HOROCYCLE FLOW ON FLAT PROJECTIVE BUNDLES:
FROM TOPOLOGY TO MEASURES

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Abstract. In this paper we study some topological and measurable aspects
of the dynamics of the foliated horocycle flow on flat projective bundles over
hyperbolic surfaces. If \( \rho : \Gamma \to \mathrm{PSL}(n+1, \mathbb{R}) \) is a representation of a non-
elementary Fuchsian group \( \Gamma \), the unit tangent bundle \( Y \) associated to the
flat projective bundle defined by \( \rho \) admits a natural action of the affine group
\( B \) obtained by combining the foliated geodesic and horocycle flows. If the
image \( \rho(\Gamma) \) satisfies Conze-Guivarc'h conditions, namely strong irreducibility
and proximality, the dynamics of the \( B \)-action is captured by the proximal
dynamics of \( \rho(\Gamma) \) on \( \mathbb{R}P^n \) (Theorem A). In fact, the dynamics of the foliated
horocycle flow on the unique \( B \)-minimal subset of \( Y \) can be described in terms
of dynamics of the horocycle flow on its non-wandering set in the unit tangent
bundle \( X \) of the base surface \( S = \Gamma \backslash \mathbb{H} \) (Theorem B). Assuming the existence
of a continuous limit map induced by \( \rho \), an one-to-one correspondence between
conservative ergodic invariant measures can be stated for the foliated horocycle
flow on the proximal part of the non-wandering set in \( Y \) and the horocycle flow
on the non-wandering set in \( X \) (Theorem C).

1. Introduction

In the 1930s G.A. Hedlund [18] proved the minimality of the right action of the
unipotent subgroup
\[
U = \{ \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \mid s \in \mathbb{R} \}
\]
of \( \mathrm{PSL}(2, \mathbb{R}) = \{ \pm \text{Id} \} \backslash \mathrm{SL}(2, \mathbb{R}) \) on the quotient \( X = \Gamma \backslash \mathrm{PSL}(2, \mathbb{R}) \) by a cocompact
Fuchsian group \( \Gamma \). Later H. Furstenberg [10] obtained a stronger result, namely the
\( U \)-action is uniquely ergodic. Identifying \( \mathrm{PSL}(2, \mathbb{R}) \) and the unit tangent bundle of
the hyperbolic plane \( \mathbb{H} \) with the Poincaré metric, when \( \Gamma \) is torsion-free, the
quotient \( X \) becomes the unit tangent bundle of the hyperbolic surface \( S = \Gamma \backslash \mathbb{H} \).
In this geometric setting, the \( U \)-action on \( X \) identifies with the horocycle flow and we write
\[
h_s(\Gamma u) = \Gamma u \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}
\]
for all \( u \in \mathrm{PSL}(2, \mathbb{R}) \) and all \( s \in \mathbb{R} \). Hedlund’s and Furstenberg’s results have been extended to the case where \( \Gamma \) is finitely generated, but replacing \( X \) by the non-wandering set \( \Omega_X \) of the \( U \)-action.

Theorem. Let \( \Gamma \) be a finitely generated Fuchsian group.
(1) For any \( x \in \Omega_X \), either \( xU \) is periodic or \( xU = \Omega_X \) [13, 15, 18].
(2) For any ergodic \( U \)-invariant Radon measure \( \mu \) supported by \( \Omega_X \), either \( \mu \) is supported by a periodic orbit or \( \mu \) is the Burger-Roblin measure up to a multiplicative constant [10, 25, 26, 27, 28].
As explained in \cite{13}, it turns out that property (1) is true if and only if $\Gamma$ is finitely generated, but the topological dynamics of the $U$-action on $\Omega_X$ is not well understood otherwise. On the other hand, it follows from Ratner’s work that the measure $\mu$ in property (2) is finite if and only if $\mu$ supported by a periodic orbit or $\mu$ is the Haar measure (up to a constant) and in this case the surface $S$ has finite volume. Recently, O. Landesberg and E. Lindenstrauss \cite{21} have given a description of ergodic $U$-invariant Radon measures on $X$ for hyperbolic surfaces $S$ whose injectivity radius is uniformly bounded away from 0.

In this paper we start by studying the topological dynamics of the foliated horocycle flow on flat projective bundles over hyperbolic surfaces. Let $\Gamma$ be a non-elementary Fuchsian group. We consider a representation $\rho : \Gamma \to \text{PSL}(n+1, \mathbb{R})$ with $n \geq 1$. The subgroup $\Gamma_\rho = \{ (\gamma, \rho(\gamma)) \mid \gamma \in \Gamma \}$ of $\text{PSL}(2, \mathbb{R}) \times \text{PSL}(n+1, \mathbb{R})$ acts properly discontinuously on $\tilde{Y} = \text{PSL}(2, \mathbb{R}) \times \mathbb{R}P^n$. As this action preserves the product structure of $\tilde{Y}$, the projective bundle $Y = \Gamma_\rho \backslash \tilde{Y}$ over $X = \Gamma \backslash \text{PSL}(2, \mathbb{R})$ admit a foliation transverse to the fibration $\pi : Y \to X$ given by $\pi(\Gamma_\rho(u, x)) = \Gamma u$. The leaves are 3-manifolds endowed with a natural $\text{PSL}(2, \mathbb{R})$-geometric structure.

Clearly the $U$-action on $\tilde{Y}$ defined by right translation on the first factor induces an $U$-action on $Y$ preserving each leaf. This action corresponds to a foliated horocycle flow as introduced in \cite{22} and studied in \cite{2, 3, 4}.

Assuming $Y$ compact and $\rho(\Gamma)$ Zariski dense, C. Bonatti, A. Eskin and A. Wilkinson proved that the $U$-action on $Y$ is uniquely ergodic \cite[Corollary 2.4]{8}. They actually prove this result under weaker conditions generalising an earlier result by C. Bonatti and J. Gómez-Mont \cite{9} for representations of $\text{PSL}(2, \mathbb{R})$-lattices into $\text{PSL}(n, \mathbb{C})$ with $n = 2, 3$.

Here we do not suppose $\Gamma$ is cocompact or having finite covolume. Moreover the Zariski density of $\rho(\Gamma)$ is also replaced by the following weaker conditions, called Conze-Guivarc’h conditions \cite{12}:

(CG1) $\rho(\Gamma)$ is strongly irreducible,

(CG2) $\rho(\Gamma)$ contains a proximal element.

Both conditions will be explained and commented below.

The affine group

$$B = \{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid b \in \mathbb{R}, a \in \mathbb{R}_+^* \}$$

also acts on $X$ preserving the non-wandering set $\Omega_X$. Clearly $\Omega_X$ is the unique $B$-minimal set. This property extends to the $B$-action on $Y$:

**Theorem A.** Let $\Gamma$ be a non-elementary Fuchsian group and $\rho : \Gamma \to \text{PSL}(n+1, \mathbb{R})$ be a representation satisfying conditions (CG1) and (CG2). Then there is a unique $B$-minimal set $M_B \subset Y$, i.e. $M_B$ is a non-empty closed $B$-invariant set such that $yB = M_B$ for all $y \in M_B$.

**Theorem B.** Under the same assumptions of Theorem A for each point $y \in M_B$, we have:

$$\overline{yU} = M_B \iff \overline{\pi(y)U} = \Omega_X.$$

**Corollary 1.** Let $\Gamma$ be a non-elementary Fuchsian group and $\rho : \Gamma \to \text{PSL}(n+1, \mathbb{R})$ be a representation satisfying conditions (CG1) and (CG2). Then $M_B$ is $U$-minimal (i.e. $\overline{yU} = M_B$ for every $y \in M_B$) if and only if $\Gamma$ is convex-cocompact (i.e. finitely generated without parabolic elements).
When \( \rho \) satisfies (CG1) and (CG2), J.-P. Conze and Y. Guivarc'h proved the existence of a unique minimal set \( L(\rho(\Gamma)) \subset \mathbb{R}P^n \) for the projective action of \( \rho(\Gamma) \) \cite{12}. Namely \( L(\rho(\Gamma)) \) is the closure of the dominant directions of the proximal elements of \( \rho(\Gamma) \). According to \cite{20}, we say \( \rho \) satisfies Nielsen’s condition if (N) \( \rho \) induces a continuous map \( \varphi : L(\Gamma) \to L(\rho(\Gamma)) \), called limit map, such that \( \varphi _\ast \gamma = \rho(\gamma) \varphi \) for all \( \gamma \in \Gamma \).

As \( L(\rho(\Gamma)) \) is minimal, the map \( \varphi \) is always surjective. If we denote

\[
Y_{\text{prox}} = \Gamma \backslash \text{PSL}(2, \mathbb{R}) \times L(\rho(\Gamma)),
\]

the closed \( B \)-invariant set \( \Omega_{\text{prox}} = Y_{\text{prox}} \cap \pi^{-1}(\Omega_X) \) inherits from \( Y \) a natural structure of \( L(\rho(\Gamma))\text{-fibre bundle over } \Omega_X \). Clearly \( \mathcal{M}_B \subset \Omega_{\text{prox}} \). Condition (N) gives arise to a continuous section \( \Phi : \Omega_X \to \mathcal{M}_B \) for the fibration \( \pi : \Omega_{\text{prox}} \to \Omega_X \).

**Theorem C.** Let \( \Gamma \) be a non-elementary Fuchsian group and \( \rho : \Gamma \to \text{PSL}(n+1, \mathbb{R}) \) be a representation satisfying conditions (CG1), (CG2) and (N). If \( m \) is a conservative ergodic \( U \)-invariant Radon measure on \( \Omega_{\text{prox}} \), then \( m = \Phi _\ast \mu \) where \( \mu \) is a conservative ergodic \( U \)-invariant Radon measure on \( \Omega_X \).

**Corollary 2.** Under the conditions of Theorem C, assume \( \Gamma \) is finitely generated. Then there is a unique (up to a multiplicative constant) conservative ergodic \( U \)-invariant Radon measure \( m \) on \( \Omega_{\text{prox}} \) if and only if \( \Gamma \) is convex-cocompact. In particular, there is a unique \( U \)-invariant probability measure \( m \) on \( Y_{\text{prox}} \) if and only if \( \Gamma \) is cocompact.

In general, except when \( \Gamma \) has finite covolume, Theorem C deals with infinite measures and does not intersect theorems from \cite{8} and \cite{9}. Otherwise, in addition to an integrability condition (automatically verified in the cocompact case), Bonatti, Eskin and Wilkinson replace the conditions (CG1) and (CG2) by the irreducibility and the simplicity of the largest Lyapunov exponent for the action of the diagonal group

\[
D = \{ \begin{pmatrix} t^2 & 0 \\ 0 & e^{-t} \end{pmatrix} \mid t \in \mathbb{R} \}
\]

to deduce unique ergodicity. Using this condition, they construct a measurable section \( \sigma : X \to \mathcal{M}_B \) which allow them to lift the Haar measure \( \mu \) on \( X \) into the unique \( U \)-invariant probability measure \( m \) supported by \( \mathcal{M}_B \).

**2. Preliminaries**

A matrix \( A \in \text{SL}(n+1, \mathbb{R}) \) is said to be proximal if \( A \) admits a simple dominant real eigenvalue \( \lambda_A \). Let \( w_A \in \mathbb{R}^{n+1} \) be an eigenvector associated to \( \lambda_A \) and \( \chi_A \in \mathbb{R}P^n \) its direction, also called dominant for \( A \). Further, we have the decomposition

\[
\mathbb{R}^{n+1} = \mathbb{R}w_A \oplus \mathcal{W} \]

where

\[
\mathcal{W} = \{ w \in \mathbb{R}^{n+1} \mid \lambda_A^{-k} A^k w \to 0 \text{ as } k \to +\infty \}.
\]

**Definition 1.** Let \( G \) be a subgroup of \( \text{SL}(n+1, \mathbb{R}) \). We say \( G \) is:

(CG1) strongly irreducible if there does no exist any proper non-trivial subspace of \( \mathbb{R}^{n+1} \) invariant by the action of a subgroup of finite index of \( G \);

(CG2) proximal if \( G \) contains a proximal element \( A \).

Both conditions will be called Conze-Guivarc'h conditions

Conditions (CG1) and (CG2) are satisfied by \( G \) if and only if they are satisfied by its Zariski closure in \( \text{SL}(n+1, \mathbb{R}) \) \cite{12}. But these conditions do not imply that \( G \) is Zariski dense in \( \text{SL}(n+1, \mathbb{R}) \) since \( \text{SO}(n, 1) \) satisfies (CG1) and (CG2).
Proposition 1 (\cite{12}). Let $G$ be a subgroup of $\text{SL}(n+1, \mathbb{R})$ satisfying (CG1) and (CG2). Then

$$L(G) = \{ \chi_A \in \mathbb{R}P^n \mid A \in G \text{ proximal} \}.$$ 

is the unique $G$-minimal set $L(G) \subset \mathbb{R}P^n$. \hfill \square

Assume $G \subset \text{PSL}(n + 1, \mathbb{R})$ is discrete and consider the $G$-action induced on $L^c(G) = \mathbb{R}P^n - L(G)$. For $n = 1$, as this action is properly discontinuous, the set $L(G)$ is the non-wandering set for the $G$-action (i.e. $\xi \in L(G)$ if and only if there are a non stationary sequence $g_k$ in $G$ and a sequence $\xi_k \to \xi$ such that $g_k \xi_k \to \xi$) and also a $G$-attractor (i.e. for any point $\xi \in \mathbb{R}P^1$ and for any non stationary sequence $g_k$ in $G$, the condition $g_k \xi \to \xi'$ implies $\xi' \in L(G)$) which captures the proximal dynamics of $G$. When $n \geq 2$, these properties are not true in general:

Example 1. Consider $\mathbb{R}^3$ equipped with the Lorentz quadratic form

$$q(x) = x_1^2 + x_2^2 - x_3^2.$$ 

For $i = -1, 0, 1$, denote $H_i = \{ x \in \mathbb{R}^3 \mid q(x) = i \}$ and let $p : \mathbb{R}^3 - \{0\} \to \mathbb{R}P^2$ be the canonical projection. Let $G$ be a discrete subgroup of $\text{SO}^+(2, 1)$, namely the connected component of the identity in the group $\text{SO}(2, 1)$ of orientation-preserving linear isometries of $q$. If $G$ contains a non abelian free subgroup, then $G \setminus H_+^*$ is isometric to a hyperbolic surface $S$, where $H_+ = H_\infty \cap \{ x \in \mathbb{R}^3 \mid x_3 \geq 0 \}$ (see \cite{13}). The limit set $L(G)$ is contained into $p(H_0^* - \{0\})$. For any vector $x \in H_1$, the orthogonal plane (with respect to $q$) intersects $H_0$ along two lines $D_1(x)$ and $D_2(x)$. Let $H_1(G)$ be the set of vectors $x \in H_1$ such that the directions represented by $D_1(x)$ and $D_2(x)$ belong to $L(G)$. This is a $G$-invariant closed subset of $\mathbb{R}^3 - \{0\}$. According to \cite{13} Proposition VI.2.5, the dynamics of the $G$-action on $H_1(G)$ is dual to that of the geodesic flow on the non-wandering set of $T^1S$. In particular, the $G$-action on $H_1(G)$ has dense orbits (see \cite{13} Property VI.2.12]), as well many non-empty proper minimal sets, and hence the $G$-action on the closure $F(G)$ of $p(H_1(G))$ in $\mathbb{R}P^2$ also has dense orbits, as well many non-empty invariant closed sets $F \subset F(G)$ such that $L(G) \subset F$. The $G$-action on $L^c(G)$ is absolutely not discontinuous.

Let $\Gamma$ be a non-elementary Fuchsian group and $\rho : \Gamma \to \text{SL}(n+1, \mathbb{R})$ a representation satisfying conditions (CG1) and (CG2). Theorems \ref{A} and \ref{A*} will be proved using the dual approach based on \cite{16}. The linear action of $\Gamma$ on $E = (\pm I) \setminus \mathbb{R}^2 - \{0\}$ is conjugated to the $\Gamma$-action on $PSL(2, \mathbb{R})/U$. This action also induces a projective action of $\Gamma$ on $\mathbb{R}P^1$ conjugated to the $\Gamma$-action on $PSL(2, \mathbb{R})/B$. Both actions are dual to the $U$-action and the $B$-action on $X = \Gamma \setminus PSL(2, \mathbb{R})$ respectively.

In our case, the linear and projective actions extend to actions of

$$\Gamma_\rho = \{ (\gamma, \rho(\gamma)) \mid \gamma \in \Gamma \}$$

on $E \times \mathbb{R}P^n$ and $\mathbb{R}P^1 \times \mathbb{R}P^n$. As before, they are dual to the $U$-action and the $B$-action on the flat projective bundle $Y = \Gamma^\rho \setminus \tilde{Y}$ over $X$ where $\tilde{Y} = PSL(2, \mathbb{R}) \times \mathbb{R}P^n$.

From a geometrical point of view, $Y$ is the unitary tangent bundle to the foliation by hyperbolic surfaces on $\Gamma_\rho \setminus \mathbb{H} \times \mathbb{R}P^n$ which is induced by the horizontal foliation on $\mathbb{H} \times \mathbb{R}P^n$.

Theorem $A^{*}$. Under assumptions of Theorem $A$ there is a unique non-empty minimal closed $\Gamma_\rho$-invariant set $M \subset \mathbb{R}P^1 \times \mathbb{R}P^n$. Moreover $M \subset L(\Gamma) \times L(\rho(\Gamma))$.

The relation between the sets $M_B$ and $M$ considered in Theorems $A$ and $A^{*}$ is given by

$$M_B = \{ \Gamma_\rho(u, \chi) \in Y \mid (u(+\infty), \chi) \in M \},$$

where $u(+\infty)$ is the endpoint of the geodesic ray associated to $u \in T^1\mathbb{H}$. 


For each vector \( v \in E \), denote \( \bar{v} \in \mathbb{RP}^1 \) its direction. Clearly the set
\[
E(\Gamma) = \{ v \in E \mid \bar{v} \in L(\Gamma) \}
\]
is dual to \( \Omega_X \) and the set
\[
E(\mathcal{M}) = \{ (v, \chi) \in E \times \mathbb{RP}^n \mid (\bar{v}, \chi) \in \mathcal{M} \},
\]
is dual to \( \mathcal{M}_B \).

**Theorem B*. Under assumptions of Theorem A for each pair \( (v, \chi) \in E(\mathcal{M}) \), we have:
\[
\Gamma_\rho(v, \chi) = E(\mathcal{M}) \iff \Gamma v = E(\Gamma).
\]

3. Proof of Theorems A and B

Let \( \Gamma \) be a non-elementary Fuchsian group and \( \rho : \Gamma \to \text{PSL}(n+1, \mathbb{R}) \) be a representation satisfying (CG1) and (CG2). Take \((\gamma, A) \in \Gamma_\rho \) with \( A \) proximal and denote \( \chi_A \in \mathbb{RP}^n \) the dominant direction of \( A \). Since \( A \) has infinite order, \( \gamma \) is hyperbolic or parabolic. Consider \( \gamma^+ = \lim_{k \to +\infty} \gamma^k(z) \) for any \( z \in \mathbb{H} \).

**First Key Lemma.** For any non-empty closed \( \Gamma_\rho \)-invariant set \( F \subset \mathbb{RP}^1 \times \mathbb{RP}^n \), we have \((\gamma^+, \chi_A) \in F \).

**Proof.** Since \( \mathbb{RP}^1 \) is compact, \( F \) projects on a closed \( \rho(\Gamma) \)-invariant subset of \( \mathbb{RP}^n \) containing \( L(\rho(\Gamma)) \). It follows that there exists \( \xi \in \mathbb{RP}^1 \) such that \((\xi, \chi_A) \in F \). If \( \xi \neq \lim_{k \to +\infty} \gamma^{-k}(z) \), then \( \lim_{k \to +\infty} \gamma^k(\xi) = \gamma^+ \) and hence
\[
\lim_{k \to +\infty} \left( \gamma^k(\xi), \rho(\gamma^k(\xi)) \right) = \left( \gamma^+(\xi), A^k \chi_A \right) = (\gamma^+, \chi_A) \in F.
\]
Otherwise, by the irreducibility condition (CG1), there exists \( \gamma' \in \Gamma^- < \gamma > \) such that \( \rho(\gamma') \chi_A \) does not belong to the projection \( \overline{W}_A \) of \( W_A \) into \( \mathbb{RP}^n \). Since \( \Gamma \) is discrete, we have \( \gamma'(\xi) \neq \xi \). As a consequence, we have:
\[
\lim_{k \to +\infty} \left( \gamma^k(\xi), A^k \rho(\gamma') \chi_A \right) = (\gamma^+, \chi_A) \in F.
\]

**Proof of the Theorem A** By the First Key Lemma, the intersection of all non-empty closed \( \Gamma_\rho \)-invariant sets contains
\[
\mathcal{M} = \{ (\gamma^+, \chi_A) \mid \gamma \in \Gamma, A = \rho(\gamma) \text{ proximal} \} \subset L(\Gamma) \times L(\rho(\Gamma)).
\]
Thus \( \mathcal{M} \) becomes the unique minimal set for the \( \Gamma_\rho \)-action on \( \mathbb{RP}^1 \times \mathbb{RP}^n \). \( \square \)

**Remark 1 (on the shape of \( \mathcal{M} \)).** (1) If \( \rho \) is not injective, then \( \mathcal{M} = L(\Gamma) \times L(\rho(\Gamma)) \) because \( N = \text{Ker} \rho \) is a normal subgroup of \( \Gamma \) and then \( L(\Gamma) = L(N) \).

(2) A similar conclusion holds if \( \rho \) is indiscrete (in the sense that \( \rho(\Gamma) \) is not discrete).

Indeed, let \( \gamma_k \) be a non-stationary sequence of elements of \( \Gamma \) such that \( \rho(\gamma_k) \to \text{Id} \).

Using [2, Lemma 2.2] and passing to a subsequence if necessary, there exist two points \( \xi^+ \) and \( \xi^- \) in \( \mathbb{RP}^1 \) such that
\[
\lim_{k \to +\infty} \gamma^k(\xi) = \xi^+
\]
for any \( \xi \neq \xi^- \). For any \( \chi \in L(\rho(\Gamma)) \), take an element \((\xi, \chi) \in \mathcal{M} \) such that \( \xi \notin \Gamma \xi^- \). Since \( \Gamma \) is non-elementary, such element always exists. For any \( \gamma \in \Gamma \), we have:
\[
\lim_{k \to +\infty} \left( \gamma \gamma_k \gamma^{-1}(\xi), \rho(\gamma \gamma_k \gamma^{-1}) \chi \right) = (\gamma(\xi^+), \chi).
\]
Therefore \( \Gamma_\rho(\gamma(\xi^+), \chi) \subset \mathcal{M} \) and hence \( L(\Gamma) \times L(\rho(\Gamma)) = \mathcal{M} \).

(3) In the opposite side, if \( n = 1 \) and \( \rho \) is the natural inclusion of \( \Gamma \) into \( \text{PSL}(2, \mathbb{R}) \), then \( \mathcal{M} = \{ (\xi, \xi) \mid \xi \in L(\Gamma) \} \).

Two new lemmas are needed to prove Theorem B*. 

Lemma 1. Let $\Gamma$ be a non-elementary Fuchsian group and $\rho : \Gamma \to \text{PSL}(n+1, \mathbb{R})$ be a representation satisfying (CG1) and (CG2). There are two hyperbolic elements $\gamma_1$ and $\gamma_2$ of $\Gamma$ such that
1) the dominant eigenvalues $\lambda_1$ and $\lambda_2$ generate a dense subgroup of the positive multiplicative group $\mathbb{R}^*_+$,
2) $A_1 = \rho(\gamma_1)$ and $A_2 = \rho(\gamma_2)$ are proximal.

Proof. Under conditions (CG1) and (CG2), the group $\rho(\Gamma)$ contains two elements $A_1$ and $A_2$ which generate a non-abelian free group containing only proximal elements \[ \text{[Lemma 3.9]} \](see also \[12\] and \[17\] Lemma 3]). Let $\gamma_1$ and $\gamma_2$ be two elements of $\Gamma$ such that $\rho(\gamma_1) = A_1$ and $\rho(\gamma_2) = A_2$. Reasoning as in \[14\], we can replace $\gamma_1$ and $\gamma_2$ with two hyperbolic elements of $\Gamma$ whose dominant eigenvalues $\lambda_1$ and $\lambda_2$ generate a dense subgroup of $\mathbb{R}^*_+$. \[ \square \]

For each hyperbolic element $\gamma$ of $\Gamma$, we denote $v_\gamma$ the unit eigenvector in $E$ associated to dominant eigenvalue $\lambda_\gamma$. Clearly $v_\gamma \in E(\Gamma)$ since its direction $v_\gamma = \gamma^+ \in L(\Gamma)$. From Theorem \[A\] it is also clear that
\[ E(\mathcal{M}) \subset E(\Gamma) \times L(\rho(\Gamma)). \]

Lemma 2. Let $(v, \chi) \in E(\mathcal{M})$ such that \[ \Gamma v = E(\Gamma). \] For any hyperbolic element $\gamma \in \Gamma$ such that $A = \rho(\gamma)$ is proximal, there exists $\alpha \in \mathbb{R}^*$ such that
\[ (\alpha v_\gamma, \chi_A) \in \overline{E(\rho(v, \chi))}. \]

Proof. Assuming \[ \Gamma v = E(\Gamma), \] there exists a sequence of elements $\gamma_k \in \Gamma$ such that the norms $\|\gamma_k v\|$ converge to 0 as $k \to +\infty$. Since $\Gamma$ is non elementary and $\rho(\Gamma)$ is irreducible, replacing $\gamma_k$ by $\gamma' \gamma_k$ for some $\gamma' \in \Gamma$, up to take a subsequence, we can suppose:
1) $\gamma_k v = a_k v_\gamma + b_k v_{\gamma_k-1}$ where $a_k \neq 0$ for any $k$,
2) $\rho(\gamma_k) \chi \to \chi_0 \notin \overline{V_A}$.

Let $p_k$ an increasing sequence of integers converging to $+\infty$ such that $\chi_A^{p_k} a_k$ converges to some $\alpha \neq 0$. Then we have $\gamma_k \gamma_k v \to \alpha v_\gamma$. Let us prove that
\[ A^{p_k} \rho(\gamma_k) \chi \to \chi_A. \tag{3.1} \]
Since $\chi_0 \notin \overline{V_A}$, there exist an open neighbourhood $V(\chi_A)$ of $\chi_A$ containing $\chi_0$, an integer $N \gg 0$ and a constant $0 < c < 1$ satisfying \[ \text{[Lemma 3]} \]:
i) $A^{NK}(V(\chi_A)) \subset V(\chi_A)$ for all $k \geq 0$,
ii) $\delta(A^{NK} \chi_1, A^{NK} \chi_2) \leq c^k \delta(\chi_1, \chi_2)$ for all $\chi_1, \chi_2 \in V(\chi_A)$ and for all $k \geq 0$.

For $k \geq 0$ large enough, we have $\rho(\gamma_k) \chi \in V(\chi_A)$. Assuming $p_k = Nq_k + r_k$ with $0 \leq r_k < N$, the inequality
\[ \delta(A^{Nq_k} \rho(\gamma_k) \chi, \chi_A) \leq c^{p_k} \delta(\rho(\gamma_k) \chi, \chi_A) \]
implies
\[ \lim_{k \to +\infty} \delta(A^{Nq_k} \rho(\gamma_k) \chi, \chi_A) = 0 \]
and hence
\[ \lim_{k \to +\infty} \delta(A^{p_k} \rho(\gamma_k) \chi, \chi_A) = 0. \]
This proves \[3.1\]. Finally, we deduce:
\[ \lim_{k \to +\infty} \left( \gamma_k \gamma_k v, A^{p_k} \rho(\gamma_k) \chi \right) = (\alpha v_\gamma, \chi_A) \in \overline{E(\rho(v, \chi))}. \] \[ \square \]
Proof of the Theorem Let \((v, \chi) \in E(M)\) with \(\Gamma v = E(\Gamma)\). Take \(\gamma_1, \gamma_2 \in \Gamma\) given by Lemma and its images \(A_1 = \rho(\gamma_1)\) and \(A_2 = \rho(\gamma_2)\). Applying Lemma there exists a real number \(\alpha_1 \neq 0\) such that \((\alpha_1 \gamma_1 v, \chi_{A_1}) \in \Gamma_\rho(v, \chi)\) and hence
\[
(\alpha_1 \gamma_1^p v, \chi_{A_1^p}) \in \Gamma_\rho(v, \chi)
\] (3.2)
for any \(p \in \mathbb{Z}\). Since \(\Gamma \gamma_1 = E(\Gamma)\) [13 Theorem V.3.1], by the same argument, we obtain another real number \(\alpha_2 \neq 0\) such that
\[
(\alpha_1 \alpha_2 \gamma_2^p v, \chi_{A_2^p}) \in \Gamma_\rho(v, \chi)
\] (3.3)
for any pair \(p, q \in \mathbb{Z}\). As \(\lambda_1\) and \(\lambda_2\) generate a dense subgroup of \(\mathbb{R}_+\) by Lemma we deduce from (3.2) and (3.3) that
\[
(\lambda \gamma_2 v, \chi_{A_2}) \in \Gamma_\rho(v, \chi)
\]
for any \(\lambda > 0\).

For any \((v', \chi') \in E(M)\), since \((v', \chi')\) and \((\gamma_2 v, \chi_{A_2})\) belong to the minimal set \(M\), there exists a sequence \(\gamma_k \in \Gamma\) such that
\[
\gamma_k \gamma_2 v, \chi_{A_2} \rightarrow v' \quad \text{and} \quad \rho(\gamma_k) \chi_{A_2} \rightarrow \chi'.
\]
It follows there exists a sequence \(\lambda_k \in \mathbb{R}\) such that
\[
\lambda_k \gamma_k v, \chi_{A_2} \rightarrow \alpha v'
\] (3.4)
for some \(\alpha \neq 0\). As \((\lambda_k \gamma_k v, \chi_{A_2}) \in \Gamma_\rho(v, \chi)\), we deduce \((\alpha v', \chi') \in \Gamma_\rho(v, \chi)\). The same argument applies when multiply the two terms of (3.4) by a real number \(\lambda > 0\).

We deduce from Theorem that \(E(M)\) is a non-empty minimal closed \(\Gamma_\rho\)-invariant set if and only if \(E(\Gamma)\) is a minimal closed \(\Gamma\)-invariant set. Since this condition is satisfied if and only if \(\Gamma\) is convex-compact [13 Proposition V.4.3], we retrieve Corollary

Corollary 3. The set \(M_B\) is \(U\)-minimal if and only if \(\Gamma\) is convex-cocompact

More generally, since \(\mathbb{R}P^n\) is compact, any non-empty minimal \(\Gamma_\rho\)-invariant closed subset \(F \subset E \times \mathbb{R}P^n\) projects onto a non-empty minimal \(\Gamma\)-invariant closed subset \(p_1(F) \subset E\). If \(\Gamma\) is finitely generated, then either \(F\) projects onto a closed \(\Gamma\)-orbit or \(F = E(M)\) and \(\Gamma\) is convex-compact [13 Theorem V.4.1]. On the contrary, if \(\Gamma\) is not finitely generated, then there exist examples where \(E(\Gamma)\) does not admit any non-empty minimal \(\Gamma\)-invariant closed subset [5, 19, 23, 24].

Corollary 4. There exist infinitely generated Fuchsian groups \(\Gamma\) such that for any representations \(\rho: \Gamma \to \mathrm{PSL}(n+1, \mathbb{R})\) satisfying conditions (CG1) and (CG2), the projective bundle \(Y\) does not admit any non-empty \(U\)-minimal subset of \(\pi^{-1}(\Omega_X)\).

4. Proof of Theorem

In this section, we restrict our attention to the space
\[
Y_{\text{prox}} = \Gamma_\rho \backslash \mathrm{PSL}(2, \mathbb{R}) \times L(\rho(\Gamma)).
\]
This space is a closed \(\mathrm{PSL}(2, \mathbb{R})\)-invariant subset of \(Y\) for which the induced \(\mathrm{PSL}(2, \mathbb{R})\)-action is minimal. From a geometrical point of view, \(Y_{\text{prox}}\) is the unit tangent bundle to a minimal lamination by hyperbolic surfaces [5]. Intersecting with \(\pi^{-1}(\Omega_X)\), we obtain a closed \(B\)-invariant set
\[
\Omega_{\text{prox}} = Y_{\text{prox}} \cap \pi^{-1}(\Omega_X)
\]
such that:

(i) \(\Omega_{\text{prox}}\) is included in the non-wandering set for the \(U\)-action on \(Y_{\text{prox}}\).
(ii) $\Omega_{\text{prox}}$ inherits from $Y$ a natural structure of $L(\rho(\Gamma))$-fibre bundle over $\Omega_X$ with projection $\pi : \Omega_{\text{prox}} \to \Omega_X$.

By duality, $U$-orbits in $\Omega_{\text{prox}}$ are in one-to-one correspondance with $\Gamma_\rho$-orbits in $E(\Gamma) \times L(\rho(\Gamma))$. Note that $\mathcal{M}_B \subset \Omega_{\text{prox}}$ is the unique non-empty minimal closed $B$-invariant subset of $\Omega_{\text{prox}}$.

We also add a new condition on the representation $\rho : \Gamma \to \text{PSL}(n+1, \mathbb{R})$, which we called condition de Nielsen:

(N) there exists a continuous map

$$\varphi : L(\Gamma) \to L(\rho(\Gamma)),$$

called limit map, such that $\varphi \circ \gamma = \rho(\gamma) \circ \varphi$ for all $\gamma \in \Gamma$.

Conditions (CG1), (CG2) and (N) imply $\rho$ is injective and discrete and $\varphi$ is surjective.

A wide family of representations $\rho$ satisfying conditions (CG1), (CG2) and (N) can be find in the litterature: for $\rho(\Gamma) \subset SO(n,1)$ see [30] and for $\rho(\Gamma) \subset SL(n+1,\mathbb{R})$ Anosov see [20]. In general, even if $\Gamma$ is finitely generated, $\varphi$ is not necessarily injective. This is the case for example if $\gamma$ is hyperbolique and $\rho(\gamma)$ is parabolic [29]. One of the most surprising example is a discrete faithful representation $\rho : \Gamma \to SO(3,1)$ of a torsion-free cocompact Fuchsian group $\Gamma$ that gives raise to sphere-filling map $\varphi : S^1 \to S^2$ [11].

Under condition (N), the graph of the limit map $\varphi$ is clearly a minimal $\Gamma_\rho$-invariant closed subset of $\mathbb{R}P^1 \times \mathbb{R}P^n$.

**Lemma 3.** Let $\Gamma$ be a non-elementary Fuchsian group and $\rho : \Gamma \to \text{PSL}(n+1, \mathbb{R})$ be a representation satisfying conditions (CG1), (CG2) and (N). Then the unique $B$-minimal set $\mathcal{M}_B$ is the image of continuous section $\Phi : \Omega_X \to \Omega_{\text{prox}}$ given by

$$\Phi(\Gamma u) = \Gamma_u(u, \varphi(u(+\infty))),$$

namely

$$\mathcal{M}_B = \{ \Gamma_u(u, \varphi(u(+\infty))) \in Y \mid u \in \text{PSL}(2, \mathbb{R}), u(+\infty) \in L(\Gamma) \}.$$ 

Proof. Applying Theorem A we deduce the unique $\Gamma$-minimal set $\mathcal{M} \subset \mathbb{R}P^1 \times \mathbb{R}P^n$ coincides with the graph of $\varphi$. The lemma follows by duality.

**Second Key Lemma.** The $B$-minimal set $\mathcal{M}_B$ is a $U$-attractor relative to $\Omega_{\text{prox}}$, i.e. for any point $y \in \Omega_{\text{prox}}$ and for any sequence $s_k \to +\infty$, we have:

$$h_{s_k}(y) \to y' \Rightarrow y' \in \mathcal{M}_B.$$ 

In particular, the set of $U$-recurrent points in $\Omega_{\text{prox}}$

$$\mathcal{R}_{\text{prox}} = \{ y \in \Omega_{\text{prox}} \mid \exists s_n \to +\infty : h_{s_n}(y) \to y \}$$

is contained in $\mathcal{M}_B$.

**Proof.** Take $y = \Gamma_{\rho}(u,\chi) \in \Omega_{\text{prox}}$ and assume $h_{s_k}(y) \to y'$ for some sequence $s_k \to +\infty$. Since $\Omega_{\text{prox}}$ is $U$-invariant, $y' = \Gamma_{\rho}(u',\chi')$ with $u'(+) \in L(\Gamma)$ and $\chi' \in L(\rho(\Gamma))$. As $y \in \mathcal{M}_B$ implies $y' \in \mathcal{M}_B$, the proof reduces to the case where $\chi = \varphi(\xi) \neq \varphi(u(+) \in L(\Gamma))$. By construction, there exists a sequence $\gamma_k \in \Gamma$ such that

$$\gamma_k u \begin{pmatrix} 1 & s_k \\ 0 & 1 \end{pmatrix} \to u' \quad \text{and} \quad \rho(\gamma_k)\chi \to \chi'.$$

Let us return to the hyperbolic point of view, identifying $\text{PSL}(2, \mathbb{R})$ with the unit tangent bundle $T^1\mathbb{H}$. In this model, each element $u \in \text{PSL}(2, \mathbb{R})$ identifies with $u = (u(0), \tilde{u}) \in T^1\mathbb{H}$ where $u(0)$ is a point of $\mathbb{H}$ and $\tilde{u}$ is a unit tangent vector to
H at \( u(0) \). Denoting \( B_{u(\pm \infty)}(i, u(0)) \) the Busemann cocycle centred at \( u(\pm \infty) \) and calculated at \( i \) and \( u(0) \), we have the following conditions [13] Chapter V:

1. \( \gamma_k(u(\pm \infty)) \to u'(\pm \infty) \),
2. \( B_{\gamma_k(u(\pm \infty))}(i, \gamma_k(u(0))) \to B_{u'(\pm \infty)}(i, u'(0)) \),
3. \( \rho(\gamma_k)\chi \to \chi' \).

Properties (1) and (2) imply

\[
\lim_{k \to \infty} \gamma_k(u(0)) = u'(\pm \infty).
\]

Since \( B_{\gamma_k(u(\pm \infty))}(i, \gamma_k(u(0))) = B_{u'(\pm \infty)}(\gamma_k^{-1}(i), u(0)) \), applying again Property (2), we deduce:

\[
\lim_{k \to \infty} \gamma_k^{-1}(i) = u(\pm \infty).
\]

As a consequence, since \( \xi \neq u(\pm \infty) \), we have (see [2, Lemma 2.2]):

\[
\gamma_k(\xi) \to u'(\pm \infty).
\]

By continuity of \( \varphi \), it follows:

\[
\rho(\gamma_k)\chi = \varphi(\gamma_k(\xi)) \to \varphi(u'(\pm \infty)).
\]

Property (3) implies \( \chi' = \varphi(u'(\pm \infty)) \) and hence \( y' \in \mathcal{M}_B \).

\[\square\]

**Remark 2.** Example [11] shows that we cannot expect \( \mathcal{M}_B \) to be a global \( U \)-attractor in general. This is why we introduced the laminated space \( \Omega_{\text{prox}} \).

As a consequence of Second Key Lemma, we have:

\[\textbf{Corollary 5.} \text{ Let } \Gamma \text{ be a non-elementary Fuchsian group and } \rho : \Gamma \to \text{PSL}(n+1, \mathbb{R}) \text{ be a representation satisfying conditions (CG1), (CG2) and (N). Then } \mathcal{M}_B \text{ is the unique } U \text{-minimal set in } \Omega_{\text{prox}}. \quad \square\]

Let \( m \) be a \( U \)-invariant (non necessarily finite) Radon measure on \( \Omega_{\text{prox}} \). If \( m \) is conservative, then Poincaré’s Recurrence Theorem (see [11] Theorem 1.1.5) for the discrete version) implies that the set of \( U \)-recurrent points \( R_{\text{prox}} \) and hence the \( U \)-attractor \( \mathcal{M}_B \) has full-measure (in the sense that \( m(\Omega_{\text{prox}} - \mathcal{M}_B) = 0 \)). This remark allows us to prove Theorem [10].

**Proof of the Theorem [4]** Under conditions (CG1), (CG2) and (N), the continuous section \( \Phi \) constructed in Lemma [3] sends homeomorphically \( \Omega_X \) onto \( \mathcal{M}_B \). Thus, any (conservative) ergodic \( U \)-invariant measure \( \mu \) on \( \Omega_X \) can be lifted to a (conservative) ergodic \( U \)-invariant measure \( \Phi_*\mu \) on \( \mathcal{M}_B \). If \( m \) is a conservative \( U \)-invariant Radon measure, then \( m \) is supported by the image \( \mathcal{M}_B \) of the section \( \Phi \) as commented above. Its push-forward by \( \pi \) defines a \( U \)-invariant measure \( \mu \) on \( \Omega_X \), which is also conservative, such that \( \Phi_*\mu = m \). As before, \( m \) is ergodic if and only if \( \mu \) is ergodic.

\[\square\]

If \( \Gamma \) is finitely generated, as we recall in the introduction, any ergodic \( U \)-invariant measure \( \mu \) either is supported by a closed orbit or is equal to the Burger-Roblin measure [10, 28] up to a multiplicative constant. In the last case, \( \mu \) is conservative, so we have the following result:

\[\textbf{Corollary 6.} \text{ Under the conditions of Theorem [4], there is a unique conservative ergodic } U \text{-invariant Radon measure } m \text{ on } \Omega_{\text{prox}} \text{ (defined up to a multiplicative constant and supported by the unique } U \text{-minimal set } \mathcal{M}_B \text{ in } \Omega_{\text{prox}} \text{) if and only if } \Gamma \text{ is convex-cocompact}. \]


References

[1] Jon Aaronson, An Introduction to Infinite Ergodic Theory, Mathematical Surveys and Monographs, vol. 50, American Mathematical Society, 1997.

[2] Fernando Alcalde Cuesta and Françoise Dal’Bo, Remarks on the dynamics of the horocycle flow for homogeneous foliations by hyperbolic surfaces, Expo. Math. 33 (2015), no. 4, 431–451.

[3] Fernando Alcalde Cuesta, Françoise Dal’Bo, Matilde Martínez, and Alberto Verjovsky, Minimality of the horocycle flow on laminations by hyperbolic surfaces with non-trivial topology, Discrete Contin. Dyn. Syst. 36 (2016), no. 9, 4619–4635.

[4] ______, Unique ergodicity of the horocycle flow on riemannian foliations, Ergodic Theory and Dynamical Systems 40 (2020), no. 6, 1459–1479.

[5] Alexandre Bellis, Étude topologique du flot horocyclique : le cas des surfaces géométriquement infinies, Ph.D. Thesis, Université de Rennes 1, 2018.

[6] Pierre-Louis Blayac, Aspects dynamiques des structures projectives convexes, Ph.D. Thesis, Université Paris-Saclay, 2021.

[7] Christian Bonatti, Alex Eskin, and Amie Wilkinson, Projective cocycles over $SL(2,\mathbb{R})$ actions: measures invariant under the upper triangular group, Astérisque 415 (2020), 157–180.

[8] Christian Bonatti and Xavier Gómez-Mont, Sur le comportement statistique des feuilles de certains feuilletages holomorphes, Essays on geometry and related topics, Vol. 1, 2, Monogr. Enseign. Math., vol. 38, Enseignement Math., Geneva, 2001, pp. 15–41.

[9] Marc Burger, Horocycle flow on geometrically finite surfaces, Duke Mathematical Journal 61 (1990), no. 3, 779 – 803.

[10] James W. Cannon and William P. Thurston, Group invariant Peano curves, Geom. Topol. 11 (2007), 1315–1355.

[11] Jean-Pierre Conze and Yves Guivarc’h, Limit sets of groups of linear transformations, Sankhyā: The Indian Journal of Statistics, Series A (1961-2002) 62 (2000), no. 3, 367–385.

[12] Matilde Martínez, Shigenori Matsumoto, and Alberto Verjovsky, Horocycle flows for laminations by hyperbolic Riemann surfaces and Hedlund’s theorem, J. Mod. Dyn. 10 (2016), 113–134.

[13] Marina Ratner, Strict measure rigidity for unipotent subgroups of solvable groups, Inventiones Mathematicae 101 (1990), no. 1, 449–482.

[14] Gaye Masseye and Cheikh Lo, Sur l’existence d’ensembles minimaux pour le flot horocyclique, Confluentes Mathematicae 9 (2017), 95–104.

[15] Shigenori Matsumoto, Horocycle flows without minimal sets, J. Math. Sci. Univ. Tokyo 23 (2016), no. 3, 661–673.

[16] Marina Ratner, Strict measure rigidity for unipotent subgroups of solvable groups, Inventiones Mathematicae 101 (1990), no. 1, 449–482.
[27] Invariant measures and orbit closures for unipotent actions on homogeneous spaces, Geom. Funct. Anal. 4 (1994), no. 2, 236–257.

[28] Thomas Roblin, Ergodicité et équidistribution en courbure négative, Mémoires de la Société Mathématique de France, vol. 95, Société Mathématique de France, 2003.

[29] Pekka Tukia, A remark on a paper by Floyd, Holomorphic Functions and Moduli II, Mathematical Sciences Research Institute Publications, vol. 11, Springer, New York, NY, 1988, pp. 165–172.

[30] The limit map of a homomorphism of discrete Möbius groups, Publications Mathématiques de l’IHÉS 82 (1995), 97–132.

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