Combinatorial properties of Temperley–Lieb algebras

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A.A. 2010/2011

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Introduction

The Temperley–Lieb algebra $TL(X)$ is a quotient of the Hecke algebra $\mathcal{H}(X)$ associated to a Coxeter group $W(X)$, $X$ being an arbitrary Coxeter graph. It first appeared in [TL71], in the context of statistical mechanics (see, e.g., [Jon85]). The case $X = A$ was studied by Jones (see [Jon87]) in connection to knot theory. For an arbitrary Coxeter graph, the Temperley–Lieb algebra was studied by Graham. More precisely, in [Gra] Graham showed that $TL(X)$ is finite dimensional whenever $X$ is of type $A, B, D, E, F, H$ and $I$. If $X \neq A$ then $TL(X)$ is usually referred to as the generalized Temperley–Lieb algebra.

The algebra $TL(X)$ has many properties similar to the Hecke algebra $\mathcal{H}(X)$. In particular, it is shown in [GL99] that $TL(X)$ inherits an involution from $\mathcal{H}(X)$ and that it always has a basis, indexed by the fully commutative elements of $W(X)$, with some remarkable properties, called an IC basis (see [Du96] and [GL99]). Thus, one has two families of polynomials, indexed by pairs of fully commutative elements of $W(X)$, which are analogous to the Kazhdan–Lusztig and $R$–polynomials of $\mathcal{H}(X)$. Although the Kazhdan–Lusztig and $R$–polynomials have been extensively studied (see, e.g., [Jon09a], [Gre09], [Woo09], [Jon09b], [Bre09a], [Inc07], [Del06], [Inc06], [BI06], [BCM06], [Xi06], [Bre03], [Bre04], [dC02], [War01], [Bre00], [dC99], [Pol99a], [Bre98c], [Bre97a], [dC96], [Tag95a], [Las95], [Bre02a], [Bre98a], [Boe88a], [BW01], [BW03], [Boe88b], [Bre98b], [BS00], [Bre02a], [Car94], [Cas03], [Deo85], [Deo90], [Dye93], [Dye97], [KL00], [LS81], [LT00], [Mar02a], [Pol99b], [SSV98], [Tag95b], [Zel83]) and $TL(X)$ plays an important role in several areas and has also been extensively studied (see, e.g., [Gra], [Jon87], [FG97], [Gre98], [Fan96], [GL00], [Wes95], [Fan95], [Mar88], [Lev90], [Mar90], [Deg91], [Lic92], [Tem93], [Lin96], [CJ03], [KT03], [Vin06], [Nic06], [Zha06], [Abr08], [KL08], [FK09], [Zha09]), these polynomials have not been investigated very much. Our purpose in this work is to begin the study of these polynomials from a combinatorial point of view. More precisely we obtain recursions, non–recursive formulas, symmetry properties, and expressions for the constant terms, of these polynomials. To do this, we need to study some aux-
iliary polynomials (which have no analogue in $H(X)$, and which in some sense express the relationship between $H(X)$ and $TL(X)$) which were first defined in [GL99]. Most of our results hold for every finite irreducible or affine non–branching Coxeter graph different from $\tilde{F}_4$, although some hold in full generality. Our results show that there is a close relationship between Kazhdan–Lusztig and $R$–polynomials and their analogues in $TL(X)$.

The organization of the thesis is as follows.

In Chapter 1 we fix the notation and recall the standard definitions and results needed in the sequel. In particular, we introduce the notion of Coxeter group, Bruhat order and present some basic examples. Finally we introduce the concept of pattern–avoiding element. In Chapter 2 we briefly reproduce part of theory developed in the celebrated work [KL79] concerning the Hecke algebra associated to a Coxeter group. In Chapter 3 we define the generalized Temperley–Lieb algebra $TL(X)$ associated to a Coxeter group $W(X)$ and introduce the $D$–polynomials, which are a central tool in the study of the combinatorial properties of the Temperley–Lieb algebra. Then we define two other families of polynomials, namely $\{a_{x,w}\}$ and $\{L_{x,w}\}$, that arise naturally in $TL(X)$. In fact, they are the analogous of the well–known $R$–polynomials and Kazhdan–Lusztig polynomials of $H(X)$. Both these polynomials can be studied in terms of $D$–polynomials. In Chapter 4 we prove a recurrence relation for the $D$–polynomials, which holds in general, and we derive some closed formulas in type $A$. Next, we prove our main result on $D$–polynomials, which holds for every finite irreducible or affine non–branching Coxeter graph $X$, except $\tilde{F}_4$. We do the same for $\{a_{x,w}\}$ and $\{L_{x,w}\}$. For each family of polynomials we give recursive formulas. The rest of the chapter deals with lots of consequences of the main result for these polynomials, including symmetry properties, and some combinatorial non–recursive formulas.

Acknowledgements

I am grateful to my Ph.D. advisor, Francesco Brenti. He introduced me to algebraic combinatorics. I wish to thank him for many fruitful and interesting discussions on research topics. Without his constant encouragement and generous guidance this thesis would not exist.
Chapter 1

Coxeter groups

1.1 Notation

We collect here some notation that is adhered to throughout the book.

\[ \mathbb{P} \quad \text{the positive integers} \]
\[ \mathbb{N} \quad \text{the non–negative integers} \]
\[ \mathbb{Z} \quad \text{the integers} \]
\[ [n] \quad \text{the set \{1, 2, \ldots, n\}} \]
\[ [\pm n] \quad \text{the set \{-1, -2, \ldots, n\}} \]
\[ |A| \quad \text{the cardinality of a set } A \]
\[ R[q] \quad \text{ring of polynomials with coefficients in } R \]
\[ p_i \quad \text{the coefficient of } q^i \text{ in } P \]
\[ \delta_{i,j} \quad \text{the Kronecker delta: } \delta_{i,j} \overset{\text{def}}{=} \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{if } i \neq j. \end{cases} \]

We write \( \overset{\text{def}}{=} \) if we are defining the left hand side by the right hand side. The symbol \( \square \) denotes the end of a proof or an example. Moreover, we put a \( \square \) at the end of the statement of a result if the result is obvious at that stage of reading.
1.2 Definition of Coxeter group

Let $S$ be a finite set and let $m : S \times S \rightarrow \mathbb{P} \cup \{\infty\}$ such that

1. $m(s_i, s_j) = m(s_j, s_i)$, for every $s_i, s_j \in S$;
2. $m(s_i, s_j) \geq 2$ if $i \neq j$;
3. $m(s_i, s_i) = 1$, for every $s_i \in S$.

Then $m$ is called a Coxeter matrix. Equivalently, $m$ can be represented by a Coxeter graph whose node set is $S$ and whose edges are the unordered pairs \{$(s_i, s_j)$\} such that $m(s_i, s_j) \geq 3$. The edges with $m(s_i, s_j) \geq 4$ are labeled by the number $m(s_i, s_j)$.

**Example 1.2.1.** Let $S = \{s_1, s_2, s_3\}$ and denote by $m$ the Coxeter matrix

$$
\begin{bmatrix}
1 & 4 & 2 \\
4 & 1 & 3 \\
2 & 3 & 1
\end{bmatrix}
$$

Then, $m$ corresponds to the Coxeter graph $B_3$ (cf. Appendix A).

![Figure 1.1: Coxeter graph B_3.](image)

A Coxeter system is a pair $(W, S)$ where $W$ is a group with set of generators $S = \{s_1, \cdots, s_n\}$ subject to the relations

$$(s_is_j)^{m(s_i, s_j)} = e$$

where $e$ denotes the identity element of $W$. Equivalently, $W$ can be viewed as the quotient $F_S/N$ where $F_S$ is the free group generated by $S$ and $N$ is the normal subgroup generated by $\{(s_is_j)^{m(s_i, s_j)} : s_i, s_j \in S, m(s_i, s_j) < \infty\}$. The cardinality of $S$ is called the rank of $(W, S)$.  

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In the previous example, the group $W$ determined by the Coxeter matrix $m$ has the presentation

$$\begin{align*}
s_1^2 &= s_2^2 = s_3^2 = e, &\leftrightarrow & m(s_i, s_i) = 1 \\
s_1s_2s_1s_2 &= s_2s_1s_2s_1, &\leftrightarrow & m(s_1, s_2) = 4 \\
s_3s_2s_3 &= s_2s_3s_2, &\leftrightarrow & m(s_2, s_3) = 3 \\
s_1s_3 &= s_3s_1, &\leftrightarrow & m(s_1, s_3) = 2
\end{align*}$$

There exists a bijection between Coxeter system $(W, S)$ and the Coxeter graph $X$ (or, equivalently, the Coxeter matrix $m$) associated to $W$ (see, e.g., [BB05, Theorem 1.1.2]). Therefore we say that $(W, S)$ is of type $X$. A Coxeter system is said to be finite if $|W| < \infty$ and irreducible if its Coxeter graph is connected. In Appendix A we list all the finite irreducible and affine Coxeter systems. A standard reference for these results is [Hum90].

### 1.3 Length function and Bruhat order

Let $(W, S)$ be a Coxeter system. Each element $v \in W$ can be written as product of generators $s_i \in S$. We denote by $\ell(v)$ the minimal integer $k$ such that $v$ can be written as a product of $k$ generators. If $v = s_{i_1} \cdots s_{i_k}$ and $\ell(v) = k$ then $k$ is called the length of $v$ and $s_{i_1} \cdots s_{i_k}$ is called a reduced expression of $v$. We list some basic properties on the length function (see, e.g., [BB05, Proposition 1.4.2]).

**Proposition 1.3.1.** Let $u, v \in W$. Then

(i) $\ell(e) \overset{\text{def}}{=} 0$;

(ii) $\ell(u) = 1 \iff u \in S$;

(iii) $\ell(u) = \ell(u^{-1})$;

(iv) $\ell(uv) \equiv \ell(u) + \ell(v) \pmod{2}$;

(v) $\ell(us) = \ell(u) \pm 1$ for all $s \in S$.
Denote by $T$ the set $\{vsv^{-1}, v \in W, s \in S\}$. An element $t \in T$ is called a reflection. Write $u \rightarrow v$ if there exists $t \in T$ such that $ut = v$, with $\ell(u) < \ell(v)$. The Bruhat graph of $(W,S)$ is the directed graph whose nodes are the elements of $W$. An ordered pair $(u,v)$ is an edge if and only if $u \rightarrow v$. It is possible to define a partial order relation on $W$.

**Definition 1.3.2.** Let $u, v \in W$. Then $u \leq v$ in the Bruhat order if there exist $u_i \in W$ such that

$$u = u_0 \rightarrow u_1 \rightarrow \cdots \rightarrow u_k = v.$$ 

Observe that the Bruhat order relation is the transitive closure of $\rightarrow$. The set $\{u \leq y \leq v : y \in W\}$ is customarily denoted by $[u,v]$.

**Example 1.3.3.** Denote by $A_2$ the Coxeter graph with node set $S = \{s_1, s_2\}$ and unlabelled edge $\{s_1, s_2\}$. Then the group $W(A_2)$ has the diagram shown in Figure 1.2 under the Bruhat order. To obtain the Bruhat graph of $A_2$,

![Figure 1.2: Bruhat order of $A_2$.](image)

direct all edges of Figure 1.2 upward and add the edge $e \rightarrow s_1s_2s_1.$

The following fundamental property characterizes the Bruhat order relation (see, e.g., [BB05, Theorem 2.2.2]).

**Proposition 1.3.4 (Subword Property).** Let $v = s_1 \cdots s_r$ be a reduced expression. Then $u \leq v$ if and only if there exists a reduced expression $s_{i_1} \cdots s_{i_k}$ of $u$ such that $1 \leq i_1 \leq \cdots \leq i_k \leq r$. 

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Another property which characterizes the Bruhat order is the following, which will be very useful in the sequel (see, e.g., [BB05, Proposition 2.2.7]).

**Lemma 1.3.5 (Lifting Property).** Let $u, v \in W$ be such that $u < v$ and suppose $s \in S$ such that $vs < v$ and $us > u$. Then $u \leq vs$ and $us \leq v$.

The Bruhat order relation gives $W$ the structure of graded partially ordered set, with the length function $\ell$ as rank function (see, e.g., [BB05 §2.2]).

A proof of the next result can be found in [BB05 Corollary 2.2.5 and Proposition 2.3.4].

**Proposition 1.3.6.** Let $W$ be a finite Coxeter group with maximum element $w_0$. Then the following are equivalent:

(i) $u \leq v$;

(ii) $u^{-1} \leq v^{-1}$;

(iii) $w_0uw_0 \leq w_0vw_0$.

In other words, Proposition 1.3.6 states that the mappings $\phi : u \mapsto u^{-1}$ and $\psi : u \mapsto w_0uw_0$ are automorphism of the Bruhat order.

1.4 The word property

Let $(W, S)$ be a Coxeter system with associated Coxeter matrix $[m(s_i, s_j)]$. Define

$$\alpha_{s_i, s_j} \overset{\text{def}}{=} \underbrace{s_is_js_is_j \cdots}_{m(s_i, s_j) \text{ factors}}.$$
Let \( u, v \in W \). We say that \( u \) and \( v \) are linked by a braid–move if there exist \( s_i, s_j \in S \) such that \( u \) can be obtained from \( v \) by substituting \( \alpha_{s_i,s_j} \) for \( \alpha_{s_j,s_i} \). Moreover we say that \( u \) and \( v \) are linked by a nil–move if \( u \) can be obtained from \( v \) by deleting a factor of the form \( s_i s_i \). In both cases we write \( u \sim v \). The problem of deciding whether two expressions in the alphabet \( S \) represents the same element in \( W \) is not trivial at all. A complete answer to this question is given by the following general result (see, e.g., [BB05, Theorem 3.3.1]).

**Theorem 1.4.1 (Word Property).** Let \( v \in W \). Then any reduced expression for \( v \) can be obtained from any other by applying a finite sequence of braid–moves and nil–moves.

**Corollary 1.4.2.** Let \( s_1 \cdots s_r \) be an expression of \( v \in W \). Then any reduced expression for \( v \) can be obtained from \( s_1 \cdots s_r \) by applying a finite sequence of braid–moves and nil–moves.

**Example 1.4.3.** Let us consider the Coxeter graph drawn in Example 1.2.1. Then, we have nil–relations

\[
s_1^2 = s_2^2 = s_3^2 = e
\]

and braid–relations

\[
s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1, \quad s_3 s_2 s_3 = s_2 s_3 s_2, \quad s_1 s_3 = s_3 s_1.
\]

Hence, a reduced expression for \( v = s_2 s_3 s_2 s_1 s_2 s_3 s_2 s_1 \) can be obtained by a sequence of three nil-moves and one braid-move, as shown below:

\[
s_2 s_3 s_2 s_1 s_3 s_2 s_1 \sim s_2 s_3 s_2 s_1 s_3 s_2 s_1 \sim s_3 s_2 s_3 s_2 s_1 \sim s_3 s_2 s_1 \sim s_3 s_1.
\]

\( \square \)

### 1.5 The symmetric group

For a proof of the results given in this section we refer to [BB05, §1.5].

The symmetric group \( S_n \) is the group of bijections of \([n]\) to itself. An element
\( \tau \in S_n \) is called a permutation. We consider different standard notations for a permutation \( \tau \). The first one is the matrix notation
\[
\begin{pmatrix}
1 & 2 & 3 & \cdots & n \\
\tau_1 & \tau_2 & \tau_3 & \cdots & \tau_n
\end{pmatrix}
\]
and it means that \( \tau : i \mapsto \tau_i \). By taking only the second row of the above matrix we get the one-line notation and we write \( \tau = \tau_1 \cdots \tau_n \). Another notation is the disjoint-cycles notation. For instance, if \( \tau = 432516 \) then we write \( \tau = (1, 4, 5)(2, 3) \) and omit the 1-cycles of \( \tau \).

The product \( \sigma \tau \) is defined as the function composition \( \sigma \circ \tau \). For example, \((1, 3, 2)(3, 2) = (1, 3)\).

Consider as set of generators for \( S_n \) the set of the adjacent transpositions \( S = \{(i, i+1), i \in [n-1]\} \), and set \( s_i \overset{\text{def}}{=} (i, i+1), \) for every \( i \in [n-1] \). Denote by \( A_{n-1} \) the Coxeter graph having nodes \( \{s_1, \cdots, s_{n-1}\} \) and unlabelled edges \( \{s_i, s_{i+1}\} \), for all \( i \in [n-1] \) (cf. Appendix [A]).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{image}
\caption{Coxeter graph \( A_{n-1} \).}
\end{figure}

**Proposition 1.5.1.** \((S_n, S)\) is a Coxeter system of type \( A_{n-1} \).

In fact, it is straightforward to verify that the generators \( s_1, s_2, \cdots, s_{n-1} \) satisfy the Coxeter relations
\[
\begin{cases}
  s_i^2 = e, \\
  s_isjsi = s_jsis_j \quad \text{if } |i-j| = 1, \\
  sisj = sjsi \quad \text{if } |i-j| \geq 2.
\end{cases}
\]

Let \( \tau \in S_n \) and define
\[
\text{inv}(\tau) \overset{\text{def}}{=} |\{(i, j) \in [n]^2 : i < j, \tau_i > \tau_j\}|,
\]
called the number of inversions of \( \tau \). The length and the number of inversions of a permutation in the symmetric group are closely related, as explained by the following result.

**Proposition 1.5.2.** Let \( \tau \in S_n \). Then \( \ell(\tau) = \text{inv}(\tau) \).

Hence, the longest element, with respect to the length function, of \( S_n \) is \( w_0 = nn - 1 \cdots 321 \).
1.6 The hyperoctahedral group

For a proof of the results given in this section we refer to [BB05 §8.1].

The hyperoctahedral group \( S_n^B \) is the group of bijections \( \sigma \) of \([\pm n]\) to itself such that \( \sigma(-i) = -\sigma(i) \), for all \( i \in [n] \). An element in \( S_n^B \) is called a signed permutation. We consider different standard notations for an element \( \sigma \in S_n^B \). The first one is the window notation: 
\[
\sigma = [\sigma_1, \sigma_2, \cdots, \sigma_n]
\]
means that \( \sigma(i) = \sigma_i \) for every \( i \in [n] \). A more compact notation is often used: we write \( \sigma = |\sigma_1| |\sigma_2| \cdots |\sigma_n| \) and put bar over an element with a negative sign. For instance, \( 35412 \) corresponds to the signed permutation \([3, -5, 4, 1, -2]\). Another notation is the disjoint–cycles notation. For instance, if \( \tau = [4, -3, -2, 5, 1, 6] \) then we write \( \tau = (1, 4)(5)(2, -3) \), and we omit the 1–cycles of \( \tau \). The product \( \sigma \tau \) is defined as the function composition \( \sigma \circ \tau \). Consider as set of generators for \( S_n^B \) the set 
\[
S = \{ (1, -1) \} \cup \{ (i, i+1)(-i, -i-1), i \in [n-1] \},
\]
and set \( s_0 \) \( \overset{\text{def}}{=} \) \( (1, -1) \), \( s_i \) \( \overset{\text{def}}{=} \) \( (i, i+1)(-i, -i-1) \), for every \( i \in [n-1] \). Denote by \( B_n \) the Coxeter graph having nodes \( \{ s_0, \cdots, s_{n-1} \} \), unlabelled edges \( \{ s_i, s_{i+1} \} \) for all \( i \in [n-2] \) and the edge \( \{ s_0, s_1 \} \) labelled by 4 (cf. Appendix [A]).

![Figure 1.5: Coxeter graph \( B_n \), \( n \geq 2 \).](image)

**Proposition 1.6.1.** \((S_n^B, S)\) is a Coxeter system of type \( B_n \).

In fact, it can be easily checked that the generators \( s_1, s_2, \cdots, s_{n-1} \) satisfy the Coxeter relations
\[
\begin{align*}
s_i^2 &= e, \\
s_0s_1s_0s_1 &= s_1s_0s_1s_0, \\
s_is_js_i &= s_js_is_j & \text{if } |i - j| = 1 \text{ and } i, j \in [n-1], \\
s_is_j &= s_js_i & \text{if } |i - j| \geq 2.
\end{align*}
\]

Let \( \sigma \in S_n^B \) and define
\[
inv(\sigma) \overset{\text{def}}{=} |\{ (i, j) \in [\pm n]^2 : i < j, \sigma_i > \sigma_j \}|,
\]

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called the number of inversions of $\sigma$. The length function $\ell_B$ of $S_n^B$ may be expressed in many ways. A simple combinatorial description of $\ell_B$ is given in [Bre94b, Proposition 3.1].

**Proposition 1.6.2.** Let $\sigma \in S_n^B$. Then
\[
\ell_B(\sigma) = \text{inv}(\sigma) - \sum_{j \in [n]: \sigma(j) < 0} \sigma(j).
\]
Hence, the longest element of $S_n^B$ is $w_0 = 1\, 2\, \cdots \, n$.

## 1.7 Restricted permutations

Two numerical sequences $\alpha = (\alpha_1, \cdots, \alpha_m)$ and $\beta = (\beta_1, \cdots, \beta_m)$ are said to be order–isomorphic if $\alpha_i < \alpha_j \iff \beta_i < \beta_j$, for all $i, j \in [m]$. Let $\pi \in S_n$ and $\tau \in S_k$. An occurrence of $\tau$ in $\pi$ is a subsequence $1 \leq i_1 \leq \cdots \leq i_k \leq n$ such that $(\pi_{i_1}, \cdots, \pi_{i_k})$ is order–isomorphic to $(\tau_1, \cdots, \tau_k)$. In this context $\tau$ is usually called a pattern and we say that $\pi$ is $\tau$–avoiding if there is no occurrence of $\tau$ in $\pi$. Denote by $S_n(\tau)$ the set $\{\sigma \in S_n : \sigma$ is $\tau$–avoiding$\}$. For an arbitrary finite collection of patterns $T$, we say that $\pi$ avoids $T$ if $\pi$ avoids any $\tau \in T$ and denote by $S_n(T)$ the corresponding subset of $S_n$.

Throughout this work the 321–avoiding permutations of $S_n$ will be of particular interest.

**Definition 1.7.1.** An element $w \in W$ is fully commutative if any reduced expression for $w$ can be obtained from any other by applying braid–moves that involve only commuting generators. We let
\[
W_c \overset{\text{def}}{=} \{ w \in W : w$ is a fully commutative element$\}.
\]

We will denote by $[x, w]_c$ the set $\{y \in [x, w] : y \in W_c\}$. As we have seen in Section 1.5 if $X = A_{n-1}$ then $W = W(X) \simeq S_n$. In this case $W_c(A_{n-1})$ may be described as the set of elements of $W(A_{n-1})$ all of whose reduced expressions avoid substrings of the form $s_is_{i\pm 1}s_i$, for all $s_i \in S$ (see [Ste96] Proposition 1.1]). The notion of fully commutative element in type $A$ can be reformulated in terms of pattern–avoidance. A proof of the next result can be found in [BJS93 Theorem 2.1]).
Proposition 1.7.2. The sets $S_n(321)$ and $W_c(A_{n-1})$ coincide. Moreover $|W_c(A_{n-1})| = C_n$, where $C_n \overset{\text{def}}{=} \frac{1}{n+1} \binom{2n}{n}$ denotes the $n$–th Catalan number.

A similar result holds for $W_c(B_n)$. Since the general definition of signed pattern–avoidance in type $B$ is quite complicated, we prefer to explain it by examples. For instance, we say that an element $w = w_1w_2\cdots w_n \in S^n_B$ avoids the pattern $231$ if there is no triple $i < j < k$ such that $-w_j > w_i > -w_k > 0$. For a proof of the following result we refer to [Ste97, Theorem 5.1 and Proposition 5.9].

Proposition 1.7.3. The sets $S^n_B(\{12;321;231;\overline{321};\overline{231}\})$ and $W_c(B_n)$ coincide. Moreover $|W_c(B_n)| = (n + 2)C_n - 1$, where $C_n \overset{\text{def}}{=} \frac{1}{n+1} \binom{2n}{n}$ denotes the $n$–th Catalan number.

Let us now look at the maps $\phi : u \mapsto u^{-1}$ and $\psi : u \mapsto w_0uw_0$ defined in Proposition 1.3.6.

Lemma 1.7.4. Let $W(X)$ be a finite Coxeter group. Then

$$u \in W_c(X) \iff \phi(u) \in W_c(X) \iff \psi(u) \in W_c(X),$$

for all $u \in W(X)$.

Proof. The maps $u \mapsto w_0uw_0$ and $u \mapsto u^{-1}$ are Bruhat order automorphisms (see [BB05] Proposition 2.3.4 and Corollary 2.3.6]). Moreover, both these maps send Coxeter generators to Coxeter generators, since $\ell(u) = \ell(u^{-1}) = \ell(w_0uw_0)$ (see [BB05] Corollary 2.3.3]).
Chapter 2

Hecke algebras

2.1 Definition of Hecke algebra

In this section we recall some basic facts about Hecke algebras $\mathcal{H}(X)$, $X$ being any Coxeter graph. A standard reference for this topic is [Hum90]. Let $W(X)$ be the Coxeter group having $X$ as Coxeter graph and $S(X)$ as set of generators. Throughout this work we will denote by $\varepsilon_w$ the constant $(-1)^{\ell(w)}$, for every $w \in W(X)$. Let $A$ be the ring of Laurent polynomials $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$. The Hecke algebra $\mathcal{H}(X)$ associated to $W(X)$ is an $A$–algebra with linear basis $\{T_w : w \in W(X)\}$. For all $w \in W(X)$ and $s \in S(X)$ the multiplication law is determined by

$$T_w T_s = \begin{cases} T_{ws} & \text{if } \ell(ws) > \ell(w), \\ qT_{ws} + (q-1)T_w & \text{if } \ell(ws) < \ell(w), \end{cases} \quad (2.1)$$

where $\ell$ denotes the usual length function of $W(X)$ (see Section 1.3). We refer to $\{T_w : w \in W(X)\}$ as the $T$–basis for $\mathcal{H}(X)$.

One easily checks that $T_s^2 = (q-1)T_s + qT_e$, being $T_e$ the identity element, and so

$$T_s^{-1} = q^{-1}(T_s - (q-1)T_e). \quad (2.2)$$
It follows that all the elements $T_w$ are invertible, since, if $w = s_1 \cdots s_r$ and $\ell(w) = r$, then $T_w = T_{s_1} \cdots T_{s_r}$.

Define a map $\iota : \mathcal{H} \to \mathcal{H}$ such that $\iota(T_w) = (T_{w^{-1}})^{-1}$, $\iota(q) = q^{-1}$ and extend by linearity.

**Proposition 2.1.1.** The map $\iota$ is a ring homomorphism of order 2 on $\mathcal{H}(X)$.

**Proof.** It is straightforward to prove that $\iota^2(T_s) = T_s$, for all $s \in S(X)$. We will show that $\iota$ is a ring homomorphism. First, we show that

$$\iota(T_s T_w) = \iota(T_s) \iota(T_w), \text{ for all } w \in W(X) \text{ and } s \in S(X). \tag{2.3}$$

There are two cases to study: if $\ell(sw) > \ell(w)$ then

$$\iota(T_s T_w) = \iota(T_sw) = (T_{w^{-1}s})^{-1} = (T_{w^{-1}1})^{-1} = (T_s)^{-1} (T_{w^{-1}})^{-1} = \iota(T_s) \iota(T_w).$$

If $\ell(sw) < \ell(w)$ then let $v \overset{\text{def}}{=} sw$, so that $w = sv$. Therefore $T_w = T_s T_v$. By (2.1), we obtain

$$\iota(T_s T_w) = \iota(q T_sw + (q - 1) T_w) = q^{-1} (T_{w^{-1}})^{-1} \cdot (q^{-1} - 1) (T_{w^{-1}})^{-1}.$$

On the other hand

$$\iota(T_s) \iota(T_w) = (T_s)^{-1} (T_{w^{-1}})^{-1} = q^{-1} (T_s + 1 - q) (T_{w^{-1}})^{-1} = q^{-1} T_s (T_{w^{-1}})^{-1} + (q^{-1} - 1) (T_{w^{-1}})^{-1}.$$

But $(T_{w^{-1}})^{-1} = T_s (T_{w^{-1}})^{-1} \iff T_{w^{-1}} = (T_{w^{-1}}) T_s$, which is obviously true by definition. Hence $\iota(T_s T_w) = \iota(T_s) \iota(T_w)$. Now we ready to prove that $\iota(T_x T_w) = \iota(T_x) \iota(T_w)$ for all $x, w \in W(X)$. Proceed by induction on $\ell(x)$. If $\ell(x) = 1$ then $x = s' \in S(X)$. If $\ell(x) > 1$ then there exists $s \in S(X)$ such that $\ell(xs) < \ell(x)$ and, by induction,

$$\iota(T_x T_w) = \iota(T_{xs} T_{sw}) = \iota(T_{xs}) \iota(T_{sw}).$$

Finally, by applying (2.3) and the induction hypothesis we get

$$\iota(T_{xs}) \iota(T_s T_w) = \iota(T_{xs}) \iota(T_s) \iota(T_w) = \iota(T_x) \iota(T_w),$$

as desired. \qed
2.2 $R$–polynomials

In Proposition 2.1.1 we stated the existence of the involution $\iota$ in $\mathcal{H}(X)$. To express the image of $T_w$ under $\iota$ as a linear combination of elements in the $T$–basis, one defines the so–called $R$–polynomials.

**Theorem 2.2.1.** Let $w \in W(X)$. Then there exists a unique family of polynomials $\{R_{x,w}\}_{x \in W(X)} \subseteq \mathbb{Z}[q]$ such that

\[
(T_w^{-1})^{-1} = \varepsilon_w q^{-\ell(w)} \sum_{x \leq w} \varepsilon_x R_{x,w} T_x, \tag{2.4}
\]

where $R_{w,w} = 1$ and $R_{x,w} = 0$ if $x \not\leq w$.

**Proof.** First, we prove the existence of the $R$–polynomials. If $w = s \in S(X)$, then the statement follows by (2.2) and so we set $R_{e,s} \overset{\text{def}}{=} q - 1$. Now proceed
by induction on \( \ell(w) \). Set \( w = sv \), so that \( \varepsilon_w = -\varepsilon_v \) and \( q^{-\ell(w)} = q^{-\ell(v)} \).

\[
(T_{w^{-1}})^{-1} = (T_s)^{-1}(T_{v^{-1}})^{-1}
\]
\[
= \frac{1}{q}(T_s - (q - 1)T_e)(\varepsilon_vq^{-\ell(v)} \sum_{y \leq v} \varepsilon_y R_{y,v}T_y)
\]
\[
= -\varepsilon_vq^{-\ell(v)-1} \left( (q - 1) \sum_{y \leq v} \varepsilon_y R_{y,v}T_y - \sum_{y \leq v, sg > y} \varepsilon_y R_{y,v}T_{sy} \right)
\]
\[
= \varepsilon_wq^{-\ell(w)} \left( (q - 1) \sum_{y \leq v} \varepsilon_y R_{y,v}T_y - \sum_{y \leq v, sg > y} \varepsilon_y R_{y,v}T_{sy} \right) +
\]
\[
- \varepsilon_wq^{-\ell(w)} \left( \sum_{y \leq v, sg > y} (q \varepsilon_y R_{y,v}T_{sy} + (q - 1)\varepsilon_y R_{y,v}T_y) \right)
\]
\[
= \varepsilon_wq^{-\ell(w)} \left( (q - 1) \sum_{y \leq v} \varepsilon_y R_{y,v}T_y - q \sum_{y \leq v, sg > y} \varepsilon_y R_{y,v}T_{sy} \right) +
\]
\[
- \varepsilon_wq^{-\ell(w)} \left( \sum_{y \leq v, sg > y} \varepsilon_y R_{y,v}T_{sy} \right)
\]
\[
= \varepsilon_wq^{-\ell(w)} \left( \sum_{x \leq v, sx > x} \varepsilon_x((q - 1)R_{x,v} + qR_{sx,v})T_x + \sum_{x \leq v, sx > x} \varepsilon_x R_{sx,v}T_x \right),
\]
where we set \( sx = y \). If \( sx < x \), then we can define \( R_{x,w} \equiv R_{sx,sw} \). Otherwise, we set \( R_{x,w} \equiv qR_{sx,sw} + (q - 1)R_{x,sw} \). In both cases, \( R_{x,w} \) satisfies the condition of the theorem.

The uniqueness of the \( R \)-polynomials is trivial. \( \Box \)

We remark that in the proof of Theorem 2.2.1 is given an inductive method to compute the \( R \)-polynomials.

**Corollary 2.2.2.** Let \( x, w \in W(X) \) such that \( x \leq w \) and \( s \in S(X) \) such that \( \ell(sw) < \ell(w) \). Then

\[
R_{x,w} = \begin{cases} 
R_{sx,sw}, & \text{if } \ell(sw) < \ell(x), \\
qR_{sx,sw} + (q - 1)R_{x,sw}, & \text{otherwise.} 
\end{cases}
\]

(2.5)
Obviously, there exists a right version of Corollary 2.2.2 with $s$ occurring on the right instead of the left.

Next, we state some basic facts about the $R$–polynomials.

**Corollary 2.2.3.** Let $x, w \in W(X)$, $x \leq w$. Then, $R_{x,w}$ is a monic polynomial of degree $\ell(w) - \ell(x)$ such that $R_{x,w}(0) = \varepsilon_x \varepsilon_w$ and $R_{x,w}(1) = \delta_{x,w}$.

**Proof.** The statement follows from (2.5), by induction on $\ell(w)$. □

**Proposition 2.2.4.** Let $x, w \in W(X)$. Then

$$R_{x,w}(q^{-1}) = \varepsilon_x \varepsilon_w q^{\ell(x) - \ell(w)} R_{x,w}(q).$$

**Proof.** We proceed by induction on $\ell(w)$. Let $\ell(w) > 0$ and let $s \in S(X)$ be such that $sw < w$. By (2.5), $sx < x$ implies $R_{x,w}(q) = R_{sx,sw}(q)$ and, by induction, we have

$$R_{sx,sw}(q^{-1}) = \varepsilon_{sx} \varepsilon_{sw} q^{\ell(sx) - \ell(sw)} R_{sx,sw}(q) = \varepsilon_x \varepsilon_w q^{\ell(x) - \ell(w)} R_{x,w}(q).$$

The case $sx > x$ implies

$$R_{x,w}(q^{-1}) = \varepsilon_x \varepsilon_w q^{\ell(x) - \ell(w)} (q R_{sx,sw}(q) + (q - 1)R_{x,sw}(q)).$$

By induction, we achieve

$$R_{x,w}(q^{-1}) = q^{-1} R_{sx,sw}(q^{-1}) + (q^{-1} - 1) R_{x,sw}(q^{-1})$$

$$= q^{-1} \varepsilon_{sx} \varepsilon_{sw} q^{\ell(sx) - \ell(sw)} R_{sx,sw}(q) + (q^{-1} - 1) \varepsilon_x \varepsilon_w q^{\ell(x) - \ell(sw)} R_{x,sw}(q)$$

$$= q^{-1} \varepsilon_x \varepsilon_w q^{\ell(x) - \ell(w)} + 2 R_{sx,sw}(q) - (1 - q) \varepsilon_x \varepsilon_w q^{\ell(x) - \ell(w)} R_{x,sw}(q),$$

$$= \varepsilon_x \varepsilon_w q^{\ell(x) - \ell(w)} (q R_{sx,sw}(q) + (q - 1)R_{x,sw}(q)),$$

as desired. □

**Proposition 2.2.5.** Let $\sigma : \mathcal{H} \to \mathcal{H}$ be such that $\sigma(T_w) = \varepsilon_w q^{-\ell(w)} T_w$ and $\sigma(q) = q^{-1}$. Then $\sigma$ is an involution and $\iota \sigma = \sigma \iota$. 17
Proof. The proof that $\sigma$ is a ring homomorphism of order 2 is similar to the one given in Proposition 2.3 for $\iota$.
Let us now prove that $\iota \sigma = \sigma \iota$. By definition, $\sigma$ coincides with $\iota$ on $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$. Moreover

$$\iota \sigma(T_w) = \iota(\varepsilon_w q^{-\ell(w)} T_w) = \varepsilon_w q^{\ell(w)} (T_{w^{-1}})^{-1} = \sum_{x \leq w} \varepsilon_x R_{x,w}(q) T_x.$$ 

On the other hand,

$$\sigma \iota(T_w) = \sigma(\varepsilon_w q^{-\ell(w)} \sum_{x \leq w} \varepsilon_x R_{x,w}(q) T_x) = \varepsilon_w q^{\ell(w)} \sum_{x \leq w} q^{-\ell(x)} R_{x,w}(q^{-1}) T_x,$$

and the statement follows by applying Proposition 2.2.4.

The next property is needed in Proposition 4.2.9.

**Proposition 2.2.6.** Let $w \in W(X)$. Then $\sum_{x \leq w} R_{x,w} = q^{\ell(w)}$.

Proof. We proceed by induction on $\ell(w)$. The case $\ell(w) \leq 1$ is trivial. Hence, suppose $\ell(w) > 1$ and $s \in S(X)$ such that $sw < w$. Then, by Corollary 2.2.2 and Lemma 1.3.5 we obtain

$$\sum_{x \leq w} R_{x,w} = \sum_{x \leq w, sx < x} R_{x,w} + \sum_{x \leq w, sx > x} R_{x,w}$$

$$= \sum_{x \leq w} R_{sx,sx} + \sum_{x \leq w} q R_{sx,sw} + (q - 1) R_{sx,sw}$$

$$= \sum_{x \leq w} R_{sx,sx} + \sum_{y \leq w} q R_{sy,sw} + (q - 1) R_{sy,sw}$$

$$= q \sum_{x \leq w} R_{sx,sx} + q \sum_{y \leq w} R_{y,sw}$$

$$= q \sum_{y \leq w} R_{y,sw} + q \sum_{y \leq w} R_{y,sw}$$

$$= q \sum_{y \leq w} R_{y,sw} = q \cdot q^\ell(sw) = q^{\ell(w)}.$$ 

□

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2.3 Kazhdan–Lusztig polynomials

In this section we prove the existence and uniqueness of an \( \iota \)-invariant basis for \( \mathcal{H}(X) \).

**Theorem 2.3.1.** There exists a unique basis \( \{ C_w : w \in W(X) \} \) for \( \mathcal{H}(X) \) such that the following properties hold:

(i) \( \iota(C_w) = C_w \),

(ii) \( C_w = \varepsilon_w q^{\ell(w) \over 2} \sum_{x \leq w} \varepsilon_x q^{-\ell(x)} P_{x,w}(q^{-1}) T_x \),

where \( \{ P_{x,w}(q) \} \subseteq \mathbb{Z}[q] \), \( P_{w,w}(q) = 1 \) and \( \deg(P_{x,w}(q)) \leq \frac{1}{2}(\ell(w) - \ell(x) - 1) \) if \( x < w \).

**Proof.** First, we prove the existence of \( C_w \) by induction on \( \ell(w) \). Assume \( w \neq e \) and suppose that \( C_x \) has already been constructed, for every \( x < w \). Obviously, \( C_e = T_e \). Let \( s \in S(X) \) be such that \( sw < w \) and set \( w = sv \). We define

\[
C_w \overset{\text{def}}{=} C_s C_v - \sum_{sz < z} \mu(z,v) C_z,
\]

(2.6)

where \( \mu(z,v) \overset{\text{def}}{=} [q^{\ell(v) - \ell(z) - 1 \over 2}] P_{z,v}(q) \). Observe that (2.6) implies \( \iota(C_w) = C_w \).

In fact,

\[
C_s = q^{\frac{1}{2}} T_s - q^{\frac{1}{2}} T_e.
\]

On the other hand,

\[
\iota(C_s) = \iota(q^{\frac{1}{2}} T_s - q^{\frac{1}{2}} T_e) = q^{\frac{1}{2}} (T_s)^{-1} - q^{\frac{1}{2}} T_e = q^{\frac{1}{2}} (q^{-1} (T_s - (q - 1) T_e)) - q^{\frac{1}{2}} T_e = q^{\frac{1}{2}} T_s - q^{\frac{1}{2}} T_e
\]

and the invariance follows by induction on \( \ell(w) \). Next, extract and equate the coefficient of \( T_x \) on both sides of (2.6). A careful analysis of the left hand side of (2.6) shows that the coefficient of \( T_x \) in \( C_s C_v \) is

\[
[T_x]\{ C_s C_v \} = \begin{cases} 
\varepsilon_x \varepsilon_w q^{-\ell(x)} q^{\ell(w) \over 2} (P_{sx,w}(q^{-1}) + q^{-1} P_{x,w}(q^{-1})), & \text{if } \ell(sx) < \ell(x), \\
\varepsilon_x \varepsilon_w q^{-\ell(x)} q^{\ell(w) \over 2} (q^{-1} P_{sx,w}(q^{-1}) + P_{x,w}(q^{-1})), & \text{otherwise},
\end{cases}
\]
and that

\[ [T_x] \left( \sum_{sz < z} \mu(z, v) C_z \right) = \varepsilon_x \varepsilon_w q^{-\ell(x)} q^{\ell(w)} \sum_{sz < z} \mu(z, v) q^{-\ell(z)} q^{-\ell(z)} P_{x,z}(q^{-1}). \]

Combining these information we can express a basis element in the form

\[ C_w = \varepsilon_w q^{\ell(w)} \sum_{x \leq w} \varepsilon_x q^{-\ell(x)} P_{x,w}(q^{-1}) T_x, \quad (2.7) \]

where we set

\[ P_{x,w}(q) \overset{\text{def}}{=} q^{1-c} P_{sx,sw}(q) + q^c P_{x,sw}(q) - \sum_{\{z : sz < z\}} q^{\ell(z)} \mu(z, sw) P_{x,z}(q), \quad (2.8) \]

with \( c = 1 \) if \( sx < x \) and \( c = 0 \) otherwise. This is routine to check that the polynomials defined by (2.8) satisfy the degree bound stated in the theorem.

Now we deal with the uniqueness part, assuming the existence and the invariance of \( C_w \). The uniqueness of the element in (2.7) is equivalent to saying that there is a unique choice for polynomials \( P_{x,w} \). We proceed by induction on \( \ell(w) - \ell(x) \), where \( w \in W(X) \) is fixed. Assume that the \( P_{y,w} \) can be chosen in a unique way, for all \( x < y \leq w \). We will show that \( P_{x,w} \) is uniquely determined. Combining (2.7) with (2.4), we achieve

\[ \ell(C_w) = \varepsilon_w q^{\ell(w)} \sum_{y \leq w} \varepsilon_y q^{\ell(y)} P_{y,w}(q)(T_{y-1})^{-1} \]

\[ = \varepsilon_w q^{\ell(w)} \sum_{y \leq w} \varepsilon_y q^{\ell(y)} P_{y,w}(q) \left( \varepsilon_y q^{-\ell(y)} \sum_{x \leq y} \varepsilon_x R_{x,y} T_x \right) \]

\[ = \varepsilon_w q^{\ell(w)} \sum_{x \leq w} \left( \sum_{x \leq y \leq w} \varepsilon_x R_{x,y} P_{y,w} \right) T_x. \quad (2.9) \]

On the other hand, \( \ell(C_w) = C_w \). Therefore, equating the coefficient of \( T_x \) in (2.9) with the one in (2.7) we obtain

\[ \varepsilon_x \varepsilon_w q^{-\ell(x)} q^{\ell(w)} P_{x,w}(q^{-1}) = \varepsilon_w q^{-\ell(w)} \sum_{x \leq y \leq w} \varepsilon_x R_{x,y} P_{y,w}. \quad (2.10) \]

Multiplying both sides of (2.10) by \( q^{\ell(x)} \) and moving the term for \( y = x \) to the left, we get

\[ q^{-\ell(x)} q^{\ell(w)} P_{x,w}(q^{-1}) - q^{\ell(x)} q^{\ell(w)} P_{x,w}(q) = q^{-\ell(x)} q^{\ell(w)} \sum_{x < y \leq w} R_{x,y} P_{y,w}. \quad (2.11) \]
The bound on the degree of $P_{x,w}$ implies that no cancellation occurs on the left hand side of (2.11). Hence, we conclude that the polynomial $P_{x,w}$ satisfying (2.11) is unique.

The polynomials $\{P_{x,w}\}_{x,w \in W(X)}$ are the so–called Kazhdan–Lusztig polynomials of $W(X)$. In [Hum90, §7.9] it is shown that one can substitute the basis $\{C_w : w \in W(X)\}$ with the equivalent basis $\{C'_w : w \in W(X)\}$, where

$$C'_w \overset{\text{def}}{=} q^{-\ell(w)} \sum_{x \leq w} P_{x,w}(q)T_x. \quad (2.12)$$

We will refer to the latter basis as the Kazhdan–Lusztig basis for $\mathcal{H}(X)$. Observe that $C'_w = \varepsilon_w \sigma(C_w)$, so that, by Proposition 2.2.5, we achieve

$$\iota(C'_w) = \varepsilon_w \iota(C_w) = \varepsilon_w \sigma(C_w) = \varepsilon_w \sigma(C_w) = C'_w. \quad (2.13)$$

The Kazhdan–Lusztig polynomials can be computed inductively by appealing to the following result.

**Proposition 2.3.2.** Let $x, w \in W(X)$ be such that $x \leq w$. Then

$$q^{\ell(w) - \ell(x)} P_{x,w}(q^{-1}) = \sum_{y \in [x,w]} R_{x,y}(q) P_{y,w}(q).$$

**Proof.**

$$\iota(C'_w) = q^{-\ell(w)} \sum_{x \leq w} P_{x,w}(q^{-1}) (T_x^{-1})^{-1} = q^{-\ell(w)} \sum_{x \leq w} P_{x,w}(q^{-1}) \left( \varepsilon_x q^{-\ell(x)} \sum_{y \leq x} \varepsilon_y R_{y,x}(q) T_y \right) = q^{\ell(w)} \sum_{y \leq w} \left( \sum_{x \in [y,w]} \varepsilon_x q^{-\ell(x)} R_{y,x}(q) P_{x,w}(q^{-1}) \right) T_y. \quad (2.14)$$

The relation (2.13) implies that the coefficient of $T_y$ in (2.14) and that of $T_y$ in (2.12) are the same, that is

$$q^{-\ell(w)} P_{y,w}(q) = \sum_{x \in [y,w]} \varepsilon_x q^{-\ell(x)} R_{y,x}(q) P_{x,w}(q^{-1}).$$
Therefore,
\[
\iota(q^{-\ell(w)} P_{y,w}(q)) = q^{\ell(w)} P_{y,w}(q^{-1}) = \sum_{x \in [y,w]} \varepsilon_x q^{\ell(x)} R_{y,x}(q^{-1}) P_{x,w}(q),
\]
and the statement follows by applying Proposition 2.2.4.

It is a routine exercise to prove the following properties (see, e.g., [BB05, §5, Exercise 3 and Exercise 7(a)]).

**Lemma 2.3.3.** Let \( x, w \in W(X) \) be such that \( x \leq w \). If \( \ell(w) - \ell(x) \leq 2 \) then

(i) \( R_{x,w} = (q - 1)^{\ell(w) - \ell(x)} \);  

(ii) \( P_{x,w} = 1 \).

**Definition 2.3.4.** Let \( x, w \in W(X) \) such that \( x < w \). Define \( \mu(x, w) \in \mathbb{Z} \) to be the top coefficient of \( P_{x,w} \), namely
\[
\mu(x, w) \overset{\text{def}}{=} [q^{\ell(w) - \ell(x) - 1}] P_{x,w}.
\]

If \( \mu(x, w) \neq 0 \) then we write \( x \prec w \).

In the symmetric group \( S_4 \) there are exactly two pairs of element \((x, w)\) such that \( x \prec w \). They are \([(1, 3, 2, 4), [3, 4, 1, 2]] \) and \([(2, 1, 4, 3), [4, 2, 3, 1]] \) (see [Hum90, §7.12]). In both cases, by means of Proposition 2.3.2 one easily checks that \( P_{x,w} = q + 1 \).

The product of two Kazhdan–Lusztig basis elements may be computed by means of the following well-known formula, which is implicit in the proof of Theorem 2.3.1

**Proposition 2.3.5.** Let \( s, w \in W(X) \), with \( s \in S(X) \). Then
\[
C'_s C'_w = \begin{cases} 
C'_{sw} + \sum_{x \prec w} \mu(x, w) C'_x & \text{if } \ell(sw) > \ell(w); \\
(q^{\frac{1}{2}} + q^{-\frac{1}{2}}) C'_w & \text{otherwise}.
\end{cases}
\]

\[\Box\]
Moreover, the proof of Theorem 2.3.1 results in the following interesting relation.

**Theorem 2.3.6.** Let $x, w \in W(X)$ be such that $x \leq w$, and let $s \in S(X)$ be such that $sw < w$. Then,

$$P_{x,w} = q^{1-c}P_{sx,sw} + q^cP_{x,sw} - \sum_{\{z: sz < z\}} q^{\ell(w)-\ell(z)} \mu(z, sw)P_{x,z}, \quad (2.15)$$

with $c = 1$ if $sx < x$ and $c = 0$ otherwise. $$\square$$

An important consequence of Theorem 2.3.6 follows.

**Corollary 2.3.7.** Let $x, w \in W(X)$ be such that $x \leq w$, and let $s \in S(X)$ be such that $sw < w$. Then $P_{x,w} = P_{sx,w}$.

**Proof.** We proceed by induction on $\ell(w)$. If $\ell(w) \leq 1$ then $P_{x,w} = 1$ by Lemma 2.3.3. Let $\ell(w) > 1$ and suppose that $sx < x$. Then, by (2.15), we get

$$P_{sx,w} = qP_{x,sw} + P_{sx,sw} - \sum_{\{z: sz < z\}} q^{\ell(w)-\ell(z)} \mu(z, sw)P_{sx,z}$$

$$= qP_{x,sw} + P_{sx,sw} - \sum_{\{z: sz < z\}} q^{\ell(w)-\ell(z)} \mu(z, sw)P_{x,z}$$

$$= P_{x,w}$$

since $sz < z \leq sw$. The case $x < sx$ follows similarly. $$\square$$

The following is needed in Proposition 4.1.9 (see [BB05, §5, Exercise 17]).

**Proposition 2.3.8.** Let $x, w \in W(X)$. Then

$$\sum_{x \leq w} \varepsilon_x P_{x,w} = \delta_{e,w}. $$
**Proof.** The case \( w = e \) is trivial. Suppose \( w \neq e \). Let \( s \in S(X) \) such that \( sw < w \). Combining Proposition 2.3.7 and Lemma 1.3.5, we get

\[
\sum_{x \leq w} \varepsilon_x P_{x,w} = \sum_{x \leq w, sx < w} \varepsilon_x P_{x,w} + \sum_{x \leq w, sx > x} \varepsilon_x P_{x,w} = \sum_{x \leq w, sx < x} \varepsilon_x P_{x,w} + \sum_{y \leq w, sy < y} \varepsilon_y P_{y,w} = 0.
\]
Chapter 3

The generalized Temperley–Lieb algebra $TL(X)$

3.1 Definition of generalized Temperley–Lieb algebra

Throughout this section we will denote by $X$ an arbitrary Coxeter graph. Let $s_i, s_j \in S(X)$ and denote by $\langle s_i, s_j \rangle$ the subgroup of $W(X)$ generated by $s_i$ and $s_j$. Following [Gra], we consider the two–sided ideal $J(X)$ generated by all elements of $\mathcal{H}(X)$ of the form

$$
\sum_{w \in \langle s_i, s_j \rangle} T_w,
$$

where $(s_i, s_j)$ runs over all pairs of non–commuting elements in $S(X)$ such that the order of $s_i s_j$ is finite.

**Definition 3.1.1.** The generalized Temperley–Lieb algebra is

$$
TL(X) \overset{\text{def}}{=} \mathcal{H}(X)/J(X).
$$
When \( X \) is of type \( A \), we refer to \( TL(X) \) as the Temperley–Lieb algebra.
If we project the \( T \)-basis of \( \mathcal{H}(X) \) to the quotient \( \mathcal{H}(X)/J(X) \) we obtain a basis for \( TL(X) \). Let \( t_w = \sigma(T_w) \), where \( \sigma: \mathcal{H} \to \mathcal{H}/J \) is the canonical projection. A proof of the following can be found in [GL99].

**Theorem 3.1.2.** The Temperley–Lieb algebra admits an \( A \)-basis of the form \( \{ t_w : w \in W_c(X) \} \).

We call \( \{ t_w : w \in W_c(X) \} \) the \( t \)-basis of \( TL(X) \). By (2.1), it satisfies

\[
t_w t_s = \begin{cases} 
  t_{ws} & \text{if } \ell(ws) > \ell(w), \\
  qt_{ws} + (q - 1)t_w & \text{if } \ell(ws) < \ell(w).
\end{cases}
\] (3.1)

Observe that it may be the case that \( ws \notin W_c(X) \). We will see in Proposition 3.2.1 how to handle this case.

The next result appears in [GL99, Lemma 1.4].

**Lemma 3.1.3.** The involution \( \iota \) fixes the ideal \( J(X) \).

*Proof.* Let \( w \in \langle s_i, s_j \rangle \), with \( s_i, s_j \in S(X) \) such that \( 2 < m(s_i, s_j) < +\infty \). Therefore \( w^{-1} \leq w_0(s_i, s_j) \), where \( w_0(s_i, s_j) \) denotes the longest element in \( \langle s_i, s_j \rangle \). Hence, there exists \( u \in \langle s_i, s_j \rangle \) such that \( w_0(s_i, s_j) = w^{-1}u \) and \( \ell(w_0(s_i, s_j)) = \ell(w^{-1}) + \ell(u) \). Observe that the map \( w^{-1} \mapsto u \) is a bijection of \( \langle s_i, s_j \rangle \). We have \( T_{w_0(s_i, s_j)} = T_{w^{-1}}T_u \), that is \( (T_{w^{-1}})^{-1} = T_u(T_{w_0(s_i, s_j)})^{-1} \). Then

\[
\iota \left( \sum_{w \in \langle s_i, s_j \rangle} T_w \right) = \sum_{w \in \langle s_i, s_j \rangle} (T_{w^{-1}})^{-1} = \left( \sum_{w \in \langle s_i, s_j \rangle} T_u \right) (T_{w_0(s_i, s_j)})^{-1} \in J(X).
\]

Therefore, we conclude that \( \iota(J(X)) \subseteq J(X) \) and the statement follows. \( \square \)

From Lemma 3.1.3 it follows that \( \iota \) induces an involution on \( TL(X) \), which we still denote by \( \iota \), if there is no danger of confusion.

**Proposition 3.1.4.** The map \( \iota \) is a ring homomorphism of order 2 such that \( \iota(t_w) = (t_{w^{-1}})^{-1} \) and \( \iota(q) = q^{-1} \). \( \square \)
3.2 The polynomials $D_{x,w}$

For the theory developed in this section we refer to [GL99] and to [Bre].

**Proposition 3.2.1.** Let $w \in W(X)$. Then there exists a unique family of polynomials $\{D_{x,w}\}_{x \in W_c(X)} \subseteq \mathbb{Z}[q]$ such that

$$t_w = \sum_{x \in W_c(X), x \leq w} D_{x,w} t_x,$$

where $D_{w,w} = 1$ if $w \in W_c(X)$. Furthermore, $D_{x,w} = 0$ if $x \not\leq w$.

**Proof.** We proceed by induction on $\ell(w)$. First, observe that if $x, w \in W_c(X)$ then the statement is trivially true and this covers the case $\ell(w) \leq 1$. Now, denote by $w_0(s_i, s_j)$ the longest element in $\langle s_i, s_j \rangle$, for all $s_i, s_j \in S(X)$ such that $m(s_i, s_j) < \infty$. Let $\ell(w) \geq 2$ such that $w \not\in W_c(X)$. Then there exist $s_i, s_j \in S(X)$ and $u, v \in W(X)$ such that $m(s_i, s_j) < \infty$ and $w = uw_0(s_i, s_j)v$, with $\ell(w) = \ell(u) + \ell(w_0(s_i, s_j)) + \ell(v)$. Hence

$$t_w = t_ut_{w_0}t_v = t_u \left(- \sum_{x < w_0} t_x \right) t_v = \sum_{x < w_0} t_u t_xt_v.$$

Hence, $t_w$ is a linear combination of element $t_y = t_ut_xt_v$, where $y < w$. By the induction hypothesis, each term $t_y$ can be expressed as a $\mathbb{Z}[q]$-linear combination of elements $t_z$, with $z < y$, $z \in W_c(X)$ and the statement follows. \qed

To get a better feeling for how $D$-polynomials were defined, we compute $D_{x,w}$ step by step, in the case $w = s_1s_2s_3s_2s_1 \in W(A_3)$.

**Example 3.2.2.** Let $w = s_1s_2s_3s_2s_1 = 1, 2, 3, 2, 1 \in W(A_3)$. To compute $D_{x,w}(q)$ we need to combine (3.1) with the relation

$$t_{s_is_{i+1}s_i} = -t_{s_i} - t_{s_{i+1}} - t_{s_i,s_{i+1}} - t_{s_{i+1},s_i}.$$
\[
t_{1,2,3,2,1} = t_1 \cdot t_{2,3,2} \cdot t_1 \\
= t_1 \cdot (-t_e - t_2 - t_3 - t_{2,3} - t_{3,2}) \cdot t_1 \\
= -t_1 \cdot t_{1,2,1} - t_{1,3} \cdot t_1 - t_{1,2,3,1} - t_{1,3,2,1} \\
= -(qt_e + (q - 1)t_1) - t_{1,2,1} - (qt_3 + (q - 1)t_{1,3}) - t_{1,2,1,3} - t_{1,3,1,2} \\
= -qt_e - (q - 1)t_1 + t_1 + t_3 + t_{1,2} + t_{1,3} - qt_3 - (q - 1)t_{1,3} + \\
+ t_3 + t_{1,3} + t_{3,2} + t_{1,2,3} + t_{2,1,3} + t_3 + t_{1,3} + t_{3,2} + t_{3,2,1} + t_{3,1,2} \\
= (1 - q)t_e + (2 - q)t_1 + t_2 + (2 - q)t_3 + (3 - q)t_{1,3} + t_{1,2} + \\
+ t_{2,1} + t_{3,2} + t_{3,2} + t_{1,3} + t_{3,2,1} + t_{1,3} + t_{3,2}.
\]

Therefore we get \(D_{e,w} = 1 - q, \ D_{s_1,w} = D_{s_3,w} = 2 - q \) and \(D_{s_1,s_3,w} = 3 - q\). Moreover, \(D_{x,w} = 1 \) for the rest of the elements \(x \leq w\). \(\square\)

The following proposition mirrors Theorem 2.2.1 in the context of the Temperley–Lieb algebra.

**Proposition 3.2.3.** Let \(w \in W_c(X)\). Then there exists a unique family of polynomials \(\{a_{y,w}\}_{y \in W_c(X)} \subseteq \mathbb{Z}[q]\) such that

\[
(t_{w^{-1}})^{-1} = q^{-t(w)} \sum_{y \in W_c(X) \atop y \leq w} a_{y,w} t_y,
\]

where \(a_{w,w} = 1\) and \(a_{y,w} = 0\) if \(y \not\leq w\).

**Proof.** By Lemma 3.1.3 we get

\[
(t_{w^{-1}})^{-1} = \iota(t_w) = \iota(\sigma(T_w)) = \iota(T_w + J) = \iota(T_w) + J = T_{w^{-1}}^{-1} + J.
\]
Therefore, from Theorem 2.2.1 we achieve

\[ T_{w-1}^{-1} + J = \varepsilon_w q^{-\ell(w)} \sum_{x \leq w} \varepsilon_x R_{x,w} T_x + J \]

\[ = \varepsilon_w q^{-\ell(w)} \sum_{x \leq w} \varepsilon_x R_{x,w} t_x \]

\[ = \varepsilon_w q^{-\ell(w)} \sum_{x \leq w} \varepsilon_x R_{x,w} \sum_{y \in W_c(X)} D_{y,x} t_y \]

\[ = \sum_{y \in W_c(X)} q^{-\ell(w)} \left( \sum_{y \leq x \leq w} \varepsilon_x \varepsilon_w R_{x,w} D_{y,x} \right) t_y, \]

\[ = q^{-\ell(w)} \sum_{y \in W_c(X)} \sum_{y \leq w} a_{y,w} t_y, \]

since the expression in the round brackets is a polynomial with integer coefficients, depending only on the elements \( y \) and \( w \).

The polynomials \( \{a_{x,w}\} \) associated to \( TL(X) \) play the same role as the polynomials \( \{R_{x,w}\} \) associated to \( \mathcal{H}(X) \). They both represent the coordinates of elements of the form \( \iota(t_w) \) (respectively \( \iota(T_w) \)) with respect to the \( t \)–basis (respectively \( T \)–basis). The next result is the analogue of a well–known result for the \( R \)–polynomials (see [Hum90, §7.8]). We follow the proof given in [Bre].

**Proposition 3.2.4.** Let \( y, w \in W_c(X) \) be such that \( y \leq w \). Then

\[ \sum_{x \in [y,w]} q^{\ell(w) - \ell(x)} a_{y,x}(q)a_{x,w}(q^{-1}) = \delta_{y,w}. \]
Proof. From Proposition 3.2.3 we get

\[ \tau(t_w) = \tau(t_{w^{-1}}) = \tau \left( q^{-\ell(w)} \sum_{x \in W_c(X) \atop x \leq w} a_{x,w}(q)t_x \right) \]

\[ = q^{\ell(w)} \sum_{x \in W_c(X) \atop x \leq w} a_{x,w}(q^{-1})\tau(t_x) \]

\[ = q^{\ell(w)} \sum_{x \in W_c(X) \atop x \leq w} a_{x,w}(q^{-1}) \cdot q^{-\ell(x)} \sum_{y \in W_c(X) \atop y \leq x} a_{y,x}(q)t_y \]

\[ = \sum_{y \in W_c(X) \atop y \leq w} \left( \sum_{x \in [y,w]} q^{\ell(w)-\ell(x)} a_{y,x}(q) a_{x,w}(q^{-1}) \right) t_y. \quad (3.2) \]

Hence, the expression in the round brackets in (3.2) is equal to 1 if \( y = w \) and 0 otherwise. \qed

The generalized Temperley–Lieb algebra admits a canonical basis \( \{ c_w : w \in W_c(X) \} \) that is analogous to the Kazhdan–Lusztig basis \( \{ C'_w : w \in W(X) \} \) of \( \mathcal{H}(X) \). To introduce this new basis we need some definitions. Let \( \mathcal{L} \) be the free \( \mathbb{Z}[q^{-\frac{1}{2}}] \)-module of \( TL(X) \) with basis \( \{ q^{-\frac{\ell(w)}{2}} t_w : w \in W_c(X) \} \) and let \( \pi : \mathcal{L} \to \mathcal{L}/q^{-\frac{1}{2}} \mathcal{L} \) be the canonical projection.

**Definition 3.2.5.** If there exists a unique basis \( \{ c_w : w \in W_c(X) \} \) for \( \mathcal{L} \) such that \( c_w \) is \( \iota \)-invariant and \( \pi(c_w) = \pi(q^{-\frac{\ell(w)}{2}} t_w) \), then the basis \( \{ c_w : w \in W_c(X) \} \) is called an IC basis for \( TL(X) \) with respect to the triple \( (\{ q^{-\frac{\ell(w)}{2}} t_w \}, \iota, \mathcal{L}) \).

The next results will enable us to state the existence of the IC basis for \( TL(X) \), \( X \) being any Coxeter graph. Observe that the existence of such a basis was established in [GL99].

Here we introduce a family of polynomials in a purely combinatorial way, as explained in [Bre].

**Theorem 3.2.6.** Let \( w \in W_c(X) \). Then there exists a unique family of polynomials \( \{ L_{x,w} \}_{x \in W_c(X)} \subseteq q^{-\frac{1}{2}} \mathbb{Z}[q^{-\frac{1}{2}}] \) such that
(i) $L_{x,w} = 0$ if $x \not\leq w$;

(ii) $L_{x,x} = 1$;

(iii) $L_{x,w} \in q^{-\frac{1}{2}}\mathbb{Z}[q^{-\frac{1}{2}}]$ if $x < w$;

(iv) $L_{x,w}(q^{-\frac{1}{2}}) = \sum_{y \in [x,w]} q^{\frac{(\ell(x) - \ell(y))}{2}} a_{x,y}(q)L_{y,w}(q^{\frac{1}{2}})$.

**Proof.** For the existence part, we proceed by induction on $\ell(w) - \ell(x)$. Let $\ell(w) - \ell(x) > 0$ and define $p \overset{\text{def}}{=} \iota(L_{x,w} - \iota(L_{x,w}))$. Then

$$p = \iota \left( \sum_{y \in [x,w]} q^{\frac{(\ell(x) - \ell(y))}{2}} a_{x,y}(q)\iota(L_{y,w}) \right)$$

$$= \sum_{y \in [x,w]} q^{\frac{(\ell(x) - \ell(y))}{2}} a_{x,y}(q^{-1})L_{y,w}$$

$$= \sum_{y \in [x,w]} q^{\frac{(\ell(x) - \ell(y))}{2}} a_{x,y}(q^{-1}) \left( \sum_{z \in [y,w]} q^{\frac{(\ell(y) - \ell(z))}{2}} a_{y,z}(q)\iota(L_{z,w}) \right)$$

$$= \sum_{z \in [x,w]} \iota(L_{z,w})q^{\frac{(\ell(z) - \ell(x))}{2}} \left( \sum_{y \in [x,z]} q^{\frac{(\ell(y) - \ell(z))}{2}} a_{x,y}(q^{-1})a_{y,z}(q) \right)$$

$$= \sum_{z \in [x,w]} \iota(L_{z,w})q^{\frac{(\ell(z) - \ell(x))}{2}} (-q^{\ell(x) - \ell(z)}a_{x,z}(q))$$

$$= - \sum_{z \in [x,w]} \iota(L_{z,w})q^{\frac{(\ell(z) - \ell(x))}{2}} a_{x,z}(q) = -\iota(p).$$

Hence, $p$ is antisymmetric and the existence statement follows.

Next we prove the uniqueness part, assuming the existence of $L_{x,w}$. We proceed by induction on $\ell(w) - \ell(x)$, where $w \in W(X)$ is fixed. Assume that the $L_{y,w}$ can be chosen in a unique way, for all $x < y \leq w$. We will show that $L_{x,w}$ is uniquely determined. Consider equation (iv) and move the term for $y = x$ to the left, so that

$$L_{x,w}(q^{-\frac{1}{2}}) - L_{x,w}(q^{\frac{1}{2}}) = \sum_{y \in [x,w]} q^{\frac{(\ell(x) - \ell(y))}{2}} a_{x,y}(q)L_{y,w}(q^{\frac{1}{2}}).$$

Observe that no cancellation occurs on the left hand side of (2.11), since $L_{x,w}(q^{-\frac{1}{2}}) \in q^{-\frac{1}{2}}\mathbb{Z}[q^{-\frac{1}{2}}]$. Hence, we conclude that the polynomial $L_{x,w}$ satisfying (iv) is unique. \hfill $\Box$
Theorem 3.2.7. Let $X$ be an arbitrary Coxeter graph. Let $w \in W_c(X)$ and define

$$c_w \overset{\text{def}}{=} \sum_{x \in W_c(X)} q^{-\ell(x)/2} L_{x,w}(q^{-1/2})t_x.$$ 

Then $\{c_w : w \in W_c(X)\}$ is an IC basis for $TL(X)$.

Proof. We have to show that every basis element $c_w$ satisfies the conditions of Definition 3.2.5. First, $\pi(c_w) = \pi(q^{-\ell(w)/2}t_w)$ since $L_{x,w} = 1 \iff x = w$ and $L_{x,w}(q^{-1/2}) \in q^{-1/2}\mathbb{Z}[q^{-1/2}]$ if $x < w$ (see Theorem 3.2.6). Second, we check that $c_w$ is $\iota$–invariant.

$$\iota(c_w) = \sum_{x \in W_c(X)} q^{-\ell(x)/2} (L_{x,w}(q^{1/2})) \iota(t_x)$$

$$= \sum_{x \in W_c(X)} q^{-\ell(x)/2} \left( \sum_{y \in [x,w)_{c}} q^{-\ell(y)/2} a_{x,y}(q^{-1}) L_{y,w}(q^{-1/2}) \right) \iota(t_x)$$

$$= \sum_{x \in W_c(X)} \left( \sum_{y \in [x,w)_{c}} q^{-\ell(y)/2} a_{x,y}(q^{-1}) L_{y,w}(q^{-1/2}) \right) q^{-\ell(x)} \sum_{z \leq x} a_{z,x}(q) t_z$$

$$= \sum_{z \in W_c(X)} \left( \sum_{y \in W_c(X)} q^{-\ell(y)/2} \left( \sum_{x \in [z,y)_{c}} q^{-\ell(y) - \ell(x)} a_{z,x}(q^{-1}) a_{x,y}(q^{-1}) L_{y,w}(q^{-1/2}) \right) \right) t_z.$$ 

Finally, by applying Proposition 3.2.4 we get

$$\sum_{z \in W_c(X)} \left( \sum_{y \in W_c(X)} q^{-\ell(y)/2} \delta_{z,y} L_{y,w}(q^{-1/2}) \right) t_z = \sum_{z \in W_c(X)} q^{-\ell(z)/2} L_{z,w}(q^{-1/2}) t_z$$ 

as desired. 

Corollary 3.2.8. There exists an IC basis for $TL(X)$ with respect to the triple $(\{q^{-\ell(w)/2}t_w\}, \iota, \mathcal{L})$. 

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Comparing the definition of $c_w$ (see Theorem 3.2.7) with that of $C'_w$ (see relation 2.12), we notice that the polynomials $L_{x,w}$ play the same role as $\frac{q}{\ell(x)-\ell(w)} P_{x,w}$, where $P_{x,w}$ are the Kazhdan–Lusztig polynomials defined in Theorem 2.3.1.

### 3.3 The projection property

Since the Kazhdan–Lusztig basis and the canonical basis are both $\iota$–invariant and since $\iota(J) = J$, it is natural to ask to what extent $\{\sigma(C'_w) : w \in W(X)\}$ coincides with $\{c_w : w \in W_c(X)\}$. Denote by $\mathcal{C}$ the set $\{C'_w : w \in W_c(X)\}$.

**Definition 3.3.1.** We say that a Coxeter graph $X$ satisfies the projection property if

$$\sigma(\mathcal{C}) = \{c_w : w \in W_c(X)\}.$$ 

A sufficient condition for a Coxeter graph to have the projection property is given in [GL00, Proposition 1.2.3].

**Proposition 3.3.2.** Let $\sigma : \mathcal{H}(X) \to \mathcal{H}(X)/J(X) = TL(X)$ be the canonical projection. If $\text{Ker}(\sigma)$ is spanned by the basis element $C'_w$ that it contains, then $X$ satisfies the projection property.

The kernel of the canonical projection $\sigma : \mathcal{H}(A_n) \to \mathcal{H}(A_n)/J(A_n)$ is spanned by all elements $C'_w$ such that $w \notin W_c(A_n)$ (see [FG97, Proposition 3.1.1]). Therefore, type $A$ has the projection property and the same argument holds for types $B$ and $I_2(m)$ (see [GL01, Theorem 3.1.1] and [Gre07, Proposition 6.14]). This fact was also verified for types $H_3$, $H_4$, and $F_4$, by means of computer calculations (see [GL01, §3]). The converse of Proposition 3.3.2 is not true in general. The following counterexample is given in [Los00, Example 2.5], where Losonczy shows that type $D_n$, with $n \geq 4$, has the projection property, but Proposition 3.3.2 does not apply.

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Example 3.3.3. Let $W = W(D_4)$ with set of generators $\{s_1, s_2, s_3, s_4\}$, where $s_3$ corresponds to the branch node, as shown in Figure 3.1. On the one hand, if $s_2s_3s_4s_3s_1s_2s_3$ is a reduced expression of $w \notin W_c(D_4)$ then $C'_w \in J$. Hence $C'_s C'_w \in J$. On the other hand, by Proposition 2.3.5 we get

$$C'_s C'_w = C'_s s_2s_3s_4s_3s_1s_2s_3 + C'_s s_1s_2s_3s_4.$$

Therefore there exists a nonzero $A$–linear combination of elements of $\mathcal{C}$ that belongs to $J$. But $\sigma(\mathcal{C}) = \{\sigma(C'_w) : w \in W_c(X)\}$ is a basis for $TL(D_4)$, so we conclude that $J(D_4)$ is not spanned by the Kazhdan–Lusztig basis elements that it contains.

\[\square\]

Figure 3.1: Coxeter graph $D_4$.

Similar problems arise whenever $X$ is a Coxeter graph that contains a vertex connected to at least three other vertices. Graphs having this property are sometimes called branching graphs. Some examples of branching graphs are types $D$, $E_6$, $E_7$, $E_8$, and in these cases Proposition 3.3.2 does not apply (see [GL01, Corollary 3.1.3]). However, no example of a Coxeter group that fails to satisfy the projection property is known and even in type $E$ it is an open problem (see [GL01, §2]).

We remark that Example 3.3.3 can be translated in the “language” of Kazhdan–Lusztig cells. We stick to the definitions given in [GL01, Definition 1.2.2].

Definition 3.3.4. Let $x, w \in W(X)$. If there exists a chain $x = x_0, x_1, \cdots, x_k = w$, $k \geq 0$, such that for every $i < k$, $C'_{x_i}$ occurs with non–zero coefficient in the linear expansion of $C'_s C'_{x_{i+1}}$ for some $s \in S(X)$ such that $sx_{i+1} > x_{i+1}$, then we write $x \leq w$.

Define the equivalence relation $\sim_L$ as follows: $x \sim_L w \iff x \leq_L w$ and $w \leq_L x$. The equivalence classes with respect to $\sim_L$ are called left cells of $W(X)$. We
write $x \leq w \iff x^{-1} \leq w^{-1}$. Finally, we set $x \leq w \iff x \sim w$ and $x \leq w$. The equivalence classes with respect to the equivalence relation $\sim$ (respectively $\simLR$) are called right cells (respectively two–sided cells).

By Definition 3.3.4, we can restate Example 3.3.3 as follows: $s_1 s_2 s_3 s_4 \leq s_2 s_3 s_4 s_3 s_1 s_2 s_3$. Therefore, $W(D_4) \setminus W_c(D_4)$ is not closed under $\leq$.

Observe that Proposition 3.3.2 is equivalent to asking that $\sigma(C'_w) = 0$, for all elements $w \notin W_c(X)$ (see [GL01, Theorem 2.2.3]). This is a key observation in order to study the $D$–polynomials introduced in Proposition 3.2.1. In particular, one may wonder whether the map $\sigma : \mathcal{H}(X) \to \mathcal{H}(X)/J(X)$ satisfies

$$\sigma(C'_w) = \begin{cases} 
  c_w & \text{if } w \in W_c(X), \\
  0 & \text{if } w \notin W_c(X).
\end{cases} \quad (3.3)$$

The answer is affirmative in types $A, B, I_2(m), F_4, H_3$ and $H_4$, and negative for types $D, E_6, E_7$ and $E_8$ (for a complete discussion of these results, see [Gre07] and [GL01]). More generally, if $X$ is a finite irreducible or affine Coxeter graph, relation (3.3) holds if and only if $W_c(X)$ is closed under $\leq$ or, equivalently, if and only if $W_c(X)$ is a union of two–sided Kazhdan–Lusztig cells (see [Shi05, Theorem 2.1] and [GL01, Theorem 2.2.3]). On the other hand, in [Shi03] it is shown that $W_c(X)$ is a union of two–sided Kazhdan–Lusztig cells if and only if $X$ is non–branching and $X \neq \widetilde{F}_4$.

**Example 3.3.5.** Let $W = W(\widetilde{F}_4)$ with set of generators $\{s_0, s_1, s_2, s_3, s_4\}$, where $m(s_2, s_3) = 4$, as shown in Figure 3.2. On the one hand, $s_0 s_2 s_4$ is a fully commutative element in $W(\widetilde{F}_4)$ and $s_0 s_1 s_0 \notin W_c(\widetilde{F}_4)$. On the other hand, in [Shi03, §3.7] Shi states that $s_0 s_2 s_4 \simLR s_0 s_1 s_0$. Therefore, $W_c(\widetilde{F}_4)$ is not a union of two–sided Kazhdan–Lusztig cells.

![Figure 3.2: Coxeter graph \(\widetilde{F}_4\).](image)

**Theorem 3.3.6.** Let $X$ be a finite irreducible or affine Coxeter graph. Then, relation (3.3) holds if and only if $X$ is non–branching and $X \neq \widetilde{F}_4$. □
Chapter 4

Combinatorial properties of TL($X$)

4.1 Combinatorial properties of $D_{x,w}$

In the first part of this section we study the $D$–polynomials defined in Proposition 3.2.1. More precisely, we obtain a recurrence relation for the polynomials \{$D_{x,w}$\}$_{x \in \mathcal{W}_c(X), w \in \mathcal{W}(X)} \subseteq \mathbb{Z}[q]$, $X$ being an arbitrary Coxeter graph. Then we will focus on the Coxeter graph satisfying equation (3.3) and derive some results concerning symmetry properties, the value of the constant term and explicit formulas for $D_{x,w}$ when the Bruhat interval $[x, w]$ has a particular structure. Throughout this chapter $\ell(x, w)$ will denote the difference $\ell(w) - \ell(x)$.

**Proposition 4.1.1.** Let $X$ be an arbitrary Coxeter graph. Let $w \not\in \mathcal{W}_c(X)$ and $s \in S(X)$ be such that $ws \not\in \mathcal{W}_c(X)$, with $\ell(ws) < \ell(w)$. Then, for all $x \in \mathcal{W}_c(X)$, $x \leq w$, we have

$$D_{x,w} = \tilde{D}_{x,w} + \sum_{y \in \mathcal{W}_c(X), ys \not\in \mathcal{W}_c(X)} \sum_{ys > y} D_{x,ys}D_{y,ws}.$$
where

\[
\tilde{D}_{x,w} = \begin{cases} 
D_{xs,ws} + (q-1)D_{x,ws} & \text{if } xs < x, \\
qD_{xs,ws} & \text{if } x < xs \in W_c(X), \\
0 & \text{if } x < xs \notin W_c(X),
\end{cases}
\]

Proof. On the one hand, by Proposition 3.2.1, we have

\[
t_w = \sum_{x \in W_c(X)} D_{x,w} t_x.
\]

On the other hand, letting \( v \overset{\text{def}}{=} ws \),

\[
t_w = t_v t_s = \left( \sum_{y \in W_c(X) \atop y \leq v} D_{y,v} t_y \right) t_s
\]

\[
= \sum_{y \in W_c(X), ys \in W_c(X) \atop y \leq v, ys > y} D_{y,v} t_{ys} + \sum_{y \in W_c(X), ys \notin W_c(X) \atop y \leq v, ys > y} D_{y,v} (qt_{ys} + (q-1)t_y)
\]

\[
= \sum_{y \in W_c(X), ys \in W_c(X) \atop y \leq v, ys > y} D_{y,v} t_{ys} + \sum_{y \in W_c(X), ys \notin W_c(X) \atop y \leq v, ys > y} D_{y,v} (qt_{ys} + (q-1)t_y)
\]

\[
= \sum_{y \in W_c(X), ys \in W_c(X) \atop y \leq v, ys > y} D_{y,v} t_{ys} + \sum_{y \in W_c(X) \atop y \leq v, ys < y} D_{y,v} qt_{ys}
\]

\[
+ \sum_{y \in W_c(X) \atop y \leq v, ys > y} (q-1)D_{y,v} t_y + \sum_{y \in W_c(X), ys \notin W_c(X) \atop y \leq v, ys > y} D_{y,v} \left( \sum_{x \in W_c(X) \atop x \leq ys} D_{x,ys} t_x \right)
\]

\[
= \sum_{x \in W_c(X) \atop xs < x \leq w} D_{xs,ws} t_x + \sum_{x \in W_c(X) \atop x \leq w, xs > x} qD_{xs,ws} t_x
\]

\[
+ \sum_{x \in W_c(X) \atop xs < x \leq w} (q-1)D_{x,ws} t_x + \sum_{x \in W_c(X) \atop y \leq ys} D_{y,ws} D_{y,ws} t_x
\]

\[
= \sum_{x \in W_c(X) \atop xs < x \leq w} D_{xs,ws} t_x + qD_{xs,ws} t_x
\]

\[
+ \sum_{x \in W_c(X) \atop xs < x \leq w} (q-1)D_{x,ws} t_x + \sum_{y \in W_c(X), ys \notin W_c(X) \atop y \leq ys} D_{y,ws} D_{y,ws} t_x
\]

\[
= \sum_{x \in W_c(X) \atop xs < x \leq w} D_{xs,ws} t_x + qD_{xs,ws} t_x
\]

\[
+ \sum_{x \in W_c(X) \atop xs < x \leq w} (q-1)D_{x,ws} t_x + \sum_{y \in W_c(X), ys \notin W_c(X) \atop y \leq ys} D_{y,ws} D_{y,ws} t_x
\]

\[
= \sum_{x \in W_c(X) \atop xs < x \leq w} D_{xs,ws} t_x + qD_{xs,ws} t_x
\]

\[
+ \sum_{x \in W_c(X) \atop xs < x \leq w} (q-1)D_{x,ws} t_x + \sum_{y \in W_c(X), ys \notin W_c(X) \atop y \leq ys} D_{y,ws} D_{y,ws} t_x
\]
(note that \(xs \in W_c(X), x < xs \Rightarrow x \in W_c(X)\)). Extracting the coefficient of \(t_x\) we get

\[
D_{x,w} = \begin{cases} 
D_{xs,ws} + (q - 1)D_{x,ws} + b(x,w) & \text{if } xs < x, \\
qD_{xs,ws} + b(x,w) & \text{if } x < xs \in W_c(X), \\
b(x,w) & \text{if } x < xs \notin W_c(X),
\end{cases}
\]

where

\[
b(x,w) = \sum_{y \in W_c(X) : ys < y < ys} D_{x,ys}D_{y,ws},
\]

as desired.

It is interesting to note that the recursion in Proposition 4.1.1 is similar to the one for the parabolic Kazhdan–Lusztig polynomials (see [Deo87]). The preceding recursion can sometimes be solved explicitly. In the proof of the next result we need the notion of Grassmannian and bi–Grassmannian elements (see, e.g., [LS96, §3] and [BB05, §5, Exercise 39]).

**Definition 4.1.2.** Let \(w \in W(A_{n-1})\) and define \(D_R(w) \overset{\text{def}}{=} |\{s \in S(X) : ws < w\}|\). The permutation \(w\) is called Grassmannian if \(|D_R(w)| \leq 1\) and bi–Grassmannian if \(|D_R(w)| = |D_R(w^{-1})| = 1\).

As a consequence of [BJS93 Theorem 2.1], if \(w \in W(A_{n-1})\) is Grassmannian then \(w \in W_c(A_{n-1})\).

**Corollary 4.1.3.** Let \(X\) be of type \(A\) and let \(x_0 \in W_c(X)\) be a bi–Grassmannian element. If \(x_0\) is a maximal element in the Bruhat order of \(W_c(X)\), then \(D_{x_0,w} = \varepsilon_{x_0}\varepsilon_w\), for all elements \(w \geq x_0\).

**Proof.** If \(w \in W_c(X)\) then the result is trivial. Suppose \(w \notin W_c(X)\). Observe that if \(s \in S(X)\) is such that \(x_0s > x_0\), then \(x_0s \notin W_c(X)\). Moreover, if \(x_0s > x_0\) and \(y \in W_c(X)\) is such that \(ys > x_0\) then \(y \geq x_0\) by Lemma 1.3.5 so \(y = x_0\). Hence

\[
\{y \in W_c(X) : ys > x_0\} = \{x_0\},
\]

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for any \( s \not\in D_R(x_0) \). Choosing \( s \) such that \( x_0s > x_0, ws < w \) (there exists such an \( s \) since \( x_0 \) is a bi-Grassmannian element, while \( w \not\in W_c(X) \) is not Grassmannian), the third case of Theorem 4.1.1 applies, so

\[
D_{x_0,w} = D_{x_0,x_0s}D_{x_0,ws}.
\]

Define \( \ell(x, w) \) \( \overset{\text{def}}{=} \ell(w) - \ell(x) \), and proceed by induction on \( \ell(x_0, w) \). Suppose \( \ell(x_0, w) = 1, \) with \( w \not\in W_c(A_{n-1}). \) By Proposition 1.3.4 it follows that \( w \) admits a reduced expression of the form \( x_1s_is_{i+1}s_ix_2 \), with \( x_1, x_2 \in W_c(A_{n-1}) \), \( s_i, s_{i+1} \in S(A_{n-1}) \), and \( x_0 \) admits a reduced expression of the form \( \hat{x}_1\hat{s}_i\hat{s}_{i+1}s_i\hat{x}_2 \) or \( x_1s_is_{i+1}\hat{s}_i\hat{x}_2 \), since \( x_0 \in W_c(A_{n-1}) \). Therefore

\[
t_w = t_{x_1}t_{s_iss_{i+1}s_i}t_{x_2} = t_{x_1}(-t_{s_iss_{i+1}} - t_{s_{i+1}s_i} - t_{s_i} - t_e)t_{x_2},
\]

and the statement follows by applying (3.1). If \( \ell(x_0, w) > 1, \) then

\[
D_{x_0,w} = D_{x_0,x_0s}D_{x_0,ws} = -D_{x_0,ws} = -\varepsilon_{x_0}\varepsilon_{ws} = \varepsilon_{x_0}\varepsilon_w.
\]

Observe that a maximal element in the Bruhat order of \( W_c(A_{n-1}) \) is an element whose one–line notation is of the form \([k+1, k+2, \cdots, n, 1, 2, \cdots, k]\), with \( k \in [n - 1] \).

From here to the end of this section we will denote by \( X \) a Coxeter graph satisfying (3.3). Observe that \( D_{x,w} = \delta_{x,w} \) if \( x, w \in W_c(X) \).

**Lemma 4.1.4.** For all \( x \in W_c(X) \) and \( w \not\in W_c(X), \) we have

\[
\sum_{x \leq y \leq w} D_{x,y}P_{y,w} = 0.
\]

**Proof.** Let \( w \in W(X) \). Then, by Proposition 3.2.1,

\[
\sigma(C'_w) = q^{-\ell(w)/2} \sum_{y \leq w} P_{y,w} \sigma(T_y)
\]

\[
= q^{-\ell(w)/2} \sum_{y \leq w} P_{y,w} \left( \sum_{x \in W_c(X)} D_{x,y}t_x \right)
\]

\[
= q^{-\ell(w)/2} \sum_{x \in W_c(X)} \left( \sum_{x \leq y \leq w} D_{x,y}P_{y,w} \right) t_x.
\]
When \( w \notin W_c(X) \) we get \( \sigma(C_w') = 0 \), so the expression in round brackets must vanish and the statement follows.

The next result will be useful in §4.2 and §4.3.

**Lemma 4.1.5.** Let \( x \in W_c(X) \) be such that \( xs \notin W_c(X) \) and let \( w \notin W_c(X) \) be such that \( w > ws \in W_c(X) \). Then

\[
D_{x,w} = -\delta_{x,ws}.
\]

**Proof.** We proceed by induction on \( \ell(x,w) \). If \( \ell(x,w) = 1 \), then \( D_{x,w} = D_{x,xs} = -1 = -\delta_{x,ws} \). Suppose \( \ell(x,w) > 1 \). From Lemma 4.1.4 and Corollary 2.3.7 we get

\[
D_{x,w} = -P_{x,w} - \sum_{t \notin W_c(X), x < t < w} D_{x,t}P_{t,w} = -P_{x,w} - D_{x,xs}P_{xs,w} - \sum_{t \notin W_c(X), t \neq xs, x < t < w} D_{x,t}P_{t,w} = -P_{x,w} - D_{x,xs}P_{x,w} - \sum_{t \notin W_c(X), t \neq xs, x < t < w} D_{x,t}P_{t,w} = - \sum_{t \notin W_c(X), t \neq xs, x < t < w} D_{x,t}P_{t,w},
\]

By induction hypothesis, the term \( D_{x,t} \) in the first sum is equal to \( -\delta_{x,ts} \), since \( \ell(x,t) < \ell(x,w) \). Therefore, the first sum is zero. On the other hand, the second and the third sum can be written as

\[
- \sum_{t > ts \in W_c(X), t \neq xs, x < t < w} D_{x,t}P_{t,w} - \sum_{t > ts \in W_c(X), x < t < w} D_{x,t}P_{t,w} - \sum_{t < ts \in W_c(X), x < t < w} D_{x,t}P_{t,w},
\]

since \( ts > t \notin W_c(X) \) implies \( ts \notin W_c(X) \). To prove the statement we have to show that the term (4.1) is zero. First, observe that \( \ell(x,z) < \ell(x,w) \), since
Lemma 1.3.5 implies $z \leq w$, but $z \notin W_c(X)$ and $w \in W_c(X)$. Moreover, by Proposition 1.1.1 and by induction hypothesis, we achieve

$$D_{x,zs} = \sum_{u \in W_c(X), u < us \notin W_c(X)} D_{x,us}D_{u,z} = \sum_{u \in W_c(X), u < us \notin W_c(X)} (-\delta_{x,u})D_{u,z} = -D_{x,z}. \quad (4.2)$$

We conclude that $D_{x,zs} + D_{x,z} = 0$, for all $z \notin W_c(X)$ such that $x < z < zs < ws$, so the sum in (4.1) is zero.

The following is the main result of this section.

**Theorem 4.1.6.** Let $X$ be such that equation (3.3) holds. For all $x \in W_c(X)$ and $w \notin W_c(X)$ such that $x < w$, we have

$$D_{x,w} = \sum (-1)^k \prod_{i=1}^k P_{x_{i-1},x_i},$$

where the sum is taken over all the chains $x = x_0 < x_1 < \cdots < x_k = w$ such that $x_i \notin W_c(X)$ if $i > 0$, and $1 \leq k \leq \ell(x,w)$.

**Proof.** We proceed by induction on $\ell(x,w)$.

If $\ell(x,w) = 1$ then, from Lemma 4.1.4 we get $D_{x,w} = -P_{x,w}$, proving the claim in this case. If $\ell(x,w) > 1$ then, from Lemma 4.1.4 and our induction
hypothesis, we have

\[
D_{x,w} = -P_{x,w} - \sum_{t \in W_c(X) \atop x \leq t < w} D_{t,w} P_{t,w}
\]

\[
= -P_{x,w} - \sum_{t \in W_c(X) \atop x < t < w} \sum_{k=1}^{\ell(x,t)} \left( \sum_{x=x_0 < ... < x_{k-1} = t} (-1)^k \prod_{i=1}^k P_{x_{i-1},x_i} \right)
\]

\[
= -P_{x,w} + \sum_{t \in W_c(X) \atop x < t < w} \left( \sum_{k=0}^{\ell(x,t)} \left( \sum_{x=x_0 < ... < x_{k+1} = w \atop x_k = t \text{ if } k \neq 0} (-1)^{k+1} \prod_{i=1}^{k+1} P_{x_{i-1},x_i} \right) \right)
\]

\[
= \sum_{k \geq 0} \sum_{t \in W_c(X) \atop x < t < w} \left( \sum_{x=x_0 < ... < x_{k+1} = w \atop x_k = t \text{ if } k \neq 0} (-1)^{k+1} \prod_{i=1}^{k+1} P_{x_{i-1},x_i} \right)
\]

\[
= \sum_{k \geq 0} \sum_{x=x_0 < ... < x_{k+1} = w} (-1)^{k+1} \prod_{i=1}^{k+1} P_{x_{i-1},x_i},
\]

as desired. \( \square \)

Theorem 4.1.6 shows that the \( D \)-polynomials are intimately related to the Kazhdan–Lusztig polynomials, which is not at all obvious from their definition.

We now derive some consequences of Theorem 4.1.6. First we obtain some symmetry properties of the polynomials \( \{D_{x,w}\} \).

**Corollary 4.1.7.** Let \( x \in W_c(X), \ w \not\in W_c(X) \) and \( x < w \). Then

(i) \( D_{x,w} = D_{x-1,w-1} \); 

(ii) \( D_{x,w} = D_{w_0x,w_0w_0x,w_0w_0} \).
Proof. By Lemma 1.7.4, \( x^{-1} \in W_c(X) \) for every \( x \in W_c(X) \). Therefore we get

\[
D_{x^{-1}, w^{-1}} = \sum_{x^{-1} = x_0 < x_1 < \ldots < x_k = w^{-1}} (-1)^k \prod_{i=1}^{k} P_{x_{i-1}, x_i}.
\]

\[
= \sum_{x = x_0 < x_1 < \ldots < x_k = w} (-1)^k \prod_{i=1}^{k} P_{x_{i-1}, x_i}.
\]

\[
= \sum_{x = x_0 < x_1 < \ldots < x_k = w} (-1)^k \prod_{i=1}^{k} P_{x_{i-1}, x_i}.
\]

\[
= \sum_{x = y_0 < y_1 < \ldots < y_k = w} (-1)^k \prod_{i=1}^{k} P_{y_{i-1}, y_i}.
\]

\[
= D_{x, w},
\]

where we have used a well-known property of the Kazhdan–Lusztig polynomials (see, e.g., [BB05, §5, Exercise 12]). The same holds for \( D_{w_0 x w_0, w_0 w w_0} \), using the properties in [BB05, §5, Exercise 13(c)].

Next, we compute the constant term of the polynomials \( D_{x, w} \).

**Corollary 4.1.8.** For all \( x \in W_c(X) \) and \( w \not\in W_c(X) \) such that \( x < w \), we have

\[
D_{x, w}(0) = \sum_{x = x_0 < \ldots < x_k = w} (-1)^k,
\]

where \( x_i \not\in W_c(X) \) if \( i > 0 \), and \( 1 \leq k \leq \ell(x, w) \).

**Proof.** The statement follows immediately from Theorem 4.1.6 and the well-known fact that \( P_{x, w}(0) = 1 \) for all \( x, w \in W(X) \) such that \( x \leq w \) (see, e.g., [BB05, Proposition 5.1.5]).

By [Sta97, Proposition 3.8.5], Corollary 4.1.8 asserts that \( D_{x, w}(0) \) equals the Möbius function \( \mu(\bar{0}, w) \) in the poset \( \{ y \in W(X) \setminus W_c(X) : y \in [x, w] \} \cup \{ \bar{0} \} \).
This suggests the study of the partial order induced on $W(X) \setminus W_c(X)$ by the Bruhat order.

We now derive an interesting property for $D$–polynomials.

**Proposition 4.1.9.** Let $w \in W(X)$. Then

$$\sum_{x \in W_c(X) \mid x \leq w} \varepsilon_x D_{x,w} = \varepsilon_w.$$ 

**Proof.** We proceed by induction on $\ell(w)$. The proposition is trivial if $w \in W_c(X)$, which covers the case $\ell(w) \leq 2$. Suppose that $w \notin W_c(X)$. Then, by Lemma 4.1.4 we have

$$\sum_{x \in W_c(X) \mid x \leq w} \varepsilon_x D_{x,w} = \sum_{x \in W_c(X) \mid x \leq w} \varepsilon_x D_{x,w} = -\sum_{x \in W_c(X) \mid x \leq w} \varepsilon_x P_{x,w} - \sum_{t \in W_c(X) \mid t < w} P_{t,w} \left( \sum_{x \in W_c(X) \mid x < t} \varepsilon_x D_{x,t} \right)$$

$$= -\sum_{x \in W_c(X) \mid x < w} \varepsilon_x P_{x,w} - \sum_{t \in W_c(X) \mid t < w} P_{t,w} \varepsilon_t$$

$$= -\sum_{x \in W_c(X) \mid x < w} \varepsilon_x P_{x,w},$$

and the statement follows from Proposition 2.3.8.

**Lemma 4.1.10.** Let $x \in W_c(X), w \notin W_c(X)$ be such that $x < w$. If $\ell(x, w) = 1$ then $D_{x,w} = -1$. If $\ell(x, w) = 2$, then

$$D_{x,w} = \begin{cases} 1 & \text{if } k = 2, \\ 0 & \text{if } k = 1, \\ -1 & \text{if } k = 0, \end{cases}$$

with $k \stackrel{\text{def}}{=} |\{y \notin W_c(X) : x < y < w\}|$.

**Proof.** By Lemma 2.3.3, $P_{x,w} = 1$ for all $x, w \in W(X)$ such that $\ell(x, w) \leq 2$. If $\ell(x, w) = 1$, then Theorem 4.1.6 implies $D_{x,w} = -P_{x,w} = -1$. If $\ell(x, w) =$
2, then the interval \([x, w]\) is isomorphic to the boolean lattice \(B_2\) (see, e.g., \cite{BB05} Lemma 2.7.3), so \(k \in \{0, 1, 2\}\). Moreover, \(P_{x,y} = P_{y,w} = 1\) for all \(y \not\in W_c(X)\) such that \(x < y < w\), since \(\ell(x, y) = \ell(y, w) = 1\). From Theorem 4.1.6 we get

\[
D_{x,w} = \sum (-1)^k \prod_{i=1}^k P_{x_{i-1},x_i}
\]

and the statement follows.

Using Theorem 4.1.6 we obtain an upper bound for the degree of \(D_{x,w}\) that is also used in 4.3.

**Proposition 4.1.11.** For all \(x \in W_c(X), w \not\in W_c(X)\) such that \(x < w\), we have \(\text{deg}(D_{x,w}) \leq \frac{1}{2}(\ell(w) - \ell(x) - 1)\).

**Proof.** Recall from \cite{KL79} Theorem 1.1] that \(\text{deg}(P_{x,w}) \leq \frac{1}{2}(\ell(w) - \ell(x) - 1)\) if \(x < w\). By Theorem 4.1.6 we know that

\[
D_{x,w} = \sum (-1)^k \prod_{i=1}^k P_{x_{i-1},x_i},
\]

where the sum runs over all the chains \(x = x_0 < x_1 < \cdots < x_k = w\) such that \(x_i \not\in W_c(X)\) if \(i > 0\), and \(1 \leq k \leq \ell(x, w)\). Each term \(\prod_{i=1}^k P_{x_{i-1},x_i}\) has degree

\[
\sum_{i=1}^k \text{deg}(P_{x_{i-1},x_i}) \leq \sum_{i=1}^k \frac{1}{2}(\ell(x_i) - \ell(x_{i-1}) - 1) = \frac{1}{2}(\ell(x_k) - \ell(x_0) - k).
\]

Since \(k \geq 1\), the statement follows.
We end this section by deriving from Theorem 4.1.6 a closed formula for the polynomials $D_{x,w}$ indexed by elements $x \in W_c(X)$ and $w \not\in W_c(X)$ such that $([x, w] \cap (W(X) \setminus W_c(X))) \cup \{x\} = [x, w] \cong B_{l(x,w)}$. In type $A$ it is easy to realize this case. Let $x \in W(A_n)$. Recall that $x$ is said to be a Coxeter element if $s_{\sigma(1)} \cdots s_{\sigma(n)}$ is a reduced expression for $x$, for some $\sigma \in S_n$. It is clear that a Coxeter element is always a fully commutative element.

Theorem 4.1.12. Let $s_1 s_2 \cdots s_n \cdots s_2 s_1$ be a reduced expression for $w \in W(A_n)$ and let $x \in W(A_n)$ be a Coxeter element. Then the following hold:

(i) $x \leq w$;

(ii) $[x, w] \cong B_{l(x,w)}$;

(iii) $([x, w] \cap (W(A_n) \setminus W_c(A_n))) \cup \{x\} = [x, w]$.

Proof. (i) We find a reduced expression for $x$ that is a subexpression of $s_1 s_2 \cdots s_n \cdots s_2 s_1$.

From [Shi97, Theorem 1.5] there is a bijection between the set of Coxeter elements and the acyclic orientations of the Coxeter graph $A_n$. Let $A_n^x$ be the acyclic orientation of the graph $A_n$ associated to $x$. We say that $s_i$ is on the left (respectively on the right) of $s_{i+1}$ in $x = s_{\sigma(1)} \cdots s_{\sigma(n)}$ if $s_i \rightarrow s_{i+1}$ (respectively $s_i \leftarrow s_{i+1}$) in $A_n^x$. Therefore we are able to produce a reduced expression for $x$ from $A_n^x$ in the following way: set $x_n := s_n$ and juxtapose $s_{n-1}$ to the left (respectively to the right) of $x_n$ if $s_{n-1} \rightarrow s_n$ (respectively $s_{n-1} \leftarrow s_n$). Set $x_{n-1} := s_{n-1} x_n$ (respectively $x_n s_{n-1}$). Repeat the same process with $x_{n-1}$ and $s_{n-2}$, and so on. The process ends when we get $x_1$. In fact, $x_1$ is a reduced expression for $x$ and $x_1$ is, by construction, a subexpression of $w$. Hence (i) follows from Theorem 1.3.4.

(ii) By Theorem 1.3.4 every element $y \in [x, w]$ admits (at least) one reduced expression that is a subexpression of $s_1 s_2 \cdots s_n \cdots s_2 s_1$. Let $r(y)$ be one of these reduced expressions. Observe that the reduced expression $x_1$ obtained in (i) is a possible choice for $r(x)$. Consider the map $\phi : [x, w] \rightarrow A$, with
\( \mathcal{A} = \{(\alpha_1, \ldots, \alpha_{n-1}) : \alpha_i \in \{1, 2\}\} \), such that \( \phi(y) = (\alpha_1, \ldots, \alpha_{n-1}) \) if and only if \( r(y) \) has \( \alpha_i \) occurrences of the generator \( s_i \). By [Mar06, Corollary 3.3], the map \( \phi \) is well-defined. We claim that \( \phi \) is a bijection.

First, we prove the surjectivity. Fix \( (\alpha_1, \ldots, \alpha_{n-1}) \in \mathcal{A} \). We describe an algorithm to construct (a reduced expression \( r(y) \) for) an element \( y \in W(A_n) \) such that \( y \in [x, w] \) and \( \phi(y) = (\alpha_1, \ldots, \alpha_{n-1}) \) in the following way: set \( y_n := s_n \). If \( \alpha_{n-1} = 2 \) then set \( y_{n-1} := s_{n-1}y_n s_{n-1} \). Otherwise, proceed as in the proof of point (i), that is, juxtapose \( s_{n-1} \) to the left (respectively, to the right) of \( y_n \), if \( s_{n-1} \rightarrow s_n \) (respectively, \( s_{n-1} \leftarrow s_n \)) and set \( y_{n-1} := s_{n-1}y_n \) (respectively, \( y_n s_{n-1} \)). Repeat the same process with \( y_{n-1} \) and \( \alpha_{n-2} \), and so on. The process ends when we get \( y_1 \). In fact, \( r(x) = x_1 \) is a subexpression of \( y_1 \) by construction. Next, we show that \( y_1 \) is a reduced expression. Observe that if \( y_j \) is reduced then \( y_j s_{j-1} > y_j \) and \( s_{j-1} y_j > y_j \), since there is no occurrence of \( s_{j-1} \) in \( y_j \). Now, we proceed by contradiction to prove that \( s_{j-1} y_j s_{j-1} \) is reduced. Suppose \( \ell(s_{j-1} y_j s_{j-1}) < \ell(y_j) + 2 \), i.e., \( \ell(s_{j-1} y_j s_{j-1}) \leq \ell(y_j) \). By applying Lemma 1.5.5 with \( x = s_{j-1} y_j s_{j-1} \) and \( w = y_j s_{j-1} \) we get \( s_{j-1} y_j = y_j s_{j-1} \), and \( s_{j-1} y_j, y_j s_{j-1} \) are both reduced. Hence, [Mar06, Lemma 3.1] implies that \( s_{j-1} \) commutes with each generator in \( y_j \), which is absurd, since \( s_j \leq y_j \) by construction. Therefore \( \ell(s_{j-1} y_j s_{j-1}) = \ell(y_j) + 2 \).

We conclude that \( y_1 \) is a reduced expression by induction on \( n - i \), with \( i = 0 \cdots n - 1 \).

Denote by \( y \in W(A_n) \) the element that admits \( y_1 \) as a reduced expression. Then \( y \) has the desired properties.

For the injectivity we proceed by contradiction. Suppose that \( u, v \in [x, w] \) are such that \( \phi(u) = \phi(v) = (\alpha_1, \ldots, \alpha_{n-1}) \), with \( u \neq v \). Denote by \( r(u) \) (respectively, \( r(v) \)) the reduced expression of \( u \) (respectively, \( v \)) obtained by applying the algorithm described above. Then \( u \neq v \) implies \( r(u) \neq r(v) \), that is, there exists an index \( i \in [n - 1] \) such that \( \alpha_i = 1 \) and the position of the factor \( s_i \) in \( r(u) \) and \( r(v) \) is different. Denote by \( j \) be the minimum among these indices. Therefore, for every \( h < j \) such that \( \alpha_h = 1 \), \( s_h \) appears on the same side in \( r(u) \) as \( r(v) \). Suppose that \( \alpha_{j+1} = 1 \) and, for instance, that

\[
 r(u) = y_1 s_j s_{j+1} \cdots s_n s_{j+1} s_j y_2, 
\]
where \( y_1 \leq s_1 s_2 \cdots s_{j-1} \) and \( y_2 \leq s_{j-1} s_{j-2} \cdots s_1 \). Then

\[
r(v) = y_1 \hat{s}_j s_{j+1} \cdots s_n \cdots s_{j+1} s_j y_2 \quad \text{or} \quad r(v) = y_1 \hat{s}_j s_{j+1} \cdots s_n \cdots s_{j+1} s_j y_2.
\]

In both cases, \( r(v) \) is a reduced expression such that \( s_j \) is on the right of \( s_{j+1} \). On the other hand, \( r(u) \) is a reduced expression of \( u \) such that \( s_j \) is on the left of \( s_{j+1} \). Hence, Theorem 1.3.4 implies that \( x \) is uniquely determined by the relations \( s_i \rightarrow s_{i+1} \) or \( s_i \leftarrow s_{i+1} \). The same conclusion holds if we consider different deletions of \( s_j \) and \( s_{j+1} \).

In the case \( \alpha_{j+1} = 2 \), we may assume that

\[
r(u) = y_1 s_j s_{j+1} \cdots s_n \cdots s_{j+1} \hat{s}_j y_2 \quad \text{and} \quad r(v) = y_1 \hat{s}_j s_{j+1} \cdots s_n \cdots s_{j+1} s_j y_2.
\]

Observe that \( r(v) \) (respectively, \( r(u) \)) is a reduced expression such that \( s_j \) is on the right (respectively, on the left) of \( s_{j+1} \). Therefore, we reach the same contradiction that we obtained in the previous case.

(iii) Let \( y \in (x, w] \) and \( \phi(y) = (\alpha_1, \cdots, \alpha_{n-1}) \). Let \( j \) be the maximum of the \( i \in [n-1] \) such that \( \alpha_i = 2 \). Then \( r(y) \) contains the braid \( s_j s_{j+1} s_j \), so\( y \notin W_c(X) \).

\begin{corollary}
Let \( s_1 s_2 \cdots s_n \cdots s_2 s_1 \) be a reduced expression for \( w \in W(A_n) \) and let \( x \in W(A_n) \) be a Coxeter element. Then \( D_{x,w} = \varepsilon_x \varepsilon_w \).
\end{corollary}

\begin{proof}
If \( n = 1 \) then \( x = w \) and the statement follows trivially. Suppose \( n > 1 \). If \( u, v \in W(A_n) \) are such that \([u, v] \simeq B_{\ell(u,v)}\), then \( P_{u,v} = 1 \) (see [Bre97b, Corollary 4.12]). Therefore, Theorem 4.1.12 implies that \( P_{u,v} = 1 \), for all \( u, v \in [x, w] \). Hence, from Theorem 4.1.6 and Theorem 4.1.12 we achieve

\[
D_{x,w} = \sum \left( (-1)^k \prod_{i=1}^{k} P_{x_{i-1},x_i} \right) = \sum (-1)^k,
\]

where the sum runs over all the chains \( x = x_0 < x_1 < \cdots < x_k = w \) such that \( 1 \leq k \leq \ell(x, w) \). Therefore, \( D_{x,w} \) equals the alternating sum

\[
\sum_{k=1}^{\ell(x,w)} (-1)^k c_k,
\]

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where \( c_k \) denotes the number of chains \( x = x_0 < x_1 < \cdots < x_k = w \). By \([\text{Sta97}, \text{Proposition 3.8.5}]\), we get

\[
\sum_{k=1}^{\ell(x,w)} (-1)^k c_k = \mu(x, w),
\]

where \( \mu \) denotes the Möbius function on the poset induced by the Bruhat order on \([x, w]\). On the other hand, if \( \mathcal{P} = (\mathcal{P}, \leq) \) is a boolean poset and \( U, V \in \mathcal{P} \) are such that \( U \leq V \), then \( \mu_\mathcal{P}(U, V) = (-1)^{|V-U|} \), where \(|V-U|\) denotes the length of the interval \([U, V]\) (see, e.g., \([\text{Sta97}, \text{Example 3.8.3}]\)). Finally, observe that the length of a Bruhat interval \([x, w]\) is \( \ell(x, w) \) (see, e.g., \([\text{BB05, Theorem 2.2.6}]\)) and the statement follows. \( \square \)

For instance, in Example \([1.4.3]\) we showed that \( D_{x,w} = \varepsilon_x \varepsilon_w = 1 \), for every \( x \in \{s_1s_2s_3, s_1s_3s_2, s_2s_1s_3, s_3s_2s_1\} \), with \( w = s_1s_2s_3s_2s_1 \).
4.2 Combinatorial properties of $a_{x,w}$

In this section we study the family of polynomials $\{a_{x,w}\}_{x,w \in W_c(X)} \subseteq \mathbb{Z}[q]$, which express the involution $\iota$ in terms of the $t$–basis (see Proposition 3.2.3). More precisely, we obtain a recurrence relation for $a_{x,w}$, $X$ being an arbitrary Coxeter graph. Then we will focus on the Coxeter graph satisfying equation (3.3) and derive some results concerning symmetry properties, and the value of the constant term of $a_{x,w}$.

**Proposition 4.2.1.** Let $X$ be an arbitrary Coxeter graph. Let $w \in W_c(X)$ and $s \in S(X)$ be such that $w > ws \in W_c(X)$. Then, for all $x \in W_c(X)$, $x \leq w$, we have

$$a_{x,w} = \tilde{a}_{x,w} + \sum_{y \in W_c(X), y < y} D_{x,y,s}a_{y,ws},$$

where

$$\tilde{a}_{x,w} \overset{\text{def}}{=} \begin{cases} a_{xs,ws} & \text{if } x > xs, \\ qa_{xs,ws} + (1 - q)a_{x,ws} & \text{if } x < xs \in W_c(X), \\ (1 - q)a_{x,ws} & \text{if } x < xs \not\in W_c(X). \end{cases}$$

**Proof.** On the one hand, by Proposition 3.2.3, we have

$$(t_w^{-1})^{-1} = q^{-\ell(w)} \sum_{y \in W_c(X), y \leq w} a_{y,w}t_y.$$
On the other hand, letting \( v \overset{\text{def}}{=} w s \), we get

\[
(t_{w^{-1}})^{-1} = (t_{v^{-1}})^{-1}(t_s)^{-1}
= q^{-\ell(w)} \sum_{y \in W_e(X)} a_{y,v} t_y \cdot q^{-1}(t_s - (q - 1)t_e)
= q^{-\ell(w)} \left( \sum_{y \in W_e(X)} a_{y,v} t_y t_s - (q - 1) \sum_{y \in W_e(X)} a_{y,v} t_y \right)
+ q^{-\ell(w)} \left( \sum_{y \in W_e(X), ys \in W_e(X)} a_{y,v} t_{ys} + \sum_{y \in W_e(X), ys \in W_e(X)} a_{y,v} t_{ys} \right)
+ q^{-\ell(w)} \left( \sum_{y \in W_e(X)} a_{y,v} (q t_{ys} + (q - 1)t_y) - (q - 1) \sum_{y \in W_e(X)} a_{y,v} t_y \right)
= q^{-\ell(w)} \left( \sum_{y \in W_e(X)} a_{y,v} t_{ys} + \sum_{y \in W_e(X), ys \in W_e(X)} a_{y,v} \left( \sum_{z \in W_e(X)} D_{z,ys} t_z \right) \right)
+ q^{-\ell(w)} \left( \sum_{y \in W_e(X)} a_{y,v} q t_{ys} + (q - 1) \sum_{y \in W_e(X)} a_{y,v} t_y - (q - 1) \sum_{y \in W_e(X)} a_{y,v} t_y \right)
\]

Observe that

\[
(q - 1) \sum_{y \in W_e(X)} a_{y,v} t_y - (q - 1) \sum_{y \in W_e(X)} a_{y,v} t_y = (1 - q) \sum_{y \in W_e(X)} a_{y,v} t_y.
\]
To sum up,

\[(t_{w-1})^{-1} = q^{-\ell(w)} \left( \sum_{x \in W_c(X)} a_{x,v} t_x + \sum_{x < vs} \left( \sum_{x' < x} D_{x,y} a_{y,v} \right) t_x \right) \]

\[+ q^{-\ell(w)} \left( q \sum_{xs \in W_c(X)} a_{xs,v} t_x + (1 - q) \sum_{x < xs} a_{x,v} t_x \right), \]

and the statement follows by extracting the coefficient of \( t_x \).

From now on, we assume \( X \) to be such that equation (3.3) holds.

\textbf{Corollary 4.2.2.} Let \( x, w \in W_c(X) \). If there exists \( s \in S(X) \) such that \( ws < w \) and \( x < xs \notin W_c(X) \), then

\[ a_{x,w} = -qa_{x,ws}. \]

\textbf{Proof.} By applying Proposition 4.2.1 and Lemma 4.1.5, we have

\[ a_{x,w} = (1 - q)a_{x,ws} + \sum_{y \in W_c(X), y < ws} D_{x,y} a_{y,ws} \]

\[ = (1 - q)a_{x,ws} + \sum_{y \in W_c(X), y < ws} (-\delta_{x,y}) a_{y,ws} \]

\[ = (1 - q)a_{x,ws} - a_{x,ws}, \]

and the statement follows.

Next we obtain, using the results in \S 4.1, a non–recursive formula for polynomials \( \{a_{x,w}\} \), an expression for their constant term, and symmetry properties.

\textbf{Proposition 4.2.3.} Let \( x, w \in W_c(X) \) be such that \( x \leq w \). Then

\[ a_{x,w} = \varepsilon_x \varepsilon_w R_{x,w} + \sum_{y \in W_c(X), x < y < w} \varepsilon_y \varepsilon_w R_{y,w} \left( \sum_{i=1}^k (-1)^i \prod_{i=1}^k P_{x_{i-1},x_i} \right), \]

where the second sum runs over all the chains \( x = x_0 < \cdots < x_k = y \) such that \( x_i \notin W_c(X) \) if \( i > 0 \).
Proof. From Proposition 3.2.3 we get
\[ a_{x,w} = \sum_{x \leq y \leq w} \varepsilon_y \varepsilon_w R_{y,w} D_{x,y}, \]
for all \( x, w \in W_c(X) \) such that \( x \leq w \). Since \( D_{x,w} = \delta_{x,w} \) if \( x, w \in W_c(X) \), we have
\[ a_{x,w} = \varepsilon_x \varepsilon_w R_{x,w} + \sum_{y \in W_c(X) \atop x < y < w} \varepsilon_y \varepsilon_w R_{y,w} D_{x,y}, \]
so the statement follows immediately from Theorem 4.1.6. \( \square \)

The recursion given in Corollary 4.2.2 can sometimes be solved explicitly.

**Proposition 4.2.4.** Let \( w = s_{i_1} s_{i_2} \cdots s_{i_k} \cdot s_{i_{j-1}} s_{i_j} \cdots s_{i_k} \in W(A_n) \) and let \( x = s_{i_1} s_{i_2} \cdots s_{i_k} \in W(A_n) \), with \( i \in [2,n] \), \( k \in [1,n-i] \), \( j \in [1,i-1] \). Then
\[ a_{x,w} = (-q)^k (1-q)^j. \]

**Proof.** Note that the one–line notation for \( w \), as a permutation in \( S_{n+1} \), is
\[ w = [w_1, w_2, \cdots, w_{i+k-1}, i - j, i, w_{i+k+2}, \cdots, w_{n+1}], \]
with \( w_1 < w_2 < \cdots < w_{i+k-1} < w_{i+k+2} < \cdots < w_{n+1} \). Hence, \( w \) avoids the pattern 321, that is \( w \in W_c(A_n) \) (see Proposition 1.7.2). Moreover \( inv(w) = 2k + j + 1 \). On the other hand, \( \ell(w) = inv(w) \) (see Proposition 1.5.2), so \( w \) is reduced.

Observe that \( x < x s_{i+h} \not\in W_c(X) \), for every \( h \in [0,k-1] \) and that \( w s_{i+k-1} < w \). By applying Corollary 4.2.2 to the triple \((x, w, s_{i+k-1})\) we get \( a_{x,w} = -qa_{x,w s_{i+k-1}} \). Repeat the same process with the triple \((x, w s_{i+k-1}, s_{i+k-2})\), and so on. After \( k \) iteration of the process we get \( a_{x,w}(q) = (-q)^k a_{x,w s_{i+k-1} \cdots s_i} = (-q)^k a_{x,w'}(q) \), where we set
\[ w' = s_{i_1} s_{i_2} \cdots s_{i_k} s_{i_{j-1}} s_{i_j} \cdots s_{i_k}. \]
To conclude, we show that \( a_{x,w'} = (1-q)^j \). Observe that \([x, w'] \simeq B_{\ell(w')-\ell(x)} \), so \( R_{x,w'} = (q-1)^{\ell(w')-\ell(x)} \) (see [Bre97b, Corollary 4.10]). On the other hand, Proposition 1.2.3 implies \( a_{x,w'} = \varepsilon_x \varepsilon_w R_{x,w'} \), since \( \{y \in [x,w'] : y \not\in W_c(X)\} = \emptyset \). Therefore \( a_{x,w'} = \varepsilon_x \varepsilon_w (q-1)^{\ell(w')-\ell(x)} = (1-q)^j \), as desired. \( \square \)
Example 4.2.5. Let $w = s_6s_7s_8s_9s_4s_6s_7s_8 \in W(A_{10})$ and $x = s_6s_7s_8s_9 \in W(A_{10})$. Then $n = 10, i = 6, k = 3, j = 2$ and $\underline{w} = [1, 2, 3, 5, 7, 8, 9, 10, 4, 6, 11]$. By Proposition 4.2.4 $a_{x,w} = -q^3(q^2 - 2q + 1)$.

Proposition 4.2.3 allows us to compute the constant term of the polynomials $\{a_{x,w}\}_{x,w \in W_c(X)}$.

Corollary 4.2.6. For all $x, w \in W_c(X)$ such that $x < w$ we have

\begin{enumerate}[(i)]
\item $a_{x,w}(1) = 0$;
\item $a_{x,w}(0) = \sum (-1)^k$,
\end{enumerate}

where the sum runs over all the chains $x = x_0 < x_1 < \cdots < x_{k+1} = w$ such that $x_i \notin W_c(X)$ if $1 \leq i \leq k$, and $0 \leq k \leq \ell(x, w) - 1$.

Proof. The statement follows from (4.4) by applying Corollary 2.2.3 and Corollary 4.1.8.

Again, we deduce from Proposition 4.2.3 the following symmetry properties of the polynomials $\{a_{x,w}\}_{x,w \in W_c(X)}$.

Corollary 4.2.7. Let $x, w \in W_c(X)$. Then

\begin{enumerate}[(i)]
\item $a_{x,w} = a_{x^{-1},w^{-1}}$;
\item $a_{x,w} = a_{w_0xw_0,w_0w_0}$.
\end{enumerate}

Proof. By Lemma 1.7.4 and by (4.3) we get

$$a_{x^{-1},w^{-1}} = \sum_{x^{-1} \leq y \leq w^{-1}} \varepsilon_y \varepsilon_{w^{-1}} R_{y,w^{-1}} D_{x^{-1},y}$$

$$= \sum_{x^{-1} \leq z \leq w^{-1}} \varepsilon_z \varepsilon_{w^{-1}} R_{z,w^{-1}} D_{x^{-1},z^{-1}}$$

$$= \sum_{x \leq z \leq w} \varepsilon_z \varepsilon_{w} R_{z,w} D_{x,z}$$

$$= a_{x,w},$$

where we used Corollary 4.1.7 (i) and the property $R_{x,w} = R_{x^{-1},w^{-1}}$, for all $x, w \in W(X)$ (see, e.g., [BB05, §5, Exercise 10(a)]). The same holds for $a_{w_0xw_0,w_0w_0}$, using Corollary 4.1.7 (ii) and [BB05, §5, Exercise 10(b)].
Corollary 4.2.8. Let $x, w \in W_c(X)$ and $x \leq w$. Then $a_{x,w}$ has degree $\ell(x, w)$ and leading term $\varepsilon_x \varepsilon_w$.

Proof. The statement follows from Proposition 4.2.3 combining Proposition 4.1.11 with Corollary 2.2.3. □

Next, we obtain a property for polynomials $\{a_{x,w}\}$ that will be required in Section 4.3.

Proposition 4.2.9. Let $w \in W_c(X)$. Then

$$\sum_{x \in W_c(X)} \sum_{x \leq w} \varepsilon_x \varepsilon_w a_{x,w} = q^{\ell(w)}.$$

Proof. By combining equation (4.4) with Proposition 4.1.9 we get

$$\sum_{x \in W_c(X)} \sum_{x \leq w} \varepsilon_x \varepsilon_w a_{x,w} = \sum_{x \in W_c(X)} \sum_{x \leq w} \varepsilon_x \varepsilon_w R_{x,w} + \sum_{y \in W_c(X)} \sum_{x < y < w} \varepsilon_y \varepsilon_w R_{y,w} D_{x,y}$$

$$= \sum_{x \in W_c(X)} R_{x,w} + \sum_{y \in W_c(X)} \sum_{x < y < w} \varepsilon_x \varepsilon_y R_{y,w} D_{x,y}$$

$$= \sum_{x \in W_c(X)} R_{x,w} + \sum_{y \in W_c(X)} \sum_{x < y < w} \varepsilon_x \varepsilon_y R_{y,w} D_{x,y}$$

$$= \sum_{x \leq w} R_{x,w}$$

and the statement follows from Proposition 2.2.6. □

We refer to $\{y \in [x, w] : \ell(y) = \ell(x) + 1\}$ as the set of atoms in $[x, w]$ and denote by $a(x, w)$ the number of atoms in $[x, w]$. 55
Corollary 4.2.10. Let \( w \in W_e(X) \) be such that \( \ell(w) > 1 \). Then,

\[
[q^{\ell(w)-1}]a_{e,w} = -\varepsilon_w a(e, w).
\]

Proof. By Proposition 4.2.9, we get \( \varepsilon_w a_{e,w} = q^{\ell(w)} - \sum_{e < y \leq w} \varepsilon_y \varepsilon_w a_{y,w} \). Therefore, by Corollary 4.2.8, we achieve

\[
[q^{\ell(w)-1}]a_{e,w} = -\varepsilon_w \sum_{y \in (e,w]} 1 = -\varepsilon_w a(e, w).
\]

\[\square\]

Lemma 4.2.11. Let \( x, w \in W_e(X) \) be such that \( x < w \). If \( \ell(x,w) = 1 \) then \( a_{x,w} = 1 - q \). If \( \ell(x,w) = 2 \), then

\[
a_{x,w} = \begin{cases} 
q^2 - 1 & \text{if } k = 2, \\
q^2 - q & \text{if } k = 1, \\
q^2 - 2q + 1 & \text{if } k = 0,
\end{cases}
\]

with \( k = |\{ y \not\in W_e(X) : x < y < w \}| \).

Proof. By Lemma 2.3.3, \( R_{x,w} = (q - 1)^{\ell(x,w)} \) for all \( x, w \in W(X) \), \( x \leq w \) such that \( \ell(x,w) \leq 2 \). If \( \ell(x,w) = 1 \), then Equation (4.4) implies \( a_{x,w} = \varepsilon_x \varepsilon_w R_{x,w} = 1 - q \). If \( \ell(x,w) = 2 \), then the interval \( [x,w] \) is isomorphic to the boolean lattice \( B_2 \) (see, e.g., [BB05, Lemma 2.7.3]). Moreover \( R_{y,w} = q - 1 \) and \( D_{x,y} = -1 \) for all \( y \not\in W_e(X) \) such that \( x < y < w \), since \( \ell(x,y) = \ell(y,w) = 1 \) (see Lemma 4.1.10). From Equation (4.4) we get

\[
a_{x,w} = \varepsilon_x \varepsilon_w R_{x,w} + \sum_{y \not\in W_e(X)} \varepsilon_y \varepsilon_w R_{y,w} D_{x,y}
\]

\[
= (q - 1)^2 + \sum_{y \not\in W_e(X)} (q - 1)
\]

\[
= (q - 1)^2 + k(q - 1),
\]

and the statement follows. //
4.3 Combinatorial properties of $L_{x,w}$

In this section we study the polynomials $\{L_{x,w}\}_{x,w \in W_c(X)}$ which play the same role, in $TL(X)$, as the Kazhdan–Lusztig polynomials play in $H(X)$. First, we derive a recursive formula for $L_{x,w}$ by means of some results in [Gre07]. Then, using the results in Section 4.1, we obtain a non–recursive formula, symmetry properties, expressions for the constant term, and bounds for the degrees, for these polynomials. All the results stated in this section hold for every Coxeter graph $X$ satisfying (3.3).

It is known that the terms of maximum possible degree in the $L$–polynomials and in the Kazhdan–Lusztig polynomials coincide (see [Gre07, Theorem 5.13]).

**Proposition 4.3.1.** For $x, w \in W_c(X)$ let $M(x, w)$ be the coefficient of $q^{-\frac{1}{2}}$ in $L_{x,w}$ and let $\mu(x, w)$ be the coefficient of $q^{\ell(w) - \ell(x) - \frac{1}{2}}$ in $P_{x,w}$. Then $M(x, w) = \mu(x, w)$.

The product of two IC basis elements can be computed by means of the following formula (see [Gre07, Theorem 5.13]). Recall that if $\mu(x, w) \neq 0$ then we write $x \prec w$ (see Definition 2.3.4).

**Proposition 4.3.2.** Let $s \in S(X)$ and $w \in W_c(X)$. Then

$$c_sc_w = \begin{cases} 
c_{sw} + \sum_{x < w, x \prec w} \mu(x, w)c_x & \text{if } \ell(sw) > \ell(w); \\
(q^{\frac{1}{2}} + q^{-\frac{1}{2}})c_w & \text{otherwise,}
\end{cases}$$

where $c_x \overset{\text{def}}{=} 0$ for every $x \not\in W_c(X)$.

**Corollary 4.3.3.** Let $s \in S(X)$ and $w \in W_c(X)$. Then

$$t_sc_w = \begin{cases} 
-c_w + q^{\frac{1}{2}}\left(c_{sw} + \sum_{x < w, x \prec w} \mu(x, w)c_x\right) & \text{if } \ell(sw) > \ell(w); \\
qc_w & \text{otherwise.}
\end{cases}$$

**Proof.** Observe that $t_s = q^{\frac{1}{2}}c_s - c_e$. So $t_sc_w = q^{\frac{1}{2}}c_sc_w - c_w$ and the statement follows by applying Proposition 4.3.2. \qed
Theorem 4.3.4. Let \( x, w \in W_c(X) \) be such that \( sx \in W_c(X) \) and \( sw < w \). Then

\[
L_{x,w} = L_{sx,sw} + q^{-\frac{1}{2}}L_{x,sw} - \sum_{z \in [sx,sw]_c} \mu(z,sw)L_{x,z}
\]

\[+ q^{-\frac{1}{2}} \sum_{sz \not\in W_c(X) \atop z \in [x,w]_c} q^{r(z) - r(s)} D_{x,sz} L_{z,sw},
\]

where \( c = 1 \) if \( sx < x \) and \( 0 \) otherwise.

Proof. Let \( w = sv \). By Proposition 4.3.2, we have

\[
c_w = c_{sv} = c_sc_v - \sum_{sz < z} \mu(z,sw)c_z.
\] (4.5)

Recall that \( c_s = q^{-\frac{1}{2}} (t_s + t_c) \), hence we get

\[
c_s c_v = q^{-\frac{1}{2}} c_v + q^{-\frac{1}{2}} t_s c_v
\]

\[
= q^{-\frac{1}{2}} c_v + \sum_{x \in W_c(X) \atop x \leq sw} q^{-\frac{r(x)}{2}} L_{x,sw} t_s x
\]

\[
= q^{-\frac{1}{2}} \left( c_v + \sum_{sx \in W_c(X) \atop x < sx} q^{r(x)} L_{x,sw} t_s x + \sum_{sx < x} q^{-\frac{r(x)}{2}} L_{x,sw} (qt_{sx} + (q-1)t_x) \right)
\]

\[+ q^{-\frac{1}{2}} \sum_{sx \not\in W_c(X) \atop x < sx} q^{r(x)} L_{x,sw} \left( \sum_{y \in W_c \atop y < sx} D_{y,sx} t_y \right) \]

\[
= q^{-\frac{1}{2}} \left( c_v + \sum_{sx \in W_c(X) \atop x < sx} q^{r(x)} L_{x,sw} t_s x + \sum_{sx < x} q^{-\frac{r(x)}{2}} L_{x,sw} (qt_{sx} + (q-1)t_x) \right)
\]

\[+ q^{-\frac{1}{2}} \left( \sum_{y \in W_c(X) \atop y \leq w} \left( \sum_{sx \not\in W_c(X) \atop x < sx} q^{r(x)} D_{y,sx} L_{x,sw} \right) t_y \right).
\]

Suppose that \( su > u \) and extract the coefficient of \( t_{su} \) on both sides of (4.5). It follows that

\[
L_{su,w} = L_{u,sw} + q^{\frac{1}{2}} L_{su,sw} + \sum_{sx \not\in W_c(X) \atop x < sz} q^{r(x) - r(s)} D_{su,sz} L_{z,sw} - \sum_{z \in [u,w]_c \atop sz < z} \mu(z,sw)L_{su,z}.
\]

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Otherwise, if $su < u$ then

$$L_{su,w} = L_{u,sw} + q^{-\frac{1}{2}} L_{su,sw} + q^{-1} \sum_{s \in \mathcal{W}_c(X)} \mu(s) L_{su,z} - \mu(z, sw) L_{su, z},$$

and the statement follows by applying the substitution $x = su$.

\[ \square \]

**Theorem 4.3.5.** Let $x, w \in \mathcal{W}_c(X)$ be such that $x < w$. If there exists $s \in S(X)$ such that $sw < w$ and $x < sx \in \mathcal{W}_c(X)$, then

$$L_{x,w} = q^{-\frac{1}{2}} L_{sx,w} - \sum_{s \in \mathcal{W}_c(X)} q^{\ell(x) - \ell(x)} D_{sx,sz} L_{x,w}.$$ 

**Proof.** By Corollary 4.3.3 we get $t_s c_w = q c_w$, since $\ell(sw) < \ell(w)$ by hypothesis. Furthermore, if $x < sx \in \mathcal{W}_c(X)$ then $[t_{sx}](qc_w) = q \cdot q^{-\frac{1}{2}} L_{sx,w}$. On the other hand, by Theorem 3.2.7 we get

$$t_s c_w = \sum_{x \in \mathcal{W}_c(X)} q^{\ell(x)} L_{x,w} t_s t_x$$

$$= \sum_{s \in \mathcal{W}_c(X)} q^{\ell(x)} L_{x,w} t_{sx} + \sum_{s \in \mathcal{W}_c(X)} q^{\ell(x)} L_{x,w} t_{sz} + \sum_{x \in \mathcal{W}_c(X)} q^{\ell(x)} L_{x,w} (qt_{sx} + (q - 1)t_x)$$

$$= \sum_{s \in \mathcal{W}_c(X)} q^{\ell(x)} L_{x,w} t_{sx} + \sum_{s \in \mathcal{W}_c(X)} q^{\ell(x)} L_{x,w} \left( \sum_{y \in \mathcal{W}_c(X)} D_{y,sz} t_y \right) +$$

$$+ q \sum_{s \in \mathcal{W}_c(X)} q^{\ell(x)} L_{sx,w} t_z + (q - 1) \sum_{s \in \mathcal{W}_c(X)} q^{\ell(x)} L_{sz,w} t_{sz}$$

$$= \sum_{s \in \mathcal{W}_c(X)} q^{\ell(x)} L_{x,w} t_{sx} + q^2 q^{\ell(x)} L_{sx,w} t_x + q^2 q^{\ell(x)} L_{sx,w} t_{sx} +$$

$$- q^{-\frac{1}{2}} q^{\ell(x)} L_{sx,w} t_{sx} + \sum_{x \in \mathcal{W}_c(X)} q^{-\frac{1}{2}} D_{x,sz} L_{x,w} t_x. \ (4.6)$$

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By extracting the coefficient of \( t_{sx} \) in (4.6) we obtain
\[
q^{\frac{1}{2}}q^{-\ell(x)} L_{sx,w} = q^{-\ell(x)} L_{x,w} + q^{\frac{1}{2}} q^{-\ell(x)} L_{sx,w} - q^{-\ell(x)} L_{sx,w} + \sum_{sz \in W_c(X) \atop z \in [x,w]} q^{-\ell(z)} D_{sx,sz} L_{z,w},
\]
and the statement follows.

The following result was inspired by a similar property for the Kazhdan–Lusztig polynomials (see, e.g., [BB05, §5, Exercises 16]).

**Proposition 4.3.6.** Let \( w \in W_c(X) \) and define
\[
F_w(q^{-\frac{1}{2}}) \overset{\text{def}}{=} \sum_{x \in W_c(X) \atop x \leq w} \varepsilon_x q^{-\ell(x)} L_{x,w}(q^{-\frac{1}{2}}).
\]
Then \( F_w(q^{-\frac{1}{2}}) = \delta_{e,w} \).

**Proof.** The case \( w = e \) is trivial. Suppose \( w \neq e \). Combining Theorem 3.2.6 (iv) with Proposition 4.2.9 we have
\[
F_w(q^{-\frac{1}{2}}) = \sum_{u \in W_c(X) \atop u \leq w} \varepsilon_u q^{-\ell(u)} \left( \sum_{x \in W_c(X) \atop u \leq x \leq w} q^{-\ell(u)-\ell(x)} a_{u,x}(q)L_{x,w}(q^{-\frac{1}{2}}) \right)
\]
\[
= \sum_{x \in W_c(X) \atop x \leq w} \varepsilon_x q^{-\ell(x)} L_{x,w}(q^{-\frac{1}{2}}) \left( \sum_{u \in W_c(X) \atop u \leq x} \varepsilon_u a_{u,x}(q) \right)
\]
\[
= \sum_{x \in W_c(X) \atop x \leq w} \varepsilon_x q^{-\ell(x)} L_{x,w}(q^{\frac{1}{2}}) q^{\ell(x)}
\]
\[
= \sum_{x \in W_c(X) \atop x \leq w} \varepsilon_x q^{-\ell(x)} L_{x,w}(q^{\frac{1}{2}})
\]
\[
= F_w(q^{\frac{1}{2}}).
\]
This implies that $F_w(q^{-\frac{1}{2}})$ is constant. On the other hand, the constant term in $F_w(q^{-\frac{1}{2}})$ is zero since $L_{x,w} \in q^{-\frac{1}{2}}\mathbb{Z}[q^{\frac{1}{2}}]$ by Theorem 3.2.6 and the statement follows.

The next result is a restatement of the well–known property $\mu(e,w) = 0$, for every $w \in W(X)$ such that $\ell(e,w) > 1$ (see, e.g., [BB05, Proposition 5.1.9]).

**Corollary 4.3.7.** Let $w \in W_c(X)$. Then

$$[q^{-\frac{1}{2}}]L_{e,w} = \begin{cases} 1 & \text{if } \ell(w) = 1, \\ 0 & \text{if } \ell(w) \neq 1. \end{cases}$$

**Corollary 4.3.8.** Let $w \in W_c(X)$. Then

$$[q^{-1}]L_{e,w} = \begin{cases} 0 & \text{if } \ell(w) < 2, \\ 1 & \text{if } \ell(w) = 2, \\ \sum_{s \in S(X)} \mu(s,w) & \text{if } \ell(w) > 2. \end{cases}$$

**Proof.** The case $\ell(w) \leq 1$ is trivial. If $\ell(w) = 2$ then $L_{e,w} = q^{-1}$ as explained at the end of this section. Suppose $\ell(w) > 2$. Then

$$[q^{-1}](L_{e,w}) = -\sum_{x \in W_c(X), \ell(x) = 1} \varepsilon_x[q^{-\frac{1}{2}}]L_{x,w} - \sum_{x \in W_c(X), \ell(x) = 2} \varepsilon_x[q^0]L_{x,w} - q^{-2}$$

$$= \sum_{s \in S(X), s \leq w} [q^{-\frac{1}{2}}]L_{s,w},$$

and the statement follows by Proposition 4.3.1.

The previous results would suggest that $[q^k]L_{e,w} \geq 0$, for every $k \in \mathbb{Q}$. However, this is not true even in $W(A_3)$. For instance, computer calculations show that $L_{e,s_2s_1s_3s_2} = q^{-1} - q^{-2}$.

**Theorem 4.3.9.** For all elements $x, w \in W_c(X)$ such that $x < w$ we have

$$L_{x,w} = q^{\frac{\ell(x) - \ell(w)}{2}} \sum \left( (-1)^k \prod_{i=1}^{k+1} P_{x_{i-1},x_i} \right),$$

where the sum runs over all the chains $x = x_0 < x_1 < \cdots < x_{k+1} = w$ such that $x_i \notin W_c(X)$ if $1 \leq i \leq k$, and $0 \leq k \leq \ell(x,w) - 1$. 

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Proof. On the one hand, from Proposition 3.2.1 and the definition of the t–basis we get
\[ \sigma(C'_{w}) = q^{-\ell(w)/2} \sum_{y \leq w} P_{y,w} \sigma(T_y) \]
\[ = q^{-\ell(w)/2} \sum_{y \leq w} P_{y,w} \left( \sum_{x \in W_c(X)} \left( \sum_{x \leq y \leq w} D_{x,y} t_x \right) \right) \]
\[ = q^{-\ell(w)/2} \sum_{x \in W_c(X)} \left( \sum_{x \leq y \leq w} D_{x,y} P_{y,w} \right) t_x. \] (4.7)

On the other hand, by Theorem 3.2.7,
\[ c_w = \sum_{x \in W_c(X)} q^{-\ell(x)/2} L_{x,w} t_x. \] (4.8)

Therefore, equation (3.3) implies that the coefficient of \( t_x \) in (4.8) and in (4.7) are equal, that is
\[ q^{-\ell(x)/2} L_{x,w} = q^{-\ell(w)/2} \sum_{x \leq y \leq w} D_{x,y} P_{y,w}. \]

Since \( D_{x,y} = \delta_{x,y} \) if \( y \in W_c(X) \) and \( x \leq y \), we achieve
\[ L_{x,w} = q^{-\ell(x) - \ell(w)/2} \left( P_{x,w} + \sum_{y \in W_c(X)} \sum_{x < y < w} D_{x,y} P_{y,w} \right). \] (4.9)

Combining (4.9) and Theorem 4.1.6 we get
\[ L_{x,w} = q^{-\ell(x) - \ell(w)/2} \left( P_{x,w} + \sum_{y \in W_c(X)} \sum_{x < y < w} \left( -1 \right)^k \prod_{i=1}^{k} P_{x_i-1,x_i} \right) P_{y,w} \]
\[ = q^{-\ell(x) - \ell(w)/2} \left( \sum_{x = x_0 < \cdots < x_{k+1} = w} \left( -1 \right)^k \prod_{i=1}^{k+1} P_{x_i-1,x_i} \right), \]

where \( x_i \notin W_c(X) \) if \( 1 \leq i \leq k \). \( \square \)
The previous theorem shows that the $L$-polynomials depend only on the Kazhdan–Lusztig polynomials and the poset structure induced by the Bruhat order on $\{x, w\} \cup (\{(x, w) \setminus (x, w)\} \cup x, w \cap (x, w))$, where $(x, w)_c = \{y \in (x, w) : y \in W_c(X)\}$.

**Lemma 4.3.10.** Let $x, w \in W_c(X)$. If there exists $s \in S(X)$ such that $sw < w$ and $x < sx \not\in W_c(X)$, then $L_{x, w} = 0$.

**Proof.** By (4.9) we get

$$L_{x, w} = q^{\ell(x) - \ell(w)} \left( P_{x, w} + \sum_{y \in W_c(X), x < y < w} D_{x, y} P_{y, w} \right)$$

$$= q^{\ell(x) - \ell(w)} \left( P_{x, w} + D_{x, sx} P_{sx, w} + \sum_{y \in W_c(X), x < y < w} D_{x, y} P_{y, w} \right)$$

$$= q^{\ell(x) - \ell(w)} \left( \sum_{y \in W_c(X), x < y < w} D_{x, y} P_{y, w} \right). \quad (4.10)$$

Denote by $(\ast)$ the expression in round brackets in (4.10). We show that $(\ast)$ is zero and the statement follows. In fact, by applying relation (4.12) and Lemma 4.1.5, we get

$$(\ast) = \sum_{y \not\in W_c(X)} D_{x, y} P_{y, w} + \sum_{y \in W_c(X), y \neq sx} D_{x, y} P_{y, w}$$

$$= \sum_{y \in W_c(X), y < sx \not\in W_c(X)} D_{x, y} P_{y, w} + \sum_{y \not\in W_c(X), y \neq sx} D_{x, y} P_{y, w}$$

$$= \sum_{y \not\in W_c(X), y \neq sx} (D_{x, sy} + D_{x, y}) P_{y, w} + \sum_{y \not\in W_c(X), y \neq sx} D_{x, y} P_{y, w}$$

$$= \sum_{y \not\in W_c(X), y \neq sx} (\delta_{x, sy}) P_{y, w} = 0,$$

as desired. \[\square\]
The next result mirrors a well-known property of the Kazhdan–Lusztig polynomials (see, e.g., [BB05 Proposition 5.1.8]).

**Corollary 4.3.11.** Let $x, w \in W_c(X)$ be such that $x < w$. If there exists $s \in S(X)$ such that $sw < w$ and $x < sx \in W_c(X)$, then $L_{x,w} = q^{-\frac{1}{2}}L_{sx,w}$.

**Proof.** The result follows by combining Theorem 4.3.5 with Lemma 4.3.10. □

In the same way that Corollary 4.1.7 follows from Theorem 4.1.6 we deduce from Theorem 4.3.9 the following symmetry properties of the $L$-polynomials, whose proof we omit.

**Corollary 4.3.12.** Let $x, w \in W_c(X)$. Then

(i) $L_{x,w} = L_{x^{-1}, w^{-1}}$;

(ii) $L_{x,w} = L_{w_0xw_0, w_0xw_0}$.

□

**Lemma 4.3.13.** Let $x, w \in W_c(X)$ be such that $x < w$. If $\ell(x, w) = 1$ then $L_{x,w} = q^{-\frac{1}{2}}$. If $\ell(x, w) = 2$, then

$$L_{x,w} = \begin{cases} 
- q^{-1} & \text{if } k = 2, \\
0 & \text{if } k = 1, \\
q^{-1} & \text{if } k = 0,
\end{cases}$$

with $k = |\{y \in W_c(X) : x < y < w\}|$.

**Proof.** If $\ell(x, w) = 1$, then $P_{x,w} = 1$ (see Lemma 2.3.3). Equation (4.9) then implies that $L_{x,w} = q^{-\frac{1}{2}}P_{x,w} = q^{-\frac{1}{2}}$. If $\ell(x, w) = 2$, then from Equation (4.9)
and Lemma 4.1.10 we get

\[
L_{x,w} = q \frac{d(x)-d(w)}{2} \left( P_{x,w} + \sum_{y \in W(x), \ x < y < w} D_{x,y} P_{y,w} \right)
\]

\[
= q^{-1} \left( 1 + \sum_{y \in W(x), \ x < y < w} (-1) \right)
\]

\[
= q^{-1} (1 - k)
\]

and the statement follows. \qed
Appendix A

Coxeter systems

A.1 Finite irreducible Coxeter systems

\[ s_1 s_2 s_3 \cdots s_{n-1} s_n \]

\[ A_n \quad (n \geq 1) \]

\[ s_0 4 s_1 s_2 \cdots s_{n-2} s_{n-1} \]

\[ B_n \quad (n \geq 2) \]

\[ s_2 s_3 s_4 \cdots s_{n-1} s_n \]

\[ D_n \quad (n \geq 4) \]

Figure A.1: Finite irreducible Coxeter systems (part I).
Figure A.2: Finite irreducible Coxeter systems (part II).
A.2 Affine Coxeter systems

\[ \tilde{A}_1 \]

\[ \tilde{A}_{n-1} \quad (n \geq 3) \]

\[ \tilde{B}_n \quad (n \geq 3) \]

\[ \tilde{C}_n \quad (n \geq 2) \]

\[ \tilde{D}_n \quad (n \geq 4) \]

Figure A.3: Affine Coxeter systems (part I).
Figure A.4: Affine Coxeter systems (part II).
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