Production of genuine entangled states of four atomic qubits

Gui-Yun Liu and Le-Man Kuang

Key Laboratory of Low-Dimensional Quantum Structures and Quantum Control of Ministry of Education, and Department of Physics, Hunan Normal University, Changsha 410081, People’s Republic of China

E-mail: lmkuang@hunnu.edu.cn

Received 20 February 2009, in final form 29 April 2009
Published 27 July 2009
Online at stacks.iop.org/JPhysB/42/165505

Abstract
We propose an optical scheme to generate genuine entangled states of four atomic qubits in optical cavities using a single-photon source, beam splitters and single-photon detectors. We demonstrate how to generate deterministically 16 orthonormal and independent genuine entangled states of four atomic qubits. It is found that the 16 genuine entangled states form a new type of representation of the four-atomic-qubit system, i.e. the genuine entangled-state representation. This representation brings a new interesting insight into better understanding of multipartite entanglement.

1. Introduction

Quantum entanglement plays an important role in quantum information processing and quantum mechanics. While bipartite entanglement is well understood, multipartite entanglement is still under intensive research. Multipartite entanglement is thus not a straightforward extension of bipartite entanglement and gives rise to new phenomena which can be exploited in quantum information and quantum computing processes. For example, there are quantum communication protocols that require multiparty entanglement such as universal error correction [1], quantum secret sharing [2] and telecloning [3]. Also, highly entangled multiparticle states are needed for one-way quantum computing [4]. In fact, all known quantum algorithms do work with multipartite entanglement. On the other hand, multipartite entangled states provide a stronger test of local realism. As a general rule, one can say that the more particles are entangled, the more clearly nonclassical effects are exhibited, and the more useful the states are for quantum applications. In addition, multipartite entanglement is expected to play a key role on quantum-phase transition phenomena [5].

Although multipartite entanglement is ubiquitous in many-body quantum systems, it is very difficult both to characterize and to quantify it. Up to now, there is no unique way to define multipartite entanglement, even in the simplest case of pure states. The presence of multipartite entanglement clearly depends on the partitioning that one imposes in order to group the individual subsystems into parties. Furthermore, given a fixed partition, one can single out a hierarchy of different levels of multipartite entanglement which establishes a smooth connection between the two limiting cases of a fully separable state where the parties are all disentangled, and of fully inseparable states where entanglement exists across any global bisection, and the parties are supposed to share genuine multipartite entanglement. Genuine multipartite entanglement is distinguished from other types of entanglement by the participation of all parties in quantum correlations, and it is particularly distinct from biseparable entanglement. Yeo and Chua [6] indicated that an arbitrary two-qubit state can be faithfully teleported by the use of a genuine four-qubit entangled state. Wang and Yang [7] presented a simple scheme for creating such genuine four-qubit entangled states in an ion-trap system. It was found that a genuine 2N-qubit entangled state can be used to teleport an arbitrary N-qubit state [8], and a genuine (2N + 1)-qubit entangled state can realize controlled teleportation of an arbitrary N-qubit state [9]. One of the important issues regarding many-body quantum systems is to generate and to verify genuine multipartite entanglement among parties. The purpose of this paper is to propose an optical scheme to produce genuine entangled states (GESs) of four atomic qubits in optical cavities. This paper is organized as follows. In section 2, we propose our theoretical model. In section 3, we show how to create 16 orthonormal and
independent genuine entangled states of four atomic qubits. We shall conclude our paper with discussions and remarks in the last section.

2. Theoretical model

The basic setup for genuine entanglement generation of four atomic qubits is indicated in figure 1 where we make use of a Mach–Zehnder (MZ) interferometer consisting of two 50/50 optical beam splitters (BSs). The input light field is bifurcated at the first BS, guided to interact sequentially with the atomic qubits which are placed in four separate high-finesse optical cavities in which four atomic qubits are placed, and then recombined at the second BS. We consider the atomic qubits based on degenerate hyperfine states denoted by |0\rangle_i and |1\rangle_i (i = 1, 2, 3, 4). A single pulse of light is passed through an optical interferometer. Lights in different arms of the interferometer have different polarization states. Photons in different polarization states interact with different internal atomic states through the mechanism proposed in [11]. The interaction between the polarized photons and atomic qubits can be achieved by choosing different degenerate ground states of atomic qubits. For instance, we may choose the atomic qubit to be the Zeeman sublevels of an F = 1/2 ground state, or m = ±1 states of an F = 1 ground state in a Λ-type level configuration [12]. For both cases, the arms of the interferometer would correspond to orthogonal circular polarization states. Under the condition of the large detuning, the atom–photon evolution is governed by the effective polarization-dependent unitary transformation [13]

\[ \hat{U}_i = \exp\left[-i\phi(\hat{a}_{UL}^\dagger \hat{a}_{UL}|0\rangle_i(0) + \hat{a}_{UL}^\dagger \hat{a}_{UL}|1\rangle_i(1))\right], \]  

where \( \hat{a}_{UL}^\dagger \) and \( \hat{a}_{UL} \) are the creation and annihilation operators of the polarization light fields corresponding to the upper arm and the lower arm, respectively. |0\rangle_i and |1\rangle_i are the atomic degenerate hyperfine states corresponding to the four atomic qubits (i = 1, 2, 3, 4). The interaction is governed by the phase shift \( \phi = \frac{d^2 E \tau}{\hbar \Delta} \), where \( d \) is the electrical dipole moment and \( E = \sqrt{\hbar \omega/2\Delta V} \) is the electrical field of a photon for laser with the frequency \( \omega \) and the mode volume \( V \), \( \tau \) is the atom–photon interaction time, \( \Delta \) is the detuning between the atomic resonance and laser frequencies. The unitary evolution of the whole system indicated in figure 1 can be described by the following unitary transformation:

\[ \hat{U} = \hat{U}_{BS2} \hat{U}_4 \hat{U}_3 \hat{U}_2 \hat{U}_1 \hat{U}_{BS1}. \]  

where \( \hat{U}_{BSi} \) (i = 1, 2) is the 50/50 polarization BS operator given by

\[ \hat{U}_{BSi} = \exp\left[-i\frac{\pi}{4}(\hat{a}_{UL}^\dagger \hat{a}_{UL} + \hat{a}_{UL} \hat{a}_{UL}^\dagger)\right]. \]  

In the following we will show the genuine entangled states of four atomic qubits by using above unitary transformation and single-photon detections.

3. Genuine entangled states of four atomic qubits

In this section, we show how to prepare genuine entangled states of four atomic qubits using the setup indicated in figure 1. Let us consider the case of the single-photon input. In this case, the upper channel is in the single-photon state |1⟩ while the lower channel is in the vacuum state |0⟩. Then the initial state of the optical field is |\Psi_i⟩_0 = |10⟩ while the initial state of the four atomic qubits can be supposed as

\[ |\Psi_i⟩_A = |\Phi_1⟩ \otimes |\Phi_2⟩ \otimes |\Phi_3⟩ \otimes |\Phi_4⟩, \]  

where the initial state of the single atomic qubit is given by

\[ |\Phi_i⟩ = \cos \theta |0⟩_i + \sin \theta |1⟩_i \quad (i = 1, 2, 3, 4), \]  

which implies that the initial state of the whole system is |\Psi_i⟩_0 = |\Psi_i⟩_A. The initial state given by equation (5) can be prepared through applying proper classical fields to the related atom in the ground state [10].

The state of the system at the output of the MZ interferometer is then given by |\Psi_f⟩ = \hat{U}|\Psi_i⟩, where the unitary transformation \( \hat{U} \) is given by equation (3). It is easy to find that the output state |\Psi_f⟩ has the following expression:

\[ |\Psi_f⟩ = |01⟩ \otimes |\chi'(\phi)⟩ + |10⟩ \otimes |\chi''(\phi)⟩, \]  

where we have got rid of a global phase factor, |01⟩ and |10⟩ are the quantum states of two light fields, |\chi'(\phi)⟩ and |\chi''(\phi)⟩ are the quantum states of the four atomic qubits given by

\[ |\chi'(\phi)⟩ = \cos 2\phi |A_+⟩ + \cos \phi (|B⟩ + |C⟩) + |D⟩, \]  

\[ |\chi''(\phi)⟩ = \sin 2\phi |A_-⟩ + \sin \phi (|B⟩ - |C⟩), \]  

which are generally entangled states. Here we have introduced

\[ |A_+⟩ = c_1 c_2 c_3 c_4 |0000⟩ \pm s_1 s_2 s_3 s_4 |1111⟩, \]  

\[ |B⟩ = c_1 c_2 s_3 c_4 |0010⟩ + c_1 s_2 c_3 c_4 |0100⟩ + s_1 c_2 c_3 c_4 |1000⟩ + c_1 c_2 s_3 s_4 |0001⟩, \]  

Figure 1. Schematic setup to generate genuine four atomic qubits. The input single-photon field is bifurcated at the first BS, guided to interact sequentially with the atomic qubits placed in four separate high-finesse optical cavities, and then recombined at the second BS. Each cavity has an atomic qubit. D1 and D2 are two single-photon detectors. (This figure is in colour only in the electronic version)
The amount of entanglement between any two atomic–qubit pairs in four-atom-qubit entangled states \(|\chi^{\prime}(\pi/2)\rangle\) and \(|\chi^{\prime\prime}(\pi/2)\rangle\) can be measured in terms of the von Neumann entropy [6]. We find that the von Neumann entropy for any two qubit pairs in four-atom-qubit entangled states \(|\chi^{\prime}(\pi/2)\rangle\) and \(|\chi^{\prime\prime}(\pi/2)\rangle\) is given by

\[
S(|\chi^{\prime}\rangle) = 1 - \Phi(1 + \delta_{\pi}) \log_{2}(1 + \delta_{\pi}) + (1 - \delta_{\pi}) \log_{2}(1 - \delta_{\pi}),
\]

(23)

where \(\delta_{\pi} = \frac{\cos 2\theta_{1}\cos 2\theta_{2} + \cos 2\theta_{1}\sin 2\theta_{1}}{1 + \cos 2\theta_{1}\cos 2\theta_{2} \cos 2\theta_{3} \cos 2\theta_{4}}\).

A further calculation indicates that the amount of entanglement between any one atomic qubit and the other three atomic qubits in four-atom-qubit entangled states \(|\chi^{\prime}(\pi/2)\rangle\) and \(|\chi^{\prime\prime}(\pi/2)\rangle\) is the same as that of any two–atom–qubit pairs.

In particular, from equations (13) and (17) we can see that when \(\theta_{1} = \theta_{2} = \theta_{3} = \theta_{4} = \pm \pi/4\), the resultant entangled states become

\[
|\chi^{-}\rangle = \frac{1}{\sqrt{2}}[(0000) + |1111\rangle - |0110\rangle - |1100\rangle - |0101\rangle - |0011\rangle - |0001\rangle],
\]

(26)

\[
|\chi^{0}\rangle = \frac{1}{\sqrt{2}^2}[(1111) + |0111\rangle + |1011\rangle + |1101\rangle - |0100\rangle - |0010\rangle - |0000\rangle].
\]

(27)

For the above entangled states, from equations (21)–(25) we find that \(C(|\chi^{\prime}\rangle) = C(|\chi^{\prime\prime}\rangle) = 0\) and \(S(|\chi^{\prime}\rangle) = S(|\chi^{\prime\prime}\rangle) = 1\). This implies that there is absolutely zero entanglement between any two atomic qubits, and the entanglement is purely between pairs of atomic qubits. Therefore, these entangled states given by equations (26) and (27) are GESs for four atomic qubits.

We then have a look at the probability of success to get above GESs. From equation (6), it is straightforward to see that when the second photodetector D2 clicks, at the same time the photodetector D1 clicks, at the same time the photodetector D2 detects the null result, the atomic-qubit state will collapse onto the GES \(|\chi^{-}\rangle\) with the probability of success being 1/2, and when the first photodetector D1 clicks, at the same time the photodetector D2 detects the null result, the atomic-qubit state will collapse onto the GES \(|\chi^{0}\rangle\) with the probability of success being 1/2 too. It is interesting to note that the GES \(|\chi^{-}\rangle\) (or \(|\chi^{0}\rangle\)) can be obtained by applying the Pauli operator \(\hat{\sigma}_{z}^{1}\) to the fourth atomic qubit of the GES \(|\chi^{-}\rangle\) (or \(|\chi^{0}\rangle\)), i.e.

\[
\hat{\sigma}_{z}^{1}|\chi^{-}\rangle = |\chi^{0}\rangle, \quad \hat{\sigma}_{z}^{1}|\chi^{0}\rangle = |\chi^{-}\rangle.
\]

(28)
which implies that the GESs $|\chi\rangle$ and $|\overline{\chi}\rangle$ can be produced with the probability of success being 1 through making use of $\hat{\sigma}^z$ operation on the fourth atomic qubit. Hence, in our scheme the GESs $|\chi\rangle$ and $|\overline{\chi}\rangle$ can be produced deterministically through making single-photon detections of output fields and unitary transformation ($\hat{\sigma}^z$) upon the fourth atomic qubit.

It is interesting to see that starting with the GES $|\chi\rangle$ or $|\overline{\chi}\rangle$ we can generate a basis of 16 orthonormal states by applying Pauli operators of atomic qubits to the GES $|\chi\rangle$ or $|\overline{\chi}\rangle$. For instance, for the GES $|\overline{\chi}\rangle$ we can obtain the following 16 GESs:

$$|\psi_1\rangle_\mu = \sigma_1^\mu \sigma_2^\mu |\overline{\chi}\rangle,$$
$$|\psi_2\rangle_\mu = \sigma_1^\mu \sigma_3^\mu |\overline{\chi}\rangle,$$
$$|\psi_3\rangle_\mu = \sigma_2^\mu \sigma_3^\mu |\overline{\chi}\rangle,$$
$$|\psi_4\rangle_\mu = \sigma_2^\mu \sigma_3^\mu |\overline{\chi}\rangle,$$

where $\sigma_\mu^\mu$ ($\mu = 0, 1, 2, 3$) denotes the $\mu$th component of the Pauli operator for the $\mu$th qubit.

The above 16 GESs can be explicitly expressed as

$$|\psi_{1,2}\rangle_0 = \frac{1}{\sqrt{8}}(|0000\rangle + |1111\rangle + |0110\rangle - |1010\rangle - |0011\rangle - |1101\rangle + |1001\rangle - |0101\rangle),\tag{31}$$
$$|\psi_{3,4}\rangle_0 = \frac{1}{\sqrt{8}}(|0000\rangle + |1111\rangle + |0110\rangle + |1010\rangle - |0011\rangle - |1101\rangle + |1001\rangle - |0101\rangle),\tag{32}$$
$$|\psi_{1,2}\rangle_1 = \frac{1}{\sqrt{8}}(|1110\rangle - |0111\rangle - |1101\rangle + |0101\rangle + |0010\rangle - |1000\rangle + |0100\rangle - |1010\rangle),\tag{33}$$
$$|\psi_{3,4}\rangle_1 = \frac{1}{\sqrt{8}}(|1110\rangle + |0111\rangle + |1101\rangle + |0101\rangle - |0010\rangle - |1000\rangle - |0100\rangle - |1010\rangle),\tag{34}$$
$$|\psi_{1,2}\rangle_2 = \frac{1}{\sqrt{8}}(|1110\rangle + |0111\rangle + |1101\rangle + |0101\rangle + |0010\rangle + |1000\rangle + |0100\rangle + |1010\rangle),\tag{35}$$
$$|\psi_{3,4}\rangle_2 = \frac{1}{\sqrt{8}}(|1110\rangle + |0111\rangle - |1101\rangle - |0101\rangle - |0010\rangle - |1000\rangle - |0100\rangle + |1010\rangle),\tag{36}$$
$$|\psi_{1,2}\rangle_3 = \frac{1}{\sqrt{8}}(|0000\rangle + |1111\rangle + |0110\rangle + |1010\rangle + |0011\rangle + |1101\rangle + |0101\rangle + |1001\rangle),\tag{37}$$
$$|\psi_{3,4}\rangle_3 = \frac{1}{\sqrt{8}}(|0000\rangle + |1111\rangle - |0110\rangle - |1010\rangle - |0011\rangle - |1101\rangle - |0101\rangle + |1001\rangle),\tag{38}$$

It is straightforward to check that $\langle \psi_{\mu} | \psi_{\nu}\rangle_{\nu'} = \delta_{\mu \nu} \delta_{\nu' \nu}$ and $\sum_{\mu \nu} |\psi_{\mu} \rangle_{\nu} \langle \psi_{\mu} | = 1$. This implies that the above 16 states form an orthonormal and complete Hilbert space of the four-qubit system. In fact, they build a new type of representation of the four-qubit system, i.e. a genuine entangled-state representation. In this representation an arbitrary state of the four-qubit system can be expressed in terms of the basis of the representation.

Finally, we consider the influence of the imperfection of photon detections in the present scheme. An ideal photon detection with quantum efficiency $\eta = 1$ can be described by the positive operator-valued measure (POVM) of each detector $\{\Pi_0 = |0\rangle \langle 0|, \Pi_1 = I - \Pi_0\}$. In the realistic case, an incoming photon cannot be detected with the probability of success being 1. If the quantum efficiency of the photodetector is $\eta$, the POVM is given by $\{\Pi_0(\eta) = \sum_{i=0}^1 (1 - \eta)^i |i\rangle \langle i|, \Pi_1(\eta) = I - \Pi_0(\eta)\}$. From which it is straightforward to find that the inefficiency of photodetectors does not affect the quality of the generated entangled states, but it decreases the success probability. In our scheme, the success probability of the state to obtain the GESs $|\chi\rangle$ and $|\overline{\chi}\rangle$ are $\eta^2$ when the quantum efficiency of each photodetector is $\eta$. 

4. Concluding remarks

In conclusion, we have proposed a theoretical scheme to generate genuine entangled states of four atomic qubits in separated optical cavities using the atom–light interaction under the condition of the large detuning and single-photon detections. We have shown that GESs of four atomic qubits can be produced deterministically. Starting with one prepared GES we have found the 16 orthonormal and independent GESs. We have shown that these 16 GESs build a new type of representation for the four-qubit system, the genuine entangled-state representation. This representation provides us with new interesting insight into better understanding of multipartite entanglement. We have considered the influence of the imperfection of photodetectors in the present scheme, and indicated that the inefficiency of photodetectors does not affect the quality of the generated entangled states, but it decreases the success probability. In contrast to the iontrap scheme [7], our scheme is operated on demand and is scalable. Our proposed scheme is based on the state-of-the-art in cavity quantum electrodynamics [18, 19], can be readily scaled up to many atomic qubits and can be integrated with protocols for the realization of quantum networks. In fact, in our approach, arbitrary single-qubit and multi-qubit operations can be realized through controlling sequences of light pulses guided among the atomic qubits since single-atom qubits are held in isolated cavities. We believe that the GESs created in the present scheme provide new entanglement sources to realize quantum information processing.

Acknowledgments

This work was supported by the National Fundamental Research Program grant no 2007CB925204, the National Natural Science Foundation under grant nos 10775048 and 10325523, and the Education Committee of Hunan Province under grant no 08W012.

References

[1] Shor P W 1995 Phys. Rev. A 52 R2493
[2] Hillery M, Bužek V and Berthiaume A 1999 Phys. Rev. A 59 1829
[3] Murao M, Jonathan D, Plenio M B and Vedral V 1999 Phys. Rev. A 59 156
[4] Rauschenfeld R and Briegel H J 2001 Phys. Rev. Lett. 86 5188
[5] de Oliveira T R, Rigolin G and de Oliveira M C 2006 Phys. Rev. A 73 010305
[6] Yeo Y and Chua W K 2006 Phys. Rev. Lett. 96 060502
[7] Wang X W and Yang G J 2008 Phys. Rev. A 78 024301
[8] Chen P X, Zhu S Y and Guo G C 2006 Phys. Rev. A 74 032324
[9] Man Z X, Xia Y J and N B An 2007 Phys. Rev. A 75 052306
[10] Devitt S J, Greentree A D, Ionicioiu R, O'Brien J L, Munro W J and Hehlenber L C L 2007 Phys. Rev. A 76 052312
[11] Duan L M, Cirac J I, Zoller P and Polzik E S 2000 Phys. Rev. Lett. 85 5643
[12] Guo Y, Zhou L, Kuang L M and Sun C P 2008 Phys. Rev. A 78 013833
[13] Huang Y P and Moore M G 2008 Phys. Rev. A 77 032349
[14] Wootters W K 1998 Phys. Rev. Lett. 80 2245
[15] Kok P, Munro W J, Nemoto K, Ralph T C, Dowling J P and Milburn G J 2007 Rev. Mod. Phys. 79 135
[16] Kuang L M, Chen Z B and Pan J W 2007 Phys. Rev. A 76 052324
[17] Olivares S, Paris M G A and Bonifacio R 2003 Phys. Rev. A 67 032314
[18] McKeever J, Buck J R, Boozer A D, Kuzmich A, Nagerl H C, Stamper-Kurn D M and Kimble H J 2003 Phys. Rev. Lett. 90 133602
[19] Mabuchi H and Doherty A C 2002 Science 298 1372