GENERALIZED NOTIONS OF AMENABILITY FOR A CLASS OF MATRIX ALGEBRAS

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Abstract. We investigate the notions of amenability and its related homological notions for a class of $I \times I$-upper triangular matrix algebra, say $UP(I, A)$, where $A$ is a Banach algebra equipped with a non-zero character. We show that $UP(I, A)$ is pseudo-contractible (amenable) if and only if $I$ is singleton and $A$ is pseudo-contractible (amenable), respectively. We also study the notions of pseudo-amenability and approximate biprojectivity of $UP(I, A)$.

1. Introduction and Preliminaries

B. E. Johnson studied the class of amenable Banach algebras. Indeed a Banach algebra $A$ is amenable if every continuous derivation $D : A \to X^*$ is inner, for every Banach $A$-bimodule $X$, that is, there exists $x_0 \in X^*$ such that
\[ D(a) = a \cdot x_0 - x_0 \cdot a \quad (a \in A). \]
He also showed that $A$ is amenable if and only if there exists a bounded net $(m_\alpha)$ in $A \otimes_p A$ such that
\[ a \cdot m_\alpha - m_\alpha \cdot a \to 0, \quad \pi_A(m_\alpha) a \to a \quad (a \in A), \]
where $\pi_A : A \otimes_p A \to A$ is given by $\pi_A(a \otimes b) = ab$ for every $a, b \in A$, see [16]. About the same time A. Ya. Helemskii defined the homological notions of biflatness and biprojectivity for Banach algebras. In fact a Banach algebra $A$ is called biflat (biprojective), if there exists a bounded $A$-bimodule morphism $\rho : A \to (A \otimes_p A)^{**}$ ($\rho : A \to A \otimes_p A$) such that $\pi_A^{**} \circ \rho$ is the canonical embedding of $A$ into $A^{**}$ ($\rho$ is a right inverse for $\pi_A$), respectively see [13]. Note that a Banach algebra $A$ is amenable if and only if $A$ is biflat and $A$ has a bounded approximate identity. It is known that for a locally compact group $G$, $L^1(G)$ is biflat (biprojective) if and only if $G$ is amenable(compact), respectively. Amenability of some matrix algebras studied by Esalamzadeh [9] and also biflatness and biprojectivity of some semigroup algebras related to matrix algebras investigated by Ramsden in [19].

Recently approximate versions of amenability and homological properties of Banach algebras have been under more observations. In [24] Zhang introduced the notion of approximately biprojective Banach algebras, that is, $A$ is approximately biprojective if there exists a net of $A$-bimodule morphism $\rho_\alpha : A \to A \otimes_p A$ such that
\[ \pi_A \circ \rho_\alpha (a) \to a \quad (a \in A). \]

Author with A. Pourabbas investigated approximate biprojectivity of $2 \times 2$ upper triangular Banach algebra which is a matrix algebra, also we characterized approximate biprojectivity of Segal algebras and weighted group algebras. We show that a Segal algebra $S(G)$ is approximately biprojective if and only if

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$G$ is compact and also we show that $L^1(G, w)$ is approximately biprojective if and only if $G$ is compact, provided that $w \geq 1$ is a continuous weight function, see [21] and [23].

Approximate amenable Banach algebras have been introduced by Ghahramani and Loy. Indeed a Banach algebra $A$ is approximate amenable if for every continuous derivation $D : A \to X^*$, there exists a net $(x_\alpha)$ in $X^*$ such that

$$D(a) = \lim_{\alpha} a \cdot x_\alpha - x_\alpha \cdot a \quad (a \in A).$$

Other extensions of amenability are pseudo-amenability and pseudo-contractibility. A Banach algebra $A$ is pseudo-amenable (pseudo-contractible) if there exists a not necessarily bounded net $(m_\alpha)$ in $A \otimes_p A$ such that

$$a \cdot m_\alpha - m_\alpha \cdot a \to 0, \quad (a \cdot m_\alpha = m_\alpha \cdot a), \quad \pi_A(m_\alpha)a \to a \quad (a \in A).$$

For more information about these new concepts the reader referred to [12], [10] and [11]. Recently in [7] and [8] pseudo-amenability and pseudo-contractibility of certain semigroup algebras, using the properties of matrix algebras, have been studied.

In this paper, we investigate amenability and its related homological notions for a class of matrix algebras. We show that for a Banach algebra $A$ with a non-zero character, $I \times I$ upper triangular Banach algebra $UP(I, A)$ is pseudo-contractible (amenable) if and only if $I$ is singleton and $A$ is pseudo-contractible (amenable), respectively. Also we characterize whether $UP(I, A)$ is approximate amenable, pseudo-amenable and approximate biprojective. The paper concluded by studying amenability and approximate biprojectivity of some semigroup algebras related to a matrix algebra.

We remark some standard notations and definitions that we shall need in this paper. Let $A$ be a Banach algebra. Throughout this paper the character space of $A$ is denoted by $\Delta(A)$, that is, all non-zero multiplicative linear functionals on $A$. Let $A$ be a Banach algebra. The projective tensor product $A \otimes_p A$ is a Banach $A$-bimodule via the following actions

$$a \cdot (b \otimes c) = ab \otimes c, \quad (b \otimes c) \cdot a = b \otimes ca \quad (a, b, c \in A).$$

Let $A$ be a Banach algebra and $I$ be a non-empty set. $UP(I, A)$ is denoted for the set of all $I \times I$ upper triangular matrices which entries come from $A$ and

$$||(a_{i,j})_{i,j \in I}|| = \sum_{i,j \in I} ||a_{i,j}|| < \infty.$$ 

With the usual matrix operations and $|| \cdot ||$ as a norm, $UP(I, A)$ becomes a Banach algebra.

2. A CLASS OF MATRIX ALGEBRAS AND GENERALIZED NOTIONS OF AMENABILITY

In this section we investigate generalized notions of amenability for upper triangular Banach algebras.

We remind that a Banach algebra $A$ with $\phi \in \Delta(A)$ is called left(right) $\phi$-contractible, if there exists $m \in A$ such that $am = \phi(a)m (ma = \phi(a)m)$ and $\phi(m) = 1$ for every $a \in A$, respectively. For more information the reader referred to [18].

**Theorem 2.1.** Let $I$ be a non-empty set and $A$ be a unital Banach algebra with $\Delta(A) \neq \emptyset$. $UP(I, A)$ is pseudo-contractible if and only if $I$ is singleton and $A$ is pseudo-contractible.
Proof. Let $UP(I,A)$ be pseudo-contractible. Then $UP(I,A)$ has a central approximate identity, say $(e_{\alpha})$. Put $F_{i,j}$ for a matrix belongs to $UP(I,A)$ which $(i,j)$-th entry is $e_{\alpha}$ and others are zero, where $e_{\alpha}$ is an identity of $A$. Thus $F_{i,j}e_{\alpha} = e_{\alpha}F_{i,j}$ for every $i,j \in I$. This equation implies that the entries on main diagonal of $e_{\alpha}$ is equal. Suppose conversely that $I$ is infinite. Since the entries on main diagonal of $e_{\alpha}$ are equal, it implies that $\|e_{\alpha}\| = \infty$ or the main diagonal of $e_{\alpha}$ is zero. In the case $\|e_{\alpha}\| = \infty$, $e_{\alpha}$ does not belong to $UP(I,A)$ which is impossible. Otherwise if the main diagonal of $e_{\alpha}$ is zero, then $e_{\alpha}F_{i,i} = 0$. Thus $0 = e_{\alpha}F_{i,i} \rightarrow F_{i,i}$ which is impossible, hence $I$ must be finite. Suppose that $I = \{i_{1},i_{2},...,i_{n}\}$ and $\phi \in \Delta(A)$. Define $\psi \in \Delta(UP(I,A))$ by $\psi((a_{i,j}),i,j \in I) = \phi(a_{i_{1},i_{n}})$ for every $(a_{i,j}) \in UP(I,A)$. Since $UP(I,A)$ is pseudo-contractible, by [2, Theorem 1.1] $UP(I,A)$ is left and right $\psi$-contractible. Set

$$J = \{(a_{i,j}) \in UP(I,A)|a_{i,j} = 0, \text{for all } j \neq i_{n}\}.$$ 

It is clear $J$ is a closed ideal of $UP(I,A)$ and $\psi|_{J} \neq 0$, hence by [13, Proposition 3.8] $J$ is left and right $\psi$-contractible. So there exist $m_{1},m_{2} \in J$ such that $jm_{1} = \psi(j)m_{1}$ and $m_{2}j = \psi(j)m_{2}$ and also $\psi(m_{1}) = \psi(m_{2}) = 1$ for each $j \in J$. Set $m = m_{1}m_{2} \in J$. Clearly we have

$$jm = mj = \psi(j)m, \quad \psi(m) = \psi(m_{1}m_{2}) = \psi(m_{1})\psi(m_{2}) = 1, \quad (j \in J).$$

(2.1) Suppose conversely that $|I| > 1$. Set $m$ for the matrix with $n$-th columns $(x_{1},x_{2},...,x_{n})^{t}$, where $x_{i} \in A$ for each $i \in \{1,2,...,n\}$. Let $a$ be an element of $J$ which its $n$-th columns has the form $(0,0,...,a_{n})^{t}$ for an arbitrary element $a_{n} \in A$. Applying (2.1) we have

$$x_{1}a_{n} = x_{2}a_{n} = ... = x_{n-1}a_{n} = 0, \quad \phi(a_{n})x_{1} = \phi(a_{n})x_{2} = ... = \phi(a_{n})x_{n-1} = 0,$$

and also

$$a_{n}x_{n} = x_{n}a_{n} = \phi(a_{n})x_{n}, \quad \phi(x_{n}) = 1.$$ 

Pick an element $a_{n} \in A$ such that $\phi(a_{n}) = 0$. Applying (2.1) follows that $x_{1} = x_{2} = ... = x_{n-1} = 0$. Then $m$ becomes a matrix which its $n$-th columns has the form $(0,0,...,0,x_{n})^{t}$. Set $b$ for a matrix in $J$ which its $n$-th columns has the form $(b_{1},b_{2},...,b_{n-1},b_{n})^{t}$, where $b_{n} \in \ker \phi$ and $\phi(b_{1}) = \phi(b_{2}) = ... = \phi(b_{n-1}) = 1$. Applying (2.1) we have $a_{1}x_{n} = 0$. Taking $\phi$ on this equation we have $0 = \phi(a_{1}x_{n}) = \phi(a_{1})\phi(x_{n}) = 1$ which is a contradiction. Therefore $I$ must be singleton. So $A$ is pseudo-contractible.

Converse is clear. □

Suppose that $A$ is a Banach algebra and $\phi \in \Delta(A)$. $A$ is called (approximately) left $\phi$-amenable if there exists (a not necessarily) bounded net $(m_{\alpha})$ in $A$ such that

$$am_{\alpha} - \phi(a)m_{\alpha} \rightarrow 0 \quad \phi(m_{\alpha}) \rightarrow 1 \quad (a \in A),$$

respectively. Right cases define similarly. For more information about these new concepts of amenability and its related homological notions see [1], [17], [14] and [22].

**Theorem 2.2.** Let $I$ be an ordered set with an smallest element. Also let $A$ be a Banach algebra with a left unit such that $\Delta(A) \neq \emptyset$. $UP(I,A)$ is pseudo-amenable (approximate amenable) if and only if $I$ is singleton and $A$ is pseudo-amenable(approximate amenable), respectively.
Proof. Here we proof the pseudo-amenable case, approximate amenability is similar. Suppose that $UP(I, A)$ is pseudo-amenable. Then there exists a net $(\alpha_n)$ in $UP(I, A) \otimes_p UP(I, A)$ such that

$$a \cdot m_\alpha - m_\alpha \cdot a \to 0, \quad \pi_{UP(I,A)}(m_\alpha)a \to a \quad (a \in UP(I,A)).$$

Let $i_0$ be a smallest element of $I$. It is easy to see that $\psi$ given by $\psi(a) = \phi(a_{i_0,i_0})$ is a character on $UP(I, A)$, for each $a = (a_{i,j}) \in UP(I, A)$. Define

$$T : UP(I, A) \otimes_p UP(I, A) \to UP(I, A)$$

by $T(a \otimes b) = \psi(a)b$ for each $a, b \in UP(I, A)$. It is easy to see that $T$ is a bounded linear map which satisfies the following:

$$T(a \cdot x) = \psi(a)T(x), \quad T(x \cdot a) = T(x)a, \quad \psi \circ T(x) = \psi \circ \pi_{UP(I,A)}(x),$$

for each $a, b \in UP(I, A)$ and $x \in UP(I, A) \otimes_p UP(I, A)$. Thus we have

$$\psi(a)T(m_\alpha) - T(m_\alpha)a = T(a \cdot m_\alpha - m_\alpha \cdot a) \to 0$$

and $\psi \circ T(m_\alpha) = \psi \circ \pi_{UP(I,A)}(m_\alpha) \to 1$. Hence $UP(I, A)$ is approximately right $\psi$-amenable. Using the same arguments as in the proof of Theorem 2.1 and applying [20, Proposition 5.1] one can see that $I$ is singleton and $A$ is pseudo-amenable.

Converse is clear. \qed

Let $A$ be a Banach algebra and $a \in A$. By $a_{i,j}$ we mean a matrix belongs to $UP(I, A)$ with $(i,j)$-th place is $a$ and zero elsewhere.

**Theorem 2.3.** Let $I$ be non-empty set and $A$ be a Banach algebra such that $\Delta(A) \neq \emptyset$. $UP(I, A)$ is amenable if and only if $I$ is singleton and $A$ amenable.

**Proof.** Let $UP(I, A)$ be amenable. Then $UP(I, A)$ has a bounded approximate identity, say $(E_\alpha)$. Let $M > 0$ be a bound for $(E_\alpha)$. We claim that $A$ has a bounded left approximate identity. To see this, fix $k, l \in I$. Then for each $a \in A$, we have

$$0 = \lim_\alpha \|E_\alpha a_{i,k,l} - a\| = \lim_\alpha \|\sum_{i,j} E_\alpha^{i,j}a_{i,j} - a\| = \lim_\alpha \|\sum_i E_i a_{i,l} - a\| + \|E_k a - a\|.\quad(2.2)$$

Thus $e_\alpha = E_k^{i,j}$ is a left approximate identity of $A$. It is easy to see that $\|e_\alpha\| \leq \|E_\alpha\| \leq M$. So $(e_\alpha)$ is a bounded left approximate identity for $A$. We claim that $I$ is finite. Suppose conversely that $I$ is infinite. Pick $a \in A$ such that $\|a\| = 1$. Since $(e_\alpha)$ is a bounded left approximate identity for $A$, then $\lim_\alpha e_\alpha a = a$, for each $a \in A$. Thus there exists a $\alpha_{i,k}$ such that $\alpha \geq \alpha_{i,k}$ such that $\frac{1}{\alpha} < \|e_\alpha a\|$. Hence for $\alpha \geq \alpha_{i,k}$ we have

$$\frac{1}{2} < \|e_\alpha a\| \leq \|e_\alpha\| = \|E_\alpha^{i,k}\|.\quad(2.3)$$
Since $I$ is infinite we can choose $N \in \mathbb{N}$ such that $N > 2M$. Then choose distinct $k_1, l_1, k_2, l_2, ..., k_N, l_N$ in $I$ and $\alpha \geq \alpha_{k_i, l_i}, i = 1, 2, ..., N$. Using (2.3) one can see that
\[ M < \frac{1}{2} N = \sum_{i=1}^{N} ||E_{k_i, l_i}^\alpha|| \leq \sum_{i,j \in I} ||E_{i,j}^\alpha|| \leq M, \]
which is a contradiction. So $I$ is finite.

Applying the same method as in the proof of previous Theorem, it is easy to see that $I$ must be singleton, then $A$ is amenable. □

3. A class of Matrix algebra and approximate biprojectivity

In this section we study approximate biprojectivity of some matrix algebra. We also investigate the relation of approximate biprojectivity and discreteness of maximal ideal space of a Banach algebra.

**Theorem 3.1.** Let $I$ be an ordered set with an smallest element. Also let $A$ be a Banach algebra with a right identity such that $\Delta(A) \neq \emptyset$. $UP(I, A)$ is approximately biprojective if and only if $I$ is singleton and $A$ is approximately biprojective.

**Proof.** Let $i_0$ be smallest element of $I$. Define $\psi \in \Delta(UP(I, A))$ by $\psi(a) = \phi(a_{i_0, i_0})$, where $a = (a_{i,j}) \in UP(I, A)$. Suppose that $UP(I, A)$ is approximately biprojective. Since $A$ has a right identity, by [20, Lemma 5.2], $UP(I, A)$ has a right approximate identity. Applying [23, Theorem 3.9], $UP(I, A)$ is right $\psi$-contractible. Using the same arguments as in the proof of the Theorem [21], $I$ is singleton and $A$ is approximately biprojective.

Converse is clear. □

**Remark 3.2.** Let $A$ be a Banach algebra with a left approximate identity and $I$ be a finite set which has at least two elements. Then $UP(I, A)$ is never approximately biprojective. To see this, since $I = \{i_1, i_2, ..., i_n\}$ is finite then left approximate identity of $A$ gives a left approximate identity for $UP(I, A)$. Define $\psi \in \Delta(UP(I, A))$ by $\psi(a) = \phi(a_{i_n, i_n})$ for every $a = (a_{i,j}) \in UP(I, A)$. By [23, Theorem 3.9] approximate biprojectivity of $UP(I, A)$ implies that $UP(I, A)$ is left $\psi$-contractible, then the rest is similar to the proof of Theorem [21].

**Proposition 3.3.** Let $A$ be a Banach algebra with a left approximate identity and $\Delta(A)$ be a non-empty set. If $A$ is approximately biprojective, then $\Delta(A)$ is discrete with respect to the $w^*\text{-topology.}$

**Proof.** Since $A$ is an approximately biprojective Banach algebra with a left approximate identity, by [23, Theorem 3.9] $A$ is left $\phi$-contractible for every $\phi \in \Delta(A)$. Applying [4, Corollary 2.2] one can see that $\Delta(A)$ is discrete. □

**Corollary 3.4.** Let $A$ be a Banach algebra with a left identity, $\phi \in \Delta(A)$ and let $I$ be a non-empty set. If $UP(I, A)$ is approximate biprojective, then $\Delta(UP(I, A))$ is discrete with respect to the $w^*\text{-topology.}$

**Proof.** Note that, since $\phi \in \Delta(A)$, $\Delta(UP(I, A))$ is a non-empty set. Existence of left identity for $A$ implies that $UP(I, A)$ has a left approximate identity, see [20, Lemma 5.2]. Applying previous Proposition one can see that $\Delta(UP(I, A))$ is discrete with respect to the $w^*\text{-topology.}$ □
Let $A$ be a Banach algebra and $\phi \in \Delta(A)$. $A$ is $\phi$-inner amenable if there exists a bounded net $(a_\alpha)$ in $A$ such that

$$aa_\alpha - a_\alpha a \to 0, \quad \phi(a_\alpha) \to 1 \quad (a \in A).$$

For more information about $\phi$-inner amenability, see [15].

**Lemma 3.5.** Let $A$ be a Banach algebra and $\phi \in \Delta(A)$. Suppose that $A$ has an approximate identity. Then approximate biprojectivity of $A$ implies that $A$ is $\phi$-inner amenable.

**Proof.** Suppose that $A$ is approximate biprojective. Using [23, Theorem 3.9], existence of approximate identity implies that $A$ is left and right $\phi$-contractible. Then there exist $m_1$ and $m_2$ in $A$ such that

$$am_1 = \phi(a)m_1(m_2a = \phi(a)m_2) \quad \phi(m_1) = \phi(m_2) = 1 \quad (a \in A),$$

respectively. Since

$$m_1 = \phi(m_2)m_1 = m_2m_1 = \phi(m_1)m_2 = m_2,$$

one can see that

$$am_1 = m_1a = \phi(a)m_1 \quad \phi(m_1) = 1, (a \in A).$$

It follows that $A$ is $\phi$-inner amenable. \qed

**Remark 3.6.** There exists a matrix algebra which is approximate biprojective but it is not $\phi$-inner amenable. Then the converse of previous Lemma is not always true.

To see this, let $A = \begin{pmatrix} 0 & \mathbb{C} \\ 0 & \mathbb{C} \end{pmatrix}$ and also let $a_0 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$. Define $\rho : A \to A \otimes_p A$ by $\rho(a) = a \otimes a_0$ for every $a \in A$. It is easy to see that $\rho$ is a bounded $A$-bimodule morphism and

$$\pi_A \circ \rho(a) = a, \quad (a \in A).$$

Then $A$ is biprojective and it follows that $A$ is approximate biprojective. Set $\phi\left(\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}\right) = b$ for every $a, b \in \mathbb{C}$. It is easy to see that $\phi \in \Delta(A)$. We claim that $A$ is not $\phi$-inner amenable. We suppose conversely that $A$ is $\phi$-inner amenable. Then there exists a bounded net $(a_\alpha)$ in $A$ such that

$$aa_\alpha - a_\alpha a \to 0, \quad \phi(a_\alpha) \to 1 \quad (a \in A).$$

It is easy to see that $ab = \phi(b)a$ for every $a \in A$. Hence we have

$$0 = \lim_\alpha a_0a_\alpha - a_\alpha a_0 = \lim_\alpha \phi(a_\alpha)a_0 - \phi(a_0)a_\alpha = \lim_\alpha a_0 - a_\alpha,$$

It follows that $a_0 = \lim_\alpha a_\alpha$. Hence for each $a \in A$, we have

$$aa_0 = a_0a, \quad \phi(a_0) = 1.$$

It follows that $a = \phi(a)a_0$. Thus $\dim A = 1$ which is a contradiction.
4. Examples of semigroup algebras related to the matrix algebras

Example 4.1. Suppose that $A$ is a Banach algebra and $I$ is a non-empty set. Put $B = UP(I, A)$. It is obvious that $B$ with matrix multiplication can be observed as a semigroup. Equip this semigroup with the discrete topology and denote it with $S_B$. Suppose that $A$ has a non-zero idempotent. We claim that $\ell^1(S_B)$ is not amenable, whenever $I$ is an infinite set. Suppose conversely that $\ell^1(S_B)$ is amenable. Let $e$ be an idempotent for $A$. $E_{i,i}$ for a matrix belongs to $B$ which its $(i,i)$-th entry is $e$, otherwise is $0$. It is easy to see that $E_{i,i}$ is an idempotent for the semigroup $S_B$, for every $i \in I$. So the set of idempotents of $S_B$ is infinite, whenever $I$ is infinite. Thus by [5, Theorem 2] $\ell^1(S_B)$ is not amenable which is contradiction.

Suppose that $A$ is a Banach algebra with a left identity, also suppose that $I$ is an ordered set with smallest element. We also claim that $\ell^1(S_B)$ is never approximate biprojective. To see this suppose conversely that $\ell^1(S_B)$ is approximately biprojective. We denote augmentation character on $\ell^1(S_B)$ by $\phi_{S_B}$. It is easy to see that $\delta_{i,j} \in S_B$ and $\phi_{S_B}(\delta_{i,j}) = 1$, where $\hat{0}$ is denoted for the zero matrix belongs to $S_B$. One can see that the center of $S_B$, say $Z(S_B)$, is non-empty, because $\hat{0}$ belongs to $Z(S_B)$. Using [23, Proposition 3.1(ii)], one can see that $\ell^1(S_B)$ is left $\phi_{S_B}$-contractible. Let $i_0$ be an smallest element of $I$. Define

$$J = \{(a_{i,j}) \in S_B | a_{i,j} = 0, \text{ for all } i \neq i_0\},$$

it is easy to see that $J$ is an ideal of $S_B$, then by [5, page 50] $\ell^1(J)$ is a closed ideal of $\ell^1(S_B)$. Since $\phi_{S_B}|_{\ell^1(J)}$ is non-zero, $\ell^1(J)$ is left $\phi_{S_B}$-contractible. Thus there exists $m \in \ell^1(J)$ such that $am = \phi_{S_B}(a)m$ and $\phi_{S_B}(m) = 1$, for every $a \in A$. On the other hand since $A$ has a left identity, then $J$ has a left identity. Thus by the same argument as in the proof of [23, Proposition 3.1(ii)] we have

$$m(j) = m(e_{i,j}) = \delta_{j}m(e_{i}) = \phi_{S_B}(\delta_{j})m(e_{i}) = m(e_{i}) \quad (j \in J),$$

where $e_i$ is a left unit for $J$. It follows that $m$ is a constant function belongs to $\ell^1(J)$. Since $\phi_{S_B}(m) = 1$, then $m \neq 0$ which implies that $J$ is finite which is impossible.

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