Research Article

Optimal Weak Type Estimates for the P-ADIC Hardy Type Operators on Higher-Dimensional Product Spaces

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In this paper, we introduce the fractional p-adic Hardy operators and its conjugate operators and obtain its optimal weak type estimates on the p-adic Lebesgue product spaces.

1. Introduction

In recent years, p-adic analysis has been widely used in quantum mechanics, the probability theory, and the dynamical systems [1, 2]. Meanwhile, there is an increasing attention in pseudo-differential equations, wavelet theory, and harmonic analysis (see [3–8]).

For a prime number p, let \( \mathbb{Q}_p \) be the field of p-adic numbers, a nonzero rational number \( x \) is represented as \( x = p^m y \), where \( y \) is an integer and the integers \( m, n \) are not divisible by \( p \). Then, the norm is defined as \( |x|_p = p^{-m} \), and it is easy to see that the norm satisfies the following properties:

1. \( |x|_p \geq 0, \forall x \in \mathbb{Q}_p, |x|_p = 0 \Leftrightarrow x = 0 \)
2. \( |x y|_p = |x|_p |y|_p, \forall x, y \in \mathbb{Q}_p \)
3. \( |x + y|_p \leq \max\{ |x|_p, |y|_p \}, \forall x, y \in \mathbb{Q}_p, \) in the case when \( |x|_p \neq |y|_p \), we have \( |x + y|_p = \max\{ |x|_p, |y|_p \} \)

It is well known that \( \mathbb{Q}_p \) is a typical model of non-Archimedean local fields. From the standard p-adic analysis, any \( x \in \mathbb{Q}_p \setminus \{0\} \) can be uniquely represented as a canonical form

\[
x = p^r \sum_{k=0}^{\infty} a_k p^k,
\]

where \( a_k, y \in \mathbb{Z}, a_0 \neq 0 \leq a_k < p \), note that the series (1) converges with respect to the norm \( |x|_p \) because one has \( |p^i a_k p^k|_p = p^{r-k} \). The space \( \mathbb{Q}_p^n \) consists of elements \( x = (x_1, x_2, \ldots, x_n) \), where \( x_i \in \mathbb{Q}_p, \) \( i = 1, 2, \ldots, n \). The norm in this space is

\[
|x|_p = \max_{1 \leq i \leq n} \{ |x_i|_p \}, \quad x \in \mathbb{Q}_p^n.
\]

The symbols \( B_x(a) \) and \( S_x(a) \) represent, respectively, the ball and the sphere with center at \( a \in \mathbb{Q}_p^n \) and radius \( p^i \), defined by

\[
B_x(a) = \{ x \in \mathbb{Q}_p^n : |x - a|_p \leq p^i \}, \quad S_x(a) = \{ x \in \mathbb{Q}_p^n : |x - a|_p = p^i \}.
\]

It is clear that \( S_y(a) = B_y(a)/B_{y-1}(a) \), and we set \( B_y(0) = B_y \) and \( S_y(0) = S_y \).

As \( \mathbb{Q}_p^n \) is a locally compact commutative group with respect to addition, there exists a Harr measure \( dx \) on \( \mathbb{Q}_p^n \), which is unique up to a positive constant factor and is translation invariant, that is, \( d(x + a) = dx \). We normalize the measure \( dx \) such that

\[
\int_{B_0(0)} dx = |B_0(0)|_H = 1,
\]

where \( |B|_H \) denotes the Harr measure of a measure subset \( B \) of \( \mathbb{Q}_p^n \). By simple calculation, we can obtain that

\[
|B_y(a)|_H = p^{in}, \quad |S_y(a)|_H = p^i(1 - p^{-n}).
\]

The classical Hardy operator
defines the fractional p-adic Hardy operator on higher-dimensional product spaces as follows:

$$H \mathcal{H} f (x) := \frac{1}{x} \int_0^x f (t) \, dt, \quad x > 0$$  \quad (6)

was introduced by Hardy in [9], and a celebrated integral inequality states that

$$\| \mathcal{H} f \|_{L^q (\mathbb{R}^n)} \leq \frac{q}{q - 1} \| f \|_{L^p (\mathbb{R}^n)}, \quad 1 < q < \infty.$$  \quad (7)

The fractional Hardy operator and its adjoint are defined and studied in [19–26]. Faris in [10] and Christ and Grafakos in [11] proposed an extension of (1) and its adjoint to the n-dimensional Euclidean spaces \( \mathbb{R}^n \) of which the equivalent forms are

$$H f (x) = \frac{1}{|x|} \int_{|y| \leq |x|} f (y) \, dy, \quad H^* f (x) = \frac{1}{|x|} \int_{|y| \geq |x|} f (y) \, dy,$$  \quad (8)

$$H^p_{\beta} f (x) = \frac{1}{|x|^{\frac{n-\beta}{p}}} \int_{|y| \leq |x|} f (y) \, dy, \quad H^{p,*}_{\beta} f (x) = \frac{1}{|x|^{\frac{n-\beta}{p}}} \int_{|y| \geq |x|} f (y) \, dy,$$  \quad (9)

when \( \beta = 0 \), the fractional p-adic Hardy and adjoint Hardy operator reduces to p-adic Hardy and adjoint Hardy operator. Some other papers showing the boundedness of p-adic Hardy-type operators are included [19–26].

In 2020, Li et al. [27] introduced the definition of the fractional Hardy operator on higher-dimensional product spaces as follows:

$$H_{\beta_1, \ldots, \beta_m} f (x) := \left( \prod_{i=1}^m \frac{1}{|B (0, |x_i|)|^{1-\beta_i/m_i}} \right) \int_{|y_1| < |x_1|} \cdots \int_{|y_m| < |x_m|} f (y_1, \ldots, y_m) \, dy_1 \cdots dy_m,$$  \quad (10)

where \( f \) be a nonnegative integrable function on \( \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_m}, \quad x_i \in \mathbb{R}^{n_i}, \quad m \in \mathbb{N}, 0 \leq \beta_i < n_i, \quad i = 1, \ldots, m, \quad x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_m}, \) with \( \prod_{i=1}^m |x_i| \neq 0 \). Furthermore, the corresponding operator norm on the weak Lebesgue product spaces was obtained.

Next, we will introduce the definition of the fractional Hardy operator on higher-dimensional p-adic product spaces and obtain sharp weak bounds.

$$H^p_{\beta_1, \ldots, \beta_m} f (x) := \frac{1}{|Q_p^{n_1} \times \cdots \times Q_p^{n_m}|^{1-\beta_i/n_i}} \int_{|y_1| < |x_1|} \cdots \int_{|y_m| < |x_m|} f (y_1, \ldots, y_m) \, dy_1 \cdots dy_m,$$  \quad (11)

where \( x = (x_1, x_2, \ldots, x_m) \in Q_p^{n_1} \times Q_p^{n_2} \times \cdots \times Q_p^{n_m} \) with \( \prod_{i=1}^m |x_i|_{p_i} \neq 0 \). It was also shown that the constant factor \( q! (q - 1) \) is optimal, knowing its importance in analysis.

Faris in [10] and Christ and Grafakos in [11] proposed an extension of (1) and its adjoint to the \( n \)-dimensional Euclidean spaces \( \mathbb{R}^n \) of which the equivalent forms are

$$H f (x) = \frac{1}{|x|} \int_{|y| \leq |x|} f (y) \, dy, \quad H^* f (x) = \frac{1}{|x|} \int_{|y| \geq |x|} f (y) \, dy,$$  \quad (8)

$$H^p_{\beta} f (x) = \frac{1}{|x|^{\frac{n-\beta}{p}}} \int_{|y| \leq |x|} f (y) \, dy, \quad H^{p,*}_{\beta} f (x) = \frac{1}{|x|^{\frac{n-\beta}{p}}} \int_{|y| \geq |x|} f (y) \, dy,$$  \quad (9)

when \( \beta = 0 \), the fractional p-adic Hardy and adjoint Hardy operator reduces to p-adic Hardy and adjoint Hardy operator. Some other papers showing the boundedness of p-adic Hardy-type operators are included [19–26].

In 2020, Li et al. [27] introduced the definition of the fractional Hardy operator on higher-dimensional product spaces as follows:

$$H_{\beta_1, \ldots, \beta_m} f (x) := \left( \prod_{i=1}^m \frac{1}{|B (0, |x_i|)|^{1-\beta_i/m_i}} \right) \int_{|y_1| < |x_1|} \cdots \int_{|y_m| < |x_m|} f (y_1, \ldots, y_m) \, dy_1 \cdots dy_m,$$  \quad (10)

Definition 1. Let \( m, n_i \in \mathbb{N}, x_i \in Q_p^{n_i}, 0 \leq \beta_i < n_i, \ i = 1, \ldots, m, \) and \( f \) be a nonnegative integrable function on \( Q_p^{n_1} \times Q_p^{n_2} \times \cdots \times Q_p^{n_m} \). Define the fractional p-adic Hardy operator on higher-dimensional product spaces by

$$H^p_{\beta_1, \ldots, \beta_m} f (x) := \frac{1}{|Q_p^{n_1} \times \cdots \times Q_p^{n_m}|^{1-\beta_i/n_i}} \int_{|y_1| < |x_1|} \cdots \int_{|y_m| < |x_m|} f (y_1, \ldots, y_m) \, dy_1 \cdots dy_m,$$  \quad (11)

where \( x = (x_1, x_2, \ldots, x_m) \in Q_p^{n_1} \times Q_p^{n_2} \times \cdots \times Q_p^{n_m} \) with \( \prod_{i=1}^m |x_i|_{p_i} \neq 0 \). In 2020, Wang et al. [28] gave the definition of fractional conjugate Hardy operator on higher-dimensional product spaces as follows:
where \( f \) be a nonnegative integrable function on \( \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_m} \), \( x_i \in \mathbb{R}^{n_i}, \ m \in \mathbb{N}, \ n \leq \beta < n_i, \ i = 1, \ldots, m, \) with \( \prod_{i=1}^{m} |x_i| \neq 0 \), and they also got the corresponding operator norm on the weak Lebesgue product spaces.

Next, we will give a higher-dimensional version of the fractional p-adic conjugate Hardy operator and obtain sharp weak bounds.

\[
H_{p_1, \ldots, p_m}^* f(x) = \int_{|y_1| > |x_1|} \cdots \int_{|y_m| > |x_m|} \left( \prod_{i=1}^{m} \frac{f(y_1, \ldots, y_m)}{|B(0, |y_i|)|^{1/p_i/m}} \right) dy_1 \cdots dy_m,
\]

\( \beta_1, \ldots, \beta_m \) are positive numbers.

2. Sharp Weak Bounds for Fractional Hardy Operators

This section considers the problem of obtaining optimal weak bounds for \( H_{p_1, \ldots, p_m}^* \) and our results as follows.

**Theorem 1.** Set \( 0 < \beta < n_i \), let \( Q = (n_1/(n_1 - \beta_1), \ldots, n_m/(n_m - \beta_m)) \), \( i = 1, \ldots, m \). If \( f \in L^1(Q_p^n \times Q_p^{n_2} \times \cdots \times Q_p^{n_m}) \), then we have

\[
\left\| H_{p_1, \ldots, p_m}^* f \right\|_{wL^1(Q_p^n \times Q_p^{n_2} \times \cdots \times Q_p^{n_m})} \leq \left\| f \right\|_{L^1(Q_p^n \times Q_p^{n_2} \times \cdots \times Q_p^{n_m})}.
\]

Furthermore,

\[
H_{p_1, \ldots, p_m}^* f \rightarrow_{L^1(Q_p^n)} = 1.
\]

To obtain the desired result, we need the following lemma.

**Lemma 1.** Suppose that \( 0 \leq \beta < n_i \), if \( f \in L^1(Q_p^n) \), then for any \( \lambda > 0 \),

\[
\left\| H_{p_1} f \right\|_{L^1, x_1 < (\lambda |x_1|)^{1/n}} \leq \left\| f \right\|_{L^1(Q_p^n)}.
\]

Moreover,

\[
\left\| H_{p_1} f \right\|_{L^1, x_1 < (\lambda |x_1|)^{1/n}} \leq \left\| f \right\|_{L^1(Q_p^n)}^\lambda.
\]

**Definition 2.** Let \( m \in \mathbb{N}, n_i \in \mathbb{N}, x_i \in Q_p^n, \ 0 \leq \beta_i < n_i, i = 1, \ldots, m, \) and \( f \) be a nonnegative integrable function on \( Q_p^n \times Q_p^{n_2} \times \cdots \times Q_p^{n_m} \). Define the fractional p-adic conjugate Hardy operator on higher-dimensional product spaces by
On the other hand, we let \( f_0(x) = \chi_{\{|x| \leq 1\}}(x) \), then
\[
\|f_0\|_{L^1(Q^c_\rho)} = \int_{Q^c_\rho} \chi_{\{|x| \leq 1\}}(x) \, dx = 1. \tag{22}
\]
Also,
\[
|H^p_\beta f_0(x)| = \frac{1}{|x|^p} \int_{|y| \leq |x|} f_0(y) \, dy
= \frac{1}{|x|^p} \int_{|y| \leq |x|} \chi_{\{|x| \leq 1\}} \, dy
= \frac{1}{|x|^p} \left\{ \begin{aligned}
&|x|^p, \quad |x| \leq 1, \\
&|x|^{p-n}, \quad |x| > 1.
\end{aligned} \right. \tag{23}
\]
\[
\left\|H^p_\beta f_0\right\|_{L^n(\mathbb{R}^n)} = \sup_{0 < \lambda < \infty} \lambda \left( \int_{Q^c_\rho} \chi_{\{|x| \leq 1/\lambda^{1/(p-n)}\}}(x) \, dx \right)^{n/(n-p)} \nonumber
= \sup_{0 < \lambda < \infty} \lambda \left( \int_{1/\lambda^{1/(p-n)}}^{\infty} \right) = \sup_{0 < \lambda < \infty} \left( \lambda^{p/(n-p)} \int_{\lambda}^{\infty} \right) \nonumber
= (1 - p^{-n})^{(n-p)/n} \sup_{0 < \lambda < 1} \left( \frac{1}{1 - p^{-n}} \right) \nonumber
= (1 - p^{-n})^{(n-p)/n} \sup_{0 < \lambda < 1} \left( \frac{\lambda^{n(1/(p-n))}}{1 - p^{-n}} \right) \nonumber
= \|f_0\|_{L^1(Q^c_\rho)}. \tag{27}
\]
Thus, as above, we get
\[
\left\|H^p_\beta\right\|_{L^1(Q^c_\rho)} \rightarrow L^{n-p,n}(Q^c_\rho) = 1. \tag{28}
\]
Now let us prove Theorem 1.

---

**Proof.** Without loss of generality, we consider only the situation when \( m = 2 \), and then, the case \( m \geq 3 \) is just a repetition of the case \( m = 2 \). For \( m = 2 \), the operator \( H^p_\beta \) can be written as...
\[
\left( H^p_{\beta_1, \beta_2} f \right)(x_1, x_2) = \frac{1}{|B(0, |x_1|_p)|_{H}} \int_{y_1 < |x_1|_p} f(y_1, y_2) \, dy_2
\] (29)

When \( f \in L^1(Q^0_p \times Q^0_p) \), we get
\[
\frac{1}{|B(0, |x_2|_p)|_{H}} \int_{y_1 < |x_2|_p} f(y_1, y_2) \, dy_2 \in L^1(Q^0_p), \quad \text{for all } x_2 \in Q^0_p.
\] (30)

Using Fubini theorem, we obtain that
\[
\left\| \left( H^p_{\beta_1, \beta_2} f \right)(\cdot, x_2) \right\|_{L^{n-\beta_1/m_1}(Q^0_p)} = \sup_{\lambda_1 > 0} \left\{ \lambda_1 \left| \left\{ x_1 : \left( H^p_{\beta_1, \beta_2} f \right)(x_1, x_2) > \lambda_1 \right\} \right| \right\}^{n_2-\beta_2/m_2}
\] (31)

\[
\leq 1 \left\| \frac{1}{|B(0, |x_2|_p)|_{H}} \int_{y_1 < |x_2|_p} f(y_1, y_2) \, dy_2 \right\|_{L^1(Q^0_p)}.
\]

Obviously, \( \int_{Q^0_p} f(y_1, y_2) \, dy_1 \in L^1(Q^0_p) \), if \( f \in L^1(Q^0_p \times Q^0_p) \). Then, applying the lemma again, we get
\[
\left\| \frac{1}{|B(0, |x_2|_p)|_{H}} \int_{y_1 < |x_2|_p} \left( \int_{Q^0_p} |f(y_1, y_2)| \, dy_2 \right) \, dy_1 \right\|_{L^{n-\beta_1/m_1}(Q^0_p)}
\]

\[
= \sup_{\lambda_2 > 0} \left\{ \lambda_2 \left| \left\{ x_2 : \frac{1}{|B(0, |x_2|_p)|_{H}} \int_{y_1 < |x_2|_p} \left( \int_{Q^0_p} |f(y_1, y_2)| \, dy_2 \right) \, dy_1 > \lambda_2 \right\} \right| \right\}^{n_2-\beta_2/m_2}
\] (33)

Combining (31)–(33), we get
\[
\left\| H^p_{\beta_1, \beta_2} f \right\|_{L^1(Q^0_p \times Q^0_p)} \leq 1 \cdot \left\| f \right\|_{L^1(Q^0_p \times Q^0_p)}.
\] (34)

Conversely, to prove that the constant 1 is optimal, we took
\[ f_0(x) = \chi_{\{x \in Q_p^n : |x|_p \leq 1\}}(x). \] (35)

And choosing \( F(x_1, x_2) = f_0(x_1)f_0(x_2) \), where \( x_1 \in Q_p^n, x_2 \in Q_p^n \), we get from the definition of \( H^p_{\beta_1, \beta_2} \) that

\[
H^p_{\beta_1, \beta_2} F(x_1, x_2) = \left( \prod_{i=1}^{2} \left( \frac{1}{B_0, |x_i|_p} \right)^{1 - \beta_i/\beta_i} \right) \int_{|x_1|_p < |x_1|_p} \int_{|x_1|_p < |x_1|_p} f_0(y_1)f_0(y_2) \, dy_1 \, dy_2
\]

\[
= H^p_{\beta_1} f_0(x_1) H^p_{\beta_2} f_0(x_2).
\] (36)

Also,

\[
\left| H^p_{\beta_1} f_0(x_1) \right| = \frac{1}{B(0, |x_1|_p)^{1 - \beta_i/\beta_i}} \int_{|x_1|_p < |x_1|_p} f_0(y_1) \, dy
\]

\[
= \frac{1}{|x_1|_p^{\beta_i - \beta_i}} \int_{|x_1|_p < 1} \int_{|x_1|_p \geq 1} \, dy,
\]

\[
= \left\{ \begin{array}{ll}
|x_1|_p^{\beta_i} & |x_1|_p < 1, \\
|x_1|_p^{\beta_i - \beta} & |x_1|_p \geq 1.
\end{array} \right.
\] (37)

We now that let \( L = H^p_{\beta_1} f_0(x_2) \), for \( 0 < \lambda_1 < L \), then combining both the cases, we get

\[
\left| \left\{ x_1 \in Q_p^n : H^p_{\beta_1, \beta_2} F(x_1, x_2) > \lambda_1 \right\} \right|
\]

\[
= \left| \left\{ x_1 \in Q_p^n : \lambda_1^{1/\beta} L > \lambda_1 \right\} + \left\{ x_1 \in Q_p^n : |x_1|_p \geq 1 : |x_1|_p^{\beta_i - \beta} L > \lambda_1 \right\} \right|
\]

\[
= \left| \left\{ x_1 \in Q_p^n : \left( \frac{\lambda_1}{L} \right)^{1/\beta} < |x_1|_p < 1 \right\} + \left\{ x_1 \in Q_p^n : 1 \leq |x_1|_p < \left( \frac{L}{\lambda_1} \right)^{1/\beta} \right\} \right|
\]

\[
= \left| \left\{ x_1 \in Q_p^n : \lambda_1^{1/\beta} |x_1|_p < \left( \frac{L}{\lambda_1} \right)^{1/\beta} \right\} \right|.
\] (38)
Therefore, we have

\[
\| H^p_{\beta_1,\beta_2} F(x_1, x_2) \|_{L^p} = \frac{n_1}{n_1 - \beta_1} \cdot \sup_{0 < \lambda_1 < 1} \left( \lambda_1 \int_{Q_0^n} \chi_{(L/\lambda_1)^{(n_1, \beta_1)}}(x_1) \, dx_1 \right)^{n_1 - \beta_1/n_1}
\]

\[
= \left( 1 - p^{-n_1} \right)^{(n_1 - \beta_1)/n_1} \sup_{0 < \lambda_1 < 1} \left( \lambda_1 \sum_{j = -\log_p(n_1, \beta_1) + 1}^{\log_p(n_1, \beta_1)} p^j \right)^{n_1 - \beta_1/n_1}
\]

\[
= 1 - L.
\]

For \(0 < \lambda_2 < 1\), we also divide \(x_2\) into two cases \(|x_2|_p < 1\) and \(|x_2|_p \geq 1\). As above, we get

\[
\sup_{0 < \lambda_2 < 1} \left\| x_2 \in Q_0^n : |H^p_{\beta_1, \beta_2} f_0(x_2) > \lambda_2 \right\|_{L^p} = 1 \times 1.
\]

(40)

Since \(\|F\|_{L^1(Q_0^n, Q_0^n)} = 1 \times 1\), by combining (37) with (38),

\[
\| H^p_{\beta_1, \beta_2} F \|_{L^1(Q_0^n, Q_0^n)} = 1 \cdot \| F \|_{L^1(Q_0^n, Q_0^n)}.
\]

(41)

This completes the proof. \(\square\)

3. Sharp Weak Bounds for Fractional Conjugate Hardy Operators

Likewise, this section contains the results having sharp weak bounds for fractional p-adic conjugate Hardy operators, and our results are as follows.

**Theorem 2.** Set \(0 < \beta_1 < \beta_2\) let \(Q = \frac{n_1}{(n_1 - \beta_1)} \times \cdots \times \frac{n_m}{(n_m - \beta_m)}\), \(i = 1, \ldots, m\). If \(f \in L^1(Q_0^n, Q_0^n, \ldots, Q_0^n)\), then we have

\[
\left\| H^p_{\beta_1, \beta_2} f \right\|_{L^1(Q_0^n, Q_0^n, \ldots, Q_0^n)} \leq 1 \cdot \| f \|_{L^1(Q_0^n, Q_0^n, \ldots, Q_0^n)}.
\]

(42)

Furthermore,

\[
\left\| H^p_{\beta_1, \beta_2} f \right\|_{L^1(Q_0^n, Q_0^n, \ldots, Q_0^n)} \leq 1. \]

(43)

In order to prove our theorem, we need the following lemma.

**Lemma 2.** Suppose that \(0 \leq \beta < n\), if \(f \in L^1(Q_0^n)\), then for any \(\lambda > 0\,

\[
\left\| H^p_{\beta, \beta} f \right\|_{L^1(Q_0^n, Q_0^n)} \leq 1 \cdot \| f \|_{L^1(Q_0^n)}.
\]

(44)

Moreover,

\[
\left\| H^p_{\beta, \beta} f \right\|_{L^1(Q_0^n, Q_0^n)} \leq 1. \]

(45)

The proof of this result is almost the same as Lemma 1; here, we omit the proof details. Next, we give the proof of Theorem 2.
Proof. Without loss of generality, we only discuss the case $m = 2$, and then, the case $m \geq 3$ is just a repetition of the case $m = 2$. When $m = 2$, the operator $H^{p,\ast}_{p,0,2}$ can be written as

\[
(H^{p,\ast}_{p,0,2} f)(x_1, x_2) = \int_{|y_1| > |x_1|} \int_{|y_2| > |x_2|} \frac{f(y_1, y_2)}{B(0, |y_1|)^{1-\beta_1/m_1}} |y_1|^1/H \, dy_1 \, dy_2.
\]

Using Lemma 2 and Fubini theorem, it implies that

\[
\left\|H^{p,\ast}_{p,0,2} f \right\|_{L^{p/n_2}(Q_p^n)} = \sup_{\lambda_2 \geq 0} \left\{ x_2 \in Q_p^n \right\} \left\{ \int_{|y_1| > |x_1|} \int_{|y_2| > |x_2|} f(y_1, y_2) \, dy_1 \, dy_2 \right\}.
\]

We conclude that

\[
\left\|H^{p,\ast}_{p,0,2} f \right\|_{L^{p/n_2}(Q_p^n)} = \left\| \mathcal{F} f \right\|_{L^{p/n_2}(Q_p^n)}
\]

Consequently, combining (45) and (46), we get

\[
\left\|H^{p,\ast}_{p,0,2} f \right\|_{L^{p/n_2}(Q_p^n \times Q_p^n)} \leq 1 \cdot \left\| \mathcal{F} f \right\|_{L^{p/n_2}(Q_p^n \times Q_p^n)}.
\]

On the other hand, for any $0 < \epsilon < 1$, we took

\[
f_{\epsilon}(x) = \begin{cases} \frac{|x|^{p-\epsilon(n-1)}}{p} & |x|_p \geq 1, \\ 0 & |x|_p < 1. \end{cases}
\]

Let $F(x_1, x_2) = f_{\epsilon_1}(x_1)f_{\epsilon_2}(x_2)$, where $x_1 \in Q_p^n$, $x_2 \in Q_p^n$, then

\[
\begin{align*}
\|F\|_{L^1(Q_p^n \times Q_p^n)} &= \int_{Q_p^n \times Q_p^n} f_{\epsilon_1}(x_1)f_{\epsilon_2}(x_2) \, dx_1 \, dx_2 \\
&= \int_{Q_p^n \times Q_p^n} f_{\epsilon_1}(x_1)f_{\epsilon_2}(x_2) \, dx_1 \, dx_2
\end{align*}
\]
Set \( C_{e_i} = (1 - p^{-n_i})/p^{((\beta + n_i)\epsilon_i - \beta)} - 1 \) and \( M = (C_{e_i} H_{p_i}^{\beta} f_2(x_2)/\lambda_1)^{1/((\beta + n_i)\epsilon_i - \beta)} \), we obtain that

\[
\left| \{ x_1 \in Q_{p_i}^n : |H_{p_i}^{\beta} f(\lambda_1, x_1, x_2)| > \lambda_1 \} \right|
= \left| \{ |x_1| p < 1 : C_{e_i} H_{p_i}^{\beta} f_2(x_2) > \lambda_1 \} \right| + \left| \{ |x_1| p \geq 1 : C_{e_i} |x_1|^{\beta - (\beta + n_i)\epsilon_i} H_{p_i}^{\beta} f_2(x_2) > \lambda_1 \} \right|
\]

\[
= \left| \{ x_1 \in Q_{p_i}^n : |x_1|^{(\beta + n_i)\epsilon_i - \beta} < C_{e_i} H_{p_i}^{\beta} f_2(x_2) / \lambda_1 \} \right|.
\]

Notice that when \( \lambda_i > C_{e_i}, \{ x_i \in Q_{p_i}^n : |H_{p_i}^{\beta} f(x_2)| > \lambda_i \} = \emptyset \), if \( \epsilon_i \) is small enough, \( C_{e_i} \) tends to zero; therefore, when \( \epsilon_i \) is small enough, we get

\[
I_0 := \sup_{0 < \lambda_i < H_{p_i}^{\beta} f_2(x_2)} \lambda_i \left| \{ x_1 \in Q_{p_i}^n : |H_{p_i}^{\beta} f_2(x_2)| > \lambda_i \} \right|^{n_i/((n_i - \beta))}
= \sup_{0 < \lambda_i < H_{p_i}^{\beta} f_2(x_2)} \lambda_i \left( \int |x_1|^{(\beta + n_i)\epsilon_i - \beta} < C_{e_i} H_{p_i}^{\beta} f_2(x_2) / \lambda_i \right) \, dx_1
= \sup_{0 < \lambda_i < H_{p_i}^{\beta} f_2(x_2)} \lambda_i \left( \sum_{j=-\infty}^{\log_{p_i} n_i} \int \, dx_1 \right)
= (1 - p^{-n_i})^{n_i/((n_i - \beta))} \sup_{0 < \lambda_i < H_{p_i}^{\beta} f_2(x_2)} \lambda_i \left( \sum_{j=-\infty}^{\log_{p_i} n_i} \right)
= \sup_{0 < \lambda_i < H_{p_i}^{\beta} f_2(x_2)} \lambda_i (M^{n_i})^{n_i/((n_i - \beta))}
\]

\[
= \sup_{0 < \lambda_i < H_{p_i}^{\beta} f_2(x_2)} \lambda_i \left( C_{e_i} H_{p_i}^{\beta} f_2(x_2) / \lambda_1 \right)^{1/((\beta + n_i)\epsilon_i - \beta))} \right)^{n_i/((n_i - \beta))}
= C_{e_i} H_{p_i}^{\beta} f_2(x_2)^{1/((n_i - \beta)) ((\beta + n_i)\epsilon_i - \beta))}
\]

\[
\times \sup_{0 < \lambda_i < \min\left\{ H_{p_i}^{\beta} f_2(x_2) / \lambda_1 \right\}} \lambda_i^{1/((n_i - \beta)) ((\beta + n_i)\epsilon_i - \beta))}
= C_{e_i} H_{p_i}^{\beta} f_2(x_2)^{1/((n_i - \beta)) ((\beta + n_i)\epsilon_i - \beta))}
\]

Using the same method for \( x_2 \), we obtain that
\[
\sup_{\lambda_2 > \lambda_2} \sup_{\lambda_2 > \lambda_2} \left\{ x_2 \in \Omega_\beta^2 \mid I_\beta \geq \lambda_2 \right\}^{\beta/2} \bigg( \frac{p-\beta}{p-\beta} \bigg)^{\epsilon_1-\lambda_2} \| f \|_L^p \bigg( \Omega_\beta^2 \bigg)
\]

\[
= C_{\epsilon_2} \left( \sup_{\lambda_2 > \lambda_2} \left\{ x_2 \in \Omega_\beta^2 \mid I_\beta \geq \lambda_2 \right\}^{\beta/2} \bigg( \frac{p-\beta}{p-\beta} \bigg)^{\epsilon_1-\lambda_2} \| f \|_L^p \bigg( \Omega_\beta^2 \bigg) \right)
\]

\[
\times \sup_{\lambda_2 > \lambda_2} \left\{ x_2 \in \Omega_\beta^2 \mid I_\beta \geq \lambda_2 \right\}^{\beta/2} \bigg( \frac{p-\beta}{p-\beta} \bigg)^{\epsilon_1-\lambda_2} \| f \|_L^p \bigg( \Omega_\beta^2 \bigg)
\]

\[
= C_{\epsilon_2} \left( \sup_{\lambda_2 > \lambda_2} \left\{ x_2 \in \Omega_\beta^2 \mid I_\beta \geq \lambda_2 \right\}^{\beta/2} \bigg( \frac{p-\beta}{p-\beta} \bigg)^{\epsilon_1-\lambda_2} \| f \|_L^p \bigg( \Omega_\beta^2 \bigg) \right)
\]

\[
\times \left( \sup_{\lambda_2 > \lambda_2} \left\{ x_2 \in \Omega_\beta^2 \mid I_\beta \geq \lambda_2 \right\}^{\beta/2} \bigg( \frac{p-\beta}{p-\beta} \bigg)^{\epsilon_1-\lambda_2} \| f \|_L^p \bigg( \Omega_\beta^2 \bigg) \right)
\]

(55)

Let \( \epsilon_1 \rightarrow 0^+ \) and \( \epsilon_2 \rightarrow 0^+ \), it implies that

\[
\left\| H_{p_1, p_2}^{\Delta_n} \right\|_{L^p \rightarrow L^p} \left( \Omega_\beta^2 \times \Omega_\beta^2 \right) \geq 1 \cdot \left\| F \right\|_L^p \left( \Omega_\beta^2 \times \Omega_\beta^2 \right).
\]

(56)

This finishes the proof of Theorem 2. \( \square \)

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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