Hierarchy-Problem and a Bound State of 6 t and 6 \bar{t}^* 

C. D. Froggatt  

Department of Physics and Astronomy  
University of Glasgow, Glasgow G12 8QQ, Scotland

H. B. Nielsen  

Niels Bohr Institute,  
Blegdamsvej 17-21, DK 2100 Copenhagen, Denmark

L. V. Laperashvili  

Institute of Theoretical and Experimental Physics,  
Cheremushkinskaya ulica 25, Moscow

Abstract

We propose a unification of some fine-tuning problems – really in this article only the problem of why the weak scale is so small in energy compared to a presumed fundamental scale, being say the Planck scale – by postulating the zero or very small value of the cosmological constant not only for one but for several vacua. This postulate corresponds to what we have called the Multiple Point Principle, namely that there be many “vacuum” states with the same energy density. We further assume that 6 top quarks and 6 anti-top quarks can bind by Higgs exchange so strongly as to become tachyonic and form a condensate. This gives rise to the possibility of having a phase transition between vacua with and without such a condensate. The two vacua distinguished by such a condensate will have the same cosmological constant provided the top Yukawa coupling is about $1.1 \pm 0.2$, in good correspondence with the experimental value. The further requirement that this value of the Yukawa coupling, at the weak scale, be compatible with the existence of a third vacuum, with a Higgs field expectation value of the order of the fundamental scale, enforces a hierarchical scale ratio between the fundamental and weak scales of order $10^{16} - 10^{20}$.

*We dedicate this article to Paul Frampton, our friend for many years, on the occasion of his sixtieth birthday. It is to be published in the Proceedings of the Coral Gables Conference on High Energy Physics and Cosmology, Fort Lauderdale, Florida, 17 - 21 December 2003.
1 Introduction

The present article has the purpose of suggesting the following two relatively new ideas:

1) A unified fine-tuning principle, according to which there exist several vacuum states having zero or approximately zero value for the cosmological constant.

2) The existence of a bound state of six top quarks and six anti-top quarks whose binding, due to Higgs particle exchange, is so strong that a condensate of such bound states could form and make up a phase in which essentially tachyonic bound states of this type fill the vacuum.

There are several fine-tuning problems in the Standard Model (SM): the tiny values of the cosmological constant and the strong CP violating parameter $\Theta_{QCD}$, the small hierarchy problem of the Yukawa couplings and the large hierarchy problem of the weak to fundamental (Planck) scale ratio. However these are only fine-tuning problems and do not necessarily require a modification of the SM. They could a priori be resolved by a general fine-tuning principle, which we take in the form of a zero cosmological constant postulate combined with our so-called Multiple Point Principle [1] of degenerate vacua. Thus, much like in supersymmetry, our idea is to postulate that there be many vacuum states all having zero or rather approximately zero cosmological constant, but without having supersymmetry! We do not speculate here on the underlying mechanism responsible for such degenerate vacua, but it seems likely that some kind of non-locality is required [1].

In this article we point out the possible existence of at least 3 degenerate vacua in the pure SM, which could be responsible for the hierarchy between the fundamental and weak scales. In particular we emphasize our second idea that one of these vacua is due to the condensation of an exotic meson consisting of 6 $t$ and 6 $\bar{t}$ quarks [2]. The reason that such a strongly bound exotic meson has been overlooked until now is that its binding is based on the collective effect of attraction between several quarks due to Higgs exchange. The effect builds up for many (here 12) particles in an analogous way to that of the universal gravitational force of attraction, as we now describe.

\[\text{Of course neutrino masses indicate some new physics at the see-saw scale, which requires a minor modification of the SM.}\]
2 Proposed bound state of 6 top quarks and 6 anti-top quarks

As emphasized above, the virtual exchange of the Higgs particle between two quarks, two anti-quarks or a quark anti-quark pair yields an attractive force in each case. We now consider putting more and more $t$ and $\bar{t}$ quarks together in the lowest energy relative S-wave states. The Higgs exchange binding energy for the whole system becomes proportional to the number of pairs of constituents, rather than to the number of constituents. So a priori, by combining sufficiently many constituents, the total binding energy could exceed the constituent mass of the system! However we can put a maximum of $6t + 6\bar{t}$ quarks into the ground state S-wave. So let us now estimate the binding energy of such a 12 particle bound state.

As a first step we consider the binding energy $E_1$ of one of them to the remaining 11 constituents treated as just one particle analogous to the nucleus in the hydrogen atom. We assume that the radius of the system turns out to be reasonably small, compared to the Compton wavelength of the Higgs particle, and use the well-known Bohr formula for the binding energy of a one-electron atom with atomic number $Z = 11$ to obtain the crude estimate:

$$E_1 = -\left(\frac{11g_t^2/2}{4\pi}\right)^2 \frac{11m_t}{24}. \quad (1)$$

Here $g_t$ is the top quark Yukawa coupling constant, in a normalisation in which the top quark mass is given by $m_t = g_t \times 174$ GeV.

The non-relativistic binding energy $E_{binding}$ of the 12 particle system is then obtained by multiplying by 12 and dividing by 2 to avoid double-counting the pairwise binding contributions. This estimate only takes account of the $t$-channel exchange of a Higgs particle between the constituents. A simple estimate of the $u$-channel Higgs exchange contribution \[2\] increases the binding energy by a further factor of $(16/11)^2$, giving:

$$E_{binding} = \left(\frac{11g_t^4}{\pi^2}\right) m_t \quad (2)$$

We have so far neglected the attraction due to the exchange of gauge particles. So let us estimate the main effect coming from gluon exchange$^2$ with

---

$^2$Note that we here improve our earlier estimates \[2\].
a QCD fine structure constant \( \alpha_s(M_Z) = g_s^2(M_Z)/4\pi = 0.118 \), corresponding to an effective gluon \( t - \bar{t} \) coupling constant squared of:

\[
e^2_{tt} = \frac{4}{3} g_s^2 \simeq \frac{4}{3} 1.5 \simeq 2.0
\] (3)

For definiteness, consider a \( t \) quark in the bound state; it interacts with 6 \( \bar{t} \) quarks and 5 \( t \) quarks. The 6 \( \bar{t} \) quarks form a colour singlet and so their combined interaction with the considered \( t \) quark vanishes. On the other hand the 5 \( t \) quarks combine to form a colour anti-triplet, which together interact like a \( \bar{t} \) quark with the considered \( t \) quark. So the total gluon interaction of the considered \( t \) quark is the same as it would have with a single \( \bar{t} \) quark. In this case the \( u \)-channel gluon contribution should equal that of the \( t \)-channel. Thus we should compare the effective gluon coupling strength \( 2 \times e^2_{tt} \simeq 2 \times 2 = 4 \) with \( (16/11) \times Z g_t^2 / 2 \simeq 16 \times 1.0 / 2 = 8 \) from the Higgs particle. This leads to an increase of \( E_{binding} \) by a factor of \( (4 + 8)^2 = (3/2)^2 \), giving our final result:

\[
E_{binding} = \left( \frac{99 g_t^4}{4\pi^2} \right) m_t
\] (4)

We are now interested in the condition that this bound state should become tachyonic, \( m_{bound}^2 < 0 \), in order that a new vacuum phase could appear due to Bose-Einstein condensation. For this purpose we consider a Taylor expansion in \( g_t^2 \) for the mass squared of the bound state, crudely estimated from our non-relativistic binding energy formula:

\[
m_{bound}^2 = (12m_t)^2 - 2 (12m_t) \times E_{binding} + ... \]

\[
= (12m_t)^2 \left( 1 - \frac{33}{8\pi^2} g_t^4 + ... \right)
\] (5)

(6)

Assuming that this expansion can, to first approximation, be trusted even for large \( g_t \), the condition \( m_{bound}^2 = 0 \) for the appearance of the above phase transition with degenerate vacua becomes to leading order:

\[
g_t|_{phase \ transition} = \left( \frac{8\pi^2}{33} \right)^{1/4} \simeq 1.24
\] (7)

We have of course neglected several effects, such as weak gauge boson exchange, \( s \)-channel Higgs exchange and relativistic corrections. In particular quantum fluctuations in the Higgs field could have an important effect
in reducing $g_t|_{\text{phase transition}}$ by up to a factor of $\sqrt{2}$, as discussed in Section 4.3. It is therefore quite conceivable that the value of the top quark running Yukawa coupling constant, predicted from our vacuum degeneracy fine-tuning principle, could be in agreement with the experimental value $g_t(\mu_{\text{weak}})^{\text{exp}} \approx 0.95 \pm 0.03$. Assuming this to be the case, we now make a further application of our fine-tuning principle to postulate the existence of a third degenerate vacuum, in which the SM Higgs field has a vacuum expectation value of order the fundamental scale $\mu_{\text{fundamental}}$.

3 Three degenerate vacua and the huge scale ratio

In this section, we explain how our degenerate vacuum fine-tuning principle can be used to derive the huge ratio, $\mu_{\text{fundamental}}/\mu_{\text{weak}}$, between the fundamental and weak scales. The basic idea is to use this principle to tune the value of the running top quark Yukawa coupling $g_t(\mu)$ both at the weak scale, as described above, and at the fundamental scale. Since running couplings vary logarithmically with scale, the predicted values $g_t(\mu_{\text{weak}})$ and $g_t(\mu_{\text{fundamental}})$ can easily imply an exponentially large scale ratio.

In order to tune the value of $g_t(\mu_{\text{fundamental}})$ we postulate the existence of a third degenerate vacuum, in which the SM Higgs field has a vacuum expectation value of order $\mu_{\text{fundamental}}$. For large values of the SM Higgs field $\phi \sim \mu_{\text{fundamental}} \gg \mu_{\text{weak}}$, the renormalisation group improved effective potential is well approximated by

$$V_{\text{eff}}(\phi) \simeq \frac{1}{8}\lambda(\mu = |\phi|)|\phi|^4$$

and the degeneracy condition means that $\lambda(\mu_{\text{fundamental}})$ should vanish to high accuracy. The effective potential $V_{\text{eff}}$ must also have a minimum and so its derivative should vanish. Therefore the vacuum degeneracy requirement means that the Higgs self-coupling constant and its beta function should vanish near the fundamental scale:

$$\lambda(\mu_{\text{fundamental}}) = \beta_{\lambda}(\mu_{\text{fundamental}}) = 0$$

This leads to the fine-tuning condition

$$g_t^4 = \frac{1}{48} \left(9g_2^4 + 6g_2^2g_1^2 + 3g_1^4 \right)$$

4
relating the top quark Yukawa coupling $g_t(\mu)$ and the electroweak gauge coupling constants $g_1(\mu)$ and $g_2(\mu)$ at $\mu = \mu_{\text{fundamental}}$. We must now input the experimental values of the electroweak gauge coupling constants, which we evaluate at the Planck scale using the SM renormalisation group equations, and obtain our prediction:

$$g_t(\mu_{\text{fundamental}}) \simeq 0.39.$$  \hspace{1cm} (11)

However we note that this value of $g_t(\mu_{\text{fundamental}})$, determined from the right hand side of Eq. (10), is rather insensitive to the scale, varying by approximately 10% between $\mu = 246$ GeV and $\mu = 10^{19}$ GeV.

We now estimate the fundamental to weak scale ratio by using the leading order SM beta function for the top quark Yukawa coupling $g_t(\mu)$:

$$\beta_{g_t} = \frac{dg_t}{d\ln \mu} = \frac{g_t}{16\pi^2} \left( \frac{9}{2}g_t^2 - 8g_3^2 - \frac{9}{4}g_2^2 - \frac{17}{12}g_1^2 \right)$$  \hspace{1cm} (12)

where the $SU(3) \times SU(2) \times U(1)$ gauge coupling constants are considered as given. It should be noticed that, due to the relative smallness of the fine structure constants $\alpha_i = g_i^2/4\pi$ and particularly of $\alpha_3(\mu_{\text{fundamental}})$, the beta function $\beta_{g_t}$ is numerically rather small at the fundamental scale. Hence we need many $e$-foldings between the two scales, where $g_t(\mu_{\text{fundamental}}) \simeq 0.39$ and $g_t(\mu_{\text{weak}}) \simeq 1.24$. The predicted scale ratio is quite sensitive to the input value of $\alpha_3(\mu_{\text{fundamental}})$. If we input the value of $\alpha_3 \simeq 1/54$ evaluated at the Planck scale in the SM, we predict the scale ratio to be $\mu_{\text{fundamental}}/\mu_{\text{weak}} \sim 10^{16} - 10^{20}$. We note that, as the rate of logarithmic running of $g_t(\mu)$ increases as $\alpha_3$ increases, the value of the weak scale is naturally fine-tuned to be a few orders of magnitude above the QCD scale. We also predict $M_H = 135 \pm 9$ GeV.

4 Phenomenology of the bound state

4.1 Rho parameter

Strictly speaking, it is a priori not obvious within our scenario in which of the two degenerate vacua discussed in Section 2 we live. There is however one argument that we live in the phase without a condensate of new bound states rather than in the one with such a condensate. The reason is that such a condensate is not invariant under the $SU(2) \times U(1)$ electroweak gauge group
and would contribute to the squared masses of the $W^\pm$ and $Z^0$ gauge bosons. Although these contributions are somewhat difficult to calculate, preliminary calculations indicate it is unlikely that, by some mathematical accident, they should be in the same ratio as those from the SM Higgs field. This ratio is essentially the $\rho$-parameter, which has been measured to be in accurate agreement with the SM value without a new bound state condensate. So we conclude that we live in a phase without a condensate of new bound states.

We have previously \cite{2} given a weak argument in favour of the phase with a condensate emerging in the early Universe out of the Big Bang. However, even if it were valid, one could imagine that a phase transition occurred, in our part of the Universe, from a metastable phase with a bound state condensate into the present one without a condensate. A phenomenological signal for such a phase transition would be the slight variation of various coupling constants, on a very large scale, from region to region in cosmological space and time.

4.2 **Seeing a bound state of $6t + 5 \bar{t}$?**

We expect the new bound state to be strongly bound and very long lived in our vacuum; it could only decay into a channel in which all 12 constituents disappeared together. The production cross-section of such a particle would also be expected to be very low, if it were just crudely related to the cross section for producing $6t$ and $6\bar{t}$ quarks. It would be weakly interacting and difficult to detect. There would be a better chance of observing an effect, if we optimistically assume that the mass of the bound state is close to zero (i.e. very light compared to $12m_t \approx 2$ TeV and possibly a dark matter candidate) even in the phase in which we live. In this case the bound state obtained by removing one of the 12 quarks would also be expected to be light. These bound states with radii of order $1/m_t$ might then be smaller than or similar in size to their Compton wavelengths and so be well described by effective scalar and Dirac fields respectively. The $6t + 6\bar{t}$ bound state would couple only weakly to gluons whereas the $6t + 5\bar{t}$ bound state would be a colour triplet and be produced like a fourth generation top quark at the LHC. If these 11 constituent bound states were pair produced, they would presumably decay into the lighter (undetected) 12 constituent bound states with the emission of a $t$ and a $\bar{t}$ quark.
4.3 Fine-tuning the top mass; Higgs field fluctuations

The crucial phenomenological test of our fine-tuning principle is of course that it correctly predicts the experimental values of physical parameters. The predicted existence of a new phase at the weak scale and the value of the top Yukawa coupling $g_t$ at the phase transition provide, in principle, a very clean test, since it only involves SM physics. However, in practice, the calculation of the binding energy of the proposed 6 $t + 6 \bar{t}$ bound state is hard and indeed Eq. (7) overestimates $g_t$. So here we consider a potentially large correction due to quantum fluctuations in the Higgs field.

The fluctuations in the average of the Higgs field over the interior of the bound state get bigger and bigger, as the top Yukawa coupling is increased and the size of the bound state diminishes. There is then a significant chance that the average value would turn out to be negative compared to the usual vacuum value. By thinking of the top quark Dirac sea configuration in the bound state, we see that for a sign-inverted Higgs field this configuration becomes just the vacuum state. Such a sign-inverted configuration may perhaps best be described by saying that neither the non-relativistic kinetic term for the quarks nor their mass energy are present, both being in these situations approximated by zero. Let us denote by $P_v$ the probability of fluctuating into such a vacuum configuration. The most primitive way to take the effect of these fluctuations into account is to correct the constituent mass in the bound state from $m_t$ to $(1 - P_v)m_t$, and the non-relativistic kinetic term for the same constituents from $\vec{p}^2/(2m_t)$ to $(1 - P_v)\vec{p}^2/(2m_t)$.

It is the kinetic term which determines the binding energy and the above correction corresponds to increasing $m_t$ by a factor $1/(1 - P_v)$ in the binding energy. Therefore the binding energy, which for dimensional reasons is proportional to the $m_t$ occurring in the kinetic term (for fixed $g_t$), will increase by this factor $1/(1 - P_v)$. On the other hand the constituent masses are corrected the opposite way, meaning that they decrease from $m_t$ to $(1 - P_v)m_t$. So the ratio of the binding energy to the constituent energy – the binding fraction one could say – increases by the square of the factor $1/(1 - P_v)$.

In principle we should now calculate the probability $P_v$ of a sign fluctuation as a function of $g_t$. The probability $P_v$ is expected to increase as a function of $g_t$ for two reasons: the reduction of the Higgs field inside the bound state and its decreasing radius. We note that both these effects are more important for a bound state of 12 constituents than for, say, toponium. We can, however, not expect the fluctuation probability to go beyond $P_v = 1/2$. So, for a
crude orientation, let us calculate the correction in this limiting case. In this case the ratio of the binding energy to the constituent energy, which is proportional to $g^4$, should be increased by the factor $(\frac{1}{1-P_v})^2 = 4$. Applying this correction to Eq. (7), we obtain the limiting value $g_t|_{\text{phase transition}} \simeq 1.24/4^{1/4} = 0.88$. This value corresponds to the largest possible correction from fluctuations and so we take:

$$g_t|_{\text{phase transition}} = 1.06 \pm 0.18$$  \hspace{1cm} (13)

as our best estimate, which is in good agreement with the experimental value $g_t(\mu_{\text{weak}})_{\text{exp}} \simeq 0.95$ determined from the physical top quark mass.

References

[1] D.L. Bennett, C.D. Froggatt and H.B. Nielsen, in *Proc. of the 27th International Conference on High Energy Physics*, p. 557, ed. P. Bussey and I. Knowles (IOP Publishing Ltd, 1995); *Perspectives in Particle Physics ’94*, p. 255, ed. D. Klabučar, I. Picek and D. Tadić (World Scientific, 1995) [arXiv:hep-ph/9504294].

[2] C.D. Froggatt and H.B. Nielsen, *Surv. High Energy Phys.* **18**, 55 (2003) [arXiv:hep-ph/0308144]; *Proc. to the Euroconference on Symmetries Beyond the Standard Model*, p. 73 (DMFA, Založnictvo, 2003) [arXiv:hep-ph/0312218].

[3] C.D. Froggatt and H.B. Nielsen, *Phys. Lett.* **B368**, 96 (1996) [arXiv:hep-ph/9511371].