QUANTUM $SL_2$, INFINITE CURVATURE AND PITMAN’S 2M-X THEOREM

FRANÇOIS CHAPON AND REDA CHHAIBI

Abstract. It is understood that Pitman’s theorem in probability theory is intimately related to the representation theory of $U_q(sl_2)$, in the so-called crystal regime $q \to 0$. This relationship has been explored by Biane and then Biane-Bougerol-O’Connell at several levels. On the other hand, Bougerol and Jeulin showed the appearance of the Pitman transform in the infinite curvature limit $r \to \infty$ of Brownian motion on the symmetric space $SL_2(\mathbb{C})/SU_2$.

In order to understand this phenomenon, we exhibit a presentation of the Jimbo-Drinfeld quantum group $U_q^{\hbar}(sl_2)$ which isolates the role $r$ of curvature and that of the Planck constant $\hbar$. The simple relationship between parameters is $q = e^{-r}$. The semi-classical limits $\hbar \to 0$ are the Poisson-Lie groups dual to $SL_2(\mathbb{C})$ with varying curvatures $r \in \mathbb{R}_+$. We also construct classical and quantum random walks, drawing a full picture which includes Biane’s quantum walks and the construction of Bougerol-Jeulin. The curvature parameter $r$ leads to both to the crystal regime at the level of representation theory ($\hbar > 0$) and to the Bougerol-Jeulin construction in the classical world ($\hbar = 0$).

All these results are neatly in accordance with the philosophy of Kirillov’s orbit method.

To our teachers,
Philippe Biane and Philippe Bougerol.

Contents

Notation 2
1. Statement of the problem 2
1.1. Bougerol-Jeulin’s approach via curvature deformation 4
1.2. Kirillov’s orbit method 6
1.3. Biane’s quantum walks 7
1.4. Quantum groups and crystals 10
2. Definitions and informal statements of the main results 11
2.1. Complex Poisson-Lie groups with varying curvatures $r \geq 0$ 13
2.2. Real forms 14
2.3. Structure of the paper 15
3. A heterodox presentation of the Jimbo-Drinfeld quantum group 15
3.1. Relevant literature 17
3.2. Representations 18
4. Static semi-classical limits 18
4.1. Uniform measure on a dressing orbit and Archimedes’ Theorem 19
4.2. Proof of (4.1): One orbit 20
4.3. Proof of (4.2): Convolution of two orbits 21
5. Semiclassical limits: from quantum tensor dynamics to convolution dynamics 22
5.1. Quantum walk on the quantum group 23

Date: 1st of April 2019.

Key words and phrases. Orbit method, Jimbo-Drinfeld’s quantum groups, Non-commutative (=quantum) probability, Quantum random walks, Brownian motion on $SL_2(\mathbb{C})/SU_2$, Infinite curvature.
5.2. Brownian motion on $$(SU_2)_r$$

Appendix A. Non-commutative topological considerations

References

NOTATION

$\mathcal{L}(X)$ is the probability measure which is the law of a random variable $X$. Equality in law between two random variables $X$ and $Y$ is written $X \overset{\mathcal{L}}{=} Y$. If $X = (X_t)_{t \in \mathbb{T}}$ is a process indexed by $\mathbb{T}$, then its natural filtration is denoted $\mathcal{F}^X$. We will only consider $\mathbb{T} = \mathbb{N}$ for discrete time, and $\mathbb{T} = \mathbb{R}_+$ for continuous time.

If $V$ is a finite dimensional vector space, then $\text{Tr} : \text{End}(V) \to \mathbb{C}$ is the usual trace, while $\text{tr} := \frac{\text{Tr}}{\text{dim} V}$ is the normalized trace. Moreover, if $G$ is a group, then its Lie algebra i.e the tangent space at the identity is denoted $T_0 G = g$.

Throughout the paper, $\hbar > 0$ will denote a positive real, which will play the role of Planck constant. For any $\Lambda \in \mathbb{R}$, we write $\Lambda^{\hbar} := \lfloor \Lambda / \hbar \rfloor \in \mathbb{Z}$.

1. STATEMENT OF THE PROBLEM

In order to state the problem at the end, the role of this Section is two-fold. We will present the related body of work and distill the necessary representation-theoretic and geometric notions as we progress. To that endeavor, we adopt the informal style of a survey, which will not be faithful nor all-encompassing.

Indeed, we shall only provide the computations which we deem important. Also, certain prior results will be slightly reformulated, in order to reflect a personal point of view and lay the groundwork for this paper.

While we are focused on relating the representation theory of quantum groups and (possibly non-commutative) geometry, our starting point is Pitman’s theorem from probability theory [Pit75]. It will play the role of Ariadne’s thread while navigating through the mathematical maze created by the interaction of all these fields.

**Theorem 1.1** (Pitman’s 2M-X Theorem, Discrete version). Let $(X_t; t \in \mathbb{N})$ be a simple random walk in $\mathbb{Z}$, i.e increments are independent and

$$\forall t \in \mathbb{N}_+, \; \mathbb{P}(X_{t+1} - X_t = 1) = 1 - \mathbb{P}(X_{t+1} - X_t = -1) = \frac{1}{2}.$$ 

Then the process $(\Lambda^\infty_t; t \in \mathbb{R}_+)$ defined as

$$\Lambda^\infty_t := X_t - 2 \inf_{0 \leq s \leq t} X_s$$

is a Markov chain on $\mathbb{N}$ with transition kernel given by $Q$:

$$(1.1) \quad Q(\lambda, \lambda + 1) = \frac{\lambda + 2}{2(\lambda + 1)}, \quad Q(\lambda, \lambda - 1) = \frac{\lambda}{2(\lambda + 1)}.$$ 

Moreover, the missing information is stationary and equidistributed in law in the sense that for all $t \in \mathbb{N}$:

$$\mathcal{L}(X_t | F^\Lambda_t, \Lambda^\infty_t = \lambda) \overset{\mathcal{L}}{=} \frac{1}{\lambda + 1} \sum_{-\lambda \leq k \leq \lambda} \delta_k.$$ 

**Remark 1.2.** The reader trained in probability theory knows that the Markov property is very fragile and can be easily broken, while $(-\inf_{0 \leq s \leq t} X_s; t \in \mathbb{N})$ is the archetype of
non-Markovian behavior. As such, Pitman’s theorem is rather peculiar. It is also very rigid, as

\[ (X_t - k \inf_{0 \leq s \leq t} X_s; t \geq 0) \]

enjoys the Markov property only for \( k = 0, 1, \) and \( 2 ; \) the latter case being by far the most interesting.

The original proof uses the combinatorics of random walks and is formulated in terms of the running maximum instead of the running infimum. Both are equivalent upon replacing \( X \) by \( -X \), hence the common name of “Pitman’s 2M – X Theorem”, where the capital \( M \) stands for “Maximum”. Of course, a simple application of Donsker’s invariance principle yields a Brownian version.

**Theorem 1.3** (Pitman’s 2M-X Theorem, Continuous version). Let \( (X_t; t \in \mathbb{R}_+) \) be a standard Brownian motion. Then the process \( (\Lambda_\infty^t; t \in \mathbb{R}_+) \) defined as

\[ \Lambda_\infty^t := X_t - 2 \inf_{0 \leq s \leq t} X_s \]

is a Bessel 3 process, that is to say it has the same distribution as

\[ \Lambda_0^t := \sqrt{X_t^2 + Y_t^2 + Z_t^2} , \]

where \( (X, Y, Z) \) is a Euclidean Brownian motion on \( \mathbb{R}^3 \).

Moreover, the missing information is stationary and equidistributed in law in the sense that:

\[ \mathcal{L} \left( X_t \mid \mathcal{F}_\Lambda, \Lambda_\infty = \lambda \right) \equiv \frac{1}{2\lambda} 1_{[-\lambda, \lambda]}(x)dx . \]

In fact, direct proofs at the level of continuous-time stochastic processes are available. Jeulin [RY13] has an approach that uses filtration enlargement techniques and [RPS81] makes use of intertwinings of Markov kernels. These two proofs led to a flurry of very interesting probabilistic developments. For example, see the essay [Nik06] for filtration enlargement, and [DM09] for intertwining.

If other proofs and generalizations abound, we want to focus on two specific approaches where the complex group \( SL_2(\mathbb{C}) \) plays an important role. One approach by Bougerol and Jeulin [BJ02] is based on a geometric construction while the other approach by Bi健康 [Bia06, Bia09] is based on the representation theory of that group. The Lie algebra of \( SU_2 \) is

\[ \mathfrak{su}_2 := T_e SU_2 = \text{Span}_\mathbb{R} (X_g, Y_g, Z_g) \]

(1.2)

where \( (X_g, Y_g, Z_g) \) is the basis of anti-Hermitian matrices:

\[ X_g = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} ; Y_g = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} ; Z_g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \]

Our use of the subscript \( g \) is due to the fact that the symbols \( X, Y \) and \( Z \) will often be used to refer to other objects. The complexification of \( \mathfrak{su}_2 \) is the Lie algebra of \( SL_2(\mathbb{C}) \):

\[ \mathfrak{sl}_2 := T_e SL_2(\mathbb{C}) = \mathfrak{su}_2 \otimes \mathbb{C} = \text{Span}_\mathbb{C} (E, F, H) \]

(1.3)

where:

\[ H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ; E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} ; F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} . \]
1.1. Bougerol-Jeulin’s approach via curvature deformation. In [BJ02], Bougerol and Jeulin take a parameter $r > 0$ and consider a left-invariant process $(g^r_t ; t \geq 0)$ on the symmetric space $SL_2(\mathbb{C})/SU_2$. Because of the Gram-Schmidt decomposition, we make the identification $SL_2(\mathbb{C})/SU_2 \approx NA$, where $NA$ is the subgroup of lower triangular matrices, with positive diagonals. More precisely:

$$A := \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathbb{R}^*_+ \right\}, \quad \text{and} \quad N := \left\{ \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \mid b \in \mathbb{C} \right\}.$$

The Lie algebras are denoted by $\mathfrak{a} := T_eA$ and $\mathfrak{n} := T_eN$.

In that identification, the process $g^r$ satisfies the left-invariant stochastic differential equation (SDE)

$$d g^r_t = \begin{pmatrix} \frac{1}{2} r d X_t \\ r (d Y_t + i Z_t) \end{pmatrix} \circ g^r_t,$$

where $(X, Y, Z)$ is a standard Euclidean Brownian motion on $\mathbb{R}^3$. Here, the symbol $\circ$ refers to the Stratonovich integration convention. Solving explicitly the SDE yields:

$$g^r_t = Id + r \left( \begin{array}{cc} 0 & X_t \\ Y_t + i Z_t & 0 \end{array} \right) + o(r) =: Id + r x^0_t + o(r),$$

and obtains a three dimensional Brownian motion $(x^0_t ; t \geq 0)$ on $\mathfrak{n} \oplus \mathfrak{n} \approx \mathbb{R}^3$, which is a flat space. Because of Brownian motion’s time-scaling properties, rescaling $r$ amounts to speeding up the Brownian motion and hence the associated vector fields. As the process $g^r$ moves more erratically as $r > 0$ grows larger, the non-commutativity of the underlying space $NA$ becomes more apparent. One could say that the space increases in curvature.

The result by Bougerol-Jeulin holds for all complex semi-simple groups $G$, but in the context of $G = SL_2(\mathbb{C})$, we have:

**Theorem 1.4** (Bougerol-Jeulin, [BJ02]). Let $\frac{1}{2} r \Lambda^r_t$ be the radial part of $g^r_t$, i.e $\exp (r \Lambda^r_t)$ is the largest singular value of $g^r_t$ or equivalently that $\Lambda^r \geq 0$ and there exists $(k_1, k_2) \in SU_2 \times SU_2$ such that

$$g^r_t = k_1 \begin{pmatrix} e^{\frac{1}{2} r \Lambda^r_t} & 0 \\ 0 & e^{-\frac{1}{2} r \Lambda^r_t} \end{pmatrix} k_2.$$

Then, $\Lambda^r$ is a process whose distribution does not depend on $r \geq 0$. It is explicitly given by:

$$\Lambda^r_t = \frac{1}{r} \text{Argch} \left[ \frac{1}{2} r^2 \left| e^{\frac{1}{2} r X_t} \int_0^t e^{-r X_s} (dY_s + i dZ_s) \right|^2 + \cosh (r X_t) \right],$$

where $\text{Argch}(x) = \log \left( x + \sqrt{x^2 - 1} \right)$ is the inverse of $\cosh : \mathbb{R}_+ \rightarrow [1, \infty)$. In particular:

$$\begin{align*}
\Lambda^r_t = & \sqrt{X^2_t + Y^2_t + Z^2_t}, \\
\Lambda^r_t = & X_t - 2 \inf_{0 \leq s \leq t} X_s.
\end{align*}$$

**Sketch of proof.** Because Bougerol and Jeulin treat the general case, for a general complex semi-simple group $G$, extracting the above statement is not a trivial task. We provide a rather complete sketch of proof for the reader’s convenience, and give a few illuminating computations as well.
The fact that the distribution of $\Lambda^r$ does not depend on $r > 0$ is rather subtle, and is [BJ02, Proposition 3.2]. We will only give the general idea. One starts by the fact that the canonical Brownian motion on $SL_2(\mathbb{C})/SU_2 \approx AN$, has a special drift on the diagonal - given by the Weyl vector. Upon a change of measure, whose essential ingredient is the Harish-Chandra spherical function, one obtains $g_t^r=1$ and by computing the generators, $\Lambda_t^{r=1}$ has the same distribution as a Bessel 3 process, which has the Brownian scaling. Then using the Brownian scaling $g^r$ has a distribution independent of $r > 0$.

For the first expression, from Eq. (1.4) write:

$$\cosh (r\Lambda_t^r) = \frac{1}{2} \text{Tr} g_t g_t^* = \cosh(rX_t) + \frac{1}{2} r^2 \left| e^{\frac{1}{2} r X_t} \int_0^t e^{-rX_s} (dY_s + idZ_s) \right|^2.$$  
Taking the inverse of cosh yields the result.

Now, let us prove the existence of the limits $r \to 0$ and $r \to \infty$ for $\Lambda^r$. The following derivation is complete except two technical points labelled by * and **. These technical aspects are handled by classical applications of Burkholder-Davis-Gundy inequalities and $L^p(\Omega, P)$ estimates, which we omit.

The computation of $\Lambda_{r=0}$ goes via the following limiting argument. Since $\text{Argch}(1+x) \sim \sqrt{2x}$ for $x \to 0$, we have as $r \to 0$ and uniformly in $t$ inside a compact interval:

$$\Lambda_t^r = (1 + o(1)) \sqrt{X_t^2 + \left| e^{\frac{1}{2} r X_t} \int_0^t e^{-rX_s} (dY_s + idZ_s) \right|^2},$$

hence the result for $\Lambda_{r=0}$.

For the computation of $\Lambda_{r=\infty}$, the crux of the argument is that for $x \to \infty$, $\text{Argch}(x) \sim \log 2x$, hence as $r \to \infty$:

$$\Lambda_t^r = (1 + o(1)) \frac{1}{r} \log \left( 2 \cosh(rX_t) + r^2 \left| e^{\frac{1}{2} r X_t} \int_0^t e^{-rX_s} (dY_s + idZ_s) \right|^2 \right),$$

$$= (1 + o(1)) \frac{1}{r} \log \left( e^{rX_t} + e^{-rX_t} \left| e^{rX_t} \int_0^t e^{-rX_s} (dY_s + idZ_s) \right|^2 \right).$$

Now, because of the Dambis-Dubins-Schwartz theorem (see [RY13]), there exists a complex Brownian motion $\beta^C$ such that:

$$\int_0^t e^{-rX_s} (dY_s + idZ_s).$$

Adding to that the tropicalization trick:

$$\forall (a, b) \in \mathbb{R} \times \mathbb{R}, \ \lim_{r \to \infty} \frac{1}{r} \log (e^{ra} + e^{rb}) = \max(a, b),$$

we have:

$$\Lambda_t^r = (1 + o(1)) \max \left( |X_t|, X_t + \frac{1}{r} \log \left| \int_0^t e^{-rX_s} (dY_s + idZ_s) \right|^2 \right).$$
\[(1 + o(1)) \max \left( |X_t|, X_t + \frac{1}{r} \log \int_0^t e^{-2rX_s} ds + o(1) \right) \]
\[= (1 + o(1)) \left( X_t - 2 \inf_{0 \leq s \leq t} X_s \right). \]

In the last step, we invoked the Laplace method, which states that for all continuous functions \(f\):
\[
\lim_{r \to \infty} \frac{1}{r} \log \int_0^t e^{-rf(s)} ds = -\inf_{0 \leq s \leq t} f(s). \]

The important remark is that the Pitman transform shows up in infinite curvature, while the norm process in \(\mathbb{R}^3\) appears in flat curvature. The interpretation of the parameter \(r\) as a curvature is mainly absent from the literature except in the very astute remark in the final paragraphs of [BJ02, Section 1].

Now, before describing Biane's approach, we present the correct general framework to understand this group-theoretic story.

1.2. Kirillov’s orbit method. As explained in [Kir99], the orbit method is more of a philosophy, with merits and demerits. For a Lie group \(G\) with Lie algebra \(\mathfrak{g} = T_eG\), it is well-known that the study of representation theory for \(G\) is equivalent to its local version, that is the study of the representations of the Lie algebra \(\mathfrak{g}\). Equivalently, one prefers to work with the universal enveloping algebra which is defined as
\[U_\hbar(\mathfrak{g}) = T(\mathfrak{g}) / \{ x \otimes y - y \otimes x - \hbar [x, y] \},\]
where \(T(\mathfrak{g})\) is the tensor algebra. The fundamental idea behind the orbit method is that the representation theory \(G\) should be seen as the quantization of a certain Poisson manifold. Already, one sees that the algebra \(U_\hbar(\mathfrak{g})\) degenerates as \(\hbar \to 0\) to the algebra \(S(\mathfrak{g}^*)\) of symmetric tensors, which is canonically identified with \(\mathbb{C}[\mathfrak{g}^*]\), the algebra of polynomials on \(\mathfrak{g}^*\). There are two interesting structures on \(\mathbb{C}[\mathfrak{g}^*]\), whose combination is referred to as the trivial Poisson-Lie structure on \(\mathfrak{g}^*\). Basically, we are only saying that \(\mathfrak{g}^*\) has to be seen as a flat Poisson manifold, once endowed with the canonical Kirillov-Kostant-Souriau (KKS) Poisson bracket. The formal definition is as follows, which will fix the notations for later use.

On the one hand, let \(\mathcal{C}^\infty(\mathfrak{g}^*)\) be the algebra of smooth functions on \(\mathfrak{g}^*\) and we have the inclusion of sub-algebras \(\mathbb{C}[\mathfrak{g}^*] \hookrightarrow \mathcal{C}^\infty(\mathfrak{g}^*)\). As a semi-classical limit, \(\mathcal{C}^\infty(\mathfrak{g}^*)\) becomes a Poisson algebra once endowed with the KKS bracket \(\{ \cdot, \cdot \}_0 : \mathcal{C}^\infty(\mathfrak{g}^*) \times \mathcal{C}^\infty(\mathfrak{g}^*) \to \mathcal{C}^\infty(\mathfrak{g}^*)\).

By definition, a Poisson bracket is a derivation in both variables. Therefore, because of the Leibniz rule, the Poisson bracket is entirely determined by its values on linear functions:
\[
\forall X \in \mathfrak{g} \approx (\mathfrak{g}^*)^*, \quad f_X(\cdot) := \langle X, \cdot \rangle. \]

On linear forms, KKS bracket is defined as:
\[
\{f_X, f_Y\}_0 := f_{[X,Y]} = \langle [X,Y], \cdot \rangle. \tag{1.5} \]

On the other hand, recall that if \((\mathcal{G}, *_{\mathcal{G}})\) is a group, then the group law \(*_{\mathcal{G}}\) can be encoded thanks to a coproduct on algebras of functions. By definition, the coproduct \(\Delta\) associated to \((\mathcal{G}, *_{\mathcal{G}})\) is the map:
\[
\Delta : \mathcal{C}^\infty(\mathcal{G}) \to \mathcal{C}^\infty(\mathcal{G} \times \mathcal{G}) \quad f \mapsto ((g_1, g_2) \mapsto f(g_1 *_{\mathcal{G}} g_2)). \tag{1.6} \]
If \( \mathcal{A} \subset C^\infty(\mathcal{G}) \) is a dense sub-algebra such that for all \( f \in \mathcal{A} \), \( \Delta(f) \) is a separable function, which is written in Sweedler’s notation:

\[
\Delta(f)(g_1, g_2) = \sum_{(f)} f_1(g_1) f_2(g_2),
\]

then we write \( \Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A} \), which is the customary choice. Here, consider \((g^*, +)\) to be an Abelian group, which amounts to the trivial coproduct defined on linear functions \( X \in g \approx (g^*)^* \) via:

\[
\Delta_0 : C[g^*] \to C[g^*] \otimes C[g^*], \quad X \mapsto X \otimes 1 + 1 \otimes X.
\]

We give the following trivial, yet key, example.

**Example 1.5.** Consider the Abelian group \((\mathbb{R}^3, +)\). Its coordinate algebra is the polynomial algebra in three variables \(\mathbb{C}[X, Y, Z]\), and \((X, Y, Z)\) are linear forms on \(\mathbb{R}^3\).

We have for \( S \in \{X, Y, Z\}\):

\[
\Delta_0(S) \left( \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \right) \overset{\text{def}}{=} S \left( \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix} \right) = S \left( \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \right) + S \left( \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \right),
\]

which is more compactly written \(\Delta_0(S) = S \otimes 1 + 1 \otimes S\), with the convention that tensors with index \(i = 1, 2\) are functions of the \(i\)-th variable. Since \(\Delta_0\) is a morphism of algebras, it is entirely determined by its values on generators \((X, Y, Z)\).

The simplest illustration of the orbit method is the Heisenberg Lie algebra, which is the Lie algebra with two generators \(x\) and \(p\), along with the commutation relation \([x, p] = \hbar \text{id}\). The orbit method morally says that "the quantum mechanics of one particle on the real line is the representation theory of the Heisenberg algebra".

In the end, implementing the orbit method consists in drawing correspondences between the two worlds: Unitary representations should correspond to orbits, characters should correspond to orbital integrals, tensor products should correspond to convolutions of orbital measures etc... For an extensive dictionary, we refer again to [Kir99]. In that sense, some of our results will be implementations of the orbit method and the general philosophy will be our inspiration.

Nevertheless, for the purposes of this paper, we will only consider \(SL_2(\mathbb{C})\) and groups related to it.

### 1.3. Biane’s quantum walks

In accordance with the orbit method, Biane considers in [Bia91] \(L^b(\mathfrak{su}_2)\) as an algebra of observables for a non-commutative probability space. This space is the quantization of \(\mathfrak{su}_2^* \approx \mathbb{R}^3\) endowed with the KKS structure, and as such, one should think of a three dimensional space where we cannot measure the directions \((X, Y, Z)\) independently. The measurement operators \((X_g, Y_g, Z_g)\) do not commute, with exactly the relations given by the Lie bracket of \(\mathfrak{su}_2\). Although it is never stated explicitly, studying quantum systems in this space is equivalent to the representation theory of \(\mathfrak{su}_2\).

In a sense, Biane’s work gives a dynamical flavor to Kirillov’s orbit method. More precisely, Biane considers the following quantum dynamical system or quantum random walk using the framework of non-commutative probability. For a comprehensive survey, we recommend [Bia08]. For the purposes of this paper, we want to push for the idea that Biane’s construction is about quantum mechanics on the flat geometric space \(\mathfrak{su}_2^* \approx \mathbb{R}^3\), or equivalently non-commutative probability on the Abelian Lie group \(\mathfrak{su}_2^* \approx \mathbb{R}^3\).
**Notions of non-commutative probability:** If classical probability theory uses the algebra $L^\infty(\Omega)$ and the linear form $E$, non-commutative probability relies on a possibly non-commutative $\ast$-algebra $\mathcal{A}$ endowed with a state $\tau : \mathcal{A} \to \mathbb{C}$. A state is a positive linear form, i.e. $\varphi(aa^*) \geq 0$ for all $a \in \mathcal{A}$, and plays the role of expectation. The elements of $\mathcal{A}$ are non-commutative random variables. If $(a_i)_{i \in I}$ is a family of non-commutative random variables, then their joint distribution is defined as the collection of $\ast$-moments:

$$(\varphi(a_{i_1}^{s_1}a_{i_2}^{s_2} \cdots a_{i_k}^{s_k}) ; k \in \mathbb{N} \text{ and } \forall j = 1, \ldots, k; \ (i_j, \varepsilon_j) \in I \times \{1, \ast\}) .$$

Convergence in distribution is defined as the convergence in $\ast$-moments.

Now, as a guiding example, let us construct explicitly the probability space underlying a random walk in $\mathbb{R}^d$ - with say, $d = 3$ - and independent identically distributed increments sharing a common distribution. The independent increments are defined on the classical probability space $\Omega = (\mathbb{R}^d)^\mathbb{N}$ endowed with an infinite product measure $\mathbb{P}$. Writing everything in terms of functions, which are referred to as observables, we need to have a dual point of view. Since an infinite product is in fact a projective limit of spaces, the dual notion will be an inductive limit of functions. Given the natural inclusion $\mathcal{F}(\mathbb{R}^d)^\otimes n \hookrightarrow \mathcal{F}(\mathbb{R}^d)^\otimes (n+1)$, one realizes that a convenient algebra of functions is

$$\mathcal{F}(\Omega) = \lim_{n \to \infty} \mathcal{F}(\mathbb{R}^d)^\otimes n .$$

This is the inductive limit of polynomial functions depending on a finite number of increments. It is nothing but the polynomial algebra in infinitely variables and, if increments are bounded, we have the natural inclusion $\mathcal{F}(\Omega) \hookrightarrow L^\infty(\Omega, \mathbb{P})$. In the non-commutative setting, the analogue of the algebra of observables depending on a single increment is $\mathcal{U}(\mathfrak{sl}_2)$.

From the previous discussion, it is natural to consider:

$$\mathcal{A} := \lim_{n \to \infty} \mathcal{U}(\mathfrak{sl}_2)^\otimes n ,$$

as the algebra of all observables. The $\ast$ involution makes the elements in $\mathfrak{su}_2$ self-adjoint. The natural inclusion $\mathcal{U}(\mathfrak{sl}_2)^\otimes n \hookrightarrow \mathcal{U}(\mathfrak{sl}_2)^\otimes (n+1)$ is $x \mapsto x \otimes 1$. In particular, in $\mathcal{A}$, we identify $x_1 \otimes \cdots \otimes x_k$ with $x_1 \otimes \cdots \otimes x_k \otimes 1^\infty$. As a state $\tau$, we take a product state on pure tensors i.e for all $x_1 \otimes \cdots \otimes x_k \in \mathcal{U}(\mathfrak{sl}_2)^\otimes k$:

$$\tau(x_1 \otimes \cdots \otimes x_k) = \prod_{i=1}^{k} \tau(x_i) ,$$

and for every single elementary observable $x \in \mathcal{U}(\mathfrak{sl}_2)$, we have:

$$\tau(x) = \text{tr } \rho(x) ,$$

where $\rho : \mathcal{U}(\mathfrak{sl}_2) \to \text{End}(\mathbb{C}^2)$ is the natural representation. Recall that $\rho$ is the representation with highest weight $\lambda = 1$, and could easily be replaced by another representation.

Also, for the sake of simpler exposition, we do not detail the matters of completion in this section. Indeed, at this point, $\mathcal{A}$ is the non-commutative analogue of a polynomial algebra. In order to have functional calculus available and carry certain analytic arguments, $\mathcal{A}$ needs to be completed into a Von Neumann algebra. We refer to Appendix A for that matter.
Random walks: Now, the crucial point is that $\mathcal{U}(\mathfrak{sl}_2)$ is endowed with a product $\Delta_0 : \mathcal{U}(\mathfrak{sl}_2) \rightarrow \mathcal{U}(\mathfrak{sl}_2) \otimes \mathcal{U}(\mathfrak{sl}_2)$ that is exactly the same on $\mathfrak{sl}_2$ as the trivial coproduct (1.7). This allows to construct a random walk whose algebra of observables is not commutative or quantum random walk for short. We stress that the underlying space has to be seen as an Abelian group because of the choice of coproduct, but it is the algebra of observables that is not commutative. In this construction, we consider measurement operators, which measure for every time $t \in \mathbb{N}$, the observable $x \in \mathcal{U}(\mathfrak{sl}_2)$ applied to the quantum random walk. As such, one exhibits a morphism of algebras $M_t : \mathcal{U}(\mathfrak{sl}_2) \rightarrow \mathcal{A}$ as follows:

$$
\begin{cases}
M_0 &= 1, \\
M_t &= (M_{t-1} \otimes 1) \circ \Delta.
\end{cases}
$$

We extend the definition from $t \in \mathbb{N}$ to $t \in \mathbb{R}_+$ by defining:

$$
\forall t \in \mathbb{R}_+, \ M_t := M_{[t]}.
$$

Because $M_t$ is a morphism of algebras, the operators $(M_t(x) : x \in \mathcal{U}(\mathfrak{sl}_2))$ have the same commutation relations as the enveloping algebras, and hence are truly quantum observables, on the non-commutative space $\mathcal{U}(\mathfrak{sl}_2)$ which is the quantization of $\mathfrak{su}_2^* \approx \mathbb{R}^3$. The discrepancy between $\mathfrak{sl}_2$ and $\mathfrak{su}_2$ is due to an implicit complexification, which will explained upon discussing real forms.

**Theorem 1.6** (Biane). Consider $C_\theta := \sqrt{\frac{1}{2} + X^2_\theta + Y^2_\theta + Z^2_\theta}$ to be the Casimir operator associated to $\mathcal{U}(\mathfrak{sl}_2)$. Then define for $t \in \mathbb{N}$:

$$
\begin{cases}
(X_t, Y_t, Z_t) := (M_t(X_\theta), M_t(Y_\theta), M_t(Z_\theta)) , \\
\Lambda_t := M_t(C_\theta) = \sqrt{\frac{1}{2} + X^2_t + Y^2_t + Z^2_t}.
\end{cases}
$$

The triple $((X_t, Y_t, Z_t) : t \in \mathbb{N})$ is non-commutative process, with each coordinate being a simple random walk. Furthermore, the pair $(X, \Lambda)$ forms a classical Markov chain on the state space:

$$\{(\omega, \lambda) \in \mathbb{Z} \times \mathbb{N} \mid \omega \in \{-\lambda, \lambda + 2, \ldots, \lambda - 2, \lambda\}\}$$

and with transitions:

$$
\begin{align*}
p((\omega, \lambda), (\omega + 1, \lambda + 1)) &= \frac{\lambda + \omega + 2}{2(\lambda + 1)}, \\
 p((\omega, \lambda), (\omega - 1, \lambda + 1)) &= \frac{\lambda - \omega + 2}{2(\lambda + 1)}, \\
 p((\omega, \lambda), (\omega + 1, \lambda - 1)) &= \frac{\lambda - \omega}{2(\lambda + 1)}, \\
 p((\omega, \lambda), (\omega - 1, \lambda - 1)) &= \frac{\lambda + \omega}{2(\lambda + 1)}.
\end{align*}
$$

In particular, $X$ is a simple random walk, while $\Lambda$ follows the same transitions as (1.1).

**Pointers to the proof.** We recall the construction for the reader’s convenience and to emphasize similarities with the more complicated cases.

In order to see that $S \in \{X, Y, Z\}$ is in fact a classical simple random walks, we unroll the recurrence Eq. (1.8) with the fact that $\Delta(S)$ is given by the trivial coproduct (1.7). Then for $t \in \mathbb{N}$:

$$
S_t = M_t(S) := \sum_{s=0}^{t} 1^\otimes_s \otimes S \in \mathcal{A}.
$$

Now notice that the increments $(1^\otimes_s \otimes S : s \in \mathbb{N})$ are commuting operators. By computing moments thanks to the state $\tau$, each has spectral measure $\frac{1}{2} \delta_{-1} + \frac{1}{2} \delta_1$. This is indeed equivalent to saying that $S$ is a simple random walk.
Moreover, one can prove that $\Lambda_t = \sqrt{\frac{1}{2} + X_t^2 + Y_t^2 + Z_t^2}$ is the dynamic “read” by the Casimir and which follows a classical Littlewood-Richardson process:

$$\mathbb{E}(f(\Lambda_{t+1}) \mid \Lambda_t) = \frac{\Lambda_t + 2}{2(\Lambda_t + 1)} f(\Lambda_t + 1) + \frac{\Lambda_t}{2(\Lambda_t + 1)} f(\Lambda_t - 1).$$

The transitions probabilities in Eq. (1.10) are computed through Clebsch-Gordan coefficients at the end of [Bia06, Section 3]

If the pair $(X, \Lambda)$ has coordinate-wise exactly the same dynamic as in Pitman’s theorem, the joint dynamic is different: $X$ is a simple random walk while $\Lambda$ is the quantized analogue of a Euclidean norm - not the Pitman transform of $X$! This is even more apparent upon taking the following semi-classical limit.

**Corollary 1.7 (Biane).** In the sense of $\ast$-commutative moments, we have the convergence in law:

$$(hM_{t/\mathbb{R}^2}(X_\mathbb{R}), M_{t/\mathbb{R}^2}(Y_\mathbb{R}), M_{t/\mathbb{R}^2}(Z_\mathbb{R}); \ t \geq 0) \overset{h \to 0}{\to} ((X_t, Y_t, Z_t); \ t \geq 0),$$

where $(X, Y, Z)$ is a Euclidean Brownian motion on $\mathbb{R}^3$. Moreover, jointly with the above convergence $hM_{t/\mathbb{R}^2}(C_\mathbb{R})$ converges to the Euclidean norm i.e a Bessel-3 process.

The story stopped there in the 90s. In order to see the relation with the approach of Bougerol and Jeulin, we make the double identification $su_2^s \approx \mathbb{R}^3 \approx \mathfrak{n} \oplus \mathfrak{a}$ and reformulate the above result as the convergence:

$$(hM_{t/\mathbb{R}^2}(F); F \in \mbox{Fix}(U^h(\mathfrak{sl}_2), \ast), F^\ast = F, \ t \geq 0) \overset{h \to 0}{\to} \left(f(x_0^0) = f \left( \left( \frac{1}{2}X_t \quad Y_t + iZ_t \quad -\frac{1}{2}X_t \right) \right); \ f = \pi(F), \ t \geq 0 \right).$$

Here $\mbox{Fix}(U^h(\mathfrak{sl}_2), \ast)$ are the elements fixed by the $\ast$ involution, i.e the self-adjoint operators and $\pi : \mbox{Fix}(U^h(\mathfrak{sl}_2), \ast) \rightarrow \mathbb{C}[su_2^s] \approx \mathbb{C}[\mathfrak{n} \oplus \mathfrak{a}]$ is the quotient map mod $h$, which consists in seeing any non-commutative monomial as a commutative one.

### 1.4. Quantum groups and crystals.

The mismatch between Pitman’s theorem and the previous section is fixed upon considering quantum groups. The classical presentation of a quantum group is:

$$U_q(\mathfrak{sl}_2) := \langle K = q^H, K^{-1}, E, F \rangle / \mathcal{R},$$

where $q = e^h$, and $\mathcal{R}$ is the two-sided ideal of relations:

$$KEK^{-1} = q^2 E, \ KFK^{-1} = q^{-2} F, \ EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

Here, $h$ should not be seen as the true Planck constant. It is a deformation parameter such that formally $U_q(\mathfrak{sl}_2) \rightarrow U(\mathfrak{sl}_2)$. This can be seen from Taylor expanding the relations up to order 1. For example, upon writing $q = 1 + h + o(h)$, the first relation becomes:

$$E + h[H, E] + o(h) = E + h2E + o(h),$$

and therefore one recovers the classical commutation relation $[H, E] = 2E$.

As $q \rightarrow 0$, the algebra structure breaks down but a combinatorial structure called crystals remains at the level of the representation theory. In the years 2000, Biane understood that it is the combinatorics of crystals that is lurking behind [Bia09]. The generalization which consists in tensoring by other representations and in general Lie type is developed in [LLP12] and [LLP13]. In fact, they revisited the works of [BBO05] and [BBO09] where continuous crystals where directly constructed. The following statement is extracted from [Bia09]:
Theorem 1.8. There is a classical walk \((X, \Lambda)\) on \(U_q(\mathfrak{sl}_2)\), whose transition probabilities are:

\[
\begin{align*}
    p((\omega, \lambda), (\omega + 1, \lambda + 1)) &= \frac{q^{\lambda-\omega} - q^{2(\lambda+1)}}{2(1 - q^{2(\lambda+1)})}, \\
    p((\omega, \lambda), (\omega - 1, \lambda + 1)) &= 1 - q^{\lambda-\omega} \\
    p((\omega, \lambda), (\omega + 1, \lambda - 1)) &= 1 - q^{\lambda-\omega} \\
    p((\omega, \lambda), (\omega - 1, \lambda - 1)) &= \frac{q^{\lambda-\omega} - q^{2(\lambda+1)}}{2(1 - q^{2(\lambda+1)})},
\end{align*}
\]

with the convention that \(0^0 = 1\). In particular, the limit \(q \to 1\) coincides with the result of the previous section, while \(q \to 0\) coincides with Pitman’s theorem.

We conclude this subsection by stating that Pitman’s theorem, in its discrete version, has to do with quantum random walks on \(U_q(\mathfrak{sl}_2)\) and taking \(q\) from \(q = 1\) to \(q = 0\), where crystals do appear. In fact, everything can be conveniently recast in terms of the Littelmann path model, which is a combinatorial model for crystals. The random walks at hand are readily identified with crystal elements. For an overview, see the introduction of one of the author’s PhD thesis [Chh13].

We are ready to state the problematic that is addressed in the paper:

**Question 1.9.** The Pitman transform is understood to be intimately related to crystals, which appear at the representation theory of \(U_q(\mathfrak{sl}_2)\) at \(q = 0\). Why would there be crystal-like phenomena by taking curvature \(r \to \infty\) in a symmetric space \(SL_2(\mathbb{C})/SU_2 \approx NA\)?

It is certainly desirable to have single global picture, with an interplay between both the representation of \(U_q(\mathfrak{sl}_2)\), as \(q > 0\) varies, and the geometry of the symmetric space \(SL_2(\mathbb{C})/SU_2\) with varying curvatures \(r > 0\).

2. Definitions and informal statements of the main results

At this point, let us summarize the landscape:

- On the one hand, at \(q = 1\), there is Biane’s construction of quantum random walks [Bia91]. The diffusive limit is Brownian motion on the space \(\mathfrak{su}_2^*\), which can be seen as a flat space with curvature \((r = 0)\).
- On the other hand, at \(q = 0\), using Kashiwara crystals, for example in the path model form, one recovers Pitman’s theorem. The latter is also recovered upon taking a Brownian motion on the symmetric space \(SL_2(\mathbb{C})/SU_2\) and taking curvature to infinity \((r \to \infty)\).

Thus, we want to interpolate the two different regimes, and perhaps reinterpret the parameter \(q\) in quantum groups as a curvature parameter. The most fruitful idea in trying to answer Question 1.9 is to discard the idea that \(q = e^h\) in the Drinfeld-Jimbo quantum group \(U_q(\mathfrak{sl}_2)\), with \(h\) a Planck constant. In fact, we have to revisit the presentation given in 1.11. First hand, let us present two arguments in favor of our seemingly heterodox point of view since this goes against many classical textbooks [CP95, Chapter 6], [KS12, Chapter 3], [Kas12].

- On the one hand, \(q = e^h\) and \(U_q(\mathfrak{sl}_2) \xrightarrow{h \to 0} U(\mathfrak{sl}_2)\), with \(h\) a Planck constant, does not make sense from the point of view of the orbit method.
Quantum mechanics, Representation theory  
\[\mathcal{U}_q(sl_2)\] (Quantum group)  
\[h \to 0\] or \[q \to 1\] ?  
\[\mathcal{U}(sl_2) \approx \mathcal{U}^h(sl_2)\] (Enveloping algebra)  
\[\mathbb{C}[su^*_2]\] (Coordinate algebra of \(su^*_2 \approx \mathbb{R}^3\).)  

\begin{center}
\textbf{Figure 1. An incomplete picture}
\end{center}

As shown in the incomplete picture drawn in Fig. 1, how can it be \(\mathcal{U}_q(sl_2)\) is a quantum deformation of \(\mathcal{U}(sl_2)\), since that from the point of view of the orbit method, \(\mathcal{U}(sl_2)\) already belongs to the quantum world?

- On the other hand, since \(sl_2\) is (semi-)simple, Drinfeld's rigidity [CP95, Theorem 6.1.8] states that the deformation theory of \(\mathcal{U}(sl_2)\) is trivial, in the sense that every one-parameter deformation of \(\mathcal{U}(sl_2)\) using \(h\) is isomorphic to \(\mathcal{U}(sl_2[[h]]\) as an algebra. One could take a pessimistic stance on that theorem and decide that there is no point in trying to find presentations that are different from (1.11).

We take an optimistic stance and argue that there is an interesting presentation that allows us to answer Question (1.9).

Quoting Kirillov [Pra05, p.305], who attributes the statement to Drinfeld, the first approximation to quantum groups as classical objects are Poisson-Lie groups. The following conversation will take us back to the genesis of quantum groups, which we feel is necessary in order to really distinguish what is quantum and what is not.

Following the ideas of quantization, Biane’s random walk has to be understood as follows. \((X_g, Y_g, Z_g)\) in \(su_2\) are self-adjoint measurement operators which do not commute and the full algebra of observables is \(\mathcal{U}(sl_2)\). Complexification is implicit when considering all operators, beyond the self-adjoint ones. This remains true at the level of semi-classical limits: \(sl_2\) has to be seen as the space of complex linear forms on \(su^*_2\), while the observables in \(\mathbb{C}[su^*_2]\) can be seen as the ”self-adjoint elements” of \(S[sl_2] = \mathbb{C}[sl^*_2]\). Because working with complex Lie algebras is easier, we will consider \(sl_2 = su_2 \otimes \mathbb{C}\) before discussing the matters of choosing real forms.

Notice that the space \(su^*_2\) has no reason of having a Lie bracket, hence the trivial Lie bracket on \(sl^*_2\). It is natural to interpret a trivial Lie bracket as zero curvature, since that for Lie groups equipped with an invariant metric, the curvature tensor is basically equivalent to the bracket. Adding to that the KKS Poisson bracket \(\{\cdot, \cdot\}_{0}\), which is nothing but the Lie bracket of \(sl_2\), one says that

\[\left(su^*_2, 0 = [\cdot, \cdot]_{sl^*_2}, \{\cdot, \cdot\}_{0}\right)\]

is a Lie bialgebra.

We want a curved version of this statement. By that we mean a non-commutative probability space, whose semi-classical limit is a non-trivial Poisson-Lie group. The infinitesimal version is a bialgebra structure:

\[\left(su^*_2, r[\cdot, \cdot], \{\cdot, \cdot\}_{r}\right)\]

where \(r\) plays the role of curvature, dilating the Lie bracket on \(sl^*_2\), and \(\{\cdot, \cdot\}_{r}\) has to be a compatible Poisson bracket.
2.1. **Complex Poisson-Lie groups with varying curvatures** \( r \geq 0 \). Consider \( B \subset SL_2(\mathbb{C}) \) as the Borel subgroup:
\[
B := \left\{ \begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix} \mid a \in \mathbb{C}^*, \ b \in \mathbb{C} \right\},
\]
while \( B^+ \) is the transpose. If \( b \in B \cup B^+ \), then \([b]_0\) denotes the projection to the diagonal. The following complex group will play an important role:
\[
(2.1) \quad SL_2^r := \{ (b,b^+) \in B \times B^+ \mid [b]_0 = [b^+]_0^{-1} \},
\]
which is called the Poisson-Lie group dual to \( SL_2(C) \) as the polynomial algebra generated by the variables \( e \), \( r \).

**Remark 2.2** (Coproduct is not quantum). Then using the definition of \( \Delta \), we extract from the above expressions exactly the rescaling of the original bracket by a factor \( r \geq 0 \). Clearly, as \( r \rightarrow 0 \), the group \( ((SL_2^r)_{r}, *) \) becomes the Abelian group \( (sl_2^r, +) \).

In order to interpolate with the trivial Poisson-Lie group, notice that the exponential map is a diffeomorphism \( : sl_2^r \xrightarrow{\sim} SL_2^r \). As such, we start by identifying \( sl_2^r \) and \( SL_2^r \) as topological spaces. Then we define a group law with a parameter \( r \geq 0 \) via:
\[
(2.3) \quad \forall (X,Y) \in sl_2^r \times sl_2^r, \ X \ast_r Y := \frac{1}{r} \log \left( e^{X} e^{rY} \right).
\]

In turn, these variables are defined by writing:
\[
(2.4) \quad \forall g \in (SL_2^r), \ g^{(r)} = \begin{pmatrix} e^{\frac{1}{2r}H} & 0 \\ 2rF & e^{-\frac{1}{2r}H} \end{pmatrix}, \begin{pmatrix} e^{-\frac{1}{2r}H} & 2rE \\ 0 & e^{\frac{1}{2r}H} \end{pmatrix}.
\]

**Lemma 2.1.** The coproduct on \( C [(SL_2^r)] \) is given by:
\[
\Delta_r \left( e^{\frac{1}{2r}H} \right) = e^{\frac{1}{2r}H} \otimes e^{\frac{1}{2r}H}, \quad \Delta_r (F) = F \otimes e^{\frac{1}{2r}H} + e^{-\frac{1}{2r}H} \otimes F, \quad \Delta_r (E) = E \otimes e^{\frac{1}{2r}H} + e^{-\frac{1}{2r}H} \otimes E.
\]

**Proof.** Writing two elements \( g_i \in (SL_2^r), \ i = 1,2 \) as in (2.4) and computing the product, we have that \( g_1 g_2 = (b,b^+) \) with:
\[
\begin{align*}
& b = \begin{pmatrix} e^{\frac{1}{2r}(H_1+H_2)} & 0 \\ 2r \left(F_1 e^{\frac{1}{2r}H_2} + e^{\frac{1}{2r}H_1} F_2 \right) & e^{-\frac{1}{2r}(H_1+H_2)} \end{pmatrix}, \\
& b^+ = \begin{pmatrix} e^{-\frac{1}{2r}(H_1+H_2)} & 2r \left(E_1 e^{\frac{1}{2r}H_2} + e^{-\frac{1}{2r}H_1} E_2 \right) \\ 0 & e^{\frac{1}{2r}(H_1+H_2)} \end{pmatrix}.
\end{align*}
\]

Then using the definition of \( \Delta_r \) in (1.6), we extract from the above expressions exactly the announced expressions for the coproduct. \( \square \)

**Remark 2.2** (Coproduct is not quantum). The reader familiar with the topic will recognize the coproduct of quantum groups upon setting \( K = e^{\frac{1}{2r}H} \). Already we see there is nothing quantum about that!
Remark 2.3 (The constant $r$ is a curvature parameter). Following [GHL90, §3.17], if $G$ is a Lie group with invariant metric, then the $(0, 3)$ curvature tensor is

$$R(X, Y, Z) = \frac{1}{4}[X, [Y, Z]],$$

where $(X, Y, Z)$ represent invariant vector fields. In our case, the parameter $r \geq 0$ controls the curvature of $(\mathbb{SL}_2^*)_r$ via:

$$R(X, Y, Z) = r^2 \frac{[X, [Y, Z]]}{4}.$$

Now, by using the same trick as in the definition of the KKS structure, we identify $X \in \mathfrak{sl}_2$ with a linear forms $f_X$ on $(\mathbb{SL}_2^*)_r$. It is an easy computation to check that

Lemma 2.4. There exists a legitimate Poisson bracket defined by:

$$\{f_H, f_E\}_r := 2f_E,$$

$$\{f_H, f_F\}_r := -2f_F,$$

$$\{f_E, f_F\}_r := \frac{e^{rf_H} - e^{-rf_H}}{2r}.$$

Proof. A Poisson bracket is determined by its value on linear forms because of the Leibniz rule. Thus we only have to check the Jacobi identity:

$$0 = \{A, \{B, C\}_r\}_r + \{B, \{C, A\}_r\}_r + \{C, \{A, B\}_r\}_r$$

for all elements $A, B, C$ that are linear forms. Also, because $\{\cdot, \cdot\}_r$ is anti-symmetric, there is no need to check all the possibilities, only $(A, B, C) = (f_H, f_E, f_F)$ i.e

$$0 = \left\{f_H, \frac{e^{rf_H} - e^{-rf_H}}{2r}\right\}_r + \{f_E, 2f_F\}_r + \{f_F, 2f_E\}_r$$

This last term is indeed zero, as the Poisson bracket is a derivation in each variable. □

As we shall see, this is the semi-classical limit of (a different presentation of) the Drinfeld-Jimbo quantum group.

2.2. Real forms. We consider the following antimorphism $*$ acting on points of $(\mathbb{SL}_2^*)_r$ as follows. If:

$$x = \begin{pmatrix} e^{\frac{1}{2}rH} & 0 & e^{-\frac{1}{2}rH} \\ 2rF & e^{\frac{1}{2}rH} & 0 \\ e^{-\frac{1}{2}rH} & 0 & e^{\frac{1}{2}rH} \end{pmatrix},$$

then, we define:

$$x^* = \begin{pmatrix} e^{\frac{1}{2}\pi} & 0 & e^{-\frac{1}{2}\pi} \\ 2rF^* & e^{\frac{1}{2}\pi} & 0 \\ e^{-\frac{1}{2}\pi} & 0 & e^{\frac{1}{2}\pi} \end{pmatrix}.$$
Because of the standard duality between $\mathfrak{sl}_2$ and $\mathfrak{sl}_2^\ast$ (see [CP95]), we transport $\ast$ to the Lie algebra $\mathfrak{sl}_2$. One finds that $\ast$ fixes $\mathfrak{su}_2$, and thus $\mathfrak{su}_2^\ast$ can indeed be identified with lower triangular matrices. At the group level, one finally finds the compact form $SU_2$:

$$SU_2 := \{ x \in SL_2(\mathbb{C}) \mid xx^\ast = \text{id} \} .$$

whose Poisson-Lie dual is $(SU_2)_r = NA$. The subscript $r$ indicates that we renormalize the group law thanks to the parameter $r$.

The dressing action of $SU_2$ on $(SU_2)_r = NA$ is exactly the curved version of the coadjoint action (see [KS97]). It is defined as:

$$SU_2 \times NA \to NA$$

$$(k, b) \mapsto k \cdot b ,$$

where $k \cdot b = b'k'$ is the lower triangular matrix obtained via the Gram-Schmidt decomposition $kb = b'k'$, with $k' \in SU_2$.

2.3. Structure of the paper. In Section 3, we exhibit a presentation $U^h_q(\mathfrak{sl}_2)$ of the Jimbo-Drinfeld quantum group with two parameters $\hbar > 0$ and $q = e^{-r}$. It is tailored so that the commutative diagram in Figure 2 completes Figure 1.

In fact, we shall give a precise meaning to the commutative diagram. To do so, we will have to develop a notion of "convergence of Hopf algebras" with parameters, which is not formal. From this presentation, one could argue that the classical name of QUE algebras which stands for "Quantum Universal Enveloping algebras" is misleading. It is perhaps more appropriate to refer to the Drinfeld-Jimbo quantum group as a "quantized Poisson-Lie function algebra", which is Drinfeld’s original point of view or a "curved universal enveloping algebra".

In Section 4, we prove in Theorem 4.1 that quantum observables in large representations become classical observables. This is the implementation of the orbit method, which proves quantitatively that large representations behave like symplectic leaves (here dressing orbits), and that tensor product behaves like convolution of orbital measures.

Finally, in Section 5, we formulate a commutative diagram which mirrors Figure 2 for the dynamics. There, Theorem 5.1 gives a global picture where Pitman’s Theorem is related to both the geometric construction of Bougerol-Jeulin and Biane’s quantum walks.

3. A Heterodox Presentation of the Jimbo-Drinfeld Quantum Group

We define $U^h_q(\mathfrak{sl}_2)$ with $q = e^{-r}$ as follows. As explained before, $r$ has to be understood as curvature and $\hbar > 0$ is a Planck constant. We set

$$U^h_q(\mathfrak{g}) := \langle K = q^H, E, F \rangle / \mathcal{R}$$

FIGURE 2. The complete commutative diagram
where \( \mathcal{R} \) is the two-sided ideal of generated by the relations:

\[
[E, F] = \hbar \frac{K^{-1} - K}{2r},
\]

\[
KEK^{-1} = q^{2\hbar}E,
\]

\[
KFK^{-1} = q^{-2\hbar}F.
\]

It is also endowed with the same coproduct \( \Delta_r \) as \((SL^*_2)_r\) (See Lemma 2.1).

**The compact real form:** The analogue of choosing a real form for a Lie algebra in the context of Hopf algebras is choosing \(*\)-structures ([KS12, Section 1.2.7]). Here, \( * \) is an algebra anti-involution and we consider the \(*\)-fixed structures. The \(*\) anti-involution on the quantum group \( U^h_q(\mathfrak{sl}_2) \) is:

\[
K^* = K, \ E^* = F.
\]

This will correspond to the chosen real form in the Poisson-Lie group \((SL^*_2)_r\).

**Isomorphism:** There is a Hopf algebra isomorphism \( \Phi : U_q(\mathfrak{g}) \to U_{q,h}(\mathfrak{g}) \), between the classical presentation of Drinfeld-Jimbo and ours, such that:

\[
\begin{align*}
\Phi : & \quad U_q(\mathfrak{g}) \to U_{q,h}(\mathfrak{g}) \\
& \quad K \mapsto K \\
& \quad q \mapsto q^h \\
& \quad E \mapsto E \sqrt{\frac{2r}{h(q^{-h} - q^h)}} \\
& \quad F \mapsto F \sqrt{\frac{2r}{h(q^{-h} - q^h)}}
\end{align*}
\]

As such, the usual Casimir element, which generates the center \( Z[U_q(\mathfrak{g})] \):

\[
EF + \frac{q^{-1}K + qK^{-1}}{(q - q^{-1})^2}
\]

maps to a rescaling of

\[
4r^2EF + \left(q^{-h}K + q^hK^{-1}\right) \frac{2r\hbar}{(q^{-h} - q^h)} \in Z[U_q^h(\mathfrak{g})].
\]

Note that by a general result of Drinfeld (see his foundational papers), deformation theory says there is essentially only one quantum group. Our take is that there is a nicer presentation, which renders transparent the fact that semi-classical limits of quantum groups have to be Poisson-Lie groups with different curvatures.

**Degenerations** \( r \to 0 \) and \( \hbar \to 0 \): Clearly \( h \to 0 \) degenerates to a commutative algebra with three free generators. As a Poisson algebra, it is exactly the coordinate algebra \( \mathbb{C}[(SL^*_2)_r] \). Moreover, the Casimir becomes:

\[
4r^2EF + \left(e^{rH} + e^{-rH}\right) = \text{tr}(xx^*)
\]

which is exactly the invariant for a dressing orbit. Finally \( q \to 1 \) or \( r \to 0 \) gives the flat \( U^h(\mathfrak{sl}_2) \).

All these considerations hint to the following theorem. A detailed proof will be given in subsequent versions.
Theorem 3.1 (A commutative diagram for spaces). The following diagram between Hopf algebras commutes:

\[
\begin{array}{c}
(U_q^h(\mathfrak{sl}_2), \mu^h_q, \Delta_r) \xrightarrow{\hbar \to 0} (\mathbb{C}[(SU_2)^r], \{\cdot, \cdot\}_r, \Delta_r) \\
\downarrow r \to 0 \quad \downarrow r \to 0
\end{array} \quad \begin{array}{c}
(U_q^h(\mathfrak{sl}_2), \mu^h_q, \Delta_0) \xrightarrow{\hbar \to 0} (\mathbb{C}[\mathfrak{su}_2^*], \{\cdot, \cdot\}_0, \Delta_0)
\end{array}
\]

We arrive now to a proper implementation of the orbit method. We also have a convergence of quantum observables to classical observables for all \( r > 0 \). In fact, as vector spaces:

\[ U_q^h(\mathfrak{sl}_2) \approx \mathbb{C}[(SU_2)^r][[\hbar]] . \]

and:

\[ \mathbb{C}[(SU_2)^r] \approx U_q^h(\mathfrak{sl}_2) \mod \hbar. \]

3.1. Relevant literature. A point of view similar to this paper is developed in [KT00] and [BCDO09].

3.1.1. Kassel and Turaev’s biquantization. In [KT00], Kassel and Turaev present a commutative diagram that intriguingly similar to ours, as it uses two parameters denoted \( u \) and \( v \). Furthermore, their result, termed as biquantization of Lie bialgebras, holds for all Lie algebras. In fact, even at an advanced stage of our paper, it was not obvious to us that their construction matches ours. And it does.

The deformed law (2.3) is considered in [KT00, Eq. (2.6)], and defined formally from the Baker-Campbell-Hausdorff formula. The Baker-Campbell-Hausdorff formula converges in our case. Dually, one obtains a coproduct \( \Delta_r \) on the coordinate algebra of \((SL_2^*)_r\). Also, it is made into a Poisson algebra thanks to the Poisson bivector \( \pi_r \), which comes from the bracket of \( \mathfrak{sl}_2 \) (see [KT00, Eq. (2.9)]).

Notice that unlike [KT00], we are not working at the formal level. Also, referring to a two-parameter deformation as a biquantization gives symmetric roles to these parameters, which is not the case when trying to implement the orbit method. Indeed, in our case the first parameter \( \hbar \geq 0 \) controls the non-commutativity in the algebra of observables, in accordance with the principles of quantum mechanics, while \( r \geq 0 \) controls the non-commutativity of invariant vector fields, in the underlying space and thus controls the curvature.

Poisson-Lie duality reverses the roles of \( \hbar \) and \( r \), and therefore gives an impression of symmetry. However, the geometry and the representation theory of the underlying groups change entirely! This justifies our choice of vocabulary: ”curvature” replaces ”lack of non-cocommutativity” while ”positive Planck constant” replaces ”non-commutativity” in our treatment of quantum groups.

3.1.2. Ballesteros, Celeghini and Del Olmo’s point of view. In a very insightful comment, the authors of [BCDO09] argue that quantum groups and quantum algebras need to be viewed as abstract Hopf algebras since that, in many physical cases, the parameter \( q \) is completely independent from the truly quantum \( \hbar \) constant. An example they give, among others, is that \( q \) clearly plays the role of anisotropy parameter in the context of the Heisenberg XXZ spin chain.

A similar commutative diagram as ours appears in their paper, and in [BCDO09, Section 3.1], Ballesteros, Celeghini and Del Olmo introduce the parameter \( z \) instead of \( r \) at exactly the same place as us, in the standard Poisson-Lie structure.
3.2. **Representations.** Since Theorem 3.1 deals with another presentation of the Drinfeld-Jimbo quantum group, the representation theory is essentially unchanged. Of course, the constants $\hbar$ and $r$ appear at various places and the goal of this subsection is to record these unessential changes.

Recall from [KS12, Page 61] that $\mathcal{U}_q(\mathfrak{sl}_2)$, with the usual presentation 1.6, that for every $\lambda \in \mathbb{N}$, there exists a representation $V_\lambda$ of dimension $\lambda + 1$ and with basis labelled
\[
(\epsilon_k ; k = -\lambda, -\lambda + 2, \ldots, \lambda - 2, \lambda)
\]
The action on generators is given in terms of $q$-integers $[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}}$ by:
\[
\begin{cases}
K \cdot \epsilon_k = q^k \epsilon_k, \\
E \cdot \epsilon_k = \sqrt{\left[\frac{1}{2} \lambda + \frac{1}{2} k \right]_q \left[\frac{1}{2} \lambda + \frac{1}{2} k + 1 \right]_q} \epsilon_{k+2} = \frac{\sqrt{q^{\lambda+1} - q^{-1}} - q^{k+1} - q^{-k-1}}{q - q^{-1}} \epsilon_{k+2}, \\
F \cdot \epsilon_k = \sqrt{\left[\frac{1}{2} \lambda + \frac{1}{2} k \right]_q \left[\frac{1}{2} \lambda - \frac{1}{2} k + 1 \right]_q} \epsilon_{k-2} = \frac{\sqrt{q^{\lambda+1} - q^{-1}} - q^{k-1} - q^{-k+1}}{q - q^{-1}} \epsilon_{k-2}.
\end{cases}
\]

Upon rescaling thanks to the isomorphism (3.2), the weight $\mathbb{Z}$ lattice is rescaled to $\hbar \mathbb{Z}$, and highest weights are therefore of the form $\Lambda^\hbar$ for $\Lambda \geq 0$. The action on generators of $\mathcal{U}_q^\hbar(\mathfrak{sl}_2)$ becomes:
\[
\begin{cases}
K \cdot \epsilon_k = e^{-r\hbar \epsilon_k}, \\
E \cdot \epsilon_k = \sqrt{e^{h(\Lambda^\hbar + 1)} + e^{-h(\Lambda^\hbar + 1)} - e^{h(k+1)} + e^{-h(k+1)}} \epsilon_{k+2}, \\
F \cdot \epsilon_k = \sqrt{e^{h(\Lambda^\hbar + 1)} + e^{-h(\Lambda^\hbar + 1)} - e^{h(k-1)} + e^{-h(k-1)}} \epsilon_{k-2}.
\end{cases}
\]

Likewise, instead of the usual Casimir (3.3), which acts on $V_\lambda$ as the constant:
\[
(C_q)_{V_\lambda} := \frac{q^{\frac{1}{2} \lambda + 1} + q^{-\frac{1}{2} \lambda + 1}}{(q - q^{-1})^2},
\]
our Casimir (3.4) acts on $V_{\Lambda^\hbar}$ like the constant:
\[
(C_q^\hbar)_{V_{\Lambda^\hbar}} := \frac{e^{\frac{1}{2} r\hbar \Lambda^\hbar + r \hbar} + e^{-\frac{1}{2} r\hbar \Lambda^\hbar + r \hbar}}{(q - q^{-1})^2}.
\]

4. **Static semi-classical limits**

The main theorem of this Section deals with quantum observables converging to classical observables.

**Theorem 4.1.** Let $\pi := \text{mod} \ h$ be the quotient map and for $\Lambda > 0$ let
\[
\mathcal{O}_r(\Lambda) := SU_2 \ast_r \Lambda \subset (SU_2)^r,
\]
be a dressing orbit under the action of $SU_2$. Here $\Lambda > 0$ is identified with the diagonal matrix $\begin{pmatrix} \Lambda & 0 \\ 0 & -\Lambda \end{pmatrix}$. The orbit $\mathcal{O}_r(\Lambda)$ is endowed with a natural invariant measure $\omega$, induced by the Haar measure on $SU_2$.

Then the following limits hold for all $F \in \mathcal{U}_q^\hbar(\mathfrak{sl}_2)$, upon writing $f = \pi(F)$:
\[
\begin{align*}
\frac{\text{Tr}_{V^\hbar_{\pi(\Lambda)}}(q^{-H} F)}{\text{Tr}_{V^\hbar_{\pi(\Lambda)}}(q^{-H})} \xrightarrow{\hbar \to 0} \int_{\mathcal{O}_r(\Lambda)} f(p) \omega(dp) &= \int_{SU_2} f(k \ast_r \Lambda) dk, \\
\frac{\text{Tr}_{V^\hbar_{\pi(\Lambda_1)} \otimes V^\hbar_{\pi(\Lambda_2)}}(q^{-H} F)}{\text{Tr}_{V^\hbar_{\pi(\Lambda_1)} \otimes V^\hbar_{\pi(\Lambda_2)}}(q^{-H})} \xrightarrow{\hbar \to 0} \int_{\mathcal{O}_r(\Lambda_1) \times \mathcal{O}_r(\Lambda_2)} f(p \ast_r q) \omega(dp) \omega(dq) &= \int_{SU_2 \times SU_2} f(k_1 \Lambda_1 \ast_r k_2 \Lambda_2) dk_1 dk_2.
\end{align*}
\]

The proof is carried in the next three subsections.
4.1. Uniform measure on a dressing orbit and Archimedes’ Theorem. The following Lemma describes, in coordinates, the uniform measure on dressing orbits \( O_r(\Lambda) = SU_2 \ast_r \Lambda \). By uniform measure, we mean of course, the measure induced by the Haar measure.

Proposition 4.2. A uniform element in \( O_r(\Lambda) = SU_2 \ast_r \Lambda \) is uniquely written:

\[
g = \begin{pmatrix} e^{\frac{1}{2} r \mu} e^{i \Theta} \sqrt{e^{r \Lambda} + e^{-r \Lambda} - e^{r \mu} - e^{-r \mu}} & 0 \\ e^{-\frac{1}{2} r \mu} & e^{-r \mu} \end{pmatrix},
\]

with \( \mu \) and \( \Theta \) independent random variables. The phase \( \Theta \) is uniform on \([0, 2\pi]\) and \( \mu \) follows the distribution:

\[
P(\mu \in dx) = \frac{r e^{r x} dx}{e^{r \Lambda} - e^{-r \Lambda}} \mathbb{1}_{[-\Lambda, \Lambda]}.
\]

In fact, the law of \( \mu \) is exactly encoded by the Harish-Chandra spherical function, but we will prefer an elementary derivation based on the following result, which dates back to Archimedes’ work “On the sphere and cylinder”. We provide the proof for the sake of completeness.

Theorem 4.3 (Archimedes’ Theorem). Consider the unit sphere \( S^2 \subset \mathbb{R}^3 \). Then:

- (Global version): If the sphere is inscribed inside a cylinder, the sphere has exactly the same surface area as the lateral side of the cylinder.
- (Local version): Considering the uniform measure on the sphere, the image measure through the projection along any axis is uniform on \([-1, 1]\).

Proof. The global version is naturally deduced from the local version by integration along the axis. One can also invoke the nowadays known area formulas. This is almost Archimedes’ original statement as he included the bases of the cylinder, leading him to the celebrated formulation that “the surface of the cylinder is half as large again as the surface of the sphere” [H+02].

For the local version, one can perform a Jacobian computation but we prefer the following probabilistic derivation. It generalizes easily to any dimension, showing that projections along an axis are Beta distributions. Now, because of the rotation invariance of the Gaussian distribution, a uniform random variable on the unit sphere is obtained by normalizing a standard Gaussian vector \((\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3)\). Therefore, the measure of interest is the law of

\[
X := \frac{\mathcal{N}_1}{\sqrt{\mathcal{N}_1^2 + \mathcal{N}_2^2 + \mathcal{N}_3^2}}.
\]

We need to show that:

\[
P(X \in dx) = \mathbb{1}_{[-1, 1]} \frac{dx}{2}.
\]

We have that \( \varepsilon = \text{sgn}(X) \) is a Bernoulli random variable and \(|X|\) is independent, by symmetry of the Gaussian distribution. Therefore, we write:

\[
X^2 = \frac{\mathcal{N}_1^2}{\mathcal{N}_1^2 + \mathcal{N}_2^2 + \mathcal{N}_3^2} = \frac{\gamma_1}{\gamma_\frac{1}{2} + \gamma_1},
\]

where the \( \gamma_k \)'s are independent Gamma variables with parameter \( k \). Because of the Beta-Gamma algebra identities, \( X^2 \) has a Beta distribution with parameters \((\frac{1}{2}, 1)\). As such \( P(X^2 \in dx) = \mathbb{1}_{[0, 1]} \frac{dx}{2 x^\frac{1}{2}} \), which implies by change of variables that \( P(|X| \in dx) = \mathbb{1}_{[0, 1]} dx \).

The result follows. \( \square \)
Proof of Proposition 4.2. Using notation (2.4), we have:
\[ e^{r\Lambda} + e^{-r\Lambda} = \text{tr}(gg^*) = e^{rH} + e^{-rH} + 4r^2 |F|^2. \]
The expression (4.3) then follows, by setting \( H = \mu \) and \( 2rF = e^{i\Theta}|2rF| \). Uniqueness is obvious. Let us now compute the joint distribution of \( \Theta \) and \( \mu \).

Since that for any \( \theta \in \mathbb{R} \), \( t = \left( \begin{array}{cc} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{array} \right) \in SU_2 \) and

\[ \text{tgt}^{-1} = \left( \begin{array}{cc} e^{i\frac{r}{2}\mu} & \sqrt{e^{r\Lambda} + e^{-r\Lambda} - e^{\mu} - e^{-\mu}} \\ e^{-i\frac{r}{2}\mu} & 0 \end{array} \right) \]
still belongs to the orbit and is also distributed according to the uniform measure on \( \mathcal{O}_r(\Lambda) \). As such for all \( \theta \in \mathbb{R} \), we have the equality in law

\[ (\mu, \Theta) \overset{\text{L}}{=} (\mu, \Theta - 2\theta \mod 2\pi). \]

Necessarily \( \Theta \) is uniform on \([0, 2\pi]\) and is independent of \( \mu \). In order to track the distribution of \( \mu \), let us write:

\[ g = k_1 \left( \begin{array}{cc} e^{\frac{r}{2}\Lambda} & 0 \\ 0 & e^{-\frac{r}{2}\Lambda} \end{array} \right) k_2 \]
with \( k_1 \) Haar distributed on \( SU_2 \), and \( k_2 \) in such a way that \( g \in NA \). Thus:

\[ gg^* = k_1 \left( \begin{array}{cc} e^{r\Lambda} & 0 \\ 0 & e^{-r\Lambda} \end{array} \right) k_1^* \]

\[ = \left( \begin{array}{cc} e^{i\theta}e^{\frac{r}{2}\mu} & \sqrt{e^{r\Lambda} + e^{-r\Lambda} - e^{\mu} - e^{-\mu}} \\ e^{-i\theta}e^{-\frac{r}{2}\mu} & 0 \end{array} \right) \left( e^{r\Lambda} + e^{-r\Lambda} - e^{\mu} - e^{-\mu} + e^{-r\mu} \right), \]

and therefore \( e^{r\mu} \) is the top-left coefficient of a Hermitian matrix, whose spectrum is \([e^{-r\Lambda}, e^{r\Lambda}]\), and whose eigenvectors are Haar distributed. The space of such matrices is a sphere with principal diameter \([e^{-r\Lambda}, e^{r\Lambda}]\), and the induced measure is uniform. Hence, by Archimedes’ Theorem 4.3, \( e^{r\mu} \) is uniform on \([e^{-r\Lambda}, e^{r\Lambda}]\).

We conclude by the following computation. For all bounded measurable function \( \varphi : \mathbb{R} \to \mathbb{R} \), the last paragraph yields:

\[ \mathbb{E}(\varphi(\mu)) = \int_{e^{r\Lambda}}^{e^{-r\Lambda}} \frac{du}{e^{r\Lambda} - e^{-r\Lambda}} \varphi \left( \frac{1}{r} \log u \right) \]
\[ = \int_{-\Lambda}^{\Lambda} re^{rx} dx e^{r\Lambda} \varphi(x). \]

\[ \square \]

4.2. Proof of (4.1): One orbit. In this case \( \Lambda \in \mathbb{R}_+^* \) and \( V(\Lambda^h) \) has basis

\[ (e_k ; k = -\Lambda^h, -\Lambda^h + 2, \ldots, \Lambda^h - 2, \Lambda^h). \]

Because of the PBW basis theorem, it suffices to prove the result when \( \mathcal{F} = E^a F^b H^c \) for natural numbers \( a, b \) and \( c \). Thanks to the structure of the representation given in (3.5), we have that:

\[ \mathcal{F} e_k \in \mathbb{C} e_{k+2(a-b)}. \]

As such, excepted for a bounded number of indices \( k \), we have that \( \langle \mathcal{F} e_k, e_k^* \rangle = 0 \) if \( a \neq b \). Upon computing the coefficients, we have as \( h \to 0 \):

\[ \langle q^{-H} \mathcal{F} e_k, e_k^* \rangle = (1 + o(1)) \delta_{a,b} e^{r(hk)} (hk)^c \left| e^{r\Lambda} + e^{-r\Lambda} - e^{rk} - e^{-rk} \right|^a, \]
and each of these terms is bounded. As such:

\[ \text{tr}_{V(\Lambda^h)}(q^{-H} F) = \sum_{k=-\Lambda^h}^{\Lambda^h} \langle q^{-H} F e_k, e_k^* \rangle \]

\[ = \mathcal{O}(1) + \sum_{k=-\Lambda^h}^{\Lambda^h} \delta_{a,b} (1 + o(1)) e^{r(hk)} (\hbar k)^c \left| e^{rA} + e^{r-hk} - e^{-r} \right|^a \]

\[ = \mathcal{O}(1) + \delta_{a,b} \frac{1 + o(1)}{\hbar} \int_{-\Lambda}^{\Lambda} e^{rx} x^c \left| e^{rA} + e^{-rA} - e^{rx} - e^{-rx} \right|^a dx , \]

where we recognized a Riemann sum at the last step. Therefore, if we divide by

\[ \text{tr}_{V(\Lambda^h)}(q^{-H}) = \mathcal{O}(1) + \frac{1 + o(1)}{r \hbar} (e^{rA} - e^{-rA}) , \]

we obtain:

\[ \frac{\text{tr}_{V(\Lambda^h)}(q^{-H} F)}{\text{tr}_{V(\Lambda^h)}(q^{-H})} = \mathcal{O}(1) + \delta_{a,b} (1 + o(1)) \int_{-\Lambda}^{\Lambda} e^{rx} (e^{rA} - e^{-rA} - e^{rx} - e^{-rx})^a dx , \]

Now, thanks to Proposition 4.2, any element \( x \) which is uniform in \( \mathcal{O}_r(\Lambda) = SU_2 \cdot \Lambda \) can be written:

\[ x = \left( \begin{pmatrix} e^{\frac{i}{2} r^H} & 0 \\ 2rF & e^{-\frac{i}{2} r^H} \end{pmatrix}, \begin{pmatrix} e^{\frac{i}{2} r^H} & 2rE \\ 0 & e^{-\frac{i}{2} r^H} \end{pmatrix} \right) \]

where \( E^a(x) = F(x) = e^{i\Theta} \sqrt{e^{rA} + e^{-rA} - e^{rH} - e^{-rH}} \), \( \Theta \) is uniform on \( [0, 2\pi] \) and \( H(x) \) follows the prescribed distribution. As such:

\[ \frac{\text{tr}_{V(\Lambda^h)}(q^{-H} F)}{\text{tr}_{V(\Lambda^h)}(q^{-H})} = \mathcal{O}(1) + (1 + o(1)) \mathbb{E} (E(x)^a F(x)^b H(x)^c) = o(1) + \int_{SU_2} f(k \cdot \Lambda) dk . \]

This concludes the proof of Eq. (4.1) for one orbit. Notice that the proof holds for all \( r \geq 0 \).

4.3. Proof of (4.2): Convolution of two orbits. Sweedler’s notation for the coproduct \( \Delta_r \) on \( \mathcal{U}_q^h(sl_2) \) is written:

\[ \Delta_r \mathcal{F} = \bigoplus_{(\mathcal{F})} \mathcal{F}_1 \otimes \mathcal{F}_2 . \]

Hence, since \( q^{-H} \) is group-like and \( \Delta_r \) is a morphism:

\[ \Delta_r (q^{-H} \mathcal{F}) = \bigoplus_{(\mathcal{F})} (q^{-H} \mathcal{F}_1) \otimes (q^{-H} \mathcal{F}_2) . \]

This gives:

\[ \text{Tr}_{V(\Lambda^h_1) \otimes V(\Lambda^h_2)}(q^{-H} \mathcal{F}) = \sum_{i,j} \langle \Delta_r q^{-H} \mathcal{F} \cdot e_i \otimes e_j, e_i \otimes e_j \rangle \]

\[ = \sum_{(\mathcal{F})} \sum_{i,j} \langle q^{-H} \mathcal{F}_1 e_i, e_i \rangle \langle q^{-H} \mathcal{F}_2 \cdot e_j, e_j \rangle \]

\[ = \sum_{(\mathcal{F})} \text{Tr}_{V(\Lambda^h_1)}(q^{-H} \mathcal{F}_1) \text{Tr}_{V(\Lambda^h_2)}(q^{-H} \mathcal{F}_2) . \]

Again, because \( q^{-H} \) is group-like, we have:

\[ \text{Tr}_{V(\Lambda^h_1) \otimes V(\Lambda^h_2)}(q^{-H}) = \text{Tr}_{V(\Lambda^h_1)}(q^{-H}) \text{Tr}_{V(\Lambda^h_2)}(q^{-H}) . \]
and upon dividing by that quantity, we obtain the equality:

\[
\frac{\text{Tr}_{V(\Lambda_1) \otimes V(\Lambda_2)} \left( q^{-\hbar} F \right)}{\text{Tr}_{V(\Lambda_1^0) \otimes V(\Lambda_2^0)} (q^{-\hbar})} = \sum_{(F)} \frac{\text{Tr}_{V(\Lambda_1^0)} \left( q^{-\hbar} F \right)}{\text{Tr}_{V(\Lambda_1^0)} (q^{-\hbar})} \frac{\text{Tr}_{V(\Lambda_2^0)} \left( q^{-\hbar} F \right)}{\text{Tr}_{V(\Lambda_2^0)} (q^{-\hbar})} .
\]

Using the result (4.1) for one orbit, one has as \( \hbar \to 0: \)

\[
= o(1) + \sum_{(F)} \int_{O_r(\Lambda_1)} \pi(F_1)(x) \omega(dx) \int_{O_r(\Lambda_2)} \pi(F_2)(y) \omega(dy)
\]

\[
= o(1) + \int_{O_r(\Lambda_1) \times O_r(\Lambda_2)} \left( \sum_{(F)} \pi(F_1)(x) \pi(F_2)(y) \right) \omega(dx) \omega(dy).
\]

Now, recall from Remark 2.2 that the coproduct \( \Delta_r \) on has nothing inherently quantum! It is the coproduct associated to the dual Poisson-Lie group \((SL_2)_r\) and has such, by definition:

\[
\sum_{(F)} \pi(F_1)(x) \pi(F_2)(y) = \pi(F)(x * y).
\]

The result (4.2) follows.

5. **Semiclassical limits: from quantum tensor dynamics to convolution dynamics**

**Theorem 5.1** (The commutative diagram for dynamics). There exist random walks (tensor dynamics in the quantum case, convolution dynamics in the classical case) such that the following convergences in law hold.

\[
\begin{align*}
\Lambda_n \xrightarrow{x_n \to 0} & \quad \left( r e^{\frac{1}{2} r X_t} \int_0^t e^{-r X_s} d(Y_s + iZ_s) \right) \\
\text{Quantum random walks} & \quad \text{The convolution dynamic of Bougerol-Jeulin} \\
\downarrow & \quad \text{Flat Brownian Motion on } \mathbb{R}^2 \\
\Lambda_n = \sqrt{\frac{1}{2} + X_n^2 + Y_n^2 + Z_n^2} (X_n, Y_n, Z_n) & \quad \Lambda_t = \sqrt{X_t^2 + Y_t^2 + Z_t^2} \\
\downarrow & \quad \text{Convolution dynamic/} \\
\text{Biane’s quantum random walks} & \quad \text{Flat Brownian Motion on } \mathfrak{su}_2^* \approx \mathbb{R}^3 \\
\end{align*}
\]

In order to construct the quantum walk on \( U_q^h(\mathfrak{sl}_2) \), this latter algebra is taken as the algebra of observables for one increment. Thanks to the framework detailed in Subsection 1.3 the relevant algebra is the inductive limit

\[
\mathcal{A} = \lim_{n \to \infty} (U_q^h(\mathfrak{sl}_2))^{\otimes n}.
\]

Again, the state is a product state using the representation \( V_{\lambda=1} \). And the measurement operators are exactly the same as Eq. (1.8). The goal of this section is to describe different
These different processes that appear are portrayed in the diagram:

\[ e^{h\Lambda_n} + e^{-h\Lambda_n} \in A \xrightarrow[h \to 0]{} x_n \xrightarrow[r \to 0]{} \Lambda_n = \sqrt{\frac{1}{2} X_n^2 + Y_n^2 + Z_n^2} \in A \xrightarrow[h \to 0]{} x_t = \left( \frac{1}{2} X_t, 0, -\frac{1}{2} X_t \right) \]

5.1. **Quantum walk on the quantum group.** In this subsection, one needs to have in mind the presentation of \( U_q^h(\mathfrak{sl}_2) \) made in Section 3.

The quantum dynamics of \( H_t = M_t(H) \) and \( \Lambda_t = M_t(C_q^h) \) are easily deduced from the earlier sections. The behavior is the same as in the flat case, because

- \( H \) is primitive. And representations have the same weight multiplicities. Therefore, in this case, \( H \) is a Bernoulli random walk.
- \( C_q^h \) generating the center, \( \{\Lambda_t; t \in \mathbb{N}\} \) is a classical process. Moreover, because of the rigidity of quantum groups, the Littlewood-Richardson rule is unchanged.

On the other hand, the other generators of the quantum group yield a different dynamic. By abuse of notation, define the Bernoulli random walks:

\[ M_n(\Re \beta) = \sum_{k=0}^{n} 1^\otimes k \otimes \Re F, \]
\[ M_n(\Im \beta) = \sum_{k=0}^{n} 1^\otimes k \otimes \Im F, \]

with the obvious convention that for operators \( \Re F = \frac{1}{2}(F + F^*) \). This is an abuse of notation because the RHS has no reason to belong to the image of \( M_n \). Moreover, it is different from \( M_n(F) \) as \( F \) is not primitive, hence the reason why this is an abuse of notation.

We can express \( M_n(E) \) and \( M_n(F) \) in terms of these Bernoulli random walks. Using the expression of coproducts:

\[ M_{n+1}(F) = (M_n \otimes 1)(\Delta F) = M_n(F) \otimes K + M_n(K^{-1}) \otimes F, \]

As such:

\[ M_n(F) = M_n(K)M_n(K^{-1})M_n(F) \]
\[ = M_n(K) \sum_{s=0}^{n} (M_s(K^{-1})M_s(F) - M_{s-1}(K^{-1})M_{s-1}(F)) \]
\[ = M_n(K) \sum_{s=0}^{n} [M_s(K^{-1}) \cdot (M_{s-1}(F) \otimes K + M_{s-1}(K^{-1}) \otimes F) - M_{s-1}(K^{-1})M_{s-1}(F)] \]
\[ = M_n(K) \sum_{s=0}^{n} M_{s-1}(K^{-1})M_{s-1}(F) + M_{s-1}(K^{-2}) \otimes K^{-1}F - M_{s-1}(K^{-1})M_{s-1}(F) \]
= M_n(K) \sum_{s=0}^{n} M_{s-1}(K^{-2}) \otimes K^{-1} F.

This expression has been tailored to look similar to the computation of the $F$ coefficient via matrix products. As such one can see the above expression as a (non-commutative) martingale transform or discrete stochastic integral.

5.2. Brownian motion on $(SU_2^\ast)_r$. For the semi-classical limit, it is clear from the convergence of Bernoulli random walks that

\begin{equation}
\left(\hbar M_{t/\hbar^2}(H), \hbar M_{t/\hbar^2}(\beta); t \geq 0 \right) \xrightarrow{\hbar \to 0} \left(H_t, \beta^C_t; t \geq 0 \right).
\end{equation}

Now, $H_t$ a Brownian motion, which is the diagonal driving the matrix process on $K^\ast \approx NA$:

$$x^r_t = \begin{pmatrix} e^{2rH_t} & 0 \\ r^2 e^{2rH_t} \int_0^t e^{-rH_s} d\beta^C_s & e^{-2rH_t} \end{pmatrix}. $$

In order to justify the convergence of the full non-commutative process to $(x_t, t \geq 0)$, all that is required is to prove that, jointly with the rest:

$$\left(F^h_t := \hbar M_{t/\hbar^2}(F); t \geq 0 \right) \xrightarrow{\hbar \to 0} \begin{pmatrix} 1^{s-u/\hbar^2} \otimes K^{-1} F \end{pmatrix}.$$

This is basically realizing that the last expression of the previous paragraph is a discrete and quantum version of the stochastic integral.

**Proposition 5.2.** The above convergence holds.

**Proof.** Let $D$ be a subdivision of the segment $[0, t]$. The maximal meshsize is denoted $|D|$. Associated to $D$ and $t$, one defines the sums:

$$F^{h,D}_t := \hbar M_{t/\hbar^2}(K) \sum_{[u,v] \in D} \sum_{s=u/\hbar^2}^{v/\hbar^2-1} M_{u/\hbar^2-1}(K^{-2}) \otimes 1^{s-u/\hbar^2} \otimes K^{-1} F.$$

$$= \hbar M_{t/\hbar^2}(K) \sum_{[u,v] \in D} M_{u/\hbar^2-1}(K^{-2}) \sum_{s=u/\hbar^2}^{v/\hbar^2-1} 1^{s-1} \otimes K^{-1} F.$$

**Step 1:** Using the weak convergence of Eq. (5.1), one has the weak convergence jointly with the walks:

$$\left(F^{h,D}_t; t \geq 0 \right) \xrightarrow{\hbar \to 0} \begin{pmatrix} e^{2rH_t} \sum_{[u,v] \in D} e^{-rH_u} \left( \beta^C_v - \beta^C_u \right); t \geq 0 \right).$$

Indeed even when considering different times, $F^{h,D}_t$ is polynomial function of the random walks. We are simply applying a polynomial version the mapping theorem, in the non-commutative setting.

**Step 2:** Let us prove that for every fixed $h_0$, $T > 0$ and $p > 1$

$$\limsup_{|D| \to \infty} \sup_{\hbar \in [0,h_0]} \sup_{t \in [0,T]} \left|F^h_t - F^{h,D}_t\right|_{L^p(M,\tau)} = 0.$$ We start by noticing that $M_{t/\hbar^2}(K)$ is uniformly bounded in all $L^p$ as Bernoulli walks have exponential moments of all order. Then using the Cauchy-Schwartz inequality:

$$\left|F^h_t - F^{h,D}_t\right|^2_{L^p(M,\tau)}$$
\[ \leq \left| M_{t/h^2}(K) \right|_{L^{2p}(M,\tau)} \hbar \sum_{[u,v] \in D} \sum_{s = u/h^2}^{v/h^2-1} (M_{s-1}(K^-) - M_{u/h^2-1}(K^-) \otimes 1^{s-u/h^2}) \otimes K^{-1} F \left|_{L^{2p}(M,\tau)} \right. \]

\[ \leq h \sum_{[u,v] \in D} \sum_{s = u/h^2}^{v/h^2-1} \left( M_{s-1}(K^-) - M_{u/h^2-1}(K^-) \otimes 1^{s-u/h^2} \right) \otimes K^{-1} F \left|_{L^{2p}(M,\tau)} \right. \]

Now, thanks to the non-commutative Burkholder-Davis-Gundy inequality [PX97, Theorem 2.1] established by Pisier-Xu, the above sum is controlled by non-commutative brackets:

\[ \left| F_t^h - F_t^{h,D} \right|_{L^p(M,\tau)}^2 \leq \left( \sum_{[u,v] \in D} \sum_{s = u/h^2}^{v/h^2-1} h^2 \left( M_{s-1}(K^-) - M_{u/h^2-1}(K^-) \right)^2 \right)^{1/2} \]

\[ \leq \left( \sum_{[u,v] \in D} \sum_{s = u/h^2}^{v/h^2-1} e^{r\hbar H_{s-1}} - e^{r\hbar H_{u/h^2-1}} \right)^2 \left|_{L^p(M,\tau)} \right. \]

Now, we need an estimate showing that \( e^{r\hbar H_{s-1}} e^{r\hbar H_{u/h^2-1}} \right|_{L^{2p}(M,\tau)}^2 \leq (s h^2 - u) \), using the classical BDG inequalities, since we are dealing with a classical random walk. As such:

\[ \left| F_t^h - F_t^{h,D} \right|_{L^p(M,\tau)}^2 \leq \left( \sum_{[u,v] \in D} \sum_{s = u/h^2}^{v/h^2-1} h^2 (s h^2 - u) \right)^{1/2} \]

\[ = \left( \sum_{[u,v] \in D} \sum_{s = 0}^{v/h^2 - u/h^2 - 1} h^2 (s h^2) \right)^{1/2} \]

\[ = \left( \sum_{[u,v] \in D} \frac{1}{2} h^4 (v/h^2 - u/h^2 - 1)(v/h^2 - u/h^2) \right)^{1/2} \]

\[ = \left( \sum_{[u,v] \in D} \frac{1}{2} (v - u - h^2)(v - u) \right)^{1/2} \]

\[ \leq \left( \sum_{[u,v] \in D} (v - u - h^2)(v - u) \right)^{1/2} \]

\[ \leq |D|^{1/2} \]

all implicit constants being uniform in \( t \in [0,T] \) and \( h \).
Step 3: By the same computation, in the easier classical setting, any fixed moment of \( \left( e^{tH_1} \sum_{[u,v] \in D} e^{-tH_u} (\beta_v^C - \beta_u^C) : t \geq 0 \right) \) is close to the moment of \( \left( e^{tH_1} \int_0^t e^{-sH_1} d\beta_s^C : t \geq 0 \right) \), the error depending only on \(|D|\). Also one can invoke the convergence in probability and boundedness in every \( L^p \).

Step 4: To conclude, any moments of \( F_t^{(h,D)} \) are

- close to the moments of \( F_t^{(h,D)} \) via Step 2.
- which are close to the moments of \( e^{tH_1} \sum_{[u,v] \in D} e^{-tH_u} (\beta_v^C - \beta_u^C) \) via Step 1.
- which are close to the moments of \( e^{tH_1} \int_0^t e^{-sH_1} d\beta_s^C \) via Step 3.

\( \square \)

The quantum Casimir becomes the invariant:

\[
\frac{1}{2} \text{tr} x_t x_t^* = \frac{1}{2} \left( e^{\Lambda t} + e^{-\Lambda t} \right) = 2r^2 |F_t|^2 + \frac{1}{2} \left( e^{rH_1} + e^{-rH_1} \right)
\]

And as such:

\[
\Lambda_t = \frac{1}{r} \text{Argch} \left[ 2r^2 |F_t|^2 + \cosh(H_1) \right].
\]

Appendix A. Non-commutative topological considerations

Throughout the paper, we avoided the matters of completions of algebras. Since this is definitely not the main focus of the paper, we chose to postpone these topological considerations to this appendix.

Already, let us explain why, in most cases, completion into a Von Neuman algebra is an overkill. In the entire paper, one can perform functional calculus at the level of the matrix algebras obtained after representation. For example, the first instance where we invoked elements belonging to a completion was Theorem 1.6 where the Casimir element \( C_g \) is defined as a square-root. The simplest way of defining the object is the following. One has to remember that, in the computation of non-commutative moments, \( C_g^2 \) is represented as a Hermitian matrix before taking the trace. At that level, functional calculus is available for Hermitian matrices and the square-root is perfectly well-defined.

Nevertheless, since the machinery exists (see [PX97], [PX03] and references therein), it is possible to have a more intrinsic point of view and complete any of the algebras \( \mathcal{A} \) considered in the paper into a \( C^* \) algebra or a Von Neuman algebra. The general technique relies on the state \( \tau \). In order to form a Banach algebra, we define the norm \( d \) given by:

\[
\forall X \in \text{End}(V), \quad d(X) = \lim_{p \to \infty} \tau \left( (XX^*)^p \right)^\frac{1}{p},
\]

then we complete \( \mathcal{A} \) thanks to \( d \). In order to form a Von Neuman algebra, there is the Gelfand-Naimark-Segal construction (GNS for short). This is done in several steps which we detail for the quantum group \( U_q^h(sl_2) = \mathcal{A} \). The other algebras considered in the paper are tensor products or degenerations.

Step 1: Linear structure. Consider a representation \( \rho : \mathcal{A} = U_q^h(sl_2) \to \bigoplus_{n=0}^\infty \text{End}(V_n) \), which can be taken to be the faithful Peter-Weyl isomorphism. For shorter notations, let \( \rho_n = \rho|_{\text{End}(V_n)} \) be the restriction to the \( n \)-th component. Then define a trace via

\[
\tau(a) = \sum_{n=0}^\infty d_n \frac{\text{tr}(\rho_n(a))}{\dim V_n},
\]

where

\[
\sum_{n=0}^\infty d_n = 1.
\]
From this normalized trace $\tau$, one forms the scalar product $\langle a, b \rangle := \tau(a^\dagger b)$ and consider the completion into a Hilbert space $H$. Thus we embed the algebra into a linear space, but the multiplicative structure is missing.

**Step 2: Multiplicative structure.** The algebra $\mathcal{A} \subset H$ acts on $H$ via multiplication. Moreover, seeing $A \in \mathcal{A}$ as an operator on $H$, we write for $b \in H$:

$$A(b) := A \times b,$$

and from the Cauchy-Schwartz inequality:

$$\|A(b)\|_H = \sqrt{\tau[\rho(A^\dagger A)b^\dagger b]} \leq \sqrt{\tau[\rho(A^\dagger A)]}\|b\|_H.$$  

Hence the $A$ acts necessarily as a bounded operator. As such the algebra $\mathcal{A}$ is identified to a subalgebra of $B(H)$, the algebra of bounded operators on $H$.

**Step 3: Completions.** Upon completion for the operator norm, one obtains a $C^*$ algebra. Upon completion with respect to the weak-* topology, one obtains a Von Neuman algebra. Going even further, one obtains the unbounded operators on $H$ affiliated to the algebra $\mathcal{A}$.

**References**

[BBO05] Philippe Biane, Philippe Bougerol, and Neil O’Connell. Littelmann paths and Brownian paths. *Duke Math. J.*, 130(1):127–167, 2005.

[BBO09] Philippe Biane, Philippe Bougerol, and Neil O’Connell. Continuous crystal and Duistermaat-Heckman measure for Coxeter groups. *Adv. Math.*, 221(5):1522–1583, 2009.

[BCDO09] A Ballesteros, E Celeghini, and MA Del Olmo. Poisson–hopf limit of quantum algebras. *Journal of Physics A: Mathematical and Theoretical*, 42(27):275202, 2009.

[Bia91] Philippe Biane. Quantum random walk on the dual of SU(n). *Probab. Theory Related Fields*, 89(1):117–129, 1991.

[Bia06a] Philippe Biane. Le théorème de Pitman, le groupe quantique $SU_q(2)$, et une question de P. A. Meyer. In In memoriam Paul-André Meyer: Séminaire de Probabilités XXXIX, volume 1874 of *Lecture Notes in Math.*, pages 61–75. Springer, Berlin, 2006.

[Bia06b] Philippe Biane. Introduction to random walks on noncommutative spaces. In *Quantum potential theory*, pages 61–116. Springer, 2008.

[Bia09] Philippe Biane. From Pitman’s theorem to crystals. In *Noncommutativity and singularities*, volume 55 of *Adv. Stud. Pure Math.*, pages 1–13. Math. Soc. Japan, Tokyo, 2009.

[BJ02] Philippe Bougerol and Thierry Jeulin. Paths in weyl chambers and random matrices. *Probability Theory and Related Fields*, 124(4):517–543, 2002.

[Chh13] Reda Chhaibi. Littelmann path model for geometric crystals, whittaker functions on lie groups and brownian motion. *arXiv preprint arXiv:1302.0902*, 2013.

[CP95] Vyjayanthi Chari and Andrew N Pressley. *A guide to quantum groups*. Cambridge university press, 1995.

[DM90] Persi Diaconis and Laurent Miclo. On times to quasi-stationarity for birth and death processes. *Journal of Theoretical Probability*, 22(3):558–586, 2009.

[GHL90] Sylvestre Gallot, Dominique Hulin, and Jacques Lafontaine. *Riemannian geometry*, volume 3. Springer, 1990.

[GZB91] MD Gould, RB Zhang, and AJ Bracken. Generalized gel’fand invariants and characteristic identities for quantum groups. *Journal of mathematical physics*, 32(9):2298–2303, 1991.

[H⁺02] Thomas Little Heath et al. *The works of Archimedes*. Courier Corporation, 2002.

[Kas12] Christian Kassel. *Quantum groups*, volume 155. Springer Science & Business Media, 2012.

[Kir99] A. A. Kirillov. Merits and demerits of the orbit method. *Bull. Amer. Math. Soc. (N.S.)*, 36(4):433–488, 1999.

[KS97] Yvette Kosmann-Schwarzbach. Lie bialgebras, poisson lie groups and dressing transformations. In *Integrability of nonlinear systems*, pages 104–170. Springer, 1997.

[KS12] Anatoli Klimyk and Konrad Schmüdgen. *Quantum groups and their representations*. Springer Science & Business Media, 2012.

[KT00] Christian Kassel and Vladimir Turaev. Biquantization of lie bialgebras. *Pacific J. Math.*, 195(2):297–369, 2000.
[LLP12] Cédric Lecouvey, Emmanuel Lesigne, and Marc Peigné. Random walks in Weyl chambers and crystals. *Proc. Lond. Math. Soc. (3)*, 104(2):323–358, 2012.

[LLP13] Cédric Lecouvey, Emmanuel Lesigne, and Marc Peigné. Conditioned random walks from Kac-Moody root systems. *Transactions of the AMS (accepted)*, pages 1–30, 2013, arXiv:1306.3082.

[Nik06] Ashkan Nikeghbali. An essay on the general theory of stochastic processes. *Probab. Surveys*, 3:345–412, 2006.

[Pit75] James W Pitman. One-dimensional brownian motion and the three-dimensional bessel process. *Advances in Applied Probability*, 7(3):511–526, 1975.

[Pra05] Victor Prasolov. *Surveys in modern mathematics*, volume 321. Cambridge University Press, 2005.

[PX97] Gilles Pisier and Quanhua Xu. Non-commutative martingale inequalities. *Communications in mathematical physics*, 189(3):667–698, 1997.

[PX03] Gilles Pisier and Quanhua Xu. Non-commutative lp-spaces. *Handbook of the geometry of Banach spaces*, 2:1459–1517, 2003.

[RP81] L. Chris G Rogers and JW Pitman. Markov functions. *The Annals of Probability*, pages 573–582, 1981.

[RY13] Daniel Revuz and Marc Yor. *Continuous martingales and Brownian motion*, volume 293. Springer Science & Business Media, 2013.