Existence and nonexistence of solutions for a singular $p$-Laplacian Dirichlet problem

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Abstract

We study the existence of positive radially symmetric solution for the singular $p$-Laplacian Dirichlet problem, $-\Delta_p u = \lambda |u|^{p-2}u - \gamma u^{-\alpha}$ where $\lambda > 0, \gamma > 0$ and, $0 < \alpha < 1$, are parameters and $\Omega$, the domain of the equation, is a ball in $\mathbb{R}^N$. By using some variational methods we show that, if $\lambda$ is contained in some interval, then the problem has a radially symmetric positive solution on the ball. Moreover, we obtain a nonexistence result, whenever $\lambda \leq 0, \gamma < 0$ and $\Omega$ is a bounded domain, with smooth boundary.

Key words: Nonlinear elliptic problem, radially symmetric solution, nonexistence result.

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1 Introduction

In this paper we study the singular $p$-Laplacian Dirichlet problem

$$
- 
\Delta_p u = \lambda |u|^{p-2}u - g(u) \quad \text{in } \Omega,
$$

$$
u = 0 \quad \text{on } \partial \Omega,
$$

(1.1)

where $\Omega$ is a ball with center 0 in $\mathbb{R}^N$, $N \geq 2$ and $g: (0, \infty) \to (0, \infty)$ is a function satisfying $g(\tau) \to \infty$ as $\tau \to 0$.

Indeed, we obtain existence and nonexistence results under some assumptions on $N, p, g, \lambda$ and $\Omega$. Chen in [1], in the case $p = 2$ and $g(\tau) = \frac{\tau^{1+a}}{1+a}$ for $\tau > 0$ and $\Omega = \{x \in \mathbb{R}^N : |x| < R\}$, by using the shooting method obtained the following results:

- There are real numbers $R_1 > R_2 > 0$ such that the problem (1.1) has a radially symmetric, positive solution if $R_1 \geq R > R_2$. Besides, if $u$ is a radially symmetric, positive solution for the problem in the case of $R = R_1$, then $\frac{\partial u}{\partial r} = 0$ on $\partial \Omega$, where $\frac{\partial}{\partial r}$ is the outward normal derivative.

In order to show the existence of solutions, we use the variational methods by considering the following functional:

$$
F(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p dx - \lambda \int_{\Omega} |u|^p dx + \int_{\Omega} \int_0^{u(x)} g(\tau) d\tau dx, \quad u \in W^{1,p}_0(\Omega),
$$

(1.2)

associated with the problem. Since this functional is not even Gâteaux differentiable, we cannot use the deformation argument. Neither can we use the strong maximum principle because of the property of the nonlinear term $g$. Here, we will show that if $u$ is a function which is a minimax value of $F$, then $u$ is a radially symmetric, positive solution of the problem.

For the nonexistence result we use the Pohozaev identity which is introduced in [2] and we show that if $\lambda \leq 0$, we may have no positive solution in $W^{2,p_1}(\Omega) \cap W^{1,p_1}_0(\Omega)$, $p_1 > N$. In this case we assume that $\Omega$ is a bounded domain and its boundary, $\partial \Omega$ has the following property:

There exists a unit normal vector $v(x) = (v_1(x), \ldots, v_N(x))$ at every point $x \in \partial \Omega$ such that

$$
\sum_{i=1}^N x_i v_i(x) \geq 0.
$$

2 Existence Result

In this section we prove the existence of radially symmetric positive solution for the problem (1.1) in the following theorem.
Theorem 2.1. Let $\Omega$ be a ball in $\mathbb{R}^N$ with center 0, $0 < \alpha < 1$, $p \geq 2$ and $N \geq 2$. Suppose $g$ is a $c^\infty(0, \infty)$ function with $g(\tau) > 0$ and
\[ \frac{d}{d\tau}(g(\tau)) < 0, \]  \hfill (2.3)
moreover
\[ m_1\tau^{-\alpha} \leq g(\tau) \leq m_2\tau^{-\alpha} \]  \hfill (2.4)
for some positive constants $m_1$ and $m_2$ with $\frac{pm_1}{1-\alpha} > m_2 \geq m_1$. If
\[ \lambda - \frac{\lambda p(1-\alpha)m_1}{pNm_2 - (N-p)(1-\alpha)m_1} \leq \lambda_1 < \lambda, \]  \hfill (2.5)
where $\lambda_1$ is the first eigenvalue of the operator $-\Delta_p$ with homogenous Dirichlet boundary condition, then problem (1.1) has a radically symmetric positive solution.

In order to the proof this Theorem we need some preliminary lemmas. The following sets will be used in our proofs.

\[ U = \{ u \in W^{1,p}_0(\Omega) : u \text{ is radially symmetric} \}, \]
and
\[ W = \{ u \in U : \int_{\Omega} |\nabla u|^p dx < \lambda \int |u|^p dx \}. \]

Note that for a function $u \in W$, we may regard it as a one variable function $u(r)$, where $r = |x|$ with $x \in \Omega$. Also note that $W$ is not empty, since $\lambda_1 < \lambda$ and the eigenfunctions of $-\Delta_p$ with homogenous Dirichlet boundary condition for $\lambda_1$ are radially symmetric.

In the following Lemmas we assume that all of the conditions of Theorem 2.1 hold. Moreover, we assume that $g$ is defined on $\mathbb{R}$ with $g(0) = 0$ and $g(t) = -g(-t)$ for $t < 0$.

Lemma 2.2. Let $u \in W$, then $\int_{\Omega} g(tu)\frac{u}{t^{p-1}} dx \to +\infty$ as $t \to 0^+$, $\int_{\Omega} \frac{g(tu)u}{t^{p-1}} dx \to 0$ as $t \to +\infty$ and the function $t \to \int_{\Omega} \frac{g(tu)u}{t} dx$ is strictly decreasing for $t > 0$. Especially, there exists a unique $t > 0$ such that
\[ \int_{\Omega} |\nabla tu|^p dx + \int_{\Omega} g(tu)tudx = \lambda \int |tu|^p dx, \]
which is equivalent to $F(tu) = \max_{s > 0} F(su)$. 

3
Proof. From (2.4), we have
\[ m_1 t^{-(p-1)-\alpha} \int_\Omega |u|^{1-\alpha} dx \leq \int_\Omega g(tu) \frac{u}{tp-1} dx \leq m_2 t^{-(p-1)-\alpha} \int_\Omega |u|^{1-\alpha} dx, \]
for every \( t > 0 \). Thus we obtain
\[ \int_\Omega g(tu) \frac{u}{tp-1} dx \to +\infty \quad \text{as} \quad t \to 0^+, \]
\[ \int_\Omega g(tu) \frac{u}{tp-1} dx \to 0 \quad \text{as} \quad t \to \infty. \]
From (2.3), we see that the function \( t \to \int_\Omega g(tu) \frac{u}{tp-1} dx \) is strictly decreasing for \( t > 0 \).

We define a subset \( V \) of \( W \) by
\[ V = \{ u \in W : \int_\Omega |\nabla u|^p dx + \int_\Omega g(u) u dx = \lambda \int_\Omega |u|^p dx \}. \]
The previous lemma says that for every \( u \in W \), there exists a unique \( t > 0 \) with \( tu \in V \). We will show that if \( u \in V \), \( u \geq 0 \) and \( F(u) = \min_{v \in V} F(v) = \min_{u \in W} \max_{s > 0} F(sv) \) then \( u \) is a solution for our problem.

Lemma 2.3. There exists \( u \in V \) such that \( F(u) = \min_{v \in V} F(v) \).

Proof. Let \( \{u_n\} \) be a sequence in \( V \) with \( F(u_n) \downarrow \inf_{v \in V} F(v) \). Notice that we may assume \( u_n \geq 0 \). We set \( t_n = (\int_\Omega |\nabla u_n|^p dx)^\frac{1}{p} \) and \( w_n = u_n/t_n \) for every \( n \in \mathbb{N} \). We may assume that \( \{w_n\} \) converges weakly in \( V \) to some \( w \in V \) and by Rellich theorem \( \{w_n\} \) converges strongly to \( w \) in \( L^p(\Omega) \). Moreover, by the Vitali convergence theorem \( \int_\Omega |w_n|^{1-\alpha} dx \to \int_\Omega |w|^{1-\alpha} dx \).

We may assume \( t_n \to t > 0 \), indeed, if \( t_n \to 0 \), then we have
\[ \lambda \int_\Omega |w_n|^p dx = 1 + \int_\Omega \frac{g(t_n w_n) w_n}{t_n^p} dx \geq 1 + \frac{m_1}{t_n^{\alpha+p-1}} \int_\Omega |w_n|^{1-\alpha} dx \to +\infty, \]
moreover if \( t_n \to \infty \), then we must have
\[ F(u_n) = \int_\Omega \left( \int_0^{u_n(x)} g(\tau) d\tau - \frac{1}{p} g(u_n) u_n \right) dx \geq \left( \frac{m_1}{1-\alpha} - \frac{m_2}{p} \right) t_n^{1-\alpha} \int_\Omega |w_n|^{1-\alpha} \to +\infty. \]

Thus a subsequence of \( \{t_n\} \) converges to a positive number \( t \), then we have,
\[ 1 + \int_\Omega \frac{g(tw) w}{tp-1} dx = \lambda \int_\Omega |w|^p dx. \]
Now, we will show \( \int_\Omega |\nabla w|^p dx = 1 \). Suppose not, then \( \int_\Omega |\nabla w|^p dx < 1 \). By Lemma 2.2, there is \( s \in (0,t) \) such that \( sw \in V \). From (2.3) it follows that

\[
\inf_{v \in V} F(v) = \lim_{n \to \infty} F(u_n) = \int_\Omega \left( \int_0^{tw(x)} g(\tau) d\tau - \frac{1}{p} g(tw) tw \right) dx
= \int_\Omega \int_0^{tw(x)} ((1 - \frac{1}{p}) g(\tau) - 1/p g'(\tau) \tau) d\tau dx
\]

\[
> \int_\Omega \int_0^{sw(x)} (1 - \frac{1}{p}) g(\tau) - 1/p g'(\tau) \tau d\tau dx = F(sw),
\]

which is a contradiction. Therefore \( \int_\Omega |\nabla w|^p dx = 1 \), and hence \( \{w_n\} \) converges strongly to \( w \) in \( V \). This means that \( tw \in V \) and \( F(tw) = \inf_{v \in V} F(v) \). □

Now, we fix \( u \in V \) with \( F(u) = \min_{v \in V} F(v) \). Since \( |u| \in V \) and \( g \) is an odd function then \( F(u) = F(|u|) \). Hence, we may assume \( u \geq 0 \).

In this step, we show that \( u > 0 \) in \( \Omega \), which ensures existence of the Gâteaux derivative of \( F \) at \( u \) in the direction of every \( v \in C_0^\infty(\Omega) \cap U \).

**Lemma 2.4.** If there is \( x_0 \in \Omega - \{0\} \) such that \( u(x_0) = 0 \) then \( u_1 \equiv 0 \), or \( u_2 \equiv 0 \), where

\[
u_1(x) = \begin{cases}
 u(x) & |x| \leq |x_0|, \\
 0 & |x| \geq |x_0| \end{cases}
\]

and

\[
u_2(x) = \begin{cases}
 0 & |x| \leq |x_0|, \\
 u(x) & |x| \geq |x_0|. \end{cases}
\]

**Proof.** Suppose that the conclusion does not hold, i.e., there is \( x_0 \in \Omega - \{0\} \) such that \( u(x_0) = 0 \), \( u_1 \not\equiv 0 \) and \( u_2 \not\equiv 0 \). From the definition of the set \( V \) we may assume \( \int_\Omega |\nabla u_1|^p dx + \int_\Omega g(u_1) u_1 dx \leq \lambda_1 \int |u_1|^p dx \). By Lemma 2.2, there is \( s \in (0,1) \) with \( su_1 \in V \). Then (2.3) and \( u \not\equiv u_1 \), implies that \( F(u) > F(su_1) \), which is a contradiction. □

**Lemma 2.5.** There is no \( x_0 \in \Omega - \{0\} \) such that \( u(x) = 0 \) for every \( x \in \Omega \) with \( |x| \geq |x_0| \).

**Proof.** Suppose that the conclusion does not hold. Notice that \( u \) is not an eigenfunction of \( -\Delta_p \) with homogeneous Dirichlet boundary condition for \( \lambda_1 \), thus \( \int_\Omega |\nabla u|^p > \lambda_1 \int |u|^p \). Let \( \epsilon \) be a positive real number and sufficiently small. For \( s \in [1, 1+\epsilon) \) we can define \( u_s \in W \) by \( u_s(x) = u(x/s) \) for \( x \in \Omega \). We set

\[
\varphi(t, s) = \frac{t^p}{p} (s^{N-p} \int_\Omega |\nabla u|^p dx - \lambda s^N \int_\Omega |u|^p dx) + s^N \int_0^{t \omega(x)} g(\tau) \tau d\tau,
\]

5
and
\[
\psi(t, s) = t^p (s^{N-p} \int_\Omega |\nabla u|^p dx - \lambda s^N \int_\Omega |u|^p dx) + s^N \int_\Omega g(tu)tdx,
\]
for every \( t, s \geq 0 \). Notice that for \( t > 0 \) and \( s \in [1, 1 + \epsilon) \) we will have \( \varphi(t, s) = F(tu_s) \) and
\[
\psi(t, s) = \int_\Omega |\nabla tu_s|^p dx - \lambda \int_\Omega |tu_s|^p dx + \int_\Omega g(tu_s)tu_s dx.
\]
From \( u \in V \) and (2.3), we obtain
\[
\frac{T \partial \psi}{\partial t}(1, 1) = p \int_\Omega |\nabla u|^p dx - \lambda \int_\Omega |u|^p dx + \int_\Omega (g'(u)u^2 + ug(u)) dx
\]
\[
= \int_\Omega g'(u)u^2 + (1 - p)g(u) udx < 0.
\]
Hence, the implicit function theorem implies that \( \psi(t, s) = 0 \) defines a continuously differentiable function, \( t = t(s) \) with \( \psi(t_s(s), s) = 0 \) near \( s = 1 \). On the other hand \( \varphi(1, 1) = \min \{ \varphi(t(s), s) 1 \leq s < 1 + \epsilon \} \), therefore
\[
0 \leq \frac{T \partial \varphi}{\partial t}(1, 1) \frac{T dt}{ds}(1) + \frac{T \partial \varphi}{\partial s}(1, 1) = \frac{T \partial \varphi}{\partial s}(1, 1)
\]
\[
= \frac{1}{p} (N - p) \int_\Omega |\nabla u|^p dx - \lambda \frac{N}{p} \int_\Omega |u|^p dx + N \int_\Omega \int_0^{u(x)} g(\tau) d\tau dx
\]
\[
\leq \left( \frac{N - p}{N - \frac{Nm_2}{m_1(1 - \alpha)}} \right) \int_\Omega |\nabla u|^p dx + \lambda N \left( \frac{m_2}{m_1(1 - \alpha)} - \frac{1}{p} \right) \int_\Omega |u|^p dx
\]
\[
< \left( \frac{N - p}{N - \frac{Nm_2}{m_1(1 - \alpha)}} + \frac{\lambda N}{\lambda_1} \left( \frac{m_2}{m_1(1 - \alpha)} - \frac{1}{p} \right) \right) \int_\Omega |\nabla u|^p dx.
\]
Thus, we must have
\[
\lambda_1 < \lambda - \frac{\lambda p(1 - \alpha)m_1}{pNm_2 - (N - p)(1 - \alpha)m_1},
\]
which contradicts (2.5). This completes the proof. \( \square \)

**Lemma 2.6.** There is no \( x_0 \in \Omega \) such that \( u(x) = 0 \) for every \( x \in \Omega \) with \( |x| \leq |x_0| \).

**Proof.** Let \( \Omega = \{ x \in \mathbb{R}^N : |x| < R_1 \} \). Suppose that the conclusion does not hold. If \( M \) is the maximum value of \( u \) and \( R \) is a point in \((0, R_1)\) with \( u(R) = M \). Then for \( s \in [0, \epsilon) \), where \( \epsilon \) is a sufficiently small positive real number, we can define \( u_s \in W \) by
\[
u_s(r) = \begin{cases} u(r + s) & 0 \leq r \leq R - s, \\ M & R - s \leq r \leq R, \\ u(r) & R \leq r \leq R_1. \end{cases}
\]
Now, we define

\[ \varphi(t, s) = \frac{t^p}{p} \left( \int_0^R |u'|^p (r-s)^N dr + \int_{R_1}^R |u'|^{p} r^{N-1} dr \right) - \lambda \int_{s}^{R} |u|^p (r-s)^{N-1} dr - \frac{\lambda M^p}{N} (R^N - (R-s)^N) + \int_{R_1}^R |u|^{p} r^{N-1} dr \]

and

\[ \psi(t, s) = t^p \left( \int_0^R |u'|^p (r-s)^N dr + \int_{R_1}^R |u'|^{p} r^{N-1} dr \right) - \lambda \int_{s}^{R} |u|^p (r-s)^{N-1} dr - \frac{\lambda M^p}{N} (R^N - (R-s)^N) - \int_{s}^{R} g(tu) tu(r-s)^{N-1} dr + \int_{R_1}^R g(tu) tu r^{N-1} dr, \]

Notice that \(|S| \varphi(t, s) = F(tu_s)\) and

\[ |S| \psi(t, s) = \int_\Omega |\nabla (tu_s)|^p dx - \lambda \int_\Omega |tu_s|^p dx + \int_\Omega g(tu_s) tu_s dx \]

for \(t > 0\) and \(s \in [0, \epsilon)\), where \(|S|\) is the measure of the surface of the unit sphere \(S\) in \(\mathbb{R}^N\). From \(u \in V, \int_\Omega |\nabla u|^p dx > \lambda_1 \int_\Omega |u|^p dx\) and (2.4), we get

\[ \lambda \int_\Omega |u|^p = \int_\Omega |\nabla u|^p dx + \int_\Omega g(u) u dx > \lambda_1 \int_\Omega |u|^p dx + \frac{m_1}{M^{p-1+\alpha}} \int_\Omega |u|^p dx, \]

which implies \(\lambda - \lambda_1 > \frac{m_1}{M^{p-1+\alpha}}\). From \(\frac{\partial \varphi}{\partial t}(1, 0) < 0\) and \(\varphi(1, 0) = \min \{\varphi(t(s)), s\} : 0 \leq s < \epsilon\}, \) we obtain

\[ 0 \leq \lim \frac{\partial \varphi}{\partial t}(t(s), s) \frac{dt(s)}{ds} + \frac{\partial \varphi}{\partial s}(t(s), s) = \lim \frac{\partial \varphi}{\partial s}(t(s), s) \]

\[ = \frac{1}{p}(-(N-1) \int_0^R |u'|^{p} r^{N-2} dr + (N-1)\lambda \int_0^R |u'|^{p} r^{N-2} dr - \lambda M^p R^{N-1}) \]

\[ - (N-1) \int_0^{\tau} \int_0^M g(\tau) d\tau r^{N-2} dr + R^{N-1} \int_0^M g(\tau) d\tau \]

\[ \leq \frac{1}{p}(-(N-1) \lambda \int_0^R |u'|^{p} r^{N-2} dr - \lambda M^p R^{N-1}) \]

\[ - (N-1) \int_0^{\tau} \int_0^M g(\tau) d\tau r^{N-2} dr + R^{N-1} \int_0^M g(\tau) d\tau. \]
Since \( \lambda - \lambda_1 > \frac{m_2}{m_2 - (1 - \alpha)m_1} \) and \( H(\tau) = \frac{1}{\tau} \int_0^\tau g(p)dp \) is decreasing for \( \tau > 0 \), we have

\[
-(N-1) \int_0^R \left| \frac{u}{M} \right|^{pN-2} + R^{N-1} \leq \frac{p}{\lambda M^{p}} \left( -(N-1) \int_0^R \int_0^{u(\tau)} g(\tau)d\tau r^{N-2}d\tau + R^{N-1} \int_0^M g(\tau)d\tau \right) < \frac{p(\lambda - \lambda_1)}{\lambda M^{1-\alpha}m_1} \int_0^M g(t)dt \left( -(N-1) \int_0^R \left| \frac{u}{M} \right|^{pN-2} + R^{N-1} \right) .
\]

Then, we obtain

\[
1 < \frac{p(\lambda - \lambda_1)}{\lambda M^{1-\alpha}m_1} \int_0^M g(t)dt \leq \frac{p(\lambda - \lambda_1)}{\lambda M(1-\alpha)m_1 \alpha M^{1-\alpha}} = \frac{p(\lambda - \lambda_1)m_2}{\lambda(1-\alpha)m_1},
\]

or

\[
\lambda > \frac{p\lambda_1m_2}{pm_2 - (1 - \alpha)m_1}.
\]

On the other hand

\[
\lambda - \frac{\lambda(1-\alpha)m_1}{pm_2} \leq \lambda - \frac{\lambda p(1-\alpha)m_1}{pN m_2 - (1 - \alpha)(N - p)m_1} \leq \lambda_1.
\]

Therefore

\[
\lambda \leq \frac{p\lambda_1m_2}{pm_2 - (1 - \alpha)m_1},
\]

which is a contradiction. This complete the proof. \( \square \)

**Corollary 2.7.** For all \( x \in \Omega \), \( u(x) \neq 0 \).

**Proof.** It is a direct consequence of Lemmas 2.4, 2.5 and 2.6. \( \square \)

Now, we are ready to prove Theorem 2.1.

**Proof of Theorem 2.1.** By Corollary 2.6 we have \( u > 0 \) on \( \Omega \). Now, we will show that \( u \) is a weak solution of (1.1). In order to do this, we fix \( v \in c_0^\infty(\Omega) \cap U \) and define,

\[
\varphi(t,s) = \frac{t^p}{p} \left( \int_\Omega |\nabla (u + sv)|^p dx - \lambda \int_\Omega |u + sv|^p dx + \int_\Omega \int_0^{t(u(x)+sv(x))} g(t)dt dx \right),
\]

and

\[
\psi(t,s) = \frac{t^p}{p} \left( \int_\Omega |\nabla (u + sv)|^p dx - \lambda \int_\Omega |u + sv|^p dx + \int_\Omega \int_0^{t(u(x)+sv(x))} g(t)dt dx \right),
\]

8
\[ \int_{\Omega} g(t(u + sv))t(u + sv)dx, \]

for \( t, s \in \mathbb{R} \). From \( u \in V \) and (2.3), we have \( \frac{\partial \psi}{\partial t}(1, 0) < 0 \). By implicit function theorem, \( \psi(t, s) = 0 \) defines a continuously differentiable function \( t = t(s) \) with \( \psi(t(s), s) = 0 \) near \( s = 0 \). Since for some \( \epsilon > 0 \), \( u \geq \epsilon \) on the support of \( v \), the function \( F \) is Gâteaux differentiable at \( u \) in the direction \( v \). This means that \( \frac{\partial \varphi}{\partial s}(1, 0) \) exists. Since \( \varphi(1, 0) = \min \{ \varphi(t(s), s) : s \) sufficiently close to 0\}, we have

\[ 0 = \frac{\partial \varphi}{\partial t}(1, 0) \frac{dt}{ds}(0) + \frac{\partial \varphi}{\partial s}(1, 0) = \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v dx - \lambda \int_{\Omega} |u|^{p-2}u.v dx + \int_{\Omega} g(u)v dx. \]

Hence \( u \) is a weak solution of problem (1.1). \( \square \)

### 3 Nonexistence result

Let \( \Omega \subset \mathbb{R}^N \) be a bounded domain with the boundary \( \partial \Omega \) which has the following property:

There exists a unit normal vector \( v(x) = (v_1(x), \ldots v_N(x)) \) at every point \( x \in \partial \Omega \) and

\[ \sum_{i=1}^{N} x_i v_i(x) \geq 0 \quad (3.6) \]

for every \( x = (x_1, \ldots, x_N) \in \partial \Omega \). Let us consider the boundary value problem

\[ - \Delta_p u = \lambda |u|^{p-2}u + g(u) \quad \text{in} \ \Omega, \]
\[ u = 0 \quad \text{on} \ \partial \Omega \quad (3.7) \]

Here, we will show that this problem does not have a positive solution in \( W^{2,p_1}_0(\Omega), (p_1 > N) \).

In order to see this claim, let \( u \in W^{2,p_1}_0(\Omega), (p_1 > N) \) be a positive solution of this problem. By the Pohozaev identity introduced in [?], we must have

\[ \frac{N - p}{p} \int_{\Omega} |\nabla u|^p dx - \frac{\lambda N}{p} \int_{\Omega} |u|^p dx - N \int_{\Omega} \int_0^{u(x)} g(t) dt = \]
\[ -(1 - \frac{1}{p}) \int_{\partial \Omega} |\nabla u|^p \sum_{i=1}^{N} x_i v_i dx. \]

9
On the other hand
\[
\int_{\Omega} |\nabla u|^p \, dx - \lambda \int_{\Omega} |u|^p \, dx + \int_{\Omega} g(u) u \, dx = 0.
\]
From the above two identities, we see that the following identity holds for every \( \beta \in \mathbb{R} \),
\[
\left( \frac{N-p}{p} + \beta \right) \int_{\Omega} |\nabla u|^p \, dx - \lambda \left( \frac{N}{p} + \beta \right) \int_{\Omega} |u|^p \, dx - N \int_{\Omega} \int_0^{u(x)} g(t) \, dt + \beta \int_{\Omega} g(u) u \, dx = -(1 - \frac{1}{p}) \int_{\partial \Omega} |\nabla u|^p \sum_{i=1}^N x_i v_i \, ds.
\]
Hence
\[
\left( \frac{N-p}{p} + \beta \right) \int_{\Omega} |\nabla u|^p \, dx - \lambda \left( \frac{N}{p} + \beta \right) \int_{\Omega} |u|^p \, dx - \frac{N m_2}{1 - \alpha} + \beta m_1 \int_{\Omega} u^{1-\alpha} \, dx \leq -(1 - \frac{1}{p}) \int_{\partial \Omega} |\nabla u|^p \sum_{i=1}^N x_i v_i(u) \, ds.
\] (3.8)

Now, it follows from (3.8) that the following inequalities
\[
\frac{N-p}{p} + \beta \geq 0, \quad (3.9)
\]
\[
-\lambda \left( \frac{N}{p} + \beta \right) \geq 0, \quad (3.10)
\]
\[
-(\frac{N m_2}{1 - \alpha} + \beta m_1) \geq 0, \quad (3.11)
\]
cannot hold simultaneously with at least one strict inequality sign. Thus, we have the following nonexistence result.

**Theorem 3.1.** Let \( N \geq 2, p > 1, 0 < \alpha < 1, m_1 > 0, m_2 > 0 \) be real numbers such that (3.9), (3.10) and (3.11) hold with at least one strict inequality sign. Then the boundary-value problem (3.7), has no positive solution in \( W^{2,p_1}(\Omega) \cap W_0^{1,p_1}(\Omega) \) for \( p_1 > N \).

**References**

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