QUASI-AFFINENESS AND 1-RESOLUTION PROPERTY

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Abstract. We prove that, under moderate hypothesis, any normal algebraic space which satisfies 1-resolution property is quasi-affine. More generally, we show that for algebraic stacks satisfying similar hypothesis, 1-resolution property implies separatedness.

1. Introduction

The resolution property for an algebraic stack is the statement that every coherent sheaf is the quotient of a vector bundle. Although it is always satisfied for quasi-projective schemes, it is not well-understood in general. Totaro has proved that any (noetherian normal) stack which has the resolution property is the quotient of a quasi-affine scheme by some \(GL(n)\) (see [4]). This has been extended further by Gross quasi-compact and quasi-separated algebraic stacks (see [2]). For further details on the resolution property, we refer the reader to [2], [4], [11].

In this paper, we will consider a special case of the resolution property where every coherent sheaf is generated by a single vector bundle in the following sense:

Definition 1.1. An algebraic stack \(X\) is said to have 1-resolution property if it admits a vector bundle \(V\) such that every coherent sheaf on \(X\) is a quotient of \(V^{\oplus n}\) for some \(n\). We will say that such a \(V\) is special.

We note at the outset that all our stacks will be quasi-separated. More precisely, they will have separated, quasi-compact diagonals (see [10, Def. 4.1]).

In [3], Hall and Rydh ask the following question about algebraic stack with 1-resolution property:

Question 1.2. (see [3, 7.4]) Does every algebraic stack with 1-resolution property admit a finite flat covering by a quasi-affine scheme? Moreover, is every algebraic space with 1-resolution property quasi-affine?

The purpose of this paper is to analyse a few situations where the answer to the above questions is indeed positive.

We begin by recalling (see [6, Tag 07R3]) that a locally noetherian scheme \(S\) is said to satisfy J-2 if for any locally finite type \(S\)-scheme \(X\), the regular locus of \(X\) is open in \(X\). All fields, \(\mathbb{Z}\), noetherian complete local rings, or schemes locally of finite type over these rings, give examples of J-2 schemes.

Theorem 1.3. Let \(X\) be a noetherian normal algebraic space which is finite type over an affine scheme \(S\), and suppose that \(S\) satisfies J-2. If \(X\) has 1-resolution property, then \(X\) is quasi-affine.

Using the fact that every normal algebraic space \(X\) is a quotient of a normal scheme \(Y\) by a finite group [10, 16.6.2], we quickly reduce the proof of Theorem 1.3 to the case where \(X\) is a scheme. The only reason why we need \(S\) to satisfy J-2 is that we need the regular locus of this scheme to be open. Next we prove the theorem, first in the case where \(X\) is a regular scheme and next in the case where \(X\) is a normal scheme whose singular locus is contained in an affine open. The general case is then deduced from this by using Zariski’s main theorem [7, IV,8.12.6].

Recall the following theorem of Totaro.

Theorem 1.4. [4, Prop. 1.3] Let \(\mathcal{X}\) be a noetherian algebraic stack whose stabilizer groups at closed points are affine. If \(\mathcal{X}\) has the resolution property, then the diagonal morphism \(\mathcal{X} \to \mathcal{X} \times_{\mathcal{X}} \mathcal{X}\) is affine.
Although we work entirely in the noetherian situation, we would like to point out that the above theorem works without the noetherian hypothesis (see [2]). Moreover, it is because of this theorem that all schemes and stacks considered in this paper will automatically have affine diagonal. One could interpret this theorem as saying that a stack with resolution property is very close to being separated. Indeed, to our knowledge there are no known examples of stacks with affine diagonal which do not have the resolution property. Although there are simple examples of schemes which admit resolution property but are not separated (see Example 2.5), our second main theorem shows that a stack with 1-resolution property is necessarily separated.

**Theorem 1.5.** Let $\mathcal{X}$ be an algebraic stack whose stabilizer groups at closed points are affine. Assume $\mathcal{X}$ is finite type over a noetherian affine scheme $S$. If $\mathcal{X}$ has 1-resolution property, then $\mathcal{X}$ is separated (i.e. $\mathcal{X}/\text{Spec} (\mathbb{Z})$ is separated).

The assumption that the stabilizers are affine is reasonable because as noted by Totaro, the resolution property for stacks with non-affine stabilizer groups is not interesting [4, Remark 1.1].

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2. Examples and Counterexamples

Before we present our main results we would like to mention a few examples and counterexamples of 1-resolution property.

**Example 2.1.** If $G$ is a finite group over $k$ then we claim that $BG$ has 1-resolution property. This follows from the fact that the map $\text{Spec} (k) \rightarrow BG$ is a finite, faithfully flat cover and from [2, Prop. 2.13].

**Example 2.2.** Consider the multiplicative group $\mathbb{G}_m$ over a field $k$. The classifying stack $BG_m$ does not have 1-resolution property.

Note that any representation $V$ of $\mathbb{G}_m$ decomposes as a direct sum, $V \cong \bigoplus_n \chi_n^{\otimes n}$, where $\chi_n$ are irreducible representations given by

$$\chi_n : \mathbb{G}_m \rightarrow \mathbb{G}_m$$

$$t \mapsto t^n$$

Assume (if possible) that $BG_m$ has 1-resolution property. Let $V$ be a special vector bundle on $BG_m$. This means that $V$ is a $G_m$-representation with the property that for any representation $W$ of $G_m$, one can find a surjective map,

$$(1) \quad V^{\otimes m} \rightarrow W$$

of $G_m$-representations. But since $V$ decomposes as a direct sum, $V \cong \bigoplus_n \chi_n^{\otimes n}$, for any $\chi_r$ not appearing in the decomposition of $V$, equation (1) fails. This contradicts our initial assumption.

**Example 2.3.** Let $G_a$ be the additive group over a field $k$. We will show that $BG_a$ does not have the 1-resolution property. The key idea is to identify an invariant $\eta(V) \in \mathbb{N}$, for every finite dimensional representation $V$ of $G_a$ such that

1. $\eta(V) = \eta(V^{\oplus r}) \forall r \in \mathbb{N}$.
2. For all surjections $V \rightarrow W$, $\eta(W) \leq \eta(V)$.
3. For all large enough integers $m$, there exists a $V$ with $\eta(V) = m$.

We claim that the very existence of an invariant $\eta$ as above shows that $BG_a$ cannot have 1-resolution property. Since otherwise, there would exist a vector bundle $V$ on $BG_a$ such that every vector bundle $W$ is a quotient of $V^{\oplus r}$ for some $r \in \mathbb{N}$. This would give us, by (1) and (2) above, that $\eta(W) \leq \eta(V)$ for all $W$, which contradicts (3).
Case 1: \( \text{char}(k) = 0 \). In this case, representations of \( \mathbb{G}_a \) are precisely pairs \((V, \rho)\) where \( \rho \) is a nilpotent endomorphism of \( V \) (see [5, Example 4.9]). The map \( \mathbb{G}_a \to GL(V) \) is given by \( t \mapsto \exp(\rho t) \). Thus any such representation \((V, \rho)\) admits a filtration by subrepresentations,

\[
V \supseteq \rho(V) \supseteq \rho^2(V) \cdots \supseteq \rho^n(V) = \{0\},
\]

with the property that the successive quotients are trivial representations. We denote by \( l(V) \) the smallest integer \( m \) such that \( \rho^m = 0 \). We claim that this invariant \( l \) satisfies the above three properties. Given two representations, \((V, \rho) \) and \((W, \xi)\), we see that \( l(V \oplus W) = \max\{\rho, \xi\} \). Thus, \( l \) satisfies conditions (1). That condition (2) is satisfied is straightforward. Moreover, by taking \( \rho \) to be the \( n \times n \) matrix with 1’s on the superdiagonal (and all other entries zero), it is easy to construct \( V \) such that \( l(V) = n \) for every \( n \in \mathbb{N} \). This shows condition (3).

Case 2: \( \text{char}(k) = p > 0 \). In this case, it is well known (see [5, Example 4.9]) that a \( \mathbb{G}_a \)-representation \( V \) given by a sequence \( \{s_i\}_{i \geq 0} \) of endomorphisms of \( V \) such that \( s_i \circ s_j = s_j \circ s_i \) and \( s_i^p = 0 \). The map \( \mathbb{G}_a \to GL(V) \) is then given by

\[
t \mapsto \prod_i \exp(s_it)
\]

where

\[
\exp(A) := 1 + A + \frac{A^2}{2!} \cdots + \frac{A^{(p-1)}}{(p-1)!}.
\]

Since these are finite dimensional representations, all but finitely many of the \( s_i \)'s are zero. We define our invariant to be the largest integer \( n \) such that \( s_n \) is non-zero, and denote it by \( \gamma(V) \). Clearly, \( \gamma(V \oplus W) = \max\{\rho, \xi\} \) showing that condition (1) is satisfied. To see condition (3) for any \( n \in \mathbb{N} \), take any non-zero nilpotent endomorphism \( s \) of \( V \), and consider the representation given by the sequence where \( s_i = s \text{ } \forall i \leq n \) and zero otherwise. Clearly \( \gamma(V) = n \). We only need to show that condition (2) is satisfied. This is due to the following lemma:

**Lemma 2.4.** Let \( V \) be a representation of \( \mathbb{G}_a \) and \( W \subseteq V \) a \( k \)-subspace. Then \( W \) is a subrepresentation if and only if \( s_i(W) \subseteq W \) for each \( i \).

**Proof.** Let \( \{s_i\}_{i \geq 0} \) be a representation of \( V \). Let \( W \subseteq V \) be a subrepresentation, and \( \{r_i\}_{i \geq 0} \) be the corresponding endomorphisms of \( W \). Then we have commutative diagram of comodule maps,

\[
\begin{array}{ccc}
W & \longrightarrow & W \otimes k[t] \\
\downarrow & & \downarrow \bigtriangleup \\
V & \longrightarrow & V \otimes k[t]
\end{array}
\]

Comparing coefficients of powers of \( t \), we see that \( s_i|_W = r_i \). So, \( s_i \)'s restrict to endomorphism of \( W \). \( \square \)

**Example 2.5.** A DVR with a double point does not have 1-resolution property. Let \( R \) be a discrete valuation ring with function field \( K \) and valuation \( \nu \). Let \( Y \) denote the scheme obtained by gluing two copies of \( \text{Spec}(R) \) along \( \text{Spec}(K) \). We will call this a **DVR with a double point**.

We will denote by \( V_1 \) and \( V_2 \) the two copies of \( \text{Spec}(R) \) in \( Y \), and let \( U := \text{Spec}(K) \) be the open point. Note that \( U = V_1 \cap V_2 \).

Although we will show that \( Y \) does not have 1-resolution property, it is easy to see that every coherent sheaf on \( Y \) is the quotient of some vector bundle, i.e, it has the resolution property. If \( \mathcal{F} \) is a coherent sheaf on \( Y \), then on each of the \( V_i \)'s, \( \mathcal{F}|_{V_i} := \mathcal{F}_i \) is just a finitely generated module over a PID. So, it decomposes into a free part and a torsion part. We have, \( \mathcal{F}_i \cong \mathcal{F}_{i,\text{free}} \oplus \mathcal{F}_{i,\text{tor}} \), where \( \mathcal{F}_{i,\text{free}} \) and \( \mathcal{F}_{i,\text{tor}} \) are finitely generated free and torsion modules over \( R \). Further, restricting \( \mathcal{F}_i \) to \( U \) gives us the transition map,

\[
A : \mathcal{F}_1|_U \to \mathcal{F}_2|_U
\]
where $A$ is a $K$-linear map. Since the torsion part of $F_i$ vanishes on restriction to $U$, the transition map $A$ is completely determined by the free part of $F_i$. Hence, to produce a surjection from a vector bundle to $F_i$, we take an $n$ large enough so that on each $V_i$, there is a surjection, $R^n \to F_{i,tor}$ onto the torsion part of $F_i$. This gives a map,

$$F_{i,free} \oplus R^n \to F_{i,free} \oplus F_{i,tor},$$

for each $i$. Glueing these maps on $U$ gives the required surjection. Thus, $Y$ has the resolution property.

To see that it does not have 1-resolution property, assume the contrary and let $E$ be a special vector bundle. Any vector bundle on $Y$ can described by a pair $(n, A)$, where $n$ is the rank and $A \in GL(n, K)$ is the gluing map. In coordinates, let $A = (a_{ij}) \in GL(n, K)$ be the gluing map $A : K^n \to K^n$ of $E$ on $U$. Let $\alpha := min\{\nu(a_{ij})\}$ be minimum of the valuations of the entries of $A$. Consider a line bundle $L_\lambda$ such that the gluing map for $L_\lambda$ is given by $\lambda \in K^\times$ with $\nu(\lambda) < \alpha$.

As $E$ is a special vector bundle, there exists a surjection $\phi : E^{\oplus m} \to L_\lambda$, for some $m$. Restricting $\phi$ on each $V_i$, gives us the maps (written in coordinates as) $\phi_i : R^N \to R$, with $N := nm$. Let $e_i$ be the chosen basis of $R^N$, and let $r_i := \phi_1(e_i)$ and $s_i := \phi_2(e_i)$. We may assume that $\nu(\phi_1(e_1)) = 0$. Otherwise, for any linear combination $\sum_{i=1}^n a_i e_i$, the valuation of its image along $\phi_i$ would be

$$\nu(\sum_{i=1}^n a_i \phi_1(e_i)) \geq min\{\nu(a_i e_i)\} > 0.$$

In this situation, the image would only generate an ideal of $R$, contradicting surjectivity.

Since $\phi$ is a homomorphism of sheaves, restricting $\phi_1$ and $\phi_2$ to $U$ gives the following commutative diagram,

$$\begin{array}{ccc}
K^N & \xrightarrow{\phi_1} & K \\
\downarrow{A^{\oplus m}} & & \downarrow{\lambda} \\
K^N & \xrightarrow{\phi_2} & K
\end{array}$$

Note that along these maps, the chosen basis $e_i$ of $R^N$ restricts to a basis of $K^N$. So, by the above diagram we must have,

$$\lambda \phi_1(e_1) = \phi_2(A^{\oplus m}(e_1)),$$

or in coordinates, $\lambda r_1 = \sum_{i=1}^n a_i e_i$. But as $\nu(s_i) \geq 0$, comparing valuations of the left and right hand-side tells us that,

$$\nu(\sum_{i=1}^n s_i a_{ij}) \geq min\{\nu(s_i a_{ij})\} \geq min\{\nu(a_{ij})\} > \nu(\lambda).$$

This contradicts commutativity of the above diagram.

**Example 2.6.** Let $X$ be (quasi-)projective over a field $k$ then $X$ has 1-resolution property if and only if $X$ is quasi-affine. If $V$ is a special vector bundle, then it follows from [8, II.5] that there exists an $n_0$ such that $\oplus_N O_X(n_0) \to V$ is surjective. Hence, for any coherent sheaf $F(n_0)$, there is a surjection $\oplus_M O_X(n_0) \to F(n_0)$. Untwisting by $O(-n_0)$, tells us that every coherent sheaf is globally generated, i.e, $X$ is quasi-affine.

3. 1-Resolution for Schemes

The goal of this section is to prove Theorem 1.3. We first note the following simple lemma.

**Lemma 3.1.** Let $X$ be a quasi-compact scheme with 1-resolution property. Let $E$ be a special vector bundle. Then $X$ is quasi-affine if and only if $E$ is globally generated.

**Proof.** A scheme is quasi-affine iff every coherent sheaf on it is globally generated (see [7, II, 5.1.2]). However since every coherent sheaf on $X$ is a quotient of $E^{\oplus r}$ for some $r \in \mathbb{N}$, it is clear that $X$ is quasi-affine if and only if $E$ is globally generated. \qed
We will first prove the Theorem 1.3 in the special case when $X$ is a regular scheme. Note that in this case we do not use the assumption that $X$ is of finite type over a J-2 scheme.

**Lemma 3.2.** Let $X$ be a noetherian regular scheme. If $X$ has 1-resolution property, then $X$ is quasi-affine.

**Proof.** Let $X = \bigcup_{i=1}^{n} U_i$ be an affine covering of $X$. We prove the theorem by induction on $n$. Then we may write

$$X = U \cup V$$

where $V := \bigcup_{i=1}^{n-1} U_i$ and $U := U_n$ is affine. Since $V$ is covered by $n - 1$ affines, it is quasi-affine by induction on $n$.

Let $Z_U = X \setminus U$ and $Z_V = X \setminus V$ denote the complements of $U$ and $V$, respectively. Since $U$ is affine, $Z_U$ supports an effective Cartier divisor $D_U$, [6, Tag 0BCU]. Let $D_V \subset Z_V$ be union of irreducible components of $Z_V$ of codimension 1. Then $D_V$ is an effective cartier divisor. Let $W := X \setminus D_V$. Note that $V$ is contained in $W$ and its complement (in $W$) has codimension at least 2.

**Step 1:** Let $\mathcal{E}$ be a special vector bundle on $X$. We claim that for all $m >> 0$ there exists an integer $r > 0$ and a surjection

$$\bigoplus_{i=1}^{r} (\mathcal{O}_X(-mD_U) \oplus \mathcal{O}_X(-mD_V)) \to \mathcal{E}.$$  

Since $U$ is affine, $\mathcal{E}|_U$ is globally generated. We can choose sections $s_1, \ldots, s_r \in \Gamma(U, \mathcal{E})$ which generate $\mathcal{E}$ on $U$. Similarly since $V$ is quasi-affine, we can choose sections $t_1, \ldots, t_r \in \Gamma(V, \mathcal{E})$ which generate $E|_V$. Without loss of generality, we may assume $r_1 = r_2 = r$. Since $V$ is regular and since complement of $V$ in $W$ has codimension at least 2, the restriction map

$$\Gamma(W, \mathcal{E}) \to \Gamma(V, \mathcal{E})$$

is bijective. Hence $t_i$'s extend uniquely to give sections (denoted again by $t_i$) of $\mathcal{E}$ on $W$.

By [8, II.5.14], there exists an integer $m_U$, such that for each $m > m_U$, the $s_i$'s lift to global sections of $\mathcal{E} \otimes \mathcal{O}_X(mD_U)$ thus giving a map

$$\bigoplus_{i=1}^{r} \mathcal{O}_X(-mD_U) \to \mathcal{E}$$

which is surjective on $U$.

Similarly, there exists an integer $m_V$, such that for every $m > m_V$ the $t_i$'s in $\Gamma(W, \mathcal{E})$ lift to global sections of $\mathcal{E} \otimes \mathcal{O}_X(mD_V)$, thereby giving a map

$$\bigoplus_{i=1}^{r} \mathcal{O}_X(-mD_V) \to \mathcal{E}$$

which is surjective on $V$. Taking direct sum of the above maps, proves the claim.

**Step 2:** In this step we will show that there exists a line bundle $\mathcal{L}$ such that both $\mathcal{L} \oplus \mathcal{O}_X$ and $\mathcal{L}^2 \oplus \mathcal{O}_X$ are special vector bundles. By the above step, we may find an integer $m$ such that we have surjections

$$\bigoplus_{i=1}^{r} (\mathcal{O}_X(-mD_U) \oplus \mathcal{O}_X(-mD_V)) \to \mathcal{E},$$

$$\bigoplus_{i=1}^{r} (\mathcal{O}_X(-2mD_U) \oplus \mathcal{O}_X(-2mD_V)) \to \mathcal{E}.$$  

We claim that $\mathcal{L} = \mathcal{O}_X(m(D_V - D_U))$ satisfies our requirement. Tensoring the first map by $\mathcal{O}_X(mD_V)$ we get a surjective map

$$\bigoplus_{i=1}^{r} (\mathcal{L} \oplus \mathcal{O}_X) \to \mathcal{E} \otimes \mathcal{O}_X(mD_V).$$

It is clear that tensor product of a special vector bundle with any line bundle is also a special vector bundle. Thus $\mathcal{E} \otimes \mathcal{O}_X(mD_V)$ is special which tells us that so is $\mathcal{L} \oplus \mathcal{O}_X$. 


Similarly tensoring the second surjection above by \( \mathcal{O}_X(2mDV) \) we get a surjection
\[
\oplus_{i=1}^n (\mathcal{L}^2 \oplus \mathcal{O}_X) \rightarrow \mathcal{E} \otimes \mathcal{O}_X(2mDV),
\]
thus showing \( \mathcal{L}^2 \oplus \mathcal{O}_X \) is also special.

**Step 3:** We now claim that \( \mathcal{L}^2 \oplus \mathcal{O}_X \) is globally generated. This will finish the proof of the theorem, thanks to Lemma 3.1. Since \( \mathcal{L} \oplus \mathcal{O}_X \) is special, we have a surjection
\[
\Phi : \oplus_{i=1}^n (\mathcal{L} \oplus \mathcal{O}_X) \twoheadrightarrow \mathcal{L}^2.
\]

Hence, assume, if possible, that \( p \in X \) is a point in the base locus of \( \mathcal{L}^2 \). Since the base locus of \( \mathcal{L} \) contains that of \( \mathcal{L}^2 \), \( p \) is also in the base locus of \( \mathcal{L} \). The above surjection has as its summands, the maps,
\[
\mathcal{L} \xrightarrow{\phi_i} \mathcal{L}^2, \quad \mathcal{O}_X \xrightarrow{\psi_i} \mathcal{L}^2.
\]

As \( p \) is in the base locus, for all \( i, \psi_{i,p} : \mathcal{O}_{X,p} \to \mathcal{L}^2 \) is the zero map.

Thus, we have an \( i_0 \) such that \( \phi_{i_0,p} : \mathcal{O}_{\mathcal{L},mDV} \to \mathcal{L}^2 \) is an isomorphism. Untwisting the map \( \phi_{i_0} \) by \( \mathcal{L} \), gives us a section of \( \mathcal{L} \) which does not vanish at \( p \), contradicting the initial assumption.

\[\square\]

**Lemma 3.3.** Let \( X \) be finite type over a noetherian affine scheme \( S \). If \( X \) has 1-resolution property, then \( X \) is separated.

**Proof.** Note that as \( S \) is affine, if \( X/S \) is separated, then \( X \) is separated (over \( \mathbb{Z} \)). If \( X \) is not separated then by valuative criteria, there exists a discrete valuation ring \( R \), and two maps \( f_1, f_2 : \text{Spec} (R) \rightarrow X \), lifting the diagram:
\[
\begin{array}{ccc}
\text{Spec} (K) & \xrightarrow{x} & X \\
\downarrow & & \downarrow \\
\text{Spec} (R) & \xrightarrow{f_1, f_2} & S
\end{array}
\]

where \( \text{Spec} (K) \) is the generic point of \( \text{Spec} (R) \).

Let \( Y \) be the scheme obtained by gluing two copies of \( \text{Spec} (R) \) along \( \text{Spec} (K) \). We will call this the DVR with a double point. Note that \( Y \) is regular. Consider the map \( f : Y \rightarrow X \), which restricts to \( f_1 \) and \( f_2 \) on each of the copies of \( \text{Spec} (R) \), respectively. Thus, \( f \) is quasi-affine. By [2, Prop. 2.8 (v)], \( Y \) has 1-resolution property. This leads to contradiction by Lemma 3.2 (or by Example 2.5), since \( Y \) is not separated.

In the sequel, we will repeatedly rely on [2, Prop. 2.8 (v)], and so we state it here for convenience.

**Theorem 3.4.** [2, Prop. 2.8 (v)] Let \( X \rightarrow Y \) be a quasi-affine morphism of algebraic stacks. If \( Y \) has the 1-resolution property then so does \( X \).

**Lemma 3.5.** Let \( X \) be a noetherian normal scheme over an affine base scheme \( S \). Suppose that \( X/S \) is finite type, \( S \) satisfies J-2, and the singular locus \( B \) of \( X \) is contained in an affine open \( U \). Then, if \( X \) has 1-resolution property, it is quasi-affine.

**Proof.** Let \( X_{\text{reg}} \) denote the regular locus of \( X \). Since the singular locus \( B \) is contained in the affine open \( U \), we have \( X = X_{\text{reg}} \cup U \). \( X_{\text{reg}} \) is quasi-affine by Lemma 3.2. Let \( Z_U := X \setminus U \). As \( U \) is affine, \( Z_U \) is of pure codimension 1 [6, Tag 0BCU]. Moreover, \( Z_U \) lies entirely in \( X_{\text{reg}} \), and hence supports an effective cartier divisor \( D_U \).

Let \( \mathcal{E} \) be a special vector bundle on \( X \). Then, as in Step 1 of Lemma 3.2, since \( U \) is affine, we can choose sections \( s_1, \ldots, s_{r_1} \in \Gamma(U, \mathcal{E}) \) which generate \( \mathcal{E} \) on \( U \). Similarly, as \( X_{\text{reg}} \) is quasi-affine, we can choose sections \( t_1, \ldots, t_{r_2} \in \Gamma(X_{\text{reg}}, \mathcal{E}) \) which generate \( \mathcal{E} \) on \( X_{\text{reg}} \). We may assume, without loss
of generality, that \( r_1 = r_2 = r \). By, normality of \( X \), the restriction map \( \Gamma(X, \mathcal{E}) \to \Gamma(X_{\text{reg}}, \mathcal{E}) \) is a bijection. Thus the sections \( t_1, ..., t_r \) extend uniquely to sections over whole of \( X \) giving us a map
\[
\bigoplus_{i=1}^r \mathcal{O}_X \to \mathcal{E}
\]
which is surjective on \( X_{\text{reg}} \).

By [8, II.5.14], there exists an integer \( m_U \), such that for each \( m > m_U \), the \( s_i \)'s lift to global sections of \( \mathcal{E} \otimes \mathcal{O}_X(mD_U) \) thus giving a map
\[
\bigoplus_{i=1}^r \mathcal{O}_X(-mD_U) \to \mathcal{E}
\]
which is surjective on \( U \). Thus, as in Step 1 of the proof of Lemma 3.2, we get a surjective map
\[
\bigoplus_{i=1}^r \left( \mathcal{O}_X(-mD_U) \oplus \mathcal{O}_X \right) \to \mathcal{E}.
\]
The remaining argument is very similar to that of Lemma 3.2 and hence we skip details. Choose an \( m_0 > m_U \), and let \( \mathcal{L} := \mathcal{O}_X(-m_0D_U) \). Then, as in Step 2 of Lemma 3.2, the above equations tell us that both \( \mathcal{L} \oplus \mathcal{O}_X \) and \( \mathcal{L}^2 \oplus \mathcal{O}_X \) are special vector bundles. As in Step 3 of Lemma 3.2, this proves that \( \mathcal{L}^2 \) is globally generated and hence proves the result.

\[ \square \]

**Theorem 3.6.** Let \( X \) be a noetherian normal scheme over an affine base scheme \( S \). Suppose that \( X/S \) is finite type, and that \( S \) satisfies J-2. If \( X \) has 1-resolution property, then \( X \) is quasi-affine.

**Proof.** We will show that the canonical map
\[
\gamma : X \to \text{Spec} \left( \Gamma(X, \mathcal{O}_X) \right),
\]
is quasi-finite and separated. This will finish the proof as by Zariski’s Main Theorem [7, IV,8.12.6], \( \gamma \) will be quasi-affine, thus showing that \( X \) itself is quasi-affine.

That \( \gamma \) is separated is clear from Lemma 3.3.

To show that \( \gamma \) is quasi-finite, we let \( X_{\text{reg}} \) and \( B \) denote the regular and singular loci of \( X \), respectively. Let \( B \subset \bigcup_{i=1}^n U_i \) be a covering, with \( U_i \) affine in \( X \), and \( Y_i := X_{\text{reg}} \cup U_i \). Then, each \( Y_i \) satisfies the hypothesis of the previous lemma, and hence is quasi-affine. Thus, the canonical map
\[
\gamma : X \to \text{Spec} \left( \Gamma(X, \mathcal{O}_X) \right)
\]
is an open immersion on each \( Y_i \). As \( X \) is of finite type, \( \gamma \) is quasi-finite.

\[ \square \]

**Proof of 1.3.** Any normal algebraic space over \( S \) is the quotient of a scheme by a finite group [10, 16.6.2]. So, we have a finite map, \( U \to X \) with \( U \) normal and finite type. By Theorem 3.4, \( U \) has 1-resolution property. Hence, it is quasi-affine by Theorem 3.6, which implies that so is \( X \).

\[ \square \]

4. 1-Resolution for Stacks

The goal of this section is to prove Theorem 1.5. By Lemma 3.3 we already know that any noetherian scheme \( X \) which satisfies 1-resolution property is separated.

Recall the following definition of a finite type point of a stack.

**Definition 4.1.** (see [6, Tag 06FY]) We say that \( \xi \) is a point of finite type if there exists a representative \( x : \text{Spec}(k) \to \mathcal{X} \) such that \( x \) is a morphism locally of finite type. For a stack \( \mathcal{X} \), we let \( \mathcal{X}_{\text{ft-pts}} \) denote the subset of points of finite type of \( \mathcal{X} \).

The significance of the notion of finite type points in this paper comes from the following two results.

**Lemma 4.2.** [6, Tag 06G2] Let \( \mathcal{X} \) be an algebraic stack. For any locally closed subset \( T \subset |\mathcal{X}| \) we have
\[
T \neq \emptyset \Rightarrow T \cap \mathcal{X}_{\text{ft-pts}} \neq \emptyset.
\]

By [9, Def. 3.34], this is the statement that \( \mathcal{X}_{\text{ft-pts}} \) is very dense in \( |\mathcal{X}| \).
Lemma 4.3. Let $X$ be any noetherian stack and $\xi \in |X|$ be a point of finite type. Let $i : G_\xi \hookrightarrow X$ be the inclusion of the residual gerbe at $\xi$. Then $i$ is quasi-affine.

Proof. By [6, Tag 06G3], the morphism $i : G_\xi \hookrightarrow X$ is locally of finite type. As $|G_\xi|$ is a singleton and $X$ is quasi-separated, $i$ is also quasi-compact. If $U \to X$ is any morphism with $U$ affine, then the base change $G_\xi \times_X U \to U$ is a finite type monomorphism. Thus it is quasi-finite and separated, and therefore, by Zariski’s Main Theorem (see [7, IV,8.12.6]), $G_\xi \times_X U$ is a quasi-affine scheme. Hence $i : G_\xi \hookrightarrow X$ is quasi-affine. \qed

The following proposition can be considered as a special case of Theorem 1.5.

Proposition 4.4. Let $G$ be an affine algebraic group over $k$. Then, $BG$ has 1-resolution property if and only if $\dim(G) = 0$.

Proof. ($\Leftarrow$) If $\dim(G) = 0$, the map $\text{Spec}(k) \to BG$ is finite and faithfully flat. Thus, the 1-resolution property for $BG$ follows from [2, Prop. 2.13].

($\Rightarrow$) Assume $BG$ has 1-resolution property. Let $T \subset G$ be a maximal torus. The following diagram is cartesian,

\[
\begin{array}{ccc}
G/T & \longrightarrow & \text{Spec}(k) \\
\downarrow & & \downarrow \\
BT & \longrightarrow & BG
\end{array}
\]

As the scheme $G/T$ is affine, it follows from fppf descent that the morphism $BT \to BG$ is affine. Hence by Theorem 3.4, $BT$ has 1-resolution property. However this is impossible if $T$ is a positive dimensional torus (see Example 2.2). Thus $G$ must be unipotent. Assume $\dim(G) > 0$, if possible. By using the derived series of $G$, one can find a closed subgroup of $G$ isomorphic to $G_a$. The inclusion $G_a \hookrightarrow G$ gives rise to a quasi-affine morphism $BG_a \to BG$. Again, by Theorem 3.4, we deduce that $BG_a$ has 1-resolution property, which is impossible (see Example 2.3). Thus $\dim(G)$ must be 0. \qed

Theorem 4.5. Let $X$ be a noetherian algebraic stack whose stabilizers at closed points are affine. If $X$ has the 1-resolution property, then the inertia stack $f : I_X \to X$ is quasi-finite. Equivalently, the diagonal $\Delta : X \to X \times X$ is quasi-finite.

Proof. The inertia stack is of finite presentation. This is because the diagonal $\Delta$ is a finite type morphism (see [10, 4.2]) and $X$ is noetherian. Thus to show $f$ is quasi-finite, it is enough to show that it has finite fibres. Moreover, since the set of points of $X$ over which $f$ has finite fibres is a constructible set by [9, Prop. 10.96], it is enough to show that $f$ has finite fibres over a very dense set, or in particular over points of finite type (see [9, Remark 10.15] and Lemma 4.2). Let $\xi : \text{Spec}(k) \to X$ be a finite type point. We need to show that the stabilizer group at $\xi$ is finite. Let $i : G_\xi \hookrightarrow X$ be the residual gerbe at $\xi$. By Lemma 4.3, $i$ is quasi-affine and hence by Theorem 3.4, $G_\xi$ has 1-resolution property. Now, finiteness of the stabilizer group at $\xi$ follows directly from Proposition 4.4. \qed

Lemma 4.6. Let $X$ be a noetherian algebraic stack and let $a : X \to X$ be a finite surjective map from a scheme $X$. Then $X$ is separated implies so is $X$.

Proof. Assume, if possible that $X$ is not separated. Then by valuative criteria there exists a valuation ring $R$, with two distinct maps $f_1, f_2 : \text{Spec}(R) \to X$ making the following diagram commute:

\[
\begin{array}{ccc}
\text{Spec}(K) & \xrightarrow{\pi} & X \\
\downarrow & & \downarrow \\
\text{Spec}(R) & \xrightarrow{f_1, f_2} & X
\end{array}
\]
Here, $K \supseteq R$ is the field of fractions of $R$. Without loss of generality, by going to a finite extension of $K$, we may assume that the map $x : \text{Spec}(K) \to \mathcal{X}$ lifts to give a map $\tilde{x} : \text{Spec}(K) \to X$. Since the map $a : X \to \mathcal{X}$ is finite and hence is proper and representable, the maps $f_1, f_2$ also lift uniquely to give a commutative diagram

$$
\begin{array}{ccc}
\text{Spec}(K) & \xrightarrow{\tilde{x}} & X \\
\downarrow & \searrow & \downarrow f_1 \\
\text{Spec}(R) & \xrightarrow{f_2} & \mathcal{X}
\end{array}
$$

However this contradicts the separatedness of $X$. Hence $\mathcal{X}$ must be separated.

Finally, we prove Theorem 1.5.

**Proof of 1.5.** Since the diagonal $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is quasi-finite by Theorem 4.5, $\mathcal{X}$ admits a finite surjective $a : X \to \mathcal{X}$ where $X$ is a scheme (see [1, Theorem 2.7]). Then, by Theorem 3.4, $X$ also has 1-resolution property and hence is separated by Lemma 3.3. This shows that $\mathcal{X}$ is separated by Lemma 4.6.

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