A NOTE ON A LÉVY INSURANCE RISK MODEL UNDER PERIODIC DIVIDEND DECISIONS

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Abstract. In this paper, we consider a spectrally negative Lévy insurance risk process with a barrier-type dividend strategy. In contrast to the traditional barrier strategy in which dividends are payable to the shareholders immediately when the surplus process reaches a fixed level \( b \) (as long as ruin has not yet occurred), it is assumed that the insurer only makes dividend decisions at some discrete time points in the spirit of [1]. Under such a dividend strategy with Erlang inter-dividend-decision times, expressions for the Gerber-Shiu expected discounted penalty function proposed in [24] and the moments of total discounted dividends payable until ruin are derived. The results are expressed in terms of the scale functions of a spectrally negative Lévy process and an embedded spectrally negative Markov additive process. Our analyses rely on the introduction of a potential measure associated with an Erlang random variable. Numerical illustrations are also given.

1. Introduction. In this paper, the surplus of an insurance company (before dividends) is modeled by a spectrally negative Lévy process \( X = \{X_t\}_{t \geq 0} \) described as follows (see [27]). For \( x \in \mathbb{R} \), let \( \mathbb{P}_x \) be the probability law under which \( X \) starts from \( x \). Denote by \( \mathbb{E}_x \) the expectation operator associated with \( \mathbb{P}_x \), and for convenience we shall write \( \mathbb{E} = \mathbb{E}_0 \). The process \( X \) is uniquely characterized by the Laplace exponent

\[
\psi(s) = \frac{1}{t} \ln \mathbb{E} e^{sX_t} = cs + \frac{\sigma^2}{2} s^2 + \int_{(0,\infty)} \left( e^{-sx} - 1 + sx 1_{(x<1)} \right) \nu(dx).
\]

(1.1)

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Here \( \sigma \geq 0 \), and \( \nu \) is a nonnegative measure on \((0, \infty)\) satisfying the usual condition 
\[
\int_{(0,\infty)} (1 \wedge x^2) \nu(dx) < \infty
\]
and the additional condition 
\[
\int_{(1,\infty)} x \nu(dx) < \infty
\]
which ensures that \( X \) has finite expectation. Note that the Laplace exponent is well defined at least for \( s \) with \( \Re(s) \geq 0 \). Throughout this paper, we assume that the constant \( c \) in (1.1) is such that the positive security loading condition, namely 
\[
\psi'(0+) = \mathbb{E}X_1 > 0,
\]
holds true. To avoid trivialities, it is natural to assume that \( X \) does not have monotone sample paths.

The class of Lévy processes represents a popular family of stochastic processes that arises in many fields of applied probability, such as queueing theory, branching processes and storage models. In particular, Lévy insurance risk models have received lots of attention in recent years. The advances in fluctuation theory ([27]), along with the use of scale functions and potential measures, play an important role in the analyses of various ruin-related quantities in these risk models. The expected discounted penalty function proposed by [24] and its generalizations in Lévy risk processes have been analyzed by e.g. [7, 10, 19]; whereas the study of Lévy models with dividend strategies can be found in e.g. [30, 37, 29]. Furthermore, Albrecher et al. [5] and Kyprianou and Zhou [33] considered Lévy models with taxation; while Czarna and Palmowski [14] and Loeffen et al. [36] looked at the Parisian ruin problem. Interested readers are also referred to e.g. [17, 20, 40, 43, 26, 21] for earlier study of (some special cases of) Lévy insurance risk processes.

In the traditional dividend barrier strategy (e.g. [22, 34, 16]), dividends are payable immediately once the insurer’s surplus reaches a fixed level \( b > 0 \) (as long as ruin has not occurred in the interim). In other words, it is implicitly assumed that dividend decisions are made continuously. Recently, Albrecher et al. [1, 2] proposed the idea of ‘randomized observations’ under the compound Poisson setting with a barrier, where the process is only ‘observed’ at Erlang distributed intervals. At these ‘observation time points’, any excess of the surplus over \( b \) is paid as lump sum dividends and ruin is declared if the surplus is observed to be negative. However, in practice the financial position of an insurer is typically monitored much more frequently than dividend decisions are made. This motivated [9] (albeit in the dual risk model) to propose a variant where solvency is monitored continuously as usual. Since then, such a variant has been analyzed under perturbed compound Poisson and Markov additive insurance risk models by [44] and [45]. We remark that the reverse case with continuous dividend decisions and Poissonian monitoring of ruin was also studied by [4].

In this paper, we shall consider the modifications as in [9] and [45] in the context of a spectrally negative Lévy risk process \( X \). The dynamics of the modified surplus process, namely \( X^b = \{X^b_t\}_{t \geq 0} \), under a barrier-type strategy are described below. Let \( \{L_i\}_{i=1}^{\infty} \) be the sequence of dividend decision times. Without loss of generality, it is assumed that time 0 is not a dividend decision time, i.e. the process \( X^b \) is allowed to stay above the barrier \( b \) at time 0 without immediate dividend payment (see Remark 1). For convenience, we define \( L_0 = 0 \) and introduce the auxiliary processes \( X^{(i)} = \{X^{(i)}_t\}_{t \geq L_{i-1}} \) for \( i = 1, 2, \ldots \), which can be described jointly with \( X^b \) via the coupled equations

\[
X^{(i)}_t = \begin{cases} 
X_t, & i = 1; \\
X^b_{L_{i-1}} + X_t - X_{L_{i-1}}, & i = 2, 3, \ldots; \ t \geq L_{i-1},
\end{cases}
\]

and

\[
\int_{(0,\infty)} (1 \wedge x^2) \nu(dx) < \infty
\]
and the additional condition 
\[
\int_{(1,\infty)} x \nu(dx) < \infty
\]
which ensures that \( X \) has finite expectation. Note that the Laplace exponent is well defined at least for \( s \) with \( \Re(s) \geq 0 \). Throughout this paper, we assume that the constant \( c \) in (1.1) is such that the positive security loading condition, namely 
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\[
X^{(i)}_t = \begin{cases} 
X_t, & i = 1; \\
X^b_{L_{i-1}} + X_t - X_{L_{i-1}}, & i = 2, 3, \ldots; \ t \geq L_{i-1},
\end{cases}
\]
\[ X^b_t = \begin{cases} X^{(i)}_t, & L_{i-1} < t < L_i, \\ \min(X^{(i)}_{L_i}, b), & t = L_i, \end{cases} \]

The above definition can be interpreted as follows. Equation (1.2) means that \(X^{(i)}\) is the same as the baseline (dividend-free) process \(X\); whereas for each \(i = 2, 3, \ldots\) the process \(X^{(i)}\) starts with the post-dividend level \(X^b_{L_{i-1}}\) at time \(L_{i-1}\), and then its increment \(X_t^{(i)} - X^b_{L_{i-1}}\) in the time period \(\{L_{i-1}, t\}\) is identical to that of \(X\) in the same period as if there are no further dividend modifications. Then (1.3) essentially puts the pieces \(X^{(i)}\) (for \(i = 1, 2, \ldots\)) between the dividend decision times together and defines the appropriate post-dividend surplus level at each time point \(L_i\). In particular, (1.3) first defines the modified process \(X^b\) to be identical to \(X\) before the first dividend decision time \(L_1\), i.e. \(X^b_0 = X_t\) for \(0 \leq t < L_1\) (and no dividend is payable at time 0 even if \(X^b_0 = X_0 > b\)). For \(i = 2, 3, \ldots\), the process \(X^b\) evolves like \(X^{(i)}\) in the time period \(\{L_{i-1}, L_i\}\), with the initial level at time \(L_{i-1}\) given by \(\min(X^{(i-1)}_{L_i}, b)\) and ending level at time \(L_i\) being \(\min(X^{(i)}_{L_i}, b)\). This means that at the dividend decision time \(L_i\) (for \(i = 1, 2, \ldots\)), if the observed pre-dividend surplus level \(X^{(i)}_{L_i}\) is above \(b\) then the excess is paid as a dividend.

Letting \(T_i = L_i - L_{i-1}\) be the \(i\)-th inter-dividend-decision time for \(i = 1, 2, \ldots\), we assume that \(\{T_i\}_{i=1}^{\infty}\) form an independent and identically distributed (i.i.d.) sequence that is independent of (the attributes of) \(X\). Motivated by the concept of Erlangization first introduced in risk theory by [8], it is assumed that each \(T_i\) has the same distribution as a generic Erlang \((m, \beta)\) random variable \(T\) with density

\[
f(t; m, \beta) = \frac{\beta^m t^{m-1} e^{-\beta t}}{(m-1)!}, \quad t > 0,
\]

where \(m\) is the shape parameter that is a positive integer, and \(\beta > 0\) is the scale parameter. The reasons for the choice of Erlang \((m, \beta)\) distribution are two-fold. First, if one lets \(\beta \to \infty\) for a fixed \(m\) (typically taken to be 1), then \(T\) is essentially a point mass at 0, and hence the traditional barrier strategy (i.e. the case of continuous dividend decisions) is retrieved. Second, if one increases \(m\) and \(\beta\) simultaneously so as to keep the mean \(ET = m/\beta = h\) fixed, then \(T\) converges in distribution to a point mass at \(h\), thereby approximating the situation of deterministic inter-dividend-decision times (e.g. quarterly or semi-annual dividends) and the procedure is known as ‘Erlangization’. In the ruin theory literature, the Erlangization technique has been mostly applied in finite-time ruin problems (e.g. [8, 41, 42, 38]), and it has recently gained popularity in risk processes with periodic decisions as well (e.g. [2, 9, 45, 47]). We also refer interested readers to e.g. [11, 32] for some early applications of Erlangization in option pricing.

To define the quantities of interest, we first let \(\varsigma_0 = \inf\{t > 0 : X^b_t < 0\}\) be the time to ruin of the process \(X^b\) with the usual convention \(\inf \emptyset = \infty\). Gerber and Shiu [24] proposed an expected discounted penalty function (now commonly known as the Gerber-Shiu function) which can be used to analyze jointly the time to ruin \(\varsigma_0\), the surplus immediately before ruin \(X^b_{\varsigma_0}\) and the deficit at ruin \(|X^b_{\varsigma_0}|\). Letting \(\delta \geq 0\) be the force of interest or a Laplace transform argument, for a given penalty function \(w : [0, \infty) \times [0, \infty) \to [0, \infty)\) the Gerber-Shiu function is defined by (see Remark 2)

\[
\phi_\delta(w; b) = \mathbb{E}_a[e^{-\delta \varsigma_0} w(X^b_{\varsigma_0}, |X^b_{\varsigma_0}|); \varsigma_0 < \infty]
\]
\[ w(0, 0) \phi_{d, \delta}(u; b) + \phi_{w, \delta}(u; b), \quad u > 0, \]  

(1.5)

where

\[ \phi_{d, \delta}(u; b) = \mathbb{E}_u[e^{-\delta \varsigma_0}; \varsigma_b < \infty, X^b_{\varsigma_b} = 0] \]  

(1.6)

is the Laplace transform of the ruin time when ruin is caused by oscillation (due to Brownian motion), and

\[ \phi_{w, \delta}(u; b) = \mathbb{E}_u[e^{-\delta \varsigma_0} w(X^b_{\varsigma_b-1}, |X^b_{\varsigma_b}|); \varsigma_b < \infty, X^b_{\varsigma_b} < 0] \]  

(1.7)

is the expected discounted penalty function when ruin is caused by claims (due to downward jumps). Since the process \( X^b \) can never be above the level \( b \) at the dividend decision times \( \{L_i\}_{i=1}^\infty \), ruin of \( X^b \) occurs almost surely and thus the event \( \{\varsigma_b < \infty\} \) can indeed be omitted from the above definitions. Under the present barrier-type strategy, the total amount of discounted dividends payable until ruin is given by

\[ D_{b, \delta} = \sum_{i=1}^\infty e^{-\delta L_i} (X^b_{L_i-} - b) \mathbf{1}_{\{X^b_{L_i-} > b, L_i < \infty\}} \]

under a force of interest \( \delta \geq 0 \). In addition to the Gerber-Shiu function, we are also interested in, for \( n = 1, 2, \ldots \), the \( n \)-th moment of \( D_{b, \delta} \), namely

\[ V_{n, \delta}(u; b) = \mathbb{E}_u[D^\delta_{b, n}], \quad u \geq 0. \]

The convention that \( V_{0, \delta}(u; b) \equiv 1 \) is adopted. While the expectation \( V_{1, \delta}(u; b) \) represents the value of the firm in corporate finance for which the shareholders of the company should aim at maximizing (e.g. [22]), the shareholders may also be concerned with the variability of \( D_{b, \delta} \). This led to various studies in the higher moments of discounted dividends (e.g. [37, 23, 3]). See e.g. [13] for dividend problems in minimizing the coefficient of variation of dividends. Note that in general it is very difficult to obtain the exact distribution of \( D_{b, \delta} \) unless \( \delta = 0 \) (e.g. see Section 3 in [16] and our Proposition 7), but in principle the distribution of a random variable can be estimated from a finite number of moments (see e.g. [39]).

**Remark 1.** On the other hand, one may additionally define time 0 to be a dividend decision time. In this case, if the initial surplus \( u \) is such that \( u \geq b \), then a dividend of size \( u - b \) is immediately payable at time 0 and the process starts at level \( b \) afterwards. Therefore, denoting the resulting Gerber-Shiu function and the \( n \)-th dividend moment by \( \phi_{d, \delta}^*(u; b) \) and \( V_{n, \delta}^*(u; b) \) respectively, it is immediate to see that \( \phi_{d, \delta}^*(u; b) = \phi_{d, \delta}(u; b) \) and \( V_{n, \delta}^*(u; b) = V_{n, \delta}(u; b) \) for \( 0 < u \leq b \) while for \( u > b \) one has \( \phi_{d, \delta}^*(u; b) = \phi_{d, \delta}(b; b) \) and

\[ V_{n, \delta}^*(u; b) = \sum_{k=0}^n \binom{n}{k} (u - b)^{n-k} V_{k, \delta}(b; b) \]

using a binomial expansion. In other words, \( \phi_{d, \delta}^*(u; b) \) and \( V_{n, \delta}^*(u; b) \) are simply expressible in terms of the Gerber-Shiu function and the dividend moments in the model where time 0 is not a dividend decision time. \( \square \)

**Remark 2.** Note that we write the domain of the initial surplus as \( u > 0 \) instead \( u \geq 0 \) in the definition (1.5). This is because when \( \sigma = 0 \) and \( X \) has unbounded variation, the existing results (2.4)-(2.6) in the literature concerning Gerber-Shiu functions are only valid for \( u > 0 \), but the case \( u = 0 \) is not of interest as ruin occurs immediately (see Lemma 8.6 in [27]). Nonetheless, if (i) \( X \) has bounded variation; or (ii) \( X \) has unbounded variation due to Brownian motion but not jumps, then
the subsequent main results such as (3.1), (3.2), (4.28) and (4.29) also hold true for \( u = 0 \). See also Remark 4.2 on p.385 of [7] for related comments. \( \square \)

This paper is organized as follows. Section 2 first introduces a few notations and preliminary results in Lévy processes that will be used later on. In Section 3, assuming that the inter-dividend-decision times are exponentially distributed, the Gerber-Shiu functions \( \phi_{d,b}(u; b) \) and \( \phi_{w,b}(u; b) \) as well as the dividend moment \( V_{n,b}(u; b) \) are derived in terms of the scale function and a potential measure pertaining to the spectrally negative Lévy process \( X \). Section 4 extends the results to case where the inter-dividend-decision times follow Erlang\((m, \beta)\) distribution. This relies on a new potential measure associated with an Erlang random variable and the introduction of an embedded spectrally negative Markov additive process. In Section 5, we take an alternative look at the problem by considering the first overshoot of \( X \) over the barrier level \( b \) at a dividend decision time avoiding ruin enroute. Section 6 ends the paper with some numerical examples. It is instructive to note that the techniques in this paper are different from those in [45] who analyzed the perturbed risk model with Markovian claim arrivals in contrast to the present Lévy model. These two classes of models do not belong to each other, although they both contain e.g. the Brownian motion risk model and compound Poisson model as special cases. While Zhang and Cheung [45] obtained the ruin-related quantities by solving integro-differential equations with the use of matrix Dickson-Hipp operators (see [15, 18]), this paper expresses the results only in terms of scale functions and potential measures.

2. Preliminaries. Throughout the paper, \( \mathcal{L} \) represents the Laplace transform operator and \( \mathcal{L}^{-1} \) is its inverse operator. Define the right inverse of the Laplace exponent \( \psi \) by

\[
\Phi(q) = \sup\{ s : \psi(s) = q \}, \quad q \geq 0.
\]

Because of the assumption \( \psi'(0+) > 0 \), it is known from e.g. [27] that \( \Phi(q) \) is the unique solution to \( \psi(s) = q \) in \([0, \infty)\). For \( q \geq 0 \), the \( q \)-scale function associated with \( X \), namely \( W(q) \), is a strictly increasing and continuous function on \([0, \infty)\) such that

\[
\mathcal{L}W(q)(s) = \frac{1}{\psi(s) - q}, \quad s > \Phi(q), \quad (2.1)
\]

and \( W(q)(x) = 0 \) for \( x < 0 \). See Theorem 8.1(i) in [27].

For \( x \in \mathbb{R} \), define the first exit times

\[
\tau^+_x = \inf\{ t > 0 : X_t > x \}, \quad \text{and} \quad \tau^-_x = \inf\{ t > 0 : X_t < x \}.
\]

From Theorem 8.1(iii) in [27], one has that for any \( q \geq 0 \),

\[
\mathbb{E}_u[e^{-q\tau^+_u} \tau^-_u < \tau^-_0] = \frac{W(q)(u)}{W(q)(b)}, \quad u \leq b. \quad (2.2)
\]

Furthermore, the \( q \)-potential measure of \( X \) killed on exiting \([0, \infty)\) is defined by, for \( q \geq 0 \),

\[
\mathcal{R}^{(q)}(u, dx) = \int_0^\infty e^{-qt} \mathbb{P}_u(X_t \in dx, \tau^-_0 > t) dt, \quad u, x \geq 0.
\]

It follows from the proof of Theorem 8.7 in [27] that there exists a density \( r^{(q)}(u, x) \) which is such that \( \mathcal{R}^{(q)}(u, dx) = r^{(q)}(u, x) dx \) and is given by

\[
r^{(q)}(u, x) = e^{-\Phi(q)x} W(q)(u) - W(q)(u - x). \quad (2.3)
\]
The \( q \)-scale function and the \( q \)-potential measure are known to play an important role in the representation of the Gerber-Shiu function in the (barrier-free) \( \text{Lévy risk process} \ X \). With a slight abuse of notation, one has

\[
\phi_{d, \delta}(u; \infty) := \mathbb{E}_u [e^{-\delta \tau_0^\delta}; \tau_0^- < \infty, X_{\tau_0^-} = 0] = \frac{\sigma^2}{2} [W^{(\delta)}(u) - \Phi(\delta) W^{(\delta)}(u)], \quad u > 0, \tag{2.4}
\]

and

\[
\phi_{w, \delta}(u; \infty) := \mathbb{E}_u [e^{-\delta \tau_0^\delta} w(X_{\tau_0^-}; |X_{\tau_0^-}|); \tau_0^- < \infty, X_{\tau_0^-} < 0] = \int_0^\infty \int_0^\infty w(x, y) r^{(\delta)}(u, x) \nu(x + dy) dx, \quad u > 0. \tag{2.5}
\]

See e.g. [7], Chapter XII.4.

Another useful tool is the \( q \)-potential measure of \( X \), killed on exiting \([0, b]\), which is defined by, for \( q \geq 0 \),

\[
\mathcal{R}^{(q)}(u; dx; b) = \int_0^\infty e^{-q t} \mathbb{P}_u(X_t \in dx, \tau_0^- + \tau_0^+ > t) dt, \quad u, x \in [0, b].
\]

From Theorem 8.7 in [27], the density \( r^{(q)}(u; x; b) \) such that \( \mathcal{R}^{(q)}(u; dx; b) = r^{(q)}(u, x; b) dx \) exists and is equal to

\[
r^{(q)}(u, x; b) = \frac{W^{(q)}(u) W^{(q)}(b) - W^{(q)}(b - x)}{W^{(q)}(b)} - W^{(q)}(u - x).
\]

Since \( W^{(q)}(b - x)/W^{(q)}(b) \to e^{-\Phi(q)x} \) as \( b \to \infty \) from the proof of Corollary 8.9 in [27], \( r^{(q)}(u, x) \) is in fact the limit of \( r^{(q)}(u, x; b) \) as \( b \to \infty \). It follows from the proof of Theorem 5.5 in [28] that the Gerber-Shiu function for \( X \) due to claims avoiding level \( b \) enroute admits the representation

\[
\mathbb{E}_u [e^{-\delta \tau_0^\delta} w(X_{\tau_0^-}; |X_{\tau_0^-}|); \tau_0^- < \tau_0^+, X_{\tau_0^-} < 0] = \int_0^b \int_0^\infty w(x, y) r^{(\delta)}(u, x; b) \nu(x + dy) dx, \quad u \in (0, b]. \tag{2.6}
\]

3. Exponential inter-dividend-decision times. In this entire section, it is assumed that the generic inter-dividend-decision time \( T \) is exponentially distributed with density \( f(t; 1, \beta) = \beta e^{-\beta t} \). The following Proposition gives the expressions of the Gerber-Shiu functions \( \phi_{d, \delta}(u; b) \) and \( \phi_{w, \delta}(u; b) \) defined by (1.6) and (1.7) respectively.

**Proposition 1.** Suppose that the inter-dividend-decision times are exponentially distributed with mean \( 1/\beta \). Then, we have

\[
\phi_{d, \delta}(u; b) = \beta \phi_{d, \delta}(b; b) \int_0^b \left( \frac{W^{(\delta)}(x)}{W^{(\delta)}(b)} 1_{(0 < x < b)} + 1_{(x \geq b)} \right) r^{(\beta+\delta)}(u, x) dx + \frac{\sigma^2}{2} \beta \int_0^b \left( W^{(\delta)}(x) - \frac{W^{(\delta)}(b)}{W^{(\delta)}(x)} W^{(\delta)}(b) \right) r^{(\beta+\delta)}(u, x) dx + \frac{\sigma^2}{2} W^{(\beta+\delta)}(u) - \Phi(\beta + \delta) W^{(\beta+\delta)}(u), \quad u > 0, \tag{3.1}
\]

and

\[
\phi_{w, \delta}(u; b) = \beta \phi_{w, \delta}(b; b) \int_0^\infty \left( \frac{W^{(\delta)}(x)}{W^{(\delta)}(b)} 1_{(0 < x < b)} + 1_{(x \geq b)} \right) r^{(\beta+\delta)}(u, x) dx.
\]
where
\[
\phi_{d,\delta}(b; b) = \frac{\beta \int_0^b \int_0^b \int_0^\infty w(z, y) r^{(\delta)}(x, z; b) r^{(\beta + \delta)}(u, x) \nu(z + dy) \, dz \, dx}{1 - \beta \int_0^\infty \frac{W^{(\delta)}(x)}{W^{(\beta + \delta)}(b)} W^{(\beta + \delta)}(b) \, dx}
\]
(3.3)
and
\[
\phi_{w,\delta}(b; b) = \frac{\beta \int_0^b \int_0^b \int_0^\infty w(z, y) r^{(\delta)}(x, z; b) r^{(\beta + \delta)}(u, x) \nu(z + dy) \, dz \, dx}{1 - \beta \int_0^\infty \frac{W^{(\delta)}(x)}{W^{(\beta + \delta)}(b)} W^{(\beta + \delta)}(b) \, dx}
\]
(3.4)

Proof. We first analyze $\phi_{d,\delta}(u; b)$. By considering whether or not ruin occurs before the first dividend decision time $L_1 = T_1$ and noting that $X^b$ is identical to $X$ for $t < L_1$, we obtain for $u > 0$
\[
\begin{align*}
\phi_{d,\delta}(u; b) &= \int_0^\infty \beta e^{-(\beta + \delta) t} \int_0^\infty \left[ \phi_{d,\delta}(x; b) 1_{(0 < x < b)} + \phi_{d,\delta}(b; b) 1_{(x \geq b)} \right] P_u(X_t \in dx, \tau^-_0 > t) \, dt \\
&\quad + \int_0^\infty \beta e^{-\delta t} E_u[e^{-\delta \tau^-_0}; \tau^-_0 < t, X^-_0 = 0] \, dt \\
&= \beta \int_0^\infty \left[ \phi_{d,\delta}(x; b) 1_{(0 < x < b)} + \phi_{d,\delta}(b; b) 1_{(x \geq b)} \right] r^{(\beta + \delta)}(u, x) \, dx + \phi_{d,\beta + \delta}(u; \infty).
\end{align*}
\]
(3.5)
Because of the strong Markov property of $X$, one has
\[
\begin{align*}
\phi_{d,\delta}(x; b) &= E_x[e^{-\delta \tau^-_0}; \tau^-_0 < \tau^+_0] \phi_{d,\delta}(b; b) \\
&\quad + E_x[e^{-\delta \tau^-_0}; \tau^-_0 < \tau^+_0, X^-_0 = 0], \quad x \in (0, b).
\end{align*}
\]
(3.6)
The second expectation, namely the Gerber-Shiu function for $X$ due to oscillation avoiding level $b$ en route, can be evaluated using the strong Markov property of $X$ along with (2.2) and (2.4). This results in
\[
\begin{align*}
E_x[e^{-\delta \tau^-_0}; \tau^-_0 < \tau^+_0, X^-_0 = 0] \\
&= E_x[e^{-\delta \tau^-_0}; \tau^-_0 < \infty, X^-_0 = 0] - E_x[e^{-\delta \tau^-_0}; \tau^-_0 < \tau^+_0 < \infty, X^-_0 = 0] \\
&= E_x[e^{-\delta \tau^-_0}; \tau^-_0 < \infty, X^-_0 = 0] - E_x[e^{-\delta \tau^-_0}; \tau^-_0 < \tau^+_0] E_b[e^{-\delta \tau^-_0}; \tau^-_0 < \infty, X^-_0 = 0] \\
&= \frac{\sigma^2}{2} \left[ W^{(\delta)'}(x) - \Phi(\delta) W^{(\delta)}(x) \right] - \frac{\sigma^2}{2} \frac{W^{(\delta)}(x)}{W^{(\delta)}(b)} \left[ W^{(\delta)'}(b) - \Phi(\delta) W^{(\delta)}(b) \right] \\
&= \frac{\sigma^2}{2} \left( W^{(\delta)'}(x) - \frac{W^{(\delta)}(x)}{W^{(\delta)}(b)} W^{(\delta)'}(b) \right), \quad x \in (0, b].
\end{align*}
\]
(3.7)
Application of (2.2) and (3.7) leads (3.6) to
\[
\phi_{d,\delta}(x; b) = \frac{W^{(\delta)}(x)}{W^{(\delta)}(b)} \phi_{d,\delta}(b; b) + \frac{\sigma^2}{2} \left( W^{(\delta)'}(x) - \frac{W^{(\delta)}(x)}{W^{(\delta)}(b)} W^{(\delta)'}(b) \right), \quad x \in (0, b].
\]
Substitution of the above result and (2.4) into (3.5) yields (3.1). Finally, putting $u = b$ in (3.1) and solving for $\phi_{d,\delta}(b; b)$, we can obtain $\phi_{d,\delta}(b; b)$ as in (3.3).

Next, we consider $\phi_{w,\delta}(u; b)$. Via exactly the same arguments leading to (3.5), we arrive at

$$
\phi_{w,\delta}(u; b) = \beta \int_0^\infty \left[ \phi_{w,\delta}(x; b) 1_{(0 < x < b)} + \phi_{w,\delta}(b; b) 1_{(x \geq b)} \right] r^{(\beta + \delta)}(u, x) \, dx 
+ \phi_{w,\delta}(u; \infty), \quad u > 0,
$$

(3.8)

where for $x \in (0, b]$

$$
\phi_{w,\delta}(x; b) = \frac{W^{(\delta)}(x)}{W^{(\delta)}(b)} \phi_{w,\delta}(b; b) + \mathbb{E}_x [e^{-\delta \tau^-} w(X_{\tau^-}, |X_{\tau^-}|); \tau^-_0 < \tau^+_0, X_{\tau^-} < 0].
$$

Further use of (2.5) and (2.6) proves (3.2). In particular, plugging $u = b$ into (3.2) concludes that $\phi_{w,\delta}(b; b)$ is given by (3.4), and the proof is complete.

The following Proposition is concerned with the moments of discounted dividends before ruin.

**Proposition 2.** Suppose that the inter-dividend-decision times are exponentially distributed with mean $1/\beta$. Then, we have, for $n = 1, 2, \ldots$,

$$
V_{n,\delta}(u; b) = \beta V_{n,\delta}(b; b) \int_0^\infty \left( \frac{W^{(\delta)}(x)}{W^{(\delta)}(b)} 1_{(0 < x < b)} + 1_{(x \geq b)} \right) r^{(\beta + n\delta)}(u, x) \, dx
+ \beta \sum_{k=0}^{n-1} \binom{n}{k} V_{k,\delta}(b; b) \int_b^\infty (x - b)^{n-k} r^{(\beta + n\delta)}(u, x) \, dx, \quad u \geq 0,
$$

(3.9)

where $V_{n,\delta}(b; b)$ is given recursively in $n$ via

$$
V_{0,\delta}(b; b) = \frac{\sum_{k=0}^{n-1} \binom{n}{k} V_{k,\delta}(b; b) \frac{2 W^{(\delta)}(b) e^{-\delta \tau^-}}{\Phi^{(\beta + n\delta)}}}{1 - \beta \int_0^\infty \frac{W^{(\delta)}(x)}{W^{(\delta)}(b)} 1_{(0 < x < b)} + 1_{(x \geq b)} \right) r^{(\beta + \delta)}(b, x) \, dx
$$

(3.10)

with the starting point $V_{0,\delta}(b; b) = 1$.

**Proof.** Note that no dividends will ever be payable if ruin occurs before the first dividend decision time $L_1$. Conditioning on $L_1$ along with a binomial expansion yields

$$
V_{n,\delta}(u; b)
= \int_0^\infty \beta e^{-(\beta + n\delta)t} \int_0^b V_{n,\delta}(x; b) \mathbb{P}_u(X_t \in dx, \tau^-_0 > t) \, dt
+ \beta \sum_{k=0}^{n-1} \binom{n}{k} V_{k,\delta}(b; b) \int_0^\infty \beta e^{-(\beta + n\delta)t} \int_0^b (x - b)^{n-k} \mathbb{P}_u(X_t \in dx, \tau^-_0 > t) \, dt
= \beta \int_0^b V_{n,\delta}(x; b) r^{(\beta + n\delta)}(u, x) \, dx
+ \beta \sum_{k=0}^{n-1} \binom{n}{k} V_{k,\delta}(b; b) \int_b^\infty (x - b)^{n-k} r^{(\beta + n\delta)}(u, x) \, dx, \quad u \geq 0.
$$

(3.11)
Due to strong Markov property of $X$ and (2.2), one has

\[
V_{n, \delta}(x; b) = \mathbb{E}_x[e^{-n\delta \tau^+_b}; \tau^+_b < \tau_0] V_{n, \delta}(b; b) = \frac{W^{(n\delta)}(x)}{W^{(n\delta)}(b)} V_{n, \delta}(b; b), \quad x \in [0, b].
\]  

(3.12)

Substitution of (3.12) into (3.11) followed by separation of the last term $k = n$ in the summation yields (3.9). Finally, with the use of (2.3) and the fact that $W^{(\beta+n\delta)}(\cdot) = 0$ for negative argument, we arrive at

\[
\int_b^\infty (x-b)^{n-k} W^{(\beta+n\delta)}(b, x) \, dx = \int_b^\infty (x-b)^{n-k} e^{-\Phi(\beta+n\delta)x} W^{(\beta+n\delta)}(b) \, dx
\]

\[
= \left( \int_0^\infty x^{n-k} e^{-\Phi(\beta+n\delta)x} \, dx \right) W^{(\beta+n\delta)}(b) e^{-\Phi(\beta+n\delta)b}
\]

\[
= \frac{(n-k)!}{[\Phi(\beta + n\delta)]^{n-k+1}} W^{(\beta+n\delta)}(b) e^{-\Phi(\beta+n\delta)b}.
\]

Hence, setting $u = b$ in (3.9) and solving for $V_{n, \delta}(b; b)$ yields (3.10). \qed

**Remark 3.** One can also study the asymptotic behaviors of $\phi_{d, \delta}(u; b)$, $\phi_{w, \delta}(u; b)$ and $V_{n, \delta}(u; b)$ as $u \to \infty$ by applying the final value theorem of Laplace transforms. First, for $s > \Phi(q)$ we have that

\[
\mathcal{L}_r(q)(s, x) = \int_0^\infty e^{-su}[e^{-\Phi(q)x} W(q)(u) - W(q)(u-x)] \, du = \frac{e^{-\Phi(q)x} - e^{-sx}}{\psi(s) - q}.
\]  

(3.13)

Here we adopt the convention that any Laplace transform operator $\mathcal{L}$ is taken with respect to the first argument of the function in question. Hence, for $s > \Phi(\beta + \delta)$ taking Laplace transforms on both sides of (3.5) gives

\[
\mathcal{L} \phi_{d, \delta}(s; b) = \frac{\beta}{\psi(s) - (\beta + \delta)} \int_0^\infty [\phi_{d, \delta}(x; b)1_{(0<x<b)} + \phi_{d, \delta}(b; b)1_{(x\geq b)}] [e^{-\Phi(\beta + \delta)x} - e^{-sx}] \, dx
\]

\[
+ \mathcal{L} \phi_{d, \beta+\delta}(s; \infty).
\]

By analytical continuation of Laplace transforms, the above equation holds true for all $\Re(s) > 0$. Thus, application of the final value theorem of Laplace transforms to the above equation leads to

\[
\lim_{u \to \infty} \phi_{d, \delta}(u; b) = \lim_{s \to 0} s \mathcal{L} \phi_{d, \delta}(s; b)
\]

\[
= \lim_{s \to 0} \frac{\beta}{\beta + \delta - \psi(s)} \phi_{d, \delta}(b; b) \int_b^\infty se^{-sx} \, dx
\]

\[
+ \lim_{s \to 0} \frac{\beta}{\psi(s) - (\beta + \delta)} \phi_{d, \delta}(x; b) \int_0^b [e^{-\Phi(\beta + \delta)x} - e^{-sx}] \, dx
\]

\[
+ \lim_{s \to 0} \frac{\beta}{\psi(s) - (\beta + \delta)} \phi_{d, \delta}(b; b) \int_b^\infty e^{-\Phi(\beta + \delta)x} \, dx + \lim_{s \to 0} s \mathcal{L} \phi_{d, \beta+\delta}(s; \infty).
\]  

(3.14)

Because $\phi_{d, \delta}(x; b) \leq 1$ and

\[
\lim_{s \to 0} s \mathcal{L} \phi_{d, \beta+\delta}(s; \infty) = \lim_{u \to \infty} \mathbb{E}_u[e^{-(\beta + \delta)\tau^+_0}; \tau^+_0 < \infty, X_{\tau^+_0} = 0] = 0,
\]

the only non-zero term in (3.14) is the first limit. Therefore, we arrive at

\[
\lim_{u \to \infty} \phi_{d, \delta}(u; b) = \frac{\beta}{\beta + \delta} \phi_{d, \delta}(b; b).
\]  

(3.15)
Similarly, we can obtain from (3.8) and (3.11) respectively the asymptotics
\[ \phi_{w,\delta}(u; b) \sim \frac{\beta}{\beta + \delta} \phi_{w,\delta}(b; b) \]  
(3.16)
and
\[ V_{n,\delta}(u; b) \sim \frac{\beta}{\beta + n\delta} u^n, \]  
(3.17)
where for two functions \( g_1(u) \) and \( g_2(u) \), the notation \( g_1(u) \sim g_2(u) \) means that \( \lim_{u \to \infty} [g_1(u)/g_2(u)] = 1 \). The probabilistic interpretations of (3.15)-(3.17) are similar to those in Remarks 2.3 and 3.2 in [9], and are omitted here. \( \square \)

4. Erlang\((m, \beta)\) inter-dividend-decision times. In this section, we study the Gerber-Shiu functions and the dividend moments when the inter-dividend-decision times are Erlang\((m, \beta)\) distributed with density (1.4). The analysis in this general situation is considerably more complicated, and we need to first introduce a potential measure associated with the Erlang\((m, \beta)\) distribution in the following subsection.

4.1. A potential measure. First, we note from the proof of Theorem 8.7 in [27] that, for \( q > 0 \),
\[ R^{(q)}(u, dx) = \frac{1}{q} P_u(X_{n+1} \in dx, \tau_0^- > X_q), \quad u, x \geq 0, \]
where \( X_q \), independent of \( X \), is an exponential random variable with mean \( 1/q \). Let \( \{X_q\}_{k=1}^{\infty} \) be i.i.d. copies of \( X_q \) that are also independent of \( X \). For \( k = 1, 2, \ldots \) and \( q > 0 \), we define the potential measure
\[ R_k^{(q)}(u, dx) = \frac{1}{q} P_u \left( X_{\sum_{j=1}^{k} X_{q,j}} \in dx, \tau_0^- > \sum_{j=1}^{k} X_{q,j} \right), \quad u, x \geq 0. \]  
(4.1)
Clearly, we have \( R_1^{(q)}(u, dx) = R^{(q)}(u, dx) \). For \( k = 2, 3, \ldots \), the Markov property implies that
\[ R_k^{(q)}(u, dx) = \int_{y \in (0, \infty)} R^{(q)}(u, dy) R_{k-1}^{(q)}(y, dx) = \int_{y \in (0, \infty)} R_{k-1}^{(q)}(u, dy) R^{(q)}(y, dx). \]
Because the density \( r^{(q)}(u, x) = R^{(q)}(u, dx)/dx \) exists (and is given by (2.3)), there exists a density \( r_k^{(q)}(u, x) \) such that \( R_k^{(q)}(u, dx) = r_k^{(q)}(u, x) dx \). More specifically, one has that, for \( k = 2, 3, \ldots \),
\[ r_k^{(q)}(u, x) = \int_0^\infty r_1^{(q)}(u, y) r_{k-1}^{(q)}(y, x) dy = \int_0^\infty r_{k-1}^{(q)}(u, y) r_1^{(q)}(y, x) dy, \]  
(4.2)
where \( r_1^{(q)}(u, x) = r^{(q)}(u, x) \). Letting \( W^{(q)\times 1}(x) = W^{(q)}(x) \), we further define the convolution, for \( k = 2, 3, \ldots \),
\[ W^{(q)\times k}(x) = \int_0^x W^{(q)\times (k-1)}(y) W^{(q)}(x - y) dy, \quad x \geq 0, \]
with \( W^{(q)\times k}(x) = 0 \) for \( x < 0 \). For general positive integer \( k = 1, 2, \ldots \), the following Proposition shows how \( r_k^{(q)}(u, x) \) can be expressed in terms of \( \{W^{(q)\times j}(x)\}_{j=1}^k \).
Proposition 3. For \( k = 1, 2, \ldots \), the density \( r_k^{(q)}(u, x) = \frac{\mathcal{L}_k^{(q)}(u, dx)}{dx} \) can be represented as
\[
r_k^{(q)}(u, x) = (-1)^k [W^{(q)\ast k}(u - x) - e^{-\Phi(q)x} W^{(q)\ast k}(u)] \\
- \sum_{i=1}^{k-1} (-1)^i W^{(q)\ast i}(u) \sum_{j=1}^{k-1} c_{k-i,j}(q) x^j e^{-\Phi(q)x}, \quad u, x \geq 0,
\] (4.3)
with the understanding that an empty sum equals 0. The coefficients \( \{c_{i,j}(q)\}_{j=1}^{i} \) can be evaluated recursively in \( i \) (\( i = 1, 2, \ldots \)) via
\[
c_{i,j}(q) = \sum_{l=1}^{i-j} b_{i,l}(q) c_{i-l,j}(q), \quad j = 1, 2, \ldots, i - 1,
\] (4.4)
and
\[
c_{i,i}(q) = \frac{1}{i! [\psi'(\Phi(q))]^i},
\] (4.5)
with \( b_{i,l}(q) \) given explicitly by
\[
b_{i,l}(q) = \frac{1}{i! [\psi'(\Phi(q))]^i} \frac{d^i}{ds^i} [q - \psi(s)]^{i-l} \bigg|_{s=\Phi(q)}.
\] (4.6)

Proof. To prove (4.3), we proceed by first deriving the Laplace transform \( \mathcal{L}_k^{(q)}(s, x) = \int_{0}^{\infty} e^{-sx} r_k^{(q)}(u, x) \, du \) and then inverting it. Note that \( \mathcal{L}_1^{(q)}(s, x) \) is given by (3.13), which can be rewritten as
\[
\mathcal{L}_1^{(q)}(s, x) = \frac{1}{q - \psi(s)} [e^{-sx} - e^{-\Phi(q)x}].
\]
Hence, taking Laplace transforms in the first equality of (4.2) with respect to \( u \) yields, for \( k = 2, 3, \ldots \),
\[
\mathcal{L}_k^{(q)}(s, x) = \frac{1}{q - \psi(s)} \mathcal{L}_{k-1}^{(q)}(s, x) - \frac{1}{q - \psi(s)} \mathcal{L}_{k-1}^{(q)}(\Phi(q), x).
\]
Iteratively, we deduce that, for \( k = 1, 2, \ldots \),
\[
\mathcal{L}_k^{(q)}(s, x) = \left( \frac{1}{q - \psi(s)} \right)^k [e^{-sx} - e^{-\Phi(q)x}] - \sum_{l=1}^{k-1} \left( \frac{1}{q - \psi(s)} \right)^l \mathcal{L}_{k-l}^{(q)}(\Phi(q), x).
\] (4.7)
Because of (2.1), one has that, for \( k = 1, 2, \ldots \),
\[
\mathcal{L}^{-1} \left( \frac{1}{q - \psi(s)} \right)^k = (-1)^k W^{(q)\ast k}(u)
\]
and
\[
\mathcal{L}^{-1} \left( \frac{1}{q - \psi(s)} \right)^k e^{-sx} = (-1)^k W^{(q)\ast k}(u - x).
\]
Therefore, by performing Laplace transforms inversion in (4.7), we arrive at
\[
r_k^{(q)}(u, x) = (-1)^k [W^{(q)\ast k}(u - x) - e^{-\Phi(q)x} W^{(q)\ast k}(u)] \\
- \sum_{l=1}^{k-1} (-1)^l W^{(q)\ast l}(u) \mathcal{L}_{k-l}^{(q)}(\Phi(q), x), \quad u, x \geq 0.
\] (4.8)
It remains to determine $L_{r_i}^{(q)}(\Phi(q), x)$ for $i = 1, 2, \ldots$. For $i = 1$, application of L'Hospital's rule to (4.13) yields

$$L_{r_1}^{(q)}(\Phi(q), x) = \lim_{s \to \Phi(q)} \frac{e^{-\Phi(q)x} - e^{-sx}}{\psi(s) - \psi(\Phi(q))} = \frac{xe^{-\Phi(q)x}}{\psi'(\Phi(q))} = c_{1,1}(q) xe^{-\Phi(q)x}, \quad (4.9)$$

where $c_{1,1}(q)$ is defined according to (4.5). For $i = 2, 3, \ldots$, we rewrite (4.7) (with $k$ replaced by $i$) in the form

$$L_{r_i}^{(q)}(s, x) = \frac{[e^{-sx} - e^{-\Phi(q)x}] - \sum_{j=1}^{i-1} \frac{q_j}{q_j^i} \cdot \frac{q_j!}{q_j^i} \cdot [-\psi'(\Phi(q))]^i}{[q - \psi(s)]^i}, \quad (4.10)$$

where the denominator has a zero of $\Phi(q)$ with multiplicity $i$. Since $L_{r_i}^{(q)}(s, x)$ is an analytic function of $s$ for $\Re(s) > 0$, the quantity $\Phi(q)$ must also be a zero of the numerator in (4.10) with multiplicity at least $i$. Owing to $\psi(\Phi(q)) = q$, it is easily verified that

$$\lim_{s \to \Phi(q)} \frac{d^i}{ds^i}[q - \psi(s)]^i = i! \cdot [-\psi'(\Phi(q))]^i.$$

By taking the limit $s \to \Phi(q)$ in (4.10) and using L'Hospital's rule, we obtain

$$L_{r_i}^{(q)}(\Phi(q), x) = \frac{(-x)^i e^{-\Phi(q)x} - \sum_{j=1}^{i-1} \left( \lim_{s \to \Phi(q)} \frac{d^i}{ds^i}[q - \psi(s)]^{i-j} \right) L_{r_{i-j}}^{(q)}(\Phi(q), x)}{i! \cdot [-\psi'(\Phi(q))]^i}$$

$$= \frac{1}{i! \cdot [-\psi'(\Phi(q))]^i} x^i e^{-\Phi(q)x}$$

$$- \sum_{j=1}^{i-1} \frac{1}{i! \cdot [-\psi'(\Phi(q))]^i} \left( \frac{d^i}{ds^i}[q - \psi(s)]^{i-j} \right) \bigg|_{s=\Phi(q)} L_{r_{i-j}}^{(q)}(\Phi(q), x)$$

$$= c_{i,i}(q) x^i e^{-\Phi(q)x} + \sum_{j=1}^{i-1} b_{i,j}(q) L_{r_{i-j}}^{(q)}(\Phi(q), x), \quad (4.11)$$

where $c_{i,i}(q)$ and $b_{i,j}(q)$ are defined in (4.5) and (4.6) respectively. On the grounds of the above iterative equation along with the starting point (4.9), one easily deduces that $L_{r_i}^{(q)}(\Phi(q), x)$ is of the form, for $i = 1, 2, \ldots,$

$$L_{r_i}^{(q)}(\Phi(q), x) = \sum_{j=1}^{i} c_{i,j}(q) x^j e^{-\Phi(q)x}, \quad (4.12)$$

for some constants $\{c_{i,j}(q)\}_{j=1}^{i}$. Back substitution of (4.12) into both sides of (4.11) followed by a change of order of summations on the right-hand side leads to

$$\sum_{j=1}^{i} c_{i,j}(q) x^j e^{-\Phi(q)x} = c_{i,i}(q) x^i e^{-\Phi(q)x} + \sum_{j=1}^{i-1} \left( \sum_{\ell=1}^{i-j} b_{i,j}(q) c_{i-\ell,j}(q) \right) x^j e^{-\Phi(q)x}.$$

Since the above equation is true for all $x \geq 0$, equating the coefficients of $x^j e^{-\Phi(q)x}$ for $j = 1, 2, \ldots, i - 1$ results in the recursion (4.4). To conclude, (4.3) follows from (4.8) and (4.12).
4.2. Gerber-Shiu functions. For the remainder of this paper, we shall use \( I \) to denote the identity matrix of dimension \( m \). Furthermore, \( \mathbf{e}_i \) denotes the \( m \)-dimensional column vector with the \( j \)-th element being 1 and other elements being 0. For any matrix \( \mathbf{A} \), its \((i,j)\)-th element is denoted by \([\mathbf{A}]_{ij}\), and its transpose is represented as \( \mathbf{A}^\top \).

To study the Gerber-Shiu functions, it will be convenient to define some auxiliary functions as follows. First, for \( j = 1, 2, \ldots, m \), we let \( \phi_{d,\delta,1}(u; b) \) and \( \phi_{w,\delta,1}(u; b) \) respectively be the Gerber-Shiu functions \((1.6)\) and \((1.7)\) computed under the same conditions except that the time until the first dividend decision is Erlang\((m - j + 1, \beta)\) distributed. Clearly, one has \( \phi_{d,\delta,1}(u; b) = \phi_{d,\delta}(u; b) \) and \( \phi_{w,\delta,1}(u; b) = \phi_{w,\delta}(u; b) \). For later use, we also define the vectors \( \hat{\mathbf{\phi}}_{d,\delta}(u; b) = (\phi_{d,\delta,1}(u; b), \ldots, \phi_{d,\delta,m}(u; b))^\top \) and \( \hat{\mathbf{\phi}}_{w,\delta}(u; b) = (\phi_{w,\delta,1}(u; b), \ldots, \phi_{w,\delta,m}(u; b))^\top \).

Adopting the notation \((1.4)\) regarding an Erlang density, we start the analysis by noting that

\[
\int_0^\infty e^{-\delta t} \mathbb{P}_u(X_t \in dx, \tau^-_0 > t) \frac{f(t; k, \beta)}{\beta} dt = \left( \frac{\beta}{\beta + \delta} \right)^k \int_0^\infty \mathbb{P}_u(X_t \in dx, \tau^-_0 > t) \frac{f(t; k, \beta + \delta)}{\beta} dt
\]

\[
= \beta^k r_k((\beta + \delta)\mathbf{u}, \mathbf{x}) dx, \quad u, x \geq 0, \tag{4.13}
\]

where the definition \((4.1)\) has been used. Then, again by distinguishing whether ruin occurs or not before the first dividend decision time as in \((3.5)\), we arrive at, for \( j = 1, 2, \ldots, m \),

\[
\phi_{d,\delta,j}(u; b) = \int_0^\infty e^{-\delta t} \int_0^\infty \left[ \phi_{d,\delta,1}(x; b) \mathbf{1}_{(0 < x < b)} + \phi_{d,\delta,1}(b; b) \mathbf{1}_{(x \geq b)} \right] \times \mathbb{P}_u(X_t \in dx, \tau^-_0 > t) \frac{f(t; m - j + 1, \beta)}{\beta} dt
\]

\[
+ \int_0^\infty \mathbb{E}_u[e^{-\delta \tau^-_0}; \tau^-_0 < t, X^-_{\tau^-_0} = 0] \frac{f(t; m - j + 1, \beta)}{\beta} dt
\]

\[
= \beta^{m - j + 1} \int_0^b \phi_{d,\delta,1}(x; b) r_{m - j + 1}(\beta + \delta)(\mathbf{u}, \mathbf{x}) dx
\]

\[
+ \phi_{d,\delta,1}(b; b) \beta^{m - j + 1} \int_0^\infty r_{m - j + 1}(\beta + \delta)(\mathbf{u}, \mathbf{x}) dx + \varphi_{d,\delta,m-j+1}(u), \quad u > 0, \tag{4.14}
\]

where, for \( i = 1, 2, \ldots, m \),

\[
\varphi_{d,\delta,i}(u) = \mathbb{E}_u\left[ e^{-\delta \tau^-_0}; \tau^-_0 < \sum_{k=1}^i \mathcal{E}_{\beta, k}, X^-_{\tau^-_0} = 0 \right], \quad u > 0,
\]

is just the Gerber-Shiu function in the barrier-free process \( X \) when ruin occurs by oscillation within an independent Erlang\((i, \beta)\) time horizon. In particular, for \( i = 1 \), it is clear that

\[
\varphi_{d,\delta,1}(u) = \mathbb{E}_u\left[ e^{-\delta \tau^-_0}; \tau^-_0 < \mathcal{E}_{\beta,1}, X^-_{\tau^-_0} = 0 \right] = \phi_{d,\beta}(u; \infty). \tag{4.15}
\]

For \( i = 2, 3, \ldots, m \), we can write

\[
\varphi_{d,\delta,i}(u) = \mathbb{E}_u\left[ e^{-\delta \tau^-_0}; \tau^-_0 < \mathcal{E}_{\beta,1}, X^-_{\tau^-_0} = 0 \right]
\]
\[
\begin{align*}
&+ \sum_{l=2}^{i} \mathbb{E}_u \left[ e^{-\delta \tau_0} ; \sum_{k=1}^{l-1} \mathcal{E}_{\beta,k} \leq \tau_0^- < \sum_{k=1}^{l} \mathcal{E}_{\beta,k}, X_{\tau_0^-} = 0 \right] \\
&= \phi_{d,\beta+\delta}(u; \infty) + \sum_{l=2}^{i} \int_{0}^{\infty} \int_{0}^{\infty} e^{-\delta t} \mathbb{E}_x \left[ e^{-\delta \tau_0^-} ; \tau_0^- < \mathcal{E}_{\beta,l}, X_{\tau_0^-} = 0 \right] \\
&\quad \times \mathbb{P}_u (X_{t} \in dx, \tau_0^- > t) f(t; l-1, \beta) \, dt \\
&= \frac{\sigma^2}{2} \left[ W^{(\beta+\delta)'}(u) - \Phi(\beta + \delta)W^{(\beta+\delta)}(u) \right] \\
&\quad + \frac{\sigma^2}{2} \beta^{l-1} \int_{0}^{\infty} \left[ W^{(\beta+\delta)'}(x) - \Phi(\beta + \delta)W^{(\beta+\delta)}(x) \right] r_{l-1}^{(\beta+\delta)}(u,x) \, dx,
\end{align*}
\]

where (2.4), (4.13) and (4.15) have been applied. With an empty sum being 0, the above expression also holds true for \( i = 1 \).

Applying similar arguments leading to (4.14) and (4.16) gives rise to, for \( j = 1, 2, \ldots, m, \)
\[
\phi_{w,\delta,j}(u; b) = \beta^{m-j+1} \int_{0}^{b} \phi_{w,\delta,1}(x; b) r_{m-j+1}^{(\beta+\delta)}(u,x) \, dx \\
+ \phi_{w,\delta,1}(b; b) \beta^{m-j+1} \int_{b}^{\infty} r_{m-j+1}^{(\beta+\delta)}(u,x) \, dx + \varphi_{w,\delta,m-j+1}(u), \quad u > 0, \quad (4.17)
\]

where, for \( i = 1, 2, \ldots, m, \)
\[
\varphi_{w,\delta,i}(u) = \mathbb{E}_u \left[ e^{-\delta \tau_0} w(X_{\tau_0^-}; X_{\tau_0^-}); \tau_0^- < \sum_{k=1}^{i} \mathcal{E}_{\beta,k}, X_{\tau_0^-} < 0 \right] \\
= \int_{0}^{\infty} \int_{0}^{\infty} w(x,y) i_{l}^{(\beta+\delta)}(u,x) \nu(x + dy) \, dx \\
+ \sum_{l=2}^{i} \beta^{l-1} \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} w(z,y) r_{l-1}^{(\beta+\delta)}(u,x) r_{l}^{(\beta+\delta)}(x,z) \nu(z + dy) \, dz \, dx \\
= \sum_{l=1}^{i} \beta^{l-1} \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} w(x,y) r_{l}^{(\beta+\delta)}(u,x) \nu(x + dy) \, dx \quad u > 0. \quad (4.18)
\]

Note that the last line above follows by using the second equality of (4.2) at \( k = l \) and combining terms.

It follows from (4.14) and (4.17) that further information about \( \phi_{d,\delta,1}(x; b) \) and \( \phi_{w,\delta,1}(x; b) \) for \( x \in (0, b) \) is required. Similar to the proof of Proposition 1, level crossing arguments will be used. However, with Erlang\((m, \beta)\) inter-dividend-decision times, one needs to additionally keep track of the phase of the Erlang random variable. To this end, we introduce an \( m \)-state continuous-time Markov chain \( J = \{J_t\}_{t \geq 0} \) with transition rate matrix
for sufficiently large $s$. As in Chapter XI.2 in \[\text{[6]}\] with matrix cumulant generating function

$$G(s) = Q + \text{diag}(\psi(s), \ldots, \psi(s)).$$

As in [31, 39], for $q \geq 0$, the $q$-scale matrix $W(r) \cdot$ defined on $[0, \infty)$ is characterized by the Laplace transform

$$\mathcal{L}W (r)(s) = (G(s) - qI)^{-1}$$

for sufficiently large $s$. From Theorem 3(iv) in [31], it is known that

$$\mathbb{E}_v\left[e^{-xT^+; \tau^+_b < \tau^+_0, J^+_x = j; J_0 = i}\right] = \left[W(r)(x) W(r)(b)^{-1}\right]_{ij}, \quad x \in [0, b],$$

(4.20)

With the above definitions, conditioning on the first exit time of $X$ from the interval $[0, b]$ as well as the resulting state of $J$ (in the case where the exit is via level $b$) leads to

$$\phi_{d, \delta, 1}(x; b) = \mathbb{E}_x\left[e^{-\delta \tau^+_0; \tau^+_0 < \tau^+_b; X_{\tau^+_0} = 0}\right]$$

$$\quad + \sum_{k=1}^{m} \mathbb{E}_x\left[e^{-\delta \tau^+_0; \tau^+_0 < \tau^+_b; J_{\tau^+_0} = k; J_0 = 0}\right] \phi_{d, \delta, k}(b; b)$$

$$\quad = \frac{\sigma^2}{2} \left(W(\delta)^{\prime}(x) - \frac{W(\delta)(x)}{W(\delta)(b)} W(\delta)^{\prime}(b)\right)$$

$$\quad + \sum_{k=1}^{m} \left[W(\delta)(x) W(\delta)(b)^{-1}\right]_{1k} \phi_{d, \delta, k}(b; b), \quad x \in (0, b],$$

(4.21)

where (3.7) and (4.20) have been used in the last equality. Similarly, for $\phi_{w, \delta, 1}(x; b)$ one has

$$\phi_{w, \delta, 1}(x; b) = \mathbb{E}_x\left[e^{-\delta \tau^+_0; w(X_{\tau^+_0}; X_{\tau^+_0}) < \tau^+_b; X_{\tau^+_0} < 0}\right]$$

$$\quad + \sum_{k=1}^{m} \mathbb{E}_x\left[e^{-\delta \tau^+_0; \tau^+_0 < \tau^+_b; J_{\tau^+_0} = k; J_0 = 0}\right] \phi_{w, \delta, k}(b; b)$$

$$\quad = \int_0^b \int_0^\infty w(z, y) \nu(\delta)(x, z; b) \nu(z + dy) dz$$

$$\quad + \sum_{k=1}^{m} \left[W(\delta)(x) W(\delta)(b)^{-1}\right]_{1k} \phi_{w, \delta, k}(b; b), \quad x \in (0, b].$$

(4.22)

Now, by plugging (4.21) and (4.22) into (4.14) and (4.17) respectively, the results can be neatly expressed in matrix form as

$$\phi_{d, \delta}(u; b) = A_\delta(u; b) \phi_{d, \delta}(b; b) + \gamma_{d, \delta}(u; b), \quad u > 0,$$

(4.23)

and

$$\phi_{w, \delta}(u; b) = A_\delta(u; b) \phi_{w, \delta}(b; b) + \gamma_{w, \delta}(u; b), \quad u > 0,$$

(4.24)
where $A_\delta(u; b)$ is an $m$-dimensional square matrix with $(j, k)$-th element
\[
[A_\delta(u; b)]_{jk} = \beta^{m-j+1} \int_0^b r_{m-j+1}^{(\beta+\delta)}(u, x) \left[ W^{(\delta)}(x) W^{(\delta)}(b)^{-1} \right]_{1k} dx
+ 1_{(k=1)} \beta^{m-j+1} \int_b^\infty r_{m-j+1}^{(\beta+\delta)}(u, x) dx, \quad u \geq 0.
\] (4.25)

In addition, $\mathcal{G}_{d,\delta}(u; b)$ and $\mathcal{G}_{w,\delta}(u; b)$ are $m$-dimensional column vectors with $j$-th elements
\[
\mathcal{G}_{d,\delta}(u; b) = \frac{\beta^{m-j+1}}{2} \int_0^b r_{m-j+1}^{(\beta+\delta)}(u, x) \left( W^{(\delta)}(x) - \frac{W^{(\delta)}(x)}{W^{(\delta)}(b)} W^{(\delta)}(b) \right) dx
+ \phi_{d,\delta,m-j+1}(u), \quad u > 0,
\] (4.26)
and
\[
\mathcal{G}_{w,\delta}(u; b) = \beta^{m-j+1} \int_0^b r_{m-j+1}^{(\beta+\delta)}(u, x) \int_0^\infty \int_0^\infty w(z, y) r^{(\delta)}(x, z; b) \nu(z + dy) dz dx
+ \phi_{w,\delta,m-j+1}(u), \quad u > 0,
\] (4.27)
respectively. The following Proposition is a direct consequence.

**Proposition 4.** Suppose that the inter-dividend-decision times are Erlang($m, \beta$) distributed with density (1.4). Then, we have
\[
\tilde{\phi}_{d,\delta}(u; b) = A_\delta(u; b) \left[ I - A_\delta(b; b) \right]^{-1} \mathcal{G}_{d,\delta}(b; b) + \mathcal{G}_{d,\delta}(u; b), \quad u > 0,
\] (4.28)
and
\[
\tilde{\phi}_{w,\delta}(u; b) = A_\delta(u; b) \left[ I - A_\delta(b; b) \right]^{-1} \mathcal{G}_{w,\delta}(b; b) + \mathcal{G}_{w,\delta}(u; b) \quad u > 0.
\] (4.29)

**Proof.** To prove (4.28), we first put $u = b$ into (4.23) and solve for $\tilde{\phi}_{d,\delta}(b; b)$. This results in
\[
\tilde{\phi}_{d,\delta}(b; b) = \left[ I - A_\delta(b; b) \right]^{-1} \mathcal{G}_{d,\delta}(b; b),
\] (4.30)
given that $I - A_\delta(b; b)$ is non-singular. To see this, it suffices to verify that $I - A_\delta(b; b)$ is strictly diagonally dominant. For each fixed $j = 1, 2, \ldots, m$, applying (4.13), (4.20) and (4.25) leads to
\[
\sum_{k=1}^m [A_\delta(b; b)]_{jk} = \beta^{m-j+1} \int_0^b r_{m-j+1}^{(\beta+\delta)}(b, x) \mathbb{E}_x \left[ e^{-\delta \tau_b^+}; \tau_b^+ < \tau_0^+ | J_0 = 1 \right] dx
+ \beta^{m-j+1} \int_b^\infty r_{m-j+1}^{(\beta+\delta)}(b, x) dx
\leq \beta^{m-j+1} \int_0^\infty r_{m-j+1}^{(\beta+\delta)}(b, x) dx
= \int_0^\infty \int_0^\infty e^{-\delta t} \mathbb{P}_b(X_t \in dx, \tau_0^- > t) f(t; m - j + 1, \beta) dt
= \int_0^\infty e^{-\delta t} \mathbb{P}_b(\tau_0^- > t) f(t; m - j + 1, \beta) dt < 1,
\]
from which strict diagonal dominance follows. Back substitution of (4.30) into (4.23) gives (4.28). Next, (4.29) can be obtained from (4.24) in identical manner. \(\square\)

It is instructive to note that in Proposition 4 above, the Gerber-Shiu functions are only expressed in terms of the scale function $W^{(\delta)}(\cdot)$, the scale matrix $W^{(\delta)}(\cdot)$, and the newly introduced potential measure $r_{k}^{(q)}(\cdot, \cdot)$ via the definitions (4.25)-(4.27).
4.3. Moments of discounted dividends. This subsection is concerned with the moments of discounted dividends payable until ruin. Since the analyses are very similar to those in Section 4.2, we only present the key steps and the main results.

For $j = 1, 2, \ldots, m$, let $V_{n, \delta, j}(u; b)$ be the $n$-th dividend moment given that the time until the first dividend decision time follows Erlang($m - j + 1, \beta$) distribution. Then $V_{n, \delta, 1}(u; b) = V_{n, \delta}(u; b)$. By conditioning on the first dividend decision time, we arrive at

$$V_{n, \delta, j}(u; b) = \int_0^\infty e^{-n\delta t} \int_0^b V_{n, \delta, 1}(x; b) \mathbb{P}_u(X_t \in dx, \tau_0^- > t) f(t; m - j + 1, \beta) dt$$

$$+ \sum_{k=0}^n \binom{n}{k} V_{k, \delta, 1}(b; b) \int_0^\infty e^{-n\delta t} \int_b^\infty (x - b)^{n-k} \times \mathbb{P}_u(X_t \in dx, \tau_0^- > t) f(t; m - j + 1, \beta) dt$$

$$= \beta^{m-j+1} \int_0^b V_{n, \delta, 1}(x; b) r_{m-j+1}(u, x) dx$$

$$+ \beta^{m-j+1} \sum_{k=0}^n \binom{n}{k} V_{k, \delta, 1}(b; b) \int_b^\infty (x - b)^{n-k} r_{m-j+1}(u, x) dx. \quad (4.31)$$

On the other hand, the strong Markov property of $(J, X)$ ensures that

$$V_{n, \delta, 1}(x; b) = \sum_{k=1}^m \left[ \mathbf{W}(n\delta)(x) \mathbf{W}(n\delta)(b)^{-1} \right]_{1k} V_{n, \delta, k}(b; b), \quad x \in [0, b]. \quad (4.32)$$

Substitution of (4.32) into (4.31) gives rise to

$$\tilde{V}_{n, \delta}(u; b) = A_{n\delta}(u; b) \tilde{V}_{n, \delta}(b; b) + \kappa_{n, \delta}(u; b), \quad u \geq 0, \quad (4.33)$$

where

$$\tilde{V}_{n, \delta}(u; b) = (V_{n, \delta, 1}(u; b), \ldots, V_{n, \delta, m}(u; b))^\top,$$

$$\kappa_{n, \delta}(u; b) = (\kappa_{n, \delta, 1}(u; b), \ldots, \kappa_{n, \delta, m}(u; b))^\top,$$

with

$$\kappa_{n, \delta, j}(u; b) = \beta^{m-j+1} \sum_{k=0}^{n-1} \binom{n}{k} V_{k, \delta, 1}(b; b) \int_b^\infty (x - b)^{n-k} r_{m-j+1}(u, x) dx, \quad u \geq 0.$$

The use of (4.33) leads to the following Proposition.

**Proposition 5.** Suppose that the inter-dividend-decision times are Erlang($m, \beta$) distributed with density (1.4). Then, we have, for $n = 1, 2, \ldots$,

$$\tilde{V}_{n, \delta}(u; b) = A_{n\delta}(u; b) [I - A_{n\delta}(b; b)]^{-1} \kappa_{n, \delta}(b; b) + \kappa_{n, \delta}(u; b), \quad u \geq 0,$$

with the starting point $V_{0, \delta, 1}(b; b) = 1$. 

5. An alternative view: First overshoot at a dividend decision time. In this section, a different approach will be used to analyze the Gerber-Shiu functions and the discounted dividends until ruin. More specifically, we shall consider the first overshoot of the process $X$ over the barrier level $b$ at a dividend decision time. Define $\epsilon_1 = \inf\{k \geq 1 : X_{L_k} > b\}$, and for $i = 2, 3, \ldots$ let $\epsilon_i = \inf\{k > \epsilon_{i-1} : X_{L_k} - X_{L_{k-1}} > 0\}$. Then, for $i = 1, 2, \ldots$, the variable $L_{\epsilon_i}$ represents the time when the $i$-th (positive) dividend is paid in the modified surplus process $X^b$, as long as ruin has not occurred in $X^b$. Because of the positive security loading condition $EX_1 > 0$, one has that $L_{\epsilon_i} < \infty$ almost surely. Note that after each time $L_{\epsilon_i}$, the process $X^b$ returns to level $b$. Furthermore, $(\{L_{\epsilon_{i+1}} - L_{\epsilon_i}, X_{L_{\epsilon_{i+1}}} - X_{L_{\epsilon_i}}\})_{i=1}^{\infty}$ form an i.i.d. sequence of bivariate random vectors.

5.1. Gerber-Shiu functions. To study the Gerber-Shiu functions, we begin by defining a few auxiliary functions as follows. Let

$$\chi_{d,\delta}(u; b) = \mathbb{E}_u[e^{-\delta \tau_0^\beta}; \tau_0^\beta < L_{\epsilon_1}, X_{\tau_0^\beta} = 0], \quad u > 0,$$

and

$$\chi_{w,\delta}(u; b) = \mathbb{E}_u[e^{-\delta \tau_0^\beta} w(X_{\tau_0^\beta}, |X_{\tau_0^\beta}|); \tau_0^\beta < L_{\epsilon_1}, X_{\tau_0^\beta} < 0], \quad u > 0,$$

be the Gerber-Shiu functions for $X$ due to oscillation and due to claims respectively, both with ruin occurring before $L_{\epsilon_1}$. Further define, for $i = 0, 1, \ldots$,

$$\zeta_{i,\delta}(u; b) = \mathbb{E}_u[e^{-\delta \tau_{i+1}} (X_{L_{\epsilon_{i+1}}} - b)^i; L_{\epsilon_i} < \tau_0^\beta], \quad u \geq 0,$$

which is the $i$-th moment of the first positive dividend payment. Then, by strong Markov property of $X$, we have

$$\phi_{d,\delta}(u; b) = \chi_{d,\delta}(u; b) + \zeta_{0,\delta}(u; b) \phi_{d,\delta}(b; b), \quad u > 0,$$

and

$$\phi_{w,\delta}(u; b) = \chi_{w,\delta}(u; b) + \zeta_{0,\delta}(u; b) \phi_{w,\delta}(b; b), \quad u > 0.$$

Plugging $u = b$ into the above equations, solving for $\phi_{d,\delta}(b; b)$ and $\phi_{w,\delta}(b; b)$ and then performing back substitutions, we arrive at

$$\phi_{d,\delta}(u; b) = \chi_{d,\delta}(u; b) + \frac{\zeta_{0,\delta}(u; b)}{1 - \zeta_{0,\delta}(b; b)} \chi_{d,\delta}(b; b), \quad u > 0,$$

and

$$\phi_{w,\delta}(u; b) = \chi_{w,\delta}(u; b) + \frac{\zeta_{0,\delta}(u; b)}{1 - \zeta_{0,\delta}(b; b)} \chi_{w,\delta}(b; b), \quad u > 0.$$

With (5.2) and (5.3), it remains to determine $\chi_{d,\delta}(u; b), \chi_{w,\delta}(u; b)$ and $\zeta_{0,\delta}(u; b)$ under Erlang$(m, \beta)$ inter-dividend-decision times. (For later use concerning the dividend moments, we shall study the more general quantity $\zeta_{i,\delta}(u; b)$.) Similar to Section 4, we define $\chi_{d,\delta,j}(u; b), \chi_{w,\delta,j}(u; b)$ and $\zeta_{i,\delta,j}(u; b)$ respectively to be modifications of $\chi_{d,\delta}(u; b), \chi_{w,\delta}(u; b)$ and $\zeta_{i,\delta}(u; b)$ when the time until the first dividend decision is Erlang$(m - j + 1, \beta)$ distributed. Thus, one has $\chi_{d,\delta,1}(u; b) = \chi_{d,\delta}(u; b), \chi_{w,\delta,1}(u; b) = \chi_{w,\delta}(u; b)$ and $\zeta_{i,\delta,1}(u; b) = \zeta_{i,\delta}(u; b)$. First, we consider $\chi_{d,\delta,j}(u; b)$. Application of the procedure leading to (4.14) and (4.21) yields, for $j = 1, 2, \ldots, m$,

$$\chi_{d,\delta,j}(u; b) = \beta^{m-j+1} \int_0^b \chi_{d,\delta,1}(x; b) r_{m-j+1}^{(\beta+\delta)}(u, x) dx + \phi_{d,\delta,m-j+1}(u), \quad u > 0,$$

(5.4)
Combining the above two formulas, it is found that
\[
I
\]
where the strict diagonal dominance (and hence invertibility) of \( I \) can be proved in the same manner as in Proposition 4. Back substitution of (5.8) into (5.6) yields
\[
\hat{x}_{d,\delta}(u; b) = C_{\delta}(u; b) [I - C_{\delta}(b; b)]^{-1} \hat{y}_{d,\delta}(b; b), \quad u > 0.
\]
In particular, (5.6) at \( u = b \) gives
\[
\hat{x}_{d,\delta}(b; b) = [I - C_{\delta}(b; b)]^{-1} \hat{y}_{d,\delta}(b; b),
\]
where the strict diagonal dominance (and hence invertibility) of \( I - C_{\delta}(b; b) \) can be proved in the same manner as in Proposition 4. Back substitution of (5.8) into (5.6) yields
\[
\hat{x}_{d,\delta}(u; b) = C_{\delta}(u; b) [I - C_{\delta}(b; b)]^{-1} \hat{y}_{d,\delta}(b; b) + \hat{y}_{d,\delta}(u; b), \quad u > 0.
\]
Similarly, we obtain
\[
\hat{x}_{w,\delta}(u; b) = C_{\delta}(u; b) [I - C_{\delta}(b; b)]^{-1} \hat{y}_{w,\delta}(b; b) + \hat{y}_{w,\delta}(u; b), \quad u > 0,
\]
where \( \hat{x}_{w,\delta}(u; b) = (\chi_{w,\delta,1}(u; b), \ldots, \chi_{w,\delta,m}(u; b))^\top \). Finally, concerning the quantity defined by (5.1), one has that, for \( j = 1, 2, \ldots, m, \)
\[
\zeta_{i,\delta,j}(u; b) = \beta^{m-j+1} \int_{0}^{b} \zeta_{i,\delta,1}(x; b) \tau_{m-j+1}^{(\beta+\delta)}(u, x) dx
\]
\[
+ \beta^{m-j+1} \int_{b}^{\infty} (x - b)^{\delta} \tau_{m-j+1}^{(\beta+\delta)}(u, x) dx, \quad u \geq 0,
\]
and
\[
\zeta_{i,\delta,1}(x; b) = \sum_{k=1}^{m} [W^{(\delta)}(x) W^{(\delta)}(b)^{-1}]_{1k} \zeta_{i,\delta,k}(b; b), \quad x \in [0, b].
\]
Combining the above two formulas, it is found that
\[
\tilde{\zeta}_{i,\delta}(u; b) = C_{\delta}(u; b) \tilde{\zeta}_{i,\delta}(b; b) + \tilde{\theta}_{i,\delta}(u; b), \quad u \geq 0,
\]
which has solution
\[
\tilde{\zeta}_{i,\delta}(u; b) = C_{\delta}(u; b) [I - C_{\delta}(b; b)]^{-1} \tilde{\theta}_{i,\delta}(b; b) + \tilde{\theta}_{i,\delta}(u; b), \quad u \geq 0.
\]
Here \( \tilde{\zeta}_{i,\delta}(u; b) = (\zeta_{i,\delta,1}(u; b), \ldots, \zeta_{i,\delta,m}(u; b))^\top \), and \( \tilde{\theta}_{i,\delta}(u; b) \) is an \( m \)-dimensional column vector with \( j \)-th element
\[
\theta_{i,\delta,j}(u; b) = \beta^{m-j+1} \int_{b}^{\infty} (x - b)^{\delta} \tau_{m-j+1}^{(\beta+\delta)}(u, x) dx, \quad u \geq 0.
\]
Hence, with the solutions (5.9), (5.10) and (5.12) obtained, the Gerber-Shiu functions (5.2) and (5.3) are completely determined.
Remark 4. Although the expressions (5.2) and (5.3) look different from those in Proposition 4, it can be shown the results are in agreement for $u > 0$. We only demonstrate the equivalence for $\phi_{d,\delta}(u; b)$, since the case of $\phi_{w,\delta}(u; b)$ can be studied in the same manner. First, we extract the first element of $\tilde{\chi}_{d,\delta}(u; b)$ by pre-multiplying both sides of (5.6) with $\tilde{e}_1^T$. This results in

$$\chi_{d,\delta}(u; b) = \tilde{e}_1^T \tilde{\chi}_{d,\delta}(u; b) = \tilde{e}_1^T C_{\delta}(u; b) \tilde{\chi}_{d,\delta}(b; b) + \tilde{e}_1^T \tilde{\gamma}_{d,\delta}(u; b).$$

Then (5.2) can be rewritten as

$$\phi_{d,\delta}(u; b) = \tilde{e}_1^T \tilde{\chi}_{d,\delta}(u; b) + \frac{\tilde{e}_1^T \tilde{\zeta}_{0,\delta}(u; b)}{1 - \tilde{e}_1^T \tilde{\zeta}_{0,\delta}(b; b)} \tilde{e}_1^T \tilde{\chi}_{d,\delta}(b; b)$$

$$= \tilde{e}_1^T \left( C_{\delta}(u; b) + \frac{\tilde{e}_1^T \tilde{\zeta}_{0,\delta}(u; b)}{1 - \tilde{e}_1^T \tilde{\zeta}_{0,\delta}(b; b)} I \right) \tilde{\chi}_{d,\delta}(b; b) + \tilde{e}_1^T \tilde{\gamma}_{d,\delta}(u; b)$$

$$= \tilde{e}_1^T \left( C_{\delta}(u; b) + \frac{\tilde{e}_1^T \tilde{\zeta}_{0,\delta}(u; b)}{1 - \tilde{e}_1^T \tilde{\zeta}_{0,\delta}(b; b)} I \right) [I - C_{\delta}(b; b)]^{-1} \tilde{\gamma}_{d,\delta}(b; b) + \tilde{e}_1^T \tilde{\gamma}_{d,\delta}(u; b),$$

where the last equality follows from the use of (5.8). On the other hand, Proposition 4 asserts that

$$\phi_{d,\delta}(u; b) = \tilde{e}_1^T A_{\delta}(u; b) [I - A_{\delta}(b; b)]^{-1} \tilde{\gamma}_{d,\delta}(b; b) + \tilde{e}_1^T \tilde{\gamma}_{d,\delta}(u; b).$$

Hence, comparing the above two equations, it suffices to verify that

$$\tilde{e}_1^T \left( C_{\delta}(u; b) + \frac{\tilde{e}_1^T \tilde{\zeta}_{0,\delta}(u; b)}{1 - \tilde{e}_1^T \tilde{\zeta}_{0,\delta}(b; b)} I \right) [I - C_{\delta}(b; b)]^{-1} [I - A_{\delta}(b; b)] = \tilde{e}_1^T A_{\delta}(u; b).$$

(5.14)

From the definitions (4.25), (5.7) and (5.13) at $i = 0$, it is easily seen that

$$A_{\delta}(u; b) = C_{\delta}(u; b) + \tilde{\theta}_{0,\delta}(u; b) \tilde{e}_1^T,$$

(5.15)

leading to

$$[I - C_{\delta}(b; b)]^{-1} [I - A_{\delta}(b; b)] = I - [I - C_{\delta}(b; b)]^{-1} \tilde{\theta}_{0,\delta}(b; b) \tilde{e}_1^T = I - \tilde{\zeta}_{0,\delta}(b; b) \tilde{e}_1^T$$

thanks to (5.11) in the last step. Using the above result, the left-hand side of (5.14) can be expressed as

$$\tilde{e}_1^T \left( C_{\delta}(u; b) + \frac{\tilde{e}_1^T \tilde{\zeta}_{0,\delta}(u; b)}{1 - \tilde{e}_1^T \tilde{\zeta}_{0,\delta}(b; b)} I \right) [I - C_{\delta}(b; b)]^{-1} [I - A_{\delta}(b; b)]$$

$$= \tilde{e}_1^T \left( C_{\delta}(u; b) - \tilde{e}_1^T C_{\delta}(u; b) \tilde{\zeta}_{0,\delta}(b; b) \tilde{e}_1^T \right)$$

$$+ \frac{\tilde{e}_1^T \tilde{\zeta}_{0,\delta}(u; b)}{1 - \tilde{e}_1^T \tilde{\zeta}_{0,\delta}(b; b)} \tilde{e}_1^T - \frac{\tilde{e}_1^T \tilde{\zeta}_{0,\delta}(u; b)}{1 - \tilde{e}_1^T \tilde{\zeta}_{0,\delta}(b; b)} \tilde{e}_1^T \tilde{\zeta}_{0,\delta}(b; b) \tilde{e}_1^T$$

$$= \tilde{e}_1^T C_{\delta}(u; b) - \tilde{e}_1^T C_{\delta}(u; b) \tilde{\zeta}_{0,\delta}(b; b) \tilde{e}_1^T + \tilde{e}_1^T \tilde{\zeta}_{0,\delta}(u; b) \tilde{e}_1^T$$

$$= \tilde{e}_1^T C_{\delta}(u; b) + \tilde{e}_1^T \tilde{\theta}_{0,\delta}(u; b) \tilde{e}_1^T = \tilde{e}_1^T A_{\delta}(u; b),$$

where last two steps follow from (5.11) and (5.15). Hence, (5.14) is proved. □
5.2. Discounted dividends until ruin. With the function \( \zeta_{i,\delta}(u; b) \) defined in (5.1), the dividend moments can be readily obtained as follows. If ruin of the process \( X_t \) (or \( X \)) has not occurred before time \( L_{\epsilon_1} \), then a dividend of size \( X_{L_{\epsilon_1}} - b \) will be payable to the shareholders at time \( L_{\epsilon_1} \). In this case, there will also be potential future dividends. Thus, by applying a binomial expansion to these two contributions, we arrive at

\[
V_{n,\delta}(u; b) = \sum_{k=0}^{n} \binom{n}{k} E_u[e^{-n\delta L_{\epsilon_1}} (X_{L_{\epsilon_1}} - b)^{n-k}; L_{\epsilon_1} < \tau_0^-] V_{k,\delta}(b; b)
\]

\[
= \sum_{k=0}^{n} \binom{n}{k} \zeta_{n-k, \delta}(u; b) V_{k,\delta}(b; b), \quad u \geq 0.
\]  \( (5.16) \)

The above equation is expressed in terms of the unknown quantities \( \{ V_{k,\delta}(b; b) \}_{k=0}^{\infty} \). Setting \( u = b \) in (5.16) yields the recursive formula in \( n \), namely

\[
V_{n,\delta}(b; b) = \frac{\sum_{k=0}^{n-1} \binom{n}{k} \zeta_{n-k, \delta}(b; b) V_{k,\delta}(b; b)}{1 - \zeta_{0, \delta}(b; b)},
\]  \( (5.17) \)

with the starting value \( V_{0,\delta}(b; b) = 1 \). Hence, \( V_{n,\delta}(u; b) \) is completely determined. In particular, the expected value of discounted dividends until ruin (i.e. \( n = 1 \)) is given by

\[
V_{1,\delta}(u; b) = \zeta_{1,\delta}(u; b) + \zeta_{0,\delta}(u; b) \frac{\zeta_{1,\delta}(b; b)}{1 - \zeta_{0,\delta}(b; b)}, \quad u \geq 0.
\]  \( (5.18) \)

Next, one may also be interested in the discounted dividends up to and including the \( i \)-th (positive) dividend payment before ruin, which can be represented as, for \( i = 1, 2, \ldots, \)

\[
D_{\delta, b, i} = e^{-\delta L_{\epsilon_1}}(X_{L_{\epsilon_1}} - b)1_{(L_{\epsilon_1} < \varsigma_0)} + \sum_{j=2}^{i} e^{-\delta L_{\epsilon_j}}(X_{L_{\epsilon_j}} - X_{L_{\epsilon_{j-1}}})1_{(L_{\epsilon_j} < \varsigma_0)}
\]

\[
= e^{-\delta L_{\epsilon_1}}(X_{L_{\epsilon_1}} - b)1_{(L_{\epsilon_1} < \varsigma_0)}
\]

\[
+ \sum_{j=2}^{i} \prod_{k=2}^{j-1} e^{-\delta (L_{\epsilon_k} - L_{\epsilon_{k-1}})} e^{-\delta (L_{\epsilon_j} - L_{\epsilon_{j-1}})}(X_{L_{\epsilon_j}} - X_{L_{\epsilon_{j-1}}})1_{(L_{\epsilon_j} < \varsigma_0)},
\]

with an empty product taken to be 1. Denoting its expectation by \( V_{1,\delta}(u; b, i) = E_u[D_{\delta, b, i}] \), one has

\[
V_{1,\delta}(u; b, i) = \zeta_{1,\delta}(u; b) + \zeta_{0,\delta}(u; b) \sum_{j=2}^{i} [\zeta_{0,\delta}(b; b)]^{j-2} \zeta_{1,\delta}(b; b)
\]

\[
= \zeta_{1,\delta}(u; b) + \frac{1 - [\zeta_{0,\delta}(b; b)]^{i-1}}{1 - \zeta_{0,\delta}(b; b)} \zeta_{0,\delta}(u; b) \zeta_{1,\delta}(b; b), \quad u \geq 0.
\]

Obviously, with \( \zeta_{0,\delta}(b; b) < 1 \), it is immediate that \( V_{1,\delta}(u; b, i) \rightarrow V_{1,\delta}(u; b) \) as \( i \rightarrow \infty \), where \( V_{1,\delta}(u; b) \) is given by (5.18).

For the remainder of the paper, it is assumed that the inter-dividend-decision times are exponentially distributed with mean \( 1/\beta \), and we consider initial surplus such that \( X_0 = u \in (0, b] \). Under these simplifying assumptions, the distributions of the first overshoot and the total dividends paid until ruin can be explicitly identified. For later use, define the quantity

\[
B_{\delta}(u, x; b) = E_u[e^{-\delta L_{\epsilon_1}}; X_{L_{\epsilon_1}} - b > x, L_{\epsilon_1} < \tau_0^-], \quad u \in (0, b]; x \geq 0.
\]  \( (5.19) \)
which can be regarded as the discounted survival function of the first overshoot at a dividend decision time avoiding ruin enroute. We have the following two Propositions.

**Proposition 6.** Suppose that the inter-dividend-decision times are exponentially distributed with mean $1/\beta$. The quantity defined by (5.19) has solution

$$B_b(u, x; b) = \zeta_0, \delta(u; b) e^{-\Phi(\beta + \delta)x}, \quad u \in (0, b]; x \geq 0.$$  

(5.20)

**Proof.** First, for $i = 1, 2, \ldots$, let

$$\Omega_i = \left\{ \inf_{0 \leq t \leq L_i} X_t \geq 0, \sup_{1 \leq k \leq i-1} X_{L_k} \leq b \right\}.$$  

Then, we proceed by conditioning on the number of dividend decision times before the first positive dividend payment and applying the strong Markov property of $X$. This leads to

$$B_b(u, x; b) = \int_0^\infty \beta e^{-\beta x + \beta \delta t} P_u(X_t - b > x, \tau_0^- > t) dt$$

$$+ \sum_{i=1}^\infty \int_0^b \int_0^\infty \beta e^{-\beta x + \beta \delta t} P_u(X_t - b > x, \tau_0^- > t) dt E_u[e^{-\delta L_i}; \Omega_i, X_{L_i} \in dy]$$

$$= \beta \int_{b+x}^\infty r^\delta(u, z) dz + \sum_{i=1}^\infty \beta \int_0^b \int_{b+x}^\infty r^\delta(y, z) dz E_u[e^{-\delta L_i}; \Omega_i, X_{L_i} \in dy].$$

(5.21)

One can indeed interpret $\sum_{i=1}^\infty E_u[e^{-\delta L_i}; \Omega_i, X_{L_i} \in dy]/dy$ as the discounted kernel density of the process $X$ being at level $y \in (0, b]$ at the times $\{L_i\}_{i=1}^\infty$ without falling below zero and without being observed to be above $b$ in the interim. See also Lemma 1 in [12] for a similar type of kernel density arising in ruin theory. Using the potential measure (2.3) and noting that the integration variable $z$ in (5.21) is such that $z \geq b + x \geq u$ (as we restrict $u \in (0, b]$ and $x \geq 0$), we arrive at

$$B_b(u, x; b) = \frac{\beta}{\Phi(\beta + \delta)} e^{-\Phi(\beta + \delta)(b+x)}$$

$$\times \left( W^{\delta}(u) + \sum_{i=1}^\infty \int_0^b W^{\delta}(y) E_u[e^{-\delta L_i}; \Omega_i, X_{L_i} \in dy] \right).$$

(22)

In the above expression, the dependence on $x$ only appears through $e^{-\Phi(\beta + \delta)x}$, and therefore

$$B_b(u, x; b) = B_b(u, 0; b) e^{-\Phi(\beta + \delta)x}. $$

(5.23)

Finally, comparison of the definition (5.19) at $x = 0$ with (5.1) at $i = 0$ gives

$$B_b(u, 0; b) = E_u[e^{-\delta L_1}; L_{e_1} < \tau_0^-] = \zeta_0, \delta(u; b),$$

and hence (5.20) follows from (5.23). \qed

From Proposition 6, it is clear that for $u \in (0, b]$ and $x > 0$

$$E_u[e^{-\delta L_1}; X_{L_1} - b \in dx, L_{e_1} < \tau_0^-] = \zeta_0, \delta(u; b) \Phi(\beta + \delta) e^{-\Phi(\beta + \delta)x} dx.$$

Thus, according to the definition (5.1), one has that, for $i = 0, 1, \ldots$,

$$\zeta_i, \delta(u; b) = \zeta_0, \delta(u; b) \Phi(\beta + \delta) \int_0^\infty x^i e^{-\Phi(\beta + \delta)x} dx.$$
Under exponential inter-dividend-decision times with initial surplus \( u \in (0, b] \), it follows from (5.24), (5.16) and (5.17) that the dividends moments only depend on the function \( \zeta_0 \).

Another implication of Proposition 6 is that the amount of the first overshoot \( X_{L_{e_1} - b} \), conditional on the event that ruin does not occur in the interim, is exponentially distributed with mean \( 1/\Phi(\beta) \), if the process starts with \( X_0 = u \in (0, b] \). This is evident from

\[
\mathbb{P}_u(X_{L_{e_1} - b > x | L_{e_1} < \tau_0}) = \mathbb{P}_u(X_{L_{e_1} - b > x, L_{e_1} < \tau_0}) = \frac{B_0(u, x; b)}{B_0(u, 0; b)} = e^{-\Phi(\beta)x}, \quad u \in (0, b]; x > 0.
\]

This in turn implies that every dividend payment is exponentially distributed with mean \( 1/\Phi(\beta) \) given that there is such a payment, resulting in the following Proposition.

**Proposition 7.** Suppose that the inter-dividend-decision times are exponentially distributed with mean \( 1/\beta \), and the process starts at \( X_0 = u \in (0, b] \). Without discounting (i.e. \( \delta = 0 \)), the distribution of total amount of dividends payable until ruin \( D_{0,b} \) is a mixture of (1) a point mass at 0 with weight \( 1 - \zeta_{0,0}(u; b) \); and (2) an exponential distribution with mean \( 1/\{ \Phi(\beta) [1 - \zeta_{0,0}(b; b)] \} \) and weight \( \zeta_{0,0}(u; b) \).

**Proof.** It is obvious that no dividends will ever be paid if the process \( X \) does not overshoot level \( b \) (avoiding ruin enroute) at any dividend decision time. Therefore, \( D_{0,b} = 0 \) with probability \( 1 - \zeta_{0,0}(u; b) \). For \( i = 1, 2, \ldots \), with probability \( \zeta_{0,0}(u; b) [\zeta_{0,0}(b; b)]^{i-1} [1 - \zeta_{0,0}(b; b)] \) there will be \( i \) times that a (positive) dividend is made, and each of these dividend payments is exponential with mean \( 1/\Phi(\beta) \) since \( X_0 = u \in (0, b] \). Consequently, the Laplace transform of \( D_{0,b} \) can be evaluated as

\[
\mathbb{E}_u[e^{-sD_{0,b}}] = 1 - \zeta_{0,0}(u; b) + \sum_{i=1}^{\infty} \zeta_{0,0}(u; b) [\zeta_{0,0}(b; b)]^{i-1} [1 - \zeta_{0,0}(b; b)] \left( \frac{\Phi(\beta)}{\Phi(\beta) + s} \right)^i
\]

\[
= 1 - \zeta_{0,0}(u; b) + \zeta_{0,0}(u; b) \frac{\Phi(\beta)}{\Phi(\beta) [1 - \zeta_{0,0}(b; b)] + s},
\]

which is exactly the Laplace transform of the mixed distribution in the statement of the Proposition.

6. **Numerical illustrations.** This section aims to present some numerical examples in connection to the Gerber-Shiu function and the moments of discounted dividends prior to ruin under the proposed periodic dividend barrier strategy. We shall look at the following two classes of Lévy risk models, namely

- a Brownian motion risk model with Laplace exponent \( \psi(s) = 0.5s + s^2 \); and
- compound Poisson risk models with Laplace exponents (i) \( \psi(s) = 1.5s - 1 + \frac{3}{5}s^2 - \frac{3}{5}s^4 \); (ii) \( \psi(s) = 1.5s - 1 + \frac{1}{4}s^2 + \frac{3}{4}s^4 \); and (iii) \( \psi(s) = 1.5s - 1 + \frac{3}{2}s^2 + \frac{1}{2}s^4 \).

The above compound Poisson models have also been considered by e.g. [1] and [46]. The corresponding claim size distributions are (i) a sum of two exponentials with means \( 1/3 \) and \( 2/3 \); (ii) an exponential distribution with mean 1; and (iii) a mixture of an exponential with mean 2 (mixing probability \( 1/3 \)) and another exponential with mean 1/2 (mixing probability 2/3). These claim amounts have common mean
from Equations (3.7) and (2.11) in [58] expected present value of dividend payments under the traditional barrier. It follows that $V_t$ and $\phi$ for reference). In each case, we note that with different claim distributions (and the Brownian motion model is also provided in Figure 1 against the initial surplus $u$ that of the compound Poisson model (ii) with exponential claims). In the sequel, we always use the force of interest $\delta = 0.01$ and assume the penalty function $w \equiv 1$ (so that the Gerber-Shiu function $\phi_{\delta}(u; b)$ reduces to the Laplace transform of the ruin time).

We begin by considering the Brownian motion model. The periodic dividend barrier strategy with exponential inter-dividend-decision times (i.e. $m = 1$) under different dividend decision frequency $\beta$ will be studied, and this will also be compared with the traditional barrier strategy (i.e. continuous decision). Let $\phi_{c,\delta}(u; b)$ and $V_{c,\delta}(u; b)$ respectively be the Laplace transform of the ruin time and the expected present value of dividend payments under the traditional barrier strategy. This explains the observation from Figure 1 that $\phi_{\delta}(u; b)$ is decreasing in $u$ while $V_{1,\delta}(u; b)$ is increasing in $u$ (see also Figure 2). Clearly, a higher initial surplus keeps the process further away from ruin and hence dividends are more likely to be paid (and the dividend payment especially at the beginning of the surplus process.

In addition, it can be seen from Figure 1(a) that $\phi_{\delta}(u; b)$ is an increasing function of $\beta$. This is because a larger $\beta$ leads to more frequent check of the surplus process for dividends (recall that the post-dividend surplus level cannot exceed $b$), resulting in earlier ruin and thereby increasing $\phi_{\delta}(u; b)$. In contrast, Figure 1(b) shows that $V_{1,\delta}(u; b)$ is decreasing in $\beta$. In general, there are two opposing effects to the dividend payments as $\beta$ increases. First, more frequent dividend decision tends to increase the amount of dividends especially at the beginning of the surplus process. On the other hand, earlier occurrence of ruin also means that the insurer is less likely to pay much dividend at later times since dividend payments cease after ruin.

Figure 1(b) suggests that the second effect dominates under the current parameter set. Note that as $\beta$ increases further and further, the inter-dividend-decision times becomes shorter and shorter and ultimately our model becomes the traditional barrier strategy. This explains the observation from Figure 1 that $\phi_{\delta}(u; b) \rightarrow \phi_{c,\delta}(u; b)$ and $V_{1,\delta}(u; b) \rightarrow V_{c,\delta}(u; b)$ as $\beta \rightarrow \infty$.

Next, we study the impact of the claim distribution on the ruin-related quantities in the compound Poisson model. Let $b = 8$, $m = 4$ and $\beta = 1.6$ (so that the inter-dividend-decision times have mean $ET = m/\beta = 2.5$). Figure 2 depicts the behaviour of both $\phi_{\delta}(u; b)$ and $V_{1,\delta}(u; b)$ for the three compound Poisson models with different claim distributions (and the Brownian motion model is also provided for reference). In each case, we note that $\phi_{\delta}(u; b)$ is almost a constant while $V_{1,\delta}(u; b)$
is almost linear as \( u \) becomes large, which are in agreement with formulas (3.16) and (3.17) respectively. Furthermore, it is observed that a higher variance of the claim amount results in larger \( \phi_\delta(u;b) \) but smaller \( V_{1,\delta}(u;b) \). Intuitively, a compound Poisson risk process with a larger claim variance (with the mean being fixed) tends to have more extremes and is therefore more likely to ruin early, and this in turn reduces the amount of dividend payments.

Finally, we are interested in the optimal dividend barrier, namely \( b^* \) if it exists, that maximizes the expected discounted dividends before ruin \( V_{1,\delta}(u;b) \). Let \( m = 3 \) and \( \beta = 1.2 \) (so that \( E[T] = 2.5 \)). We plot \( V_{1,\delta}(u;b) \) as a function of \( b \) for the fixed initial surplus levels \( u = 1, 5, 10, 15 \) in the Brownian motion model (Figure 3(a)) and the compound Poisson model (ii) with exponential claims (Figure 3(b)). In each model, \( V_{1,\delta}(u;b) \) first increases and then decreases in \( b \), and the optimal barrier \( b^* \) exists and is independent of the initial surplus \( u \). In particular, one has \( b^* = 10.858 \) in the Brownian motion and \( b^* = 13.270 \) in the compound Poisson model. To
Figure 3. Expected discounted dividends $V_{1,\delta}(u; b)$ as a function of $b$. (a) Brownian motion model. (b) Compound Poisson model with exponential claims.

Further demonstrate the effect of Erlangization (i.e. approximation of constant inter-dividend-decision times by increasing $m$ while keeping $m/\beta$ fixed), we consider $m = 1, 2, \ldots, 8$ and fix $m/\beta = 2.5$. Some exact values of $b^*$, $V_{1,\delta}(u; b^*)$ and $SD_\delta(u; b^*)$ are presented in Tables 1 and 2, where $SD_\delta(u; b) = \sqrt{V_{2,\delta}(u; b) - (V_{1,\delta}(u; b))^2}$ is the standard derivation of the discounted dividends until ruin. Numerically we always find that $b^*$ is independent of $u$ (but dependent on $m$), but it appears very difficult to prove such a property analytically due to the complexity of our formulas (see also Section 6 of [1]). Across each row of Tables 1 and 2, a converging trend is observed when $m$ increases as Erlangization is in effect. Note that no dividend is payable in the Brownian motion model when the initial surplus is zero because ruin occurs immediately by diffusion. Therefore, the values of $V_5(0; b^*)$ and $SD_5(0; b^*)$ are omitted in Table 1 as they are all identical to zero.

Table 1. Exact values of $b^*$, $V_{1,\delta}(u; b^*)$ and $SD_\delta(u; b^*)$ in the Brownian motion model.

| $m$  | 1     | 2     | 3     | 4     | 5     | 6     | 7     | 8     |
|------|-------|-------|-------|-------|-------|-------|-------|-------|
| $b^*$| 10.698| 10.816| 10.858| 10.879| 10.892| 10.900| 10.906| 10.911|
| $V_{1,\delta}(5; b^*)$ | 39.9857| 40.1571| 40.2133| 40.2412| 40.2579| 40.2690| 40.2769| 40.2828|
| $SD_\delta(5; b^*)$ | 15.9415| 16.0128| 16.0385| 16.0478| 16.0577| 16.0627| 16.0647| 16.0660|
| $V_{1,\delta}(10; b^*)$ | 47.0084| 47.2091| 47.2758| 47.3082| 47.3283| 47.3419| 47.3508| 47.3574|
| $SD_\delta(10; b^*)$ | 12.1893| 12.2434| 12.2672| 12.2727| 12.2828| 12.2921| 12.2919| 12.2908|
| $V_{1,\delta}(5; b^*)$ | 47.7133| 47.9345| 48.1452| 48.1979| 48.2329| 48.2543| 48.2688| 48.2802|
| $SD_\delta(5; b^*)$ | 12.1021| 12.2434| 12.2672| 12.2727| 12.2828| 12.2921| 12.2919| 12.2908|

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Table 2. Exact values of $b^*$, $V_{1,\delta}(u; b^*)$ and $SD_{\delta}(u; b^*)$ in the compound Poisson model (ii).

| m   | 1    | 2    | 3    | 4    | 5    | 6    | 7    | 8    |
|-----|------|------|------|------|------|------|------|------|
| $b^*$ | 13.036 | 13.209 | 13.270 | 13.300 | 13.319 | 13.332 | 13.339 | 13.349 |
| $V_{1,\delta}(0; b^*)$ | 13.1645 | 13.2114 | 13.2266 | 13.2341 | 13.2386 | 13.2416 | 13.2437 | 13.2454 |
| $SD_{\delta}(0; b^*)$ | 19.6093 | 19.6802 | 19.7034 | 19.7146 | 19.7215 | 19.7263 | 19.7293 | 19.7314 |
| $V_{1,\delta}(5; b^*)$ | 36.2764 | 36.4057 | 36.4477 | 36.4684 | 36.4808 | 36.4890 | 36.4948 | 36.4993 |
| $SD_{\delta}(5; b^*)$ | 17.2879 | 17.3529 | 17.3748 | 17.3846 | 17.3911 | 17.3961 | 17.3986 | 17.3996 |
| $V_{1,\delta}(10; b^*)$ | 43.8083 | 43.9645 | 44.0151 | 44.0402 | 44.0551 | 44.0650 | 44.0721 | 44.0775 |
| $SD_{\delta}(10; b^*)$ | 13.5952 | 13.6492 | 13.6681 | 13.6758 | 13.6813 | 13.6864 | 13.6879 | 13.6877 |
| $V_{1,\delta}(b^*; b^*)$ | 13.0272 | 13.0611 | 13.0776 | 13.0796 | 13.0908 | 13.0968 | 13.0962 | 13.0907 |
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