The Spin Group in Superspace

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Abstract

There are two well-known ways of describing elements of the rotation group $\text{SO}(m)$. First, according to the Cartan-Dieudonné theorem, every rotation matrix can be written as an even number of reflections. And second, they can also be expressed as the exponential of some anti-symmetric matrix.

In this paper, we study similar descriptions of a group of rotations $\text{SO}_0$ in the superspace setting. This group can be seen as the action of the functor of points of the orthosymplectic supergroup $\text{OSp}(m|2n)$ on a Grassmann algebra. While still being connected, the group $\text{SO}_0$ is thus no longer compact. As a consequence, it cannot be fully described by just one action of the exponential map on its Lie algebra. Instead, we obtain an Iwasawa-type decomposition for this group in terms of three exponentials acting on three direct summands of the corresponding Lie algebra of supermatrices.

At the same time, $\text{SO}_0$ strictly contains the group generated by super-vector reflections. Therefore, its Lie algebra is isomorphic to a certain extension of the algebra of superbivectors. This means that the Spin group in this setting has to be seen as the group generated by the exponentials of the so-called extended superbivectors in order to cover $\text{SO}_0$. We also study the actions of this Spin group on supervectors and provide a proper subset of it that is a double cover of $\text{SO}_0$. Finally, we show that every fractional Fourier transform in n bosonic dimensions can be seen as an element of this spin group.

Keywords. Spin groups, symplectic groups, Clifford analysis, bivectors, superspace

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1 Introduction

Supermanifolds and in particular superspaces play an important rôle in contemporary theoretical physics, e.g. in the particle theory of supersymmetry, supergravity or superstring theories, etc. Superspaces are equipped with both a set of commuting co-ordinates and a set of anti-commuting co-ordinates. From the mathematical point of view, they have been studied using algebraic and geometrical methods. Some pioneering references in the development of analysis on superspace are [2, 15, 24, 25, 26]. For more modern treatments we address the reader to [4, 14, 27, 31]. More recently, harmonic and Clifford analysis have been extended to superspace by introducing some important differential operators (such as Dirac and Laplace operators) and by studying special functions and orthogonal polynomials related to these operators, see e.g. [3, 6, 7, 8, 9, 10].

In classical harmonic and Clifford analysis in $\mathbb{R}^m$, the most important invariance group is the set of rotations $\text{SO}(m)$, i.e. the connected group of $m \times m$ real matrices leaving invariant the Euclidean inner product $\langle x, y \rangle = -\frac{1}{2} (xy^T + yx^T) = \sum_{j=1}^{m} x_j y_j$, $x, y \in \mathbb{R}^m$. Every rotation in $\text{SO}(m)$ can be written as the exponential of some anti-symmetric matrix and vice versa, each one of such exponentials is a rotation in $\mathbb{R}^m$. The Lie algebra of $\text{SO}(m)$ is given by the set $\mathfrak{so}(m)$ of anti-symmetric matrices.

On the other hand, the group $\text{SO}(m)$ can also be described by means of the spin group $\text{Spin}(m) := \left\{ \prod_{j=1}^{2k} w_j : k \in \mathbb{N}, w_j \in \mathbb{S}^{m-1} \right\}$, where $\mathbb{S}^{m-1} = \{ w \in \mathbb{R}^m : w^2 = -1 \}$ denotes the unit sphere in $\mathbb{R}^m$. The

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relation between Spin$(m)$ and SO$(m)$ is easily seen through the Lie group representation $h : \text{Spin}(m) \to \text{SO}(m)$

$$h(s)[x] = s \bar{x}, \quad s \in \text{Spin}(m), \ x \in \mathbb{R}^m,$$

which describes the action of every element of SO$(m)$ in terms of Clifford multiplication. It easily follows from the above representation that Spin$(m)$ constitutes a double cover of SO$(m)$. Such a description of the spin group follows from the Cartan-Dieudonné theorem which states that every orthogonal transformation in an $m$-dimensional symmetric bilinear space can be written as the composition of at most $m$ reflections. Basic references for this setting are [13, 19].

In this paper we study the similar situation in the framework of Clifford analysis in superspace, where the Cartan-Dieudonné theorem is no longer valid. The main goal is to properly define a spin group as a set of elements describing every super-rotation through Clifford multiplication. To that end, we consider linear actions on supervector variables using both commuting and anti-commuting coefficients in a Grassmann algebra $\Lambda(f_1, \ldots, f_N)$. This makes it possible to study the group of supermatrices leaving the inner product invariant and to define in a proper way the spin group in this case. It is worth noticing that the superstructures are absorbed by the Grassmann algebras leading to classical Lie groups and Lie algebras instead of supergroups or superalgebras.

Finite dimensional Grassmann algebras are the most simple coefficient algebras to consider if one wants to study linear actions mixing bosonic and fermionic variables. Moreover, the use of this coefficient algebra is already rich enough for the purposes of Clifford analysis. Indeed, in [12] it was shown that this group theoretical approach using Grassmann coefficients offers a new way of describing the $\mathfrak{osp}(m|2n)$-invariance of the super Dirac operator (see also [5]). This approach also leads to new insights in the study of linear actions (supermatrices) on supervectors. For example, the rotation group is no longer compact and it strictly contains the subgroup generated by an even number of supervector reflections (see Sections 4 and 5).

In a more algebraic geometric approach, one can generalize these groups by using the language of categories. For example, one can define the group SO$_0$ of superrotations via the functor of $A$-points that maps the category $\mathbb{R}$-$\text{Salg}$ of real commutative superalgebras to the category $\text{Grp}$ of groups as follows

$$\mathbb{R}$-$\text{Salg} \to \text{Grp} : A \mapsto \text{SO}_0(A), \quad A \in \mathbb{R}$-$\text{Salg}.$$

Here, SO$_0(A)$ denotes the group of $(m|2n)$-supermatrices satisfying the defining conditions of SO$_0$ (see Section 4.2) but with entries in the superalgebra $A$. This functor can be proved to be representable since superrotations are solutions of some set (super) polynomial equations (see e.g. [4, Chpt. 11] and [1] for other examples of representations of affine supergroups). In fact, this functor is isomorphic to the functor of points of the orthosymplectic supergroup $\text{OSp}(m|2n)$. However, the representability of the functor associated to the Spin group defined in this paper (see Section 5) seems to be more difficult to prove. We shall address this problem in future work.

The paper is organized as follows. We start with some preliminaries on Grassmann algebras, Grassmann envelopes and supermatrices in Section 2. In particular, we carefully recall the notion of an exponential map for Grassmann numbers and supermatrices as elements of finite dimensional associative algebras. In Section 3 we briefly describe the Clifford setting in superspace leading to the introduction of the Lie algebra of superbivectors. An extension of this algebra plays an important rôle in the description of the spin group in this setting. The use of the exponential map in such an extension necessitates the introduction of the corresponding tensor algebra. Section 4 is devoted to the study of the invariance of the “inner product” of supervectors. There, we study several groups of supermatrices and in particular, the group of superrotations SO$_0$ and its Lie algebra $\mathfrak{so}_0$, which combine both orthogonal and symplectic structures. We prove that every superrotation can be uniquely decomposed as the product of three exponentials acting in some special subspaces of $\mathfrak{so}_0$. Finally, in section 5 we study the problem of defining the spin group in this setting and its differences with the classical case. We show that the compositions of even numbers of vector reflections are not enough to fully describe SO$_0$ since they only show an orthogonal structure and do not include the symplectic part of SO$_0$. Next we propose an alternative, by defining the spin group through the exponential of extended superbivectors and show that they indeed cover the whole set of superrotations. In particular, we explicitly describe a subset $\Xi$ which is a double covering of SO$_0$ and contains in particular every fractional Fourier transform.
2 Preliminaries

In this section we provide some preliminaries on the Grassmann algebra of coefficients that is going to be used to define linear actions on supervector variables. The definitions and properties contained in this section are well known material and can be found for example in [2, 4, 14, 31]. We provide this summary to increase the readability of this manuscript.

2.1 Grassmann algebras and Grassmann envelopes

Let $Λ_N$ be the Grassmann algebra of order $N ∈ N$ over the field $K \ (K = \mathbb{R} \text{ or } \mathbb{C})$ with canonical generators $f_1, ..., f_N$ which are subject to the multiplication rules $f_j f_k + f_k f_j = 0$ implying in particular that $f_j^2 = 0$. A basis for $Λ_N$ consists of elements of the form $f_0 = 1$, $f_A = f_{j_1} ... f_{j_k}$ for $A = \{j_1, ..., j_k\}$ $(1 ≤ j_1 < ... < j_k ≤ N)$. Hence an arbitrary element $a ∈ Λ_N$ has the form $a = \sum_{A \subseteq \{1, ..., N\}} a_A f_A$ with $a_A ∈ K$. We define the space of homogeneous elements of degree $k$ by $Λ_N^{(k)} = \text{span}_K \{f_A : |A| = k\}$, where in particular $Λ_N^{(k)} = \{0\}$ for $k > N$. It then easily follows that $Λ_N = \bigoplus_{k=0}^N Λ_N^{(k)}$ and $Λ_N^{(k)} Λ_N^{(ℓ)} ⊂ Λ_N^{(k+ℓ)}$.

The projection of $Λ_N$ on its $k$-homogeneous part is denoted by $[^k] : Λ_N → Λ_N^{(k)}$, i.e. $[a]_k = \sum_{|A|=k} a_A f_A$. In particular we denote $[a]_0 = a_0 = : a_0$. It is well-known that $Λ_N$ shows a natural $\mathbb{Z}_2$-grading. In fact, defining $Λ_N^{(ev)} = \bigoplus_{k≥0} Λ_N^{(2k)}$ and $Λ_N^{(odd)} = \bigoplus_{k≥0} Λ_N^{(2k+1)}$ as the spaces of homogeneous even and odd elements respectively, we obtain the superalgebra structure $Λ_N = Λ_N^{(ev)} ⊕ Λ_N^{(odd)}$. We recall that $Λ_N$ is graded commutative in the sense that

$$vw = vw, \quad uv = vu, \quad uvw = -wvu, \quad v, w ∈ Λ_N^{(ev)}, \quad u, v, w ∈ Λ_N^{(odd)}.$$ 

Every $a ∈ Λ_N$ may be written as the sum $a = a_0 + a$ of a number $a_0 := a_0 ∈ K$ and a nilpotent element $a = \sum_{|A|≤1} a_A f_A$ (in particular $a^{N+1} = 0$). The elements $a_0, a$ are called the body and the nilpotent part of $a ∈ Λ_N$, respectively. The subalgebra of all nilpotent elements is denoted by $Λ_N^{(n)} := \bigoplus_{k=1}^N Λ_N^{(k)}$. It is easily seen that the projection $[^0] : Λ_N → K$ is an algebra homomorphism, i.e. $[ab]_0 = a_0 b_0$ for $a, b ∈ Λ_N$. In particular the following property holds.

**Lemma 1.** Let $a ∈ Λ_N$ such that $a^2 ∈ K \setminus \{0\}$. Then $a ∈ K$.

The algebra $Λ_N$ is a $K$-vector space of dimension $2^N$. As every finite dimensional $K$-vector space, $Λ_N$ becomes a Banach space with the introduction of an arbitrary norm, all norms being equivalent. In particular, the norm $\| \cdot \|_Λ$ defined on $Λ_N$ by $\|a\|_Λ = \sum_{A \subseteq \{1, ..., N\}} |a_A|$ satisfies

$$\|ab\|_Λ ≤ \|a\|_Λ \|b\|_Λ, \quad \text{for every } a, b ∈ Λ_N.$$ 

The exponential of $a ∈ Λ_N$, denoted by $e^a$ or $\exp(a)$, is defined by the power series

$$e^a = \sum_{j=0}^\infty a_j j!.$$ 

This series converges for every $a ∈ Λ_N$ and defines a continuous function in $Λ_N$.

Now consider the graded vector space $K^p,q$ with standard homogeneous bases $e_1, ..., e_p, e_{p+1}, ..., e_q$, i.e. $K^p,q = K^p,0 ⊕ K^{0,q}$ where $\{e_1, ..., e_p\}$ is a basis for $K^p,0$ and $\{ e_{p+1}, ..., e_q\}$ is a basis for $K^{0,q}$. Elements in $K^p,0$ are called even homogeneous elements while elements in $K^{0,q}$ are called odd homogeneous elements. In [2, p. 91], the Grassmann envelope $K^p,q(Λ_N)$ was defined as the set of Grassmann supervectors

$$w = w + w^c = \sum_{j=1}^p w_j e_j + \sum_{j=1}^q w_j^c e_j, \quad \text{where } w_j ∈ Λ_N^{(ev)}, \quad w^c_j ∈ Λ_N^{(odd)}.$$ 

(1)

**Remark 2.1.** The Grassmann envelope of a general graded $K$-vector space $V = V_T ⊕ V_T^c$ is similarly defined as

$$V(Λ_N) = (V ⊕ Λ_N)_0 = \left( Λ_N^{(ev)} ⊕ V_T^c \right) ⊕ \left( Λ_N^{(odd)} ⊕ V_T \right).$$
The set $\mathbb{K}^{p,q}(\Lambda_N)$ is a $\mathbb{K}$-vector space of dimension $2^{N-1}(p+q)$, inheriting the $\mathbb{Z}_2$-grading of $\mathbb{K}^{p,q}$, i.e.

$$\mathbb{K}^{p,q}(\Lambda_N) = \mathbb{K}^{p,0}(\Lambda_N) \oplus \mathbb{K}^{0,q}(\Lambda_N),$$

where $\mathbb{K}^{p,0}(\Lambda_N)$ denotes the subspace of vectors of the form (1) with $w_j = 0$, and $\mathbb{K}^{0,q}(\Lambda_N)$ denotes the subspace of vectors of the form (1) with $w_j = 0$. The subspaces $\mathbb{K}^{p,0}(\Lambda_N)$ and $\mathbb{K}^{0,q}(\Lambda_N)$ are called the Grassmann envelopes of $\mathbb{K}^{p,0}$ and $\mathbb{K}^{0,q}$, respectively.

In $\mathbb{K}^{p,q}(\Lambda_N)$, there exists a subspace which is naturally isomorphic to $\mathbb{K}^{p,0}$. It consists of vectors (1) of the form $w = \sum_{j=1}^n w_j e_j$ with $w_j \in \mathbb{K}$. The map $[\cdot]_0 : \mathbb{K}^{p,q}(\Lambda_N) \rightarrow \mathbb{K}^{p,0}$ defined by $[w]_0 = \sum_{j=1}^n [w_j]_0 e_j$ will be useful.

The standard basis of $\mathbb{K}^{p,q}$ can be represented by the columns $e_j = (0, \ldots, 1, \ldots, 0)^T$ (1 on the $j$-th place from the left) and $\bar{e}_j = (0, \ldots, 0, \ldots, 1, \ldots, 0)^T$ (1 on the $(p+j)$-th place from the left). In this basis, elements of $\mathbb{K}^{p,q}(\Lambda_N)$ take the form $w = (w_1, \ldots, w_p, \bar{w}_1, \ldots, \bar{w}_q)^T$.

### 2.2 Supermatrices

The $\mathbb{Z}_2$-grading of $\mathbb{K}^{p,q}$ yields the $\mathbb{Z}_2$-grading of the space $\text{End}(\mathbb{K}^{p,q})$ of endomorphisms on $\mathbb{K}^{p,q}$. When seen as a Lie superalgebra, this space is denoted by $\text{gl}(p|q)(\mathbb{K})$. The super Lie bracket on $\text{gl}(p|q)(\mathbb{K})$ is given by $[X,Y] = XY - (-1)^{|X||Y|} YX$ where $X, Y \in \text{gl}(p|q)(\mathbb{K})$ are homogeneous elements, i.e. elements in the even or in the odd subalgebra. Here the grading function $|X|$ is defined as 0 if $X$ is even and 1 if $X$ is odd.

It is easily seen that the Grassmann envelope of any Lie subsuperalgebra of $\text{gl}(p|q)(\mathbb{K})$ is a classical Lie algebra. The Grassmann envelope of $\text{gl}(p|q)(\mathbb{K})$ is denoted by $\text{Mat}(p|q)(\Lambda_N)$. Elements in $\text{Mat}(p|q)(\Lambda_N)$ are called supermatrices and are of the form

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} + \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \quad (2)$$

where $A$ is a $p \times p$ matrix with entries in $\Lambda_N^{(ev)}$, $B$ is a $p \times q$ matrix with entries in $\Lambda_N^{(odd)}$, $C$ is a $q \times p$ matrix with entries in $\Lambda_N^{(odd)}$, and $D$ is a $q \times q$ matrix with entries in $\Lambda_N^{(ev)}$. Let $\text{Mat}(p|q)(\Lambda_N)$ be the space of homogeneous supermatrices of degree $k$. These subspaces define a grading in $\text{Mat}(p|q)(\Lambda_N)$ by $\text{Mat}(p|q)(\Lambda_N) = \bigoplus_{k=0}^N \text{Mat}(p|q)(\Lambda_N^k)$ and $\text{Mat}(p|q)(\Lambda_N^k) \subset \text{Mat}(p|q)(\Lambda_N^{k+1})$.

Every supermatrix $M$ can be written as the sum of a $\text{body}$ matrix $M_0 \in \text{Mat}(p|q)(\Lambda_N^0)$ and a $\text{nilpotent}$ element $M \in \text{Mat}(p|q)(\Lambda_N^+) := \bigoplus_{k=1}^N \text{Mat}(p|q)(\Lambda_N^k)$. We also define the algebra homomorphism $[\cdot]_0 : \text{Mat}(p|q)(\Lambda_N) \rightarrow \text{Mat}(p|q)(\Lambda_N^0)$ as the projection:

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \rightarrow \begin{pmatrix} A_0 & 0 \\ 0 & D_0 \end{pmatrix} = M_0 = [M]_0$$

where $A_0$ and $D_0$ are the $\text{body}$ projections of $A$ and $D$ on $\mathbb{K}^{p \times p}$ and $\mathbb{K}^{q \times q}$ respectively. We recall that $\text{Mat}(p|q)(\Lambda_N^0)$ is equal to the even subalgebra of $\text{gl}(p|q)(\mathbb{K})$. Given a set of supermatrices $\mathbf{S}$ we define $[\mathbf{S}]_0 = \{[M]_0 : M \in \mathbf{S}\}$.

Let $\text{GL}(p|q)(\Lambda_N)$ be the Lie group of all invertible elements of $\text{Mat}(p|q)(\Lambda_N)$. A supermatrix $M \in \text{Mat}(p|q)(\Lambda_N)$ of the form (2) is invertible if and only if the blocks $A$ and $D$ are invertible (or equivalently, $A_0$ and $D_0$ are invertible), see e.g. [2]. The inverse of every $M \in \text{GL}(p|q)(\Lambda_N)$ is given by

$$M^{-1} = \begin{pmatrix} (A - B D^{-1} C)^{-1} & -A^{-1} B (D - C A^{-1} B)^{-1} \\ -D^{-1} C (A - B D^{-1} C)^{-1} & (D - C A^{-1} B)^{-1} \end{pmatrix}. \quad (3)$$

The $\text{supertranspose}$ of $M \in \text{Mat}(p|q)(\Lambda_N)$ is defined by $M^{ST} = \begin{pmatrix} A^T & C^T \\ -B^T & D^T \end{pmatrix}$, where $^T$ denotes the usual matrix transpose. It easily follows that $(M L)^{ST} = L^{ST} M^{ST}$.

The $\text{supertrace}$ is defined as the map $\text{str} : \text{Mat}(p|q)(\Lambda_N) \rightarrow \Lambda_N^{(ev)}$ given by $\text{str}(M) = \text{tr}(A) - \text{tr}(D)$. It follows that $\text{str}(M L) = \text{str}(L M)$ for any pair $M, L \in \text{Mat}(p|q)(\Lambda_N)$. 


The superdeterminant or Berezinian is a function from $\text{GL}(p|q)(A_N)$ to $A_N^{(ev)}$ defined by
\[\text{sdet}(M) = \det (A - B D^{-1} C) \det(D)^{-1} = \det (D - C A^{-1} B)^{-1} \det(A).\]
In particular, one has that $\text{sdet}(ML) = \text{sdet}(M) \text{sdet}(L)$ and $\text{sdet}(M^{ST}) = \text{sdet}(M)$.

In the vector space $\text{Mat}(p|q)(A_N)$ we introduce the norm $\|M\| = \sum_{j,k=1}^{p+q} \|m_{j,k}\|_{\Lambda}$, where $m_{j,k} \in A_N$ $(j,k = 1, \ldots, p+q)$ are the entries of $M \in \text{Mat}(p|q)(A_N)$. As was the case in $A_N$, also this norm satisfies the inequality $\|ML\| \leq \|M\|\|L\|$ for every pair $M, L \in \text{Mat}(p|q)(A_N)$, leading to the absolute convergence of the series
\[\exp(M) = \sum_{j=0}^{\infty} \frac{M^j}{j!}\]
and hence, the continuity of the exponential map in $\text{Mat}(p|q)(A_N)$. The supertranspose, the supertrace and the superdeterminant are also continuous maps. In particular, one has that (see e.g. [2])
\[\text{sdet}(e^M) = e^{\text{str}(M)}, \quad M \in \text{Mat}(p|q)(A_N).\] (4)
Moreover, $e^{tM}$ $(t \in \mathbb{R})$ is a smooth curve with $\frac{d}{dt} e^{tM} = M e^{tM} = e^{tM} M$ and $\frac{d}{dt} e^{tM}|_{t=0} = M$.

The logarithm for a supermatrix $M \in \text{Mat}(p|q)(A_N)$ is defined by $\ln(M) = \sum_{j=1}^{\infty} (-1)^{j+1} \frac{(M-I_{p+q})^j}{j}$ wherever it converges. This series converges and yields a continuous function near $I_{p+q}$.

**Proposition 1.** In $\text{Mat}(p|q)(A_N)$, let $U$ be a neighbourhood of $I_{p+q}$ on which $\ln$ is defined and let $V$ be a neighbourhood of $0$ such that $\exp(V) := \{e^M : M \in V\} \subset U$. Then $e^{\ln(M)} = M$, $\forall M \in U$, and $\ln(e^L) = L$, $\forall L \in V$.

The exponential of a nilpotent matrix $M \in \text{Mat}(p|q)(A_N^+)$ reduces to a finite sum, yielding the bijective mapping
\[\exp : \text{Mat}(p|q)(A_N^+) \to I_{p+q} + \text{Mat}(p|q)(A_N^+)\]
with inverse
\[\ln : I_{p+q} + \text{Mat}(p|q)(A_N^+) \to \text{Mat}(p|q)(A_N^+),\]
since also the second expansion only has a finite number of non-zero terms, whence problems of convergence do not arise. We recall that a supermatrix $M$ belongs to $\text{GL}(p|q)(A_N)$ if and only if its body $M_0$ has an inverse. Then $M = M_0 (I_{p+q} + M_0^{-1} M) = M_0 \exp(L)$, for some unique $L \in \text{Mat}(p|q)(A_N^+)$. 

3 The algebra $A_{m,2n} \otimes A_N$

3.1 The Clifford setting in superspace

In order to set up the Clifford analysis framework in superspace, take $p = m$, $q = 2n$ $(m, n \in \mathbb{N})$ and $\mathbb{K} = \mathbb{R}$. The canonical homogeneous basis $e_1, \ldots, e_m, \epsilon_1, \ldots, \epsilon_{2n}$ of $\mathbb{R}^{m,2n}$ can be endowed with an orthogonal and a symplectic structure by the multiplication rules
\[e_j e_k + e_k e_j = -2 \delta_{j,k}, \quad e_j \epsilon_k + \epsilon_k e_j = 0, \quad e_j \epsilon_k - \epsilon_k e_j = g_{j,k},\] (5)
where the symplectic form $g_{j,k}$ is defined by
\[g_{j,k} g_{j',k'} = g_{j'-1,k} = 0, \quad g_{j,-1,k} = -g_{2k,j} = \delta_{j,k}, \quad j,k = 1, \ldots, n.\]
Following these relations, elements in $\mathbb{R}^{m,2n}$ generate an infinite dimensional algebra denoted by $\mathcal{C}_{m,2n}$.

The definition of the Clifford supervector variable follows from a representation of the so-called radial algebra, see e.g. [8, 11, 28, 30]. Given a set $S = \{x, y, \ldots\}$ of $\ell > 1$ abstract vector variables we define the radial algebra $R(S)$ as the associative algebra over $\mathbb{R}$ freely generated by $S$ and subject to the defining axiom
\[\text{(A1)} \quad \{x,y,z\} = 0 \quad \text{for any } x,y,z \in S,\]
where \( \{a, b\} = ab + ba \) and \( \{a, b\} = ab - ba \). A radial algebra representation is an algebra homomorphism \( \Psi : R(S) \to \mathfrak{A} \) from \( R(S) \) into an algebra \( \mathfrak{A} \). The term representation also refers to the range \( \Psi(R(S)) \subset \mathfrak{A} \) of that mapping. The easiest and most important example of radial algebra representation is the algebra generated by standard Clifford vector variables. In that way, \( R(S) \) describes the main algebraic properties of the Clifford function theory. For a detailed study on radial algebras and its representations we refer the reader to [8, 11, 28, 30].

The representation of the radial algebra in superspace is defined by the mapping

\[
x \mapsto x = \bar{x} + \bar{\bar{x}} = \sum_{j=1}^{m} x_j e_j + \sum_{j=1}^{2n} \bar{x}_j \bar{e}_j, \quad x \in S,
\]

between \( S \) and the set of independent supervector variables \( S = \{x : x \in S\} \). For each \( x \in S \) we consider in (6) \( m \) bosonic (commuting) variables \( x_1, \ldots, x_m \) and \( 2n \) fermionic (anti-commuting) variables \( \bar{x}_1, \ldots, \bar{x}_{2n} \). The projections \( \bar{x} = \sum_{j=1}^{m} x_j e_j \) and \( \bar{\bar{x}} = \sum_{j=1}^{2n} \bar{x}_j \bar{e}_j \) are called the bosonic and fermionic vector variables, respectively.

Let us define the sets \( \text{VAR} \) and \( \text{VAR}' \) of bosonic and fermionic variables

\[
\text{VAR} = \bigcup_{x \in S} \{x_1, \ldots, x_m\}, \quad \text{VAR}' = \bigcup_{x \in S} \{\bar{x}_1, \ldots, \bar{x}_{2n}\}
\]

respectively, where the sets \( \{x_1, \ldots, x_m\} \) and \( \{\bar{x}_1, \ldots, \bar{x}_{2n}\} \) correspond to the coordinates of the bosonic and the fermionic vector variables associated to each \( x \in S \) through the correspondences (6). In this way, \( \text{VAR} \) contains \( m \ell \) bosonic variables and \( \text{VAR}' \) contains \( 2n \ell \) fermionic variables. They give rise to the algebra of super-polynomials \( \mathcal{V} = \text{Alg}_\mathbb{R}\{\text{VAR} \cup \text{VAR}'\} \) which is extended to the algebra of Clifford-valued super-polynomials

\[
\mathcal{A}_{m,2n} = \mathcal{V} \otimes \mathcal{C}_{m,2n},
\]

where the elements of \( \mathcal{V} \) commute with the elements of \( \mathcal{C}_{m,2n} \).

The algebra \( \mathcal{V} \) clearly is \( \mathbb{Z}_2 \)-graded. Indeed, \( \mathcal{V} = \mathcal{V}_\mathbb{E} \oplus \mathcal{V}_\mathbb{O} \) where \( \mathcal{V}_\mathbb{E} \) consists of all commuting super-polynomials and \( \mathcal{V}_\mathbb{O} \) consists of all anti-commuting super-polynomials in \( \mathcal{V} \). It is easily seen that the fundamental axiom of the radial algebra is fulfilled in this representation since for every pair \( x, y \in S \)

\[
\langle x, y \rangle := -\frac{1}{2} \{x, y\} = \sum_{j=1}^{m} x_j y_j - \frac{1}{2} \sum_{j=1}^{2n} (\bar{x}_{2j-1} \bar{y}_{2j} - \bar{x}_{2j} \bar{y}_{2j-1}) \in \mathcal{V}_\mathbb{E}.
\]

The above definition \( \langle x, y \rangle \) will be used as generalized inner product. As mentioned before, one of the goals of this paper is to study the invariance under linear transformations of this inner product of supervector variables. In order to study linear actions on the algebra \( \mathcal{A}_{m,2n} = \mathcal{V} \otimes \mathcal{C}_{m,2n} \) we must consider a suitable set of coefficients. Observe that the field of numbers \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{C} \) is too limited for that aim since it does not lead to any interaction between even and odd elements. For instance, multiplication by real or complex numbers leaves the decomposition \( \mathcal{V} = \mathcal{V}_\mathbb{E} \oplus \mathcal{V}_\mathbb{O} \) of the algebra of super-polynomials invariant.

The study of linear actions on \( \mathcal{A}_{m,2n} \) requires of a set including both commuting and anti-commuting elements. In this paper we consider the most simple set of such coefficients, i.e. the Grassmann algebra \( \Lambda_N \) generated by odd independent elements \( f_1, \ldots, f_N \). This leads to the \( \mathbb{Z}_2 \)-graded algebra of super-polynomials with Grassmann coefficients \( \mathcal{V} \otimes \Lambda_N \), generated over \( \mathbb{R} \) by the set of \( m \ell \) commuting variables \( \text{VAR} \) and the set of independent \( 2n \ell + N \) anti-commuting symbols \( \text{VAR}' \cup \{f_1, \ldots, f_N\} \). In general we consider the algebra

\[
\mathcal{A}_{m,2n} \otimes \Lambda_N = \mathcal{V} \otimes \Lambda_N \otimes \mathcal{C}_{m,2n},
\]

of super-polynomials with coefficients in \( \Lambda_N \otimes \mathcal{C}_{m,2n} \). Here elements of \( \mathcal{V} \otimes \Lambda_N \) commute with elements in \( \mathcal{C}_{m,2n} \).

In the set of coefficients \( \Lambda_N \otimes \mathcal{C}_{m,2n} \) one has a radial algebra representation by considering supervectors \( w \in \mathbb{R}^{m,2n}(\Lambda_N) \), i.e.

\[
w = w + \bar{w} = \sum_{j=1}^{m} w_j e_j + \sum_{j=1}^{2n} \bar{w}_j \bar{e}_j, \quad w_j \in \Lambda_N^{(ev)}, \quad \bar{w}_j \in \Lambda_N^{(odd)},
\]
where clearly the basis elements \( e_1, \ldots, e_m, \hat{e}_1, \ldots, \hat{e}_{2n} \) of \( \mathbb{R}^{m,2n} \) have to satisfy to the multiplication rules (5). Indeed, the anti-commutator of two constant supervectors \( w, v \in \mathbb{R}^{m,2n}(\Lambda_N) \) clearly is a central element in \( \Lambda_N \otimes C_{m,2n} \), i.e. \( [w,v] = 2 \sum_{j=1}^{m} w_j v_{\bar{j}} + \sum_{j=1}^{n} (w_{2j-1} v_{2\bar{j}} - w_{2j} v_{2\bar{j}-1}) \in \Lambda_N^{(ev)} \).

The subalgebra generated by the Grassmann envelope \( \mathbb{R}^{m,2n}(\Lambda_N) \) of constant supervectors is called the radial algebra embedded in \( \Lambda_N \otimes C_{m,2n} \). This algebra is denoted by \( \mathbb{R}_{m|2n}(\Lambda_N) \). Observe that \( \mathbb{R}_{m|2n}(\Lambda_N) \) is a finite dimensional vector space since it is generated by the set

\[
\{ f_A e_j : A \subset \{1, \ldots, N\}, |A| \text{ even}, j = 1, \ldots, m \} \cup \{ f_A \hat{e}_j : A \subset \{1, \ldots, N\}, |A| \text{ odd}, j = 1, \ldots, 2n \},
\]

and there is a finite number of possible products amongst these generators.

Every element in \( \Lambda_N \otimes C_{m,2n} \) can be written as a finite sum of terms of the form \( a e_{j_1} \cdots e_{j_k} \hat{e}_{j_1} \cdots \hat{e}_{j_2n} \), where \( a \in \Lambda_N \), \( 1 \leq j_1 \leq \ldots \leq j_k \leq m \) and \((\alpha_1, \ldots, \alpha_{2n}) \in (\mathbb{N} \cup \{0\})^{2n}\) is a multi-index. In this algebra we consider the corresponding generalization of the projection \( | \cdot |_0 \) which now goes from \( \Lambda_N \otimes C_{m,2n} \) to \( C_{m,2n} \) and is defined by \([a e_{j_1} \cdots e_{j_k} \hat{e}_{j_1} \cdots \hat{e}_{j_{2n}}]|_0 = [a] e_{j_1} \cdots e_{j_k} \hat{e}_{j_1} \cdots \hat{e}_{j_{2n}}\).

We now can define linear actions on supervector variables \( x \in \mathbb{S} \) by means of supermatrices \( M \in \text{Mat}(m|2n)(\Lambda_N) \). We recall that the basis elements \( e_1, \ldots, e_m, \hat{e}_1, \ldots, \hat{e}_{2n} \) can be written as column vectors. Then, by writing the \( x = \bar{x} + \hat{x} \in \mathbb{S} \) in its column representation we obtain,

\[
Mx = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \bar{x} \\ \hat{x} \end{pmatrix} = \begin{pmatrix} Ax + B\hat{x} \\ C\bar{x} + D\hat{x} \end{pmatrix}.
\]

This action produces a new supervector variable \( Mx = (y_1, \ldots, y_m, \hat{y}_1, \ldots, \hat{y}_{2n})^T \) where the \( y_j \) are even elements of \( \mathcal{V} \otimes \Lambda_N \) while the \( \hat{y}_j \) are odd ones. It is clear that \( (Mx)^T = x^T M^{ST} \).

### 3.2 Superbivectors

Superbivectors in \( \Lambda_N \otimes C_{m,2n} \) play a very important rôle when studying the invariance of the inner product (8). Following the radial algebra approach, the space of bivectors is generated by the wedge product of superbivectors of \( \mathbb{R}^{m,2n}(\Lambda_N) \), i.e.

\[
w \wedge v = \frac{1}{2}[w,v] = \sum_{1 \leq j < k \leq m} (w_j v_k - w_k v_j) e_j e_k + \sum_{1 \leq j \leq k \leq 2n} (w_j \hat{v}_k - w_k \hat{v}_j) \hat{e}_j \hat{e}_k + \sum_{1 \leq j < k \leq 2n} (w_j \hat{v}_k + w_k \hat{v}_j) \hat{e}_j \hat{e}_k,
\]

where \( \hat{e}_j \hat{e}_k = \frac{1}{2}(\hat{e}_j, \hat{e}_k) \). Hence, the space \( \mathbb{R}^{(2)}_{m|2n}(\Lambda_N) \) of superbivectors consists of elements of the form

\[
B = \sum_{1 \leq j < k \leq m} b_{j,k} e_j e_k + \sum_{1 \leq j \leq k \leq 2n} b_{j,k} \hat{e}_j \hat{e}_k + \sum_{1 \leq j < k \leq 2n} B_{j,k} \hat{e}_j \hat{e}_k,
\]

where \( b_{j,k} \in \Lambda_N^{(ev)}, b_{j,k} \in \Lambda_N^{(odd)} \) and \( B_{j,k} \in \Lambda_N^{(ev)} \cap \Lambda_N^+ \). Observe that the coefficients \( B_{j,k} \) are commuting but nilpotent since they are generated by elements of the form \( \hat{e}_j \hat{v}_k + \hat{v}_k \hat{e}_j \) that belong to \( \Lambda_N^+ \). This constitutes an important limitation for the space of superbivectors because it means that \( \mathbb{R}^{(2)}_{m|2n}(\Lambda_N) \) does not allow for any other structure than the orthogonal one. In fact, the real projection \( [B]_0 \) of every superbivector \( B \) is just the classical Clifford bivector:

\[
[B]_0 = \sum_{1 \leq j < k \leq m} [b_{j,k}]_0 e_j e_k \in \mathbb{R}^{(2)}_{0,m}.
\]

Hence it is necessary to introduce an extension \( \mathbb{R}^{(2)}_{m|2n}(\Lambda_N) \) of \( \mathbb{R}^{(2)}_{m|2n}(\Lambda_N) \) containing elements \( B \) of the form (10) but with \( B_{j,k} \in \Lambda_N^{(ev)} \). This extension enables us to consider two different structures in the same element \( B \): the orthogonal and the symplectic one. In fact, in this case we have

\[
[B]_0 = \sum_{1 \leq j < k \leq m} [b_{j,k}]_0 e_j e_k + \sum_{1 \leq j < k \leq 2n} [B_{j,k}]_0 \hat{e}_j \hat{e}_k.
\]
Remark 3.1. Observe that \( \mathbb{R}^{(2)}_{m|2n}(\Lambda_N) \) and \( \mathbb{R}^{(2)}_{m|2n}(\Lambda_N) \) are finite dimensional real vector subspaces of \( \Lambda_N \otimes \mathbb{C}_{m,2n} \) with

\[
\dim \mathbb{R}^{(2)}_{m|2n}(\Lambda_N) = 2^{N-1} \frac{m(m-1)}{2} + 2^{N-1} 2mn + (2^{N-1} - 1) n(2n+1),
\]

\[
\dim \mathbb{R}^{(2)}_{m|2n}(\Lambda_N) = 2^{N-1} \frac{m(m-1)}{2} + 2^{N-1} 2mn + 2^{N-1} n(2n+1).
\]

The extension \( \mathbb{R}^{(2)}_{m|2n}(\Lambda_N) \) of the superbivector space clearly lies outside the radial algebra \( \mathbb{R}_{m|2n}(\Lambda_N) \), and its elements generate an infinite dimensional algebra. Elements in \( \mathbb{R}^{(2)}_{m|2n}(\Lambda_N) \) are called extended superbivectors. Both superbivectors and extended superbivectors preserve several properties of classical Clifford bivectors.

Proposition 2. The space \( \mathbb{R}^{(2)}_{m|2n}(\Lambda_N) \) is a Lie algebra. In addition, \( \mathbb{R}^{(2)}_{m|2n}(\Lambda_N) \) is a Lie subalgebra of \( \mathbb{R}^{(2)}_{m|2n}(\Lambda_N) \).

Proof. We only need to check that the Lie bracket defined by the commutator in the associative algebra \( \mathbb{A}_{m,2n} \otimes \Lambda_N \) is an internal binary operation in \( \mathbb{R}^{(2)}_{m|2n}(\Lambda_N) \) and \( \mathbb{R}^{(2)}_{m|2n}(\Lambda_N) \). Direct computation shows that for \( a, b \in \Lambda_N \) and \( \Lambda_N \) we get:

\[
[a e_j e_k, b e_r e_s] = ab (2 \delta_{j,s} e_r e_k - 2 \delta_{j,k} e_r e_s + 2 \delta_{r,k} e_j e_s),
\]

\[
[a e_j e_k, b e_r e_s] = ab (2 \delta_{r,j} e_s e_k - 2 \delta_{r,k} e_j e_s),
\]

\[
[a e_j e_k, b e_r \odot e_s] = 0,
\]

\[
[a e_j e_k, b e_r \odot e_s] = ab (2 \delta_{j,r} e_k \odot e_s + (1 - \delta_{j,r}) g_s, k e_j e_r),
\]

\[
[a e_j e_k, b e_r \odot e_s] = ab (g_j, s e_r e_k + g_k, r e_j e_s),
\]

\[
[a e_j \odot e_k, b e_r \odot e_s] = ab (g_j, s e_r e_k + g_k, r e_j e_s),
\]

\[
[a e_j \odot e_k, b e_r \odot e_s] = ab (g_j, e_r \odot e_k + g_k, e_r \odot e_j + g_j, r e_k \odot e_s + g_k, r e_j \odot e_s).
\]

\[\square\]

It is well known from the radial algebra framework that the commutator of a bivector with a vector always yields a linear combination of vectors with coefficients in the scalar subalgebra. Indeed, for the abstract vector variables \( x, y, z \in S \) we obtain:

\[
[x \wedge y, z] = \frac{1}{2} [x, y, z] = \frac{1}{2} [(x, y) - (x, y), z] = [x, y, z] = \{x, z\} x - \{x, z\} y.
\]

This property can be easily generalized to \( \mathbb{R}^{(2)}_{m|2n}(\Lambda_N) \) by straightforward computation. In particular, the following results hold.

Proposition 3. Let \( x \in S \) be a supervector variable, let \( \{b_1, \ldots, b_{2n-1}\} \) be a basis for \( \Lambda_N \) and let \( \{b_1, \ldots, b_{2n-1}\} \) be a basis for \( \Lambda_N \). Then:

\[
[b_r e_j e_k, x] = 2 b_r (x_j e_k - x_k e_j),
\]

\[
[b_r e_j e_{2k-1}, x] = b_r (2 x_j e_{2k-1} + x_{2k} e_j),
\]

\[
[b_r e_j \odot e_k, x] = b_r (2 x_j e_{2k} - x_{2k-1} e_j),
\]

\[
[b_r \odot e_k, x] = b_r (2 x_j e_{2k} + x_{2k-1} e_j).
\]

Clearly, the above computations remain valid when replacing \( x \) by a fixed supervector \( w \in \mathbb{R}^{m,2n}(\Lambda_N) \).

3.3 Tensor algebra and exponential map

Since \( \Lambda_N \otimes \mathbb{C}_{m,2n} \) is infinite dimensional, the definition of the exponential map by means of a power series is not as straightforward as it was for the algebras \( \Lambda_N \) or \( \text{Mat}(p|q)(\Lambda_N) \). A correct definition of the exponential map in \( \Lambda_N \otimes \mathbb{C}_{m,2n} \) requires the introduction of the tensor algebra. More details about the general theory of tensor algebras can be found in several basic references, see e.g. [16, 23, 32].
Let $V$ be the $\mathbb{R}$-vector space with basis $B_V = \{f_1, \ldots, f_N, e_1, \ldots, e_m, \hat{e}_1, \ldots, \hat{e}_{2n}\}$ and let $T(V)$ be its tensor algebra. This is $T(V) = \bigoplus_{j=0}^{\infty} T^j(V)$ where $T^j(V) = \text{span}_{\mathbb{R}} \{v_1 \otimes \cdots \otimes v_j : v \in B_V\}$ is the $j$-fold tensor product of $V$ with itself. Then $\Lambda_N \otimes \mathcal{C}_{m,2n}$ can be seen as a subalgebra of $T(V)/I$ where $I \subset T(V)$ is the two-sided ideal generated by the elements:

$$f_j \otimes f_k + f_k \otimes f_j, \quad e_j \otimes e_k - e_k \otimes e_j, \quad e_j \otimes \hat{e}_k - \hat{e}_k \otimes \hat{e}_j - g_{j,k}.$$

Indeed, $T(V)/I$ is isomorphic to the extension of $\Lambda_N \otimes \mathcal{C}_{m,2n}$ which also contains infinite sums of arbitrary terms of the form $ae_{j_1} \cdots e_{j_k} \hat{e}_1^{\alpha_1} \cdots \hat{e}_{2n}^{\alpha_{2n}}$ where $a \in \Lambda_N$, $1 \leq j_1 \leq \ldots \leq j_k \leq m$ and $(\alpha_1, \ldots, \alpha_{2n}) \in (\mathbb{N} \cup \{0\})^{2n}$ is a multi-index.

The exponential map $\exp(x) = e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!}$ is known to be well defined in the tensor algebra $T(V)$, see e.g. [16], whence it also is well defined in $T(V)/I$. This approach to the exponential map will be enough for the purposes of this paper. For a more detailed treatment on non flat supermanifolds, we refer the reader to [18]. The following mapping properties hold

$$\exp : \Lambda_N \otimes \mathcal{C}_{m,2n} \to T(V)/I, \quad \exp : \mathbb{R}_{m|2n}(\Lambda_N) \to \mathbb{R}_{m|2n}(\Lambda_N).$$

The first statement directly follows from the definition of $T(V)/I$, while the second one can be obtained following the standard procedure established for $\Lambda_N$ and $\text{Mat}(p|q)(\Lambda_N)$, since the radial algebra $\mathbb{R}_{m|2n}(\Lambda_N) \subset \Lambda_N \otimes \mathcal{C}_{m,2n}$ is finite dimensional.

4 The orthosymplectic structure in $\mathbb{R}^{m,2n}(\Lambda_N)$

4.1 Invariance of the inner product

The inner product (8) can be easily written as

$$\langle x, y \rangle = x^T Q y$$

in terms of the supermatrix $Q = \begin{pmatrix} I_m & 0 \\ 0 & -\frac{1}{2} \Omega_{2n} \end{pmatrix}$, where $\Omega_{2n} = \text{diag} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. In order to find all supermatrices $M \in \text{Mat}(m|2n)(\Lambda_N)$ leaving the inner product $\langle \cdot, \cdot \rangle$ invariant, we observe that

$$\langle Mx, My \rangle = \langle x, y \rangle \iff (Mx)^T Q M y = x^T Q y \iff x^T (M^{ST} Q M - Q) y = 0,$$

whence the desired set is given by

$$O_0 = O_0(m|2n)(\Lambda_N) = \{ M \in \text{Mat}(m|2n)(\Lambda_N) : M^{ST} Q M - Q = 0 \},$$

**Remark 4.1.** It is clear that elements in the above set of supermatrices also leave the same bilinear form in $\mathbb{R}^{m,2n}(\Lambda_N)$ invariant, i.e. $-\frac{1}{2} \{ M \hat{w}, M \hat{v} \} = -\frac{1}{2} \{ \hat{w}, \hat{v} \}$, for $M \in O_0$ and $\hat{w}, \hat{v} \in \mathbb{R}^{m,2n}(\Lambda_N)$. In general, every property that holds for supermatrix actions on supervector variables $x \in \mathcal{S}$ also holds for the same actions on fixed supervectors $w \in \mathbb{R}^{m,2n}(\Lambda_N)$.

**Theorem 1.** The following statements hold:

(i) $O_0(m|2n)(\Lambda_N) \subset \text{GL}(m|2n)(\Lambda_N)$.

(ii) $O_0(m|2n)(\Lambda_N)$ is a group under the usual matrix multiplication.

(iii) $O_0(m|2n)(\Lambda_N)$ is a closed subgroup of $\text{GL}(m|2n)(\Lambda_N)$.

**Summarizing,** $O_0(m|2n)(\Lambda_N)$ is a Lie group.

**Proof.**
Proposition 4. The following statements hold:

(i) A supermatrix \( M \in \text{Mat}(m|2n)(\Lambda_N) \) of the form (2) belongs to \( O_0 \) if and only if

\[
\begin{align*}
A^T A - \frac{1}{2} C^T \Omega_{2n} C &= I_m, \\
A^T B - \frac{1}{2} C^T \Omega_{2n} D &= 0, \\
B^T B + \frac{1}{2} D^T \Omega_{2n} D &= \frac{1}{2} \Omega_{2n}.
\end{align*}
\]

(ii) \( \text{sdet}(M) = \pm 1 \) for every \( M \in O_0 \).

(iii) \( [O_0]_0 = O(m) \times \text{Sp}_\Omega(2n) \).

Remark 4.2. As usual, \( O(m) \) is the classical orthogonal group in dimension \( m \) and \( \text{Sp}_\Omega(2n) \) is the symplectic group associated to the antisymmetric matrix \( \Omega_{2n} \), i.e. \( \text{Sp}_\Omega(2n) = \{ D \in \mathbb{R}^{2n \times 2n} : D^T \Omega_{2n} D = \Omega_{2n} \} \).

Proof.

(i) The relation \( M^{ST} Q M = Q \) can be written in terms of \( A, B, C, D \) as:

\[
\begin{pmatrix}
A^T A - \frac{1}{2} C^T \Omega_{2n} C & A^T B - \frac{1}{2} C^T \Omega_{2n} D \\
-B^T A - \frac{1}{2} D^T \Omega_{2n} C & -B^T B - \frac{1}{2} D^T \Omega_{2n} D
\end{pmatrix} = \begin{pmatrix}
I_m & 0 \\
0 & -\frac{1}{2} \Omega_{2n}
\end{pmatrix}.
\]

(ii) The relation \( M^{ST} Q M = Q \) implies that \( \text{sdet}(M)^2 \text{sdet}(Q) = \text{sdet}(Q) \), whence \( \text{sdet}(M)^2 = 1 \). The statement then follows from Lemma 1.

(iii) See the proof of Theorem 1 (i). \( \square \)

4.2 Group of superrotations \( \text{SO}_0 \).

As in the classical way, we now can introduce the set of superrotations by

\[ \text{SO}_0 = \text{SO}_0(m|2n)(\Lambda_N) = \{ M \in O_0 : \text{sdet}(M) = 1 \} \].

This is easily seen to be a Lie subgroup of \( O_0 \) with real projection equal to \( \text{SO}(m) \times \text{Sp}_\Omega(2n) \), where \( \text{SO}(m) \subset O(m) \) is the special orthogonal group in dimension \( m \). In fact, the conditions \( M^{ST} Q M = Q \) and \( \text{sdet}(M) = 1 \) imply that \( M_0^{ST} Q M_0 = Q \) and \( \text{sdet}(M_0) = 1 \), whence

\[
M_0 = \begin{pmatrix}
A_0 & 0 \\
0 & D_0
\end{pmatrix}
\]

with \( A_0^T A_0 = I_m \), \( D_0^T \Omega_{2n} D_0 = \Omega_{2n} \) and \( \text{det}(A_0) = \text{det}(D_0) \). But \( D_0 \in \text{Sp}_\Omega(2n) \) implies \( \text{det}(D_0) = 1 \), yielding \( \text{det}(A_0) = 1 \) and \( A_0 \in \text{SO}(m) \).

The following proposition states that, as in the classical case, \( \text{SO}_0 \) is connected and in consequence, it is the identity component of \( O_0 \).
Proposition 5. $SO_0$ is a connected Lie group.

Proof. Since the real projection $SO(m) \times Sp_{2n}(2n)$ of $SO_0$ is a connected group, it suffices to prove that for every $M \in SO_0$ there exist a continuous path inside $SO_0$ connecting $M$ with its real projection $M_0$. To that end, let us write $M = \sum_{j=0}^N [M]_j$, where $[M]_j$ is the projection of $M$ on $Mat(m|2n)(\Lambda_N^j)$ for each $j = 0, 1, \ldots, N$. Then, observe that

$$M^{ST}QM = Q \iff \left( \sum_{j=0}^N [M^{ST}]_j \right) Q \left( \sum_{j=0}^N [M]_j \right) = Q \iff \sum_{k=0}^N \left( \sum_{j=0}^k [M^{ST}]_j \right) [M]_{k-j} = Q$$

$$\iff M_0^TQM_0 = Q, \quad \text{and} \quad \sum_{j=0}^k [M^{ST}]_j [M]_{k-j} = 0, \quad k = 1, \ldots, N.$$

Let us now take the path $M(t) = \sum_{j=0}^N t^j [M]_j$. For $t \in [0, 1]$ this is a continuous path with $M(0) = M_0$ and $M(1) = M$. In addition, $M(t)_0^TQM(t)_0 = M_0^TQM_0 = Q$ and for every $k = 1, \ldots, N$ we have,

$$\sum_{j=0}^k [M(t)^{ST}]_j [M(t)]_{k-j} = t^k \sum_{j=0}^k [M^{ST}]_j [M]_{k-j} = 0.$$

Hence, $M(t)^{ST}QM(t) = Q$, $t \in [0, 1]$. Finally, observe that $s\det(M(t)) = 1$ for every $t \in [0, 1]$, since $s\det(M(0)) = s\det(M_0) = 1$.

We will now investigate the corresponding Lie algebras of $O_0$ and $SO_0$.

Theorem 2.

(i) The Lie algebra $so_0 = so_0(m|2n)(\Lambda_N)$ of $O_0$ coincides with the Lie algebra of $SO_0$ and is given by the space of all "super anti-symmetric" supermatrices

$$so_0 = \{ X \in Mat(m|2n)(\Lambda_N) : X^{ST}Q + QX = 0 \}.$$

(ii) A supermatrix $X \in Mat(m|2n)(\Lambda_N)$ of the form (2) belongs to $so_0$ if and only if

$$\begin{align*}
A^T + A &= 0, \\
B - \frac{1}{2}C^T\Omega_{2n} &= 0, \\
D^T\Omega_{2n} + \Omega_{2n}D &= 0.
\end{align*}$$  \hspace{1cm} (12)

(iii) $[so_0]_0 = so(m) \oplus sp_{2n}(2n)$.

Remark 4.3. As usual, $so(m) = \{ A_0 \in \mathbb{R}^{m \times m} : A_0^T + A_0 = 0 \}$ is the special orthogonal Lie algebra in dimension $m$ and $sp_{2n}(2n) = \{ D_0 \in \mathbb{R}^{2n \times 2n} : D_0^T\Omega_{2n} + \Omega_{2n}D_0 = 0 \}$ is the symplectic Lie algebra defined through the antisymmetric matrix $\Omega_{2n}$.

Proof.

(i) If $X \in Mat(m|2n)(\Lambda_N)$ is in the Lie algebra of $O_0$ then $e^{tX} \in O_0$ for every $t \in \mathbb{R}$, i.e. $e^{tX^{ST}}Qe^{tX} - Q = 0$. Differentiating at $t = 0$ we obtain $X^{ST}Q + QX = 0$. On the other hand, if $X \in Mat(m|2n)(\Lambda_N)$ satisfies $X^{ST}Q + QX = 0$, then $X^{ST} = -QXQ^{-1}$. Computing the exponential of $tX^{ST}$ we obtain

$$e^{tX^{ST}} = \sum_{j=0}^{\infty} \frac{(tXQ^{-1})^j}{j!} = Qe^{-tX}Q^{-1},$$

which implies that $e^{tX^{ST}}Qe^{tX} - Q = 0$, i.e. $e^{tX} \in O_0$. Then $so_0$ is the Lie algebra of $O_0$.

From (4) it easily follows that the Lie algebra of $SO_0$ is given by

$$\{ X \in Mat(m|2n)(\Lambda_N) : X^{ST}Q + QX = 0, \text{str}(X) = 0 \}.$$
But \( X^{ST}Q + QX = 0 \) implies \( \text{str}(X) = 0 \). In fact, the condition \( X^{ST} = -QXQ^{-1} \) implies
\[
\text{str}(X^{ST}) = -\text{str}(QXQ^{-1}) = -\text{str}(X),
\]
yielding \( \text{str}(X) = \text{str}(X^{ST}) = -\text{str}(X) \) and \( \text{str}(X) = 0 \). Hence, the Lie algebra of \( \text{SO}_0 \) is \( \mathfrak{s o}_0 \).

(ii) Observe that the relation \( X^{ST}Q + QX = 0 \) can be written in terms of \( A, B, C, D \) as follows:
\[
\begin{pmatrix}
A^T + A & -\frac{1}{2}C^T\Omega_{2n} + B \\
-B^T - \frac{1}{2}D^T\Omega_{2n} & -\frac{1}{2}D^T\Omega_{2n} + \frac{1}{2}B + \frac{1}{2}C^T\Omega_{2n}
\end{pmatrix} = 0.
\]

(iii) Let \( X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{s o}_0 \), then \( X_0 = [X]_0 = \begin{pmatrix} A_0 & 0 \\ 0 & D_0 \end{pmatrix} \) satisfies \( X_0^{ST}Q + QX_0 = 0 \). Using (ii) we obtain \( A_0^T + A_0 = 0 \) and \( D_0^T\Omega_{2n} + \Omega_{2n}D_0 = 0 \) which implies that \( A_0 \in \mathfrak{s o}(m) \) and \( D_0 \in \mathfrak{s p}_n(2n) \).

\[\square\]

**Remark 4.4.** The group \( \text{SO}_0 \) is isomorphic to the group obtained by the action of the functor of points of \( \text{OSP}(m|2n) \) on \( \Lambda_N \). We will explicitly describe this isomorphism in terms of the corresponding Lie algebras. Indeed, the Lie algebra \( \mathfrak{s o}_0 \) constitutes a Grassmann envelope of the orthosymplectic Lie superalgebra \( \mathfrak{osp}(m|2n) \). Here we define \( \mathfrak{osp}(m|2n) \), in accordance with [5], as the subsuperalgebra of \( \mathfrak{gl}(m|2n)(\mathbb{R}) \) given by,
\[
\mathfrak{osp}(m|2n) := \{X \in \mathfrak{gl}(m|2n)(\mathbb{R}) : X^{ST}G + GX = 0\}, \quad \text{with} \quad G = \begin{pmatrix} I_m & 0 \\ 0 & J_{2n} \end{pmatrix}, \quad J_{2n} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.
\]

It suffices to note that \( \mathfrak{s o}_0 \) is the Grassmann envelope of
\[
\mathfrak{s o}_0(m|2n)(\mathbb{R}) := \{X \in \mathfrak{gl}(m|2n)(\mathbb{R}) : X^{ST}Q + QX = 0\},
\]
which is isomorphic to \( \mathfrak{osp}(m|2n) \). In order to explicitly find this isomorphism we first need the matrix
\[
R = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix} \in \text{O}(2n),
\]
which satisfies \( R^TJ_{2n}R = \Omega_{2n} \). Then the mapping \( \Phi : \mathfrak{s o}_0(m|2n)(\mathbb{R}) \to \mathfrak{osp}(m|2n), \) given by
\[
\Phi(X) = R^{-1}XR, \quad \text{with} \quad R = \begin{pmatrix} I_m & 0 \\ 0 & i\sqrt{2}R^T \end{pmatrix},
\]
is easily seen to be a Lie superalgebra isomorphism. Indeed, the matrix \( R \) is such that \( R^{ST}QR = G \). As a consequence, for every \( X \in \mathfrak{s o}_0(m|2n)(\mathbb{R}) \) one has that
\[
\Phi(X)^{ST}G + G\Phi(X) = R^{ST}X^{ST}(R^{-1})^{ST}G + GR^{-1}XR = R^{ST}(X^{ST}Q + QX)R = 0.
\]

The use of Grassmann envelopes allows to study particular aspects of the theory of Lie superalgebras in terms of classical Lie algebras and Lie groups. The \( \mathfrak{osp}(m|2n) \)-invariance of the super Dirac operator \( \partial_X \) used in [5] has been obtained in [12] in terms of the invariance of \( \partial_X \) under the action of the Grassmann envelope \( \mathfrak{s o}_0 \) (or equivalently, under the action of the group \( \text{SO}_0 \)).

The connectedness of \( \text{SO}_0 \) allows to write any of its elements as a finite product of exponentials of supermatrices in \( \mathfrak{s o}_0 \), see [20, p. 71]. In the classical case, a single exponential suffices for such a description since \( \text{SO}(m) \) is compact and in consequence \( \exp : \mathfrak{s o}(m) \to \text{SO}(m) \) is surjective, see Corollary 11.10 [20, p. 314]. This property, however, does not hold in the group of superrotations \( \text{SO}_0 \), since the exponential
map from \(\mathfrak{sp}_\Omega(2n)\) to the non-compact Lie group \(\text{Sp}_\Omega(2n)\) is surjective, whence not every element in \(\text{SO}_0\) can be written as a single exponential of a supermatrix in \(\mathfrak{so}_0\). Nevertheless, it is possible to find a decomposition for elements of \(\text{SO}_0\) in terms of a fixed number of exponentials of \(\mathfrak{so}_0\) elements.

Every supermatrix \(M \in \text{SO}_0\) has a unique decomposition \(M = M_0 + M = M_0(I_m+2n + L)\) where \(M_0\) is its real projection, \(M \in \text{Mat}(m(2n))(\Lambda^+_N)\) its nilpotent projection and \(L = M_0^{-1}M\). We will now separately study the decompositions for \(M_0 \in \text{SO}(m) \times \text{Sp}_\Omega(2n)\) and \(I_m+2n + L \in \text{SO}_0\).

First consider \(M_0 \in \text{SO}(m) \times \text{Sp}_\Omega(2n)\). We already mentioned that \(\exp : \mathfrak{so}(m) \to \text{SO}(m)\) is surjective, while \(\exp : \mathfrak{sp}_\Omega(2n) \to \text{Sp}_\Omega(2n)\) is not. However, it can be proven that

\[
\text{Sp}_\Omega(2n) = \exp(\mathfrak{sp}_\Omega(2n)) \cdot \exp(\mathfrak{sp}_\Omega(2n)),
\]

invoking the following polar decomposition for real algebraic Lie groups, see Proposition 4.3.3 in [21].

**Proposition 6.** Let \(G \subset \text{GL}(p)\) be an algebraic Lie group such that \(G = G^T\) and let \(\mathfrak{g}\) be its Lie algebra. Then every \(A \in G\) can be uniquely written as \(A = R\exp(X)\), \(R \in G \cap O(p), X \in \mathfrak{g} \cap \text{Sym}(p)\), where Sym(p) is the subspace of all symmetric matrices in \(\mathbb{R}^{p \times p}\).

**Remark 4.5.** A subgroup \(G \subset \text{GL}(p)\) is called algebraic if there exists a family \(\{p_j\}_{j \in \mathbb{Y}}\) of real polynomials

\[
p_j(M) = p_j(m_{11}, m_{12}, \ldots, m_{pp}) \in \mathbb{R}[m_{11}, \ldots, m_{pp}]
\]

in the entries of the matrix \(M \in \mathbb{R}^{p \times p}\) such that \(G = \{M \in \text{GL}(p) : p_j(M) = 0, \forall j \in \mathbb{Y}\}\). See [21, p. 73] for more details. Obviously, the groups \(O(m), \text{SO}(m), \text{Sp}_\Omega(2n)\) are algebraic Lie groups.

Taking \(p = 2n\) and \(G = \text{Sp}_\Omega(2n)\) in the above proposition we get that every symplectic matrix \(D_0\) can be uniquely written as \(D_0 = R_0\exp(Z)\) with \(R_0 \in \text{Sp}_\Omega(2n) \cap \text{O}(2n)\) and \(Z_0 \in \mathfrak{sp}_\Omega(2n) \cap \text{Sym}(2n)\). But the group \(\text{Sp}_\Omega(2n) \cap \text{O}(2n)\) is isomorphic to the unitary group \(\text{U}(n) = \{L_0 \in \mathbb{C}^{n \times n} : (L_0^1)^*L_0 = I_n\}\) which is connected and compact. Then the exponential map from the Lie algebra \(\mathfrak{sp}_\Omega(2n) \cap \mathfrak{so}(2n) \cong \mathfrak{u}(n)\) is surjective on \(\text{Sp}_\Omega(2n) \cap \text{O}(2n)\) where \(\mathfrak{u}(n) = \{L_0 \in \mathbb{C}^{n \times n} : (L_0^1)^* + L_0 = 0\}\) is the unitary Lie algebra in dimension \(n\). This means that \(D_0 \in \text{Sp}_\Omega(2n)\) can be written as \(D_0 = \exp\exp(Z_0)\) with \(Y_0 \in \mathfrak{sp}_\Omega(2n) \cap \mathfrak{so}(2n)\) and \(Z_0 \in \mathfrak{sp}_\Omega(2n) \cap \text{Sym}(2n)\). Hence, the supermatrix \(M_0 \in \text{SO}(m) \times \text{Sp}_\Omega(2n)\) can be decomposed as

\[
M_0 = \begin{pmatrix}
e^{X_0} & 0 & 0 \\
0 & e^{X_0} & 0 \\
0 & 0 & e^{Z_0}
\end{pmatrix} = \begin{pmatrix}
e^{X_0} & 0 \\
0 & e^{Y_0} \\
0 & 0 & e^{Z_0}
\end{pmatrix} = e^X e^Y,
\]

where \(X = \begin{pmatrix}X_0 & 0 \\
0 & Y_0
\end{pmatrix} \in \mathfrak{so}(m) \times \mathfrak{sp}_\Omega(2n) \cap \mathfrak{so}(2n)\) and \(Y = \begin{pmatrix}0 & 0 \\
0 & Z_0
\end{pmatrix} \in \{0_{m}\} \times \mathfrak{sp}_\Omega(2n) \cap \text{Sym}(2n)\).

Now consider the element \(I_m+2n + L \in \text{SO}_0\). As shown at the end of Section 2, the function \(\exp : \text{Mat}(m(2n))(\Lambda^+_N) \to I_m+2n + \text{Mat}(m(2n))(\Lambda^+_N)\) is a bijection with the logarithmic function as its inverse. Then the supermatrix \(Z = \text{ln}(I_m+2n + L)\) satisfies \(e^Z = I_m+2n + L\) and is nilpotent. Those properties suffice for proving that \(Z \in \mathfrak{so}_0\). From now on we will now write the set \(\mathfrak{so}_0 \cap \text{Mat}(m(2n))(\Lambda^+_N)\) of nilpotent elements of \(\mathfrak{so}_0\) by \(\mathfrak{so}_0(m(2n))(\Lambda^+_N)\).

**Proposition 7.** Let \(Z \in \text{Mat}(m(2n))(\Lambda^+_N)\) such that \(e^Z \in \text{SO}_0\). Then \(Z \in \mathfrak{so}_0\).

**Proof.**

Since \(e^Z \in \text{SO}_0\), it is clear that \(e^Z \in \mathfrak{so}_0\) for every \(t \in \mathbb{R}\). Let us prove that the same property holds for every \(t \in \mathbb{R}\). The expression \(e^{tZ^{ST}}Qe^Z - Q\) can be written as the following polynomial in the real variable \(t\).

\[
P(t) = e^{tZ^{ST}}Qe^Z - Q = \left[\sum_{j=0}^{N} \frac{t^j(Z^{ST})^j}{j!}\right] Q \left[\sum_{k=0}^{N} \frac{t^kZ^k}{k!}\right] - Q
\]

\[
= \sum_{k=1}^{N} \sum_{j=0}^{k} \frac{t^j(Z^{ST})^j}{j!} \frac{k^j}{(k-j)!} = \sum_{k=1}^{N} \sum_{j=0}^{k} \frac{k^j}{j!} \left(\frac{Z^{ST}}{j!}\right)^j Z^{k-j} = \sum_{k=1}^{N} \frac{t^k}{k!} P_k(Z),
\]

13
where \( P_k(Z) = \sum_{j=0}^{k} \binom{k}{j} (Z^{ST})^j QZ^{k-j} \). If \( P(t) \) is not identically zero, i.e. not all the \( P_k(Z) \) are 0, we can take \( k_0 \in \{1,2,\ldots,N\} \) to be the largest subindex for which \( P_{k_0}(Z) \neq 0 \). Then,

\[
\lim_{t \to \infty} \frac{1}{t^{k_0}} P(t) = \frac{P_{k_0}(Z)}{k_0!} \neq 0,
\]

contradicting that \( P(Z) = \{0\} \). So \( P(t) \) identically vanishes, yielding \( e^tZ \in \text{SO}_0 \) for every \( t \in \mathbb{R} \). □

In this way, we have proven the following result.

**Theorem 3.** Every supermatrix in \( \text{SO}_0 \) can be written as

\[
M = e^X e^Y e^Z, \quad \text{with} \quad \begin{aligned}
X & \in \mathfrak{so}(m) \times [\mathfrak{sp}_0(2n) \cap \mathfrak{so}(2n)], \\
Y & \in \{0_m\} \times [\mathfrak{sp}_0(2n) \cap \text{Sym}(2n)], \\
Z & \in \mathfrak{so}_0(m|2n)(\Lambda_N^\circ).
\end{aligned}
\]

Moreover, the elements \( Y \) and \( Z \) are unique.

### 4.3 Relation with superbivectors.

Theorem 2 allows to compute the dimension of \( \mathfrak{so}_0 \) as a real vector space.

**Corollary 1.** The dimension of the real Lie algebra \( \mathfrak{so}_0 \) is \( 2^{N-1} \left( \frac{m(m-1)}{2} + 2mn + n(2n+1) \right) \).

**Proof.**

Since \( \mathfrak{so}_0 \) is the direct sum of the corresponding subspaces of block components \( A, B, C \) and \( D \) respectively, it suffices to compute the dimension of each one of them. According to Theorem 2 (iii) we have:

\[
V_1 = \left\{ \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} : A^T = -A, \ A \in \Lambda_N^{(ev)} \right\} \cong \Lambda_N^{(ev)} \otimes \mathfrak{so}(m),
\]

\[
V_2 = \left\{ \begin{pmatrix} 0 & \frac{1}{2} CT \Omega_{2n} \\ C & 0 \end{pmatrix} : C \in \Lambda_N^{(odd)} \right\} \cong \Lambda_N^{(odd)} \otimes \mathbb{R}^{2n \times m},
\]

\[
V_3 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} : D^T \Omega_{2n} + \Omega_{2n} D = 0, \ D \in \Lambda_N^{(ev)} \right\} \cong \Lambda_N^{(ev)} \otimes \mathfrak{sp}_0(2n).
\]

This leads to \( \dim V_1 = 2^{N-1} \frac{m(m-1)}{2} \), \( \dim V_2 = 2^{N-1}m2n \) and \( \dim V_3 = 2^{N-1}n(2n+1) \). □

Comparing this result with the one in Remark 3.1 we obtain that \( \dim \mathbb{R}^{(2E)}_{m|2n}(\Lambda_N) = \dim \mathfrak{so}_0 \). This means that both vector spaces are isomorphic. This isomorphism also holds on the Lie algebra level. Following the classical Clifford approach, the commutator \([B, x] \), with \( B \in \mathbb{R}^{(2E)}_{m|2n}(\Lambda_N) \) and \( x \in \mathfrak{S} \), should be the key for the Lie algebra isomorphism. Proposition 3 shows that this commutator defines a linear action on the supervector variable \( x \in \mathfrak{S} \) that can be represented by a supermatrix in \( \text{Mat}(m|2n)(\Lambda_N) \), see (9).

**Lemma 2.** The map \( \phi : \mathbb{R}^{(2E)}_{m|2n}(\Lambda_N) \rightarrow \text{Mat}(m|2n)(\Lambda_N) \) defined by

\[
\phi(B)x = [B, x] \quad \text{for} \quad B \in \mathbb{R}^{(2E)}_{m|2n}(\Lambda_N), \ x \in \mathfrak{S},
\]

takes values in \( \mathfrak{so}_0 \). In particular, if we consider \( \{b_1, \ldots, b_{2N-1}\} \) and \( \{b_1, \ldots, b_{2N-1}\} \) to be the canonical
basis of $\Lambda_N^{(e)}$ and $\Lambda_N^{(odd)}$ respectively, we obtain the following basis for $\mathfrak{so}_0$.

$\phi(b_r e_j e_k) = 2b_r \begin{pmatrix} E_{k,j} - E_{j,k} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, \quad 1 \leq r \leq 2^{N-1}, \quad 1 \leq j < k \leq m,

$\phi(b_r e_j \hat{e}_{2k-1}) = b_r \begin{pmatrix} 0 & E_{j,2k} & 0 \\ 0 & 0 & 0 \end{pmatrix}$, \quad 1 \leq r \leq 2^{N-1}, \quad 1 \leq j \leq m, \quad 1 \leq k \leq n,

$\phi(b_r e_j \hat{e}_{2k}) = b_r \begin{pmatrix} 0 & 0 & -E_{j,2k-1} \\ 2E_{2k-1,j} & 0 & 0 \end{pmatrix}$, \quad 1 \leq r \leq 2^{N-1}, \quad 1 \leq j \leq m, \quad 1 \leq k \leq n,

$\phi(b_r \hat{e}_{2j} \odot \hat{e}_{2k}) = -b_r \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, \quad 1 \leq r \leq 2^{N-1}, \quad 1 \leq j \leq k \leq n,

$\phi(b_r \hat{e}_{2j-1} \odot \hat{e}_{2k-1}) = b_r \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, \quad 1 \leq r \leq 2^{N-1}, \quad 1 \leq j \leq k \leq n,

$\phi(b_r \hat{e}_{2j-1} \odot \hat{e}_{2k}) = b_r \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, \quad 1 \leq r \leq 2^{N-1}, \quad 1 \leq j \leq k \leq n,

$\phi(b_r \hat{e}_{2j} \odot \hat{e}_{2k-1}) = b_r \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, \quad 1 \leq r \leq 2^{N-1}, \quad 1 \leq j < k \leq n,

where $E_{j,k}$ denotes the matrix in which only the element on the crossing of the $j$-th row and the $k$-th column equals 1 and all the other entries are zero. The order of $E_{j,k}$ should be deduced from the context.

**Proof.**

The above equalities can be directly obtained from Proposition 3, whence we should only check that all supermatrices obtained above form a basis for $\mathfrak{so}_0$. The matrices $E_{j,k}$ satisfy the relations

$E_{j,k}^T = E_{k,j}, \quad E_{j,2k-1}J_{2n} = E_{j,2k}, \quad E_{j,2k}J_{2n} = -E_{j,2k-1}, \quad J_{2n}E_{j,k} = E_{2j-1,k}, \quad J_{2n}E_{2j-1,k} = -E_{2j,k}$.

Then

- for $\phi(b_r e_j e_k)$ we have $A = 2b_r (E_{k,j} - E_{j,k})$, $B = 0$, $C = 0$ and $D = 0$, whence $A^T = 2b_r (E_{j,k} - E_{k,j}) = -A$;

- for $\phi(b_r e_j \hat{e}_{2k-1})$ we have $A = 0$, $B = b_r E_{j,2k}$, $C = 2b_r E_{2k-1,j}$ and $D = 0$, whence

$$\frac{1}{2} C^T \Omega_{2n} = b_r E_{j,2k-1} \Omega_{2n} = b_r E_{j,2k} = B;$$

- for $\phi(b_r e_j \hat{e}_{2k})$ we have $A = 0$, $B = -b_r E_{j,2k-1}$, $C = 2b_r E_{2k,j}$ and $D = 0$, whence

$$\frac{1}{2} C^T \Omega_{2n} = b_r E_{j,2k} \Omega_{2n} = -b_r E_{j,2k-1} = B;$$

- for $\phi(b_r \hat{e}_{2j} \odot \hat{e}_{2k})$ we have $A = 0$, $B = 0$, $C = 0$ and $D = -b_r (E_{2j,2k-1} + E_{2k,2j-1})$, whence

$$D^T \Omega_{2n} + \Omega_{2n} D = -b_r (E_{2k-1,2j} \Omega_{2n} + E_{2j-1,2k} \Omega_{2n} + \Omega_{2n} E_{2j,2k-1} + \Omega_{2n} E_{2k,2j-1}) = -b_r (-E_{2k-1,2j-1} - E_{2j-1,2k-1} + E_{2j-1,2k-1} + E_{2k-1,2j-1}) = 0;$$

- for $\phi(b_r \hat{e}_{2j-1} \odot \hat{e}_{2k-1})$ we have $A = 0$, $B = 0$, $C = 0$ and $D = b_r (E_{2j-1,2k} + E_{2k-1,2j})$, whence

$$D^T \Omega_{2n} + \Omega_{2n} D = b_r (E_{2j,2k-1} \Omega_{2n} + E_{2j-1,2k} \Omega_{2n} + \Omega_{2n} E_{2j,2k-1} + \Omega_{2n} E_{2k,2j-1}) = b_r (E_{2k,2j} + E_{2j,2k} - E_{2j,2k} - E_{2k,2j}) = 0;$$

- for $\phi(b_r \hat{e}_{2j-1} \odot \hat{e}_{2k})$ we have $A = 0$, $B = 0$, $C = 0$ and $D = b_r (E_{2j,2k} - E_{2k-1,2j-1})$, whence

$$D^T \Omega_{2n} + \Omega_{2n} D = b_r (E_{2j,2k} \Omega_{2n} - E_{2k-1,2j-1} \Omega_{2n} + \Omega_{2n} E_{2k,2j} - \Omega_{2n} E_{2j-1,2k-1}) = b_r (-E_{2j,2k-1} - E_{2k-1,2j} + E_{2j,2k-1} + E_{2k,2j}) = 0.
The above computations show that all supermatrices obtained belong to \( \mathfrak{so}_0 \). Direct verification shows that they form a set of \( 2^{N-1} \frac{m(m-1)}{2} + 2^{N-1} 2mn + 2^{N-1} n(2n+1) \) linearly independent elements, i.e., a basis of \( \mathfrak{so}_0 \).

\[ \text{□} \]

\textbf{Theorem 4.} The map \( \phi: \mathbb{R}^{(2)E}_{m|2n}(\Lambda_N) \rightarrow \mathfrak{so}_0 \) defined in (13) is a Lie algebra isomorphism.

\textbf{Proof.} From Lemma 2 it follows that \( \phi \) is a vector space isomorphism. In addition, due to the Jacobi identity in the associative algebra \( \mathcal{A}_{m,2n} \otimes \Lambda_N \) we have for all \( B_1, B_2 \in \mathbb{R}^{(2)E}_{m|2n}(\Lambda_N) \) and \( x \in S \) that

\[ [\phi(B_1), \phi(B_2)] \mathbf{x} = [B_1, [B_2, \mathbf{x}]] + [B_2, [\mathbf{x}, B_1]] = [[B_1, B_2], \mathbf{x}] = \phi([B_1, B_2]) \mathbf{x}, \]

yielding \( [\phi(B_1), \phi(B_2)] = \phi([B_1, B_2]), \) i.e., \( \phi \) is a Lie algebra isomorphism.

\[ \text{□} \]

\textbf{Remark 4.6.} In virtue of Remark 4.4, the algebra \( \mathbb{R}^{(2)E}_{m|2n}(\Lambda_N) \) is a Grassmann envelope of \( \mathfrak{osp}(m|2n) \).

\section{The Spin group associated to SO\(_0\)}

So far we have seen that the Lie algebra \( \mathfrak{so}_0 \) of the Lie group of superrotations \( \text{SO}_0 \) has a realization in \( \Lambda_N \otimes \mathcal{C}_{m,2n} \) as the Lie algebra of extended superbivectors. In this section, we discuss the proper way of defining the corresponding realization of \( \text{SO}_0 \) in \( T(V)/I \), i.e., the analogue of the Spin group in this framework.

\subsection{Supervector reflections}

The group of linear transformations generated by the supervector reflections was briefly introduced in [29] using the notion of the unit sphere in \( \mathbb{R}^{m,2n}(\Lambda_N) \) defined as \( \mathbb{S}(m|2n)(\Lambda_N) = \{ \mathbf{w} \in \mathbb{R}^{m,2n}(\Lambda_N) : \mathbf{w}^2 = -1 \} \). The reflection associated to the supervector \( \mathbf{w} \in \mathbb{S}(m|2n)(\Lambda_N) \) is defined by the linear action on supervector variables

\[ \psi(\mathbf{w})[\mathbf{x}] = \mathbf{w}\mathbf{x}\mathbf{w}, \quad \mathbf{x} \in S. \]

It is known from the radial algebra setting that \( \psi(\mathbf{w})[\mathbf{x}] \) yields a new supervector variable. Indeed, for \( x, y \in S \) one has \( xy = \{ x, y \} y - y^2 x = \{ x, y \} y + x \). Every supervector reflection can be represented by a supermatrix in \( \text{Mat}(m|2n)(\Lambda_N) \).

\textbf{Lemma 3.} Let \( \mathbf{w} = \mathbf{w} + \mathbf{w}' = \sum_{j=1}^m w_j e_j + \sum_{j=1}^{2n} \tilde{w}_j \tilde{e}_j \in \mathbb{S}(m|2n)(\Lambda_N) \). Then, the linear transformation (14) can be represented by a supermatrix

\[ \psi(\mathbf{w}) = \begin{pmatrix} \mathbf{A}(\mathbf{w}) & \mathbf{B}(\mathbf{w}) \\ \mathbf{C}(\mathbf{w}) & \mathbf{D}(\mathbf{w}) \end{pmatrix} \in \text{Mat}(m|2n)(\Lambda_N) \]

with \( \mathbf{A}(\mathbf{w}) = -2 \mathbf{D}_m E_{m \times m} D_{\mathbf{w}} + \mathbf{I}_m \), \( \mathbf{B}(\mathbf{w}) = \mathbf{D}_m E_{m \times 2n} D_{\mathbf{w}} \Omega_{2n} \), \( \mathbf{C}(\mathbf{w}) = -2 \mathbf{D}_m E_{2n \times m} D_{\mathbf{w}} \), and finally \( \mathbf{D}(\mathbf{w}) = \mathbf{D}_m E_{2n \times 2n} D_{\mathbf{w}} \Omega_{2n} + \mathbf{I}_{2n} \), where

\[ D_m = \begin{pmatrix} \mathbf{w}_1 \\ \vdots \\ \mathbf{w}_m \end{pmatrix}, \quad D_{\mathbf{w}} = \begin{pmatrix} \mathbf{w}_1 \\ \vdots \\ \mathbf{w}_{2n} \end{pmatrix}, \quad E_{p \times q} = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & 1 \end{pmatrix} \in \mathbb{R}^{p \times q}. \]

\textbf{Proof.} Observe that \( \psi(\mathbf{w})[\mathbf{x}] = \mathbf{w}\mathbf{x}\mathbf{w} = \{ \mathbf{x}, \mathbf{w} \} \mathbf{x} + \mathbf{x} = \sum_{k=1}^m y_k e_k + \sum_{k=1}^{2n} \tilde{y}_k \tilde{e}_k \), where

\[ y_k = -2 \left( \sum_{j=1}^m w_j w_k x_j \right) + x_k + \sum_{j=1}^n \tilde{w}_{2j-1} \tilde{w}_k \tilde{x}_{2j} - \tilde{w}_{2j} \tilde{w}_k \tilde{x}_{2j-1}, \]

\[ \tilde{y}_k = -2 \left( \sum_{j=1}^m w_j \tilde{w}_k \tilde{x}_j \right) + \tilde{x}_k + \sum_{j=1}^n -\tilde{w}_{2j-1} \tilde{w}_k \tilde{x}_{2j} + \tilde{w}_{2j} \tilde{w}_k \tilde{x}_{2j-1}. \]
Then, \( \psi(w)x = \begin{pmatrix} A(w) & B(w) \\ C(w) & D(w) \end{pmatrix} \begin{pmatrix} x \\ \psi \end{pmatrix} \), where,

\[
A(w) = -2 \begin{pmatrix} w_1^2 & w_1 w_1 & \cdots & w_m w_1 \\
w_1 w_2 & w_2^2 & \cdots & w_m w_2 \\
\vdots & \vdots & \ddots & \vdots \\
w_1 w_m & w_2 w_m & \cdots & w_m^2 \end{pmatrix} + I_m = -2D_{w}E_{m \times m}D_{w} + I_m,
\]

\[
B(w) = \begin{pmatrix} -w_2 w_1 & w_1 w_1 & \cdots & w_2 w_{2n-1} w_1 \\
-w_2 w_1 & w_1 w_2 & \cdots & w_2 w_{2n-2} w_2 \\
\vdots & \vdots & \ddots & \vdots \\
-w_2 w_m & w_1 w_m & \cdots & w_2 w_{2n-1} w_m \end{pmatrix} = D_{w}E_{2n}D_{w}J_{2n},
\]

\[
C(w) = -2 \begin{pmatrix} w_1 w_1 & w_2 w_1 & \cdots & w_m w_1 \\
w_1 w_2 & w_2 w_2 & \cdots & w_m w_2 \\
\vdots & \vdots & \ddots & \vdots \\
w_1 w_{2n} & w_2 w_{2n} & \cdots & w_m w_{2n} \end{pmatrix} = -2D_{w}E_{2n \times m}D_{w},
\]

\[
D(w) = \begin{pmatrix} w_1 2 w_1 & w_1 w_1 & \cdots & w_1 w_{2n-1} w_1 \\
w_2 w_2 & w_2 w_2 & \cdots & w_2 w_{2n-2} w_2 \\
\vdots & \vdots & \ddots & \vdots \\
w_2 w_{2n} & w_2 w_{2n} & \cdots & w_2 w_{2n-1} w_{2n} \end{pmatrix} + I_{2n} = D_{w}E_{2n \times 2n}D_{w}J_{2n} + I_{2n}.
\]

Algebraic operations with the matrices \( A(w), B(w), C(w), D(w) \) are easy since

\[
E_{p \times m}D_{w}^{2}E_{m \times q} = \left( \sum_{j=1}^{m} w_{j}^{2} \right) E_{p \times q}, \quad E_{p \times 2n}D_{w}J_{2n}D_{w}E_{2n \times q} = 2 \left( \sum_{j=1}^{n} w_{2j-1} w_{2j} \right) E_{p \times q}. \quad (15)
\]

**Proposition 8.** Let \( w \in S(m|2n)(\Lambda_N) \). Then \( \psi(w) \in O_{0} \) and \( \text{sdet}(\psi(w)) = -1 \).

**Proof.**

In order to prove that \( \psi(w) \in O_{0} \) it suffices to prove that \( A(w), B(w), C(w), D(w) \) satisfy (11). This can be easily done using (15) and the identity \(-1 = w^{2} = -\sum_{j=1}^{m} w_{j}^{2} + \sum_{j=1}^{n} w_{2j-1} w_{2j} \). In fact, we have

\[
A(w)^{T}A(w) = 4D_{w}E_{m \times m}D_{w}^{2}E_{m \times m}D_{w} - 4D_{w}E_{m \times m}D_{w} + I_{m}
= 4 \left( \sum_{j=1}^{m} w_{j}^{2} \right) D_{w}E_{m \times m}D_{w} - 4D_{w}E_{m \times m}D_{w} + I_{m},
\]

and \( C(w)^{T}O_{2n}C(w) = 4D_{w}E_{m \times 2n}D_{w}O_{2n}D_{w}E_{2n \times m}D_{w} = 8 \left( \sum_{j=1}^{n} w_{2j-1} w_{2j} \right) D_{w}E_{m \times m}D_{w} \). Then,

\[
A(w)^{T}A(w) - \frac{1}{2}C(w)^{T}O_{2n}C(w) = 4 \left[ \sum_{j=1}^{m} w_{j}^{2} - 1 - \sum_{j=1}^{n} w_{2j-1} w_{2j} \right] D_{w}E_{m \times m}D_{w} + I_{m} = I_{m}. \]

Also,

\[
A(w)^{T}B(w) = -2D_{w}E_{m \times m}D_{w}E_{m \times 2n}D_{w}O_{2n} + D_{w}E_{m \times 2n}D_{w}O_{2n}
= -2 \left( \sum_{j=1}^{m} w_{j}^{2} \right) D_{w}E_{m \times 2n}D_{w}O_{2n} + D_{w}E_{m \times 2n}D_{w}O_{2n},
\]

and

\[
C(w)^{T}O_{2n}D(w) = -2D_{w}E_{m \times 2n}D_{w}O_{2n}D_{w}E_{2n \times m}D_{w}O_{2n} - 2D_{w}E_{m \times 2n}D_{w}O_{2n}
= -4 \left( \sum_{j=1}^{n} w_{2j-1} w_{2j} \right) D_{w}E_{m \times 2n}D_{w}O_{2n} - 2D_{w}E_{m \times 2n}D_{w}O_{2n}.
\]
Hence, $A(w)^T B(w) - \frac{1}{2} C(w)^T \Omega_{2n} D(w) = 2 \left[ - \sum_{j=1}^m w_j^2 + 1 + \sum_{j=1}^n w_{2j-1} w_{2j} \right] D_w E_{m \times 2n} D_w \Omega_{2n} = 0$. In the same way we have

$$B(w)^T B(w) = - \Omega_{2n} D_w E_{2n \times m} D_w^2 E_{m \times 2n} D_w \Omega_{2n} = - \left( \sum_{j=1}^m w_j^2 \right) \Omega_{2n} D_w E_{2n \times 2n} D_w \Omega_{2n},$$

and

$$D(w)^T \Omega_{2n} D(w) = \Omega_{2n} D_w E_{2n \times 2n} D_w \Omega_{2n} + 2 \Omega_{2n} D_w I_{2n \times 2n} D_w \Omega_{2n} = \Omega_{2n},$$

whence

$$B(w)^T B(w) + \frac{1}{2} D(w)^T \Omega_{2n} D(w) = - \left( \sum_{j=1}^m w_j^2 + 1 + \sum_{j=1}^n w_{2j-1} w_{2j} \right) \Omega_{2n} D_w E_{2n \times 2n} D_w \Omega_{2n} + \frac{1}{2} \Omega_{2n} = \frac{1}{2} \Omega_{2n}.$$
It is clearly seen that the restriction of $\psi$ to the bosonic spin group, defined as
\[
\text{Spin}_b(m|2n)(\Lambda_N) = \{ w_1 \cdots w_{2k} : w_j \in S(m|2n)(\Lambda_N), k \in \mathbb{N} \},
\]
takes values in the subgroup $SO_0 \subset O_0$.

In the classical case, the Pin group and the Spin group are double coverings of the groups $O(m)$ and $SO(m)$ respectively. A natural question in this setting is whether $\text{Pin}_b(m|2n)(\Lambda_N)$ and $\text{Spin}_b(m|2n)(\Lambda_N)$ cover the groups $O_0$ and $SO_0$. The answer to this question is negative and the main reason for this is that the real projection of every vector $w \in S(m|2n)(\Lambda_N)$ is in the unitary sphere $\mathbb{S}^{m-1}$ of $\mathbb{R}^m$, i.e.,
\[
[w]_0 = \sum_{j=1}^{m} [w_j]_0 e_j \quad \text{and} \quad [w]_0^2 = -1.
\]

Then, the real projection of $\psi(\text{Pin}_b(m|2n)(\Lambda_N))$ is just $O(m)$, while $[O_0]_0 = O(m) \times \text{Sp}_\Omega(2n)$. This means that these bosonic versions of Pin and Spin do not describe the symplectic parts of $O_0$ and $SO_0$. This phenomenon is due to the natural structure of supervectors: their real projections belong to a space with an orthogonal structure while the symplectic structure plays no rôlé. Up to a nilpotent vector, they are classical Clifford vectors, whence it is impossible to generate by this approach the real symplectic geometry that is also present in the structure of $O_0$ and $SO_0$. That is why we have chosen the name of "bosonic" Pin and "bosonic" Spin groups. This also explains why we had to extend the space of superbivectors in Section 3.2. The ordinary superbivectors in $\Lambda_N \otimes C_{m,2n}$ are generated over $\Lambda^{ev}_N$ by the wedge product of supervectors. Then, they can only describe $\mathfrak{so}(m)$ and not $\mathfrak{sp}_\Omega(2n)$ and in consequence, they do not cover $\mathfrak{so}_0$.

As in the classical setting (see [17]), it is possible to obtain the following result that shows, from another point of view, that $\text{Pin}_b(m|2n)(\Lambda_N)$ cannot completely describe $O_0$.

**Proposition 9.** The Lie algebra of $\text{Pin}_b(m|2n)(\Lambda_N)$ is included in $\mathbb{R}^{(2)}_{m|2n}(\Lambda_N)$.

**Proof.**
Let $\gamma(t) = w_1(t) \cdots w_k(t)$ be a path in $\text{Pin}_b(m|2n)(\Lambda_N)$ with $w_j(t) \in S(m|2n)(\Lambda_N)$ for every $t \in \mathbb{R}$ and $\gamma(0) = 1$. The tangent to $\gamma$ at $t = 0$ is $\frac{d\gamma}{dt}|_{t=0} = \sum_{j=1}^{k} w_1(0) \cdots w_j'(0) \cdots w_k(0)$. We will show that each summand of $\frac{d\gamma}{dt}|_{t=0}$ belongs to $\mathbb{R}^{(2)}_{m|2n}(\Lambda_N)$.

For $j = 1$ we have $w_1'(0)w_2(0) \cdots w_k(0) = -w_1'(0)w_1(0)$. But $w_1(t)w_1(t) \equiv -1$ implies
\[
\{ w_1'(0), w_1(0) \} = w_1'(0)w_1(0) + w_1(0)w_1'(0) = 0.
\]
Then $w_1'(0)w_1(0) = \frac{1}{2} \{ w_1'(0), w_1(0) \} + w_1'(0) \wedge w_1(0) = w_1'(0) \wedge w_1(0) \in \mathbb{R}^{(2)}_{m|2n}(\Lambda_N)$. For $j = 2,$
\[
w_1(0)w_2(0) \cdots w_k(0) = w_1(0)w_2'(0)w_2(0)w_1(0) = -[w_1(0)w_2'(0)w_1(0)] [w_1(0)w_2(0)w_1(0)]
= -\psi(w_1(0)) [w_2'(0)] \psi(w_1(0))[w_2(0)].
\]
But $\psi(w_1(0)) \in O_0$ preserves the inner product (see remark 4.1), so
\[
w_1(0)w_2'(0) \cdots w_k(0) = \psi(w_1(0))[w_2'(0)] \wedge \psi(w_1(0))[w_2(0)] \in \mathbb{R}^{(2)}_{m|2n}(\Lambda_N).
\]
We can proceed similarly for every $j = 3, \ldots, k$. \hfill $\Box$

### 5.2 A proper definition for the group $\text{Spin}(m|2n)(\Lambda_N)$

The above approach shows that the radial algebra setting does not contain a suitable realization of $SO_0$ in the Clifford superspace framework. Observe that the Clifford representation of $\mathfrak{so}_0$ given by $\mathbb{R}^{(2)}_{m|2n}(\Lambda_N)$ lies outside of the radial algebra $\mathbb{R}^{(2)}_{m|2n}(\Lambda_N)$, which suggests that something similar should happen with the corresponding Lie group $SO_0$. In this case, a proper definition for the Spin group would be generated by the exponentials (in general contained in $T(V)/I$) of all elements in $\mathbb{R}^{(2)}_{m|2n}(\Lambda_N)$, i.e.
\[
\text{Spin}(m|2n)(\Lambda_N) := \{ e^{B_1} \cdots e^{B_k} : B_1, \ldots, B_k \in \mathbb{R}^{(2)}_{m|2n}(\Lambda_N), k \in \mathbb{N} \},
\]

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and the action of this group on supervector variables $x \in S$ is given by the group homomorphism $h : \text{Spin}(m|2n)(\Lambda_N) \to \text{SO}_0$ defined by

$$h(e^B)[x] = e^Bxe^{-B}, \quad B \in \mathbb{R}^{(2)}_{m|2n}(\Lambda_N), \quad x \in S.$$  \hspace{1cm} (16)

In fact, for every extended superbivector $B$, $h(e^B)$ maps supervector variables into new supervector variables and admits a supermatrix representation in $\text{Mat}(m|2n)(\Lambda_N)$ belonging to $\text{SO}_0$. This is summarized below.

**Proposition 10.** Let $B \in \mathbb{R}^{(2)}_{m|2n}(\Lambda_N)$. Then, $h(e^B)[x] = e^\phi(B)x$ for every $x \in S$.

**Proof.**

In the associative algebra $\mathcal{A}_{m,2n} \otimes \Lambda_N$, the identity $[B,[B,[B,x]...]] = \sum_{j=0}^{k} \binom{k}{j} B^j x (-B)^{k-j}$ holds.

Then,

$$h(e^B)[x] = e^Bxe^{-B} = \left( \sum_{k=0}^{\infty} \frac{B^k}{k!} \right) x \left( \sum_{k=0}^{\infty} \frac{(-B)^k}{k!} \right) = \sum_{k=0}^{\infty} \frac{B^k}{k!} \left( \sum_{j=0}^{k} \frac{(B_x)^{k-j}}{(k-j)!} \right)$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \left( \sum_{j=0}^{k} \binom{k}{j} B^j x (-B)^{k-j} \right) = \sum_{k=0}^{\infty} \frac{1}{k!} \left( [B,[B,[B,x]...]] \right) = \sum_{k=0}^{\infty} \frac{\phi(B)^k x}{k!} = e^{\phi(B)}x.$$

\[\square\]

**Remark 5.1.** Proposition 10 means that the Lie algebra isomorphism $\phi : \mathbb{R}^{(2)}_{m|2n}(\Lambda_N) \to \mathfrak{so}_0$ is the derivative at the origin (or infinitesimal representation) of the Lie group homomorphism $h : \text{Spin}(m|2n)(\Lambda_N) \to \text{SO}_0$, i.e.,

$$e^{t\phi(B)} = h(e^{tB}) \quad \forall t \in \mathbb{R}, \quad B \in \mathbb{R}^{(2)}_{m|2n}(\Lambda_N). \hspace{1cm} (17)$$

On account of the connectedness of $\text{SO}_0$ it can be shown that the group $\text{Spin}(m|2n)(\Lambda_N)$ is a realization of $\text{SO}_0$ in $T(V)/I$ through the representation $h$.

**Theorem 5.** For every $M \in \text{SO}_0$ there exists an element $s \in \text{Spin}(m|2n)(\Lambda_N)$ such that $h(s) = M$.

**Proof.**

Since $\text{SO}_0$ is a connected Lie group (Proposition 5), for every supermatrix $M \in \text{SO}_0$ there exist $X_1,\ldots,X_k \in \mathfrak{so}_0$ such that $e^{X_1}\cdots e^{X_k} = M$, see Corollary 3.47 in [20]. Taking $B_1,\ldots,B_k \in \mathbb{R}^{(2)}_{m|2n}(\Lambda_N)$ such that $\phi(B_j) = X_j, \; j = 1,\ldots,k$, we obtain

$$Mx = e^{X_1}\cdots e^{X_k} x = e^{\phi(B_1)}\cdots e^{\phi(B_k)} x = h(e^{B_1}) \circ \cdots \circ h(e^{B_k})[x] = h(e^{B_1}\cdots e^{B_k})[x].$$

Then, $s = e^{B_1}\cdots e^{B_k} \in \text{Spin}(m|2n)(\Lambda_N)$ satisfies $h(s) = M$. \[\square\]

The decomposition of $\text{SO}_0$ given in Theorem 3 provides the exact number of exponentials of extended superbivectors to be considered in $\text{Spin}(m|2n)(\Lambda_N)$ in order to cover the whole group $\text{SO}_0$. If we consider the subspaces $\Xi_1,\Xi_2,\Xi_3$ of $\mathbb{R}^{(2)}_{m|2n}(\Lambda_N)$ given by

$$\Xi_1 = \phi^{-1}(\mathfrak{so}(m) \times [\mathfrak{sp}_{2n}(2n) \cap \mathfrak{so}(2n)]), \quad \text{dim } \Xi_1 = \frac{m(m-1)}{2} + n^2,$$

$$\Xi_2 = \phi^{-1}([0,m] \times [\mathfrak{sp}_{2n}(2n) \cap \text{Sym}(2n)]), \quad \text{dim } \Xi_2 = n^2 + n,$$

$$\Xi_3 = \phi^{-1}(\mathfrak{so}_0(m|2n)(\Lambda_N)), \quad \text{dim } \Xi_3 = \text{dim } \mathfrak{so}_0 - \frac{m(m-1)}{2} - n(2n + 1), \hspace{1cm} (18)$$

we get the decomposition $\mathbb{R}^{(2)}_{m|2n}(\Lambda_N) = \Xi_1 \oplus \Xi_2 \oplus \Xi_3$, leading to the subset

$$\Xi = \text{exp}(\Xi_1) \exp(\Xi_2) \exp(\Xi_3) \subset \text{Spin}(m|2n)(\Lambda_N),$$

which suffices for describing $\text{SO}_0$. Indeed, from Theorem 3 it follows that the restriction $h : \Xi \to \text{SO}_0$ is surjective. We now investigate the explicit form of the superbivectors in each of the subspaces $\Xi_1$, $\Xi_2$ and $\Xi_3$. 

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Proposition 11. The following statements hold.

(i) A basis for \( \Xi_1 \) is
\[
\begin{align*}
&\{e_{j,k}, 1 \leq j < k \leq m, \\
&\hat{e}_{2j-1} \odot \hat{e}_{2k-1} + \hat{e}_{2j} \odot \hat{e}_{2k}, 1 \leq j \leq k \leq n, \\
&\hat{e}_{2j-1} \odot \hat{e}_{2k} - \hat{e}_{2j} \odot \hat{e}_{2k-1}, 1 \leq j < k \leq n.
\end{align*}
\]

(ii) A basis for \( \Xi_2 \) is:
\[
\begin{align*}
&\{\hat{e}_{2j-1} \odot \hat{e}_{2j}, 1 \leq j \leq n, \\
&\hat{e}_{2j-1} \odot \hat{e}_{2k} - \hat{e}_{2j} \odot \hat{e}_{2k}, 1 \leq j \leq k \leq n, \\
&\hat{e}_{2j-1} \odot \hat{e}_{2k} + \hat{e}_{2j} \odot \hat{e}_{2k-1}, 1 \leq j < k \leq n.
\end{align*}
\]

(iii) \( \Xi_3 \) consists of all elements of the form (10) with \( b_{j,k}, B_{j,k} \in \Lambda_N^{(ev)} \cap \Lambda_N^+ \) and \( \bar{b}_{j,k} \in \Lambda_N^{(odd)} \).

Proof.

We first recall that a basis for the Lie algebra \( \mathfrak{sp}_N(2n) \) is given by the elements
\[
A_{j,k} := E_{2j,2k-1} + E_{2k,2j-1}, 1 \leq j \leq k \leq n, \quad B_{j,k} := E_{2j-1,2k} + E_{2k-1,2j}, 1 \leq j \leq k \leq n, \quad D_{j,k} := E_{2j,2k} - E_{2k-1,2j-1}, 1 \leq j \leq k \leq n,
\]
where the matrices \( E_{j,k} \in \mathbb{R}^{n \times n} \) are defined as in Lemma 2. It holds that \( A^T_{j,k} = B_{j,k} \) for \( 1 \leq j \leq k \leq n \), \( C^T_{j,k} = D_{j,k} \) for \( 1 \leq j < k \leq n \) and \( C^T_{j,j} = C_{j,j} \) for \( 1 \leq j \leq n \). Hence, for every \( D_0 \in \mathfrak{sp}_N(2n) \) we have
\[
D_0 = \sum_{1 \leq j \leq k \leq n} (a_{j,k} A_{j,k} + b_{j,k} B_{j,k} + c_{j,k} C_{j,k}) + \sum_{1 \leq j \leq k \leq n} d_{j,k} D_{j,k},
\]

\[
D^T_0 = \sum_{1 \leq j \leq k \leq n} (a_{j,k} B_{j,k} + b_{j,k} A_{j,k}) + \sum_{1 \leq j < k \leq n} (c_{j,k} D_{j,k} + d_{j,k} C_{j,k}) + \sum_{j=1}^n c_{j,j} C_{j,j},
\]

where \( a_{j,k}, b_{j,k}, c_{j,k}, d_{j,k} \in \mathbb{R} \).

(i) From the previous equalities we get that \( D^T_0 = -D_0 \) if and only if
\[
D_0 = \sum_{1 \leq j \leq k \leq n} a_{j,k} (A_{j,k} - B_{j,k}) + \sum_{1 \leq j \leq k \leq n} c_{j,k} (C_{j,k} - D_{j,k}).
\]

Then, a basis for \( \mathfrak{sp}_N(2n) \cap \mathfrak{so}(2n) \) is \( \{A_{j,k} - B_{j,k} : 1 \leq j \leq k \leq n\} \cup \{C_{j,k} - D_{j,k} : 1 \leq j < k \leq n\} \).

The remainder of the proof directly follows from Lemma 2.

(ii) In this case we have that \( D^T_0 = D_0 \) if and only if
\[
D_0 = \sum_{1 \leq j \leq k \leq n} a_{j,k} (A_{j,k} + B_{j,k}) + \sum_{1 \leq j < k \leq n} c_{j,k} (C_{j,k} + D_{j,k}) + \sum_{j=1}^n c_{j,j} C_{j,j},
\]

whence a basis for \( \mathfrak{sp}_N(2n) \cap \mathfrak{sym}(2n) \) is
\[
\{A_{j,k} + B_{j,k} : 1 \leq j \leq k \leq n\} \cup \{C_{j,j} : 1 \leq j \leq n\} \cup \{C_{j,k} + D_{j,k} : 1 \leq j < k \leq n\}.
\]

The remainder of the proof directly follows from Lemma 2.

(iii) This trivially follows from Lemma 2. \( \square \)

5.3 Spin covering of the group \( \text{SO}_0 \)

It is a natural question in this setting whether the spin group still is a double covering of the group of rotations, as it is in classical Clifford analysis. In other words, we will investigate how many times \( \Xi \subset \text{Spin}(m|2n)(\Lambda_N) \) covers \( \text{SO}_0 \), or more precisely, we will determine the cardinality of the set \( \{s \in \Xi : h(s) = M\} \) given a certain fixed element \( M \in \text{SO}_0 \).

From Proposition 10 we have that the representation \( h \) of an element \( s = e^{B_1}e^{B_2}e^{B_3} \in \Xi, B_j \in \Xi_j \), has the form \( h(s) = e^{ \phi(B_1)}e^{ \phi(B_2)}e^{ \phi(B_3)} \). Following the decomposition \( M = e^Xe^Ye^Z \) given in Theorem
3 for $M \in SO_0$, we get that $h(s) = M$ if and only if $e^{\phi(B_1)} = e^X$, $B_2 = \phi^{-1}(Y)$ and $B_3 = \phi^{-1}(Z)$. Then, the cardinality of $\{ s \in \Xi : h(s) = M \}$ only depends on the number of extended superbivectors $B_1 \in \Xi_1$ that satisfy $e^{\phi(B_1)} = e^X$. It reduces our analysis to finding the kernel of the restriction $h_{\text{exp}(\Xi_1)} : \text{exp}(\Xi_1) \to SO(m) \times [\mathfrak{sp}_\Omega(2n) \cap SO(2n)]$ of the Lie group homomorphism $h$ to exp$(\Xi_1)$. This kernel is given by ker $h_{\text{exp}(\Xi_1)} = \{ e^B : e^{\phi(B)} = I_{m+2n}, B \in \Xi_1 \}$.

We recall, from Proposition 11, that $B \in \Xi_1$ may be written as $B = B_0 + B_1$ where $B_0 \in \mathbb{R}_{0,m}^{(2)}$ is a classical real bivector and $B_1 \in \phi^{-1}(\{0_m\} \times [\mathfrak{sp}_\Omega(2n) \cap \mathfrak{so}(2n)])$. The components $B_0, B_1$ commute and in consequence, $e^B = e^{B_0} e^{B_1}$. Consider the projections $\phi_0$ and $\phi_1$ of $\phi$ over the algebra of classical bivectors $\mathbb{R}_{0,m}^{(2)}$ and over the algebra $\phi^{-1}(\{0_m\} \times [\mathfrak{sp}_\Omega(2n) \cap \mathfrak{so}(2n)])$ respectively, i.e.

$$\phi_0 : \mathbb{R}_{0,m}^{(2)} \to \mathfrak{so}(m), \quad \phi_1 : \phi^{-1}(\{0_m\} \times [\mathfrak{sp}_\Omega(2n) \cap \mathfrak{so}(2n)]) \to \mathfrak{sp}_\Omega(2n) \cap \mathfrak{so}(2n),$$

where $\phi(B) = \left( \begin{array}{cc} \phi_0(B_0) & 0 \\ 0 & \phi_1(B_1) \end{array} \right)$ for $B \in \Xi_1$. Or equivalently,

$$\begin{cases} \phi_0(B_0) \left[ \begin{array}{c} x \\ \hat{x} \end{array} \right] = \left[ B_0, x \right], \\ \phi_1(B_1) \left[ \begin{array}{c} x \\ \hat{x} \end{array} \right] = \left[ B_1, x \right], \end{cases} \quad x = x + \hat{x} \in \mathcal{S}.$$  

Hence $e^{\phi_0(B)} = I_{m+2n}$ if and only if $e^\phi_0(B_0) = I_m$ and $e^\phi_1(B_1) = I_{2n}$. For the first condition, we know from classical Clifford analysis that Spin$(m) = \{ e^B : B \in \mathbb{R}_{0,m}^{(2)} \}$ is a double covering of SO$(m)$ and in consequence $e^\phi_0(B_0) = I_m$ implies $e^{B_0} = \pm 1$. Let us now compute all possible values for $e^{B_1}$ for which $e^\phi_1(B_1) = I_{2n}$. To that end, we need the following linear algebra result.

**Proposition 12.** Every matrix $D_0 \in \mathfrak{so}(2n) \cap \mathfrak{sp}_\Omega(2n)$ can be written in the form $D_0 = R \Sigma R^T$ where $R \in SO(2n) \cap Sp_\Omega(2n)$ and

$$\Sigma = \begin{pmatrix} 0 & \theta_1 \\ -\theta_1 & 0 \\ & \ddots \\ & & 0 & \theta_n \\ & & -\theta_n & 0 \end{pmatrix}, \quad \theta_j \in \mathbb{R}, \quad j = 1, \ldots, n. \quad (19)$$

**Proof.**

The map $\Psi(D_0) = \frac{1}{2} Q D_0 \left( Q^T \right)^c$, where

$$Q = \begin{pmatrix} 1 & i & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & i \end{pmatrix} \in \mathbb{C}^{n \times 2n},$$

is a Lie group isomorphism between SO$(2n) \cap Sp_\Omega(2n)$ and $U(n)$. It is easily proven that $\Psi$ is its own infinitesimal representation on the Lie algebra level, and in consequence, a Lie algebra isomorphism between $\mathfrak{so}(2n) \cap \mathfrak{sp}_\Omega(2n)$ and $\mathfrak{u}(n)$. The inverse of $\Psi$ is given by $\Psi^{-1}(L) = \frac{1}{2} \left[ (Q^T)^c L Q + Q^T L^c Q \right]$. For every $D_0 \in \mathfrak{so}(2n) \cap \mathfrak{sp}_\Omega(2n)$, let us consider the skew-Hermitian matrix $L = \Psi(D_0) \in \mathfrak{u}(n)$. It is known that every skew-Hermitian matrix is unitarily diagonalizable and all its eigenvalues are purely imaginary, see [22]. Hence, $L = \Psi(D_0)$ can be written as $L = U \mathcal{D} (U^T)^c$ where $U \in U(n)$ and $\mathcal{D} = \text{diag}(-i\theta_1, \ldots, -i\theta_n)$, $\theta_j \in \mathbb{R}$. Then, $D_0 = \Psi^{-1}(L) = R \Sigma R^T$ where $R = \Psi^{-1}(U) \in SO(2n) \cap Sp_\Omega(2n)$ and $\Sigma = \Psi^{-1}(\mathcal{D})$ has the form (19).

Since $\phi_1(B_1) \in \mathfrak{so}(2n) \cap \mathfrak{sp}_\Omega(2n)$, we have $\phi_1(B_1) = R \Sigma R^T$ as in the previous proposition. Hence, $e^{\phi_1(B_1)} = R e^\Sigma R^T$ where $e^\Sigma$ is the block-diagonal matrix

$$e^\Sigma = \text{diag}(e^{\theta_1 \Omega_2}, \ldots, e^{\theta_n \Omega_2}) \quad \text{with} \quad e^{\theta_j \Omega_2} = \cos \theta_j I_2 + \sin \theta_j \Omega_2.$$  

Hence $e^{\phi_1(B_1)} = I_{2n}$ if and only if $e^\Sigma = I_{2n}$, which is seen to be equivalent to $\theta_j = 2k_j \pi$, $k_j \in \mathbb{Z}$ ($j = 1, \ldots, n$), or to

$$\Sigma = \sum_{j=1}^n 2k_j \pi (E_{2j-1,2j} - E_{2j,2j-1}), \quad k_j \in \mathbb{Z} \ (j = 1, \ldots, n).$$
Now, $\text{SO}(2n) \cap \text{Sp}_{\mathbb{H}}(2n)$ being connected and compact, there exists $B_R \in \phi^{-1}(\mathfrak{so}(2n) \cap \mathfrak{sp}_{\mathbb{H}}(2n))$ such that $R = e^{\phi(B_R)}$. We recall that the $h$-action leaves any multivector structure invariant, in particular, $h[e^R] \left( \mathbb{P}^{(2n)}_{m;2n}(\Lambda N) \right) \subset \mathbb{P}^{(2n)}_{m;2n}(\Lambda N)$ for every $B \in \mathbb{P}^{(2n)}_{m;2n}(\Lambda N)$. Then, using the fact that $\phi$ is the derivative at the origin of $h$, we get that the extended supervector $h(e^{B_R})[\phi^{-1}(\Sigma)] = e^{B_R} \phi^{-1}(\Sigma) e^{-B_R}$ is such that

$$
\phi(e^{B_R} \phi^{-1}(\Sigma) e^{-B_R}) = e^{\phi(B_R)} \pi \phi^{-1}(\Sigma) e^{-\phi(B_R)} = R \Sigma R^T = \phi(B_R),
$$

implying that $B_R = e^{B_R} \phi^{-1}(\Sigma) e^{-B_R}$. Then, in order to compute $e^{B_R} = e^{B_R} e^{\phi^{-1}(\Sigma)} e^{-B_R}$, we first have to compute $e^{\phi^{-1}(\Sigma)}$. Following the correspondences given in Lemma 2 we get

$$
\phi^{-1}(\Sigma) = \sum_{j=1}^{n} 2k_j \pi \phi^{-1}((E_{2j-1,2j} - E_{2j,2j+1}) = \sum_{j=1}^{n} k_j \pi (e^{2}_{2j-1} + e^{2}_{2j})
$$

and, in consequence

$$
e^{\phi^{-1}(\Sigma)} = \exp \left( \sum_{j=1}^{n} k_j \pi (e^{2}_{2j-1} + e^{2}_{2j}) \right) = \prod_{j=1}^{n} \exp \left[ k_j \pi (e^{2}_{2j-1} + e^{2}_{2j}) \right]. \tag{20}
$$

Let us compute $\exp \left[ \pi (e^{2}_{2j-1} + e^{2}_{2j}) \right]$, $j \in \{1, \ldots, n\}$. Consider $a = e^{2}_{2j-1} - i e^{2}_{2j}$ and $b = e^{2}_{2j-1} + i e^{2}_{2j}$ where $i$ is the usual imaginary unit in $\mathbb{C}$. It is clear that $ab = e^{2}_{2j-1} + e^{2}_{2j} + i(e^{2}_{2j-1} - e^{2}_{2j}) = e^{2}_{2j-1} + e^{2}_{2j} + i$ and $[a, b] = 2i$ which is a commuting element. Then, $\exp \left[ \pi (e^{2}_{2j-1} + e^{2}_{2j}) \right] = \exp (\pi ab - i\pi) = -\exp (\pi ab)$ in order to compute $\exp (\pi ab)$ we first prove the following results.

**Lemma 4.** For every $k \in \mathbb{N}$ the following relations hold.

(i) $[b^k, a] = -2ik b^{k-1}$,  
(ii) $a^k b^k a b = a^{k+1} b^{k+1} - 2ik a^k b^k$.

**Proof.**

(i) We proceed by induction. For $k = 1$ we get $[b, a] = -2i$ which obviously is true. Now assume that (i) is true for $k \geq 1$, then for $k + 1$ we get

$$
b^{k+1} a = b (b^k a) = b a b^k - 2ik b^k = (ab - 2i) b^k - 2ik b^k = ab^{k+1} - 2i (k + 1) b^k. $$

(ii) From (i) we get $a^k b^k a b = a^k (ab - 2ik b^{k-1}) b = a^{k+1} b^{k+1} - 2i k a^k b^k$. \hfill $\square$

**Lemma 5.** For every $k \in \mathbb{N}$ it holds that $[ab]^k = \sum_{j=1}^{k} (-2i)^{k-j} S(k, j) a^j b^{k-j}$, where $S(n, j)$ is the Stirling number of the second kind corresponding to $n$ and $j$.

**Remark 5.2.** The Stirling number of the second kind $S(k, j)$ is the number of ways of partitioning a set of $k$ elements into $j$ non-empty subsets. We recall the following properties of the Stirling numbers,

$$
S(k, 1) = S(k, k) = 1, \quad S(k+1, j+1) = S(k, j) + (j + 1) S(k, j + 1), \quad \sum_{k=0}^{\infty} S(k, j) \frac{x^k}{k!} = \frac{(e^x - 1)^j}{j!}.
$$

**Proof of Lemma 5.**

We proceed by induction. For $k = 1$ the statement clearly is true. Now assume it to be true for $k \geq 1$. Using Lemma 4, we have for $k + 1$ that

$$
(ab)^{k+1} = \sum_{j=1}^{k} (-2i)^{k-j} S(k, j) a^j b^{k-j} = \sum_{j=1}^{k} (-2i)^{k-j} S(k, j) a^{j+1} b^{j+1} + (-2i)^{k+1-j} S(k, j) a^j b^j
$$

$$
= (-2i)^k ab + \sum_{j=1}^{k-1} (-2i)^{k-j} [S(k, j) + (j + 1) S(k, j + 1)] a^j b^{j+1} + a^{k+1} b^{k+1}
$$

$$
= \sum_{j=1}^{k+1} (-2i)^{k+1-j} S(k + 1, j) a^j b^j,
$$
which proves the lemma.

Then we obtain

\[
e^{ab} = \sum_{k=0}^{\infty} \frac{\pi^k}{k!} (ab)^k = 1 + \sum_{j=1}^{\infty} \sum_{k=j}^{\infty} \frac{\pi^k}{k!} (-2i)^{k-j} S(k, j) a^j b^j
\]

\[
= 1 + \sum_{j=1}^{\infty} \sum_{k=j}^{\infty} \frac{\pi^k}{k!} (-2i)^{k-j} S(k, j) a^j b^j = 1 + \sum_{j=1}^{\infty} (-2i)^{j-1} \left[ \sum_{k=j}^{\infty} \frac{(-2i)^k}{k!} S(k, j) \right] a^j b^j
\]

\[
= 1 + \sum_{j=1}^{\infty} (-2i)^{j-1} \left( e^{\frac{\pi i}{2}} - 1 \right) a^j b^j = 1,
\]

from which we conclude that \( \exp \left[ \pi \left( e^{2\theta - 1} + e^{2\theta} \right) \right] = -1 - \exp(ab) = -1 \).

**Remark 5.3.** Within the algebra \( C_{0,2n} = \text{Alg}_R \{ e^1, \ldots, e_{2n} \} \) the elements \( e_{2j-1}, e_{2j} \) may be identified with the operators \( e^{\frac{\pi i}{2}} \partial_{a_j} e^{-\frac{\pi i}{2}} a_j \), respectively, the \( a_j \)'s being real variables. Indeed, these identifications immediately lead to the Weyl algebra defining relations

\[
e^{\frac{\pi i}{2}} \partial_{a_j} e^{-\frac{\pi i}{2}} a_k - e^{-\frac{\pi i}{2}} a_k e^{\frac{\pi i}{2}} \partial_{a_j} = \partial_{a_j} a_k - a_k \partial_{a_j} = \delta_{j,k}.
\]

Hence \( e_{2j-1}^2 + e_{2j}^2 \) may be identified with the harmonic oscillator \( i \left( \partial_{a_j}^2 - a_j^2 \right) \) and in consequence, the element \( \exp \left[ \pi \left( e_{2j-1}^2 + e_{2j}^2 \right) \right] \) corresponds to \( \exp \left[ \pi i \left( \partial_{a_j}^2 - a_j^2 \right) \right] \). We recall that the classical Fourier transform in one variable can be written as an operator exponential

\[
\mathcal{F}[f] = \exp \left( \frac{\pi i}{4} \right) \exp \left( \frac{\pi i}{4} \left( \partial_{a_j}^2 - a_j^2 \right) \right) [f].
\]

Hence, \( \exp \left[ \pi i \left( \partial_{a_j}^2 - a_j^2 \right) \right] = -\mathcal{F}^4 = -\text{id} \), where \( \text{id} \) denotes the identity operator.

Going back to (20) we have \( e^{\phi^{-1}(\Xi)} = \prod_{j=1}^{n} \exp \left[ \pi \left( e_{2j-1}^2 + e_{2j}^2 \right) \right] \), whence \( e^{B_5} = \pm 1 \). Then, for \( B = B_o + B_s \in \Xi_1 \) such that \( e^{B} = B_o e^{B_s} = \pm 1 \), we have \( e^B = e^{B_o} e^{B_s} = \pm 1 \), i.e. \( \ker h|_{\exp(\Xi_1)} = \{-1, 1\} \).

In this way, we have proven the following result.

**Theorem 6.** The set \( \Xi = \exp(\Xi_1) \exp(\Xi_2) \exp(\Xi_3) \) is a double covering of \( \text{SO}_0 \).

**Remark 5.4.** As shown before, every extended superbivector of the form \( B = \sum_{j=1}^{n} \theta_j \pi \left( e_{2j-1}^2 + e_{2j}^2 \right) \), \( \theta_j \in \mathbb{R} \), belongs to \( \Xi_1 \). Then, through the identifications (21) we can see all operators

\[
\exp \left[ \sum_{j=1}^{n} \theta_j \frac{\pi i}{2} \left( \partial_{a_j}^2 - a_j^2 \right) \right] = \prod_{j=1}^{n} \exp \left[ \theta_j \frac{\pi i}{2} (\partial_{a_j}^2 - a_j^2) \right] = \prod_{j=1}^{n} \exp \left( -\theta_j \frac{\pi i}{2} \right) \mathcal{F}_{a_j}^{2\theta_j},
\]

as elements of the Spin group in superspace. Here, \( \mathcal{F}_{a_j}^{2\theta_j} \) denotes the one-dimensional fractional Fourier transform of order \( 2\theta_j \) in the variable \( a_j \).

### 6 Conclusions

In this paper we have shown that supervector reflections are not enough to describe the set of linear transformations leaving the inner product invariant. This constitutes a very important difference with the classical case in which the algebra of bivectors \( x \wedge y \) is isomorphic to the special orthogonal algebra \( \mathfrak{so}(m) \). Such a property is no longer fulfilled in this setting. The real projection of the algebra of superbivectors \( \mathbb{R}^{2m}_{m\geq 2n}(\Lambda_N) \) does not include the symplectic algebra structure which is present in the Lie algebra of supermatrices \( \mathfrak{so}_0 \), corresponding to the group of super rotations.

That fact has an major impact on the definition of the Spin group in this setting. The set of elements defined through the multiplication of an even number of unit vectors in \( \mathbb{R}^{m,2n}(\Lambda_N) \) does not
suffice for describing $\text{Spin}(m|2n)(\Lambda_N)$. A suitable alternative, in this case, is to define the (super) spin elements as products of exponentials of extended superbivectors. Such an extension of the Lie algebra of superbivectors contains, through the corresponding identifications, harmonic oscillators. This way, we obtain the Spin group as a cover of the set of superrotations $\text{SO}_\Lambda$ through the usual representation $h$. In addition, every fractional Fourier transform can be identified with a spin element.

In the related paper [12], we have already proven the invariance of the (super) Dirac operator $\partial_x$ under the corresponding actions of this Spin group. We have also studied there the invariance of the Hermitian system under the action of the corresponding Spin subgroup.

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