Abstract
Let \((M, I)\) be a compact Kähler manifold admitting a hypercomplex structure \((M, I, J, K)\). We show that \((M, I, J, K)\) admits a natural HKT-metric. This is used to construct a holomorphic symplectic form on \((M, I)\).

1 Introduction
1.1 Hypercomplex manifolds
Let \((M, I, J, K)\) be a manifold equipped with an action of the quaternion algebra \(\mathbb{H}\) on \(T M\). The manifold \(M\) is called \textbf{hypercomplex} if the operators \(I, J, K \in \mathbb{H}\) define integrable complex structures on \(M\). As Obata proved ([Ob]), this condition is satisfied if and only if \(M\) admits a torsion-free connection \(\nabla\) preserving the quaternionic action:
\[
\nabla I = \nabla J = \nabla K = 0.
\]

Such a connection is called \textbf{an Obata connection on} \((M, I, J, K)\). It is necessarily unique ([Ob]).

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Hypercomplex manifolds were defined by C.P. Boyer ([Bo]), who gave a classification of compact hypercomplex manifolds for $\dim H = 1$.

If the Obata connection $\nabla$, in addition, preserves a quaternionic Hermitian¹ metric $g$ on $M$, then $(M, I, J, K, g)$ is called hyperkähler. This definition is equivalent to the standard one, see e.g. [Bes].

It is unknown precisely which complex manifold admit hypercomplex structures.

**Question 1.1:** Consider a compact complex manifold $(M, I)$. Describe the set of hypercomplex structures $(I, J, K)$ compatible with the given complex structure on $M$.

A similar question about hyperkähler structures is easily answered by the Calabi-Yau theorem. Recall that a hyperkähler manifold is holomorphically symplectic. Indeed, consider the 2-forms $\omega_J(\cdot, \cdot) = g(J\cdot, \cdot)$, $\omega_K(\cdot, \cdot) = g(K\cdot, \cdot)$; then

$$\Omega := \omega_J + \sqrt{-1} \omega_K$$

(1.1)
is a nowhere degenerate holomorphic $(2,0)$-form on $(M, I)$ ([Bes]). A converse result is implied by Calabi-Yau theorem: a holomorphically symplectic compact Kähler manifold is necessarily hyperkähler.

**Theorem 1.2:** Let $(M, I)$ be a compact holomorphically symplectic manifold with a Kähler form $\omega$. Then there exists a unique hyperkähler metric $g$ on $M$, with the same Kähler class as $\omega$.

**Proof:** See [Bes]. $lacklozenge$

We have no similar description of complex manifolds admitting a hypercomplex structure. In this paper we study the following problem.

**Question 1.3:** Let $(M, I)$ be a compact complex manifold of Kähler type². When $(M, I)$ admits a hypercomplex structure?

¹A metric $b$ is called quaternionic Hermitian if

$$g(Ix, Iy) = g(Jx, Jy) = g(Kx, Ky) = g(x, y)$$

for all $x, y \in TM$.

²That is, admitting a Kähler metric.
The following theorem gives an answer.

**Theorem 1.4:** Let \((M, I, J, K)\) be a compact hypercomplex manifold. Assume that \((M, I)\) admits a Kähler structure. Then \((M, I)\) is holomorphically symplectic.

**Proof:** In Subsection 1.2 we deduce Theorem 1.4 from Theorem 1.9, Theorem 1.10 and Theorem 1.11, which are proven in Sections 2, 3 and 4.

**Remark 1.5:** By Calabi-Yau theorem (Theorem 1.2), a holomorphically symplectic manifold admits a hyperkähler structure. However, the hypercomplex structure \((M, I, J, K)\) on \(M\) can a priori have a different nature. The manifold \((M, I, J, K)\) is hyperkähler if and only if the Obata connection \(\nabla\) preserves a metric. However, if the holonomy of \(\nabla\) is non-unitarian, such a metric does not exist.

**Definition 1.6:** Let \((M, I)\) be a compact holomorphically symplectic Kähler manifold, and \((M, I, J, K)\) a hypercomplex structure on \((M, I)\). Then \((M, I, J, K)\) is called exotic if \((M, I, J, K)\) is not hyperkähler, that is, if the holonomy of its Obata connection is not unitarian.

We conjecture that exotic hypercomplex structures do not exist.

### 1.2 HKT metrics and the canonical class

Let \(M\) be a hypercomplex manifold. A “hyperkähler with torsion” (HKT) metric on \(M\) is a special kind of a quaternionic Hermitian metric, which became increasingly important in mathematics and physics for the last 7 years.

HKT-metrics were introduced by P.S. Howe and G. Papadopoulos ([HP]) and much discussed in physics literature since then. For an excellent survey of these works written from a mathematician’s point of view, the reader is referred to the paper of G. Grantcharov and Y. S. Poon ([GP]).

The term “hyperkähler metric with torsion” is actually misleading, because an HKT-metric is not hyperkähler. This is why we prefer to use the abbreviation “HKT-manifold”.

Let \((M, I, J, K)\) be a hypercomplex manifold, \(g\) a quaternionic Hermitian form, and \(\Omega\) the \((2, 0)\)-form on \((M, I)\) constructed from \(g\) as in (1.1). The hyperkähler condition can be written down as \(d\Omega = 0\) ([Bes]). The HKT condition is weaker:
**Definition 1.7:** A quaternionic Hermitian metric is called an HKT-metric if

$$\partial(\Omega) = 0,$$

(1.2)

where $\partial : \Lambda^2_{2,0}(M) \rightarrow \Lambda^3_{3,0}(M)$ is the Dolbeault differential on $(M, I)$, and $\Omega$ the $(2,0)$-form on $(M, I)$ constructed from $g$ as in (1.1).

It was shown in [HP], [GP], that this condition is in fact independent from the choice of the triple of complex structures $(I, J, K)$, $IJ = -JI = K$ in $\mathbb{H}$. In particular, we could replace the hypercomplex structure $(M, I, J, K)$ with $(M, J, K, I)$. We obtain the following trivial claim

**Claim 1.8:** Let $(M, I, J, K)$ be a hypercomplex manifold, $g$ a quaternionic Hermitian metric. Consider $g$ as a quaternionic Hermitian metric on a hypercomplex manifold $(M, J, K, I)$. Then $g$ satisfies the HKT-condition on $(M, I, J, K)$ if and only if $g$ satisfies the HKT-condition on $(M, J, K, I)$.

HKT-metrics play in hypercomplex geometry the same role as the Kähler metrics play in complex geometry ([V1]).

The proof of [Theorem 1.3] is split onto three steps, as follows.

**Theorem 1.9:** Let $(M, I, J, K)$ be a compact hypercomplex manifold. Assume that $(M, I)$ admits a Kähler structure. Then the ge exists a finite non-ramified covering $\tilde{M} \rightarrow M$ such that the canonical bundle of $(\tilde{M}, I)$ is trivial as a holomorphic vector bundle.

**Proof:** See Section 2.

**Theorem 1.10:** Let $(M, I, J, K)$ be a hypercomplex manifold. Assume that $(M, I)$ admits a Kähler metric $g$. Then $(M, I, J, K)$ admits an HKT-metric $g_1$. Moreover, $g_1$ can be obtained by averaging $g$ with the $SU(2)$-action induced by quaternions.

**Proof:** See Section 3.

**Theorem 1.11:** Let $(M, I, J, K)$ be a compact hypercomplex manifold admitting an HKT-metric. Assume that $(M, I)$ admits a Kähler structure.
Assume, moreover, that there exists a finite non-ramified cover \( \tilde{M} \to M \) such that the canonical bundle \( K(\tilde{M}, I) \) has a holomorphic trivialization. Then \( (M, I) \) is holomorphically symplectic.

**Proof:** See Section 4. ■

Theorem 1.11 concludes the proof of Theorem 1.4. Indeed, consider a compact hypercomplex manifold \( (M, I, J, K) \), and assume that \( (M, I) \) admits a Kähler metric. By Theorem 1.9 the canonical class of \( (M, I) \) is trivial, by Theorem 1.10 \( (M, I, J, K) \) is HKT. We arrive at assumptions of Theorem 1.11 obtaining immediately that \( (M, I) \) is holomorphically symplectic.

2 Calabi-Yau theorem and triviality of canonical bundle

The following proposition is elementary.

**Proposition 2.1:** Let \( (M, I, J, K) \), \( \dim_{\mathbb{H}} M = n \) be a hypercomplex manifold, and

\[
\gamma_1(M, I) \in H^2(M, \mathbb{Z})
\]

the first Chern class of \( (M, I) \). Then \( \gamma_1(M, I) = 0 \).

**Proof:** Let \( SU(2) \subset \mathbb{H}^* \) be the group of unitary quaternions, acting on \( TM \). A Riemannian metric \( g \) on \( M \) is quaternionic Hermitian if and only if \( g \) is \( SU(2) \)-invariant. Taking an arbitrary Riemannian metric and averaging over \( SU(2) \), we obtain a quaternionic Hermitian metric. We proved the following trivial claim

**Claim 2.2:** Let \( M \) be a hypercomplex manifold. Then \( M \) admits a quaternionic Hermitian metric. ■

Return to the proof of Proposition 2.1. To show that \( \gamma_1(M, I) = 0 \), we need to construct a continuous trivialization of the canonical bundle \( K(M, I) = \Lambda^{2n,0}(M, I) \), where \( 2n = \dim_{\mathbb{C}} M \). Let \( g \) be a quaternionic Hermitian metric on \( M \), and

\[
\Omega := g(J \cdot, \cdot) + \sqrt{-1} g(K \cdot, \cdot)
\]
the corresponding non-degenerate \((2,0)\)-form on \((M, I)\). Then

\[ \Omega^n \in \Lambda^{2n,0}_I(M) \]

is a non-degenerate smooth section of the canonical bundle of \(\Lambda^{2n,0}_I(M) = K(M, I)\) of \((M, I)\). Therefore, this bundle is topologically trivial. This gives \(c_1(M, I) = 0\). □

The classification of Kähler manifolds with vanishing \(c_1\) ([Bo], [Bea], [Bes]) easily implies the following result.

**Theorem 2.3:** Let \((M, I)\) be a compact Kähler manifold with

\[ c_1(M, I) = 0. \]

Then there exists a finite non-ramified covering \(\tilde{M} \rightarrow M\) such that the canonical bundle \(K(\tilde{M}, I)\) is trivial.

□

Combining Proposition 2.1 and Theorem 2.3, we obtain Theorem 1.9.

**Remark 2.4:** For a typical non-hyperkaehler compact hypercomplex manifold \((M, I, J, K)\), the complex manifold \((M, I)\) admits no Kähler metrics, and the Calabi-Yau theorem cannot be applied. The canonical bundle \(K(M, I)\) is trivial topologically by Proposition 2.1. However, it is in most cases non-trivial as a holomorphic vector bundle, even if one passes to a finite covering. It is possible to show that \(K(M, I)\) is non-trivial for all hypercomplex manifold \((M, I, J, K)\) such that \((M, I)\) is a principal toric fibration over a base which has non-trivial canonical class; these include quasiregular Hopf manifolds and semisimple Lie groups with hypercomplex structure constructed by D. Joyce ([J]).

## 3 Kähler metrics and HKT metrics

Let \((M, I, J, K)\) be hypercomplex manifold. Since \(J\) and \(I\) anticommute, \(J\) maps \((p, q)\)-forms on \((M, I)\) to \((q, p)\)-forms:

\[ J : \Lambda^{p,q}_I(M) \rightarrow \Lambda^{q,p}_I(M). \]

**Definition 3.1:** Let \(\eta \in \Lambda^{2,0}_I(M)\) be a \((2,0)\)-form on \((M, I)\). Then \(\eta\) is called \(J\)-real if \(J(\eta) = \overline{\eta}\), and \(J\)-positive if for any \(x \in T^{1,0}(M, I)\),
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η(x, J(x)) ⩾ 0. We say that η is strictly J-positive if this inequality is strict for all x ≠ 0.

Denote the space of J-real, strictly J-positive (2, 0)-forms by Λ^{2,0}_{>0}(M, I).

We need the following linear-algebraic lemma, which is well known (see e.g. [V2]).

Lemma 3.2: Let M be a hypercomplex manifold. Then Λ^{2,0}_{>0}(M, I) is in one-to-one correspondence with the set of quaternionic Hermitian metrics g on M. This correspondence is given by

\[ g \rightarrow g(J \cdot, \cdot) + \sqrt{-1} g(K \cdot, \cdot), \]

and the inverse correspondence by

\[ \Omega \rightarrow g(x, y) := \Omega(x, J(y)). \quad (3.1) \]

Lemma 3.3: Let (M, I, J, K) be a hypercomplex manifold, g_1 a Hermitian metric on (M, J), \( \omega_1 = g_1(\cdot, J\cdot) \) the corresponding differential 2-form, and \( \Omega_1 \) the \( \Lambda^2_{I>0}(M) \)-part of \( \omega_1 \). Then \( \Omega_1 \) is strictly J-positive and J-real.

Proof: Since \( \omega_1 \) is a (1, 1)-form on (M, J), we have \( J(\omega_1) = \omega_1 \). Therefore, \( J(\Omega_1) = \overline{\Omega_1} \), and \( \Omega_1 \) is J-real.

Given \( x \in T^1_{I>0}(M), x \neq 0 \), the number \( \Omega_1(x, J(x)) \) is real because \( \Omega_1 \) is J-real. On the other hand,

\[ \Omega_1(x, J(x)) = \omega_1(x, J(x)) = g_1(x, x) > 0, \]

because \( \Omega_1 \) is a (2, 0)-part of \( \omega_1 \) and \( x, J(x) \) are (1, 0)-vector fields. We have shown that \( \Omega_1 \) is strictly J-positive. This proves Lemma 3.3. \[ \square \]

We also have the following trivial claim

Claim 3.4: In assumptions of Lemma 3.3 let

\[ \partial : \Lambda^{p,q}_{I}(M) \rightarrow \Lambda^{p+1,q}_{I}(M) \]

denote the standard Dolbeault differential \( \partial \) on (M, I). Then \( \partial \Omega_1 \) is the (3, 0)-part of \( d\omega_J \). In particular, if \( g_1 \) is Kähler on (M, J) then \( \partial \Omega_1 = 0. \)
Proof: By definition, $\Omega_1$ is the $(2,0)$-part of $\omega_J$, and $\partial\Omega_1$ is the $(3,0)$-part of $d\Omega_1$. ■

Remark 3.5: Let $\varphi$ be a Kähler potential for the Kähler form $\omega_1$ on $(M, J)$. By Claim 2.3 of [V2], on $(M, I)$ we have $\Omega_1 = \partial \partial_J \varphi$, where $\partial_J = -J \circ \partial \circ J$. The function $\varphi$ satisfying $\Omega_1 = \partial \partial_J \varphi$ for an HKT-form $\Omega_1$ is called an HKT-potential for an HKT-form $\Omega_1$.

Now, let $(M, I, J, K)$ be a hypercomplex manifold, and $g_1$ a Kähler metric on $(M, J)$. Consider the form $\Omega_1 \in \Lambda^2_{I}(M)$ constructed above. Then $\Omega_1$ is strictly $J$-positive and $J$-real by Lemma 3.3 and hence corresponds to a quaternionic Hermitian metric $g$ on $(M, I, J, K)$. By Claim 3.4, $\partial\Omega_1 = 0$, hence $g$ is HKT. Doing all calculations explicitly, a reader can show that $g$ is obtained from $g_1$ by averaging over $SU(2)$ (we shall not use this claim). This proves Theorem 1.10. Indeed, in assumptions of Theorem 1.10 we are given a Kähler metric on $(M, I)$, so the above argument gives an HKT-metric on the hypercomplex manifold $(M, J, K, I)$; this is equivalent to having an HKT-metric on $(M, I, J, K)$, as Claim 1.8 implies.

4 Supersymmetry on HKT-manifolds with trivial canonical class

Let $(M, I, J, K, g)$ be an HKT-manifold, and $g_1$ a Kähler metric on $(M, J)$. Consider the form $\Omega_1 \in \Lambda^2_{J}(M)$ constructed above. Then $\Omega_1$ is strictly $J$-positive and $J$-real by Lemma 3.3 and hence corresponds to a quaternionic Hermitian metric $g$ on $(M, I, J, K)$. By Claim 3.4, $\partial\Omega_1 = 0$, hence $g$ is HKT. Doing all calculations explicitly, a reader can show that $g$ is obtained from $g_1$ by averaging over $SU(2)$ (we shall not use this claim). This proves Theorem 1.10. Indeed, in assumptions of Theorem 1.10 we are given a Kähler metric on $(M, I)$, so the above argument gives an HKT-metric on the hypercomplex manifold $(M, J, K, I)$; this is equivalent to having an HKT-metric on $(M, I, J, K)$, as Claim 1.8 implies.

In [V1] we proved the following theorem, which is implied by an analogue of the Lefschetz-type $\mathfrak{sl}(2)$-action in the HKT setting.

Theorem 4.1: Let $(M, I, J, K)$ be a compact HKT-manifold, dim$_{\mathbb{H}} M = n$, and $K^{1/2}$ the square root of a canonical bundle $K(M, I)$ constructed as above. Consider the Dolbeault class $[\Omega] \in H^{0,2}_{\partial}(M, I) = H^2(\mathcal{O}(M, I))$ of $\Omega$,
where $\Omega \in \Lambda^2_I(M)$ is the HKT-form of $M$, and let

$$H^l(K^{1/2}) \Lambda^{n-l} \rightarrow H^{2n-l}(K^{1/2})$$

(4.1)

be the corresponding multiplicative map on the holomorphic cohomology of $K^{1/2}$. Then (4.1) is an isomorphism.

**Proof:** In [V1] it was shown that the natural operator

$$L_\Omega : H^l(K^{1/2}) \rightarrow H^{l+2}(K^{1/2})$$

belongs to an $\mathfrak{sl}(2)$-triple. This is used in [V1] to obtain Theorem 4.1 in the same way as one obtains a similar result for the cohomology of a Kähler manifold. ■

When $K(M, I)$ is a trivial holomorphic bundle, $K^{1/2}$ is also a trivial bundle. We obtain that

when $K(M, I)$ is trivial, $[\Omega]^n$ is a generator of $H^{2n}(K^{1/2}) \cong H^0(K^{1/2})^* = \mathbb{C}$

(4.2)

(the last isomorphism is provided by the Serre’s duality, using the triviality of the canonical bundle). Now we can prove Theorem 1.11.

Let $(M, I, J, K)$ be a compact HKT-manifold, with $M$ a non-ramified finite covering of $M$ with the canonical bundle $K(M, I)$ trivial. Assume that $(M, I)$ admits a Kähler metric. By Calabi-Yau theorem ([Yau], (M, I) admits a Ricci-flat Kähler metric $h$. Let $\Omega_h \in \Lambda^{0,2}_I(M)$ be a harmonic representative of the cohomology class $[\Omega] \in H^{0,2}(M, I)$ under $h$. Since $(M, I)$ is Kähler, the harmonic $(2, 0)$-form $\Omega_h$ is holomorphic. By Bochner-Lichnerowicz theorem ([Bos]), this implies

$$\nabla_h \Omega_h = 0,$$

(4.3)

where $\nabla_h$ is the Levi-Civita connection of $h$ (this is true for any holomorphic form $\Omega_h$ on a Ricci-flat compact Kähler manifold). Let $\Omega_h$, $\Omega$ be $\Omega_h$, $\Omega$ lifted to $\tilde{M}$. By (4.2), $\tilde{\Omega}^n$, and hence $\tilde{\Omega}_h^n$, represents non-zero class in cohomology of $(\tilde{M}, I)$. This implies $\tilde{\Omega}_h^n \neq 0$. By (4.3), we also have $\nabla_h \tilde{\Omega}_h^n = 0$, hence $\tilde{\Omega}_h^n$ trivializes $\Lambda^{2n,0}_I(M)$. We obtain that $\Omega_h$ is a non-degenerate holomorphic symplectic form on $(M, I)$. This proves Theorem 4.1. We finished the proof of Theorem 1.11.

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MISHA VERBITSKY

UNIVERSITY OF GLASGOW, DEPARTMENT OF MATHEMATICS,
15 UNIVERSITY GARDENS, GLASGOW G12 8QW, SCOTLAND,

INSTITUTE OF THEORETICAL AND EXPERIMENTAL PHYSICS
B. CHEREMUSHKINSKAYA, 25, MOSCOW, 117259, RUSSIA

verbit@maths.gla.ac.uk, verbit@mccme.ru