A Liquid drop Falling in Another Fluid: A Two-Phase Flow Phenomenon

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Abstract. Liquid drops exist everywhere in daily life. However, such a common phenomenon involves complex mechanisms, and therefore has never been investigated thoroughly—the achievements of the studies are limited and unsatisfactory. In this article, the phenomenon of a liquid drop falling in another immiscible fluid (mostly its settling speed) was investigated. A series of experiments, in which liquid drops emerging from a syringe falls inside a measuring cylinder filled another fluid, was performed. Fluids used in the experiment include ethanol solutions and rice bran oil; for rice bran oil, its temperature-dependent viscosity was approximated by a function, which was proved effective on other similar oils. The dependence of the settling speed of the liquid drop on the density difference between the liquid drop and the surrounding fluid, as well as on the temperature of the surrounding fluid, was measured. Finite-Element Modeling (FEM) simulations were then carried out to model these situations, and lead to results that agreed with the experiments. It also visualized the flow field and revealed more details of the two-phase flow that were not detectable by our devices. Furthermore, inspired by some previous formulae, quantitative equations were derived semi-analytically by modifying an approximate drag formula originally developed for a rigid object at moderate Reynolds numbers. The form of the modification was never introduced in any other previous works. The equations were validated by simulations under ideal conditions.

1. Introduction
Two immiscible fluids often generate liquid drops. From cooking to crude oil leakage, liquid drops exist everywhere in our daily life. Although numerous investigators have devoted themselves to this day-to-day phenomenon in the past century, calculating the settling speed of falling liquid drops has long been a concern. This dilemma is encountered not only in liquid drops, but also in solid and gaseous ones. Two major factors are involved in determining the settling speed: the body force of the liquid drop, and the drag force exerted on the liquid drop by the other fluid.

For very low Reynolds numbers, Stokes gave an analytical solution—using the Navier-Stokes equation—to calculate the drag force encountered by a solid sphere. However, flow fields with higher Reynolds number have different properties. Scientists have made empirical modifications to the equation, including Oseen’s [1] and Goldstein’s [2] efforts. But a perfect analytical equation, describing the drag coefficient and applicable in a wide range of Reynolds numbers, has never been developed.

Many scholars, including A R Khan and J F Richardson, claimed that “outside the region where Stokes’ Law applies, a satisfactory theoretical form of this function does not exist. Therefore, it is necessary to apply some form of empirical treatment to correlate and interpret experimental data.” [3]
This paper presents both experimental data and simulation results. The experimental data were collected through an experiment in which a liquid came out slowly from a metal needle immersed in a less dense liquid, formed a liquid drop and fell. The simulation results were obtained from FEM (finite element model). This was made possible by the nowadays developed software, while back in the era where computer technologies were not as sophisticated, such approach was not available to the pioneers in this field. In the simulation, the viscosity curve of the oil was approximated by a hyperbola. Detailed discussions will be presented in later sections.

The paper first examines the factors influencing the settling speed: density and viscosity. Then, it takes a close look at the patterns of the flow field in the model and analyses various factors and phenomena impacting the drag force, such as speed gradients, vortexes, separation points. Finally, it investigates one of the most ancient but effective and commonly used empirical formulae describing the velocity of a falling rigid sphere. By analysing the relationship between solid and liquid drops, the paper presents a new method to find a solid equivalent for a liquid drop. Thus, equations that are applicable to solid spheres can be extended to liquid ones in a wide range of Reynolds numbers.

2. Method

2.1. Experiment Setup
The experiment involved a denser liquid coming out from a metal needle immersed in another liquid and forming a liquid drop. The liquid drop gradually increased until its gravity became greater than the buoyant force and the surface tensions clinging to the needle. The liquid drops then fell and accelerated until reaching its settling speed.

The experiment investigated the impact of the density difference and the temperature on the settling speed of the falling liquid drop.

![Figure 1](image)

Figure 1. Front view of the experiment. The injector injects a liquid drop through a metallic needle into another fluid. The two colors, yellow and blue, mark the liquid drop fluid and the outer fluid, respectively.

The pump was placed on top of the cylinder and vertically, with the needle pointing at the center line of the measuring cylinder (see Figure 1). A tripod was used to support the pump. The tripod’s
height was carefully chosen so that the needle tip was 1cm below the uppermost line of the graduated cylinder, where the level of the other liquid would be. The pump was set to pump at a rate of 5mm·s⁻¹. The diameter of the tube in which the piston moved was 15mm.

Therefore, the cross-sectional area was about 177mm², and the liquid was ejected at a rate of 0.88mL·s⁻¹.

Figure 2. Video Analysis with Logger Pro

The camera was placed around 35cm away from the cylinder. It filmed the falling process of the liquid drop. Using the video and the video analysis software, Logger Pro (see Figure 2), the height of the liquid drop (with respect to the lines on the cylinder) could be plotted in relation to time. The settling velocity of the liquid drop was then calculated and recorded.

For the investigation of density, another 100mL measuring cylinder and an electronic balance were applied to produce ethanol solutions of different concentration and thus different densities. Then, either the oil drop was dripped into the ethanol solution (when the ethanol solution was less dense than the rice bran oil), or the ethanol solution drop was dripped into the oil (when the ethanol solution was denser than the rice bran oil).

For the investigation of temperature, the measuring cylinder, filled with a little water at the bottom and the rice bran oil on top of it, was placed in a pot of hot water until being heated to the target temperature. An electronic thermometer was placed inside the cylinder to check the temperature. This method made it heat up uniformly, and prevented the water at the bottom from boiling and forming gaseous bubbles.

2.2. Simulation

2.2.1. Model
A finite element model was built in Comsol 5.4 to simulate the process of a liquid drop falling in another fluid.
The liquid drop fell along the center line of the cylinder, and the fluids were uniformly distributed in the cylinder. It was assumed that the velocity field was symmetric, without a Kármán vortex street. For simplicity, the model was axisymmetric, and this allowed a finer mesh to be set under a given amount of calculation.

The geometry of this model (see Figure 3) consisted of three parts: a rectangle representing the two fluids inside the cylinder, a line that marked the interface of the two fluids, and a circular liquid drop. Due to surface tension, the shape of the liquid drop when it had just left the needle tip in the experiment was a sphere.

The width of the rectangle was 10.92 mm, equal to the radius of the measuring cylinder in the experiment. The height of the rectangle was less than the actual value. This was to include less nodes, simplifying the calculation. A test was run to show that the liquid drop, once reaching its maximum speed, held that speed until it met the interface. Therefore, the distance between the liquid drop and the bottom did not affect its speed much.

The radius of the circle, representing the radius of the liquid drop, varied across experiments, but all the datapoints on the trend lines had radii 2.5 mm. The mechanism of liquid drop radii changes will be viewed in the results section.

The material properties were manually assigned according to the actual value. However, there was not much literature value for the viscosity of rice bran oil, and an approximation was required. It will be shown in details in the next section.

Since only the settling speeds of the liquid drops were recorded, the initial speed of the liquid drop set in the model did not affect the outcome. It was set to 0.

The upper and lower liquid surfaces were set as walls. The lower surface was a wall with no slip because the bottom of the measuring cylinder is solid. The upper surface was a wall with slip, ignoring air viscosity. The outside edge was set as a wetted wall because it was submerged in fluids in the experiment when the liquid drop came down.
The model involved the Earth’s gravitational field. Thus, the gravitational constant was \( g = 9.81 \text{m} \cdot \text{s}^2 \).

The model solved the Navier-Stokes equations for the conservation of momentum,

\[
\frac{d\vec{V}}{dt} = f - \frac{1}{\rho} \nabla p + \frac{\mu}{\rho} \nabla^2 \vec{V} + \frac{1}{3} \mu \nabla (\nabla \cdot \vec{V})
\]

(1)

and a continuity equation for the conservation of mass,

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0
\]

(2)

Here \( \vec{V} \) is the velocity field, \( \frac{d\vec{V}}{dt} \) is the material derivative, \( t \) represents time, \( f \) is the body force, \( \rho \) is the density of the fluid, \( \nabla \) is the divergence, \( \mu \) is the viscosity, and \( p \) is the pressure. The surface tension on the interface was included so that the geometry of the liquid drop would be accurately simulated. Without surface tension, the liquid drop would deform quickly.

2.2.2. Approximation of the Viscosity of Rice Bran Oil

The available literature values, shown in Table I, only included the viscosities of rice bran oil under 26°C, 38°C and 50°C, measured by Diamante and Lan [4]. However, the viscosity curves of some other similar cooking oils were also available, as shown in Table II, according to the study of Sahasrabudhe et al. [5]

Table 1. The literature viscosity of rice bran oil.

| \( T/\text{°C} \) | \( \mu_{\text{RiceBranOil}}/(\text{Pa} \cdot \text{s}) \) |
|-----------------|--------------------------|
| 26              | 0.0593 ± 0.0006          |
| 38              | 0.0398 ± 0.0001          |
| 50              | 0.0280 ± 0.0000          |

Table 2. The literature viscosity of canola oil and olive oil.

| \( T/\text{°C} \) | \( \mu_{\text{oil}}(\text{mPa} \cdot \text{s}) \) |
|-----------------|--------------------------|
| 22 ± 1          | 63.5 ± 1.6               | 74.1 ± 2.2 |
| 40              | 34.9 ± 0.9               | 40.1 ± 1.5 |
| 60              | 18.8 ± 0.3               | 21.1 ± 0.9 |
| 80              | 11.8 ± 0.4               | 13.4 ± 0.9 |
| 100             | 8.2 ± 0.4                | 9.6 ± 0.8  |
| 120             | 5.5 ± 0.2                | 6.1 ± 0.2  |
| 140             | 4.4 ± 0.1                | 4.8 ± 0.2  |
| 160             | 3.7 ± 0.2                | 4.0 ± 0.1  |
| 180             | 3.0 ± 0.2                | 3.3 ± 0.1  |
| 200             | 2.6 ± 0.3                |            |

It was discovered that the following equation could be used to fit the viscosity curve of oils:

\[
\mu = \frac{a_1}{T+a_2} + a_3
\]

(3)

Where \( T \) is the temperature, and \( k_1, k_2 \) and \( k_3 \) are constants specific to the type of oil.
Figure 4. Approximation of the viscosity of (a) canola oil and (b) olive oil. The blue asterisks are the literature values, and the orange lines are the fitting lines determined by Equation (3) and the parameters in the curly brackets.

Figure 4(a) shows the approximation of the viscosity of canola oil using Equation (3). The blue line represents the literature value, while the orange line is the line of best-fit. Figure 4(b) shows the approximation of olive oil. The error of the approximation is relatively low. Apply Equation (3) to rice bran oil, and the result is shown in Figure 5.

Figure 5. Approximation of the viscosity of rice bran oil. The blue asterisks are the literature values at 26°C, 38°C and 50°C, and the orange line is the best-fit line based on Equation (3).

The viscosity of rice bran oil is

\[ \mu_{RiceBranOil} = \frac{2.9154}{T(\text{in } ^\circ\text{C})+10.7792} - 0.0200 \] \hspace{1cm} (4)

3. Results

In this section, we shall first review the basic theory of drag. We will then move on to discuss our experimental results.

3.1. Theories of drag of an object in fluid

The experiment involved a denser liquid coming out from a metal needle immersed in another liquid and forming a liquid drop. The liquid drop gradually increased until its gravity became greater than the
buoyant force and the surface tensions clinging to the needle. The liquid drops then fell and accelerated until reaching its settling speed.

The liquid drop reached its settling speed with the drag balanced with the liquid drop’s body force. The body force \( F \) is defined as the difference between the liquid drop’s weight and buoyant force.

\[
F = W - F_{\text{buoyant}}
\]  
(5)

where \( W \) is the weight of the liquid drop, and \( F_{\text{buoyant}} \) is the buoyant force. Therefore,

\[
F = (\rho - \rho_0)Vg
\]  
(6)

where \( \rho \) is the density of the liquid drop, \( \rho_0 \) is the density of the outer fluid, \( V \) is the volume of the liquid drop, and \( g \) is the gravitational constant. Also, the drag force \( F_D \) of an object is given by Equation (7):

\[
F_D = \frac{1}{2} \rho_0 V^2 C_D A
\]  
(7)

Here \( v \) is the speed of the object, \( C_D \) is the drag coefficient, and \( A \) is the cross-sectional area.

The calculation of \( C_D \) is comparatively more complicated. For rigid spheres, it depends solely on the Reynolds number, which is defined as

\[
Re = \frac{\rho v L}{\mu}
\]  
(8)

Stokes obtained an analytical solution applicable for rigid spheres at very small Reynolds numbers by solving the equation of Navier-Stokes. [6]

\[
C_D = \frac{48}{Re}
\]  
(9)

However, as mentioned before, liquid drops have internal flows. Equation (9) needs some modification to become applicable to non-rigid spheres: [7]

\[
C_D = \frac{24}{Re} \frac{2 + 3(\mu_D/\mu_C)}{3 + 3(\mu_D/\mu_C)}
\]  
(10)

Here \( \mu_D \) is the viscosity of the spherical liquid drop and \( \mu_C \) is the viscosity of the outer fluid. Note that here \( C_D \) is already multiplied by the coefficient of \( \frac{1}{2} \) in Equation (7). Thus, for solid spheres, Equation (10) coincides with Equation (9). Equation (10) is indeed a desirable approximation at low Reynolds numbers. “However, the experimentally obtained \( C_D \) values have shown different trends in variation with \( Re \), which has encouraged many investigators to propose various formulation for \( C_D \).” For higher \( Re \), one of the most commonly used equation to calculate the drag coefficient of a rigid sphere was proposed by Schiller and Naumann: [8]

\[
C_D = \frac{24}{Re} (1 + 0.15 Re^{0.687})
\]  
(11)

(For a rigid sphere with \( 0.2 < Re < 800 \))

Same as Equation (10), here \( C_D \) is already multiplied by the coefficient of \( \frac{1}{2} \) in Equation (7). Also, for \( A \) in Equation (7) and \( V \) in Equation (6), since the liquid drop is a sphere,
\[ A = \pi \left( \frac{L}{2} \right)^2 \]  
(12)

\[ V = \frac{4}{3} \pi \left( \frac{L}{2} \right)^3 \]  
(13)

where \( L \) is the diameter of the liquid drop. In conclusion, when the liquid drop is in equilibrium,

\[ F_D = F \]  
(14)

Thus,

\[ 3\pi \mu L \left[ 1 + 0.15 \left( \frac{\rho_0 V L}{\mu} \right)^{0.687} \right] = \frac{4}{3} \pi \left( \frac{L}{2} \right)^3 (\rho - \rho_0)g \]  
(15)

(For a rigid sphere with \( 0.2 < Re < 800 \))

This section discusses the determinants of drag force and presented a detailed discussion of the equations obtained by prior investigators. Their equations successfully describe the drag force of a liquid drop at a low Reynolds number and the drag force of a solid sphere at a higher Reynolds number. However, the equation describing the settling speed of a liquid drop at a higher Reynolds number is absent.

In the following sections, a qualitative analysis will be presented to show the impact of various factors on the settling speed of the liquid drop. Then, based on that, a quantitative equation will be deduced.

### 3.2. Simulation and Experiment Results

#### 3.2.1. Vortex

It was discovered that there was a vortex behind oil drops. Figure 6(a) shows an ethanol solution drop whose density was \( 930 \text{ kg m}^{-3} \) falling through the oil, and Figure 6(b) shows an oil drop falling through an ethanol solution with density \( 870 \text{ kg m}^{-3} \). The reason was that, for the oil drop, the Reynolds number was higher. The definition of the Reynolds number is shown in Equation (8).

![Figure 6](image-url)  
Figure 6. The ethanol solution drop (a) and the oil drop (b) with a vortex behind. Contour: speed field; interval: \( 1 \text{ mm} \cdot \text{s}^{-1} \).
Under a relatively high $Re$, the fluid separated from the interface and created a vortex behind the liquid drop. According to Johnson and Patel [9], for a flow past a solid sphere, steady and symmetrical vortexes exist when $Re$ exceeds 20. Also, asymmetrical but steady vortexes exist when $210 < Re < 270$. Afterwards, when $Re > 270$, the vortexes become unsteady. In the investigation of density, $Re$ ranges from 0.2600 to 1.646 for ethanol drops and ranges from 89.31 to 368.2 for oil drops. Therefore, all the oil drops had vortexes behind them, while all the ethanol drops did not. Note that when $Re > 270$, the simulation only provided an approximation. In the investigation of temperature, $Re$ was large enough to generate vortexes, but it never exceeded 210, either in simulation or in the actual experiment.

3.2.2. Liquid Drop Size

The surface tensions of the liquid drop-outer fluid interface, the liquid drop-needle interface and the outer fluid-needle interface kept the liquid drop on the needle tip. When the liquid drop was large enough to overcome the surface tensions, the liquid drop fell.

The diameter of the liquid drop varied across experiments. However, the actual liquid drop diameter could hardly be measured because it was two orders of magnitudes less than the height of the measuring cylinder. The resolution of the video analysis application was not high enough to determine the precise size of the liquid drops. However, qualitative comparisons could still be made. Figure 7 shows the pictures captured by a separate camera. The length between two graduation lines on the measuring cylinder was 2.62mm.

![Water and oil drops under different density differences and temperatures](image)

Figure 7. Water and oil drops under different density differences and temperatures. The length between two graduation lines on the measuring cylinder was 2.62mm.

Due to refraction, the images of the liquid drops are horizontally stretched, but their vertical lengths do not change.

The first column of Figure 7 suggests that the water drop became smaller as the temperature increases. This was because the temperature negatively impacted the surface tensions. However, the
impact was comparatively insignificant. The second column shows that the diameter increased as the density difference decreases. As the density ratio decreased, the liquid drop had to be increasingly large to overcome the surface tensions. Also, the first row shows that, in the experiment, the oil drops were much smaller than ethanol solution drops, ceteris paribus.

Since the diameters of water drops under different temperatures were approximately two intervals (5.24mm) long, the liquid drop diameters in the simulations were always 5mm. It was a potential source of error, and its impact will be discussed.

3.2.3. The Impact of Density

In Figure 8, the vertical speed of the liquid drop is plotted in relation to the density difference of the liquid drop to the surrounding fluid. The blue datapoints represent the data acquired through experiments, while the red datapoints represent the simulation results. All the simulation results were acquired with a liquid drop diameter $L$ of 5mm. Dots indicated that the liquid drops consisted of ethanol solution, while triangles indicated that the liquid drops were rice bran oil.

The datapoints on the x-axis mark the situations where the liquid drop suspended in the surrounding fluid or fell at undetectable speed.

Both trendlines increased as density difference increased. This is because the body force $F$ is proportionate to $(\rho - \rho_0)$, while the drag force is constant when $\rho_0, \nu$ and $L$ are constant.

The trendline of oil drops slightly concaves down when the density ratio is between 1.01 and 1.06, and becomes almost linear thereafter. However, the experimental data concaved down much more. When $\rho - \rho_0 > 60$, Re exceeds 270, so some inaccuracy is comprehensible. However, this is also because that the liquid drop diameter $L$ decreased as the density difference $(\rho - \rho_0)$ increased in the actual experiment. According to Equation (7), the drag force $F_D$ is proportional to $L$ if $C_D$ is constant. However, the body force $F$ of the liquid drop was proportional to $L^3$. A larger radius would therefore result in a higher vertical speed. As the density difference increased, the size of the liquid drop decreased, so that its vertical velocity did not increase as rapid as the simulated trendline, which held $L$ constant. Simulation shows that, when oil drops fall in an ethanol solution with density 875kg ⋅ m$^{-3}$, a liquid drop 6mm in diameter is $4.178mm \cdot s^{-1}$ faster than a liquid drop 5mm in diameter.
The trendline of ethanol solution drops increased almost linearly. The experimental values exhibited similar behavior, only that it concaved down slightly, which was explained in the previous paragraph. Note that the simulated datapoints were consistently below the trendline, which was probably caused by an error in liquid drop radius measurement, rice bran oil viscosity data, or a systematic error in the simulation. However, the trend of the lines still matches closely.

It is suspected that the error was caused by the literature value of rice bran oil viscosity. The settling speed of an ethanol drop would be severely affected but that of an oil drop would not, if the oil had a different viscosity as the literature value, since the drag mainly came from the viscous force of the outer fluid. This, however, is very likely, given the difference in ingredients and processing across different kitchen oil manufacturers. The black line represents the simulation results after calibration. This hypothesis will be validated in the investigation of temperature.

Also, the oil drop trendline is higher than the ethanol solution drop trendline. Apart from the difference in $\rho$, which directly impacted the drag, it should be attributed to the impact of viscosity.

### 3.2.4. Liquid Drop Size

This section studies the impact of viscosity of the outer fluid on the terminal speed of the liquid drop. The most convenient way to change the viscosity of a fluid is usually changing the temperature. Therefore, in the experiment, water drops fell in rice bran oils with different temperatures, and their settling speeds were recorded (see Figure 9). The simulations held the densities constant.

![Figure 9. Settling Speed of water drop falling in oil as a function of temperature. The blue dots are experimental values, while the red line and the black line are the simulation results and the simulation results after calibration, respectively.](image)

In Figure 9, the red datapoints represent simulation results, while the blue datapoints represent experimental values. Both of the lines increase as temperature increases and oil viscosity decreases.

Meanwhile, the simulation results are always greater than the experimental results. Suppose that the error is systematic and caused by the inaccuracy in the literature values of the viscosity of rice bran oil, which is a hypothesis mentioned before in the investigation of density. Thus, the simulation results can be calibrated by multiplying a coefficient. It is worth mentioning that the calibration is only valid when the changes in Reynolds number and the internal flow of the liquid drop are negligible, according to Equation (15)—when the Reynolds number is constant, $C_D$ is constant; when the internal flow field of the liquid drop does not change, the liquid drop can be analogized with a solid sphere (further discussions will be presented later,) so that Equation (15) is applicable.

The coefficient is determined by averaging the ratios of experimental results to simulation results in Figure 8 whose $(\rho - \rho_0) = 70, 80$ and $100 \text{kg} \cdot \text{m}^{-3}$; they, too, represent the settling speed of an oil
drop, and they have similar Reynolds numbers as the datapoints in Figure 8. The result is $\frac{1}{1.78}$. In Figure 8 and Figure 9, the black lines are the calibrated lines.

From 30°C to 60°C, the calibrated line increases more rapidly than the experimental result. This is due to its constant, rather than decreasing, liquid drop radius.

From 60°C to 70°C, the experimental result increases dramatically. This was possibly because the actual Reynolds number increased as $\mu$ decreases. Thus, $C_D$ decreased. Another possible cause was that not only the scale but also the shape of the actual $\mu_{\text{RiceBranOil}} - T$ curve was different from the literature values.

From 70°C to 90°C, the experimental values become almost flat. One of the possible causes was, in fact, a technical difficulty: since that in the experiment, the injector connected to the pump was plastic, the water could not be heated. The water drop was at room temperature when it came into the hot oil. It cooled the surrounding oil down and increased oil viscosity. The second possible cause was the convection inside the measuring cylinder: in the experiment, the heated measuring cylinder was exposed in room temperature when the water drops fell through the oil. Therefore, the oil near the wall of the cylinder was cooler than the oil in the center. The cooler oil was denser and thus sank, while the warmer oil in the center rose. The water drop falling through the center line met this resistant force and slowed down. However, in the simulation, the wall was insulated. The higher the oil temperature, the faster it would cool down, the stronger the convection.

In conclusion, the simulation results, especially those after calibration, are close to the results attained in the experiments. It proves the effectiveness and accuracy of the simulation.

4. Discussions

4.1. The Impact of Viscosities

To fit Equation (9) on liquid drops, Equation (10) focused on the viscosity ratio of the inner and outer fluids. This proved the viscosities of both fluids a vital factor in determining the settling speed of a liquid drop. This section will dig deeper into exactly how the viscosity of the two fluids influence the falling speed of the liquid drop, respectively.

![Figure 10. The speed field of a falling liquid drop in a measuring cylinder. The length of an arrow is proportional to the speed at its midpoint.](image-url)
In Figure 10, the arrows represent the velocity of the fluids. For better clarification, the reference point of the velocity field is the uppermost point of the liquid drop, so the figure looked as if the liquid drop remained stationary and the outer fluid flows by. The liquid drop fluid near the interface flows upward along the interface due to the viscous force of the outer fluid, while the liquid drop fluid near the axis of symmetry flows downward to maintain the conservation of mass.

Only two factors can affect the drag of the liquid drop: the viscous force the outer fluid exerts on the liquid drop, and the vortex behind the liquid drop (not shown on Figure 10, but exists when Re becomes higher.)

4.1.1. The Effect of the Viscosities on the Viscous Force

The Newton’s law of viscosity demonstrated that the viscous force depends on the speed gradient, since both fluids are Newtonian fluids:

\[
\tau_{y,x} = -\mu \frac{du_x}{dy}
\]  

(16)

Here \( \tau \) is the shear stress, \( u_x \) is the x-component of the velocity of the fluid, and \( y \) is the y-coordinate. Therefore, the steeper the speed gradient of the outer fluid near the interface, the stronger the viscous force it exerts on the liquid drop.

For a rigid sphere, it is straightforward that the speed gradient depends only on the viscosity of the outer fluid and the speed at which the liquid drop falls. However, the liquid drop itself is a fluid, and therefore flow exists on the interface of the liquid drop and inside the liquid drop. The faster the superficial speed the liquid drop has, the less speed difference in the boundary layer is, and the less steep the speed gradient of the outer fluid is. Thus, the drag decreases.

Simulation supports that changes in liquid drop viscosity affects the drag. It shows that the oil drop fell 23.3% faster (from 78.18 \( \text{mm} \cdot \text{s}^{-1} \) to 96.31 \( \text{mm} \cdot \text{s}^{-1} \)) when oil temperature increases from 30\( ^\circ \text{C} \) to 90\( ^\circ \text{C} \). (However, the velocities of water drops under 30\( ^\circ \text{C} \) and 90\( ^\circ \text{C} \) reveal that the liquid drop’s viscosity does not contribute much change to the superficial speed of water drops, since water’s viscosity changes little in relation to temperature.)

Simulation (see Figure 11) further shows how the viscosity of the liquid drop affects the speed gradient of both fluids. To magnify the effect of liquid drop viscosity, it is better to use oil drops, whose viscosity changes dramatically with respect to temperature. The simulation involves two rice bran oil drops that are under 30\( ^\circ \text{C} \) (Figure 11(a)) and 90\( ^\circ \text{C} \) (Figure 11(b)) and fall through two ethanol solutions with densities of 810 kg \( \cdot \text{m}^{-3} \) and 835 kg \( \cdot \text{m}^{-3} \), respectively. The resultant speed of the liquid drops is held constant at 78 mm \( \cdot \text{s}^{-1} \). The reference point of the speed field is the uppermost point of the liquid drop, still.

Figure 11. Two rice bran oil drops that are under 30\( ^\circ \text{C} \) (a) and 90\( ^\circ \text{C} \) (b) falling through two ethanol solutions with densities of 810 kg \( \cdot \text{m}^{-3} \) and 835 kg \( \cdot \text{m}^{-3} \), respectively, at the same speed of 78 mm \( \cdot \text{s}^{-1} \). Contour: speed. Interval: 3 mm \( \cdot \text{s}^{-1} \).
Figure 11 shows that the less viscous liquid drop allows a steeper speed gradient inside the liquid drop, resulting in a higher superficial speed.

In fact, the ratio of the viscosity of the inner fluid to that of the outer fluid is a good approximation of the speed gradient distribution inside and outside the liquid drop (see Figure 12), suppose that the gradient is linear and the acceleration of the superficial fluid is negligible. In Figure 10, both liquid drops have a point where the speed is zero. It is represented by the blue circle in Figure 12. Also, the interface is marked by the blue vertical line. The speed gradient outside the liquid drop gives the interface a positive viscous force, and the speed gradient inside the liquid drop gives the interface a negative viscous force. For these two forces to be balanced, the ratio of velocity gradient must be the inverse of the ratio of the viscosity ratio. Thus Equation (10) makes sense.

Figure 12. The speed field inside and near the interface of the liquid drop. The blue circle is the point where speed is zero (inside the liquid drop.) The blue line is the interface; the red arrows mark the speed field.

In conclusion, the drag force has a positive causal relationship with both the viscosities of the inner and outer fluids. The ratio between the viscosities can approximate the distribution of speed gradient near the interface, both inside and outside.

4.1.2. The Effect of the Viscosities on the Vortex
The drag of the liquid drop has a positive causal relationship with the area of vortex behind it. However, as shown in the following discussions, the viscosities of the inner and outer fluids influence the separation point differently.

1) Outer Fluid
When $\mu$ increases, the Reynolds number $Re$ decreases. The separation point, which is solely determined by $Re$, shifts upstream thus.

2) Inner Fluid
A more viscous liquid drop is more resistant to the flow, flowing with less speed on the interface. This is in equivalent with the outer fluid flowing at a higher speed, which would also cause an increase in the Reynolds number. Thus, the separation point shifts upstream. This is also supported by the simulation result shown in Figure 11.

3) Conclusion
The viscosities of the liquid drop and the outer fluid have opposite effects on the separation point. Under the same liquid drop speed, the separation point moves upstream when the liquid drop’s viscosity increases and the outer fluid’s viscosity decreases. Further, when temperature increases, the
viscosities of both fluids increase, and the shift of the separation point depends on whose viscosity change is more dominant.

4.2. Effective Liquid Drop Diameter

Prolong the speed gradient shown in Figure 13, it reaches a surface where the speed is zero, as shown in Figure 13.

Figure 13. Prolonged speed gradient. The dashed line is the imaginary surface where the speed is zero.

This is particularly significant because, if the surface of the liquid drop is moved inwards to the black dashed line, it will have no superficial flow, and therefore it can be regarded as a solid-liquid interface. That is to say, every liquid drop has a smaller solid equivalent. The diameter of the solid equivalent is defined as the effective liquid drop diameter, \( L_{\text{effective}} \). Note that this is a simplified model. In reality, the speed gradient of the outer fluid is not linear, and it is not constant: on the bottom and the top of the liquid drop, it is less steep.

Another method to find a solid equivalent of a liquid drop is simply subtracting the superficial speed of the liquid drop from the speed of the outer fluid. These two methods have different impacts on the flow field:

4.2.1. Decreasing the Diameter

It maintains the speed but decreases the area of vortex behind the liquid drop. It maintains the speed so it does not change the speed \( v \) in Equation (7). Also, as shown in Figure 11, the speed gradient is less steep near the bottom of the liquid drop, meaning the point with zero speed is further inside the liquid drop. That is to say, a uniform \( L_{\text{effective}} \) decreases the viscous force acting on the liquid drop more than it decreases the viscous force at the bottom of the liquid drop.

4.2.2. Liquid Drop Size

It maintains the area of vortex but decreases its speed. It maintains the diameter so it does not change the reference area \( A \) in Equation (7). Also, decreasing the speed of the outer fluid lessens the viscous force by roughly the same proportion over all of the interface. Finally, it can wipe out the systematic error in the simulation, if any.

Therefore, a combined method will probably be the most optimal. Both \( L_{\text{effective}} \) and another constant \( b \) are appended to the Equation (15) to obtain Equation (17)

\[
3\pi \mu (v - b) L_{\text{effective}} \left[ 1 + 0.15 \left( \frac{\rho (v - b) L_{\text{effective}}}{\mu} \right)^{0.687} \right] = \frac{4}{3} \pi \left( \frac{L}{2} \right)^3 (\rho - \rho_0) g
\]  
(17)
Where $L_{\text{effective}}$ is in meters, and $b$ is in meters per second. For better clarification, all variables, including $v$, are now presented in SI units.

The data in Figure 7 were used to test Equation (17). When $\begin{cases} L_{\text{effective}} = 0.0051m \\ b = 0.002m \cdot s^{-1} \end{cases}$, the predicted value accurately described the curve when other parameters were constant: $\begin{cases} \mu = 0.001043Pa \cdot s \\ L = 0.005m \end{cases}$. The result is shown in Figure 14.

![Figure 14. Simulation results and predicted values (after fitting.) The orange asterisks are the simulation results, and the blue line is the line fitted by Equation (17).](image)

4.3. Determining $L_{\text{effective}}$ and $b$

4.3.1. The Relationship Between $L_{\text{effective}}$ and $b$

As illustrated, $L_{\text{effective}}$ and $b$ are both used to find the solid equivalent of a liquid drop. Therefore, they should be positively related to each other—when $L_{\text{effective}}$ does not increase the liquid drop speed much, $b$ should be larger to compensate, vice versa. To validate this hypothesis, six best-fit lines are drawn. Each line approximates a $v - (\rho - \rho_0)$ curve under a particular $\mu$. Meanwhile, $L$ and $\rho$ are constant: $\begin{cases} L = 0.005m \\ \rho = 900kg \cdot m^{-3} \end{cases}$.

To further guarantee the validity and accuracy of the simulation, $\mu$ has relatively high values in this investigation, so that $Re$ does not exceed 210. Thus, the vortexes, if any, are always symmetrical and steady, matching the simulation settings.

The results are shown in Table 3, and the graphs are shown in Figure 17 in the Appendix. The error bounds are determined when the fitted line clearly visibly diverges from the simulation datapoints (see Figure 18 and Figure 19 in the Appendix.)

| No. | $\mu(Pa \cdot s)$ | $L_{\text{effective}}(m \pm 0.0001m)$ | $b(m \cdot s^{-1} \pm 0.0004m \cdot s^{-1})$ |
|-----|------------------|-------------------------------------|--------------------------------------------|
| (a) | 0.015            | 0.0032                              | $-0.0016$                                  |
| (b) | 0.01             | 0.0034                              | $-0.001$                                   |
| (c) | 0.006            | 0.00375                             | 0.0003                                     |
| (d) | 0.004            | 0.0041                              | 0.0017                                     |
| (e) | 0.0035           | 0.0042                              | 0.0021                                     |
| (f) | 0.0025           | 0.0045                              | 0.0033                                     |

While fitting the line, it was discovered that $L_{\text{effective}}$ mainly adjusted the gradient and curvature of the curve, while $b$ shifted the curve (usually upwards.) This matched the analysis that $L_{\text{effective}}$ and $b$ transformed the liquid drop in different ways to approach its solid equivalent. Also, as predicted,
$L_{\text{effective}}$ is always smaller than the actual $L$ in the table. However, $b$ is not always nonnegative—when $L_{\text{effective}}$ increases the liquid drop speed too much, $b$ has to be smaller to compensate, according to the analysis at the beginning of this section. In fact, $L_{\text{effective}}$ may be larger than $L$ if $b$ compensates the slowing. This is the case in Figure 14. In conclusion, each of $L_{\text{effective}}$ and $b$ does not necessarily decrease the drag or increase the settling speed, but together they must increase the settling speed of the liquid drop.

$b$ is plotted in relation to $L_{\text{effective}}$ (see Figure 13), and it is found that the relationship can be approximated by a second-degree polynomial function:

$$b = b_1 L_{\text{effective}}^2 + b_2 L_{\text{effective}} + b_3$$

(18)

The result is graphed in Figure 15.

![Figure 15. $b$ for rice bran oil drops as a function of $L_{\text{effective}}$. Blue asterisks are the simulation results, and the orange line is the line fitted by Equation (18).](image)

4.3.2. The Effect of the Viscosities on the Vortex

When the liquid drop viscosity is unchanged, a greater $\mu$ makes the liquid drop viscosity less significant. Therefore, when $\mu$ increases, the liquid drop diverges more from the solid sphere. Thus, it is hypothesized that $L_{\text{effective}}$ will decrease if $\mu$ increases, ceteris paribus:

$$L_{\text{effective}} = \frac{k_1}{\mu + k_2} + k_3$$

(19)

The result is shown in Figure 16.
Figure 16. $L_{\text{effective}}$ for rice bran oil drops as a function of the viscosity of the outer fluid, $\mu$. Blue asterisks are the simulation results, and the orange line is the line fitted by Equation (19).

According to the equation, $L_{\text{effective}}$ would be $0.0062m$ if $\mu = 0$, which somewhat diverges from the actual $L$ of $0.005m$. However, in this case, $Re$ would be infinitely large, and the equation would no longer be applicable.

4.4. Equations for a Liquid Drop Falling in Fluid

Substitute Equation (18) and Equation (19) into Equation (15):

$$3\pi \mu \left[ v - b_1 \left( \frac{k_1 + k_3}{\mu + k_2} \right)^2 - b_2 \left( \frac{k_1 + k_3}{\mu + k_2} \right) - b_3 \left( \frac{k_1 + k_3}{\mu + k_2} \right) \right]$$

$$\left\{ 1 + 0.15 \left[ \rho_0 \left( v - b_1 \frac{k_1 + k_3}{\mu + k_2} \right)^2 - b_2 \frac{k_1 + k_3}{\mu + k_2} - b_3 \left( \frac{k_1 + k_3}{\mu + k_2} \right)^{0.687} \right] \right\} = \frac{4}{3} \pi \left( \frac{L}{2} \right)^3 (\rho - \rho_0) g \quad (20)$$

Here $k_1$, $k_2$, $k_3$, $b_1$, $b_2$, and $b_3$ depend only on the viscosity and diameter of the liquid drop. Therefore, they are termed as the “liquid drop-specific constants.” They do not depend on the density of the liquid drop, $\rho$, which is already taken into consideration in the equation.

For rice bran oil drops at $30^\circ C$ (i.e., with a viscosity of $0.057477 \text{Pa} \cdot \text{s}$) and with diameter $5\text{mm}$,

$$\begin{align*}
k_1 &= 9.5358 \times 10^{-6} \\
k_2 &= 0.0027 \\
k_3 &= 0.0027 \\
b_1 &= 351.2 \\
b_2 &= 1.1123 \\
b_3 &= -0.0088
\end{align*}$$
This investigation introduced a new form of modification that was different from the density-ratio approach adopted previously in Equation (10). Equation (20) can be used to deduce the settling speed of a liquid drop in any other immiscible liquids, when the liquid drop-specific constants \( k_1, k_2, k_3, b_1, b_2, \) and \( b_3 \) are given. The applicable range of this equation is yet to be investigated. So far, the equations are mainly validated with simulation with \( Re < 210 \).

Although the significance of \( L_{\text{effective}} \) and \( b \) was clearly explained, the physical significance of the fitting constants and the form of the equation remain unclear. Further investigations may give an explanation to this, or may even reveal the mechanisms beyond it.

5. Conclusion
The process of a liquid drop coming out from a needle and falling through another immiscible liquid was investigated. The experiment used a measuring cylinder with diameter 10.92mm, and oil, ethanol and water as the fluids. The experiment and simulation results were examined, and the conclusions are summarized as follows:

- The viscosity curves of oils can be approximated by hyperbolas.
- The settling speed of the liquid drop is positively related with its diameter \( L \), density \( \rho \), and negatively related with its viscosity and the outer fluid’s viscosity \( \mu \).
- The diameter \( L \) of the liquid drop is positively related with the perimeter of the needle and negatively related to the density difference of the liquid drop and the outer fluid, \((\rho - \rho_0)\).
- The simulation has a systematic error because of the literature value of the viscosity of rice bran oil. But when the oil viscosity is calibrated, or when the viscosities are manually set, the simulation is highly accurate.
- An increase in temperature \( T \) causes the separation point to shift upstream when the outer fluid viscosity \( \mu \) is dominant and causes the separation point to shift downstream when the liquid drop viscosity is dominant. It should be mentioned that, when \( T \) increases and the liquid drop becomes less viscous, although the area of vortex is larger, the decrease in viscous force is more significant, causing the liquid drop to fall faster.
- The equation describing the drag force experienced by a falling solid sphere also applies to liquid drops with some modifications. Equations describing the settling velocity of a liquid drop are derived and shown in details in Equation (18), Equation (19) and Equation (20).

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