Harmonic Decompositions of Convolutional Networks

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Abstract

We consider convolutional networks from a reproducing kernel Hilbert space viewpoint. We establish harmonic decompositions of convolutional networks, that is expansions into sums of elementary functions of increasing order. The elementary functions are related to the spherical harmonics, a fundamental class of special functions on spheres. The harmonic decompositions allow us to characterize the integral operators associated with convolutional networks, and obtain as a result statistical bounds for convolutional networks.

1 Introduction

The renewed interest in convolutional neural networks \cite{lecun2015deep,krizhevsky2012imagenet} in computer vision and signal processing has lead to a major leap in generalization performance on common task benchmarks, supported by the recent advances in graphical processing hardware and the collection of huge labelled datasets for training and evaluation. Convolutional neural networks pose major a challenge to statistical learning theory. First and foremost a convolutional network learns from data, jointly, both a feature representation through its hidden layers and a prediction function through its ultimate layer. A convolutional neural network implements a function unfolding as a composition of basic functions (respectively nonlinearity, convolution, and pooling), which appear to model well visual information in images. Yet the relevant function spaces to analyze their statistical performance remain unclear.

The analysis of convolutional neural networks (CNNs) has been an active research topic. Different viewpoints have been developed. A straightforward viewpoint is to dismiss completely the grid- or lattice-structure of images and analyze a multi-layer perceptron (MLP) instead acting on vectorized images, which has the downside the set aside the most interesting property CNNs which is to model well images that is data with a 2D lattice structure.

The scattering transform viewpoint and the \textit{i}-theory viewpoint \cite{aharoni2010scattering,balavoine2018decompositions,jacques2016efficient,kowalski2018i,kowalski2018low} keeps the triad of components nonlinearity-convolution-pooling and their combination in a deep architecture and characterize the group-invariance properties and compression properties of convolutional neural networks. Recent work \cite{feng2019risk} considers risk bounds involving appropriately-defined spectral norms for convolutional kernel networks acting on continuous-domain images. We present in this paper the construction of a function space containing the function implemented by a convolutional network. Doing so we characterize the sequence of eigenvalues an eigenfunctions of the related integral operator, hence shedding light on the harmonic structure of the function space of a convolutional neural network. Indeed the eigenvalue decay controls the statistical convergence rate. Thanks to this spectral characterization, we establish high-probability statistical bounds, relating the eigenvalue decay and the convergence rate.

We show that a convolutional network function admits a decomposition whose structure is related to a functional tensor-product space ANOVA model decomposition \cite{scetbon2020}. Such models extend the popular additive models in order to capture interactions of any order between covariates. Indeed a tensor-product space ANOVA (TPS-ANOVA) model decomposes a high-dimensional multivariate function as a sum of one-dimensional functions (main effects), two-dimensional functions (two-way interactions), and so on.
A remarkable property of TPS-ANOVA models is their statistical convergence rate, which is within a log factor of the rate in one dimension, under appropriate assumptions. We bring to light a similar TPS-ANOVA structure in the decomposition of a convolutional network function. This structure plays an essential role in the convergence rates we present. This suggests that an important component of the modeling power of a convolutional network is to capture spatial interactions between sub-images or patches.

This work makes the following contributions.

- We construct the kernel and the corresponding reproducing kernel Hilbert space (RKHS) of convolutional networks (CNNs), for networks that may have an arbitrary number of filters per layer. Moreover we give a sufficient condition for the kernel to be universal.

- We establish an explicit, analytical, Mercer decomposition of the multi-layer kernel associated to this RKHS. We show that CNNs are related to Tensor-Product ANOVA models, in that a sum-product structure involving interactions between sub-images or patches underlies the CNN models.

- We obtain convergence rates of the learned function to the Bayes classifier when minimizing the least-squares loss and express the convergence rate in terms of the eigenvalue decay rate of the associated integral operator.

2 Basic Notions and Notations

We consider the standard nonparametric learning framework \[13, 27\], where the goal is to learn, from independent and identically distributed examples \( z = \{(x_1, y_1), \ldots, (x_\ell, y_\ell)\} \) from an unknown distribution \( \rho \), a functional dependency \( f_z : \mathcal{X} \to \mathcal{Y} \) between input \( x \in \mathcal{X} \) and output \( y \in \mathcal{Y} \). We adopt the same framework as \[24\], which we reproduce here for convenience. The joint distribution \( \rho(x, y) \), the marginal distribution \( \rho_X \), and the conditional distribution \( \rho(y|x) \), are related through \( \rho(x, y) = \rho_X(x)\rho(y|x) \). We call the \( f_z \) the learning method or the estimator and the learning algorithm is the procedure that, for any sample size \( \ell \in \mathbb{N} \) and training set \( z \in \mathcal{Z}_\ell \), yields the learned function or estimator \( f_z \). If the output space \( \mathcal{Y} \subset \mathbb{R} \), given a function \( f : \mathcal{X} \to \mathcal{Y} \), the ability of \( f \) to describe the distribution \( \rho \) is measured by its expected risk

\[
R(f) := \int_{\mathcal{X} \times \mathcal{Y}} (f(x) - y)^2 \, d\rho(x, y).
\]

The minimizer over the space of measurable \( \mathcal{Y} \)-valued functions on \( \mathcal{X} \) is the function

\[
f_\rho(x) := \int_{\mathcal{Y}} y \, d\rho(y|x).
\]

The final aim of learning theory is to find an algorithm such that \( R(f_z) \) is close to \( R(f_\rho) \) with high probability. Let us now introduce the regularized least-squares algorithm. Consider as hypothesis space a Hilbert space \( H \) of functions \( f : \mathcal{X} \to \mathcal{Y} \). For any regularization parameter \( \lambda > 0 \) and training set \( z \in \mathcal{Z}_\ell \), the Regularized Least-Square (RLS) estimator \( f_{H, z, \lambda} \) is the solution of

\[
\min_{f \in H} \left\{ \frac{1}{\ell} \sum_{i=1}^\ell (f(x_i) - y_i)^2 + \lambda \|f\|_H^2 \right\}.
\]

**Image Space.** An image is viewed as a collection of normalized sub-images or patches. The sub-image or patch representation is standard in image processing and computer vision, and encompasses the pixel representation as a special case. Note that the framework presented here can readily apply to signals and any grid or lattice-structured data with obvious changes in indexing. We focus on the case of images as it is currently a popular application of convolutional networks.

Denote \( \mathcal{X} \) the space of images. Let \( h, w \geq 1 \) respectively the length and width of the images and \( \min(h^2, w^2) \geq d \geq 2 \) the size of each square patch. We define for each \( (i, j) \in \{1, \ldots, h - \sqrt{d} + 1\} \times \{1, \ldots, w - \sqrt{d} + 1\} \) the set of patches:

\[
\mathcal{X}_{ij} = \{ (x, y) \in \mathcal{X} : (x_i, y_j) \in \mathcal{X} \},
\]

where \( x_i = i - \sqrt{d} + 1 \) and \( y_j = j - \sqrt{d} + 1 \) are the coordinates of the patch in the image. The set of all such patches is denoted by \( \mathcal{X} \).

In addition, we have a set of filters \( \mathcal{F} \) on \( \mathcal{X} \) such that \( \mathcal{F} \cap \mathcal{X} = \emptyset \) and \( \mathcal{F} \cap \mathcal{X} = \mathcal{X} \). Let \( f : \mathcal{X} \to \mathcal{Y} \) be a function on \( \mathcal{X} \) and let \( f^\prime : \mathcal{F} \to \mathcal{Y} \) be a function on \( \mathcal{F} \). We define the convolution of \( f \) and \( f^\prime \) as the function \( (f * f^\prime)(x) \) for all \( x \in \mathcal{X} \) such that

\[
(f * f^\prime)(x) = \int_{\mathcal{F}} f(x + \xi) f^\prime(\xi) \, d\mu(\xi),
\]

where \( \mu \) is a measure on \( \mathcal{F} \).
\[ \sqrt{d + 1} \) the patch extraction operator at location \((i,j)\) as \( P_{i,j}(X) := (X_{i+t,j+k})_{t,k \in \{1, \ldots, \sqrt{d}\}} \in \mathbb{R}^{d}\) where \( X \in \mathbb{R}^{h \times w}. \) Moreover let \( 1 \leq n \leq (h - \sqrt{d} + 1)(w - \sqrt{d} + 1) \) and let \( A \subseteq \{1, \ldots, h - \sqrt{d} + 1\} \times \{1, \ldots, w - \sqrt{d} + 1\} \) such that \(|A| = n.\)

Let us now define the initial space of images as \( E_{A} := \{X \in \mathbb{R}^{h \times w}, \|P_{z}(X)\|_2 = 1 \text{ for } z \in A\} \) where each patch considered has been normalized. Since \( \{i + \ell, j + k\} : (i,j) \in A \text{ and } \ell, k \in \{1, \ldots, \sqrt{d}\} = \{1, \ldots, h - \sqrt{d} + 1\} \times \{1, \ldots, w - \sqrt{d} + 1\}, \) the mapping

\[ \phi : \mathbb{R}^{h \times w} \rightarrow \mathbb{R}^{d} \times \ldots \times \mathbb{R}^{d} \]

\[ X \rightarrow (P_{z}(X))_{z \in A} \]

is injective. Assuming the condition above, the mappings \( I := \phi(E_{A}) \) and \( E_{A} \) are isomorphic and we shall work from now on with \( I \) as the image space.

We have by construction that \( I \subseteq \prod_{i=1}^{n} S^{d-1} \) the \( n \)th Cartesian power of \( S^{d-1}, \) where \( S^{d-1} \) is the unit sphere of \( \mathbb{R}^{d}. \) Moreover, as soon as the patches considered are disjoint, we have that \( I = \prod_{i=1}^{n} S^{d-1}. \) In order to simplify the notation, we shall always consider the case where \( I = \prod_{i=1}^{n} S^{d-1} \) where \( d \) is the dimension of the square patches and \( n \) is the number of patches considered. In the following we denote for any \( q \geq 1 \) and set \( \mathcal{A}, \) the \( q \)-ary Cartesian power \( \prod_{i=1}^{q} \mathcal{A} := (\mathcal{A})^{q}. \) Moreover if \( X \in (\mathcal{A})^{q}, \) we denote \( X := (X_{i})_{i=1}^{q} \) where each \( X_{i} \in \mathcal{A}. \)

Let \( P_{m}(d) \) be the space of homogeneous polynomials of degree \( m \) in \( d \) variables with real coefficients, \( H_{m}(d) \) be the space of harmonics polynomials defined by

\[ H_{m}(d) := \{P \in P_{m}(d)|\Delta P = 0\} \]

\[ H_{m}(S^{d-1}) \) the space of real spherical harmonics of degree \( m, \) be the set of restrictions of harmonic polynomials in \( H_{m}(d) \) to \( S^{d-1}, \) \( L^{2}_{d^{d-1}}(S^{d-1}) \) be the space of (real) square-integrable functions on the sphere \( S^{d-1} \) endowed with its induced Lebesgue measure \( d\sigma_{d-1} \) and \( |S^{d-1}| \) the surface area of \( S^{d-1}. \) Moreover, \( L^{2}_{d^{d-1}}(S^{d-1}) \) endowed with its natural inner product is a separable Hilbert space. The family of spaces \( (H_{m}(S^{d-1}))_{m \geq 0}, \) yields a direct sum decomposition

\[ L^{2}_{d^{d-1}}(S^{d-1}) = \bigoplus_{m \geq 0} H_{m}(S^{d-1}) \]

which means that the summands are closed and pairwise orthogonal. Moreover, each \( H_{m}(S^{d-1}) \) has a finite dimension \( \alpha_{m,d} \) with \( \alpha_{0,d} = 1, \alpha_{1,d} = d \) and for \( m \geq 2 \)

\[ \alpha_{m,d} = \binom{d - 1 + m}{m} - \binom{d - 1 + m - 2}{m - 2} \]

Therefore for all \( m \geq 0, \) given any orthonormal basis of \( H_{m}(S^{d-1}), \) \( (Y_{m}^{1}, \ldots, Y_{m}^{\alpha_{m,d}}), \) we can build an Hilbertian basis of \( L^{2}_{d^{d-1}}(S^{d-1}) \) by concatenating these orthonormal basis. Let \( L_{2}(I) := L_{2}^{\otimes_{i=1}^{n} d\sigma_{d-1}}(I) \) be the space of (real) square-integrable functions on \( I \) endowed with the \( n \)-tensor product measure \( \otimes_{i=1}^{n} d\sigma_{d-1} := d\sigma_{d-1} \otimes \ldots \otimes d\sigma_{d-1} \) and let us define the integral operator on \( L_{2}(I) \) associated with a positive semi-definite kernel \( K \) on \( I \)

\[ T_{K} : L_{2}(I) \rightarrow L_{2}(I) \]

\[ f \rightarrow \int_{I} K(x,x) f(x) \otimes_{i=1}^{n} d\sigma_{d-1}(x). \]

As soon as \( \int_{I} K(x,x) d\sigma_{d-1} \otimes \ldots \otimes d\sigma_{d-1}(x) \) is finite, which is clearly satisfied when \( K \) is continuous, \( T_{K} \) is well defined, self-adjoint, positive semi-definite and trace-class.

We approach here the modeling of interactions through functional ANOVA models. Let us first recall basic notions in tensor product space of functional Hilbert spaces. For a Hilbert space \( E_{1} \) of functions of \( X_{1} \) and a Hilbert space \( E_{2} \) of functions of \( X_{2}, \) the tensor product space \( E_{1} \otimes E_{2} \) is defined as the completion of the class of functions of the form

\[ \sum_{i=1}^{k} f_{i}(X_{1})g_{i}(X_{2}) \]
where \( f_i \in E_1 \), \( g_i \in E_2 \) and \( k \) is any positive integer, under the norm induced by the norms in \( E_1 \) and \( E_2 \). The norm in \( E_1 \otimes E_2 \) satisfies
\[
\langle f_1(X_1)g_1(X_2), f_2(X_1)g_2(X_2) \rangle_{E_1 \otimes E_2} = \langle f_1(X_1), f_2(X_1) \rangle_{E_1} \langle g_1(X_2), g_2(X_2) \rangle_{E_2}
\]
where for \( i = 1, 2, \langle \cdot, \cdot \rangle_{E_i} \) denote the inner product in \( E_i \). A tensor product ANOVA or functional ANOVA model captures interactions between covariates as follows. Let \( D \) be the highest order of interaction in the model. A functional ANOVA model assumes that the high-dimensional function to be estimated is a sum of one-dimensional functions, two-dimensional functions, and so on. That is, the \( n \)-dimensional function \( f \) decomposes as
\[
f(x_1, \ldots, x_n) = \text{Constant} + \sum_{i=1}^{n} f_i(x_i) + \sum_{i<j} f_{i,j}(x_i, x_j) + \ldots
\]
where the sum is truncated with at most \( D \) interactions. After determining the function space of each main effect, this strategy models an interaction as lying in the tensor product space of the function spaces of the interacting main effects. In other words, if we assume \( f_1(X_1) \) to be in a Hilbert space \( E_1 \) of functions of \( X_1 \) and \( f_2(X_2) \) be in a Hilbert space \( E_2 \) of functions of \( X_2 \), then we can model \( f_{12} \) as in \( E_1 \otimes E_2 \), the tensor product space of \( E_1 \) and \( E_2 \). Higher order interactions are modeled similarly. In [16], the author considers the case where the main effects are univariate functions living in a Sobolev-Hilbert space with order \( m \geq 1 \) and domain \([0, 1]\), denoted \( H^m([0, 1]) \), defined as
\[
\left\{ f : f^{(\nu)} \text{ abs. cont.}, \nu = 0, \ldots, m - 1; f^{(m)} \in L^2 \right\}
\]
More generally, functional ANOVA models assume that the main effects are univariate functions living in a RKHS.

3 Convolutional Networks and Multi-Layer Kernels

We consider the case of a convolutional network. Let \( N \) be the number of hidden layers, \( (\sigma_i)_{i=1}^N \), \( N \) real-valued functions defined on \( \mathbb{R} \) be the activation functions at each hidden layer, \( (d_i)_{i=1}^N \) the sizes of square patches at each hidden layer, \( (p_i)_{i=1}^N \) the number of filters at each hidden layer and \( (n_i)_{i=1}^N \) the number of patches at each hidden layer. As our input space is \( \mathcal{I} = (S^{d-1})^n \), we set \( d_0 = d \), \( p_0 = 1 \), \( n_0 = n \). Moreover as the prediction layer is a linear transformation of the \( N \)th layer, we do not need to extract patches from the \( N \)th layer, and we set \( d_N = n_{N-1} \) such that the only “patch” extracted for the prediction layer is the full “image” itself. Therefore we can also set \( n_N = 1 \). Then, any function defined by a convolutional neural network is parametrized by a sequence \( W := (W^0, \ldots, W^N) \) where for \( 0 \leq k \leq N - 1 \), \( W^k \in \mathbb{R}^{p_{k+1} \times d_k p_k} \) and \( W^N \in \mathbb{R}^{d_N p_N} \) for the prediction layer. Indeed let denote for \( k \in \{0, \ldots, N - 1\} \), \( W^k := (w^k_1, \ldots, w^k_{p_{k+1}})^T \) where for all \( j \in \{1, \ldots, p_{k+1}\} \), \( w^k_j \in \mathbb{R}^{d_k p_k} \) and let us define for all \( k \in \{0, \ldots, N - 1\} \), \( j \in \{1, \ldots, p_{k+1}\} \) and \( q \in \{1, \ldots, n_{k+1}\} \) the following sequence of operators.

Convolution Operators.
\[
C^k : \mathbb{Z} \in (\mathbb{R}^{d_k p_k})^{n_k} \longrightarrow C^k_j(\mathbb{Z}) := \left( (\mathbb{Z}_i, w^k_j) \right)^{n_k}_{i=1} \in \mathbb{R}^{n_k}
\]

Non-Linear Operators.
\[
M_k : \mathbb{X} \in \mathbb{R}^{n_k} \longrightarrow M_k(\mathbb{X}) := (\sigma_k(\mathbb{X}_i))^{n_k}_{i=1} \in \mathbb{R}^{n_k}
\]

Pooling Operators. Let \( (\gamma^k_{i,j})^{n_k}_{i=1} \) the pooling factors at layer \( k \) (which are often assumed to be decreasing with respect to the distance between \( i \) and \( j \)).
\[
A_k : \mathbb{X} \in \mathbb{R}^{n_k} \longrightarrow A_k(\mathbb{X}) := \left( \sum_{j=1}^{n_k} \gamma^k_{i,j} \mathbb{X}_j \right)^{n_k}_{i=1} \in \mathbb{R}^{n_k}
\]
Patch extraction Operators.

\[ P_{q}^{k+1} : (\mathbb{R}^{p_{k+1}})^{n_{k}} \rightarrow \mathbb{R}^{p_{k+1}d_{k+1}} \]
\[ U \rightarrow P_{q}^{k+1}(U) := (U_{q+i})_{i=0}^{d_{k+1}-1} \]

Notice that as we set \( d_{N} = n_{N-1} \) and \( n_{N} = 1 \), therefore when \( k = N - 1 \), there is only one patch extraction operator which is \( \hat{P}_{N}^{N} = \text{Id} \).

Then \( \mathcal{N} \) can be obtained by the following procedure: let \( X^{0} \in \mathcal{I} \), then we can denote \( X^{0} = (X_{i}^{k})_{i=1}^{n_{1}} \) where for all \( i \in [1, n_{1}] \), \( X_{i}^{0} \in S^{d-1} \). Therefore we can build by induction the sequence \( (X_{i}^{k})_{k=0}^{N} \) by doing the following operations starting from \( k = 0 \) until \( k = N - 1 \):

\[ C_{j}^{k}(X) = ((X_{i}^{k}, w_{j}^{k}))_{i=1}^{n_{k}} \]
\[ M_{k}(C_{j}^{k}(X)) = (\sigma_{k}((X_{i}^{k}, w_{j}^{k})))_{i=1}^{n_{k}} \]
\[ A_{k}(M_{k}(C_{j}^{k}(X))) = \left( \sum_{i=1}^{n_{k}} \gamma_{i,q} \sigma_{k}((X_{i}^{k}, w_{j}^{k})) \right)_{i=1}^{n_{k}} \]
\[ Z^{k+1}(i,j) = A_{k}(M_{k}(C_{j}^{k}(X)))_{i} \]
\[ X^{k+1} = (Z_{k+1}(i,1),...,Z_{k+1}(i,p_{k+1}))_{i=1}^{n_{k}} \]
\[ P_{q}^{k+1}(X) = (P_{q}^{k+1}(X))_{q=1}^{n_{k+1}} \]

Finally the function defined by a convolutional network is \( \mathcal{N}_{W}(X^{0}) := (X^{N}, W^{N})_{\mathbb{R}^{p_{N}n_{N-1}}} \). In the following, we denote \( \mathcal{F}_{(\sigma_{i})_{i=1}^{N},(p_{i})_{i=1}^{N}} \) the function space of all the functions \( \mathcal{N}_{W} \) defined as above on \( \mathcal{I} \) for any choice of \( (W_{k})_{k=0}^{N} \) such that for \( 0 \leq k \leq N - 1 \), \( W_{k} \in \mathbb{R}^{p_{k+1}d_{k}p_{k}} \) and \( W^{N} \in \mathbb{R}^{d_{N} \times p_{N}} \). We omit the dependence of \( \mathcal{F}_{(\sigma_{i})_{i=1}^{N},(p_{i})_{i=1}^{N}} \) with respect to \( (d_{i})_{i=1}^{N} \) and \( (n_{i})_{i=1}^{N} \) to simplify the notations. We shall also consider the union space

\[ \bigcup_{(p_{1},...,p_{N}) \in \mathbb{N}_{N}} \mathcal{F}_{(\sigma_{i})_{i=1}^{N},(p_{i})_{i=1}^{N}} \]

Example. Let us consider the case where at each layer the number of filters is 1 which corresponds to the case where for all \( k \in [1, N] \), \( p_{k} = 1 \). Therefore we can omit the dependence in \( j \) of the convolution operators defined above. At each layer \( k \), \( \hat{X}^{k+1} \in \mathbb{R}^{n_{k}} \) is the new image obtained after a convolution, a nonlinear and a pooling operation with \( n_{k} \) pixels which is the number of patches that has been extracted from the image \( \hat{X}^{k} \) at layer \( k - 1 \). Moreover \( X^{k+1} \) is the decomposition of the image \( \hat{X}^{k+1} \) in \( n_{k+1} \) patches obtained thanks to the patch extraction operators \( (P_{q}^{k+1})_{q=1}^{n_{k+1}} \). Finally after \( N \) layers, we obtain that \( X^{N} = X^{N} \in \mathbb{R}^{n_{N-1}} \) which is the final image with \( n_{N} \) pixels obtained after repeating \( N \) times the above operations. Then the prediction layer is just a linear combination of the coordinates of the final image \( X^{N} \) from which we can finally define for all \( X^{0} \in \mathcal{I} \), \( \mathcal{N}_{W}(X^{0}) := (X^{N}, W^{N})_{\mathbb{R}^{n_{N-1}}} \).

We show in the following Proposition that there exists a RKHS which contains the space of functions \( \mathcal{F}_{(\sigma_{i})_{i=1}^{N}} \) for any activation functions, \( (\sigma_{i})_{i=1}^{N} \), which admits a Taylor decomposition on \( \mathbb{R} \). See proof of Proposition [1] in Appendix A.1. Moreover we show that for well chosen nonlinear functions, the kernel is a c-universal kernel on \( \mathcal{X} \).

**Definition 3.1.** (c-universal [22]) A continuous positive semi-definite kernel \( k \) on a compact Hausdorff space \( \mathcal{X} \) is called c-universal if the RKHS, \( H \) induced by \( k \) is dense in \( C(\mathcal{X}) \) w.r.t. the uniform norm.

**Proposition 1.** Let \( N \geq 2 \) and \( (\sigma_{i})_{i=1}^{N} \) be a sequence of \( N \) functions which admits a Taylor decomposition on \( \mathbb{R} \). Moreover let \( (f_{i})_{i=1}^{N} \) be the sequence of functions such that for every \( i \in \{1,...,N\} \)

\[ f_{i}(x) = \sum_{t \geq 0} \frac{\sigma_{i}^{(t)}(0)}{t!} x^{t} \]
Then the following application defined on $\mathcal{I} \times \mathcal{I}$

$$K_N(X, X') := f_N \circ ... \circ f_2 \left( \sum_{i=1}^{n} f_1 (\langle X(i), X'(i) \rangle_{\mathbb{R}^d}) \right)$$

is a positive definite kernel on $\mathcal{I}$, and the RKHS associated $H_N$ contains $\mathcal{F}_{\{\sigma_i\}_{i=1}^{N+1}}$, the function space generated by convolutional neural networks. Moreover as soon as $\sigma_i^{(t)}(0) \neq 0$ for all $i \geq 1$ and $t \geq 0$, then $K_N$ is a c-universal kernel on $\mathcal{I}$.

**Arbitrary Width Network.** It is worthwhile to emphasize that the definition of the RKHS $H_N$ does not depend on the number of filters $(p_j)_{j=2}^{N+1}$ considered at each hidden layer.

For suitably chosen activation functions, the kernel $K_N$ defined above is actually universal. Therefore the RKHS $H_N$ associated approximate the Bayes risk of a large class of loss, in particular the one of interest here the least-squares loss

$$\inf_{f \in H} E[(f(X) - Y)^2] = R^*$$

where $R^*$ is the Bayes risk. See Corollary 5.29 [27]. For instance, if at each layer the nonlinear function is $\sigma_{\exp}(x) = \exp x$, then the RKHS becomes universal. There are other examples of activation functions satisfying the assumptions of the Proposition 4.1 such as the square activation $\sigma_2(x) = x^2$, the smooth hinge activation $\sigma_{sh}$, close to the rectifier linear unit (ReLU) activation, or a sigmoid-like function such as $\sigma_{erf}$, similar to the sigmoid function.

$$\sigma_{erf}(x) = \frac{1}{2} \left( 1 + \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} dt \right)$$

$$\sigma_{sh}(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{x} xe^{-t^2} dt + \frac{\exp(-\sqrt{\pi}x^2)}{2\pi}$$

In the following section, we study in more detail the properties of the kernel $K_N$. In particular we show an explicit Mercer decomposition of the kernel from which we deduce the close relation existing between convolutional networks and functional ANOVA models.

### 4 Spectral Analysis of Convolutional Networks

In this section we exhibit a Mercer decomposition of the kernel introduced in Proposition 4.1. Moreover we show that the multivariate function space generated by a convolutional networks is closely related to functional TPS-ANOVA models where the highest order of interaction is controlled by the nonlinear functions $(\sigma_i)_{i=1}^{N}$ involved in the construction of the network. See Appendix B for the proof.

**Theorem 4.1.** Let $N \geq 2$, $f_1$ a real value function that admits a Taylor decomposition around 0 on $[-1, 1]$ with non-negative coefficients and $(f_i)_{i=2}^{N}$ a sequence of real value functions such that $f_2 \circ ... \circ f_2$ admits a Taylor decomposition around 0 on $\mathbb{R}$ with non-negative coefficients $(a_q)_{q \geq 0}$. Let us denote for all $k_1, ..., k_n \geq 0$, $(l_{k_1}, ..., l_{k_n}) \in \{1, ..., \alpha_{k_1,d}\} \times ... \times \{1, ..., \alpha_{k_n,d}\}$ and $X \in \mathcal{I}$,

$$e_{(k_1, ..., k_n)}(X) := \prod_{i=1}^{n} Y_{l_{ki}}(X_i)$$

Then each $e_{(k_1, ..., k_n)}(X)$ is an eigenfunction of $T_{K_N}$ the integral operator associated to the kernel $K_N$, with associated eigenvalue given by the formula:

$$\mu_{(k_1, ..., k_n)} := \sum_{q \geq 0} a_q \sum_{\alpha_1, ..., \alpha_n \geq 0} \binom{q}{\alpha_1, ..., \alpha_n} \prod_{i=1}^{n} \lambda_{k_i, \alpha_i}$$
where for each $i \in \{1, \ldots, n\}$, $k_i \geq 0$ and $\alpha_i \geq 0$ we have
\[
\lambda_{k_i, \alpha_i} = \frac{|S|^{d-2} \Gamma((d-1)/2)}{2^{k_i+1}} \sum_{s \geq 0} \left[ \frac{d^{2s+k_i}}{d!^{2s+k_i}} \right] \frac{f_0^n(t)}{(2s+k_i)!} \frac{\Gamma(s+1/2)}{\Gamma(s+k_i+d/2)}
\]
Moreover we have
\[
K_N(X, X') = \sum_{k_1, \ldots, k_n \geq 0} \mu_{(k_1, l_{k_1})}^{n_1} \ldots \mu_{(k_n, l_{k_n})}^{n_n} (X) \delta_{(k_1, l_{k_1})}^{n_1} (X')
\]
where the convergence is absolute and uniform.

Therefore the sequence of positive eigenvalues of $T_{K_N}$ with their multiplicities is exactly the subsequence of positive eigenvalues in \( \left( \mu_{(k_i, l_{k_i})}^{n_i} \right) \).

From this Mercer decomposition we deduce a special decomposition of the multivariate functions generated by a convolutional networks closely related to TPS-ANOVA models. Indeed let us denote
\[
L_{2,0}^{d\sigma_{d-1}}(S^{d-1}) \subset \{1\} \bigoplus L_{2,0}^{d\sigma_{d-1}}(S^{d-1})
\]
where $L_{2,0}^{d\sigma_{d-1}}(S^{d-1})$ is the subspace orthogonal to \( \{1\} \). Thus we have
\[
\bigotimes_{i=1}^n L_{2,0}^{d\sigma_{d-1}}(S^{d-1}) = \bigotimes_{i=1}^n \left[ \{1\} \bigoplus L_{2,0}^{d\sigma_{d-1}}(S^{d-1}) \right].
\]
Identify the tensor product of \( \{1\} \) with any Hilbert space with that Hilbert space itself, then $\bigotimes_{i=1}^n L_{2,0}^{d\sigma_{d-1}}(S^{d-1})$ is the direct sum of all the subspaces of the form \( L_{2,0}^{d\sigma_{d-1}}(X_{j_1}) \otimes \ldots \otimes L_{2,0}^{d\sigma_{d-1}}(X_{j_k}) \) and \( \{1\} \) where \( \{j_1, \ldots, j_k\} \) is a subset of \( \{1, \ldots, n\} \) and the subspaces in the decomposition are all orthogonal to each other. From this decomposition we want to be able to characterise functions following the exact same strategy as TPS-ANOVA models but where we allow the main effects to live in an Hilbert space which is not necessary a RKHS of univariate functions. Therefore we introduce the following definition:

**Definition 4.1.** ANOVA-like Decomposition Let $f$ a real valued function defined on $I$. We say that $f$ admits an ANOVA-like decomposition of order $r$ if $f$ can be written as
\[
f(X_1, \ldots, X_n) = C + \sum_{k=1}^r \sum_{A \subset \{1, \ldots, n\} \setminus |A|=k} f_A(X_A)
\]
where $C$ is a constant, for all $k \in \{1, \ldots, r\}$ and $A = \{j_1, \ldots, j_k\} \subset \{1, \ldots, n\}$ $X_A = (X_{j_1}, \ldots, X_{j_k})$, $f_A \in L_{2,0}^{d\sigma_{d-1}}(X_{j_1}) \otimes \ldots \otimes L_{2,0}^{d\sigma_{d-1}}(X_{j_k})$ and the decomposition is unique.

Indeed here the main effects live in $L_{2,0}^{d\sigma_{d-1}}(S^{d-1})$ which is a Hilbert space of multivariate functions.

As $(Y^r_{m_{T'}})_{m_{T'}}$ is an Hilbert basis of $L_{2,0}^{d\sigma_{d-1}}(S^{d-1})$, we obtain that $(\epsilon_{(k_i, l_{k_i})})_{i=1}^n$ is an orthonormal basis of $\bigotimes_{i=1}^n L_{2,0}^{d\sigma_{d-1}}(S^{d-1})$ (see Proposition 7.14 [11]). Moreover from the Mercer decomposition, we have also that the subsequence of $(\epsilon_{(k_i, l_{k_i})})_{i=1}^n$ associated with the subsequence of positive eigenvalues $\left( \mu_{(k_i, l_{k_i})}^{n_i} \right)$ is an orthogonal basis of the RKHS $H_N$ associated to the kernel $K_N$. Therefore any multivariate functions generated by a convolutional networks admits such a decomposition and the depth of the network give an explicit control of the highest order of interactions allowed by the model:
Corollary 4.1. Let $N \geq 2$, $f_1$ a real value function that admits a Taylor decomposition around 0 on $[-1, 1]$ with non-negative coefficients and $f_N \circ \cdots \circ f_2$ a polynomial of degree $D \geq 1$. Then by denoting $d^* := \min(D, n)$, for any $f \in \mathcal{F}_{(\sigma_i)}_{i=1}^N$, $q > d^*$ and $\{j_1, \ldots, j_q\} \subset \{1, \ldots, n\}$, we have

$$f \in \left(L^2_{\sigma_1} (X_{j_1}) \otimes \cdots \otimes L^2_{\sigma_n} (X_{j_q}) \right)^{-1}$$

Proof. Let $1 \leq D < n$, $f \in \mathcal{F}_{(\sigma_i)}_{i=1}^N$, $q > d^*$ and $\{j_1, \ldots, j_q\} \subset \{1, \ldots, n\}$. Without loss of generality we can only consider the case $\{j_1, \ldots, j_q\} = \{1, \ldots, q\}$. As $(Y_m^I)_{m \geq 1, I}$ is an Hilbertian basis of $L^2_{\sigma_i} (\mathcal{S}^{d-1})$, we have that

$$L^2_{\sigma_i} (X_1) \otimes \cdots \otimes L^2_{\sigma_i} (X_q) = \bigoplus_{k_1, \ldots, k_q \geq 1, 1 \leq l_i \leq \alpha_{k_i, d}} \text{Vect} \left( \epsilon(k_i, l_i)_{i=1}^n \right)$$

Therefore to show the result, thanks to Theorem 4.1, we just need to show that $\mu(k_i, l_i)_{i=1}^n = 0$ as soon as $k_1, \ldots, k_q \geq 1, k_{q+1}, \ldots, k_n = 0$ and $1 \leq l_i \leq \alpha_{k_i, d}$. Indeed we have

$$\mu(k_i, l_i)_{i=1}^n := \sum_{j=0}^D a_j \sum_{\alpha_1, \ldots, \alpha_n \geq 0} \left( \alpha_j \prod_{i=1}^n \lambda_{k_i, \alpha_i} \right)$$

Let $j \in \{0, D\}$ and $\alpha_1, \ldots, \alpha_n \geq 0$ such that $\sum_{i=1}^n \alpha_i = j$. As $j \leq D$ we have that:

$$\{|i: \alpha_i = 0, i = 1, \ldots, n\} \geq n - D$$

But as $q \geq D + 1$, there exists $\ell \in \{0, q\}$ such that $\alpha_\ell = 0$. But as $k_\ell \geq 1$, it is easy to check that $\lambda_{k_\ell, 0} = 0$, therefore all the terms in the sum are null, and the result follows.

5 Regularized Least-Squares for CNNs

Given a dataset $z = (x_i, y_i)_{i=1}^m$ independently sampled from an unknown distribution $\rho(x, y) = \rho_I(x) \rho(y|x)$ on $Z := \mathcal{I} \times \mathcal{Y}$ where $\rho_I$ is the marginal distribution on $\mathcal{I}$ and $\rho(.|x)$ is the conditional distribution of $y$ given $x \in \mathcal{I}$, the goal least-squares regression is to estimate the conditional mean function $f_\rho : \mathcal{I} \rightarrow \mathbb{R}$ given by $f_\rho(x) := E(Y|X = x)$. Before stating the statistical bound for a convolutional network, we recall some basic definitions in order to clarify what we mean by asymptotic upper rate, lower rate and minimax rate optimality. We want to track the precise behavior of these rates and the effects of adding layers in a convolutional neural network. More precisely, we consider a class of Borel probability distributions $\mathcal{P}$ on $\mathcal{I} \times \mathbb{R}$ satisfying basic general assumptions. We consider rates of convergence according to the $L^2_{\rho} \otimes \mathcal{Z}$ norm denoted $||.||_\rho$. 8
Definition 5.1. (Upper Rate of Convergence) A sequence \((a_\ell)_{\ell \geq 1}\) of positive numbers is called upper rate of convergence in \(L_2^{d\rho} \) norm over the model \(\mathcal{P}\), for the sequence of estimated solutions \((f_{x,\lambda_\ell})_{\ell \geq 1}\) using regularization parameters \((\lambda_\ell)_{\ell \geq 0}\) if
\[
\lim_{\tau \to +\infty} \limsup_{\ell \to \infty} \sup_{\rho \in \mathcal{P}} \rho^{\ell} (z : \|f_{x,\lambda_\ell} - f_\rho\|_\rho^2 > \tau a_\ell) = 0
\]

Definition 5.2. (Minimax Lower Rate of Convergence) A sequence \((w_\ell)_{\ell \geq 1}\) of positive numbers is called minimax lower rate of convergence in \(L_2^{d\rho} \) norm over the model \(\mathcal{P}\) if
\[
\lim_{\tau \to 0^+} \liminf_{\ell \to \infty} \inf_{f_\rho \in \mathcal{P}} \rho^{\ell} (z : \|f_x - f_\rho\|_\rho^2 > \tau w_\ell) = 1
\]

where the infimum is taken over all measurable learning methods with respect to \(\mathcal{P}\).

In the following we call such sequences \((w_\ell)_{\ell \geq 1}\) (minimax) lower rates. Obviously, every sequence \((\hat{w}_\ell)_{\ell \geq 1}\) which decreases at least with the same speed as \((w_\ell)_{\ell \geq 1}\) is also a lower rate for this set of probability measures and on every larger set of probability measures at least the same lower rate holds. The meaning of a lower rate \((w_\ell)_{\ell \geq 1}\) is, that no measurable learning method can fulfill a \(L_2^{d\rho} \) (\(\mathcal{I}\))-learning rate (\(a_\ell)_{\ell \geq 1}\) in the sense of Definition 5.1 that decreases faster than \((w_\ell)_{\ell \geq 1}\). In the case where the learning rate of the sequence of estimated solutions coincides with the minimax lower rates, we say that it is optimal in the minimax sense.

**Setting.** Here the hypothesis space considered is the RKHS \(H_N\) associated to the Kernel \(K_N\) introduced in Proposition 5 where, \(N \geq 2\), \(f_1\) a function which admits a Taylor decomposition on \([-1,1]\) with non-negative coefficients \((b_m)_{m \geq 0}\) and \((f_i)_{i=2}^N\) a sequence of real valued functions such that \(g := f_N \circ \ldots \circ f_2\) admits a Taylor decomposition on \(\mathbb{R}\) with non-negative coefficients. In the following, we denote by \(T_{\rho N}\) the integral operator on \(L_2^{d\rho} (\mathcal{I})\) associated with \(K_N\) defined as
\[
T_{\rho N} : L_2^{d\rho} (\mathcal{I}) \to L_2^{d\rho} (\mathcal{I})
\]

Let us now introduce the general assumptions on the class of probability measures considered. Let us denote \(d\nu := \otimes_{i=1}^n d\nu_{x_i}d\nu_{x_{i-1}}\) and for \(\omega \geq 1\), we denote by \(\mathcal{W}_\omega\) the set of all probability measures \(\nu\) on \(\mathcal{I}\) satisfying \(\frac{d\nu}{d\rho} < \omega\). Furthermore, we introduce for a constant \(\omega \geq 1 > h > 0\), \(\mathcal{W}_{\omega, h} \subset \mathcal{W}_\omega\) the set of probability measures \(\mu\) on \(\mathcal{I}\) which additionally satisfy \(\frac{d\mu}{d\rho} > h\).

**Assumptions (probability measures on \(\mathcal{I} \times \mathcal{Y}\)).** Let \(B, B_\infty, L, \sigma > 0\) be some constants and \(0 < \beta \leq 2\) a parameter. Then we denote by \(\mathcal{F}_{B, B_\infty, L, \sigma, \beta} (\mathcal{P})\) the set of all probability measures \(\rho\) on \(\mathcal{I} \times \mathcal{Y}\) with the following properties

- \(\rho \in \mathcal{P}\), \(\int_{\mathcal{I} \times \mathcal{Y}} y^2 \rho (x, y) < \infty\) and \(\|f_\rho\|_{L_2^{d\rho}} \leq B_\infty\)
- There exist \(g \in L_2^{d\rho}(\mathcal{I})\) such that \(f_\rho = T_{\rho N}^{\beta/2} g\) and \(\|g\|_\rho^2 \leq B\)
- There exist \(\sigma > 0\) and \(L > 0\) such that \(\int_{\mathcal{I}} \|y - f_\rho(x)\|^m \rho (y|x) \leq \frac{1}{2} \sigma! L^{m-2}\)

A sufficient condition for the last assumption is that \(\rho\) is concentrated on \(\mathcal{I} \times [-M, M]\) for some constant \(M > 0\). In the following we denote \(\mathcal{G}_{\omega, \beta} := \mathcal{F}_{B, B_\infty, L, \sigma, \beta} (\mathcal{W}_\omega)\) and \(\mathcal{G}_{\omega, h, \beta} := \mathcal{F}_{B, B_\infty, L, \sigma, \beta} (\mathcal{W}_{\omega, h})\).

Our main result is given in the following theorem. See Appendix C.2 for the proof.

**Theorem 5.1.** Let us assume there exists \(1 > r > 0\) and \(c_1 > 0\) a constant such that \((b_m)_{m \geq 0}\) satisfies for all \(m \geq 0\) we have \(b_m \leq c_1 r^{m}\). Moreover let us assume that \(f_N \circ \ldots \circ f_2\) is a polynomial of degree \(D \geq 1\) and let us denote \(d^* := \min(D, n)\). Let also \(w_\geq 1\) and \(0 < \beta \leq 2\). Then there exists \(A, C > 0\) some constants independent of \(\beta\) (see Appendix C for their definitions) such that for any \(\rho \in \mathcal{G}_{\omega, \beta}\) and \(\tau \geq 1\) we have:
exist a constant \(0\) we have that for any \(w\)

Under the exact same assumptions of Theorem 5.1, and if we assume additionally that there
Theorem 5.2.

Appendix 6 for the proof.

of the patches \(d\) the rate is very close to the optimal rate for
coincide with the minimax lower rates and therefore are optimal in the minimax sense. Notice this optimal

In order to investigate the optimality of the convergence rates, let us take a look at the lower rates. See

if one of the following conditions hold

• \(\beta > 1\), then for \(\lambda_\ell = \frac{1}{\ell^{1/\beta}}\) and
  \(\ell \geq \max \left( e^\beta, \left( \frac{A}{2^{\alpha}} \right)^\frac{1}{(d-1)d^*} \log(\ell) \left( \frac{(d-1)d^*}{\beta^2} \right) \right)\), with a \(\rho^\ell\)-probability \(\geq 1 - e^{-4\tau}\) it holds

\[
\|f_{\theta_N, \lambda_\ell} - f_\rho\|_\rho^2 \leq 3C \tau^2 \frac{\log(\ell)^{(d-1)d^*}}{\ell}
\]

• \(\beta = 1\), then for \(\lambda_\ell = \frac{\log(\ell)\mu}{\ell}\), \(\mu > (d-1)d^* > 0\) and \(\ell \geq \max \left( \exp \left( (\sqrt{z}) \frac{1}{\max(1, d, d^*)} \right) \right), e^1 \log(\ell)^\mu\), with a \(\rho^\ell\)-probability \(\geq 1 - e^{-4\tau}\) it holds

\[
\|f_{\theta_N, \lambda_\ell} - f_\rho\|_\rho^2 \leq 3C \tau^2 \frac{\log(\ell)^\mu}{\ell^\beta}
\]

• \(\beta < 1\), then for \(\lambda_\ell = \frac{\log(\ell)\mu}{\ell}\) and
  \(\ell \geq \max \left( \exp \left( (\sqrt{z}) \frac{1}{\max(1, d, d^*)} \right) \right), e^1 \log(\ell)^\mu\), with a \(\rho^\ell\)-probability \(\geq 1 - e^{-4\tau}\) it holds

\[
\|f_{\theta_N, \lambda_\ell} - f_\rho\|_\rho^2 \leq 3C \tau^2 \frac{\log(\ell)^{(d-1)d^*}}{\ell^\beta}
\]

In fact from the above theorem, we can deduce asymptotic upper rate of convergence. Indeed we have

\[
\lim_{\tau \to +\infty} \limsup_{\ell \to \infty} \sup_{\rho \in \mathcal{G}_{\omega, \beta}} \rho^\ell \left( \mathcal{Z} : \|f_{\theta, \lambda_\ell} - f_\rho\|_\rho^2 > \tau a_\ell \right) = 0
\]

if one of the following conditions hold

• \(\beta > 1\), \(\lambda_\ell = \frac{1}{\ell^{1/\beta\tau}}\) and \(\ell \geq \max \left( e^\beta, \left( \frac{A}{2^{\alpha}} \right)^\frac{1}{(d-1)d^*} \log(\ell) \left( \frac{(d-1)d^*}{\beta^2} \right) \right)\), with a \(\rho^\ell\)-probability \(\geq 1 - e^{-4\tau}\) it holds

\[
\|f_{\theta_N, \lambda_\ell} - f_\rho\|_\rho^2 \leq 3C \tau^2 \frac{\log(\ell)^{(d-1)d^*}}{\ell}
\]

Theorem 5.2. Under the exact same assumptions of Theorem 5.1 and if we assume additionally that there
exist a constant \(0 < c_2 < c_1\) such that for all \(m \geq 0\):

\[
c_2^m \leq b_m
\]

we have that for any \(0 < \beta \leq 2\) and \(\omega \geq 1 > h > 0\) such that \(W_{\omega, h}\) is not empty

\[
\lim_{\tau \to 0^+} \liminf_{\ell \to \infty} \inf_{f_\rho} \sup_{\rho \in \mathcal{G}_{\omega, h, \beta}} \rho^\ell \left( \mathcal{Z} : \|f_{\theta, \lambda_\ell} - f_\rho\|_\rho^2 > \tau w_\ell \right) = 1
\]

where \(w_\ell = \frac{\log(\ell)^{(d-1)d^*}}{\ell}\). The infimum is taken over all measurable learning methods with respect to \(\mathcal{G}_{\omega, h, \beta}\).

Therefore when \(\beta > 1\) the learning rates of the regularized least-squares estimator stated in Theorem 5.1
coincide with the minimax lower rates and therefore are optimal in the minimax sense. Notice this optimal
rate is very close to the optimal rate for \(d\)-dimensional model in regression setting, where \(d\) is the dimension
of the patches

\[
\|f_{\theta, \lambda_\ell} - f_\rho\|_\rho^2 \leq 3C \tau^2 \frac{\log(\ell)^{d-1}}{\ell}.
\]
obtains similar results for TPS-ANOVA model in the case where the main effects live in $H^m([0,1])$. Indeed, by denoting $d^*$ the highest order of interaction in the model, the regularized least-squares algorithm gives an optimal rate of convergence which is within a log factor of the one-dimensional optimal rate

$$\|f_{x,\lambda_e} - f_{\rho}\|_p \leq 3C\tau^2 \left( \frac{\log(\ell) d^*-1}{\ell} \right)^{\frac{2m}{d^*+1}}.$$  

Proof. (Sketch) To show these results, we need to control the rate of decay of the eigenvalues ranked in the non-decreasing order with their multiplicities associated with the integral operator $T_{KN}$. This control gives us a notion of the complexity of the model from which upper and lower rates are obtained. To do so we first control for $\alpha \geq 1$ and $m \geq 0 \lambda_{m,\alpha}$ introduced in theorem 4.1 and obtain for some constants $C_{1,\alpha}, C_{2,\alpha} > 0$ that (see Proposition C.1 in Appendix C.2)

$$C_{2,\alpha}(r/4)^m \leq \lambda_{m,\alpha} \leq C_{1,\alpha}(m+1)^{\alpha-1} r^m.$$  

From this control and the Mercer decomposition obtained in Proposition 4.1, we derive the rate of decay of the spectrum associated with $T_{KN}$ and obtain for some constants $C_3, C_4 > 0$ and $0 < \gamma < \eta$ that (see Appendix C.2 for the proof)

$$C_4 e^{-\gamma m^{\frac{1}{\alpha-\frac{1}{d^*}}}} \leq \eta_m \leq C_3 e^{-\gamma m^{\frac{1}{\alpha-\frac{1}{d^*}}}}.$$

The rates in Theorem 5.1 highlight two important aspects of the behavior of CNNs. First, the highest order of interactions, given by the network depth, controls the statistical performance of such models. If the order is small, we obtain optimal rates which are close to the optimal rate for estimating multivariate functions in $d$ dimensions where $d$ is the patch size. Moreover, adding layers makes the eigenvalue decay decrease slower and even as soon as $(\sigma_i)_i=2$ is an arbitrary polynomial functions with degrees higher than $n$, then the optimal rates will be exactly the same as the one obtain for a polynomial function of degree $n$. There is thus a regime in which adding layers does not affect the convergence rate of convergence, and allows the function space of target functions to grow. Indeed the eigenvalue decay gives a concrete notion of the complexity of the function space considered. Given an eigensystem $(\mu_m)_{m \geq 0}$ and $(\epsilon_m)_{m \geq 0}$ of positive eigenvalues and eigenfunctions respectively of the integral operator $T_{\rho}$, associated with the Kernel $K_N$, defined on $L_2(I)$, the RKHS $H_N$ associated is defined as:

$$H_N = \left\{ f \in L_2(I): \ f = \sum_{m \geq 0} a_m \epsilon_m \ \text{with} \ \left( \frac{a_m}{\sqrt{\mu_m}} \right) \in \ell_2 \right\}$$

endowed with the following inner product:

$$\langle f, g \rangle = \sum_{m \geq 0} \frac{a_m b_m}{\mu_m}$$

From this definition, we see immediately that as the eigenvalues of the integral operator decreases slower, the RKHS becomes larger. Therefore composing layers allows the function space generated by the network to grow.

Illustration. We investigate the relation between network depth and classification accuracy on dataset CIFAR-10 [14]. We consider CNNs with depth varying from 1 to 8 layers. We replicate the infinitely wide networks by setting the number of filters to be large and equal across hidden layers. We consider cases where the number of filters is either 128, 256 or 512 at each layer. At each layer $k$ the size of square patches is $d_k = 3 \times 3$. At the first layer, we consider overlapping patches where each patch has been normalized to be in a subset of the $n$-ary Cartesian power of $S_d-1$, where $n$ is the total number of patches considered. Figure 1 shows results after 20 epochs. In each case, as the number of layers increases, the classification accuracy increases. The networks therefore get better and better at approximating the target function as the number of layers increases.
6 Related works

In [3], the author considers a single-hidden layer neural network with affine transforms and homogeneous functions acting on vectorial data. In this particular case, the author provides a detailed theoretical analysis of generalization performance. See e.g. [4, 1, 19] for classical and [28, 29] for recent related approaches.

Recent works [5, 20, 6] studied various kinds of bounds for multi-layer perceptrons, and in the particular case of [4], convolutional kernel networks. These analyses result in bounds scaling in the product of spectral norms of weights of layers. Putting these bounds in the context of our analysis, these bounds do not involve the full eigenspectrum of the integral operator of each layer.

We considered here a multiple-layer convolutional neural network acting on image data. Note that a similar analysis holds for signal data, with sub-signals/windows in place of sub-images/patches, and any lattice-structure data (including e.g. voxel data) in general.

Conclusion. We have presented a function space in which one can embed a convolutional network. The function space is defined through a multi-layer kernel. The construction uncovers an interesting sum-product structure which shed light on the types of functions learned by convolutional networks.

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In Section \[\text{A}\] we build the reproducing kernel Hilbert space (RKHS) of a convolutional network and establish its universality properties. In Section \[\text{B}\] we exhibit a Mercer decomposition of the kernel and highlight the relationship between convolutional neural networks and tensor product space ANOVA models. In Section \[\text{C}\] we prove statistical bounds. Finally in Sections \[\text{D-E}\] we collect useful technical results and basic notions.

We first recall basic definitions and notions used throughout the proofs. Consider a class of Borel probability distributions \(\mathcal{P}\) on \(\mathcal{I} \times \mathbb{R}\). We shall state rates of convergence in \(L_2^{d\sigma}\) \[\cite{27}\].

**Definition 1.** (Upper Rate of Convergence) A sequence \((a_t)_{t \geq 1}\) of positive numbers is called upper rate of convergence in \(L_2^{d\sigma}\) norm over the model \(\mathcal{P}\), for the sequence of estimated solutions \((f_{x,\lambda_t})_{t \geq 1}\) using regularization parameters \((\lambda_t)_{t \geq 0}\) if

\[
\lim_{\tau \to +\infty} \limsup_{t \to \infty} \sup_{\rho \in \mathcal{P}} \rho^t (z : \|f_{x,\lambda_t} - f_{\rho}\|_p^2 > \tau a_t) = 0
\]

**Definition 2.** (Minimax Lower Rate of Convergence) A sequence \((w_t)_{t \geq 1}\) of positive numbers is called minimax lower rate of convergence in \(L_2^{d\sigma}\) norm over the model \(\mathcal{P}\) if

\[
\lim_{\tau \to 0^+} \liminf_{t \to \infty} \inf_{f_s \rho \in \mathcal{P}} \rho^t (z : \|f_s - f_{\rho}\|_p^2 > \tau w_t) = 1
\]

where the infimum is taken over all measurable learning methods with respect to \(\mathcal{P}\).

In order to obtain such rates, we ought to control the model complexity. This boils down in our framework to the control of the eigenvalue decay of the integral operator

\[
T_{K_N} : L_2(\mathcal{I}) \rightarrow L_2(\mathcal{I})
\]

\[
f \rightarrow \int_{\mathcal{I}} K_N(x,.) f(x) \otimes_{m=1}^n d\sigma_{d-1}(x).
\]

As \(K_N\) is bounded, \(T_{K_N}\) is self-adjoint, positive semi-definite and trace-class; see \[\cite{8} \cite{27}\]. The spectral theorem for compact operators implies that, for an for most countable index set \(I\), a positive, decreasing sequence \((\mu_i)_{i \in I} \in \ell_1(I)\) and a family \((e_i)_{i \in I} \subset H_N\), such that \((\mu_i^{1/2} e_i)_{i \in I}\) is an orthonormal system in \(H_N\) and \((\varepsilon_i)_{i \in I}\) is an orthonormal system in \(L_2(\mathcal{I})\) with

\[
T_{K_N} = \sum_{i \in I} \mu_i \langle ., e_i \rangle L_2(\mathcal{I}) e_i.
\]

In fact, we have an explicit formulation of the eigensystem associated with \(T_{K_N}\) which leads us to an explicit Mercer decomposition of the kernel of interest \(K_N\). Moreover, in our case, the Mercer decomposition is in fact related to Tensor-Product ANOVA decompositions in additive modeling and nonparametric learning \[\cite{16}\]. Indeed any function generated by a convolutional network admits what we call an ANOVA-like Decomposition.

**Definition 3. ANOVA-like Decomposition** Let \(f\) a real valued function defined on \(\mathcal{I}\). We say that \(f\) admits an ANOVA-like Decomposition of order \(r\) if \(f\) can be written as

\[
f(X_1, ..., X_n) = C + \sum_{k=1}^{r} \sum_{A \subset \{1, ..., n\}} \sum_{|A| = k} f_A(x_A)
\]

where \(C\) is a constant, for all \(k \in \{1, ..., r\}\) and \(A = \{j_1, ..., j_k\} \subset \{1, ..., n\}\) \(x_A = (x_{j_1}, ..., x_{j_k})\), \(f_A \in L_2^{d\sigma_{d-1}}(X_{j_1}) \otimes ... \otimes L_2^{d\sigma_{d-1}}(X_{j_k})\) and the decomposition is unique.

In the following , for any \(q \geq 1\) and set \(\mathcal{X}\) if \(X \in \mathcal{X}^q\), we denote \(X := (X(i))^q_{i=1}\) where each \(X(i) \in \mathcal{X}\).
A Convolutional Networks and Multi-Layer Kernels

Let us first recall the various operators involved in a convolutional neural network. Let $N$ be the number of hidden layers, $(\sigma_i)_{i=1}^N$, $N$ real-valued functions defined on $\mathbb{R}$ be the activation functions at each layer, $(d_i)_{i=1}^N$ the sizes of square patches at each layer, $(p_i)_{i=1}^N$ the number of channels at each layer and $(n_i)_{i=1}^N$ the number of patches at each layer with $d_1 = d$, $p_1 = 1$, $n_1 = n$. Let also define $p_{N+1} \geq 1$, $n_{N+1} = 1$ and $d_{N+1} = n_N$ respectively the number of channels, the number of patches and the size of the patch for the prediction layer. Then, any function defined by a convolutional neural network is parameterized by a sequence $\mathcal{W} := (W^k)_{k=1}^{N+1}$ where for $1 \leq k \leq N$, $W^k \in \mathbb{R}^{p_{k+1} \times d_k p_k}$ and $W^{N+1} \in \mathbb{R}^{d_{N+1} \times p_{N+1}}$ for the prediction layer. Indeed let denote for $k \in \{1, ..., N\}$, $W^k := (w^k_i, ..., w^k_{p_{k+1}})$ where for all $j \in \{1, ..., p_{k+1}\}$, $w^k_j \in \mathbb{R}^{d_k p_k}$ and let us first define for all $k \in \{1, ..., N\}$, $j \in \{1, ..., n_{k+1}\}$ and $q \in \{1, ..., n_k\}$ the sequence of the following operators.

**Convolution Operators.**

$$C^k_j : \mathbb{Z} \in (\mathbb{R}^{d_k p_k})^{n_k} \rightarrow C^k_j(\mathbb{Z}) := ((\mathbb{Z}_i, w^k_j)_i^{n_k}) \in \mathbb{R}^{n_k}$$

**Non-Linear Operators.**

$$M_k : \mathbb{X} \in \mathbb{R}^{n_k} \rightarrow M_k(\mathbb{X}) := (\sigma_k(\mathbb{X}_i))_i^{n_k} \in \mathbb{R}^{n_k}$$

**Pooling Operators.** Let $(\gamma^k_{i,j} \mid i,j=1$ the pooling factors at layer $k$ (which are often assumed to be decreasing with respect to the distance between $i$ and $j$)

$$A_k : \mathbb{X} \in \mathbb{R}^{n_k} \rightarrow A_k(\mathbb{X}) := \left(\sum_{j=1}^{n_k} \gamma^k_{i,j} \mathbb{X}_j\right)_{i=1}^{n_k} \in \mathbb{R}^{n_k}$$

**Patch extraction Operators.**

$$P^k_{q+i} : (\mathbb{R}^{p_{k+1}})^{n_k} \rightarrow (\mathbb{R}^{p_{k+1}})^{d_{k+1}}$$

$$U \rightarrow P^k_{q+i}(U) := (U_{q+i})_{i=0}^{d_{k+1} - 1}$$

Then $\mathcal{N}$ can be obtained by the following procedure: let $\mathbb{X}^1 \in \mathcal{I}$, then we can denote $\mathbb{X}^1 = (\mathbb{X}^1_i)_{i=1}^{n_1}$ where for all $i \in \{1, ..., n_1\}$, $\mathbb{X}^1_i \in \mathbb{R}^{d_1}$. Therefore we can build by induction the sequence $(\mathbb{X}^k)_{k=1}^N$ by doing the following operations starting from $k = 1$ until $k = N$

$$C^k_j(\mathbb{X}^k) = ((\mathbb{X}^k_i, w^k_j)_i^{n_k})$$

$$M_k(C^k_j(\mathbb{X}^k)) = (\sigma_k((\mathbb{X}^k_i, w^k_j)_i^{n_k})$$

$$A_k(M_k(C^k_j(\mathbb{X}^k))) = \left(\sum_{q=1}^{n_k} \gamma^k_{i,j} \sigma_k((\mathbb{X}^k_i, w^k_j)_i^{n_k})$$

$$Z^{k+1}(i,j) = A_k(M_k(C^k_j(\mathbb{X}^k)))_i$$

$$\mathbb{X}^{k+1} = (Z^{k+1}(i,1), ..., Z^{k+1}(i,p_{k+1}))_i^{n_k}$$

Finally the function defined by a Convolutional network is $\mathcal{N}_W(\mathbb{X}^1) := (\mathbb{X}^{N+1}, W^{N+1})_{\mathbb{R}^{p_{N+1} \times d_{N+1}}}^1$.

A.1 Proof of Proposition

**Proof.** Let $N \geq 0$ be the number of layers and let $(\sigma_i)_{i=1}^N$ be a sequence of $N$ functions which admits a Taylor decomposition around $0$ on $\mathbb{R}$ such that for every $i \in \{1, ..., N\}$ and $x \in \mathbb{R}$

$$\sigma_i(x) = \sum_{t \geq 0} a_{i,t} x^t$$

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We can now define the sequence \((f_i)_{i=1}^N\) such that for every \(i \in \{1, ..., N\}\) and \(x \in \mathbb{R}\)

\[f_i(x) := \sum_{t \geq 0} |a_{i,t}|x^t\]

Let us now introduce two sequence of functions \((\phi_i)_{i=1}^N\) and \((\psi_i)_{i=1}^N\) such that for all \(i \in \{1, ..., N\}\) and \(x \in \ell_2\)

\[
\phi_i(x) := \left(\sqrt{|a_{i,t}|x_{k_1}...x_{k_t}}\right)_{t \in \mathbb{N}, k_1,...,k_t \in \mathbb{N}}
\]

\[
\psi_i(x) := \left(\frac{a_{i,t}}{\sqrt{|a_{i,t}|}}x_{k_1}...x_{k_t}\right)_{t \in \mathbb{N}, k_1,...,k_t \in \mathbb{N}}
\]

with the convention that \(\frac{0}{0} = 0\). Moreover as a countable union of countable sets is countable and \((\sigma_i)_{i=1}^N\)
are defined on \(\mathbb{R}\), we have that for all \(x \in \ell_2\) and \(i \in \{1, ..., N\}\), \(\phi_i(x), \psi_i(x) \in \ell_2\). Indeed there exists a bijection \(\mu : \mathbb{N} \rightarrow \bigcup_{i \geq 0} \mathbb{N}^i\), therefore we can denote for all \(i \in \{1, ..., N\}\) and \(x \in \ell_2\), \(\phi_i(x) = (\phi_i(x)_{\mu(j)})_{j \in \mathbb{N}}\) and \(\psi_i(x) = (\psi_i(x)_{\mu(j)})_{j \in \mathbb{N}}\). We have then

\[
\langle \phi_i(x), \phi_i(x') \rangle_{\ell_2} = \sum_{j \in \mathbb{N}} \phi_i(x)_{\mu(j)} \phi_i(x')_{\mu(j)}
\]

(20)

\[
= \sum_{t \geq 0} |a_{i,t}| \sum_{k_1,...,k_t} x_{k_1}...x_{k_t} x'_{k_1}...x'_{k_t}
\]

(21)

\[
= \sum_{t \geq 0} |a_{i,t}| \langle x, x' \rangle_{\ell_2}^t
\]

(22)

\[
= f_i(\langle x, x' \rangle_{\ell_2})
\]

(23)

Moreover the same calculation method leads also to the fact that

\[
\langle \psi_i(x), \psi_i(x') \rangle_{\ell_2} = f_i(\langle x, x' \rangle_{\ell_2}) .
\]

Therefore \(\phi_i\) and \(\psi_i\) are feature maps of the positive semi-definite kernel \(k_i : x, x' \in \ell_2 \times \ell_2 \rightarrow f_i(\langle x, x' \rangle_{\ell_2})\).

Let us now define the following kernel on \(\mathcal{I}\)

\[
K_1(X, X') = \sum_{i=1}^n f_i(\langle X(i), X'(i) \rangle_{\mathbb{R}^d})
\]

As any vectors of \(\mathbb{R}^d\) can be seen as an element of \(\ell_2\), we have that

\[
K_1(X, X') = \sum_{i=1}^n f_i(\langle X(i), X'(i) \rangle_{\ell_2})
\]

\[
= \sum_{i=1}^n \langle \phi_i(X(i)), \phi_i(X'(i)) \rangle_{\ell_2}
\]

Defining \(\Phi(X) := (\phi_1(X(i)))_{i=1}^n \in \ell_2\), we have then

\[
K_1(X, X') = \langle \Phi(X), \Phi(X') \rangle_{\ell_2}.
\]

Let \((W^k)_{k=1}^{N+1}\) be any sequence such that for \(1 \leq k \leq N\), \(W^k \in \mathbb{R}^{p_{k-1} \times d_k p_k}\) and for the prediction layer \(W^{N+1} \in \mathbb{R}^{d_{N+1} \times p_{N+1}}\). Moreover let \(N\) the function in \(\mathcal{F}_{\langle \sigma_i \rangle_{i=1}^N}\) associated. Let \(X^1 \in \mathcal{I}\) and let us now denote for \(k \in \{1, ..., N\}, i \in \{1, ..., n_k\}\) and \(j \in \{1, ..., p_{k+1}\}\)

\[
\Psi^k_{i,j}(X^1) := A_k(M_k(C^k_j(X^k)))(i)
\]

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Let us now show by induction on $k \in \{1, ..., N\}$ that for all $i \in \{1, ..., n_k\}$ and $j \in \{1, ..., p_{k+1}\}$ there exists $\mathbf{Z}_{i,i,j}^k \in \ell_2$ such that we have

$$
\Psi_{i,j}^k(\mathbf{X}^1) = \langle \Phi(\mathbf{X}^1), \mathbf{Z}_{i,j}^k \rangle_{\ell_2} \quad \text{if} \quad k = 1
$$

$$
\Psi_{i,j}^k(\mathbf{X}^1) = \langle \phi_k \circ \cdots \phi_2 \circ \Phi(\mathbf{X}^1), \mathbf{Z}_{i,j}^k \rangle_{\ell_2} \quad \text{if} \quad k \geq 2
$$

For $k = 1$, let $i \in \{1, ..., n\}$ and $j \in \{1, ..., p_2\}$, we have by considering $\mathbb{R}^d \subset \ell_2$

$$
\Psi_{i,j}^k(\mathbf{X}^1) = \sum_{q=1}^{n} \gamma_{i,q}^1 \sigma_1((\mathbf{X}^1(q), \mathbf{w}_j^1))_{\ell_2}
$$

Moreover we remark that for any $x, w \in \ell_2$

$$
\sigma_i((x, w)) = \sum_{t \geq 0} a_{i,t} (x, w)^t
$$

$$
= \sum_{t \geq 0} a_{i,t} \sum_{k_1, ..., k_t} x_{k_1} \cdots x_{k_t} w_{k_1} \cdots w_{k_t}
$$

$$
= \sum_{t \geq 0} \sum_{k_1, ..., k_t} \sqrt{a_{i,t}} a_{i,t} x_{k_1} \cdots x_{k_t} \frac{w_{k_1} \cdots w_{k_t}}{\sqrt{a_{i,t}}}
$$

Therefore we obtain that

$$
\sigma_i((x, w)) = \langle \phi_i(x), \psi_i(w) \rangle_{\ell_2}
$$

And we have

$$
\Psi_{i,j}^k(\mathbf{X}^1) = \sum_{q=1}^{n} \gamma_{i,q}^1 \langle \phi_1(\mathbf{X}^1(q)), \psi_1(\mathbf{w}_j^1) \rangle_{\ell_2}
$$

$$
= \sum_{q=1}^{n} \langle \phi_1(\mathbf{X}^1(q)), \gamma_{i,q}^1 \psi_1(\mathbf{w}_j^1) \rangle_{\ell_2}
$$

$$
= \langle \Phi(\mathbf{X}^1), \mathbf{Z}_{i,j}^1 \rangle_{\ell_2}
$$

with $\mathbf{Z}_{i,j}^1 = (\gamma_{i,q}^1 \psi_1(\mathbf{w}_j^1))_{q=1}^n \in \ell_2$. Let us now assume the result for $1 \leq k \leq N - 1$, therefore we have

$$
\mathbf{X}^{k+1} = \left(\Psi_{i,1}^k(\mathbf{X}^1), ..., \Psi_{i,p_{k+1}}^k(\mathbf{X}^1)\right)_{i=1}^{n_k}
$$

$$
\mathbf{X}^{k+1} = \left(P_q^k(\mathbf{X}^{k+1})\right)_{q=1}^{n_{k+1}}
$$

Therefore by denoting for all $i \in \{1, ..., n_k\}$, $\Psi_i^k(\mathbf{X}^1) := \left(\Psi_{i,1}^k(\mathbf{X}^1), ..., \Psi_{i,p_{k+1}}^k(\mathbf{X}^1)\right)$ we have that

$$
\mathbf{X}^{k+1} = \left(P_q^k(\mathbf{X}^{k+1})\right)_{q=1}^{n_{k+1}}
$$

$$
= \left(P_q^k(\mathbf{X}^{k+1})\right)_{q=1}^{n_{k+1}} = \left((\Psi_{i,q}^k(\mathbf{X}^1))_{i=1}^{n_k} \right)_{q=1}^{n_{k+1}}
$$

Let $i \in \{1, ..., n_{k+1}\}$ and $j \in \{1, ..., p_{k+1}\}$, we have that

$$
\Psi_{i,j}^{k+1}(\mathbf{X}^1) = A_{k+1}(M_{k+1}(C_{j}^{k+1}(\mathbf{X}^{k+1}))) (i)
$$

$$
= \sum_{q=1}^{n_{k+1}} \gamma_{i,q}^{k+1} \sigma_{k+1}(\left(P_q^{k+1}(\mathbf{X}^{k+1}), \mathbf{w}_{i,j}^{k+1}\right)_{\ell_2})
$$

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where the last equality is obtained by applying the formula (24). Therefore we have:

\[
\sigma_{k+1} \left( (P_{q}^{k+1}(X^{k+1}), w_{j}^{k+1})_{H_{k+1}} \right) = \sigma_{k+1} \left( \sum_{\ell=1}^{d_{k+1}} w_{j}^{k+1}(\ell) \Psi_{q+\ell}(X^{1}) \right)
\]

and we have for all \( q \in \{1, ..., n_{k+1} \} \)

\[
\Psi_{i,j}^{k+1}(X^{1}) = \sum_{q=1}^{n_{k+1}} \gamma_{i,q}^{k+1} \left( (P_{q}^{k+1}(X^{k+1}), w_{j}^{k+1})_{H_{k+1}} \right)
\]

Then by induction we have

\[
\sigma_{k+1} \left( (P_{q}^{k+1}(X^{k+1}), w_{j}^{k+1})_{H_{k+1}} \right) = \sigma_{k+1} \left( (D_{q}^{k+1}(X^{k+1}), w_{j}^{k+1})_{H_{k+1}} \right)
\]

where the last equality is obtained by applying the formula (24). Therefore we have:

\[
\Psi_{i,j}^{k+1}(X^{1}) = \sum_{q=1}^{n_{k+1}} \gamma_{i,q}^{k+1} \left( (P_{q}^{k+1}(X^{k+1}), w_{j}^{k+1})_{H_{k+1}} \right)
\]

and the result follows. Finally at the prediction layer, if \( N \geq 2 \) we just have

\[
N(X^{1}) = \sum_{i=1}^{N_{n+1}} \sum_{j=1}^{n_{n+1}} w^{N+1}(i,j) \Psi_{i,j}^{N}(X^{1})
\]

Then by induction we have that:

\[
K_{N}(X, X') = f_{N} \circ ... \circ f_{2} \left( (\Phi(X^{1}), \Phi(X'^{1}))_{\ell_{2}} \right)
\]

Finally as \( K_{1}(X, X') = (\Phi(X), \Phi(X'))_{\ell_{2}} \) we obtain that:

\[
K_{N}(X, X') = f_{N} \circ ... \circ f_{2} \left( \sum_{j=1}^{n} f_{1}(X(i), X'(i))_{\ell_{2}} \right)
\]
For $N = 1$ the result is clear from the result above. Moreover let us now assume that $N \geq 2$ and $\sigma_i(t)(0) \neq 0$ for all $i \geq 1$ and $t \geq 0$ and let us show that $K_N$ is a c-universal Kernel on $I$. Thanks to theorem \[2\] it suffices to show that $\Phi$ is a continuous and injective mapping and that the coefficients of the Taylor decomposition of $f_N \circ \ldots \circ f_1$ are positive. For that purpose let $k$ be the kernel on $S^{d-1}$ defined by

$$k(x, x') := f_1((x, x'))$$

Therefore $k$ is clearly a continuous kernel and thanks to Lemma \[3\] $\phi_1$ is continuous. Moreover, as for all $q \geq 0$, $f_1^{(q)}(0) > 0$, then thanks to theorem \[3\] $k$ is a c-universal kernel on $S^{d-1}$. Therefore thanks to lemma \[4\] $\phi_1$ is then also injective. Therefore $\Phi : X \in I \rightarrow (\phi_1(X(i)))_{i=1}^n \in \ell_2$ is then injective and continuous from $I$ to $\ell_2$. Moreover we have by construction that:

$$K_N(X, X') = f_N \circ \ldots \circ f_2((\Phi(X), \Phi(X')))_{\ell_2}$$

Therefore we now just need to show that the coefficients in the Taylor decomposition of $f_N \circ \ldots \circ f_2$ are positive and the result will follow from Theorem \[2\]. In fact we have the following lemma (see proof in section E.1).

**Lemma 1.** Let $(f_i)_{i=1}^N$ a family of functions that can be expanded in their Taylor series in 0 on $\mathbb{R}$ such that for all $k \in \{1, \ldots, N\}$, $(f_i^{(k)}(0))_{n \geq 0}$ are positive. Let us define also $\phi_1, \ldots, \phi_{N-1} : \mathbb{N}^2 \rightarrow \mathbb{R}_+$ such that for every $k \in \{1, \ldots, N-1\}$ and $l, m \geq 0$

$$\phi_k(l, m) := \frac{d^m}{dt^m}|_{t=0} f_k^{(l)}(t)$$

Then $g := f_N \circ \ldots \circ f_1$ can be expanded in its Taylor series on $\mathbb{R}$ such that for all $t \in \mathbb{R}$

$$g(t) = \sum_{l_1, \ldots, l_N \geq 0} \frac{f_N^{(l_N)}(0)}{l_N!} \times \phi_{N-1}(l_N, l_{N-1}) \ldots \times \phi_1(l_2, l_1) t^{l_1}$$

Moreover $(g^{(k)}(0))_{n \geq 0}$ is a positive sequence.

Therefore the coefficients in the Taylor decomposition of $f_N \circ \ldots \circ f_2$ are positive and the result follows. □

**B Spectral Analysis of Convolutional Networks**

**B.1 Proof of Theorem 4.1**

*Proof.* Let $g$ a function which admits a Taylor decomposition around 0 on $[-1, 1]$ such that $(g^{(m)})_{m \geq 0}$ are non-negative. By denoting $(b_m)_{m \geq 0}$ its coefficients, we can define the following dot product kernel $k_g$ on $S^{d-1}$ associated:

$$k_g(x, x') := g((x, x') \mathbb{R}_d) = \sum_{m \geq 0} b_m ((x, x') \mathbb{R}_d)^m$$

(25)

Moreover thanks to theorem \[4\] we have an explicit formula of the eigenvalues of the integral operator associated with the kernel $k_g$ defined on $L^2_{2d-1}(S^{d-1})$

$$\lambda_k = \frac{|S^{d-2}| \Gamma((d-1)/2)}{2^{k+1}} \sum_{s \geq 0} b_{2s+k} \frac{(2s+k)!}{(2s)!} \frac{\Gamma(s+1/2)}{\Gamma(s+k+d/2)}$$

(26)

where each spherical harmonics of degree $k$, $Y_k \in H_k(S^{d-1})$, is an eigenfunction of the integral operator with associated eigenvalue $\lambda_k$. Therefore $(Y_k^{(l_k)})_{k,l_k}$ is an orthonormal basis of eigenfunctions of $C_{k_g}$ associated.
with the non-negative eigenvalues \((\lambda_{k,d})_{k,d}\) such that for all \(k \geq 0\) and \(1 \leq l_k \leq \alpha_{k,d}\), \(\lambda_{k,d} := \lambda_k \geq 0\) where \(\lambda_k\) is given by the formula (26). And by Mercer theorem \([10]\) we have for all \(x, x' \in S^{d-1}\):

\[
k_g(x, x') = \sum_{k \geq 0} \sum_{l_k = 1}^{\alpha_{k,d}} \lambda_k Y_{k,l_k}^k(x) Y_{k,l_k}^k(x')
\]

where the convergence is absolute and uniform. Let now \(q \geq 1\), then we have:

\[
K_1(X, X')^q = \left( \sum_{i=1}^{n} f_1((X(i), X'(i))_{\mathbb{R}^d}) \right)^q
\]

\[
= \sum_{j_1, \ldots, j_q = 1}^{n} \prod_{k=1}^{q} f_1((X(j_k), X'(j_k))
\]

\[
= \sum_{\alpha_1, \ldots, \alpha_n \geq 0} \left( \sum_{k=1}^{\alpha_{k,d}} \lambda_k \right)^n \prod_{k=1}^{q} f_1((X(k), X'(k)))^n
\]

where \(\frac{q!}{\alpha_1! \cdots \alpha_n!} = q!\). The formula above hold even when \(q = 0\). But thanks to the Cauchy Product formula (see Theorem 5), we have that for all \(\alpha \geq 0\), \(f_1^\alpha\) admits a Taylor decomposition on \([-1, 1]\) with non-negative coefficients, and by denoting \(k f_1^\alpha\) the dot product kernel associated to \(f_1^\alpha\), we have that for all \(x, x' \in S^{d-1}\):

\[
f_1^\alpha(\langle x, x' \rangle) = \sum_{k \geq 0} \sum_{l_k = 1}^{\alpha_{k,d}} \lambda_k Y_{k,l_k}^k(x) Y_{k,l_k}^k(x')
\]

where the notation \((\lambda_{k,d})_{d \geq 0}\) reflects the fact that the eigenvalues given by the formula (26) depends on the coefficients of the Taylor decomposition of \(f_1^\alpha\). Let now \(q \geq 0\) and \(\alpha_1, \ldots, \alpha_q \geq 0\) such their sum is equal to \(q\). Then we have:

\[
\prod_{k=1}^{n} (f_1((X(k), X'(k)))^n = \left( \sum_{k \geq 0} \sum_{l_k = 1}^{\alpha_{k,d}} \lambda_{k,\alpha_k} Y_{k,l_k}^k(X(k)) Y_{k,l_k}^k(X'(k)) \right)^n
\]

\[
= \sum_{k_1, \ldots, k_n \geq 1} \sum_{1 \leq k_i \leq \alpha_{k_i,d}} \prod_{i=1}^{n} \lambda_{k_i, \alpha_i} \prod_{i=1}^{n} Y_{k_i}^{l_{k_i}}(X(i)) \prod_{i=1}^{n} Y_{k_i}^{l_{k_i}}(X'(i))
\]

Therefore we have:

\[
K_1(X, X')^q = \sum_{\alpha_1, \ldots, \alpha_n \geq 0} \sum_{\sum_{i=1}^{n} \alpha_i = q} \left( \sum_{k = 1}^{\alpha_{k,d}} \left( f_1((X(k), X'(k))) \right)^n \right)
\]

\[
= \sum_{\alpha_1, \ldots, \alpha_n \geq 0} \sum_{\sum_{i=1}^{n} \alpha_i = q} \left[ \left( \sum_{k_1, \ldots, k_n \geq 1} \sum_{1 \leq k_i \leq \alpha_{k_i,d}} \prod_{i=1}^{n} \lambda_{k_i, \alpha_i} \prod_{i=1}^{n} Y_{k_i}^{l_{k_i}}(X(i)) \prod_{i=1}^{n} Y_{k_i}^{l_{k_i}}(X'(i)) \right) \right]
\]

\[
= \sum_{k_1, \ldots, k_n \geq 1} \sum_{1 \leq k_i \leq \alpha_{k_i,d}} \sum_{\sum_{i=1}^{n} \alpha_i = q} \left[ \prod_{i=1}^{n} \lambda_{k_i, \alpha_i} \prod_{i=1}^{n} Y_{k_i}^{l_{k_i}}(X(i)) \prod_{i=1}^{n} Y_{k_i}^{l_{k_i}}(X'(i)) \right]
\]

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Let us now denote \((a_q)_{q \geq 0}\) the non-negative coefficients of the Taylor decomposition of \(f_N \circ \ldots \circ f_2\) such that for all \(t \in \mathbb{R}\):

\[
f_N \circ \ldots \circ f_2(t) = \sum_{q \geq 0} a_q t^q
\]

Finally we obtain that:

\[
K_N(X, X') = \sum_{q \geq 0} a_q K_1((X, X'))^q
\]

\[
= \sum_{q \geq 0} a_q \sum_{k_1, \ldots, k_n \geq 0} 1 \leq l_{k_i} \leq \alpha_{k_i,d} \left[ \sum_{i=1}^n \sum_{\alpha_1, \ldots, \alpha_n \geq 0} \left( \sum_{\sum_{i=1}^n \alpha_i = q} \prod_{i=1}^n \lambda_{\alpha_i} \right) \prod_{i=1}^n Y_{l_{k_i}}(X(i)) \prod_{i=1}^n Y_{l_{k_i}}(X'(i)) \right]
\]

Moreover \(\prod_{i=1}^n Y_{l_{k_i}}(X(i))\) is clearly an orthonormal system (ONS) of \(L_2^\infty; \mathbb{P} \) with the following:

\[
e_{(k_i,l_{k_i})}_{i=1}^n(X) := \prod_{i=1}^n Y_{l_{k_i}}(X(i))
\]

\[
\mu_{(k_i,l_{k_i})}_{i=1}^n := \sum_{q \geq 0} a_q \sum_{\alpha_1, \ldots, \alpha_n \geq 0} \left( \sum_{\sum_{i=1}^n \alpha_i = q} \prod_{i=1}^n \lambda_{\alpha_i} \right) \prod_{i=1}^n Y_{l_{k_i}}(X(i)) \prod_{i=1}^n Y_{l_{k_i}}(X'(i))
\]

We have:

\[
K_N(X, X') = \sum_{k_1, \ldots, k_n \geq 0} \sum_{1 \leq l_{k_i} \leq \alpha_{k_i,d}} \mu_{(k_i,l_{k_i})}_{i=1}^n \cdot e_{(k_i,l_{k_i})}_{i=1}^n(X) e_{(k_i,l_{k_i})}_{i=1}^n(X')
\]

where the convergence is absolute and uniform. Therefore \(\left( e_{(k_i,l_{k_i})}_{i=1}^n \right)_{k_i,l_{k_i}}\) is also an orthonormal system of eigenfunctions of \(T_{K_N}\) associated with the non-negative eigenvalues \(\left( \mu_{(k_i,l_{k_i})}_{i=1}^n \right)_{k_i,l_{k_i}}\). Moreover the sequence of positive eigenvalues of \(T_{K_N}\) with their multiplicities must be a subsequence of \(\left( \mu_{(k_i,l_{k_i})}_{i=1}^n \right)_{k_i,l_{k_i}}\).

\[\square\]

C Regularized Least-Squares for CNNs

C.1 Notations

Let \((\mathcal{X}, \mathcal{B})\) a measurable space, \(\mathcal{Y} = \mathbb{R}\) and \(H\) be an infinite dimensional separable RKHS on \(\mathcal{X}\) with respect to a bounded and measurable kernel \(k\). Furthermore, let \(C, \gamma > 0\) be some constants and \(\alpha > 0\) be a parameter. By \(\mathcal{P}_{H,C,\gamma,\alpha}\) we denote the set of all probability measures \(\nu\) on \(\mathcal{X}\) with the following:
Then we have that for all $\alpha$ where we denote by $f$ the solution of the following minimizing problem:

$$
\min_{f \in H} \left\{ \frac{1}{\ell} \sum_{i=1}^{\ell} (f(x_i) - y_i)^2 + \lambda \|f\|_H^2 \right\}
$$

### C.2 Proof of Theorem 5.1 and 5.2

Here the main goal is to control the rate of decay of the eigenvalues associated with the integral operator $T_\rho$. To do so, let us first show the following proposition:

**Proposition C.1.** If there exist $1 > r > 0$ and $0 < c_2 \leq c_1$ constants such that for all $m \geq 0$:

$$
c_{2}r^{m} \leq b_{m} \leq c_{1}r^{m}
$$

then for all $\alpha \geq 1$, there exists $C_{1,\alpha}, C_{2,\alpha} > 0$ constants depending only on $\alpha$ and $d$ such that for all $m \geq 0$:

$$
C_{2,\alpha} \left( \frac{r}{4} \right)^{m} \leq \lambda_{m,\alpha} \leq C_{1,\alpha}(m+1)^{\alpha-1}r^{m}
$$

where

$$
\lambda_{m,\alpha} = \frac{|S^{d-2}|\Gamma((d-1)/2)}{2^{m+1}} \sum_{s \geq 0} \left[ \frac{d^{2s+m}}{d^{2s+m}|t=0} \right] \frac{f_{\alpha}^{d}(t)}{(2s+m)!} \frac{(2s+m)!}{\Gamma(s+1/2)} \frac{s + m + d/2}{(2s)!} \Gamma(s + m + d/2)
$$

**Proof.** Let us first introduce the following lemma (see proof section E.2):

**Lemma 2.** If there exists $1 > r > 0$ and $c_1 \geq c_2 > 0$ there for all $m \geq 0$:

$$
c_{2}r^{m} \leq b_{m} \leq c_{1}r^{m}
$$

Then we have that for all $\alpha \geq 1$ and $m \geq 0$:

$$
ce_{2}^{\alpha}r^{m} \leq \frac{d^{m}}{dt^{m}|t=0} \frac{f_{\alpha}}{m!} \leq c_{1}^{\alpha}(m+1)^{\alpha-1}r^{m}
$$
Let now \( \alpha \geq 1 \) and \( m \geq 0 \). By definition of \( \lambda_{m,\alpha} \), we have:

\[
\lambda_{m,\alpha} = \frac{|S^d-2|\Gamma((d-1)/2)}{2^{m+1}} \sum_{s \geq 0} \left[ \frac{d^{2s+m}}{dt^{2s+m}} \big|_{t=0} f_s^\alpha(t) \right] \frac{(2s+m)!}{(2s+1)!} \frac{\Gamma(s+1/2)}{\Gamma(s+m+d/2)}
\]

In the following we denote \( b_{2s+m,\alpha} := \frac{d^{2s+m}}{dt^{2s+m}} \big|_{t=0} f_s^\alpha(t) \) and \( \theta_{s,m,\alpha} = b_{2s+m,\alpha} \frac{(2s+m)!}{(2s+1)!} \frac{\Gamma(s+1/2)}{\Gamma(s+m+d/2)} \). Therefore we have:

\[
\theta_{s,m,\alpha} = b_{2s+m,\alpha} \frac{(2s+m)...(2s+1)}{(s+m+\frac{d}{2})...(s+\frac{1}{2})} = b_{2s+m,\alpha} \frac{(2s+m)...(2s+1)}{(2s+2m+d-2)...(2s+1)} \times 2^{m+\frac{d}{2}-1}
\]

Moreover \( \frac{(2s+m)...(2s+1)}{(2s+2m+d-2)...(2s+1)} \leq 1 \) and thanks to the upper bound given in Lemma 2 we have:

\[
\theta_{s,m,\alpha} \leq 2 \frac{d}{2} e_1^\alpha (m+2s+1)^{\alpha-1}(2r)^m r^{2s} \\
\leq 2 \frac{d}{2} e_1^\alpha (m+1)^{\alpha-1}(2r)^m (2s+1)^{\alpha-1} r^{2s}
\]

Therefore we have:

\[
\sum_{s \geq 0} \theta_{s,m,\alpha} \leq (m+1)^{\alpha-1}(2r)^m \left[ 2 \frac{d}{2} e_1^\alpha \sum_{s \geq 0} (2s+1)^{\alpha-1} r^{2s} \right]
\]

And:

\[
\lambda_{m,\alpha} = \frac{|S^d-2|\Gamma((d-1)/2)}{2^{m+1}} \sum_{s \geq 0} \theta_{s,m,\alpha} \leq (m+1)^{\alpha-1} r^{2m} \left[ \frac{|S^d-2|\Gamma((d-1)/2)}{2^{m+1}} \frac{2^{\frac{d}{2}} e_1^\alpha}{2} \sum_{s \geq 0} (2s+1)^{\alpha-1} r^{2s} \right]
\]

Moreover we have:

\[
\lambda_{m,\alpha} = \frac{|S^d-2|\Gamma((d-1)/2)}{2^{m+1}} \sum_{s \geq 0} \theta_{s,m,\alpha} \geq \frac{|S^d-2|\Gamma((d-1)/2)}{2^{m+1}} \theta_{0,m,\alpha}
\]

\[
\geq b_{m,\alpha} \frac{|S^d-2|\Gamma((d-1)/2)}{2^{m+1}} \frac{m!\Gamma(1/2)}{\Gamma(m+d/2)}
\]

\[
\geq \frac{|S^d-2|\Gamma((d-1)/2)\Gamma(1/2)\frac{e_1^\alpha}{2} \left( \frac{r}{2} \right)^m m!}{\Gamma(m+d/2)}
\]

The last inequality comes from the lower bound given in in eq. 27. Moreover thanks to the Stirlings approximation formula we have:

\[
\Gamma(x) \sim \sqrt{2\pi} x^{x-1/2} e^{-x}
\]

which leads to

\[
\frac{m!}{\Gamma(m+d/2)} \sim e^{d/2} \left( 1 - \frac{d/2}{m+d/2} \right)^m \frac{m^{1/2}}{(m+d/2)^{d-1/2}}
\]

Finally we obtain

\[
\frac{m!}{\Gamma(m+d/2)} \sim \frac{1}{m^{d-1/2}}
\]
Therefore there exists a constant $C > 0$ such that for all $m \geq 0$ we have:

$$\frac{m!}{\Gamma(m + d/2)} \geq C \frac{1}{2^m}$$

Finally we have:

$$\lambda_{m,\alpha} \geq |S^{d-2}|\Gamma((d - 1)/2)\Gamma(1/2)C_{m,\alpha}^2 \left(\frac{r}{4}\right)^m$$

We can now derive a sharp control of the eigenvalues of $T_{K_N}$:

**Proposition C.2.** Let us assume that $f_N \circ \ldots \circ f_2$ is a polynomial of degree $D \geq 1$ and let $a := \min(D, n)$. Let $(\eta_m)_{m=0}^M$ be the positive eigenvalues of the integral operator $T_{K_N}$ associated to the kernel $K_N$ ranked in a non-increasing order with their multiplicities, where $M \in \mathbb{N} \cup \{+\infty\}$. If there exist $1 > r > 0$ and $0 < c_2 \leq c_1$ constants such that for all $m \geq 0$:

$$c_2r^m \leq b_m \leq c_1r^m$$

then $M = +\infty$ and by denoting $d^* := \min(D, n)$, there exists $C_3, C_4 > 0$ and $0 < \gamma < q$ constants such that for all $m \geq 0$:

$$C_4e^{-q} \leq \eta_m \leq C_3e^{-\gamma m}$$

**Proof.** Let us first recall that the positive eigenvalues of $T_{K_N}$ are exactly the subsequence of positive eigenvalues in $\mu_{(k_1, l_1), \ldots, (k_n, l_n)}$. Moreover the assumption on $(b_m)_{m \geq 0}$ guarantees that $b_m > 0$ for all $m \geq 0$, and thanks to the formula of [26] we deduce that that $M = +\infty$.

Moreover we have for all $k_1, \ldots, k_n \geq 0$, and $(l_1, \ldots, l_n) \in \{1, \ldots, \alpha_{k_1, d}\} \times \ldots \times \{1, \ldots, \alpha_{k_n, d}\}$:

$$\mu_{(k_1, l_1), \ldots, (k_n, l_n)} := \sum_{q=0}^{D} a_q \sum_{\alpha_1, \ldots, \alpha_n \geq 0} \left(\sum_{\sum_{i=1}^{n} \alpha_i = q} \lambda_{k_i, \alpha_i}\right)^{n}$$

We first remark that if $\alpha = 0$, then we have:

$$\lambda_{m,\alpha} = \begin{cases} 0 & \text{if } m \geq 1 \\ \frac{|S^{d-2}|\Gamma((d - 1)/2)\Gamma(1/2)}{2^m} & \text{if } m = 0 \end{cases}$$

Moreover thanks to the Proposition [C.1] if $\alpha \geq 1$, there exists $C_{1,\alpha}, C_{2,\alpha} > 0$ constants depending only on $\alpha$ such that for all $m \geq 0$:

$$C_{2,\alpha} \left(\frac{r}{4}\right)^m \leq \lambda_{m,\alpha} \leq C_{1,\alpha} (m + 1)^{\alpha-1}r^m$$

Let $\lambda > 0$, therefore to obtain the rate of convergence of the positive eigenvalues with their multiplicities ranked in the decreasing order of $T_{K_N}$, we need to find the number of eigenvalues which are bigger than $\lambda$, that is to say the cardinal of

$$E^\lambda := \left\{((k_1, l_1), \ldots, (k_n, l_n)) : \mu_{(k_1, l_1), \ldots, (k_n, l_n)} \geq \lambda, \ k_1, \ldots, k_n \geq 0, \ l_i \in \{1, \ldots, \alpha_{k_i, d}\} \text{ for } i \in \{1, \ldots, n\}\right\}$$

For $q \in \{1, \ldots, D\}$ and let us define:

$$\mu_{(k_1, l_1), \ldots, (k_n, l_n)} := \sum_{\alpha_1, \ldots, \alpha_n \geq 0} \left(\sum_{\sum_{i=1}^{n} \alpha_i = q} \lambda_{k_i, \alpha_i}\right)^{n}$$

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and

\[ E^\lambda_q := \left\{ (k_1, l_{k_1}), \ldots, (k_n, l_{k_n}) : \mu_{(k_i, l_{k_i})_{i=1}^n} \geq \lambda, \quad k_1, \ldots, k_n \geq 0, \quad l_{k_i} \in \{1, \ldots, \alpha_{k_i, d}\} \quad \text{for } i \in \{1, \ldots, n\} \right\} \]

Therefore by denoting

\[ c := \max_{q=1,\ldots,D} a_q \]

we have that:

\[ E^{\hat{\lambda}, D} \subset E^\lambda \subset \bigcup_{q=1}^D E^{\hat{\lambda}, q} \]

Let \( q \in \{1, \ldots, D\} \) and let us denote \( a_q = \min(q, n) \). To obtain the cardinality of \( E^\lambda_q \), we first define for all \( k_1, \ldots, k_n \geq 0 \) the following set:

\[ A(k_1, \ldots, k_n) := \{i : k_i \geq 1\} \]

Let us now define the following partition of \( E^\lambda_q \):

\[ E^\lambda_{a_q+1} := \left\{ (k_1, l_{k_1}), \ldots, (k_n, l_{k_n}) : \mu_{(k_i, l_{k_i})_{i=1}^n} \geq \lambda \quad \text{and} \quad |A_q(k_1, \ldots, k_n)| \geq a_q + 1 \right\} \]

And for \( w \in \{0, \ldots, a_q\} \), we define:

\[ E^\lambda_w := \left\{ (k_1, l_{k_1}), \ldots, (k_n, l_{k_n}) : \mu_{(k_i, l_{k_i})_{i=1}^n} \geq \lambda \quad \text{and} \quad |A_q(k_1, \ldots, k_n)| = w \right\} \]

But as for all \( ((k_1, l_{k_1}), \ldots, (k_n, l_{k_n})) \in E^\lambda_{a_q+1} \) either there exist \( j \in \{1, \ldots, n\} \) such that \( k_j \geq 1 \) and \( \alpha_j = 0 \), therefore \( \mu_{(k_i, l_{k_i})_{i=1}^n} = 0 \) or \( a_q + 1 \geq n + 1 \). Therefore we always have \( E^\lambda_{a_q+1} = \emptyset \) and we have the following partition:

\[ E^\lambda_q = \bigcup_{w=0}^{a_q} E^\lambda_w \]

Moreover if \( k_i = 0 \) then \( l_{k_i} = 0 \), therefore each \( E^\lambda_w \) is a disjoint union of \( \binom{n}{w} \) sets which have all the same cardinality as:

\[ E^\lambda_w^{\lambda, \text{Id}} := \left\{ (k_1, l_{k_1}), \ldots, (k_n, l_{k_n}) : \mu_{(k_i, l_{k_i})_{i=1}^n} \geq \lambda, \quad k_1, \ldots, k_w \geq 1 \quad \text{and} \quad k_{w+1} = \ldots = k_n = 0 \right\} \]

Indeed we have:

\[ E^\lambda_w = \bigcup_{\sigma \in S_{w,n}} E^\lambda_w^{\lambda, \sigma} \]

where \( S_{w,n} \) is the set of class of injective functions from \( \{1, \ldots, w\} \) to \( \{1, \ldots, n\} \) such that \( \sigma \sim \sigma' \) if and only if \( \sigma(\{1, \ldots, w\}) = \sigma'(\{1, \ldots, w\}) \) and:

\[ E^\lambda_w^{\lambda, \sigma} := \left\{ (k_{\sigma(1)}, l_{k_{\sigma(1)}}, \ldots, (k_{\sigma(w)}, l_{k_{\sigma(w)}}), (0, 0), \ldots, (0, 0)) : \mu_{(k_{\sigma(i)}, l_{k_{\sigma(i)}})_{i=1}^n} \geq \lambda \quad \text{and} \quad k_{\sigma(1)}, \ldots, k_{\sigma(w)} \geq 1 \right\} \]

Therefore we have:

\[ |E^\lambda_q| = \binom{n}{w}|E^\lambda_w^{\lambda, \text{Id}}| \]

Let \( ((k_1, l_{k_1}), \ldots, (k_n, l_{k_n})) \in E^\lambda_w^{\lambda, \text{Id}} \) and let \( \alpha_1, \ldots, \alpha_n \geq 0 \) such that \( \sum_{i=1}^n \alpha_i = q \). If there exist \( j \in \{1, \ldots, w\} \) such that \( \alpha_j = 0 \), then:

\[ \prod_{i=1}^n \lambda_{k_i, \alpha_i} = 0 \]
Therefore we have:

\[
\mu(k_i,l_i)_{i=1}^n = \sum_{\alpha_1, \ldots, \alpha_w \geq 1 \atop \alpha_{w+1}, \ldots, \alpha_n \geq 0 \atop \sum_{i=1}^n \alpha_i = q} \binom{q}{\alpha_1, \ldots, \alpha_n} \prod_{i=1}^n \lambda_{k_i, \alpha_i}
\]

Let now \(\alpha_1, \ldots, \alpha_w \geq 1\) and \(\alpha_{w+1}, \ldots, \alpha_n \geq 0\) such that \(\sum_{i=1}^n \alpha_i = q\). Therefore we have:

\[
\prod_{i=1}^n \lambda_{k_i, \alpha_i} = \prod_{i=1}^w \lambda_{k_i, \alpha_i} \prod_{i=w+1}^n \lambda_{0, \alpha_i}
\]

And:

\[
\prod_{i=1}^w C_{2, \alpha_i} \left(\frac{r}{4}\right)^{k_i} \prod_{i=w+1}^n \lambda_{0, \alpha_i} \leq \prod_{i=1}^n \lambda_{k_i, \alpha_i} \leq \prod_{i=1}^w C_{1, \alpha_i} (k_i + 1)^{\alpha_i - 1} r^{k_i} \prod_{i=w+1}^n \lambda_{0, \alpha_i}
\]

Therefore by denoting \(v_w := \sum_{i=1}^w k_i\) we obtain that:

\[
\left[\prod_{i=1}^w C_{2, \alpha_i} \prod_{i=w+1}^n \lambda_{0, \alpha_i}\right] \left(\frac{r}{4}\right)^{v_w} \leq \prod_{i=1}^n \lambda_{k_i, \alpha_i} \leq \left[\prod_{i=1}^w C_{1, \alpha_i} \prod_{i=w+1}^n \lambda_{0, \alpha_i}\right] v_w r^{v_w}
\]

Let us denote

\[C_{1, q} := \max_{w \in \{0, \ldots, n\}} \max_{\alpha_1, \ldots, \alpha_w \geq 1 \atop \alpha_{w+1}, \ldots, \alpha_n \geq 0 \atop \sum_{i=1}^n \alpha_i = q} \prod_{i=1}^w C_{1, \alpha_i} \prod_{i=w+1}^n \lambda_{0, \alpha_i}\]

and

\[C_{2, q} := \min_{w \in \{0, \ldots, n\}} \min_{\alpha_1, \ldots, \alpha_w \geq 1 \atop \alpha_{w+1}, \ldots, \alpha_n \geq 0 \atop \sum_{i=1}^n \alpha_i = q} \prod_{i=1}^w C_{2, \alpha_i} \prod_{i=w+1}^n \lambda_{0, \alpha_i}\]

Therefore we have:

\[C_{2, q} \left(\frac{r}{4}\right)^{v_w} \leq \prod_{i=1}^n \lambda_{k_i, \alpha_i} \leq C_{1, q} v_w r^{v_w}\]

Then we obtain that:

\[
\sum_{\alpha_1, \ldots, \alpha_n \geq 0 \atop \sum_{i=1}^n \alpha_i = q} \binom{q}{\alpha_1, \ldots, \alpha_n} C_{2, q} \left(\frac{r}{4}\right)^{v_w} \leq \mu(k_i,l_i)_{i=1}^n q \leq \sum_{\alpha_1, \ldots, \alpha_n \geq 0 \atop \sum_{i=1}^n \alpha_i = q} \binom{q}{\alpha_1, \ldots, \alpha_n} C_{1, q} v_q r^{v_w}
\]

\[
C_{2, q} \left(\frac{r}{4}\right)^{v_w} \leq \mu(k_i,l_i)_{i=1}^n q \leq C_{1, q} q^w r^{v_w}
\]

Let \(1 > r' > r > 0\) and let \(Q_q := \max_{w \geq 1} \frac{w^{v_w}}{r'^w}\). Therefore by denoting \(C'_{1, q} := C_{1, q} q^w Q_q\) we obtain that:

\[C_{2, q} \left(\frac{r}{4}\right)^{v_w} \leq \mu(k_i,l_i)_{i=1}^n q \leq C'_{1, q} r^{v_w}\]
Therefore we have:
\[
\mu(k_i, l_{ki}) \geq \lambda \Rightarrow C_{1,q}^{r_{wu}} \geq \lambda
\]
\[
\Rightarrow v_w \leq \frac{\log(C_{1,q}^r/\lambda)}{\log(1/r')}
\]
\[
\Rightarrow k_i \leq \frac{\log(C_{1,q}^r/\lambda)}{\log(1/r')} \text{ for } i \in \{1, \ldots, w\}
\]

Therefore we obtain that:
\[
|E_{w}^{\lambda,q,Id}| \leq \left\{ ((k_1, l_{k_1}, \ldots, (k_w, l_{k_w}), (0,0), \ldots, (0,0)) : 0 \leq k_i \leq \frac{\log(C_{1,q}^r/\lambda)}{\log(1/r')} \text{ for } i \in \{1, \ldots, w\} \right\}
\]

Moreover as we have that for all \(M \geq 2\):
\[
\alpha_{M,d} = \binom{d - 1 + M}{M} - \binom{d - 1 + M - 2}{M - 2}
\]

Then we have that:
\[
\sum_{i=0}^{M} \alpha_{i,d} \sim \frac{2M^{d-1}}{(d-1)!}
\]

and there exist \(Q_2 > 1 > Q_1 > 0\) constants such that:
\[
Q_1 M^{d-1} \leq \sum_{i=0}^{M} \alpha_{i,d} \leq Q_2 M^{d-1}
\]

Finally by considering the case where \(M = \frac{\log(C/\lambda)}{\log(1/r')}\) we obtain that:
\[
|E_{w}^{\lambda,q,Id}| \leq \prod_{i=1}^{w} Q_2 \left( \frac{\log(C/\lambda)}{\log(1/r')} \right)^{d-1}
\]
\[
\leq Q_2^{a_q} \left( \frac{\log(C_{1,q}^r/\lambda)}{\log(1/r')} \right)^{(d-1)w}
\]

Finally we obtain that:
\[
|E_{w}^{\lambda,q}| = \left( \binom{n}{w} \right) |E_{w}^{\lambda,q,Id}| \leq \left( \binom{n}{w} \right) Q_2^{a_q} \left( \frac{\log(C_{1,q}^r/\lambda)}{\log(1/r')} \right)^{(d-1)w}
\]  (29)

Moreover we have also:
\[
C_{2,q} \left( \frac{r}{4} \right)^{v_w} \geq \lambda \Rightarrow \mu(k_i, l_{ki}) \geq \lambda
\]

But we have that:
\[
C_{2,q} \left( \frac{r}{4} \right)^{v_w} \geq \lambda \Leftrightarrow v_w \leq \frac{\log(C_{2,q}/\lambda)}{\log(4/r)}
\]

Then we have that:
\[
|E_{w}^{\lambda,q,Id}| \geq \left\{ ((k_1, l_{k_1}, \ldots, (k_w, l_{k_w}), (0,0), \ldots, (0,0)) : 0 \leq k_i \leq \frac{\log(C_{2,q}/\lambda)}{\log(4/r)(w+1)} \text{ for } i = 1, \ldots, w \right\}
\]
And by the same reasoning as above we obtain that:

\[
|E_{w}^{\lambda, q, 1d}| \geq \prod_{i=1}^{w} Q_1 \left( \frac{\log(C_{2,q}/\lambda)}{\log(4/r)(w+1)} \right)^{d-1} \geq Q_1^{\alpha_q} \left( \frac{\log(C_{2,q}/\lambda)}{\log(4/r)(w+1)} \right)^{(d-1)w} \tag{30}
\]

Moreover thanks to eq. 29, 30 we obtain that:

\[
\sum_{w \in \{0,\ldots,a_q\}} \binom{n}{w} Q_1^{a_q} \left( \frac{\log(C_{2,q}/\lambda)}{\log(4/r)(w+1)} \right)^{(d-1)w} \leq |E_{w}^{\lambda, q}| \leq \sum_{w \in \{0,\ldots,a_q\}} \binom{n}{w} Q_2^{a_q} \left( \frac{\log(C_{1,q}/\lambda)}{\log(1/r')} \right)^{(d-1)w} \tag{31}
\]

Finally we have that:

\[
|E_{w}^{\lambda}| = \binom{n}{w} |E_{w}^{\lambda, q, 1d}| \geq \binom{n}{w} Q_1^{a_q} \left( \frac{\log(C_{2,q}/\lambda)}{\log(4/r)(w+1)} \right)^{(d-1)w}
\]

By denoting \(K_D := \max_{q=1,\ldots,D} C_{1,q}^{+}\) we finally obtain that:

\[
|E_{w}^{\lambda}| \leq 2^n Q_2^{\alpha q} D \left( \frac{\log((K_D cD)/\lambda)}{\log(1/r')} \right)^{(d-1)d^*}
\]

And also

\[
|E_{w}^{\lambda}| \geq E_{w}^{\lambda, D} \geq Q_1^{\alpha q} \left( \frac{\log(C_{2,D} d^*)/\lambda}{\log(4/r)(d^*+1)} \right)^{(d-1)d^*}
\]

Let now \(m \geq 1\) and let \(\lambda_m\) such that:

\[
2^n Q_2^{\alpha q} D \left( \frac{\log((K_D cD)/\lambda_m)}{\log(1/r')} \right)^{(d-1)d^*} = m
\]

Therefore by denoting \(\gamma = \frac{\log(1/r')}{(2^n Q_2^{\alpha q} D)^{(d-1)d^*}}\) and \(C_3 = K_D cD\), we obtain that:

\[
\lambda_m = C_3 e^{-\gamma m^{1/(d^*)}}
\]

And by definition of \(\eta_m\) we obtain that:

\[
\eta_m \leq C_3 e^{-\gamma m^{1/(d^*)}}
\]

Moreover by the exact same reasoning we obtain that:

\[
\eta_m \geq C_4 e^{-\gamma m^{1/(d^*)}}
\]

where \(q = \frac{\log(4/r)(d^*+1)}{Q_1^{\alpha q}/\log(1/r')}\) and \(C_4 = C_{2,D} d^*\). and the result follows. \(\square\)
We can now derive a sharp control of the eigenvalues of $T_\rho$ denoted $(\mu_m)_{m \geq 0}$ in the following. Let us first recall the two key assumptions to obtain a control on the eigenvalues of $T_\rho$. Indeed we have assumed that

$$\frac{d\nu}{\otimes_{i=1}^n d\sigma_{d-1}} < \omega \quad \text{and} \quad \frac{d\nu}{\otimes_{i=1}^n d\sigma_{d-1}} > h \quad (32)$$

Let $\rho \in \mathcal{G}_{\omega,\beta}$ and $\rho_v$ its marginal on $\mathcal{I}$. Let us first show that $I = \mathbb{N}$. Indeed as $\mathcal{I}$ is compact and $K_N$ continuous, the Mercer theorem guarantees that $H_N$ and $L^2_{\rho_v}(I)$ are isomorphic. Let us now define

$$T_\omega : L^2_{\rho_v}(\mathcal{I}) \rightarrow f \rightarrow \omega \int_{\mathcal{I}} K_N(x,.)f(x) \otimes_{i=1}^n d\sigma_{d-1}(x) - \int_{\mathcal{I}} K_N(x,.)f(x)d\rho_v(x)$$

Let us denote $E^k$, the span of the greatest $k$ eigenvalues strictly positive of $T_{\rho_v}$ with their multiplicities. Thanks to the min-max Courant-Fischer theorem we have that:

$$\mu_k = \max_{V \subset G_k} \min_{x \in V \setminus \{0\}} \langle T_{\rho_v}x, x \rangle_{L^2_{\rho_v}(\mathcal{I})}$$

where $G_k$ is the set of all s.e.v of dimension $k$ in $L^2_{\rho_v}(\mathcal{I})$. Therefore we have:

$$\eta_k \geq \frac{1}{\omega} \min_{x \in E^k(\{0\})} \frac{\langle \omega \times T_{K_N}x, x \rangle_{L^2_{\rho_v}(\mathcal{I})}}{\|x\|=1} = \frac{1}{\omega} \min_{x \in E^k(\{0\})} \left\{ \langle T_{\rho_v}x, x \rangle_{L^2_{\rho_v}(\mathcal{I})} + \langle T_\omega x, x \rangle_{L^2_{\rho_v}(\mathcal{I})} \right\} \geq \frac{1}{\omega} \min_{x \in E^k(\{0\})} \langle T_{\rho_v}x, x \rangle_{L^2_{\rho_v}(\mathcal{I})} + \frac{1}{\omega} \min_{x \in E^k(\{0\})} \langle T_\omega x, x \rangle_{L^2_{\rho_v}(\mathcal{I})}$$

Then if $T_\omega$ is positive we obtain that:

$$\eta_k \geq \frac{1}{\omega} \mu_k$$

Let us now show the positivity of $T_\omega$. Thanks to the assumption we have that for all $f \in L^2_{\rho_v}(\mathcal{I})$

$$T_\omega(f) = \int_{\mathcal{I}} \left[ \omega - \frac{d\rho_v}{\otimes_{i=1}^n d\sigma_{d-1}} \right] K_N(x,.)f(x) \otimes_{i=1}^n d\sigma_{d-1}(x)$$

Therefore $v := \omega - \frac{d\rho_v}{\otimes_{i=1}^n d\sigma_{d-1}(x)}$ is positive and by denoting $M = \int_{\mathcal{I}} v(x) \otimes_{i=1}^n d\sigma_{d-1}(x)$ and by re-scaling the above equality by $\frac{1}{M}$, we have that $V : x \rightarrow \frac{v(x)}{M}$ is a density function and by denoting $d\Gamma = V \otimes_{i=1}^n d\sigma_{d-1}$ we have:

$$\frac{1}{M} \times T_\omega(f) = \int_{\mathcal{I}} K_N(x,.)f(x)d\Gamma(x)$$

Therefore $T_\omega$ is positive and thanks to Proposition we have:

$$\mu_m \leq \omega \eta_m \leq \omega C_3 e^{-\gamma m \|\sigma_{d-1}\|_{\Gamma}}$$
Moreover if we assume in addition that the assumption 32, we obtain by an analogue reasoning that for all $k \geq 0$:

$$\eta_k \leq \frac{1}{h} \mu_k$$

And we have that for all $m \geq 0$:

$$hC_4 e^{-\gamma m \frac{1}{\|x\|^2}} \leq h\eta_m \leq \mu_m \leq \omega \eta_m \leq \omega C_3 e^{-\gamma m \frac{1}{\|x\|^2}} \quad (33)$$

**Upper rate.** Let us now show prove theorem 5.1. Let $w \geq 1$ and $0 < \beta \leq 2$ and let us denote $\alpha = (d - 1) \star d^*$ and $C_0 = \omega C_3$. From eq. (33) we have that for any $\rho \in G_{\omega,\beta}$, the eigenvalues, $(\mu_i)_{i \geq 0}$, of the integral operator $T_{\rho z}$ associated with $K_N$ fulfill the following upper bound for all $i$:

$$\mu_i \leq C_0 e^{-\gamma i / \alpha}$$

Therefore $G_{\omega,\beta} \subset F_{H_N,\alpha,\beta}$ and the result follows from Theorem 6.

**Lower rate.** Moreover let $0 < h < 1 \leq \omega$. To show the minimax-rate obtained in theorem 5.2, by denoting $c = hC_4$ we have in addition that for any $\rho \in G_{\omega,h,\beta}$, the eigenvalues, $(\mu_i)_{i \geq 0}$ of the integral operator $T_{\rho z}$ associated with $K_N$ fulfill the following lower bound for all $i$:

$$\mu_i \geq ce^{-\gamma i / \alpha}$$

then $G_{\omega,h,\beta} \subset F_{H_N,\alpha,q,\beta}$ and the result follows from theorem 7.

### D Useful Theorems

**Theorem 1.** [23] Let $\phi : X \to H$ be a feature map to a Hilbert space $H$, and let $K(z,z') := \langle \phi(z), \phi(z') \rangle_H$ a positive semi-definite kernel on $X$. Then $\mathcal{H} := \{f_\alpha : z \in X \to \langle \alpha, \phi(z) \rangle_H, \quad \alpha \in H\}$ endowed with the following norm:

$$\|f_\alpha\|^2 := \inf_{\alpha' \in H} \{\|\alpha'\|^2 \quad s.t \quad f_{\alpha'} = f_\alpha\}$$

is the RKHS associated to $K$.

**Theorem 2.** [9] Let $X$ be a compact metric space and $H$ be a separable Hilbert space such that there exists a continuous and injective map $\phi : X \to H$. Furthermore, let $f : \mathbb{R} \to \mathbb{R}$ be a function of the form:

$$f(x) = \sum_{m=0}^{\infty} a_m x^m. \quad (34)$$

If $a_m > 0$ for all $m \in \mathbb{N}$, then the following application:

$$k(x,x') := f(\langle \phi(x), \phi(x') \rangle_H) = \sum_{m \geq 0} a_m (\langle \phi(x), \phi(x') \rangle_H)^m$$

defines a c-universal kernel on $X$.

**Theorem 3.** [26] Let $0 < r \leq +\infty$ and $f : (-r,r) \to \mathbb{R}$ be a $C^\infty$-function that can be expanded into its Taylor series in 0, i.e.

$$f(x) = \sum_{m=0}^{\infty} a_m x^m.$$ 

Let $X := \{x \in \mathbb{R}^d : \|x\|_2 < \sqrt{r}\}$. If we have $a_n > 0$ for all $n \geq 0$ then $k(x,y) := f(x,y)$ defines a universal kernel on every compact subset of $X$. 

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Theorem 4. Each spherical harmonics of degree m, $Y_m \in H_m(S^{d-1})$, is an eigenfunction of $D_k$ with
associated eigenvalue given by the formula:

$$\lambda_m = \frac{d-2}{2^{n+1}} \sum_{s \geq 0} b_{2s+m} \frac{(2s+m)!}{(2s)!} \frac{\Gamma(s+1/2)}{\Gamma(s+m+d/2)}$$

Theorem 5. Consider the power series $\sum_{n \geq 0} a_n x^n$ with a radius of convergence $R_1$, and the power series
$\sum_{n \geq 0} b_n x^n$ with a radius of convergence $R_2$. Then whenever both of these power series convergent we have that

$$(\sum_{n \geq 0} a_n x^n)(\sum_{n \geq 0} b_n x^n) = \sum_{n \geq 0} c_n x^n$$

where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$. This power series has a radius of convergence $R$ such that $R \geq \min(R_1, R_2)$.

Theorem 6. Let $H$ be a separable RKHS on $X$ with respect to a bounded and measurable kernel $k$, $\alpha > 0$ and $2 \leq \beta > 0$. Then for any $f \in \mathcal{F}_{H,\alpha,\beta}$ and $\tau \geq 1$ we have:

- If $\beta > 1$, then for $\ell \geq \max\left(e^\beta, \frac{N}{\beta^2} \tau^\beta 2^\frac{\beta}{\beta-1} \log(\frac{\alpha}{\beta})\right)$ and $\lambda_\ell = \frac{1}{\ell^{1/\tau}}$, with a $\rho^\ell$-probability $\geq 1 - e^{-4\tau}$ it holds

  $$\|f_{H,\xi,\lambda} - \rho\|^2 \leq 3C\tau^2 \frac{\log(\ell)^\alpha}{\ell}$$

- If $\beta = 1$, then for $\ell \geq \max\left(\exp\left((N\tau)^\frac{1}{\alpha-\beta}\right), e^{1/\log(\ell)\lambda_\ell}\right)$ and $\mu > 0$, with a $\rho^\ell$-probability $\geq 1 - e^{-4\tau}$ it holds

  $$\|f_{H,\xi,\lambda_\ell} - \rho\|^2 \leq 3C\tau^2 \frac{\log(\ell)^\mu}{\ell^{1/2}}$$

- If $\beta < 1$, then for $\ell \geq \max\left(\exp\left((N\tau)^\frac{1}{\alpha-\beta}\right), e^{1/\log(\ell)\lambda_\ell}\right)$, with a $\rho^\ell$-probability $\geq 1 - e^{-4\tau}$ it holds

  $$\|f_{H,\xi,\lambda_\ell} - \rho\|^2 \leq 3C\tau^2 \frac{\log(\ell)^\beta}{\ell^{1/2}}$$

where $N = \max(256KQ, 16K, 1)$, $C = 2 \max(B, 128V \max(5Q, K)), V = \max(L^2, \sigma^2, 2BK + 2B_8)$, $Q = \left(\frac{1}{\lambda}\right)^\alpha \left[1 + C_0 \int_1^\infty \frac{(\log(u)+1)^{\alpha-1}}{C_0 u + u^2} \, du\right]$ and $K = \sup_{x \in X} k(x, x)$.

Theorem 7. Let $H$ be a separable RKHS on $X$ with respect to a bounded and measurable kernel $k$, $q \geq \gamma > 0$, $\alpha > 0$, $0 < \beta \leq 2$ such that $\mathcal{P}_{H,\alpha,\beta}$ is not empty. Then it holds

$$\lim_{\tau \to 0^+} \lim_{\ell \to \infty} \inf_{\rho \in \mathcal{F}_{H,\alpha,\beta}} \sup_{z : \|f_z - \rho\|^2 > \tau b_\ell} \rho^\ell(z) = 1$$

where $b_\ell = \frac{\log(\ell)^\alpha}{\ell}$. The infimum is taken over all measurable learning methods with respect to $\mathcal{F}_{H,\alpha,\beta}$.

E Technical Lemmas

Lemma 3. Let $k$ be a kernel on the metric space $(X, d)$ and $\phi : X \to H$ be a feature map of $k$. Then $k$
is continuous if and only if $\phi$ is continuous.

Lemma 4. Every feature map of a universal kernel is injective.
E.1 Proof of Lemma \[\text{[1]}\]

Proof. Let us show the result by induction on \(N\). For \(N = 1\) the result is clear as \(f_1\) can be expand in its Taylor series in 0 on \(\mathbb{R}\) with positive coefficients. Let \(N \geq 2\), therefore we have

\[g(t) = (f_N \circ ... \circ f_2) \circ (f_1(t))\]

By induction, we have that for all \(t \in \mathbb{R}\):

\[f_N \circ ... \circ f_2(t) = \sum_{l_2, ..., l_N \geq 0} \frac{f_N^{(l_N)}(0)}{l_N!} \times \phi_{N-1}(l_N, l_{N-1})... \times \phi_2(l_3, l_2)(f_1(t))^{l_2}\]

Therefore we have that:

\[g(t) = \sum_{l_2, ..., l_N \geq 0} \frac{f_N^{(l_N)}(0)}{l_N!} \times \phi_{N-1}(l_N, l_{N-1})... \times \phi_2(l_3, l_2)(f_1(t))^{l_2}\]

Moreover, for all \(n \geq 0\), \(f_1^n\) can be expand in its Taylor series in 0 on \(\mathbb{R}\) with non-negative coefficients, and we have that for all \(n \geq 0\) and \(t \in \mathbb{R}\):

\[(f_1(t))^n = \sum_{l_1 \geq 0} \phi_1(n, l_1)t^{l_1}\]

And we obtain that:

\[g(t) = \sum_{l_1, ..., l_N \geq 0} \frac{f_N^{(l_N)}(0)}{l_N!} \times \phi_{N-1}(l_N, l_{N-1})... \times \phi_1(l_2, l_1)t^{l_1}\]

Finally we have by unicity of the Taylor decomposition that for all \(l_1 \geq 0\):

\[\frac{g^{(l_1)}(0)}{l_1!} = \sum_{l_2, ..., l_N \geq 0} \frac{f_N^{(l_N)}(0)}{l_N!} \times \phi_{N-1}(l_N, l_{N-1})... \times \phi_1(l_2, l_1)\]

Moreover let \(k \in \{1, ..., N - 1\}\), \(l \geq 1\) and let us denote \((a_k^l)_{i \geq 0}\) the coefficients in the Taylor decomposition of \(f_k\). Then we have:

\[f_k^l(t) = \sum_{n_1, ..., n_i \geq 0} \prod_{i=1}^l (a_k^{n_i}) x^{n_1 + ... + n_i}\]

\[= \sum_{q \geq 0} \left[ \sum_{n_1, ..., n_i \geq 0} \prod_{i=1}^l (a_k^{n_i}) \right] x^q\]

But as \(a_k^l > 0\) for all \(i \geq 0\) and by unicity of the Taylor decomposition, we obtain that for all \(m \geq 0\):

\[\phi_k(l, m) > 0\]

and the result follows.
E.2 Proof of Lemma 2

Proof. Recall that for all \( m \geq 0 \), \( b_m := \frac{d^m}{dt^m}|_{t=0} f_1^m \). Let us now show the result by induction on \( \alpha \). For \( \alpha = 1 \), the result comes directly from eq. 28. Let now \( \alpha \geq 1 \) and \( m \geq 0 \), then we have:

\[
\frac{d^m}{dt^m}|_{t=0} f_1^{\alpha+1} = \frac{1}{m!} m \sum_{k=0}^{m} \frac{(m)_k}{k!} \frac{d^k}{dt^k}|_{t=0} f_1^\alpha \frac{d^k}{dt^k}|_{t=0} f_1
\]

Moreover by induction we have for all \( 1 \leq q \leq \alpha \) and \( k \geq 0 \):

\[
c_2^{q} r^k \leq \frac{d^k}{dt^k}|_{t=0} f_1^q \leq 2c_1^{q}(k+1)^{-1} r^k
\]

Therefore we have:

\[
\sum_{k=0}^{m} c_2^{\alpha-1} r^k \leq \frac{d^m}{dt^m}|_{t=0} f_1^{\alpha+1} \leq \sum_{k=0}^{m} c_1^{\alpha}(k+1)^{\alpha-1} r^k \leq c_1^{\alpha+1} r^m \sum_{k=0}^{m} (k+1)^{\alpha-1} \leq c_1^{\alpha+1} r^m \sum_{k=0}^{m+1} k^{\alpha-1}
\]

Moreover a clear induction give us for all \( m \geq 1 \) and \( \alpha \geq 1 \)

\[
\sum_{k=1}^{m} k^{\alpha-1} \leq m^\alpha
\]

Indeed for \( \alpha = 1 \) the result is clear and for all \( m \geq 1 \) and \( \alpha \geq 1 \) we have

\[
\sum_{k=1}^{m} k^{\alpha} \leq m \times \sum_{k=1}^{m} k^{\alpha-1}
\leq m^{\alpha+1}
\]

The last inequality is obtained by induction on \( \alpha \geq 1 \). Therefore we have for all \( m \geq 0 \)

\[
c_2^{\alpha+1} r^m \leq \frac{d^m}{dt^m}|_{t=0} f_1^{\alpha+1} \leq c_1^{\alpha+1} (m+1)^\alpha r^m
\]

and the result follows. \( \square \)