcal{L}_\xi + \lambda_{\omega} + \xi \omega] = \delta_{\lambda + \xi \omega}$, and we get $B_{6\epsilon} = -\kappa' \varepsilon_{abcd} e^a e^b (\delta \lambda^{cd} + \xi_{\omega} \omega^{cd})$. Thus the spinor field does not contribute through $B_{6\epsilon}$, nor $\tilde{Q}_{\epsilon}$ in the general formula (I.14), it only enters through the extra boundary term in (I.16). The complete surface charge density for the Einstein-Cartan-Dirac theory is

$$
k_{\epsilon} = -\varepsilon_{abcd} \left( \kappa' \left( \lambda^{ab} \delta (e^c e^d) - \delta \omega^{ab} \xi_{\omega} (e^c e^d) \right) + i \alpha_{\psi} \xi_{\omega} e^a e^b e^\delta \left( \bar{\psi} \gamma^{d} \gamma_{5} \psi \right) \right).
$$

(I.17)

We stress that the addition of a spinorial mass term in the action does not change the surface charge formula. Then, this result is already useful enough to compute charges for this theory, its massive spinor equivalent, or even with an additional cosmological constant term. But we can go further. Let us split the gravity terms as we did at the beginning, (I.3), then

$$
k_{\epsilon} = \tilde{k}_{\epsilon} + \bar{k}_{\epsilon} + k_{\epsilon} \psi
$$

(I.18)
We can compute $\tilde{k}_\gamma$ explicitly, as given by (I.6), by solving the contorsion $\tilde{\omega}^a b$ from the torsion equation of motion. We do it step by step. The equation we need is

$$\varepsilon_{abcd} T^{e d} = \frac{i}{12} \alpha \psi \varepsilon_{cdmn} \varepsilon^{e d n} \bar{\psi}(\delta^m_a \gamma_b - \delta^m_b \gamma_a) \gamma_5 \psi,$$

(1.20)

with $T^c = d_c \bar{e}^c = \tilde{\omega}^c_f e^f = \tilde{\omega}^a f g \bar{e}^g e^f$, we have

$$\varepsilon_{abcd} \tilde{\omega}^a f g e^f e^d = \frac{i}{12} \alpha \psi \varepsilon_{cdmn} \varepsilon^{e d n} \bar{\psi}(\delta^m_a \gamma_b - \delta^m_b \gamma_a) \gamma_5 \psi,$$

(1.21)
or its dual equation

$$\varepsilon_{abcd} \varepsilon^{g f d h} \bar{\omega}^f_{g h} = \frac{i}{12} \alpha \psi \varepsilon_{cdmn} \varepsilon^{e d n} \bar{\psi}(\delta^m_a \gamma_b - \delta^m_b \gamma_a) \gamma_5 \psi.$$

(1.22)

Note that in the l.h.s. $\varepsilon_{abcd} \varepsilon^{g f d h} = -(-3! \delta^g_{[a} \delta^f_{b} \delta^h_{c]} - 3! \delta^g_{[a} \delta^f_{b} \delta^h_{c]}) = -6! \delta^g_{[a} \delta^f_{b} \delta^h_{c]}$, where we have to be careful with the extra minus sign because we raise indices with the flat metric $\eta^{ab}$. We also use in the r.h.s. $\varepsilon_{cdmn} \varepsilon^{e d n} = -\varepsilon_{cdmn} \varepsilon^{e d n} = -(-6! \delta^g_{[a} \delta^f_{b} \delta^h_{c]} = 6! \delta^g_{[a} \delta^f_{b} \delta^h_{c]}$. Therefore

$$2 \bar{\omega}^h_{ba} + \delta^h_b \omega^c_{eb} - \delta^h_b \omega^e_{cb} = \frac{i}{2} \alpha \psi \bar{\psi}(\delta^m_a \gamma_b - \delta^m_b \gamma_a) \gamma_5 \psi,$$

(1.23)

To completely solve this equation we have to contract it and then replace the result in itself. Contracting $h = b$ we get $\tilde{\omega}^c_{ca} = \frac{3}{2} \alpha \psi \bar{\psi} \gamma_5 \psi$. Then

$$\bar{\omega}^h_{ba} = \frac{i}{2} \alpha \psi \bar{\psi}(\delta^h_b \gamma_a - \delta^h_a \gamma_b) \gamma_5 \psi,$$

(1.24)
or equivalently

$$\tilde{\omega}^{ab}_f = \frac{i}{2} \alpha \psi \bar{\psi}(\eta^{ab} \gamma_f - \delta^{a b}_{[a} \gamma_{b]} \gamma_5 \psi.$$

(1.25)

We are ready to replace this into the expression for $\tilde{k}_\gamma$, (I.6). We do it by parts. First note that the following combination simply vanishes

$$\varepsilon_{abcd} \tilde{\omega}^a b e^c (\xi \bar{e}^d \delta e^d + \xi \bar{e}^d \delta e^d) = \frac{i}{2} \alpha \psi \varepsilon_{abcd} \bar{\psi}(\eta^{ab} \gamma_f - \delta^{a b}_{[a} \gamma_{b]} \gamma_5 \psi) \varepsilon^{e d n} \bar{\psi}(\delta^m_a \gamma_b - \delta^m_b \gamma_a) \gamma_5 \psi = 0.$$

Then

$$\tilde{k}_\psi = -2 \varepsilon_{abcd} \delta \bar{\omega}^a b e^c (\xi \bar{e}^d \delta e^d + \xi \bar{e}^d \delta e^d) = i \alpha \psi \varepsilon_{abcd} \bar{\psi} \xi \bar{e}^d \delta (\bar{\psi} \gamma^d \gamma_5 \psi),$$

(1.26)

this term is exactly $-k^\psi_\gamma$ as in (1.19). Therefore, as it might have been suspected, for the surface charge density, we have an exact cancellation of all the terms concerning the spinor field

$$\tilde{k}_\gamma = \tilde{k}_\gamma + k^\psi_\gamma = 0,$$

(1.27)
or equivalently, this means that the full surface charge density for the Einstein-Cartan-Dirac theory is simply $k_\gamma = \tilde{k}_\gamma$ as in equation (I.4). In particular, this implies that in a spacetime with a spinor field living on it, as far as exact symmetries are satisfied, it is not needed to have the explicit solution for the spinor field to compute charges.
**J D-dimensional Chern-Simons form**

A Chern-Simons (CS) Lagrangian in $D = 2n + 1$ dimensions is a local function of a one-form gauge connection, $A$, valued on a Lie algebra. That is $A = A^i \tau_i = A^i \tau_i e^a$ with $e^a$ the one-form frame field, $\tau_i$ the generators of the group algebra, $[\tau_i, \tau_j] = f_{ij}^k \tau_k$, and $f_{ij}^k$ the algebra structure constants. The full CS Lagrangian can be expressed in a very compact form using the trick of an integral over an auxiliary variable $t$ [43] (see also [44]),

$$L^{(2n+1)}[A] = \kappa_n \int_0^1 dt \langle AF_t^n \rangle, \quad (J.1)$$

where $F_t \equiv dA_t + A_t \wedge A_t$, $A_t = tA$, and $\kappa_n = \kappa_{CS}(n + 1)$ with $\kappa_{CS}$ the CS level. Notice that the one-form nature of $A$ induces the algebra commutator on the $A_t^2$ term, explicitly

$$A_t^2 = (tA) \wedge (tA) = t^2 A \wedge A = \frac{1}{2} t^2 A^i \wedge A^j [\tau_i, \tau_j].$$

The angled bracket $\langle \cdot \rangle$ denotes the symmetric invariant polynomial on the algebra such that for any two algebra valued forms, says the $p$-form $P$ and the $q$-form $Q$ the usual commutation properties are respected, namely

$$\langle \cdots PQ \cdots \rangle = (-1)^{pq} \langle \cdots QP \cdots \rangle. \quad (J.2)$$

The CS theory possesses two symmetries, in fact the Lagrangian is invariant under diffeomorphisms and also quasi-invariant (up to a boundary term/exact form) under gauge symmetries, these are the key gauge symmetries that push us to compute surface charges. The trick to have a compact expression for the Lagrangian allows us to perform all calculations directly, we show them in detail. Let us start with a general variation of (J.1), we have

$$\delta L^{(2n+1)}[A] = \kappa_n \int_0^1 dt \langle \delta AF_t^n + nA \delta F_t F_t^{n-1} \rangle$$

$$= \kappa_n \int_0^1 dt \langle \delta AF_t^n + nA d_{A_t}(\delta A_t) F_t^{n-1} \rangle$$

$$= \kappa_n \int_0^1 dt \langle \delta AF_t^n + \frac{d}{dt} \delta A_t \rangle$$

$$F_t^n - \delta AF_t^n - n d(A \delta A_t F_t^{n-1}) \rangle$$

$$= \kappa_n \langle \delta AF^n \rangle - d\Theta(\delta A), \quad (J.3)$$

where we used, $\delta F_t = d\delta A_t + [A_t, \delta A_t] = d_{A_t} \delta A_t$ with the notation $d_{A_t}$ for the exterior covariant derivative for the connection $A_t$, we used also the Leibniz’s rule for $d_{A_t}$, that $d_{A_t} F_t = 0$, the identity $\frac{d}{dt} F_t = d_{A_t} A$, integration by parts in the variable $t$, that $\frac{d}{dt} \delta A_t = \delta A$, and the invariance property of the symmetric polynomial $\langle \cdot \rangle$. Then, the equations of motion and the boundary term are

$$\langle F^n \rangle = 0, \quad (J.4)$$

$$\Theta(\delta A) = -n\kappa_n \int_0^1 dt \langle \delta A_t AF_t^{n-1} \rangle. \quad (J.5)$$

Notice that we defined the boundary term with an overall minus sign, this convention save us of carrying a minus sign in the following calculations. This is conventional, remember that surface charge densities are defined up to overall factors.
Later we will also need the linearized equation of motion, namely

\[ \delta(F^n) = n(\langle \delta F \rangle F^{n-1}) = n(\langle d_A(\delta A) F^{n-1} \rangle) = 0, \]  

(J.6)

where \( d_A(\cdot) \equiv d(\cdot) + [A, \cdot] \) denotes the covariant exterior derivative for the connection \( A \).

To obtain the surface charge density we first compute the symplectic structure density using the boundary term. With two independent general variations on the phase space, say \( \delta_1 \) and \( \delta_2 \), the symplectic structure density reads

\[ \Omega(\delta_1, \delta_2) = \delta_1 \Theta(\delta_2 A) - \delta_2 \Theta(\delta_1 A) - \Theta([\delta_1, \delta_2] A) \]

\[ = -n\kappa_n \int_0^1 dt \langle 2\delta_2 A_t \delta_1 A F_t^{n-1} + \delta_2 A_t \delta_1 F_t^{n-1} - \delta_1 A_t \delta_2 F_t^{n-1} \rangle. \]  

(J.7)

Now, the key to get a more tractable expression is to rewrite the second term as

\[
\langle \delta_2 A_t \delta_1 A F_t^{n-1} \rangle = (n - 1) \langle \delta_2 A_t d_A(\delta_1 A_t) F_t^{n-2} \rangle \\
= (n - 1) \langle d(\delta_2 A_t \delta_1 A_t F_t^{n-2} - d_A(\delta_2 A_t) A \delta_1 A_t F_t^{n-2} + \delta_2 A_t d_A \delta_1 A_t F_t^{n-2} \rangle \\
= (n - 1) \langle d(\delta_2 A_t \delta_1 A_t F_t^{n-2} - \delta_2 F_t A \delta_1 A_t F_t^{n-2} + \frac{d}{dt}(F_t) \delta_1 A_t F_t^{n-2} \rangle \\
= (n - 1) \langle d(\delta_2 A_t \delta_1 A_t F_t^{n-2} - \delta_2 F_t A \delta_1 A_t F_t^{n-2}) + \frac{d}{dt} \langle \delta_2 A_t \delta_1 A_t F_t^{n-1} \rangle - 2 \langle \delta_2 A_t \delta_1 A F_t^{n-1} \rangle \rangle
\]  

(J.8)

where we used in the second line the Leibniz's rule for the covariant derivative \( d_A \), and the identity \( d_A F_t^{n-2} = 0 \). In the third line, \( d_A(\delta_2 A_t) = \delta_2 F_t \) and \( \frac{d}{dt} F_t = d_A A \). In the fourth line, we introduce a total derivative in \( t \) we use that all the expression is inside the bracket \( \langle \cdot \rangle \) to perform commutations of the algebra valued forms, and used also that \( \frac{d}{dt} \delta A_t = \delta A \).

Now, replacing back, the second and fourth terms of (J.8) cancel exactly the first and third terms of (J.7), respectively. We are left with a total derivative in \( t \) which we can integrate trivially, and also an exact form. Then, the result is a symplectic structure density composed by a piece that could have been expected plus another piece which is an exact form

\[ \Omega^{(2n+1)}(\delta_1, \delta_2) = n\kappa_n \langle \delta_1 A \delta_2 A F^{n-1} \rangle - n(n + 1)\kappa_n \frac{d}{dt} \langle \delta_2 A_t \delta_1 A_t F_t^{n-2} \rangle \)  

(J.9)

We observe that unlike other theories, the symplectic structure density for CS is not gauge invariant due to the last term, this is expected because the Lagrangian as well as the boundary term \( \Theta(\delta A) \) are not gauge invariant forms. Remember that the theory is just quasi-invariant.

Now, we combine infinitesimal diffeomorphism and gauge transformations for the connection to write an improved general infinitesimal symmetry transformation as

\[ \delta_c A = \mathcal{L}_\xi A - d_A \lambda' = \xi J F - d_A \lambda, \]  

(J.10)
where as usual we select the parameter as $\lambda = \lambda + \xi \cdot A$ in order to define an overall homogeneous infinitesimal transformation. Remember that $\lambda = \lambda' \tau_i = (\lambda' - \xi \mu A_i) \tau_i$ is an algebra valued gauge parameter which is also field dependent.

Then, we evaluate the symplectic structure density, (J.9), such that one of its entries is an improved symmetry transformation, $\delta_2 A \rightarrow \delta_\epsilon A$ as in (J.10) (and $\delta_1 A \rightarrow \delta A$). Using the equation of motion (J.4), and also the linearized equation of motion (J.6) (varied e.o.m. on phase space), it is straightforward to show that the first term becomes also an exact form

$$\Omega^{(2n+1)}(\delta, \delta_\epsilon) = n\kappa_n d\langle \lambda \delta A F^{n-1} \rangle - n(n + 1)\kappa_n d\left( \int_0^1 dt \langle \delta_\epsilon AA A_t \delta A F^{n-2} \rangle \right). \quad \text{(J.11)}$$

When the exact symmetry condition is satisfied: $\delta_\epsilon A = 0$ the symplectic structure density simply vanishes and it also vanishes the integral second term of the last expression. Therefore, we conclude that the surface charge density for a $D$-dimensional Chern-Simons theory, that satisfies the conservation law (i.e. is closed $dk = 0$), is

$$k_\epsilon^{(2n+1)} = n\kappa_n \langle \lambda \delta A F^{n-1} \rangle. \quad \text{(J.12)}$$

This is the main result of this appendix and it could have been expected by symmetry considerations. In fact, with only a connection at disposal there are no other ways to write a $(D - 2)$-form which is also a variation (or one-form in field space) and at the same time a gauge invariant expression.

Note that for $n = 1$ we recover the standard $D = 3$ dimensional surface charge for a CS theory computed in the main text.

We remark that in the last step we needed to invoke the exact symmetry condition to get rid of the integral term in the symplectic structure density, (J.11). This is not usually the case. For all other theories worked out through these notes the surface charge density is read directly once we replace the symmetry transformation as one of the entries of the symplectic structure. Instead, here there is this extra exact form, expressed as an integral in $t$. As stressed before this is related with the quasi-invariance of the CS theory and that our prescription to define the symplectic structure relies on the Lagrangian.\footnote{In the method based on the contracting homotopy operator this is not the case and the symplectic structure is defined directly from the equations of motion. Because of this reason this alternative prescription is sometimes called \textit{invariant} symplectic structure [45].}

Having said that, at this stage it should be already clear through our discussions that it is only for those cases, when the exact symmetry condition holds, that the surface charge density is closed and therefore it becomes a meaningful formula to compute true charges.

\subsection{J.1 D-CS surface charge from the contracting homotopy operator}

As a final remark of this appendix we note that the surface charge density formula for CS in $D = 2n + 1$ could had been easily obtained using the corresponding contracting homotopy operator. For a CS theory we can sketch the operator as

$$I_{\delta A} \equiv \delta A \frac{\partial}{\partial F}. \quad \text{(J.13)}$$
Now the Noether identity implies that $-\langle \kappa_n \delta_\epsilon AF^n \rangle = dS_\epsilon + \mathcal{N}_\epsilon^0$ with $S_\epsilon = \kappa_n \langle \lambda AF^n \rangle$. Thus, for the surface charge density

$$k_\epsilon^{(2n+1)} = I_{\delta \lambda} S_\epsilon = n\kappa_n \langle \lambda \delta AF^{n-1} \rangle.$$  \hspace{1cm} (J.14)

and we directly recover (J.12). This short calculation shows the power of the contracting homotopy operator approach. Of course the procedure to obtain (J.13) as a rigorous expression for the operator is the missing part here but, as we checked, its naive application it is certainly powerful enough.

K BF Theory

K.1 From Einstein-Cartan-$\Lambda$ to Chiral BF

In the following we show how the Einstein-Cartan-$\Lambda$ action in four dimensions can be written as a chiral theory (for instructive talks see [35]). Afterwards, we go further and rewrite it as a chiral BF theory.

The first observation is that the Einstein-Cartan-$\Lambda$ Lagrangian in (4.1) can be supplemented with two particular extra terms of the form $e_a e_b R^{ab}$ and $e_a e_b e^c e^b$ that do not modify the equations of motion. The later term is trivially zero, but the former term is part of the Nieh-Yan topological density [37]: $d(e_a T^a) = T^a T_a - e_a e_b R^{ab}$, therefore $e_a e_b R^{ab}$ is not a boundary term but can be traded, up to a boundary term, by $T^a T_a$. Still, if we add this term to the action, at the level of the equations of motion it can be checked that they are exactly equivalent to the Einstein-Cartan-$\Lambda$ equations of motion. This observation is also the basis of the so-called Holst action [38].

Consider the modified Einstein-Cartan-$\Lambda$ action

$$S[e^a, \omega^{ab}] = \kappa' \int_M \varepsilon_{abcd} e^a e^b \left( R^{cd} \pm \frac{1}{2\ell^2} e^c e^d \right) + 2i\eta_{[a} \varepsilon_{b]cd} e^a e^b \left( R^{cd} \pm \frac{1}{2\ell^2} e^c e^d \right) \hspace{1cm} (K.1)$$

$$= 4i\kappa' \int_M e_a e_b P_{+ cd} R^{ab} \hspace{1cm} (K.2)$$

$$= 4i\kappa' \int_M \left( e_a e_b R^{ab} (\omega^+) \pm \frac{1}{12\ell^2} \varepsilon_{abcd} e^a e^b e^c e^d \right), \hspace{1cm} (K.3)$$

in the first line we added both mentioned terms with a specific imaginary coefficient to the action. The compact notation in the second line uses the (anti)-de Sitter curvature $R^{ab} = R^{ab} \pm \frac{1}{2\ell^2} e^a e^b$ and the definition of the projectors $P_{+ cd} = \frac{1}{2} \left( \delta^{ab}_{cd} \pm \frac{1}{2} \varepsilon^{ab}_{cd} \right)$. In the third line we use the projector to define $\omega^{ab} = P_{+ cd} \omega^{cd}$ and a short calculation shows that the curvature splits $R(\omega) = R(\omega^+ + \omega^-) = R(\omega^+) + R(\omega^-)$ such that $P_+ R(\omega) = R(\omega^+)$. Thus, we have written an equivalent action that depends only on half of the connection components: The $\omega^{ab}$ are absent. This is a chiral Einstein-Cartan action that in fact still encodes the full GR dynamics.

With the action in this form we select an internal time direction and make an explicit split of the internal Lorentz group to define the Plebanski variables [39] as
\[
\Sigma^i(e) = \frac{1}{2i} P^c_{+cd} e^c e^d = i e^0 \wedge e^i - \frac{1}{2} \varepsilon^{ijk} e^j \wedge e^k \tag{K.4}
\]

\[
A^i(\omega) = \frac{1}{2i} P^c_{+cd} \omega^c e^d = i \omega^0 - \frac{1}{2} \varepsilon^{ijk} \omega^j e^k, \tag{K.5}
\]

where the summation rule on the internal indices \(i, j, k = 1, 2, 3\) is assumed, and because the Euclidean metric for the space indices \(\eta_{ij} = \delta_{ij}\) no distinction between upstairs and downstairs indices is needed in the following \(e^i = e_i\). With these definitions at hand the chiral action is rewritten as

\[
S[e^a, \omega^{ab}] = 4i \kappa' \int_M \left( \Sigma^i F^i(A) \pm \frac{1}{2 \ell^2} \Sigma^i \Sigma^j \right), \tag{K.6}
\]

where the field strength is

\[
F^i = dA^i + \frac{1}{2} \varepsilon^{ijk} A^j \wedge A^k \text{ (remember } \varepsilon^0_{ijk} = \varepsilon_{ijk} = \varepsilon^{ijk}\).
\]

To get a BF formulation, here we follow [40], we note that \(\Sigma^i\) is a basis for self-dual two-forms in four dimensions. Therefore, if we have a collection of three arbitrary two-form self-dual fields named \(B^i\) we have

\[
B^i = \sigma b^i_j \Sigma^j, \tag{K.7}
\]

where the \(b^i_j\) are the components on the basis, and \(\sigma = \pm\) is a sign. As explained in [40] because of the conformal invariance of the self-dual property we can always choose the \(3 \times 3\) matrix as unimodular: \(\det b^i_j = 1\). In order to write the action in terms of \(B^i\) fields as true variables and still have a GR theory we have to impose a constraint on the \(B^i\) fields to fix \(b^i_j\), such that the \(b^i_j\) matrices are just the identity or at most a 3D rotation, that is

\[
b^i_k b^j_l \delta_{kl} = \delta_{ij}. \tag{K.8}
\]

Now, note that the Plebanski variables satisfy

\[
\Sigma^i \wedge \Sigma^j \sim \delta^i_j, \tag{K.8}
\]

inspired by previous equation, a constraint that does the job is \(B^i \wedge B^j \sim \delta^i_j\), or more explicitly

\[
B^i \wedge B^j = \frac{1}{3} \delta^i_j B_k \wedge B^k, \tag{K.9}
\]

To implement the constraint into the action we introduce a term with a Lagrange multiplier \(\chi^i_\jmath \ B^i \wedge B^j\) that has a traceless condition \(\chi^i_\jmath \delta^i_j = 0\). Then, we have the following chiral BF action

\[
S[B^i, A^i, \chi^i_\jmath] = \int_M \left( B^i F^i(A) \pm \frac{1}{2 \ell^2} B^i B^j + \frac{1}{2} \chi^i_\jmath B^i B^j \right). \tag{K.10}
\]

The overall factor \(4i \kappa'\) is dropped because it does not affect the equations of motion and here we will not consider the coupling with non-gravitational fields. The constraint ensures us that the \(B^i\) has a Plebanski variable form in terms of the vielbein and therefore a GR metric is implicit. However, it is interesting to note that the metric components can be obtained at once through the Urbantke’s formula [41]

\[
\sqrt{g} g_{\mu \nu} = \frac{1}{12} \varepsilon_{ijk} \delta^\alpha_\beta \delta^\gamma_\delta B^i_{\mu \alpha} B^j_{\beta \gamma} B^k_{\nu \delta}, \tag{K.11}
\]

This is the equivalent of the equation \(g_{\mu \nu} = e^a_{\mu} e^b_{\nu} \eta_{ab}\) for the \(B^i\) field.

Here we have explained the equivalence of Einstein-Cartan-\(\Lambda\) and chiral BF theories at the level of the action. It is also instructive to recover the Einstein field equations in its metric form directly from the chiral BF equations of motion, for the interested reader we suggest [40] and [46].
K.2 Surface Charges for a Chiral BF Theory

Consider the chiral BF action (K.10). It should be noted that the connection \( A^i \) defines, as usual, a covariant exterior derivative \( d_A(\cdot) = d(\cdot) + [A, (\cdot)] \). The variation of the Lagrangian is straightforward

\[
\delta L = \delta B^i E_{(b)}^i + \delta A^i E_{(a)}^i + \delta \chi^{ij} E_{(x)}^{ij} + d \Theta(\delta A), 
\]  

(K.12)

with the three equations of motion and the boundary term given by

\[
E_{(b)}^i = F^i \pm \frac{1}{\ell^2} B^i + \chi^{ij} B^j = 0 
\]  

(K.13)

\[
E_{(a)}^i = d_A B^i = 0 
\]  

(K.14)

\[
E_{(x)}^{ij} = 0 \rightarrow B^i \wedge B^j = \frac{1}{3} \delta^{ij} B_k \wedge B^k 
\]  

(K.15)

\[
\Theta(\delta A) = B^i \delta A^i. 
\]  

(K.16)

The boundary term is used to compute the symplectic structure density

\[
\Omega(\delta_1, \delta_2) = \delta_1 B^i \wedge \delta_2 A^i - \delta_2 B^i \wedge \delta_1 A^i. 
\]  

(K.17)

The improved infinitesimal gauge plus diffeomorphism transformations are

\[
\delta_\epsilon B^i = \xi \cdot d_A B^i + d_A \xi \cdot B^i + [\lambda, B]^i 
\]  

(K.18)

\[
\delta_\epsilon A^i = \xi \cdot F^i - d_A \lambda^i, 
\]  

(K.19)

then, as usual, we evaluate the symplectic structure density on those transformations

\[
\Omega(\delta, \delta_\epsilon) = \delta B^i (\xi \cdot F^i - d_A \lambda^i) - (d_A \xi \cdot B^i + \xi \cdot d_A B^i + [\lambda, B]^i) \delta A^i 
\]  

(K.20)

\[
= -d(\delta B^i \lambda^i + \xi \cdot B^i \delta A^i) + \delta B^i \xi \cdot F^i + d_A \delta B^i \lambda^i - \xi \cdot d_A B^i \delta A^i 
\]  

(K.21)

\[
-\xi \cdot \delta B^i d_A \delta A^i - [\lambda, B]^i \delta A^i 
\]  

(K.22)

\[
= -d(\delta B^i \lambda^i + \xi \cdot B^i \delta A^i), 
\]  

(K.23)

where to get the second line we use the Leibniz’s rule for the exterior covariant derivative to express the second and third terms of the first line as exact forms (plus the rest). Then, we use the equation of motion and the linearized equations of motions. With this, all the five non-exact forms on the third line cancel. In particular note \( \delta F^i = d_A \delta A^i \) and the linearized equation of motion: \( \delta F^i \pm \frac{1}{\ell^2} \delta B^i - \chi^{ij} \delta B^j = 0 \). Hence, the surface charge density formula for the chiral BF theory is simply

\[
k_\epsilon = -\delta B^i \lambda^i + \delta A^i \xi \cdot B^i. 
\]  

(K.24)

For the surface charge, built as a closed integral of the previous formula, to be conserved we require, as usual, the exact symmetry condition to hold: \( \delta_\epsilon B^i = 0 \) and \( \delta_\epsilon A^i = 0 \) as in (K.18) and (K.19), respectively. We stress that still from the condition for \( B^i \) it is possible to directly solve the parameter \( \lambda^i \), we leave it as an exercise.
K.3 Jackiw-Teitelboim model as a BF theory

Here we show how pure gravity in 1 + 1 dimensions admits a BF-like formulation [47] and then we compute the corresponding surface charge density.

Analogous to the BF formulation of General Relativity in four dimensions, it is possible to have a similar construction for a gravity theory on a line, i.e., lineal gravity. Recently, this topic has active interest due to its use as a 1 + 1 model in the context of the AdS$_2$/CFT$_1$ conjecture [48] and also as a low-energy limits model of near-extremal black holes [49]; in both cases this is a toy model with the attractiveness of being simple enough such that many computation are doable even in a quantum regime.

The Einstein-Hilbert action in 1 + 1 is an uninteresting theory of gravity because the action is just a boundary term. In [50, 51] Jackiw and Teitelboim established a new model—the Jackiw-Teitelboim (JT) model—to describe the dynamics of gravity in (1 + 1)-dimensions.

They solved the problem by simply introducing an additional variable, a Lagrange multiplier field $\eta$, as follows

$$S[g_{\mu\nu}, \eta] = \int_{\mathcal{M}} d^2x \sqrt{-g} \eta (R - \Lambda),$$  \hspace{1cm} (K.25)

where $\Lambda$ is the cosmological constant.

Here we are interested in the fact that this JT action admits a BF-like formulation [47], where now the $B$ field is a zero-form that plays the role of a Lagrange multiplier. The first order action to consider is

$$S[B^i, A^i] = \int_{\mathcal{M}} \langle B, F(A) \rangle,$$  \hspace{1cm} (K.26)

with the field strength $F = dA + A \wedge A$, and $A$ an algebra valued gauge connection $A = A^i X_i$, with $X_i$ the algebra generators. The $B$ field is also algebra valued $B = B^i X_i$ and the $\langle \cdot, \cdot \rangle$ denotes the non-degenerate bilinear form compatible with the algebra.

In order to get a gravitational model we choose our variables valued on the anti-de Sitter algebra $\mathfrak{so}(2,1)$ given by

$$[P_a, P_b] = \varepsilon_{ab} J, \quad [J, P_a] = \varepsilon^b_a P_b,$$  \hspace{1cm} (K.27)

where we have hidden the cosmological constant by the defining the generator $P_a = \frac{1}{\sqrt{\Lambda}} \tilde{P}_a$.

This algebra admits a non-degenerate invariant bilinear form (in this subsection the indices run as $a, b, c, ... = 0, 1$)

$$\langle P_a, P_b \rangle = \eta_{ab}, \quad \langle J, J \rangle = 1.$$  \hspace{1cm} (K.28)

Explicitly $A = A^i X_i = e^a P_a + \omega J$ and $B = B^i X_i = B^a P_a + \tilde{B} J$, with $P_a$ the translations and $J$ the rotation algebra generators. The $e^a$ field corresponds to the one-form zweibein (related with a metric field $g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$), and $\omega = \tilde{\varepsilon}_{ab} \omega^{ab}$ is the dual spin connection ($\varepsilon_{01} = -\varepsilon_{10} = 1$). With the bilinear form of the anti-de Sitter algebra the (K.26) action expands as

$$S[B, e^a, \omega] = \int_{\mathcal{M}} \langle B, F \rangle = \int_{\mathcal{M}} B^i F^j (X_i, X_j) = \int_{\mathcal{M}} \left( B_a (d e^a + \varepsilon^a_{b k} \omega^b) + \tilde{B} (d \omega + \varepsilon_{ab} e^a e^b) \right),$$  \hspace{1cm} (K.29)

---

21 The Einstein-Hilbert action $\int d^2x \sqrt{-g} R$ is a surface term, it is in fact the topological Euler invariant and therefore does not lead to sensitive equations of motion.
where the $B^i$ field decomposes in the Lagrange multipliers: $B^a$ enforcing the vanishing of torsion and $\tilde{B}$, equivalent to the field $\eta$, enforcing the vanishing of the (anti-de Sitter) curvature. The torsionless equation allows us to solve the spin connection in terms of the geometric field $\epsilon^a$, by replacing it back into the action (K.29) it is possible to obtain a second order formulation of the action which is nothing but the JT model (K.25).

Now we compute the surface charge density. We start by varying the action (K.26) to read the symplectic potential

$$\Theta(\delta A) = \langle B, \delta A \rangle = B^i \delta A^i \langle X_i, X_j \rangle = B^a \delta e_a + \tilde{B} \delta \omega.$$  \hspace{1cm} (K.30)

Then, the symplectic structure density yields

$$\Omega(\delta, \delta \epsilon) = \langle \delta_1 B, \delta_2 A \rangle - \langle \delta_2 B, \delta_1 A \rangle = \delta_1 B^a \delta_2 e_a - \delta_2 B^a \delta_1 e_a + \delta_1 \tilde{B} \delta_2 \omega - \delta_2 \tilde{B} \delta_1 \omega.$$  \hspace{1cm} (K.31)

The action (K.29) is invariant under diffeomorphism and gauge transformations. The infinitesimal transformation of the field, in their improved version, are

$$\delta_\epsilon A = \xi \lrcorner F - d_A \lambda,$$
$$\delta_\epsilon B = \xi \lrcorner d_A B + d_A \xi \lrcorner B + \{\lambda, B\},$$  \hspace{1cm} (K.32)

with $\xi$ a vector field and $\lambda = \lambda^i X_i = \lambda^a P_a + \lambda J$ a gauge parameter valued in the $\mathfrak{so}(2,1)$ algebra. Then, plugging (K.32) into (K.31), and using the equations of motion and the linearized equations of motions we have

$$\Omega(\delta, \delta \epsilon) = dk_\epsilon,$$  \hspace{1cm} (K.33)

with the surface charge density

$$k_\epsilon = -\langle \delta B, \lambda \rangle = -\delta B^i \lambda^j \langle X_i, X_j \rangle = -\delta \tilde{B} \lambda - \delta B^a \lambda_a.$$  \hspace{1cm} (K.34)

Notice that, as it might have guessed, this expression corresponds to the formula (4.35), where $\xi \lrcorner B^i = 0$ because in this context it is a zero form.

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