Analysis of Linear Difference Schemes in the Sparse Grid Combination Technique

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Sparse grids are tailored to the approximation of smooth high-dimensional functions. On a d-dimensional cube, the number of grid points is \( N = \mathcal{O}(h^{-1}\log h^{d-1}) \) with a mesh size parameter \( h \). The so-called combination technique, based on hierarchical decomposition, facilitates the numerical solution of partial differential equations on these grids. Key to the convergence analysis are specific multivariate error expansions, which we derive in a generic way for linear difference schemes through an error correction technique employing semi-discretisations. We obtain error formulae of the structure \( \varepsilon = \mathcal{O}(h^{p}\log h^{d-1}) \) and illustrate the convergence by numerical examples.

Key words. Sparse grids, error expansions, error correction, method of lines, elliptic equations, finite difference schemes

AMS subject classifications. 65N06, 65N12, 65N15, 65N40

1 Introduction

1.1 Background

High-dimensional models play an important role in various applications. A prominent example from finance are option pricing problems, which typically involve computation of the expectation of the pay-off with respect to a number of underlying stochastic factors (for instance equities or interest rates). Commonly used ways to accomplish this are numerical integration through deterministic cubature rules or stochastic simulation, or the solution of the corresponding (potentially high-dimensional, if the number of components is large) Feynman-Kac PDE.

In quantum-mechanics, high-dimensional eigenvalue problems for the Schrödinger equation govern the density functions of the states of large molecular systems. More examples with high-dimensional state spaces are found in population dynamics or data mining, to name but a few.

Classical grid based methods on a d-dimensional cube, with mesh width \( h \), say, suffer from the curse of dimensionality: the number of unknowns \( N \) required to achieve a prescribed error \( \varepsilon \) by an order \( p \) method grows exponentially like

\[
N(\varepsilon) = \mathcal{O}(\varepsilon^{-d/p}).
\]

The analysis in \cite{2} shows that for a hierarchical Tensor product basis the contribution of single piecewise polynomial basis functions (to the approximation of sufficiently smooth functions) is proportional to the measure of their support. Optimal approximation for a given number of degrees of freedom is attained for so-called sparse grids. Their relevance for the solution of PDEs was first revealed by Zenger in \cite{14}, and subsequently error bounds for finite element methods for
elliptic problems were derived in detail in [1], [2]. Recently, optimal convergence rates were shown for parabolic equations under very weak assumptions on the regularity of the initial condition for a sparse wavelet method and \( hp \) discontinuous Galerkin time stepping in [13].

In contrast to Galerkin-type methods, the combination technique – first introduced in [7] – decomposes the solution into contributions from Tensor product grids. This has the practical advantage that only numerical approximations on relatively small conventional grids need to be computed. These can be obtained independently and are superposed subsequently, which lends itself to very efficient parallel implementations. This concept has been successfully used in a number of applications, e. g. computational fluid dynamics [8], quantum mechanics [4] and computational finance [12].

Theoretical results for this extrapolation scheme, however, which inevitably rely on the expansion of the hierarchical surplus in terms of the grid sizes, have so far only been obtained for simple model problems. Bungartz et al. [3] obtain error bounds in terms of the Fourier coefficients for a central difference scheme for the two-dimensional Laplace equation. Pflaum [9, 10] employs Sobolev space techniques for the combination solution of a finite element method and prove asymptotic errors of the form \( h^2 \log h \) in the \( L_2 \)-norm for general linear elliptic equations in two dimensions, and first order convergence in the \( H^1 \)-norm. A similar result is also shown for the Poisson equation in higher dimensions. Common to these two approaches is the use of semi-discrete solutions to derive expansions of the hierarchical surplus. In the first paper Fourier representations for semi-discrete solutions to the Laplace problem are introduced as stepping stone from the continuous to the discrete version. This corresponds to variational formulations with semi-discrete conforming subspaces in [9, 10].

We introduce a new framework to derive error bounds for general difference schemes in arbitrary dimensions. As in the aforementioned articles, again semi-discrete problems will play an important role, but in the somewhat different setting of an error correction scheme that determines a suitable error expansion by a simple generic recursion. This will be detailed with the central difference stencil for the Poisson equation and with upwinding for the advection equation. For these two examples sparse grid error bounds are then derived in terms of the dimension of the problem and the smoothness of the solution (measured in \( C^k \)-Norms). It will become evident how this extends to more general linear elliptic and parabolic equations. We conclude with a thorough analysis of carefully chosen numerical examples. In particular, we highlight the dependence of the error on the smoothness of the solution and the problem dimension.

1.2 Hierarchical surplus and combination technique

Consider the \( d \)-dimensional unit cube \( I^d = [0, 1]^d \) and a Cartesian grid with mesh sizes \( h_i = 2^{-i} \) (corresponding to a level \( i \in \mathbb{N}_0 \) in direction \( i = 1, \ldots, d \)).

For a vector \( \mathbf{h} = (h_i)_{1 \leq i \leq d} \) we denote by \( u_\mathbf{h} \) the approximation of a function on this grid with points \( x_h = (ix_j h_j)_{0 \leq i_j \leq N_j, 1 \leq j \leq d} \). Hereby \( N_j = 1/h_j = 2^{l_j} \). We will ultimately be concerned with a setting where \( u_h \) is the finite difference solution to a scalar PDE, in which case \( u_h \) is different from the vector obtained by evaluating the exact solution to the PDE, \( u \), at the grid points. Therefore we denote the latter by \( u(x_h) \).

We now define the family \( U \) of solutions corresponding to these grids (see Fig. 1) by \( U = \{U(i)\}_{i \in \mathbb{N}_0^d} \)

\[
U(i) := u_{2^{-i}}
\]

i. e. as the family of numerical approximations \( u_h \) on tensor product grids with \( h_k = 2^{-i_k} \). Then the \( \text{hierarchical surplus} \) is the sequence

\[
\delta U := \delta_1 \ldots \delta_d U
\]
Figure 1: Full grid $\mathcal{M}_4$, sparse grid $\mathcal{M}_4$, and a possible realisation of a dimension adaptive sparse grid $\mathcal{M}_4$.

with

$$\delta_k U(i) := \begin{cases} U(i) - U(i - e_k) & i_k > 0, \\ U(i) & i_k = 0. \end{cases}$$

(2)

and $e_k$ the $k$-th unit vector. Obviously the difference operators commute.

Given a sequence of index sets $\mathcal{M}_n \subset \mathbb{N}_d^d$, the approximation on level $n$ is now defined as

$$u_n := \sum_{i \in \mathcal{M}_n} \delta U(i).$$

(3)

If this series converges in absolute terms, the error

$$\|u_n - u\| = \left\| \sum_{i \notin \mathcal{M}_n} \delta U(i) \right\| \leq \sum_{i \notin \mathcal{M}_n} \|\delta U(i)\|$$

in a suitable norm $\|\cdot\|$ is therefore bounded by the remaining surpluses. The choice of an optimal refinement strategy is thus thrown back to the control of the surplus. If this is done a posteriori, dimension adaptive schemes are obtained. This requires that hierarchical surpluses on finer levels are estimated from coarser levels. A heuristic example for integration problems is found in [5]. Conversely, an a priori analysis requires analytic estimates for the surplus in terms of the grid sizes $h_1, \ldots, h_d$ and is the focus of this paper.

In [7], a splitting into lower-dimensional contributions of the form

$$u - u_h = \sum_{m=1}^{d} \sum_{\{j_1, \ldots, j_m\} \subset \{1, \ldots, d\}} \gamma_{j_1, \ldots, j_m}(\cdot; h_{j_1}, \ldots, h_{j_m})h_{j_1}^{p_{j_1}} \cdots h_{j_m}^{p_{j_m}}$$

(4)

was proposed to analyse the combination technique. The ‘·’ stands for the arguments of $u$. As motivation, before we turn to the derivation of such expansions, we briefly sketch the immediate consequences for the combination technique.
If we view $u$ as a sequence that is constant over all refinement levels, and therefore set $\delta u = 0$, then under assumption (4)

$$\delta U(i) = \delta (U(i) - u) = O\left(2^{-p|i|}\right)$$

(5)

with $|i| = |i|_1 := \sum_{k=1}^{d} i_k$, because all lower order terms cancel out through the difference operator. Thus an asymptotically optimal choice for the index set is

$$\mathcal{M}_n = \{i \in \mathbb{N}_0^d : |i| \leq n\}.$$ 

The number of elements of $\mathcal{M}_n$ is then given through

$$|\mathcal{M}_n / \mathcal{M}_{n-1}| = \left(\frac{n + d - 1}{d - 1}\right) \Rightarrow |\mathcal{M}_n| = \sum_{l=0}^{n} \left(\frac{l + d - 1}{d - 1}\right) = O(n^{d-1}),$$

the number of nodes in the grid corresponding to an index $i$ is bounded by $\prod_{k=1}^{d} (2^{i_k} + 1) = O\left(2^{|i|}\right)$. For this choice of grid we conclude from (5)

$$\|u - u_n\| \leq \sum_{i \notin \mathcal{M}_n} \|\delta U(i)\| \leq \sum_{|i| > n} O(2^{-p|i|}) = O(n^{d-1}2^{-pn}).$$

(6)

The significance of the result (6) is that although the number of degrees of freedom is only $N_{dof} = O\left(n^{d-1}2^n\right)$ compared to $2^{dn}$ on the full grid, the error only deteriorates by a factor of order $n^{d-1}$ compared to the full grid result $2^{-pn}$.

We illustrate the combination formula by a two-dimensional example:

$$
\begin{align*}
    u_0 &= u(0, 0) \\
    u_1 &= [u(1, 0) - u(0, 0)] + [u(0, 1) - u(0, 0)] + u_0 \\
    &= [u(1, 0) + u(0, 1)] - u(0, 0) \\
    u_2 &= [u(2, 0) - u(1, 0)] + [u(1, 1) - u(1, 0) - u(0, 1) + u(0, 0)] + [u(0, 2) - u(0, 1)] + u_1 \\
    &= [u(2, 0) + u(1, 1) + u(0, 2)] - [u(1, 0) + u(0, 1)].
\end{align*}
$$

The general structure in two dimensions is

$$u_n = \sum_{l=0}^{n} U(l, n-l) - \sum_{l=0}^{n-1} U(l, n-1-l).$$

In one dimension obviously the sparse grid is identical to the full grid. In general, for $d \geq 1$ and $n > d - 1$, the combined solution can be written as

$$
\begin{align*}
    u_n &= |\delta_1^{d-1} S|(n), \\
    S(n) &= \sum_{|i|=n} U(i),
\end{align*}
$$

where the difference operator $\delta_1$, applied $d - 1$ times to the index $n$, is the one-dimensional version of (1), (2). Evaluating the coefficients explicitly gives

$$u_n = \sum_{l=n-d+1}^{n} a_{n-l} S(n)$$

(7)

with

$$a_i := (-1)^{d-1-i} \left(\frac{d-1}{i}\right) \quad 0 \leq i \leq d - 1.$$
1.3 Notation and outline

We see that the error analysis falls into two main parts: Firstly, an expansion like (4) has to be shown for the problem at hand. We consider linear PDEs on \( I^d = [0,1]^d \) and denote by \( \partial_i u \) the partial derivative of \( u \) with respect to \( x_i \). For a multi-index \( \alpha = (\alpha_1, \ldots, \alpha_d) \) let \( D^\alpha u = \partial_1^{\alpha_1} \ldots \partial_d^{\alpha_d} u \). The appropriate function spaces are those with bounded mixed derivatives

\[
X^d_\alpha := \{ u : [0,1]^d \to \mathbb{R} : D^\beta u \in C_0(I^d) \ \forall \beta \leq \alpha \}
\]

\[
X^d_\alpha(K) := \{ u \in X^d_\alpha : \| D^\beta u \| \leq K \ \forall \beta \leq \alpha \}
\]

\[
X^d_k := \{ u : [0,1]^d \to \mathbb{R} : D^\alpha u \in C_0(I^d) \ \forall \alpha \in \{0, \ldots, k\}^d \}
\]

\[
X^d_k(K) := \{ u \in X^d_k : \| D^\alpha u \| \leq K \ \forall \alpha \in \{0, \ldots, k\}^d \}.
\]

By \( C_0(I^d) \) we denote the space of continuous functions which are zero at the boundary, so we require for \( X^d_k(K) \) that all mixed derivatives up to order \( k \) vanish at the boundary.

A detailed error representation will be derived for the finite difference discretisation of the Poisson problem and the advection equation. For the Poisson problem

\[
\Delta u = \sum_{i=1}^{d} \partial_i^2 u = f \quad \text{in} \ I^d,
\]

\[
u = 0 \quad \text{on} \ \partial I^d,
\]

we assume sufficiently smooth \( f \) such that \( u \in X^d_4 \) for the solution of (9). For the advection equation

\[
u_t + \sum_{i=1}^{d-1} \partial_i u = 0 \quad \forall x \in I^{d-1} , \ t \in [0,1]
\]

\[
u(x,0) = u_0(x) \quad \forall x \in I^{d-1}
\]

\[
u(x,t) = u_1(x, t) \quad \forall x \in I^{d-1} , \ \exists i : x_i = 0 , \ t \in [0,1]
\]

we take for simplicity of notation the velocity constant and equal to one in each direction, but it will be obvious how the result generalises to the case with variable velocity. We can also identify \( t \) with \( x_d \) in a combined space-time formulation to write \( \sum_{i=1}^{d} \partial_i u = 0 \) with boundary condition \( u(x) = g(x) \ \forall x \) s.t. \( \exists i : x_i = 0 \). Here \( u_0, u_1 \) (or \( g \), respectively) are required to fulfils some compatibility conditions such that \( u \in X^d_5 \) for the solution \( u \) of (11).

With one-sided differences

\[
\delta^+_{k,h_k} u = \frac{1}{h_k} [u(\cdot + e_k h_k) - u], \quad \delta^-_{k,h_k} u = \frac{1}{h_k} [u - u(\cdot - e_k h_k)]
\]

\((e_k \ \text{the unit vector in direction} \ k)\) we consider central differences \( \delta^+_{k,h_k} \delta^-_{k,h_k} \) for the Poisson equation and upwinding with left-sided differences \( \delta^-_{k,h_k} \) and implicit time-stepping for the advection equation. The solution is then extended to \( I^d \) by multi-linear interpolation. For the latter instationary problem we will illustrate the difference between a sparse grid in the ‘space’ coordinates only and a ‘space-time sparse grid’.

We write the discretised systems in matrix notation

\[
A_h u_h = f_h,
\]

with the discrete solution vector \( u_h \), and write \( u(x_h) \) for the (vector representing the) exact solution at the grid points \( x_h \), such that the truncation error can be written as \( A_h u(x_h) - f_h \).

In order to split up the truncation error, we define according semi-discrete operators \( A_h^{(i_1,\ldots,i_m)} \) in directions \( i_1, \ldots, i_m \) by

\[
A_h^{(i_1,\ldots,i_m)} u = \sum_{i \in \{i_1,\ldots,i_m\}} \delta^+_{i_h,i_h} \delta^-_{i_h,i_h} u + \sum_{i \notin \{i_1,\ldots,i_m\}} \partial_i^2 u.
\]
for the Poisson problem and likewise for the advection equation. In contrast to the fully discrete solutions \( u_h \) satisfying (14), this leads to a system of PDEs

\[
A_h^{(i_1, \ldots, i_m)} u_h^{(i_1, \ldots, i_m)} = f_{h}^{(i_1, \ldots, i_m)}
\]

(with respective boundary conditions), where \( u_h^{(i_1, \ldots, i_m)} \) are defined on hyper-planes

\[
I^{(i_1, \ldots, i_m)} = \{ x \in \mathbb{R}^d : x_{i_k} \in \{ j h_{i_k} : 0 \leq j \leq h_{i_k}^{-1} \}, 1 \leq k \leq m \}
\]

and \( f_{h}^{(i_1, \ldots, i_m)} = R_h^{(i_1, \ldots, i_m)} f \) is the restriction of \( f \) to \( I^{(i_1, \ldots, i_m)} \). Denote by \( R_h := R_h^{(1, \ldots, d)} \) the restriction to the grid. Let \( C(I^{(i_1, \ldots, i_m)}) \) denote the space of continuous functions on these planes with maximum norm \( \| \cdot \|_\infty \) and derivatives are defined in directions along the planes.

The rest of the paper is organised as follows. Section 2 contains the main step of the analysis, in which we prove the expansion (11) for the error \( u(x_h) - u_h \) at the grid points \( x_h \). We then extend the solution to \( I^d \) by multilinear interpolation (section 3) and show that the expansion (11) is maintained pointwise for the error between the exact solution \( u(x) \) and the interpolated approximation \( (I u_h)(x) \). The latter result is interesting in its own right as it yields expansions for the interpolation error of sufficiently smooth functions on sparse grids and cubature formulae that are based on such interpolation properties will be obtained en passant.

In the second step of the analysis (section 4), these error representations on Cartesian grids are used to estimate the error of the combined solution by combinatorial formulae. Exact asymptotic expansions in terms of the smoothness of the solution will be given. In section 5 numerical results will illustrate the theory. Section 6 discusses these results and points out future directions, extensions to other problem classes and possible applications.

The author wishes to thank Mike Giles for very illuminating discussions on the subject, in particular for pointing him towards the concept of adjoint error correction [6], which proved vital for the theory of section 2.

2 Error expansion for finite difference schemes

The goal of this section is to establish error representations of the form

\[
u(x_h) - u_h = \sum_{m=1}^{d} \sum_{\{j_1, \ldots, j_m \} \subseteq \{1, \ldots, d\}} w_{j_1, \ldots, j_m}(x_h; h_{j_1}, \ldots, h_{j_m}) h_{j_1}^{p_1} \cdots h_{j_m}^{p_m}
\]

between the finite difference solution \( u_h \) and the exact solution \( u \) evaluated on a grid \( x_h \).

We have in mind linear (possibly degenerate) elliptic equations of the type

\[
Au = \sum_{i,j=1}^{d} a_{ij} \partial_i \partial_j u + \sum_{j=1}^{d} b_j \partial_j u + cu = f
\]

with \( \sum_{i,j=1}^{d} n_i n_j a_{ij} \geq 0 \), and view parabolic equations as a special case, in which we identify the ‘time’ coordinate with \( x_d \) and set \( b_d = -1, b_i = 0 \) for \( i = 1, \ldots, d-1 \), and \( a_{i,d} = a_{d,i} = 0 \) for \( i = 1, \ldots, d \). Upwinding in direction \( d \) is then equivalent to fully implicit time-stepping.

It seems instructive to first look at the main principles of the proof, which are generic and can be applied to a large class of problems. We restrict this sketch to two space dimensions in order to avoid cumbersome notation. We then extend this to higher dimensions and specify the requirements for particular examples (Poisson, advection equation), derive the coefficients in the expansion (16) and give sharp bounds.
2.1 Outline of proof in two dimensions

Starting point is a consistency assumption of order $p$ of the form

$$A_h u - f = h^p \cdot \tau^{(0)}_1(\cdot, h_1) + h^p \cdot \tau^{(0)}_2(\cdot, h_2) + h^p h^p \cdot \tau^{(0)}_{1,2}(\cdot, h_1, h_2).$$

(17)

This will typically be straightforward to see by Taylor expansion. In so doing we are assuming that the solution is sufficiently smooth, and we are making such an assumption throughout the following considerations. Note that often, e.g. for the Poisson problem, $\tau^{(0)}_{1,2}(\cdot, h_1, h_2) = 0$, but this term will be present for mixed derivatives.

To derive the respective convergence order, one would usually proceed to write

$$u(x_h) - A^{-1}_h f_h = h_i \cdot A^{-1}_h \tau^{(0)}_1(x_h, h_1) + h_i \cdot A^{-1}_h \tau^{(0)}_2(x_h, h_2) + h^p h^p \cdot A^{-1}_h \tau^{(0)}_{1,2}(x_h, h_1, h_2)$$

(18)

and deduce from the boundedness of $A^{-1}_h$ convergence of the scheme. In this context, however, since $A^{-1}_h$ depends on all grid sizes $h_1, \ldots, h_d$, it is not possible to derive an error expansion of the form (18) in this way, unless $A^{-1}_h$ is known explicitly in terms of the grid sizes and an expansion of $A^{-1}_h$ can be derived. This effectively happens in [3, where Fourier series of continuous and discrete solutions to the Laplace equation are used. Note that this is generally rather cumbersome and only possible in special cases where such representations are known.

This is where the concept of error correction comes into play, and we determine the error terms by solving the auxiliary semi-discrete problems

$$A^{(1)}_h w_1(\cdot, h_1) = \tau^{(0)}_1(\cdot, h_1)$$

(19)

$$A^{(2)}_h w_2(\cdot, h_2) = \tau^{(0)}_2(\cdot, h_2)$$

(20)

on the hyper-planes $I^{(1)}_h$ and $I^{(2)}_h$, respectively (for the definition of these operators see the preceding section). Note that $w_1(\cdot, h_1)$ and $w_2(\cdot, h_2)$ indeed only depend on $h_1$ and $h_2$ respectively, because they are the solution of a semi-discretisation as in [5] and not of the fully discrete equation.

Under suitable regularity assumptions, the first terms on the right-hand side of

$$A_h (u - h^p w_1(\cdot, h_1) - h^p w_2(\cdot, h_2)) = h^p \left( A^{(1)}_h w_1(\cdot, h_1) - A_h w_1(\cdot, h_1) \right) + h^p \left( A^{(2)}_h w_2(\cdot, h_2) - A_h w_2(\cdot, h_2) \right) + h^p h^p \tau_{1,2}^{(0)}(\cdot, h_1, h_2)$$

can be absorbed in the higher order terms by further expanding

$$\left[ A^{(1)}_h - A_h \right] w_1(\cdot; h_1) = h^p \sigma_{1,2}(\cdot; h_1; h_2)$$

(21)

$$\left[ A^{(2)}_h - A_h \right] w_2(\cdot; h_2) = h^p \sigma_{2,1}(\cdot; h_2; h_1),$$

(22)

so if we define

$$\tau^{(1)}_{1,2} = \tau^{(0)}_{1,2} + \sigma_{1,2}(\cdot; h_1; h_2) + \sigma_{2,1}(\cdot; h_2; h_1),$$

we get

$$A_h (u - h^p w_1(\cdot, h_1) - h^p w_2(\cdot, h_2)) = h^p h^p \tau^{(1)}_{1,2}(\cdot, h_1, h_2).$$

Now we can do the final step as we would have done in [5] to obtain

$$u(x_h) - u_h = h^p w_1(x_h; h_1) + h^p w_2(x_h; h_2) + h^p h^p w_{1,2}(x_h; h_1, h_2)$$

(23)

with

$$w_{1,2}(x_h; h_1, h_2) = A^{-1}_h \tau^{(1)}_{1,2}(x_h; h_1, h_2).$$
In higher dimensions, this procedure will be applied inductively. The start of the induction, \( m = 1 \), is always the consistency assumption (17). The final step \( m = d + 1 \) is always essentially equivalent to (23).

The central proofs in this article go along these lines and it will become clear how this framework can be applied to other settings. The main gap that has to be filled concerns the existence of an expansion (21), (22), and the boundedness of its coefficients, which relies on the smoothness of the solutions of (19), (20). In the following two sections we detail the regularity requirements and give explicit bounds on the coefficients in the expansion for the Poisson and the advection equation.

### 2.2 Detailed analysis for the Poisson equation

As an example, we consider central differences for the Poisson problem (9). In this case \( p = 2 \).

#### 2.2.1 Regularity, stability

Standard elliptic regularity results do not apply here due to possible singularities in the corners. Also, we need regularity of the solutions to semi-discrete problems, which are systems of elliptic equations of this type, and it is not obvious how estimates independent of the grid sizes can be obtained, which is crucial in our context. Therefore we formulate the following lemma.

**Lemma 2.1.** Let \( u \) be a solution of \( \Delta u = f \) with homogeneous Dirichlet data and \( u_h^{(i_1, \ldots, i_m)} \) the solution of \( A_h^{(i_1, \ldots, i_m)} u_h^{(i_1, \ldots, i_m)} = f_h^{(i_1, \ldots, i_m)} \) (with the definitions from section 1.3). Then

1. \( \|u\|_\infty \leq \frac{1}{2} \|f\|_\infty \)
2. \( \|u_h^{(i_1, \ldots, i_m)}\|_\infty \leq \frac{1}{8} \|f\|_\infty \)

**Proof.**

1. Let \( |f| < \bar{f} \) and \( v := -\frac{1}{2} f x_1 (1 - x_1) \). Then \( \Delta (u + v) = f + \bar{f} > 0 \) and hence \( u + v < 0 \) (maximum at the boundary), i.e. \( u < -v \leq \bar{f}/8 \). Similarly \( u \geq -\bar{f}/8 \).

2. Again a semi-discrete maximum principle holds and the analogous result follows by considering \( A_h^{(i_1, \ldots, i_m)} v = \bar{f} \) (central differences are exact for quadratic functions).

Similar estimates for the derivatives of the solution can be obtained by differentiating the equation, and using the fact that we are considering function spaces for which derivatives of sufficiently high order vanish at the boundaries.

#### 2.2.2 Consistency

For completeness we state the expansions (17) and (21), (22) in detail. Note that the truncation error is defined continuously and not just at the grid points.

**Lemma 2.2** (Truncation error of difference stencil). 1. Let \( u \in X_d^d(K) \), then

\[
(A - A_h)u = \sum_{k=1}^{d} h_k^2 \tau_k (: h_k)
\]

for some \( \tau_k \) with

\[
\|D^\alpha \tau_k\|_\infty \leq \frac{1}{12} K
\]

for \( \alpha \in \{0, 4\}^d, \alpha_k = 0 \).
2. Let \( u \in X^d_\beta(K) \) with \( \beta_i = \alpha \forall \beta_i \notin \{i_1, \ldots, i_m\} \), then

\[
(A_h^{(i_1, \ldots, i_m)} - A_h)u = \sum_{k \notin \{i_1, \ldots, i_m\}} h_k^2 \tau_k(; h_k),
\]

again with (24), but now for \( \alpha \in \{0, 4\}^d \), \( \alpha_i = 0 \ \forall i \in \{i_1, \ldots, i_m\} \cup \{k\} \) and \( \alpha_i \leq \beta_i \) otherwise.

**Proof.** Standard Taylor expansion in one variable. \( \Box \)

### 2.2.3 Error expansion on grid

We first prove an error expansion for the finite difference solution at the grid points.

**Theorem 2.3.** Let \( u \in X^d_\beta(K) \) be the solution of the Poisson equation and \( u_h \) the finite difference approximation on a grid \( x_h \). Then

\[
u(x_h) - u_h = \sum_{m=1}^d \sum_{\{j_1, \ldots, j_m\} \subset \{1, \ldots, d\}} w_{j_1, \ldots, j_m}(x_h; h_{j_1}, \ldots, h_{j_m}) h_{j_1}^2 \ldots h_{j_m}^2
\]

with \( w_{j_1, \ldots, j_m} \in C^1(1_{h}^{j_1, \ldots, j_m}) \) and

\[
\|w_{j_1, \ldots, j_m}(; h_{j_1}, \ldots, h_{j_m})\|_\infty \leq K \frac{m!}{96^m}.
\]

**Proof.** We prove by induction for \( 1 \leq m \leq d \)

\[
A_h \left( u(x_h) - \sum_{k=1}^{m-1} \sum_{\{i_1, \ldots, i_k\} \subset \{1, \ldots, d\}} h_{i_1}^2 \ldots h_{i_k}^2 w_{i_1, \ldots, i_k}(x_h; h_{i_1}, \ldots, h_{i_k}) \right) - f_h = \sum_{\{i_1, \ldots, i_m\} \subset \{1, \ldots, d\}} h_{i_1}^2 \ldots h_{i_m}^2 \tau_{i_1, \ldots, i_m}(x_h; h_{i_1}, \ldots, h_{i_m})
\]

where \( w_{i_1, \ldots, i_k}(; h_{i_1}, \ldots, h_{i_k}) \) and \( \tau_{i_1, \ldots, i_m}(; h_{i_1}, \ldots, h_{i_m}) \) are functions defined on the hyper-planes \( 1_{h}^{j_1, \ldots, j_k} \), for which the estimates

\[
\|D^\alpha \tau_{i_1, \ldots, i_m}(; h_{i_1}, \ldots, h_{i_m})\|_\infty \leq m! 8^{-(m-1)} 12^{-m} K, \tag{27}
\]

\[
\|D^\alpha w_{i_1, \ldots, i_k}(; h_{i_1}, \ldots, h_{i_k})\|_\infty \leq k! 8^{-k} 12^{-k} K \quad \text{for } 1 \leq k \leq m-1 \tag{28}
\]

hold if \( \alpha \in \{0, 4\}^d \) with \( \alpha_{i_j} = 0, 1 \leq j \leq k \) (that is along the planes where partial derivatives are defined).

The case \( m = 1 \),

\[
A_h u(x_h) - f_h = \sum_{k=1}^{d} h_k^2 \tau_k(x_h, h_k),
\]

follows from Lemma 2.2, 1. with \( \|D^\alpha \tau_k\|_\infty \leq \frac{1}{4^\alpha} K \) for \( \alpha \in \{0, 4\}^d \) with \( \alpha_k = 0 \). Now assume 26 for \( m \geq 1 \) with the bound (27). Then the solution \( w_{i_1, \ldots, i_m} \) of

\[
A_h^{(i_1, \ldots, i_m)} w_{i_1, \ldots, i_m} = \tau_{i_1, \ldots, i_m}(; h_{i_1}, \ldots, h_{i_m}) \tag{29}
\]
with homogeneous Dirichlet boundary conditions satisfies (Lemma 2.1).

\[
\|D^\alpha w_{i_1,\ldots,i_m}(\cdot; h_{i_1},\ldots,h_{i_m})\|_{\infty} \leq m!8^{-m}12^{-m}K
\]

with \(\alpha \in \{0, 4\}^d\) and \(\alpha_{i_j} = 0, 1 \leq j \leq k\) and therefore, for \(m < d\), from Lemma 2.2, there exist \(\sigma_{i_1,\ldots,i_m;k}(\cdot; h_{i_1},\ldots,h_{i_m}; h_k)\) such that

\[
A_{h_{i_1,\ldots,i_m}} w_{i_1,\ldots,i_m} - A_h w_{i_1,\ldots,i_m} = \sum_{k \notin \{i_1,\ldots,i_m\}} h_k^2 \sigma_{i_1,\ldots,i_m;k}(\cdot; h_{i_1},\ldots,h_{i_m}; h_k)
\]

and

\[
\|D^\alpha \sigma_{i_1,\ldots,i_m;k}\|_{\infty} \leq m!8^{-m}12^{-(m+1)}K
\]

for \(\alpha_{i_j} = 0, 1 \leq j \leq m\) and \(\alpha_k = 0\). For \(m = d\) obviously \(A_{h_{i_1,\ldots,i_m}} - A_h = 0\). So define

\[
\tau_{i_1,\ldots,i_{m+1}}(\cdot; h_{i_1},\ldots,h_{i_{m+1}}) := \sum_{j_1,\ldots,j_{m}, j \text{ s.t. }\{j_1,\ldots,j_m\} \cup \{j\} = \{i_1,\ldots,i_{m+1}\}} \sigma_{j_1,\ldots,j_m;j}(\cdot; h_{j_1},\ldots,h_{j_m}, h_j)
\]

and since the sum has \(m + 1\) terms

\[
\|D^\alpha \tau_{i_1,\ldots,i_{m+1}}\|_{\infty} \leq (m + 1)!8^{-m}12^{-m-1}K
\]

for \(\alpha \in \{0, 4\}^d\) with \(\alpha_{i_j} = 0, 1 \leq j \leq m + 1\). By induction (25) follows for all \(m\) and (26) is obtained directly with \(m = d\).

From the proof of Theorem 2.3 we see immediately the following result:

**Corollary 2.4.** The weights \(w_{j_1,\ldots,j_m}(x_{h,k}; h_{j_1},\ldots,h_{j_m})\) in (26) are the restriction of functions \(w_{j_1,\ldots,j_m}(\cdot; h_{j_1},\ldots,h_{j_m}); \|\cdot\|_{\infty} \leq k!96^{-k}K\)

hold for \(\alpha \in \{0, 4\}^d\), \(\alpha_i = 0\) for all \(i \in \{i_1,\ldots,i_k\}\).

### 2.3 Advection equation and extensions

We now turn to the upwind discretisation of equations (11)–(13) with fully implicit time-stepping, because unconditional stability in the maximum norm is easy to establish here. Analogous results to Lemmas 2.1 and 2.2 can be derived and give the following expansion.

**Theorem 2.5.** Let \(u \in X_2(K)\) be the solution of the advection equation and \(u_h\) the finite difference approximation. Then

\[
u(x_h) - u_h = \sum_{m=1}^{d} \sum_{\{j_1,\ldots,j_m\} \subseteq \{1,\ldots,d\}} w_{j_1,\ldots,j_m}(x_{h,k}; h_{j_1},\ldots,h_{j_m})h_{j_1}\ldots h_{j_m}
\]

(30)

with \(w_{j_1,\ldots,j_m} \in C_T^{(j_1,\ldots,j_m)}\) and

\[
\|w_{j_1,\ldots,j_m}(\cdot; h_{j_1},\ldots,h_{j_m})\|_{\infty} \leq K \frac{m!}{2^m}.
\]

**Proof.** First derive an expression for the truncation error (the equivalent of Lemma 2.2),

\[(A - A_h) u = \sum_{k=1}^{d} h_k \tau_k(\cdot; h_k)\]

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with
\[ \| D^{\alpha} \tau_k \|_\infty \leq \frac{1}{2} K \]
for \( \alpha \in \{0, 2\}^d \), \( \alpha_k = 0 \), and a stability result (the equivalent of Lemma 2.1)
\[ \| u_h^{(i_1, \ldots, i_m)} \|_\infty \leq \max(\| u_0 \|_\infty, \| u_1 \|_\infty). \]
The rest follows by the same steps as for the Poisson problem.

We can combine the above results for the Poisson problem and the advection equation to derive error formulae and estimates for an upwind discretisation of advection-diffusion equations of the form
\[ b \cdot \nabla u = \Delta u - cu. \]
The truncation error is additive, regularity and stability apply similarly under the assumptions made above. The derivation can therefore follow the same steps.

Instationary problems
\[ u_t + b \cdot \nabla u = \Delta u - cu \]
or
\[ u_t + \sum_{i=1}^d b_i \partial_i u = \sum_{i=1}^d a_i \partial_i^2 u - cu \]
can be seen as a degenerate case with vanishing diffusion in one direction, in which case upwinding in this coordinate is equivalent to fully implicit time-stepping (see comment at beginning of the section). We can therefore construct a ‘space-time’ sparse grid, which splits up both space and time in a hierarchical way. For other time discretisations, e. g. the second order Crank-Nicolson scheme, which has a time step constraint for stability in the maximum norm, time has to be treated separately: a sparse grid is used for the space-like coordinates, while the time step has to be chosen to satisfy the appropriate stability criterion on the highest space refinement level.

For non-diagonal diffusion-tensors, it seems difficult to construct finite difference schemes, for which discrete maximum principles hold on anisotropic grids. In this case the above analysis is not applicable. We discuss this in further detail in section 4.

More generally, all linear problems that admit smooth solutions and satisfy stability properties that can be carried over to the discrete case, fit into this framework.

3 Multilinear interpolation

For the definition of an approximation on a sparse grid it is necessary to extend the finite difference solution by interpolation. We will first show that the interpolation error for a sufficiently smooth function by piecewise multilinear splines on a Cartesian grid has an error expansion of the form (31). Subsequently we show that the difference between the numerical approximation of the PDE (the interpolated finite difference result) and the exact solution still has such an expansion.

As a central ingredient we first derive a particular partial Taylor expansion. This resembles the semi-discretisations of the previous section.

Lemma 3.1 (Expansion of functions with bounded mixed derivatives). Let \( u \in X_2^d \), \( x = (x_1, \ldots, x_d) \), and introduce \( x^{(i_1, \ldots, i_k)} \in \mathbb{R}^d \) with \( x_j^{(i_1, \ldots, i_k)} = x_j \) if \( j \in \{i_1, \ldots i_k\} \), 0 otherwise, and likewise for...
\[ u(x) = u(0) + \sum_{i=1}^{d} x_i \partial_i u(x - x^{(i)}) + \sum_{\substack{i, j = 1 \\
i \neq j}}^{d} x_i x_j \partial_i \partial_j u(x - x^{(i,j)}) + \ldots \]

\[ + x_1 \cdot \ldots \cdot x_d \partial_1 \ldots \partial_d u(0) + \sum_{i=1}^{d} \int_{0}^{x_i} (x_i - s_i) \partial^2_i u(s^{(i)}) \, ds_i + \ldots \]

\[ + \int_{0}^{x_1} \ldots \int_{0}^{x_d} (x_1 - s_1) \cdot \ldots \cdot (x_d - s_d) \partial^2_i \ldots \partial^2_d u(s) \, ds_d \ldots ds_1. \]

**Proof.** See Appendix A. \(\Box\)

The obvious, but crucial point is that all terms that are linear in one or more directions are interpolated exactly (in these directions).

**Theorem 3.2** (Interpolation of functions with bounded mixed derivatives). Assume \(u \in X_2^d\) and let \(I u(x_h)\) be the multilinear interpolating function on a grid \(x_h\). Then

\[ u(x) - (I u(x_h))(x) = \sum_{m=1}^{d} \sum_{\{(j_1, \ldots, j_m) \subset \{1, \ldots, d\}} \alpha_{j_1, \ldots, j_m}(x; h_{j_1}, \ldots, h_{j_m}) h_{j_1}^2 \ldots h_{j_m}^2 \quad (31) \]

with

\[ \| \alpha_{j_1, \ldots, j_m}(\cdot; h_{j_1}, \ldots, h_{j_m}) \|_\infty \leq \left( \frac{4}{2^d} \right)^m \| \partial_{j_1} \ldots \partial_{j_m} u \|_\infty. \]

**Proof.** See Appendix A. \(\Box\)

**Remark** (Cubature). From approximation results of this form error expansions for cubature formulae are obtained directly. Since the trapezoidal rule is exact on piecewise multilinear functions, the integration error is just the integral of the error terms over \([0, 1]^d\).

To see that the error expansion (30) is preserved under interpolation of the finite difference solution from the grid to \([0, 1]^d\), we split

\[ u(x) - (I u_h)(x) = u(x) - (I u_h)(x) + (I u_h)(x) - (I u_h)(x) \]

\[ = u(x) - (I u_h)(x) + (I u_h) - u_h)(x). \quad (32) \]

The first term \(u(x) - (I u_h)(x)\) is the interpolation error of the exact solution and is given by (31). The second term, the interpolation of the discretisation error

\[ u(x_h) - u_h = \sum_{k=1}^{d} \sum_{\{(j_1, \ldots, j_m) \subset \{1, \ldots, d\}} h_{j_1}^2 \ldots h_{j_m}^2 w_{j_1, \ldots, j_m}(x_h; h_{j_1}, \ldots, h_{j_m}) \quad (33) \]

on the grid \(x_h\), is problem dependent (see Theorem 2.3) and needs to be evaluated separately. It is straightforward to see that the linear interpolant of the error is bounded by the error at the grid points, but it is essential, and more involved, to derive the exact form of the expansion.

We show this again for the Poisson problem first.

**Theorem 3.3.** Let \(u \in X_2^d(K)\) be the solution to the Poisson problem and \(u_h\) the numerical solution with central differences, \(I u_h\) its multilinear interpolation. Then

\[ u - I u_h = \sum_{m=1}^{d} \sum_{\{(j_1, \ldots, j_m) \subset \{1, \ldots, d\}} v_{j_1, \ldots, j_m}(\cdot; h_{j_1}, \ldots, h_{j_m}) h_{j_1}^2 \ldots h_{j_m}^2 \quad (34) \]
with
\[ \|v_{j_1, \ldots, j_m}(\cdot; h_{j_1}, \ldots, h_{j_m})\|_\infty \leq C \cdot K \cdot \frac{m!}{96^m}, \]
and \( C < 150188 \) depends on neither the dimension nor the data.

Proof. See Appendix A.

Similarly one gets for the advection equation the following result.

**Theorem 3.4.** Let \( u \in X^d_2(K) \) be the solution of the advection equation and \( u_h \) the numerical solution with upwinding and implicit Euler time-stepping. Then
\[ u - T u_h = \sum_{m=1}^{d} \sum_{\{j_1, \ldots, j_m\} \subset \{1, \ldots, d\}} v_{j_1, \ldots, j_m}(\cdot; h_{j_1}, \ldots, h_{j_m}) h_{j_1} \cdot \ldots \cdot h_{j_m} \]
and \( C < 3/2 \) depends on neither the dimension nor the data.

Proof. See Appendix A.

## 4 Combined error bounds and asymptotics

We study now in detail the extrapolation effect that the combination formula has on error terms by means of combinatorial relations. This leads to error bounds for the combination solution of the order seen in (6). Starting point is the pointwise expansion of the error on tensor product grids with mesh sizes \( h = (h_1, \ldots, h_d) \),
\[ u - u_h = \sum_{m=1}^{d} \sum_{\{j_1, \ldots, j_m\} \subset \{1, \ldots, d\}} v_{j_1, \ldots, j_m}(\cdot; h_{j_1}, \ldots, h_{j_m}) h_{j_1}^p \cdot \ldots \cdot h_{j_m}^p \]
(e. g. as shown in the preceding sections for the Poisson and advection equation), where
\[ |v_{j_1, \ldots, j_m}| \leq K \quad \forall 1 \leq m \leq d \quad \forall \{j_1, \ldots, j_m\} \subset \{1, \ldots, d\}. \]

Obviously this second step of the analysis is independent of the problem, given that (36) holds.

### 4.1 Error bounds

In [7] Griebel et al. derive from (36) and (37) in the two- and three-dimensional case for \( p = 2 \) the bounds
\[ |u - u_n| \leq K 2^{-2n} \left( 1 + \frac{5}{4} n \right) \]
and
\[ |u - u_n| \leq K 2^{-2n} \left( 1 + \frac{65}{32} n + \frac{25}{32} n^2 \right), \]
respectively. This section is devoted to a generalisation of (38) and (39) to error bounds of the form
\[ |u - u_n| \leq K c(d, p)n^{d-\frac{1}{2} - pn}, \]
for arbitrary dimension and arbitrary order. In addition the asymptotic limit
\[ \lim_{n \to \infty} \frac{|u - u_n|}{n^{d-\frac{1}{2} - pn}} \]
will be given. We will use the representation of the combined solution
\[ u_n = [\delta_1^{d-1} S](n), \]  
\[ S(n) = \sum_{|i|=n} U(i) \]  
(40)  
(41)

(see section 1.2) in terms of an iterated application of the one-dimensional difference operator \( \delta_1 \).

The proof requires a few combinatorial equalities. The following formula for iterated differences of products — a discrete version of the product rule for differentiation — will be useful here.

**Proposition 4.1.** Let \( f, g \in \mathbb{R}^{N_0} \). Then
\[ \delta_1^k (fg) = \sum_{j=0}^{k} \binom{k}{j} \delta_1^{k-j} f(\cdot + j) \delta_1^j g \quad \forall k \in \mathbb{N}_0. \]  
(42)

The straightforward proof (based on induction) is omitted here, but see [11].

Since the number of grids on level \( n \) that are involved in the combination solution in dimension \( d \) is given by
\[ N(n, d) = \binom{n+d-1}{d-1}, \]  
(43)

the following Lemma 4.2 states a necessary condition for the consistency of the combination technique (i.e. the sum of coefficients of all grids is 1).

**Lemma 4.2 (Consistency).** With \( N \) from (43) \( \forall d \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq d - 1 \)
\[ \delta_1^{d-1} N(n, d) = 1. \]

**Proof.** By induction one proves for \( 0 \leq k \leq d - 1 \)
\[ \delta_1^k N(n, d) = \binom{n+d-1}{d-k-1}. \]

The following formula forms a key to the proof of the main result in this section, Theorem 4.4. To see why, look at \( n \) as the sparse grid level and \( s_l 2^{-pl} \) as the error of order \( p \) on level \( l \), with some coefficient \( s_l \) that is collected from the number of grids given by the binomial term. Then Lemma 4.3 says that only the highest order terms \( 2^{-p(n+d-1)} \) are left in the sparse grid solution, whereas the combination formula cancels out all lower order terms that come from the larger mesh sizes on the anisotropic grids. This is a more quantitative version of (5).

**Lemma 4.3 (Error representation formula).** Let \( m, d \geq 1, v : \mathbb{R}_m^+ \rightarrow \mathbb{R} \) and for \( n \in \mathbb{N}_0 \)
\[ F(n) := \sum_{\substack{i \in N_0^d \\mid |i| = n}} v(2^{-i_1}, \ldots, 2^{-i_m}) 2^{-pi_1} \cdots 2^{-pi_m}. \]

Then for \( d \in \mathbb{N} \)
\[ \delta_1^{d-1} F(n) = 2^{-p(n+d-1)} \sum_{i=0}^{m-1} s_{n+d-i-1} \binom{m-1}{i} (-2)^{pi} \]
with
\[ s_l := \sum_{\substack{i \in N_0^d \\mid |i| = l}} v(2^{-i_1}, \ldots, 2^{-i_m}). \]

**Proof.** See Appendix B.
After these preparations, the error terms can be estimated conveniently.

**Theorem 4.4 (Error bounds).** Assume for all \( u_h \) a pointwise error expansion of the form \((37)\) with \((38)\). Then the combination solution \((40)\) fulfills the error estimate

\[
|u - u_n| \leq \frac{2K}{(d-1)!} \left( \frac{2p+1}{2^p-1} \right)^{d-1} (n + 2(d-1))^{d-1} 2^{-pn}.
\]

**Proof.** Because of Lemma 4.2 (the exact solution is constant over the grid levels)

\[
\delta_1^{d-1} \sum_{|i|=n} u = u \delta_1^{d-1} \sum_{|i|=n} 1 = u \delta_1^{d-1} N(n, d) = u
\]

and one may write

\[
u - u_n = \delta_1^{d-1} \sum_{|i|=n} (u - U(i)).
\]

The error terms in

\[
u - U(i) = \sum_{m=1}^d \sum_{\{j_1, \ldots, j_m\} \subset \{1, \ldots, d\}} v_{j_1, \ldots, j_m}(\cdot; 2^{-i_1}, \ldots, 2^{-i_m}) 2^{-pi_{j_1}} \ldots 2^{-pi_{j_m}}
\]

will now be studied separately with the help of Lemma 4.3 applied to

\[
F_{j_1, \ldots, j_m}(n) := \sum_{|i|=n} v_{j_1, \ldots, j_m}(\cdot; 2^{-i_1}, \ldots, 2^{-i_m}) 2^{-pi_{j_1}} \ldots 2^{-pi_{j_m}}
\]

in

\[
u - u_n = \sum_{m=1}^d \sum_{\{j_1, \ldots, j_m\} \subset \{1, \ldots, d\}} \delta_1^{d-1} F_{j_1, \ldots, j_m}(n).
\]

The point is that with Lemma 4.3 the factors \( h_i \) then no longer appear separately, but only in their highest order \( h_1 \cdot \ldots \cdot h_m = 2^{-n} \). It remains to estimate the coefficients, which leads to the polynomial terms in \( n \). From \( |v_{j_1, \ldots, j_m}| \leq K \) follows

\[
|s_i| \leq K \left( \frac{l + m - 1}{m - 1} \right) \Rightarrow \max_{i=0}^{m-1} |s_{n+d-i-1}| \leq K \left( \frac{n + d + m - 2}{m - 1} \right),
\]

therefore

\[
\left| \sum_{i=0}^{m-1} s_{n+d-i-1} \left( \frac{m-1}{i} \right) (-2)^i \right| \leq K \left( \frac{n + d + m - 2}{m - 1} \right) \sum_{i=0}^{m-1} \left( \frac{m-1}{i} \right) 2^i \leq K \left( \frac{n + d + m - 2}{m - 1} \right) (2^p + 1)^{m-1},
\]

and with Lemma 4.3

\[
|\delta_1^{d-1} F_{j_1, \ldots, j_m}(n)| \leq 2^{-p(n+d-1)} (2^p + 1)^{m-1} K \left( \frac{n + d + m - 2}{m - 1} \right)
\]

\[
\leq 2^{-p(n+d-1)} (2^p + 1)^{d-1} K \left( \frac{n + 2(d - 1)}{d - 1} \right)
\]

\[
< 2^{-pn} \left( \frac{2^p + 1}{2^p} \right)^{d-1} K \left( \frac{n + 2(d - 1)}{d - 1} \right)^{d-1}.
\]
Since the number of terms in (45) is $2^d - 1$, we get
$$\|u - u_n\| < \left(\frac{2^p + 1}{2^{p-1}}\right)^{d-1} \frac{2K}{(d-1)!} (n + 2(d-1))^{d-1} 2^{-pn}.$$ 

Let us study equation (44) for $p = 2$ and small $d$. For $d = 1$, where the sparse grid is identical to the full grid, the estimate reduces to $\|u - u_n\| \leq 2K4^{-n}$ (the substitution $2^d - 1$ by $2^d$ in the end of the above proof explains the unnecessary factor 2). In the case $d = 2$ the leading term is given by (49), such that in the highest power of $n$

$$\|u - u_n\| \sim K \frac{5}{4} n 4^{-n},$$

in accordance with (38). Similarly, in three dimensions one gets

$$\|u - u_n\| \sim K \frac{5}{2} \left(\frac{5}{4}\right)^2 4^{-n} = K \frac{25}{32} n^2 4^{-n}$$

and recovers (39). Lower order terms differ due to the numbering of the grids and in fact they were not estimated separately from (48) onwards.

More interesting, however, is the dependence of (44) on the dimension $d$ for large $d$. Keeping $n$ fixed, one sees from Stirling’s formula,

$$k! \sim \sqrt{2\pi k} \left(\frac{k}{e}\right)^k,$$

that the factor depending explicitly on $d$ grows asymptotically like $(5e)^d / \sqrt{d}$. It is an interesting and practically very relevant question, whether the bounds in (44) are sharp asymptotically or the exponentially growing constants can be omitted by subtle treatment of the error terms.

4.2 Improved error bounds and exact asymptotics

The questions raised at the end of the previous section are addressed in the following corollaries.

**Corollary 4.5** (to Theorem 4.4). Under the assumptions of Theorem 4.4 the sharper bound

$$\|u - u_n\| \leq 2K \left(\frac{2^p + 1}{2^{p-1}}\right)^{d-1} \left(1 + \frac{n + (d - 1) + \ln(d - 1)}{d - 1}\right)^{d-1} 2^{-pn}$$

holds for $d \geq 2$.

Proof. See Appendix B.

**Corollary 4.6** (to Theorem 4.4). If additionally $v_1, \ldots, v_d$ in (36) is continuous in $0 \in \mathbb{R}^d$ with $\bar{v} := v_1, \ldots, v_d(:0, \ldots, 0) \neq 0$, then the asymptotic behaviour is

$$u - u_n = \bar{v} \left(\frac{2^p - 1}{2^p}\right)^{d-1} \frac{n^{d-1}}{(d-1)!} 2^{-pn} + O\left(n^{d-2} 2^{-pn}\right).$$

(50)

Proof. See Appendix B.

Remark. In the case $\bar{v} = 0$ the leading term proportional to $n^{d-1} 2^{-pn}$ vanishes. This is for example the case if the solution has a low superposition dimension, i.e. is a sum of functions that depend only on a subset of the coordinates.

Remark. (50) shows that for fixed $n$ the coefficients of $2^{-pn}$ tend to 0 for $d \to \infty$. This does not mean, however, that the approximation is better in higher dimensions. On the contrary, this asymptotic approximation was achieved under the assumption that $n$ is large compared to $d$ and the asymptotic range is reached for increasing $n$ in higher dimensions. This behaviour is also related to the fact that the number of levels contained in the combination solution grows linearly in $d$.

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4.3 Sparse grid error bounds for model problems

Finally we can collect the results from the previous sections to obtain error bounds and asymptotic error formulae with explicitly computable constants.

4.3.1 Poisson equation

We combine the results from Theorems 4.4 and 3.3 to obtain the following.

**Theorem 4.7.** Let \( u \in X_d^4 \) be a solution of the Poisson problem and \( u_n \) the sparse grid solution on level \( n \). Then

\[
|u - u_n| \leq c \cdot \sup_{|\alpha| \leq 4} \|D^\alpha u\|_\infty \cdot d \cdot \left( \frac{5}{2} \right)^d \cdot (n + 2(d - 1))^{d-1} \cdot 4^{-n}
\]

with \( c \leq 121000 \) and

\[
u - u_n = \tilde{c} \cdot \frac{d}{4^d} \cdot n^{d-1} \cdot 4^{-n} + O(n^{d-2}4^{-n}),
\]

where \( \tilde{c} \leq c \|D^\alpha u\|_\infty \) with \( \alpha = (4, \ldots, 4) \).

The dependence on dimensionality and smoothness becomes apparent.

4.3.2 Advection equation

A similar result follows for the advection equation from Theorems 4.4 and 3.4.

**Theorem 4.8.** Let \( u \in X_d^2 \) be a solution of the advection equation and \( u_n \) the sparse grid solution on level \( n \). Then

\[
|u - u_n| \leq c \cdot \sup_{|\alpha| \leq 2} \|D^\alpha u\|_\infty \cdot d \cdot \left( \frac{3}{2} \right)^d \cdot (n + 2(d - 1))^{d-1} \cdot 2^{-n}
\]

with \( c \leq 2 \) and

\[
u - u_n = \tilde{c} \cdot \frac{d}{4^d} \cdot n^{d-1} \cdot 2^{-n} + O(n^{d-2}2^{-n}),
\]

where \( \tilde{c} \leq c \|D^\alpha u\|_\infty \) with \( \alpha = (2, \ldots, 2) \).

5 Numerical results

We illustrate the theoretical findings by numerical experiments, and pay particular attention to the asymptotic convergence order, the dependence of the error on the dimension and on the smoothness of the solution, as reflected in (51) and (53).

5.1 Elliptic problems

Consider central differences for

\[
\begin{align*}
\Delta u &= f & \text{in } [0,1]^d \\
u &= g & \text{on } \partial[0,1]^d.
\end{align*}
\]

We choose the data such that the solution is

\[
u(x) = \exp \left( -\frac{1}{2} \sum_{i=1}^d \lambda_i (x_i - p_i)^2 \right)
\]

(56)
\( \lambda_i \geq 0 \) and \( p \in [0,1]^d \), that is \( f(x) = \sum_{i=1}^d \lambda_i \left( -1 + \lambda_i y_i^2 \right) \cdot u(x) \). This allows us to control all derivatives in the light of (51). In particular

\[
\| D^{2\alpha} u \|_\infty = |D^{2\alpha} u(p)| = \prod_{i=1}^d \lambda_i^{\alpha_i}.
\]

We choose \( p_1 = 0.22081976 \), \( p_2 = 0.29072005 \), \( p_3 = 0.28051979 \), \( p_4 = 0.27032006 \), \( p_5 = 0.24122005 \), \( p_6 = 0.17071947 \), \( p_7 = 0.10101947 \), \( p_8 = 0.09021981 \) to avoid symmetry effects.

Another obvious choice for a suitable test case would be a combination of trigonometric functions. We omit such an example as it basically just reproduces the Fourier analysis of sparse grid solutions (see [3]) numerically.

### 5.1.1 Convergence order and dimensionality

We start by looking at the case (56) with \( d \geq 1 \) and \( \lambda_i = 1, i = 1, \ldots, d \). Then from above

\[
\max_{|\alpha| \leq 4} \| D^\alpha u \|_\infty = 1 \text{ for all } d.
\]

Figure 2 shows a logarithmic plot of the error

\[
\epsilon_n := |u(x^*) - u_n(x^*)|
\]

for the sparse grid solution \( u_n \) at level \( n \), evaluated at a fixed point \( x^* \). Here \( x^* = (1/2, \ldots, 1/2) \), the centre point.

From the result (52) we know that

\[
\epsilon_n = O(n^q 2^{-pn})
\]

with \( q = d - 1 \) and \( p = 2 \). We therefore fit the function

\[
l(n, p, q, r) = -r + q \log_2 n - pn
\]

(58) to \( \log_2 \epsilon_n \) and determine \( r, p \) and \( q \) by least squares. The results shown in Table 1 correspond very well to the theoretical values. Note that it is very hard to estimate the exponent of the logarithm properly.

We perform the same exercise for the error versus the number of grid points \( N \) and fit

\[
l(N, \tilde{p}, \tilde{q}, \tilde{r}) = -\tilde{r} + \tilde{q} \log_2 N - \tilde{p}N
\]

(59)
Figure 3: A similar plot to Fig. 2 but comparing the sparse grid in dimensions 1, 3 and 5 (upper graphs in the left plot and lower graphs on the right) to the full grid. The full grid error on a given level does not depend significantly on the dimension (three lines on top of each other), whereas the sparse grid requires more refinements to achieve the same accuracy. The right plot, however, shows the superior complexity in terms of error reduction per unknown.

Table 1: Coefficients as in (58), fitted to the computed errors by regression. In parenthesis see the values from the theory. The corresponding curves are plotted with the data in Fig. 2.

| $d$ | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $p$ | 2.000 (2) | 1.938 (2) | 1.905 (2) | 1.970 (2) | 1.871 (2) | 1.901 (2) | 1.944 (2) | 1.982 (2) |
| $q$ | -0.001 (0) | 0.478 (1) | 1.44 (2) | 2.86 (3) | 3.42 (4) | 4.75 (5) | 6.24 (6) | 7.76 (7) |
| $r$ | 11.43 | 3.97 | 3.97 | 5.31 | 6.04 | 8.42 | 11.61 | 15.28 |

Table 2: Coefficients as in (59), fitted to the computed errors by regression.

| $d$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----|---|---|---|---|---|---|---|---|
| $\tilde{p}$ | 2.000 | -1.889 | 1.815 | 1.846 | 1.715 | 1.702 | 1.689 | 1.668 |
| $\tilde{q}$ | -0.002 | 2.23 | 5.08 | 8.74 | 10.6 | 13.7 | 16.7 | 19.6 |
| $\tilde{r}$ | 2.43 | 4.75 | 9.48 | 18.5 | 24.9 | 35.5 | 47.1 | 59.2 |

5.1.2 Dependence on derivatives

It remains to study the effect of smoothness and (an-)isotropy on the convergence. We consider the three-dimensional case and vary $\lambda_1$, $\lambda_2$ and $\lambda_3$. The leading error term is proportional to

$$\max_{x \in [0,1]^d} \frac{\partial^{12} u}{\partial x_1^4 \partial x_2^4 \partial x_3^4}(x) = \lambda_1^2 \lambda_2^2 \lambda_3^2$$

The evaluation point $x^*$ is chosen equal to the point $p$. Fig. 4 shows how the error increases with increasing (mixed) derivatives.

5.2 Parabolic and hyperbolic problems

We consider

$$u_t - \nu \Delta u + b \cdot \nabla u = 0$$
in two dimensions for different values of \( \nu \). The case \( \nu = 0 \) resembles the advection equation.

To adjust the smoothness, we choose an initial profile

\[
  u(x_1, x_2, 0) = \begin{cases} 
    \arctan \left( \tan^k \left( \frac{\pi}{2} \frac{|x - \tau|}{\epsilon} \right) \right) & r < \tau \\
    1 & \tau \leq r \leq \tau \\
    0 & r > \tau 
  \end{cases}
\]

where

\[
  r = \sqrt{(x_1 - m_1)^2 + (x_2 - m_2)^2},
\]

\( \tau = (1 - \epsilon)\tau \) for \( \epsilon \in [0, 1] \) and \( k \geq 1 \) (see Fig. 5, left, for \( \epsilon = 0.9 \)).

The value is 1 in an inner circle with centre \((m_1, m_2)\) and radius \( \tau \), and changes to 0 within a distance of \( \epsilon \). The regions are joined together such that \( k - 1 \) is the order of differentiability for \( \epsilon > 0 \). \( \epsilon = 0 \) is the discontinuous limit. We choose \( k = 5 \), because from the analysis it is known that mixed derivatives of order up to 4 are required for upwinding on a sparse grid. Homogeneous Dirichlet conditions \( u = 0 \) are set at the boundary. To avoid any effects arising from the velocity being aligned with the grid, we choose \( b_1 = 0.31415926535897932385 \), \( b_2 = -0.27182818284590452354 \), and furthermore set \( m_1 = 0.5(1 - b_1), m_2 = 0.5(1 - b_2) \) and \( \tau = 0.5|b|_2 \), such that in the non-diffusive case the profile starts from the upper left quarter of the unit square, with the outer circle going through the centre (Fig. 5, left, for \( \epsilon = 0.9 \)), and is propagated down to the lower right quarter, touching the centre from the other side at \( T = 1 \). We evaluate the solution at the centre \( x_*= (1/2, \ldots, 1/2) \) and since the exact solution is unknown, we study the surplus

\[
  \hat{\epsilon}_n = |u_{n+1}(x_*) - u_n(x_*)|.
\]  

(60)

We first consider the case \( \nu = 0.1, \epsilon = 0.9 \). The numerical solution is shown in Fig. 5. First order convergence is observed as illustrated by Fig. 6.

We now take \( \epsilon = 0 \) and change \( \nu = 0.1, 0.01, 0.001 \). (Fig. 7, left). Due to the discontinuous initial condition, the problem does not fit into the theoretical framework. Nonetheless the smoothing property of the diffusion operator suffices to provide sufficient regularity for \( t > 0 \).

Alternatively, taking \( \nu = 0 \), we let the smooth transition collapse to \( \epsilon = 0.1, 0.01, 0.001 \) (Fig. 7, right).
Figure 5: Sparse grid solution at level $n = 15$ for $\nu = 0.1, \epsilon = 0.9$ at $t = 0$ (left) and $t = 1$ (right).

Figure 6: $\hat{\epsilon}_n$ as in (60) and convergence rate $r_n$ at refinement level $n$ for $\nu = 0.1, \epsilon = 0.9$.

Figure 7: Convergence of the sparse grid solution for discontinuous initial data for small diffusivity (left) and smooth initial data without diffusion (right) for with $\hat{\epsilon}_n$ from (60).

Although the problem is sufficiently smooth, the scales involved are not resolved properly and the
Figure 8: Sparse grid solution at level $n = 15$ for $\nu = 0, \epsilon = 0$ at $t = 0$ (left) and $t = 1$ (right).

Figure 9: $\hat{\epsilon}_n$ as in (60) and convergence rate $r_n$ at refinement level $n$ for discontinuous initial data without diffusion.

| $n$ | $r_n = \epsilon_n / \epsilon_{n+1}$ |
|-----|-----------------------------------|
| 4   | 2.0381                           |
| 5   | 1.0974                           |
| 6   | 0.85633                          |
| 7   | -0.61664                         |
| 8   | 1.7166                           |
| 9   | -6.8473                          |
| 10  | 0.29471                          |
| 11  | 0.87042                          |
| 12  | 1.9361                           |
| 13  | 2.4390                           |
| 14  | -5.2997                          |
| 15  | -0.64137                         |
| 16  | -0.43160                         |
| 17  | 0.38301                          |
| 18  | 1.58094                          |

(mixed) derivatives are too large to reach the asymptotic range at feasible refinement levels.

Finally, consider the limiting case $\nu = 0, \epsilon = 0$, i.e., a discontinuous initial profile without diffusion.

Already in Fig. 8 the problems of the sparse grid become apparent. It is noticeable how the anisotropic elements in the sparse grid fail to capture the discontinuous transition at the circumference of the circle. Fig. 9 confirms that convergence (if any) is slow and erratic.

6 Discussion

In this article we give explicit error bounds for the sparse grid solution to model problems, and at the same time provide a framework that can be applied to more general linear PDEs. The ingredients are

1. sufficiently smooth data to yield solutions with bounded mixed derivatives of required order,
2. a discretisation scheme that provides a truncation error of mixed order,

3. stability of the discretisation scheme, that is a bounded inverse in the maximum norm.

In most cases the truncation error can be assessed easily by Taylor expansion. The boundedness of the inverse will be harder to establish, but note that no additional requirements to the corresponding full grid case are necessary here. Examples that fall into this category (where discrete maximum principles are known) are elliptic and parabolic equations. The numerical results reproduce the theoretical findings nicely.

Limitations we encountered concern non-smooth problems, dimensions in excess of eight, and non-diagonal diffusion tensors. The latter can often be resolved in practice by diagonalising the diffusion tensor and performing a principal component or asymptotic analysis, see e. g. [12].

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A Proofs from Section 3

Proof of Lemma 3.1 Define

\[ v := u(x_1, \ldots, x_d) - u(0, \ldots, 0) - \sum_{i=1}^{d} x_i \partial_i u(x - x^{(i)}) - \sum_{i, j = 1 \atop i \neq j}^{d} x_i x_j \partial_i \partial_j u(x - x^{(i,j)}) - \ldots - \]

\[ - x_1 \cdot \ldots \cdot x_d \partial_1 \ldots \partial_d u|_{x_1, \ldots, x_d = 0} - \sum_{i=1}^{d} \int_{0}^{x_i} (x_i - s_i) \partial_i^2 u(s^{(i)}) \, ds_i - \ldots - \]

\[ - \sum_{\{i_1, \ldots, i_{d-1}\} \subset \{1, \ldots, d\}} \int_{0}^{x_{i_1}} \ldots \int_{0}^{x_{i_{d-1}}} (x_{i_1} - s_{i_1}) \cdot \ldots \cdot (x_{i_{d-1}} - s_{i_{d-1}}) \partial_{i_1}^2 \ldots \partial_{i_{d-1}}^2 u(s^{(i_1, \ldots, i_{d-1})}) \, ds_{i_1} \ldots ds_{i_{d-1}}. \]

It is straightforward to see that

\[ v(x_1, \ldots, x_d) = 0 \quad \text{if} \quad \exists k : x_k = 0 \]

and also

\[ D^{\alpha} v(x_1, \ldots, x_d) = 0 \quad \text{if} \quad \exists k : x_k = 0 \wedge \alpha_k = 1. \]

From

\[ \partial_1^2 \ldots \partial_d^2 v = 0 \]

one gets \( v = 0 \) from which the result follows. \( \square \)

Proof of Theorem 3.2 Without loss of generality consider a point \( x \) in the box \( B := [0, h_1] \times \ldots \times [0, h_d] \). For all corner points \( p = (p_1, \ldots, p_d) \in B_c := \times_{i=1}^{d} \{0, h_i\} \) use Lemma 3.1 to express

\[ u(p) = u(x) + \sum_{i=1}^{d} (p_i - x_i) \partial_i u(x^{(i)}) + \sum_{i, j = 1 \atop i \neq j}^{d} (p_i - x_i)(p_j - x_j) \partial_i \partial_j u(x^{(i,j)}) + \ldots + \]

\[ + (p_1 - x_1) \cdot \ldots \cdot (p_d - x_d) \partial_1 \ldots \partial_d u(x) + \sum_{i=1}^{d} \int_{x_i}^{p_i} (p_i - s_i) \partial_i^2 u(s^{(i)}) \, ds_i + \ldots + \]

\[ + \int_{x_1}^{p_1} \ldots \int_{x_d}^{p_d} (p_1 - s_1) \cdot \ldots \cdot (p_d - s_d) \partial_1^2 \ldots \partial_d^2 u(s) \, ds_d \ldots ds_1. \]

(61)

Here \( x = (x_1, \ldots, x_d) \), and introduce \( x^{(i_1, \ldots, i_k)} \), \( s^{(i_1, \ldots, i_k)} \) with \( x_j^{(i_1, \ldots, i_k)} = x_j \) if \( j \notin \{i_1, \ldots, i_k\} \), \( p_j \) otherwise, and \( s_j^{(i_1, \ldots, i_k)} = s_j \) if \( j \in \{i_1, \ldots, i_k\} \), \( x_j \) otherwise.

Then insert in the multilinear approximation

\[ (\mathcal{I} u(x_h))(x) = \sum_{p \in B_c} \frac{|p_1 - x_1|}{h_1} \cdot \ldots \cdot \frac{|p_d - x_d|}{h_d} \cdot u(p) = u(x) + \sum_{m=1}^{d} \sum_{\{i_1, \ldots, i_m\} \subset \{1, \ldots, d\}} \sum_{p \in B_c} w_{i_1, \ldots, i_m}(p, x) \]

with

\[ w_{i_1, \ldots, i_m}(p, x) = \left( \prod_{k=1}^{d} \frac{|p_k - x_k|}{h_k} \right) \int_{x_{i_1}}^{p_{i_1}} \ldots \int_{x_{i_m}}^{p_{i_m}} \left( \prod_{k=1}^{m} (p_{i_k} - s_{i_k}) \right) \partial_{i_1}^2 \ldots \partial_{i_m}^2 u(s^{(i_1, \ldots, i_m)}) \, ds_{i_m} \ldots ds_{i_1}, \]

because all the terms in (61) that are multilinear are exactly represented. By inserting the points \( p \in B_c \) one shows after some calculation the representation

\[ \alpha_{j_1, \ldots, j_m}(x; h_{j_1}, \ldots, h_{j_m}) = \prod_{k=1}^{m} \frac{x_{j_k}}{h_{j_k}} \left( 1 - \frac{x_{j_k}}{h_{j_k}} \right) \sum_{y \cdot z = 1}^{m} \prod_{k=1}^{m} \left( \frac{x_{j_k}}{h_{j_k}} \right)^{y_k} \left( 1 - \frac{x_{j_k}}{h_{j_k}} \right)^{z_k} u_{y,z}(x; h) \]

(62)
with \( h = (h_{i_1}, \ldots, h_{i_m}) \) and

\[
w_{x}(x; h) = \frac{1}{\prod_{k=1}^{d} 2^{y_k} (h_{j_k} - x_j)^2 z_k} \int_{x_1}^{z_{k_1}} \cdots \int_{x_m}^{z_{k_m}} \left( \prod_{k=1}^{m} (z_k h_{j_k} - s_k) \right) \partial^2_{y_1} \cdots \partial^2_{y_m} u(s_{j_1}, \ldots, j_m) \, ds_m \cdots ds_1.
\]

The rest follows because the maximum of \( \xi^2 (1 - \xi) \) on \([0, 1]\) is \( \frac{1}{27} \).

**Proof of Theorem 3.4.** From Corollary 2.2, \( w_{i_1 \ldots, i_k} (x; h_{i_1}, \ldots, h_{i_k}) \) (in 3.3) is the restriction of a function \( w_{i_1 \ldots, i_k} \) from hyperplanes \( \Gamma_h \) to the grid \( x_h \), where for the derivatives in the continuous directions

\[
\| D^\alpha w_{i_1 \ldots, i_k} (\cdot; h_{i_1}, \ldots, h_{i_k}) \|_0 \leq \frac{k!}{48^k} \| D^\theta u \|_\infty
\]

with \( \alpha_i = 4, i \notin \{i_1, \ldots, i_k\} \beta_i = 4 \) holds.

By Theorem 3.2 for points \( x \) on the hyper-planes \( \Gamma_h \)

\[
(\mathcal{I} w_{i_1 \ldots, i_k} (x; h_{i_1}, \ldots, h_{i_k}))(x) = w_{i_1 \ldots, i_k} (x; h_{i_1}, \ldots, h_{i_k}) + \sum_{\{j_1, \ldots, j_m\} \cap \{i_1, \ldots, i_k\} = \emptyset} \gamma_{i_1, \ldots, i_k; j_1, \ldots, j_m} (x, h_{i_1}, \ldots, h_{i_k}; h_{j_1}, \ldots, h_{j_m}) h_{j_1}^2 \cdots h_{j_m}^2
\]

(63)

with \( |\gamma_{i_1, \ldots, i_k; j_1, \ldots, j_m} (x; h_{i_1}, \ldots, h_{i_k}; h_{j_1}, \ldots, h_{j_m})| \leq \frac{d^m}{27^m} \| D^\alpha w \|_\infty \) where \( \alpha_{i_1} = 4, \alpha_{j_m} = 2 \). Values between the hyperplanes are obtained by multilinear interpolation, which introduces no new maxima, and thus

\[
\mathcal{I} (u(x_h) - u_h) = \sum_{\{i_1, \ldots, i_k\} \subset \{1, \ldots, k\}} \beta_{i_1, \ldots, i_k} (\cdot; h_{i_1}, \ldots, h_{i_k}) h_{i_1}^2 \cdots h_{i_k}^2 + \sum_{\{j_1, \ldots, j_m\} \cap \{i_1, \ldots, i_k\} = \emptyset} \beta_{i_1, \ldots, i_k; j_1, \ldots, j_m} (\cdot; h_{i_1}, \ldots, h_{i_k}; h_{j_1}, \ldots, h_{j_m}) h_{j_1}^2 \cdots h_{j_m}^2
\]

(64)

with \( \|\beta_{i_1, \ldots, i_k; j_1, \ldots, j_m} (\cdot; h_{i_1}, \ldots, h_{i_k}; h_{j_1}, \ldots, h_{j_m})\|_\infty \leq K \frac{k!}{96^m} \frac{4^m}{27^m} \).

Putting (63) and (64) together one gets

\[
u - \mathcal{I} u_h = \sum_{m=1}^{d} \sum_{\{j_1, \ldots, j_m\} \subset \{1, \ldots, d\}} v_{j_1, \ldots, j_m} (\cdot; h_{j_1}, \ldots, h_{j_m}) h_{j_1}^2 \cdots h_{j_m}^2
\]

with

\[
\|v_{j_1, \ldots, j_m} (\cdot; h_{j_1}, \ldots, h_{j_m})\|_\infty \leq K \frac{4^m}{27^m} + K \frac{m!}{96^m} + K \sum_{l=1}^{m-1} \left( \begin{array}{c} m \\ l \\ \end{array} \right) \left( \frac{4}{27} \right)^l \frac{(m-l)!}{96^{m-l}} = K \frac{m!}{96^m} \sum_{l=0}^{m} \left( \begin{array}{c} m \\ l \\ \end{array} \right) \left( \frac{384}{27} \right)^l < K \frac{m!}{96^m} e^{384/27} < 150188 K \frac{m!}{96^m} \]

\( \square \)

**Proof of Theorem 3.4.** Follows by combining Theorems 2.3 and 3.2 similarly to the Poisson case by observing that one can write second order terms (from the bilinear interpolation) as first order terms by defining

\[
\hat{\beta}_{i_1, \ldots, i_k} (\cdot; h_{i_1}, \ldots, h_{i_k}) h_{i_1}^2 \cdots h_{i_k}^2 =: \hat{\beta}_{i_1, \ldots, i_k} (\cdot; h_{i_1}, \ldots, h_{i_k}) h_{i_1} \cdots h_{i_k}
\]
Therefore it follows from Proposition 4.1 that

\[ \|v_{j_1, \ldots, j_m}(\cdot; h_{j_1}, \ldots, h_{j_m})\|_\infty \leq K \frac{m!}{2^n} \sum_{l=0}^m \frac{1}{l!} \left( \frac{8}{2^l} \right) \leq K \frac{m!}{2^n} e^{8/27} \leq \frac{3}{2} K \frac{m!}{2^n}. \]

\square

### B Proofs from Section 4

**Proof of Lemma 4.3** Inserting gives

\[ F(n) = \sum_{|i|=n} v(2^{-i_1}, \ldots, 2^{-i_m}) 2^{-p_{i_1}} \ldots 2^{-p_{i_m}} = \]

\[ = \sum_{|i|=n} v(2^{-i_1}, \ldots, 2^{-i_m}) 2^{-p \sum_{k=1}^m i_k} \]

\[ = \sum_{l=0}^n \sum_{\sum_{k=1}^m i_k = l} v(2^{-i_1}, \ldots, 2^{-i_m}) 2^{-pl} \sum_{k=m+1}^{l=n} 1 \]

\[ = \sum_{l=0}^n s_l 2^{-pl} \left( \frac{n-l + d - m - 1}{d - m - 1} \right). \]

From Lemma B.1 below, (\( d \to d - m, f_l \to s_l 2^{-pl} \)) one obtains

\[ \delta_1^{d-m} F(n) = \delta_1^{d-m} \sum_{|i|=n} v(2^{-i_1}, \ldots, 2^{-i_m}) 2^{-p_{i_1}} \ldots 2^{-p_{i_m}} = s_{n+d-m} 2^{-p(n+d-m)}. \] (65)

For all \( j \geq 0 \)

\[ \delta_1^j 2^{-pn} = (2^{-p} - 1)^j 2^{-pn}, \]

as one sees inductively \((j \to j + 1)\) by

\[ \delta_1^{j+1} 2^{-pn} = (2^{-p} - 1)^j \left( 2^{-p(n+1)} - 2^{-pn} \right) = (2^{-p} - 1)^{j+1} 2^{-pn}. \]

Therefore it follows from Proposition 4.1 that

\[ \delta_1^{m-1} \left[ s_{n+d-m} 2^{-p(n+d-m)} \right] = \]

\[ = \sum_{j=0}^{m-1} \binom{m-1}{j} \delta_1^j s_{n+d-m} \delta_1^{m-1-j} 2^{-p(n+d-m+j)} \]

\[ = \sum_{j=0}^{m-1} \binom{m-1}{j} \delta_1^j s_{n+d-m} (2^{-p} - 1)^{m-1-j} 2^{-p(n+d-m+j)} \]

\[ = 2^{-p(n+d-1)} \sum_{j=0}^{m-1} \binom{m-1}{j} (1 - 2^p)^{m-1-j} \delta_1^j s_{n+d-m} \]

\[ = 2^{-p(n+d-1)} (\delta_1 - 2p + 1)^{m-1} s_{n+d-m} \]

\[ = 2^{-p(n+d-1)} \sum_{i=0}^{m-1} s_{n+d-m+i} \binom{m-1}{i} (-2)^{p(m-1-i)} \]

\[ = 2^{-p(n+d-1)} \sum_{i=0}^{m-1} s_{n+d-i-1} \binom{m-1}{i} (-2)^{pi}. \]

\square
Lemma B.1 (Differencing formula). Let $d \in \mathbb{N}$ and $f, F \in \mathbb{R}^{\mathbb{N}_0}$ s. t.

$$F(n) = \sum_{l=0}^{n} f_l \left( \frac{n - l + d - 1}{d - 1} \right).$$

1. Then for $0 \leq k < d$

$$\delta_1^k F(n) = G^k(n) + H^k(n),$$

where

$$G^k(n) := \begin{cases} 0 & k = 0 \\ \sum_{j=1}^{k} f_{n+j} \left( \frac{d - j - 1}{k - j} \right) & k \geq 1 \end{cases},$$

$$H^k(n) := \sum_{l=0}^{n} f_l \left( \frac{n - l + d - 1}{d - k - 1} \right).$$

Proof. 1. Induction in $k$: Clearly $(k = 0)$

$$\delta_1^0 F(n) = F(n) = H^0(n) = G^0(n) + H^0(n).$$

Since $F(n)$ has the form $F(n) = \sum_{l=0}^{n} f(l, n),$

$$\delta_1 F(n) = f(n + 1, n + 1) - \sum_{l=0}^{n} (f(l, n + 1) - f(l, n)),$$

hence

$$\delta_1 F(n) = f_{n+1} + \sum_{l=0}^{n} f_l \left[ \left( \frac{n - l + d}{d - 1} \right) - \left( \frac{n - l + d - 1}{d - 1} \right) \right]$$

$$= f_{n+1} + \sum_{l=0}^{n} f_l \left( \frac{n - l + d - 1}{d} \right)$$

$$= G^1(n) + H^1(n),$$

i. e. the case $k = 1$.

For $k > 1$ the contributions $(k \to k + 1)$ are

$$\delta_1 G^k(n) = \sum_{j=1}^{k} (f_{n+j+1} - f_{n+j}) \left( \frac{d - j - 1}{k - j} \right)$$

$$= f_{n+k+1} - f_{n+1} \left( \frac{d - 2}{k - 1} \right) + \sum_{j=2}^{k} f_{n+j} \left[ \left( \frac{d - j}{k - j + 1} \right) - \left( \frac{d - j - 1}{k - j} \right) \right]$$

$$= \sum_{j=2}^{k+1} f_{n+j} \left( \frac{d - j - 1}{k - j + 1} \right) - f_{n+1} \left( \frac{d - 2}{k - 1} \right)$$

$$= G^{k+1}(n) - f_{n+1} \left( \frac{d - 2}{k} \right) - f_{n+1} \left( \frac{d - 2}{k - 1} \right)$$

$$= G^{k+1}(n) - f_{n+1} \left( \frac{d - 1}{k} \right).$$
and in $H^k(n)$

$$
\delta_1 H^k(n) = f_{n+1} \left( \frac{d-1}{d-k-1} \right) + \sum_{l=0}^{n} f_l \left[ \left( \frac{n-l+d}{d-k-1} \right) - \left( \frac{n-l+d-1}{d-k-1} \right) \right]
$$

$$
= f_{n+1} \left( \frac{d-1}{k} \right) + \sum_{l=0}^{n} f_l \left( \frac{n-l+d-1}{d-k-2} \right)
$$

$$
= H^{k+1}(n) + f_{n+1} \left( \frac{d-1}{k} \right).
$$

This completes the result as

$$
\delta^{k+1}_1 F(n) = \delta_1 \delta^{k}_1 F(n) = \delta_1 (G^k(n) + H^k(n)) = \delta_1 G^k(n) + \delta_1 H^k(n)
$$

$$
= G^{k+1}(n) + H^{k+1}(n).
$$

2. From [1] one obtains for $k = d - 1$

$$
\delta^{d-1}_1 F(n) = \sum_{j=1}^{d-1} f_{n+j} + \sum_{l=0}^{n} f_l
$$

and

$$
\delta^{d}_1 F(n) = \sum_{j=1}^{d-1} (f_{n+j+1} - f_{n+j}) + f_{n+1} = f_{n+d}.
$$

Proof of Lemma 4.5 A binomial of the form (47) or (48), respectively, can be written as ($l = n + d - 1$, $k = m - 1$)

$$
\binom{l+k}{k} = \frac{\prod_{j=1}^{k} (l+j)}{\prod_{j=1}^{k} j} = \prod_{j=1}^{k} \left( 1 + \frac{l}{j} \right).
$$

From the inequality between the arithmetic and geometric mean one gets

$$
\sqrt[k]{\prod_{j=1}^{k} \left( 1 + \frac{l}{j} \right)} < \frac{1}{k} \left[ k + l \left( \sum_{j=1}^{k} \frac{1}{j} \right) \right],
$$

furthermore

$$
\sum_{j=1}^{k} \frac{1}{j} < 1 + \int_{1}^{k} \frac{dx}{x} = 1 + \ln k,
$$

because the first sum is a lower sum for the integral. This gives

$$
\binom{l+k}{k} < \left[ 1 + \frac{1 + \ln k}{k} \right]^k
$$

and

$$
\binom{n+d+m-2}{m-1} < \left[ 1 + (n + d - 1) \frac{1 + \ln(m-1)}{m-1} \right]^{m-1}.
$$

The rest follows as in the proof of Theorem 4.4.
Proof of Lemma 4.6. For continuous $v_1, \ldots, v_m$ we can first show
\[
\lim_{n \to \infty} \frac{s_n}{n + m - 1} = v_1, \ldots, v_m(0, \ldots, 0) =: v_0.
\] (66)

In the following the index of $v_1, \ldots, v_m$ is omitted for simplicity of notation. Let $\epsilon > 0$ and
\[
\epsilon := \sup_{0 \leq h_1, \ldots, h_m \leq 1} |v(h_1, \ldots, h_m) - v_0|.
\]
Choose $n_0$ such that
\[
\forall k \geq n_0, 1 \leq k \leq m : |v(2^{-i_1}, \ldots, 2^{-i_m}) - v_0| \leq \frac{\epsilon}{2}
\]
and then $n$ sufficiently large such that
\[
N_0 := \left( n - kn_0 + m - 1 \right) / \left( n + m - 1 \right) \geq \left( 1 - \frac{\epsilon}{c} \right) \left( n + m - 1 \right) =: N
\]
The latter is possible, because
\[
\left( n - kn_0 + m - 1 \right) / \left( n + m - 1 \right) = \prod_{j=1}^{m-1} \left( 1 - \frac{kn_0}{n + j} \right) \to 1
\]
for $n \to \infty$. The idea is now to show that the contribution of the terms that do not lie in an $\epsilon$-ball around $v_0$ can be neglected. This motivates the splitting
\[
s_n = \sum_{\sum_{k} i_k = n} v(2^{-i_1}, \ldots, 2^{-i_m}) + \sum_{\sum_{k} i_k = n} v(2^{-i_1}, \ldots, 2^{-i_m}).
\]
Because of
\[
\sum_{\sum_{k} i_k = n} 1 = \sum_{\sum_{k} i_k = n - kn_0} 1 = N_0
\]
one sees
\[
s_n - v_0 N = \left[ \sum_{\sum_{k} i_k = n} + \sum_{\sum_{k} i_k = n - kn_0} \right] (v(2^{-i_1}, \ldots, 2^{-i_m}) - v_0)
\]
and consequently
\[
|s_n - v_0 N| \leq \frac{\epsilon}{2} N + c(N - N_0) \leq \frac{\epsilon}{2} N + c \frac{\epsilon}{2 \epsilon} N = \epsilon N.
\]
Division by $N$ leads to (66).
Asymptotically one gets instead of (66) for $v_0 \neq 0$
\[
\sum_{i=0}^{m-1} s_{n+d-i-1} \left( \begin{array}{c} m - 1 \\ i \end{array} \right) (-2)^{pi} \sim v_0 N \sum_{i=0}^{m-1} \left( \begin{array}{c} m - 1 \\ i \end{array} \right) (-2)^{p(m-1-i)} = v_0 \left( \begin{array}{c} n + m - 1 \\ m - 1 \end{array} \right) (1 - 2^p)^{m-1},
\]
in other words $\forall \epsilon > 0 \ \exists N \ \forall n \geq N$
\[
\left| \sum_{i=0}^{m-1} s_{n+d-i-1} \left( \begin{array}{c} m - 1 \\ i \end{array} \right) (-2)^{pi} \right| \leq v_0 \left( \begin{array}{c} n + m - 1 \\ m - 1 \end{array} \right) (2^p - 1)^{m-1}(1 + \epsilon).
\]
The rest follows as in Theorem 4.4. \qed