Cosmological Standard Timers from Primordial Black Hole Binaries

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We propose that primordial black hole (PBH) binary systems can work as standard timers in tracking the evolution of the Universe. Through gravitational waves from monochromatic PBH binaries, the probability distribution on major axis and eccentricity from the same redshift is obtained. By studying the dynamical evolution of PBH binaries from the initial probability distribution to observed redshifted ones, the redshift-time calibration can be extracted, which can constrain cosmological models. A general formalism of the standard timer is further concluded based on the evolution of statistical distribution in dynamical systems.

I. INTRODUCTION

With the development of modern cosmology, various cosmic properties have been found in observations, e.g., the cosmic microwave background (CMB) [1–4], large scale structure (LSS) [5], and cosmic accelerating expansion [6, 7]. Lambda-cold dark matter ($\Lambda$CDM) model is therefore established [8–11] as the standard model in cosmology.

For precise understanding in evolution of the Universe, various observational technologies have been proposed in studying the cosmological distance-redshift relation. Type Ia supernovae (SNe) produce consistent peak luminosity. The luminosity distance can be obtained by comparing the absolute and apparent magnitude of them, which works as the standard candle to provide luminosity distance-redshift relation [12, 13]. Baryon acoustic oscillations (BAO) determine a fixed sound horizon, by observing sound horizons at different redshifts. The calibration between angular diameter and redshift is extracted, serving sound horizons at different redshifts. The calibration between angular diameter and redshift is extracted, which serves as standard rulers [4, 14–16]. Gravitational waves (GWs) from binary systems and their electromagnetic counterparts provide the calibration between the luminosity distance and redshift as standard sirens [17]. However, there still exists puzzles in $\Lambda$CDM model, e.g., the Hubble tension [18–20] and the $\sigma_8$ tension [4]. Such tensions could be caused by unrecognized systematic uncertainty in measurements [21, 22] or hidden new physics [23–29]. New observational methods are needed in cross checking the observed tensions.

Intuitively, the evolution of the Universe can be briefly characterized by the scale factor $a(t)$ in the Friedmann-Robertson-Walker (FRW) metric, which is related to the cosmological redshift $z$ by $1 + z(t) = a_0/a(t)$, where $a_0$ is the present scale factor. Tracking $z(t)$ provides another perspective in studying cosmological evolution (also used in the determination of the age of the Universe, see [30–33] for details), which could be achieved in the cosmological dynamical systems. Following their intrinsic dynamics, the physical evolution time from the initial state to the later state is attained. Meanwhile, redshift is decoded from their observable. Hence, redshift-time calibration could be constructed in cosmological dynamic systems. This approach is known as standard timers.

In the first study on standard timers [34], we have shown that through the Hawking radiation emitted from light primordial black hole (PBH) clusters, PBH stellar bubbles [35] can be used as standard timers. Due to the primordial origin of PBHs, the initial mass function of PBHs in clustering should be the same. With the emission of Hawking radiation, PBHs evaporate which deforms the mass function of PBHs. By studying the evolution of PBH mass function, its physical evolution time is attained. While observing the gamma ray spectrum from PBH stellar bubbles gives the redshifted PBH mass function where redshift is encoded. Hence, the calibration between redshift and physical evolution time of PBH mass function is constructed.

In this paper, we note that PBH binary systems can be used as standard timers. Under the assumption of random distribution of PBHs in space [36–38], the PBH binaries could decouple from the Hubble flow and form an identical initial probability distribution on the major axis $a$ and the eccentricity $e$ [39, 40]. With the evolution of PBH binaries, the later probability distribution in PBH binary systems evolves accordingly. Through gravitational waves (GWs) emitted from PBH binaries, the probability distribution on major axis and eccentricity from the same redshift is obtained. By studying the evolution of PBH binaries from the initial probability distribution to later ones, cosmological redshift and physical evolution time in PBH binary systems are connected, which can constrain cosmological models. Considering the next generation of GW detectors, such as the Einstein Telescope [41], Laser Interferometer Space Antenna [42], can detect GWs from high redshift ($z > 20$), where PBH binaries may dominate the binary systems [43]. Detecting GWs from PBH binary systems at high redshift would own a pure GW background, which can construct the high-precision standard timer.

This paper is organized as follows. In Sec. II, we show how to construct the standard timer in PBH binary systems, including single parameter PBH binary systems in Sec. II A, multi-parameter PBH binary systems in Sec. II B and PBH binary systems without the
initial probability distribution in Sec. II.C. In Sec. III, the conclusion and discussions about the standard timer from PBH binary systems are given. In Appendix A & B, a formalism on constructing standard timers in general dynamic systems is shown.

II. THE STANDARD TIMER IN PBH BINARY SYSTEMS

In constructing the standard timer in PBH binary systems, two essential requirements are needed. One is the identical initial state of PBH binary systems. The other one is their later evolved state from the same redshift can be obtained. The former of requirements is achieved in the identical initial probability distribution on major axis and eccentricity [39, 40] under the assumption of random distribution of PBHs in the space. The latter can be easily realized in local signal sources, such as the PBH cluster [35, 44], where the redshift of signals from a local source should be the same. However, redshifts of PBH binaries are hardly classified, due to their being globally distributed in the Universe.

The potential monochromatic mass spectrum in PBH scenarios changes the story. It introduces PBH binary systems in the standard timer through GW channels. After collecting many GW signals from different redshifts, we can extract the redshifted chirp mass $M_z$ and mass ratio $q$ from the GW waveform. Due to the unknown intrinsic chirp mass $M$, which follows $M = M_z/(1+\zeta)$ [45], the redshift of binary systems cannot be determined. However, if GW signals come from PBH binaries, under the assumption that the mass of PBHs is monochromatic, the mass ratio of PBH binaries follows $q = 1$ (also a number of BH binaries with $q = 1$ are found in GWTC-2, see [46] for details), and GW signals emitted from PBH binaries at the same redshift can give an identical redshifted chirp mass. Then, PBH binaries can be classified into different redshift shells based on their redshifted chirp mass. As a result, the standard timer can be constructed by comparing the initial probability distribution on major axis $a$ and eccentricity $e$ and later ones from the same redshift.

A. A toy model in single parameter PBH binary systems

By assuming the mass of PBHs is monochromatic, the state of PBH binary systems can be described by a probability distribution on major axis $a$ and eccentricity $e$, which is $dP/da de$. For an intuitive understanding on how standard timers work in PBH binary systems, we start with circular binary systems. Then states of PBH binary systems only depend on the single parameter major axis $a$, which is $dP/da$. In studying the evolution of its probability distribution, we have

$$S(a; t) = \frac{dP}{da} = \frac{dP}{da_1} \frac{da_1}{da_0},$$

where, $a_0$ and $a_1$ denote the major axis at the physical time $t_1$ and later physical time $t$, respectively. $S(a; t)$ denotes the probability distribution on major axis $a$ at physical time $t$ and $dP/da$ is the initial probability distribution $S(a; t_1)$. In connecting $S(a; t)$ with $S(a; t_1)$, we consider time evolution of the major axis, following [47]

$$\frac{da}{dt} = -\frac{128}{5} \frac{G^3 M^3_{PBH}}{c^5 a^5},$$

where $M_{PBH}$ is the monochromatic mass of PBHs, $G$ and $c$ are the Newton’s constant and the speed of light, respectively. Integrating Eq. (2) from the initial physical time $t_1$ to later physical time $t$, we have $a_1^4 = a_0^4 + \delta^4(\Delta t)$, where $\delta^4$ is defined as $\delta^4(\Delta t) = 512 G^3 M^3_{PBH} \Delta t / 5 c^5$ and $\Delta t$ is the physical evolution time $\Delta t = t - t_1$. Fixing the evolution time $\Delta t$, the evolution of major axis $da_1/da_0$ can be obtained, then the probability distribution of single parameter PBH binary systems from an identical redshift shell can be expressed as following

$$S(a; t) = \frac{dP}{da_0} \frac{a_2^4}{(a_1^4 + \delta^4(\Delta t))^{3/4}}.$$  

With the expansion of the Universe, the observed major axis $a_o$ is redshifted by $a_o = (1+z)a$, which can be found in the Kepler’s third law $a_o \sim (GM_z)^{1/3} f_2^{-2/3} \sim (1+z)(GM)^{1/3} f_2^{-2/3}$, where $M_z$ and $f_2$ are the observed mass and GW frequency in binary systems, respectively. Therefore, the observed probability distribution follows

$$S_o(a_o; t) = \frac{dP}{da_0(z)} \frac{a_2^4}{(a_1^4 + \delta^4(\Delta t_2))^{3/4}}.$$  

Here, subscript $o$ denotes the observational quantity. As above, we have the relation $a_2^4 (z) = a_1^4 + \delta^4(\Delta t_2)$, where $\Delta t_2$ depends on the redshift of PBH binaries. Therefore, a correct redshift should be firstly obtained in calibrating the redshift-time relation from the observed probability distribution. We consider the condition $a_1^4 \gg \delta^4(\Delta t_2)$, which infers $a_1(z) \simeq a_o(z)$. In this large major axis limit, the observed probability distribution becomes

$$S_o(a_o; t) \simeq S_o(a_1(1+z); t) = \frac{dP}{da_0(z)} = \frac{1}{1+\zeta} \frac{dP}{da_1}.$$  

Then the redshift can be numerically resolved from the equation $(1+z)S_o(a(1+z); t) = S(a; t_1)$, where $a$ is chosen in the large major axis limit. After obtaining the redshift, probability distributions can recover from redshifted ones by $S(a; t) = (1+z)S_o(a(1+z); t)$. Then, physical evolution time can be extracted in the condition $a_1^4 \ll \delta^4(\Delta t)$, which indicates $a_1 \simeq \delta(\Delta t)$. In this small major axis limit, we have the log probability distribution from Eq. (3) as following

$$\log S(a; t) = \log \frac{S(\delta(\Delta t); t_1)}{\delta^3(\Delta t)} + 3 \log a_1.$$
Then $\delta(\Delta t)$ can be extracted from $S(\delta(\Delta t); t_1)/\delta^3(\Delta t)$ in Eq. (6). $\delta(\Delta t)$ depends on the mass of PBHs $M_{PBH}$ and physical evolution time $\Delta t$. In above calculation, redshift is obtained from the probability distribution in the large major axis limit, which helps determine the mass of PBHs from the redshifted mass $M_{PBH} = M_\odot/(1 + z)$. Then, physical evolution time $\Delta t$ can be resolved from obtained $\delta(\Delta t)$. After obtaining the redshift and physical evolution time, the calibration between redshift and time can be constructed in single parameter PBH binary systems.

In general, the standard timer requires two properties in the evolution of dynamical systems, we take the single parameter PBH binary systems as an example in Fig. 1. Typical properties in $da_i/da$ are the flat constant part in the large major axis tail and the rapid evolution part in the small major axis tail. The flat constant part is the region where time evolution is negligible, the redshift can be extracted by comparing the redshifted and the initial probability distributions in this region. After obtaining the redshift, intrinsic evolution functions (solid lines) can recover from redshifted ones (dashed lines) in Fig. 1. The rapid evolution part describes the physical time evolution effectively changes the major axis of PBH binaries, then physical evolution time can be extracted in Eq. (6).

**B. A practical model in PBH binary systems**

A practical description of monochromatic PBH binary systems needs two parameters, major axis $a$ and eccentricity $e$. In multi-parameter probability distributions, the evolution of probability distribution of PBH binary systems $dP/da_de$ from an identical redshift shell can be described as following

$$S(a, e; t) = \frac{dP}{da_i de_i} \det J(a, e, \Delta t),$$

$$J(a, e, \Delta t) = \left( \frac{\partial a_i}{\partial a}, \frac{\partial a_i}{\partial e} \right).$$

Here, $dP/da_i de_i$ is the initial probability distribution of PBH binary systems, and $J(a, e, \Delta t)$ is the Jacobian of two-parameter PBH binary systems after the evolution of physical evolution time $\Delta t$, which connects the initial and later probability distribution. In calculating $J(a, e, \Delta t)$, we consider the time evolution of parameters in PBH binaries, following [47]

$$\frac{da}{dt} = -\frac{128}{5} \frac{G \delta^3 M_{PBH}^3}{c^5 a^3 (1 - e^2)^{7/2}} \left( 1 + \frac{73}{24} e^2 + \frac{37}{96} e^4 \right),$$

$$\frac{de}{dt} = -\frac{608}{15} \frac{G \delta^3 M_{PBH}^3}{c^5 a^4 (1 - e^2)^{5/2}} \left( 1 + \frac{121}{304} e^2 \right).$$

Due to the expansion of the Universe, the cosmological redshift is introduced in the observed probability distribution, which is

$$S_o(a_z, e; t) = \frac{dP}{da_i(z) de_i} \det J(a_z, e, \Delta t_z),$$

where redshifted major axis becomes $a_z = (1 + z)a$. However, the redshift does not leave imprints on the eccentricity $e$, it can be found in time evolution of the eccentricity that the redshift effect in major axis, mass and time cancel with each other in Eq. (8), which results in no redshift effect appearing in observed eccentricity. In measuring eccentricity in binary systems, the precision is not very high due to the lack of suitable GW waveform templates [48]. Therefore, we only consider probability distributions on the major axis $dP/da$ for a practical numerical solution, which can be obtained from Eq. (9) as following

$$\frac{dP}{da_z} = \int_0^{a_{max}} \frac{dP}{da_i(z) de_i} \det J(a_z, e, \Delta t_z) de.$$  

In the numerical solution of evolution in probability distributions, we consider two types of initial probability distributions in PBH binary systems. One is the Gaussian distribution localized at a specific major axis and eccentricity. After studying the Gaussian distribution, a general distribution can be decomposed into Gaussian distributions [49, 50]. The other probability distribution in PBH binary systems is chosen from [39, 51],

$$\frac{dP}{de} = \frac{3}{4} f_{PBH} \frac{a^{1/2}}{2^{3/2} \left( 1 - e^2 \right)^{1/2}},$$

where $f_{PBH}$ is the present energy density fraction of PBHs in the dark matter and $\bar{x}$ is the physical mean separation of PBHs at matter-radiation equality. Based
on different types of initial probability distributions, a numerical study on the evolution of probability distribution of PBH binaries on the major axis $a$ following Eqs. (8)(10) is shown in Fig. 2.

As mentioned in Sec. II A, various regions of probability distributions behave differently. In the large major axis limit, the evolution of probability distribution is negligible, the redshift can be obtained by comparing the initial probability distribution (black line) and redshifted probability distributions (dashed lines) in Fig. 2, which can be further numerically resolved from the equation $(1 + z)dP/da_z = dP/da_i$. However, the large major axis tail in observed probability distributions can hardly be obtained due to its extremely low gravitational wave frequency, as shown in the right panel of Fig. 2. In this case, we can use numerical solutions as templates of probability distributions to match observational results. After obtaining the redshift, observed probability distributions $dP/da_z$ can be restored to intrinsic probability distributions $dP/da_i$ following $dP/da_i = (1 + z)dP/da_z$. Then physical evolution time $\Delta t$ can be extracted from the numerical solution in the small major axis limit, where the evolution of probability distribution dominates. Accordingly, the redshift-time calibration is constructed in multi-parameter PBH binary systems.

Generally, the mass of PBHs is very essential in describing the evolution of PBH binary systems, when the mass function of PBHs is not monochromatic. The evolution of parameters follows [47]

$$\frac{da}{dt} = -\frac{64}{5} \frac{G^2 m_1 m_2 (m_1 + m_2)}{e_0 a^4 (1 - e^2)^{3/2}} \left( 1 + \frac{73}{24} e^2 + \frac{37}{96} e^4 \right),$$

$$\frac{de}{dt} = -\frac{304}{15} \frac{G^3 m_1 m_2 (m_1 + m_2)}{e_0 a^4 (1 - e^2)^{5/2}} e \left( 1 + \frac{121}{304} e^2 \right).$$

Here, $m_1$ and $m_2$ are the mass of PBHs in the binary. Therefore the probability distribution of general PBH binary systems should be described as $dP/da$ at $z$ following $dP/da = (1 + z)dP/da_z$. Then physical evolution time $\Delta t$ can be extracted from the numerical solution in the small major axis limit, where the evolution of probability distribution dominates. Accordingly, the redshift-time calibration is constructed in multi-parameter PBH binary systems.

C. The standard timer without the initial probability distribution

In constructing standard timers in PBH binary systems, the initial probability distribution of PBH binaries...
plays an important role in extracting the redshift and physical evolution time by comparing it with observed redshifted probability distributions. However, the initial probability distribution on major axis and eccentricity of PBH binaries is indeterminate, due to the unknown mass spectrum and space distributions of PBHs, etc. A number of previous works [40, 51, 55] have been done in calculating the initial probability distribution in different scenarios.

In practice, the information about PBHs is insufficient at present, such as the PBH mass, the initial probability distribution on major axis and eccentricity, etc. However, under the assumption of random distribution of PBHs in space, initial probability distributions of PBH binaries at different redshifts are same, which ensures one-to-one correspondence between later evolution probability distributions and physical evolution time. As the result, we cannot extract the redshift by comparing the initial and redshifted probability distribution in Fig. 2. Instead, the redshift ratio \( \eta \) can be obtained from two redshifted probability distributions, following \( dP/da_0 = (1 + z_1)dP/da_{z_1} = (1 + z_2)dP/da_{z_2} \) in large major axis limit, which further gives

\[
\frac{dP}{da_{z_1}} = \frac{\eta}{dP} \frac{dP}{da_{z_2}}, \tag{13}
\]

where \( \eta \equiv (1 + z_2)/(1 + z_1) \). In order to obtain the redshift of observed GW signals, we can assume the cosmological evolution between two redshift shells follows the standard ΛCDM cosmology. Then, we assume the redshift of one observed probability distribution is \( \tilde{z}_1 \) and the redshift of the other one is \( \tilde{z}_2 = (1 + \tilde{z}_1)\eta - 1 \). Under the assumption of \( \tilde{z}_1 \) and \( \tilde{z}_2 \), the PBH mass \( M_{\text{PBH}} \) and physical evolution time \( \Delta t \) can be obtained as we have discussed in Sec. II B. The cosmological time between \( \tilde{z}_1 \) and \( \tilde{z}_2 \) can be calculated as \( t_\Delta = \int_{\tilde{z}_1}^{\tilde{z}_2} dz/H(z)(1 + z) \), where \( H(z) \) is the Hubble parameter along the line of sight. Then proper redshift \( \tilde{z}_1 \) is chosen such that physical evolution time is same as cosmological time between two redshifts as following

\[
\Delta t = t_\Delta. \tag{14}
\]

After obtaining the redshift of PBH binaries from same redshift shell \( z_0 \), the PBH mass can recover from the observed redshifted mass by \( M_{\text{PBH}} = M_z/(1 + z_0) \) and redshift of other probability distributions can be extracted by \( z_{\text{PBH}} = (1 + z_0)\eta - 1 \). Then we can numerically solve physical evolution time \( \Delta t \) between \( z_0 \) and \( z_{\text{PBH}} \) as discussed in Sec. II B. Consequently, the redshift-time calibration \( (z_0, z_{\text{PBH}}, \Delta t) \) is obtained.

Furthermore, cosmological models can be tested in standard timers. Considering the cosmological redshift-time relation \( dz/dt = -(1 + z)H(z) \), we apply the obtained redshift-time calibration \( (z_0, z_{\text{PBH}}, \Delta t) \) from standard timers, which gives

\[
\int_{z_0}^{z_{\text{PBH}}} \frac{dz}{(1 + z)H(z)} = \int_{\text{PBH}}^{t_\Delta} dt = \Delta t. \tag{15}
\]

Take the flat ΛCDM model as an example, \( H(z) = H_0/\Omega_m (1 + z)^{3/2} + \Omega_\Lambda (1 + z)^2 + \Omega_\gamma \). After constructing the redshift-time calibration in PBH binary systems from the primordial Universe to the present Universe, the Markov chain Monte Carlo (MCMC) simulation can be applied on the flat ΛCDM model in constraining the Hubble parameter \( H_0 \), energy density fraction of radiation \( \Omega_\gamma \), matter \( \Omega_m \) and cosmological constant \( \Omega_\Lambda \).

III. CONCLUSION AND DISCUSSIONS

To summarize, we propose that PBH binary systems can work as standard timers to record the evolution of the cosmological redshift \( z(t) \). Under the assumption of random distribution of PBHs in space, PBH binary systems have an identical initial probability distribution on major axis and eccentricity. By studying the evolution of the probability distribution in binary systems, the physical evolution time between the initial and later probability distribution can be extracted. Then the redshift-time relation can be constructed by studying the probability distribution of PBH binary systems at different redshifts. In order to obtain the probability distribution on major axis and eccentricity from the identical redshift shell. We assume that PBH mass is monochromatic, through GWs produced from PBH binaries, its redshifted chirp mass can be obtained in GW waveforms, the PBH binaries from the same redshift have the same redshifted chirp mass and the mass ratio follows \( q = 1 \), then we extract the redshifted probability distribution on major axis and eccentricity from the same redshift.

For demonstrating how standard timers work in PBH binary systems, we perform an analytically studied toy model in single parameter PBH binary systems where eccentricity is set as \( e = 0 \) and a numerically studied practical model in non-circular PBH binary systems. We show that the redshift can be determined by comparing the initial and redshifted probability distribution at the large major axis limit and the physical evolution time can be obtained by comparing the initial and recovered redshifted probability distribution at the small major axis limit. Considering the initial probability distribution on major axis and eccentricity in PBH binary systems is indeterminate, the redshift of observed probability distributions cannot be directly obtained. We assume cosmological time between two redshifts follows the standard cosmology, then proper redshift of PBH binary systems should be chosen when the physical evolution time between two redshifted probability distributions equals its cosmological time, which further constructs standard timers in PBH binary systems without initial conditions.

In the above discussions, we mainly focus on PBH binary systems with a monochromatic mass spectrum, which helps classify the redshift of PBH binaries. In a general description of PBH binary systems, an extended mass spectrum should be taken into consideration, which is shown at the end of Sec. II B. The standard timer can...
be constructed by a numerical study in a probability distribution on major axis, eccentricity and mass of PBHs in binaries $dP/dM$. However, an extended mass spectrum of PBHs could cause difficulties in redshift classification of PBH binaries, which needs further studies in redshift identification, e.g., [56–58] in standard sirens.

In general, the cosmological standard timer can be constructed based on dynamical systems in the Universe. Due to the same formation mechanism of dynamical systems, the statistical distribution of their initial states can be set as the standard reference, through the evolution mechanism of their statistical distribution, the elapsed time in the standard timer is evaluated. Meanwhile, cosmological redshift is encoded in the observable from dynamical systems. For signals from individual sources locally, the redshifted statistical distribution in dynamical systems from the same redshift can be obtained, and further gives their redshift by comparing with the initial state (see [34, 35] for more details). For signals from global sources, as we have discussed in this article, GWs from PBH binaries globally, the redshifted statistical distribution from the same redshift can be extracted according to their redshifted parameters, and hence obtain their redshift. Consequently, the redshift-time calibration is constructed in a general dynamic system (see Append. A & B for the detailed formalism).

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Appendix A: The standard timer from single parameter dynamical systems

In constructing standard timers in dynamical systems, we need to set a particular condition of dynamical systems as a standard reference. For example, the standard reference in standard candles is consistent peak luminosity produced by Type Ia supernovae (SNe), and the standard reference in standard rulers is the fixed baryon acoustic oscillation (BAO) scale that the sound wave can travel before the recombination. Generally, initial states of dynamical systems are uncertainty under the Gaussian distribution of perturbations. However, the statistical distribution of initial states in dynamical systems can be unique due to the same physical mechanism behind them, which can be set as a standard reference. With the standard reference, the physical evolution time and redshift can be extracted by studying the evolution of observed dynamical systems, which can help calibrate $z(t)$.

For simplicity, we start with a single parameter dynamical system, whose time evolution follows $dM/dt = -f(M)$. Here $M$ is the observable physical parameter that characterizes the dynamical system and $f(M)$ is its time derivative function. The statistical distribution of the single parameter dynamical system $S(M;t)$ can be described as

$$S(M;t) = \frac{dN}{dM_t}, \quad (A1)$$

where $M_t \equiv M(t)$, $N$ is the statistic of the distribution of dynamical systems. In order to trace the historical evolution of dynamical systems, a standard initial distribution is essential. Eq. (A1) can be written as

$$S(M;t) = \frac{dN}{dM_t} \cdot \frac{dM_t}{dM_i}. \quad (A2)$$

Here, $dN/dM_t$ is the initial statistical distribution of dynamical systems $S(M;t_i)$. $dM_t/dM_i$ describes the evolution of the dynamical system, which can be further expressed by its time evolution $dM/dt = -f(M)$, which gives

$$\int_{M_i}^{M_t} \frac{dM}{f(M)} = g(M_t) - g(M_i) = -\Delta t. \quad (A3)$$

Here, function $g(M)$ is an antiderivative of function $1/f(M)$. Then, the evolution of dynamical systems can be written as

$$\frac{dM_t}{dM_i} = g'(M_t) = g^{-1}(g(M_t) + \Delta t))/g'(g(M_t) + \Delta t)). \quad (A4)$$

Here, $g'(M) \equiv dq(M)/dM$ and $g^{-1}$ denotes the inverse function of $g(M)$. As the result, Eq. (A2) can be further expressed as

$$S(M;t) = \frac{dN}{dM_i} \cdot g'(g^{-1}(g(M_t) + \Delta t)). \quad (A5)$$

The physical evolution time $\Delta t$ can be extracted by giving an initial statistical distribution $dN/dM_i$ in Eq. (A5).

In the observational aspect, the statistical distribution of the dynamical system is deformed, due to observed physical parameter $M$ is redshifted by the cosmological expansion, which gives the observational distribution $S_o(M_z;t)$ (subscript $o$ denotes the observational quantity) as

$$S_o(M_z;t) = \frac{dN}{dM_i(z)} \cdot \frac{dM_i(z)}{dM_z} = \frac{dN}{dM_i(z)} \cdot g'(g^{-1}(g(M_z) + \Delta t_z)). \quad (A6)$$

Here, $M_z$ denotes the redshifted physical parameter, such as redshifted photon energy $E_z = E/(1+z)$ and redshifted chirp mass in binary black hole systems $M_z = (1+z)M$. $dN/dM_i(z)$ characterizes the cosmological redshift effect in the initial statistical distribution. Following Eq. (A3), we have $g(M_i(z)) = g(M_z) + \Delta t_z$.

In order to extract the redshift-time calibration, we consider two cases in Eq. (A6). For a fixed evolution
time, the first case is $g(M_z) \gg \Delta t_z$, which makes sure the
time evolution is negligible and redshift can be extracted by comparing the redshifted physical parameter
$M_z$ with the initial physical parameter $M_i$ in the initial
statistical distribution. The second case is $g(M_z) \ll \Delta t_z$, where $\Delta t_z$ dominates in the redshifted physical parameter,
which gives $g(M_z(z)) \approx \Delta t_z$. Then $\Delta t_z$ can be extracted in following expression

$$S_o(M_z;t) \simeq \begin{cases} 
\frac{dN}{dM_i(z)}, & g(M_z) \gg \Delta t_z \\
\frac{dN}{dg^{-1}(\Delta t_z) g'(g^{-1}(\Delta t_z))}, & g(M_z) \ll \Delta t_z
\end{cases} \quad (A7)$$

Above all, we have discussed the formalism of a standard timer in an observable dynamical system $S(M;t)$. However, this formalism does not apply to the case that $M$ is not an observable of dynamical systems, meanwhile, the signals produced from them is an observable, e.g., electromagnetic waves and gravitational waves. In these cases, we consider the following integral equation,

$$P(E;t) = \int_0^\infty K(E,M) S(M;t) dM. \quad (A8)$$

Here, $K(E,M)$ is the kernel function which transfers an unobservable distribution $S(M;t)$ to an observable distribution $P(E;t)$. $S(M;t)$ can be extracted by an inverse integral equation

$$S(M;t) = \int_0^\infty K^{-1}(E,M) P(E;t) dE, \quad (A9)$$

where $K^{-1}(E,M)$ is the inverse kernel function of $K(E,M)$. Due to the cosmological expansion, the observed physical parameter $E$ is redshifted to $E_z$. Therefore, the observable becomes

$$P_o(E_z;t) = \int_0^\infty K_o(E, E_z) S(M;t) dM, \quad (A10)$$

where $Z_1$ function describes the redshift effect in the observable $E_z$. In order to construct the redshift-time relation, a redshift term need to appear in $S(M;t)$, which requires the connection between $E$ and $M$ in the kernel function, for instance, the primary Hawking radiation kernel follows $H(E(1+z),M) = H(E,M(1+z)) \ [34]$. Therefore, we assume the kernel function follows

$$K_o(Z_1(E_z), M) = K_o(E_z, Z_2(M)) \quad (A11)$$

Here, $Z_1$ and $Z_2$ function describe how the redshift term transfers from $E_z$ to $M$ in kernel function. Then, Eq. (A10) can be written as

$$P_o(E_z;t) = \int_0^\infty K_o(E_z, Z_2(M)) S_o(Z_2(M);t) dZ_2(M). \quad (A12)$$

As the result, $S_o(Z_2(M);t)$ is given by

$$S_o(Z_2(M);t) = \int_0^\infty K_o^{-1}(E_z, Z_2(M)) P_o(E_z;t) dE_z. \quad (A13)$$

As we discuss in Eq. (A7), $S_o(Z_2(M);t)$ can be expressed in two conditions in Eq. (A14), which further gives the redshift-time calibration.

$$S_o(Z_2(M);t) \simeq \begin{cases} 
\frac{dN}{dZ_2(M_i)}, & g(Z_2(M)) \gg \Delta t_z \\
\frac{dN}{dg^{-1}(\Delta t_z) g'(g^{-1}(\Delta t_z))}, & g(Z_2(M)) \ll \Delta t_z
\end{cases} \quad (A14)$$

**Appendix B: The standard timer from multi-parameter dynamical systems**

In general, we consider multi-parameter dynamical systems in the Universe, whose statistical distribution can be expressed as

$$S(M;t) = \frac{dN}{d^M M(t)} \quad (B1)$$

where $M$ denotes the $n$-dimensional physical parameter vector which characterizes the dynamical system. By introducing the initial statistical distribution as the standard reference, Eq. (B1) can be written as

$$S(M;t) = \frac{dN}{d^M M(t)} \det \mathbf{J}(M,t) \quad (B2)$$

Here, $\mathbf{J}$ is the Jacobian of the dynamical system which is defined as $\mathbf{J}_{ij} = \partial M_i(t_i) / \partial M_j(t_i)$. With the reference of the initial statistical distribution, the physical evolution time $\Delta t$ can be extracted from the determinant of the Jacobian $\det \mathbf{J}(M,t)$. However, due to strong coupling among different parameter components in its time evolution $dM/dt = -\mathbf{f}(M)$, the general analytical expression of the Jacobian element $\partial M_i(t_i) / \partial M_j(t)$ cannot be found, which indicates the numerical solution of $\det \mathbf{J}(M,t)$ is essential in extracting the physical evolution time $\Delta t$.

In the observational perspective, the redshift term caused by the cosmological expansion also appears in the statistical distribution of multi-parameter dynamical systems as following

$$S_o(M_z;t) = \frac{dN}{d^M M(z)} \det \mathbf{J}(M_z,t) \quad (B3)$$

where $M_z$ denotes the redshifted $n$-dimensional physical parameter vector and $dN/d^M M_i(z)$ characterizes the redshifted initial statistical distribution.

As we have discussed in Appendix. A, we consider two cases in extracting the redshift-time calibration. One
case is that in the parameter space where the time evolution of parameters is negligible compared with their initial value, which gives \( \text{det} J(M; \Delta t) \simeq 1 \). Then the redshift \( z \) can be obtained by comparing the observed statistical distribution \( S_o(M; \Delta t) \simeq dN/d^nM(z) \) with the initial one. The other case is that in the parameter space where the time evolution of parameters dominates their initial value, where physical evolution time \( \Delta t \) can be extracted from the numerical solution.

In the scenario that \( S(M; t) \) is not observable, we consider the observable \( P(E; t) \) as following

\[
P(E; t) = \int_V K(E, M)S(M; t)d^nM ,
\]

where, \( K(E, M) \) is the transfer kernel which transfers an unobservable \( S(M; t) \) to an observable \( P(E; t) \), \( V \) is the integral region of \( n \)-dimensional parameter \( M \). With the expansion of the Universe, the redshift effect appears in the observable in the following form

\[
P_o(E_z; t) = \int_V K_o(Z_1(E_z), M)S(M; t)d^nM .
\]

As we have shown in Eq. (A11), we introduce \( Z_1 \) and \( Z_2 \) function to transfer a redshift term from \( E_z \) to \( M \), which gives

\[
P_o(E_z; t) = \int_V K_o(E_z, Z_2(M))S_o(Z_2(M); t)d^nZ_2(M) .
\]

As the result, the unobservable statistical distribution \( S_o(Z_2(M); t) \) can be obtained by an inverse kernel transformation as Eq. (A13),

\[
S_o(Z_2(M); t) = \int_0^\infty K_o(E_z, Z_2(M))^{-1}P_o(E_z; t)dE_z .
\]

However, the analytical form of the inverse kernel in multi-parameter dynamical systems \( K_o(E, M)^{-1} \) could hardly be found, which needs further numerical methods, e.g., the method for the least squares problem [59, 60]. After obtaining \( S_o(Z_2(M); t) \), the redshift-time calibration can be extracted as Eq. (A14).

[1] G. Gamow, “The origin of elements and the separation of galaxies,” *Phys. Rev.* 74 (Aug, 1948) 505–506. [https://link.aps.org/doi/10.1103/PhysRev.74.505.2](https://link.aps.org/doi/10.1103/PhysRev.74.505.2).
[2] R. A. Alpher and R. C. Herman, “On the relative abundance of the elements,” *Phys. Rev.* 74 (Dec, 1948) 1737–1742. [https://link.aps.org/doi/10.1103/PhysRev.74.1737](https://link.aps.org/doi/10.1103/PhysRev.74.1737).
[3] R. A. Alpher and R. C. Herman, “Remarks on the evolution of the expanding universe,” *Phys. Rev.* 75 (Apr, 1949) 1089–1095. [https://link.aps.org/doi/10.1103/PhysRev.75.1089](https://link.aps.org/doi/10.1103/PhysRev.75.1089).
[4] Planck Collaboration, N. Aghanim *et al.*, “Planck 2018 results. VI. Cosmological parameters,” *Astron. Astrophys.* 641 (2020) A6, arXiv:1807.06209 [astro-ph.CO]. [Erratum: Astron. Astrophys. 652, C4 (2021)].
[5] P. J. E. Peebles, “Large scale background temperature and mass fluctuations due to scale invariant primedal perturbations,” *Astrophys. J. Lett.* 263 (1982) L1–L5.
[6] Supernova Cosmology Project Collaboration, S. Perlmutter *et al.*, “Measurements of \( \Omega \) and \( \Lambda \) from 42 high redshift supernovae,” *Astrophys. J.* 517 (1999) 565–586, arXiv:astro-ph/9802133.
[7] Supernova Search Team Collaboration, A. G. Riess *et al.*, “Observational evidence from supernovae for an accelerating universe and a cosmological constant,” *Astron. J.* 116 (1998) 1009–1038, arXiv:astro-ph/9805201.
[8] V. C. Rubin and W. K. Ford, Jr., “Rotation of the Andromeda Nebula from a Spectroscopic Survey of Emission Regions,” *Astrophys. J.* 159 (1970) 379–403.
[9] S. M. Faber and J. S. Gallagher, “Masses and mass-to-light ratios of galaxies,” *Ann. Rev. Astron. Astrophys.* 17 (1979) 135–183.
[10] M. S. Turner, G. Steigman, and L. M. Krauss, “Flatness of the universe: Reconciling theoretical prejudices with observational data,” *Phys. Rev. Lett.* 52 (Jun, 1984) 2090–2093.
[11] C. S. Frenk, S. D. M. White, G. Efstathiou, and M. Davis, “Cold dark matter, the structure of galactic haloes and the origin of the hubble sequence,” *Nature* 317 no. 6038, (Oct, 1985) 505–507.
[12] J. D. Fernie, “The period-luminosity relation: A historical review,” *Publications of the Astronomical Society of the Pacific* 81 (Dec, 1969) 707.
[13] Pan-STARRS1 Collaboration, D. M. Scolnic *et al.*, “The Complete Light-curve Sample of Spectroscopically Confirmed SNe Ia from Pan-STARRS1 and Cosmological Constraints from the Combined Pantheon Sample,” *Astrophys. J.* 859 no. 2, (2018) 101, arXiv:1710.00845 [astro-ph.CO].
[14] F. Beutler, C. Blake, M. Colless, D. H. Jones, L. Staveley-Smith, L. Campbell, Q. Parker, W. Saunders, and F. Watson, “The 6df galaxy survey: baryon acoustic oscillations and the local hubble constant,” *Monthly Notices of the Royal Astronomical Society* 416 no. 4, (2011) 3017–3032.
[15] A. J. Ross, L. Samushia, C. Howlett, W. J. Percival, A. Burden, and M. Manera, “The clustering of the SDSS DR7 main Galaxy sample – I. A 4 per cent distance measure at \( z = 0.15 \),” *Mon. Not. Roy. Astron. Soc.* 449 no. 1, (2015) 835–847, arXiv:1409.3242 [astro-ph.CO].
[16] BOSS Collaboration, T. Dehnbuc et al., “Baryon acoustic oscillations in the Lyo forest of BOSS DR11 quasars,” *Astron. Astrophys.* 574 (2015) A59, arXiv:1404.1801 [astro-ph.CO].
[17] LIGO Scientific, Virgo Collaboration, B. P. Abbott *et al.*, “GW170817: Observation of Gravitational Waves from a Binary Neutron Star Inspiral,” *Phys. Rev. Lett.*
[51] K. Ioka, T. Chiba, T. Tanaka, and T. Nakamura, “Black hole binary formation in the expanding universe: Three body problem approximation,” *Phys. Rev. D* **58** (1998) 063003, arXiv:astro-ph/9807018.

[52] A. Dolgov and J. Silk, “Baryon isocurvature fluctuations at small scales and baryonic dark matter,” *Phys. Rev. D* 47 (1993) 4244–4255.

[53] B. Carr, F. Kuhnel, and M. Sandstad, “Primordial Black Holes as Dark Matter,” *Phys. Rev. D* **94** no. 8, (2016) 083504, arXiv:1607.06077 [astro-ph.CO].

[54] B. Carr, M. Raidal, T. Tenkanen, V. Vaskonen, and H. Veermäe, “Primordial black hole constraints for extended mass functions,” *Phys. Rev. D* **96** no. 2, (2017) 023514, arXiv:1705.05567 [astro-ph.CO].

[55] Z.-C. Chen and Q.-G. Huang, “Merger Rate Distribution of Primordial-Black-Hole Binaries,” *Astrophys. J.* **864** no. 1, (2018) 61, arXiv:1801.10327 [astro-ph.CO].

[56] T. Namikawa, A. Nishizawa, and A. Taruya, “Anisotropies of gravitational-wave standard sirens as a new cosmological probe without redshift information,” *Phys. Rev. Lett.* **116** no. 12, (2016) 121302, arXiv:1511.04638 [astro-ph.CO].

[57] M. Oguri, “Measuring the distance-redshift relation with the cross-correlation of gravitational wave standard sirens and galaxies,” *Phys. Rev. D* **93** no. 8, (2016) 083511, arXiv:1603.02356 [astro-ph.CO].

[58] K. Osato, “Exploring the distance-redshift relation with gravitational wave standard sirens and tomographic weak lensing,” *Phys. Rev. D* **98** no. 8, (2018) 083524, arXiv:1807.00016 [astro-ph.CO].

[59] C. L. Lawson and R. J. Hanson, *Solving least squares problems*. SIAM, 1995.

[60] S. W. Provencher, “A constrained regularization method for inverting data represented by linear algebraic or integral equations,” *Computer Physics Communications* **27** no. 3, (1982) 213–227.