GLOBAL WEAK SOLUTIONS IN A THREE-DIMENSIONAL KELLER-SEGEL-NAVIER-STOKES SYSTEM MODELING CORAL FERTILIZATION

LING LIU
Department of Basic Science, Jilin Jianzhu University
Changchun 130118, China

JIASHAN ZHENG* AND GUI BAO
School of Mathematics and Statistics Science, Ludong University
Yantai 264025, China

(Communicated by Christina Surulescu)

ABSTRACT. We consider an initial-boundary value problem for the incompressible four-component Keller-Segel-Navier-Stokes system with rotational flux

\[
\begin{align*}
    n_t + u \cdot \nabla n &= \Delta n - \nabla \cdot (n S(x, n, c) \nabla c) - nm, & x \in \Omega, t > 0, \\
    c_t + u \cdot \nabla c &= \Delta c - c + m, & x \in \Omega, t > 0, \\
    m_t + u \cdot \nabla m &= \Delta m - nm, & x \in \Omega, t > 0, \\
    u_t + \kappa (u \cdot \nabla) u + \nabla P &= \Delta u + (n + m) \nabla \phi, & x \in \Omega, t > 0, \\
    \nabla \cdot u &= 0, & x \in \Omega, t > 0
\end{align*}
\]

in a bounded domain \( \Omega \subset \mathbb{R}^3 \) with smooth boundary, where \( \kappa \in \mathbb{R} \) is given constant, \( S \) is a matrix-valued sensitivity satisfying \( |S(x, n, c)| \leq C_S (1 + n)^{-\alpha} \) with some \( C_S > 0 \) and \( \alpha \geq 0 \). As the case \( \kappa = 0 \) (with \( \alpha \geq \frac{1}{2} \) or the initial data satisfy a certain smallness condition) has been considered in [18], based on new gradient-like functional inequality, it is shown in the present paper that the corresponding initial-boundary problem with \( \kappa \neq 0 \) admits at least one global weak solution if \( \alpha > 0 \). To the best of our knowledge, this is the first analytical work for the full three-dimensional four-component chemotaxis-Navier-Stokes system.

1. Introduction. Many phenomena, which appear in natural science, especially, biology and physics, support animals' lives (see [46, 35, 8, 24]). Chemotaxis has been extensively studied in the context of modeling mold and bacterial colonies (see Hillen and Painter [10] and Bellomo et al. [1]). In order to describe this biological phenomenon in mathematics, in 1970, Keller and Segel ([15]) proposed the following system

\[
\begin{align*}
    n_t &= \Delta n - \chi \nabla \cdot (n \nabla c), \\
    c_t &= \Delta c - c + n,
\end{align*}
\]

which is called Keller-Segel system. Here \( \chi > 0 \) is called chemotactic sensitivity, \( n \) and \( c \) denote the density of the cell population and the concentration of the attracting chemical substance, respectively. Since then, there has been an enormous

2020 Mathematics Subject Classification. 35K55, 35Q92, 35Q35, 92C17.

Key words and phrases. Navier-Stokes system, Keller-Segel model, global existence, tensor-valued sensitivity.

* Corresponding author.
amount of effort devoted to the possible blow-up and regularity of solutions, as well as the asymptotic behavior and other properties (see e.g. [1]). We refer to [10, 11] and [26] for the further reading. Beyond this, a large number of variants of system (1.1) have been investigated, including the system with the logistic terms (see [2, 32, 39, 56], for instance) and the nonlinear diffusion ([29, 45, 51, 49, 50, 52]), the signal is consumed by the cells (see e.g. Tao and Winkler [30], [57]) two-species chemotaxis system (see [19, 53], for instance) and so on.

In order to discuss the coral fertilization, Kiselev and Ryzhik ([16] and [17]) investigated the important effect of chemotaxis on the coral fertilization process via the Keller-Segel type system of the form

\[
\begin{cases}
\rho_t + u \cdot \nabla \rho = \Delta \rho - \chi \nabla \cdot (\rho \nabla c) - \rho^q, \\
0 = \Delta c + \rho,
\end{cases}
\]

(1.2)

where \( \rho \) is the density of egg (sperm) gametes, \( u \) is the smooth divergence free sea fluid velocity and \( c \) denotes the concentration of chemical signal which is released by the eggs. This model (1.2) implicitly assumes that the densities of sperm and egg gametes are identical. Kiselev and Ryzhik ([16] and [17]) proved that if \( q > 2 \) and the chemotactic sensitivity \( \chi \) increases, for the associated 2D Cauchy problem of (1.2), the total mass \( \int_{\mathbb{R}^2} \rho \) can become arbitrarily small, whereas if \( q = 2 \), a corresponding weaker but yet relevant effect within finite time intervals is detected (see Kiselev and Ryzhik [17]).

In various situations, however, the interaction of chemotactic movement of the gametes and the surrounding fluid is not negligible (see Espejo and Suzuki [5], Espejo and Winkler [6]). To model such biological processes, Espejo and Suzuki ([5]) proposed the following model

\[
\begin{cases}
\rho_t + u \cdot \nabla \rho = \Delta \rho - \chi \nabla \cdot (\rho \nabla c) - \mu \rho^2, \\
c_t + u \cdot \nabla c = \Delta c - c + \rho, \\
u_t + \kappa (u \cdot \nabla)u = \Delta u - \nabla P + \rho \nabla \phi, \\
\nabla \cdot u = 0,
\end{cases}
\]

(1.3)

where \( \rho \) and \( c \) are defined as before. Here \( u, P, \phi \) and \( \kappa \in \mathbb{R} \) denote, respectively, the velocity field, the associated pressure of the fluid, the potential of the gravitational field and the strength of nonlinear fluid convection.

Recently, in order to analyze a further refinement of the model (1.3) which explicitly distinguishes between sperms and eggs, Espejo and Winkler ([6]) proposed the following four-component Keller-Segel(-Navier)-Stokes system with (rotational flux):

\[
\begin{cases}
n_t + u \cdot \nabla n = \Delta n - \nabla \cdot (nS(x, n, c) \cdot \nabla c) - nm, & x \in \Omega, t > 0, \\
c_t + u \cdot \nabla c = \Delta c - c + m, & x \in \Omega, t > 0, \\
m_t + u \cdot \nabla m = \Delta m - nm, & x \in \Omega, t > 0, \\
u_t + \kappa (u \cdot \nabla)u + \nabla P = \Delta u + (n + m) \nabla \phi, & x \in \Omega, t > 0, \\
\nabla \cdot u = 0, & x \in \Omega, t > 0, \\
(\nabla n - nS(x, n, c) \nabla c) \cdot \nu = \nabla c \cdot \nu = \nabla m \cdot \nu = 0, & u = 0, & x \in \partial \Omega, t > 0, \\
n(x, 0) = n_0(x), c(x, 0) = c_0(x), m(x, 0) = m_0(x), u(x, 0) = u_0(x), & x \in \Omega
\end{cases}
\]

(1.4)

in a domain \( \Omega \subset \mathbb{R}^N (N = 2) \), where \( u, P, \phi, \kappa \in \mathbb{R} \) and \( c \) are defined as before and \( S \) is a tensor-valued function or a scalar function which satisfies

\[
S \in C^2(\bar{\Omega} \times [0, \infty)^2; \mathbb{R}^{N \times N})
\]

(1.5)
and
\[ |S(x,n,c)| \leq C_S(1 + n)^{-\alpha} \text{ for all } (x,n,c) \in \Omega \times [0, \infty)^2 \]
with some \(C_S > 0\) and \(\alpha > 0\). Here the scalar functions \(n = n(x,t)\) and \(m = m(x,t)\) denote the population densities of unfertilized sperms and eggs, respectively. In [6], assuming that \(S(x,n,c) \equiv 1\), Espejo and Winkler showed that the 2D four-component Keller-Segel-Navier-Stokes system (1.4) possesses at least one bounded classical solution, whereas, in three dimensions, Li, Pang and Wang ([18]) showed that the four-component Keller-Segel-Stokes system (1.4) admits at least one bounded classical solution. These indeed extend and improve the recent corresponding results obtained by Li, Pang and Wang ([18]). However, it seems that the method in [55] can not deal with the full three-dimensional four-component chemotaxis-Navier-Stokes system (1.4) \((\kappa \neq 0 \text{ in (1.4))})\). Moreover, as far as we know, for the full three-dimensional four-component chemotaxis-Navier-Stokes system (1.4) \((\kappa \neq 0 \text{ in (1.4))})\) it is still not clear whether the solution of the system (1.4) exists or not. Furthermore, in [6], [18] and [55], the authors also showed that the corresponding solutions converge to a spatially homogeneous equilibrium exponentially as \(t \to \infty\) as well.

Motivated by the above works, the main objective of the paper is to investigate the four-component Keller-Segel-Navier-Stokes system (1.4) with rotational flux. We sketch here the main ideas and methods used in this article. A key role in our existence analysis is played by the observation that for appropriate positive constants \(a_i\) and \(b_i(i = 1, 2, 3)\), the functional
\[ \left\{ \begin{array}{ll}
\int_{\Omega} n_{\varepsilon}^{4\alpha+2} + a_1 \int_{\Omega} |\nabla c_{\varepsilon}|^2 + b_1 \int_{\Omega} |u_{\varepsilon}|^2 & \text{if } \alpha \neq \frac{1}{12}, \\
\int_{\Omega} n_{\varepsilon} \ln n_{\varepsilon} + a_2 \int_{\Omega} |\nabla c_{\varepsilon}|^2 + b_2 \int_{\Omega} |u_{\varepsilon}|^2 & \text{if } \alpha = \frac{1}{12},
\end{array} \right. \]
possesses a favorable entropy-like property, where \(n_{\varepsilon}, c_{\varepsilon}\) and \(u_{\varepsilon}\) are components of the solutions to (2.1). This will entail a series of a priori estimates which will derive further \(\varepsilon\)-independent bounds for spatio-temporal integrals of the approximated solutions and several \(\varepsilon\)-independent regularity features of their time derivatives (see Section 4-5). On the basis of the compactness properties thereby implied, we shall finally pass to the limit along an adequate sequence of numbers \(\varepsilon = \varepsilon_j \searrow 0\) and thereby verify the main results (see Section 6).

Before going into our mathematical analysis, we recall some important progresses on system (1.4) and its variants. In order to describe the behavior of bacteria of the species Bacillus subtilis suspended in sessile water drops, Tuval et al. ([33]) proposed the following chemotaxis-fluid model
\[ \left\{ \begin{array}{ll}
t_t + u \cdot \nabla n = \Delta n - \nabla \cdot (nS(x,n,c)\nabla c), & x \in \Omega, t > 0, \\
n_t + u \cdot \nabla c = \Delta c - nf(c), & x \in \Omega, t > 0, \\
u_t + \kappa (u \cdot \nabla)u + \nabla P = \Delta u + n\nabla \phi, & x \in \Omega, t > 0, \\
\nabla \cdot u = 0, & x \in \Omega, t > 0, 
\end{array} \right. \]
where \(f(c)\) is the consumption rate of the oxygen by the cells. The model (1.7) occurs in the modelling of the collective behaviour of chemotaxis-driven swimming aerobic bacteria.
If the chemotactic sensitivity $S(x, n, c) := S(c)$ is a scalar function, by making use of energy-type functionals, some local and global solvability of corresponding initial value problem for (1.7) in either bounded or unbounded domains have been obtained in the past years (see e.g. Chae et. al. [3], Duan et. al. [4], Liu and Lorz [21, 23], Tao and Winkler [31, 40, 41, 43], Zhang and Zheng [48], Zheng [55] and references therein).

As pointed out by Xue and Othmer in [47], the chemotactic sensitivity $S$ should be a tensor function rather than a scalar one, so that, the corresponding chemotaxis-fluid system (1.7) loses some energy-like structure, which plays a key role in the analysis of the scalar-valued case. Therefore, there are only a few works concerning chemotaxis-fluid coupled models with tensor-valued sensitivity (see Ishida [12], Wang et al. [9, 34], Winkler [42] and Zheng [55] for example).

In comparison to (1.7), if we assume that the signal is produced rather than consumed by cells, then the corresponding chemotaxis-fluid model is the Keller-Segel-fluid system of the form (see [44, 38, 36, 37, 54, 14])

\[
\begin{align*}
  n_t + u \cdot \nabla n &= \Delta n - \nabla \cdot (nS(x, n, c) \cdot \nabla c), & x \in \Omega, t > 0, \\
  c_t + u \cdot \nabla c &= \Delta c - c + n, & x \in \Omega, t > 0, \\
  u_t + \kappa (u \cdot \nabla) u + \nabla P &= \Delta u + n \nabla \phi, & x \in \Omega, t > 0, \\
  \nabla \cdot u &= 0, & x \in \Omega, t > 0.
\end{align*}
\]  

(1.8)

Over the past few years, the mathematical analysis of (1.8) (with tensor-valued sensitivity) began to flourish (see [44, 38, 36, 37, 54, 14] and references therein). In fact, if the domain $\Omega$ is further assumed to be convex, Wang, Xiang and Winkler ([38]) established the global existence and boundedness of the 2D system (1.8) under the assumption of (1.6) with $\alpha > 0$. Recently, Zheng ([55]) extends the results of [38] to the general bounded domain by some new entropy-energy estimates. Furthermore, if $S(x, n, c)$ satisfies (1.5) and (1.6) with $\alpha > \frac{1}{2}$, Wang and Xiang ([37]) proved the same result for for the three-dimensional Stokes version (1.8) (i.e., the system with $\kappa = 0$). Wang and Liu ([22]) showed that 3D Keller-Segel-Navier-Stokes ($\kappa \neq 0$ in the third equation of (1.8)) system (1.8) admits a global weak solution for tensor-valued sensitivity $S(x, n, c)$ satisfies (1.5) and (1.6) with $\alpha > \frac{3}{7}$.

More recently, Ke and Zheng ([14]) extends the result of [22] to the case $\alpha > \frac{1}{3}$, which in light of the known results for the fluid-free system mentioned above is an optimal restriction on $\alpha$. Some other results on global existence and boundedness properties have also been obtained for the variant of (1.8) obtained on replacing $\Delta n$ by nonlinear diffusion operators generalizing the porous medium-type choice $\Delta n^m$ for several ranges of $m > 1$ ([54, 25, 20, 52]).

In order to formulate our main result, we will first briefly introduce the technique framework: The initial data are assumed to be

\[
\begin{align*}
  n_0 \in C(\bar{\Omega}) & \text{ with } n_0 \geq 0 \text{ and } n_0 \not\equiv 0, \\
  c_0 \in W^{1,\infty}(\Omega) & \text{ with } c_0 \geq 0 \text{ in } \Omega, \\
  m_0 \in C(\bar{\Omega}) & \text{ with } m_0 \geq 0 \text{ and } m_0 \not\equiv 0, \\
  u_0 \in D(A_\gamma^r) & \text{ for some } \gamma \in \left(\frac{3}{4}, 1\right) \text{ and any } r \in (1, \infty),
\end{align*}
\]  

(1.9)

where $A_\gamma$ denotes the Stokes operator with domain $D(A_\gamma) := W^{2,r}(\Omega) \cap W^{1,r}_0(\Omega) \cap L^r_\gamma(\Omega)$, and $L^r_\gamma(\Omega) := \{ \varphi \in L^r(\Omega) | \nabla \cdot \varphi = 0 \}$ for $r \in (1, \infty)$ ([28]). Apart from this, we shall merely suppose that

\[
\phi \in W^{2,\infty}(\Omega).
\]  

(1.10)

Under these assumptions, our main result can be read as
Theorem 1.1. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary, (1.10) and (1.9) hold, and suppose that $S$ satisfies (1.5) and (1.6) with some
\[ \alpha > 0. \] (1.11)
Then the problem (1.4) possesses at least one global weak solution $(n, c, m, u)$ in the sense of Definition 6.1.

Remark 1.1. (i) To the best of our knowledge, this is the first analytical work for the full three-dimensional four-component chemotaxis-Navier-Stokes system.

(ii) We should point that the idea of this paper can not deal with the case $\alpha = 0$, since, it is hard to establish the $\varepsilon$-independent estimates (see the proof of Lemma 4.1). Therefore, when $N = 3$ and $\alpha = 0$, we must choose other methods. In addition, we evaluate that the global existence of system (1.4) may depend on the initial data and the $C_S$ when $\alpha = 0$. Here $C_S$ is the same as (1.6).

(iii) We have to leave open the question whether the condition (1.11) is optimal or not.

2. Preliminaries. Due to the strongly nonlinear term $\kappa(u \cdot \nabla)u$ and the presence of tensor-valued $S$ in system (1.4), we need to consider an appropriately regularized problem of (1.4) at first. According to the ideas in [43], the corresponding regularized problem is introduced as follows:

\[
\begin{cases}
\begin{aligned}
n_{\varepsilon,t} + u_{\varepsilon} \cdot \nabla n_{\varepsilon} &= \Delta n_{\varepsilon} - \nabla \cdot (n_{\varepsilon} \frac{1}{1+n_{\varepsilon}})S(x, n_{\varepsilon}, c_{\varepsilon}) \nabla c_{\varepsilon} - n_{\varepsilon} m_{\varepsilon}, \quad x \in \Omega, t > 0, \\
c_{\varepsilon,t} + u_{\varepsilon} \cdot \nabla c_{\varepsilon} &= \Delta c_{\varepsilon} - c_{\varepsilon} + m_{\varepsilon}, \quad x \in \Omega, t > 0, \\
m_{\varepsilon,t} + u_{\varepsilon} \cdot \nabla m_{\varepsilon} &= \Delta m_{\varepsilon} - n_{\varepsilon} m_{\varepsilon}, \quad x \in \Omega, t > 0, \\
u_{\varepsilon} + \nabla P_{\varepsilon} &= \Delta u_{\varepsilon} - \kappa(Y_{\varepsilon} u_{\varepsilon} \cdot \nabla)u_{\varepsilon} + (n_{\varepsilon} + m_{\varepsilon}) \nabla \phi, \quad x \in \Omega, t > 0, \\
\n&= 0, \quad x \in \Omega, t > 0, \\
\n&= \nabla \cdot u_{\varepsilon} = 0, \quad x \in \Omega, t > 0, \\
\n&= \nabla n_{\varepsilon} \cdot \nu = \nabla c_{\varepsilon} \cdot \nu = \nabla m_{\varepsilon} \cdot \nu = 0, u_{\varepsilon} = 0, \quad x \in \partial \Omega, t > 0, \\
\n&= n_{\varepsilon}(x,0) = n_0(x), c_{\varepsilon}(x,0) = c_0(x), m_{\varepsilon}(x,0) = m_0(x), u_{\varepsilon}(x,0) = u_0(x), \quad x \in \Omega,
\end{aligned}
\end{cases}
\] (2.1)

where for $\varepsilon \in (0,1)$,
\[ S_{\varepsilon}(x, n, c) = \rho_{\varepsilon}(x)S(x, n, c), \quad x \in \bar{\Omega}, \ n \geq 0, \ c \geq 0 \] (2.2)

and
\[ Y_{\varepsilon}w := (1 + \varepsilon A_2)^{-1}w \] for all $w \in L^2(\Omega)$ (2.3)
is the standard Yosida approximation. Here $(\rho_{\varepsilon})_{\varepsilon \in (0,1)} \in C^\infty(\Omega)$ be a family of standard cut-off functions satisfying $0 \leq \rho_{\varepsilon} \leq 1$ in $\Omega$ and $\rho_{\varepsilon} \not\rightarrow 1$ in $\Omega$ as $\varepsilon \not\rightarrow 0$.

By an adaptation of well-established fixed point arguments (see e.g. Lemma 2.2 of [43] as well as [42]) and a suitable extensibility criterion, one can readily verify the local solvability of (2.1).

Lemma 2.1. Assume that $\varepsilon \in (0,1)$. Then there exist $T_{\max,\varepsilon} \in (0,\infty)$ and a classical solution $(n_{\varepsilon,\varepsilon}, c_{\varepsilon,\varepsilon}, m_{\varepsilon,\varepsilon}, u_{\varepsilon,\varepsilon}, P_{\varepsilon,\varepsilon})$ of (2.1) in $\Omega \times (0,T_{\max,\varepsilon})$ such that
\[
\begin{cases}
n_{\varepsilon} \in C^0(\bar{\Omega} \times [0,T_{\max,\varepsilon})) \cap C^{2,1}(\bar{\Omega} \times (0,T_{\max,\varepsilon})), \\
c_{\varepsilon} \in C^0(\bar{\Omega} \times [0,T_{\max,\varepsilon})) \cap C^{2,1}(\bar{\Omega} \times (0,T_{\max,\varepsilon})), \\
m_{\varepsilon} \in C^0(\bar{\Omega} \times [0,T_{\max,\varepsilon})) \cap C^{2,1}(\bar{\Omega} \times (0,T_{\max,\varepsilon})), \\
u_{\varepsilon} \in C^0(\bar{\Omega} \times [0,T_{\max,\varepsilon})) \cap C^{2,1}(\bar{\Omega} \times (0,T_{\max,\varepsilon})), \\
P_{\varepsilon} \in C^{1,0}(\bar{\Omega} \times [0,T_{\max,\varepsilon})),
\end{cases}
\] (2.4)
classically solving (2.1) in $\Omega \times [0, T_{\text{max}, \varepsilon})$. Moreover, $n_\varepsilon$, $c_\varepsilon$ and $m_\varepsilon$ are nonnegative in $\Omega \times (0, T_{\text{max}, \varepsilon})$, and
\begin{equation}
\|n_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} + \|c_\varepsilon(\cdot, t)\|_{W^{1, \infty}(\Omega)} + \|m_\varepsilon(\cdot, t)\|_{W^{1, \infty}(\Omega)} + \|A^\gamma u_\varepsilon(\cdot, t)\|_{L^2(\Omega)} \to \infty,
\end{equation}
as $t \nearrow T_{\text{max}, \varepsilon}$, where $\gamma$ is given by (1.9).

3. Some basic estimates and global existence in the regularized problems.

In this section we want to ensure that the time-local solutions obtained in Lemma 2.1 are in fact global solutions. To this end, in a first step, upon a straightforward integration of the first, two and three equations in (2.1) over $\Omega$, we can establish the following basic estimates by using the maximum principle to the second and third equations. The detailed proof can be found in Lemma 2.2 of [6] (see also [18]). Therefore, we list them here without proof.

**Lemma 3.1.** There exists $\lambda > 0$ independent of $\varepsilon$ such that the solution of (2.1) satisfies
\begin{align*}
\int_\Omega n_\varepsilon + \int_\Omega c_\varepsilon + \int_\Omega m_\varepsilon + \|c_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} &\leq \lambda \text{ for all } t \in (0, T_{\text{max}, \varepsilon}) \quad (3.1) \\
\|m_\varepsilon(\cdot, t)\|_{L^2(\Omega)}^2 + 2 \int_0^t \int_\Omega |\nabla m_\varepsilon|^2 &\leq \lambda \text{ for all } t \in (0, T_{\text{max}, \varepsilon}) \quad (3.2) \\
\int_0^t \int_\Omega |\nabla c_\varepsilon|^2 &\leq \lambda \text{ for all } t \in (0, T_{\text{max}, \varepsilon}). \quad (3.3)
\end{align*}

With all the above estimates at hand, we can now establish the global existence result of our approximate solutions. If you need, refer Lemma 3.9 in [42] for Step 3 and Step 6.

**Lemma 3.2.** Let $\alpha \geq 0$. Then for all $\varepsilon \in (0, 1)$, the solution of (2.1) is global in time.

**Proof.** **Step 1. The boundedness of $\|n_\varepsilon(\cdot, t)\|_{L^2(\Omega)}$ for all $t \in (0, T_{\text{max}, \varepsilon})$.**

Multiplying the first equation in (2.1) by $n_\varepsilon$ and using $\nabla \cdot u_\varepsilon = 0$, we derive
\begin{align*}
\frac{1}{2} \frac{d}{dt} \|n_\varepsilon\|_{L^2(\Omega)}^2 + \int_\Omega |\nabla n_\varepsilon|^2 &= - \int_\Omega n_\varepsilon \nabla \cdot \left( n_\varepsilon \frac{1}{(1 + \varepsilon n_\varepsilon)} S_\varepsilon(x, n_\varepsilon, c_\varepsilon) \cdot \nabla c_\varepsilon \right) - \int_\Omega n_\varepsilon^2 m_\varepsilon \\
&\leq \int_\Omega n_\varepsilon \frac{1}{(1 + \varepsilon n_\varepsilon)} |S_\varepsilon(x, n_\varepsilon, c_\varepsilon)| |\nabla n_\varepsilon| |\nabla c_\varepsilon| \text{ for all } t \in (0, T_{\text{max}, \varepsilon}), \quad (3.4)
\end{align*}

where the last inequality we have used the nonnegativity of $n_\varepsilon$ and $m_\varepsilon$. Recalling (1.6), by Young inequality, one can see that
\begin{align*}
\int_\Omega n_\varepsilon \frac{1}{(1 + \varepsilon n_\varepsilon)} |S_\varepsilon(x, n_\varepsilon, c_\varepsilon)| |\nabla n_\varepsilon| |\nabla c_\varepsilon| &\leq \frac{1}{\varepsilon} C_S \int_\Omega |\nabla n_\varepsilon|^2 |\nabla c_\varepsilon| \\
&\leq \frac{1}{2} \int_\Omega |\nabla n_\varepsilon|^2 + C_1 \int_\Omega |\nabla c_\varepsilon|^2 \text{ for all } t \in (0, T_{\text{max}, \varepsilon}), \quad (3.5)
\end{align*}
where \( C_1 \) is a positive constant, as all subsequently appearing constants \( C_2, C_3, \ldots \) possibly depend on \( \varepsilon \). Substituting (3.5) into (3.4) and using (3.3), we derive that
\[
\int_{\Omega} n_\varepsilon^2 \leq C_2 \text{ for all } t \in (0, T_{\text{max},\varepsilon}). \tag{3.6}
\]

**Step 2. The boundedness of \( \|u_\varepsilon(\cdot, t)\|_{L^2(\Omega)} \) for all \( t \in (0, T_{\text{max},\varepsilon}) \).** Next, testing the fourth equation of (2.1) by \( u_\varepsilon \), integrating by parts and using \( \nabla \cdot u_\varepsilon = 0 \)
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |u_\varepsilon|^2 + \int_{\Omega} \nabla u_\varepsilon^2 = \int_{\Omega} (n_\varepsilon + m_\varepsilon) u_\varepsilon \cdot \nabla \phi \text{ for all } t \in (0, T_{\text{max},\varepsilon}), \tag{3.7}
\]
where we used the facts that \( \nabla \cdot u_\varepsilon \equiv 0 \) and \( \nabla \cdot (1 + \varepsilon A)^{-1} u_\varepsilon \equiv 0 \) (see also Lemma 3.5 in [43]). In light of (1.10), (3.2) and (3.9) this readily implies
\[
\int_{\Omega} |u_\varepsilon|^2 \leq C_3 \text{ for all } t \in (0, T_{\text{max},\varepsilon}) \tag{3.8}
\]
for some \( C_3 > 0 \). Relying on properties of the Yosida approximation \( Y_\varepsilon \), we can also immediately find \( C_4 > 0 \) and \( C_5 > 0 \) such that
\[
\|Y_\varepsilon u_\varepsilon\|_{L^\infty(\Omega)} = \|(I + \varepsilon A)^{-1} u_\varepsilon\|_{L^\infty(\Omega)} \leq C_4 \|u_\varepsilon(\cdot, t)\|_{L^2(\Omega)} \leq C_5 \text{ for all } t \in (0, T_{\text{max},\varepsilon}). \tag{3.9}
\]

**Step 3. The boundedness of \( \|u_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \) for all \( t \in (\frac{T}{2}, T_{\text{max},\varepsilon}) \) with any \( \tau \in (0, T_{\text{max},\varepsilon}) \).** Next, testing the projected Stokes equation \( u_{\varepsilon t} + Au_\varepsilon = P[-\kappa(Y_\varepsilon u_\varepsilon \cdot \nabla)u_\varepsilon + (n_\varepsilon + m_\varepsilon)\nabla \phi] \) by \( Au_\varepsilon \), we derive
\[
\frac{1}{2} \frac{d}{dt} \|A^{\frac{1}{2}} u_\varepsilon\|^2_{L^2(\Omega)} + \int_{\Omega} |Au_\varepsilon|^2
\]
\[
\leq \int_{\Omega} A u_\varepsilon P[-\kappa(Y_\varepsilon u_\varepsilon \cdot \nabla)u_\varepsilon] + \int_{\Omega} P[(n_\varepsilon + m_\varepsilon)\nabla \phi]Au_\varepsilon
\]
\[
\leq \frac{1}{2} \|Au_\varepsilon\|^2 + \|\nabla \phi\|^2_{L^\infty(\Omega)} \int_{\Omega} (n_\varepsilon + m_\varepsilon)^2
\]
\[
\leq \frac{1}{2} \|Au_\varepsilon\|^2 + C_6 \int_{\Omega} |\nabla u_\varepsilon|^2 + C_7 \text{ for all } t \in (0, T_{\text{max},\varepsilon}) \tag{3.10}
\]
by using (3.2) as well as (3.23) and (3.9). Hence, (3.10) implies
\[
\int_{\Omega} |\nabla u_\varepsilon|^2 \leq C_8 \text{ for all } t \in (0, T_{\text{max},\varepsilon}) \tag{3.11}
\]
by some basic calculation. Now, let \( h_\varepsilon(x, t) = P[(n_\varepsilon + m_\varepsilon)\nabla \phi - \kappa(Y_\varepsilon u_\varepsilon \cdot \nabla)u_\varepsilon] \). Then
\[
\|h_\varepsilon(\cdot, t)\|_{L^2(\Omega)} \leq C_9 \text{ for all } t \in (0, T_{\text{max},\varepsilon}) \tag{3.12}
\]
by using (3.23) and (3.11). Now, we express \( A^\gamma u_\varepsilon \) by its variation-of-constants representation and make use of well-known smoothing properties of the Stokes semigroup (see page 201 of [7]) to obtain \( C_{10} > 0 \) such that for any \( \tau \in (0, T_{\text{max},\varepsilon}) \),
\[
\|A^\gamma u_\varepsilon(\cdot, t)\|_{L^2(\Omega)} \leq C_{10} \text{ for all } t \in (\frac{T}{2}, T_{\text{max},\varepsilon}), \tag{3.13}
\]
where \( \gamma \in (\frac{3}{4}, 1) \). Since, \( D(A^\gamma) \) is continuously embedded into \( L^\infty(\Omega) \) by \( \gamma > \frac{3}{4} \), so that, (3.13) yields to for some positive constant \( C_{11} \) such that
\[
\|u_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq C_{11} \text{ for all } t \in (\frac{T}{2}, T_{\text{max},\varepsilon}). \tag{3.14}
\]
Step 4. The boundedness of \( \|c_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)} \) for all \( t \in (\frac{T}{2}, T_{max,\varepsilon}) \) with any \( \tau \in (0, T_{max,\varepsilon}) \). Now, test the second equation of (2.1) by \(-\Delta c_\varepsilon\) and obtain, upon two applications of Young’s inequality, that

\[
\frac{1}{2} \frac{d}{dt} \|\nabla c_\varepsilon\|_{L^2(\Omega)}^2 + \int_{\Omega} |\Delta c_\varepsilon|^2 + \int_{\Omega} |\nabla c_\varepsilon|^2 \leq \frac{1}{2} \int_{\Omega} |\Delta c_\varepsilon|^2 + \int_{\Omega} m_\varepsilon^2 + \|u_\varepsilon\|_{L^\infty(\Omega)}^2 \int_{\Omega} |\nabla c_\varepsilon|^2. \tag{3.15}
\]

Recalling the bounds provided by (3.14) and (3.2), this immediately implies that for any \( \tau \in (0, T_{max,\varepsilon}) \),

\[
\int_{\Omega} |\nabla c_\varepsilon(\cdot,t)|^2 \leq C_{12} \text{ for all } t \in (\frac{T}{2}, T_{max,\varepsilon}). \tag{3.16}
\]

In light of Lemma 2.1 of [13] and the Hölder inequality, we derive that,

\[
\begin{align*}
&\leq C_{13}(1 + \sup_{s \in (\frac{T}{2}, \tau)} \|m_\varepsilon(s)\|_{L^1(\Omega)}) \|u_\varepsilon(s)\|_{L^\infty(\Omega)} \|\nabla c_\varepsilon(\cdot,s)\|_{L^1(\Omega)} \\
&\leq C_{13}(1 + \sup_{s \in (\frac{T}{2}, \tau)} \|m_\varepsilon(s)\|_{L^1(\Omega)}) \|u_\varepsilon(s)\|_{L^\infty(\Omega)} \sup_{s \in (\frac{T}{2}, \tau)} \|\nabla c_\varepsilon(\cdot,s)\|_{L^1(\Omega)} \\
&\leq C_{13}(1 + \sup_{t \in (\frac{T}{2}, T_{max,\varepsilon})} \|m_\varepsilon(\cdot,t)\|_{L^1(\Omega)}) \|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \sup_{s \in (\frac{T}{2}, \tau)} \|\nabla c_\varepsilon(\cdot,s)\|_{L^1(\Omega)} \\
&\leq C_{14}(1 + \sup_{s \in (\frac{T}{2}, \tau)} \|\nabla c_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)}) \text{ for all } t \in (\frac{T}{2}, T_{max,\varepsilon}),
\end{align*}
\tag{3.17}
\]

which combined with (3.1) implies that

\[
\|c_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)} \leq C_{15} \text{ for all } t \in (\frac{T}{2}, T_{max,\varepsilon}). \tag{3.18}
\]

Step 5. The boundedness of \( \|m_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)} \) for all \( t \in (\frac{T}{2}, T_{max,\varepsilon}) \) with any \( \tau \in (0, T_{max,\varepsilon}) \).

Now, multiplying the third equation of (2.1) by \(-\Delta m_\varepsilon\) and using the Young inequality lead to that

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \|\nabla m_\varepsilon\|_{L^2(\Omega)}^2 + \int_{\Omega} |\Delta m_\varepsilon|^2 &\leq \frac{1}{2} \int_{\Omega} |\Delta m_\varepsilon|^2 + \int_{\Omega} m_\varepsilon^2 + \|u_\varepsilon\|_{L^\infty(\Omega)}^2 \int_{\Omega} |\nabla m_\varepsilon|^2, \tag{3.19}
\end{align*}
\]

which together with (3.6), (3.1) as well as (3.14) and (3.2) yields that for any \( \tau \in (0, T_{max,\varepsilon}) \),

\[
\int_{\Omega} |\nabla m_\varepsilon(\cdot,t)|^2 \leq C_{16} \text{ for all } t \in (\frac{T}{2}, T_{max,\varepsilon}). \tag{3.20}
\]
Now, multiplying the first equation in (2.1) by \( n_\varepsilon^3 \) and using \( \nabla \cdot u_\varepsilon = 0 \), we derive
\[
\frac{1}{4} \frac{d}{dt} \| n_\varepsilon \|^4_{L^4(\Omega)} + 3 \int_\Omega n_\varepsilon^2 |\nabla n_\varepsilon|^2 \leq - \int_\Omega n_\varepsilon^3 \nabla \cdot (n_\varepsilon (1 + \varepsilon n_\varepsilon) S_\varepsilon(x,n_\varepsilon,c_\varepsilon) \cdot \nabla c_\varepsilon) - \int_\Omega n_\varepsilon^4 m_\varepsilon \tag{3.21}
\]
where the last inequality we have used the nonnegativity of \( n_\varepsilon \) and \( m_\varepsilon \). This combined with (1.6) and Young inequality imply that
\[
3 \int_\Omega n_\varepsilon^3 (1 + \varepsilon n_\varepsilon) \| S_\varepsilon(x,n_\varepsilon,c_\varepsilon) \| \nabla n_\varepsilon \| \nabla c_\varepsilon \| \leq C_{18} \quad \text{for all } t \in \left( \frac{\tau}{2}, T_{max,\varepsilon} \right),
\]
where \( \tau \) is the time when \( n_\varepsilon \) and \( m_\varepsilon \) are defined. 

In view of Lemma 2.1 of [13], using the Hölder inequality and (3.14), we have
\[
\| \nabla m_\varepsilon(\cdot,t) \|_{L^\infty(\Omega)} \leq C_{19} (1 + \sup_{s \in (\frac{\tau}{2},t)} \| n_\varepsilon(\cdot,s) m_\varepsilon(\cdot,s) + u_\varepsilon(\cdot,s) \cdot \nabla m_\varepsilon(\cdot,s) - m_\varepsilon(\cdot,s) \|_{L^4(\Omega)})
\]
where \( \sup_{s \in (\frac{\tau}{2},t)} \| n_\varepsilon(\cdot,s) \|_{L^4(\Omega)} \leq C_{20}, \) and
\[
C_{21} (1 + \sup_{t \in (\frac{\tau}{2},T_{max,\varepsilon})} \| n_\varepsilon(\cdot,t) \|_{L^4(\Omega)})
\]
where \( \sup_{t \in (\frac{\tau}{2},T_{max,\varepsilon})} \| u_\varepsilon(\cdot,t) \|_{L^\infty(\Omega)} \leq C_{22}, \) and
\[
\| \nabla m_\varepsilon(\cdot,s) \|_{L^\infty(\Omega)} \leq C_{23} \quad \text{for all } t \in (\frac{\tau}{2}, T_{max,\varepsilon}),
\]
which combined with (3.1) implies that
\[
\| m_\varepsilon(\cdot,t) \|_{W^{1,\infty}(\Omega)} \leq C_{23} \quad \text{for all } t \in (\frac{\tau}{2}, T_{max,\varepsilon}). \tag{3.25}
\]

Step 6. The boundedness of \( \| n_\varepsilon(\cdot,t) \|_{L^\infty(\Omega)} \) for all \( t \in (\tau, T_{max,\varepsilon}) \) with any \( \tau \in (0, T_{max,\varepsilon}) \):

Furthermore, applying the variation-of-constants formula to the \( n_\varepsilon \)-equation in (2.1), we conclude that for any \( \tau \in (0, T_{max,\varepsilon}) \),
\[
n_\varepsilon(t) = e^{(t-\frac{\tau}{2})\Delta} n_\varepsilon(\cdot,\frac{\tau}{2}) - \int_\frac{\tau}{2}^t e^{(t-s)\Delta} \nabla \cdot (n_\varepsilon(\cdot,s)c_\varepsilon \nabla c_\varepsilon + u_\varepsilon)ds - \int_\frac{\tau}{2}^t e^{(t-s)\Delta} n_\varepsilon(\cdot,s)m_\varepsilon(\cdot,s)ds \tag{3.26}
\]
and
\[
\| \tilde{h}_\varepsilon(\cdot,t) \|_{L^\infty(\Omega)} \leq C_{24} \quad \text{for all } t \in (\frac{\tau}{2}, T_{max,\varepsilon}).
\]
As the last summand in (3.26) is nonnegative by the maximum principle, so that, we can thus estimate
\[
\|n_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq \|e^{(t-s)\frac{\varepsilon}{T}}n_\varepsilon(\cdot, \frac{\varepsilon}{T})\|_{L^\infty(\Omega)} + \int_{\frac{\varepsilon}{T}}^{t} \|e^{(t-s)\frac{\varepsilon}{T}} \nabla \cdot (n_\varepsilon(\cdot, s)\tilde{h}_\varepsilon(\cdot, s))\|_{L^\infty(\Omega)} ds \\
\leq C_{25}(t - \frac{\varepsilon}{T})\frac{\varepsilon}{T}^{-\frac{\varepsilon}{T}}\|n_0\|_{L^1(\Omega)} + C_{25} \int_{\frac{\varepsilon}{T}}^{t} (t-s)^{-\frac{\varepsilon}{T}} e^{-\lambda_1(t-s)}\|n_\varepsilon(\cdot, s)\tilde{h}_\varepsilon(\cdot, s)\|_{L^1(\Omega)} ds \\
\leq C_{26} + C_{26} \int_{\frac{\varepsilon}{T}}^{t} (t-s)^{-\frac{\varepsilon}{T}} e^{-\lambda_1(t-s)}\|n_\varepsilon(\cdot, s)\|_{L^1(\Omega)} ds \\
\leq C_{26} + C_{27} \int_{\frac{\varepsilon}{T}}^{t} (t-s)^{-\frac{\varepsilon}{T}} e^{-\lambda_1(t-s)}\|n_\varepsilon(\cdot, s)\|_{L^1(\Omega)} ds \\
\leq C_{26} + C_{28} \sup_{s \in (\frac{\varepsilon}{T}, T_{max, \varepsilon})} \|n_\varepsilon(\cdot, s)\|^\frac{3}{2}_{L^\infty(\Omega)} \text{ for all } t \in (\tau, T_{max, \varepsilon}),
\]
where
\[
C_{28} = C_{27} \int_{0}^{\infty} s^{-\frac{\varepsilon}{T}} e^{-\lambda_1 s}\|n_\varepsilon(\cdot, s)\|_{L^1(\Omega)}^{\frac{3}{2}} < +\infty
\]
and \(\lambda_1\) is the first nonzero eigenvalue of \(-\Delta\) on \(\Omega\) under the Neumann boundary condition. And thereby
\[
\|n_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq C_{29} \text{ for all } t \in (\tau, T_{max, \varepsilon})
\] (3.28)
by using the Young inequality.

Assume that \(T_{max, \varepsilon} < \infty\). In view of (3.13), (3.18) (3.25) and (3.28), we apply Lemma 2.1 to reach a contradiction. \(\square\)

4. **A priori estimates for the regularized problem (2.1) which is independent of \(\varepsilon\)**. Since we want to obtain a weak solution of (1.4) by means of taking \(\varepsilon \downarrow 0\) in (2.1), we will require regularity information which is independent of \(\varepsilon \in (0, 1)\). The main portion of important estimates will be prepared in the following section.

**Lemma 4.1.** Let \(\alpha > 0\) and \(p = 4\alpha + \frac{2}{\varepsilon}\). Then there exists \(C > 0\) independent of \(\varepsilon\) such that the solution of (2.1) satisfies
\[
\int_{\Omega} n_\varepsilon^p + \int_{\Omega} |\nabla c_\varepsilon|^2 + \int_{\Omega} |u_\varepsilon|^2 \leq C \text{ for all } t > 0.
\] (4.1)
Moreover, for \(T > 0\), it holds that one can find a constant \(C > 0\) independent of \(\varepsilon\) such that
\[
\int_{0}^{T} \left[ \int_{\Omega} |\nabla u_\varepsilon|^2 + \int_{\Omega} |\nabla c_\varepsilon|^4 + \|\nabla n_\varepsilon\|^2_{L^2(\Omega)} + \int_{\Omega} |\Delta c_\varepsilon|^2 \right] \leq C(T + 1).
\] (4.2)

**Proof.** Let \(p = 4\alpha + \frac{2}{\varepsilon}\). We first obtain from \(\nabla \cdot u_\varepsilon = 0\) in \(\Omega \times (0, T_{max, \varepsilon})\) and straightforward calculations that
\[
sign(p - 1) \frac{d}{dt} \|n_\varepsilon\|_{L^p(\Omega)} + \sign(p - 1) \frac{4(p - 1)}{p^2} \|\nabla n_\varepsilon\|^2_{L^2(\Omega)} \\
= -\sign(p - 1) \int_{\Omega} n_\varepsilon^{p-1} \nabla \cdot (n_\varepsilon \frac{1}{1 + \varepsilon n_\varepsilon}) S_\varepsilon(x, n_\varepsilon, c_\varepsilon) \cdot \nabla c_\varepsilon - \sign(p - 1) \int_{\Omega} n_\varepsilon^p m_\varepsilon \\
\leq \sign(p - 1) (p - 1) \int_{\Omega} n_\varepsilon^{p-1} \frac{1}{1 + \varepsilon n_\varepsilon} |S_\varepsilon(x, n_\varepsilon, c_\varepsilon)| \|\nabla n_\varepsilon\| \|\nabla c_\varepsilon\| \\
- \sign(p - 1) \int_{\Omega} n_\varepsilon^p m_\varepsilon
\] (4.3)
for all $t > 0$. Therefore, in light of (1.6), with the help of the Young inequality, we can estimate the right of (4.3) by following

$$ \text{sign}(p - 1)(p - 1) \int_\Omega n^{p-1}_\varepsilon \left( \frac{1}{(1 + \varepsilon_n^e)} |S_\varepsilon(x, n_\varepsilon, c_\varepsilon)| \right) |\nabla n_\varepsilon| |\nabla c_\varepsilon| $$

$$ \leq \frac{|p - 1|}{2} \int_\Omega n^{p-2}_\varepsilon |\nabla n_\varepsilon|^2 + \frac{|p - 1|}{2} C_S^2 \int n^{p-2}_\varepsilon (1 + n_\varepsilon)^{-2\alpha} |\nabla c_\varepsilon|^2 $$

$$ \leq \frac{|p - 1|}{2} \int_\Omega n^{p-2}_\varepsilon |\nabla n_\varepsilon|^2 + \frac{|p - 1|}{2} C_S^2 \int n^{p-2\alpha}_\varepsilon |\nabla c_\varepsilon|^2 $$

$$ = \frac{2|p - 1|}{p^2} \int \nabla n_\varepsilon^2 |_{L^2(\Omega)}^2 + \frac{|p - 1|}{2} C_S^2 \int n^{p-2\alpha}_\varepsilon |\nabla c_\varepsilon|^2 $$

by using the fact that $(1 + n_\varepsilon)^{-2\alpha} \leq n^{-2\alpha}_\varepsilon$ for all $\varepsilon \geq 0$, $n_\varepsilon$ and $\alpha \geq 0$. In the following we will estimate the term $\frac{|p - 1|}{2} C_S^2 \int n^{p-2\alpha}_\varepsilon |\nabla c_\varepsilon|^2$ in the right hand side of (4.4). To this end, we firstly invoke the Gagliardo-Nirenberg inequality again to obtain $C_1 > 0$ and $C_2 > 0$ such that

$$ \int_\Omega n^{2p-4\alpha}_\varepsilon = \int \left| \nabla n_\varepsilon \right|_{L^p(\Omega)} \left( \frac{4(p-2\alpha)}{L^p(\Omega)} \right) \left( \frac{4(p-2\alpha)}{L^p(\Omega)} \right) \left( \frac{2(p-2\alpha)}{L^p(\Omega)} \right) + C_1 \int \nabla n_\varepsilon^2 |_{L^2(\Omega)} $n^{2p-4\alpha}_\varepsilon + C_2 \int \nabla c_\varepsilon^2 |_{L^2(\Omega)} $$

(4.5)

by using (3.1) and $p = 4\alpha + \frac{2}{3}$. Next, recalling the Young inequality,

$$ \frac{|p - 1|}{2} C_S^2 \int n^{p-2\alpha}_\varepsilon |\nabla c_\varepsilon|^2 \leq \frac{1}{C_2} \int \nabla n_\varepsilon^2 |_{L^2(\Omega)} |\nabla c_\varepsilon| $$

(4.6)

with $C_3 = \frac{8}{8p^2} C_2 |p - 1| C_4^2$, where $C_2$ is the same as (4.5). Inserting (4.6) into (4.4), we may derive that

$$ \text{sign}(p - 1)(p - 1) \int_\Omega n^{p-1}_\varepsilon \left( \frac{1}{(1 + \varepsilon_n^e)} |S_\varepsilon(x, n_\varepsilon, c_\varepsilon)| \right) |\nabla n_\varepsilon| |\nabla c_\varepsilon| $$

$$ \leq \frac{2|p - 1|}{p^2} \int \nabla n_\varepsilon^2 |_{L^2(\Omega)}^2 + \frac{1}{C_2} \int \nabla n_\varepsilon^2 |_{L^2(\Omega)}^2 + C_3 \int |\nabla c_\varepsilon|^4 $$

(4.7)

Next, using the Gagliardo-Nirenberg inequality and (3.1), one can get

$$ \int |\nabla c_\varepsilon|^2 \leq \int |\nabla c_\varepsilon|^4 $$

(4.8)

for some positive constants $\lambda_0, \lambda_1$ and $\lambda_2$ independent of $\varepsilon$. Collecting (4.3)–(4.5) and (4.7)–(4.8), we conclude that there exist positive constants $C_4$ and $C_5$ such that

$$ \text{sign}(p - 1) \frac{1}{p} \int |\nabla n_\varepsilon|^2 + (\frac{p}{p - 1}) \int \nabla n_\varepsilon^2 |_{L^2(\Omega)} $$

$$ \leq \frac{1}{C_4} |\Delta c_\varepsilon|^2 + C_5 - \text{sign}(p - 1) \int n^{p}_\varepsilon $$

(4.9)
To track the time evolution of \( c_\varepsilon \), taking \( -\Delta c_\varepsilon \) as the test function for the second equation of (2.1) and using (3.1), we have

\[
\frac{1}{2} \frac{d}{dt} \| \nabla c_\varepsilon \|_{L^2(\Omega)}^2 + \int_\Omega |\Delta c_\varepsilon|^2 + \int_\Omega |\nabla c_\varepsilon|^2 \\
= - \int_\Omega m_\varepsilon \Delta c_\varepsilon + \int_\Omega \Delta c_\varepsilon u_\varepsilon \cdot \nabla c_\varepsilon \\
= - \int_\Omega m_\varepsilon \Delta c_\varepsilon - \int_\Omega \nabla c_\varepsilon \cdot \nabla (u_\varepsilon \cdot \nabla c_\varepsilon) \\
= - \int_\Omega m_\varepsilon \Delta c_\varepsilon - \int_\Omega \nabla c_\varepsilon \cdot (\nabla u_\varepsilon \cdot \nabla c_\varepsilon) - \int_\Omega \nabla c_\varepsilon \cdot (D^2 c_\varepsilon \cdot u_\varepsilon),
\]

which together with the fact that

\[- \int_\Omega \nabla c_\varepsilon \cdot (D^2 c_\varepsilon \cdot u_\varepsilon) = - \frac{1}{2} \int_\Omega u_\varepsilon \cdot \nabla |\nabla c_\varepsilon|^2 = 0\]

implies that

\[
\frac{1}{2} \frac{d}{dt} \| \nabla c_\varepsilon \|_{L^2(\Omega)}^2 + \frac{1}{4} \int_\Omega |\Delta c_\varepsilon|^2 + \frac{1}{4} \int_\Omega |\nabla c_\varepsilon|^2 \\
\leq \frac{1}{4} \int_\Omega |\Delta c_\varepsilon|^2 + \frac{1}{4} \int_\Omega |m_\varepsilon|^2 + \frac{1}{4\lambda_2} \int_\Omega |\nabla c_\varepsilon|^4 + \lambda_2 \int_\Omega |\nabla u_\varepsilon|^2,
\]

where \( \lambda_2 \) is the same as (4.8). This together with (4.8) yields to

\[
\frac{1}{2} \frac{d}{dt} \| \nabla c_\varepsilon \|_{L^2(\Omega)}^2 + \frac{1}{4} \int_\Omega |\Delta c_\varepsilon|^2 + \frac{1}{4} \int_\Omega |\nabla c_\varepsilon|^2 \leq \lambda_2 \int_\Omega |\nabla u_\varepsilon|^2 + C_6
\]

by using (3.2). Taking an evident linear combination of the inequalities provided by (4.9) and (4.12), we conclude

\[
\text{sign}(p-1)\frac{1}{p} \frac{d}{dt} \| n_\varepsilon \|_{L^p(\Omega)}^p + 2C_4 \frac{d}{dt} \| \nabla c_\varepsilon \|_{L^2(\Omega)}^2 \\
+ \left( \text{sign}(p-1) \frac{4(p-1)}{p^2} - \frac{5}{2p^2} \right) \| \nabla n_\varepsilon \|_{L^2(\Omega)}^2 \\
+ C_4 \int_\Omega |\Delta c_\varepsilon|^2 + 4C_4 \int_\Omega |\nabla c_\varepsilon|^2 \\
\leq \kappa_0 \int_\Omega |\nabla n_\varepsilon|^2 + C_7 - \text{sign}(p-1) \int_\Omega n_\varepsilon^p m_\varepsilon \text{ for all } t > 0,
\]

where

\[
\kappa_0 = 4\lambda_2 C_4, C_7 = 4C_6 C_4 + C_5.
\]

Now, multiplying the fourth equation of (2.1) by \( u_\varepsilon \), integrating by parts and using \( \nabla \cdot u_\varepsilon = 0 \)

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |u_\varepsilon|^2 + \int_\Omega |\nabla u_\varepsilon|^2 = \int_\Omega (n_\varepsilon + m_\varepsilon) u_\varepsilon \cdot \nabla \phi \text{ for all } t > 0.
\]
Noticing the fact $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$ in the 3D case and making use of the Hölder inequality and the Young inequality we can estimate the right hand of (4.15) as

$$
\int_\Omega (n_\varepsilon + m_\varepsilon) u_\varepsilon \cdot \nabla \phi 
\leq \|\nabla \phi\|_{L^\infty(\Omega)} \|n_\varepsilon\|_{L^2(\Omega)} \|u_\varepsilon\|_{L^6(\Omega)} + \|\nabla \phi\|_{L^\infty(\Omega)} \|m_\varepsilon\|_{L^{\frac{6}{5}}(\Omega)} \|u_\varepsilon\|_{L^6(\Omega)}
\leq C_8 \|\nabla \phi\|_{L^\infty(\Omega)} (\|n_\varepsilon\|_{L^\frac{2}{3}(\Omega)} + \|m_\varepsilon\|_{L^{\frac{6}{5}}(\Omega)}) \|\nabla u_\varepsilon\|_{L^2(\Omega)}
\leq \frac{1}{2} \|\nabla u_\varepsilon\|^2_{L^2(\Omega)} + C_9 (\|n_\varepsilon\|^2_{L^{\frac{2}{3}}(\Omega)} + 1) \text{ for all } t > 0
$$

(4.16)

by using (3.1) and (1.10). Next, the Young inequality along with the assumed boundedness of $\phi$ (see (1.10)) as well as the Gagliardo–Nirenberg inequality and (3.1) yields

$$
\int_\Omega (n_\varepsilon + m_\varepsilon) u_\varepsilon \cdot \nabla \phi 
\leq \frac{1}{2} \|\nabla u_\varepsilon\|^2_{L^2(\Omega)} + C_9 \|n_\varepsilon\|_{L^{\frac{2}{3}}(\Omega)}^\frac{1}{3} + C_9
\leq \frac{1}{2} \|\nabla u_\varepsilon\|^2_{L^2(\Omega)} + C_{10} \|n_\varepsilon\|_{L^{\frac{2}{3}}(\Omega)}^\frac{2}{3} \|\nabla \phi\|_{L^\frac{2}{3}(\Omega)}^\frac{1}{3} + C_{10} \|n_\varepsilon\|_{L^{\frac{2}{3}}(\Omega)}^\frac{2}{3} + C_9
\leq \frac{1}{2} \|\nabla u_\varepsilon\|^2_{L^2(\Omega)} + \frac{p-1}{4\kappa_0} \frac{1}{2p^2} \|\nabla n_\varepsilon\|^2_{L^2(\Omega)} + C_{11} \text{ for all } t > 0,
$$

(4.17)

where $\kappa_0$ is given by (4.14), $C_9, C_{10}$ and $C_{11}$ are positive constants which are independent of $\varepsilon$. Here the last inequality we have used the fact that

$$
\frac{2}{3p-1} < 2 \text{ by } p = 4\alpha + \frac{2}{3} > \frac{2}{3}.
$$

Now, substituting (4.17) into (4.15), one has

$$
\frac{1}{2} \frac{d}{dt} \int_\Omega |u_\varepsilon|^2 + \frac{1}{2} \|\nabla u_\varepsilon\|^2_{L^2(\Omega)} \leq \frac{|p-1|}{2p^2} \frac{1}{4\kappa_0} \|\nabla n_\varepsilon\|^2_{L^2(\Omega)} + C_{12} \text{ for all } t > 0,
$$

(4.18)

so that, which together with (4.13) implies that

$$
2\kappa_0 \frac{d}{dt} \int_\Omega |u_\varepsilon|^2 + \text{sign}(p-1) \frac{1}{p} \frac{d}{dt} \|n_\varepsilon\|^p_{L^p(\Omega)} + 2C_4 \frac{d}{dt} \|\nabla c_\varepsilon\|^2_{L^2(\Omega)}
+ \kappa_0 \|\nabla u_\varepsilon\|^2_{L^2(\Omega)} + (\text{sign}(p-1) \frac{4(p-1)}{p^2} - \frac{3(p-1)}{p^2}) \|\nabla n_\varepsilon\|^2_{L^2(\Omega)}
+ C_4 \int_\Omega |\Delta c_\varepsilon|^2 + 4C_4 \int_\Omega |\nabla c_\varepsilon|^2
\leq C_{13} - \text{sign}(p-1) \int_\Omega \|n_\varepsilon\|^2_{L^2(\Omega)} \text{ for all } t > 0.
$$

(4.19)

**Case** $p > 1$. Then $\text{sign}(p-1) = 1 > 0$. Thus, (4.19) implies that

$$
2\kappa_0 \frac{d}{dt} \int_\Omega |u_\varepsilon|^2 + \frac{1}{p} \frac{d}{dt} \|n_\varepsilon\|^p_{L^p(\Omega)} + 2C_4 \frac{d}{dt} \|\nabla c_\varepsilon\|^2_{L^2(\Omega)}
+ \kappa_0 \|\nabla u_\varepsilon\|^2_{L^2(\Omega)} + \frac{|p-1|}{p^2} \|\nabla n_\varepsilon\|^2_{L^2(\Omega)}
+ C_4 \int_\Omega |\Delta c_\varepsilon|^2 + 4C_4 \int_\Omega |\nabla c_\varepsilon|^2
\leq C_{13} \text{ for all } t > 0
$$

(4.20)
by using
\[-\text{sign}(p - 1) \int_{\Omega} n_{\varepsilon}^p m_{\varepsilon} = - \int_{\Omega} n_{\varepsilon}^p m_{\varepsilon} \leq 0.\]

Next, integrating (4.20) in time, we can obtain from (4.8) that
\[
\int_{\Omega} |u_{\varepsilon}|^2 + \int_{\Omega} n_{\varepsilon}^p + \int_{\Omega} |\nabla c_{\varepsilon}|^2 \leq C_{14} \text{ for all } t > 0
\]
(4.21)
and
\[
\int_{0}^{T} \left[ \int_{\Omega} |\nabla u_{\varepsilon}|^2 + \int_{\Omega} |\nabla c_{\varepsilon}|^4 + \|\nabla n_{\varepsilon}^p\|_{L^2(\Omega)}^2 + \int_{\Omega} |\Delta c_{\varepsilon}|^2 \right] \leq C_{14}(T + 1) \text{ for all } T \in (0, T_{\text{max}, \varepsilon})
\]
(4.22)
and some positive constant \(C_{14}\). Here we have used the Gagliardo-Nirenberg inequality.

**Case** \(p = 4\alpha + \frac{2}{3} < 1\). Then \(\text{sign}(p - 1) = -1 < 0\), therefore, by using \(p < 1\) and (3.1), we derive from the Young inequality that
\[-\text{sign}(p - 1) \int_{\Omega} n_{\varepsilon}^p m_{\varepsilon} = \int_{\Omega} n_{\varepsilon}^p m_{\varepsilon} \leq \hat{\kappa}_0
\]
for some positive constant \(\hat{\kappa}_0\), hence, integrating (4.19) in time and employing the Hölder inequality, we also conclude that there exists a positive constant \(C_{15}\) such that
\[
\int_{\Omega} |u_{\varepsilon}|^2 + \int_{\Omega} n_{\varepsilon}^p + \int_{\Omega} |\nabla c_{\varepsilon}|^2 \leq C_{15} \text{ for all } t > 0
\]
(4.23)
and
\[
\int_{0}^{T} \left[ \int_{\Omega} |\nabla u_{\varepsilon}|^2 + \int_{\Omega} |\nabla c_{\varepsilon}|^4 + \|\nabla n_{\varepsilon}^p\|_{L^2(\Omega)}^2 + \int_{\Omega} |\Delta c_{\varepsilon}|^2 \right] \leq C_{15}(T + 1) \text{ for all } T > 0
\]
(4.24)

**Case** \(p = 1\). Using the first equation of (2.1), from integration by parts and applying (1.6), we derive from (3.1) that for some positive constant \(C_{16}\),
\[
\frac{d}{dt} \int_{\Omega} n_{\varepsilon} \ln n_{\varepsilon} + \frac{1}{2} \int_{\Omega} \frac{\nabla n_{\varepsilon}}{n_{\varepsilon}} \cdot \left( 1 + \frac{n_{\varepsilon}}{n_{\varepsilon}^p} \right) S_{\varepsilon}(x, n_{\varepsilon}, c_{\varepsilon}) \cdot \nabla c_{\varepsilon}
\]
(4.25)
\[
\leq \int_{\Omega} |\nabla n_{\varepsilon}|^2 + \int_{\Omega} C_{S}(1 + n_{\varepsilon})^{-\alpha} \frac{n_{\varepsilon}}{n_{\varepsilon}^p} |\nabla n_{\varepsilon}| |\nabla c_{\varepsilon}| + C_{16} \text{ for all } t > 0,
\]
which combined with the Young inequality implies that
\[
\frac{d}{dt} \int_{\Omega} n_{\varepsilon} \ln n_{\varepsilon} + \frac{1}{2} \int_{\Omega} \frac{\nabla n_{\varepsilon}}{n_{\varepsilon}} \cdot \left( 1 + \frac{n_{\varepsilon}}{n_{\varepsilon}^p} \right) S_{\varepsilon}(x, n_{\varepsilon}, c_{\varepsilon}) \cdot \nabla c_{\varepsilon}
\]
(4.26)
On the other hand, due to $p = 1$ yields to $\frac{\alpha}{2} + \frac{2}{3} > \frac{2}{3}$, employing almost exactly the same arguments as in the proof of (4.10)–(4.22) (the minor necessary changes are left as an easy exercise to the reader), we conclude the estimate

$$
\int_{\Omega} |u_\varepsilon|^2 + \int_{\Omega} n_\varepsilon^p + \int_{\Omega} |\nabla c_\varepsilon|^2 \leq C_{17} \text{ for all } t > 0 \tag{4.27}
$$

and

$$
\int_0^T \left[ \int_{\Omega} |\nabla u_\varepsilon|^2 + \int_{\Omega} |\nabla c_\varepsilon|^4 + \|\nabla n_\varepsilon^n\|_{L^2(\Omega)}^2 + \int_{\Omega} |\Delta c_\varepsilon|^2 \right] \leq C_{17}(T + 1) \text{ for all } T > 0. \tag{4.28}
$$

#### 4.1. Further a-priori estimates

With the help of Lemma 4.1 and the Gagliardo–Nirenberg inequality, one can derive the following Lemma:

**Lemma 4.2.** Let $\alpha > 0$. Then for each $T > 0$, there exists $C > 0$ independent of $\varepsilon$ such that the solution of (2.1) satisfies

$$
\int_0^T \int_{\Omega} \left[ |\nabla n_\varepsilon|^\gamma_0 + n_\varepsilon^\alpha + 4 \right] \, dt \leq C(T + 1), \tag{4.29}
$$

where $\gamma_0 = \min\{3\alpha + 1, 2\}$.

**Proof.** Due to (3.1), (4.1) and (4.2), in light of the Gagliardo–Nirenberg inequality, for some $C_1$ and $C_2 > 0$ which are independent of $\varepsilon$, one may verify that

$$
\int_0^T \int_{\Omega} n_\varepsilon^{\frac{12\alpha+4}{12\alpha+6}} \leq C_1 \left( \int_0^T \int_{\Omega} n_\varepsilon^{\frac{6\alpha+1}{12\alpha+6}} \right)^{\frac{3\alpha+1}{12\alpha+6}} \left( \int_0^T \int_{\Omega} n_\varepsilon^{\frac{12\alpha+4}{6\alpha+3}} \right)^{\frac{1-3\alpha}{12\alpha+6}} \tag{4.30}
$$

**Case** $0 < \alpha < \frac{1}{3}$. Therefore, employing the Hölder inequality (with two exponents $\frac{2}{3\alpha+1}$ and $\frac{2}{1-3\alpha}$), we conclude that there exists a positive constant $C_3$ such that

$$
\int_0^T \int_{\Omega} |\nabla n_\varepsilon|^{3\alpha+1} \leq \left[ \int_0^T \int_{\Omega} n_\varepsilon^{\frac{12\alpha-4}{3\alpha+1}} \right]^{\frac{3\alpha+1}{2}} \left[ \int_0^T \int_{\Omega} n_\varepsilon^{\frac{12\alpha+4}{6\alpha+3}} \right]^{\frac{1-3\alpha}{2}} \tag{4.31}
$$

**Case** $\alpha \geq \frac{1}{3}$. Multiply the first equation in (2.1) by $n_\varepsilon$ and using $\nabla \cdot u_\varepsilon = 0$, we derive

$$
\frac{1}{2} \frac{d}{dt} \|n_\varepsilon\|_{L^2(\Omega)}^2 + \int_{\Omega} |\nabla n_\varepsilon|^2 = - \int_{\Omega} n_\varepsilon \nabla \cdot \left( \frac{n_\varepsilon}{(1 + \varepsilon n_\varepsilon)} \right) S_\varepsilon(x, n_\varepsilon, c_\varepsilon) \cdot \nabla c_\varepsilon - \int_{\Omega} n_\varepsilon^2 m_\varepsilon \tag{4.32}
$$

$$
\leq \int_{\Omega} n_\varepsilon \left( \frac{1}{1 + \varepsilon n_\varepsilon} \right) |S_\varepsilon(x, n_\varepsilon, c_\varepsilon)||\nabla n_\varepsilon||\nabla c_\varepsilon| \text{ for all } t > 0.
$$
Recalling (1.6) and using $\alpha \geq \frac{1}{3}$, from Young inequality again, we derive that for any $\delta_0 > 0$

$$
\int_\Omega n_\varepsilon \left(1 + \varepsilon n_\varepsilon^2\right) \left|S(x, n_\varepsilon, c_\varepsilon)\right| |\nabla n_\varepsilon| |\nabla c_\varepsilon| \\
\leq C_S \int_\Omega n_\varepsilon (1 + n_\varepsilon)^{-\alpha} |\nabla n_\varepsilon| |\nabla c_\varepsilon| \\
\leq \frac{1}{2} \int_\Omega |\nabla n_\varepsilon|^2 + \frac{C_S^2}{2} \int_\Omega n_\varepsilon^2 (1 + n_\varepsilon)^{-2\alpha} |\nabla c_\varepsilon|^2 \\
\leq \frac{1}{4} \int_\Omega |\nabla n_\varepsilon|^2 + \frac{1}{4\delta_0} \int_\Omega n_\varepsilon^2 (1 + n_\varepsilon)^{-4\alpha} + \frac{\delta_0 C_S^4}{4} \int_\Omega |\nabla c_\varepsilon|^4 	ext{ for all } t > 0.
$$

(4.33)

If $\frac{1}{3} \leq \alpha < \frac{3}{4}$, then

$$
1 < 4 - 4\alpha < 6,
$$

thus, invoking the Gagliardo-Nirenberg inequality again to obtain $C_4, C_5$ and $C_6 > 0$ such that

$$
\int_\Omega n_\varepsilon^2 (1 + n_\varepsilon)^{-4\alpha} \leq \int_\Omega n_\varepsilon^{(4-4\alpha)} \\
= \left\| n_\varepsilon \right\|_{L^{4-4\alpha}(\Omega)}^{4-4\alpha} \\
\leq C_4 \left\| \nabla n_\varepsilon \right\|_{L^{4-12\alpha}(\Omega)} \left\| n_\varepsilon \right\|_{L^4(\Omega)}^{4-4\alpha-\frac{12\alpha}{4}} + C_4 \left\| n_\varepsilon \right\|_{L^4(\Omega)}^{4-4\alpha} \\
\leq C_5 \left( \left\| \nabla n_\varepsilon \right\|_{L^4(\Omega)}^4 + 1 \right) \\
\leq C_6 \left( \left\| \nabla n_\varepsilon \right\|_{L^4(\Omega)}^2 + 1 \right)
$$

(4.34)

by using (3.1) as well as (1.6) and the Young inequality. Now, inserting (4.34) into (4.33) and choosing $\delta_0 = C_6$, one has

$$
\int_\Omega n_\varepsilon \left(1 + \varepsilon n_\varepsilon^2\right) \left|S(x, n_\varepsilon, c_\varepsilon)\right| |\nabla n_\varepsilon| |\nabla c_\varepsilon| \\
\leq C_S \int_\Omega n_\varepsilon (1 + n_\varepsilon)^{-\alpha} |\nabla n_\varepsilon| |\nabla c_\varepsilon| \\
\leq \frac{3}{4} \int_\Omega |\nabla n_\varepsilon|^2 + \frac{C_6 C_S^4}{4} \int_\Omega |\nabla c_\varepsilon|^4 + \frac{1}{4} C_6 \text{ for all } t > 0,
$$

which combined with (4.32) implies that

$$
\frac{1}{2} \frac{d}{dt} \left\| n_\varepsilon \right\|_{L^2(\Omega)}^2 + \frac{1}{4} \int_\Omega |\nabla n_\varepsilon|^2 \leq \frac{C_6 C_S^4}{4} \int_\Omega |\nabla c_\varepsilon|^4 + \frac{1}{4} C_6 \text{ for all } t > 0,
$$

(4.35)

so that, gathering (4.2) and (4.35), we have

$$
\int_\Omega n_\varepsilon^2 \leq C_7 (T + 1) \text{ for all } t > 0
$$

(4.36)

and

$$
\int_0^T \int_\Omega |\nabla n_\varepsilon|^2 \leq C_7 (T + 1) \text{ for all } T > 0.
$$

(4.37)

While if $\alpha \geq 1$, observing $s(1 + s)^{-\alpha} \leq 1$ (for any $s \geq 0$), therefore, by using (1.6) and (4.1), we may estimate the term $\int_\Omega n_\varepsilon \left(1 + \varepsilon n_\varepsilon^2\right) \left|S(x, n_\varepsilon, c_\varepsilon)\right| |\nabla n_\varepsilon| |\nabla c_\varepsilon|$ in
(4.32) by
\[
\int_{\Omega} n_\varepsilon \frac{1}{(1 + \varepsilon n_\varepsilon)} |S_\varepsilon(x, n_\varepsilon, c_\varepsilon)||\nabla n_\varepsilon||\nabla c_\varepsilon|
\leq C_8 \int_{\Omega} n_\varepsilon (1 + n_\varepsilon)^{-\alpha} |\nabla n_\varepsilon||\nabla c_\varepsilon|
\leq \frac{1}{2} \int_{\Omega} |\nabla n_\varepsilon|^2 + \frac{C_8^2}{2} \int_{\Omega} |\nabla c_\varepsilon|^2
\leq \frac{1}{2} \int_{\Omega} |\nabla n_\varepsilon|^2 + C_8 \text{ for all } t > 0.
\]

Thus, collecting (4.32) and (4.38), we have
\[
\int_{\Omega} n_\varepsilon^2 \leq C_9 (T + 1) \text{ for all } t > 0 \quad (4.39)
\]
and
\[
\int_0^T \int_{\Omega} |\nabla n_\varepsilon|^2 \leq C_9 (T + 1) \text{ for all } T > 0. \quad (4.40)
\]

Finally, if \( \frac{3}{4} \leq \alpha < 1 \), then by using (3.1) and the Young inequality, we have
\[
\int_{\Omega} n_\varepsilon^4 (1 + n_\varepsilon)^{-4\alpha} \leq \int_{\Omega} n_\varepsilon^{(4-4\alpha)} \leq C_{10} \text{ for all } t > 0,
\]
where we have used the fact that
\[ 4 - 4\alpha < 1. \]

So, inserting (4.41) into (4.32) again, one derives
\[
\int_{\Omega} n_\varepsilon^2 \leq C_{11} (T + 1) \text{ for all } t > 0 \quad (4.42)
\]
and
\[
\int_0^T \int_{\Omega} |\nabla n_\varepsilon|^2 \leq C_{11} (T + 1) \text{ for all } T > 0. \quad (4.43)
\]

In order to prove the limit functions \( n \) and \( u \) gained below (see Section 6), we will rely on an additional regularity estimate for \( n_\varepsilon u_\varepsilon \).

**Lemma 4.3.** Let \( \alpha > 0 \). Then there exists \( C > 0 \) independent of \( \varepsilon \) such that, for each \( T > 0 \), the solution of (2.1) satisfies
\[
\int_0^T \int_{\Omega} |n_\varepsilon u_\varepsilon|^{\frac{2+6\alpha}{2+3\alpha}} \leq C(T + 1). \quad (4.44)
\]

**Proof.** In view of the Hölder inequality and the Young inequality, we have
\[
\int_0^T \int_{\Omega} |n_\varepsilon u_\varepsilon|^{\frac{2+6\alpha}{2+3\alpha}} \leq \int_0^T \|n_\varepsilon\|^{\frac{2+6\alpha}{2+3\alpha}}_{L^{2+6\alpha}(2+6\alpha)\theta-1(T)} \|u_\varepsilon\|^{\frac{2+6\alpha}{2+3\alpha}}_{L^{2+6\alpha}(2+6\alpha)\theta-1(T)}
\leq C_1 \int_0^T \|n_\varepsilon\|^{\frac{2+6\alpha}{2+3\alpha}}_{L^{2+6\alpha}(2+6\alpha)\theta-1(\Omega)} \|\nabla u_\varepsilon\|^{\frac{2+6\alpha}{2+3\alpha}}_{L^2(\Omega)}
\leq C_2 \int_0^T \|n_\varepsilon\|^{\frac{2+6\alpha}{2+3\alpha}}_{L^{2+6\alpha}(2+6\alpha)\theta-1(\Omega)} + C_2 \int_0^T \|\nabla u_\varepsilon\|^2_{L^2(\Omega)} \text{ for all } T > 0,
\]
where $\theta = \frac{3(2+3\alpha)}{1+3\alpha}$, $C_1$ and $C_2$ are positive constants independent of $\varepsilon$. Next, by (3.1), we derive that

$$C_2 \int_0^T \|n_\varepsilon\|_{L^{2+6\alpha}(\Omega)}^{2+6\alpha} \, dt \leq C_2 \int_0^T \|n_\varepsilon\|_{L^{2+6\alpha}(\Omega)}^{2+6\alpha+1} \, dt \leq C_2 \int_0^T \|n_\varepsilon\|_{L^{2+6\alpha}(\Omega)}^{\frac{6\alpha+1}{3}} \frac{\frac{3(6\alpha+2)}{3(2+6\alpha)}}{3(2+6\alpha)} \, dt \leq C_4(T+1) \text{ for all } T > 0 \quad \blacksquare$$

by using (4.2).

5. **Regularity properties of time derivatives.** To prepare our subsequent compactness properties of $(n_\varepsilon, c_\varepsilon, m_\varepsilon, u_\varepsilon)$ by means of the Aubin-Lions lemma (see Simon [27]), we use Lemmas 3.1-4.2 to obtain the following regularity property with respect to the time variable.

**Lemma 5.1.** Let $\alpha > 0$, (1.10) and (1.9) hold. Then for any $T > 0$, one can find $C > 0$ independent of $\varepsilon$ such that

$$\int_0^T \|\partial_\tau n_\varepsilon(\cdot, t)\|_{W^{1,2+6\alpha}(\Omega)^*} \, dt \leq C(T+1), \quad (5.1)$$

$$\int_0^T \|\partial_\tau c_\varepsilon(\cdot, t)\|_{W^{1,4}(\Omega)^*} \, dt \leq C(T+1) \quad (5.2)$$

as well as

$$\int_0^T \|\partial_\tau m_\varepsilon(\cdot, t)\|_{W^{1,4}(\Omega)^*} \, dt \leq C(T+1) \quad (5.3)$$

and

$$\int_0^T \|\partial_\tau u_\varepsilon(\cdot, t)\|_{W^{1,4}(\Omega)^*} \, dt \leq C(T+1). \quad (5.4)$$

**Proof.** Firstly, an elementary calculation ensures that

$$1 < \frac{2+6\alpha}{2+3\alpha} < \min\{3\alpha+1, 2\} \quad \text{and} \quad \frac{2+6\alpha}{2+3\alpha} < \frac{12\alpha+4}{3\alpha+4} \quad (5.5)$$

by using $\alpha > 0$. Next, testing the first equation of (2.1) by certain $\varphi \in C^\infty(\Omega)$, we have

$$\left\| \int_\Omega (n_\varepsilon) \varphi \right\| \leq \|\partial_\tau n_\varepsilon \cdot \varphi + \frac{1}{1+\varepsilon c_\varepsilon} S_\varepsilon(x, n_\varepsilon, c_\varepsilon) \nabla c_\varepsilon \cdot \nabla \varphi \|_{L^{2+6\alpha}(\Omega)} \leq \left\{ \|\nabla n_\varepsilon\|_{L^{2+6\alpha}(\Omega)} + \|n_\varepsilon S_\varepsilon(x, n_\varepsilon, c_\varepsilon) \nabla c_\varepsilon\|_{L^{2+6\alpha}(\Omega)} \right\} \|\varphi\|_{W^{1,2+6\alpha}(\Omega)}$$

$$+ \left\{ \|n_\varepsilon u_\varepsilon\|_{L^{2+6\alpha}(\Omega)} + \|n_\varepsilon m_\varepsilon\|_{L^{2+6\alpha}(\Omega)} \right\} \|\varphi\|_{W^{1,2+6\alpha}(\Omega)} \quad (5.6)$$
for all \( t > 0 \). Along with (4.29), (3.1) and (6.19), further implies that

\[
\int_0^T \| \partial_t n_\varepsilon(\cdot, t) \|_{L^{\beta_0}(\Omega)}^{\beta_0} \, dt \\
\leq C_1 \int_0^T \left\{ \| \nabla n_\varepsilon \|_{L^{\beta_0}(\Omega)} + \left\| \frac{n_\varepsilon S_\varepsilon(x, n_\varepsilon, c_\varepsilon) \nabla c_\varepsilon}{(1 + \varepsilon n_\varepsilon)} \right\|_{L^{\beta_0}(\Omega)} \right\}^{\beta_0} \, dt \\
+ C_1 \int_0^T \left\{ \| n_\varepsilon u_\varepsilon \|_{L^{\beta_0}(\Omega)} + \| n_\varepsilon m_\varepsilon \|_{L^{\beta_0}(\Omega)} \right\}^{\beta_0} \, dt \tag{5.7}
\]

where \( \beta_0 = \frac{2+6\varepsilon}{2+5\varepsilon} \) and \( C_1, C_2 \) are positive constants independent of \( \varepsilon \). Finally, (5.1) is a consequence of (3.2), (4.29), (6.19), (5.5) and the Hölder inequality. Multiplying the second equation as well as the third equation and the fourth equation in (2.1) by \( \varphi \in C^\infty(\bar{\Omega}) \), \( \varphi \in C^\infty(\Omega) \) and \( \varphi \in C_{0,\sigma}^\infty(\bar{\Omega}; \mathbb{R}^3) \), respectively, we obtain (5.2)–(5.4) in a completed similar manner (see [43, 52] for details).

6. **The proof of Theorem 1.1.** In order to prove Theorem 1.1, we first define the weak solution of four-component Keller-Segel-Navier-Stokes system (1.4).

**Definition 6.1.** Let \( T > 0 \) and \( (u_0, c_0, m_0, u_0) \) fulfills (1.9). Then a quadruple of functions \((n, c, m, u)\) is called a weak solution of (1.4) if the following conditions are satisfied

\[
\left\{ \begin{array}{l}
n \in L_{loc}^1([0, T]; W^{1,1}(\Omega)), \\
c \in L_{loc}^1([0, T]; W^{1,1}(\Omega)), \\
m \in L_{loc}^1([0, T]; W^{1,1}(\Omega)), \\
u \in L_{loc}^1([0, T]; W^{1,1}(\Omega); \mathbb{R}^3), \\
\end{array} \right. \tag{6.1}
\]

where \( n \geq 0, c \geq 0 \) and \( m \geq 0 \) in \( \Omega \times (0, T) \) as well as \( \nabla \cdot u = 0 \) in the distributional sense in \( \Omega \times (0, T) \), moreover,

\[
u \otimes u \in L_{loc}^1(\bar{\Omega} \times [0, \infty); \mathbb{R}^{3 \times 3}) \text{ and } nm \text{ belong to } L_{loc}^1(\bar{\Omega} \times [0, \infty)),
\]

\[
\text{cu, } nu, \text{ and } nS(x, n, c)\nabla c \text{ belong to } L_{loc}^1(\bar{\Omega} \times [0, \infty); \mathbb{R}^3), \tag{6.2}
\]

and

\[
- \int_0^T \int_\Omega n \varphi_t - \int_\Omega n_0 \varphi(\cdot, 0) = - \int_0^T \int_\Omega \nabla n \cdot \nabla \varphi + \int_\Omega \int_0^T \int_\Omega nS(x, n, c)\nabla c \cdot \nabla \varphi \\
+ \int_0^T \int_\Omega nu \cdot \nabla \varphi - \int_\Omega \int_0^T nm \varphi \tag{6.3}
\]

for any \( \varphi \in C_{0,\sigma}^\infty(\bar{\Omega} \times [0, T]) \) satisfying \( \frac{\partial \varphi}{\partial n} = 0 \) on \( \partial \Omega \times (0, T) \) as well as

\[
- \int_0^T \int_\Omega c \varphi_t - \int_\Omega c_0 \varphi(\cdot, 0) = - \int_0^T \int_\Omega \nabla c \cdot \nabla \varphi - \int_0^T \int_\Omega c \varphi \\
+ \int_0^T \int_\Omega m \varphi + \int_0^T \int_\Omega cu \cdot \nabla \varphi \tag{6.4}
\]
for any \( \varphi \in C_0^\infty(\Omega \times [0, T]) \),
\[
- \int_0^T \int_\Omega m \varphi_t - \int_\Omega m \varphi(\cdot, 0) = - \int_0^T \int_\Omega \nabla m \cdot \nabla \varphi - \int_0^T \int_\Omega n \varphi + \int_0^T \int_\Omega m u \varphi
\]
(6.5)
for any \( \varphi \in C_0^\infty(\Omega \times [0, T]) \) and
\[
- \int_0^T \int_\Omega u \varphi - \int_\Omega u \varphi(\cdot, 0) - \kappa \int_0^T \int_\Omega u \otimes u \cdot \nabla \varphi = - \int_0^T \int_\Omega u \cdot \nabla \varphi + \int_0^T \int_\Omega (n + m) \nabla \phi \cdot \varphi
\]
(6.6)
for any \( \varphi \in C_0^\infty(\Omega \times [0, T); \mathbb{R}^3) \) fulfilling \( \nabla \cdot \varphi \equiv 0 \) in \( \Omega \times (0, T) \). If \( (n, c, m, u) : \Omega \times (0, \infty) \rightarrow \mathbb{R}^9 \) is a weak solution of (1.4) in \( \Omega \times (0, T) \) for all \( T > 0 \), then we call \( (n, c, m, u) \) a global weak solution of (1.4).

With the help of a priori estimates (see Lemmas 4.1–4.3 and 5.1), by extracting suitable subsequences in a standard way (see also [43]), we could see the solution of (1.4) is indeed globally solvable.

The proof of Theorem 1.1. Firstly, due to (4.1) and (4.2), in light of the Gagliardo–Nirenberg inequality, we derive that there exist positive constants \( C_1 \) and \( C_2 \) such that
\[
\int_0^T \int_\Omega |u_\varepsilon|^\frac{10}{3} \leq C_1 \int_0^T \left( \|\nabla u_\varepsilon\|_{L^2(\Omega)}^2 \|u_\varepsilon\|_{L^2(\Omega)}^2 + \|u_\varepsilon\|_{L^2(\Omega)}^4 \right)
\]
(6.7)
so that, according to Lemmas 3.1, 4.1 and 5.1, an application of the Aubin–Lions lemma (see e.g. [27]) provides a sequence \( (\varepsilon_j)_{j \in \mathbb{N}} \subset (0, 1) \) and limit functions \( n, c, m \) and \( u \) such that \( \varepsilon_j \searrow 0 \) as \( j \to \infty \) and such that
\[
n_\varepsilon \to n \text{ a.e. in } \Omega \times (0, \infty) \text{ and in } L^r_{\text{loc}}(\Omega \times [0, \infty)) \text{ with } r = \begin{cases} 3 \alpha + 1 & \text{if } 0 < \alpha < \frac{1}{3}, \\ 2 & \text{if } \alpha \geq \frac{1}{3}, \end{cases}
\]
(6.8)
\[
\nabla n_\varepsilon \rightharpoonup \nabla n \text{ in } L^r_{\text{loc}}(\overline{\Omega} \times [0, \infty)) \text{ with } r = \begin{cases} 3 \alpha + 1 & \text{if } 0 < \alpha < \frac{1}{3}, \\ 2 & \text{if } \alpha \geq \frac{1}{3}, \end{cases}
\]
(6.9)
\[
c_\varepsilon \to c \text{ in } L^2_{\text{loc}}(\overline{\Omega} \times [0, \infty)) \text{ and a.e. in } \Omega \times (0, \infty),
\]
(6.10)
\[
m_\varepsilon \to m \text{ in } L^2_{\text{loc}}(\overline{\Omega} \times [0, \infty)) \text{ and a.e. in } \Omega \times (0, \infty),
\]
(6.11)
\[
\nabla c_\varepsilon \rightharpoonup \nabla c \text{ in } L^4_{\text{loc}}(\overline{\Omega} \times [0, \infty)),
\]
(6.12)
\[
u_\varepsilon \to u \text{ in } L^2_{\text{loc}}(\overline{\Omega} \times [0, \infty)) \text{ and a.e. in } \Omega \times (0, \infty),
\]
(6.13)
\[
\nabla c_\varepsilon \rightharpoonup \nabla c \text{ in } L^2_{\text{loc}}(\overline{\Omega} \times [0, \infty)),
\]
(6.14)
\[
\nabla m_\varepsilon \rightharpoonup \nabla m \text{ in } L^2_{\text{loc}}(\overline{\Omega} \times [0, \infty))
\]
(6.15)
as well as
\[
\nabla u_\varepsilon \rightharpoonup \nabla u \text{ in } L^2_{\text{loc}}(\overline{\Omega} \times [0, \infty); \mathbb{R}^3)
\]
(6.16)
and
\[
u_\varepsilon \to u \text{ in } L^{10}_{\text{loc}}(\overline{\Omega} \times [0, \infty))
\]
(6.17)
with some quadruple \((n, c, m, u)\).
In the following, we shall prove \((n,c,m,u)\) is a weak solution of problem (1.4) in Definition 6.1. To this end, recalling (3.1), (4.1) and (4.2), we derive \((c_{\varepsilon})_{\varepsilon>0}\) is bounded in \(L^2((0,T);W^{2,2}(\Omega))\). Thus, by virtue of (5.2) and the Aubin–Lions lemma we derive that the relative compactness of \((c_{\varepsilon})_{\varepsilon>0}\) in \(L^2((0,T);W^{1,2}(\Omega))\). We can pick an appropriate subsequence which is still written as \((\varepsilon_j)_{j\in\mathbb{N}}\) such that \(\nabla c_{\varepsilon_j} \rightarrow z_1\) in \(L^2(\Omega \times (0,T))\) for all \(T \in (0,\infty)\) and some \(z_1 \in L^2(\Omega \times (0,T))\) as \(j \rightarrow \infty\). Hence \(\nabla c_{\varepsilon_j} \rightarrow z_1\) a.e. in \(\Omega \times (0,\infty)\) as \(j \rightarrow \infty\). In view of (6.14) and the Egorov theorem we conclude that \(z_1 = \nabla c\), and whence
\[
\nabla c_{\varepsilon_j} \rightarrow \nabla c \text{ a.e. in } \Omega \times (0,\infty) \tag{6.18}
\]
holds.

Next, \(\alpha > 0\) yields to
\[
r > 1,
\]
where \(r\) is given by (6.8). Therefore, with the help of (6.8)–(6.10), (6.13)–(6.16), we can derive (6.1). Now, by the nonnegativity of \(n_{\varepsilon},c_{\varepsilon}\) and \(m_{\varepsilon}\), we derive \(n,c \geq 0\) and \(m \geq 0\). Next, due to (6.16) and \(\nabla \cdot u_{\varepsilon} = 0\), we conclude that \(\nabla \cdot u = 0\) a.e. in \(\Omega \times (0,\infty)\). Now, by (1.6), (4.29) and (2.2), we derive that
\[
n_{\varepsilon} \frac{1}{1 + \varepsilon n_{\varepsilon}} S_c(x,n_{\varepsilon},c_{\varepsilon}) \leq C_S n_{\varepsilon}.
\]
It is not difficult to verify that
\[
\frac{3\alpha + 4}{12\alpha + 4} = \frac{1}{4} + \frac{3}{12\alpha + 4}.
\]
From this and by (4.29) and (4.2), and recalling the Hölder inequality, we can obtain for some positive constant \(C_1\) such that
\[
\int_0^T \int_\Omega \left[n_{\varepsilon} \frac{1}{1 + \varepsilon n_{\varepsilon}} S_c(x,n_{\varepsilon},c_{\varepsilon}) \nabla c_{\varepsilon} \right|^{\frac{12\alpha + 4}{6\alpha + 4}} \right] \leq C_1 (T + 1), \tag{6.19}
\]
so that, we conclude that
\[
n_{\varepsilon} \frac{1}{1 + \varepsilon n_{\varepsilon}} S_c(x,n_{\varepsilon},c_{\varepsilon}) \nabla c_{\varepsilon} \rightarrow z_2 \text{ in } L^{\frac{12\alpha + 4}{6\alpha + 4}}(\Omega \times (0,T);\mathbb{R}^3) \text{ as } \varepsilon = \varepsilon_j \searrow 0 \tag{6.20}
\]
for each \(T \in (0,\infty)\). On the other hand, it follows from (1.5), (2.2), (6.8), (6.10) and (6.18) that
\[
n_{\varepsilon} \frac{1}{(1 + \varepsilon n_{\varepsilon})} S_c(x,n_{\varepsilon},c_{\varepsilon}) \nabla c_{\varepsilon} \rightarrow nS(x,n,c)\nabla c \text{ a.e. in } \Omega \times (0,\infty) \text{ as } \varepsilon = \varepsilon_j \searrow 0. \tag{6.21}
\]
Again by the Egorov theorem, we gain \(z_2 = nS(x,n,c)\nabla c\), and thus (6.20) can be rewritten as
\[
n_{\varepsilon} S_c(x,n_{\varepsilon},c_{\varepsilon}) \nabla c_{\varepsilon} \rightarrow nS(x,n,c)\nabla c \text{ in } L^{\frac{12\alpha + 4}{6\alpha + 4}}(\Omega \times (0,T);\mathbb{R}^3) \text{ as } \varepsilon = \varepsilon_j \searrow 0 \tag{6.22}
\]
for each \(T \in (0,\infty)\). This together with \(\frac{12\alpha + 4}{3\alpha + 4} > 1\) (by \(\alpha > 0\)) implies the integrability of \(nS(x,n,c)\nabla c\) in (6.2) as well. It is not hard to check that
\[
\frac{2 + 6\alpha}{2 + 3\alpha} > 1 \text{ by } \alpha > 0.
\]
Thereupon, recalling (4.44), we infer that for each \(T \in (0,\infty)\)
\[
n_{\varepsilon} u_{\varepsilon} \rightarrow z_3 \text{ in } L^{\frac{2 + 6\alpha}{2 + 3\alpha}}(\Omega \times (0,T)) \text{ as } \varepsilon = \varepsilon_j \searrow 0. \tag{6.23}
\]
This, together with (6.8) and (6.13), implies
\[ n_\varepsilon u_\varepsilon \to nu \text{ a.e. in } \Omega \times (0, \infty) \text{ as } \varepsilon = \varepsilon_j \searrow 0. \tag{6.24} \]
Along with (6.23) and (6.24), the Egorov theorem guarantees that \( z_3 = nu \), where-upon we derive from (6.23) that
\[ n_\varepsilon u_\varepsilon \rightharpoonup nu \text{ in } L^{2+6\alpha/(2+3\alpha)}(\Omega \times (0, T)) \text{ as } \varepsilon = \varepsilon_j \searrow 0 \tag{6.25} \]
for each \( T \in (0, \infty) \).
Next, one can directly derive from (3.1) and (6.8) that
\[ n_\varepsilon m_\varepsilon \to nm \text{ in } L^1(\Omega \times (0, T)) \text{ as } \varepsilon = \varepsilon_j \searrow 0 \tag{6.26} \]
for each \( T \in (0, \infty) \).
As a straightforward consequence of (6.10), (6.11) and (6.13), it holds that
\[ c_\varepsilon u_\varepsilon \to cu \text{ in } L^1_{\text{loc}}(\Omega \times (0, \infty); \mathbb{R}^3) \text{ as } \varepsilon = \varepsilon_j \searrow 0 \tag{6.27} \]
and
\[ m_\varepsilon u_\varepsilon \to mu \text{ in } L^1_{\text{loc}}(\Omega \times (0, \infty); \mathbb{R}^3) \text{ as } \varepsilon = \varepsilon_j \searrow 0. \tag{6.28} \]
Thus, the integrability of \( nu, nm, mu \text{ and } cu \) in (6.2) is verified by (6.25)–(6.28). Now, following an argument from Lemma 4.1 of [43] (see also [52]), one could prove
\[ Y_\varepsilon u_\varepsilon \otimes u_\varepsilon \to u \otimes u \text{ in } L^1_{\text{loc}}(\Omega \times [0, \infty); \mathbb{R}^{3 \times 3}) \text{ as } \varepsilon = \varepsilon_j \searrow 0. \tag{6.29} \]
Finally, according to (6.8)–(6.11), (6.13)–(6.16), (6.14), (6.26)–(6.29), we may pass to the limit in the respective weak formulations associated with the the regularized system (2.1) and get the integral identities (6.3)–(6.6).

Acknowledgments. The authors are very grateful to the anonymous reviewers for their carefully reading and valuable suggestions which greatly improved this work. This work is partially supported by Shandong Provincial Science Foundation for Outstanding Youth (No. ZR2018JL005), the National Natural Science Foundation of China (No. 11601215) and Project funded by China Postdoctoral Science Foundation (No. 2019T120168, 2019M650927).

REFERENCES

[1] N. Bellomo, A. Belloquid, Y. Tao and M. Winkler, Toward a mathematical theory of Keller–Segel models of pattern formation in biological tissues, Math. Models Methods Appl. Sci., 25 (2015), 1663–1763.
[2] X. Cao, Boundedness in a quasilinear parabolic–parabolic Keller–Segel system with logistic source, J. Math. Anal. Appl., 412 (2014), 181–188.
[3] M. Chae, K. Kang and J. Lee, Global existence and temporal decay in Keller–Segel models coupled to fluid equations, Comm. Partial Differential Equations, 39 (2014), 1205–1235.
[4] R. Duan, A. Lorz and P. A. Markowich, Global solutions to the coupled chemotaxis-fluid equations, Comm. Partial Differential Equations, 35 (2010), 1635–1673.
[5] E. E. Espejo and T. Suzuki, Reaction terms avoiding aggregation in slow fluids, Nonlinear Anal. Real World Appl., 21 (2015), 110–126.
[6] E. E. Espejo and M. Winkler, Global classical solvability and stabilization in a two-dimensional chemotaxis-Navier-Stokes system modeling coral fertilization, Nonlinearity, 31 (2018), 1227–1259.
[7] Y. Giga, Solutions for semilinear parabolic equations in \( L^p \) and regularity of weak solutions of the Navier–Stokes system, J. Differential Equations, 61 (1986), 186–212.
[8] J. Gu and F. Meng, Some new nonlinear Volterra-Fredholm type dynamic integral inequalities on time scales, Appl. Math. Comput., 245 (2014), 235–242.
[9] P. He, Y. Wang and L. Zhao, A further study on a 3D chemotaxis-Stokes system with tensor-valued sensitivity, Appl. Math. Lett., 90 (2019), 23–29.
[10] T. Hillen and K. Painter, A user’s guide to PDE models for chemotaxis, J. Math. Biol., 58 (2009), 183–217.
[11] D. Horstmann, From 1970 until present: the Keller–Segel model in chemotaxis and its consequences, Jahresber. Deutsch. Math.-Verein., 105 (2003), 103–165.
[12] S. Ishida, Global existence and boundedness for chemotaxis–Navier–Stokes system with position-dependent sensitivity in 2D bounded domains, Discrete Contin. Dyn. Syst., 32 (2015), 3463–3482.
[13] S. Ishida, K. Seki and T. Yokota, Boundedness in quasilinear Keller–Segel systems of parabolic–parabolic type on non-convex bounded domains, J. Differential Equations, 256 (2014), 2993–3010.
[14] Y. Ke and J. Zheng, An optimal result for global existence in a three-dimensional Keller–Segel–Navier–Stokes system involving tensor-valued sensitivity with saturation, Calc. Var. Partial Differential Equations, 58 (2019), 27pp.
[15] E. Keller and L. Segel, Model for chemotaxis, J. Theor. Biol., 30 (1971), 225–234.
[16] A. Kiselev and L. Ryzhik, Biomixing by chemotaxis and enhancement of biological reactions, Comm. Partial Differential Equations, 37 (2012), 298–318.
[17] A. Kiselev and L. Ryzhik, Biomixing by chemotaxis and efficiency of biological reactions: The critical reaction case, J. Math. Phys., 53 (2012), 9pp.
[18] J. Li, P. Pang and Y. Wang, Global boundedness and decay property of a three-dimensional Keller–Segel–Stokes system modeling coral fertilization, Nonlinearity, 32 (2019), 2815–2847.
[19] Y. Li and Y. Li, Finite-time blow-up in higher dimensional fully-parabolic chemotaxis system for two species, Nonlinear Anal., 109 (2014), 72–84.
[20] X. Li, Y. Wang and Z. Xiang, Global existence and boundedness in a 2D Keller-Segel-Stokes system with nonlinear diffusion and rotational flux, Commun. Math. Sci., 14 (2016), 1889–1910.
[21] J.-G. Liu and A. Lorz, A coupled chemotaxis–fluid model: Global existence, Ann. Inst. H. Poincaré Anal. Non Linéaire, 28 (2011), 643–652.
[22] J. Liu and Y. Wang, Global weak solutions in a three-dimensional Keller-Segel-Navier-Stokes system involving a tensor-valued sensitivity with saturation, J. Differential Equations, 262 (2017), 5271–5305.
[23] A. Lorz, Coupled chemotaxis fluid equations, Math. Models Methods Appl. Sci., 20 (2010), 987–1004.
[24] K. Painter and T. Hillen, Volume-filling and quorum-sensing in models for chemosensitive movement, Can. Appl. Math. Q., 10 (2002), 501–543.
[25] Y. Peng and Z. Xiang, Global existence and boundedness in a 3D Keller–Segel–Stokes system with nonlinear diffusion and rotational flux, Z. Angew. Math. Phys., 68 (2017), 26pp.
[26] B. Perthame, Transport Equations in Biology, Frontiers in Mathematics, Birkhäuser Verlag, Basel, 2007.
[27] J. Simon, Compact sets in the space $L^p(0; T; B)$, Ann. Mat. Pura Appl. (4), 146 (1986), 65–96.
[28] H. Sohr, The Navier–Stokes Equations. An Elementary Functional Analytic Approach, Birkhäuser Advanced Texts: Basel Textbooks, Birkhäuser Verlag, Basel, 2001.
[29] Y. Tao and M. Winkler, Boundedness in a quasilinear parabolic–parabolic Keller–Segel system with subcritical sensitivity, J. Differential Equations, 252 (2012), 692–715.
[30] Y. Tao and M. Winkler, Locally bounded global solutions in a three-dimensional chemotaxis–Stokes system with nonlinear diffusion, Ann. Inst. H. Poincaré Anal. Non Linéaire, 30 (2013), 157–178.
[31] Y. Tao and M. Winkler, Boundedness and decay enforced by quadratic degradation in a three-dimensional chemotaxis–fluid system, Z. Angew. Math. Phys., 66 (2015), 2555–2573.
[32] J. I. Tello and M. Winkler, A chemotaxis system with logistic source, Comm. Partial Differential Equations, 32 (2007), 849–877.
[33] I. Tuval, L. Cisneros and C. Dombrowski, et al., Bacterial swimming and oxygen transport near contact lines, Proc. Natl. Acad. Sci. USA, 102 (2005), 2277–2282.
[34] Y. Wang and X. Cao, Global classical solutions of a 3D chemotaxis–Stokes system with rotation, Discrete Contin. Dyn. Syst. Ser. B, 20 (2015), 3235–3254.
[35] Y. Wang, L. Liu, X. Zhang and Y. Wu, Positive solutions of an abstract fractional semipositone differential system model for bioprocesses of HIV infection, Applied Math. Comput., 258 (2015), 312–324.
[36] Y. Wang and Z. Xiang, Global existence and boundedness in a Keller–Segel–Stokes system involving a tensor-valued sensitivity with saturation, *J. Differential Equations*, **259** (2015), 7578–7609.

[37] Y. Wang and Z. Xiang, Global existence and boundedness in a Keller–Segel–Stokes system involving a tensor-valued sensitivity with saturation: The 3D case, *J. Differential Equations*, **261** (2016), 4944–4973.

[38] Y. Wang, M. Winkler and Z. Xiang, Global classical solutions in a two-dimensional chemotaxis–Navier–Stokes system with subcritical sensitivity, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, **18** (2018), 421–466.

[39] M. Winkler, Boundedness in the higher-dimensional parabolic–parabolic chemotaxis system with logistic source, *Comm. Partial Differential Equations*, **35** (2010), 1516–1537.

[40] M. Winkler, Global large-data solutions in a chemotaxis–(Navier–)Stokes system modeling cellular swimming in fluid drops, *Comm. Partial Differential Equations*, **37** (2012), 319–351.

[41] M. Winkler, Stabilization in a two-dimensional chemotaxis–Navier–Stokes system, *Arch. Ration. Mech. Anal.*, **211** (2014), 455–487.

[42] M. Winkler, Boundedness and large time behavior in a three-dimensional chemotaxis–Stokes system with nonlinear diffusion and general sensitivity, *Calc. Var. Partial Differential Equations*, **54** (2015), 3789–3828.

[43] M. Winkler, Global weak solutions in a three-dimensional chemotaxis–Navier–Stokes system, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **33** (2016), 1329—1352.

[44] M. Winkler, Does fluid interaction affect regularity in the three-dimensional Keller-Segel System with saturated sensitivity?, *J. Math. Fluid Mech.*, **20** (2018), 1889–1909.

[45] M. Winkler and K. C. Djie, Boundedness and finite-time collapse in a chemotaxis system with volume-filling effect, *Nonlinear Anal.*, **72** (2010), 1044–1064.

[46] F. Xu and L. Liu, On the well-posedness of the incompressible flow in porous media, *J. Nonlinear Sci. Appl.*, **9** (2016), 6371–6381.

[47] C. Xue and H. G. Othmer, Multiscale models of taxis-driven patterning in bacterial populations, *SIAM J. Appl. Math.*, **70** (2009), 133–167.

[48] Q. Zhang and X. Zheng, Global well-posedness for the two-dimensional incompressible chemotaxis–Navier–Stokes equations, *SIAM J. Math. Anal.*, **46** (2014), 3078–3105.

[49] J. Zheng, Boundedness of solutions to a quasilinear parabolic–elliptic Keller–Segel system with logistic source, *J. Differential Equations*, **259** (2015), 120–140.

[50] J. Zheng, Boundedness of solutions to a quasilinear parabolic–parabolic Keller–Segel system with a logistic source, *J. Math. Anal. Appl.*, **431** (2015), 867–888.

[51] J. Zheng, A note on boundedness of solutions to a higher-dimensional quasi-linear chemotaxis system with logistic source, *ZAMM Z. Angew. Math. Mech.*, **97** (2017), 414–421.

[52] J. Zheng, Global weak solutions in a three-dimensional Keller–Segel–Navier–Stokes system with nonlinear diffusion, *J. Differential Equations*, **263** (2017), 2606–2629.

[53] J. Zheng, Boundedness in a two-species quasi-linear chemotaxis system with two chemicals, *Topol. Methods Nonlinear Anal.*, **49** (2017), 463–480.

[54] J. Zheng, An optimal result for global existence and boundedness in a three-dimensional Keller–Segel–Stokes system with nonlinear diffusion, *J. Differential Equations*, **267** (2019), 2385–2415.

[55] J. Zheng, A new result for the global boundedness and decay property of a three-dimensional Keller–Segel–Stokes system modeling coral fertilization, preprint.

[56] J. Zheng, et. al., A new result for global existence and boundedness of solutions to a parabolic–parabolic Keller–Segel system with logistic source, *J. Math. Anal. Appl.*, **462** (2018), 1–25.

[57] J. Zheng and Y. Wang, A note on global existence to a higher-dimensional quasilinear chemotaxis system with consumption of chemotacticant, *Discrete Contin. Dyn. Syst. Ser. B*, **22** (2017), 669–686.

Received June 2019; revised October 2019.

E-mail address: liuling2004@sohu.com
E-mail address: zhengjiashan2008@163.com
E-mail address: baoguigui@163.com