Monopoles in Space-Time Noncommutative Born-Infeld theory

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Abstract: We transform static solutions of space-noncommutative Dirac-Born-Infeld theory (DBI) into static solutions of space-time noncommutative DBI. Via Seiberg-Witten map we match this symmetry transformation with a corresponding symmetry of commutative DBI. This allows to: 1) study new BPS type magnetic monopoles, with constant electric and magnetic background and describe them both in the commutative and in the noncommutative setting; 2) relate by S-duality space-noncommutative magnetic monopoles to space-noncommutative electric monopoles.

1 Introduction

Dirichlet branes effective actions can be described by noncommutative gauge theories, the noncommutativity arising from a nonzero constant background NS \( B \) field, see \cite{1} and references therein. In fact, when \( B \neq 0 \), the effective physics on the D-brane can be described both by a commutative gauge theory \( \mathcal{L}(F + B) \) and by a noncommutative one \( \hat{\mathcal{L}}(\hat{F}) \), where

\[
\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - i[\hat{A}_\mu \star \hat{A}_\nu]
\]

and \( \star \) is the Moyal star product; on coordinates \([x^\mu \star x^{\nu}] = x^\mu \star x^{\nu} - x^{\nu} \star x^\mu = i\Theta^{\mu\nu}\). The noncommutativity parameter \( \Theta \) depends on \( B \) and on the metric on the D-brane. The commutative/noncommutative descriptions are related by Seiberg-Witten map (SW map) \cite{4}. Initially space-noncommutativity has been considered (\( \Theta^{ij} \neq 0 \) i.e. \( B_{ij} \neq 0 \)) then theories also with time noncommutativity (\( \Theta^{0i} \neq 0 \) i.e \( B_{0i} \neq 0 \)) have been studied \cite{2}. It turns out that unitarity of noncommutative Yang-Mills theory (NCYM) holds only if \( \Theta \) is space-like or light-like i.e. only if the electric and magnetic components of \( \Theta \) (or \( B \)) are perpendicular and if the electric component is not bigger in magnitude than the magnetic component. These are precisely the NCYM theories that can be obtained from open strings in the decoupling limit \( \alpha' \rightarrow 0 \) \cite{3}. In this talk we consider these two kinds of space-time noncommutativity.

In Section 2 we show that for any space-noncommutative static solution we can turn on time-noncommutativity and obtain a static solution with space-time noncommutativity. This holds in particular for solutions of noncommutative Dirac-Born-Infeld-theory (NCDBI). Via SW map we obtain the nontrivial action of this symmetry in the corresponding commutative DBI theory and show that it is a rotation (boost) between the time component \( A_0 \) of the gauge potential and the worldvolume D-brane coordinates \( x^i \).

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This boost, first studied in [4], is similar to the target space rotation that relates the linear monopole to the nonlinear monopole [3], [4]. In Section 3 we study BPS solutions of NCDBI and DBI theories with both electric and magnetic background. In [3, 4] solutions to the BPS equations of noncommutative electromagnetism (NCEM), and of NCDBI, with just space-noncommutativity are found. The solution in [3] describes a smeared monopole connected with a string-like flux tube and is interpreted as a D1-string ending on a D3-brane with constant magnetic field background. We see that this solution remains a BPS solution also when we turn on time-noncommutativity. The corresponding commutative configuration is also found: It is a new BPS configuration. Its energy is the energy of the initial BPS monopole with only magnetic field plus the energy of a constant electric field in DBI theory. The new BPS configuration describes a monopole plus string in a background that is both electric and magnetic. The monopole has the fundamental magneton charge and the string tension is that of a D1-string, this strongly suggests that we have a D1-D3 brane system with both electric and magnetic background. The D1-string tension is also matched with the string tension of the corresponding space-time noncommutative BPS monopole.

Finally in Section 4 we address the issue of duality rotations in NCDBI and NCEM [8, 9]. At the field theory level duality is present only if Θ is light-like i.e. the magnetic and electric component of Θ are perpendicular and equal in magnitude [10, 11]. For Θ space-like as shown in [8] we do not have noncommutative gauge theory self-duality and the S-dual of space-like NCEM is a noncommutative open string theory decoupled from closed strings. A main point here is that under S-duality the magnetic background is mapped into an electric one, and this background does not lead to a field theory in the $\alpha' \rightarrow 1$ limit. Since Θ must be light-like, it may seem that duality rotations have a very restricted range of application. This is not the case because the symmetry we present in Sections 2 and 3 changes the background too. It turns out that it is possible to compose duality rotations with this symmetry, we are thus able to consider duality rotations with background fixed and arbitrary. In particular we briefly discuss the S-dual of the D1-string D3-brane configuration of [6]; it describes an electric monopole plus string in a magnetic background, possibly a fundamental string ending on a D3-brane in the presence of a constant magnetic field.

## 2 Gauge theory with space-time noncommutativity

We use the following notations: Θ is a generic constant noncommutativity tensor, we have $[x^\mu \star x^\nu] = i\Theta^{\mu\nu}$; $\theta$ is just a space-noncommutativity tensor $\theta^{ij}, \theta^{0i} = 0$; $\theta^e$ is a space-time noncommutativity tensor obtained from $\theta$ adding electric components, $\theta^{eij} = \theta^{ij}, \theta^{0i}$. In three vector notation the electric and magnetic components of $\theta$, respectively $\theta^e$, are $(0, \theta)$ and $(\mathcal{E}, \theta)$. The background fields corresponding to $\Theta, \theta, \theta^e$ are $B, b, b^e$.

Consider a noncommutative Lagrangian $\hat{\mathcal{L}}^e = \hat{\mathcal{L}}(\hat{F}, \hat{\phi}, G, \star_{\theta})$ where $\hat{\phi}$ are scalar fields and $G$ is the metric. The equations of motion (EOM) for $\hat{\mathcal{L}}^e$ read

$$f_\alpha(\hat{F}, \hat{\phi}, G, \star_{\theta}) = 0$$

where $f_\alpha$ are functions of the noncommutative fields and their derivatives. We notice that a static solution of the $\hat{\mathcal{L}}^e$ EOM (2) is also a static solution of (2) with $\theta^e$ instead of $\theta$, i.e. it is a static solution of the $\hat{\mathcal{L}}^e$ EOM. Indeed the star products $\star_{\theta}$ and $\star_{\theta^e}$ act in the same way on time independent fields. Moreover the energy and charges of the solution
are invariant. A similar $\theta$-$\theta^C$ symmetry property holds if the fields are independent from a coordinate $x^\mu$ (not necessarily $t$). This $\theta$-$\theta^C$ symmetry property of static solutions can be used to construct moving solutions of a space-noncommutative theory $\mathcal{L}^\theta$ from static solutions of the same theory $\hat{\mathcal{L}}^\theta$. Indeed, given $\theta$, if we turn on an electric component such that $\mathbf{E} \perp \theta$ and $|\mathbf{E}| < |\theta|$, then with a Lorentz boost we can transform this new $\theta^C$ into a space-like $\theta'$ proportional to the initial $\theta$. Rescaling $\theta' \rightarrow \theta$ we thus obtain a solution (moving with constant velocity) of the space-noncommutative Lagrangian $\hat{\mathcal{L}}^\theta$.

We now use SW map and study how the $\theta$-$\theta^C$ symmetry acts in the commutative theory. We have to consider the two SW maps $SW^\theta$ and $SW^{\theta^C}$. In general a static solution $\hat{\phi}, \hat{A}_\mu$ is mapped by $SW^\theta$ and $SW^{\theta^C}$ into two different commutative solutions, however if $\hat{A}_0 = 0$ then $SW^\theta = SW^{\theta^C}$. This can be seen from the index structure of SW map. In general we have

$$A_\mu = \hat{A}_\mu + \sum_{n \geq 0} (\Theta^{(n)} \partial^{(n+s)} \hat{A}^{(n-s)} \hat{A})_\mu$$

$$\phi = \hat{\phi} + \sum_{n \geq 0} \Theta^{(n)} \partial^{(n+s)} \hat{A}^{(n-s)} \hat{\phi}$$

where the number of times $n, n+s, n-s$ that $\Theta, \partial, \hat{A}$ appear is dictated by dimensional analysis. In (3) we do not specify which $\partial$ acts on which $\hat{A}$ and we do not specify the coefficients of each addend. Because of the index structure we notice that $\Theta^0$ never enters (3) if $\phi, \hat{A}$ are time independent and $\hat{A}_0 = 0$. The commutative fields $\phi, A_i$ corresponding to $\hat{\phi}, \hat{A}_i$ ($i \neq 0$) are solution of both $\mathcal{L}^\theta$ and $\mathcal{L}^{\theta^C}$. Here $\mathcal{L}^\theta$ and $\mathcal{L}^{\theta^C}$ are the commutative Lagrangians corresponding to $\hat{\mathcal{L}}^\theta$ and $\hat{\mathcal{L}}^{\theta^C}$ via SW map. In the case of the DBI Lagrangian with a scalar field $\phi$, $\mathcal{L}^\theta$ and $\mathcal{L}^{\theta^C}$ reads

$$\mathcal{L}_{DBI}(F + b, \phi, g, g_s) = -\frac{1}{\alpha'^2 g_s} \sqrt{-\text{det}(g + \alpha'(F + b) + \alpha'^2 \partial^2 \phi_\alpha \partial^2 \phi^\alpha)}$$

and $\mathcal{L}_{DBI}(F + b^C, \phi, g^C, g_s)$. We should write $g_s^C$ instead of $g_s$ in this last expression, however we can rescale $G_s$ and thus impose the invariance of the closed string coupling constant $g_s$. The relation between closed and open string parameters is given by (see [1]):

$$(g + \alpha'B)^{-1} = G^{-1} + \Theta/\alpha', \ G_s = g_s \sqrt{\text{det}G \text{det}(g + \alpha'B)^{-1}}.$$
For time independent fields we have that the EOM imply $\mathcal{H}' = -\nabla' \chi'$ and $\mathcal{E}' = -\nabla' \psi'$, $(\psi' = -\mathcal{A}_0')$. As shown in [4] it follows that $\tilde{H}(\psi', \chi', \phi')$ is the action of a space-like 3-brane immersed in a target space of coordinates $X^A = \{\alpha' \psi', \alpha' \chi', \alpha' \phi', x'^i\}$ and metric $\eta = \text{diag}(-1, -1, 1, 1, 1)$

$$\int d^3 x' \tilde{H} = -\frac{1}{g_s \alpha'^d} \int d^3 x' \sqrt{\det \left( \eta_{AB} \frac{\partial X^A}{\partial x'^i} \frac{\partial X^B}{\partial x'^j} \right)}.$$ (7)

It is the $SO(2, 4)$ symmetry [4] of this static gauge action that is relevant in our context: Consider the Lorentz transformation $Y^A = \Lambda^A_{\beta} X^B$ (where $X'^i = x'^i$) and express $Y^A$ as $Y^A = Y^A(y^i)$ (where $Y^i = y^i$) so that we are still in static gauge; the action (7) is invariant under $X'^A(x'^i) \rightarrow Y^A(y^i)$. In particular a boost in the $\alpha' \psi'$, $x'_2$ plane with velocity $\beta = -\alpha' e''$ gives (7) [11].

### 3 BPS solutions for (NC)DBI with a scalar field

A BPS solution of DBI theory in the $x$-reference system with metric $g$ and background $b = \theta/(\alpha^2 + \theta^2)$, $\theta = -\theta^{12}$, all others $\theta^{\mu\nu} = 0$ is given by

$$\phi = -\frac{\theta}{\alpha^2} x^3 - \frac{1}{2r}, \quad B_i = -\partial_i \phi; \quad r^2 = g_{ij} x^i x^j = (x^3)^2 + (x^1)^2 + (x^2)^2 \frac{1 + \theta^2/\alpha^2}{\alpha^2}$$ (8)

where $\frac{1}{2r} = \frac{q_m}{2m}$ with $q_m$ the magneton charge $q_m = 2\pi$. This solution describes a D1-string ending on a D3-brane. Because of the magnetic background field $b$ on the brane the string is not perpendicular to the brane, in (8) the string is vertical and the brane is tilted w.r.t. the horizontal direction. The magnetic force acting on the end of the D1-string is compensated by the tension of the D1-string.

A BPS solution of NCDBI with space-like noncommutativity given by $\theta^{12} = -\theta$, (all others $\theta^{\mu\nu} = 0$) has been studied in [6]. It is a static solution with $\hat{A}_0 = 0$. This noncommutative BPS solution is characterized by a noncommutative string tension and a magneton charge. It is expected to correspond, via SW map (and the target space rotation relating the linear monopole to the nonlinear one [3]), to (8). A first evidence is the correspondence between the tension of the noncommutative string and that of the D1-string [3]; then in [6] the spectrum of small fluctuations around (a limit of) this solution is studied and found in agreement with the expectations from string theory.

We now discuss what happens to these commutative/noncommutative BPS solutions when we apply the $\theta^{\mu\nu}$ symmetry. The result [11] is that we obtain two new BPS solutions that describe a D1-string ending on a D3-brane with both an electric and magnetic background. The noncommutative one is obtained simply writing $\theta^{\mu\nu}$ instead of $\theta$ (and rescaling the open string coupling constant $G_s \rightarrow G_s^e$ since we keep $g_s$ invariant), it still satisfies the noncommutative BPS equations $\dot{B}_j = -\dot{D}_j \phi$ with $\dot{D}_j \phi = \partial_j \phi - i [\hat{A}_j \phi]$. The commutative solution is most easily written in the orthonormal reference system $x''$:

$$\mathcal{B}''_2 = -\gamma \partial'_2 \phi''$$
$$\mathcal{B}''_q = -\gamma^{-1} \partial'_q \phi''$$
$$\mathcal{E}''_2 = e''$$
$$\mathcal{E}''_q = 0$$ (9)

with $\gamma^{-1} = \sqrt{1 - \alpha'^2 e''}$ and

$$\phi'' = -\gamma b'' x'^3 - \frac{1}{2R}, \quad R^2 \equiv (x''^1)^2 + \gamma^{-2} (x''^2)^2 + (x''^3)^2$$ (10)
Eq.s (1) are obtained from the (x'-reference system) BPS equations \( B'_i = -\partial_i \phi' \) via the boost \( \Upsilon \), cf. (3). The nonvanishing components of \( b'_{\mu \nu} \) and \( \theta^{\mu \nu} \) are \( \epsilon'' = -b''_{02} = -\frac{\varepsilon}{\sqrt{\alpha' + \theta'}} \), \( \theta'' = b''_{12} = \frac{\theta}{\alpha' + \theta'} \). A solution of (1) has an energy

\[
\Sigma'' = \Sigma' + \frac{1}{g_s} \int d^3 x'' \mathcal{E}_n'' D''^i
\]

where \( \Sigma' \) is the energy of the corresponding solution of \( B'_i = \partial_i \phi' \). We see that the energy is of BPS type, indeed it is the sum of the old BPS energy \( \Sigma' \) plus the topological charge \( Z'' = \frac{1}{g_s} \int d^3 x'' \mathcal{E}_n'' D''^i = -\frac{1}{g_s} \int d^3 x'' \partial_i \phi'' D''^i \). The explicit value of \( \mathcal{D}''^i = g_s \frac{\partial \mathcal{L}_{DBI}}{\partial B''^i} \) is \( \mathcal{D}''^2 = e'' \gamma \), \( \mathcal{D}''^1 = \mathcal{D}''^3 = 0 \). We can also write \( \Sigma'' = \frac{1}{g_s} \int d^3 x'' + |Z''| + \frac{1}{\beta} \int d^3 x'', \) and recognize the brane tension, the topological charge \( Z'' = \int d^3 x'' \partial_i \phi'' B''^i = \int d^3 x' \partial_i \phi'' B''^i = Z'' \) and the energy of just the electric field \( e'' \) in DBI theory. We also have that the magnetic charge and the string tension associated to solution (1), (12) are those of a D1-string as we expect from a BPS state. Notice that the shape of the funnel representing this D1-string is no more symmetric in the \( x''_1, x''_2 \) directions. A section determined by \( \phi'' = \text{const} \), \( x''_3 = 0 \) is an ellipse in the \( x''_1, x''_2 \) plane. The ratio between the ellipse axes is given by \( \gamma \). One can project the D1-string on the D3-brane and consider the tension of this projected string: it matches the tension associated to the corresponding noncommutative BPS solution.

4 Dual string-brane configuration

If we duality rotate the D1-D3 brane configuration (9), (10) we obtain a soliton solution that describes a fundamental string ending on a D3-brane with electric and magnetic background. Under a \( \pi/2 \) duality rotation we have (we set \( 2\pi = 1 \), recall also \( g''_{\mu \nu} = \eta_{\mu \nu} \))

\[
g''_s = \frac{1}{g_s} , \quad g''_{\mu \nu} = \frac{1}{g_s} \eta_{\mu \nu} , \quad \phi''_D = \left( \frac{1}{g_s} \right)^2 \phi''
\]

the dual of solution (9), (10) is given by (12) and

\[
\mathcal{A}''_0 = -\left( \frac{1}{g_s} \right)^2 \phi''_D , \quad \mathcal{A}''_1 = -\frac{1}{g_s} \gamma e'' x''_3 , \quad \mathcal{A}''_2 = \mathcal{A}''_3 = 0 .
\]

Is there a noncommutative field theory description of the F1-D3 system? Since NCDBI and its \( \alpha' \rightarrow 1 \) limit, NCEM, admit duality rotations only if \( \theta \) is light-like, it seems that we have a F1-D3 system only if we consider a light-like background. This light-like condition may appear a strong constraint. Actually, using the \( \theta-\theta' \) symmetry we are not bound to consider only this restrictive case of light-like background. Indeed for any space-noncommutative static solution we can turn on time-noncommutativity and obtain a static solution with light-like noncommutativity. We can then apply a duality rotation, switch off the time-noncommutativity and thus obtain a new solution of the original pure space-noncommutative theory.

In particular in order to obtain the duality rotated configuration of the one described in (3), we consider the corresponding commutative configuration (8), that in the \( x' \) orthonormal frame reads \( B'_i = -\partial'_i \phi', \phi' = -\frac{1}{2} \theta x'^3 - \frac{1}{2} \theta', \gamma'^2 \equiv (x'^1)^2 + (x'^2)^2 + (x'^3)^2 \). In order to have a light-like background we turn on a constant electric field keeping here fixed \( G_s \) besides the open string metric \( G = \eta \) (therefore here \( g_s \rightarrow g'_s \neq g_s \)). We then
obtain (9) and (10) with $e'' = -b''$ (i.e. $\theta = \varepsilon$). Next we duality rotate this solution and obtain (12) and (13) with $g^\varepsilon_s, g^s_s$ instead of $g^D_s, g^s_s$. Finally we go back to the original $x$-reference system (the $x \to x''$ coordinate transformation commutes with duality rotations), we apply SW map and arrive at the noncommutative fields $\hat{A}^\nu, \hat{\phi}_D$ with open string coupling constant, metric and light-like noncommutativity given by

$$G_s^\nu = \frac{1}{G_s}, \quad G_s^{\mu
u} = \frac{1}{G_s} \eta_{\mu
u}, \quad \theta^\varepsilon_{\mu\nu} = \frac{1}{2} G_s \epsilon^{\mu\nu\rho\sigma} \theta^\varepsilon_{\rho\sigma}. \quad (14)$$

The noncommutative fields $\hat{A}^\nu, \hat{\phi}_D$ correspond to a fundamental string ending on a D3-brane with light-like background. These fields solve also $\hat{L}_{DBI}(\hat{F}, \hat{\phi}, G^D_s, G^s_s, \ast \theta_D)$ where $\theta_D (\theta^D_{13} = G_s \theta, \text{all others } \theta^\varepsilon_{\mu\nu} = 0)$ is just the space part of $\theta^\varepsilon_D$; they describe an electric monopole with a string attached. Since the noncommutative string tension and charges are invariant under $\theta^\varepsilon_D \to \theta_D$, the $\hat{A}^\nu, \hat{\phi}_D$ fields are a good candidate to describe an F-string ending on a D3-brane with constant magnetic background.

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