On pion and kaon decay constants and chiral SU(3) extrapolations

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Abstract

We consider the dependence of the pion and kaon decay constants on the up, down and strange quark masses in QCD with strict isospin symmetry. The role of dynamical vector meson degrees of freedom is scrutinized in terms of an effective chiral Lagrangian for vector mesons. Applying a set of low-energy parameters as determined previously from QCD lattice data on the masses of the light vector mesons from PACS-CS, QCDSF-UKQCD and HSC we compute its implications on the pion and kaon decay constants for QCD lattice ensembles of HPQCD, CLS and ETMC. It is shown that with Gasser-Leutwyler $L_4$ and $L_5$ parameters fixed to the empirical decay constants an accurate reproduction of their values at unphysical quark masses as computed by HPQCD, CLS and ETMC is achieved. Results for the masses of the light vector meson, the $\omega - \phi$ mixing angles and the quark mass ratios for the ensembles used by HPQCD, CLS and ETMC are discussed.

1. Introduction

So far any chiral extrapolation attempt for the decay constants of the Goldstone bosons of QCD that is based on the chiral Lagrangian formulated for three light flavours appeared futile because of the rather large strange quark mass. By today most lattice collaborations abandoned the use of three flavour extrapolations and consider chiral extrapolations significant only in the small up and down quark masses of QCD \cite{1}. In turn the flavour SU(3) limit value, $f$, of the pion and kaon decay constants is poorly known at present. Given the fact that $\Lambda_\chi = 4 \pi f$ sets the chiral symmetry breaking scale of QCD with three light flavours this is quite unfortu-
nate. The parameter $f$ is of fundamental importance in hadronic interactions since it drives any application of the three-flavour chiral Lagrangian of QCD [2, 3, 4, 5, 6, 7, 8, 9, 10]. The root of this unpleasant situation lies in the rather poor convergence properties of a strict $\chi$PT expansion based on the three-flavour chiral Lagrangian [11, 12, 13, 14, 15, 16].

In this work we present a remedy of this issue by two unconventional ingredients. First we recast one-loop expressions derived from a chiral Lagrangian in terms of physical meson masses, while keeping the renormalization scale invariance of the expressions [17, 18, 19]. Second, we consider a chiral Lagrangian with explicit vector meson degrees of freedom [20, 21, 19]. Its low-energy constants have been determined recently at the one-loop level in [22] from lattice data on meson masses from PACS-CS, QCDSF-UKQCD and HSC [23, 24, 25, 26]. In particular a rather small value for $f = (70.5 \pm 3.0)$ MeV was obtained compatible with the range suggested by the two-loop estimates of $\chi$PT in [15]. This study left undetermined the two low-energy parameters $L_4$ and $L_5$ only, which enter the computation of the pion and kaon decay constants. It is the purpose of this work to present results for the decay constants on the ensembles used by HPQCD, CLS and ETMC [27, 28, 29].

2. Chiral dynamics for the pion and kaon decay constants

Our analysis on the pion and kaon decay constants is based on the chiral Lagrangian with dynamical vector meson degrees of freedom as further developed recently in [19, 22]. For the specifics of the Lagrangian and explicit expressions valid at the one-loop level properly derived in a finite volume we refer to our previous works [19, 22]. Note that as in [19] we do not yet consider the explicit effects of the $\eta'$ field [21].

Here we recall only the expressions for the decay constants. The one-loop contributions from the Goldstone bosons and the light vector mesons in (1) drive the decay constants $f_P$ away from their chiral limit value $f$. Altogether, for the decay constants we use the simple result

\[
 f_P = f_P^{\chi PT} + \left( \sqrt{Z_P^{\text{bubble}}} - 1 \right) f - \frac{f}{\sqrt{Z_P^{\text{bubble}}}} \frac{\Pi_P^{\text{bubble}}}{m_P^2},
\]

\[
 Z_P^{\text{bubble}} = 1 + \frac{\partial}{\partial m_P^2} \frac{\Pi_P^{\text{bubble}}(m_P^2)}{m_P^2},
\]
Figure 1: Our results from Fit 2 of [22] for the pion and kaon decay constants in the infinite volume limit as a function of the quark mass ratio $m_s/m$ at physical value for $m$. While the solid lines are from (1), the dashed ones from (2) where the contributions of vector meson loops are ignored. Further results are shown for the masses and mixing angles of the vector mesons.

where we split the loop contributions into a conventional part $f_f^{\chi-PT}$ and terms that reflect the presence of dynamical vector meson degrees of freedom. The contribution from the latter was derived previously in [19] at $Z_P^{\text{bubble}} = 1$. Here we consider the effect from the wave function renormalization in addition. The conventional part

$$f f_{\pi}^{\chi-PT} = f^2 - \bar{I}_\pi - \frac{1}{2} \bar{I}_K + 4 m_\pi^2 L_5 + 4 (2 m_K^2 + m_\pi^2) L_4 ,$$

$$f f_{K}^{\chi-PT} = f^2 - \frac{3}{8} \bar{I}_\pi - \frac{3}{4} \bar{I}_K - \frac{3}{8} \bar{I}_\eta + 4 m_K^2 L_5 + 6 (m_\eta^2 + m_\pi^2) L_4 ,$$

$$\bar{I}_P = \frac{m_P^2}{(4\pi)^2} \log \frac{m_P^2}{\mu^2} + \text{finite volume terms} ,$$

(2)

involves the low-energy constants $L_4$ and $L_5$ of Gasser and Leutwyler and
the tadpole integrals $\tilde{I}_P$ properly evaluated in a finite volume. The latter depends on the meson mass $m_P$, the renormalization scale $\mu$ and the lattice volume only (see e.g [30]). Our expressions differ from the traditional results to the extent that our form does not involve any explicit dependence on the quark masses. Still the expressions do not depend on the renormalization scale $\mu$, strictly. This reflects our strategy that hadron masses inside loop expressions should take their on-shell values. We assure that upon a further chiral expansion of $f_P^{\chi-PT}$ the traditional form is recovered identically. The contributions of the vector mesons are implied by a bubble loop contributions to the polarization function $\Pi_P(s = m_P^2)$ as was derived in [19, 22]. The latter involves the masses of the Goldstone bosons, $m_P$, and the vector mesons, $M_V$, in their isospin limit.

In Fig. 1 the effect of dynamical vector mesons is illustrated as it comes in the infinite volume limit. In the upper panel the pion and kaon decay constants are shown as a function of the quark mass ratio $r = m_s/m$ at physical value of $m = m_u = m_d$. While at the flavour limit point with $r = 1$, the results from (1) and (2) almost agree, at the physical ratio with $r \simeq 26.5$ we find a significant effect from the vector meson loop contributions. We note a discontinuous dependence on the quark mass ratio. A small but significant jump in the kaon decay constant is visible at $r \simeq 20.5$. The main driving source of this effect is traced to a striking discontinuous behaviour of the mass dependent $\omega - \phi$ mixing angle. Of physical relevance is the mixing angle evaluated at either the $\omega$ or $\phi$ on-shell mass only, which we denote by $\epsilon_\omega$ and $\epsilon_\phi$ respectively. The latter are shown in the lower panels of Fig. 1 which also presents our results for the vector meson masses. Here the $\omega$ and $\phi$ meson masses exhibit a visible jump at the same critical value of the quark-mass ratio. It would be important to scrutinize our prediction by suitable QCD lattice simulations for the quark-mass dependence of the $\omega - \phi$ mixing angles. The possibility of such a discontinuous behaviour opens once on-shell masses inside loop expressions are used [31]. This implies that a set of non-linear and coupled equations have to be considered. We emphasize that such a phase transition cannot be ruled out from first principals in QCD. Similar transitions, but in different systems, were observed in previous works [31, 32].

In the determination of our low-energy constants [22] from QCD lattice data we considered finite-box energy levels from PACS-CS, QCDSF-UKQCD and HSC [23, 24, 25, 26]. For any lattice ensemble of a given finite-box size we took the pion and kaon masses as input parameters. The set of nine
coupled and non-linear mass equations is solved in terms of the two quark masses, $B_0 m, B_0 m_s$, the remaining 5 meson masses, $m_\eta, M_\rho, M_\omega, M_{K^*}, M_\phi$ and the $\omega - \phi$ mixing angles $\epsilon_\omega$ and $\epsilon_\phi$. We apply the evolutionary algorithm of GENEVA 1.9.0-GSI \cite{33} with runs of a population size 1500 on 300 parallel CPU cores. The two parameters $L_4$ and $L_5$ can be dialed always as to recover the PDG values of $f_\pi = (92.1 \pm 1.2)$ MeV and $f_K = (110.0 \pm 0.3)$ MeV. The challenge is to describe then the decay constants at unphysical quark masses as provided by QCD lattice computations. We emphasize that the presence of the vector meson contributions as given here and \cite{22} does not renormalize either $L_4$ or $L_5$. Given our renormalization scheme the vector meson contributions are at least of quadratic order in the quark masses. The decay constants depend on $f, L_4, L_5$ and the meson masses only. No further explicit unknown parameter dependence is encountered in our one-loop framework.

Following the strategy of our previous works \cite{17,18,22} we use the empirical values of the meson masses and together with the pion and kaon decay constants from the Particle Data Group \cite{34} as additional constraints in our fit scenarios. The main target of our studies is to derive its low-energy representation in terms of hadronic degrees of freedom. In this case it is of advantage to perform a non-standard scale setting in terms of a larger set of observable quantities. The lattice scale of all ensembles at a given $\beta$ value is considered as a free parameter to be determined from the lattice data set together with the chosen set of quantities from the PDG.

In Tab. 1 of \cite{22} three sets of LEC are collected. While Fit 1 is based on the meson masses only, the other two scenarios considered the pion and kaon decay constants as measured by HPQCD and CLS on their lattice ensembles \cite{27,28}. Our Fit 3 considers in addition the pion decay constants from ETMC \cite{29}. Here we do not take into account their kaon decay constants,
\begin{table}
\begin{tabular}{c|cc|c}
 & Fit 2 & Fit 3 & Lattice \\
\hline
$a_{\text{HPQCD}}^{=5.8}$ [fm] & 0.1535 & 0.1524 & 0.1509 - 0.1543 \\
$\chi^2/N$ & 1.11 & 1.50 & \cite{27} \\
\hline
$a_{\text{HPQCD}}^{=6.0}$ [fm] & 0.1230 & 0.1222 & 0.1212 - 0.1241 \\
$\chi^2/N$ & 0.93 & 1.39 & \cite{27} \\
\hline
$a_{\text{HPQCD}}^{=6.3}$ [fm] & 0.0890 & 0.0887 & 0.0879 - 0.0907 \\
$\chi^2/N$ & 1.18 & 1.37 & \cite{27} \\
\hline
$a_{\text{CLS}}^{=3.40}$ [fm] & 0.0786 & 0.0778 & I: 0.08636(98)(40) \\
$\chi^2/N$ & 0.33 & 0.51 & II: 0.0790(11) \cite{28} \\
\hline
$a_{\text{CLS}}^{=3.46}$ [fm] & 0.0715 & 0.0706 & I: 0.07634(92)(31) \\
$\chi^2/N$ & 0.20 & 0.15 & II: 0.071(2) \cite{28} \\
\hline
$a_{\text{CLS}}^{=3.55}$ [fm] & 0.0603 & 0.0598 & I: 0.06426(74)(17) \\
$\chi^2/N$ & 1.00 & 1.30 & II: 0.0613(9) \cite{28} \\
\hline
$a_{\text{CLS}}^{=3.70}$ [fm] & 0.0475 & 0.0471 & I: 0.04981(56)(10) \\
$\chi^2/N$ & 0.23 & 0.27 & II: 0.0481(8) \cite{28} \\
\hline
$a_{\text{ETM}}^{=1.95}$ [fm] & 0.0830 & 0.0830 & 0.0815(30) \\
$\chi^2/N$ & 20.86 & 7.06 & \cite{29} \\
\hline
$a_{\text{ETM}}^{=2.10}$ [fm] & 0.0610 & 0.0610 & 0.0619(18) \\
$\chi^2/N$ & 1.79 & 0.98 & \cite{29} \\
\end{tabular}
\caption{Partial $\chi^2/N$ values and lattice scales for the various ensembles as implied by the fit scenario 2 and 3.}
\end{table}

since they are affected by a wave function factor that is subject to significant uncertainties. We will return to this issue below.

In Tab. 1 the result of our three scenarios for $f$, $L_4$ and $L_5$ are displayed. A rough estimate of the uncertainties in the LEC is suggested by the spread of the latter in the three scenarios. Four significant digits are shown in
order to permit a numerical reproduction. The values in $f$ and $L_4, L_5$ show moderate variations. A rather small value for $L_4$ is obtained always as was expected by its suppression in the large-$N_c$ limit of QCD. We observe that the strong anti-correlation of the parameters $f$ and $L_4$ in the decay constants as emphasized in [11, 13], is lifted significantly in our approach since it considers QCD lattice data at unphysical quark masses. We find most remarkable the small values for $L_5$ in the three scenarios, which are in striking conflict with the range provided by the conventional approach based on the chiral SU(3) Lagrangian at the two-loop level [15]. The consideration of dynamical vector meson degrees of freedom causes a significant change in $L_5$, driving it to a value that is almost compatible with zero at the given renormalization scale. This was anticipated already in our previous work [19] and should be scrutinized in further dedicated lattice studies.

Since a large set of lattice data is fitted the propagated statistical error on any of the low-energy constants is very small and insignificant. Any uncertainty in the latter stems from systematical deficiencies underlying our approach, like the neglect of discretization effects, or the impact of two-loop diagrams in our scheme. Such a systematic study is much beyond the scope of the present work. Note that the small uncertainties in the empirical values for the vector meson masses or the decay constants do not propagate to any significant uncertainty in our scheme.

At this stage we assume universal systematical errors for the vector meson masses and the decay constants. Our fits are based on an asymmetric error in the vector meson masses of $^{+10}_{-20}$ MeV together with a symmetric error of $^{\pm 125}$ MeV for the decay constants $\sqrt{2} f_P$. This implies the $\chi^2/N$ values collected in Tab. 2 and also in Tab. 1 of [22], where $N$ is always the number of fitted lattice data points. The lattice data sets from HPQCD and CLS on the pion and kaon decay constants are well reproduced in Fit 2 and Fit 3. Our results from Fit 2 are visualized by Fig. 2, in which we show the decay constants in units of the lattice scale for all considered ensembles of HPQCD and CLS. With typical values $\chi^2/N \sim 1$ the decay constants are recovered with an uncertainty of about 0.9 MeV.

In Tab. 2 we provide also our results for the lattice scales of HPQCD and CLS at different $\beta$ values as they are a consequence of our fit strategy. While our scale setting for the three $\beta$ values considered by HPQCD appears compatible with the analysis in [27], we observe a disagreement with the preferred scale setting for the four $\beta$ values of CLS in [28]. The authors
Figure 2: Our results from Fit 2 for the pion and kaon decay constants in lattice units, where the corresponding lattice scales are collected in Tab. 2. The lattice results are given by green, blue, and red filled symbols, where statistical errors are shown only. They are compared to the chiral extrapolation results in open symbols, which are always displayed on top of the lattice symbols.

We report on two distinct methods. Their method I, which has small statistical errors only, is in conflict to our results. However, their second method, which comes with somewhat larger statistical errors, predicts lattice scales that are quite compatible with our values. Both values are recalled in the last column of Tab. 2. While two different scale setting methods need not lead to identical lattice scales, the size of discretization effects in the observable quantities may be distinct in the two methods. Our conclusion on the CLS ensembles would be that their second method, may have larger statistical errors, however, it comes with smaller discretization errors, and therefore is more convenient to use.

We turn to the decay constants of ETMC [29]. In Fit 3 the pion decay constants are included in the total chisquare function. The chisquare values
Figure 3: Our results from Fit 3 for the pion and kaon decay constants in lattice units. The lattice results are given by blue and red filled symbols, where statistical errors are shown only. They are compared to the chiral extrapolation results in open symbols, which are always displayed on top of the lattice symbols. We use $Z = 0.6884$ and $Z = 0.7428$ for the $\beta = 1.95$ and $\beta = 2.10$ ensembles respectively. Only statistical error bars are shown.

in Tab. 2 are with respect to the latter only. The values are a bit larger than those for HPQCD and CLS. This may hint at somewhat larger discretization effects on the ETMC ensembles, in particular on the ensembles with $\beta = 1.95$. We note that our lattice scale determination for ETMC in Fit 3 is in the range suggested in [29]. The chisquare values for Fit 2, which did not consider constraints from ETMC, are based on our lattice scales of Fit 3. In Fig. 3 we show the pion and kaon decay constants in lattice units. The points for the kaon decay constants in the figure are 'pseudo' data obtained from [29] upon a tuning of their wave function factor $Z$. Since only the leading impact of $Z$ on the kaon decay constants is accessible from [29] our points in the figure are subject to additional moderate changes. Our estimates for $Z$ are in the range defined by the values in [29] based on two distinct methods. We find amazing that given our estimates for the wave function we match the quark-mass ratios as provided by ETMC quite accurately. Such ratios are not included in our chisquare function. This is illustrated with Fig. 4 which shows in addition all vector meson masses and the $\omega - \phi$ mixing angles. Like in our previous work on the PACS-CS, QCDSF-UKQCD and HSC ensembles we foresee a significant quark mass dependence of the $\omega - \phi$ mixing angles also on the ETMC ensembles. Our predictions for the finite-box vector meson energy levels in Fig. 4 await direct computations of the latter on the ETMC
Figure 4: Our results from Fit 3 for the vector meson masses, $\omega - \phi$ mixing angles and quark mass ratios on the ETM ensembles. The lattice results are given by blue and red filled symbols. They are compared to the chiral extrapolation results in open symbols, which are always displayed on top of the lattice symbols. We use yellow or grey colour filled symbols for the cases where there is no corresponding lattice point available yet.

We should caution the reader against the case where a given ensemble
Figure 5: Our results from Fit 2 for the vector meson masses, the $\omega - \phi$ mixing angles and quark mass ratios on the HPQCD ensembles. The lattice results are given by green, blue, and red filled symbols, where statistical errors are shown only. They are compared to the chiral extrapolation results in open symbols, which are always displayed on top of the lattice symbols. We use yellow or grey colour filled symbols for the cases where there is no corresponding lattice point available yet.
leads to more than one zero-momentum energy level relevant for a vector meson mass determination. Our self-consistency condition is set up, at this stage, only for a single relevant finite-box energy level. A generalization to more than one level, as it is implied by large volume lattice simulations necessarily, is feasible but beyond the scope of the present work. In order to delve into this issue we systematically show the first unperturbed two-body scattering levels. From the positions of the latter in Fig. 4 we conclude that our predictions for the energy levels of the vector mesons on the ETMC ensembles are sound.

Our results for the ensembles of HPQCD are shown in Fig. 5. Note that for some ensembles a \( \phi \) meson energy level is available from HPQCD \cite{35}. Such levels were considered in Fit 2 and Fit 3 and are accurately reproduced. In various cases the scattering levels turn important for the \( \rho \) and \( K^* \) and therefore our results on such ensembles have to be taken with a grain of salt. In Fig. 5 we show the first few unperturbed scattering levels embedded into a shaded area. While we cannot describe the set of expected energy levels in this case, our results are still significant. This is so since our self-consistency condition implies an average over the distributed levels. We expect this average to lie close to the most relevant energy level.

We turn to the CLS ensembles as scrutinized in Fig. 6. The recent results on the \( \rho \) meson energy levels in \cite{36} were not considered in any of our fits. From the six ensembles analyzed in \cite{36} we considered the four cases C101, D200, N200 and J303. For the two ensembles N200 and J303 there is one significant level for the \( \rho \) meson at rest only, and indeed here our 'predictions' are in line with the levels given in \cite{36}. For the remaining ensembles C101 and D200 two significant energy levels are reported on in \cite{36}. Those are connected by a solid line in Fig. 6. Our predicted effective levels are close to the upper one in both cases. Indeed, according to \cite{36} the latter are close to the nominal \( \rho \) meson mass. This confirms our expectation that in the presence of more than one relevant energy level we still arrive at significant results. However, to further improve the accuracy of our results it may be useful to generalize the approach and implement the self-consistency condition for a set of \( \rho \) meson energy levels.

In Fig. 5 and Fig. 6 we show also our predictions for the \( \omega - \phi \) mixing angles and the quark mass ratios on the ensembles of HPQCD and CLS. We confirm our previous claim on a striking quark-mass dependence of those mixing angles. For both collaborations we recover their quark mass ratios on all considered ensembles quite accurately, despite the fact that none of them
Figure 6: Our results from Fit 2 for the vector meson masses, the $\omega - \phi$ mixing angles and quark mass ratios on the CLS ensembles. The lattice results are given by purple, green, blue and red filled symbols, where statistical errors are shown only. They are compared to the chiral extrapolation results in open symbols, which are always displayed on top of the lattice symbols. We use yellow or grey colour filled symbols for the cases where there is no corresponding lattice point available yet.
was considered in any of our chisquare functions.

3. Summary

In this work we considered the pion and kaon decay constants, \( f_\pi \) and \( f_K \), as evaluated from a chiral SU(3) Lagrangian with dynamical vector meson fields. Our results are based on the one-loop level and the strict isospin limit. It was shown that with Gasser and Leutwyler’s \( L_4 \) and \( L_5 \) parameters adjusted to the empirical values of \( f_\pi \) and \( f_K \), corresponding lattice results from HPQCD, CLS and ETMC on ensembles with unphysical quark masses can be reproduced accurately once the effect of dynamical vector meson degrees of freedom are taken into account. At the renormalization point \( \mu = 770 \) MeV our predicted value for \( L_5 = (2 - 4) \times 10^{-5} \), is in striking contradiction to conventional estimates based on \( \chi \)PT studies at the two-loop level. This supports our previous claim that the presence of dynamical vector meson can not be absorbed convincingly into low-orders \( \chi \)PT approaches.

Our previous result for the chiral SU(3) limit value of the decay constants, which was obtained by global fits to lattice data on the vector meson masses from PACS-CS, QCDSF-UKQCD and HSC, was shown to be consistent with the lattice data on the decay constants from HPQCD, CLS and ETMC. Our best estimate is \( f = (69.1 \pm 1.6) \) MeV. Quantitative results for the masses of the light vector meson as well as the quark mass ratios for the ensembles used by HPQCD and CLS are predicted.

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References

[1] S. Aoki, et al., Review of lattice results concerning low-energy particle physics, Eur. Phys. J. C77 (2) (2017) 112. [arXiv:1607.00299](https://arxiv.org/abs/1607.00299) doi:[10.1140/epjc/s10052-016-4509-7](https://doi.org/10.1140/epjc/s10052-016-4509-7)

[2] G. Ecker, J. Gasser, A. Pich, E. de Rafael, The role of resonances in chiral perturbation theory, Nucl. Phys. B321 (1989) 311. [doi:10.1016/0550-3213(89)90346-5](https://doi.org/10.1016/0550-3213(89)90346-5)
[3] E. E. Jenkins, A. V. Manohar, M. B. Wise, Chiral perturbation theory for vector mesons, Phys. Rev. Lett. 75 (1995) 2272–2275. arXiv:hep-ph/9506356, doi:10.1103/PhysRevLett.75.2272.

[4] F. Klingl, N. Kaiser, W. Weise, Effective Lagrangian approach to vector mesons, their structure and decays, Z. Phys. A356 (1996) 193–206. arXiv:hep-ph/9607431, doi:10.1007/s002180050167.

[5] M. C. Birse, Effective chiral Lagrangians for spin 1 mesons, Z. Phys. A355 (1996) 231–246. arXiv:hep-ph/9603251, doi:10.1007/s002180050105.

[6] J. Bijnens, P. Gosdzinsky, P. Talavera, Vector meson masses in chiral perturbation theory, Nucl. Phys. B501 (1997) 495–517. arXiv:hep-ph/9704212, doi:10.1016/S0550-3213(97)00391-X.

[7] J. Bijnens, P. Gosdzinsky, P. Talavera, Matching the heavy vector meson theory, JHEP 01 (1998) 014. arXiv:hep-ph/9708232, doi:10.1088/1126-6708/1998/01/014.

[8] V. Cirigliano, G. Ecker, H. Neufeld, A. Pich, Meson resonances, large N(c) and chiral symmetry, JHEP 06 (2003) 012. arXiv:hep-ph/0305311, doi:10.1088/1126-6708/2003/06/012.

[9] I. Rosell, J. J. Sanz-Cillero, A. Pich, Quantum loops in the resonance chiral theory: The Vector form-factor, JHEP 08 (2004) 042. arXiv:hep-ph/0407240, doi:10.1088/1126-6708/2004/08/042.

[10] P. C. Bruns, L. Greil, A. Schäfer, Chiral behavior of vector meson self energies, Phys. Rev. D88 (2013) 114503. doi:10.1103/PhysRevD.88.114503.

[11] S. Descotes-Genon, N. H. Fuchs, L. Girlanda, J. Stern, Resumming QCD vacuum fluctuations in three flavor chiral perturbation theory, Eur. Phys. J. C34 (2004) 201–227. arXiv:hep-ph/0311120, doi:10.1140/epjc/s2004-01601-4.

[12] J. Soto, P. Talavera, J. Tarrus, Chiral Effective Theory with A Light Scalar and Lattice QCD, Nucl. Phys. B866 (2013) 270–292. arXiv:1110.6156, doi:10.1016/j.nuclphysb.2012.09.005.
[13] G. Ecker, P. Masjuan, H. Neufeld, Approximating chiral $SU(3)$ amplitudes, Eur. Phys. J. C74 (2) (2014) 2748. arXiv:1310.8452, doi:10.1140/epjc/s10052-014-2748-z.

[14] Z.-H. Guo, J. J. Sanz-Cillero, Resonance effects in pion and kaon decay constants, Phys. Rev. D89 (9) (2014) 094024. arXiv:1403.0855, doi:10.1103/PhysRevD.89.094024.

[15] J. Bijnens, G. Ecker, Mesonic low-energy constants, Ann. Rev. Nucl. Part. Sci. 64 (2014) 149–174. arXiv:1405.6488, doi:10.1146/annurev-nucl-102313-025528.

[16] C. Terschlüsen, S. Leupold, Contributions of loops with dynamical vector mesons to masses and decay constants of pseudoscalar mesons and their quark mass dependence arXiv:1604.01682.

[17] M. F. M. Lutz, Y. Heo, X.-Y. Guo, On the convergence of the chiral expansion for the baryon ground-state masses, Nucl. Phys. A977 (2018) 146–207. arXiv:1801.06417, doi:10.1016/j.nuclphysa.2018.05.007.

[18] X.-Y. Guo, Y. Heo, M. F. M. Lutz, On chiral extrapolations of charmed meson masses and coupled-channel reaction dynamics, Phys. Rev. D98 (1) (2018) 014510. arXiv:1801.10122, doi:10.1103/PhysRevD.98.014510.

[19] R. Bavontaweepanya, X.-Y. Guo, M. F. M. Lutz, On the chiral expansion of vector meson masses, Phys. Rev. D98 (5) (2018) 056005. arXiv:1801.10522, doi:10.1103/PhysRevD.98.056005.

[20] M. F. M. Lutz, S. Leupold, On the radiative decays of light vector and axial-vector mesons, Nucl. Phys. A813 (2008) 96–170.

[21] C. Terschlüsen, S. Leupold, M. F. M. Lutz, Electromagnetic Transitions in an Effective Chiral Lagrangian with the $\eta'$ and Light Vector Mesons, Eur. Phys. J. A48 (2012) 190. doi:10.1140/epja/i2012-12190-6.

[22] X.-Y. Guo, M. F. M. Lutz, On light vector mesons and chiral SU(3) extrapolations. arXiv:1810.07078.
[23] S. Aoki, et al., 2+1 Flavor Lattice QCD toward the Physical Point, Phys. Rev. D79 (2009) 034503. arXiv:0807.1661, doi:10.1103/PhysRevD.79.034503

[24] W. Bietenholz, et al., Flavour blindness and patterns of flavour symmetry breaking in lattice simulations of up, down and strange quarks, Phys. Rev. D84 (2011) 054509. arXiv:1102.5300, doi:10.1103/PhysRevD.84.054509

[25] H.-W. Lin, et al., First results from 2+1 dynamical quark flavors on an anisotropic lattice: Light-hadron spectroscopy and setting the strange-quark mass, Phys. Rev. D79 (2009) 034502. arXiv:0810.3588, doi:10.1103/PhysRevD.79.034502

[26] J. J. Dudek, R. G. Edwards, P. Guo, C. E. Thomas, Toward the excited isoscalar meson spectrum from lattice QCD, Phys. Rev. D88 (9) (2013) 094505. arXiv:1309.2608, doi:10.1103/PhysRevD.88.094505

[27] R. J. Dowdall, C. T. H. Davies, G. P. Lepage, C. McNeile, Vus from pi and K decay constants in full lattice QCD with physical u, d, s and c quarks, Phys. Rev. D88 (2013) 074504. arXiv:1303.1670, doi:10.1103/PhysRevD.88.074504

[28] M. Bruno, T. Korzec, S. Schaefer, Setting the scale for the CLS 2 + 1 flavor ensembles, Phys. Rev. D95 (7) (2017) 074504. arXiv:1608.08900, doi:10.1103/PhysRevD.95.074504

[29] K. Ottnad, C. Urbach, Flavor-singlet meson decay constants from $N_f = 2 + 1 + 1$ twisted mass lattice QCD, Phys. Rev. D97 (5) (2018) 054508. arXiv:1710.07986, doi:10.1103/PhysRevD.97.054508

[30] M. F. M. Lutz, R. Bavontaweepanya, C. Kobdaj, K. Schwarz, Finite volume effects in the chiral extrapolation of baryon masses, Phys. Rev. D90 (5) (2014) 054505. doi:10.1103/PhysRevD.90.054505

[31] A. Semke, M. F. M. Lutz, Discontinuous quark-mass dependence of baryon octet and decuplet masses, Nucl. Phys. A789 (2007) 251–259. arXiv:nucl-th/0606027, doi:10.1016/j.nuclphysa.2007.02.011

[32] F. K. Guo, C. Hanhart, F. J. Llanes-Estrada, U. G. Meissner, When hadrons become unstable: A novel type of non-analyticity in chiral...
extrapolations, Phys. Lett. B703 (2011) 510–515. arXiv:1105.3366, doi:10.1016/j.physletb.2011.08.022

[33] R. Berlich, S. Gabriel, A. Garcia, M. Kunze, Distributed Parametric Optimization with the Geneva Library. Data Driven e-Science, Conference proceedings of ISGC 2010, Springer New York (2010) 303. URL www.gemfony.eu

[34] M. Tanabashi, et al., Review of Particle Physics, Phys. Rev. D98 (3) (2018) 030001. doi:10.1103/PhysRevD.98.030001

[35] B. Chakraborty, C. T. H. Davies, G. C. Donald, J. Koponen, G. P. Lepage, Nonperturbative comparison of clover and highly improved staggered quarks in lattice QCD and the properties of the $\phi$ meson, Phys. Rev. D96 (7) (2017) 074502. arXiv:1703.05552, doi:10.1103/PhysRevD.96.074502

[36] C. Andersen, J. Bulava, B. Hrz, C. Morningstar, The $I = 1$ pion-pion scattering amplitude and timelike pion form factor from $N_f = 2 + 1$ lattice QCD, Nucl. Phys. B939 (2019) 145–173. arXiv:1808.05007, doi:10.1016/j.nuclphysb.2018.12.018