A GEOMETRICAL CORRESPONDENCE BETWEEN MAXIMAL SURFACES IN ANTI-DE SITTER SPACE-TIME AND MINIMAL SURFACES IN $H^2 \times \mathbb{R}$

FRANCISCO TORRALBO

ABSTRACT. A geometrical correspondence between maximal surfaces in anti-De Sitter space-time and minimal surfaces in the Riemannian product of the hyperbolic plane and the real line is established. New examples of maximal surfaces in anti-De Sitter space-time are obtained in order to illustrate this correspondence.

1. Introduction

The study of minimal surfaces in product spaces $M^2 \times \mathbb{R}$ was initiated by Rosenberg and Meeks [Ros02, MR05] and has been very active since then. Among that spaces, there are three homogeneous Riemannian manifolds: $\mathbb{R}^3$, where the classical theory of minimal surfaces has been developed, and $S^2 \times \mathbb{R}$ and $H^2 \times \mathbb{R}$, where many authors have been actively working. Giving a complete list of references in the subject is far from being possible so we will only mention a few of them: Nelli and Rosenberg [NR02] proved a Jenkins-Serrin-type theorem in $H^2 \times \mathbb{R}$, Hauswirth [Hau06] constructed minimal examples or Riemann type, Sá Earp and Tobiana [ST04] investigated the screw motion invariant surfaces in $H^2 \times \mathbb{R}$, Daniel [Dan09] and Hauswirth, Sá Earp and Tobiana [HST08] showed, independently, the existence of an associated family of minimal immersions for simply connected minimal surfaces in $S^2 \times \mathbb{R}$ and $H^2 \times \mathbb{R}$, Urbano and the author [TU13] tackled a general study of minimal surfaces in $S^2 \times S^2$ with applications to $S^2 \times \mathbb{R}$, and very recently Manzano, Plehnert and the author [MPT13] constructed orientable and non-orientable even Euler characteristic embedded minimal surfaces in the quotient $S^2 \times S^1$ and Martín, Mazzeo and Rodríguez [MMR14] constructed the first examples of complete, properly embedded minimal surfaces in $H^2 \times \mathbb{R}$ with finite total curvature and positive genus.

In this paper we are going to show a geometric relation between maximal surfaces in anti-De Sitter space-time $H^3_1$ and minimal immersions in $H^2 \times \mathbb{R}$. It is well-known that the Gauss map of a spacelike maximal immersion in $H^3_1$ is always a minimal Lagrangian immersion in $H^2 \times H^2$. 

2000 Mathematics Subject Classification. Primary 53C42; Secondary 53C40.

Key words and phrases. Surfaces, minimal, complex surfaces.

Research partially supported by a MCyT-Feder research project MTM2011-22547, Junta Andalucía Grants P09-FQM-5088 and P09-FQM-4496 and Belgian Interuniversity Attraction Pole P07/18 (Dygest).
2 FRANCISCO TORRALBO

(see [Tor07] and also [CU07] where an analogous case for Lagrangian minimal immersions in $S^2 \times S^2$ is studied). We are going to get new minimal immersions in $H^2 \times H^2$ by pairing different components of the Gauss map of two suitable maximal immersions in anti-De Sitter space-time (see Theorem [1]). This construction is in the same spirit as in [TU13]. From that, and under an appropriate choice of one element in the pair, we will establish a conformal correspondence between maximal immersions in anti-De Sitter space-time and minimal immersions in $H^2 \times R$ (see Corollary [1]). We will also show that this result admits a local converse, i.e. that roughly speaking every minimal surface in $H^2 \times R$ is locally the Gauss map of a maximal immersion in anti-De Sitter space (see Theorem [2]).

Finally, we will illustrate this geometric correspondence by showing new examples of maximal surfaces in anti-De Sitter space in Proposition [1] and computing their Gauss map (in the sense of Corollary [1]). This will provide us with two 1-parameter families of minimal examples in $H^2 \times R$.

We want to point out that, although the constructed maximal immersions in $H^3_1$ are non-complete (see Proposition [1]), their corresponding minimal immersions in $H^2 \times R$ induce complete metrics in the surface. Moreover, they are invariant by screw motions and they were first described in [ST04].

The structure of the paper is the following: Section 2 introduces both anti-De Sitter space-time $H^3_1$ and the Riemannian products $H^2 \times H^2$ and $H^2 \times R$, where $H^2$ stands for the hyperbolic plane. In Section 3, we will briefly present some basic facts about maximal surfaces in anti-De Sitter space as well as some examples. Section 4 contains the main theorems that are illustrated in Section 5. Finally, Section 6 contains an analysis of the solutions to the sinh-Gordon equation that only depend on one variable.

2. Preliminaries

2.1. The hyperbolic plane and the product manifold $H^2 \times H^2$. Let $H^2$ be the hyperbolic plane and $\langle , \rangle$ its metric. Although for all computations we will use the hyperboloid model of $H^2$, i.e. $H^2 = \{ p \in \mathbb{R}^3 : \langle p, p \rangle = -1, p_1 > 0 \}$, where $\langle x, y \rangle = -x_1 y_1 + x_2 y_2 + x_3 y_3$, the Poincaré disc model is also considered in Figures 1 and 2.

We endow $H^2 \times H^2$ with the product metric, also denoted by $\langle , \rangle$. So $H^2 \times H^2$ is an Einstein manifold with constant scalar curvature $-4$.

We will consider $H^2 \times R$ as the totally geodesic submanifold of $H^2 \times H^2$ given by the image of the map $i : H^2 \times R \to H^2 \times H^2$ defined by $i(p, t) = \{ p, (\sinh(t), 0, \cosh(t)) \}$.

2.2. The anti-De Sitter 3-space. The anti-De Sitter 3-space, that it is usually denoted by $H^3_1$, is a Lorentz manifold of dimension 3 and constant curvature $-1$. It is defined as a quadric in a vector space. More precisely, let $\mathbb{R}^4_2$ be the euclidean 4-space endow with the metric $\langle u, v \rangle = u_1 v_1 + u_2 v_2 - u_3 v_3 - u_4 v_4$. Then $H^3_1 = \{ p \in \mathbb{R}^4_2 : \langle p, p \rangle = -1 \}$. Moreover, the map $\pi : H^3_1 \subset \mathbb{R}^4_2 \equiv \mathbb{C}^4_1 \to H^2(-2)$, where $H^2(c)$ stands for the
hyperbolic plane of constant curvature $c < 0$, given by
\[ \pi(z, w) = \left( zw, \frac{1}{2}(|z|^2 + |w|^2) \right), \]
is a semi-Riemannian submersion with totally geodesic fibers generated by the unit temporal vector field $\xi(z, w) = (iz, iw)$. The fiber of a point $(z_0, w_0) \in H^3_1$ is the circle $(z_0t, w_0t)$.

2.3. The Gauss map. Let $\phi : \Sigma \to \mathbb{R}^4_1$ be a spacelike immersion of an oriented surface $\Sigma$. Its Gauss map assigns to each point of the surface its mean curvature, of an oriented surface $\Sigma$ (see for instance [Pal90]) that the 2-differential $\Theta_{\phi}(z) = \theta(z)dz \otimes dz = \langle \xi(z), N(z) \rangle dz \otimes dz$ is holomorphic, where $N$ is the (timelike) unit normal vector field to $\phi$ such that $\{\phi_x, \phi_y, \phi, N\}$ is a positively oriented frame in

\[ g(v \wedge w, v' \wedge w') = \langle (v, v') \rangle \langle (w, w') \rangle - \langle (v, v') \rangle \langle (w, w') \rangle. \]
The star operator $\ast : \Lambda^2 \mathbb{R}^4_2 \to \Lambda^2 \mathbb{R}^4_2$ defined by $a \wedge (\ast b) = g(a, b)\Omega$ for all $a, b \in \Lambda^2 \mathbb{R}^4_2$, where $\Omega$ is the orientation form of $\mathbb{R}^4$, is an automorphism of $\Lambda^2 \mathbb{R}^4_2$ with eigenvalues $\pm 1$. Consider $\Lambda^2_\pm \mathbb{R}^4_2$ the eigenspaces of $\ast$ associated to the eigenvalues $\pm 1$. Observe that we can decompose $\Lambda^2 \mathbb{R}^4_2 = \Lambda^2_+ \mathbb{R}^4_2 \oplus \Lambda^2_- \mathbb{R}^4_2$. Let $\{e_1, e_2, e_3, e_4\}$ be an oriented orthonormal frame of $\mathbb{R}^4_2$, i.e. $|e_1|^2 = |e_2|^2 = - |e_3|^2 = - |e_4|^2 = 1$ and $\langle (e_i, e_j) \rangle = 0$, $i \neq j$.

The frame $\{E_j : j = 1, 2, 3\}$ given by:
\[ E_1 = \frac{1}{\sqrt{2}} (e_1 \pm e_2 \pm e_4 \pm e_3), \]
is an orthonormal oriented reference in $\Lambda^2_\pm \mathbb{R}^4_2$, i.e. $g(E_i \wedge E_j) = \delta_{ij}$, where $e_1 = -1$ and $e_2 = e_3 = 1$. Hence each $\Lambda^2_\pm \mathbb{R}^4_2$ is isometric to the Lorentz-Minkowski 3-space. We denote by $H^1_\pm$ the hyperbolic plane in the 3-space $\Lambda^2_\pm \mathbb{R}^4_2$.

Finally, if $\{v, w\}$ is an oriented orthonormal frame of a plane $P \in G^+_\ast(2, 4)$, then the map $G^+_\ast(2, 4) \to H^1_+ \times H^1_-$ given by
\[ P \mapsto \frac{1}{\sqrt{2}} [v \wedge w + \ast (v \wedge w), v \wedge w - \ast (v \wedge w)] \]
is a diffeomorphism.

3. Maximal surfaces in anti-De Sitter space

Let $\phi : \Sigma \to H_1$ be a spacelike maximal immersion, i.e. with zero mean curvature, of an oriented surface $\Sigma$ and $N$ a unit normal vector field to $\phi$. Given a conformal parameter $z = x + iy$ on $\Sigma$, it is well-known (see for instance [Pal90]) that the 2-differential $\Theta_{\phi}(z) = \theta(z)dz \otimes dz = \langle \xi(z), N(z) \rangle dz \otimes dz$ is holomorphic, where $N$ is the (timelike) unit normal vector field to $\phi$ such that $\{\phi_x, \phi_y, \phi, N\}$ is a positively oriented frame in
\( \mathbb{R}^4_2 \) (we are using subscripts to indicate derivatives). The associated conformal factor \( e^{2\varphi} \) satisfies \( \sigma_{zz} + e^{-2\varphi} |\Theta_{\varphi}|^2 - \frac{1}{4} e^{2\varphi} = 0 \). Moreover, the Frenet equations of the immersion are given by
\[
\phi_{zz} = 2\varphi_z \varphi_z + \theta N, \quad \phi_{zz} = \frac{1}{2} e^{2\varphi} \varphi_z, \quad N_z = 2e^{-2\varphi} \theta \varphi_z.
\] (3.1)

Conversely, we get the following result (see [Pal90, Proposition 2.1] and also [Per09, Lemma 3.3]):

For any solution \( \varphi : D \to \mathbb{R}, D \subset \mathbb{C} \), to the equation \( \sigma_{zz} - \frac{1}{2} \sinh(2\varphi) = 0 \) there exists a 1-parameter family \( \phi_t : \mathbb{C} \to \mathbb{H}^3_1 \) of maximal immersions whose induced metric is \( e^{2\varphi} |dz|^2 \) and whose Hopf differential is \( \Theta_{\varphi_t}(z) = \frac{i}{2} e^{\varphi_t} dz \otimes dz \).

In this section we are going to present some examples of spacelike maximal surfaces in \( \mathbb{H}^3_1 \) that will be useful in the sequel. The first simple example is the totally geodesic embedding of the hyperbolic plane \( \mathbb{H}^2 \) into \( \mathbb{H}^3_1 \) given by \( \mathcal{B} = \{ (z, w) \in \mathbb{R}^4_2 \cong \mathbb{C}^2 : \text{Im}(w) = 0 \} \), up to isometries of \( \mathbb{H}^3_1 \).

The second example, that will play an important role in the following section (see Corollary [1]), is the so-called hyperbolic cylinder
\[
\mathcal{C} = \{ (z, w) \in \mathbb{R}^4_2 \cong \mathbb{C}^2 : \text{Re}(z)^2 - \text{Re}(w)^2 = \text{Im}(z)^2 - \text{Im}(w)^2 = -\frac{1}{2} \}.
\]

It is a complete spacelike maximal surface with vanishing Gauss curvature, constant principal curvatures \( \lambda_1 = -\lambda_2 = 1 \) and the norm of the second fundamental form is \( |\sigma|^2 = 2 \). It was characterized by Ishihara [Ish88] as the only complete maximal surface in \( \mathbb{H}^3_1 \), up to rigid motions, with \( |\sigma|^2 = 2 \). We can parametrize the hyperbolic cylinder \( \mathcal{C} \) by
\[
(3.2) \quad \psi_t(x, y) = \frac{1}{\sqrt{2}} (\sinh a_t(x, y), \sinh b_t(x, y), \cosh a_t(x, y), \cosh b_t(x, y)),
\]
where
\[
a_t(x, y) = (x + y) \cos \frac{t}{2} + (x - y) \sin \frac{t}{2}, \quad b_t(x, y) = (y - x) \cos \frac{t}{2} + (x + y) \sin \frac{t}{2}.
\]

Then \( \psi_t(\mathbb{R}^2) = \mathcal{C}, z = x + iy \) is a conformal parameter with conformal factor \( e^{2u(x,y)} \) where \( u(x,y) = 0 \), and the associated Hopf differential is \( \Theta_{\psi_t} = \frac{i}{2} e^{u} dz \otimes dz \).

Next we are going to show examples invariant under a 1-parameter group of isometries of \( \mathbb{H}^3_1 \). In that case, the equation for its conformal factor \( \sigma_{zz} - \frac{1}{2} \sinh(2\varphi) = 0 \) becomes an ordinary differential equation
\[
(3.3) \quad \varphi''(x) - 2 \sinh(2\varphi(x)) = 0, \quad \text{with energy } E = \frac{1}{2} \varphi'(x)^2 - \cosh(2\varphi(x)).
\]

It is possible to integrate explicitly the Frenet system (3.1) for some values of \( E \) obtaining the following result (see also Section [9] where we get all the solutions to the previous equation in terms of Jacobi elliptic functions).
Proposition 1. Let $v : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a solution to (3.3) with energy $E$. Then the map $\phi_E : \Sigma_v = (I \times \mathbb{R}, e^{2v}g_0) \rightarrow H^3_1$ given by:

$$
\phi_E(x, y) = \frac{1}{\sqrt{2E}} \left( e^{v(x)} \cos(\sqrt{2Ey}), -e^{v(x)} \sin(\sqrt{2Ey}), -\sqrt{2E + e^{2v(x)} \cos(\sqrt{2EG(x)})}, -\sqrt{2E + e^{2v(x)} \sin(\sqrt{2EG(x)})} \right), \quad E > 0,
$$

$$
\phi_E(x, y) = \frac{1}{\sqrt{-2E}} \left( \sqrt{-2E - e^{2v(x)} \sinh(\sqrt{-2EG(x)})}, e^{v(x)} \sinh(\sqrt{-2Ey}) \right), \quad E < 0,
$$

is an isometric maximal immersion with associated Hopf differential $\Theta(z) = \frac{1}{2} dz \otimes dz$, where $G(x) = \int_0^x \frac{dt}{\sqrt{2E + e^{2v(t)}}}$ and $g_0$ is the Euclidean metric in $\mathbb{R}^2$.

Moreover, all the surfaces $\Sigma_v$, except the one associated to the trivial solution $v(x) = 0$ which is the hyperbolic cylinder, are not complete.

Remark 1. The obtained examples in Proposition 1 are invariant by the following 1-parameter group of isometries depending on the sign of $E$:

$$
E > 0: \quad \begin{pmatrix} \cos \theta & -\sin \theta & 0 & 0 \\ \sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad E < 0: \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cosh \theta & \sinh \theta & 0 \\ 0 & \sinh \theta & \cosh \theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

Besides, when $E < 0$ the immersion $\phi_E$ is only defined for $2E < -e^{2v(x)}$.

A deep analysis of the solutions (see Section 6) shows that there is always a non-empty interval $I' \subseteq I$ where this happens, being $I$ the maximal interval of definition of $v$ (see also the proof of Proposition 1).

Remark 2. It is not surprising that the obtained examples are not complete. In [Per09], the author pointed out the difficulties of getting complete maximal immersions in anti-De Sitter space-time and he also provided with complete examples looking for radial solutions of the sinh-Gordon equation. Palmer in [Pal90, Theorem 1] also provide complete minimal examples in $H^3_1$ in terms of holomorphic quadratic differentials.

Proof. Taking equation (3.3) into account, it is straightforward to check that $\phi_E$ is a maximal isometric immersion. We will now show that the metric $e^{2v}g_0$ is non-complete except for the trivial solution $v = 0$, by finding a divergent curve $\gamma$ in $\Sigma$ with finite length. Notice that we can consider the solutions to (3.3) given in Lemma 1 for $a_0 = 0$, which are always defined in an interval $I = [0, \ell]$. Observe that if $v$ is a solution then $-v$ is also a solution (see Section 9) so we have to deal with both cases.

If $E > -1$, thanks to the symmetries of the solutions (see Remark 5.1), $v$ and $-v$ are symmetric so, considering the one given in Lemma 1 we have that $e^{v(x)} \leq 1$ in $[0, \ell/2]$. Hence the curve $\gamma : [0, \ell/2] \rightarrow \Sigma$, given by $\gamma(t) = \frac{\ell}{2} - t$, diverges in $\Sigma$ but has finite length.

If $E = -1$ we have three different solutions: (1) $v(x) = 0$, which produces the hyperbolic cylinder which is complete; (2) $v(x) = \log \tanh(x)$,
in this case $\Sigma = (\mathbb{R}^+ \times \mathbb{R}, \tanh^2(x)g_0)$ and so the curve $\gamma(t) = (a - t, 0)$, $t \in ]0, a]$, $a \in \mathbb{R}$ arbitrary, diverges in $\Sigma$ but has finite length; and (3) $v(x) = \log \cotanh(x)$. In this case the immersion is only defined when $\cosh^2(x) < 2$ (see Remark 1), i.e. $\Sigma = (]0, \cosh(\sqrt{2})[ \times \mathbb{R}, \cosh^2(x)g_0)$. Hence the curve $\gamma(t) = (t, 0)$, $t \in ]\frac{1}{2}, \cosh(\sqrt{2})[\)$ diverges in $\Sigma$ but has finite length.

Finally, if $E < -1$ we have two different types of solutions, namely $v_1(x) = -\log(\frac{1}{t} \sinh(\lambda x))$ and $v_2(x) = -v_1(x)$ (see Lemma 1). In the first case the immersion is only defined when $2E + e^{2v_1(x)} < 0$, i.e. for $x \in J = [c, \ell - c[$ where $c = \frac{1}{\lambda} \arcsinh(\frac{\lambda}{\sqrt{-2E}})$ (see Lemma 1) for the definition of $\lambda$ and $\mu$). But $e^{v_1(x)} \leq \sqrt{-2E}$ for $x \in ]c, \ell - c[$ and so $\Sigma$ is also incomplete in this case.

In the second case, $2E + e^{2v_2(x)} < 0$ so $\Sigma = (]0, \ell[ \times \mathbb{R}, e^{2v_2(x)}g_0)$, but $e^{2v_2(x)} \leq \frac{1}{t}$ so the curve $\gamma(t) = (\ell/2 - t, 0)$, $t \in ]0, \ell/2[\) diverges in $\Sigma$ and has finite length. \hfill \Box

4. The Gauss map of a pair of maximal surfaces in $\mathbb{H}^3_3$

Let $\phi : \Sigma \rightarrow \mathbb{H}^2_3 \subset \mathbb{R}^4_2$ be a spacelike immersion of an oriented surface $\Sigma$. The Gauss map of $\phi : \Sigma \rightarrow \mathbb{H}^2_3$ is the map $\nu_\phi = (\nu^+_\phi, \nu^-_\phi) : \Sigma \rightarrow \mathbb{H}^2_+ \times \mathbb{H}^-_3$ defined by

$$\nu^\pm_\phi(p) = \frac{1}{\sqrt{2}}[e_1 \wedge e_2 \pm N(p) \wedge \phi(p)],$$

where $\{e_1, e_2\}$ is an oriented orthonormal basis in $T_p\Sigma$ and $N$ is the unit (timelike) normal vector field to the immersion $\phi$ such that $\{e_1, e_2, \phi(p), N_p\}$ is oriented in $\mathbb{R}^4_2$ (see Section 2.3).

If $\phi$ is maximal then its Gauss map is a Lagrangian minimal immersion in $\mathbb{H}^2_+ \times \mathbb{H}^-_3$ (see [Tor07]). For instance:

- The Gauss map of the totally geodesic embedding of the hyperbolic plane in $\mathbb{H}^3_3$ given in Section 3.5 is the diagonal map $\nu : \mathbb{H}^2 \rightarrow \mathbb{H}^2_+ \times \mathbb{H}^-_3$, $\nu(p) = (p, p)$.
- The Gauss map of the hyperbolic cylinder is the product of two geodesics of $\mathbb{H}^2$.

**Theorem 1.** Let $\Sigma$ be a Riemann surface and $\phi, \psi : \Sigma \rightarrow \mathbb{H}^3_3$ two conformal spacelike maximal immersions with the same Hopf differentials $\Theta_\phi = \Theta_\psi$. Then $$\nu_{\{\phi, \psi\}} : (\nu^+_\phi, \nu^-_\phi) : \Sigma \rightarrow \mathbb{H}^2_+ \times \mathbb{H}^-_3$$ is a conformal minimal immersion. Moreover, the induced metric by $\nu_{\{\phi, \psi\}}$ is

$$g = \frac{1}{2} \left[ (2 + |\sigma_\phi|^2)g_\phi + (2 + |\sigma_\psi|^2)g_\psi \right],$$

where $g_\phi$ and $g_\psi$ are the induced metrics on $\Sigma$ by $\phi$ and $\psi$, respectively. Here $|\sigma_\phi|$ and $|\sigma_\psi|$ are the lengths of the second fundamental forms of $\phi$ and $\psi$ in $\mathbb{H}^3_3$, computed with respect to $g_\phi$ and $g_\psi$, respectively.

**Remark 3.**

1. If $\phi = \psi$, then $\nu_{\{\phi, \psi\}} = \nu_\phi$ is the Gauss map of $\phi$. 
(2) Given a maximal immersion \( \phi : \Sigma \to \mathbb{H}^3 \), its polar immersion (possibly branched) is \( N : \Sigma \to \mathbb{H}^3 \), where \( N \) is a unit normal vector field to \( \phi \). \( N \) is also a maximal conformal immersion with the same Hopf differential as \( \phi \). Nevertheless, \( \nu_{(\phi,N)} \) is congruent to \( \nu_{\phi} \), the Gauss map of \( \phi \).

(3) Given \( A \in O_2(4) \), then it is easy to check that \( \nu_{(A\phi,\psi)} \) is congruent to \( \nu_{(\phi,\psi)} \).

**Proof.** The immersion \( \phi : \Sigma \to \mathbb{R}^4_1 \) has parallel mean curvature vector because it is contained in \( \mathbb{H}^3_1 \) as a maximal surface. From [Pal91, Theorem 3.2] we deduce that each \( \nu_{\phi} \) is a harmonic map. Analogously \( \nu_{\psi} : \Sigma \to \mathbb{H}^2 \) are also harmonic maps. Hence \( \nu_{(\phi,\psi)} = (\nu_{\phi}^{+},\nu_{\phi}^{-}) : \Sigma \to \mathbb{H}^2 \times \mathbb{H}^2 \) is a harmonic map.

It remains to check that \( \nu \) is conformal (and so minimal). Let \( z = x + iy \) a conformal parameter over \( \Sigma \) and \( N_{\phi}, N_{\psi} \) the temporal unit normal vector field to \( \phi \) and \( \psi \) respectively such that \( \{\phi_x, \phi_y, \phi, N_{\phi}\} \) and \( \{\psi_x, \psi_y, \psi, N_{\psi}\} \) are oriented references in \( \mathbb{R}^4_1 \). Since \( \phi \) and \( \psi \) are conformal immersions the induced metrics by \( \phi \) and \( \psi \) in \( \Sigma \) are given by \( g_{\phi} = e^{2u}|dz|^2 \) and \( g_{\psi} = e^{2w}|dz|^2 \) for certain functions \( u \) and \( w \). Moreover, \( \Theta_{\phi} = \Theta_{\psi} = \theta dz \otimes dz \) for some function \( \theta(z) \) by hypothesis.

We can express the component of the Gauss map \( \nu_{(\phi,\psi)} \) as

\[
\nu_{\phi}^{+}(z) = \frac{1}{\sqrt{2}}(-2ie^{-2u}\phi_z \wedge \phi - \phi \wedge N_{\phi}), \\
\nu_{\phi}^{-}(z) = \frac{1}{\sqrt{2}}(-2ie^{-2w}\psi_z \wedge \psi + \psi \wedge N_{\phi}).
\]

Taking the Frenet equations (3.1) of \( \phi \) and \( \psi \) and Section 2.3 into account, we easily get that

\[
(v^{\phi}_{\phi})_z = \frac{1}{2}e^{2u}(-i + 2\theta e^{-2u})E_2(z) + \frac{i}{2}e^{2u}(i + 2\theta e^{-2w})E_3(z), \\
(v^{\phi}_{\psi})_z = \frac{1}{2}e^{2w}(-i - 2\theta e^{-2w})E_2(z) + \frac{1}{2}e^{2w}(i + 2\theta e^{-2w})E_3(z).
\]

Then we deduce from the previous equations that

\[
(\langle (v^{\phi}_{\phi})_z, (v^{\phi}_{\psi})_z \rangle) = -2i\theta, \\
(\langle (v^{\phi}_{\phi})_z, (v^{\phi}_{\phi})_z \rangle) = 2i\theta,
\]

\[
|v^{\phi}_{\phi}|^2 = \frac{1}{2}(e^{2u} + 4e^{-2w}|\theta|^2), \\
|v^{\phi}_{\psi}|^2 = \frac{1}{2}(e^{2w} + 4e^{-2w}|\theta|^2).
\]

Finally, \( \langle \nu_z, \nu_z \rangle = 0 \) and so \( \nu = \nu_{(\phi,\psi)} \) is a conformal map. Moreover, from

\[
8|\theta|^2 = e^{4u}|c_{\phi}|^2 = e^{4w}|c_{\psi}|^2
\]

we get the expression of the induced metric on \( \Sigma \) by \( \nu \).

Now, let \( \phi : \Sigma \to \mathbb{H}^3_1 \) be a conformal maximal immersion. There is no loss of generality in assuming that locally the Hopf differential \( \Theta(z) = \frac{i}{2}e^{i\theta}dz \otimes dz \) (observe that either \( \Theta = 0 \) and so the immersion is totally geodesic or the zeroes of \( \Theta \) are isolated and we can locally normalize \( \Theta \) away from the zeroes). Then \( \phi \) and \( \psi_z \), the immersion of the hyperbolic cylinder given in Section 3, are two conformal maximal immersions with
the same Hopf differentials. Then, thanks to the previous theorem, \( \hat{v}_{\phi} = v_{(\phi, \psi)} : \Sigma \to H^2 \times H^2 \) is a minimal immersion that we call the modified Gauss map of \( \phi \). Now, the Gauss map of the hyperbolic cylinder \( \psi_t \) is the product of two geodesics in \( H^2 \times H^2 \) so its second component can be viewed as a map from \( \Sigma \) to \( R \). Hence the modified Gauss map of a maximal immersion \( \phi : \Sigma \to H^3_1 \) is a conformal minimal immersion \( \hat{v}_{\phi} : \Sigma \to H^2 \times R \).

We get the following result:

**Corollary 1.** Let \( u : D \subseteq \mathbb{C} \to R \) be a solution of \( u_{zz} - \frac{1}{2} \sinh(2u) = 0 \). Then the 1-parameter family of minimal immersions \( \Phi_t : (D, 4 \cosh^2 u |dz|^2) \to H^2 \times R \) with Hopf differential \( \Theta = e^u dz \otimes dz \) (see [HST08, Corollary 10]) associated to \( u \) is given by:

\[
\Phi_t(z) = (v_{\phi_t}^{-1}(z), 2 \text{Im}(ze^{it/2})),
\]

where \( \phi_t : (D, e^{2u} |dz|^2) \to H^3_1 \) is the 1-parameter family of immersions associated to \( u \) with Hopf differential \( \Theta_{\phi_t} = \frac{1}{2}e^{it}dz \otimes dz \).

**Proof.** Let \( \psi_t = (C, |dz|^2) \to H^3_1 \) the immersion of the hyperbolic cylinder given in equation (3.2). Then \( \phi_t, \psi_t : D \subseteq \mathbb{C} \to H^3_1 \) are conformal maximal immersions with the same Hopf differentials. Hence, applying the previous theorem we get that \( \nu_t = v_{(\phi_t, \psi_t)} \) is a conformal minimal immersion in \( H^2 \times H^2 \) with induced metric \( 4 \cosh^2 u |dz|^2 \).

Furthermore, a straightforward computation shows that

\[
v_{\psi_t}(z) = \cosh[2 \text{Im}(ze^{it/2})] E_1 - \sinh[2 \text{Im}(ze^{it/2})] E_3,
\]

where \( \{E_1, E_2, E_3\} \) is the orthonormal reference in \( \Lambda^2 \mathbb{R}^2 \) associated with the canonical base of \( \mathbb{R}^3_1 \) (see Section 2.3). Hence, as we have mentioned above, \( v_{\psi_t}(D) \) is contained in a geodesic of \( H^2 \). Considering \( \mathbb{R} \) embedded in \( H^2 \) as such geodesic we get the result. Finally, it is easy to check that \( \Theta_{\nu_t} = -2i\Theta_{\phi_t} = \Theta \).

The next result is a local converse of Theorem 1 in the special case of surfaces immersed in \( H^2 \times R \).

**Theorem 2.** Let \( \phi : \Sigma \to H^2 \times R \) an isometric minimal immersion of a simply connected Riemann surface \( \Sigma \) satisfying \( \nu^2 < 1 \), where \( \nu = \langle N, \bar{\partial} \rangle \). Then, there exists a conformal maximal immersion \( \psi : \Sigma \to H^3_1 \) such that \( \phi = v_{\psi} \), up to an ambient isometry.

**Proof.** Let \( w \) a conformal parameter over \( \Sigma \). Then \( Y(w) = \langle \phi_w, \bar{\partial} \rangle dw \) is a holomorphic 1-form without zeroes (note that \( |Y|^2 = \frac{1}{4}(1 - \nu^2) > 0 \) by assumption). Then we can always find another conformal parameter \( z \) such that \( Y = dz \). The conformal factor induced by \( \phi \) in this new parameter is \( 4 \cosh^2 u \), where \( u = \tanh(\nu) \) satisfies \( u_{zz} - \frac{1}{2} \sinh(2u) = 0 \). Moreover, the fundamental data of the immersion \( \phi \) can be expressed in terms of \( u \) (see [FM10 Theorem 2.3]).

Let \( \psi : \Sigma \to H^3_1 \) be the maximal conformal immersion associated to \( u \) with Hopf differential \( \Theta_{\psi}(z) = \frac{1}{2}dz \otimes dz \) (see Section 3). Then, \( \nu_{\psi} : \Sigma \to \)}
$\mathbb{H}^2 \times \mathbb{R}$ is a minimal immersion with the same fundamental data as $\phi$ and so both immersions differ in an ambient isometry.

Remark 4. It is possible to get a similar result for minimal immersion of $\mathbb{H}^2 \times \mathbb{H}^2$ without complex points as in [TU13, Theorem 3], that is, every minimal immersion in $\mathbb{H}^2 \times \mathbb{H}^2$ without complex points is locally congruent to the Gauss map of the pair of two maximal immersion in the anti-de Sitter space-time.

5. Examples

In this section we are going to use Corollary [1] to compute the minimal immersions associated to the maximal immersions in $\mathbb{H}^3$ given by Proposition [1]. As we shall see, the obtained examples are invariant by 1-parameter groups of isometries of $\mathbb{H}^2 \times \mathbb{R}$, namely, elliptic and hyperbolic screw motions (see figures [1] and [2]). Moreover, although the considered maximal immersions in $\mathbb{H}^3$ are not complete (see Remark [1]), their Gauss maps, in the sense of Corollary [1], are complete immersions.

Let $v : I \subseteq \mathbb{R} \to \mathbb{R}$ be a solution of $v''(x) - 2\sinh(2v) = 0$ with energy $E$ (cf. equation (3.3)). Then, the map $\Phi_E : \Sigma = (I \times \mathbb{R}, 4\cosh^2(v)g_0) \to \mathbb{H}^2 \times \mathbb{R}$ given by:

$$
\Phi_E(x, y) = \begin{cases}
\frac{1}{\sqrt{2E}} (v'(x), \sqrt{2E}e^{-v(x)} \cos \sqrt{2E} (y-G(x)) - v'(x)e^{v(x)} \sin \sqrt{2E} (y-G(x))), & E > 0,

\frac{1}{\sqrt{2E}} (v'(x), \sqrt{2E}e^{-v(x)} \sin \sqrt{2E} (y-G(x)) + v'(x)e^{v(x)} \cos \sqrt{2E} (y-G(x))), & E > 0,

\frac{1}{\sqrt{-2E}e^{v(x)}} (\sqrt{-2E}e^{-v(x)} \cosh \sqrt{-2E} (y-G(x)) - v'(x)e^{v(x)} \sinh \sqrt{-2E} (y-G(x))), & E < 0,

\frac{1}{\sqrt{-2E}e^{v(x)}} (\sqrt{-2E}e^{-v(x)} \sinh \sqrt{-2E} (y-G(x)) + v'(x)e^{v(x)} \cosh \sqrt{-2E} (y-G(x))), & E < 0,
\end{cases}
$$

is an isometric minimal immersion with associated Hopf differential $\Theta = -idz \otimes dz$, where $G(x) = \int_0^x \frac{dt}{2Ee^{v(t)}}$ and $g_0$ stands for the Euclidean metric in $\mathbb{R}^2$.

This can be checked directly from the definition or, taking into account Corollary [1], computing the first component of the Gauss map of the maximal immersion $\phi_E$ given in Proposition [1].

These examples are invariant by the 1-parameter group of isometries $A_\theta \times \tau_0$ of $\mathbb{H}^2 \times \mathbb{R}$, where $A_\theta$ is an isometry of $\mathbb{H}^2$ given by:

$$
A_\theta = \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \theta & -\sin \theta \\
0 & \sin \theta & \cos \theta
\end{pmatrix}
$$

and $\tau_0 : \mathbb{R} \to \mathbb{R}$ is given by $\tau_0(t) = t + \frac{\theta}{\sqrt{E}}$.

The complete classification of constant mean curvature surfaces (in particular the minimal ones) invariant by a 1-parameter group of $\mathbb{H}^2 \times \mathbb{R}$ can be found in [Onn08] and the references therein.
Figure 1. From left to right, typical solutions for positive energy $E = 4, 1, 0.1$ in $\mathbb{H}^2 \times \mathbb{R}$ being $\mathbb{H}^2$ the disc model. Below each surface the top view has been drawn. The boundary of $\mathbb{H}^2$ is drawn to help the visualization.

Figure 2. From left to right, typical solutions for negative energy $E = -0.5, -1, -6$ in $\mathbb{H}^2 \times \mathbb{R}$ being $\mathbb{H}^2$ the disc model. Below each surface the top view has been drawn.
6. Appendix

In this section we will exhibit explicit solutions for the equation $\Delta v - 2 \sinh(2v) = 0$. We will restring ourselves to the simplest case, that is, when the function only depends on one variable, i.e. $v = v(x)$. In that case it is easy to find a first integral of the equation, namely the energy $E = (v')^2/2 - \cosh(2v)$ is constant for every solution $v$ (cf. (3.3)). Moreover, if $v$ is a solution then $u(x) = -v(x)$ and $w(x) = v(-x)$ are also solutions with the same energy of $v$. Hence, we only need to consider initial conditions $v(0) = v_0 \geq 0$ and $v'(0) = \sqrt{2(E + \cosh(2v_0))} \geq 0$ (note that $E + \cosh(2v(x)) \geq 0$ by the definition of $E$). Thus we are interested in solving the following initial value problem:

$$
\begin{align*}
\mathcal{F}(\varphi, \mu) &= \int_{0}^{\pi/2} \frac{d\varphi}{\sqrt{1 - \mu \sin^2 \varphi}}, \\
&0 \leq \mu \leq 1.
\end{align*}
$$

It is possible to obtain all the solutions to that problem in terms of Elliptic Jacobi functions (see for instance [BF71] for further details). Let

$$
\begin{align*}
\text{sn}_\mu(x) &= \sin \text{am}_\mu(x), \\
\text{cn}_\mu(x) &= \cos \text{am}_\mu(x), \\
\text{dn}_\mu(x) &= \sqrt{1 - \mu \sin^2 \text{am}_\mu(x)}. \\
\text{sn}_\mu(x) &= \text{cn}_\mu(x) \text{dn}_\mu(x), \\
\text{cn}_\mu(x) &= -\text{sn}_\mu(x) \text{dn}_\mu(x), \\
\text{dn}_\mu(x) &= \text{cn}_\mu(x) \\
\text{sn}_\mu(x) &= \text{dn}_\mu(x), \\
\text{cn}_\mu(x) &= -\text{sn}_\mu(x) \text{dn}_\mu(x), \\
\text{dn}_\mu(x) &= -\mu \text{sn}_\mu(x) \text{cn}_\mu(x).
\end{align*}
$$

The basic properties of these functions are:

$$
\begin{align*}
\text{sn}_\mu(x)^2 + \text{cn}_\mu(x)^2 &= 1, \\
\mu \text{sn}_\mu(x)^2 + \text{dn}_\mu(x)^2 &= 1, \\
\text{sn}_\mu(x + 2K(\mu)) &= -\text{sn}_\mu(x), \\
\text{cn}_\mu(x + 2K(\mu)) &= -\text{cn}_\mu(x), \\
\text{dn}_\mu(x + 2K(\mu)) &= \text{dn}_\mu(x), \\
\text{sn}_\mu(x) &= \text{dn}_\mu(x), \\
\text{sn}_\mu(x) &= -\text{sn}_\mu(x) \text{dn}_\mu(x), \\
\text{cn}_\mu(x) &= -\text{sn}_\mu(x) \text{dn}_\mu(x), \\
\text{dn}_\mu(x) &= -\mu \text{sn}_\mu(x) \text{cn}_\mu(x).
\end{align*}
$$

where $K(\mu) = F(\pi/2, \mu)$ is the complete elliptic integral of the first kind. Moreover, the derivatives of the Jacobi elliptic functions are:

$$
\begin{align*}
\text{sn}_\mu'(x) &= \text{cn}_\mu(x) \text{dn}_\mu(x), \\
\text{cn}_\mu'(x) &= -\text{sn}_\mu(x) \text{dn}_\mu(x), \\
\text{am}_\mu'(x) &= \text{dn}_\mu(x), \\
\text{dn}_\mu'(x) &= -\mu \text{sn}_\mu(x) \text{cn}_\mu(x).
\end{align*}
$$
Lemma 1. The solution \( v : I \to \mathbb{R} \) of the initial value problem (6.1) and its maximal definition interval \( I \) are given, in terms of the energy \( E \), by:

\[
\begin{align*}
[E > 1] & \quad v(x) = \log(\lambda \tanh(\lambda^{-1}x + a_0)), & \mu = 1 - \lambda^4, \\
I &= \left| -a_0, \lambda K(\mu) - a_0 \right|, & a_0 = \arctan(\lambda^{-1}e^{\nu_0}), \\
[|E| \leq 1] & \quad v(x) = \log(\tanh(x + a_0) \, \text{d}n(\lambda x + a_0)), & \mu = \frac{1 - \lambda}{2}, \\
I &= \left| -a_0, K(\mu) - a_0 \right|, & a_0 = \frac{1}{2} \arccosh(\frac{\tanh(v_0)}{\lambda}), \\
[E < -1] & \quad v(x) = -\log(\lambda^{-1} \text{sn}(\lambda x + a_0)), & \mu = \lambda^{-4}, \\
I &= \left| -a_0, 2\lambda^{-1}K(\mu) - a_0 \right|, & a_0 = \arccosh(\lambda e^{-\nu_0})
\end{align*}
\]

where \( \lambda^2 = |E - \sqrt{E^2 - 1}| \) for \(|E| > 1\).

Proof. It is a direct computation taking into account the aforementioned properties of the Jacobi elliptic functions. \(\square\)

Remark 5. In the special cases \( E = 1 \) and \( E = -1 \) we get solutions in terms of elementary functions, namely, \( v_1(x) = \log \tan(x) \), \( v_{-1}(x) = \log \cotanh(x) \) as well as the constant solution \( v(x) = 0 \) (also with \( E = -1 \)).

On the one hand, the solutions of the sinh-Gordon equation with energy \( E > -1 \) are symmetric with respect to the middle point of the maximal interval of definition. On the other hand, the solutions \( v \) with energy \( E < -1 \) never vanish and are symmetric with respect to the vertical line passing through the middle point of the maximal interval of definition.

References

[BF71] Byrd, P. F., and Friedman, M. D. Handbook of elliptic integrals for engineers and scientists New York: Springer-Verlag, 1971.

[CU07] Castro, I., and Urbano, F. Minimal Lagrangian surfaces in \( S^2 \times S^2 \). Comm. Anal. Geom. 15 (2007), 217–248.

[Dan09] Daniel, B. Isometric immersions into \( S^n \times \mathbb{R} \) and \( H^n \times \mathbb{R} \) and applications to minimal surfaces. Trans. Amer. Math. Soc., 361 (12) (2009), 6255–6282.

[FM10] Fernández, I. and Mira, P. A characterization of constant mean curvature surfaces in homogeneous 3-manifolds. Differential Geom. Appl., 25 (3) (2007), 281–289.

[Hau06] Hauswirth, L. Minimal surfaces of Riemann type in three-dimensional product manifolds. Pacific J. Math. 224 (2006), 91–117.

[HST08] Hauswirth, L., Sá Earp, R. and Toubiana, E. Associate and conjugate minimal immersions in \( M \times \mathbb{R} \). Tohoku Math. J., 60 (2) (2008), 267–286.

[Ish88] Ishihara, T. Maximal spacelike submanifolds of a pseudo-Riemannian space of constant curvature. The Michigan Mathematical Journal, 35 (3) (1991), 345–352.

[MPT13] Manzano, J. M., Plehnert, J. and Torralbo, F. Compact embedded minimal surfaces in \( S^3 \times S^1 \). Preprint. arXiv: 1311.2500 [math.DG]

[MMR14] Martin, F., Mazzeo, R. and Rodriguez M. Minimal surfaces with positive genus and finite total curvature in \( H^2 \times \mathbb{R} \). Geometry & Topology, 18 (2014), 141–177.

[MR05] W. H. Meeks and H. Rosenberg. The theory of minimal surfaces in \( M \times \mathbb{R} \). Comment. Math. Helv., 80 (4) (2005), 811–858.

[NR02] Nelli, B. and Rosenberg, H. Minimal surfaces in \( H^{2} \times \mathbb{R} \). Bull. Braz. Math. Soc. 33 (2) (2002), 263–292.
GEOMETRICAL CORRESPONDENCE BETWEEN MINIMAL SURFACES IN $H^3_1$ AND $H^3_2 \times \mathbb{R}^1$  

[Onn08] Onnis, I. I. . Invariant surfaces with constant mean curvature in $H^2_2 \times \mathbb{R}$. Annali Di Matematica Pura Ed Applicata, 187 (4) (2008), 667–682.  

[Pal90] Palmer, B. Spacelike constant mean curvature surfaces in pseudo-Riemannian space forms Ann. Global Anal. Geom. 8, no. 3 (1990), 217–226.  

[Pal91] Palmer, B. Surfaces in Lorentzian hyperbolic space. Ann. Global Anal. Geom. 9 (1991), 117–128.  

[Per09] Perdomo, O. New examples of maximal space like surfaces in the anti-de Sitter space. J. Math. Anal. Appl. 353 (2009), 403–409.  

[Ros02] Rosenberg, H. Minimal surfaces in $M^2 \times \mathbb{R}$. Illinois J. of Math., 46 (2002), 1177–1195  

[ST04] Sá Earp, R. and Toubiana, E. Screw motion surfaces in $H^2_2 \times \mathbb{R}$ and $S^2 \times \mathbb{R}$ Illinois J. Math. 49 (4) (2005), 1323–1362.  

[Tor07] Torralbo, F. Minimal Lagrangian immersions in $RH^2_2 \times RH^2_2$. Symposium on the Differential Geometry of Submanifolds, 217–219, Valenciennes 2007. isbn: 978–1–8479–9016–7.  

[TU13] Torralbo, F. and Urbano, F. Minimal surfaces in $S^2 \times S^2$. To appear in J. Geom. Anal., doi: 10.1007/s12220-013-9460-3.  

DEPARTEMENT WISKUNDE. KU LEUVEN. CELESTIJNENLAAN 200B, B-3001 LEUVEN, BELGIUM

E-mail address: francisco.torralbo@wis.kuleuven.be