A CONJECTURAL PETERSON ISOMORPHISM IN $K$-THEORY

THOMAS LAM, CHANGZHENG LI, LEONARDO C. MIHALCEA, AND MARK SHIMOZONO

Abstract. We state a precise conjectural isomorphism between localizations of the equivariant quantum $K$-theory ring of a flag variety and the equivariant $K$-homology ring of the affine Grassmannian, in particular relating their Schubert bases and structure constants. This generalizes Peterson’s isomorphism in (co)homology. We prove a formula for the Pontryagin structure constants in the $K$-homology ring, and we use it to check our conjecture in few situations.

1. The $K$-Peterson conjecture

The goal of this manuscript is to present a precise conjecture which asserts the coincidence of the Schubert structure constants for the Pontryagin product in $K$-homology of the affine Grassmannian, with those for the quantum $K$-theory of the flag manifold. This is a $K$-theoretic analogue of the celebrated Peterson isomorphism between the homology of the affine Grassmannian and the quantum cohomology of the flag manifold [Pet, LS, LL].

Let $G$ be a simple and simply-connected complex Lie group with chosen Borel subgroup $B$ and maximal torus $T$, Weyl group $W$ and affine Weyl group $W_{af} = W \ltimes Q'$ where $Q'$ denotes the coroot lattice. Let $\Lambda$ denote the weight lattice of $G$, so that the representation ring $R(T)$ of $T$ is given by $R(T) \cong K^*_T(pt) \cong \mathbb{Z}[\Lambda]$. The torus-equivariant $K$-homology $K^*_T(Gr)$ of the affine Grassmannian $Gr = Gr_G$ of $G$ has as basis over $\mathbb{Z}[\Lambda]$ the Schubert classes $O_x$ of structure sheaves of Schubert varieties in $Gr$, where $x$ varies over the set $W_{af} \subseteq W_{af}$ of affine Grassmannian elements.

Conjecture 1. Let $ut_\lambda, vt_\mu, wt_\nu \in W_{af}^-$, and let $\eta \in Q'$. Assume that $\nu = \lambda + \mu$. Then

$$c^{wt_\nu + \eta}_{ut_\lambda, vt_\mu} = N^{\nu, \eta}_{\nu, \nu} \quad \text{in } K^*_T(pt)$$

where $c$’s are the structure constants in $K^*_0(Gr)$ with respect to $O_x$ and $N$’s are the structure constants in $QK_T(G/B)$ with respect to $O^w$.

Conjecture 1 implies that the multiplication in the ring $QK_T(G/B)$ is finite, and thus it is possible to define it over $\mathbb{Z}[q]$ instead of $\mathbb{Z}[[q]]$. On the affine side, it implies that we have the formula $O_{wt_\lambda} : O_{t_\nu} = O_{wt_\lambda + \nu}$ in the $K$-homology ring $K^*_T(Gr)$ endowed with the Pontryagin product. Conjecture 1 can then be alternatively formulated as follows.

Conjecture 2. The $R(T)$-module homomorphism

$$\Psi : K^*_0(Gr)[O^{-1}]_{t_\nu} \longrightarrow QK_T(G/B)[q^{-1}_{t_\nu}]$$

$$O_{wt_\lambda} \cdot O_{t_\nu}^{-1} \longmapsto q_{\lambda - \nu} O^w$$

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is an isomorphism of $R(T)$-algebras.

The remainder of this article makes our conventions precise. We will give the geometric meaning of [LSS, Theorems 5.3 and 5.4] in Theorem 1 and provide a precise combinatorial formula of the aforementioned structure constants $c$’s in Theorem 2. This leads to computational evidence for these conjectures.

Remark 1. Ikeda, Iwao, and Maeno [IIM] have recently shown that the $K$-homology ring $K_0(\text{Gr}_{SL_n})$ is isomorphic to Kirillov-Maeno’s conjectural presentation of the quantum $K$-theory $\mathcal{Q}K(\text{Fl}_n)$ of complete flag manifold $\text{Fl}_n$ after localization. Their approach is via the relativistic Toda lattice, and the behavior of Schubert classes under their isomorphism is also studied.

Remark 2. Braverman and Finkelberg [BF] showed that the coefficients of Givental’s $K$-theoretic $J$-function [Giv] for a flag variety are the equivariant characters of the polynomial functions on a Zastava space, which consists of based quasimaps to the flag variety. Moreover, in each homogeneous degree, the functions on a Zastava space are isomorphic to the functions on a transverse slice of a $G$-stable stratum inside another $G$-stable stratum in the affine Grassmannian. Together with the $K$-theoretic reconstruction theorems [LP, IMT], this provides a conceptual connection between quantum $K$-theory of flag varieties and $K$-homology of affine Grassmannians.

Remark 3. In [HL, Corollary 5.10], it is shown that the $K$-homology Schubert structure constants determine the 3-point $K$-theoretic Gromov-Witten invariants of a cominuscule flag variety $G/P$. However, a direct formula relating the two sets of invariants is not given.

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2. Quantum $K$-theory of flag varieties

Let $G$ be a complex, simple, simply-connected Lie group and $B,B^- \subset G$ is a pair of opposite Borel subgroups containing the fixed torus $T := B \cap B^-$. For each element $w \in W$ in the (finite) Weyl group there are the Schubert cells $X(w)^o := BwB/B$, $Y(w) := B^-wB/B$ and the Schubert varieties $X(w) := BwB/B$ and $Y(w) := B^-wB/B$ in the flag manifold $X := G/B$. Then $\dim_{\mathbb{C}} X(w) = \text{codim}_{\mathbb{C}} Y(w) = \ell(w)$ (the length of $w$). The boundary of the Schubert varieties is defined by $\partial X(w) = X(w) \setminus X(w)^o$ and $\partial Y(w) := Y(w) \setminus Y(w)^o$. The boundary is generally a reduced, Cohen-Macaulay, codimension 1 subscheme of the corresponding Schubert variety.

We briefly recall the relevant definitions regarding the equivariant $K$-theory ring, following e.g. [CG]. For any (complex) projective variety $Z$ with an algebraic action of a torus $T$, one can define the equivariant $K$-theory ring $K^T(Z)$. This is the ring generated by symbols $[E]_T$ of $T$-equivariant vector bundles $E \to Z$ subject to the relations $[F]_T +$
There is a pairing $\langle \cdot, \cdot \rangle : K^T(Z) \otimes K^T(Z) \to K^T(pt) = R(T)$, where $R(T)$ is the representation ring of $T$, given by

$$\langle [E]_T, [F]_T \rangle = \chi_T(Z; E \otimes F) = \sum_{i=0}^{\dim Z} (-1)^i \text{ch}_T(H^i(Z; E \otimes F));$$

here $\chi_T$ denotes the equivariant Euler characteristic and $\text{ch}_T \in R(T)$ denotes the character of a $T$-module. If in addition $Z$ is smooth then any $T$-equivariant coherent sheaf $F$ on $Z$ has a finite resolution by equivariant vector bundles, and thus there is a well defined class $[F]_T \in K^T(Z)$. This identifies the Grothendieck group $K_T(Z)$ of equivariant coherent sheaves with $K^T(Z)$. For any $T$-equivariant map of projective varieties $f : Z_1 \to Z_2$, there is a well defined push-forward $f_* : K_T(Z_1) \to K_T(Z_2)$ given by $f_*[F]_T = \sum_{i \geq 0} (-1)^i [R^if_*F]_T$; in this language the pairing above is given by $\langle [E]_T, [F]_T \rangle = \pi_*(\langle [E]_T, [F]_T \rangle)$ where $\pi : Z \to pt$ is the structure map.

The maximal torus $T$ acts on $X = G/B$ by left multiplication and the Schubert varieties $X(w), Y(w)$ are $T$-stable. Then the structure sheaves of the Schubert varieties determine the Grothendieck classes $O_w := [O_X(w)]_T$ and $O_w^* := [O_Y(w)]_T$ in the $T$-equivariant $K$-theory ring $K_T(X)$. We will also need the ideal sheaf classes $\xi_w := [O_X(w)(-\partial X(w))]_T$ and $\xi_w^* := [O_Y(w)(-\partial Y(w))]_T$ determined by the boundaries of the corresponding Schubert varieties. The ideal sheaf classes are duals of the Schubert classes:

$$\langle O_w, \xi_v^* \rangle = \langle O_w, \xi_v \rangle = \delta_{w,v},$$

where $\delta$ is the Kronecker delta symbol. We refer to [Bri05 §3.3] or [AGM] for the proofs of this.

Motivated by the relation between quantum cohomology and Toda lattice discovered by Givental and Kim [GK, Kim], Givental and Lee [Giv, Lee] defined a ring which deforms both the (equivariant) $K$-theory and the quantum cohomology rings for the flag manifold $X$. This is the equivariant quantum $K$-theory ring $QK_T(X)$. Additively, $QK_T(X)$ is the free module over the power series ring $K_T(pt)[[q]] = R(T)[[q_1, \ldots, q_r]]$ which has a $R(T)[[q]]$-basis given by Schubert classes $O_w$ (or $O^*_w$) as $w$ varies in $W$.

Here $r$ denotes the rank of $H_2(X)$, which for $X = G/B$ is the same as the number of simple reflections $s_i \in W$. The multiplication is determined by Laurent polynomials $N_{u,v}^{w,d} \in R(T)$ such that

$$(1) \quad O_u \ast O^v = \sum_{w \in W} N_{u,v}^{w,d} q^d O_w$$

where the sum is over effective degrees $d = \sum_{i=1}^{r} m_i [X(s_i)] \in H_2(X)_{\geq 0}$ (i.e. each $m_i \geq 0$), and $q^d = \prod_{i=1}^{r} q_i^{d_i}$. The precise definition of $N_{u,v}^{w,d}$ requires taking Euler characteristic of certain $K$-theory classes on the Kontsevich moduli space of stable maps $\overline{\mathcal{M}}_{0,3}(X, d)$ and over some of its boundary components $\overline{\mathcal{M}}_{0,3}(X) :$

$$N_{u,v}^{w,d} = \sum_{i=1}^{d} (-1)^i \chi_{\overline{\mathcal{M}}_{0,3}(X)}(\text{ev}_1^*(O^u) \cdot \text{ev}_2^*(O^v) \cdot \text{ev}_3^*(\xi_w)).$$
(This is unlike the case of quantum cohomology, where boundary components do not contribute to the structure constants.) Because boundary strata are fiber products of moduli spaces with 2 or 3 marked points, a standard calculation (see e.g. [BM11, §5]) shows that:

\[ N_{u,v}^{w,d} = \sum (O^u, O^v, \xi_{\sigma_0})_{d_0} \cdot (O^{\sigma_0}, \xi_{\sigma_1})_{d_1} \cdot \ldots \cdot (O^{d_{j-1}}, \xi_{\sigma_{j-1}})_{d_{j-1}} \cdot (O^{\sigma_{j-1}}, \xi_w)_{d_j}, \]

where the sum is over \( \sigma_0, \ldots, \sigma_j \in W \) and multidegrees \( d = (d_0, \ldots, d_j) \) such that \( d_i \in H_2(X) \) are effective and \( d_i \neq 0 \) for \( i > 0 \). The notation \( (O^u, O^v, \xi_u)_{d} \) stands for the (equivariant) 3-point K-theoretic Gromov-Witten (KGW) invariant

\[ (O^u, O^v, \xi_u)_{d} = \chi_M(X_d)(ev_1^*(O^u) \cdot ev_2^*(O^v) \cdot ev_3^*(\xi_u)), \]

where \( ev_i : M_{0,3}(X, d) \to X \) are the evaluation maps. If one integrates over \( M_{0,2}(X, d) \), it gives the 2-point invariants \( (O^u, \xi_v)_{d} \). Geometrical properties of the evaluation maps studied in [BCMP13, §3] imply that the 2-point KGW invariants \( (O^u, \xi_v)_{d} \) are always 0 or 1. Formulas for these invariants, using combinatorially explicit recursions to calculate curve neighborhoods of Schubert varieties, can be found in [BM15, Rmk. 7.5].

If one declares \( \deg q_i = c_1(T_X) \cap [X(s_i)] = 2 \) then the quantum \( K \)-theory algebra \( QK_T(X) \) has a filtration by degrees, and its associated graded algebra is naturally isomorphic to the quantum cohomology algebra. Because KGW invariants are non-zero for infinitely many degrees (e.g. \( (O^{\ast d}, O^{\ast d}, O^{\ast d})_{d} \) is the trivial 1-dimensional \( T \)-representation for any degree \( d \)), it is unclear whether the expansion of the product \( O^u \ast O^v \in QK_T(X) \) has finitely many terms. This was conjectured to be true for any flag manifold \( G/P \) by Buch, Chaput, Mihalcea and Perrin. The conjecture is true in the case of cominuscule Grassmannians [BCMP13] and for partial flag manifolds \( G/P \) with \( P \) a maximal parabolic group [BCMP16].

While there is no algorithm to calculate the structure constants \( N_{u,v}^{w,d} \) for \( X = G/B \) or for arbitrary flag manifolds \( G/P \), there are several particular instances where algorithms are available. In the case of a cominuscule Grassmannian, a “quantum = classical” statement, calculating KGW invariants in terms of certain K-theoretic intersection numbers on two-step flag manifolds was obtained in [BM11]; in Lie types different from type A, this uses rationality results from [CP11]. As a result, a Chevalley formula, which calculates the multiplication by a divisor class, was obtained in [BM11] for the type A Grassmannians, and it was recently extended in [BCMP16+] to all cominuscule Grassmannians. In the equivariant context, this formula determines an algorithm to calculate any product of Schubert classes, generalizing the result from quantum cohomology [Mih]. Formulas to calculate \( N_{u,v}^{w,d} \) for \( X = G/B \) and \( d \) a “line class”, i.e. \( d = [X(s_i)] \), were obtained in [LM] by making use of the geometry of lines on flag manifolds. In this case they also proved that \( N_{u,v}^{w,d} \) satisfy the same positivity property as the one proved by Anderson, Griffeth and Miller [AGM] for the structure constants in the (ordinary) equivariant \( K \)-theory of \( X \). There are also algorithms based on reconstruction formula [LP] [INT] which in principle can be used to calculate KGW invariants. In practice, however, these lead to quantities which quickly become unfeasible for explicit calculations.
3. \(K\)-homology of the affine Grassmannian

3.1. \(K\)-groups for thick and thin affine Grassmannians. The foundational reference for the thick affine Grassmannian is \([\text{Kas}]\) and for the thin affine Grassmannian we use \([\text{Kum02}]\) and \([\text{Kum15}]\).

We use notation from Section 1. The (thin) affine Grassmannian \(\text{Gr}\) is a \(\text{ind-finite}\) scheme: it is the union of finite-dimensional projective Schubert varieties \(X_w, \text{ for } w \in W_{\text{af}}\) (in analogy with the Schubert varieties \(X(w)\) for \(G/B\)). The dimension of the complex projective variety \(X_w\) is equal to the length \(\ell(w)\). Let \(K_T^T(\text{Gr})\) be the Grothendieck group of the category of \(T\)-equivariant finitely-supported (that is, supported on some \(X_w\)) coherent sheaves on \(\text{Gr}\). We have

\[
K_T^T(\text{Gr}) \cong \bigoplus_{w \in W_{\text{af}}} R(T) \cdot \mathcal{O}_w
\]

where \(\mathcal{O}_w = [\mathcal{O}_{X_w}]_T\) denotes the class of the structure sheaf of \(X_w\) (cf. \([\text{Kum15}],\) Section 3). We call the \(R(T)\)-module \(K_T^T(\text{Gr})\) the \((\text{equivariant}) K\)-homology of \(\text{Gr}\). We notice that \(\xi^\text{Gr}_w := [\mathcal{O}_{X_w}(-\partial X_w)]_T, \text{ for } w \in W_{\text{af}}, \) form another \(R(T)\)-basis of \(K_T^0(\text{Gr})\), which we simply denote as \(\xi_w\) whenever it is clear from the context.

The thick affine Grassmannian \(\overline{\text{Gr}}\) is an infinite-dimensional non quasicompact scheme: it is a union of finite-codimensional Schubert varieties \(X_w, \text{ for } w \in W_{\text{af}}\), of codimension \(\ell(w)\). Let \(K_T^0(\overline{\text{Gr}})\) be the Grothendieck group of the category of \(T\)-equivariant coherent sheaves on \(\overline{\text{Gr}}\), defined for example in \([\text{LSS},\) Section 3.2\]. We have

\[
K_T^0(\overline{\text{Gr}}) \cong \prod_{w \in W_{\text{af}}} R(T) \cdot \mathcal{O}^w
\]

where \(\mathcal{O}^w = [\mathcal{O}_{X^w}]_T\) denotes the class of the coherent structure sheaf of \(X^w\).

Let \(\overline{\text{Fl}}\) denote the (ind-)affine flag manifold, and \(\overline{\text{Fl}}\) denote the thick version. As for the affine Grassmannian, one defines Schubert varieties \(X_w \subset \overline{\text{Fl}}\) and \(X^w \subset \overline{\text{Fl}}\) such that \(\text{dim } X(w) = \text{codim } X^w = \ell(w)\). In this case \(w\) varies in the affine Weyl group \(W_{\text{af}}\). Let \(\mathcal{O}_w = [\mathcal{O}_{X_w}]_T \in K_T^0(\overline{\text{Fl}})\) and \(\mathcal{O}^w = [\mathcal{O}_{X^w}]_T \in K_T^0(\overline{\text{Fl}})\); we refer to \([\text{KaSh}]\) or \([\text{Kum15}]\) for the (rather delicate) details. There are \(T\)-equivariant projection maps \(\pi : \overline{\text{Fl}} \to \overline{\text{Gr}}\) and (abusing notation) \(\pi : \overline{\text{Fl}} \to \overline{\text{Gr}}\) which are locally trivial \(G/B\)-bundles. In particular they are flat, and

\[
\pi^* \mathcal{O}^w_{\overline{\text{Gr}}} = \mathcal{O}^w_{\overline{\text{Fl}}}
\]

for any \(w \in W_{\text{af}}\). Further, similar arguments to those in the finite case show that for any \(w \in W_{\text{af}}\),

\[
\pi_* \mathcal{O}_w^{\overline{\text{Fl}}} = \mathcal{O}_w^{\overline{\text{Gr}}}_{\pi(w)}
\]

where \(\pi(w)\) denotes the image of \(w\) in \(W_{\text{af}}\) under the projection map. (See e.g. \([\text{BK05},\) Thm. 3.3.4\] for a proof based on Frobenius splitting; or \([\text{BCMP13},\) Prop. 3.2\] for an argument based on a theorem of Kollár.) There is a pairing \(\langle \cdot, \cdot \rangle_{\overline{\text{Fl}}} : K_T^0(\overline{\text{Fl}}) \otimes K_T^0(\overline{\text{Fl}}) \to\)
for any classes $[\mathcal{F}], [\mathcal{G}]$ such $\mathcal{F}$ is a $T$-equivariant sheaf on $\overline{\text{Fl}}$ and $\mathcal{G}$ is a $T$-equivariant sheaf supported on a finite dimensional stratum $(\text{Fl})_n$ of the ind-variety Fl. By [Kum15, Lemma 3.4] this pairing is well defined. In fact, the definition of this pairing extends in an obvious way to any partial flag variety, in particular to the affine Grassmannian $\text{Gr}$. It was proved in [BK, Prop. 3.9] that the pairing satisfies the property $(\mathcal{O}^{u}_{\text{Fl}}, \xi_{\text{Fl}}) = \delta_{u,v}$. We will need the following additional properties of this pairing.

**Lemma 1.** Consider the pairing $\langle \cdot, \cdot \rangle_X$ and take any $u, v \in W_{af}$ in the case when $X = \text{Fl}$ and $u, v \in W_{af}^{-}$ for $X = \text{Gr}$. Then

$$\langle \mathcal{O}^{u}_{\text{Fl}}, \mathcal{O}_{v} \rangle_{X} = \begin{cases} 1 & \text{if } u \leq v; \\ 0 & \text{otherwise.} \end{cases}$$

**Proof.** Consider first $X = \text{Fl}$. By definition we have

$$\langle \mathcal{O}^{u}_{\text{Fl}}, \mathcal{O}_{v} \rangle_{X} = \sum_{i} (-1)^{i} \chi_{T}(X_{v}, \text{Tor}_{i}^{\mathcal{O}_{\text{Gr}}^{T}}(\mathcal{O}_{X^{u}_{v}}, \mathcal{O}_{X_{u}})).$$

According to [Kum15, Lemma 5.5] all Tor sheaves are 0 for $i > 0$, and by definition $\text{Tor}_{i}^{\mathcal{O}_{\text{Gr}}^{T}}(\mathcal{O}_{X^{u}_{v}}, \mathcal{O}_{X_{v}}) = \mathcal{O}_{X_{v}}$ where $X^{u}_{v} := X^{u} \cap X_{v}$ is the Richardson variety (cf. e.g. 12 of loc.cit). According to [KuSc, Cor. 3.3], the higher cohomology groups $H^{i}(X^{u}_{v}, \mathcal{O}_{X^{u}_{v}}) = 0$ for $i > 0$ and since $X^{u}_{v}$ is irreducible $H^{0}(X^{u}_{v}, \mathcal{O}_{X^{u}_{v}}) = \mathbb{C}$. It follows that the sheaf Euler characteristic $\chi_{T}(X^{u}_{v}, \mathcal{O}_{X^{u}_{v}}) = 1$.

We now turn to the situation when $X = \text{Gr}$. Let $u, v \in W_{af}^{-}$. The same argument as before reduces the statement to the calculation of $\chi_{T}(X^{u}_{v}, \mathcal{O}_{X^{u}_{v}})$ where $X^{u}_{v} \subset \text{Gr}$ is the Richardson variety. By definition of Schubert varieties, $\pi^{-1}(X^{u}_{v}) = X^{u}_{v0} \subset \text{Fl}$ where $w_{0} \in W$ is the longest element in the finite Weyl group, and $\pi^{-1}(X^{u}) = X^{u}$. It follows that the preimage of the grassmannian Richardson variety is $\pi^{-1}(X^{u}_{v}) = X^{u}_{v0}$. Then a standard argument based on the Leray spectral sequence (taking into account that the fiber of $\pi : \pi^{-1}(X^{u}_{v}) \to X^{u}_{v}$ is the finite flag manifold $G/B$, and that $H^{i}(G/B, \mathcal{O}_{G/B}) = 0$ for $i > 0$) gives that $H^{i}(X^{u}_{v0}, \mathcal{O}_{X^{u}_{v0}}) = H^{i}(X^{u}_{v}, \mathcal{O}_{X^{u}_{v}})$ for all $i$, thus the required Euler characteristic equals 1, as needed. \[\square\]

**Lemma 2.** For any $u, v \in W_{af}^{-}$, we have

(i) $\langle \mathcal{O}^{u}_{\text{Gr}}, \xi_{v} \rangle_{\text{Gr}} = \delta_{u,v};$

(ii) $\mathcal{O}_{v} = \sum_{w \leq v; w \in W_{af}^{-}} \xi_{w}.$

**Proof.** (i) The statement follows from the same arguments as for Fl in [BK, Prop. 3.9].

To prove (ii), we write $\mathcal{O}_{v} = \sum_{w} a_{w,v} \xi_{w}$, which is a finite sum because the class $\mathcal{O}_{v}$ is supported on a finite-dimensional variety. By statement (i) and Lemma 1,

$$a_{w,v} = \langle \mathcal{O}^{w}_{\text{Gr}}, \mathcal{O}_{v} \rangle_{\text{Gr}} = \begin{cases} 1 & \text{if } w \leq v, \\ 0 & \text{otherwise.} \end{cases}$$

\[\square\]
3.2. *K*-Peterson algebra. The *K*-groups \( K_0^T(\text{Gr}) \) and \( K_0^0(\text{Gr}) \) acquire dual Hopf algebra structures from the homotopy equivalence \( \text{Gr} \cong \Omega K \), where \( K \subset G \) is a maximal compact subgroup and \( \Omega K \) is the group of based loops into \( K \). An algebraic model for these Hopf algebras is constructed in \cite{LSS}. Only the product structure of \( K_0^T(\text{Gr}) \), arising from the Pontryagin product \( \Omega K \times \Omega K \to \Omega K \) will be of concern to us.

We consider a variation of Kostant and Kumar’s *K*-nilHecke ring, the “small torus” affine *K*-nilHecke ring of \cite{LSS}, which was inspired by the homological analogue \cite{Pet}.

The affine Weyl group \( W_{af} \) acts on the weight lattice \( \Lambda \) of \( T \) by the level-zero action (that is, we take the null root \( \delta = 0 \))

\[
wt_\mu \cdot \lambda = w \cdot \lambda \quad \text{for } w \in W, \mu \in Q^+ \text{ and } \lambda \in \Lambda.
\]

Let \( I_{af} \) denote the vertex set of the affine Dynkin diagram. Abusing notation, we denote by \( \{\alpha_i \mid i \in I_{af}\} \) the images of the simple affine roots in \( \Lambda \). In particular, \( \alpha_0 = -\theta \in \Lambda \) where \( \theta \) is the highest root of \( G \). Let \( Q(T) = \text{Frac}(R(T)) \) and equip \( \mathbb{K}_Q = Q(T) \otimes_{R(T)} \mathbb{Q}[W_{af}] \) with product \( (p \otimes v)(q \otimes w) = p(v \cdot q) \otimes vw \) for \( p, q \in Q(T) \) and \( v, w \in W_{af} \).

Define

\[
T_i = (1 - e^{\alpha_i})^{-1}(s_i - 1) \quad \text{for } i \in I_{af}.
\]

The \( T_i \) satisfy \( T_i^2 = -T_i \) and the same braid relations as the \( s_i \) do. Therefore for a reduced expression \( w = s_i s_{i_2} \cdots s_{i_\ell} \in W \) there are well defined elements \( T_w = T_{i_1} T_{i_2} \cdots T_{i_\ell} \). Let \( \mathbb{K} \) be the subring generated by \( T_i \) for \( i \in I_{af} \) and \( R(T) \). We call it the small-torus affine *K*-nilHecke ring.

Let \( \mathbb{L} \subset \mathbb{K} \) be the centralizer of \( R(T) \) in \( \mathbb{K} \); this is called the *K*-Peterson subalgebra. The following theorem clarifies the geometric meaning of \cite{LSS} Theorems 5.3 and 5.4.

Recall that the ideal sheaf basis \( \{\xi_w \mid w \in W_{af}\} \) in \( K_0^T(\text{Gr}) \) are the unique elements characterized by \( \langle \mathcal{O}^w, \xi_w \rangle_{\text{Gr}} = \delta_{vw} \).

**Theorem 1.** There is an isomorphism of \( R(T) \)-Hopf algebras \( k : K_0^T(\text{Gr}) \cong \mathbb{L} \) such that for every \( w \in W_{af}^{-} \)

(a) the element \( k_w := k(\xi_w) \) is the unique element in \( \mathbb{L} \) of the form

\[
k_w = T_w + \sum_{x \in W_{af} \setminus W_{af}^{-}} k_x T_x
\]

where \( k_x \in R(T) \), and

(b) the element \( l_w := k(\mathcal{O}_w) \) is given by

\[
l_w = \sum_{v \leq w} k_v.
\]

**Proof.** In \cite{LSS}, a *K*-homology Hopf algebra \( K_0^T(\text{Gr}) \) was constructed as a Hopf dual to \( K_0^0(\text{Gr}) \). In \cite{LSS} Theorem 5.3, an isomorphism \( K_0^T(\text{Gr}) \cong \mathbb{L} \) is constructed, and the \( R(T) \)-bilinear pairing \( \langle \cdot, \cdot \rangle_{\mathbb{L}} : K_0^0(\text{Gr}) \times \mathbb{L} \) is given by

\[
\langle \mathcal{O}^w, a \rangle_{\mathbb{L}} = a_w,
\]

where \( w \in W_{af}^{-} \) and \( a = \sum_{v \in W_{af}} a_v T_v \in \mathbb{L} \subset \mathbb{K} \) with \( a_v \in R(T) \); see \cite{LSS} §2.4, especially equation (2.10). The uniqueness of the elements \( k_w \) given by \cite{S} is \cite{LSS} Theorem 5.4.
We now identify the $L$ with $K_T^G(Gr)$ via (6) and (10). It follows from [LSS, Theorem 5.4] that under the resulting isomorphism $k : K_T^G(Gr) \cong L$, we have $k(\xi_w) = k_w$. Statement (b) follows immediately from Lemma 2. □

3.3. Closed formula for structure constants. For $x, y, z \in W_{af}$, define the structure constants $c_{x,y}^z$ by

$$O_x \cdot O_y = \sum_{z \in W_{af}} c_{x,y}^z O_z$$

with the product structure given by the isomorphism of Theorem 1. We now give a closed formula for $c_{x,y}^z$ in terms of equivariant localizations.

Define the elements $y_i = 1 + T_i$ for $i \in I_{af}$. Then $y_i^2 = y_i$ and the $y_i$ satisfy the braid relations so that for $w \in W_{af}$ we can define $y_w \in K$. The $\{y_w \mid w \in W_{af}\}$ form a $R(T)$-basis of $K$. For any $q \in Q(T)$, we have $q y_{s_i} = y_{s_i}(s_i q) + \frac{q - s_i q}{1 - e^{-\alpha_i}} y_{id}$. Define $b_{w,u} \in Q(T)$ and $e_{w,u} \in Q(T)$ respectively by

$$y_w = \sum_{u \in W_{af}} b_{w,u} u, \quad w = \sum_{u \in W_{af}} e_{w,u} y_u. \quad (11)$$

The matrix $(b_{w,u})$ is invertible, and its inverse is given by $(e_{w,u})$.

**Proposition 1.** Let $u, v \in W_{af}$. Let $u = s_{\beta_1} \cdots s_{\beta_m}$ be a reduced expression of $u$. We have

$$b_{u,v} = \sum_{\epsilon = 0}^{m} \prod_{k=1}^m \left( s_{\beta_1}^{\epsilon_1} \cdots s_{\beta_{k-1}}^{\epsilon_{k-1}} \frac{(-e^{-\beta_k})^{\epsilon_k}}{1 - e^{-\beta_k}} \right)_{\alpha_0 = -\theta}, \quad (12)$$

the summation over all $(\epsilon_1, \cdots, \epsilon_m) \in \{0,1\}^m$ satisfying $s_{\beta_1}^{\epsilon_1} \cdots s_{\beta_m}^{\epsilon_m} = v$.

Denote $\gamma_j = s_{\beta_1} \cdots s_{\beta_{j-1}}(\beta_j)$ for each $j$. Then we have

$$e_{u,v} = \sum_{\epsilon = 0}^{m} \prod_{k=1}^m \left( (1 - e^{\gamma_k}) + \epsilon_k (1 - e^{\gamma_k}) \right)_{\alpha_0 = -\theta}, \quad (13)$$

the summation over all $(\epsilon_1, \cdots, \epsilon_m) \in \{0,1\}^m$ satisfying $y_{s_{\beta_1}}^{\epsilon_1} \cdots y_{s_{\beta_m}}^{\epsilon_m} = y_v$.

In the present work, we work in $Q(T_{af})$, where $T \subset G$ is the finite torus. Our proof below holds in $Q(T_{af})$ where $T_{af}$ denotes the affine torus.

**Proof.** The formula for $b_{u,v}$ follows immediately from the definitions.

The formula for $e_{u,v}$ holds by showing that both sides satisfy the same recursive formulas. Precisely, let $\tilde{e}_{u,v}$ denote the RHS of (13). We shall show that $e_{u,v}$ and $\tilde{e}_{u,v}$

\[^1\text{In the notation of [KK90], they are denoted as } b_{u^{-1},v^{-1}} \text{ and } e^{v^{-1},u^{-1}} \text{ respectively.}\]
satisfy the same recursions. We have

\[ s_i u = (e^{\alpha_i} y_{id} + (1 - e^{\alpha_i}) y_{s_i}) \sum_v e_{u,v} y_v \]

\[ = \sum_v e^{\alpha_i} e_{u,v} y_v + \sum_v (1 - e^{\alpha_i}) y_{s_i} e_{u,v} y_v \]

\[ = \sum_v e^{\alpha_i} e_{u,v} y_v + \sum_v (1 - e^{\alpha_i}) (s_i(e_{u,v}) y_{s_i} - s_i(e_{u,v}) - e_{u,v} y_{id}) y_v \]

\[ = \sum_v e^{\alpha_i} e_{u,v} y_v + \sum_v (1 - e^{\alpha_i}) s_i(e_{u,v}) y_v + \sum_v e^{\alpha_i} (s_i(e_{u,v}) - e_{u,v}) y_v \]

\[ = \sum_{v:s_i u < v} (e^{\alpha_i} e_{u,v} + (1 - e^{\alpha_i}) s_i(e_{u,v}) + (1 - e^{\alpha_i}) s_i(e_{u,s_i v}) + e^{\alpha_i} (s_i(e_{u,v}) - e_{u,v})) y_v \]

\[ + \sum_{v:s_i u > v} (e^{\alpha_i} e_{u,v} + e^{\alpha_i} (s_i(e_{u,v}) - e_{u,v})) y_v \]

\[ = \sum_{v:s_i u < v} (s_i(e_{u,v}) + (1 - e^{\alpha_i}) s_i(e_{u,s_i v})) y_v + \sum_{v:s_i u > v} e^{\alpha_i} (s_i(e_{u,v}) y_v) \]

That is, for \( s_i u < u \), we have

\[ e_{s_i u,v} = \begin{cases} 
  s_i(e_{u,v}) + (1 - e^{\alpha_i}) s_i(e_{u,s_i v}), & \text{if } s_i u < v, \\
  e^{\alpha_i} s_i(e_{u,v}), & \text{if } s_i u > v.
\end{cases} \] (14)

It follows directly from (13) that \( e_{s_i u,v} \) satisfies the same recursive rule. Moreover, \( e_{id,v} = \delta_{id,v} = \tilde{e}_{id,v} \) for any \( v \). Therefore the statement follows. \( \square \)

Consider the left \( Q(T) \)-module homomorphism \( \kappa : Q(T) \otimes_{R(T)} \mathbb{K} \to Q(T) \otimes_{R(T)} \mathbb{L} \) defined by

\[ \kappa(t_\lambda w) = t_\lambda \quad \text{for } w \in \mathbb{W} \text{ and } \lambda \in Q^\vee. \]

**Proposition 2.** The map \( \kappa \) restricts to a \( R(T) \)-module map \( \kappa : \mathbb{K} \to \mathbb{L} \), and

\[ \kappa(T_u) = 0 \quad \text{if } u \in W_{af} \setminus W^-_{af}, \]

\[ \kappa(T_u) = k_u \quad \text{if } u \in W^-_{af}, \]

\[ \kappa(y_u) = l_u \quad \text{if } u \in W^-_{af}. \]

**Proof.** The first claim follows from the three formulas. From the definition, \( \kappa(T_i) = 0 \) for \( i \neq 0 \). It follows easily that \( \kappa(T_u) = 0 \) if \( u \notin W^-_{af} \). By [LSS, (5.1)], the element \( k_u \in \mathbb{L} \) can be characterized as follows. Let \( T_u = \sum_{x \in W_{af}} a_x x \) for \( a_x \in Q(T) \) and \( k_u = \sum_{\lambda \in Q^\vee} a'_{\lambda,\lambda} t_\lambda \) for \( a'_{\lambda,\lambda} \in Q(T) \). Then for any function \( f : W_{af} \to R(T) \) satisfying \( f(x) = f(xv) \) for \( v \in \mathbb{W} \), we have

\[ \sum_{x \in W_{af}} a_x f(x) = \sum_{\lambda \in Q^\vee} a'_{\lambda,\lambda} f(t_\lambda). \]

It follows that \( \kappa(T_u) = k_u \). The last claim follows from the first two formulas, the equality \( y_w = \sum_{v \in W_{af}, v \leq w} T_v \) and (9). \( \square \)
Denote
\[ b_{x,[y]} := \sum_{z \in y W} b_{x,z}, \quad e_{x,[y]} := \sum_{z \in y W} e_{x,z}. \]

**Theorem 2.** For any \( x, y, z \in W^- \), the coefficient \( c^z_{x,y} \) is given by
\[
(15) \quad c^z_{x,y} = \sum_{t_1, t_2 \in Q^V} b_{x,[t_1]} b_{y,[t_2]} c_{t_1 t_2, [z]}.
\]

**Proof.** By Theorem 1 we have \( l_x l_y = \sum_{z \in W^-} c^z_{x,y} l_z \). By Proposition 2 we have
\[
l_x l_y = \sum_{u, v \in W^-} \kappa(b_{x,u} u) \kappa(b_{y,v} v)
= \sum_{t_1, t_2 \in Q^V} \sum_{u, v \in W^-} b_{x,t_1 u} b_{y,t_2 v} \kappa(t_1 u) \kappa(t_2 v)
= \sum_{t_1, t_2 \in Q^V} \sum_{u, v \in W^-} b_{x,t_1 u} b_{y,t_2 v} t_1 t_2
= \sum_{z \in W^-} \sum_{t_1, t_2 \in Q^V} b_{x,[t_1]} b_{y,[t_2]} c_{t_1 t_2, [z]} l_z. \quad \square
\]

### 3.4. Geometric remarks.
We will provide a brief geometric interpretation of the previous approach. There is an \( R(T) \)-module identification \( \mathbb{K} = K_0^T(\text{Fl}) \) and an \( R(T) \)-Hopf algebra identification \( \mathbb{L} = K_0^T(\text{Gr}) \). The classes \( y_w \in \mathbb{K} \) play two roles: on one side \( y_w = O_w \) are the structure (finite dimension) Schubert structure sheaves on the affine flag manifold \( \text{Fl} \); on the other side they act \( \partial_w \) in the affine Grassmannian. In particular, Proposition 2 states that \( \psi_w \in K_0^T(\text{Gr}) \) correspond to the ideal sheaves \( \xi_w \) on \( K_0(\text{Fl}) \), or to the BGG-type operators \( \partial_w - id \). The map \( \kappa : \mathbb{K} \to \mathbb{L} \) is the K-theoretic projection map \( \pi_* : K_0^T(\text{Fl}) \to K_0^T(\text{Gr}) \), and the classes \( k_w \) and \( l_w \) (for \( w \in W^- \)) correspond respectively to the ideal sheaves and Schubert structure sheaves in the affine Grassmannian. In particular, Proposition 2 states that
\[
\pi_*(\xi_{w,\text{Fl}}) = \begin{cases} 
\xi_w & \text{if } w \in W^- \\
0 & \text{otherwise}
\end{cases}; \quad \pi_*(O_{u,\text{Fl}}) = O_u \text{ for } u \in W^-.
\]

It is not difficult to prove these identities directly, using Lemma 3 and identities (4), (5).

For each of \( K_0^T(\text{Fl}) \) and \( K_0^T(\text{Gr}) \), there is a third basis \( \{ \iota_w \} \), indexed respectively by \( W^- \) and \( W^- / W \), called the localization basis. If \( w \in W^- \), then \( \iota_w \in K_0^T(\text{Fl}) \) is the map \( \iota_w : K_0^T(\text{Fl}) \to R(T) \) defined by sending the \( K \)-cohomology class \( O^a \) to its localization to the fixed point \( w \). Then equation (11) above corresponds to expanding the structure sheaf basis into localization basis and vice versa. A key observation from [LL] and [LSS], which is used in the proof of Theorem 2, is that the Pontryagin multiplication on \( K_0^T(\text{Gr}) \) is easy to write in the localization basis: if \( \lambda, \mu \in Q^V \) and \( \iota_{\lambda}, \iota_{\mu} \in K_0^T(\text{Gr}) \) are the corresponding localization elements, then \( \iota_{\lambda} \cdot \iota_{\mu} = \iota_{\lambda+\mu} \); see [LSS] Lemma 5.1.
4. Data and Evidence

As we observed above, the cohomological versions of Conjectures 1 and 2 were proved in [LS]. In the K-theoretic version, we can verify Conjecture 1 when the degree \( d \) in \( N_{\mu,\nu}^{w,d} \) is \( d = 0 \) or \( d = \alpha_i^\vee \) is a simple coroot. Our arguments are similar to those in [LL], but are quite involved, even in these situations. It would be desirable to find more conceptual explanations. Next we provide two computational examples.

4.1. Conjecture is true for \( G = SL_3 \). The complete flag manifold \( SL_3/B \) is the complex projective line \( \mathbb{P}^1 \). The Weyl group \( W = \mathbb{Z}_2 \) is generated by the simple reflection \( s_1 = s_\alpha \) of the unique simple root \( \alpha = \alpha_1 \). The equivariant quantum \( K \)-theory \( QK_T(\mathbb{P}^1) \) has an \( R(T)[q] \)-basis \( \{O^d, O^s_1\} \). As shown in [BMT], the only nontrivial quantum product is given by \(^2\)

\[
O^s_1 \star O^s_1 = (1 - e^{-\alpha})O^s_1 + e^{-\alpha}q. 
\]

On the affine side, we notice that \( s_0 = s_1t_{-\alpha^\vee} \) and that \( W_\af = \{id\} \cup \{wt_{n\alpha^\vee} \mid n \in \mathbb{Z}_{<0}, w = id \text{ or } s_1\} \).

Let \( g_m \) be the unique element of \( W_\af \) of length \( m \) for \( m \geq 0 \). Let \( h_m \) be the unique element of \( W_\af \backslash W_\af^\vee \) of length \( m \) for \( m \geq 1 \). For example, \( g_3 = s_0s_1s_0 \) and \( h_4 = s_0s_1s_0s_1 \). Notice that \( T_i f = s_i(f)T_i + T_i(f) \) and \( T_i^2 = -T_i \) for any \( f \in R(T) \) and \( i \in \{0, 1\} \).

**Lemma 3.** We have \( k_{id} = 1 \). For \( r \geq 1 \), we have

\[
k_{g_{2r-1}} = T_{g_{2r-1}} + T_{h_{2r-1}} + (1 - e^{-\alpha})T_{h_{2r}}, \quad \text{and} \quad k_{g_{2r}} = T_{g_{2r}} + e^{-\alpha}T_{h_{2r}}.
\]

**Proof.** Denote by \( \tilde{k}_{g_m} \) the expected formula. By Theorem [a], it suffices to show \( \tilde{k}_{g_m} \in \mathbb{L} \), or equivalently, \( k_{g_m} e^{-\alpha} = e^{-\alpha} \tilde{k}_{g_m} \). Clearly, this holds when \( m = 0 \). It also holds for \( m \in \{1, 2\} \) by direct calculations. In particular we have

\[
(T_0 + T_1 + (1 - e^{-\alpha})T_{01})e^{\pm\alpha} = e^{\pm\alpha}(T_0 + T_1 + (1 - e^{-\alpha})T_{01});
\]

\[
(T_{10} + e^{-\alpha}T_{01})e^{\pm\alpha} = e^{\pm\alpha}(T_{10} + e^{-\alpha}T_{01}).
\]

Assume that it holds for \( m \leq 2r \) where \( r \geq 1 \). Then we have

\[
\tilde{k}_{g_{2r+1}} e^{-\alpha} = (T_{g_{2r+1}} + T_{h_{2r}} + (1 - e^{-\alpha})T_{01}T_{h_{2r}})e^{-\alpha}
\]

\[
= T_{g_{2r+1}} e^{-\alpha} + (T_1 + (1 - e^{-\alpha})T_{01})e^{\alpha}e^{-\alpha}T_{h_{2r}} e^{-\alpha}
\]

\[
= T_{g_{2r+1}} e^{-\alpha} + (T_0 + T_1 + (1 - e^{-\alpha})T_{01})e^{-\alpha}(T_{g_{2r}} + e^{-\alpha}T_{h_{2r}}) - T_{g_{2r}} e^{-\alpha}
\]

\[
= T_{g_{2r+1}} e^{-\alpha} + (T_0 + T_1 + (1 - e^{-\alpha})T_{01})T_{g_{2r}} e^{-\alpha} - e^{\alpha}(T_{01} + (1 - e^{-\alpha})T_{01})T_{h_{2r}}
\]

\[
- T_0 (T_{g_{2r}} + e^{-\alpha}T_{h_{2r}}) + T_0 e^{\alpha} T_{g_{2r}} e^{-\alpha} - e^{\alpha}(T_{01} + (1 - e^{-\alpha})T_{01})T_{g_{2r}} e^{-\alpha}
\]

\[
= e^{-\alpha} \tilde{k}_{g_{2r+1}} + T_{g_{2r}} + e^{-\alpha}T_{01}T_{h_{2r}} - T_{01} T_{h_{2r}} e^{-\alpha} - e^{\alpha} T_{1} T_{g_{2r}} e^{-\alpha}
\]

\[
= e^{-\alpha} \tilde{k}_{g_{2r+1}} - (T_{g_{2r}} + e^{-\alpha}T_{h_{2r}}) e^{\alpha} e^{-\alpha} + T_{h_{2r}} e^{-\alpha} + e^{\alpha} T_{g_{2r}} e^{-\alpha}
\]

\[
= e^{-\alpha} \tilde{k}_{g_{2r+1}}
\]

Similarly, we can show \( \tilde{k}_{g_{2r+2}} e^{-\alpha} = e^{-\alpha} \tilde{k}_{g_{2r+2}} \). Thus the statement follows. \( \square \)

\(^2\)We use the opposite identification \( e^{\alpha} = -[C_{e_i}] \in R(T) \) compared with [BMT] Section 5.5.
The following result follows from Lemma 3 and Theorem 1(b).

**Lemma 4.** We have $l_{id} = 1$. For $r \geq 1$, we have

\[ \ell_{g_{2r-1}} = (1 - e^{-\alpha}) T_{h_{2r}} + \sum_{v \in W_{af}, \ell(v) \leq 2r-1} T_v \quad \text{and} \quad \ell_{g_{2r}} = \sum_{v \in W_{af}, \ell(v) \leq 2r} T_v. \]

**Proposition 3.** For $x \in W_{af}$ and $n \in \mathbb{Z}_{<0}$, we have in $K_T^*(\text{Gr}_{SL_2})$

\[ O_x \cdot O_{t_{na^\vee}} = O_{xt_{na^\vee}}. \]

**Proof.** It suffices to prove the statement for $n = -1$. Notice that $t_{-\alpha^\vee} = s_1s_0 = g_2$ and $x = g_m$ for some $m \in \mathbb{Z}_{\geq 0}$. By Theorem 1, we just need to show $l_{g_m}l_{g_2} = l_{g_{m+2}}$. This follows from Lemma 4 and mathematical induction on $m$. \(\square\)

Thanks to the above formula, it remains to compute $O_{s_1t_{-\alpha^\vee}} \cdot O_{s_1t_{-\alpha^\vee}}$. For $x = s_1t_{-\alpha^\vee} = s_0 = g_1$, by direct calculations we have $l_{g_1}^2 = e^{-\alpha}l_{g_2} + (1 - e^{-\alpha})l_{g_3}$. Therefore

\[ (17) \quad O_{s_1t_{-\alpha^\vee}} \cdot O_{s_1t_{-\alpha^\vee}} = (1 - e^{-\alpha}) O_{s_1t_{-2\alpha^\vee}} + e^{-\alpha}O_{t_{-\alpha^\vee}}. \]

**Remark 4.** We can also calculate the above product by using Theorem 2. For instance, for $z = s_1s_0 = t_{-\alpha^\vee}$, all the terms in the formula (15) for $c^z_{x,x}$ vanish unless $t_1 = t_2 = t_{\alpha^\vee}$. Therefore

\[ c^z_{x,x} = b_{s_0,[t_{\alpha^\vee}]} b_{t_{2\alpha^\vee},[s_1s_0]} = \left( \frac{-e^\alpha}{1 - e^\alpha} \right)^2 e^{-\alpha} (1 - e^{-\alpha})^2 = e^{-\alpha}. \]

Formulas (16) and (17), together with Proposition 3, implies that Conjectures 1 and 2 hold when $G = SL_2$.

### 4.2. Multiplication for $\text{Gr}_{SL_3}$

The complete flag manifold $SL_3/B = \text{Fl}_3 = \{ V_1 \subset V_2 \subset \mathbb{C}^3 \mid \dim V_1 = 1, \dim V_2 = 2 \}$ parameterizes complete flags in $\mathbb{C}^3$. The Weyl group $W$ is the permutation group $S_3$ of three objects generated by simple reflections $s_1, s_2$. We have the highest root $\theta = \alpha_1 + \alpha_2$ and coroot $\theta^\vee = \alpha_1^\vee + \alpha_2^\vee$. By calculations using Theorem 2, we obtain $O_{w_{1\alpha^\vee}}O_{t_{\alpha^\vee}} = O_{w_{-1\alpha^\vee}}$ in $K_T^*(\text{Gr}_{SL_3})$ for any $w \in W$, in addition to the following multiplication table.
The remaining products are read off immediately from the above table by the symmetry of the Dynkin diagram of Lie type $A_2$.

Comparing the above table with the appendix in [LM], we conclude that Conjecture [4] holds whenever the degree $d$ in $N_{u,v}^w$ is given by $(0,0), (1,0)$ or $(0,1)$.

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Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA
E-mail address: tfylam@umich.edu

School of Mathematics, Sun Yat-sen University, Guangzhou 510275, P.R. China
E-mail address: lichangzh@mail.sysu.edu.cn

460 McBryde Hall, Department of Mathematics, Virginia Tech, Blacksburg, VA 24061, USA
E-mail address: lmihalce@math.vt.edu

460 McBryde Hall, Department of Mathematics, Virginia Tech, Blacksburg, VA 24061, USA
E-mail address: mshimo@math.vt.edu