YETTER-DRINFELD MODULES FOR TURAEV CROSSED STRUCTURES

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Abstract. We provide an analog of the Joyal-Street center construction and of the Kassel-Turaev categorical quantum double in the context of the crossed categories introduced by Turaev. Then, we focus or attention to the case of categories of representation. In particular, we introduce the notion of a Yetter-Drinfeld module over a crossed group coalgebra $H$ and we prove that both the category of Yetter-Drinfeld modules over $H$ and the center of the category of representations of $H$ as well as the category of representations of the quantum double of $H$ are isomorphic as braided crossed categories.

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Introduction

Recently, Turaev [17, 18] (see also [8] and [20]) generalized quantum invariants of 3-manifold to the case of a 3-manifold $M$ endowed with a homotopy class of maps $M \to K(\pi, 1)$, where $\pi$ is a group (such homotopy classes of maps $M \to K(\pi, 1)$ classify principal flat $\pi$-bundles over $M$).

One of the key points of the theory [18] is a generalization of the definition of a tensor category to the notion of a \textit{crossed} $\pi$-\textit{category}, here called a Turaev category or, briefly, a $T$-\textit{category}. The algebraic counterpart is the generalization of the definition of Hopf algebra to the notion of a \textit{crossed Hopf} $\pi$-\textit{coalgebra}, here called a Turaev coalgebra or, briefly, a $T$-\textit{coalgebra}. As the category of representations of a Hopf algebra has a structure of a tensor category, the category of representations of...
a T-coalgebra has a structure of a T-category. Concepts like braiding (universal R-matrix), ribbon, and modularity can be extended to the crossed case. Ribbon and modular T-categories play a central role in the construction of the new invariants.

Roughly speaking, a T-coalgebra $H$ is a family $\{H_\alpha\}_{\alpha \in \pi}$ of algebras endowed with a comultiplication $\Delta_{\alpha, \beta}: H_{\alpha \beta} \to H_\alpha \otimes H_\beta$, a counit $\varepsilon: k \to H_1$ (where 1 is the neutral element of $\pi$), and an antipode $s_\alpha: H_\alpha \to H_{-\alpha}$. It is required that $H$ satisfies axioms that generalize those of a Hopf algebra. It is also required that $H$ is endowed with a family of algebra isomorphisms $\varphi_{\beta}^\alpha : H_\alpha \to H_{\beta \alpha \beta}$, the conjugation, compatible with the above structures and such that $\varphi_{\beta \gamma} = \varphi_{\beta} \circ \varphi_{\gamma}$. In particular, when $\pi = 1$, we recover the usual definition of a tensor category. If $H$ is a T-coalgebra, then the disjoint union $\bigoplus_{\alpha \in \pi} \text{Rep}(H_{\alpha})$ of the categories $\text{Rep}(H_{\alpha})$ has a structure of a T-category.

In [21] we provide an analog of the Drinfeld quantum double (see [3]) and of the ribbon extension (see [12]), that is we showed how to obtain a quasi-triangular and a ribbon T-coalgebra starting from any finite-type T-coalgebra. In this article we focus our attention on the corresponding categorical constructions and the relations between these and the algebraic one.

In the first part of the article, we study how to obtain braided or ribbon T-categories starting from a T-category that is not braided. Firstly, given any T-category $T$, we obtain a braided T-category $\mathcal{Z}(T)$, the center of $T$ (Theorem 5.1). When $\pi = 1$, we recover the definition of the center of a tensor category, see [4]. Given a braided T-category $T'$, by generalizing the constructions in [14] and [7], we obtain a ribbon T-category $\mathcal{N}(\mathcal{Z}(T')^2)$. We can apply this construction when $T = \mathcal{Z}(T)$ so that for any T-category $T$ we obtain a ribbon T-category $\mathcal{D}(T) = \mathcal{N}(\mathcal{Z}(T)^2)$.

In the second part of the article, we consider the case of category of representations of T-coalgebras and the algebraic constructions that we previously introduced in [21]. Firstly, we introduce the fundamental notion of a Yetter-Drinfeld module over a T-coalgebra $H$, or, briefly, a YD-module, as a module $V$ over a component $H_\alpha$ of $H$ endowed with a family of $k$-linear morphisms $\Delta_{\beta}: V \to V \otimes H_\beta$ (for any $\beta \in \pi$) satisfying axioms that generalize the usual definition of a Yetter-Drinfeld module over a Hopf algebra. We state now the main result relating our double constructions for crossed categories and for crossed Hopf group coalgebras.

**Theorems 8.1 and 10.1.** Let $H$ be a T-coalgebra of finite type. We have the isomorphisms of braided T-categories

$$\mathcal{Z}(\text{Rep}(H)) = \mathcal{YD}(H) = \text{Rep}(\overline{D}(H)),$$

where $\mathcal{YD}(H)$ is the category of Yetter-Drinfeld modules over $H$ and $\overline{D}(H)$ is the mirror of the quantum double $D(H)$ of $H$ (see [21]).

We can describe Theorems 8.1 and 10.1 via the commutative diagram

$$\begin{array}{cccc}
\text{Rep}(H) & \xrightarrow{Z} & \mathcal{Z}(\text{Rep}(H)) & \xrightarrow{\mathcal{YD}} & \text{Rep}(\overline{D}(H)) \\
\text{Rep} & \downarrow & \text{Rep} & \downarrow & \text{Rep} \\
H & \xrightarrow{\overline{D}} & \overline{\mathcal{D}(H)} & & \overline{D}(H)
\end{array}$$
Similar results are obtained for ribbon structures.

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1. Tensor categories

To fix our notations, let us recall few basic definitions. A tensor category $C = (C, \otimes, a, l, r)$ (see [10, 3]), also called a monoidal category, is a category $C$ endowed with a functor $\otimes: C \times C \to C$ (the tensor product), an object $I \in C$ (the tensor unit) and natural isomorphisms $a = a_{U,V,W}: (U \otimes V) \otimes W \to U \otimes (V \otimes W)$ for all $U,V,W \in C$ (the associativity constraint) and $l = l_U: I \otimes U \to U$, $r = r_U: U \otimes I \to U$, for any $U \in C$ (the left unit constraint and the right unit constraint, respectively) such that, for all $U,V,W,X \in C$, the two identities $a_{U,V,W} \circ a_{U \otimes V,W,X} = (U \otimes a_{V,W,X}) \circ a_{U,V \otimes W,X}$ (called the associativity pentagon) and $(U \otimes I) \circ (r_U \otimes V) = a_{U,V,0}$ are satisfied. A tensor category $C$ is strict when all the constraints are identities.

Given two tensor categories $C$ and $D$, a tensor functor $F = (F_2, F_0): C \to D$ consists of the following items.

- A functor $F: C \to D$.
- A natural family $\{F_2(U,V): F(U) \otimes F(V) \to F(U \otimes V)\}_{U,V \in C}$ of isomorphisms in $D$ such that $F(a_{U,V,W}) \circ F_2(U \otimes V, W) \circ F_2(U,V) \otimes F(W) = F_2(U \otimes V \otimes W) \circ (F(U) \otimes F_2(V,W)) \circ a_{U,V,W}$, for any $U,V,W \in C$.
- An isomorphism $F_0: I \to F(1)$ in $D$ such that $F(r_U) \circ F_2(U) \circ F_0 = r_{F(U)}$ and $F(l_U) \circ F_2(1,U) \circ (F_0 \otimes F(U)) = l_{F(U)}$, for any $U \in C$.

$F$ is said strict when $F_0$ and all the $F_2(U,V)$ are identities.

Remark 1.1. Let $C$ be a tensor category. We recall [10] that $C$ is equivalent to a strict tensor category $S(C)$ via a tensor functor $F: S(C) \to C$ and a tensor functor $G: C \to S(C)$. More precisely, the category $S(C)$ and the functors $F$ and $G$ can be obtained as follows.

- The objects of $S(C)$ are all the finite sequences $u = (U_1, \ldots, U_n)$ of objects $U_1, \ldots, U_n \in C$. Also the empty sequence, denoted $u_0$, is an object in $S(C)$.
- For any $u \in S(C)$, the object $F(u)$ is given by

\[
F(u) = \begin{cases} 
U_1 & \text{if } u = u_0, \\
\left(\cdots ((U_1 \otimes U_2) \otimes U_3) \otimes \cdots \right) \otimes U_n & \text{if } u = (U_1, \ldots, U_n), \text{ with } n \in \mathbb{N} \setminus \{0\},
\end{cases}
\]

where on the right all pairs of parenthesis begin if front. For any $u, v \in S(C)$, the arrows from $u$ to $v$ in $C$ are given by $S(C)(u, v) = C(F(v), F(u))$. In that way, with the composition induced by $C$, we obtain the tensor category $S(C)$.

- $S(C)$ becomes a tensor category with the tensor product of objects given by the concatenation product and the tensor product of two arrows $f \in S(C)(u,v)$ and $g \in S(C)(u',v')$ given by the composite $F_2(v,v') \circ (f \otimes g) \circ F_2^{-1}(u,u')$, where, for any $w, w' \in S(C)$, the arrow $F_2(w, w')$ is the canonical isomorphism in $C$ from $F(w) \otimes F(w')$ to $F(w \otimes w')$ obtained iterating the associativity constraint $a$ (well defined by the coherence theorem in [9]).
The definition of the functor $F$ is completed by setting $F(f) = f$ for any arrow $f \in S(C)(u, v) = C(F(u), F(v))$, with $u, v \in S(C)$. $F$ becomes a tensor functor by defining $F_2(\cdot, \cdot)$ as above and $F_0 = I_d$.

The category $C$ can be embedded in $S(C)$ by identifying $C$ with the full subcategory of $S(C)$ given by the sequences of length one. The functor $G$ is given by the immersion of $C$ in $S(C)$. $G$ becomes a tensor functor by setting $G_2(U, V) = a_{U, V}$ for any $U, V \in C$, and $G_0 = I_d \in S(C)(\alpha, I) = C(I, I)$.

DUALITIES. Let $C$ be a tensor category. For simplicity, allowed by Remark [4], we suppose that $C$ is strict. Given $U, V \in C$, a pairing between $V$ and $U$ is an arrow $d: V \otimes U \rightarrow I_\alpha$ in $C$. If, for any arrow $f: X \rightarrow U \otimes Y$ in $C$, we set $d^f(f) = \left( V \otimes X \xrightarrow{V \otimes f} V \otimes U \otimes Y \xrightarrow{d \otimes Y} Y \right)$, then we obtain an application $d^f: C(X, U \otimes Y) \rightarrow C(V \otimes X, Y)$. The pairing $d$ is exact when $d^f$ is bijective for any $X, Y \in C$, i.e., if we have an adjunction of functors $V \otimes \cdot \rightarrow U \otimes \cdot$. It follows that $d$ is exact if and only if there exists a map $b: 1 \rightarrow U \otimes V$ (that is, $b = (d^f)^{-1}(I_d)$) such that the relations (called adjunction triangles or duality relations) $(U \otimes d) \circ (b \otimes U) = U$ and $(d \otimes V) \circ (V \otimes b) = V$ hold. When the pairing is exact, we say that the pair $(b, d)$ is an adjunction or a duality between $V$ and $U$.

We also say that $V$ is left adjoint or left dual to $U$, that $U$ is right adjoint or right dual to $V$, and we write $(b, d): V \rightarrow U$. We call $b$ the unit and $d$ the counit of the adjunction.

Given two adjunctions $(b_1, d_1): V_1 \rightarrow U_1$ and $(b_2, d_2): V_2 \rightarrow U_2$ in $C$, we have a bijection $\alpha = \alpha^{-1}: C(V_1, V_2) \rightarrow C(U_2, U_1)$ with inverse $\beta = \beta^{-1}: C(U_2, U_1) \rightarrow C(V_1, V_2)$, obtained by setting, for any $f \in C(V_1, V_2)$ and $g \in C(U_2, U_1)$, $\tilde{g} = (V_2 \otimes b_1) \circ (V_2 \otimes g \otimes V_1)$ and $\tilde{f} = (b_2 \otimes U_1) \circ (U_2 \otimes f \otimes U_1) \circ (U_2 \otimes d_1)$. Then $g = f \circ \tilde{g}$. We say that $C$ is left (respectively, right) autonomous when any object has a left (respectively, right) dual. We say that $C$ is autonomous if it is both left and right autonomous.

When $C$ is left autonomous, choose an adjunction $(b_V, d_V): V^* \rightarrow V$ for any $V \in C$, we get a functor $\alpha^*: C \rightarrow C^*$, defined on $f \in C(V, U)$ by the condition $f^* \circ \alpha f_\alpha \cdot$. Fix two adjunctions $(b_U, d_U): U^* \rightarrow U$ and $(b_V, d_V): V^* \rightarrow V$ in a tensor category $C$. Given $f \in C(X \otimes U, V \otimes Y)$, the mate $f^\alpha$ of $f$ is the arrow

$$f^\alpha = \left( V^* \otimes X \xrightarrow{V^* \otimes f \otimes U^*} V^* \otimes V \otimes U \otimes U^* \xrightarrow{d_V \otimes Y \otimes U^*} V \otimes Y \otimes U^* \right).$$

2. T-CATEGORIES

Let $\pi$ be a group. A T-category $\mathcal{T}$ (over $\pi$) is given by the following data.

- A tensor category $\mathcal{T}$.
- A family of subcategories $\{\mathcal{T}_\alpha\}_{\alpha \in \pi}$ such that $\mathcal{T}$ is disjoint union of this family and that $U \otimes V \in \mathcal{T}_{\alpha \beta}$, for any $\alpha, \beta \in \pi$, $U \in \mathcal{T}_\alpha$, and $V \in \mathcal{T}_\beta$.
- Denoted $\text{aut}(\mathcal{T})$ the group of the invertible strict tensor functors from $\mathcal{T}$ to itself, a group homomorphism $\varphi: \pi \rightarrow \text{aut}(\mathcal{T})$, $\alpha \mapsto \varphi_\alpha$, the conjugation, such that $\varphi_\beta(\alpha) = \mathcal{T}_{\alpha \beta -1}$ for any $\alpha, \beta \in \pi$.

In the terminology of [13], a T-category is called a crossed group category. Differently from [13], we do not require that a T-category is a linear category. Given $\alpha \in \pi$, the subcategory $\mathcal{T}_\alpha$ is called the $\alpha$-th component of $\mathcal{T}$ while the functors $\varphi_\beta$
are called conjugation isomorphisms. \( T \) is called \emph{strict} when it is strict as a tensor category. When \( \pi = 1 \), then \( \mathcal{T} \) is nothing but a tensor category.

Given two \( T \)-categories \( \mathcal{T} \) and \( \mathcal{T}' \), a \( T \)-functor \( F: \mathcal{T} \to \mathcal{T}' \) is a tensor functor from \( \mathcal{T} \) to \( \mathcal{T}' \) that satisfies the following two conditions.

1. For any \( \alpha \in \pi \), \( F(\mathcal{T}_\alpha) \subset \mathcal{T}'_\alpha \).
2. \( F \) commutes with the conjugation isomorphisms.

Two \( T \)-categories \( \mathcal{T} \) and \( \mathcal{T}' \) are \emph{equivalent as \( T \)-categories} if they are equivalent as categories via a \( T \)-functor \( F: \mathcal{T} \to \mathcal{T}' \) and a \( T \)-functor \( G: \mathcal{T}' \to \mathcal{T} \).

\( \text{LEFT INDEX NOTATION.} \) Given \( \beta \in \pi \) and an object \( V \in \mathcal{T}_\beta \), the functor \( \varphi_\beta \) will be denoted \( \nabla(\cdot) \), as in \([18]\), or also \( \beta(\cdot) \). We introduce the notation \( \nabla^{-1}(\cdot) \) for \( \beta^{-1}(\cdot) \). Since \( \nabla(\cdot) \) is a functor, for any object \( U \in \mathcal{T} \) and for any couple of composable arrows \( \nabla \circ f \circ \nabla \) in \( \mathcal{T} \), we obtain \( \nabla \operatorname{Id}_U = \operatorname{Id}_{\nabla U} \) and \( \nabla (g \circ f) = \nabla g \circ \nabla f \). Since the conjugation \( \varphi: \pi \to \operatorname{aut}(\mathcal{T}) \) is a group homomorphism, for any \( V, W \in \mathcal{T} \), we have \( \nabla \otimes \nabla (\cdot) = \nabla (\nabla (\cdot)) \) and \( \nabla (\cdot) = \nabla (\nabla (\cdot)) = \operatorname{Id}_\mathcal{T} \). Since, for any \( V \in \mathcal{C} \), the functor \( \nabla (\cdot) \) is strict, we have \( \nabla (f \otimes g) = \nabla f \otimes \nabla g \), for any arrow \( f \) and \( g \) in \( \mathcal{T} \), and \( \nabla \mathbb{I} = \mathbb{I} \).

\( \text{STRICT EQUIVALENCE FOR \( T \)-CATEGORIES.} \)

**Theorem 2.1.** Let \( \mathcal{T} \) be a \( T \)-category. \( \mathcal{T} \) is equivalent as a \( T \)-category to a strict \( T \)-category \( \mathcal{S}(\mathcal{T}) \).

**Proof (sketch).** Define the category \( \mathcal{S}(\mathcal{T}) \) and the functors \( F \) and \( G \) as in Remark \([18]\).

- Let \( u = (U_1, \ldots, U_n) \) be in \( \mathcal{S}(\mathcal{T}_\alpha) \), with \( n \geq 1 \) and let \( U_1 \in \mathcal{T}_{\alpha_1}, U_2 \in \mathcal{T}_{\alpha_2}, \ldots, U_n \in \mathcal{T}_{\alpha_n} \). We set \( m(u) = \alpha_1 \alpha_2 \cdots \alpha_n \) and \( m(u_0) = 1 \), where \( u_0 \) is the empty sequence. Any \( \alpha \)-th component of \( \mathcal{S}(\mathcal{T}) \) is defined as the full subcategory \( \mathcal{S}_\alpha(\mathcal{T}) \) whose objects are the objects \( u \) of \( \mathcal{S}(\mathcal{T}) \) such that \( m(u) = \alpha \).
- The conjugation \( \varphi^\text{str} \) of \( \mathcal{S}(\mathcal{T}) \) is obtained by setting, for any \( \alpha \in \pi \), \( \varphi^\text{str}_\alpha(u) = \varphi^\text{str}_\alpha(U_1, \ldots, U_n) = (\varphi_\alpha(U_1), \ldots, \varphi_\alpha(U_n)) \), for any \( u = (U_1, \ldots, U_n) \in \mathcal{S}(\mathcal{T}) \), and \( \varphi^\text{str}_\alpha(u_0) = u_0 \). The definition is completed by setting \( \varphi^\text{str}_\alpha(f) = \varphi_\alpha(f) \), for any arrow \( f \in \mathcal{S}(\mathcal{T}) \).

It is easy to prove that, in that way, \( \mathcal{S}(\mathcal{T}) \) becomes a \( T \)-category and \( F \) and \( G \) become \( T \)-functors. Notice that the hypothesis that the functor \( \varphi_\alpha \) (\( \alpha \in \pi \)) is strict is essential to obtain the functor \( \varphi^\text{str}_\alpha \).

In virtue of Theorem \([2.1]\), often we will consider only strict \( T \)-categories.

**ADJUNCTIONS IN A \( T \)-CATEGORY.** A \emph{left autonomous \( T \)-category} \( \mathcal{T} = (\mathcal{T}, (\cdot)^*) \) is a \( T \)-category endowed with a choice of left dualities \((\cdot)^*\) satisfying the following two conditions.

- If \( U \) is an object in \( \mathcal{T}_\alpha \) (with \( \alpha \in \pi \)), then \( U^* \) is an object in \( \mathcal{T}_{\alpha^{-1}} \).
- The conjugation preserve the chosen dualities, i.e., \( \varphi_\beta(b_U) = b_{\varphi_\beta(U)} \) and \( \varphi_\beta(d_U) = d_{\varphi_\beta(U)} \) for any \( \beta \in \pi \) and \( U \in \mathcal{T} \).

Similarly, it is possible to introduce the notion of a right autonomous \( T \)-category. An \emph{autonomous \( T \)-category} is a \( T \)-category that is both left and right autonomous.

Given two left autonomous \( T \)-categories \( \mathcal{T} \) and \( \mathcal{T}' \), a \emph{left autonomous \( T \)-functor} \( F: \mathcal{T} \to \mathcal{T}' \) is a \( T \)-functor from \( \mathcal{T} \) to \( \mathcal{T}' \) that preserves the dualities, that is \( F(b_U) = b_{F(U)} \) and \( F(d_U) = d_{F(U)} \), for any \( U \in \mathcal{T} \). Two left autonomous \( T \)-categories \( \mathcal{T} \) and \( \mathcal{T}' \) are \emph{equivalent as left autonomous \( T \)-categories} if they are equivalent as categories via a left autonomous \( T \)-functor \( F: \mathcal{T} \to \mathcal{T}' \) and a left autonomous \( T \)-functor \( G: \mathcal{T} \to \mathcal{T}' \). Similarly, it is possible to introduce the notions of a right autonomous
T-functor and of an autonomous T-functor and the notions of equivalence of right autonomous T-categories and of autonomous T-categories.

Remark 2.2. Let \( \mathcal{T} \) be a left autonomous T-category. Define the T-category \( S(\mathcal{T}) \) and the T-functors \( F \) and \( G \) as in Theorem 2.1. Given \( u \in S(\mathcal{T}) \), if we set \( u^* = G(F(u)^*) \), then the exact pairing \( F(u^* \otimes u) \overset{\cong}{\to} F(u)^* \otimes F(u) \overset{d_{F(u)}}{\to} \mathbb{I} \) in \( \mathcal{T} \) gives also an exact pairing \( u^* \otimes u \to \mathbb{I} \) under the identification \( S(\mathcal{T})(u^* \otimes u, u_0) = \mathcal{T}(F(u^* \otimes u), \mathbb{I}) \). It is easy to check that \( S(\mathcal{T}) \) becomes a left autonomous T-category and that \( \mathcal{T} \) is equivalent to \( S(\mathcal{T}) \) as a left autonomous T-category via \( F \) and \( G \).

Stable Left Duals. Let \( \mathcal{T} \) be a T-category and let \( U \in \mathcal{T}_\alpha \) (with \( \alpha \in \pi \)) be an object endowed with an adjunction \((b_U, d_U): U^* \overset{\cong}{\longrightarrow} U\). We say that \((b_U, d_U)\) is a stable adjunction and \( U^* \) a stable left dual of \( U \) when, for any \( \beta \in \pi \) that commutes with \( \alpha \), if \( \varphi_\beta(U) = U \) then \((\varphi_\beta(b_U), \varphi_\beta(d_U)) = (b_U, d_U)\). If we set \( \Phi(U) = \{\varphi_\beta(U)\}_{\beta \in \pi} \), then, given \( V \in \Phi(U) \) and \( \beta \in \pi \) such that \( V = \varphi_\beta(U) \), the stable adjunction \((b_V, d_V)\) gives rise to another stable adjunction \((\varphi_\beta(b_U), \varphi_\beta(d_U)) : \varphi_\beta(U^*) \longrightarrow V\) that does not depend on \( \beta \).

Lemma 2.3. A T-category \( \mathcal{T} \) admits a structure of left autonomous T-category if and only if, for any \( U \in \mathcal{T} \), there exists an object \( U_0 \in \Phi(U) \) endowed with a stable adjunction \((b_0, d_0): U_0^* \longrightarrow U_0\).

Remark 2.4. The terminology concerning a category with dualities is not completely fixed. In particular, if some authors [3] only require that any object \( V \) in a left autonomous category admits an exact pairing, other authors [3, 13] also require the choice of a pairing, i.e., they consider a fixed adjunction for any object of the category. To be coherent with the definition in [13], we choose the second convention. This will also be useful in the second part of the article, since the considered categories will be endowed with a natural choice of stable dualities. However, we will see that, starting from a T-category \( \mathcal{T} \) endowed with a twist \( \theta \), it is possible to obtain a ribbon subcategory \( \mathcal{N}(\mathcal{T}) \) of \( \mathcal{T} \), i.e., a subcategory of \( \mathcal{T} \) endowed with stable dualities compatible with the twist. With the exception of the trivial case in which we just know that \( \mathcal{T} \) is ribbon (so that we have \( \mathcal{N}(\mathcal{T}) = \mathcal{T} \)), there is no natural way to obtain a canonical duality \((\cdot)^*\) for \( \mathcal{N}(\mathcal{T}) \).

Braiding. A braiding for a T-category \( \mathcal{T} \) is a family of isomorphisms

\[
c = \left\{ c_{U,V} \in \mathcal{T}\left( U \otimes V, (\triangledown V) \otimes U \right) \right\}_{U,V \in \mathcal{T}}
\]

satisfying the following conditions.

- For any arrow \( f \in \mathcal{T}_\alpha(U, U') \) (with \( \alpha \in \pi \), \( g \in \mathcal{T}(V, V') \)) we have
  \[
  \left( \triangledown g \otimes f \right) \circ c_{U,V} = c_{U',V'} \circ (f \otimes g).
  \]
  \[
  \tag{2a}
  \]

- For any \( U, V, W \in \mathcal{T} \), we have
  \[
  c_{U \otimes V, W} = a_{U \otimes V,W, U,V} \circ (c_{U,V,W} \otimes V) \circ a_{U,V,W, U,V}^{-1} \circ (U \otimes c_{V,W}) \circ (c_{U,V,W} \otimes V) \circ (a_{U,V,W}^{-1} \circ c_{U,V,W} \otimes V).
  \]
  \[
  \tag{2b}
  \]
  and
  \[
  \tag{2c}
  c_{U,V \otimes W} = a_{U,V \otimes W, U,V}^{-1} \circ \left( \triangledown V \otimes c_{U,W} \right) \circ a_{U,V,W} \circ (c_{U,V,W} \otimes W) \circ a_{U,V,W}^{-1} \circ c_{U,V,W} \otimes W.
  \]

- For any \( U, V \in \mathcal{T} \) and \( \beta \in \pi \), we have
  \[
  \varphi_\beta(c_{U,V}) = c_{\varphi_\beta(U), \varphi_\beta(V)}.
  \]
  \[
  \tag{2d}
  \]
A T-category endowed with a braiding is called a \textit{braided T-category}. In particular, when \( \pi = 1 \), we recover the usual definition of a braided tensor category \([3]\).

Given two braided T-category \( T \) and \( T' \), a \textit{braided T-functor} \( F: T \to T' \) is T-functor from \( T \) to \( T' \) that preserves the braiding, i.e., such that \( F(c_{U,V}) = c_{F(U),F(V)} \) for any \( U, V \in T \). Two braided T-categories \( T \) and \( T' \) are \textit{equivalent as braided T-categories} if they are equivalent as T-categories via a braided T-functor \( F: T \to T' \) and a braided T-functor \( G: T' \to T \).

**Remark 2.5.** Let \( T \) be a braided T-category and define \( S(T), F, \) and \( G \) as in Theorem 2.4. The family of arrows

\[
e_{u,v} = \left( F(u \otimes v) \xrightarrow{\cong} F(u) \otimes F(v) \xrightarrow{c_{F(u),F(v)}} \left( m(u)F(v) \right) \otimes F(u) \xrightarrow{\cong} F \left( (u) \otimes u \right) \right)
\]

(for any \( u, v \in S(T) \)) is a braiding in \( S(T) \). With this structure of braided T-category on \( S(T) \), the functors \( F \) and \( G \) become braided T-functors and so \( T \) is equivalent to \( S(T) \) as a braided T-category.

**The morphism \( \omega \).** Let \( U \) be an object in a braided T-category \( T \) endowed with a left dual \( U^* \) via an adjunction \((b_U, d_U)\). We set

\[
\omega_U = (d_{u \otimes U^*} \otimes U) \circ \left( \left( U \otimes U^* \right) \otimes c_{U^*,U \otimes U} \right) \circ \left( (c_{U^*,U} \circ b_U) \otimes U \otimes U \right).
\]

**Lemma 2.6.** \( \omega_U \) is independent from the choice of the stable left adjunction of \( U \).

Since the proof is relatively long, it is omitted.

**Twist.** A \textit{twist} for a braided T-category \( T \) is a family of isomorphisms

\[
\theta = \{ \theta_U: U \to U^* \}_{U \in T}
\]

satisfying the following conditions.

- \( \theta \) is \textit{natural}, i.e., for any \( f \in T_{\alpha}(U,V) \) (with \( \alpha \in \pi \)),

\[
\theta_V \circ f = (\circ f) \circ \theta_U.
\]

(4a)

- For any \( U \in T_{\alpha} \) and \( V \in T_{\beta} \) (with \( \alpha, \beta \in \pi \)),

\[
\theta_{U \otimes V} = c_{U \otimes V, U} \circ c_{U, V} \circ (\theta_U \otimes \theta_V).
\]

(4b)

- For any \( U \in T \) and \( \alpha \in \pi \),

\[
\varphi_{\alpha}(\theta_U) = \theta_{\varphi_{\alpha}(U)}.
\]

(4c)

A braided T-category endowed with a twist is called a \textit{balanced T-category}. In particular, for \( \pi = 1 \) we recover the usual definition of a balanced tensor category \([3]\).

Given two balanced T-categories \( T \) and \( T' \), a balanced T-functor \( F: T \to T' \) is a \textit{balanced T-functor} if it preserves the twist, i.e., if \( F(\theta_U) = \theta_{F(U)} \) for any \( U \in T \). Two T-categories \( T \) and \( T' \) are \textit{equivalent as balanced T-categories} if they are equivalent as T-category via balanced T-functor \( F: T \to T' \) and a balanced T-functor \( G: T' \to T \).

A \textit{ribbon T-category} \( T \) is a balanced T-category that is also a left autonomous T-category such that for any \( U \in T_{\pi} \) (with \( \alpha \in \pi \)),

\[
\left( (U^*) \otimes \theta_{U} \right) \circ b_U = (\theta_{U} \otimes U^*) \circ b_U.
\]

(5)

For \( \pi = 1 \) we recover the usual definition of a ribbon category \([12, 16]\) also called \textit{tortile tensor category} \([5, 4, 13]\).

Given two ribbon T-categories \( T \) and \( T' \), a T-functor \( F: T \to T' \) that is at the same time a balanced T-functor and a left autonomous T-functor is called \textit{ribbon T-functor}. Two ribbon T-categories \( T \) and \( T' \) are \textit{equivalent as ribbon T-categories}
if they are equivalent as T-categories via a ribbon T-functor \( F: \mathcal{T} \to \mathcal{T}' \) and a ribbon T-functor \( G: \mathcal{T}' \to \mathcal{T} \).

**Remark 2.7.** Let \( \mathcal{T} \) be a balanced T-category and define the equivalent braided T-category \( \mathcal{S}(\mathcal{T}) \) and the functors \( F \) and \( G \) as in Theorem 2.4 and Remark 2.4. The family of arrows \( \theta_u = \theta_{F(u)} \) (with \( u \in \mathcal{S}(\mathcal{T}) \)) is a twist in \( \mathcal{S}(\mathcal{T}) \) such that \( \mathcal{T} \) is equivalent to \( \mathcal{S}(\mathcal{T}) \) as a balanced T-category via \( F \) and \( G \). If \( \mathcal{T} \) is ribbon, then, with the structure of left autonomous T-category on \( \mathcal{S}(\mathcal{T}) \) provided in Remark 2.2, \( \mathcal{T} \) is equivalent to \( \mathcal{S}(\mathcal{T}) \) as a ribbon T-category.

**Mirror T-category.** Let \( \mathcal{T} \) be T-category. The mirror \( \overline{\mathcal{T}} \) of a T-category \( \mathcal{T} \) (see [18] and the description of the corresponding algebraic notion of mirror T-coalgebra below) is the T-category defined as follows.

- For any \( \alpha \in \pi \), we set \( \overline{\mathcal{T}}_\alpha = \mathcal{T}_{\alpha^{-1}} \) as a category. So, as a category, \( \overline{\mathcal{T}} = \mathcal{T} \).
- The tensor product \( U \otimes V \) in \( \overline{\mathcal{T}} \) of \( U \in \overline{\mathcal{T}}_\alpha = \mathcal{T}_{\alpha^{-1}} \) and \( V \in \overline{\mathcal{T}}_\beta = \mathcal{T}_{\beta^{-1}} \) (with \( \alpha, \beta \in \pi \)) is given by \( U \otimes V = \varphi_{\beta^{-1}}(U) \otimes V \in \mathcal{T}_{\alpha \beta} \). Given an arrow \( f \) in \( \mathcal{T}_\alpha \) and an arrow \( g \) in \( \mathcal{T}_\beta \) (with \( \alpha, \beta \in \pi \)), the tensor product \( f \otimes g \) of \( f \) and \( g \) in \( \mathcal{T} \) is given by \( \varphi_{\beta^{-1}}(f) \otimes g \).
- The associativity constraint is given by \( \pi_{U,V,W} = \alpha_{\varphi_{\alpha^{-1}}(U), \varphi_{\alpha^{-1}}(V), W} \), for any \( U \in \mathcal{T}_\alpha \), \( V \in \mathcal{T}_\beta \), and \( W \in \mathcal{T}_\gamma \) (with \( \alpha, \beta, \gamma \in \pi \)).
- The left unit constraint and the right unit constraint are given by \( \overline{l}_U = l_U \) and \( \overline{r}_U = r_U \), for any \( U \in \mathcal{T} \).
- The conjugation is given by \( \overline{\varphi}_\alpha = \varphi_\alpha \).

When \( \mathcal{T} \) is a left autonomous, \( \overline{\mathcal{T}} \) is left autonomous by setting, for any \( U \in \overline{\mathcal{T}}_\alpha \) (with \( \alpha \in \pi \)), \( \overline{l}_U = \varphi_\alpha(b_U) \) and \( \overline{r}_U = d_U \). When \( \mathcal{T} \) is braided, \( \overline{\mathcal{T}} \) is braided via \( \overline{\tau}_{U,V} = (c_{V,U})^{-1} \), for any \( U, V \in \mathcal{T} \). When \( \mathcal{T} \) is balanced (respectively, ribbon), \( \overline{\mathcal{T}} \) is also balanced (respectively, ribbon) via \( \overline{\theta}_U = (\theta_{\varphi_\alpha(U)})^{-1} \), for any \( U \in \mathcal{T}_\alpha \) (with \( \alpha \in \pi \)). The mirror construction is involutive, i.e., we have \( \overline{\overline{\mathcal{T}}} = \mathcal{T} \).

Two \([\)left/right\)] autonomous, braided, balanced or ribbon\] T-categories \( \mathcal{T} \) and \( \mathcal{T}' \) are mirror equivalent \([\)as \( \)left/right\)] autonomous, braided, balanced or ribbon T-categories\] if \( \mathcal{T} \) is equivalent to \( \overline{\mathcal{T}'} \) as \([\)left/right\)] autonomous, braided, balanced or ribbon\] T-category.

## 3. T-CATEGORIES OF REPRESENTATIONS

**T-coalgebras.** Let \( k \) be a commutative field and let \( \pi \) be a group. A T-coalgebra \( H \) \((\)over \( \pi \) \)and \( k \)) \([\)is \]given by the following data.

- For any \( \alpha \in \pi \), an associative \( k \)-algebra \( H_\alpha \), called the \( \alpha \)-th component of \( H \). The multiplication is denoted \( \mu_\alpha: H_\alpha \otimes H_\alpha \to H_\alpha \) and the unit is denoted \( \eta_\alpha: k \to H_\alpha \), with \( 1_\alpha = \eta_\alpha(1) \).
- A family of algebra morphisms \( \Delta = \{ \Delta_{\alpha,\beta}: H_\alpha \otimes H_\beta \to H_{\alpha \beta} \}_{\alpha,\beta \in \pi} \), called comultiplication, that is coassociative in the sense that, for any \( \alpha, \beta, \gamma \in \pi \), we have \( (H_\alpha \otimes H_\beta \otimes H_\gamma) \circ \Delta_{\alpha,\beta,\gamma} = (H_\alpha \otimes H_\gamma) \circ \Delta_{\alpha,\beta,\gamma} \).
- An algebra morphism \( \varepsilon: H_1 \to k \), called counit, such that, for any \( \alpha \in \pi \), we have \( (\varepsilon \otimes H_\alpha) \circ \Delta_1,\alpha = 1d \) and \( (H_\alpha \otimes \varepsilon) \circ \Delta_{\alpha,1} = 1d \).
- A set of algebra isomorphisms \( \varphi = \{ \varphi_\alpha: H_\alpha \to H_{\alpha^{-1}} \}_{\alpha \in \pi} \), called conjugation. When not strictly necessary, the upper index will be omitted. We require that \( \varphi \) satisfies the following conditions.
  - \( \varphi \) is multiplicative, i.e., for any \( \beta, \gamma \in \pi \), we have \( \varphi_\beta \circ \varphi_\gamma = \varphi_{\beta \gamma} \). It follows that, for any \( \alpha \in \pi \), we have \( \varphi_{\alpha} = 1d_{H_\alpha} \).
  - \( \varphi \) is compatible with \( \Delta \), i.e., for any \( \alpha, \beta, \gamma \in \pi \), we have \( (\varphi_\gamma \otimes \varphi_\gamma) \circ \Delta_{\alpha,\beta} = \Delta_{\gamma \alpha \gamma^{-1}, \gamma \beta \gamma^{-1}} \circ \varphi_\gamma \).
We say that $H$ is of finite-type when any component $H_{\alpha}$ (with $\alpha \in \pi$) is a finite-dimensional $k$-vector space. We say that $H$ is totally-finite when $\dim_k \bigoplus_{\alpha \in \pi} H_{\alpha} < \infty$, i.e., when $H$ is of finite-type and almost all the $H_{\alpha}$ are zero. It was proved in [14] that the antipode of a finite-type T-coalgebra is always bijective. We observe that the component $H_1$ of a T-coalgebra $H$ is a Hopf algebra in the usual sense.

**COOPPOSITE T-COALGEBRA.** Let $H$ be a T-coalgebra with invertible antipode. The coopposite T-coalgebra $H^{\text{cop}}$ is the T-coalgebra defined as follows.

- For any $\alpha \in \pi$, we set $H_{\alpha}^{\text{cop}} = H_{\alpha^{-1}}$ as an algebra.
- The comultiplication $\Delta^{\text{cop}}$ is obtained by setting, for any $\alpha, \beta \in \pi$,
  \[
  \Delta^{\text{cop}}_{\alpha, \beta} = \left( H_{\alpha \beta}^{\text{cop}} = H_{\beta^{-1} \alpha^{-1}} \Delta_{\beta^{-1} \alpha^{-1}} \right) \rightarrow H_{\beta^{-1}} \otimes H_{\alpha^{-1}} \rightarrow H_{\alpha^{-1}} \otimes H_{\beta^{-1}} = H_{\alpha}^{\text{cop}} \otimes H_{\beta}^{\text{cop}}.
  \]
- The counit is given by $\varepsilon^{\text{cop}} = \varepsilon$.
- The antipode $s^{\text{cop}}$ is obtained by setting $s^{\text{cop}}_{\alpha} = s_{\alpha^{-1}}$, for any $\alpha \in \pi$.
- The conjugation $\varphi^{\text{cop}}$ is obtained by setting $\varphi^{\text{cop}} = \varphi$, for any $\beta \in \pi$.

**HEYNEMANN-SWEEDLER NOTATION.** The coassociativity of $H$ allows us to introduce an analog of the Heynemann-Sweedler notation [15]. Given $\alpha_1, \ldots, \alpha_n \in \pi$ and defined

\[
\Delta_{\alpha_1, \alpha_2, \ldots, \alpha_n} = \left( H_{\alpha_1 \alpha_2 \ldots \alpha_n} \Delta_{\alpha_1, \alpha_2 \ldots, \alpha_n} \rightarrow H_{\alpha_1} \otimes H_{\alpha_2} \otimes \cdots \otimes H_{\alpha_n} \right)
\]

for any $h \in H_{\alpha_1 \alpha_2 \ldots \alpha_n}$, we set $h'_{(\alpha_1)} \otimes h'_{(\alpha_2)} \otimes \cdots \otimes h'_{(\alpha_n)} = \Delta_{\alpha_1, \alpha_2, \ldots, \alpha_n}(h)$. Let $H$ be a T-coalgebra over a group $\pi$ and a field $k$ and let $M$ be a $k$-vector space. Suppose that $f : H_{\alpha_1} \otimes H_{\alpha_2} \otimes \cdots \otimes H_{\alpha_n} \rightarrow M$ is a $k$-multilinear map. Denoted $\hat{f}$ the tensor lift of $f$, we introduce the notation $\hat{f} \left( h'_{(\alpha_1)} \otimes h'_{(\alpha_2)} \otimes \cdots \otimes h'_{(\alpha_n)} \right)$.

For simplicity, we also suppress the subscript “$(\alpha_i)$” when $\alpha_i = 1$.

**QUASITRIANGULAR T-COALGEBRAS.** A quasitriangular T-coalgebra (see [18]) is a T-coalgebra $H$ endowed with a family $R = \{ R_{\alpha \beta} = \xi_{(\alpha), i} \otimes \zeta_{(\beta), i} \in H_{\alpha} \otimes H_{\beta} \mid \alpha, \beta \in \pi \}$, the universal R-matrix, such that $R_{\alpha \beta}$ is invertible for any $\alpha, \beta \in \pi$ and the following conditions are satisfied.

- For any $\alpha, \beta \in \pi$ and $h \in H_{\alpha \beta}$,
  \[
  R_{\alpha \beta} \Delta_{\alpha, \beta}(h) = (\sigma \circ (\varphi_{\alpha^{-1} \otimes \alpha_1} \circ \Delta_{\alpha_2, \alpha_2}))(h) R_{\alpha \beta}.
  \]

- For any $\alpha, \beta, \gamma \in \pi$,
  \[
  (H_{\alpha} \otimes \Delta_{\beta, \gamma}) (R_{\alpha \beta}) = (R_{\alpha \gamma}) \Delta_{\beta, \gamma}.
  \]

where, given two vector spaces $P$ and $Q$ over $k$, for any $x = p_i \otimes q_i \in P \otimes Q$ we set $x_{13} = p_i \otimes 1_3 \otimes q_i$ and $x_{12} \gamma = p_i \otimes q_i \otimes 1_\gamma$.

- For any $\alpha, \beta, \gamma \in \pi$,
  \[
  (\Delta_{\alpha, \beta} \otimes H_{\gamma}) (R_{\alpha \beta, \gamma}) = ((\varphi_{\beta, \gamma} \otimes H_{\gamma}) (R_{\beta^{-1} \alpha \beta, \gamma})) \Delta_{\beta, \gamma}.
  \]

where, given two vector spaces $P$ and $Q$, for any $x = p_i \otimes q_i \in P \otimes Q$ we set $x_{13} = 1_\alpha \otimes p_i \otimes q_i$.

- $R$ is compatible with $\varphi$, in the sense that, for any $\alpha, \beta, \gamma \in \pi$, we have
  \[
  (\varphi_{\alpha} \otimes \varphi_{\beta})(R_{\beta \gamma}) = R_{\alpha \beta^{-1} \alpha \gamma^{-1}}.
  \]
Notice that \((H_1, R_{1,1})\) is a quasitriangular Hopf algebra in the usual sense. For any \(\alpha, \beta \in \pi\), we introduce the notation \(\xi_{(\alpha)}.i \otimes \zeta_{(\beta)}.i = \tilde{R}_{\alpha, \beta} R^{-1}_{\alpha, \beta}\).

Following [13], we set \(u_\alpha = (s_{\alpha^{-1}} \circ \varphi_\alpha)(\xi_{(\alpha^{-1}).i}) \xi_{(\alpha)}.i \in H_\alpha\) and \(u = \{u_\alpha\}_{\alpha \in \pi}\). The \(u_\alpha\) are called Drinfeld elements of \(H\). When \(\pi = 1\), we recover the usual definition of Drinfeld element of a quasitriangular Hopf algebra. The following properties of \(u\) are proved in [13]. Let \(\alpha\) and \(\beta\) be in \(\pi\) and let \(h\) be in \(H_\alpha\).

(7a) \(u_1 = s_1(\zeta_{(1)}.i)\xi_{(1)}.i\).

(7b) \(u_\alpha\) is invertible with inverse \(u_\alpha^{-1} = s_\alpha^{-1}(\tilde{\zeta}_{(\alpha^{-1}).i}) \tilde{\zeta}_{(\alpha)}.i\). Moreover we have

\[
u_1 \xi_{(\alpha)}.i = (\xi_{(\alpha)}.i \circ s_{\alpha^{-1}} \circ s_\alpha)(\zeta_{(\alpha)}.i).
\]

(7c) \((u_{\alpha\beta})'_{(\alpha)} \otimes (u_{\alpha\beta})'_{(\beta)} = \xi_{(\alpha)}.i \zeta_{(\alpha)}.j \nu_\alpha \otimes \zeta_{(\beta)}.i \nu_{\alpha^{-1}}(\xi_{(\alpha\beta\alpha^{-1}).j}) \nu_\beta\).

(7d) \(c(u_1) = 1\).

(7e) \((s_{\alpha^{-1}} \circ s_\alpha)(\varphi_\alpha)(h) = u_\alpha h u_\alpha^{-1}\).

(7f) \(\varphi_{\alpha \beta} = u_{\beta\alpha \beta^{-1}}\).

(7g) \((s_{\alpha^{-1}} \circ s_\alpha \circ \varphi_\alpha)(h) = u_{\alpha} h u_{\alpha}^{-1}\).

\(u_{\alpha} s_{\alpha^{-1}} s_{\alpha^{-1}} = (u_{\alpha})^{-1}\).

\(u_\alpha s_{\alpha^{-1}} s_{\alpha^{-1}} = (u_\alpha)^{-1}\).

The mirror T-coalgebra. Let \(H = (H, R)\) be a quasitriangular Hopf algebra (with \(R = \xi_i \otimes \zeta_i\) and \(R^{-1} = \tilde{R} = \zeta_i \otimes \xi_i\)) By replacing \(R\) with \(\tilde{R} = \sigma(R) = \zeta_i \otimes \xi_i\) we obtain another quasitriangular structure \(\tilde{H} = (H, \tilde{R})\). This means that, in the category of representations of \(H\), we replace the braiding \(c_R\) provided by \(R\) by the braiding \(c_{\tilde{R}}\) provided by \(\tilde{R}\). When \(H\) is a T-coalgebra, the family \(R_{\alpha, \beta}^{-1} = \tilde{\zeta}_{\alpha}.i \otimes \tilde{\zeta}_{\beta}.i\) is not a universal \(R\)-matrix for \(H\). Nevertheless, it is still possible to generalize the definition of \(\tilde{H}\). Let \(H\) be a T-coalgebra. The T-coalgebra \(\tilde{H}\), called mirror of \(H\) [13], is defined as follows.

- For any \(\alpha \in \pi\), we set \(\tilde{H}_\alpha = H_{\alpha^{-1}}\).
- For any \(\alpha, \beta \in \pi\), the component \(\tilde{\Sigma}_{\alpha, \beta}\) of the comultiplication \(\tilde{\Sigma}\) of \(\tilde{H}\) is

\[
\tilde{\Sigma}_{\alpha, \beta}(h) = (\varphi_{\alpha \beta} \otimes H_{\beta^{-1}}) \circ \Delta_{\beta^{-1} \alpha \beta^{-1}}(h) \in H_{\alpha^{-1}} \otimes H_{\beta^{-1}} = \tilde{H}_{\alpha} \otimes \tilde{H}_{\beta},
\]

for any \(h \in H_{\beta^{-1} \alpha^{-1}} = \tilde{H}_{\alpha \beta}\). If, we set \(h'_{(\alpha)} \otimes h'_{(\beta)} = \tilde{\Sigma}_{\alpha, \beta}(h)\), then [8] can be written in the form \(h'_{(\alpha)} \otimes h'_{(\beta)} = \varphi_{\beta}(h_{(\beta^{-1} \alpha^{-1})}) \otimes h_{(\beta^{-1} \alpha^{-1})}\). The counit of \(\tilde{H}\) is given by \(\varepsilon \in H_{1}^{*} = \tilde{H}_{1}\).

- For any \(\alpha \in \pi\), the \(\alpha\)-th component of the antipode \(\tilde{\sigma}\) of \(\tilde{H}\) is given by

\[
\tilde{\sigma}_{\alpha} = \varphi_{\alpha} \circ s_{\alpha^{-1}} : \tilde{H}_{\alpha} = H_{\alpha^{-1}} \rightarrow H_{\alpha} = \tilde{H}_{\alpha^{-1}}.
\]

- Finally, for any \(\alpha \in \pi\), we set \(\tilde{\varphi}_{\alpha} = \varphi_{\alpha}\).

If \(H\) is quasitriangular, then \(\tilde{H}\) is also quasitriangular with universal \(R\)-matrix

\[
\tilde{\Sigma}_{\xi(\alpha),i} \otimes \tilde{\zeta(\beta),i} = \tilde{\Sigma}_{\alpha,\beta} = (\sigma(R_{\beta^{-1} \alpha^{-1}}))^{-1} \in H_{\alpha^{-1}} \otimes H_{\beta^{-1}} = \tilde{H}_{\alpha} \otimes \tilde{H}_{\beta}
\]

for any \(\alpha, \beta \in \pi\).

Ribbon T-coalgebras. We say that a quasitriangular T-coalgebra \(H\) is ribbon when it is endowed with a family \(\{\theta_\alpha|H_\alpha\}_{\alpha \in \pi}\) such that \(\theta_\alpha\) is invertible for any \(\alpha \in \pi\) and the following conditions are satisfied for any \(\alpha, \beta \in \pi\) and \(h \in H_\alpha\).

1. \(\varphi_{\alpha}(h) = \theta_{\alpha^{-1}} h \theta_{\alpha}\).
2. \(s_{\alpha}(\theta_{\alpha}) = \theta_{\alpha^{-1}}\).
3. \(\theta_{\alpha \beta}((\theta_{\alpha \beta})''_{(\beta)}) = \theta_{\alpha \beta} \xi_{(\alpha),i} \xi_{(\alpha),j} \otimes \theta_{\beta} \varphi_{\alpha^{-1}}(\xi_{(\alpha \beta \alpha^{-1}).i}) \zeta_{(\beta),j}\).
4. \(\varphi_{\alpha}(h) = \theta_{\alpha \beta^{-1}}\).

Notice that \((H_1, R_{1,1}, \theta_1)\) is a ribbon Hopf algebra in the usual sense.

T-categories of representations. Let \(H\) be a T-coalgebra over a field \(k\). The T-category \(\text{Rep}(H)\) (see [13]) is defined as follows.

- For any \(\alpha \in \pi\), the \(\alpha\)-th component of \(\text{Rep}(H)\), denoted \(\text{Rep}_\alpha(H)\), is the category of representations of the algebra \(H_\alpha\).
The tensor product $U \otimes V$ of $U \in \text{Rep}_\alpha(H)$ and $V \in \text{Rep}_\beta(H)$ (with $\alpha, \beta \in \pi$) is obtained by the tensor product of $k$-vector spaces $U \otimes_k V$ endowed with the action of $H_{\alpha \beta}$ given by $h(u \otimes v) = \Delta_{\alpha, \beta}(h)(u \otimes v) = h'(\alpha)u \otimes h''(\beta)v$, for any $h \in H_{\alpha \beta}$, $u \in U$, and $v \in V$.

The tensor product of two arrows $f \in \text{Rep}_\alpha(H)$ and $g \in \text{Rep}_\beta(H)$ is given by the tensor product of $k$-linear morphisms, i.e., the forgetful functor from $\text{Rep}(H)$ to the category of vector spaces over $k$ is faithful.

The unit $I$ is the ground field $k$ with the action of $H_1$ provided by $\varepsilon$.

• Given $\beta \in \pi$, we need to define the functor $\beta(\cdot)$. To avoid confusion, in this context we reserve the notation $\varphi_\beta$ for the isomorphism of algebras $\varphi_\beta : H_\alpha \to H_{\beta \alpha \beta^{-1}}$. Let $U$ be in $\text{Rep}_\alpha(H)$, with $\alpha \in \pi$. The object $\beta U$ has the same underlying vector space of $U$. Given $u \in U$, we denote $\beta u$ the corresponding element in $\beta U$. The action of $H_{\beta \alpha \beta^{-1}}$ on $\beta U$ is given by

$$h^\beta u = \beta(\varphi_{\beta^{-1}}(h)u)$$

for any $u \in U$ and $h \in H_{\beta \alpha \beta^{-1}}$.

The objects of $\text{Rep}(H)$ are called representations of $H$.

When $H$ is quasitriangular, $\text{Rep}(H)$ is braided by setting, for any $u \in U$, $v \in V$, $U \in \text{Rep}_\alpha(H)$, $V \in \text{Rep}_\beta(H)$, and $\alpha, \beta \in \pi$,

$$c_{U,V} : U \otimes V \to (U^\alpha) \otimes (\alpha \otimes v) \otimes U : u \otimes v \mapsto (\alpha(\varphi_\beta h), v) \otimes (\alpha, h) u.$$

Let us consider the full subcategory $\text{Rep}_1(H)$ of finite-dimensional representations of $H$, i.e., of representations $U$ of $H$ such that $\text{dim}_k U \in \mathbb{N}$. The T-category $\text{Rep}_1(H)$ has a structure left autonomous T-category obtained in the following way. For any $U \in \mathcal{T}_\alpha$ we set $U^* = \text{Hom}_k(U, k)$, with the action of $H_{\alpha^{-1}}$ on $U^*$ given by

$$\langle hf, u \rangle = (\langle f, s_\alpha^{-1}(h) u \rangle)$$

for any $h \in H_{\alpha^{-1}}$, $f \in U^*$ and $u \in U$. Then, $U^*$ is a left dual of $U$ via

$$b_U : 1 \to U \otimes U^* : 1 \mapsto e_i \otimes e^i$$

(where $\{e_i\}$ is a basis of $U$ as a $k$-vector space and $\{e^i\}$ its dual basis), and

$$d_U : U^* \otimes U \to k : f \otimes u \mapsto (f, u) = f(u)$$

for any $f \in U^*$ and $u \in U$).

If $H$ is endowed with a twist $\{\theta_\alpha \in H_{\alpha}\}_{\alpha \in \pi}$, then $\text{Rep}(H)$ is a balanced T-category, with the twist $\theta_U : U \to U^\alpha : u \mapsto \alpha(\theta_\alpha u)$, for any $u \in U$, with $U \in \text{Rep}_\alpha(H)$, and $\alpha \in \pi$. Similarly, $\text{Rep}_1(H)$ is a ribbon T-category.

Notice that $\text{Rep}(H)$ is isomorphic to $\text{Rep}(\overline{H})$. Similarly, $\text{Rep}_1(H) = \text{Rep}_1(\overline{H})$.

4. THE CENTER OF A T-CATEGORY

We generalize the center construction of a tensor category described in [3] to the case of a T-category $\mathcal{T}$ (that, for simplicity, we suppose strict), obtaining a braided T-category $Z(\mathcal{T})$ in the following way.

• The objects of $Z(\mathcal{T})$, called half-braidings, are the pairs $(U, \zeta_u)$ satisfying the following conditions.

- $U$ is an object of $\mathcal{T}$.
- $\zeta_u$ is a natural isomorphism from the functor $U \otimes -$ to the functor $U \otimes (\zeta_u) \otimes U$ such that for any $X, Y \in \mathcal{T}$, we have

$$\zeta_{X \otimes Y} = \left( (U^X) \otimes \zeta_Y \right) \circ (\zeta_X \otimes Y).$$
The arrows in \( Z(T) \) from an object \((U, \zeta)\) to an object \((V, \alpha)\) are the arrows \( f \in T(U, V) \) such that, for any \( X \in T \), we have

\[
(12b) \quad \left( (U X) \otimes f \right) \circ \xi_X = \alpha_X \circ (f \otimes X).
\]

The composition of two arrows in \( Z(T) \) is given by the composition in \( T \), i.e., by requiring that the forgetful \( Z(T) \rightarrow T \): \((U, \zeta) \rightarrow U\) is faithful.

- Given \( Z = (U, \zeta), Z' = (U', \zeta') \in Z(T) \), their tensor product \( Z \otimes Z' \) in \( Z(C) \) is the couple \((U \otimes U', (\zeta \square \zeta')_X)\), where \((\zeta \square \zeta')_X\) is obtained by setting, for any \( X \in T \),

\[
(12c) \quad (\zeta \square \zeta')_X = \zeta_{U' \otimes U} \circ (U \otimes \zeta_X).
\]

- The tensor unit of \( Z(T) \) is the couple \( Z_1 = (I, \text{Id}_X) \), where \( I \) is the tensor unit of \( T \).

- For any \( \alpha \in \pi \), the \( \alpha \)-th component of \( Z(T) \), denoted \( \mathcal{Z}_\alpha(T) \), is the full subcategory of \( Z(T) \) whose objects are the pairs \((U, \zeta)\) with \( U \in T_\alpha \).

- For any \( \beta \in \pi \), the automorphism \( \varphi_{Z, \beta} \) is obtained by setting, for any \((U, \zeta) \in \mathcal{Z}(T)\),

\[
(12d) \quad \varphi_{Z, \beta}(U, \zeta) = (\varphi_\beta(U), \varphi_{Z, \beta}(\zeta)),
\]

where \( \varphi_{Z, \beta}(\zeta)_X = \varphi_\beta(\zeta_{\varphi_{\beta}^{-1}(X)}) \), for any \( X \in T \). The definition of \( \varphi_\beta \) is completed by requiring that the forgetful \( Z(T) \rightarrow T \) is a T-functor.

- The braiding \( c \) in \( Z(T) \) is obtained by setting, \( c_{Z, Z'} = c_{U'} \), for any \( Z = (U, \zeta), Z' = (U', \zeta') \in Z(T) \).

**Theorem 4.1.** \( Z(T) \) is a braided T-category.

The proof of Theorem [4, 3] and of most of the results in the next three sections can be obtained by introducing a graphical calculus as is [4, 4] and in [5] and then to follow, mutatis mutandis, the computation in [6], with the main difference that we do not obtain the categorical quantum double directly, but in more steps, as described before. Alternatively, it is possible to make every computation algebraically. This has the advantage that you can consider T-categories that are not equivalent to strict T-categories. In particular, you can consider T-categories such that the conjugate automorphism \( \varphi \) are not strict (see note at the end of the proof of Theorem [4, 3]). However, in that way, computations become very long.

**Remark 4.2.** Bruguïères [6] noticed that, if \( C \) is an autonomous tensor category, then \( Z(C) \) is autonomous too. He also noticed that this result is still true if we replace \( C \) with a T-category \( T \).

**Remark 4.3.** The definition of the center given here generalizes the most usual convention adopted also in [3, 4]. However, in [3], the center of a tensor category is constructed in a similar way, but considering the natural transformation of the kind \( \zeta \otimes U \rightarrow U \otimes \zeta \). The choice in [3] seems more appropriate in some context, e.g., in the construction of the isomorphism between the center of the category of representations of a Hopf algebra \( H \) and the category of representations of \( D(H) \).

5. **The twist extension of a braided T-category**

Let \( T \) be a braided T-category (again, for simplicity, we suppose \( T \) strict). Generalizing the construction described in [3, 4], we obtain a balanced T-category \( T^Z \), the twist extension of \( T \). Even if we do not have, in general, an embedding \( T \hookrightarrow T^Z \), the name is justified by the observation that we still have an embedding \( T_1 \hookrightarrow T^Z_1 \). We will see that, if \( H \) is a T-coalgebra and \( T = \text{Rep}(H) \) or \( T = \text{Rep}_1(H) \), then there is an embedding \( T \hookrightarrow T^Z \).
The objects of $\mathcal{T}^Z$ are the pairs $T = (U, t)$, where $U \in \mathcal{T}$ and $t \in \mathcal{T}(U, U')$ is invertible.

For any $T_1 = (U_1, t_1), T_2 = (U_2, t_2) \in \mathcal{T}^Z$, the arrows from $T_1$ to $T_2$ in $\mathcal{T}^Z$ are the arrows $f \in \mathcal{T}(U_1, U_2)$ such that

$$(U^f) \circ t_1 = t_2 \circ f.$$  

The composition is given by the composition in $\mathcal{T}$, i.e., we require that the forgetful functor from $\mathcal{T}^Z \to \mathcal{T}: (U, t) \mapsto U$ is faithful.

The tensor product of $T_1 = (U_1, t_1), T_2 = (U_2, t_2) \in \mathcal{T}^Z$ is the couple $T_1 \boxtimes T_2 = (U_1 \otimes U_2, t_1 \boxtimes t_2)$, where

$$t_1 \boxtimes t_2 = c_{U \otimes U', U'} \circ c_{U', U'} \circ (t_1 \otimes t_2).$$

The tensor unit of $\mathcal{T}^Z$ is the tensor product of the units of $\mathcal{T}$.

The tensor product of two arrows in $\mathcal{T}^Z$ is given by the tensor product of arrows in $\mathcal{T}$.

For any $\alpha \in \pi$, the component $\mathcal{T}^Z_\alpha = (\mathcal{T}^Z)_\alpha$ is the full subcategory of $\mathcal{T}^Z$ whose objects are the pairs $(U, t)$ with $U \in \mathcal{T}_\alpha$.

For any $\beta \in \pi$, the functor $\varphi^Z_\beta$ is obtained by setting, for any $(U, t) \in \mathcal{T}^Z$,

$$\varphi^Z_\beta(U, t) = (\varphi(U), \varphi(t))$$

and $\varphi^Z(f) = \varphi(f)$, for any arrow $f$ in $\mathcal{T}^Z$.

The braiding in $\mathcal{T}^Z$ is obtained by requiring that the forgetful functor from $\mathcal{T}^Z$ to $\mathcal{T}$ is braided.

The twist $\theta$ of $\mathcal{T}^Z$ is obtained by setting $\theta_T = t$, for any $T = (U, t) \in \mathcal{T}^Z$.

**Theorem 5.1.** $\mathcal{T}^Z$ is a balanced $\mathcal{T}$-category.

**DUALITIES IN $\mathcal{T}^Z$.** Even when $\mathcal{T}$ is left autonomous, an object in $\mathcal{T}^Z$ not necessarily admits a left dual. So, in particular, $\mathcal{T}^Z$ is not necessarily ribbon. The following lemma gives a characterization of the objects in $\mathcal{T}^Z$ endowed with a stable left dual.

**Lemma 5.2.** Let $T = (U, t)$ and $T^* = (U^*, \tau)$ be objects in $\mathcal{T}^Z$. Then, $T^*$ is a stable left dual of $T$ with unit $b_T$ and counit $d_T$ if and only if

- $U^*$ is a stable left dual of $U$ in $\mathcal{T}$ via unit $b_U = b_T$ and counit $d_U = d_T$ and
- $\tau = U^* t^*$, where $t \in \mathcal{T}(U, U')$ satisfies $t^{-1} \circ U^* t^{-1} = \omega_U$.

6. **DUALITIES IN A BALANCED T-CATEGORY**

Let $\mathcal{T}$ be a balanced T-category. Generalizing some results in [4, 5, 10, 12], to the case of a T-category, we study the properties of dualities in $\mathcal{T}$. In particular, this will allow us to obtain a full subcategory $N(\mathcal{T})$ of $\mathcal{T}$ that will be the biggest ribbon category included in $\mathcal{T}$. This is the analog, in the case of a T-category, of the construction given in [4] in the case of a tensor category.

**Reflexive objects.** Let $\mathcal{T}$ be a balanced T-category and $U \in \mathcal{T}$. We set

$$\theta_U^2 = \left(U \xrightarrow{\theta_U} UU \xrightarrow{\theta_U \otimes U} U \otimes UU\right)$$

and $\theta_U^{-2} = (\theta_U^2)^{-1}$. We say that $U$ is reflexive if it is endowed with a stable left dual $U^*$, such that $\theta_U^{-2} = \omega_U$.

**Lemma 6.1.** If $U \in \mathcal{T}$ has a stable left dual $U^*$ such that the ribbon condition [4] is satisfied, then $U$ is reflexive. In particular, any object in a ribbon T-category is reflexive.
Reversed duality. Let $U$ be a reflexive object in $\mathcal{T}$. We set

$$
\begin{align*}
  b_U' &= \left(\mathbb{I} \xrightarrow{U^* b_U} U^* U \otimes U^* \xrightarrow{\varepsilon_U^*} U^* U \xrightarrow{\varepsilon_U^* b_U} U^* U \right) \\
  d'_U &= \left(U \otimes U^* \xrightarrow{\delta_U \otimes U^*} U U \otimes U^* \xrightarrow{c_{U^* U}^*} U U^* \otimes U \xrightarrow{d_U} \mathbb{I} \right).
\end{align*}
$$

Lemma 6.2. $U$ is a left dual of $U^*$ via $(b_U', d'_U)$, i.e., $(d'_U \otimes U) \circ (U \otimes b_U') = U$ and $(U^* \otimes d'_U) \circ (b'_U \otimes U^*) = U^*$, that is, $U$ is a stable left dual of $U^*$ with unit $b'_U$ and counit $d'_U$.

The adjunction $(b'_U, d'_U)$ is said reversed adjunction of $(b_U, d_U)$.

Good left duals. Let $U$ be a reflexive object. We say that $U^*$ is a good left dual if further we have

$$
\theta_{U^*} = U^* \theta_U.
$$

Lemma 6.3. Let $U$ be an object in a balanced $\mathcal{T}$-category $\mathcal{T}$ endowed with a stable adjunction $(b_U, d_U) : U^* \rightarrow U$. The ribbon condition (13) is satisfied if and only if (14) is satisfied. In particular, $\mathcal{T}$ is ribbon if and only if any object $U \in \mathcal{T}$ satisfies (14).

Lemma 6.4. Let $U^*$ be a good left dual of $U \in \mathcal{T}$. If we set $U^{**} = U$ via the reversed adjunction $(b'_U, d'_U)$, then we have $b'_U = b_U$ and $d'_U = d_U$. If $V^*$ is a good left dual of $V \in \mathcal{T}$ and we set $V^{**} = V$ via the reversed adjunction $(b'_V, d'_V)$, then, for any $f \in \mathcal{T}(U, V)$, we have $f^{**} = f$.

The category $\mathcal{N}(\mathcal{T})$. Let $\mathcal{T}$ be a balanced $\mathcal{T}$-category. By definition, $\mathcal{N}(\mathcal{T})$ is the full subcategory of $\mathcal{T}$ of the object $U \in \mathcal{T}$ that admits a good left dual. For any class $\Phi(U)$ in $\mathcal{N}(\mathcal{T})$ we also fix an object $U_0 \in \Phi(U)$ and a good left dual $U_0^*$ of $U_0$, obtaining, in that way, a good left dual $V^*$ for any $V \in \Phi(U)$.

Theorem 6.5. $\mathcal{N}(\mathcal{T})$ has a structure of balanced $\mathcal{T}$-category. Moreover, $\mathcal{N}(\mathcal{T})$ is a ribbon $\mathcal{T}$-category and any other ribbon subcategory of $\mathcal{T}$ is included in $\mathcal{N}(\mathcal{T})$.

Proof. The proof that $\mathcal{N}(\mathcal{T})$ has a structure of balanced $\mathcal{T}$-category is given in Lemma 6.3. The proof that $\mathcal{N}(\mathcal{T})$ is autonomous is given in Lemma 6.8. Since, by hypothesis, any object of $\mathcal{N}(\mathcal{T})$ satisfies (13), then, by Lemma 6.3, $\mathcal{N}(\mathcal{T})$ is ribbon. The fact that any other ribbon $\mathcal{T}$-category included in $\mathcal{T}$ is also included in $\mathcal{N}(\mathcal{T})$ follows by Lemma 6.3 and Lemma 6.8.

To prove that $\mathcal{N}(\mathcal{T})$ is a tensor category, we need the following observation.

Lemma 6.6. Let $U$ and $V$ be objects in $\mathcal{T}$, let $U^*$ be a stable left dual of $U$, and let $V^*$ be a stable left dual of $V$. Consider the dual $(U \otimes V)^* = V^* \otimes U^*$ of $U \otimes V$ via the unit $b_{U \otimes V} = (U \otimes b_V \otimes U^*) \circ b_U$ and the counit $d_{U \otimes V} = d_U \circ (U^* \otimes d_V \otimes U)$. We have $c_{V^*, U^*} = c_{V, U}^*$.

Lemma 6.7. The structure of balanced $\mathcal{T}$-category of $\mathcal{T}$ induces a structure of balanced $\mathcal{T}$-category on $\mathcal{N}(\mathcal{T})$.

Proof. The only non trivial part is to show that $\mathcal{N}(\mathcal{T})$ is a tensor category. Since $\mathcal{N}(\mathcal{T})$ is a full subcategory of $\mathcal{T}$, we only need to show that the tensor product of $U, V \in \mathcal{N}(\mathcal{T})$ lies in $\mathcal{N}(\mathcal{T})$, i.e., that $U \otimes V$ admits a good left dual. Let $U^*$ be a good left dual of $U$ and $V^*$ a good left dual $V$. Given $V^* \otimes U^*$ as a stable left dual of $U \otimes V$ with unit $b_{U \otimes V}$ and counit $d_{U \otimes V}$ as in Lemma 6.4, then $V^* \otimes U^*$ is a
Proof. Given the ribbon condition (5). So, by Lemma 6.1, \( U \) has a good left dual of \( D \) via the reversed duality, then (14) is satisfied. Now, by Lemma 6.4, we only need to check (15)

\[
\theta_U = U \theta_U^*.
\]

Applying \( U(\cdot) \) to (14), we get \( U \theta_U^* = \theta^*_U \). By duality, we find (15).

\[ \square \]

**Lemma 6.8.** \( \mathcal{N}(\mathcal{T}) \) is an autonomous \( \mathcal{T} \)-category.

Proof. Given \( U \in \mathcal{N}(\mathcal{T}) \) and a good left dual \( U^* \) of \( U \), we need to prove that also \( U^* \) is an object in \( \mathcal{N}(\mathcal{T}) \). Since \( U^* \) is a good left dual of \( U \), by Lemma 6.3, it satisfies the ribbon condition \( \square \). So, by Lemma 6.1, \( U^* \) is reflexive and so \( U \) is a stable left dual of \( U^* \) under the reversed duality. We only need to show that, if we set \( U^{**} = U \) via the reversed duality, then (14) is satisfied. Now, by Lemma 6.4, we have \( (\theta_U^*)^* = \theta_U \), so we only need to check

\[
(15) \quad \theta_U = U \theta_U^*.
\]

Applying \( U(\cdot) \) to (14), we get \( U \theta_U^* = \theta^*_U \). By duality, we find (15).

\[ \square \]

7. THE QUANTUM DOUBLE OF A \( \mathcal{T} \)-CATEGORY

Let \( \mathcal{T} \) be a \( \mathcal{T} \)-category (that again, for simplicity, we suppose strict). Apply the center construction obtaining the braided \( \mathcal{T} \)-category \( \mathcal{Z}(\mathcal{T}) \). Then consider its twist extension \( (\mathcal{Z}(\mathcal{T}))^Z \). Finally, consider its maximal ribbon subcategory \( \mathcal{D}(\mathcal{T}) = \mathcal{N}(\mathcal{Z}(\mathcal{T}))^Z \). Starting from \( \mathcal{T} \), we obtained a ribbon \( \mathcal{T} \)-category. However, generalizing the construction \( \square \), we can directly construct a ribbon \( \mathcal{T} \)-category \( \mathcal{D}(\mathcal{C}) \), the quantum double of \( \mathcal{T} \), isomorphic to \( \mathcal{N}(\mathcal{Z}(\mathcal{T}))^Z \). One of the advantages is that a choice of dualities in \( \mathcal{T} \), i.e., a structure of autonomous \( \mathcal{T} \)-category, induces a choice of dualities in \( \mathcal{D}(\mathcal{T}) \).

- The objects of \( \mathcal{D}(\mathcal{T}) \) are the triples \( D = (U, \epsilon, t) \), where
  - \( U \) is an object in \( \mathcal{T} \),
  - \( \epsilon : U \otimes \cdot \rightarrow U(\cdot) \otimes U \) is a natural isomorphism that satisfies the half-braiding axiom (12a),
  - \( t \in \mathcal{T}(U, U^*) \) is an isomorphism such that \( (U_1^t \circ t)^{-1} = \omega_U \).
- Given two objects \( D_1 = (U_1, \epsilon_1, t_1), D_2 = (U_2, \epsilon_2, t_2) \in \mathcal{D}(\mathcal{T}) \), an arrow \( f \in \mathcal{D}(\mathcal{T})(D_1, D_2) \) is an arrow \( f \in \mathcal{T}(U_1, U_2) \) that is compatible with the half-braidings and the twist, i.e., it satisfies (12b) and (13a).
- The tensor product of two objects \( D_1 = (U_1, \epsilon_1, t_1), D_2 = (U_2, \epsilon_2, t_2) \in \mathcal{D}(\mathcal{T}) \), is the triple \( D_1 \otimes D_2 = (U_1 \otimes U_2, (\epsilon \otimes \epsilon)_1 \otimes t_2 \otimes t_2) \), where \( \otimes \) is defined as in (12a) and \( \Box \) as in (13b). The tensor product of arrows is obtained by requiring that the functor \( \mathcal{D}(\mathcal{T}) \rightarrow \mathcal{T} : (U, \epsilon, t) \rightarrow U \) is a tensor functor.
• The conjugation is given by \( \varphi_{D, \beta}(U, \zeta, t) = (\varphi_{\beta}(U), \varphi_{\beta}(Z, \zeta), \varphi_{\beta}(t)) \), for any \( \beta \in \pi \) and \( (U, \zeta, t) \in D(T) \), where \( \varphi_{Z, \beta}(\zeta) \) is defined as in the case of the center of \( T \) (see [12a]). For any arrow \( f \in D(C) \), we set \( \varphi_{D, \beta}(f) = \varphi_{\beta}(f) \).

• Let \( D = (U, \zeta, \theta_D) \) be an object in \( D(T) \). We obtain a stable left dual \( D^* \) of \( D \) in \( D(T) \) by setting \( D^* = (U^*, \zeta^*, U^*)^T \) and \( b_D = b_U, d_D = d_U \), where \( b_U \) and \( d_U \) are the unit and the counit of \( U \) in \( T \).

• The braiding is obtained by \( c_{D_1, D_2} = c_{U_2} \) for any \( D_1 = (U_1, \zeta_1, t_1), D_2 = (U_2, \zeta_2, t_2) \in D(T) \).

• The twist is obtained by setting \( \theta_D = t \) for any \( D = (U, \zeta, t) \in D(T) \).

**Theorem 7.1.** \( D(T) \) is a ribbon \( T \)-category isomorphic to \( \mathcal{N}((Z(T))^Z) \) as balanced \( T \)-category.

8. **Yetter-Drinfeld modules and the center of \( \text{Rep}(H) \)**

We give the definition of a Yetter-Drinfeld module over a \( T \)-coalgebra \( H \) and we prove that the category \( YD(H) \) of Yetter-Drinfeld modules over \( H \) is a braided \( T \)-category isomorphic to \( Z(\text{Rep}(H)) \).

**Definition of a Yetter-Drinfeld module.** Let us fix \( \alpha \in \pi \). An \( \alpha \)-Yetter-Drinfeld module or, simply, a \( YD_\alpha \)-module is a couple \( V = (V, \Delta_V = \{\Delta_{V, \lambda}\}_{\lambda \in \pi}) \), where \( V \) is an \( H_\alpha \)-module and, for any \( \lambda \in \pi, \Delta_{V, \lambda} : V \rightarrow V \otimes H_\lambda \) is a \( \mathbb{k} \)-linear morphism such that the following conditions are satisfied.

- \( V \) is **coassociative** in the sense that, for any \( \lambda_1, \lambda_2 \in \pi \), we have
  \[
  (V \otimes \Delta_{\lambda_1, \lambda_2}) \circ \Delta_{V, \lambda_1 \lambda_2} = (\Delta_{V, \lambda_1} \otimes H_{\lambda_2}) \circ \Delta_{V, \lambda_2}.
  \]

- \( V \) is **counitary** in the sense that
  \[
  (V \otimes \varepsilon) \circ \Delta_{V, 1} = \text{Id}.
  \]

- \( V \) is **crossed** in the sense that, for any \( \lambda \in \pi \), the diagram

\[
\begin{array}{ccc}
H_\alpha \otimes H_\lambda \otimes V & \xrightarrow{\Delta_{\alpha, \lambda} \otimes \Delta_{V, \lambda}} & H_\alpha \otimes V \\
& \downarrow \Delta_{\alpha, \lambda_1} \otimes V & \downarrow \mu_V \otimes \mu_{\lambda} \\
H_{\alpha \lambda_1} \otimes V & \xrightarrow{H_{\alpha \lambda_1} \otimes V} & V \otimes H_{\lambda_1} \\
& \downarrow \sigma & \downarrow \Delta_{V, \lambda} \otimes \varphi_{\alpha^{-1}} \\
V \otimes H_{\alpha \lambda_1^{-1}} & \xrightarrow{V \otimes H_{\alpha \lambda_1^{-1}}} & V \otimes H_{\lambda_1}
\end{array}
\]

commutes (\( \mu_{\lambda} \) is the product of \( H_\lambda \) while \( \mu_V \) is the \( H_\alpha \)-module structural map of \( V \)).

If, for any \( v \in V \), we set
\[
(v_{(V)} \otimes v_{(\lambda)}) = \Delta_{V, \lambda}(v),
\]
then we can rewrite axiom \([16a] \), as \((v_{(V)})(v_{(V)})(v_{(V)})_{(\lambda_1)}(v_{(V)})_{(\lambda_2)} = v_{(V)} (v_{(V)})(v_{(V)})(v_{(V)})_{(\lambda_1)}(v_{(V)})(v_{(V)})(v_{(V)})_{(\lambda_2)} \) Similarly, we can rewrite axiom \([16b] \), as \( \varepsilon(v_{(V)}) v_{(V)} = v \). Finally,
we can rewrite the commutativity of (16) as \( h'_\alpha v_\lambda \otimes h''_\lambda v_\lambda = (h'_\alpha v_\lambda) \otimes (h''_\lambda v_\lambda) \), for any \( \lambda \in \pi \) and \( h \in H_\alpha\). 

Given two YD\(\alpha\)-modules \((V, \Delta_V)\) and \((W, \Delta_W)\), a morphism of YD\(\alpha\)-modules \(f: (V, \Delta_V) \to (W, \Delta_W)\), is a \(H_\alpha\)-linear map \(f: V \to W\) such that, for any \( \lambda \in \pi\),

\[
\Delta_{W,\lambda} \circ f = (f \otimes H_\lambda) \circ \Delta_{V,\lambda}.
\]

With the notation provided in (17), axiom (18) can be rewritten as

\[
f_{(v_\lambda)} = (f(v))_\lambda \otimes f(v_\lambda), \quad \text{for any } v \in V.
\]

We complete the structure of the category YD\(\alpha\)(\(h\)) by defining the composition of morphisms of YD\(\alpha\)-modules via the standard composition of the underlying linear maps, i.e., by requiring that the forgetful functor YD\(\alpha\)(\(h\)) \to Rep\(\alpha\)(\(h\)): \((V, \Delta_V) \to V\) is faithful.

Let YD\((H)\) be the disjoint union of the categories YD\(\alpha\)(\(h\)) for all \( \alpha \in \pi \). The category YD\((H)\) admits a structure of braided T-category as follows.

- The tensor product of a YD\(\alpha\)-module \((V, \Delta_V)\) and a YD\(\beta\)-module \((W, \Delta_W)\) (with \( \alpha, \beta \in \pi \)) is the YD\(\alpha\)\(\beta\)-module \((V \otimes W, \Delta_{V \otimes W})\), where, for any \( v \in V, w \in W \), \( \lambda \in \pi \), we have

\[
\Delta_{V \otimes W,\lambda}(v \otimes w) = v_\lambda \otimes w_\lambda \otimes \varphi_{\beta^{-1}}(v_\lambda w_\beta^{-1}).
\]

The tensor unit of YD\((H)\) is the couple \( \mathbb{I}_{YD} = (k, \Delta_k) \), where, for any \( \lambda \in \pi \) and \( k \in k \), we have \( \Delta_{V,\lambda}(k) = k \otimes 1_\lambda \).

Finally, the tensor product of arrows is given by the tensor product of \( k \)-linear maps, i.e., by requiring that the forgetful functor YD\((H)\) \to Rep\((H)\): \((V, \Delta_V) \to V\) is a tensor functor.

- Given \( \beta \in \pi \), the conjugation functor \( \beta(-) \) is obtained as follows. Let \( a \) be in \( \pi \) and let \((V, \Delta_V)\) be a YD\(\alpha\)-module. We set \( \beta(V, \Delta_V) = \left(\beta V, \Delta_{\alpha\beta}\right) \), where, for any \( \lambda \in \pi \) and \( w \in \beta V \),

\[
\Delta_{\alpha\beta}(w) = \left(\beta^{-1} \left(\left(\beta^{-1} w\right)_\lambda \right)\right) \otimes \varphi_{\beta}(\left(\beta^{-1} w\right)_{\beta^{-1}\lambda\beta}).
\]

Given a morphism \( f: (V, \Delta_V) \to (W, \Delta_W) \) of YD-modules, for any \( v \in V \), we set \( \left(\beta f\right)(v_\lambda) = \beta(f(v))_\lambda \), i.e., we require that the forgetful functor from YD\((H)\) \to Rep\((H)\) is a T-functor.

- The braiding \( c \) is obtained by setting, for any YD\(\alpha\)-module \((V, \Delta_V)\), any YD\(\beta\)-module \((W, \Delta_W)\), and any \( v \in V \) and \( w \in W \),

\[
c_{(V, \Delta_V), (W, \Delta_W)}(v \otimes w) = \alpha_\beta^{-1}(v_\lambda w_\beta^{-1}) \otimes v_\lambda.
\]

To prove that YD\((H)\) is a T-category and that it is braided, we prove before that YD\((H)\) is isomorphic to \( Z(\text{Rep}(H)) \) as a category.

**Theorem 8.1.** The category YD\((H)\) is isomorphic to the category \( Z(\text{Rep}(H)) \). This isomorphism induces on YD\((H)\) the structure of crossed T-category described above.

Firstly, we construct two functors \( F_1: Z(\text{Rep}(H)) \to YD(H) \), and \( \tilde{F}_1: YD(H) \to Z(\text{Rep}(H)) \). Then we prove \( F_1 \circ \tilde{F}_1 = \text{Id}_{YD(H)} \) and \( \tilde{F}_1 \circ F_1 = \text{Id}_{Z(\text{Rep}(H))} \). Via this isomorphism, YD\((H)\) becomes a braided T-category. We complete the proof of Theorem 8.1 by proving that this structure of T-category is the structure described above.
The functor $F_1$. Let $\alpha$ be in $\pi$ and let $(V, c_V)$ be an object in $Z_\alpha(\text{Rep}(H))$. For any $\lambda \in \pi$, we set

$$\Delta_{V, \lambda}(v) = c^{-1}_{H_\lambda}\left(\alpha_\lambda \otimes v\right).$$

**Lemma 8.2.** The couple $(V, \Delta_V = \{\Delta_{V, \lambda}\}_{\lambda \in \pi})$ is a $YD_\alpha$-module. In that way, we obtain a structure of $YD$-module for any object in the center of $\text{Rep}(H)$. With respect to this natural structure, any morphism in the center of $\text{Rep}(H)$ is also a morphism of $YD$-modules. By setting $F_1(V, \epsilon_\lambda) = (V, \Delta_V)$ and $F_1(f) = f$, we obtain a functor $F_1: Z(\text{Rep}(H)) \to YD(H)$.

To prove Lemma 8.2, we need some preliminary results.

**Remark 8.3.** Given $\lambda \in \pi$, the algebra $H_\lambda$ is a left module over itself via the action provided by the multiplication. Similarly, $H_{\alpha^{-1}\lambda}$ is a left module over itself. By definition (10) of the action of $H_{\alpha^{-1}\lambda}$ on the module $\alpha^{-1}H_\lambda$, the $k$-linear map

$$\hat{\phi}_\alpha : H_{\alpha^{-1}\lambda} \to \alpha^{-1}H_\lambda : h \mapsto \alpha^{-1}\left(\hat{\phi}_\alpha(h)\right) = h\left(\alpha^{-1}1_\lambda\right).$$

is $H_{\alpha^{-1}\beta\alpha}$-linear and so it is an isomorphism of $H_{\alpha^{-1}\beta\alpha}$-modules. Notice that $\alpha\left(\hat{\phi}_\alpha(h)\right) = \varphi_\alpha(h)$ and that, for any $\alpha_1, \alpha_2 \in \pi$, we have $\hat{\varphi}_{\alpha_1\alpha_2} = \left(\alpha^{-1}_1\hat{\phi}_{\alpha_1}\right) \circ \hat{\phi}_{\alpha_2}$.

Let $X$ be an $H_\lambda$-module (with $\lambda \in \pi$) and let $\hat{x} : H_\lambda \to X$ be the unique $H_\lambda$-linear map sending $1_\lambda$ to $x$. We set

$$\hat{x}^{(\alpha)} : H_{\alpha\lambda\alpha^{-1}} \overset{\hat{\phi}_{\alpha^{-1}}}{\longrightarrow} \alpha H_\lambda \overset{\alpha\hat{x}}{\longrightarrow} \alpha X.$$  

Since, for any $h \in H_{\alpha\lambda\alpha^{-1}}$, $\hat{x}^{(\alpha)}(h) = \alpha\left(\left(\hat{x} \circ \hat{\phi}_{\alpha^{-1}}\right)(h)\right) = \hat{x} h$, we have that $\hat{x}^{(\alpha)}$ is the unique $H_{\alpha\beta\alpha^{-1}}$-linear map sending $1_{\alpha\lambda\alpha^{-1}}$ to $x \in \alpha X$, i.e., $\hat{x}^{(\alpha)} = \alpha\hat{x}$.

**Lemma 8.4.** Let $V$ be a $YD_\alpha$-module. For any $v \in V$ and $x \in X$ we have

$$c_X^{-1}(y \otimes v) = v(y) \otimes c_\lambda^{-1}(x).$$

**Proof.** The proof follows by the commutativity of the diagram

$$\xymatrixrowsep{2pc} \xymatrixcolsep{2pc} \xymatrix{ V \ar[r]_{v \mapsto 1_{\alpha\lambda\alpha^{-1}} \otimes v} & H_{\alpha\lambda\alpha^{-1}} \otimes V \ar[r]_{\hat{x}^{(\alpha)} \otimes V} & (\alpha H_\lambda) \otimes V \ar[r]_{\alpha \hat{x}} & \alpha X \otimes V \ar[r]_{c_X^{-1}} & (\alpha X) \otimes V.}$$

for $x = \alpha^{-1}y$. The top triangle commutes by definition of $\Delta_{V, \lambda}$. The bottom triangle commutes by definition of $\hat{x}^{(\alpha)}$. The square commutes because $c_\lambda$ is an isomorphism of functors. \hfill $\Box$

**Proof of Lemma 8.2.** We check that $(V, \Delta_V)$ satisfies the axioms of $YD_\alpha$-module, then we conclude the proof of Lemma 8.2 with the part concerning morphisms.

**Coassociativity.** Let $X_1$ be a $H_{\lambda_1}$-module and let $X_2$ be a $H_{\lambda_2}$-module, with $\lambda_1, \lambda_2 \in \pi$. By (12a), we have $c_{X_1 \otimes X_2} = c_{X_1}^{-1} \otimes c_{X_2}^{-1} \circ (\alpha_{X_1} \otimes \alpha_{X_2})$, so, for any $v \in V$, $x_1 \in X_1$ and $x_2 \in X_2$, we get $v(y) \otimes (v(x_1 \otimes x_2))_{(\lambda_1)}^{-1} \otimes (v(x_1 \otimes x_2))_{(\lambda_2)}^{-1} = v(y) \otimes (v(x_1 \otimes x_2))_{(\lambda_1)}^{-1} \otimes (v(x_1 \otimes x_2))_{(\lambda_2)}^{-1}$. The rest of the proof follows by the commutativity of the diagram.
For any \( V, \lambda \) and \( (\lambda, \lambda) \mod \), if we evaluate this formula for \( X_1 = H_{\lambda_1}, X_2 = H_{\lambda_2}, x_1 = \alpha_1_{\lambda_1}, \) and \( x_2 = \alpha_1_{\lambda_2}, \) then we get
\[
(v_{(V)}(X_1) \otimes (v_{(V)})_{(lambda)}) x_1 \otimes (v_{(lambda)} x_2).
\]
We can now apply the counitary property. Since we have \( v_{(V)}(X_1) \otimes (v_{(V)})_{(lambda)} = v \otimes 1 \), we get \( \varepsilon(v_{(lambda)} x) = \varepsilon(v) = v = 1 \).

**Counit.** Let \( X \) be a \( H\lambda \)-module. For any \( v \in V \) and \( x \in X \)
we have \( h_{X}^{-1}(\alpha x) \otimes v = \Delta_{\lambda}(h)(v_{(V)} \otimes (v_{(V)})_{(lambda)}) x = (h'_{(\alpha v)} \otimes h_{(\alpha v)}^\mu x) \) and \( \chi_{X}^{-1}(\alpha x) \otimes v = \chi_{X}^{-1}(h'(\alpha v) \otimes h_{(\alpha v)}^\mu x). \) We can now apply the Counit property. Let \( \varphi \) be any \( H\lambda \)-module, so the crossing property (16c) follows by the \( H\lambda \)-linearity of \( \chi_{X}^{-1} \). This completes the proof that \( (V, \Delta, v) \) is a \( YD\alpha \)-module.

**Morphisms.** Let \((\hat{W}, \hat{\alpha}) \) be another object in \( Z_{\alpha}(\text{Rep}(H)) \). Define \( \hat{\Delta} \) as above for \( \Delta \). Given any \( f: V \to W \) in \( Z(\text{Rep}(H)) \), we prove that \( f \) gives rise to a morphism of \( YD\alpha \)-modules, i.e., that (18) is satisfied. By the commutativity of \( \hat{\Delta} \hat{\Delta} \lambda \) (\( (\alpha H\lambda) \otimes f) = (f \otimes H\lambda) \circ \hat{\Delta} \lambda \) for any \( f \in \text{Rep}(H) \), we have \( ((f \otimes H\lambda) \circ \Delta \lambda)(v) = (f \otimes H\lambda) \circ \hat{\Delta} \lambda(v) = (f \otimes 1_H)(\alpha H\lambda(v)). \) We can now apply the Counit property. Let \( \varphi \) be any \( H\lambda \)-module, so the crossing property (16c) follows by the \( H\lambda \)-linearity of \( \chi_{X}^{-1} \). This completes the proof that \( (V, \Delta, v) \) is a \( YD\alpha \)-module.

The proof that \( \hat{F}_1 \) is a functor is now trivial.

**The functor \( \hat{F}_1 \).** Let \( (V, \Delta, v) \) be any \( YD\alpha \)-module. Given \( \lambda \in \pi \), for any representation \( X \) of \( H\lambda \) set
\[
\chi_{X}^{-1}(\alpha x) = v_{(V)} \otimes (v_{(V)})_{(lambda)}(\alpha^{-1} y).
\]

*Lemma 8.5.* The couple \( (V, \chi) \) is an object in \( Z(\text{Rep}(H)) \). In particular,
\[
\chi_{X}^{-1}(\alpha x) = v_{(V)} \otimes (v_{(V)})_{(lambda)}(\alpha^{-1} y)
\]
for any \( y \in \alpha X \) and \( v \in V \). With respect to this natural structure, any morphism of \( YD\alpha \)-modules gives rise to an arrow in \( Z(\text{Rep}(H)) \). By setting \( \hat{F}_1(V, \Delta) = (V, \chi) \) and \( \hat{F}_1(f) = f \), we obtain a functor from \( YD(H) \) to \( Z(\text{Rep}(H)) \). The functors \( \hat{F}_1 \) and \( \hat{F}_1 \) are mutually inverses.

To prove Lemma 8.5, we need another preliminary lemma.

*Lemma 8.6.* For any \( v \in V \) we have \( (v_{(V)})_{(lambda)}(s_{(\lambda-1)}(v_{(lambda)})) = v \otimes 1 \lambda \)
and \( (v_{(V)})_{(lambda)}(s_{(\lambda-1)}(v_{(lambda)})) = v \otimes 1 \lambda \).

*Proof.* Since \( \Delta \) is counitary, the proof follows by the commutativity of the two squares on the right-hand side are commutative because of the coassociativity of \( \Delta \), while the two squares on the left-hand side are commutative because \( s \) is the antipode of a T-coalgebra.
Proof of Lemma \textsc{X.1}. Let us check that \((V, \zeta)\) is an object in \(Z_{\alpha}(\text{Rep}(H))\).

**Invertibility.** Let \(X\) be a representation of \(H_\lambda\), with \(\lambda \in \pi\). We set
\[
\hat{c}_X : (\alpha X) \otimes V \to V \otimes X : y \otimes v \mapsto v(\lambda) y \otimes v(V) \cdot (\alpha^{-1} y).
\]
Let us prove that \(\hat{c}_X\) is the inverse of \(c_X\). For any \(v \in V\) and \(x \in X\) we have
\[
v \otimes x \xrightarrow{\hat{c}_X} \left(\alpha (s_{\lambda^{-1}}(v(\lambda^{-1})x))\right) \otimes v(V) \xrightarrow{c_X} \left((v(V)) \otimes (v(V)) \cdot (\lambda^{-1}) v(\lambda^{-1}) \right) x = v \otimes x
\]
where the last passage follows by Lemma \textsc{X.0}. Similarly, for any \(v \in V\) and \(y \in \alpha X\) we have
\[
y \otimes v \xleftarrow{\hat{c}_X} v(V) \otimes v(\lambda) \left(\alpha^{-1} y\right) \xleftarrow{c_X} s_{\lambda^{-1}} ((v(V)) \cdot (\alpha^{-1}) v(\lambda)) \otimes v(V) = y \otimes \varepsilon(v(\lambda)) v(V) = y \otimes v.
\]

**Linearity.** Let \(X\) be a representation of \(H_\lambda\), with \(\lambda \in \pi\). It is a bit easier to prove that \(\hat{c}_X\) (instead of \(c_X\)) is \(H_{\alpha \lambda}\)-linear. For any \(v \in V\), \(y \in \alpha^{-1} X\), and \(h \in H_{\alpha \lambda}\), we have
\[
h \cdot (\hat{c}_X v(y \otimes v)) = h \left(v(V) \otimes v(\lambda) \left(\alpha^{-1} y\right)\right) = h'(\alpha) v(V) \otimes h''(\lambda) v(\lambda) \left(\alpha^{-1} y\right)
\]
and
\[
\hat{c}_X \left(h(v(y \otimes v))\right) = \hat{c}_X \left(h'(\alpha \lambda^{-1}) v(y) \otimes h''(\lambda) v(\lambda)\right) = (h'(\alpha) v(V)) \otimes (h''(\lambda) v(\lambda) \varphi \alpha^{-1} h'(\lambda)(\alpha \lambda^{-1})) v(\lambda).
\]
By the crossing property \textbf{[L22]} of \((V, \Delta_V)\), these two expressions are equal.

**Naturality.** Let us check that \(\zeta\) is a natural transformation from the functor \(V \otimes \cdot\) to the functor \(V(\lambda) \otimes V\). Given two representations \(X_1\) and \(X_2\) of \(H_\lambda\) and a \(H_{\alpha \lambda}\)-linear map \(f : X_1 \to X_2\), for any \(v \in V\) and \(x \in X_1\) we have
\[
\left((\alpha f) \otimes V \right) \circ c_{X_1} (v \otimes x) = \left((\alpha f) \otimes V\right) \left((\alpha s_{\lambda^{-1}}(v(\lambda^{-1})x))\right) \otimes v(V) = \left((\alpha s_{\lambda^{-1}}(v(\lambda^{-1})f(x)))\right) \otimes v(V) = \left(c_{X_2} \circ (V \otimes f)\right) (v \otimes x).
\]

**Half-braiding axiom.** We still have to check that \((V, \zeta)\) satisfies the half-braiding axiom \textbf{[L23]}. Let \(X_1\) be a \(H_{\lambda_1}\)-module and let \(X_2\) be a \(H_{\lambda_2}\)-module, with \(\lambda_1, \lambda_2 \in \pi\). We want
\[
c_{X_1 \otimes X_2} (v \otimes x_1 \otimes x_2) = \left((\alpha X_1) \otimes c_{X_2} \right) \circ (c_{X_1} \otimes X_2) (v \otimes x_1 \otimes x_2).
\]
for any \(x_1 \in X_1\), \(x_2 \in X_2\), and \(v \in V\). We have
\[
c_{V, X \otimes X_1} (v \otimes x_1 \otimes x_2) = \left((\alpha s_{\lambda^{-1}}((v((\lambda_1 \lambda_2)^{-1})x_1))\right) \otimes
\]
\[
\otimes \left((\alpha s_{\lambda_2}^{-1}((v((\lambda_1 \lambda_2)^{-1})x_2))\right) \otimes v(V)
\]
and
\[
\left((\alpha X_1) \otimes c_{X_2} \right) \circ (c_{X_1} \otimes X_2) (v \otimes x_1 \otimes x_2) = \left((\alpha s_{\lambda^{-1}}(v(\lambda_1^{-1})x_1))\right) \otimes
\]
\[
\otimes \left((\alpha s_{\lambda_2}^{-1}((v(V))_{(\lambda_1^{-1})x_2}))\right) \otimes (v(V)) v(V).
\]
By the coassociativity of \(\Delta_V\) these two expressions are equal.
This concludes the proof that \((V, \zeta)\) is an object in \(Z_{\alpha}(\text{Rep}(H))\).

**Morphisms.** Let \((W, \zeta')\) be another object in \(Z_{\alpha}(\text{Rep}(H))\). Define \(\Delta_W\) as for \(\Delta_V\) above. Given \(f : V \to W\) in \(Z(\text{Rep}(H))\), we prove that \(f\) gives rise to a morphism of \(YD_\alpha\)-modules. Let \(X\) be a \(H_\lambda\)-module, with \(\lambda \in \pi\). Given \(v \in \alpha X\), the map \(\hat{c}_X \circ (v(V) \otimes f(v(V))) \otimes (v(V)) v(V) = (f(v(V))) v(V)
\]

\[
\left((\alpha X_1) \otimes c_{X_2} \right) \circ (c_{X_1} \otimes X_2) (v \otimes x_1 \otimes x_2) = \left((\alpha s_{\lambda^{-1}}(v(\lambda_1^{-1})x_1))\right) \otimes
\]
\[
\otimes \left((\alpha s_{\lambda_2}^{-1}((v(V))_{(\lambda_1^{-1})x_2}))\right) \otimes (v(V)) v(V).
\]
By the coassociativity of \(\Delta_V\) these two expressions are equal.
This concludes the proof that \((V, \zeta)\) is an object in \(Z_{\alpha}(\text{Rep}(H))\).
$V$ and $x \in X$ we have $(\epsilon_W \circ (f \otimes X))(v \otimes x) = \left( \alpha (s_{\lambda^{-1}}(v_{(\lambda^{-1})}x) \otimes f(v_{(\lambda)}) = \left( (\alpha X) \otimes f \right) \left( (\alpha s_{\lambda^{-1}}(v_{(\lambda^{-1})}x) \otimes f(v_{(\lambda)}) \right) \otimes v_{(\lambda)}} = \left( (\alpha X) \otimes f \right) \circ \epsilon_W(v \otimes x), \text{ where in the second passage we used } [18].$

The proof that $\hat{F}_1$ is a functor is now trivial. We still have to check that $F_1$ and $\hat{F}_1$ are mutually inverse.

**Isomorphism.** Let us prove that $\hat{F}_1 \circ F_1 = \text{Id}_{Z(\text{Rep}(H))}$. Let $(V, \epsilon_v)$ be an object in $Z(\text{Rep}(H))$, with $\alpha \in \pi$. We have $F_1(V, \epsilon_v) = (V, \Delta_v)$, where, for any $\lambda \in \pi$, $v_{(\lambda)} \otimes v_{(\lambda)} = c_{H_\lambda}^{-1} \left( (\alpha 1_\lambda) \otimes v \right)$. We set $(V, \hat{\epsilon}_v) = (\hat{F}_1 \circ F_1)(V, \epsilon_v) = \hat{F}_1(V, \Delta_v)$. Given any $H_\lambda$-module $X$, with $\lambda \in \pi$, for any $v \in V$ and $x \in X$ we have $c_X^{-1} \left( (\alpha x) \otimes v \right) = v_{(\lambda)} \otimes v_{(\lambda)} x = c_{H_\lambda}^{-1} \left( (\alpha 1_\lambda) \otimes v \right) x = c_X^{-1} \left( (\alpha x) \otimes v \right)$, where the last passage follows by the commutativity of the square in diagram (22).

Let us prove that $F_1 \circ \hat{F}_1 = \text{Id}_{YD(H)}$. Given $\alpha \in \pi$ and a $YD_\alpha$-module $X$, we have $F_1(V, \Delta_v) = (V, \epsilon_v)$, where, for any representation $X$ of $H_\lambda$ (with $\lambda \in \pi$) and for any $v \in V$ and $x \in X$, we have $\epsilon_v(v \otimes x) = \left( (\alpha s_{\lambda^{-1}}(v_{(\lambda^{-1})}x) \otimes v(v) \right)$. If we set $(V, \hat{\Delta}_v) = (F_1 \circ \hat{F}_1)(V, \Delta_v) = (F_1(V, \epsilon_v)$, then we obtain $\hat{\Delta}_v(v) = c_{H_\lambda}^{-1} \left( (\alpha 1_\lambda) \otimes v \right) = v_{(\lambda)} \otimes v_{(\lambda)} 1_\lambda = v_{(\lambda)} \otimes v_{(\lambda)} = \Delta_\lambda(v)$, where the second passage follows by [8,5].

This concludes the proof of Lemma 8.5. □

**Proof of Theorem 8.4.** By Lemma 8.5 the categories $Z(\text{Rep}(H))$ and $YD(H)$ are isomorphic via the functor $F_1$ and the functor $\hat{F}_1$. This isomorphism induces on $YD(H)$ a structure of a strict $T$-category.

**Components.** Let $\alpha$ be in $\pi$. Since $YD_\alpha(H) = (F_1 \circ \hat{F}_1)(YD_\alpha(H))$, the $\alpha$-th component of $YD(H)$ is $YD_\alpha(H)$.

**Tensor Category Structure.** Let $(V, \Delta_v)$ be a $YD_\alpha$-module and let $(W, \Delta_w)$ be a $YD_\beta$-module. Suppose $(V, \epsilon_v) = \hat{F}_1(V, \Delta_v)$ and $(W, \epsilon_w) = \hat{F}_1(W, \Delta_w)$. We set $(V, \Delta_v) \otimes (W, \Delta_w) = F_1(\hat{F}_1(V, \Delta_v) \otimes \hat{F}_1(W, \Delta_w)) = F_1(V \otimes V', (\epsilon \otimes \delta)_\omega)$. Since, for any $v \in V$, $(\epsilon_{H_\lambda})^{-1} \left( (\alpha \otimes 1_\lambda) \otimes v \right) = v_{(\lambda)} \otimes \varphi_{\beta^{-1}}(\varphi_{\beta^{-1}}(v_{(\beta \lambda \beta^{-1})})$, we obtain $\Delta_v \otimes w_\lambda (v \otimes w) = ((\epsilon \otimes \epsilon_{H_\lambda})^{-1} \left( (\alpha \otimes 1_\lambda) \otimes v \otimes w \right) = ((\epsilon \otimes \delta_\omega)^{-1} (V \otimes (\otimes H_\lambda)^{-1} \otimes W) \left( (\alpha \otimes 1_\lambda) \otimes v \otimes w \right) = v_{(\lambda)} \otimes w_{(\lambda)} \otimes w_{(\lambda)} \varphi_{\beta^{-1}}(v_{(\beta \lambda \beta^{-1})}, \text{ for any } v \in V \text{ and } w \in W$. The part concerning the tensor unit of $YD(H)$ is trivial.

**Conjugation.** The T-category structure of $YD(H)$ is completed by setting, for any $\beta \in \pi$, $\delta(\cdot) = \left( YD(H) \xrightarrow{\hat{F}_1} Z(\text{Rep}(H)) \xrightarrow{\delta(\cdot)} Z(\text{Rep}(H)) \xrightarrow{F_1} YD(H) \right)$. In particular, given $\alpha \in \pi$ and a $YD_\alpha$-module $(V, \Delta_v)$, if $(V, \epsilon_v) = \hat{F}_1(V, \Delta_v)$, then,
for any \( \lambda \in \pi \) and \( v \in V \), we get \( \Delta_{\nu;\lambda}(\beta v) = (\beta \epsilon)_H \left( \left( \beta_{\alpha\beta^{-1}\lambda} \right) \otimes (\beta v) \right) = \beta(v(V) \otimes \phi_\beta(v(\beta^{-1}\lambda \beta))) = \beta(v(V)) \otimes v(\beta^{-1}\lambda \beta). \) By setting \( w = \beta v \), we get (19b).

**Braiding.** The braiding in \( YD(H) \) is obtained by setting \( c(v,\Delta v),(W,\Delta W) = F_1(\xi_1(V,\Delta v),\xi_1(W,\Delta W)) = c_W \), for any \( (V,\Delta V),(W,\Delta W) \in YD(H) \), where \( (V,\xi_1) = F_1(V,\Delta V) \). By definition (22) of \( \varphi_\xi \), we get (19c).

This concludes the proof of the theorem. \( \square \)

9. The \( T \)-coalgebra \( \overline{D}(H) \) and its representations

We define a quasitriangular \( T \)-coalgebra \( \overline{D}(H) \). This \( T \)-coalgebra is the mirror of the \( T \)-coalgebra \( D(H) \) defined in [21]. To do this, we need to introduce another \( T \)-coalgebra \( H^\ast \) (the mirror of \( H^\ast \)-module \( \overline{D}(H) \)) defined in [21]. Then, we discuss the structure of a module over \( \overline{D}(H) \). More in detail, we prove that a \( k \)-vector space \( V \) is a \( \overline{D}(H) \)-module if and only if it is both a \( H \)-module and a \( H^\ast \)-module and the actions of \( H \) and \( H^\ast \) satisfy a compatibility condition. Finally, we prove that \( \mathcal{Z}(\text{Rep}(H)) \) and \( \text{Rep}(\overline{D}(H)) \) are isomorphic as braided \( T \)-categories.

**Definition of \( H^\ast \).** The \( T \)-coalgebra \( H^\ast \) is defined as follows.

- For any \( \alpha \in \pi \), the component \( H^\ast_\alpha \) is equal to \( \bigoplus_{\beta \in \pi} H^\ast_\beta \) as a vector space, with the product of \( f \in H^\ast_\gamma \) and \( g \in H^\ast_\delta \) (with \( \gamma, \delta \in \pi \)) given by the linear map \( fg \in H^\ast_{\gamma\delta} \) defined by \( \langle f, x \rangle = \langle f, x' \rangle \langle g, x'' \rangle \) for any \( x \in H_{\gamma\delta} \). The unit of \( H_\alpha \) is the morphism \( \varepsilon \in H^\ast_1 \subset H^\ast_\pi \).
- The comultiplication \( \Delta^\ast \) is defined by setting, for any \( \alpha, \beta, \gamma \in \pi \) and \( f \in H^\ast_\gamma \), \( \Delta^\ast_\alpha\beta(f) = \Delta_\beta(f) \in H^\ast_{\gamma\beta^{-1}} \otimes H^\ast_\beta \), where \( \langle \Delta_\beta(f), x \otimes y \rangle = \langle f, y\varphi_{\beta^{-1}}(x) \rangle \) for any \( x \in H_{\beta\gamma^{-1}} \). We introduce the notation \( f_{\beta,\gamma} \otimes f_{\beta,\gamma^{-1}} = \Delta_\beta(f) \).

The counit \( \varepsilon^\ast : H^\ast_1 \rightarrow k \) is given by \( \langle \varepsilon^\ast, f \rangle = f, 1 \rangle \), for any \( f \in H^\ast_1 \), with \( \gamma \in \pi \).

- For any \( \alpha \in \pi \), the component \( s^\ast_\alpha \) of the antipode \( s^\ast \) of \( H^\ast \) sends \( f \in H_\gamma \) to \( s^\ast_\alpha(f) = (f, x^\ast \varphi_{\alpha^{-1}}(x^\ast_{\gamma\alpha^{-1}})) \in H^\ast_{\gamma^{-1}\alpha^{-1}} \).
- Finally, for any \( \beta \in \pi \), we set \( \varphi_\beta^\ast = \varphi_{\beta^{-1}} \).

**The \( T \)-coalgebra \( \overline{D}(H) \).** The \( T \)-coalgebra \( \overline{D}(H) \) is defined as follows (see the detailed description of the mirror \( D(H) \) of \( \overline{D}(H) \) in [21]).

- For any \( \alpha \in \pi \), the \( \alpha \)-th component of \( \overline{D}(H) \), denoted \( \overline{D}_\alpha(H) \), is equal to \( H_\alpha \otimes \bigoplus_{\beta \in \pi} H^\ast_\beta \) as a vector space. Given \( h \in H_\alpha \) and \( F \in \bigoplus_{\beta \in \pi} H^\ast_\beta \), the element in \( \overline{D}_\alpha(H) \) corresponding to \( h \otimes F \) is denoted \( h \oplus F \). The product in \( \overline{D}_\alpha(H) \) is given by

\[
(h \oplus f)(k \oplus g) = h''_\alpha k \oplus f(g, s^{-1}_\delta(h'''_{(\delta-1)}) \omega_{\alpha^{-1}}(h'_{(\alpha\delta^{-1})}))
\]

for any \( h, k \in H_\alpha \), \( f \in H^\ast_\alpha \), and \( g \in H^\ast_\delta \), with \( \gamma, \delta \in \pi \). \( H_\alpha \) has unit \( 1_\alpha \otimes \varepsilon \).

The algebra structure of \( D_\alpha(H) \) is uniquely defined by the condition that the inclusions \( H_\alpha, H^\ast_\alpha \rightarrow D_\alpha(H) \) are algebra morphisms and that the relations

\[
(24a) \quad (1_\alpha \otimes f)(h \oplus \varepsilon) = h \oplus f
\]

and

\[
(24b) \quad (h \oplus \varepsilon)(1_\alpha \otimes f) = h''_{(\alpha)} \oplus \langle f, s^{-1}_\gamma(h'''_{(\gamma^{-1})}) \omega_{\alpha^{-1}}(h'_{(\alpha\gamma^{-1})}) \rangle,
\]

(for any \( h \in H_\alpha \) and \( f \in H^\ast_\alpha \), with \( \gamma \in \pi \)) are satisfied.
Theorem 9.1. \( \text{Rep}_\alpha(H, H^\ast, \otimes) \) is isomorphic to \( \text{Rep}\left( \overline{T}(H) \right) \) as a braided T-category.

Proof. The simplest way to prove the theorem is to construct an isomorphism of categories \( F_3 : \text{Rep}\left( \overline{T}(H) \right) \rightarrow \text{Rep}_\alpha(H, H^\ast, \otimes) \) such that \( F_3 \) induces on \( \text{Rep}\left( \overline{T}(H) \right) \) the structure of braided T-category described above.

Let \( V \) be a \( D_\alpha(H) \)-module, with \( \alpha \in \pi \). Since both \( H_\alpha \) and \( H^\ast \) can be identified with subalgebras of \( D_\alpha(H) \) via the canonical embeddings, \( V \) has both a natural structure of left \( H_\alpha \)-module and a natural structure of left \( H^\ast \)-module. Explicitly, for any \( v \in V, h \in H_\alpha \), and \( f \in H^\ast \), with \( \gamma \in \pi \), we set

\[
\begin{align*}
hv &= (h \otimes 1) v \\
v \triangleright f &= (1_\alpha \otimes f) v.
\end{align*}
\]

\[\begin{align*}
(h \otimes F_\alpha^{\beta}(u)) \otimes (h \otimes F_\beta^{\gamma}(v)) &= (h^{\alpha}_{\beta}(u) \otimes F_{1,\beta}) \otimes (h^{\gamma}_{\beta}(v) \otimes F_{1,\beta}),
\end{align*}\]

for any \( \alpha, \beta \in \pi \), \( h \in H_{\alpha\beta} \) and \( F \in H^\ast \). The counit is given by \( (\varepsilon, h \otimes f) = (\varepsilon, f)(1, 1) \) for any \( h \in 1 \) and \( f \in H^\ast \), with \( \gamma \in \pi \).

The antipode is given by \( S_\alpha(h \otimes f) = (s_\alpha(h) \otimes \varepsilon) (1_\alpha \otimes s^\ast_\alpha(f)) \), for any \( \alpha \in \pi \), \( h \in H_\alpha \), and \( f \in H^\ast \).

The conjugation is obtained in the obvious way by the conjugation of \( H \), \( H^\ast \), and \( (e, i) \) dual basis. The inverse of \( R_{\alpha,\beta} = R_{\alpha,\beta} \) is \( \xi(\alpha,i) \otimes \xi(\beta,i) = 1_\alpha \otimes e^{\beta - 1,i} \otimes \varepsilon \).

The category \( \text{Rep}(H, H^\ast, \otimes) \) of \( (H, H^\ast, \otimes)_\alpha \)-module is a \( k \)-vector space \( V \) endowed with both a structure of left module over \( H_\alpha \) and a structure of left \( H^\ast \)-module over \( H^\ast_\alpha = H^\ast_1 \) (via an action denoted \( \triangleright \)) satisfying the compatibility condition

\[
h(f \triangleright v) = \langle f, s^{-1}_{\gamma_1}(h^{\gamma_1}_{\gamma_1 - 1}) \phi_{\alpha_1}^{-1}(h^{\gamma_1}_{\gamma_1 \alpha_1 - 1}) \rangle \triangleright (h_{\alpha}^{\alpha} v).
\]

for any \( \alpha \in \pi \), \( h \in H_\alpha \), and \( f \in H^\ast_\gamma \), with \( \gamma \in \pi \). A morphism of \( (H, H^\ast, \otimes)_\alpha \)-modules is a morphisms that is both a morphism of \( H_\alpha \)-modules and a morphism of \( H^\ast_\alpha \)-modules. In that way, with the obvious composition, we obtain the category \( \text{Rep}_\alpha(H, H^\ast, \otimes) \) of \( (H, H^\ast, \otimes)_\alpha \)-modules. The disjoint union \( \text{Rep}(H, H^\ast, \otimes) \) of \( \text{Rep}_\alpha(H, H^\ast, \otimes) \) for all \( \alpha \in \pi \) is a braided T-category as follows.

• \( \text{Rep}_\alpha(H, H^\ast, \otimes) \) is the \( \alpha \)-th component of \( \text{Rep}(H, H^\ast, \otimes) \).

• Given \( \alpha, \beta \in \pi \), let \( U \) be an object in \( \text{Rep}_\alpha(H, H^\ast, \otimes) \) and let \( V \) be an object in \( \text{Rep}_\beta(H, H^\ast, \otimes) \). The tensor product \( U \otimes V \) of \( (H, H^\ast, \otimes)_\alpha \)-modules is given by the tensor product of \( U \) and \( V \) as both \( H_\alpha \)-modules and \( H^\ast_\beta \)-modules, i.e., given, \( u \in U \) and \( v \in V \), the action of \( h \in H_{\alpha\beta} \) and, respectively, \( f \in H^\ast_\gamma \) (with \( \gamma \in \pi \)) on \( u \otimes v \) given by \( h(u \otimes v) = h_{\alpha}(u) \otimes h_{\beta}(v) \) and \( f \triangleright (u \otimes v) = f_{1,\beta} \triangleright u \otimes f_{1,\beta} \triangleright v \).

• The conjugation is obtained in the obvious way by the conjugation of \( \text{Rep}(H) \) and the conjugation of \( \text{Rep}(H^\ast) \).

• The braiding is obtained by setting,

\[
c_{U,V} : U \otimes V \rightarrow \left( \alpha V \right) \otimes \alpha : u \otimes v \mapsto \alpha \left( s_{\beta - 1}(e_{\beta - 1,i}) v \right) \otimes e_{\beta - 1,i} \triangleright u
\]

for any \( U \in \text{Rep}_\alpha(H, H^\ast, \otimes) \) and \( V \in \text{Rep}_\beta(H, H^\ast, \otimes) \), with \( \alpha, \beta \in \pi \).
Let us prove that the compatibility condition (22) is satisfied. By the associativity of the action of $\overline{D}(H)$ on $V$ and by (24) we get $h(f \triangleright v) = (h \circ \varepsilon)\left((1_\alpha \oplus f) v\right)$ $= ((h \circ \varepsilon)(1_\alpha \oplus f)) v = \left(h''_{(\alpha)} \otimes \langle f, s_{\gamma^{-1}}(h''_{(\gamma^{-1})}) \triangleright \varphi_{\alpha^{-1}}(h''_{(\alpha \gamma^{-1})}) \rangle v\right)$ $= (1_\alpha \oplus \langle f, s_{\gamma^{-1}}(h''_{(\gamma^{-1})}) \triangleright \varphi_{\alpha^{-1}}(h''_{(\alpha \gamma^{-1})}) \rangle v) = (1_\alpha \oplus \langle f, s_{\gamma^{-1}}(h''_{(\gamma^{-1})}) \triangleright \varphi_{\alpha^{-1}}(h''_{(\alpha \gamma^{-1})}) \rangle \otimes \varepsilon) v = \langle f, s_{\gamma^{-1}}(h''_{(\gamma^{-1})}) \triangleright \varphi_{\alpha^{-1}}(h''_{(\alpha \gamma^{-1})}) \rangle \triangleright (h''_{(\alpha)} v)$. We set $F_3(V)$ equal to $V$ endowed with the structure of $(H, H^\bullet, \otimes)_\alpha$-module described above.

Given another $\overline{D}_\alpha(H)$-module $V$ and a $k$-linear morphism $f : V \rightarrow W$, it is easy to prove that $f$ is a morphism of $\overline{D}_\alpha(H)$-modules if and only if it is both a morphism of $H_\alpha$-modules and a morphisms of $H^\bullet_\alpha$-modules. By setting $F_3(f) = f$, we obviously obtained a functor.

Let us prove that $F_3$ is invertible. Given a $(H, H^\bullet, \otimes)_\alpha$-module $W$, we define an action of $\overline{D}_\alpha(H)$ on $W$ via the tensor lift of the linear map $H_\alpha \times H^\bullet_\alpha \times W \rightarrow W : (h, f, v) \rightarrow f \triangleright (hv)$, we have to prove that we obtained a $\overline{D}_\alpha(H)$-module. For any $h, k \in H_\alpha$, $f \in H^\bullet_\alpha$, and $g \in H^\bullet_\alpha$, with $\gamma, \delta \in \pi$, we have $\langle 1_\alpha \otimes \varepsilon \rangle v = (1_\alpha \otimes \varepsilon) v = \varepsilon \triangleright v = v$ and $(h \otimes f)(k \otimes g) v = (h \otimes f)(g \otimes (kv)) = f \triangleright (h(g \otimes (kv))) = f \triangleright (g, s_{\delta^{-1}}(h''_{(\delta^{-1})}) \triangleright \varphi_{\alpha^{-1}}(h''_{(\alpha \delta^{-1})}) \triangleright (h''_{(\alpha)} k v) = \left(h_{(\alpha)} k \otimes \langle f, s_{\gamma^{-1}}(h''_{(\gamma^{-1})}) \triangleright \varphi_{\alpha^{-1}}(h''_{(\alpha \gamma^{-1})}) \rangle v\right) = ((h \oplus f)(k \otimes g)) v$.

To prove that $F_3$ is invertible and to complete the proof of the theorem is now trivial.

\[10. \quad \mathcal{Z}\left(\text{Rep}(H)\right) \text{ and } \text{Rep}\left(\overline{D}(H)\right) \text{ are isomorphic} \]

In this section we prove that $\mathcal{Z}(\text{Rep}(H))$ and $\text{Rep}\left(\overline{D}(H)\right)$ are isomorphic as braided $\mathbb{T}$-categories. We start by defining a braided $\mathbb{T}$-functor $F_2 : YD(H) \rightarrow \text{Rep}(H, H^\bullet, \otimes)$. After that, we set $G = F_3 \circ F_2 \circ F_1 : \mathcal{Z}(\text{Rep}(H)) \rightarrow \text{Rep}(\overline{D}(H))$ and prove that $G$ is invertible.

**Theorem 10.1.** $\mathcal{Z}(\text{Rep}(H))$ and $\text{Rep}(\overline{D}(H))$ are isomorphic braided $\mathbb{T}$-categories.

**The functor $F_2.$** To prove Theorem 10.1, we start by constructing the functor $F_2$. For this, we need two preliminary lemmas.

**Lemma 10.2.** Let $(V, \Delta_V)$ be a $YD_\alpha$-module (with $\alpha \in \pi$). Given $f \in H^\bullet_\alpha$, with $\gamma \in \pi$, for any $v \in V$ we set

\[ f \triangleright v = \langle f, v_{(\gamma)} \rangle v_{(V)}. \tag{27} \]

With this action, $V$ becomes a $H^\bullet_\alpha$-module and a $(H, H^\bullet, \otimes)_\alpha$-module.

**Proof.** Let us prove that the action $\triangleright$ is associative and unitary.

**Associativity.** Given $f \in H^\bullet_\gamma$ and $g \in H^\bullet_\delta$, with $\gamma, \delta \in \pi$, for any $v \in V$, we have $f \triangleright (g \triangleright v) = f \triangleright (g, v_{(\delta)}) v_{(V)} = \langle g, v_{(\delta)} \rangle \langle f, v_{(\gamma)} \rangle v_{(V)}$ and $f \triangleright (g \triangleright v) = (f, v_{(\gamma)}) v_{(V)} = \langle f, v_{(\gamma)} \rangle (g, v_{(\gamma)}) v_{(V)}$. By the coassociativity of a YD-module, these two expressions coincide.

**Unit.** By (16b), for any $v \in V$ we have $\varepsilon \triangleright v = \langle \varepsilon, v_{(1)} \rangle v_{(V)} = v$, i.e., $\triangleright$ is unitary.
Compatibility condition (22). Given $h \in H_\alpha$ and $f \in H_\gamma^\ast$, with $\gamma \in \pi$, for any $v \in V$, by using (16c) we get
\[
\begin{align*}
    h(f \triangleright v) &= \langle f, v(\gamma) \rangle h v(\gamma) = \langle f, v(\gamma) \rangle (\varepsilon, h''_\gamma) h'_\alpha v(\gamma) = \langle f, (\varepsilon, h''_\gamma) v(\gamma) \rangle h'_\alpha v(\gamma) \\
    &= \langle f, s^{-1}_\gamma (h''_\gamma) h'_\alpha v(\gamma) \rangle h'_\alpha v(\gamma) \\
    &= \langle f, s^{-1}_\gamma (h''_\gamma) h'_\alpha v(\gamma) \rangle v(\gamma) = \langle f, (s^{-1}_\gamma (h''_\gamma) h'_\alpha) v(\gamma) \rangle (h'_\alpha v(\gamma)) \\
    &\quad \text{(by the crossing property (16c))} \\
    &= \langle f, s^{-1}_\gamma (h''_\gamma) (h'_\alpha v(\gamma) \gamma_\alpha v(\gamma)) h'_\alpha v(\gamma) \rangle (h'_\alpha v(\gamma)) \\
    &= \langle f, s^{-1}_\gamma (h''_\gamma) (h'_\alpha v(\gamma)) v(\gamma) \rangle (h'_\alpha v(\gamma)) \\
    &= \langle f, s^{-1}_\gamma (h''_\gamma) (h'_\alpha v(\gamma)) v(\gamma) \rangle v(\gamma) = \langle f, (s^{-1}_\gamma (h''_\gamma) h'_\alpha) v(\gamma) \rangle (h'_\alpha v(\gamma)).
\end{align*}
\]

\[\Box\]

Lemma 10.3. Take two YD-modules $(V, \Delta_V)$ and $(W, \Delta_W)$ and define the action of $H_\ast^\pi$ on both $V$ and $W$ via (10.2). A morphism of YD-modules $f : V \to W$ is also a morphism of $(H, H_\ast^\pi, \otimes)$-modules.

Proof. We only need to show that $f$ preserves the action of $H_\ast^\pi$. Let $v \in V$ and $g \in H_\gamma^\ast$, with $\gamma \in \pi$. Since $f$ is a morphism of YD-modules, we have $g \triangleright f(v) = \langle g, v(\gamma) \rangle f(v) = \langle g, v(\gamma) \rangle f(v) = f(g \triangleright v) \triangleright v.$

\[\Box\]

Lemma 10.4. For any YD-module module $(V, \Delta_V)$, set $F_2(V, \Delta_V) = (V, \triangleright)$, with the action $\triangleright$ of $H_\ast^\pi$ on $V$ defined as in (10.2). For any morphism $f$ of YD-modules, set $F_2(f) = f$. In that way, we obtain a braided $T$-functor $F_2 : YD(H) \to \text{Rep}(H, H_\ast^\pi, \otimes)$.

Proof. By Lemma 10.2 and Lemma 10.3, $F_2$ is well defined. The proof that it is a functor (i.e., that preserves identities and composition), is trivial. We have to check that it is a tensor functor, that it commutes with the conjugation and that it is braided.

Tensor product. Given $\alpha, \beta \in \pi$, let $(V, \Delta_V)$ be a YD$_\alpha$-module and let $(W, \Delta_W)$ be a YD$_\beta$-module module. By the definition (19b) of the tensor product in $YD(H)$, the action $\triangleright$ of $H_\ast^\pi$ of $V \otimes W$ is given by $f \triangleright (v \otimes w) = \langle f, (v \otimes w)(\gamma) \rangle = \langle f, \phi_\gamma (v(\beta_\beta^{-1}) \otimes w(\gamma)) \rangle \otimes \omega_\gamma = \langle f, (\beta_{\beta^{-1}} \triangleright v) \otimes (f_{\beta} \triangleright w) \rangle$, i.e., $F_2$ preserves the tensor product. The fact that $F_2$ preserves the tensor unit is trivial.

Crossing. Let $\alpha$ and $\beta$ be in $\pi$ and let $(V, \Delta_V)$ be a YD$_\alpha$-module. The action of $H_\ast^\pi$ on $\beta(F_2(V, \Delta_V))$ is given by $f \triangleright w = \langle f, \varphi_{\beta^{-1}}(f) \phi_{\beta^{-1}}(w) \rangle = \langle f, \varphi_{\beta^{-1}}(f) \phi_{\beta^{-1}}(\varphi_{\beta^{-1}}(w)) \rangle = \langle f, \varphi_{\beta^{-1}}(\varphi_{\beta^{-1}}(w)) \rangle = \langle f, \varphi_{\beta^{-1}}(\varphi_{\beta^{-1}}(w)) \rangle$, for any $f \in H_\gamma^\ast$, with $\gamma \in \pi$, and $w \in \beta V$. By (19b), both $\beta(F_2(V, \Delta_V))$ and $F_2(\beta(V, \Delta_V))$ has the same structure of $H_\ast^\pi$-module and so of $(H, H_\ast^\pi, \otimes)$-module. Since both $\beta(\cdot)$ and $F_2$ are the identity on the morphisms, we conclude that $F_2$ commute with the conjugation and that it is a $T$-functor.

Braiding. Let $(V_1, \Delta_{V_1})$ be a YD$_{\alpha_1}$-module and let $(V_2, \Delta_{V_2})$ be a YD$_{\alpha_2}$-module module. By (20), for any $v_1 \in V_1$ and $v_2 \in V_2$ we have
\[
\begin{align*}
    c_{F_2(V_1, \Delta_{V_1}), F_2(V_2, \Delta_{V_2})}(v_1 \otimes v_2) &= a_1(s_{\alpha_2^{-1}}, e_{\alpha_2^{-1}}, v_2) \otimes e_{\alpha_2^{-1}} \triangleright v_1 \\
    &= a_1(s_{\alpha_2^{-1}}, e_{\alpha_2^{-1}}, v_2) \otimes (e_{\alpha_2^{-1}} \triangleright (v_1(\alpha_2^{-1})))(v_1(\alpha_2^{-1})) \\
    &= a_1(s_{\alpha_2^{-1}}, (v_1(\alpha_2^{-1}))v_2) \otimes (v_1(\alpha_2^{-1})).
\end{align*}
\]
By (26) we have
\[ c_{F_2(V_1, \Delta V_1), F_2(V_2, \Delta V_2)} = c_{(V_1, \Delta V_1), (V_2, \Delta V_2)} = F_2(c_{(V_1, \Delta V_1), (V_2, \Delta V_2)}). \]

\[ \Box \]

**Proof of Theorem 10.1.** To prove Theorem 10.1, we need a preliminary lemma.

**Lemma 10.5.** For any \( f \in H^*_\gamma \), with \( \gamma \in \pi \), we have \( f = \langle f, e(\gamma), i_\gamma \rangle e(\gamma, i). \)

**Proof.** By evaluating \( \langle f, e(\gamma), i_\gamma \rangle e(\gamma, i) \) against a generic \( h \in H_\gamma \) we obtain
\[ \langle \langle f, e(\gamma), i_\gamma \rangle e(\gamma, i), h \rangle = \langle f, e(\gamma), i_\gamma \rangle \langle e(\gamma, i), h \rangle = \langle f, h \rangle. \]

\[ \Box \]

**Proof (of Theorem 10.4).** Let us set \( G = F_3 \circ F_2 \circ F_1 : Z(\text{Rep}(H)) \to \text{Rep}(\mathcal{D}(H)) \).

Since both \( F_1 \) and \( F_2 \) as well as \( F_3 \) are braided T-functors, \( G \) is a braided T-functor.

To complete the proof of Theorem 10.4, we need only to show that \( G \) is invertible.

Given a \( \mathcal{D}(H) \)-module \( V \), we set \( G(V) = (V, c_V) \). Of course, \( G(V) \) is an half-braiding and by setting \( \hat{G}(f) = f \), for any morphism \( f \) of \( YD \)-modules, we obtain a functor \( \hat{G} : \text{Rep}(\mathcal{D}(H)) \to \mathcal{Z}(\text{Rep}(H)) \). Let us prove that \( \hat{G} \) and \( G \) are mutually inverses.

\[ \hat{G} \circ G = \text{Id}. \] Let \((V, \zeta)\) be an object in \( \mathcal{Z}(\text{Rep}(H)) \). Since \( c_{G(V, \zeta), \zeta} = G(c_{(V, \zeta), \zeta}) = G(\zeta) = \zeta \), we get \((\hat{G} \circ G)(V, \zeta) = (V, c_G(V, \zeta)) = (V, \zeta)\).

\[ G \circ \hat{G} = \text{Id}. \] Let \( V \) be a \( \mathcal{D}(H) \)-module, with \( \alpha \in \pi \). Clearly, \((G \circ \hat{G})(V)\) and \( V \) have the same structure of \( k \)-vector spaces and the same structure of \( H_\alpha \)-module (via the embedding \( H_\alpha \hookrightarrow \mathcal{D}(H) \)). To prove \( G \circ \hat{G} \) have the same structure of \( H_\alpha \)-module, we only need to check that the action \( \triangleright \) of \( H_\alpha^* \) on \( V \) and the action \( \triangleright \) of \( H_\alpha^* \) on \((G \circ \hat{G})(V)\) are the same.

Let \( f \) be in \( H^*_\gamma \), with \( \gamma \in \pi \). By observing that, for any \( v \in V \), \( e^{-1}_{(\gamma), i} \zeta_{(\gamma), i} \gamma = e(\gamma, i) \triangleright v \triangleright e(\gamma, i) \gamma = \langle f, v(\gamma) \rangle v V = \langle e(\gamma), i_\gamma \rangle e(\gamma, i). \]

\[ \Box \]

**Corollary 10.6.** \( \mathcal{Z}(\text{Rep}(H)), \text{YD}(H), \text{Rep}(H, H^*, \oplus), \text{and Rep}(\mathcal{D}(H)) \) are isomorphic braided T-categories.

**Proof.** We have seen that both the functor \( F_1 \) and the functor \( F_3 \) are isomorphisms of braided T-categories. By Lemma 10.4, \( F_2 \) is an braided T-functor and, by Theorem 10.1, \( F_2 \) is invertible with inverse \( F_1 = F_2 \circ G \).

Let \( \text{YD}(H) \) be the category of finite-dimensional \( \text{YD} \)-modules, i.e., the category of \( \text{YD} \)-modules \((V, \zeta)\) such that \( \dim_k V \in \mathbb{N} \), and let \( \text{Rep}(H, H^*, \oplus) \) be the category of finite-dimensional \((H, H^*, \oplus)\)-modules.

**Corollary 10.7.** \( \mathcal{Z}(\text{Rep}(H)), \text{YD}(H), \text{Rep}(H, H^*, \oplus), \text{and Rep}(\mathcal{D}(H)) \) are isomorphic braided T-categories.

**Proof.** The functor \( F_1 \) sends the full subcategory \( \mathcal{Z}(\text{Rep}(H)) \) of \( \mathcal{Z}(\text{Rep}(H)) \) to the full subcategory \( \text{YD}(H) \) of \( \text{YD}(H) \). Similarly, the functor \( F_2 \) sends \( \text{YD}(H) \) to the full subcategory \( \text{Rep}(H, H^*, \oplus) \) of \( \text{Rep}(H, H^*, \oplus) \) and the functor \( F_3 \) sends \( \text{Rep}(H, H^*, \oplus) \) to the full subcategory \( \text{Rep}(\mathcal{D}(H)) \) of \( \text{Rep}(\mathcal{D}(H)) \).
Remark 10.8 (modular T-categories). The categorical analog of the notion of modular Hopf algebra is the notion of modular category [12, 16]. A T-category $\mathcal{T}$ is modular when the component $\mathcal{T}_f$ is modular as a tensor category [18].

Let $\mathcal{R}$ be a semisimple tensor category. It was proved by Müger [11] that, under certain conditions on $\mathcal{R}$, the center of $\mathcal{R}$, is modular. We expect that it will be possible to generalize the result to the crossed case when $\pi$ is finite. On the contrary, when $\pi$ is not finite, since the quantum double of a semisimple T-coalgebra is not modular, the theory fails to be applicable to the crossed case. However, in some case, for instance when the isomorphism classes of the $\mathcal{H}_\alpha$ (for all $\alpha \in \pi$) are finite, $\mathcal{Z}(\mathcal{R})$ should be modular, or at least, premodular in the sense of Bruguières and, in that case, they should give rise to a modular category.

11. Ribbon structures

We conclude by discussing the relation between algebraic and categorical ribbon extensions. Let $H$ be T-coalgebra (not necessarily of a finite-type). Firstly, we recall the definition of the ribbon T-coalgebra $RT(H)$ (see [2]). Then, we prove that the categories $Rep_f(RT(H))$ and $\mathcal{N}\left(\left(Rep_f(H)\right)^N\right)$ are isomorphic as balanced T-categories. To prove this statement we start by introducing an auxiliary ribbon T-category Rib ($H$). Then we prove that $Rep_f(RT(H))$ and Rib ($H$) are isomorphic as ribbon T-categories while Rib ($H$) that $\mathcal{N}\left(\left(Rep_f(H)\right)^N\right)$ are isomorphic as balanced T-categories. Finally, we prove that, if $H'$ is a T-coalgebra of finite-type, then $Rep_f\left(RT(D(H'))\right)$ and $D(Rep_f(H'))$ are isomorphic ribbon T-categories.

The T-coalgebra $RT(H)$. The ribbon T-coalgebra $RT(H)$ is defined as follow.

- For any $\alpha \in \pi$, the $\alpha$-th component of $RT(H)$, denoted $RT_\alpha(H)$, is the vector space whose elements are formal expressions $h + kv_\alpha$, with $h, k \in H_\alpha$. The sum is given by $(h + kv_\alpha) + (h' + k'v_\alpha) = (h + h') + (k + k')v_\alpha$, for any $h, h', k, k' \in H_\alpha$. The multiplication is obtained by requiring $v_{\alpha}^2 = u_\alpha s_{\alpha-1}(u_{\alpha-1})$, i.e., by setting, for any $h, h', k, k' \in H_\alpha$, $(h + kv_\alpha)(h' + k'v_\alpha) = hh' + k\varphi_\alpha(h')v_\alpha + k'\varphi_\alpha(h)u_\alpha s_{\alpha-1}(u_{\alpha-1}) = (hh' + k\varphi_\alpha(k')u_\alpha s_{\alpha-1}(u_{\alpha-1})) + (kk' + k'\varphi_\alpha(k'))v_\alpha$.

We identify $H_\alpha$ with the subset $\{h + 0v_\alpha | h \in H_\alpha\}$ of $RT_\alpha(H)$. The algebra $RT_\alpha(H)$ is unitary with unit $1_\alpha = 1 + 0v_\alpha$. Moreover, for any $\alpha, \beta \in \pi$, we have $R_{\alpha, \beta} \in H_\alpha \otimes H_\beta \subset RT_\alpha(H) \otimes RT_\beta(H)$.

- The comultiplication is given by $\Delta_{\alpha, \beta}(h + kv_\alpha) = \left(h'_{(\alpha)} + k'_{(\alpha)}\tilde{\xi}_{(\alpha), i}\xi_{(\alpha), j}v_\alpha\right) \otimes \left(h''_{(\beta)} + k''_{(\beta)}\tilde{\xi}_{(\beta), i}\varphi_{\alpha-1}(\tilde{\xi}_{(\alpha), (\alpha-1), j})v_\beta\right)$, for any $h, k \in H_\alpha$, and $\alpha, \beta \in \pi$.

Further, the counit is given by $\langle \varepsilon, h + kv_\alpha \rangle = \langle \varepsilon, h \rangle + \langle \varepsilon, k \rangle$, for any $h, k \in H_1$.

- The antipode is given by $s_\alpha(h + kv_\alpha) = s_\alpha(h) + (s_\alpha \circ \varphi_{\alpha-1})(k)v_{\alpha-1}$, for any $h, k \in H_\alpha$ and $\alpha \in \pi$.

Finally, the conjugation is given by $\varphi_\beta(h + kv_\alpha) = \varphi_\beta(h) + \varphi_\beta(k)v_{\beta, \alpha, \beta-1}$, for any $h, k \in H_\alpha$ and $\alpha, \beta \in \pi$.

The category Rib ($H$). The ribbon T-category Rib ($H$), is defined as follows.

- For any $\alpha \in \pi$, the objects of the component Rib$_\alpha(H)$ of Rib ($H$) are the couples $(M, t)$, where $M$ is a finite-dimensional representation of $H_\alpha$ and $t : M \to \overline{M}$ is a $H_\alpha$-linear isomorphism such that, if we set

$$t^2 = \left(M \xrightarrow{t} \overline{M} \xrightarrow{M} \overline{M} \overline{M}\right)$$
and $t^{-2} = (t^2)^{-1}$, then we have
\[
(28a) \quad t^{-2}(α^2 m) = u_α s_{α^{-1}}(u_{α^{-1}})m
\]
for any $m \in M$ (where $u_α$ is the $α$-th Drinfeld element).

- Given two objects $(M_1, t_1), (M_2, t_2) \in \text{Rib}_α(H)$, a morphism $f: (M_1, t_1) \to (M_2, t_2)$ is a $H_α$-linear map $f: M_1 \to M_2$ such that $t_2 \circ f = (α f) \circ t_1$.
- The composition of morphisms in $\text{Rib}_α(H)$ is obtained in the obvious way via the compositions of $H_α$-modules.
- The tensor product of two objects $(M, t), (M', t') \in \text{Rib}(H)$, is given by
\[
(28b) \quad (M, t) \otimes (M', t') = (M \otimes M', t \boxtimes t'),
\]
where we recall that
\[
(28c) \quad t \boxtimes t' = \left(\left(\left(M M'\right)_t\right) \otimes M'_t\right) \circ c_{M,M'} \circ c_{M,M'},
\]
where $c$ is the standard braiding in $\text{Rep}_1(H)$.
- The tensor unit of $\text{Rib}(H)$ is the couple $(k, \text{Id}_k)$, where $k$ is a $H_1$-module via the counit of $H$.
- The action of the crossing on an object $(M, t) \in \text{Rib}(H)$ is obtained by setting $β(M, t) = (β M, β f)$ for any $β \in π$, while the action of the crossing on morphisms in obtained by requiring that the forgetful functor $\text{Rib}(H) \to \text{Rep}_1(H): (M, t) \to M$ is a $T$-functor.
- The braiding is given by $c_{(M,t),(M',t')} = c_{M,M'}$ for any $(M, t), (M', t') \in \text{Rib}(H)$.
- The twist is given by $θ(M, t) = t$ for any $(M, t) \in \text{Rib}(H)$.
- The duality in $\text{Rib}(H)$ is obtained as follows. Let $(M, t)$ be an object in $\text{Rib}(H)$. The dual object of $(M, t)$ is given by the couple $(M^*, M^t)$ where $M^*$ is the dual $H$-module of $M$ (via unit the $b_M$ and the counit $d_M$ defined in $[11]$). Finally we set $b_{(M, t)} = b_M$ and $d_{(M, t)} = d_M$.

**Theorem 11.1.** $\text{Rib}(H)$ is a ribbon $T$-category isomorphic to $\text{Rep}_1(\text{RT}(H))$. Moreover, $\text{Rib}(H)$ is isomorphic to $N\left(\left(\text{Rep}_1(H)\right)^N\right)$ as a balanced $T$-category.

To prove Theorem 11.1 we need three preliminary lemmas.

**Lemma 11.2.** For any $α \in π$ we have
\[
(29) \quad s_{α^{-1}}(u_{α^{-1}}) = ξ(α, i)s_{α^{-1}}(ξ(α^{-1}, i)).
\]

*Proof.** By observing that $s_α(ξ(α, i)) \otimes ζ(β, i) = ϕ_α(ξ(α^{-1}, i)) \otimes ζ(β^{-1}, i)$ (see $[19]$), we have $ξ(α^{-1}, i) \otimes ζ(α, i) = (s_{α^{-1}} \circ ϕ_{α^{-1}})(ξ(α, i)) \otimes s_{α^{-1}}(ξ(α^{-1}, i))$, so we get $u_{α^{-1}} = (s_α \circ ϕ_{α^{-1}})(ξ(α, i))s_{α^{-1}}(ξ(α^{-1}, i))$. By composing both sides by $s_{α^{-1}}$ we get $[29]$. \hfill \□

**Lemma 11.3.** For any $α \in π$ and $h \in H_α$, we have
\[
(30) \quad s_{α^{-1}}(u_{α^{-1}})h = (s_{α^{-1}} \circ s_{α^{-1}} \circ ϕ_α)(h)s_{α^{-1}}(u_{α^{-1}}).
\]

*Proof.** Let $k$ be in $H_{α^{-1}}$. By $(29)$, we have $(s_α \circ s_{α^{-1}} \circ ϕ_{α^{-1}})(k) = u_{α^{-1}} ku_{α^{-1}}^{-1}$. By composing both sides by $s_{α^{-1}}$ and observing that, by $(29)$, $s_{α^{-1}}(u_{α^{-1}}) = s_{α^{-1}}(u_{α^{-1}})$, we get $s_{α^{-1}}(u_{α^{-1}})s_{α^{-1}}(ϕ_{α^{-1}})(k) = s_{α^{-1}}(k)s_{α^{-1}}(u_{α^{-1}})$. For $k = (ϕ_α \circ s_{α^{-1}})(h)$, we get $(30)$. \hfill \□

**Lemma 11.4.** Let $M$ be a finite-dimensional representation of $H$ and let $ω_M$ defined as in $[3]$. For any $m \in M$ we have $Ω_M\left(α^2 m\right) = u_α s_{α^{-1}}(u_α)m$.
The proof is a long but not difficult computation and is omitted.

Proof of Theorem 11.4. Rib(\(H\)) is obviously a well defined category. We start by proving that Rib(\(H\)) is isomorphic to Rep(\(RT(H)\)) as a category. Let M be a finite-dimensional representation of \(RT(H)\). Set \(\theta_M : M \to M : x \mapsto M_1(\theta x)\). Since \(\theta^{-2} = s_{\alpha^{-1}}(u_{\alpha^{-1}})u_\alpha\) (see \([9]\)), the couple \((M, \theta_M)\) is an object in Rib(\(H\)). Conversely, let \((N, t)\) be an object in Rib_{\alpha}(H), with \(\alpha \in \pi\). Define the action of RT_{\alpha}(H) on N via

\[
(h + kv_\alpha)n = hn + kt^{-1}(\alpha n)
\]

for any \(h, k \in H\) and \(n \in N\). Let us check that the action defined in (31) is RT_{\alpha}(H)-linear, i.e., that we provided \(N\) of a structure of RT_{\alpha}(H)-module. For any \(h_1, k_1, h_2, k_2 \in H\) and \(n \in N\), we have \(((h_1 + k_1 v_\alpha)(h_2 + k_2 v_\alpha))n = (h_1h_2 + k_1\varphi_\alpha(k_2)u_\alpha s_{\alpha^{-1}}(u_{\alpha^{-1}}))n + (h_1k_2 + k_1\varphi_\alpha(h_2))t^{-1}(\alpha n)\) and \((h_1 + k_1 v_\alpha)(h_2 + k_2 v_\alpha)n = h_1h_2n + h_1k_2t^{-1}(\alpha n) + k_1t^{-1}(\alpha (h_2n)) + k_1t^{-1}(\alpha (k_2t^{-1}(\alpha n)))\).

By observing that

\[
k_1t^{-1}(\alpha (h_2n)) = k_1t^{-1}(\varphi_\alpha(h_2)\alpha n) = k_1\varphi_\alpha(h_2)t^{-1}(\alpha n)
\]

and that

\[
k_1t^{-1}(\alpha (k_2t^{-1}(\alpha n))) = k_1t^{-1}(\varphi_\alpha(k_2)(t^{-1}(\alpha n))) = k_1\varphi_\alpha(k_2)(t^{-1} \circ \alpha t^{-1})(\alpha n)
\]

we obtain

\[
(h_1 + k_1 v_\alpha)(h_2 + k_2 v_\alpha)n = (h_1h_2 + k_1\varphi_\alpha(k_2)u_\alpha s_{\alpha^{-1}}(u_{\alpha^{-1}}))n + (h_1k_2 + k_1\varphi_\alpha(h_2))t^{-1}(\alpha n),
\]

i.e., the action defined in (31) is RT_{\alpha}(H)-linear. To complete the proof that Rib(\(H\)) and Rep(\(RT(H)\)) are isomorphic categories is now trivial.

Since Rib(\(H\)) and Rep(\(RT(H)\)) are isomorphic, the ribbon T-category structure of Rep(\(RT(H)\)) induces a ribbon T-category structure on Rib(\(H\)). This is the structure described above. The only nontrivial point is to show that the tensor product induced in Rib(\(H\)) is the same given in (28). Let \((M_1, t_1)\) be an object in Rib_{\alpha}(H) and let \((M_2, t_2)\) be objects in Rib_{\beta}(H), with \(\alpha, \beta \in \pi\). To prove that \(\theta_{M_1 \otimes M_2} = t_1 \otimes t_2\), an easy but long computation shows that, for any \(m_1 \in M_1\) and \(m_2 \in M_2\), both \(\theta_{M_1 \otimes M_2}(m_1 \otimes m_2)\) and \((t_1 \otimes t_2)(m_1 \otimes m_2)\) are equal to

\[
\alpha^\beta((\theta_\alpha \zeta(\alpha, i)\xi(\alpha, j)m_1) \otimes \alpha^\beta(\theta_\beta \varphi_\alpha^{-1}(\xi(\alpha\beta^{-1}, i))\zeta(\beta, j)m_2)).
\]

Let us prove that Rib(\(H\)) and \(N\left(\{\text{Rep}_1(\tilde{H})\}^{N}\right)\) are isomorphic. If \((M, \theta_M)\) is an object in Rib_{\alpha}(H), with \(\alpha \in \pi\), then, by Lemma 11.4, for any \(m \in M\) we have \(\Omega_M(\alpha^{-2}m) = u_\alpha \theta_M^{-2}(m)\), so that \((M, \theta_M)\) is an object in \(N\left(\{\text{Rep}_1(\tilde{H})\}^{N}\right)\).

Conversely, if \((M, t)\) is an object in \(N\left(\{\text{Rep}_1(\tilde{H})\}^{N}\right)\), then, by Lemma 6.1 and Lemma 11.4, for any \(m \in M\) we have \(t(\alpha^{-2}m) = u_\alpha \theta_M^{-2}(m)\), i.e., \((M, t)\) is an object in Rib(\(H\)). The rest follows easily.

Since, by Theorem 11.4, \(N\left(\{\text{Rep}_1(\tilde{H})\}^{N}\right)\) is isomorphic to Rep_{1}(RT(H)), the balanced T-category \(N\left(\{\text{Rep}_1(\tilde{H})\}^{N}\right)\) has also a natural structure of ribbon T-category. In particular, when \(H\) is the quantum double of a finite-type T-coalgebra \(H'\), this structure of a ribbon T-category is the same induced by the isomorphism between Rep_{1}(RT(D(H'))) and D(Rep_{1}(H')), so that we obtain the following corollary.
Corollary 11.5. If $H'$ is a finite-type $T$-algebra, then $\text{Rep}_f(\mathcal{R}(\mathcal{D}(H'))) \text{ and } \mathcal{D}(\text{Rep}_f(H'))$ are isomorphic as ribbon $T$-categories.

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