Many-body quantum metrology with scalar bosons in a single potential well

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We theoretically explore the possibility of a precise measurement of acceleration by using an ultracold gas of scalar bosons in a single potential well. Inspired by the fact that the ground state represents a fragmented condensate for sufficiently large interaction, we develop a protocol to realize a Mach-Zehnder interferometer for the estimation of an acceleration of the system. The splitting process into two modes is in our model entirely caused by the interaction, and is neither due to an externally imposed double-well potential nor due to populating a spinor degree of freedom. We investigate numerically the useful range of parameters for the quantum Fisher information (QFI), both when maximized over the set of initial states and when starting the protocol with the fragmented ground state. It is then demonstrated, both for a fragmented state with relatively low degree of fragmentation and for a coherent state, that the two-body interaction significantly enhances the QFI over the one of an ideal interferometer, i.e., an interferometer in which beam splitting and phase accumulation are completely separated. In particular, we show how a renormalization of the single-particle energy difference between the modes due to the two-body interaction can serve this purpose.

I. INTRODUCTION

In the last few decades quantum metrology has been applied to a wide range of physical models. Due to the probabilistic nature of quantum theory, it has been possible to apply — and extend — results from statistics to what is now known as quantum parameter estimation theory, whose aim is to investigate questions related to the estimation of a classical and deterministic parameter — say $\lambda$ — using a quantum system \([1]\). The main paradigm is, having $\nu$ copies of a $\lambda$-dependent quantum state $\rho_\lambda$ at our disposal, to find what the best possible precision is, according to quantum mechanics, for the estimation of $\lambda$. The quantum Cramér-Rao bound (QCRB) answers this question by relating the variance of any unbiased estimator $\lambda_{\text{est}}$ to the quantum Fisher information (QFI) \([2–4]\):

$$\text{Var}[\lambda_{\text{est}}] \geq \frac{1}{\nu I_\lambda}.$$  \hfill (1)

Crucially, this bound can always be saturated in the limit of large $\nu$, therefore making the QFI $I_\lambda$ a proper metrological figure of merit for the precision which can in principle be attained.

In general, a protocol for quantum metrology is four-step: The preparation of an initial state, the imprinting of the parameter during the evolution, the measurement of the final state, and eventually — after repeating these three steps $\nu$ times — the estimation of the parameter from the measurement results \([5]\). While originally developed in the field of optics, interferometric techniques have been applied to atomic systems as well. Original realizations for atomic interferometry include Young’s slit experiment \([6]\), Raman interferometers that use internal degrees of freedom \([7]\), Sagnac interferometers \([8]\), and “three-diffraction-grating” interferometers \([9]\).

An important class of interferometers is represented by the Mach-Zehnder interferometer (MZI) type. In the optical case, light is sent to one or both input ports, passes through a first beam splitter, accumulates a phase in one arm, passes through a second beam splitter, and then the measurement is performed. The crucial point that brought so much attention to quantum metrology is that, depending on the initial state, the scaling of the QFI — and thus of the precision — with the number of particles — photons for an optical interferometer, but atoms in the case we will be interested in — can be tremendously improved by designing proper quantum protocols. In a classical setup, one is constrained by the shot noise limit (SNL) scaling, which corresponds to a QFI scaling linearly with the number of particles $I_\lambda \propto N$. This scaling can be interpreted as a simple result of the central limit theorem. The key point is that using properly optimized states, one can reach in the quantum realm a quadratic scaling of the QFI with the number of particles $I_\lambda \propto N^2$, the so-called Heisenberg limit (HL) scaling \([10–13]\).

Despite the similarities, there is a crucial difference between photonic and atomic systems, namely whether interaction between the constituents exists. Indeed, interaction is naturally present in atomic systems, while it should be engineered in photonic systems [e.g. by using nonlinear (Kerr) media]. In many-body physics, the role of interaction is often witnessed in the ground state of the Hamiltonian, which can play a role in quantum metrology in at least two ways. On one hand, we can directly perform a measurement on the ground state to estimate a parameter present in the Hamiltonian \([14–18]\). This approach has been successfully applied to study phase transitions \([19, 20]\). On the other hand, we can use the ground state of a Hamiltonian as an initial state for a metrological protocol. Much attention has therefore been devoted to quantum metrology using atoms instead of light \([21–25]\), and experiments have already been performed \([26, 30]\). In particular, quantum metrology with
bosons has been thoroughly studied \cite{31,33}, with a particular
attention on the double-well case \cite{34,32}. For scalar bosons in a single trap where — contrary to the
double-well case in which fragmentation is caused by an externally imposed spatial separation — the fragmenta-
tion of the ground state is purely due to the presence of
two-body interaction \cite{33}.

In this paper, we investigate how scalar bosons in a sin-
gle trap can be used for quantum metrology. The ques-
tion of the optimal estimation of each parameter of the
Hamiltonian of the bosons was recently addressed in \cite{44}. Here we make a step towards a more realistic implementa-
tion and show how this system can be used to realize an
MZI in order to estimate an externally imposed acceler-
ation. We first note that not being able to cancel the
energy difference between the two modes deteriorates the
precision of the interferometer, as is the case for bosons
in a double well \cite{39,45}. This is best understood using
the tools of Hamiltonian parameter estimation, in par-
ticular the channel QFI, equal to the QFI maximized
over the set of initial states \cite{45,53}. However, while in
the double well we obtain a simple nonlinear term from
interactions, in our single potential well the interaction produces a renormalization of the single particle term, in
addition to nonlinear terms. These more complex rami-
fications of the two-body interaction will be shown to be
useful to improve the QFI, even over the one obtained for
a standard ideal Mach-Zehnder interferometric sequence,
in particular for states with a low (and thus physically
realistic) degree of fragmentation.

II. SCALAR BOSONS IN A SINGLE WELL AND
FRAGMENTED GROUND STATE

A. System Hamiltonian

It is well established that many-body correlations are
strongest for one-dimensional (1D) systems of interacting
particles, when compared to their higher-dimensional cousins. Our many-body figure of merit, the fragmenta-
tion degree of the ground state, see below, is therefore
generally largest in 1D \cite{51}. Therefore, to be specific as
regards our setup we consider a harmonically trapped
ultracold 1D bosonic gas, with short-ranged two-body
interactions of coupling strength $g_{1D}$. Its Hamiltonian is

$$
\hat{H}_{\text{sys}} = \frac{1}{2} \sum_{i=1}^{N} \left[ -\frac{\partial^2}{\partial x_i^2} + \omega^2 x_i^2 \right] + g_{1D} \sum_{\langle j,k \rangle} \delta(x_j - x_k),
$$

setting the Planck constant $\hbar$ and atomic mass $m$ both
to unity, and where the coordinates of atoms are denoted
by $x_i$; $\langle j,k \rangle$ denotes summation over pairs of atoms. For
reasons of convenience, we rescale the Hamiltonian by
$l^{-2}$. The length unit, $l$, which can for example be chosen
to represent a typical interparticle separation, thus de-
fines the units we employ; all quantities in what follows
are dimensionless (given in units of powers of $l$).

Introducing the bosonic field operator $\hat{\psi}(x)$ that satis-
fies the commutation relations $[\hat{\psi}(x), \hat{\psi}^\dagger(x')] = \delta(x - x')$ and $[\hat{\psi}(x), \hat{\psi}(x')] = [\hat{\psi}^\dagger(x), \hat{\psi}^\dagger(x')] = 0$ gives the field-
quantized form of $\hat{H}_{\text{sys}}$:

$$
\hat{H}_{\text{sys}} = \frac{1}{2} \int dx \hat{\psi}^\dagger(x) \left[ -\frac{\partial^2}{\partial x^2} + \omega^2 x^2 \right] \hat{\psi}(x)
+ \frac{g_{1D}}{2} \int dx \hat{\psi}^\dagger(x) \hat{\psi}(x) \hat{\psi}(x) \hat{\psi}(x). \tag{3}
$$

The weakly interacting bosonic gas, represented by the
limit of small $g_{1D}$, resides in the mean-field regime, in
which all atoms occupy a single mode $\psi_0(x)$. As $g_{1D}$
increases, additional higher modes should be included in
the field operator expansion. For very large $g_{1D}$, the gas
enters the Tonks-Girardeau regime, in which the atoms
become localized in space \cite{52}. In this extreme limit of
(infinitely) large $g_{1D}$ (or at vanishingly small densities),
the number of orbitals occupied equals the number of
particles. Here, focusing on relatively moderate values of
$g_{1D}$, cf. \cite{53}, we restrict ourselves to a simple two-mode
model to investigate the metrological properties of the
fragmented ground state introduced in \cite{33}.

Writing the two-mode truncated field operator as
$\hat{\psi}(x) = \psi_0(x) \hat{a}_0 + \psi_1(x) \hat{a}_1$ with annihilator
operators $\hat{a}_0$ and $\hat{a}_1$, we obtain \cite{33}

$$
\hat{H}_{\text{sys}} = \frac{1}{2} \sum_{i=0}^{1} \epsilon_i \hat{a}_i^\dagger \hat{a}_i + A_1 \hat{a}_0^\dagger \hat{a}_0 \hat{a}_0 \hat{a}_0 + A_2 \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1 + h.c.
+ \frac{A_3}{2} \hat{a}_0^\dagger \hat{a}_0 \hat{a}_1 \hat{a}_1 + \frac{A_4}{2} \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1 \hat{a}_1 \hat{a}_1, \tag{4}
$$

where the single-particle energies are given by $\epsilon_i =
\frac{1}{2} \int dx \psi_i^*(x) \left[ -\frac{\partial^2}{\partial x^2} + \omega^2 x^2 \right] \psi_i(x)$, and the interaction
couplings read $A_1 = V_{0000}, A_2 = V_{1111}, A_3 = V_{0011},$ and $A_4 = V_{1010} + V_{1010} + V_{1001} + V_{0110}$, where $V_{ijkl} =
g_{1D} \int dx \psi_i^*(x) \psi_j^*(x) \psi_k(x) \psi_l(x)$.

$A_1$ (resp. $A_2$) is the interaction energy of atoms in
$\psi_0(x)$ [resp. $\psi_1(x)$], $A_3$ is the
interaction energy of atoms in $\hat{\psi}_0(x)$ [resp. $\hat{\psi}_1(x)$],
and $A_4$ is the
interaction energy of atoms in $\hat{\psi}_1(x)$ [resp. $\hat{\psi}_0(x)$].

To make the physics of our model invariant with respect
to the number of particles, in the sense that single-
particle and interaction terms retain the same impor-
tance in the limit of large $N$, also cf. \cite{54}, we define the
additional parameter

$$
g = Ng_{1D}. \tag{5}
$$

In the following, our metrological protocol is designed
to estimate the value of an externally imposed acceleration which generates the Stark-type potential $\chi \sum_{i=1}^{N} x_i$ in the Hamiltonian, where $\chi$ is the force. Its field-quantized form is

$$\hat{H}_{\text{acc}} = \chi \gamma (\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_0^\dagger \hat{a}_0),$$

with $\gamma = \langle 0|\hat{x}|1 \rangle = \langle 1|\hat{x}|0 \rangle = \int dx \psi_0^*(x) x \psi_1(x)$.

There exists an important connection between a spin system and a two-mode bosonic system. Namely, the latter is formally equivalent to a spin of size $J = N/2$. In the Schwinger representation, we define the SU(2) operators as $\hat{J}_x = (\hat{a}_0^\dagger \hat{a}_1 + \hat{a}_1^\dagger \hat{a}_0)/2$, $\hat{J}_y = (\hat{a}_0^\dagger \hat{a}_1 - \hat{a}_1^\dagger \hat{a}_0)/2i$, $\hat{J}_z = (\hat{a}_0^\dagger \hat{a}_0 - \hat{a}_1^\dagger \hat{a}_1)/2$, and $\hat{J}_0 = (\hat{a}_0^\dagger \hat{a}_0 + \hat{a}_1^\dagger \hat{a}_1)/2 = \hat{N}/2$. Then, the Hamiltonian of the bosons in a single trap within the two-mode truncation is recast as

$$\hat{H}_{\text{sys}} = q \hat{J}_z + g \frac{N}{2} (\hat{J}_x^2 + \xi \hat{J}_y^2),$$

where, due to particle number conservation, we dropped terms proportional to $\hat{N}$. In terms of the parameters introduced previously in [4], we have

$$q = -\delta_1 - g \frac{(N - 1)}{N} \frac{\delta_4}{2},$$

$$\eta = \frac{\delta_4 + 2 \delta_4 - \sigma_4}{2},$$

$$\xi = \frac{\sigma_4 + 2 \sigma_4 - \delta_4}{\sigma_4 - (2 \delta_4 + \delta_4)},$$

with $\delta_1 = \epsilon_1 - \epsilon_0$, $\delta_4 = \delta_1 - \delta_2$, $\sigma_4 = \delta_1 + \delta_2$, and $\delta_4 = \delta_4/\gamma_{1D}$. Enforcing the reality of mode functions restricts the available range of each parameter: $\delta_4 \geq 0$, $\delta_4 = 4 \delta_4$, and $\sigma_4 \geq 2 \delta_4$. Note that in a double well, renormalization of the linear term is zero, $\delta_4 = 0$, and both $A_4$ and $A_4$ are exponentially small.

The relations above imply that $\eta$ and $\xi$ have opposite signs, and $\xi$ is negative or larger than unity. We assume that there are enough degrees of freedom in the mode functions such that the $A_4s$ is can be considered independent, and hence so are the parameters $\{\delta_4, \delta_4, \eta, \xi\}$. The Hamiltonian [7] is known as the anisotropic Lipkin-Meshkov-Glick (LMG) model [55]. Also, the SU(2) representation of $\hat{H}_{\text{acc}}$ in [6] is

$$\hat{H}_{\text{acc}}(\lambda) = \lambda \hat{J}_x,$$

with $\lambda = 2 \chi \gamma$. In the following $\lambda$ will be the parameter we want to estimate and is referred to as the acceleration.

Within the Schwinger representation, it is also possible to use the language of spin squeezing to study our Hamiltonian. To create a spin-squeezed state from a coherent state, Kitagawa and Ueda introduced two Hamiltonians [50] termed one-axis twisting and two-axis twisting Hamiltonians. One-axis twisting takes the typical form $\hat{J}_x^2$, while two-axis twisting takes the typical form $\hat{J}_x^2 - \hat{J}_y^2$ (the choice of axis for twisting is arbitrary). The LMG Hamiltonian [7] realizes both kinds of twisting: For $\xi = -1$, we obtain a two-axis twisting Hamiltonian, while for $\xi = 1$ and $\xi = 0$ we obtain — up to an irrelevant term proportional to the number of particles — a one-axis twisting Hamiltonian [57] [58].

### B. Fragmented ground state of the Hamiltonian

It was shown analytically (in the large $N$ limit), and confirmed numerically, that the ground state of the Hamiltonian [7] is fragmented when $A_4 > 0$ [33]. The fragmentation of a given many-body state $|\psi\rangle$ is defined by the single particle density matrix $\rho(x, x') = \langle\psi|\psi(x)\psi(x')|\psi\rangle$. For a singly condensed system, a simple Bose-Einstein condensate, $\rho(x, x')$ has only one macroscopic eigenvalue, i.e., which is $\mathcal{O}(N)$. If there is more than one such macroscopic eigenvalue, the state is a fragmented condensate. We note that this formal definition must obviously remain somewhat vague when $N$ remains large but finite, but this does not usually lead to practical difficulties [61]. The components of the single particle density matrix are represented by $\rho_{ij} = \langle\psi|a_i^\dagger a_j|\psi\rangle$. When $\lambda_0$ and $\lambda_1$ are the two eigenvalues of this matrix (in two-mode truncation), the degree of fragmentation can be defined by $\mathcal{F} = 1 - |\lambda_0 - \lambda_1|/N$.

The stability of the ground state against various perturbations was verified in [62]. It was further shown that, introducing a phase state formalism [63], the ground state can be recast as a superposition of two spin coherent states, and that the fragmentation can be detected by analyzing density-density correlations. The explicit representation of the ground state that we will use consists, for sufficiently large $N$, of the superposition of two spin-coherent states according to [62] [63] (also cf. [64])

$$|\psi_k(\theta)\rangle = \frac{1}{\sqrt{2}} \left( |N, \theta, \pi/2 \rangle + |N, \theta, 3\pi/2 \rangle \right),$$

where the spin coherent state, of phase angle $\phi$, is defined as $|N, \theta, \phi\rangle = \frac{1}{\sqrt{N}} \left[ \cos(\theta/2) \hat{a}_0^\dagger + e^{i\phi} \sin(\theta/2) \hat{a}_1^\dagger \right]^N |0, 0\rangle$ with $[0, 0]$ the vacuum state. The angle $\theta$ parametrizes the degree of fragmentation, and therefore the ground state, according to $\mathcal{F} = 2 \sin^2(\theta/2)$ for $0 \leq \theta \leq \pi/2$ or $\mathcal{F} = 2 - 2 \sin^2(\theta/2)$ for $\pi/2 \leq \theta \leq \pi$.

### III. NONINTERACTING INTERFEROMETER

#### A. Protocol for ideal interferometer

The implementation of an MZI requires at least two steps: The beam splitting part and the phase accumulation part. For scalar bosons in a double well, the two operators used are $\delta_1 \hat{J}_x$, coming from the energy offset of the wells, and $\omega \hat{J}_x$, coming from the tunneling between wells. If one wants to estimate $\delta_1$, the beam splitting
will be done by tunneling, tuned by lowering the barrier between wells, and the phase accumulation by letting the bosons evolve without tunneling. In what we call an idealMZI, the two steps can be completely decoupled—which is what happens in the conventional quantum optics setup—and the evolution operator reads $e^{-i\lambda \hat{J}_z} e^{i \pi/2 \hat{J}_x}$. Notice that in general an MZI incorporates a last beam splitter with an opposite phase to the first one ($-\pi/2$ rotation). Since we use as our metrological figure of merit the QFI, this last stage is not relevant for us. Indeed the QFI is invariant under parameter-independent unitary transformations (and decreases for more general evolutions, a property known as monotonicity [65]). For convenience, it is easier to rewrite the evolution as $e^{i \pi/2 \hat{J}_x} e^{-i \frac{\pi}{4} \hat{J}_x} e^{-i \lambda \hat{J}_z} e^{i \frac{\pi}{4} \hat{J}_x} = e^{i \frac{\pi}{2} \hat{J}_x} e^{i \lambda \hat{J}_z}$. In a fine, due to the monotonicity we can remove the first unitary and we are left with the effective transformation $e^{i \lambda \hat{J}_z}$. Notice that the result is unchanged upon inverting the role of the operators $\hat{J}_z$ and $\hat{J}_x$—in this inverted case the estimated parameter is then the tunneling strength.

Within our model, the parameter we want to estimate is an acceleration, associated with the operator $\hat{J}_z$. To perform the beam splitting we thus need to apply a $\hat{J}_x$ operator. The problem that appears is that we cannot turn on and off at will the different terms of the natural Hamiltonian. It may be possible to turn on and off the $\hat{J}_x$ term, being the external force applied to the system. Using Feshbach resonances it is also in principle possible to tune the interaction strength $g$ and to possibly even suppress it completely. But even then there is still the $\hat{J}_z$ term that cannot be suppressed during the phase accumulation. As a pedagogical example, and to retrieve some basic results of Mach-Zehnder interferometry, we first consider the ideal case where there is no interaction and the operator $\hat{J}_z$ can be turned on and off.

Starting with the ground state (12), the final state at the output of the ideal MZI is

$$e^{-i \lambda \hat{J}_z} e^{i \frac{\pi}{2} \hat{J}_x} |\psi_g(\theta)\rangle.$$  (13)

In this case, the calculation of the QFI is particularly simple. The QFI is equal to the variance of the generator in the initial state. Using the SU(2) algebra, we obtain

$$I_\lambda = 4 (\langle \psi_g(\theta) | \hat{J}_x^2 | \psi_g(\theta) \rangle - (\langle \psi_g(\theta) | \hat{J}_y | \psi_g(\theta) \rangle)^2),$$  (14)

which here results in the simple expression

$$I_\lambda = t^2 (N^2 \sin^2 \theta + N \cos^2 \theta) .$$  (15)

This shows that the QFI is the sum of two terms, a SNL term—proportional to $N$—and a HL term—proportional to $N^2$. The weights are simply given by the degree of fragmentation through $\theta$. Especially, as mentioned before, for maximal degree of fragmentation (unity) we reach the HL, $N^2 t^2$, with no SNL term, which is optimal. Conversely, for a coherent state, which has a vanishing degree of fragmentation, we reach the SNL.

**B. Channel QFI and dynamical generator**

In the previous section, we have been dealing with a very simple Hamiltonian used to imprint the parameter in the state: a generator $\hat{G} = \hat{J}_z$ multiplied by the parameter $\lambda$. We refer to this form of Hamiltonian as a phase-shift Hamiltonian or multiplicative Hamiltonian. Quantum metrology with unitary evolution has traditionally been focusing on such phase-shift Hamiltonians. While phase-shifts are ubiquitous in quantum optics, where the Hamiltonian is often the one of a noninteracting field, for atomic systems interactions are intrinsically present. Recently, the community has increasingly been investigating quantum metrology with general Hamiltonians $H(\lambda)$, a field known as Hamiltonian parameter estimation [48–49].

A useful tool to study general Hamiltonians is the channel QFI (cQFI). It is defined as the maximal QFI one can obtain for a given quantum channel describing a unitary or dissipative evolution. For general quantum channels it requires to use a state in a Hilbert space with doubled dimension [46], but when dealing with unitary transformations, the cQFI, $C_\lambda$, is simply obtained by maximizing the QFI over the set of states in the original Hilbert space

$$C_\lambda = \max_{|\psi_0\rangle} I_\lambda (e^{-i t \hat{H}(\lambda)} |\psi_0\rangle),$$  (16)

where we made explicit the dependence of the QFI on the initial state $|\psi_0\rangle$. The most important feature of the cQFI is that it focuses only on the Hamiltonian, the state dependency being removed.

The cQFI is particularly suited to investigate the role of interaction in quantum metrology due to its exclusively Hamiltonian-based properties. One of the obvious ways to go away from a phase-shift is to take a multiplicative one-body Hamiltonian and to add an interaction to it. This problem was studied by Boixo et al. in the context of achieving super-Heisenberg scaling [47]. It is known that by estimating a multiplicative parameter in front of a k-body generator, the QFI can scale as $N^{2k}$ for NOON type states and as $N^{2k-1}$ for separable states [17, 65–68]. The authors then found that if the multiplicative parameter is linked to a one-body term, adding a k-body operator cannot improve the scaling. To show this fact they introduced some very useful tools that we will also be using here. Namely, one can express the cQFI as the semi-norm $\| \bullet \|_{SN}$—defined as the difference between the maximal and the minimal eigenvalues of the operator — of the so-called dynamical generator $\hat{\mathcal{H}}$:

$$C_\lambda = \| \hat{\mathcal{H}} \|_{SN}^2 ,$$  (17)

with $\hat{\mathcal{H}} = i \hat{U} \hat{\partial}_\lambda \hat{U}$ where $\hat{U}$ is the evolution operator.
\( \hat{U} = e^{-iH(\lambda)} \). Then we can put an upper bound on the cQFI, to obtain \( C_\lambda \leq \ell^2 \| \partial_\lambda \hat{H}(\lambda) \|_{\text{SN}}^2 \). Notice that such an upper bound can be generalized to time-dependent Hamiltonians \([45]\). The dynamical generator has also been introduced in the context of double-well metrology \([15]\), and a closed form of it depending on the spectral decomposition of the Hamiltonian has been found \([48, 49]\), allowing to infer some properties especially in terms of time scaling (see for a detailed discussion below). The dynamical generator can also be used to express the QFI conveniently:

\[
I_\lambda = 4\langle \langle \psi_0 | \mathcal{H}^2 | \psi_0 \rangle - \langle \psi_0 | \mathcal{H} | \psi_0 \rangle^2 \rangle .
\]  

(18)

To illustrate the use of cQFI we move away from the ideal MZI case and work out the more realistic nonideal case, for which the \( \hat{J}_z \) term is not canceled during the phase accumulation stage — still assuming no interaction is present. Now the transformation of the initial states obeys \( e^{-i(\lambda J_z - \delta J_z)} e^{i \frac{\eta}{2} \hat{J}_x} \). In particular, we see that the beam splitting is left untouched (the effect of interaction during the beam splitting stage was studied in \([70]\)) and that during the phase accumulation the \( \hat{J}_x \) operator is present, to which we refer in what follows as the detrimental term. This situation has been considered for a double-well system (the roles of \( \hat{J}_z \) and \( \hat{J}_x \) therein being inverted in comparison to our study), both by using the QFI \([40]\) and the cQFI \([45]\).

In our model, the relevant Hamiltonian (used for phase accumulation) is written as \( \hat{H}(\lambda) = \hat{H}_{\text{sys}} + \hat{H}_{\text{acc}}(\lambda) \). The upper bound to the cQFI results in the HL expression \( \ell^2 \| \partial_\lambda \hat{H}(\lambda) \|_{\text{SN}}^2 = \ell^2 \| \hat{J}_x \|_{\text{SN}}^2 = \ell^2 N^2 \). The cQFI for the noninteracting Hamiltonian is

\[
C_\lambda = N^2 \left\{ \frac{\ell^2 \lambda^2}{\lambda^2 + \delta^2} + \left( \frac{2\delta}{\lambda^2 + \delta^2} \right)^2 \sin^2 \left( \frac{\lambda^2}{2} \sqrt{\lambda^2 + \delta^2} \right) \right\} .
\]

(19)

There is no SNL term remaining in this cQFI: It scales purely quadratically with particle number. As expected, for \( \delta = 0 \) the cQFI is equal to the HL, \( N^2 \ell^2 \). In the low \( \lambda \) limit (\( \lambda \ll \delta \)) the cQFI gives up to first order \( 4N^2 \sin^2(t\delta/2)/\delta^2 + O(\lambda^2) \). The presence of a zeroth-order term shows that even for \( \lambda = 0 \) the cQFI does not vanish \([45, 48, 50]\).

We have up to now focused our attention on the scaling with the number of particles. But the scaling with (phase accumulation) time is equally important in quantum metrology. Already in Eq. \( (15) \) we see that the QFI scales quadratically with time. This constitutes a significant advantage over classical metrology, where the scaling in time is linear. Notice that the time behavior and the particle number behavior can be related when going from a sequential scheme to a parallel one \([16, 17]\). In the context of Hamiltonian parameter estimation, the time scaling of the cQFI is well understood \([48, 49]\). In full generality, the cQFI exhibits a term which is quadratic in time and a periodic term. The quadratic one finds its origin on the parameter dependence of the eigenvalues of the Hamiltonian, while the periodic behavior originates in the parameter dependence of the eigenvectors of the Hamiltonian. This explains why, for a phase-shift Hamiltonian only the quadratic term survives — as the eigenvectors are clearly parameter-independent there.

We represent on the right hand plot in the Fig. \( (\) the cQFI as a function of \( \delta \), for different values of \( \lambda \). As pointed out before, for small \( \delta \), the cQFI decreases, but for larger \( \delta \), values the cQFI is not a monotonous function of \( \delta \), anymore (see the black straight line corresponding to \( \lambda = 1 \)). Such nonmonotonic behavior — meaning that far from the limiting point, we do not know if it is better to increase or reduce the detrimental term — has been observed previously in \([50, 72]\). In the left hand plot the cQFI is represented as a function of time. We see that for larger values of \( \delta \), the quadratic prefactor gets reduced, and that a periodic part appears in the cQFI. This is in line with the general analysis presented above: Without \( \hat{J}_z \) during phase accumulation, we have a phase-shift Hamiltonian and a purely quadratic time scaling, the presence of \( \hat{J}_z \) introduces a parameter dependence in the eigenvectors and therefore leads to a periodic term in the cQFI.

### IV. EFFECT OF INTERACTIONS

In the absence of interaction, we have seen, using the cQFI, how the presence of \( \hat{J}_z \) — to which we referred as the detrimental one-body term — during the phase accumulation stage leads to a decreased precision. In this section we investigate the effect of the interaction on the precision. It was shown in the double-well case that interaction can help to counterbalance the effect of the detrimental one-body term in the cQFI \([15]\). Within our model, the effect of interaction is however much richer and more subtle than in the double-well case. We have both a renormalization of the one-body operator via \( \delta \) and the nonlinearities. Moreover the nonlinear operator assumes a different form depending on the value of \( \xi \), while in the double-well case its form is fixed.

We aim at finding the values of \( \delta, \eta, \xi, \) and \( g \)
parametrizing the Hamiltonian (7) that lead to a high λ-parameter estimation precision. To that end, we start by studying the cQFI as it sets a fundamental limit on the QFI as well. Then we move on to the study of the QFI itself in the regions with a high cQFI being present. In particular, we are interested in the behavior of the QFI for the experimentally realizable low values of the degree of fragmentation.

A. Channel QFI

As mentioned in the above, the effect of interaction on the cQFI has previously been studied for a double well in [15]. The authors considered the estimation of the energy difference, which corresponds effectively in our model to setting ξ = 0 (in the sense that then the nonlinear term equals the square of the generator). The main analytical result then is that for large interaction strength the cQFI saturates its upper bound, the HL. Such results have been generalized in the context of Hamiltonian parameter estimation [50]. In general, if and only if the eigenvectors — assumed nondegenerate — with maximal and minimal eigenvalues of ∂_λ H(λ) are also eigenvectors of H(λ), the cQFI saturates its upper bound. Let us introduce a simple model \( H = \lambda \hat{G} + \hat{F} \), where \( \hat{G} \) is the generator and \( \hat{F} \) is an arbitrary extra term, which is considered as detrimental. We can think of at least two methods to saturate the upper bound: (i) Adding another extra operator equal to \( -\hat{F} \), in such a way that we exactly retrieve a phase-shift; (ii) Adding an extra large dominant operator, say \( \alpha \hat{K} \), designed in such a way that the extremal eigenvectors of \( \hat{G} \) are also eigenvectors of \( \alpha \hat{K} + \hat{H} \) for large \( \alpha \).

Due to the rich structure of our Hamiltonian, we can make in principle use of both methods. On one hand the renormalization of the one-body term allows to implement the method (i). Then we need to tune \( g \approx 2\delta /\delta_\lambda \) as well as \( \eta = 0 \) (or \( \xi = 0 \)). On the other hand, we can take a very large \( g \) with \( \xi = 0 \), resulting in a nonlinear term which equals \( \hat{J}_z^2 \), to implement the method (ii). However, due to the potential difficulty to fine-tune \( \eta \) and \( \xi \), it appears more realistic not to insist on one of the methods, but rather to consider nonoptimal values of \( \eta \) and \( \xi \) and investigate numerically the behavior of the cQFI, in an attempt to reconcile both methods.

To do so, we first fix \( \lambda = 1 \) and \( \delta_\lambda = 10 \) to be in a regime where the detrimental one-body operator \( \hat{J}_z \) really deteriorates the cQFI and study the effect of the nonlinearity. In Fig. 2 (top row), we represented the cQFI as a function of both \( \eta \) and \( \xi \), which control the nonlinear term, for different values of interaction strength (from left to right, \( g = -10, 10, 15, 20, 25 \)). It is important to remember that only the upper left and the bottom right quarters of these plots are accessible since \( \eta \) and \( \xi \) have opposite sign. Also, the stripe \( \xi \in [0, 1] \) is excluded. That being said we see that the most interesting regime corresponds to \( \eta > 0 \) and \( \xi < 0 \) (upper left quarter) which corresponds to the signs we obtain by for example assuming harmonic oscillator ground and first excited state mode functions. For negative interaction, there is a larger gap around \( \eta = 0 \) due to the renormalization of the one-body term: The absolute value of \( q \) gets larger due to interaction. For low values of interaction — either positive or negative — the cQFI remains very low. Only for \( g \gtrsim 20 \) the cQFI rises again, with a reduced gap around \( \eta = 0 \). Now that we identified the appropriate region for \( \eta \) and \( \xi \), we look at the influence of the other terms in the Hamiltonian and take \( \eta = 1 \), \( \xi = -0.5 \), and \( N = 50 \). In the upper left hand plot of Fig. 2 we depicted the cQFI as a function of time. We clearly see the increase of the cQFI for higher values of interaction. In comparison to

FIG. 2. Channel QFI (first row) calculated according to Eq. (17), and QFI according to Eq. (18) for fragmented state (second row) and for coherent state (third row). The axes are the interaction Hamiltonian parameters \( \eta \) and \( \xi \). The other parameters are \( N = 50, t = 1, \lambda = 1, \delta_\lambda = 10, \delta_\lambda = 1 \). From left to right the coupling increases according to \( g = -10, 10, 15, 20, 25 \). The horizontal plane in the second (third) row shows the QFI using an ideal MZI fed with a fragmented (coherent) state.
the noninteracting case (dashed-dotted line on the left hand plot in Fig. 1) we see that the periodic behavior is not observed anymore for \( g = 8 \) and \( g = 10 \), and that the quadratic scaling is restored. On the upper right hand plot in Fig. 3 we look at the \( \delta_\varepsilon \) dependence for different values of \( \lambda \) (compare to the right hand plot in Fig. 1). Again, due to the interaction the cQFI remains high even for larger values of \( \delta_\varepsilon \), this being particularly true for the parameter value \( \lambda = 1 \). We also observe some nontrivial behavior, as for \( \delta_\varepsilon \simeq 10 \) the cQFI is higher for \( \lambda = 1 \) than for \( \lambda = 5 \) and \( \lambda = 10 \). Note that the maximum is not attained anymore for \( \delta_\varepsilon = 0 \) but for \( \delta_\varepsilon \simeq 5 \). This is due to the above discussed renormalization of the one-body operator: For \( \delta_\varepsilon \simeq 5 \), we have \( g \simeq 0 \).

The bottom left hand plot in Fig. 3 illustrates again the effect of renormalization. For different values of interaction, the optimal \( \delta_\lambda \) follows roughly \( \delta_\lambda \approx 2\delta_\varepsilon/g \). Finally, the last plot in Fig. 3 shows the behavior of the cQFI as a function of the interaction strength itself. Once more the effect of renormalization can be verified by looking at the maximum of the curves. We also notice that there exist threshold values for \( g \), at which the cQFI increases very fast.

### B. Fragmented and coherent initial state

The cQFI gives us precious information as it tells us, for a given Hamiltonian, what the highest QFI one can optimally get is, independent of the input state. Still, it is important to check what the actual performance of various and physically implementable initial states is. We thus study the QFI for the family of initial states defined by the ground state and parametrized by \( \theta \), which is directly linked to the degree of fragmentation. We will focus on the regime of low degrees of fragmentation, which is the most realistic one for our system, operating within the realm of validity of a two-mode approximation. In the extreme case of very small coupling \( g \) used for the ground state preparation, no fragmentation remains and a coherent state obtains. There is two obvious points of comparison for the QFI. On one hand, we can compare to the cQFI and see how close our state is from optimality. On the other hand, we can make a less ambitious comparison and simply compare to the QFI obtained for an ideal interferometer. While it is excluded that one beats the ideal interferometer using the cQFI (in the ideal case the cQFI saturates the upper bound), this is not the case for the QFI: The non-optimality of the state leaves room to witness an increase of the QFI above the ideal case.

In the second and third row of Fig. 2 we plotted the QFI for a fragmented (\( \theta = 0.5 \), \( \mathcal{F} \simeq 0.12 \)) and a coherent state as a function of \( \eta \) and \( \xi \). We focus on the region \( \eta > 0 \) and \( \xi < 0 \) and start by analyzing the fragmented state (middle row). For negative interaction (\( g = -10 \)), the behavior of the QFI follows the one of the cQFI, but with a reduced value. For moderate positive interactions (\( g = 10 \)), while the cQFI saturates almost its upper bound, the QFI stays extremely low. We need to reach values around \( g = 20 \) and higher to retrieve a high QFI (roughly equal to half of the cQFI). For the coherent state (bottom row), the global behavior of the QFI is similar to the fragmented case, but with an amplitude reduced by more than a factor of two. Still we see that the QFI displays strong variations both in the \( \eta \) and \( \xi \) direction, while the fragmented state shows a smooth behavior. In particular, the regions of high QFI are much more narrow and would require fine tuning of \( \eta \) and \( \xi \). In the plots of the second and third row, the horizontal planes show, as a benchmark, the value of the QFI using the same initial state in an ideal interferometer. For the fragmented case there is a gain above that “ideal” case, even if moderate. But for the coherent state the relative gain is much higher, as coherent states lead to the SNL.

The effect of fragmentation is more clearly demonstrated in Fig. 4, where the QFI as a function of the...
degree of fragmentation is displayed. The thick dotted line shows the QFI for an ideal MZI, while the other dotted line corresponds to the cQFI for the various values of $g$. On the left hand plot (negative interactions and $\eta = 3$, $\xi = -0.5$), we see that for both $g = -10$ and $g = -15$, the QFI almost reaches the cQFI, indicating that the corresponding state is almost optimal. We observe that these states do not correspond to fully fragmented states. For lower interaction $g = -5$, the QFI remains low for any degree of fragmentation. Of particular interest is the steep slope of the QFI for low values of degree of fragmentation, meaning that the cost of increasing the interaction to increase the QFI is moderate. Also, in this region there is a clear gain from the ideal interferometer. The right hand plot (positive interactions and $\eta = 1$, $\xi = -0.5$) shows a different behavior. The maximum is obtained for high values of degree of fragmentation and the gain over the ideal MZI is less striking. Still the QFI slope for low degrees of fragmentation remains high.

Finally, in Fig. [5] we illustrate more in depth the exact role of nonlinearities in the Hamiltonian. In the above subsection [IV A] we explained how by adding a large, dominant, extra operator $\alpha K$, properly designed, we could saturate the upper bound [the so-called method (ii)]. In our protocol the generator is $\hat{J}_x$, and if we add a large dominant $\hat{J}_x^2$, we are able to saturate the upper bound $[73]$. But one should not be tempted to believe that the increase in the cQFI is due to nonlinearity: If instead of $\hat{J}_x^2$ we add simply $\hat{J}_z$, we can also saturate the upper bound.

To illustrate this fact, in the left hand plot of Fig. [5] are represented the cQFIs for a nonlinear Hamiltonian $\lambda \hat{J}_x + q \hat{J}_z + g \hat{J}_x^2 / N$ and a linearized Hamiltonian $\lambda \hat{J}_x + q \hat{J}_z + g \hat{J}_x$, where we replaced $\hat{J}_x^2$ by $N \hat{J}_x$ (the factor $N$ being included to make the comparison as fair as possible). We see that the linear Hamiltonian performs as good — and often even better than — as the nonlinear one, demonstrating the nonessential character of the nonlinearities in the implementation of method (ii). This is explained by the optimized character of the cQFI. Being the QFI for an optimal state, all the correlations needed are already in this optimal state. This is in stark contrast to the right hand plot, where we represented the QFI for a coherent state, using both the linear and the nonlinear Hamiltonian. There we see that the nonlinear Hamiltonian leads to a larger QFI than the linear one for large enough interactions. Indeed, with the linear Hamiltonian we are limited to the SNL. This was to be expected: Since there are no useful correlations in a coherent state to begin with, we need the nonlinearities to beat the SNL.

V. CONCLUSION

We investigated the realization of an interferometric protocol using scalar bosons in a single potential well. Two-mode interferometry with bosons had so far focused on double wells and on bosons with an inner (spinor) degree of freedom [35–42]. For scalar bosons in a single potential well, the splitting comes only from interaction [43]. Our model therefore represents a first step towards a form of many-body quantum interferometry in which the interferometer operates purely with interaction-triggered field-operator splitting into macroscopically occupied modes.

Using the SU(2) representation of two-mode bosons, the description of the MZI is straightforward: The energy difference between each mode produces a $\hat{J}_z$ term used for beam splitting, and an external acceleration produces the phase we want to estimate, attached to a $\hat{J}_x$ operator. The main point to be made here is that during the phase accumulation stage the term $\hat{J}_z$ does not disappear, and reduces the QFI, as it is also the case in a double well [46–49]. In distinction to the double well, our system however demonstrates a richer Hamiltonian. In particular, the interaction on one hand generally leads to a renormalization of the linear term related to energy level splitting. On the other hand, the nonlinear terms display a greater parameter freedom, which comes from the fact that pair-exchange ($A_2$) and density-density interactions ($A_4$) occur in the original two-mode Hamiltonian.

At first we employed the QFI for an optimal state, a metrological figure of merit known as the channel QFI, to focus on the properties of the Hamiltonian itself. We then demonstrated how we can harness the effect of interaction in our model to counter the detrimental effect of the $\hat{J}_z$ term during phase accumulation. This result is obtained by using both the interaction-induced renormalization of the detrimental one-body term and the nonlinear term. Then we study the QFI using as our initial states the family of fragmented states, focusing on the (more realistic) case of low degrees of fragmentation. Due to the non-optimality of the initial state, we can use interaction to not only compensate the negative effects of the detrimental one-body term, but even reach a QFI higher than the one obtained in an ideal (with no detrimental term.
during the phase accumulation) interferometer. The beneficial effect of interaction is therefore twofold: First in the very preparation of a fragmented ground state, and second in the dynamical evolution of this ground state.

Our work can be extended in its range of applicability with respect to both of its two main pillars, the metrological and the many-body pillar.

Regarding the many-body aspect, our study assumes fixed mode functions (orbitals), from which the Hamiltonian in this given basis is derived, and hence also the many-body state in Fock space is obtained. The multiconfigurational time-dependent Hartree-Fock method \cite{schwenke1992}, on the other hand, determines orbitals and Fock space occupation distribution self-consistently. This self-consistency in solving the problem would potentially lead to new insights for many-body quantum metrology, as well as provide additional degrees of freedom for tuning towards optimality.

On the metrological side, it would be of interest to investigate the performance of practical measurements to see how well we can approach the QFI. This question has been extensively studied for the double-well scenario \cite{schwenke1992} \cite{vittorio2008} \cite{pichler2008}. The question of the very measurement is crucial in quantum metrology, as in general the optimal measurement that saturates the QFI is not accessible to experiment. Interaction can also help at the measurement stage. The so-called interaction-based readout scheme uses an interaction before the measurement stage to solve the problem of single particle detection \cite{vittorio2008} \cite{pichler2008}. Another important issue is the robustness of the protocol towards decoherence. It is well known that decoherence can have a strongly detrimental effect on the potential precision improvements afforded by quantum metrology with highly entangled states \cite{zyczkowski2009} \cite{sedlmayr2009}, and much effort has therefore been devoted to finding protocols not primarily based on entanglement \cite{zyczkowski2009}. In our model, decoherence could for example take the form of dephasing or particle loss. Investigating whether our protocol for many-body metrology remains viable under the presence of such dissipative channels represents a subject for future study. We mention in this regard that for macroscopic superpositions of spin-coherent states, particle loss has been considered in \cite{zyczkowski2009}, and it was found that the achievable precision can still beat the SNL.

ACKNOWLEDGMENTS

We thank Seunglee Bae for contributions at the early stages of this work. This research was supported by the NRF of Korea, Grant Nos. 2014R1A2A2A01006535 and 2017R1A2A2A05001422.

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