One dimensional solutions of the $\lambda$-self shrinkers

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Abstract

In this paper, we study the solution to the 1-dimensional $\lambda$-self shrinkers and show that for certain $\lambda < 0$, there are some closed, embedded solutions other than the circle.

1 Introduction

The self-shrinker equation

$$H = \frac{\langle x, N \rangle}{2}$$

comes from the self-similar solutions of mean curvature flow which move by scaling with respect to the origin. These solutions are models for singularities and are important in the study of mean curvature flow. For 1-dimensional self-shrinker in $\mathbb{R}^2$, Abresch and Langer proved the circle is the only closed, embedded solution.

In this paper, we consider a similar equation

$$H = \frac{\langle x, N \rangle}{2} + \lambda,$$  \hspace{1cm} (1)

where $\lambda$ is a constant. We show that there are some closed embedded solution other than the circle. This equation arises in the Gaussian isoperimetric problem: In $\mathbb{R}^{n+1}$, we can consider the weighted volume element $dV_{\mu} = \exp(-\frac{|x|^2}{4})dV$ and the area element $dA_{\mu} = \exp(-\frac{|x|^2}{4})dA$, where $dV$, $dA$ is the volume element and area element induced by the Euclidean metric on $\mathbb{R}^{n+1}$. The gaussian isoperimetric problem asks that among every region with weighted volume $V_0$, which one has the least weighted boundary area. The answer to this problem is given in [2], [6] that the half space minimizes the weighted boundary area.
Now, we can consider the local version of the Gaussian isoperimetric problem: Find a region $\Omega$ such that among all the region $\Omega'$ closed to $\Omega$ with the same weighted volume, the weighted boundary area for $\Omega$ is the smallest. Let $\partial \Omega = \Sigma$, $F : \Sigma \times [-\epsilon, \epsilon] \to \mathbb{R}^{n+1}$ be a smooth variation with $\langle \partial_t F(x,0), N(x) \rangle = u$, $\Sigma_t = F(\sigma,t)$, $\Omega_t$ be the region enclosed by $\Sigma_t$. Compute the first order derivative of the weighted area functional and the weighted volume functional for the variation with compact support on $\Sigma$, we have

$$
\partial_t V_\mu(\Omega_t) = \int_\Sigma u dA_\mu,
$$
$$
\partial_t A_\mu(\Sigma_t) = \int_\Sigma u(H - \frac{\langle x, N \rangle}{2}) dA_\mu.
$$

For all the compact variation that fixes the volume, we have $\int_\Sigma u dA_\mu = 0$. The first derivative of the weighted boundary area should also be 0 under such variation. Therefore, the boundary $\Sigma$ must satisfies the equation $[1]$. This equation is defined on $\Sigma$ locally and it can be studied even if $\Sigma$ does not actually come from a boundary of some open region. For more detail, the reader can refer to [2]. When $\lambda = 0$, the equation $[1]$ is the self-shrinker equation since the self-shrinkers can also be realized as the critical point of the weighted area functional in Gaussian space.

To simplify the equation, we scale the solution to drop the constant $\frac{1}{2}$ and use the one dimensional curvature $k$ in the place of $H$. The equation now becomes

$$
k = -\langle x, N \rangle + \lambda.
$$

in this paper. It’s interesting to compare the result with the isoperimetric problem in the Euclidean space, where the critical surface to the area functional should have constant mean curvature. Therefore, the only 1-dimensional solution in $\mathbb{R}^2$ is the circle.

This paper will be structured as following: In section 2, start from the defining equation, we derive an ODE system for the 1-dimensional $\lambda$-self shrinkers and define some quantity and formula for the later use. In section 3, we analyze the behavior of the solution prove that under the restriction $k > 0$. There are no solution other than the circle when $\lambda > 0$. For $\frac{2}{\sqrt{3}} < \lambda < 0$ or $\lambda < \frac{-7}{2\sqrt{2}}$, there are closed embedded solutions other than the circle. In section 4, some of the solutions are graphed by the computer for a better understanding.
2 Setting up the ODE system

For a curve \( x(s) \in \mathbb{R}^2 \), where \( s \) is the arc length of the curve, we have the following:

\[
\frac{d}{ds} x = T, \\
\frac{d}{ds} T = kN.
\]

Note that for any curve in \( \mathbb{R}^2 \), we have two possible choices of \( N \): either rotate \( T \) clockwise by \( \frac{\pi}{2} \) or \( -\frac{\pi}{2} \). If we let \( N^- = -N, k^- = -k \), we have \( kN = k^- N^- \). Therefore, we have

\[
k^- = -k = \langle x, N \rangle - \lambda = -\langle x, N^- \rangle - \lambda.
\]

This tells us the change of selection of \( N \) will change the sign of \( k \) and give us a solution corresponding to \( -\lambda \).

Use the method in [4], put \( \tau = \langle x, T \rangle, \nu = \langle x, N \rangle \), where \( T = \frac{d}{ds} x \) is the unit tangent of the curve and \( kN = \frac{d}{ds} T \) is the curvature vector. We can get the system

\[
\frac{d}{ds} \tau = 1 + k\nu = 1 - \nu^2 + \lambda \nu \\
\frac{d}{ds} \nu = -k\tau = \nu \tau - \lambda \tau.
\]

The equilibrium is given by solving \( \frac{d}{ds} \tau = \frac{d}{ds} \nu = 0 \). It’s given by \((0, \nu^+)\), where \( \nu^\pm \) are the positive and the negative solution of the equation of \( \nu^2 - \lambda \nu - 1 = 0 \), respectively. At the equilibrium, the curvature is a nonzero constant. It corresponds to the circle centered at the origin. For \((0, \nu^+)\), it’s a circle of radius \( \nu^+ \) with the normal pointed outward and \( k < 0 \). For \((0, \nu^-)\), it’s a circle of radius \( -\nu^- = |\nu^-| \) with the normal pointed inward and \( k > 0 \).

Also, note that \((\tau, \nu) = (s, \lambda)\) is a solution which correspond to a line with the minimum distance to the origin equal to \( \lambda \). From now on, without lose of generality, we only consider the solution with \( k \geq 0 \). That’s the half plane in \( \tau - \nu \) plane which is below the \( \{\nu = \lambda\} \) line. For the part \( k < 0 \), we can choose the opposite normal and consider it as the solution corresponding to \( -\lambda \) with positive \( k \).

2.1 Periodicity of the solution

On each of the solution curve, if we consider the function

\[
F(\tau, \nu) = (\lambda - \nu) \exp(-\frac{\nu^2 + \tau^2}{2}),
\]

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note that in the \( \{ \nu \leq \lambda \} \) half plane, we always have \( F \geq 0 \). Differentiate with respect to the arclength \( s \), we have

\[
\frac{d}{ds}F = \left( - \frac{d}{ds} \nu - (\lambda - \nu)(\nu \frac{d}{ds} \nu + \tau \frac{d}{ds} \tau) \right) \exp\left(-\frac{\nu^2 + \tau^2}{2}\right)
\]

\[
= \left( - (\nu - \lambda)\tau + (\nu - \lambda)(\nu(\nu - \lambda)\tau + \tau(1 - \nu(\nu - \lambda))) \right) \exp\left(-\frac{\nu^2 + \tau^2}{2}\right)
\]

\[
= 0.
\]

Therefore, every solution lies in a level set of \( F \). Since all the level set of \( F \) is a simple closed curve except \( \{ F = 0 \} \), which correspond to the line mentioned before, we can get a uniform lower bound away from 0 about the speed of the \((\tau(s), \nu(s))\) curve on each level set and conclude that the solution \((\tau(s), \nu(s))\) should be periodic in \( s \).

**Remark 1.** Note that if \( x(s) \) is periodic, we have \( \tau, \nu \) are periodic. But we don’t have the opposite. Starting from periodic \((\tau(s), \nu(s))\), even though we can reconstruct \( x(s) \) by this and the initial condition, the resulting \( x(s) \) is periodic only when the change of angle in a period can be express as \( \frac{\kappa}{2}\pi \), where \( k, l \) are relatively prime positive integers. In this case, the period of \( x(s) \) is \( l \) times the period of \((\tau(s), \nu(s))\) and we can get a closed solution.

### 2.2 Change of angle in a period

Now, since \( k \) is more directly related to the geometric behavior than \( \nu \), we use \((\tau, k)\) as the variable instead of \((\tau, \nu)\). Also, from now on, unless otherwise specified, we use \( ' \) for \( \frac{d}{ds} \). Plug \( \nu = \lambda - k \). The ODE system becomes

\[
\tau' = 1 + \lambda k - k^2,
\]

\[
k' = k\tau.
\]

and the \( \{ \nu \leq \lambda \} \) half plane becomes \( \{ k \geq 0 \} \) in \( \tau - k \) plane. Note that under this change of variable, we still have the equilibrium at \((0, k^\pm)\), where \( k^\pm = \nu^\pm \) because they satisfy exactly the same equation. However, \((0, k^\pm)\) correspond to \((0, \nu^\pm)\), respectively. The \( \nu = \lambda \) line in \( \tau - \nu \) space now becomes \( k = 0 \) line in \( \tau - k \) space.

In the \( \{ k > 0 \} \) half space, set \( B = 2 \log k \). We have \( B' = 2\tau \) and

\[
B'' = 2\tau' = 2 + 2k(\lambda - k) = 2 + 2\lambda e^\frac{B}{2} - 2e^B.
\]

Multiply both side by \( B' \) and integrate with respect to \( s \), we get

\[
\frac{1}{2}(B')^2 + 2e^B - 4\lambda e^\frac{B}{2} - 2B = -4 \log F - 2\lambda^2.
\]
If we define $F_\lambda = F \cdot \exp \frac{\lambda^2}{2}$, $V(B) = e^B - 2\lambda e^{\frac{B}{2}} - B$, we have

$$\frac{1}{2}(B')^2 + 2V(B) = -4\log F_\lambda.$$ 

The minimum of $V(B)$ is attained when $\frac{d}{dB} V(B) = 0$. $e^\frac{B}{2} = k^+$. This corresponds to the equilibrium at $(0, k^+)$ and $\min V(B) = -\lambda k^+ - 2\log k^+ + 1$. Now, for any $\eta > \min V(B)$, we can find the solutions $B^-_\eta < B^+\eta$ of $V(B) = \eta$. Consider the differential equation of $B$, we get

$$\frac{1}{2}(B')^2 + 2V(B) = 2\eta,$$

$$B' = \pm 2\sqrt{\eta - V(B)}.$$ 

Therefore, the length of the curve in a period is given by

$$\oint ds = 2 \int_{B^-\eta}^{B^+\eta} \frac{dB}{ds}^{-1} dB = \int_{B^-\eta}^{B^+\eta} \frac{1}{\sqrt{\eta - V(B)}} dB,$$

and the change of the angle in a period is given by

$$\Delta \theta = \oint k ds = \int_{B^-\eta}^{B^+\eta} \frac{e^{\frac{B}{2}}}{\sqrt{\eta - V(B)}} dB.$$

In order to simplify the calculation, let $u = e^\frac{B}{2}$, $u^-\eta = e^\frac{B^-\eta}{2}$, $u^+\eta = e^\frac{B^+\eta}{2}$ respectively, $V(u) = u^2 - 2\lambda u - 2\log u$. The change of angle, $\Delta \theta$ can be expressed as

$$\Delta \theta = \int_{u^-\eta}^{u^+\eta} \frac{2du}{\sqrt{\eta - V(u)}}.$$

### 3 The behavior of the solutions

Now, we will focus on the behavior of $\Delta \theta$ when the energy $\eta$ varies from $\min V(B)$ to $\infty$.

#### 3.1 The behavior of the solution when $\eta$ is near $\min V(B)$

**Lemma 2.** For any potential function $V \in C^2$, at a local minimum $x_0$ with positive second derivative, let $u^-\eta$, be the largest solution of $V(u) = \eta$ which is below $x_0$, $u^+\eta$, be the smallest solution of $V(u) = \eta$ which is above $x_0$, we have

$$\lim_{\eta \to V(x_0)^+} \int_{u^-\eta}^{u^+\eta} \frac{du}{\sqrt{\eta - V(u)}} = \sqrt{\frac{2}{V''(x_0)}} \pi.$$
Proof. First, note that for the case in which the potential is quadratic, \( V(u) = V(x_0) + \frac{V''(x_0)(x-x_0)^2}{2} \), a simple calculation shows that

\[
\int_{u^-}^{u^+} \frac{du}{\sqrt{\eta - V(u)}} = \sqrt{\frac{2}{V''(x_0)}} \pi
\]

for any \( \eta > V(x_0) \) and is independent of \( \eta \).

For general potential function \( V \in C^2 \), for any \( \epsilon > 0 \), we can find \( \delta > 0 \) such that for all \( V(x_0) < \eta < V(x_0) + \delta \), we have \( |V''(u) - V''(x_0)| < \epsilon \) for \( u \in [u^-, u^+] \). Let \( V_\pm \) be the quadratic function which pass through \((u^-, \eta), (u^+, \eta)\), \( V''_\pm = V''(x_0) \pm \epsilon \). We have \( V_- < V < V_+ \) in \((u^-, u^+)\). Therefore, we have

\[
\sqrt{\frac{2}{V''(x_0)}} + \epsilon \pi = \int_{u^-}^{u^+} \frac{du}{\sqrt{\eta - V(u)}} \leq \int_{u^-}^{u^+} \frac{du}{\sqrt{\eta - V_-(u)}} \leq \int_{u^-}^{u^+} \frac{du}{\sqrt{\eta - V_+(u)}} \leq \sqrt{\frac{2}{V''(x_0)}} - \epsilon \pi.
\]

Letting \( \epsilon \) goes to 0 yields the desired result. \( \square \)

**Proposition 3.** When \( \eta \to \min V(B)^+ \), \( \triangle \theta \) approaches \( \pi \sqrt{2} \sqrt{\frac{\lambda}{\sqrt{\lambda^2 + 4}}} + 1 \).

Moreover, \( \triangle \theta \) is decreasing in a neighborhood of \( \min V(B) \).

**Proof.** Let \( \eta \to \min V(B)^+ \). The derivatives of \( V(u) \) with respect to \( u \) at the minimum point is

\[
V^{(2)}(k^+) = 2 + 2(k^+)^{-2},
V^{(3)}(k^+) = -4(k^+)^{-3},
V^{(4)}(k^+) = 12(k^+)^{-4}.
\]

Therefore, from the lemma above and also recall that \( k^+ = \frac{\lambda + \sqrt{\lambda^2 + 4}}{2} \), we have

\[
\lim_{\eta \to \min V(B)^+} \triangle \theta = \lim_{\eta \to \min V(B)^+} \int_{u^-}^{u^+} \frac{2du}{\sqrt{\eta - V(u)}} = 2\pi \cdot \frac{2(k^+)^2}{2(k^+)^2 + 2} = \pi \sqrt{2} \sqrt{\frac{\lambda}{\sqrt{\lambda^2 + 4}}} + 1.
\]

From the result of \( \beth \), since

\[
5(V^{(3)})^2 - 4V^{(2)}V^{(4)} = 80(k^+)^{-6} - 96((k^+)^{-6} + (k^+)^{-4}) = -16(k^+)^{-6} - 96(k^+)^{-4} < 0,
\]

the function \( \triangle \theta \) is monotone decreasing near \( \min V(B) \) with respect to \( \eta \). \( \square \)
Remark 4. For the original self shrinker case ($\lambda = 0$), we have $\Delta \theta \to \sqrt{2}\pi$, as the result in [11]. This function is strictly increasing with respect to $\lambda$. When $\lambda$ approaches $\infty$, $\Delta \theta$ approaches $2\pi$. When $\lambda$ approaches $-\infty$, $\Delta \theta$ approaches 0.

3.2 The behavior of the solution when $\eta$ is near infinity

Now, we turn our attention to the behavior of $\Delta \theta$ when the energy approaches infinity. The upperbound of $\Delta \theta$ is given by the following proposition.

Proposition 5. For any $L > 1$, we have

$$
\Delta \theta \leq \pi + 2(\lambda - 1 + \sqrt{\frac{L}{L-1}}) \frac{1}{\sqrt{\eta}} + o\left(\frac{1}{\sqrt{\eta}}\right)
$$

as $\eta$ goes to infinity.

Proof. In order to get the upper bound of $\Delta \theta$, separate the integration into two terms,

$$
\Delta \theta = \int_{u_{\eta}^-}^{u_{\eta}^+} \frac{2du}{\sqrt{\eta - V(u)}} = \int_{1}^{u_{\eta}^-} \frac{2du}{\sqrt{\eta - V(u)}} + \int_{1}^{u_{\eta}^+} \frac{2du}{\sqrt{\eta - V(u)}}
$$

When $1 \leq u \leq u_{\eta}^+$, let $\bar{u}_{\eta}^+$ be the positive solution of $\eta = u^2 - 2\lambda u$. Note that $\bar{u}_{\eta}^+ < u_{\eta}^+$. Set $\bar{V}(u) = (\frac{\bar{u}_{\eta}^+}{u_{\eta}^+} - 1)(u - 1 + 1)^2 - 2\lambda(\frac{\bar{u}_{\eta}^+}{u_{\eta}^-} - 1)(u - 1 + 1)$. At $u = u_{\eta}^+$, $V(u) = \bar{V}(u) = \eta$. At $u = 1$, $V(u) = \bar{V}(u) = 1 - 2\lambda$. The second derivative of $\bar{V}(u) - V(u)$ is

$$
\left(\bar{V}(u) - V(u)\right)^{''} = 2(\frac{\bar{u}_{\eta}^+}{u_{\eta}^+} - 1)^2 - (2 + 2\frac{1}{u^2}) < 0.
$$

We can conclude

$$
\bar{V}(u) - V(u) \geq 0
$$

for any $1 < u < u_{\eta}^+$. Therefore, we have

$$
\int_{1}^{u_{\eta}^+} \frac{2du}{\sqrt{\eta - V(u)}} \leq \int_{1}^{u_{\eta}^+} \frac{2du}{\sqrt{\eta - \bar{V}(u)}}
$$

$$
= \frac{u_{\eta}^+ - 1}{\bar{u}_{\eta}^+ - 1} \int_{1}^{\bar{u}_{\eta}^+} \frac{2dv}{\sqrt{\eta - v^2 + 2\lambda v}}
$$

$$
= 2\frac{u_{\eta}^+ - 1}{u_{\eta}^+ - 1}\left(\frac{\pi}{2} - \sin^{-1}\frac{1 - \lambda}{\sqrt{\eta + \lambda^2}}\right).
$$
We need to get an upper bound for \( \frac{u_\eta^+ - 1}{\bar{u}_\eta^+ - 1} \). Start from \( \bar{u}_\eta^+ = \lambda + \sqrt{\lambda^2 + \eta} \), \( V(\bar{u}_\eta^+) = \eta - 2 \log \bar{u}_\eta^+ \), \( V(u_\eta^+) = \eta \) and \( V'(u) \geq 2 \bar{u}_\eta^+ - 2\lambda - 2 \frac{1}{\bar{u}_\eta^+} \) for \( \bar{u}_\eta^+ < u < u_\eta^+ \), we have
\[
\sqrt{\eta - V(u)} = \eta - 2 \log \bar{u}_\eta^+ \leq C \log \eta \leq O(\eta^{-\frac{1}{2}} \log \eta)
\]
for \( \eta \) large enough. Hence,
\[
\frac{u_\eta^+ - 1}{\bar{u}_\eta^+ - 1} = 1 + \frac{u_\eta^+ - \bar{u}_\eta^+}{\bar{u}_\eta^+ - 1} = 1 + O(\eta^{-1} \log \eta).
\]
Therfore,
\[
\int_1^{u_\eta^+} \frac{2du}{\sqrt{\eta - V(u)}} \leq \frac{2L}{\eta^2} \int_{\bar{u}_\eta^+}^{u_\eta^+} \frac{2du}{u_\eta^+ - 1} = \frac{\pi}{2} - \sin^{-1} \left( \frac{1}{\sqrt{\eta} + \lambda} \right) = \pi + 2 \frac{\lambda - 1}{\sqrt{\eta}} + o(\frac{1}{\sqrt{\eta}}).
\]
Now, we are going to estimate the other term. For all \( L > 1 \), let \( \eta^{-} = \exp(-\frac{L}{2\eta} + \frac{1}{2} + |\lambda|) \). Note that when \( \eta \) is large enough, \( \eta^{-} < 1 \) and \( V(\eta^{-}) < \frac{\eta}{L} \).
\[
\int_{\eta^{-}}^{1} \frac{2du}{\sqrt{\eta - V(u)}} = \int_{\eta^{-}}^{\eta^{-}} \frac{2du}{\eta^{-} - V(u)} + \int_{\eta^{-}}^{1} \frac{2du}{\eta^{-} - V(u)}.
\]
For the first term, since \( V(\eta^{-}) = \eta \), \( V(\eta^{-}) < \frac{\eta}{L} \), \( V'' > 0 \), we have
\[
V(u) < \frac{\eta}{L} + \left( \frac{L - 1}{L} \right) \eta \frac{u - \eta^{-}}{\eta^{-} - \eta^{-}}
\]
for \( \eta^{-} < u < \eta^{-} \) and
\[
\int_{\eta^{-}}^{\eta^{-}} \frac{2du}{\eta^{-} - V(u)} \leq \int_{\eta^{-}}^{\eta^{-}} \frac{2du}{\sqrt{\frac{L - 1}{L} \eta^{-} (\frac{u - \eta^{-}}{\eta^{-} - \eta^{-}})}} = (\eta^{-} - \eta^{-}) \int_{0}^{1} \frac{2dv}{\sqrt{\frac{L - 1}{L} \eta^{-} v}} \leq \eta^{-} \int_{0}^{1} \frac{2dv}{\sqrt{\frac{L - 1}{L} \eta^{-}}} = \eta^{-} \sqrt{\frac{L}{L - 1} \frac{4}{\eta^{-}}}.
\]
The second term can be bounded by the following,
\[
\int_{\eta^{-}}^{1} \frac{2du}{\eta^{-} - V(u)} \leq \int_{\eta^{-}}^{1} \frac{2du}{\sqrt{\frac{L - 1}{L} \eta^{-} v}} \leq \sqrt{\frac{L}{L - 1} \frac{2}{\eta^{-}}}.
\]
Therefore, we get
\[
\int_{u_\eta}^{1} \frac{2du}{\sqrt{\eta - V(u)}} \leq \sqrt{\frac{L}{L - 1}} \frac{2}{\sqrt{\eta}} + o\left(\frac{1}{\sqrt{\eta}}\right).
\]

Combine the estimation of both terms, we have
\[
\triangle \theta \leq \pi + 2\left(\lambda - 1 + \frac{1}{\sqrt{\eta}}\right) + o\left(\frac{1}{\sqrt{\eta}}\right)
\]

Now, the lower bound of \(\triangle \theta\) is given by the following:

**Proposition 6.** We have
\[
\triangle \theta \geq \pi + \sin^{-1} \frac{\lambda - u_\eta^-}{\sqrt{\eta + 2 \log u_\eta^+ + \lambda^2}}.
\]

**Proof.** To get the lower bound of \(\triangle \theta\), use \(\log u \leq \log u_\eta^+\) when \(u_\eta^- \leq u \leq u_\eta^+\).
\[
\triangle \theta = \int_{u_\eta^-}^{u_\eta^+} \frac{2du}{\sqrt{\eta - V(u)}}
\]
\[
\geq \int_{u_\eta^-}^{u_\eta^+} \frac{2du}{\sqrt{\eta - u^2 + 2\lambda u + 2 \log u_\eta^+}}
\]
\[
= \int_{u_\eta^-}^{u_\eta^+} \frac{2du}{\sqrt{(\eta + 2 \log u_\eta^+ + \lambda^2) - (u - \lambda)^2}}
\]
\[
= \pi + \sin^{-1} \frac{\lambda - u_\eta^-}{\sqrt{\eta + 2 \log u_\eta^+ + \lambda^2}}.
\]

Now combine both the upper bound and the lower bound, we can get the limit of \(\triangle \theta\) when the energy \(\eta\) goes to infinity.

**Proposition 7.** When the energy \(\eta\) goes to infinity, we have
\[
\lim_{\eta \to \infty} \triangle \theta = \pi.
\]

**Proof.** As \(\eta\) goes to infinity, we have \(u_\eta^+\) goes to infinity and \(u_\eta^-\) goes to zero. Therefore, the upper bound and the lower bound both goes to \(\pi\) as \(\eta\) goes to infinity.

\[
\square
\]
For the case $\lambda < 0$, combining the behavior of $\triangle \theta$ near $\min V(B)$ and infinity, we can guarantee the existence of embedded solution for some $\lambda$.

**Corollary 8.** When $\lambda < 0$, $\triangle \theta < \pi$ for $\eta$ large enough.

**Proof.** Since

$$\triangle \theta \leq \pi + 2\left(\lambda - 1 + \sqrt{\frac{L}{L - 1}}\right)\frac{1}{\sqrt{\eta}} + o\left(\frac{1}{\sqrt{\eta}}\right)$$

for arbitrary $L > 1$, choose $L$ large enough so that $\lambda - 1 + \sqrt{\frac{L}{L - 1}} < 0$. $\square$

**Corollary 9.** For $\frac{-2}{\sqrt{3}} < \lambda < 0$, there exists embedded solution with 2-symmetry.

**Proof.** For any $\lambda$ in this range, when $\eta \to \min V(B)+$, the limit of $\triangle \theta$ is greater than $\pi$. Since $\triangle \theta$ is a continuous function of $\eta$ and in this case, $\triangle \theta < \pi$ when $\eta$ is large enough, there exists $\eta$ such that $\triangle \theta$ is exactly $\pi$. $\square$

**Corollary 10.** There exist $\delta > 0$ such that for $\lambda < \frac{-7}{2\sqrt{2}} + \delta$, there exists embedded solution with $k$-symmetry, $k > 2$.

**Proof.** For any $\lambda < \frac{-7}{2\sqrt{2}}$, when $\eta \to \min V(B)+$, the limit of $\triangle \theta$ is less than $\frac{2\pi}{k}$. When $\eta$ is large enough, $\triangle \theta$ approaches $\pi$. Since $\triangle \theta$ is a continuous function of $\eta$, there exist $\eta$ such that $\triangle \theta$ is exactly $\frac{2\pi}{k}$, for some $k > 2$.

Since $\triangle \theta$ is decreasing when $\eta$ is near $\min V(B)$. When $\lambda = \frac{-7}{2\sqrt{2}}$, $\min \eta \triangle \theta < \frac{2\pi}{3}$. From the continuity, the result above can be extend a little higher than $\frac{-7}{2\sqrt{2}}$. $\square$

### 3.3 Relation between $\lambda$ and $\triangle \theta$

For the case $\lambda > 0$, the behavior is similar to the original case for self shrinking curve in Abresch and Langer’s paper. We want to compare the change of angle with the case $\lambda = 0$.

In order to understand the behavior of $\triangle \theta$ with respect to $\lambda$, we do the following: For simplicity, use $k$ for $k^+$ which is depend on $\lambda$. Move the minimum point of $V^\lambda(u)$ to the origin, define $\tilde{V}^\lambda(u) = V^\lambda(u + k) - \min V^\lambda$, where $\min \tilde{V}^\lambda = \tilde{V}^\lambda(k) = k^2 - 2\lambda k - 2\log k$. Let $\tilde{\eta} = \eta - \min \tilde{V}^\lambda$ be the energy relative to the minimum. We have the following theorem:

**Theorem 11.** If we set everything as above, write $\triangle \theta = \triangle \theta(\tilde{\eta}, \lambda)$, then when we fix $\tilde{\eta}$, $\triangle \theta$ is increasing with respect to $\lambda$.  

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Proof. Note that in this setting,

$$\Delta \theta = \int_{u_{\tilde{\eta},\lambda}^-}^{u_{\tilde{\eta},\lambda}^+} \frac{2du}{\sqrt{\tilde{\eta} - \hat{V}(u)}},$$

where $u_{\tilde{\eta},\lambda}^\pm$ are the positive and negative solution of $\tilde{\eta} = \hat{V}_\lambda(u)$ respectively.

Now, fix $\tilde{\eta}$, we want to know the relation between $\lambda$ and $u$ when $\tilde{\eta} = \hat{V}_\lambda(u)$. Differentiate the equation with respect to $\lambda$, we have

$$0 = 2\left(\frac{(u + k) - \lambda - \frac{1}{u + k}}{u + k}\right)\left(\frac{du}{d\lambda} + \frac{dk}{d\lambda}\right) - 2(u + k) - 2(k - \lambda - \frac{1}{k})\frac{dk}{d\lambda} + 2k$$

$$= 2\frac{(u + k)^2 - \lambda(u + k) - 1}{u + k}\left(\frac{du}{d\lambda} + \frac{dk}{d\lambda}\right) - 2u$$

$$= 2\frac{u^2 + 2ku - \lambda u}{u + k}\left(\frac{du}{d\lambda} + \frac{dk}{d\lambda}\right) - 2u$$

$$= 2u\left[\frac{u + 2k - \lambda}{u + k}\left(\frac{du}{d\lambda} + \frac{dk}{d\lambda}\right) - 1\right].$$

Therefore,

$$\frac{du}{d\lambda} + \frac{dk}{d\lambda} = \frac{u + k}{u + 2k - \lambda}.$$

Since $k = \frac{\lambda + \sqrt{\lambda^2 + 4}}{2}$, we have

$$\frac{dk}{d\lambda} = \frac{1 + \frac{\lambda}{\sqrt{\lambda^2 + 4}}}{2} = \frac{\sqrt{\lambda^2 + 4} + \lambda}{2\sqrt{\lambda^2 + 4}} = \frac{k}{2k - \lambda},$$

and

$$\frac{du}{d\lambda} = \frac{u + k}{u + 2k - \lambda} - \frac{k}{2k - \lambda} = \frac{(k - \lambda)u}{(u + 2k - \lambda)(2k - \lambda)}.$$

Since $k - \lambda$, $2k - \lambda$, $u + k$ are all positive, $\frac{du}{d\lambda}$ has the same sign as $u$. It means $\frac{\partial u_{\tilde{\eta},\lambda}^+}{\partial \lambda} > 0$, $\frac{\partial u_{\tilde{\eta},\lambda}^-}{\partial \lambda} < 0$.

Now, starting from $\hat{V}'_\lambda(u) = 2(u + k - \lambda - \frac{1}{u + k})$, when we fix $\tilde{\eta}$, we want to know the change of the slope of $\hat{V}'_\lambda$ at $u_{\tilde{\eta},\lambda}^\pm$ with respect to $\lambda$. Differentiate the equation with respect to $\lambda$, we have

$$\frac{d}{d\lambda}(\hat{V}'_\lambda(u)) = 2\left((1 + \frac{1}{(u + k)^2})\left(\frac{du}{d\lambda} + \frac{dk}{d\lambda}\right) - 1\right)$$

$$= 2\left((1 + \frac{1}{(u + k)^2})\frac{u + k}{u + 2k - \lambda} - 1\right)$$

$$= \frac{(\lambda - k)u}{(u + k)(u + 2k - \lambda)}.$$
Note that $\lambda - k < 0$, therefore $\frac{d}{du}(\tilde{V}'_\eta(u))$ and $u$ have the opposite sign.

Now, fix $\tilde{\eta}, \lambda_1 < \lambda_2$. Since $\frac{d}{du}$ and $u$ has the same sign, we have $u_{\tilde{\eta},\lambda_2} < 0 < u_{\tilde{\eta},\lambda_1} < u_{\tilde{\eta},\lambda_2}$. Consider the function $\tilde{V}_{\lambda_1}(u)$ and $\tilde{V}_{\lambda_2}(u + u_{\tilde{\eta},\lambda_2} - u_{\tilde{\eta},\lambda_1})$. Both of them have the same value $\tilde{\eta}$ at $u = u_{\tilde{\eta},\lambda_1}$. Now, for all fixed $\tilde{\eta} \in (0, \tilde{\eta})$, $d\lambda(\tilde{V}'_\eta(u))$ and $u$ have the opposite sign, we have

$$\frac{\partial u_{\tilde{\eta},\lambda_1}}{\partial \tilde{\eta}} = \frac{1}{\tilde{V}'_{\lambda_1}(u_{\tilde{\eta},\lambda_1})} < \frac{1}{\tilde{V}'_{\lambda_2}(u_{\tilde{\eta},\lambda_2})} = \frac{\partial u_{\tilde{\eta},\lambda_2}}{\partial \tilde{\eta}}.$$ 

Therefore, for any $\tilde{\eta} \in (0, \tilde{\eta})$, we have $u_{\tilde{\eta},\lambda_2} > u_{\tilde{\eta},\lambda_1} > (u_{\tilde{\eta},\lambda_2} - u_{\tilde{\eta},\lambda_1})$, i.e. the graph of $(u_{\tilde{\eta},\lambda_1}, \tilde{\eta})$ lies on the right of the graph of $(u_{\tilde{\eta},\lambda_2} - u_{\tilde{\eta},\lambda_1}, \tilde{\eta})$.

Since $\tilde{V}'_{\lambda_2}(u_{\tilde{\eta},\lambda_2}) > 0$, it implies $\tilde{V}_{\lambda_1}(u) < \tilde{V}_{\lambda_2}(u + u_{\tilde{\eta},\lambda_2} - u_{\tilde{\eta},\lambda_1})$ for $u \in (0, u_{\tilde{\eta},\lambda_1})$. We can do the same argument for the negative part and get $\tilde{V}_{\lambda_1}(u) < \tilde{V}_{\lambda_2}(u + u_{\tilde{\eta},\lambda_2} - u_{\tilde{\eta},\lambda_1})$ for $u \in (u_{\tilde{\eta},\lambda_1}, 0)$.

Therefore,

$$\triangle \theta(\tilde{\eta}, \lambda_1) = \int_{u_{\tilde{\eta},\lambda_1}}^{u_{\tilde{\eta},\lambda_2}} \frac{2du}{\sqrt{\tilde{\eta} - \tilde{V}_{\lambda_1}(u)}} = \int_{0}^{u_{\tilde{\eta},\lambda_1}} \frac{2du}{\sqrt{\tilde{\eta} - \tilde{V}_{\lambda_1}(u)}} + \int_{0}^{u_{\tilde{\eta},\lambda_2}} \frac{2du}{\sqrt{\tilde{\eta} - \tilde{V}_{\lambda_2}(u)}}$$

$$< \int_{0}^{u_{\tilde{\eta},\lambda_1}} \frac{2du}{\sqrt{\tilde{\eta} - \tilde{V}_{\lambda_2}(u + u_{\tilde{\eta},\lambda_2} - u_{\tilde{\eta},\lambda_1})}} + \int_{0}^{u_{\tilde{\eta},\lambda_2}} \frac{2du}{\sqrt{\tilde{\eta} - \tilde{V}_{\lambda_2}(u + u_{\tilde{\eta},\lambda_2} - u_{\tilde{\eta},\lambda_1})}}$$

$$= \int_{u_{\tilde{\eta},\lambda_2}}^{u_{\tilde{\eta},\lambda_1}} \frac{2du}{\sqrt{\tilde{\eta} - \tilde{V}_{\lambda_2}(u)}} + \int_{u_{\tilde{\eta},\lambda_2}}^{u_{\tilde{\eta},\lambda_1}} \frac{2du}{\sqrt{\tilde{\eta} - \tilde{V}_{\lambda_2}(u)}}$$

$$< \int_{u_{\tilde{\eta},\lambda_2}}^{u_{\tilde{\eta},\lambda_1}} \frac{2du}{\sqrt{\tilde{\eta} - \tilde{V}_{\lambda_2}(u)}} = \triangle \theta(\tilde{\eta}, \lambda_2).$$

Now, we can deal with the case $\lambda > 0$ by compare it with the self shrinker case.

**Corollary 12.** When $\lambda > 0$, for any $\tilde{\eta} > 0$, $\triangle \theta > \pi$.

**Proof.** Use the result in [1] that when $\lambda = 0$, $\triangle \theta(\tilde{\eta}, 0)$ is a decreasing function of $\tilde{\eta}$, $\lim_{\tilde{\eta} \to 0} \triangle \theta(\tilde{\eta}, 0) = \sqrt{2\pi}$, $\lim_{\tilde{\eta} \to -\infty} \triangle \theta(\tilde{\eta}, 0) = \pi$, we have for all $\tilde{\eta}$, $\triangle \theta(\tilde{\eta}, 0) > \pi$. Let $\lambda_1 = 0$, $\lambda_2 = \lambda$ in the previous theorem, we get $\triangle \theta(\tilde{\eta}, \lambda) > \triangle \theta(\tilde{\eta}, 0) > \pi$ for $\lambda > 0$. 

\[\square\]
Remark 13. Since $\Delta \theta > \pi$ when $\lambda > 0$, there is no embedded solution when $\lambda > 0$.

4 Simulation of the curves

By using some computer program, we are able to use numerical method to see what should the solution be like.

When $\lambda > 0$, the range for $\Delta \theta$ contains $(\pi, \pi\sqrt{2}\frac{\sqrt{\lambda}}{\sqrt{\lambda^2 + 1}} + 1]$. In this case, $\Delta \theta > \pi$ and there will not be embedded solutions. The following is some of the closed solutions for the case $\lambda = 0.19$ and $\lambda = 0.726$. The energy $\eta$ increases from left to right. Note that at some point, the solution passes the origin. If we keep increase $\eta$, unlike the case where $\lambda = 0$, the origin will not be on the same side of the solution anymore. Also, we can conjecture that $\Delta \theta$ is decreasing when $\eta$ is increasing, as in the case for self-shrinkers.

![Figure 1: Solutions for $\lambda = 0.19$, $\Delta \theta = \frac{10\pi}{7}, \frac{4\pi}{3}, \frac{5\pi}{4}, \frac{7\pi}{6}$, respectively.](image)

![Figure 2: Solutions for $\lambda = 0.726$, $\Delta \theta = \frac{8\pi}{5}, \frac{3\pi}{2}, \frac{10\pi}{7}, \frac{4\pi}{3}$, respectively.](image)

The case when $\lambda < 0$ is much more interesting. For each $-\frac{2}{\sqrt{3}} < \lambda < 0$, there exist $\eta$ such that $\Delta \theta = \pi$ and the solution is embedded and have 2-symmetry. The following are some of the examples. From left to right, $\lambda = -0.2, -0.3, -0.4, -0.5, -0.6, -0.7, -0.8, -0.9$, respectively.

For $\lambda \leq -\frac{2}{\sqrt{3}}$, we don’t have embedded solutions with 2-symmetry. However, as $\lambda < -\frac{7}{2\sqrt{2}}$, we have embedded solution with $k$-symmetry, $k > 2$. The
following are the case where $\lambda = -3, -5$. The energy $\eta$ increases from left to right. Unlike the case $\lambda > 0$, when $\lambda \leq -\frac{2}{\sqrt{3}}$, even though $\Delta \theta$ should be decreasing near $\min V(B)$, we can observe that for the larger $\eta$, $\Delta \theta$ should be increasing while $\eta$ is increasing.

Figure 4: Solutions for $\lambda = -3$, $\Delta \theta = \frac{3\pi}{5}, \frac{2\pi}{3}, \frac{5\pi}{7}, \frac{4\pi}{5}$, respectively.

Figure 5: Solutions for $\lambda = -5$, $\Delta \theta = \frac{2\pi}{5}, \frac{\pi}{2}, \frac{4\pi}{7}, \frac{2\pi}{3}$, respectively.

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