Functorial filtrations for homotopy categories of some generalisations of gentle algebras

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Abstract
We consider algebras defined over a complete, local and noetherian ground ring. They are gentle algebras in case the ground ring is a field. The unbounded homotopy category of complexes of projective modules is considered. Complexes with finitely-generated homogeneous components are shown to be isomorphic to direct sums of indecomposable string and band complexes. The corresponding isoclasses are described, and the Krull-Remak-Schmidt-Azumaya property is verified. This classification problem is solved using the idea of functorial filtrations.

1. Introduction.

Gentle algebras, as introduced by Assem and Skowroński [3], have the form $\Gamma = kQ/J$ where $k$ is a field, $Q$ is a quiver, $kQ$ is the path algebra and the ideal $J \subseteq kQ$ is generated by length 2 paths. The combinatorial restrictions [3, §1.1, Proposition, R1, R2] on $J$ and $Q$ ensure $\Gamma$ is a finite-dimensional special biserial algebra, in the sense of Skowroński and Waschbühl [31, §1, (SP)]. By a result of Wald and Waschbühl [32, Proposition 2.3], if $\Omega$ is a special biserial algebra, then indecomposables in the category $\Omega$-$\text{mod}$ of finite-dimensional modules are string modules, band modules or non-uniserial projective-injectives. Since $J$ is generated by paths, indecomposable projective-injective $\Gamma$-modules are uniserial.

Various authors have also capitalised on the aforementioned restrictions of $Q$ and $J$ to study derived categories of gentle algebras. Schröer and Zimmermann [30, Corollary 1.2] have shown gentle algebras are closed under derived equivalence. Bekkert and Merklen [5, Theorem 3] have shown indecomposable objects in the bounded derived category $D^b(\Gamma$-$\text{mod})$ are (what we call) string complexes or band complexes. The repetitive algebra $\Omega$ of $\Gamma$, as introduced by Hughes and Waschbühl [19], was shown by Pogorzaly and Skowroński [26, Lemma 8] to be special biserial. Bobiński [8, §5, §6] has studied connections between string and band modules over $\Omega$ and string and band complexes over $\Gamma$, describing almost split triangles in the category of perfect complexes. Geiß and Reiten [17, Theorem 3.4] have shown gentle algebras are Iwanaga-Gorenstein rings. Kalck [21, Theorem 2.5(b)] has since described the singularity category. Arnesen, Laking and Pauksztello [11, Theorem A] have described all morphisms between string and band complexes, and these authors together with Prest [2, Theorem B] described all indecomposable pure-injective objects in the unbounded homotopy category $K(\Gamma$-$\text{Proj})$.

In this article we consider more generally complete gentle algebras, defined using a quiver $Q$ with relations, and a complete, local and noetherian ground ring $(R, \mathfrak{m}, k)$. See [2] for details.

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For some prime number $p > 0$, let $R$ be the ring $\widehat{\mathbb{Z}}_p$ of $p$-adic integers. A notable mixed-characteristic example of a complete gentle algebra is the $R$-algebra

$$
\mathcal{Y}_p = \left\{ \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix} \in M_2(\widehat{\mathbb{Z}}_p) \mid \gamma_{11} - \gamma_{22}, \; \gamma_{12} \in p\widehat{\mathbb{Z}}_p \right\}
$$

(1.1)

If $R$ is a field, then by [6 Corollary 1.2.11] any complete gentle $R$-algebra is a finite-dimensional gentle algebra in the sense of [3]. Cases where $R$ strictly contains a field give interesting infinite-dimensional examples of complete gentle algebras. To see an example, consider the completed string algebras in the sense of Ricke [28], which were defined as quotients $\Theta = \overline{kQ}/\langle \rho \rangle$ of the completed path algebra of a quiver $Q$. If the pair $(Q, \rho)$ satisfies gentle conditions (see Definition 2.3) then $\Theta$ is a complete gentle $k[[t]]$-algebra (see Remark 2.6). For example, if $Q$ consists of two loops $a$ and $b$ and $\rho = \{ab, ba\}$ then $\Theta$ is the local $k[[t]]$-algebra $k[[x, y]]/(xy)$, where $t$ acts as $x + y + (xy)$ (see [6 Example 1.2.30] for details). The complex of free $k[[x, y]]/(xy)$-modules

$$
P = \cdots \xrightarrow{x^2} \Lambda \xrightarrow{y} \Lambda \xrightarrow{x^2} \Lambda \xrightarrow{y^3} \Lambda \xrightarrow{x^2} \Lambda \xrightarrow{y} \Lambda \xrightarrow{x} \Lambda \rightarrow 0 \rightarrow \cdots
$$

is an example of a string complex. The name refers to the depiction of $P$ by the schema

```
\begin{verbatim}
\cdots \xrightarrow{x^2} \Lambda \xrightarrow{y} \Lambda \xrightarrow{x^2} \Lambda \xrightarrow{y} \Lambda \xrightarrow{x} \Lambda \rightarrow 0 \rightarrow \cdots
\end{verbatim}
```

Band complexes are defined similarly, but the corresponding schema is finite and joins up with itself. String complexes are indexed by a homotopy word $C$ which is aperiodic, and band complexes by a periodic homotopy word together with an indecomposable $R[T, T^{-1}]$-module $V$ which is free over $R$. See [3] for details. Given a complete gentle algebra $\Lambda$ let $\mathcal{K}(\Lambda\text{-}\text{Proj})$ (respectively $\mathcal{K}(\Lambda\text{-}\text{proj})$) be the unbounded homotopy category of complexes of (finitely generated) projective modules. Theorem [1.1] describes objects in $\mathcal{K}(\Lambda\text{-}\text{proj})$. Theorems 1.2 and 1.3 explain the way in which this description is unique.

**Theorem 1.1.** [6 Theorem 2.0.1] Let $\Lambda$ be a complete gentle algebra.

(i) Every object in $\mathcal{K}(\Lambda\text{-}\text{proj})$ is isomorphic to a (possibly infinite) direct sum of shifts of string complexes $P(C)$ and shifts of band complexes $P(C, V)$.

(ii) Each shift of a string or band complex is an indecomposable object in $\mathcal{K}(\Lambda\text{-}\text{Proj})$. 


A well-known result of Kaplansky [22] Theorem 2] says that any projective $R$-module is free. Consider the category $R[T,T^{-1}]$-$\text{Mod}$ of $R[T,T^{-1}]$-modules, and the full subcategory $R[T,T^{-1}]$-$\text{Mod}_{R,\text{Proj}}$ of $R[T,T^{-1}]$-modules which are free as $R$-modules. Note that there is a natural isomorphism $k[T,T^{-1}] \otimes_{R[T,T^{-1}]} - \simeq k \otimes_R -$ of functors $R[T,T^{-1}]$-$\text{Mod}_{R,\text{Proj}} \rightarrow k[T,T^{-1}]$-$\text{Mod}$. Let $i$ define the $R$-algebra involution of $R[T,T^{-1}]$ which exchanges $T$ and $T^{-1}$. Define a functor $\text{res}_{i,R} : R[T,T^{-1}]$-$\text{Mod}_{R,\text{Proj}} \rightarrow R[T,T^{-1}]$-$\text{Mod}_{R,\text{Proj}}$ by setting $\text{res}_{i,R}(V)$ to have underlying $R$-module structure $V$ but where $T$ acts on $v \in \text{res}_{i,R}(V)$ by $v \mapsto T^{-1}v$.

The homotopy letters which make up a homotopy word $C$ are indexed using a subset $I_C$ of $\mathbb{Z}$. The inverse $C^{-1}$ defines a new homotopy word by inverting the letters and reversing their order. Similarly the shift $C[t]$ ($t \in \mathbb{Z}$) is defined by reindexing the letters by $i \mapsto i - t$. Any $i \in I_C$ defines a vertex $v_C(i)$ of the quiver $Q$ defining $\Lambda$, and in this way the set $I_C$ defines the projective indecomposable summands of the underlying module of the string or band complex defined by $C$. Furthermore there is a function $\mu_C : I_C \rightarrow \mathbb{Z}$ which detects the homogeneous degree of each aforementioned summand. See Definitions 3.1 and 3.3 for details. Theorem 1.2 characterises when two shifts of string or band complexes are isomorphic.

**Theorem 1.2.** [6 Theorem 2.0.4] Let $\Lambda$ be a complete gentle algebra. Let $C$ and $E$ be homotopy words, let $V$ and $W$ be objects of $R[T,T^{-1}]$-$\text{Mod}_{R,\text{Proj}}$ and let $n \in \mathbb{Z}$.

(i) If $C$ and $E$ are aperiodic, then $P(C)[n] \simeq P(E)$ in $K(\Lambda$-$\text{Proj})$ if and only if:
(a) we have $I_C = \{0, \ldots, m\}$ and $(I_E,E,n) = (I_C,C,0)$ or $(I_C,C^{-1},\mu_C(m))$;
(b) we have $I_C = \pm \mathbb{N}$ and $(I_E,E,n) = \pm \mathbb{N},C,0)$ or $(\mp \mathbb{N},C^{-1},0)$;
(c) we have $I_C = \mathbb{Z}$ and $(I_E,E,n) = (\mathbb{Z},C^{t+1}[t],\mu_C(\pm t))$ for some $t \in \mathbb{Z}$.

(ii) If $C$ and $E$ are periodic, then $P(C,V)[n] \simeq P(E,W)$ in $K(\Lambda$-$\text{Proj})$ if and only if:
(a) we have $E = C[t]$, $k \otimes_R V \simeq k \otimes_R W$ and $n = \mu_C(t)$ for some $t \in \mathbb{Z}$;
(b) we have $E = C^{-1}[t]$, $k \otimes_R V \simeq k \otimes_R \text{res}_{i,R} W$ and $n = \mu_C(-t)$ for some $t \in \mathbb{Z}$.

(iii) If $C$ is aperiodic and $E$ is periodic, then $P(C)[n] \not\simeq P(E,V)$ in $K(\Lambda$-$\text{Proj})$.

**Theorem 1.3.** [6 Theorem 2.0.5] Let $\Lambda$ be a complete gentle algebra. If two direct sums of shifts of string and band complexes are isomorphic in $K(\Lambda$-$\text{proj})$, then there is an isoclass preserving bijection between the summands.

For a finite-dimensional gentle algebra $\Gamma$, the classification of objects in $\mathcal{D}^{b}(\Gamma$-$\text{mod})$ is due to Bekkert and Merklen [5]. These authors constructed a functor to a category of square-zero block matrices which preserves and respects indecomposables. Objects in this category were classified by Bondarenko using the matrix reductions method [9] Theorem 1. Bekkert, Drozd and Furtorny [4] Theorem 2.7 note that the technique of reducing to a matrix problem used in [5] may be employed to the class of complete gentle $k[[t]]$-algebras $\Theta = kQ/\rho$ discussed before Equation 1.2 (see [4] Definition 2.5) and Remark 2.6. Theorems 1.1, 1.2 and 1.3 together with Remark 2.6 give a new approach to the aforementioned classification problems solved by Bekkert and Merklen and Bekkert, Drozd and Furtorny (see [6] Corollaries 2.7.2 and 2.7.6).

In our proof of Theorems 1.1, 1.2 and 1.3 we work directly with the category $K(\Lambda$-$\text{Proj})$ and avoid reducing to a matrix problem. The classification method we adapt is called the functorial filtrations method. This approach yields results which appear to be new: our results hold for mixed-characteristic complete gentle algebras, and so Theorems 1.1, 1.2 and 1.3 strictly generalise [4] Theorem 2.7; string and band complexes whose homogeneous components may be infinitely-generated are indecomposable by Theorem 1.1(ii), which generalises the indecomposability statement in [2] Corollary 3.5; in Theorem 1.1(i) we classify complexes in $K(\Lambda$-$\text{proj})$ which need not satisfy any bounded-ness conditions; and in Theorem 1.3 the summands of any direct sum of shifts of string and band complexes have been identified.
Functorial filtrations have been written in Mac Lane’s language of linear relations [25], and were used in the past to classify modules (with certain finiteness conditions) up to isomorphism. Gel’fand and Ponomarev [18] seem to be the first to use this method, during a classification Harish-Chandra modules for the Lorentz group. Their work was interpreted in the language of functors by Gabriel [16]. Ringel [29] then used this approach to describe indecomposable representations of the dihedral 2-group. Since then several classes of rings have had their modules classified by authors adapting the method from [18] and [29]: Brauer graph algebras by Donovan and Freislich [15]; locally bounded string algebras by Butler and Ringel [11]; clannish algebras [12], semidihedral algebras [13] and string algebras which may be both infinite-dimensional and unital [14] by Crawley-Boevey; and completed string algebras by Ricke [28]. Functorial filtrations have also been used to classify representations of significance outside representation theory. Prest and Puninski [27] classified pure-injective indecomposable modules over domestic string algebras using linear relations that correspond to subgroups of finite definition. Such objects arise from the model theory of modules. In joint work [15] with Crawley-Boevey we employed similar ideas, adapting the focus of the functorial filtrations method in [14] to classify Σ-pure-injective modules.

The results in this article form part of the author’s PhD thesis [6]. The article is organised as follows. In §2 we define complete gentle algebras. In §3 we define string and band complexes. In §4 we explain how our classification works by providing a categorical blueprint. In §5 we consider linear relations over the ring R, and introduce the notion of a reduction which meets in the maximal ideal m. In §6 we explain how each homotopy word gives rise to a functor on the category of complexes, each of which is defined by such linear relations. In the remaining sections of the article we: check compatibility conditions between (string and band complexes) and (linear relations given by homotopy words); we verify that our setting fits the blueprint discussed in §3 and we provide proofs of Theorems 1.1, 1.2 and 1.3.

2. Some generalisations of gentle algebras.

Assumption 2.1. For the remainder of the article we fix:

(i) a commutative, noetherian and local ring R with maximal ideal m and k = R/m;
(ii) a finite quiver Q with path algebra $RQ$ and an ideal $\mathcal{J} \triangleleft RQ$ generated by a set paths in Q of length at least 2; and
(iii) a surjective R-algebra homomorphism $\vartheta : RQ \to \Lambda$ where $\mathcal{J} \subseteq \ker(\vartheta)$ and $\vartheta(p) \neq 0$ for any path $p \notin \mathcal{J}$. Notation is abused by writing p for $\vartheta(p)$.

Notation 2.2. Let $P$ be the set of non-trivial paths $p \notin J$ with head $h(p)$ and tail $t(p)$. For each $t > 0$ and each vertex $v$ let $P(t, v \to)$ (respectively $P(t, \to v)$) be the set of paths $p \in P$ of length t with $t(p) = v$ (respectively $h(p) = v$). Let $A$ be the set of arrows in $Q$, $A(v \to) = P(1, v \to)$ and $A(\to v) = P(1, \to v)$. The composition of $a \in A(\to v)$ and $b \in A(u \to)$ is $ba$ if $u = v$, and 0 otherwise. If X is a finite set, let $\#X$ be the cardinality of X.

Definition 2.3. [31, §1, (SP)] We say that the pair $(Q, \mathcal{J})$ satisfies special conditions if: 
(SPI) if v is a vertex then $\#A(v \to) \leq 2$ and $\#A(\to v) \leq 2$; and
(SPII) if $y \in A$ then $\#\{x \in A(h(y) \to) \mid xy \in P\} \leq 1$ and $\#\{z \in A(t(y) \to) \mid yz \in P\} \leq 1$.

We say $(Q, \mathcal{J})$ satisfies gentle conditions if it satisfies special conditions and:

(GI) any path $p \notin P$ has a subpath $q \notin P$ of length 2; and
(GII) if $y \in A$ then $\#\{x \in A(h(y) \to) \mid xy \notin P\} \leq 1$ and $\#\{z \in A(t(y) \to) \mid yz \notin P\} \leq 1$. 

Example 2.4. [6] Example 1.1.6] Let $R = \hat{\mathbb{Z}}_p$ and $m = p\hat{\mathbb{Z}}_p$. Let $Q$ be the quiver given by two loops $\alpha$ and $\beta$ at a single vertex $v$, and let $J = \langle \alpha^2, \beta^2 \rangle$. Let $\Lambda$ be the subring of $2 \times 2$ matrices whose $ij$-entry $\gamma_{ij} \in R$ satisfies $p \mid \gamma_{11} - \gamma_{22}, \gamma_{12}$, and so $\Lambda = \Upsilon_p$ from Equation 1.1. Define $\vartheta : RQ \to \Lambda$ by (multiplicatively and $R$-linearly) extending the assignments

$$\vartheta(\alpha) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \vartheta(\beta) = \begin{pmatrix} 0 & p \\ 0 & 0 \end{pmatrix}.$$ 

That $(Q, J)$ satisfies gentle conditions is clear. Furthermore $\text{rad}(\Lambda)$ is the subset of matrices $(\gamma_{ij})$ with $p \mid \gamma_{11}, \gamma_{22}$, which is generated as a two-sided ideal by $\alpha$ and $\beta$. Note $\alpha \beta + \beta \alpha = p$ in $\Lambda$, and so $(\text{rad}(\Lambda))^3 \subseteq \Lambda p \subseteq \text{rad}(\Lambda)$. In particular the ideal $\text{rad}(\Lambda)/\Lambda p$ of $\Lambda/\Lambda p$ is nilpotent.

Definition 2.5. [6] Definition 1.1.19] We say the ring $\Lambda$ is $\text{rad}$-nilpotent modulo $m$ if $\text{rad}(\Lambda)^n \subseteq \Lambda m \subseteq \text{rad}(\Lambda)$ for some $n \geq 1$ (and so the ideal $\text{rad}(\Lambda)/\Lambda m$ of $\Lambda/\Lambda m$ is nilpotent). [6] Definition 1.1.21] We say that $\Lambda$ is a quasi-bounded gentle $R$-algebra if:

(i) the pair $(Q, J)$ satisfies gentle conditions (SPI, SPII, GI and GII);
(ii) the ring $\Lambda$ is $\text{rad}$-nilpotent modulo $m$;
(iii) if $a \in A$ then the $R$-modules $\Lambda a$ and $a \Lambda$ are finitely generated;
(iv) the ideal $\text{rad}(\Lambda)$ of $\Lambda$ is generated by $A$; and
(v) if $a, a' \in A$ and $a \neq a'$ then $\Lambda a' \cap \Lambda a = 0 = a \Lambda \cap a' \Lambda$.

It is straightforward to check the mixed-characteristic ring $\Upsilon_p$ from Equation 1.1 and Example 2.4 satisfies conditions (i), (ii), (iii), (iv) and (v) from Definition 2.5. Hence $\Upsilon_p$ is a quasi-bounded gentle algebra over the $p$-adic integers. We now add detail to the discussion between Equations 1.1 and 1.2.

Remark 2.6. Suppose $k$ is a field and $(Q, J)$ satisfies gentle conditions. The completed path algebra $\widehat{kQ}$ consists of (possibly infinite) sums $\sum \lambda_p p$ where $\lambda_p \in k$ and $p$ runs once through each of the distinct paths in $Q$. Note that $\text{rad}(kQ)$ is generated by $A$ [6] Proposition 1.2.23]. Let $z$ be the sum of the cycles $\sigma \in P$ which are not non-trivial powers of other cycles, and such that $\sigma^n \in P$ for all integers $n > 0$. Note $z$ acts centrally on $kQ/J$, and $kQ/J$ is module finite over $k[z]$ [14] Lemmas 3.1 and 3.2]. Furthermore there exists $n > 0$ with $A^n \subseteq \langle z \rangle$, so $kQ/J$ is the $A$-adic completion of the ring $kQ/J$, where $J$ is the ideal in $kQ$ generated by $J$ [28] Proposition 3.2.7] (see also [6] Proposition 1.2.25, Lemma 1.2.27]). Altogether the above shows that $\Lambda = \widehat{kQ/J}$ is a quasi-bounded gentle $k[[z]]$-algebra [6] Corollary 1.2.29]. Note that $\Lambda$ is a finite-dimensional gentle algebra over $k$ if and only if there are no cycles $c \in P$ as above.

In the notation from Remark 2.6, note that if $(Q, J)$ satisfies gentle conditions then Bekkert, Drozd and Futorny [14] call $\widehat{kQ/J}$ a gentle algebra. Ricke [28], who uses a more general definition, calls $\widehat{kQ/J}$ a completed string algebra.

Example 2.7. Let $J = \langle lr, rq, tf, gs, eb, dc, ba \rangle$ where $Q$ is the quiver

\[
\begin{array}{c}
0 \overset{p}{\rightarrow} 1 \overset{q}{\rightarrow} 2 \\
\uparrow{\scriptstyle n} \quad \overset{r}{\downarrow} \quad \overset{f}{\downarrow} \\
6 \overset{l}{\rightarrow} 7 \overset{t}{\rightarrow} 8 \overset{g}{\rightarrow} 9 \\
\end{array}
\quad
\begin{array}{c}
3 \overset{e}{\leftarrow} 4 \overset{b}{\leftarrow} 5 \\
\downarrow{\scriptstyle d} \quad \overset{a}{\downarrow} \\
10 \overset{c}{\leftarrow} 11 \\
\end{array}
\]

It is clear that $(Q, J)$ satisfies gentle conditions. Here the primitive cycles are: $tsr$ (at 2); $edgf$ (at 3); $dgfe$ (at 4); $rts$ (at 8); $srt$ and $fedg$ (at 9); and $gfed$ (at 10). This shows that the sum $z$ of these paths annihilates $e_u$ provided $u$ is 0, 1, 5, 6, 7 or 11.
Note that the \( k[z] \)-module \( (kQ/\mathcal{J})e_9 \) is generated by \( e_9, t, g, rt, dg, srt = ze_9 - f \) and \( edg \). Moreover, any path in \( Q \) of length 6 lying outside \( \mathcal{J} \) must have the form \( \mu z \) where \( \mu \) is a nontrivial path of length 2 or 3 lying outside \( \mathcal{J} \), and so \( A^6 \subseteq (z) \).

**Remark 2.8.** There is an embedding of the \( \hat{\mathbb{Z}}_p \)-algebra \( \Upsilon_p \) (from Equation \([1,1]\)) into the hereditary order of \( 2 \times 2 \) matrices \( (\lambda_{ij}) \) with \( p \mid \lambda_{12} \), and so \( \Upsilon_p \) is a nodal ring in the sense of Burban and Drozd \([10, \text{Definition 2.1}]\). Like-wise, there is an embedding of the \( k[[z]] \)-algebra \( kQ/\mathcal{J} \) from Example \([2.1]\) into the hereditary algebra \( k\Gamma \) where \( \Gamma \) is the quiver

\[
\begin{array}{cccccccc}
0 & p & 1 & q & 2' & 2'' & 3 & 4' & 5' & 5'' \\
\downarrow n & & & & & & & & & \\
6 & m & 7 & l & 8' & 8'' & f & d & 9' & 9'' & 10'' & \leftarrow e & 11
\end{array}
\]

This embedding is defined by the construction from the proof of \([34, \text{Theorem}]\), in which Zemlyak proves that any finite-dimensional gentle algebra is a nodal algebra. Since \( kQ/\mathcal{J} \) is infinite-dimensional and has minimal ideals, it is not nodal in either sense.

**Remark 2.9.** Suppose \( \Lambda \) is a quasi-bounded gentle \( R \)-algebra. By Definition \([2.5(iv)]\), for each vertex \( v \) the left \( \Lambda \)-module \( \Lambda e_v \) (respectively right \( \Lambda \)-module \( e_v\Lambda \)) has a unique maximal submodule generated by \( A(v \to) \) (respectively \( A(\to v) \)). By \([6, \text{Corollary 1.1.17}]\) this shows, for any \( p \in P \), that: every proper non-zero submodule of \( \Lambda p \) (respectively \( p\Lambda \)) has the form \( \Lambda qp \) (respectively \( pq\Lambda \)) with \( q \in P \); \( \Lambda p \) is local with \( \text{rad}(\Lambda p) = \text{rad}(\Lambda)p \); if \( \text{rad}(\Lambda p) \neq 0 \) then \( \text{rad}(\Lambda p) = \Lambda p \) for an arrow \( a \) with \( ap \in P \); and if \( p' \in P \), \( \Lambda p' = \Lambda p \) and \( f(p) = f(p') \) (respectively \( \Lambda p = \Lambda p' \) and \( l(p) = l(p') \)) then \( p = p' \). In particular, for any vertex \( v \), the left \( \Lambda \)-module \( \text{rad}(\Lambda e_v) \) (respectively right \( \Lambda \)-module \( \text{rad}(e_v\Lambda) \)) is the sum of at most two uniserial modules, and this sum is direct by Definition \([2.5(v)]\).

Furthermore the elements \( e_v \) form a complete set of orthogonal local idempotents in the ring \( \Lambda \). By \([6, \text{Corollary 1.25}]\) this means that the \( R \)-modules \( k, \Lambda e_v/\text{rad}(\Lambda e_v) \) and \( e_v\Lambda/\text{rad}(e_v\Lambda) \) are isomorphic: and furthermore \( \Lambda \) is a noetherian semilocal ring which is module finite over \( R \). By the intersection theorem of Krull (see for example \([24, \text{Exercise 4.23}]\)), if \( M \) is a finitely generated \( \Lambda \)-module, then the intersection of \( (\text{rad}(\Lambda))^nM \) over all \( n > 0 \) is trivial.

**Notation 2.10.** \([6, \text{Definition 1.1.13}]\) Any non-trivial path \( p \) in \( Q \) has a first arrow \( f(p) \) and a last arrow \( l(p) \) satisfying \( l(p)p' = p = p''f(p) \) for some (possibly trivial) paths \( p' \) and \( p'' \).

The proof of Corollary \([2.11]\) uses the results discussed in Remark \([2.9]\).

**Corollary 2.11.** \([6, \text{Corollary 1.2.14, 1.2.18 and 1.2.21}]\) Let \( v \) be a vertex, let \( t > 0 \) be an integer and let \( q \in P \). Let \( \Lambda \) be a quasi-bounded gentle \( R \)-algebra.

(i) If \( v = h(q), \lambda \in \Lambda e_v \) and \( \lambda q = 0 \), then \( \Lambda e_v \in \bigoplus_{a \in A(v \to)} \Lambda a \).

(ii) If \( v = t(q), \lambda \in e_v\Lambda \) and \( q\lambda = 0 \), then \( e_v\Lambda \in \bigoplus_{a \in A(\to v)} \Lambda a \).

(iii) We have \( \text{rad}(\Lambda q) \cap l(q) \Lambda \subseteq q \text{rad}(\Lambda) \) and \( \text{rad}(\Lambda q) \cap \Lambda l(q) \subseteq \text{rad}(\Lambda)q \).

(iv) If \( \sum pr_p \neq 0 \) where \( r_p \in R \) as \( p \) runs over \( \{h \in P \mid hl(q) \in P\} \), then \( \sum pr_p q \neq 0 \).

(v) If \( \sum pr_p \neq 0 \) where \( r_p \in R \) as \( p \) runs over \( \{h \in P \mid f(q)h \in P\} \), then \( \sum pr_p q \neq 0 \).

(vi) If \( R \) is an \( m \)-adically complete ring then \( \Lambda \) is a semiperfect ring.

**Definition 2.12.** \([6, \text{Definition 1.2.19}]\) By a complete gentle algebra we mean a quasi-bounded gentle \( R \)-algebra where \( R \) is \( m \)-adically complete.
ASSUMPTION 2.13. For the remainder of the article let $\Lambda$ be a complete gentle $R$-algebra.

3. String and band complexes.

In what follows we define a new system of words to adapt the functorial filtrations method, as in [11] to classify complexes. To do so we modify the alphabet used by Bekkert and Merklen [5] to define generalised strings and bands.

**Definition 3.1.** [6] Definition 1.3.26] A homotopy letter $q$ is one of $\gamma$, $\gamma^{-1}$, $d_a$, or $d_a^{-1}$ for $\gamma \in P$ and an arrow $a$. Those of the form $\gamma$ or $d_a$ will be called direct, and those of the form $\gamma^{-1}$ or $d_a^{-1}$ will be called inverse. The inverse $q^{-1}$ of a homotopy letter $q$ is defined by setting $\gamma^{-1} = \gamma^\ast$, $(\gamma^{-1})^{-1} = \gamma$, $(d_a)^{-1} = d_a^\ast$ and $(d_a^{-1})^{-1} = d_a$.

Let $I$ be one of the sets $\{0, \ldots, m\}$ (for some $m \geq 0$), $\mathbb{N}$, $-\mathbb{N} = \{-n \mid n \in \mathbb{N}\}$, or $Z$. For $I \neq \{0\}$ a homotopy $I$-word is a sequence of homotopy letters

$$C = \begin{cases} l_1^{-1}r_1 \ldots l_m^{-1}r_m & \text{(if } I = \{0, \ldots, m\}) \\ l_1^{-1}r_1l_2^{-1}r_2 & \text{(if } I = \mathbb{N}) \\ \ldots l_i^{-1}r_{i-1}l_0^{-1}r_0 & \text{(if } I = -\mathbb{N}) \\ \ldots l_i^{-1}r_{i-1}l_0^{-1}r_0 | l_1^{-1}r_1l_2^{-1}r_2 & \text{(if } I = Z) \end{cases}$$

(which will be written as $C = \ldots l_i^{-1}r_i \ldots$ to save space) such that:

(i) any homotopy letter in $C$ of the form $l_i^{-1}$ (respectively $r_i$) is inverse (respectively direct);

(ii) any sequence of 2 consecutive letters in $C$, which is of the form $l_i^{-1}r_i$, is one of $\gamma^{-1}d_i(\gamma)$ or $d_i^{-1}(\gamma)$ for some $\gamma \in P$; and

(iii) any sequence of 4 consecutive letters in $C$ of the form $l_i^{-1}r_il_{i+1}^{-1}r_{i+1}$ is one of

(a) $\gamma^{-1}d_i(\gamma)d_i^{-1}(\lambda)$ where $h(\gamma) = h(\lambda)$ and $l(\gamma) \neq l(\lambda)$;

(b) $d_i^{-1}(\gamma)d_i^{-1}(\lambda)$ where $t(\gamma) = t(\lambda)$ and $f(\gamma) = f(\lambda)$;

(c) $d_i(\gamma)^{-1}d_i^{-1}(\lambda)$ where $t(\gamma) = t(\lambda)$ and $f(\gamma) = f(\lambda)$;

(d) $\gamma^{-1}d_i(\gamma)^{-1}d_i^{-1}(\lambda)$ where $h(\gamma) = t(\lambda)$ and $f(\gamma)l(\lambda) \in J$.

For $J = \{0\}$ there are trivial homotopy words $1_{v,1}$ and $1_{v,-1}$ for each vertex $v$.

[6] Definition 1.3.29] The head and tail of any path $\gamma \in P$ are already defined and we extend this by setting $h(d_a^{\pm 1}) = h(a)$ for any arrow $a$ and $h(q^{-1}) = t(q)$ for all homotopy letters $q$. For each $i \in I$ there is an associated vertex $v_C(i)$ defined by: $v_C(i) = t(l_{i+1})$ for $i \leq 0$ and $v_C(i) = t(r_i)$ for $i > 0$ provided $C = \ldots l_i^{-1}r_i \ldots$ is non-trivial; and $v_{1_{v,\pm 1}}(0) = v$ otherwise.

If $\gamma \in P$ and $a = l(\gamma)$ let $H(\gamma^{-1}d_a) = -1$ and $H(d_a^{-1}\gamma) = 1$. Let $\mu_C(0) = 0$ and

$$\mu_C(i) = \begin{cases} H(l_1^{-1}r_1) + \ldots + H(l_i^{-1}r_i) & \text{(if } 0 < i \in I) \\ -H(l_0^{-1}r_0) + \ldots + H(l_{i+1}^{-1}r_{i+1}) & \text{(if } 0 > i \in I) \end{cases}$$

[5] Definition 2] (see also [6] Definition 1.3.34].) For $n \in \mathbb{Z}$ let $P^n(C)$ be the sum $\bigoplus \Lambda e_{v_C(i)}$ over $i \in \mu_C^{-1}(n)$. For each $i \in I$ let $b_{i,C}$ denote the coset of $e_{v_C(i)}$ in $P(C)$ (in degree $\mu_C(i)$). If the dependency on $C$ is irrelevant let $b_{i,C} = b_i$. We define the complex $P(C)$ by extending the assignment $d_{P(C)}(b_i) = b_i^+ + b_i^-$ linearly over $\Lambda$ for each $i \in I$, where

$$b_i^+ = \begin{cases} ab_{i+1} & \text{(if } i + 1 \in I, l_{i+1}^{-1}r_{i+1} = d_i^{-1}(\alpha) \} \\ 0 & \text{(otherwise)} \end{cases}$$

$$b_i^- = \begin{cases} b_{i-1} & \text{(if } i - 1 \in I, l_i^{-1}r_i = \beta^{-1}d_i(\beta) \} \\ 0 & \text{(otherwise)} \end{cases}$$
Let \([C]_i = [\gamma^{-1}]\) if \(I_{i-1} r_i = d_i^{-1}(\gamma)\) and \([C]_i = [\gamma]\) if \(I_{i-1} r_i = \gamma^{-1}d_i(\gamma)\). Then \([C] = \cdots [C]_i \cdots\) defines a generalised string or a generalised band as in Bekkert and Merklen [5] §4.1.

**Example 3.2.** (See also [6] Examples 1.3.27 and 1.3.35). Consider the finite-dimensional gentle algebra \(\Lambda = kQ/J\) given by \(J = \langle gf, hg, fh, sr, ts, rt \rangle\) where \(Q\) is the quiver

![Diagram](image)

Let \(C = d_r^{-1}rhd_g^{-1}gd_f^{-1}fr^{-1}ds^{-1}ds^{-1}d_x^{-1}x\). We may depict the string complex \(P(C)\) by

![Diagram](image)

where an arrow \(\Lambda e_v \to \Lambda e_u\) labelled by a path \(p\) with head \(v\) and tail \(u\) indicates right-multiplication by \(p\). The corresponding generalised string is \([C] = [[(rh)^{-1}]][g^{-1}][f^{-1}]|r|s|x^{-1}]\).

**Definition 3.3.** [6] Definitions 1.3.26, 1.3.32 and 1.3.42 Let \(C\) be a homotopy word. Write \(I_C\) for the subset of \(Z\) where \(C\) is a homotopy \(I_C\)-word. Let \(t \in Z\). We say \(C\) has controlled homotopy if the preimage \(\mu_C^{-1}(n) = \{i \in I_C \mid \mu_C(i) = n\}\) is a finite set for each \(n \in Z\).

Let \(t \in Z\). If \(I_C = Z\) we let \(C[t] = \cdots I_{l-1} r_l \mid I_{l+1} r_{l+1} \cdots\). That is, in the language of generalised strings and bands, if \(I_C = Z\) let \([C[t]]_i = [C]_{i+t}\). If instead \(I_C \neq Z\) we let \(C = C[t]\).

The inverse \(C^{-1}\) of \(C\) is defined by \((1_{x, \delta})^{-1} = 1_{x, \delta}^{-1}\) if \(I = \{0\}\), and otherwise inverting the homotopy letters and reversing their order. Note the homotopy \(Z\)-words are indexed so that

\[
(\cdots l_{-1}^{-1}r_{-1}^{-1}l_{0}^{-1}r_{0} \mid l_{1}^{-1}r_{1}l_{2}^{-1}r_{2} \cdots)^{-1} = \cdots r_{2}^{-1}l_{2}r_{1}^{-1}l_{1} \mid r_{0}^{-1}l_{0}r_{-1}^{-1}l_{-1} \cdots
\]

**Lemma 3.4.** [6] Lemma 1.3.33, Corollary 1.3.43 Let \(C\) be a homotopy \(I\)-word and \(i \in I\).

(i) If \(I = \{0, \ldots, m\}\) then \(v_{C^{-1}}(i) = v_{C}(m - i), \mu_{C^{-1}}(i) = \mu_{C}(m - i) - \mu_{C}(m)\) and there is an isomorphism of complexes \(P(C^{-1}) \to P(C)[\mu_{C}(m)]\) given by \(b_{i,C} \mapsto b_{C^{-1},-m-i}\).

(ii) If \(I\) is infinite then \(v_{C^{-1}}(i) = v_{C}(-i), \mu_{C^{-1}}(i) = \mu_{C}(-i)\) and there is an isomorphism of complexes \(P(C^{-1}) \to P(C)\) given by \(b_{i,C} \mapsto b_{-i,C^{-1}}\).

(iii) If \(I = Z\) and \(t \in Z\) then \(v_{C}(i + t) = v_{C[t]}(i), \mu_{C}(i + t) = \mu_{C[t]}(i) + \mu_{C}(t)\) and there is an isomorphism of complexes \(P(C[t]) \to P(C)[\mu_{C}(t)]\) given by \(b_{i,C} \mapsto b_{C[t],i-t}\).

**Definition 3.5.** [6] Definitions 1.3.42 and 1.3.45 We say \(C\) is periodic if \(I_C = Z, C = C[p]\) and \(\mu_{C}(p) = 0\) for some \(p > 0\). In this case the minimal such \(p\) is the period of \(C\), and we say \(C\) is \(p\)-periodic. We say \(C\) is aperiodic if \(C\) is not periodic.

If \(C\) is periodic of period \(p\) then by Lemma 3.4, \(P^n(C)\) is a \(\Lambda\)-\(R[T,T^{-1}]\)-bimodule where \(T\) acts on the right by \(b_i \mapsto b_{i-p}\). By translational symmetry the map \(d^n_{P(C)} : P^n(C) \to P^{n+1}(C)\) is \(\Lambda \otimes_R R[T,T^{-1}]\)-linear. For an \(R[T, T^{-1}]\)-module \(V\) we define \(P^n(C, V)\) by \(P^n(C, V) = P^n(C) \otimes_{R[T,T^{-1}]} V\) and \(d^n_{P(C,V)} = d^n_{P(C)} \otimes id_V\) for each \(n \in Z\).
Lemma 3.6. Let $n \in \mathbb{Z}$ and $C$ be a $p$-periodic homotopy word, and let $V$ be an $R[T, T^{-1}]$-module which is free as an $R$-module.

(i) Letting $\langle n, p \rangle = \mu_C^{-1}(n) \cap [0, p-1]$ gives $\mu_C^{-1}(n) = \{ j + ps \mid j \in \langle n, p \rangle, s \in \mathbb{Z} \}$.

(ii) Let $L$ be the $\Lambda$-module $\bigoplus_{j \in \langle n, p \rangle} \Lambda b_j \otimes_R V$. The map $P^n(C) \times V \to L$ given by

$$\left( \sum_{i \in \mu_C^{-1}(n)} \lambda_i b_i, v \right) \mapsto \sum_{j \in \langle n, p \rangle} \left( \sum_{s \in \mathbb{Z}} \lambda_{j+p-s} b_j \otimes T^{-s} v \right)$$

is $R[T, T^{-1}]$-balanced, and defines a $\Lambda$-module isomorphism $\kappa_n : P^n(C, V) \to L$.

(iii) The $\Lambda$-module $P^n(C, V)$ is projective. Moreover, if $V$ has an $R$-basis $\{ v_\lambda \mid \lambda \in \Omega \}$ then the underlying $\Lambda$-module of $P(C, V)$ is generated by $\{ b_i \otimes v_\lambda \mid 0 \leq i \leq p-1, \lambda \in \Omega \}$.

Definition 3.7. If $C$ is a $p$-periodic homotopy word then $C = \ldots EE \ldots$ for some unique homotopy word $E = l_1^{-1} r_1 \ldots l_p^{-1} r_p$. As in Lemma 3.6, let $V$ be an $R[T, T^{-1}]$-module with basis $\{ v_\lambda \mid \lambda \in \Omega \}$, and for each $n \in \mathbb{Z}$ let $\langle n, p \rangle = \mu_C^{-1}(n) \cap [0, p-1]$. Let $T^{\pm 1} v_\lambda = \sum_{\mu} a_{\mu} v_\mu$ for some $a_{\mu} \in R$.

Fix $n \in \mathbb{Z}$. Let $P^n(E, V)$ be the $\Lambda$-module $\bigoplus_{\Omega} \oplus_{j} \Lambda b_{j,C}$ where $j$ runs through $\langle n, p \rangle$. Let $c_{j,\lambda}$ denote the copy of $b_j C$ in the summand indexed by $(j)$ and $\lambda \in \Omega$. Define the $\Lambda$-module map $d_{\mu}^{P(E,V)} : P^n(E, V) \to P^{n+1}(E, V)$ by sending $c_{j,\lambda} \mapsto c_{j,\lambda}^+ + c_{j,\lambda}^-$ where

$$c_{j,\lambda}^+ = \begin{cases} \alpha c_{j+1,\lambda} & \text{(if } j < p-1, l_{j+1}^{-1} r_{j+1} = d_{i(\alpha)}^{-1}) \\ \alpha(\sum_{\mu} a_{\mu} c_{,\mu}^{-}) & \text{(if } j = p-1, l_{p}^{-1} r_{p} = d_{i(\alpha)}^{-1}) \\ 0 & \text{(otherwise)} \end{cases}$$

$$c_{j,\lambda}^- = \begin{cases} \beta c_{j-1,\lambda} & \text{(if } j > 0, l_{j}^{-1} r_{j} = \beta^{-1} d_{i(\beta)}^{-1}) \\ \beta(\sum_{\mu} a_{\mu} c_{p-1,\mu}^{-}) & \text{(if } j = 0, l_{p}^{-1} r_{p} = \beta^{-1} d_{i(\beta)}^{-1}) \\ 0 & \text{(otherwise)} \end{cases}$$

Remark 3.8. Fixing the notation from Definition 3.7, it is straightforward to check that $P(E, V)$ defines a complex of projective modules. Furthermore, it is straightforward to check that the composition of the isomorphism $\kappa_n$ from Lemma 3.6(ii) together with the canonical isomorphism $\bigoplus \Lambda b_j C \otimes_R V \to P^n(E, V)$ of $\Lambda$-modules (where $j$ runs through $\langle n, p \rangle$) defines an isomorphism of complexes $P(C, V) \to P(E, V)$. In particular, this means that the complex $P(E, V)$ is defined, up to isomorphism, independently of the choice of an $R$-basis for $V$.

Example 3.9. (See also Examples 1.3.41 and 1.3.46). Let $\Lambda = k[[x, y]]/(xy)$. Note that

$$C = \ldots y^{-3} d_y d_x^{-1} x y^{-2} d_y d_x^{-1} x^5 y^{-3} d_y d_x^{-1} x \mid y^{-2} d_y d_x^{-1} x^5 y^{-3} d_y d_x^{-1} x y^{-2} d_y d_x^{-1} x^5 \ldots$$

is 4-periodic. The action of $T$ on $P^0(C)$ and $P^1(C)$ may be depicted by dashed arrows such as

```
\begin{tikzpicture}
  \node (A) at (0,0) {$\Lambda$};
  \node (B) at (1,0) {$\Lambda$};
  \node (C) at (2,0) {$\Lambda$};
  \node (D) at (3,0) {$\Lambda$};
  \node (E) at (4,0) {$\Lambda$};
  \node (F) at (5,0) {$\Lambda$};
  \node (G) at (6,0) {$\Lambda$};
  \node (H) at (7,0) {$\Lambda$};

  \draw[->] (A) -- (B);
  \draw[->] (B) -- (C);
  \draw[->] (C) -- (D);
  \draw[->] (D) -- (E);
  \draw[->] (E) -- (F);
  \draw[->] (F) -- (G);
  \draw[->] (G) -- (H);

  \draw[dashed,->] (A) -- (B);
  \draw[dashed,->] (B) -- (C);
  \draw[dashed,->] (C) -- (D);
  \draw[dashed,->] (D) -- (E);
  \draw[dashed,->] (E) -- (F);
  \draw[dashed,->] (F) -- (G);
  \draw[dashed,->] (G) -- (H);

  \node at (-1,0) {$\ldots$};
  \node at (1,0) {$\Lambda$};
  \node at (2,0) {$\Lambda$};
  \node at (3,0) {$\Lambda$};
  \node at (4,0) {$\Lambda$};
  \node at (5,0) {$\Lambda$};
  \node at (6,0) {$\Lambda$};
  \node at (7,0) {$\Lambda$};

  \node at (-1,1) {$\ldots$};
  \node at (1,1) {$\Lambda$};
  \node at (2,1) {$\Lambda$};
  \node at (3,1) {$\Lambda$};
  \node at (4,1) {$\Lambda$};
  \node at (5,1) {$\Lambda$};
  \node at (6,1) {$\Lambda$};
  \node at (7,1) {$\Lambda$};

  \node at (8,0) {$\ldots$};
\end{tikzpicture}
```
Define the \( k[[t]] [T, T^{-1}]\)-module \( V = k[[t]] \oplus k[[t]] \) by \( T(f(t), g(t)) = (f(t) - tg(t), tf(t) + \lambda g(t)) \) where \( 0 \neq \lambda \in k \). Note that \( P(E, V) \simeq P(C, V) \) where \( P(E, V) \) is depicted by

\[
A = \begin{pmatrix}
1 & -t \\
t & \lambda
\end{pmatrix}^{-1}
\]

where \( E = y^{-2}d_0x^{-1}x^5y^{-3}d_0x^{-1}x \) (so that \( C = \infty \)). Note that the matrix \( A^{-1} \) defines the action of \( T \) on \( V \). Furthermore \( A^{-1} \) has determinant \( \lambda + t^2 \), and the inverse of \( \lambda + t^2 \) is \( \sum_{n=0}^{\infty} (-1)^{n} \lambda^{-(n+1)} t^{2n} \in k[[t]] \).

**Lemma 3.10.** Let \( C \) be a \( p \)-periodic homotopy word of the form \( C = \infty \) where \( E = l_p^{-1}r_1 \ldots l_p^{-1}r_p \) is a homotopy word. Suppose \( l_p^{-1}r_1^{-1}r_1 = d_{(\alpha)}^{-1} \alpha \beta^{-1} d_{(\beta)} \) for some \( \alpha, \beta \in \mathbb{P} \). Let \( U \) and \( V \) be \( R[T, T^{-1}] \) modules which are free as \( R \) modules. If \( k \otimes_R U \simeq k \otimes V \) as \( k[T, T^{-1}] \) modules then \( P(E, U) \simeq P(E, V) \) as complexes.

Before giving the proof of Lemma 3.10 we recall why, over a semiperfect ring, any homomorphism between projective modules, which becomes an isomorphism upon factoring the radicals, must have been an isomorphism.

**Remark 3.11.** Let \( \Gamma \) be a semiperfect ring, and so \( \Gamma = \bigoplus_s \Gamma e_s = \bigoplus_s e_s \Gamma \) for some finite set \( E = \{ e_s \mid s \in S \} \) of idempotents where the left (respectively right) \( \Gamma \) module \( \Gamma e \) (respectively \( e \Gamma \)) is local with unique maximal submodule \( \text{rad}(\Gamma)e \) (respectively \( \text{rad}(\Gamma)) \). A \( \Gamma \)-module is said to be quasi-free if it is a direct sum of modules of the form \( \Gamma e_s \). We write \( \Gamma \text{-quas} \) for the full subcategory of \( \Gamma \text{-Mod} \) consisting of finitely generated quasi-free modules. We say a full subcategory \( \mathcal{A} \) of \( \Gamma \text{-Mod} \) of all \( \Gamma \) modules reflects monomorphisms (respectively epimorphisms, respectively isomorphisms) modulo the radical provided that for any homomorphism \( f : N \to L \) in \( \mathcal{A} \), if the induced morphism \( f : N/\text{rad}(N) \to L/\text{rad}(L) \) is a monomorphism (respectively epimorphism, respectively isomorphism) then \( f \) is a monomorphism (respectively epimorphism, respectively isomorphism). Note that every quasi-free \( \Gamma \)-module is a union of its finitely-generated quasi-free summands [6 Lemma 3.1.40]. So, by [6 Lemma 3.1.39], if \( \Gamma \text{-quas} \) reflects monomorphisms (respectively epimorphisms, respectively isomorphisms) modulo the radical, then so does the full subcategory \( \Gamma \text{-Proj} \) of \( \Gamma \text{-Mod} \) consisting of projective modules.

We now explain why \( \Gamma \text{-Proj} \) reflects isomorphisms modulo the radical. By the above, assuming \( f : N \to L \) is a homomorphism in \( \Gamma \text{-quas} \) such that \( \tilde{f} : N/\text{rad}(N) \to L/\text{rad}(L) \) is an isomorphism, it suffices to show \( f \) is an isomorphism. Write \( h \) for the inverse of \( f \). Since \( N \) and \( L \) are quasi-free the canonical epimorphisms \( N \to N/\text{rad}(N) \) and \( L \to L/\text{rad}(L) \) are projective covers. Since \( \tilde{f} \) is a section, and since \( N \) and \( L \) are (finitely generated and) projective, \( f \) is a section, by (the proof of) [20 Lemma 2.2]. Write \( g \) for the corresponding retraction. Since \( gf \) is the identity on \( N \), \( g \tilde{f} \) is the identity on \( N/\text{rad}(N) \), and so \( h = g \tilde{f} \) (note inverses of isomorphisms are unique). This means \( g \tilde{f} \) is an isomorphism, and so as above \( g \) is a section. Again, since inverses are unique, this means \( f \) is an isomorphism.

**Proof of Lemma 3.10.** Let \( \tilde{r} : k \otimes_R U \to k \otimes V \) be the said isomorphism of \( k[T, T^{-1}] \) modules, that is, an isomorphism of \( k \) vector spaces which respects the action of \( T \). For any \( R \)-basis \( (u_\lambda \mid \lambda \in \Omega) \) of a free \( R \)-module \( W \) let \( \tilde{u}_\lambda = 1 \otimes_R u_\lambda \) for each \( \lambda \), which defines a \( k \)-basis \( (\tilde{w}_\lambda \mid \lambda \in \Omega) \) of \( k \otimes_R W \). If \( W \) is also an \( R[T, T^{-1}] \) module, then by Remark 3.5 any choice of an \( R \)-basis defines \( P(E, W) \) up to isomorphism. Altogether we can choose \( R \)-bases \( (u_\lambda \mid \lambda \in \Omega) \) and \( (v_\lambda \mid \lambda \in \Omega) \) of \( U \) and \( V \) respectively such that \( \tilde{r}(T^{-1} \tilde{u}_\lambda) = T^{-1} \tilde{v}_\lambda \) for each \( \lambda \).
Let $c_{j,\lambda}$ (respectively $e_{j,\lambda}$) denote the copy of $b_{j,c}$ in the summand of $P^n(E, U)$ (respectively $P^n(E, V)$) indexed by $j$ and $\lambda$. Let $\phi(c_{j,\lambda}) = e_{j,\lambda}$ for all $\lambda$ and all $j$ with $0 < j \leq p - 1$. To define a morphism $\phi : P(E, U) \rightarrow P(E, V)$ of complexes it suffices to define $\phi(c_{0,\lambda}) \in P^0(E, V)$ for all $\lambda$, and explain why extending these assignments (linearly over $\Lambda$) satisfies $\phi(d_{P(E, U)}(c_{j,\lambda})) = d_{P(E, V)}(\phi(c_{j,\lambda}))$ for all $j$ and $\lambda$. We begin by defining $\phi(c_{0,\lambda})$.

For each $\lambda$ let $\tau(T^{n+1}_0) = \sum_\mu s^+_\mu \bar{v}_\mu$ and $T^{n+1}_0 = \sum_\mu \bar{v}_\mu$ for some $s^+_\mu, r^+_{\mu} \in R$. Since $\tau(T^{-1}_0) = T^{-1}_0 \tilde{v}_\lambda$ for each $\lambda$, this means that $\sum_\mu (s^+_{\mu} - r^+_{\mu})v_\mu = \sum_\mu z_{\mu}z_{\mu}v_\mu$ for some $z_{\mu} \in \mathfrak{m}$. Since $V$ has $R$-basis $(v_\lambda | \lambda \in \Omega)$ this means $s^+_{\mu} - r^+_{\mu} = z_{\mu}v_\mu$ for all $\mu$. Since $EE$ is a homotopy word, $t(E) = h(E)$ and $-s(E^{-1}) = s(E)$, which means $t(\alpha) = t(\beta)$ and $t(\alpha) \neq t(\beta)$.

Let $w = t(\alpha)$. Since $\Lambda$ is rad-nilpotent modulo $\mathfrak{m}$ we have $e_\omega z_{\mu}v_\mu \in \mathfrak{a} \Lambda$ where the sum runs over $\alpha \in \Lambda(\rightarrow w)$. Without loss of generality we assume $\Lambda(\rightarrow w) = \{x, y\}$ where $x \neq y$.

Hence for all $\mu, \lambda \in \Omega$ there exists some $\gamma_{\mu, x, \mu, \lambda, y} \in \Lambda$ such that $e_\omega z_{\mu}v_\mu = x^\gamma_{\mu, x, \mu, \lambda, y}$. Without loss of generality $\alpha x, \beta y \not\in \mathfrak{p} \supset \beta x, \alpha y$. For all $\lambda \in \Omega$ let $\delta_{\lambda,\lambda} = 1$ and $\delta_{\lambda, \mu} = 0$ for all $\mu \in \Omega$ with $\mu \neq \lambda$. For each $\mu \in \mathfrak{m}$ extend the assignment $\phi(c_{0,\mu}) = \sum_\eta (\delta_{\mu, \eta} + r^+_{\eta}\gamma_{\eta,\mu,\eta})e_{\eta,\mu}$ linearly over $\Lambda$, where the sum runs over $\eta \in \Omega$. We now check $\phi(d_{P(E, V)}(c_{j,\lambda})) = d_{P(E, V)}(\phi(c_{j,\lambda}))$ for all $j$ and all $\lambda$. By our assumption on the form of the homotopy word $E$ we have that: $c_{0,\lambda} = 0$ and $e_{0,\lambda} = 0$; if $j > 1$ then $\phi(c_{j,\lambda}) = e_{j,\lambda}$; and if $j < p - 1$ then $\phi(c_{j,\lambda}) = e_{j,\lambda}$. In case $j = 1$ then

$$
\phi(c_{1,\lambda}) = \frac{1}{\beta}(\beta e_{0,\lambda}) = \beta(e_{0,\lambda}) = \beta(\sum_\eta (\delta_{\eta, \eta} + r^+_{\eta}\gamma_{\eta,\eta,\eta})e_{\eta,\eta}) = \sum_\eta \delta_{\eta,\eta}e_{\eta,\eta} + \sum_\eta \beta y_{\eta,\eta,\eta}e_{\eta,\eta} = \beta e_{\eta,\eta} = e_{1,\lambda}
$$

In case $j = 0$ then

$$
\phi(c_{0,\lambda}) = \alpha(\sum_\eta r^-_{\eta}\gamma_{\eta,\eta,\eta})e_{\eta,\eta} = \sum_\eta r^-_{\eta}\gamma_{\eta,\eta,\eta}e_{\eta,\eta} = \sum_\eta \alpha r_{\eta,\eta,\eta}e_{\eta,\eta} = \sum_\eta \alpha r^+_{\eta}e_{\eta,\eta} = \sum_\eta \alpha r_{\eta,\eta,\eta}e_{\eta,\eta} = \sum_\eta \alpha r_{\eta,\eta,\eta}e_{\eta,\eta}
$$

Altogether we have shown that $\phi(c_{j,\lambda}) = e_{j,\lambda}$, and so $\phi$ defines a morphism of complexes. It suffices to explain why each map $\phi^n$ is an isomorphism of $\Lambda$-modules. By construction, and by Remark 3.14 to show each $\phi^n$ is an isomorphism it will be enough to explain why the induced morphism $\tilde{\phi} : P^n(E, U) \rightarrow P^n(E, V)$ is an isomorphism. By construction, $\tilde{\phi}$ maps $c_{1,\lambda} + \text{rad}(P^n(E, U))$ to $e_{1,\lambda} + \text{rad}(P^n(E, V))$ (for all appropriate $j$ and $n$), which is clearly an isomorphism.

**Definition 3.12.** [6 Definition 1.3.48] A string complex has the form $P(C)$ where $C$ is aperiodic. If $V$ is an $R[T, T^{-1}]$-module we call $P(C, V)$ a band complex provided $C$ is a periodic homotopy $\mathbb{Z}$-word, $V$ is an indecomposable $R[T, T^{-1}]$-module and $V$ is free as an $R$-module.

**4. How the classification works.**

**Notation 4.1.** If $A$ is an abelian category let $\mathcal{P}_A$ be the full subcategory of $\mathcal{A}$ consisting of the projective objects and let $\mathcal{C}(\mathcal{P}_A)$ be the category of complexes in $\mathcal{P}_A$. We say $A$ has **enough radicals** if any object $X$ of $\mathcal{A}$ has a set of maximal subobjects, whose infimum, the radical $\text{rad}(X)$ of $X$, exists in $\mathcal{A}$. If $A$ is an abelian category with enough radicals let $\mathcal{C}_{\text{min}}(\mathcal{P}_A)$ and $\mathcal{K}_{\text{min}}(\mathcal{P}_A)$ be the full subcategories of $\mathcal{C}(\mathcal{P}_A)$ and $\mathcal{K}(\mathcal{P}_A)$ consisting of homotopically minimal complexes: objects $M$ in $\mathcal{C}(\mathcal{P}_A)$ such that $\text{im}(d^n_M) \subseteq \text{rad}(M^{n+1})$ for all $n \in \mathbb{Z}$. Here let $\Xi_A : \mathcal{C}_{\text{min}}(\mathcal{P}_A) \rightarrow \mathcal{K}_{\text{min}}(\mathcal{P}_A)$ be the restriction of the quotient functor $\mathcal{C}(\mathcal{P}_A) \rightarrow \mathcal{K}(\mathcal{P}_A)$.
ASSUMPTION 4.2. In \([3]\) we let: \(M\) be an abelian category with enough radicals and small coproducts; \(\mathcal{N}\) be an abelian subcategory of \(M\); \(\mathcal{J}\) be an index set; and \(S_i : A_i \to C_{\min}(\mathcal{P}_M)\) and \(F_i : K_{\min}(\mathcal{P}_M) \to X_i\) be additive functors for all \(i \in \mathcal{J}\).

To prove Theorems \([11, 12, 13]\) and \([24, 25, 26]\) we choose \(M = \Lambda - \text{Mod}\) and \(\mathcal{N} = \Lambda - \text{mod}\). Note \(\mathcal{N}\) has enough projective covers by Corollary \([2, 11]\)vi. In Lemma \([4, 5]\) we adapt an interpretation by Ringel \([29]\) §3, p. 22, Lemma] of a result by Gabriel \([16]\) §4, Structure theorem].

COROLLARY 4.3. \([6]\) Corollary 3.2.25] (see also \([23]\) Proposition B.2]). If \(\mathcal{N}\) has enough projective covers then the subcategory \(K_{\min}(\mathcal{P}_\mathcal{N})\) of \(K(\mathcal{P}_\mathcal{N})\) is dense.

We now provide the categorical blueprint that was discussed at the end of \([1]\).

DEFINITION 4.4. \([6]\) Definition 2.6.4] Suppose \(\mathcal{N}\) has enough projective covers. We say that the collection \(\{(S_i, F_i) \mid i \in \mathcal{J}\}\) detects the objects in \(K(\mathcal{P}_\mathcal{N})\) if the following statements hold.

(i) For any \(i \in \mathcal{J}\):
  (FFI) the functor \(F_i \Xi_M S_i\) is dense and reflects isomorphisms;
  (FFII) \(F_i \Xi_M S_i \simeq 0\) for each \(j \in \mathcal{J}\) with \(j \neq i\);
  (FFIII) \(F_i\) preserves small coproducts; and
  (FFIV) for each object \(M \in K_{\min}(\mathcal{P}_\mathcal{N})\) there exists an object \(A_{i,M} \in A_i\) and a morphism \(\gamma_{i,M} : \Xi_N(S_i(A_{i,M})) \to M\) in \(K_{\min}(\mathcal{P}_M)\) such that \(F_i(\gamma_{i,M})\) is an isomorphism.

(ii) For all morphisms \(\varphi : N \to M \in C_{\min}(\mathcal{P}_M)\):
  (FFV) if \(M\) lies in \(C_{\min}(\mathcal{P}_\mathcal{N})\) and \(F_i(\Xi_M(\varphi))\) is epic for all \(i \in \mathcal{I}\) then each \(\varphi^n\) is epic; and
  (FFVI) if \(N = \bigoplus_{i \in \mathcal{I}} S_i(A_i)\) and \(F_i(\Xi_M(\varphi))\) is monic for each \(i \in \mathcal{I}\) then each \(\varphi^n\) is monic.

LEMMA 4.5. \([6]\) Lemma 2.6.5] Suppose that \(\mathcal{N}\) has enough projective covers, and that \(\{(S_i, F_i) \mid i \in \mathcal{J}\}\) detects the objects in \(K(\mathcal{P}_\mathcal{N})\).

(i) Any object \(N\) of \(K(\mathcal{P}_\mathcal{N})\) is isomorphic to \(\bigoplus_{i \in \mathcal{I}} \Xi_N(S_i(A_{i,M}))\) for some \(M\) in \(K_{\min}(\mathcal{P}_\mathcal{N})\).

(ii) If \(M\) is an indecomposable object of \(K(\mathcal{P}_\mathcal{N})\) then there is some \(i \in \mathcal{J}\) and some indecomposable object \(A\) of \(A_i\) such that \(M \simeq \Xi_N(S_i(A))\).

(iii) If \(i \in \mathcal{J}\) and \(A\) is an indecomposable object of \(A_i\) then \(\Xi_M(S_i(A))\) is indecomposable.

Proof. (i) Since \(\mathcal{N}\) has projective covers we have \(N \simeq M\) for some object \(M\) of \(K_{\min}(\mathcal{P}_\mathcal{N})\) by Corollary \([4, 5]\) Let \(N_i = \Xi_M(S_i(A_{i,M}))\) for each \(i \in \mathcal{J}\). Fix \(l \in \mathcal{J}\) and let \(\iota_l : N_l \to \bigoplus_{i \in \mathcal{J}} N_i\) be the monic in the coproduct. For each \(i\) let \(g_i\) be the morphism defined by the identity on \(N_l\) if \(i = l\), and \(N_l \to 0\) otherwise. By the universal property of the coproduct there exists \(\pi_i : \bigoplus_{i \in \mathcal{J}} N_i \to N_l\) with \(\pi_i \iota_l = \text{id}\).

By FFI, for any set \(A\) and any collection \(X = \{X_a \mid a \in A\}\) of objects in \(K_{\min}(\mathcal{P}_M)\) there is an isomorphism \(\sigma_X : \bigoplus_a F_i(X_a) \to F(\bigoplus_a X_a)\) such that \(F_i(\bigoplus_a F(a)\sigma_X = \sigma_Y \bigoplus_a F_i(F(a))\) for any collection of morphisms \(\{f_a : X_a \to Y_a \mid a \in A\}\). Let \(A = \mathcal{J}\), and for each \(i \in A\) let \(f_i\) be the morphism defined by the identity on \(N_l\) if \(i = l\), and \(0 \to N_l\) otherwise. By FFI and the above there are isomorphisms \(\sigma_X : F_i(N_l) \to F_i(N_l)\) and \(\sigma_Y : F_i(\bigoplus_{i \in \mathcal{J}} N_l) \to F_i(N_l)\) with \(\sigma_Y F_i(\iota_l) = F_i(\text{id})\sigma_X\), so \(F_i(\iota_l) F_i(\pi_i) = \text{id}\).

By FFV and the coproduct universal property there exists \(\theta : \Xi_M(\bigoplus_{i \in \mathcal{J}} S_i(A_{i,M})) \to M\) in \(K_{\min}(\mathcal{P}_M)\) satisfying \(\theta_i = \gamma_{i,M}\) for each \(i \in \mathcal{J}\). Since \(\Xi_N\) is dense there is an object \(L\) in \(C_{\min}(\mathcal{P}_\mathcal{N})\) and an isomorphism \(\psi : \Xi_N(L) \to M\) in \(K_{\min}(\mathcal{P}_\mathcal{N})\). Since \(\Xi_M\) is full we have a morphism \(\varphi : \bigoplus_i S_i(A_{i,M}) \to L\) in \(C_{\min}(\mathcal{P}_M)\) with \(\Xi(\varphi) = \psi^{-1}\theta\).
By the above we have $F_i(\theta) = F_i(\gamma_{i,M})F_i(\pi_i)$ which is an isomorphism for each $i$ by previous paragraph, and so $F_i(\Xi_M(\varphi))$ is an isomorphism for each $i \in I$. By FFV and FFVI this shows $\varphi''$ is an isomorphism for each $n \in \mathbb{Z}$, so $\theta$ is an isomorphism. As $\Xi_M$ preserves small coproducts, altogether we have $M \simeq \bigoplus_{i \in I} \Xi_M(S_i(A_{i,M}))$.

(ii) By (i) $M \simeq \bigoplus_{i \in I} \Xi_N(S_i(A_{i,M}))$ and so $\Xi_N(S_i(A_{i,M})) = 0$ apart from when $i = t$ for some $t \in J$. Hence $M \simeq \Xi_N(S_t(A_t(M)))$. Given objects $X$ and $Y$ of $A_t$ for which $A_tM = X \oplus Y$ we have $\Xi_M(S_t(X)) = 0$ without loss of generality. This means $F_t(\Xi_M(S_t(0)))$ is an isomorphism where $0 : X \to 0$ in $A_t$. Since $F_t\Xi_MS_t$ reflects isomorphisms by FFII, $X = 0$.

(iii) If $F_i(\Xi_M(S_i(A))) = 0$ then $A = 0$ since $F_i\Xi_MS_i$ is dense and reflects isomorphisms by FFII. Hence $\Xi_M(S_i(A)) \neq 0$ and we suppose $\Xi_M(S_i(A)) = X' \oplus Y'$ for objects $X'$ and $Y'$ of $K_{\min}(P_M)$. So there are objects $X$ and $Y$ in $C_{\min}(P_M)$ with $S_i(A) = X \oplus Y$, $\Xi(X) = X'$ and $\Xi(Y) = Y'$. Hence, without loss of generality, $F_i(X') = 0$.

Label the monics of the coproduct $\iota_X : X \to X \oplus Y$ and $\iota_Y : Y \to X \oplus Y$, and epis of the product $\pi_X : X \oplus Y \to X$ and $\pi_Y : X \oplus Y \to Y$. For $j \in J$ with $j \neq i$ we have $F_j(X') \oplus F_j(Y') = F_j(\Xi(S_i(A))) = 0$ by FFII. Hence for any $j$ we have $F_j(\Xi_M(\iota_X)) = 0$ and so $F_j(\Xi_M(\iota_Y))$ is an isomorphism with inverse $F_j(\Xi_M(\iota_Y))$. Since $\pi_Y$ is an epimorphism in $C_{\min}(P_M)$ of the form $S_i(A) \to Y$ this means $\pi_Y^{t'}$ is monic for each $n \in \mathbb{Z}$ by FFVI. Since each $\pi_Y^{t'}$ is also an epimorphism, we have $X' = 0$.

In [17] and [18] we define functors $S_{B,D,n}$ and $F_{B,D,n}$ for each integer $n$ and certain pairs $B,D$ of homotopy words. This is done in a way so that the collection of $(S_{B,D,n}, F_{B,D,n})$ detects the objects in $K(\Lambda\text{-proj})$ (see Proposition [13.1]).

5. Linear relations and reductions

In previous applications of the functorial filtrations method to module classifications, so called refined functors were defined as quotients of subfunctors of the forgetful functor (sending a module over a $k$-algebra to its underlying vector space). These subfunctors were constructed using the language of linear relations studied by Mac Lane [25, §2]. To adapt the functorial filtrations method to classify complexes we use a similar approach.

Given $R$-modules $L$ and $M$ an $R$-linear relation from $L$ to $M$ (or on $M$ if $L = M$) is an $R$-submodule $C$ of the direct sum $L \oplus M$. We will often write relation to mean $R$-linear relation. This notion generalises the graph of an $R$-linear map $L \to M$. Let $Cm = \{n \in M : (m,n) \in C\}$ for any $m \in M$, and for a subset $S \subseteq M$ let $CS$ be the union $\bigcup CS$ over $s \in S$. When $C$ is the graph of a map $f$ then $CS$ is the image of $S$ under $f$. If $W$ is a relation from a module $L$ to $M$ the composition $CW$ is the relation from $L$ to $N$ given by pairs $(l,n) \in L \oplus N$ such that $(m,n) \in C$ and $(l,m) \in W$ for some $m \in M$. Now suppose $C$ is a relation on $M$. For any integer $n > 0$ let $C^n$ be the $n$-fold composition of $C$ with itself (so $C^1 = C$ and $C^2 = CC$). Let $C' = C'' \cap (C^{-1})''$, $C'' = C'' \cap (C')' \cap (C^{-1})''$ where, for a relation $D$ on $M$,

$$D' = \bigcup_{n>0} U^n, \quad D'' = \{m \in M \mid \exists m_0, m_1, m_2, \ldots \in M : m_0 = m, m_i \in Dm_{i+1} \forall i\}$$

The following result from [14] was written only in the context where $R$ is a field. The proof does not make use of this assumption, and generalises with no complication.

**Lemma 5.1.** [14] Lemmas 4.4 and 4.5 For any relation $C$ on $M$ we have: $C^2 \subseteq CC^2$; $C^0 = C^2 \cap C C^0$; $C^2 \subseteq C^{-1} C^2$; $C^0 = C^2 \cap C^{-1} C^2$. Consequently there is an $R$-module automorphism $\theta$ on $C^2/C^0$ defined by $\theta(m + C^0) = m' + C^0$ if and only if $m' \in C^2 \cap (C^0 + Cm)$.
Definition 5.2. A reduction of a relation $C$ on an $R$-module $M$ is a pair $(U, f)$ where: $U$ is an $R[T, T^{-1}]$-module; $U$ is free over $R$; $f : U \to M$ is $R$-linear; $C^d = \text{im}(f) + C^o$; and $f(Tu) \in Cf(u)$ for all $u \in U$. We say a reduction $(U, f)$ of $C$ meets in $m$ if $\{ u \in U : f(u) \in C^o \} \subseteq mU$.

When $R$ is a field, if $(U, f)$ is a reduction of a relation $C$ on $M$ which meets in 0, then by Corollary 1.4.33 $C$ is split in the sense of [14] p. 9: that is, there is an $R$-linear subspace $W$ of $M$ such that $C^d = W \oplus C^o$ and $#Cm \cap W = 1$ for each $m \in W$. In our setting $C^o$ need not have an complement as an $R$-submodule of $C^d$ (see [6] Example 1.4.35)). So, the following is a generalisation of [14] Lemma 4.6).

Lemma 5.3. Let $M$ be an $R$-module and $C$ a relation on $M$ such that $C^d/C^o$ is a finite-dimensional $R/m = k$-vector space. Then there is a reduction $(U, f)$ of $C$ which meets in $m$ where $U$ has free $R$-rank $\text{dim}_k(C^d/C^o)$.

Proof. Let $\theta$ be the induced $R$-module automorphism of $C^d/C^o$ from Lemma [5.1]. Let $A = (a_{ij})$ be the matrix of $\theta$ (with entries from $k$) with respect to a $k$-basis $v_1, \ldots, v_d$ of $C^d/C^o$. For each $i$ choose $v_i \in C^d$ such that $\overline{v_i} = v_i + C^o$ and for each $j$ choose $a_{ij} \in R$ such that $\overline{a_{ij}} = a_{ij} + m$. As $A \in \text{GL}_d(k)$, $\det(A) \neq 0$ and so if we let $A$ be the matrix $(a_{ij})$ we have $\det(A) \notin m$ and so $A \in \text{GL}_d(R)$ as $R$ is local. Now fix $j \in \{1, \ldots, d\}$.

Write $A = (a_{ij})$ be the matrix of $\theta$ (with entries from $k$) with respect to a $k$-basis $v_1, \ldots, v_d$ of $C^d/C^o$. For each $i$ choose $v_i \in C^d$ such that $\overline{v_i} = v_i + C^o$ and for each $j$ choose $a_{ij} \in R$ such that $\overline{a_{ij}} = a_{ij} + m$. As $A \in \text{GL}_d(k)$, $\det(A) \neq 0$ and so if we let $A$ be the matrix $(a_{ij})$ we have $\det(A) \notin m$ and so $A \in \text{GL}_d(R)$ as $R$ is local. Now fix $j \in \{1, \ldots, d\}$.

We have $\theta(\overline{v_i}) = \sum_{i=1}^d a_{ij}v_i + C^o$ as $mC^d \subseteq C^d$, and so by definition there is some $w_j \in Cw_j$ for which $\sum_{i=1}^d a_{ij}v_i = w_i \in C^o$. Let $z_j = w_j - \sum_{i=1}^d a_{ij}v_i$. Write $z_j = z_j^+ + z_j^-$ for elements $z_j^+ \in (C^{-1})^+ \cap C^d$ and $z_j^- \in (C^{-1})^- \cap C^d$.

Hence there are some integers $n_-$ and $n_+$, and a collection $\{ z_{j,n}^+ \mid n \in \mathbb{Z} \} \subseteq M$ for which: $z_{j,n}^+ \in Cz_{j,n+1}^+$ for each $n \in \mathbb{Z}$; $z_{j,n}^- = 0$ for each $n > n_-$; and $z_{j,n}^+ = 0$ for each $n < n_+$. Now for each $n \in \mathbb{Z}$ define the matrices $L_{+,n}$ by

$$L_{+,n} = \begin{cases} 0 & \text{if } n > 0 \\ (A^{-1})^{-1} & \text{otherwise} \end{cases} \quad L_{-,n} = \begin{cases} 0 & \text{if } n > 0 \\ -A^{-1} & \text{otherwise} \end{cases}$$

Write $L^\pm,n = (m_{ij}^\pm,n)_{i,j}$ for elements $m_{ij}^\pm,n \in R$. Note that $(m_{ij}^+,n)_{i,j}$ and $(m_{ij}^-,n)_{i,j}$ are finite. As in the proof of [14] Lemma 4.6] this gives $\sum_{j=1}^d a_{ij}u_i \in Cu_j$ where for each $i$ we let

$$u_i = v_i + \sum_{n \in \mathbb{Z}} \left( \sum_{k=1}^d m_{ki}^+ z_{k,n}^+ + \sum_{n \in \mathbb{Z}} \left( \sum_{k=1}^d m_{ki}^- z_{k,n}^- \right) \right).$$

Let $U = \bigoplus_{i=1}^d R$, $T(r_i) = (\sum_{j=1}^d a_{ij}r_j)$, and $f((r_i)) = \sum_{i=1}^d r_i u_i$. Since $C^d/C^o$ has $k$-basis $v_1, \ldots, v_d$, for any $m \in C^d$ there are elements $s_1, \ldots, s_d \in R$ such that writing $\overline{s_i} = s_i + m$ for each $i$ gives $m + C^o = \sum_i \overline{s_i}(v_i + C^o)$ which equals $\sum_i s_i u_i + C^o$. There is an element $x = \sum_i s_i u_i = f((s_i)) \in \text{im}(f)$ and an element $c \in C^o$ with $m = x + c$. This shows $C^d \subseteq \text{im}(f) + C^o$ and as $u_i \in C^d$ for each $i$ we have equality.

Since $mC^d \subseteq C^o$ we have $mU \subseteq \{ u \in U : f(u) \in C^o \}$. Conversely if $f(u) \in C^o$ where $f(u) = \sum_{i=1}^d r_i u_i$ then $0 = \sum_{i=1}^d \overline{r_i} u_i = \sum_{i=1}^d \overline{r_i} v_i$, and as $\overline{r_i} \in \mathbb{Z}$, $\overline{r_i}$ is an $R/m$-basis for $C^d/C^o$, this means $r_i + m = 0$ in $k$ (and hence $r_i \in m$) for each $i$. Hence $mU \supseteq \{ u \in U : f(u) \in C^o \}$. Now fix $u = (r_i) \in U$. By definition, $Tu = (\sum_{j=1}^d a_{ij}r_j)$, and so $f(Tu) = \sum_{j=1}^d r_j \sum_{i=1}^d a_{ij} u_i = \sum_{j=1}^d r_j Cu_j \subseteq C(f(u))$.}

In [6] we provide examples of $R$-linear relations.
6. Homotopy words and relations.

Assumption 6.1. In [6] we fix a homotopically minimal complex of projective \( \Lambda \)-modules \( M \). Hence \( M^i \) is a projective \( \Lambda \)-module and \( \text{im}(d^i_M) \subseteq \text{rad}(M^{i+1}) \) for each integer \( i \).

Notation 6.2. We abuse notation writing \( M \) for the projective \( \Lambda \)-module \( \bigoplus_{i \in \mathbb{Z}} M^i \), and let \( d_M \) be the \( \Lambda \)-module endomorphism \( \bigoplus_{i \in \mathbb{Z}} d_M \) of \( M \) sending \( \sum_i m_i \) to \( \sum_i d_M(m_i) \). If \( L \) is an \( R \)-module let \( \text{End}_{R}(L) \) be the ring of \( R \)-module endomorphisms of \( L \). For each vertex \( v \) define \( d_{v,M} \in \text{End}_{R}(e_v M) \) by the restriction of \( d_M \).

Lemma 6.3. Fix \( M \) as in Assumption 6.1

(i) \([6] \text{Lemma } 2.1.1 \) (see also [5] Lemma 5). If \( a, b \in A \):

(a) if \( v = h(b) = t(a) \) and \( ab \in P \), \( abm = 0 \) implies \( bm = 0 \) for all \( m \in M \);

(b) if \( v = t(a) \) then \( \{ m' \in e_v M \mid am' = 0 \} = \sum_{b' \in A(\rightarrow v)} ab' \in P \)'s \( b'M \); and

(c) if \( v = h(b) = h(a) \) and \( a \neq b \) the sum \( aM + bM \) is direct.

(ii) \([6] \text{Lemma } 2.1.2 \) For each \( \alpha \in A \) there exists \( d_{\alpha,M} \in \text{End}_{R}(e_{h(\alpha)} M) \) such that \( d_{v,M} = \sum_{\beta \in A(\rightarrow v)} d_{\beta,M} \) for any vertex \( v \). Furthermore for all \( \tau \in \mathbf{P} \) and all \( x \in e_{(\tau)} M \):

(a) if \( \exists \sigma \in A \) such that \( \tau \sigma \notin \mathbf{P} \), then \( d_{(\tau)M} (\tau x) = d_{\sigma,M} (x) \); and

(b) if \( \tau \sigma \notin \mathbf{P} \) for all \( \sigma \in A \), then \( d_{(\tau)M} (\tau x) = 0 \); and

(c) if \( h(\theta) = h(\tau) \) for some arrow \( \theta \notin \mathbf{P} \), then \( d_{\beta,M} (\tau x) = 0 \); and

(d) if \( h(\phi) = h(\tau) \) for some arrow \( \phi \), then \( d_{\beta,M} (\tau x) = 0 \); and

(e) if \( \tau x \in \text{im}(d_{(\tau)M}) \) then \( d_{\sigma,M} (x) = 0 \) for any arrow \( \sigma \) such that \( \tau \sigma \in \mathbf{P} \).

The proof of Lemma 6.3(i) is straightforward and omitted. The proof Lemma 6.3(ii) involves the definition of the \( R \)-module endomorphisms \( d_{\alpha,M} : e_{h(\alpha)} M \to e_{h(\alpha)} M \) which motivate the introduction of homotopy letters of the form \( d^\alpha_{\gamma} \).

Proof of Lemma 6.3(ii). Since \( M \) is projective we have \( \text{rad}(M) = \text{rad}(\Lambda) M \) (see for example [24] Theorem 24.7)], and by Assumption 6.1 \( \text{im}(d_{v,M}) \subseteq e_v \text{rad}(\Lambda) M \). Note \( e_v \text{rad}(\Lambda) = \bigoplus \beta A \) where \( \beta \) runs through \( \mathbf{A}(\rightarrow v) \), and so \( e_v \text{rad}(M) = \sum \beta M \) which is a direct sum by part (iv). For any \( \gamma \in \mathbf{A}(\rightarrow v) \) let \( \pi_{\gamma} : \bigoplus \beta M \to \gamma M \) and \( \tau_{\gamma} : \gamma M \to \bigoplus \beta M \) be the natural projections and inclusions of \( R \)-modules. Define \( d_{\alpha,M} \) by \( d_{\alpha,M} (m) = \iota_{\alpha} \pi_{\gamma} (d_{v,M} (m)) \). The proof of parts (a), (b), (c), (d) and (e) of (ii) follow from part (i).

Definition 6.4. [6] Example 1.4.2, Definitions 1.4.9 and 2.1.5 \( \) If \( p \in \mathbf{P} \) and \( a \in A \) let \( \text{rel}^p(M) = \{ (m, pm) \mid m \in e_{t(p)} M \} \) and \( \text{rel}^p_{\alpha}(M) = \{ (m, d_{\alpha,M}(m)) \mid m \in e_{h(\alpha)} M \} \).

If \( v \) is a vertex and \( C = 1_{v,\pm} \) let \( \text{rel}^C(M) \) be the relation \( \{ (m, m) \mid m \in e_v M \} \). Now let \( C = l_1^{-1} r_1 \cdots l_n^{-1} r_n \) be a non-trivial homotopy word. For each \( i \) with \( 0 < i < n \) let

\[
\text{rel}^C_i(M) = \begin{cases} \text{rel}^\gamma_i(M) \text{rel}^l_i(M) \text{rel}^d_i(M) & \text{if } l_i^{-1} r_i = \gamma^{-1} d_i(M), \\ \text{rel}^\gamma_i(M) \text{rel}^l_i(M) \text{rel}^d_i(M) & \text{if } l_i^{-1} r_i = d_i(M). \end{cases}
\]

and let \( \text{rel}^C(M) = \text{rel}^C_1(M) \cdots \text{rel}^C_n(M) \), the \( n \)-fold composition of these relations.

Let \( \gamma \) be a homotopy letter (that is, one of \( \gamma, \gamma^{-1}, d_{\alpha} \) or \( d_{\alpha}^{-1} \) for some path \( \gamma \in \mathbf{P} \) or some arrow \( \alpha \)). If \( U \) is a subset of \( e_{t(q)} M \) then define the subset \( qU \) of \( e_{t(q)} M \) by

\[
\gamma U = \{ \gamma m \in e_{h(\gamma)} M \mid m \in U \}, \quad \gamma^{-1} U = \{ m \in e_{t(q)} M \mid \gamma m \in U \}, \quad d_{\alpha} U = \{ d_{\alpha,M}(m) \in e_{h(\alpha)} M \mid m \in U \}, \quad d_{\alpha}^{-1} U = \{ m \in e_{h(\alpha)} M \mid d_{\alpha,M}(m) \in U \}.
\]

For any vertex \( v \) and any subset \( U \) of \( e_v M \) let \( 1_{v,\pm} U = U \). When \( U = e_{t(q)} M \) we let \( qM = qU \). Similarly when \( U = e_{t(q)} \text{rad}(M) \) we let \( q rad(M) = qU \).
 Furthermore, given an arrow \( \alpha \) lies in that if \( b \) ranges over all arrows of homotopy letters of the form \( \alpha \) (for \( \alpha \in \mathbf{A} \)) indicate the action of \( \Lambda \). Their shape indicate the indecomposable summands \( \Lambda \)-\( \mathbf{e}_7 \), \( \Lambda \)-\( \mathbf{e}_2 \), \( \Lambda \)-\( \mathbf{e}_4 \) and \( \Lambda \)-\( \mathbf{e}_6 \) of \( \mathcal{P}(C) \). The dotted arrows indicate the action of \( d_M \). Note \( d_M(m_2) = tm_1 + hm_3 \). For a subset \( U \subseteq \mathbf{e}_7 \), \( CU \) is the set of \( m_0 \in \mathbf{e}_6 \) above where \( m_3 \in U \).

**Example 6.5.** (See also [6] Examples 2.1.6 and 2.1.8]). Consider the finite-dimensional gentle algebra \( \Lambda = kQ/J \) from Example 6.2. Let \( C = s^{-1}d_{s^{-1}}d_{s^{-1}}r_{h} \). Then \( ref^C(M) = \{ (m_3,m_0) \mid sm_0 = d_{s,M}(m_1), tm_1 = d_{t,M}(m_2), d_{h,M}(m_2) = hm_3 \text{ for some } m_1, m_2 \} \).

The elements \( m_0, \ldots, m_3 \) may be arranged using schemas from [6] Remark 1.3.36, as follows

Only the elements \( m_0, m_1, m_2 \) and \( m_3 \) are needed to describe the set \( ref^C(M) \). The bold arrows labelled by homotopy letters of the form \( \alpha \) (for \( \alpha \in \mathbf{A} \)) indicate the action of \( \Lambda \). Their shape indicate the indecomposable summands \( \Lambda \)-\( \mathbf{e}_7 \), \( \Lambda \)-\( \mathbf{e}_2 \), \( \Lambda \)-\( \mathbf{e}_4 \) and \( \Lambda \)-\( \mathbf{e}_6 \) of \( \mathcal{P}(C) \). The dotted arrows indicate the action of \( d_M \). Note \( d_M(m_2) = tm_1 + hm_3 \). For a subset \( U \subseteq \mathbf{e}_7 \), \( CU \) is the set of \( m_0 \in \mathbf{e}_6 \) above where \( m_3 \in U \).

**Corollary 6.6.** [6] Corollary 2.1.9] If \( a \) is an arrow then \( a^{-1}d_a \text{ rad}(M) \subseteq e_{l(a)} \text{ rad}(M) \). Furthermore, given an arrow \( b \) with \( ab \in \mathbf{P} \) we have \( (ab)^{-1}adbM = b^{-1}dbM \).

**Proof.** From the definition of \( d_{a,M} \) from Lemma 6.3(ii) we have \( a^{-1}d_a \text{ rad}(M) = a^{-1}d_a \alpha aM \). By parts (a) and (b) of Lemma 6.3(ii) we have that if \( b \) exists then \( a^{-1}d_a \alpha aM = a^{-1}adbM \), and otherwise \( a^{-1}d_a \alpha aM = a^{-10} \). If \( b \) exists then by Lemma 6.3(iia) any \( m \in a^{-1}d_a \alpha aM \) satisfies \( am = ad_{b,M}(m') \text{ for some } m' \in e_{h(b)}M \), and so \( m - d_{b,M}(m') \in a^{-10} = \sum b'M \) where the sum ranges over all arrows \( b' \) with \( ab' \notin \mathbf{P} \), by Lemma 6.3(b). As \( im(d_{b,M}) \subseteq \text{ rad}(M) \) this shows that if \( b \) exists then \( a^{-1}d_a \alpha aM \subseteq e_{l(a)} \text{ rad}(M) \). If \( b \) does not exist then \( a^{-1}d_a \alpha aM = a^{-1}d_a0 = a^{-1} = \sum b'M \), concluding that \( a^{-1}d_a \alpha aM \subseteq e_{l(a)} \text{ rad}(M) \).

Now assume \( b \) exists. If \( m \in b^{-1}a^{-1}adbM \) there exists \( m' \in e_{h(b)}M \) such that \( bm - d_{b,M}(m') \) lies in \( a^{-10} \). As \( bm - d_{b,M}(m') = bm'' \) for some \( m'' \in M \), we have \( abm'' = 0 \) which means \( bm'' = 0 \) by Lemma 6.3(iia). This gives \( b^{-1}a^{-1}adbM \subseteq b^{-1}dbM \). The reverse inclusion is clear.

**Corollary 6.7.** [6] Corollary 2.1.10] If \( \alpha, \beta, \gamma, \sigma, \alpha \beta \in \mathbf{P} \), \( h(\gamma) = h(\sigma) \) and \( l(\gamma) \neq l(\sigma) \) then we have the following inclusions

\[
\beta^{-1}d_{l(\beta)}M \subseteq (\alpha \beta)^{-1}d_{l(\alpha)}M, \quad d_{l(\alpha)}^{-1} \alpha \beta M \subseteq d_{l(\alpha)}^{-1} \alpha M,
\alpha^{-1}d_{l(\alpha)}M \subseteq d_{l(\beta)}^{-1} \beta 0, \quad \gamma M \subseteq d_{l(\gamma)}^{-1} \sigma 0, \quad d_{l(\sigma)}M \subseteq d_{l(\alpha)}^{-1} \sigma 0.
\]

In Definition 6.12 we define functors \( C^\pm : \mathcal{C}_{\text{min}}(\mathbf{A-Proj}) \rightarrow \mathcal{R-Mod} \) (see Corollary 6.13). To do so we adapt ideas used by Ringel [29] which were developed by Crawley-Boevey [14].

**Definition 6.8.** Choose a sign \( s(q) \in \{ \pm 1 \} \) for each homotopy letter \( q \) in the set \( \mathbf{A}^\pm \) of homotopy letters of the form \( \alpha \) or \( \alpha^{-1} \) for some \( \alpha \in \mathbf{A} \), such that if distinct \( q, q' \in \mathbf{A}^\pm \)
satisfy $h(q) = h(q')$, then $s(q) = s(q')$ if and only if $\{q, q'\} = \{\alpha^{-1}, \beta\}$ with $\alpha \beta \notin \mathcal{P}$. Now let $s(\gamma) = s(l(\gamma)), \ s(\gamma^{-1}) = s(f(\gamma)^{-1})$, and $s(d_\alpha^{+1}) = -s(\alpha)$ for each $\gamma \in \mathcal{P}$ and each $\alpha$.

For a (non-trivial finite or $\mathbb{N}$)-homotopy word $C$ we let $h(C)$ and $s(C)$ be the head and sign of the first letter of $C$. For any vertex $v$ let $s_0(v, \pm 1) = \pm 1$ and $h_0(v, \pm 1) = v$.

Let $D$ and $E$ be homotopy words where $I_{D^{-1}} \subseteq \mathbb{N}$ and $I_{E} \subseteq \mathbb{N}$. If $u = h(D^{-1})$ and $\epsilon = -s(D^{-1})$ let $D_1 = D$. If $v = h(E)$ and $\delta = s(E)$ we let $l(v, \delta) = E$. The composition $DE$ is the concatenation of the homotopy letters in $D$ with those in $E$. By Corollary Proposition 6.11, $DE$ is a homotopy word if and only if $h(D^{-1}) = h(E)$ and $s(D^{-1}) = -s(E)$. If $D = \ldots l_1^{-1} r_1 l_0^{-1} r_0$ is an $\mathbb{N}$-word and $E = l_1^{-1} r_1 l_2^{-1} r_2 \ldots$ is an $\mathbb{N}$-word, write $DE = \ldots l_0^{-1} r_0 l_1^{-1} r_1 \ldots$.

**Corollary 6.9.** [Corollary 2.1.15] Let $a, b \in \mathcal{A}$ and $Ca^{-1}d_a, Cd_b^{-1}b$ be homotopy words.

(i) If $\gamma \in \mathcal{P}$ then $C\gamma^{-1}d_l(\gamma)$ is a homotopy word if and only if $f(\gamma) = a$.

(ii) If $\tau \in \mathcal{P}$ then $Cd_l(\tau)$ is a homotopy word if and only if $l(\tau) = b$.

(iii) If $\gamma' \in \mathcal{P}$ is longer than $\gamma \in \mathcal{P}$ and $f(\gamma') = f(\gamma) = a$ then $C\gamma'^{-1}d_l(\gamma')M \subseteq C\gamma^{-1}d_l(\gamma)M$.

(iv) If $\tau' \in \mathcal{P}$ is longer than $\tau \in \mathcal{P}$ and $l(\tau') = l(\tau) = b$ then $Cd_l^{-1}(\tau') \tau'M \subseteq Cd_l^{-1}(\tau)M$.

**Example 6.10.** [Example 2.1.31] For the complete gentle algebra $k[[x, y]]/(xy)$ the iterated inclusions given by Corollaries 6.7 and 6.9 may be used to construct intervals such as

\[
\begin{array}{c}
M \quad d_y^{-1} y M \quad d_y^{-1} y^2 d_x^{-1} x M
\end{array}
\]

\[
\begin{array}{c}
d_y^{-1} y^2 M \quad d_y^{-1} y^2 d_x^{-1} x^2 M
\end{array}
\]

\[
\begin{array}{c}
d_y^{-1} y^3 M
\end{array}
\]

\[
\begin{array}{c}
\ldots \ldots \ldots
\end{array}
\]

\[
\begin{array}{c}
(1, 1)^+(M) = \bigcap_{n>0} d_y^{-1} y^n M \quad (d_y^{-1} y^2)^+(M) = \bigcap_{n>0} d_y^{-1} y^2 d_x^{-1} x^n M
\end{array}
\]

\[
\begin{array}{c}
(1, 1)^-(M) = \bigcup_{n>0} y^{-n} d_y M \quad (d_y^{-1} y^2)^-(M) = \bigcup_{n>0} d_y^{-1} y^2 x^{-n} d_x M
\end{array}
\]

\[
\begin{array}{c}
x M + d_y M \quad x M + d_y M
\end{array}
\]

\[
\begin{array}{c}
d_y^{-1} y^3 M \quad d_y^{-1} y^2 (y M + d_x M)
\end{array}
\]

We define the sets $(1, 1)^\pm(M)$ and $(d_y^{-1} y^2)^\pm(M)$ in general in Definition 6.12.

**Notation 6.11.** [Definition 2.1.17] If $C = \ldots l_i^{-1} r_i \ldots$ is a homotopy word and $i \in I_C$ is arbitrary, let $C_i = l_i^{-1} r_i$ and $C_{\leq i} = \ldots l_i^{-1} r_i$ given $i - 1 \in I_C$, and otherwise $C_i = C_{\leq i} = 1_{h(C), s(C)}$. Similarly let $C_{> i} = l_i^{-1} r_i + \ldots$ given $i + 1 \in I_C$ and otherwise $C_{> i} = 1_{h(C^{-1}), s(C^{-1})}$. So, $C_{< i}$ and $C_{\geq i}$ are the unique homotopy words with $C_{< i} = C_{< i} C_i$ and $C_i C_{\geq i} = C_{\geq i}$. For any vertex $v$ and $\delta = \pm 1$, let $W_{v, \delta}$ be the set of homotopy $I$-words with $I \subseteq \mathbb{N}$, head $v$ and sign $\delta$. 

Definition 6.12. [6 Definition 2.1.17] Let $C \in \mathcal{W}_{v,\delta}$. Suppose $I_C$ is finite. If $a$ is an arrow and $Cd_a^{-1}a$ is a homotopy word let $C^+(M)$ be the intersection $\bigcap \beta Cd_a^{-1}\beta \text{rad}(M)$ over $\beta \in \mathcal{P}$ with $l(\beta) = a$. By [6 Lemma 2.1.19], if there are finitely many such $\beta$ then $C^+(M) = Cd_a^{-1}$, and otherwise $C^+(M) = \bigcap \beta Cd_a^{-1}\beta M$. If there is no arrow $a$ such that $Cd_a^{-1}a$ is a homotopy word, then we let $C^+(M) = CM$.

If there exists an arrow $b$ where $Cb^{-1}db$ is a homotopy word let $C^-(M)$ be the union $\bigcup \gamma Ca^{-1}\gamma d_M$ over all $\alpha \in \mathcal{P}$ with $l(\alpha) = b$. Otherwise let $C^-(M) = C(\sum d_{c,+}M + \sum c(-)M)$ where $c(\pm)$ runs through all arrows with head $h(C^-(1))$ and sign $\pm(s(C^-(1))$.

Suppose instead $I_C = \mathbb{N}$. In this case let $C^+(M)$ be the set of all $m \in e_v M$ with a sequence of elements $(m_i) \in \prod_{i \in \mathbb{N}} e_{v \mathcal{C}(i)}M$ satisfying $m_0 = m$ and $m_i \in l^{-1}r_i m_{i+1}$ for each $i \geq 0$, and let $C^-(M)$ be the subset of $C^+(M)$ where each sequence $(m_i)$ is eventually zero.

Corollary 6.13. Let $C \in \mathcal{W}_{v,\delta}$.

(i) The assignments $M \mapsto C^+(M)$, $M \mapsto C^-(M)$ and $(M \mapsto CM$ for when $C$ is finite) respectively define subfunctors $C^+$, $C^-$ and $(C$ for when $C$ is finite) of the forgetful functor $C_{\min}(\Lambda-Proj) \to R$-Mod. Furthermore $C^- \leq C^+$.

(ii) If $I_C$ is finite then the functor $C$ preserves small direct sums and small direct products.

(iii) [6 Corollary 2.1.20, Lemma 2.1.21] The functors $C^+$ preserve small direct sums.

Proof. (i) We show that, if $C$ is finite, then $\text{im}(g|_{C(M)}) \subseteq C(N)$ for any morphism $g : M \to N$ in $C_{\min}(\Lambda-Proj)$. Since $g$ is $A$-linear, for any $p \in \mathcal{P}$ we have $(g(m), g(m')) \in \text{rel}(p)$ for all $(m, m') \in \text{rel}(M)$. Next we show, for any $a \in A$, that $(g(m), g(m')) \in \text{rel}(p)$ assuming $(m, m') \in \text{rel}(M)$. For $v = h(a)$ we have $gd_{v,M} = d_{v,N}g$ and so $\sum \beta g(d_{v,M}(m)) - d_{v,N}(g(m)) = 0$ where $\beta$ runs through a direct sum by Lemma [6.3].

This shows that, if $C$ is finite, $C$ is functorial. From this, and by Definition [6.12] we have that $\text{im}(g|_{C^+(M)}) \subseteq C^+(N)$ for any $C$ and for any morphism $g : M \to N$ in $C_{\min}(\Lambda-Proj)$. This shows $C^+$ is functorial. The proof that $C^-(M) \subseteq C^+(M)$ follows by applying Corollaries [6.7] and [6.9] to Definition [6.12] (see the proof of [6] Corollary 2.1.20(ii) for details).

(ii) Fix a set $S$ and a collection $(M(s) | s \in S)$ of objects in $C_{\min}(\Lambda-Proj)$. Recall the direct product $\prod M(s)$ (respectively direct sum $\bigoplus M(s)$) over this collection is the complex whose homogeneous part in degree $n \in \mathbb{Z}$ is $\prod_{s \in S} M(s)^n$ (respectively $\bigoplus_{s \in S} M(s)^n$), and whose differential is defined by $d_M((m_s)) = (d_{M(s)}(m_s))$. In general, considering $M(s)$ as a $\Lambda$-module for each $s$, we have $\bigoplus \text{rad}(M(s)) = \text{rad}(\bigoplus M(s))$. Similarly note $\text{rad}(\prod M(s)) = \prod \text{rad}(M(s))$, which we now prove is an equality. Since $\Lambda$ is semilocal we have $\text{rad}(X) = \text{rad}(\Lambda)X$ for any $\Lambda$-module $X$ (see for example [24] Proposition 24.4)), and so $\text{rad}(\prod M(s)) = \text{rad}(\Lambda)\prod M(s)$ and $\text{rad}(M(s)) = \text{rad}(\Lambda)M(s)$ for each $s$. Hence one can show $\text{rad}(\prod M(s)) \subseteq \text{rad}(\Lambda)\prod M(s)$, since $\text{rad}(\Lambda)$ is finitely generated by $A$.

Fix some $\gamma \in \mathcal{P}$ with $a = l(\gamma)$ and $u = t(\gamma)$. Now let $N(s)$ be an $R$-submodule of $e_v M(s)$ for each $s$. Since $e_v \text{rad}(\prod M(s)) = \prod e_v \text{rad}(M(s))$ we have $d_n \text{rad}(\prod M(s)) = (d_{M(s)}(m_s))$. Similarly since $e_v \text{rad}(\prod M(s)) = \prod e_v \text{rad}(M(s))$ we have $d_n \text{rad}(\prod M(s)) = \prod d_n e_v \text{rad}(M(s))$. Altogether we have $l^{-1}r(\prod N(s)) = \prod l^{-1}r N(s)$ and $l^{-1}r(\prod N(s)) = \prod l^{-1}r N(s)$ whenever $l^{-1}r$ is a homotopy $\{0,1\}$-word. Let $C = l^{-1}r_1 \ldots l^{-1}r_n$, then by iterating the above equalities (in case $n > 0$) we have

$$C(\bigoplus M(s)) = l^{-1}r_1 \ldots l^{-1}r_{n-1} l^{-1}r_n M(s) = \cdots = \bigoplus CM(s).$$

Similarly one can show $C(\prod M(s)) = \prod CM(s)$.
(iii) Suppose firstly that $I_C$ is finite. We start by showing $C^-(\bigoplus M(s)) = \bigoplus C^-(M(s))$. Let $C^{-1}d_b$ be a homotopy word for some $b \in A$. By part (ii) we have

$$C^-(\bigoplus M(s)) = \bigcup_{\alpha} C\alpha^{-1}d_b(\bigoplus_s M(s)) = \bigcup_{\alpha} \bigoplus_s C\alpha^{-1}d_b M(s) \subseteq \bigoplus C^-(M(s))$$

where the union runs over all $\alpha \in P$ with $f(\alpha) = b$. Suppose conversely that $(m_s) \in \bigoplus C^-(M(s))$, and so for each $s$ there is some $\alpha(s) \in P$ with $f(\alpha(s)) = b$ such that $m_s \in C\alpha(s)^{-1}d_s M(s)$. Since $m_s = 0$ (and so $\alpha(t) = b$) for all but finitely many $t$, we can choose $\gamma$ of maximal length among $(\alpha(s))$. This gives $(m_s) \in C\gamma^{-1}d_b(\bigoplus_s M(s)) \subseteq C^-\bigoplus M(s)$.

Assume instead $C^{b^{-1}}d_b$ is not a homotopy word for all $b \in A$. By definition $C^-(\bigoplus M(s)) = C(\sum d_{c(+)}(\bigoplus M(s) + \sum c(\pm) (\bigoplus M(s))$ where $c(\pm)$ runs through all arrows with head $h(C^{-1})$ and sign $\pm s(C^{-1})$. As above, and by part (ii), we have $\sum d_{c(+)}(\bigoplus M(s) = \bigoplus (\sum d_{c(+)M(s)})$ and $\sum c(\pm) (\bigoplus M(s) = \bigoplus (\sum c(\pm) M(s))$. Altogether this shows $C^-\bigoplus M(s) = C^-\bigoplus M(s)$. We now show $C^+(\bigoplus M(s)) = \bigoplus C^+(M(s))$. If $C^{b^{-1}}a$ is not a homotopy word for all $a \in A$ then $C\bigoplus M(s)$ is not a homotopy word and so $C^+(\bigoplus M(s)) = \bigoplus C^+(M(s))$. Suppose instead $C^{b^{-1}}a$ is a homotopy word for some $a \in A$. Then by part (ii) we have

$$C^+(\bigoplus M(s)) = \bigcap_{\alpha} C\alpha^{-1}d_b(\bigoplus_s M(s)) = \bigcap_{\alpha} \bigoplus_s C\alpha^{-1}d_b M(s) = \bigoplus C^+(M(s))$$

where the intersection runs over all $\beta \in P$ with $f(\beta) = a$.

Suppose finally that $C$ is a homotopy N-word. By definition, $(m_s) \in C^+(\bigoplus M(s))$ if and only if there is a sequence $(m_{0,s}), (m_{1,s}), \ldots \in \bigoplus M(s)$ with $m_s = m_{0,s}$ for all $s$ and $(m_{n-1,s}) \in l_n^{-1}r_n(m_{n,s})$ for all $n > 0$. As in the proof of (ii) we have $d_{a,M(s)}((m_s)) = (d_{a,M(s)}(m_s))$ for any $a \in A$, and so $(m_{n-1,s}) \in l_n^{-1}r_n(m_{n,s})$ if and only if $m_{n-1,s} \in l_n^{-1}r_n m_{s,n,s}$ for all $s$. Altogether this shows $C^+\bigoplus M(s) = \bigoplus C^+ M(s)$.

For each $s$ let $m_s \in C^-(M(s))$. So there is a sequence $(m_{0,s}), (m_{1,s}), (m_{2,s}), \ldots \in \bigoplus M(s)$ such that $m_s = m_{0,s}$, $(m_{n-1,s}) \in l_n^{-1}r_n(m_{n,s})$ for all $n > 0$ and $m_{n,s} = 0$ for all $n \geq l(s)$ for some $l(s) \in N$. We can assume $l(s) = 0$ for all $s$ with $m_s = 0$, which holds for all but finitely many $s$. Hence there exists $r > 0$ with $r > l(s)$ for all $s$. So we have $(m_{0,s}), (m_{1,s}), \ldots \in \bigoplus M(s)$ such that $(m_s) = (m_{0,s}), (m_{n-1,s}) \in l_n^{-1}r_n(m_{n,s})$ for all $n > 0$ and $(m_{n,s}) = 0$ for all $n \geq r$. This shows $\bigoplus C^- M(s) \subseteq C^-\bigoplus M(s)$. The reverse inclusion is straightforward. \hfill $\square$

7. \textit{Refined functors.}

\textbf{Definition 7.1.} Fix a vertex $v$ and $\delta = \pm 1$. Below we define a total order $< \subset W_{v,\delta}$.

(i) \textbf{Lemma 2.1.22, Definition 2.1.23} For distinct $l^{-1}r, l^{-1}r' \in W_{v,\delta}$ let $l^{-1}r < l^{-1}r'$ if:

\begin{itemize}
\item[(a)] $l^{-1}r = d^{-1}_{l(\gamma)}(\gamma) and l^{-1}r' = d^{-1}_{l(\gamma)}(\gamma)$ for some $\gamma, \nu \in P$ such that $\gamma \nu \in P$; or
\item[(b)] $l^{-1}r = \mu^{-1} d_{l(\gamma)}(\mu) and l^{-1}r' = \mu^{-1} d_{l(\gamma)}(\mu) for some $\mu, \eta \in P$ such that $\mu \eta \in P$; or
\item[(c)] $l^{-1}r = \gamma^{-1} d_{l(\gamma)}(\gamma) and l^{-1}r' = \gamma^{-1} d_{l(\gamma)}(\gamma)$ for some $\kappa, \lambda \in P$ such that $\kappa \lambda \in P$.
\end{itemize}

(ii) \textbf{Definition 2.1.26, Definition 2.1.27} For distinct $C, C' \in W_{v,\delta}$ let $C < C'$ if:

\begin{itemize}
\item[(a)] there are homotopy letters $l, l', r and r'$ and homotopy words $B, D, D'$ for which $C = B \gamma^{-1} D, C' = B' \gamma^{-1} D' and l^{-1}r < l^{-1}r'$; or
\item[(b)] there is some $\beta \in P$ for which $C' = C d^{-1}_{l(\gamma)}(\beta) E$ for some homotopy word $E$; or
\item[(c)] there is some $\alpha \in P$ for which $C = C' \alpha^{-1} d_{l(\gamma)}(\alpha) E'$ for some homotopy word $E'$.
\end{itemize}

\textbf{Lemma 7.2.} Fix a vertex $v$ and $\delta = \pm 1$.

(i) \textbf{Lemma 2.1.29} If $(\alpha \in P, s(\alpha) = \delta, v = h(\alpha) and C \in W_{v,\delta}) then l(\alpha) M \subseteq C^-(M)$.

(ii) \textbf{Lemma 2.1.28} If $l^{-1}r D, l^{-1}r' D' \in W_{v,\delta}$ for homotopy words $D, D'$ and homotopy letters $l, l', r, r'$ where $l^{-1}r < l^{-1}r'$, then $(l^{-1}r D)^+(M) \subseteq (l^{-1}r' D')^-(M)$.

(iii) \textbf{Proposition 2.1.30} For all $C, C' \in W_{v,\delta}$ with $C < C'$ we have $C^+(M) \subseteq C'-(M)$. 

Proof. (i) Suppose firstly that $C$ is trivial. If $C \beta^{-1}d_l(\beta)$ is a homotopy word for some $\beta \in P$ then $h(\beta^{-1}) = h(C^{-1}) = h(1)(\alpha)$ so $t(\beta) = h(\alpha)$. Similarly $s(\beta^{-1}) = s(1)(\alpha)$ which gives $f(\beta)(\alpha) \notin P$, so $l(\alpha)M \subseteq \beta^{-1}0$ which lies in $C^{-1}(M)$. Otherwise there is no such $\beta$, and so $l(\alpha)M \subseteq C^{-1}(M)$ by Definition 6.12.

Suppose $C$ is non-trivial. If $C = \beta^{-1}d_l(\beta)D$ for some homotopy word $D$ and some $\beta \in P$ then $t(\beta) = h(C) = h(\alpha)$ and $s(\beta^{-1}) = s(\alpha) = s(1)(\alpha)$. So as before $f(\beta)(\alpha) \notin P$ and again $l(\alpha)M \subseteq \beta^{-1}0 \subseteq C^{-1}(M)$. Suppose $C = d_{(\gamma)}^{-1}E$, for some homotopy word $E$ and $\gamma \in P$. Here $h(\gamma) = h(\alpha)$ and $s(l(\alpha)) = -s(l(\gamma))$. So $l(\alpha) \neq l(\gamma)$, and so $l(\alpha)M \subseteq d_{(\gamma)}^{-1}0$ by Lemma 6.3(ii), which means $l(\alpha)M \subseteq d_{(\gamma)}^{-1}E^{-1}(M) = C^{-1}(M)$. This shows that, in any case, $l(\alpha)M \subseteq C^{-1}(M)$.

(ii), (iii) Suppose $l^{-1}r = d_{(\gamma)}^{-1}\gamma\nu \text{ and } l^{-1}r = d_{(\gamma)}^{-1}\gamma$ for some $\gamma, \nu \in P$ such that $\gamma\nu \in P$. Here $(l^{-1}r)(\gamma)^{-1}(M) \subseteq d_{(\gamma)}^{-1}(\nu)(\gamma)M$ and $d_{(\gamma)}^{-1}(\nu)(\gamma)(D)^{-1}(M) = (l^{-1}r)^{-1}(D)(M)$. Since $\gamma\nu \in P$ we have that $f(\gamma)(\nu) \in P$ and so $s(l(\nu)) = -s(f(\gamma)^{-1})$. Since $d_{(\gamma)}^{-1}(\nu)D^\prime$ is a homotopy word we have that $s(d_{(\gamma)}^{-1}(\nu)^{-1}) = -s(D^\prime)$, and so $l(\nu) = s(D^\prime)$. By part (i) this means $l(\nu)M \subseteq (D^\prime)^{-1}(M)$. For the proof of (ii), the other cases of $l^{-1}r < l^{-1}r^\prime$ are similar and omitted. The proof of part (iii) follows from part (i) by considering cases (iiia), (iiib) and (iic) of Definition 7.1. 

Assumption 7.3. In the remainder of §7 let $M$ and $N$ be objects in $\mathcal{C}_{\text{min}}(\Lambda \text{-Proj})$, and let $B$ and $D$ be homotopy words with head $v$ such that $B = B^{-1}D$ is a homotopy word.

Corollary 7.4. [6] Corollary 2.2.3 Let $n \in \mathbb{Z}$, $I_C = \{0, \ldots, t\}$ and $X^i$ and $Y^i$ be $R$-submodules of $e_{I(C)}M^i$ and $e_{h(C)}M^i$ for all $i \in \mathbb{Z}$. Then $Y^n \cap C(\bigoplus_i X^i) = Y^n \cap CX^{n+\mu C(t)}$.

Proof. Both the map $d_{a,M}$ (for $a \in A$), and the map given by multiplication by $\gamma \in P$, are homogeneous: and so $Y^n \cap C(\bigoplus_i X^i) \subseteq Y^n \cap CX^{n+\mu C(t)}$. The reverse inclusion is clear. 

Lemma 7.5. [6] Lemma 2.2.4 We have the following inclusions:

(i) $B^+(M) \cap D^+(M) \cap e_v \text{rad}(M) \subseteq (B^+(M) \cap D^-(M) + (B^-(M) \cap D^+(M))$;

(ii) $(B^-(M) + D^+(M) \cap B^+(M)) \cap e_v \text{rad}(M) \subseteq (B^-(M) + D^-(M) \cap B^+(M))$;

(iii) $B^+(M) \cap D^+(M) + e_v \text{rad}(M) \subseteq (B^+(M) + e_v \text{rad}(M)) \cap (D^+(M) + e_v \text{rad}(M)))$.

Proof. Without loss of generality we assume that $s(B) = 1$, $s(D) = -1$ and that $A(v) = \{x(\pm), x(\mp)\}$ where $s(x(\pm)) = \pm 1$.

(i) Since $\Lambda$ is semilocal we have $\text{rad}(M) = \text{rad}(\Lambda)M$. So for any $m \in e_v \text{rad}(M)$ there are some $m_+, m_- \in M$ for which $m = x(\pm)m_+ + x(\pm)m_-$. By Lemma 7.2(i) we have that $x(\pm)m_+ \in B^-(M)$ and $x(\pm)m_- \in D^-(M)$. If also $m \in B^+(M) \cap D^+(M)$ we have $x(\pm)m_+ \in D^+(M) \cap B^-(M)$ as $x(\pm)m_+ = m - x(\pm)m_- - D^-(M) \subseteq D^+(M)$. By symmetry $x(\pm)m_- \in B^+(M) \cap D^-(M)$.

(ii) If $m \in (B^+(M) + D^+(M) \cap B^+(M)) \cap e_v \text{rad}(M)$ then $m = x(\pm)m_+ + x(\pm)m_-$ for some $m_+, m_- \in M$ as above. We also have $m = m' + m''$ where $m' \in B^-(M)$ and $m'' \in D^+(M) \cap B^+(M)$, and so $x(\pm)m_+ = m' + m'' - x(\pm)m_+ \in B^+(M)$.

(iii) Clearly $B^+(M) \cap D^+(M) + e_v \text{rad}(M)$ is contained in the intersection of $B^+(M) + e_v \text{rad}(M)$ and $D^+(M) + e_v \text{rad}(M)$. Any $m$ from this intersection may be written as: $m_B + x(-)m_- + x(+)m_+$ for $m_B \in B^+(M)$ and $m_{\pm} \in M$; and as $m_D + x(-)m'_+ + x(+)m'_-$ for $m_D \in D^+(M)$ and $m'_{\pm} \in M$. So, $x(+)m_+, x(+)m'_- \in B^-(M)$ and $x(-)m_-, x(-)m'_+ \in D^-(M) \subseteq D^+(M)$. This means $m_B + x(+)m_+ - x(+)m'_- = x(-)m'_+ - x(-)m_- + m_D$ lies in $D^+(M)$. This shows $m = (m_B + x(+)m_+ - x(+)m'_+) + (x(+)m'_+ + x(-)m_-)$ lies in $B^+(M) \cap D^+(M) + e_v \text{rad}(M)$.

\[\square\]
Moreover, if morphism $F$ is a homotopy word (equivalently \((\Lambda - \text{Proj}), \mod{\Lambda} \) and \(\text{rad}(M)\)) then since \(\Lambda\) is periodic then there is a homotopy word

\[ A = A^+ + a_{k(c)} \text{rad}(M) \]  

for any homotopy word $A$ with $I_A \subseteq \mathbb{N}$.

**Remark 7.7.** By Lemma [7.5](#) we have

\[ F_{B,D,n}^+ (M) = e_v M^n \cap (B^+(M) \cap D^+(M)), \]

\[ F_{B,D,n}^- (M) = e_v M^n \cap ((B^+(M) \cap D^-(M)) \cap (B^-(M) \cap D^+(M))), \]

and

\[ G_{B,D,n}^+ (M) = e_v M^n \cap (B^-(M) \cap D^+(M) \cap B^+(M)). \]

Consequently $F_{B,D,n}$, $\bar{F}_{B,D,n}$, $G_{B,D,n}$, and $\bar{G}_{B,D,n}$ all define naturally isomorphic additive functors $K_{\min}(\Lambda - \text{Proj}) \to k\text{-Mod}$ Corollary 2.2.8.

Moreover, if $M$ is a complex of finitely generated projectives, then since $\Lambda$ is $R$-module finite, $F_{B,D,n}(M)$ and $G_{B,D,n}(M)$ are finite-dimensional Corollary 2.2.9.

**Definition 2.2.10** Assume $C$ is $p$-periodic, so there is a homotopy word $E = l_1^{-1} r_1 \ldots l_p^{-1} r_p$ with $C = \infty E \infty$. For each $n \in \mathbb{Z}$ let $E(n)$ be the relation on $e_v M^n$ given by all \((m, m') \in e_v M^n \oplus e_v M^n \) with $m \in E M'$. Then: $E(n)^2 = F_{B,D,n}^+ (M)$ and $E(n)^3 = F_{B,D,n}(M)$; and by Lemma [5.1](#) there is a $k$-vector space automorphism of $E(n)^3/ E(n)^5$ defined by sending $m + E(n)^5$ to $m' + E(n)^5$ if and only if $m' \in E(n)^3 \cap E(n)^5 + E(n)m$ (see [6](#) Lemma 2.2.11).

Consequently, as above, if $C$ is periodic then $F_{B,D,n}$, $\bar{F}_{B,D,n}$, $G_{B,D,n}$, and $\bar{G}_{B,D,n}$ all define naturally isomorphic functors $K_{\min}(\Lambda - \text{Proj}) \to k[T, T^{-1}] \cdot \text{Mod}$ Furthermore these functors take objects in $K_{\min}(\Lambda - \text{proj})$ to objects in $k[T, T^{-1}] \cdot \text{Mod}_{k\text{-mod}}$, the full subcategory of $k[T, T^{-1}] \cdot \text{Mod}$ consisting of finite-dimensional modules Corollary 2.2.12.

8. Natural isomorphisms and constructive functors.

**Definition 8.1.** Let $\Sigma$ be the set of all triples $(B, D, n)$ where $B^{-1} D$ is a homotopy word (equivalently $(B, D) \in W_{v, \pm 1} \times W_{v, \mp 1}$) and $n$ is an integer.

**Assumption 8.2.** In $\Sigma$ fix $(B, D, n), (B', D', n') \in \Sigma$ and let $C = B^{-1} D$ and $C' = B'^{-1} D'$.

**Definition 8.3.** We write $C \sim C'$ if and only if $C' = C^{\pm[1]} [t]$ for some $t \in \mathbb{Z}$, So either $(C' = C^{\pm[1]} \text{ and } I_C \neq \mathbb{Z} \neq I_C')$ or $(C' = C^{\pm[1]} [t] \text{ and } I_C = I_C')$ (see [13](#) Lemma 2.1). Define the axis $a_{B,D} \in \mathbb{Z}$ of $(B, D)$ by $C_{< a_{B,D}} = B^{-1}$ and $C_{> a_{B,D}} = D$. If $I_C = \{0, \ldots, m\}$ then $a_{D,B} = m - a_{B,D}$; if $I_C = \pm \mathbb{N}$ then $a_{D,B} = -a_{B,D}$; and if $I_C = \mathbb{Z}$ then $a_{B,D} = 0$ (see [6](#) Lemma 2.2.15).
Definition 2.2.18] If \( C \sim C' \) let
\[
\begin{align*}
r(B, D; B', D') = \begin{cases} 
\mu_C(a_{B', D'}) - \mu_C(a_{B, D}) & (\text{if } C' = C \text{ is not a homotopy } \mathbb{Z}\text{-word}) \\
\mu_C(a_{D', B'}) - \mu_C(a_{B, D}) & (\text{if } C' = C^{-1} \text{ is not a homotopy } \mathbb{Z}\text{-word}) \\
\mu_C(\pm t) & (\text{if } C' = C^{\pm 1}[t] \text{ is a homotopy } \mathbb{Z}\text{-word})
\end{cases}
\end{align*}
\]
We write \((B, D, n) \sim (B', D', n')\) if and only if \(B^{-1}D \) and \(B'^{-1}D'\) are equivalent and \(n' - n = r(B, D; B', D')\).

By [6 Lemma 2.2.19] we have that \(r(B, D; B', D') = -r(B', D'; B, D)\) and \(r(B, D; B'', D'') = r(B, D; B', D') + r(B', D'; B'', D'')\) for all \((B, D', D'') \in \Sigma\) with \(B''^{-1}D'' \sim B'^{-1}D'\), and so \(\sim\) is an equivalence relation.

Recall the involution \(\text{res}_{i,R}\) of \(R[T, T^{-1}]-\text{Mod}\), which swaps the action of \(T\) and \(T^{-1}\), and which restricts to an involution of \(R[T, T^{-1}]-\text{Mod}_{R\text{-Proj}}\).

**Lemma 8.4.** Fix \((B, D, n), (B', D', n'), C\) and \(C'\) as in Assumption [8.2]

(i) [6 Lemma 2.2.21] (see also [14 Lemma 7.1]).
   (a) If \(C\) is aperiodic then \(F_{B, D, n} \simeq F_{D, B, n};\) and
   (b) If \(C\) is periodic then \(F_{B, D, n} \simeq \text{res}_{i,k} F_{D, B, n}\).

(ii) [6 Corollary 2.2.24] Suppose \((B, D, n) \sim (B', D', n')\) in \(\Sigma\).
   (a) If \(C\) is aperiodic then \(G_{B, D, n} \simeq G_{B', D', n'}\).
   (b) If \(C\) is periodic and \(C' = C[t]\) for some \(t \in \mathbb{Z}\) then \(G_{B, D, n} \simeq G_{B', D', n'}\).
   (c) If \(C\) is periodic and \(C' = C^{-1}[t]\) for some \(t \in \mathbb{Z}\) then \(G_{B, D, n} \simeq \text{res}_{i,k} G_{B', D', n'}\).

**Proof.** (i) Let \(M\) be an object in \(K_{\min}(A\text{-Proj})\). If \(C\) is aperiodic then \(F_{B, D, n}^\pm(M) = F_{D, B, n}^\pm(M)\) as \(R\)-modules; and if \(C\) is periodic the action of \(T^{\pm 1}\) on \(F_{B, D, n}(M)\) depends on the order of \(B\) and \(D\), and must be exchanged with \(T^{\mp 1}\) if and only if \(B\) and \(D\) are swapped.

(ii) Let \(A, E, \gamma\), \((d_{i}(\gamma))^{-1}\), \(A' = \gamma^{-1}d_{i}(\gamma) E\) and \(E' = d_{i}(\gamma) E\) be homotopy words. Then \(G_{A, A', 0} \simeq L \simeq G_{E', E, -1}\) where \(L(M) = L^+(M) / L^-(M)\) and \(L^\pm(M) = e_{n} M^{n+1} \gamma A^{-1}(M) + d_{i}(\gamma) E\).

**Definition 8.5.** [6 Definition 2.2.20] Let \(\Sigma(s)\) be the set of \((B, D, n) \in \Sigma\) where \(B^{-1}D\) is aperiodic, and \(\Sigma(b)\) the set of such \((B, D, n)\) where \(B^{-1}D\) is periodic. Note that the relation \(\sim\) on \(\Sigma\) restricts to an equivalence relation \(\sim_{s}\) (respectively \(\sim_{b}\)) on \(\Sigma(s)\) (respectively \(\Sigma(b)\)). Let \(I(s) \subseteq \Sigma(s)\) (respectively \(I(b) \subseteq \Sigma(b)\)) denote a chosen collection of representatives \((B, D, n)\), one for each equivalence class of \(\Sigma(s)\) (respectively \(\Sigma(b)\)). Let \(I = I(s) \cup I(b)\).

**Definition 2.2.25** Let \((B, D, n) \in I, C = B^{-1}D, P = P(C)|\mu_C(a_{B, D}) - n|, V, V'\) be free \(R\)-modules with bases \((v_{\lambda} | \lambda \in \Omega)\) and \((v'_{\lambda} | \lambda' \in \Omega')\) and let \(f : V \to V'\) be \(R\)-linear. In this notation define \(f_{\lambda, \lambda'} \in P R\) by \(f(v_{\lambda}) = \sum_{\lambda'} f_{\lambda, \lambda'} v_{\lambda'}\). If \((B, D, n) \in I(b)\) then \(a_{B, D} = 0\), \(f\) is \(R[T, T^{-1}]\)-linear and \(T\) defines automorphisms \(\varphi_{V}\) of \(V\) and \(\varphi_{V'}\) of \(V'\) with \(f \varphi_{V} = \varphi_{V'} f\).

If \((B, D, n) \in I(s)\) (respectively \(I(b)\)) we use \(S_{B, D, n} R\)-modules to denote a functor \(R\text{-Proj} \to C_{\min}(A\text{-Proj})\) (respectively \(R[T, T^{-1}]-\text{Mod}_{R\text{-Proj}} \to C_{\min}(A\text{-Proj})\)), defined as follows. On objects \(V\) define the homogenous component \(S_{B, D, n}^{m}(V)\) of the complex \(S_{B, D, n}(V)\) in degree \(m \in \mathbb{Z}\) by \(P^{m} \otimes_{R} V\) (respectively \(P^{m} \otimes_{R[T, T^{-1}]} V\)). Define the corresponding differential \(d^{m}_{S_{B, D, n}(V)}\) by \(d^{m}_{P} \otimes_{R} \text{id}_{V}\) (respectively \(d^{m}_{P} \otimes_{R[T, T^{-1}]} \text{id}_{V}\)) in degree \(m \in \mathbb{Z}\). Similarly we can define the map \(S_{B, D, n}^{m}(f)\) of the image \(S_{B, D, n}(f)\) of \(S_{B, D, n}\) on a morphism \(f\) by \(\text{id}_{P}^{m} \otimes_{R} f\) (respectively \(\text{id}_{P}^{m} \otimes_{R[T, T^{-1}]} f\)).
Corollary 8.6. Suppose \((B, D, n) \sim (B', D', n')\) in \(\Sigma\).
(i) If \(C\) is aperiodic then \(S_{B,D,n} \simeq S'_{B',D',n'}\).
(ii) If \(C\) is periodic and \(C' = C[t]\) for some \(t \in \mathbb{Z}\) then \(S_{B,D,n} \simeq S'_{B',D',n'}\).
(iii) If \(C\) is periodic and \(C' = C^{-1}[t]\) for some \(t \in \mathbb{Z}\) then \(S_{B,D,n} \simeq S'_{B',D',n'}\).

Proof. Recall \(b_{i,C}\) denotes the coset of \(e_{v_C(i)}\) in \(P(C)\). By Lemma 3.4 there is a canonical bijection \(\omega : I_{C'} \to I_C\) such that \(b_{i,C} \to b_{\omega(i),C'}\) gives a natural isomorphism \(\theta\) of complexes \(P(C'[\mu_C(a_{B',D'}) - n']) \to P(C[\mu_C(a_{B,D}) - n])\). If \(C\) is aperiodic this shows \(S_{B,D,n} \simeq S'_{B',D',n'}\). If instead \(C\) is periodic we have \((\theta(b_{i,C}T) = b_{\omega(i),C'T}\) when \(C' = C[t]\) and \((\theta(b_{i,C}T^{-1}) = b_{\omega(i),C'T}\) when \(C' = C^{-1}[t]\), as required.

Corollary 8.7. Let \((B, D, n) \sim (B', D', n')\), let \(C\) be \(p\)-periodic and let \(V\) and \(W\) be objects in \(R[T, T^{-1}]\)-Mod\(_R\)Proj. If \(k \otimes_R V \simeq k \otimes W\) in \(k[T, T^{-1}]\)-Mod then \((S_{B,D,n}(V) \simeq S'_{B',D',n'}(W)\) if \(C' = C[t]\) and \((S_{B,D,n}(V) \simeq S'_{B',D',n'}(\text{res}_R(W))\) if \(C' = C^{-1}[t]\).}

Proof. Since \(p > 0\), and by the definition of \(\mu_C\), there exists \(s \in \mathbb{Z}\) such that
\[
C[s] = \cdots l_2^{-1}r_2^{-1} l_1^{-1} r_1^{-1} d_l^{-1} \alpha | \beta d_l^{-1} d_l^{-1} r_2 l_3^{-1} r_3 \cdots
\]
for some \(\alpha, \beta \in \mathfrak{P}\). Note that \(C[s]\) is also \(p\)-periodic, and so \(C[s] = \infty E\infty\) where \(E = l_1^{-1} r_1 \cdots l_p^{-1} r_p\) is a homotopy word. Since \(l_1^{-1} r_1 \cdots l_p^{-1} r_p\) is homotopy word, by Lemma 3.10 we have \(P(E, V) \simeq P(E, W)\) as complexes. By Remark 3.3 we have \(P(C[s], V) \simeq P(E, V)\) and \(P(C[s], W) \simeq P(E, W)\) as complexes. Setting \(H = (E^{-1})\infty\) and \(L = E\infty\) gives \((B, D, n) \sim (H, L, n + \mu_C(s))\) (since \(H^{-1}L = C[s]\), and so we have \(S_{B,D,n} \simeq S_{H,L,n+\mu_C(s)}\) by Corollary 8.6 ii). Altogether we have
\[
S_{B,D,n}(V) \simeq S_{H,L,n+\mu_C(s)}(V) = P(C[s], V)(n + \mu_C(s))
\]
\[
\simeq P(C[s], W)[n + \mu_C(s)] = S_{H,L,n+\mu_C(s)}(W) \simeq S_{B,D,n}(W),
\]
and so the result follows by parts (ii) and (iii) of Corollary 8.6.

9. Relations on complexes.

Assumption 9.1. Let \(C\) be any homotopy I-word. We fix a transversal \(S\) of \(m\) by choosing a lift \(s \in R\) for each element of \(k = R/m\) so that \(S \cap (r + m)\) has precisely one element for each \(r \in R\). We also assume \(S \cap m = \{0\}\) and \(S \cap (1 + m) = \{1\}\).

Notation 9.2. Let \(i \in I\). Recall the symbol \(b_i\) denotes the coset of \(e_{v_C(i)}\) in the summand \(\Lambda e_{v_C(i)}\) of \(P_{v_C(i)}(C)\). Let \(P[i]\) be the set \(P(v_C(i))\) of all non-trivial paths \(\sigma \notin J\) with tail \(v_C(i)\). Let \(x \in A\) let \(P[x, i] = \{\sigma \in P[i] \mid \|\sigma\| = x\} \).

Recall \(\text{Am} \subseteq \text{rad}(\Lambda)\). So, if \(m \in P(C)\) then \(m = \sum_i (\eta_i b_i + \sum_{\sigma \in P[i]} r_{i\sigma} \sigma b_i)\) where \(\eta_i \in \mathfrak{S}\) and \(r_{i\sigma}, i \in R\) for each \(i\). \(r_{i\sigma} = 0\) for all but finitely many \(i\), and given \(i \in I\) fixed we have \(r_{i\sigma} = 0\) for all but finitely many \(\sigma\). In this case notation \(\{m\} = \sum_i \eta_i b_i\) and \([m] = \sum_{i\sigma} r_{i\sigma} \sigma b_i\). Let \(t \in I\). Let \(\psi_t\) denote the \(A\)-module epimorphism \(P(C) = \bigoplus_{i} \Lambda e_{v_C(i)} \to \Lambda e_{v_C(t)}\) sending \(m = \sum_i m_i b_i\) to \(m_t\). For \(m = \sum_{i\sigma} (\eta_i b_i + \sum_{\sigma \in P[i]} r_{i\sigma} \sigma b_i)\) as above this gives \(\psi_t(\gamma[m]) = \eta_t \gamma\) and \(\psi_t(\gamma[m]) = \sum_{i\sigma} r_{i\sigma} \gamma\sigma\).
Let \( m_{x,t} = \lfloor m_{x,t} \rfloor + \lceil m_{x,t} \rceil \) and \( m \rfloor_{x,t} = m \rfloor_{x,t} + m \lceil_{x,t} \) where
\[
\lfloor m_{x,t} \rfloor = \begin{cases} \eta_{t+1} \alpha & \text{if } t + 1 \in I \text{ and } l_{t+1}^{-1} r_{t+1} = \alpha^{-1} d_x \\ 0 & \text{otherwise} \end{cases}\\
\lceil m_{x,t} \rceil = \begin{cases} \eta_{t-1} \beta & \text{if } t - 1 \in I \text{ and } l_t^{-1} r_t = d_x^{-1} \beta \\ 0 & \text{otherwise} \end{cases}
\]
\[
\lfloor m_{x,t} \rfloor = \begin{cases} \sum_{\sigma \in P(x,t+1)} r_{\sigma,t+1} \sigma \kappa & \text{if } t + 1 \in I \text{ and } l_{t+1}^{-1} r_{t+1} = \kappa^{-1} d_{(i)}(\nu) \\ 0 & \text{otherwise} \end{cases}
\]
\[
\lceil m_{x,t} \rceil = \begin{cases} \sum_{\sigma \in P(x,t-1)} r_{\sigma,t-1} \zeta & \text{if } t - 1 \in I \text{ and } l_t^{-1} r_t = d_t^{-1} \zeta \\ 0 & \text{otherwise} \end{cases}
\]

For a vertex \( v \) recall the sum \( \sum_{\sigma \in A(v \to v)} \Lambda a \) is direct. For any arrow \( y \) with tail \( v \)
let \( \theta_y : \bigoplus_\Lambda a \Lambda \to \Lambda y \) be the canonical projection. For \( m' \in P(C) \) write \( m' = \sum_i \eta_i b_i + \sum_i \sigma \in P[i] \sum_{i} r'_{i \sigma} \sigma b_i \) where \( \eta_i \in \mathbb{S} \) and \( r'_{i \sigma} \in R \) have finite support (as above).

**Lemma 9.3.** [6] Lemma 2.3.5] Fix \( m, m' \in P(C) \) as in Notation 9.2. Let \( x \in A \) and \( t \in I \).

(i) We have \( \psi_t (d_{x,P(C)}(\lfloor m \rfloor)) = \lfloor m_{x,t} \rfloor + m_{x,t} \) and \( \psi_t (d_{x,P(C)}(\lceil m \rceil)) = \lceil m_{x,t} \rceil + m_{x,t} \).

(ii) If \( \gamma \in P[x,t] \) and \( m \in \gamma^{-1} d_x' m' \) then \( \theta_{(i)}(\psi_t (d_{x,P(C)}(m'))) = \eta_{t+1} \gamma \in \gamma \text{rad}(\Lambda) \).

(iii) If \( l_{t+1}^{-1} r_{t+1} = \gamma^{-1} d_{(i)}(\nu) \) then \( \theta_{(i)}(m_{i \nu(t)}, \nu) - \eta_{t+1} \gamma \in \gamma \text{rad}(\Lambda) \).

(iv) If \( l_t^{-1} r_t = d_{(i)}^{-1} \gamma \) then \( \theta_{(i)}(m_{i \nu(t)}, - \eta_t \gamma) \in \gamma \text{rad}(\Lambda) \) and \( \theta_{(i)}(m_{i \nu(t)}, \nu) = 0 \).

**Proof.** (i) Let \( t_x : xP(C) \to \bigoplus yP(C) \) and \( \pi_x : \bigoplus yP(C) \to xP(C) \) respectively denote the natural \( R \)-module inclusions and projections. Note \( d_{x,P(C)}(\sum \eta \nu_i b_i) = \sum \eta l_{x} \sigma (\pi_x(b_i + b_i')) \) and if \( \psi_t (b_i \neq 0) \) then \( i = t \pm 1 \). By case analysis \( \psi_t (d_{x,P(C)}(\lfloor m \rfloor)) = \lfloor m_{x,t} \rfloor + m_{x,t} \) and by Lemma 9.3. \( d_{x,P(C)}(\sum_i \sigma r_{\sigma,t} \sigma b_i) = \sum_i (r_{\sigma,t} \sigma d_{(i)}(\nu)(b_i)) = \sigma d_{(i)}(\nu) (b_i) \) for each \( i \) where the triples \( (\sigma, z, i) \) run through all \( \sigma \in P[x, i] \) and all arrows \( z \) with \( f(\sigma) z \in P \). For any such triple \( (\sigma, z, i) \) we have \( \sigma d_{(i)}(\nu)(b_i) = \sigma d_{z,P(C)}(b_i) \) and so \( \psi_t (d_{x,P(C)}(\lceil m \rceil)) = \lceil m_{x,t} \rceil + m_{x,t} \).

(ii) Since \( \sum_{\sigma \in P} r_{\sigma,t} \sigma \in \gamma \text{rad}(\Lambda) \) we have \( \theta_{(i)}(m) - \eta_{t+1} \gamma \in \gamma \text{rad}(\Lambda) \) and so applying \( \theta_{(i)}(\gamma) \) to either side of \( \gamma = d_{x,P(C)}(m') \) gives \( \theta_{(i)}(\psi_t (d_{x,P(C)}(m'))) = \eta_{t+1} \gamma \in \gamma \text{rad}(\Lambda) \).

(iii) If \( l_{t+1}^{-1} r_{t+1} = \gamma^{-1} d_{(i)}(\nu) \) then \( m_{i \nu(t), \nu} = 0 \) unless \( l_t^{-1} r_t = d_{(i)}^{-1} \gamma \) in which case \( f(\gamma) \neq f(\gamma) \) since \( d_{(i)}^{-1} \gamma \neq d_{(i)}(\nu) \). Furthermore \( m_{i \nu(t), \nu} = \eta_{t+1} \gamma \) and \( m_{i \nu(t), \nu} = 0 \) is equal to the sum over \( \sigma \in P[[\gamma], t + 1] \) of the terms \( r_{\sigma,t} \sigma \gamma \). Hence \( m_{i \nu(t), \nu} \in \gamma \text{rad}(\Lambda) \), \( \gamma \in \gamma \text{rad}(\Lambda) \) which is contained in \( \gamma \text{rad}(\Lambda) \) by Corollary 2.11(iii). For (iv) apply the above to \( D = C^{-1} \).

**Notation 9.4.** [6] Definition 2.3.7] For each \( i \) the words \( C_{i,k} \) and \( C_{i,k}^{-1} \) have head \( v_{C_i(i)} \) and opposite sign, and we let \( C(i, \delta) \) be the one with sign \( \delta \). If \( C(i, \delta) = C_{i,k} \) then let \( d_t(C(i, \delta), \delta) = 1 \) and otherwise \( C(i, \delta) = C_{i,k}^{-1} \) in which case we let \( d_t(C(i, \delta), \delta) = -1 \). \( C(i, \delta) \) and \( s + 1 \in I_{C(i, \delta)} \) then \( C(i, \delta) + 1 \) and \( C(i, \delta) + 1 \) is equal to the sum over \( \sigma \in P[[\gamma], t + 1] \) of the terms \( r_{\sigma,t} \sigma \gamma \). Hence \( m_{i \nu(t), \nu} \in \gamma \text{rad}(\Lambda) \), \( \gamma \in \gamma \text{rad}(\Lambda) \) which is contained in \( \gamma \text{rad}(\Lambda) \) by Corollary 2.11(iii). For (iv) apply the above to \( D = C^{-1} \).

**Corollary 9.5.** [6] Corollary 2.3.8] Fix \( m, m' \in P(C) \) as in Notation 9.2. Let \( i \in I \) and \( \delta = 1 \) and let \( d_t(C(i, \delta)) = 0 \) as in Notation 9.3

(i) If \( n - 1, n \in I_{C(i, \delta)} \) and \( m \in C(i, \delta), m' \) then \( \eta_{i+n+1} \delta = 0 \).

(ii) If \( I_{C(i, \delta)} = \{ 0, \ldots, h \} \), \( C(i, \delta) \mathbb{I}_{1, \mathbb{e}, \epsilon} = C(i, \delta) \) and \( m \in \mathbb{I}_{1, \mathbb{e}, \epsilon}(P(C)) \) then \( \eta_{i+n+1} \delta = 0 \).

(iii) If \( m \in C(i, \delta) \) then \( \theta_{(i)} = 0 \).

**Proof.** (i) Let \( C(i, \delta) = \gamma^{-1} d_{(i)}(\nu), t = i + d(n - 1) \) and \( x = 1 \) so that \( \gamma \in P[x, t] \). By Lemma 9.3. we have \( \theta_{(i)}(\psi_{i+n+1}(d_{x,P(C)}(m'))) = \theta_{(i)}(\psi_{i+n+1}(d_{x,P(C)}(m'))) = \theta_{(i)}(m_{x,i+d(n-1)}(i,e)) + \theta_{(i)}(m'_{x,i+d(n-1)}(i,e)) \).

By Lemma 9.3. we have \( \theta_{(i)}(\psi_{i+n+1}(d_{x,P(C)}(m'))) = \eta_{i+n+1} \gamma \in \gamma \text{rad}(\Lambda) \).
If $d = 1$ then $l_{i+n}^{-1}r_{i+n} = \gamma^{-1}d_{i+1}^{-1}$, and applying Lemma 9.3(iii) (where $t = i + n - 1$ and $m$ is replaced with $m'$) gives $\theta_{i+1}(m')_{x,i+n-1} = \eta_{i+n} \gamma \in \gamma\ker(A)$ and $\theta_{i+1}(m')_{x,i+n-1} = 0$. This gives $\theta_{i+1}(\psi_{i+n-1}(d_{x,P(C)}(m))) = \eta_{i+n} \gamma \in \gamma\ker(A)$, and so $(\eta_{i+n} - \eta_{i+n-1})\gamma \in \gamma\ker(A)$. In case $d = -1$ we similarly have $(\eta_{i-n} - \eta_{i-1-n})\gamma \in \gamma\ker(A)$, by applying Lemma 9.3(iv) where $t = i - n + 1$.

In either case $(\eta_{i+n} - \eta_{i+n-1})\gamma \in \gamma\ker(A)$, and if also $\eta_{i+n} - \eta_{i+n-1}$ lies outside $m$ then $\gamma\ker(A) = \gamma\ker(A)$ which contradicts Remark 2.9. Hence $\eta_{i+n} - \eta_{i+n-1} \in m$, and as $S$ is a transversal in $R$ with respect to $m$, this means $\eta_{i} \in \eta_{i+n-1}$. For the case where $C(i,d) = l_{i-1}^{-1}$ the proof is similar: when we use Lemma 9.3(ii) we set $t = i + d$, and swap $m$ and $m'$, and when we use Lemma 9.3(iii) and Lemma 9.3(iv) we set $t = i$ and $t = i + n$ respectively.

(ii) It suffices to prove $\eta_{i+d} \in m$, since $S \cap m = 0$. If there is no $b \in P$ for which $1_{u,e}^{-1}d_{i}(\beta)$ is a word then $1_{u,e}^{-1}(P(C)) \subseteq \gamma\ker(A)$ and so $\psi_{i+d}(m) \in \gamma\ker(A)$. Since $\eta_{i} \in \gamma\ker(A)$, this gives $\eta_{i} \in m$ as $\eta_{i} \in \gamma\ker(A)$ is local. Suppose instead there is some $b \in P$ for which $1_{u,e}^{-1}d_{i}(\beta)$ is a word. By definition $m \in \gamma^{-1}d_{i}(\gamma)$ for some $m' \in P(C)$ and some $\gamma \in P$ such that $C(i,d) \gamma^{-1}d_{i}(\gamma)$ is a homotopy word.

Let $x \in I(\gamma)$. By Lemma 9.3(iii) $\psi_{i+d}(d_{x,P(C)}(m')) = [m'_{x,i+d}, m']_{x,i+d}$, since $i + d(h + 1) \notin I$ and $d$ is $1$ implies $m'_{x,i+h} = 0$ and $d = 1$ implies $m'_{x,i+h} = 0$. If $d = 1$ and $m'_{x,i+h} \neq 0$ then $i + h - 1 \in I$ and $l_{i-1}^{-1}r_{i-1}^{-1} \gamma^{-1}d_{x,P(C)}(m')$ which means $d_{y}^{-1}r_{x,P(C)}(m')$ is a homotopy word and hence $\theta_{i+1}(m') = 0$. Similarly $\theta_{i+1}(m')_{x,i-h} = 0$ when $d = -1$, and altogether this gives $\theta_{i+1}(\psi_{i+d}(d_{x,P(C)}(m'))) = 0$ so $\eta_{i+d} \gamma \in \gamma\ker(A)$ by Lemma 9.3(ii). As above this shows $\eta_{i+d} \in m$, completing the proof of (ii).

(iii) Choose $h \geq 0$ such that $(I_C(i,d) = \emptyset$ and $m \in C(i,d)_{x,i+h} = C(i,d)$ for a vertex $u$ and $c \in \{1,2\}$). We have elements $m_{j,1} = \sum_{i,1} \eta_{i,j} \psi_{i+1}(d_{x,P(C)}(m'))$, and we can consider $C(i,d)_{x,i+h}$ for any $h$. If $x \rightarrow_{\delta} i$. Applying (i) to each natural number $j \leq 1$ gives $\eta_{i+d,h} = \eta_{i+d,h+1}$, and together this shows $\eta_{i+d,h} = \eta_{i+d,h+1} = \cdots = \eta_{i+d,h-1},h-1 = 0$.

**COROLLARY 9.6.** [6 Corollary 2.3.9] We have $b_{i} \in C(i,d)^{+}(P(C))$ for any $i \in I_{C}$.

Proof. Let $d = d_{i}(C,d)$. It is straightforward to show that, if $n-1, n \in I_{C(i,d)}$ then $b_{i+n}(P(C)) = C(i,d)_{x,i+n}$, and in particular, if $I_{C(i,d)} = \emptyset$ then $b_{i} \in C(i,d)_{x,i+n}$, and so on. So, if $I_{C(i,d)} = \emptyset$ then the existence of the $N$-sequence $(b_{i} | i \in \mathbb{N})$ in $P(C)$ shows that $b_{i} \in C(i,d)^{*}(P(C))$.

Now consider the case $I_{C(i,d)} = \emptyset$ and $C(i,d)_{x,i+n} = C(i,d)$. As above we have $b_{i} \in C(i,d)_{x,i+n}$, and so it suffices to show $b_{i+n} \in C(i,d)_{x,i+n}$. It is enough to assume $C(i,d)_{x,i+n}$ is a homotopy word for some arrow $x$, as otherwise $e_{u}P(C) = 1_{u,e}^{+}P(C)$. Note $d_{P(C)}(b_{i+n}) = b_{i+n}^{+}$, where $d = \mp 1$, and if $l_{i+n}^{-1}r_{i+n} = \gamma^{-1}d_{i+n}(\beta)$ then $l(\beta) \neq x$. So $d_{x,P(C)}(b_{i+n}) = 0$ which shows $b_{i+n} \in d_{x}^{-1}x \subseteq 1_{u,e}^{+}P(C)$.

**NOTATION 9.7.** [6 Definitions 2.3.10 and 2.3.12] Let $V$ be a $R[T,T^{-1}]$-module with free $R$-basis $(v_{\lambda} | \lambda \in \Omega)$, $C = \infty E \infty$ periodic of period $p$ and $E = l_{1}^{-1}r_{1} \cdots l_{p}^{-1}r_{p}$. If $\lambda \in \Omega$ and $0 \leq i \leq p - 1$ let $b_{i,\lambda} = b_{i} \otimes v_{\lambda}$. So $q \in P(C,V)$ gives $q = \sum_{\lambda} q_{i,\lambda} b_{i,\lambda}$ where $q_{i,\lambda} = \eta_{i,\lambda} + \sum_{\sigma} r_{\sigma,i,\lambda} \sigma, \eta_{i,\lambda} \in S, r_{\sigma,i,\lambda} \in R$ and $r_{\sigma,i,\lambda} = 0$ for all but finitely many $\sigma \in P[i]$. 

Let $[q] = \sum i, \lambda \eta_i, \lambda b_i, \lambda$, $[q] = \sum i, \lambda \sum r_{\sigma, i, \lambda} \sigma b_i, \lambda$. For any $x \in A$ we define $[q_{x,t}] = [q_{x,t} + [q_{x,t}]_{x,t}$. For any $x \in A$ we define $[q_{x,t}] = [q_{x,t} + [q_{x,t}]_{x,t}$ by setting

$$
[q]_{x,t} = \begin{cases} 
\sum \eta_{t+1, \lambda} \eta \otimes \nu_{\lambda} & (0 \leq t < p - 1 \text{ and } l_{t+1}^{-1} r_{t+1} = \alpha^{-1} d_{z}) \\
\sum \eta_{0, \lambda} \otimes T \nu_{\lambda} & (t = p - 1 \text{ and } l_{p}^{-1} r_{p} = \alpha^{-1} d_{z}) \\
\sum \eta_{p-1, \lambda} \otimes T^{-1} \nu_{\lambda} & (t = 0 \text{ and } l_{0}^{-1} r_{0} = \alpha^{-1} \beta) \\
0 & \text{(otherwise)}
\end{cases}
$$

$$
[q_{x,t}] = \begin{cases} 
\sum \sigma \in \mathcal{P}[x,t] r_{\sigma, t+1, \lambda} \sigma \nu \otimes \nu_{\lambda} & (0 \leq t < p - 1 \text{ and } l_{t+1}^{-1} r_{t+1} = \kappa^{-1} d_{(i)}) \\
\sum \sigma \in \mathcal{P}[x,0] r_{\sigma, 0, \lambda} \sigma \nu \otimes T \nu_{\lambda} & (t = p - 1 \text{ and } l_{p}^{-1} r_{p} = \kappa^{-1} d_{(i)}) \\
\sum \sigma \in \mathcal{P}[x,p-1] r_{\sigma, p-1, \lambda} \sigma \nu \otimes T^{-1} \nu_{\lambda} & (t = 0 \text{ and } l_{0}^{-1} r_{0} = \kappa^{-1} d_{(i)}) \\
0 & \text{(otherwise)}
\end{cases}
$$

If $0 \leq t < p$ let $\varphi_t : P(C, V) \to \Lambda e_{v(c)}(i) \otimes_R V$ be the composition $\omega_t \kappa$ where $\kappa : P(C, V) \to \bigoplus_{t=0}^{p-1} \Lambda e_{v(c)}(i) \otimes_R V$ is the isomorphism from Lemma 3.6 and $\omega_t$ is the canonical projection. If $m \in P(C)$ and $v \in V$ then, in the sense of Notation 9.2 we have $\varphi_0(d_{P(C,V)}(m \otimes v)) = \psi_0(d_{P(C,V)}(m)) \otimes v + \psi_{p-1}(d_{P(C,V)}(m)) \otimes v$ and $\varphi_{p-1}(d_{P(C,V)}(m \otimes v)) = \psi_{p-1}(d_{P(C,V)}(m)) \otimes v + \psi_{p-1}(d_{P(C,V)}(m)) \otimes v$ [Lemma 2.3.11]. If $y \in A(v) \to$ let $\phi_y : (\bigoplus_{a \in A(v \to) \Lambda a) \otimes_R V \to \Lambda a \otimes_R V$ be the natural $\Lambda$-module projection.

The proofs of Lemma 9.8 and Lemma 9.9 are omitted. The proof of Lemma 9.8 involves a straightforward application of Lemmas 3.6 and 9.3. The proof of Lemma 9.9 uses Lemma 9.8 and is similar to the proof of Corollaries 9.5 and 9.6. See [3] §2.3.1 for details.

**Lemma 9.8.** [6] Lemma 2.3.13] Let $x \in A$, $0 \leq t < p$ and $M = \text{rad}(\Lambda) \otimes_R V$. Fix $q \in P(C, V)$ and consider Notation 9.7.

(i) We have $\varphi_t(d_{P(C,V)}([q])) = [q_{x,t}] + [q_{x,t}]_{x,t}$ and $\varphi_t(d_{P(C,V)}([q])) = [q_{x,t}] + [q_{x,t}]_{x,t}$.

(ii) If $\gamma \in P[x, t]$ and $q \in \gamma^{-1} d_z q'$ then $\phi_{(i)}(\varphi_t(d_{P(C,V)}([q]))) = \sum \lambda \eta_{t, \gamma} \otimes \nu \in \gamma M$.

(iii) If $l_{t+1}^{-1} r_{t+1} = \gamma^{-1} d_{z}$ then $\phi_{(i)}([q_{x,t}]) = 0$.

(iv) If $l_{t}^{-1} r_{t} = d_{z}^{-1} \gamma$ then $\phi_{(i)}([q_{x,t}]) = 0$.

(v) If $l_{t+1}^{-1} r_{t+1} = \gamma^{-1} d_{z}$ and $t < p - 1$ then $\phi_{(i)}([q_{x,t}]) - \sum \lambda \eta_{t+1, \gamma} \otimes \nu \in \gamma M$.

(vi) If $l_{t}^{-1} r_{t} = d_{z}^{-1} \gamma$ and $0 < t$ then $\phi_{(i)}([q_{x,t}]) - \sum \lambda \eta_{t-1, \gamma} \otimes \nu \in \gamma M$.

(vii) If $l_{p}^{-1} r_{p} = \gamma^{-1} d_{z}$ then $\phi_{(i)}([q_{x,p-1}]) - \sum \lambda \eta_{p-1, \gamma} \otimes T \nu \in \gamma M$.

(viii) If $l_{0}^{-1} r_{0} = d_{z}^{-1} \gamma$ then $\phi_{(i)}([q_{x,0}]) - \sum \lambda \eta_{0, \gamma} \otimes T^{-1} \nu \in \gamma M$.

**Lemma 9.9.** [6] Lemmas 2.3.14 and 2.3.15] Let $i, n \in \mathbb{Z}$, $0 \leq i \leq p - 1$, $1 \leq n \leq p$, $\delta = \pm 1$, $d = d_{i}(C, \delta)$ and let $\mu \in \Omega$. Fix $q, q' \in P(C, V)$ with $q \in C(i, \delta) q'$ and consider Notation 9.7.

(i) If $(i < p - n$ and $d = 1)$ or $(i > n - 1$ and $d = -1)$ then $\eta_{i+n}(d_{i-1}, \mu) = \eta_{i+n}(d_{i-1}, \mu')$.

(ii) If $(i > p - n$ and $d = 1)$ or $(i < n - 1$ and $d = -1)$, then $\eta_{i+n}(d_{i-1}, \mu) = \eta_{i+n}(d_{i-1}, \mu')$.

(iii) If $i = p - n$ and $d = 1$ then $\eta_{0, i} = 0$ for all $\lambda$ if and only if $\eta_{p-1, \lambda} = 0$ for all $\lambda$.

(iv) If $i = n - 1$ and $d = -1$ then $\eta_{0, i} = 0$ for all $\lambda$ if and only if $\eta_{p-1, \lambda} = 0$ for all $\lambda$.

Consequently $b_{i, \lambda} \in C(i, \delta)_{+}(P(C, V))$, and if $q \in C(i, \delta)^{\circ}(P(C, V))$ then $\{\eta_{i, \lambda} \mid \lambda \in \Omega\} = \{0\}$. 


10. Refining complexes.

Assumption 10.1. As in Assumption 9.1 in §10 we let \( C \) be a homotopy \( I \)-word, and we let \( S \) be a transversal of \( \mathbf{m} \) in \( R \) such that \( \mathbf{S} \cap \mathbf{m} = \{0\} \) and \( \mathbf{S} \cap (1 + \mathbf{m}) = \{1\} \).

Corollary 10.2. Let \( A \in \mathcal{W}_{v,\delta} \) and \( (I, A, +) \) (respectively \( (I, A, -) \)) be the set of \( i \in I \) with \( v_C(i) = v \) and \( C(i, \delta) \leq A \) (respectively \( C(i, \delta) < A \)).

(i) Let \( \Phi \) be a natural isomorphism

\[
\Phi(\mathbf{m}) \Rightarrow \Phi(\mathbf{n})
\]

Then \( \Phi \) is a straightforward application of part (ii), noting that there is a natural isomorphism \( \mathcal{C}(\mathbf{m}) \Rightarrow \mathcal{C}(\mathbf{n}) \).

(ii) Assume (i) holds and \( \mathbf{m} \) and \( \mathbf{n} \) are \( \mathbf{m} \)-words.

Notation 10.3. \( \mathcal{C}(\mathbf{m}) \Rightarrow \mathcal{C}(\mathbf{n}) \) be the restriction of the quotient functor \( \mathcal{C}(\mathbf{m}) \Rightarrow \mathcal{C}(\mathbf{n}) \).

Recall the set \( \sum \) of \((B, D, n)\) where \( n \in \mathbb{Z} \) and \( B, D \) and \( B^{-1}D \) are homotopy words. Recall the functors \( \mathcal{S}_{B, D, n} \) and \( \mathcal{F}_{B, D, n} \) (\((B, D, n) \in \sum\)). See §7 and §8 for details.

Lemma 10.4. \( \mathcal{C}(\mathbf{m}) \Rightarrow \mathcal{C}(\mathbf{n}) \) be a natural isomorphism between \( \mathcal{C}(\mathbf{m}) \) and \( \mathcal{C}(\mathbf{n}) \) for each \( \lambda \in \Omega \). By Corollary 10.2 we have \( \mathbf{m} \cap (1 + \mathbf{m}) = \{1\} \). Similarity \( \mathbf{m} \cap (1 + \mathbf{m}) = \{1\} \).

Proof. Let \( \lambda \in \Omega \). By Corollary 10.2 we have \( \mathbf{m} \cap (1 + \mathbf{m}) = \{1\} \). Similarity \( \mathbf{m} \cap (1 + \mathbf{m}) = \{1\} \).
Lemma 10.5. (Lemma 2.3.20) (see also [14] Lemma 8.2). Let \((B, D, n), (B', D', n') \in \Sigma\) such that \(C = B^{-1}D\) is aperiodic. Let \(C' = (B')^{-1}D'\) and \(P = P(C)\).

(i) If \(i \in \mathbb{I}\) then \(\tilde{F}_{C(i),1},C(i,-1),n}(P[\mu_C(i) - n]) = \tilde{F}_{C(i),1},C(i,-1),n}(P[\mu_C(i) - n]) + \sum_R b_i\).

(ii) If \(C' = C\) and \(n - n' = \mu_C(a_{B,D}) - \mu_C(a_{B',D'})\) then \(k \otimes_R \simeq F_{B',D',n',n'}(B, D, n).

(iii) If \((B, D, n)\) is not equivalent to \((B', D', n')\) then \(\tilde{F}_{B',D',n',n'}(P[\mu_C(a_{B,D}) - n]) = 0.\)

Proof. (i) Let \(M = P[\mu_C(i) - n]\) and \(v = v_C(i)\). Let \(A = C(i, 1)\) and \(E = C(i, -1)\). Let \((i, 1, \pm) = (i, A, \pm)\) and \((i, -1, \pm) = (i, E, \pm)\) as in the proof of Lemma 10.4. By Corollary 10.2 \(\tilde{F}_{A,E,n}(M) = \tilde{F}_{A,E,n}(M) + \tilde{R}_b\). Now let \(m \in \tilde{F}_{A,E,n}(M)\). By assumption and by Corollary 10.2 we may write \(m = \sum_{j} \eta_j b_j + m_0\) for \(\eta_j \in S\) and some \(m_0 \in e_n \text{rad}(M)\), where \(\eta_j = 0\) whenever \(C(j, 1) > A\) or \(C(j, -1) > E\). Since \(m \in M^\infty = P[\mu_C(i)](C)\) we have \(\eta_j = 0\) for all \(j \in I\) with \(\mu_C(j) \neq \mu_C(i)\). So \(\sum_j \eta_j b_j \in \sum R_{b_j}\) where \(t\) runs through \((i, 1, +) \cap (i, -1, +)\). Let \((i, \delta, \pm) = \{j \in I | C(j, \delta) = C(i, \delta)\}\). Then \((i, 1, +) \cap (i, -1, +)\) is the union of \((i, 1, +) \cap (i, -1, -), (i, 1, -) \cap (i, -1, +)\) and \((i, 1, =) \cap (i, -1, =)\). So by Corollary 10.2 \(\sum \eta_j b_j\) lies in \(F_{A,E,n}(M) + \sum R_{b_t}\) where \(t\) runs over \((i, 1, =) \cap (i, -1, =)\) with \(\mu_C(j) = \mu_C(i)\). It suffices to assume \(t \in I, C(t, 1) = A, C(t, -1) = C(i, -1)\) and \(\mu_C(t) = \mu_C(i)\); and show \(t = i\). If \(C_{\geq 1} = (C_{< 1}^{-1})^{-1} = C_{> 1}\) then \(C[t] = C^{-1}[t]\) which contradicts Lemma 2.2.17. Hence \(C_{\leq i} = C_{> i}\) and \(C_{\leq i} = C_{< i}\) which shows \(C = C[t - i]\). Applying Lemma 3.4(iii) twice yields \(\mu_C(t - i) = 0\), which means \(t = i\) since \(C\) is aperiodic.

(ii) Any element of \(S_{B,D,n}(V)\) may be written as the coket of a sum of pure tensors \(\sum_{j=1}^n r_j b_j \otimes v + \xi(S_{B,D,n}(V))\) for some \(v_1, ..., v_n \in V\). Hence the \(k\)-linear embedding \(\Phi_V\) from Lemma 10.4(i) is surjective, and so \(\Phi\) from Lemma 10.4(ii) is a natural isomorphism. Since \((B, D, n) \sim (B', D', n')\) we have \(F_{B',D',n',n'}(P(C)|\mu_C(a_{B,D} - n)) = 0\). It suffices to let \(s(B') = 1\) and \(s(D') = -1\), and show \((B, D, n) \sim (B', D', n')\) assuming \(F_{B',D',n',n'}(P) = 0\). By Corollary 10.2 the \(R\)-submodules \(G_{B,D,n}(P)\) are spanned by sets of elements of the form \(b_{i,t}\) together with \(\text{rad}(P)\), and hence, as in the proof of [14] Lemma 8.2], the existence of \(b_i\) shows that \(B' = C(i, 1)\) and \(D' = C(i, -1)\), so \(B' - D' = C\). Since \(b_i\) lies in both \(P\) and \(P'\) we have \(\mu_C(i) = n' + \mu_C(a_{B,D} - n)\) and so \((B, D, n) \sim (B', D', n')\).

Lemma 10.6. (Lemma 2.3.21) (see also [14] Lemma 8.5). Let \((B, D, n), (B', D', n') \in \Sigma\) where \(C = B^{-1}D\) is periodic and \(i \in \{0, ..., p - 1\}\). If \(P = P(C, V)\) then:

(i) \(\tilde{F}_{C(i,1),C(i,-1),n}(P[\mu_C(i) - n]) = \tilde{F}_{C(i,1),C(i,-1),n}(P[\mu_C(i) - n]) + \sum_R b_i\).

(ii) If \(C' = C\) and \(n - n' = \mu_C(m)\) then \(k[T, T^{-1}][R[T, T^{-1}]] - \simeq F_{B',D',n',n'}(B, D, n, S_{B,D,n})\) and \(F_{B',D',n',n'}(P[-n]) = 0.\)

11. Compactness and Covering.

Assumption 11.1. In [11] we let \(v\) be a vertex and fix any homotopically minimal complex \(M\) of projective \(A\)-modules. Use \(M\) to denote the underlying \(A\)-module or underlying \(R\)-module.

The topology we refer to will be the \(m\)-adic topology. Let \(m^0 = R\). Recall a base of open sets for an \(R\)-module \(N\) with this topology is the collection of cosets \(m + m^nU\) where \(n \in \mathbb{N}\) and \(U\) is an \(R\)-submodule of \(N\).

Later in this article we apply the results in [11] in the context where the object \(M\) of \(K_{\text{min}}(A \text{-}\text{Proj})\) lies in \(K_{\text{min}}(A \text{-}\text{proj})\) (and so \(M^f\) is finitely generated over \(A\) for each \(i \in \mathbb{Z}\)).
Remark 11.2. Recall from Remark 2.9 the, by Definition 2.3, $\Lambda$ is a finitely generated $R$-module. Consequently, under Assumption 11.1 if $M$ lies in $K_{\text{min}}(\Lambda-\text{proj})$ each $R$-module of the form $e_v M'$ is finitely generated.

Definition 11.3. [33 §1, p. 80, Definition] Let $L$ be a subset of an $R$-module $N$. We write $L \subseteq N$ if and only if $L$ is closed. We say $L$ is a linear variety if $L = U + m \subseteq N$ for some $R$-submodule $U$ of $N$. We say $N$ is linearly compact if any collection of linear varieties in $N$ with the finite intersection property must have a non-void intersection.

Example 11.4. For each non-zero $n \in \mathbb{N}$ the $R$-module $R/m^n$ has the minimum condition on closed submodules, and so it is linearly compact by [33 Proposition 5]. Since $R$ is m-adically complete, $R$ (as a module over itself) is isomorphic to the inverse limit of a system of linearly compact $R$-modules. By [33 Proposition 4] this means $R$ is linearly compact. By [33 Proposition 1] this means that any finitely generated free $R$-module is linearly compact.

Lemma 11.5. [6 Lemma 2.4.1]. Suppose $M$ lies in $K_{\text{min}}(\Lambda-\text{proj})$. Let $r \in \mathbb{Z}$ and $\delta = \pm 1$. Let $U$ be an $R$-submodule of $e_v M'$ with $e_v \text{rad}(M') \subseteq U$.

(i) (See also [14 Lemma 10.4]). If $H$ is a linear variety in $e_v M'$ and $m \in H \setminus U$, then there is a homotopy word $C \in W_{e_v, \delta}$ such that $H \cap (U + m)$ meets $C^+(M)$ but not $C^-(M)$.

(ii) (See also [14 Lemma 10.5]). If $m \in e_v M' \setminus U$ then there are words $B \in W_{e_v, \delta}$ and $D \in W_{e_v, -\delta}$ such that $U + m$ meets $G_{B,D,r}(M)$ but not $G_{B,D,r}(M)$.

The proof of Lemma 11.5 is given at the end of §.

Lemma 11.6. [6 Lemma 2.4.8] (see also [14 Lemma 10.3]). Fix an integer $r$ and some $\delta \in \{\pm 1\}$. For any non-empty subset $S$ of $e_v M'$ which does not meet $\text{rad}(M)$ there is a homotopy word $C \in W_{e_v, \delta}$ such that either:

(i) $C$ is finite and $S$ meets $C^+(M)$ but not $C^-(M)$; or

(ii) $C$ is a homotopy $\mathbb{N}$-word and $S$ meets $C_{\leq n}M$ but not $C_{\leq n} \text{rad}(M)$ for each $n \geq 0$.

In Lemma 11.6 we do not require that $M$ is an object of $K_{\text{min}}(\Lambda-\text{proj})$. In Lemma 11.10 we do consider this setting, and show $S \cap C^+(M) \neq \emptyset = S \cap C^-(M)$ in case (ii) of Lemma 11.6.

Proof of Lemma 11.6 We assume either $S \cap B^+(M) = \emptyset$ or $S \cap B^-(M) \neq \emptyset$ for any finite homotopy word $B \in W_{e_v, \delta}$. We now construct a homotopy $\mathbb{N}$-word $C$ iteratively from $C_{<0} = 1_{v, \delta}$, such that $S$ meets $C_{\leq n}M$ but not $C_{\leq n} \text{rad}(M)$ for each $n \geq 0$. If $n = 0$ there is nothing to prove. Assume $S$ meets $C_{\leq m}M$ but not $C_{\leq m} \text{rad}(M)$ for some fixed $m \geq 0$. It suffices to choose $l_{m+1}$ and $r_{m+1}$ such that $S$ meets $C_{\leq m} l_{m+1}^{-1} r_{m+1} M$ but not $C_{\leq m} l_{m+1}^{-1} r_{m+1} \text{rad}(M)$.

Suppose $S \cap (C_{\leq m} M)^{\perp} \neq \emptyset$. We can assume there exists $y \in A$ where $C_{\leq m} y^{-1} d_y$ is a homotopy word, since otherwise $(C_{\leq m} M)^{\perp} \cap S \subseteq C_{\leq m} \text{rad}(M) \cap S = \emptyset$. As $S$ meets $(C_{\leq m} M)^{\perp}$ there exists $\gamma \in \mathbf{P}$ of minimal length for which $S$ meets $C_{\leq m} \gamma^{-1} d_{l(\gamma)} M$. Let $l_{m+1} = \gamma$ and $r_{m+1} = d_{l(\gamma)}$. For case (ii) it suffices to show $S$ does not meet $C_{\leq m} \gamma^{-1} d_{r(\gamma)} \text{rad}(M)$.

If $\gamma \in A$ then $\gamma^{-1} d_{r(\gamma)} \text{rad}(M) \subseteq e_{l(\gamma)} \text{rad}(M)$ by Corollary 6.6 and so $S$ does not meet $C_{\leq m} \gamma^{-1} d_{r(\gamma)} \text{rad}(M)$. So we can assume $\gamma = l(\gamma) \alpha$ for some $\alpha \in \mathbf{P}$. By Corollary 6.6 we also have $\alpha^{-1} d_{l(\alpha)} M = \alpha^{-1} l(\gamma)^{-1} l(\gamma) d_{l(\alpha)} M$ and by Lemma 6.3 (ii) we have $l(\gamma) d_{l(\alpha)} M = d_{l(\gamma)} l(\gamma) M = d_{l(\gamma)} \text{rad}(M)$. The minimality of the length of $\gamma$ shows that $S$ does not meet $C_{\leq m} \alpha^{-1} d_{l(\alpha)} M$ and altogether this shows $S$ does not meet $C_{\leq m} \gamma^{-1} d_{l(\gamma)} \text{rad}(M)$. 


Suppose instead \( S \cap (C_{\leq m})^{-}(M) = \emptyset \). Here \( S \cap (C_{\leq m})^{+}(M) = \emptyset \) by the assumption at the beginning of the proof, so there is some arrow \( x \) for which \( C_{\leq m}d_{x}^{-1}x \) is a homotopy word.

By definition \( d_{x,M} \) sends elements of \( e_{h(x)}M \) to \( xM \), and so \( S \) meets \( C_{\leq m}d_{x}^{-1}xM \). Suppose the set \( L \) of \( \lambda \in P \) where \( (C_{\leq m}d_{x}^{-1}\lambda) \) is a homotopy word is infinite. By Lemma 2.1.19 we have \( \bigcap \lambda C_{\leq m}d_{x}^{-1}\lambda M = (C_{\leq m})^{+}(M) \) which does not meet \( S \). By Corollary 6.7 there is some maximal length \( \mu \in L \) for which \( S \) meets \( C_{\leq m}d_{x}^{-1}\mu M \). As \( L \) is infinite \( \mu \eta \in L \) for some arrow \( \eta \) in which case \( C_{\leq m}d_{x}^{-1}\mu \eta \mu M \) which does not meet \( S \) by construction. In this case it is sufficient to let \( l_{m+1} = d(\mu) \) and \( r_{m+1} = \mu \). Otherwise \( L \) is finite with longest path \( \mu' \), in which case it suffices to let \( l_{m+1} = d(\mu') \) and \( r_{m+1} = \mu' \).

**Lemma 11.7.** [Lemma 2.4.4] Suppose \( M \) lies in \( K_{\min}(A-proj) \). Let \( i \in \mathbb{Z} \) and \( m \in e_{v}M^{i} \).

(i) The \( R \)-module \( L = e_{v}M^{i} \) is linearly compact.

(ii) If \( U \subseteq e_{v}M^{i} \) with \( e_{v}m^{n}M^{j} \subseteq U \) for some \( n > 0 \) then \( U + m \subseteq e_{v}M^{i} \).

(iii) For any \( m \in e_{v}M^{i} \) we have \( \{m\} = 0 + m \subseteq e_{v}M^{i} \).

**Proof.** (i) By Remark 11.2 \( L \) is finitely generated. By Example 11.4 \( L \) is linearly compact.

(ii), (iii) Let \( l \) be a limit point of \( U + m \). So for all \( t \geq 0 \) there exists \( u_{t} \in U \) with \( u_{t} + m \neq l \) and \( u_{t} + m \in l + m^{t}M \). So there exist \( u_{n+1} \in U \) and \( x_{t+1} \in m^{n+1}M \) with \( u_{n+1} + m = l + x_{n+1} \). Since \( e_{v}(u_{n+1} + m - l) = u_{n+1} + m - l \) we have \( x_{n+1} \in e_{v}M \subseteq U \), and so \( l = (u_{n+1} - x_{n+1}) + m \in U + m \). This gives (ii). For (iii), any neighborhood of \( l \) contains \( m \), so \( m - l \in e_{v}m^{t}M \) for all \( t \geq 0 \). Hence \( m - l \in \bigcap_{t \geq 0} e_{v}m^{t}M^{i} = 0 \) by the last statement in Remark 2.9.

**Remark 11.8.** For any \( \gamma \in P \) and any \( \alpha \in A \) we have that: if \( U \subseteq e_{\eta(\gamma)}M^{i} \) then \( \gamma U \subseteq e_{h(\gamma)}M^{i} \); if \( V \subseteq e_{h(\alpha)}M^{i} \) then \( d_{\alpha}V \subseteq e_{h(\alpha)}M^{i+1} \) and \( d_{\alpha}^{-1}V \cap e_{h(\alpha)}M^{i-1} \subseteq e_{h(\alpha)}M^{i-1} \); and if \( W \subseteq e_{h(\gamma)}M^{i} \) then \( \gamma^{-1}W \cap e_{\eta(\gamma)}M^{i} \subseteq e_{\eta(\gamma)}M^{i} \) (see 6 Corollary 2.4.5 for details).

**Corollary 11.9.** [Corollary 2.4.6] Suppose \( M \) lies in \( K_{\min}(A-proj) \), \( N \) is an \( R \)-submodule of \( M \) and that \( C \) is a homotopy \( \{0, \ldots, t\} \)-word. If \( M^{i} \cap CN \subseteq e_{h(C)}M^{i} \) then \( e_{h(C)}M^{i} \cap CN \subseteq e_{h(C)}M^{i} \).

**Proof.** By Corollary 7.3 we have \( e_{h(\gamma)}M^{i} \cap C_{\leq n}N = e_{h(\gamma)}M^{i} \cap C(M^{i} \cap CN \subseteq e_{h(C)}M^{i}) \).

**Lemma 11.10.** [Lemma 2.4.7] Suppose \( M \) lies in \( K_{\min}(A-proj) \). Let \( \gamma \in P \), \( i \in \mathbb{Z} \), \( v(+) = t(\gamma) \) and \( v(-) = h(\gamma) \). Let \( C = L_{t(\gamma)}^{-1}r_{1} \ldots \) be a homotopy \( I \)-word where \( \emptyset \neq I \subseteq \mathbb{N} \).

(i) If \( m \in e_{h(\gamma)}M^{i} \) then \( \gamma^{-1}d(\gamma)M^{i+1} \cap e_{v(\pm)}M^{i+1} \) is a linear variety if it is non-empty.

(ii) If \( 0 < t \in I \) and \( m \in M^{i} \cap r_{i-1}(C_{t}^{-1}) \leq_{n} M \) then \( M^{i} \cap r_{i-1}(C_{t}^{-1}) \leq_{n} M \) meets the intersection \( \bigcap_{n \in I, n \geq t+1}(C_{n})_{\leq_{n} M} \).

(iii) If \( I = \mathbb{N} \) and \( S \subseteq e_{v}M^{i} \) then \( S \cap C^{+}(M) = \bigcap_{n \geq 0} S \cap C_{\leq n}M \).

**Proof.** (i) Let \( P = \gamma^{-1}d(\gamma)M^{i+1} \cap e_{v(\pm)}M^{i+1} \). If \( x \in P \) then \( P^{x} = e_{\eta}M^{i+1} \cap \gamma^{-1}M^{i} \). If \( y \in P \) then \( y = x \in P^{x} \) as \( \gamma y = d(\gamma), m(\gamma) = (\gamma x) \). By Lemma 11.7(iii), Remark 11.8 and Corollary 11.9 \( P \) is closed. This gives the case \( v(\pm) = v(+) \). The proof for the case \( v(\pm) = v(-) \) is similar.

(ii) For each \( n \geq t \) we have \( m \in (C_{n})_{\leq n}M \) and so there is some \( u_{n} \in M^{i} \cap r_{i-1}(C_{n}^{-1}) \leq_{n} M \) for which \( u_{n} \in (C_{n})_{\leq n}M \). Let \( \Delta \) be the collection of \( M^{i} \cap (C_{n})_{\leq n}M \) \( (n \geq t) \) together with \( M^{i} \cap r_{i-1}(C_{n}^{-1}) \leq_{n} M \) for each \( n \geq 0 \).
Clearly $\Delta$ has finite intersections. Each member of $\Delta$ lies in $e_{v_C(t)}M^{i+\mu_C(t)}$, which is linearly compact by Lemma 11.7(i). By (i) we have $M^{i+\mu_C(t)} \cap r^{-1}_{i,m}e_{v_C(t)}M^{i+\mu_C(t)}$, and so by Corollary 11.9 the collection $\Delta$ consists of linear varieties. Hence $\bigcap_{n \in \mathbb{N}} V_n = 0$.

(iii) The argument in the proof of [7] Lemma 3.1] adapts with few complications.

**Proof of Lemma 11.7** (i) Let $S = H \cap (U + m)$. Note $S \cap \text{rad}(M) = \emptyset$ since $e_x \text{rad}(M') \subseteq U$ and $m \notin U$. So by Lemma 11.7(iii) there is a homotopy word $C$ such that either $C$ is finite and $S \cap C^+(M) \neq \emptyset = S \cap C^-(M)$, or $C$ is a homotopy $\mathbb{N}$-word and for all $n \geq 0$ we have $S \cap C^M_n M \neq \emptyset = S \cap C^M_n \text{rad}(M)$. We assume $I_C = \mathbb{N}$ as otherwise there is nothing to prove. Note that the collection $(S \cap C^M_n M \mid n \geq 0)$ has the finite intersection property. By Lemma 11.7(ii) and Corollary 11.9 each $S \cap C^M_n M$ is a linear variety in $e_x M'$. By Lemma 11.7(i) $e_x M'$ is linearly compact, and so $\bigcap_{n \geq 0} S \cap C^M_n M \neq \emptyset$. This shows $S \cap C^+(M) \neq \emptyset$ by Lemma 11.10(iii). Since $S \cap C^M_n \text{rad}(M) = \emptyset$ for all $n$, $S \cap C^-(M) \subseteq \bigcup S \cap C^M_n \text{rad}(M) = \emptyset$.

(ii) The argument in the proof of [14] Lemma 10.5] adapts with few complications.

12. Local and global mapping properties.

**Assumption 12.1.** In [12] fix an object $M$ of $K_{\text{min}}(\Lambda\text{-Proj})$.

Recall: $\Sigma$ is the set of triples $(B, D, n)$ where $B^{-1}D$ is a homotopy word and $n \in \mathbb{Z}$; $\Sigma(s)$ (respectively $\Sigma(b)$) is the set of $(B, D, n)$ where $B^{-1}D$ is aperiodic (respectively periodic); $\mathcal{I}(s) \subseteq \Sigma(s)$ (respectively $\mathcal{I}(b) \subseteq \Sigma(b)$) is a collection of representatives $(B, D, n)$, one for each equivalence class of $\Sigma(s)$ (respectively $\Sigma(b)$); and $\mathcal{I} = \mathcal{I}(s) \cup \mathcal{I}(b)$ (see Definition 8.5).

**Assumption 12.2.** In Lemmas 12.3 and 12.3 fix $(B, D, n) \in \Sigma$ and let $C = B^{-1}D$. If $j \in I_C$, $i = a_{B,D}$ and $t = i - j$ then $v_C(j) = v_B(t)$ and $\mu_C(j) = \mu_C(i) + \mu_B(t)$ if $t \geq 0$, and $v_C(j) = v_D(-t)$ and $\mu_C(j) = \mu_C(i) + \mu_D(-t)$ if $t < 0$ [6] Lemma 2.5.1. If $I_B \neq \{0\}$ let $B = l_i^{-1}r_1 \ldots$ and if $I_D \neq \{0\}$ let $D = l_i^{-1}r_1' \ldots$

**Lemma 12.3.** [6] Lemma 2.5.2] (see also [14] Lemma 8.3]). If $(B, D, n) \in \mathcal{I}(s)$ and $B = (\overline{u}_{\lambda} \mid \lambda \in \Omega)$ is a $k$-basis of $F_{B,D,n}(M)$ then there is a morphism of complexes $\theta_{B,D,n,m} : \bigoplus \mathcal{P}(C)[\mu_C(a_{B,D}) - n] \rightarrow M$ such that $F_{B,D,n}(\theta_{B,D,n,m})$ is an isomorphism.

**Proof.** Let $i = a_{B,D}$. For each $\lambda$ choose a lift $u_{\lambda} \in F^+_1(B,D,n,M) \setminus F^+_1(B,D,n,M)$ of $\overline{u}_{\lambda}$. Since $u_{\lambda} \in B^+(M)$ we have, by Corollary 12.4 that for all $s \in I_B$ there exists $u^s_{\lambda} \in e_B(s)$ in degree $n + \mu_B(s)$ where $u^0_{\lambda} = u_{\lambda}$ and $u^s_{\lambda} \in l^{-1}_{s+t}r_{s+t+1}u^t_{\lambda}$ whenever $s + 1 \in I_B$. Similarly there are $u^t_{\lambda} \in e_D(t)M$ in degree $n + \mu_D(t)$ where $u^0_{\lambda} = u_{\lambda}$ and $u^t_{\lambda} \in l^{-1}_{t+s}r_{t+s}u^s_{\lambda}$ whenever $t + 1 \in I_D$. Let $u_{\lambda} = u^s_{\lambda} - u^t_{\lambda}$ if $j < i$ and $u_{\lambda} = u^s_{\lambda} - u^t_{\lambda}$ if $j \geq i$.

By [6] Lemma 2.5.1] (see Assumption 12.2) setting $\theta_{B,D,n,m}(b_{j,\lambda}) = u_{\lambda}$ (for all $j$ and $\lambda$) defines a degree 0 graded $\Lambda$-module map into $M$. Note $d_M(u_{\lambda}) = u^{s+1}_{\lambda} + u^t_{\lambda}$ where $u^{s+1}_{\lambda} = \sum_{s,t} d_M(u_{\lambda})$ and $\sigma^{s+1}$ runs through the arrows with head $v_C(j)$ and sign $\pm 1$. It is straightforward to show $u^{s+1}_{\lambda} = \theta_{B,D,n,m}(b^s_{j,\lambda})$ by separating the cases $j \in I$ and $j \notin I$.

Together this gives $\theta_{B,D,n,m}(b^s_{j,\lambda} + b^t_{j,\lambda}) = d_M(u_{\lambda})$ and so $\theta_{B,D,n,m}$ is a morphism of complexes. By Lemma 10.5(i) the elements $c_{i,\lambda} = b_{i,\lambda} + F_{B,D,n}(P(C)[\mu_C(i) - n]) (\lambda \in \mathcal{B})$ together define a basis of the $k$-vector space $F_{B,D,n}(\bigoplus \mathcal{P}(C)[\mu_C(i) - n])$. Since $\theta_{B,D,n,m}(b_{i,\lambda}) = u_{i,\lambda}$ we have that $F_{B,D,n}(\theta_{B,D,n,m})(c_{i,\lambda}) = \overline{u}_{i,\lambda} = \overline{u}_{\lambda}$ so $F_{B,D,n}(\theta_{B,D,n,m})$ is an isomorphism. \(\square\)
Recall $E(n)$ consists of pairs $(m, m')$ with $m, m' \in e_*M^n$ and $m \in E_m'$, and so $E(n)^2 = F^{+}_{B,D,n}(M)$ and $E(n)^0 = F^{-}_{B,D,n}(M)$ (see Remark 77).

**Lemma 12.4.** (6 Lemma 2.5.4) (see also [14 Lemma 8.6]). Suppose $(B, D, n) \in \mathcal{I}(b)$, say where $C = \infty \in \infty$ is periodic of period $p > 0$ and $E = l_1^{-1}r_1 \ldots l_p^{-1}r_p$. If there is a reduction $(U, f)$ of the relation $E(n)$ on $e_*M^n$ then there is a morphism $\theta_{B,D,n,M} : P(C, U)[-n] \to M$ of complexes such that $F_{B,D,n}(\theta_{B,D,n,M})$ is an isomorphism.

**Proof.** Let $F_{B,D,n}(M) = V$. Choose an $R$-basis $(u_\lambda \mid \lambda \in \Omega)$ of $U$. Since $(U, f)$ is a reduction, $\text{im}(f) \subseteq E(n)^0$ and so $f(u_\lambda) \in F^{+}_{B,D,n}(M)$ for each $\lambda$. Similarly $f(Tu_\lambda) \in E f(u_\lambda)$, and so there are elements $u_{n,\lambda}, u_{p,\lambda} M$ where $u_{n,\lambda} = g(u_\lambda), u_{0,\lambda} = g(Tu_\lambda)$, and $u_{j,\lambda} \in l_j^{-1}r_j u_{j-1,\lambda}$ given $j > 0$. By Lemma [3.6] to define a $\Lambda$-module map $\theta_{B,D,n,M} : P(C, U)[-n] \to M$ it is enough to extend $\theta_{B,D,n,M}(b_j \otimes u_\lambda) = u_{j,\lambda}$ linearly over $\Lambda$ for each $\lambda$ and each $j$ with $0 \leq j \leq p - 1$. By Corollary 7.4 we have that $u_{n,\lambda} \in e_{v_\lambda(j)}M^{n+u_\lambda(j)}$ for each $\lambda$, and so $\theta_{B,D,n,M}$ is homogeneous of degree 0. As in the proofs of [14 Lemma 8.6] and [29 §5, Proposition] the morphism $P(C) \times U \to M$ given by $((b_j, u_\lambda) \mapsto u_{j,\lambda})$ is $\mathcal{R}(T, T^{-1})$-balanced. To show $\theta_{B,D,n,M}$ is a morphism of complexes one uses Lemma [6.3(ii)], separating the cases $j = 0, j = p - 1$ and $p - 1 \neq j \neq 0$.

We now show that $F_{B,D,n}(\theta_{B,D,n,M})$ is an isomorphism. Write $e_{p,\lambda}$ for the coset $b_p \otimes u_\lambda + F^{+}_{B,D,n}(P(C, U)[-n])$. By Lemma 10.6(ii) the elements $(e_{p,\lambda} \mid \lambda \in \Omega)$ give a $k$-basis of $F_{B,D,n}(P(C, U)[-n]) \cong k \otimes R V$. Let $u_{p,\lambda} + F^{+}_{B,D,n}(P(C, U)[-n]) = \bar{u}_{p,\lambda}$ for each $\lambda \in \Omega$.

Since $F_{B,D,n}(\theta_{B,D,n,M})(e_{p,\lambda}) = \bar{u}_{p,\lambda}$, to prove $F_{B,D,n}(\theta_{B,D,n,M})$ is an isomorphism we need only show $\text{Ker}(e_{p,\lambda} \mid \lambda \in \Omega)$ is a $k$-linearly independent subset of $V = F_{B,D,n}(M)$. If we have $\sum \lambda (r_\lambda + m) u_{p,\lambda} = 0$ in $V$ for some $(r_\lambda \mid \lambda \in \Omega) \in \bigoplus R$ then $\sum \lambda (r_\lambda u_\lambda) \in E(n)^2$. Since the reduction $(U, f)$ meets in $m$, we have $\sum \lambda r_\lambda u_\lambda \in \bigoplus m u_i$, as required.

**Lemma 12.5.** (6 Lemma 2.5.5) (see also [14 Lemma 10.5] and [11 p. 163]). Let $\theta : P \to M$ be a morphism in $k_{\min}(\Lambda-\text{Proj})$, and suppose $M^i$ is finitely generated for each $i$. If $F_{B,D,n}(\theta)$ is surjective for each $(B, D, n) \in \Sigma$ then $\theta^i$ is surjective for each $i$.

**Proof.** For a contradiction suppose that $\theta^i$ is not surjective for some $i \in \mathbb{Z}$. Since $\text{rad}(M^i)$ is a superfluous submodule of $M^i$, $e_*\text{im}(\theta^i) + e_*\text{rad}(M^i)$ is contained in a maximal $R$-submodule $U$ of $e_*M^i$. Since $e_*\text{rad}(M^i) \subseteq U$ and $U \neq e_*M^i$, by Lemma 11.5(ii) for some element $m \in e_*M^i \setminus U$ there are homotopy words $B \in W_{s,\delta}$ and $D \in W_{s,\delta}$ for which $(B^{-1}D$ is a homotopy word) and $U + m$ meets $G^+_{B,D,i}(M)$ but not $G^-_{B,D,i}(M)$. From here one can show $F_{B,D,i}(\theta)$ is not surjective by adapting the argument from the proof of [14 Lemma 10.6].

**Assumption 12.6.** For the remainder of §12 we fix a direct sum $N$ of shifts of string and band complexes as follows. Let $\mathcal{S}$ and $\mathcal{B}$ be sets, $\{t(\sigma), s(\beta) \mid \sigma \in \mathcal{S}, \beta \in \mathcal{B}\}$ be a set of integers, $\{V^\beta \mid \beta \in \mathcal{B}\}$ be a set of objects from $R[T, T^{-1}]$-$\text{Mod}_{R-\text{Proj}}$ and $\{A(\sigma), E(\beta) \mid \sigma \in \mathcal{S}, \beta \in \mathcal{B}\}$ be a set of homotopy words, where each $A(\sigma)$ is aperiodic and each $E(\beta)$ is $p\beta$-periodic. Let

$$N = \left( \bigoplus_{\sigma \in \mathcal{S}} P(A(\sigma))[-t(\sigma)] \right) \oplus \left( \bigoplus_{\beta \in \mathcal{B}} P(E(\beta), V^\beta)[-s(\beta)] \right)$$

**Definition 12.7.** The homotopy words $(A(\sigma)_{\leq 0})^{-1}$ and $A(\sigma)_{> 0}$ (respectively $(E(\beta)_{< 0})^{-1}$ and $E(\beta)_{> 0}$) have the same head and opposite sign, and we let $A(\sigma, \pm)$ (respectively $E(\beta, \pm)$) be the one with sign $\pm1$. 

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LEMMA 12.8. [6 Lemma 2.5.6] (see also [14 Lemma 9.4]). Let \( N \) be the direct sum of string and band complexes from Assumption 12.6. Let \( \theta : N \to M \) be a map in \( K_{\min}(\Lambda\text{-}\text{Proj}) \) where \( F_{B,D,n}(\theta) \) is injective for all \( (B,D,n) \in \mathcal{I} \). Then each \( \theta^i \) is injective.

Proof. Assume there is some \( h \in \mathbb{Z} \) for which \( \theta^h \) is not injective. Since \( N^h \) and \( M^h \) are projective \( \Lambda \)-modules, we have rad\((N^h) = \text{rad}(\Lambda)N^h \) and rad\((M^h) = \text{rad}(\Lambda)M^h \) (see example [24 Theorem 24.7]). Since \( \Lambda/\text{rad}(\Lambda) \) is semisimple, \( N^h/\text{rad}(\Lambda)^h \) is an injective \( \Lambda/\text{rad}(\Lambda) \)-module. Hence the induced map \( \tilde{\theta}^h : N^h/\text{rad}(\Lambda)^h \to M^h/\text{rad}(\Lambda)^h \) is not injective, as otherwise it must be a section, which would mean \( \theta^h \) was injective by [20 Lemma 2.2].

So there is a vertex \( v \) and non-zero \( n \in e_v \text{rad}(N^h) \) with \( \theta^h(n) \in e_v \text{rad}(M^h) \). Since \( n \) has finite support over the summands of \( N \), we can assume \( S = \{1, \ldots, m\} \) and \( B = \{1, \ldots, q\} \).

For each \( \sigma \in S \), and each \( i \in I_{A(\sigma)} \), let \( b_i^\sigma = b_i,A(\sigma) \). For each \( \beta \in B \), fix an \( R \)-basis \( (v_\lambda^\beta | \lambda \in \Omega(\beta)) \) of \( V^\beta \), and let \( b_{j,\lambda}^\beta = b_j,E(\beta) \otimes v_\lambda^\beta \). Fix a transversal \( S \) of \( m \) in \( R \).

\[
n = x + \sum_{\sigma=1}^{m} \sum_{i \in I_{A(\sigma)}} \eta_{\sigma,i}b_i^\sigma + \sum_{\beta=1}^{q} \sum_{j \in I_{E(\beta)}} \sum_{\lambda \in \Omega(\beta)} \eta_{\beta,j,\lambda}b_{j,\lambda}^\beta
\]

for some \( x \in e_v \text{rad}(N^h) \) and some \( \eta_{\sigma,i}, \eta_{\beta,j,\lambda} \in \mathbb{S} \). After reordering we can assume \( A(\sigma,1) \leq A(\sigma',1) \) for \( \sigma \leq \sigma' \) and \( E(\beta,1) \leq E(\beta',1) \) for \( \beta \leq \beta' \). Let \( B \) be the largest of \( A(m,1) \) and \( E(q,1) \). If \( B = A(m,1) \) let \( D \) be the largest homotopy word \( A(\sigma',1) \) among \( A(1,1), \ldots, A(m,1) \) for which \( A(\sigma',1) = A(m,1) \). In this case there is some such \( i \) with \( v_{A(\sigma)}(i) = v \) and \( \mu_{A(\sigma)}(i) = h - t(\sigma) \); and so \( n_{\sigma,i}b_i^\sigma = x \in G_{B,D,h}^+(N) \setminus G_{B,D,h}^-(N) \).

13. Completing the proofs of the main theorems.

We now verify the hypotheses of Lemma 14.5 in our setting. Recall, from Definition 8.3, the equivalence relation on the triples \((B,D,n)\) where \( B^{-1}D \) is a homotopy word and \( n \in \mathbb{Z} \). Recall, from Definition 8.5, that \( \mathcal{I} = \mathcal{I}(s) \cup \mathcal{I}(b) \) where \( \mathcal{I}(s) \) (respectively \( \mathcal{I}(b) \)) denotes a chosen set of representatives \((B,D,n)\) such that \( B^{-1}D \) is aperiodic (respectively periodic).

Recall that if \((B,D,n)\) lies in \( \mathcal{I}(s) \) (respectively \( \mathcal{I}(b) \)) then the functor \( S_{B,D,n} \) has the form \( R_{-}\text{-\text{Proj}} \to C_{\min}(\Lambda\text{-\text{Proj}}) \) (respectively \( R[T,T^{-1}]-\text{Mod}_{R_{-}\text{-\text{Proj}}} \to C_{\min}(\Lambda\text{-\text{Proj}}) \)), and the functor \( F_{B,D,n} \) has the form \( K_{\min}(\Lambda\text{-\text{Proj}}) \to k_{-}\text{-\text{Mod}} \) (respectively \( K_{\min}(\Lambda\text{-\text{Proj}}) \to k[T,T^{-1}]-\text{Mod} \)). Recall Definition 4.4.

PROPOSITION 13.1. [6 Proposition 2.6.6] (see also [11] p. 163, Proposition]). Let \( M = \Lambda\text{-\text{Mod}} \) and \( \mathcal{I} = \mathcal{I}(s) \cup \mathcal{I}(b) \); and for each \( i = (B,D,n) \in \mathcal{I} \) let

\[
(\mathfrak{A}_i, \mathfrak{X}_i) = \begin{cases} (R_{-}\text{-\text{Proj}}, k_{-}\text{-\text{Mod}}) & \text{if } B^{-1}D \text{ is aperiodic,} \\ (R[T,T^{-1}]-\text{Mod}_{R_{-}\text{-\text{Proj}}}, k[T,T^{-1}]-\text{Mod}) & \text{if } B^{-1}D \text{ is periodic.} \\
\end{cases}
\]

(i) The category \( \mathcal{N} = \Lambda\text{-mod} \) has enough projective covers.

(ii) The collection \( \{(S_{B,D,n}, F_{B,D,n}) \mid (B,D,n) \in \mathcal{I}\} \) detects the objects in \( K_{\min}(\Lambda\text{-\text{Proj}}) \).

In the proof of Proposition 13.1 ii) we verify the conditions FFI, FFII, FFIII, FFIV, FFV and FFVI from Definition 4.3. Later we use Proposition 13.1 in the context of Lemma 4.5.
Proof.  (i) By Corollary 2.11 (vi) $\Lambda$ is a semiperfect ring, and so every finitely generated $\Lambda$-module has a projective cover (see for example [24 Proposition 24.12]).

(ii) FFI) Recall $k[T, T^{-1}] \otimes_{R[T, T^{-1}]} \times k \otimes_R$ as functors $R[T, T^{-1}]$-$\text{Mod}_{R}[\text{Proj}] \rightarrow k[T, T^{-1}]$-$\text{Mod}$. Let $(B, D, n) \in \Sigma$ and $B^{-1} D = C$. If $C$ is aperiodic (respectively periodic) then by Lemma 10.3 ii) (respectively Lemma 10.0 ii)) we have $F_{B, D, n} \oplus S_{B, D, n} \approx k \otimes_R$ as functors $\mathcal{A}_{B, D, n} \rightarrow \mathcal{X}_{B, D, n}$. We show $k \otimes_R - : \mathcal{A}_{B, D, n} \rightarrow \mathcal{X}_{B, D, n}$ is dense and reflects isomorphisms.

Recall that any local ring (such as $R$) is a semiperfect ring. If $g$ is a morphism in $\mathcal{A}_{B, D, n}$ then $g$ is homomorphism of free $R$-modules, and $g$ is an isomorphism in $\mathcal{A}_{B, D, n}$ if and only if $g$ is bijective. Similarly any isomorphism in $\mathcal{X}_{B, D, n}$ is an isomorphism of vector spaces. Altogether, by Remark 3.11 if $g$ is a morphism in $\mathcal{A}_{B, D, n}$ such that $k \otimes g$ is an isomorphism in $\mathcal{X}_{B, D, n}$, then $g$ was an isomorphism. Hence $k \otimes_R -$ reflects isomorphisms.

Note that for any vector space $V$ with basis $(v_\lambda \mid \lambda \in \Omega)$ over $k$ there is an isomorphism $f : V \rightarrow k \otimes_R F$ of $k$-vector spaces where $F = \bigoplus_{\lambda} R a_\lambda$ is the free $R$-module with $R$-basis $(a_\lambda \mid \lambda \in \Omega)$, such that $f(v_\lambda) = (1 + m) \otimes a_\lambda$. This shows $k \otimes_R -$ is dense when $(B, D, n)$ lies in $\Sigma$. Suppose instead that $(B, D, n)$ lies in $\mathcal{I}(b)$, and that $V$ is a $k[T, T^{-1}]$-module. There is an $R$-module endomorphism of $F$ given by $a_\lambda \mapsto \sum_{\mu} r_{\mu \lambda} a_\mu$ for each $\lambda$, where $Tv_\lambda = \sum_{\mu} (r_{\mu \lambda} + m)v_\mu$ for some $r_{\mu \lambda} \in R$. As above, by Remark 3.11 this endomorphism is an isomorphism, and so $F$ is an $R[T, T^{-1}]$-module where $T a_\lambda = \sum_{\mu} r_{\mu \lambda} a_\mu$. By construction $f$ is $k[T, T^{-1}]$-linear.

FFII) Let $(B', D', n') \in \mathcal{I}$. If $(B', D', n') \neq (B, D, n) \in \mathcal{I}(s)$ then $F_{B', D', n'}(P(C)[\mu C(a, b, D) - n]) = 0$ by Lemma 10.3 iii) where $C = B^{-1} D$. This shows $F_{B', D', n'} \oplus S_{B, D, n} = 0$ since $F_{B', D', n'} \approx F_{B', D', n'}$. If $(B, D, n) \in \mathcal{I}(b)$ then the proof is similar and uses Lemma 10.6 iii).

FFIII) By Corollary 6.13 each of the subfunctors $C^\pm$ of the forgetful functor $C_{\text{min}}(\Lambda-\text{Proj}) \rightarrow \text{R-Mod}$ commutes with direct sums. It follows that $F_{B, D, n}^\pm$ commutes with direct sums of objects in $\text{K}_{\text{min}}(\Lambda-\text{Proj})$ (see [6, Lemma 2.1.21] for details).

FFIV) Let $M$ be an object in $\text{K}_{\text{min}}(\Lambda-\text{Proj})$, which means $F_{B, D, n}(M)$ is finite-dimensional as a $k$-vector space. If $(B, D, n)$ lies in $\mathcal{I}(s)$, by Lemma 12.3 there is a free $R$-module $U$ of $R$-rank $\dim_k(F_{B, D, n}(M))$ and a morphism $\theta_{B, D, n} : S_{B, D, n}(U) \rightarrow M$ for which $F_{B, D, n}(\theta_{B, D, n})$ is an isomorphism. Now suppose instead that $(B, D, n)$ lies in $\mathcal{I}(b)$, say where $B^{-1} D = \infty E$ is periodic of period $p > 0$, and where $E = t_1^{-1} r_1 \ldots t_p^{-1} r_p$. Note $F_{B, D, n}(M) = E(n)^b$ and $F_{B, D, n}^+(M) = E(n)^b$, so $E(n)^b / E(n)^b = F_{B, D, n}(M)$. By Lemma 6.3 there is a (unique) $U$ of $E(n)$ which meets in $\mathfrak{m}$. The required morphism exists by Lemma 12.4.

FFIV, FFVI) Let $\theta : N \rightarrow M$ be an morphism in the category $\text{K}_{\text{min}}(\Lambda-\text{Proj})$. If $M$ is a complex in $\text{K}_{\text{min}}(\Lambda-\text{proj})$ and $F_{B, D, n}(\theta)$ is epic for all $(B, D, n) \in \mathcal{I}$ then $\theta^e$ is epic for each $n \in N$ by Lemma 12.5. This shows FFV holds, and similarly FFVI holds by Lemma 12.8.

Proof of Theorem 1.1) Parts (i) and (ii) of Theorem 1.1 are precisely parts (i) and (iii) of Lemma 4.5 after applying Proposition 13.1.

Note that, in the context of the proof of Theorem 1.1) Lemma 4.3 ii) says that any indecomposable object in $\text{K}_{\text{min}}(\Lambda-\text{proj})$ is the shift of a string or band complex.
Let \((B, D, n) \in \Sigma\). If \((B, D, n) \in \Theta(s)\) let \(S(B, D, n)\) be the set of \(\sigma\) such that \(t(\sigma) - n = r(B, D, C(1), C(-1))\) and \(A(\sigma) \sim B^{-1}D\). If \((B, D, n) \in \Theta(b)\) let \(B^\pm(B, D, n)\) be the set of \(\beta\) such that \(s(\beta) - n = \mu_{B^{-1}D}(\pm m)\) and \(E(\beta) = (B^{-1}D)^{\pm[1]}\). For \(\beta \in B(B, D, n)^\pm\) and \(\beta' \in B'(B, D, n)^\pm\) define \(V_+(\beta), V_-(\beta) = k \otimes R V^\beta, W_+(\beta') = k \otimes R W^{\beta'}\) by

\[
V_+(\beta) = V^\beta, \quad V_-(\beta) = \text{res}_{i,R} V^\beta, \quad W_+(\beta') = W^{\beta'}, \quad W_-(\beta') = \text{res}_{i,R} W^{\beta'}.
\]

**Lemma 13.3.** [6, Lemma 2.5.8] Let \(B, B', D, D'\) be homotopy words such that \(C = B^{-1}D\) and \(C' = B'^{-1}D'\) are homotopy words. Let \(n\) and \(n'\) be integers.

(i) If \(n \neq n'\) then \(S(B, D, n) \cap S(B, D, n') = \emptyset = B(B, D, n)^\pm \cap B(B, D, n')^\pm\).

(ii) If \(B^{-1}D \sim B'^{-1}D'\) then \(S(B, D, n) \cap S(B', D', n) = \emptyset = B(B, D, n)^\pm \cap B(B', D', n)^\pm\).

(iii) We have \(\bigcup_{n \in Z} S(B(D, D, t)) = \{ \sigma \in S \mid A(\sigma) \sim C \} \).

(iv) We have \(\bigcup_{n \in Z} B(B, D, t)^+ \cup B(B, D, t)^- = \{ \beta \in B \mid E(\beta) \sim C \} \).

**Theorem 13.4.** [6, Theorem 2.5.9] (see also [14, Theorem 9.1]). Let \((B, D, m) \in \Sigma\).

(i) If \(C = B^{-1}D\) is aperiodic then in \(k\text{-Mod}\) we have isomorphisms

\[
F_{B, D, m}(N) \cong \bigoplus_{\sigma \in S(B, D, m)} k, \quad \text{and} \quad F_{B, D, n}(N) \cong \bigoplus_{\sigma \in S \mid A(\sigma) = C^{\pm 1}} k.
\]

(ii) If \(C\) is periodic then in \(k[T, T^{-1}]\text{-Mod}\) we have isomorphisms

\[
F_{B, D, m}(N) \cong \bigoplus_{\beta \in B(B, D, m)^+} \hat{V}_+(\beta) \oplus \bigoplus_{\beta' \in B(B, D, m)^-} \hat{V}_-(\beta'), \quad \text{and}
\]

\[
\bigoplus_{n \in Z} F_{B, D, n}(N) \cong \bigoplus_{\beta \in B \mid E(\beta) = C^{[t]} \text{ for some } t} \hat{V}_+(\beta) \oplus \bigoplus_{\beta' \in B \mid E(\beta) = C^{[t]} - 1} \hat{V}_-(\beta').
\]

**Proof.** (i) By Corollary 6.13 the functor \(F_{B, D, m}\) preserves small coproducts. This together with Lemma 10.5(iii) shows \(F_{B, D, m}(N) \cong \bigoplus_{\sigma \in S} F_{B, D, m}(P(A(\sigma))[-t(\sigma)])\) as \(C\) is aperiodic. If \(\sigma \in S(B, D, m)\) then \((A(\sigma), 1), (A(\sigma), -1), t(\sigma)) \sim (B, D, m)\) and so by Lemma 8.3 and Lemma 10.5(ii) we have \(F_{B, D, m}(P(A(\sigma))[-t(\sigma)]) \cong F_{B, D, m}(S(B, D, m)(R)) \cong k\). Otherwise \(\sigma \notin S(B, D, m)\) and so as above \(F_{B, D, m}(P(A(\sigma))[-t(\sigma)]) = 0\). By Lemma 13.3(i) \(\sum_{n \in Z} \# S(B, D, n) = \sum_{n \in Z} S(B, D, n)\). By Lemma 13.3(iii) this completes the proof of (i).

(ii) Similar to the above, \(F_{B, D, m}(N)\) is isomorphic to \(\bigoplus_{\beta \in B} F_{B, D, m}(P(\beta, V^\beta)[-s(\beta)])\) by Lemma 10.6(iii). If \(\beta \in B(B, D, m)^\pm\) then by Lemma 8.3 and Lemma 10.6(ii) we have that \(F_{B, D, m}(P(\beta, V^\beta)[-s(\beta)])\) is isomorphic to \(\hat{V}_+(\beta)\) as above. If \(\beta \notin B(B, D, m)^+ \cup B(B, D, m)^-\) then \(F_{B, D, m}(P(\beta, V^\beta)[-s(\beta)]) = 0\) by Lemma 10.6(iii). By Lemma 13.3(ii) this shows \(F_{B, D, m}(N) \cong (\bigoplus_\beta \hat{V}_+(\beta)) \oplus (\bigoplus_\beta \hat{V}_-(\beta))\) where \(\hat{V}_+\) runs through \(B(B, D, m)^+\).

**Proof of Theorem 13.2.** Recall Definition 12.7. Let \(C\) and \(E\) be homotopy words and let \(n \in Z\). We assume \(s((E_{\leq 0})^{-1}) = 1\) and \(s(E_{> 0}) = -1\). The case where \(s((E_{\leq 0})^{-1}) = 1\) and \(s(E_{> 0}) = 1\) is similar, and omitted. Recall that homotopy words \((C_{\leq 0})^{-1}\) and \(C_{> 0}\) have the same head and opposite sign. We let \(C(\pm 1)\) be the one with sign \(\pm 1\). Choose \(\delta \in \{ -1, 1 \}\) such that \(C(-\delta) = (C_{\leq 0})^{-1}\) and \(C(\delta) = C_{> 0}\). Let \(V\) and \(W\) be objects of \(R[T, T^{-1}]\text{-Mod}_{R_{\text{proj}}}\).

(i) Let \(C\) and \(E\) be aperiodic. Assume momentarily that: \(I_C = \{ 0, \ldots, m \}\) and \((I_E, E, n) = (I_C, C(1), \mu_C([m]))\); or that \(I_C = \pm N\) and \((I_E, E, n) = (\pm N, C(0))\) or \((I_E, E, n) = (\pm N, C(-1), 0)\); or that \(I_C = Z\) and \((I_E, E, n) = (Z, C^{\pm 1}[t], \mu_C(\pm t)))\) for some \(t \in Z\).

In each of the cases (respectively) there is an isomorphism \(P(C)[n] \cong P(E)\) in \(K(A_{\text{proj}})\) given by parts (i), (ii) and (iii) of Lemma 6.4.
For the converse suppose \( P(C)[n] \simeq P(E) \) in \( K(\text{A-Proj}) \). In the notation of Assumption 13.2 we have \( N = P(C)[n] \) and \( N' = P(E) \) where \( B = B' = \emptyset, S = \{ \sigma \}, S' = \{ \sigma' \}, A(\sigma) = C, B(\sigma') = E, l(\sigma) = -n \) and \( w(\sigma') = 0 \). By Theorem 13.4 (i) we have that, for any \( (H, L, q) \in \Sigma \), \( F_{H,L,q}(N) \) (respectively \( F_{H,L,q}(N') \)) is isomorphic to \( \bigoplus k \) where the direct sum runs through all \( \sigma \in S \) (respectively \( \sigma' \in S' \)) such that \( n - q = r(H, L; C(1), C(-1)) \) (respectively \( 0 - q = r(H, L; (E_{<0})^{-1}, E_{>0}) \)) and \( C \simeq H^{-1}L \) (respectively \( E \sim H^{-1}L \)).

Applying \( F_{H,L,0} \) to \( N \simeq N' \) in case \( (H, L) = ((E_{<0})^{-1}, E_{>0}) \) gives \( F_{H,L,0}(N) \simeq k \), which shows that \( n = r((E_{<0})^{-1}, E_{>0}; C(1), C(-1)) \) and \( C \sim E \). By definition this means that: \( C = E \) which is not a homotopy \( Z \)-word, and so \( E_{>0} = C \) and \( s(C) = 1 \), and so \( n = \mu_E(0) = 0 \); or \( C = H^{-1}L \) which is not a homotopy \( Z \)-word, and so \( E_{\leq m} = C^{-1} \) and \( s(C) = -1 \), and so \( n = \mu_E(m) \) where \( I_C = \{0, \ldots, m\} \) or \( (I_C = \pm N \text{ and } m = 0) \); or \( E = E^{+\{1\}} \) is a homotopy \( Z \)-word and \( n = \mu_E(\pm t) \). This shows one of the conditions (a), (b) or (c) must hold.

(ii) Let \( C \) and \( E \) be periodic. Assume momentarily, for some \( t \in \mathbb{Z} \), that \( E = C[t] \) and \( n = \mu_C(t) \); or that \( E = C^{-1}[t] \) and \( n = \mu_C(-t) \). By definition this means \( (H, L, 0) \sim (C(1), C(-1), -n) \) where \( H = (E_{<0})^{-1} \) and \( L = E_{>0} \). By Corollary 5.7 we have that: if \( E = C[t] \), \( k \otimes_R V \simeq k \otimes_R W \) and \( n = \mu_C(t) \) then \( S_{C(1),C(-1),-n}(V) \simeq H_{L,0}(W) \); and if \( E = C^{-1}[t], k \otimes_R V \simeq k \otimes_R W \) and \( n = \mu_C(-t) \) then \( S_{C(1),C(-1),-n}(V) \simeq H_{L,0}(W) \). Since \( \mu_{E,R}(id) \) have \( P(C, V)[n] \simeq P(E, W) \) in \( K(\text{A-Proj}) \). As above we have \( N' = P(C, V)[n] \) and \( N = P(E, W) \) where \( S = S' = \emptyset, B = \{ \beta \}, B' = \{ \beta' \}, E(\beta) = E, D(\beta') = C, s(\beta) = 0 \) and \( r(\beta') = -n \). By Theorem 13.4(ii) we have that, for any \( (H, L, q) \in \Sigma \), \( F_{H,L,q}(N) \simeq \bigoplus W \) (respectively \( F_{H,L,q}(N') \simeq \bigoplus V \)) in \( k[T, T^{-1}]-\text{Mod} \) where the sum runs through all \( \beta \in B \) (respectively \( \beta' \in B' \)) with \( -q = r(H, L; (E_{<0})^{-1}, E_{>0}) \) (respectively \( n = r(H, L; (E_{<0})^{-1}, E_{>0}) \)) and \( E \sim H^{-1}L \) (respectively \( C \sim H^{-1}L \)).

Applying \( F_{H,L,n} \) to the isomorphism \( N \simeq N' \) in case \( (H, L) = ((C(1), C(-1)) \) gives \( F_{H,L,n}(N) \simeq k \otimes_R V \), which shows that \( n = r((E_{<0})^{-1}, E_{>0}; C(1), C(-1)) \) and \( C \sim E \). By definition this means that: \( E = C[t] \) and \( n = \mu_C(t) \), in which case \( F_{H,L,n}(N) \simeq k \otimes_R W \); or that \( E = C^{-1}[t] \), in which case \( F_{H,L,n}(N) \simeq k \otimes_R W \).

(iii) Using similar ideas to the above, one can show that if \( P(C)[n] \simeq P(E, V) \) then we must have \( C \simeq E \), which is impossible if \( C \) is not periodic, yet \( E \) is periodic.

Proof of Theorem 13.2 Let \( (B, D, n) \in \mathcal{T} \). Suppose \( N \simeq N' \) in the notation of Assumption 13.2 We define the bijection \( \chi : S \cup B \to S' \cup B' \) as follows.

Suppose firstly that \( (B, D, n) \in \mathcal{T}(s) \). As in the proof of Theorem 13.2 by Theorem 13.4(i) there is a bijection \( \varphi : S(B, D, n) \to S'(B, D, n) \). By Lemma 8.3 if \( \varphi(\sigma) = \sigma' \) then both \( P(A(\sigma))[-t(\sigma)] \) and \( P(B(\sigma'))[-u(\sigma')] \) are isomorphic to \( P(C)[-n] \). If instead \( (B, D, n) \in \mathcal{T}(b) \) then, as \( F_{B,D,n}(N) \simeq P_{B,D,n}(N) \), by Theorem 13.4(ii) we have

\[
\bigoplus_{\beta_+} V_+(\beta_+ \cup \bigoplus_{\beta_-} W_-(\beta_-) \simeq \bigoplus_{\beta_+} V_+(\beta_+) \cup \bigoplus_{\beta_-} W_-(\beta_-)
\]

where \( \beta_+ \) (respectively \( \beta_- \)) runs through \( B(B, D, n)^\pm \) (respectively \( B'(B, D, n)^\pm \)). As in the proof of Theorem 13.2 by Theorem 13.4(ii) and by the Krull-Remak-Schmidt property for \( k[T, T^{-1}]-\text{Mod}_{k,mod} \) there is an isomorphism class preserving bijection

\[
\psi : B(B, D, n)^- \cup B(B, D, n)^+ \to B'(B, D, n)^- \cup B'(B, D, n)^+.
\]

Provided \( \psi(\beta) = \beta' \) for some \( \beta \in B(B, D, n)^\pm \), note that: if \( \beta' \in B'(B, D, n)^\pm \) then \( V_+ (\beta) \simeq W_+(\beta') \); and if \( \beta' \in B'(B, D, n)^\pm \) then \( V_+(\beta) \simeq W_+(\beta') \). By Corollary 8.7 if \( \psi(\beta) = \beta' \) then

\[
P(\psi(\beta), V_+[\beta])[-s(\beta)] \simeq S_{B,D,n}(V_+(\beta)) \simeq P(D(\beta'), W^+[\beta])[-r(\beta')]
\]

Define \( \chi : S \cup B \to S' \cup B' \) by \( \chi(\alpha) = \alpha' \) if and only if \( (\alpha \in S(B, D, n) \text{ and } \alpha' = \varphi(\alpha)) \) or \( (\alpha \in B(B, D, n)^\pm \cup B(B, D, n)^- \text{ and } \alpha' = \psi(\alpha)) \) for some \( (B, D, n) \in \mathcal{T} \). Note that \( \chi \) is well defined and bijective by Lemma 13.3.

\[
\square
\]
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