A Neural Network Anomaly Detector Using the Random Cluster Model

Robert A. Murphy, Ph.D.

e-mail: robert.a.murphy@wustl.edu

Abstract: The random cluster model is used to define an upper bound on a distance measure as a function of the number of data points to be classified and the expected value of the number of classes to form in a hybrid K-means and regression classification methodology, with the intent of detecting anomalies. Conditions are given for the identification of classes which contain anomalies and individual anomalies within identified classes. A neural network model describes the decision region-separating surface for offline storage and recall in any new anomaly detection.

Keywords and phrases: probability, neural network, anomaly, detector, random cluster, K-means, machine learning, classification.

1. Introduction and Related Work

In $K$-means classification, a set of data will form clusters, i.e. classes, if the measured distances between data points (or some common point in each class) are below a certain threshold. With the assumption that the data points are randomly generated throughout some bounded region according to a certain probability distribution, we identify an upper bound on a chosen distance measure of regularity such that clusters of data points form when their measured distances from a common weighted-center is less than the upper bound.

In [15], Lee and Xiang propose an information-theoretic measure of regularity for anomaly detection of in-network traffic patterns, among other test cases. They posit that there is a set of intrinsic characteristics in the measured data that identifies regularity and these characteristics can be measured as a count of the number of bits required to accurately distinguish one class of data from another distinct class. The particular class differentiator used is a measure of entropy such that when particular attributes/variables/columns are identified as being most responsible for the formation of a particular class, then a particular data sample is anomalous with respect to a given class, if entropy, as a function of the identified attributes, is increased as a result of adding the sample to the given class.

With the choice of a distance measure of regularity, the data points are assumed to be generated into a bounded subset $\mathcal{B}' \subset \mathbb{R}^L$. Let $\mathcal{B} \subset \mathbb{R}^2$ be a bounded region of unit area. Using graph-based methods, we can define a bijective projection mapping of the data points in $\mathcal{B}'$ into $\mathcal{B}$ by partitioning $\mathcal{B}$ into a fixed number of structures of the same size and shape such that disjoint class regularities are maintained, and each data point in $\mathcal{B}'$ maps to exactly one structure in the partition of $\mathcal{B}$. From this partition, we can calculate the maximum size of each structure such that disjoint class regularities are maintained after projection.

In [22], Saligrama and Zhao propose a graph-based method of anomaly detection which relies upon a temporal and spatial composite score using distance measures. They show that the scoring structure is asymptotically optimal in terms of reducing the occurrence of false alarms as the number of samples increases.
With [22] as motivation, we assume a finite, independent sampling from the normal distribution, which is then ordered by applying some suitable order statistic, producing a dependent, beta-distributed sampling [16]. The ordered sample is projected into \( B \), while maintaining disjoint class regularities. We then define a composite score for classes and an individual score for data points within a class, as an indication of an anomaly, by calculating the measured distance of the center of a class (respectively, data point within a class) from a regression hyperplane through the sampling in \( B' \).

Motivation for the use of the measured distances of class centers and data points within a class from the regression hyperplane to indicate an anomaly comes from [1], where the linear regression model is an estimate of the data points which minimizes the mean squared error between the model and the data points. As such, any class with a center that has a distance from the regression hyperplane which is greater than the computed bound is defined to be an anomalous class. Likewise, any data point within a class that has a distance from the regression hyperplane which is greater than the computed bound is defined to be an anomalous data point. By detailing a certain shift and reflection of the regression hyperplane, we show that the calculated weights can be used in the activation function for a two-class discriminating neural network for anomaly detection.

2. Setup

We note the arguments presented in Murphy [21, Section (2)] for obtaining an estimate for the expected number of clusters to form, \( K = N^2 \).

Let \( T^2 \) be the total number of sampled data points, with each data point having \( L \) attributes. The attributes of each of the \( T^2 \) are normalized to have values in the interval \([0, 1]\). What we now have are \( T^2 \) data points, each with \( L \) attributes, which are represented by \( T^2 \) points in an \( L \)-dimensional hypercube of unit area. Let \( J = M^2 << T^2 \) and assume that we have a smaller dataset of size \( M^2 \), which is further sampled from the \( T^2 \) data set.

Let \( B \) be the unit square in the 2-dimensional plane. Theorem (7.3.38) from [20] gives the minimum number of hexagons required to partition \( B \) into hexagons of equal size such that there are \( K = N^2 \) disjoint partitions totaling \( J = M^2 \) hexagons.

**Theorem 1** Assume that there are \( J = M^2 \) samples and \( K = N^2 \) classifications for the samples. The minimum number of hexagons required to partition the unit square into \( K = N^2 \) disjoint regions such that \( J = M^2 \) is the sum total of all hexagons in the disjoint regions is given by

\[
S(M, N) = M^2 + (N - 1)^2 + 2MN. \tag{1}
\]

The idea is to use the result of the theorem to calculate, as a function of \( M \) and \( N = N(M) \), an estimate of the radial size of a circumscribing circle of a prototypical hexagon which will be used to partition \( B \) is given in Murphy [21] as

\[
R(M, N) = \frac{1}{2\sqrt{S(M, N)}}, \tag{2}
\]

thereby necessarily indicating that

\[
B(M, N) = 2 \times R(M, N)
\]

is the diameter of the circle. We have the following result from Murphy [21, Lemma (2)].
Lemma 2 \( R(M, N) \) is decreasing for increasing \( M \) and \( N \).

Theorem 3 Denote the critical probability as \( p_c \). With probability 1, every node in \( \mathcal{B} \) is contained within one contiguous cluster if and only if the probability of any one hexagon containing a node, exceeds \( p_c \). Otherwise, all clusters are disjoint with probability 1.

In order to not exceed the critical probability, while maintaining the \( K = N^2 \) classes of \( J = M^2 \) data points, the common radial size \( r_0 \) of each circle must be less than \( R(M, N) \). By thm. (3), the clusters will be disjoint with probability 1. The following corollary to thm. (1) follows from these statements and lemma (5).

Corollary 4 Let \( H \) be a hexagon of size such that a circle of radius \( r_0 = r_0(M, N) \) can be circumscribed, where

\[
r_0 \leq R(M, N).
\]

If \( \mathcal{B} \) is partitioned into copies of \( H \), then with probability 1, it follows that \( K = N^2 \) is the mean number of disjoint clusters of contiguous hexagons in the region \( \mathcal{B} \) that are occupied by the \( J = M^2 \) data points.

With \( r_0 \) given by corollary (4), the size of the prototypical hexagon can be calculated for repartitioning \( \mathcal{B} \) through each classification of the \( T^2 \) data set. Furthermore, corollary (4) guarantees that the classes will remain distinct, with probability 1, through each new classification. We have the following result from Murphy [21, Lemma (5)].

Lemma 5 For a fixed number \( (J = M^2) \) of uniformly distributed data points in \( \mathcal{B} \) and for any \( \rho \in (0, p_c] \), with \( p_c = 1 - 2\sin(\pi/18) \),

\[
\frac{M^2}{S(M, N)} = \frac{M^2}{M^2 + (N - 1)^2 + 2MN} = \rho
\]

(3)
determines the expected number \( K = N^2 \) of disjoint classes to form such that \( J = M^2 \) is the total of all occupied hexagons across all classes.

3. Anomaly Detection

From lemma (2), recall that \( R(M, N) \) is decreasing for increasing \( M \). As such, once \( N_0^2 \) is known, then given \( M^2 << T^2 \), for classifications of the ordered data set, it follows that \( R(M, N_0) \) constitutes the actual upper bound on the distance that any class member is allowed to differ from the center of its respective class. Since the initial classification of the ordered set was done with \( R(M, N) \) as the upper bound, then class members are candidate anomalies when they fall outside the new upper bound \( R(M, N_0) \) in distance from the linear average of the respective class, as given by the segment of the regression hyperplane through the class in question. Furthermore, the same restriction is applied to the class centers, whereby, given that the regression hyperplane is the linear average of the data points used in its definition, non-anomalous classes are those with centers that do not deviate from their respective sections of the regression hyperplane by more than a distance of \( R(M, N_0) \).

3.1. Definitions

Definition 6 (Macro Anomaly Detection) Let \( N_0^2 \) be the actual number of classes to form after classification of all data points has completed. Suppose the regression hyperplane \( H_{[M, N_0]}^\theta \) is given

\[
\text{Lemma 2 } R(M, N) \text{ is decreasing for increasing } M \text{ and } N.
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\text{Theorem 3 Denote the critical probability as } p_c. \text{ With probability } 1, \text{ every node in } \mathcal{B} \text{ is contained within one contiguous cluster if and only if the probability of any one hexagon containing a node, exceeds } p_c. \text{ Otherwise, all clusters are disjoint with probability } 1.
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\]

\[
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3.1. Definitions

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\]
by \((w_{[M,N]}^{\theta})^t\) \(z = \theta\), for some real vector \(w_{[M,N]}^{\theta}\) and some constant \(\theta \in \mathbf{R}\). Let \(h \in [1, N_0^2]\) and \(l_h\) be the segment of the regression hyperplane through class \(C_h\). Then, \(C_h\) is anomalous, if \(x_{(h)}\) is the mean of \(C_h\) and \(d(x_{(h)}, l_h) \geq R(M, N_0)\). Otherwise, \(C_h\) is NON-anomalous.

**Definition 7** (Micro Anomaly Detection) Let \(N_0^2\) and \(l_h\) be the same as in def. (6). \(x \in C_h\) is anomalous, if \(d(x, l_h) \geq R(M, N_0)\) and NON-anomalous otherwise.

### 3.2. Anomaly Segregation and The Decision Boundary

The combined definitions of macro and micro anomaly detection given in defs. (6) and (7) simply states that the non-anomalous data should all be tightly wrapped in the interior of \(N_0^2\) hyperspheres of radius no more than \(2 \times R(M, N_0)\), with each of the weighted centers being of distance no more than \(R(M, N_0)\) from the regression hyperplane, once classification has ceased.

#### 3.2.1. Normality Characterized

**Lemma 8** Class \(C_h\) is NON-anomalous if and only if \(d(x_{(h)}, l_h) < R(M, N_0)\). Furthermore, if \(C_h\) is NON-anomalous, then \(d(x, x_{(h)}) < 2 \times R(M, N_0)\) for all \(x \in C_h\).

**Proof** The first part follows from the definition of an anomalous class given in section (6). For the second part, the triangle inequality and the above definitions of anomalous class and anomalous data point in sections (6) and (7), respectively, are used to obtain

\[
d(x, x_{(h)}) \leq d(x, l_h) + d(l_h, x_{(h)}) < R(M, N_0) + R(M, N_0) = 2 \times R(M, N_0).
\]

#### 3.2.2. An Anomaly Segregation Theorem

**Theorem 9** If class \(C_h\) is anomalous, then there exists at least one anomalous data point, \(x \in C_h\).

**Proof** Let \(l_h\) be defined as before and define \(X_{[M,N]}^{(h)} = \{x \in C_h \mid d(x, l_h) \geq R(M, N_0)\}\). Seeking a contradiction, suppose \(X_{[M,N]}^{(h)} = \emptyset\). Then, for every \(x \in C_h\), it is true that \(d(x, l_h) < R(M, N_0)\). Therefore, the contradiction is obtained and the theorem is proven, if it can be shown that there exists \(x \in C_h\) such that \(d(x_{(h)}, l_h) \leq d(x, l_h)\), which implies \(d(x_{(h)}, l_h) < R(M, N_0)\), the sought contradiction to class \(C_h\) being anomalous. Thus, if \(|C_h| = 1\), then \(x_{(h)} = x \in C_h\). Otherwise, suppose that \(|C_h| > 1\) and assume that \(d(x_{(h)}, l_h) > d(x, l_h)\) for all \(x \in C_h\). Let \(v_h\) be a vector normal to \(l_h\) and \(v_h^T\) its transpose. If \(\hat{x} \in C_h\) is such that \(d(\hat{x}, l_h) \geq d(x, l_h)\) for all \(x \in C_h\), then

\[
d(x_{(h)}, l_h) = \frac{|v_h^t x_{(h)}|}{\|v_h\|} = \frac{|v_h^t \sum_{x \in C_h} x|}{\|v_h\|} = \|v_h\|^{-1} \frac{\sum_{x \in C_h} v_h^t x}{|C_h|}
\]

(4)
and

\[
d(\hat{x}, l_h) = \frac{|v_h^j \hat{x}|}{\|v_h\|} = \|v_h\|^{-1} \sum_{x \in C_h} |v_h^j \hat{x}| / |C_h|
\]

so that eq. (5), together with \(d(x, l_h) > d(x, l_h)\) and the triangle inequality applied to eq. (4) implies \(d(x, l_h) = \|v_h\|^{-1}|v_h^j x| > \|v_h\|^{-1}|v_h^j \hat{x}| = d(\hat{x}, l_h)\) for all \(x \in C_h\). This contradicts \(d(\hat{x}, l_h) \geq d(x, l_h)\) for all \(x \in C_h\). Thus, \(d(x, l_h) \leq d(x, l_h)\) for at least one \(x \in C_h\), which is the originally-sought contradiction.

Theorem (9) provides a means for segregating all anomalous data points from designated anomalous classes, leaving only classes consisting of non-anomalous data points. For each \(h \in [1, N_0^2]\), let \(X^{(h)}_{[M,N_0]} = \{x \in C_h \mid d(x, l_h) \geq R(M, N_0)\}\), as in the proof of theorem (9). Then, \((X^{(h)}_{[M,N_0]^c}) = \{x \in C_h \mid d(x, l_h) < R(M, N_0)\}\) is a class of non-anomalous data points for each \(h \in [1, N_0^2]^c\). Define

\[
X_{[M,N_0]} = \bigcup_{h=1}^{N_0^2} X^{(h)}_{[M,N_0]}.
\]

**Definition 10**: \(\Omega = X^{(h)}_{[M,N_0]} \cup X^{(h)}_{[M,N_0]}\) is pointwise linearly separable, if there exists \(x \in \Omega\) and a subset \(A_x \subset \Omega\) such that \(w_x^j y \leq \theta\) for all \(y \in A_x\) and \(w_x^j y > \theta\) for all \(y \in \Omega \setminus A_x\), where \(w_x \in \mathbb{R}^L\) and \(\theta \in \mathbb{R}\).

**Theorem 11**: If \(X^{(h)}_{[M,N_0]} \neq \emptyset\), then \(\Omega\) is pointwise linearly separable into \(X^{(h)}_{[M,N_0]}\) and \(X^{(h)}_{[M,N_0]}\) for some \(x \in X^{(h)}_{[M,N_0]} \subset \Omega\).

**Proof**: For some \(h \in [1, N_0^2]\), suppose \(x \in X^{(h)}_{[M,N_0]} \subset X^{(h)}_{[M,N_0]}\). The idea is to shift \(H^{\theta}_{[M,N_0]}\) by a specific length along the vector normal to \(H^{\theta}_{[M,N_0]}\). Thus, without loss of generality, suppose \(x\) lies to one side of \(H^{\theta}_{[M,N_0]}\) so that \(\left[w^{\theta}_{[M,N_0]} \right]^t x \leq \theta\) is a hyper half-plane. Define \(\hat{y} \in X^{(h)}_{[M,N_0]}\) to be a vector such that \(d(\hat{y}, H^{\theta}_{[M,N_0]}) \geq d(y, H^{\theta}_{[M,N_0]})\) for all \(y \in X^{(h)}_{[M,N_0]}\), where \(d(y, H^{\theta}_{[M,N_0]})\) is the length of the projection of \(y\) onto the vector normal to \(H^{\theta}_{[M,N_0]}\), and is given by

\[
d(y, H^{\theta}_{[M,N_0]}) = \|w^{\theta}_{[M,N_0]}\|^{-1} \left(w^{\theta}_{[M,N_0]} \right)^t y.
\]

Define

\[
d^{\theta}_{[M,N_0]} = \min_{x \in X^{(h)}_{[M,N_0]}} (d(x, H^{\theta}_{[M,N_0]}) - d(\hat{y}, H^{\theta}_{[M,N_0]})).
\]

Suppose \(x\) is a vector such that eq. (6) holds and \(\gamma \in (0, 1)\). Let \(w^{\theta x}_{[M,N_0]}\) be a real vector such that \(\left(w^{\theta x}_{[M,N_0]} \right)^z = \theta - \theta^\gamma\) is the hyperplane

\[
H^{\theta x}_{[M,N_0]} = H^{\theta}_{[M,N_0]} + \left(\theta^\gamma \times \frac{w^{\theta}_{[M,N_0]}}{\|w^{\theta}_{[M,N_0]}\|}\right),
\]

\[
\]
where
\[ \theta_\gamma^x = d(x, H_{[M,N_0]}^\theta) - \gamma d_{[M,N_0]}^\theta, \] (8)
with the right side of eq. (7) being a vector sum for each vector in \( H_{[M,N_0]}^\theta \). Thus, by copying and shifting the regression hyperplane \( H_{[M,N_0]}^\theta \) along the direction of its normal vector in order to obtain \( H_{[M,N_0]}^{\theta_\gamma^x} \) and by considering the reflection of the shifted hyperplane across the regression hyperplane, it now follows that
\[ \left( w_{[M,N_0]}^{\theta_\gamma^x} \right)^t y \leq (\theta + \theta_\gamma^x), \] (9)
for all \( y \in X_{[M,N_0]} \) such that \( \left( w_{[M,N_0]}^\theta \right)^t y \leq \theta \) and
\[ \left( w_{[M,N_0]}^{\theta_\gamma^x} \right)^t y > (\theta + \theta_\gamma^x) \] (10)
for all \( y \in X_{[M,N_0]} \) such that \( \left( w_{[M,N_0]}^\theta \right)^t y > \theta \), with the opposite of inequalities (9) and (10) otherwise, for \( y \in X_{[M,N_0]}^c \). Given \( M^2, T^2, N_0^2, \gamma \in (0, 1) \) and some fixed \( \theta \in \mathbb{R} \), take \( w_x = w_{[M,N_0]}^{\theta_\gamma^x} \) for \( x \in X_{[M,N_0]} \) which satisfies eq. (6).

Theorem (11) provides the means for identifying the decision boundary to be used when determining if certain data points are anomalous, with inequality (9) or (10) defining the shifted regression hyperplane.

3.2.3. The Neural Network Anomaly Detector

By theorem (11), with the anomalous data points segregated and collected into the set \( X_{[M,N_0]} \), it’s now possible to store the anomaly detector offline as the set of synaptic weights of a two-class discriminating neural network, which can be designed as a perceptron with a single \( L \)-dimensional input layer used to compute the synaptic weights \( w_{[M,N_0]}^{\theta_\gamma^x} \) associated with copying and shifting the regression hyperplane \( H_{[M,N_0]}^\theta \), given by \( \left( w_{[M,N_0]}^\theta \right)^t z = \theta \), in the direction of the normal vector to \( H_{[M,N_0]}^\theta \), for some \( x \in X_{[M,N_0]} \).

**Theorem 12 (Neural Network Anomaly Detector)** Suppose \( X_{[M,N_0]} \neq \emptyset \) and let \( x \in X_{[M,N_0]} \) satisfy eq. (6), with \( w_{[M,N_0]}^{\theta_\gamma^x} \) defined by
\[ w_{[M,N_0]}^{\theta_\gamma^x} = w_{[M,N_0]}^\theta + \left( \theta_\gamma^x \times \frac{w_{[M,N_0]}^\theta}{\|w_{[M,N_0]}^\theta\|} \right), \] (11)
for some chosen \( \gamma \in (0, 1) \). For all newly sampled data points \( y \in \Omega \), define \( \phi_{[M,N_0]}^{\theta_\gamma^x} : \Omega \rightarrow \mathbb{R} \) as
\[ \phi_{[M,N_0]}^{\theta_\gamma^x}(y) = \left( w_{[M,N_0]}^{\theta_\gamma^x} \right)^t y - (\theta + \theta_\gamma^x). \] (12)

Then, the activation function \( \phi_{[M,N_0]}^{\theta_\gamma^x} \), along with the synaptic weight vector \( w_{[M,N_0]}^{\theta_\gamma^x} \), defines a two-class discriminating neural network such that \( y \in \Omega \) is anomalous if for some \( \theta_\gamma^x \in \mathbb{R} \), the reflection
of \( \Phi_{[M,N]}^{\theta_x} \) across \( w_{[M,N]}^\theta \) = \( \theta \), given by \( \Phi_{[M,N]}^{\hat{\theta}_x} \), satisfies \( \Phi_{[M,N]}^{\hat{\theta}_x}(y) > 0 \) whenever \( (w_{[M,N]}^\theta)^t y \leq \theta \)
or if \( \Phi_{[M,N]}^{\hat{\theta}_x}(y) \leq 0 \) whenever \( (w_{[M,N]}^\theta)^t y > \theta \). Otherwise, \( y \in \Omega \) is non-anomalous.

**Proof** The synaptic weight vector \( w_{[M,N]}^{\theta_x} \) given in eq. (11) follows since \( H_{[M,N]}^{\gamma} \) given in eq. (7) is uniquely determined by shifting the vector \( w_{[M,N]}^\theta \) normal to \( H_{[M,N]}^\theta \) at the origin, in the direction which is determined by inequality (9). Without loss of generality, suppose \( \theta = 0 \) and further suppose \( (w_{[M,N]}^\theta)^t y \leq \theta \). The neural network anomaly detector, given by the activation function \( \Phi_{[M,N]}^{\theta_x} \) in eq. (12), gives

\[
\Phi_{[M,N]}^{\theta_x}(y) = (w_{[M,N]}^\theta)^t y - (\theta + \theta_x)
\]

\[
= (w_{[M,N]}^\theta)^t y + \left( \frac{w_{[M,N]}^\theta}{\|w_{[M,N]}^\theta\|} \right) - (\theta + \theta_x)
\]

\[
= (w_{[M,N]}^\theta)^t y - \theta + \left( \frac{w_{[M,N]}^\theta}{\|w_{[M,N]}^\theta\|} \right) - \theta_x
\]

\[
\leq \left( \frac{w_{[M,N]}^\theta}{\|w_{[M,N]}^\theta\|} \right) - \theta_x \quad \text{(13)}
\]

\[
\leq 0, \quad \text{(14)}
\]

where inequality (13) follows by the assumption that \( (w_{[M,N]}^\theta)^t y \leq \theta \) and inequality (14) follows since \( \theta_x \geq 0 \) and since, by assumption, \( (w_{[M,N]}^\theta)^t y \leq \theta = 0 \). It follows that \( \Phi_{[M,N]}^{\hat{\theta}_x}(y) > 0 \).

Similarly, \( \Phi_{[M,N]}^{\hat{\theta}_x}(y) \leq 0 \) whenever \( (w_{[M,N]}^\theta)^t y > \theta \). All other cases result in \( y \) being non-anomalous.

**Corollary 13** \( \hat{\theta}_x = -\theta_x \) and \( \Phi_{[M,N]}^{\hat{\theta}_x} = \Phi_{[M,N]}^{\theta_x} \).

**Proof** Follows directly from theorem (12) along with eqs. (11) and (12).

**Remark 14** The shifted regression hyperplane, \( \Phi_{[M,N]}^{\theta_x}(y) = 0 \), and its reflection across \( (w_{[M,N]}^\theta)^t y = \theta \) given by \( \Phi_{[M,N]}^{\hat{\theta}_x}(y) = 0 \), combine to segregate anomalous data points from non-anomalous data points.

4. Conclusions

By deriving an upper bound on a distance measure between points in a bounded region or some common point such that when the measured distances fall below the derived threshold, segregated classes of correlated points form. Using the centers of the correlated classes of points which form in a K-means classification process, we showed that calculating the distances from the centers of
the classes to a regression hyperplane is tantamount to identifying where to look for anomalous data points and that there must be at least one anomalous data point in an identified anomalous class. Finally, by shifting and reflecting the regression hyperplane along its unit normal vector, we showed that we can define the activation function for a two-class discriminating neural network, which identifies anomalous data points in a data set.
References

[1] Alpaydin, E. (2010), *Introduction to Machine Learning, Second Edition*, The MIT Press.
[2] Barnes, R.; Burkett, T. (2010), *Structural Redundancy and Multiplicity in Corporate Networks*, International Network for Social Network Analysis (INSNA), Volume 30, Issue 2, pp. 4 - 20, 2010
[3] Carroll, D.E.; Goel, A. (2004), *Lower Bounds for Embedding into Distributions over Excluded Minor Graph Families*, Lecture Notes in Computer Science, Volume 3221, pp. 146-156, 2004
[4] Chandola, V.; Banerjee, A.; Kumar, V. (2009), *Anomaly Detection: A Survey*, ACM Computing Surveys, Volume 41, Issue 3, Article 15, 2009
[5] Davidson, I.; Ward, M. (2001), *A Particle Visualization Framework for Clustering and Anomaly Detection*, ACM KDD Workshop on Visual Data Mining, 2001
[6] Dempster, A.P.; Laird, N.M.; Rubin, D.B. (1977), *Maximum Likelihood for Incomplete Data via the EM Algorithm*, Journal of the Royal Statistical Society, Series B (Methodological), Volume 39, No. 1., pp. 1 - 38, 1977
[7] Grimmett, Geoffrey (1999), *Percolation*, Springer-Verlag.
[8] Grimmett, Geoffrey (2006), *The Random-Cluster Model*, Springer-Verlag.
[9] Guthrie, D.; Guthrie, L.; Allison, B.; Wilks, Y. (2007), *Unsupervised Anomaly Detection*, IJCAI-07, pp. 1624 - 1628, 2007
[10] Guyon, Xavier. (1995), *Random Fields on a Network: Modeling, Statistics and Applications*, Springer.
[11] Haykin, S. (1994), *Neural Networks, A Comprehensive Foundation*, Macmillan College Publishing Company, Inc.
[12] Hogg, R.V.; McKean, J.W.; Craig, A.T. (2005), *Introduction to Mathematical Statistics*, Pearson Prentice Hall.
[13] Huang, L.; Nguyen, X.L.; Garofalakis, M.; Jordan, M.; Joseph, A.D.; Taft, N. (2006), *Network PCA and Anomaly Detection*, NIPS, 2006
[14] Kar, A. (2003), *Weyl’s Equidistribution Theorem*, Resonance, pp. 30 - 37, 2003
[15] Lee, W.; Xiang, D. (2001), *Information-Theoretic Anomaly Detection*, Proceedings. 2001 IEEE Symposium on Security and Privacy, pp. 130 - 143, 2001
[16] Mameli, V.; Musio, M. (2013), *A Generalization of the Skew-Normal Distribution: The Beta Skew-Normal*, Communications in Statistics - Theory and Methods, Volume 42, pp. 2229-2244, 2013
[17] Maselli, G.; Deri, L.; Suin, S. (2003), *Design and Implementation of an Anomaly Detection System: An Empirical Approach*, Proceedings of Terana Networking Conference, Zagreb Croatia, 2003
[18] Meester, Ronald; Roy, Rahul (1996), *Continuum Percolation*, Cambridge University Press.
[19] Miller, W.T.; Sutton, R.S.; Werbos, P.J. (1990), *Neural Networks for Control*, The MIT Press.
[20] Murphy, Robert (2011), *Partial Connectivity in Wireless Sensor Networks with Applications*, UMI Proquest.
[21] Murphy, R. (2015), “Estimating the Mean Number of K-Means Clusters to Form”, *arXiv*, ID 1503.03488, 2015
[22] Saligrama, V.; Zhao, M. (2012), *Local Anomaly Detection*, Journal of Machine Learning Research, W&CP, Volume 22, pp. 969 - 983, 2012
[23] Song, X.; Wu, M.; Jermaine, C.; Ranka, S. (2007), *Conditional Anomaly Detection*, IEEE
Transactions on Knowledge and Data Engineering, Volume 19, Issue 5, pp. 631 - 635, 2007

[24] Steinwart, I.; Hush, D.; Scovel, C. (2005), A Classification Framework for Anomaly Detection, Journal of Machine Learning Research, Volume 6, pp. 211 - 232, 2005