Which NP-Hard SAT and CSP Problems Admit Exponentially Improved Algorithms?

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Abstract

We study the complexity of SAT(\Gamma) problems for potentially infinite languages \Gamma closed under variable negation, which we refer to as sign-symmetric languages \Gamma. Via an algebraic connection, this reduces to the study of restricted partial polymorphisms we refer to as pSDI-operations (for partial, self-dual and idempotent), under which the language \Gamma is invariant. First, we focus on the language classes themselves. We classify the structure of the least restrictive pSDI-operations, corresponding to the most powerful languages \Gamma, and find that these operations can be divided into levels, corresponding to a rough notion of difficulty, where every level \(k\) has an easiest language class, containing the language for \((k-1)\)-SAT, and a hardest language class, containing (among other things) constraints encoded as roots of multivariate polynomials of degree \((k-1)\). Particular classes in each level correspond to the natural partially defined versions of previously studied total algebraic invariants. In particular, the easiest class on level \(k \geq 3\) corresponds to the partial \(k\)-ary near-unanimity \((k\text{-NU})\) operation, and a larger class corresponds to the partial \(k\)-edge operation. The largest class at each level corresponds to a partial operation \(u_k\) we call \(k\)-universal. Furthermore, every sign-symmetric language \Gamma not preserved by \(u_k\) implements all \(k\)-clauses, hence SAT(\Gamma) is at least as hard as \(k\)-SAT; and if \Gamma is not preserved by \(u_k\) for any \(k\), then SAT(\Gamma) is trivially SETH-hard (i.e., takes time \(O^{*}(2^n)\) under SETH).

Second, we consider implications of this for the complexity of SAT(\Gamma). We find that particular classes in the hierarchy correspond to previously known algorithmic strategies. In particular, languages preserved by the partial 2-edge operation can be solved via \texttt{Subset Sum}\textsuperscript{-style} meet in the middle, and languages preserved by the partial 3-NU operation can be solved via fast matrix multiplication. These results also hold for the corresponding non-Boolean CSP problems. We also find that symmetric 3-edge languages reduce to finding a monochromatic triangle in an edge-coloured graph, which can be done using algorithms for sparse matrix multiplication; and if the sunflower conjecture holds for sunflowers with \(k\) petals, then the partial \(k\)-NU language has an improved algorithm via Schöning-style local search.

Complementing this, we show a lower bound, showing that for every level \(k\) there is a constant \(c_k\) such that for every partial operation \(p\) on level \(k\), the problem SAT(\Gamma) with \(\Gamma = \text{Inv}(p)\) cannot be solved faster than \(O^{*}(c_k^n)\) unless SETH fails. In particular, when \(\Gamma = \text{Inv}(2\text{-edge})\), this gives us the first NP-hard SAT problem which simultaneously has non-trivial upper and lower bounds on the running time, assuming SETH. Finally, we note a possible conjecture: It is consistent with our present knowledge that SAT(\Gamma) admits an improved algorithm if and only if \Gamma is preserved by \(u_k\) for some constant \(k\). However, to show this in the positive poses some significant difficulty.

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1 Introduction

Significant attention has been paid to the exact time complexity of SAT and its various restrictions; in particular CNF-SAT and $k$-SAT, but also other restrictions such as NOT-ALL-EQUAL SAT, 1-in-$k$ SAT, and several more cases [15, 23, 25, 42, 49]. The usual focus is on an improved algorithm for some particular variant, i.e., showing that the problem can be solved in time $O^*(c^n)$ for some $c < 2$, or, in some cases, that such an improvement is not feasible, up to our current knowledge (i.e., it would require disproving the strong exponential-time hypothesis, SETH; see below). Here, and in the sequel, the parameter $n$ will in this context always denote the number of variables in a given instance. But what is the general rule for when a SAT problem admits such an improved algorithm? And can we say anything at all about lower bounds on such improvements?

To refine the question, let us recall some terminology. A constraint language is a (possibly infinite) set $\Gamma$ of finitary relations $R \subseteq D^{\text{ar}(R)}$ over some domain $D$, where $\text{ar}(R)$ denotes the arity of $R$. We will mainly focus on the Boolean case, i.e., $D = \{0, 1\}$. Then SAT($\Gamma$), occasionally called the parameterized satisfiability problem, is the SAT problem where the constraints of the instance are applications of relations from $\Gamma$, i.e., the constraints are statements that $R(x_1, \ldots, x_r)$ must hold, for some $R \in \Gamma$ and some variables $x_1, \ldots, x_r$ from the variable set (where we do allow repetitions of a variable). The multi-valued generalization of SAT, the constraint satisfaction problem over $\Gamma$ (CSP($\Gamma$)) is defined in essentially the same way, except that $\Gamma$ may be non-Boolean. Full definitions of the problems under consideration follow in Section 2. Thus, for example, 3-SAT corresponds to SAT($\Gamma^3_{\text{SAT}}$) where $\Gamma^3_{\text{SAT}}$ for each 3-clause in $\{(x \lor y \lor z), \ldots, (\neg x \lor \neg y \lor \neg z)\}$ contains the relation excluding only the tuple forbidden by that particular clause. Similarly, for $k \geq 3$ let $\Gamma^k_{\text{SAT}}$ denote the constraint language of all $k$-clauses, i.e., SAT($\Gamma^k_{\text{SAT}}$) is equivalent to $k$-SAT.

Let us also tentatively define $c(\Gamma)$ as the infimum over all constants $c > 1$ such that SAT($\Gamma$) can be solved in $O(c^n)$ on $n$ variables. Then the exponential time hypothesis (ETH), due to Impagliazzo and Paturi, states that $c(\Gamma^3_{\text{SAT}}) > 1$ for every $k$, and was shown to be equivalent to the statement that $c(\Gamma^3_{\text{SAT}}) > 1$ [25]. It has also been shown to be equivalent to the statement that $c(\Gamma) > 1$ for every $\Gamma$ such that SAT($\Gamma$) is NP-hard [31]. The strong exponential time hypothesis (SETH) is the statement that $\lim_{n \to \infty} c(\Gamma^k_{\text{SAT}}) = 2$ [10, 25]. Then our main research question can be rephrased as, for which constraint languages $\Gamma$ is $c(\Gamma) < 2$, respectively, when would $c(\Gamma) < 2$ contradict SETH? We say that SAT($\Gamma$) allows an improved algorithm in the former case, and that it is SETH-hard in the latter. Hence, our main interest is in exponential improvements rather than subexponential improvements of the form $O(2^{n-o(n)})$ which have been proven to exist for CNF-SAT [17].

Before we discuss our approach for the general case, we consider a few examples. First of all, the algorithms for $k$-SAT imply that $c(\Gamma) < 2$ for every finite language $\Gamma$. However, such bounds are also known for some infinite languages. One example is Exact SAT, the language of 1-in-$k$-clauses of all arities, which admits an improved algorithm [53]. As has been shown more recently, so does the problem where constraints are encoded as the roots of bounded-degree multivariate polynomials over a finite field [42]. Thus, we need a way to discuss properties of infinite arbitrary languages, and we need to consider the representation of constraints from such a language. We address these issues in Section 1.2.

Lower bounds on $c(\Gamma)$ for some $\Gamma$ have been significantly harder to come by. Some SAT problems have been shown to be SETH-hard, in particular Not-All-Equal SAT and problems related to SAT such as Hitting Set [15]. It is also known that assuming ETH, the value of $c(\Gamma^k_{\text{SAT}})$ increases infinitely often [25]. However, we do not even have conjectural evidence against any particular value of $c(\Gamma)$ for any language $\Gamma$ such that SAT($\Gamma$) is not SETH-hard, other than for trivial cases\footnote{By trivial cases, we mean problems where the natural search space is smaller than $2^n$ but otherwise unrestricted. Consider a language where every variable is involved in a disequality, e.g., the language of relations $R'(x_1, \ldots, x_{2k}) \equiv (x_1 \not= x_{k+1}) \land \ldots \land (x_k \not= x_{2k}) \land R(x_1, \ldots, x_k)$ for arbitrary relations $R$. It is easy to see that under SETH, this problem has $c(\Gamma) = 2^{1/2}$.}. We also are not aware of any previous attempts to engage with the question of what makes a SAT problem SETH-hard or not in general.

In this paper, we study these questions using tools from universal algebra. It is known that the value
Theorem 1. Let $\Gamma$ and $\Delta$ be finite constraint languages over a finite domain $D$. If $\text{Pol}(\Delta) \subseteq \text{Pol}(\Gamma)$, then $\text{CSP}(\Gamma)$ is polynomial-time many-one reducible to $\text{CSP}(\Delta)$. 

1.1 Universal algebraic aspects of SAT problems

To make the discussion of our approach more precise, we need to review some notions from universal algebra. This is simply intended as an introduction and overview to make the extended abstract self-contained; full definitions follow later in the paper in Section 2. The universal algebraic approach to problem complexity originates in research into the constraint satisfaction problem (CSP) [29]. Recall the definitions of a constraint language $\Gamma$ and the problem CSP$(\Gamma)$ from the preceding section. Clearly, the complexity of CSP$(\Gamma)$ varies as a function of $\Gamma$: if $\Gamma$ is simple enough, then CSP$(\Gamma)$ is in P; and if $\Gamma$ is rich enough, then CSP$(\Gamma)$ is NP-complete. The dichotomy conjecture, first posed by Feder and Vardi [19], states that these are the only two cases and that no NP-intermediate CSP problems exist: for every fixed language $\Gamma$, CSP$(\Gamma)$ is either in P or is NP-complete. This conjecture has been the subject of intense research and the piece remaining to complete the puzzle was recently resolved by two independent authors [7, 57].

The algebraic approach turned out to be central in this research programme. In short, this approach boils down to the realization that properties of constraint languages can be expressed by properties of their polymorphisms. Informally, a polymorphism of a constraint language $\Gamma$ is an operation which yields a method to combine satisfying assignments of instances of CSP$(\Gamma)$. The algebraic reformulation of the CSP dichotomy theorem then states that CSP$(\Gamma)$ is tractable if there exists a non-trivial method to combine solutions, and is NP-complete otherwise. More formally, we may define polymorphisms as follows. First, let $R \subseteq D^n$ be a relation on $D$, and let $p : D^r \rightarrow D$ be an $r$-ary operation over $D$. We can then generalise $p$ to an operation $(D^n)^r \rightarrow D^n$ on tuples over $D$ by $p(x_1, \ldots, x_r)[i] = p(x_1[i], \ldots, x_r[i])$ for every position $i \in [n]$ (where $x_j[i]$ denotes the $i$th element of the tuple $x_j$). Then $p$ is a polymorphism of $R$ if this generalised operation preserves $R$, i.e., if $p(x_1, \ldots, x_r) \in R$ for any $x_1, \ldots, x_r \in R$. Note that if $p$ is a projection, i.e., $p(t_1, \ldots, t_r) = t_i$ for some $i \in [r]$, then $p$ preserves every possible relation. The notion of a polymorphism easily extends to constraint languages, and we say that $p$ is a polymorphism of the constraint language $\Gamma$ if $p$ is a polymorphism of $R$ for every relation $R \in \Gamma$, and let $\text{Pol}(\Gamma)$ denote this set. It is then known that the complexity of CSP$(\Gamma)$, up to polynomial-time many-one reductions, is determined entirely by $\text{Pol}(\Gamma)$ [28].

Theorem 1. Let $\Gamma$ and $\Delta$ be finite constraint languages over a finite domain $D$. If $\text{Pol}(\Delta) \subseteq \text{Pol}(\Gamma)$, then CSP$(\Gamma)$ is polynomial-time many-one reducible to CSP$(\Delta)$. 

of $c(\Gamma)$ is determined by algebraic invariants of $\Gamma$ known as partial polymorphisms [31]. It is not difficult to prove that if $\Gamma$ has no interesting partial polymorphisms, then SAT$(\Gamma)$ is trivially SETH-hard. We study the converse to this question, to essentially ask, does the existence of even a single relevant partial polymorphism $p$ imply that SAT$(\Gamma)$ has an improved algorithm? In particular, is it possible to design an algorithm with an exponentially improved running time, whose correctness depends only on $p$? One of the main strengths of using such an algebraic approach is that it makes the task of identifying languages $\Gamma$ such that $c(\Gamma) < 2$ considerably easier. In fact, as we discuss in Section 1.1, these languages can be succinctly classified according to the expressive power of individual partial operations.

Our paper has two main contributions. First, we characterize the structure of the weakest non-trivial invariants $p$. In this, we restrict ourselves to sign-symmetric languages (see below). This reveals a characterization of problem complexity, with close ties to several previously studied problems and algorithm classes. Second, we use the framework to provide both upper and lower bounds on $c(\Gamma)$ for the corresponding languages $\Gamma$, under SETH. We show that algorithms from the literature can be extended to work for every language having a certain partial polymorphism $p$. In the negative direction, we are able to prove lower bounds on $c(\Gamma)$ for every language $\Gamma$ characterised purely by its invariants. As a result, we produce the first language $\Gamma$ such that $c(\Gamma)$ has both non-trivial upper and lower bounds under SETH. Finally, we make connections between these SAT$(\Gamma)$ problems and some problems in polynomial-time fine-grained complexity.

Our approach also implies some results for CSPs on a non-Boolean domain, but our main focus in the present paper lies in studying the Boolean case.
At this stage this result may seem slightly puzzling since we do not yet have a clear correspondence between polymorphisms and their implications on constraint languages. However, there exists a dual concept to polymorphisms on the relational side called implementations. Given a set of relations $\Gamma$ over a domain $D$, a $k$-ary relation $R$ is definable by a primitive positive implementation over $\Gamma$ (pp-definable) if there exists a first-order formula making use of existential quantification and conjunctive constraints over $\Gamma$ such that the set of models of this formula is precisely $R$. Given a constraint language $\Gamma$ we then let $\langle \Gamma \rangle$ be the smallest set of relations containing $\Gamma$ and which is closed under taking pp-definitions. The polymorphisms of $\Gamma$ then characterize the power of pp-definitions over $\Gamma$ in the following sense.

**Theorem 2** ([5][6][21]). Let $\Gamma$ and $\Delta$ be two constraint languages. Then $\Gamma \subseteq \langle \Delta \rangle$ if and only if $\text{Pol}(\Delta) \subseteq \text{Pol}(\Gamma)$.

This duality has two implications. First, note that an instance of CSP($\Gamma$) can be viewed as a special case of a pp-definition over $\Gamma$, hence the polymorphisms of $\Gamma$ describe closure properties for the whole CSP($\Gamma$) problem, and can be used to design polynomial-time algorithms. This is in line with the intuition that a polymorphism yields a method for combining satisfying assignments. Second, if $R$ has a pp-definition in $\Gamma$ then there is a polynomial-time many-one reduction from CSP($\Gamma \cup \{R\}$) to CSP($\Gamma$); essentially, the pp-definition describes a classical “gadget reduction” between the problems obtained by replacing constraints over $R$ by the collection of constraints over $\Gamma$ prescribed by the pp-definition. Therefore, dually to the previous point, the absence of sufficiently interesting polymorphisms for $\Gamma$ would imply a polynomial-time reduction from an NP-hard problem CSP($\Gamma'$), e.g., 3-SAT, to CSP($\Gamma$).

In practice, for CSPs beyond the Boolean domain, the complexity landscape gets very complex and one needs to apply a richer algebraic toolbox to make progress. However, it was realized early that not only does the complexity of CSP($\Gamma$) depend on Pol($\Gamma$), but in fact only the identities satisfied by the operations in Pol($\Gamma$) [9]. In technical terms this means that the complexity of CSP($\Gamma$) only depends on the variety generated by Pol($\Gamma$). We will not define these concepts formally since they are not needed to present the main results; it is sufficient to know that the complexity of CSP($\Gamma$) only depends on the identities satisfied by the operations in Pol($\Gamma$). For example, CSP($\Gamma$) is solvable using $k$-consistency if Pol($\Gamma$) contains a majority operation, i.e., a ternary operation $m$ satisfying the identities $m(x, y, y) = y$, $m(y, x, y) = y$, $m(y, y, x) = y$ [29]. Moreover, all operations resulting in tractable CSPs can be characterized using such identities.

It is worth remarking that for the Boolean domain the situation is considerably simplified due to Post’s classification of Boolean Pol($\Gamma$) [46], and a large range of such problems have been proven to admit dichotomies [14]. For example, Schaefer’s dichotomy theorem for SAT($\Gamma$) [48] can be proven in an extremely straightforward manner using this approach. However, for our purposes the above methods are too coarse-grained, since the precise running time $O^*(c^n)$ for a problem SAT($\Gamma$) is not preserved by the introduction of existentially quantified variables. Hence, we are in need of more fine-grained algebraic tools than usual, which can be applied as follows.

A **partial operation** over $D$ (of some arity $r$) is an operation $p : X \rightarrow D$ for some domain $X \subseteq D^r$. Similar to the total case we again extend it to a partial operation on tuples over $D$: for $x_1, \ldots, x_r \in D^n$, we let $p(x_1, \ldots, x_r)[i] = p(x_1[i], \ldots, x_r[i])$ if this is defined for every position $i \in [n]$; otherwise $p(x_1, \ldots, x_r)$ is undefined. Then $p$ is a partial polymorphism of a relation $R \subseteq D^n$ if, for any $x_1, \ldots, x_r \in R$ such that $p(x_1, \ldots, x_r)$ is defined we have $p(x_1, \ldots, x_r) \in R$. We will occasionally also say that $R$ is invariant under the partial operation $p$. A partial projection is a subfunction of a projection; such an operation preserves every possible relation. A partial polymorphism of a constraint language $\Gamma$ is a partial polymorphism of every relation $R \in \Gamma$ and we let PP($\Gamma$) denote the set of all partial polymorphisms of $\Gamma$. Similarly, given a set of partial operations $P$ we write Inv($P$) to denote the set of relations invariant under $P$, and if $P = \{p\}$ is singleton we write Inv($p$) instead of Inv($\{p\}$). Dually to this relaxed notion of a polymorphism, we have a strengthened notion on the relational side: a **quantifier-free primitive positive definition** (qfpp-definition).
over \( \Gamma \) is a pp-definition without existential quantification. We let \( \langle \Gamma \rangle_{\exists} \) denote the smallest set of relations containing \( \Gamma \) and which is closed under qfpp-definitions, and then obtain the following correspondence.

**Theorem 3** ([21][47]). \( \Gamma \subseteq \langle \Delta \rangle_{\exists} \) if and only if \( \text{pPol}(\Delta) \subseteq \text{pPol}(\Gamma) \) for any constraint languages \( \Gamma \) and \( \Delta \).

With the help of this correspondence Jonsson et al. [31] proved that partial polymorphisms indeed can be used for studying the fine-grained complexity of SAT and CSP.

**Theorem 4.** Let \( \Gamma \) and \( \Delta \) be two finite constraint languages. If \( \text{pPol}(\Gamma) \subseteq \text{pPol}(\Delta) \) then there is a polynomial-time many-one reduction from \( \text{CSP}(\Delta) \) to \( \text{CSP}(\Gamma) \) which does not increase the number of variables.

Unfortunately, this theorem is difficult to apply in practice since it requires a good understanding of the structure of the closed sets \( \text{pPol}(\Gamma) \) for all possible choices of \( \Gamma \). Despite advances made by several different researchers [12][13][35][51], no such classification is known even for Boolean \( \Gamma \), and even less is known for \( \Gamma \) such that \( \text{SAT}(\Gamma) \) is NP-hard. Hence, we propose a method inspired by the rich algebraic toolbox developed for studying the classical complexity of CSP: does the SETH-hardness of \( \text{SAT}(\Gamma) \) and \( \text{CSP}(\Gamma) \) only depend on the identities satisfied by the partial polymorphisms of \( \Gamma \)? On the one hand, it is easily verified that if the only partial polymorphisms of \( \Gamma \) are the partial projections, then \( \Gamma \) can qfpp-define all \( k \)-clauses for every \( k \geq 1 \), and \( \text{SAT}(\Gamma) \) is SETH-hard. On the other hand, we would have to show that every non-trivial partial polymorphism \( p \) allows the design of an algorithm that solves \( \text{SAT}(\Gamma) \) in \( O^*(c^n) \) time for some \( c < 2 \).

One issue which speaks against the feasibility of this approach is that individual partial polymorphisms are very weak restrictions. For one thing, it is known that for every finite set \( P \) of partial operations (that does not imply any non-trivial total operation), the set \( \text{Inv}(P) \) of all relations that are invariant under \( P \) contains a double-exponential number of relations as a function of the arity \( n \) [37], Lemma 35. Note that for a finite language such as \( k \)-SAT, there are in contrast only \( 2^{O(n^k)} \) distinct instances on \( n \) variables. Hence, languages \( \text{Inv}(p) \) for a single partial operation \( p \) would be much richer than previously studied problems. Very similarly, in a related study [35], it was shown that the existence of so-called polynomial kernels for \( \text{SAT}(\Gamma) \) cannot be characterised by such a finite set \( P \), whereas every finite problem, as well as \( \text{Exact SAT} \) and problems defined via bounded-degree polynomials, have polynomial kernels [27].

Nevertheless, contrary to these earlier results, we will prove that the presence of certain individual partial polymorphisms can be used to design improved algorithms for \( \text{SAT} \) problems. As a starting point we in the first hand consider the partial analogues of well-studied polymorphisms resulting in tractable CSPs. For example, a **Mal'tsev operation** is a ternary operation \( \phi \) satisfying the two identities \( \phi(x, x, y) = y \) and \( \phi(y, x, x) = y \), and is well-known to result in tractable CSPs due to the algorithm by Bulatov and Dalmau [8]. We may then define the **partial Mal’tsev operation** over a domain \( D \) as the unique partial operation which for all \( x, y \in D \) satisfies these two identities, but which is undefined otherwise. Similarly, it is possible to define partial variants of \( k \)-ary near unanimity \( (k\text{-NU}) \) and \( k \)-ary edge \( (k\text{-edge}) \) operations. These classes of operations are formally defined in Section 2.5 and at the moment we will simply regard them as well-behaved operations resulting in tractable CSPs, but we remark that a 2-edge operation is equivalent to a Mal’tsev operation and that a ternary NU-operation is nothing else than a majority operation. It may also be interesting to observe that the partial operations defined in this manner are unique for every fixed domain, even though there may exist a large number of total operations satisfying the identities.

**1.2 Our results and structure of the paper**

For a partial polymorphism \( p \), let \( \text{Inv}(p)\text{-SAT} \) refer to the problem \( \text{SAT}(\Gamma) \) where \( \Gamma = \text{Inv}(p) \). Hence, in this problem every involved relation is invariant under the given partial operation \( p \). We will sometimes also refer to the CSP-variants of these problems and denote these by \( \text{Inv}(p)\text{-CSP} \) (and tacitly assume that the domain of the operation \( p \) is clear from the context, or is not relevant). We look at three related aspects of the complexity of these problems. Let us first discuss our model more carefully.
Our questions and model. Since $\Gamma$ is infinite we first need to fix a constraint representation. Let $R \subseteq \{0,1\}^r$ be a relation. An explicit representation of $R$ is a list of all tuples $t \in R$. For infinite languages the explicit representation is not always the most natural one since a relation may contain exponentially many tuples with respect to the arity. This is particulary troublesome when proving lower bounds for Inv(p)-SAT since we may not be able to construct relations of arbitrary arity in the required time bound. Hence, we also consider an implicit representation. In this model of representation a contraint $R(x_1, \ldots, x_r)$ is represented by an oracle consisting of a computable function which, given an assignment to variables $X \subseteq \{x_1, \ldots, x_r\}$, can determine if this assignment can be extended to an assignment to $\{x_1, \ldots, x_r\}$ consistent with $R$.

Example 5. For each $r \geq 3$ consider the relation $R^r = \{(x_1, \ldots, x_r) \in \{0,1\}^r \mid x_1 + \ldots + x_r \text{ is even}\}$. Even though $|R^r|$ is exponential with respect to $r$ it is not difficult to see that constraints over $R^r$ can be implicitly represented by computing the parity of the given assignment.

Given these definitions, we consider the following three notions of improved algorithms.

**Definition 6.** Let $\Gamma$ be an infinite constraint language.

1. SAT$(\Gamma)$ admits a non-uniform improved algorithm with running time $O^*(c^n)$, $c < 2$, if for every finite $\Gamma' \subseteq \Gamma$ the problem SAT$(\Gamma')$ can be solved in $O^*(c^n)$ time.

2. SAT$(\Gamma)$ admits an improved algorithm in explicit representation if SAT$(\Gamma)$ admits an algorithm for the problem variant where every relation is provided in explicit representation.

3. SAT$(\Gamma)$ admits an improved algorithm in the oracle model if SAT$(\Gamma)$ admits an improved algorithm when constraints are provided only as extension oracles.

Note that for a non-uniform improved algorithm, the representation does not matter. Also note that these are gradually stronger requirements, and that in these terms, SETH states that CNF-SAT does not admit even a non-uniform improved algorithm. On the other hand, allowing constraints of unbounded arity via oracle access can be useful; for example, the $n$-ary constraint $(\sum_{i=1}^n x_i = k)$ has a simple extension oracle, and if included in the language, can be used to phrase optimisation problems as oracle-access SAT problems.

To restrict our scope, we focus on constraint languages that are closed under variable negation. Informally, this means that whenever $R \in \Gamma$, in addition to constraints $R(x_1, \ldots, x_r)$ on only positive variables, we are also allowed to impose constraints such as $R(x_1, \ldots, \neg x_i, \ldots, x_r)$ with some occurrences of variables $x_i$ negated in the constraint. More formally, it means that for every $R \in \Gamma$, and for every subset $S \subseteq \{\text{ar}(R)\}$ of positions of $R$, the relation produced by negating every tuple $t \in R$ in positions $S$ is also contained in $\Gamma$. In this case, we say that $\Gamma$ is sign-symmetric. This is a natural restriction which holds for many well-studied constraint language, e.g., the languages corresponding to $k$-SAT, $1$-$n$-k-SAT and the roots of bounded-degree polynomials are all sign-symmetric. Furthermore, it is known that the expressive power of a sign-symmetric constraint language is characterised by a restricted kind of partial polymorphism which we refer to as pSDI-operations (for partial, self-dual and idempotent) [34][33]. Thus, the restriction to sign-symmetric languages corresponds directly to a restriction on the algebraic level. Most importantly, the Boolean partial operations arising from system of identities of the form considered in Section 1.1 are guaranteed to be pSDI.

The fine-grained structure of NP-hard SAT problems. The first part of the paper, Section 3 is dedicated to explaining the the structure of pSDI-operations. Due to the algebraic correspondence between partial polymorphisms and qfpp-definability this also serves as a classification of the NP-hard SAT problems we need to consider for constructing improved algorithms.

First, we study the structure of single pSDI-operations $p$ that impose some non-trivial restrictions on the expressive power of $\Gamma$. We particularly consider the weakest such operations, i.e., such that the language
\[ \Gamma = \text{Inv}(p) \] is as rich as possible. In particular, we consider \( p \) such that every subfunction of \( p \) which is pSDI is a partial projection. Let us refer to such an operation as being minimal. For example, the partial variants of Maltsev, \( k\text{-NU} \), and \( k\text{-edge} \) operations are all minimal. Equipped with this notion we then show that minimal pSDI-operations are naturally organised into levels, with a structure as follows.

- There is a single minimal operation on level 2, which is the partial Maltsev, or, equivalently, the partial 2-edge operation. This is also equivalent to the 2-universal operation defined below.
- For every other minimal pSDI-operation \( p \), there is a unique largest constant \( k \) such that \( p \) is implied by the partial \( k\text{-NU} \) operation \( \nu_k \). We refer to this as the level of \( k \). Thus, the partial \( k\text{-NU} \) operation is the strongest operation on level \( k \geq 3 \).
- For every level \( k \geq 2 \), there is also a unique weakest pSDI-operation \( \nu_k \) which we refer to as the \( k\text{-universal} \) operation, such that \( \nu_k \) is implied by every operation on level \( k \).
- The language \( \Gamma^k_{\text{SAT}} \) corresponding to \( k\text{-SAT} \) is preserved by the partial \( (k + 1)\text{-NU} \) operation, but not by any operation on a previous level; and every sign-symmetric language \( \Gamma \) that is not preserved by the \( k\text{-universal} \) operation can qfpp-define \( \Gamma^k_{\text{SAT}} \).
- Finally, as an interesting case, roots of polynomials of degree at most \( d \) are preserved by the \( (d + 1)\text{-universal} \) operation, but not by any other operation on a level up to \( d + 1 \).

Thus, the levels of minimal pSDI-operations correspond to a natural notion of difficulty. It also follows that if a sign-symmetric language \( \Gamma \) is not preserved by the \( k\text{-universal} \) operation for any constant \( k \), then \( \text{SAT}(\Gamma) \) is trivially SETH-hard, whereas every other language \( \Gamma \) has some kind of restriction on its expressive power. We also note that there is no known case of a problem known to be SETH-hard, which fits into a framework where the particular choice of representation does not matter. We then obtain the following results.

- When \( p \) is the partial 2-edge operation, we refer to \( \text{Inv}(p) \)-SAT as \( 2\text{-\textsc{edge}}\)-SAT. We show that \( 2\text{-\textsc{edge}}\)-SAT can be solved in \( O^\ast(2^{n^r}) \) time in the oracle setting using a meet-in-the-middle strategy combined with the computation of a kind of canonical labels for partial assignments, similarly to the \( O^\ast(2^{n^d}) \)-time algorithm for \( \text{\textsc{Subset Sum}} \) with \( n \) integers [24]. A similar improved algorithm is possible for the generalisation to \( 2\text{-\textsc{edge-CSP}} \), i.e., for fixed non-Boolean domains. Furthermore, if \( c(\text{Inv}(p)) < 2^{1/2} \)
in the extension oracle setting, then \textsc{Subset Sum} can be solved in $O^*(2^{\left(\frac{1}{2}\right)(\log n)^k})$ for some $\varepsilon > 0$, which is a long-standing open problem.

- When $p$ is the partial $k$-NU operation, we refer to \textsc{Inv($p$)}-\textsc{SAT} as $k$-NU-SAT. For $k = 3$, this problem is equivalent to 2-SAT, and hence in P, but the generalisation 3-NU-CSP to larger fixed domains is NP-hard and admits an improved algorithm using fast matrix multiplication, similarly to the well-known algorithm for the CSP problem over binary constraints.

- For $k > 3$, we show two conditional connections. First, if the $(k, k - 1)$-\textsc{Hyperclique} problem for hypergraphs with ground set of size $n$ can be solved in time $O(n^{k-\varepsilon})$ for any $\varepsilon > 0$, then both $k$-NU-SAT and $k$-NU-CSP admit improved algorithms in the oracle setting. Second, if the Erdős-Rado \textit{sunflower conjecture} [18] holds for sunflowers with $k$ sets, then $k$-NU-SAT admits an improved algorithm via a local search strategy in the explicit representation, similar to Schöning’s algorithm for $k$-\textsc{SAT} [52].

- We also investigate the case that $p$ is the partial 3-edge operation $e_3$, and give a partial result. Assume that every relation $R$ in the input is either preserved by the partial 2-edge relation, or by $n_{13}$, or $R$ is symmetric and preserved by $e_3$—i.e., whether $t \in R$ depends only on the Hamming weight of $t$. Then the SAT problem has an improved algorithm via a reduction to the problem of finding monochromatic triangles in an edge-coloured graph, which in turn can be solved using fast algorithms for triangle finding in sparse graphs. We do not know whether this strategy generalises to non-symmetric relations.

For further classes, we note that $\textsc{SAT}((\Gamma))$ contains some highly challenging special cases. In particular an algorithm for the $k$-universal languages for $k > 2$ would need to generalise the algorithm of Lokshtanov et al. for bounded-degree polynomials [42], while only using the abstract properties guaranteed by $u_k$.

Finally, we show lower bounds in the oracle extension model: for every minimal pSDI-operation $p$, we get a concrete lower bound $c(\textsc{Inv}(p)) \geq c_k > 1$ assuming the randomized SETH, where $k$ is the level of $p$. That is, unless SETH is false, no algorithm can solve $\textsc{Inv}(p)$-\textsc{SAT} in time $O^*(c_k^{1-\varepsilon}n)$ for any $\varepsilon > 0$ and any $p$ at level $k$. The bound $c_k$ converges to 2 at a rate of $2 - c_k = \Theta\left(\frac{\log k}{k}\right)$.

\textbf{A connection to polynomial-time problems.} Finally, we make some connections between the $\textsc{Inv}(p)$-\textsc{SAT} and $\textsc{Inv}(p)$-\textsc{CSP} problems and some problems in polynomial-time algorithms. We show that the minimal pSDI-operations generalise not only to CSP problems on fixed domains, but to abstract conditions on “CSP-like” problems on a domain of size $n$ and with $d = \Theta(1)$ variables. We refer to this as the \emph{abstract $\textsc{Inv}(p)$-problem}. Any solution to such a problem that runs in time $O(n^{d-\varepsilon})$ for any $\varepsilon > 0$ implies an improved algorithm for the corresponding $\textsc{Inv}(p)$-\textsc{SAT} and $\textsc{Inv}(p)$-\textsc{CSP} problems in the oracle setting for every fixed domain. This lies behind the improved algorithms for 2-\textsc{EDGE-CSP} and 3-\textsc{NU-CSP}.

However, there is some indication that these problems may be tougher than the original problems, since the reduction loses a significant amount of instance structure (e.g., the local search strategy for $k$-NU-SAT cannot be lifted to the abstract problem). In fact, there are conjectures that would prevent improved algorithms for most cases of the abstract problem considered in this article:

- The abstract $k$-NU problem is equivalent to $(k, k - 1)$-\textsc{Hyperclique}, i.e., the problem of finding a $k$-hyperclique in a $(k - 1)$-regular hypergraph. Thus, it has an improved algorithm for $k = 3$ but the status is unknown for $k > 3$. Moreover, the general $(l, k)$-hyperclique problem for $l > k$ has been conjectured to require $n^{l-o(1)}$ time [40].

- The abstract 3-universal problem contains the problem of finding a zero-weight triangle in an edge-weighted graph with arbitrary edge weights. This does not admit an improved algorithm unless the 3-SUM conjecture fails (but SETH-hardness is not known) [56].

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Considering the connections, we still consider it useful to ask which minimal pSDI-operations $p$ suffice to guarantee an improved algorithm for the abstract $\text{Inv}(p)$-problem. We leave this question for future work.

1.3 Technical notes and proof methods

Let us now give a few more details about the proofs of the above results.

The structural characterisation builds on a description of minimal non-trivial pSDI-operations (Lemma 27) — they are precisely the operations produced by padding the partial $k$-NU operation by additional arguments. The weakest and strongest operations on each level follow from this almost by definition. It also follows that the operations on each level $k$ are characterized by the presence or absence of each of roughly $2^k$ possible types of padding argument. Note that such a padding makes an operation weaker; e.g., in order to apply the partial majority operation to a sequence of tuples $t_1, \ldots, t_k \in R$ for some relation $R$, in a padded version of arity $r$ we require that $R$ further contains a sequence of tuples $t_{k+1}, \ldots, t_r$ determined by the padding arguments from the tuples $t_1, \ldots, t_k$.

This also provides a way to think about the consequences of not being preserved by such an operation. Assume e.g. that a relation $R$ is not preserved by $u_k$. Then by definition there are $t_1, \ldots, t_k \in R$ such that $u_k(t_1, \ldots, t_k) = t$ is defined, and by sign-symmetry we may assume that $t$ is the constant 0-tuple $0^{\ar(R)}$. Then the witness produces a partition of the arguments of $R$, in a way which can be used to implement a relation $R'$ of arity $k$ which accepts every tuple of weight 1 but none of weight 0. However, we have no information at this point about the remaining tuples in $R'$. Continuing this line of reasoning to derive a consequence for an infinite sign-symmetric language $\Gamma$ with $u_k \notin \text{pPol}(\Gamma)$ for every $k$, we first observe that we can define a symmetric relation $R'' \notin \text{Inv}(u_k)$ as a conjunction of $k!$ applications of $R'$ under argument permutation, then (as announced) analyse the possibilities for families of such relations using Szemerédi’s theorem. In particular, a broken arithmetic progression of $i$ accepted weights in such a relation implies that we can qpp-define an $i + 1$-clause using $R$.

By contrast, if $u_k \notin \text{pPol}(R)$, then the tuples $t_1, \ldots, t_{2^k-1} \in R$ required by the arguments of $u_k$ imply that such a relation $R'$ must have $|R'| = 2^k - 1$, i.e., it must be the relation corresponding to a $k$-clause.

Moving on to the algorithmic applications, most of the positive results are relatively straight-forward applications of known ideas; the interesting aspect is that the applicability of these ideas follows from such simple conditions as the minimal pSDI-operations. Here, we particularly wish to highlight the conjectural connection to local search. Recall that Schöning’s algorithm [52] reduces $k$-SAT to several applications of local search, i.e., given a starting point $x \in \{0, 1\}^n$ and a parameter $t$, find a satisfying assignment within Hamming distance $t$ of $x$. By sign-symmetry, for our problem this reduces to the case $x = 0^n$ (alternatively, one could use monotone local search; cf. Fomin et al. [20]). Now, consider the set of all minimal tuples in any relation $R \in \text{Inv}(u_k)$ with $0^{\ar(R)} \notin R$. It is easy to see that by the $u_k$-condition, this set does not contain a sunflower of $k$ sets, and by the sunflower conjecture, this implies that for every $i$ there are at most $C_i$ such minimal tuples in $R$ of weight $i$ for some $C$. A simple computation shows that a recursive algorithm that finds an unsatisfied relation $R$, enumerates minimal tuples in it, and recursively proceeds from every such tuple yields a total searching time of $2^{O(i)}$, which would be precisely sufficient to yield an improved algorithm for $k$-NU-SAT. This algorithm uses the explicit representation in order to be able to enumerate such minimal tuples. It is an interesting open question whether this can be achieved efficiently in the oracle setting.

Finally, we move on to our lower bounds. These are of two kinds, a reduction from Subset Sum to 2-edge-SAT, and the generic lower bound under SETH against any problem $\text{Inv}(p)$-SAT. For the former, recall that the partial 2-edge operation is equivalent to $u_2$, and thus contains all constraints which can be phrased as linear equations, e.g., Subset Sum instances. But we are also required to provide an extension oracle for each constraint, which is clearly infeasible if we plug in the Subset Sum equation as-is. However, this is easily solved by splitting the binary expansion of the target number into $O(\sqrt{n})$ blocks of $O(\sqrt{n})$. 

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bits each. With some moderate guessing, each block reduces to one linear equation, and via the tabulation algorithm for Subset Sum an extension oracle each such block can be produced with a query time of $2^{O(\sqrt{m})}$.

The generic bounds, in turn, work via a generic padding argument: we show that for every level $k$, and any set $X$ of $n$ variables, there is a universal padding formula $R(X, Y)$ on $|Y| = \Theta(n)$ additional variables such that $R^*(X, Y) \equiv R'(X) \land R(X, Y)$ is $k$-NU for any relation $R'(X)$. Furthermore, random parity-check variables suffice to produce this padding formula, allowing for an efficient extension oracle for the relation $R'(X, Y)$. Finally, by the regularity of the padding formula, we can reuse the same variables $Y$ for all constraints in an input instance of $q$-SAT, for any $q$, and only pay with $|Y| = \Theta(n)$ extra variables in total.

The fact that some such padding existed was previously known \[57\]. Recall that every operation $p$ considered has at least one tuple of values for which it is undefined. Then, if we add enough random variables, for every attempt $p(t_1, \ldots, t_r)$ of finding a valid application of $p$ on a relation $R$ there will be a padding variable $j$ such that $(t_1[j], \ldots, t_r[j])$ takes the values of such a tuple, and $p$ is undefined. The fact that parity-check variables suffice in our case follows from the fact that $p$ contains $k$ arguments that form a partial $k$-NU operation. It is easy to check that almost all parity-check variables form an undefined tuple of values already over these arguments. This construction could be derandomized using a universal hash family, possibly at the cost of a larger constant $|Y|/|X|$, but we do not pursue this.

### 1.4 Related work

Our work can be seen as an amalgamation of the following areas: fine-grained time complexity and lower bounds under the SETH, and the algebraic approach for studying classical complexity of CSP.

Concerning the former, SETH has turned out to be a highly useful conjecture for exact algorithms since a relative lower bound from SETH shows that any further improvements also implies a breakthrough speed-up for SAT. Many different problems have been shown to admit lower bounds via the SETH, but in the current context of SAT, in addition to the foundational works of Impagliazzo et al. \[25, 26\] it is worth mentioning the lower bound for \textsc{not-all-equal SAT} (NAE-SAT) by Cygan et al. \[15\] and the lower bound for \textsc{Π2-3-SAT} by Calabro et al. \[11\]. However, to the best of our knowledge, all concrete lower bounds using SETH for exponential-time algorithms falls into one of the following cases: either the lower bound matches the running time of a trivial algorithm, as in the case of \textsc{Hitting Set}, NAE-SAT, and \textsc{Π2-3-SAT}, showing that no improvement is possible; or the lower bounds are with respect to a much more permissive complexity parameter than $n$, such as treewidth \[41\]. The one other example we are aware of is from the study of infinite-domain CSPs by Jonsson and Lagerkvist \[30\], who obtained upper bounds of the form $O^*(2^{f(n)})$ for non-linear functions $f$ and a lower bound stating that the CSPs are not solvable in $O(c^n)$ time for any constant $c$. These bounds are therefore in a sense closer to non-subexponentiality results usually obtained from the ETH. SETH and other conjectures have also seen significant applications over recent years in producing conditional lower bounds for polynomial-time solvable problems, but these are only tangentially relevant here.

With regards to the algebraic approach we wish to highlight a few related but different results. Partial polymorphisms and the link to \textsc{qfpp}-definitions were first introduced to the CSP community by Schnoor & Schnoor \[50\] even though these notions were well-known in the algebraic community much longer \[21, 47\]. However, the principal motivation by Schnoor & Schnoor was to obtain dichotomy theorems for CSP-like problems incompatible with existential quantification, and the explicit connection to fine-grained time complexity of CSP was not realized until later by Jonsson et al. \[41\]. This work utilized a \textit{lattice-informed} approach which exploited the structure of the inclusion structure of closed sets of partial polymorphisms, in order to identify an NP-complete \textsc{SAT}(\Gamma) problem such that $c(\Gamma) \leq c(\Delta)$ for every other NP-complete \textsc{SAT}(\Delta). This problem was referred to as the \textit{easiest NP-complete SAT problem} and was later generalized to a broad class of finite-domain CSPs \[32\]. However, continued advancements in understanding this inclusion structure revealed that even severely restricted classes of constraint languages had a very complicated
structure \[12, 35\]. In a similar vein of negative results it was also proven that (1) \( p\text{Pol}(\Gamma) \) cannot be generated by any finite set of partial operations whenever \( \Gamma \) is finite and \( \text{SAT}(\Gamma) \) is NP-hard, and (2) if \( P \) is a finite set of partial operations such that \( \text{Inv}(P) \text{-SAT} \) is NP-hard, then any pp-definable relation over \( \text{Inv}(P) \) can be transformed into a pp-definition using only a linear number of existentially quantified variables \[37\]. In plain language, these results show that finite constraint languages result in complex partial polymorphisms, and that simple partial polymorphisms result in complex constraint languages. A previous attempt at grappling with this difficulty provided closure operators that generate \( p\text{Pol}(\Gamma) \) for a finite \( \Gamma \) from a finite basis \[34\], but this intrinsically uses that \( \Gamma \) is finite, and is not applicable in the current paper.

Our approach in this paper avoids the pitfalls of the lattice-informed approach since it is sufficient to understand the behaviour of individual pSDI-operations. This is in line with how the research programme of classifying the complexity of finite-domain CSPs evolved into a project of describing properties of operations defined by system of identities (see the survey by Barto et al. for more details \[3\]).

Another related paper by the present authors investigates the existence of polynomial (or linear) kernels for problems \( \text{SAT}(\Gamma) \), using ideas of extending the language \( \Gamma \) into a tractable CSP on a larger domain \[36\], including extensions into 2-edge (i.e., Maltsev) and \( k \)-edge languages. However, there is no concrete technical connection between that paper and this one, as having polynomial kernels turns out to be a much more restricted property than admitting improved algorithms.

### 1.5 Concluding remarks and open questions

Our principal motivation in this paper is to study the SETH-hardness of the parameterized \( \text{SAT}(\Gamma) \) problem. To simplify our study we restricted our focus to sign-symmetric constraint languages, which is a common assumption for SAT problems studied in practice. Moreover, due to the connection between sign-symmetric constraint languages and pSDI-operations, understanding the inclusion structure between sign-symmetric constraint languages is tantamount to describing the expressive power of pSDI-operations. Even better, pSDI-operations can in many cases be understood as the partial analogues of well-studied operations such as Maltsev operations, NU-operations and edge-operations, making them easier to reason with.

The main open question is whether our results can be strengthened into a dichotomy for sign-symmetric SAT problems. One direction is already clear: if \( \Gamma \) is not preserved by any \( k \)-universal operation then \( \text{SAT}(\Gamma) \) is SETH-hard and does not admit an improved algorithm without breaking the SETH. The other direction is harder and requires a substantially better understanding of languages invariant under a given \( k \)-universal operation; such languages include, but are not limited to, relations expressible as roots of polynomial equations of degree at most \( k + 1 \), where an improved algorithm is known \[42\]. It is not clear at this point how much richer the set \( \text{Inv}(u_k) \) is, compared to this class of problems. Existing (conjectured) lower bounds against polynomial-time problems captured by abstract \( \text{Inv}(p) \)-problems also indicate that the problem might be more difficult for remaining cases. We also proved that the SAT problems under consideration admit lower bounds under the SETH. To the best of our knowledge, this is the first result showcasing both a non-trivial upper bound and a concrete lower bound under the SETH in terms of a natural parameter \( n \). These bounds were obtained in the extension oracle setting and it is currently unclear if matching bounds can also be obtained if constraints are represented explicitly. The padding construction is still valid in this setting, but it is a challenge to apply it without creating constraints with exponentially many tuples.

Last, our approach easily extends to finite-domain CSPs, as evidenced by the improved algorithms for 2-edge-CSP and 3-NU-CSP. The notion of a pSDI-operation is only relevant in the Boolean domain, but a similar notion can likely be defined for arbitrary finite domains. For example, instead of self-duality, essentially meaning that the partial operation is closed under negation, we would require that the operation is closed under every unary operation over the domain. However, it is not clear if the inclusion structure of such generalized pSDI-operations can be characterized in a similar hierarchy as the Boolean pSDI-operations.
2 Preliminaries

A k-ary relation over a domain $D$ is a subset of $D^k$. If $t = (x_1, \ldots, x_n)$ is a k-ary tuple we for every $1 \leq i \leq k$ let $t[i] = x_i$, and if $i_1, \ldots, i_k \in [k] = \{1, \ldots, k\}$ we write $\text{Proj}_{i_1, \ldots, i_k}(t) = (t[i_1], \ldots, t[i_k])$ for the projection of $t$ on the coordinates $i_1, \ldots, i_k$. This notation easily extends to relations and we write $\text{Proj}_{i_1, \ldots, i_k}(R)$ for the relation $\{\text{Proj}_{i_1, \ldots, i_k}(t) \mid t \in R\}$.

A set of relations is called a constraint language, or simply a language, and will usually be denoted by $\Gamma$ and $\Delta$. We will typically define relations either by their defining logical formulas or by their defining equations. For example, the relation $R_{1/3} = \{(0, 0, 1), (0, 1, 0), (1, 0, 0)\}$ may be defined by the expression $R_{1/3} \equiv x_1 + x_2 + x_3 = 1$. However, we will not always make a sharp distinction between relations and their defining logical formulas and will sometimes treat e.g. a $k$-clause as a relation. We write $\text{ar}(R)$ for the arity of a relation $R$, and use the notation $\text{Eq}_D$ to denote the equality relation $\{(x, x) \mid x \in D\}$ over $D$.

A k-ary relation $R$ is said to be totally symmetric, or just symmetric, if there exists a set $S \subseteq [k] = \{1, \ldots, k\}$ such that $(x_1, \ldots, x_k) \in R$ if and only if $x_1 + \ldots + x_k \in S$. For example, $R_{1/3}$ is totally symmetric as witnessed by the set $S = \{1\}$. Symmetric relations will prove to be useful since it is sometimes considerably simpler to describe the symmetric relations invariant under a partial operation.

2.1 The parameterized SAT and CSP Problems

Let $\Gamma$ be a Boolean constraint language. The parameterized satisfiability problem over $\Gamma$ (SAT($\Gamma$)) is the computational decision problem defined as follows.

**Instance:** A set $V$ of variables and a set $C$ of constraint applications $R(v_1, \ldots, v_k)$ where $R \in \Gamma$, $\text{ar}(R) = k$, and $v_1, \ldots, v_k \in V$.

**Question:** Is there a function $f : V \rightarrow \{0, 1\}$ such that $(f(v_1), \ldots, f(v_k)) \in R$ for each $R(v_1, \ldots, v_k)$ in $C$?

The constraint satisfaction problem over a constraint language $\Gamma$ (CSP($\Gamma$)) is defined analogously with the only distinction that $\Gamma$ is not necessarily Boolean. We write $(d,k)$-CSP for the CSP problem over a domain with $d$ elements where each constraint has arity at most $k$.

2.2 The extension oracle model

Recall from Section 2 that we consider two distinct representations of SAT and CSP instances. We now define these in more detail. In the first representation each relation $R$ occurring in a constraint $R(x_1, \ldots, x_k)$ is represented as a list of tuples. We call this representation the explicit representation. This is one of the most frequently occurring representation methods in the algebraic approach to CSP, but it is fair to say that it is not convenient in any practical application since a relation may contain exponentially many tuples with respect to the number of arguments. We therefore consider a more implicit representation where each constraint is represented by a procedure which can verify whether a partial assignment of its variables is consistent with the constraint.

**Definition 7.** Let $R$ be an $n$-ary relation over a set $D$. A computable function which given indices $i_1, \ldots, i_{n'} \in [n]$ and $t \in D^{n'}$ answers yes if and only if $t \in \text{Proj}_{i_1, \ldots, i_{n'}}(R)$ is called an extension oracle representation of $R$.

Hence, given a constraint $R(x_1, \ldots, x_n)$ and a partial truth assignment $f : X \rightarrow D$, $X \subseteq \{x_1, \ldots, x_n\}$, the extension oracle representation can be used to decide whether $f$ can be completed into a satisfying assignment of $R(x_1, \ldots, x_n)$.

**Example 8.** CNF-SAT can be succinctly represented in the extension oracle model. Consider e.g. a positive clause $(x_1 \lor \ldots \lor x_n)$ and a partial truth assignment $f$ on $\{x_1, \ldots, x_n\}$. We can then answer yes if and only if not every variable $x_i$ occurring in the clause is assigned the value 0.
2.3 Sign-symmetric constraint languages

An \( n \)-ary sign pattern is an tuple \( s \) where \( s[i] \in \{+, -\} \) for each \( 1 \leq i \leq n \). If \( t \) is an \( n \)-ary Boolean tuple and \( s \) an \( n \)-ary sign pattern then we let \( t^s \) be the tuple where \( t^s[i] = t[i] \) if \( s[i] = + \) and \( t^s[i] = 1 - t[i] \) if \( s[i] = - \). Similarly, if \( R \) is a Boolean relation and \( s \) an \( n \)-ary sign pattern we by \( R^s \) denote the relation \( \{t^s \mid t \in R\} \). Last, for \( 1 \leq i \leq n \) and \( c \in \{0, 1\} \) we let \( R_{i=c} = \{t \mid t \in R, t[i] = c\} \) be the relation resulting from freezing the \( i \)-th argument of \( R \) to \( c \).

**Definition 9.** A Boolean constraint language \( \Gamma \) is said to be sign-symmetric if (1) \( R^s \in \Gamma \) for every \( n \)-ary \( R \in \Gamma \) and every \( n \)-ary sign pattern \( s \) and (2) \( R_{i=c} \in \Gamma \) for every \( c \in \{0, 1\} \) and every \( 1 \leq i \leq n \).

2.4 Partial polymorphisms and quantifier-free primitive positive definitions

Let \( D \) be a finite set of values. A \( k \)-ary partial operation, or a partial function, \( f \) over \( D \) is a mapping \( X \to D \) where \( X \subseteq D^k \). The set \( X \) is said to be the domain of \( f \) and we let \( \text{domain}(f) = X \) denote this set and \( \text{ar}(f) = k \) denote the arity of \( f \). If \( f \) and \( g \) are two \( n \)-ary partial operations over \( D \) such that \( \text{domain}(g) \subseteq \text{domain}(f) \) and \( g(x_1, \ldots, x_n) = f(x_1, \ldots, x_n) \) for every \( (x_1, \ldots, x_n) \in \text{domain}(g) \) then \( g \) is said to be a subfunction of \( f \). For \( n \geq 1 \) the \( i \)-ary projection, \( 1 \leq i \leq n \), is the operation \( \pi_i^g(x_1, \ldots, x_i, \ldots, x_n) = x_i \) and a partial projection is any subfunction of a total projection.

If \( R \) is an \( n \)-ary relation over \( D \) and \( f \) a \( k \)-ary partial operation over \( D \) we say that \( f \) is a partial polymorphism of \( R \), that \( R \) is invariant under \( f \), or that \( f \) preserves \( R \), if \( f(t_1, \ldots, t_k) \in t \) or \( f(t_1, \ldots, t_k) \) is undefined, for each sequence of tuples \( t_1, \ldots, t_k \). We let \( \text{pPol}(R) \) be the set of all partial polymorphisms of the relation \( R \), and if \( \Gamma \) is a constraint language we let \( \text{pPol}(\Gamma) \) denote the set of partial operations preserving each relation in \( \Gamma \). The notion of a total polymorphism can be defined simply by requiring that \( f \) is total, i.e., \( \text{domain}(f) = D^k \), and we let \( \text{Pol}(\Gamma) \) be the set of all total polymorphisms of the constraint language \( \Gamma \). Similarly, if \( P \) is a set of partial operations we let \( \text{Inv}(P) \) be the set of all relations invariant under \( P \). Each set of partial operations \( P \) naturally induces a SAT problem \( \text{SAT}(\text{Inv}(P)) \), where each relation involved in a constraint is preserved by every partial operation in \( P \). Recall from Section 1.2 that we as a shorthand denote this problem by \( \text{Inv}(P) \)-SAT. The two operators \( \text{Inv}(-) \) and \( \text{pPol}(-) \) are related by the following Galois connection.

**Theorem 10 ([21][47]).** Let \( \Gamma \) and \( \Delta \) be two constraint languages. Then \( \Gamma \subseteq \text{Inv}(\text{pPol}(\Delta)) \) if and only if \( \text{pPol}(\Delta) \subseteq \text{pPol}(\Gamma) \).

The applicability of partial polymorphism in the context of fine-grained time complexity might not be evident from these definitions. However, sets of the form \( \text{Inv}(P) \), called weak systems or weak co-clones, are closed under certain restricted first-order formulas which are highly useful in this context. Say that a \( k \)-ary relation \( R \) has a quantifier-free definition (qfpp-definition) over a constraint language \( \Gamma \) over a domain \( D \) if \( R(x_1, \ldots, x_k) \equiv R_1(x_1) \land \ldots \land R_m(x_m) \) where each \( R_i \in \Gamma \cup \{E_{l_D}\} \) and each \( x_i \) is a tuple of variables of length \( \text{ar}(R_i) \). It is then known that \( \text{Inv}(P) \) for any set of partial operations \( P \) is closed under taking qfpp-definitions. With this property the following theorem is then a straightforward consequence.

**Theorem 11.** Let \( \Gamma \) and \( \Delta \) be two finite constraint languages. If \( \text{pPol}(\Gamma) \subseteq \text{pPol}(\Delta) \) then there exists a polynomial-time many-one reduction from \( \text{SAT}(\Delta) \) to \( \text{SAT}(\Gamma) \) which maps an instance \( (V, C) \) of \( \text{SAT}(\Delta) \) to an instance \( (V', C') \) of \( \text{SAT}(\Gamma) \) where \( |V'| \leq |V| \) and \( |C'| \leq c|C| \), where \( c \) depends only on \( \Gamma \) and \( \Delta \).

In particular this implies that if \( \text{CSP}(\Gamma) \) is solvable in \( O(e^n) \) time and \( \text{pPol}(\Gamma) \subseteq \text{pPol}(\Delta) \) then \( \text{CSP}(\Delta) \) is solvable in \( O(e^n) \) time, too. We will now briefly describe the closure properties of \( \text{pPol}(\Gamma) \), which are usually called strong partial clones. First, if \( f, g_1, \ldots, g_m \in \text{pPol}(\Gamma) \) where \( f \) is \( m \)-ary and each \( g_i \) is \( n \)-ary, then the composition \( f \circ g_1, \ldots, g_m(x_1, \ldots, x_n) = f(g_1(x_1, \ldots, x_n), \ldots, g_m(x_1, \ldots, x_n)) \) is also included
in \( \text{pPol}(\Gamma) \). This operation will be defined on a tuple \((x_1, \ldots, x_n) \in D^n\) if and only if each \(g_t(x_1, \ldots, x_n)\) is defined and the resulting application over \(f\) is defined. Second, \(\text{pPol}(\Gamma)\) contains every partial projection, which is known to imply that \(\text{pPol}(\Gamma)\) is closed under taking subfunctions (i.e., if \(f \in \text{pPol}(\Gamma)\) then every subfunction of \(f\) is included in \(\text{pPol}(\Gamma)\)). If \(P\) is a set of partial operations we write \([P]_s = \text{pPol}(\text{Inv}(P))\) for the smallest strong partial clone containing \(P\).

### 2.5 Polymorphism patterns

In this section we describe a method for constructing partial polymorphisms that have a strong connection to the sign-symmetric constraint languages defined in Section 2.3. As a shorthand we will sometimes denote the \(k\)-ary constant tuple \((d, \ldots, d)\) by \(d^k\).

**Definition 12.** Let \(f\) be a Boolean partial operation. We say (1) that \(f\) is self-dual if \(\bar{x} \in \text{domain}(f)\) for every \(x \in \text{domain}(f)\) and \(f(x) = 1 - f(\bar{x})\), where \(\bar{x}\) denotes the complement of the tuple \(x\), and (2) that \(f\) is idempotent if \(d^k \in \text{domain}(f)\) and \(f(d^k) = d\) for every \(d \in D\).

In the sequel, we will call a Boolean partial operation which is both self-dual and idempotent a \(\text{pSDI-operation}\), short for partial, self-dual, and idempotent operation. Let a polymorphism pattern of arity \(r\) be a set of pairs \((t, x)\) where \(t\) is an \(r\)-ary tuple of variables and where \(x\) occurs in \(t\). We say that a \(r\)-ary partial operation \(f\) over a set of values \(D\) satisfies an \(r\)-ary polymorphism pattern \(P\) if

\[
\text{domain}(f) = \{(\tau(x_1), \ldots, \tau(x_r)) \mid ((x_1, \ldots, x_r), x) \in P, \tau : \{x_1, \ldots, x_r\} \rightarrow D\}
\]

and \(f(\tau(x_1), \ldots, \tau(x_r)) = \tau(x)\) for every \(((x_1, \ldots, x_r), x) \in P\) and every \(\tau : \{x_1, \ldots, x_r\} \rightarrow D\).

A Boolean operation is \(\text{pSDI}\) if and only if it satisfies a polymorphism pattern. To see this, note that if \(f\) is \(\text{pSDI}\), then it is easy to create a polymorphism pattern \(P\) by letting each tuple \(t \in \text{domain}(f)\) such that \(f(t) = 0\) correspond to an entry in \(P\). Similarly, it is not difficult to show that any partial operation satisfying a polymorphism pattern must be self-dual and idempotent. We then have the following link between sign-symmetric constraint languages and partial operations satisfying polymorphism patterns.

**Theorem 13.** \([38]\) Let \(f\) be a \(\text{pSDI-operation}\). Then \(\text{Inv}(f)\) is sign-symmetric.

Hence, \(\text{pSDI}\)-operations provide a straightforward way to describe broad classes of sign-symmetric constraint languages. It is also known that if \(\Gamma\) is sign-symmetric and \(\text{SAT}(\Gamma)\) is NP-hard, then every partial polymorphism of \(\Gamma\) is a subfunction of a \(\text{pSDI}\)-operation preserving \(\Gamma\) \([38]\)[Theorem 3] (see Lagerkvist \([34]\) for a full proof). We will now define the \(\text{pSDI}\)-operations that will play a central role in our current pursuit.

**Definition 14.** Let \(k \geq 2\). A \((k+1)\)-ary partial operation is a partial \(k\)-edge operation if it satisfies the pattern consisting of \(((x, x, y, y, \ldots, y, y, x, y)\), \(((x, y, x, y, y, \ldots, y, y, y)\), and for each \(i \in \{4, \ldots, k+1\}\), the tuple \(((y, \ldots, y, x, y, y, \ldots, y)\), where \(x\) appears in position \(i\).

We will typically denote partial \(k\)-edge operations by \(e_k\), and, if the underlying set \(D\) is important, by \(e_k^D\). A partial 2-edge operation will sometimes be called a partial Mal'tsev operation.

**Definition 15.** Let \(k \geq 3\). A \(k\)-ary partial operation is a partial \(k\)-ary near-unanimity operation (partial \(k\)-NU operation) if it satisfies the pattern which for each \(i \in \{1, \ldots, k\}\) contains \(((x, x, \ldots, x, y, x, \ldots, x)\), \(x\), where \(y\) occurs in position \(i\).

We write \(\text{nu}_k^D\) to denote this operation over the domain \(D\), and \(\text{nu}_k\) if the domain is clear from the context, or not relevant. Ternary partial NU-operations will sometimes be called partial majority operations. Note that the partial majority operation is total in the Boolean domain but is properly partial for every larger domain. Last, we define the following class of self-dual partial operations. Say that the argument \(i\) of a \(k\)-ary partial operation \(f\) is redundant if there exists \(j \neq i\) such that \(t[i] = t[j]\) for every \(t \in \text{domain}(f)\).
Definition 16. Let \( k \geq 2 \). The \( k \)-universal operation \( u_k \) is the Boolean \((2^k - 1)\)-ary pSDI-operation defined on \( 2k + 2 \) tuples such that (1) \( u_k \) is not a partial projection and (2) \( u_k \) does not have any redundant arguments.

While not immediate from the definition, the operation \( u_k \) is in fact unique up to permutation of arguments. To see this, simply take the \( k \) non-constant tuples \( t_1, \ldots, t_k \in \text{domain}(u_k) \) such that \( u_k(t_1) = \ldots = u_k(t_k) = 0 \). Since \( u_k \) is not a projection and is pSDI, it follows that there cannot exist \( i \in [2^k - 1] \) such that \( (t_1[i], \ldots, t_k[i]) = 0^k \). Hence, since \( u_k \) does not have any redundant arguments, there for every \( t \in \{0, 1\}^k \setminus \{0^k\} \) must exist a unique \( i \in [2^k - 1] \) such that \( (t_1[i], \ldots, t_k[i]) = t \).

Last, we remark that there is a connection between our notion of polymorphism patterns and the operations studied in connection to the CSP dichotomy (see e.g. the survey by Barto et al. [3]). In technical terms polymorphism patterns essentially matches strong Mal'tsev conditions where the right-hand side is restricted to a single variable. Similar restrictions, called height-1 identities, have been considered earlier and it is known that the complexity of a CSP(\( \Gamma \)) problem only depends on the height-1 identities satisfied by the operations in \( \text{Pol}(\Gamma) \) [33].

3 Structure of Constraint Languages under Minimal Restrictions

We now properly begin the first part of the paper, investigating the structure of maximally expressive, yet restricted sign-symmetric constraint languages. This investigation is performed via the study of the weakest non-trivial pSDI-operations, including the operations defined in Section 2.5. As a preview of the structure, and of some of the included problems, we refer to Figure 1. The problem and language inclusions illustrated in this figure will be shown across the next two subsections.

More precisely, by “weakest” pSDI-operations, we mean partial operations that are minimal in the following sense. Recall that for every pSDI-operation \( f \) and every subfunction \( f' \) of \( f \), we have \( \text{Inv}(f) \subseteq \text{Inv}(f') \). This motivates the following definition.
Definition 17. Let $f$ be a pSDI-operation. We say that $f$ is trivial if it is a subfunction of a projection, and a minimal non-trivial pSDI-operation if $f$ is non-trivial but every proper subfunction $f'$ of $f$ which is a pSDI-operation is trivial.

Our study in this section is focused on constraint languages $\Gamma = \text{Inv}(f)$ where $f$ is a single minimal non-trivial pSDI-operation, since these are the most expressive sign-symmetric constraint languages that are still restricted in expressive power. We begin by giving some examples for the particular classes of $k$-NU, $k$-edge and $k$-universal partial operations defined in Section 2.5.

3.1 Properties of specific sign-symmetric constraint languages

In this section, we provide some illustrative examples of languages included in $\text{Inv}(f)$ for particular pSDI-operations $f$. We first recall the following result from Lagerkvist & Wahlström.

Theorem 18. [37] Let $F$ be a finite set of partial operations such that $\text{Inv}(F)$-SAT is NP-complete. Then any $n$-ary Boolean relation has a pp-definition over $\text{Inv}(F)$ using at most $O(n)$ existentially quantified variables.

In effect, this implies that any constraint language $\text{Inv}(F)$, where $F$ is a finite set of pSDI-operations, is extremely expressive. One direct consequence is that $\text{Inv}(F)$ contains at least $2^{2^n}$ $n$-ary relations for some constant $0 < c \leq 1$. This makes such constraint languages markedly different from finite constraint languages, since for any finite constraint language $\Gamma$, the number of $n$-ary qfpp-definable relations over $\Gamma$ is bounded by $O(2^{p(n)})$ for a polynomial $p$ depending on $\Gamma$. This also implies that there cannot exist a finite $\Gamma$ such that $\text{pPol}(\Gamma) = [F]_s$. In fact, the relations of $\text{Inv}(F)$ for such an $F$ are dense enough that for any $n$-ary relation $R$, a random padding of $R$ by $O(n)$ parity-check variables is enough to create a variable in $\text{Inv}(F)$ with high probability. This fact will be exploited in Section 3.3.

Despite this, we will see that the pSDI-operations defined in Section 2.5 do correspond roughly to natural restrictions on the expressive power of a language $\Gamma$. We now illustrate the classes with a few examples. In the process will occasionally refer to the language inclusions illustrated in Figure 1. Proofs of these inclusions is given in Theorem 29 in Section 3.2. Let us now begin with a basic example.

Lemma 19. $R \in \text{Inv}(\text{nu}_k)$ for every $(k-1)$-ary relation $R$, $k \geq 3$.

Proof. Let $t_1, \ldots, t_k \in R$ be such that $\text{nu}_k(t_1, \ldots, t_k)$ is defined, and for $i \in [k-1]$ let $t^{(i)} = (t_1[i], \ldots, t_k[i])$. For every $i \in [k-1]$, either $t^{(i)}$ is constant or there is a single index $j$ where $t^{(i)}[j]$ deviates from its other entries. By the pigeonhole principle, there is at least one index $j \in [k]$ such that $t^{(i)}[j]$ does not deviate from the majority for any $i \in [k-1]$. Then we have $\text{nu}_k(t_1, \ldots, t_k) = t_j$. □

We also show a corresponding negative statement. By the inclusions shown in the next section, this will imply that a $k$-clause is not preserved by any operation at “level $k$” of the hierarchy in Figure 1.

Lemma 20. Let $R \subset \{0, 1\}^k$ be a $k$-clause, i.e., $|R| = 2^k - 1$, $k \geq 2$. Then $R$ is not preserved by the partial $k$-universal operation.

Proof. By sign-symmetry, we assume that $R = \{0, 1\}^k \setminus \{0^k\}$. Let $t_1, \ldots, t_k$ be the non-constant tuples in $\text{domain}(\text{nu}_k)$ such that $\text{nu}_k(t_i) = 0$ for each $i \in [k]$. Then for each $i \in [2^k - 1]$, the tuple $t^{(i)} = (t_1[i], \ldots, t_k[i])$ defines a tuple of $R$; thus the application $\text{nu}_k(t^{(1)}, \ldots, t^{(2^k-1)}) = 0^k$ is defined and shows that $R \notin \text{Inv}(\text{nu}_k)$. □

Next, we consider a canonical example of a useful relation preserved by the partial 2-edge operation.
Lemma 21. Let \( R(x_1, \ldots, x_n) \subseteq \{0, 1\}^n \) be defined via a linear equation
\[
\sum_{i=1}^{n} \alpha_i x_i = \beta
\]
evaluated over a finite field \( \mathbb{F} \). Then \( R \in \text{Inv}(e_2) \).

Proof. This is a special case of the notion of a Mal'tsev embedding of \( R \) previously investigated by the authors [36]. It is known that a relation with a Mal'tsev embedding is closed under a family of partial operations, of which \( e_2 \) is the simplest. \( \square \)

A particular example of such relations is the Exact SAT problem. We show that its 1-in-\( k \) relations are also not closed under \( \nu_k \).

Lemma 22. Let \( R_{1/k} = \{(x_1, \ldots, x_k) \in \{0, 1\}^k \mid x_1 + \ldots + x_k = 1 \} \), and \( \Gamma_{\text{XSAT}} = \{ R_{1/k}^s \mid k \geq 1, s \) is a \( k \)-ary sign-pattern \}. Then \( \Gamma_{\text{XSAT}} \subseteq \text{Inv}(e_2) \) but is not preserved by \( \nu_k \) for any \( k \).

Proof. The positive direction follows from Lemma 21 since \( R_{1/k} \) can be phrased as a linear equation over the integers mod \( p \), for \( p \geq k + 1 \). The negative direction is immediate: let \( R_{1/k} = \{ t_1, \ldots, t_k \} \). Then \( \nu_k(t_1, \ldots, t_k) \) is defined and equals \( 0^k \).

Another example of a problem with the character of linear equations is Subset Sum. Even though an instance of Subset Sum is defined by just a single linear equation rather than as a SAT(\( \Gamma \)) instance, we show in Section 5.2 that the complexity of 2-edge-SAT and Subset Sum are closely connected. As for the class \( \text{Inv}(e_2) \) for \( k \geq 3 \), the inclusions illustrated in Figure 7 imply that this class contains both relations with linear equation extensions and all \((k - 1)\)-clauses.

Finally, we show two examples for the partial \( k \)-universal operation \( \nu_k \). The first is a previously studied class of Lokshnanov et al. [42]. Note this problem does admit an improved algorithm.

Definition 23. Let \( P_d \) denote the set of Boolean relations such that each \( n \)-ary \( R \in P_d \) is the set of roots of an \( n \)-variate polynomial equation where each polynomial has degree at most \( d \).

Lemma 24. Let \( R \in P_d \) be an \( n \)-ary relation. Then \( R \) is preserved by \( \nu_{d+1} \), but not by any other non-trivial pSDI-operation of domain size at most \( 2d + 2 \).

Proof. For the first direction, let \( P(x_1, \ldots, x_n) \) be the polynomial defining \( R \), and let \( t_1, \ldots, t_r \in R \) be such that \( \nu_{d+1}(t_1, \ldots, t_r) = t' \) is defined. Since the set of relations representable by bounded-degree polynomials is sign-symmetric, we may assume for simplicity that \( t' = 1^n \). The tuples \( (t_1, \ldots, t_r) \) define a new polynomial of degree at most \( d \) and with at most \( d + 1 \) variables, defined by identifying all pairs of variables \( x_i \) and \( x_j \) that have the same pattern in \( (t_1, \ldots, t_r) \), i.e., if \( t_a[i] = t_a[j] \) for every \( a \in [r] \). We also eliminate any variable \( x_i \) such that \( t_j[i] = 1 \) for every \( j \in [r] \) by replacing \( x_i \) by the constant 1 in \( P \). Let \( P' \) be the resulting polynomial, and let \( R' \) be the corresponding relation. If \( \text{ar}(R') \leq d + 1 \), then by Lemma 20 \( R' \) is preserved by \( \nu_{d+1} \) and thus by \( \nu_{d+1} \) as well (see Theorem 29). Otherwise, for each \( I \subseteq [d + 1] \) let \( \alpha_I \) be the coefficient of the monomial \( \prod_{i \in I} x_i \) in \( P' \), and let \( \chi_I \in \{0, 1\}^{d+1} \) be the tuple such that \( \chi_I[i] = 1 \) if and only if \( i \in I \). Note that \( P'(\chi_I) = \sum_{I \subseteq J} \alpha_J \). We find that \( \alpha_I = 0 \) for every \( I \). Indeed, \( \alpha_0 = 0 \) since \( 0^{d+1} \in R' \); and \( \alpha_{\{i\}} = 0 \) for every \( i \) since \( P'(\chi_{\{i\}}) = \alpha_{\{i\}} + \alpha_0 = \alpha - \{i\} = 0 \); and so on, in order of increasing cardinality of \( I \). Then \( P' \) is the constantly-zero polynomial, and \( 1^{d+1} \in R' \), hence \( t' = 1^n \in R \). We have thus shown that relations defined as roots of polynomials of degree \( d \) are preserved by the \((d+1)\)-universal operation.

In the other direction, the same argument will show that for any pSDI-operation \( f \) with \( |\text{domain}(f)| \leq 2d + 2 \) other than the \((d+1)\)-universal operation, it is possible to define a polynomial on \((|\text{domain}(f)| - 2)/2 \)
variables and of degree at most \( d \) such that the corresponding relation is not preserved by \( f \). Indeed, let \( n = (|\text{domain}(f)| - 2)/2 \) and \( r = \ar(f) \), and let \( t_1, \ldots, t_r \) be tuples of arity \( n \) such that no tuple \((t_1[i], \ldots, t_r[i])\) is constant and \( f(t_1, \ldots, t_r) = 1^n \) is defined. If \( n \leq d \), then we may simply consider the polynomial \( P(x_1, \ldots, x_n) = \prod_{i \in [n]} x_i \), whose corresponding relation \( R \) is not preserved by \( f \). Otherwise, let \( I \subset [d + 1] \) be such that \( \chi_I \notin \{t_1, \ldots, t_r\} \); this exists since \( f \) is not the \((d + 1)\)-universal partial operation. Let \( P' \) be the \( d + 1 \)-variate polynomial with coefficients \( \alpha_j = 0 \) if \( I \not\subseteq J \), and with \( \alpha_j = (-1)^{|J| \cdot |I|} \) otherwise, for all \( J \subset [d + 1] \). Then \( P'(t_J) = 1 \), and it can be verified that \( P'(t_J) = 0 \) for every \( J \subset [d + 1] \), \( J \neq I \), whereas \( P'(1^{d+1}) = -(-1)^{d+1-|I|} \). Hence the relation corresponding to \( P' \) is not preserved by \( f \).

Finally, we give one example of a symmetric relation in \( \text{Inv}(u_3) \) that has no obvious connection to roots of polynomials. A Sidon set is a set \( S \subseteq \{0, \ldots, n\} \) in which all sums \( i + j, i, j \in S \) are distinct.

**Lemma 25.** Let \( S \subseteq \{0, \ldots, n\} \) be a Sidon set, and define a relation \( R(x_1, \ldots, x_n) \subseteq \{0, 1\}^n \) as

\[
R(x_1, \ldots, x_n) \equiv \left( \sum_{i=1}^n x_i \in S \right).
\]

Then \( R \) is preserved by \( u_3 \).

**Proof.** Assume that there exists \( t_1, \ldots, t_7 \in R \) such that \( u_3(t_1, \ldots, t_7) = t \not\in R \). For \( i \in [n] \), let \( x_i = (t_1[i], \ldots, t_7[i]) \) be the tuple of values taken by argument \( i \) of \( R \) in these tuples. Then the tuples \( x_i \) take up to \( 8 \) different values, partitioned as two constant tuples and three pairs of complementary tuples. Let \( X_j \) for \( j = 1, 2, 3 \) be the set of arguments \( i \in [n] \) such that the tuple \( x_i \) belongs to the \( j \)-th of these pairs, and let \( n_j \) be the difference in Hamming weight compared to \( t \) if flipping all values belonging to \( X_j \). Let \( W \) be the Hamming weight of \( t \). Then \( S \) contains the values \( W + n_1, W + n_2, W + n_1 + n_3 \) and \( W + n_2 + n_3 \), forming two pairs of weights with common difference \( n_3 \). Since \( n_3 \neq 0 \), we must have \( n_1 = n_2 \). By symmetry, we have \( n_1 = n_2 = n_3 \). But then \( S \) contains the values \( W + n_1, W + n_1 + n_2 = W + 2n_1, \) and \( W + n_1 + n_2 + n_3 = W + 3n_1 \), which is a contradiction. Thus \( n_j = 0 \) for at least one \( j \), hence \( W \in S \) and \( t \in R \), contradicting the original assumption.

### 3.2 Structure of minimal non-trivial pSDI-operations

Note that if \( f \) is a pSDI-operation, then \( |\text{domain}(f)| = 2k + 2 \) for some \( k \), since \( f \) is defined on the two constant tuples and since the tuples of the domain can be paired up as \((t, \bar{t})\) where \( \bar{t} \) is the complement of \( t \). Hence, we define the level of a minimal non-trivial pSDI operation \( f \) as \((|\text{domain}(f)| - 2)/2 \). We find no examples on level 0 or 1, and the only non-trivial example on level 2 is the 2-edge operation. At each level \( k \geq 3 \) the partial \( k \)-NU and \( k \)-universal operations are the unique strongest and weakest minimal non-trivial pSDI-operation, respectively, whereas the \( k \)-edge operation is intermediate. This structure is also illustrated in Figure[1]. We also find that the \( k \)-universal operations \( u_k \) are maximally weak in the sense that any non-trivial pSDI-operation with a domain of size \( 2k + 2 \) can define \( u_k \).

We begin with the following lemma, which formalizes one of the main methods of constructing a \((k + 1)\)-ary partial operation from a \( k \)-ary partial operation. We refer to \( g \) as an argument padding of \( f \).

**Lemma 26.** Let \( f \) be a \( k \)-ary partial operation and let \( g \) be a \((k + 1)\)-ary partial operation such that (1) \( \text{Proj}_{i=k+1}(\text{domain}(g)) = \text{domain}(f) \) and (2) \( f(x_1, \ldots, x_k) = g(x_1, \ldots, x_k, x_{k+1}) \) for every \((x_1, \ldots, x_k, x_{k+1}) \in \text{domain}(g) \). Then \( g \in [f]_s \).

**Proof.** Let \( f \) and \( g \) be as in the statement, and first construct the \((k + 1)\)-ary partial operation

\[
f'(x_1, \ldots, x_k, x_{k+1}) = f(\pi^{k+1}_1(x_1, \ldots, x_k, x_{k+1}), \ldots, \pi^{k+1}_k(x_1, \ldots, x_k, x_{k+1})).
\]

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Clearly, \( f' \in [f]_s \), since it is a composition of \( f \) and the projections \( \pi_{1}^{k+1}, \ldots, \pi_{k}^{k+1} \), and it is not difficult to see that \( \text{Proj}_{1,\ldots,k}(\text{domain}(f')) = \text{domain}(f) \) and that \( g \) can be obtained as a subfunction of \( f' \). Since \([f]_s \) is closed under taking subfunctions it follows that \( g \in [f]_s \).

The following will aid us in reasoning about minimal non-trivial pSDI-operations.

**Lemma 27.** Let \( f \) be a pSDI-operation with \( |\text{domain}(f)| = 2k + 2, k \geq 3 \). Then \( f \) is a minimal non-trivial operation if and only if \( f \) is an argument padding of \( u_k \).

**Proof.** In the one direction, assume that \( f \) is a padding of \( u_k \). It is not hard to verify that every subfunction \( f' \) of \( f \) which is pSDI is a partial projection, and that \( f \) is non-trivial. Thus, \( f \) is minimal non-trivial. In the other direction, assume that \( f \) is minimal and non-trivial, and let \( r = \ar(f) \). Let \( t_1, \ldots, t_k \) be the non-constant tuples such that \( f(t_1) = 0 \) is defined. For each \( i \in [k] \), let \( j_i \in [r] \) be such that making \( f \) undefined on \( t_i \) and its complement \( \overline{t}_i \) leaves a subfunction of \( \pi_{r_i} \). It follows that for all \( a \in [k], t_a[j_i] \neq 0 \) if and only if \( a = i \). Then the arguments \( j_1, \ldots, j_k \) of \( f \) define the partial \( k\)-NU operation, and \( f \) is a padding of it.

Our claims about the weakest and strongest operations follows from this.

**Lemma 28.** The following hold.

1. The unique non-trivial non-total pSDI-operation at level \( k < 3 \) is the partial 2-edge operation.
2. For any minimal non-trivial pSDI-operation \( f \) at level \( k \geq 3 \), we have \( \text{Inv}(u_k) \subseteq \text{Inv}(f) \subseteq \text{Inv}(u_k) \).
3. There are at most \( 2^{2^k-k-1} \) distinct minimal non-trivial pSDI-operations at level \( k \).

**Proof.** 1. It is easy to verify that no non-trivial operation is possible on level 1. Let \( f \) be a non-trivial pSDI-operation on level 2, and let \( t_1, t_2 \in \text{domain}(f) \) be the non-constant tuples such that \( f(t_1) = 0 \). Consider the options for the pairs \((t_1[i], t_2[i])\) for \( i \in [\ar(f)] \). If two distinct positions \( i, i' \) give identical pairs, then \( t[i] = t[i'] \) for every \( t \in \text{domain}(f) \) and \( i \) and \( i' \) are redundant arguments in \( f \), which we may assume does not occur. If \( t_1[i] = t_2[i] = 0 \) for some \( i \in [\ar(f)] \) then \( f \) is a partial projection. This leaves three possible arguments, and unless all three exist, \( f \) will be a total operation. The remaining case is that \( f = e_2 \).

2. By Lemma 27 \( f \) is a padding of \( u_k \), which provides the first inclusion. For the second, we may assume that \( f \) has no redundant arguments, since otherwise \( f \) is equivalent to an operation with fewer arguments. But then by design, \( u_k \) is a padding of \( f \), and the second inclusion follows.

3. By Lemma 27, we can restrict our attention to paddings of \( u_k \). Since \( f \) is a pSDI-operation, it is defined by the values of the \( k \) non-constant tuples \( t \) in the domain with \( f(t) = 0 \). Let \( t_1, \ldots, t_k \) be those tuples, and for \( i \in [\ar(f)] \) let \( t^{(i)} = (t_1[i], \ldots, t_k[i]) \). As above, we may assume that \( t^{(i)} \neq t^{(j)} \) for all distinct \( i, j \in [\ar(f)] \). This leaves at most \( 2^k \) possible arguments. Furthermore, \( t^{(i)} \) cannot be all-zero unless \( f \) is a partial projection, and \( k \) arguments are determined by \( u_k \). This leaves \( 2^k - k - 1 \) arguments, whose presence or absence defines \( f \).

The inclusion structure between the \( k\)-NU, \( k\)-edge and \( k\)-universal partial operations are now straightforward to prove with these results.

**Theorem 29.** Let \( k \geq 3 \). Then the following inclusions hold.

1. \( \text{Inv}(e_2) \subseteq \text{Inv}(e_k) \),
2. \( \text{Inv}(u_k) \subseteq \text{Inv}(e_k) \subseteq \text{Inv}(u_k) \).
3. Inv(\nu_k) \subset Inv(\nu_{k+1}),

4. Inv(e_k) \subset Inv(e_{k+1}), and

5. Inv(u_k) \subset Inv(u_{k+1}).

Proof. For the inclusions, the second item follows from Lemma 28 and every other inclusion follows from Lemma 26. Indeed, it is readily verified that for every \( k \geq 3 \), \( e_k \) is an argument padding of \( e_{k-1} \) and \( u_{k+1} \) is an argument padding of \( u_k \). For the universal operations, let \( t_1, \ldots, t_{k+1} \) be the non-constant tuples of domain(\( u_{k+1} \)) such that \( u_{k+1}(t_i) = 0, i \in [k+1] \). Then the tuples \( t^{(i)} = (t_1[i], \ldots, t_{k+1}[i]), i \in [2^{k+1} - 1] \) spell out all \((k + 1)\)-tuples except \( 0^{k+1}, \) without repetition. Consider the subset \( I \subset [\text{ar}(u_{k+1})] \) consisting of indices \( i \) such that \( t_{k+1}[i] = 0 \). Note that \( t^{(i)} \) for \( i \in I \) enumerates all \( k \)-tuples except \( 0^k \), padded with a 0. It follows that \( \text{Proj}_j(u_{k+1}) = \text{domain}(u_k) \) and that \( u_{k+1} \) is an argument padding of \( u_k \). By Lemma 26 the inclusion follows.

To show that the inclusions are strict, consider the following: a \( k \)-clause is preserved by \( u_{k+1} \) (Lemma 19) but not by \( u_k \) (Lemma 20); a 1-in-\( k \) constraint is preserved by \( e_2 \) but not by \( u_k \) (Lemma 22); and the language \( \mathcal{P}_{k-1} \) of roots of polynomials of degree at most \( k - 1 \) is preserved by \( u_k \) but not by any other operation on level \( k \) by Lemma 24.

Finally, we have an easy consequence in more general terms.

Corollary 30. Let \( f \) be a pSDI-operation with \( |\text{domain}(f)| = 2k + 2 \). Then Inv(f) \( \subseteq \) Inv(u_k).

Proof. Let \( f' \) be an arbitrary minimal pSDI-operation that is a subfunction of \( f \). Then \( f' \) belongs to some level \( k' \leq k \), hence Inv(f) \( \subseteq \) Inv(u_{k'}) \( \subseteq \) Inv(u_k) by Lemma 28 and Theorem 29.

3.3 Complementary consequences

We now consider some dual questions, i.e., what consequences can we (in general) draw from the information that some sign-symmetric language \( \Gamma \) is not preserved by \( f \), for some pSDI-operation \( f \)? We begin with an easy result, which forms the building block of later results.

Lemma 31. Let \( \Gamma \) be a sign-symmetric language which is not preserved by \( u_k \), for some \( k \geq 3 \). Then \( \Gamma \) can qfpp-define a \( k \)-ary symmetric relation \( R \) such that \( R \) does not contain tuples of weight 0, but does contain tuples of weight 2.

Proof. Let \( k \geq 3 \) be an arbitrary constant, and let \( R \in \Gamma \) be a relation not preserved by \( u_k \) of some arity \( n = \text{ar}(R) \). Let \( t_1, \ldots, t_k \in R \) be witnesses to this, i.e., \( t = u_k(t_1, \ldots, t_k) \) is defined and \( t \notin R \). Define \( t^{(i)} = (t_1[i], \ldots, t_k[i]) \).

By sign-symmetry, we may assume that \( t = 0^n \). Furthermore, if there is an argument \( i \in [n] \) such that \( t^{(i)} = 0^k \), then we can find a smaller counterexample by fixing argument \( i \) of \( R \) to be constantly 0. Thus, for every \( i \in [n] \), the tuple \( t^{(i)} \) now contains precisely one non-zero value. Let us define a new relation \( R'(x_1, \ldots, x_k) \) of arity \( k \) by identifying arguments according to this, i.e., for every position \( i \in [n] \) such that \( t^{(i)} \) is non-zero in position \( j \in [k] \), insert variable \( x_j \) in position \( i \) in \( R \). Additionally define \( R'' \) as the result of the conjunction of all \( k! \) applications of \( R' \) with permuted argument order. Then \( R'' \) is a symmetric relation which contains all tuples of weight 1 but none of weight 0. Thus, \( \Gamma \) qfpp-defines a relation \( R_k = R'' \) as described of every arity \( k \geq 3 \).

By a similar strategy, we have an important result about languages not preserved by the \( k \)-universal operation.
Theorem 33. Let \( \Gamma \) be a sign-symmetric language not preserved by \( u_k \) for some \( k \geq 2 \). Then \( \Gamma \) can \( \text{afpp} \)-define all \( k \)-clauses.

Proof. Let \( R \in \Gamma \) be a relation not preserved by \( u_k \), and let \( n = \text{ar}(R) \) and \( r = 2^k - 1 \) be the arity of \( u_k \). Let \( t_1, \ldots, t_r \in R \) be such that \( u_k(t_1, \ldots, t_r) = t \) is defined and \( t \notin R \). By sign-symmetry of \( \Gamma \), we may assume \( t = 0^p \). Create a new relation by identifying all variables \( x_i \) and \( x_j \) in \( R(x_1, \ldots, x_n) \) for which \( t_a[i] = t_a[j] \) for every \( a \in [r] \). Also assume that there is no variable \( x_i \) such that \( t_a[i] = 0 \) for every \( a \in [r] \), or else replace \( x_i \) by the constant 0 in \( R \) (again by sign-symmetry). This defines a new relation \( R' \) of arity at most \( k \). Since \( t \notin \{t_1, \ldots, t_r\} \), we find that \( R' \) has arity precisely \( k \) and contains every possible \( k \)-tuple except \( 0^k \), i.e., \( R' \) \( \text{afpp} \)-defines a \( k \)-clause. By sign-symmetry, \( \Gamma \) \( \text{afpp} \)-defines all \( k \)-clauses.

### 3.3.1 Infinitary case

Finally, we consider consequences of a language not being preserved by any operation in a family of operations.

**Theorem 33.** Let \( \Gamma \) be a sign-symmetric language that is not preserved by the partial \( k \)-NU operation, for any \( k \). Then one of the following holds.

1. \( \Gamma \) can \( \text{afpp} \)-define all \( k \)-clauses for every \( k \).
2. \( \Gamma \) can \( \text{afpp} \)-define 1-in-\( k \)-clauses for every \( k \).
3. There is a fixed prime \( p \) such that \( \Gamma \) can \( \text{afpp} \)-define relations

\[
\sum_{i=1}^{k} x_i \equiv a \pmod{p}
\]

for every \( 0 \leq a < p \), of every arity \( k \).

Before we proceed with the proof, let us make a simple observation about \( \text{afpp} \)-definitions among symmetric relations.

**Lemma 34.** Let \( R \) be a symmetric \( n \)-ary relation, including tuples of weights \( S \subseteq \{0, \ldots, n\} \). Using \( R \), we can \( \text{afpp} \)-define symmetric relations of the following descriptions.

1. Shift down: a relation of arity \( n - 1 \) accepting values \( S' = \{x - 1 \mid x \in S, x > 0\} \).
2. Truncate: a relation of arity \( n - 1 \) accepting values \( S' = \{x \in S \mid x < n\} \).
3. Grouping: for any integer \( p > 1 \), a relation of arity \( \lfloor n/p \rfloor \) accepting values \( S' = \{x' \mid x'p \in S\} \).

**Proof.** These are implemented by, respectively, fixing an argument to 1 in \( R \); fixing an argument to 0 in \( R \); and grouping arguments of \( R \) in groups of size \( p \) (after truncating \( \text{ar}(R) \) to an even multiple of \( p \)).

We can now show the result.

**Proof of Theorem 33**. Let \( k \geq 3 \) be an arbitrary constant. By Szemerédi’s theorem [54], there is a constant \( n = N(2k, 1/(2k+1)) \) such that every set \( S \subseteq [n] \) with \( |S| \geq n/(2k+1) \) contains an arithmetic progression \( a, a + p, \ldots \) of at least \( 2k \) items. Let \( R_n \) be a relation produced by Lemma 34 of arity \( n \), and let \( S \) be the accepted weights for \( R_n \). Say that an arithmetic progression \( a, a + p, \ldots \) is complete in \( S \) if \( S \) contains all values \( \{x \in \{0, \ldots, n\} \mid x \equiv a \pmod{b}\} \). We consider a few cases.
We may safely assume \( n \) contains no pairs weight holds with only 1-in-\( k \). In summary of this section, towards the purpose of discussing sign-symmetric languages

Section summary. In summary of this section, towards the purpose of discussing sign-symmetric languages \( \Gamma \) such that \( \text{SAT}(\Gamma) \) does, or does not, admit an improved algorithm under SETH, we conclude the following. Recall that \( \Gamma_{\text{SAT}}^k \) denotes the language of all \( k \)-clauses. We find that \( \Gamma_{\text{SAT}}^k \) is preserved by every minimal...
operation on level \( k' > k \) (in particular, by \( nu_{k+1} \)); not preserved by any operation on a level \( k' \leq k \); and that any sign-symmetric language \( \Gamma \) which is not preserved by the \( k \)-universal partial operation \( u_k \) can \( qfpp \)-define \( \Gamma_{\text{SAT}}^{k} \). Assuming \( \text{SETH} \), the minimal non-trivial \( \text{pSDI} \)-operations that preserve \( \Gamma \) therefore appear to be reasonable proxies for the complexity of \( \text{SAT}(\Gamma) \).

Finally, for each level \( k \), there is a language – namely the language of roots of polynomials of degree less than \( k \) – which is preserved by \( u_k \) but not by any other operation at level \( k' \leq k \), and which does admit an improved algorithm \([42]\). This shows that any “dichotomy” characterizing sign-symmetric languages \( \Gamma \) for which \( \text{SAT}(\Gamma) \) admits an improved algorithm under \( \text{SETH} \), cannot require a minimal non-trivial \( \text{pSDI} \)-operation other than \( u_k \) for some \( k \).

It remains to show that these very mild restrictions, of requiring only the presence of a single non-trivial \( \text{pSDI} \)-operation \( f \) preserving \( \Gamma \), can be powerful enough to ensure that \( \text{SAT}(\Gamma) \) admits an improved algorithm. This is our topic of study for the next section.

### 4 Upper bounds for sign-symmetric satisfiability problems

In this section, we consider the feasibility of designing an improved algorithm directly for \( \text{Inv}(f) \)-\( \text{SAT} \) and \( \text{Inv}(f) \)-\( \text{CSP} \) for a minimal non-trivial \( \text{pSDI} \)-operation \( f \), i.e., an improved algorithm that only uses the abstract properties guaranteed by such an operation \( f \).

We show this unconditionally for \( f = e_2 \) and for \( f = nu_3 \), over arbitrary finite domains (where the latter result is only interesting for the non-Boolean case, since the Boolean case is in \( \text{P} \)). The algorithms for these cases use, respectively, a \( \text{SUBSET SUM} \)-style meet-in-the-middle algorithm and fast matrix multiplication over exponentially large matrices. These algorithms all work in the extension oracle model.

We also show conditional or partial results. We show two conditional results for partial \( k \)-\( \text{NU} \) operations, showing that \( k \)-\( \text{NU} \)-\( \text{CSP} \) admits an improved algorithm in the oracle model if the \((k, k - 1)\)-\text{HYPERCLIQUE} problem admits an improved algorithm, and that \( k \)-\( \text{NU} \)-\( \text{SAT} \) admits an improved algorithm in the explicit representation model if the Erdős-Rado sunflower conjecture \([18]\) holds for sunflowers with \( k \) sets. The first of these results is a direct generalisation of the matrix multiplication strategy; the second uses fast local search in the style of Schöning \([52]\). Finally, we also consider the symmetric special case of \( 3 \)-\text{EDGE} \(-\text{SAT} \), and show that this problem reduces to a problem of finding a unit-coloured triangle in an edge-coloured graph. This, in turn, follows from fast algorithms for sparse triangle detection. Several of the algorithms we reduce to have a running time that depends on the matrix multiplication exponent \( \omega \); the best currently known value is \( \omega < 2.373 \) \([39, 55]\).

Before we begin, we need the following lemma, which shows that if a relation is preserved by a \( \text{pSDI} \)-operation, then it is possible to view the relation as a relation of smaller arity over a larger domain, which is preserved by the corresponding partial operation over the larger domain.

**Lemma 35.** Let \( R \) be an \( n \)-ary relation over a set of values \( D \), \( P \) a polymorphism pattern, and \( f \) a partial operation preserving \( R \) and satisfying \( P \). Let \( I_1, \ldots, I_m \) be a partition of \([n]\), and \( R_{I_1,\ldots,I_m} \) the \( m \)-ary relation

\[
R_{I_1,\ldots,I_m} = \{(\text{Proj}_{I_1}(t), \ldots, \text{Proj}_{I_m}(t)) \mid t \in R\}
\]

over the set of values \( \{\text{Proj}_{I_1}(R) \cup \cdots \cup \text{Proj}_{I_m}(R)\} \). Then every partial operation \( f' \) satisfying \( P \) over \( \{\text{Proj}_{I_1}(R) \cup \cdots \cup \text{Proj}_{I_m}(R)\} \) preserves \( R_{I_1,\ldots,I_m} \).

**Proof.** Let \( k = \text{ar}(f') = \text{ar}(f) \). Let \( t_1, \ldots, t_k \in R \) and let \( t'_1, \ldots, t'_k \in R_{I_1,\ldots,I_m} \) be the corresponding tuples of \( R_{I_1,\ldots,I_m} \). Assume that \( f'(t_1, \ldots, t_k) \) is defined, i.e., \( (t_1[j], \ldots, t_k[j]) \) is in domain(\( f' \)) for each \( j \in [k] \). Let \( i \in [n] \) and let \( I_j \) be the index set such that \( i \in I_j \). Since \( f'(t_1[j], \ldots, t_k[j]) \) is defined it must be an instantiation of a tuple \( p \in P \). It follows that \( (t'_1[i], \ldots, t'_k[i]) \) must be an instantiation of \( p \) as well, implying that \( f(t'_1[i], \ldots, t'_k[i]) \) is defined. Hence, \( f' \) preserves \( R_{I_1,\ldots,I_m} \). \( \square \)
4.1 An $O^*(|D|^{\frac{n}{2}})$ algorithm for 2-edge-CSP

Given a binary relation $R$ one can construct a bipartite graph where two vertices $x$ and $y$ have an edge between them if and only if $(x,y) \in R$. Formally, the vertices $V_1 \cup V_2$ of this graph will consist of the disjoint union of $\text{Proj}_1(R)$ and $\text{Proj}_2(R)$, i.e., $V_1 = \{(1,x) \mid x \in \text{Proj}_1(R)\}$ and $V_2 = \{(2,x) \mid x \in \text{Proj}_2(R)\}$. However, whenever convenient, we will not make this distinction and instead assume that $V_1 = \text{Proj}_1(R)$ and $V_2 = \text{Proj}_2(R)$. We say that a binary relation $R$ is rectangular if its bipartite graph representation is a disjoint union of bicliques.

**Lemma 36.** Let $\phi_D$ be the partial Maltsev operation over a domain $D$. Then every binary relation preserved by $\phi_D$ is rectangular.

**Proof.** The proof is very similar to the total case, which is essentially folklore in universal algebra. First note that $R$ is rectangular if and only if a path of length 4 between nodes $x,x',y,y'$ implies that there is an edge between $x$ and $y'$. Therefore, let $(x,y),(x',y),(x',y') \in R$. But then $\phi_D((x,y),(x',y),(x',y')) = (\phi_D(x,x'),\phi_D(y,y,y')) = (x,y')$, implying that $(x,y') \in R$ since $R$ is preserved by $\phi_D$. Hence, $R$ is rectangular.

If $R$ is an $n$-ary relation, $I_1 \cup I_2$ a partition of $[n]$, and $s \in \text{Proj}_{I_1}(R)$, $t \in \text{Proj}_{I_2}(R)$, we write $s \times t$ to denote the $n$-ary tuple in $R$ satisfying $\text{Proj}_{I_1}(s \times t) = s$ and $\text{Proj}_{I_2}(s \times t) = t$. Let $D = \{d_0, d_1, \ldots, d_k\}$ be a finite set of values. We can then order $D$ according to a total order $<$, by letting $d_0 < d_1 < \ldots < d_k$. This order easily extends to $n$-ary tuples $s$ and $t$ over $D$ by letting $s < t$ if and only if there exists an $i \in [n]$ such that $\text{Proj}_{I_1}(s) = \text{Proj}_{I_1}(t)$ and $s[i+1] < t[i+1]$. Given a relation $R$ we say that the tuple $t$ is lex-min if $t \in R$ and there does not exist any $t' \in R$ such that $t' \neq t$ and $t' < t$.

**Lemma 37.** Let $R$ be an $n$-ary relation preserved by $\phi_D$ and let $I_1 \cup I_2$ be a partition of $[n]$. Then there exists a bipartite graph $(V,E)$ where $V$ is the disjoint union of $\text{Proj}_{I_1}(R)$ and $\text{Proj}_{I_2}(R)$ such that

1. $(V,E)$ is a disjoint union of bicliques,
2. $\{s,t\} \in E$ if and only if $s \times t \in R$,
3. for every $s \in V$ occurring in a biclique $C_1 \cup C_2$ a pair $s_0 \in C_1, t_0 \in C_2$ such that $s_0$ is lex-min in $C_1$ and $t_0$ lex-min in $C_2$ can be computed in $O(\text{poly}(n,|D|))$ time in the extension oracle model.

**Proof.** Consider the binary relation $R_{I_1,I_2} = \{(\text{Proj}_{I_1}(t),\text{Proj}_{I_2}(t)) \mid t \in R\}$ over the set of values $\text{Proj}_{I_1}(R) \cup \text{Proj}_{I_2}(R)$. By Lemma 35 this relation is preserved by $\phi$ over the larger domain, and Lemma 36 then implies that $R_{I_1,I_2}$ is rectangular. Take the bipartite graph representation $(V_1 \cup V_2,E)$ of $R_{I_1,I_2}$ (which by the rectangularity property is a disjoint union of bicliques), and thus satisfies property (1). Property number (2) then follows easily from the construction of the bipartite graph $(V_1 \cup V_2,E)$ since two vertices $s$ and $t$ are connected with an edge if and only if $(s,t) \in R_{I_1,I_2}$, which holds if and only if $s \times t \in R$.

For property (3) we need to show that we, given $s \in V$, can compute lex-min representatives of the biclique $C_1 \cup C_2$ containing $s$, in polynomial time with respect to $n$ and $|D|$. Assume without loss of generality that $s \in V_1$, and order $I_2$ in ascending order as $i_1,\ldots,i_{|I_2|}$. Then determine the smallest value $d_1 \in D$ such that $s \times \{i_1\} \in \text{Proj}_{I_1 \cup \{i_1\}}(R)$. This can be computed in polynomial time using the extension oracle. Then continue, by for each $i_2,\ldots,i_j$ determine the smallest $d_j \in D$ such that $s \times \{i_1,\ldots,i_j\} \in \text{Proj}_{I_1 \cup \{i_1,\ldots,i_j\}}(R)$. Let $t_0$ denote the resulting tuple, and observe that $t_0 \in C_2$ and that $\{s,t_0\} \in E$. We then repeat this using the index set $I_1$ in order to obtain a lex-min tuple $s_0$ such that $\{s_0,t_0\} \in E$, which again can be done in polynomial time in the extension oracle model.

**Theorem 38.** 2-edge-CSP is solvable in $O^*(|D|^{\frac{n}{2}})$ time in both the extension oracle model and the explicit representation.
Proof. Let $(V, C)$ be an instance of 2-edge-CSP, where $V = \{x_1, \ldots, x_n\}$ and $C = \{C_1, \ldots, C_m\}$. Assume without loss of generality that $n$ is even, and let $I = \lfloor \frac{n}{2} \rfloor$ and $J = [n] \setminus I$. Consider two sets $P$ and $Q$ constructed as follows. Initially we let $P$ and $Q$ consist of all $\frac{n}{2}$-ary tuples over $D$. Then, for each $p \in P$, $q \in Q$ we enumerate each constraint in the instance containing only variables indexed by $I$ or $J$ and check whether $p$ or $q$ is contradicted by the constraint. If this is the case we remove $p$ from $P$ or $q$ from $Q$. More formally, if $p \in P$ and $R_i(x_{i_1}, \ldots, x_{i_k}) \in C$, $k = ax(R_i)$, such that $\{i_1, \ldots, i_k\} \subseteq I$, we check whether $\text{Proj}_{i_1,\ldots,i_k}(p) \in \text{Proj}_{i_1,\ldots,i_k}(R_i)$, and similarly for $q \in Q$. Each such step can be done in $O(\text{poly}(k))$ time in the extension oracle model and in $O(k + |R_i|)$ time if constraints are explicitly represented. By repeating this for all elements in $P$ and $Q$ we will therefore obtain two sets of partial assignments that do not directly contradict individual constraints in the input instance.

Next, for each $p \in P$ and $q \in Q$ create two $m$-ary tuples $p'$ and $q'$. By using Lemma 37 we for each constraint $C_i \in C$ will associate the $i$th element of $p'$ and $q'$ with a representative of the $i$th element of $p$ and $q$. Hence, let $C_i = R_i(x_{i_1}, \ldots, x_{i_k}) \in C$, $k = ax(R_i)$, be a constraint. We distinguish between two cases. First, assume that $\{i_1, \ldots, i_k\} \subseteq I$ or that $\{i_1, \ldots, i_k\} \subseteq J$. In this case we for every $t \in P \cup Q$ let $t'[i] = 1$. Second, assume that $i_1, \ldots, i_k \in I \cup J$ but that $\{i_1, \ldots, i_k\} \not\subseteq I$ and $\{i_1, \ldots, i_k\} \not\subseteq J$. In other words the constraint contains variables indexed by members of both $I$ and $J$. For every $p \in P$ compute the lex-min representatives $p_0$ and $q_0$ of the biclique containing $p$, with respect to the two index sets $P_i = \{j \mid i_j \in I\}$ and $Q_i = \{j \mid i_j \in J\}$. This can be done in polynomial time via Lemma 37. Assign the $i$th value to the tuple $p'$ the value $(p_0, q_0)$, and then repeat this for every $q \in Q$.

Let $P' = \{p' \mid p \in P\}$ and $Q' = \{q' \mid q \in Q\}$ be the sets resulting from repeating this for every constraint in the instance. We observe that the combination of $p \in P$ and $q \in Q$ satisfies a constraint $R_i(x_{i_1}, \ldots, x_{i_k}) \in C$ if and only if $p'[i] = q'[i]$, due to property (2) in Lemma 37. Hence, the instance is satisfiable if and only if the two sets $P'$ and $Q'$ intersect. Since $P'$ and $Q'$ contain at most $|D|^{\frac{n}{2}}$ tuples, each of length $m$, this test can easily be accomplished in $O^*(|D|^\omega)$ time using standard algorithms. □

4.2 An $O^*(|D|^\omega)$ algorithm for 3-NU-CSP

The algorithm in Section 4.1 used the rectangularity property of binary relations in order to obtain an improved algorithm for 2-edge-CSP. In this section we will devise an $O^*(|D|^{\omega_3})$ time algorithm for 3-NU-CSP by exploiting a structural property that is valid for all ternary relations preserved by $\nu_3$. Here, $\omega < 2.373$ is the matrix multiplication exponent. We will need the following definition.

Definition 39. An n-ary relation $R$ over $D$ is $k$-decomposable if there for every $t \notin R$ exists an index set $I \subseteq [n]$, $|I| \leq k$, such that $\text{Proj}_I(t) \notin \text{Proj}_I(R)$.

In the total case it is known that $R$ is $k$-decomposable if $R$ is preserved by a total $k$-ary NU-operation [29]. In general, this is not true for partial NU-operations, but we still obtain the following result.

Lemma 40. Let $R$ be a $k$-ary relation preserved by $\nu_k$. Then $R$ is $(k - 1)$-decomposable.

Proof. Let $t$ be a $k$-ary tuple not included in $R$. Assume that $\text{Proj}_I(t) \in \text{Proj}_I(R)$ for every index set $I \subseteq [k]$, $|I| < k$. But then there must exist $t_1, \ldots, t_k \in R$ such that each $t_i$ differ from $t$ in at most one position. This furthermore implies that $\nu_k(t_1, \ldots, t_k)$ is defined, and therefore also that $\nu_k(t_1, \ldots, t_k) = t \notin R$. This contradictions the assumption that $\nu_k$ preserves $R$, and we therefore conclude that there must exist an index set $I \subseteq [k]$ of size at most $k - 1$, such that $\text{Proj}_I(t) \notin \text{Proj}_I(R)$. □

Theorem 41. 3-NU-CSP is solvable in $O^*(|D|^{\omega_3})$ time in both the extension oracle model and the explicit representation, where $\omega < 2.373$ is the matrix multiplication exponent.
Proof. Let \((V, C)\) be an instance of 3-NU-CSP where \(V = \{x_1, \ldots, x_n\}\) and \(C = \{C_1, \ldots, C_m\}\). Partition \([n]\) into three sets \(I_1, I_2, I_3\) such that \(|I_i| = \frac{n}{3}\) (or, if this is not possible, as close as possible). Let \(F_1, F_2, F_3\) denote the set of all partial truth assignments corresponding to \(I_1, I_2, I_3\), and observe that \(|F_i| \leq |D|^{\frac{n}{3}}\). First, for each partial truth assignment \(f \in F_i\), remove it from the set \(F_i\) if there exists a constraint in the instance which is not satisfied by \(f\). This can be done in polynomial time with respect to the number of constraints in the instance, using an extension oracle query for each constraint. Second, construct a 3-partite graph where the node set is the disjoint union of \(F_1, F_2, F_3\), and add an edge between two nodes in this graph if and only if the combination of this partial truth assignment is not contradicted by any constraint in the instance. Last, answer yes if and only if the resulting graph contains a triangle.

We begin by proving correctness of this algorithm and then analyse its complexity. We first claim that if the combination of \(f_1 \in F_1, f_2 \in F_2, f_3 \in F_3\) does not satisfy a constraint in the instance, then there exists \(g_1, g_2 \in F_1 \cup F_2 \cup F_3\) which do not satisfy the instance either. Hence, take a constraint \(R(x_{i_1}, \ldots, x_{i_k}) \in C\), \(k = ar(R)\), which is not satisfied by the combination of \(f_1, f_2, f_3\). Let \(I_1' = \{j \mid i_j \in I_1\}\), \(I_2' = \{j \mid i_j \in I_2\}\), and \(I_3' = \{j \mid i_j \in I_3\}\) and consider the relation \(R_{I_1', I_2', I_3'} = \{(\text{Proj}_{I_1}(t), \text{Proj}_{I_2}(t), \text{Proj}_{I_3}(t)) \mid t \in R\}\) over the set of values \(\text{Proj}_{I_1}(R) \cup \text{Proj}_{I_2}(R) \cup \text{Proj}_{I_3}(R)\). By Lemma 35 this relation is preserved by the \(n\)-ary operation over the larger domain, and it then follows from Lemma 40 that this relation is 2-decomposable. But then it is easy to see that there must exist partial truth assignments \(y_1, y_2 \in F_1 \cup F_2 \cup F_3\) such that \(y_1\) and \(y_2\) do not satisfy \(R(x_{i_1}, \ldots, x_{i_k})\). Hence, if \((V, C)\) is satisfiable, then there clearly exists a triangle in the 3-partite graph, and if there exists a triangle, then by following the reasoning above, the instance must be satisfiable.

For the complexity, we begin by enumerating the three sets of partial truth assignments, which takes \(O(|D|^{\frac{n}{3}})\) time. We then remove any partial truth assignment which is not consistent with the instance, which increases this by a polynomial factor, depending only on the number of constraints and the extension queries for each constraint. Similarly, when constructing the 3-partite graph we enumerate all binary combinations of partial truth assignments from the three sets and check whether they are consistent. After this we check for the existence of a triangle in the resulting graph with \(O(|D|^{\frac{n}{3}})\) nodes, which can be solved in \(O(|D|^{\frac{n}{3}+\omega})\) time for \(\omega < 2.373\), using fast matrix multiplication.

4.3 Strategies for \(k\)-NU-SAT

It is easy to see that the strategy used in Theorem 41 extends to reducing \(k\)-NU-CSP problems to \((k, k-1)\)-Hyperclique, i.e., the problem of finding a \(k\)-vertex hyperclique in a \((k-1)\)-regular hypergraph. Thus we get the following.

Lemma 42. Assume that \((k, k-1)\)-Hyperclique on \(n\) vertices can be solved in time \(O^*(n^{k-\varepsilon})\) for some \(\varepsilon > 0\). Then \(k\)-NU-CSP admits an improved algorithm in the extension oracle model, i.e., an algorithm running in time \(O^*\left(|D|^{(1-\varepsilon')n}\right)\) on domain size \(D\) and on \(n\) variables, for some \(\varepsilon' > 0\).

However, it should be noted that this is a notoriously difficult problem, and there is some evidence against such results 40. Thus, we also investigate a less general algorithm that rests on a milder assumption.

4.3.1 \(k\)-NU-SAT via local search

We show that subject to a popular conjecture, \(k\)-NU-SAT admits an improved algorithm in the explicit representation model via a local search strategy. To state this we need a few basic definitions. A sunflower (with \(k\) sets) is a collection of \(k\) sets \(S_1, \ldots, S_k\) with common intersection \(S = S_1 \cap \ldots \cap S_k\), called the core, such that for every pair \(i, j \in [k], i \neq j\), we have \(S_i \cap S_j = S\). Note that we may have \(S = \emptyset\). The sunflower conjecture 18, in the form we will need, states that for every \(k\) there is a constant \(C_k\) such that for every \(n\), every collection of at least \(C_k^n\) sets of cardinality \(n\) contains a sunflower with \(k\) petals. This
Theorem 45. Assume that the sunflower conjecture holds for sunflowers with $k$ sets and relations $R \in \text{Inv}(\mu_k)$. For convenience, for a set $S \subseteq [n]$ we denote by $\chi_S^n$ the tuple $t \in \{0, 1\}^n$ such that for each $i \in [n]$, $t[i] = 1$ is $i \in S$ and $t[i] = 0$ otherwise.

Lemma 43. Let $R \subseteq \{0, 1\}^n$ be a relation with $0^n \notin R$. Say that a tuple $t = \chi_S^n$ is minimal in $R$ if $t \in R$ but for every $S' \subset S$ we have $\chi_{S'}^n \notin R$. For $i \in [n]$, let $F_i$ be the set of minimal tuples in $R$ of Hamming weight $i$. If $R$ is preserved by $\mu_k$, then $F_i$ does not contain a sunflower of $k$ sets.

Proof. Let $F_i$ be as in the statement, and assume that $R$ is preserved by $\mu_k$. Assume that there are distinct sets $S_1, \ldots, S_k$ forming a sunflower with some core $S$, such that $\chi_{S_j} \in F_i$ for every $j \in [k]$. But then $\mu_k(\chi_{S_1}, \ldots, \chi_{S_k})$ is defined, and produces the tuple $\chi_S$. This contradicts that the tuples are minimal in $R$. \hfill $\square$

We show that the sunflower conjecture is sufficient to allow an improved algorithm.

Lemma 44. Assume that the sunflower conjecture holds for sunflowers with $k$ sets, with some constant $C_k$. Let $\Gamma$ be a sign-symmetric language preserved by $\mu_k$. Assume that for every $n$-ary relation $R \in \Gamma$ and every $p \in [n]$, the minimal tuples in $R$ of Hamming weight at most $p$ can be enumerated in time $O^*(2^{O(p)})$. Then SAT($\Gamma$) admits an improved algorithm.

Proof. We first show that the assumptions are sufficient to allow a solution for the local search problem for SAT($\Gamma$), in the following form. Let an instance $(V, C)$ of SAT($\Gamma$) with $|V| = n$, a tuple $t \in \{0, 1\}^n$, and an integer $p \in [n]$ be provided. We can in $O^*(2^{O(p)})$ time decide whether there is a tuple $t' \in \{0, 1\}^n$ with Hamming distance at most $p$ from $t$ that satisfies $(V, C)$.

For this, we repeatedly perform the following procedure. Verify whether the present tuple $t$ satisfies $(V, C)$, and if not, let $R(X)$ be a constraint in $C$ falsified by $t$, and let $I \subseteq [n]$ be the set of indices corresponding to the set of variables $X$. Let $s$ be the sign pattern such that $(\text{Proj}_I(t))^s = 0^{|X|}$. Note that $R^s \in \Gamma$ by assumption. We then enumerate the minimal tuples in $R^s$ of Hamming weight at most $p$, and for every such tuple $t'$, of weight $i$, let $t''$ be the tuple $t$ with bits flipped according to $t'$, and recursively solve the local search problem from tuple $t''$ with new parameter $p - i$. Correctness is clear, since the search is exhaustive (because we loop through all minimal tuples). We argue that this solves the local search problem itself in $O^*(2^{O(p)})$ time. For the running time, assume for simplicity that producing the tuples takes $O^*(c^p)$ time and, for the same constant $c$, there are at most $c^i$ minimal tuples of weight $i$ (by Lemma 43). Up to polynomial factors, the running time is then bounded by a recurrence

$$T(p) = c^p + \sum_{i=1}^{p} c^i T(p - i),$$

which is bounded as $T(p) \leq (2c)^p$.

From here on, well-known methods can be used to complete the above into an improved algorithm; cf. Schöning’s algorithm for $k$-SAT \[52\] and its derandomization \[16\], or even restrict the above to monotone local search instead of arbitrary local search and apply the method of Fomin et al. \[20\]. \hfill $\square$

In particular, this is allows for an algorithm in the explicit representation model.

Theorem 45. Assume that the sunflower conjecture holds for sunflowers with $k$ sets. Then $k$-NU-SAT admits an improved algorithm in the explicit representation model.
We leave it as an open question whether access to an extension oracle (also known as an interval oracle) suffices to solve the local search problem in single-exponential time. The problem, of course, is that the bounds above only apply to the minimal tuples, and while it is easy to find a single minimal tuple using an extension oracle, it is less obvious how to test for the existence of a minimal tuple within a given interval. Meeks [43] showed how a similar result is possible, but her method would require an oracle for finding minimal satisfying tuples of weight exactly $i$, which is also not clear how to do.

### 4.3.2 $k$-NU-SAT and bounded block sensitivity

Finally, we briefly investigate connections between the $\mu_k$ partial operation and a notion from Boolean function analysis known as block sensitivity, introduced by Nisan [44]. See also the book by O’Donnell [45].

We first introduce some temporary notation. For any relation $R \subseteq \{0,1\}^n$, let $f_R : \{0,1\}^n \rightarrow \{0,1\}$ be a function defined as $f_R(t) = [t \in R]$, i.e., $f_R(t) = 1$ if $t \in R$ and $f_R(t) = 0$ otherwise. For a tuple $t \in \{0,1\}^n$ and a set $S \subseteq [n]$, let $t^S$ denote the tuple $t$ with the bits of $S$ flipped. A function $f : \{0,1\}^n \rightarrow \{0,1\}$ has block sensitivity bounded by $b$ if for every $t \in \{0,1\}^n$ there are at most $b$ disjoint sets $S_1, \ldots, S_b \subseteq [n]$ such that $f(t^S_i) \neq f(t)$ for every $i \in [b]$. We show that $\mu_k$ can be seen as a one-sided version of block sensitivity.

**Lemma 46.** Let $R \subseteq \{0,1\}^n$ be a relation. Then $f_R$ has block sensitivity less than $k$ if and only if both $R$ and its complement $\overline{R} := \{0,1\}^n \setminus R$ are preserved by $\mu_k$.

**Proof.** In the first direction, assume that $f$ has block sensitivity at least $k$. Let $t \in \{0,1\}^n$ be a tuple and let $[n] = X_0 \cup \ldots \cup X_k$ be a partition of $[n]$ into blocks such that for each $1 \leq i \leq k$, we have $f(t^{X_i}) \neq f(t)$. Then if $f(t) = 1$, then the tuples $t^{X_i}$ form a witness that $R$ is not preserved by $\mu_k$, and if $f(t) = 0$ they form a witness against $\overline{R}$ being preserved by $\mu_k$. In the other direction, let $t_1, \ldots, t_k \in R$ be such that $\mu_k(t_1, \ldots, t_k) = t$ is defined and $t \notin R$. For $i \in [k]$, let $X_i$ be the positions $j$ where $t[j] \neq t_i[j]$. Then $X_1 \cup \ldots \cup X_k$ forms a subpartition of $[n]$, showing that $f$ has block sensitivity at least $k$. The case that $\overline{R}$ is not preserved by $\mu_k$, instead of $R$, is completely dual.

It is known that a block sensitivity of at most $b$ implies a certificate complexity of at most $b^2$, i.e., for any relation $R \in \text{Inv}(\mu_k)$ and any tuple $t \in \overline{R}$, there are at most $b^2$ bits in $t$ that certify that $t \notin R$ [44]. This suggests a branching or local search algorithm for SAT($\Gamma$) where $\Gamma$ contains such relations. However, more strongly, it implies that $R$ has a decision tree of bounded depth [44], and thus, since $k$ is a constant, that $R$ only depends on constantly many arguments. Thus, block sensitivity is a significantly stronger restriction than what $\mu_k$ imposes.

However, one related question remains. Assume that $R$ is an $n$-ary relation preserved by $\mu_k$, and which does depend on all its arguments. Is there a non-trivial upper bound on $|R|$, e.g., does it hold that $|R| \leq (2 - \varepsilon_k)^n$ for some $\varepsilon_k$ depending on $k$? A positive answer to this question would imply a trivial improved algorithm for $k$-NU-SAT via enumeration of satisfying assignments, constraint by constraint.

### 4.4 Symmetric 3-edge-SAT

We finish this section with a result showing that a number of special cases of 3-EDGE-CSP admits an improved algorithm via sparse triangle finding. The class in particular contains 3-EDGE-SAT for symmetric relations $R \in \text{Inv}(e_3)$. We begin by characterising the symmetric relations in $\text{Inv}(e_3)$.

**Lemma 47.** Let $R \subseteq \{0,1\}^n$ be a symmetric relation preserved by $e_3$. Let $S \subseteq \{0, \ldots, n\}$ be the weights accepted by $R$. Then either $S$ is a complete arithmetic progression (possibly a trivial one, of length 1), or $S = \{a, a + b\}$ or $S = \{n - a, n - a - b\}$ for some $a < b$. 

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Proof. Let us first make a simpler claim: If $a, a + b \in S$ is a pair that does not extend to a complete progression in $S$, then either $a - b < 0$ or $a + 2b > n$.

To see this, let $a, a + b \in S$, and assume $a + 2b \notin S, a + 2b \leq n$. First assume $a \geq b$. We subpartition $[n]$ into one set $T_0$ of size $a - b \geq 0$ and three sets $T_i$ of size $b$, $i = 1, 2, 3$. This is possible since $a - b + 3b = a + 2b \leq n$. Let $t = \chi_{T_0 \cup \ldots \cup T_3}$ and for $i = 1, 2, 3$ let $t_i = t|_{T_i}$. Finally, let $t_4 = t_{T_1 \cup T_2}$. Then $e_3(t_1, \ldots, t_4)$ is defined and produces $t$. Thus we conclude $a < b$, i.e., $a - b < 0$. By the symmetric argument, if $a, a + b \in S$ with $a - b \geq 0$ and $a - b \notin S$, then $a + 2b > n$. This finishes the claim.

Next, assume that $|S| \geq 2$ and that $S$ contains some pair $a, a + b$ such that the progression does not continue. Let $b > 0$ be the smallest value such that such a pair exists, and again by symmetry assume that $a + 2b \leq n$; thus $a - b < 0$. Let $c \in S \setminus \{a, a + b\}$. First assume $c > a + 2b$. Then we may, similarly to above, pack sets with $|T_0| = a, |T_1| = |T_2| = b$, and $|T_3| = c - a - 2b$, and we have a witness showing $a + 2b \in S$. But in the remaining cases, $c$ must be involved in a complete progression with either $a$ or $a + b$, by the choice of $a$ and $b$. It is easy to check that this implies the existence of a value $c' \in S$ with $a < c' < a + b$, and that iterating the claim eventually produces an arithmetic progression of step size dividing $b$, covering $a$ and $a + b$, contradicting the assumption that $a + 2b \notin S$. Thus $|S| = 2$, i.e., $S = \{a, a + b\}$. 

In particular, this lemma shows that every symmetric relation in $\Inv(e_2)$ is a simple arithmetic progression. It also shows that $R$ has a simple-to-compute 2-edge embedding, i.e., $\tilde{R} \supseteq R, \tilde{R} \cap \{0, 1\}^{|R|} = R$, and $\tilde{R}$ is preserved by a total 2-edge operation \cite{36}, produced by extending $S$ into a complete progression.

We now describe the algorithm. Let $R$ be a relation with arguments $X$. For a partition $X = X_1 \cup X_2$ and an assignment $f$ to $X_1$, we refer to the 2-edge label of $f$ as the pair $(f_0, g_0)$ produced by first extending $f$ to a lex-min assignment $g_0$ such that $(f, g_0) \in R$, then extending $g_0$ to a lex-min assignment $f_0$ such that $(f_0, g_0) \in R$. Note that this is the same procedure used in the algorithm for 2-EDGE-CSP.

We extend this to 3-partite graphs as follows. Let the variable set be partitioned as $[n] = X \cup Y \cup Z$, and define a graph $G = (V, E)$ with partition $V = V_X \cup V_Y \cup V_Z$, where the nodes of each part represent partial assignments as in Section \ref{28}. For each edge, verify that the corresponding partial assignment is consistent with each relation in the input instance. We proceed to give labels to edges of $G$ for each relation $R$ as follows. We assume that for each relation, the “type” of $R$ is known to us (2-edge, 3-NU, or symmetric 3-edge). If $R \in \Inv(\mu_3)$, all edges get the same label. Otherwise, let $\tilde{R} \supseteq R$ be the 2-edge-embedding of $R$ (with $\tilde{R} = R$ if $R$ is already 2-edge). Let $pq$ be an edge in $G$, corresponding to partial assignments $p, q$. If one of these assignments, say $p$, is an assignment to $X$, then we set the label of $pq$ to the 2-edge label of $p$ in the partition $X \cup (Y \cup Z)$. Otherwise, $p \cup q$ is an assignment to $Y \cup Z$, and we set the label of $pq$ to the 2-edge label of this assignment in $X \cup (Y \cup Z)$. We show that this label scheme captures our language.

Lemma 48. Let $R$ be a relation with arguments $U$, for some $U \subseteq [n]$, and let $G = (V, E)$ and $X \cup Y \cup Z$ be as above. If either $R \in \Inv(e_2)$, or $R \in \Inv(\mu_3)$, or $R$ is Boolean, symmetric and $R \in \Inv(e_3)$, then a triple $(f, g, h)$ with $f \in V_X, g \in V_Y, h \in V_Z$ satisfies $R$ if and only if $fgh$ is a triangle in $G$ where the edges $fg, fh, gh$ all have the same label.

Proof. Refer to a triangle $fgh$ with all edge labels identical as a single-label triangle. We will also slightly abuse notation by treating $R$ as a 3-ary relation taking values from $V_X \times V_Y \times V_Z$. First assume that $R \in \Inv(e_2)$, and recall that $R$ is rectangular. Let $fgh$ be a single-label triangle with shared label $L = (f_0, g_0, h_0)$; we show that $(f, g, h) \in R$. Since $L$ is the label of the edge $gh$, it must be that $(f_0, g, h), (f_0, g_0, h_0) \in R$, and by the edges $fg$ and $fh$ it must be that $(f, g_0, h_0) \in R$ as well. By the partial 2-edge operation, this implies $(f, g, h) \in R$. Thus every single-label triangle corresponds to a satisfying assignment.

In the other direction, let $(f, g, h) \in R$. Since $R$ is rectangular, there is a unique lex-min pair $(f_0, g_0, h_0)$ in the biclique containing $(f, gh)$, and both extensions $(f, g_0, h_0)$ and $(f_0, gh)$ are compatible with $R$. Thus all three edges get the same label and the algorithm works for $R \in \Inv(e_2)$.
The case $R \in \text{Inv}(e_2)$ is trivial. Since such a relation is 2-decomposable, the entire verification of $R$ happens in the stage where edges are filtered, and in the remaining graph, every triangle represents a satisfying assignment and every triangle is single-label.

Finally, assume $R \in \text{Inv}(e_3)$ and is symmetric. If $R \in \text{Inv}(e_2)$, then we argue as above. Otherwise, by Lemma 47 either $S = \{a, a + b\}$ or $S = \{n - a, n - a - b\}$ for $a < b$, and $\hat{R}$ verifies that each assignment $(f, g, h)$ has the correct weight when computed mod $b$. First assume that $fgh$ is a single-label triangle in $G$. First assume $S = \{a, a + b\}$. By the edge-filtering step, we know that for each of the edges $fg, gh, fh$ the corresponding partial assignment has weight at most $a + b$. Thus the total weight of $(f, g, h)$ is at most $(a + b)(3/2) \leq a + b + (a + b)/2 < a + 2b$. Dually, assume $S = \{n - a - b, n - a\}$. No edge in $fgh$ has more than $a + b$ zeroes, thus the total assignment has weight greater than $n - a - 2b$. In both cases, since the edge-labels work to verify the value mod $b$, we conclude $(f, g, h) \in R$.

On the other hand, assume $(f, g, h) \in R$. Since the edge labels verify the more permissive relation $\hat{R}$, the triangle $fgh$ is a single-label triangle. □

The remaining problem can now be solved via algorithms for triangle-finding in sparse graphs.

**Theorem 49.** Assume a CSP or SAT problem with the following characteristic: for every relation $R$, either $R \in \text{Inv}(e_2)$ and $R$ is labelled with type $e_2$, or $R \in \text{Inv}(e_3)$ and $R$ is labelled with type $e_3$, or the language is Boolean, $R$ is a symmetric relation in $\text{Inv}(e_2)$ and $R$ is labelled with type $e_3$. This problem can be solved in time $O^*(|D|^{|1 + 2\omega/3|})$ in the extension oracle model, where $\omega < 2.373$ is the matrix multiplication exponent.

**Proof.** By the description above, we create a 3-partite graph $G$ on $3|D|^{n/3}$ vertices (where $|D| = 2$ in the Boolean case), and for every edge in $G$ we give it a vector of labels, one label per relation in the input instance. We refer to this vector as the *colour* of the edge. Note that a symmetric relation $R$ can be “inspected” using its extension oracle to find out the set $S$ of accepted weights. By Lemma 48, the instance has a satisfying assignment if and only if $G$ has a triangle where all edges have the same colour.

This we solve as follows. For every colour $c$ used by an edge in $G$, we generate the graph $G_c$ consisting of all edges of colour $c$. Let $m_c$ be the number of edges of $G_c$, and let $N \leq 3|D|^{n/3}$ be the number of vertices in $G$. We check if $G_c$ contains a triangle. If $G_c$ is dense enough, then we use the usual triangle-finding algorithm for this, with running time $O^*(N^{\omega})$, otherwise we use an algorithm for triangle finding in sparse graphs. Alon, Yuster and Zwick [2] show such an algorithm with running time $O(n_{\omega/(\omega + 1)}^{2\omega/3})$, where $\omega < 2.373$ is the matrix multiplication exponent. Hence, the crossover point at which we use the dense algorithm is $m_c \geq N^{(\omega + 1)/2} = N^\alpha$. Summing over all colours, we have $\sum_c m_c \leq N^2$. Since the algorithm for sparse graphs has a super-linear running time, the worst case is when we are at the crossover density and use the sparse algorithm $N^{2-\alpha}$ times for a cost of $O(N^\omega)$ each time. This works out to a total running time $O(N^{(\omega + 3)/2})$ for triangle-finding, i.e., the CSP is solved in time $O^*(|D|^{(\omega + 3)n/6}) = O^*(|D|^{0.896n})$ using $\omega = 2.373$.

We do not know whether this strategy can be extended to arbitrary relations $R \in \text{Inv}(e_3)$, even for a non-uniform algorithm.

**Section summary.** We have proven that it is indeed feasible to construct improved algorithms for $\text{Inv}(p)$-SAT and $\text{Inv}(p)$-CSP for individual pSDI-operations $p$. A crucial step for constructing algorithms of this form is first to identify non-trivial properties of relations invariant under $p$, which for the partial 2-edge operation turned out be rectangularity, and for the partial 3-NU operation 2-decomposability. However, it might not always be the case that every invariant relation satisfies such a clear-cut property, and for 3-edge-SAT we had to settle for an improved algorithm for symmetric relations.
For \( k \)-NU-CSP and \( k \)-NU-SAT we also gave conditional improvements in terms of \((k, k-1)\)-HYPERCLIQUE and the sunflower conjecture. At the present, it is too early to say whether these algorithms constitute the only source of improvement or if more direct arguments are applicable.

## 5 Lower Bounds

In this section we turn to the problem of proving lower bounds for sign-symmetric SAT problems.

### 5.1 Lower bounds based on \( k \)-SAT

As an easy warm-up, we first consider languages \( \Gamma \) such that \( \text{SAT}(\Gamma) \) is at least as hard as \( k \)-SAT for some \( k \). For each \( k \geq 3 \) let \( c_k \geq 0 \) denote the infimum of the set \( \{ c \mid k \text{-SAT is solvable in } O(2^{cn}) \text{ time} \} \). Under the ETH, \( c_k > 0 \) for each \( k \geq 3 \), and for each \( k \geq 3 \) there exists \( k' > k \) such that \( c_{k'} > c_k \) [25]. The best known upper bounds yield \( c_k \leq 1 - \Theta(1/k) \), but no methods for lower-bounding the values \( c_k \) are known.

Recall that Lemma 32 gives a condition under which a language \( \Gamma \) can qfpp-define all \( k \)-clauses. We observe the immediate consequence of this.

**Lemma 50.** Let \( \Gamma \) be a sign-symmetric constraint language not preserved by the \( k \)-universal partial operation. Then \( \text{SAT}(\Gamma) \) cannot be solved in time \( O^*(2^{cn}) \) for any \( c < c_k \), even in the non-uniform model.

**Proof.** By Lemma 32, \( \Gamma \) can qfpp-define all \( k \)-clauses. More concretely, there is a finite set \( \Gamma' \subseteq \Gamma \) of relations such that every \( k \)-clause has a fixed, finite-sized gadget implementation over \( \Gamma' \). Thus, given a \( k \)-SAT instance on \( n \) variables, we can produce an equivalent instance of \( \text{SAT}(\Gamma') \) in linear time, with the same variable set.

As a consequence, \( c_k \) is also a lower bound on the running time for \( \text{Inv(f)} \)-SAT for every minimal pSDI-operation at level \( k + 1 \) and higher. However, this above lemma applies to any sign-symmetric constraint language, and not just to the special case when \( \Gamma = \text{Inv(f)} \). We can also observe a similar consequence for SETH-hardness.

**Corollary 51.** Let \( \Gamma \) be a sign-symmetric constraint language not preserved by the \( k \)-universal partial operation for any \( k \). Then assuming SETH, \( \text{SAT}(\Gamma) \) does not admit an improved algorithm, even in the non-uniform model.

**Proof.** By SETH, there is for every \( \varepsilon > 0 \) a constant \( k \) such that \( k \)-SAT cannot be solved in \( O^*((2 - \varepsilon)^n) \) time. By Lemma 50 there is a reduction from \( k \)-SAT to \( \text{SAT}(\Gamma) \) for this \( k \). Thus, \( \text{SAT}(\Gamma) \) does not admit an improved non-uniform algorithm.

### 5.2 2-edge-SAT and Subset Sum

Next, we sharpen the connection between \textsc{Subset Sum} and 2-\textsc{edge-SAT}. Recall that an instance of \textsc{Subset Sum} consists of a set \( S = \{x_1, \ldots, x_n\} \) of \( n \) numbers and a target integer \( t \), with the question of whether there is a set \( X' \subseteq S \) such that \( \sum X' = t \). This can also be phrased as asking for \( z_1, \ldots, z_n \in \{0, 1\} \) such that

\[
\sum_{i=1}^{n} z_i x_i = t.
\]

Also recall from Lemma 21 that such a relation is contained in \text{Inv}(e_2). However, this does not by itself imply a problem reduction, since an instance or 2-\textsc{edge-SAT} assumes the existence of an extension oracle for every constraint. We show that such a reduction can be implemented by splitting the above equation apart into several equations, based on the bit-expansion of \( t \).
Theorem 52. If 2-\text{EDGE-SAT} is solvable in $O(2^n)$ time for $c > 0$ in the extension oracle model, then \text{Subset Sum} is solvable in $O(2^{(c+\varepsilon)n})$ time for every $\varepsilon > 0$.

Proof. Let $x_1, \ldots, x_n, t \in \mathbb{N}$ be the input to a \text{Subset Sum} instance. We will reduce this instance in subexponential time to a disjunction over 2-\text{EDGE-SAT} instances on $n$ variables each.

We proceed as follows. Harnik and Naor \cite{22} give a randomized procedure for this that reduces a \text{Subset Sum} instance to bit length at most $2n + \log \ell$, where $\ell$ is the bit length of the input. If $\ell \geq 2^n$, then we solve the instance by brute force in time polynomial in the input length, otherwise we are left with an instance of \text{Subset Sum} of size at most $2^n$, for which we replace the original equation $S/u.sc/b.sc/s.sc/e.sc/t.sc S/u.sc/m.sc$ $O a$ single block there are $O(\ell)$ options for the contribution within the block. We get at most $2^{\ell} \leq 3n$ guesses in total, after which we have replaced the original equation by the conjunction of $2^{\ell}$ linear equations, each with a target integer of $O(\sqrt{n})$ bits. This allows us to implement an extension oracle for every such constraint with a running time of $2^{O(\sqrt{n})}$, using the well-known tabulation approach.

This encodes an instance of 2-\text{EDGE-SAT} in the extension oracle model with $n$ variables. Using an algorithm for this problem, and multiplying its running time by the time required for answering an oracle query, yields the claimed running time for \text{Subset Sum}.

Given that the running time for 2-\text{EDGE-SAT} in the extension oracle model given in this paper matches the best known running time for \text{Subset Sum}, and given that improving the latter is a long-open problem, it seems at the very least that an improvement to 2-\text{EDGE-SAT} would require significant new ideas.

5.3 Padding formulas

We now give a combinatorial interlude, showing how relations $R \subseteq \{0, 1\}^n$ can be padded with additional variables such that the new relation lies in $\text{Inv}(f)$, for any non-total partial operation $f$. This will be leveraged in the next section to finally provide concrete lower bounds on the running time of $\text{Inv}(f)$-\text{SAT} for pSDi-operations $f$.

For a partial operation $p$, say of arity $k$, and a sequence of tuples $t_1, \ldots, t_k$, we say that $p(t_1, \ldots, t_k)$ is a projective application if $p(t_1, \ldots, t_k)$ is either undefined or $p(t_1, \ldots, t_k) \in \{t_1, \ldots, t_k\}$. Similarly, if $p(t_1, \ldots, t_k)$ is defined and $p(t_1, \ldots, t_k) \notin \{t_1, \ldots, t_k\}$ we call $p(t_1, \ldots, t_k)$ a non-projective application.

Definition 53. Let $R \subseteq \{0, 1\}^n$ be a relation and $P$ a set of Boolean partial operations. A padding of $R$ with respect to $P$ is an $(n + m)$-ary relation $\mathcal{P}_R$ such that (1) $\text{Proj}_{1, \ldots, n}(\mathcal{P}_R) = R$, (2) $|\mathcal{P}_R| = |R|$, and (3) $\mathcal{P}_R \in \text{Inv}(P)$. A universal padding formula for $n \geq 1$ with respect to $P$ is an $(n + m)$-ary relation $U\mathcal{P}_P$ which (1) is a padding of the relation $\{0, 1\}^n$ and (2) $p(t_1, \ldots, t_{\text{at}(p)})$ is a projective application for every partial operation $p \in P$ and every sequence of tuples $t_1, \ldots, t_{\text{at}(p)} \in U\mathcal{P}_P$.

Note that if $R$ is a relation and $p$ a $k$-ary partial operation such that $p(t_1, \ldots, t_k)$ is a projective application for every sequence $t_1, \ldots, t_k \in R$, then $R \in \text{Inv}(P)$. In particular this implies that $U\mathcal{P}_P \in \text{Inv}(P)$ for every universal padding formula $U\mathcal{P}_P$ of $P$. Also, critically, if $U\mathcal{P}_P$ is an $(n + m)$-ary universal padding formula for a set of partial operations $P$, and $R$ is an $n$-ary relation, then the relation $R'(x_1, \ldots, x_n, y_1, \ldots, y_m) \equiv R(x_1, \ldots, x_n) \land U\mathcal{P}_P(x_1, \ldots, x_n, y_1, \ldots, y_m)$ is a padding formula for $R$. Hence, a universal padding formula can be viewed as a blueprint which can be applied to obtain a concrete padding formula for any relation. It is known that if $P$ contains no total operation, then a universal padding formula can be constructed using a universal hash family \cite{31}.
Lemma 54. Let $P$ be a finite set of partial operations such that the only total functions in $[P]_n$ are projections. For every $n \geq 1$ there exists an $(n + m)$-ary universal padding formula $\mathcal{UP}_P$ such that $m \leq c \cdot n$, for a constant $c$ depending on $P$.

Proof. See Lagerkvist & Wahlström [37, Lemma 35].

A quick note is in place on the role of universal padding formulas in obtaining lower bounds for $\text{Inv}(P)$-SAT, when $P$ is a finite set of partial operations. Note that in a standard “gadget” reduction from CNF-SAT to some problem $\text{SAT}(\Gamma)$, one would introduce some number of local variables for every clause of the input, to create an equivalent output formula that only uses constraints from $\Gamma$. The existence of padding formulas does allow us to do this for $\text{Inv}(P)$-SAT, but for lower bounds under $\text{SETH}$ this is not useful since we have no control over the number of additional variables created this way. However, the universality property of universal padding formulas allow us to reuse the padding variables between different constraints, to produce an output which only has $n + m = O(n)$ variables in total. The details are given in the next section, but first we investigate concrete values of the constant $c$ for partial $k$-edge and $k$-NU operations.

Lemma 55. Let $X = \{x_1, \ldots, x_n\}$ be a set of variables, and let $y = \bigoplus_{i \in S} x_i$ be the parity sum for a set $S \subseteq [n]$ chosen uniformly at random. For any tuple $t \in \{0, 1\}^n$, let $t'$ be $t$ padded by $y$. Let $p$ be a partial operation as specified below, let $r = \text{ar}(p)$, and let $(t_1, \ldots, t_r)$ be a sequence of tuples in $\{0, 1\}^n$ such that $p(t_1, \ldots, t_r)$ is a non-projective application. Then the following hold.

1. If $p$ is the partial $2$-edge operation, with $r = 3$, then the probability that $p(t'_1, t'_2, t'_3)$ is defined is $3/4$.

2. If $p$ is the partial $3$-edge operation, with $r = 4$, then the probability that $p(t'_1, \ldots, t'_4)$ is defined is $1/2$.

3. If $p$ is the partial $k$-NU operation, $k \geq 4$, then the probability that $p(t'_1, \ldots, t'_r)$ is defined is $(2k+2)/2^k$.

For every weaker operation, e.g., for the partial $k$-edge or $k$-universal operations, the probability is at most this high.

4. If $p$ is the partial $k$-universal operation, $k \geq 3$, then the probability that $p(t'_1, \ldots, t'_r)$ is defined is $(k + 1)/2^k$.

Proof. Throughout the proof, we write $y(t) = \bigoplus_{i \in S} t[i]$. Let us consider each case in turn.

1. We have $\text{ar}(p) = 3$. Let $I$ respectively $J$ be the set of indices $i \in [n]$ such that $t_1[i] = t_2[i] \neq t_3[i]$, respectively, $t_1[i] \neq t_2[i] = t_3[i]$. Note that both $I$ and $J$ are non-empty since $p(t_1, t_2, t_3)$ is a non-projective application. Then $p(y(t_1), y(t_2), y(t_3))$ is undefined if and only if the parity of $S \cap I$ and $S \cap J$ are both odd. Since $I$ and $J$ are disjoint, the probability of this is exactly $1/4$.

2. For the partial $3$-edge operation, recall from Theorem [28] that $p$ can be constructed by adding a fictitious argument to the partial $3$-NU operation. Hence, the arguments $i \in [n]$ such that $(t_1[i], \ldots, t_4[i])$ is non-constant partition into three sets $I_1, I_2, I_3 \subseteq [n]$, and since $p(t_1, \ldots, t_4)$ is a non-projective application, all three sets must be non-empty. It can be verified that $p(y(t_1), \ldots, y(t_4))$ is defined if and only if $S \cap I_1$ is odd for at most one $i \in [3]$. This happens with exactly $1/2$ probability.

3. For the partial $k$-NU operation, we have $\text{ar}(p) = k$; let $p(t_1, \ldots, t_k) = t$. There are $k$ non-empty pairwise disjoint sets $I_1, \ldots, I_k$ such that $t_i[j] \neq t$ if and only if $j \in I_i$, for each $i \in [k], j \in [n]$. The tuple $(y(t_1), \ldots, y(t_k))$ has one value, say $b$, in every row $i \in [k]$ where $S \cap I_i$ is odd, and another value, $1 - b$, in every row $i$ where $S \cap I_i$ is odd. Thus $p(y(t_1), \ldots, y(t_k))$ is defined if either $S \cap I_i$ is odd for at most one index or $S \cap I_i$ is even for at most one index; these are $2k + 2$ possibilities. For all other $2^k - (2k + 2)$ possibilities, the operation is undefined. Note that all these possibilities happen with equal probability, since the sets $I_i$ are non-empty and pairwise disjoint.

4. We have $\text{ar}(p) = 2^k - 1 = r$, with the non-constant parts of $\text{domain}(p)$ partitioned into $k$ pairs. Let $I_i, i \in [k]$ be the sets of indices $j \in [n]$ such that $(t_1[j], \ldots, t_r[j])$ belongs to the $i$th of these pairs, in some
enumeration. We claim that \( p(y(t_1), \ldots, y(t_r)) \) is defined if and only if \( S \cap I_i \) is odd for at most one \( i \in [k] \). On the one hand, if this holds, then \( (y(t_1), \ldots, y(t_r)) \) is contained in pair number \( i \) or is constant, and it is clear that the operation is defined. Otherwise, let \( S \cap I_i \) and \( S \cap I_j \) both be odd, \( i \neq j \). Let \( t = p(t_1, \ldots, t_r) \); let \( a \in [r] \) be the argument such that \( t_a[i] \neq t[i] \) if and only if \( i \in I_i \); let \( b \in [r] \) be the argument such that \( t_b[i] \neq t[i] \) if and only if \( i \in I_j \); and let \( c \in [r] \) be the argument such that \( t_c[i] \neq t[i] \) if and only if \( i \in I_i \cup I_j \). Then the three positions \( y(t_a), y(t_b), y(t_c) \) have a pattern that is not compatible with any domain element of \( p \). It follows that the probability that \( p(t_1, \ldots, t_r) \) is defined is exactly \((k+1)/2^k\).

**Lemma 56.** Let \( p \) be a partial operation. There are \(|\text{domain}(p)|\) \( n \) sequences \( (t_1, \ldots, t_{ar(p)}) \) of tuples in \( \{0, 1\}^n \) such that \( p(t_1, \ldots, t_{ar(p)}) \) is defined.

**Proof.** For every argument \( i \in [n] \), we choose which element from \( \text{domain}(p) \) the tuple \((t_1[i], \ldots, t_{ar(p)}[i])\) will correspond to. Every such choice results in a distinct sequence of tuples.

**Lemma 57.** Let \( R(x_1, \ldots, x_n, y_1, \ldots, y_m) \) be a padding formula for \( \{0, 1\}^n \), where each \( y_i \) is a parity bit over \( \{x_1, \ldots, x_n\} \) chosen uniformly at random. Then the following hold.

1. For the partial 2-edge operation, \( R(x_1, \ldots, x_n, y_1, \ldots, y_m) \) is a universal padding formula with probability at least \( 1 - \varepsilon \) if \( m \geq 6.23 n + \log(1/\varepsilon) \).

2. For the partial 3-edge operation, \( R(x_1, \ldots, x_n, y_1, \ldots, y_m) \) is a universal padding formula with probability at least \( 1 - \varepsilon \) if \( m \geq 3n + \log(1/\varepsilon) \).

3. For the partial \( k \)-NU operation, \( k \geq 4 \), and for any operation weaker than it, \( R(x_1, \ldots, x_n, y_1, \ldots, y_m) \) is a universal padding formula with exponentially small failure probability if \( m = \Omega\left(\frac{\log k}{k} n\right)\).

**Proof.** 1. By Lemma 56 there are \( 6^n \) triples such that \( p \) is defined. For each such triple such that the application of \( p \) is non-projective, the probability that it remains defined after the addition of a single random parity bit is \( 3/4 \). Thus after adding \( t \) parity bits, the expected number of non-projective triples is at most

\[
6^n (3/4)^t = 2^{n \log 6 - t \log(4/3)}.
\]

With \( t = (n \log 6)/(\log(4/3)) + d \), this number equals \( 1/2^d \), which means that with probability at least \( 1 - 1/2^d \), no defined triples remain. The constant factor works out to \((\log 6)/(\log(4/3)) = (1 + \log 3)/(2 - \log 3) < 6.23\).

2. There are \( 8^n \) tuples \((t_1, \ldots, t_4)\) such that \( p \) is defined, and for each of them which is non-projective the probability of remaining defined after the addition of a single parity bit is \( 1/2 \). Thus adding \( 3n + d \) parity bits leaves in expectation at most

\[
8^n (1/2)^{3n+d} = 2^{-d}
\]

non-projective tuples, and the probability that no non-projective tuples remain is at least \( 1 - 1/2^d \).

3. In the general case, there are \( (2k+2)^n = 2^{(1+\log(k+1))n} \) defined tuples, and the probability of a non-projective tuple remaining defined after the addition of a random parity bit is \( O(k/2^k) \). Note that \((ck/2^k)^t = 2^{(\log c + \log k - k)t}\). Thus the expected number of non-projective tuples after \( t \) parity bits is at most

\[
2^{(1+\log(k+1))n - (k - \log k - c)t},
\]

and it suffices to let \( t = \Omega\left(\frac{\log k}{k} n\right)\).

We remark that with a padding strategy other than simple parity bits, a significantly lower scaling ratio may be possible for the partial \( k \)-universal operation. However, the advantage of padding with parity bits is that the padding can be efficiently inverted, allowing for efficient extension oracles for the padded relation.
5.4 Lower bounds in the extension oracle model

In this section we use the bounds obtained in Section 5.3 to obtain lower bounds for \( \text{Inv}(P) \)-SAT in the extension oracle model.

**Lemma 58.** Let \( \mathcal{U} \mathcal{P}_P \) be an \((n + m)\)-ary universal padding formula via the construction in Lemma 57. Let \( R = \{0, 1\}^k \setminus \{t\} \) for a \( k \)-ary tuple \( t \in \{0, 1\}^k \). Then there is a polynomial-time extension oracle for \( R(x_1, \ldots, x_k) \land \mathcal{U} \mathcal{P}_P(x_1, \ldots, x_n, y_1, \ldots, y_m) \).

**Proof.** Let \( \alpha : X \to \{0, 1\} \), \( X \subseteq \{x_1, \ldots, x_k, y_1, \ldots, y_m\} \), be a partial truth assignment. We need to show that we can decide if \( \alpha \) is consistent with \( R(x_1, \ldots, x_k) \land \mathcal{U} \mathcal{P}_P(x_1, \ldots, x_n, y_1, \ldots, y_m) \) in polynomial time. First, we check whether \( \alpha \) is consistent with the constraint \( R(x_1, \ldots, x_k) \), which is easy to do due to the representation of \( R \). Second, recall that there for each \( y_i \) exists an index set \( S_i \) such that \( y_i = \bigoplus_{s \in S_i} x_s \). Hence, the partial assignment \( \alpha \) together with \( R(x_1, \ldots, x_k) \land \mathcal{U} \mathcal{P}_P(x_1, \ldots, x_n, y_1, \ldots, y_m) \) induces a system of linear equations over GF(2) where the unknown variables are those unassigned by \( \alpha \). We may thus solve this system and check whether it has any solution \( f \) where \( f[i] \neq t[i] \) for some \( i \in [k] \).

**Theorem 59.** Let \( P \) be a set of partial operations, and set \( m \geq cn + \log n \) such that a random parity-padded formula \( \mathcal{U} \mathcal{P}_P(x_1, \ldots, x_n, y_1, \ldots, y_m) \) is a universal padding formula with high probability. Then \( \text{Inv}(P) \)-SAT cannot be solved in time \( O((2^{1/(c+1) - \varepsilon})^n) \) for any \( \varepsilon > 0 \), assuming the randomized version of SETH is true. In particular, we have the following lower bounds for specific problems:

1. 2-edge-SAT cannot be solved in \( O(2^{(c-\varepsilon)n}) \) time for any \( \varepsilon > 0 \), where \( c \approx 1/7.28 \).
2. 3-edge-SAT cannot be solved in \( O(2^{(c-\varepsilon)n}) \) time for any \( \varepsilon > 0 \), where \( c = 1/3 \).
3. For \( k \geq 4 \), \( k \)-NU-SAT cannot be solved in \( O(2^{(c-\varepsilon)n}) \) time for any \( \varepsilon > 0 \), where \( c = 1 - \Theta(\log k) \), and the same bound holds for the harder problems \( k \)-edge-SAT and \( k \)-universal SAT.

**Proof.** Let \( \mathcal{F} \) be a CNF-SAT instance on variable set \( X \), \( |X| = n \), and compute a random padding formula \( \mathcal{U} \mathcal{P}_P(x_1, \ldots, x_n, y_1, \ldots, y_m) \), with \( m \) as stated. We assume that the construction is successful, i.e., that the resulting relation is a universal padding formula with respect to \( P \). For every clause in the input, defined on a tuple of variables \( (x_{i_1}, \ldots, x_{i_r}) \), let \( R(x_{i_1}, \ldots, x_{i_r}) \) be the corresponding relation, and let \( R'(x_{i_1}, \ldots, x_{i_r}) \land \mathcal{U} \mathcal{P}_P(x_1, \ldots, x_n, y_1, \ldots, y_m) \) be the relation as in Lemma 58 (up to the ordering of variables). Note that we do not need to explicitly enumerate the tuples in this relation, since we may simply provide the extension oracle proven to exist in Lemma 58. Then the output is a conjunction of \( \text{Inv}(P) \)-SAT relations, with a polynomial-time extension oracle for each one, and the resulting instance is equivalent to \( \mathcal{F} \). Since the output instance has \( n + m = (c + 1) \cdot n \) variables, an algorithm solving \( \text{Inv}(P) \)-SAT faster than the time stated would imply an improved algorithm for CNF-SAT. The bounds for specific problems follow from the bounds for universal padding formulas computed in Lemma 57.

Finally, we note that the convergence of the lower bounds for \( k \)-NU-SAT towards \( 2^n \), assuming SETH, is at a slower rate than the upper bounds for the best known algorithms for \( k \)-SAT, which scale as \( c_k \leq 1 - \Theta(1/k) \) [25]. There are also significant differences in problem model (finite language versus infinite language, and concrete constraints versus extension oracles). It would be interesting to improve these results, to either improve the convergence rate or provide bounds in some explicit representation model, assuming SETH.
Section summary. We have proven lower bounds under SETH. The bounds obtained in Theorem [59] are only valid in the extension oracle model, and it does not appear entirely straightforward to extend them to the explicit representation. However, for 2-edge-SAT we also gave a lower bound subject to the Subset Sum problem, which as remarked is strong evidence that the $O^*(2^{n^2})$ algorithm from Theorem [38] is the best we could reasonably hope for.

6 Discussions and Conclusions

We have investigated the structure of constraint languages under fine-grained reductions, with a focus on sign-symmetric Boolean languages, and applied the results to an analysis of the time complexity of NP-hard SAT problems, in a general setting.

The structural analysis uses an algebraic connection to analyse constraint languages via their partial polymorphisms. Thereby the structural conclusions are relevant for any problem that takes as input a constraint formula over some fixed constraint language, under just a few assumptions: (1) that the constraints in the formula are “crisp” rather than soft, and are required to all be satisfied (as opposed to problems such as MAX-SAT, where a feasible solution may falsify some constraints); (2) that there are no structural restrictions of the formula itself (e.g., no bounds on the number of occurrences per variable); and (3) that the constraint language is sign-symmetric, i.e., allows the free application of negated variables and the use of constants in constraints. Thus it naturally applies to SAT($\Gamma$) problems, but would also be relevant for the analysis of problems such as #SAT and optimisation problems, or even parameterized problems such as LOCAL SEARCH SAT($\Gamma$) – is there a solution within distance $k$ of a given non-satisfying assignment $t$?

Structural results. The expressive power of sign-symmetric languages is characterised by the restricted partial polymorphisms in this paper referred to as pSDI-operations. We characterise the structure of all minimal non-trivial pSDI-operations, and find that they are organised into a hierarchy, whose levels correspond to the problem complexity, with close connections to being able to express the $k$-SAT languages. Moreover, we described the weakest and strongest operations on each level. We find that particular families of pSDI-operations correspond to partially defined versions of well-known algebraic conditions from the study of CSPs; in particular, the strongest operation at each level $k$ corresponds to the $k$-NU condition. Finally, we also give a result in the “vertical” direction of the hierarchy, giving a simple characterisation of languages not preserved by the partial $k$-NU operation for any $k$. By the above discussion, this result should be of interest also for other inquiries.

Complexity of SAT($\Gamma$) problems. We apply our results to an analysis of the fine-grained time complexity of SAT($\Gamma$) for sign-symmetric languages, under SETH. We consider previously studied languages with improved algorithms – i.e., such that SAT($\Gamma$) can be solved in time $O^*(c^n)$ for some $c < 2$ – and find that they correspond well to particular classes of the hierarchy. Conversely, every known language $\Gamma$ such that SAT($\Gamma$) is SETH-hard – i.e., admits no improved algorithm assuming SETH – lives entirely outside of the hierarchy. We also show the feasibility of giving improved algorithms whose correctness relies only and directly on the above-mentioned pSDI-operations, by showing that known algorithmic strategies such as fast matrix multiplication and (conjecturally) fast local search can be extended to work for such classes.

Finally, we give complementary lower bounds – for every invariant $f$ as above, there is a constant $c_f$ such that Inv($f$)-SAT cannot be solved in $O^*(c^n)$ time for any $c < c_f$, assuming SETH. These results are arguably the first of their kind; every previously known concrete lower bound under SETH has either been for showing that a problem admits no non-trivial algorithm, or has been applied to problems analysed under more permissive parameters such as treewidth. In particular, 2-edge-SAT is the first SAT problem which simultaneously has non-trivial upper and lower bounds on the running time under SETH.
6.1 The abstract problem and polynomial-time connections

Finally, let us make a short detour to consider what we may call the abstract problem. We have noted that for every Boolean pSDI-operation \( f \), there is a set of equational conditions that characterise \( f \), similarly to definitions of varieties in universal algebra, and for every larger domain \( D \), these conditions will uniquely determine a partial operation over the domain \( D \). Furthermore, these conditions are preserved under taking powers of the domain, which we have exploited for particular cases of Inv\((f)\)-SAT and Inv\((f)\)-CSP to reduce input instances to instances of polynomial-time solvable problems on exponentially many variables.

These polynomial-time problem will in general be search problems, like CSPs, and will be preserved by the same type of operation \( f \), but have a fixed number of variables \( d \) and with an unbounded domain size \( n \).

Let us refer to this as the abstract Inv\((f)\)-problem. The question can be raised, for which pSDI-operations \( f \) does such a problem allow improved polynomial-time algorithms?

We refrain from phrasing the question formally, because the polynomial-time complexity may be strongly affected by details such as constraint representation, but we note that the class of problems defined this way, unlike the original problems SAT\((\Gamma)\), contain several problems conjectured not to have such an improvement.

First, we note that every constraint of arity less than \( d \) is preserved by the \( k\)-NU-type partial operation with \( k \geq d \). This in particular includes the \( k\)-hyperclique problem for \((k-1)\)-uniform hypergraphs, which has been conjectured not to be solvable in time \( O(n^{k-\varepsilon}) \) for any \( \varepsilon > 0 \) and \( k > 3 \) [40]. Thus the abstract \( d\)-NU problem does not admit an improved algorithm for \( d > 3 \) under this conjecture.

Second, it can be verified that the problem of finding a zero-weight triangle, under arbitrary large edge weights, if viewed as a single constraint of arity \( d = 3 \), is preserved by the corresponding 3-universal partial operation. It is known that subject to the 3SUM conjecture, this problem cannot be solved in \( O(n^{3-\varepsilon}) \) for any \( \varepsilon > 0 \) [56].

If we restrict ourselves to the minimal non-trivial pSDI-operations \( f \) defined for the Boolean domain in this paper, this leaves only a small number of concrete problems open under the above conjectures. By the inclusions we have established, any operation \( f \) at a level \( k > 3 \) yields an abstract problem as hard as the \( k\)-NU operation. Furthermore, the abstract 3-NU problem does admit an improved algorithm via fast matrix multiplication. It can be easily checked that up to argument permutation, there are only eight distinct pSDI-operations \( f \) at level 3 of the hierarchy; and by the above discussion, the easiest and the hardest are (conjecturally) resolved. We consider it an interesting question to investigate the complexity of the problem for these remaining cases.

6.2 Regarding a dichotomy for sign-symmetric SAT problems

Ignoring for the moment the lower bounds discussed in the previous section, the results throughout our paper suggest a simple potential dichotomy between NP-complete SAT problems solvable in \( O(2^{cn}) \) time for \( c < 1 \) and SAT problems not solvable in \( O(2^{cn}) \) time for any \( c < 1 \) unless SETH fails. We can formulate this conjecture as follows. To simplify the conjecture we restrict ourselves to the non-uniform model.

**Conjecture 60.** Let \( \Gamma \) be a possibly infinite sign-symmetric Boolean constraint language such that SAT\((\Gamma)\) is NP-complete. Then SAT\((\Gamma)\) admits a non-uniform algorithm with running time in \( O(2^{cn}) \) time for \( c < 1 \) if and only if \( \Gamma \) is preserved by a non-trivial pSDI-operation.

Note that by Corollary 51, the negative direction of this conjecture is already known, up to SETH. It thus remains to consider whether \( k\)-UNIVERSAL SAT admits a non-uniform improved algorithm for every \( k \).

Furthermore, as discussed in the Introduction, the class of constraints definable as the roots of bounded-degree multivariate polynomials represents an example which by Lemma 24 is directly associated with \( k\)-UNIVERSAL SAT, and which has an improved algorithm by Lokshtanov et al. [42]. Thus, the above conjecture at least represent a kind of Occam’s razor-type extrapolation of least mathematical surprise.
However, at the moment this conjecture seems difficult to settle. An extreme negative result, such as the conclusion that the full problem $\text{Inv}(f)$-SAT admits an improved algorithm only when the abstract $\text{Inv}(f)$-problem does, would by Theorem 45 need to refute the sunflower conjecture. A full positive resolution would need to generalise the result of Lokshatov et al. [42] to apply based only on a weak abstract condition, whereas their present algorithm strongly uses properties specific to polynomials. Intermediate outcomes are of course possible, but would raise further questions of which pSDI-operations $f$ are powerful enough to guarantee the existence of an improved algorithm.

6.3 Future work

The investigations in this paper leave several concrete open questions, and significant avenues for future work, regarding all parts of the paper. Let us highlight a few.

Structural aspects. Assuming that the class of partial $k$-edge operations turn out to be relevant for the analysis of future problems, it would be valuable to have a set of canonical consequences to a language not being preserved by any partial $k$-edge operation, similarly to Theorem 33. To this aim, it may also be enlightening to fully describe the symmetric relations contained in various classes in the hierarchy.

Another concrete question is regarding the structure of $\text{Inv}(\nu_k)$ for $k > 3$. Assume that $R \in \text{Inv}(\nu_k)$ is an $n$-ary Boolean relation, which depends on every argument. Is there a non-trivial upper bound on $|R|$?

Extension to CSPs. Many questions remain regarding an extension of the project to CSPs on non-Boolean domains. While the minimal non-trivial pSDI-operations defined in this paper do have higher-domain analogues, via polymorphism patterns, and while these analogues do in some cases have useful consequences for the complexity of the corresponding CSP, it is not clear that they are in general the only kind of condition that is relevant for the fine-grained complexity of CSPs. In particular, in the Boolean domain there is a known correspondence between pSDI-operations and sign-symmetric languages. No such correspondence has been shown for CSPs in general.

In a different vein, for higher-domain CSPs there are also classes of NP-hard problems whose time complexity is far better than $O^*(|D|^n)$, e.g., $k$-Colouring corresponds to a CSP of domain size $|D| = k$ and can be solved in $O^*(2^n)$ time for every $k$ [4]. Arguably, we do not have a good understanding of when this occurs in general, and we cannot claim that an $O(c^n)$ time algorithm for $c < |D|$ is necessarily an improvement. A reasonable starting point to mitigate some of these technical difficulties is to initially only consider constraint languages whose total polymorphisms are the projections.

Problems. Let us mention a few concrete algorithmic questions. First of all, by Lemma 25, symmetric relations defined by Sidon sets are preserved by the $3$-universal operation, but they do not seem to be captured by currently known algorithms for problems in this class. Does the language consisting of all such relations admit an improved algorithm?

Another problem is to find a generalisation of the algorithm for constraints defined via bounded-degree polynomials [42], without explicitly using properties specific to polynomials. A different generalisation of this class was considered by the present authors (see the arXiv version of [36]), in the form of relations with bounded-degree Maltsev embeddings. Since this properly generalises bounded-degree polynomials, it is natural to ask whether this class admits an improved algorithm.

More broadly, as remarked earlier, the classification of the expressiveness of sign-symmetric constraint languages may be of interest for questions other than just satisfiability. The algorithm for $2$-EDGE-SAT, for instance, can be used to solve the corresponding counting problem, showing that pSDI-operations may be powerful enough also in other settings. Concrete questions to consider here include improved algorithms for the counting problem $\#\text{SAT}(\Gamma)$ and the parameterized problem $\text{LOCAL search SAT}(\Gamma)$.

Lower bounds. Can the padding scheme be improved to give better asymptotics with respect to the level $k$? Recall that the lower bound behaves as a bound of $2 - \Theta((\log k)/k)$, whereas all known algorithmic strategies yield running times of the form $(2 - \Theta(1/k))^n$. 

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It would also be very interesting to have a SETH-based lower bound in the explicit representation model. As discussed earlier the padding construction is valid also in this representation, but is difficult to implement in practice since the resulting relations may contain exponentially many tuples with respect to the number of variables.

References

[1] N. Alon, A. Shpilka, and C. Umans. On sunflowers and matrix multiplication. Computational Complexity, 22(2):219–243, 2013.

[2] N. Alon, R. Yuster, and U. Zwick. Finding and counting given length cycles. Algorithmica, 17(3):209–223, 1997.

[3] L. Barto, A. Krokhin, and R. Willard. Polymorphisms, and How to Use Them. In A. Krokhin and S. Zivny, editors, The Constraint Satisfaction Problem: Complexity and Approximability, volume 7 of Dagstuhl Follow-Ups, pages 1–44. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, Dagstuhl, Germany, 2017.

[4] A. Björklund, T. Husfeldt, and M. Koivisto. Set partitioning via inclusion-exclusion. SIAM Journal on Computing, 39(2):546–563, 2009.

[5] V. G. Bodnarchuk, L. A. Kaluzhnin, V. N. Kotov, and B. A. Romov. Galois theory for Post algebras. I. Cybernetics, 5:243–252, 1969.

[6] V. G. Bodnarchuk, L. A. Kaluzhnin, V. N. Kotov, and B. A. Romov. Galois theory for Post algebras. II. Cybernetics, 5:531–539, 1969.

[7] A. Bulatov. A dichotomy theorem for nonuniform CSPs. In Proceedings of the 58th Annual Symposium on Foundations of Computer Science (FOCS-2017). IEEE Computer Society, 2017.

[8] A. Bulatov and V. Dalmau. A simple algorithm for Mal’tsev constraints. SIAM Journal On Computing, 36(1):16–27, 2006.

[9] A. Bulatov, P. Jeavons, and A. Krokhin. Classifying the complexity of constraints using finite algebras. SIAM Journal on Computing, 34(3):720–742, Mar. 2005.

[10] C. Calabro, R. Impagliazzo, and R. Paturi. The complexity of satisfiability of small depth circuits. In Parameterized and Exact Computation, 4th International Workshop (IWPEC 2009), pages 75–85, 2009.

[11] C. Calabro, R. Impagliazzo, and R. Paturi. On the exact complexity of evaluating quantified k-CNF. Algorithmica, 65(4):817–827, Apr 2013.

[12] M. Couceiro, L. Haddad, V. Lagerkvist, and B. Roy. On the interval of Boolean strong partial clones containing only projections as total operations. In Proceedings of the 47th International Symposium on Multiple-Valued Logic (ISMVL-2017), pages 88–93. IEEE Computer Society, 2017.

[13] M. Couceiro, L. Haddad, K. Schölzel, and T. Waldhauser. Relation graphs and partial clones on a 2-element set. In Proceedings of the 44th International Symposium on Multiple-Valued Logic (ISMVL-2014), pages 161–166. IEEE Computer Society, 2014.
[14] N. Creignou and H. Vollmer. Boolean constraint satisfaction problems: When does Post's lattice help? In N. Creignou, P. G. Kolaitis, and H. Vollmer, editors, Complexity of Constraints, volume 5250 of Lecture Notes in Computer Science, pages 3–37. Springer Berlin Heidelberg, 2008.

[15] M. Cygan, H. Dell, D. Lokshtanov, D. Marx, J. Nederlof, Y. Okamoto, R. Paturi, S. Saurabh, and M. Wahlström. On problems as hard as CNF-SAT. ACM Transactions on Algorithms, 12(3):41:1–41:24, 2016.

[16] E. Dantsin, A. Goerdt, E. A. Hirsch, R. Kannan, J. M. Kleinberg, C. H. Papadimitriou, P. Raghavan, and U. Schöning. A deterministic $(2 - 2/(k + 1))^n$ algorithm for k-SAT based on local search. Theoretical Computer Science, 289(1):69–83, 2002.

[17] E. Dantsin and A. Wolpert. Derandomization of Schuler's algorithm for SAT. In Proceedings of Theory and Applications of Satisfiability Testing (SAT-2004), pages 80–88, 2005.

[18] P. Erdős and R. Rado. Intersection theorems for systems of sets. Journal of the London Mathematical Society, s1-35(1):85–90, 1960.

[19] T. Feder and M. Vardi. The computational structure of monotone monadic SNP and constraint satisfaction: A study through datalog and group theory. SIAM Journal on Computing, 28(1):57–104, 1998.

[20] F. V. Fomin, S. Gaspers, D. Lokshtanov, and S. Saurabh. Exact algorithms via monotone local search. In Proceedings of the 48th Annual ACM SIGACT Symposium on Theory of Computing (STOC 2016), pages 764–775, 2016.

[21] D. Geiger. Closed systems of functions and predicates. Pacific Journal of Mathematics, 27(1):95–100, 1968.

[22] D. Harnik and M. Naor. On the compressibility of NP instances and cryptographic applications. SIAM Journal on Computing, 39(5):1667–1713, 2010.

[23] T. Hertli. 3-SAT faster and simpler - unique-SAT bounds for PPSZ hold in general. SIAM Journal on Computing, 43(2):718–729, 2014.

[24] E. Horowitz and S. Sahni. Computing partitions with applications to the knapsack problem. Journal of the ACM, 21(2):277–292, Apr. 1974.

[25] R. Impagliazzo and R. Paturi. On the complexity of k-SAT. Journal of Computer and System Sciences, 62(2):367 – 375, 2001.

[26] R. Impagliazzo, R. Paturi, and F. Zane. Which problems have strongly exponential complexity? Journal of Computer and System Sciences, 63:512–530, 2001.

[27] B. M. P. Jansen and A. Pieterse. Optimal sparsification for some binary CSPs using low-degree polynomials. In Proceedings of the 41st International Symposium on Mathematical Foundations of Computer Science (MFCS-2016), volume 58, pages 71:1–71:14, 2016.

[28] P. Jeavons. On the algebraic structure of combinatorial problems. Theoretical Computer Science, 200:185–204, 1998.

[29] P. Jeavons, D. Cohen, and M. Gyssens. Closure properties of constraints. Journal of the ACM, 44(4):527–548, July 1997.
[30] P. Jonsson and V. Lagerkvist. An initial study of time complexity in infinite-domain constraint satisfaction. *Artificial Intelligence*, 245:115–133, 2017.

[31] P. Jonsson, V. Lagerkvist, G. Nordh, and B. Zanuttini. Strong partial clones and the time complexity of SAT problems. *Journal of Computer and System Sciences*, 84:52 – 78, 2017.

[32] P. Jonsson, V. Lagerkvist, and B. Roy. Time complexity of constraint satisfaction via universal algebra. In *Proceedings of the 42nd International Symposium on Mathematical Foundations of Computer Science (MFCS-2017)*, pages 17:1–17:15, 2017.

[33] M. P. L. Barto, J. Oprsal. The wonderland of reflections. *Israel Journal of Mathematics*. To appear.

[34] V. Lagerkvist. *Strong Partial Clones and the Complexity of Constraint Satisfaction Problems: Limitations and Applications*. PhD thesis, Linköping University, The Institute of Technology, 2016.

[35] V. Lagerkvist and B. Roy. A Preliminary Investigation of Satisfiability Problems Not Harder than 1-in-3-SAT. In *Proceedings of the 41st International Symposium on Mathematical Foundations of Computer Science (MFCS-2016)*, pages 64:1–64:14, 2016.

[36] V. Lagerkvist and M. Wahlström. Kernelization of constraint satisfaction problems: A study through universal algebra. In *Principles and Practice of Constraint Programming - 23rd International Conference (CP 2017)*, pages 157–171, 2017.

[37] V. Lagerkvist and M. Wahlström. The power of primitive positive definitions with polynomially many variables. *Journal of Logic and Computation*, 27(5):1465–1488, 2017.

[38] V. Lagerkvist, M. Wahlström, and B. Zanuttini. Bounded bases of strong partial clones. In *Proceedings of the 45th International Symposium on Multiple-Valued Logic (ISMVL-2015)*, pages 189–194, 2015.

[39] F. Le Gall. Powers of tensors and fast matrix multiplication. In *Proceedings of the International Symposium on Symbolic and Algebraic Computation (ISSAC-2014)*, pages 296–303, 2014.

[40] A. Lincoln, V. Vassilevska Williams, and R. Williams. Tight hardness for shortest cycles and paths in sparse graphs. In *Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms (SODA-2018)*, pages 1236–1252, 01 2018.

[41] D. Lokshtanov, D. Marx, and S. Saurabh. Known algorithms on graphs of bounded treewidth are probably optimal. In *Proceedings of the Twenty-second Annual ACM-SIAM Symposium on Discrete Algorithms (SODA-2011)*, pages 777–789, 2011.

[42] D. Lokshtanov, R. Paturi, S. Tamaki, R. R. Williams, and H. Yu. Beating brute force for systems of polynomial equations over finite fields. In P. N. Klein, editor, *Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms (SODA 2017)*, pages 2190–2202, 2017.

[43] K. Meeks. Randomised enumeration of small witnesses using a decision oracle. In *11th International Symposium on Parameterized and Exact Computation (IPEC 2016)*, pages 22:1–22:12, 2016.

[44] N. Nisan. CREW PRAMs and decision trees. *SIAM Journal On Computing*, 20(6):999–1007, 1991.

[45] R. O’Donnell. *Analysis of Boolean Functions*. Cambridge University Press, 2014.

[46] E. Post. The two-valued iterative systems of mathematical logic. *Annals of Mathematical Studies*, 5:1–122, 1941.
[47] B. Romov. The algebras of partial functions and their invariants. Cybernetics, 17(2):157–167, 1981.

[48] T. Schaefer. The complexity of satisfiability problems. In Proceedings of the 10th Annual ACM Symposium on Theory Of Computing (STOC-1978), pages 216–226. ACM Press, 1978.

[49] D. Scheder and J. P. Steinberger. PPSZ for general k-SAT - making Hertli’s analysis simpler and 3-SAT faster. In Proceedings of the 32nd Computational Complexity Conference (CCC-2017), pages 9:1–9:15, 2017.

[50] H. Schnoor and I. Schnoor. Partial polymorphisms and constraint satisfaction problems. In N. Creignou, P. G. Kolaitis, and H. Vollmer, editors, Complexity of Constraints, volume 5250 of Lecture Notes in Computer Science, pages 229–254. Springer Berlin Heidelberg, 2008.

[51] K. Schölzel. Dichotomy on intervals of strong partial Boolean clones. Algebra Universalis, 73(3-4):347–368, 2015.

[52] U. Schöning. A probabilistic algorithm for k-SAT and constraint satisfaction problems. In Proceedings of the 40th Annual Symposium on Foundations of Computer Science (FOCS-1999), pages 410–414, 1999.

[53] R. Schroeppel and A. Shamir. A $T = O(2^{n/2})$, $S = O(2^{n/4})$ algorithm for certain NP-complete problems. SIAM Journal On Computing, 10(3):456–464, 1981.

[54] E. Szemerédi. On sets of integers containing no k elements in arithmetic progression. Acta Arithmetica, 27:199–245, 1975.

[55] V. V. Williams. Multiplying matrices faster than Coppersmith-Winograd. In Proceedings of the 44th Symposium on Theory of Computing Conference (STOC 2012), pages 887–898, 2012.

[56] V. V. Williams and R. Williams. Finding, minimizing, and counting weighted subgraphs. SIAM Journal On Computing, 42(3):831–854, 2013.

[57] D. Zhuk. The proof of CSP dichotomy conjecture. In Proceedings of the 58th Annual Symposium on Foundations of Computer Science (FOCS-2017). IEEE Computer Society, 2017.