ON THE PROP CORRESPONDING TO BIALGEBRAS

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Résumé. Un PROP $A$ est une catégorie monoïdale symétrique stricte avec la propriété suivante (cf. [3]): les objets de $A$ sont les nombres naturels et l’opération monoïdale est l’addition sur les objets. Une algèbre sur $A$ est un foncteur monoïdal strict de $A$ vers la catégorie tensorielle $\text{Vect}$ des espaces vectoriels sur un corps commutatif $k$. On construit le PROP $QF(\text{as})$ et on montre que les algèbres sur $QF(\text{as})$ sont exactement les bigèbres.

1 Introduction

A PROP is a permutative category $(A, \boxtimes)$, whose set of objects is the set of natural numbers and on objects the monoidal structure is given by the addition. An $A$-algebra is a symmetric strict monoidal functor to the tensor category of vector spaces.

It is well-known that there exists a PROP whose category of algebras is equivalent to the category of bialgebras (= associative and coassociative bialgebras). In [14] there is a description of this PROP in terms of generators and relations. Here we give a more explicit construction of the same object. Our construction uses the Quillen’s $Q$-construction for double categories given in [7].

The paper is organized as follows: In Section 2 we recall the definition of PROP and show how to obtain commutative algebras as $F$-algebras. Here $F$ is the PROP of finite sets. In the next section following to [7] we construct the PROP of noncommutative sets denoted by $F(\text{as})$ and we show that $F(\text{as})$-algebras are exactly associative algebras. The material of the Sections 2 and 3 are well known to experts. In Section 4 we generalize the notion of Mackey functor for double categories and in Section 5 we describe our hero $QF(\text{as})$, which is the PROP, with the property that $QF(\text{as})$-algebras are exactly bialgebras. By definition of PROP the category $QF(\text{as})$ encodes the natural transformations $H^\otimes n \to H^\otimes m$ and relations between them. Here $H$ runs over all bialgebras. As a sample we give the following application. For any bialgebra $H$, any natural number $n \in \mathbb{N}$ and any permutation $\sigma \in \mathfrak{S}_n$, we let

$$\Psi(n,\sigma) : H \to H$$
be the composition $\mu^n \circ \sigma \circ \Delta : H \to H$, where $\Delta : H \to H \otimes n$ is the $(n-1)$-th iteration of the comultiplication $\Delta : H \to H \otimes H$, $\sigma : H \otimes n \to H \otimes n$ is induced by the permutation $\sigma$, that is

$$\sigma(x_1 \otimes \cdots \otimes x_n) = x_{\sigma 1} \otimes \cdots \otimes x_{\sigma n}$$

and $\mu^n : H \otimes n \to H$ is the $(n-1)$-th iteration of the multiplication $\mu : H \otimes H \to H$. Moreover let $\Phi : S_n \times S_m \to S_{nm}$ be the map constructed in Proposition 5.3. Then it is a consequence of our discussion in Section 5, that for any permutations $\sigma \in S_n$ and $\tau \in S_m$ one has the equality

$$\Psi^{(n,\sigma)} \circ \Psi^{(m,\tau)} = \Psi^{(nm,\Phi(\sigma,\tau))}.$$ 

Let us note that if $\sigma$ is the identity, then $\Psi^{(n,id)}$ is nothing but the Adams operation \[11\] and hence our formula gives the rule for the composition of Adams operations.

### 2 Preliminaries on PROP’s

Recall that a symmetric monoidal category is a category $S$ with a unit $0 \in S$ and a bifunctor

$$\square : S \times S \to S$$

together with natural isomorphisms

$$a_{X,Y,Z} : X\square(Y\square Z) \to (X\square Y)\square Z,$$

$$l_X : X\square 0 \to X, r_X : 0\square X \to X, e_{X,Y} : X\square Y \to Y\square X$$

satisfying some coherent conditions (see \[8\]). If in addition $a_{X,Y,Z}, l_X, r_X$ are identity morphism then, $S$ is called a permutative category. If $S$ and $S_1$ are symmetric monoidal categories, then a functor $M : S \to S_1$ is a symmetric monoidal functor if there exist isomorphisms

$$u_{X,Y} : M(X)\square M(Y) \to M(X\square Y)$$

satisfying the usual associativity and unit coherence conditions (see for example \[8\]). A symmetric monoidal functor is called strict if $u_{X,Y}$ is identity for all $X, Y \in S$. According to \[13\] a PROP is a permutative category $(A, \square)$, with the following property: $A$ has a set of objects equal to the set of natural
numbers and on objects the bifunctor $\Box$ is given by $m \Box n = m + n$. An $A$-algebra is a symmetric strict monoidal functor from $A$ to the tensor category $\text{Vect}$ of vector spaces over a field $k$.

**Examples.** 1) Let $\mathcal{F}$ be the category of finite sets. For any $n \geq 0$, we let $\underline{n}$ be the set $\{1, \ldots, n\}$. Hence $\underline{0}$ is the empty set. We assume that the objects of $\mathcal{F}$ are the sets $\underline{n}$, $n \geq 0$. The disjoint union makes the category $\mathcal{F}$ a PROP. It is well-known that the category of algebras over $\mathcal{F}$ is equivalent to the category of commutative and associative algebras with unit. Indeed, if $A$ is a commutative algebra, then the functor $L^*(A) : \mathcal{F} \to \text{Vect}$ is a $\mathcal{F}$-algebra. Here the functor $L^*(A)$ is given by

$$L^*(A)(\underline{n}) = A^\otimes n.$$ 

For any map $f : \underline{n} \to \underline{m}$, the action of $f$ on $L^*(A)$ is given by

$$f_*(a_1 \otimes \cdots \otimes a_n) = b_1 \otimes \cdots \otimes b_m,$$

where

$$b_j = \prod_{f(i) = j} a_i, \quad j = 1, \ldots, n.$$ 

Conversely, assume $T$ is a $\mathcal{F}$-algebra. We let $A$ be the value of $T$ on $\underline{1}$. The unique map $\underline{2} \to \underline{1}$ yields a homomorphism

$$\mu : A \otimes A \simeq T(2) \to T(1) = A.$$ 

On the other hand the unique map $\underline{0} \to \underline{1}$ yields a homomorphism $\eta : k = T(\underline{0}) \to T(\underline{1}) = A$. The pair $(\mu, \eta)$ defines on $A$ a structure of commutative and associative algebra with unit. One can use the fact that $T$ is a symmetric strict monoidal functor to prove that $T \simeq L^*(A)$.

2) Let us note that the opposite of a PROP is still a PROP with the same $\Box$. Hence the disjoint union yields also a structure of PROP on $\mathcal{F}^{op}$. The category of $\mathcal{F}^{op}$-algebras is equivalent to the category of cocommutative and coassociative coalgebras with counit. For any such coalgebra $C$ we let $L^*(C) : \mathcal{F}^{op} \to \text{Vect}$ be the corresponding $\mathcal{F}^{op}$-algebra. On objects we still have $L^*(C)(\underline{n}) = C^\otimes n$.

3) We let $\Omega$ be the subcategory of $\mathcal{F}$, which has the same objects as $\mathcal{F}$, but morphisms are surjections. Clearly $\Omega$ is a subPROP of $\mathcal{F}$ and $\Omega$-algebras are (nonunital) commutative algebras.
4) We let \( \textbf{Mon} \) be the category of finitely generated free monoids, which is a PROP with respect to coproduct. Similarly the category \( \textbf{Abmon} \) of finitely generated free abelian monoids, the category \( \textbf{Ab} \) of finitely generated free abelian groups and the category \( \textbf{Gr} \) of finitely generated free groups are PROP’s with respect to coproducts. For the category of algebras over these PROP’s see Theorem 5.2 and Remark 1 at the end of the paper.

5) Any algebraic theory in the sense of Lawvere \cite{2} gives rise to a PROP. This generalizes the examples from 1) and 4).

In the next section we give a noncommutative generalization of Examples 1)-3).

3 Preliminaries on noncommutative sets

In this section following to \cite{1} we introduce the PROP \( \mathcal{F}(\text{as}) \) with property that \( \mathcal{F}(\text{as}) \)-algebras are associative algebras with unit. As a category \( \mathcal{F}(\text{as}) \) is described in \cite{1}, p.191 under the name “symmetric category”. It is also isomorphic to the category \( \Delta S \) considered in \cite{10}, \cite{7}. Objects of \( \mathcal{F}(\text{as}) \) are finite sets. So \( \text{Ob}(\mathcal{F}) = \text{Ob}(\mathcal{F}(\text{as})) \). A morphism from \( n \) to \( m \) is a map \( f : n \rightarrow m \) together with a total ordering on \( f^{-1}(j) \) for all \( j \in m \). By abuse of notation we will denote morphisms in \( \mathcal{F}(\text{as}) \) by \( f, g \) etc. Moreover sometimes we write \( |f| \) for the underlying map of \( f \in \mathcal{F}(\text{as}) \). We will also say that \( f \) is a noncommutative lifting of a map \( |f| \). In order to define the composition in \( \mathcal{F}(\text{as}) \) we recall the definition of ordered union of ordered sets. Assume \( \Lambda \) is a totally ordered set and for each \( \lambda \in \Lambda \) a totally ordered set \( X_\lambda \) is given. Then \( X = \bigsqcup X_\lambda \) is the disjoint union of the sets \( X_\lambda \) which is ordered as follows. If \( x \in X_\lambda \) and \( y \in X_\mu \), then \( x \leq y \) in \( X \) iff \( \lambda < \mu \) or \( \lambda = \mu \) and \( x \leq y \) in \( X_\lambda \).

If \( f \in \text{Hom}_{\mathcal{F}(\text{as})}(n, m) \) and \( g \in \text{Hom}_{\mathcal{F}(\text{as})}(m, k) \), then the composite \( gf \) is \( |g| \circ |f| \) as a map, while the total ordering in \( (gf)^{-1}(i) \), \( i \in k \) is given by the identification

\[
(gf)^{-1}(i) = \bigsqcup_{j \in g^{-1}(i)} f^{-1}(j).
\]

Clearly one has the forgetful functor \( \mathcal{F}(\text{as}) \rightarrow \mathcal{F} \). A morphism \( f \) in \( \mathcal{F}(\text{as}) \) is called a surjection if the map \( |f| \) is a surjection. An elementary surjection is a surjection \( f : n \rightarrow m \) for which \( n - m \leq 1 \).
Since any injective map has the unique noncommutative lifting, we see that the disjoint union, which defines the symmetric monoidal category structure in \( \mathcal{F} \) has the unique lifting in \( \mathcal{F}(\text{as}) \). Hence \( \mathcal{F}(\text{as}) \) is a PROP.

We claim that the category \( \mathcal{F}(\text{as}) \)-algebras is equivalent to the category of associative algebras with unit. The only point here is the following. Let us denote by \( \prod_{i \in I} x_i \) the product of the elements \( x_i \in A \) where \( I \) is a finite totally ordered set and the ordering in the product follows to the ordering \( I \). Here \( A \) is an associative algebra. Now we have a \( \mathcal{F}(\text{as}) \)-algebra \( X^*(A) : \mathcal{F}(\text{as}) \to \text{Vect} \). Here the functor \( X^*(A) \) is given by the same rule as \( \mathcal{L}^*_s(A) \) in the previous section, but to take \( \prod^< \) in the definition of \( b_j \). For example, if \( f : 4 \to 3 \) is given by \( f(1) = f(2) = f(4) = 3, \ f(3) = 1 \) and the total ordering in \( f^{-1}(3) \) is \( 2 < 4 < 1 \) then \( f^* : A^{\otimes 4} \to A^{\otimes 3} \) is nothing but \( a_1 \otimes a_2 \otimes a_3 \otimes a_4 \mapsto a_3 \otimes 1 \otimes a_2 a_4 a_1 \).

We let \( \Omega(\text{as}) \) be the subcategory of \( \mathcal{F}(\text{as}) \), which has the same objects as \( \mathcal{F}(\text{as}) \), but morphisms are surjections. Clearly \( \Omega(\text{as}) \) is a subPROP of \( \mathcal{F}(\text{as}) \) and \( \Omega(\text{as}) \)-algebras are (nonunital) associative algebras.

Quite similarly, for any coassociative coalgebra \( C \) with counit one has a \( \mathcal{F}(\text{as})^{\text{op}} \)-algebra \( X^{*}(C) : \mathcal{F}(\text{as})^{\text{op}} \to \text{Vect} \) with \( X^{*}(C)(\mathfrak{U}) = C^{\otimes n} \) and the category \( \mathcal{F}(\text{as})^{\text{op}} \)-algebras is equivalent to the category of coassociative coalgebras with counit.

In order to put bialgebras in the picture we need the language of Mackey functors.

4 On double categories and Mackey functors

Let us recall that a double category consists of objects, a set of horizontal morphisms, a set of vertical morphisms and a set of bimorphisms satisfying natural conditions [4] (see also [7]). If \( D \) is a double category, we let \( D^h \) (resp. \( D^v \)) be the category of objects and horizontal (resp. vertical) morphisms of \( D \).

A Janus functor \( M \) from a double category \( D \) to \( \text{ Vect } \) is the following data

i) a covariant functor \( M_* : D^h \to \text{ Vect } \)

ii) a contravariant functor \( M_* : (D^v)^{\text{op}} \to \text{ Vect } \)

such that for each object \( S \in D \) one has \( M_*(S) = M^*(S) = M(S) \). A Mackey functor \( M = (M_*, M^*) \) from a double category \( D \) to \( \text{ Vect } \) is a Janus functor.
$M$ from a double category $D$ to $\text{Vect}$ such that for each bimorphism in $D$

\[
\begin{array}{ccc}
U & f_1 & S \\
\alpha = & \downarrow & \phi_1 & \phi & \downarrow \\
T & f & V
\end{array}
\]

the following equality holds:

\[M^*(\phi)M_*(f) = M_*(f_1)M^*(\phi_1).\]

**Examples**

1) Let $C$ be a category with pullbacks. Then one has a double category whose objects are the same as $C$. Moreover $\text{Mor}^v = \text{Mor}^h = \text{Mor}(C)$, while bimorphisms are pullback diagrams in $C$. In this case the notion of Mackey functors corresponds to pre-Mackey functors from [5]. By abuse of notation we will still denote this double category by $C$. In what follows $\mathcal{F}$ is equipped with this double category structure.

2) Now we consider a double category, whose objects are still finite sets, but $\text{Mor}^v = \text{Mor}^h = \text{Mor}(\mathcal{F}(\text{as}))$, where $\mathcal{F}(\text{as})$ was introduced in Section 3. By definition a bimorphism is a diagram in $\mathcal{F}(\text{as})$

\[
\begin{array}{ccc}
U & f_1 & S \\
\alpha = & \downarrow & \phi_1 & \phi & \downarrow \\
T & f & V
\end{array}
\]

such that the following holds:

i) the image $|\alpha|$ of $\alpha$ in $\mathcal{F}$ is a pullback diagram of sets,

ii) for all $x \in T$ the induced map $f_* : \phi_1^{-1}(x) \to \phi^{-1}(fx)$ is an isomorphism of ordered sets

iii) for all $y \in S$ the induced map $\phi_* : f_1^{-1}(y) \to f^{-1}(\phi_1y)$ is an isomorphism of ordered sets.

Let us note that for a bimorphism $\alpha$ in $\mathcal{F}(\text{as})$ in general $f \circ \phi_1 \neq \phi \circ f_1$. By abuse of notation we will denote this double category by $\mathcal{F}(\text{as})$. It is different from a double category considered in [6], which is also associated to the category $\mathcal{F}(\text{as})$.

One observes that for any arrows $f : T \to V$, $\phi : S \to V$ in $\mathcal{F}(\text{as})$ there exists a bimorphism $\alpha$ which has $f$ and $\phi$ as edges and it is unique up to
natural isomorphism. Indeed, as a set we take $U$ to be the pullback and then we lift set maps $f_1$ and $\phi_1$ to the noncommutative world according to the properties ii) and iii). Clearly such lifting exists and it is unique.

3) We can also consider the double category $F(as)_1$ whose objects are still finite sets, vertical arrows are set maps, while horizontal ones are morphisms from $F(as)$. The bimorphisms are diagrams similar to the diagrams in Example 2) but such that $\phi$ and $\phi_1$ are set maps, while $f$ and $f_1$ are morphisms from $F(as)$. Furthermore the conditions i) and iii) from the previous example hold. We need also a double category $F(as)_2$ which is defined similarly, but now vertical arrows are morphisms from $F(as)$ and horizontal ones are set maps.

We have the following diagram of double categories, where arrows are forgetful functors

$$
\begin{array}{ccc}
F(as)_1 & \rightarrow & F(as)_2 \\
\uparrow & & \downarrow \\
F(as) & \rightarrow & F \\
\downarrow & & \uparrow \\
F(as)_2 & \rightarrow & 
\end{array}
$$

Let $D$ be one of the double categories considered in (4.0). A bimorphism $\alpha$ is called elementary if both $f$ and $\phi$ are elementary surjections. The following Lemma for $D = F$ was proved in [1]. The proof in other cases is quite similar and hence we omit it.

**Lemma 4.1** Let $D$ be one of the double categories considered in (4.0). Then a Janus functor $M$ is a Mackey functor iff the following two conditions hold

i) for any injection $g : A \rightarrow B$ one has $M^*(g)M_*(g) = \text{id}_A$

ii) for any elementary bimorphism $\alpha$ one has

$$M^*(\phi)M_*(f) = M_*(f_1)M^*(\phi_1).$$

**Theorem 4.2** Let $V$ be a vector space, which is equipped simultaneously with the structure of associative algebra with unit and coassociative coalgebra with counit. Then $V$ is a bialgebra iff

$$\mathcal{X}(V) = (\mathcal{X}_*(V), \mathcal{X}^*(V)) : F(as) \rightarrow \text{Vect}$$
is a Mackey functor.

**Proof.** One observes that the condition 1) of the previous lemma always holds. On the other hand the diagram

\[
\begin{array}{ccc}
4 & \xrightarrow{p} & 2 \\
\downarrow \quad q & \quad f & \downarrow \\
2 & \xrightarrow{f} & 1
\end{array}
\]

is a bimorphism. Here \( f^{-1}(1) = \{1 < 2\}, p^{-1}(1) = \{1 < 2\}, p^{-1}(2) = \{3 < 4\}, q^{-1}(1) = \{1 < 3\} \) and \( q^{-1}(2) = \{2 < 4\} \). Clearly \( f^*: V^\otimes 2 \to V \) is the multiplication \( \mu \) on \( V \) and \( f^*: V \to V^\otimes 2 \) is the comultiplication \( \Delta \) on \( V \), while \( p_* = (\mu \otimes \mu) \circ \tau_{2,3} \) and \( q^* = \tau_{2,3} \circ \Delta \otimes \Delta \), where \( \tau_{2,3}: V^\otimes 4 \to V^\otimes 4 \) permutes the second and the third coordinates. Hence \( V \) is a bialgebra iff the condition ii) of the previous lemma holds for \( \alpha \). Since both \( \mathcal{X}(V) \) and \( \mathcal{X}^*(V) \) send disjoint union to tensor product the result follows from Lemma 4.1.

**Addendum.** For a cocommutative bialgebra \( C \) the Mackey functor \( \mathcal{X}(C) \) factors through the double category \( \mathcal{F}(as)_1 \), for a commutative bialgebra \( A \) the Mackey functor \( \mathcal{X}(A) \) factors through \( \mathcal{F}(as)_2 \) and in the case of commutative and cocommutative bialgebra \( H \) one has the Mackey functor \( \mathcal{L}(H): \mathcal{F} \to \text{Vect} \).

## 5 The construction of \( Q\mathcal{F}(as) \)

Let \( D \) be one of the double categories considered in Examples 1)-3). Clearly categories \( D^v \) and \( D^h \) have the same class of isomorphisms, which we call **isomorphisms of \( D \)**. We let \( QD \) be the category whose objects are finite sets, while the morphisms from \( T \) to \( S \) are equivalence classes of diagrams:

\[
\begin{array}{ccc}
U & \xrightarrow{f} & S \\
\downarrow \quad \phi & & \\
T
\end{array}
\]

Here \( f \in D^h \) is a horizontal morphism and \( \phi \in D^v \) is a vertical morphism. For simplicity such data will be denoted by \( T \xleftarrow{\phi} U \xrightarrow{f} S \). Two diagrams
$T \xleftarrow{\phi} U \xrightarrow{f} S$ and $T \xleftarrow{\phi_1} U_1 \xrightarrow{f_1} S$ are equivalent if there exists a commutative diagram

\[
\begin{array}{ccc}
T & \xleftarrow{\phi} & U \xrightarrow{f} S \\
\| & h \downarrow & \| \\
T & \xleftarrow{\phi_1} & U_1 \xrightarrow{f_1} S
\end{array}
\]

such that $h$ is an isomorphism. The composition of $T \xleftarrow{\phi} U \xrightarrow{f} S$ and $S \xleftarrow{\psi} V \xrightarrow{g} R$ in $\mathcal{Q} \mathcal{D}$ is by definition $T \xleftarrow{\psi_1 \phi} W \xrightarrow{gf_1} R$, where

\[
\begin{array}{ccc}
W & \xrightarrow{f_1} & V \\
\downarrow & \psi_1 & \psi \downarrow \\
U & \xrightarrow{f} & S.
\end{array}
\]

is a bimorphism in $\mathcal{D}$. One easily checks that $\mathcal{Q} \mathcal{D}$ is a category and for any object $S$ the diagram $S \xleftarrow{1_s} S \xrightarrow{1_s} S$ is an identity morphism in $\mathcal{Q} \mathcal{D}$.

Clearly the disjoint union yields a structure of PROP on $\mathcal{Q} \mathcal{D}$ and $\emptyset$ is not only a unit object with respect to this monoidal structure, but also a zero object.

For a horizontal morphism $f : S \rightarrow T$ in $\mathcal{D}$ we let $i_*(f) : S \rightarrow T$ be the following morphism in $\mathcal{Q} \mathcal{D}$:

\[
S \xleftarrow{1_s} S \xrightarrow{f} T.
\]

Similarly, for a vertical morphism $\phi : S \rightarrow T$ we let $i^*(f) : T \rightarrow S$ be the following morphism in $\mathcal{Q} \mathcal{D}$:

\[
T \xleftarrow{f} S \xrightarrow{1_s} S.
\]

In this way one obtains the morphisms of PROP’s: $i_* : \mathcal{D} \rightarrow \mathcal{Q} \mathcal{D}$ and $i^* : \mathcal{D}^{op} \rightarrow \mathcal{Q} \mathcal{D}$.

**Remark.** The construction of $\mathcal{Q} \mathcal{D}$ is a particular case of the generalized Quillen $\mathcal{Q}$-construction [16] considered by Fiedorowicz and Loday in [7]. The following lemma is a variant of a result of [16].
Lemma 5.1 The category of Mackey functors from $D$ to $Vect$ is equivalent to the category of functors $M : QD \to Vect$.

Proof. Let $M : QD \to Vect$ be a functor. For any arrow $f : S \to T$ we put $M_*(f) := M(i_*(f))$ and $M^*(f) := M(i^*(f))$. In this way we get a Mackey functor on $D$. Conversely, if $M$ is a Mackey functor on $D$, then we put

$M(S \xymatrix{ & V \ar[r]^f & T}) = M_*(f)M^*(g)$.

One easily shows that in this way we get a covariant functor $QD$ to $Vect$ and the proof is finished.

By applying the $Q$-construction to the diagram (4.0) one obtains the following (noncommutative) diagram of PROP’s:

\[
\begin{array}{ccc}
Q(F(as)_{11}) & & Q(F(as)_{12}) \\
& \uparrow & \\
Q(F(as)) & & Q(F) \\
& \downarrow & \\
& Q(F(as)_2) & \uparrow \\
\end{array}
\]

The following theorem gives the identification of the terms involved in the diagram, except for $Q(F(as))$.

Theorem 5.2 i) The category of $Q(F(as))$-algebras is equivalent to the category of bialgebras.

ii) The category $Q(F(as)_{11})$-algebras is equivalent to the category of cocommutative bialgebras and $Q(F(as)_{11})$ is isomorphic to the PROP $\text{Mon}^{op}$.

iii) The category of $Q(F(as)_2)$-algebras is equivalent to the category of commutative bialgebras and $Q(F(as)_2)$ is isomorphic to the PROP $\text{Mon}$.

iv) The category of $Q(F)$-algebras is equivalent to the category of cocommutative and commutative bialgebras and $Q(F)$ is isomorphic to the PROP $\text{AbMon}$.

Proof. Theorem 4.2 together with Lemma 5.1 shows that any bialgebra $V$ gives rise to $X(V)$-algebra. Conversely assume $M$ is a $Q(F(as))$-algebra and let $V = M(\mathbf{1})$. Then $M \circ i_*$ is a $F(as)$-algebra and $M \circ i^*$ is a $F(as)^{op}$-algebra.
Thus $M$ carries natural structures of associative algebra and coassociative coalgebra. Since $M = (M \circ i_*, M \circ i^*)$ is a Mackey functor on $\mathcal{F}(\text{as})$, it follows from Theorem 4.2 that $V$ is indeed a bialgebra. To prove the remaining parts of the theorem, let us observe that $(\mathcal{Q}(\mathcal{F}(\text{as})_2))^{op} \cong \mathcal{Q}(\mathcal{F}(\text{as})_1)$, where equivalence is identity on objects and sends $T \xleftarrow{\phi} U \xrightarrow{f} S$ to $S \xleftarrow{f} U \xrightarrow{\phi} T$.

We now show that $\mathcal{Q}(\mathcal{F}(\text{as})_2) \cong \text{Mon}$. The main observation here is the fact that if $f : X \rightarrow S_1 \coprod S_2$ is a morphism in $\mathcal{F}(\text{as})$ then $f = f_1 \coprod f_2$ in the category $\mathcal{F}(\text{as})$, where $f_i$ as a map is the restriction of $f$ on $f^{-1}(S_i), i = 1, 2$. Since $f_i^{-1}(y) = f^{-1}(y)$ for all $y \in f^{-1}(S_i)$ we can take the same total ordering in $f_i^{-1}(y)$ to turn $f_i$ into a morphism in $\mathcal{F}(\text{as})$. A conclusion of this observation is the fact that disjoint union defines not only a symmetric monoidal category structure but it is the coproduct in $\mathcal{Q}(\mathcal{F}(\text{as})_2)$. Clearly $\mathbb{1}$ is an $n$-fold coproduct of $\mathbb{1}$. On the other hand, we may assume that the objects of $\text{Mon}$ are natural numbers, while the set of morphisms from $k$ to $n$ is the same as $\text{Hom}_{\text{Mon}}(F_k, F_n)$, where $F_n$ is the free monoid on $n$ generators. This set can be identified with the set of $k$-tuples of words on $n$ variables $x_1, \ldots, x_n$. Since $\mathcal{Q}(\mathcal{F}(\text{as})_2)$ and $\text{Mon}$ are categories with finite coproducts and any object in both categories is a coproduct of some copies of $\mathbb{1}$, we need only to identify the set of morphisms originating from $\mathbb{1}$. A morphism $\mathbb{1} \rightarrow n$ in $\mathcal{Q}(\mathcal{F}(\text{as})_2)$ is a diagram $\mathbb{1} \xleftarrow{\phi} U \xrightarrow{f} n$, where $\phi$ is a map of noncommutative sets. We can associate to this morphism a word $w$ of length $m$ on $n$ variables $x_1, \ldots, x_n$. Here $m = \text{Card}(U)$ and the $i$-th place of $w$ is $x_{f(y_i)}$, where $U = \{y_1 < \cdots < y_m\}$. In this way one sees immediately that this correspondence defines the equivalence of categories $\mathcal{Q}(\mathcal{F}(\text{as})_2) \cong \text{Mon}$. We refer the reader to [1] for the fact that $\mathcal{Q}(\mathcal{F})$ is equivalent to $\text{Abmon}$. Argument in this case is even simpler than the previous one and can be sketched as follows. Since the PROP $\mathcal{Q}(\mathcal{F})$ is isomorphic to its opposite disjoint union yields not only the coproduct in $\mathcal{Q}(\mathcal{F})$ but also the product. Next, morphisms $\mathbb{1} \rightarrow \mathbb{1}$ in $\mathcal{Q}(\mathcal{F})$ are diagrams of maps $\mathbb{1} \leftarrow U \rightarrow \mathbb{1}$, whose equivalence class is completely determined by the cardinality of $U$. This gives identification of morphisms from $\mathbb{1} \rightarrow \mathbb{1}$ with natural numbers and the proof is done.
Thus the above diagram of PROP’s is equivalent to the diagram
\[
\begin{array}{ccc}
\text{Mon}^{op} & \to & \text{Abmon} \\
\uparrow & & \downarrow \\
Q(\mathcal{F}(\text{as})) & \to & \text{Mon}
\end{array}
\]

Here \(\text{Mon} \to \text{Abmon}\) is given by abelization functor. Let us note that \(Q(\mathcal{F}(\text{as}))\) and \(\text{Abmon}\) are self dual PROP’s, and the arrows are surjection on morphisms. If one looks at endomorphisms of \(1\) we see that the endomorphism monoid \(\text{End}_C(1)\) for \(C = \text{Mon}^{op}, \text{Mon, Abmon}\) is isomorphic to the multiplicative monoid of natural numbers. This corresponds to the fact that the operations \(\Psi^{(n,\sigma)}\) from the introduction for commutative or cocommutative bialgebras are independent of \(\sigma\) and \(\Psi^n \circ \Psi^m = \Psi^{nm} \square\).

The following proposition describes the endomorphism monoid \(\text{End}_C(1)\) for \(C = Q(\mathcal{F}(\text{as}))\).

Let \(n \in \mathbb{N}\) be a natural number and let \(\sigma \in \mathcal{S}_n\) be a permutation. Here \(\mathcal{S}_n\) is the group of permutations on \(n\) letters. We let \([\sigma]\) be the morphism \(n \to 1\) in \(\mathcal{F}(\text{as})\) corresponding to the ordering \(\sigma(1) < \sigma(2) < \cdots < \sigma(n)\). For example \([\text{id}_n]\), or simply \([\text{id}]\) denotes the morphism \(n \to 1\) in \(\mathcal{F}(\text{as})\) corresponding to the ordering \(1 < 2 < \cdots < n\). Moreover we let \((n, \sigma) : 1 \to 1\) be the morphism in \(Q(\mathcal{F}(\text{as}))\) corresponding to the diagram \(1 \xrightarrow{[\sigma]} n \xrightarrow{[\text{id}]} 1\).

**Proposition 5.3** The monoid of endomorphisms of \(1 \in Q(\mathcal{F}(\text{as}))\) is isomorphic to the monoid of pairs \((n, \sigma)\), where \(\sigma \in \mathcal{S}_n\) and \(n \in \mathbb{N}\), with the following multiplication

\[(n, \sigma) \circ (m, \tau) = (nm, \Phi(\sigma, \tau)).\]

Here

\[\Phi : \mathcal{S}_n \times \mathcal{S}_m \to \mathcal{S}_{nm}\]

is a map, which is defined by

\[\Phi(\sigma, \tau)(x) = \tau(p + 1) + m(\sigma(q) - 1), \quad 1 \leq x \leq nm,\]

where \(x = pn + q\), \(1 \leq q \leq n\) and \(0 \leq p \leq m - 1\).
Proof. A morphism 1 → 1 in Q(F(as)) is a diagram 1 ← φ U → 1, where φ and f are morphisms of noncommutative sets. Hence U has two total orderings corresponding to φ and f. We will identify U to n via ordering corresponding to f. Here n is the cardinality of U. We denote the first (resp. the second, · · · ) element in the ordering corresponding to φ by σ(1) (resp. σ(2), · · · ). In this way we get a permutation σ ∈ Sn. Thus any morphism 1 → 1 in Q(F(as)) is of the form (n, σ). In order to identify the composition law it is enough to note the following two facts:

i) The diagram
\[
\begin{array}{ccc}
nm & \xrightarrow{f} & n \\
g \downarrow & & \downarrow [\sigma] \\
m & \xrightarrow{id} & 1
\end{array}
\]
is a bimorphism in Q(F(as)). Here f and g are given by
\[
f^{-1}(j) = \{1 + (j - 1)m < 2 + (j - 1)m < \cdots < (m - 1) + (j - 1)m < jm\},
\]
\[
g^{-1}(i) = \{i + (\sigma(1) - 1)m < i + (\sigma(2) - 1)m < \cdots < i + (\sigma(n) - 1)m\},
\]
for i ∈ m and j ∈ n.

ii) One has [Φ(σ, τ)] = [τ] ◦ g and [idn] ◦ f = [idnm].

We now give an alternative description of the function Φ. Let
\[
(5.4) \quad \gamma : S(n) \times S(m_1) \times \cdots \times S(m_n) \to S(m_1 + \cdots + m_n)
\]
be a map given by
\[
\gamma(\sigma; \sigma_{m_1}, \ldots, \sigma_{m_n}) = \sigma(m_1, \ldots, m_n) \circ (\sigma_1 \coprod \cdots \coprod \sigma_{m_n}),
\]
where σ(m_1, · · · , m_n) permutes the n blocks according to σ. Moreover, for any integers n and m we let
\[
I : nm \to n \times m
\]
be the bijection corresponding to the following ordering of the Cartesian product:
\[
(i, j) < (s, t) \text{ iff } i < s \text{ or } i = s \text{ and } j < t.
\]
Similarly, we let
\[
II : nm \to n \times m
\]
be the bijection corresponding to the following ordering of the Cartesian product:

\[ (i, j) < (s, t) \text{ iff } j < t \text{ or } j = t \text{ and } i < s. \]

Then we put \( \Phi(n, m) := I^{-1} \circ II \in \mathcal{S}_{nm} \). It is not too difficult to see that \( \Phi(n, m) = \Phi(1^n, 1^m) \) and

\[ \Phi(\sigma, \tau) = \Phi(n, m) \circ \gamma(\tau, \sigma, \cdots, \sigma). \]

**Remarks.** 1) It is well known that the PROP corresponding to cocommutative Hopf algebras is \( \text{Gr}^{op} \) (see next remark), the PROP corresponding to commutative Hopf algebras is \( \text{Gr} \), while the PROP corresponding to commutative and cocommutative Hopf algebras is \( \text{Ab} \). Of course the category of Hopf algebras are also algebras over some PROP, which can be easily described via generators and relations \([14]\). An explicit description of this particular PROP will be the subject of a forthcoming paper.

2) Let \( A \) be a cocommutative Hopf algebra. Since \( \otimes \) is a product in the category \( \text{Coalg} \) of cocommutative coalgebras, \( A \) is a group object in this category. On the other hand any group object in any category \( \mathcal{A} \) with finite products gives rise to the model in \( \mathcal{A} \) of the algebraic theory of groups in the sense of Lawvere \([8]\). But the algebraic theory of groups is nothing but \( \text{Gr}^{op} \) and hence we have the functor \( \mathcal{X}(A) : \text{Gr}^{op} \to \text{Coalg} \), which assigns \( A^{\otimes n} \) to \( < n > \). Here \( < n > \) is a free group on \( x_1, \ldots, x_n \). Moreover it assigns \( \mu \) to the morphism \( < 1 > \to < 2 > \) given by \( x_1 \mapsto x_1x_2 \). Similarly \( \mathcal{X}(A) \) assigns \( \Delta \) to the homomorphism \( < 2 > \to < 1 > \) given by \( x_1, x_2 \mapsto x_1 \). Of course it assigns the antipode \( S : A \to A \) to \( x_1 \mapsto x_1^{-1} \). Having these facts in mind one easily describes the action of \( \mathcal{X}(A) \) on more complicated morphisms. For example one checks that \( \mathcal{X}(A) \) assigns

\[ (\mu, \mu) \circ (\mu, id, \mu, id) \circ (S, id_{A^{\otimes 1}}) \circ \tau_{2,3} \circ (id_{A^{\otimes 3}}, \Delta, id) \circ (\Delta, \Delta, id) \]

to the morphism \( < 2 > \to < 3 > \) corresponding to the pair of words \( (x_1^{-1}x_2x_1, x_1^2x_3) \). Here \( \tau_{2,3} \) permutes the second and third coordinates. Conversely any linear map \( A^{\otimes n} \to A^{\otimes m} \) constructed using the structural data of a cocommutative Hopf algebra \( A \) is coming in this way. Hence to check whether a complicated diagram involving such maps commutes it is enough to look to the corresponding diagram in \( \text{Gr} \), which is usually simpler to handle.

3) It is well known that the morphism \( n \to m \) in \( \text{Abmon} \) can be identified with \( (m \times n) \)-matrices over natural numbers. Under this identification the
equivalence $Q(\mathcal{F}) \cong \textbf{Abmon}$ is given by assigning the matrix whose $(i,j)$-component is the cardinality of $f^{-1}(j) \cap g^{-1}(i)$, $1 \leq i \leq m$, $1 \leq j \leq n$ to the diagram $\mathfrak{n} \xleftarrow{f} X \xrightarrow{g} \mathfrak{m}$. It is less known that the morphisms $\mathfrak{n} \to \mathfrak{m}$ in $\textbf{Mon}$ can be described via shuffles. In order to explain this connection let us start with particular case. Consider a word $x^2yxy^3x^2$ of bidegree $(5,4)$. It defines a morphism $1 \to 2$ in $\textbf{Mon}$. One associates a $(5,4)$-shuffle $(1,2,4,8,9,3,5,6,7)$ to this word, whose first five values are just the numbers of places where $x$ lies. Similarly morphisms $\mathfrak{n} \to \mathfrak{m}$ in $\textbf{Mon}$ are in 1-1-correspondence with collections $\{A = (a_{ij}), (\varphi_1, \ldots, \varphi_n)\}$, where $A$ is an $(m \times n)$-matrix over natural numbers and $\varphi_i$ is a $(a_{i1}, \ldots, a_{in})$-shuffle, $i = 1, \ldots, n$. The functor $\textbf{Mon} \to \textbf{Abmon}$ corresponds to forgetting the shuffles. Now combine this observation with Proposition 5.3 to get the description of morphisms $\mathfrak{n} \to \mathfrak{m}$ in $Q(\mathcal{F}(\text{as}))$ as collections $\{A = (\alpha_{ij}), (\varphi_1, \ldots, \varphi_n)\}$, where $\alpha = (a_{ij}, \sigma_{ij})$ and $a_{ij}$ is a natural number, while $\sigma_{ij} \in \mathfrak{S}_{a_{ij}}$ is a permutation and finally $\varphi_i$ is a $(a_{i1}, \ldots, a_{im})$-shuffle.

4) Recently Sarah Whitehouse ([??], [??]) defined the action of $\mathfrak{S}_{k+1}$ on $A^{\otimes k}$ for any commutative or cocommutative Hopf algebra $A$. Actually she implicitly constructed the group homomorphism

$$\xi_k : \mathfrak{S}_{k+1} \to \mathfrak{S}_k,$$

where $\mathfrak{S}_k$ is the automorphism group of $< k >$. Then the action of $x \in \mathfrak{S}_{k+1}$ on $A^{\otimes k}$ is obtained by applying the functor $\mathcal{X}(A)$ to $\xi_k(x)$. The homomorphism $\xi_k$ is given by

$$\sigma_1(x_1) = x_1^{-1}, \sigma_1(x_2) = x_1x_2, \sigma_1(x_i) = x_i, \quad i \geq 2$$

$$\sigma_i(x_{i-1}) = x_{i-1}x_i, \quad \sigma_i(x_i) = x_i^{-1}, \quad \sigma_i(x_{i+1}) = x_ix_{i+1}, \quad \sigma_i(x_j) = x_j,$$

for $1 < i < k$, $j \neq i-1, i, i+1$ and

$$\sigma_k(x_{k-1}) = x_{k-1}x_k, \quad \sigma_k(x_k) = x_k^{-1}, \quad \sigma_k(x_j) = x_j \text{ if } j < n-1.$$

Here $\sigma_i \in \mathfrak{S}_{k+1}$ is the transposition $(i, i+1)$, $1 \leq i \leq k$. The homomorphisms $\xi_k$, $k \geq 0$ are restrictions of a functor $\xi : \mathcal{F} \to \textbf{Gr}$, which is given as follows. For a set $X$ the group $\xi(X)$ is generated by symbols $< x, y >$, $x, y \in X$ modulo the relations

$$< x, y > < y, z > = < x, z >, \quad x, y, z \in X.$$
6 Generalization for operads

Let $\mathcal{P}$ be an operad of sets [12]. Let us recall that then $\mathcal{P}$ is a collection of $\mathfrak{S}_n$-sets $\mathcal{P}(n)$, $n \geq 0$ together with the composition law

$$\gamma : \mathcal{P}(n) \times \mathcal{P}(m_1) \times \cdots \times \mathcal{P}(m_n) \to \mathcal{P}(m_1 + \cdots + m_n)$$

and an element $e \in \mathcal{P}(1)$ satisfying some associativity and unite conditions [12]. We will assume that $\mathcal{P}(0) = \ast$. Any set $X$ gives rise to an operad $\mathcal{E}_X$, for which $\mathcal{E}_X(n) = \text{Maps}(X^n, X)$. A $\mathcal{P}$-algebra is a set $X$ together with a morphism of operads $\mathcal{P} \to \mathcal{E}_X$. We let $\mathcal{P}$-$\text{Alg}$ be the category of $\mathcal{P}$-algebras. The forgetful functor $\mathcal{P}$-$\text{Alg} \to \text{Sets}$ has the left adjoint functor $F_\mathcal{P} : \text{Sets} \to \mathcal{P}$-$\text{Alg}$ which is given by

$$F_\mathcal{P}(X) = \coprod_{n \geq 0} \mathcal{P}(n) \times \mathfrak{S}_n X^n.$$ 

We let $\text{Free}(\mathcal{P})$ be the full subcategory of $\mathcal{P}$-$\text{Alg}$ whose objects are $F_\mathcal{P}(n)$, $n \geq 0$.

Now we introduce the category $\mathcal{F}(\mathcal{P})$. For any map $f : n \to m$ one puts

$$\mathcal{P}_f = \prod_{i=1}^m \mathcal{P}(| f^{-1}(i) |).$$

Here $| S |$ denotes the cardinality of a set $S$. The category $\mathcal{F}(\mathcal{P})$ has the same objects as $\mathcal{F}$, while the morphisms from $\underline{n}$ to $\underline{m}$ in $\mathcal{F}(\mathcal{P})$ are pairs $(f, \omega^f)$, where $f : n \to m$ is a map and $\omega^f = (\omega^f_1, \cdots, \omega^f_m) \in \mathcal{P}_f$. If $(f, \omega^f)$ and $(g, \omega^g) : m \to k$ are morphisms in $\mathcal{F}(\mathcal{P})$ then the composition $(g, \omega^g) \circ (f, \omega^f)$ is a pair $(h, \omega^h)$, where $h = gf$ and for each $1 \leq i \leq k$ one has

$$\omega^h_i = \gamma(\omega^g_j; \omega^f_{j_1}, \cdots, \omega^f_{j_s}).$$

Here $g^{-1}(i) = \{j_1, \cdots, j_s\}$. This construction goes back to May and Thomason [13].

One observes that if $\mathcal{P} = \text{as}$, then $\mathcal{F}(\mathcal{P})$ is nothing but $\mathcal{F}(\text{as})$, while $\text{Free}(\mathcal{P})$ is equivalent to the category of finitely generated free monoids. Here $\text{as}$ is the operad given by $\text{as}(n) = \mathfrak{S}_n$ for all $n \geq 0$ and $\gamma$ is the same as in (5.4). Thus $\text{as}$-algebras are associative monoids. We now show how to generalize Theorem 5.2 ii) for arbitrary operads.
Let \( \mathcal{F}(\mathcal{P})_2 \) be the double category, whose objects are sets, horizontal arrows are set maps and vertical arrows are morphisms from \( \mathcal{F}(\mathcal{P}) \). Double morphisms are pullback diagrams of sets

\[
\begin{array}{ccc}
U & \overset{p}{\rightarrow} & S \\
\downarrow & & \downarrow \\
T & \overset{q}{\rightarrow} & U
\end{array}
\]

\( \alpha = \begin{array}{lcl} \end{array} \)

\( g & \downarrow & f \\
\end{array} \)

\[ T \overset{q}{\rightarrow} U \]

\( \text{together with lifting of } g \text{ and } f \text{ in } \mathcal{F}(\mathcal{P}). \) Hence the elements \( \omega^f \in \mathcal{P}_f \) and \( \omega^g \in \mathcal{P}_g \) are given. One requires that these elements are compatible

\[ \omega^g_t = \omega^f_{qt}, \ t \in T. \]

We claim that the category \( \mathcal{Q}(\mathcal{F}(\mathcal{P})_2) \) and \( \text{Free}(\mathcal{P}) \) are equivalent. On objects one assigns \( F_P(n) \) to \( n \). Both categories in the question possess finite coproducts and thus one needs only to identify morphisms from \( 1 \). Let \( 1 \overset{\omega}{\rightarrow} m \overset{f}{\rightarrow} X \) be a morphism in \( \mathcal{Q}(\mathcal{F}(\mathcal{P})_2) \). By definition \( \omega \in \mathcal{P}(m) \) and \( f \in X^n \). Thus it gives an element in \( F_P(X) \) and therefore a morphism \( F_P(1) \rightarrow F_P(X) \) in \( \text{Free}(X) \). It is clear that in this way one obtains expected equivalence of categories.

Any set operad \( \mathcal{P} \) gives rise to the linear operad \( k[\mathcal{P}] \), which is spanned on \( \mathcal{P} \). Clearly the disjoint union yields a structure of PROP on \( \mathcal{F}(\mathcal{P}) \) and \( \mathcal{F}(\mathcal{P}) \)-algebras are nothing but \( k[\mathcal{P}] \)-algebras in the tensor category \( \text{Vect} \).

We leave as an exercise to the interested readers to show that the \( \mathcal{Q} \)-construction of Section 5 and the notion of the bialgebra have the canonical generalizations for any operad \( \mathcal{P} \).

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