Flavor-Mixing Effects on the QCD Phase Diagram at non-vanishing Isospin Chemical Potential: One or Two Phase Transitions?

M. Frank\textsuperscript{a,b}, M. Buballa\textsuperscript{a,c}, and M. Oertel\textsuperscript{d}

\textsuperscript{a} Institut für Kernphysik, TU Darmstadt, Schlossgartenstr. 9, 64289 Darmstadt, Germany
\textsuperscript{b} Fachbereich Mathematik, TU Darmstadt, Schlossgartenstr. 7, 64289 Darmstadt, Germany
\textsuperscript{c} Gesellschaft für Schwerionenforschung (GSI), Planckstr. 1, 64291 Darmstadt, Germany
\textsuperscript{d} IPN-Lyon, 43 Bd du 11 Novembre 1918, 69622 Villeurbanne Cédex, France

Abstract

We investigate effects of a fixed nonzero isospin chemical potential on the $\mu_B$-$T$ phase diagram of strongly interacting matter using a Nambu–Jona-Lasinio-type four fermion interaction. We focus on the influence of a flavor-mixing interaction induced by instantons. We find that already for rather moderate values of the coupling strength in the flavor-mixing channel the recent findings of two separate phase transitions do not persist.

The structure of the phase diagram of QCD at nonzero temperature and baryon chemical potential has been intensively studied throughout the last decade, as well on theoretical as on experimental side. In the baryon chemical potential-temperature ($\mu_B$-$T$) plane roughly speaking three domains can be distinguished: the hadronic phase at low temperature and density, a quark matter phase at low temperature and high baryon density, which is most probably a color superconductor \cite{1,2}, and the quark gluon plasma at high temperature. For two flavors at zero chemical potential the hadronic phase is expected to be separated by a crossover from the quark-gluon-plasma \cite{3}. The fact that at zero temperature a first-order transition to quark matter is expected \cite{4} implies the existence of a second-order endpoint somewhere in the $\mu_B$-$T$-plane. This second-order endpoint is one of the features of the QCD phase diagram which could be detected in heavy-ion collisions \cite{5}.

Up to now most theoretical investigations of the phase diagram and are restricted to zero isospin chemical potential $\mu_I$ (see, e.g., Refs.\cite{1,2,4,5}). The influence of a non-zero $\mu_I$ has been studied in more detail in the context of quark matter at high baryon density and low temperature, e.g., \cite{6,7,8}, as well as for vanishing $\mu_B$ \cite{9,10}.

Recently the $\mu_B$-$T$ phase diagram for a fixed $\mu_I \neq 0$ has been studied using a random matrix model \cite{11} and a Nambu–Jona-Lasinio (NJL) model \cite{12}. The authors of Refs.\cite{11,12} find striking effects: There are two first-order phase transitions at low temperature and high baryon chemical potential and thus two second-order endpoints. This could be important, e.g., for heavy-ion collisions where $\mu_I$ is supposed to be nonzero. In this letter
we will argue that the existence of two separate phase transitions becomes unlikely, once flavor-mixing effects due to instantons are taken into account.

Our starting point is the following Lagrangian for two flavors:

\[ \mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 , \]

with a free part

\[ \mathcal{L}_0 = \bar{q} (i\partial - m) q , \]

and two different interaction parts (see, e.g., [13, 14]),

\[ \mathcal{L}_1 = G_1 \left\{ (\bar{q}q)^2 + (\bar{q} \gamma_5 q)^2 + (\bar{q} i\gamma_5 \bar{q}q)^2 \right\} \]

and

\[ \mathcal{L}_2 = G_2 \left\{ (\bar{q}q)^2 - (\bar{q} \gamma_5 q)^2 - (\bar{q} i\gamma_5 \bar{q}q)^2 + (\bar{q} i\gamma_5 \bar{q}q)^2 \right\} , \]

where \( q = (u, d)^T \). \( G_1 \) and \( G_2 \) are coupling constants of dimension energy\(^{-2} \) and \( m = \text{diag}(m_u, m_d) \) contains the current quark masses. For simplicity we assume \( m_u = m_d \).

The interaction is invariant under \( SU_L(2) \times SU_R(2) \times U_V(1) \) transformations. \( \mathcal{L}_1 \) exhibits an additional \( U_A(1) \) symmetry, whereas this is not true for \( \mathcal{L}_2 \). \( \mathcal{L}_2 \) has the structure of a 't-Hooft determinant in flavor space [15]. This interaction can be interpreted as induced by instantons and reflects the \( U_A(1) \)-anomaly of QCD. The reason for this particular choice of the two interaction parts will become clear below.

We now want to study the properties of the system at temperature \( T \) and the up and down quark chemical potentials

\[ \mu_u = \mu + \delta \mu , \quad \mu_d = \mu - \delta \mu , \]

which are in general different. Here \( \mu = \mu_B / 3 \) is the quark number chemical potential and \( \delta \mu = \mu_I / 2 \) is proportional to the isospin chemical potential \(*\). To obtain the mean field thermodynamic potential \( \Omega(T, \mu, \delta \mu) \), we linearize the Lagrangian in the presence of the following quark condensates:

\[ \phi_u = \langle \bar{u}u \rangle , \quad \phi_d = \langle \bar{d}d \rangle , \]

which a priori can be different. We do not consider \( \rho = \frac{1}{2} (\langle \bar{u} \gamma_5 d \rangle - \langle \bar{d} \gamma_5 u \rangle) \), i.e., we exclude the possibility of pion condensation. This restricts our model to values of \( |\delta \mu| < m_\pi / 2 \simeq 70 \text{ MeV} \). For the moment we also do not consider color superconducting phases. This will, however, not change our results qualitatively.

For our further proceeding it is convenient to introduce constituent quark masses:

\[ M_i = m_i - 4 G_1 \phi_i - 4 G_2 \phi_j , \quad i \neq j \in \{ u, d \} . \]

\(*\)This follows from the definitions of the baryon density, \( n_B = \frac{1}{3} (n_u + n_d) \), and the isospin density \( n_I = \frac{1}{2} (n_u - n_d) \) together with the thermodynamic relation \( n_i = -\partial \Omega / \partial \mu_i \) for \( i = u, d, B, I \). Note that often different definitions are used. For instance, \( \mu_B \) and \( \mu_I \) of Refs. [8, 12] correspond to our \( \mu \) and \( \delta \mu \).
Using this definition and Eq. (4) for the chemical potentials of up and down quarks, respectively, we obtain for the mean field thermodynamic potential:

$$\Omega(T, \mu_u, \mu_d) = \sum_{f=u,d} \Omega_0(T, \mu_f; M_f) + 2 G_1 (\phi_u^2 + \phi_d^2) + 4 G_2 \phi_u \phi_d,$$

(8)

where $$\Omega_0(T, \mu_f; M_f)$$ corresponds to the contribution of a gas of quasiparticles of flavor $$f$$:

$$\Omega_0(T, \mu_f; M_f) = -\frac{3}{\pi^2} \int dp p^2 \left\{ E_f + T \ln \left( 1 + \exp \left( -\frac{1}{T}(E_f - \mu_f) \right) \right) + T \ln \left( 1 + \exp \left( -\frac{1}{T}(E_f + \mu_f) \right) \right) \right\},$$

(9)

with $$E_f = \sqrt{M_f^2 + p^2}$$. In order to determine the physical solutions, we have to look for the stationary points of the thermodynamic potential with respect to the two condensates $$\phi_u$$ and $$\phi_d$$. This leads to the following gap equations which have to be solved self-consistently:

$$\phi_f = -\frac{3}{\pi^2} \int dp p^2 \frac{M_f}{E_f} \left\{ 1 - n(E_f) - \bar{n}(E_f) \right\},$$

(10)

where $$n(E_f) = 1/(\exp((E_f - \mu_f)/T) + 1)$$ and $$\bar{n} = 1/(\exp((E_f + \mu_f)/T) + 1)$$ are Fermi occupation numbers. Here we have to keep in mind that via Eq. (8), the constituent mass $$M_i$$ for one flavor depends in general on both condensates and therefore the two flavors are coupled. At this point our separation of the interaction part becomes clear: If we for the moment neglect $$\mathcal{L}_2$$, i.e., $$G_2 = 0$$, the two flavors decouple. That means $$M_i$$ only depends on the condensate of the same flavor $$\phi_i$$ and the mixed contribution to $$\Omega$$ (last term of Eq. (8)) vanishes. In this limit we recover the expression for the thermodynamic potential of Ref. [12] for $$\rho = 0$$. In the opposite limit, i.e., $$G_1 = 0$$, we have “maximal” mixing: The constituent mass of flavor $$i$$ only depends on the condensate of the same flavor $$\phi_i$$ with $$i \neq j$$. For $$G_1 = G_2$$ we recover the original Lagrangian proposed by Nambu and Jona-Lasinio [16], implying $$M_u = M_d$$.

To study the effects of flavor mixing, let us now write

$$G_1 = (1 - \alpha) G_0, \quad G_2 = \alpha G_0,$$

(11)

and calculate the phase diagram for fixed $$G_0$$ but different values of $$\alpha$$. The amount of flavor mixing is thereby controlled by the particular value of $$\alpha$$ while the values of the vacuum constituent quark masses $$M_{\text{vac}}$$ are kept constant.

For our numerical studies we use the following set of parameters: $$m_u = m_d = 6 \text{ MeV}$$, a three-dimensional sharp cutoff $$\Lambda = 590 \text{ MeV}$$ and $$G_0 \Lambda^2 = 2.435$$, corresponding to vacuum constituent masses $$M_{\text{vac}} = 400 \text{ MeV}$$. With these parameters we obtain reasonable values for the pion mass, decay constant and the quark condensate in the vacuum: $$m_\pi = 140.2 \text{ MeV}$$, $$f_\pi = 92.6 \text{ MeV}$$ and $$\langle \bar{u}u \rangle = (-241.5 \text{ MeV})^3$$.

We begin our discussion with the results at $$T = 0$$. In Fig. 4 we display the values of $$M_u$$ and $$M_d$$ as functions of the quark number chemical potential $$\mu$$ for fixed $$\delta \mu = 30 \text{ MeV}$$.

\footnote{Following common practice (e.g., [6, 8, 12]) we take a positive value of $$\delta \mu$$, although for the description of heavy-ion collisions $$\delta \mu < 0$$ would be more appropriate. However, since changing the sign of $$\delta \mu$$ does only interchange the roles of up and down quarks, this does not alter our conclusions.}
Figure 1: Constituent quark masses $M_u$ (solid) and $M_d$ (dashed) at $T = 0$ as functions of quark number chemical potential $\mu$ for $\delta \mu = 30$ MeV and $\alpha = 0$ (left), $\alpha = 0.05$ (center), and $\alpha = 0.11$ (right).

The left panel corresponds to $\alpha = 0$, i.e., to the case without flavor mixing. We observe two distinct phase transitions at $\mu = 353$ MeV for the up quarks and at $\mu = 413$ MeV for the down quarks. This behavior is easily understood when we recall that at $\alpha = 0$ the up and down quark contributions to the thermodynamic potential completely decouple. Hence, if we had plotted $M_u$ and $M_d$, in terms of the corresponding flavor chemical potential $\mu_u$ and $\mu_d$, respectively, we would have found two identical functions with a phase transition at $\mu_f = 383$ MeV. This is basically the result reported in Refs. [11, 12].

Now we want to study the influence of a non-vanishing flavor mixing. In the central panel of Fig. 1 we show the behavior of the constituent quark masses for $\alpha = 0.05$. The situation remains qualitatively unchanged, i.e., we still find two distinct phase transitions. However, because $M_d$ now also depends on $\phi_u$ (and thus on $M_u$), and vice versa, both constituent masses drop at both critical chemical potentials. Moreover, this small amount of flavor mixing already diminishes the difference between the two critical quark number chemical potentials considerably. Finally, for $\alpha$ larger than a critical value of 0.104 we find only one single first-order phase transition. This is illustrated in the right panel of Fig. 1 which corresponds to $\alpha = 0.11$.

Next, we extend our analysis to non-vanishing temperature. The phase diagrams in the $\mu$-$T$ plane for fixed $\delta \mu = 30$ MeV and three different values of $\alpha$ are shown in Fig. 2. At $\alpha = 0$ (left panel) we qualitatively reproduce the results discussed in Refs. [11, 12], i.e., two separate first-order phase boundaries which end in two second-order endpoints. Again, since for $\alpha = 0$ the up and down quarks decouple completely, we would obtain two identical phase diagrams if we plotted the phase structure of flavor $f$ in the $\mu_f$-$T$ plane. In the central panel of Fig. 2 we consider $\alpha = 0.11$, i.e., slightly larger than the critical value $\alpha_c(T = 0) = 0.104$ for a single phase transition at $T = 0$. Accordingly, there is only one phase boundary at low temperatures, but at $T = 25$ MeV it splits into two lines which end at two different second-order endpoints. The two branches are, however, very close to each other and already at $\alpha = 0.12$ we find only one phase boundary with a single endpoint. This is illustrated by the diagram on the right, which corresponds to $\alpha = 0.15$. 
Figure 2: Phase diagrams in the $\mu$-$T$-plane for $\delta \mu = 30$ MeV and $\alpha = 0$ (left), $\alpha = 0.11$ (center), and $\alpha = 0.15$ (right). The lines correspond to first-order phase boundaries which end in second-order endpoints.

In our example a rather small amount of flavor mixing is sufficient to remove the existence of the second phase transition: Of course, there must be a single phase transition at $\alpha = 0.5$, where $M_u$ and $M_d$ are equal (see Eq. 11). (This was the case studied in Ref. [8].) However, the critical value $\alpha_c \approx 0.12$ we found in our example is much smaller. At $T = 0$, a rough, but perhaps more general estimate for the critical $\alpha$ can be obtained from the observation that the phase transition takes place when the chemical potential of quark $f$ comes close to its constituent mass, i.e., $\mu_f \approx M_i$. Applying this condition to the $u$ quark we expect the first phase transition to take place at $\mu_u \approx M_{vac}$, i.e., at $\mu \approx M_{vac} - \delta \mu$. At this point $M_u$ drops and, according to Eq. 7, $M_d$ drops as well. Neglecting the current quark mass, we find

$$M_d \approx -(1 - \alpha) 4G_0 \phi_d \lesssim (1 - \alpha) M_{vac}.$$  

(12)

If this value becomes smaller than the value of $\mu_d$, we expect also the down quarks to exhibit a phase transition. Hence, we estimate

$$\alpha^c(T = 0) \lesssim \frac{2\delta \mu}{M_{vac}}.$$  

(13)

Note that this estimate would not be affected by a possible restoration of the $U_A(1)$ symmetry at the phase boundary. Obviously, if $G_2$ goes to zero, $M_d$ would drop as well.

For our example, Eq. 13 gives $\alpha^c(T = 0) \approx 0.15$. Comparing this value with the numerical result $\alpha^c(T = 0) = 0.104$, we see that Eq. 13 is a quite conservative estimate. This is easily understood, since in the second step of Eq. 12 we have neglected the fact, that $\phi_d$ also becomes smaller. Our estimate does also not include the observation, that the critical chemical potential for the first phase transition rises with $\alpha$. In any case, we have to admit that our arguments cannot explain quantitatively why Eq. 13 seems to hold even for temperatures approaching the critical endpoint where the quark masses do no longer drop discontinuously.
At this point one can ask, which value of $\alpha$ is “realistic”. To answer this question it is helpful to have a look at the 3-flavor NJL model, where the strength of the ’t Hooft interaction has been fitted by several authors to describe the $\eta-\eta'$-splitting [17, 18]. For three flavors, the ’t Hooft determinant is a six-point interaction [15], and the constituent quark masses in a 3-flavor NJL model are given by

$$M_i = m_i - 4 G \phi_i + 2 K \phi_j \phi_k, \quad i \neq j \neq k \neq i \in \{u, d, s\}. \quad (14)$$

When we compare this with Eq. (7) we can identify $G_1 = G$ and $G_2 = -\frac{1}{2} K \phi_s$ and thus

$$\alpha = \frac{-\frac{1}{2} K \phi_s}{G - \frac{1}{2} K \phi_s}. \quad (15)$$

If we take, for instance, the values of Ref. [17], $\Lambda = 602.3$ MeV, $G A^2 = 1.835$, $K A^5 = 12.36$, and $\phi_s = (-257.7$ MeV$)^3$, we find $\alpha \simeq 0.21$. For the parameters of Ref. [18] we get a somewhat smaller value, $\alpha \simeq 0.16$. On the other hand, the success of the instanton liquid model to describe hadronic correlation functions [19] would suggest that $L_2$ is the dominant part of our Lagrangian, i.e., $\alpha \simeq 1$. Anyway, in all cases we would find only one phase transition for $\delta \mu = 30$ MeV. Typical values of $|\delta \mu|$ in heavy-ion collisions are likely to be smaller than that. (A simple estimate, assuming the density ratio $n_u : n_d = 290 : 334$ as in $^{208}$Pb, and the approximate relation $n_u : n_d \approx (\mu_u : \mu_d)^3$ yields $\delta \mu \approx -10$ MeV for $\mu = 400$ MeV. Empirically, one finds $\mu_t = 2 \delta \mu = -5$ MeV at chemical freeze-out for Pb-Pb collisions at SPS [20] and $\mu_t = -12$ MeV for Si+Au collisions at AGS [21].)

However, before drawing quantitative conclusions, we should be aware of several shortcomings of the present model. First, the description of the “hadronic phase” as a gas of quarks, rather than hadrons, is certainly unrealistic. In fact, it is quite obvious that any prediction of the critical endpoint(s) in non-confining mean-field models should not be trusted. (Lattice calculations, although not yet generally accepted, indicate that the critical endpoint of QCD at $\delta \mu = 0$ is located at $\mu_B = 3 \mu \simeq 725 \pm 35$ MeV and $T = 160 \pm 3.5$ MeV [22], i.e., at lower chemical potential and higher temperature than our or other NJL results, e.g., [12, 13].) Moreover, as already mentioned, we have neglected the possibilities of pion and diquark condensation. Since pion condensation, although interesting by itself, does only occur for $\delta \mu \gtrsim 70$ MeV, it is irrelevant for the present discussion. A possible diquark condensation in the 2SC phase [11] would further favor a single phase transition, since this phase requires an approximately equal number of up and down quarks. An extension of the present model to include these possibilities is straightforward.

Keeping the limitations of our model in mind, our results show that flavor-mixing effects cannot be neglected in the discussion of the phase diagram. The pure existence of these effects is related to instantons and the $U_A(1)$-anomaly of QCD. Of course their magnitude is a matter of debate, but they probably cancel the interesting phenomena discussed in Refs. [11, 12].

**Acknowledgments:**
We thank P. Braun-Munzinger for pointing out Ref. [21] to us. M.O. acknowledges support from the Alexander von Humboldt foundation as a Feodor-Lynen fellow.
References

[1] K. Rajagopal and F. Wilczek, hep-ph/0011333, and references therein.

[2] M. Alford, Ann. Rev. Nucl. Part. Sci. 51 (2001) 131.

[3] F. Karsch, AIP Conf. Proc. 602 (2001) 323;
   J.B. Kogut, hep-lat/0208077

[4] M.A. Halasz, A.D. Jackson, R.E. Shrock, M.A. Stephanov, and J.J. Verbaarschot,
   Phys. Rev. D 58 (1998) 096007.

[5] M. Stephanov, K. Rajagopal, and E. Shuryak, Phys. Rev. D 60 (1999) 114028.

[6] P.F. Bedaque, Nucl. Phys. A 697 (2002) 569.

[7] M.G. Alford, J.A. Bowers, and K. Rajagopal, Phys. Rev. D 63 (2001) 074016.

[8] O. Kiriyama, S. Yasui, and H. Toki, Int. J. Mod. Phys. E 10 (2002) 501.

[9] D.T. Son and M. Stephanov, Phys. Rev. Lett. 86 (2001) 592.

[10] J.B. Kogut and D.K. Sinclair, Phys. Rev. D 66 (2002), 014508; ibid. 034505.
[11] B. Klein, D. Toublan, and J.J.M. Verbaarschot, hep-ph/0301143

[12] D. Toublan, J.B. Kogut, hep-ph/0301183

[13] M. Asakawa and K. Yazaki, Nucl. Phys. A 504 (1989) 668.

[14] S.P. Klevansky, Rev. Mod. Phys. 64 (1992) 3.

[15] G. ’t Hooft, Phys. Rev. D 14 (1976) 3432; Phys. Rep. 142 (1986) 357.

[16] Y. Nambu and G. Jona-Lasinio, Phys. Rev. 122 (1961) 345; 124 (1961) 246.

[17] P. Rehberg, S.P. Klevansky and J. Hüfner, Phys. Rev. C 53 (1996) 410.

[18] T. Kunihiro, Phys. Lett. B 219 (1989) 363.

[19] T. Schäfer and E. Shuryak, Rev. Mod. Phys. 70 (1998) 323.

[20] P. Braun-Munzinger, I. Heppe, and J. Stachel, Phys. Lett. B 465 (1999) 15.

[21] I. Heppe, Diplomarbeit, Universität Heidelberg 1998, unpublished
    (http://www.physi.uni-heidelberg.de/physi/ceres/publications/IHeppe-diplom.ps).

[22] Z. Fodor and S.D. Katz, JHEP 0203 (2002) 014.