Several Separations Based on a Partial Boolean Function*

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Abstract

We show a partial Boolean function \( f \) together with an input \( x \in f^{-1}(\ast) \) such that both \( C_{0}(f, x) \) and \( C_{1}(f, x) \) are at least \( C(f)^{2-o(\log \log n)} \). Due to recent results by Ben-David, Göös, Jain, and Kothari, this result implies several other separations in query and communication complexity. For example, it gives a function \( f \) with \( C(f) = \Omega(\deg(f)) \) where \( C \) and \( \deg \) denote certificate complexity and polynomial degree of \( f \). (This is the first improvement over a separation between \( C(f) \) and \( \deg(f) \) by Kushilevitz and Nisan in 1995.) Other implications of this result are an improved separation between sensitivity and polynomial degree, a near-optimal lower bound on conondeterministic communication complexity for Clique vs. Independent Set problem and a near-optimal lower bound on complexity of Alon–Saks–Seymour problem in graph theory.

1 The puzzle

Recently Ben-David, Göös, Jain, and Kothari published a paper [2] demonstrating that several separation problems can be reformulated (disguised) as one of three equivalent puzzles, hinting that such formulations may be more seductive for tricking more people into trying to solve them. We report that they have indeed succeeded and show an optimal solution to one of the puzzles.

We use the following formulation from [2]. Consider a partial Boolean function \( f : \{0, 1\}^{n} \rightarrow \{0, 1, \ast\} \) where some of the inputs are undefined, \( f(x) = \ast \). Let \( \Sigma \subseteq \{0, 1, \ast\} \) be a subset of output symbols. Denote by \( 0, 1, 0, 1 \) the output sets \( \{0\}, \{1\}, \{1, \ast\}, \{0, \ast\} \). A partial input \( \rho \in \{0, 1, \ast\}^{n} \) is a \( \Sigma \)-certificate for \( x \in \{0, 1\}^{n} \) if \( \rho \) is consistent with \( x \) (i.e., for each entry in \( x \), the corresponding entry of \( \rho \) contains the same symbol or \( \ast \)) and for every input \( x' \) consistent with \( \rho \) we have \( f(x') \in \Sigma \). The size of \( \rho \), denoted \( |\rho| \), is the number of its non-\( \ast \) entries. The \( \Sigma \)-certificate complexity of \( x \), denoted \( C_{\Sigma}(f, x) \), is the least size of a \( \Sigma \)-certificate for \( x \). The \( \Sigma \)-certificate complexity of \( f \), denoted \( C_{\Sigma}(f) \), is the maximum of \( C_{\Sigma}(f, x) \) over all \( x \in f^{-1}(\Sigma) \).

Finally, we define the certificate complexity \( C(f) \) as \( \max\{C_{0}(f), C_{1}(f)\} \).

**Puzzle.** For \( \alpha > 1 \), does there exist a partial function \( f \) together with an \( x \in f^{-1}(\ast) \) such that both \( C_{0}(f, x) \) and \( C_{1}(f, x) \) are at least \( C(f)^{\alpha-o(\log \log n)} \)?

Abusing the terminology, instead of a single Boolean function actually an infinite sequence of functions \( f_{n} \) satisfying \( C(f_{n}) \rightarrow \infty \) as \( n \rightarrow \infty \) is meant. It is known that a solution with \( \alpha = 2 \) would be optimal. In [2] a simple function with \( \alpha = 1.5 \) inspired by the board game Hex is constructed. It is also conjectured that the puzzles are soluble with exponent 2. Indeed, this is the case and we demonstrate a function achieving the optimal \( \alpha = 2 \).

*Supported by the project “Quantum algorithms: from complexity theory to experiment” funded under ERDF programme 1.1.1.5.
2 The solution

Our contribution is as follows.

**Theorem 1.** There exists a monotone partial Boolean function \( f \) and an input \( x \in f^{-1}(\ast) \) such that both \( C_0(f, x) \) and \( C_1(f, x) \) are at least \( C(f)^{2-o(1)} \).

We denote by \([n]\) the set \( \{1, 2, \ldots, n\} \).

Let \((r_k : [n] \times [n] \rightarrow [n])_{k \in [\ell]}\) be a collection of \( \ell \) independent random functions where the output is chosen uniformly from \([n]\).

**Lemma 1.** Let \( \ell > 4 \). Consider an arbitrary \( S \subseteq [n] \) with \(|S| = m \leq n^{\frac{1+\ell}{2}} \). With probability \(1-o(1)\),

\[ |\{(i, j) \mid i, j \in S \land \forall k \in [\ell] r_k(i, j) \in S\}| \leq \ell \cdot n. \]

**Proof.** First, let us consider a random \( S \subseteq [n] \). Define a random variable

\[ Z_{i,j}^S = \begin{cases} 1, & \text{if } \forall k \in [\ell] r_k(i, j) \in S \\ 0, & \text{otherwise} \end{cases}. \]

\[ E[Z_{i,j}^S] = Pr[Z_{i,j}^S = 1] = \left(\frac{m}{n}\right)^{\ell}. \]

Let \( Z^S = \sum_{i,j \in S} Z_{i,j}^S \).

\[ E[Z^S] = \sum_{i,j \in S} E[Z_{i,j}^S] \]
\[ = m^2 \cdot \left(\frac{m}{n}\right)^{\ell} \]
\[ \leq \left(\frac{n^{\ell+1}}{n}\right)^2 \cdot \left(\frac{n^{\ell+1}}{n}\right)^{\ell} \]
\[ = n^{2 \ell+2} \cdot n^{\ell+1} \]
\[ = n. \]

Therefore, \( \mu_u = n \) is an upper bound on \( E[Z^S] \).

By the Chernoff inequality (see e.g. [5]) \( Pr[Z^S \geq (1 + \epsilon)\mu_u] \leq \exp\left(-\frac{\epsilon^2}{2} \mu_u\right) \).

Therefore,

\[ Pr[Z^S \geq \ell \cdot n] = Pr[Z^S \geq (1 + (\ell - 1))\mu_u] \]
\[ \leq \exp\left(-\frac{(\ell - 1)^2}{2 + \ell - 1} n\right) \]
\[ = \exp\left(-\frac{(\ell - 1)^2 - 4 + 4}{\ell + 1} n\right) \]
\[ \leq \exp\left(-\frac{(\ell - 1)^2 - 4}{\ell + 1} n\right) \]
\[ = \exp\left(-\frac{(\ell - 1 - 2)\ell - 1 + 2}{\ell + 1} n\right) \]
\[ = \exp(-(\ell - 3)n). \]
Now, let us calculate the probability that there exists such $S$ that violates the inequality in the Lemma statement.

$$
\Pr \left[ \exists S : Z^S \geq \ell \cdot n \right] \leq \frac{2^n}{e^{(\ell-1)n}} \leq \frac{e^n}{e^{(\ell-1)n}} = \frac{1}{e^{(\ell-3)n}} = o(1).
$$

Consider an input consisting of $2n^2$ variables $x_{i,j,b} \in \{0,1\}$ with $i,j \in [n]$ and $b \in [2]$. The input is interpreted as an $n \times n$ matrix containing pairs of Boolean values as entries. The variable pair $(x_{i,j,1}, x_{i,j,2})$ is the entry in the $i$-th row and $j$-th column. We refer to the $i$-th row by $x_i$.

We call two entries $(a_1, a_2)$ and $(b_1, b_2)$ matching if $(a_1 \land b_1) \lor (a_2 \land b_2)$. We call two distinct rows $x_i$ and $x_j$ matching if in each column they have matching entries.

We call a row bad if it contains an entry $(0,0)$.

For every pair $(i_1, i_2)$, we call the rows $x_{r_1,(i_1, i_2)}, \ldots, x_{r_{\ell(\delta, i_2)}}$ associated with the rows $x_{i_1}, x_{i_2}$.

Define $f(z) = 1$ if there exist two matching rows $x_{i_1}, x_{i_2}$, and none of the associated rows $x_{r_{\ell(\delta, i_2)}}$ are bad.

Notice that a bad row dismisses its chance to be matching with any other row, as well as spoils every pair for which it is associated.

Define $f(x) = 0$ if there exists a certificate on at most $(2\ell + 2)n$ variables which certifies that $f(x) \neq 1$.

Otherwise, define $f(x) = \ast$.

More formally,

$$
f(x) = \begin{cases} 
1, & \exists i_1, i_2 \in [n] : (i_1 \neq i_2) \\
\land \left( \forall j \in [n] \exists b \in [2] \left( x_{i_1,j,b} \land x_{i_2,j,b} \right) \right) \\
\land \left( \forall k \in [\ell] \forall j \in [n] \exists b \in [2] x_{r_{\ell(\delta, i_2)}, j,b} \right) \\
0, & \text{a certificate with } \leq (2\ell + 2)n \text{ variables exists certifying that } f(x) \neq 1 \\
\ast, & \text{otherwise}
\end{cases}.
$$

C1($f$) $\leq 2n \cdot (\ell + 2)$ because the two matching rows $x_{i_1}, x_{i_2}$ together with the associated rows $x_{r_{\ell(\delta, i_2)}}$ certify that $f(x) = 1$.

C0($f$) $\leq (2\ell + 2)n$ by definition.

Consider the input $z$ in which $z_{i_1,i_1} = 1, z_{i_1,i_2} = 0$, and $z_{i,j,1} = 0, z_{i,j,2} = 1$ for $i \neq j$. I.e., the diagonal entries are $(1,0)$, and all other entries are $(0,1)$. Clearly, $f(z) \neq 1$ as every pair of rows are non-matching, due to the diagonal entries.

$$
C_1(f,z) \geq \frac{n^{(n-1)}}{2} \text{ because there are no bad rows and every pair of rows } z_{i_1}, z_{i_2} \text{ could be made matching by setting } z_{i_1,i_1} = z_{i_1,i_2} = 1, \text{ therefore any certificate certifying that } f(z) \neq 1 \text{ should contain at least one of } z_{i_1,i_1} \text{ and } z_{i_2,i_2} \text{ for every } i_1, i_2 \in [n] \ (i_1 \neq i_2).
$$

Claim 1. $C_0(f,z) > n^{\frac{\ell^2 + 3}{2}}$.

Proof. Let $\rho$ be a partial input consistent with $z$ that has size $n^{\frac{\ell^2 + 3}{2}}$. We will construct a $0$-certificate $\sigma$ consistent with $\rho$ with size $\leq (2\ell + 2)n$. By averaging, there exists a column in which $\rho$ has read (one or both variables) from $m \leq n^{\frac{\ell^2 + 2}{2}}$ entries. In this column in the unseen entries we write $(0,0)$ in $\sigma$, therefore making the corresponding rows bad, hence unfit for being matching. Here we have used no more than $2n$ variables in $\sigma$.

Now we have to spoil the remaining $m(m-1)$ possible pairs. Notice that, by association, most of these pairs are already spoiled. By Lemma there are at most $\ell \cdot n$ possibly matching unspoiled pairs of rows (the rest are spoiled by having at least one bad row associated with
them). We spoil each of them by exposing in $\sigma$ two zeros that make these rows non-matching. With this, the 0-certificate $\sigma$ is complete – every pair of rows are shown to be non-matching or having an associated bad row. In this step we have used at most $2\ell n$ variables in $\sigma$, and at most $(2\ell + 2)n$ in total.

Therefore, we have constructed a 0-certificate that is consistent with $\rho$ and has size $\leq (2\ell + 2)n$. Therefore, $\rho$ cannot be a $\bar{0}$-certificate, i.e., any $\bar{0}$-certificate must have size $> n^{2\ell + 3\ell/\ell+2}$. □

By setting $\ell = \log n$, we have

$$C_1(f) = \Omega(n)$$
$$C_0(f) = \Omega(n)$$
$$C_1(f, z) = n^{2-o(1)}$$
$$C_0(f, z) = n^{2-o(1)}$$

and Theorem 4 follows.

Notice that $f$ is monotone, i.e., flipping any bit in an input $z$ from 0 to 1 can only change $f(z)$ from 0 to * or 1, or from * to 1. This is no coincidence, because, in fact, $f$ was derived from a somewhat more complex function by a transformation inspired by [2, Remark 15] which transforms a function into a monotone one.

### 3 The implications

In this section we list the main bounds and separations arising from our result. All of them are noted in [2]. As they cover a wide range of concepts and contain no new contributions from our side, we restrict ourselves to only listing them and indeed even do not define all the terminology used for stating them, but refer the reader to [2] and other mentioned sources instead.

The following two corollaries follow from the other two formulations of equivalent puzzles in [2].

**Corollary 1.** There exists a Boolean function $f$ with $C_0(f) \geq \Omega((f)^2 - o(1))$.

**Corollary 2.** There exists an intersecting hypergraph $G = (V, E)$ together with a colouring $c : V \to \{0, 1\}$ such that every $c$-monochromatic hitting set has size at least $r(G)^{2-o(1)}$.

The next corollary follows from [6] and gives a near-optimal lower bound for the complexity of the Clique vs. Independent Set problem by Yannakakis [8].

**Corollary 3.** There exists a graph $G$ such that the CIS$_G$ requires $\Omega\left(\log^{2-o(1)} n\right)$ bits of conondeterministic communication.

Equivalently (see, e.g., [4]), the same gap applies to the graph-theoretic Alon–Saks–Seymour problem.

**Corollary 4.** There exists a graph $G$ such that $\chi(G) \geq \exp\left(\Omega\left(\log^{2-o(1)} bp(H)\right)\right)$.

The next two separations follow from the cheat sheet constructions [3, 4]. They improve the power-2.5 separation due to [2] and the power-1.63 separation due to Nisan, Kushilevitz, and Wigderson [7], respectively.
Corollary 5. There exists a Boolean function $f$ with $C(f) \geq \Omega\left(s(f)^{3-o(1)}\right)$.

Corollary 6. There exists a Boolean function $f$ with $C(f) \geq \Omega\left(\deg(f)^{2-o(1)}\right)$.

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