\[(q, t)\text{-}KZ\text{ equation for Ding-Iohara-Miki algebra}\]

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Abstract

We derive the generalization of the Knizhnik-Zamolodchikov equation (KZE) associated with the Ding-Iohara-Miki (DIM) algebra \(U_{q,t}(\hat{\mathfrak{gl}}_1)\). We demonstrate that certain refined topological string amplitudes satisfy these equations and find that the braiding transformations are performed by the \(R\)-matrix of \(U_{q,t}(\hat{\mathfrak{gl}}_1)\). The resulting system is the uplifting of the \(\hat{u}_1\) Wess-Zumino-Witten model. The solutions to the \((q, t)\)-KZE are identified with the (spectral dual of) building blocks of the Nekrasov partition function for 5d linear quiver gauge theories. We also construct an elliptic version of the KZE and discuss its modular and monodromy properties, the latter being related to a dual version of KZE.

1 Introduction

The AGT relations \([1,2]\) connect two different worlds: that of 2d conformal field theories [3] and of instanton (SUSY ADHM) moduli spaces \([4,5,6,7]\). The natural quantities in the instanton world count embeddings of certain worldsheet manifold \(\Sigma\) into the moduli space \(M_k\) of instantons (e.g. for \(\Sigma\) being Riemann surface, they are related to Gromov-Witten invariants \([8]\)). In the simplest case of \(\Sigma\) being a point, the generating functions of embeddings reduce to (super)volucmes and are expressed via Nekrasov functions \([7,9]\). The quantities at the CFT side are various conformal blocks. According to the AGT relations \([1]\):

\[
\text{Conformal blocks} = \text{Nekrasov functions}\]

More generally, one can consider embeddings of arbitrary worldsheets which naturally carry more information about the moduli space (e.g. quantum multiplication).

Let us briefly remind the gauge theory interpretation of these quantities. The path integral of the 4d \(\mathcal{N} = 2\) gauge theory localizes on the instanton moduli space and can thus be reduced to the finite dimensional LMNS integral over the ADHM data. This integral gives the (regularized) volume of the moduli space and can itself be
localized by putting the theory in the $\Omega$-background $\mathbb{R}^4_{q,t-1}$. This gives the Nekrasov function, i.e. the cleverly regularized volume (more precisely, equivariant cohomology) of the ADHM space:

$$Z_{4d} = \sum_k \zeta^k \int_{\mathcal{M}_k} 1 = Z_{\text{Nekr}}$$

Consider now the 5d $\mathcal{N} = 1$ uplift of the 4d theory to $\mathbb{R}^4_{q,t-1} \times S^1$. Configurations, on which the path integral localizes, are now the maps from $S^1$ to the instanton moduli space. The integral over them is the index of supersymmetric quantum mechanics with the target space $\mathcal{M}_k$, which can be again localized to a finite dimensional LMNS integral, but the measure is generalized from rational to trigonometric. The index can also be understood in terms of equivariant $K$-theory of the ADHM moduli space:

$$Z_{5d} = \sum_k \zeta^k \# (S^1 \to \mathcal{M}_k) = \sum_k \zeta^k \text{Ind}_{\mathcal{M}_k}(q) = Z_{\text{Nekr}}^{K-\text{theor}}(q)$$

We can now generalize the uplifting procedure to higher dimensions. The partition function of the 6d theory on $T^2 \times \mathbb{R}^4_{q,t-1}$ naturally counts maps from $T^2$ to the instanton moduli space. The partition function is given by the elliptic genus of $\mathcal{M}_k$:

$$Z_{6d} = \sum_k \zeta^k \# (T^2 \to \mathcal{M}_k) = \sum_k \zeta^k \text{Ind}_{\mathcal{M}_k}^{\text{ell}}(q,p) = Z_{\text{Nekr}}^{\text{ell}}(q,p)$$

One can consider further generalizations, with curved spaces and embedded branes and more complicated subvarieties of different dimensions, but the pattern is already clear: the higher-dimensional partition functions (at least their low-energy limits) can be thought of as generating functions of mappings from some target to the ADHM moduli space.

Of course, the same is true if we extend our theories to liftings from lower dimensions: this reveals new structures hidden in the naive 4d consideration. One can consider the 5d theory on $\mathbb{R}^4_{q,t} \times S^1$ as a 3d theory with worldsheet $S^1 \times \mathbb{R}^2_q$ and target space being the moduli space $\mathcal{Y}_k$ of 2d vortices. This gives the 3d–5d correspondence between the gauge theories [10]. The resulting integrals over $\mathcal{Y}_k$ are manifestly equivalent to Dotsenko-Fateev (DF) integrals in $q$-deformed CFT. Combining this equivalence with spectral duality [11] of conformal blocks we get the AGT duality between 5d gauge theories and $q$-deformed CFT correlators.

We can summarize these basic relations between the CFT and ADHM objects in the following table:

| Dotsenko-Fateev integrals | $\text{AGT} \leftrightarrow \text{ADHM}$ | $\text{ADHM}$ spaces $\mathcal{M}_k$ |
|--------------------------|-----------------------------------------------|--------------------------------------|
| $\langle \prod \Psi^* \exp (\oint J) \rangle$ | $J = \sum_k \zeta^k \cdot \# (\Sigma \to \mathcal{M}_k)$ | $\Sigma = S^1 \times \mathbb{R}^2_q$ $\implies$ The setup of [12] |
| $W_{q,t}(\hat{g}_1)$ conformal block | $\Sigma = S^1 \times \mathbb{R}^2_q$ $\implies$ | Genus-zero Gromov-Witten invariants of $\mathcal{M}_k$ |
| $W_{\beta}(\hat{g}_1)$ conformal block | $\Sigma = S^2$ $\implies$ | |
| single-line BNM $\subset$ satisfies $(q,t)$-KZE | Balanced network models $\cup$ | $\Sigma = \text{point} \implies$ Volume (equivariant cohomology) of $\mathcal{M}_k$ expressed through Nekrasov functions |
| $W_{q,t}(\hat{g}_1)$ conformal blocks | $\Sigma = S^1$ $\implies$ $K$-theory of $\mathcal{M}_k$ |
| WZW$_{u_1}$ $\subset$ satisfies KZE | Lioville/W models $\leftrightarrow$ | |

The top line in the table can be thought of as the partition function of the 7d gauge theory on $S^1 \times \mathbb{R}^6_{q,t-1,2}$ (we remind that the $\Omega$-background in the space of dimension $2n$ is related to the action of $T^n$ on $\mathbb{C}^n$ and depends on $n$ deformation parameters). Equivalently it corresponds to the 5d theory with matter content given by the quiver, which represents the moduli for 2d vortices: one adjoint hypermultiplet plus several fundamental ones. There is a third way of looking at this theory: it can be understood as the 3d gauge sigma model on the ADHM moduli space. It is this last option, that is reflected in the table. All the three (3d–5d–7d) ways of looking at the theory are complementary and useful in different situations. For example, from the 7d perspective one can naturally see the symmetry between the three equivariant parameters: two from the $\Omega$-background $\mathbb{R}^4_{q,t-1}$, in
which the ADHM instantons live, and one from $S^1 \times \mathbb{R}^2_Q$. There are several limits of this construction which are also listed in the table.

The symmetry, which controls the properties of the models in this pattern is the DIM algebra \cite{16,15} and its further generalizations. Already the simplest DIM algebra $U_{q,t}(\widehat{sl}_1)$ involves a $(q,t)$-deformation at least, which means that in the instanton (ADHM) moduli space story, we make a lift from the level of $\Sigma = \text{point}$ and the volume (equivariant cohomology) of the moduli space to, at least, that of $\Sigma = S^1$ ($K$-theory of the ADHM space). Further lifting to $\Sigma = \text{Riemann surface}$ should correspond to further (elliptic) generalizations of DIM.

Pagoda algebra $U_{q,t,p}(\widehat{sl}_1)$ from \cite{14} is expected to be related to the most general system of this kind.

Since the left low corner of this pattern is best understood, it is most natural to look at the structures which are well known there and try to extend them to other places. From this perspective, the network model generalizes the Dotsenko-Fateev (DF) realization \cite{17,18} of 2d conformal blocks as correlators of the free field vertex operators and screening charges, which commute with the operator algebra (Virasoro, $W$-operators) and can be expressed as integrals of screening currents. Balanced network model \cite{19,15} is built from a trivalent graph with lines of rational slopes in such a way that all external ends of the graph are either strictly horizontal or strictly vertical. Generally the lines are associated with Fock representations of DIM algebra with the central charges $(0, 1)$ (vertical lines) or $(1, N)$ (horizontal lines). The vertex operators, standing at the vertices, intertwine these representations of DIM algebra \cite{16,15} and are expressed through $(q,t)$-oscillators. Moreover, exponentiated screening charges appear automatically incorporated into the vertex operators, so that the right number of screening charges is selected by conservation laws for any concrete network.

**The present paper** addresses the question what happens at the DIM level, when the Liouville/$W_N$ model is substituted by the Wess-Zumino-Witten (WZW) model \cite{20,21}. Usually, this means that the operator algebra is extended from Virasoro or $W_N$ to a Kac-Moody algebra \widehat{g}, e.g. \widehat{su}_N. Such an extension of the symmetry implies a reduction of the set of screening charges: they should now commute not only with the stress-energy tensor and with $W$-operators, but also with all the currents of \widehat{g}. In the result, the set of WZW correlation functions is a small subset of those in a generic CFT of the Liouville/$W_N$ type. As a manifestation of this, the correlators in WZW model satisfy the universal first order Knizhnik-Zamolodchikov equations \cite{22,23} (KZE), relating the shift in position of vertex operators to a transformation in the group space, that is, to action of the $R$-matrix. KZE for DIM algebra can be schematically written as follows:

\[
\left( \frac{2}{t} \right)^{z_{\alpha^*}} \prod_{\alpha^*} \langle V_1(z_1) \cdots V_N(z_N) \rangle = \prod_{i \neq k} R_{i,k} \langle V_1(z_1) \cdots V_N(z_N) \rangle \right) \tag{5}
\]

where $V_i(z_i)$ are the primary fields and $R_{i,j}$ is the $R$-matrix acting on the $i$-th and $j$-th fields.

Let us recall that in general, the number of free fields in the bosonization of the WZW model is $\dim(su_N) \sim N^2$ \cite{21}. The usual $W_N$-models are, in fact, reductions of $WZW_{su_N}$. For lowest Kac-Moody central charge $k = 1$, the number of the free fields in the bosonization reduces to $(N - 1)$. In such a description the WZW screening currents are restricted to $exp(\alpha_i \cdot \vec{\phi})$ with $\alpha_i$ being the simple roots, while the Virasoro screenings are restricted by the only condition $\vec{\alpha}^2 = 2$.

Another thing to keep in mind is that the simplest DIM algebra considered in \cite{15}, was $U_{q,t}(\widehat{sl}_1)$ rather than \widehat{g} for a simple g like $su_N$. Thus, it retains some Abelian traits, which simplify the WZW theory considerably. In the Kac-Moody case, the “Abelian” $WZW_{\widehat{sl}_1}$ model associated with the (non-Abelian) algebra $\widehat{u}_1$ is nearly trivial: only free field correlators without any screenings are picked up and satisfy the corresponding KZE (see sec. 3 for this archetypical example) though the KZE itself does not look much simpler in this case. The lift of this “Abelian” model to the DIM level is large enough to include all the $W_N$ models with any $N$ and with all their generalized hypergeometric correlators. In this sense the introduction of an additional loop is equivalent to considering all $W_N$ in a unified way. However, in the balanced network model there is a remnant of $N$: the number of horizontal lines, see Fig. 1.

The number of lines fixes the number of Virasoro/$W_N$-screenings, which are actually associated with the vertical segments between the horizontal lines. Therefore, in the model with just a single horizontal line there are no screenings at all, and thus it is the proper lifting ($(q,t)$-deformation) of the $WZW_{\widehat{sl}_1}$. It is natural then to expect that it satisfies a $(q,t)$-deformed version of KZE. In sec. 3 we demonstrate that this is indeed the case.

Moreover, one can derive the $(q,t)$-KZBE for conformal blocks on an elliptic curve to obtain a counterpart of the Knizhnik-Zamolodchikov-Bernard equation (KZBE) \cite{24}. Technically, this is done by considering the traces (partition functions) instead of matrix elements (correlation functions) \cite{25} and by applying a counterpart of the “averaging” procedure \cite{26}, which ultimately leads to the emergence of the elliptic $R$-matrix. We deal with the KZBE in section 3 and derive, along with the KZBE, also a monodromy equation, which turns out to be dual to the KZE (cf. the quantum affine case in \cite{25,27}).
For the correlator of primary fields of the current algebra, the insertion of the current is expressed through the stress tensor $T(z)$ through the currents $J^a(z)$ of $\hat{g}$:

$$T(z) = \frac{1}{2} : J^a(z) :$$

One can notice that the $L_{-1}$-generator acts as a derivative on the fields: $L_{-1} = \oint dz T(z) \to \partial / \partial z$. Hence, one can insert into the correlator the integral $\oint dz \ldots$ of equation (7) and obtain some relation between correlators.

Step 2. For the correlator of primary fields of the current algebra, the insertion of the current is expressed through the correlator itself and described by the Ward identity [22]

$$\langle J^a(z)V_i(z_i) \rangle = \sum_i \rho_i(t^a) \langle J^a(z)V_i(z_i) \rangle$$

where $\rho_i(t^a)$ are generators of $\hat{g}$ acting on the $i$-th primary field $V_i$ which transforms in the representation $\rho_i$. Thus, for the correlators of this kind, (7) gives a closed equation, which is called KZE.

Step 3. The KZE can be solved with the help of the free field formalism of [21], so that the primary fields are exponentials of free fields. However, it is a quite poor class of solutions. In order to have an ample class of solutions, one has to apply the Dotsenko-Fateev trick [17] and insert into the correlator a series of screening charges, the operators commuting with any generators (currents) $J^a(z)$ of $\hat{g}$, not only with the stress tensor. In order to construct them, one may take the screening currents, their integrals being screening charges. Then, the solutions to the KZE is given by Dotsenko-Fateev integrals made from $\dim(g)$ free fields, the integrands being constructed from exponentials of these fields [34].

From above explanations, it is clear that this $(q,t)$-KZE is not a generalization of “truly non-Abelian” equations, neither the original (classical) KZE, nor its quantum $(q)$ deformation from [23]: in order to derive their liftings, one should consider $\hat{g}$ with non-Abelian $g$. However, $U_{q,t}(\hat{g})$ is large enough to possess a non-trivial $R$-matrix [28] [29], and the $(q,t)$-KZE satisfied by the correlators in this case is also non-trivial. A reminiscence of its Abelian origin is simplicity of the $R$-matrix: in fact, though being infinite-dimensional, it is diagonal and belongs to the set of trigonometric $R$-matrices, where spectral parameter can not be excluded without trivializing the matrix itself. As a manifestation of this, such $R$-matrices can not be used in knot calculus: they satisfy the relation [29]

$$R^{-1}(z,w) = R(w,z)$$

which make the associated knot polynomials trivial. Thus, for knot theory purposes further lifting to $\hat{g}$ is essential. However, the $(q,t)$-KZE [57]–[58] by itself is a nice non-trivial equation.

It deserves noting that the difference equations derived by A. Okounkov and A. Smirnov [30] [12] [31] for the entries at the upper right corner of the table above are quite different and involve very different $R$-matrices. A much simpler and elementary equations (57)-(58) do not seem to appear in those papers. It is also an interesting open problem to clarify the relation of the equations from [30, 12, 31] to the (q,t)-KZE from the viewpoint of AGT relation.

For the detailed description of network models we refer the reader to [19] [32] [14] [15], and for the associated RTT-relations to [33].

2 Warm-up: KZE for the Kac-Moody algebra $\hat{u}_1$

Let us explain the logic of constructing the KZE and its solutions in the simplest example of the Kac-Moody algebra $\hat{u}_1$ [22]. The whole story consists of three steps.

• Step 1. The derivation of KZE is based on the Sugawara construction, which expresses the stress tensor $T(z)$ through the currents $J^a(z)$ of $\hat{g}$:

$$T(z) = \frac{1}{2} : J^a(z) :$$

One can notice that the $L_{-1}$-generator acts as a derivative on the fields: $L_{-1} = \oint dz T(z) \to \partial / \partial z$. Hence, one can insert into the correlator the integral $\oint dz \ldots$ of equation (7) and obtain some relation between correlators.

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• Step 3. The KZE can be solved with the help of the free field formalism of [21], so that the primary fields are exponentials of free fields. However, it is a quite poor class of solutions. In order to have an ample class of solutions, one has to apply the Dotsenko-Fateev trick [17] and insert into the correlator a series of screening charges, the operators commuting with any generators (currents) $J^a(z)$ of $\hat{g}$, not only with the stress tensor. In order to construct them, one may take the screening currents, their integrals being screening charges. Then, the solutions to the KZE is given by Dotsenko-Fateev integrals made from $\dim(g)$ free fields, the integrands being constructed from exponentials of these fields [34].
For the Kac-Moody algebra \( \mathfrak{g} = \hat{\mathfrak{u}}_N \) with the central charge \( k = 1 \), the number of free fields can be reduced to \( \text{rank}(\mathfrak{g}) = N - 1 \) and the screening currents are just \( e^{\hat{a}_i \hat{\delta}} \) with \( \hat{a}_i \) being simple roots of \( \mathfrak{g} \), while any \( \oint e^{\hat{a}_i \hat{\delta}} \) with \( \hat{\delta}^2 = 2 \) commutes with \( T \), i.e. can be used as a Virasoro screening charge. Other (not only simple) roots can also be used, but they do not produce new (independent) correlators.

In the simplest case of \( \mathfrak{g} = \hat{\mathfrak{u}}_1 \), there are no currents commuting with the \( u_1 \)-current, and all the correlators in WZW \( u_1 \) are just \( \prod_{i<j} (z_i - z_j)^{\alpha_i \alpha_j} \), while those of the single free field (Liouville model) can be multiple integrals of such quantities, i.e. generalized hypergeometric functions.

Indeed, the KZE in this case are

\[
\frac{\partial F\{z|\alpha\}}{\partial z_i} = \sum_{j \neq i} \frac{\alpha_i \alpha_j}{z_i - z_j} \cdot F\{z|\alpha\}
\]

where \( F\{z|\alpha\} \) is a correlator of \( V_{\alpha_i}(z_i) \) \( \hat{\mathfrak{u}}_1 \)-primaries. The most general solution to these equations has the form

\[
F\{z|\alpha\} \sim \prod_{i<j} (z_i - z_j)^{\alpha_i \alpha_j} \sim \langle e^{\alpha_i \phi(z_i)} \rangle
\]

i.e. they do not admit any screening charges.

Note that the usual correlators in the Liouville theory are constructed using the Virasoro screening charge \( e^{b \theta(x)} \), where \( b \) parameterizes the central charge \( c = 1 - 6(b - 1/b)^2 \):

\[
F\{z|\alpha\} = \left\langle \prod_{i} e^{\alpha_i \phi(z_i)} \prod_{\mu} \oint e^{b \phi(x_{\mu})} dx_{\mu} \right\rangle \sim \prod_{i<j} (z_i - z_j)^{\alpha_i \alpha_j} \cdot \oint dx_{\mu} \prod_{\mu > \nu} (x_{\mu} - x_{\nu})^b \prod_{i,\mu} (z_i - x_{\mu})^{b \alpha_i}
\]

This correlator, however, does not satisfy the KZE because the variation of \( z_i \) is induced by the action of the stress tensor \( T(z) \), which has a peculiar form of the square of the \( \hat{\mathfrak{u}}_1 \) current \( J(z) = \partial \phi(z) \):

\[
\frac{\partial F}{\partial z_i} = \left\langle \oint_{z_i} T(y) dy \prod_{j} e^{\alpha_j \phi(z_j)} \prod_{\mu} \oint e^{b \phi(x_{\mu})} dx_{\mu} \right\rangle = - \sum_{k \neq i} \left\langle \oint_{z_k} T(y) dy \prod_{j \neq k} e^{\alpha_j \phi(z_j)} \prod_{\mu} \oint e^{b \phi(x_{\mu})} dx_{\mu} \right\rangle = - \sum_{k \neq i} \sum_{j \neq k} \frac{\alpha_j \alpha_k}{z_j - z_k} \cdot F + \text{pairings with screenings}
\]

In fact, the last contribution cannot be neglected in the \( \hat{\mathfrak{u}}_1 \) case: already the hypergeometric \( F \) (for one screening) satisfies a second order differential equation rather than the first order KZE. Thus, the WZW model for \( \mathfrak{g} = \hat{\mathfrak{u}}_1 \) has only correlators of the form \([10]\] and not the generic Liouville theory correlators \([11]\]. At the same time, the Sugawara equation \([7]\) being inserted into \([11]\) gives rise to some relations between different correlators in the generic Liouville theory.

What happens for simple algebras \( \mathfrak{g} \) is that there is a restricted, still non-empty set of screenings, which commute with the currents, not only with the stress tensor \([21]\], and the analogue of the last contribution in \([12]\] is absent for them. This selects correlators of the WZW model, which is a subset of the correlators of the associated \( W(\mathfrak{g}) \) (Toda) model.

One can similarly derive the KZE on torus. In the \( \hat{\mathfrak{u}}_1 \) case, the equation has the form

\[
\frac{\partial F\{z|\alpha\}}{\partial z_i} = \sum_{j \neq i} \alpha_i \alpha_j r(z_i - z_j) \cdot F\{z|\alpha\}
\]

where \( r(z) \) is the logarithmic derivative of the \( \theta \)-function: \( r(z) = (\log \theta(z))' \) and the solution is rather trivial:

\[
F\{z|\alpha\} \sim \prod_{i<j} \theta(z_i - z_j)^{\alpha_i \alpha_j}
\]

The extension of this equation to arbitrary \( \hat{\mathfrak{g}} \) and the Riemann surface of arbitrary genus is rather involved and sometimes called KZBE \([24]\).

The \((q-)\)Knizhnik-Zamolodchikov equation extends these results from \( \hat{\mathfrak{u}}_1 \) to arbitrary Kac-Moody algebra, classical and quantum. The present paper initiates the program of extending it further to the DIM algebra.
To this end, we need to translate the above properties into a pure algebraic language, with vertex operators interpreted as intertwiners of Fock representations and construct a counterpart of the Sugawara relation. However, the stress tensor and operator expansions are, in fact, not necessarily needed: one needs just the shift operator expressed through currents and appropriate commutation relations. Moreover, the substitute of the Sugawara construction for the shift operator is just a decomposition

$$\Psi(pz) = T_+ \Psi(z) T_-$$

(15)

where the subscripts ± denote the parts with only positive or negative powers of the spectral parameter. In practice, they are realized by taking the DIM $T$-operator at special values of spectral parameters. Let us stress here that $T$ is the resolved conifold, i.e. a bilinear combination of two topological vertices (intertwiners) and is not the energy-momentum tensor.

Eq. (15) is the DIM counterpart of $L_{-1} = J^a\alpha J^a$. Hence, with this equation we only have step 1 done. In order to make step 2, we find the commutation relations of $T$ and $\Psi$, and, as step 3, one obtains the algebraic solutions similarly to the $u_1$ case. Similarly to the $u_1$ case, there are no nontrivial screening operators for $U_{q,t}(\widehat{gl}_1)$.

3 Extension to DIM algebra $U_{q,t}(\widehat{gl}_1)$

There are two (different) directions to generalize the story of $u_1$ WZW model, or Abelian KZ equation. One is to the quantum affine algebra $U_q(\widehat{sl}_N)$ and the other is to DIM algebra $U_{q,t}(\widehat{gl}_1)$. In the first case non-Abelian nature of $\widehat{sl}_N$ allows the screening operators which make the correlation functions of hypergeometric type. On the other hand in the second case we keep Abelian nature of the algebra. The structure of correlation functions is of free field type typically obtained by use of the Wick contraction, but the building block (the "propagator") becomes rather non-trivial and interesting. Still there are several common features. We have a free field realization of the Drinfeld currents in both cases. The quantum deformation of KZ equation for the correlation functions of the intertwining operators (vertex operators) takes the similar form which involves a product of $R$ matrices. Hence, we can expect these two cases (addition of the second loop and non-Abelian generalization) are unified by considering the quantum toroidal algebra $U_{q,t}(\widehat{g})$ for non-Abelian algebra $g$.

In this section, we derive and solve the KZ equation for the DIM algebra $U_{q,t}(\widehat{gl}_1)$. We proceed in steps. First, in sec. 3.2 we rewrite the shift operator acting on the vertical spectral parameter of the intertwiner $\Psi$ in terms of the $T$-operators, schematically

$$\frac{q}{t}z, \lambda = \left(\frac{z, \lambda}{\sqrt{t^2 z, \lambda}}\right)^{-1}$$

(16)

(17)

where the $T(z)$ operators are evaluated for certain concrete values of the spectral parameters depicted on the figure. This can be viewed as the DIM version of the Sugawara construction which relates the conformal and algebraic properties of the intertwiners. Then in sec. 3.3 we derive the commutation relations between the $T$-operators and the intertwiners, which look like:

$$\mathcal{T} \Psi = R \Psi \mathcal{T}$$

(18)

with $R$ representing the $R$-matrix. In sec. 3.3 we combine the obtained relations and write down the $(q,t)$-KZ equation for the vacuum matrix element of the product of several intertwiners of the form

$$\langle \emptyset | \Psi \cdots P \left(\frac{q}{t}z\right) \cdots \Psi | \emptyset \rangle = \langle \emptyset | \Psi \cdots T \Psi(z) T^{-1} \cdots \Psi | \emptyset \rangle = \left(\prod \mathcal{R}\right) \langle \emptyset | \mathcal{T} \Psi \cdots \Psi(z) \cdots \mathcal{T}^{-1} | \emptyset \rangle = \left(\prod \mathcal{R}\right) \langle \emptyset | \Psi \cdots \Psi(z) \cdots \Psi | \emptyset \rangle$$

(19)

where in the last line we have used the fact that the $T$ operators which appear for the special values of the spectral parameters act trivially on the vacuum vectors. Finally, in sec. 3.4 we give explicit solutions to the DIM
\((q, t)\)-KZ equation. This solution is similar to the solution of \(\tilde{u}_1\) KZE. It is a product of two-point correlators (again, schematically)
\[
\langle \varnothing | \Psi(z_1) \cdots \Psi(z_n) | \varnothing \rangle = \prod_{i<j} \langle \varnothing | \Psi(z_i) \Psi(z_j) | \varnothing \rangle \tag{20}
\]
or, more generally,
\[
\langle \varnothing | \Psi^*(w_1) \cdots \Psi^*(w_m) \Psi(z_1) \cdots \Psi(z_n) | \varnothing \rangle = \prod_{k<l} \langle \varnothing | \Psi^*(w_k) \Psi^*(w_l) | \varnothing \rangle \prod_{i<j} \langle \varnothing | \Psi(z_i) \Psi(z_j) | \varnothing \rangle \prod_{k,i} \langle \varnothing | \Psi^*(w_k) \Psi(z_i) | \varnothing \rangle \tag{21}
\]

### 3.1 Difference operator

Let us first derive the operator which produces the shift in the vertical spectral parameters \((z, w)\) of the intertwiners \(\Psi^\lambda\) and \(\Psi^\mu\), where \(\lambda\) and \(\mu\) are partitions attached to the vertical line. The intertwiners also depend on the horizontal spectral parameter \(u\) and on the slope parameter \(N\). Let us recall that the explicit expression for them is given in \([16, 15]\),

\[
\Psi^\lambda(N, u|z) = \frac{z, \lambda}{(0, 1)} \quad (1, N + 1) \quad u \quad (1, N) = \langle z \rangle^{-N|\lambda|} u^{|\lambda|} f^N_{\lambda} \frac{q^N}{C_{\lambda}} \prod_{n \geq 1} \left( -\frac{1}{1 - q^n} + (1 - t^{-n}) C_{\lambda}(q^n, t^n) \right) a_{-n} \tag{22}
\]

and

\[
\Psi^\mu^*(N, u|w) = \frac{(1, N - 1)}{w, \mu} \quad u \quad (0, 1) = \langle -w \rangle^{-N|\mu|} u^{-|\mu|} f^N_{\mu} \frac{q^N}{C_{\mu}} \times \prod_{n \geq 1} \left( -\frac{1}{1 - q^n} - (1 - t^n) C_{\lambda}(q^n, t^n) \right) a_{-n} \tag{23}
\]

Here and henceforth, we use the following notations:

\[
|\lambda| = \sum_i \lambda_i \tag{24}
\]

\[
f_{\lambda} = (-1)^{|\lambda|} q^{n(\lambda^T) + |\lambda|} t^{-n(\lambda) - |\lambda|}, \quad n(\lambda) = \sum_{(i,j) \in \lambda} (i - 1) = \frac{||\lambda^T||^2 - |\lambda|}{2} \tag{25}
\]

\[
C_{\lambda} = \prod_{(i,j) \in \lambda} \left( 1 - q^{\lambda_i - j} t^{\lambda_j - i} \right), \quad C_{\lambda}(q, t) = \sum_{(i,j) \in \lambda} q^{i-1} t^{l-1} \tag{26}
\]

where we identify a partition \(\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \lambda_i \geq \cdots)\) with a Young diagram and \((i, j) \in \lambda\) means a box in the diagram. The Heisenberg algebra is

\[
[a_n, a_m] = \frac{1 - q^n}{1 - t^n} \delta_{m+n, 0} \tag{27}
\]

We will omit the slope labels \((0, 1)\) on the vertical lines and label only the horizontal ones.

For general complex number \(p\), one can write two difference equations for the intertwiner: one in the horizontal and one in the vertical spectral parameters. The former is trivial:

\[
\Psi^\lambda(N, pu|z) = p^{\lambda^T} \Psi^\lambda(N, u|z) \tag{28}
\]

It can be understood as the action of the vertical grading operator \(p^{\lambda^T}\) on the Fock representations being intertwined \([16]\). This operator shifts the spectral parameters of the two horizontal legs of the intertwiner and counts the level of the basis vectors in the vertical leg, hence Eq. (28).
The second equation is more involved and will be the one used in the derivation of the \((q,t)\)-KZ equation. For any \(p\), one has

\[
\Psi^\lambda(N, u|pz) = p^{-N|\lambda|} \exp \left[ \sum_{n \geq 1} \frac{z^n}{n} (p^n - 1) \left( -\frac{1}{1 - q^n} + (1 - t^{-n}) \text{Ch}_\lambda(q^n, t^n) \right) a_{-n} \right] \Psi^\lambda(N, u|z) \times \exp \left[ \sum_{n \geq 1} \frac{z^{-n}}{n} (p^{-n} - 1) \left( -\frac{q^n}{1 - q^n} - (1 - t^n) \text{Ch}_\lambda(q^{-n}, t^{-n}) \right) a_n \right]
\]

(29)

It will be crucial for us that, at the special values of \(p = \left( \frac{q}{t} \right)^{\pm 1}\), the two exponentials appearing in the r.h.s. can be recast into the \(T\)-operators, bilinear combinations of the intertwiners which geometrically correspond to resolved conifolds (at the level of refined amplitudes) and satisfy the \(RTT\) relations with the \(\mathcal{D}\) algebra \(\mathcal{R}\)-matrix, see s.3.2 below and [33][29]. For definiteness, consider \(p = \frac{q}{t}\).

The general expression that defines the \(T\)-operator is

\[
T^\lambda(N, u|z, w) = \begin{pmatrix} z, \lambda \end{pmatrix}_{(1, N)} \begin{pmatrix} u \end{pmatrix}_{(1, N)} \exp \left[ \sum_{n \geq 1} \frac{1}{n} \left( \left( \begin{array}{c} z \\ qtw \end{array} \right) + \left( \begin{array}{c} t \\ qwu \end{array} \right) \right) a_n \right] \times \exp \left[ \sum_{n \geq 1} \frac{1}{n} \left( \left( \begin{array}{c} q^n \\ w \end{array} \right) + \left( \begin{array}{c} t^{-n} \\ u \end{array} \right) \right) a_{-n} \right]
\]

\[
\times \exp \left[ \sum_{n \geq 1} \frac{1}{n} \left( \left( \begin{array}{c} q^n \\ w \end{array} \right) + \left( \begin{array}{c} t^{-n} \\ u \end{array} \right) \right) a_n \right] \times \exp \left[ \sum_{n \geq 1} \frac{1}{n} \left( \left( \begin{array}{c} q^n \\ w \end{array} \right) + \left( \begin{array}{c} t^{-n} \\ u \end{array} \right) \right) a_{-n} \right]
\]

(30)

where

\[
C_{\alpha\beta}(x) = \prod_{(i,j) \in \alpha} \left( 1 - x q^{-\alpha_i^+ - j} t^{-\alpha_j^+ - 1} \right) \prod_{(i,j) \in \beta} \left( 1 - x q^{-\beta_i^+ - j} t^{-\alpha_j^+ - 1} \right)
\]

(31)

is the standard Nekrasov factor.

For \(p = \left( \frac{q}{t} \right)^{\pm 1}\) and \(\lambda = \mu\), (30) simplifies as follows:

\[
T^\lambda(N, u|z, \sqrt{q/t} w) = \left( \frac{q}{t} \right)^{-\frac{1}{2}(N+1)|\lambda|} q^{|\lambda|^2} C_{\lambda\lambda}(q/t)^{1/2} \exp \left[ \sum_{n \geq 1} \frac{1}{n} \left( \frac{1}{1 - q^n} \right) (1 - t^n) \text{Ch}_\lambda(q^n, t^n) a_{-n} \right]
\]

\[
\times \exp \left[ \sum_{n \geq 1} \frac{z^n}{n} (1 - \frac{t}{q} n) \left( -\frac{1}{1 - q^n} + (1 - t^n) \text{Ch}_\lambda(q^n, t^n) \right) a_n \right]
\]

(32)

\[
T^\lambda(N, u|z, \sqrt{q/t} w) = \left( \frac{q}{t} \right)^{-\frac{1}{2}(N+1)|\lambda|} q^{|\lambda|^2} C_{\lambda\lambda}(q/t) \exp \left[ \sum_{n \geq 1} \frac{1}{n} \left( \frac{q^n}{1 - q^n} \right) (1 - t^n) \text{Ch}_\lambda(q^n, t^n) a_{-n} \right]
\]

\[
\times \exp \left[ \sum_{n \geq 1} \frac{z^n}{n} (1 - \frac{t}{q} n) \left( -\frac{1}{1 - q^n} + (1 - t^n) \text{Ch}_\lambda(q^n, t^n) \right) a_n \right]
\]

(33)

Notice that the \(T\)-operators for the special values of parameters in Eq. (32) (resp. (33)) contain only bosonic
annihilation (resp. creation) operators. Therefore, they act trivially on the vacuum vectors:

\[
\sum_{n \geq 1} \frac{1}{n} \left( \frac{\sqrt{q}}{t} \right)^n \exp \left[ \sum_{n \geq 1} \frac{1}{n} \left( \frac{\sqrt{q}}{t} \right)^n \frac{1}{(1-q^n)(1-t^n)} \right] \left( T^\lambda \right)_{1}^{N} \langle \emptyset | (1, N) \rangle = \left( \frac{\sqrt{q}}{t} \right)^n G(q, t)_{\lambda \lambda} (q/t) \left( T^\lambda \right)_{1}^{N} \langle \emptyset | (1, N) \rangle \]

Combining Eqs. (32) and (33) with Eq. (29), one finds that

\[
\Psi^\lambda \left( N, \sqrt[2]{q} \sqrt{q} t | u | q \sqrt{q} t \right) = \exp \left[ \sum_{n \geq 1} \frac{1}{n} \left( \frac{1}{1-q^n} \right) \right] \times \left( T^\lambda \right)_{0}^{N} \langle \emptyset | (1, N) \rangle \langle \emptyset | (1, N) \rangle = \left( \frac{\sqrt{q}}{t} \right)^n G(q, t)_{\lambda \lambda} (q/t) \left( T^\lambda \right)_{0}^{N} \langle \emptyset | (1, N) \rangle \langle \emptyset | (1, N) \rangle
\]

or, pictorially,

\[
\begin{align*}
\sqrt[2]{q} z, \lambda & \quad \sqrt[2]{q} z, \lambda \\
-\sqrt[2]{q} u z & \quad \sqrt[2]{q} u z \\
(1, N+1) & \quad (1, N+1)
\end{align*}
\]

We have thus expressed the shift operator in terms of the \( T \)-operators, algebraic objects with known properties [33, 29]. Notice that the r.h.s. of Eq. (36) is automatically normal ordered since the left (resp. right) \( T \)-operator contains only the creation (resp. annihilation) operators. The horizontal spectral parameters of the intertwiners are arranged so that they act consistently on the Fock spaces. However, this is inessential, since the \( T \)-operators in the r.h.s. are actually independent of the horizontal spectral parameter, as can be seen from Eqs. (32), (33).

The dual intertwiner \( \Psi^* \) satisfies similar difference equations:

\[
\Psi^* \left( N, pu \right) | w \rangle = p^{-|w|} \Psi^* \left( N, u \right) | w \rangle
\]

\[
\Psi^* \left( N, \sqrt[2]{q} \sqrt{q} t | u | q \sqrt{q} t | w \right) = \exp \left[ -\sum_{n \geq 1} \frac{1}{n} \left( \frac{1}{1-q^n} \right) \right] \times \left( T^\mu \right)_{0}^{N} \langle \emptyset | (1, N) \rangle \langle \emptyset | (1, N) \rangle = \left( \frac{\sqrt{q}}{t} \right)^n G_{\lambda \lambda} (q/t) \left( T^\mu \right)_{0}^{N} \langle \emptyset | (1, N) \rangle \langle \emptyset | (1, N) \rangle
\]

\[\Psi^* \left( N, \sqrt[2]{q} \sqrt{q} t | u | q \sqrt{q} t | w \right) = \exp \left[ -\sum_{n \geq 1} \frac{1}{n} \left( \frac{1}{1-q^n} \right) \right] \times \left( T^\mu \right)_{0}^{N} \langle \emptyset | (1, N) \rangle \langle \emptyset | (1, N) \rangle = \left( \frac{\sqrt{q}}{t} \right)^n G_{\lambda \lambda} (q/t) \left( T^\mu \right)_{0}^{N} \langle \emptyset | (1, N) \rangle \langle \emptyset | (1, N) \rangle
\]

\[\Psi^* \left( N, \sqrt[2]{q} \sqrt{q} t | u | q \sqrt{q} t | w \right) = \exp \left[ -\sum_{n \geq 1} \frac{1}{n} \left( \frac{1}{1-q^n} \right) \right] \times \left( T^\mu \right)_{0}^{N} \langle \emptyset | (1, N) \rangle \langle \emptyset | (1, N) \rangle = \left( \frac{\sqrt{q}}{t} \right)^n G_{\lambda \lambda} (q/t) \left( T^\mu \right)_{0}^{N} \langle \emptyset | (1, N) \rangle \langle \emptyset | (1, N) \rangle
\]

---

1One may find the identity:

\[G_{\lambda \lambda}^{(q, t)} \left( \frac{q}{t} \right) = \left( \frac{q}{t} \right)^{|\lambda|} G_{\lambda \lambda}^{(q, t)} (1)\]

useful in the derivation of Eq. (36).
If one takes a product of several intertwiners \( \Psi \) and \( \Psi^* \) ending with the vacuum vectors on the both ends, after shifting one of the vertical spectral parameters, one is able to move the two \( T \)-operators in Eq. (36) to the left and to the right correspondingly, where they eventually annihilate the vacua. In the course of doing this, one has to use the commutation relations between the \( T \)-operators and the intertwiners, which we describe in the next section.

### 3.2 Commutation of vertex and \( T \)-operators

Let us prove the commutation relations which we use in the derivation of the \((q,t)\)-KZ equation. By direct computation, one gets

\[
\Psi^*_\mu^1(N + 1, -u z_1|w) = \Psi^\lambda(N - 1, -u z|w) \Psi^\mu(N, u|w) \Upsilon_{q,t} \left( \sqrt{q} \frac{z}{t} \right) (40)
\]

\[
\Psi^\lambda_i(N + 1, -u z_2|z_1) = \Psi^\lambda_2(N + 1, -u z_1|z_2) \Psi^\lambda_1(N, u|z_1) R_{\lambda_2 \lambda_1} \left( \frac{z_2}{z_1} \right) \Upsilon_{q,t} \left( \frac{q}{t} \frac{z_1}{z_2} \right) (41)
\]

\[
\Psi^*_{\mu^2}(N - 1, -u w_1|w_2) = \Psi^*_{\mu^2}(N, u|w_2) \Psi^*_{\mu^1}(N, u|w_1) \Upsilon_{q,t} \left( \frac{1}{w_1 w_2} \right) \frac{R_{\mu^2 \mu^1}}{R_{\mu^1 \mu^2}} \left( \frac{w_2}{w_1} \right) (42)
\]

where

\[
R_{\alpha\beta}(x) = \left( \frac{q}{t} \right)^{1/2} \frac{G_{\alpha\beta}(q,t)}{G_{\alpha\beta}(q,t)} \left( \frac{q}{t} x \right), \\
\Upsilon_{q,t}(\alpha|x) = \exp \left[ \sum_{n \geq 1} \frac{a^n}{n} \left( x^n - \frac{x}{q^n} \right) \right] (43)
\]

We see that the commutation relations are in fact very simple. Apart from the “anomalous factors” \( \Upsilon_{q,t} \), they are equivalent to two copies of the Zamolodchikov algebra [35] satisfied separately by \( \Psi \) and \( \Psi^* \). The pictorial description of the commutation relations (40)-(42) is drawn as follows:

\[
\text{Figures 44-46}
\]

Here the rectangular block denotes the \( R \)-matrix in the following convention:

\[
\delta_{\mu_2 \mu_1} \delta_{\lambda_2 \lambda_1} \left( \frac{z_2}{z_1} \right) (47)
\]
Using Eqs. (40)-(42), one immediately gets the commutation relations for the intertwiners \( \Psi, \Psi^* \) and the \( T \)-operator:

\[
T^\mu_{\nu}(N+1,-uy|z,w)\Psi^\lambda(N,u|y) = R_{\lambda\mu} \left( \frac{y}{z} \right) \Psi^\lambda \left( N, u \frac{z}{w} \right) \quad T^\nu_{\nu}(N,u|z,w)Y_{q,t} \left( \frac{q}{t} \right) \quad (48)
\]

\[
T^\mu_{\nu} \left( N-1,-\frac{u}{y}|z,w \right) \Psi^\lambda_{\mu_1} \left( N, u \frac{z}{w} \right) = \frac{1}{R_{\lambda\nu}} \Psi^\lambda_{\mu_1} \left( N, u \frac{z}{w} \right) \quad T^\nu_{\nu}(N,u|z,w)Y_{q,t} \left( 1 \frac{u}{y} \right) \quad (49)
\]

### 3.3 \((q,t)-KZ\) equation for DIM

Let us introduce a compact notation for the vacuum correlation matrix element of the product of \( \Psi \) and \( \Psi^* \) intertwiners:

\[
G^\lambda_{\mu_1 \cdots \mu_m} (u_n | w_1, \ldots, w_m) \overset{\text{def}}{=} \langle \varnothing | \Psi^*_{\mu_1} (n-m+1, v_1) \cdots \Psi^*_{\mu_m} (n, v_m | w_m) \Psi^{\lambda_1} (n-1, u_1 | z_1) \cdots \Psi^{\lambda_n} (0, u_n | z_n) | \varnothing \rangle \quad (50)
\]

One can draw \( G \) as follows:

\[
G^\lambda_{\mu_1 \mu_2} (u_2 | z_1, z_2) = \langle \varnothing | v_0 | (1,1) \quad v_1 = u_0 | (1,2) \quad u_1 | (1,1) \quad u_2 | (1,0) \rangle \quad (51)
\]

Here the only independent horizontal spectral parameter is \( u_n \) and all other horizontal spectral parameters are determined by the relations

\[
u_i = -z_{i+1} u_{i+1}, \quad i = 1 \ldots (n-1), \quad v_m = -z_1 u_1, \quad j = 1 \ldots (m-1).
\]

Without loss of generality, we chose the slope of the rightmost horizontal leg to be \((1,0)\). General slopes of the form \((1,N)\) can be obtained by the application of the automorphism \( \Sigma = (\frac{1}{0} \frac{1}{0}) \in SL(2,\mathbb{Z}) \) of DIM. This automorphism transforms the horizontal representations changing their slopes in the natural way, \((n_1, n_2) \xrightarrow{T} (n_1, n_1 + n_2)\). It acts diagonally on the vertical representation with matrix elements in the Macdonald basis given by:

\[
\Xi^\lambda_{\mu} = \langle M_{\lambda}, z | M_{\mu}, z \rangle = \delta_{\lambda\mu} (-z)^{\lambda} f_{\lambda}
\]

Due to the intertwining property of \( \Psi \) and \( \Psi^* \), one can rewrite the change of slope of the horizontal legs in terms of the action of \( T \) on the vertical leg:

\[
\Psi^\lambda(N+1,u|z) = \Xi^{-1} \Psi^\lambda(N,u|z) \Xi = \sum_{\mu} (\Xi^{-1})^\lambda_{\mu} \Psi^\mu(N, u|z) = (-z)^{\lambda} f_{\lambda}^{-1} \Psi^\lambda(N, u|z)
\]

\[
\Psi^*_{\mu}(N+1,u|w) = \Xi^{-1} \Psi^*_{\mu}(N,u|w) \Xi = \sum_{\nu} \Xi^\nu_{\nu} \Psi^*_{\mu}(N,u|w) = (-w)^{\mu} f_{\nu} \Psi^*_{\mu}(N,u|w)
\]

Therefore, changing the overall slope in \( G \) amounts to the multiplication by certain simple factors.

Let us act on the correlator \( G \) with the difference operator \( (\frac{z}{f})^{z_1 z_2} \). According to Eq. (48), this creates a pair of \( T \)-operators surrounding the intertwiner \( \Psi \). We then push the \( T \)-operators to the ends of the correlator using the commutation relations, (48), (49). Eventually, the \( T \)-operators act on the vacuum giving the prefactor, which cancels the prefactor in (36) and leaving behind the original correlator \( G \).
To write down the resulting equation, it is convenient to introduce the modified $\mathcal{R}$-matrix $\tilde{\mathcal{R}}_{\lambda\mu}(x)$ differing from $\mathcal{R}_{\lambda\mu}(x)$ by a scalar factor:

$$\tilde{\mathcal{R}}_{\lambda\mu}(x) \equiv \mathcal{R}_{\lambda\mu}(x) \frac{\mathcal{Y}_{q,t}(1|x)}{\mathcal{Y}_{q,t}(\sqrt{t}|\sqrt{t}x)} = \mathcal{R}_{\lambda\mu}(x) \exp \left[ -\sum_{n \geq 1} \frac{x^{-n}}{n} \frac{1 - \left( \frac{q}{t} \right)^n}{(1 - q^n)(1 - t^{-n})} \right]$$ (56)

The ($q,t$)-KZ equation for DIM intertwiners is then written as follows:

$$\left( q^{\lambda(k)} + \frac{1}{t} q_{\lambda(k)} \right) \mathcal{G}^{\lambda_1\cdots\lambda_n}_{\mu_1\cdots\mu_m}(u_n | z_1,\ldots,z_n) = \left( q^{\lambda(\mu)} + \frac{1}{t} q_{\lambda(\mu)} \right) \mathcal{G}^{\lambda_1\cdots\lambda_n}_{\mu_1\cdots\mu_m}(u_n | w_1,\ldots,w_m)$$ (57)

Here the extra shift of $u_n$ by $\sqrt{\frac{q}{t}}$ was introduced to conform with Eq. (28). A similar equation holds for the shift in the vertical spectral parameters of the dual intertwiners $\Psi^*$:

$$\left( q^{\lambda(k)} \frac{w_k}{t} + \frac{1}{t} q_{\lambda(k)} \frac{w_k}{t} \right) \mathcal{G}^{\lambda_1\cdots\lambda_n}_{\mu_1\cdots\mu_m}(u_n | z_1,\ldots,z_n) = \left( q^{\lambda(\mu)} \frac{w_k}{t} + \frac{1}{t} q_{\lambda(\mu)} \frac{w_k}{t} \right) \mathcal{G}^{\lambda_1\cdots\lambda_n}_{\mu_1\cdots\mu_m}(u_n | w_1,\ldots,w_m)$$ (58)

These are the main results of this section. We will now give explicit solutions to these equations, which are equal to refined topological amplitudes on the strip geometry.

3.4 Explicit solution to DIM ($q,t$)-KZ

As we stressed earlier, the DIM algebra $U_{q,t}(\hat{g}_1)$ is quasi-Abelian, which means that there are no screening charges in this case. Indeed, as we explained in [15], the screening charges are associated with the network models containing two ($q$-Virasoro) or more ($q$-$W$-algebra) horizontal lines. In fact, the number of horizontal lines corresponds to the first central charge: if it is equal to an integer $k$, we have $q$-$W^{(k)}$-algebra. In particular, the deformation of WZW model is described by one horizontal line (see, e.g., the figure in [35] or Fig.1). Thus, in this case, there are no screening charges, and the “Abelian” ($q,t$)-KZ equation has a solution in terms of products of propagators. In our case, the role of the propagators is played by the functions

$$F_{\lambda\mu}(x) \equiv x^{-|\mu|} G_{\lambda\mu}(x) \exp \left[ \sum_{n \geq 1} \frac{1}{n} \left( \frac{t}{q} x \right)^{-n} \frac{1}{(1 - q^n)(1 - t^{-n})} \right].$$ (59)

Notice that the exponential factor is nothing but the generating function of equivariant Euler characters of the Hilbert scheme $\text{Hilb}(\mathbb{C}^2)$, [37] or the refined topological string amplitude of the resolved conifold.

The function $F_{\lambda\mu}(x)$ satisfies the following fundamental difference equation:

$$\frac{F_{\lambda\mu}(q x)}{F_{\lambda\mu}(x)} = \left( \frac{q^{\lambda(\mu)}}{t} \right) \frac{1}{\tilde{\mathcal{R}}_{\lambda\mu}(x)}.$$ (60)

In terms of the function $F_{\lambda\mu}(x)$, the solution reads as follows:

$$\mathcal{G}^{\lambda_1\cdots\lambda_n}_{\mu_1\cdots\mu_m}(u_n | z_1,\ldots,z_n) = \prod_{k=1}^m u_{n_1}^{\sum_{j=1}^n |\lambda_j|-\sum_{j=1}^m |\mu_j|} \frac{\prod_{i=1}^n F_{\mu_i \lambda_i} \left( \sqrt{\frac{t}{q}} \frac{u_k}{z_i} \right)}{\prod_{k<l} F_{\mu_k \mu_l} \left( \frac{q_{\mu_k \mu_l}}{t} \frac{u_k}{u_l} \right) \prod_{i<j} F_{\lambda_i \lambda_j} \left( \frac{z_i}{z_j} \right)}.$$ (61)

---

2In the notation of [36], our $\mathcal{R}$-matrix $\mathcal{R}_{\lambda\mu}(x)$ is denoted by $R_{\mathcal{F},\mathcal{F}}(x)$ and the $\mathcal{R}$-matrix $\tilde{\mathcal{R}}_{\lambda\mu}(x)$ with the additional scalar factor introduced in Eq. (60) is denoted simply by $\mathcal{R}(x)$.
The solution is of course defined up to multiplication by a \((\frac{2}{\pi})\)-periodic function of the spectral parameters \(z_i\) and \(w_j\). It is curious to notice that the form of the solution resembles the free fermion correlator.

The structure on the r.h.s. of Eq. (61) is consistent with the fact that by the spectral duality \([11]\) it gives a building block of the Nekrasov partition function for \(N = 2\) linear quiver gauge theory. That is, a pair of edges on the opposite sides of the (horizontal) line gives the bifundamental matter contribution, while a pair on the same side gives the gauge (root) boson contribution.

The braiding transformations \(\tilde{B}_{i,j+1}^i\) and \(\tilde{B}_{j+1}^j\) (permuting the positions of \(\Psi\) and \(\Psi^*\) intertwiners respectively) of the solution are defined as follows:

\[
\tilde{B}_{i,j+1}^i G_{\mu_1 \cdots \mu_m}^{\lambda_1 \cdots \lambda_n} (u_n | z_1, \ldots, z_m, w_1, \ldots, w_m) \equiv G_{\mu_1 \cdots \mu_m}^{\lambda_1 \lambda_{i+1} \cdots \lambda_n} (u_n | z_1, \ldots, z_{i+1}, z_i, \ldots, z_m, w_1, \ldots, w_m),
\]

\[
\tilde{B}_{j+1}^j G_{\mu_1 \cdots \mu_m}^{\lambda_1 \cdots \lambda_n} (u_n | z_1, \ldots, z_m, w_1, \ldots, w_m) \equiv G_{\mu_1 \cdots \mu_{j+1} \mu_j \cdots \mu_m}^{\lambda_1 \cdots \lambda_n} (u_n | z_1, \ldots, z_{j+1}, z_j, \ldots, z_m, w_1, \ldots, w_m).
\]

Evidently, the two sets of operators commute, \([\tilde{B}_{i,j+1}^i, \tilde{B}_{j+1}^j] = 0\) and separately satisfy the braid group relations. The explicit computation shows that the braiding is performed by the DIM \(\mathcal{R}\)-matrix with the anomalous factor as in Eqs. (41), (42):

\[
\tilde{B}_{i,j+1}^i G_{\mu_1 \cdots \mu_m}^{\lambda_1 \cdots \lambda_n} (u_n | z_1, \ldots, z_m, w_1, \ldots, w_m) = \frac{f_{\lambda_i}}{f_{\lambda_{i+1}}} \mathcal{R}_{\lambda_i, \lambda_{i+1}} \left( \frac{z_i}{z_{i+1}} \right) \mathcal{Y}_{q,t} \left( q^{\frac{w_{i+1}}{q}}, \frac{z_{i+1}}{z_i} \right) G_{\mu_1 \cdots \mu_m}^{\lambda_1 \cdots \lambda_n} (u_n | z_1, \ldots, z_{i+1}, z_i, \ldots, z_m, w_1, \ldots, w_m),
\]

\[
\tilde{B}_{j+1}^j G_{\mu_1 \cdots \mu_m}^{\lambda_1 \cdots \lambda_n} (u_n | z_1, \ldots, z_m, w_1, \ldots, w_m) = \left( \frac{q}{t} \right)^{|\mu_j| - |\mu_{j+1}|} \frac{f_{\mu_j}}{f_{\mu_{j+1}}} \mathcal{R}_{\mu_j, \mu_{j+1}} \left( \frac{w_{j+1}}{w_j} \right) G_{\mu_1 \cdots \mu_m}^{\lambda_1 \cdots \lambda_n} (u_n | z_1, \ldots, z_{j+1}, z_j, \ldots, z_m, w_1, \ldots, w_m).
\]

Extra constant factors are not captured by our solution (which is defined up to a \((\frac{2}{\pi})\)-periodic function) and appear in the r.h.s. of Eqs. (64), (65). They can be eliminated by a simple change of the normalization of solutions.

### 4 Elliptic \((q,t)\)-KZ equation for DIM

Using the difference operator from sec. 3.1 and commutation relations from sec. 3.2 one can also derive elliptic \((q,t)\)-KZ equation for the trace of a product of intertwiners, which is a DIM counterpart of the KZBE on torus (see a similar equation for the quantum affine algebra in [25]). As in the previous section, we first sketch the main idea of the derivation and then give the precise expressions with correct arguments and prefactors.

We start with the trace of the following form

\[
\text{"Tr} \left( Q^d Q^d_{\perp} \Psi \cdots \Psi \right) ^n
\]

Here we added the grading operators \(d\) and \(d_{\perp}\) for the sake of generality. In a moment, we will see that they are in fact customary to get a meaningful equation for the trace. Proceeding as in the previous section, we rewrite the shift operator in terms of the \(T\)-operators as sketched in Eq. (16) and move the \(T\)-operators to the left and right of \(\Psi\). This gives a product of \(\mathcal{R}\)-matrices as in Eq. (19) of the previous section:

\[
\text{"Tr} \left( Q^d Q^d_{\perp} \Psi \cdots \Psi \left( \frac{q}{t} \right) \cdots \Psi \right) = \text{Tr} \left( Q^d Q^d_{\perp} \Psi \cdots \mathcal{T}(z) \Psi(z) \mathcal{T}^{-1}(z) \cdots \Psi \right) = \left( \prod \mathcal{R} \right) \text{Tr} \left( Q^d Q^d_{\perp} \mathcal{T}(z) \Psi(z) \cdots \Psi \mathcal{T}^{-1}(z) \cdots \Psi \right)
\]

However, now the \(T\)-operators act not on the vacuum vectors but under the trace. Therefore, using the cyclic property of the trace, we move them further \emph{through each other} so that they reappear on the opposite side of the expression. Taking a full circle, we obtain the trace which looks similar to the initial one with the arguments of the \(T\)-operators shifted by \(Q\) (note that the products of \(\mathcal{R}\)-matrices in \((67)\) and \((68)\) are different):

\[
\text{"Tr} \left( Q^d Q^d_{\perp} \Psi \cdots \Psi \left( \frac{q}{t} \right) \cdots \Psi \right) = \left( \prod \mathcal{R} \right) \text{Tr} \left( Q^d Q^d_{\perp} \mathcal{T}(Qz) \Psi(z) \mathcal{T}^{-1}(Q^{-1}z) \cdots \Psi \right)
\]

We repeat the cyclic transfer of the \(T\)-operators so that their arguments are shifted by higher and higher powers of \(Q\). At the same time, one gets more and more \(\mathcal{R}\)-matrices at the r.h.s. of Eq. (68). The crucial point is that the two \(T\)-operators featuring in Eq. (68) are \emph{Taylor} (not Laurent) series in \(Q\) and \(Q^{-1}\) respectively. If we assume that \(|Q| < 1\), then for high enough positive powers of \(Q\) the \(T\)-operators converge to a \emph{scalar factor}. 13
After that we are left with an infinite product of $\mathcal{R}$-matrices, which can be packed into the finite product of the new elliptic $\mathcal{R}$-matrices which we denote by the gothic letter $\mathfrak{R}$:

$$"\text{Tr} \left( Q^d Q^d_{\perp} \Psi \cdots \Psi \left( \frac{q}{t} z \right) \cdots \Psi \right) = \left( \prod \mathfrak{R} \right) \text{Tr} \left( Q^d Q^d_{\perp} \Psi \cdots \Psi \left( z \right) \cdots \Psi \right) "$$

(69)

This equation is the DIM version of the elliptic $(q,t)$-KZ equation usually satisfied by torus blocks of the affine algebra. The procedure itself is by essence the "averaging" procedure of constructing elliptic $\mathcal{R}$-matrices [25]. Let us now proceed to the detailed derivation.

4.1 Matching the Fock spaces

To take a trace of a product of intertwiners, one has to ensure that this product is an endomorphism of some Fock space, i.e. that the Fock spaces on the incoming and outgoing horizontal legs coincide.

First of all, the slopes should match. In the previous section, we adopted the convention that the rightmost intertwiner acts on the space with slope $(1,0)$. After $n$ $\Psi$ intertwiners and $m$ $\Psi^*$ acted on the space, the slope changes to $(1,n-m)$. We therefore conclude that under the trace the number of $\Psi$ should be equal to the number of $\Psi^*$, i.e. exactly the case of balanced network, $n = m$.

Secondly, the spectral parameters on the left and right should coincide. Let the rightmost intertwiner act on $\mathfrak{g}$

$$z_1, \lambda_1 \quad z_2, \lambda_2 \quad w_1, \mu_1 \quad w_2, \mu_2$$

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Secondly, the spectral parameters on the left and right should coincide. Let the rightmost intertwiner act on the Fock space with the horizontal spectral parameter $u_n$. Then, using relations [52], we obtain the horizontal spectral parameter $v_0$ to the left of all the intertwiners:

$$v_0 = u_n \prod_{i=1}^{n} \frac{z_i}{w_i}$$

(70)

Notice, however, that the operator $Q^d_{\perp}$ shifts the horizontal spectral parameter from $v_0$ to $Q^d v_0$. To be able to take the trace we, therefore, will henceforth set

$$Q^d_{\perp} = \prod_{i=1}^{n} \frac{w_i}{z_i}$$

(71)

in all expressions. As in the previous section, we introduce the notation for the trace of intertwiners:

$$G^{\lambda_1 \cdots \lambda_n}_{\mu_1 \cdot \cdot \cdot \mu_n} (u_n|Q| z_1, \ldots, z_n) = \text{Tr} \left\{ Q^d Q^d_{\perp} \Psi_{\mu_1} (1, 1|v_1) \cdots \Psi_{\mu_n} (n, n|w_n) \psi_{\lambda_1} (n-1, u_1|z_1) \cdots \psi_{\lambda_n} (0, u_n|z_n) \right\}$$

(72)

We can draw the corresponding picture denoting the trace pairing as wavy lines and the grading operator insertions as boxes:

$$G^{\lambda_1 \lambda_2}_{\mu_1, \mu_2} (u_2|Q| z_1, z_2) = \begin{array}{c}
\chi \hline
u_2 \\
1, 0
\end{array} \begin{array}{c}
Q^d Q^d_{\perp} \\
(1, 1)
\end{array} \begin{array}{c}
v_0 \\
(1, 0)
\end{array} \begin{array}{c}
v_1 \\
(1, 1)
\end{array} \begin{array}{c}
v_2 = u_0 \\
(1, 2)
\end{array} \begin{array}{c}
u_1 \\
(1, 1)
\end{array} \begin{array}{c}
u_2 \\
(1, 0)
\end{array} \begin{array}{c}
w_1, \mu_1 \\
(1, 0)
\end{array} \begin{array}{c}
w_2, \mu_2 \\
(1, 0)
\end{array} \begin{array}{c}
z_1, \lambda_1 \\
z_2, \lambda_2
\end{array}$$

(73)

The parameter $Q$ remains free and can be interpreted as the exponentiated modular parameter of the torus on which the conformal block of DIM lives. The results of the previous section are recovered in the limit $Q \to 0$, when the torus degenerates into a cylinder.

4.2 $q$-KZ equation for the trace of intertwiners

The derivation of the $q$-KZ equation for the trace of intertwiners goes along the same lines as that of the equation for the vacuum matrix element. We express the shift operator through the $T$-operators using Eq. [50]. We then use the commutation relations for the intertwiners and $T$-operators [48] to move the $T$-operators cyclically under the trace. After each cycle the vertical spectral parameters of the $T$-operators are multiplied by $Q$:

$$Q^d T^\lambda (N, u|z, w) = T^\lambda (N, u|Qz, Qw) Q^d$$

(74)
From Eqs. (32)–(33) assuming $|Q|<1$ in the limit of large number $k$ of cycles, one has
\begin{equation}
\mathcal{T}_N^J \left( N, u \big| Q^{-k}z, \sqrt{\frac{q}{t}} Q^{-k}z \right) \xrightarrow{k \to \infty} \left( \frac{q}{t} \right)^{\frac{1}{2}N(N+1)\lambda\lambda} \left( \frac{q}{t} \right)^{\frac{1}{2}N(N+1)\lambda\lambda} \frac{q^{||\lambda||^2}G_{\lambda\lambda}^{(q,t)}(1)}{C_{\lambda\lambda}^{2}} \exp \left[ \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{q}{1 - q^n} \right)^n \frac{1}{(1 - t^n) (1 - t^{-n})} \right] \tag{75}
\end{equation}

Thus, after making a sufficient number of cycles, the $\mathcal{T}$-operators essentially disappear and we are left with the infinite number of $\mathcal{R}$-matrices. We pack the infinite product into the combination of *three* different elliptic $\mathcal{R}$-matrices differing by slight shifts of arguments:

\begin{align*}
\tilde{\mathcal{R}}_{\lambda\mu}^{(1)}(x\mid Q) & \overset{\text{def}}{=} \prod_{k \geq 0} \frac{\mathcal{R}_{\lambda\mu}(xQ^{k+1})}{\mathcal{R}_{\lambda\mu}(x^{-1}Q^k)} \exp \left[ \sum_{n=1}^{\infty} \frac{1 - \left( \frac{q}{t} \right)^n}{n} \frac{(Qx)^n - x^n}{(1 - q^n)(1 - t^n)(1 - Q^n)} \right], \\
\tilde{\mathcal{R}}_{\lambda\mu}^{(2)}(x\mid Q) & \overset{\text{def}}{=} \prod_{k \geq 0} \frac{\mathcal{R}_{\lambda\mu}\left( \frac{q}{t} xQ^{k+1} \right)}{\mathcal{R}_{\lambda\mu}(x^{-1}Q^k)} \exp \left[ \sum_{n=1}^{\infty} \frac{1 - \left( \frac{q}{t} \right)^n}{n} \frac{(\frac{q}{t}Qx)^n - x^n}{(1 - q^n)(1 - t^n)(1 - Q^n)} \right], \\
\tilde{\mathcal{R}}_{\lambda\mu}^{(3)}(x\mid Q) & \overset{\text{def}}{=} \prod_{k \geq 0} \frac{\mathcal{R}_{\lambda\mu}\left( \frac{q}{t} xQ^{k+1} \right)}{\mathcal{R}_{\lambda\mu}(x^{-1}Q^k)} \exp \left[ \sum_{n=1}^{\infty} \frac{1 - \left( \frac{q}{t} \right)^n}{n} \frac{(1/Qx)^n - x^n}{(1 - q^n)(1 - t^n)(1 - Q^n)} \right].
\end{align*}

These $\mathcal{R}$-matrices should be associated with the elliptic DIM algebra \cite{38}. Eventually, we obtain elliptic KZ equations for the shifts of $\Psi$ and $\Psi^*$ intertwiners:

\begin{align}
\left( \frac{q}{t} \right)^{z_k \partial_{x_k} - \frac{4}{t} u_n \partial_{u_n}} \mathfrak{S}_{\mu_1 \cdots \mu_n}^{1 \cdots \lambda_n}(u_n \mid Q \mid z_1, \ldots, z_n) &= \prod_{k=1}^{n} \tilde{\mathcal{R}}_{\mu_1 \lambda_k}^{(1)} \left( \sqrt{\frac{t}{q}} \frac{u}{z_k} Q \right) \prod_{i < k} \tilde{\mathcal{R}}_{\lambda_i \lambda_k}^{(2)} \left( \frac{z_k}{z_j} \right) \mathfrak{S}_{\mu_1 \cdots \mu_n}^{\lambda_1 \cdots \lambda_n}(u_n \mid Q \mid z_1, \ldots, z_n) \tag{78} \\
\left( \frac{q}{t} \right)^{w_k \partial_{x_k} + \frac{4}{t} u_n \partial_{u_n}} \mathfrak{S}_{\mu_1 \cdots \mu_n}^{1 \cdots \lambda_n}(u_n \mid Q \mid z_1, \ldots, z_n) &= \prod_{k=1}^{n} \tilde{\mathcal{R}}_{\mu_k \lambda_k}^{(3)} \left( \frac{u_k}{w_k} Q \right) \prod_{j > k} \tilde{\mathcal{R}}_{\mu_k \mu_j}^{(4)} \left( \frac{w_k}{t w_j} Q \right) \prod_{i=1}^{k} \tilde{\mathcal{R}}_{\mu_i \lambda_k}^{(1)} \left( \sqrt{\frac{t}{q}} \frac{w_k}{z_i} Q \right) \mathfrak{S}_{\mu_1 \cdots \mu_n}^{\lambda_1 \cdots \lambda_n}(u_n \mid Q \mid z_1, \ldots, z_n) \tag{79}
\end{align}

The form of KZBE \cite{78, 79} is very similar to that of the KZE \cite{57, 58}, the only important difference being the elliptic $\mathcal{R}$-matrices. In the next subsection, we find the explicit solution to the elliptic KZ equations.

### 4.3 Explicit solution for elliptic $(q,t)$-KZ equation

Proceeding as in the case of $(q,t)$-KZE, we introduce three propagators corresponding to the three $\mathcal{R}$-matrices \cite{77} (again the difference is in slight shifts of the parameters):

\begin{align*}
\Theta_{\lambda\mu}(x\mid Q) &= \prod_{k \geq 0} G_{\lambda\mu}(xQ^k) \exp \left[ \sum_{n=1}^{\infty} \frac{x^n + (\frac{q}{t} Q)^n x^{-n}}{n(1 - q^n)(1 - t^n)(1 - Q^n)} \right], \\
\Phi_{\lambda\mu}(x\mid Q) &= \prod_{k \geq 0} G_{\lambda\mu}(\frac{q}{t} xQ^k) \exp \left[ \sum_{n=1}^{\infty} \frac{x^n + Q^n x^{-n}}{n(1 - q^n)(1 - t^n)(1 - Q^n)} \right], \\
\Psi_{\lambda\mu}(x\mid Q) &= \prod_{k \geq 0} G_{\lambda\mu}(xQ^k) \exp \left[ \sum_{n=1}^{\infty} \frac{x^n + Q^n x^{-n}}{n(1 - q^n)(1 - t^n)(1 - Q^n)} \right].
\end{align*}
These functions obey the following basic difference equations:

\[ \Theta_{\lambda \mu}(x = t^{-1}q) \Theta_{\lambda \mu}(x | Q) = \tilde{\mathcal{R}}_{\mu \lambda}^{(1)}(x^{-1} | Q) \Theta_{\lambda \mu}(x | Q), \]

(81)

\[ \Phi_{\lambda \mu}(x = t^{-1}q) \Phi_{\lambda \mu}(x | Q) = \tilde{\mathcal{R}}_{\mu \lambda}^{(2)}(t^{-1}q) \Phi_{\lambda \mu}(x | Q), \]

(82)

\[ \bar{\Phi}_{\lambda \mu}(x = t^{-1}q) \bar{\Phi}_{\lambda \mu}(x | Q) = \tilde{\mathcal{R}}_{\mu \lambda}^{(3)}(x^{-1} | Q) \bar{\Phi}_{\lambda \mu}(x | Q). \]

(83)

An explicit solution of the elliptic KZ equation is up to a periodic function given by

\[ \Theta_{\lambda_1 \cdots \lambda_n}(u_n | Q | z_1, \ldots, z_n) = \frac{\Pi_{i < j} \Theta_{\lambda_i \lambda_j} \left( \frac{\sqrt{u_i}}{u_j} | Q \right) \Pi_{k < l} \bar{\Phi}_{\mu_k \mu_l} \left( \frac{w_k}{w_l} | Q \right)}{\Pi_{i < j} \Phi_{\lambda_i \lambda_j} \left( \frac{z_i}{z_j} | Q \right) \Pi_{k < l} \bar{\Phi}_{\mu_k \mu_l} \left( \frac{w_k}{w_l} | Q \right)}. \]

(84)

The three different propagators appear in the elliptic case, because the pairings \( \Psi \Psi, \Psi \Psi^* \) and \( \Psi^* \Psi^* \) are all different and cannot be expressed through a single function.

### 4.4 Braiding, dual KZE and modular properties

The braiding operators \( \mathcal{B}_{i,i+1}, \mathcal{B}_{j,j+1} \) defined by Eq. (62) are local in the sense that their action depends only on the parameters of the two legs being exchanged. Therefore, they are not modified by taking trace.

There is, however, one more braiding operator arising in the elliptic case:

\[ \mathcal{B}_{n+1,1} \Theta_{\mu_1 \cdots \mu_n}(u_n | Q | z_1, \ldots, z_n) = \mathcal{B}_{n,1} \Theta_{\mu_1 \cdots \mu_n}(u_n | Q | z_1, \ldots, z_n), \]

(85)

\[ \mathcal{B}_{n+1,1} \Phi_{\mu_1 \cdots \mu_n}(u_n | Q | z_1, \ldots, z_n) = \mathcal{B}_{n,1} \Phi_{\mu_1 \cdots \mu_n}(u_n | Q | z_1, \ldots, z_n). \]

(86)

It acts in the following way on the solutions of the elliptic KZ equation:

\[ \frac{\mathcal{B}_{n+1,1} \Theta_{\mu_1 \cdots \mu_n}(u_n | Q | z_1, \ldots, z_n)}{\Theta_{\mu_1 \cdots \mu_n}(u_n | Q | z_1, \ldots, z_n)} = \mathcal{R}_{\lambda_1 \lambda_n} \left( \frac{z_n}{z_1} \right) \gamma_{q,t} \left( \frac{q}{t}, \frac{z_1}{z_n} \right) \prod_{i=1}^{n} \gamma_{q,t} \left( \sqrt{\frac{u_i}{u_1}} \right), \]

(87)

\[ \frac{\mathcal{B}_{n+1,1} \Phi_{\mu_1 \cdots \mu_n}(u_n | Q | z_1, \ldots, z_n)}{\Phi_{\mu_1 \cdots \mu_n}(u_n | Q | z_1, \ldots, z_n)} = \gamma_{q,t} \left( 1, \frac{w_n}{w_1} \right) \prod_{j=1}^{n} \gamma_{q,t} \left( \sqrt{\frac{z_j}{z_{n+1}}} \right). \]

(88)

Notice that the anomaly factors in Eqs. (87), (88) differ from those in Eqs. (64), (65). The complete set of braiding transformations satisfies the affine braid group relations of type \( \hat{A}_n \). There is a set of distinguished \emph{period} elements \( \hat{P}^k \), of the affine braid group, which carry the variable \( z_k \) over the full cycle under the trace:

\[ \hat{P}^k = \mathcal{B}^{(k,k+1)}(1) \mathcal{B}^{(k+1,k+2)} \cdots \mathcal{B}^{(n,1)} \mathcal{B}^{(1,2)} \cdots \mathcal{B}^{(k-1,k)} \]

(89)

\[ \hat{P}_k = \mathcal{B}(k,k+1) \mathcal{B}(k+1,k+2) \cdots \mathcal{B}(n,1) \mathcal{B}(1,2) \cdots \mathcal{B}(k-1,k) \]

(90)

The operator \( \hat{P}^k \) effectively shifts the variable \( z_k \) by \( Q \):

\[ \hat{P}^k \Theta_{\mu_1 \cdots \mu_n}(u_n | Q | z_1, \ldots, z_n) = \Theta_{\mu_1 \cdots \mu_n}(u_n | Q | z_1, \ldots, z_{k-1}Qz_k, z_{k+1}, \ldots, z_n), \]

(91)

\[ \hat{P}_k \Theta_{\mu_1 \cdots \mu_n}(u_n | Q | z_1, \ldots, z_n) = \Theta_{\mu_1 \cdots \mu_n}(u_n | Q | z_1, \ldots, z_{k-1}, z_k, z_{k+1}, \ldots, z_n). \]

(92)

One can evaluate the action of the period elements explicitly using Eqs. (64), (65), (87) and (88) and arrive at

\[ \hat{P}^k \Theta_{\mu_1 \cdots \mu_n}(u_n | Q | z_1, \ldots, z_n) = \frac{1}{\prod_{i=1}^{n} \gamma_{q,t} \left( \sqrt{\frac{Q z_i}{w_1}} \right) \Pi_{i < k} \mathcal{R}_{\lambda_i \lambda_k} \left( \frac{z_i}{z_k} \right) \Theta_{\mu_1 \cdots \mu_n}(u_n | Q | z_1, \ldots, z_n)}, \]

(93)

\[ \hat{P}_k \Theta_{\mu_1 \cdots \mu_n}(u_n | Q | z_1, \ldots, z_n) = \frac{1}{\prod_{j=1}^{n} \gamma_{q,t} \left( \sqrt{\frac{Q z_j}{w_n}} \right) \Pi_{j > k} \mathcal{R}_{\lambda_j \lambda_k} \left( \frac{Q z_k}{z_j} \right) \Theta_{\mu_1 \cdots \mu_n}(u_n | Q | z_1, \ldots, z_n)}. \]

(94)
Here we introduced the dual $\mathcal{R}$-matrices differing from the DIM $\mathcal{R}$-matrix \cite{13} by simple scalar factors

$$\mathcal{R}_{\lambda\mu}(x) = \frac{f_{x}}{f_{\mu}} \mathcal{R}_{\lambda\mu}(x) \mathcal{Y}_{q,t}(\frac{Q}{t} \big| x^{-1}), \quad \mathcal{R}_{\mu\lambda}(x^{-1}) = (\mathcal{R}_{\lambda\mu}(x))^{-1},$$

(95)

$$\mathcal{R}_{\lambda\mu}(x) = \left( \frac{Q}{t} \right)^{|\lambda|-|\mu|} \frac{f_{x}}{f_{\lambda}} \mathcal{R}_{\lambda\mu}(x) \mathcal{Y}_{q,t}(1 \big| x), \quad \mathcal{R}_{\mu\lambda}(x^{-1}) = (\mathcal{R}_{\lambda\mu}(x))^{-1}$$

(96)

Eqs. (93) are dual KZE satisfied by the trace of intertwiners. They are very similar to the $(q,t)$-KZ equations \cite{52, 58}, except for the shifts in the dual picture which are not $\frac{Q}{t}$, but $Q$! Moreover, the $\mathcal{R}$-matrices in the dual equation are not elliptic, but trigonometric. It is natural to compare this with the analogous results for quantum affine algebras \cite{25, 27} where both the $q$-KZ and dual $q$-KZ equations were elliptic.

Let us also briefly discuss the modular properties of solutions that we have just obtained. We focus on the simplest case of $n = 1$, i.e. $\mathcal{G}_{\mu}(u|Q, \tilde{\omega})$, and then give some comments about the general solutions. We use the following identity:\cite{48}

$$\Theta_{\lambda\mu}(x|Q) = \prod_{k \geq 1} (1 - Q^{k})^{-|\lambda|-|\mu|} \prod_{(i,j) \in \lambda} \theta_{Q}(xq^{\lambda_{i} - j + \mu_{j} + 1}) \prod_{(i,j) \in \mu} \theta_{Q}(xq^{j - \mu_{i} - 1 + \lambda_{i}}) \prod_{k \geq 1} \frac{\theta_{Q}(qx^{k})}{\prod_{k \geq 1} (1 - Q^{k})}$$

(97)

where $\theta_{Q}(x) = \prod_{k \geq 0} (1 - Q^{k+1})(1 - Q^{k}x)(1 - x^{-1}Q^{k+1})$. We use the modular properties of the theta-function and the Dedekind function:

$$\theta_{Q}(\tilde{x}) = (-i\tau)^{\frac{1}{2}} e^{\frac{i\pi}{2}(1 - \frac{1}{2})^{2}} \theta_{Q}(x)$$

$$\prod_{k \geq 1} (1 - \tilde{Q}^{k}) = (-i\tau)^{\frac{1}{2}} Q^{\frac{x}{2} + \frac{1}{2}} \prod_{k \geq 1} (1 - Q^{k}),$$

(98)

(99)

where

$$Q = e^{2\pi i \tau}, \quad \tilde{Q} = e^{-2\pi i \tau},$$

$$x = e^{2\pi i a}, \quad \tilde{x} = e^{2\pi i a}$$

(100)

(101)

We note that $\Theta_{\emptyset\emptyset}(x|Q)$ is nothing but the inverse of the double elliptic gamma function $G_{2}(x|q, t^{-1}, Q)$ defined by

$$G_{2}(x|q_{0}, q_{1}, q_{2}) \overset{\text{def}}{=} \prod_{i,j,k=0}^{\infty} (1 - xq_{i}^{|i+j+k|}(1 - x^{-1}q_{i}^{j}q_{j}^{k}).$$

(102)

This is the third member of a hierarchy of meromorphic functions\cite{9} $G_{r}(x|q_{0}, \ldots, q_{r})$ starting from a Jacobi theta function $\theta_{0}(z, \tau) = G_{0}(e^{2\pi iz}|e^{2\pi i\tau})$. The elliptic gamma function $\Gamma(z, \tau, \sigma) = G_{1}(e^{2\pi iz}|e^{2\pi i\tau}, e^{2\pi i\sigma})$ was introduced in \cite{39} and implicitly appeared in the formula of the free energy of the eight-vertex model \cite{10}. It also appears in the solutions to the elliptic $q$-KZB equations \cite{11, 12}. The modular property of the elliptic gamma function is discussed in \cite{43}. The function $G_{r}(x|q_{0}, \ldots, q_{r})$ are called multiple elliptic gamma functions and regarded as an elliptic generalization of the Barnes multiple gamma function \cite{44}. The useful formulas for $G_{r}(x|q_{0}, \ldots, q_{r})$ including the modular property are summarized in \cite{45}. More recently the full partition function of 5d $\mathcal{N} = 2^{*}$ gauge theory with a hypermultiplet in the adjoint representation was expressed using the double elliptic gamma function $G_{2}(x|q_{0}, q_{1}, q_{2})$ \cite{46, 47}. Finally there is a closely related function called multiple sine function $S_{r}(z|q_{0}, \ldots, q_{r})$. It is amusing that some solutions of the $q$-KZ equation for XXZ model (or $U_{q}(\mathfrak{sl}_{2})$) are given by the double sine function \cite{48}.

Solution (84) to the elliptic $(q,t)$-KZ equations have quite involved modular behaviour because of the factor

\footnotetext{The infinite product in the r.h.s. requires some regularization.}

\footnotetext{These functions $G_{r}(x|q) = (x|q)^{(-1)^{r}}(x^{-1}q_{0}\cdots q_{r}|q)_{\infty}$ are defined using the multiple $q$-Pochhammer symbol $(x|q)_{\infty} = \prod_{j=0}^{\infty} (1 - xq_{j}^{0}\cdots q_{r}^{j})$.}
We have considered a natural question within the context of the network models with underlying DIM symmetry:

\[
\Theta_{\mathcal{G}}(x|\Omega) \text{.}
\]

However, one can obtain a relatively simple transformation law for the ratio of two solutions:

\[
\frac{\Theta_{\mu}(u|\tilde{Q}, \tilde{q}, \tilde{t})}{\Theta_{\nu}(u|Q, q, t)} = \exp \left[ \frac{i\pi}{\tau} \sum_{(i,j) \in \lambda} \left( a - b + \epsilon_2 (j - \lambda_i + 1) + \epsilon_1 (\mu_j - i + 1) - i \right) \frac{1}{2} - \frac{\tau}{2} \right]^2 \times\]

\[
\Theta_{\lambda}(\sqrt{\frac{i}{\tau}} \tilde{Q}, \tilde{q}, \tilde{t}) \frac{\Theta_{\mu}(\sqrt{\frac{i}{\tau}} Q, q, t)}{\Theta_{\nu}(\sqrt{\frac{i}{\tau}} Q, q, t)}
\]

(103)

where

\[
q = e^{-2\pi i \epsilon_2}, \quad \tilde{q} = e^{-2\pi i \frac{\epsilon_2}{2}}
\]

(104)

\[
t = e^{2\pi i \epsilon_1}, \quad \tilde{t} = e^{2\pi i \frac{\epsilon_1}{2}}
\]

(105)

\[
z = e^{2\pi i \epsilon_a}, \quad \tilde{z} = e^{2\pi i \frac{\epsilon_a}{2}}
\]

(106)

\[
w = e^{2\pi i \epsilon_b}, \quad \tilde{w} = e^{2\pi i \frac{\epsilon_b}{2}}
\]

(107)

We can see that the modular transformation is diagonal and given by multiplication with a simple factor. As usual, one can restore the exact modular invariance by adding an antiholomorphic term in the solution. The solution will then satisfy the holomorphic anomaly equation.

For higher-point correlators, i.e. for \( n \geq 1 \) the transformation properties of the solution are more involved. In particular, the \( \Phi \) and \( \bar{\Phi} \) propagators are not modular invariant. The transformation matrix is nontrivial and is no longer given by a simple overall factor. It would be very interesting to compute this matrix exactly.

Let us mention that our results are parallel to the properties of the M-strings partition function [49]. In fact, solutions of the elliptic \( (q,t) \)-KZ equations can be naturally understood within this context.

We would like to point out an interesting observation given in [42] that the group of modular transformations acting on solutions to the elliptic \( q \)-KZ is generalized from \( SL(2, \mathbb{Z}) \) to \( SL(3, \mathbb{Z}) \). It is tempting to assume that in the DIM case this symmetry is enhanced even further. This can be seen from the fact that the modular group acting on the function \( G_r \) is in general \( SL(r+2, \mathbb{Z}) \). Since the solution [84] is related to \( G_2 \) (see Eq. 102), it naturally transforms under \( SL(4, \mathbb{Z}) \).

5 Conclusion

We have considered a natural question within the context of the network models with underlying DIM symmetry: the way this symmetry is actually realized on the correlators. Since the original studies of WZW model, it is traditional to describe this action in terms of the Knizhnik-Zamolodchikov and Knizhnik-Zamolodchikov-Bernard equations, which express the variation of conformal blocks under the change of moduli through the action of the symmetry. Equations of similar type for the network models were already considered in [12], but the emphasis there was on the dependence on Kähler structures, while the ordinary KZ equation was largely ignored. The reason for this is probably that the models currently studied have the \( G_3 \) symmetry, which is essentially Abelian, and this leads to an extraordinary simplification of the ordinary KZ making it look too trivial. Still, the equation exists, should be written and investigated, and this is the purpose of the present text.

From two most popular approaches to the KZE, the free field and the abstract group theory approaches, we took the former one. The latter approach is also interesting, but considerably different and will be addressed elsewhere. In the free field formalism [21], the main origin of non-trivial (transcendental) solutions to the KZE are insertions of peculiar screening charges, which commute with the chiral algebra. While screening charges exist in all network models [15], in the case of \( \tilde{u}_1 \) they commute only with the Virasoro/W\( _{\mathcal{N}} \)-operators but not with the generators of the DIM algebra. In terms of the balanced network models, this means that the KZE appears only for the single horizontal line networks, where screening charges are not allowed since they are associated with the vertical edges connecting different horizontal lines. The corresponding averages (network conformal blocks) are algebraic, direct counterparts of \( \prod_{i<j} (z_i - z_j)^{\alpha_i \alpha_j} \) rather than of the Schechtman-Varchenko “hypergeometric” functions, obtained by integration over some of the \( z \)-variables. This is the property shared by all the solutions to the ordinary KZE for the WZW models with \( \tilde{u}_1 \) symmetry.
Let us notice that the combinations of intertwiners which we consider in this paper also appear as building blocks of the Nekrasov partition functions. In particular, the trigonometric block corresponds to the bifundamental matter contribution in 5d theory (see Fig. 1), while the elliptic block gives rise to the 6d version. By the spectral duality, the 6d partition function with bifundamental matter can be thought of as the 5d adjoint one. The crucial point is that though the individual conformal blocks satisfy the \((q,t)\)-KZ equations, their convolution featuring in the Nekrasov function does not. The reason is that the \(\mathcal{R}\)-matrices in the r.h.s. of the \((q,t)\)-KZ equations for the two horizontal strands in Fig. 1 do not cancel.

Still, even in the simplest Abelian case there are many questions to ask and directions to explore. For example, in the elliptic generalization, where the modular group acts, we provide only preliminary results about the emerging modular matrices. These matrices might be useful in the construction of refined knot invariants. Another kind of questions is about emerging integrable structures.

As is well-known, the solutions to KZ equation in the semiclassical limit are related to solutions of the Bethe equations for the corresponding quantum integrable systems [50]. The same relation should be valid in the DIM case. It will be studied in future publications.

However, the most interesting directions are the study of generic representations (associated with plane rather than ordinary partitions) and most importantly the lifting from \(\hat{u}_1\) to \(\hat{g}\) with a simple Lie algebra \(g\), where generalizations of the true non-Abelian KZ equation emerge, even for the Fock representations. In the free field approach, this requires lifting of the entire technique of [21] to the level of DIM, which is straightforward but cumbersome. Further lifting to generic (plane-partition) representations is a real challenge requiring the “double loop” chiral free fields, which is a separate very interesting problem with many promising ideas coming from attempts in different branches of theoretical physics. This paper is just the first step on the long road to full understanding of symmetries of the network models and their realization through the variety of KZ and KZB equations.

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