Super-Relativity and State-Dependent Gauge Fields

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Abstract

Objective interpretation of the quantum states leads to the enhancement of the principle of ‘relativity to the measuring device’ to the principle of ‘relativity to embedding into environmental field’ and further - to the super-relativity. In fact the geometric phase works rely on tacitly thinking version of such principle. The geometric phase approach refers to the symplectic invariants of the state space (or parameter space) independent on dynamical (in the space-time) behavior of “macroscopic” quantum systems. As far as understood the geometric phase concept is applicable to the really “microscopic” quantum systems like ‘elementary particles’, due to a generality, as well. However unification of the fundamental interactions including gravity requires the geometric (in the state space) description of quantum dynamical variables. Therefore the state space geometry should be non-distinguishable from the quantum dynamics. The new kinds of dynamical invariants (quantum integrals) associated with symmetric part of Kähler structure arise in such description. These integrals are relating to some analog of non-Abelian Yang-Mills unified fields.

Key words: geometric phase, state-dependent gauge principle, ‘functional’ non-Abelian gauge fields, relativity to the measuring device, cosmic potential, tunneling in the state space

1 Introduction

The eyewitness role is seldom achievable for a detective. Hence, only some traces of a crime are really observable. Therefore it would be very strange to demand from the
detective to operate with observable facts only during the crime investigation. He should, of course, use some evidences and facts, but the most important to be able to connect these fragments into the entire ‘real’ picture of the crime. Physicist frequently acts like the detective trying to find the rational explanation of an observable phenomenon. That is why Einstein emphasized that it would be incorrect to base physical theory on the observable quantities only. Therefore, the observability principle, from this point of view, should give up one’s seat to the objective principle of the relativity to the embedding into environmental field as an enhancement of the ‘relativity to the measuring device’ principle. ‘Principle relativity to the measuring device’ was declared by Fock in quantum theory as the generalization of relativity to the reference frame in classical physics [1]. We will avoid here the criticism of Fock’s philosophy which was in the contradiction with (correct from our point of view) Dirac’s objective interpretation of quantum state of a system [2]. Such comprehension of the quantum state leads to the consistent development of Fock idea of the ‘relativity to the measuring device’;- namely, relativity to embedding into environmental field.

During last years I have understood that super-relativity principle [3, 4, 5, 6] may be understood as the principle of relativity to embedding into environmental field in a ‘strong’ interpretation. The ‘weak’ interpretation is simply trivial statement, since our scientific paradigm based on the assumption that different physical devices (let’s say accelerators in Protvino and in the CERN, i.e. different background fields) give us qualitatively identical results on proton’s collisions. But this form of the invariance principle is very restrictive since there is the energetic threshold for external field beyond that we have new generations of the leptons, etc. In order to avoid this limitation, one should admit the particle identification as a function of the measuring equipment (background field): different choice of the measuring (generating) device (i.e. ‘external fields’) may lead to the different combination of internal degrees of freedom, i.e. to the quantum particle mutation. It is clear that the ‘strong’ invariance principle may be realized on very deep level. Presumably it may be Plank’s scale level. Realistic physical picture on this level is absent, but at least in one thing we can be sure that the electrodynamics itself is not acceptable on this level as a paragon gauge theory. Namely, pointwise charged particles is very deep defect of the theory capable to spoil any efforts in mimic generalization of the gauge theory [7]. However, extended (in the space-time) quantum particles are localizable in a functional state space. Therefore in the state space we have to have a state-dependent gauge principle [8]. Herein more general kind of gauge invariance has been proposed than one mentioned in Jones’es work [8]. It will applied to the fundamental fields (FF) defined on projective Hilbert space CP(N − 1) [8]. In principle, one can use
$CP(\infty)$ but frequently finite dimension projective space is quite enough for practical targets.

Some aspects of $CP(N - 1)$ using as a phase space discussed already widely, however, for different physical aims (see for example well known articles in this direction \[9, 10, 11, 12\]). These approaches should be changed for our targets as follows:

1. The ordinary quantum mechanics is obviously linear with very high degree of accuracy. It is clear that the foundations of quantum mechanics are rooted into the nonlinear quantum field theory of the ‘elementary’ particles. Therefore the linear character of quantum mechanics is an approximation of the more deep theory but of course is not a particular case of classical physics (even advanced one) \[10\]. We propose $CP(N - 1)$ as an underlying unified physical construction for the field foundation of quantum theory. Namely, the points of projective Hilbert space represent states of all conceivable background fields (matter) created by self-consistent Universe cosmic potential (see below). From the mathematical point of view projective Hilbert space is the base of tangent fiber bundle. In each tangent space when local coordinates are close to zero, the curvature is negligible and one has the linear picture of the ordinary quantum mechanics (the tangent space approximation).

2. Geometric description of the gravity is achievable due to the fact of the independence of accelerations of freely falling masses. The space-time behavior of quantum particles is definitely different. Hence, a different criterion than accelerations equivalence, and different arena rather the space-time should be used for the geometric unification of the fundamental interactions in the quantum case. First of all the notion of a quantum state should be used instead of material point notion. Then, I assume that desirable unification of all fundamental interactions may be achieved if we take into account the universal projective Hilbert space geometry of quantum states. In particular, the problem of the comparison of two quantum states in the different background fields leads to the ‘parallel transport’ in some generalized sense and newly understood affine connection. In fact $CP(N - 1)$ geometry is the unique common property of all known quantum particles since their rays of states obey to identical geometric laws under unitary dynamics. Local coordinates in the projective Hilbert space will correspond boson fundamental field (FF) \[6\]. It looks like the phenomenological Lagrangians method but boson fields presented by the points of the $CP(N - 1)$, are now coordinates of dynamical field variables like $D^i_\alpha$ (see below) realizing the nonlinear representation $SU(N)$ group.

3. Riemannian structure treated in the works \[1, 11, 12\] as an evidence of the statistical character theory. We develop the deterministic foundation of quantum theory assuming that statistical properties arise under space-time embedding of underlying
nonlinear (Kählerian) pre-dynamics.

4. Due to the positive curvature of the projective Hilbert state space these “geometric bosons” are effectively interacting and being condensed (like the spin complex) obey nonlinear differential equations of motion. Presumably one can find some Skyrmion-like (soliton) solution of these equations determined by the projective structure of $CP(N-1)$. Further they may be quantized like boson or fermion [13].

5. I will discuss here the vacuum excitations modes of the “fundamental field” [6]. We assume that the FF may be identified with the field of a self-consistent global Universe potential $U_{global}$ and, hence, may treated as a vacuum for each concrete quantum system. The concrete quantum system will be defined in tangent space $T_{\pi}CP(N-1)$ at the origin point $\pi = (\pi^1, ..., \pi^{N-1})$ by the set of dynamical variables (generators of the $SU(N)$).

6. Poincaré group arises as covering group for “question” dynamical variables (logical spin 1/2) treated as a local space-time coordinates.

7. The area of pulse (with the action dimension) transmitted to the quantum system will be treated herein as the invariant measure of the fundamental interaction. In order to demolish the ambiguous of the external field one should demolish the difference between environmental field and ‘particles’. In fact quantum field theory made already important steps in that direction. Namely, all fields should be ‘secondly quantized’ i.e. represented as a result of the action by specified creation operators on the standard vacuum state $|V>$. Procedure of the second quantization has been proposed by Dirac. This consist of the prescription that the Fourier components of the wave function (c-numbers) should be changed by operators (q-numbers) with definite commutation relations. However there are a lot of problems with such approach, namely: all operators corresponding to particles one should be introduced ‘by hand’; commutation relations between these operators should be given a priori (boson or fermion); such commutation relations lead to highly singular functions and problematic description suffering on UV divergences. Therefore some different version of the ‘second quantization’ as unified quantum dynamics we must build. I will establish in the present work only general framework for such kind of quantization.
2 Vacuum and Self-Consistent Potential of Universe

Nobody knows the real self-consistent Universe potential $U_{\text{global}}$ and corresponding vacuum state $|V\rangle$. However we have some hints appeared from the foundation of general and special relativity.

It is well known that postulate of light’s velocity invariance in special relativity, was rejected by Einstein during development of the general relativity. Namely, light’s velocity in the general relativity is a function of the gravitation potential. Since all material objects have energy (mass) they produce local in Universe perturbation of the global potential and, hence, light’s velocity.

I will assume there is the global (gravitation or, probably,- universal?) self-consistent potential of the Universe $U_{\text{global}}$ with the value $c^2$. It is clear that now $E = mc^2$ looks like potential internal energy, since the relativistic Lagrangian

$$\mathcal{L} = -mc^2\sqrt{1 - \frac{v^2}{c^2}} - mG_N \frac{M}{r} = -mc^2\sqrt{1 - \frac{v^2}{c^2}} - m\phi_{\text{local}},$$

may be approximately represented as follows:

$$L_a = T - V = \frac{mv^2}{2} - mc^2 - m\phi_{\text{local}},$$

where $V = m(c^2 + \phi_{\text{local}}) = m(U_{\text{global}} + \phi_{\text{local}})$. The question is: how the special relativity formula $E = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}}$ anticipates the global gravitation potential $c^2$? Probably, because the paragon theory has such ability. For example Dirac’s relativistic equation for electron predicts automatically spin degrees of freedom and positrons; Maxwell electrodynamics gives different example of the similar kind of anticipation: it has Poincaré invariant form before Einstein’s recognition of the physical sense of this invariance.

So, we will assume that just fundamental constant $c^2$ is defined by self-consistent potential of the whole Universe, not a mass in spite of Mach-Einstein assumption \[5\]. This global potential has presumably a microscopic exhibition (‘tension’ of the vacuum) too. Now we will assume that universal character of the potential $U_{\text{global}} = c^2$ connected not only with the space-time geometry, but with the geometry of quantum state space. One reasonable hypothesis is that the constant sectional curvature of
$CP(N-1)$ should contain this world constant. This assumption leads to far-going consequence: fundamental interactions presumably determined by dimensionless holomorphic sectional curvature $\mathcal{K}$ of the projective Hilbert space $CP(N-1)$, using as the carrier of self-consistent vacuum states, should be proportional to the fine structure constant $\mathcal{K} = a\alpha = a\frac{e^2}{\hbar c}$ (see, for a comparison, the arguments by \cite{[10, 11]} for the $\hbar$ introducing in the sectional curvature of $CP(N-1)$). The universal and small dimensionless parameter $\mathcal{K}$ will be used in the next work for the unified interaction problem.

Einstein emphasized that it is impossible to distinguish the “kinetic energy” from the potential energy that associated with $mc^2$ in the formula $E = mc^2 \sqrt{1 - \frac{v^2}{c^2}}$. It is because Einstein dealt with closed system of light source and electromagnetic wave packets in his thought experiment. The global potential makes any system open, therefore interpretation $mc^2$ as a potential energy is possible. In quantum theory due to the vacuum fluctuations, any system is open and isolated quantum object is an important approximation. It is interesting to find a pure quantum analog of the formula for energy of some quantum model so, that it is possible to distinguish kinetic energy and potential only approximately as well as in the classical formula for the material point. It may be the energy of soliton solution for a non-linear wave equation. So-called ‘chiral models’ give us the examples of such kind. One of the well known ‘chiral model’ is $CP(N-1)$ model.

3 Underlying Field CP(N-1) Structure

Super-relativity principles tells that there exist some fundamental invariants (like electric charge) concerning internal state space geometry that do not depend on embedding of a quantum system into background field. In fact this principle has been established in particular cases in so-called geometric phases works. Mainly these themes associated with works of Berry \cite{[14, 15]}, Aharonov-Anandan \cite{[16, 17]}, and Wilczek-Zee \cite{[18]}. In particular Aharonov and Anandan emphasized there exist (infinite) class of equivalent Hamiltonians (i.e. fields) “that generate the motion in $\mathcal{H}$ that project to the same close curve $\mathcal{C}$ in $\mathcal{P}$”. The geometric phase $\beta$ depends on the $\mathcal{C}$ only, i.e. it is invariant (Chern class) associated with connection and curvature in the fiber bundle. In fact only simplectic structure of the state space has been taken into account in these works. We will use the complex rays space $CP(N-1)$ like \cite{[10, 17]} instead of parameter space of Hamiltonian family \cite{[14, 15]}. Berry \cite{[13]} already
pointed out that the symmetric part of the Kähler structure may have essential physical meaning. This is quantum geometric tensor defining Riemannian structure on that manifold \([9]\). I will discuss here only the case of projective Hilbert space.

The using \(CP(\infty)\) or \(CP(N-1)\) to the coherent states description is quite habitual. In our interpretation \(CP(N-1)\) is newly defined as a “space of shapes” \([19]\) of the ‘elementary particles’ (elementary FF excitations), i.e. the space of representation of the coset manifold \(SU(N)/S[U(1) \times U(N-1)]\). It physically means that there are some set of geometric invariants of the quantum state \(|\psi\rangle = \psi^i|i\rangle\) with which it is possible to associate some ‘shape’ of quantum state under unitary dynamics. All conceivable states of the elementary vacuum excitations in the Universe consist of ‘shapes’ cum location in space-time. Location in space-time is represented by the local variations of the global potential \(<V|U_{\text{global}}|V> = c^2\). In accordance with our assumption the unitary super-multiplete \(\Psi^k\) is capable to represent any quantum particle in any background field. Therefore, there is the omnipresence of the local unitary transformations \(U_{(AB)} \in SU(N-1)\) capable to transform particle A into particle B: \(|\Psi(B)\rangle = U_{(AB)}|\Psi(A)\rangle\). This give us a possibility looking for pre-dynamics in the compact manifold \(CP(N-1)\) of \(SU(N)\) group representation instead of space-time itself. We will looking for only relative amplitudes \(\pi^k = \frac{\psi^k}{\bar{\psi}^k}\) presenting points of \(CP(N-1)\) and tangent fiber bundle over them. Variations of points in \(CP(N-1)\) correspond to local variations of the global potential (initial conditions). The measure of the distance between two coherent states in \(CP(N-1)\) may be express through the quantum metric tensor

\[
G_{ik^*} = \mathcal{K}^{-1}(1 + \sum |\pi^s|^2)\delta_{ik} - \pi^{i^*}\pi^k
\]

(3.1)

Now we should generalize Pancharatnam problem of the comparison of two polarized light beams. Namely, how one can say that two quantum system are ‘in phase’, i.e. coherent? First Einstein in his works in ideal quantum gas predicting Bose-Einstein condensation realized that the conditions of the constructive interference of de Broglie waves in the ideal gas, are identical masses of atoms and their velocities \([20, 21]\). The problem of comparison of identical quantum particles (their dynamical variables in the different background fields) leads to the problem of the parallel transport of tangent vector fields corresponding to different quantum dynamics. The affine connection \(\Gamma^i_{mn}\) defined as follows

\[
\Gamma^i_{mn} = \frac{1}{2}G^{ip^*}(\frac{\partial G_{mp^*}}{\partial \pi^n} + \frac{\partial G_{p^*n}}{\partial \pi^m})
\]
\[
\delta m^i \pi^m^* + \delta n^i \pi^m^* \\
1 + \sum |\pi|^2,
\]
where
\[
G^{ik^*} = K(1 + \sum |\pi|^2)(\delta^{ik} + \pi^i \pi^k^*)
\]
for the parallel transport agreed with Fubini-Study metric should be taken into account. The Riemannian curvature
\[
R_{klm^*}^i = \frac{-\partial \Gamma_{kl}^i}{\partial \pi^m^*}
\]
will be associated with the Yang-Mills fields of the new kind defined on \( CP(N - 1) \).

4 Quantum Pre-Dynamics of the Fundamental Field

4.1 Generalized Coherent States and Gauge Principle

In the absence (at this stage) of space-time coordinates one need some ‘hidden’ quantum pre-dynamics. However, dynamics should be exhibited in a sub-manifold of quantum states especially identifying with ordinary space-time. Due to the nonlinear character of \( CP(N - 1) \) (curvature) we have self-interacting quanta of the vacuum excitations. The gauge principle is the simplest way to establish field equations corresponding to such interaction. Initially it will be equation in the state space and only after special ‘reduction’ one could obtain space-time representation of the field equations.

Pursuing to realize the ‘strong relativity to embedding fields’ principle, one should admit a wide class of the functional frames capable to represent such fields. Furthermore, dynamical variables should be covariant operators relative functional frame variations. That is one have to have a covariant ‘functional calculus’ [22]. We will use such covariant calculus in the projective Hilbert state space \( CP(N - 1) \) relative Fubini-Study metric. The discussion of the convergence problem in the case \( CP(\infty) \) will be postponed for the future work.
Locally (in the state space) the gauge transformation mechanism may be described by the connection form in $CP(N - 1)$

$$\Omega^i_k = \Gamma^i_{km}\delta\pi^m. \quad (4.1)$$

Different choice of the independent field variables $\pi^i$ (holomorphic diffeomorphism of a bounded domain $D \subset C^N$ onto a bounded domain $D' \subset C^N$, say transition for a different vacuum $\psi^k \neq 0$),

$$\pi^i = \tilde{\pi}^i(\pi^1, ..., \pi^{N-1}), \quad (4.2)$$

such that

$$\frac{\partial\pi^i}{\partial\pi^m} = 0 \quad (4.3)$$

corresponding to the variation of representation (variation of the functional reference frame), entails the non-Abelian gauge transformations law of the connection form $\Omega^m_n$ of the well known kind:

$$\Omega^i_k = J^{-1}_{m} \Omega^m_n J^n_k + J^{-1}_{t} \delta J^t_k, \quad (4.4)$$

where

$$J^i_m = \frac{\partial\pi^i}{\partial\pi^m} \quad (4.5)$$

is the matrix with holomorphic Jacobian. Transformations of the connection form should be accompanied by the transformations of tangent vectors fields taking the place of the state vectors in the tangent spaces intrinsically defined by the dynamical variables in our interpretation

$$T^i = J^i_m T^m = \frac{\partial\pi^i}{\partial\pi^m} T^m. \quad (4.6)$$

Hence, in general, the gauge transformations belongs to $GL(N - 1, \mathbb{C})$ but in physical applications this group should be reduced to the $U(N - 1)$. In particular, an important example of such transformations applicable to the $\Phi^i_\sigma$ (see below).

$$\Phi^i_\sigma = J^i_m \Phi^m_\sigma = \frac{\partial\pi^i}{\partial\pi^m} \Phi^m_\sigma. \quad (4.7)$$
I would like to emphasize here that non-Abelian gauge transformation only looks like Wilczek-Zee gauge potential but is not identical to this one. Our non-Abelian potential has geometric origin in cotangent space $T^*_\pi CP(N-1)$ (affine connection form), whereas Wilczek-Zee potential originated by the instant Hamiltonian with degenerated spectrum in Hilbert space $C^N$. Jacobian factors $J^\tilde{n}_k$ are akin here to Shapere-Wilczek gauge transformations in the space of shapes of deformable bodies. However, one has simpler situation here since we deal not with arbitrary deformations of the swimming body but with the so-called ‘generalized solid body’ motion with deformations of the ellipsoid of polarization restricted by the coset structure $CP(N-1) = SU(N)/S[U(1) \times U(N-1)]$ of the dynamical group $SU(N)$. The tangent vectors to $CP(N-1)$ determinate the velocities of entanglement of infinitesimally close amplitudes i.e. variations of vacuum state. For example the infinitesimal shifts generated by the one-parameter groups in $1 \leq \sigma \leq N^2-1$ directions of $AlgSU(N)$ defined by coefficient functions as follows:

$$
\Phi^i_\sigma = \lim_{\epsilon \to 0} \epsilon^{-1} \{ \frac{\exp(i\epsilon \lambda_\sigma)}{i m} \Psi^m - \frac{\Psi^i}{i m} \} = \lim_{\epsilon \to 0} \epsilon^{-1} \{ \pi^i(\epsilon \lambda_\sigma) - \pi^i \}.
$$

(4.8)

Dynamical variables $D_\sigma = \Phi^i_\sigma \frac{\partial}{\partial \pi^i} + c.c.$ play role of quantum states in the local tangent space. We will distinguish two kinds of quantum states: ‘coherent quantum state’ defined by $\pi^i$ local coordinates of FF in $CP(N-1)$ and ‘dynamical quantum state’ in tangent space defined by the linear combinations of $SU(N)$ generators $\Psi(\pi) = a^\sigma D^\sigma$ in local coordinates. Thereby in the framework of the local state-dependent approach one can formulate a quantum scheme with help more flexible mathematical structure than matrix formalism. It means that matrix elements of transitions between two arbitrary far states are associated with, in fact, bi-local dynamical variables that bring a lot of technical problems in quantum field area. (See notes in this spirit in \textnormal{[9]}). However the infinitesimal local dynamical variables related to deformations of quantum states are well defined in projective Hilbert space as well as quantum states itself. They are local tangent vector fields to the projective Hilbert space $CP(N-1)$ which correspond to the group variation of the relative ‘Fourier components’, i.e. generators (differential operators of first order) \textnormal{[3, 4, 5, 6]}. In the local coordinates $\pi^i = \frac{\phi^i}{\Psi^i}$ one can build the infinitesimal generators of the Lie algebra $AlgSU(N)$. Then one has to use explicit form $\Phi^i_\sigma$ for $N^2 - 1$ of infinitesimal generators of the Lie algebra $AlgSU(N)$. For example for the three-level system, algebra $SU(3)$ has 8 infinitesimal generators which are given by the vector fields:

$$
D_1(\lambda) = i \frac{\hbar}{2} \left[ [1 - (\pi^1)^2] \frac{\delta}{\delta \pi^1} - \pi^1 \pi^2 \frac{\delta}{\delta \pi^2} - [1 - (\pi^{1*})^2] \frac{\delta}{\delta \pi^{1*}} + \pi^{1*} \pi^{2*} \frac{\delta}{\delta \pi^{2*}} \right],
$$

10
\[ D_2(\lambda) = -\frac{\hbar}{2} \left[ [1 + (\pi^1)^2] \frac{\delta}{\delta \pi^1} + \pi^1 \pi^2 \frac{\delta}{\delta \pi^2} + [1 + (\pi^{1*})^2] \frac{\delta}{\delta \pi^{1*}} + \pi^{1*} \pi^{2*} \frac{\delta}{\delta \pi^{2*}} \right], \]
\[ D_3(\lambda) = -i h [\frac{\pi^1}{\delta \pi^1} + \frac{\pi^2}{\delta \pi^2} + \pi^{1*} \frac{\delta}{\delta \pi^{1*}} + \frac{\pi^{2*}}{\delta \pi^{2*}}], \]
\[ D_4(\lambda) = i \frac{\hbar}{2} \left[ [1 - (\pi^2)^2] \frac{\delta}{\delta \pi^2} - \pi^1 \pi^2 \frac{\delta}{\delta \pi^1} - [1 - (\pi^{2*})^2] \frac{\delta}{\delta \pi^{1*}} + \pi^{1*} \pi^{2*} \frac{\delta}{\delta \pi^{2*}} \right], \]
\[ D_5(\lambda) = -\frac{\hbar}{2} \left[ [1 + (\pi^2)^2] \frac{\delta}{\delta \pi^2} + \pi^1 \pi^2 \frac{\delta}{\delta \pi^1} + [1 + (\pi^{2*})^2] \frac{\delta}{\delta \pi^{1*}} + \pi^{1*} \pi^{2*} \frac{\delta}{\delta \pi^{2*}} \right], \]
\[ D_6(\lambda) = i \frac{\hbar}{2} \left[ \pi^2 \frac{\delta}{\delta \pi^1} + \pi^1 \frac{\delta}{\delta \pi^2} - \pi^{2*} \frac{\delta}{\delta \pi^{1*}} - \pi^{1*} \frac{\delta}{\delta \pi^{2*}} \right], \]
\[ D_7(\lambda) = -\frac{3i}{2} \frac{\hbar}{\delta \pi^2} \left[ \pi^2 \frac{\delta}{\delta \pi^1} - \pi^1 \frac{\delta}{\delta \pi^2} + \pi^{2*} \frac{\delta}{\delta \pi^{1*}} - \pi^{1*} \frac{\delta}{\delta \pi^{2*}} \right], \]
\[ D_8(\lambda) = -\frac{1}{2i} \frac{\hbar}{\delta \pi^2} \left[ \pi^2 \frac{\delta}{\delta \pi^1} - \pi^1 \frac{\delta}{\delta \pi^2} - \pi^{2*} \frac{\delta}{\delta \pi^{1*}} - \pi^{1*} \frac{\delta}{\delta \pi^{2*}} \right], \quad (4.9) \]

In general, such dynamical variables \( \Phi^i_\alpha, \Phi^i_\beta \) define the curvature in 2-dimension direction \((\alpha, \beta)\)

\[ R(D_\alpha, D_\beta) X^k = [\nabla_{D_\alpha}, \nabla_{D_\beta}] X^k - \nabla_{[D_\alpha, D_\beta]} X^k \]
\[ = \{(D_\alpha \Phi^i_\alpha - D_\beta \Phi^i_\beta) \Gamma^k_m - (\Phi^i_\beta \Phi^i_\alpha^{*}) \frac{\partial}{\partial \pi^m} \}\frac{\partial \Gamma^k_n}{\partial \pi^m} \]
\[ + \Phi^m_\alpha \Gamma^k_m \Phi^i_\beta \Gamma^p_i - \Phi^m_\beta \Gamma^k_m \Phi^i_\alpha \Gamma^p_i - C^\gamma_{\alpha \beta} \Phi^i_\gamma \Gamma^k_{in} \} X^n \]
\[ = \{(D_\alpha \Phi^i_\alpha - D_\beta \Phi^i_\beta) \Gamma^k_m - (\Phi^i_\beta \Phi^i_\alpha^{*}) \} R^k_{i*mn} \]
\[ + \Phi^m_\alpha \Gamma^k_m \Phi^i_\beta \Gamma^p_i - \Phi^m_\beta \Gamma^k_m \Phi^i_\alpha \Gamma^p_i - C^\gamma_{\alpha \beta} \Phi^i_\gamma \Gamma^k_{in} \} X^n. \quad (4.10) \]

Here \( C^\gamma_{\alpha \beta} \) are of the \( SU(N) \) group structure constants \[ \Box. \] Now we can introduce the follows expression

\[ F_{\alpha \beta} = R(D_\alpha, D_\beta) X^k \frac{\partial}{\partial \pi^k} + c.c., \quad (4.11) \]

which is the analog of Yang-Mills fields (with the unknown \( X^k \)) of the gauge potential associated with the intrinsic affine \( CP(N - 1) \) connection

\[ \Omega^i_k = \Gamma^i_{km} \delta \pi^m = \frac{1}{\hbar} \Gamma^i_{km} \Phi^m_\alpha(\pi) \delta a^\alpha, \quad (4.12) \]

where \( \delta a^\alpha \) is a variation of the action. In comparison with Berry singular potential \( A_\gamma(R) = \langle n(R) | \nabla_R n(R) \rangle \) this potential is not singular and the source of non-Abelian field is the curvature of \( CP(N - 1) \). This non-Abelian field is expressed in
local ‘slow’ coordinates $\pi^i$ as well as Berry “monopole” field in terms of $R$- coordinates. I treat these functional fields as a pre-dynamical, initial condition variations, or tunneling processes, in spite of the ordinary gauge fields constructions where fields are functions of the space-time coordinates.

So, the super-relativity principle formulated as the covariance of the field theory relative holomorphic local coordinates $\pi^i$ variations is realized by the state-dependent gauge principle concerning initially interacting quantum objects [8]. But in our case we have much more general kind of the non-Abelian gauge transformations [4,2]. Here one has the literally covariant derivatives associated with Fubini-Study metric [3,1] in spite of the commonly used generalized gauge theory notions. The hermitian hamiltonian defined by dynamical variables (tangent vectors) $T^p(\pi)$ may then be expressed as follows:

$$H^{ps^*} = G^{ik^*}(\frac{\partial T^p}{\partial \pi^i} + \Gamma^p_{in} T^n)(\frac{\partial T^{ss^*}}{\partial \pi^{k^*}} + \Gamma_{k^*q^*}^{s^*} T^{q^*}) = G^{ik^*} T^p T^{s^*}. \quad (4.13)$$

It is similar to the scalar composition

$$\mathcal{H} = G^{ik^*} \frac{\partial U}{\partial \pi^i} \frac{\partial U}{\partial \pi^{k^*}} \quad (4.14)$$

generalizing Weinberg’s multiplication $a \ast b$ in his nonliner modification of the quantum mechanics [23]. In fact, however, one has here the formulation of the covariant variational problem for the functional vector field instead of ‘scalar’-functional of the traditional variational calculus. So, hamiltonian created by the tangent vector field in the $\sigma$ - direction of iso-space of adjoint representation looks like

$$H^{ps^*_\sigma} = G^{ik^*} \Phi^p_{\sigma i} \Phi^{s^*_k}. \quad (4.15)$$

Then in the “b-direction” [3,5], i.e. along the direction of the geodesic flow generators, one has

$$H^{ps^*_b} = G^{ik^*} \Phi^p_{b i} \Phi^{s^*_k} = 0, \quad (4.16)$$

since these tangent vectors fields are parallel transported along geodesics. Since all geodesics in $CP(N - 1)$ issued from the original point $\pi^k_0$ are mutually transforming by the gauge group $H = U(1) \times U(N - 1)$, it is enough to proof it for $CP(1)$. In the local coordinates $\pi^k$ equation for geodesics is as follows:

$$\frac{d^2 \pi}{ds^2} - \frac{2 \pi^*}{1 + \pi^2} (\frac{d \pi}{ds})^2 = 0, \quad (4.17)$$
where $s$ is the length of a curve in $CP(1)$. It has solution $\pi(s) = \exp i\alpha \tan s$ with arbitrary constant $\alpha$.

Let’s put $\alpha = 0$, then $\pi(s) = \tan s$ and $\frac{d\pi}{ds} = \frac{1}{\cos^2 s}$. On the other hand corresponding vector field representing the generator ‘OY-rotation’ is $1 + \pi^2 = 1 + \tan^2 s = \frac{1}{\cos^2 s}$. It is clear we have the coincidence (parallel transport) of the generator of rotation (see two-level analog of 4.9 in [3]) and the tangent vector field anywhere along corresponding geodesic of $CP(1)$. If one put $\alpha = i\pi/2$ it is easy to see that the similar calculations give us $\frac{d\pi}{ds} = \frac{i}{\cos^2 s}$ that corresponds to the parallel transporting generator of the ‘OX-rotation’: $i(1 - \pi^2) = i(1 - \exp i\pi \tan^2 s) = \frac{i}{\cos^2 s}$. So, one has $2s = N - 1$ complex vector fields generated zero-hamiltonian in the strong, i.e. in the non-average value sense. Then one has in the $4s^2 = (N - 1)^2$ real “h-directions” generated by the vector fields $D_h = \Phi^i_h \frac{\partial}{\partial \pi^i} + c.c.$ It gives the finite value hermitian matrix

$$H^{ps*}_{h} = G^{ik*}_{h;\ell} \Phi^{p}_{\ell;h} \Phi^{s*}_{h;k*} \neq 0,$$

representing the class of the covariant local Hamiltonians in the tangent space. I would like to note for N-level system not only dipole fields (like $H_x, H_y, H_z$) are actual but multipole components (like $Q_{xx}, Q_{xy}$), etc., too. The origin of these multipole field components for spin is anisotropy of magnetic materials. The eight gluons fields take the place of these multipole ‘angle-field’ components for the color quarks of QCD, etc. Hence, pulse acquired by the quantum N-level system during an interaction process, has, in general, complicated dynamical structure [24]. One can think even that quantum particles themselves represent the multipole action pulse of the FF vacuum excitations. Our target is to find some dynamical covariant and invariant variables associated with the geometry of the N-level coherent state space.

### 4.2 Quantum Mechanics As The Tangent Approximation

Let us discuss now briefly construction of the liner quantum systems of the ordinary quantum mechanics in the tangent fiber bundle over $CP(N - 1)$. As before, we will discuss now only finite dimension version of the theory, infinite dimension case will be postponed to the special publication.
I will use for the simplicity the vacuum vector in follows form:

\[
|V > = \begin{pmatrix}
R_{\text{vac}} \exp (i\gamma) \\
0 \\
0 \\
\vdots \\
0
\end{pmatrix}.
\]  (4.19)

Corresponding local coordinates in \(CP(N - 1)\) is simply \((\pi^1, ..., \pi^i, ..., \pi^{N-1}) = (0, ..., 0)\). Any tangent vector corresponding some dynamical variable (a self vector of the hermitian Hamiltonian, for example), we take in the form

\[
|\xi > = \begin{pmatrix}
0 \\
\xi^1 \\
\xi^2 \\
\vdots \\
\xi^{N-1}
\end{pmatrix}.
\]  (4.20)

It is clear that they are orthogonal relative the standard scalar product in ‘surrounding’ Hilbert space \(C^N\): \(< V|\xi > = 0\). In the standard perturbation theory \(|V >\) corresponds to the unperturbed solution, and \(|\xi >\) corresponds to one of the approximation solutions orthogonal to the vacuum. If one looks on \(|\xi >\) as a finite variation of the vacuum (‘tangent extrapolation’ of the infinitesimal generator), than the sum is the state vector

\[
|\phi > = |V > + |\xi > = \begin{pmatrix}
R_{\text{vac}} \exp (i\gamma) \\
\xi^1 \\
\xi^2 \\
\vdots \\
\xi^{N-1}
\end{pmatrix}.
\]  (4.21)

For us will be interesting not this ‘imaginary’ state vector, but its projection on \(CP(N - 1)\). In order to do it one has to normalize \(|\phi >\) to the ‘vacuum radius’ \(R_{\text{vac}}\).
It is easy to see that coordinates is as follows:

\[
|\phi_N> = \mathcal{N} \left( \begin{array}{c} R_{vac} \exp(i\gamma) \\ \xi^1 \\ \xi^2 \\ \vdots \\ \vdots \\ \xi^{N-1} \end{array} \right),
\]

(4.22)

where \( \mathcal{N} = \frac{R_{vac}}{\sqrt{R_{vac}^2 + \sum |\xi|^2}} \). The local coordinates of \( |\phi_N> \) in the chart originated at the new “physical vacuum” one will find \( \pi^i = \frac{\xi^i \exp(-i\gamma)}{R_{vac}} \). In fact the ‘tangent extrapolation’ is illegal since an infinitesimal operator (‘Hamiltonian of perturbation’) does not define the finite unitary (norm-preserving) variation of the vacuum vector. Only infinite set of the infinitesimal, continuous, successive unitary transformations lead to the state with the local coordinates \( \pi^i \). Furthermore, there are continuum pathes in the Hilbert projective space connecting the vacuum and the perturbed state. In particular, one can chose the geodesic in \( CP(N-1) \) connecting the point \( (\pi^1, ..., \pi^i, ..., \pi^{N-1}) = (0, ..., 0) \) with the point \( \pi'^i = \frac{\xi^i \exp(-i\gamma)}{R_{vac}} \). Probably the geodesic motion in projective Hilbert space \( CP(N-1) \) physically corresponds to the tunneling process going in a ‘tunneling time’. It may demolish the necessity in wormholes in space-time since quantum tunneling goes in the state space acquiring thereby physical reality.

Therefore, a realistic physical problem with interacting terms should be formulated in the spirit of the sketch given above. In particular, dynamical quantum state creeps along geodesic of \( CP(N-1) \) from one vacuum to another and will belongs to the different tangent Hilbert spaces at each instant of the ‘tunneling time’ [25].

It is absolutely clear that any quantum setup for generating or registration of the quantum objects is built from physical fields and, hence, its coordinates correspond to some point of \( CP(N-1) \). What we see in the space-time (say, cloud chamber) is merely very poor picture. Deep processes going in the state space. It looks like any quantum object is placed in the self-consistent global (cosmic) potential \( <V|U_{global}|V> = c^2 \) and all attempts to disturb this equilibrium state leads to the ‘reaction’ which manifestation is the inertia of the quantum object. Reaction of the quantum setup on the some particle emission is an example of such kind. New coordinates will correspond after the emission process to the particle and to the setup too. One may assume that there is ‘inertia’ of the quantum state and its elasticity...
under deformation. This elasticity presumably determines the inertia (mass) of the quantum system [3]. The structure of this state has some quantum integrals of motion (‘charges’) and, hence, should be invariant under embedding into an external field.

### 4.3 Topological and Tunneling Integrals

There are two kinds of the physically interesting integrals arising over base manifold of the tangent fiber bundle \( CP(N - 1) \). The first kind is the characteristic classes by Chern of the coholonomy groups \( c_k \in H^{2k}(CP(N - 1)) \). Corresponding topological invariants are integrals

\[
I_k = \int_{M_{2k} \subset CP(N-1)} \tilde{c}_k
\]

from the forms

\[
\tilde{c}_k = \frac{1}{2\pi i} Tr(R(D_{\alpha_1}, D_{\beta_1}) \wedge R(D_{\alpha_2}, D_{\beta_2}) \wedge ... \wedge R(D_{\alpha_k}, D_{\beta_k}))
\]

representing a new kind of geometric phases. The formal ‘Yang-Mills equations’ obtained as a requirement of the elimination of the covariant derivative from the analog of Yang-Mills field

\[
\nabla_{\beta} R(D_{\alpha}, D_{\beta}) X^k = 0
\]

should be analyzed carefully in the future investigations from the physical point of view.

The second kind is the ‘tunneling integrals’, i.e. invariants connected with Fubini-Study metric [3,4]. Any scalar product \( (\theta(\pi)|\xi(\pi)) = G_{ik}(\pi)\theta^i(\pi)\xi^k(\pi) \), where \( |\xi(\pi), |\theta(\pi) \) belong to the tangent space at the origin point, is invariant relative the parallel transport in \( CP(N - 1) \). Besides such kind of the local invariants should exist global invariants due to the global subgroup \( H = U(1) \times U(N - 1) \) of the gauge symmetry on the \( CP(N - 1) \). This group transforms one geodesic to another, therefore states belonging to these geodesics should have some invariant characteristics. One can associate that state characteristics with the shape of the “ellipsoid of polarization”. The full set of the ellipsoid parameters comprises of state vector \( 4s \) real variables itself and the \( 4s^2 \) real parameters of “orientation” of the ellipsoid relative the quantum frame. Here we derive corresponding formulas for such ellipsoid. First of all let us remind formulas for the simplest case of the two level system \( N = 2; N = 2s + 1 \). Let us discuss some two level quantum system in the basis
|1>, |2> with the coefficients \( \psi^1 = |\psi^1| e^{i\alpha_1} = |\psi^1| e^{i\alpha'_1 + \tau} \), \( \psi^2 = |\psi^2| e^{i\alpha_2} = |\psi^1| e^{i\alpha'_2 + \tau} \).

If one defines \( \rho^k = Re \psi^k \) then it is easy to verify that follows equation of ellipse takes the place:

\[
\left( \frac{\rho^1}{|\psi^1|} \right)^2 + \left( \frac{\rho^2}{|\psi^2|} \right)^2 - 2 \frac{\rho^1}{|\psi^1|} \frac{\rho^2}{|\psi^2|} \cos \delta = \sin^2 \delta, \tag{4.26}
\]

where \( \delta = \alpha'_2 - \alpha'_1 \). This ellipse (its shape) is invariant relative isotropy group of the spin \( s = 1/2 \) ground state (vacuum)

\[
|1> = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{4.27}
\]

Our target now is to get general invariant for N-level system. This problem was risen by V.Ostrovskii in the private talk in the connection with his interesting article [24]. I think that it is natural to build the follows quadric formula:

\[
\sum_{i=1}^{N} \left( \frac{\rho^i}{|\psi^i|} \right)^2 - \frac{2}{N-1} \sum_{i<j}^{(N^2-N)/2} \frac{\rho^i}{|\psi^i|} \frac{\rho^j}{|\psi^j|} \cos (\delta_{ij}) = \frac{1}{N-1} \sum_{i<j}^{(N^2-N)/2} \sin^2 (\delta_{ij}) \tag{4.28}
\]

where \( \delta_{ij} = \alpha_i - \alpha_j \). I have found that it is ellipsoid, but I could not to prove that its shape is really invariant relative the isotropy group \( H = U(1) \times U(N-1) \) of the ground state

\[
|1'> = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix}. \tag{4.29}
\]

It is important to verify the invariance fact of the ellipsoid shape. I will sincerely grateful for somebody who will solve this problem. If my assumption is correct, it lead to the mechanism of the reconstruction of the unitary symmetry \( SU(N) \). Such reconstruction of the unitary symmetry represents the concrete mechanism of symmetry breakdown by the coset transformations \( G/H = SU(N+1)/S[U(1) \times U(N)] \) up to isotropy group \( H = U(1) \times U(N) \). Therefore the shape of the ellipsoid of polarization will a new integral invariant of unitary quantum dynamics. Then will be possible to try to connect this mechanism of the unitary symmetry breakdown with the problem of mass split effects in the unitary fundamental multiplet of “elementary particles”.

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4.4 Quantization in CP(N−1)

Let us use the local projective representation of $SU(N)$ generators acting upon the smooth (holomorphic) functions of the coherent state instead of matrices acting upon N-component original state vector $|\psi>$ of FF. These $D_{\sigma}$ generators are the tangent vector fields to the $CP(N−1)$ [3, 6, 25]. Hereby the boson ‘second quantization’ over $CP(N−1)$ takes the place quite naturally. But one should take into account the ambiguity of quantization axes notion for $SU(N)$. Since the rank of $AlgSU(N)$ is $r = N − 1 = 2S$ and number of the local independent charts for the covering $SU(N)$ is $l = r + 1 = N = 2S + 1$. Single axes of quantization there is only for a two level system ($S = 1/2$). The choice of the chart marks the ‘open channel’ ($\psi^j \neq 0$) in a ‘filter’ experiment, and it determines a local ‘vacuum’ of our model. On the pre-dynamics level one has not a possibility to measure space-time or energy-momentum values. It means that there exist only the operator of action

$$A = \hbar \pi^{s_1} \frac{\partial}{\partial \pi^{s_1}},$$

which creates the quanta of action. Then the character of the distribution of these action quanta among degrees of freedom will be realized as specific quantum particles.

Commutation relations for “geometrical bosons” is as follows:

$$\left[ \frac{\partial}{\partial \pi^k}, \pi^i \right]_− = \delta^i_k.$$  

The standard Fock representation is achievable under the definition of vacuum state by a holomorphic function $F_{vac}$. Over compact connected manifold like $CP(N−1)$ such function is a constant. Then, since $\frac{\partial F_{vac}}{\partial \pi^k} = 0$ one can introduce the function of excitations of different degrees of freedom $F(s_1, ..., s_N) = \pi^{*s_1} \pi^{*s_2} ... \pi^{*s_N} F_{vac}$ and the function of a multifold excited degree of freedom $F(s; N) = (\pi^{*s})^N F_{vac}$. For this function one has the equidistant spectrum of action:

$$AF(s; 0) = 0;$$
$$AF(s; 1) = \hbar F(s; 1);$$
$$AF(s; 2) = 2\hbar F(s; 2);$$

$$...$$

$$AF(s; N) = N\hbar F(s; N).$$  

$$\text{(4.32)}$$

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