ON THE FILLING INVARIANTS AT INFINITY
OF HADAMARD MANIFOLDS

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Abstract. We study the filling invariants at infinity $\text{div}_k$ for Hadamard manifolds defined by Brady and Farb in [BF98]. Among other results, we give a positive answer to the question they posed: whether these invariants can be used to detect the rank of a symmetric space of noncompact type.

1. Introduction

Asymptotic invariants have been used previously as basic tools to study large scale geometry of various spaces. Precedents include Gromov in [Gro93], and Block and Weinberger in [BW97]. In [Kni97] Knieper, generalizing the work of Margulis in the case of negative curvature, studied the growth function of the volume of distance spheres in Hadamard manifolds and found an unexpected relation to the rank of the manifold.

In [Ger94b] Gersten studied the divergence of geodesics in CAT(0) spaces, and gave an example of a finite CAT(0) 2-complex whose universal cover possesses two geodesic rays which diverge quadratically and such that no pair of geodesics diverges faster than quadratically. Adopting a definition of divergence which is a quasi-isometry invariant, Gersten introduced a new invariant for geodesic metric spaces, which we refer to as $\text{div}_0$. In [Ger94a] Gersten used the $\text{div}_0$ invariant to distinguish the quasi isometry type of graph manifolds among all closed Haken 3-manifolds.

Using the same trick introduced by Gersten in [Ger94b] to get a quasi-isometry invariant, Brady and Farb in [BF98] introduced a family of new quasi-isometry invariants $\text{div}_k(X)$ for $0 \leq k \leq n-2$, for a Hadamard manifold $X^n$. These invariants were meant to be a finer measure of the spread of geodesics in $X$.

The precise definition will be given below, but roughly the definition of $\text{div}_k(X)$ is as follows: Find the minimum volume of a $(k+1)$-ball which is needed to fill a $k$-sphere which lies on the $(n-1)$-dimensional distance sphere $S_r(x_0)$ in $X$. The filling is required to lie outside the $n$-dimensional open ball $B^o_r(x_0)$. The $\text{div}_k(X)$ invariant measures the asymptotic behavior of the volume of the filling as $r \to \infty$ when the volume of the $k$-sphere grows polynomially in $r$.

2000 Mathematics Subject Classification. Primary 53C35; Secondary 53C23, 20F67.
After computing some of these invariants for certain Hadamard manifolds, Brady and Farb posed the following two questions.

**Question 1.** Can the $\text{div}_k(X)$ invariants be used to detect the rank of a noncompact symmetric space $X$?

**Question 2.** What symmetric spaces can be distinguished by the invariants $\text{div}_k$?

In this paper we study the $\text{div}_k$ invariants for various Hadamard manifolds including symmetric spaces of noncompact type. The first result we obtain is the following theorem.

**Theorem 1.1.** If $X$ is an $n$-dimensional symmetric space of nonpositive curvature and of rank $l$, then $\text{div}_k(X)$ has a polynomial growth of degree at most $k + 1$ for every $k \geq l$.

Brady and Farb showed in [BF98] that $\text{div}_{k-1}(X)$ is exponential when $X = H^{m_1} \times \cdots \times H^{m_k}$ is a product of $k$ hyperbolic spaces. The idea was to show that there are quasi-isometric embeddings of $H^{(m_1+\cdots+m_k)-k+1}$ in $X$ and use that to show an exponential growth of the filling of a $(k-1)$-sphere which lies in a $k$-flat, and therefore proving that $\text{div}_{k-1}(X)$ is indeed exponential. For more details, see section 4 in [BF98].

The same idea was taken further by Leuzinger in [Leu00] to generalize the above result to any rank $k$ symmetric space $X$ of nonpositive curvature by showing the existence of an embedded submanifold $Y \subset X$ of dimension $n - k + 1$ which is quasi-isometric to a Riemannian manifold with strictly negative sectional curvature, and which intersects a maximal flat in a geodesic.

Combining Theorem 1.1 with Leuzinger’s result mentioned above, we obtain the following corollary which gives a positive answer to question 1.

**Corollary 1.2.** The rank of a nonpositively curved symmetric space $X$ can be detected using the $\text{div}_k(X)$ invariants.

Brady and Farb in [BF98] suspected that $\text{div}_1$ has exponential growth for the symmetric space $SL_n(\mathbb{R})/SO_n(\mathbb{R})$. That was proved by Leuzinger to be true for $n = 3$, see [Leu00]. We show that the same result does not hold anymore for $n > 3$. More generally we prove,

**Theorem 1.3.** If $X$ is a symmetric space of nonpositive curvature and rank $k \geq 3$, then $\text{div}_1(X)$ has a quadratic polynomial growth.

After studying the case of symmetric spaces, we turn our focus to the class of Hadamard manifolds of pinched negative curvature. We show,

**Theorem 1.4.** If $X$ is a Hadamard manifold with sectional curvature $-b^2 \leq K \leq -a^2 < 0$, then $\text{div}_k(X)$ has a polynomial growth of degree at most $k$ for every $k \geq 1$.

A natural question at this point would be whether the same result holds if we weaken the assumption on the manifold $X$ to be a rank 1 instead of being negatively curved.
We give a negative answer to that question. We give an example of a nonpositively curved graph manifold of rank 1 where \( \text{div}_1 \) is exponential.

All the previously known results about the filling invariants, see [BF98, Ger94a, Leu00], as well as the new results in this paper suggest a connection between these invariants and the connectedness of the Tits boundary of a Hadamard manifold. Theorem 1.3 is one example. The non-simply connected Tits boundary of a symmetric space of rank 2 is reflected in the exponential growth of \( \text{div}_1 \), while the quadratic polynomial growth of \( \text{div}_1 \) when the rank is bigger than 2 is a direct consequence of the simple connectedness of the Tits boundary.

Except in a very few special cases, very little is known in general about which part of the Tits geometry of a nonpositively curved space is preserved under a quasi isometry. Further investigation of the connection between these invariants and the connectedness of the Tits boundary may shed some light on that question.

The paper will be organized as follows: Section 1 is an introduction. In section 2 we give the precise definition of \( \text{div}_k \). In section 3 the proof of Theorem 1.1 in the simple case of rank 1 symmetric spaces is given where the basic idea is illustrated. In section 4 we prove Theorem 1.1 for higher rank symmetric spaces. The proof of Theorem 1.3 is give in section 5. In section 6 we prove Theorem 1.4. In section 7 we give the graph manifold example mentioned above. In appendix A we give a different and shorter proof of Leuzinger’s Theorem, which was proved in [Leu00].

Acknowledgments. The author would like to thank his advisor Chris Croke for many helpful discussions during the development of this paper, and also like to thank the University of Bonn, in particular Werner Ballmann for the opportunity to visit in the summer of 2003 when his interest began in the filling invariants.

2. Definitions and Background

Let \( X^n \) be an \( n \)-dimensional Hadamard manifold, by that we mean a complete simply connected Riemannian manifold with nonpositive sectional curvature. By Cartan-Hadamard Theorem \( X^n \) is diffeomorphic to \( \mathbb{R}^n \). In fact the \( \exp_{x_0} \) map at any point \( x_0 \in X \) is a diffeomorphism. For a standard source on Hadamard manifolds we refer the reader to [BGS85].

We denote the ideal boundary of \( X \) by \( X(\infty) \). For any two different points \( p, q \in X \), \( \overrightarrow{pq}, \overrightarrow{pq} \) denote respectively the geodesic segment connecting \( p \) to \( q \), and the geodesic ray staring at \( p \) and passing through \( q \). By \( \overrightarrow{pq}(\infty) \) we denote the limit point in \( X(\infty) \) of the ray \( \overrightarrow{pq} \).

Let \( \overline{X} = X \cup X(\infty) \), if \( x_0 \in X \) and \( p, q \in \overline{X} \setminus \{x_0\} \) then \( \angle_{x_0}(p, q) \) is the angle between the unique geodesic rays connecting \( x_0 \) to \( p \) and \( q \) respectively. If \( p, q \in X(\infty) \), then \( \angle(p, q) = \sup_{x_0 \in X} \angle_{x_0}(p, q) \) denotes the Tits angle between \( p \) and \( q \). If \( \angle_{x_0}(p, q) = \angle(p, q) < \pi \) for some \( x_0 \in X \) then the geodesic rays connecting \( x_0 \) to \( p \) and
\( q \) respectively bound a flat sector. Conversely if those rays bound a flat sector then 
\[ \angle_x(p, q) = \angle(p, q). \]

Let \( S_r(x_0), B_r(x_0) \) and \( B_r^c(x_0) \) denote respectively the distance sphere, the distance ball and the open distance ball of radius \( r \) and center \( x_0 \). Let \( S^k \) and \( B^{k+1} \) denote respectively the unit sphere and the unit ball in \( \mathbb{R}^{k+1} \). Let \( C_r(x_0) = X \setminus B_r^c(x_0) \).

Projection along geodesics of \( C_r(x_0) \) onto the sphere \( S_r(x_0) \) is a deformation retract, which decreases distances, since the ball \( B_r(x_0) \) is convex and the manifold is nonpositively curved. Any continuous map \( f : S^k \to S_r(x_0) \) admits a continuous extension, \("filling", \( \hat{f} : B^{k+1} \to C_r(x_0) \), and the extension could be chosen to be Lipschitz if \( f \) is Lipschitz.

Lipschitz maps are differential almost everywhere. Let \( |D_x(f)| \) denote the Jacobian of \( f \) at \( x \). The \( k \)-volume of \( f \) and the \((k+1)\)-volume of \( \hat{f} \) are defined as follows,

\[
\begin{align*}
\text{vol}_k(f) &= \int_{S^k} |D_x f|, \\
\text{vol}_{k+1}(\hat{f}) &= \int_{B^{k+1}} |D_x \hat{f}|.
\end{align*}
\]

Let \( 0 < A \) and \( 0 < \rho \leq 1 \) be given. A Lipschitz map \( f : S^k \to S_r(x_0) \) is called \( A \)-admissible if \( \text{vol}_k(f) \leq A r^k \) and a Lipschitz filling \( \hat{f} \) is called \( \rho \)-admissible if \( \hat{f}(B^{k+1}) \subset C_{\rho r}(x_0) \). Let

\[
\delta_k^{\rho, A} = \sup_f \inf_{\hat{f}} \text{vol}_{k+1}(\hat{f}),
\]

where the supremum is taken over all \( A \)-admissible maps and the infimum is taken over all \( \rho \)-admissible fillings.

**Definition 2.1.** The invariant \( \text{div}_k(X) \) is the two parameter family of functions

\[ \text{div}_k(X) = \{ \delta_k^{\rho, A} \mid 0 < \rho \leq 1 \text{ and } 0 < A \}. \]

Fix an integer \( k \geq 0 \), for any two functions \( f, g : \mathbb{R}^+ \to \mathbb{R}^+ \), we write \( f \preceq_k g \) if there exist two positive constants \( a, b \) and a polynomial \( p_{k+1}(x) \) of degree at most \( k+1 \) with a positive leading coefficient such that \( f(x) \leq ag(bx) + p_{k+1}(x) \). Now we write \( f \asymp_k g \) if \( f \preceq_k g \) and \( g \preceq_k f \). This defines an equivalence relation.

We say that \( \text{div}_k \preceq \text{div}'_k \) if there exist \( 0 < \rho_0, \rho'_0 \leq 1 \) and \( A_0, A'_0 > 0 \) such that for every \( \rho < \rho_0 \) and \( A > A_0 \) there exist \( \rho' < \rho'_0 \) and \( A' > A'_0 \) such that \( \delta_k^{\rho, A} \preceq \delta_k^{\rho', A'} \). We define \( \text{div}_k \sim \text{div}'_k \) if \( \text{div}_k \preceq \text{div}'_k \) and \( \text{div}'_k \preceq \text{div}_k \). This is an equivalence relation and under this identification \( \text{div}_k \) is a quasi-isometry invariant (see [BF98] for details).

**Remark 2.2.** As a quasi-isometry invariant, a polynomial growth rate of \( \text{div}_k \) is only defined up to \( k + 1 \). And the reader should view the \( k \) polynomial growth rate of \( \text{div}_k \) in Theorem 1.4 accordingly.
The polynomial bound on $\text{vol}_k(f)$ in the definition above is essential to prevent the possibility of constructing exponential $k$-volume maps requiring exponential volume fillings, and therefore making the filling invariants always exponential. On the other hand, allowing the filling to be “slightly” inside the ball, i.e. inside $C_{pr}(x_0)$ for some fixed $0 < \rho$ is needed since continuous quasi-isometries map spheres to distorted spheres.

Remark 2.3. Through the paper we will use the following cone construction. Given an $n$-dimensional Hadamard manifold $X$ and a Lipschitz map $f: S^k \to X$, if we cone $f$ from a point $x_0 \in X$ we obtain a Lipschitz extension $\hat{f}: D^{k+1} \to X$. Using comparison with Euclidean space, it is clear that $\text{vol}_{k+1}(\hat{f}) \leq \text{vol}_k(f) \sup_{p \in S^k} d(x_0, f(p))$. If $X$ has sectional curvature $K \leq -a^2 < 0$, then there exists a constant $C = C(a, n)$ which does not depend on $x_0$ nor $f$ such that $\text{vol}_{k+1}(\hat{f}) \leq C \text{vol}_k(f)$. If $a = 1$ this constant can be chosen to be 1.

3. RANK ONE SYMMETRIC SPACES

The $\text{div}_0(X)$ of a Hadamard manifold $X$ is the same as the “rate of divergence” of geodesics. Therefore for rank 1 symmetric spaces it is exponential since they have pinched negative curvature.

In this section we calculate $\text{div}_k$ where $k \geq 1$ for any rank one symmetric space. First we start with a lemma which will be needed in the proof of Theorem 3.3.

Lemma 3.1. Given a Lipschitz function $f: S^k \to \mathbb{R}^n$, where $1 \leq k < n$, there exists a constant $c = c(n)$ such that for any ball $B_r(u_0)$, we can find a map $g: S^k \to \mathbb{R}^n$ which satisfies the following conditions:

i. $f$ is homotopic to $g$.
ii. $\text{vol}_k(g) \leq c \text{vol}_k(f)$.
iii. $g(S^k) \cap B^*_r(u_0) = \emptyset$.
iv. $f$ and $g$ agree outside $B_r(u_0)$.
v. The $(k + 1)$-volume of the homotopy is bounded by $cr \text{vol}_k(f)$.

Proof. The proof follows the proof of Theorem 10.3.3 in [ECH+92]. The idea is to project the part inside the ball $B_r(u_0)$ to the sphere $S_r(u_0)$. We will project from a point within the ball $B_{r/2}(u_0)$, but since projecting from the wrong center might increase the volume by a huge factor, we average over all possible projections and prove that the average is under control, and therefore we have plenty of centers from which we can project and still have the volume of the new function under control.

Let $\omega_{n-1}$ be the volume of the unit $(n - 1)$-sphere in $\mathbb{R}^n$. Let $|D_x f|$ represent the Jacobian of $f$ at the point $x$. Let $\pi_u$ be the projection map from the point $u \in B_{r/2}(u_0)$
to the sphere $S_r(u_0)$.

\begin{equation}
\int_{B_{r/2}(u_0)} \text{vol}_k(\pi_u \circ f) \, du \leq \int_{f^{-1}(B_r(u_0))} \frac{|D_x f| (2r)^k}{\|f(x) - u\|^k \cos \theta} \, dx + \text{vol}_k(f),
\end{equation}

where $\theta$ is the angle between the ray connecting $u$ to $f(x)$ and the outward normal vector to the sphere at the point of intersection of the ray and the sphere $S_r(u_0)$. If $u \in B_{r/2}(u_0)$ then it is easy to see that $\theta \leq \pi/6$ and therefore $1/\cos \theta \leq 2/\sqrt{3} \leq 2$.

By integrating (3.1) over the ball $B_{r/2}(u_0)$ we get,

\begin{equation}
\int_{B_{r/2}(u_0)} \text{vol}_k(\pi_u \circ f) \, du \leq \int_{B_{r/2}(u_0)} \int_{f^{-1}(B_r(u_0))} \frac{|D_x f| (2r)^k}{\|f(x) - u\|^k \cos \theta} \, dx \, du
+ \text{vol}(B_{r/2}(u_0)) \, \text{vol}_k(f)
\leq \int_{f^{-1}(B_r(u_0))} |D_x f| \int_{B_{r/2}(u_0)} \frac{2 (2r)^k}{\|f(x) - u\|^k} \, du \, dx
+ \text{vol}(B_{r/2}(u_0)) \, \text{vol}_k(f)
\leq 2^{k+1} r^k \int_{f^{-1}(B_r(u_0))} |D_x f| \int_{B_{r/2}(u_0)} \frac{1}{\|u\|^k} \, du \, dx
+ \text{vol}(B_{r/2}(u_0)) \, \text{vol}_k(f)
= \frac{2^{k+1} 3^{n-k} \rho_n^{-1} \omega_{n-1}}{2^{n-k}(n-k)} \int_{f^{-1}(B_r(u_0))} |D_x f| \, dx
+ \text{vol}(B_{r/2}(u_0)) \, \text{vol}_k(f)
\leq \frac{2^{2k+1} 3^{n-k}}{n-k} + 1 \, \text{vol}(B_{r/2}(u_0)) \, \text{vol}_k(f)
\leq [2^{2n-1} 3^{n-1} n + 1] \, \text{vol}(B_{r/2}(u_0)) \, \text{vol}_k(f).
\end{equation}

The $(k+1)$-volume of the homotopy is bounded by $2r[2^{2n-1} 3^{n-1} n + 1] \, \text{vol}_k(f)$. We take $c = 2[2^{2n-1} 3^{n-1} n + 1]$, which now satisfies all the requirements of the lemma. \qed

Using the standard notation, we denote by $s_{nk}$ respectively $c_{nk}$ the solution to the differential equation $x''(t) + kx(t) = 0$ with the initial conditions $x(0) = 0$ and $x'(0) = 1$ respectively $x(0) = 1$ and $x'(0) = 0$. We also set $c_{nk} = c_{sk} / s_{nk}$.

Now we use Lemma 3.1 to prove a similar result for any Riemannian manifold with bounded geometry which we will call the “Deformation Lemma.”
Lemma 3.2 (Deformation Lemma). Given a Riemannian manifold \( M \), with sectional curvature \( L \leq K \leq H \) and injectivity radius \( \text{inj}(M) \geq \epsilon \), then there exist constants \( \delta = \delta(L, K, \epsilon) > 0 \) and \( c = c(n, L, K, \epsilon) \), which do not depend on \( M \) such that for any Lipschitz map \( f: S^k \rightarrow M \) and any ball \( B_r(p) \subset M \) with radius \( r \leq \delta \), we can find a map \( g: S^k \rightarrow M \) which satisfies the following conditions:

i. \( f \) is homotopic to \( g \).

ii. \( \text{vol}_k(g) \leq c \text{vol}_k(f) \).

iii. \( g(S^k) \cap B^p_r(p) = \emptyset \).

iv. \( f \) and \( g \) agree outside \( B_r(p) \).

v. The \((k+1)\)-volume of the homotopy is bounded by \( c \text{vol}_k(f) \).

Proof. Take \( \delta = \min\{ \epsilon/2, \pi/2\sqrt{H} \} \). By comparison with the spaces of constant curvature \( L \) and \( H \), we have the following bounds within \( B_\delta(p) \).

\[
\| D \exp_p^{-1} \| \leq A = \max\{1, \delta/\text{sn}_H(\delta)\},
\]

\[
\| D \exp_p \| \leq B = \max\{1, \text{sn}_L(\delta)/\delta\}.
\]

See Theorem 2.3 and Corollary 2.4 in chapter 6 of [Pet98] for details but the reader should be aware that the estimate for \( \| D \exp_p^{-1} \| \) is stated incorrectly there. Note that \( A \) and \( B \) only depend on \( L, K \) and \( \epsilon \) but not on \( p \) nor \( M \).

Now we use \( \exp_p^{-1} \) to lift the map \( f \) locally, i.e. in \( B_\delta(p) \), to \( T_pM \) then use Lemma 3.1 to deform this lifted map to a map that lands outside the ball \( B_\delta(0) \subset T_pM \) and then project back using \( \exp_p \) to the manifold \( M \). The \( k \)-volume of the new map and the \((k+1)\)-volume of the homotopy will be controlled because of the bounds on \( \| D \exp_p \| \) and \( \| D \exp_p^{-1} \| \) given by equations (3.3) and (3.4) and the estimates given in Lemma 3.1. This finishes the proof of the lemma. \( \square \)

Theorem 3.3. If \( X \) is a rank 1 symmetric space of noncompact type, then \( \text{div}_k(X) \) has a polynomial growth of degree at most \( k \) for every \( k \geq 1 \).

Proof. Let \( x_0 \) be any point in \( X \), \( S_r(x_0) \) the distance sphere of radius \( r \) centered at \( x_0 \), and \( d_{S_r(x_0)} \) the Riemannian distance function on the sphere.

We will show that there exists \( 0 < \rho < 1 \) such that there is a filling of any admissible \( k \)-sphere on \( S_r(x_0) \) outside \( B^p_{\rho r}(x_0) \) which grows polynomially of degree at most \( k \) as \( r \rightarrow \infty \).

Let \( \pi^R_r: S_R(x_0) \rightarrow S_r(x_0) \) be the radial projection from \( S_R(x_0) \) to \( S_r(x_0) \), and let \( \lambda^R_r: S_r(x_0) \rightarrow S_R(x_0) \) be its inverse. Assume that the metric on \( X \) is normalized such that the sectional curvature is bounded between \(-4\) and \(-1\). By comparison with the spaces of constant curvatures \(-1\) and \(-4\), it is easy to see that the map \( \pi^R_r \) decreases distance by at least a factor of \( \sinh R/\sinh r \), and \( \lambda^R_r \) increases distance at most by a factor of \( \sinh 2R/\sinh 2r \).

Fix \( A > 0 \) and let \( f: S^k \rightarrow S_r(x_0) \) be a Lipschitz map such that \( \text{vol}_k(f) \leq Ar^k \). Then \( \text{vol}_k(\pi^R_{\rho r} \circ f) \leq Ar^k (\sinh \rho r/\sinh r)^k \).
Since horospheres are Lie groups with left invariant metrics, the curvature is bounded above and below and there is a lower bound on the injectivity radius. This puts a uniform bound on the curvature as well as the injectivity radius of distance spheres $S_r(x_0)$ as long as $r$ is big enough. Because of that and by using the Deformation Lemma, we could deform $\pi_{\rho r} \circ f$ on $S_{\rho r}(x_0)$ to a new function $g$ which misses a ball, on the sphere $S_{\rho r}(x_0)$, of radius $\delta$ and has $k$-volume $\leq c \text{vol}_k(\pi_{\rho r} \circ f)$ where $\delta$ and $c$ are the constants given by the Deformation Lemma. And such that the homotopy between $\pi_{\rho r} \circ f$ and $g$ has $(k+1)$-volume $\leq c \text{vol}_k(\pi_{\rho r} \circ f)$.

Let $p \in S_{\rho r}(x_0)$ be the center of this ball and let $q$ be its antipodal point. We will cone $g$ from $q$, inside the sphere, and then project the cone from $x_0$ back to the sphere $S_{\rho r}(x_0)$. Because the curvature is less than $-1$ the $(k+1)$-volume of the cone is smaller than $\text{vol}_k(g) \leq c \text{vol}_k(\pi_{\rho r} \circ f)$ (see Remark 2.3).

To estimate the volume of the projection of the cone, we need to find a lower bound on the distance from $x_0$ to the image of the cone. Let $x \in S_{\rho r}(x_0)$ such that $d_{S_{\rho r}(x_0)}(x, p) = \delta$. Let $x'$ be the intersection point of $S_{2\rho r}(q)$ and the ray $\overrightarrow{q \cdot t}$. For large values of $r$, $x$ and $x'$ are close, and therefore we have $d_{S_{2\rho r}(q)}(p, x') \geq \delta/2$. By comparison to the space of constant curvature $-4$ we see that $\theta = \angle_q(p, x) \geq \delta/(2\sinh 4\rho r)$. Now by comparison with Euclidian space we have $d(x_0, \overrightarrow{q \cdot t}) \geq \rho r \sin \theta \geq \rho r \sin(\delta/(2\sinh 4\rho r))$. So the cone from $q$ misses a ball of radius $\rho r \sin(\delta/(2\sinh 4\rho r))$ around $x_0$. Projecting the cone which lies inside $S_{\rho r}(x_0)$ from $x_0$ to $S_{\rho r}(x_0)$ gives us a filling for $g$ of volume $\leq cA \rho^k (\sinh \rho r / \sinh r)^k \left(\frac{\sinh 2\rho r}{\sinh(2\rho r \sin(\delta/2\sinh 4\rho r))}\right)^{k+1}$.

Notice that $\sinh t$ behaves like $e^t/2$ as $t \to \infty$ and like $t$ as $t \to 0$, while $\sin t$ behaves like $t$ as $t \to 0$. Because of that the above estimate of the filling of $g$ behaves like $cA \rho^k \left(e^{\rho r / \rho r}\right)^k \left(\frac{e^{2\rho r / 2}}{2\rho r}ight)^{k+1} = \frac{cA}{(4\rho r)^{k+1}} e^{(7k+6)\rho - k}$. Clearly we can choose $\rho$ small enough to make $(7k+6)\rho - k$ negative. Therefore there exists $r_0 > 0$ such that for all $r \geq r_0$ the $(k+1)$-volume of the filling of $g$ will be smaller than 1. Now the filling of the original map $f$ consists of three parts.

i. The radial projection of $f$ to the sphere $S_{\rho r}(x_0)$, which has $(k+1)$-volume $\leq A \rho^k$.

ii. The homotopy used to deform the map $\pi_{\rho r} \circ f$ to the new map $g$, which has $(k+1)$-volume $\leq c \text{vol}_k(\pi_{\rho r} \circ f) \leq c \text{vol}_k(f) \leq cA \rho^k$.

iii. The filling of $g$, which has $(k+1)$-volume $< 1$.

This filling of $f$ is Lipschitz and lies outside $B_{\rho r}^o(x_0)$, and is therefore $\rho$-admissible, and it has $(k+1)$-volume smaller than $A(c+1)\rho^k + 1$. This finishes the proof of the theorem.

\[\square\]

4. Higher Rank Symmetric Spaces

In this section we give the proof for Theorem 1.1. The idea is similar to the one used in the proof of Theorem 3.3, namely project the Lipschitz map you wish to fill, to a
smaller sphere to make the \( k \)-volume of the projection small and then fill the projection on the smaller sphere.

For the rank one symmetric space case, the projection decreases distances exponentially in all direction, which is no longer true for the higher rank case. Nevertheless, we still have \( n - l \) directions in which the projection, to smaller spheres, decreases distance by an exponential factor. Hence, the projection will decrease the volume of a Lipschitz map \( f \) by an exponential factor as long as the dimension of the domain of \( f \) is bigger than \( l - 1 \), to include at least one of the exponentially decreasing directions. See Lemma 4.1 for details.

We start by recalling some basic facts about symmetric spaces. For a general reference of symmetric spaces of nonpositive curvature and for the proofs of the facts used in this section see chapter 2 in [Ebe96].

Let \( X = G/K \) be a symmetric space of nonpositive curvature of dimension \( n \) and rank \( l \). Fix a point \( x_0 \in X \) and let \( g = \mathfrak{k} + \mathfrak{p} \) be the corresponding Cartan decomposition, where \( \mathfrak{k} \) is the Lie algebra of the isotropy group \( K \) at \( x_0 \). Fix a maximal abelian subspace \( \mathfrak{a} \subset \mathfrak{p} \) and let \( F \) be the flat determined by \( \mathfrak{a} \). We identify \( \mathfrak{p} \) with \( T_{x_0}X \) in the usual way. For each regular vector \( v \in \mathfrak{a} \), let \( R_v : v^+ \rightarrow v^+ \) denote the curvature tensor where \( R_v(w) = R(w, v)v \). Let \( \lambda_1(v), \ldots, \lambda_{n-1}(v) \) be the eigenvalues of \( R_v \) with corresponding eigenvectors \( E_1, \ldots, E_{n-1} \) such that \( E_1, \ldots, E_{l-1} \) are tangent to the \( l \)-flat \( F \). Therefore \( \lambda_1(v) = \cdots = \lambda_{l-1}(v) = 0 \) while \( \lambda_j(v) < 0 \) for \( j \geq l \). Since for every \( v_1, v_2 \in \mathfrak{a} \), \( R_{v_1} \) and \( R_{v_2} \) commute and therefore can be simultaneously diagonalized, \( E_{l+1}, \ldots, E_{n-1} \in (T_{x_0}F)^\perp \) can be chosen not to depend on the choice of \( v \) and the \( \lambda_j(v) \) are continuous functions in \( v \).

Let \( E_j(t) \) be the parallel translates along the geodesic \( \gamma_v(t) \) with \( \gamma_v(0) = x_0 \) and \( \dot{\gamma}_v(0) = v \). Then \( J_j(t) = f_j(t)E_j(t) \) determine an orthogonal basis for the space of Jacobi fields along \( \gamma_v(t) \) which vanishes at \( x_0 \) and are orthogonal to \( \gamma_v(t) \), where \( f_j(t) = t \) if \( \lambda_j(v) = 0 \) (i.e. \( 1 \leq j \leq l - 1 \)) and \( f_j(t) = 1/\sqrt{-\lambda_j(v)}\sinh \sqrt{-\lambda_j(v)}t \) if \( \lambda_j(v) < 0 \) (i.e. \( l \leq j \leq n - 1 \)).

Let \( \mathcal{C} \) be the collection of all Weyl chambers of the first type (i.e. Weyl chambers defined on \( S_{x_0}X \)), where \( S_{x_0}X \subset T_{x_0}X \) is the sphere of unit vectors in the tangent space \( T_{x_0}X \). We will often identify \( S_{x_0}X \) with \( S_r(x_0) \) in an obvious way. Let \( \mathcal{C}(v) \) be the Weyl chamber containing \( v \) for any regular vector \( v \). Let \( v_\mathcal{C} \) be the algebraic centroid of the Weyl chamber \( \mathcal{C} \). Let \( \mathcal{B} \) be the set of the algebraic centroids of all Weyl chambers. By the identification of \( S_{x_0}X \) and \( S_r(x_0) \), we will think of \( \mathcal{B} \) as a set of points on \( S_r(x_0) \) a set of vectors of length \( r \) in \( T_{x_0}X \) and a set of points in \( X(\infty) \). The meaning will be clear from the context. Recall that the subgroup of isometries \( K \) fixing \( x_0 \) acts transitively on \( \mathcal{B} \).

Assume \( v \in \mathcal{B} \) belongs to the \( l \)-flat \( F \). Let \( \mathcal{C}(v) \) be the unique Weyl chamber containing \( v \). Since \( v \) is a regular vector then \( \lambda_l(v), \ldots, \lambda_{n-1}(v) \) are negative. By the continuity of \( \lambda_j \), there exist \( \epsilon > 0 \) and \( \delta > 0 \) such that for every \( w \in N_\epsilon(v) \), \( \lambda_j(w) \leq -\delta^2 \)
for every \( j \geq l \), where \( \mathcal{N}_\epsilon(v) = \{ w \in \mathcal{C}(v) \mid \angle_{x_0}(v, w) = \angle(v, w) \leq \epsilon \} \). We choose \( \epsilon \) small enough such that \( \mathcal{N}_\epsilon(v) \) is contained in the interior of \( \mathcal{C}(v) \) and away from the boundary of \( \mathcal{C}(v) \). By the transitive action of \( K \) on \( \mathcal{B} \) we have such neighborhood around each element of \( \mathcal{B} \). Let \( \mathcal{N}_\epsilon \) be the union of these neighborhoods. Recall that any two Weyl chambers at infinity are contained in the boundary at infinity of a flat in \( X \). Note that if \( v \in \mathcal{B} \) and \( v' \) is the centroid of an opposite Weyl chamber to \( \mathcal{C}(v) \) then \( \angle(v, v') = \pi \). And therefore, for any \( v \in \mathcal{B} \) and any \( w \notin \mathcal{N}_\epsilon \), we have \( \angle(v, w) < \pi - \epsilon \).

From the above discussion we have immediately the following lemma.

**Lemma 4.1.** Let \( 0 < \rho \leq 1 \), \( k \geq l \) and \( \pi_{pr}^r : S_r(x_0) \to S_{pr}(x_0) \) be the projection map. If \( f : S^k \to S_r(x_0) \) is a Lipschitz map then \( \text{vol}_k(\pi_{pr}^r \circ f \cap \mathcal{N}_\epsilon) \leq \text{vol}_k(f \cap \mathcal{N}_\epsilon) \left( \sinh \rho r / \sinh \delta r \right) \leq \text{vol}_k(f) \left( \sinh \rho r / \sinh \delta r \right) \).

Before starting the proof of Theorem 1.1 we need the following lemma.

**Lemma 4.2.** Let \( X \) be a Hadamard manifold, and let \( x_0 \in X \). For every \( \epsilon > 0 \) there exists \( \eta > 0 \) such that for any two points \( p_1, p_2 \in S_r(x_0) \) with \( \angle(x_0, p_1, p_2) \leq \pi - \epsilon \) we have \( d(x_0, \overline{p_1 p_2}) \geq \eta r \).

**Proof.** Let \( \gamma_i = x_0 \overrightarrow{p_i}(t) \) and \( z_i = \gamma_i(\infty) \) for \( i = 1, 2 \). Let \( \alpha_i(t) = \angle \gamma_i(t)(x_0, \gamma_j(t)) \) for \( j \neq i \).

Note that \( \pi - (\alpha_1(t) + \alpha_2(t)) \) is an increasing function of \( t \) which converges to \( \angle(z_1, z_2) \) as \( t \to \infty \). Therefore \( \epsilon \leq \alpha_1(r) + \alpha_2(r) \). Without any loss of generality we may assume that \( \epsilon/2 \leq \alpha_1(r) \). Using the first law of cosines we have \( r^2 + d(p_1, p_2)^2 - 2r d(p_1, p_2) \cos(\epsilon/2) \leq r^2 \). Therefore \( d(p_1, p_2) \leq 2r \cos(\epsilon/2) \). Let \( m \) be the closest point on \( \overline{p_1 p_2} \) to \( x_0 \), and without loss of generality assume that \( d(m, p_1) \leq d(m, p_2) \). By taking \( \eta = 1 - \cos(\epsilon/2) \), we have

\[
d(x_0, \overline{p_1 p_2}) = d(x_0, m) \\
\geq d(x_0, p_1) - d(p_1, m) \\
\geq r - r \cos(\epsilon/2) \\
= \eta r,
\]

which finishes the proof of the lemma. \( \square \)

**Proof of Theorem 1.1.** Let \( -\lambda^2 \) be a lower bound on the sectional curvature of \( X \). Fix \( A > 0 \) and let \( f : S^k \to S_r(x_0) \) be a Lipschitz map such that \( \text{vol}_k(f) \leq Ar^k \). By Lemma 4.1, \( \text{vol}_k(\pi_{pr}^r \circ f \cap \mathcal{N}_\epsilon) \leq \text{vol}_k(f) \left( \sinh \rho r / \sinh \delta r \right) \).

Fix \( q \in S_{pr}(x_0) \) to be the algebraic centroid of a Weyl chamber, and let \( p \in S_{pr}(x_0) \) be its antipodal point. Since curvature and injectivity radii of large spheres are again controlled, using the Deformation Lemma as in the proof of Theorem 3.3, we can assume that \( \pi_{pr}^r \circ f \) misses a ball \( B_\mu(p) \) on \( S_{pr}(x_0) \) and \( \mu \) does not depends on \( r \) as long as \( r \) is large enough.
Now cone the image of $\pi_{pr}^r \circ f$ from $q$ inside the sphere $S_{pr}(x_0)$. Let $C_f$ be the image of the cone. Let $C_1$ be the part of the cone coming from $\pi_{pr}^r \circ f \cap \mathcal{N} \epsilon$ and $C_2 = C_f \setminus C_1$. By Lemma 4.1, $\text{vol}_{k+1}(C_1) \leq 2\rho Ar^{k+1}(\sinh \rho \delta r / \sinh \delta r) \leq 2Ar^{k+1}(\sinh \rho \delta r / \sinh \delta r)$ and $\text{vol}_{k+1}(C_2) \leq 2\rho Ar^{k+1} \leq 2Ar^{k+1}$. Taking $\eta$ to be the constant given by Lemma 4.2, we see that $C_2 \cap B^o_{\eta pr}(x_0) = \emptyset$.

By comparison with the space of constant curvature $-\lambda^2$ and arguing as in the proof of Theorem 3.3, it is easy to see that $C_1$ misses a ball of radius $\rho r \sin(\mu / (2 \sinh 2\lambda \rho r))$ around $x_0$. Now we project from $x_0$ the part of $C_1$ in $B_{\eta pr}(x_0)$ to the sphere $S_{\eta pr}(x_0)$ to obtain a filling $g$ of $\pi_{pr}^r \circ f$ lying outside $B^o_{\eta pr}(x_0)$. Now

$$\text{vol}_{k+1}(g) \leq \text{vol}_{k+1}(C_2) + \text{vol}_{k+1}(C_1) \left( \frac{\sinh \lambda \eta \rho r}{\sinh(\lambda \rho r \sin(\mu / (2 \sinh 2\lambda \rho r)))} \right)^{k+1} \leq 2Ar^{k+1} \left[ 1 + \left( \frac{\sinh \rho \delta r}{\sinh \delta r} \right) \left( \frac{\sinh \lambda \eta \rho r}{\sinh(\lambda \rho r \sin(\mu / (2 \sinh 2\lambda \rho r)))} \right)^{k+1} \right].$$

(4.2)

This estimate behaves like $2Ar^{k+1}[1 + e(\rho(\delta + \lambda(k+1)(2 + \eta)) - \delta)r / (2\lambda \rho \mu)^{k+1}]$ as $r \to \infty$. Clearly we can choose $\rho$ small enough to make the exponent $\rho(\delta + \lambda(k+1)(2 + \eta)) - \delta$ negative and therefore the $(k+1)$-volume of the $\eta \rho$-admissible filling of $\pi_{pr}^r \circ f$ will be bounded by $4Ar^{k+1}$ and the $(k+1)$-volume of the $\eta \rho$-admissible filling of $f$ is bounded by $5Ar^{k+1}$. This finishes the proof of the theorem where $\eta \rho$ is taken in place of $\rho$. □

5. First Filling Invariant for Symmetric Spaces

In this section we prove Theorem 1.3. The proof consists of two steps.

The first step is to deform the closed curve we wish to fill to a new curve which is continuous when viewed as a curve at infinity with respect to the Tits metric. The length of the new curve and the area of the deformation will have to be under control. The proof is valid for any symmetric space of higher rank. We will establish this in Lemma 5.1.

In the second step we will use the assumption that the rank is bigger than or equal 3, to fill the new curve with a disk with controlled area. This step will be done in Proposition 5.2.

We denote by $\lambda_\infty^\times \colon S_r(x_0) \to (X(\infty), Td)$ the map obtained by sending a point $x \in S_r(x_0)$ to the point $\bar{x}_0 \hat{x}(\infty) \in X(\infty)$. Notice that this map is almost never continuous.

Lemma 5.1. Let $X$ be a symmetric space of noncompact type and rank $k \geq 2$. There exist two constants $c > 0$ and $0 < \rho \leq 1$ which depend only on $X$ such that, for every Lipschitz curve $f \colon S^1 \to S_r(x_0)$, there exists a new curve $g \colon S^1 \to S_r(x_0)$ which satisfies the following conditions:

i. $\lambda_\infty^\times \circ g$ is continuous with respect to the Tits metric.

ii. There exists a homotopy between $f$ and $g$ which lies outside $B^o_{\eta pr}(x_0)$. 
iii. The area of the homotopy is bounded above by $c r \text{vol}_1(f)$.

iv. $\text{vol}_1(g) \leq c \text{vol}_1(f)$.

**Proof.** Let $\delta = \min(1, \eta/2)$, where $\eta$ is the constant given by Lemma 4.2 for $\epsilon = \pi/2$. Divide $f$ into pieces each of length $\delta r$, the last piece possibly could be shorter than $\delta r$. If that is the case we will call it “short” and all the other pieces “long”.

Let $c_j: [0, 1] \rightarrow S_r(x_0)$ be one of these pieces. Let $C_i$ be a Weyl chamber containing $c_j(i)$ for $i = 0, 1$. Let $\mathcal{A}$ be an apartment containing $C_0$ and $C_1$. Let $\gamma_j: [0, 1] \rightarrow S_r(x_0)$ be a geodesic (with respect to the Tits metric) in the apartment $\mathcal{A}$ connecting $c_j(0)$ to $c_j(1)$. The goal is to replace the piece $c_j$ with the geodesic $\gamma_j$. The homotopy between them which leaves the end points $c_j(0)$ and $c_j(1)$ fixed will be through geodesics in $X$ connecting $c_j(t)$ to $\gamma_j(t)$ for every $0 \leq t \leq 1$. The length of these geodesics is no longer than $2r$, since they lie inside the ball $B_r(x_0)$.

The new curve $g$ will be formed by replacing each piece $c_j$ with the curve $\gamma_j$. Notice that the length of each $\gamma_j$ is no bigger than $\pi r$. If $\text{vol}_1(f) < \delta r$, i.e. we have no “long” pieces then $g$ is just a point and the statement trivially follows. If $\text{vol}_1(f) \geq \delta r$, i.e. we have at least one long piece then it is not hard to see that $\text{vol}_1(g) \leq 2 \pi \text{vol}_1(f)/\delta$.

Since $X$ is nonpositively curved, the area of the homotopy between $c_j$ and $\gamma_j$ is no bigger than $2\pi r^2$. And the area of the homotopy between $f$ and $g$ is no bigger than $c r \text{vol}_1(f)$, where $c = 4\pi/\delta$.

To finish the proof we need to show that the homotopy lies outside $B^o_{\rho r}(x_0)$. So we need to show that the distance between $x_0$ and the geodesic, in $X$, connecting $c_j(t)$ and $\gamma_j(t)$ is at least $\rho r$ for every $0 \leq t \leq 1$. Assume that $t \leq 1/2$, the other case is similar. Notice that $\angle(c_j(0), \gamma_j(t)) \leq \pi/2$. Applying Lemma 4.2, we get that $d_X(x_0, c_j(0)\gamma_j(t)) \geq \eta r$, where $\eta$ is the constant in Lemma 4.2.

Recall that $d_X(c_j(0), c_j(t)) \leq \delta r$. For every point $s$ on the geodesic $c_j(t)\gamma_j(t)$ there exists a point $s'$ on the geodesic $\gamma_j(t)c_j(0)$ such that $d_X(s, s') \leq \delta r$. Therefore $d_X(x_0, s) \geq (\eta - \delta) r \geq \eta r/2$. By taking $\rho = \eta/2$, the image of the homotopy lies outside $B^o_{\rho r}(x_0)$. This finishes the proof of the lemma. \qed

Now we proceed to the second step of the proof. We prove a more general result.

**Proposition 5.2.** Let $\Delta$ be a spherical building with a re-scaled metric such that each apartment is isometric to $S^{n-1}(r)$, the round sphere of radius $r$. There exists a constant $c > 0$ which depends only on $\Delta$ but not $r$ such that for any Lipschitz function $g: S^k \rightarrow \Delta$, where $k < n-1$, we can extend $g$ to a new function $\hat{g}: B^{k+1} \rightarrow \Delta$ such that $\text{vol}_{k+1}(\hat{g}) \leq c r \text{vol}_k(g)$.

**Proof.** Fix a point $p$ to be the center of a Weyl chamber $C$. Let $\mathcal{A}$ be the collection of all opposite Weyl chambers to $C$, and let $\mathcal{B}$ be the collection of all antipodal points to $p$. Let $\epsilon > 0$ be the largest positive number such for any $q \in \mathcal{B}$ the ball $B_{\epsilon r}(q)$ is
contained in the interior of the Weyl chamber containing \( q \). Notice that \( \epsilon \) only depends on \( \Delta \).

The proof of Lemma 3.1 can be easily modified to deform the function \( g \) to miss the ball \( B_\epsilon r(q) \) for every \( q \in \mathcal{B} \). Now we cone the new deformed function from the point \( p \) to obtain the desired extension.

**Proof of Theorem 1.3.** The proof is immediate. We deform the function \( f \) we wish to fill to a new function \( g \) using Lemma 5.1. We fill \( g \) with a disc by invoking Proposition 5.2, for \( k = 1 \), where we identify \( \Delta \) with the Tits building structure on the sphere \( S_\epsilon(x_0) \) induced from \( (X(\infty), Td) \). The filling of \( g \) lies on \( S_\epsilon(x_0) \) and therefore outside \( B_\epsilon r(x_0) \).

**Remark 5.3.** We expect a similar result to Theorem 1.3 to hold for a larger class than Symmetric spaces. One candidate is the class of Hadamard manifolds whose boundary at infinity is simply connected with respect to the Tits metric.

### 6. Riemannian Manifolds of Pinched Negative Curvature

In this section we give the proof of Theorem 1.4. All the steps of the proof of Theorem 3.3 carry over automatically to our new setting, except the uniform bounds on the curvature and the injectivity radius of distance spheres \( S_\epsilon(x_0) \). By uniform we mean independent of \( r \) for large values of \( r \). These bounds will be established in Lemma 6.3 and Lemma 6.4 below. The proofs are straightforward and we include them for completeness. First we recall proposition 2.5 in chapter IV from [Bal95] concerning estimates on Jacobi fields.

**Proposition 6.1.** Let \( \gamma : \mathbb{R} \rightarrow M \) be a unit speed geodesic and suppose that the sectional curvature of \( M \) along \( \gamma \) is bounded from below by a constant \( \lambda \). If \( J \) is a Jacobi field along \( \gamma \) with \( J(0) = 0 \), \( J'(0) \perp \dot{\gamma}(0) \) and \( \| J'(0) \| = 1 \), then

\[
\| J(t) \| \leq \sin \lambda(t) \quad \text{and} \quad \| J'(t) \| \leq c \lambda(t) \| J(t) \| ,
\]

if there is no pair of conjugate points along \( \gamma \mid [0, t] \).

Notice that the estimate on \( \| J'(t) \| \) is still valid without requiring that \( \| J'(0) \| = 1 \). From the proposition we have the following immediate corollary.

**Corollary 6.2.** Let \( X \) be a Hadamard manifold, with sectional curvature \(-b^2 \leq K \leq -a^2 < 0\), if \( \gamma \) is a unit speed geodesic, \( J(t) \) is a Jacobi field along \( \gamma \) with \( J(0) = 0 \), \( J(t) \perp \dot{\gamma}(t) \) and \( \| J(r) \| = 1 \) then \( \| J'(r) \| \leq c \lambda(t) \| J(t) \| \), and therefore the upper bound goes uniformly, independent of \( \gamma \), to \( b \) as \( r \rightarrow \infty \).

We use the estimate on the derivative of Jacobi fields to put an estimate on the second fundamental form of distance spheres, and hence bounds on the sectional curvature.
Lemma 6.3. Let $X$ be a Hadamard manifold, with sectional curvature $-b^2 \leq K \leq -a^2 < 0$, and $x_0 \in X$. There exists a number $H = H(a, b) > 0$ such that the absolute value of the sectional curvature of $S_r(x_0)$ is bounded by $H$ for every $r \geq 1$.

Proof. The lemma is immediate using the Gauss formula for the sectional curvature of hypersurfaces, namely

$$K(Y, Z) - \overline{K}(Y, Z) = \langle B(Y, Y), B(Z, Z) \rangle - \|B(Y, Z)\|^2.$$

Where $Y$ and $Z$ are orthogonal unit vectors tangent to the hypersurface.

Notice that for $S_r(x_0)$ we have $\|B(Y, Z)\| \leq \|J'_r(r)\|$, with $J(0) = 0$ and $J_Y(r) = Y$, for every two unit vectors $Y$ and $Z$ tangent to the sphere. But $\|J'_r(r)\|$ is bounded by $b$ for large values of $r$ by Corollary 6.2.

In the next lemma we establish a lower bound on the injectivity radius $\text{inj}$ of distance spheres.

Lemma 6.4. Let $X$ be a Hadamard manifold, with sectional curvature $-b^2 \leq K \leq -a^2 < 0$, and $x_0 \in X$. There exists a number $\delta > 0$ such that $\text{inj}(S_r(x_0)) \geq \delta$ for every $r \geq 1$.

Proof. Using Lemma 6.3 we have a uniform upper bound $H$ on the sectional curvature of distance spheres $S_r(x_0)$ for $r \geq 1$ and therefore a lower bound $\pi/\sqrt{H}$ on the conjugate radius.

Our plan is to choose $\delta$ small enough and prove that if $\text{inj}(S_r(x_0)) \geq \delta$ then $\text{inj}(S_{r+s}(x_0)) \geq \delta$ for all $0 \leq s \leq 1$. Using the lower bound $-b^2$ on the sectional curvature of $X$ and by comparison with the space of constant curvature $-b^2$, there exists a constant $0 < B = B(b) < 1$ independent of $r$ and $0 \leq s \leq 1$ such that $\pi_r^{r+s}: S_{r+s}(x_0) \rightarrow S_r(x_0)$ satisfies the following inequality

$$Bd(x, y) \leq d(\pi_r^{r+s}(x), \pi_r^{r+s}(y)) \leq d(x, y), \quad \forall x, y \in S_{r+s}(x_0).$$

Assume $\delta < \min\{\text{inj}(S_1(x_0)), B\pi/2\sqrt{H}\}$. Let us assume that $\text{inj}(S_r(x_0)) \geq \delta$ but $\text{inj}(S_{r+s}(x_0)) < \delta$. Since $\delta \leq \pi/2\sqrt{H}$, there exists two points $p, q \in S_{r+s}(x_0)$ and two minimizing geodesic (with respect to the induced Riemannian metric on $S_{r+s}(x_0)$) $\gamma_1$, $\gamma_2$ connecting $p$ to $q$. Moreover $d(p, q) < \delta$. By Klingenberg’s Lemma, see [dC92], any homotopy from $\gamma_1$ to $\gamma_2$ with the end points fixed must contain a curve which goes outside $B_\delta^{\pi/\sqrt{H}}(p)$. Therefore these two curves are not homotopic inside the ball $B_\delta^{\pi/\sqrt{H}}(p)$. Using (6.1) we have,

$$B_\delta^{\pi/\sqrt{H}}(\pi_r^{r+s}(p)) \sube \pi_r^{r+s}(B_\delta^{\pi/\sqrt{H}}(p)).$$

Notice that $\pi_r^{r+s} \circ \gamma_i$ are contained in $B_\delta^{\pi/\sqrt{H}}(\pi_r^{r+s}(p))$, but not homotopic to each other within that ball, which is a topological ball since $\text{inj}(S_r(x_0)) \geq \delta$ and this is a contradiction. This finishes the proof of the lemma. □
7. The Graph Manifold Example

In this section we give the example mentioned in the introduction showing that Theorem 1.4 is false if we relaxed the condition on the manifold from being negatively curved to being merely rank 1.

Our example will be a graph manifold. Graph manifolds of nonpositive curvature form an interesting class of 3-dimensional manifolds for various reasons. They are the easiest nontrivial examples of rank 1 manifolds whose fundamental group is not hyperbolic. They are rank 1, yet still have a lot of 0-curvature. In fact every tangent vector \( v \in T_p X \) is contained in a 2-plane \( \sigma \subset T_p X \) with curvature \( K(\sigma) = 0 \).

They were used by Gromov in [Gro78] to give examples of open manifolds with curvature \( -a^2 \leq K \leq 0 \) and finite volume which have infinite topological type, contrary to the case of pinched negative curvature. The compact ones have been extensively studied by Schroeder in [Sch86].

Since we are mainly interested in giving a counter example, and for the sake of clarity, we will consider the simplest possible graph manifold. Even though the same idea works for a large class of graph manifolds.

We start by giving a description of the manifold. Let \( W_1 \) and \( W_2 \) be two tori with one disk removed from each one of them. Let \( B_i = W_i \times S^1 \). Each \( B_i \) is called a block. The boundary of each block is diffeomorphic to \( S^1 \times S^1 \). The manifold \( M \) is obtained by gluing the two blocks \( B_1 \) and \( B_2 \) along the boundary after interchanging the \( S^1 \)-factors.

Since the invariants \( \text{div}_k \) do not depend on the metric, we choose a metric which is convenient to work with. Take the flat torus, corresponding to the lattice \( \mathbb{Z} \times \mathbb{Z} \subset \mathbb{R}^2 \). Let \( \beta_1 \) and \( \beta_2 \) be the unique closed geodesics of length 1. We picture the torus as the unit square \( [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}] \subset \mathbb{R}^2 \) with the boundary identified. Remove a small disk in the middle and pull the boundary up, such that the metric is rotationally symmetric around the \( z \)-axis. It is easy to see we can obtain a metric on \( W_i \) with curvature \( -b^2 \leq K \leq 0 \), and make it product near the boundary which is a closed geodesic. Rescale the metric on \( W_i \) to make the curvature \( -1 \leq K \leq 0 \). The closed curves \( \beta_1 \) and \( \beta_2 \) are still closed geodesics, and generate the fundamental group of the punctured torus. Take the metric on each block to be the product of this metric with a circle of length equal to the length of the boundary of \( W_i \). Gluing the two metrics on the two blocks together gives a smooth metric on \( M \) with curvature \( -1 \leq K \leq 0 \).

Let \( X \) be the universal covering space of \( M \), and \( \pi \colon X \to M \) be the covering map. We show that \( \text{div}_1(X) \) is exponential.

Let \( Y \) be a connected component of \( \pi^{-1}(B_1) \). Since \( B_1 \) is a convex subset of \( M \), it is easy to see that \( Y = \mathbb{Z} \times \mathbb{R} \) is the universal covering space of \( B_1 \), where \( \mathbb{Z} \) is the universal covering space of \( W_1 \), and the restriction of \( \pi \) to \( \mathbb{Z} \times \{0\} \) is the covering map onto \( W_1 \). We will identify \( \mathbb{Z} \times \{0\} \) with \( \mathbb{Z} \). Clearly \( \pi_1(B_1) = \pi_1(W_1) \times \mathbb{Z} \) and
\(\pi_1(W_1) = \mathbb{Z} \ast \mathbb{Z}\) is a free group generated by the closed geodesics \(\beta_1\) and \(\beta_2\). Moreover the universal covering space \(Z\) is a thickening of the Cayley graph of \(\mathbb{Z} \ast \mathbb{Z}\), and it retracts to it.

Let \(w_0\) be the unique intersection point of \(\beta_1\) and \(\beta_2\). Take \(w_0\) to be the base point of the fundamental group of \(W_1\). Let \(a_1, a_2 \in \pi_1(W_1, w_0)\) represent the elements corresponding to \(\beta_1\) and \(\beta_2\).

Let \(\psi: Z \rightarrow W_1\) be the covering map which is the restriction to \(Z = \mathbb{Z} \times \{0\}\) of the covering map \(\pi: X \rightarrow M\). Fix \(p_0 \in Z\) such that \(\psi(p_0) = w_0\). We denote the deck transformation corresponding to any element \(s \in \pi_1(W_1, w_0)\) by \(\phi_s\). Let \(\gamma_i\) be the lift of \(\beta_i\) to a geodesic in \(Z\) starting at \(p_0\). Notice that \(\phi_{a_i}\) is a translation along the geodesic \(\gamma_i\). Take \(F = \gamma_1 \times \mathbb{R}\), which is a totally geodesic submanifold isometric to \(\mathbb{R}^2\). And let \(x_0 = (p_0, 0) \in Z \times \mathbb{R}\), where \(p_0 = \gamma_1(0) = \gamma_2(0)\). Let \(S_F(r) = S_r(x_0) \cap F\), be the 1-sphere in \(F\) with center \(x_0\) and radius \(r\), \(A_Z(r) = S_r(x_0) \cap Z\) and \(B_Z(r) = B_r(x_0) \cap Z\).

Let \(f_r: S^1 \rightarrow S_F(r)\) be the canonical diffeomorphism with constant velocity. Notice that \(f_r\) is a Lipschitz map and \(\text{vol}_1(f_r) = 2\pi r\) and therefore an admissible map.

Our goal is to show that for every fixed \(0 < \rho \leq 1\) the infimum over all \(\rho\)-admissible fillings of \(f_r\) grows exponentially as \(r \to \infty\). We show this first for \(\rho = 1\) and then the general case will follow easily from that.

Notice that since the curvature is nonpositive then any filling of \(f_r\) outside \(B_r^c(x_0)\) can be made smaller by radial projection. Therefore a smallest filling of \(f_r\) outside \(B_r^c(x_0)\) would actually lie on the sphere \(S_r(x_0)\), so we will only look at those fillings which lie on the sphere. \(S_r(x_0)\) is a 2-dimensional sphere and \(f_r\) is a simple closed curve on it, therefore it divides the sphere into two halves \(H_1, S\) and \(H_2, S\). Any filling of \(f_r\) has to cover one of these two halves. So it is enough to show that \(\text{vol}_2(H_1, S)\) grows exponentially with \(r\) for \(i = 1, 2\).

We will estimate the area of each half from below by estimating the area of the part which lies inside \(Y\). The geodesic \(\gamma_1\) divides \(Z\) into two halves \(H_1\) and \(H_2\). We concentrate on one of them, say \(H_1\). Let \(b(r) = \text{vol}_2(B_Z(r) \cap H_1)\), which is an increasing function.

The portion of the half sphere \(H_1, S\) inside \(Y\) has area bigger than the area of \(B_Z(r) \cap H_1\). That is easy to see since vertical projection of that part will cover \(B_Z(r) \cap H_1\), and the projection from \(Y = Z \times \mathbb{R}\) onto \(Z\) decreases distance since the metric is a product. So it is enough to show that \(b(r) = \text{vol}_2(B_Z(r) \cap H_1)\) grows exponentially.

It is easy to see that \(\text{vol}_2(B_Z(r))\) grows exponentially with \(r\). One way to see it is to consider the covering map \(\psi: Z \rightarrow W_1\), and look at all lifts, under deck transformations, of a small ball around \(w_0 \in W_1\). The number of these disjoint lifted balls which are contained in \(B_Z(r)\) increases exponentially because the fundamental group is free and therefore has exponential growth.
We need to show that the number of these lifted balls in each half $H_1$ or $H_2$ increases exponentially. We show that for $H_1$.

Without loss of generality we assume that $\phi a_2(p_0) \in H_1$. We claim that under the deck transformations corresponding to the subset $S = \{sa_2 \mid sa_2$ is a reduced word$\} \subset \pi_1(W_1, w_0)$, the image of $p_0$ is in $H_1$. Recall that $Z$ is a thickening of the the Cayley graph of $\mathbb{Z} \ast \mathbb{Z}$ and the action of the deck transformation corresponding to the action of the free group on its Cayley graph. Now the statement follows since the Cayley graph is a tree. But the number of elements in $S$ of length less than or equal $m$ increases exponentially as $m \to \infty$. This finishes the proof of the claim. Therefore $\text{vol}_2(H_1S) \geq b(r)$ grows exponentially and that finishes the proof for $\rho = 1$.

Now we turn to the general case where $0 < \rho \leq 1$. We showed that there exists $0 < \epsilon$ such that for large values of $r$, $e^{\epsilon r} \leq \inf(\text{vol}_2(\hat{f}_r))$, where the infimum is taken over all 1-admissible fillings, i.e. the fillings which lie in $C_r(x_0) = X \setminus B_0^*(x_0)$. Fix $0 < \rho \leq 1$ and let $\pi^r_{\rho r}: S_r(x_0) \to S_{\rho r}(x_0)$, be the projection map. Let $g$ be any $\rho$-admissible filling of $f_r$. Clearly $\pi^r_{\rho r} \circ g$ is a 1-admissible filling of $\pi^r_{\rho r} \circ f_r = f_{\rho r}$. Therefore $\text{vol}_2(g) \geq \text{vol}_2(\pi^r_{\rho r} \circ g) \geq e^{\epsilon \rho r}$, as long as $r$ is big enough. And this finishes the proof.

Remark 7.1. In [Ger94a] Gersten studied the growth rate of the $\text{div}_0$ invariant for a large class of 3-manifolds including graph manifolds. Gersten showed in Theorem 5 that a closed Haken 3-manifold is a graph manifold if and only if the $\text{div}_0$ invariant has a quadratic growth. The author would like to thank Bruce Kleiner for bringing this paper to his attention.

Appendix A.

In this appendix we give a different proof of the following theorem.

**Theorem A.1** (Leuzinger [Leu00]). If $X$ is a rank $k$ symmetric space of nonpositive curvature, then $\text{div}_{k-1}(X)$ has exponential growth.

**Proof.** We use the same notation as in section 4. Fix a maximal $k$-flat $F$ passing through $x_0$. Fix a Weyl chamber $C$ in $F$. Let $v \in C$ be the algebraic centroid of $C$. We identify $X(\infty)$ and $S_r(x_0)$. Since the set of regular points is an open subset of $S_r(x_0)$ with the cone topology, we can find $0 < \eta$ such that the set $S = \{w \in S_r(x_0) \mid \angle_{x_0}(v, w) < \eta\}$ does not contain any singular point. Moreover by choosing $\eta$ small enough we may assume that $S \subset N_c$. To see this it is enough to show a small neighborhood (with respect to the cone topology) of $v$ is contained in $N_c$. Take any sequence $\{v_n\}$ which converges to $v$ in the cone topology. We may assume that $v_n$ is a regular point for each $n$ since the set of regular points is open. Any open Weyl chamber is a fundamental domain of the action of $K$ on the set of regular points in $X(\infty)$, see proposition 2.17.24 in [Ebe96]. Note that the algebraic centroid of a Weyl
chamber is mapped to the algebraic centroid of another Weyl chamber under the action of $K$. Therefore for large values of $n$, $v_n$ would be close to the algebraic centroid of the unique Weyl chamber containing $v_n$, and therefore contained in $\mathcal{N}_\epsilon$.

Let $f_r: S^{k-1} \to S_r(x_0) \cap F$ be a diffeomorphism. We will show that any filling of $f_r$ grows exponentially with $r \to \infty$. Assume not, then for each $n \in \mathbb{N}$ there is a filling $\hat{f}_n$ for $f_n$ such that the vol$_k(\hat{f}_n)$ grows sub-exponentially. Let $\pi_n^1: S_n(x_0) \to S_1(x_0)$ be the projection map. By Lemma 4.1 vol$_k(\pi_n^1 \circ \hat{f}_n \cap \mathcal{N}_\epsilon) \leq$ vol$_k(\hat{f}_n)(\sinh \delta / \sinh \delta n)$, which goes to zero as $n \to \infty$. Let $\phi$ be the projection to the flat $F$. And $g_n$ be the projection of $\pi_n^1 \circ \hat{f}_n$ to $F$. The image of $g_n$ has to cover the closed unit ball in $F$.

For every $w \notin \mathcal{N}_\epsilon$, $\angle x_0(v, w) \geq \eta$ and therefore $d(\phi(w), v) \geq 2a = 1 - \cos \eta$. Let $A = B_a(v) \cap B_1(x_0)$ be the part of the $k$-ball in the flat $F$ centered at $v$ with radius $a$ which lies inside the unit closed $k$-ball in $F$. Now it is easy to see that the only part of $g_n$ which will lie inside $A$ would be coming from the portion inside $\mathcal{N}_\epsilon$. The $k$-volume of that part goes to zero as $n \to \infty$, nevertheless it has to cover $A$ which is a contradiction. Therefore the filling of $f_r$ grows exponentially. \hfill $\square$

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