FERRETING OUT THE FLUFFY BUNNIES:
ENTANGLEMENT CONSTRAINED BY GENERALIZED
SUPERSELECTION RULES

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Entanglement is a resource central to quantum information (QI). In particular, entanglement shared between two distant parties allows them to do certain tasks that would otherwise be impossible. In this context, we study the effect on the available entanglement of physical restrictions on the local operations that can be performed by the two parties. We enforce these physical restrictions by generalized superselection rules (SSRs), which we define to be associated with a given group of physical transformations. Specifically the generalized SSR is that the local operations must be covariant with respect to that group. Then we operationally define the entanglement constrained by a SSR, and show that it may be far below that expected on the basis of a naive (or “fluffy bunny”) calculation. We consider two examples. The first is a particle number SSR. Using this we show that for a two-mode BEC (with Alice owning mode $A$ and Bob mode $B$), the useful entanglement shared by Alice and Bob is identically zero. The second, a SSR associated with the symmetric group, is applicable to ensemble QI processing such as in liquid-NMR. We prove that even for an ensemble comprising many pairs of qubits, with each pair described by a pure Bell state, the entanglement per pair constrained by this SSR goes to zero for a large ensemble.

1. Introduction

Entanglement is profoundly important in quantum information (QI) \(^1\), and has been much studied in recent years. Surprisingly, it is still a contro-
versial topic, even for pure states. For example, consider a Bose-Einstein condensate (BEC) containing \(N\) atoms, suddenly split into two modes (say two internal states, or two wells).

Sørenson, Duan, Cirac and Zoller\(^2\) claimed that after some evolution (inter-mode cycling and intra-mode collisions), there would be entanglement between the particles. They showed that this would be useful for precision measurement, exactly as in non-condensed spin-squeezing\(^3\). Hines, McKenzie and Milburn\(^4\), however, criticized this characterization of entanglement, saying that “the decomposition of the system . . . into subsystems made up of individual bosons is not physically realizable, due to the indistinguishibility of the bosons within the condensates.”

The power of entanglement is seen most strikingly in its form as a non-local resource, allowing two distant parties to do certain tasks that would otherwise be impossible. In this context, one can understand the criticism by Hines et al. of the formalism of Sørenson et al. As an alternative, Hines et al. propose calculating the entanglement between the two modes, which is maximal immediately after the split. However, we argue, this entanglement can only be accessed by the two parties if they violate fundamental conservation laws. Thus, in both of the above formalisms, the calculated entanglement is useless as a non-local resource for QI tasks. In this context, both are examples of what Burnett has called fluffy bunny entanglement\(^5\).

In this work we show how to calculate entanglement between two distant parties in the presence of physical restrictions on the local operations they can perform. The first step is to give an operational definition of entanglement in the presence of physical constraints\(^6\) (Sec. II). The next is to define generalized superselection rules (SSRs) as a way to formulate a wide variety of constraints\(^7\) (Sec. III). For such constraints an expression for the entanglement can then be derived and, for pure states, simply evaluated (Sec. IV). The two main examples we have considered are the SSR associated with particle number conservation\(^6\) (Sec. V) and the SSR associated with particle permutation covariance\(^7\) (Sec. VI). We conclude in Sec. VII with a summary and a discussion of future investigations.

2. Operational Definition of Bipartite Entanglement

Say two distant parties, Alice and Bob, share some quantum system \(S\) which for simplicity we will for the moment assume to be in a pure state \(\ket{\psi_{AB}}\). Bipartite entanglement is a resource that enables Alice and Bob to do certain quantum tasks, such as teleportation\(^1\), independent of the
medium that holds that entanglement. We thus assume that, in addition to $S$, Alice and Bob each have a conventional quantum register, initially pure. We now define the entanglement in $S$ to be the maximal amount of entanglement which Alice and Bob can produce between their quantum registers by physically allowed local operations $O$. With no physical constraints, the entanglement would simply be

$$E(|\psi_{AB}\rangle) \equiv S(\rho_A),$$

where $S(\rho) = -\text{Tr}[\rho \log_2 \rho]$, and $\rho_A = \text{Tr}_B[|\psi_{AB}\rangle\langle\psi_{AB}|]/\langle\psi_{AB}|\psi_{AB}\rangle$. Here $\text{Tr}_B$ denotes the trace over Bob’s Hilbert space, and for convenience we are allowing for unnormalized states. With physical constraints, it may be impossible to transfer this entanglement to the registers, so the constrained entanglement will be less than $E(|\psi_{AB}\rangle)$. A simple example to which we will return is the one-particle state $|0_A\rangle|1_B\rangle + |1_A\rangle|0_B\rangle$. This appears to contain 1 ebit of entanglement. But to transfer this into the conventional register would require a $\text{SWAP}$ gate $^1$. Such a gate between this system and a conventional register is physically forbidden, because it could act as follows:

$$\text{SWAP}(\alpha|0_{A}^{\text{reg}}\rangle + \beta|1_{A}^{\text{reg}}\rangle)|0_{A}^{\text{sys}}\rangle = |0_{A}^{\text{reg}}\rangle(\alpha|0_{A}^{\text{sys}}\rangle + \beta|1_{A}^{\text{sys}}\rangle).$$

Creating such a superposition, of the vacuum and one particle, would violate fundamental conservation laws, such as those for charge, lepton number et cetera, and hence is forbidden by superselection rules (SSRs).

### 3. Physical Constraints as Generalized SSRs

We define a SSR to be a restriction (fundamental or practical) on the allowed local operations on a system. It is not, in our view, a restriction on its allowed states. Note that “operations” includes measurements as well as unitaries $^1$. More particularly, we define a SSR to be a rule associated with a group $G$ of local physical transformations $g$. The rule, which we denote the $G$-SSR, is that operations must be $G$-covariant. Here we define an operation $O$ to be $G$-covariant if

$$\forall \rho \text{ and } \forall g \in G, \ O[\hat{T}(g)\rho\hat{T}^\dagger(g)] = \hat{T}(g)[O\rho]\hat{T}^\dagger(g),$$

where $\hat{T}(g)$ is a unitary representation of the group $G$.

Traditionally $^8$ one talks of a SSR for an operator, rather than a SSR associated with a group of transformations. For example, a SSR for local charge $\hat{Q}$ means that it is forbidden to create a superposition of states with different $Q$-values. This can be derived from the conservation of global charge, an assumption that the initial state of the Universe had definite
charge, plus the fact that global charge is the sum of local charges. However, as we will see, other SSRs may be practical rather than fundamental constraints. Note that it is pointless to talk of a SSR for global charge because that is a conserved quantity. Local charge is a non-conserved quantity so a SSR for it is meaningful.

The terminology “SSR for \( \hat{Q} \)” or “\( \hat{Q} \)-SSR” is compatible with our above definition if it is read as “SSR associated with the Lie group \( G \) generated by \( \hat{Q} \).” That is, the operations that cannot create local charge superpositions satisfy \( \forall \rho \) and \( \forall \xi \in \mathbb{R}, \mathcal{O}[e^{-i\xi \hat{Q}}\rho e^{i\xi \hat{Q}}] = e^{-i\xi \hat{Q}}[\mathcal{O}[\rho]e^{i\xi \hat{Q}}]. \)

If a SSR associated with \( G \) is in force, then all experimental predictions are unchanged if a state \( \rho \) is replaced by the state \( \hat{T}(g)\rho \hat{T}^\dagger(g) \) for any \( g \in G \). The most mixed state (that is, the state containing no irrelevant information) with which \( \rho \) is physically equivalent is

\[
\mathcal{G}_\rho = \begin{cases} (\dim G)^{-1} \sum_{g \in G} \hat{T}(g)\rho \hat{T}^\dagger(g), & \text{finite groups} \\ \int_G d\mu_{\text{Haar}}(g) \hat{T}(g)\rho \hat{T}^\dagger(g), & \text{Lie groups} \end{cases}.
\]

We call this the \( G \)-invariant state, as \( \forall g \in G, \hat{T}(g)[\mathcal{G}_\rho] \hat{T}^\dagger(g) = \mathcal{G}_\rho \).

**4. Entanglement Constrained by a \( G \)-SSR**

Having precisely defined SSRs, we can now generalize (and specialize) our operational definition of entanglement to

\[
E_{G,\text{SSR}}(\rho_{AB}^{\text{sys}}) = \max_{\mathcal{O}} E_D(\text{Tr}_{\text{sys}}[\mathcal{O}(\rho_{AB}^{\text{sys}} \otimes \varrho_{AB}^{\text{reg}})]).
\]

The generalization is that we have allowed for a mixed system state \( \rho_{AB}^{\text{sys}} \).

As a consequence the entanglement is not uniquely defined, so we have to specify the entanglement measure. Since we are interested in how much useful entanglement ends up in the registers, the entanglement of distillation \( E_D \) is a natural choice. The initial register state \( \varrho_{AB}^{\text{reg}} \) is still pure and separable. The specialization of our previous definition is that the physical restrictions are those enforced by a generalized SSR. That is, the operations \( \mathcal{O} \) are \( G \)-covariant local operations. We can now prove the following:

**Theorem:** The SSR can be enforced by removing all irrelevant information from \( \rho \) by the decoherence process \( \rho \rightarrow \mathcal{G}_\rho \). That is,

\[
E_{G,\text{SSR}}(\rho_{AB}) = E_D(\mathcal{G}_\rho \rho_{AB}).
\]

**5. Example: Particle Number SSR**

As for charge, global conservation laws and suitable initial conditions lead to a SSR for local particle number \( \hat{N} \), which is however not conserved.
Because they are not subject to a number SSR, quanta of excitation such as photons or excitons are not particles by our meaning of the word. But electrons, protons, and Rubidium atoms in a specified electronic state are.

Note that the global conservation law need not be for particle number; the total number of Rb atoms in the universe is not conserved. However there are conservation laws for lepton number, baryon number and so on that ensure that there is a Rb atom number SSR.

In this case the entanglement constrained by the $\hat{N}$-SSR is

$$E_{\hat{N} - \text{SSR}}(\rho_{AB}) = E_D(\hat{N}\rho_{AB})$$

(7)

where the operation $\hat{N}$ destroys coherence between eigenspaces of $\hat{\Pi}_n$, having different local particle number $n$:

$$\hat{N}\rho_{AB} = \sum_n \hat{\Pi}_n \rho_{AB} \hat{\Pi}_n.$$ 

(8)

Here, “local” could mean either Alice’s or Bob’s; it makes no difference.

Consider a simple pure-state example. We use the notation of separating the occupation numbers of Alice’s mode(s) from those of Bob’s mode(s) by a comma, as in $|n_A, n_B\rangle$. For a particle in a mode split between Alice and Bob, the state is $|\psi\rangle = |0,1\rangle + |1,0\rangle$: a superposition, apparently with one e-bit of entanglement. But the equivalent invariant state is $\hat{N}(|\psi\rangle\langle\psi|) = |0,1\rangle\langle0,1| + |1,0\rangle\langle1,0|$, an unentangled mixture.

For general pure states (which we here assume to be normalized), the entanglement constrained by the $\hat{N}$-SSR is

$$E_{\hat{N} - \text{SSR}}(|\psi_{AB}\rangle) = \sum_n \langle\psi_{AB}|\hat{\Pi}_n|\psi_{AB}\rangle S\left(\text{Tr}_B \left[ \hat{\Pi}_n |\psi_{AB}\rangle\langle\psi_{AB}|\hat{\Pi}_n \right] \right).$$

(9)

Some pure state examples are given in Table 1.

| Description          | State                     | $E_{\text{Hines}}$ | $E_{\text{Sør}}$ | $E_{\hat{N} - \text{SSR}}$ |
|----------------------|---------------------------|---------------------|------------------|---------------------------|
| split particle       | $|0,1\rangle + |1,0\rangle$ | 1                   | -                | 0                         |
| Hines-entangled “BEC”| $|0,2\rangle + \sqrt{2}|1,1\rangle + |2,0\rangle$ | 3/2                 | 0                | 0                         |
| Sørenson-entangled “BEC” | $|1,1\rangle$ | 0                   | 1                | 0                         |
| either-entangled “BEC”  | $|0,2\rangle + |2,0\rangle$ | 1                   | 1                | 0                         |
| 2 split particles    | $|0,1\rangle + |1,0\rangle)^{\otimes 2}$ | 2                   | 1                | 1/2                       |
| Bell pair            | $|01,10\rangle + |10,01\rangle$ | 1                   | 2                | 1                         |
| ?                    | $|11,00\rangle + |00,11\rangle$ | 1                   | 2                | 0                         |
| $M$ split particles  | $(|0,1\rangle + |1,0\rangle)^{\otimes M}$ | $M$                 | ?                | $\sim M$                  |

Table 1. Entanglement of various states according to the measure of Hines et al., Sørenson et al., and the present work.
We now discuss some properties of $E_{N-SSR}$ illustrated by these examples. The first is super-additivity $^6$:

$$E_{N-SSR}(|\psi\rangle \otimes |\phi\rangle) \geq E_{N-SSR}(|\psi\rangle) + E_{N-SSR}(|\phi\rangle).$$

(10)

All standard measures of entanglement are sub-additive $^{10}$. One could attribute this anomaly to the fact that for identical particles one $|\psi\rangle$ is not truly independent of another $|\phi\rangle$.

The second property is asymptotic recovery of standard entanglement $^6$. For a large number $C$ of copies of a state $|\psi\rangle$,

$$E_{N-SSR}(|\psi\rangle^{\otimes C}) \sim CE(|\psi\rangle) - \frac{1}{2} \log_2(V_\psi C) + O(1),$$

(11)

$$\lim_{C\to\infty} E_{N-SSR}(|\psi\rangle^{\otimes C}) / E(|\psi\rangle^{\otimes C}) = 1.$$  

(12)

6. Example: Ensemble QIP

Ensemble QI processing means (i) there are $N \gg 1$ identical copies of a system ("a molecule") containing $M$ qubits, and (ii) all operations are collective (i.e., affect each molecule identically). For example, in NMR QIP $^{11}$ each molecule contains $M$ atoms having a spin-$\frac{1}{2}$ nucleus. The collective operations use (in general) spatially uniform RF magnetic pulses for unitaries and fixed external antennae for measurements. In liquid-NMR the molecules can only be prepared in highly mixed states. We show that even if pure states could be produced, the above restrictions imply that the useful entanglement per molecule goes to zero as $N \to \infty$.

The restriction on operations $O$ can be formulated as the SSR

$$O[\hat{T}(p)\rho\hat{T}^\dagger(p)] = \hat{T}(p)[O\rho]\hat{T}^\dagger(p)$$

(13)

Here $p$ is a permutation of the $N$ molecules and $\hat{T}(p)$ is the unitary operator that implements that permutation. We call this the $S_N$-SSR, as the $N!$ permutations $p$ form a group called the Symmetric group $S_N$. We define the $S_N$-invariant (that is, randomly permuted) state

$$P\rho \equiv \frac{1}{N!} \sum_{p \in S_N} \hat{T}(p)\rho\hat{T}^\dagger(p).$$

(14)

To understand the above, consider a simple example. Say $M = 2$ (two nuclei, $A$ and $B$, per molecule) and $N = 2$ (there are two molecules, 1 and 2), and the state is

$$|\psi\rangle = |\uparrow_A^1 \downarrow_B^1 \rangle |\uparrow_A^2 \downarrow_B^2 \rangle.$$  

(15)
We consider that the $A$s belong to Alice and the $B$s to Bob, and the $S_2$-SSR applies independently to Alice and to Bob. Now if Alice’s local operations (acting only on nucleus $A$) cannot distinguish molecules 1 and 2, then this state is equivalent to
\[
\hat{T}_A(1 \leftrightarrow 2)|\psi\rangle = |\downarrow_A^1 \uparrow_A^1 \downarrow_B^2 \uparrow_B^2\rangle. \tag{16}
\]
Under the action of $\mathcal{P}_A$ (or $\mathcal{P}_B$), $|\psi\rangle$ goes to an equal mixture of these two states, and all correlations are lost.

Now consider a more interesting example, where $N = 2J$ and each molecule of the above sort is prepared in a pure Bell state
\[
|\Phi\rangle = |\uparrow_A \uparrow_B\rangle + |\downarrow_A \downarrow_B\rangle. \tag{17}
\]
How much entanglement do Alice and Bob have at their disposal? The naïve answer (no restrictions) is $N$ ebits — 1 per molecule. By contrast, the constrained entanglement is
\[
E_{S_N-SSR}(|\Phi\rangle^\otimes N) = \sum_{j=0}^{J} \frac{(2j + 1)^2}{2^{2j}(J + j + 1)} \left( \frac{2J}{J - j} \right) \log_2(2j + 1) \sim \frac{1}{2} \log_2 N
\]
so the entanglement per molecule goes to zero as $N \to \infty$.

7. Discussion

In this work we have argued as follows. Bipartite entanglement is a resource that enables the two parties to do certain quantum tasks, independent of the medium that holds it. For many systems there are restrictions upon the physical operations, so naïve calculations of entanglement may overestimate it. For such systems we operationally define the entanglement as the amount of distillable entanglement that can be produced between two conventional (i.e. unrestricted) registers. If the restrictions can be formulated as a generalized superselection rule (which we have defined) then we can derive an explicit expression for this entanglement.

We have considered two SSRs in detail, those associated with the group generated by local particle-number, and the group of local permutations of particles. We have applied the first to a two-mode BEC (with Alice owning mode $A$ and Bob mode $B$), and find that the entanglement shared by Alice and Bob is identically zero. We have applied the second to a pure NMR “ensemble Bell-state” (with Alice owning nucleus $A$ and Bob nucleus $B$), and find that their entanglement per molecule is asymptotically zero.

There are many other aspects of our work discussed in the papers \[.\] Our work also opens up many avenues of future investigation. First, there
are the relations with reference frames and quantum communication \(^{12}\), and with quantum nonlocality \(^{13}\). Second, there is the question of how to treat physical restrictions not expressible as a SSR (as defined by us). In particular, QIP in NMR is *more restricted* than implied by our SSR because there are no controllable inter-molecular interactions. Thirdly, there is an apparent duality between our conclusion, that in the presence of a SSR, a *nonseparable* \(\rho\) *does not imply* that the system is *entangled*, with the conclusion of Verstraete and Cirac \(^{14}\), that in the presence of a SSR, a *separable* \(\rho\) *does not imply* that the system is *locally preparable*.

It is clear that SSRs place severe constraints on QI processing, and our operational definitions of SSRs and entanglement constrained by them provide a new understanding and a valuable tool to QI science.

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