Multicomponent theory of buoyancy instabilities in magnetized astrophysical plasmas: MHD analysis revisited

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ABSTRACT

We develop a theory of buoyancy instabilities of the electron-ion plasma with the heat flux based on not the MHD equations, but using the multicomponent plasma approach. We investigate a geometry in which the background magnetic field, gravity, and stratification are directed along one axis. No simplifications usual for the MHD-approach in studying these instabilities are used. The background electron thermal flux and collisions between electrons and ions are included. We derive the simple dispersion relation, which shows that the thermal flux perturbation generally stabilizes an instability. There is a narrow region of the temperature gradient, where an instability is possible. This result contradicts to a conclusion obtained in the MHD-approach. We show that the reason of this contradiction is the simplified assumptions used in the MHD analysis of buoyancy instabilities and the role of the longitudinal electric field perturbation, which is not captured by the MHD equations. Our dispersion relation also shows that a medium with the electron thermal flux can be unstable, if the temperature gradient of ions and electrons have the opposite signs. The results obtained can be applied to ICM and clusters of galaxies.

Subject headings: convection - instabilities - magnetic fields - plasmas - waves
1. INTRODUCTION

Various instability mechanisms are studied to understand some of the main features and processes in the astrophysical objects depending on their physical properties. Convective or buoyancy instability arising as a result of stratification is among those instabilities that may operate under different circumstances from the stellar interiors (e.g., Schwarzschild 1958), accretion disks (Balbus 2000, 2001), and neutron stars (Chang & Quataert 2009) to the hot accretion flows (e.g., Narayan et al. 2000, 2002) and even galaxy clusters and intercluster medium (ICM) (e.g., Quataert 2008; Sharma et al. 2009; Ren et al. 2009). Analogous instabilities also exist in the neutral atmosphere of the Earth and ocean (Gossard & Hooke 1975; Pedlosky 1982). Diversity of the astrophysical objects, in which convective instabilities may have a significant role, leading to turbulence and anomalous energy and matter transport, is a good motivation to explore this instability either through linear analytical analysis or by direct numerical simulations from different physical point of views. Although the significant role of convection in the transport of energy in stellar interiors is a well-known physical process, theoretical efforts to understand convective energy transport in the tenuous and hot plasmas such as ICM (Sarazin 1988) have lead to some results over recent years.

According to the standard Schwarzschild criterion, a thermally stratified fluid is convectively unstable when the entropy increases in the direction of gravity (Schwarzschild 1958). By taking into account the anisotropic heat flux in plasmas where the mean free path of ions and electrons is much larger than their Larmor radius, one obtains additional instabilities for short wave numbers with larger growth rates than that without thermal flux. These instabilities have been shown to arise when the temperature increases in the direction of gravity at the absence of the background thermal flux (the magnetothermal instability (MTI)) (Balbus 2000, 2001) and when the temperature decreases along gravity at the presence of the latter (the heat buoyancy instability (HBI)) (Quataert 2008). Both MTI and HBI have been simulated in 2D and 3D by many authors over recent years (e.g., Parrish et al. 2008; Parrish & Quataert 2008; Parrish et al. 2009). Following recent achievements in the convective theory, it has attracted attention of the authors for analyzing its possible role in ICM after a long time discounting. Majority of the mass of a cluster of galaxies is in the dark matter. However, around 1/6 of its mass consists of a hot, magnetized, and low density plasma known as ICM. The electron density is $n_e \approx 10^{-2}$ to $10^{-1}$ cm$^{-3}$ at the central parts of ICM. The electron temperature $T_e$ is measured of the order of a few keV, though the ion temperature $T_i$ has not yet been measured directly (e.g., Fabian et al. 2006; Sanders et al. 2010). The magnetic field strength $B$ in ICM is estimated to be in the range 0.1-10 $\mu$G depending on where the measurement is made (Carilli & Taylor 2002) which implies a dynamically weak magnetic field with $\beta = 8\pi n_e T_e / B^2 \approx 200 - 2000$. Thus, ICM with the ion Larmor radius $10^{8-9}$ cm ($T_i \sim T_e$) and the mean free path $10^{22-23}$ cm is classified as a weakly collisional
plasma (Carilli & Taylor 2002). In simulating ICM, it is important to consider anisotropic viscosity as well because the Reynolds number is very low (Lyutikov 2007, 2008). Another important physical agent is cosmic rays. Recent studies show that centrally concentrated cosmic rays have a destabilizing effect on the convection in ICM (Chandran & Dennis 2006; Rasera and Chandran 2008).

Theoretical models applied for study of buoyancy instabilities are based on the ideal magnetohydrodynamic (MHD) equations (Balbus 2000, 2001; Quataert 2008, Chang & Quataert 2009; Ren et al. 2009). Using of these equations permits us comparatively easily to consider different problems. However, the ideal MHD does not capture some important effects. One of the such effects is the nonzero longitudinal electric field perturbation along the background magnetic field. As we show here, the contribution of currents due to this small field to the dispersion relation can be of the same order of magnitude as that due to other electric field components. Besides, the MHD equations do not take into account the very existence of various charged and neutral species with different masses and electric charges and their collisions between each others and therefore can not be applied to multicomponent systems. On the contrary, the plasma $E$-approach deals with dynamical equations for each species. From Faraday’s and Ampere’s laws one obtains equations for the electric field components. Such an approach allows us to follow the movement and changing of parameters of each species separately and obtain rigorous conditions of consideration and physical consequences in specific cases. This approach permits us to include various species of ions and dust grains having different charges and masses. In this way, streaming instabilities of rotating multicomponent objects (accretion disks, molecular clouds and so on) have been investigated by Nekrasov (e.g., 2008, 2009 a, 2009 b), which have growth rates much larger than that of the magnetorotational instability (Balbus 1991). In some cases, the standard methods used in MHD leads to conclusions that are different from those obtained by the method using the electric field perturbations. One of a such example is considered in Nekrasov (2009 c).

In this paper, we apply a multicomponent approach to study buoyancy instabilities in magnetized electron-ion astrophysical plasmas with the background electron thermal flux. We include collisions between electrons and ions. However, we adopt here that cyclotron frequencies of species are much larger than their collision frequencies. Such conditions are typical for ICM and galaxy clusters. In this case, as it is known, the heat flux is anisotropic and directed along the magnetic field lines (Braginskii 1965). We consider a geometry in which gravity, stratification, and the background magnetic field are all directed along one ($z$-) axis. In our approach, it is important to obtain exact expressions for species’ velocities in an inhomogeneous medium. We give main equations and results. However, for those who are not interested in the mathematical details, they can directly refer to Sections 7 and 8.
The dispersion relation is obtained for cases, in which the background heat flux is absent or present. This gives a possibility to compare two cases. Solutions of the dispersion relation are discussed.

The paper is organized as follows. In Section 2, the fundamental equations are given. An equilibrium state is considered in Section 3. Perturbed ion velocity, number density, and thermal pressure are obtained in Section 4. In Section 5, we consider the perturbed velocity and temperature for electrons. Components of the dielectric permeability tensor are found in Section 6.

2. BASIC EQUATIONS

We start with the following equations for ions:

\[
\frac{\partial \mathbf{v}_i}{\partial t} = -\frac{\nabla p_i}{m_i n_i} + g + \frac{q_i}{m_i} \mathbf{E} + \frac{q_i}{m_i c} \mathbf{v}_i \times \mathbf{B} - \nu_{ie} (\mathbf{v}_i - \mathbf{v}_e),
\]

the momentum equation,

\[
\frac{\partial n_i}{\partial t} + \nabla \cdot n_i \mathbf{v}_i = 0,
\]

the continuity equation, and

\[
\frac{\partial p_i}{\partial t} + \mathbf{v}_i \cdot \nabla p_i + \gamma p_i \nabla \cdot \mathbf{v}_i = 0,
\]

the pressure equation. The corresponding equations for electrons are:

\[
0 = -\frac{\nabla p_e}{n_e} + q_e \mathbf{E} + \frac{q_e}{c} \mathbf{v}_e \times \mathbf{B} - m_e \nu_{ei} (\mathbf{v}_e - \mathbf{v}_i),
\]

\[
\frac{\partial n_e}{\partial t} + \nabla \cdot n_e \mathbf{v}_e = 0,
\]

\[
\frac{\partial p_e}{\partial t} + \mathbf{v}_e \cdot \nabla p_e + \gamma p_e \nabla \cdot \mathbf{v}_e = \lambda - (\gamma - 1) \nabla \cdot \mathbf{q}_e,
\]

\[
\frac{\partial T_e}{\partial t} + \mathbf{v}_e \cdot \nabla T_e + (\gamma - 1) T_e \nabla \cdot \mathbf{v}_e = \frac{\lambda}{n_e} - (\gamma - 1) \frac{1}{n_e} \nabla \cdot \mathbf{q}_e,
\]

the temperature equation, where \( \mathbf{q}_e \) is the electron heat flux (Braginskii 1965). We neglect inertia of the electrons. In Equations (1)-(7), \( q_j \) and \( m_j \) are the charge and mass of species \( j = i, e \), \( \mathbf{v}_j \) is the hydrodynamic velocity, \( n_j \) is the number density, \( p_j = n_j T_j \) is the thermal pressure, \( T_j \) is the temperature, \( \nu_{ie} \) (\( \nu_{ei} \)) is the collision frequency of ions (electrons) with electrons (ions), \( g \) is gravity, \( \mathbf{E} \) and \( \mathbf{B} \) are the electric and magnetic fields, \( c \) is the speed of
light in vacuum, and $\gamma$ is the adiabatic constant. We assume the electrons to be magnetized when their cyclotron frequency $\omega_{ce} = q_e B/m_e c \gg \nu_{ee}$, where $\nu_{ee}$ is the electron-electron collision frequency. In this case, the electron thermal flux is mainly directed along the magnetic field,

$$q_e = -\chi_e b \cdot \nabla T_e,$$

where $\chi_e$ is the electron thermal conductivity coefficient and $b = B/B$ is the unit vector along the magnetic field (Braginskii 1965). The term $\lambda$ compensates the temperature change as a result of the equilibrium heat flux. We take only into account the electron thermal conductivity by equation (8), because the corresponding ion conductivity is considerably smaller (Braginskii 1965).

Electromagnetic equations are Faraday’s law

$$\nabla \times E = -\frac{1}{c} \frac{\partial B}{\partial t},$$

(9)

and Ampere’s law

$$\nabla \times B = 4\pi \frac{j}{c},$$

(10)

where $j = \sum_j q_j n_j v_j$. We consider the wave processes with typical time-scales much larger than the time the light spends to cover the wavelength of perturbations. In this case, one can neglect the displacement current in Equation (10) that results in quasineutrality both in electromagnetic and purely electrostatic perturbations. The magnetic field $B$ includes the background magnetic field $B_0$, the magnetic field $B_{0cur}$ of the background current (when it presents), and the perturbed magnetic field.

3. EQUILIBRIUM STATE

At first, we consider an equilibrium state. We assume that background velocities are absent. In this paper, we study configuration, in which the background magnetic field, gravity, and stratification are directed along the $z$-axis. Let, for definiteness, $g$ be $g = -zg$, where $g > 0$ and $z$ is the unit vector along the $z$-direction. Then, Equations (1) and (4) give

$$g_i = -\frac{1}{m_i n_{i0}} \frac{\partial p_{i0}}{\partial z} = g - \frac{q_i}{m_i} E_0,$$

(11)

$$g_e = -\frac{1}{m_i n_{e0}} \frac{\partial p_{e0}}{\partial z} = \frac{q_i}{m_i} E_0,$$

(12)

where (and below) the index 0 denotes equilibrium values. Here and below we assume that $q_i = -q_e$. We see that equilibrium distributions of ions and electrons influence each other
through the background electric field \( E_0 \). In the case \( n_{i0} = n_e0 \) (this equality is satisfied for the two-component plasma) and \( T_{i0} = T_{e0} \), we obtain \( g_i = g_e = g/2 \). Thus, we have \( E_0 = m_i g/2q_i \). The presence of the third component, for example, of the cold dust grains with the charge \( q_d \) and mass \( m_d \gg m_i \) results in other value of \( E_0 = m_d g/q_d \). In this case, the ions and electrons are in equilibrium under the action of the thermal pressure and equilibrium electric field, being \( g_i \simeq -g_e \).

**4. LINEAR ION PERTURBATIONS**

Let us write Equations (1)-(3) for ions in the linear approximation,

\[
\frac{\partial v_{i1}}{\partial t} = -\frac{\nabla p_{i1}}{m_i n_{i0}} + \frac{\nabla p_{i0}}{m_i n_{i0} n_{i0}} + F_{i1} + \frac{q_i}{m_i c} v_{i1} \times B_0, \quad (13)
\]

\[
\frac{\partial n_{i1}}{\partial t} + v_{i1z} \frac{\partial n_{i0}}{\partial z} + n_{i0} \nabla \cdot v_{i1} = 0, \quad (14)
\]

\[
\frac{\partial p_{i1}}{\partial t} + v_{i1z} \frac{\partial p_{i0}}{\partial z} + \gamma p_{i0} \nabla \cdot v_{i1} = 0, \quad (15)
\]

where

\[
F_{i1} = \frac{q_i}{m_i} E_1 - \nu_{ie} (v_{i1} - v_{e1}), \quad (16)
\]

and the index 1 denotes the perturbed variables. Below, we solve these equations to find the perturbed velocity of ions in an inhomogeneous medium.

**4.1. Perturbed velocity of ions**

Applying the operator \( \partial / \partial t \) to Equation (13) and using Equations (14) and (15), we obtain

\[
\frac{\partial^2 v_{i1}}{\partial t^2} = -g_i \nabla v_{i1z} + \frac{1}{m_i n_{i0}} [(\gamma - 1) (\nabla p_{i0}) + \gamma p_{i0} \nabla] \nabla \cdot v_{i1} + \frac{\partial F_{i1}}{\partial t} + \frac{q_i}{m_i c} \frac{\partial v_{i1}}{\partial t} \times B_0. \quad (17)
\]

We can find solutions for the components of \( v_{i1} \). For simplicity, we assume that \( \partial / \partial x = 0 \), because a system is symmetric in the transverse direction relative to the \( z \)-axis. The \( x \)-component of Equation (17) has the form

\[
\frac{\partial v_{i1x}}{\partial t} = F_{i1x} + \omega_{ci} v_{i1y}, \quad (18)
\]
where $\omega_{ci} = q_i B_0 / m_i c$ is the ion cyclotron frequency. For the $y$-component of Equation (17), we obtain:

$$\frac{\partial^2 v_{i1y}}{\partial t^2} = -g_i \frac{\partial v_{i1z}}{\partial y} + c_{si}^2 \frac{\partial}{\partial y} \nabla \cdot v_{i1} + \frac{\partial F_{i1y}}{\partial t} - \omega_{ci} \frac{\partial v_{i1x}}{\partial t}. \quad (19)$$

Here, $c_{si} = (\gamma T_i / m_i)^{1/2}$ is the ion sound velocity.

Using Equation (18), a relation for $v_{i1y}$ is given from Equation (19) as follows:

$$\left(\frac{\partial^2}{\partial t^2} + \omega_{ci}^2 \right) v_{i1y} - Q_{i1y} = \frac{\partial P_{i1}}{\partial y}. \quad (20)$$

Then from Equation (18), we obtain

$$\frac{\partial}{\partial \omega_{ci}} \left[ \left( \frac{\partial^2}{\partial t^2} + \omega_{ci}^2 \right) v_{i1x} - Q_{i1x} \right] = \frac{\partial P_{i1}}{\partial y}. \quad (21)$$

Here, the following notations are introduced:

$$P_{i1} = -g_i v_{i1z} + c_{si}^2 \nabla \cdot v_{i1}, \quad (22)$$

$$Q_{i1x} = \omega_{ci} F_{i1y} + \frac{\partial F_{i1x}}{\partial t}, \quad (23)$$

$$Q_{i1y} = -\omega_{ci} F_{i1x} + \frac{\partial F_{i1y}}{\partial t}. \quad (24)$$

The value $P_{i1}$ defines the pressure perturbation (Eq. [15]). We see from Equation (21) that when $\partial / \partial t \ll \omega_{ci}$ the thermal pressure effect on the velocity $v_{i1x}$ is much larger than that on $v_{i1y}$. The $z$-component of Equation (17) takes the form

$$\frac{\partial}{\partial t} \left( \frac{\partial v_{i1z}}{\partial t} - F_{i1z} \right) = -g_i \frac{\partial v_{i1z}}{\partial z} + \left[ (1 - \gamma) g_i + c_{si}^2 \frac{\partial}{\partial z} \right] \nabla \cdot v_{i1}. \quad (25)$$

Let us now find $\nabla \cdot v_{i1}$ through $v_{i1z}$. Differentiating Equation (20) with respect to $y$ and using expression (22), we obtain

$$L_1 \nabla \cdot v_{i1} = L_2 v_{i1z} + \frac{\partial Q_{i1y}}{\partial y}, \quad (26)$$

where the following operators are introduced:

$$L_1 = \frac{\partial^2}{\partial t^2} + \omega_{ci}^2 c_{si}^2 \frac{\partial^2}{\partial y^2}, \quad (27)$$

$$L_2 = \left( \frac{\partial^2}{\partial t^2} + \omega_{ci}^2 \right) \frac{\partial}{\partial z} - g_i \frac{\partial^2}{\partial y^2}. \quad (28)$$
We can derive an equation for the longitudinal velocity $v_{i1z}$, substituting $\nabla \cdot \mathbf{v}_{i1}$ found from Equation (26) in Equation (25),

$$L_3 v_{i1z} = L_1 \frac{\partial F_{11z}}{\partial t} + L_4 \frac{\partial Q_{i1y}}{\partial y}, \quad (29)$$

where operators $L_3$ and $L_4$ have the form

$$L_3 = \left( \frac{\partial^2}{\partial t^2} + \omega_{ci}^2 \right) \frac{\partial^2}{\partial y^2} - c_{si}^2 \left( \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \frac{\partial^2}{\partial t^2} - c_{si}^2 \omega_{ci}^2 \frac{\partial^2}{\partial z^2} \right)$$

$$+ \gamma g_i \left( \frac{\partial^2}{\partial t^2} + \omega_{ci}^2 \right) \frac{\partial}{\partial z} + c_{si}^2 \frac{\partial L_1}{L_1 \partial z} L_2 + (1 - \gamma) g_i^2 \frac{\partial^2}{\partial y^2}, \quad (30)$$

$$L_4 = (1 - \gamma) g_i + c_{si}^2 \left( \frac{\partial}{\partial z} - \frac{\partial L_1}{L_1 \partial z} \right). \quad (31)$$

For obtaining expression (30), we have used expressions (27) and (28).

It is easy to see that at the absence of the background magnetic field and without taking into account electromagnetic perturbations (the right hand-side of Eq. [29]), the equation $L_3 v_{i1z} = 0$ describes the ion sound and internal gravity waves. In this case, a sum of the last two terms on the right hand-side of expression (30) is equal to $-c_{si}^2 \omega_{bi}^2 \frac{\partial^2}{\partial y^2}$, where $\omega_{bi}$ is the (ion) Brunt-Väisälä frequency equal to

$$\omega_{bi}^2 = \frac{g_i}{c_{si}^2} \left( \gamma - 1 \right) g_i + \frac{\partial c_{si}^2}{\partial z}. \quad (32)$$

However, we see the existence of the background magnetic field considerably modifies the operator $L_3$. Note the right hand-side of Equation (29) describes a connection between ions and electrons through the electric field $\mathbf{E}_1$ and collisions.

### 4.2. Specific case for ions

So far, we have not made any simplifications and all the equations and the expressions are given in their general forms. Now, we consider further perturbations with a frequency much lower than the ion cyclotron frequency and the transverse wavelengths much larger than the ion Larmor radius. Such conditions are typical for the astrophysical plasmas. Besides, we investigate a part of the frequency spectrum in the region lower than the ion sound frequency. Thus, we set

$$\omega_{ci}^2 \gg \frac{\partial^2}{\partial t^2} , c_{si}^2 \frac{\partial^2}{\partial y^2} , c_{si}^2 \frac{\partial^2}{\partial z^2} \gg \frac{\partial^2}{\partial t^2}. \quad (33)$$
In this case, operators (27), (28), (30), and (31) take the form
\[ L_1 \simeq \omega_{ci}^2, \quad L_2 \simeq \omega_{ci}^2 \frac{\partial}{\partial z}, \quad \]
\[ L_3 = -\omega_{ci}^2 \left( c_{si}^2 \frac{\partial}{\partial z} - \gamma g_i \right) \frac{\partial}{\partial z} - \frac{\partial^2}{\partial t^2}, \quad \]
\[ L_4 = (1 - \gamma) g_i + c_{si}^2 \frac{\partial}{\partial z}. \]

Also, the operator \( L_3 \) can be written for a case in which
\[ \frac{\partial^2}{\partial t^2} \gg \frac{\partial c_{si}^2}{\partial z}. \]

The small corrections in operators \( L_3 \) and \( L_4 \) are needed to be kept because some main terms in expressions for ion and electron velocities are equal each other (see below). Therefore, when calculating the electric current these main terms will be canceled and small corrections to velocities will only contribute to the current.

For the cases represented by inequalities (33) and (35) when the operators have the form (34), the equations for \( v_{i1z} \) and \( \nabla \cdot v_{i1} \) become
\[ \left[ \left( c_{si}^2 \frac{\partial}{\partial z} - \gamma g_i \right) \frac{\partial}{\partial z} - \frac{\partial^2}{\partial t^2} \right] v_{i1z} = -\frac{\partial F_{i1z}}{\partial t} - \left[ (1 - \gamma) g_i + c_{si}^2 \frac{\partial}{\partial z} \right] \frac{\partial Q_{i1y}}{\omega_{ci}^2 \partial y}, \]
\[ \nabla \cdot v_{i1} \simeq -\frac{\partial v_{i1z}}{\partial z} + \frac{\partial Q_{i1y}}{\omega_{ci}^2 \partial y}. \]

### 4.3. Ion velocity in the Fourier transform

Calculations show that some main terms in expressions for \( v_{i1z} \) (when calculating the current), \( \nabla \cdot v_{i1} \) and \( P_{i1} \) are canceled. Therefore, the small terms proportional to inhomogeneity must be taken into account. To make this correctly, we can not make the Fourier transformation in Equations (36) and (37) to find the perturbed ion pressure \( P_{i1} \). However, firstly, we should apply the operator \( \partial / \partial z \) to this variable for using Equation (36). It is analogous to obtain the term \( \partial c_{si}^2 / \partial z \) in expression (32) for the Brunt-Väisälä frequency. After that, we can apply in a local approximation the Fourier transformation assuming the linear perturbations to be proportional to \( \exp(ikr - i\omega t) \). As a result, we obtain for the Fourier-components \( v_{i1zk}, k \cdot v_{i1k} \) and \( P_{i1k} \), where \( k = (k, \omega) \), the following expressions:
\[ v_{i1zk} = -i \frac{\omega}{k_z^2 c_{si}^2} \left( 1 - i \frac{\gamma g_i}{k_z c_{si}^2} \right) F_{i1zk} = \frac{k_y}{k_z \omega_{ci}^2} \left( 1 - i \frac{g_i}{k_z c_{si}^2} \right) Q_{i1yk}, \]
\[ k \cdot v_{1k} = -i \frac{\omega}{k_z c_{s_i}} \left( 1 - i \frac{\gamma g_i}{k_z c_{s_i}^2} \right) F_{1z k} + i \frac{k_y g_i}{k_z c_{s_i}^2 \omega_{ci}^2} Q_{1yk}, \]  
\[ P_{1k} = \frac{\omega}{k_z} F_{1z k} - i \frac{\omega}{k_z c_{s_i}^2} \left[ (\gamma - 1) g_i + \frac{\partial c_{s_i}^2}{\partial z} \right] F_{1z k} + \frac{k_y g_i}{k_z c_{s_i}^2 \omega_{ci}^2} \left[ (\gamma - 1) g_i + \frac{\partial c_{s_i}^2}{\partial z} - \omega^2 c_{s_i}^2 g_i \right] Q_{1yk}. \]  

In expressions (38) and (39), we have omitted additional small terms at \( Q_{1yk} \), which are needed for calculation of \( P_{1k} \). When calculating the current along the \( z \)-axis, the main term \( \sim Q_{1yk} \) in Equation (38) will be canceled. The contribution of the first term \( \sim F_{1z k} \) to this current has, as we shall show below, the same order of magnitude for the buoyancy instabilities as that of the term \( \sim g_i Q_{1yk} \). The same relates to expressions (39) and (40). Thus, the longitudinal electric field perturbations must be taken into account. However, in the ideal MHD, this field is absent. We see from expressions (38) and (39) that \( \nabla \cdot v_{1} \sim (g_i/c_{s_i}^2) v_{1z} \). This relation is the same as that for the internal gravity waves in the Earth’s atmosphere (e.g., Nekrasov 1994). Using expression (40), we obtain velocities \( v_{1yk} \) and \( v_{1z k} \) from Equations (20) and (21), correspondingly.

### 4.4. Perturbed ion number density and pressure

It is followed from above that \( \nabla \cdot v_{1} \neq 0 \). Let us find the perturbed ion number density and pressure in the Fourier-representation. From Equations (14), (38) and (39), we obtain

\[ \frac{n_{1k}}{n_{0}} = -i \frac{1}{k_z c_{s_i}^2} F_{1z k} - i \frac{k_y g_i}{k_z c_{s_i}^2 \omega_{ci}^2} \left[ (\gamma - 1) g_i + \frac{\partial c_{s_i}^2}{\partial z} \right] Q_{1yk}. \]  

Equation (15) gives \( \partial p_{1}/\partial t = -m_i n_{0} P_{1} \). Thus, we obtain, using Equation (40),

\[ \frac{p_{1k}}{p_{0}} = -i \frac{\gamma}{k_z c_{s_i}^2} F_{1z k} + \frac{\gamma k_y g_i}{k_z c_{s_i}^2 \omega_{ci}^2} \left[ (\gamma - 1) g_i + \frac{\partial c_{s_i}^2}{\partial z} - \omega^2 c_{s_i}^2 g_i \right] Q_{1yk}. \]  

Comparing Equations (41) and (42), we see that the relative perturbation of the pressure due to the transverse electric force \( Q_{1yk} \) is much smaller than the relative perturbation of the number density. However, these perturbations as a result of the action of the longitudinal electric force \( F_{1z k} \) have the same order of magnitude. Thus, \( p_{1k}/p_{0} \sim n_{1k}/n_{0} \). This result contradicts a supposition \( p_{1k}/p_{0} \ll n_{1k}/n_{0} \) adopted in the MHD analysis of buoyancy instabilities (Balbus 2000, 2001; Quataert 2008) because the latter does not take into account the longitudinal electric field perturbations. From the results given below, it is followed that,
as we have already noted above, the both terms on the right hand-side of Equation (41) have the same order of magnitude.

5. LINEAR ELECTRON PERTURBATIONS

Equations for the electrons in the linear approximation are the following:

\[
0 = -\frac{\nabla p_{e1}}{n_{e0}} + \frac{\nabla p_{e0}}{n_{e0}} \frac{n_{e1}}{n_{e0}} + F_{e1} + \frac{q_{e}}{c} v_{e1} \times B_{0},
\]

\[
\frac{\partial n_{e1}}{\partial t} + v_{e1z} \frac{\partial n_{e0}}{\partial z} + n_{e0} \nabla \cdot v_{e1} = 0,
\]

\[
\frac{\partial p_{e1}}{\partial t} + v_{e1z} \frac{\partial p_{e0}}{\partial z} + \gamma p_{e0} \nabla \cdot v_{e1} = - (\gamma - 1) \nabla \cdot q_{e1},
\]

\[
\frac{\partial T_{e1}}{\partial t} + v_{e1z} \frac{\partial T_{e0}}{\partial z} + (\gamma - 1) T_{e0} \nabla \cdot v_{e1} = - (\gamma - 1) \frac{1}{n_{e0}} \nabla \cdot q_{e1},
\]

\[
q_{e1} = -b_{1} \frac{\partial T_{e0}}{\partial z} - b_{0} \frac{\partial T_{e1}}{\partial z} - b_{0} \chi_{e1} \frac{\partial T_{e0}}{\partial z},
\]

\[
F_{e1} = q_{e} E_{1} - m_{e} v_{ei} (v_{e1} - v_{i1}).
\]

Here, \(\chi_{e1} = 5 \gamma_{e0} T_{e1}/2 T_{e0}\) (and \(\chi_{e} \sim T_{e}^{5/2}\), see Spitzer (1962)) is the perturbation of the thermal flux conductivity coefficient. The perturbation of the unit magnetic vector \(b_{1}\) is equal to \(b_{1x,y} = B_{1x,y}/B_{0}\) and \(b_{1z} = 0\). The thermal flux in equilibrium is \(q_{e0} = -b_{0} \chi_{e0} \frac{\partial T_{e0}}{\partial z}\).

We have seen above at consideration of the ion perturbations that the terms \(\sim 1/H^{2}\), where \(H\) is the typical scale height, are needed to be kept (see the last term in Equation (40)). Therefore, these terms are kept also for the electrons.

5.1. Equation for the electron temperature perturbation

Let us find equation for the electron temperature perturbation. The expression \(\nabla \cdot q_{e1}\), where \(q_{e1}\) is defined by (47), is given by

\[
\nabla \cdot q_{e1} = \frac{\partial q_{e1y}}{\partial y} + \frac{\partial q_{e1z}}{\partial z} = -\chi_{e0} \frac{\partial T_{e0}}{\partial z} 1 \frac{\partial B_{1y}}{\partial y} = -\chi_{e0} \frac{\partial T_{e1}}{\partial z} 2 \frac{\partial T_{e0}}{\partial z} 2 - 2 \frac{\partial T_{e0}}{\partial z} 2 - 2 \frac{\partial T_{e0}}{\partial z} 2.\]

(49)
Substituting this expression into Equation (46), we obtain

$$D_1T_{e1} = -v_{e1z} \frac{\partial T_{e0}}{\partial z} - (\gamma - 1) \frac{T_{e0}}{n_{e0}} \frac{\partial B_{1y}}{B_0 \partial y},$$

(50)

where the operator $D_1$ is defined as

$$D_1 = \left[ \frac{\partial}{\partial t} - (\gamma - 1) \frac{1}{n_{e0}} \left( \chi_{e0} \frac{\partial^2}{\partial z^2} + 2 \frac{\partial \chi_{e0}}{\partial z} \frac{\partial}{\partial z} + \frac{\partial^2 \chi_{e0}}{\partial z^2} \right) \right].$$

(51)

5.2. Perturbed velocity and temperature of electrons

We find now equations for components of the perturbed velocity of electrons. The $x$-component of Equation (43) has a simple form, i.e.

$$v_{e1y} = -\frac{1}{m_e \omega_{ce}} F_{e1x},$$

(52)

where $\omega_{ce} = q_e B_0 / m_e c$. Applying the operator $\partial / \partial t$ to the $y$-component of Equation (43) and using Equations (45) and (49), we obtain

$$\frac{\partial}{\partial t} \left( v_{e1x} - \frac{1}{m_e \omega_{ce}} F_{e1y} \right) = -\frac{1}{\omega_{ci}} \frac{\partial P_{e1}}{\partial y} - (\gamma - 1) \frac{\chi_{e0}}{m_e \omega_{ce} n_{e0}} \frac{\partial T_{e0}}{\partial z} \frac{\partial^2 B_{1y}}{B_0 \partial y^2}$$

$$+ \frac{1}{m_e \omega_{ce}} \left( D_1 - \frac{\partial}{\partial t} \right) \frac{\partial T_{e1}}{\partial y},$$

(53)

where

$$P_{e1} = -g_e v_{e1z} + c_{se}^2 \nabla \cdot v_{e1}$$

(54)

and $c_{se}^2 = \gamma p_{e0} / m_i n_{e0}$. The variable $P_{e1}$ is analogous to $P_{i1}$ (see Eq. [22]), which defines the ion pressure perturbation. But for electrons, their pressure perturbation is also affected by the thermal conductivity (see Eq. [45]).

Let us express $\nabla \cdot v_{e1}$ through $v_{e1z}$, using Equation (52),

$$\nabla \cdot v_{e1} = \frac{\partial v_{e1z}}{\partial z} - \frac{1}{m_e \omega_{ce}} \frac{\partial F_{e1z}}{\partial y}.$$  

(55)

The $z$-component of Equation (43) takes the form

$$0 = -\frac{1}{n_{e0}} \frac{\partial p_{e1}}{\partial z} + \frac{1}{n_{e0}} \frac{\partial p_{e0} n_{e1}}{\partial z} + F_{e1z},$$

(56)
We consider further perturbations with the dynamic frequency $\partial/\partial t$ satisfying the following conditions:

$$\frac{\chi_{e0}}{n_{e0}} \frac{\partial^2}{\partial z^2} \gg \frac{\partial}{\partial t} \gg \frac{1}{n_{e0}} \frac{\partial \chi_{e0}}{\partial z} \frac{\partial}{\partial z}.$$  \hspace{1cm} (57)

In this case, the terms proportional to $\partial \chi_{e0}/\partial z$ in the temperature equation (50) (see [51]) are unimportant because the necessary small corrections proportional to $\partial/\partial t$ in this equation will be larger than that $\sim \partial \chi_{e0}/\partial z$. Thus, an inhomogeneity of the thermal flux conductivity coefficient and its perturbation can be neglected. We further apply the operator $\partial/\partial t$ to Equation (56) and use Equations (44), (45), (49), and (55). As a result, we obtain

$$\left( c_{se} \frac{\partial}{\partial z} - \gamma g_e \right) \frac{\partial v_{e1z}}{\partial z} = - \frac{\partial F_{e1z}}{m_i \partial t} + \left[ (1 - \gamma) g_e + c_{se}^2 \frac{\partial}{\partial z} \right] \frac{1}{m_{e\omega_{ce}}} \frac{\partial F_{e1x}}{\partial y}$$

$$+ \left( \gamma - 1 \right) \frac{\chi_{e0}}{m_i n_{e0}} \left( \frac{\partial T_{e0}}{\partial z} \frac{1}{B_0} \frac{\partial^2 B_{1y}}{\partial y \partial z} + \frac{\partial^3 T_{e1}}{\partial z^3} \right).$$

Equation for the temperature perturbation under conditions (57) has the form

$$\left[ (\gamma - 1) \frac{\chi_{e0}}{n_{e0}} \frac{\partial^2}{\partial z^2} - \frac{\partial}{\partial t} \right] T_{e1} = v_{e1z} \frac{\partial T_{e0}}{\partial z} + (\gamma - 1) T_{e0} \left( \frac{\partial v_{e1z}}{\partial z} - \frac{1}{m_{e\omega_{ce}}} \frac{\partial F_{e1x}}{\partial y} \right)$$

$$- (\gamma - 1) \frac{\chi_{e0}}{n_{e0}} \frac{\partial T_{e0}}{\partial z} \frac{\partial B_{1y}}{B_0 \partial y},$$

where we have used Equation (55). Substituting $T_{e1}$ in Equation (58) and carrying out some transformations, we find equation for the longitudinal velocity $v_{e1z}$

$$\frac{\partial^2 v_{e1z}}{\partial z^2} = - \frac{\partial^2 F_{e1z}}{T_{e0} \partial z \partial t} \frac{n_{e0}}{\chi_{e0}} \left( \frac{\partial}{\partial z} \right)^{-1} \frac{\partial^2 F_{e1z}}{T_{e0} \partial t^2} + \frac{1}{m_{e\omega_{ce}}} \frac{\partial^3 F_{e1x}}{\partial y \partial z^2}$$

$$+ \frac{1}{c_{se}^2} \left( \gamma g_e + \frac{\partial c_{se}}{\partial z} \right) \frac{1}{m_{e\omega_{ce}}} \frac{\partial^2 F_{e1x}}{\partial z \partial y \partial z} - \frac{\partial T_{e0}}{T_{e0} \partial z \partial B_0 \partial y \partial t}.$$ \hspace{1cm} (60)

The correction proportional to $\partial F_{e1x}/\partial t$ is absent. The last term on the right-hand side of Equation (60) is connected with the background electron thermal flux (Quataert 2008).

From Equations (59) and (60), we can find equation for the temperature perturbation

$$\left( \gamma - 1 \right) \frac{\chi_{e0}}{n_{e0}} \frac{\partial}{\partial z} \left( \frac{\partial^2 T_{e1}}{\partial z^2} + \frac{\partial T_{e0}}{\partial z} \frac{\partial B_{1y}}{B_0 \partial y} \right) = \frac{\gamma T_{e0}}{c_{se}^2} \left[ (\gamma - 1) g_e + \frac{\partial c_{se}}{\partial z} \right] \frac{1}{m_{e\omega_{ce}}} \frac{\partial F_{e1x}}{\partial y}$$

$$- (\gamma - 1) \frac{\partial F_{e1x}}{\partial t} - \gamma \frac{n_{e0}}{\chi_{e0}} \frac{\partial}{\partial z} \left( \frac{\partial}{\partial z} \right)^{-1} \frac{\partial^2 F_{e1x}}{\partial t^2}$$

$$- \frac{\partial T_{e0}}{\partial z} \left( \frac{\partial}{\partial z} \right)^{-1} \frac{\partial^2 B_{1y}}{B_0 \partial y \partial t}.$$
It is followed from results obtained below that all terms on the right-hand side of Equation (61) (except the correction \( \sim \partial^2 F_{e_1z} / \partial t^2 \)) have the same order of magnitude (see Section [4.3]). The left-hand side of this equation is larger (see conditions [57]). Thus, the temperature perturbation in the zero order of magnitude can be found by equaling the left part of Equation (61) to zero. However, the right side is necessary for finding the transverse velocity perturbation \( v_{e_1x} \).

To find the velocity \( v_{e_1x} \), we need to calculate the value \( P_{e_1} \) (see Eqs. [53] and [54]). Performing calculations in the same way as that for ions (see Section 4.3), we obtain

\[
\frac{c_{se}^2 \partial^2 P_{e_1}}{\partial z^2} = \left[ c_{se}^2 \frac{\partial}{\partial z} + (\gamma - 1) g_e + \frac{\partial c_{se}^2}{\partial z} \right] \left( -\frac{\partial F_{e_1z}}{m_i \partial t} + \frac{\partial V_{e_1}}{\partial z} \right) + g_e \left( (\gamma - 1) g_e + \frac{\partial c_{se}^2}{\partial z} \right) \frac{1}{m_e \omega_{ce}} \frac{\partial F_{e_1x}}{\partial y},
\]

where we have introduced the notation connected with the thermal flux,

\[
V_{e_1} = (\gamma - 1) \frac{\lambda_{e0}}{m_i n_{e0}} \left( \frac{\partial T_{e0}}{\partial z} \frac{1}{B_0} \frac{\partial B_{1y}}{\partial y} + \frac{\partial^2 T_{e1}}{\partial z^2} \right).
\]

Equation (62) can be re-written in the form, which is convenient for finding the velocity \( v_{e_1x} \). Using Equation (61), we obtain

\[
\frac{\partial^2}{\partial z^2} \left( P_{e_1} - V_{e_1} \right) = -\frac{\partial^2 F_{e_1z}}{m_i \partial z \partial t} - \frac{\gamma}{c_{se}^2} \left[ (\gamma - 1) g_e + \frac{\partial c_{se}^2}{\partial z} \right] \frac{\partial F_{e_1z}}{m_i \partial t} + \frac{1}{c_{se}^2} \left[ (\gamma - 1) g_e + \frac{\partial c_{se}^2}{\partial z} \right] \left( \gamma g_e + \frac{\partial c_{se}^2}{\partial z} \right) \frac{1}{m_e \omega_{ce}} \frac{\partial F_{e_1x}}{\partial y}
\]

\[- \left[ (\gamma - 1) g_e + \frac{\partial c_{se}^2}{\partial z} \right] \frac{\partial T_{e0}}{T_{e0} \partial z} \left( \frac{\partial}{\partial z} \right)^{-1} \frac{\partial^2 B_{1y}}{\partial y \partial t}.
\]

It is easy to see that Equation (53) has the form

\[
\frac{\partial}{\partial t} \left( v_{e_1x} - \frac{1}{m_e \omega_{ce}} F_{e_1y} \right) = -\frac{1}{\omega_{ei} \partial y} \frac{\partial}{\partial y} \left( P_{e_1} - V_{e_1} \right).
\]

Thus, the main contribution of the flux described by Equation (63) does not influence on the electron dynamics. Applying to Equation (65) the operator \( \partial^2 / \partial z^2 \) and using Equation (64), we find an equation for the velocity \( v_{e_1x} \).
6. FOURIER CURRENT COMPONENTS

6.1. Fourier velocity components of ions and electrons

Let us give velocities of ions and electrons in the Fourier-representation. From Equations (20), (21), and (40), we have

\[ v_{i1xk} = \frac{1}{\omega_{ci}^2} \left( 1 + \frac{\omega^2}{\omega_{ci}^2} \right) Q_{i1xk} + i \frac{k_y}{k_z^2} \frac{\omega^2 - g_i a_i}{\omega_{ci}^2} Q_{i1y}k_y - \frac{1}{\omega_{ci}} \frac{a_i}{k_z} F_{i1z}, \]  

(66)

\[ v_{i1yk} = \frac{1}{\omega_{ci}^2} \left[ 1 + \frac{(k_y^2 \omega^2 - k_z^2 g_i a_i)}{k_z^2 \omega_{ci}^2} \right] Q_{i1yk} + \frac{\omega}{\omega_{ci}} \frac{k_y}{k_z} \left( 1 - i \frac{a_i}{k_z} \right) F_{i1z}. \]  

(67)

Here and below, we have introduced notations

\[ a_{i,e} = \frac{1}{c_{si,e}^2} \left[ (\gamma - 1) g_{i,e} + \frac{\partial c_{si,e}^2}{\partial z} \right]. \]  

(68)

The velocity \( v_{i1z} \) is given by Equation (38).

From Equations (64) and (65), we find

\[ v_{e1xk} = -i \frac{a_e c_{se}^2 k_y}{\omega_{ci}^2 k_z^2} \left( b_e \frac{1}{m_e \omega_{ce}} F_{e1xk} + \omega \frac{\partial T_{e0}}{k_z T_{e0} \partial z} \frac{B_{1yk}}{B_0} \right) \]  

(69)

\[ + \frac{1}{m_e \omega_{ce}} F_{e1y}k_y - \frac{k_y}{k_z} \left( 1 - i \gamma a_e \right) \frac{1}{m_e \omega_{ce}} F_{e1z}, \]

where the following notation is introduced:

\[ b_e = \frac{1}{c_{se}^2} \left( \gamma g_e + \frac{\partial c_{se}^2}{\partial z} \right). \]  

(70)

Equation (60) also gives us

\[ v_{e1zk} = \frac{k_y}{k_z m_e \omega_{ce}} F_{e1zk} - i \frac{k_y}{k_z^2} \left( b_e \frac{1}{m_e \omega_{ce}} F_{e1xk} + \omega \frac{\partial T_{e0}}{k_z T_{e0} \partial z} \frac{B_{1yk}}{B_0} \right) \]  

(71)

\[ - i \frac{\omega}{k_z^2 T_{e0}} \left( 1 + i \omega \frac{n_{e0}}{\chi_{e0} k_z^2} \right) F_{e1z}. \]

The velocity \( v_{e1y} \) is defined by Equation (52).
6.2. Fourier electron velocity components at the absence of heat flux

To elucidate the role of the electron thermal flux, we also consider the dispersion relation when the flux is absent. Therefore, we give here the corresponding electron velocity components:

\[ v_{e1x} = -i k_y g_e a_e k_z^2 \frac{1}{\omega_{ce}} F_{e1x} + 1 m_e \omega_{ce} F_{e1y} - k_y \left( 1 - i a_e \right) k_z^2 \frac{1}{m_e \omega_{ce}} F_{e1z}, \]

\[ v_{e1y} = k_y \left( 1 - i g_e k_z^2 \right) \frac{1}{m_e \omega_{ce}} F_{e1x} - i \frac{\omega}{k_z^2 \omega_{ce}} n_i \left( 1 - i \gamma g_e k_z^2 \right) F_{e1z}. \]

Comparing expressions (69) and (71) with these equations, we see that the thermal flux under conditions (57) essentially modifies the small terms in the electron velocity.

6.3. Fourier components of the current

We find now the Fourier components of the linear current \( j_1 = q_i n_0 v_{i1} + q_e n_0 v_{e1} \). It is convenient to consider the value \( 4\pi i j_1 / \omega \). Using expressions (38), (52), and (66)-(71), we obtain the following current components:

\[ \frac{4\pi i}{\omega} j_{1x} = a_{xx} E_{1x} + i a_{xy} E_{1y} - a_{xz} E_{1z} - b_{xx} (v_{i1x} - v_{e1x}) - i b_{xy} (v_{i1y} - v_{e1y}) + b_{xz} (v_{i1z} - v_{e1z}); \]

\[ \frac{4\pi i}{\omega} j_{1y} = -i a_{yx} E_{1x} + a_{yy} E_{1y} - a_{yz} E_{1z} + i b_{yx} (v_{i1x} - v_{e1x}) - b_{yy} (v_{i1y} - v_{e1y}) + b_{yz} (v_{i1z} - v_{e1z}); \]

\[ \frac{4\pi i}{\omega} j_{1z} = -a_{xx} E_{1x} - a_{xy} E_{1y} + a_{xz} E_{1z} + b_{xx} (v_{i1x} - v_{e1x}) + b_{xy} (v_{i1y} - v_{e1y}) - b_{xz} (v_{i1z} - v_{e1z}). \]

When obtaining expressions (74)-(76), we have used notations (16), (23), (24), and (48) and equalities \( q_e = -q_i, n_{e0} = n_{i0}, m_e \nu_{ei} = m_i \nu_{ie} \). We have also substituted \( B_{1y} \) by \( (k_z c / \omega) E_{1x} \) (see below). The following notations are introduced above:

\[ a_{xx} = \frac{\omega^2}{\omega_{ci}^2} k^2 \left( 1 - \frac{k^2}{k_z^2} g_i a_i + \frac{k^2}{k_z^2} \omega^2 \frac{\partial T_{e0}^*}{\partial T_{e0}^*} \right), \]
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\[ a_{xy} = a_{yx} = \frac{\omega_{pi} k^2}{\omega_{ci}^2} \left( 1 - \frac{k_y^2}{k^2} \right) + \frac{\omega_{pi} k_y}{\omega_{ci} k_z} (a_i - \gamma a_e), \]

\[ a_{yy} = \frac{\omega_{pi}^2}{\omega_{ci}^2}, a_{yz} = a_{zy} = \frac{\omega_{pi} k_y}{\omega_{ci} k_z} (b_e - \frac{g_i}{c_{si}^2} + \frac{\partial T_{e0}^*}{T_{e0} \partial z}), \]

\[ a_{zz} = \frac{\omega_{pi}^2}{k_z^2} \left( \frac{\gamma}{c_{se}^2} + \frac{1}{c_{si}^2} \right) \]

and

\[ b_{x x} = \frac{\omega_{pi} \nu_{ie} m_i}{k_y^2} \left( 1 - \frac{k_y^2}{k^2} \right) + \frac{\omega_{pi} k_y}{k_z^2} \left( b_e - \frac{g_i}{c_{si}^2} \right) m_i \nu_{ie}, \]

\[ b_{ij} = a_{ij} \frac{m_i}{q_i} \nu_{ie}. \]

Here \( \omega_{pi} = (4 \pi n_0 q_i^2/m_i)^{1/2} \) is the plasma frequency and \( k^2 = k_y^2 + k_z^2 \). The terms proportional to \( T_{e0}^* \) are connected with the background electron thermal flux.

Calculations show that to obtain expressions for \( a_{ij} \) without thermal flux, using electron velocities (72) and (73), we must change \( b_e \) by \( g_e/c_{se}^2 \), put \( T_{e0}^* = 0 \), and take \( \gamma = 1 \) in terms \( a_{xz} \) and \( a_{zz} \).

### 6.4. Simplification of collision contribution

From the formal point of view, an assumption that electrons are magnetized has only been involved in neglecting the transverse electron thermal flux. In other respects, a relationship between \( \omega_{ce} \) and \( \nu_{ie} \) or \( \omega_{ci} \) and \( \nu_{ie} \) (that is the same) can be arbitrary in Equations (74)-(76). We further proceed by assuming that \( \omega \ll \omega_{ci} \). In this case, we can neglect the collisional terms proportional to \( b_{xy} \) and \( b_{yx} \). However, the system of Equations (74)-(76) stays sufficiently complex to find \( j_1 \) through \( E_1 \). Therefore, we further consider the specific case in which the frequency \( \omega \) and wave numbers satisfy the following conditions:

\[ \frac{\omega_{ci}^2 k_y^2}{\nu_{ie}^2 k_z^2} \gg \frac{\omega}{\nu_{ie}} \gg \frac{1}{k_z^4 H^2 \omega_{ci}^2}, \]

where

\[ c_s^2 = \frac{c_{si}^2 c_{se}^2}{\gamma c_{si}^2 + c_{se}^2}. \]
It is clear that conditions (79) can easily be realized. In this case, the current components are equal to

\[
\frac{4\pi i}{\omega} j_{1xk} = \varepsilon_{xx} E_{1xk} + i \varepsilon_{xy} E_{1yk} - \varepsilon_{xz} E_{1zk}, \\
\frac{4\pi i}{\omega} j_{1yk} = -i \varepsilon_{yx} E_{1xk} + \varepsilon_{yy} E_{1yk} - \varepsilon_{yz} E_{1zk}, \\
\frac{4\pi i}{\omega} j_{1zk} = -\varepsilon_{zx} E_{1xk} - \varepsilon_{zy} E_{1yk} + \varepsilon_{zz} E_{1zk}.
\]

Components of the dielectric permeability tensor \( \varepsilon_{ij} \) are the following:

\[
\varepsilon_{xx} = a_{xx} + \frac{\nu_{ie} k_y}{\omega c} \left( a_i - \gamma a_e \right) a_{xx}, \varepsilon_{xy} = a_{xy} + \frac{\nu_{ie} k_y}{\omega c} \left( a_i - \gamma a_e \right) a_{zy}, \\
\varepsilon_{xz} = \frac{a_{xz}}{(1 - id_z)}, \varepsilon_{yx} = a_{yx} - \frac{\omega \nu_{ie} k_y}{\omega c} \frac{a_{xx}}{k_z} (1 - id_z), \varepsilon_{yy} = a_{yy}, \\
\varepsilon_{yz} = \frac{a_{yz}}{(1 - id_z)}, \varepsilon_{zx} = \frac{a_{zx}}{(1 - id_z)}, \varepsilon_{zy} = \frac{a_{zy}}{(1 - id_z)}, \varepsilon_{zz} = \frac{a_{zz}}{(1 - id_z)},
\]

where we have used notations (78)

\[
d_z = \frac{\omega \nu_{ie}}{k^2 c^2}.
\]

Parameter \( d_z \) defines the collisionless, \( d_z \ll 1 \), and collisional, \( d_z \gg 1 \) regimes. Below, we derive the dispersion relation.

7. DISPERSION RELATION

From Equations (9) and (10) in the Fourier-representation and using system of equations (81), we obtain the following equations for the electric field components:

\[
\left( n^2 - \varepsilon_{xx} \right) E_{1xk} - i \varepsilon_{xy} E_{1yk} + \varepsilon_{xz} E_{1zk} = 0, \\
i \varepsilon_{yx} E_{1xk} + \left( n_z^2 - \varepsilon_{yy} \right) E_{1yk} + \left( -n_y n_z + \varepsilon_{yz} \right) E_{1zk} = 0, \\
\varepsilon_{zx} E_{1xk} + \left( -n_y n_z + \varepsilon_{zy} \right) E_{1yk} + \left( n_z^2 - \varepsilon_{zz} \right) E_{1zk} = 0,
\]

where \( \mathbf{n} = \mathbf{k} \omega / |\omega| \). The dispersion relation can be found by setting the determinant of the system (84) equal to zero. In our case, the terms proportional to \( \varepsilon_{xy} \) and \( \varepsilon_{yx} \) can be neglected. As a result, we have

\[
\left( n^2 - \varepsilon_{xx} \right) \left[ n_y^2 \varepsilon_{yy} + \left( n_z^2 - \varepsilon_{yy} \right) \varepsilon_{zz} - n_y n_z \left( \varepsilon_{yz} + \varepsilon_{zy} \right) + \varepsilon_{yz} \varepsilon_{zy} \right] = 0,
\]
The above dispersion relation can be studied for different cases. In subsequent sections, we consider both the collisionless and collisional cases.

7.1. Collisionless case

We assume now that the condition

$$\frac{\omega_{ve}}{k^2 c_s^2} \ll 1,$$  \hspace{1cm} (86)

is satisfied. Then, using notations (77) and (82), the dispersion relation (85) becomes

$$\left(\omega^2 - k^2 c_A^2\right) \left(\omega^2 - k^2 c_A^2 - \Omega^2 \frac{k^2_y}{k^2}\right) = 0,$$  \hspace{1cm} (87)

where $c_A = B_0/(4\pi m_i n_i)^{1/2}$ is the Alfvén velocity and

$$\Omega^2 = g_i a_i + c_{se} a_e b_e + c_{se} a_e \frac{\partial T_{e0}^*}{T_{e0} \partial z} + c_s^2 (a_i - \gamma a_e) \left( b_e - \frac{g_i}{c_{si}^2} + \frac{\partial T_{e0}^*}{T_{e0} \partial z} \right).$$  \hspace{1cm} (88)

For obtaining Equation (87), we have used the condition $k^2_y c_s^2/\omega_{ci}^2 \ll 1$. We see that there are two wave modes. The first wave mode, $\omega^2 = k^2 c_A^2$, is the Alfvén wave with a polarization of the electric field mainly along the $y$-axis (the wave vector $k$ is situated in the $y-z$ plane). This wave does not feel the inhomogeneity of the medium. The second wave has a polarization of the magnetosonic wave, i.e. its electric field is directed mainly along the $x$-axis (see below). This wave is undergone by the action of the medium inhomogeneity effect. The corresponding dispersion relation is

$$\omega^2 = k^2 c_A^2 + \Omega^2 \frac{k^2_y}{k^2}.$$  \hspace{1cm} (89)

The expression (88) can be further simplified using equations (11), (12), (68), (70), and (80). As a result, we obtain

$$\Omega^2 = \frac{\gamma}{\gamma c_{si}^2 + c_{se}^2 m_i^2} \left[(\gamma - 1) m_i g + \gamma \frac{\partial (T_{i0} + T_{e0})}{\partial z}\right] \left[m_i g + \frac{\partial (T_{e0} + T_{e0}^*)}{\partial z}\right].$$  \hspace{1cm} (90)

We have pointed out at the end of Section (6.3) what changes must be done in expressions (77) and (78) to consider the case without heat flux. This case follows from Equation (90), if we omit the term $\partial (T_{e0} + T_{e0}^*)/\partial z$ and put $\gamma = 1$ in the first multiplier. Then $\Omega^2$ becomes...
\[ \Omega^2 = \frac{g}{c_{si}^2 + c_{se}^2} \left[ (\gamma - 1) g + \frac{\partial (c_{si}^2 + c_{se}^2)}{\partial z} \right]. \]  

(91)

This is the Brunt-Väisälä frequency. Comparing (90) and (91), we see that the heat flux stabilizes the unstable stratification. The presence of the background heat flux does not play of principle role. If the temperature decreases in the direction of gravity \((\partial T_{ie0}/\partial z > 0)\), a medium is stable. Solution (90) describes an instability regime only when

\[ \frac{\gamma - 1}{2\gamma} m_i g < -\frac{\partial T_0}{\partial z} < \frac{1}{2} m_i g. \]

where \((T_{i0} \sim T_{e0} = T_0)\). We also note that \(\Omega^2\) can be negative if gradients of \(T_{i0}\) and \(T_{e0}\) have different signs.

For a comparison, we give here the corresponding dispersion relation by Quataert (2008)

\[ \omega^2 \simeq -g \left( \frac{d \ln T_0}{dz} \right) \frac{k_y^2}{k_z^2}, \]

which is discussed in Section 8.

### 7.2. Collisional case

We proceed with the collisional case when

\[ \frac{\omega \nu_{ie}}{k_z^2 c_s^2} \gg 1. \]  

(92)

In this limiting case, we obtain again Equation (89).

### 7.3. Polarization of perturbations

Let us neglect in the system of equations (84) the small contributions given by \(\varepsilon_{xy}\) and \(\varepsilon_{yx}\). Then, for example, in the collisionless case, we obtain for the second wave \(\omega^2 \neq k_z^2 c_A^2\),

\[ E_{1yk} = \frac{k_y}{k_z} E_{1zk}, \]

\[ E_{1zk} = \frac{\varepsilon_{xx}}{\varepsilon_{zz}} E_{1zk} \ll E_{1zk}. \]  

(93)

Thus, the second wave has a polarization of the electric field mainly along the \(x\)-axis. In spite of that the component \(E_{1zk} \ll E_{1zk}\), it is multiplied by a large coefficient in the first
equation of the system (84). As a result, the contribution of this term is the same on the order of magnitude as that of the first term.

In the collisional case, the component $E_{1z}^k$ is also defined by Equation (93). However, its contribution to the first equation of the system (84) can be neglected.

8. DISCUSSION

Dispersion relation (87) with $\Omega^2$ defined by Equations (88) or (90) considerably differs from that given in (Quataert 2008) for the case of our geometry. The reason goes back to the assumptions made in the MHD analysis of buoyancy instabilities $p_1/p_0 \ll \rho_1/\rho_0$, where $p$ and $\rho$ denote the pressure and mass density of fluid, and the condition of incompressibility $\nabla \cdot \mathbf{v}_1 = 0$, where $\mathbf{v}_1 = \mathbf{v}_{i1}$ is the perturbed fluid velocity. We now shortly show how one can obtain the result of Quataert (2008) in our geometry, using these assumptions. We sum Equations (13) and (43) and use the Ampere’s law (10). The components of the equations become,

$$\frac{\partial v_{i1x}}{\partial t} = \frac{B_0}{4\pi\rho_0} \frac{\partial B_{1x}}{\partial z},$$

$$\frac{\partial v_{i1y}}{\partial t} = -\frac{\partial p_1}{\rho_0 \partial y} - \frac{B_0}{4\pi\rho_0} \left( \frac{\partial B_{1z}}{\partial y} - \frac{\partial B_{1y}}{\partial z} \right),$$

$$\frac{\partial v_{i1z}}{\partial t} = -\frac{\partial p_1}{\rho_0 \partial y} - gn_{e1} \frac{n_{\infty}}{n_0},$$

where $n_{\infty} = n_{e0} = n_0$, $\rho_0 = m_\infty n_0$, and we have used $n_{i1} = n_{e1}$. The components of the ideal magnetic induction equation are the following:

$$\frac{\partial B_{1x}}{\partial t} = B_0 \frac{\partial v_{i1x}}{\partial z},$$

$$\frac{\partial B_{1y}}{\partial t} = B_0 \frac{\partial v_{i1y}}{\partial z},$$

$$\frac{\partial B_{1z}}{\partial t} = -B_0 \frac{\partial v_{i1y}}{\partial y}.$$
and $\partial^3/\partial y\partial z\partial t$ to the second and third equations of the systems of equations (94), correspondingly, using equation $\nabla \cdot \mathbf{v}_{i1} = 0$ and the second equation (95), and subtracting one equation from another, we obtain
\[
\left( \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \frac{\partial^2 v_{i1y}}{\partial t^2} = c_A \left( \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \frac{\partial^2 v_{i1y}}{\partial z^2} + \frac{g}{n_0} \frac{\partial^3 n_{e1}}{\partial y \partial z \partial t}. \tag{96}
\]
Also, from Equation (61), we have (see conditions [57]),
\[
\frac{\partial^2 T_{e1}}{\partial z^2} = - \frac{\partial T_{e0}}{\partial z} \frac{\partial B_{1y}}{B_0 \partial y}. \tag{97}
\]
Taking into account that $T_{e1}/T_{e0} = -n_{e1}/n_{e0}$, differentiating Equation (97) over $t$, using the second equation (95), and substituting the equation obtained in Equation (96), we find
\[
\left( \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \frac{\partial^2 v_{i1y}}{\partial t^2} = c_A \left( \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \frac{\partial^2 v_{i1y}}{\partial z^2} + \frac{g}{T_{e0}} \frac{\partial T_{e0}}{\partial z} \frac{\partial^2 v_{i1y}}{\partial y^2}. \tag{98}
\]
By neglecting the contribution of the magnetic field, this equation coincides in the Fourier-representation with Equation (22) in Quataert (2008).

However, the presence of the longitudinal electric field perturbations $E_{1z}$ results in that $p_{i,e1k}/p_{i,e0} \sim n_{i,e1k}/n_{i,e0}$ (for ions, see Section [4.4]). Besides, Equation (97) together with equation $\nabla \cdot \mathbf{v}_{i1} = 0$ lead to the nonphysical equation
\[
\frac{\partial T_{e1}}{\partial t} = v_{e1z} \frac{\partial T_{e0}}{\partial z}. \tag{98}
\]
Thus, Equation (98) is incorrect.

From the dispersion relation (85), we see the necessity of involving the contribution of values $\varepsilon_{xz}$, $\varepsilon_{zx}$, and $\varepsilon_{zz}$ in the collisionless case (86) (values $\varepsilon_{xx}$ and $\varepsilon_{zx}$ give the last term on the right hand-side of Eq. [88]). This means that contribution of currents $j_{1z} \sim E_{1z}$ and $j_{1z} \sim E_{1x}$, $E_{1z}$ must be taken into account. However, the role of the longitudinal electric field $E_{1z}$ in the MHD equations is not clear. The same also relates to the collisional case (92). In the current $j_{1zk}$, we must take into consideration the contribution of the current $j_{1zk}$ as a result of collisions, which is proportional to $E_{1zk}$ (see Eqs. [74] and [76]).

Thus, the standard MHD equations with simplified assumptions are not applicable for the correct theory of buoyancy instabilities. Such a theory can only be given by the multi-component approach used in this paper.

The results following from Equation (90) show that the thermal flux stabilizes the buoyancy instability. The instability is only possible in the narrow region of the temperature
gradient (see Section [7.1]). The presence of the background electron thermal heat (the term \(\sim T_{e0}\)) does not play an essential role. An instability is also possibly, if the temperature gradients of ions and electrons have the opposite signs.

The contribution of collisions between electrons and ions depends on the parameter \(d_z\) defined by Equation (83). In the both limits (86) \((d_z \ll 1)\) and (92) \((d_z \gg 1)\), the dispersion relation has the same form.

We would like to say a few words about the Schwarzschild criterion of the buoyancy instability. It is generally accepted that this instability is possible, if the entropy increases in the direction of gravity. From a formal point of view, it is correct, if we take the Brunt-Väisälä frequency \(N\) in the form (e.g. Balbus 2000),

\[
N^2 = -\frac{1}{\gamma \rho} \frac{\partial p}{\partial z} \frac{\partial \ln \rho p^{-\gamma}}{\partial z}.
\]

However, this expression can easily be transformed to expression (32). Thus, we see that for instability to exist, the temperature must increase along gravity and exceed the threshold.

9. CONCLUSION

In this paper, we have investigated buoyancy instabilities in magnetized electron-ion astrophysical plasmas with the background electron thermal flux, using the E-approach when dynamical equations for the ions and the electrons are solved separately via electric field perturbations. We have included the background electron heat flux and collisions between electrons and ions. The important role of the longitudinal electric field perturbations, which are not captured by the MHD equations, has been shown. We showed that the previous MHD result for the growth rate in the geometry considered in this paper when all background quantities are directed along the one axis is incorrect. The reason of this has been shown to be in simplified assumptions made in the MHD analysis of the buoyancy instabilities.

We have adopted that cyclotron frequencies of species are much larger than their collision frequencies that is typical for ICM and galaxy clusters. The dispersion relation obtained shows that the anisotropic electron heat flux, including the background one, stabilizes the unstable stratification except the narrow region of the temperature gradient. However, when gradients of the ion and electron temperatures have opposite signs, the medium becomes unstable.

Results obtained in this paper are applicable to the magnetized weakly collisional stratified objects and can be useful for searching sources of turbulent transport of energy and
matter. It has been suggested that buoyancy instability can act as a driving mechanism to generate turbulence in ICM and this extra source of the heating may help to resolve cooling flow problem. However, all previous analytical or numerical studies are restricted to the MHD approach. Our study shows that when the true multifluid nature of the system with the electron heat flux is considered, one can not expect buoyancy instability unless for a very limited range of the gradient of the temperature or when the gradients of the temperature of the electrons and ions have opposite signs which both cases are very unlikely. However, in the case when the heat flux does not play the role, the system can be unstable according to the Schwarzschild criterion.

The current linear analysis is for simplified initial conditions, in which the background magnetic field, temperature gradient, and gravity are along the same direction. However, another configuration should also be examined using the $E$-approach, in which the initial magnetic field is perpendicular to the direction of gravity. This will be done in the forthcoming paper.

9.1. References

Balbus, S. A. 2000, ApJ, 534, 420
Balbus, S. A. 2001, ApJ, 562, 909
Balbus, S. A., & Hawley, J. F. 1991, ApJ, 376, 214
Braginskii, S. I. 1965, Rev. Plasma Phys., 1, 205
Carilli, C. L., & Taylor, G. B. 2002, ARA&A, 40, 319
Chandran, B. D., & Dennis, T. J. 2006, ApJ, 642, 140
Chang, P., & Quataert, E. 2009, 0909.3041. (submitted to MNRAS)
Fabian, A. C., Sanders, J. S., Taylor, G. B., Allen, S. W., Crawford, C. S., Johnstone, R. M.,
& Iwasawa, K. 2006, MNRAS, 366, 417
Gossard, E.E., & Hooke, W.H. 1975, Waves in the Atmosphere (Amsterdam: Elsevier Scientific Publishing Company)
Lyutikov, M. 2007, ApJL, 668, L1
Lyutikov, M. 2008, ApJL, 673, L115
Narayan, R., Igumenshchev, I. V., & Abramowicz, M. A. 2000, ApJ, 539, 798
Narayan, R., Quataert, E., Igumenshchev, I. V., & Abramowicz, M. A. 2002, ApJ, 577, 295
Nekrasov, A. K. 1994 J. Atmos.Terr. Phys., 56, 931
Nekrasov, A. K. 2008, Phys. Plasmas, 15, 032907
Nekrasov, A. K. 2009 a, Phys. Plasmas, 16, 032902
Nekrasov, A. K. 2009 b, ApJ, 695, 46
Nekrasov, A. K. 2009 c, ApJ, 704, 80
Parrish, I. J., & Quataert, E. 2008, ApJL, 677, L9
Parrish, I. J., Stone, J. M., & Lemaster, N. 2008, ApJ, 688, 905
Parrish, I. J., Quataert, E., & Sharma, P. 2009, ApJ, 703, 96
Pedlosky, J. 1982, Geophysical Fluid Dynamics, (New York: Springer-Verlag)
Quataert, E. 2008, ApJ, 673, 758
Rasera, Y., & Chandran, B. 2008, ApJ, 685, 105
Ren, H., Wu, Z., Cao, J., & Chu, P. K. 2009, Phys. Plasmas, 16, 102109
Sanders, J. S., Fabian, A. C., Frank, K. A., Peterson, J. R., & Russell, H. R. 2010, MNRAS, 402, 127
Sarazin, C. L. 1988, X-Ray Emission from Clusters of Galaxies (Cambridge: Cambridge Univ. Press)
Schwarzschild, M. 1958, Structure and Evolution of the Stars (New York: Dover)
Sharma, P., Chandran, B. D. G., Quataert, E., & Parrish, I. J. 2009, ApJ, 699, 348
Spitzer, L., Jr. 1962, Physics of Fully Ionized Gases (2d ed.; New York: Interscience)