\( \mathcal{N} = 2 \) **SUPER-TEICHMÜLLER THEORY**

IVAN C. H. IP, ROBERT C. PENNER, AND ANTON M. ZEITLIN

**Abstract.** Based on earlier work of the latter two named authors on the higher super-Teichmüller space with \( \mathcal{N} = 1 \), a component of the flat \( OSp(1|2) \) connections on a punctured surface, here we extend to the case \( \mathcal{N} = 2 \) of flat \( OSp(2|2) \) connections. Indeed, we construct here coordinates on the higher super-Teichmüller space of a surface \( F \) with at least one puncture associated to the supergroup \( OSp(2|2) \), which in particular specializes to give another treatment for \( \mathcal{N} = 1 \) simpler than the earlier work. The Minkowski space in the current case, where the corresponding super Fuchsian groups act, is replaced by the superspace \( \mathbb{R}^{2|4} \), and the familiar lambda lengths are extended by odd invariants of triples of special isotropic vectors in \( \mathbb{R}^{2|4} \) as well as extra bosonic parameters, which we call ratios, defining a flat \( \mathbb{R}_+ \)-connection on \( F \). As in the pure bosonic or \( \mathcal{N} = 1 \) cases, we derive the analogue of Ptolemy transformations for all these new variables.

**Contents**

1. Introduction
   1.1. Brief overview of Teichmüller and super-Teichmüller theory
   1.2. Outline of the paper
2. Light cone and \( \tilde{SL}(1|2) \)-action
3. Orbits of \( \tilde{SL}(1|2) \) in the light cone
4. Orbits of pairs and triples in the light cone
5. Basic calculation
6. Spin structures and fatgraph connections
7. Decorated super-Teichmüller space
8. Ptolemy transform
   8.1. Even Ptolemy transformations
   8.2. Odd Ptolemy transformation
Appendix A. \( SL(1|2) \): Notation and conventions
Appendix B. Ptolemy transformation in the special cases
References

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1. Introduction

1.1. Brief overview of Teichmüller and super-Teichmüller theory. The three cases \( \mathcal{N} = 0, 1, 2 \), where \( \mathcal{N} \) is half the number of odd generators, correspond to the respective Lie (super) groups \( G = PSL_2(\mathbb{R}), OSp(1|2) \) and \( OSp(2|2) \), with \( PSL_2(\mathbb{R}) \) denoting the Möbius group and \( OSp \) the orthosymplectic groups \([8, 6]\). The (super) Teichmüller space is defined as a component of the moduli space \( \text{Hom}(\pi_1(F), G)/G \) of flat \( G \)-connections on the surface \( F \), namely,

\[
T_G(F) = \text{Hom}'(\pi_1(F), G)/G,
\]

where \( G \) acts naturally by conjugation and \( \text{Hom}' \) denotes those faithful (i.e., injective) representations \( \rho : \pi_1(F) \rightarrow G \) onto a discrete subgroup of the Lie (super) group \( G \) (i.e., the identity of \( G \) is isolated in \( \rho(\pi_1(F)) \)) that satisfy the further condition that the value of \( \rho \) on a loop that becomes null homotopic upon replacing a puncture must be a parabolic element (i.e., have trace \( \pm 2 \)) in the case of \( G = PSL_2(\mathbb{R}) \) and project to such a parabolic Fuchsian transformation in case \( G = OSp(1|2), OSp(2|2) \).

In fact, it was a useful innovation already in the bosonic case \( \mathcal{N} = 0 \) in \([10]\) to require at least one puncture and to work in Minkowski space \( \mathbb{R}^{2,1} \), where \( PSL_2(\mathbb{R}) \approx SO_+(2,1) \) agrees with the component of the identity in the group of Minkowski isometries. Upon identifying the open positive light-cone with the space of all horocycles, the \( \lambda \)-length, i.e., the square root of the inner product of two isotropic vectors, is precisely the square root of the exponential of signed hyperbolic lengths between corresponding horocycles. In case \( \mathcal{N} = 1 \), the analogous action of \( OSp(1|2) \) as a subgroup of the Minkowski isometries \( OSp(2,1|2) \) acting on \( \mathbb{R}^{2,1|2} \) was exploited in \([12]\). Here we study an \( OSp(2|2) \) subgroup of \( OSp(2,2|4) \) acting on the superspace \( \mathbb{R}^{2,2|4} \). An ideal triangulation \( \Delta \) of \( F \) (i.e., a collection of embedded arcs decomposing \( F \) into triangles with vertices at the punctures) provides the convenient index set for the collection of \( \lambda \)-lengths with different choices of ideal triangulation playing the role of different bases for the coordinate system; flips, (i.e., remove an edge \( e \) of \( \Delta \) to produce a complementary quadrilateral with frontier \( a, b, c, d \), and replace with the other diagonal \( f \) of this quadrilateral) provide the basic transformations relating ideal triangulations with the bosonic lambda lengths governed by the Ptolemy relation \( ef = ac + bd \).

Dual to \( \Delta \) (with one trivalent vertex for each triangle in \( F - \cup \Delta \) and one edge connecting each pair of vertices whose corresponding triangles share an edge), there is a fatgraph \( \tau \) embedded in \( F \). Part of the utility of \( \lambda \)-lengths comes from their role in a recursive construction of the so-called lift \( \ell : \bar{\Delta}_\infty \rightarrow \mathcal{L}_0 \), where \( \bar{\Delta}_\infty \) denotes the set of ideal points at infinity of a lift of \( \Delta \) to the universal cover of \( F \) and \( \mathcal{L}_0 \) denotes an appropriate set of isotropic vectors in Minkowski space. This paradigmatic role in case \( \mathcal{N} = 0 \) carries over to \( \mathcal{N} = 1, 2 \) albeit in the more complicated circumstance of dependence on spin structures on \( F \), indeed on \( \mathcal{N} \) many spin structures insofar as a component

\footnote{This condition requires the image of \( \pi_1(F) \) in the \( \mathcal{N} = 2 \) case to belong to the subgroup of \( OSp(2|2) \) with the same Lie algebra, which we denote as \( \hat{SL}(1|2) \). The reason for this notation is that this subgroup is isomorphic to the semidirect product of a certain involution on the Lie algebra \( sl(1|2) \approx \mathfrak{osp}(2|2) \) and the connected component of identity of supergroup \( SL(1|2) \) (for more details see Appendix A).}
of $T_{OSp(1|2)}(F)$ is determined by a spin structure and a component of $T_{OSp(2|2)}(F)$ by two spin structures.

In each case of $\mathcal{N} = 0, 1, 2$, our coordinates are defined on a $\mathbb{R}^s_+$-bundle, the decorated Teichmüller space $\tilde{T}(F)$ over $T(F)$, which amounts to the set of all possible Fuchsian lifts. It is only in the bosonic case $\mathcal{N} = 0$ that these extra parameters, the decorations, admit an explicit geometric interpretation as $s$-tuples of lengths of horocycles, one horocycle about each puncture. However, in all three case $\mathcal{N} = 0, 1, 2$, one can pass from $\lambda$-lengths to cross ratios $\chi = \frac{ac}{bd}$ on each edge of $\Delta$ to describe coordinates on the undecorated (super) Teichmüller space itself, where the product of cross ratios about each puncture must equal unity. There is also the notion of “surfaces with holes”, where one drops this condition on cross ratios and the punctures can open to oriented boundary components, cf. [11].

1.2. Outline of the paper. Section 2 is devoted to the description of superspace $\mathbb{R}^{2,2|4}$ modeled as the space of adjoint representation of supergroup $\tilde{SL}(1|2) \subset OSp(2|2)$. We fix notations and describe explicitly the action of certain important generators of $\tilde{SL}(1|2)$.

Again, as in [12], we take the orbit of the highest weight vector in adjoint representation as the light cone $\mathcal{L}_0$. We derive necessary functional relations on the components of the light-cone vectors. This is the subject of Section 3.

In Section 4 we describe orbits of pairs and triples of vectors in $\mathcal{L}_0$. For the pairs such orbits are classified by a single even invariant $\lambda$-length, corresponding to the pairing of those two vectors. It turns out that the orbit of triple of linearly independent vectors in $\mathcal{L}_0$ is described by means of three $\lambda$-lengths and two odd parameters $(\theta_1, \theta_2)$ modulo permutation and rescaling $(\theta_1, \theta_2) \rightarrow (a\theta_1, a^{-1}\theta_2)$, where $a$ is any even number.

The invariants of a quadruple of points include one more even parameter in addition to 2 pairs of odd invariants and 5 $\lambda$-lengths. This extra invariant can be heuristically interpreted as a “ratio” of the odd data for the two triples of points combined into the quadruple. In Section 5 we describe in detail the orbits of 4 points, its invariants and related ambiguities for putting these 4 points in certain standard position.

In Section 6 we return to the geometry of triangulated Riemann surfaces, recall and reformulate necessary facts about spin structures and connections on fatgraphs. In particular, we recall the description of spin structures given in [12] based on equivalence classes of orientations on fatgraphs.

Section 7 is devoted to the main subject of the paper: description of coordinates on the $\mathcal{N} = 2$ super-Teichmüller space. Let $F := F_g^s$ be a Riemann surface with genus $g \geq 0$ and $s \geq 1$ punctures such that $2g + s - 2 > 0$. As usual, the construction of such coordinates involves lifting certain data assigned to the triangulation $\Delta$, or equivalently to the fatgraph $\tau$ dual to $\Delta$, to the light cone $\mathcal{L}_0$. The data giving the coordinate system $\tilde{C}(F, \Delta)$ are as follows:

- we assign to each edge of $\Delta$ a positive even coordinate $e$;
- we assign to each triangle of $\Delta$ two odd coordinates $(\theta_1, \theta_2)$;
we assign to each edge $e$ of $\Delta$ a positive even coordinate $h_e$, called the ratio, such that if $h_e$ and $h'_e$ are assigned to two triangles sharing the same edge $e$, then we have $h_e h'_e = 1$.

The odd coordinates are defined up to an overall sign changes $\theta_i \leftrightarrow -\theta_i$, as well as an overall involution $(\theta_1, \theta_2) \leftrightarrow (\theta_2, \theta_1)$.

In particular, this assignment implies that the ratios \{h_i\} uniquely define an $\mathbb{R}_+$-graph connection on $\tau$. One can define the following action of the gauge transformations on the set of coordinates: if $h_a, h_b, h_c$ are ratios assigned to a triangle $T$ with odd coordinate $(\theta_1, \theta_2)$, then a vertex rescaling at $T$ is the following transformation:

\[
(h_a, h_b, h_c, \theta_1, \theta_2) \mapsto (\alpha h_a, \alpha h_b, \alpha h_c, \alpha^{-1} \theta_1, \alpha \theta_2)
\]  

(1.1)

for some $\alpha > 0$, and all other coordinates fixed. We say that two coordinate vectors of $\tilde{C}(F, \Delta)$ are equivalent if they are related by a finite number of such vertex rescalings (i.e., gauge transformations). In particular, the underlying $\mathbb{R}_+$-graph connections on $\tau$ are equivalent.

Let $C(F, \Delta) := \tilde{C}(F, \Delta)/\sim$ be the equivalence classes of coordinate vectors. Then it can be represented by coordinates with $h_ah_bh_c = 1$ for the ratios of a common triangle. This implies that

\[
C(F, \Delta) \simeq \mathbb{R}^{8g+4s} / \mathbb{Z}_2 \times \mathbb{Z}_2
\]

(1.2)

in accordance with the dimension for $\mathcal{N} = 2$ super-Teichmüller space given in [9] (with an extra $s$-dimension given by decorations).

We can state the first main result of the paper:

**Theorem 1.1.** Fix $F, \Delta, \tau$ as before. Let $\omega_\sigma$ and $\omega_i$ be orientations on $\tau$ representing respective spin structures $s_\sigma$ and $s_i$ on $F$. Given a coordinate vector $\tilde{\mathbf{c}} \in \tilde{C}(F, \Delta)$, there exists a map called the lift,

\[
\ell = \ell_{\omega_\sigma, \omega_i} : \tilde{\Delta}_\infty \rightarrow \mathcal{L}_0,
\]

from the vertices $\tilde{\Delta}_\infty$ at infinity of a lift of $\Delta$ to the universal cover of $F$ to an appropriate set $\mathcal{L}_0$ of isotropic vectors in Minkowski space, where $\ell$ is uniquely determined up to post-composition by $\tilde{SL}(1|2)$ under certain admissibility conditions and only depends on the equivalence class $C(F, \Delta)$ of the coordinates. Moreover, there is a representation $\hat{\rho} : \pi_1 := \pi_1(F) \rightarrow \tilde{SL}(1|2)$, uniquely determined up to conjugacy by an element of $\tilde{SL}(1|2)$, such that

1. $\ell$ is $\pi_1$-equivariant, i.e., $\hat{\rho}(\gamma)(\ell(a)) = \ell(\gamma(a))$ for each $\gamma \in \pi_1$ and $a \in \tilde{\Delta}_\infty$;

2. $\hat{\rho}$ is a super Fuchsian representation, i.e., the natural projection

\[
\rho : \pi_1 \hat{\rightarrow} \tilde{SL}(1|2) \rightarrow SL(2, \mathbb{R}) \rightarrow PSL(2, \mathbb{R})
\]

is a Fuchsian representation;

3. the lift $\hat{\rho} : \pi_1 \hat{\rightarrow} \tilde{SL}(1|2) \rightarrow SL(2, \mathbb{R})$ of $\rho$ does not depend on $\omega_i$, and the space of all such lifts is in one-to-one correspondence with the spin structures $[\omega_\sigma] \in \mathcal{O}(\tau)$ as in section 4.
We call the space of all such lifts as in the theorem above the decorated $\mathcal{N} = 2$ super-Teichmüller space $S\tilde{T}(F)$ to distinguish it from the decorated $\mathcal{N} = 1$ super-Teichmüller space studied in [12].

In the construction of the lift $\ell$, which depends on two spin structures, we note that one spin structure controls the sign change for the odd invariants of attached triangles and the other one controls the change of the order of fermions. It is also important to mention that proving a similar result in [12] we relied on an ad hoc construction involving bipartite graphs. The current result can be easily reduced to $\mathcal{N} = 1$ case, by considering lifts with equal value of odd coordinates per triangle and therefore abolishing the dependence on the second spin structure. The proof in this paper follows the general constructions of [12] now in this case $\mathcal{N} = 2$ but is different conceptually from the one in [12]: we no longer require the bipartite fatgraph as the starting point of the construction. Instead we use any fatgraph and given classes of orientations in order to construct the lift, so that $\mathcal{N} = 2$ is in fact more closely analogous to the bosonic case $\mathcal{N} = 0$ in [10] than the treatment for $\mathcal{N} = 1$ in [12].

The next theorem, which follows from the one above, describes coordinates on $S\tilde{T}(F)$.

**Theorem 1.2.** The components of $S\tilde{T}(F)$ are determined by two spin structures $s_\sigma, s_\iota \in \mathcal{O}(\tau)$. For fixed representatives of the spin structures, $C(F, \Delta)$ provides global analytic coordinates on each component of $\mathbb{R}_{+}^{8\ell_{g}+4s_{g_{-}}-7|8\ell_{g}+4s_{g_{+}}-8}/\mathbb{Z}_{2} \times \mathbb{Z}_{2}$.

In Section 8 we describe our second main result. We show explicitly, how the coordinates behave under triangulation change and, in other words, give the description of generalized Ptolemy transformations.

**Theorem 1.3.** In the generic situation where all $a, b, c, d$ are different edges of the triangulations of $F$, the Ptolemy transformations, corresponding to Figure 1 below are as follows:

\[
e f = (ac + bd) \left( 1 + \frac{h_{e}^{-1}\sigma_{1}\theta_{2}}{2(\sqrt{\chi} + \sqrt{-\chi})} + \frac{h_{e}\sigma_{2}\theta_{1}}{2(\sqrt{\chi} + \sqrt{-\chi})} \right),
\]

\[
\mu_{1} = \frac{h_{e}\theta_{1} + \sqrt{\chi}\sigma_{1}}{D}, \quad \mu_{2} = \frac{h_{e}^{-1}\theta_{2} + \sqrt{\chi}\sigma_{2}}{D},
\]

\[
\nu_{1} = \frac{\sigma_{1} - \sqrt{\chi}h_{e}\theta_{1}}{D}, \quad \nu_{2} = \frac{\sigma_{2} - \sqrt{\chi}h_{e}^{-1}\theta_{2}}{D},
\]

\[
h_{a}' = \frac{h_{a}}{h_{e}c_{\theta}}, \quad h_{b}' = \frac{h_{b}c_{\theta}}{h_{e}}, \quad h_{c}' = \frac{h_{c}c_{\theta}}{c_{\mu}}, \quad h_{d}' = h_{d}c_{\nu}, \quad h_{f}' = \frac{c_{\sigma}}{c_{\theta}},
\]

where

\[
D := \sqrt{1 + \chi + \frac{\sqrt{\chi}}{2}(h_{e}^{-1}\sigma_{1}\theta_{2} + h_{e}\sigma_{2}\theta_{1})}
\]

and $c_{\theta} := 1 + \frac{\theta_{1}\theta_{2}}{6}$, while the signs of the fermions follow from the construction associated to the spin graph evolution rule in Figure 2, where $\epsilon_{i}$ denote the orientations of the edges.
In Appendix [B] we also write down formulas for flip transformation in the other cases with some edges being identified. In particular we write explicit formulas in the case of the 1-punctured torus and the 3-punctured sphere.

One prominent structure from [12] which we did not generalize to $N = 2$ case is the analogue of the Weil-Petersson 2-form. We leave this and many other questions for subsequent publications.

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2. Light cone and $\tilde{SL}(1|2)$-action

Let $SL(1|2)$ be the $(2|1) \times (2|1)$ supermatrices with superdeterminant equal to 1, and $\mathfrak{sl}(1|2)$ the corresponding Lie superalgebra described in Appendix [A]. Also let $\tilde{SL}(1|2)$ be the semidirect product $\Psi \ltimes SL(1|2)_0$ of the involution $\Psi$ and the component $SL(1|2)_0$ described in Definition [A.5]. Consider the adjoint action of $SL(1|2)$ acting on the pure even elements of the form

$$M := x_1 E - x_2 F - y H + \xi_1^+ e_1^+ + \xi_2^+ e_2^+ + \xi_1^- e_1^- + \xi_2^- e_2^- + z (h_1 + h_2)$$

$$= \begin{pmatrix} z - y & \xi_1^+ & x_1 \\ \xi_1^- & 2z & \xi_2^+ \\ -x_2 & \xi_2^- & z + y \end{pmatrix}$$

$$:= (x_1, x_2, y, z | \xi_1^+, \xi_2^+, \xi_1^-, \xi_2^-) \in \mathbb{R}^{2|2},$$

where the invariant quadratic form providing the Minkowski space structure $\mathbb{R}^{2|2}$ is given by the supertrace:

$$\langle M, M \rangle := -\frac{1}{2} str(M^2)$$

$$= -\frac{1}{2} (-2x_1 x_2 + 2y^2 - 2\xi_1^+ \xi_1^- - 2\xi_2^+ \xi_2^- - 2z^2)$$

$$= x_1 x_2 - y^2 + \xi_1^+ \xi_1^- - \xi_2^+ \xi_2^- + z^2. \quad (2.1)$$

**Definition 2.1.** The light cone $\mathcal{L} \subset \mathbb{R}^{2|2}$ is defined to be the set

$$\mathcal{L} = \{(x_1, x_2, y, z | \xi_1^+, \xi_2^+, \xi_1^-, \xi_2^-) \in \mathbb{R}^{2|2} : x_1 x_2 - y^2 + \xi_1^+ \xi_1^- - \xi_2^+ \xi_2^- + z^2 = 0\}, \quad (2.2)$$

i.e., the points $M \in \mathbb{R}^{2|2}$ such that $\langle M, M \rangle = 0$.

In particular, we have an inner product

$$\langle M, M' \rangle = \frac{1}{2} (x_1 x_2' + x_2 x_1') - yy' + \frac{1}{2} (\xi_1^+ \xi_1'^- - \xi_2^+ \xi_2'^- - \xi_1^- \xi_1'^+ + \xi_2^- \xi_2'^+) + zz', \quad (2.3)$$

and we will refer to the square root of $\langle M, M' \rangle$ as the $\lambda$-length between the points $M$ and $M'$.

**Definition 2.2.** We denote certain elements of $SL(1|2)$ as follows:

$$D_{a,c} := \begin{pmatrix} a & 0 & 0 \\ 0 & ac & 0 \\ 0 & 0 & c \end{pmatrix}, \quad D_a := D_{a,-1}, \quad Z_a := D_{a,a}, \quad J := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

$$U_a := \begin{pmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad V_\beta := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \beta \\ 0 & 0 & 0 \end{pmatrix}, \quad W_b = \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. $$
Lemma 2.3. The adjoint actions on $\mathcal{L}$ by conjugation is given by

$$D_{a,c} \cdot (x_1, x_2, y, z|\xi_1^+, \xi_2^+, \xi_1^-, \xi_2^-) = (ac^{-1}x_1, a^{-1}cx_2, y, z|c^{-1}\xi_1^+, a\xi_2^+, c\xi_1^-, a^{-1}\xi_2^-),$$
$$D_a \cdot (x_1, x_2, y, z|\xi_1^+, \xi_2^+, \xi_1^-, \xi_2^-) = (a^2x_1, a^{-2}x_2, y, z|a\xi_1^+, a\xi_2^+, a^{-1}\xi_1^-, a^{-1}\xi_2^-),$$
$$Z_a \cdot (x_1, x_2, y, z|\xi_1^+, \xi_2^+, \xi_1^-, \xi_2^-) = (x_1, x_2, y, z|a^{-1}\xi_1^+, a\xi_2^+, a\xi_1^-, a^{-1}\xi_2^-),$$
$$J \cdot (x_1, x_2, y, z|\xi_1^+, \xi_2^+, \xi_1^-, \xi_2^-) = (x_2, x_1, -y, z|\xi_1^-, -\xi_1^+, \xi_2^-, -\xi_2^+).$$

Proof. The actions of bosonic elements follows from usual matrix calculation. Let us explicate the action for $U_\alpha$. The case for $V_\beta$ is similar. We have $U^{-1}_\alpha = \begin{pmatrix} 1 & -\alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, hence following the convention in (A.3),

$$U_\alpha \cdot (x_1, x_2, y, z|\xi_1^+, \xi_2^+, \xi_1^-, \xi_2^-) = \begin{pmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} z - y & \xi_1^+ & x_1 \\ \xi_1^- & 2z & \xi_2^+ \\ -x_2 & \xi_2^- & z + y \end{pmatrix} \begin{pmatrix} 1 & -\alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} z - y - \alpha\xi_1^- & \xi_1^+ + 2z\alpha & x_1 - \alpha\xi_2^+ \\ \xi_1^- & 2z & \xi_2^+ \\ -x_2 & \xi_2^- & z + y \end{pmatrix} \begin{pmatrix} 1 & -\alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} z - y - \alpha\xi_1^- & \xi_1^+ + (y + z)\alpha & x_1 - \alpha\xi_2^+ \\ \xi_1^- & 2z + \xi_1^- \alpha & \xi_2^+ \\ -x_2 & \xi_2^- + x_2\alpha & z + y \end{pmatrix}$$

$$= (x_1 - \alpha\xi_2^+, x_2, y + \frac{\alpha\xi_1^-}{2}, z - \frac{\alpha\xi_1^-}{2}|\xi_1^+ + (y + z)\alpha, \xi_2^+, \xi_1^-, \xi_2^- + x_2\alpha).$$

□

Let us also consider the action of the involution $\Psi$ defined in Proposition A.4

Lemma 2.4. The involution $\Psi$ acts on $\mathcal{L}$ by

$$\Psi \cdot (x_1, x_2, y, z|\xi_1^+, \xi_2^+, \xi_1^-, \xi_2^-) = (x_1, x_2, y, -z|\xi_2^+, \xi_1^-, \xi_2^+, -\xi_1^-).$$
The lower Borel elements and their inverses are

\[
B = \begin{pmatrix}
a & 0 & 0 \\
\alpha & f & 0 \\
c & \beta & d
\end{pmatrix}, \quad B^{-1} = \begin{pmatrix}
a^{-1} & 0 & 0 \\
-\alpha f & f^{-1} & 0 \\
-\beta f + \alpha \beta & -\beta d & d^{-1}
\end{pmatrix},
\]

where by definition of the superdeterminant we have \( f = ad \). Finally we also embed the \( SL(2) \) subgroup as

\[
g = \begin{pmatrix}
a & 0 & b \\
0 & 1 & 0 \\
c & 0 & d
\end{pmatrix} \in SL(2) \hookrightarrow SL(1|2),
\]

where \( ad - bc = 1 \).

3. Orbits of \( \widetilde{SL}(1|2) \) in the light cone

In this section we would like to describe a particular orbit of \( \widetilde{SL}(1|2) \), namely the orbit of

\[ e_0 := E = (1, 0, 0, 0|0, 0, 0) \in \mathcal{L} \]

under the adjoint action.

**Proposition 3.1.** The elements \((x_1, x_2, y, z|\xi_1^+, \xi_2^+, \xi_1^-, \xi_2^-)\) of the light cone \( \mathcal{L} \) such that \( x_1, x_2 \) have non-negative bodies, and satisfying

\[
\begin{align*}
\xi_1^- &= -\frac{y}{x_1} \xi_2^+, \\
\xi_2^- &= \frac{y}{x_1} \xi_1^+, \\
\xi_2^+ &= -\frac{y}{x_2} \xi_1^-, \\
\xi_1^+ &= \frac{y}{x_2} \xi_2^-, \\
z &= \frac{\xi_1^+ \xi_2^+}{2x_1} \quad \text{if } x_1 \neq 0, \\
z &= \frac{\xi_1^- \xi_2^-}{2x_2} \quad \text{if } x_2 \neq 0,
\end{align*}
\]

constitute the orbit \( \mathcal{L}_0 \) of \( e_0 = (1, 0, 0, 0|0, 0, 0) \).

Following [12], we refer to \( \mathcal{L}_0 \) as the special light cone.

**Proof.** Note that if an element \( M \in \mathcal{L} \) satisfies \(3.1\), then for \( x_1 \neq 0 \) we have

\[
0 = x_1 x_2 - y^2 + \xi_1^+ \xi_1^- - \xi_2^+ \xi_2^- + z^2
= x_1 x_2 - y^2 - \frac{y}{x_1} \xi_1^+ \xi_2^+ - \frac{y}{x_1} \xi_2^+ \xi_1^- + \left(\frac{\xi_1^+ \xi_2^+}{4x_1^2}\right)^2
= x_1 x_2 - y^2,
\]

and similarly for the case of \( x_2 \neq 0 \). Hence we have \( y^2 = x_1 x_2 \) for these elements.

Now consider \( M = (x_1, x_2, y, z|\xi_1^+, \xi_2^+, \xi_1^-, \xi_2^-) \) that satisfies \(3.1\). Acting on it by an element \( g \in SL(2) \), we can assume that \( x_1 \neq 0, x_2 \neq 0 \). Then acting on it by the diagonal matrix \( D_{a,c} \) with \( a = \sqrt{\frac{x_2}{x_1}}, c = 1 \) we can assume \( x_1 = x_2 = t \). Since \( y^2 = x_1 x_2 = t^2 \) we have \( y = \pm t \). By the action of \( J \) we can further assume \( y = t \).
Now consider the action of the lower Borel acting on $e_0$:

$$B \cdot e_0 = \begin{pmatrix} a & 0 & 0 \\ \alpha & f & 0 \\ c & \beta & d \end{pmatrix} \begin{pmatrix} a^{-1} & 0 & 0 \\ -\frac{\alpha f}{f^2} & f^{-1} & 0 \\ -\frac{\xi f}{f^2} + \frac{\alpha \beta}{f^2} & -\frac{\beta}{f} & d^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{c}{f} + \frac{\alpha \beta}{f^2} & -\frac{\alpha \beta}{f^2} & \frac{\alpha}{f} \\ -\frac{\alpha f}{f^2} & \frac{\alpha \beta}{f^2} & \frac{\alpha}{f} \\ -\frac{c}{f} - \frac{\alpha \beta}{f^2} & \frac{\alpha}{f} & \frac{\alpha}{f} \end{pmatrix}. \quad (3.2)$$

Hence solving for $B \cdot e_0 = M$ for $y = t$, one of the solutions is given by

$$a = t, \quad d = 1, \quad f = t, \quad c = \frac{\xi_1^+ \xi_2^+}{2t} + t, \quad \alpha = \xi_2^+, \quad \beta = -\xi_1^+. \quad (3.1)$$

Hence an element satisfying $\xi_1^+ \neq 0$ lies in the orbit of $e_0$.

Next, we note that for any connected Lie (super)group, a neighborhood of the identity generates the whole group, therefore, to show that the orbit of $e_0$ satisfies $\xi_1^+ \neq 0$, it suffices to show that the relations $\xi_1^+ \neq 0$ are preserved under the action of the infinitesimal group transformations and the involution $\Psi$ (since $\Psi$ gives a second connected component of $SL(1|2)$). From (3.2) we see easily that the relations $\xi_1^+ \neq 0$ are preserved under the lower Borel action. Also by Lemma 2.3, the relations are clearly preserved by the involutions $J$ and $\Psi$. Hence it suffices to consider the action of $U_\alpha, V_\beta$. By the action of $J$ which interchanges $x_1$ and $x_2$, it suffices to consider the case for $x_2 \neq 0$.

By Lemma 2.3 the action of $U_\alpha$ is given by

$$U_\alpha \cdot (x_1, x_2, y, z|\xi_1^+, \xi_2^+, \xi_1^-, \xi_2^-)$$

$$= (x_1 - \alpha \xi_2^+, x_2, y + \frac{\alpha \xi_1^-}{2}, z - \frac{\alpha \xi_1^-}{2}(\xi_1^+ + (y + z)\alpha, \xi_2^+, \xi_1^-, \xi_2^- + x_2\alpha))$$

$$=: (x_1', x_2', y', z'|\xi_1'^+, \xi_2'^+, \xi_1'^-, \xi_2'^-).$$

Then

$$\frac{y'}{x_2'} \xi_1'^- = (-\frac{y}{x_2} - \frac{\alpha \xi_1^-}{2x_2}) \xi_1^- = -\frac{y}{x_2} \xi_1^- = \xi_2^- = \xi_2^+,$$

$$\frac{y'}{x_2'} \xi_2'^- = (\frac{y}{x_2} + \frac{\alpha \xi_1^-}{2x_2})(\xi_1^- + x_2\alpha) = \frac{y}{x_2} \xi_1^- + y\alpha + \frac{\alpha \xi_1^- \xi_2^-}{2x_2} = \xi_1^+ + y\alpha + z\alpha = \xi_1^+,$$

$$\frac{\xi_1^- \xi_2'^-}{2x_2'} = \frac{\xi_1^- (\xi_1^- + x_2\alpha)}{2x_2} = \frac{\xi_1^- \xi_2^-}{2x_2} - \frac{\alpha \xi_1^-}{2} = z'.$$

Similarly for the action of $V_\beta$ we have

$$V_\beta \cdot (x_1, x_2, y, z|\xi_1^+, \xi_2^+, \xi_1^-, \xi_2^-)$$

$$= (x_1 - \beta \xi_1^+, x_2, y - \frac{\beta \xi_2^-}{2}, z - \frac{\beta \xi_2^-}{2}(\xi_1^+ + (y - z)\beta, \xi_1^- - x_2\beta, \xi_2^-))$$

$$=: (x_1'', x_2'', y'', z''|\xi_1''^+, \xi_2''^+, \xi_1''^-, \xi_2''^-).$$
Then
\[ \frac{y''}{x''} \xi''_{1} = \left( y - \frac{\beta \xi^{-}_{2}}{2x_{2}} \right) (\xi^{-}_{1} - x_{2} \beta) = \frac{y \xi^{-}_{1}}{x_{2}} - y \beta + \frac{\beta \xi^{-}_{1} \xi^{-}_{2}}{2x_{2}} = -\xi^{+}_{2} - y \beta + z \beta = -\xi^{+} \]
\[ \frac{y''}{x''} \xi''_{2} = \left( y - \frac{\beta \xi^{-}_{2}}{2x_{2}} \right) \xi_{2} = \frac{y}{x_{2}} \xi_{2} = \xi^{+}_{1} = \xi''_{1} \]
\[ \frac{\xi''_{1} \xi''_{2}}{2x_{2}''} = \left( \frac{\xi^{-}_{1}}{x_{2}} - x_{2} \beta \right) \xi_{2} = \frac{\xi^{-}_{1} \xi_{2}^{-}}{2x_{2}} - \frac{\beta \xi_{2}^{-}}{2} = z'' \]
and this completes the proof. \( \square \)

4. ORBITS OF PAIRS AND TRIPLES IN THE LIGHT CONE

In this section we study the space of \( \tilde{SL}(1|2) \)-orbits of linearly independent ordered pairs and positively oriented triples of points in the special light cone \( L_{0} \).

**Proposition 4.1.** For two points on \( L_{0} \) which are linearly independent (i.e., the pairing has non-zero body), the space of orbits is described by one parameter, which will be called the \( \lambda \)-length.

**Proof.** Let \( v_{1}, v_{2} \) be two vectors on \( L_{0} \). By means of the action of \( SL(1|2) \), we can bring \( v_{1} = -F \), i.e., \( x_{2} = 1 \) and all other components zero. Since we assume \( \langle v_{1}, v_{2} \rangle \neq 0 \), the \( x_{1} \)-component of \( v_{2} \) is non-zero. Therefore as in the previous section one can use an element \( B \) of the lower Borel subgroup of \( SL(1|2) \) in order to reduce it to the vector \( v_{2} = E \), i.e., \( x_{1} = 1 \) and all other components zero. At the same time, \( B \cdot v_{1} = \lambda^{2} v_{1} \) for some scalar \( \lambda > 0 \) since \( F \) is a lowest weight vector of the adjoint action. This scalar multiple is exactly the pairing \( \langle v_{1}, v_{2} \rangle \). \( \square \)

**Definition 4.2.** We call an ordered triple of points \( (A, B, C) := \Delta ABC \) on \( L_{0} \) positively ordered if the triple of points are linearly independent, and the bodies of the underlying bosonic triples \( (x_{1}, x_{2}, y) \) constitute a positively oriented basis of \( \mathbb{R}^{2,1} \).

**Definition 4.3.** A positively ordered triple \( \Delta ABC \subset L_{0} \) is said to be in standard position with odd parameters \( \theta := (\theta_{1}, \theta_{2}) \) if the points comprising the triple are of the form
\[ A = r(0, 1, 0, 0|0, 0, 0, 0), \quad (4.1) \]
\[ B = t(1, 1, 1, \frac{\theta_{1} \theta_{2}}{2}|\theta_{1}, \theta_{2}, -\theta_{2}, \theta_{1}), \quad (4.2) \]
\[ C = s(1, 0, 0, 0|0, 0, 0, 0), \quad (4.3) \]
for some positive numbers \( r, s, t \in \mathbb{R}_{>0} \) and a pair of fermions \( \theta_{1}, \theta_{2} \). We denote this by \( \Delta ABC \in S^{\theta} \).

**Proposition 4.4.** \( \Delta ABC \in S^{\theta} \) and \( \Delta A'B'C' \in S^{\theta'} \) are in the same \( SL(1|2) \)-orbit if and only if \( (\theta_{1}', \theta_{2}') = (a \theta_{1}, a^{-1} \theta_{2}) \) for some \( a \neq 0 \). In particular the stabilizers of \( \Delta ABC \in S^{\theta} \) are of the form \( Z_{a} \in Z \) where \( a \theta_{i} = \theta_{i} \) for \( i = 1, 2 \).

**Proof.** It is easy to check that the stabilizers for both \( A \) and \( C \) are the \( \mathbb{Z} \)-subgroup elements. Hence the result follows by the action of \( Z \) given in Lemma 2.3. \( \square \)
Proposition 4.5. The space of $\widetilde{SL}(1|2)$-orbits of positively ordered triples of points on $\mathcal{L}_0$ is parametrized by 3 positive numbers $r, s, t \in \mathbb{R}_{>0}$ and an unordered pair of fermions $\{\theta_1, \theta_2\}$ modulo the transformation $\{\theta_1, \theta_2\} \sim \{a\theta_1, a^{-1}\theta_2\}$ where $a$ is invertible. We call the equivalence class $[\theta]$ of $\{\theta_1, \theta_2\}$ the odd invariant of the ordered triple.

Proof. Consider the positively ordered triple $\Delta ABC$. Let us put $C$ into the position $C = s(1, 0, 0|0, 0, 0, 0)$ by an $SL(1|2)$ transform. Using Lemma 2.3 one sees that $U_\alpha, V_\beta$ and the $SL(2)$ element $W_b$ lie in the stabilizer of $C$. Now given $A = (x_1, x_2, y, z|\xi^+_1, \xi^+_2, \xi^-_1, \xi^-_2)$, we can assume $x_2 \neq 0$ (since otherwise the even part will be parallel to $C$), we can apply $V_\beta U_\alpha$ to $A$ where $\alpha = -\frac{\xi^-_2}{x_2}$ and $\beta = \frac{\xi^+_2}{x_2}$ and get $A' = (x'_1, x'_2, y', 0|0, 0, 0, 0)$. Finally applying $W_b$ where $b = -\frac{x'_2}{x_2}$ we bring $A$ into the form $r(0, 1, 0, 0|0, 0, 0, 0)$.

Now $B$ will be in the general form $(x_1, x_2, y, z|\xi^+_1, \xi^+_2, \xi^-_1, \xi^-_2)$ with $x_1 \neq 0$. Applying the diagonal transformation $D_{a,a^{-1}}$ with $a = \sqrt{\frac{x_2}{x_1}}$ we can bring $B$ to the form $t(1, 1, y, z|\theta_1, \theta_2, \theta'_1, \theta'_2)$. Since $y^2 = x_1 x_2$ on the orbit, we have $y = 1$. From the relations (3.1) we see that $z = \frac{\theta_1 \theta_2}{2}, \theta'_1 = -\theta_2, \theta'_2 = \theta_1$. Hence the ordered triple $\Delta ABC \in S^6$ is brought into standard position.

By Lemma 2.3 $Z_a$ acts on the triple by mapping the odd parameters $(\theta_1, \theta_2) \mapsto (a^{-1}\theta_1, a\theta_2)$, hence the parameters are determined up to rescaling. Finally the involution acts by interchanging $\theta_1$ and $\theta_2$, hence the orbit does not depend on the order of the fermions. $\square$

Proposition 4.6. The odd invariant $[\theta]$ does not depend on the order of the vertices in standard position.

Proof. We will show that the choice of the vertex $B$ is not relevant for $\theta_1, \theta_2$ modulo rescaling. To do that we will construct the analogue of the “prime transformation” from [12] in this case by rotating the vertices. Recall from (3.2), that we have

$$U(1, 0, 0, 0|0, 0, 0, 0)U^{-1} = (1, 1, 1, \frac{\theta_1 \theta_2}{2}|\theta_1, \theta_2, -\theta_2, \theta_1),$$

where

$$U = \begin{pmatrix} 1 & 0 & 0 \\ \theta_2 & 1 & 0 \\ 1 + \frac{\theta_2}{\theta_1} & -\theta_1 & 1 \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -\theta_2 & 1 & 0 \\ \frac{\theta_2}{\theta_1} & -1 & \theta_1 \end{pmatrix}. $$

Let $\Delta ABC \in S^6$. Applying $U^{-1}$ to $\Delta ABC$ gives

$$A \mapsto r(0, 1, 0, 0|0, 0, 0, 0),$$

$$B \mapsto t(1, 0, 0, 0|0, 0, 0, 0),$$

$$C \mapsto s(1, 1, 1, \frac{\theta_1 \theta_2}{2}|-\theta_1, -\theta_2, -\theta_2, \theta_1).$$
Finally apply $J$ from Lemma 2.3 which maps

$A \mapsto r(1, 0, 0, 0|0, 0, 0, 0)$,
$B \mapsto t(0, 1, 0, 0|0, 0, 0, 0)$,
$C \mapsto s(1, 1, 1, \frac{\theta_1\theta_2}{2}|\theta_1, \theta_2, -\theta_2, \theta_1)$.

Hence the ordered triple $\Delta BCA$ is now in standard form with the same odd parameters.

The prime transformation $P' \in SL(1|2)$ is therefore given by

$$P' = J \circ U^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -\theta_2 & 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{\theta_1\theta_2}{2} & -1 & \theta_1 & 1 \\ -\theta_2 & 1 & 0 & 1 \\ -1 & 0 & 0 & 1 \end{pmatrix}.$$

However, one checks that actually

$$P'^3 = \begin{pmatrix} 1 - \frac{\theta_1\theta_2}{2} & 0 & 0 & 0 \\ 0 & 1 - \theta_1\theta_2 & 0 & 0 \\ 0 & 0 & 1 - \frac{\theta_1\theta_2}{2} & 0 \end{pmatrix} = Z_{1 - \frac{\theta_1\theta_2}{2}} \in \mathbb{Z}.$$

First let us introduce the following notations:

**Definition 4.7.** For notational convenience, we write $\theta := (\theta_1, \theta_2)$, $a\theta := (a\theta_1, a^{-1}\theta_2)$ for $a \neq 0$, and $\theta^{op} := (\theta_2, \theta_1)$. We will write $a^{\pm i} := a^{3-2i} = \left\{ \begin{array}{ll} a & i = 1 \\ a^{-1} & i = 2 \end{array} \right.$.

We will denote the constant

$$c_\theta := \frac{1 + \theta_1\theta_2}{6}.$$  \hspace{2cm} (4.4)

Then $c_\theta$ is invariant under the rescaling $\theta \rightarrow a\theta$.

Finally we write $\theta \in [\Delta ABC]$ if $\theta$ is a representative of the odd invariant $[\theta]$ of the ordered triple $\Delta ABC$.

Now we can define the following more general prime transformations. Let $h_A, h_B, h_C$ be some positive numbers.

**Definition 4.8.** The general prime transformation $P_{h_B, h_C}^\theta$, which transforms $\Delta ABC \in S_{h_B\theta}^\theta$ to $\Delta BCA \in S_{h_C\theta}^\theta$ is given by

$$P_{h_B, h_C}^\theta := Z_{c_\theta} \circ Z_{h_C^{-1}} \circ P' \circ Z_{h_B}.$$

**Proposition 4.9.** We have the relations

$$P_{h_A, h_B}^\theta \circ P_{h_C, h_A}^\theta \circ P_{h_B, h_C}^\theta = 1.$$

By direct calculation we obtain
Lemma 4.10. The action of the general prime transformation $P_{h_B,h_C}^\theta$ on $\mathcal{L}_0$ is given by

$$P_{h_B,h_C}^\theta \cdot (x_1, x_2, y, z|\xi_1^+, \xi_2^+, \xi_1^-, \xi_2^-) = (x_1', x_2', y, z'|\xi_1'^+, \xi_2'^+, \xi_1'^-, \xi_2'^-)$$

where

$$x_2' := x_1,$$

$$z' := z + \frac{1}{2}(h_B^{-1}\theta_2 \xi_1^+ - h_B \theta_1 \xi_2^+ + \theta_1 \theta_2 x_1),$$

$$y' := -y + \frac{1}{2}(h_B^{-1}\theta_2 \xi_1^+ + h_B \theta_1 \xi_2^+) + x_1,$$

$$\xi_1'^- := c_\theta h_C^{-1}(h_B \xi_2^+ - \theta_2 x_1),$$

$$\xi_2'^- := c_\theta^{-1} h_C(-h_B^{-1} \xi_1^+ + \theta_1 x_1),$$

and by (3.1) we have

$$x_1' = \frac{y'^2}{x_2'}, \quad \xi_1'^+ = \frac{y'}{x_2'} \xi_2'^-, \quad \xi_2'^+ = -\frac{y'}{x_2'} \xi_1'^-.$$

For notational convenience, we will denote edges as well as their $\lambda$-lengths by the same (Latin lowercase) letter as long as it does not lead to confusion. The real positive numbers $h$, corresponding to the given vertex will from now on be labeled by the edge opposite to that vertex. We will sometimes denote

$$P_{h_a,h_b}^{\theta,+} := P_{h_a,h_b}^\theta, \quad P_{h_a,h_b}^{\theta,-} := (P_{h_b,h_a}^\theta)^{-1}. \quad (4.5)$$

By definition,

$$P_{h_a,h_b}^{\theta,+} = Z_{c_a} \circ Z_{h_b}^{-1} \circ P' \circ Z_{h_a},$$

$$P_{h_a,h_b}^{\theta,-} = Z_{c_a}^{-1} \circ Z_{h_b}^{-1} \circ P'^{-1} \circ Z_{h_a},$$

which denote respectively the clockwise and anti-clockwise prime transformations.

5. Basic calculation

In this section we deal with orbits of 4-tuples $(A, B, C, D) := \Diamond ABCD$ in the special light cone $\mathcal{L}_0$ such that $\Delta ABC$ and $\Delta CDA$ are positively ordered triples of points.

Recall the pairing (2.3) that is invariant under the action of $SL(1|2)$:

$$\langle M, M' \rangle = \frac{1}{2}(x_1 x_2' + x_2 x_1') - y y' + \frac{1}{2}(\xi_1^+ \xi_1'^- - \xi_2^+ \xi_2'^- - \xi_1^- \xi_1'^+ + \xi_2^- \xi_2'^+) + z z'. \quad (5.1)$$

Let $\Delta ABC \in S^\theta$ be in standard position

$$A = r(0, 1, 0, 0|0, 0, 0, 0),$$

$$B = t(1, 1, \frac{\theta_1 \theta_2}{2}|\theta_1, \theta_2, -\theta_2, \theta_1),$$

$$C = s(1, 0, 0, 0|0, 0, 0, 0).$$
Then (the squares of) $\lambda$-lengths of the edges by definition are given by

\[ a^2 = \langle A, B \rangle = \frac{rt}{2}, \quad b^2 = \langle B, C \rangle = \frac{st}{2}, \quad e^2 = \langle A, C \rangle = \frac{rs}{2}. \]  \hfill (5.2)

By abuse of notation, we will index the edge with the same label as the $\lambda$-length.

Let $\triangle CDA$ be another positively ordered triple, then $D$ is of the form

\[ D = (x_1, x_2, -y, z|\xi_1^+, \xi_2^+, \xi_1^-, \xi_2^-) \]

with $y > 0$. We have

\[ c^2 = \langle C, D \rangle = \frac{sx_2}{2}, \quad d^2 = \langle A, D \rangle = \frac{rx_1}{2}. \]  \hfill (5.3)

As in the bosonic case, we can solve for the even parameters from the $\lambda$-lengths themselves.

**Lemma 5.1.** Denote the cross-ratio by

\[ \chi = \frac{ac}{bd}. \]  \hfill (5.4)

Then we have

\[ r = \sqrt{2 \frac{ae}{b}}, \quad s = \sqrt{2 \frac{be}{a}}, \quad t = \sqrt{2 \frac{ab}{e}}, \quad x_1 = \sqrt{2 \frac{bd^2}{ae}} = \sqrt{2 \frac{cd}{e}} \chi^{-1}, \quad x_2 = \sqrt{2 \frac{ac^2}{be}} = \sqrt{2 \frac{cd}{e}} \chi, \quad y = \sqrt{2 \frac{cd}{e}}. \]

In particular we have

\[ \sqrt{\frac{x_2}{x_1}} = \frac{ac}{bd} = \chi. \]  \hfill (5.5)

Next let us calculate the odd invariants $[\sigma]$ of the triple $\triangle CDA$. 

---

**Figure 3.** Labelling with $A, B, C$ in standard position
Proposition 5.2. \( \Delta CDA \) can be transformed into standard position with
\[
C \mapsto \hat{r}(0, 1, 0, 0|0, 0, 0, 0), \quad D \mapsto \hat{t}(1, 1, 1, \frac{\sigma_1 \sigma_2}{2}|\sigma_1, \sigma_2, -\sigma_2, \sigma_1), \quad A \mapsto \hat{s}(1, 0, 0, 0|0, 0, 0, 0),
\]
where
\[
\hat{r} = \sqrt{x_1 x_2}, \quad \hat{s} = \frac{4}{\sqrt{x_1 x_2}}, \quad \hat{t} = \frac{4}{\sqrt{x_1 x_2}}, \quad (5.6)
\]
\[
\sigma_1 = \frac{\xi_1}{\sqrt{x_2}} \sqrt{x_1 x_2}, \quad \sigma_2 = \frac{\xi_2}{\sqrt{x_1 x_2}}, \quad (5.7)
\]
\[
\xi_1^+ = -\sqrt{\chi \sqrt{\frac{2}{e} d}} \sigma_1, \quad \xi_2^+ = -\sqrt{\chi \sqrt{\frac{2}{e} d}} \sigma_2, \quad \xi_1^- = -\sqrt{\chi \sqrt{\frac{2}{e} d}} \sigma_1, \quad \xi_2^- = -\sqrt{\chi \sqrt{\frac{2}{e} d}} \sigma_2, \quad (5.8)
\]
Proof. Acting on \( \Delta CDA \) by the diagonal transformation \( D_a \) with \( a = \sqrt{\frac{2}{e} d} = \sqrt{\chi} \) sends
\[
C \mapsto \hat{r}(0, 1, 0, 0|0, 0, 0, 0), \quad D \mapsto \hat{t}(1, 1, 1, -\frac{\sigma_1 \sigma_2}{2}|-\sigma_1, -\sigma_2, -\sigma_2, \sigma_1), \quad A \mapsto \hat{s}(0, 1, 0, 0|0, 0, 0, 0).
\]
Applying \( J \) then sends \( \Delta CDA \) to standard position and the result follows. \( \square \)

Corollary 5.3. Using Lemma 5.1 we can express all variables in terms of \( \lambda \)-lengths as follows
\[
\hat{r} = \sqrt{2 \frac{d e}{c}}, \quad \hat{s} = \sqrt{2 \frac{e c}{d}}, \quad \hat{t} = \sqrt{2 \frac{c d}{e}} = y,
\]
\[
\xi_1^+ = -\sqrt{\chi \sqrt{\frac{2}{e} d}} \sigma_1, \quad \xi_2^+ = -\sqrt{\chi \sqrt{\frac{2}{e} d}} \sigma_2, \quad (5.9)
\]
\[
\xi_1^- = -\sqrt{\chi \sqrt{\frac{2}{e} d}} \sigma_1, \quad \xi_2^- = -\sqrt{\chi \sqrt{\frac{2}{e} d}} \sigma_2, \quad (5.10)
\]
where we have used (3.1) from Proposition 3.1 for the expression of \( \xi_1^+, \xi_2^+, \xi_1^- \) and \( z \).

Recall that the odd invariants are defined modulo a rescaling. Let us introduce the following terminology.

Definition 5.4. A quadrilateral \( \Diamond ABCD \) is said to be in the standard \((\theta, \sigma)\)-position with ratio \( h_e \) if
\begin{itemize}
  \item The ordered triple \( \Delta ABC \in S^{h_e \theta} \) for \( h_e > 0 \),
  \item The triangle \( \Delta CDA \) is positively oriented, and \((J \circ D_{\sqrt{\chi} \sigma} Z_{h_e}) \cdot \Delta CDA \in S^{h_e^{-1} \sigma}\).
\end{itemize}
We will denote such a standard quadrilateral by
\[ \vartriangle ABCD \in S_{\theta e}^{\alpha, \sigma} \]
and call the transformation
\[ \Upsilon_{\theta e}^e := J \circ D\sqrt{\chi} \circ Z_{\theta e} \]
the upside-down transformation of the quadrilateral \( \vartriangle ABCD \). If the odd parameters \( \theta, \sigma \) have already been fixed, we will simply say \( \vartriangle ABCD \) has ratio \( e \) with \( \vartriangle CDA \) across the edge \( c \).

By Proposition 5.2 if \( D = (x_1, x_2, -y, z|\xi^+, \xi^0, \xi^-) \) for \( y > 0 \), then \( (\sigma_1, \sigma_2) = (-\xi^+ / \sqrt{y^2 + 1}, -\xi^- / \sqrt{y^2 + 1}) \). Note that the \( \lambda \)-lengths uniquely defines \( 4 \) points \( \vartriangle ABCD \in S_{\theta e}^{\alpha, \sigma} \) on the light cone by Lemma 5.1 and Corollary 5.3.

**Lemma 5.5.** Let \( \vartriangle ABCD \in S_{\theta e}^{\alpha, \sigma} \) with cross-ratio \( \chi \). Then
1. \( Z_{-1} \cdot \vartriangle ABCD \in S_{\theta e}^{\alpha, -\sigma} \) where \( -\theta := (-\theta_1, -\theta_2) \);
2. \( \Psi \cdot \vartriangle ABCD \in S_{\theta e}^{\alpha, \sigma_{\op}} \) where \( \sigma_{\op} := (\theta_2, \theta_1) \);
3. Let \( \vartriangle A'B'C'D' := \Upsilon_{\theta e}^e \cdot \vartriangle ABCD \). Then \( \vartriangle C'D'A'B' \in S_{h^{-1}_e}^{\alpha, \sigma} \).

**Proof.** (1) and (2) follows from the definition of the action.

(3) By definition \( \vartriangle C'D'A' \in S_{h^{-1}_e}^{\alpha, \sigma} \), while \( B' \) is of the form
\[ B \mapsto t(a^2, a^{-2}, -1, \frac{\theta_1 \theta_2}{2}|a \theta_1, a \theta_2, a^{-1} \theta_2, -a^{-1} \theta_1), \]
where \( a = \sqrt{\chi}^{-1} \). Hence \( \sigma_i^{new} := -\frac{\xi_i^+}{\sqrt{y^2 + 1}} = -\frac{a \theta_i}{\sqrt{1 + a^2}} = -\theta_i \) or \( \sigma_i^{new} = -\theta_i \).

**Proposition 5.6.** If \( \theta \in [\vartriangle ABC] \) and \( \sigma \in [\vartriangle CDA] \) for two positively ordered triples \( \vartriangle ABC \) and \( \vartriangle CDA \), then there exists \( g \in SL(1|2) \) such that \( g \cdot \vartriangle ABCD \in S_{\theta e}^{\alpha, \sigma} \) for some \( h_e > 0 \), where \( \sigma = \pm \sigma \) or \( \pm \sigma_{\op} \). The choice of \( \sigma \) is uniquely determined, while \( h_e \) is uniquely defined by \( \theta \) and \( \sigma \) up to multiplication by constants \( c_1 \) and \( c_2 \) such that \( c_1 \theta = \theta \) and \( c_2 \sigma = \sigma \).

**Proof.** By Proposition 4.5 there exists \( g \in SL(1|2) \) such that \( g \cdot \vartriangle ABCD \in S_{\theta e}^{\alpha, \sigma} \). Then \( (\Upsilon_{\theta e}^e \circ g) \cdot \vartriangle CDA \) is in standard position hence belongs to \( S_{\sigma'} \) for some \( \sigma' \in [\vartriangle CDA] \), therefore equals \( a \sigma \) or \( a \sigma_{\op} \) with \( a \neq 0 \). By Proposition 4.4 the stabilizer for \( \vartriangle ABC \) is given by \( Z_{\sigma} \) for some \( c > 0 \), hence the sign of \( a \) and the order \( \sigma \) or \( \sigma_{\op} \) is determined uniquely. Let \( a = \pm h_e c_0 \) with \( h_e > 0 \) and \( \sigma := h_e \sigma' \), applying \( Z_{h^{-1}_e} \in SL(1|2) \) we see that \( \vartriangle ABCD \in S_{h^{-1}_e}^{\alpha, \sigma} \) as required.

Finally, if \( \vartriangle ABCD \in S_{h^{-1}_e}^{\alpha, \sigma} \), and \( Z_{c} \cdot \vartriangle ABCD \in S_{h^{-1}_e}^{\alpha, \sigma} \), then \( c e \theta = h_e c_0 \theta \) and \( c \sigma = \sigma \), which implies \( h'_e = c_1 c_2 \) satisfying the condition above.

**Corollary 5.7.** The stabilizers of \( \vartriangle ABCD \in S_{\theta e}^{\alpha, \sigma} \) are of the form \( Z_{\sigma} \) for some even \( c \) such that \( c - 1 \) is an annihilator of \( \theta_i, \sigma_i, i = 1, 2 \).

**Proof.** From the proof above, we have \( c e \theta = \theta \) and \( c \sigma = \sigma \), hence the conclusion follows.
6. Spin structures and fatgraph connections

Let \( \tau \) be a trivalent fatgraph spine that is dual to some triangulation \( \Delta \) of the surface \( F := F_g^s \) determined up to isotopy. Let \( \tau_0, \tau_1 \) denote the set of vertices and edges of \( \tau \) respectively. In particular each vertex \( v \in \tau_0 \) corresponds to a triangle in \( \Delta \). Let \( \omega \) be an orientation on the edges \( \tau_1 \subset \tau \). As in [12], we define a fatgraph reflection at a vertex \( v \) of \( (\tau, \omega) \) to reverse the orientations of \( \omega \) on every edge of \( \tau \) incident to \( v \).

**Definition 6.1.** We define \( O(\tau) \) to be the equivalence classes of orientations on a trivalent fatgraph \( \tau \) spine of \( F \), where the equivalence relation is given by \( \omega_1 \sim \omega_2 \) iff \( \omega_1 \) and \( \omega_2 \) differ by finite number of fatgraph reflections. It is an affine \( H^1 \)-space where the cohomology group \( H^1 := H^1(F; \mathbb{Z}_2) \) acts on \( O(\tau) \) by changing the orientation of the edges along cycles.

In [12], various realizations of the spin structures on the surface \( F \) are described.

**Definition 6.2.** [9] A spin structure is determined by a lift \( \tilde{\rho} : \pi_1(F) \longrightarrow SL(2, \mathbb{R}) \) of the Fuchsian representation \( \rho : \pi_1(F) \longrightarrow PSL(2, \mathbb{R}) \) familiar from Teichmüller theory.

In fact given a spin structure, one can define a quadratic form \( q : H_1(F; \mathbb{Z}_2) \longrightarrow \mathbb{Z}_2 \) on simple cycles \( \gamma \in \pi_1(F) \) by

\[
q([\gamma]) = \text{sign} \text{Tr}(\tilde{\rho}(\gamma)),
\]

and extend to all of \( H_1(F; \mathbb{Z}_2) \) by \( q(a + b) = q(a) + q(b) + a \cdot b \) where \( a \cdot b \) denotes the intersection form. Then we have

**Proposition 6.3.** [2] [12] The set of spin structures is isomorphic to the space of quadratic forms \( Q(F) \) on \( H_1(F; \mathbb{Z}_2) \), and also isomorphic to \( O(\tau) \), as affine \( H^1 \)-spaces.

In terms of oriented fatgraphs, under the chosen isomorphism in [12] between \( O(\tau) \) and \( Q(F) \), the evolutions for spin structure under a change of the triangulations generated by flips that preserves the corresponding quadratic forms are described. By choosing appropriate representatives of the orientation classes, the flip transformation is equivalent to the rule depicted in Figure 4 i.e., the middle arrow changes from pointing up to pointing left, and the upper right arrow changes orientation, while the other branches remain the same.

![Figure 4. Spin graph evolution with \( \epsilon_i \) denote orientations](image)

A slight modification of the rule has to be made if some of the branches are identified, e.g., in the cases \( F_1^1 \) and \( F_0^3 \). This will be described in Appendix [13].
In the next section, starting from an element \( s \in \mathcal{O}(\tau) \), we will construct the one-to-one correspondence providing the lift \( \tilde{\rho} \) as above. Henceforth we will also refer to elements of \( \mathcal{O}(\tau) \) as spin structures as well.

**Definition 6.4.** Fix a fatgraph \( \tau \), we denote by \( o_\omega(e) \) the orientation of the edge \( e \in \tau \) in the orientation \( \omega \). We define

\[
\delta_{\omega_1, \omega_2}(e) := \begin{cases} +1 & o_{\omega_1}(e) = o_{\omega_2}(e), \\ -1 & o_{\omega_1}(e) \neq o_{\omega_2}(e), \end{cases}
\]

which defines an element in \( H^1(F; \mathbb{Z}_2) \).

**Definition 6.5.** Let \( G \) be a group. A \( G \)-graph connection on \( \tau \) is the assignment \( h_e \in G \) to each oriented edge \( e \) of \( \tau \) so that \( h_\tau = h_e^{-1} \) if \( \tau \) is the opposite orientation to \( e \). Two assignments \( \{h_e\}, \{h'_e\} \) are equivalent iff there are \( t_v \in G \) for each vertex \( v \) of \( \tau \) such that \( h'_e = t_v h_e t_w^{-1} \) for each oriented edge \( e \in \tau_1 \) with initial point \( v \) and terminal point \( w \).

As an immediate consequence of Proposition 6.3, we obtain

**Corollary 6.6.** The space of spin connections on \( F \) is identified with \( \mathbb{Z}_2 \)-graph connections on a given fatgraph \( \tau \) of \( F \).

**Proof.** Pick an orientation \( \omega \) on fatgraph \( \tau \). For any orientation \( \omega' \), we label the edge by \( +1 \) if they agree on that edge and by \( -1 \) if they do not agree. Classes of orientations give rise to classes of \( \mathbb{Z}_2 \)-connections, since the reflection of orientation at the vertices corresponds to multiplication by \( -1 \). \( \square \)

For a Lie (super) group \( G \), the following statement is true (see e.g., [4]).

**Proposition 6.7.** The moduli space of flat \( G \)-connections on \( F \) is isomorphic to the space of equivalence classes of \( G \)-graph connections on \( \tau \).

The case which will be exploited fully in the next section is when \( G = \mathbb{R}^+ \), i.e., the set of all even numbers in \( S \) (underlying the Grassmann algebra over \( \mathbb{R} \)) with positive bodies.

Since the product of \( h_e \) around each cycle gives the monodromies defining the flat connection, the Proposition implies that one can always rescale \( h_e \) around vertices in such a way such that if \( a, b, e \) are three edges oriented towards a vertex \( v \), we have \( h_a h_b h_e = 1 \), and this uniquely defines the set of equivalence classes of graph connections on \( \tau \). In particular for \( F^s_g \) it is known that the moduli space of flat \( G \)-connections has dimension \( (2g + s - 1) \dim(G) \).

### 7. Decorated Super-Teichmüller Space

In this section we describe the coordinates of the decorated super-Teichmüller space using the basic calculation from Section 5.

We fix a surface \( F := F^s_g \) with genus \( g \geq 0 \) and \( s \geq 1 \) punctures where \( 2g + s - 2 > 0 \). We fix a triangulation \( \Delta \) of \( F \) and the corresponding trivalent fatgraph \( \tau \subset F \). We consider the lift of \( \Delta \) to an ideal triangulation \( \tilde{\Delta} \) with corresponding fatgraph \( \tau \) of the universal cover \( \tilde{\pi} : \tilde{F} \rightarrow F \) where \( \tilde{F} \) is topologically equivalent to the unit disk with
hyperbolic metric, and where the ideal vertices $\Delta_\infty$ of $\tilde{\Delta}$ lie on the boundary circle $S^1$ at infinity.

**Definition 7.1.** Given $F, \Delta$ as above. We define the coordinate system $\tilde{C}(F, \Delta)$ as follows:

- we assign to each edge $e$ of $\Delta$ a positive even coordinate called the $\lambda$-length, also denoted by $e$;

- we assign to each triangle of $\Delta$ two ordered odd coordinates $(\theta_1, \theta_2)$;

- we assign to each edge $e$ of a triangle of $\Delta$ a positive even coordinate $h_e$, called the ratios, such that if $h_e$ and $h'_e$ are assigned to two triangles sharing the same edge $e$ we have $h_e h'_e = 1$.

The odd coordinates are defined up to an overall sign changes $\theta_i \leftrightarrow -\theta_i$, as well as an overall involution $(\theta_1, \theta_2) \leftrightarrow (\theta_2, \theta_1)$.

Note that the ratios $\{h_e\}$ uniquely define an $\mathbb{R}_+$-graph connection on $\tau$, where the oriented edge of $\tau$ pointing towards a triangle crossing the edge $e$ will have the value $h_e$ of that triangle assigned.

**Definition 7.2.** Let $\vec{c} \in \tilde{C}(F, \Delta)$ be a coordinate vector. If $h_a, h_b, h_e$ are ratios assigned to a triangle $T$ with odd coordinates $(\theta_1, \theta_2)$, then a vertex rescaling at $T$ of $\vec{c}$ is the new coordinate vector obtained by changing

$$ (h_a, h_b, h_e, \theta_1, \theta_2) \mapsto (\alpha h_a, \alpha h_b, \alpha h_e, \alpha^{-1} \theta_1, \alpha \theta_2) \quad (7.1) $$

for some $\alpha > 0$, and all other coordinates fixed. Two coordinate vectors of $\tilde{C}(F, \Delta)$ are said to be equivalent if they are related by a finite number of vertex rescalings. In particular the underlying graph connections on $\tau$ are equivalent (cf. Definition 6.5).

Let $C(F, \Delta) := \tilde{C}(F, \Delta)/\sim$ be the equivalence classes of coordinate vectors. Then by Proposition 6.7 it can be represented by coordinates with $h_a h_b h_e = 1$ for the ratios of the same triangle. We can easily deduce that

$$ C(F, \Delta) \simeq \mathbb{R}^{8g+4s-7|8g+4s-8}/\mathbb{Z}_2 \times \mathbb{Z}_2 \quad (7.2) $$

in accordance with the dimension given in [9] (with an extra $s$-dimension given by decorations). The coordinates naturally pullback to coordinates on the ideal triangulations $\tilde{\Delta}$ of the universal cover $\tilde{F}$ by $\pi$.

**Definition 7.3.** For each fatgraph $\tau \subset F$ and each spin structure $s \in O(\tau)$, we will fix one representative orientation which will be denoted by $\omega_{s,\tau}$ such that $[\omega_{s,\tau}] = s \in O(\tau)$.

As we have seen in Proposition 5.6, the transformations involving ordered triples and quadrilaterals in $L_0$ possess some freedom involving the $\mathbb{Z}$-subgroup elements. In order to fix this degree of freedom (which will be absorbed into the ratios $h_e$), we introduce the following definition.

**Definition 7.4.** Given a coordinate vector $\vec{c} \in \tilde{C}(F, \Delta)$, a transformation $g \in \tilde{SL}(1|2)$ is called $\vec{c}$-admissible if $g$ is a composition only of
prime transformations $P^0_{h_a,h_b}$;

- the upside-down transformations $\Upsilon^\chi_{h_c} := J \circ D \sqrt{\tau} \circ Z_{h_c}$;

- sign changes $J^2 = Z_{-1}$ and involution $\Psi$,

where $h_a,h_b$ are ratios of the triangle with odd coordinates $\theta$, and $\chi,h_c$ are the cross-ratio and ratio of the diagonal of the quadrilateral to be affected by the upside-down transformation.

Now we can state the first main result of the paper.

**Theorem 7.5.** Fix $F,\Delta,\tau$ as before. Let $\omega_\sigma := \omega_{s_\sigma,\tau}$ corresponds to a specified spin structure $s_\sigma$ of $F$, and let $\omega_i := \omega_{s_i,\tau}$ be the representative of another spin structure $s_i$. Given a coordinate vector $\vec{c} \in \tilde{C}(F,\Delta)$, there exists a map called the lift,

$$\ell_{\omega_\sigma,\omega_i} : \tilde{\Delta}_\infty \rightarrow \mathcal{L}_0,$$

such that if $\Diamond ABCD$ is the image of a quadrilateral from $\tilde{\Delta}_\infty$ with coordinates labeled as in Figure 3, $\Delta ABC$ has ratio $h_e$ with $\Delta CD A$ across the edge $e$, and $t$ is the edge of the fatgraph $\tau$ with orientation $\omega$ pointing from $\Delta CDA \rightarrow \Delta ABC$, then

$$t : \Delta_{ABC} \leftarrow \Delta_{CDA} \Rightarrow \left\{ \begin{array}{ll} g \cdot \Diamond ABCD \in S_{\ell_{\omega_\sigma,\omega_i}}^{0,\sigma} & \text{if } \delta_{\omega_\sigma,\omega_i}(t) = 1, \\
 g \cdot \Diamond ABCD \in S_{h_c}^{0,\sigma} & \text{if } \delta_{\omega_\sigma,\omega_i}(t) = -1 \end{array} \right. \quad (7.3)$$

for some $g \in \tilde{SL}(1|2)$ that is $\vec{c}$-admissible.

The lift is uniquely determined up to post-composition by $\tilde{SL}(1|2)$, and only depends on the equivalent classes $C(F,\Delta)$ of the coordinates.

**Theorem 7.6.** Fixing a coordinate vector $\vec{c} \in \tilde{C}(F,\Delta)$ and given a lift $\ell := \ell_{\omega_\sigma,\omega_i} : \tilde{\Delta}_\infty \rightarrow \mathcal{L}_0$ constructed in Theorem 7.5, there is a representation $\hat{\rho} : \pi_1 := \pi_1(F) \rightarrow \tilde{SL}(1|2)$ uniquely determined up to conjugacy by an element of $\tilde{SL}(1|2)$ such that

1. $\ell$ is $\pi_1$-equivariant, i.e., $\hat{\rho}(\gamma)(\ell(a)) = \ell(\gamma(a))$ for each $\gamma \in \pi_1$ and $a \in \tilde{\Delta}_\infty$;

2. $\hat{\rho}$ is a super Fuchsian representation, i.e., the natural projection

   $$\rho : \pi_1 \xrightarrow{\hat{\rho}} \tilde{SL}(1|2) \rightarrow SL(2,\mathbb{R}) \rightarrow PSL(2,\mathbb{R})$$

   is a Fuchsian representation;

3. the lift $\tilde{\rho} : \pi_1 \xrightarrow{\tilde{\rho}} \tilde{SL}(1|2) \rightarrow SL(2,\mathbb{R})$ of $\rho$ does not depend on $\omega_i$, and the space of all such lifts is in one-to-one correspondence with the spin structures $[\omega_\sigma] \in \mathcal{O}(\tau)$.

**Definition 7.7.** The space of $\tilde{SL}(1|2)$-orbits of lifts $\ell : \tilde{\Delta}_\infty \rightarrow \mathcal{L}_0$ that is $\pi_1$-equivariant for some super Fuchsian representation $\hat{\rho} : \pi_1 \rightarrow \tilde{SL}(1|2)$ is called the decorated super Teichmüller space $\tilde{ST}(F)$.

**Proof.** (Theorem 7.5) The construction is similar to the one presented in [12] but with some modifications. In particular our construction of the lift here will directly incorporate the spin structures by determining explicitly the signs and orders of the odd coordinates $\theta = (\theta_1, \theta_2)$ for each triangle.
First we note that by Lemma 5.5, if \( t \) has the opposite orientation instead, then (7.3) implies

\[
\begin{align*}
t : \Delta_{ABC} &\rightarrow \Delta_{CDA} \\
g \cdot \Diamond ABCD &\in S_{h_{e}}^{\theta,-\sigma} \quad \text{if } \delta_{\omega_{e},\omega_{i}}(t) = 1, \\
g \cdot \Diamond ABCD &\in S_{h_{e}}^{\theta,-\sigma_{op}} \quad \text{if } \delta_{\omega_{e},\omega_{i}}(t) = -1
\end{align*}
\]  

(7.4)

for some admissible \( g \in \tilde{SL}(1|2) \).

To begin our construction, we first choose a distinguished triangle-edge pair \((T, e)\) in \( \tilde{\Delta}_{\infty} \) with specified \( \lambda \)-lengths, odd coordinates \( \theta \), and ratio \( h_{e} \). Then according to Lemma 5.1 there is a unique triangle \( \Delta_{ABC} \subset \mathcal{L}_{0} \) realizing the \( \lambda \)-lengths and satisfying \( \Delta_{ABC} \in S_{h_{e}}^{\theta} \) where \( BC \) the image of the distinguished edge. We define \( \Delta_{ABC} \) to be the lift of \( T \) and call \( T \) the base triangle of our lift.

Consider the adjacent triangle \( T' \) opposite to \( e \) with odd coordinates \( \sigma \). Then we define \( \Delta_{CDA} \) to be the lift of \( T' \), where \( D \in \mathcal{L}_{0} \) is the unique point by Proposition 5.2 that realizes the \( \lambda \)-lengths, and satisfies (7.3) or (7.4) with \( g = 1 \). We define \( \Delta_{CDA} \) to be the image of \( T' \).

Now consider the lift \( \Delta AD'B \) of the adjacent triangle of \( T \) with odd coordinates \( \sigma' \) and ratio \( h_{a} \) at the edge \( AB \) of \( \Delta_{ABC} \). Applying \( P_{h_{e},h_{a}}^{\theta} \), we bring \( \Delta BCA \) to standard position. Then the point \( D' \in \mathcal{L}_{0} \) is uniquely determined by \( \Diamond BCD' \in S_{h_{a}}^{\theta',\sigma'} \) where \( \tilde{\sigma}' = \pm \sigma' \) or \( \pm \sigma'_{op} \) according to (7.3) or (7.4). If we now apply the upside-down transformation \( \Upsilon_{h_{a}}^{\chi'} \) where \( \chi' \) is the cross-ratio for \( \Diamond BCD' \), and if necessary the sign change \( J_{2} \) and involution \( \Psi \), we bring \( \Delta AD'B \in S_{h_{a}}^{-1}\sigma' \) and we can continue our inductive process of lifting other triangles.

More precisely, fixing \( T \) as the base triangle of our lift, the lift \( \Delta A'B'C' \) of a triangle \( T' \) with specified \( \lambda \)-lengths, odd coordinates \( \theta' \) and ratio \( h' \) across \( A'C' \) is defined to be \( g^{-1} \cdot \Delta_{g'} \), where \( \Delta_{g'} \in S_{h'}^{\theta'\sigma'} \) is determined by the unique ordered triple in \( \mathcal{L}_{0} \) with specified fermions and \( \lambda \)-lengths of \( T' \), and \( g \in \tilde{SL}(1|2) \) is the unique admissible transformation constructed above bringing \( \Delta A'B'C' \) to standard position along the path of \( \tilde{\tau} \) joining \( T \) and \( T' \) in the universal cover.

We completely define the lift \( \ell : \tilde{\Delta}_{\infty} \rightarrow \mathcal{L}_{0} \) in this manner, which satisfies the statement of the Theorem by construction. Choosing a different base triangle and edge pair amounts to an overall action by \( g \in \tilde{SL}(1|2) \) on \( \mathcal{L}_{0} \), where \( g \) is the unique admissible transformation that brings the image of this new triangle in the original lift to standard position, i.e., \( \ell'_{new} = g \cdot \ell \). Hence the lift \( \ell \) is uniquely defined up to post-composition with an \( \tilde{SL}(1|2) \) element.

Finally, under a vertex rescaling \( \alpha \) of the coordinate at some triangle \( T_{\theta} \) that is not the base triangle, the admissible transformation along the portion of the path
$T_{\sigma} \rightarrow T_{\theta} \rightarrow T_{\sigma'}$ that passes through $T_{\theta}$ changes as (cf. (1.5))

$$g_{\alpha} = \ldots \circ P_{\alpha^{-1}h_{\epsilon}} \circ \gamma_{h_{\epsilon}} \circ P_{\alpha h_{\epsilon}} \circ \gamma_{h_{\epsilon}} \circ \ldots$$

where we used the fact that $P_{\alpha^{-1}h_{\epsilon}} = P_{h_{\epsilon}}$. Hence the definition of our lift does not change. The case with sign changes and involutions are similar. However if $T_{\theta}$ is our base triangle, then the lift differs by an overall conjugation of $Z_{\alpha} \in \widetilde{SL}(1|2)$. \hfill $\square$

Proof. (Theorem 7.6) The construction is a simplified version of [12] without the need to modify the signs of $\tilde{\rho}$ for the spin structures since they are already incorporated in the construction of the lift.

As in [12], we fix a base triangle $T$ for our lift and choose a connected fundamental domain $D \subset \tilde{F}$ for the action of $\pi_{1}$ that contains $T$. Then $D$ is a $4g + 2s$-sided ideal polygon, and in particular, the frontier edges of $D$ in $\tilde{F}$ arise in pairs $c, c'$ together with an abstract identification $c' = \gamma(c)$ induced by some $\gamma \in \pi_{1}$. We let $c'_{i} = \gamma(c_{i})$ enumerate the collection of these edges pairing, where $i = 1, ..., 2g + s$. Then it is known that $\pi_{1}$ is the free group generated by these $2g + s$ elements $\gamma_{i}$.

To determine the image of $\tilde{\rho}(\gamma_{i}) \in \widetilde{SL}(1|2)$, let $\Delta ABC$ and $\Delta A'B'C'$ be the lift of the unique pair of triangles such that $BC = \ell(c_{i}), B'C' = \ell(c'_{i}), \ell^{-1}(ABC) \subset D$, and $\ell^{-1}(A'B'C') \not\subset D$. Then by definition of $\ell$, there are unique admissible transformations $g, g'$ bringing standard position from $\Delta ABC$ and $\Delta A'B'C'$ respectively to $\ell(T)$. We define

$$\tilde{\rho}(\gamma_{i}) := g'^{-1}g \in \widetilde{SL}(1|2). \quad (7.5)$$

Since $\pi_{1}$ is a free group this defines our representation $\tilde{\rho}$.

More explicitly, let $\gamma_{i}$ be homotopically represented by a path in $\tau$. Then $\tilde{\rho}(\gamma_{i})$ is a composition of the form

$$\tilde{\rho}(\gamma_{i}) = \prod_{k} \gamma_{h'_{k}} \circ P_{h_{k},h_{k}} \in \widetilde{SL}(1|2) \quad (7.6)$$

that brings standard position from $\Delta ABC$ to $\Delta A'B'C'$ moving along the edge of $\tau$, where $P_{h_{k},h_{k}}$ is the prime transformation corresponding to turning left (+) or right (−) at the vertex of $\tau$ with starting ratio $h_{k}$ and ending ratio $h'_{k}$. The signs $Z_{\pm}$ of $\Psi_{\pm} \in \{Id, \Psi\}$ correspond to the orientations $\omega_{\sigma}$ and $\omega_{\iota}$ respectively according to the rule (7.3) and (7.4).

The same argument as in [12] shows that choosing a different initial triangle defining our lift $\ell$, and choosing a different fundamental domain $D'$ for our construction, amounts again to an overall conjugation of $\tilde{\rho}$ by an admissible element of $\widetilde{SL}(1|2)$. (1) follows directly from the definition of the construction of $\tilde{\rho}$. By an argument in [10, 11], (2) follows from the fact that the bosonic reduction $\rho(\pi_{1}) \subset PSL(2, \mathbb{R})$ by construction leaves invariant the tessellation $\tilde{\Delta} \subset \tilde{F} \simeq \mathbb{D}$. 

$\mathcal{N} = 2$ SUPER-TEICHMULLER THEORY
Finally to show (3), recall that the projection $\tilde{SL}(1|2) \to SL(2, \mathbb{R})$ maps $\Psi \mapsto 1$, hence $\tilde{\rho}$ does not depend on $\omega_i$. Now we note that if $\omega$ and $\omega'$ are related by a fatgraph reflection at the vertex $v$, then the orientation of either zero or two edges of $\tau$ are flipped along each cycle. Since the projection $Z_{-1} \to (-1)$ to $SL(2, \mathbb{R})$ commutes with every operators in (7.6), the projection $\tilde{\rho}(\gamma_i) \in SL(2, \mathbb{R})$ remains unchanged under fatgraph reflection. In particular if $[\omega]$ and $[\omega']$ defines the same lift $\tilde{\rho}$ to $SL(2, \mathbb{R})$, then all cycles of $\pi_1$ differ in orientations at even number of edges of $\tau$, which implies $[\omega] = [\omega'] \in \mathcal{O}(\tau)$, thus proving the one-to-one correspondence between lifts $\tilde{\rho}$ and spin structures. \hfill $\square$

**Corollary 7.8.** Given an oriented simple cycle $\gamma \in \pi_1(F)$ homotopic to a path on the fatgraph $\tau$ with orientation class $[\omega]$, the quadratic form corresponding to the lift $\tilde{\rho} : \pi_1 \to SL(2, \mathbb{R})$ according to (6.1) is given by

$$q([\gamma]) = (-1)^{L_\gamma} (-1)^{N_\gamma} = (-1)^{R_\gamma} (-1)^{\overline{N}_\gamma},$$

(7.7)

where $L_\gamma$ (resp. $R_\gamma$) is the number of left (resp. right) turns of $\gamma$ on the fatgraph $\tau$, and $N_\gamma$ (resp. $\overline{N}_\gamma$) is the number of edges of $\tau$ such that $\gamma$ and $\omega$ have the same (resp. opposite) orientation. Moreover, under the flip transformation, the spin structure changes according to the rule expressed in the Figure 4.

**Proof.** According to (6.1), we need to calculate the sign of the trace of $\tilde{\rho}(\gamma)$. By the explicit expression (7.6), under the projection to $SL(2, \mathbb{R})$, the $Z_{h_k}$ elements for $h_k > 0$ become $h_k I_{2 \times 2}$ hence does not affect the sign. Therefore the projection of the representation of $\tilde{\rho}(\gamma)$ is given by $N_\gamma$ number of signs

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

and compositions of left and right turns:

$$L_k := \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a_k & 0 \\ 0 & a_k^{-1} \end{pmatrix} = \begin{pmatrix} -a_k & -a_k^{-1} \\ 0 & -a_k \end{pmatrix},$$

$$R_k := \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a_k & 0 \\ 0 & a_k^{-1} \end{pmatrix} = \begin{pmatrix} a_k & 0 \\ a_k & a_k^{-1} \end{pmatrix},$$

where $a_k > 0$. Then it can easily be shown by induction that multiplication by $L_k$ change the sign of trace while $R_k$ does not, hence the trace depends on the number of left turns $L_\gamma$ made by $\gamma$. Finally let $|\gamma|$ be the length of the cycle, then we have $R_\gamma = |\gamma| - L_\gamma$ and $N_\gamma = |\gamma| - \overline{N}_\gamma$ hence the second equation follows.

In order to prove the last part of the statement, one just needs to follow the same route as in (12), namely consider various pieces of curves running along the part of the fatgraph which is affected by the flip and compare the contributions to the quadratic form from those before and after the flip. \hfill $\square$

**Remark 7.9.** One can show that using the special Kasteleyn orientation and the canonical dimer described in (12), the formula in (2) produces a quadratic form that differs from (7.7) by an overall change of signs for each simple cycle. Hence if we modify the definition of $\tilde{\rho}(\gamma_i)$ by composition with $Z_{-1}$, our construction will agree with the isomorphisms in Proposition 6.3 described previously in the literature.
Theorem 7.10. The components of $\tilde{S}(F)$ are determined by two spin structures $s_\sigma, s_\iota \in O(\tau)$. For fixed representatives of the spin structures, $C(F,\Delta)$ provides an analytic homeomorphism from each component to $\mathbb{R}^{8g+4s-7} / \mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. By the uniqueness of the basic calculation from Lemma 5.1 and Proposition 5.2, given a lift $\ell \in \tilde{S}(F)$ for fixed representatives of the spin structures, the coordinates are uniquely determined since they are defined intrinsically in $L_0$.

If $\omega_\sigma$ and $\omega_\iota$ are representatives of $s_\sigma$ (resp. $s_\iota$) that differ by a fatgraph reflection, then from the construction of the lift $\ell$ we see that changing the coordinates from $C(F,\Delta)_{\omega_\sigma}$ to $C(F,\Delta)_{\omega_\iota}$ by $\theta \rightarrow -\theta$ (resp. $\theta \rightarrow \theta^{op}$) of the corresponding vertex defines the same lift. Hence the components of $\tilde{S}(F)$ depends only on the equivalence classes $s_\sigma$ and $s_\iota$. □

Remark 7.11. Removal of the decoration, i.e., the construction of the coordinates on the $N=2$ super-Teichmüller space follows standard bosonic case. From the construction of the lift, we obtain that the representation $\tilde{\rho}$ of $\tilde{SL}(1|2)$ depends on $\lambda$-lengths only through cross-ratios $\chi$. Therefore, passing from $\lambda$-lengths to cross-ratios for every edge, we obtain a coordinate system for $T(F)$, either as a subspace determined by one constraint for each puncture of $F$ or more generally for $F$ as a surface with holes.

Remark 7.12. One can obtain the coordinates on the (decorated) $N=1$ Teichmüller space by considering lifts such that $h_e = 1$ for every edge $e$ and $\theta_1 = \theta_2$ for every triangle. That precisely corresponds to the reduction $\tilde{SL}(1|2) \rightarrow OSp(1|2)$.

8. Ptolemy transform

In this section, we describe the Ptolemy transform on the coordinate system $\tilde{C}(F,\Delta)$ of $\tilde{S}(F)$. In particular, our result gives the mapping class group action on the super Teichmüller space. More precisely,

Definition 8.1. If $\Delta$ and $\Delta'$ differ by a flip of the diagonal in one quadrilateral, and the underlying fatgraphs $\omega_\sigma, \omega_\iota$ evolve according to the rule given by Figure 7, then the local coordinate transformation $\tilde{C}(F,\Delta) \rightarrow \tilde{C}(F,\Delta')$ around the quadrilateral:

$$\begin{align*}
(\ldots, a, b, c, d, e, h_a, h_b, h_c, h_d, h_e, \ldots, \theta_1, \theta_2, \sigma_1, \sigma_2, \ldots) & \mapsto (\ldots, a, b, c, d, f, h'_a, h'_b, h'_c, h'_d, h_f, \ldots, \mu_1, \mu_2, \nu_1, \nu_2, \ldots) \\
\text{(and all other coordinates fixed)}
\end{align*}$$

(8.1)

(8.2)

Since the definition of flip depends only on the equivalence classes $C(F,\Delta)$ of the coordinate vectors, the Ptolemy transform descends to a coordinate transformation $C(F,\Delta) \rightarrow C(F,\Delta')$. Since all the ratios under the projection from $\tilde{C}(F,\Delta)$ to $C(F,\Delta)$ will be rescaled, the transformation is no longer local.

As we shall see, the calculations for the $\lambda$-lengths are similar to [12]. However, the calculation of the odd coordinates and the ratios are significantly different due to the rescaling properties.
8.1. Even Ptolemy transformations. Since the $\lambda$-lengths are invariant under the action of $\widetilde{SL}(1|2)$, let us assume the quadrilateral in question is in standard position.

Theorem 8.2. Assume $\diamond ABCD \in \mathcal{S}^{\theta,\sigma}_{h_e}$ labeled as in Figure 5 and also $f^2 := \langle B, D \rangle$. Then we have the (bosonic) Ptolemy relations

$$ef = (ac + bd) \left( 1 + \frac{h_e^{-1} \sigma_1 \theta_2}{2(\sqrt{\chi} + \sqrt{\chi}^{-1})} + \frac{h_e \sigma_2 \theta_1}{2(\sqrt{\chi} + \sqrt{\chi}^{-1})} \right). \quad (8.3)$$

The Ptolemy relation is invariant under the change of representatives $\theta, \sigma$ and the corresponding $h_e$.

Proof. By definition we have the coordinates

$$B = t(1, 1, 1, \frac{\theta_1 \theta_2}{2} | h_e \theta_1, h_e^{-1} \theta_2, -h_e^{-1} \theta_2, h_e \theta_1),$$

$$D = (x_1, x_2, -y, z, \xi_1^+, \xi_1^-, \xi_2^+, \xi_2^-).$$

Figure 5. Ptolemy transformation of $\diamond ABCD$
Writing all the variables in terms of $\lambda$-lengths, we have
\[
f^2 = \langle B, D \rangle = \frac{b^2d^2}{e^2} + \frac{a^2c^2}{e^2} + \frac{2abcd}{e^2} + \frac{abcd}{e^2} (h_e^{-2}\sqrt{\chi}\sigma_2 - h_e^{-2}\sqrt{\chi}\sigma_1 - h_e^{-2}\sqrt{\chi}^{-1}\theta_2\sigma_1 - h_e^{-2}\sqrt{\chi}^{-1}\theta_1\sigma_2) + \frac{abcd}{2e^2} \sigma_1\sigma_2\theta_1\theta_2
\]
\[
= \frac{(ac + bd)^2}{e^2} + \frac{abcd}{e^2} (h_e^{-2}(\sqrt{\chi} + \sqrt{\chi}^{-1})\sigma_1\theta_2 + h_e^2(\sqrt{\chi} + \sqrt{\chi}^{-1})\sigma_2\theta_1 + \frac{1}{2}\sigma_1\sigma_2\theta_1\theta_2).
\]
Hence
\[
e^2f^2 = (ac + bd)^2 + R,
\]
where
\[
R = abcd \left( h_e^{-2}(\sqrt{\chi} + \sqrt{\chi}^{-1})\sigma_1\theta_2 + h_e^2(\sqrt{\chi} + \sqrt{\chi}^{-1})\sigma_2\theta_1 + \frac{1}{2}\sigma_1\sigma_2\theta_1\theta_2 \right).
\]
Note that
\[
R^2 = 2(abcd)^2(\sqrt{\chi} + \sqrt{\chi}^{-1})^2\sigma_1\sigma_2\theta_1\theta_2
\]
and $R^n = 0$ for $n \geq 3$. Also we have
\[
(ac + bd)^2 = (\sqrt{\chi} + \sqrt{\chi}^{-1})^2(abcd).
\]
Hence we have
\[
ef = (ac + bd) \sqrt{1 + \frac{R}{(ac + bd)^2}}
\]
\[
= (ac + bd) \left( 1 + \frac{1}{2} \frac{R}{(ac + bd)^2} - \frac{1}{8} \frac{R^2}{(ac + bd)^4} \right)
\]
\[
= (ac + bd) \left( 1 + \frac{h_e^{-2}(\sqrt{\chi} + \sqrt{\chi}^{-1})\sigma_1\theta_2 + h_e^2(\sqrt{\chi} + \sqrt{\chi}^{-1})\sigma_2\theta_1 + \frac{1}{2}\sigma_1\sigma_2\theta_1\theta_2}{2(\sqrt{\chi} + \sqrt{\chi}^{-1})^2}
\]
\[
- \frac{\sigma_1\sigma_2\theta_1\theta_2}{4(\sqrt{\chi} + \sqrt{\chi}^{-1})^2} \right)
\]
\[
= (ac + bd) \left( 1 + \frac{h_e^{-2}\sigma_1\theta_2}{2(\sqrt{\chi} + \sqrt{\chi}^{-1})} + \frac{h_e^2\sigma_2\theta_1}{2(\sqrt{\chi} + \sqrt{\chi}^{-1})} \right).
\]

By Proposition 5.6 for a different choice of representative $\theta', \sigma'$, there exists $g' \in Z \subset \hat{SL}(1|2)$ such that $g' \cdot \diamond ABCD \in S_{h_e}^{\theta', \sigma'}$. Hence formula (8.3) remains invariant since the pairing is invariant under the $Z$-action. \hfill $\square$
8.2. **Odd Ptolemy transformation.** Since the (equivalence classes of) odd invariants do not change under the $\widetilde{SL}(1|2)$ action, let us first calculate the Ptolemy action on fermions. Assume $\Diamond ABCD \in S^\theta_{h_a}$. We would like to bring $\Diamond DABC \in S^\mu_{h_b}$ by the $\widetilde{SL}(1|2)$ action and calculate the new odd parameters $\mu, \nu$ which comprise the odd Ptolemy transformation. On the other hand, with the new configuration we will have new ratios $h'_a, h'_b, h'_c, h'_d$ with the attaching 4 triangles outside $\Diamond DABC$. Recall that the lift is the same under vertex rescaling, so we have a freedom to rescale the fermions $\mu$ and $\nu$, which we will do to fix the ratio $h'_a$ and $h'_b$ in the sequel that is defined up to certain constants according to Proposition 5.6.

Here we will assume that no two sides of $\Diamond ABCD$ are identified with one other under the projection to the surface $\widetilde{F} \rightarrow F$, so that the attaching triangles at the sides $a, b, c, d$ do not change under Ptolemy transformation of $\Diamond ABCD$. The special cases when sides are identified is treated in Appendix B.

We start with $\Diamond ABCD$ in standard position

\begin{align*}
A &= r(0, 1, 0, 0, 0, 0, 0, 0), \\
B &= t(1, 1, 1, \frac{\theta_1 \theta_2}{2}, h_e \theta_1, h_e^{-1} \theta_2, -h_e^{-1} \theta_2, h_e \theta_1), \\
C &= s(1, 0, 0, 0|0, 0, 0, 0), \\
D &= (x_1, x_2, -y, z; \xi_1^+, \xi_2^+, \xi_1^-, \xi_2^-).
\end{align*}

First using the prime transformation $P_{h_e, h_a}^\theta$, we bring $\Delta BCA$ into standard position:

\begin{align*}
A &= r(0, 1, 0, 0, 0|0, 0, 0, 0), \\
B &= t(0, 1, 0, 0|0, 0, 0, 0), \\
C &= s(1, 1, 1, \frac{\theta_1 \theta_2}{2}, h_a \theta_1, h_a^{-1} \theta_2, -h_a^{-1} \theta_2, h_a \theta_1),
\end{align*}

while transforming $D$ by Lemma 4.10 into $(x_1', x_2', y', z'|\xi_1'^+, \xi_2'^+, \xi_1'^-, \xi_2'^-)$: where

\begin{align*}
x_2' &= x_1, \\
z' &= z + \frac{1}{2}(h_e^{-1} \theta_2 \xi_1^+ - h_e \theta_1 \xi_2^+ + \theta_1 \theta_2 x_1), \\
y' &= y + \frac{1}{2}(h_e^{-1} \theta_2 \xi_2^+ + h_e \theta_1 \xi_2^+) + x_1, \\
\xi_1'^- &= c_\theta h_a^{-1}(h_e \xi_2^+ - \theta_2 x_1), \\
\xi_2'^- &= c_\theta h_a(-h_e^{-1} \xi_1^+ + \theta_1 x_1),
\end{align*}

and $x_1', \xi_1'^+, \xi_2'^-$ can be recovered by 3.1.

Now by a diagonal transform $D_a$ with $a = \sqrt{\frac{x_2'}{x_1'}}$, we bring $\Delta BDA$ into standard position with odd parameters $(\bar{\mu}_1, \bar{\mu}_2)$ where

\begin{align*}
\bar{\mu}_1 &= \frac{\xi_2'^-}{\sqrt{x_2'y'}} = \frac{h_a}{c_\theta} \frac{-h_e \xi_1^+ + \theta_1 x_1}{\sqrt{x_1(y + \frac{1}{2}(h_e^{-1} \theta_2 \xi_1^+ + h_e \theta_1 \xi_2^+) + x_1)}}.
\end{align*}
Now rewriting using the cross-ratios from Lemma 5.1 we find:

\[ x_1 = y\chi^{-1}, \quad \xi_1^+ = -\sqrt{\chi^{-1}}y\sigma_1, \quad \xi_2^+ = -\sqrt{\chi^{-1}}y\sigma_2 \]

hence

\[
\tilde{\mu}_1 = \frac{h_{ac}\theta}{c\theta} \frac{h_e^{-1}\sqrt{\chi^{-1}}y\sigma_1 + \chi^{-1}y\theta_1}{\sqrt{\chi^{-1}}y\sqrt{y}y - \frac{1}{2}(h_e^{-1}\sqrt{\chi^{-1}}y\theta_2\sigma_1 + h_e\sqrt{\chi^{-1}}y\theta_1\sigma_2) + \chi^{-1}y} = \frac{h_{ac}\theta}{h_{cc}\theta} \frac{h_e^{-1}\theta_2 + \sqrt{\chi}\sigma_2}{\left(1 + \frac{\sqrt{\chi}}{2}\right)(h_e^{-1}\sigma_1\theta_2 + h_e\sigma_2\theta_1) + \chi}.
\]

Similarly we have

\[
\tilde{\mu}_2 = \frac{h_{ec}\theta}{h_{ac}\theta} \frac{h_e^{-1}\theta_2 + \sqrt{\chi}\sigma_2}{\left(1 + \frac{\sqrt{\chi}}{2}\right)(h_e^{-1}\sigma_1\theta_2 + h_e\sigma_2\theta_1) + \chi}.
\]

Let us denote the denominator by

\[
\mathcal{D} := \sqrt{1 + \frac{\sqrt{\chi}}{2}\left(h_e^{-1}\sigma_1\theta_2 + h_e\sigma_2\theta_1\right) + \chi} \tag{8.4}
\]

for simplicity.

**Definition 8.3.** We will fix the choice of the odd parameters \(\mu_1, \mu_2\) of the triangle \(\Delta ABD\) to be

\[
\mu_1 = \frac{h_e\theta_1 + \sqrt{\chi}\sigma_1}{D}, \quad \mu_2 = \frac{h_e^{-1}\theta_2 + \sqrt{\chi}\sigma_2}{D}, \tag{8.5}
\]

so that \((\tilde{\mu}_1, \tilde{\mu}_2) = (h'_a\mu_1, h'_a^{-1}\mu_2)\) with \(h'_a = \frac{h_{ac}}{h_{cc}}\).

Notice that according to Proposition 5.6 \(h'_a\) is unique up to multiplication by a constant \(c\) with \(c\mu = \mu\). However, since we are working in \(\tilde{C}(F, \Delta)\), we have a freedom to rescale the ratios. Hence we will fix our choice here as \(h'_a = \frac{h_{ac}}{h_{cc}}\).

Similarly, using instead the inverse prime transform \(P_{h_e,h_b}\) on \(\Delta ABC\), and again applying a diagonal transformation, we bring \(\Delta CDB\) into standard position with odd parameters \((\tilde{\nu}_1, \tilde{\nu}_2)\) with

\[
\tilde{\nu}_1 = \frac{h_{bc}\theta}{h_e} \frac{\sqrt{\chi}h_e\theta_1 - \sigma_1}{D}, \quad \tilde{\nu}_2 = \frac{h_e}{h_{bc}\theta} \frac{\sqrt{\chi}h_e^{-1}\theta_2 - \sigma_2}{D}.
\]

Now recall that we want the configuration \(\Diamond DABC \in S_{h_f}^{\mu,\nu}\), hence according to the spin graph evolution rule from Figure 4, we need to change the signs with respect to the attaching triangle at \(BC\).

**Definition 8.4.** We will fix the choice of the odd parameters \(\nu_1, \nu_2\) of the triangle \(\Delta CDB\) to be

\[
\nu_1 = \frac{\sigma_1 - \sqrt{\chi}h_e\theta_1}{D}, \quad \nu_2 = \frac{\sigma_2 - \sqrt{\chi}h_e^{-1}\theta_2}{D}, \tag{8.6}
\]

so that \((\tilde{\nu}_1, \tilde{\nu}_2) = (-h'_b\nu_1, -h'_b^{-1}\nu_2)\) with \(h'_b = \frac{h_{bc}}{h_e}\).
Again there is a freedom of choice of \( h' \) up to a multiplication by a constant \( c \) with \( cv = \nu \), and we still have a freedom to rescale the ratio at \( \nu \). Hence we will fix our choice here as \( h' = \frac{h_a}{h_c} \). Now this fixes all the rescaling of the remaining variables.

**Lemma 8.5.** We have

\[
c_\mu c_\nu = c_\theta c_\sigma.
\]

**Proof.** Expanding in terms of \( \theta \) and \( \sigma \) and simplifying, we get

\[
c_\mu c_\nu = \left(1 + \frac{\mu_1 \mu_2}{6}\right)\left(1 + \frac{\nu_1 \nu_2}{6}\right)
\]

\[
= \left(1 + \frac{(1 + \chi)\theta_1 \theta_2}{6D^2}\right)\left(1 + \frac{(1 + \chi)\sigma_1 \sigma_2}{6D^2}\right).
\]

Note that

\[
\frac{1 + \chi}{D^2} = \frac{1 + \chi}{1 + \chi + \frac{\sqrt{\chi}}{2}(h_c \sigma_2 \theta_1 + h_c^{-1} \sigma_1 \theta_2)} = 1 + P(\sigma_2 \theta_1, \sigma_1 \theta_2)
\]

for some polynomials \( P \). Hence

\[
\frac{1 + \chi}{D^2} \theta_1 \theta_2 = \theta_1 \theta_2, \quad \frac{1 + \chi}{D^2} \sigma_1 \sigma_2 = \sigma_1 \sigma_2,
\]

and we have

\[
c_\mu c_\nu = c_\theta c_\sigma.
\]

□

Now we are in a position to compare the two lifts under Ptolemy transform. We consider 4 attaching triangles of \( \diamond ABCD \) as in Figure 6 in the universal covering \( \tilde{F} \). Note that the two configurations share the same triangle \( \Delta ABD_1 \) attached to \( \Delta ABC \) and \( \Delta ABD \) respectively.

![Figure 6. Labelling of the 4 attached triangles](image)

In particular, consider the lift of the two configurations with \( \Delta AD_1 B \) as the image of the base triangle. Then by definition the lifts differ by a post composition with \( Z_{\frac{\mu}{\sigma}} \).
Since the Ptolemy transform is local, $h'_a$ does not depend on $\sigma_1$, and this uniquely determines the relation between the two lifts. Then all the points on $L_0$ are uniquely defined by the lifts and also differ by $Z_{\mu}$ for the image of both lifts.

Let us denote by $\chi_i$ the cross-ratio of the outside triangle containing $D_i$ with the attaching triangle in the quadrilateral before the flip, and $\chi'_i$ the corresponding cross-ratio after the flip, with the common edge as the diagonal. For example, $\chi_1, \chi'_1$ are the cross-ratios of $\Diamond AD_1 BC, \Diamond AD_1 BD$ respectively.

Assume the spin structures $w_\sigma$ and $w_\ell$ coincide for simplicity. Also assume the lift of a point $X$ is defined by admissible transformation where the path goes through $\Delta BD_2 C$. Then by definition the image of the point $X$ is defined by the admissible transformations of the form

$$g \circ P^{\sigma_2, \pm}_{h^\prime_{b^{-1}}, h_a} \circ \Upsilon_{h_b} \circ P^\theta_{h_a h_b} \circ \Upsilon_{h_a^{-1}}$$

for the original lift with some $g$ defined outside of these triangles, and

$$g \circ P^{\sigma_2, \pm}_{h^\prime_{b^{-1}}, h_a} \circ Z_{-1} \circ \Upsilon_{h_b} \circ P^\nu_{h_f^{-1}, h_b} \circ \Upsilon_{h_f^{-1}} \circ P^\mu_{h_a h_f} \circ \Upsilon_{h_a^{-1}}$$

for the lift flipped diagonal, where $g$ is the same as before and the $Z_{-1}$ term comes from the spin graph evolution. Since the lift differs by $Z_{\mu}$, we have the equality:

$$g \circ P^{\sigma_2}_{h^\prime_{b^{-1}}, h_a} \circ \Upsilon_{h_b} \circ P^\theta_{h_a h_b} \circ \Upsilon_{h_a^{-1}} = g \circ P^{\sigma_2}_{h^\prime_{b^{-1}}, h_a} \circ Z_{-1} \circ \Upsilon_{h_b} \circ P^\nu_{h_f^{-1}, h_b} \circ \Upsilon_{h_f^{-1}} \circ P^\mu_{h_a h_f} \circ \Upsilon_{h_a^{-1}} \circ Z_{\mu}.$$

Now recall that the above admissible transformations are compositions of

$$\Upsilon_{h_b} \circ P^\theta_{h_b} = (JD_{\sqrt{\chi_k}} Z_{\theta_k} \circ (Z_{c_{\theta_k}} Z^{-1} \Upsilon_{\theta_k} Z_{\theta_k}) = D_{\sqrt{\chi_k}} Z_{-c_{\theta_k}} \Upsilon_{\theta_k}^{-1} Z_{\theta_k},$$

where $\Upsilon_{\theta_k}$ are lower Borel subgroup elements with 1 at the diagonals. In particular, we can commute all the Borel elements to one side, again with 1 at the diagonals. Hence the diagonal of the product of admissible transformation depends only on the diagonal transformations $D, Z$, and the number of $J$'s changing the signs, which is taken care of automatically by the choice of the spin graph evolution. In particular, by looking at the diagonal entry and using the fact that the $Z$-subgroup commutes with $J$ and $D$, we immediately obtain the following relation:

$$Z_{h_b}^{-1} D_{\sqrt{\chi_2}} Z_{c_b} D_{\sqrt{\chi_1}} = Z_{h'_b}^{-1} D_{\sqrt{\chi'_2}} Z_{c'_b} Z_{h'_f}^{-1} D_{\sqrt{\chi'_1}} Z_{c'_{\mu}} D_{\sqrt{\chi'_1}} Z_{\mu}. $$

Using the fact that $\chi_2 \chi_1 = \chi'_2 \chi^{-1} \chi'_1$, we obtain

$$h'_b = h'_b \frac{c_{\mu} c_{\mu}}{h_a h_f c_p} = h'_a \frac{c_{\mu}}{h_a h_f}.$$

Since $h'_a = \frac{h_a}{h_c}$ and $h'_b = \frac{h_c c_{\mu}}{h_a}$, we get $h_f = c_{\mu}$. Using $\chi'_1 = \chi_1 D$ which can be derived from \((8.3)\), we can check that the off diagonal entries are also identities.
Using this trick, it is easy to obtain the other relations by just looking at the diagonal entries. We obtain the relations:

\[
\begin{align*}
\frac{h'_d}{h_d} &= \frac{h_a}{h_a} e c_{\nu}, \\
\frac{h'_c}{h_c} &= \frac{h'_b}{h_b} h_c c_{\mu},
\end{align*}
\]

and we get finally our second main result of the paper:

**Theorem 8.6.** The Ptolemy transformation is given by

\[
\begin{align*}
\mu_1 &= \frac{h_e \theta_1 + \sqrt{\chi} \sigma_1}{D}, \\
\mu_2 &= \frac{h_e^{-1} \theta_2 + \sqrt{\chi} \sigma_2}{D} \\
\nu_1 &= \frac{\sigma_1 - \sqrt{\chi} h_e \theta_1}{D}, \\
\nu_2 &= \frac{\sigma_2 - \sqrt{\chi} h_e^{-1} \theta_2}{D}
\end{align*}
\]

(8.7)

\[
\begin{align*}
h'_a &= \frac{h_a}{h_e c_0}, \\
h'_b &= \frac{h_b c_0}{h_e}, \\
h'_c &= h_c c_{\mu}, \\
h'_d &= h_d c_0, \\
h_f &= \frac{c_0}{c_0}
\end{align*}
\]

(8.8)

where

\[
D = \sqrt{1 + \chi + \frac{\chi}{2} (h_e^{-1} \theta_2 + h_e \sigma_2 \theta_1)}
\]

and \( c_0 = 1 + \frac{\theta_1 \theta_2}{6} \), while the signs of the fermions follow the spin graph evolution rule as in Figure 4.

After flipping twice, we arrive at the same equivalence class of coordinate under vertex rescaling, or in other words

\[
(h''_a, h''_b, h''_c, h''_d, h''_e | \theta'', \sigma'') = (h_a \alpha_1, h_b \alpha_1, h_c \alpha_2, h_d \alpha_2, h_e^{-1} \alpha_1^{-1} \alpha_2^{-1} \sigma, -\alpha_1^{-1} \theta),
\]

where double prime denote the variables after application of Ptolemy transformation twice, \( \alpha_1 = \frac{c_0^2}{h_e c_0}, \alpha_2 = \frac{c_0^2}{c_0^2} \), and the new coordinate tuple corresponds to the triangulation where the quadrilateral is turned upside down, altering the numeration. In particular by an upside-down transformation, we get back the same configuration.

In the cases \( F = F_1, F_3 \) when some of the sides of \( \diamond ABCD \) are identified, the Ptolemy transform of the ratios can be calculated by looking at the representation of \( \hat{\rho}(\gamma) \) directly as in Appendix B.

**Corollary 8.7.** If two triangulations \( \Delta \) and \( \Delta' \) with fixed representative of the spin graphs are related by finite sequence of flips, then the super Ptolemy transformations in Theorem 8.2 and Theorem 8.6 provide a coordinate transformation from \( C(F, \Delta) \) to \( C(F, \Delta') \). In particular the mapping class group \( MC(F) \) acts naturally on \( S \hat{T}(F) \) and permuting the spin components.

**Proof.** The argument for the mapping class group action follows as in [12].

**APPENDIX A. SL(1|2): Notation and Conventions**

Following [12], we will be working over the Grassmann algebra \( S_\mathbb{R} \) over the field \( \mathbb{R} \) with possibly infinitely many generators, with standard decomposition \( S_\mathbb{R} = S_0 \oplus S_1 \) into even and odd elements. Given \( a \in S_\mathbb{R} \) there is a projection \( S_\mathbb{R} \rightarrow \mathbb{R} \) to its constant term \( a \mapsto a_\# \) called the body of \( a \). By abuse of notation, unless otherwise specified we
will write \( a \neq 0 \) if the body of \( a \) is non-zero and \( a > 0 \) if the body of \( a \) is positive. Then \( a \) is invertible if and only if \( a \neq 0 \). For simplicity we write \( \mathbb{R}_+ \) for the subspace of \( S_0 \) with positive bodies.

**Definition A.1.** The super vector space \( \mathbb{R}^{m|n} \) is defined to be the space

\[
\mathbb{R}^{m|n} := \{(z_1, z_2, ..., z_m|\theta_1, \theta_2, ..., \theta_n) : z_i \in S_0, \theta_j \in S_1\},
\]

and \( \mathbb{R}^{m|n}_+ \) a subspace of \( \mathbb{R}^{m|n} \) such that all even coordinates \( z_i \) have positive bodies.

Let \( g \) be a Lie superalgebra, one can consider its Grassmann envelope \( g(S_{\mathbb{R}}) := S_{\mathbb{R}} \otimes g \) so that one can construct a representation of \( g(S_{\mathbb{R}}) \) in the space \( S_{\mathbb{R}} \otimes \mathbb{R}^{m|n} \) from a given representation of \( g \) in \( \mathbb{R}^{m|n} \). One can then produce a representation of the corresponding Lie supergroup \( G(S) \) by exponentiating pure even elements from \( g(S_{\mathbb{R}}) \) in \( S_{\mathbb{R}} \otimes \mathbb{R}^{m|n} \).

In this paper we will only consider matrix representations on \( \mathbb{R}^{2|1} \). For conventional conveniences, we will choose the basis of \( \mathbb{R}^{2|1} \) such that the matrix elements of a pure even matrix are of the form

\[
\begin{pmatrix}
  b & f & b \\
  f & b & f \\
  b & f & b
\end{pmatrix},
\]

where \( b \) and \( f \) stand for even (bosonic) and odd (fermionic) parity respectively. The supermatrix multiplication is then given by

\[
\begin{pmatrix}
  a & \gamma & b \\
  \alpha & f & \delta \\
  c & \beta & d
\end{pmatrix}
\begin{pmatrix}
  a' & \gamma' & b' \\
  \alpha' & f' & \delta' \\
  c' & \beta' & d'
\end{pmatrix} :=
\begin{pmatrix}
  aa' - \gamma\alpha' + bc' & a\gamma' + \gamma f' + b\beta' & ab' - \gamma\delta' + bd' \\
  \alpha a' + f \alpha' + \delta c' & -\alpha\gamma' + f f' - \delta \beta' & \alpha b' + f \delta' + \delta d' \\
  ca' - \beta \alpha' + dc' & c\gamma' + \beta f' + d\beta' & cb' - \beta \delta' + dd'
\end{pmatrix}.
\]

**Definition A.2.** The superdeterminant or Berezinian of an even matrix \( M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \) with \( D \) invertible is defined as

\[
\text{sdet}(M) = (\det D)^{-1} \det(A + BD^{-1}C).
\]

The supertrace of a supermatrix is given by

\[
\text{str} \begin{pmatrix}
  a & * & * \\
  * & b & * \\
  * & * & c
\end{pmatrix} := a + c - b.
\]

Using our convention, the superdeterminant of diagonal matrix is given by

\[
\text{sdet} \begin{pmatrix}
  a & 0 & 0 \\
  0 & f & 0 \\
  0 & 0 & d
\end{pmatrix} = \frac{ad}{f}.
\]

**Definition A.3.** \( \mathfrak{sl}(1|2) \cong \mathfrak{osp}(2|2) \) is the Lie superalgebra spanned by four odd generators \( e_1^+, e_2^+ \) and four bosonic generators \( E, F, h_1, h_2 \). Elements \( e_1^+, e_2^+ \) are the Chevalley generators and satisfy the relations of \( A(0, 1) \) superalgebra where both simple roots are chosen to be grey \([6, 8]\). One can find explicit commutation relations in \([7]\). In
particular it can be realized as the algebra of supertraceless \((2|1) \times (2|1)\) supermatrices so that

\[
\begin{align*}
    h_1 &= \begin{pmatrix}
    1 & 0 & 0 \\
    0 & 1 & 0 \\
    0 & 0 & 0
    \end{pmatrix}, &
    h_2 &= \begin{pmatrix}
    0 & 0 & 0 \\
    0 & 1 & 0 \\
    0 & 0 & 1
    \end{pmatrix}, \\
    e_1^+ &= \begin{pmatrix}
    0 & 1 & 0 \\
    0 & 0 & 0 \\
    0 & 0 & 0
    \end{pmatrix}, &
    e_2^+ &= \begin{pmatrix}
    0 & 0 & 0 \\
    0 & 0 & 1 \\
    0 & 0 & 0
    \end{pmatrix}, \\
    e_1^- &= \begin{pmatrix}
    0 & 0 & 0 \\
    1 & 0 & 0 \\
    0 & 0 & 0
    \end{pmatrix}, &
    e_2^- &= \begin{pmatrix}
    0 & 0 & 0 \\
    0 & 0 & 1 \\
    0 & 1 & 0
    \end{pmatrix}, \\
    E &= \begin{pmatrix}
    0 & 0 & 1 \\
    0 & 0 & 0 \\
    0 & 0 & 0
    \end{pmatrix}, &
    F &= \begin{pmatrix}
    0 & 0 & 0 \\
    0 & 0 & 0 \\
    1 & 0 & 0
    \end{pmatrix}.
\end{align*}
\]

Let us also define

\[
H := h_1 - h_2 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -1
\end{pmatrix}, &
Z := h_1 + h_2 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

Then \(\{H, E, F\}\) is an \(\mathfrak{sl}(2)\)-triple, \(\{h_i, e_i^+, e_i^-\}_{i=1,2}\) are \(\mathfrak{gl}(1|1)\) triples, and \(Z\) generates the subgroup \(\mathbb{Z}\) that commutes with the bosonic generators.

The corresponding supergroup \(SL(1|2)\) can be faithfully realized as \((2|1) \times (2|1)\) supermatrices with \(sdet(g) = 1\). In particular a diagonal matrix \(\begin{pmatrix}
a & 0 & 0 \\
0 & f & 0 \\
0 & 0 & d
\end{pmatrix}\) belongs to \(SL(1|2)\) if and only if \(ad = f\).

**Proposition A.4.** There exists an involution \(\Psi\) given by

\[
\begin{align*}
    e_1^+ &\mapsto e_2^+, & e_1^- &\mapsto -e_2^-, \\
    e_2^+ &\mapsto e_1^+, & e_2^- &\mapsto -e_1^-, \\
    h_1 &\mapsto -h_2, & h_2 &\mapsto -h_1, \\
    E &\mapsto E, & F &\mapsto F, & H &\mapsto H.
\end{align*}
\]

It preserves the \(\mathfrak{sl}(2)\) triple, and interchanges the two \(\mathfrak{gl}(1|1)\) triples.

**Definition A.5.** We will denote by

\[
\tilde{SL}(1|2) := \Psi \ltimes SL(1|2)_0,
\]

where \(SL(1|2)_0\) is the component of \(SL(1|2)\) with \(f > 0\). In particular, there is a canonical projection

\[
\tilde{SL}(1|2) \to SL(2, \mathbb{R})
\]
given by sending $\Psi \rightarrow I_{2 \times 2}$ and \[
\begin{pmatrix}
    a & b \\
    c & d
\end{pmatrix} \rightarrow \frac{1}{\sqrt{f}} \begin{pmatrix}
    a\# & b\# \\
    c\# & d\#
\end{pmatrix}.
\]

Remark A.6. In principle, instead of taking $\widetilde{SL}(1|2)$, we could consider, from the group-theoretic point of view, a more natural embedding of $\widetilde{SL}(1|2)$ into $OSp(2|2)$, the supergroup of $(2|2) \times (2|2)$ supermatrices preserving a certain bilinear form, so that the involution $\Psi$ is described by a certain $(2|2) \times (2|2)$ supermatrix. However, it is much easier to work with $(1|2) \times (1|2)$ supermatrices and that explains our choice.

One can recover that $(2|2) \times (2|2)$ realization of $SL(1|2)$ from the action of $\widetilde{SL}(1|2)$ on the rays of the light cone $L_0$, which coincides with the action of fractional linear transformations on $\mathbb{R}^{1|2}$ (see e.g., [9]) realized as a subgroup of $OSp(2|2)$. This gives the relation of our constructions with the geometry of uniformized $N = 2$ Riemann surfaces [3, 5, 1].

Appendix B. Ptolemy transformation in the special cases

In this section, we will deal with the Ptolemy transformation of the ratio coordinates where some of the sides of the quadrilateral $\diamond ABCD$ are identified in the projection $\tilde{F} \rightarrow F$ to the surface. There are basically two cases, where two adjacent sides are identified, or two opposite sides are identified. Similar to the argument in the general case, it suffices to look at the admissible transformation relating the sides. In particular, these are exactly the representations of the simple cycle $\gamma \in \pi_1$ that joins the corresponding edge. We note that $\mu, \nu$ are given by the same formula as before, and by fixing the rescaling, we can also let $h = \frac{c}{\sigma c^2 \theta}$ be the same expression as before.

In all the cases, we let $\Delta BCA$ be the base triangle of the lift before the flip, and $\Delta BDA$ be the base triangle after the flip. Then by assumption these two lifts are related by the post-composition with a diagonal transformation $D = D_\gamma = D_{\gamma_1}$. 

\[\text{Figure 7. Case 1: Adjacent sides identified, with the path of admissible transformation}\]
Figure 8. Case 2: Opposite sides identified, with the path of admissible transformation

Case 1: Two adjacent sides. Assume $\overrightarrow{AB}$ and $\overrightarrow{CB}$ are identified. Let $g_1$ and $g_2$ be the representations of the cycles joining the edge before and after the lift. Then

$$g_1 = \Upsilon_{h_a}^{\chi_2} \circ P_{h_a h_a^{-1}}^\theta$$

and

$$g_2 = Z_{-1} \circ \Upsilon_{h_a^{-1}}^{\chi_2'} \circ P_{h_a^{-1} h_a'}^{-1} \circ \Upsilon_{h_f}^{\chi_2^{-1}} \circ P_{h_a h_f}^\mu$$

with $g_1 = D \sqrt{D} \circ g_2 \circ D^{-1} \sqrt{D}$. Note that we have 2 $J$’s in $g_1$ and 4 $J$’s in $g_2$, hence we see that the sign $Z_{-1}$ is needed for the conjugation. In particular the orientation of the branch corresponding to $h_a$ is flipped in the spin graph evolution.

Since we only have the (clockwise) prime transformation $P^+$, the same argument as in the general case shows that we only need to look at the diagonal entries. Using $\chi_2^{-1} \chi_2' = \chi_2 = 1$, we get

$$Z_{c_0} Z_{h_a} = Z_{c_0} Z_{h_f}^{-1} Z_{c_0} Z_{h_a}$$

or

$$h_a' = \frac{c_0}{c_\mu c_\nu} h_f h_a = \frac{h_f}{c_\sigma} h_a = \frac{h_a}{c_\sigma}.$$

If $\overrightarrow{BC}$ and $\overrightarrow{DC}$ are identified instead, the inverse argument shows that the rule is given by

$$h_b' = \frac{c_\mu}{c_\nu} h_b,$$

and again an extra sign is needed which determines the spin graph evolution.

**Corollary B.1.** In the case of $F_0^3$, where $\overrightarrow{AB}$ is identified with $\overrightarrow{CB}$, and $\overrightarrow{BC}$ is identified with $\overrightarrow{DC}$, the Ptolemy transform $\mathbb{R}^{6|4} \rightarrow \mathbb{R}^{6|4}$ is given by

$$(a, b, c, h_a, h_b, h_e | \theta_1, \theta_2, \sigma_1, \sigma_2) \mapsto (a, b, f, \frac{h_a}{c_\sigma}, \frac{c_\mu}{c_\nu} h_b, \frac{c_\sigma}{c_\theta} | \mu_1, \mu_2, \nu_1, \nu_2),$$
where \( f \) is given by (8.3), \( \mu, \nu \) are given by (8.7)-(8.8), and the spin graph evolves as in Figure 4 but flipping all four outside branches. Furthermore, flipping twice return to the same configuration.

**Case 2:** Two opposite sides. Assume \( \overrightarrow{AB} \) and \( \overrightarrow{DC} \) are identified. Then

\[
g_1 = \Upsilon_{h_{a}^{-1}} \circ P_{h_{e}^{-1},h_{a}^{-1}}^{\sigma} \circ \Upsilon_{h_{e}} \circ P_{h_{e},h_{a}}^{\theta,-}
\]

and

\[
g_2 = \Upsilon_{h_{a}^{-1}}' \circ P_{h_{f}^{-1},h_{a}^{-1}}' \circ \Upsilon_{h_{f}}' \circ P_{h_{f}',h_{a}}^{\mu,+}
\]

where again \( g_1 = D_{\sqrt{D}} \circ g_2 \circ D_{\sqrt{D}}^{-1} \). This time however, we see that ignoring the diagonal elements, the Borel parts are of the form

\[
g_1 \sim U \circ J \circ U \circ J^{-1}, \quad g_2 \sim Z^{-1} \circ J \circ U \circ J \circ U.
\]

In particular we see that no extra signs are needed, i.e., the spin graph evolution stays the same on the outside branches. However, the same argument as in the general case does not work anymore here since the diagonal entries are a lot more complicated.

However, by commuting the Borel part across the \( J \)’s, they are of the form

\[
g_1 \sim U \circ V \quad \text{and} \quad g_2 \sim V \circ U,
\]

where \( V \) is upper Borel. Hence we only need to look at the off diagonal even entry (i.e., top right corner) instead, which is very simple.

By carefully expanding the matrix, the top right entries are given by

\[
(g_1)_{1,3} = \frac{h_{e} \chi_{3}}{c_{\sigma} h_{a} c_{\theta}^{2}}
\]

\[
(D_{\sqrt{D}} \circ g_2 \circ D_{\sqrt{D}}^{-1})_{1,3} = \frac{h_{a}' h_{f} \sqrt{\chi_{3} \chi}}{c_{\sigma} c_{\mu} h_{e}}
\]

Now using the fact that \( \chi_{3} = \frac{ba}{c^2}, \chi_{3}' = \frac{c^2}{ba} \), and \( \frac{c^2}{ba} = D \) so that \( \sqrt{\chi_{3}} = \sqrt{\chi_{3} D} \), we get

\[
\frac{h_{e} \chi_{3}}{c_{\sigma} h_{a} c_{\theta}^{2}} = \frac{h_{a}' h_{f} \sqrt{\chi_{3} \chi}}{c_{\sigma} c_{\mu} h_{e}}.
\]

With \( h_{f} = \frac{c_{\sigma}}{c_{\theta}} \) and \( c_{\sigma} c_{\theta} = c_{\mu} c_{\nu} \) we finally obtain the relation:

\[
h_{a}' = \frac{h_{f} c_{\theta}}{h_{e} c_{\sigma} h_{a}} = \frac{c_{\mu} h_{a}}{h_{e}^{2}}
\]

Again if \( \overrightarrow{BC} \) and \( \overrightarrow{AD} \) are identified, a similar argument shows that the rule is given by

\[
h_{b}' = \frac{h_{b} c_{\sigma}}{h_{e} h_{f} c_{\nu}} = \frac{c_{\theta}^{2} h_{b}}{c_{\nu} h_{e}},
\]

and no extra signs are needed.

**Corollary B.2.** In the case of \( F_{1}^{-1} \), where \( \overrightarrow{AB} \) is identified with \( \overrightarrow{DC} \), and \( \overrightarrow{BC} \) is identified with \( \overrightarrow{AD} \), the Ptolemy transform \( \mathbb{R}^{6|4} \rightarrow \mathbb{R}^{6|4} \) is given by

\[
(a, b, e, h_{a}, h_{b}, h_{e}; \theta_{1}, \theta_{2}, \sigma_{1}, \sigma_{2}) \mapsto (a, b, f, \frac{c_{\mu} h_{a}}{h_{e}}, c_{\theta} h_{e}, \frac{c_{\sigma} h_{b}}{c_{\nu} h_{e}}, c_{\theta}^{2} \mu_{1}, \mu_{2}, \nu_{1}, \nu_{2}),
\]
where $f$ is given by (8.3), $\mu, \nu$ are given by (8.7) – (8.8), and the spin graph evolves as in Figure 4 but all four outside branches are not flipped. Furthermore, flipping twice return to the same configuration.

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