Conformal Mapping of Circular Quadrilaterals and Weierstrass Elliptic Functions

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Abstract. Numerical and theoretical aspects of conformal mappings from a disk to a circular-arc quadrilateral, symmetric with respect to the coordinate axes, are developed. The problem of relating the accessory parameters (prevertices together with coefficients in the Schwarzian derivative) to the geometric parameters is solved numerically, including the determination of the parameters for univalence. The study involves the related mapping from an appropriate Euclidean rectangle to the circular-arc quadrilateral. Its Schwarzian derivative involves the Weierstrass \( \wp \)-function, and consideration of this related mapping problem leads to some new formulas concerning the zeroes and the images of the half-periods of \( \wp \).

Keywords: conformal mapping, accessory parameter, Schwarzian derivative, symmetric circular quadrilateral, Weierstrass elliptic function, Sturm-Liouville problem, spectral parameter power series

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0 Introduction

Conformal mappings of the unit disk onto a symmetric circular arc quadrilateral (s.c.q.) with right angles at the vertices were investigated in [3, 4, 15]. The space of conformal classes of such quadrilaterals is two-dimensional. One parameter may be taken to be the conformal module \( \Pi \) of the quadrilateral, and the other as a real factor \( \lambda \) appearing in the Schwarzian derivative of the conformal mapping, which is a rational function whose general form is classical. The articles cited are principally concerned with the question of calculating these parameters numerically when given the specific geometry of the quadrilateral, and the range of values for which the Schwarzian derivative defines a univalent function.

The aim of this article is to introduce an additional parameter in the problem by allowing a variable angle of \( \alpha \pi \) radians at each vertex, where \( 0 \leq \alpha \leq 2 \). The numerical computations and the univalence criteria are generalized to this context and refined.

An innovation here is to take into account the conformal mapping from a rectangle (rather than the unit disk) to the s.c.q. The advantage of this approach is that the Schwarzian derivative of the mapping takes the simple form \((1/2 - 2\alpha^2)\wp - 2\mu\).

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and the Weierstrass $\wp$-function has well known properties. The parameter $\mu \in \mathbb{R}$ plays the same role as the parameter $\lambda$ in [15]: in rough terms, either of these parameters controls the bending (i.e., curvatures) of the circular arcs bounding the s.c.q. We work out the explicit relationship between $\lambda$ and $\mu$ by making use of the transformation (elliptic integral) of the unit disk onto a rectangle. Let $\pm \exp(\pm it)$ be the prevertices in the unit circle which map to the vertices of the rectangle; it is perhaps surprising that the expression for $\mu$ in terms of $\lambda$ depends on both $t$ and $\alpha$ (see (15) below).

Our use of the $\wp$-function to clarify some aspects of the conformal mapping problem of s.c.q.s happily also sheds some light on the nature of the $\wp$-function itself. It is interesting, and apparently not well known, that the constants $e_1$, $e_2$, and $e_3$ related to the $\wp$-function have simple expressions in terms of trigonometric ratios of $t$ (see (10) below). Further, we make some observations (in 2.2) on the location of the zeros of $\wp$.

It is well known that the conformal mapping of the unit disk onto a specified circular arc polygon can be expressed as a ratio of linearly independent solutions of a second order ordinary differential equation, with some parameters that have to be determined if the polygon has more than three boundary arcs. If the prevertices and the value of $\alpha$ are fixed, then the conformal mapping onto an s.c.q. has one unknown parameter. In [15] the second order differential equation is expressed as a Sturm-Liouville differential equation with parameter $\lambda$, and the spectral parameter power series (SPPS) method is introduced to solve the differential equation. Here we use the SPPS method with parameter $\mu$ for computing the conformal mapping from a rectangle. A ratio of linearly independent solutions of the differential equation is in general (i.e., for any value of $\mu$) conformal locally, but need not be univalent. In Section 4 below we show how the Sturm comparison theorem can be used to find bounds for $\mu$ which are necessary for univalence of the conformal mapping. This makes it possible to find a good “seed value” for the SPPS method, and as explained and demonstrated in [15] it is then possible to find numerically the exact bounds for $\mu$ for which the conformal mapping is univalent.

As an application of the SPPS method (with $\alpha = 0$) we compute the Fuchsian group for the universal covering map of a twice punctured disk. More precisely, for a unit disk punctured at points $-a$ and $a$ for any given value $0 < a < 1$ we are able to find the Möbius transformations of the Fuchsian group by calculating the value of a parameter $s$ related to $a$. This is described in detail in Section 5 below. As an application of the explicit calculation of conformal mappings, some plots are given of geodesics surrounding the two punctures for different values of $a$. 

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1 Description of problem

In this article a symmetric circular quadrilateral (s.c.q.) is a domain $D$ in the complex plane $\mathbb{C}$ bounded by a Jordan curve $\partial D$ which is formed joining four vertices $v, \overline{v}, -v, -\overline{v}$ by circular arcs with common internal angle $\alpha \pi$ at each vertex (Figure 1). By definition an s.c.q. is symmetric in the real and imaginary axes.

1.1 Parameters of s.c.q.s

First we need to describe the geometry of an s.c.q. $D$. Denote by $p_1, p_2$ the midpoints of the rightmost and the uppermost edges of $D$. Let $2\theta_1, 2\theta_2$ be the angles subtended by these edges with respect to their centers $O_1, O_2$, and let $r_1, r_2$ be the corresponding radii. Necessarily

$$0 \leq \alpha \pi \leq 2\pi, \quad 0 < \theta_i \leq \pi. \quad (1)$$

Mappings for $\alpha = 1/2$ were studied in [3, 4, 15], and for $\alpha = 0$ in [21].

Let $p_1, \theta_1$, and $r_1$ be given. To find $v$ we need to know whether the right edge is convex. Let $\kappa_1, \kappa_2$ denote the signed curvatures of the edges, that is,

$$\kappa_1 = \frac{1}{p_1 - O_1}, \quad \kappa_2 = \frac{i}{p_2 - O_2},$$

![Figure 1: Geometric parameters defining symmetric circular quadrilaterals.](image-url)
so \( r_i = 1/|\kappa_i| \). If \( O_i = \infty \) then \( \kappa_i = 0 \) while \( \theta_i \) is not defined.

Let \( \alpha \) also be given. Suppose first that \( \kappa_1 > 0 \). The tangent direction of the right edge at \( v \) leading towards \( p_1 \) is \( e^{i(\theta_1 - \pi/2)} \) (necessarily pointing into the right half-plane), so the direction of the upper edge at \( v \) leading towards \( p_2 \) is \( e^{i(\theta_1 - \pi - \alpha \pi)} \). It follows that the nature of bending of the upper edge is given by \( \text{sgn} \kappa_2 = \text{sgn} (\sin(\theta_1 - \pi/2 - \alpha \pi)) \) (although we cannot yet calculate \( \kappa_2 \)). If \( \text{sgn} \kappa_2 < 0 \), then \( O_2 \) lies on the ray from \( v \) in the direction of \( e^{-i(\theta_1 + \pi/2 + \alpha \pi)} \), whereas if \( \text{sgn} \kappa_2 > 0 \), the direction is \( e^{i(\theta_1 - \alpha \pi)} \). In either event this ray meets the imaginary axis in \( O_2 = i(\text{Im} v - \text{Re} v \tan(\theta_1 - \alpha \pi)) \), and finally the values \( r_2 = |v - O_2| \) and then \( \kappa_2 = (\text{sgn} \kappa_2)/r_2 \) are determined.

Suppose on the other hand that \( \kappa_1 < 0 \). Then the tangent direction of the right edge at \( v \) leading towards \( p_1 \) is \( e^{-i(\theta_1 + \pi/2)} \) (necessarily pointing into the left half-plane), so the direction of the upper edge at \( v \) leading towards \( p_2 \) is now \( e^{-i(\theta_1 + \pi/2 + \alpha \pi)} \). Then \( \text{sgn} \kappa_2 \) is the sign of the imaginary part of this number, and \( O_2 \) is calculated similarly. In summary, starting from

\[
\begin{align*}
  r_1 &= \frac{1}{|\kappa_1|}, \\
  O_1 &= p_1 - \frac{1}{\kappa_1}, \\
  v &= O_1 + r_1 e^{i(\pi/2 + (\text{sgn} \kappa_1)(\theta_1 - \pi/2))},
\end{align*}
\]

one can calculate successively the parameters

\[
\begin{align*}
  \text{sgn} \kappa_2 &= \text{sgn} \sin((\text{sgn} \kappa_1)\theta_1 - (1/2 + \alpha)\pi), \\
  O_2 &= (\text{Im} v - \text{Re} v \tan((\text{sgn} \kappa_1)\theta_1 + \frac{\text{sgn} \kappa_2 - 1}{2} - \alpha \pi)) i, \\
  \kappa_2 &= \frac{\text{sgn} \kappa_2}{|v - O_2|}, \\
  p_2 &= O_2 + \frac{1}{\kappa_2} i, \\
  \theta_2 &= (\text{sgn} \kappa_2) \arg \frac{p_2 - O_2}{v - O_2}.
\end{align*}
\]

If one knows only \( O_2 \) and \( \theta_2 \), there may be two possible upper boundary edges for \( D \), which are complementary arcs of the circle centered at \( O_2 \) and passing through \( v \). Thus \( \text{sgn} \kappa_2 \) is an essential datum. These formulas can also be applied when \( \kappa_1 = 0 \), with the convention that \( \text{sgn} 0 = 0 \).

We have shown that \( D \) is determined by \( p_1, \kappa_1, \theta_1, \) and \( \alpha \). Further, if \( p_1 \) and \( \kappa_1 \) are replaced by 1 and \( p_1 \kappa_1 \), the figure is rescaled. Thus for purposes of defining conformal equivalence classes we may normalize \( p_1 = 1 \) and conclude that there is a three-dimensional space of conformally distinct s.c.q.s.

However, arbitrary values of the parameters given as above will not necessarily define an s.c.q. To begin with, \( \kappa_1 \) and \( \theta_1 \) must be such that the open right edge lies
entirely in the right half plane; for $\kappa_1 > 0$ this requirement is $p_1 - r_1(1 - \cos \theta_1) \geq 0$, while for $\kappa_1 < 0$ any value of $\theta_1$ in $(0, \pi]$ is possible. Further, $\alpha$ must be chosen so that $p_2$ does not lie in the lower half-plane. When $\kappa_1 > 0$, the upper edge leaving $v$ must not point simultaneously into the right half-plane and the lower half-plane, so if $\kappa_1 > 0$, the largest $\alpha \pi$ can be is $3\pi/2 + \theta_1$, in which case the upper edge of $D$ goes horizontally from $v$ to $+\infty$ and continues from $-\infty$ to $-\tau$. When $\kappa_1 < 0$, there are two cases. When $\alpha < 1/2$, $\theta_1$ may attain its maximum value of $\pi$, in which case $v$ coincides with the vertex $\tau$. When $\alpha > 1/2$ the largest possible value of $\theta_1$ is $3\pi/2 - \alpha \pi$, and again the upper edge of $D$ goes horizontally through $\infty$. We will loosely refer to these s.c.q.s as “of extremal type”, when either $v = \tau \in \mathbb{R}$ or $\Im v = \Im p_2$.

1.2 Schwarzian derivative and accessory parameters

Let $\mathbb{D} = \{|z < 1\}$ denote the unit disk. Let $D$ be an s.c.q. and let $f: \mathbb{D} \to D$ be a conformal mapping such that $f(0) = 0$, $f'(0) > 0$. The preimages of the vertices of $D$ are represented as $\pm e^{\pm it}$ where $0 < t < \pi/2$. We will write $\mathbb{D}_t$ when we wish to emphasize the topological quadrilateral whose boundary is $\partial \mathbb{D}$ with these four prevertices prescribed, in the context of conformal mappings from the disk to another topological quadrilateral.

The Schwarzian derivative \[18\] $S_f$ of $f$ can be expressed as $S_f(z) = R_{\alpha,t,\lambda}(z)$ where

\begin{align*}
R_{\alpha,t,\lambda}(z) &= 4(1 - \alpha^2) \psi_{0,t}(z) - 2\lambda \psi_{1,t}(z), \\
\psi_{0,t}(z) &= \frac{(\cos 2t)z^4 - 2z^2 + \cos 2t}{(z^4 - 2(\cos 2t)z^2 + 1)^2}, \\
\psi_{1,t}(z) &= \frac{2 \sin 2t}{z^4 - 2(\cos 2t)z^2 + 1}.
\end{align*}

(in the Appendix of \[21\] the justification for general circle-arc polygons is worked out in detail; however due to a typographic error $(1 - \alpha)^2$ appears in place of $(1 - \alpha^2)$.) Observe that the parameters $\alpha$ and $\lambda$ in $R_{\alpha,t,\lambda}(z)$ separate nicely in the formula (2); this fact will be important in the sequel.

A basic question is to investigate the set of combinations of accessory parameters $(\alpha, t, \lambda)$ for which $f$ is a univalent mapping. The first parameter is the easiest to relate to the geometry of the s.c.q. $\tilde{D}$, being simply $1/\pi$ times the internal angle at the vertices. We will see that relationship between $t, \lambda$ and the other geometric parameters (for instance $\kappa_1, \theta_1$) also involves $\alpha$. It is convenient to do as much as possible with $t$ alone, without considering $\alpha$ and $\lambda$; this will be accomplished in the next section.
2 Mapping from a rectangle

In this section we carry out a change of variable, writing \( f(z) = g(w) \) where \( g: R \to D \) is a conformal mapping from a Euclidean rectangle \( R \) to an s.c.q. \( D \). Taking the domain to be \( R \) instead of \( D \) naturally changes the behavior of the mapping at the vertices. In [20] such mappings from a rectangle were studied for \( \alpha = 0 \) but without consideration of their relationship to the corresponding maps from a disk.

We will fix the prevertices of \( f \) choosing \( 0 < t < \pi/2 \). For \( z \in D \) define the complex elliptic integral

\[
E(z) = \int_0^z \frac{dz}{\sqrt{(z - e^{it})(z + e^{-it})(z + e^{it})(z - e^{-it})}}. \tag{3}
\]

Note that the values \( \omega_1 = E(1) \) and \( \omega_2 = E(i) \) are respectively real and imaginary; it is well known that the image

\[ R = E(D_t) = \{ w: -\omega_1 < \text{Re} \ w < \omega_1, \ -|\omega_2| < \text{Im} \ w < |\omega_2| \} \tag{4} \]

is a rectangle with vertices at \( \pm \omega_3, \pm \overline{\omega_3} \), where \( \omega_3 = \omega_1 + \omega_2 \). We define

\[ g = f \circ E^{-1}, \tag{5} \]

the unique conformal mapping from \( R \) to \( D \) such that \( g(0) = 0, \ g'(0) > 0 \). By construction \( g \) takes vertices of \( R \) to vertices of \( D \).

2.1 The Schwarzian derivative and the \( \wp \)-function

Our main task is to calculate the Schwarzian derivative of \( f \), which by the Chain Rule [18] is equal to

\[ S_f = (S_g \circ E)E'' + S_E. \tag{6} \]

When \( f \) is continued analytically by reflection of \( z \) across edges of \( D_t \) (i.e., maximal arcs of \( \partial D - \{ \text{prevertices} \} \)), the image value \( w = E(z) \) is reflected along edges of \( R \). Even numbers of reflections return \( z \) to its original value and effect translations of \( w \). Since the Schwarzian derivative annihilates Möbius transformations, \( S_E \) is a single valued function on the Riemann sphere and must therefore be a rational function.

To describe \( S_g \) and \( S_f \) explicitly, let us introduce the auxiliary function

\[ \varphi(z) = \wp(E(z) + \omega_3) \tag{7} \]

where \( \wp = \wp_{\omega_1, \omega_2} \) is the Weierstrass \( \wp \)-function [23] with primitive periods \( 2\omega_1, 2\omega_2 \).
As customary, write \( e_i = \varphi(\omega_i), \ i = 1, 2, 3 \). These values are real and satisfy
\[
e_2 < e_3 < e_1. \tag{8}
\]

Let \( Q \) denote the quarter-disk \( Q = \{ z \in \mathbb{D} : \pi < \arg z < 3\pi/2 \} \), considered as a topological quadrilateral with vertices \((-e^{it}, -i, 0, -1)\). Since \( \varphi \) sends the subrectangle of \( R \) with vertices at \((0, \omega_1, \omega_3, \omega_2)\) to the lower half-plane, it follows from (7) that \( \varphi(Q) \) is also the lower half-plane. By applying the Reflection Principle one sees that \( \varphi \) must be a rational function of order 4 and so we can write it as
\[
\varphi(z) = a_0 + \frac{a_1}{z - e^{it}} + \frac{a_2}{z + e^{it}} + \frac{a_3}{z + e^{-it}} + \frac{a_4}{z - e^{-it}}.
\]

Clearly \( \varphi(-z) = \varphi(E(-z) + \omega_3) = \varphi(-E(z) - \omega_3) = \varphi(E(z) + \omega_3) = \varphi(z) \) and similarly \( \varphi(\overline{z}) = \varphi(z) \). From these symmetries we find that \( a_3 = -a_1, \ a_4 = -a_2, \ a_2 = -\overline{a_1} \). Further,
\[
a_0 = \lim_{z \to \infty} \varphi(z) = \varphi(E(\infty) + \omega_3) = \varphi(\omega_3) = e_3
\]

since \( E(\infty) \) is a period. We have so far
\[
\varphi(z) = e_3 + 4\frac{-z^2 \text{Re}(a_1 e^{it}) + \text{Re}(a_1 e^{-it})}{z^4 - 2(\cos 2t)z^2 + 1}.
\]

Since \( \varphi(0) = \varphi(\omega_3) = e_3 \) we have \( \text{Re}(a_1 e^{it}) = 0 \). From this it follows that \( \varphi''(0) = 8 \text{Re}(a_1 e^{it}) \). On the other hand, differentiating the definition (7) and using the classical facts \( \varphi'(\omega_3) = 0, \ \varphi''(\omega_3) = 2(e_3 - e_1)(e_3 - e_2) \) together with \( E'(0) = 1, \ E''(0) = 0 \) we arrive at
\[
\varphi(z) = e_3 - \frac{(e_1 - e_3)(e_3 - e_2)z^2}{z^4 - 2(\cos 2t)z^2 + 1}. \tag{9}
\]

Since \( \varphi \) is completely determined once \( t \) is specified, we are led to seek a formula for \( \varphi(z) \) which does not involve \( e_1, e_2, e_3 \). By the symmetry of \( E \) with respect to reflections in the coordinate axes, we know that \( \varphi(-i) = e_1, \ \varphi(-1) = e_2 \), which reduces to
\[
e_1 = e_3 + 4\sin^2 t,
\]
\[
e_2 = e_3 - 4\cos^2 t.
\]

The resulting relation \( e_1 - e_2 = 4 \), while perhaps surprising, is a consequence of the normalization we have implicitly applied to the periods of \( \varphi \). From the well-known fact \( e_1 + e_2 + e_3 = 0 \) we deduce that
\[
e_1 = 2 - \frac{2}{3} \cos 2t, \tag{10}
e_2 = -2 - \frac{2}{3} \cos 2t,
e_3 = \frac{4}{3} \cos 2t,
\]
and (9) takes the definitive form

\[
\varphi(z) = \frac{4}{3} \cos 2t - \frac{4(\sin^2 2t)z^2}{z^4 - 2(\cos 2t)z^2 + 1}.
\] (11)

We observe that the half-period \( \omega_1 = K \) can be identified with the complete elliptic integral in the terminology of Jacobian elliptic functions [23], and that \((e_3 - e_2)/(e_1 - e_2) = \cos^2 t = k^2\) is the modulus of the function \(\text{sn}(z|k^2)\) which is related to \(\varphi\) by

\[
\varphi(z) = e_2 + (e_1 - e_2)/\text{sn}^2(\sqrt{e_1 - e_2}z)
\] (12)

which in turn by (10) is equal to \((-2 - (2/3) \cos 2t) + 4/\text{sn}^2 2z\).

Next we observe that \(g\) extends by reflection along the edges of \(R\) and is also symmetric under reflection in the coordinate axes. From this it follows that when \(\omega\) is a period of \(\varphi\), the value \(g(\zeta + \omega)\) is related to \(g(\zeta)\) by a Möbius transformation. Therefore \(S_g\) is doubly-periodic with periods \(2\omega_1\) and \(2\omega_2\). A simple local calculation verifies that \(S_g\) has double poles at the vertices of \(R\), and since \(g\) converts the right angles there into angles of \(\alpha \pi\), we conclude

\[
S_g(\zeta) = \frac{1 - 4\alpha^2}{2} \varphi(\zeta + \omega_3) - 2\mu
\] (13)

for some constant \(\mu \in \mathbb{R}\). It is essential for us to see how \(\mu\) depends on \(t\) and on the geometry of the s.c.q. \(D\). From (3) we find that

\[
S_E(z) = \frac{2(\cos 2t)z^4 + (\cos 4t - 5)z^2 + 2 \cos 2t}{(z^4 - 2(\cos 2t)z^2 + 1)^2}.
\]

From this and (6), (7), we obtain

\[
S_f = \frac{1 - 4\alpha^2}{2} \varphi - 2\mu \left( \frac{2}{3} \right) E' + S_E
\] (14)

\[
= \frac{2}{3} \left( 1 - 4\alpha^2 \right) \cos 2t - 2\mu
\]

\[
= \frac{2(\cos 2t)z^4 + (\cos 4t - 2(1 - 4\alpha^2) \sin^2 2t - 5)z^2 + 2 \cos 2t}{(z^4 - 2(\cos 2t)z^2 + 1)^2}
\]

after some manipulation. By comparison with (2) we discover that

\[
\mu = \frac{2(\alpha^2 - 1)}{3} \cos 2t + 2\lambda \sin 2t.
\] (15)

Thus for fixed \(\alpha\) and \(t\), the correspondence \(\lambda \leftrightarrow \mu\) is linear, and \(\lambda, \mu\) increase simultaneously inasmuch as \(0 < t < \pi/2\).

When one has generated a mapping \(f\) via parameters \(\alpha, t, \lambda, \mu\), the corresponding parameters for \(g\) in (13) are obtained by keeping the same value of \(\alpha\), using \(\mu\)
given by (15), and evaluating the integrals $E(1), E(i)$ to determine the defining periods for $\wp$. On the other hand, when working with (13) the data may be $\alpha, \omega_2/\omega_1$ and $\mu$. Then one may proceed as follows.

We will say that a pair of half-periods $\omega_1 \in \mathbb{R}, \omega_2 \in i\mathbb{R}$ is normalized when the corresponding values of the $\wp$-function satisfy $e_1 - e_2 = 4$. A given ratio $\tau = \omega_2/\omega_1 \in i\mathbb{R}$ is realized by a unique normalized pair $(\omega_1, \omega_2)$. To calculate it, one considers the $\wp$-function with (non-normalized) primitive periods $1, \tau$ evaluated at the half-periods and calculates $r = \sqrt{\wp(1/2) - \wp(\tau/2)/4}$, and then takes $\omega_1 = r$, $\omega_2 = r\tau$. This may be done easily via Jacobian theta functions, using the formulas

\[
\wp(1/2) = \frac{\pi^2}{3}(\vartheta_3(0)^4 + \vartheta_4(0)^4),
\]

\[
\wp(\tau/2) = -\frac{\pi^2}{3}(\vartheta_3(0)^4 + \vartheta_2(0)^4),
\]

in the notation of [23], with the elliptic modulus $q = e^{i\pi\tau}$.

We observe that with normalized periods, as $\tau$ runs through the imaginary axis, the triple $(e_1, e_2, e_3) = (2 - e_3/2, -2 - e_3/2, e_3)$ traces the straight segment in $\mathbb{R}^3$ from $(4/3, -8/3, 4/3)$ to $(8/3, -4/3, -4/3)$.

Our point of view for the conformal mapping problem is that although the Schwarzian derivative (13) is a transcendental function, in some respects it is simpler to study than the rational function (2).

### 2.2 The zeros of $\wp$ expressed as elliptic integrals

We digress to observe that this setting sheds some light on the location of the zeros of $\wp$. In [6] a formula for the zeros of $\wp$ is given in terms of generalized hypergeometric functions. The starting point for the derivation of this formula is a formula in [7] which is an improper integral involving the well known modular forms $E_6(\tau)$ and $\Delta(\tau)$. The latter formula may in turn be derived from the elliptic integral

\[
z_0(\tau) = \pm \int_0^\infty \frac{dx}{\sqrt{4x^3 - g_2x - g_3}}
\]

for the zeros $z_0(\tau)$ of $\wp_\tau$, where $4x^3 - g_2x - g_3 = 4(x - e_1)(x - e_2)(x - e_3)$.

We make the observation (not mentioned in [6] and [7]) that another formula for $z_0$ in terms of an elliptic integral is

\[
z_0(\tau) = \pm \frac{1}{\sqrt{e_1 - e_2}} \int_0^{2\sqrt{-1/e_2}} \frac{dv}{\sqrt{1 - v^2} \sqrt{1 - \left(\frac{e_1 - e_2}{e_1 - e_2}\right)v^2}}
\]

(17)
This formula is an immediate consequence of the formula (12). We derive yet another formula for \(z_0\) as an elliptic integral: it follows from (7) that if \(\pi/4 < t < \pi/2\) then the assertion \(\wp(z_0) = 0\) can be expressed as \(\wp(-iy_0) = 0\), where

\[
z_0 = E(-iy_0) + \omega_1 + \omega_2,
\]

for some real number \(y_0\) in the interval \(0 < y_0 < 1\). Thus by (11),

\[
\frac{4}{3} \cos 2t + \frac{4(\sin^2 2t)y_0}{y_0^4 + 2(\cos 2t)y_0^2 + 1} = 0,
\]

which implies

\[
y_0^2 = \frac{-3 + \cos^2 2t + (\sin 2t)\sqrt{9 - \cos^2 2t}}{2 \cos 2t}.
\] (18)

We therefore have

\[
\begin{align*}
z_0 &= \omega_1 + \omega_2 - i \int_0^{y_0} \frac{dx}{\sqrt{x^4 + 2(\cos 2t)x^2 + 1}} \\
&= \omega_1 + \omega_2 - i \int_0^{2y_0/(1+y_0)} \frac{dv}{\sqrt{1 - v^2} \sqrt{1 - (\sin^2 t)v^2}}. \\
\end{align*}
\] (19)

By (18) we find that

\[
\frac{4y_0^2}{(1 + y_0^2)^2} = \frac{4 \cos 2t}{\cos^2 2t + 2 \cos 2t - 3} = \frac{16e_3}{(e_3 + 4)(3e_3 - 4)}.
\]

Consequently, (19) becomes

\[
z_0 = \omega_1 + \omega_2 - i \int_0^{4\sqrt{e_3/((e_3 + 4)(3e_3 - 4))}} \frac{dv}{\sqrt{1 - v^2} \sqrt{1 - (\frac{e_1 - e_3}{4})v^2}}. \\
\] (20)

By analytic continuation, the formulas (17) and (20) are valid in general; i.e., \(\wp(z_0(\tau)|\tau) = 0\) is valid for \(z_0(\tau)\) computed using (17) or (20) with \(\text{Im}\ \tau > 0\), since \(e_1(\tau), e_2(\tau), e_3(\tau)\) and \(z_0(\tau)\) are analytic functions of \(\tau\).

3 Calculation of the image quadrilateral as a function of the accessory parameter

In this section we will carry further the method initiated in [15] by which geometric characteristics of the image s.c.q. are represented as power series in the accessory parameter \(\lambda\) of (2) or \(\mu\) of (13). This is a straightforward application of the recent Spectral Parameter Power Series (SPPS) method for solution of Sturm-Liouville differential equations [14], which we restate here briefly to make the presentation self-contained.
3.1 SPPS method

We will work on the real interval $0 \leq z \leq 1$. One defines the sequence $I_n$ of iterated integrals generated by an arbitrary pair of functions $(q_0, q_1)$ recursively by setting $I_0 = 1$ identically and for $n \geq 1$,

$$I_n(z) = \int_0^z I_{n-1}(\zeta) q_{n-1}(\zeta) d\zeta$$

(21)

where $q_{n+2j} = q_n$ for $j = 1, 2, \ldots$

Proposition 3.1 Let $\psi_0$ and $\psi_1$ be given, and suppose that $y_\infty$ is a nonvanishing solution of

$$y_\infty'' + \psi_0 y_\infty = \lambda_\infty \psi_1 y_\infty$$

on the interval $[0, 1]$. Choose $q_0 = 1/y_\infty^2$, $q_1 = \psi_1 y_\infty^2$ and define $X^{(n)}$, $\tilde{X}^{(n)}$ to be the sequences of iterated integrals generated by $(q_0, q_1)$ and by $(q_1, q_0)$ respectively. Then for each $\lambda \in \mathbb{C}$ the functions

$$y_1 = y_\infty \sum_{k=0}^{\infty} (\lambda - \lambda_\infty)^k \tilde{X}^{(2k)},$$

$$y_2 = y_\infty \sum_{k=0}^{\infty} (\lambda - \lambda_\infty)^k X^{(2k+1)}$$

are linearly independent solutions of the equation

$$y'' + \psi_0 y = \lambda \psi_1 y$$

(22)

on $[0, 1]$. Further, the series for $y_1$ and $y_2$ converge uniformly on $[0, 1]$ for every $\lambda$.

Thus two linearly independent solutions of the second order ordinary differential equation (22), which depend on the parameter $\lambda$, are expressed as explicitly calculable analytic functions of $\lambda$.

3.2 $p_2$ and $\kappa_1$

We now concentrate on the geometric parameters $p_2$ and $\kappa_1$ described in section 1 where the s.c.q. $D$ is normalized by $p_1 = 1$. Suppose that $\alpha, t, \lambda$ are given. The formulation we give here for general $\alpha$ is analogous to the case $\alpha = 1/2$ as presented in [15]. Let $f$ be the solution of $S f = R_{\alpha, t, \lambda}$ in $\mathbb{D}$ (recall (2)) normalized by

$$f(0) = 0, \quad f'(0) = 1, \quad f''(0) = 0,$$

(23)
and suppose that \( f(\mathbb{D}) \) is an s.c.q. similar to \( D \), specifically \( f(\mathbb{D}) = w_1D \) where \( w_1 > 0 \). Further, as is common, take \( y_1 \) to be a solution of (22) in \( \mathbb{D} \) with the normalization \( y_1(0) = 1, y_1'(0) = 0 \), where we use \( \psi_{0,t}, \psi_{1,t} \) in place of \( \psi_0, \psi_1 \) in Proposition 3.1. Then one may calculate the right edge midpoint \( w_1 = f(1) \) via either of the expressions in

\[
f(z) = \int_0^z \frac{1}{y_1(u)^2} du = \frac{y_2(z)}{y_1(z)},
\]

where \( y_2 \) is the solution of (22) normalized by \( y_2(0) = 0, y_2'(0) = 1 \). Similarly, the upper midpoint \( w_2 = f(i) \) can be calculated by integrating along the segment from 0 to \( i \), or more easily, by using \( R_{\alpha,\pi/2-t,-\lambda} \) instead of \( R_{\alpha,t,\lambda} \) and integrating from 0 to 1. We will refer to the corresponding solutions of this latter integration as \( y_1^*, y_2^* \). Now midpoint of the upper edge is seen to be

\[
p_2 = \frac{w_2}{w_1} = \frac{y_1(1)y_2^*(1)}{y_1^*(1)y_2(1)}.
\]

It was observed for example in [4, 15] that the curvature of the right edge of \( f(\mathbb{D}) = w_1D \) is equal to \( y_1(1)^2 - 2y_1(1)y_1'(1) \). This can be verified by writing out the relationship \( |f(z) - O_1||f(z^*) - O_1| = 1/(\text{curvature})^2 \) which relates points \( z, z^* \) symmetric with respect to the unit circle, and examining the limit as \( z \to 1, z^* \to 1 \). The result holds for zero curvature as well although the proof must be modified for that case. The curvature of the right edge of \( D \) is \( w_1 \) times the value for \( w_1D \),

\[
\kappa_1 = y_1(1)y_2(1) - 2y_2(1)y_1'(1).
\]

Let \( \alpha, t \) be fixed. Then how do \( p_2, \kappa_1 \) depend upon \( \lambda \)? Suppose we know a parameter value \( \lambda_\infty \) such that the conformal mapping \( f_{\alpha,t,\lambda_\infty} \) whose Schwarzian derivative is \( R_{\alpha,t,\lambda_\infty} \) is univalent. (The question of finding such \( \lambda_\infty \) will be discussed in section 4.) Then it follows that the solution \( y_\infty \) of (22) with \( \lambda = \lambda_\infty \) and with the normalization \( y_\infty(0) = 1, y_\infty'(0) = 0 \) is nonvanishing in \( \mathbb{D} \), and hence in \([0,1]\). This \( y_\infty \) may be used to define the power series \( y_1[\lambda], y_2[\lambda] \) of Proposition 3.1. The notation \( [\lambda] \) refers here to evaluation at \( z = 1 \) of the function whose parameter is \( \lambda \). Then (25) and (26) give us

\[
p_2(\lambda) = \frac{y_1[\lambda]y_2'[\lambda]}{y_1'[\lambda]y_2[\lambda]} \quad \text{(27)}
\]
\[
\kappa_1(\lambda) = y_1[\lambda]y_2[\lambda] - 2y_2[\lambda]y_1'[\lambda] \quad \text{(28)}
\]

For evaluation of \( p_2(\lambda) \) at a single \( \lambda \) it one may prefer to evaluate the four power series on the right side of (27) separately rather than to work out a single series.

For solving equations of the form \( p_2(\lambda) = c \), one may rewrite this equation as \( y_1[\lambda]y_2'[\lambda] - cy_1'[\lambda]y_2[\lambda] = 0 \), thus avoiding division of power series.
Now we consider the map $g: R \rightarrow D$ given by (5), defined in the rectangle $R$. Let $\eta_1$ and $\eta_2$ be the solutions of

$$\eta''(\zeta) + \left(\frac{1}{4} - \alpha^2\right)\varphi(\zeta + e_3) - \mu \eta(\zeta) = 0$$

with normalizations $(\eta(0), \eta'(0)) = (1, 0)$ and $(0, 1)$ respectively, so

$$g = \frac{\eta_2}{\eta_1}$$

because $g'(0) = f'(0) = 1$ (recall that $E'(0) = 1$ in section 2). We have $g(\omega_1) = w_1, g(\omega_2) = w_2$ so it is easy to express $p_2$. As to the curvature, we note that for $\zeta \in R$, the symmetry is

$$|g(\zeta) - O_1| |g(2\omega_1 - \zeta) - O_1| = \frac{1}{\text{curvature}^2}$$

from which we find the curvature of the right edge of $w_1D$ to be $2\eta_1(\omega_1)\eta_1'(\omega_1)$, which is a simpler expression than for the mapping $f$ defined in the disk. Thus

$$\kappa_1 = \frac{\eta_2(\omega_1)}{\eta_1(\omega_1)}(2\eta_1(\omega_1)\eta_1'(\omega_1)) = 2\eta_2(\omega_1)\eta_1'(\omega_1).$$

Suppose that $\alpha$ and the normalized pair of half-periods $\omega_1, \omega_2$ (or their ratio $\tau$) are given. Analogously to the case of mapping from $\mathbb{D}$, if we have a parameter $\mu_\infty$ for which $g$ is univalent, we can form power series in $\mu$

$$p_2(\mu) = \frac{\eta_2[\mu]\eta_2^*[\mu]}{\eta_1[\mu]\eta_1^*[\mu]}, \quad (31)$$

$$\kappa_1(\mu) = 2\eta_2[\mu]\eta_1'[\mu]. \quad (32)$$

centered at $\mu_\infty$, where $\eta_1[\mu], \eta_2[\mu]$ refer to the values obtained by integrating the differential equation vertically from 0 to the endpoint $\omega_2$.

From the above we are able to calculate $\kappa_1, p_2$ numerically given the accessory parameters of the Schwarzian derivative, both for maps of $\mathbb{D}$ and of $R$. Further, if we assume that $\alpha, t$ or $\alpha, \tau$ are fixed and one of the values $\kappa_1$ or $p_2$ is specified, then we can approximate the parameter $\lambda$ or $\mu$ arbitrarily closely by truncating the corresponding power series to a polynomial of sufficiently high degree and calculating an appropriate real zero. Finally, when the full geometry of $D$ is specified via $\alpha, \kappa_1$, and $p_2$, we can calculate $(t, \lambda)$ or $(\tau, \mu)$ by an iterative procedure in which a guessed value for the modulus $t$ or $\tau$ is corrected by solving for $\lambda$ or $\mu$ to achieve either one of $\kappa_1$ or $p_2$, and then adjusting to make the other value come out correct as well. There is no essential difference to the way this was done in [15] for $\alpha = 1/2$, so we will not give further details here.
Figure 2: Extremal domains for $\alpha = 0.2$, $t = \pi/6$, with $\lambda_{\min} = -0.479608$ (left) and $\lambda_{\max} = 1.30611$ (right). Values near the vertices were plotted by extrapolation.

4 Parameters for univalence

4.1 Interval of univalence for fixed $\alpha, t$

Following the line of investigation of [3, 4, 15], let us assume that $\alpha, t$ are given. We investigate the interval $\lambda_{\min} \leq \lambda \leq \lambda_{\max}$ of values of $\lambda$ for which the map $f = f_{\alpha, t, \lambda}$ is univalent. A picture of the set of univalence pairs $(t, \lambda)$ corresponding to $\alpha = 1/2$ was given in [15].

By the symmetry of s.c.q.s, a univalent $f$ cannot have a pole in $\mathbb{D}$. As an aid to the discussion we introduce the interval $\lambda_{\min}^* < \lambda < \lambda_{\max}^*$ of values of $\lambda$ for which $f$ has no poles in $\mathbb{D}$. Thus

$$\lambda_{\min}^* \leq \lambda_{\min} < \lambda \leq \lambda_{\max}^*.$$  \hspace{1cm} (33)

Recalling the discussion of domains of extremal type in section [1.1] we first

Figure 3: Extremal domains for $\alpha = 1.3$, $t = 3\pi/8$, with $\lambda_{\min} = -1.28089$ (left) and $\lambda_{\max} = 1.84854$ (right).
suppose that $\alpha \leq 1/2$. Then $\lambda = \lambda_{\text{max}}$ produces an extremal domain when the vertices $v = f(e^{i\theta})$ and $f(e^{-i\theta})$ coincide on the positive real axis. When this happens, $v = w_1 + 2r_1$ where as previously, $w_1 = f(1)$ and $\kappa_1 = -1/r_1 < 0$. From elementary geometry it is seen that the center of the upper edge of the s.c.q. $D$ is $O_2 = w_2 - ir_2$ and that $\arg(v - O_2) = -\alpha \pi$. From this it follows that

$$\sin \pi \alpha = \frac{|w_2| - r_2}{r_2}$$

which says

$$|w_2| \kappa_2 - \sin \alpha \pi - 1 = 0$$

since $\kappa_2 = 1/r_2 > 0$. By section 3 we know how to express the left hand side of this equation as a power series in $\lambda$, which we may solve numerically for $\lambda_{\text{max}}$. (The art of locating the zeroes of a polynomial with real coefficients is quite well developed and we will not discuss it here.) Similarly, the value $\lambda = \lambda_{\text{min}}$ for which the upper and lower edges of $D$ degenerate into full circles may be calculated by applying the same procedure using $\pi/2 - t$, $-\lambda$ in place of $t, \lambda$. Once $\lambda_{\text{max}}$ (or $\lambda_{\text{min}}$) is found, it is easy to calculate the conformal mapping. Examples are given in Figure 2 ($\alpha < 1/2$) and Figure 3 ($\alpha > 1/2$).

The remaining case $\alpha > 1/2$ is actually simpler. The upper and lower edges of $D$ degenerate into horizontal rays meeting at $\infty$ precisely when $f(i) = \infty$, i.e., we have $\lambda^\ast_{\text{max}} = \lambda_{\text{max}}$. The degeneration of the right and left edges, $f(1) = \infty$, corresponds to $\lambda^\ast_{\text{min}} = \lambda_{\text{min}}$.

### 4.2 Estimates for $\lambda^\ast_{\text{min}}, \lambda^\ast_{\text{max}}$

There are certain advantages to working with the rectangle map $g$ because the transcendental function $S_g$ in many ways behaves more nicely than the rational function $S_f$. Thus we will investigate the values $\mu^\ast_{\text{min}}, \mu^\ast_{\text{max}}$ which correspond to $\lambda^\ast_{\text{min}}, \lambda^\ast_{\text{max}}$ by (15). Since $\wp$ is real on the coordinate axes as well as on the boundary of the rectangle with vertices $\pm \omega_3, \pm \omega_1$, and since $\wp'$ only vanishes at half-periods which are not periods, it follows from (13) that the real function $S_g$ is one-to-one on the horizontal segment from 0 to $\omega_1$ and also on the vertical segment from 0 to $\omega_2$. This can be made more precise noting that $\wp$ decreases from $e_3$ to $e_2$ on $[\omega_3, \omega_3 + \omega_1]$ while it increases from $e_3$ to $e_1$ on $[\omega_3, \omega_3 + \omega_2]$. In contrast, it appears to be much more difficult to determine bounds for $S_f$ on the coordinate axes in $\mathbb{D}$.

**Lemma 4.1** Let $\eta''(\zeta) + a(\zeta) \eta(\zeta) = 0$ on a real interval $[\zeta_0, \zeta_0 + L]$, and let $\eta(\zeta_0) = 1, \eta'(\zeta_0) = 0$. (i) Suppose $a(\zeta) \leq 0$ for all $\zeta \in [\zeta_0, \zeta_0 + L]$. Then $\eta$ never vanishes. (ii) Suppose $a(\zeta) \geq (\pi/(2L))^2$ for all $\zeta$. Then $\eta$ vanishes at least once in the interval $[\zeta_0, \zeta_0 + L]$.
Proof. The proof is immediate from the Sturm comparison theorem [8]. For (i), note that the similarly normalized solution of 
\[ \eta''(\zeta) + 0 \cdot \eta(\zeta) = 0 \]
is \[ \eta \equiv 1, \]
which never vanishes. For (ii), use comparison with \[ \eta''(\zeta) + (\pi/(2L))^2 \cdot \eta(\zeta) = 0, \]
whose solution \[ \cos((\pi/(2L))(\zeta - \zeta_0)) \] vanishes at \( \zeta_0 + L. \)

Proposition 4.2 Let the accessory parameters \( \alpha \) and \( t \) (or equivalently \( \tau \)) be fixed. (i) If \( \alpha \leq 1/2 \), then

\[
\left( \frac{1}{4} - \alpha^2 \right) e_2 - \left( \frac{\pi}{2\omega_1} \right)^2 \leq \mu_{\text{min}}^*,
\]

\[
\mu_{\text{max}}^* \leq \left( \frac{1}{4} - \alpha^2 \right) e_1 + \left( \frac{\pi}{2|\omega_2|} \right)^2.
\]

(ii) If \( \alpha \geq 1/2 \), then

\[
\left( \frac{1}{4} - \alpha^2 \right) e_3 - \left( \frac{\pi}{2\omega_1} \right)^2 \leq \mu_{\text{min}}^*,
\]

\[
\mu_{\text{max}}^* \leq \left( \frac{1}{4} - \alpha^2 \right) e_3 + \left( \frac{\pi}{2|\omega_2|} \right)^2.
\]

Proof. From (13) we are led to look at 
\[ g = \eta_2/\eta_1 \]
with \( \eta_1 \) the solution of

\[
\eta''(\zeta) + (\frac{1}{4} - \alpha^2) \varphi(\zeta + \omega_3) - \mu \eta(\zeta) = 0 \tag{34}
\]
on the interval \( 0 \leq \zeta \leq \omega_1 \), normalized as in Lemma 4.1. First suppose that \( \alpha \leq 1/2 \). Then by (8) the coefficient of \( \eta \) in (34) satisfies the bounds

\[
\left( \frac{1}{4} - \alpha^2 \right) e_2 - \mu \leq \left( \frac{1}{4} - \alpha^2 \right) \varphi(\zeta + \omega_3) - \mu \leq \left( \frac{1}{4} - \alpha^2 \right) e_3 - \mu,
\]

so \( \eta_1 \) must vanish at some point if \( (\frac{1}{4} - \alpha^2) e_2 - \mu > (\pi/(2\omega_1))^2 \), which tells us that \( \mu \leq \mu_{\text{min}}^* \) for such values of \( \mu \). Thus the first inequality of (i) is established. Further, we note that according to the Lemma, \( \eta_1 \) will never vanish if \( \mu \geq (\frac{1}{4} - \alpha^2) e_3 \).

To examine the behavior in the imaginary direction it is helpful to introduce \( \hat{\eta}(u) = \eta(\zeta) \) where \( \zeta = iu \) for \( 0 \leq u \leq |\omega_2| \). The differential equation is

\[
\hat{\eta}''(u) + (\mu - (\frac{1}{4} - \alpha^2) \varphi(iu + \omega_3)) \hat{\eta}(u) = 0, \tag{35}
\]

so the normalized solution \( \hat{\eta}_1 \) must vanish if \( \mu - (\frac{1}{4} - \alpha^2) e_1 > (\pi/(2|\omega_2|))^2 \), which tells us that \( \mu \geq \mu_{\text{max}}^* \) for such values of \( \mu \). Thus the final inequality of (i) is established. Further, we note that \( \hat{\eta}_1 \) will never vanish if \( \mu \leq (\frac{1}{4} - \alpha^2) e_3 \).

Part (i) has been proved. The proof of (ii) is similar; the main difference is that the inequalities derived from (8) are reversed when \( 1/4 - \alpha^2 \) is negative. \( \square \)
From two facts which we noted in the course of the proof, \( \eta \) and \( \tilde{\eta} \) do not both vanish for a common value of \( \mu \); i.e., \( g \) does not simultaneously have poles on both coordinate axes. The relationship of the value \( (\frac{1}{4} - \alpha^2)e_3 \) to \( \mu_{\text{min}}^* \), \( \mu_{\text{max}}^* \), and univalence appears to merit further investigation.

From (33) we have immediately the following bounds on \( \mu_{\text{min}}, \mu_{\text{max}} \).

**Corollary 4.3** For \( \alpha \leq 1/2, \)

\[
(\frac{1}{4} - \alpha^2)e_2 - \left( \frac{\pi}{2\omega_1} \right)^2 \leq \mu_{\text{min}}, \quad \mu_{\text{max}} \leq (\frac{1}{4} - \alpha^2)e_1 + \left( \frac{\pi}{2|\omega_2|} \right)^2; 
\]

whereas for \( \alpha \geq 1/2, \)

\[
(\frac{1}{4} - \alpha^2)e_3 - \left( \frac{\pi}{2\omega_1} \right)^2 \leq \mu_{\text{min}}, \quad \mu_{\text{max}} \leq (\frac{1}{4} - \alpha^2)e_3 + \left( \frac{\pi}{2|\omega_2|} \right)^2. 
\]

It follows from (33) that \( (1/4 - \alpha^2)e_2 - (\pi/(2\omega_1))^2 \leq \mu_{\text{min}} \) and \( \mu_{\text{max}} \leq (1/4 - \alpha^2)e_1 + (\pi/(2|\omega_2|))^2 \) for all \( \alpha \). The bounds can easily be converted to bounds for \( \lambda_{\text{min}}, \lambda_{\text{max}} \) by means of (15).

Bounds for \( \lambda_{\text{min}}, \lambda_{\text{max}} \) may also be derived in another way, by using Nehari’s theorem [17] which affirms that \( f \) holomorphic in \( \mathbb{D} \) is not univalent when

\[
\sup_{z \in \mathbb{D}} (1 - |z|^2)|S_f(z)| > 6. \tag{36} 
\]

Thus a necessary condition for univalence is \( |S_f(0)| \leq 6 \), but by (14) we have \( S_f(0) = 2(1 - \alpha^2)e_3 - 2\mu \), so we arrive at the following result.

**Proposition 4.4** \( (1 - \alpha^2)e_3 - 3 \leq \mu_{\text{min}} \) and \( \mu_{\text{max}} \leq (1 - \alpha^2)e_3 + 3. \)

Numerically we can find values of \( \alpha, t \) to make the bounds of either one of Propositions 4.2 and 4.4 better than the other (an example is shown in Figure 4). None of the above arguments can be expected to provide sufficient conditions for univalence, since our reasoning only samples a small subset of the domain of the mapping.

A natural candidate for a value of \( \lambda \) within the range of univalence is the average of the upper and lower bounds. Numerical experiments were made with the average of the bounds given by Propositions 4.2 and 4.4, as well as averaging the combined bounds (the maximum of the lower bounds and the minimum of the upper bounds). All of these appear to provide univalent mappings over a wide range of values of \( \alpha \) and \( t \), the best results coming from the Nehari bounds of Proposition 4.4.

Once one has succeeded in locating a value \( \lambda_{\infty} \) (or \( \mu_{\infty} \)) within the range of univalence, i.e., for which \( y_1 \) is a nonvanishing solution of the second order linear
differential equation, the SPPS method enables calculation of \( \kappa_1, w_1, \) etc., to considerable accuracy. (Actually, for the SPPS method to work it is not necessary to be within the domain of univalence, but only for the mapping to have no poles on the axis on which one is integrating.) Thus it is a simple matter to determine precise values of \( \lambda_{\min}, \lambda_{\max} \) numerically, as the nearest zeros of an appropriate power series (approximated by a polynomial) centered at \( \lambda_{\infty} \). This was done in [15] for \( \alpha = 1/2 \) and there is no essential difference for general \( \alpha \).

In Figure 5 we show the \( \lambda \)-domains of univalence for \( \alpha = 0.0, 0.2, 0.4, 0.6, 0.8, \) and 1.0, together with the bounds described above. (The pictures for \( \alpha > 1 \) turn out similar to \( \alpha = 1 \)) Following the presentation in [15], we have applied the transformation \( \arccot 4 \lambda \) to make the domain finite. For \( \alpha > 1/2 \) the bounds become difficult to distinguish from the contours of the domains. Domains for \( \alpha > 1 \) are similar and are not shown. The median curves dividing the these domains of univalence in symmetric halves were devised as follows: The slopes of the boundary curves at the corners \((0,0)\) and \((\pi/2,2\pi)\) were estimated numerically based on the calculated values of \( \lambda_{\min} \) or \( \lambda_{\max} \); write \( \beta \) for the slope at \((0,0)\). Then the median curve is given by the formula

\[
\pi + \beta(t - \frac{\pi}{4}) - \pi(1 - \frac{\beta}{4}) \cos 2t.
\]

This appears nicer than a simple average of the upper and lower contours, and generalizes the dividing curve for \( \alpha = 1/2 \) which is a straight segment as shown in [15]. One may use this median curve as starting values \( \lambda_{\infty} \) for the SPPS method.
Figure 5: Values of \( \text{arccot} \, 4\lambda \) for which the mapping \( f_{\alpha,t,\lambda} (0 < t < \pi/2) \) is univalent in the unit disk. Bounds derived from \( \mu_{\min}^*, \mu_{\max}^* \) are shown with a solid gray line; bounds derived from Nehari’s theorem with a dashed line.

5 Universal cover of a doubly-punctured disk

As an application of the calculation of the accessory parameters of mappings of s.c.q.s, we consider the triply-connected plane domain \( G_a = \mathbb{D} - \{-a, a\} \) where the value \( 0 < a < 1 \) is fixed. The universal covering map \( H: \mathbb{D} \to G_a \) has been studied in [9, 10, 22]. We can assume \( H(0) = 0 \) and \( H'(0) > 0 \). The first question is to calculate or estimate \( H'(0) \). Another is to calculate the generators of the covering group of \( G_a \) corresponding to the covering \( H \). We will use many basic facts about Fuchsian groups and covering spaces of Riemann surfaces, for which background information may be found in [16].

The mapping scheme is depicted in Figure 6. Write

\[
T_1(z) = \frac{iz + 1}{z + i}
\]
Then $T_1$ sends the upper half of $\mathbb{D}$ to its lower half, and when

$$t = \sin^{-1}\left(\frac{1 - a^2}{1 + a^2}\right),$$

(37)

$0 < t < \pi/2$, we have $T_1(e^{it}) = a$. It can thus be seen that $T_1$ sends the upper half plane, punctured at the two points $e^{it}$ and $-e^{-it}$, onto $G_a$.

Now we consider the mapping problem of the disk $\mathbb{D}_t$ to an s.c.q., taking $\alpha = 0$ and $t$ as just defined. Let $\lambda_{\max}$ be the parameter for which $f(e^{it})$ meets $f(e^{-it})$ as in section 4.1, where $f = f_{0,t,\lambda_{\max}}$. Recall that this common vertex may be expressed as $f(e^{it}) = w_1 - 2/\kappa_1$ (note that $\kappa_1 = -|\kappa_1|$ is negative) so we may divide by this value to obtain a mapping sending $e^{it}$ to 1 and sending 1 to $1/(w_1 - 2/\kappa_1)$.

When we apply $T_1$ to this rescaled s.c.q., the part in the upper half-plane is sent to a region in the lower half-plane which may be described as follows. It is the lower half of a domain $D_s \subseteq \mathbb{D}$ which is symmetric in the real axis and bounded by two arcs of $\partial \mathbb{D}$ ending at $\pm e^{\pm is}$ and four arcs orthogonal to $\partial \mathbb{D}$. The latter arcs join 1 to $e^{\pm is}$ and join $-1$ to $-e^{\pm is}$, with tangencies at 1 and $-1$. By the above construction, we can deduce that

$$e^{-is} = T_1\left(\frac{w_1}{w_1 - 2/\kappa_1}\right).$$

(38)

Finally, we define the conformal mapping $H: D_s \rightarrow G_a$ as the composition

$$H(\zeta) = T_1\left(f^{-1}\left((w_1 - \frac{2}{\kappa_1})T_1^{-1}(\zeta)\right)\right).$$

Figure 6: Universal covering mapping of doubly-punctured disk.
From the above discussion, \( H \) sends the upper and lower arcs of \( \partial D \) bounding the circular hexagon \( D_s \) to the upper and lower semicircle of \( \partial D \) respectively, and sends the four remaining arcs into the segments \([-1, -a] \) and \([a, 1]\). Thus \( H \) may be extended by reflection to all of \( D \), and \( D_s \) together with one of its reflected images forms a fundamental domain for a Fuchsian group of the second kind \([16]\) acting on \( D \), generated by two Möbius transformations \( A_1, A_2 \) which preserve \( D \) and satisfy

\[
\text{trace } A_1 = \pm 2, \\
A_1(1) = 1, \\
A_1(e^{-is}) = e^{is}, \\
A_2(-\zeta) = -A_1(\zeta).
\]

The quotient of \( D \) by this Fuchsian group is conformally equivalent to \( G_a \), and the conformal mapping \( H \) (extended to all of \( D \) by reflection) is the covering map.

The questions posed at the beginning of this section may now be answered as follows. Given \( a \), we calculate \( t \) by \([37]\), and find \( \lambda_{\max} \) as in section \([1,1]\). Based on this parameter we determine \( w_1, \kappa_1 \) as described in \([3,2]\). This permits us to obtain \( s \) via \([38]\), and then it is straightforward to find \( A_1 \) and \( A_2 \). Note that \( H(0) = 0 \), because

\[
T^{-1}_1(0) = i, \quad f^{-1}(i(w_1 - 2/\kappa_1)) = i, \quad T_1(i) = 0,
\]

and since

\[
(T_1)'(0) = 2, \quad (f^{-1})'(i(w_1 - 2/\kappa_1)) = \frac{1}{f'(i)}, \quad (T^{-1}_1)'(i) = \frac{1}{2}.
\]
the Chain Rule gives us
\[ H'(0) = \frac{w_1 - 2/\kappa_1}{f'(i)}. \] (39)

In the notation of (24), \( f'(i) = y_1(i)^{-2} \) which we have seen also can be expressed in terms of the power series produced by the SPPS method. Thus we may calculate \( H'(0) \).

There is a unique simple geodesic in the hyperbolic metric of \( G_a \) which loops once around the points \( \pm a \). This geodesic is covered by the deck transformation \( A_1 \circ A_2^{-1} \) of \( \mathbb{D} \), which fixes the points
\[ -1 + e^{i \theta} \pm \sqrt{2 + 2e^{2i \theta}} \]
\[ 1 + e^{i \theta} \] (40)
on \( \partial \mathbb{D} \). The trace of a hyperbolic M"{o}bius transformation serves as a measure of the length of the invariant geodesic which it covers. These data can be calculated for any \( 0 < a < 1 \); sample values are given in Table 1. It is not difficult to see that as \( a \to 1 \), \( A_1 \circ A_2^{-1} \) and the length of the geodesic tend to \( \infty \). However, \( \lim_{a \to 1} H'(0) \) apparently is finite. It would be interesting to see if the values we have calculated could be obtained as limiting cases of the corresponding values for \( \mathbb{D} \) minus two disks (pair of pants) as the radii of these disks tend to zero.

| \( a \) | \( t \) | \( s/\pi \) | \( H'(0) \) | \( \text{tr}A_1 \circ A_2^{-1} \) |
|---|---|---|---|---|
| 0.1 | 1.37146 | 0.32213 | 0.00865 | 11.027 |
| 0.2 | 1.17601 | 0.26104 | 0.58362 | 19.171 |
| 0.3 | 0.98788 | 0.21213 | 2.04028 | 31.388 |
| 0.4 | 0.80978 | 0.17018 | 3.73469 | 51.328 |
| 0.5 | 0.64350 | 0.13330 | 5.31735 | 86.582 |
| 0.6 | 0.48996 | 0.10052 | 6.61668 | 155.776 |
| 0.7 | 0.34934 | 0.07123 | 7.57377 | 314.856 |
| 0.8 | 0.22131 | 0.04497 | 8.19717 | 797.411 |
| 0.9 | 0.10517 | 0.02132 | 8.53008 | 3560.92 |

Table 1: Numerical solution to universal cover of double punctured disk. (About 2.5 seconds of CPU time to compute the entire table).

Figure 8: Simple hyperbolic geodesic surrounding the two punctures of a disk.

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It would also be interesting to determine the maximum of the crossing point of these geodesics with the imaginary axis (Figure 8).

Conformal mapping of s.c.q.s can be useful for investigating many types of finitely generated Fuchsian groups. In general, such groups may define branched coverings rather than true universal coverings; a ramification of order $n$ corresponds to a cycle of vertices in the fundamental domain whose total angle is $2\pi/n$. Thus it is of interest to map circle domains with diverse angles at the vertices.

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