STABILITY ON KÄHLER-RICCI FLOW, I

XIAOHUA ZHU

Abstract. In this paper, we prove that Kähler-Ricci flow converges to a Kähler-Einstein metric (or a Kähler-Ricci soliton) in the sense of Cheeger-Gromov as long as an initial Kähler metric is very closed to \(g_{KE}\) (or \(g_{KS}\)) if a compact Kähler manifold with \(c_1(M) > 0\) admits a Kähler Einstein metric \(g_{KE}\) (or a Kähler-Ricci soliton \(g_{KS}\)). The result improves Main Theorem in [TZ3] in the sense of stability of Kähler-Ricci flow.

0. Introduction

The Ricci flow was first introduced by R. Hamilton in [Ha]. If the underlying manifold \(M\) is Kähler with positive first Chern class \(c_1(M) > 0\), it is more natural to study the following Kähler-Ricci flow (normalized),

\[
\frac{\partial g(t, \cdot)}{\partial t} = -\text{Ric}(g(t, \cdot)) + g(t, \cdot),
\]

\(g(0, \cdot) = g,\)

(0.1)

where \(g\) is an initial Kähler metric with its Kähler form \(\omega_g \in 2\pi c_1(M) > 0\).

It can be shown that (0.1) preserves the Kähler class. Moreover, (0.1) has a global solution \(g_t = g(t, \cdot)\) for any \(t > 0\) ([Ca]). So, the main interest and difficulty of (0.1) is to study the limiting behavior of \(g_t\) as \(t\) tends to \(\infty\) (cf. [CT1], [CT2], [TZ3], etc.).

In this paper, we study a stability problem of Kähler-Ricci flow (0.1), namely, we assume that \(M\) admits a Kähler-Einstein metric or a Kähler-Ricci soliton, and then we analysis the behavior of evolved Kähler metrics \(g_t\) of (0.1). We shall prove

Theorem 0.1 (Main Theorem). Let \(M\) be a compact Kähler manifold with \(c_1(M) > 0\) which admits a Kähler Einstein metric \(g_{KE}\) (or a Kähler-Ricci soliton \((g_{KS}, X_0)\) with respect some holomorphic vector field \(X_0\) on \(M\)) with its Kähler form in \(2\pi c_1(M)\). Let \(\psi\) be a Kähler potential of an initial metric \(g\) of (0.1) and \(\varphi = \varphi_t\) be a family of Kähler potentials of evolved metrics

---

1991 Mathematics Subject Classification. Primary: 53C25; Secondary: 53C55, 58E11.

Key words and phrases. Kähler-Ricci flow, Kähler-Einstein metric, Kähler-Ricci soliton.

Partially supported by NSF10425102 in China.
of (0.1), i.e., \( \omega = \omega_{KE} + \sqrt{-1} \partial \bar{\partial} \psi \) (or \( \omega = \omega_{KS} + \sqrt{-1} \partial \bar{\partial} \psi \)) and 
\( \omega = \omega_{KE} + \sqrt{-1} \partial \bar{\partial} \varphi \) (or \( \omega = \omega_{KS} + \sqrt{-1} \partial \bar{\partial} \varphi \)), where \( \omega_{g}, \omega_{\varphi} = \omega_{g}, \) and 
\( \omega_{KE} \) (or \( \omega_{KS} \)) denote Kähler forms of \( g, \varphi \) and \( g_{KE} \) (or \( g_{KS} \)), respectively. Then there exists a small \( \epsilon \) such that if 
\[
\| \psi - \tilde{\psi} \|_{C^{2, \alpha}} \leq \epsilon,
\]
where \( \tilde{\psi} = \frac{1}{\int_{M} \omega_{KE}^{n}} \int_{M} \psi \omega_{KE}^{n} \) (or \( \tilde{\psi} = \frac{1}{\int_{M} \omega_{KS}^{n}} \int_{M} \psi \omega_{KS}^{n} \)), then there exist a family of holomorphisms \( \sigma = \sigma_{t} \) on \( M \) such that Kähler potentials \( (\varphi_{\sigma} - \varphi) \) are \( C^{k} \)-norm uniformly bounded, where \( \varphi_{\sigma} = \sigma^{*} \varphi + \rho \) and \( \rho = \rho_{t} \) are Kähler potentials defined by \( \rho^{*}(\omega_{KE}) = \omega_{KE} + \sqrt{-1} \partial \bar{\partial} \rho \) and \( \int_{M} e^{-\rho} \omega_{KE}^{n} = \int_{M} \omega_{KE}^{n} \)
(or \( \rho^{*}(\omega_{KS}) = \omega_{KS} + \sqrt{-1} \partial \bar{\partial} \rho \) and \( \int_{M} e^{-\rho} X_{0}(\rho) \omega_{KS}^{n} = \int_{M} \omega_{KS}^{n} \)). As a consequence, \( g_{t} \) converge to \( g_{KE} \) (or \( g_{KS} \)) smoothly in the sense of Cheeger-Gromov.

The main step in the proof of Theorem 0.1 is to obtain a decay estimate for \( \varphi \) and \( \varphi \) both when one studies the convergence of Kähler-Ricci flow as in [CT2], [PS], [TZ3] etc. In case that \( M \) admits a Kähler Einstein metric or \( M \) admits a Kähler-Ricci soliton and an initial potential \( \psi \) is \( K_{X_{0}} \)-invariant, we can obtain an exponential decay estimate for both \( \varphi \) and \( \varphi \), so we can improve that Kähler potentials \( (\sigma^{*} \varphi + \rho) \) in the theorem exponentially converge to zero as long as \( \| \psi - \tilde{\psi} \|_{C^{2, \alpha}} \) is small, where \( K_{X_{0}} \)
is an one-parameter compact subgroup generated by the imaginary part \( X_{0} \) of \( X_{0} \) ([TZ1],[TZ2]). This result is also obtained in [TZ3] where a crucial step is to use the monotonicity and the properness of the Mabuchi’s K-energy on a Kähler-Einstein manifold with \( c_{1}(M) > 0 \) (or the monotonicity and the properness of the generalized K-energy on a compact Kähler manifold which admits a Kähler-Ricci soliton , cf. [CTZ]). But at the present paper, we avoid to use these energies in our case of the stability problem. This advantage allows us to remove the \( K_{X_{0}} \)-invariant condition for the initial potential \( \psi \) in case of Kähler-Ricci soliton in Theorem 0.1, although we need more careful computations than the case of Kähler-Einstein metric. Basically, we shall use the generalized Futaki-invariant and the Gauge Transformation induced by the reductive subgroup \( \text{Aut}_{r}(M) \) of holomorphisms transformation group \( \text{Aut}(M) \) on \( M \) to control the modified Kähler potentials \( (\sigma^{*} \varphi + \rho) \) along the Kähler-Ricci flow. We note that the definition of generalized Futaki-invariant is independent of the choice of Kähler metric, which needs no \( K_{X_{0}} \)-invariant condition ([TZ2]). Unfortunately, we could not improve the convergence of \( (\sigma^{*} \varphi + \rho) \) exponentially without the assumption of \( K_{X_{0}} \)-invariant condition. But we believe that it is still true if one can extend the Gauge Transformation \( \text{Aut}_{r}(M) \) to \( \text{Aut}(M) \) (cf. Proposition 2.10).
Theorem 0.1 will be proved in Section 1 and Section 2 while in Section 1 we consider the case of Kähler-Einstein metric and in Section 2, we consider the case of Kähler-Ricci soliton. The rest of paper is as follows: In Section 3, we prove a uniqueness result for the limit of Kähler-Ricci flow as an application of Theorem 0.1; Section 4 and Section 5 are two appendixes, one is a lemma about a $W^{k,2}$-estimate for $\dot{\varphi}_t$ and another is a lemma about the existence of almost orthonormality of a Kähler potential to the space of first eigenvalue-functions of operator $(P,\omega_{KS})$ defined in Lemma 2.2 in Section 2.

The author would like to thank professor Gang Tian and professor Xiuxiong Chen for valuable discussions.

1. In case of Kähler-Einstein metric

In this section, we assume that $M$ admits a Kähler Einstein metric $g_{KE}$ with its Kähler form $\omega_{KE} \in 2\pi c_1(M)$. For simplicity, we set a class of Kähler potentials by

$$\mathcal{M}(\omega_{KE}) = \{ \phi \in C^\infty(M,\mathbb{R}) \mid \omega_{KE} + \sqrt{-1}\partial\bar{\partial}\phi > 0 \}.$$  

Then for any Kähler metric $g$ with its Kähler form $\omega_g \in 2\pi c_1(M)$, we have $\omega_g = \omega_{KE} + \sqrt{-1}\partial\bar{\partial}\psi$ for some $\psi \in \mathcal{M}(\omega_{KE})$ and Kähler-Ricci flow (0.1) is equivalent to a parabolic equation of complex Monge-Ampère type for Kähler potentials $\varphi_t = \varphi(t,\cdot)$ with $\omega_{g_t} = \omega_{KE} + \sqrt{-1}\partial\bar{\partial}\varphi_t$,

$$\frac{\partial \varphi}{\partial t} = \log \frac{\omega_\varphi^n}{\omega_{KE}^n} + \varphi,$$

$$\varphi(0) = \psi - \psi,$$

(1.1)

where $\psi = \frac{1}{V} \int_M \psi \omega_{KE}^n$ and $V = \int_M \omega_{KE}^n$.

Set a Hölder space by

$$\mathcal{K}(\epsilon_0) = \{ \phi \in \mathcal{M}(\omega_{KE}) \mid \| \phi - \varphi \|_{C^{2,\alpha}} \leq \epsilon_0 \}.$$  

Let $\text{Aut}_0(M)$ be the connected component of holomorphisms transformation group of $M$ which contains the identity map of $M$. Then we shall prove

**Theorem 1.1.** There exists a small $\epsilon$ such that for any initial data $\psi \in \mathcal{K}(\epsilon)$ in equation (1.1), $\| \varphi - \varphi \|_{C^{2,\alpha}}$ are uniformly bounded, where $\varphi = \varphi_t = \varphi(t,\cdot)$ are evolved Kähler potentials of (1.1). Moreover, there exist a family of $\sigma = \sigma_t \in \text{Aut}_0(M)$ such that Kähler potentials $(\varphi_\sigma - \varphi_\sigma)$ converge exponentially to 0 as $t \to \infty$, where $\varphi_\sigma = (\sigma^* \varphi + \rho)$, and $\rho = \rho_t$ are Kähler potentials.
defined by

\[ \sigma^*(\omega_{KE}) = \omega_{KE} + \sqrt{-1} \partial \bar{\partial} \rho, \]

(1.2)

\[ \int_M e^{-\rho} \omega^n_{KE} = \int_M \omega^n_{KE}. \]

As a consequence, Kähler metrics \( \sigma^*(\omega_\phi) \) converge exponentially to \( \omega_{KE} \).

We need several lemmas to prove Theorem 1.1. Let \( \Lambda_1(M,\omega_{KE}) \) be a finite dimensional linear space of the first eigenvalue-functions of Laplace operator \( \Delta_{\omega_{KE}} \) associated to the metric \( \omega_{KE} \). Then by using the Bochner formula, it is well-known that the first non-zero eigenvalue is 1 and \( \Lambda_1(M,\omega_{KE}) = \text{span}\{\theta_X| X \in \eta(M)\} \), where \( \eta(M) \) is a linear space consisting of holomorphic vector fields on \( M \) which is isomorphic to the Lie algebra of \( \text{Aut}_0(M) \) and \( \theta_X \) is a potential of \( X \) defined by

\[ \sqrt{-1} \partial \theta_X = i_X(\omega_{KE}), \]

(1.3)

\[ \int_M \theta_X \omega^n_{KE} = 0. \]

By using the continuity of eigenvalues of Laplacian operators, one sees

Lemma 1.2. Let \( \lambda_1(\omega_\phi) \) and \( \lambda_2(\omega_\phi) \) be the first and the second eigenvalues of Laplacian operator associated to Kähler metric \( \omega_\phi \), respectively. Then there exists a \( \delta_0 \) such that for any \( \phi \in K(\epsilon_0) \), we have

\[ \lambda_1(\omega_\phi) \geq 1 + \delta_0, \text{ if } \eta(M) = 0, \]

\[ \lambda_2(\omega_\phi) \geq 1 + \delta_0, \text{ if } \eta(M) \neq 0, \]

where \( \epsilon_0 \) is a small positive number.

Fix a large number \( T \) and \( N \), we can choose a sufficient small \( \epsilon \) depends on \( T \), \( \epsilon_0 \) and \( N \) such that for any \( t \leq T \), evolved Kähler potentials \( \varphi_t \) of (1.1) lie in \( K(\frac{\epsilon_0}{2^N}) \) and satisfy

\[ |\varphi_t - c(t)|_{C^0} \leq \left(\frac{\epsilon_0}{2N}\right)^2, \text{ and } \text{osc}(\varphi_t) \leq \frac{\epsilon_0}{4N}, \]

(1.4)

whenever \( \|\psi - \psi\|_{C^2,0} \leq \epsilon \). Here \( c(t) = \frac{1}{V} \int_M \varphi_t \omega^n_{\varphi_t} \). Choose a maximal \( \delta(T) \) such that \( \varphi_t \in K(\epsilon_0) \) for any \( t < T + \delta(T) \). We shall show that \( \delta(T) \) must be the infinity whenever \( T \) and \( N \) are large enough. First we prove

Lemma 1.3. Let \( H(t) = \frac{1}{V} \int_M |\varphi_t - c(t)|^2 \omega^n_{\varphi_t} \). Then for any \( t \in [0, T + \delta(T)] \), there exists a \( \theta > 0 \) such that

\[ H(t) \leq H(0)e^{-\theta t}. \]

(1.5)
Proof. For simplicity, we let \( \phi = \varphi_t \). By (1.1), one sees
\[
|\dot{\varphi}| \leq 3\epsilon_0, \quad \forall t \in [0, T + \delta(T)).
\]
Since \( \varphi \) satisfies,
\[
(1.6) \quad \ddot{\varphi} = \Delta \dot{\varphi} + \dot{\varphi},
\]
then by a direct computation, we have
\[
\frac{d}{dt} H_0(t) = 2 \frac{1}{V} \int_M (\dot{\varphi} - c(t))(\dot{\varphi} - c(t))\omega^n_\varphi + \frac{1}{V} \int_M (\dot{\varphi} - c(t))^2 \Delta \varphi \dot{\varphi} \omega^n_\varphi
\]
\[
= 2 \frac{1}{V} \int_M (\dot{\varphi} - c(t))(\Delta \varphi \dot{\varphi} + \dot{\varphi})\omega^n_\varphi + \frac{1}{V} \int_M (\dot{\varphi} - c(t))^2 \Delta \varphi \dot{\varphi} \omega^n_\varphi
\]
\[
= -2 \frac{1}{V} \int_M |\nabla (\dot{\varphi} - c(t))|^2 \omega^n_\varphi + 2 \frac{1}{V} \int_M (\dot{\varphi} - c(t))^2 \dot{\varphi} \omega^n_\varphi
\]
\[
- 2 \frac{1}{V} \int_M (\dot{\varphi} - c(t))|\nabla (\dot{\varphi} - c(t))|^2 \omega^n_\varphi
\]
\[
= 2H_0(t) - 2 \frac{1}{V} \int_M (1 + \dot{\varphi} - c(t))|\nabla (\dot{\varphi} - c(t))|^2 \omega^n_\varphi
\]
\[
(1.7) \quad \leq 2H_0(t) - 2(1 - 6\epsilon_0) \frac{1}{V} \int_M |\nabla (\dot{\varphi} - c(t))|^2 \omega^n_\varphi.
\]

Case 1, \( \eta(M) = 0 \). Then by (1.7) and Lemma 1.1, we have
\[
\frac{d}{dt} H_0(t) \leq -[\frac{-2 + 2(1 - 4\epsilon_0)(1 + \delta_0)}{1 + \delta_0}] H_0(t).
\]

By choosing \( \theta = -2 + 2(1 - 4\epsilon_0)(1 + \delta_0) \geq \delta_0 \), we will get
\[
(1.8) \quad H_0(t) \leq H_0(0)e^{-\theta t}.
\]

Case 2, \( \eta(M) \neq 0 \). Since the Futaki-invariant vanishes, for any \( X \in \eta(M) \), we have
\[
\int_M \Delta (\theta_X + X(\varphi))(\dot{\varphi} - c(t))\omega^n_\varphi = 0,
\]
where \( \theta_X \) is the potential of \( X \) defined by (1.3) and \( X(\phi) \) is the derivative of \( \phi \) along \( X \). It follows
\[
|\int_M \theta_X (\dot{\varphi} - c(t))\omega^n_\varphi| \leq C\epsilon_0 \int_M |\dot{\varphi} - c(t)|\omega^n_\varphi,
\]
for any \( X \in \eta(M) \) with satisfying \( \int_M \|X\|_{\omega_{KE}}^2 \omega^n_{KE} = 1 \). Here we used an estimate
\[
\|\varphi - \varphi_c\|_{C^3} = O(\epsilon_0)
\]
by the regularity of Kähler potentials \( \varphi \in \mathcal{K}(\epsilon_0) \), which can be obtained by the Implicity Functional Theorem for equation (1.22) (cf. an argument at the last paragraph of this section) with the help of \( W^{k,2} \)-estimate

\[
\| \dot{\varphi} - c(t) \|_{W^{k,2}} = O(\epsilon_0)
\]

for \( \dot{\varphi} \) (cf. the argument in Appendix 4.1). Thus by using the continuity of the eigenvalue functions, one sees

\[
\int_M \psi_i(\dot{\varphi} - c(t))\omega^n_n \leq C'\epsilon_0 \int_M |\dot{\varphi} - c(t)|\omega^n_n,
\]

where \( \psi_i \) are the first eigenvalue functions of the Laplacian operator associated to the metric \( \omega_\varphi \), which satisfy \( \int_M |\psi_i|^2\omega^n_n = 1 \).

Let \( \Lambda_1(M,\omega_\varphi) \) be a linear space spanned by a basis \( \{\psi_i\} \) and \( \Lambda^\perp_1(M,\omega_\varphi) \) be a subspace of \( L^2 \)-integral functions which are orthogonal to \( \Lambda_1(M,\omega_\varphi) \cup \mathbb{R} \). Then we can decompose \( \dot{\varphi}_t - c(t) \) as

\[
\dot{\varphi}_t - c(t) = \phi + \phi^\perp,
\]

with \( \phi \in \Lambda_1(M,\omega_\varphi) \) and \( \phi^\perp \in \Lambda^\perp_1(M,\omega_\varphi) \). Thus by (1.9), we get

\[
\int_M |\phi|^2\omega^n_n \leq A\epsilon_0^2 \int_M (\dot{\varphi}_t - c(t))^2\omega^n_n,
\]

for some uniform constant \( A \). It follows

\[
\int_M \phi^\perp |^2\omega^n_n \geq (1 - A\epsilon_0^2) \int_M (\dot{\varphi}_t - c(t))^2\omega^n_n.
\]

Hence by Lemma 1.1, we get

\[
\int_M \|\nabla(\dot{\varphi}_t - c(t))\|^2\omega^n_n \geq \int_M \|\nabla\phi^\perp\|^2\omega^n_n \geq (1 + \sigma_0)(1 - A\epsilon_0^2) \int_M (\dot{\varphi}_t - c(t))^2\omega^n_n.
\]

By choosing \( \theta = 2(1 - 6\epsilon_0)(1 + \sigma_0)(1 - C\epsilon_0^2) - 2 \geq \sigma_0 \), we obtain from (1.7),

\[
\frac{dH_0(t)}{dt} \leq -\theta H_0(t).
\]

As a consequence, we have

\[
H_0(t) \leq H_0(0)e^{-\theta t}.
\]

\( \square \)

Next we want to use Perelman’s deep estimates for the gradient of \( \dot{\varphi}_t \) and the non-collapsing result for metric \( \omega_{\varphi_t} \) to get a \( C^0 \)-estimate of \( \dot{\varphi}_t \) with help of Lemma 1.3. Let’s state the Perelman’s result (a detailed proof can be found in [ST]).
Lemma 1.4. (Perelman) Let $g_t$ be the evolved Kähler metrics of (0.1) and $\varphi = \varphi_t$ be Kähler potentials of $g_t$. Then there exists a uniform constant $C$ independent of $t$ (just depending on the initial metric $g$) such that the following two facts hold,

i) $\|\nabla \varphi\|_{\omega_0} \leq C$;

ii) for $x \in M$ and $0 < r \leq \text{diam}(M, g(t))$, $\int_{B_r(x)} \omega^n > C^{-1}r^{2n}$, where $\text{diam}(M, g(t))$ denote the diameters of $(M, g(t))$ which are uniformly bounded.

Lemma 1.5. For any $t \in [0, T + \delta(T))$, we have

\begin{equation}
|\varphi_t - c(t)| \leq \min\{\left(\frac{\epsilon_0}{2N}\right)^2, Ce^{-\frac{\theta}{2(n+1)}}t\}
\end{equation}

and

\begin{equation}
||\varphi_t - c(t)||_{C^\alpha} \leq C\min\{\left(\frac{\epsilon_0}{2N}\right)^2, Ce^{-\frac{\theta}{2(n+1)}}t\}\frac{1}{\alpha}.
\end{equation}

Here $\alpha \leq \frac{1}{4}$ and $C$ depends only on the constant in Lemma 1.4.

Proof. We suffice to consider the case of $\alpha = \frac{1}{4}$. Let $x_0$ be the point where $|\tilde{h}| = |\tilde{h}_t| = |\varphi_t - c(t)|$ achieves its maximum. Choose a small ball $B_r(x_0)$ for $r = \min\{\text{diam}(M, g(t)), e^{-\frac{\theta}{2(n+1)}}t\}$. So we have for any $x \in B_r(x_0)$,

\begin{equation}
0 \leq |\tilde{h}(x_0)| \leq |\tilde{h}(x)| + \|\nabla \tilde{h}\|r = |\tilde{h}(x)| + \|\nabla \tilde{h}\|r.
\end{equation}

Case 1), $e^{-\frac{\theta}{2(n+1)}}t \geq \text{diam}(M, g(t))$. Then by Lemma 1.4, one sees

$$
\int_M |\tilde{h}(x_0)|^2 \omega^n_\varphi \leq 2 \int_M |\tilde{h}(x)|^2 \omega^n_\varphi + 2V \text{diam}(M, g(t))^2 \|\nabla \tilde{h}\|^2
\leq Ce^{-\frac{\theta}{n+1}t}.
$$

Thus

\begin{equation}
|\tilde{h}(x_0)| \leq Ce^{-\frac{\theta}{2(n+1)}}t.
\end{equation}

Case 2), $e^{-\frac{\theta}{2(n+1)}}t \leq \text{diam}(M, g(t))$. Then

$$
\frac{1}{V(B_r(x_0))} \int_{B_r(x_0)} |\tilde{h}(x_0)|^2 \omega^n_\varphi \leq \frac{2}{V(B_r(x_0))} \int_{B_r(x_0)} |\tilde{h}(x)|^2 \omega^n_\varphi
+ \frac{2}{V(B_r(x_0))} \int_{B_r(x_0)} \|\nabla \tilde{h}\|^2 r^2 \omega^n_\varphi.
$$

Thus by Lemma 1.4, we get

$$
|\tilde{h}(x_0)|^2 \leq Ce^{-\frac{\theta}{2(n+1)}t} \int_M |\tilde{h}(x)|^2 \omega^n_\varphi + Cr^2
\leq Ce^{-\frac{\theta}{n+1}t}.
$$

It follows

\begin{equation}
|\tilde{h}(x_0)| \leq Ce^{-\frac{\theta}{2(n+1)}t}.
\end{equation}
Therefore, both (1.15) and (1.16) give the estimate (1.12).

For any \( x, y \in M \), by (1.12), we have: if \( \text{dist}(x, y) = \|x - y\|_{\omega_\varphi} = e^{-\frac{\theta}{2(n+1)} t} \),

\[
\frac{|\tilde{h}(x) - \tilde{h}(y)|}{\|x - y\|_{\omega_\varphi}^{\frac{1}{2}}} \leq 2 \frac{|\tilde{h}(x) - \tilde{h}(y)|}{\|x - y\|_{\omega_\varphi}^{\frac{1}{2}}}
\]

\[
\leq 2 |\tilde{h}(x) - \tilde{h}(y)| \frac{1}{\|x - y\|_{\omega_\varphi}^{\frac{1}{2}}}
\]

(1.17)

\[
\leq C \min\{\left(\frac{\epsilon_0}{2N}\right)^2, Ce^{-\frac{\theta}{2(n+1)} t}\} \frac{1}{2};
\]

if \( \text{dist}(x, y) \leq e^{-\frac{\theta}{2(n+1)} t} \),

\[
\frac{|\tilde{h}(x) - \tilde{h}(y)|}{\|x - y\|_{\omega_\varphi}^{\frac{1}{2}}} \leq 2 \frac{|\tilde{h}(x) - \tilde{h}(y)|}{\|x - y\|_{\omega_\varphi}^{\frac{1}{2}}}
\]

\[
= 2 |\tilde{h}(x) - \tilde{h}(y)| \frac{1}{\|x - y\|_{\omega_\varphi}^{\frac{1}{2}}}
\]

(1.18)

\[
\leq C' \min\{\left(\frac{\epsilon_0}{2N}\right)^2, Ce^{-\frac{\theta}{2(n+1)} t}\} \frac{1}{2}.
\]

Here we used Perelman’s estimates again. (1.17) and (1.18) give the estimate (1.13).

\[\square\]

**Remark 1.6.** We can avoid to use Perelman’s estimates to prove Lemma 1.5 by replacing to estimate the \( W^{k, 2} \)-norm of \( \hat{\varphi}_t \). See Appendix 1 in this paper.

**Proposition 1.7.** Choose some large \( T \) such that

\[
C \frac{4(n+1)}{\theta} e^{-\frac{\theta}{2(n+1)} t} \leq \frac{\epsilon_0}{4N},
\]

where \( C \) is the constant chosen in Lemma 1.5. Then

(1.19)

\[
|\hat{\varphi}| \leq \frac{3\epsilon_0}{4N}, \quad \forall \ t \in [0, T + \delta(T)),
\]

where \( \hat{\varphi} = \hat{\varphi}_t = \varphi(t) - \frac{1}{V} \int_M \varphi \omega_\varphi^n \).

**Proof.** Notice that

\[
\frac{d}{dt} \hat{\varphi} = \tilde{h} - \frac{1}{V} \int_M \tilde{h} \Delta \varphi \omega_\varphi^n.
\]
Then by Lemma 1.5, we have
\[
\tilde{\varphi} = \tilde{\varphi}_T + \int_T^{T+\delta(T)} \tilde{h} dt - \frac{1}{V} \int_M \tilde{h} \triangle \varphi \omega^n dt
\]
\[
\leq \frac{\epsilon_0}{2N} + C \int_T^{T+\delta(T)} e^{-\frac{\theta}{2(n+1)^t}} dt + 2C \epsilon_0 \int_T^{T+\delta(T)} e^{-\frac{\theta}{2(n+1)^t}} dt
\]
\[
\leq \frac{\epsilon_0}{2N} + 2C \frac{2(n+1)}{\theta} e^{-\frac{\theta}{2(n+1)^T}}.
\]

□

**Proof of Theorem 1.1.** First we want to show that \( \varphi_t \in K(\epsilon_0) \) for any \( t > 0 \).

By the contradiction, we may assume that there exists a number \( \delta(T) < \infty \) such that \( \varphi_t \in K(\epsilon_0) \) for any \( t < T + \delta(T) \) and there exists a sequence of \( t_i \to T + \delta(T) \) such that

\[
\|\varphi_{t_i} - \varphi_{t} \|_{C^2,\alpha} \to \epsilon_0.
\]

Let \( b_t \) be a constant so that \( \varphi_t = \tilde{\varphi} + b_t \). Then by Proposition 1.7, it is easy to see \( b_t \leq \frac{2\epsilon_0}{N} \).

Decompose \( \varphi = \varphi_\perp \), where \( \varphi \in \Lambda_1(M, \omega_{KE}) \) and \( \varphi_\perp \in \Lambda_1^\perp(M, \omega_{KE}) \), where \( \Lambda_1^\perp(M, \omega_{KE}) \) is a subspace of \( L^2 \)-integral functions which are orthogonal to \( \Lambda_1(M, \omega_{KE}) \cup \mathbb{R} \). Thus \( \varphi = \sum a_i \theta_i \) for some constants \( a_i \), where \( \theta_i \) is a basis of the space \( \Lambda_1(M, \omega_{KE}) \). As a consequence, by Proposition 1.7, we have \( |a_i| \leq \frac{2\epsilon_0}{N} \), so

\[
\|\varphi\|_{C^2,\alpha} \leq \frac{A_0 \epsilon_0}{N},
\]

for some uniform constant \( A_0 \).

By equation (1.1), we have

\[
\omega_{\varphi}^n = \omega_{KE}^n \tilde{h} + \varphi_\perp + a,
\]

where \( \tilde{h} = \dot{\varphi}_t - c_t \) and \( a = a_t \) are constants. By Lemma 1.5 and Proposition 1.7, it is easy to see that \( |a| \leq \frac{4A_0 \epsilon_0}{N} \). Let \( P \) be a projection from Banach space \( H^{2,\alpha}(M) \) to Banach space \( H^{\alpha}(M) \cap \Lambda_1^\perp(M, \omega_{KE}) \). Then \( \varphi_\perp \) is a solution of equation

\[
P[\log(\frac{\omega_{\phi_\perp}^n}{\omega_{KE}^n})] - \varphi_\perp = P(\tilde{h} + a),
\]

where \( \phi \) and \( \tilde{h} + a \) are regarded as two perturbation functions. On the other hand, by Lemma 1.5, we have

\[
\|P(\tilde{h} + a)\|_{C^\alpha} = \|P(\tilde{h})\|_{C^\alpha} \leq C \min\{\left(\frac{2\epsilon_0}{N}\right)^2, C e^{-\frac{\theta}{2(n+1)^T}}\}^{\frac{1}{2}}.
\]

Thus by using the Implicity Functional Theorem, we get

\[
\|\varphi_\perp\|_{C^2,\alpha} \leq c = O(\frac{\epsilon_0}{N}),
\]
where constant $c$ is independent of $t$ and $\epsilon_0$ and goes to zero as $N \to \infty$. Consequently, $c \leq \frac{\epsilon_0}{2}$ by choosing a large number $N$. Hence by combining (1.21) and (1.23), we obtain

$$\|\varphi\|_{C^{2,\alpha}} \leq \frac{\epsilon_0}{2} \forall t \in [T, T + \delta(T)).$$

But this is impossible according to (1.20). Therefore we prove that $\varphi_t \in K(\epsilon_0)$ for any $t > 0$.

By the above argument and lemma 1.5 and Proposition 1.7, we conclude that there exists an $\epsilon$ such that if $\|\psi - \underline{\psi}\|_{C^{2,\alpha}} \leq \epsilon$, then for any $t > 0$, we have

a) $\varphi_t \in K(\epsilon_0),$

b) $|\tilde{\varphi}| \leq \frac{3\epsilon_0}{4N},$

c) $\|\tilde{h}\|_{C^\alpha} \leq C\{\min\{\frac{\epsilon_0}{2N}, Ce^{-\frac{\theta}{2(n+1)}t}\}\}^{\frac{1}{2}}.$

On the other hand, according to [BM], one can choose an element $\sigma_t \in \text{Aut}_0(M)$ for each $\varphi$ such that potential $(\varphi_\sigma - \underline{\varphi}_\sigma)$ lies in $\Lambda^+_{\mathfrak{g}}(M, \omega_{KE})$, where $\varphi_\sigma = \varphi_{\sigma_t} = \varphi_t(\sigma_t(.)) + \rho_t(.)$ and $\rho_t$ is Kähler potential defined by (1.2). Furthermore, by the fact $\varphi \in K(\epsilon_0)$, one can prove easily

$$\text{dist}(\sigma, Id) \leq 1.$$  

Consequently, by (1.27), we have

$$\|\tilde{h}(\sigma_t(.))\|_{C^\alpha} \leq Ce^{-\frac{\theta}{(n+1)}t}.$$  

Thus by applying the Implicity Functional Theorem to the modified equation of (1.22),

$$\omega_{\varphi_\sigma}^n = \omega_{KE}^n e^{\tilde{h}(\sigma_t(.))-\varphi_\sigma + a},$$

we have

$$\|\varphi_\sigma - \underline{\varphi_\sigma}\|_{C^{2,\alpha}} \leq C(\|\tilde{h}(\sigma_t(.))\|_{C^\alpha}).$$

Furthermore, one can get an explicit estimate

$$\|\varphi_\sigma - \underline{\varphi_\sigma}\|_{C^{2,\alpha}} \leq 2\|\tilde{h}(\sigma_t(.))\|_{C^\alpha} \leq C'e^{-\frac{\theta}{(n+1)}t}.$$  

To get higher-order estimates for the modified Kähler potentials $\varphi_\rho$, one can use Lemma 4.1 in Appendix 1 and the embedding theory of Sobolev spaces to obtain

$$\|\tilde{\varphi}\|_{C^{k,\alpha}} \leq C_k e^{-\frac{\theta}{(n+1)}t}, \forall t > 0,$$

where constants $C_k$ depends only on $k, \epsilon_0$ and higher-order derivatives of the initial Kähler potential $\psi$ (we may assume that $\psi$ is smooth since we
can replace it by an evolved Kähler metric $\varphi_{t=1}$). Then by the Implicity Functional Theorem as the above, we derive
\[
\|\varphi_{\sigma} - \varphi_{\sigma}\|_{C^{k+2,\alpha}} \leq 2\|\tilde{\varphi}\|_{C^{k,\alpha}} \leq C_k e^{-\theta n t}.
\]
Therefore we prove that Kähler metrics $\sigma^*(\omega)_{\phi}$ converge exponentially to $\omega_{KE}$.

\[\square\]

2. IN CASE OF KÄHLER-RICCI SOLITON

In this section, we assume that $M$ admits a Kähler Ricci soliton $(\omega_{KS}, X_0)$ with some holomorphic vector field $X_0$ on $M$, i.e., $(\omega_{KS}, X_0)$ satisfies equation,
\[
\text{Ric}(\omega_{KS}) - \omega_{KS} = L_{X_0} \omega_{KS},
\]
where $L_{X_0}$ is a Lie derivative along the vector field $X_0$. By the Hodge theorem, one can define a real-valued potential $\theta_X$ of $X_0$ by
\[
L_{X_0} \omega = \sqrt{-1} \partial \bar{\partial} (\theta_X + X_0(\phi)),
\]
and so
\[
L_X \omega = \sqrt{-1} \partial \bar{\partial} (\theta_X + X(\phi))
\]
and
\[
L_{X'} \omega = \sqrt{-1} \partial \bar{\partial} (X'(\phi)).
\]
(2.1) also implies that for any $\psi \in C^\infty(M)$ it holds
\[
< \bar{\partial}(\theta_X + X_0(\phi)), \bar{\partial} \psi >_{\omega_{\phi}} = X_0(\psi) = X(\psi) + \sqrt{-1} X'(\psi).
\]
Thus
\[
| < \nabla(\theta_X + X(\phi)), \nabla \psi >_{\omega_{\phi}} - X(\psi)| \leq |X'(\phi)| \|\nabla \psi\|_{\omega_{\phi}}.
\]
(2.2)

We now consider a modified equation of (0.1),
\[
\frac{\partial g(t, \cdot)}{\partial t} = -\text{Ric}(g) + g + L_X g,
\]
(2.3)

\[g(0) = g.\]
Then (2.3) is equivalent to a parabolic equation of complex Monge-Ampère type,

$$
\frac{\partial \varphi}{\partial t} = \log \frac{\omega^n_\varphi}{\omega^n_{KS}} + \varphi + X(\varphi),
$$

(2.4)

$$
\varphi(0) = \psi - \overline{\psi},
$$

where $\psi = \frac{1}{V} \int_M \psi \omega^n_{KS}$, $V = \int_M \omega^n_{KE}$, and $\varphi = \varphi_t$ are potentials of evolved Kähler metrics $g_t$ of (2.3). Let $K_{X_0}$ be an one-parameter compact subgroup of $\text{Aut}_0(M)$ generated by the imaginary part $X'$ of $X_0$. By choosing a reductive subgroup $\text{Aut}_r(M)$ of $\text{Aut}_0(M)$ such that $\text{Aut}_r(M)$ contains $K_{X_0}$, we can prove

**Theorem 2.1.** Let $M$ be a compact Kähler manifold $M$ with $c_1(M) > 0$ which admits a Kähler-Ricci soliton $\omega_{KS}$. Then there exists a small $\epsilon$ such that for any initial data, potential $\psi \in \mathcal{M}(\omega_{KS})$ in equation (2.4) with $\psi \in K(\epsilon)$, there exist a family of $\sigma = \sigma_t \in \text{Aut}_r(M)$ for evolved Kähler potentials $\varphi = \varphi_t$ of (2.4) at $t$ such that Kähler potentials $(\varphi_{\sigma} - \varphi_{\sigma})$ are $C^k$-norm uniformly bounded, where $\varphi_{\sigma} = \sigma^* \varphi + \rho$ and $\rho = \rho_t$ are Kähler potentials defined by

$$
\rho^*(\omega_{KS}) = \omega_{KS} + \sqrt{-1} \partial \overline{\partial} \rho,
$$

(2.5)

$$
\int_M e^{-\rho - \overline{X_0(\rho)}} \omega^n_{KS} = \int_M \omega^n_{KS}.
$$

As a consequence, evolved Kähler metrics $g_t$ of (2.3) converge to $g_{KS}$ smoothly in the sense of Cheeger-Gromov. Furthermore, if in addition that $\psi$ is $K_{X_0}$-invariant, then there exist a family of $\sigma = \sigma_t \in \text{Aut}_r(M)$ such that $(\varphi_{\sigma} - \varphi_{\sigma})$ converge exponentially to 0 as $t \to \infty$, and consequently Kähler metrics $\sigma^*(\omega_{\varphi})$ converge exponentially to $\omega_{KS}$.

As in Section 1, to prove Theorem 2.1, we need to estimate $\dot{\varphi}$ of Kähler potentials $\varphi = \varphi_t$ of (2.4). We introduce a modified functional of $H_0(t)$ by

$$
\tilde{H}_0(t) = \frac{1}{V} \int_M (\dot{\varphi} - c(t))^2 e^h \omega^n_\varphi,
$$
where \( c(t) = \int_M \phi e^{\tilde{h}} \omega^n_\phi \) is a constant, \( \tilde{h} = \tilde{h}_t = \theta_X + X(\varphi) - \phi \) and \( V = \int_M \omega^n_{KS} \). By a direct computation, one shows

\[
\frac{d}{dt} \tilde{H}_0(t) = 2 \frac{1}{V} \int_M (\dot{\phi} - c(t))(\dot{\varphi} - \dot{c}(t)) e^{\tilde{h}} \omega^n_\phi
\]

\[+ \frac{1}{V} \int_M (\dot{\phi} - c(t))^2 (\triangle \phi \dot{\phi} + X(\phi) - \dot{\varphi}) e^{\tilde{h}} \omega^n_\phi
\]

\[= 2 \frac{1}{V} \int_M (\phi - c(t))(\triangle \phi \dot{\phi} + X(\phi) - \dot{\varphi}) e^{\tilde{h}} \omega^n_\phi
\]

\[+ \frac{1}{V} \int_M (\dot{\phi} - c(t))^2 (\triangle \phi \dot{\phi} + X(\phi) - \dot{\varphi}) e^{\tilde{h}} \omega^n_\phi
\]

\[= 2 \frac{1}{V} \int_M (\dot{\phi} - c(t))(\triangle \phi \dot{\phi} + X(\phi) - \dot{\varphi}) e^{\tilde{h}} \omega^n_\phi
\]

\[+ 2 \frac{1}{V} \int_M (\phi - c(t))^2 (1 + \dot{\phi}) e^{\tilde{h}} \omega^n_\phi.
\]

(2.6)

On the other hand, by (2.2), we see

\[
\int_M (\dot{\phi} - c(t))(\triangle \phi \dot{\phi} + X(\phi) - \dot{\varphi}) e^{\tilde{h}} \omega^n_\phi
\]

\[= - \int_M \|\nabla (\dot{\phi} - c(t))\|^2 e^{\tilde{h}} \omega^n_\phi
\]

\[+ \int_M (\dot{\phi} - c(t))[X(\phi) - \nabla (\theta_X + X(\phi) - \dot{\varphi}) \nabla \phi > e^{\tilde{h}} \omega^n_\phi
\]

\[\leq - \int_M \|\nabla (\dot{\phi} - c(t))\|^2 e^{\tilde{h}} \omega^n_\phi + V|\dot{\phi} - c(t)||X'(\phi)||\nabla \dot{\phi}||
\]

Thus inserting the above inequality into (2.6), we get

\[
\frac{d}{dt} \tilde{H}_0(t) \leq 2 \frac{1}{V} \int_M (\phi - c(t))^2 (1 + \dot{\phi}) e^{\tilde{h}} \omega^n_\phi
\]

\[- 2 \frac{1}{V} \int_M \|\nabla (\dot{\phi} - c(t))\|^2 e^{\tilde{h}} \omega^n_\phi
\]

\[+ 2|\dot{\phi} - c(t)||X'(\phi)||\nabla \dot{\phi}||
\]

(2.7)

We shall estimate the \( L^2 \)-integral of \( \nabla \dot{\phi} \) and need the following lemma,

**Lemma 2.2.** Let \( P = (P, \omega_\phi) \) be an elliptic operator on \( C^{k,\alpha}(M) \) defined by

\[ P \psi = \triangle \psi + \psi + Re \langle \overline{\partial h}, \overline{\partial \psi} \rangle > \omega_\phi,
\]

where \( \triangle \) is the Lapalace operator with respect to a Kähler metric \( \omega_\phi \) and \( h \) is a Ricci potential of \( \omega_\phi \). Then \( ker(P, \omega_\phi) \subset \eta_r(M) \), where \( \eta_r(M) \) is a reductive part of Lie algebraic \( \eta(M) \) consisting of all holomorphic vector fields on \( M \). Moreover, if \( \omega_\phi = \omega_{KS} \), then \( ker(P, \omega_{KS}) \equiv \eta_r(M) \).
Proof. Let $L \psi = \Delta \psi + \psi + \langle \partial h, \bar{\partial} \psi \rangle_{\omega_{\phi}}$ and $\bar{\mathcal{L}} \psi = \Delta \psi + \psi + \langle \partial h, \bar{\partial} \psi \rangle_{\omega_{\phi}}$, where $h$ is a Ricci potential of the metric $\omega_{\phi}$. Then by the Bochner formula, one can show (cf. Lemma 3.1 in [TZ3]),

$$\int_M - (L \psi) \psi e^h \omega^n = \int_M (\|\nabla \psi\|^2 - \psi^2) e^h \omega^n \geq 0,$$

and

$$\int_M - (\bar{\mathcal{L}} \psi) \psi e^h \omega^n = \int_M (\|\nabla \psi\|^2 - \psi^2) e^h \omega^n \geq 0,$$

Moreover, the equality (2.8) or (2.9) holds if and only if the corresponding vector field of $(0,1)$-form $\bar{\partial} \phi$ is holomorphic. Thus

$$-2 \int_M (P \psi) \psi e^h \omega^n = - \int_M (L \psi + \bar{\mathcal{L}} \psi) \psi e^h \omega^n,$$

and the equality holds if and only if the corresponding vector field of $(0,1)$-form $\bar{\partial} \psi$ is holomorphic. Since $\psi$ is a real-valued function the corresponding vector field must lie in $\eta_r(M)$. Furthermore, if one defines a potential $\theta'_Y$ by

$$L_Y \omega_{KS} = \sqrt{-1} \partial \bar{\partial} \theta'_Y \text{ and } \int_M \theta'_Y e^{\theta_Y} \omega^n_{KS} = 0,$$

for an element $Y$ in $\eta_r(M)$, then in case of $\omega_{\phi} = \omega_{KS}$, by using the fact that $X_0$ is an element of center of $\eta_r(M)$ [TZ1] and $h = \theta_{X_0}$, one can show $\theta'_Y$ must be in $\ker(P, \omega_{KS})$. \hfill \square

Set a Banach space by

$$\mathcal{K}(\epsilon_0) = \{ \phi \in \mathcal{M}(\omega_{KS}) \mid \| \phi - \phi \|_{C^{2,\alpha}} \leq \epsilon_0 \}.$$  

Then we have

**Lemma 2.3.** Let $\varphi = \varphi_t$ be an evolved Kähler potential of (2.4) at $t$ and $\theta'_Y \in \ker(P, \omega_{\phi})$ be a potential of $Y \in \eta_r(M)$ with $\int_M \|Y\|^2 \omega^n_{KS} = 1$. If $\varphi \in \mathcal{K}(\epsilon_0)$, then there exist two uniform constants $C_1$ and $C_2$ such that

$$\left| \int_M \theta'_Y (\varphi_t - c(t)) e^{\theta_Y + X_0(\phi)} \omega^n_{\phi} \right|$$

$$\leq C_1 \epsilon_0 \int_M |\varphi_t - c(t)| e^{\theta_Y + X_0(\phi)} \omega^n_{\phi} + C_2 \epsilon_0^2.$$  

**Proof.** Recall a generalized Futaki-invariant defined in [TZ2] by

$$F_{X_0}(Y) = \int_M Y [h_{\omega_{\phi}} - (\theta_X + X_0(\phi))] e^{\theta_X + X_0(\phi)} \omega^n_{\phi}, \forall Y \in \eta(M).$$
It was proved that the invariant is independent of the choice of Kähler metric $\omega_\phi$ on $M$ and the invariant vanishes if $M$ admits the Kähler-Ricci soliton $(\omega_{KS}, X_0)$. So we have

$$F_{X_0}(Y) \equiv 0, \forall \ Y \in \eta(M).$$

By applying the metrics $\omega_\phi$ to the above identity, one sees

$$\int_M Y [\dot{\varphi} - c(t) - \sqrt{-1}X'(\varphi)] e^{\theta_X + X_0(\varphi)} \omega_\phi^n = 0, \forall \ Y \in \eta(M).$$

It follows

$$|\text{Re} \left( \int_M Y [\dot{\varphi} - c(t) - \sqrt{-1}X'(\varphi)] e^{\theta_X + X(\varphi) + \ln \cos(X'(\varphi))} \omega_\phi^n \right)|$$

(2.12) \leq A_0 \|\varphi\|_{C^2} |X'(\varphi)|.

On the other hand, by using the Stoke's formula, we have

$$\int_M Y [\dot{\varphi} - c(t) - \sqrt{-1}X'(\varphi)] e^{\theta_X + X(\varphi) + \ln \cos(X'(\varphi))} \omega_\phi^n$$

$$= - \int_M (\dot{\varphi} - c(t)) - \sqrt{-1}X'(\varphi))$$

$$\times \left[ \Delta (\theta_Y + Y(\varphi)) + \langle \partial \theta_Y + Y(\varphi), \overline{\partial} (\theta_X + X(\varphi) + \ln \cos(X'(\varphi))) \rangle \right]$$

$$\times e^{\theta_X + X(\varphi) + \ln \cos(X'(\varphi))} \omega_\phi^n + O(\epsilon_0^2)$$

$$= \int_M (\dot{\varphi} - c(t) - \sqrt{-1}X'(\varphi))(\triangle_{\omega_{KS}} \theta_Y + \langle \overline{\partial} \theta_Y, \overline{\partial} \theta_X \rangle$$

$$\times e^{\theta_X + X(\varphi) + \ln \cos(X'(\varphi))} \omega_\phi^n + O(\epsilon_0^2).$$

(2.13)

Note that

$$\langle \overline{\partial} \theta_Y, \overline{\partial} \theta_X \rangle_{\omega_{KS}} = Y(\theta_X) = X(\theta_Y) = \langle \overline{\partial} \theta_X, \overline{\partial} \theta_Y \rangle_{\omega_{KS}}$$

is a real-valued function ([TZ1]). Thus

$$\triangle_{\omega_{KS}} \theta_Y + \langle \overline{\partial} \theta_Y, \overline{\partial} \theta_X \rangle_{\omega_{KS}} = \triangle_{\omega_{KS}} \theta_Y + \langle \overline{\partial} \theta_X, \overline{\partial} \theta_Y \rangle_{\omega_{KS}} = -\theta_Y.$$

Therefore, inserting (2.13) into (2.12), one will get (2.11). \(\square\)

By using Lemma 2.2 and Lemma 2.3, we can complete the $L^2$-estimate of $\dot{\varphi}$.

**Lemma 2.4.** Let $\epsilon_0 << 1$. Then

$$\tilde{H}_0(t) \leq \tilde{H}_0(0) e^{-\theta t} + \frac{B_0}{\theta} \epsilon_0^3, \forall \ t \in [0, T),$$

(2.14)
if $\varphi_t$ lies in $\mathcal{K}(\epsilon_0)$ and $\tilde{H}_0(t) \geq \frac{B_0}{\theta} \epsilon_0^3$ for any $t$ in $[0, T)$, where the constant $B_0 = B_0(\|X\|_{C^0})$ depends only on $\|X\|_{C^0}$ and the constant $\theta > 0$ depends only on the gap of the first two eigenvalues of the operator $P$ associated to the metric $\omega_{KS}$ in Lemma 2.2.

**Proof.** Let $\psi^i$ be the first eigenvalue functions of the operator $(P, \omega_\varphi)$ with respect to the metric $\omega_\varphi$ with satisfying $\int_M |\psi^i|^2 e^{\tilde{h}} \omega_\varphi^n = 1$. Then by the continuity of eigenvalue functions and (2.11), one sees that there exists two constants $C$ and $A_0$ such that

$$|\int_M \psi^i(\dot{\varphi}_t - c(t)) e^{\tilde{h}} \omega_\varphi^n| \leq C \epsilon_0 \int_M |\dot{\varphi}_t - c(t)| e^{\tilde{h}} \omega_\varphi^n + A_0 \epsilon_0^2.$$  

(2.15)

Now as same as in the proof of Lemma 1.3, we decompose $\dot{\varphi}_t - c(t)$ as $\psi + \psi^\perp$ with $\psi \in \Lambda_1(M, \omega_\varphi)$ and $\psi^\perp \in \Lambda^\perp_1(M, \omega_\varphi)$, where $\Lambda_1(M, \omega_\varphi)$ is a linear space spanned by a basis $\{\psi^i\}$ and $\Lambda^\perp_1(M, \omega_\varphi)$ be a subspace of $L^2$-weighted integral functions which are orthogonal to $\Lambda_1(M, \omega_\varphi) \cap \mathbb{R}$ in the sense of

$$\int_M \psi \psi^\perp e^{\tilde{h}} \omega_\varphi^n = 0, \quad \forall \; \psi \in \Lambda_1(M, \omega_\varphi), \psi^\prime \in \Lambda^\perp_1(M, \omega_\varphi).$$

Then we get

$$\int_M |\psi|^2 e^{\tilde{h}} \omega_\varphi^n \leq C' \epsilon_0^2 \int_M (\dot{\varphi} - c(t))^2 e^{\tilde{h}} \omega_\varphi^n + n A_0^2 \epsilon_0^4,$$

and so

$$\int_M |\psi^\perp|^2 e^{\tilde{h}} \omega_\varphi^n \geq (1 - C' \epsilon_0^2) \int_M (\dot{\varphi} - c(t))^2 e^{\tilde{h}} \omega_\varphi^n - n A_0^2 \epsilon_0^4.$$  

(2.16)

On the other hand, by using the continuity of eigenvalues and Lemma 2.2, there exists a number $\delta_0 > 0$ (compared to Lemma 1.2), which depends only on the gap of the first two eigenvalues of the operator $(P, \omega_{KS})$ with respect to the metric $\omega_{KS}$ in Lemma 2.2, such that for any $\varphi \in \mathcal{K}(\epsilon_0)$, we have

$$\int_M \|\nabla \psi^\perp\|^2 e^{\tilde{h}} \omega_\varphi^n \geq (1 + \delta_0) \int_M (\dot{\varphi}^\perp)^2 e^{\tilde{h}} \omega_\varphi^n.$$  

Thus by (2.16), we get

$$\int_M \|\nabla (\dot{\varphi}_t - c(t))\|^2 e^{\tilde{h}} \omega_\varphi^n \geq (1 + \delta_0)(1 - C' \epsilon_0^2) \int_M (\dot{\varphi}_t - c(t))^2 e^{\tilde{h}} \omega_\varphi^n - n A_0^2 \epsilon_0^4.$$  

(2.17)
By inserting (2.17) into (2.7), we obtain
\[
\frac{d}{dt} \tilde{H}_0(t) \
\leq -2[(1 - 2\epsilon_0)(1 + \delta_0)(1 - C'\epsilon_0^2) - (1 + \epsilon_0)]\tilde{H}_0(t) + B_0\epsilon_0^3 \\
\leq -\theta \tilde{H}_0(t) + B_0\epsilon_0^3,
\]
where the constant \( B_0 = B_0(\|X'\|_{C^0}) \) depends only on \( \|X'\|_{C^0} \) and \( \theta = 2[(1 - 2\epsilon_0)(1 + \delta_0)(1 - C'\epsilon_0^2) - (1 + \epsilon_0)] \geq \delta_0 \) as \( \epsilon_0 \) is small enough.

By (2.18), we have
\[
\frac{d}{dt}(\tilde{H}_0(t) - \frac{B_0\epsilon_0^3}{\theta}) \leq -\theta(\tilde{H}_0(t) - \frac{B_0\epsilon_0^3}{\theta}).
\]
Since \( \tilde{H}_0(t) \geq \frac{B_0\epsilon_0^3}{\theta} \), we get
\[
\tilde{H}_0(t) \leq e^{-\theta t}(\tilde{H}_0(0) - \frac{B_0\epsilon_0^3}{\theta}) + \frac{B_0\epsilon_0^3}{\theta} \leq e^{-\theta t}\tilde{H}_0(0) + \frac{B_0\epsilon_0^3}{\theta}.
\]

\[\square\]

**Remark 2.5.** From (2.7) and (2.12), we see that if in addition that the initial Kähler potential \( \psi \) is \( K_{X_0} \)-invariant, then (2.11) can be improved as
\[
\tilde{H}_0(t) \leq \tilde{H}_0(0)e^{-\theta t}, \forall t \in [0,T)
\]
whenever \( \varphi_t \) lies in \( \overline{K}(\epsilon_0) \).

To get a \( C^0 \)-estimate and \( C^\alpha \)-estimate for \( \dot{\varphi} \), we use a method as in Appendix to estimate \( W^{k,2} \)-estimates \( (k \geq 1) \) for \( \dot{\varphi} \). Let
\[
\tilde{H}_k(t) = \int_M \|\nabla^k\dot{\varphi}\|^2 e^{\tilde{h}}\omega_\psi^n.
\]
Then we have

**Proposition 2.6.** Let \( \epsilon_0 << 1 \). Then under the same condition in Lemma 2.4, there exist two uniform constants \( \theta', B > 0 \) which depend only on the metric \( \omega_{KS} \) and integer number \( k \) such that
\[
\tilde{H}_k(t) \leq e^{-\theta' t}(\tilde{H}_k(0) + B\tilde{H}_0(0)) + \frac{B_0B}{\theta'}\epsilon_0^3, \forall t \in [0,T),
\]
if \( \tilde{H}_k(t) + B\tilde{H}_0(t) \geq \frac{B_0B}{\theta'}\epsilon_0^3 \) for any \( t \leq T \), where \( B_0 \) is the constant determined in Lemma 2.4.

**Proof.** First note that similarly to (4.1) in Appendix 1, we can obtain
\[
\frac{d}{dt}\|\nabla^k\dot{\varphi}\|^2 \\
\leq -2\|\nabla^{k+1}\dot{\varphi}\|^2 + C_1\|\nabla^k\dot{\varphi}\|^2 + C_2\|\dot{\varphi} - c(t)\|^2.
\]
It follows
\[
\frac{d\tilde{H}_k(t)}{dt} = \int_M d\|\nabla^k \tilde{\varphi}\|^2 e^{\tilde{h}_\omega^k} + \int_M \|\nabla^k \tilde{\varphi}\|^2 (\triangle \tilde{\varphi} + X(\tilde{\varphi}) - \tilde{\varphi}) e^{\tilde{h}_\omega^k}
\]
\[
\leq -2\tilde{H}_{k+1}(t) + C'_1\tilde{H}_k(t) + C'_2\|\tilde{\varphi} - c(t)\|^2
\]
\[
\leq -\theta'\tilde{H}_k(t) + C_3\tilde{H}_0(t).
\]
On the other hand, by \((2.18)\), we have
\[
\frac{d\tilde{H}_0(t)}{dt} \leq -\theta\tilde{H}_0(t) + B_0\epsilon_0^3, \quad \forall t \in [0, T),
\]
since we may also assume that \(\tilde{H}_0(t) \geq \frac{B_k}{\theta_0}\epsilon_0^3\) for any \(t \in [0, T)\), Thus combining the above two inequalities, we get
\[
\frac{d(\tilde{H}_k(t) + B\tilde{H}_0(t))}{dt} \leq -\theta'(\tilde{H}_k(t) + B\tilde{H}_0(t)) + B_0B\epsilon_0^3
\]
\[
(2.20)
\]
where \(B\) is a sufficiently large number independent of \(\epsilon_0\). From \((2.20)\), one can easily get
\[
(\tilde{H}_k(t) + B\tilde{H}_0(t)) \leq e^{-\theta t}(\tilde{H}_k(0) + B\tilde{H}_0(0)) + \frac{B_0B}{\theta'}\epsilon_0^3, \quad \forall t \in [0, T),
\]
and so \((2.19)\) is true. \(\Box\)

By the embedding theory of Sobolev spaces, we get

**Corollary 2.7.** Let \(\epsilon_0 << 1\). Then under the same condition in Lemma 2.4, there exist two uniform constants \(\theta_0, C_0 > 0\) which depend only on the metric \(\omega_{KS}\) such that
\[
\|\tilde{\varphi}_t\|_{C^\alpha} \leq C_0e^{-\theta_0 t}\|\psi - \overline{\psi}\|_{C^{2,\alpha}} + \epsilon_0^3, \quad \forall t \in [0, T),
\]
if \(\|\tilde{\varphi}_t\|_{C^\alpha} \geq C_0\epsilon_0^3\) for any \(t \leq T\).

**Remark 2.8.** By Remark 2.5, according to the proof of Proposition 2.6, we see that if in addition that the initial Kähler potential \(\psi\) is \(K_{X_{\phi}}\)-invariant, then \((2.19)\) can be improved as
\[
\tilde{H}_k(t) \leq (\tilde{H}_0(0) + B\tilde{H}_k(0))e^{-\theta' t}, \quad \forall t \in [0, T)
\]
whenever \(\varphi_t\) lies in \(\overline{\mathcal{K}}(\epsilon_0)\). Thus \((2.20)\) can be improved as
\[
\|\tilde{\varphi}_t\|_{C^\alpha} \leq C_0e^{-\theta_0 t}\|\psi - \overline{\psi}\|_{C^{2,\alpha}}.
\]

The following lemma can be easily proved by using apriori estimates for solution \(\varphi(t, \cdot)\) of \((2.4)\) at finite time (cf [TZ3]).
Lemma 2.9. Let $\psi \in K(\frac{\omega}{N})$. Then there exists $T = T_N$ such that evolved Kähler potentials $\varphi_t$ of (2.4) with $\psi$ as an initial potential lies in $K(\epsilon_0)$ for any $t < T$.

We are now going to do a key estimate for the proof of Theorem 2.1.

Proposition 2.10. There exist a small $\epsilon_0$ and a large number $N$ such that if the initial data $\psi \in M(\omega_{KS})$ in (2.4) satisfies $\|\psi - \frac{\omega}{N}\|_{C^{2,\alpha}} \leq \frac{\epsilon_0}{N}$, then there exist a family of $\sigma_t \in \text{Aut}_r(M)$ such that

$$\|\varphi_{\sigma_t} - \varphi_{\sigma_t}\|_{C^{2,\alpha}} \leq \epsilon_0, \quad \forall \; t > 0,$$

(2.22) where $\varphi_{\sigma_t} = (\sigma_t)^{*} \varphi_t + \rho_t$ and $\rho_t$ are Kähler potentials defined by (2.5) in Theorem 2.1.

Proof. The proof is a modification of one of Theorem 1.1. Let $N_0$ be a very big number and choose another big number $N$ with $N_0 \leq N \leq \frac{1}{\epsilon_0^4}$ such that $C_0 e^{-\theta T_N} \leq \frac{1}{N_0}$, where $C_0$ and $T_N$ are two uniform numbers determined in Corollary 2.7 and Lemma 2.9, respectively. Now we consider the solution $\varphi = \varphi_{T_N}$ of (2.4) at time $T_N$. By Lemma 5.1 in Appendix 2, we see that there exists $\sigma = \sigma_{T_N}$ such that for any $Y \in \eta_r(M)$ with $\int_M \|Y\|^2 \omega_{KS} = 1,$

$$|\int_M \theta_Y \varphi e^{\theta X} \omega_{KS}^n| \leq O(\epsilon_0^2),$$

(2.23) where $\varphi = \sigma^* \varphi + \rho_\sigma$. By adding a constant to $\varphi_\sigma$ so that $\widetilde{\varphi}_\sigma = \varphi_\sigma + \text{const.}$ satisfies $\int_M \widetilde{\varphi}_\sigma e^{\theta X} \omega_{KS}^n = 0$, then we can decompose $\widetilde{\varphi}_\sigma$ into $\widetilde{\varphi}_\sigma = \phi + \phi^\perp$ with $\phi \in \Lambda_1(M, \omega_{KE})$ and $\phi^\perp \in \Lambda_1^\perp(M, \omega_{KE})$, where $\Lambda_1^\perp(M, \omega_{KS})$ is a subspace of weighted $L^2$-integral functions which are orthogonal to $\Lambda_1(M, \omega_{KE}) \cup \mathbb{R}$. Then $\phi = \sum_i a_i \theta_i$ for some constants $a_i$, where $\theta_i$ is a basis of the space $\Lambda_1(M, \omega_{KS})$. As a consequence, by (2.23), we see that $a_i = O(\epsilon_0^2)$ and so

$$\|\phi\|_{C^{2,\alpha}} \leq O(\epsilon_0^2).$$

(2.24)

Since $\rho = \rho_\sigma$ satisfies equation,

$$\omega^\rho_n = \omega_{KS}^n e^{-\rho - X(\rho)},$$

by equation (2.4), we have

$$\omega^\rho_n \varphi = \omega_{KS}^n e^{\sigma^*(\widetilde{\varphi} - \widetilde{\varphi}_\sigma - X(\widetilde{\varphi}) + b},$$

(2.25) where $b$ is a constant. Then $\phi^\perp$ is a solutions of equation

$$P[\log\left(\frac{\omega^\rho_n}{\omega_{KS}^n}\right)^m] \varphi^\perp = \varphi^\perp + X(\phi^\perp) = P[\sigma^*(\widetilde{\varphi} + b - X(\phi))] ,$$

where $P$ is a positive operator.
where $P$ is a projection from Banach space $H^{2,\alpha}(M)$ to Banach space $H^{\alpha}(M) \cap \Lambda^1(M, \omega_{KS})$, and $\phi$ and $\sigma^*(\tilde{\phi})$ are regarded as two perturbation functions. Without generality, we may assume that

$$\|\sigma^*(\tilde{\phi})\|_{C^\alpha} \geq C_0 \epsilon_0^3,$$

where $C_0$ is the constant in Corollary 2.7. Then according to Corollary 2.6 and (2.24), we have

$$\|P[\sigma^*(\tilde{\phi}) + b - X(\phi)]\|_{C^\alpha} = \|P[\sigma^*(\tilde{\phi}) - X(\phi)]\|_{C^\alpha} \leq \frac{2\epsilon_0}{NN_0}.$$

Thus we can use the Implicit Function Theorem to get

$$\|\phi\|_{C^{2,\alpha}} \leq 2(\|P[\sigma^*(\tilde{\phi}) + b - X(\phi)]\|_{C^\alpha} + \|\phi\|_{C^{2,\alpha}}) \leq \frac{8\epsilon_0}{NN_0}.$$

(2.26) and (2.26) implies

$$\|\tilde{\phi}\|_{C^{2,\alpha}} \leq \frac{16\epsilon_0}{NN_0},$$

so $\tilde{\varphi}_\rho \in \mathcal{K}(\frac{\epsilon_0}{N}).$

At the next step (Step 2) we consider equation (2.4) with $\tilde{\varphi}_\rho$ as an initial potential to replace $\psi$. By Lemma 2.9, one sees that equation is solvable for any $t \in T_N$ with evolved K"ahler potentials $\varphi^{(2)}_t \in \mathcal{K}(\epsilon_0)$ for any $t \leq T_N$. So by the argument at the last step (Step 1), we can also show that there exists $\sigma^{(2)} = \sigma^{(2)}_{T_N} \in \text{Aut}_r(M)$ such that

$$\|\tilde{\varphi}^{(2)}_\rho\|_{C^{2,\alpha}} = \|(\sigma^{(2)}_{T_N})^*\tilde{\varphi}^{(2)}_\rho + \rho_{\sigma^{(2)}_{T_N}}\|_{C^{2,\alpha}} \leq (\frac{16}{N_0})^2 \frac{\epsilon_0}{N} < \frac{\epsilon_0}{N}.$$

(2.28)

Repeating to use the above step for finite times, we can obtain

$$\|\tilde{\varphi}^{(k)}_\rho\|_{C^{2,\alpha}} \leq C_0 \epsilon_0^\frac{3}{2}$$

for some integer $k$. Then also by using the argument in Step 1, we can find $\sigma^{(k+1)} = \sigma^{(k+1)}_{T_N} \in \text{Aut}_r(M)$ such that

$$\|\tilde{\varphi}^{(k)}_\rho\|_{C^{2,\alpha}} = O(\epsilon_0^\frac{3}{2}).$$

Now (Step 3) we considering equation (2.4) with $\tilde{\varphi}^{(k)}_\rho$ as an initial potential. Then we conclude that either evolved K"ahler potentials $\varphi^{(k+1)}_t$ lies in $\mathcal{K}(\frac{\epsilon_0}{N})$ for any $t$ or there exists some time $T$ such that $\|\tilde{\varphi}^{(k+1)}_T\|_{C^{2,\alpha}} = \frac{\epsilon_0}{N}$ for an evolved K"ahler potential $\tilde{\varphi}^{(k+1)}_T$ at time $T$. If the first case happens, then we will finish all steps. If the second case happens, then we can repeat the Step1-3 and we can finally prove that there exist a family of $\sigma_t \in \text{Aut}_r(M)$
such that (2.22) satisfies for any evolved Kähler potential $\varphi_t$ of (2.4) at $t$ as long as the initial potential $\psi$ lies in $K(\frac{\omega}{\nabla})$.

\[ \square \]

**Proof of Theorem 2.1.** We suffice to do higher-order estimates for the modified evolved Kähler potentials $((\sigma_t)^*\varphi_t + \rho_t)$ of equation (2.4) in Proposition 2.10. Here we use a trick in [CT2] to choose a modified family of holomorphic transformations $\sigma_t \in \text{Aut}_r(M)$ ($0 < t < \infty, \sigma_0 = \text{Id}$) to replace $\sigma_t$ such that for any $t \in (0, \infty)$ (cf. [TZ3]),

\[ \|\sigma_t^{-1}\sigma_t - \text{Id}\| \leq C, \]

and

\[ \|\sigma_t^{-1}\sigma_t\|_g \leq C, \]

where $(\sigma_t^{-1})\frac{\partial}{\partial t} = X_t \in \eta_r(M)$ is a family of holomorphic vector fields on $M$. Furthermore, for any $k \geq 0$, we may assume that there is a constant $C_k$ such that

\[ \|\frac{\partial^k X_t}{\partial t^k}\|_g \leq C_k. \]

Note that the choice of such $\sigma_t$ just depends on the $C^0$-estimate of $\tilde{\varphi}_\sigma = ((\sigma_t)^*\varphi_t + \rho_t)$. Thus by Proposition 2.10, we also have $(\sigma_t - \frac{1}{t} \int_M \tilde{\varphi}_t \omega^n_{KS}) \in K(\epsilon_0)$. On the other hand, by equation (2.4), the new modified potential $\tilde{\varphi} = \varphi_{\sigma_t} = (\sigma_t)^*(\varphi_t + \rho_t)$ will satisfy equation,

\[ \frac{\partial \tilde{\varphi}}{\partial t} = \log \frac{\omega}{\omega_{KS}} + \tilde{\varphi} + X(\tilde{\varphi}), \]

(2.29)

$\tilde{\varphi}(0) = \psi - \psi$.

Now for each $t$, we can consider solution $\varphi'$ of equation (2.29) on the interval $[t-1, t+1]$ with $(\tilde{\varphi}_{t-1} - \frac{1}{t} \int_M \tilde{\varphi}_{t-1} \omega^n_{KS})$ as an initial data. By the Maximal Principle, it is easy to see that both $\varphi'$ and $\varphi''$ are uniformly bounded in $[t-1, t+1]$. Since

\[ \|\tilde{\varphi}' - \frac{1}{V} \int_M \tilde{\varphi}' \omega^n_{KS}\|_{C^2, \alpha} = \|\tilde{\varphi}' - \frac{1}{V} \int_M \tilde{\varphi}' \omega^n_{KS}\|_{C^2, \alpha}, \]

by the regularity theory of parabolic equation, we get all bounded $C^k$-estimates for $\tilde{\varphi}$. This implies that all $C^k$-norms of $(\tilde{\varphi} - \frac{1}{V} \int_M \tilde{\varphi} \omega^n_{KS})$ are uniformly bounded, and so are $\tilde{\varphi}_\sigma$.

From the above estimates, we see that for any sequence of Kähler metrics $\omega_{\varphi_{\sigma_t}}$, there exists a limit Kähler metric $\omega_\infty$ of subsequence of $\omega_{\varphi_{\sigma_t}}$ in the sense of $C^k$-convergence. By applying Perelman’s $W$-function in [Pe] to the normalized Ricci equation (0.1), one concluding that $\omega_\infty$ must be a Kähler-Ricci soliton (cf. [Se]). Since the Kähler-Ricci soliton is unique, we see that there exists an element $\tau_\infty \in \text{Aut}_0(M)$ such that $\omega_\infty = \tau_\infty^* \omega_{KS}$. By using
the fact that the convergent sequence is arbitrary, the above implies that
there exists a family of $\tau = \tau_t \in \text{Aut}_0(M)$ such that evolved Kähler metrics
$\tau^*g$ converge to $g_{KS}$ smoothly.

If in addition that the initial Kähler potential $\psi$ is $K_X$-invariant, by
Remark 2.7, we can follow the argument in the proof of Theorem 1.1 to
apply the Implied Functional Theorem to equation (2.25) in the proof of
Proposition 2.10 to show that there exists a family of $\sigma = \sigma_t \in \text{Aut}_r(M)$
such that the modified solution $\varphi_{\sigma} = ((\sigma_t)^*\varphi_t + \rho_t)$ of equation (2.25) satisfy

$$\|\tilde{\varphi}_{\sigma}\|_{C^{2,\alpha}} = \|((\sigma_t)^*\varphi_t + \rho_t - \frac{1}{V}\int_M (\sigma_t)^*\varphi_t + \rho_t))\omega^{P}_{KS})\|_{C^{2,\alpha}}$$
(2.30)$$
\leq 2\|P(\sigma^*(\tilde{\varphi}))\|_{C^{\alpha}} \leq C e^{-\theta t}, \forall t > 0.
$$

Similarly, we can also prove that for any $k$ it holds

$$\|\tilde{\varphi}_{\sigma}\|_{C^{k,\alpha}} \leq C_k e^{-\theta t}, \forall t > 0,$$

since by Remark 2.8 and the embedding theory of Sobolev spaces we have

$$\|\tilde{\varphi}\|_{C^{k,\alpha}} \leq C'_k e^{-\theta t}, \forall t > 0,$$

where $C_k$ and $C'_k$ are uniform constants which depends only on $k, \epsilon_0$ and
higher-order derivatives of the initial Kähler potential $\psi$. Therefore we prove
that Kähler metrics $\sigma^*(\omega_{\varphi})$ converge exponentially to the Kähler-Ricci soli-
ton $\omega_{KS}$.

3. Uniqueness of the limit of Kähler Ricci flow

By Theorem 1.1 and Theorem 2.2 in Section 1 and 2, we complete the
proof of Theorem 0.1. As an application of Theorem 0.1, we have the following uniqueness result about the limit of Kähler-Ricci flow.

**Theorem 3.1.** Let $g_t$ be the evolved Kähler metrics of Kähler-Ricci flow
(0.1) on $M$. Suppose that there exists a sequence $g_i$ of $g_t$ and a sequence of
holomorphic transformations $\sigma_i \in \text{Aut}(M)$ such that $\sigma_i^*g_i$ converge to a limit
Kähler metric $g_{\infty}$ in the sense of $C^{2,\alpha}$-norm of Kähler potentials. Then the
Kähler-Ricci flow converges to $g_{\infty}$ smoothly in the sense of Cheeger-Gromov.

**Proof.** First we note that by applying Perelman’s $W$-function in [Pe], the
limit Kähler metric $g_{\infty}$ must be a Kähler-Ricci soliton $g_{KS}$ on $M$. On the
other hand, by the convergence of $g_i$, one sees that for any $\epsilon << 1$ there
exists a big index $i$ such that the potential $\psi = \psi_i$ of $g_i$ satisfies

$$\|\psi - \psi_i\|_{C^{2,\alpha}} \leq \epsilon,$$

where $\omega_{\psi} = \omega_{KS} + \sqrt{-1}\partial\bar{\partial}\psi$. Now we consider the Kähler-Ricci flow (0.1)
with $\omega_{g} = \omega_{\psi}$ as an initial Kähler metric. Then by Theorem 0.1, this flow
converges to $g_{KS}$ smoothly in the sense of Cheeger-Gromov, so the theorem is proved.

\[ \square \]

**Remark 3.2.** In a subsequent paper, we will prove the uniqueness of the limit of Kähler-Ricci flow in more general. Namely, Theorem 3.1 is still true if we assume that there exists a sequence $g_i$ of $g_t$ of equation (0.1) which converge to a limit Riemannian metric $g_\infty$ in $C^{2,\alpha}$-norm in the sense of Cheeger-Gromov.

4. **Appendix 1**

In this appendix, we prove a lemma about $W^{k,2}$-estimates of $\dot{\varphi}$ for evolved Kähler metrics $\varphi$ of flow (1.1) under the assumption $\varphi \in \mathcal{K}(\epsilon_0)$. Recall that a $k - \text{norm}$ $\|\nabla^k \dot{\varphi}\|^2$ is defined by

$$\|\nabla^k \dot{\varphi}\|^2 = \sum g^{i_1 j_1} ... g^{i_k j_k} \hat{\varphi}_{i_1 ... i_k} \dot{\varphi}_{j_1 ... j_k},$$

where $\hat{\varphi}_{i_1 ... i_k}$ are components of the $k$-covariant derivative of $\varphi$ with respect to $g = \omega_\varphi$ as a Riemannian metric.

Since

$$\dot{\varphi}_{i_1 ... i_k} = \frac{\partial^k \varphi}{\partial x^{i_1} ... \partial x^{i_k}} + \Phi_1(\dot{\varphi}, ..., \dot{\varphi}_{i_1 ... i_{k-1}}),$$

we have

$$\frac{d \dot{\varphi}_{i_1 ... i_k}}{dt} = \frac{\partial^k \varphi}{\partial x^{i_1} ... \partial x^{i_k}} + \frac{d \Phi_1}{dt}$$

$$= \dot{\varphi}_{i_1 ... i_k} + \Phi_2(\dot{\varphi}, ..., \dot{\varphi}_{i_1 ... i_{k-1}}) + \frac{d \Phi_1}{dt},$$

where $\Phi_1$ and $\Phi_2$ are two polynomials with variables $\dot{\varphi}_{i_1}, ..., \dot{\varphi}_{i_1 ... i_{k-1}}$ and coefficients $g_{ij}, \partial^l g_{ij}, l = 1, ..., k$. Note that $\frac{d \Phi_1}{dt}$ is uniformly bounded. Then by equations (0.1) and (1.1), one can estimate

$$\frac{d \|\nabla^k \dot{\varphi}\|^2}{dt}$$

$$= \sum_{i_1, ..., i_k} \sum_{\alpha} (R_{i_\alpha i_\alpha} - g_{i_\alpha i_\alpha}) \dot{\varphi}_{i_1 ... i_{k-1} i_\alpha} \dot{\varphi}_{i_1 ... i_{k-1} i_\alpha}$$

$$+ 2 \sum g^{i_1 j_1} ... g^{i_k j_k} \frac{d \dot{\varphi}_{i_1 ... i_k}}{dt} \dot{\varphi}_{j_1 ... j_k}$$

$$\leq C_1 \|\nabla^k \dot{\varphi}\|^2 + C_2 \|\Delta \dot{\varphi}\|^2 + 2(\dot{\varphi})_{i_1 ... i_k} \dot{\varphi}_{j_1 ... j_k}$$

$$\leq -2 \|\nabla^{k+1} \dot{\varphi}\|^2 + C_1' \|\nabla^k \dot{\varphi}\|^2 + C_2' \|\Delta \dot{\varphi}\|^2 + c(t)^2.$$

(4.1)

Let

$$H_k(t) = \int_M \|\nabla^k \dot{\varphi}\|^2 \omega_\varphi^n.$$

Then by (4.1), we have

$$\frac{d H_k}{dt} \leq -2 H_k^{k+1} + C_1' H_k^k + C_2' \Delta \omega_\varphi + c(t)^2.$$
Lemma 4.1. Let $T$ be any positive number. Suppose that $\varphi_t$ lies $\mathcal{K}(\epsilon_0)$ for any $t \in [0,T)$. Then
\begin{equation}
H_k(t) \leq C e^{-\theta' t}, \quad \forall \ t \in [0,T).
\end{equation}

Proof. By (4.1), we have
\[
\frac{dH_k(t)}{dt} = \int_M \frac{d\|\nabla^k \phi\|^2}{dt} \omega^n_{\varphi} + \int_M \|\nabla^k \phi\|^2 \Delta \phi \omega^n_{\varphi} 
\leq -2H_{k+1}(t) + C_3 H_k(t) + C_2' \|\phi - c(t)\|^2
\leq -\theta' H_k(t) + C_4 H_0(t).
\]
(4.3)

On the other hand, from the proof of Lemma 1.3, we in fact prove that
\[
\frac{dH_0(t)}{dt} \leq -\theta H_0(t), \quad \forall \ t \in [0,T),
\]
if $\varphi \in \mathcal{K}(\epsilon_0), \quad \forall \ t \in [0,T)$. Thus Combining the above inequality with (4.3), we get
\[
\frac{d(H_k(t) + A H_0(t))}{dt} \leq -\theta' [H_k(t) + \frac{(A \theta - C_4)}{\theta'} H_0(t)],
\]
where $A$ is a sufficiently large number. It follows
\[
\frac{d \ln(H_k(t) + A H_0(t))}{dt} \leq -\theta' \frac{H_k(t) + (A \theta - C_4) H_0(t)}{H_k(t) + A H_0(t)} \leq -\theta'.
\]
Thus
\[
H_k(t) + A H_0(t) \leq (H_k(0) + A H_0) e^{-\theta' t}
\]
and so (4.2) follows. \qed

5. Appendix 2

The following lemma is about the existence of almost orthonormality of a Kähler potential to the space of first eigenvalue-functions of operator $(P, \omega_{KS})$ defined in Lemma 2.2 in Section 2. The lemma is crucial in the proof of Proposition 2.10.

Lemma 5.1. Let $M$ be a compact Kähler manifold $M$ with $c_1(M) > 0$ which admits a Kähler-Ricci soliton $(\omega_{KS}, X_0)$. Then for any Kähler potential $\phi \in \mathcal{K}(\epsilon_0)$ there exists a $\sigma \in \text{Aut}_r(M)$ with bounded dist($\sigma$, Id) such that for any $Y \in \eta_r(M)$ with $\int_M \|Y\|^2 \omega^n_{KS} = 1$, it holds
\[
|\int_M \theta'_Y (\sigma^* \phi + \rho_\sigma) e^{\theta X} \omega^n_{KS}| \leq C \|X'(\phi)\|^2_{C^0} = O(\epsilon_0^2),
\]
where $\theta_Y \in \ker(P, \omega_{KS})$ and $\rho_\sigma$ is a Kähler potential defined by (2.5) in Section 2.
Proof. This lemma was proved in [TZ1] if \( \phi \) is \( K_0 \)-invariant. The key point in the proof is to use a functional defined on a space of Kähler-Ricci solitons

\[
\{ \omega'_{KS} = \sigma^*(\omega_{KE}) = \omega_{KS} + \sqrt{-1} \partial \bar{\partial} \rho_\sigma | \sigma \in \text{Aut}_\tau(M) \},
\]

which was introduced in [Zh] by

\[
(I - J)(\omega_\phi, \omega'_{KS}) = \int_0^1 dt \int_M \dot{\phi} e^{\theta_{X_0}(\phi_t)} \omega_{\phi_t}^n - \int_M (\phi + \rho) e^{\theta_{X_0}(\rho)} (\omega'_{KS})^n,
\]

where \( \phi_t \) is a \( K_{X_0} \)-invariant path in \( M(\omega_{KS}) \) which connects 0 and \( -\phi + \rho \), and \( \theta_{X_0}(\phi_t) \) are potentials of \( X_0 \) associated to metric \( \omega_{\phi_t} \) defined by (2.1). It is proved in [Zh] that this well-defined for a \( K_0 \)-invariant \( \phi \), i.e., the functional is independent of the choice of a \( K_0 \)-invariant path. But for a general Kähler potential \( \phi \), one can also show that \( (I - J)(\omega_\phi, \omega'_{KS}) \) is not well-defined (to see (5.4) below), so we shall introduce another functional defined on whole space \( M(\omega_{KS}) \) to replace it. In fact, we consider the following functional

\[
\mathcal{F}(\omega_\phi, \omega'_{KS}) = \text{Re} \left[ \int_0^1 dt \int_M (\phi + \rho) e^{\theta_{X_0}(\phi_t(\phi + \rho))} \omega_{\phi_t(\phi + \rho)}^n \right] - \int_M (\phi + \rho) e^{\theta_{X_0}(\omega'_{KS})^n}.
\]  

(5.1)

Clearly, the definition of \( \mathcal{F} \) just uses a real part of \( (I - J)(\omega_\phi, \omega'_{KS}) \) while a Kähler potentials path is chosen by \( \phi_t = t(\phi + \rho_\sigma) \). We now consider a Kähler potentials path \( \rho_t \) induced by an one-parameter subgroup \( \sigma_t \) generated by the real part of \( Y \in \eta_r(M) \), i.e. \( \rho_t \) are defined by \( \omega_t = \sigma_t^* \omega_{KS} = \omega'_{KS} + \sqrt{-1} \partial \bar{\partial} \rho_t \).

Let

\[
(5.2) \quad f_Y(t) = \text{Re} \left[ \int_0^{1+t} ds \int_M (\phi_s) e^{\theta_{X_0}(\phi_s)} \omega_{s}^n - \int_M (\phi + \rho_t) e^{\theta_{X_0}(\omega_t)} \omega_{t}^n \right],
\]

where \( \phi_s \) is a path in \( M(\omega_{KS}) \) defined by \( \phi_s = s(\phi + \rho_\sigma) \), \( \forall 0 \leq s \leq 1 \) and \( \phi_s = -\phi + \rho_\sigma + \rho_t \), \( 1 \leq s \leq 1 + t \). It is easy to see

\[
\frac{d}{dt} f_Y(t)|_{t=0} = \int_M \theta_Y^*(\varphi + \rho_\sigma) e^{\theta_{X_0}(\omega_{KS})^n}.
\]

This implies

\[
(5.3) \quad \frac{d}{dt} f_Y(t)|_{t=0} = -\int_M \theta_Y^*(\sigma^{-1} \varphi + \rho_{\sigma^{-1}}) e^{\theta_{X_0}(\omega_{KS})^n}.
\]

The gap between \( f_Y(t) \) and \( \mathcal{F}(\omega_\phi, \omega_\rho) \) can be computed as follows. Let \( \Delta = \{(\tau, \sigma) | 0 \leq \tau \leq 1, 0 \leq s \leq \tau + (1 - \tau)(1 + t)\} \) be a domain in \( \mathbb{R}^2 \). Let
\[ \Phi = \Phi(\tau, s, \cdot) \]
be Kähler potentials with two parameters \((\tau, s) \in \Delta\) which satisfy:

\[ \Phi = s(-\phi + \rho_\sigma + \rho_t), \quad 0 \leq s \leq 1, \quad \text{as } \tau = 1; \]
\[ \Phi = \phi_s, \quad 0 \leq s \leq 1 + t, \quad \text{as } \tau = 0; \]
\[ \Phi = 0, \quad \text{as } s = 0; \quad \Phi = -\phi + \rho_\sigma + \rho_t, \quad s = \tau + (1 - \tau)(1 + t). \]

Then by using the Stoke's formula, we have

\[ |f_Y(t) - \mathcal{F}(\omega_\phi, \omega_t)| = |Re\left\{ \int_{\partial\Delta} d\tau, s \Phi(\tau, s; \cdot)e^{\theta X_0(\phi_s)}\omega_{\phi_s}^n \right\}| \]
\[ = |Re\left\{ \int_{\Delta} d\tau ds \int_M \Phi_r (<\bar{\partial}\Phi_s, \bar{\partial}\theta X_0(\Phi)>) -</br>\quad <\bar{\partial}\theta X_0(\Phi), \bar{\partial}X_0(\Phi)>e^{\theta X_0(\Phi)}\omega_{\Phi}^n \right\}| \]
\[ = 2|Re\left\{ \int_{\Delta} d\tau ds \int_M \Phi_r \text{Im}(X_0(\Phi_s))e^{\theta X_0(\Phi)}\omega_{\Phi}^n \right\}| \]
\[ \leq C\|X'(\phi)\|^2_{C_0}. \quad (5.4) \]

At the last inequality, we used a fact that \(X_0(\rho_\sigma)\) and \(X_0(\rho_t)\) are both real-valued. Similarly, we can get

\[ |\frac{d}{dt}(f_Y(t) - \mathcal{F}(\omega_\phi, \omega_t))|_{t=0} \leq C\|X'(\phi)\|^2_{C_0}. \quad (5.5) \]

Next we claim

\[ \mathcal{F}(\sigma) = \mathcal{F}(\omega_\phi, \omega_{KS}^{\prime}) \geq 0. \quad (5.6) \]

To prove the claim, we let

\[ g(t) = Re\left\{ \int_0^t ds \int_M (-\phi + \rho_\sigma)e^{\theta X_0(s(-\phi + \rho_\sigma))}\omega_{s(-\phi + \rho_\sigma)}^n \right\} - \int_M (-\phi + \rho)e^{\theta X_0(t(-\phi + \rho_\sigma))}\omega_{t(-\phi + \rho_\sigma)}^n \}

Then

\[ \mathcal{F}(\sigma) = g(1) = \int_0^1 g(t)dt. \]

On the other hand, we have

\[ g(t)' = Re[n\sqrt{-1}\int_M \partial(-\phi + \rho_\sigma) \wedge \bar{\partial}(-\phi + \rho_\sigma)e^{\theta X_0(t(-\phi + \rho_\sigma))}\omega_{t(-\phi + \rho_\sigma)}^n] \]

\[ \geq 0. \]

Thus we get \(g(1) \geq 0\) and prove the claim.

By the above claim, we can take a minimizing sequence of \(\mathcal{F}(\sigma)\) in \(\text{Aut}_r(M)\) and we see that for any small \(\epsilon \leq \epsilon_0\), there exists a \(\sigma \in \text{Aut}_r(M)\) with
bounded dist(σ, Id) such that for any $Y \in \eta_r(M)$ with $\int_M \|Y\|^2 \omega_{KS}^n = 1$, we have

\begin{equation}
|DF(\sigma)(Y)| \leq \epsilon.
\end{equation}

Therefore combining (5.3), (5.5) and (5.7), we prove the lemma while $\sigma$ is replaced by $\sigma^{-1}$.

\section*{References}

[1] [BM] S. Bando and T. Mabuchi, Uniqueness of Kähler Einstein metrics modulo connected group actions, Algebraic Geometry, Adv. Stud. Pure Math. (1987), 11-40.

[2] [Ca] Cao, H.D., Deformation of Kähler metrics to Kähler-Einstein metrics on compact Kähler manifolds, Invent. Math., 81 (1985), 359-372.

[3] [CT1] Chen, X.X. and Tian, G., Ricci flow on Kähler-Einstein surfaces, Invent. Math., 147 (2002), 487-544.

[4] [CT2] Chen, X.X. and Tian, G., Ricci flow on Kähler-Einstein manifolds, Duke Math. J., 131 (2006), 17-73.

[5] [CTZ] Cao, H.D., Tian, G., and Zhu, X.H., Kähler-Ricci solitons on compact Kähler manifolds with $c_1(M) > 0$, Geom and Funct. Anal., 15 (2005), 697-619.

[6] [Ha] Hamilton, R.S., Three manifolds with positive Ricci Curvature, J. Diff. Geom., 17 (1982), 255-306.

[7] [PS] Phong, D. and Strum, J., On the stability and convergence of the Kähler-Ricci flow, J. Diff. Geom., 72 (2006), 149-168.

[8] [P1] Perelman, G., The entropy formula for the Ricci flow and its geometric applications, preprint, 2002.

[9] [P2] Perelman, G., unpublished.

[10] [Se] Sesum, N., Convergence of a Kähler-Ricci flow, Math. Res. Lett., 12 (2005), 623-632.

[11] [ST] Sesum, N. and Tian, G., Bounding scalar curvature and diameter along the Kähler-Ricci flow (after Perelman), J. Inst. Math., Jussieu, 7 (2008), 575-587.

[12] [TZ1] Tian, G. and Zhu, X.H., Uniqueness of Kähler-Ricci solitons, Acta Math., 184 (2000), 271-305.

[13] [TZ2] Tian, G. and Zhu, X.H., A new holomorphic invariant and uniqueness of Kähler-Ricci solitons, Comm. Math. Helv., 77 (2002), 297-325.

[14] [TZ3] Tian, G. and Zhu, X.H., Convergence of Kähler-Ricci flow, Journal of the Amer. Math. Soci., 20 (2007), 675-699.

[15] [Zh] Zhu, X.H., Kähler-Ricci soliton type equations on compact complex manifolds with $C_1(M) > 0$, J. Geom. Anal., 10 (2000), 759-774.

XIAOHUA ZHU, DEPARTMENT OF MATHEMATICS, PEKING UNIVERSITY, BEIJING, 100871, CHINA, xhzhu@math.pku.edu.cn