ON A SUBSPACE PERTURBATION PROBLEM

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ABSTRACT. We discuss the problem of perturbation of spectral subspaces for linear self-adjoint operators on a separable Hilbert space. Let \( A \) and \( V \) be bounded self-adjoint operators. Assume that the spectrum of \( A \) consists of two disjoint parts \( \sigma \) and \( \Sigma \) such that \( d = \text{dist}(\sigma, \Sigma) > 0 \). We show that the norm of the difference of the spectral projections \( E_A(\sigma) \) and \( E_{A+V}(\{\lambda \mid \text{dist}(\lambda, \sigma) < d/2\}) \) for \( A \) and \( A + V \) is less than one whenever either (i) \( \|V\| < \frac{2}{\pi \sigma} d \) or (ii) \( \|V\| < \frac{1}{\pi} d \) and certain assumptions on the mutual disposition of the sets \( \sigma \) and \( \Sigma \) are satisfied.

1. INTRODUCTION

It is well known (see, e.g., [10]) that if \( A \) and \( V \) are bounded self-adjoint operators on a separable Hilbert space \( \mathcal{H} \), then (the perturbation) \( V \) does not close gaps of length greater than \( 2\|V\| \) in the spectrum of \( A \). More precisely, if \( (a, b) \) is a finite interval and \( (a, b) \subset \rho(A) \), the resolvent set of \( A \), then
\[
(a + \|V\|, b - \|V\|) \subset \rho(A + sV) \quad \text{for all } s \in [-1, 1]
\]
whenever \( 2\|V\| < b - a \). Hence, under the assumption that \( A \) has an isolated part \( \sigma \) of the spectrum separated from its remainder by gaps of length greater than or equal to \( d > 0 \), the spectrum of the operators \( A + sV \), \( s \in [-1, 1] \) will also have separated components, provided that the condition
\[
\|V\| < \frac{d}{2}
\]
holds.

Our main concern is to study the variation the corresponding spectral subspace associated with the isolated part \( \sigma \) of the spectrum of \( A \) under perturbations satisfying (1.1).

For notational setup we assume the following hypothesis.

Hypothesis 1. Assume that \( A \) and \( V \) are bounded self-adjoint operators on a separable Hilbert space \( \mathcal{H} \). Suppose that the spectrum of \( A \) has a part \( \sigma \) separated from the remainder of the spectrum \( \Sigma \) in the sense that
\[
\text{spec}(A) = \sigma \cup \Sigma
\]
and
\[
\text{dist}(\sigma, \Sigma) = d > 0.
\]

Introduce the orthogonal projections \( P = E_A(\sigma) \) and \( Q = E_{A+V}(U_{d/2}(\sigma)), \) where \( U_\varepsilon(\sigma), \) \( \varepsilon > 0 \) is the open \( \varepsilon \)-neighborhood of the set \( \sigma \). Here \( E_A(\Delta) \) and \( E_{A+V}(\Delta) \) denote the spectral projections for operators \( A \) and \( A + V \), respectively, corresponding to a Borel set \( \Delta \subset \mathbb{R} \).

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In this note we address the following question: Assuming Hypothesis \([\mathcal{H}]\) does condition \([\mathcal{L}]\) imply
\[
\|P - Q\| < 1?
\]
We give a partially affirmative answer to this question. The precise statement reads as follows.

**Theorem 1.** Assume Hypothesis \([\mathcal{H}]\) and suppose that either

(i) \(\|V\| < \frac{2}{2 + \pi} d\)

or

(ii) \(\|V\| < \frac{d}{2}\)

and

\[\text{conv.} \text{hull}(\sigma) \cap \Sigma = \emptyset \quad \text{or} \quad \text{conv.} \text{hull}(\Sigma) \cap \sigma = \emptyset.\]

Then
\[
\|P - Q\| < 1.
\]

Our strategy of proof of Theorem \([\mathcal{I}]\) does not allow to relax condition
\[\|V\| < \frac{2}{2 + \pi} d\]

and just assume the natural condition \([\mathcal{L}]\) with no additional hypotheses. It is an open problem whether Hypothesis \([\mathcal{I}]\) alone and the bounds
\[\frac{2}{2 + \pi} \leq \frac{\|V\|}{d} < \frac{1}{2}\]
on the perturbation \(V\) imply \(\|P - Q\| < 1\).

For compact perturbations \(V\) satisfying inequality \([\mathcal{L}]\) we can however state that the pair \((P, Q)\) of the orthogonal projections is a Fredholm pair with zero index. Recall that the pair \((P, Q)\) of orthogonal projections is called Fredholm if the operator \(QP\) viewed as a map from \(\text{Ran} P\) to \(\text{Ran} Q\) is a Fredholm operator \([\mathcal{F}]\). The index of this operator is called the index of the pair \((P, Q)\).

**Theorem 2.** Assume Hypothesis \([\mathcal{I}]\) and suppose that \(V\) is a compact operator satisfying \([\mathcal{L}]\). Then the pair \((P, Q)\) is Fredholm with zero index. In particular, the subspaces \(\ker(PQ^\perp - I)\) and \(\ker(P^\perp Q - I)\) are finite-dimensional and
\[
\dim \ker(PQ^\perp - I) = \dim \ker(P^\perp Q - I).
\]

In the “overcritical” case \(\|V\| > d/2\), the perturbed operator \(A + V\) may not have separated parts of the spectrum at all. In this case we give an example where the spectral measure of the perturbed operator \(A + V\) is “concentrated” on the unit sphere in the space of bounded operators \(B(\mathcal{H})\) centered at the point \(P = E_A(\sigma)\), with the norm of the perturbation being arbitrarily close to \(d/2\). That is, given \(d > 0\), for any \(\varepsilon > 0\) one can find a self-adjoint operator \(A\) satisfying Hypothesis \([\mathcal{I}]\) and a self-adjoint perturbation \(V\) with \(\|V\| = d/2 + \varepsilon\) such that
\[
\|E_A(\sigma) - E_{A + V}(\Delta)\| = 1
\]
for any Borel set \(\Delta \subset \mathbb{R}\).
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2. Proof of Theorem I

Our proof of Theorem I is based on the following sharp result (see [9] and references cited therein) taken from geometric perturbation theory initiated by C. Davis [6] and developed further in [4], [5], [7], [8], [10].

Proposition 2.1. Let $A$ and $B$ be bounded self-adjoint operators and $\delta$ and $\Delta$ two Borel sets on the real axis $\mathbb{R}$. Then

$$\text{dist}(\delta, \Delta) \|E_A(\delta)E_B(\Delta)\| \leq \frac{\pi}{2} \|A - B\|.$$ 

If, in addition, the convex hull of the set $\delta$ does not intersect the set $\Delta$, or the convex hull of the set $\Delta$ does not intersect the set $\delta$, then one has the stronger result

$$\text{dist}(\delta, \Delta) \|E_A(\delta)E_B(\Delta)\| \leq \|A - B\|.$$ 

We split the proof of Theorem I into the following two lemmas.

Lemma 2.2. Assume Hypothesis I. Assume, in addition, that (1.3) holds. Then

$$\|P - Q\| < 1.$$ 

Proof. Clearly $\text{spec}(A + V) \subset \overline{U_{\|V\|}(\sigma \cup \Sigma)}$, where bar denotes the (usual) closure in $\mathbb{R}$, and then

$$Q^\perp = E_{A+V}(\overline{U_{\|V\|}(\Sigma)}).$$ 

By the first claim of Proposition 2.1,

$$\|PQ^\perp\| \leq \frac{\pi}{2} \frac{\|V\|}{\text{dist}(\sigma, U_{\|V\|}(\Sigma))}.$$ 

The distance between the set $\sigma$ and the $\|V\|$-neighborhood of the set $\Sigma$ can be estimated from below as follows,

$$\text{dist}(\sigma, U_{\|V\|}(\Sigma)) \geq d - \|V\| > 0.$$ 

Then (2.1) implies the inequality

$$\|PQ^\perp\| \leq \frac{\pi}{2} \frac{\|V\|}{d - \|V\|}.$$ 

Hence, from inequality (1.3) it follows that

$$\|PQ^\perp\| \leq \frac{\pi}{2} \frac{\|V\|}{d - \|V\|} < 1.$$ 

Interchanging the roles of $\sigma$ and $\Sigma$ one obtains the analogous inequality

$$\|P^\perp Q\| < 1.$$ 

Since

$$\|P - Q\| = \max\{\|PQ^\perp\|, \|P^\perp Q\|\}$$

(see, e.g., [2] Ch. III, Section 39]), inequalities (2.2) and (2.3) prove the assertion. \qed
Under additional assumptions on mutual disposition of the parts $\sigma$ and $\Sigma$ of the spectrum of $A$ one can relax the condition (1.3) on the norm of perturbation and replace it by the natural condition (1.1).

**Lemma 2.3.** Assume Hypothesis 4 and suppose that condition (1.1) holds.

(i) If either $\sigma \cap \text{conv.hull}(\Sigma) = \emptyset$ or $\text{conv.hull}(\sigma) \cap \Sigma = \emptyset$, then

$$\|P - Q\| < 1.$$  \hspace{1cm} (2.5)

(ii) If in addition the sets $\sigma$ and $\Sigma$ are subordinated, that is,

$$\text{conv.hull}(\sigma) \cap \text{conv.hull}(\Sigma) = \emptyset,$$

then the following sharp estimate holds

$$\|P - Q\| < \frac{\sqrt{2}}{2}.$$  \hspace{1cm} (2.6)

**Proof.** (i) The proof follows that of Lemma 2.2. Applying the second assertion of Proposition 2.1 instead of inequality (2.1), one derives the estimates

$$\|PQ^\perp\| \leq \frac{\|V\|}{\text{dist}(\sigma, U_{d/2}(\Sigma))} \leq \frac{\|V\|}{d - \|V\|} < 1,$$  \hspace{1cm} (2.7)

under hypothesis (1.4), and then the inequality $\|P^\perp Q\| < 1$, proving assertion (2.5) using (2.4).

(ii) First assume that $V$ is off-diagonal, that is,

$$E_A(\sigma)V E_A(\sigma) = E_A(\sigma)^\perp V E_A(\sigma)^\perp = 0.$$

Then the inequality $\|P - Q\| < \frac{\sqrt{2}}{2}$ follows from the $\tan 2\Theta$-Theorem proven first by C. Davis (see, e.g., [8])

$$\|P - Q\| \leq \sin \left( \frac{1}{2} \arctan \frac{2\|V\|}{d} \right) < \frac{\sqrt{2}}{2}.$$  \hspace{1cm} (2.6)

A related result can be found in [1].

The general case can be reduced to the off-diagonal one by the following trick. Assume that $V$ is not necessarily off-diagonal. Decomposing the perturbation $V$ into the diagonal $V_{\text{diag}}$ and off-diagonal $V_{\text{off}}$ parts with respect to the orthogonal decomposition $H = \text{Ran} E_A(\sigma) \oplus \text{Ran} E_A(\sigma)^\perp$ associated with the range of the projection $E_A(\sigma)$

$$V = V_{\text{diag}} + V_{\text{off}},$$

one concludes that

$$E_{A + V_{\text{diag}}}(U_{d/2}(\sigma)) = E_A(\sigma).$$

Moreover, the distance between the spectrum of the part of $A + V_{\text{diag}}$ associated with the invariant subspace $\text{Ran} E_{A + V_{\text{diag}}}(U_{d/2}(\sigma))$ and the remainder of the spectrum of $A + V_{\text{diag}}$ does not exceed $d - 2\|V_{\text{diag}}\| > 0$. Using the $\tan 2\Theta$-Theorem then yields

$$\|P - Q\| \leq \sin \left( \frac{1}{2} \arctan \frac{2\|V_{\text{off}}\|}{d - 2\|V_{\text{diag}}\|} \right) \leq \sin \left( \frac{1}{2} \arctan \frac{2\|V\|}{d - 2\|V\|} \right) < \frac{\sqrt{2}}{2},$$

completing the proof. \hspace{1cm} $\Box$

The sharpness of estimate (2.6) is shown by the following example.
Example 2.4. Let $\mathcal{H} = \mathbb{C}^2$. For an arbitrary $\varepsilon \in (0, 3/4)$ consider the $2 \times 2$ matrices

\[
A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} \frac{1}{2} - \varepsilon & \sqrt{\varepsilon}/2 \\ \sqrt{\varepsilon}/2 & -\frac{1}{2} + \varepsilon \end{pmatrix}.
\]

Let $\sigma = \{0\}$ and $\Sigma = \{1\}$. Obviously, $\text{dist}(\sigma, \Sigma) = 1$. Since

\[
\|V\| = \frac{1}{2} \sqrt{1 - 3\varepsilon + 4\varepsilon^2} < \frac{1}{2},
\]

the perturbation $V$ satisfies the hypotheses of Lemma 2.3. Simple calculations yield

\[
Q = E_{A+V}(U_{1/2}(\sigma)) = E_{A+V}((-1/2, 1/2)) = \frac{1}{1 + (2\sqrt{\varepsilon} + \sqrt{1 + 4\varepsilon})^2} \begin{pmatrix} (2\sqrt{\varepsilon} + \sqrt{1 + 4\varepsilon})^2 & -2\sqrt{\varepsilon} - \sqrt{1 + 4\varepsilon} \\ -2\sqrt{\varepsilon} - \sqrt{1 + 4\varepsilon} & 1 \end{pmatrix},
\]

and hence,

\[
\|P - Q\| = \left[1 + (2\sqrt{\varepsilon} + \sqrt{1 + 4\varepsilon})^2\right]^{-1/2} < \frac{\sqrt{2}}{2}.
\]

Taking $\varepsilon$ sufficiently small, the norm $\|P - Q\|$ can be made arbitrarily close to $\sqrt{2}/2$.

3. PROOF OF THEOREM 3

Lemma 3.1. Assume Hypothesis 4 and suppose, in addition, that $V$ is a compact operator satisfying condition (1.1). Then there is a unitary $W$ such that $Q = WPW^*$ and $W - I$ is compact.

Proof. Fix $\varepsilon > 0$ such that $(1 + \varepsilon)\|V\| < d/2$ and introduce the family of spectral projections

\[
\mathcal{P}(s) = E_{A+sv}(U_{d/2}(\sigma)), \quad s \in (-\varepsilon, 1 + \varepsilon).
\]

Clearly, $\mathcal{P}(0) = P$ and $\mathcal{P}(1) = Q$. From the analytical perturbation theory (see [10]) one concludes that the operator-valued function $\mathcal{P}(s)$ is real-analytic on $(-\varepsilon, 1 + \varepsilon)$. Moreover (see [10], Section II.4.2),

\[
\mathcal{P}(s) = X(s)\mathcal{P}(0)X(s)^*, \quad s \in [0, 1],
\]

where $X(s)$ is the unique unitary solution to the initial value problem

\[
X'(s) = H(s)X(s), \quad s \in [0, 1],
\]

\[
X(0) = I,
\]

with $H(s) = \mathcal{P}'(s)\mathcal{P}(s) - \mathcal{P}(s)\mathcal{P}'(s)$.

Let $\Gamma$ be a Jordan counterclockwise oriented contour encircling $U_{d/2}(\sigma)$ in a way such that no point of $U_{\|V\|}(\Sigma)$ lies within $\Gamma$. Then

\[
\mathcal{P}(s) = -\frac{1}{2\pi i} \int_{\Gamma} (A + sV - z)^{-1}dz, \quad s \in [0, 1],
\]

and hence,

\[
\mathcal{P}'(s) = \frac{1}{2\pi i} \int_{\Gamma} (A + sV - z)^{-1}(A + sV - z)^{-1}dz, \quad s \in [0, 1].
\]

By the hypothesis $V$ is compact, and hence, $\mathcal{P}'(s), s \in [0, 1]$ is also compact, which implies that $H(s)$ is a compact operator for $s \in [0, 1]$. 
Applying the successive approximation method
\[ X_n(s) = I + \int_0^s H(t)X_{n-1}(t)dt, \quad X_0(s) = I, \]
yields that \( X_n(s) \) converges to \( X(s) \), \( s \in [0, 1] \) in the norm topology and \( X_n(s) - I \) is compact for all \( n \in \mathbb{N} \). Thus, \( X(s) - I \) is a compact operator for all \( s \in [0, 1] \). Taking \( W = X(1) \) yields \( Q = WPW^* \), completing the proof.

Lemma 5.1 implies that the operator \( PWP \) viewed as a map from \( \text{Ran} \, P \) to \( \text{Ran} \, P \) is Fredholm with zero index. By Theorem 5.2 of [3] it follows that the pair \((P, Q)\) is Fredholm and \( \text{ind}(P, Q) = \text{ind}(PW|_{\text{Ran} \, P}) = 0 \), proving Theorem 2.

4. OVERCRITICAL PERTURBATIONS

If the perturbation \( V \) closes a gap between the separated parts \( \sigma \) and \( \Sigma \) of the spectrum of the unperturbed operator \( A \), then, necessarily, we are dealing with the case \( \|V\| \geq d/2 \). In this case one encounters a new phenomenon: It may happen that any invariant subspace of the operator \( A + V \) contains a nontrivial element orthogonal to \( \text{Ran} \, P = \text{Ran} \, E_A(\sigma) \).

To illustrate this phenomenon we need the following abstract result.

**Lemma 4.1.** Let \( A \) and \( V \) be bounded self-adjoint operators and \( \sigma \neq \emptyset \) be a finite set consisting of isolated eigenvalues of \( A \) of finite multiplicity. Assume that the spectrum of the operator \( A + V \) has no pure point component. Then for the orthogonal projection \( Q \) onto an arbitrary invariant subspace of the operator \( A + V \) the subspace \( \text{Ker}(P^\perp Q - I) \), where \( P = E_A(\sigma) \), is infinite-dimensional. In particular,

\[ \|P - Q\| = 1. \tag{4.1} \]

**Proof.** Since \( A + V \) has no eigenvalues, \( \text{Ran} \, Q \) is an infinite-dimensional subspace. By hypothesis, \( \text{Ran} \, P \) is a finite-dimensional subspace. Thus, there exists an orthonormal system \( \{f_n\}_{n \in \mathbb{N}} \) in \( \text{Ran} \, Q \) such that \( f_n \) is orthogonal to \( \text{Ran} \, P \) for any \( n \in \mathbb{N} \) and hence \( P^\perp Q f_n = f_n, \forall n \in \mathbb{N} \), proving \( \dim(\text{Ker}(P^\perp Q - I)) = \infty \). Now equality (4.1) follows from representation (2.4).

The next lemma shows that an isolated eigenvalue of the unperturbed operator \( A \) separated from the remainder of the spectrum of \( A \) by a gap of length 1 may “dissolve” in the essential spectrum of the perturbed operator \( A + V \) turning into a “resonance”, with the norm of the perturbation being larger but arbitrarily close to 1/2.

**Lemma 4.2.** Let \( \varepsilon > 0 \). Let \( A \) and \( V \) be \( 2 \times 2 \) operator matrices in \( \mathcal{S} = L^2(0, 1) \oplus \mathbb{C} \),
\[ A = \begin{pmatrix} M & 0 \\ 0 & -I_{\mathbb{C}} \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} -\left( \frac{1}{2} + \varepsilon \right) I_{L^2(0,1)} & \sqrt{\varepsilon}v \\ \sqrt{\varepsilon}v^* & (\frac{1}{2} + \varepsilon)I_{\mathbb{C}} \end{pmatrix} \]
with respect to the decomposition \( \mathcal{S} = L^2(0, 1) \oplus \mathbb{C} \). Here \( M \) denotes the multiplication operator in \( L^2(0, 1) \),
\[ (Mf)(\mu) = \mu f(\mu), \quad 0 < \mu < 1, \quad f \in L^2(0, 1), \]
and \( v \in \mathcal{B}(\mathbb{C}, L^2(0, 1)) \)
\[ (vg)(\mu) = w(\mu)g, \quad \mu \in (0, 1), \quad g \in \mathbb{C}, \]
\[ w(\mu) = \sqrt{\mu(1 - \mu)}. \]

If \( \varepsilon < 2/5 \), then the operator \( A + V \) has no eigenvalues.
Proof. Assume to the contrary that $\lambda \in \mathbb{R}$ is an eigenvalue of the perturbed operator $A + V$, that is,

$$(\mu - 1/2 - \varepsilon)f(\mu) + \sqrt{\varepsilon}w(\mu)g = \lambda f(\mu) \quad \text{a.e. } \mu \in (0, 1)$$

and

$$\sqrt{\varepsilon} \int_0^1 d\mu f(\mu)w(\mu) + (-1/2 + \varepsilon)g = \lambda g$$

for some $f \in L^2(0, 1)$ and $g \in \mathbb{C}$. In particular,

$$f(\mu) = \sqrt{\varepsilon} \frac{w(\mu)}{\lambda - (\mu - \frac{1}{2} - \varepsilon)} g,$$

and hence $f \notin L^2(0, 1)$ whenever $\lambda \in [-1/2 - \varepsilon, 1/2 - \varepsilon]$ (unless $f = 0$ and $g = 0$). Thus, the interval $[-1/2 - \varepsilon, 1/2 - \varepsilon]$ does not intersect the point spectrum of $A + V$. Moreover, $\lambda \in (-\infty, -1/2 - \varepsilon) \cup (1/2 - \varepsilon, \infty)$ is an eigenvalue of $A + V$ if and only if

$$(4.2) \quad \lambda + \frac{1}{2} - \varepsilon + \varepsilon \int_0^1 d\mu \frac{\mu(1 - \mu)}{\mu - \frac{1}{2} - \varepsilon - \lambda} = 0.$$

Elementary analysis of the graph of the function on the left-hand side of (4.2) then yields that under the condition $0 < \varepsilon < 2/5$ there is no solution of equation (4.3) in $(-\infty, -1/2 - \varepsilon) \cup (1/2 - \varepsilon, \infty)$. Thus, the point spectrum of $A + V$ is empty. \hfill \Box

Remark 4.3. We note that $\text{spec}(A) = \{-1\} \cup [0, 1]$ and hence $\text{spec}(A)$ has two components separated by a gap of length one, and the norm of the perturbation $V$ may be arbitrarily close to $1/2$ (from above):

$$(4.3) \quad \|V\| = \sqrt{\left(\frac{1}{2} + \varepsilon\right)^2 + \frac{1}{6} \varepsilon} = \frac{1}{2} + \frac{7}{6} \varepsilon + O(\varepsilon^2) \quad \text{as } \varepsilon \to 0.$$

Using scaling arguments, Remark 4.3 combined with the result of Lemma 4.1 shows that given $d > 0$, for any $\varepsilon > 0$ one can find a self-adjoint operator $A$ satisfying Hypothesis 1 and a self-adjoint perturbation $V$ with $\|V\| = d/2 + \varepsilon$ such that

$$\|E_A(\sigma) - Q\| = 1$$

for the orthogonal projection $Q$ onto an arbitrary invariant subspace of the operator $A + V$.

References

[1] V. Adamyan and H. Langer, Spectral properties of a class of rational operator valued functions, J. Operator Theory 33 (1995), 259 – 277.
[2] N. I. Akhiezer and I. M. Glazman, Theory of Linear Operators in Hilbert Space, Dover Publications, New York, 1993.
[3] J. Avron, R. Seiler, and B. Simon, The index of a pair of projections, J. Funct. Anal. 120 (1994), 220 – 237.
[4] R. Bhatia, C. Davis, and A. McIntosh, Perturbation of spectral subspaces and solution of linear operator equations, Linear Algebra Appl. 52/53 (1983), 45 – 67.
[5] R. Bhatia, C. Davis, and P. Koosis, An extremal problem in Fourier analysis with applications to operator theory, J. Funct. Anal. 82 (1989), 138 – 150.
[6] C. Davis, Separation of two linear subspaces, Acta Scient. Math. (Szeged) 19 (1958), 172 – 187.
[7] C. Davis, The rotation of eigenvectors by a perturbation. I and II, J. Math. Anal. Appl. 6 (1963), 159 – 173; 11 (1965), 20 – 27.
[8] C. Davis and W. M. Kahan, The rotation of eigenvectors by a perturbation. III, SIAM J. Numer. Anal. 7 (1970), 1 – 46.
[9] R. McEachin, *Closing the gap in a subspace perturbation bound*, Linear Algebra Appl. 180 (1993), 7 – 15.

[10] T. Kato, *Perturbation Theory for Linear Operators*, Springer–Verlag, Berlin, 1966.

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