A BELLMAN FUNCTION PROOF OF THE $L^2$ BUMP CONJECTURE

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Abstract. We approach the problem of finding the sharp sufficient condition of the boundedness of all two weight Calderón–Zygmund operators. We solve this problem in $L^2$ by writing a formula for a Bellman function of the problem.

1. Introduction

1.1. Preliminaries. In this paper we give a simple Bellman function solution of the so-called “bump conjecture” for the two weight estimates of the singular integral operators.

The original (still open) question about two weight estimates for the singular integral operators is to find a necessary and sufficient condition on the weights $w$ and $v$ such that a Calderón–Zygmund operator $T : L^p(u) \to L^p(v)$ is bounded, i.e. the inequality

$$\int |Tf|^p v dx \leq C \int |f|^p u dx \quad \forall f \in L^p(u)$$

(1.1)

holds.

In the one weight case $v = u$ the famous Muckenhoupt condition is necessary and sufficient for (1.1)

$$(A_2) \quad \sup_Q \left( |Q|^{-1} \int_Q v dx \right) \left( |Q|^{-1} \int_Q v^{-p'/p} dx \right)^{p/p'} < \infty$$

where the supremum is taken over all cubes $Q$. More precisely, this condition is sufficient for all Calderón–Zygmund operators, and is also necessary for classical (interesting) Calderón–Zygmund operators, such as Hilbert transform, Riesz transform (vector-valued, when all Riesz transforms are considered together), Beurling–Ahlfors operator.

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The inequality (1.1) is equivalent to the boundedness of the operator $M_{u^{1/p}TM_{v^{1/p}}}^1$ in the non-weighted $L^p$; here $M_\varphi$ is the multiplication operator, $M_\varphi f = \varphi f$. Denoting $w = u^{-p'/p}$ we can rewrite the problem in the symmetric form as the $L^p$ boundedness of $M_{v^{1/p}TM_{w^{1/p}}^1}$.

So the problem can be stated as: Describe all weights (i.e. non-negative functions) $v, w$ such that the operator $M_{v^{1/p}TM_{w^{1/p}}}$ is bounded in (the non-weighted) $L^p$.

Note, that this symmetric formulation is more general than (1.1), because in (1.1) it is usually assumed that $u$ and $v$ are locally integrable, but also for (1.1) to hold for interesting operators (Hilbert Transform, vector Riesz Transform, Beurling–Ahlfors Transform, etc.) the function $1/u$ also has to be locally integrable.

For the interesting operators the following two weight analogue of the $A_p$ condition is necessary for the boundedness of the operator $M_{v^{1/p}TM_{w^{1/p}}}$:

\begin{align}
\sup_Q \left( |Q|^{-1} \int_Q v dx \right) \left( |Q|^{-1} \int_Q w dx \right)^{p/p'} < \infty
\end{align}

or in the symmetric form

\begin{align}
\sup_Q \left( |Q|^{-1} \int_Q v dx \right)^{1/p} \left( |Q|^{-1} \int_Q w dx \right)^{1/p'} < \infty
\end{align}

Simple counterexamples show that this condition is not sufficient for the boundedness. So a natural way to get a sufficient condition is to replace the $L^1$ norms of $v$ and $w$ in (1.3) (or the $L^p$ and $L^{p'}$ norms of $v^{1/p}$ and $w^{1/p'}$) by some stronger Orlicz norms (“bumping” the $L^p$ norms).

Namely, given a Young function $\Phi$ and a cube $Q$ one can consider the normalized on $Q$ Orlicz space $L^\Phi(Q)$ with the norm given by

$$
\| f \|_{L^\Phi(Q)} := \inf \left\{ \lambda > 0 : \int_Q \Phi \left( \frac{f(x)}{\lambda} \right) \frac{dx}{|Q|} \leq 1 \right\}.
$$

And it was conjectured (for $p = 2$) that if the Young functions $\Phi_1$ and $\Phi_2$ satisfy the condition

\begin{align}
\int_0^{\infty} \frac{dx}{\Phi_{1,2}(x)} < \infty,
\end{align}

then the condition

\begin{align}
\sup_Q \| v \|_{L^{\Phi_1}(Q)} \| w \|_{L^{\Phi_2}(Q)} < \infty
\end{align}

implies that for any bounded Calderón–Zygmund operator $T$ the operator $M_{u^{1/2}TM_{v^{1/2}}}^1$ is bounded in $L^2$. Usually in the literature a more complicated (although equivalent) form of this conjecture was presented, but at least in the case $p = 2$ condition (1.5) seems more transparent.\footnote{The bump condition was also stated for $p \neq 2$, but in this paper we only deal with the case $p = 2$.}
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Condition (1.5) was considered in numerous papers in the attempt to prove its universal sufficiency for all Calderón–Zygmund operators. The reader can find beautiful approaches in [1], [3], [4], [5], [6], [12], [18], [19], where partial results for some Calderón–Zygmund operators were proved (note that [19] is about maximal operator and not about Calderón–Zygmund operators). Finally in [13] the sufficiency of bump condition for all Calderón–Zygmund operators to be bounded was fully proved (and even generalized to all $p \in (1, \infty)$), although in formally less general situation of the weights.

Simultaneously and independently the first version of the present paper [16] appeared. Slightly earlier the sketch of the approach (with the main ideas but without much details) was circulated as [15]. A very diligent reader will recognize that the approaches in the present paper (and thus in earlier versions [15], [16]) and in Lerner’s paper [13] are very different, but still have something in common. This very important common point is the “coupling by the same cube” feature. It is the main winning idea of [13]. And it is the feature of the present paper (and [15], [16]) as Bellman function approach automatically should have this feature.

1.2. What is done in the paper. Formally, in this paper we prove the $L^2$ case of the bump conjecture using Bellman function method. As it is now well-known, a general Calderón–Zygmund operator can be represented as an average of dyadic shift and paraproducts, so it is sufficient to prove the estimates for such operators, and that is exactly what is done in the paper.

However we think that the results obtained obtained in the paper that were used to prove the bump conjecture are of significant interest by themselves; probably they are even more interesting than the solution of the bump conjecture.

Let us shortly describe what is done in the paper.

First, the Orlicz norm is not easy to work with. We introduced a lower bound for the Orlicz norm, which gives a more tractable, in our opinion, way to “bump” the averages. In particular, it allowed us to apply the Bellman function method.

The application of the Bellman function method is pretty standard, with one little twist: one of the argument belongs to an infinite-dimensional space.

The estimates for the Haar shifts and for the paraproducts are reduced to two embedding theorems, so the operators are constructively factorized through $\ell^2$. This essentially means that the bump condition is a rather rough one, since in more delicate two weight situations no such factorization appeared to be possible (at least no factorization was found).

Namely, it is known, see [17], that in the case of power bumps ($\Phi(t) = t^{1+\varepsilon}$) one can insert a Muckenhoupt $A_p$ weight between $w^{-1}$ and $Cv$, so the boundedness follows immediately. For finer bumps such insertion of $A_p$ weights is impossible, but the constructive factorization through $\ell^2$ can be considered the best next thing.
Finally, the main estimates are proved in very general (non-homogeneous) martingale settings, and can be used in more general situations. In particular, our results give the proof of the bump onjecture for the Calderón–Zygmund operators on geometrically doubling metric spaces (equipped with a doubling measure). Indeed, since random “dyadic” lattices can be constructed on geometrically doubling metric spaces, and representation of Calderón–Zygmund operators on such spaces as an average of Haar shifts is now known, everything follows from our results (see Theorem 2.3 below).

1.3. Orlicz norms and distribution functions. Orlicz norm is not very convenient to work with, so we would like to replace it be something more tractable.

1.3.1. A lower bound for the Orlicz norm. Let $\Phi$ be a continuous non-negative increasing convex function such that $\Phi(0) = 0$ and $\int^{+\infty} \frac{dt}{\Phi(t)} < +\infty$. Define $\Psi(s)$ parametrically by $\Psi(s) = \Phi'(t)$ when $s = \frac{1}{\Phi(t)\Phi'(t)} (t > 0)$. Then $\Psi(s)$ is positive and decreasing for $s > 0$ and $s\Psi(s)$ is increasing. Moreover $\int_0^{+\infty} \frac{dt}{s\Psi(s)} < +\infty$. Indeed, using our parameterization we can rewrite the last integral as

$$\int^{+\infty} \left( \frac{1}{\Phi(t)} + \frac{\Phi''(t)}{\Phi'(t)^2} \right) dt.$$ 

The first integral converges by our assumption and the second integrand has a bounded near $+\infty$ antiderivative $\frac{1}{\Phi(t)}$.

Let $w \geq 0$ on $Q \subset \mathbb{R}^n$. Define the normalized distribution function $N$ of $w$ by

$$N(t) = N^w_Q(t) = \frac{1}{|Q|} |\{ x \in I : w(x) > t \}| \quad (1.6)$$

**Lemma 1.1.** Let $\Psi : (0,1] \to \mathbb{R}_+$ be a decreasing function such that the function $s \mapsto s\Psi(s)$ is increasing. Let $\Phi$ be a Young function and let

$$\Psi(s) \leq C\Phi'(t) \quad \text{where} \quad s = \frac{1}{\Phi(t)\Phi'(t)}$$

for all sufficiently large $t$. Then for $N = N^w_I$

$$n_\Psi(N) := \int_0^\infty N(t)\Psi(N(t)) dt \leq C\|w\|_{L^\Phi(I)} \quad (1.7)$$

**Proof.** The left hand side scales like a norm under multiplication by constants, so it is enough to show that if $\|w\|_{L^\Phi(I)} \leq 1$, i.e.,

$$\frac{1}{|I|} \int_I \Phi(w) = \int_0^\infty N(t)\Phi'(t) dt \leq 1$$

then $n_\Psi(N)$ is bounded by a constant. Since $s\Psi(s)$ increases, we may have trouble only at $+\infty$ It is cleat that it suffices to estimate the integral over the set where $\Psi(N(t)) > \Phi'(t)$ but since $\Psi$ is decreasing this means that $N(t) \leq C/(\Phi(t)\Phi'(t))$, so we get at most $\int^{+\infty} \Phi(t)^{-1} dt$ and we are done. □
Remark. In the above Lemma 1.1 we do not need the assumption that

\[ \int_0^1 \frac{1}{s \Psi(s)} ds < \infty. \]  

But in what follows this assumption will be needed, and the reasoning in the beginning of this section shows that for any Young function \( \Phi \) satisfying \( \int_0^\infty (\Phi(t))^{-1} dt < \infty \), we can find \( \Psi \) from Lemma 1.1 satisfying (1.8).

1.3.2. Examples. In the above section only the behavior of \( \Phi \) at \( +\infty \) (equivalently, the behavior of \( \Psi \) near \( 0 \)) was important, so we will concentrate our attention there.

Let \( \Phi(t) = t (\ln t)^{\alpha}, \alpha > 1 \) near \( \infty \). Then

\[ \Phi'(t) \sim (\ln t)^\alpha, \quad \Phi(t)\Phi'(t) \sim t (\ln t)^{2\alpha}, \]

so \( \Psi(s) := (\ln(1/s))^\alpha \) satisfies the assumptions of Lemma 1.1 to see that we notice

\[ \ln(\Phi(t)\Phi'(t)) \sim \ln t. \]

If \( \Phi(t) = t \ln(t) (\ln \ln t)^\alpha, \alpha > 1 \), then

\[ \Phi'(t) \sim \ln(t) (\ln \ln t)^\alpha, \quad \Phi(t)\Phi'(t) \sim t (\ln t)^2 (\ln \ln t)^{2\alpha} \]

and \( \Psi(s) = \ln(1/s)(\ln \ln(1/s))^\alpha \) works because again \( \ln(\Phi(t)\Phi'(t)) \sim \ln t \).

Note that in both examples \( \int_0^1 (s \Psi(s))^{-1} ds < \infty \).

The examples of Young functions with higher order logarithms are treated similarly.

1.4. Main result. Let \( \Psi_1, \Psi_2 : (0, 1] \to \mathbb{R}_+ \) be as above, i.e. \( \Psi_{1,2} \) are decreasing, \( s \mapsto s \Psi_{1,2}(s) \) are increasing and

\[ \int_0^1 \frac{ds}{s \Psi_{1,2}(s)} < \infty. \]

Recall that for a weight \( w \) the normalized distribution function \( N_{Q}^w \) is defined by (1.6).

**Theorem 1.2.** Let the weights \( v, w \) satisfy

\[ \sup_Q n_{\Psi_1}(N_Q^v)n_{\Psi_2}(N_Q^w) < \infty; \]

here the supremum is taken over all cubes \( Q \), and \( n_\Psi \) is defined by (1.7).

Then for any bounded Calderón–Zygmund operator \( T \) the operator

\[ M_{v^{1/2}}TM_{w^{1/2}} \]

is bounded in \( L^2 \).
2. Reductions: Haar shifts, paraproducts and embedding theorems

First, let us reduce the problem to its dyadic (martingale) analogue, i.e. to the estimates of the so-called Haar shifts and paraproducts.

Since a bounded Calderón–Zygmund operator can be represented as a weighted average (over the random dyadic grids) of Haar shifts and paraproduct and their adjoins, where the weights decay exponential in complexity of the Haar shifts, it is sufficient to get the estimates for the Haar shifts that grow sub-exponentially (for example, polynomially) in the complexity of the shifts and the estimates for the paraproducts (there is no complexity of the paraproducts).

The estimates for each operators will be in turn factored into two embedding theorem, and these embedding theorem are proved in this paper.

The embedding theorems and so the estimates of the Haar shifts and paraproducts hold in very general martingale settings,

2.1. General setup. Consider a measure space \(X\) with \(\sigma\)-finite measure \(\mu\) let \(L_k = \{Q^k_j\}_j, k \in \mathbb{Z}\) (or \(k \in \mathbb{Z}^+\)) be partitions of \(X\) into disjoint sets \(Q^k_j, 0 < \mu(Q^k_j) < \infty\).

We assume that the partition \(L_{k+1}\) is a refinement of \(L_k\).

Let \(\mathfrak{A}\) be the \(\sigma\)-algebra generated by all the partitions \(L_k\). In what follows all functions on \(X\) we consider will be assumed to be \(\mathfrak{A}\)-measurable.

With respect to this \(\sigma\)-algebras we can define martingale averaging operators \(E_k\), and martingale difference operators \(\Delta_k := -E_k + E_{k+1}\).

We adapt the following notation.

\[\text{ch} I\] The collection of children of \(I \in \mathcal{L}\), i.e. if \(I \in \mathcal{L}_n\) then \(\text{ch} I = \{J \in \mathcal{L}_{n+1}: J \subset I\}\).

\[\text{ch}_k I\] The collection of children of the order \(k\) of \(I \in \mathcal{L}\); \(\text{ch}_0(I) = \{I\}\), \(\text{ch}_{k+1}(I) = \{\text{ch}(J): J \in \text{ch}_k(I)\}\).

\(\langle f \rangle_I, \tilde{f}_I f\) The average of \(f\) over \(I\), \(\langle f \rangle_I = \mu(I)^{-1} \int_I f(x) d\mu(x)\);

\(E_I\) The averaging operator, \(E_I f := \langle f \rangle_I 1_I\); note that \(E_k = \sum_{I \in \mathcal{L}} E_I\).

\(\Delta_I\) Martingale difference operator, \(\Delta_I := -E_I + \sum_{J \in \text{ch}(I)} E_J\); note that \(\Delta_k = \sum_{I \in \mathcal{L}_k} \Delta_I\).

\(\Delta^n_I\) Martingale difference operator of order \(n\),

\[\Delta^n_I := -E_I + \sum_{J \in \text{ch}_n(I)} E_J.\]

Since the measure \(\mu\) is assumed to be fixed we sometimes will be using \(|E|\) for \(\mu(E)\) and \(dx\) for \(d\mu(x)\). We also will be using \(L^2\) for \(L^2(\mu)\).

The prototypical example is \(X = \mathbb{R}\) or \(\mathbb{R}^d\) with \(\mathcal{L}\) being a dyadic lattice \(\mathcal{D}\).

2.2. Haar shifts.
Definition 2.1. A Haar shift $S$ of complexity $n$ is given by

$$Sf = \sum_{Q \in D} S_Q \Delta^n_Q f,$$

where the operators $S_Q$ act on $\Delta^n_Q L^2$ and can be represented as integral operators with kernels $a_Q$, $\|a_Q\|_\infty \leq |Q|^{-1}$. The latter means that for all $f, g \in \Delta^n_Q L^2$

$$\langle S_Q f, g \rangle = \int_Q a_Q(x, y) f(y) g(x) dxdy.$$

This is a slightly more general definition than the one in [10], but only the estimate $\|a_Q\|_\infty \leq |Q|^{-1}$ is essential for our construction. Note also that according to the definition in [10] the complexity of the corresponding shift is $n - 1$, not $n$, which really does not matter; we just find our definition of complexity a bit more convenient.

The estimate $\|a_Q\|_\infty \leq |Q|^{-1}$ means that the operators $S_Q$ are "$L^1 \times L^1$ normalized", meaning that

$$|\langle S_Q f, g \rangle| \leq |Q| \frac{\|f\|_1 \|g\|_1}{|Q|} \quad \forall f, g \in \Delta^n_Q L^2$$

Haar shifts of complexity 1 are simply "$L^1 \times L^1$ normalized" martingale transforms; martingale transform here means in particular that the subspaces $\Delta_I$ are orthogonal, and $S$ can be represented as an orthogonal sum of the operators $S_Q$.

A Haar shift of complexity $n \geq 2$ is not generally a martingale transform, meaning that the subspaces $\Delta^n_Q$ generally intersect, so $S$ does not split into direct sum of $S_Q$.

However, if one goes with step $n$, then the corresponding operator is a martingale transform, so a Haar shift of complexity $n$ can be represented as a sum of $n$ Haar shifts of complexity 1. Namely, for $k = 1, 2, \ldots, n - 1$ define

$$\mathcal{L}^k = \{\mathcal{L}_{k+j} : j \in \mathbb{Z}\},$$

and let

$$S_k = \sum_{Q \in \mathcal{L}^k} S_Q.$$

Then $S = \sum_{k=0}^{n-1} S_k$ and each $S_k$ is a Haar shift of complexity 1 with respect to the lattice $\mathcal{L}^k$.

Therefore, uniform estimate for the Haar shifts of complexity 1 (i.e. for the "$L^1 \times L^1$ normalized" martingale transforms) gives the linear in complexity estimate for the general Haar shifts.
2.3. Paraproducts. Given the lattice $\mathcal{L}$ and a locally integrable function $b$, the paraproduct $\Pi = \Pi_b = \Pi_b(\mathcal{L})$ is defined as

$$
\Pi f := \sum_{Q \in \mathcal{L}} (E_Q f)(\Delta_Q b).
$$

The necessary and sufficient condition for the paraproduct to be bounded is that

$$
\sup_{R \in \mathcal{L}} |R|^{-1} \sum_{Q \in \mathcal{L}, Q \subset R} \|\Delta_Q b\|_2^2 < \infty.
$$

In the case of dyadic lattice in $\mathbb{R}^d$ or, more generally in the homogeneous situation, when

$$
\inf_{R \in \mathcal{L}} \inf_{Q \in \text{ch}(R)} |Q|/|R| > 0
$$

this condition is equivalent to $b$ belonging to the corresponding martingale BMO space $\text{BMO}_{\mathcal{L}}$.

2.4. Reduction to the martingale case. To reduce the problem to the martingale case we use the following result can be found in [9] and [10]:

Theorem 2.2. Let $T$ be a Calderón–Zygmund operator in $\mathbb{R}^d$. There exists a probability space $(\Omega, \mathbb{P})$ of dyadic lattices $\mathcal{D}_\omega$, such that

$$
T = C \left( \int_\Omega \sum_{n=1}^{\infty} 2^{-\varepsilon n} S_n(\omega) d\mathbb{P}(\omega) + \int_\Omega (\Pi^1(\omega) + (\Pi^2(\omega)^*) d\mathbb{P}(\omega) \right),
$$

where $S_n(\omega)$ are Haar shifts of complexity $n$ with respect to the lattice $\mathcal{D}_\omega$, $\Pi^{1,2}(\omega)$ are the paraproducts with respect to the lattice $\mathcal{D}_\omega$, $\|\Pi^{1,2}(\omega)\| \leq 1$.

The constants $C$ and $\varepsilon$ depend on $d$, $\|T\|$ and Calderón–Zygmund parameters of the kernel of $T$.

Theorem 2.2 implies immediately that the main theorem (Theorem 1.2) follows from the theorem below.

Theorem 2.3. Let the weights $v$, $w$ satisfy the assumptions of Theorem 1.2. Then

(i) For all Haar shifts $S$ of order 1 the operators $M_{v^{1/2}} SM_{w^{1/2}}$ are uniformly bounded in $L^2$, $\|M_{v^{1/2}} SM_{w^{1/2}}\| \leq C$, where $C$ depends on $\Psi_2$, $\Psi_2$, the supremum in (1.9), but not on the lattice $\mathcal{L}$.

(ii) For all Haar shifts $S_n$ the operators $M_{v^{1/2}} SSM_{w^{1/2}}$ uniformly bounded in $L^2$ by $C_n$, where $C$ is the constant from (i).

(iii) Let $\Pi = \Pi_b$ be a paraproduct such that

$$
|J|^{-1} \sum_{I \in \mathcal{L}, I \subset J} \|\Delta_I b\|_{\infty}^2 |I| \leq 1 \quad \forall J \in \mathcal{L}.
$$

Then the operator $M_{v^{1/2}} \Pi M_{w^{1/2}}$ is bounded in $L^2$ by $C$, where again $C$ depends on $\Psi_2$, $\Psi_2$, the supremum in (1.9), but not on the lattice $\mathcal{L}$. 

Remark 2.4. For the homogeneous lattices, i.e. for lattices satisfying
\[
\inf_{j \in \mathcal{L}} \inf_{I \in \text{ch}(j)} \frac{|I|}{|J|} =: \delta > 0
\]
all the normalized \(L^p\) norms \(|I|^{-1/p} \|\Delta_I g\|_p\), \(p \in [1, \infty]\) are equivalent in the sense of two sided estimates. So for such lattices condition (2.2) means that \(\|\Pi\| \leq C(\delta)\). So Theorem 2.3 gives the estimates that being fed to Theorem 2.2 imply Theorem 1.2.

As it was discussed above, (i) implies (ii). Statement (i) is obtained from the following embedding theorem:

**Theorem 2.5.** Let \(\Psi\) be as above. Then for any weight \(w\) on \(\mathcal{X}\) such that \(n_{\Psi} (N_I^w) < \infty\) for all \(I \in \mathcal{L}\)
\[
\sum_{I \in \mathcal{L}} n_{\Psi} (N_I^w)^{-1} \left( \mu(I)^{-1} \int_{\mathcal{X}} |\Delta_I(f w^{1/2})| d\mu \right)^2 \mu(I) \leq C\|f\|_{L^2}^2
\]
for all \(f \in L^2(w)\); here \(C = C(\Psi)\) and in the summation we skip \(I\) on which \(w \equiv 0\).

Let us see that this theorem implies the condition (i) of Theorem 2.3. Assume, multiplying the weights by appropriate constants that the inequality
\[
\sum_{I \in \mathcal{L}} n_{\Psi_1} (N_I^w) n_{\Psi_2} (N_I^w) \leq 1
\]
holds for all \(I \in \mathcal{L}\). Then
\[
|\langle \mathcal{S} f w^{1/2}, g v^{1/2} \rangle| \leq \sum_{I \in \mathcal{L}} |\langle \mathcal{S}_I \Delta_I(f w^{1/2}), \Delta_I(g v^{1/2}) \rangle|
\]
\[
\leq \sum_{I \in \mathcal{L}} |I|^{-1} \|\Delta_I(f w^{1/2})\|_1 \|\Delta_I(g v^{1/2})\|_1
\]
\[
\leq \sum_{I \in \mathcal{L}} |I|^{-1} \|\Delta_I(f w^{1/2})\|_1 \|\Delta_I(g v^{1/2})\|_1 \frac{1}{\left(n_{\Psi_1}(N_I^w) n_{\Psi_2}(N_I^w) \right)^{1/2}}
\]
\[
\leq \frac{1}{2} \sum_{I \in \mathcal{L}} |I|^{-1} \|\Delta_I(f w^{1/2})\|_1^2 \frac{1}{n_{\Psi_1}(N_I^w)} + \frac{1}{2} \sum_{I \in \mathcal{L}} |I|^{-1} \|\Delta_I(g v^{1/2})\|_1^2 \frac{1}{n_{\Psi_2}(N_I^w)}
\]
the second inequality here follows from “\(L^1 \times L^1\) normalization” condition (2.1), the second one from (2.4) and the last one is just the trivial inequality \(2xy \leq x^2 + y^2\).

Applying Theorem 2.5 to each sum we get that
\[
|\langle \mathcal{S} f w^{1/2}, g v^{1/2} \rangle| \leq \frac{1}{2} \left( C(\Psi_1)\|f\|_2^2 + C(\Psi_2)\|g\|_2^2 \right).
\]
Replacing \(f \mapsto tf\), \(g \mapsto t^{-1}g\), \(t > 0\) we get
\[
|\langle \mathcal{S} f w^{1/2}, g v^{1/2} \rangle| \leq \frac{1}{2} \left( t^2 C(\Psi_1)\|f\|_2^2 + t^{-2} C(\Psi_2)\|g\|_2^2 \right).
\]
Taking infimum over all $t > 0$ and recalling that $2ab = \inf_{t>0}(t^2a + t^{-2}b$ for $a, b \geq 0$ we obtain

$$|\langle Sfw^{1/2}, gv^{1/2} \rangle| \leq (C(\Psi_1)C(\Psi_2))^{1/2}\|f\|_2\|g\|_2,$$

which is exactly statement (i) of Theorem 2.3.

For the statement (iii) of Theorem 2.3 we also need another embedding theorem.

**Theorem 2.6.** Let $\Psi$ be as above. Then for any normalized Carleson sequence $\{a_I\}_{I \in D}$ $(a_I \geq 0)$, i.e. for any sequence satisfying

$$\sup_{I \in D} |I|^{-1} \sum_{I' \in D, I' \subset I} a_I |I'| \leq 1,$$

we get

$$\sum_{I \in D} \langle fw^{1/2} \rangle_I^2 \frac{n_{\Psi}(N^w_I)}{a_I |I|} \leq C\|f\|^2_{L^2},$$

where again $C = C(\Psi)$.

Let us show that this theorem together with Theorem 2.5 implies statement (iii) of Theorem 2.3. Let $a_I = \|\Delta_I b\|_\infty^2$.

Again, multiplying if necessary the weight $v w$ by appropriate constants we can assume (2.4). Then we can write

$$|\langle \Pi_b fw^{1/2}, gv^{1/2} \rangle| \leq \sum_{I \in D} |\langle fw^{1/2} \rangle_I| \cdot |\langle \Delta_I b, \Delta_I (gv^{1/2}) \rangle|$$

$$\leq \sum_{I \in D} |\langle fw^{1/2} \rangle_I| (a_I)^{1/2} \frac{\|\Delta_I (gv^{1/2})\|_1}{(n_{\Psi_1}(N^w_I)^{1/2} |I|^{1/2}) \cdot (n_{\Psi_2}(N^v_I)^{1/2} |I|^{1/2})}$$

$$\leq \left( \sum_{I \in D} |\langle fw^{1/2} \rangle_I| (a_I)^{1/2} |I| \right)^{1/2} \left( \sum_{I \in D} \frac{\|\Delta_I (gv^{1/2})\|_1^2}{n_{\Psi_1}(N^w_I)^{1/2} |I|} \cdot \frac{\|\Delta_I (gv^{1/2})\|_1^2}{n_{\Psi_2}(N^v_I)^{1/2} |I|} \right)^{1/2},$$

the second inequality holds because of (2.4), and the last one is just the Cauchy–Schwartz.

Estimating the sums in parentheses by Theorem 2.6 and 2.5 respectively we get statement (iii) of Theorem 2.3. \(\square\)

3. **Proof of (the Differential Embedding) Theorem 2.5**

3.1. **Bellman function and main differential inequality.** Let $\varphi(s) := s\Psi(s)$. Multiplying $\Psi$ by an appropriate constant we can assume without loss of generality that

$$\int_0^1 \frac{1}{\varphi(s)} ds = 1.$$
Define \( m(s) \) on \([0, 1]\) by \( m(0) = m'(0) = 0, m''(s) = 1/\varphi(s) \). Identity (3.1) implies that \( m \) is well-defined and \( m'(s) \leq 1, m(s) \leq s \). For a distribution function \( N = N_I^w \) define

\[
(3.2) \quad u(N) = \int_0^\infty (2N(t) - m(N(t)))dt = 2\langle w \rangle_I - \int_0^\infty m(N(t))dt;
\]

Note that the inequality \( m(s) \leq s \) implies that \( u(N_I^w) \geq \langle w \rangle_I \).

The functional \( u \) is defined on the convex set of distribution functions, i.e. on the set of decreasing functions \( N : [0, \infty) \to [0, 1] \) such that \( \int_0^\infty N(t)dt < \infty \).

In what follows we can consider only finitely supported functions \( N \), and then use standard approximation reasoning. Consider two distribution functions \( N \) and \( N_1 \) and let \( \Delta N = N_1 - N \). Denote also

\[
\begin{align*}
\omega &:= \int_0^\infty N(t)dt, \\
\omega_1 &:= \int_0^\infty N_1(t)dt,
\end{align*}
\]

and let

\[
\Delta \omega := \omega_1 - \omega = \int_0^\infty \Delta N(t)dt;
\]

the motivation for this notation is that if \( N \) and \( N_1 \) are the distribution functions of the weights \( \omega \) and \( \omega_1 \), then the integrals are the averages on the corresponding weights. Denote also

\[
(3.3) \quad \omega_{\Delta} := \int_0^\infty |\Delta N(t)|dt;
\]

clearly \( |\Delta \omega| \leq \omega_{\Delta} \).

Let us compute derivatives of \( u \) in the direction of \( \Delta N \). The first derivative is given by

\[
\begin{align*}
\frac{d}{d\tau} u(N + \tau \Delta N) \bigg|_{\tau=0} &= \int_0^\infty (2 - m'(N(t))) \Delta N(t)dt,
\end{align*}
\]

so, in particular

\[
|u'_{\Delta N}| \leq 2\omega_{\Delta}.
\]

Therefore we can write

\[
(3.4) \quad u'_{\Delta N} = \kappa \omega_{\Delta}, \quad \kappa = \kappa(\Delta N), \quad |\kappa| \leq 2.
\]

The second derivative in the direction \( \Delta N = N_1 - N \) is given by

\[
-\frac{d^2}{d\tau^2} u(N + \tau \Delta N) \bigg|_{\tau=0} = \int_0^\infty \varphi(N(t))^{-1}(\Delta N(t))^2 dt
\]

By Cauchy-Schwarz, the integral in the right side is at least

\[
\left[ \int_0^\infty N(t)\Psi(N(t)) dt \right]^{-1} \left[ \int_0^\infty |\Delta N(t)| dt \right]^2 = n(N)^{-1} \left[ \int_0^\infty |\Delta N(t)| dt \right]^2
\]

\[
= n(N)^{-1}(\omega_{\Delta})^2,
\]
so

\[(3.5) \quad -u''_{\Delta N}(N) \geq \frac{(w_\Delta)^2}{n(N)}\]

For the scalar variable \(f \in \mathbb{R}\) and the distribution function \(N\) define the Bellman function \(\tilde{B}(f, N) = B(f, u(N))\) where

\[B(f, u) = \frac{f^2}{u}.
\]

Computing second derivative of \(\tilde{B}\) in the direction \(\Delta = (\Delta f, \Delta N)\) we get

\[\tilde{B}_{\Delta}'' = \begin{pmatrix} \Delta f' \\ \Delta N' \end{pmatrix}^T \begin{pmatrix} B_{ff} & B_{fu} \\ B_{fu} & B_{uu} \end{pmatrix} \begin{pmatrix} \Delta f' \\ \Delta N' \end{pmatrix} + B_u u''_{\Delta N}.
\]

The Hessian is easy to compute

\[(3.6) \quad \begin{pmatrix} B_{ff} & B_{fu} \\ B_{fu} & B_{uu} \end{pmatrix} = \begin{pmatrix} \frac{2}{u} & -\frac{2f}{u^2} \\ -\frac{2f}{u^2} & \frac{2f^2}{u^3} \end{pmatrix};
\]

note that this matrix is positive semidefinite.

Since \(B_u = -f^2/u^2\), we get using (3.5)

\[B_u u''_{\Delta N} \geq \frac{f^2}{u^2 n}(w_\Delta)^2.
\]

Thus, gathering everything and using (3.4) we get

\[\tilde{B}_{\Delta}'' \geq \begin{pmatrix} \Delta f' \\ \kappa w_\Delta \end{pmatrix}^T \begin{pmatrix} \frac{2}{u} & -\frac{2f}{u^2} \\ -\frac{2f}{u^2} & \frac{2f^2}{u^3} \end{pmatrix} \begin{pmatrix} \Delta f' \\ \kappa w_\Delta \end{pmatrix} \]

The matrix here is obtained from the Hessian in (3.6) by multiplying the lower right entry by \(1 + \frac{u}{2\kappa^2 n} \geq 1\), so it has more positivity than the Hessian. In particular, if we divide the upper left entry of the matrix in (3.7) by the same quantity \(1 + \frac{u}{2\kappa^2 n}\), the matrix still be positive semidefinite. But our matrix in (3.7) has something bigger in the upper-left corner!

Therefore, since

\[1 - \left(1 + \frac{u}{2\kappa^2 n}\right)^{-1} = \frac{u}{2\kappa^2 n + u}\]

we get that

\[(3.8) \quad \tilde{B}_{\Delta}'' \geq \frac{2(\Delta f)^2}{2\kappa^2 n + u} \geq \frac{2(\Delta f)^2}{2 \cdot 2^2 n + u} \geq \frac{(\Delta f)^2}{n};
\]

the last inequality holds because \(u \leq 2w \leq Cn\).

But (3.8) is the exactly the differential form of the inequality we need!

3.2. Main inequality in the finite difference form.
3.2.1. Dyadic case.

Lemma 3.1. Let

$$f = \frac{f_1 + f_2}{2}, \quad N(t) = \frac{N_1(t) + N_2(t)}{2}.$$ 

Then

$$\frac{1}{2}(B(f_1, u(N_1)) + B(f_2, u(N_2))) - B(f, u(N)) \geq \frac{c}{4} \cdot \frac{(f_1 - f)^2}{n(N)}.$$  \hspace{1cm} (3.9)

where $c$ is the constant from (3.8). (Note that $f_1 - f = f - f_2$, so we can replace $(f_1 - f)^2$ in the right side by $(f_2 - f)^2$)

Proof. Notice that

$$\frac{s_1 + s_2}{2} \Psi \left( \frac{s_1 + s_2}{2} \right) \geq \frac{s_1 + s_2}{2} \Psi(s_1 + s_2) \geq \frac{1}{2} s_1 \Psi(s_1); $$ \hspace{1cm} (3.10)

here the first inequality holds because $\Psi$ is decreasing and the second one because $s_1 \Psi(s)$ is increasing. Of course, we can interchange $s_1$ and $s_2$ in the above inequality.

Let $\Delta f := f_1 - f$, $\Delta N := N_1 - N$. Define

$$F(\tau) = B(f + \tau \Delta f, u(N + \tau \Delta N)) + B(f - \tau \Delta f, u(N - \tau \Delta N))$$

Taylor’s formula together with the estimate \textbf{(3.8)} imply that

$$F(1) - F(0) \geq \frac{c}{2} (\Delta f)^2 \left( \frac{1}{n(N + \tau \Delta N)} + \frac{1}{n(N - \tau \Delta N)} \right)$$ \hspace{1cm} (3.11)

for some $\tau \in (0, 1)$.

Estimate \textbf{(3.10)} implies that

$$n(N) \geq \frac{1}{2} n(N \pm \tau \Delta N),$$

so

$$\left( \frac{1}{n(N + \tau \Delta N)} + \frac{1}{n(N - \tau \Delta N)} \right) \geq \frac{1}{n(N)}.$$  

Then it follows from (3.11) that

$$F(1) - F(0) \geq \frac{c}{2} \cdot \frac{(\Delta f)^2}{n(N)}.$$  

Recalling the definition of $F$ and dividing this inequality by 2 we get \textbf{(3.10)}. \hfill \Box
3.2.2. General case. Let $\varphi$ and $\tilde{B}$ be as above.

**Lemma 3.2.** Let $f, f_k \in \mathbb{R}$, $\alpha_k \in \mathbb{R}_+$ and the distribution functions $N, N_k$, $k = 1, 2, \ldots, n$ satisfy

$$f = \sum_{k=1}^{n} \alpha_k f_k, \quad N = \sum_{k=1}^{n} \alpha_k N_k, \quad \sum_{k=1}^{n} \alpha_k = 1.$$  

Then

$$-\tilde{B}(f, N) + \sum_{k=1}^{n} \alpha_k \tilde{B}(f, N_k) \geq \frac{c}{16} \cdot \frac{1}{n(N)} \left( \sum_{k=1}^{n} \alpha_k |f_k - f| \right)^2.$$  

**Proof.** The reasoning below is a “baby version” of the reasoning used to prove the main estimate (Lemma 6.1) in [22].

For a weight $\alpha = \{\alpha_k\}_{k=1}^{n}$, $\alpha_k \geq 0$, let $\ell^p(\alpha)$ be the weighted (finite-dimensional) $\ell^p$ spaces, $\|x\|_{\ell^p(\alpha)} = \sum_{k=1}^{n} \alpha_k |x_k|^p$ ($\ell^\infty(\alpha)$ is just the usual finite-dimensional $\ell^\infty$).

Let $\langle \cdot, \cdot \rangle_\alpha$ be the standard duality $\langle x, y \rangle_\alpha = \sum_{k=1}^{n} \alpha_k x_k y_k$.

Define $e \in \ell^p(\alpha)$, $e = (1, 1, \ldots, 1)$.

Consider the quotient space $X = \ell^1(\alpha)/\text{span}\{e\}$. For $x \in \ell^1(\alpha)$ let $x^0 := x - \|e\|_{\ell^1(\alpha)}^{-1} \langle x, e \rangle_\alpha e$, so $\sum_{k=1}^{n} \alpha_k x_k^0 = 0$. Then

$$\|x\|_X \leq \|x^0\|_{\ell^1(\alpha)} \leq 2 \|x\|_X.$$  

Indeed, the first inequality is trivial (follows from the definition of the norm in the quotient space). As for the second one, $|\langle x, e \rangle_\alpha| \leq \|x\|_{\ell^1(\alpha)}$, so it follows from the triangle inequality that

$$\|x^0\|_{\ell^1(\alpha)} \leq \|x\|_{\ell^1(\alpha)} + \|e\|_{\ell^1(\alpha)}^{-1} |\langle x, e \rangle_\alpha| \cdot \|e\|_{\ell^1(\alpha)} \leq 2 \|x\|_{\ell^1(\alpha)}.$$  

This inequality remains true if one replaces $x$ by $x - \alpha e$, $\alpha \in \mathbb{R}$, so the second inequality in (3.13) is proved.

The dual space $X^*$ can be identified with a subspace of $\ell^\infty = \ell^\infty(\alpha)$ consisting of $x^* \in \ell^\infty(\alpha)$ such that $\langle e, x^* \rangle_\alpha = 0$ (with the usual $\ell^\infty$-norm).

So, for the vector $x = (x_1, x_2, \ldots, x_n)$, $x_k = f_k - f$ (notice that $\langle x, e \rangle_\alpha = 0$ there is $\beta = \{\beta_k\}_{k=1}^{n}$, $|\beta_k| \leq 1$ such that $\sum_{k=1}^{n} \alpha_k \beta_k = 0$ and

$$\sum_{k=1}^{n} \alpha_k \beta_k (f_k - f) = \|x\|_X \geq \frac{1}{2} \|x\|_{\ell^1(\alpha)} = \frac{1}{2} \sum_{k=1}^{n} \alpha_k |f_k - f|.$$  

Define $f^+, f^-, N^+, N^-$ by

$$f^\pm = \sum_{k=1}^{n} \alpha_k (1 \pm \beta_k) f_k, \quad N^\pm = \sum_{k=1}^{n} \alpha_k (1 \pm \beta_k) N_k.$$  

By Lemma 3.1

$$\frac{1}{2} \left( \tilde{B}(f^+, N^+) + \tilde{B}(f^-, N^-) \right) - \tilde{B}(f, N) \geq \frac{c}{4} \frac{(f^+ - f)^2}{n(N)}$$  

(3.14)
We know that
\[ f^+ - f = \sum_{k=1}^{n} \alpha_k \beta_k f_k = \sum_{k=1}^{n} \alpha_k \beta_k (f_k - f) \geq \frac{1}{2} \sum_{k=1}^{n} \alpha_k |f_k - f| \]
(the second equality holds because \( \sum_{k=1}^{n} \alpha_k \beta_k = 0 \)), so the right side of (3.14) is estimated below by
\[ \frac{c}{16} \cdot \frac{1}{n(N)} \left( \sum_{k=1}^{n} \alpha_k |f_k - f| \right)^2 \]
Since the function \( \tilde{B} \) is convex
\[ \tilde{B}(f^+, N^+) \leq \sum_{k=1}^{n} \alpha_k (1 + \beta_k) \tilde{B}(f_k, N_k), \]
\[ \tilde{B}(f^-, N^-) \leq \sum_{k=1}^{n} \alpha_k (1 - \beta_k) \tilde{B}(f_k, N_k) \]
and adding these inequalities we can estimate above the left side of (3.14) by
\[ -\tilde{B}(f, N) + \sum_{k=1}^{n} \alpha_k \tilde{B}(f, N_k). \]

\[ \square \]

3.3. From main inequality (3.12) to Theorem 2.5. Fix an interval \( I^0 \) and let \( I_k \) be its children. Applying Lemma 3.2 with \( f_k = \langle f w^{1/2}, I_k \rangle \), \( N_k = N_{I_k}^w \) and \( \alpha_k = |I_k|/|I^0| \) we get denoting \( \tilde{f} := f w^{1/2} \)
\[ \frac{c}{16} \cdot \frac{\| \Delta_{j, I} \tilde{f} \|^2}{n(N_{I_j}^w) \cdot |I|} \leq -|I^0| \tilde{B}(\langle \tilde{f}, N_{I_j}^w \rangle) + \sum_{I \in \text{ch}(I^0)} |I| \cdot \tilde{B}(\langle \tilde{f}, N_{I_j}^w \rangle) \]
Applying this formula to all children of \( I^0 \), then to their children and adding up the inequalities we get after going \( n \) generations down that
\[ \frac{c}{16} \sum_{I \in \text{ch}_k(I^0)} \frac{\| \Delta_{j, I} \tilde{f} \|^2}{n(N_{I_j}^w) \cdot |I|} \leq -|I^0| \tilde{B}(\langle \tilde{f}, N_{I_j}^w \rangle) + \sum_{I \in \text{ch}_k(I^0)} |I| \cdot \tilde{B}(\langle \tilde{f}, N_{I_j}^w \rangle) \]
\[ \leq \sum_{I \in \text{ch}_k(I^0)} |I| \cdot \tilde{B}(\langle \tilde{f}, N_{I_j}^w \rangle). \]
We know that \( \tilde{B}(f, N) \leq C \frac{r^2}{u(N)} \), and since (see 3.2) \( u(N_{I_j}^w) \geq \langle w \rangle \) we conclude using the Cauchy–Schwartz estimate \( |\langle f w^{1/2}, I \rangle|^2 \leq \langle |f|^2 \rangle \cdot \langle w \rangle \) that
\[ |I| \cdot \tilde{B}(\langle \tilde{f}, N_{I_j}^w \rangle) \leq C |I| \langle f w^{1/2} \rangle \langle w \rangle \leq C |I| \cdot \langle |f|^2 \rangle \cdot \langle w \rangle = C \int_I |f|^2 d\mu. \]
Therefore, estimating the right side we get
\[
\frac{c}{16} \sum_{I \in \text{ch}_k(I^0)} \frac{\|\Delta_I \tilde{f}\|}{n(N_I^w) \cdot |I|} \leq C \int_{I^0} |f|^2 d\mu.
\]
Since the right side does not depend on \( n \) we can make \( n \to \infty \), and have the sum in the left side over all \( I \in \mathcal{L}, I \subset I^0 \).

Taking the sum over all \( I^0 \in \mathcal{L} \) and letting \( m \to \infty \) we get conclusion of the theorem. \( \square \)

4. Proof of (the Embedding) Theorem 2.6

4.1. An auxiliary function. Let \( \Psi \) be the function from Theorem 2.6. Define \( \varphi(s) := s \Psi(s) \).

For the numbers \( A \in [1, 2], N \in \mathbb{R}_+ \) define
\[
T(A, N) := N \int_0^{N/A} \frac{1}{\varphi(s)} ds
\]

Lemma 4.1. The function \( T \) is convex and satisfies the differential inequality
\[
-\frac{\partial T}{\partial A} \geq \frac{1}{4} \cdot \frac{N^2}{\varphi(N)}.
\]

Proof. Differentiating the integral we get
\[
\frac{\partial T}{\partial A} = \frac{N^2}{A^2 \varphi(N/A)} \geq \frac{1}{4} \cdot \frac{N^2}{\varphi(N)},
\]
since \( \varphi \) is increasing and \( A \leq 2 \).

To prove the convexity notice that \( T \) is linear on the lines \( N = kA \), so the Hessian \( \partial^2 T \) degenerates.

Differentiating (4.1) we get
\[
\frac{\partial^2 T}{\partial A^2} = N^2 \frac{2A \varphi'(N/A) - N \varphi'(N/A)}{(A^2 \varphi(N/A))^2}
\]

Note that the right side is positive if \( s \varphi'(s) < 2 \varphi(s) \) (because \( \varphi(s) > 0 \)).

But for our function even a stronger inequality \( s \varphi'(s) \leq \varphi(s) \) is satisfied! Indeed, since \( \varphi(s) = s \Psi(s) \) is increasing and \( \Psi \) is decreasing, then
\[
0 \leq (s \Psi(s))' = \Psi(s) + s \Psi'(s) \leq \Psi(s)
\]
(the second inequality holds because \( \Psi \) is decreasing). Multiplying this inequality by \( s \) we get \( s \varphi'(s) \leq \varphi(s) \).

Therefore, since \( \varphi(s) > 0 \), we conclude that \( \frac{\partial^2 T}{\partial A^2} > 0 \).

But the Hessian \( \partial^2 T \) is singular, and it is an easy exercise in linear algebra to show that a singular Hermitian \( 2 \times 2 \) matrix with a positive entry on the main diagonal is positive semidefinite. \( \square \)
4.2. Bellman function and the main differential inequality. Let now $N$ be a distribution function, and let

$$T(A, N) = \int_0^\infty T(A, N(t)) \, dt.$$  

As in Section 3.1 assume, multiplying $\Psi$ by an appropriate constant, that

$$\int_0^1 \frac{1}{\varphi(s)} \, ds = 1.$$  

Then $T(A, N(t)) \leq N(t)$, so

$$T(A, N) \leq \int_0^\infty N(t) \, dt =: w = w(N).$$

For $f \in \mathbb{R}$, $M \in [0, 1]$ and for a distribution function $N$ define the function

$$\tilde{B}(f, N, M) := B(f, u(M, N)),$$

where

$$B(f, u) = \frac{f^2}{u}$$

and

$$u = u(M, N) = 2 \int_0^\infty N(t) \, dt - T(M + 1, N)$$

$$=: 2w(N) - T(M + 1, N).$$

Note that $2w(N) \geq u(M, N) \geq w(N)$.

We claim that the function $\tilde{B}$ is convex. Indeed, fix a direction $\Delta := (\Delta f, \Delta N, \Delta M)^T$ and compute the second derivative $\tilde{B}_\Delta''$ in this direction

$$\tilde{B}_\Delta'' = \frac{d^2}{d\tau^2} \tilde{B}(f + \tau \Delta f, N + \tau \Delta N, M + \tau \Delta M) \bigg|_{\tau=0}.$$  

We get

$$\tilde{B}_\Delta'' = \begin{pmatrix} \Delta f \\ \Delta u \end{pmatrix}^T \begin{pmatrix} B_{ff} & B_{fu} \\ B_{fu} & B_{uu} \end{pmatrix} \begin{pmatrix} \Delta f \\ \Delta u \end{pmatrix} + B_u u''.$$  

The Hessian

$$\begin{pmatrix} B_{ff} & B_{fu} \\ B_{fu} & B_{uu} \end{pmatrix} = \begin{pmatrix} \frac{2}{u} & -\frac{2f}{u^2} \\ -\frac{2f}{u^2} & \frac{2f^2}{u^3} \end{pmatrix}$$

is clearly positive semidefinite, so the first term is nonnegative. For the second term notice that

$$\begin{align*}
\frac{\partial}{\partial M} B_u &= -\frac{f^2}{u^2}, \\
\frac{\partial}{\partial M} u'' &= -T'' \leq 0
\end{align*}$$

because $T$, and therefore $T$ is convex. Thus $\tilde{B}_\Delta'' \geq 0$, so $\tilde{B}$ is convex. Let us compute the partial derivative

$$-\frac{\partial \tilde{B}}{\partial M} = -B_u \frac{\partial u}{\partial M} = \frac{f^2}{u^2} \left( -\frac{\partial T}{\partial M} \right)$$
By Lemma 4.1
\[-\frac{\partial T}{\partial M} \geq \frac{1}{4} \int_0^\infty \varphi(N(t)) \frac{N(t)^2}{\varphi(N(t))} dt \]
\[\geq \frac{1}{4} \left( \int_0^\infty N(t) dt \right)^2 \left( \int_0^\infty \varphi(N(t)) dt \right)^{-1} = \frac{1}{4} \cdot \frac{w(N)^2}{n(N)};\]
the second inequality here is just the Cauchy–Schwartz. Combining with (4.3) and recalling that \(u \leq 2w\) we get
\[\frac{\partial \tilde{B}}{\partial M} \geq \frac{1}{16} \cdot \frac{f^2}{n} \]
This inequality (together with the convexity of \(\tilde{B}\)) is the main differential inequality for our function.

4.3. Finite difference form of the main inequality. Let \(X = (f, N, M)\), \(X_k = (f_k, N_k, M_k)\), \((f, f_k, f_k, f_k, f_k, f_k) \in \mathbb{R}, M, M_k \in [0, 1], N, N_k\) are the distribution functions) satisfy
\[f = \sum_{k=1}^n \alpha_k f_k, \quad N = \sum_{k=1}^n \alpha_k N_k, \quad M = a + \sum_{k=1}^n \alpha_k M_k, \quad a \geq 0,\]
where
\[\sum_{k=1}^n \alpha_k = 1, \quad \alpha_k \geq 0.\]
Then
\[\tilde{B}(X) + \sum_{k=1}^n \alpha_k \tilde{B}(X_k) \geq \frac{1}{16} \cdot \frac{af^2}{n}\]
where \(n = n(N)\).
Indeed, for \(M_0 := \sum_{k=1}^n \alpha_k M_k\) the main inequality (4.4) implies
\[\tilde{B}(f, N, M_0) - \tilde{B}(f, N, M) \geq \frac{1}{16} \cdot \frac{af^2}{n}.\]
The convexity of \(\tilde{B}\) implies that
\[\tilde{B}(f, N, M_0) \leq \sum_{k=1}^n \alpha_k \tilde{B}(X_k)\]
which together with the previous inequality gives us (4.5).
4.4. From main inequality (4.5) to Theorem 2.6 The reasoning here is almost verbatim the same as in Section 3.3.

For an interval let us \( I \in \mathcal{L} \) denote \( f_I = \langle f w^{1/2} \rangle_I, \ N_I = N^w_I, \ M_I = |I|^{-1} \sum_{I' \subset I} a_{I'}, \ w_I = \langle w \rangle_I, \ u_I = u(M_I, N_I). \)

Fix \( I^0 \in \mathcal{L} \), and let \( I_k \) be its children. Applying the inequality (4.5) with \( \alpha_k = |I_k|/|I^0|, \ f_k = f_{I_k}, \ N_k = N^w_{I_k}, \ M_k = M_{I_k} \) we get that

\[
\frac{1}{16} \sum_{I \in \text{ch}(I^0)} \frac{a_k f_k^2}{n(N_k^w)} |I| \leq -|I^0| \tilde{B}(X_{I^0}) + \sum_{I \in \text{ch}(I^0)} |I| \tilde{B}(X_I).
\]

Writing the corresponding estimates for the children of \( I^0 \), then for their children, we get after going \( n \) generations down and using the telescoping sum in the right side

\[
\frac{1}{16} \sum_{I \in \text{ch}_k(I^0)} \frac{a_k f_k^2}{n(N_k^w)} |I| \leq -|I^0| \tilde{B}(X_{I^0}) + \sum_{I \in \text{ch}_n(I^0)} |I| \tilde{B}(X_I)
\]

\[
\leq \sum_{I \in \text{ch}_n(I^0)} |I| \tilde{B}(X_I);
\]

the last inequality holds because \( \tilde{B} \geq 0 \).

Since

\[
\tilde{B}(X_I) \leq f_I^2 / u_I \leq f_I^2 / w_I
\]

(the last inequality holds because \( u \geq w \)) and by Cauchy–Schwarz

\[
|\langle f w^{1/2} \rangle_I| \leq |\langle |f|^2 \rangle_I \langle w \rangle_I|
\]

we conclude, exactly as in Section 3.3 that

\[
|I| \tilde{B}(X_I) \leq |I| |\langle |f|^2 \rangle_I| = \int_I |f|^2 d\mu,
\]

so

\[
\frac{1}{16} \sum_{I \in \text{ch}_k(I^0)} \frac{a_k f_k^2}{n(N_k^w)} |I| \leq \int_{I^0} |f|^2 d\mu.
\]

Conclusion of the proof is exactly as in Section 3.3 we first let \( n \to \infty \), and then taking the sum over \( I^0 \in \mathcal{L}_{-m} \) and letting \( m \to \infty \) get the desired estimate. \( \square \)

5. Concluding remarks and open problems

5.1. One sided bumps. The famous theorem of P. Koosis \[11\] states that given a weight \( u \) on the unit circle, one can find a non-zero weight \( v \) such that the Hilbert transform \( T \) is bounded as an operator from \( L^2(u) \) to \( L^2(v) \) if and only if \( 1/u \in L^1 \). The same result holds for the maximal function, see \[21\].
So it is possible to have a situation when one has two weight estimates, but one cannot “bump” the $L^1$-norm of $1/u$ ($w$ in our notation). This leads to a very natural (in our opinion) question “can one “bump” only one weight to get the two weight estimate?” For example, can one find a reasonable Young function $\Phi$ that the condition

$$\sup_Q \left( |Q|^{-1} \int_Q w \right) \|v\|_{L^p(Q)} < \infty$$

implies the boundedness in $L^2$ of the operator $M_{1/2} TM_{1/2}$ for all (bounded) Calderón–Zygmund operators $T$?

Note, that for maximal function a one sided bump condition (but with the bump on the “wrong” side) is sufficient. Namely, it follows from the result in [19] that if $\int \infty (\Phi(x))^{-1} dx < \infty$ and the weights $v$, $w$ satisfy

$$\sup_Q \left( |Q|^{-1} \int_Q v \right) \|w\|_{L^p(Q)} < \infty,$$

then the operator $M_{1/2} M M_{1/2}$, where $M$ is the Hardy–Littlewood maximal operator, is bounded in $L^2$. It is natural to remark here that in [16] we demonstrated this result of Pérez by almost precisely the same Bellman function that the reader saw above.

5.2. Estimates for general measures. A standard way to set up the two weight estimate problem for integral operators is to make the change of variables so in the integral operator one integrates with respect to the same measure that is used to compute the norm in the domain.

namely, if one defines measures $\mu$, $\nu$, $d\mu = wdx$, $d\nu = vdx$, then the $L^p$ boundedness of the operator $M_{1/p} TM_{1/p}$ is equivalent (at least formally) to the boundedness of the operator $T_{\mu} : L^p(\mu) \to L^p(\nu)$, where

$$T_{\mu}f(x) = \int K(x, y)f(y)d\mu(y);$$

$K$ here is the kernel of the Calderón–Zygmund operator $T$.

In fact, everything can be interpreted absolutely rigorously. The boundedness of such operators can be interpreted as uniform boundedness of the smooth truncations; in fact such uniform boundedness is equivalent to the boundedness of the bilinear form on functions with separated compact supports, i.e. to the weakest possible notion of boundedness, see [14].

This setting give the most general form of the two weight problem, since $\mu$ and $\nu$ can be general measures, not necessarily absolutely continuous, they even can be purely singular. So the question arises, “how one can bump general measures?” The approach with Orlicz spaces, or any other functions spaces works only for absolutely continuous measures.

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