A BERNSTEIN TYPE INEQUALITY

VILMOS KOMORNÍK AND PAOLA LORETI

Abstract. We formulate and discuss a conjecture which would extend a classical inequality of Bernstein.

1. A Bernstein type inequality

A classical theorem due to Bernstein [2] states that every even trigonometric polynomial $T$ of order $M$ satisfies the inequality

$$\|T'\|_{L^\infty} \leq M\|T\|_{L^\infty}.$$

His result was extended to all trigonometric polynomials by Fejér [3]. Another proof was given by M. Riesz [4], [5]; this also shows that

$$(1.1) \quad \|T'\|_{L^p(I)} \leq M\|T\|_{L^p(I)}$$

for every interval $I$ of length $|I| = 2\pi$ and for every exponent $1 \leq p \leq \infty$.

For $p = 2$ this inequality follows easily by applying Parseval’s formula. Indeed, writing

$$T(x) = \sum_{k=-M}^{M} a_k e^{ikx}$$

and using the orthogonality of the functions $e^{ikx}$ we have

$$\int_I |T'(x)|^2 \, dx - M^2 \int_I |T(x)|^2 \, dx$$

$$= \int_I \left| \sum_{k=-M}^{M} ika_k e^{ikx} \right|^2 \, dx - M^2 \int_I \left| \sum_{k=-M}^{M} a_k e^{ikx} \right|^2 \, dx$$

$$= 2\pi \sum_{k=-M}^{M} |ika_k|^2 - 2\pi M^2 \sum_{k=-M}^{M} |a_k|^2$$

$$= 2\pi \sum_{k=-M}^{M} (k^2 - M^2)|a_k|^2$$

$$\leq 0.$$
2. A conjecture

Let us introduce the function

$$H(x) := \begin{cases} \cos x & \text{if } |x| \leq \pi/2, \\ 0 & \text{if } |x| \geq \pi/2. \end{cases}$$

For any positive integer $M$, the following inequality holds:

$$\int_{-\infty}^{\infty} |(H^M)'(x)|^2 \, dx \leq M^2 \int_{-\infty}^{\infty} |H^M(x)|^2 \, dx. \tag{2.1}$$

Indeed, since

$$\int_{-\infty}^{\infty} |(H^M)'(x)|^2 \, dx = \int_{-\pi/2}^{\pi/2} |(\cos^M)'(x)|^2 \, dx = \frac{1}{2} \int_{-\pi}^{\pi} |(\sin^M)'(x)|^2 \, dx$$

and

$$\int_{-\infty}^{\infty} |H^M(x)|^2 \, dx = \int_{-\pi/2}^{\pi/2} |\cos^M x|^2 \, dx = \int_{0}^{\pi} |\sin^M x|^2 \, dx = \frac{1}{2} \int_{-\pi}^{\pi} |\sin^M x|^2 \, dx,$$

the inequality follows by applying (1.1) with $T(x) := \sin^M x$ on the interval $I = (-\pi, \pi)$.

The following conjecture is a generalization of the inequality (2.1).

**Conjecture 2.1.** Let $(\lambda_n)_{n=-\infty}^{\infty}$ be a strictly increasing sequence of real numbers, satisfying for some positive integer $M$ the gap condition

$$\lambda_{n+M} - \lambda_n \geq \pi \tag{2.2}$$

for all $n$. Then for every finite sequence $(a_n)$ of real numbers, the function

$$G(x) := \sum a_n H^M(x + \lambda_n)$$

satisfies the inequality

$$\int_{-\infty}^{\infty} |G'(x)|^2 \, dx \leq M^2 \int_{-\infty}^{\infty} |G(x)|^2 \, dx. \tag{2.3}$$

In the next sections we prove the conjecture for $M = 1$ and $M = 2$.

3. Proof of the conjecture for $M = 1$

For $M = 1$ we have even an equality. Indeed, since for $m \neq n$ the product functions

$$H(x + \lambda_m)H(x + \lambda_n) \quad \text{and} \quad H'(x + \lambda_m)H'(x + \lambda_n)$$

vanish identically by (2.2), we have

$$\int_{-\infty}^{\infty} |G(x)|^2 \, dx = \int_{-\infty}^{\infty} \left| \sum a_n H(x + \lambda_n) \right|^2 \, dx = \sum |a_n|^2 \int_{-\infty}^{\infty} |H(x + \lambda_n)|^2 \, dx$$

and

$$\int_{-\infty}^{\infty} |G'(x)|^2 \, dx = \int_{-\infty}^{\infty} \left| \sum a_n H'(x + \lambda_n) \right|^2 \, dx = \sum |a_n|^2 \int_{-\infty}^{\infty} |H'(x + \lambda_n)|^2 \, dx.$$
We conclude by observing that
\[
\int_{-\infty}^{\infty} |H(x + \lambda_n)|^2 \, dx = \int_{-\pi/2}^{\pi/2} \cos^2 x \, dx = \frac{\pi}{2}
\]
and
\[
\int_{-\infty}^{\infty} |H'(x + \lambda_n)|^2 \, dx = \int_{-\pi/2}^{\pi/2} \sin^2 x \, dx = \frac{\pi}{2}.
\]

4. Discussion of the case \( M \geq 2 \)

We begin with some discussion concerning the general case. Our first lemma allows us to reformulate the conjecture.

**Lemma 4.1.** Introducing the function
\[
g(\lambda) = g_M(\lambda) := \int_{-\infty}^{\infty} H^M(x + \lambda)H^{M-2}(x) \, dx,
\]
we have
\[
M^2 \int_{-\infty}^{\infty} |G(x)|^2 \, dx - \int_{-\infty}^{\infty} |G'(x)|^2 \, dx = \sum_{m,n=-\infty}^{\infty} g(\lambda_m - \lambda_n)a_m\bar{a}_n.
\]

**Proof.** We recall from [1] that
\[
(H^M)'(x) = -M^2H^M(x) + M(M-1)H^{M-2}(x)
\]
for all \( x \). Integrating by parts and then using this relation, we have
\[
M^2 \int_{-\infty}^{\infty} |G(x)|^2 \, dx - \int_{-\infty}^{\infty} |G'(x)|^2 \, dx = \sum_{m,n=-\infty}^{\infty} g(\lambda_m - \lambda_n)a_m\bar{a}_n.
\]

In view of this lemma it suffices to show that
\[
(4.1) \quad \sum_{m,n=-\infty}^{\infty} g(\lambda_m - \lambda_n)a_m\bar{a}_n \geq 0
\]
for all finite sequences \((a_n)\) of complex numbers.

**Remark.** It follows easily from the definition that \( g_M \) is a nonnegative, even function, vanishing outside the interval \((-\pi, \pi)\). It can be computed explicitly for any
given \( M \). For example, if \( 0 \leq x \leq \pi \), then we have
\[
4g_2(x) = 2(\pi - x) + \sin 2x,
\]
\[
32g_3(x) = 12(\pi - x) \cos x + 9 \sin x + \sin 3x,
\]
\[
192g_4(x) = 36(\pi - x) + 24(\pi - x) \cos 2x + 28 \sin 2x + \sin 4x.
\]

Indeed, for \( M = 2 \) we have
\[
4g_2(x) = \int_{\pi/2}^{\pi/2} 4 \cos^2 t \, dt = \int_{-\pi/2}^{\pi/2} 2 + 2 \cos 2t \, dt = 2(\pi - x) + \sin 2x.
\]

For \( M = 3, 4 \) the computation is similar but longer.

## 5. Proof of the conjecture for \( M = 2 \)

The proof of (4.1) for \( M = 2 \) is based on the following identity:

**Lemma 5.1.** The following identity holds:
\[
\sum_{m,n=-\infty}^{\infty} g(\lambda_m - \lambda_n) a_m a_n = \sum_{n=-\infty}^{\infty} g(\lambda_n+1 - \lambda_n)|a_n + a_{n+1}|^2
\]
\[
+ \sum_{n=-\infty}^{\infty} (g(0) - g(\lambda_n - \lambda_n) - g(\lambda_n+1 - \lambda_n))|a_n|^2.
\]

**Proof.** Writing \( h_{m,n} := g(\lambda_m - \lambda_n) \) for brevity, and using the evenness of \( g \), the following computation leads to the required identity:
\[
\sum_{m,n=-\infty}^{\infty} h_{m,n} a_m a_n
\]
\[
= \sum_{n=-\infty}^{\infty} h_{n,n} |a_n|^2 + h_{n,n+1}(a_n a_{n+1} + a_{n+1} a_n)
\]
\[
= \sum_{n=-\infty}^{\infty} h_{n,n} |a_n|^2 + h_{n,n+1}(|a_n + a_{n+1}|^2 - |a_n|^2 - |a_{n+1}|^2)
\]
\[
= \sum_{n=-\infty}^{\infty} h_{n,n+1}|a_n + a_{n+1}|^2 + (h_{n,n} - h_{n,n+1} - h_{n-1,n})|a_n|^2. \]

Since \( g \) is nonnegative, the first sum on the right side of the above identity is \( \geq 0 \). Since
\[
\lambda_{n+1} - \lambda_n \geq 0, \quad \lambda_n - \lambda_{n-1} \geq 0 \quad \text{and} \quad (\lambda_{n+1} - \lambda_n) + (\lambda_n - \lambda_{n-1}) = \lambda_{n+1} - \lambda_{n-1} \geq \pi
\]
by the gap condition (2.2), the nonnegativity of the second sum follows from the next lemma which completes the proof of (4.1).

**Lemma 5.2.** If
\[
a \geq 0, \quad b \geq 0 \quad \text{and} \quad a + b \geq \pi,
\]
then
\[
\text{then} \quad g(a) + g(b) \leq g(0).
\]
Proof. Since the functions $H^2$, $H^0$ are nonnegative and since the intervals
\[
\left( -\frac{\pi}{2}, \frac{\pi}{2} - a \right) \quad \text{and} \quad \left( -\frac{\pi}{2} + b, \frac{\pi}{2} \right)
\]
are disjoint, we have
\[
g(a) + g(b) = \int_{-\infty}^{\infty} H^2(x)H^0(x + a) \, dx + \int_{-\infty}^{\infty} H^2(x)H^0(x - b) \, dx
\]
\[
= \int_{-\infty}^{\frac{\pi}{2} - a} H^2(x) \, dx + \int_{-\infty}^{\frac{\pi}{2} + b} H^2(x) \, dx
\]
\[
\leq \int_{-\infty}^{\infty} H^2(x) \, dx
\]
\[
= g(0).
\]
\[\square\]

Remark. The above proof also shows that for $a + b = \pi$ we have $g(a) + g(b) = g(0)$, i.e.,
\[
g(x) + g(\pi - x) = g(0)
\]
for all $x \in [0, \pi]$. This can also be seen from the explicit formula of $g$.

References

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