The 2n-point renormalized coupling constants in the 3d Ising model: estimates by high temperature series to order $\beta^{17}$

P. Butera and M. Comi

Istituto Nazionale di Fisica Nucleare
Dipartimento di Fisica, Università di Milano
Via Celoria 16, 20133 Milano, Italy

Abstract

Abstract: We compute the 2n-point renormalized coupling constants in the symmetric phase of the 3d Ising model on the sc lattice in terms of the high temperature expansions $O(\beta^{17})$ of the Fourier transformed 2n-point connected correlation functions at zero momentum.

Our high temperature estimates of these quantities, which enter into the small field expansion of the effective potential for a 3d scalar field at the IR fixed point or, equivalently, in the critical equation of state of the 3d Ising model universality class, are compared with recent results obtained by renormalization group methods, strong coupling, stochastic simulations as well as previous high temperature expansions.

PACS numbers: 05.50+q, 11.15.Ha, 64.60.Cn, 75.10.Hk
I. INTRODUCTION

In recent times a considerable effort has been devoted to the evaluation of the 2n-point dimensionless renormalized coupling constants (RCC’s) at zero momentum for the Ising model in three dimensions. These quantities are of interest for constructing the field theoretic effective potential \( g \) of a 3d scalar field at the infrared fixed point or, in statistical mechanics language, for the formulation of the critical equation of state of the 3d Ising model universality class \[ 3, 4, 12–14 \]. The computational methods, which so far have been used, include various approximate forms \[ 6–11 \] of the renormalization group (RG), the field theoretic strong coupling expansion \[ 2 \], the high temperature (HT) expansion \[ 3, 4 \] and (single- or multi-cluster) Monte Carlo techniques \[ 5, 15–17 \].

In this note we want to discuss how helpful in getting a first estimate of the RCC’s in the symmetric phase, can be extensive HT expansion data published long ago \[ 18 \] and so far only partially analyzed. Indeed expansions as double series in the HT variables \( v = tanh(\beta) \) and \( \tau = exp(\beta H) \), where \( \beta \) is the inverse temperature, are available for the Ising model free energy in a magnetic field \( H \) on various 2-, 3- and 4-dimensional lattices. In particular, in the 3d case the series extend up to order \( v^{17} \) for the sc lattice, up to \( v^{13} \) for the bcc lattice and up to \( v^{10} \) for the fcc lattice. By computing the 2n-th derivative of the free energy with respect to the magnetic field at zero field we readily obtain the HT expansion of the Fourier transformed 2n-point connected correlation function at zero momentum (also called the 2n-th susceptibility)

\[
\chi_{2n}(v) = \sum_{x_2,x_3...x_{2n}} < s(0)s(x_2)s(x_3)...s(x_{2n}) >_c
\]  

These expansions together with that of the second moment correlation length

\[
\xi^2(v) = \frac{\mu_2(v)}{\mu_3(v)}
\]

are the essential ingredients for the calculation of the RCC’s.

The expansion of the second moment of the 2-point correlation function \(\mu_2(v)\) on the sc lattice has been recently extended in Ref. \[ 19 \].

In terms of these quantities, the first few RCC’s, in the symmetric phase, are defined \[ 2 \] as the values \( g_{2n}^+ \), \( n \geq 2 \), that the following expressions

\[
g_4(v) = -\frac{V}{4!} \frac{\chi_4(v)}{\xi^4(v)}
\]

\[
g_6(v) = \frac{V^2}{6!} \left( -\frac{\chi_6(v)}{\xi^6(v)} + 10 \frac{\chi_7(v)}{\xi^7(v)} \right)
\]

\[
g_8(v) = \frac{V^3}{8!} \left( -\frac{\chi_8(v)}{\xi^8(v)} + 56 \frac{\chi_9(v)}{\xi^9(v)} \chi_2^2(v) - 280 \frac{\chi_3^2(v)}{\xi^3(v)} \chi_2^3(v) \right)
\]

\[
g_{10}(v) = \frac{V^4}{10!} \left( -\frac{\chi_{10}(v)}{\xi^{10}(v)} + 120 \frac{\chi_8(v)}{\xi^8(v)} \chi_2^4(v) + 126 \frac{\chi_4^2(v)}{\xi^{12}(v)} \chi_2^5(v) - 4620 \frac{\chi_6(v)}{\xi^{12}(v)} \chi_2^6(v) + 15400 \frac{\chi_4^3(v)}{\xi^{12}(v)} \chi_2^7(v) \right)
\]

take as \( v \uparrow v_c \). The volume \( V \) per lattice site has the value 1 for the sc lattice, \( 4/3\sqrt{3} \) for the bcc lattice and \( 1/\sqrt{2} \) for the fcc lattice.
We recall that scaling implies that, as the critical temperature is approached from above, we have \( \chi_{2n} \approx B_{2n}^\pm (v_c - v)^{-\gamma - (2n-2)\Delta} \), where \( \Delta \) is the gap exponent. If we also assume the validity of hyperscaling, we have \( 2\Delta = 3\nu + \gamma \) (where \( \nu \) and \( \gamma \) are the critical exponents of \( \xi \) and \( \chi \) respectively), so that the RCC’s are finite (and universal) quantities. The quantities \( g_{2n} \) are expected \(^{20}\) to be of the form \( g_{2n}(v) \approx g_{2n}^+ + A_{2n}^+ (v_c - v)^\theta + ... \) as \( v \uparrow v_c \), where the dominant universal scaling correction exponent \( \theta \) has the value \( \theta = 0.50(2) \) \(^{23}\) for the 3d Ising model.

By changing in the functions \( g_{2n}(v) \) the variable \( v \) into \( y = \xi^2(v) \), we obtain the strong coupling expansions, through the order \( y^{17} \), of the functions \( \gamma_{2n}(y) \) \(^{24}\) whose values at \( y = \infty \) give the RCC’s.

Let us add a few comments concerning the HT and the strong coupling series coefficients of the \( \chi_{2n} \) on the sc lattice that we have tabulated, up to order \( v^{17} \), in the appendix together with the coefficients of the second moment of the correlation function \( \mu_2(v) \) in order to provide the interested reader with all data we have used and thus make our calculations easily reproducible. The expansion for \( \chi_4 \) was first computed \(^{21}\) through \( v^{17} \) using the data of Ref. \(^{18}\), but only recently we could check it against a completely independent linked-cluster computation through the same order \(^{18}\). We should only draw attention to a minor misprint in the last 2 digits of the corrections tabulated there.\(^{18}\) Concerning the strong coupling expansions, we notice that in Ref. \(^{4}\) \( \gamma_6(y) \) has been tabulated, for any space dimension, through order \( y^{11} \), while \( \gamma_8(y) \) and \( \gamma_{10}(y) \) through order \( y^7 \) only. A further significant extension of all these series can still be performed \(^{19}\): only then a complete check against an independent computation will be possible for the coefficients tabulated here.

While this work was being completed, we became aware of a related work \(^{14}\), also devoted to the analysis of the data of Ref. \(^{18}\), and where also the low temperature side of the critical region is studied. We decided therefore to present only the part of our computation, mainly concerning the higher RCC’s, which was not already covered by the very thorough discussion of Ref. \(^{14}\). In fact the availability of a longer HT expansion of \( \xi^2 \) enables us to study also individual RCC’s rather than only ratios among them, and moreover gives access to the strong coupling expansions.

II. NUMERICAL RESULTS

We shall now present our estimates of the first few RCC’s as obtained from either the HT or the strong coupling expansions and discuss various ”biased” or ”unbiased” numerical procedures.

In a first and straightforward approach we estimate \( g_{2n}^+ \) by evaluating at \( v = v_c \) \(^{22}\) near diagonal Padè approximants (PA’s) of the quantity \( f_{2n}(v) \equiv g_{2n}^{\frac{1}{2n-2}}(v) \) which has a Taylor expansion in \( v \). This procedure is not convenient for extrapolating \( g_{10}(v) \), which changes its sign at some \( 0 < v_0 < v_c \). In this case we should consider instead the expression \((\frac{v}{v_c})^8 g_{10}(v)\), which also has a Taylor expansion in \( v \). Thus, (biasing only \( v_c \)) we obtain the estimates: \( g_4^+ = 1.03(3) \), \( g_6^+ = 1.93(8) \), \( g_8^+ = 1.53(36) \), \( g_{10}^- = -2.0(9) \).

Here, as in the rest of this report, our estimates are given by a suitably weighted average over the results from the approximants using at least 14 series coefficients and the uncertainties are measured, conservatively, only on the basis of the spread of the results obtained from the highest approximants, always allowing also for the (much smaller) effects of the errors in \( v_c \) and \( \theta \).

It should be noticed that the central estimate of \( g_4^+ \) obtained above is slightly, but significantly larger than the well established RG estimate \( g_4^+ = 0.988(4) \) \(^{23}\).

This discrepancy leads us to investigate whether and to what extent these values are also
affected by a "systematic" error due to the non-analytic corrections to scaling which can spoil the convergence properties of the PA’s. It has been suggested in Ref. [20] that these corrections can be allowed for, or at least their effects can be significantly reduced, by performing the quadratic mapping \( v = v_c (1 - \frac{(1-z)^2}{(1-x)^2}) \) with \( p = 2\sqrt{2} - 1 \). Essentially the same results are also obtained by using appropriately designed first order differential approximants [27] in which we can bias both \( v_c \) and the scaling correction exponent \( \theta \). We arrive thus at our final set of estimates

\[
\begin{align*}
g_4^+ & = 0.987(4), \quad g_6^+ = 1.57(10), \quad g_8^+ = 0.90(10), \quad g_{10}^+ = -0.71(35). \quad (2)
\end{align*}
\]

While the value of \( g_4^+ \) is only slightly lowered (and thereby closely reconciled with the most accurate RG estimates), the central estimates of the higher \( g_{2n}^+ \) are significantly altered and the uncertainties are reduced. Therefore it appears that our initial very simple numerical approach was rather inadequate and moreover we infer that the amplitudes \( A_{2n}^+ \) of the scaling correction terms increase with \( n \). Finally, if we notice that the uncertainties of our estimates grow rapidly with the order of the RCC’s, it is clear why, with the presently available series, we have to restrict our calculations to the \( g_{2n}^+ \) with \( n \leq 5 \).

It is interesting also to study directly other quantities such as, for instance, appropriate ratios of the functions \( g_{2n}(v) \) which do not depend on \( \xi^2 \) and might be less sensitive to the scaling corrections, as a means to understand better the actual uncertainties of our numerical procedures. We have therefore considered the expression \( T_1^+ \equiv \frac{g_6(v)}{g_6(v)^2} |v|^4 v_c \) and we have obtained the estimate \( T_1^+ = 1.75(5) \) neglecting the confluent singularity and, otherwise, \( T_1^+ = 1.59(5) \). Analogously, we have also examined \( T_2^+ \equiv \frac{g_8(v)}{g_8(v)^2} |v|^6 v_c \) and have estimated \( T_2^+ = 1.29(43) \) by the first method and \( T_2^+ = 0.92(13) \) by the second, while for \( T_3^+ \equiv \frac{g_{10}(v)}{g_{10}(v)^2} |v|^8 v_c \) we obtain \( T_3^+ = -0.7(7) \) and \( T_3^+ = -0.35(20) \), respectively. All estimates of the \( T^+ \) are then completely consistent with the corresponding separate estimates of the \( g_{2n}^+ \). Notice that the \( T_i^+ \) are simply related to the coefficients \( F_i \) of the small field expansion of the "reduced effective potential" computed in Ref. [10] as follows: \( T_1^+ = 96F_5 \), \( T_2^+ = 1728F_7 \) and \( T_3^+ = \frac{331776}{10}F_9 \).

Let us also recall that long ago the sequence of universal amplitude combinations \( I_{2r+3}^+ \equiv \frac{\chi_2(v)^r \chi_{2r+4}(v)}{\chi_{2r+3}(v)} |v|^r v_c \), \( r \geq 1 \), which are strictly related to the \( T_i^+ \), was introduced in Ref. [28] and, by using twelve term series [12], the first few \( I_i^+ \) were estimated to be \( I_5^+ = 7.73 \), \( I_7^+ = 157.5 \), and \( I_9^+ = 6180 \) (with no indication of error). Our estimates by using the direct PA procedure are \( I_5^+ = 7.81(3) \), \( I_7^+ = 161.7(3) \), \( I_9^+ = 6395(21) \), while if we allow for the scaling corrections, we find

\[
\begin{align*}
I_5^+ & = 7.92(7), \quad I_7^+ = 165.4(4), \quad I_9^+ = 6809.120.) \quad (3)
\end{align*}
\]

As it appears from the smaller difference between the results of the two kinds of numerical procedures, the \( T_i^+ \) and especially the \( I_i^+ \) turn out to be less sensitive to the scaling corrections than the \( g_{2n}^+ \) and therefore we assume that they can be be determined with higher relative accuracy. It is therefore interesting to notice that from the above estimate of \( I_5^+ \) we get the value \( g_5^+ = 1.62(6) \). Unfortunately however, at the present level of accuracy, the other simple relations among the higher \( T_i^+ \) and the \( I_i^+ \), like \( T_2^+ = \frac{12}{35} (+I_2^+ - 56I_3^+ + 280) \) etc., which follow from the definitions of the \( g_{2n}^+ \), cannot be used for improving the estimates of the higher \( T_i^+ \), and therefore of the corresponding \( g_{2n}^+ \), by expressing them in terms of the \( I_i^+ \). For instance, \( T_2^+ \) turns out to be a small difference between large numbers and the uncertainty of \( I_3^+ \) is strongly amplified. For similar reasons it is also not useful to start directly with the critical amplitudes of the \( \chi_{2n} \).
An unbiased study of the RCC’s can be performed starting with the strong coupling expansion. In Ref. [2] an elaborate extrapolation procedure was proposed which involves the dependence of the series coefficients on the space dimensionality. We have not yet computed this dependence up to order $v^{17}$ and therefore we cannot reproduce this procedure. We can, however, try the simplest approach to evaluate $\gamma_{2n}(\infty)$, which consists in forming $[N + 1/N]$ PA’s to the quantity $y^{\frac{2n}{2n-3}}(y)$ and in dividing them by $y$. This procedure is not very efficient and the only reasonably stable results obtained are: $g_{6}^{+} = 2.1(2), g_{8}^{+} = 1.9(2)$. We can also evaluate the ratios $T_{i}^{+}$ by diagonal PA’s: again we find reasonable results only for $T_{1}^{+} = 1.81(4)$. All these values are consistent with our previous first evaluation of these quantities.

Alternatively, we can generalize a technique introduced in Ref. [29], which consists in inverting the functions $z_{2n} = \gamma_{2n}^{\frac{2}{2n-3}}(y)$ (after checking that the dependence of $z_{2n}$ on $y$ is monotonic) and in determining $g_{2n}^{+}$ from the value of $z_{2n}$ where $y = y(z_{2n})$ diverges. This is conveniently done by forming PA’s of the logarithmic derivative of $y$. The results are then: $g_{4}^{+} = 1.01(2), g_{6}^{+} = 1.63(6), g_{8}^{+} = 1.05(15)$. As indicated above, these procedures cannot be used for computing $g_{10}^{+}$.

In conclusion, we believe that the general consistency among the results obtained by applying suitable approximation procedures to various quantities with somewhat different properties corroborates our estimates in (2).

### III. A COMPARISON WITH OTHER ESTIMATES

Let us now proceed to a comparison with the results already available in the literature.

Our values in (2) for $g_{4}^{+}$ and for $g_{6}^{+}$ are not far from the estimates $g_{4}^{+} = 1.018(1)$ and $g_{6}^{+} = 1.793(16)$ obtained in Ref. [14] from the analysis of the same HT series. A similar remark applies to the estimates $g_{4}^{+} = 0.988(60)$ and $g_{6}^{+} = 1.92(24)$ obtained in Ref. [3] from a sixteen term HT series. Our result for $I_{5}^{+}$ in (3) is also not far from the recent estimate $I_{5}^{+} = 7.84(2)$ of Ref. [14].

As to the strong coupling approach, we us recall that in Ref. [2] the estimate $g_{6}^{+} = 1.2(1)$ was obtained from an eleven term strong coupling series.

It is also interesting to perform a comparison with the results obtained in the most extensive recent RG study [10] by the fixed dimension (FD) expansion [25] up to five loop order, resummed by the Borel-Leroy technique combined with an appropriate conformal mapping. The estimate of $g_{4}^{+}$ agrees perfectly with ours, the central values $g_{6}^{+} = 1.603(6)$ and $g_{8}^{+} = 0.82(8)$ agree with ours within $\approx 2\%$ and $\approx 9\%$, respectively. On the other hand the larger disagreement about the value of $g_{10}^{+}$ should not be taken too seriously because, as noted above, the uncertainty which affects the calculation grows with the order of the RCC. Let us also return to a previous remark, noticing that from the estimates of the $F_{i}$ in Ref. [25] one arrives at the values $I_{5}^{+} = 7.945(7), I_{7}^{+} = 167.45(65)$ and $I_{9}^{+} = 6718.81(1)$, in very close agreement with our estimates in (3). Unfortunately, $I_{7}^{+}$ and $I_{9}^{+}$ are actually rather insensitive to the values of $F_{7}$ and $F_{9}$.

We also ought to recall that an independent calculation in the FD scheme gave the estimates $g_{6}^{+} \simeq 1.50$ in the two loop approximation [3], $g_{6}^{+} \simeq 1.622$ at three loop order with Padè-Borel resummation [5], and $g_{8}^{+} \simeq 1.596$ at four loops [3]. On the other hand, from a 3 loop computation, values for $g_{8}^{+}$ have been obtained [5] which range from 0.68 to 2.71, depending on the resummation procedure.

The approximate truncation of the RG flow equations studied in Ref. [6] yields $g_{4}^{+} = 1.2$ and $g_{6}^{+} = 2.25$, which are both clearly larger than our values, although the ratio $g_{6}^{+}/g_{4}^{+} = 1.56$ agrees
well with our estimate. An analogous, but lower order truncation of the RG flow equations [11] had given $g^+_6 = 2.40$.

The $\epsilon = 4 - d$ expansion approach has not yet been pushed beyond order $\epsilon^3$. It has been used, in Ref. [10], to produce the (rather large) estimates $g^+_4 = 1.167$, $g^+_6 = 2.30(5)$, $g^+_8 = 1.24(8)$ and $g^+_10 = -1.97(12)$. Notice however that, since also the estimate of $g^+_4$ is unusually large, the corresponding values of the $T^+_i$ (or of the $F_i$) agree very closely with the results from the FD approach. The $\epsilon$ expansion of $T^+_i$ was examined also in Ref. [7], where by Padè-Borel resummation the estimate $T^+_i = 1.653$ was obtained.

Finally, we recall that the MonteCarlo simulations of Ref. [5] indicate $g^+_6 = 2.05(15)$, which is not very far from our estimate, while the simulations described in Ref. [16] indicate the values $g^+_6 = 2.7(2)$ and $g^+_8 = 4.3(6)$, significantly larger than both the RG results and ours.

A summary of the present situation is presented in Table 1 which collects our estimates of the RCC’s along with the corresponding ones obtained by other methods.

IV. CONCLUSIONS

We may conclude that, although the various computational approaches do not yet agree perfectly, they do appear to converge to common estimates at least for the lowest RCC’s. Therefore, in view of the difficulty of these calculations, we believe that the present residual discrepancies should not be overemphasized. The $\epsilon$ expansion is certainly still too short, and perhaps, even for the FD expansions, a further extension would be welcome. The HT series presented here are not yet long enough, the more so the higher the order of the RCC considered. Indeed, we might argue that, at the order $v^8$, the dominant contributions to the HT expansion of $\chi^{2n}(v)$ come from correlation functions of spins whose average relative distance is $\simeq s/2n$, so that present HT expansions, in some sense, still describe a rather "small" system. Analogous problems of size also occur in stochastic simulations [5,16,17]. Therefore further effort would still be welcome to improve the reliability, the precision and, as a result, the consistency of the various approaches.

ACKNOWLEDGMENTS

This work has been partially supported by MURST. We are grateful to Prof. A. I. Sokolov for a very stimulating correspondence. We also thank to Prof. A. I. Sokolov, Dr. R. Guida and Prof. J. Zinn-Justin for making their results available to us before publication.

APPENDIX A: SERIES EXPANSIONS

In the case of the sc lattice the HT expansion of the susceptibilities $\chi^{2n}$ are

$$\chi_2(v) = 1 + 6v + 30v^2 + 150v^3 + 726v^4 + 3510v^5 + 16710v^6 + 79494v^7 + 375174v^8 + 1769686v^9$$

$$+8306862v^{10} + 38975286v^{11} + 182265822v^{12} + 852063558v^{13} + 3973784886v^{14}$$

$$+18527532510v^{15} + 86228667894v^{16} + 401225368086v^{17} ...$$

$$\chi_4(v) = -2 - 48v - 636v^2 - 6480v^3 - 56316v^4 - 441360v^5 - 3208812v^6 - 22059120v^7$$
The HT expansion of the second moment of the correlation function $\mu_2$ is

$$\mu_2(v) = 6v + 72v^2 + 582v^3 + 4032v^4 + 25542v^5 + 153000v^6 + 880422v^7 + 4920576v^8$$

$$+20879670v^9 + 144230088v^{10} + 762587910v^{11} + 3983525952v^{12} + 2095680694v^{13}$$

$$+10555845736v^{14} + 536926539990v^{15} + 2713148048256v^{16} + 13630071574614v^{17}.$$
The strong coupling expansions of the $\gamma_{2n}$ to order $y^{17}$ are

\[ \gamma_4(y) = \frac{y^{-3/2}}{12} \left[ 1 + 12y + 6y^2 + 48y^3 - 630y^4 + 7272y^5 - 83292y^6 + 957312y^7 - 11035662y^8 \right. \]

\[ + 12743528y^9 - 1472947908y^{10} + 17036529504y^{11} - 197169806676y^{12} + 2283416559216y^{13} \]

\[ - 26463582511368y^{14} + 306946999598144y^{15} - 3563327123879550y^{16} \]

\[ + 4140418822684120y^{17} \ldots \]

\[ \gamma_6(y) = \frac{y^{-3}}{30} \left[ 1 + 18y + 90y^2 + 48y^3 + 8352y^4 - 630y^5 - 1528416y^7 \right. \]

\[ + 19052712y^8 - 255983472y^9 + 3240722592y^{10} - 40613845392y^{11} + 505052958360y^{12} \]

\[ - 6242882909472y^{13} + 76802505994224y^{14} - 941288338072752y^{15} + 1150115866478208y^{16} \]

\[ - 140176233789711696y^{17} \ldots \]

\[ \gamma_8(y) = \frac{y^{-9/2}}{56} \left[ 1 + 24y + 192y^2 + 576y^3 + 54y^4 + 1753632y^7 \right. \]

\[ - 25771326y^8 + 364798032y^9 - 5028161232y^{10} + 67958735808y^{11} - 90482659212y^{12} \]

\[ + 11905472505792y^{13} - 155154712361520y^{14} + 2006059450196288y^{15} \]

\[ - 25765180820314374y^{16} + 329050927608994224y^{17} \ldots \]

\[ \gamma_{10}(y) = \frac{y^{-6}}{90} \left[ 1 + 30y + 330y^2 + 1620y^3 + 1080y^4 - 67200y^6 \right. \]

\[ - 1314720y^7 \]

\[ + 22683000y^8 - 363847600y^9 + 5564033040y^{10} - 82249187520y^{11} + 1185208196160y^{12} \]

\[ - 16740515134800y^{13} + 232658153938560y^{14} - 3190497478487440y^{15} \]

\[ + 4326237773737920y^{16} - 581022341984542560y^{17} \ldots \]
REFERENCES

∗ Electronic address: butera@mi.infn.it
∗∗ Electronic address: comi@mi.infn.it

[1] G. Jona-Lasinio, Nuovo Cim. 34, 1790 (1964).
[2] C. M. Bender, F. Cooper, G.S. Guralnik, H. Moreno, R. Roskies and D.H. Sharp, Phys. Rev. Lett. 43, 537 (1979), and Phys. Rev. D 23, 2999 (1981); C. M. Bender and S. Boettcher, ibid. D 48, 4919 (1993) and ibid. D 51, 1875 (1995).
[3] D. S. Gaunt and C. Domb, J. Phys. C3, 1442 (1970).
[4] S. Milosevic and H. E. Stanley, Phys. Rev. B 6, 987 (1972).
[5] M.M. Tsypin, Phys. Rev. Lett. 73, 2015 (1994).
[6] J. Berges, N. Tetradis, and C. Wetterich, Phys. Rev. Lett. 77, 873 (1996); N. Tetradis, and C. Wetterich, Nucl. Phys. B 422, 541 (1994).
[7] A. I. Sokolov, Phys. Solid State 38, 354 (1996).
[8] A.I. Sokolov, V. A. Ul’kov and E. V. Orlov, Universal effective action for 3D scalar $\lambda\phi^4$ theory from renormalization group, report at Dubna Renormalization Group 1996 Conference, St. Petersburg Preprint October 1996.
[9] A. I. Sokolov, Phys. Lett. A 227, 255 (1997).
[10] R. Guida and J. Zinn-Justin, 3D Ising model: The scaling equation of state, CEA-Saclay Preprint SPhT/96-116 October 1996, hep-th/9610223, submitted to Nucl. Phys. B.
[11] C. Bagnuls and C. Bervillier, Phys. Rev. B 41, 402 (1990).
[12] J. W. Essam and D. L. Hunter, J. Phys. C 1, 392 (1968).
[13] T. Reisz, Nucl. Phys. B 450, 569 (1995); H. Meyer-Ortmanns and T. Reisz, Critical phenomena with linked cluster expansions in a finite volume, hep-lat/9604006.
[14] S. Y. Zinn, S. N. Lai and M. E. Fisher, Phys. Rev. E 54, 1176 (1996).
[15] J.F. Wheater, Phys. Lett. B 136, 402 (1984).
[16] J.K. Kim and L. D. Landau, MonteCarlo computation of the effective potential for the three dimensional Ising system, hep-lat/9608072 to appear in Lattice96, Nucl. Phys. B Proc. Suppl..
[17] G. A. Baker and N. Kawashima, Renormalized coupling constant in the Ising model, Los Alamos Rep. LA-UR-96-2092 June 1996, to appear in J. Phys. A; J. K. Kim and A. Patrascioiu, Phys. Rev. D 47, 2588 (1993); J. K. Kim, Phys. Rev. Lett. 76, 2402 (1996); G. A. Baker and N. Kawashima, ibid 76, 2403 (1996).
[18] S. McKenzie Can. J. Phys., 57, 1239 (1979); S. Katsura, N. Yazaki and M. Takaishi, Can. J. Phys. 55, 1648 (1977).
[19] P. Butera and M. Comi, Phys. Rev. B 52, 6185 (1995), ibid. B 54, December (1996), and to be published.
[20] F. Wegner, Phys. Rev. B 5, 4529 (1972).
[21] A. J. Guttmann, Phys. Rev. B 33 (1986) 5089.
[22] At the level of accuracy of our calculation equivalent results are obtained by using either the value $v_c = 0.2180992(26)$ of Ref. 23 or the more recent value $v_c = 0.2180943(4)$ of Ref. 24 used in Ref. 14 with $\theta = 0.54(3)$.
[23] A. M. Ferrenberg and D.P.Landau, Phys.B 44, 5081(1991).
[24] H. W. Bloete and A. L. Talapov, The magnetization of the 3D Ising model, preprint cond-mat/9603013.
[25] J. Zinn Justin, Quantum field theory and critical phenomena, (Clarendon Press, Oxford 1992); G.Parisi, J. Stat. Phys. 23, 49 (1980); G.A. Baker, B. G. Nickel and D. I. Meiron, Phys. Rev.B 17, 1365 (1978);
J. C. Le Guillou and J. Zinn Justin, Phys. Rev. B 21, 3976 (1980); J. C. Le Guillou and J. Zinn Justin, J. Physique Lett. 45, 137 (1985);
J.H. Chen, M. E. Fisher and B. G. Nickel, Phys. Rev. Lett. 48, 630 (1982); S. Y. Zinn and M. E. Fisher, Physica A 226, 168 (1996); D.B. Murray and B.G. Nickel, Revised estimates for critical exponents for the continuum N-vector model in 3 dimensions, Unpublished Guelph University report (1991).

[26] R. Roskies, Phys. Rev. B 24, 5305 (1981); B.G. Nickel and M. Dixon, Phys. Rev. B 26, 3965 (1982).

[27] D. L. Hunter and G. A. Baker, Phys. Rev. B 7, 3346,(1973); 7, 3377, (1973); 19, 3808,(1979); M. E. Fisher and H. Au-Yang, J. Phys. A 12, 1677(1979) and 13, 1517(1980); A. J. Guttmann, in Phase Transitions and Critical Phenomena, edited by C. Domb and J. Lebowitz; (Academic Press, New York 1989), Vol 13; A. J. Guttmann and G. S. Joyce, J. Phys. A 5, L81(1972); J.J. Rehr,A. J. Guttmann and G. S. Joyce, J. Phys. A 13, 1587(1980).

[28] P.G. Watson, J. Phys. C 2, 1883 (1969).

[29] B. G. Nickel and B. Sharpe, J. Phys. A 12, 1819 (1979).
### TABLE I. A summary of the estimates of $g_{2n}^+$ by various methods.

| Method and Ref. | $g_4^+$ | $g_6^+$ | $g_8^+$ | $g_{10}^+$ |
|----------------|--------|--------|--------|-----------|
| HT            | 0.987(4) | 1.57(10) | 0.90(10) | -0.71(35) |
| Strong coupl. | 1.01(1)  | 1.63(5)  | 1.05(9)  |           |
| HT            | 1.019(6) | 1.791(38) |        |           |
| HT            | 0.988(60)| 1.92(24) |        |           |
| RG FD-expans. | 0.987(2) | 1.603(6) | 0.83(8) | -1.96(1.26) |
| RG FD-expans. |        | 1.596   | 0.68 - 2.71 |        |
| RG c-expans. | 1.167   | 2.30(5)  | 1.24(8)  | -1.97(12)  |
| RG approx.   | 1.2     | 2.25     |        |           |
| RG approx.   |        | 2.40     |        |           |
| Strong coupl. | 0.986(10)| 1.2(1)   |        |           |
| MC            | 0.97(2)  | 2.05(15) |        |           |
| MC            | 1.02     | 2.7(2)   | 4.3(6)  |           |