Quantum spins on star graphs and the Kondo model

N. Crampé\textsuperscript{a,b} and A. Trombettoni\textsuperscript{c,d}

\textsuperscript{a} CNRS, Laboratoire Charles Coulomb UMR 5221, Place Eugène Bataillon - CC070, F-34095 Montpellier, France
\textsuperscript{b} Université Montpellier II, Laboratoire Charles Coulomb UMR 5221, F-34095 Montpellier, France
\textsuperscript{c} CNR-IOM DEMOCRITOS Simulation Center and SISSA, Via Bonomea 265, I-34136 Trieste, Italy
\textsuperscript{d} INFN, Sezione di Trieste, I-34127 Trieste, Italy

E-mail: nicolas.crampe@univ-montp2.fr, andreatr@sissa.it

\textbf{Abstract}

We study the XX model for quantum spins on the star graph with three legs (i.e., on a Y-junction). By performing a Jordan-Wigner transformation supplemented by the introduction of an auxiliary space we find a Kondo Hamiltonian of fermions, in the spin 1 representation of $su(2)$, locally coupled with a magnetic impurity. In the continuum limit our model is shown to be equivalent to the 4-channel Kondo model coupling spin-$1/2$ fermions with a spin-$1/2$ impurity and exhibiting a non-Fermi liquid behavior. We also show that it is possible to find a XY model such that - after the Jordan-Wigner transformation - one obtains a quadratic fermionic Hamiltonian directly diagonalizable.

Keywords: Quantum spin model; Kondo model; Jordan-Wigner transformation; Star graph

PACS: 02.03.Ik; 75.10.Dg; 75.30.Mb
1 Introduction

Spatial inhomogeneities and their role in the emergence of coherent behaviors at mesoscopic length scales are the subject of a continuing interest. In general, spatial inhomogeneities may be random, due to the presence of disorder or noise, as well as non-random, as a result of an external control on the geometry of the system: in a broad sense, their formation can be dynamically generated or induced through a suitable engineering of the system. As a consequence the effects of spatial inhomogeneities have been investigated in a variety of systems, ranging from pattern formation in systems with competing interactions \[1\] to Josephson networks with non-random, yet non-translationally invariant architecture \[2, 3\].

A paradigmatic system in which the effects of spatial inhomogeneities can be studied is provided by spin models: on the one hand, not only do spin Hamiltonians directly describe many phenomena of magnetic systems \[4\], including the effects of frustration \[5\], but they are also routinely used to model physical properties of several condensed matter systems. On the other hand, spatial inhomogeneities can be straightforwardly included in spin models, to explore the consequences of the breaking of the translational invariance and the local properties on the length scales of the inhomogeneities \[6\].

As an example of the application of the study of spatial inhomogeneities in spin systems, we mention the spin chain Kondo effect. The standard Kondo effect arises from the interactions between magnetic impurities and the electrons in a metal and it is characterized by a net increase at low temperature of the resistance \[7, 8, 9\]. The Kondo effect has been initially observed for metals, like copper, in which magnetic atoms, like cobalt, are added: however, interest in the Kondo physics persisted also because it can be studied with quantum dots \[10, 11\]. The universal low-energy/long-distance physics of the Kondo model can be simulated and studied by a magnetic impurity coupled to a gapless antiferromagnetic one-dimensional chain having nearest- and next-nearest- neighbour couplings $J_1$-$J_2$ \[12\], with the the correct scaling behavior of the single channel Kondo problem being exactly reproduced by this spin model only when $J_2$ equals a critical value \[12\]. The spin model reproducing the low-energy behaviour of the Kondo problem is defined on the half line, since the radial coordinate of the fermionic model as well varies in the half line: the rationale is that the free electron Kondo problem may be described by a one-dimensional model since only the s-wave part of the electronic wavefunction is affected by the Kondo coupling \[13\]. Another example in which the scaling behavior of a fermionic Kondo model may be well reproduced by a pertinently chosen spin model is discussed in \[14\]. Using the spin chain version of the Kondo problem, a characterization of the Kondo regime using negativity was recently presented \[14, 15\] and it was shown that long-range entanglement mediated by the Kondo cloud can be induced by a quantum quench \[16\]. It stands as a open and interesting line of research to introduce and study spin systems, eventually with suitably tailored spatial inhomogeneities, reproducing the scaling behaviour of more general systems of fermions coupled to magnetic impurities, as the general multichannel Kondo effect.

Another reason of interest for introducing spatial inhomogeneities in spin models defined on networks is given by the study of the topology of the graph on the properties of the system and of the breaking of integrability. As a main example, consider a quantum (classical) spin model which is integrable in one (two) dimensions. Techniques have been developed to deal with open boundary conditions \[17\], as for free boundaries described by algebraic curves \[18\]. However, if some vertices of the graph on which the spins are located have a number of nearest neighbours larger than all others, then integrability is in general broken. One can see this by considering a one-dimensional quantum model which can be solved by a Jordan-Wigner (JW) transformation \[19\]: intersecting the chain at one site with a finite or infinite number of other chains the usual JW transformation on the spin variables will produce a fermionic model which is in general neither quadratic nor local. We recall that the two-dimensional classical Ising model at finite temperature can be solved by writing its partition functions in terms of a suitable quantum spin model on the chain which is solved by JW transformation \[20, 21\]. Therefore, finding an effective way of performing a JW transformation in non-trivial graphs amounts to the possibility of studying and possibly solving the
Ising model in some non trivial (non two-dimensional) lattices [22].

In this paper we study the XX model on a star graph obtained by merging three chains: the standard JW transformation cannot work for a star graph since there is no natural order on it (of course, this problem would generically appear for any graph, except for the circle and the segment where it works). We rather found convenient to supplement the application of the standard JW transformation with the introduction of an auxiliary space: the procedure of adding auxiliary sites to perform a JW transformation has been recently used to study higher-dimensional systems in [23, 24]. In our case, it is the use of this auxiliary space which allows us to get a Hamiltonian that is both quadratic and local in the JW fermions. Using such exact mapping, we show that the XX model on a star graph is equivalent to a generalized Kondo model, where the JW fermions enter locally and quadratically, and are coupled to a magnetic impurity.

There are several reasons for our choice of the XX model on a star graph. On the one hand, we study the XX model since we are motivated by the need of emphasize the main point of our construction in the simplest case: for the XX model in a chain, the JW transformation gives rise to free fermions (our construction can be extended to other spin models solvable by JW transformations).

On the other hand, we decide to restrict ourselves to the study of a star graph with three legs for a twofold reason: first, it is the simplest graph which can be constructed by merging a finite number of chains and having a finite number of vertices (three in our construction, see Figure 1) with coordination number $z = 3$ different and larger than the others (having $z = 2$, with the sites at the boundaries of the chain having $z = 1$). Second, the star graph (alias, the Y-junction) has been deeply studied in different contexts from different point of views: for three Tomonaga-Luttinger liquids (TLL) crossing at a point new attractive fixed points emerge [25, 26, 27]. Regular networks of TLL, with each node described by a unitary scattering matrix, were also studied [28], obtaining the same renormalization group equations derived for a single node coupled to several semi-infinite 1D wires [25]. The transport through one-dimensional TLL coupled together at a single point has been also studied [29]. Y-junctions of superconducting Josephson junctions were as well analyzed: for suitable values of the control parameters an attractive finite coupling fixed point is found [27], displaying an emerging two-level quantum system with enhanced coherence [30]. Star graphs were studied also in connection with bosonic models: properties of an ideal gas of bosons on a star graph were investigated in [31, 32] and the possible experimental realization with ultracold bosons was discussed in [32]. The dynamics of one-dimensional Bose liquids in Y-junctions and the reflection at the center of the star was studied, discussing the emergence of a repulsive fixed point [33]. Finally, we mention that the study of different theories on a graph and, particularly, on a star graph is a very active field of research: for example, for the Laplacian operator (also called quantum graphs) [34, 35, 36, 37, 38], for the Dirac operator [39, 40], for classical field theories and soliton theories [41, 42, 43] and for quantum field theories [44].

The plan of the paper is the following: in section 2, we introduce the XX model on a star graph, and we perform the JW transformation needed to obtain a fermionic Hamiltonian. The usefulness to add auxiliary sites is motivated, and the obtained Kondo Hamiltonian derived and discussed. In section 3, we show that it is possible to find an XY model such that after the JW transformation one obtains a quadratic fermionic Hamiltonian directly diagonalizable. Finally, our conclusions are presented in section 4.

2 The XX model on a star graph

In this section, we want to obtain fermionic Hamiltonians from quantum spin models on a star graph by using a JW transformation. In particular, we point out the importance to add an auxiliary site to obtain a fermionic Hamiltonian: we show that to solve this model is equivalent to solve a generalized Kondo model. The treatment is explicitly done for the XX model to emphasize our construction in the simplest case, although the procedure can be used to study other models on the star graph.
2.1 The model and the Jordan-Wigner transformation

We introduce in this section the XX model on a three-leg star graph. The graph we consider is illustrated in Figure 1 and it is made of three chains of length $L$, each one having vertices labeled by $1, \ldots, L$; the sites 1 of each of the three chains are connected between them. In each vertex of the graph (having 3 $L$ vertices) are defined the Pauli matrices $\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. As usual we use the notation $\sigma^\pm = \frac{1}{2}(\sigma^x \pm i\sigma^y)$.

The XX model is described by the following quantum Hamiltonian acting on the Hilbert space $(\mathbb{C}^2)^\otimes 3L$:

$$
\tilde{H}_{XX} = \sum_{j=1}^{L-1} \sum_{\alpha=1}^3 \sigma^+_{\alpha}(j)\sigma^-_{\alpha}(j+1) + \rho \sum_{\alpha=1}^3 \sigma^+_{\alpha}(1)\sigma^-_{\alpha+1}(1) + h.c.,
$$

(2.1)

where $\sigma^\pm_{\alpha}(j)$ stands for the matrix $\sigma^\pm$ acting on the $\alpha$th chain (with $\alpha = 1, 2, 3$) and on the $j$th site from the vertex (the labeling of the sites is plotted in Fig. 1). In equation (2.1) we have used the convention $\sigma^\pm_{4}(1) := \sigma^\pm_{1}(1)$. The parameter $\rho$ (in general complex) entering in the definition of the Hamiltonian (2.1) is a free parameter allowing one to modify the coupling constant at the center of the star graph. In particular, for $\rho = 0$, one retrieves three independent XX models on segments with free (open) boundaries.

![Figure 1: The three-leg star graph and the labeling of the vertices.](image)

At this point, we arrive at the main ingredient of our construction. To perform a JW transformation, we introduce, instead of $\tilde{H}_{XX}^3$, a slightly different Hamiltonian $H_{XX}^3$ acting on the Hilbert space $\mathbb{C}^2 \otimes (\mathbb{C}^2)^\otimes 3L$ and defined by the following rules: $H_{XX}^3$ acts as $\tilde{H}_{XX}^3$ on the last 3$L$ $\mathbb{C}^2$-spaces and trivially on the first $\mathbb{C}^2$-space. The added space (in comparison with the Hilbert space of $\tilde{H}_{XX}^3$) is denoted 0 and is called auxiliary space. We can write the link between both Hamiltonians as follows:

$$
H_{XX}^3 = Id(0) \otimes \tilde{H}_{XX}^3,
$$

(2.2)
According to the requested property 

Proposed here may be applied to other models as the anisotropic XY model with a transverse magnetic field. Different contexts, including the 2-channel Kondo model [49], quantum wire junctions described by coupled multidimensional spin systems [23, 24].

The JW transformation we use is defined, for \( j = 1, 2, \ldots, L \), by

\[
c_1(j) = \eta^x \left( \prod_{k=1}^{j-1} \sigma_1^z(k) \right) \sigma_1^x(j), \quad c_2(j) = \eta^y \left( \prod_{k=1}^{j-1} \sigma_2^z(k) \right) \sigma_2^y(j), \quad c_3(j) = \eta^z \left( \prod_{k=1}^{j-1} \sigma_3^z(k) \right) \sigma_3^z(j),
\]

where we introduced the following operators:

\[
\eta^x = \sigma^x(0) \prod_{k=1}^L \sigma_1^z(k) \sigma_2^z(k), \quad \eta^y = \sigma^y(0) \prod_{k=1}^L \sigma_1^z(k) \sigma_3^z(k), \quad \eta^z = \sigma^z(0) \prod_{k=1}^L \sigma_1^z(k) \sigma_2^z(k).
\]

The last two factors in the r.h.s. of each of the equations (2.3) are the usual JW transformations [19] and give the anti-commutation between terms in the same leg. The anti-commutation between different legs is provided by the first factor, i.e., by the operators \( \eta^a \), with \( a = x, y, z \), defined by equations (2.4). The choice (2.4) for the operators \( \eta^a \) is due to the need to satisfy the three following requests: \( i \) the operators \( c_\alpha(j) \) have to be fermionic; \( ii \) the operator \( \eta^a \) has to be \( a \)-th component of a spin operator; \( iii \) the operators \( c_\alpha(j) \) and the operators \( \eta^a \) have to commute.

Defining as usual \( c_\alpha(j)^\dagger \) as the conjugate transpose of \( c_\alpha(j) \) (for \( \alpha = 1, 2, 3 \) and \( j = 1, 2, \ldots, L \)), one can indeed show that \( c_\alpha(j) \) and \( c_\alpha(j)^\dagger \) are fermionic operators [property \( i \)] and that they satisfy for \( \alpha, \beta = 1, 2, 3 \) and \( j, k = 1, 2, \ldots, L \) the following anti-commutation relations:

\[
\{c_\alpha(j), c_\beta(k)\} = 0, \quad \{c_\alpha(j)^\dagger, c_\beta^\dagger(k)\} = 0, \quad \{c_\alpha(j), c_\beta^\dagger(k)\} = \delta_{\alpha,\beta} \delta_{jk}
\]

(2.5)

(where \( \{\ldots\} \) stands for the anti-commutator).

Furthermore the operators \( \eta^a \) share the same relations of the the Pauli matrices [property \( ii \)], since they satisfy, for \( a, b = x, y, z \),

\[
\eta^a \eta^b = \delta_{ab} \quad \{\eta^a, \eta^b\} = 2\delta_{ab} \quad \eta^a \eta^b = i\eta^z.
\]

(2.6)

An important point is that \( \eta^x \) commutes with \( \prod_{k=1}^{j-1} \sigma_1^z(k) \sigma_1^x(j) \) but anti-commutes with \( \prod_{k=1}^{j-1} \sigma_2^z(k) \sigma_2^x(j) \).

Finally, we observe for \( a = x, y, z \), \( \beta = 1, 2, 3 \) and \( j = 1, 2, \ldots, L \), the following relations hold:

\[
[\eta^a, c_\beta(j)] = 0 \quad \text{and} \quad [\eta^a, c_\beta(j)^\dagger] = 0
\]

(2.7)

according to the requested property \( iii \).

The factor \( \eta^a \) in equations (2.3) may be viewed as a Klein factor, which has been used extensively in literature: it allows one to define correctly the bosonization [45] (see also [46, 47, 48]) and it has been used in different contexts, including the 2-channel Kondo model [49], quantum wire junctions described by coupled TLL [50, 20] or the free quantum field theory on a star graph [51].

We conclude this subsection by emphasizing that the introduction of the auxiliary site and the JW transformation (2.3) do not depend on the explicit form of the Hamiltonian. Therefore, the construction proposed here may be applied to other models as the anisotropic XY model with a transverse magnetic field.
2.2 The Kondo model

By using the result of Section 2.1, it is possible to construct a model equivalent to \( H_{3}^{XX} \) expressed in terms of fermions. Indeed, by using relations (2.3), we can express the Hamiltonian \( H_{3}^{XX} \) in terms of the operators \( c_{\alpha}(j) \), \( c_{\alpha}(j)^\dagger \) and \( \eta^{\alpha} \) as follows

\[
H_{3}^{XX} = -\sum_{j=1}^{L-1} 3 \sum_{\alpha=1}^{3} c_{\alpha}(j)^\dagger c_{\alpha}(j+1) + i\rho \left( \eta^{x} c_{1}(1)^\dagger c_{2}(1) + \eta^{y} c_{2}(1)^\dagger c_{3}(1) + \eta^{y} c_{3}(1)^\dagger c_{1}(1) \right) + \text{h.c.} \quad (2.8)
\]

To write more compactly the Hamiltonian \( \text{(2.8)} \) we introduce \( \{S^{x}, S^{y}, S^{z}\} \), the \( su(2) \) generators in the 3-dimensional representation, as

\[
S^{x} = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -i \\
i & 0 & 0
\end{pmatrix}, \quad S^{y} = \begin{pmatrix}
0 & 0 & i \\
0 & 0 & 0 \\
i & 0 & 0
\end{pmatrix}, \quad S^{z} = \begin{pmatrix}
0 & -i & 0 \\
i & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}. \quad (2.9)
\]

Then, for \( \rho \in \mathbb{R} \), the Hamiltonian \( \text{(2.8)} \) becomes

\[
H_{3}^{XX} = -\sum_{j=1}^{L-1} \sum_{\alpha=1}^{3} \left( c_{\alpha}(j)^\dagger c_{\alpha}(j+1) + c_{\alpha}(j+1)^\dagger c_{\alpha}(j) \right) - \rho \sum_{a=x,y,z} \sum_{\alpha,\beta=1,2,3} \eta^{a} c_{\alpha}(1)^\dagger (S^{a})_{\alpha\beta} c_{\beta}(1) . \quad (2.10)
\]

Finally, by introducing the vectorial notation

\[
c(j)^\dagger = (c_{1}(j)^\dagger, c_{2}(j)^\dagger, c_{3}(j)^\dagger), \quad \eta = (\eta^{x}, \eta^{y}, \eta^{z}) , \quad S = \begin{pmatrix}
S^{x} \\
S^{y} \\
S^{z}
\end{pmatrix}, \quad (2.11)
\]

the Hamiltonian \( \text{(2.10)} \) may be rewritten in a more compact way as

\[
H_{3}^{XX} = -\sum_{j=1}^{L-1} \left( c(j)^\dagger c(j+1) + c(j+1)^\dagger c(j) \right) - \rho \cdot \eta \cdot c(1)^\dagger Sc(1) . \quad (2.12)
\]

The expression \( \text{(2.10)} \) is valid for three legs and it allows us to interpret the Hamiltonian \( H_{3}^{XX} \) as the Hamiltonian of free fermions coupled with a magnetic impurity. More precisely, it is a \( su(2) \) Kondo model with free fermions in the spin 1 representation and a magnetic impurity in the fundamental representation.

The historical Kondo model \([7]\) - studied using, for example, perturbation theory \([52]\), numerical renormalization group \([53]\) or exact methods \([54, 55]\) - corresponds to spin 1/2 free fermions coupled with a spin 1/2 impurity. Different generalizations have been introduced and studied: spin S impurities \([56]\), the \( su(N) \) version, so-called the Coqblin-Schrieffer model \([57]\), the multi-channel Kondo models \([58]\) or the multi-channel \( su(N) \) fermions in the fundamental representation with a spin S impurity \([59, 60]\).

The most relevant results for our case are given in \([61, 62]\). These papers showed that the dynamics of the spin sector of the single channel Kondo model coupling spin \( j \) fermions with a spin S impurity is similar to the ones of the \( k(j) = 2j(j+1)(2j+1)/3 \) channel Kondo model coupling spin 1/2 fermions with a spin S impurity. Then, using the results for the multi-channel Kondo model obtained previously by the Bethe ansatz method \([63, 64]\) and by the conformal field theory \([65]\), the universal exponents for the thermodynamic quantities (e.g. the susceptibility or the specific heat) can be obtained. A general discussion of the mapping between different multichannel exchange models with spin \( j \) fermions, impurity spin \( S \) and channel number \( N_{f} \) is presented in \([66]\).

These results applied to our case leads to the fact that our model \( \text{(2.10)} \), in the continuum limit, is equivalent to the 4-channel Kondo model coupling spin 1/2 fermions with a spin 1/2 impurity. Then, since
the number of channels is larger than twice the impurity spin, we deduce that our system shows a non-Fermi liquid behavior \[58\] and that the impurity contributes, for example, to the susceptibility and the specific, respectively, as following:

\[
\chi_{\text{imp}} \propto T^{-1/3} \quad \text{and} \quad C_{\text{imp}} \propto T^{2/3} . \tag{2.13}
\]

We conclude this section by further commenting on the JW transformation we used, to more clearly show that the standard JW transformation does not generally give for a star graph a local and quadratic fermionic Hamiltonian and that the JW transformation (2.3) - (2.4) based on the introduction of the space of an auxiliary, fictitious site is functional to have the desired commutation relations between the fermionic operators and the spin \( \eta \). We start by observing that one may think of using the JW transformation (3.2) instead of (2.3), replacing \( \eta^a \) by \( \sigma^a(0) \). However, in this case, although we get a Hamiltonian similar to (2.8), the fermionic operators obtained do not satisfy the commutation relations (2.7) with the \( \eta \)'s. Therefore, we cannot anymore to directly interpret the model as a Kondo model. Furthermore one could also think of different JW transformations without adding an auxiliary space. However, such transformations have their drawbacks: for example, if one performs the transformation (2.3) without the first factor (the \( \eta^a \)'s), one obtains a quadratic Hamiltonian, but quadratic in hardcore bosons, not fermions. Alternatively, one could consider a JW transformations following a “spiral” according

\[
c_{j+1+a} = \left( \prod_{k=1}^{j-1} \sigma_1^z(k) \sigma_2^z(k) \sigma_3^z(k) \right) \left( \prod_{\beta=1}^{j-1} \sigma_\beta^z(j) \right) \sigma_\alpha^z(j) \quad \text{for} \quad j = 1, 2, \ldots, L ; \quad \alpha = 1, 2, 3 : \tag{2.14}
\]

the operators \( c_j \) are fermionic, however the Hamiltonian finally obtained is not quadratic in these operators.

3 Free fermions on a star graph and associated spin chains

In this section we investigate if it is possible to find a XY model on a star graph which, after a JW transformation gives only a quadratic fermionic Hamiltonian. In comparison with the previous section, we allow ourselves to modify the interaction between the spins near the vertex.

3.1 Link between Hamiltonians

Since it would be very cumbersome to explore all the interactions between spins at the vertex to find the ones providing a quadratic fermionic Hamiltonian, we proceed in the following way: we start from a quadratic fermionic Hamiltonian on the star graph and we perform a JW transformation, obtaining a quantum spin model on the star graph. In this section, we restrict ourselves to the three-leg star graph (one may extend easily the obtained results).

We start from the following quadratic fermionic Hamiltonian on a three-leg star graph

\[
\tilde{H}_3^{QF} = \sum_{j=1}^{L-1} \sum_{a=1}^{3} \left( \sigma_\alpha d_\alpha(j) \dagger d_\alpha(j+1) - \gamma d_\alpha(j) d_\alpha(j+1) \right) + i \sum_{a=1}^{3} \left( a_\alpha d_\alpha(1) \dagger d_\alpha(1) + b_\alpha d_\alpha(1) d_\alpha(1) \right) + \text{h.c.} , \tag{3.1}
\]

where \( \gamma \), \( a_\alpha \) and \( b_\alpha \) are coupling constants, \( d_\alpha(j) \) and \( d_\alpha(j) \dagger \) are fermionic operators and we have used the conventions \( d_1(1) := d_1(1) \), \( d_4(1) := d_4(1) \). As in the previous section, instead of \( \tilde{H}_3^{QF} \) we consider the Hamiltonian \( H_3^{QF} = \text{Id}(0) \otimes \tilde{H}_3^{QF} \) which trivially acts on a supplementary \( \mathbb{C}^2 \)-space, the auxiliary space denoted by 0. Then, we perform the following JW transformation:

\[
d_1(j) = \sigma^x(0) \prod_{k=1}^{j-1} \sigma_1^z(k) \sigma_1^-(j) , \quad d_2(j) = \sigma^y(0) \prod_{k=1}^{j-1} \sigma_2^z(k) \sigma_2^-(j) , \quad d_3(j) = \sigma^z(0) \prod_{k=1}^{j-1} \sigma_3^z(k) \sigma_3^-(j) . \tag{3.2}
\]
with \( j = 1, 2, \ldots, L \). A straightforward computation shows that the R.H.S. of (3.2) are fermionic operators (notice the differences with the previous Jordan-Wigner transformations (2.3)).

By using the transformations (3.2), the quadratic fermionic Hamiltonian becomes the following quantum spin chain

\[
\tilde{H}_3^{QF} = - \sum_{x=1}^{L-1} \sum_{\alpha=1}^{3} \left( \sigma^+_\alpha(x)\sigma^-_\alpha(x+1) + \gamma \sigma^-_\alpha(x)\sigma^-_\alpha(x+1) \right) - H_V + h.c. \tag{3.3}
\]

where

\[
H_V = \sigma^z(0) \left( a_1 \sigma^+_1(1)\sigma^-_2(1) + b_1 \sigma^-_1(1)\sigma^-_2(1) \right) + \sigma^x(0) \left( a_2 \sigma^+_2(1)\sigma^-_3(1) + b_2 \sigma^-_2(1)\sigma^-_3(1) \right) + \sigma^y(0) \left( a_3 \sigma^+_3(1)\sigma^-_1(1) + b_3 \sigma^-_3(1)\sigma^-_1(1) \right) \tag{3.4}
\]

This shows that it is possible to obtain a XY model on a three-leg star graph which is equivalent to a quadratic fermionic Hamiltonians. Notice, however, that the interactions between spins at the center of the star graph are not of the XY type and involve three spins.

### 3.2 Solution for \( a_\alpha = a \) and \( b_\alpha = \gamma = 0 \)

The Hamiltonian \( \tilde{H}_3^{QF} \) given in relation (3.1) is a quadratic fermionic Hamiltonian. Therefore, it can be diagonalized by usual procedures [67, 68]. Evidently, this result provides also the spectrum of the Hamiltonian (3.3) since it is the same spectrum with all the degeneracies multiplied by two.

To present a specific example, we consider in the following the case \( a_\alpha = a \in \mathbb{R}, b_\alpha = 0 \) and \( \gamma = 0 \) (for \( \alpha = 1, 2, 3 \)). We can rewrite the Hamiltonian (3.1) as

\[
\tilde{H}_3^{QF} = \sum_{i,j=1}^{3L} d_i^* A_{ij} d_j, \tag{3.5}
\]

where we have changed the numeration \( d_\alpha(j) \rightarrow d_{(\alpha-1)L+j} \) and the entries of the matrix \( A \) are 0 everywhere except

\[
\begin{align*}
A_{j,j+1} &= 1 = A_{j+1,j} \quad \text{where } j = 1, \ldots, L - 1, L + 1, \ldots, 2L - 1, 2L + 1, \ldots, 3L - 1 \tag{3.6}
A_{1,L+1} &= A_{L+1,2L+1} = A_{2L+1,1} = ia \quad \text{and} \quad A_{1,2L+1} = A_{L+1,1} = A_{2L+1,L+1} = -ia. \tag{3.7}
\end{align*}
\]

To diagonalize \( \tilde{H}_3^{QF} \) one has to diagonalize the \( 3L \) times \( 3L \) matrix \( A \): it is possible to show that the eigenvalues of \( A \) are the roots of the following three polynomials

\[
U_L(\lambda/2) \pm a\sqrt{3} U_{L-1}(\lambda/2) = 0 \quad \text{or} \quad U_L(\lambda/2) = 0 \tag{3.8}
\]

where \( U_L(x) \) is the Chebyshev polynomials of the second kind of degree \( L \). Let us remark that, for \( a = 0 \), we get three times the same equation which is expected since the system becomes three identical decoupled systems.

The Hamiltonian then becomes

\[
\tilde{H}_3^{QF} = \sum_{k=1}^{3L} |\lambda_k| \left( \xi_k^+ \xi_k - \frac{1}{2} \right), \tag{3.9}
\]

where \( \xi_k \) are fermionic operators and \( \{\lambda_k\} \) is the set of the \( 3L \) solutions of (3.8). Therefore, the spectrum of \( \tilde{H}_3^{QF} \) is the following set of \( 2^{3L} \) elements

\[
\left\{ \frac{1}{2} \sum_{k=1}^{3L} \epsilon_k |\lambda_k| \mid \text{such that } \epsilon_k = \pm \right\} \tag{3.10}
\]
and the eigenvalue of the ground state is $-\frac{1}{2} \sum_{k=1}^{3L} |\lambda_k|$.  

Finally, the problem is solved when the $3L$ solutions of (3.8) are given. It is easy to find them numerically or even analytically. Indeed, for the last equation, its roots are $2\cos(k\pi/(L+1))$ with $k = 1, 2, \ldots, L$: we present them in Figure 2 for $a = 1$ and $L = 150$ as a dispersion relation. We remark that the three sets of solutions are very similar (in Figure 2, these three sets of roots are superimposed). The main difference relies on the presence of one isolated point for each of the first two equations. As shown in subsection 3.1 this result provides the spectrum for the quantum spin model (3.3).

4 Conclusions

In this paper we studied the XX model for quantum spin model on a three-leg star graph: we showed that by introducing an auxiliary space and performing a Jordan-Wigner transformation, the model is equivalent to a generalized Kondo Hamiltonian in which the free fermions, in the spin 1 representation of $su(2)$, are coupled with a magnetic impurity. Using previous results, we deduce that it is also equivalent to a 4-channel Kondo model with spin 1/2 fermions coupled with spin 1/2 impurity and conclude that it shows a non-Fermi liquid behavior. We also showed that it is possible to find a XY model such that - after the Jordan-Wigner transformation - one obtains a quadratic fermionic Hamiltonian directly diagonalizable.

We observe that we may think of different generalizations of our method. Indeed, our method based on the Jordan-Wigner transformation (2.3) could be used for the Hamiltonian obtained replacing the XX Hamiltonian (2.1) by the anisotropic XY model with a transverse magnetic field. In perspective, one can also think to investigate more complicated graphs as the star graph with a number of legs $p > 3$ or as comb-like graphs: in order to get a Kondo-like Hamiltonian, i.e., an Hamiltonian of fermions coupled with magnetic impurities, one should identify the correct Klein factors, which in the present case $p = 3$ are given by equations (2.4). Furthermore, the properties of a quantum Ising model in a transverse magnetic field on a graph $G$ may be related to the partition function of the classical Ising model in a corresponding higher-dimensional graph. Namely, the partition function of the classical Ising model on a graph made up
of \( n \) copies of the “base” graph \( \mathcal{G} \) with couplings between corresponding sites of the adjacent copies can be written as the trace of the transfer matrix \( V \) to the power \( n \) where \( V \) is written as exponentials of terms proportional to \( \sigma^x \sigma^x \) and \( \sigma^z \). By organizing the factors in the exponentials, we may recognize the exponential of a quantum Ising model. For example, for a square (resp. cubic) lattice, the “base” graph \( \mathcal{G} \) is a line (resp. square): e.g., for the square, \( V \) can be written in terms of the quantum Ising model in a transverse magnetic field on the segment \( [20] \). Therefore getting results for Kondo Hamiltonians of type \( (2.12) \) obtained from quantum Ising models on star-like graphs may be relevant to study the classical Ising model in non trivial geometries.

The mapping presented in this paper between the XX quantum spin model on the star graph and the Kondo model illustrates that the introduction of a non trivial topology, even locally, can provide new interesting physical phenomena in comparison to models on the line or on the circle. At the same time, our results show that one may also think to use the XX model on a star graph to realize (or simulate) a Kondo model: to this respect we mention that a similar Hamiltonian, describing Majorana fermions, can be realized in a superconductor, coupled to conduction electrons \([69]\).

Acknowledgments: We would like to thank V. Caudrelier, D. Giuliano, M. Fabrizio, P. Sodano and P.B. Wiegmann for very useful discussions. Useful correspondences with B. Beri, P. Lecheminant, D.C. Mattis and A.M. Tsvelik are also gratefully acknowledged.

Note Added: After this paper was submitted, several very interesting papers on Y-systems appeared on the arXiv. In \([70]\) the problem of an Ising model in a transverse field has been studied on the star graph: in the continuum limit, close to the quantum phase transition point and for coupling \( \rho << 1 \), the effective Lagrangian was worked out and the model shown to be equivalent to the overscreened two-channel Kondo model \([70]\). As previously mentioned, the approach presented in our paper can be used for the general case of an anisotropic XY in a transverse field: it would then very interesting to study the Kondo problem in such more general model, determining in particular how the low-energy physics varies across the parameter space (i.e, varying the anisotropy and the magnetic field). A discussion of the coupling of Majorana fermions to external leads was presented in \([71]\): the Klein factors of bosonization appear as extra Majoranas hybridizing with the physical ones and a \( SO(M) \) Kondo problem was shown to arise \([71]\). In \([72]\) it was studied a setup with nanowires in proximity to a common mesoscopic superconducting island, showing that a weak finite charging energy of the center island may considerably affect the low-energy behavior of the system. Finally, in \([73]\) a general Majorana junction was considered and the conditions for even-odd parity effects in the tunnel conductance for various junction topologies were examined.

References

[1] S. Chakrabarty and Z. Nussinov, *Modulation and correlation lengths in systems with competing interactions*, Phys. Rev. B 84 (2011) 144402.

[2] R. Burioni, D. Cassi, I. Meccoli, M. Rasetti, S. Regina, P. Sodano, and A. Vezzani, *Bose-Einstein condensation in inhomogeneous Josephson arrays*, Europhys. Lett. 52 (2000) 251.

[3] P. Sodano, A. Trombettoni, P. Silvestrini, R. Russo, and B. Ruggiero, *Inhomogeneous superconductivity in comb-shaped Josephson junction networks*, New J. Phys. 8 (2006) 327.

[4] D.C. Mattis, *The theory of the magnetism made simple* (World Scientific, 2006).

[5] *Introduction to frustrated magnetism: Materials, experiments, theory*, eds. C. Lacroix, P. Mendels, and F. Mila (Heidelberg, Springer, 2011).
[6] F. Iglói, I. Peschel, and L. Turban, *Inhomogeneous systems with unusual critical behaviour*, Adv. Phys. 42 (1993) 683.

[7] J. Kondo, *Resistance minimum in dilute magnetic alloys*, Prog. Theor. Phys. 32 (1964) 37.

[8] A.C. Hewson, *The Kondo problem to heavy fermions* (Cambridge, Cambridge University Press, 1993).

[9] L. Kouwenhoven and L. Glazman, *Revival of the Kondo effect*, Phys. World 14 (2001) 33.

[10] A.P. Alivisatos, *Semiconductor clusters, nanocrystals, and quantum dots*, Science 271 (1996) 933.

[11] L. Kouwenhoven and C.M. Marcus, *Quantum dots*, Phys. World 11 (1998) 35.

[12] N. Laflorencie, E.S. Sorensen, and I. Affleck, *Kondo effect in spin chains*, J. Stat. Mech. P02007 (2008).

[13] I. Affleck, *Quantum impurity problems in condensed matter physics*, arXiv:0809.3474

[14] A. Bayat, S. Bose, P. Sodano, and H. Johannesson, *Entanglement probe of two-impurity Kondo physics in a spin chain*, Phys. Rev. Lett. 109 (2012) 066403.

[15] A. Bayat, P. Sodano, and S. Bose, *Negativity as the entanglement measure to probe the Kondo regime in the spin-chain Kondo model*, Phys. Rev. B 81 (2010) 064429.

[16] P. Sodano, A. Bayat, and S. Bose, *Kondo cloud mediated long range entanglement after local quench in a spin chain*, Phys. Rev. B 81 (2010) 100412(R).

[17] J.L. Cardy, *Scaling and renormalization in statistical physics* (Cambridge, Cambridge University Press, 1996).

[18] J.L. Cardy, *Critical behaviour at an edge*, J. Phys. A 16 (1983) 3617.

[19] P. Jordan and E. Wigner, *Über das Paulische Äquivalenzverbot*, Z. Physik 47 (1928) 631.

[20] T.D. Schultz, D.C. Mattis, and E.H. Lieb, *Two-dimensional Ising model as a soluble problem of many fermions*, Rev. Mod. Phys. 36 (1964) 856.

[21] G. Mussardo, *Statistical field theory: An introduction to exactly solved models in statistical physics* (Oxford, Oxford University Press, 2010).

[22] D.C. Mattis, *Soluble Ising model in 2+1/N dimensions and XY model in 1+1/N dimensions*, Phys. Rev. B 20 (1979) 349.

[23] V. Galitski, *Fermionization transform for certain higher-dimensional quantum spin models*, Phys. Rev. B 82 (2010) 060411.

[24] F. Verstraete and J.I. Cirac, *Mapping local Hamiltonians of fermions to local Hamiltonians of spins*, J. Stat. Mech. P09012 (2005).

[25] S. Lal, S. Rao, and D. Sen, *Junction of several weakly interacting quantum wires: A renormalization group study*, Phys. Rev. B 66 (2002) 165327.

[26] C. Chamon, M. Oshikawa, and I. Affleck, *Junctions of three quantum wires and the dissipative Hofstadter model*, Phys. Rev. Lett. 91 (2003) 206403; *Junctions of three quantum wires*, J. Stat. Mech. P02008 (2006).
[27] D. Giuliano and P. Sodano, *Y-junction of superconducting Josephson chains*, New J. Phys. **10** (2008) 093023; *Frustration of decoherence in Y-shaped superconducting Josephson networks*, Nucl. Phys. B **811** (2009) 395.

[28] K. Kazymyrenko and B. Doucot, *Regular networks of Luttinger liquids*, Phys. Rev. B **71** (2005) 075110.

[29] A. Komnik and R. Egger, *Nonequilibrium transport for crossed Luttinger liquids*, Phys. Rev. Lett. **80** (1998) 2881.

[30] A. Cirillo, M. Mancini, D. Giuliano, and P. Sodano, *Enhanced coherence of a quantum doublet coupled to Tomonaga-Luttinger liquid leads*, Nucl. Phys. B **852** (2011) 235.

[31] R. Burioni, D. Cassi, M. Rasetti, P. Sodano, and A. Vezzani, *Bose-Einstein condensation on inhomogeneous complex networks*, J. Phys. B **34** (2001) 4697.

[32] I. Brunelli, G. Giusiano, F.P. Mancini, P. Sodano, and A. Trombettoni, *Topology induced spatial Bose-Einstein condensation for bosons on star-shaped optical networks*, J. Phys. B **37** (2004) S275.

[33] A. Tokuno, M. Oshikawa, and E. Demler, *Dynamics of one-dimensional Bose liquids: Andreev-like reflection at Y junctions and the absence of the Aharonov-Bohm effect*, Phys. Rev. Lett. **100** (2008) 140402.

[34] J.P. Roth, *Le spectre du Laplacian sur un graphe*, in *Théorie du potentiel - Lecture notes in mathematics*, eds. A.D. Dold and B. Beckmann, pp. 521-539, in french (Springer-Verlag 1983).

[35] P. Exner and P. Šeba, *Free quantum motion on a branching graph*, Rep. Math. Phys. **28** (1989) 7.

[36] V. Kostrykin and R. Schrader, *Kirchhoff’s rule for quantum wires*, J. Phys. A **32** (1999) 595.

[37] P. Kuchment, *Quantum graphs I. Some basic structures*, Waves in Random Media **14** (2004) S107.

[38] V. Caudrelier and E. Ragoucy, *Direct computation of scattering matrices for general quantum graphs*, Nucl. Phys. B **828** (2010) 515.

[39] W. Bulla and T. Trenkler, *The free Dirac operator on compact and noncompact graphs*, J. Math. Phys. **31** (1990) 1157.

[40] J. Bolte and J. Harrison, *Spectral statistics for the Dirac operator on graphs*, J. Phys. A **36** (2003) 2747.

[41] Z. Sobirov, D. Matrasulov, K. Sabirov, S. Sawada, and K. Nakamura, *Soliton solutions of nonlinear Schrödinger equation on simple networks*, Phys. Rev. E **81** (2010) 066602.

[42] R.C. Cascaval and C.T. Hunter, *Linear and nonlinear Schrödinger equations on simple networks*, Libertas Mathematica **30** (2010) 85.

[43] R. Adami, C. Cacciapuoti, D. Finco, and D. Noja, *Fast solitons on star graphs*, Rev. Mat. Phys. **23** (2011) 409.

[44] B. Bellazzini, M. Burrello, M. Mintchev, and P. Sorba, *Quantum field theory on star graphs*, Proc. Symp. Pure Math. **77** (2008) 639.

[45] F.D.M. Haldane, *Coupling between charge and spin degrees of freedom in the one-dimensional Fermi gas with backscattering*, J. Phys. C **12** (1979) 4791.
[46] A.O. Gogolin, A.A. Nersesyan, and A.M. Tsvelik, *Bosonization and strongly correlated systems* (Cambridge, Cambridge University Press, 1998).

[47] J. von Delft and H. Schoeller, *Bosonization for beginners — Refermionization for experts*, Annalen Phys. 7 (1998) 225.

[48] T. Giamarchi, *Quantum physics in one dimension* (Oxford, Oxford University Press, 2004).

[49] J. von Delft, M. Fabrizio, and G. Zaránd, *Finite-size bosonization of 2-channel Kondo model: A bridge between numerical renormalization group and conformal field theory*, Phys. Rev. Lett. 81 (1998) 196.

[50] C. Nayak, M. Fisher, H. Lin, and A. Ludwig, *Resonant multilead point-contact tunneling*, Phys. Rev. B 59 (1999) 15694.

[51] B. Bellazzini, M. Mintchev, and P. Sorba, *Bosonization and scale invariance on quantum wires*, J. Phys. A 40 (2007) 2485.

[52] P.W. Anderson, D.R. Hamann, and G. Yuval, *Exact results in the Kondo problem. II. Scaling theory, qualitatively Correct solution, and some new results on one-dimensional classical statistical models*, Phys. Rev. B 1 (1970) 4464.

[53] K.G. Wilson, *The renormalization group: Critical phenomena and the Kondo problem*, Rev. Mod. Phys. 47 (1975) 773.

[54] P.B. Wiegmann, *Exact solution of s-d exchange model at T=0*, JETP Lett. 31 (1980) 364.

[55] N. Andrei, *Diagonalization of the Kondo Hamiltonian*, Phys. Rev. Lett. 45 (1980) 379.

[56] V.A. Fateev and P.B. Wiegmann, *The exact solution of the s-d exchange model with arbitrary spin S*, Phys. Lett. A 81 (1981) 179.

[57] B. Coqblin and J.R. Schrieffer, *Exchange interaction in alloys with Cerium impurities*, Phys. Rev. 185 (1969) 847.

[58] P. Nozières and A. Blandin, *Kondo effect in real metals*, J. Phys. France 41 (1980) 193.

[59] N. Andrei and P. Zinn-Justin, *The generalized multi-channel Kondo model: Thermodynamics and fusion equations*, Nucl. Phys. B 528 (1998) 648.

[60] A. Jerez, N. Andrei, and G. Zaránd, *Solution of the multichannel Coqblin-Schrieffer impurity model and application to multi-level systems*, Phys. Rev. B 58 (1998) 3814.

[61] M. Fabrizio and A.O. Gogolin, *Toulouse limit for the overscreened four-channel Kondo problem*, Phys. Rev. B 50 (1994) 17732.

[62] A.M. Sengupta and Y.B. Kim, *Overscreened single channel Kondo problem*, Phys. Rev. B 54 (1996) 14918.

[63] N. Andrei and C. Destri, *Solution of the multichannel Kondo problem*, Phys. Rev. Lett. 52 (1984) 364.

[64] A.M. Tsvelik and P.B. Wiegmann, *Solution of the n-channel Kondo problem (scaling and integrability)*, Z. Phys. B 54 (1984) 201.
[65] I. Affleck and A.W.W. Ludwig, *The Kondo effect, conformal field theory and fusion rules*, Nucl. Phys. B 352 (1991) 849; Critical theory of overscreened Kondo fixed points, Nucl. Phys. B 360 (1991) 641; Exact conformal-field-theory results on the multichannel Kondo effect: Single-fermion Greens function, self-energy, and resistivity, Phys. Rev. B 48 (1993) 7297.

[66] M. Fabrizio and G. Zaránd, *Mapping between multichannel exchange models*, Phys. Rev. B 54 (1996) 10008.

[67] E. Lieb, D.C Mattis, and T.D. Schultz, *Two soluble models of an antiferromagnetic chain*, Ann. Phys. 16 (1961) 407.

[68] E. Lieb and D.C. Mattis, *Mathematical Physics in One Dimension* (New York and London, Academic Press, 1966).

[69] B. Béri and N.R. Cooper, *Topological Kondo effect with Majorana fermions*, Phys. Rev. Lett. 109 (2012) 156803.

[70] A.M. Tsvelik, *Realization of 2-channel Kondo effect in a junction of three quantum Ising chains*, arXiv:1211.3481

[71] B. Béri, *Majorana-Klein hybridization in topological superconductor junctions*, arXiv:1212.4465

[72] A. Altland and R. Egger, *Multi-terminal Coulomb-Majorana junction*, arXiv:1212.6224

[73] A. Zazunov, P. Sodano, and R. Egger, *Even-odd parity effects in Majorana junctions*, arXiv:1301.6882