The geodesic rule and the spectrum of the vacuum.

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Abstract
We analyze the consequences of a recent argument justifying the validity of the “geodesic rule” which can be used to determine the density of global topological defects. We derive a formula that provides a rough estimate of the number of string-like defects formed in a phase transition. We apply this formula to vacua which are spheres. We provide some reasons for the deviation of our predictions from the corresponding accepted values.

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The geodesic rule [1] is a way that allows us to develop predicting schemes for the density of global topological defects arising in phase transitions [2],[3]. Two approaches exist for justifying the geodesic rule. The first, time-honored and widely accepted, relies on the minimization of the energy of the system [4],[5]. We recently proposed [6] a second approach which relies on the Markovian character of the dynamics and on the random orientation of the collisions among the expanding causally connected volumes of true vacuum. Such collisions take place inside the meta-stable vacuum background. In the present paper we follow the second approach, and propose a formula that provides a rough estimate for the number of global topological strings formed in a phase transition.

Topological strings are of great importance methodologically [2],[3] due to their non-perturbative nature. What is even more important, they have severe experimental and observational implications [2],[3],[7] in models allowing their formation. Because of their great significance, a scheme leading to calculations of their density and distribution such as the geodesic rule has to be theoretically justified and its predictions have to be thoroughly tested. Sometime after [1], the importance of the role of the temperature quench and the effects of the related critical slowing down of the underlying phase transition were pointed out in [8]-[11]. The predictions relying on this modification of the geodesic rule, the Kibble-Zurek scenario, are generally considered to be in agreement with experimental data [12]-[18] and numerical simulations [19]-[23]. Any discrepancies between such experimental/numerical data and the theoretical predictions are attributed to the details of the dynamics of the particular model under study, rather than being considered a shortcoming of the general approach. Despite all these successes, there is still some work that needs to be done, especially on the analytical side, in finding formulae expressing the density of topological defects. The current study, aims in such a direction.

Let $\mathcal{M}$ denote the vacuum of the model under study and $\phi \in \mathcal{M}$ be a parametrization of it. We arbitrarily pick $\phi = 0$ on $\mathcal{M}$ as the origin of a normal coordinate system on which the calculations rely. As was explained in [6], the stochasticity of the orientation of the causally connected volumes during their collisions and the Markovian character of their subsequent coalescence can be described by a probability distribution function
$P(\phi, t)$ which obeys the Fokker-Planck equation

$$\frac{\partial P(\phi, t)}{\partial t} = D\nabla^2 P(\phi, t)$$

(1)

Here $D$ stands for the diffusion constant of this stochastic process on $\mathcal{M}$ [6]. The solution of (1), with the initial condition

$$P(\phi, 0) = \delta(\phi)$$

(2)

is the heat kernel $K(0, \phi, t) : \mathcal{M} \times \mathcal{M} \times \mathbb{R}_+ \to \mathbb{R}$. We recall that $K(0, \phi, t)$ has the eigenfunction expansion

$$K(0, \phi, t) = \sum_{j=0}^{\infty} e^{-\lambda_j t} f_j^*(0) f_j(\phi)$$

(3)

where $*$ denotes complex conjugation, $\lambda_j, j = 0, 1, \ldots$ are the eigenvalues of $\nabla^2$ on $\mathcal{M}$. Here $f_j(\phi) : \mathcal{M} \to \mathbb{R}$ denote the normalized eigenfunctions of $\nabla^2$ on $\mathcal{M}$ with respect to the hermitian product

$$(f_j, f_l) = \int_{\mathcal{M}} f_j^*(\phi) f_l(\phi) d\phi = \delta_{jl}, \quad j, l = 0, 1, \ldots$$

(4)

induced by the Riemannian metric $g$ of $\mathcal{M}$ on the space of the square integrable functions $L^2(\mathcal{M})$. We assume, for simplicity, that the vacuum $\mathcal{M}$ is a manifold without boundary. Since $\nabla^2$ is an elliptic, positive semi-definite operator on $\mathcal{M}$, then $\lambda_j \geq 0, \ j = 0, 1, \ldots$. Moreover, since $\nabla^2$ is self-adjoint with respect to the inner product $(\cdot, \cdot)$, then $f_j(\phi) \in \mathbb{R}$, $j = 0, 1, \ldots$.

We also assume, for concreteness, that the phase transition giving rise to the topological strings is of first order and proceeds by bubble nucleation [6]. Given this assumption, consider three such bubbles with corresponding values of the order parameter $\phi_1 = 0$, $\phi_2$, $\phi_3$. When these bubbles collide the geodesic rule states that $\phi$ traverses a piecewise smooth loop $C$ on $\mathcal{M}$. Without any loss of generality assume $C$ has base point $\phi_1 = 0$. The points on $\mathcal{M}$ at which $C$ is non-smooth, with probability almost 1, are exactly $0, \phi_2, \phi_3$. If $C$ is non-contractible then the collision of the three bubbles will give rise to a topological defect. The fact that to determine the formation or not of a topological string one must follow a piecewise smooth geodesic loop $C$, shows that
the quantity most closely related to such a string formation is not the generic heat kernel $K(0, \phi, t)$ but its diagonal, namely $K(0, 0, t)$. This can be expressed as

$$K(0, 0, t) = \sum_{j=0}^{\infty} e^{-\lambda_j t} f_j(0) f_j(0)$$

(5)

Since all points $\phi \in \mathcal{M}$ are equally likely to be the base-point of $C$, we have to use in what follows instead of $K(0, 0, t)$ its average over $\mathcal{M}$. Such an average is called the partition function $Z_{(\mathcal{M}, g)}$, and is defined as [24]

$$Z_{(\mathcal{M}, g)}(t) := \frac{1}{Vol_g(\mathcal{M})} \sum_{j=0}^{\infty} e^{-\lambda_j t} \int_{\mathcal{M}} f_j(\phi) f_j(\phi) d\phi$$

(6)

In (6), $Vol_g(\mathcal{M})$ is the volume of $\mathcal{M}$ with respect to the Riemannian measure induced by $g$. Since the eigenfunctions $f_j(\phi), j = 0, 1, \ldots$ are orthonormal with respect to $(\cdot, \cdot)$, (6) simplifies to

$$Z_{(\mathcal{M}, g)}(t) = \frac{1}{Vol_g(\mathcal{M})} \sum_{j=0}^{\infty} e^{-\lambda_j t}$$

(7)

An explicit calculation of $Z_{(\mathcal{M}, g)}(t)$ amounts to determination of the spectrum $\text{Spec}(\mathcal{M}) := \{\lambda_j, j = 0, 1, \ldots\}$ of the Laplace-Beltrami operator $\nabla^2$ on functions of $\mathcal{M}$. Conversely, one can prove [24] that knowing $Z_{(\mathcal{M}, g)}(t)$ amounts to determining $\text{Spec}(\mathcal{M})$. Unfortunately, it is not known how to explicitly calculate $\text{Spec}(\mathcal{M})$ or equivalently $Z_{(\mathcal{M}, g)}(t)$ in general, except for very few cases of manifolds with very high symmetry as well as some of their quotients. It is known, for example [24], that the $n$-sphere $\mathbb{S}^n$ endowed with its round metric $g$, has

$$\text{Spec}(\mathbb{S}^n) = \{\lambda_k = k(n + k - 1), \ k \in \mathbb{N}\}$$

(8)

with corresponding multiplicities

$$m_k = \frac{(n + k - 2)!}{(n-1)! \ k!} (n + 2k - 1), \ k \geq 1 \ \text{and} \ m_0 = 1$$

(9)

Since [24]

$$Vol_g(\mathbb{S}^n) = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})}$$

(10)

we find

$$Z_{(\mathbb{S}^n, g)}(t) = \frac{\Gamma(\frac{n+1}{2} \ 2\pi^{\frac{n+1}{2}}} \left\{ 1 + \sum_{k=1}^{\infty} \frac{(n + k - 2)!}{(n-1)! \ k!} (n + 2k - 1) e^{-k(n+k-1)t} \right\}$$

(11)
Other examples of spaces that are occasionally found as vacua of classical field theories, for which the spectrum is explicitly known [24] are the real and complex projective spaces \( \mathbb{R}P^n \) and \( \mathbb{C}P^n \) respectively, the tori \( \mathbb{T}^n \), the lens spaces \( L(p, q) \), and somewhat more rarely, the Heisenberg manifolds.

Major problems exist in the calculation of the spectrum if \( (\mathcal{M}, g) \) happens to have non-positive sectional curvature everywhere. Then, according to the Hadamard-Cartan theorem [25], the universal cover of \( \mathcal{M} \) is diffeomorphic to \( \mathbb{R}^{\dim \mathcal{M}} \) from which we cannot draw enough useful information that can help determine \( \text{Spec}(\mathcal{M}) \). In most cases of physical interest though, we do not have to worry about manifolds of non-positive sectional curvature. Indeed, most frequently, the classical/non-thermal vacua \( \mathcal{M} \) happen to be compact, semi-simple Lie groups \( G \), or their homogeneous spaces endowed with the corresponding Killing-Cartan metrics or their images under Riemannian submersions [25]. These spaces have non-negative sectional curvature everywhere [26], so there is no need to worry about the implications of the Hadamard-Cartan theorem in future considerations.

In most other cases we have to settle with considerably less; within the scope of the heat-kernel methods employed in this work, the short-time behavior of the trace of the heat kernel is probably the most useful approximation. As the word states, such an asymptotic expansion is employed when \( t \to 0^+ \) and states that

\[ K(\phi, \phi, t) \sim (4\pi t)^{-\frac{\dim \mathcal{M}}{2}} \sum_{i=0}^{\infty} u_i(\phi, \phi) t^i \quad (12) \]

where \( u_i(\phi, \phi) \) are polynomials involving the Riemann tensor, its covariant derivatives and their contractions at \( \phi \in \mathcal{M} \). Substituting (12) into (7), we find the corresponding asymptotic expansion of

\[ Z_{(\mathcal{M}, g)}(t) \sim \sum_{i=0}^{\infty} a_i t^{i-\frac{\dim \mathcal{M}}{2}} \quad (13) \]

where

\[ a_{2i} = \frac{1}{(4\pi)^{\frac{\dim \mathcal{M}}{2}}} \int_{\mathcal{M}} u_i(\phi, \phi) \sqrt{\det g} \, d\phi \quad i \in \mathbb{N} \quad (14) \]

and

\[ a_{2i+1} = 0, \quad i \in \mathbb{N} \quad (15) \]
In particular,

\[ a_0 = \frac{1}{(4\pi)^{\frac{\dim M}{2}}} \text{Vol}_g(M) \]  

\[ a_2 = \frac{1}{6(4\pi)^{\frac{\dim M}{2}}} \int_M R(\phi) \sqrt{|g|} \, d\phi \]  

\[ a_4 = \frac{1}{360(4\pi)^{\frac{\dim M}{2}}} \int_M \left\{ 2||Riem(\phi)||^2 - 2||Ric(\phi)||^2 + 5R^2(\phi) \right\} \sqrt{|g|} \, d\phi \]

where \( || \cdot || \) denotes the point-wise norm of the Riemann and the Ricci tensors respectively and \( R \) denotes the Ricci scalar. The coefficient \( a_6 \) is also known [24], but its expression is too long to state and not very enlightening. The zeroth order term has a clear geometric interpretation as a multiple of \( \text{Vol}_g(M) \). The second-order term is proportional to the Euler characteristic \( \chi(M) \) when \( M \) is a Riemann surface. Unfortunately, in all other cases the non-trivial higher order coefficients \( a_i \) do not have a straightforward geometric or topological interpretation. So despite of their usefulness in this asymptotic expansion, they are not of any great help in improving our intuition about the behavior of the diagonal of the heat kernel. Subsequently their potential physical significance is also relatively limited, so we will not expand further upon them.

During consecutive bubble collisions, piece-wise smooth geodesics of all possible lengths on \( M \) describe the evolution of \( \phi \). A more appropriate expression related to the density of topological defects is then

\[ Z_{(M,g)} = \int_0^{+\infty} Z_{(M,g)}(t) \, dt \]  

Substituting (7) into (20), we find

\[ Z_{(M,g)} = \frac{1}{\text{Vol}_g(M)} \left\{ \int_0^{+\infty} e^{-\lambda_0 t} dt + \sum_{i=1}^{\infty} \frac{1}{\lambda_i} \right\} \]

In (21), the first term gives an infinite contribution since for a closed manifold like \( M \) the lowest eigenvalue of the Laplace-Beltrami operator is zero \( \lambda_0 = 0 \). An infinite
subtraction of $Z(M, g)$ is required to make sense out of (21). The second term of (21) can be recognized as the value at $t = 1$ of the zeta function $\zeta_\nabla(t)$ associated with $\nabla^2$, so we can write

$$Z(M, g) = \frac{1}{Vol_g(M)} \left\{ \zeta_\nabla(1) + \int_0^{+\infty} e^{-\lambda_0 t} dt \right\}$$  \hspace{1cm} (22)

The function $\zeta_\nabla(t)$ is meromorphic over $\mathbb{C}$ with isolated simple poles at $\frac{\dim M - j}{2} \in \mathbb{R}$ with $j = 0, 1, 2, \ldots$ and corresponding residues

$$\text{Res}_{j = j_0} = \frac{a_{j_0}}{\Gamma\left(\frac{\dim M - j_0}{2}\right)}$$  \hspace{1cm} (23)

where $\Gamma(z)$, $z \in \mathbb{C}\setminus\{0, -1, -2, \ldots\}$ denotes the Euler gamma function. If $\dim M$ is large enough so that $j_0 = \dim M - 2 \in \mathbb{N}$ namely if $\dim M \geq 3$ then $t = 1$ is a simple pole of $\zeta_\nabla(t)$ with residue $a_{\dim M - 2}$. In case $\dim M$ is an odd number, then the residue at $t = 1$ is zero in accordance with the general result stated above. If a vacuum for which $\dim M \geq 4$ happens to also satisfy $a_{\dim M - 2} = -1$, and if in addition we interpret the improper integral in (22) as the weak limit $(\lambda_0 = 0)$

$$\int_0^{+\infty} e^{-\lambda_0 t} dt = \lim_{\epsilon \to 0} \int_0^{+\infty} e^{-\epsilon t} dt$$  \hspace{1cm} (24)

then the two infinities of (22) cancel exactly each other and the final result for $Z(M, g)$ is finite. In a generic case, however, an infinite subtraction from $Z(M, g)$ is required. For the case of spheres $S^n$, (11)

$$Z(S^n, g) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{2\pi^{\frac{n+1}{2}}} \left\{ \sum_{k=1}^{\infty} \frac{(n + k - 2)!}{(n - 1)! k!} \cdot \frac{(n + 2k - 1)}{k(n + k - 1)} + \int_0^{+\infty} e^{-\lambda_0 t} dt \right\}$$  \hspace{1cm} (25)

We can check that the integrated partition functions of (25) are, with few exceptions, divergent even if we disregard the term corresponding to the zero eigenvalue $\lambda_0 = 0$. To proceed further, an infinite subtraction is required. It is not really difficult to understand the origin of this infinity. The calculation of $Z(M, g)$ takes into account the contributions of geodesic loops of any length. Each geometrically distinct geodesic loop $C$ is counted infinite times, since even upon traversing it as many times as we want, it still remains a geodesic loop (although non-minimal). The measure with which each $C$ is weighted in $Z(M, g)$ is not sufficient to outweigh the contributions of all its multiples, so the total
contribution of $C$ to $Z_{(M,g)}$ is infinite. The way around such a divergence should now be clear: For a general compact $M$, its diameter $\text{diam}(M)$ which is defined as

$$\text{diam}(M) = \sup\{d(p,q), \ p, q \in M\}$$

is finite, with $d(p,q)$ indicating the distance of $p, q \in M$ calculated with respect to $g$. As it was pointed out above, the classical/non-thermal vacua of interest have everywhere non-negative sectional curvature. If, moreover, this curvature is everywhere bounded from below, away from zero, by $\delta > 0$ then according to Myers’ theorem [25] $\text{diam}M \leq \frac{\pi}{\sqrt{\delta}}$. The curvature condition of Myers’ theorem is satisfied by spheres, by projective spaces and by the Aloff-Wallach manifolds. Even if $M$ does not fulfill the requirements of Myers’ theorem though, there is a weaker estimate for $\text{diam}M$. If the Ricci, instead of the sectional, curvature is uniformly bounded from below, away from zero, by $\rho > 0$ then Cheng’s theorem [25] states that

$$\text{diam}M \leq \pi \sqrt{\frac{\dim M - 1}{\rho}}$$

Myers’ and Cheng’s theorems are useful because they provide upper bounds to a global property of the vacuum, namely its diameter, which is used in an essential way in the rest of the argument. It is also important, from a practical standpoint, that the conditions of these theorems can be checked through local computations which can be easily performed if one knows the metric of the vacuum $g$. The length of the longest possible piecewise-smooth minimal geodesic loop $C$ that describes the evolution of the order parameter/Goldstone field $\phi$ on $M$ during the consecutive bubble collisions should depend on $\text{diam}M$. Let such a dependence be denoted by $\alpha(\text{diam}M)$. It is very difficult to find an exact expression for $\alpha$ for a generic $M$. For the apparently simpler case of $C$ being a closed geodesic, it is not even clear that such an upper bound exists. However piecewise smooth geodesic loops are less rigid objects than closed geodesics, exactly because of the almost certain lack of the differentiability of the former at the points $\phi_i \in M, \ i = 0, 1, 2, \ldots$ which are the values that the order parameter attains in the colliding bubbles. With such an upper bound on $\text{diam}(M)$, only the piecewise smooth geodesic loops $C$ with maximum length $\alpha(\text{diam}M)$ will contribute to the integrated partition function $Z_{(M,g)}$. In other words, $\alpha(\text{diam}M)$ provides a natural infrared cutoff for the allowed lengths of geodesic loops $C$ contributing to $Z_{(M,g)}$. On such, “physical
grounds”, (22) reduces to the estimate

$$Z_{(\mathcal{M},g)} \sim \int_0^{\alpha(diam \mathcal{M})} Z_{(\mathcal{M},g)}(t) \, dt \quad (28)$$

Then issues of subtractions of infinity from $Z_{(\mathcal{M},g)}$ that arose from the existence of long loops on $\mathcal{M}$ no longer persist. Substituting (7) into (28), we get

$$Z_{(\mathcal{M},g)} \sim \frac{1}{Vol_g(M)} \sum_{i=0}^{\infty} \frac{1 - e^{-\lambda_i \alpha(diam \mathcal{M})}}{\lambda_i} \quad (29)$$

In this equation we immediately see that the contribution of the zero eigenvalue $\lambda_0 = 0$ to the sum is $\alpha(diam \mathcal{M})$, which is finite, a fact which justifies some of the comments preceding (28). Whether such a series converges, or is just a formal expression whose infinities should be dealt with further, depends on $\text{Spec} \mathcal{M}$. To illustrate this point, consider $\mathcal{M}_1 = S^1$. Recalling that $\Gamma(1) = 1$ and $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$, we find

$$Z_{(S^1,g)} = \frac{\pi}{12} \quad (30)$$

By contrast, let $\mathcal{M}_2 = S^2$. Then the general term of the series is

$$\frac{2k + 1}{k(k + 1)} \quad (31)$$

from which we can show, using the comparison test, that the series diverges. For both of the above examples we disregarded the (infinite) contribution of the zero eigenvalue, as is usually done in such calculations.

To determine the density of topological strings, a set of great importance is the set of the non-contractible piecewise smooth geodesic loops $\mathcal{L}'\mathcal{M}$, which is a subset of the loop space $\mathcal{LM}$. To get an estimate for the number of string-like defects, we have to calculate the contributions of all such non-contractible piecewise-smooth geodesic loops and compare these to $Z_{(\mathcal{M},g)}$, namely we want to determine

$$d_{str} = \frac{1}{Z_{(\mathcal{M},g)}} \int_{\mathcal{L}'\mathcal{M}} Z_{(\mathcal{M},g)}(t) \, dt \quad (32)$$

The expression (32) provides an estimate rather than an exact prediction for the number density of topological strings produced in a phase transition. So, we do really expect
deviations between the result of (32) and the experimentally measured or numerically computed value of the density of topological strings. To highlight this point we notice that for \( \mathcal{M} = \mathbb{S}^n \), \( n > 1 \) the density is \( d_{\text{str}} = 0 \) as expected due to that \( \pi_1(\mathbb{S}^n) = 0 \), \( n > 1 \). For \( \mathcal{M} = \mathbb{S}^1 \) we have

\[
d_{\text{str}} = \frac{12}{\pi^2} \sum_{k=1}^{\infty} \frac{e^{-k^2 \pi}}{k^2} = 0.062
\]  

This prediction is four times smaller than \( d_{\text{str}} = 0.25 \) predicted for kinks [27],[28] (in 2 spatial dimensions), and almost thirteen times smaller than \( d_{\text{str}} = 0.88 \) predicted for strings in 3 spatial dimensions [19]. Such discrepancies are due to the following: the domain of integration of (32) should be the set of all piecewise smooth loops rather than the set of geodesic loops of \( \mathcal{M} \). In reaching (33) we integrated only over the closed geodesic loops. We missed therefore a large contribution to \( d_{\text{str}} \) coming from the piecewise-smooth geodesic loops. In general, the number of piecewise smooth loops over which we have to integrate in (32) is far higher than the number of geodesic loops, for any vacuum \( \mathcal{M} \). Also, such number of piecewise smooth geodesic loops should increases at a far higher rate than the number of geodesic loops, as the length of \( C \) increases. Therefore \( d_{\text{str}} \) as was calculated in (32) provides a rough estimate about the density of topological strings rather than an actual prediction which can be directly compared with experimental data. Such numbers will be reasonably close for simple vacua \( \mathcal{M} \) with relatively small diameters and simple topology. As the metric and topological properties of \( \mathcal{M} \) get more complicated however, we would expect increasingly larger deviations between the predictions of (32) and the experimental data.

It is probably worth mentioning, at this point, that the dimension of space-time on which the model with vacuum \( \mathcal{M} \) is defined is incorporated in the determination of \( \mathcal{M} \) itself. To find \( \mathcal{M} \) one has to carry out integrations whose value, or even finiteness, depends crucially on the space-time dimension. To carry out such a calculation at non-zero temperature one has to compute the effective potential [29] which takes into account the contributions of the thermal fluctuations to the value of the classical potential. The singularity structure of the effective potential of a model depends strongly on the space-time dimension [29], a dependence which is then inherited to \( \mathcal{M} \). Therefore the dimension of the space-time on which the model is defined is used in our considerations only very indirectly. Certainly none of our arguments rely on it. We naively expect that an increase of
the spatial dimension on which the model is defined will result in large density of defects. We can see this, heuristically, by noticing that in higher dimensions there are more bubbles available that can collide with any given bubble, so such an increase should be reflected by an increase in the density of the formed defects. Whether this argument can be formally justified or to what extent it is correct could be the subject of another investigation.

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