On Holomorphic Effective Actions of Hypermultiplets Coupled to External Gauge Superfields

I.L. Buchbinder† and I.B. Samsonov‡

† Department of Theoretical Physics, Tomsk State Pedagogical University, Tomsk 634041, Russia
‡ Department of Quantum Field Theory, Tomsk State University, Tomsk 634050, Russia

Abstract

We study the structure of holomorphic effective action for hypermultiplet models interacting with background super Yang-Mills fields. A general form of holomorphic effective action is found for hypermultiplet belonging to arbitrary representation of any semisimple compact Lie group spontaneously broken to its maximal abelian subgroup. The applications of obtained results to hypermultiplets in fundamental and adjoint representations of the $SU(n)$, $SO(n)$, $Sp(n)$ groups are considered.
One of the remarkable properties of supersymmetric field theories is an existence of two types of effective lagrangians, chiral and general or holomorphic and non-holomorphic. General or non-holomorphic lagrangian contributes to effective action in form of integral over full super-space while chiral or holomorphic lagrangian contributes to effective action in form of integral over chiral subspace. We point out that an appearance of holomorphic corrections to effective action was firstly demonstrated in refs [1]-[4] (see also [5], [6]) for \( N = 1 \) SUSY and in refs [7], [8] for \( N = 2 \) SUSY.

The modern interest to effective action in \( N = 2 \) supersymmetric theories is conditioned by the work [9] where exact instanton contribution to holomorphic effective potential has been found for \( N = 2 \) \( SU(2) \) super Yang-Mills model. Generalization of this result to theories with the gauge groups \( SU(n) \) and \( SO(n) \) have been given in refs [10], [11].

It is well known, a most simple and clear way of studying of effective action in SUSY models is based on a formulation of these models in terms of unconstrained superfields. Namely such a formulation provides a possibility to preserve a manifest supersymmetry at all stages of quantum calculations. For \( N = 2 \) SUSY the corresponding formalism was developed in refs [12] and was called harmonic superspace approach.

In this note we study a structure of holomorphic effective action for the theory of hypermultiplet coupled to external super Yang-Mills field in harmonic superspace approach. It is assumed that the hypermultiplet belongs to arbitrary representation of any semisimple compact Lie group and the external superfield lies in Cartan subalgebra of the corresponding Lie algebra.

Holomorphic effective action for hypermultiplets coupled to abelian gauge superfield has been obtained in harmonic superspace approach in refs [13], [14] as follows

\[
\Gamma_{q,\omega}[V^{++}] = \int d^4x d^4\theta \mathcal{F}_{q,\omega}(W) + \text{c.c.},
\]

\[
\mathcal{F}_{q,\omega}(W) = -\frac{k_{q,\omega}}{64\pi^2} W^2 \log \frac{W^2}{\Lambda^2}, \quad k_{q,\omega} = \begin{cases} 1 & \text{for } q-\text{hypermultiplet} \\ 2 & \text{for } \omega-\text{hypermultiplet}. \end{cases}
\]

Here \( W \) is the strength of \( N = 2 \) gauge prepotential \( V^{++} \) [12], [13]

\[
W = -\int du (D^-)^2 V^{++}(x, \theta, u), \quad W = -\int du (D^-)^2 V^{++}(x, \theta, u).
\]

The \( q- \) and \( \omega- \)hypermultiplets have been described in refs [12]. Function \( \mathcal{F}_{q,\omega}(W) \) is called the holomorphic potential.

In this note we consider the \( q- \) and \( \omega- \)hypermultiplets \( Q^+ \) and \( \Omega \) in an arbitrary representation of some semisimple compact gauge group. Let \( e_i \) \((i = 1, \ldots, m)\) be a basis in representation...
space $\mathcal{V}$, $\dim \mathcal{V} = m$, and $T_i$ be the generators of the representation forming a basis in Lie algebra $\mathcal{L}$. These hypermultiplets and non-abelian gauge superfield $V^{++}$ can be written in the forms

$$Q^+ = \sum_{i=1}^{m} q_i^+ e_i, \quad \bar{Q}^+ = \sum_{i=1}^{m} \bar{q}_i^+ e_i^\dagger, \quad \Omega = \sum_{i=1}^{m} \omega_i e_i, \quad \bar{\Omega} = \sum_{i=1}^{m} \bar{\omega}_i e_i^\dagger, \quad V^{++} = \sum_{k} V^{++}_k T_k,$$

(3)

The classical actions of the hypermultiplets coupled to the external superfield $V^{++}$ look like as follows

$$S[Q^+, \bar{Q}^+, V^{++}] = \int d\zeta (-4) \bar{Q}^+(D^{++} + iV^{++})Q^+$$

$$S[\Omega, \bar{\Omega}, V^{++}] = \int d\zeta (-4)(D^{++}\bar{\Omega} - i\bar{\Omega}V^{++})(D^{++} + iV^{++})\Omega.$$  

(5)  

(6)

We use the denotations like in refs [13], [14]. Manifest form of the terms $V^{++}Q^+$, $V^{++}\Omega$ depends on representation, e.g., in the adjoint representation these terms are understood as the commutators. The expression $\bar{Q}^+D^{++}Q^+$ means $\bar{q}_i^+D^{++}q_j^+(e_i, e_j)$ where $(e_i, e_j)$ is a scalar product of vectors in a representation space.

Further we assume that the external superfield $V^{++}$ lies in the Cartan subalgebra $\mathcal{H}$ of gauge algebra $\mathcal{L}$. Therefore

$$V^{++} = \sum_{k=1}^{l} V^{++}_k h_k,$$

(7)

where $l = \text{rank} \mathcal{L}$ and $\{h_1, \ldots, h_l\}$ is a basis in $\mathcal{H}$.

The main idea of subsequent consideration is to reduce a calculation of effective actions for the models (3) to calculation of effective actions for hypermultiplets in abelian external gauge superfield and use the result (1). To do that, we ”diagonalise” the actions (5), (6) and obtain several ”one-dimensional” hypermultiplets in abelian gauge superfields. The realization of this idea is based on use of notion of the weights of Lie group representation.

The weight subspaces $\mathcal{V}^\lambda$ of representation space $\mathcal{V}$ are defined as follows [19]

$$\mathcal{V}^\lambda = \{v \in \mathcal{V} \mid hv = \lambda(h)v, \forall h \in \mathcal{H}\}. \quad (8)$$

Let the equation (8) have $m = \dim \mathcal{V}$ roots (weights) $\lambda_i$ (some of which may be equal) and corresponding eigenvectors $e_i$ be orthogonal

$$(e_i, e_j) = \delta_{ij}. \quad (9)$$

We choose vectors $e_i$ as the basis in representation space and expand hypermultiplets (3) in this basis. With the help of relation (8) it is easy to find the action of gauge superfield $V^{++}$ on
the hypermultiplets

\[ V^{++}Q^+ = \sum_{i=1}^{m} q_i^+ \lambda_i(V^{++}) e_i, \quad V^{++} = \sum_{i=1}^{m} \omega_i \lambda_i(V^{++}) e_i. \]  

(10)

It is easy to see that functions \( \lambda_i(V^{++}) \) are linear \( \lambda_i(h) = \lambda_i(h_k) V^{++} \), so the weights \( \lambda_i(V^{++}) \) have \( U(1) \) charge +2, the same as the charge of \( V^{++} \). Substituting the hypermultiplets (3) into actions (5), (6) and using eqs (9), (10) we obtain

\[ S[\dot{Q}^+, Q^+, V^{++}] = \int d\zeta (-4) \sum_{i=1}^{m} \dot{q}_i^+ (D^{++} + i\lambda_i(V^{++})) q_i^+, \]  

(11)

\[ S[\dot{\Omega}, \Omega, V^{++}] = \int d\zeta (-4) \sum_{i=1}^{m} (D^{++} - i\lambda_i(V^{++})) \dot{\omega}_i (D^{++} + i\lambda_i(V^{++})) \omega_i. \]  

(12)

The hypermultiplets labeled by different indices in (11), (12) do not mix between themselves, therefore their classical actions are the sums of actions of non-coupled ”one-dimensional” hypermultiplets. Hence, holomorphic effective actions of models (11), (12) are represented as a sum of independent ”one-dimensional” effective actions (1) over all weights of representation

\[ \Gamma_{q,\omega}[V^{++}] = \int d^4 x d^4 \theta F_{q,\omega}(W) + \text{c.c.}, \quad F_{q,\omega}(W) = -\frac{k_{q,\omega}}{64 \pi^2} \sum_{i=1}^{m} W_i^2 \log \frac{W_i^2}{\Lambda^2}, \]  

(13)

where \( W_i \) are constructed of weights \( \lambda_i(V^{++}) \) according to (2).

Thus we have obtained the effective actions of \( q \) - and \( \omega \)-hypermultiplets in arbitrary representation of any semisimple gauge group (13), they are expressed via weights of the representation. It is well known that any irreducible representation is completely determined by its highest weight \( \alpha \) and all other weights of representation are expressed of \( \alpha \) (see [19])

\[ \lambda(h) = \alpha(h) - m_1 \beta_1(h) - m_2 \beta_2(h) - \ldots - m_k \beta_k(h), \quad h \in \mathcal{H}, \]  

(14)

where \( \beta_i(h) \) are simple roots of algebra \( \mathcal{L} \), \( m_1, \ldots, m_k \) are non-negative integers. Hence, we see the effective action (13) is actually determined by highest weight \( \alpha \).

Now we are ready to write down holomorphic effective actions of \( q \) - and \( \omega \)-hypermultiplets in the fundamental and adjoint representations of \( SU(n), SO(n), Sp(n) \) groups. To do this we have to fix Cartan subalgebra of each group, find all weights of corresponding representation, and substitute these weights into (13). Using the weights of representations and structure of the Cartan subalgebras given in the Appendix one obtain holomorphic effective actions of hypermultiplets on the base of eq. (13). Below we give a list of corresponding holomorphic effective potentials:
1. Fundamental representation

\[
\mathcal{F}_{q,\omega}^{SU(n)}(W) = -\frac{k_{q,\omega}}{64\pi^2} \sum_{i=1}^{n} W_i^2 \log \frac{W_i^2}{\Lambda^2};
\]

\[
\mathcal{F}_{q,\omega}^{SO(2n)}(W) = \mathcal{F}_{q,\omega}^{SO(2n+1)}(W) = \mathcal{F}_{q,\omega}^{Sp(n)}(W) = -\frac{k_{q,\omega}}{32\pi^2} \sum_{i=1}^{n} W_i^2 \log \frac{W_i^2}{\Lambda^2};
\]

2. Adjoint representation

\[
\mathcal{F}_{q,\omega}^{SU(n)}(W) = -\frac{k_{q,\omega}}{32\pi^2} \sum_{i<j} (W_i - W_j)^2 \log \frac{(W_i - W_j)^2}{\Lambda^2},
\]

\[
\mathcal{F}_{q,\omega}^{SO(2n)}(W) = \mathcal{F}_{q,\omega}^{SO(2n+1)}(W) = -\frac{k_{q,\omega}}{64\pi^2} \left[ \sum_{i<j} (W_i - W_j)^2 \log \frac{(W_i - W_j)^2}{\Lambda^2} + \sum_{i\leq j} (W_i + W_j)^2 \log \frac{(W_i + W_j)^2}{\Lambda^2} \right],
\]

\[
\mathcal{F}_{q,\omega}^{Sp(n)}(W) = -\frac{k_{q,\omega}}{64\pi^2} \left[ \sum_{i<j} (W_i - W_j)^2 \log \frac{(W_i - W_j)^2}{\Lambda^2} + \sum_{i\leq j} (W_i + W_j)^2 \log \frac{(W_i + W_j)^2}{\Lambda^2} \right],
\]

where \( W_i \) are \( N = 2 \) SYM strengths constructed of \( V_i^{++} \) by the rule (2).

Some of these results were obtained earlier in refs [10, 11, 16, 17, 18] by different methods. In our approach the holomorphic effective actions are easily obtained on the base of single method from the general expression (13).

To conclude, in the present paper we have considered the hypermultiplet models coupled to \( N = 2 \) nonabelian gauge superfield using the harmonic superspace formalism. We obtained the holomorphic effective actions of these models for the general case when hypermultiplets belong to arbitrary representation of any semisimple gauge group and gauge superfield lies in Cartan subalgebra of the gauge algebra. Then we applied these results to find the effective actions of \( q \)- and \( \omega \)-hypermultiplets in fundamental and adjoint representations of \( SU(n) \), \( SO(n) \), \( Sp(n) \) groups.

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**Appendix**

We formulate a list of useful properties of considered algebras such as the structures of Cartan
subalgebras and weights of fundamental and adjoint representations. These properties have been used to obtain the effective potentials (15)-(20) (see the details in refs [19], [20]).

1. Fundamental representation.

The basis in representation space $e_i$, Cartan subalgebra-valued superfield $V^{++}$ and corresponding weights of fundamental representation have been chosen in the form

$$ SU(n) \quad (e_i)_k = \delta_{ik} \quad i, k = 1, \ldots, n $$
$$ V^{++} = \text{diag}(V_1^{++}, V_2^{++}, \ldots, V_n^{++}), \quad \sum_{i=1}^n V_i^{++} = 0 $$
$$ \lambda_i(V^{++}) = V_i^{++}; $$

$$ SO(2n) \quad (e_i)_j = \frac{1}{\sqrt{2}}(i\delta_{2i-1,j} + \delta_{2i,j}), \quad (e_{-i})_j = \frac{1}{\sqrt{2}}(\delta_{2i-1,j} + i\delta_{2i,j}), \quad i, j = 1, \ldots, n $$
$$ V^{++} = \text{diag}(V_1^{++}I, \ldots, V_n^{++}I), \quad I = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} $$
$$ \lambda_i(V^{++}) = V_i^{++}; $$

$$ SO(2n+1) \quad (e_i)_j = \frac{1}{\sqrt{2}}(i\delta_{2i-1,j} + \delta_{2i,j}), \quad (e_{-i})_j = \frac{1}{\sqrt{2}}(\delta_{2i-1,j} + i\delta_{2i,j}), $$
$$ (e_0)_i = \delta_{2n+1,i}, \quad i, j = 1, \ldots, n $$
$$ V^{++} = \text{diag}(V_1^{++}I, \ldots, V_n^{++}I, 0), \quad I = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} $$
$$ \lambda_i(V^{++}) = V_i^{++} (i \neq 0), \quad \lambda_0(V^{++}) = 0; $$

$$ Sp(n) \quad (e_i)_k = \delta_{ik}, \quad i, k = 1, \ldots, 2n $$
$$ V^{++} = \text{diag}\{V_1^{++}, \ldots, V_n^{++}, -V_1^{++}, \ldots, -V_n^{++}\} $$
$$ \lambda_i(V^{++}) = V_i^{++}. $$

2. Adjoint representation.

In adjoint representation the weight subspaces coincide with the root subspaces in a gauge algebra and the weights of the representation are the roots of algebra. Therefore we have to fix Cartan-Weyl basis in each considered algebra and write down its roots.
We point out, the structure of effective potentials (16)-(20) is defined by the choice of basis vectors given above and the coefficients in effective potentials depend on the normalisation of these vectors.

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