Differentially Private Median Forests for Regression and Classification

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Abstract

Random forests are a popular method for classification and regression due to their versatility. However, this flexibility can come at the cost of user privacy, since training random forests requires multiple data queries, often on small, identifiable subsets of the training data. Differentially private approaches based on extremely random trees reduce the number of queries, but can lead to low-occupancy leaf nodes which require the addition of large amounts of noise. In this paper, we propose DiPriMe forests, a novel tree-based ensemble method for regression and classification problems, that ensures differential privacy while maintaining high utility. We construct trees based on a privatized version of the median value of attributes, obtained via the exponential mechanism. The use of the noisy median encourages balanced leaf nodes, ensuring that the noise added to the parameter estimate at each leaf is not overly large. The resulting algorithm, which is appropriate for real or categorical covariates, exhibits high utility while ensuring differential privacy.

1 Introduction

The prevalence of data has been one of the key drivers of technological innovation in the last decade. The abundance of data, allied with ever-increasing computing power, has driven the rapid development of sophisticated machine learning techniques, many of which have achieved hitherto unseen levels of performance. Data collection today is pervasive, across applications and devices. The has resulted in data privacy becoming a matter of public concern.

It has long been known that querying even aggregated or perturbed data can lead to leakage of private information [1], motivating the development of databases and algorithms that mitigate such privacy breaches. Differential privacy [2] is one of the most rigorous ways of analysing and ameliorating such privacy risks. If an algorithm is $\epsilon$-differentially private, it means we can apply a multiplicative bound to the worst-case leakage of an individual’s private information. Many algorithms have been developed with this goal in mind, such as differentially private variants of linear regression [3], k-means clustering [4, 5] and expectation maximization [6].

Such privacy guarantees come at a cost—modifying an algorithm to be $\epsilon$-differentially private typically involves adding noise to any queries made by that algorithm, which will tend to negatively
affect the algorithm’s performance. This cost will tend to be higher when privatizing more complex algorithms that require multiple queries of the data, such as non-linear regression and classification algorithms \[7, 8\]. One such family of non-linear regression and classification algorithms, that allows for flexible modeling but involves multiple queries, is the class of tree-based methods. In particular, random forests and their variants have been shown to perform well on both classification and regression problems \[9, 10\], and are used in many real-world applications. Tree-based ensemble methods make minimal assumptions on the parametric forms of relationships within the data, and can be easily applied to a mixture of continuous and categorical covariates. However, building trees that capture the appropriate structure requires many queries of the data, making them challenging to privatize. Further, since the data is partitioned into arbitrarily small subsets, the noise that must be added to each subset to ensure differential privacy can quickly swamp the signal.

We propose differentially private median (DiPriMe) forests, a novel, differentially private machine learning algorithm for nonlinear regression and classification with potentially sensitive data. Unlike previous differentially private tree-based algorithms, our approach can deal with continuous or categorical covariates, and is appropriate for either regression or classification. DiPriMe forests avoid overwhelming the signal at each leaf with noise, by using a median-based splitting criteria to encourage a balanced distribution of data across the leaf nodes \[11\]. The median is a robust statistic, making it relatively cheap to privatize, and the noise added during privatization encourages exploration of the solution space.

We begin by reviewing the concept of differential privacy, and discussing related approaches, in Section 2, before introducing DiPriMe forests in Section 3. We provide theoretical and empirical guarantees on both the privacy and the utility of our approach. In Section 4 we show that our method outperforms existing differentially private tree-based algorithms on a variety of classification and regression tasks.

2 Preliminaries

2.1 Differential Privacy

Differential privacy \[12\] is a rigorous framework for limiting the amount of information that can be inferred from an individual’s inclusion in a database, or in the training set for an algorithm. Formally, a randomized mechanism \( F \) satisfies \( \epsilon \)-differential privacy for all datasets \( D_1 \) and \( D_2 \) differing on at most one element and all \( S \subseteq \text{Range}(F) \) if

\[
\Pr[F(D_1) \in S] \leq e^{\epsilon} \Pr[F(D_2) \in S].
\]  

\[1\] implies that the inclusion of an individual’s data can change the probability of any given outcome by at most a multiplicative factor of \( e^\epsilon \). A lower value of \( \epsilon \) provides a stronger privacy guarantee, as it limits the effect the omission of a data point can have on the statistic.

Typically, the mechanism \( F \) is a randomized form of some deterministic query \( f \). To determine the degree of randomization required to satisfy \[1\], we need to know the global sensitivity \[13, 14\],

\[
\Delta(f) = \max_{D_1, D_2} \| f(D_1) - f(D_2) \|_1.
\]  

The global sensitivity tells us the maximum change in the outcome of the query \( f \) due to changing a single data point. Armed with the global sensitivity, we can use a number of approaches to ensure \( \epsilon \)-DP; we outline the two most common mechanisms below.

**Laplace mechanism** Starting with a deterministic query \( f : \mathcal{D} \rightarrow \mathbb{R}^d \), where \( \mathcal{D} \) is the space of possible data sets, we can construct an \( \epsilon \)-DP query-answering mechanism that adds appropriately scaled Laplace noise, so that \[11\]

\[
F(X) = f(X) + (Y_1, Y_2, \ldots, Y_d), \quad Y_i \overset{iid}{\sim} \text{Laplace}(0, \Delta_i(f)/\epsilon),
\]  

where \( \Delta_i(f) \) denotes the sensitivity of the \( i \)-th coordinate of the output. Note how the noise added scales as \( 1/\epsilon \) – increased privacy directly translates to increased noise variance.
Exponential mechanism The Laplace mechanism assumes that our query returns values in $\mathbb{R}^d$. A more generally applicable privacy mechanism is the exponential mechanism [15], which allows us to pick an outcome $r \in \mathcal{R}$, where $\mathcal{R}$ is some arbitrary space. We define a scoring function $q : \mathcal{D} \times \mathcal{R} \rightarrow \mathbb{R}$ with global sensitivity $\Delta(q)$, and a base measure $\mu$ on $\mathcal{R}$. For any dataset $D \in \mathcal{D}$, selecting an outcome $r$ with probability

$$\Pr[F(D) = r] \propto e^{\epsilon q(D,r)/2\Delta(q)} \times \mu(r)$$  \hspace{1cm} (4)

ensures $\epsilon$-differential privacy. Clearly, the scoring function $q$ should be constructed such that preferred outputs are assigned higher scores. We usually adopt a uniform $\mu$ [15].

Often, algorithms will involve multiple queries, requiring us to account for the overall differential privacy of the composite algorithm. We can make use of the following two composition theorems to obtain the overall privacy level.

Sequential composition tells us that a sequence of differentially private queries maintain differential privacy. Let $F_i$ each provide $\epsilon_i$-differential privacy. Then sequentially evaluating $F_i(X)$ provides $(\sum_i \epsilon_i)$-differential privacy [16].

Parallel composition tells us that the privacy guarantee for a set of queries on disjoint subsets of data is only limited by the worst-case privacy guarantee for any of the queries. If $F_i$ each provide $\epsilon_i$-differential privacy, and $D_i$ are arbitrary disjoint subsets of the dataset $D$, then the sequence of $F_i(X \cap D_i)$ provides $(\max_i \epsilon_i)$-differential privacy [16].

### 2.2 Differentially private tree-based methods

There have been several methods proposed in recent literature to learn ensembles of decision trees for classification in a differentially private manner (although, to the best of our knowledge, this paper is the first to consider regression). Learning a decision tree entails greedily choosing the best split at each internal node and storing sufficient statistics (counts, in the case of classification) at the leaf nodes. To ensure differential privacy, we must privatize both the splitting mechanism and the sufficient statistics. Privatizing the sufficient statistics is relatively straightforward via the Laplace mechanism, with only minor differences between approaches explored in the literature; the key difference between the proposed methods lies in how they privatize the splits.

One way of determining the splits in a decision tree is to choose the split with lowest entropy. [17] privatizes this by noising the node counts using the Laplace mechanism, and then selecting a binary split from a set of candidates using the exponential mechanism with entropy as the score. This is built upon in the differentially private random forest (DP-RF) algorithm [18], which builds multiple private trees using bootstrapped samples. The trees are grown iteratively until they reach a specified maximum depth or the node is pure, i.e., contains instances of a single class. The differentially private decision forest [14] and the differentially private greedy decision forest [19] both use the Gini index instead of entropy as the score function in the exponential mechanism.

A limitation of these splitting approaches based on the exponential mechanism is that they assume categorical covariates. Continuous covariates need to be discretized in some manner. Any discretization techniques that utilize information about the data must also be privatized, increasing the overall privacy budget [20]. [21] proposes a relaxation on the DP-RF approach so that the ensemble of trees preserves the variance, instead of the entire distribution of the data. This relaxation enables their method to achieve better performance, and allows for numerical covariates, at the expense of losing any claim to $\epsilon$-differential privacy.

Extremely randomized trees [10] are an alternative tree-based method where the splits are chosen randomly, without considering the data. These are the motivation for the learning scheme proposed in [13], which avoids the need to split the privacy budget across the levels of each tree by drawing completely random splits. As this choice is entirely random, none of the privacy budget is consumed when determining splits. Training the ensemble of $N_T = \log(N)$ trees then involves passing all the $N$ data instances through the trees and using the Laplace mechanism to store class counts.
3 Differentially private median forests

As we saw in Section 2.2, existing privatized versions of random forests make use of private queries at each internal node, both to determine the dimension to split the node, and the value at which to split. This very quickly eats up the privacy budget, leading to very noisy function estimates. Further, the privacy mechanisms used assume that all inputs are categorical, or have been discretized, limiting their use on real-valued or mixed real/categorical data.

Privatized versions of extremely random trees [13] involve far fewer queries, since the splits are chosen randomly. This allows us to spend more of our privacy budget on privatizing the leaf nodes. However, the differential privacy requirement means we cannot deterministically stop pruning the tree based on the number of data points at a node, and as a result we are likely to end up with many low or zero-occupancy leaves. Since the detrimental impact of privatizing the parameter estimate at a leaf node increases as the number of occupants decreases, this approach can lead to excessively noisy predictions.

We take an in-between approach, using the data to gently inform the splitting of the tree. We split with an in-between approach, using the data to gently inform the splitting of the tree. We split tree-based algorithms, rather than use a single tree, we construct an ensemble of trees. As described in Algorithm 1, we randomly partition our data into disjoint subsets of the data, we can assign $\epsilon := \rho c / d_{\max}$ to each internal split, where $d_{\max}$ is the maximum tree depth.

At each internal node, we create a candidate split for each dimension using a differentially private version of the median. Let $D_i$ be the subset of $D$ associated with node $i$, and let $N_i = |D_i|$. For numerical covariates, we order the observations and use $\epsilon_1 / 2$ of our privacy budget to select the $j$th observation, using the exponential mechanism with probability

$$Pr(j) \propto \exp \left\{ -\frac{\epsilon_1}{2} |N_i - 2j| \right\}.$$ 

We then pick the value for the split uniformly from the interval between the $j$th and $(j + 1)$th value. This scoring function penalizes imbalanced splits, as $|N_i - 2j| = |N_i - j - j|$. Similarly, for a categorical covariate we consider all possible splits, selecting split $(B, D_i \setminus B)$ with probability

$$Pr(B, D_i \setminus B) \propto \exp \left\{ -\frac{\epsilon_1}{2} N_i - 2|B| \right\}.$$ 

Having selected candidate splits from $K$ covariates, we select a covariate to split on using the negative of the mean-squared error as the scoring function. The sensitivity of the mean-squared error is $4B^2/N_i$, where $B = \max\{|B_L|, |B_U|\}$. So, we choose an attribute $\tilde{a}$ to split on with probability

$$Pr(\tilde{a}) \propto \exp \left\{ -\frac{\epsilon_1 N_i}{8B^2} (MSE(\tilde{a})) \right\}.$$ 

Forests of trees based on median splits of attributes have been proposed as a simplification of random forests, and the resulting estimators have been proven to be consistent [22]. In our setting, the median serves to partition the data into sets of similar sizes, discouraging low-occupancy leaf nodes. Both these phenomena are empirically observed in Figure 1.

Using a noisy, rather than deterministic, median increases estimate variation across trees, allowing better exploration of the solution space. Once we have reached our desired depth, we privatize the statistics at each leaf. In the classification setting, the class counts each have sensitivity 1, so we add Laplace(0, 1/$c_2$) noise to each count. In the regression setting, the sensitivity of the mean is $\Delta_i(\mu) = (B_U - B_L)N_i$, where $N_i$ is the number of data points at node $i$, so we add Laplace(0, $\Delta_i/c_2$) noise to the mean. In both cases, the quality of the privatized prediction deteriorates as $N_i$ decreases. However since we are encouraging even splits at each node, we are likely to avoid scenarios where the amount of noise is excessive relative to the signal, and we can pre-select a depth that will have no empty leaves with high probability.

We summarize the process of constructing a single DiPriMe tree in Algorithm 1. As is common in tree-based algorithms, rather than use a single tree, we construct an ensemble of trees. As described in Algorithm 2 (with $part = True$), we randomly partition our data into $N_T$ groups, and learn a DiPriMe tree on each group. The parallel composition theorem ensures that the overall privacy loss is given by the loss on a single tree. We use the following notation in Algorithms 1 and 2: $D = (X, Y)$.

\[ \text{If we have less data, we may choose not to partition, however this would multiply the privacy budget by } N_T. \]
Figure 1: Histograms were generated for 100 trees of maximum depth 5 with $\epsilon = 10$. The splits yielded by DiPriMe have around half the data instances assigned to each child as opposed to the highly imbalanced splits obtained when selecting random splitting values.

refers to the set of data points to which the tree $T$ is being fit. $X$ denotes the input features and $Y$ denotes the corresponding target values. $A$ refers to the set of features that the tree can split on with $R_A$ denoting the corresponding range of categories. $B_L$ and $B_U$ refer to the lower and upper bounds on the target values $Y$. $\epsilon$ is total privacy budget for the tree, and $\rho$ is the fraction of the privacy budget allocated to determine the median split. We include code in the supplement, and will make this public upon publication.

**Algorithm 1** Differentially Private Median (DiPriMe) Tree

```python
1: class DIPRIME_TREE (i, i_max, k)  # Initialize empty tree
2:   if i ≤ i_max then
3:     d ← i
4:     d_max ← i_max
5:     K ← k
6:   end if
7: end class
8: procedure FITTREE(T, D, A, R_A, B_L, B_U, $\epsilon$, $\rho$)
9:   D = (X, Y)
10:  N = number of data points in X, M = number of features in X
11:  $R$ ← $B_U - B_L$  # Range of target variable
12:  $\epsilon_1$ ← $\frac{\epsilon \rho}{2d_{max}}$, $\epsilon_2$ ← $\epsilon(1 - \rho)$  # Privacy budget for split and mean
13:  if $T.d = T.d_{max}$ then or $A$ is empty
14:     Store privatized mean or class counts in $T$.
15:     return $T$
16: end if
17: $B$ ← max{|$B_L$|, |$B_U$|}
18: $T.ind$, $T.val$ ← FINDSPLIT(D, R_A, B, $\epsilon$, $\rho$, T, K)  # Private median split
19: $A_R$, $R_{A_R}$, $A_L$, $R_{A_L}$ ← SPLIT_RANGE(R_A, T.ind, T.val)  # Features and ranges for children
20: $D_L$ and $D_R$ denotes all the points in $D$ assigned to left and right child nodes respectively.
21: $T_R$ ← DIPRIME_TREE(T, d + 1, T.d_{max}, T, K)
22: $T.right$ ← FITTREE(T_R, D_R, A_R, R_{A_R}, B_L, B_U, $\epsilon$, $\rho$)
23: $T_L$ ← DIPRIME_TREE(T, d + 1, T.d_{max}, T, K)
24: $T.left$ ← FITTREE(T_L, D_L, A_L, R_{A_L}, B_L, B_U, $\epsilon$, $\rho$)
25: end procedure
```
Algorithm 2 Differentially Private Median (DiPriMe) Forest

1: class DiPriMeForest \((n_T, d_{\text{max}}, k, \text{part})\) \(\triangleright\) Initialize ensemble of trees
2: \(\text{partition} \leftarrow \text{part}, N_T \leftarrow n_T, \mathcal{T} \leftarrow \{}\)
3: for \(i \leftarrow 1, n_T\) do
4: \(T \leftarrow \text{DiPriMe}(0, d_{\text{max}}, k)^{'}\)
5: Add \(T\) to \(\mathcal{T}\)
6: end for
7: end class
8: procedure \(\text{FitForest}(F, X, Y, A, R_A, B_L, B_U, \epsilon, \rho)\)
9: \(N_T = F.N_T\)
10: if \(F.\text{partition}\) then
11: \(\text{Partition } D = (X, Y) \text{ into } F.N_T \text{ parts. Denoting these partitions as } \{D_i\}_{i=1,\ldots,N_T}\)
12: \(i \leftarrow 0\)
13: for all \(T \in F.\mathcal{T}\) do
14: \(T \leftarrow \text{FitTree}(T, D_i, A, R_A, B_L, B_U, \epsilon, \rho)\)
15: \(i \leftarrow i + 1\)
16: end for
17: else
18: for all \(T \in F.\mathcal{T}\) do
19: \(T \leftarrow \text{FitTree}(T, D, A, R_A, B_L, B_U, \epsilon/N_T, \rho)\)
20: end for
21: end if
22: end procedure

3.1 Utility analysis

We consider the utility of a single, privatized DiPriMe regression tree, in comparison with a non-private tree with median splits [22]. For ease of analysis, we restrict ourselves to continuous features and assume that the dataset is perfectly divisible into the leaf nodes, i.e., \(N = 2^d\) for a depth-\(d\) tree. Hence, for a tree with median splits, we will have \(N/2^d\) data points at each leaf node. Let \(\text{Obj}_{\text{No noise}}\) be the loss under the non-private median tree, i.e., the sums of the per-leaf-node sums of squares. Let \(\text{Obj}_{\text{Median noise}}^i\) denote the sum of squares at node \(i\).

**Theorem 1.** Let \(\text{Obj}_{\text{Median noise}}\) be the loss under a tree where the splits have been randomized as described above, but where the sufficient statistics at the leaf nodes are not randomized. Let \(\tilde{N}_i\) be the number of data points at the \(i^{th}\) leaf node. Then if \(|\tilde{N}_i - N/2^d| \leq t\), then \(|\text{Obj}_{\text{Median noise}}^{\text{Median noise}} - \text{Obj}_{\text{No noise}}^{\text{No noise}}| \leq 4B^2t\), where \(B = \max\{|B_U|, |B_L|\}\).

**Corollary 1.1.** If \(\max_i |\tilde{N}_i - N/2^d| \leq t\), then \(|\text{Obj}_{\text{Median noise}}^{\text{Median noise}} - \text{Obj}_{\text{No noise}}^{\text{No noise}}| \leq 2^{d+1}B^2t\)

![Figure 2: Tail probabilities of \(|\tilde{N}_i - N/2^d|\) for \(d = 10\) are estimated from the histogram. The red line depicts the tighter of the two boundes derived in Theorem 2, the blue line depicts the empirical estimate.](image)
We consider three datasets to benchmark our method’s regression performance: the Parkinson’s Appliances Flight Delay (800K) datasets. We demonstrate the tightness of the bounds derived in Theorem 2 in Figure 2. We estimate the tail probability of \( \left| \hat{N}_i - \frac{N}{2^d} \right| \geq t \) using the tighter of:

\[
P\left( \left| \hat{N}_i - \frac{N}{2^d} \right| \geq t \right) \leq \frac{\gamma}{e^2t^2},
\]

where \( \gamma = \frac{8(1-2^{-2d})}{3} \), \( \beta^2 = \frac{1}{2}(\sqrt{1-2/e} + 1) \).

**Theorem 3.** Let \( \text{Obj}_{DP} \) be the loss due to Algorithm [4] Then, \( \mathbb{E}[\text{Obj}_{DP}] - \text{Obj}_{No\ noise} \leq \left( 2B^2 + \frac{N^2}{c_1^2(N/2^d - 1)} \right) 2^{d+1} \) with probability at least \( 1 - \zeta \), where \( P\left( \left| \hat{N}_i - \frac{N}{2^d} \right| \geq t \right) \leq 2^{1-d} \zeta \).

We demonstrate the tightness of the bounds derived in Theorem 2 in Figure 2. We estimate the tail probabilities of \( \left| \hat{N}_i - \frac{N}{2^d} \right| \) by simulating draws from our algorithm and considering the histograms of the resulting histogram of samples. The bound becomes tighter for higher values of \( t \).

## 4 Experiments

To consider the utility of our proposed algorithm, we look at the estimate qualities obtained across a range of regression and classification tasks.

### 4.1 Regression

We consider three datasets to benchmark our method’s regression performance: the Parkinson’s telemonitoring dataset \( (N = 5875) \) and the Appliance Energy Prediction dataset \( (N = 19735) \) from the UCI Machine Learning Repository [23], and the Flight Delay dataset used by [24, 25]. The UCI datasets contain a mixture of categorical and numeric features, while the Flight Delay dataset contains only numeric features. For the purposes of computational complexity, we sampled 800,000 data instances from the Flight Delay dataset for this experiment. For each dataset, we scaled the target variable to lie in \([0, 1]\), took 90% of the data as the training set and computed the mean squared error (MSE) over the test set. The DP-ERT algorithm refers to a privatized version of the Extremely Randomized Trees algorithm [10], akin to that delineated in [13] for classification (we are not aware of any differentially private tree-based methods developed explicitly for regression). The results shown in Table 1 are for \( N_T = 10 \) trees in each ensemble, with the number of covariate splits to consider set to \( K = 10 \) for all but the DP-ERT. The private methods were run for \( \epsilon = 10 \) and \( \rho = 0.5 \). The maximum depth was taken to be 5 for the two UCI datasets and 10 for the Flight Delay dataset.

|                  | Parkinson’s | Appliances | Flight Delay (800K) |
|------------------|-------------|------------|---------------------|
| Random Forest    | 2.27 × 10^{-2} | 7.21 × 10^{-3} | 2.00 × 10^{-4} |
| ERT              | 2.79 × 10^{-2} | 8.04 × 10^{-3} | 2.06 × 10^{-4} |
| Median Trees     | 2.465 × 10^{-2} | 8.16 × 10^{-3} | 2.22 × 10^{-4} |
| DP-ERT           | 3.61 × 10^{-2} | 8.95 × 10^{-3} | 2.28 × 10^{-4} |
| DiPriMe          | 3.15 × 10^{-2} | 8.48 × 10^{-3} | 2.40 × 10^{-4} |

DiPriMe clearly outperforms DP-ERT as a private tree-based ensemble learner for regression. We then compared the performance of DiPriMe with these algorithms for varying values of maximum depth, number of trees and privacy budget.

The utility of using the median to form splits is borne out by all the plots in in Figure 3 with the ensemble of median trees showing comparable performance and similar trends to both the random forest and extremely randomized trees algorithms. Figure 3(a) illustrates the inherent trade-off between learning deeper trees and utility. Deeper trees give a finer approximation of the data, demonstrated by the decreasing MSE of the non-private methods. However, this deteriorates the utility of DiPriMe by (a) reducing the privacy budget for the split at each node (b) increasing the sensitivity of the mean at the leaf nodes as there are likely to be fewer data instances in deeper nodes.
Increasing the number of trees in the ensemble results in a similar trade-off, as shown in Figure 3(b); the number of data points to learn each tree is inversely related to the number of trees in the ensemble. So, while more trees are expected to generally reduce the mean squared error, each tree has less data to learn from.

4.2 Classification

We use three datasets from the UCI Machine Learning Repository [23], the Banknote Authentication ($N = 1372$), Credit Card Default ($N = 30000$) and Wall-Following Robot Navigation ($N = 5456$) datasets, to compare the performance of our proposed algorithm to the DP-DF algorithm [14] and the DP-ERT algorithm [13]. The Credit Card Default data contains both numeric and categorical features, while the other two datasets contain only numeric features. As DP-DF requires categorical features, we bin the numeric features into 5 bins. We chose this data-agnostic discretization procedure to avoid leaking privacy. Some papers have proposed other discretization strategies [18] but these have to be privatized and hence incur additional privacy budget.

Table 2: Comparison of DiPriMe with various non-private and private tree-based ensemble methods for classification, $N_T = 10$, $K = 5$, $d_{\text{max}} = 5$, $\epsilon = 2$, $\rho = 0.5$.

|                      | Banknote Authentication | Credit Card Default | Robot Navigation |
|----------------------|------------------------|---------------------|-----------------|
| Random Forest        | 0.044                  | 0.181               | 0.0440          |
| ERT                  | 0.051                  | 0.196               | 0.190           |
| Median Trees         | 0.058                  | 0.223               | 0.126           |
| DP-DF [14]           | 0.413                  | 0.223               | 0.397           |
| DP-ERT               | 0.490                  | 0.324               | 0.450           |
| DiPriMe              | 0.072                  | 0.223               | 0.207           |

Table 2 shows the classification errors obtained by each method. The performance of median trees is comparable to that of extremely randomized trees. In contrast, DiPriMe far exceeds the performance of privatized extremely randomized trees and DP-DF. This can likely be attributed to (a) DiPriMe’s capability to directly utilize and split on numeric features without the need for prior discretization (b) the performance gain derived from learning splits. As we see in Figure 3, while DiPrime has a notable loss of accuracy compared with the nonprivate algorithms for small values of $\epsilon$, as $\epsilon$ increases we get comparable performance. By contrast, DP-ERT and DP-DF continue to underperform even as $\epsilon$ increases.

5 Discussion

We have presented a new, differentially private, tree-based method for both regression and classification, based on random forests with median splits. To the best of our knowledge, this is the first differentially private tree-based method for regression, and works with both categorical and numeric covariates. Moreover, we have demonstrated, both theoretically and empirically, that our algorithm obtains impressive utility to competing methods, while maintaining the same level of differential privacy.
Figure 4: Performance of DiPriMe, Random Forest, Extremely Randomized Trees, DP-ERT and DP-DF at various values of $\epsilon$ for the Banknote Authentication data ($d_{max} = 5, N_T = 10, \rho = 0.5$).

**Broader Impact**

The privacy implications of the use of data are beginning to be acknowledged, with states and countries introducing regulation to protect the privacy of individuals. Differential privacy remains one of the few privatization methods able to achieve theoretically guaranteed bounds on privacy loss for individuals. This work provides users a tool to perform nonlinear regression and classification, while protecting the privacy of individuals.

Like most differentially private algorithms, the privacy in DiPriMe forests comes at a cost: adding noise to ensure privacy leads to lower accuracy. This decrease in accuracy could have negative implications for those relying on predictions obtained using this algorithm.

Our work does not address any potential bias either in the data, or in the resulting algorithm. It has been shown that the goals of differential privacy are often in opposition to those of fairness [26, 27]. [28] discuss methods of post-hoc bias correction for differentially private algorithms.

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A Appendix

We derive sensitivities used in the main paper in Section A.1 and provide proofs for the theorems in Section A.2. Algorithm 3 provides additional implementation details. Section A.3 provides some additional experimental results.

We shall use the following constants in our analysis. We assume all the target values are bounded, i.e., \( y_i \in [B_L, B_U] \), \( R := B_U - B_L \) and \( B := \max\{|B_L|, |B_U|\} \).

A.1 Sensitivity analysis

We assume have a set \( \mathcal{A} \) with data points \( \{y_1, y_2, \ldots, y_N\} \) such that \( y_i \in [B_L, B_U] \). \( \mathcal{A}\backslash j \) denotes the set of all data points besides \( y_j \), i.e., \( \mathcal{A}\backslash j = \{y_1, y_2, \ldots, y_{j-1}, y_{j+1}, \ldots, y_N\} \).

A.1.1 Mean

\[
\mu_{\mathcal{A}} = \frac{1}{N} \sum_{i=1}^{N} y_i,
\]

\[
\mu_{\mathcal{A}\backslash j} = \frac{1}{N-1} \sum_{i=1, i \neq j}^{N} x_i
\]

\[
\|\mu_{\mathcal{A}} - \mu_{\mathcal{A}\backslash j}\|_1 = \left\| \frac{1}{N(N-1)} \sum_{i=1, i \neq j}^{N} y_i + \frac{y_j}{N} \right\|_1
\]

\[
\leq \frac{B_U - B_L}{N}
\]

\[\therefore \Delta(\mu_{\mathcal{A}}) = \frac{B_U - B_L}{N}\]

A.1.2 Mean squared error

The mean squared error is equivalent to the variance. Denoting the variance by \( \sigma^2 \),

\[
\sigma^2_{\mathcal{A}} = \frac{1}{N} \sum_{i=1}^{N} y_i^2 - \frac{1}{N^2} \left( \sum_{i=1}^{N} y_i \right)^2
\]

\[
\sigma^2_{\mathcal{A}\backslash j} = \frac{1}{N-1} \sum_{i=1, i \neq j}^{N} y_i^2 - \frac{1}{(N-1)^2} \left( \sum_{i=1, i \neq j}^{N} y_i \right)^2
\]

Using the triangle inequality repeatedly, we get

\[
\|\sigma^2_{\mathcal{A}} - \sigma^2_{\mathcal{A}\backslash j}\|_1 \leq \left\| \frac{N-1}{N^2} y_j^2 - \frac{2y_j}{N} \left( \sum_{i=1, i \neq j}^{N} y_i \right) \right\|_1
\]

\[
\leq \left\| \frac{N-1}{N^2} B^2 \right\|_1 + \left\| \frac{N-1}{N^2} B \right\|_1 + \left\| \frac{2(N-1)}{N^2} B^2 \right\|_1
\]

\[
\leq \frac{4B^2}{N}
\]

\[\therefore \Delta(\sigma^2_{\mathcal{A}}) = \frac{4B^2}{N}\]
A.2 Proofs

A.2.1 Theorem[1]

In a depth-d median tree, there are \( M = \frac{N}{2^d} \) data points at the \( i^{th} \) leaf node. W.l.o.g. let these be \( \mathcal{A} = \{y_1, y_2, \ldots, y_M\} \). Then, the value of our objective function is

\[
\text{Obj}_{\text{No noise}}^i = \sum_{i=1}^{M} (y_i - \bar{y}_1)^2
\]

\[
= M\sigma_A^2
\]

Assuming there be a subset \( \mathcal{B} \subset \mathcal{A} \) of \( M - \delta \) data points at node \( i \) with private median splits, our noisy objective value is

\[
\text{Obj}_{\text{Median noise}}^i = (M - \delta)\sigma_B^2
\]

Sensitivity of objective  W.l.o.g., we take excluded points to be \( \{y_1, y_2, \ldots y_{\delta}\} \). We assume all the points are bounded, i.e., \( |y_i| \leq B \).

\[
M\sigma_A^2 = \sum_{i=1}^{M} y_i^2 - \frac{1}{M} \left( \sum_{i=1}^{M} y_i \right)^2
\]

\[
(M - \delta)\sigma_B^2 = \sum_{i=\delta+1}^{M} y_i^2 - \frac{1}{M - \delta} \left( \sum_{i=\delta+1}^{M} y_i \right)^2
\]

Hence,

\[
\|M\sigma_A^2 - (M - \delta)\sigma_B^2\|_1 = \left\| \frac{\delta}{M(M - \delta)} \left( \sum_{i=\delta+1}^{M} y_i \right)^2 \right\|_1 + \left\| \sum_{i=1}^{\delta} y_i^2 - \frac{1}{M} \left( \sum_{i=1}^{\delta} y_i \right)^2 \right\|_1
\]

\[
+ \left\| \frac{2}{M} \left( \sum_{i=1}^{\delta} y_i \right) \left( \sum_{i=\delta+1}^{M} y_i \right) \right\|_1
\]

\[
\leq \frac{\delta(M - \delta)}{M} B^2 + \delta B^2 + \frac{2\delta(M - \delta)}{M} B^2
\]

\[
\leq 4B^2\delta
\]

Returning to the utility analysis,  we see that noising the median by \( \delta \) leads to a maximum change of \( 4B^2\delta \) in the objective value at node \( i \).

Extending this to tree of depth \( d \), if \( \tilde{N}_i \) denotes the number of data points at the \( i^{th} \) leaf node, then

\[
|\tilde{N}_i - N/2^d| \leq t \implies |\text{Obj}_{\text{Median noise}}^i - \text{Obj}_{\text{No noise}}^i| \leq 4B^2 t
\]

Corollary[1.1] follows from applying the above result over all the \( 2^d \) leaf nodes of a depth-d tree.

A.2.2 Theorem[2]

We consider a depth-d tree. Let \( \delta_1 \) be the noise added at the first level, \( \delta_2 \) be the noise added at the second level, and so on. In the noised tree, at the leaf node (depth \( d \)), we will therefore have

\[
\tilde{N}_1 = \frac{N}{2^d} + \sum_{i=1}^{\infty} \frac{\delta_i}{2^{d-i}}, \quad \delta_i \sim \text{Laplace}(0, 1/\epsilon_1)
\]

\[
= \frac{N}{2^d} + \sum_{i=1}^{d} \Delta_i, \quad \Delta_i \sim \text{Laplace}(0, 2^{i-d}/\epsilon_1)
\]
data points.

\[ \sum_{i=1}^{n} \Delta_i \] is unbounded, so we bound the tail probability instead, i.e., find an upper bound on

\[ P \left( \sum_{i=1}^{d} \Delta_i \geq t \right) . \]

We shall use two approaches to arrive at this bound: (a) sub-exponential random variables (b) Chebyshev’s inequality.

**Using sub-exponential random variables**

\[ M_{\Delta_i}(t) = \mathbb{E}[e^{t \Delta_i}] \]

\[ = \frac{1}{1 - b_i^2 t^2}, \quad |t| < \frac{1}{b_i}, \quad b_i = \frac{2^{i-n}}{\epsilon_1} \]

\[ \leq e^{2b^2 t^2}, \quad \forall |t| \leq \frac{1}{\alpha_i} \]

This implies that \( \Delta_i \in \text{SE}(4b_i^2, \alpha_i) \) where \( \text{SE}() \) denotes the class of sub-exponential random variables. To find \( \alpha_i \), we need to find the range of \( t \) for which

\[ \frac{1}{1 - b_i^2 t^2} \leq e^{2b^2 t^2} \]

\[ \implies b_i^2 t^2 \leq \frac{1}{2} W_0 \left( -\frac{2}{e^2} \right) + 1 \]

where \( W_0(x) \) is the principal branch of the Lambert W function. For ease of analysis, we use the lower bound on \( W_0(x) \) from [30]:

\[ W_0(x) \geq \sqrt{ex + 1} - 1 \] for \( \frac{1}{e} \leq x \leq 0 \)

to get

\[ t^2 \leq \frac{1}{2b_i^2} \left( \sqrt{1 - 2/e} + 1 \right) \]

\[ \implies \alpha_i = b_i^2, \quad \beta^2 = \frac{1}{2} \left( \sqrt{1 - 2/e} + 1 \right) \approx 0.757 \]

Let \( \gamma = \frac{8(1 - 2^{-2d})}{3} \). Then,

\[ \nu_s^2 = \frac{2\gamma}{\epsilon_1} \]

\[ \nu_s^2 = \frac{2\beta \gamma}{\epsilon_1} \]

\[ \frac{\alpha_s}{\alpha_s} = \frac{2\beta \gamma}{\epsilon_1} \]

We can now bound the tail probabilities [31] as

\[ P \left( \left| \sum_{i=1}^{d} \Delta_i \right| \geq t \right) \leq \begin{cases} 2e^{-t^2/(2\nu_s^2)}, & 0 \leq t \leq \frac{\nu_s^2}{\alpha_s} \\ 2e^{-t/(2\alpha_s)}, & t > \frac{\nu_s^2}{\alpha_s} \end{cases} \]

\[ = \begin{cases} 2e^{-t^2/(4\gamma)}, & 0 \leq t \leq \frac{2\beta \gamma}{\epsilon_1} \\ 2e^{-t/(2\epsilon_1)}, & t > \frac{2\beta \gamma}{\epsilon_1} \end{cases} \]

**Using Chebyshev’s inequality**

A simpler method of bounding the tail probabilities is to use Chebyshev’s inequality. This gives

\[ P \left( \left| \sum_{i=1}^{n} \Delta_i \right| \geq t \right) \leq \frac{\text{Var} \left( \sum_{i=1}^{d} \Delta_i \right)}{t^2} \]

\[ = \frac{\gamma}{\epsilon_1^2 t^2} \]
This is tighter than the sub-exponential bound for
\[
t \in \frac{\sqrt{\gamma}}{\epsilon_1} \left( 2 \sqrt{-W_0 \left( -\frac{1}{8} \right)}, -2 \sqrt{-W_{-1} \left( -\frac{1}{8} \right)} \right)
\]
\[
\in \frac{\sqrt{\gamma}}{\epsilon_1} (0.76, 3.61)
\]

There is a clear dependence on $1/\epsilon_1$ in the bounds which intuitively makes sense as the noise scales by that factor.

### A.2.3 Theorem 3

Let us examine the effect of noising the mean of the $i$th leaf node. We have
\[
\text{Obj}^{i}_{\text{DP}} = \text{Obj}^{i}_{\text{Median noise}} + \tilde{N} \rho_i^2
\]
where $\rho_i \sim \text{Laplace}(0, R/\tilde{N}_i \epsilon_2)$.

The conditional expectation of the perturbation due to this noise is
\[
\mathbb{E}[\tilde{N} \rho_i^2 | \tilde{N}_i] = \frac{2R^2}{\tilde{N}_i \epsilon_2^2}
\]

Note that $\tilde{N}_i \geq N/2^d - t \implies \mathbb{E}[\tilde{N} \rho_i^2 | \tilde{N}_i] \leq \frac{2R^2}{(N/2^d - t)^2}$.

As the distribution of $\tilde{N}_i$ is symmetric about $N/2^d$, we have $\mathbb{E}[\tilde{N} \rho_i^2 | \tilde{N}_i] \leq \frac{2R^2}{(N/2^d - t)^2}$ with probability $1 - \zeta_i$, where $\Pr \left( \left| \tilde{N}_i - \frac{N}{2^d} \right| \geq t \right) \leq 2\zeta_i$.

Applying the union bound on the leaf nodes, and setting $\zeta_i = 2^{1-d}\zeta_i$, we get $\sum_i \mathbb{E}[\tilde{N} \rho_i^2 | \tilde{N}_i] \leq \frac{2^{d+1} R^2}{(N/2^d - t)^2}$ with probability $1 - \zeta_i$. The result of Theorem 3 follows from combining this with that of Corollary 1.1.

### A.3 Additional results

![Graphs showing mean squared error](image)

(a) $\epsilon = 10$, $N_T = 10$
(b) $\epsilon = 10$, $d_{\text{max}} = 8$
(c) $d_{\text{max}} = 8$, $N_T = 10$

Figure 5: Mean squared error of DiPriMe (with part = False), Random Forest, Extremely Randomized Trees and DP-ERT at various values of $\epsilon$, $d_{\text{max}}$ and $N_T$ on the Appliances Energy prediction dataset.

Figure 5 displays the results for DiPriMe with part = False. The trends are similar to those seen in Figure 2. An observation of note is that the optimal maximum depth for trees fit on all the data is higher than that fit on disjoint subsets of data. This is congruent with the intuition that low-occupancy nodes are noised more heavily. Hence, trees fitted to more data can be grown deeper before suffering from a similar loss of utility. This line of reasoning leads us to believe that learning an ensemble of DiPriME trees on disjoint subsets of data will be a more powerful learner with larger amounts of training data.

Figure 6 once again exhibits improved performance with larger privacy budgets. It also shows that increasing $N_T$ improves performance only to a certain limit before the increased noise reduces the
utility of the DiPriMe trees. The key insight here is the importance of the hyperparameter $\rho$ for good performance; large values of $\rho$ leaves less privacy budget for storing the means, resulting in deterioration in the MSE. This effect is more pronounced at smaller values of $\epsilon$ as the noise scales as $1/\epsilon$.

Figure 6: Mean squared error of DiPriMe for various hyperparameter settings

Figure 7: Performance of DiPriMe (with part = False), Random Forest, Extremely Randomized Trees, DP-ERT and DP-DF at various values of $\epsilon$ for the Banknote Authentication data ($d_{max} = 5, N_T = 10, \rho = 0.5$).
Algorithm 3 Differentially Private Random Median Tree (Cont’d)

29: procedure \textsc{FindSplit}(D,A,R_A,B,\epsilon,\rho,K)
30: \hspace{1em} \textbf{D} = (X,Y)
31: \hspace{1em} N = \text{number of data points in } D
32: \hspace{1em} \text{Pick a subset } A_S \text{ of } \tilde{K} = \min\{K,|A|\} \text{ features from } A.
33: \hspace{1em} \textbf{for all } a \in A_S \textbf{ do}
34: \hspace{2em} \textbf{if } a \text{ is categorical then}
35: \hspace{3em} \textbf{for all } B \subset R_a, B \neq \phi \textbf{ do}
36: \hspace{4em} N_L \leftarrow \text{number of points with } X_a \in B
37: \hspace{4em} N_R \leftarrow N - N_L
38: \hspace{4em} q(B) \leftarrow |N_L - N_R|
39: \hspace{3em} \textbf{end for}
40: \hspace{2em} \text{Draw subset } B_a \text{ using the exponential mechanism:}
41: \hspace{3em} \Pr(B_a) \propto \exp\left(-\frac{\epsilon_1}{2} q(B_a)\right)
42: \hspace{2em} \textbf{else}
43: \hspace{3em} \textbf{Try}
44: \hspace{4em} \text{Draw index } j \text{ using the exponential mechanism:}
45: \hspace{4em} \Pr(j) \propto \exp\left(-\frac{\epsilon_1}{2} |N - 2j|\right)
46: \hspace{3em} \text{Draw } x_a \text{ randomly from between the } j^{th} \text{ and } (j+1)^{th} \text{ values of the sorted } X_a.
47: \hspace{3em} \textbf{except}
48: \hspace{4em} \text{Draw } x_a \text{ uniformly at random from } R_a. \quad \triangleright \text{In case of error, draw randomly}
49: \hspace{2em} \textbf{end try}
50: \hspace{2em} \textbf{end if}
51: \hspace{2em} \text{MSE}_a \leftarrow \text{mean squared error for chosen split } (B_a \text{ or } x_a)
52: \hspace{2em} \textbf{end for}
53: \hspace{2em} \text{Pick attribute } \tilde{a} \text{ using the exponential mechanism}
54: \hspace{2em} \Pr(\tilde{a}) \propto \exp\left(-\frac{\epsilon_1}{8B^2} |\text{MSE}_a|\right)
55: \textbf{return } \tilde{a}, B_{\tilde{a}} \text{ or } x_{\tilde{a}}
56: \textbf{end procedure}
57: \textbf{end procedure} \textsc{SplitRange}(R_A,a,V)
58: \hspace{1em} \text{Use } V \text{ to split range of } a \text{ in } R_A \text{ to get } R_{AL} \text{ and } R_{AR}.
59: \hspace{1em} \textbf{if } a \text{ in } R_{AR} \text{ cannot be split on } \textbf{then}
60: \hspace{2em} A_R \leftarrow A - a
61: \hspace{2em} \textbf{end if}
62: \hspace{1em} \textbf{else}
63: \hspace{2em} \text{if } a \text{ in } R_{AL} \text{ cannot be split on } \textbf{then}
64: \hspace{3em} A_L \leftarrow A - a
65: \hspace{2em} \textbf{else}
66: \hspace{3em} A_L \leftarrow A
67: \hspace{2em} \textbf{end if}
68: \hspace{1em} \textbf{return } A_R,R_{AR},A_L,R_{AL}
69: \textbf{end procedure}