Gauge Invariant Propagators and States in Quantum Electrodynamics

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Abstract

We study combined matter/gauge field gauge invariant states in terms of data living on the boundary of gauge invariant path-integrals. To get concrete results, this is done for scalar and spinor QED, for both ‘time-slice’, and ‘causal diamond’ boundaries. We discuss both the standard case where the gauge field falls off to zero at the spatial boundaries, and the case of ‘large gauge transformations’, where it remains finite at these boundaries. The path-integral naturally generates a specific dressing factor for states living on time-slices, without fixing any gauge, and we identify a universal contribution which depends only on the nature of the boundaries. We also derive the analogous dressing for states defined on null infinity, showing both its Coulombic parts as well as soft-photon parts.

Keywords: QED, Gauge invariance, Path-integral, Constraints, Soft Photons

1. Introduction

Traditionally, both non-relativistic quantum mechanics (QM) and relativistic Quantum field theory (QFT) have been formulated in terms of states in Hilbert space, upon which operators representing measurements of physical quantities are supposed to act at specific times or on some specific hypersurface. However the last few decades have seen a growing feeling that one needs to go beyond such a framework.

One key motivation for this has come from quantum gravity, where the difficulties in defining diffeomorphism-invariant physical quantities [1–4] have led to approaches in which states are defined in terms of information residing on boundaries [5–8]. Path integrals can be used to define ground-state wave-functions for different kinds of boundary [9] or even for spacetimes with no boundaries [10] [11]. Much of modern quantum cosmology is also formulated using path integrals [12].

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There have also been extensive efforts to look for generalizations of QM in which, eg.,
the superposition principle breaks down [13, 14]; in recent years these have focused on the
possible role of gravity in engineering this breakdown [15–20]. In all of these developments,
alternative definitions of quantum states have been sought.

That path integrals provide a more general formulation of QM states has been known
for a long time [21], and their usefulness has been seen in, eg., the Aharonov-Bohm effect
[22], and in interaction-free measurements [23]. In such examples, the time evolution of a
quantum system \( S \) depends both on what can happen along the paths \( S \) does follow, and
also those paths it does not follow. In a path integral formulation this seems quite natural -
but not if we deal entirely with the wave-function \( \langle \mathbf{r} | \Psi_S(t) \rangle \), which is zero in regions where
no paths are followed.

In more recent years path integrals have also been used to deal with topological field
theories, and in describing states with fractional statistics for many-particle systems [24–
26]; in these systems, boundaries play a key role in defining both the ground and excited
states.

What almost all of these discussions have in common is (i) their emphasis on the role
of boundaries in defining quantum states; (ii) the use of the path integral in giving this
definition; and (iii) the presence of gauge fields. This last feature adds a further complication,
since one would like to be able to define gauge invariant quantities.

A key motivating factor here has been that for general spacetimes, path integrals are
actually unavoidable. For any non-trivial spacetime in which there exist achronal regions,
one must employ path integrals to handle the dynamics of even simple particles [5, 6, 27, 28].
In these situations, a conventional Hamiltonian framework is no longer applicable, and simple
Hamiltonian evolution is undefined, whereas path integrals can still compute transition
amplitudes/probabilities.

If one is prepared to accept the existence of non-trivial topologies in quantum gravity,
then any amplitude must involve sums over them. Dealing with such sums has been a major
theme in research since the 1980’s, invoked to solve key problems such as the cosmological
constant problem [29, 30] and the black hole information loss problem [see eg. 31–33]. In
these studies, the gravitational path-integral inevitably describes a structure more general
than just “wave function & Hamiltonian” quantum mechanics [34, 35].

For all these reasons, it is clear that a ‘path-integral first’ formulation of QM and QFT,
in which boundaries play a key role, is desirable.

Much of the work in this area has been quite abstract, and has tried to deal with general
kinds of boundary and boundary information. In the work of Hartle and Hawking [10],
states are defined using path integrals in some general spacetime regions \( \mathcal{V} \), bounded by
some hypersurface \( \partial \mathcal{V} \). The states are then non-local wave functionals, over configurations
specified on all of \( \partial \mathcal{V} \). Similar ideas have motivated the “general boundary quantum field
theory framework” [7, 8]; a field configuration on \( \partial \mathcal{V} \) is again mapped to an amplitude via
a path integral over field configurations in \( \mathcal{V} \).

To test abstract work of this kind, however, one needs to work out the details in specific
cases. In studies of quantum gravity this has led to considerable discussion - an example
is provided by the ongoing debate over the validity of both no-boundary and tunneling
descriptions of the early universe [36, 38].

The complexity of the issues involved makes it clear that one should try first to understand how all of this works in simpler models. This is the question addressed in the present paper, which deals with ordinary Quantum Electrodynamics (QED). We look at states defined by propagators between hypersurfaces in 4-dimensional spacetime, and show how one can naturally define both boundary contributions and bulk contributions to these propagators.

In this way we are able to not only recover known results - which now appear in a new light - but also derive several quite new results, even in flat spacetime, in which the role of boundaries in defining the states becomes very clear. QED then acts as a blueprint for the larger project of defining quantum states in some sort of boundary quantum field framework. We arrive at the conclusion that such a non-local formulation of quantum states is inevitable, even for QED, and is more natural than the traditional Hamiltonian formulation in terms of local states defined in Hilbert space.

In this study one must immediately confront both conceptual and technical issues. These involve both the gauge invariance and the asymptotic properties of the states, and arise even in flat spacetime. For this reason, in the rest of this introduction we first introduce some of the conceptual and technical questions involved, before briefly describing the organization of the paper.

1.1. Some Key Questions

In gauge theories, a crucial role is played by constraints, and by the requirement of gauge invariance. This was recognized early on by Dirac, as part of his efforts to quantize constrained theories [39, 40]; he used operator representations of the constraints to annihilate physical states. Dirac was thereby led [41] to introduce gauge-invariant “physical states” in Quantum Electrodynamics (QED); and the constraints were then the generators of the QED gauge transformations.

One can also define gauge-invariant states prepared by a path integral, which are of course non-local objects. These states will satisfy the operator constraints provided that the action, the measure, and the set of summed paths are themselves invariant under the transformations generated by the constraints [41]. A good example, already noted above, is provided in quantum gravity by the Hartle-Hawking “no-boundary” wave-function of the universe [40], where one has a Euclidean path-integral over four-dimensional metrics. This state then satisfies the Hamiltonian constraint of Einstein gravity, in the form of the Wheeler-DeWitt equation [42].

However, in both QED and quantum gravity, one must address the following issues:

(i) Eventually one needs to explicitly address the role of the - often complicated - dynamics of real charges (in QED) or masses (in quantum gravity). It is often not clear how to separate out the ‘physical’ degrees of freedom from unphysical ones - this is particularly true when one deals with rapidly accelerating objects, where a discussion on terms of ‘near field’ and ‘far field’ zones does not help in making such a separation. One can discuss things entirely in terms of asymptotic states [43–45], but this is not of much help in dealing with phenomena in bounded regions of spacetime.
(ii) If one is dealing with state superpositions involving a large spatial separation of charge or mass, real confusion arises in discussion of what are the correct physical variables, or how to test, eg., whether or not the gravitational metric field $g^{\mu\nu}(x)$ is quantized \[46–49\]. The related question of how to properly define notions like decoherence is also unclear, with different results being derived for decoherence rates by different authors \[45, 50–53\].

(iii) While integrating separately over gauge field and matter variables in a path integral, one needs to deal properly with both the constraints and the gauge redundancy. To deal with the latter one typically uses the Faddeev-Popov technique \[54\]. This still leaves the problem of implementing the constraints in a manifestly gauge invariant way.

(iv) As soon as one goes beyond Abelian gauge theory, one runs into problems defining states, even using path integrals; in non-Abelian theories, these are connected with the Gribov ambiguity \[55\], and in quantum gravity, with functional integration over different metrics. In non-trivial spacetimes, including those containing achronal regions \[5, 6, 27, 28\], the question of how to define quantum states for physical systems is completely unresolved.

These issues are all formal in nature, and we will see that by formulating everything in terms of path integrals, once can address them (although in this paper we do not try to deal with non-Abelian or gravitational theories). There are also physical (as opposed to formal) questions we would like to see answered. These include:

(a) What sort of electromagnetic dressing is “chosen” by states defined using path integrals? And how are these states related to the states defined by Dirac \[41\], Mandelstam \[56\], and others?

(b) How do the physical states so defined depend on the spacetime boundaries and the information specified on them? What about cases involving ‘large’ gauge transformations, which also act on these boundaries?

(c) What are the physical degrees of freedom involved in gauge invariant spatial superpositions and in entangled states?

(d) What are the implications of results found for QED in the larger enterprise of defining states in quantum gravity?

At the end of this paper we will return to these questions. Note that there is one other very interesting question we will only comment on briefly in this paper. This concerns decoherence and information loss; one would like to know how to correctly calculate decoherence rates, and which states we should average over to do so. This will be dealt with elsewhere (see also refs. \[15\]).

1.2. Organization of Paper

The paper is organized as follows. In section 2 we consider QED in flat spacetime, with boundaries defined by 2 time-slices. We first recall the standard definition of quantum states for scalar QED, and then show how to derive the form of the gauge-invariant propagator between time slices. We use this simple example to highlight the gauge independence of the results, demonstrate how the boundary phases emerge without fixing a gauge beforehand, and then sketch an eikonal argument for the dressings coming from the remaining path-integral. We then introducing the boundary Faddeev-Popov trick, and show how it gives
the same results. In the remainder of this section we then show to do the same for spinor QED.

After this warm-up exercise, in section 3 we look at what happens for a more general boundary hypersurface. We derive the propagator between states on the future and past regions of a large causal diamond, again for flat space. This calculation allows one to see how to generalize to much more general boundaries - a natural separation occurs between boundary degrees of freedom and bulk degrees of freedom, and it becomes clear how to define the physical variables for the system.

Up to this point, all the discussion has been for gauge transformations which vanish at infinity. In section 4 we lift this restriction, and extend all of the previous results to the case of “large gauge transformations”. This leads to an interesting connection with the soft-photon, large gauge transformation, and dressed state literature, and allows us to clarify questions about physical decoherence in QED.

Finally, in section 5, we return to the questions and general issues posed just above, and show how they can be answered using the framework and results given here.

Throughout the paper we will use units in which \( \hbar = c = \epsilon_0 = 1 \), and a \(- + ++\) metric signature.

2. Gauge Invariant Propagators and States in Quantum Electrodynamics

In this section we treat the standard case of Quantum Electrodynamics (QED). We will not be deriving any startling new results in this section. Instead we will be showing how to derive answers in a new way, and how to re-interpret them.

To avoid needless clutter, we summarize some of the derivations in this section - more detail is given in Appendix A. We begin by recalling how states have traditionally been defined in QED, and some of the problems associated with this. We then move to a path integral description of generalized states defined via propagators. This is first discussed for scalar electrodynamics, and then for spinor QED. We show how one can derive expressions for the gauge-invariant propagator of the combined matter + EM field system between 2 time slices in flat spacetime. We show how the results can derived using a “boundary value Faddeev-Popov” method, as well as by more conventional means.

2.1. States in Quantum Electrodynamics

The question of how to define gauge invariant states in QED has a long history. Gauss’ law, that \( \nabla \cdot \mathbf{E} = \rho \), obviously does not completely specify the electric field \( \mathbf{E}(x) \). One can add to the Coulomb solution any divergence-free field. In quantum theory the Gauss law operator constraint for physical states, viz.,

\[
(\nabla \cdot \hat{\mathbf{E}} - \hat{\rho}) |\Psi\rangle = 0,
\]

also has no unique solution. In the Schrödinger picture, in the \( \hat{A}_j(x) \) field value basis, the electric field operator is conjugate to \( \hat{A}_j(x) \), ie.,

\[
\hat{E}^j(x) = i \frac{\delta}{\delta \hat{A}_j(x)}.
\]
Many years ago Dirac argued that one should write the wave-function for a static physical electron in the composite form

\[ \Psi(\mathbf{r}|A^\mu) = e^{-i\varphi(\mathbf{r}|A^\mu)}\psi(\mathbf{r}) \equiv \hat{U}_C\psi(\mathbf{r}) \]

where \( \psi(\mathbf{r}) \) is the wave-function for the ‘bare’ charge.

As Dirac recognized, the phase factor \( \varphi(\mathbf{r}|A^\mu) \) represents a “dressing” function, describing the change in the field induced by the charge. Dirac chose an intuitively obvious solution for \( \varphi(\mathbf{r}|A^\mu) \), writing

\[ \varphi(\mathbf{r}|A^\mu) = e^{-e2\partial_j A^j(\mathbf{r})} = -\frac{e}{4\pi} \int d^3r' \frac{\nabla \cdot \mathbf{A}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \]

so that the dressing was that of a Coulomb field, and automatically satisfied (1).

One can always add some divergence-free function to (4), to get a quite different form; a good example is provided by the ‘Mandelstam string’ solution \[56\], viz.,

\[ \varphi(\mathbf{r}|A^\mu) = e^{\Gamma dz \cdot A(z)} \]

where \( \Gamma \) is some spacelike path terminating at the point \( \mathbf{r} \); solutions of this form have attracted interest in various contexts \[57, 58\].

So far this seems straightforward. Suppose however we consider a simple superposition of the position eigenstates \( |\mathbf{x}_1\rangle \) and \( |\mathbf{x}_2\rangle \) for the charge. The bare particle wave-function is simply \( \psi_{12}(\mathbf{r}) = \frac{1}{\sqrt{2}}(\delta(\mathbf{r} - \mathbf{x}_1) + \delta(\mathbf{r} - \mathbf{x}_2)) \). Dirac’s physical-state wave-function is then of the form

\[ \hat{U}_C\psi_{12}(\mathbf{r}) = \frac{1}{\sqrt{2}} \left( e^{-i\varphi(\mathbf{x}_1|A^\mu)}\delta(\mathbf{r} - \mathbf{x}_1) + e^{-i\varphi(\mathbf{x}_2|A^\mu)}\delta(\mathbf{r} - \mathbf{x}_2) \right) . \]

This state is not an eigenstate of the longitudinal electric field operator, and no longer has a well defined electric field, but instead a superposition of Coulomb fields centred on \( \mathbf{x}_1 \) and \( \mathbf{x}_2 \).

These ambiguities are compounded in the general case of a moving charge. Classically, the Liénard-Wiechert solution, which makes explicit reference to the trajectory of the moving charge, can still be used. However, quantum mechanically one would describe a moving charge using a wave-packet peaked on a particular momentum and the Dirac prescription would then generate a continuous superposition of the various Coulomb fields. We clearly cannot isolate out a specific physical Coulomb field – we would have no idea which one to use! The superposition of Coulomb fields in no way represents the expected Liénard-Wiechert field. Moreover, one could add different divergence-free functions into the dressing operator acting on each branch of the superposition—the constraint equation alone cannot tell us which is the correct/physical dressing for charges, because its solutions are not unique.

Once we sum over multiple paths for a charge, in QED, we must clearly sum over multiple field configurations, some of which will involve radiation, others not - the problem seems
hopeless. Nevertheless we will now see that, in a path integral formulation, one can give a unique separation between constrained variables and unconstrained radiative variables, which is manifestly gauge-invariant. Along with other results, this separation allows one to see quite clearly how the Liénard-Wiechert field arises dynamically in QED.

2.2. Scalar Quantum Electrodynamics

In scalar QED one considers a single charged particle, without spin, coupled to the quantum electromagnetic field. We wish to consider the propagator for this system between two time slices (surfaces of constant \( t \)) in Minkowski space.

We will assume here (but not later in the paper) that the gauge field \( A_\mu \), and any possible gauge transformations of it, will vanish sufficiently quickly at spatial infinity that surface terms generated by spatial integrations by parts can be ignored. In the section 4 we will generalize the results to the case of “large” gauge transformations.

We begin by considering a non-relativistic particle with position \( q \), charge \( e \), in an external potential \( V(q, t) \), and coupled to the electromagnetic field \( A_\mu \). The extension to multiple particles is trivial. The action for the system evolving from an initial time \( t_i \) to a final time \( t_f \) is

\[
S = S_M + S_{EM},
\]

where

\[
S_M[q] = \int_{t_i}^{t_f} dt \left[ \frac{1}{2} m \dot{q}^2 - V(q, t) \right] \quad \text{(7)}
\]

describes the particle alone, and

\[
S_{EM}[q, A_\mu] = \int_{t_i}^{t_f} d^4x \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + A_\mu J^\mu \right] \quad \text{(8)}
\]

describes the electromagnetic field along with the coupling to the matter; here \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) is the electromagnetic field tensor, and \( J^0(x) = e \delta^3(x - q(t)) \) and \( J^j = e \dot{q}^j \delta^3(x - q(t)) \) are components of the charge current. Note here that the current for a charged particle is conserved even when the equations of motion are not satisfied, i.e. for a general path in the path integral.

Under gauge transformation this action transforms by a boundary term. We will assume these transformations vanish at spatial infinity, but the contribution from the space-like parts of the boundary will not vanish. We assume the spacetime region shown in Fig. 1. We denote the surface of constant time \( t = t_f \) by \( \Sigma_f \), and likewise for \( t_i, \Sigma_i \). An asymptotic timelike cylinder \( S^2 \times \mathbb{R} \) at arbitrarily large radius will be denoted by \( \Sigma_\infty \).

Our path integral is then over field configurations and particle trajectories in a region \( \mathcal{V} \) bounded by \( \partial \mathcal{V} = \Sigma_f \cup \Sigma_i \cup \Sigma_\infty \). In this notation, under the gauge transformation \( A_\mu \rightarrow A^\Lambda = A_\mu + \partial_\mu \Lambda \), the EM action then acquires a boundary term

\[
\delta_\Lambda S_{EM} = \int_{\partial \mathcal{V}} d^3x \Lambda n_\mu J^\mu = \int_{\Sigma_f} d^3x \Lambda f J^0 - \int_{\Sigma_i} d^3x \Lambda i J^0 = e (\Lambda_f(q_f) - \Lambda_i(q_i)) \quad \text{(9)}
\]
Figure 1: Depiction of the propagator $K_{fi}$ in eqn. (10) and the spacetime through which it propagates. The initial configuration is on the time-slice surface $t = t_i$, the final configuration on the surface $t = t_f$. Particle paths propagate between $q_i$ at $t_i$ and $q_f$ at $t_f$; the gauge field propagates between $A_{i}^\mu$ and $A_{f}^\mu$ on these same two time-slices.

We will choose to quantize the system on the extended configuration space, i.e., we consider all configurations of $A_\mu(x)$ before quantization rather than imposing constraints and gauge conditions at the classical level and quantizing the remaining degrees of freedom.

The path integral describing the amplitude for transition between configurations $q_i, A_{\mu i}$ and $q_f, A_{\mu f}$ is then

$$K_{fi} \equiv K(q_f, A_{\mu f}; q_i, A_{\mu i}) = \int_{q_i}^{q_f} Dq \, e^{i S_M} \int_{A_{\mu i}}^{A_{\mu f}} DA_{\mu} e^{i S_{EM}}. \quad (10)$$

Here and throughout this paper we will absorb field independent constants into the path integral measure.

The expression in (10) is obviously independent of the gauge - no choice of gauge enters into it (lest anyone doubt this, we demonstrate it explicitly in Appendix A).

Our key aim in this sub-section is to show, by two quite different methods, that this expression is completely equivalent to the expression

$$K_{fi} = e^{i \tilde{S}_C} \int_{q_i}^{q_f} Dq \, e^{i \tilde{S}_M} \int_{A_{\mu i}}^{A_{\mu f}} DA_{\mu} e^{i \tilde{S}_A}. \quad (11)$$

in which we now employ a different action $\tilde{S}$, given by

$$\tilde{S} = \tilde{S}_M + \tilde{S}_C + \tilde{S}_A \quad (12)$$

with the three new terms defined as:
(i) The new matter action $\tilde{S}_M$ incorporates a ‘Coulomb self-energy’ term, to give

$$\tilde{S}_M = S_M + \frac{1}{2} \int_{t_i}^{t_f} d^4x \, J^0 \nabla^{-2} J^0$$

(13)

with $S_M$ given by (7) as before; and we note that $\tilde{S}_M$ is gauge invariant;

(ii) The boundary term $\tilde{S}_C$ is a pure phase; we have

$$\tilde{S}_C = \int_{\partial V} \sigma \, d^3r \, J^0 \nabla^{-2} (\partial_j A^j)$$

$$= -\frac{e}{4\pi} \int_{\partial V} \sigma \, d^3r \, \frac{\nabla \cdot A(r)}{|r - q_\sigma|}$$

(14)

where $\sigma = \pm$ for future/past boundary time slices (we will henceforth omit the $\sigma$, leaving it implicit). The boundary term $\tilde{S}_C$ is just Dirac’s Coulomb phase, appearing here on each boundary of the propagator.

(iii) The ‘dynamic’ part of the EM action $\tilde{S}_A$ is

$$\tilde{S}_A = \frac{1}{2} \int_{t_i}^{t_f} d^4x \left[ -\partial_\mu A^\mu \partial_\nu A_\nu + 2A_j J^j \right]$$

(15)

and we see that, like $\tilde{S}_M$, this is gauge invariant. The 3-vectors $A^j_i$ and $A^j_f$ are the initial and final configurations of the ‘transverse’ gauge field $A^j(x)$, itself defined as

$$A_j = A_j - \partial_j \nabla^{-2}(\partial^k A_k)$$

(16)

This result, as encapsulated in eqns. (11)-(16), may seem surprising - it looks as though the Coulomb gauge has been singled out here, and the result is in apparent conflict with ideas, going back to Dirac, according to which the Coulomb dressing factor is just one from a large variety of possible gauge-equivalent dressing factors which can to describe the quantum state of a scalar electron coupled to the EM field.

Nevertheless we will now show, using two different methods, that the result (11) is not only generally correct, but is gauge-invariant; moreover our derivations will avoid any gauge-fixing. We comment on the physical meaning of this result both later in this section, and in the conclusions.

2.2.1. Method 1: Decomposition of $K_{fi}$

In the first method, one starts from the gauge-invariant expression in (10), and one transforms the action into an equivalent form written in terms of new variables, which are themselves determined by the constraint equations for the system. The key here is that the form of these constraint equations is not determined by any gauge choice - it is instead determined by the boundary conditions imposed on our spacetime domain (in this case, the boundaries shown in Fig. 1).
One begins by making the $U(1)$ gauge transformation: a phase rotation for the charged particle, and $A_\mu \to A_\mu^\Lambda$ for the gauge field, as defined above. The gauge-invariant propagator then becomes

$$ K_{fi}^\Lambda \equiv K^\Lambda(q_f, A_{\mu f}; q_i, A_{\mu i}) = e^{-i\Lambda_f(q_f)}K(q_f, A_{\mu f}^\Lambda; q_i, A_{\mu i}^\Lambda)e^{i\Lambda_i(q_i)} $$

(17)

where $\Lambda_{i(f)}$ is the gauge parameter on the initial (final) time slice.

We know from Hamiltonian dynamics that the Gauss law constraint is the generator of gauge transformations. The propagator (10) should therefore satisfy Gauss' law as an operator constraint on both $\Sigma_f$ and $\Sigma_i$. Working this through (see Appendix A), one obtains the constraint equation for $K_{fi}$ on $\Sigma_f$ as

$$ 0 = \left[ \int d^3x \partial_0 \Lambda_f \frac{\delta}{\delta A_{0 f}} - i \int d^3x \Lambda_f \left( e^{3}(q_f - x) - \partial_j \hat{E}^j \right) \right] K_{fi} $$

(18)

On this surface $\Sigma_f$, the functions $\Lambda_f$ and $\partial_0 \Lambda_f$ are independent, but arbitrary, functions vanishing at spatial infinity. As a result, the propagator then satisfies two separate local constraint equations

$$ \frac{\delta}{\delta A_{0 f}} K_{fi} = 0 $$

(19)

$$ \left( \partial_j \hat{E}^j - \hat{j}^0 \right) K_{fi} = 0. $$

(20)

where we see that $K_{fi}$ not only satisfies the Gauss law operator constraint, but is also independent of the prescribed data for $A_0$. This means that we can then freely integrate over the boundary data for $A_0$, to get

$$ K_{fi} = \int_{q_i}^{q_f} Dq e^{iS_M} \int DA_0 \int_{A_{ji}}^{A_{fi}} DA_j e^{iS_{EM}}. $$

(21)

showing that $A_0$ is not a true dynamical variable.

That $A_0$ is a redundant variable for this case is of course well known; however we would like to emphasize here that:

(i) the boundary data for $A_0$ naturally falls out of the expression as a consequence of gauge invariance - there is no need for a detour through canonical Hamiltonian quantization, or a discussion of the missing conjugate momentum $\Pi^0$ to see this point. This is because we chose to evolve between constant time slices, and the pullback of the 1-form $A_\mu dx^\mu$ to these boundaries is independent of $A_0$, making $A_0$ redundant.

(ii) As a corollary to this, we expect that in other boundary geometries, the redundant variable will not be $A_0$. In the next section we will see that it is in general just the component of $A_\mu$ normal to the boundary, just as $A_0$ is the component of $A_\mu$ normal to the time slices.

To complete the derivation of (11) we need to write $K_{fi}$ in terms of the action (12) and the transverse gauge function variable $A_j$ defined in (16). This is done by actually doing
the integration over $A_0$ in (21); since this derivation is a little long, it appears in Appendix A. The final result is that we gave above, in eqn. (11).

We see that apart from the natural definition of the transverse vector field $A^j$ as the physical EM field variable, we also obtain a Coulomb form for the dressing term in $\tilde{S}_C$ in (14), and we see that this is not a consequence of choosing a Coulomb gauge! Instead it arises naturally as a boundary term in the path integral expression for $K_{fi}$.

2.2.2. Example: Eikonal Approximation

Let us briefly discuss what one expects to find from evaluating the remaining path-integral over $A_j$ and $q$. We will not attempt to discuss detailed examples here. However, the lowest-order eikonal approximation does serve to illustrate what one can expect. We give a heuristic treatment here - more detail is found in, eg. Fradkin or Fried [45, 59, 60].

In this lowest-order eikonal approximation, fluctuations in the charge trajectory about the classical saddle point are neglected in the current. Starting from the effective field term $\tilde{S}_A$ in (15), we write it as $\tilde{S}_A = \tilde{S}_A^0 + \tilde{S}_A^\text{int}$, where the interaction term is

$$\tilde{S}_A^\text{int} = \int_{t_i}^{t_f} d^4x A_j(x) J^j(x)$$

We then expand $q(t)$ as $q(t) = q_{cl}(t) + \delta q(t)$, where the classical trajectory $q_{cl}$ is independent of $A_\mu$, so that

$$\tilde{S}_A^\text{int} = e \int_{t_i}^{t_f} dt \dot{q}_j(t) A_j(q(t), t) \equiv \int_{t_i}^{t_f} d^4x A_j(x) J^j(x). \quad (22)$$

If we were to isolate only the long wavelength parts of $A_j$, we could truncate the above derivative expansion at $n = 0$; moreover, the high frequency trajectory fluctuations would not effectively couple to these long wavelength parts of $A_j$ so the term linear in $\delta q^j$ would also be negligible. Discarding these terms is of course only valid for the long-wavelength parts of the gauge field, but if we were to simply carry this through for the whole field then we effectively perform a lowest order eikonal approximation.

The lowest-order contribution for the path-integral in (11) then comes simply from replacing the original interaction term in the action by one involving just the classical path of the particle:

$$S^{\text{eik}}_{\text{int}} = e \int_{t_i}^{t_f} dt \dot{q}_d(t) A_j(q_{cl}(t), t)$$

$$\equiv \int_{t_i}^{t_f} d^4x A_j(x) J^j_d(x). \quad (24)$$

The functional integral for the gauge field coupled to an external classical source can be done exactly, and the resulting functional dependence on $A_{j,f}$ is Gaussian. Assuming
the initial state of the gauge field is also some Gaussian state, e.g., the vacuum, then the Gaussian form remains even after using the propagator as a kernel to evolve the initial state. The Gaussian functional dependence implies that the out state will generally be a squeezed coherent state of the electric field. These linear dynamics cause no squeezing of the state, so if the initial state is not squeezed nor will be the out state. The equivalent statement of this in the canonical quantization framework is textbook material, and it is easy to demonstrate because the Heisenberg equations of motion are still exactly solvable for a free field coupled linearly to a classical source.

We conclude that if the initial state is the vacuum state then, in the lowest-order eikonal approximation, the resulting state of the electromagnetic field will be a coherent state that is peaked on the classical electric field created by the source $J_{cl}$.

This particular eikonal approximation illustrates very nicely that, aside from the universal Coulomb part of the field, coherent dressed states can be understood in terms of quantum state preparation [61, 62]. As laboratory charges are moved around, measured, and probed, they are emitting low energy photons and creating a radiative “dressing”. The eikonal approximation also gives a concrete method for computing the long-wavelength parts of the dressing resulting from a given state preparation mechanism [45]. It is interesting to apply this to real world problems, away from the idealized infinite time S-matrix scattering theory, to demonstrate how time-dependent dressings emerge dynamically. This is particularly relevant for decoherence via soft photons, since the charges are entangled with their radiative dressing.

We can now address one of the questions raised in the introduction. If one considers the dynamics of a charged quantum particle, one finds that the expectation value for the long wavelength parts of the electric field operator is precisely what is expected from the classical problem, i.e., the Liénard-Wiechert field of a moving charged particle. The dressing can be characterized as follows: the resulting state of the electromagnetic field is an eigenstate of the longitudinal electric field operator with eigenvalue corresponding to the Coulomb field, and a coherent state of the transverse electric field which is peaked on a configuration determined by the classical limit of the history of the charged particle. However this field has both longitudinal and transverse parts, and by measuring fluctuations one can see that they behave quite differently from one another in quantum theory.

2.2.3. Method 2: Boundary Faddeev-Popov Trick

There is actually no need to use the off-shell current conservation constrain we just employed in the demonstration of (11). We can instead generalize the usual Faddeev-Popov technique to what we will refer to as the boundary Faddeev-Popov (bFP) trick; this is similar to a previous technique developed in refs. [65], but generalized to include quantum matter, and to make the gauge independence clear.

We start again from from the manifestly gauge-invariant (10), and, as usual, we multiply the path integral by

$$1 = \int \mathcal{D}A \Delta[A^A] \delta(\mathcal{G}(A^A)), \quad (25)$$

where $\Delta[A^A] = |\det \delta_A \mathcal{G}(A^A)|$ is the FP determinant, $\mathcal{G}(A)$ is the gauge fixing function, and...
and again, $A^A = A_\mu + \partial_\mu \Lambda$. In our case the expression (25) involves not only integration over gauge transformations in the region $V$, but also over transformations on the boundary time slices $\Sigma_i \cup \Sigma_f$. Transformations residing on the boundaries are omitted in textbook applications of the FP trick, where one typically considers vacuum generating functionals with no explicit boundaries.

The resulting integral is then

$$K_{fi} = \int \mathcal{D}\Lambda \int_{q_i}^{q_f} \mathcal{D}q e^{i S_M} \int_{A_{\mu i}}^{A_{\mu f}} \mathcal{D}A_\mu \Delta[A^A] \delta(S(A^A)) e^{i S_{EM}[A]}$$

(26)

Under gauge transformation the FP determinant is gauge invariant, and the action transforms by a boundary term

$$S_{EM}[A] = S_{EM}[A^A] - \int_{\partial V} d^3x A J^0.$$  

(27)

so that the propagator can now be written as

$$K_{fi} = \int \mathcal{D}\Lambda e^{-i \int_{\partial V} d^3x A J^0} \int_{q_i}^{q_f} \mathcal{D}q e^{i S_M} \int_{A_{\mu i}}^{A_{\mu f}} \mathcal{D}A_\mu \Delta[A^A] \delta(S(A^A)) \delta(\partial V(S(A^A))) e^{i S_{EM}[A]}$$

(28)

where we now omit the primes in the notation, and use superscripts ($V$) and ($\partial V$) to denote quantities evaluated in the bulk and the boundary respectively.

Note that both the boundary data for the gauge field, and the delta function fixing the gauge on the boundaries, are still dependent on the gauge parameter $\Lambda$ – this of course was not changed by a change of integration variables.

In the standard application of the FP trick one would note that there was no remaining dependence in the path-integral on $\Lambda$, and the integral over the gauge group would simply be divided out as overall normalization; but clearly we can’t quite do that here. Instead we again use the boundary FP method mentioned above. The manipulations are then related to those used in Method 1, and we give them in Appendix A. The resulting expression for the propagator is

$$K_{fi} = e^{i \int_{\partial V} d^3x \nabla \cdot (\partial A_\mu) [\nabla \cdot (\partial^\mu A) \pi_{n j}]} \int_{q_i}^{q_f} \mathcal{D}q e^{i \tilde{S}_M} \int_{\tilde{A}_{n j}}^{\tilde{A}_{n f}} \mathcal{D}\tilde{A}_\mu e^{i \tilde{S}_A[\tilde{A}]}$$

(29)

in which we now write things in terms of the effective actions $\tilde{S}_M$ and $\tilde{S}_A$, as in (11). We can also rewrite this result for $K_{fi}$ in the same form as (11) - this is also shown in the Appendix.

This concludes our analysis of the propagator $K_{fi}$ for scalar QED. We stress again that the propagator is gauge-independent, and its form is a simple consequence of the gauge invariance of the effective action, rather than being imposed a priori.
2.3. Spinor Quantum Electrodynamics

We now generalize the above considerations to real QED, where Dirac spinors are coupled to the EM field. Again we will begin from a manifestly gauge invariant path-integral for $K_{fi}$, and derive the same Coulomb form for the dressing. The manipulations are similar to those for scalar electrodynamics, the only difference being that the matter field also changes under gauge transformation, and the $U(1)$ charge density in the boundary phase becomes an operator.

The gauge invariant path integral representation of the transition amplitude, the analogue of (10) for scalar electrodynamics, is

$$K_{fi} = \int_{\bar{\psi}}^{\psi} D\psi D\bar{\psi} \int_{A_{\mu i}}^{A_{\mu f}} DA_{\mu} e^{iS[A,\psi,\bar{\psi}]},$$

(30)

where $\psi, \bar{\psi}$ are Grassmann fields, and the omission of boundary data for $\bar{\psi}$ indicates that this variable is to be integrated over on the boundary—necessary because the Dirac Lagrangian has a first-order form. The action is the QED action with a single Dirac fermion field of charge $e$, viz.,

$$S[A, \psi, \bar{\psi}] = \int_{t_i}^{t_f} dt d^4x \left[ -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \bar{\psi} \left( \gamma^{\mu} \partial_{\mu} - ie\gamma^{\mu}A_{\mu} + m \right) \psi \right]$$

(31)

This action is completely invariant, without need to discard a boundary term, under the $U(1)$ gauge transformation

$$A_{\mu} \rightarrow A_{\mu} + \partial_{\mu} \Lambda$$
$$\psi \rightarrow e^{ie\Lambda} \psi,$$
$$\bar{\psi} \rightarrow e^{-ie\Lambda} \bar{\psi}.$$  

(32)

One can easily verify that the propagator (30) is gauge invariant in the same way done for expression (10) in Appendix A, i.e. by transforming its data, undoing the transformation by a change of variables in the path integral, and using the invariance of the action.

This gauge invariance of $K_{fi}$ implies that it satisfies the boundary equation

$$\int_{\partial\Sigma} d^3x \left[ i e\Lambda \psi \frac{\delta}{\delta \psi} + \partial_{\mu} \Lambda \frac{\delta}{\delta A_{\mu}} \right] K_{fi} = 0.$$  

(33)

By explicitly differentiating the path integral we can confirm that the functional derivatives are proportional to the conjugate momenta for the fields:

$$\frac{\delta}{\delta \psi_{i,f}} = \pm i \hat{\Pi}_{i,f} = \mp \hat{\psi}_{i,f}^\dagger,$$

(34)

$$\frac{\delta}{\delta A_{\mu i,f}} = \pm i \hat{\Pi}_{\mu i,f} = \pm i \hat{E}_{\mu i,f}.$$  

(35)

14
Together with the expression for the $U(1)$ charge density $J^0 = i \bar{\psi} \gamma^0 \psi = -\psi \psi^\dagger$, and the invariance condition (33), this implies the propagator satisfies the operator constraints
\[
(\partial_j \hat{E}^j - J^0) K(A, \psi) = 0 \\
\frac{\delta}{\delta A_0} K(A, \psi) = 0,
\]
on both the future and past boundary time slices.

Let us now use the bFP trick again to see how the electric field dressing of the states emerges, i.e., to see how this constraint is implemented. We again insert a gauge fixing function into $K_{fi}$ by multiplying by (25), but now we must change variables for both the gauge field and the Dirac field if the action is to be invariant:
\[
K_{fi} = \int \mathcal{D}A \int_{\psi_i}^{\psi_f} \mathcal{D}\psi \mathcal{D}\bar{\psi} \int_{A_i}^{A_f} \mathcal{D}A_{\mu} \Delta[A] \delta^V(\mathcal{S}(A)) \delta^{\Omega}(\mathcal{S}(A^4)) e^{i\hat{S}[A,\psi,\bar{\psi}]}
\]
which has the same form as (11) except that now the matter action is
\[
\hat{S}_M = \int_{t_i}^{t_f} d^4x \left[ - \bar{\psi} (\gamma^\mu \partial_\mu + m) \psi + \frac{1}{2} J^0 \nabla^{-2} J^0 \right]
\]
and the dynamic gauge field action $\hat{S}_A$ is as before (cf. eqn. (15)), except that now the matter current is $J^j = ie\bar{\psi} \gamma^j \psi$.

We can take the expression in eqn. (39) one step further if we explicitly act with the $U(1)$ transformation sitting outside the path-integral. This locally rotates the boundary data for the Dirac field by an angle which depends on the longitudinal part of the gauge field, to give
\[
K_{fi} = \int e^{-ie\nabla^{-2} \partial_0 A_j \psi_f} \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{i\hat{S}_M} \int_{A_j}^{A_f} \mathcal{D}A_j e^{i\hat{S}_A}.
\]
for the gauge invariant QED propagator on the extended configuration space, in which the limits on the integrals are explicitly given as
\[
e^{-ie\nabla^{-2} \partial_0 A_j \psi} = \exp \left( i \int d^3 y A_j(y) \frac{e^{-i(y^j - x^j)}}{4\pi \left| y - x \right|^3} \psi(x) \right).
\]

Thus the gauge invariant propagator dresses every point excitation of the Dirac field by a Coulomb electric field sourced by the corresponding point charge. This is the key result we wish to emphasize for spinor QED, and it parallels that found for scalar QED. Again, as with scalar QED, the transverse dressing will be determined dynamically by the remaining integral over gauge invariant variables.
3. Flat spacetime evolution in a causal diamond

Up to now we have dealt with the rather simple problem of QED on a flat background, defined between time slices. We now turn to more general kinds of boundary and boundary information.

The region we consider is a causal diamond in Minkowski spacetime, where the state is fixed on the null boundary hypersurface. Here there is still a natural splitting into past and future sections, and so we can define a propagator which represents a transition amplitude between states on the past and future null cones (which tend to null infinity for an infinitely large diamond).

The derivation proceeds in analogy with the work in the last selection. In section 3.A we formulate the problem and show how to transform the effective action so as to extract the boundary terms and physical variables. In section 3.B we then derive the form of the propagator in terms of these variables.

We note here that one quickly encounters a subtlety in the specification of boundary data for the path-integral. Because the conjugate momentum on a null surface involves a derivative along that surface, specifying the field configuration also specifies the conjugate momentum. Giving data on both the past and future boundaries then over-specifies the boundary data for the classical evolution, and the corresponding interpretation as a quantum amplitude is then unclear.

We fix this by specifying “half” of the field data in some chosen way [66]. We will assume throughout that it is only the positive frequency parts of the field which are specified, i.e., we interpret the amplitude in terms of coherent states in the Bargmann representation. In the following discussion we will avoid making this explicit, so as not to clutter the notation.

3.1. Formulation of the Problem

In the time-slice geometry, the variable $A_0$ was ultimately unphysical, and the remaining variables $A_j$ split into purely physical transverse and pure gauge longitudinal parts - the transverse part being divergenceless, i.e., $\partial^j A_j = 0$. For more general boundary hypersurfaces, a natural idea would be to continue to decompose the field into parts with and without divergence. This is not possible, for 2 reasons. First, as before, there is still the issue of uniqueness – given a transverse-longitudinal decomposition of the vector field, one can freely add some transverse parts onto the longitudinal part and the result still transforms correctly under gauge transformation. Second, on null hypersurfaces there is no unique notion of divergence – the induced metric is degenerate, and so there is no unique inverse metric with which to define the divergence $h^{jk} \nabla_j A_k$.

For these reasons we again use a procedure whereby the path integral is used to generate a unique decomposition into pure gauge and gauge invariant parts of the field.

We recall that for flat time-slice boundaries, the boundary data of the component $A_0$ was integrated over. The saddle point solution for this Gaussian integral, $\tilde{A}_0$, determined the g-potential $\Phi$, i.e., the functional of $A_j$ transforming as $\delta A \Phi = \Lambda$; the pure gauge part of $\tilde{A}_0$ was the time derivative of $\Phi$, and the longitudinal part $A_j$ was the gradient of $\Phi$. For more general boundaries we will then need to single out the component of $A_\mu$ normal to the
boundary hypersurface. This component will play the same role as $A_0$, and the pure gauge part of its solution will yield a corresponding g-potential.

3.1.1. Coordination specification

To implement these ideas we need to choose coordinates appropriately. We pick hypersurface adapted coordinates $x^\mu = \{S, y^k\}$ such that $S = \text{const.}$ surfaces foliate the spacetime region $\mathcal{V}$, and the boundary hypersurface $\partial \mathcal{V}$ is described by particular values, $S = S_i, S_f$. Then, using a coordinate basis it is $A_S$ which is the component generalizing $A_0$, because the pullback of $A_\mu dx^\mu$ to $\partial \mathcal{V}$ will be independent of $A_S$.

For a causal diamond in Minkowski spacetime we then need to construct coordinates adapted to the boundary null cones. The coordinates we will use are rather intuitive. Consider a sphere of radius $R$ at time $t = 0$, and from each solid angle send an inwards going radial null ray to the future and to the past. These null geodesics will converge at $r = 0$ at times $t = R$ and $t = -R$ respectively, and the surface generated by the null rays is the boundary of our causal diamond.

To construct coordinates in the interior we again start from the sphere $r = R$ at $t = 0$, and now send inwards going spacelike rays to the future and past. These spacelike rays converge at $r = 0$ but at times $t$ dictated by their “velocities”. The angles and radii of spheres are still useful coordinates, but now we will replace the time coordinate $t$ with a coordinate parameterizing the “velocity” of each ray.
Each of the rays joining $r = 0$ to $r = R$ is described by a solid angle and $t, r$ satisfying the simple relation

$$t = \eta f(r),$$

(43)

for

$$f(r) = 1 - \frac{r}{R},$$

(44)

and for some $\eta \in [-R, R]$. From this relation we can quickly verify that the surfaces $\eta = \pm R$ are the future and past null boundaries of the causal diamond.

Inside the boundary, $\eta$ parameterizes spacelike surfaces and thus serves as a useful time coordinate. Thus, as desired, we’ve found hypersurface adapted coordinates where certain values of “time” denote the boundary. We can straightforwardly compute the metric in these coordinates:

$$ds^2 = -f(r)^2d\eta^2 + 2\frac{\eta}{R}f(r)d\eta dr + \left(1 - \frac{\eta^2}{R^2}\right)dr^2 + r^2d\Omega^2$$

(45)

where $d\Omega^2$ is the standard line element on the unit 2-sphere.

It is clear from this expression that $\eta = 0$ is just a standard time slice of Minkowski spacetime and that $\eta = \pm R$ are null hypersurfaces. Since $f(r)$ vanishes at $r = R$, there is a coordinate singularity. This is obvious from Fig. (2), and indeed several components of the inverse metric will diverge as $r \to R$,

$$g^{\eta\eta} = -\left(1 - \frac{\eta^2}{R^2}\right)\frac{1}{f(r)^2}, \quad g^{\eta r} = \frac{\eta}{R} f(r),$$

$$g^{rr} = 1, \quad g^{AB} = r^{-2}q^{AB}$$

(46)

where $x^A$ are sphere coordinates, and $q^{AB}$ is the inverse metric on the unit 2-sphere.

To deal with this we need to recall why we are interested in this geometry. Ultimately we wish to take $R$ to be larger than all other length scales so that the sphere $r = R$ resembles spatial infinity, and the surfaces $\eta = \pm R$ resemble null infinity. As long as we don’t take the strict limit $R \to \infty$, we can still specify data for massive fields on the boundary. The boundary considered here then plays a role similar to null infinity, but is not obtained via conformal compactification. Timelike worldlines will be able to connect all points in the bulk to some point on the boundary.

Since the electromagnetic field is massless we expect field excitations to reach null infinity but we do not expect the same for spatial infinity. For this reason we make the assumption that all important quantities will vanish sufficiently fast for $r \to R$, while allowing for finite limits as $\eta \to \pm R$. This physical assumption ensures that the coordinate singularity from $f(r)$ as $r \to R$ does not actually cause issues during the calculation. We can see explicitly how these fields vanish by comparing the field components in these coordinates with those in the standard $(t, r)$ coordinates. For example, for a one-form $w_\mu$ we have

$$w_\eta = f(r)w_1,$$

(47)
and since physical fields are finite in \((t, r)\) coordinates, \(w_\eta\) will vanish at least as fast as \(f(r)\) as \(r \to R\).

When it is necessary, we will formally “blow up” this surface, ie., excise the sphere \(r = R\) from the boundary such that limits \(r \to R\) can be \(\eta\) dependent. Note also that the boundary region \(r = R\) is effectively a two-dimensional surface, and thus has zero measure in three-dimensions. Thus when spatially integrating by parts in a four-dimensional integral, both \(r = 0\) and \(r = R\) will be zero volume surfaces, and we can then discard any spatial surface terms. We can demonstrate this explicitly by appropriately restoring factors of \(f(r)\) in the following example integral,

\[
\int d^4 x \sqrt{g} \partial^r \nabla_r h = - \int d^4 x \sqrt{g} h \nabla_r \partial^r + \left( \int d^2 \Omega d\eta r^2 f(r) h \partial^r \right) \bigg|_0^R. \tag{48}
\]

For non-singular functions \(h\) and \(\partial^r\), the factor \(r^2 f(r)\) sets the spatial boundary contribution to zero.

In the following calculations the function \(f(r)\) arises in many places, only to be cancelled out of all final results when quantities such as \(\omega_\eta\) above are replaced by their finite parts, eg. \(\omega_t\). The charge flux density \(J^\eta\) also has an apparent divergence, \(J^\eta = \frac{1}{f(r)} J^t\), \(\tag{49}\)

however since we are assuming \(J^t\) only has support for \(r \ll R\), we effectively have \(J^\eta = J^t\). Rather than redundantly tracking the factors of \(f(r)\) through the calculation, we can simply make the \(r \ll R\) approximation at the level of the metric, effectively setting \(f(r) = 1\), and yielding the expression

\[
ds^2 = -d\eta^2 + 2\frac{\eta}{R} d\eta dr + \left( 1 - \frac{\eta^2}{R^2} \right) dr^2 + r^2 d\Omega^2. \tag{50}\]

This deals with the singular behaviour of the spatial “corner” of the boundary hypersurface, but there are still the corners at the top and bottom of the causal diamond, \(r = 0, \eta = \pm R\). We will also formally blow up these points to allow fields to take angle dependent limits as \(r \to 0\) on the boundary; see FIG. 3. In doing this, we assume nothing enters or leaves \(V\) through the strict points \(r = 0, \eta = \pm R\).

If one now considers a QED propagator with information specified on the boundary of this causal diamond, the transformation of the component \(A^\eta\) involves \(\partial_\eta \Lambda\), ie., a derivative normal to surfaces of constant \(\eta\) and thus independent of the actual pullback of \(\Lambda\) to the surface. Thus any boundary data specified for \(A_\eta\) in the path integral will be superfluous. In addition the QED Lagrangian will be quadratic in \(A_\eta\), allowing it to be integrated out via Gaussian saddle point substitution.

### 3.1.2. Transformed Effective Action

For brevity we just consider the gauge field coupled to a conserved external source \(J^\mu\); this is easily generalized to scalar charged particles or to a Dirac field by promoting \(J^\mu\) in the
resulting boundary phase to an operator. As before, we first obtain results without explicitly fixing a gauge, then discuss how the bFP trick shortcuts the computation. Expanding the action so as to explicitly write $A_\eta$ we have

$$S = -\frac{1}{2} \int d^4x \sqrt{g} \left[ F^{jk} \partial_j A_k - 2A_j J^j + F^{\eta j} \partial_\eta A_j - A_\eta \left( \frac{1}{\sqrt{g}} \partial_j (\sqrt{g} F^{\eta j}) + J^j \right) - A_\eta J^\eta \right]$$

where $\sqrt{g} = r^2 \sin \theta$, and $j = \{r, \theta, \phi\}$. In writing this we’ve already freely integrated by parts in spatial directions. To integrate out $A_\eta$, we need to solve its saddle point equation, ie.

$$\frac{1}{\sqrt{g}} \partial_j (\sqrt{g} F^{\eta j}) + J^\eta = 0.$$  

Since the metric is non-diagonal, the resulting equation is qualitatively different from the previous equation for $A_0$. In terms of $A_\eta$ the equation of motion reads

$$\partial_r (\sqrt{g} \partial_r A_\eta) - g^{\eta r} \partial_A (\sqrt{g} g^{AB} \partial_B A_\eta) = g^{\eta r} \partial_A (\sqrt{g} g^{AB} F_{Br}) + \sqrt{g} J^\eta + \partial_r (\sqrt{g} \partial_\eta A_r) - g^{\eta r} \partial_A (\sqrt{g} g^{AB} \partial_\eta A_B)$$

On the right-hand side the first two terms are obviously gauge invariant, and the last two terms together transform as required so that the solution to this equation, $\tilde{A}_\eta$, will transform as $\delta_A \tilde{A}_\eta = \partial_\eta \Lambda$.

Note that $\partial_\eta g^{\eta r} = 2\eta/R^2$, a dimensionful quantity of order $R^{-1}$. By our original assumptions, $R$ is parametrically much larger than any other dimensionful quantity and thus this entire term is sub-leading. With $R$ sufficiently large we can simply assume $\partial_\eta g^{\eta r} = 0$,
allowing (53) to be written compactly as

\[ D^j \partial_j A_\eta = \frac{1}{\sqrt{g}} g^{\nu \rho} \partial_A(\sqrt{g} g^{AB} F_{Br}) + J^\eta + \partial_\eta D^j A_j, \tag{54} \]

where we’ve defined the divergence-like differential operator \( D^j \), acting as

\[ D^j w_j = \frac{1}{\sqrt{g}} \partial_r(\sqrt{g} w_r) - g^{\eta \eta} \frac{1}{\sqrt{g}} \partial_A(\sqrt{g} g^{AB} w_B). \tag{55} \]

Now (53) can formally be solved by assuming a Green’s function \( G \) satisfying

\[ D^j \partial_j G(x, x') = \frac{\delta^3(x - x')}{\sqrt{g}}, \tag{56} \]

that is,

\[ \tilde{A}_\eta = \tilde{A}_\eta^t + g + h, \tag{57} \]

where

\[ \tilde{A}_\eta^t = \int_{\Sigma_\eta} d^3 x' \sqrt{g} G \left[ \frac{1}{\sqrt{g}} g^{\nu \rho} \partial_A(\sqrt{g} g^{AB} F_{Br}) + J^\eta \right] \tag{58} \]

\[ g = \partial_\eta \int_{\Sigma_\eta} d^3 x' \sqrt{g} G D^j A_j, \tag{59} \]

and \( h \) is a homogeneous solution \( D^j \partial_j h = 0 \). Note that The integration in these expressions is over \( \Sigma_\eta \), the constant \( \eta \) hypersurface corresponding to the time \( \eta \) at which \( \tilde{A}_\eta \) is being evaluated.

We don’t have a general expression for this Green’s function; however the results that we’re interested in will ultimately only depend on its value on the null boundary, and one can find \( G \) on this boundary as well as at \( \eta = 0 \). At \( \eta = 0 \), \( g^{\eta \eta} = -1 \), and the differential operator simplifies to

\[ D^j \partial_j f(x) \big|_{\eta=0} = \frac{1}{\sqrt{g}} \left[ \partial_r(\sqrt{g} \partial_r f(x)) + \partial_A(\sqrt{g} g^{AB} \partial_B f(x)) \right] \tag{60} \]

which is of course just the standard Laplacian in spherical coordinates. This is because the hypersurface \( \eta = 0 \) is just the hypersurface \( t = 0 \). Thus at \( \eta = 0 \) the Green’s function is given by

\[ G(x, x') \big|_{\eta=0} = -\frac{1}{4\pi} \frac{1}{|x - x'|}. \tag{61} \]

At the boundary, the operator \( D^j \partial_j \) simplifies considerably because \( g^{\eta \eta} \) vanishes; we then have

\[ D^j \partial_j f \big|_{\eta=\pm} = \frac{1}{\sqrt{g}} \partial_r(\sqrt{g} \partial_r f), \tag{62} \]

21
which can be immediately integrated to find the Green’s function
\[
G(x, x') \big|_{\eta = \pm r} = \frac{\delta^2(x^A - x'^A)}{\sin \theta} \theta(r' - r) \left[ \frac{1}{r} - \frac{1}{r'} \right].
\]
which propagates along the null generators of the boundary.

Note that the boundary condition for \( G \) is chosen so that influence propagates towards smaller radii, i.e. causally on the future portion of \( \partial \mathcal{V} \). When considering the past portion of \( \partial \mathcal{V} \) one must flip the argument of the step function appropriately.

More progress can be made when looking at the homogeneous solutions. A general homogeneous solution, \( D^j \partial_j h = 0 \), will have the form
\[
h(x) = \sum_{m,l} Y^m_l(\theta, \phi) \left[ c^1_{ml}(\eta) r^{-\frac{1}{2} + \sqrt{\frac{1}{4} - g^m_l(l + 1)}} + c^2_{ml}(\eta) r^{-\frac{1}{2} - \sqrt{\frac{1}{4} - g^m_l(l + 1)}} \right]
\]
with \( Y^m_l \) a spherical harmonic and \( c^1,^2_{ml} \) a set of time dependent coefficients.

We can immediately set \( c^2_{ml} = 0 \), since it is the coefficient of a term which will never be regular at the origin. The other term will either grow monotonically with \( r \) or be constant in \( r \). With our assumptions that the fields vanish at large \( r \), both situations are unacceptable and we can set \( c^1_{ml} = 0 \). The solution with \( h = 0 \) is then the unique solution satisfying the boundary conditions.

As an aside, note that if we relax the asymptotic spatial boundary conditions and simply demand for the fields to be finite as \( r \to \infty \), we can accept solutions that are independent of \( r \). Such solutions satisfy
\[
-g^{m}l(l + 1) = 0.
\]
For all spacelike slices, \( g^{m} < 0 \), and the only solution is \( l = 0 \), i.e. a constant function of \( \theta, \phi, r \). These are the time dependent global \( U(1) \) rotations. However on the null boundaries \( g^{m} = 0 \), and the homogeneous solution space is enlarged to include any function on the sphere. This is interesting in the context of large gauge transformations, soft photons, etc, and we will return to this point in section 5.

Returning to the solution \( h = 0 \), note that the gauge-variant part \( g \) transforms as \( \delta_A g = \partial_\eta A \). From \( h = 0 \) we see we can identify it as a \( g \)-potential of form \( g = \partial_\eta \Phi \) with \( \Phi \) given by
\[
\Phi(x) = \int_{\Sigma_\eta} d^3x' \sqrt{g} G(x, x') D^j A_j(x').
\]

For the causal diamond we can now decompose the gauge field into a gauge-invariant part \( A_j = A_j - \partial_\eta \Phi \), and a pure gauge part \( \partial_\eta \Phi \); the subsequent development then parallels that for the time slice. We substitute \( \tilde{A}_\eta \) into the action \( S \) and rewrite the action in the new variables \( A_j, \Phi \). Using current conservation, we then get an effective action
\[
\tilde{S} = \int_{\partial \Sigma} d^3x \sqrt{g} \Phi J^\eta - \frac{1}{2} \int_{\Sigma} d^4x \sqrt{g} \left[ \tilde{F}^{ij} \partial_\mu A_j - 2A_j J^i \right. - J_j \int_{\Sigma_\eta} d^3x' \sqrt{g} G \left( J^\eta + \frac{1}{\sqrt{g}} g^{\eta r} \partial_\eta (\sqrt{g} g^{AB} F_{Br}) \right) \right],
\]
with
\[ \tilde{F}^{\mu j} = \partial^\mu \tilde{A}^j - \partial^j \tilde{A}^\mu. \] (68)

Note that all of the terms involving \( \Phi \) again summed to a total time derivative, and thus formed a boundary term in the action. The remaining bulk action is written in terms of explicitly gauge invariant variables.

We can actually take this expression further because the variable \( A_j = A_j - \partial_j \Phi \) is actually transverse in the sense that \( D^j A_j = 0 \). Using this, and a few spatial integrations by parts, we expand the effective action in terms of the gauge invariant variables to get
\[ \tilde{S} = \int_{\partial V} d^3x \sqrt{g} \tilde{\Phi} J^\eta + \frac{1}{2} \int_V d^4x \sqrt{g} \left[ \partial_\eta A_\eta A_\eta - g^{\eta \mu} g^{AB} \partial_\eta A_\eta A_B - 2 g^{\eta \mu} g^{AB} (\partial_\eta A_A) F_{rB} ight. 
- F^{AB} \partial_A A_B + g^{AB} F_{rA} F_{rB} + 2 \partial_j J^j 
+ \left. \left( J^\eta + \frac{1}{\sqrt{g}} g^{\eta \nu} \partial_A (\sqrt{g} g^{AB} F_{rB}) \right) \int_{\Sigma_\eta} d^3x' \sqrt{g} G \left( J^\eta + \frac{1}{\sqrt{g}} g^{\eta \nu} \partial_C (\sqrt{g} g^{CD} F_{rD}) \right) \right] \tag{69} \]

This is the expression we will work with - although we will not actually perform computations with this action. The purpose of the derivation was rather to demonstrate that when propagators are considered for different boundary geometries, we can still unambiguously extract a boundary term describing the dressing required to make charged states gauge invariant.

As expected the action (69) is non-local in space. The “Coulomb” interaction term now contains not just the charge density \( J^\eta \) but also terms describing the magnetic field. These apparent interactions arise because our coordinates are no longer adapted to the isometries of Minkowski spacetime. One can set \( \eta = 0 \), and thus \( g^{\eta \nu} = 0 \), to verify that on a standard constant time slice, this gives the usual Lagrangian density.

3.2. Form of the Propagator

Since \( \Phi \) doesn’t appear in the integrand, the path integral over \( \Phi \) can again be removed by a Faddeev-Popov procedure. The propagator
\[ K(A_\mu, \partial V) = \int_{A_\mu, \partial V} DA_\mu e^{i \int_V d^4x \sqrt{g} \left[ -\frac{1}{2} F_{\mu \nu} F^{\mu \nu} + A_\mu J^\mu \right]} \tag{70} \]
for evolution of the gauge field coupled to a source \( J^\mu \), through a large causal diamond, is then equal to
\[ K(A_\mu, \partial V) = e^{i \int_{\partial V} J^{\eta \Phi}} \int_{A_\mu, \partial V} DA_j e^{i \tilde{S}^\star[\tilde{A}; J]}, \tag{71} \]
where the effective action \( \tilde{S} \) is given by the bulk part of (69), and the prefactor involves the generalized Coulomb dressing, in which \( \Phi \) is given by eqns. (66) and (63) evaluated on the boundary. The contribution from the future part reads
\[ \Phi|_{\partial V}(r', x'^A) = \int_{r'}^\infty dr \left( \frac{1}{r'} - \frac{1}{r} \right) \partial_r \left( r^2 A_r (r, x'^A) \right), \tag{72} \]
whereas on the past part the integration is over all \( r \) interior to \( r' \).

This dressing describes the radial electric field at each point on \( \partial V \), with a strength determined by the total charge flux through \( \partial V \) at earlier times. This is our central result for the causal diamond geometry.

We emphasize again that this result is not the result of a specific gauge choice, and that the definition of gauge-invariant variables \( A_j \) again emerged naturally from the path integral. Remarkably, our procedure succeeded even though there is no unambiguous notion of the ‘transverse’ vector field, since one cannot define an intrinsic divergence on a null boundary.

If we now give the matter current \( J^\mu \) its own dynamics, we can easily generalize the above derivation. This is possible because \( U(1) \) charge current is conserved off shell for particles. Alternatively, as before, we can go back and skip the step which invokes current conservation by using the bFP trick. The derivations are as before; for Dirac fermions we then get the gauge invariant QED amplitude on the large causal diamond to be

\[
K(A_\mu \partial_V, \psi \partial_V) = \int e^{-i\Phi} \mathcal{D}\psi \mathcal{D}\bar{\psi} \int A_j \mathcal{D}A_j e^{iS[A,J]-i \int \psi \bar{\psi} \left( \gamma^\alpha \partial_\alpha + m \right) \psi} (73)
\]

where \( \Phi \) is given by (72). Analogous to the time-slice amplitude we see a dressing of each Dirac excitation in the boundary state by a Coulombic electric field.

Since we have skipped the explicit derivation of (73) and foregone the discussion of general boundaries in curved spacetime, we should at least mention that to do the bFP trick for more general boundaries one must necessarily use generalizations of canonical conjugate momenta and commutation relations. To highlight this, for a general path integral with data specified on boundary \( \partial V \), we can consider a variation of this boundary data, viz.,

\[
\delta \int \phi \mathcal{D}\phi e^{iS[\phi]} = i \int \phi \mathcal{D}\phi e^{iS[\phi]} \delta S. (74)
\]

A general variation of the action is of the form

\[
\delta S = \int d^4 x E(\phi) \delta \phi + \int \partial_V d^3 x (\partial_\mu S) \theta^\mu (\phi, \delta \phi), (75)
\]

where \( E(\phi) \) is the scalar density equation of motion, the boundary is defined by a constant \( S \) hypersurface, and the symplectic potential current density \( \theta^\mu \) is given for a general Lagrangian in ref. [67]. For a Lagrangian density which is a function only of the fields and their first derivatives we have

\[
\theta^\mu (\phi, \delta \phi) = \frac{\partial L}{\partial \nabla_\mu \phi} \delta \phi. (76)
\]

For non-null boundaries \( \sqrt{g} \partial_\mu S \) can be related to the normal covector and intrinsic volume element for the hypersurface, but the form in (75) is more general and also applies to null boundaries.

For variations with support only on the boundary we then have the functional derivative
\[ \frac{\delta}{\delta \phi(x')} \int_{\phi(x')} \mathcal{D} \phi e^{iS[\phi]} = i \int_{\phi(x')} \mathcal{D} \phi e^{iS[\phi]} \left[ \int d^3 x' \frac{\delta S(\phi, \delta \phi)}{\delta \phi(x)} \right] \]  

(77)

Defining \( \frac{\delta}{\delta \phi(x')} \phi(x') = \delta^3(x - x')/\sqrt{g} \), the commutation relation between \( \phi \) and \( -i \delta/\delta \phi \) is obviously canonical. The functional derivatives \( -i \delta/\delta \phi \) used in the bFP trick will then be operator representations of the generalized conjugate momentum

\[ \Pi_{\phi \phi}(x) = \int_{\partial V} d^3 x' \theta^S(\phi, g^{-1/2} \delta^3(x - x')). \]  

(78)

This expression was used in deriving (73), and will be explicitly used in the following section.

4. Large Gauge Transformations and Additional Constraints

Up to now we have assumed that both \( A_\mu \) and the gauge transformations on \( A_\mu \) vanish sufficiently fast at spatial infinity that one can freely integrate by parts any expression with spatial derivatives. Energy-flux finiteness arguments lead one to expect the field strength \( F_{\mu \nu} \) to obey such asymptotic fall-off conditions, at least in many physical situations. However, it is not clear why either \( A_\mu \), or gauge transformations of \( A_\mu \), should vanish at infinity.

Gauge transformations which don’t fall off as quickly as required for the above manipulations are referred to as large gauge transformations. These have a long history, especially in gravity \[68\], and have been extensively discussed in recent years \[44, 69, 70\]. Many different choices of asymptotic fall-off conditions for \( A_\mu \) have been made in the literature.

Invariance under the set of large gauge transformations implies a further set of constraints, in addition to Gauss’ law and \( \hat{E}^0 \Psi = i \frac{\delta}{\delta A_0} \Psi = 0 \). In this section we enlarge the set of allowed gauge transformations to those which are finite and non-vanishing at the spatial boundary, and generalize the techniques used above to handle these. The invariant propagators then shed light on the constraints implied by large gauge invariance; and the path integral gives explicit solutions to the operator constraint equations.

We will treat the spatial boundary as a large sphere or cylinder of radius \( R \to \infty \), and we allow for gauge transformations which have finite asymptotic limits, viz.,

\[ \lambda(t, x^A) = \lim_{r \to R} \Lambda(t, r, x^A). \]  

(79)

With finite asymptotic limits for \( \Lambda \), we must also allow for finite asymptotic limits for the gauge field, viz.,

\[ a_\mu(t, x^A) = \lim_{r \to R} A_\mu(t, r, x^A). \]  

(80)

In what follows, we warm up by discussing, in section 4.1, large gauge transformations for propagation between time slices; we then proceed in section 4.2 to the causal diamond.
4.1. Large Gauge Transformations: Time Slicing

We would like to compute the propagator

$$K(A_\mu \partial_V) = \int_{A_{\mu \partial V}} D A_\mu e^{iS}, \quad (81)$$

where the region $\mathcal{V}$ over which we integrate is again part of Minkowski space, bounded by the constant $t$ slices $\Sigma_i, \Sigma_f$ and the large cylinder of radius $R \to \infty, \Sigma_\infty$, and the action is just (8). Again, for brevity we assume that the source is an external conserved current, but as was the case in the first section, the following manipulations easily generalize to dynamic matter fields. As just discussed, while we fix boundary data $A_\mu \partial_V$ on all of $\partial \mathcal{V}$, we now lift the restriction that $A_\mu$ vanishes at spatial infinity.

At the technical level, the new challenge is that we can no longer uniquely invert the Laplacian operator when solving the Gauss law equation as in eqn. (A.15); there is now nothing restricting the homogeneous solutions.

To proceed with the integral we need to again use the boundary Faddeev-Popov trick, in the form (25). Suppose now that one tries to fix a Coulomb gauge in the FP path-integral, i.e., write $\mathcal{G}(A) = \partial^j A_j$. However in the enlarged gauge group this choice will leave the gauge under-determined, because there are homogeneous solutions, $\nabla^2 \Lambda = 0$ which are non-vanishing at spatial infinity.

If however we restrict ourselves to gauge functions which are finite at spatial infinity, then the only remaining homogeneous solution is $\Lambda(x) = c(t)$. The only residual gauge transformations in the FP integral (25) are then time dependent global $U(1)$ rotations. These leave the spatial components $A_j$ invariant, and only shift the spatially constant part of $A_0$. To properly implement the bFP trick we then must supplement the Coulomb gauge fixing delta function with another delta function which eliminates these residual transformations.

A sufficient choice is to gauge fix the $l = 0$ spherical harmonic mode of the asymptotic gauge function $\lambda(t, x^A)$. We refer to the $l = 0$ part of a function on the sphere using a superscript “(0)”.

Up to field-independent normalization we may then write

$$1 = \int D \Lambda \delta(\partial^j A_j^\Lambda) \delta(a_0^{(0)} \Lambda), \quad (82)$$

in place of (25).

In what follows it is more clear if we explicitly separate out the asymptotic $l = 0$ part of all functions. The notation may seem heavier than necessary but it will allow for a much quicker generalization to the later treatment of the causal diamond amplitude. We will therefore write,

$$\Lambda(t, r, x^A) = \bar{\Lambda}(t, r, x^A) + \lambda^{(0)}(t), \quad (83)$$

where $\bar{\Lambda}$ has a finite asymptotic limit $\bar{\lambda}(t, x^A) = \lim_{r \to \infty} \bar{\Lambda}(t, r, x^A)$, but the function $\bar{\lambda}(t, x^A)$ has a vanishing $l = 0$ mode. We’ll use this same notation for the gauge field, in terms of which the action is simply

$$S[A] = S[A_j, \bar{A}_0] + \int_{t_i}^{t_f} dt a_0^{(0)} Q, \quad (84)$$
where \( Q = \int d^3 x J^0 \) is the total charge.

With this, we can now multiply the propagator \((81)\) by a carefully chosen factor of 1, from \((82)\), to obtain

\[
K(A_{\partial V}) = \int \mathcal{D} \Lambda d\lambda^{(0)} \int_{A_{\mu \nu V}} \mathcal{D} \bar{A}_0 \mathcal{D} A_j \; \delta(\partial^j A_j^{\Lambda}) \; \delta(a_0^{(0)}) \; e^{iS[A_j, \bar{A}_0] + i \int_{-\infty}^{0} dt a_0^{(0)}Q}. \quad (85)
\]

Now, we implement the bFP trick by changing variables, as done before (cf. eqns. \((26)-(28)\), and \((A.29)\)), to get

\[
K(A_{\partial V}) = \int \mathcal{D} \Lambda d\lambda^{(0)} \; \delta^{\partial V}(\partial^j A_j^{\Lambda} + \nabla^2 \bar{\Lambda}) \; \delta^{\partial V}(a_0^{(0)} + \partial_0 \lambda^{(0)}) \\
\times e^{-i f_{\partial V} \left[ \Lambda, j^{\rho} + \lambda^{(0)}, j^{\rho} + i \partial_0 \bar{\Lambda} \frac{j^{\rho}}{\pi_{\Lambda}^{(0)}} + i \partial_0 \lambda^{(0)} - \frac{j^{\rho}}{\pi_{\Lambda}^{(0)}} + i \partial_0 \bar{\Lambda} \frac{j^{\rho}}{\pi_{\Lambda}^{(0)}} \right]}
\times \int_{A_{\mu \nu V}} \mathcal{D} \bar{A}_0 \mathcal{D} A_j \; \delta(\partial^j A_j) \; \delta(a_0^{(0)}) \; e^{iS[A_j, \bar{A}_0] + i \int_{-\infty}^{0} dt a_0^{(0)}Q}
\]

\[
(86)
\]

In the bulk part of the path integral we have effectively inserted gauge fixing delta functions as desired. The additional gauge fixing delta function simply sets \( a_0^{(0)} = 0 \), reducing the action to its usual form. As always in the bFP trick, we’ve also obtained a number of delta functions and linear shift operators outside the path integral. The crucial observation here is that the delta functions constraining the boundary gauge transformations constrain only \( \bar{\Lambda} \) and \( \partial_0 \lambda^{(0)} \), they do not constrain the other independent functions \( \partial_0 \bar{\Lambda} \) and \( \lambda^0 \).

In factoring out the bulk gauge group integral we are then left with residual integrals over \( \partial_0 \bar{\Lambda} \) and \( \lambda^0 \). The remaining boundary integrals over \( \bar{\Lambda} \) and \( \partial_0 \lambda^{(0)} \) are trivially performed using the delta functions. The result is then

\[
K(A_{\partial V}) = \left( \int d\lambda^{(0)} e^{-i f_{\partial V} \lambda^{(0)}, j^{0}} \right) e^{i f_{\partial V} \nabla^2 A_j^{(0)}} \int_{A_{j \partial V}} \mathcal{D} \bar{A}_0 \mathcal{D} A_j \; \delta(\partial^j A_j) e^{iS[A_j, \bar{A}_0]},
\]

\[
(87)
\]

where \( A_j \) is the transverse component of \( A_j \). We can now perform the \( \bar{A}_0 \) integral and there is no ambiguity in its saddle point solution; it is again given by \( \bar{A}_0 = \nabla^{-2} J^0 \) and the homogeneous solution is necessarily zero because by definition \( A_0 \) has vanishing asymptotic \( l = 0 \) mode.

Note the remarkable feature, that the vestige of working on the configuration space for \( A^\mu \) with non-vanishing asymptotic limit is just the integral over \( \lambda^{(0)} \) on the boundary. Since \( \lambda^{(0)} \) was time-dependent but spatially constant function, these are independent integrals on each of the past and future timeslices of the charge density. This does nothing other than add a delta function enforcing charge neutrality on the boundary state. In hindsight it is completely obvious that if we demand the amplitude to be invariant under time-dependent global \( U(1) \) transformations, the state must be charge neutral - by enlarging the gauge group, we’ve simply imposed this new constraint.
In principle one could never determine whether the total charge on a spatial slice is zero, since for any region with non-zero charge there could be compensating charges arbitrarily far away, which would ensure the overall charge neutrality condition. This is always possible in $U(1)$ gauge theory, because the interaction becomes arbitrarily weak at large distance.

Rather than limiting ourselves to such a scenario with fictitious image charges as infinity, we could instead restrict the path-integral such that we consider only gauge fields which are time-independent on the spatial boundary. Classically this corresponds to enforcing the electric field to vanish as $r \to \infty$, and is thus a sensible restriction if one wants to consider quantum fluctuations about finite energy classical background configurations. In this case, the allowed gauge transformations have asymptotic limits which are simply time-independent functions on a sphere

$$\lambda(x^A) \equiv \lim_{r \to R} \Lambda(t, r, x^A).$$

(88)

Even with these time-independent asymptotic limits, there is still a residual gauge-fixing to be done in addition to specifying the Coulomb gauge. The difference now, compared with eq. (82), is that we only need to gauge fix the $l = 0$ spherical harmonic mode of the asymptotic gauge field on a single time-slice. The analogous calculations above are identical in form, so we will simply quote the final result below.

If gauge fields are allowed to reach finite, but constant-in-time, configurations at spatial infinity, then the only allowed large gauge transformation in the path-integral is just a single global $U(1)$ phase rotation. The resulting propagator is then

$$K(A_{0V}) = \left( \int d\lambda^{(0)} e^{-i\Delta Q} \right) e^{i\int_{0V} \nabla \cdot (\partial^j A_j) J^0} \int \mathcal{D}A_0 \mathcal{D}A_j \delta(\partial^j A_j) e^{iS[A_j, A_0]},$$

(89)

where

$$\Delta Q = \int_{0V} J^0 = \int_{\Sigma_f} d^3x \ J^0 - \int_{\Sigma_i} d^3x \ J^0.$$

(90)

This result is identical to the case eq. (11) where large gauge transformations were forbidden, except here total charge conservation is being explicitly enforced.

This can be summarized succinctly. Previously we had required invariance under global $U(1) \times U(1)$ transformations of our boundary data, where each the $U(1)$ acts independently on the past and future surfaces ($\Sigma_i, \Sigma_f$), and this then required the total charge on each of $\Sigma_i$ and $\Sigma_f$ to vanish. Upon restricting the allowed gauge field configurations to be time-independent on the spatial boundary $\Sigma_B$, we then only required invariance under the diagonal subgroup of global $U(1)$ rotations which act simultaneously on both $\Sigma_i$ and $\Sigma_f$, and consequently it was the difference between total charges on $\Sigma_f$ and $\Sigma_i$ which needed to vanish.

Note that in both scenarios considered above we did not need to eliminate all gauge functions which are finite asymptotically in order to fix the residual gauge freedom; it was only necessary to gauge fix that part of the gauge field which was constant on the sphere at spatial infinity. Gauge functions which approach $l \neq 0$ functions on the asymptotic sphere are still allowed; they simply do not affect time slice amplitudes. In the next subsection we
see that allowing such \( l \neq 0 \) gauge transformations actually has a nontrivial effect on the causal diamond amplitude.

4.2. Large Gauge Transformations: Causal Diamond Evolution

We would now like to consider the amplitude

\[
K(A_\mu \partial V) = \int_{A_\mu \partial V} \mathcal{D}A_\mu e^{iS},
\]

where, as before, \( S \) is given by the sourced Maxwell action and the integration region \( V \) is the causal diamond of radius \( R \to \infty \), but now we allow the gauge fields to be finite as \( r \to R \). The story is very similar to the treatment of large fields in the time slice propagator, but with an interesting additional feature.

Looking back to (54) and its solution (57), we can see that without the assumption that the gauge field vanishes as \( r \to \infty \), there are infinitely many possible homogeneous solutions. In the bulk, \( \eta \in (-R, R) \), the only acceptable homogeneous solution (64) is a time-varying \( h(\eta) \) which is constant in space.

4.2.1. Applying the bFP trick

The situation for these time-dependent global \( U(1) \) transformations is the same as considered above for \( A_0 \) in the time slice amplitude, and the same remedy applies. We must separate off the asymptotic \( l = 0 \) part, \( a^{(0)}_\eta \), and enforce an additional gauge fixing which sets \( a^{(0)}_\eta = 0 \) in the bulk. This will allow for a unique saddle point solution for the remaining field \( \bar{A}_\eta = A_\eta - a^{(0)}_\eta \). The upshot is the same as the previous case; properly treating this asymptotic \( l = 0 \) part will just introduce delta functions on the boundary which enforce overall charge neutrality \( \int_{\partial V} d^3x \sqrt{g} J^\eta = 0 \).

However there is another, more interesting, result in the causal diamond geometry. When we implement the bFP trick, we aim to introduce the Faddeev-Popov gauge fixing as in (82) above; but the integrand here still does not uniquely fix the gauge. This is because when \( \eta = \pm R \), i.e. on the boundary of the causal diamond, there are homogeneous solutions \( D^j \partial_j \Lambda = 0 \) which are arbitrary functions on the sphere. The above delta functions will uniquely determine the gauge function \( \Lambda \) in the bulk, but on each of the future and past portions of the boundary there is still a residual gauge freedom given by all functions \( \Lambda \) approaching a non-constant \( (l \neq 0) \) function on the sphere, \( \lambda^{(l \neq 0)}(x^A) \), as \( \eta \to \pm R \).

Such functions will be discontinuous at \( r = 0, \eta = \pm R \), but this is allowed since these singular points have been formally “blown up”, allowing for such angle dependent limits as \( r \to 0 \) on the boundary.

To uniquely fix the gauge we then append a further gauge fixing term on the boundary; we choose \( \nabla^B a^A_B = 0 \), where \( \nabla^B a_B \) is the vector divergence on the unit two-sphere. These choices together uniquely fix the gauge, so that we can write

\[
1 = \int \mathcal{D}\Lambda \delta(D^j A^A_j) \delta(a^{(0)}_\eta) \delta^{\partial V} (\nabla^B a^A_B),
\]
up to a field independent constant. To reduce the clutter in this and following equations, we have used the condensed notation $\delta^{\partial V}(\nabla^B a_B^A) = \delta^{\partial V^+}(\nabla^B (a_B^A))\delta^{\partial V^-}(\nabla^B (a_B^A))$, where we have introduced the notation $\partial V^\pm$ to refer to the future/past portions of the causal diamond boundary respectively. We remind the reader that

$$a_B^\pm(x^A) = \lim_{r \to R} A_B(\eta = \pm R, r, x^A)$$

is the value of the gauge field taken on the boundaries of the future/past boundaries respectively, denoted collectively by $\partial \partial V$ below.

Note that $\partial V$ is not connected because of the blow-up procedure, so $\partial \partial V$ is non-empty. Rather, these regions are approached in the $r \to \infty$ limit of the future null surface and the past null surface. These codimension-2 surfaces are effectively the past boundary of future null infinity $I^-$ and future boundary of past null infinity $I^+$ commonly discussed in the literature of this topic (see [44] and refs. therein).

Before proceeding, let us also emphasize that because the future and past portions of the boundary are not connected, it was necessary to fix this residual gauge freedom on each of the future and past surfaces separately.

We can now multiply eq. (92) into our path integral representation for the causal diamond propagator. Doing this and implementing again the bFP trick, we obtain

$$K(A_{\partial V}) = \int D\tilde{\Lambda} d\lambda(0) \delta^{\partial V}(D^j A_j + D^j \partial_j \tilde{\Lambda}) \delta^{\partial V}(a_0^0 + \partial_\eta \lambda(0)) \delta^{\partial V}(\nabla^B a_B + \nabla^B \nabla B \lambda) \times e^{-i \int_{\partial V} [\tilde{\Lambda} J^\eta + \lambda(0), J^\eta + i \partial_\eta \tilde{\Lambda} \frac{\partial^2}{\partial \eta^2} + i \partial_\eta (\frac{\partial^2}{\partial \eta^2})]}
 \times \int D\tilde{\eta} D\eta(0) \int D A_j \delta(D^j A_j) \delta(a_0^0) e^{iS[A_j, \tilde{\Lambda} \eta]} + i \int_{\partial \partial V} d\eta a_0^0 Q \quad (94)$$

The propagator (94) is structurally very similar to (86), except for the new factor we’ve introduced to gauge fix the residual transformations which are allowed on the causal diamond boundary. Again, we can freely evaluate the integral over $\partial_\eta \tilde{\Lambda}$ and $\lambda(0)$ in the boundary gauge transformations since they are not fixed by the boundary gauge fixing delta functions. We can evaluate the integrals over $\tilde{\Lambda}$ and $\partial_\eta \lambda(0)$ using the delta functions. The boundary delta functions now constrain $\tilde{\Lambda}$ to be

$$\tilde{\Lambda} = -\int_{\partial V} d^3x \sqrt{g} G D^j A_j - \int_{\partial \partial V} d^2\Omega \tilde{G} \nabla^B a_B$$

where, up to a minus sign, the first term is just $\Phi$ given in (72); $d^2\Omega$ is the area element on the unit two-sphere, and $\tilde{G}$ is the Green’s function for the Laplacian (less the $l = 0$ mode) on the unit two-sphere. Evaluating these integrals we obtain the final expression for the

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1 However, see footnote 2 and the discussion at the end of Sec. (4.2).
large gauge transformation invariant causal diamond amplitude

\[ K(A_{ov}) = \left( \int d\lambda^{(0)} e^{-i \int_{ov}^{\lambda^{(0)}} J} \right)^n e^{i \int_{ov} d^3x \sqrt{g} \left[ J_{ov} \frac{d}{d^3x} \sqrt{g} G D^j A_j + f_{ov} \frac{d^2}{d^3x} \sqrt{g} \bar{G} \nabla^B a_B \right]} J^n \]

\[ \times \int_{A_{ov}} \mathcal{D} \bar{A}_{\eta} \mathcal{D} A_j \delta(D^j A_j) e^{i S[A_j, A_{\eta}]} \]

(96)

Just as we found for the time-slice geometry, there is a \( \lambda^{(0)} \) integral which enforces total charge flux neutrality, \( Q_{\partial V^+} = Q_{\partial V^-} = 0 \),

\[ Q_{\partial V^\pm} = \int_{\partial V^\pm} J^n, \]

(97)

where \( \partial V^\pm \) denotes the future/past portion of the null boundary. As discussed at the end of Sec. (4.1), we could relax this condition down to total charge conservation \( Q_{\partial V^+} = Q_{\partial V^-} \neq 0 \) by requiring the \( l = 0 \) part of the gauge fields to be time-independent at spatial infinity. As discussed above, this global \( U(1) \) invariance is not necessarily interesting.

What is novel about the causal diamond amplitude eq. (96) is that in addition to the total charge flux constraint, we’ve found that invariance under large gauge transformations with higher spherical harmonics enforces a new constraint on the system. Since \( \bar{G} \) and \( a_B \) are just angular functions, the new boundary phase in (96) implies that a certain part of the electric field at each angle is determined solely by the net flux of charge through the boundary at each angle. To see this explicitly, we define

\[ \frac{\delta}{\delta a^\sigma_A(x^A)} a^\sigma_A(x^A) = \delta^\sigma_B q^{-1/2} \delta^2(x^A - x'^A), \]

(98)

where \( \sigma = \pm \) indexes the whether the function lives on the future/past portion of the boundary, and where \( q \) is the determinant of the metric on the unit two-sphere. Using this we can see that the gauge invariant amplitude satisfies

\[ -i \frac{\delta}{\delta a_{\eta B}(x^A)} K(A_{ov}) = \left( \int_{\partial V^\sigma} d^3x' \sqrt{g} J^n(x') \nabla^B \bar{G}(x^A, x'^A) \right) K(A_{ov}), \]

(99)

on the each of the future/part portions of the causal diamond boundary.

It remains to understand what, physically, this functional differential operator represents. We can do so by using the relationship between functional derivatives and the symplectic current density in [77]. For the gauge field \( A_B \) we have

\[ \theta^n(\mu_B, \delta A_B) = \frac{\partial L}{\partial \nabla_{\eta} A_{\eta}} \delta A_B = -\sqrt{g} F^n_{\eta B} \delta A_B, \]

(100)

and if we separate the field as \( A_B = \bar{A}_B + a_B \), where \( a_B \) is independent of \( r \) and \( \bar{A}_B \) is vanishing at spatial infinity, then by linearity we have

\[ \theta^n(\mu_B, \delta a_B) = \frac{\partial L}{\partial \nabla_{\eta} A_{\eta}} \delta a_B = -\sqrt{g} F^n_{\eta B} \delta a_B. \]

(101)
Invoking (77) we then find

\[-i \frac{\delta}{\delta a_B(x^A)} K(A_{\partial V}) = \int_{A_{\partial V}} \mathcal{D}A_\mu \epsilon^{iS[A]} \left( \int_{\partial V^+} d^3x' \sqrt{g} F_{B\eta} q^{-1/2} \delta^2(x^A - x^A') \right) = \int_{A_{\partial V}} \mathcal{D}A_\mu \epsilon^{iS[A]} \left( \int_0^\infty dr r^2 F_{B\eta}(r, x^A) \right) \right) \right). \tag{102} \]

The bFP trick has then illustrated that physical (gauge invariant) states on the boundary of the large causal diamond satisfy the eigenvalue equation

\[
\left( \int_0^\infty dr r^2 F_{B\eta}(r, x^A) \right) K(A_{\partial V}) = \left( \int_{\partial V^+} d^3x' \sqrt{g} J^B(x') \nabla B G(x^A, x^A') \right) K(A_{\partial V}), \tag{103} \]

at every angle \(x^A\) on the sphere, independently on each of the future and past parts of the boundary. This is an exact relation, irrespective of the data specified for the fields or the dynamics of the charged matter, i.e. it is kinematically required. It is a direct consequence of gauge invariance for the causal diamond path-integral on the extended configuration space when the gauge fields are allowed to take finite values at spatial infinity.

This result bears a clear resemblance to results at null infinity which have been widely discussed in the literature \[44, 45, 70\]. Indeed, since (103) holds at each angle, we can multiply it by \(\partial_{\epsilon}(x^A)\), for any function on the sphere \(\epsilon(x^A)\), and integrate over the sphere. We then obtain

\[
\int_{\partial V^+} d^3x' \sqrt{q} \left( (q^{AB} \nabla_A \epsilon(x^A)) \hat{F}_{Br} + \epsilon(x^A) (r^2 J_r) \right) K(A_{\partial V}) = 0. \tag{104} \]

If we go to complex stereographic coordinates \((z, \bar{z})\) on the unit sphere such that the metric is

\[
d\Omega^2 = 2\gamma_{zz} dzd\bar{z}, \tag{105} \]

with

\[
\gamma_{zz} = \frac{2}{(1 + z\bar{z})^2}, \tag{106} \]

we obtain the operator constraint equation

\[
\int_{\partial V^+} drd^2z \left( - \partial_z \epsilon(z, \bar{z}) \hat{F}_{zr} - \partial_{\bar{z}} \epsilon(z, \bar{z}) \hat{F}_{\bar{z}r} + \epsilon(z, \bar{z}) \gamma_{zz} (r^2 J_r) \right) K(A_{\partial V}) = 0. \tag{107} \]

If we recall that, in our coordinates, \(r\) is the affine parameter on the null boundary, then we immediately recognize the operator above, acting on \(K(A_{\partial V})\), as the large gauge charge operator \(\hat{Q}_\epsilon\) discussed in the recent literature (compare ref. \[44\], section 2.5.11, and refs. therein). Thus, writing this constraint equation as

\[
\hat{Q}_\epsilon^\pm K(A_{\partial V}) = 0, \tag{108} \]

we can say that (i) the electric part of this operator is the zero-frequency part of the leading \(\mathcal{O}(r^0)\) transverse electric field at infinity, and it thus creates soft photon states; and (ii) the
matter term is the total flux through the null surface of the leading $O(r^{-2})$ term in the
matter current at infinity.

When the matter is quantum mechanical the computation can be carried though with no
additional complications, and the result is to simply replace $J_r$ by a functional differential
operator representation of the $U(1)$ current operator $\hat{J}_r$. Thus, the amplitude we’ve derived
is annihilated by the large gauge asymptotic charges $\hat{Q}_\epsilon^\pm$, for all functions $\epsilon(z, \bar{z})$ on both
the future and past surface.

From our analysis we can see that since we’ve blown up spatial infinity, thereby allowing
the value of the gauge field at spatial infinity to be different depending on whether
approached from the future or past part of the boundary, the amplitude satisfies \[107\] on the past and future null boundaries separately. As a consequence, the states on the past
and future parts are necessarily dressed by soft photons in the way described originally
by Kibble, Chung, and Faddeev and Kulish \[74–76\] in such a way as to render scattering
amplitudes infrared-finite. This is clear from the form of eq. \[107\], which demonstrates
that a non-zero flux of charge through either of the null surfaces is necessarily accompanied
by infinite wavelength electric field excitations. The form of the dressing can also be seen
explicitly in eq. \[96\], where the exponentiated positive (negative) frequency part of the
gauge field on the past (future) null surface explicitly describes a coherent state sourced by
the total charge flux through each angle.

Let us relate this result further to the commonly used notation. To be concrete, we
consider just the future part of the boundary, and confine ourselves to massless charged
particles. In comparing our coordinates with the typical null coordinates it is clear that our
$r$ is simply (up to an irrelevant constant shift), equal to $-u$, where $u$ is the retarded time.
In terms of the retarded time and stereographic angular coordinates, eq. \[103\] is written
(now with all lowered indices) as
\[
\left( \int_{-\infty}^{\infty} du \hat{F}_{zu}^{(0)}(u, z, \bar{z}) - \int_{-\infty}^{\infty} du \int d^2 \omega \gamma_{\omega \bar{\omega}} J_u^{(2)}(u, \omega, \bar{\omega})\partial_z \bar{G}(z, \bar{z}, \omega, \bar{\omega}) \right) K(A_{\partial Y}) = 0, \quad (109)
\]
where we now use the superscripts $(n)$ to denote the leading $O(r^{-n})$ part of the operator.
The charge density describing a collection of particles, each with charge $Q_k$, reaching the
future null surface at $(u_k, z_k, \bar{z}_k)$, is given by
\[
J_u^{(2)}(u, z, \bar{z}) = \sum_{k=1}^{m} Q_k \delta(u - u_k) \gamma^{zz} \delta^{(2)}(z - z_k). \quad (110)
\]
The last element needed to complete the translation is the expression for Green’s function
on the two-sphere, viz.,
\[
\bar{G}(z, \bar{z}, \omega) = \frac{1}{2\pi} \ln |z - \omega|^2. \quad (111)
\]
Inserting eqs. \[110, 111\] into eq. \[109\] we then obtain
\[
\left( \int_{-\infty}^{\infty} du \hat{F}_{uz}^{(0)}(u, z, \bar{z}) + \sum_{k \in \text{out}}^{m} \frac{Q_k}{2\pi} \frac{1}{z - z_k} \right) K(A_{\partial Y}) = 0. \quad (112)
\]
33
A completely analogous equation also holds on the past boundary.

The first term in eq. (112) is often referred to as the soft-photon mode, \( N_\varepsilon^+ \), while the second term is referred to as the soft-factor, \( \Omega^{soft\varepsilon} \) (for example, see [44, 77, 78] and refs. therein). We can then express our result concisely as

\[
(N_\varepsilon^+ + \Omega^{soft\varepsilon})K(A_{\partial V}) = (N_\varepsilon^- + \Omega^{soft\varepsilon})K(A_{\partial V}) = 0,
\]

which is simply a restatement of eq. (108) for the particular choice \( \varepsilon = (z - \omega)^{-1} \). We have thus demonstrated that the physical states described by the propagator (96) have strictly zero charge under large gauge transformations. In a sense this result is unsurprising, given that we have started from an expression (91) which was manifestly invariant under all gauge transformations. However the main result, eq. (96), is interesting in that it explicitly demonstrates how the soft-photon dressing emerges naturally as a consequence of gauge invariance.

The relationship between the Faddeev-Kulish dressing factor and large gauge transformations has been previously been expressed clearly in the context of gravitational scattering by Choi and Akhoury [79]. They have demonstrated that scattering amplitudes between states with non-zero, but conserved, asymptotic charge are the Faddeev-Kulish dressed states. Additionally, they illustrated that there is considerable freedom in defining such states, allowing one to shuffle soft-photon contributions between the future and past states. Our result in eqs. (96, 113) is consistent with this statement; however it is more restrictive. The asymptotic charge is indeed conserved during the evolution; however we’ve also found that it is necessarily zero, and there is no freedom in shuffling around the soft-photon parts—both initial and final states of charged particles must have accompanying soft-photon dressing.

Let us understand why we have arrived at this more stringent condition. The language parallels the discussion at the end of Sec. (4.1). In the above calculations the asymptotic part of the gauge field, \( a_B \), was allowed to be different on the past and future surfaces, and as a result so too were the large gauge transformations. The amplitude was thus invariant under the group \( U(1)_{z\bar{z}} \times U(1)_{z\bar{z}} \), of angle dependent \( U(1) \) transformations which act independently on the past and future surfaces, and thus the initial and final states were independently dressed.

This should be contrasted with the conjectured antipodal “matching condition” between fields at \( I^+ \) and \( I^- \), which would imply that the asymptotic gauge symmetry is only the diagonal subgroup \( U(1)_{z\bar{z}}^0 \) which acts simultaneously on both future and past [80, 77, 80]. The Ward identity associated with this symmetry was demonstrated to be equivalent to the soft photon theorem [14, 77, 81], providing further evidence that the matching condition is correct. Furthermore, the conformal embedding of Minkowski spacetime in the Einstein static cylinder is smooth at spatial infinity \( i^0 \), and provides a natural antipodal relationship between the null generators of \( I^+ \) and \( I^- \). In flat spacetime it is then natural to assume the matching condition. However for more general spacetimes the requisite smoothness at \( i^0 \) does not hold, and the matching problem has long been an open question [71, 72]. For this reason, to allow for a generalization to gravitational theories in future work, we have explicitly blown up the region analogous to \( i^0 \) in this paper, and enforced no matching
4.2.2. Enforcing the Antipodal Matching Condition

Let us now complete the discussion by assuming that the antipodal matching condition holds, and repeat the calculation. We then impose

\[ a_B^+(z, \bar{z}) = a_B^-(z, \bar{z}), \]  

(114)

where we’ve used the antipodal mapping \( z \rightarrow -\frac{1}{\bar{z}} \) between the coordinates on the future surface relative to those on the past [44]. There are two main technical differences in the calculation, viz.:

- when modifying the gauge fixing procedure (92) we now need only to fix

\[ \nabla^B a_B^+(z, \bar{z}) = 0, \]  

(115)

- the functional derivative of the symplectic current density in eq. (77) now has support on both parts of the boundary, causing the conjugate momentum to \( a_B \) to be the difference between photon zero modes,

\[ -i \frac{\delta}{\delta a_B(x^A)} \left[ \int_0^\infty dr r^2 F^{B\eta}(r, x^A) \right]_{\partial V^+} - \left[ \int_0^\infty dr r^2 F^{B\eta}(r, x^A) \right]_{\partial V^-}. \]  

(116)

The remaining steps in the calculation are identical, and we arrive at the final expression analogous to eq. (96)

\[ K(A_{\partial V}) = \delta \left( \int_{\partial V} J^\eta \right) \exp \left[ -\frac{i}{2\pi} \int d^2 z \, a_z(z, \bar{z}) \int_{\partial V} d\lambda \, \frac{J^{(2)}_A(\lambda, \omega, \bar{\omega})}{z - \omega} - h.c. \right] e^{i\int_{\partial V} d^2 z \sqrt{g} J^\eta \bar{\eta} G^D A_j} \times \int_{A_j, \partial V} \mathcal{D} A_j \mathcal{D} A_j \delta(D^j A_j) e^{iS[A_j, \tilde{A}_j]}. \]  

(117)

where \( \lambda \) is the affine parameter along each null surface. We can now explicitly see that the dressing involves the difference between charge fluxes through each surface. Furthermore, by functionally differentiating with respect to \( a_z \) we can obtain the analogous expression to eqs. (109, 112), ie.,

\[ (N^+_z - N^-_{\bar{z}}) K(A_{\partial V}) = (\Omega^{soft-} - \Omega^{soft+}) K(A_{\partial V}). \]  

(118)

These two terms need not vanish, and so we have recovered the asymptotic charge conservation law of ref. [77], as we expected.

\[ \]  

2During peer review we became aware of several recent works [73] which prove that the conjectured matching condition will hold, for asymptotic symmetry charges, in a rigorously defined general class of asymptotically flat spacetimes. This then suggests that one should gauge fix \( \nabla^B a_B \) on only one of the future and past surfaces.
5. Conclusions

Let us begin here by summarizing what has been done, and then return to the questions asked in the introduction.

This paper has defined a manifestly gauge invariant analysis of QED amplitudes and quantum states for flat space QED. We use the modern understanding of states as data living on the closed hypersurface boundaries of path-integrals, in a ‘general boundary QFT framework’, in which the path-integral allows us to go beyond canonical quantization. The interpretation in terms of states and transition amplitudes is then secondary, and only applies to particular geometries.

The bFP trick discussed here should be applicable for general boundaries. In the 2 cases considered here, we found

(a) when \( \partial V \) consists of two finitely separated constant time slices and a time-like cylinder at spatial infinity, we can interpret \( K_{fi} \) as a conventional transition amplitude which also has a representation in terms of a Hamiltonian operator;

(b) when \( \partial V \) is the null boundary of a large causal diamond, \( K_{fi} \) is most naturally described as the path-integral. The causal diamond boundary then resembles null infinity, and the amplitude resembles a scattering amplitude.

As a consequence of the gauge invariance of the QED action, these amplitudes were gauge-invariant and independent of non-canonical variables, and were written explicitly in terms of gauge invariant variables. One obtains unique expressions for the dependence of the amplitudes on the gauge-variant parts of \( A_\mu \); this dependence arose only as a boundary phase.

Thus, rather than solving the constraint equation (which under-determines the state), we found that the path integral yields unique expressions for the boundary phases for which the constraint equation is also satisfied. The dependence of \( K_{fi} \) on the gauge-variant parts of the field was determined kinematically, whereas the dependence on the gauge-invariant parts of the field was found dynamically, by a path integral over gauge invariant variables.

In cases when gauge transformations vanished at spatial infinity one inevitably finds a universal Coulombic dressing of the charges in the boundary state, as well as some non-universal, dynamically generated, transverse contributions to the dressing. However when the gauge group was extended, the causal diamond amplitude boundary states were annihilated by the “large-gauge charge” discussed previously in the literature on null infinity [44, 69]. One can explicitly enforce an asymptotic matching condition at spatial infinity and demonstrate that this zero-charge condition weakens to the asymptotic charge conservation laws associated with the soft photon theorem. From the large causal diamond path integral an explicit expression emerges very naturally for the soft-photon dressing of states on the null boundary. The resulting expressions were not novel, but the bFP technique used to get them was, and it provided a manifestly gauge invariant derivation of the result.

Let us now turn to the questions posed in the introduction. From the foregoing discussion, we can see that a fairly complete characterization has been given here of the states, their dressing, the physical degrees of freedom, and how all these depend on the boundaries (questions (a)-(c) of the introduction).
Let us then turn to question (d), regarding the larger enterprise of quantum gravity. Here we can only indicate which direction we believe should be taken - to really deal with this, one needs to generalize the work here to much more general spacetime structures. For simple spacetimes, one can certainly extend the class of allowed gauge transformations beyond those considered here - gauge invariance would then imply new angle dependent constraints on our asymptotic states, perhaps related to the sub-leading soft photon theorems.

We note however that the methods developed here will allow discussion of a key question in current debates about quantum gravity, viz., whether the metric field really needs to be quantized at all, and how one might test this question experimentally, in, eg., “BMV” experiments [46–49]. The answer to this turns essentially on how one defines physical states for the metric field. The generalization of our methods to linearized gravity - which is all that is necessary to deal with this problem - is straightforward, and will also be discussed elsewhere.

It is also of interest to try and define states using path integrals in more general theories of quantum gravity. An example is provided by the CWL theory [18–20], in which QM breaks down because of gravitational correlations between paths. Very recently we have shown how both standard 2-path superposition experiments and “4-path” experiments of the BMV type can be analyzed [82] in CWL theory - this theory in fact lends itself very naturally to the boundary quantum field theory framework which underpins the analysis of the present paper.

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Appendix A. Gauge Transformations on $K_{fi}$

Here we give some of the formal manipulations involved in deriving the key expressions in section 2, for the gauge-invariant propagator $K_{fi}$ defined between 2 time slices in flat spacetime. This is done first for scalar electrodynamics, as defined by the action given in eqns. , and then for spinor QED, defined by the action in eqn.
Appendix A.1. Scalar Electrodynamics

In the main text we began from eqn. (10) for the combined matter/EM field propagator for scalar electrodynamics, which we repeat here;

\[ K_{fi} \equiv K(q_f, A_{\mu f}; q_i, A_{\mu i}) \]

\[ = \int_{q_i}^{q_f} Dq e^{iS_{SM}} \int_{A_{\mu i}}^{A_{\mu f}} DA_{\mu} e^{iS_{EM}}. \]  

(A.1)

which we argued could also be written in the form of eqn. (11), which we also repeat here:

\[ K_{fi} = e^{i\mathcal{S}_C} \int_{q_i}^{q_f} Dq e^{iS_{SM}} \int_{A_{\mu i}}^{A_{\mu f}} DA_{\mu} e^{i\mathcal{S}_A}. \]  

(A.2)

In what follows we will first demonstrate explicitly the gauge invariance of (A.1), and then give details of the derivation of (A.2), using the two different methods described in section 2.A.

Appendix A.1.1. Gauge Invariant Propagator

In the main text we argued that eqn. (10) for the combined propagator was manifestly gauge invariant. Here we amplify on this assertion, and demonstrate it explicitly. Under gauge transformation eqn. became (recall eqn. (17)):

\[ K^{\Lambda}_{fi} \equiv K^{\Lambda}(q_f, A_{\mu f}; q_i, A_{\mu i}) \]

\[ = e^{-ie\Lambda_f(q_f)} K(q_f, A^{\Lambda_f}_{\mu f}; q_i, A^{\Lambda_i}_{\mu i})e^{ie\Lambda_i(q_i)} \]  

(A.3)

Now this propagator, with transformed boundary data, can be expressed simply in terms of the original propagator. We take the expression

\[ K(q_f, A^{\Lambda_f}_{\mu f}; q_i, A^{\Lambda_i}_{\mu i}) = \int_{q_i}^{q_f} Dq e^{iS_{SM}} \int_{A^{\Lambda_i}_{\mu i}}^{A^{\Lambda_f}_{\mu f}} DA_{\mu} e^{iS_{EM}} \]  

(A.4)

and perform a change of variables, \( A_{\mu} = A'_{\mu} + \partial_{\mu}\Lambda \), for some time-dependent function \( \Lambda \) which takes the value \( \Lambda_{i,f} \) on \( \Sigma_{i,f} \). The boundary data for the new variable \( A'_{\mu} \) is now just the original configuration, \( A_{\mu i,f} \). The action is not invariant under this change of variables, but instead, since it is effectively just a gauge transformation, we know that the action changes by a simple boundary term. The action is expressed in terms of \( A'_{\mu} \) as

\[ S_{EM}[q, A] = S_{EM}[q, A'] + e\Lambda_f(q_f) - e\Lambda_i(q_i), \]

(A.5)

so that the propagator with transformed boundary data then reads

\[ K(q_f, A^{\Lambda_f}_{\mu f}; q_i, A^{\Lambda_i}_{\mu i}) = \int_{q_i}^{q_f} Dq e^{iS_{SM}[q]} \int_{A^{\Lambda_i}_{\mu i}}^{A^{\Lambda_f}_{\mu f}} DA'_{\mu} e^{iS_{EM}[q,A'] + e\Lambda_f(q_f) - e\Lambda_i(q_i)} \]

\[ = e^{ie\Lambda_f(q_f)} K_{fi} e^{-ie\Lambda_i(q_i)}. \]  

(A.6)

The boundary phases in (A.6) generated by the gauge-field action will then precisely cancel the phases in (A.3) arising from the \( U(1) \) transformation of the matter states, and therefore the propagator for the total system is gauge invariant.
Appendix A.1.2. Constraint Equation

Here we describe the derivation of eqn. 18 from eqn. 17 in the main text. To do this, consider a gauge transformation which vanishes on \( \Sigma_i \) but not on \( \Sigma_f \), and rewrite the transformed propagator (17) using a linear shift operator as

\[
K^\Lambda_{fi} = e^{-ie\Lambda_f(q_f)+\frac{\Lambda_f}{\nabla_\mu \Lambda_f}} K_{fi} \tag{A.7}
\]

Since the propagator is gauge invariant, this implies the following simple functional differential equation

\[
0 = \left[ -ie\Lambda_f(q_f) + \int_{\Sigma_f} d^3 x \partial_\mu \Lambda_f \frac{\delta}{\delta A_{\mu f}} \right] K_{fi} \\
= \left[ \int_{\Sigma_f} d^3 x \partial_0 \Lambda_f \frac{\delta}{\delta A_{0 f}} - i \int_{\Sigma_f} d^3 x \Lambda_f \left( e\delta^3(q_f - x) - i\partial_j \frac{\delta}{\delta A_{j f}} \right) \right] K_{fi} \tag{A.8}
\]

The remaining functional derivative of the propagator with respect to \( A_{j f} \) is just the electric field operator. This is seen in a path integral treatment by evaluating the functional derivative and using the standard expression for variations of the action endpoint in mechanics, viz.,

\[
\frac{\delta S[x]}{\delta x_f} = \frac{\partial \mathcal{L}}{\partial \dot{x}(t)} \bigg|_{t_f} . \tag{A.9}
\]

Written in terms of the electric field operator we then get the constraint equation

\[
0 = \left[ \int_{\Sigma_f} d^3 x \partial_0 \Lambda_f \frac{\delta}{\delta A_{0 f}} \right. \\
\left. - i \int_{\Sigma_f} d^3 x \Lambda_f \left( e\delta^3(q_f - x) - \partial_j \hat{E}_j \right) \right] K_{fi} \tag{A.10}
\]

which is just eqn. 18 of the main text.

Appendix A.2. Extracting the Dressing

Here we give details of (i) the integration over \( A_0 \) appearing in eqn. 21 of the main text, and (ii) the derivation of the final form (11) for the scalar QED propagator \( K_{fi} \), along with the transformed form for the effective action which appears in it (ie., eqn. 12, containing the three terms 13–15).

Appendix A.2.1. Integration over \( A_0 \)

We recall eqn. 21 of the main text for the propagator \( K_{fi} \), which we repeat here:

\[
K_{fi} = \int_{q_i}^{q_f} \mathcal{D}q e^{iSM} \int \mathcal{D}A_0 \int_{A_{ji}}^{A_{j f}} \mathcal{D}A_j e^{iS_{EM}} . \tag{A.11}
\]
We now start from (A.11), and evaluate the $A_0$ integral. No gauge-fixing is required to make this integral convergent, and the boundary data is unfixed, so we can directly go ahead and do the integration. As we will see, this rather uniquely determines how we should gauge fix the remaining $A_j$ integral, and consequentially determines the form of the dressing for the states.

We first separate out the $A_0$ dependent terms in the electromagnetic part of the action:

$$S_{EM} = \int_{t_i}^{t_f} d^4x \left[ -\frac{1}{4} F_{jk} F^{jk} + A_j J^j - \frac{1}{2} F_{j0} (\partial_j A_0 - \partial_0 A_j) + A_0 J^0 \right]$$

Integrating the spatial derivatives by parts we obtain

$$S_{EM} = \int_{t_i}^{t_f} d^4x \left[ -\frac{1}{4} F_{jk} F^{jk} + A_j J^j + \frac{1}{2} F_{j0} (\partial_0 A_j + A_0 (\partial_j J^0 + J^0)) + \frac{1}{2} A_0 J^0 \right]$$

(A.12)

The variable $A_0$ appears quadratically in the action, and since its endpoints are being integrated over in (A.11), it can be integrated out as a simple Gaussian integral. The result of course is to just substitute the saddle point solution $\tilde{A}_0$ back into (A.13).

The saddle point equation for $A_0$ is just the Gauss law Maxwell equation, viz.,

$$(\partial_j F^{j0} + J^0) = 0 \quad \Rightarrow \quad \partial_j \partial^j A_0 + \partial_0 \partial^j A_j + J^0 = 0$$

(A.14)

for which the solution is

$$\tilde{A}_0 = \nabla^{-2} J^0 + g + h,$$

(A.15)

where $g$ is given by

$$g = \partial_0 \nabla^{-2} (\partial^j A_j),$$

(A.16)

and where $h$ is an undetermined homogeneous solution to the Laplace equation. The only such solution which is both regular at the origin and vanishes at spatial infinity is the trivial solution, $h(x) = 0$, so we set $h = 0$.

The solution (A.15) is then unique, without needing to impose further gauge fixing to eliminate the homogeneous solutions (if however we allow for large gauge transformations, then non-trivial expressions for $h(x)$ arise, and further gauge fixing is required - see section 4 of the main text).

Notice that $\tilde{A}_0$ is given in terms of a gauge invariant term $\nabla^{-2} J^0$ and a gauge variant term $g$. Under gauge transformation $g$ transforms as $\delta_A g = \partial_0 \Lambda$, as it must, so that $\tilde{A}_0$ transforms appropriately.

Taking inspiration from this, we formally isolate the gauge invariant part of the components $A_j$ by defining

$$A_j = A_j + \partial_j \Phi,$$

(A.17)
where $\delta_{A_0} = 0$, and $\Phi$ is a functional of $A_j$ with the assumed transformation property $\delta_{\Lambda} \Phi = \Lambda$. The functions $(A_j, \Phi)$ are just a new choice of field variables for the path-integration. To avoid introducing a field-dependent Jacobian into the integration measure, we will assume the $g$-potential $\Phi$ to be a linear functional of the $A_j$. Note that $\Phi$ is certainly not given uniquely by the required transformation property: for now we leave it unspecified.

At this point one might assume that $A_j$ and $\partial_j \Phi$ are just the transverse and longitudinal parts of $A_j$. This is of course a valid decomposition, but not a unique one. We will instead rewrite the path-integral in terms of the new variables $A_j$ and $\partial_j \Phi$, and look for a natural decomposition of the path integral. We will see that the action, transformed to the new variables, ends up separating into a non-dynamical boundary term, plus terms which are uniquely associated with the dynamical matter field and the new field variables $A_j(x)$.

### Appendix A.2.2. New Variables for the Action and Propagator

We begin by writing the propagator $K_{fi}$ in terms of $A_j$ in (A.17) and the solution (A.15) for $\tilde{A}_0$, to get

$$K_{fi} = \int_{q_{t_i}}^{q_{t_f}} Dq \ e^{iS_{EM}} \int_{\Phi_{t_i}}^{\Phi_{t_f}} D\Phi \int_{A_{j_{t_i}}}^{A_{j_{t_f}}} D A_j \ e^{i\tilde{S}_{EM}},$$  
(A.18)

with a new electromagnetic field action $\tilde{S}_{EM}$ given by

$$\tilde{S}_{EM} = \int_{t_i}^{t_f} d^4x \left[ -\frac{1}{4} F_{j k} F^{j k} + A_j J^j + \partial_j \Phi J^j + \frac{1}{2} \tilde{F}^{j 0} \partial_0 A_j + \frac{1}{2} \tilde{F}^{j 0} \partial_0 \partial_j \Phi + \frac{1}{2} J^0 \nabla^{-2} J^0 + \frac{1}{2} g J^0 \right],$$  
(A.19)

where we’ve introduced the notation $\tilde{F}^{j 0} = \partial_j \tilde{A}_0 - \partial_0 \partial_j \Phi$. Note that $F_{j k}$ is independent of $\Phi$ by antisymmetry.

We can now integrate by parts to strip the spatial derivatives off $\Phi$, to get

$$\tilde{S}_{EM} = \int_{t_i}^{t_f} d^4x \left[ \frac{1}{2} \tilde{F}^{j 0} \partial_0 A_j - \frac{1}{4} F_{j k} F^{j k} + A_j J^j + \frac{1}{2} J^0 \nabla^{-2} J^0 - \Phi \partial_j J^j - \frac{1}{2} \partial_j \tilde{F}^{j 0} \partial_0 \Phi + \frac{1}{2} g J^0 \right].$$  
(A.20)

We then use the definition $\partial_j \tilde{F}^{j 0} = -J^0$ along with the fact that $\partial_{\mu} J^\mu$ for an arbitrary trajectory of the particle, to rewrite the action as

$$\tilde{S}_{EM} = \int_{t_i}^{t_f} d^4x \left[ \frac{1}{2} \tilde{F}^{j 0} \partial_0 A_j - \frac{1}{4} F_{j k} F^{j k} + A_j J^j + \frac{1}{2} J^0 \nabla^{-2} J^0 + \Phi \partial_0 J^0 + \frac{1}{2} J^0 \partial_0 \Phi + \frac{1}{2} g J^0 \right].$$  
(A.21)

This result reveals something remarkable—if we now make the choice $\partial_0 \Phi = g$ for $\Phi$, then the last three terms sum to a total time derivative. There is of course nothing forcing us to choose this form for $\Phi$; since we are just making a change of path-integration variable, the final result for the propagator cannot depend on which decomposition we choose and it is best to make the simplest choice, especially in the later sections with lengthier calculations.
We will make this choice, and since \( g \) itself is given as the time derivative of \( \nabla^{-2}(\partial_j A^j) \), we can simply choose
\[
\Phi = \nabla^{-2}(\partial_j A^j).
\]
so that our new field variable now becomes
\[
A_j = A_j - \partial_j \nabla^{-2}(\partial^k A_k) \tag{A.23}
\]
We will find this result ultimately corresponds to the transverse-longitudinal decomposition of \( A_j \); however, instead of assuming this from the start, we will see that this decomposition is simply dictated by the solution to the \( A_0 \) saddle point equation. This pattern of logic will be used again in later sections when we consider geometries for which it is much less clear \( a \ priori \) how to define a ‘transverse part’ of \( A_j \).

Let us now complete the process of transforming to the new form for the field action. Note first that our choice of decomposition also simplifies the expression for the electric field, to
\[
\tilde{F}_{j0} = \partial_j \tilde{A}_0 - \partial_0 \tilde{A}_j = \partial_j \nabla^{-2}J^0 - \partial_0 A_j, \tag{A.24}
\]
ie., a manifestly gauge invariant form; and it renders \( A_j \) divergenceless.

We then find that a simple integration by parts gives
\[
\int_{t_i}^{t_f} d^4x \frac{1}{2} \tilde{F}^{ij} \partial_0 A_j = \int_{t_i}^{t_f} d^4x \frac{1}{2} \partial_0 A^j \partial_0 A_j, \tag{A.25}
\]
so that the field action takes the form
\[
\tilde{S}_{EM} = \int \sigma d^3x J^0 \nabla^{-2}(\partial_j A^j) + \frac{1}{2} \int_{t_i}^{t_f} d^4x \left[ - \partial_\mu A^j \partial^\mu A_j + 2\tilde{A}_j J^j + J^0 \nabla^{-2}J^0 \right] \tag{A.26}
\]
with \( \sigma = \pm 1 \) for the future and past parts of the boundary respectively.

We can now combine this field action with the original matter action \( S_M \) in (7), to get a complete form for the transformed action, as
\[
\tilde{S} = \tilde{S}_M + \tilde{S}_C + \tilde{S}_A \tag{A.27}
\]
which is eqn. (12) of the main text, with the three new terms defined in eqns. (13)-(15).

This leads finally to the propagator \( K_{fi} \) in the form that we want. Since \( \Phi \) does not appear in the bulk action, we can freely integrate over it to yield a harmless overall (divergent) normalization. Doing this, and continuing to absorb field independent constants into the measure, we arrive at eqn. (11) of the main text, for \( K_{fi} \). We see that all of the variables in the bulk action are gauge invariant, while the boundary term transforms precisely as we determined it ought to in (A.6). Note also that the \( g \)-potential \( \Phi \) is not present in the bulk action; it appears only in the boundary term.

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One can directly check that the result does not depend on this choice by noting that
\[
g = \partial_0 \nabla^{-2}(\partial^j A_j) = \partial_0 \nabla^{-2}(\partial^j A_j) + \partial_0 \Phi \quad \text{and observing that terms involving } \Phi \text{ always sum to total derivatives.} \]
Appendix A.3. Boundary Faddeev-Popov Trick

Here we show that eqn. (28) in the main text can be transformed into eqn. (29), after integration over $A_0$.

We begin by noting that the $A_0$ integral can be performed unambiguously without need for gauge-fixing. We therefore assume a gauge-fixing function which does not involve $A_0$, and rewrite the transformed boundary data using a linear shift, using functional derivatives as we did in (A.7). We define the operator

$$\hat{L}_\Lambda = \int_{\partial V} d^3x [\Lambda J^0 + i \partial_\mu \Lambda \frac{\delta}{\delta A_\mu}]$$

which now integrates over both past and future boundaries, and get

$$K_{fi} = \int D\Lambda \delta^{\partial V}(G(A^\Lambda)) e^{-i\hat{L}_\Lambda} \int_{q_i}^{q_f} Dq e^{iS_M} \int_{A_{i,j}}^{A_{f,j}} DA_{\mu} \Delta[A] \delta^V(G(A)) e^{iS_{EM}[A]}$$

The boundary delta function depends only on $\Lambda_{i,f}$ and not time derivatives thereof. The gauge transformations of the boundary data for $A_0$ are then completely decoupled from the transformations of the remaining $A_j$. In a time-sliced discretization of the path integral, the transformation involving $\Lambda$ on the slices immediately after $\Sigma_i$ and $\Sigma_f$ will only affect the transformation of $A_0$ on the boundary. Additionally, there is no dependence in the integrand on $\Lambda$ for any intermediate times. This “bulk” integration over the gauge group can be factored out as usual, leaving a residual integration over boundary gauge transformations.

The net result is that in (A.29) we can rewrite $\hat{L}_\Lambda$ as

$$\hat{L}_\Lambda \rightarrow \int_{\partial V} d^3x \left[ \Lambda J^0 + i \partial_\mu \Lambda \frac{\delta}{\delta A_\mu} \right]$$

and omit the boundary data for $A_0$. The omission of $A_0$ boundary data dictates that its values on the boundary are integrated over.

We can use the delta function to evaluate the integral over the boundary gauge transformations, and this will fix the boundary phase. Assuming $G$ is a good gauge fixing function, it will correspond to a unique gauge parameter $\Lambda = \Lambda_G[A]$. Evaluating the integral over the boundary gauge transformation we then obtain

$$K_{fi} = e^{-i\hat{L}_{\Lambda_G}} \int_{q_i}^{q_f} Dq e^{iS_M} \int_{A_{i,j}}^{A_{f,j}} DA_{\mu} \Delta[A] \delta^V(G(A)) e^{iS_{EM}[A]}$$

where now

$$\hat{L}_{\Lambda_G} = \int_{\partial V} d^3x \Lambda_G[A] \left[ J^0 + i \partial_\mu \Lambda \frac{\delta}{\delta A_\mu} \right]$$

The difference between the bFP trick and the usual FP technique is clear from (A.31). While the path integral integrand itself is standard, the additional boundary phase effects a
particular gauge transformation of the boundary data, which depends on the choice of bulk gauge fixing function $\mathcal{G}$. This boundary phase ensures that the resulting propagator remains independent of the choice of gauge fixing; it remains a gauge invariant object.

Since the propagator is independent of gauge choice, we can choose the most convenient gauge. The argumentation is then similar to what we did earlier. We first recall that after the $A_0$ integration, and the change of variables to the invariant fields $A_j$ and $\Phi$, we’re left with an effective action (A.20). Great simplification came if we then chose $\partial_0 \Phi = g$, where $g$ given in (A.16) was the unique gauge-dependent part of the saddle point solution $\tilde{A}_0$. Additionally, a few more terms in the effective action which involved $g$ and the current summed to a total derivative after using off-shell current conservation.

We could actually skip the off-shell current conservation argument at this point, by simply choosing the Coulomb gauge $\mathcal{G}(A) = \partial_0 A_j$. The particular usefulness of this gauge choice is that it sets $g = \Phi = 0$, considerably simplifying the action. It also makes the FP determinant irrelevant, and implies

$$\Lambda_\mathcal{G}[A] = -\nabla^{-2} \partial^j A_j,$$

for our boundary phases.

The resulting expression for the propagator is

$$K_{fi} = e^{i \int_{\partial\mathcal{V}} d^3 x \nabla^{-2} (\partial^k A_k) \int_{q_i}^{q_f} Dq e^{i \hat{S}_M} \int_{\hat{A}_{jI}}^{\hat{A}_{fI}} D\hat{A}_\mu e^{i \hat{S}_A[\hat{A}]}},$$

in which we write the answer in terms of the effective actions $\hat{S}_M$ and $\hat{S}_A$, as in (11).

We can show the equivalence of this result to (11) by noting that the remaining path-integral is independent of the longitudinal part of the gauge field. In the shift operator, the functional derivative $\partial_j \delta_{A_j}$ then vanishes and we’re left with an expression for the propagator $K_{fi}$ in the same form as (11) above, but with $\hat{S}_C$ now written as

$$\hat{S}_C = \int_{\partial\mathcal{V}} d^3 x A_k \partial_k (\nabla^{-2} J^0),$$

ie., as an integration by parts away from the expression for $\hat{S}_C$ in (14). Again, we find that the charge is dressed by a Coulomb field.

Appendix A.4. Spinor Quantum Electrodynamics

Here we give the derivation of the final result (39) for $K_{fi}$ starting from eqn. (38) of the main text, ie., from

$$K_{fi} = \int \mathcal{D}A \int_{\psi^{\Lambda f}_{\psi_i}} \mathcal{D}\psi \mathcal{D}\bar{\psi} \int_{A^{\Lambda f}_{\mu i}}^{A^{\Lambda f}_{\mu i}} \mathcal{D}A_\mu \mathcal{D}[A] \delta^{\psi}(\mathcal{G}(A)) \delta^{\bar{\psi}}(\mathcal{G}(A)) e^{i \hat{S}[A, \psi, \bar{\psi}]}.$$
The $A_0$ integral in this formula can again be done without gauge fixing, and we can extract the transformations of the boundary data using exponentiations of the functional derivatives,

$$K_{fi} = \int \mathcal{D}\Lambda \delta^\Omega (\mathcal{G}(A^\Lambda)) e^{\hat{\mathcal{L}}_\Lambda} \int_{\psi_i}^{\psi_f} \mathcal{D}\psi \mathcal{D}\bar{\psi} \int_{A_{\mu i}}^{A_{\mu f}} \mathcal{D}A_\mu \Delta[A] \delta^\Omega (\mathcal{G}(A)) e^{iS[A,\psi,\bar{\psi}]} \tag{A.37}$$

in which the operator $\hat{\mathcal{L}}_\Lambda$ now takes the form

$$\hat{\mathcal{L}}_\Lambda = \int d^3x \left( i e \psi \delta \overline{\partial} + \partial_\mu \Lambda \frac{\delta}{\delta A_\mu} \right)$$

$$= \int d^3x \left( \Lambda (\partial_j \hat{E}^j - \hat{J}^0) - i \partial_0 \Lambda \frac{\delta}{\delta A_0} \right) \tag{A.38}$$

where the 2nd expression uses the relations \((34), (35)\).

From this stage onwards, the manipulations are identical to those in the previous section except that the charge density in the boundary phase is an operator rather than a c-number. The resulting expression for the propagator is

$$K_{fi} = e^{i\hat{\mathcal{L}}_{A_0}} \int_{\psi_i}^{\psi_f} \mathcal{D}\psi \mathcal{D}\bar{\psi} \int_{A_{j i}}^{A_{j f}} \mathcal{D}A_j \Delta^\Omega[A] \delta^\Omega (\mathcal{G}(A)) e^{iS[A,\psi,\bar{\psi}]} \tag{A.39}$$

where now

$$\hat{\mathcal{L}}_{A_0} = \int d^3x A_0 [\partial_j \hat{E}^j - \hat{J}^0] \tag{A.40}$$

The final expression for the propagator will of course be independent of choice of $\mathcal{G}(A)$. For formal manipulations the most convenient choice is Coulomb gauge, because this sets the g-potential $\nabla^{-2} \partial^j A_j$ to zero, leaving only the invariant field components, and $\Lambda_G[A] = -\nabla^{-2}(\partial^j A_j)$. Since this choice eliminates the dependence of the integral on the longitudinal part of $A_j$, the shift operator $\exp \left( i \int_{\partial} \Lambda \partial_j \hat{E}^j \right)$ will just give zero, and the propagator is then

$$K_{fi} = e^{i\hat{\mathcal{S}}_C} \int_{\psi_i}^{\psi_f} \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{i\hat{\mathcal{S}}_{SM}} \int_{A_{j i}}^{A_{j f}} \mathcal{D}A_j e^{i\mathcal{S}_A} \tag{A.41}$$

which is just eqn. \((39)\) of the main text, as desired.

Note that if we had chosen a different gauge fixing function $\mathcal{G}(A)$, the resulting gauge fixed action would look different, but this difference would only be temporary; the shift operator in \((A.40)\) would no longer give zero in any other gauge, instead enacting a gauge transformation which would return the action to the form \((12)\), with the three terms given by eqns. \((13)-(15)\). In this form the theory is not manifestly Lorentz invariant, but this is simply because we evaluated $K_{fi}$ between two constant $t$ surfaces. In principle, one can chose a covariant gauge to compute the path-integral as long as one also evaluates the necessary shift of the longitudinal mode in the final expression.
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