Z Distance Function for KNN Classification

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Abstract—This paper proposes a new distance metric function, called Z distance, for KNN classification. The Z distance function is not a geometric direct-line distance between two data points. It gives a consideration to the class attribute of a training dataset when measuring the affinity between data points. Concretely speaking, the Z distance of two data points includes their class center distance and real distance. And its shape looks like "Z". In this way, the affinity of two data points in the same class is always stronger than that in different classes. Or, the intraclass data points are always closer than those interclass data points. We evaluated the Z distance with experiments, and demonstrated that the proposed distance function achieved better performance in KNN classification.

Index Terms—Distance functions; class attribute; KNN classification

1 INTRODUCTION

Euclidean distance is geometrically the shortest distance between two data points, which is only a spatially direct-line measure [1]. However, this is not a real reachable distance in applications although it has been widely adopted in data analysis and processing applications. We illustrate this with two cases as follows.

Case I. There is a gap, such as a frontier, a river/sea, and a mountain, between in two points, see Figure 1. This gap can often be unbridgeable. In other words, these two points are not able to reach in Euclidean distance time from each other.

![Fig. 1. There is a river between A and B](image)

Case II. Two data points are packed in different bags/shelfs, illustrated in Figure 2. For example, in a clinic doctors always put medical records of benign tumors into a bag, and all medical records of malignant tumors into another bag. This indicates that a medical record concerning a benign tumor is very far from any medical records of malignant tumors.

From the above Cases I and II, the Euclidean distance between two data points may not be the reachable distance between in them. In other words, reachable distance between two data points has been an open problem.

In real data analysis and processing applications [2], the above cases are always ignored in data preparation stage (data collection). For example, when medical records are collected and input to the computer systems in a clinic, data collector often take all the medical records out of from their bags without distinguishing them unlike doctors. After stored in the systems, these medical records look like coming from the same bag, and the natural separation information has passed away. To make data analysis algorithms applicable, the Euclidean distance function or its variants are employed to measure the affinity of data points, whether the Euclidean distance between in two data points is the reachable distance or not.

While the natural separation and unreachable information are missed in data collection stage, we advocate to take the class center distance as the natural separation information, and design a Z distance function for KNN classification. The Z distance of two data points includes their class center distance and real distance. And its shape looks like "Z". If their class center distance is large enough, the affinity of two data points in the same class can always be stronger than that in different classes. In this way, the intraclass data points are always closer than those interclass data points in training datasets. The Z distance is evaluated with experiments, and demonstrated that the proposed distance function achieved better performance in KNN classification.

The rest of this paper is organized as follows. Related work and some concepts are recalled in Section 2. The Z distance is proposed in Section 3. The Z distance is evaluated with experi-
ments in Section 4. This paper is concluded in Section 5.

2 Preliminary

This section first recalls traditional distance functions. And then, those distance functions popularly-used in data mining are briefly discussed. Finally, it simply discussed that Euclidean distance is often not the reachable distance in real applications.

2.1 Traditional distance functions

The demand of distance function is everywhere in real life [3]. For example, when building a railway, we must roughly calculate the required construction materials according to the distance between the two places [4]. The arrival time of express delivery is often related to the distance [5]. High jumpers must take off within the prescribed distance [6]. The height of basketball frame must be unified by height calculation [7]. This shows that the distance function is very important in social life [8]. Common distance functions include Euclidean distance [9], Manhattan distance [10], Chebyshev distance [11], standardized Euclidean distance [12], Mahalanobis distance [13], Bhattacharyya distance [14], Kullback-Leibler divergence [15], Hamming distance [16] and cosine distance [17]. Next, we introduce these distance functions in detail.

Traditional distance metric is defined as follows.

$$d(a, b) = \sqrt{\sum_{j=1}^{n} (a_j - b_j)^2}$$

(1)

where a and b are two sample points and n is the dimension of each sample (i.e., the number of features). In Eq (1), when p = 1, d(a, b) is the Manhattan distance, when p = 2, d(a, b) is the classical Euclidean distance, and when p = infinity, d(a, b) is the Chebyshev distance. These three kinds of distance are the most common distance measures. Euclidean distance is a very intuitive distance measure, which has a wide range of applications. However, it is not suitable for high-dimensional data. Manhattan distance is called city block distance, which is more non intuitive than Euclidean distance, and it is not the shortest path. Chebyshev distance is the maximum distance along the coordinate axis, which can only be applied to specific situations.

In view of the different distribution of each dimension in the data, the standardized Euclidean distance improves the Euclidean distance, i.e., each feature is standardized to have the same mean variance, as shown in the following formula.

$$\bar{X} = \frac{X - \mu}{\sigma}$$

(2)

Each point in X is normalized by the Eq (2), µ is the mean, and σ is the variance. \(\bar{X}\) is the data set after standardization. The standardized Euclidean distance is as follows:

$$d(a, b) = \sqrt{\sum_{j=1}^{n} (\frac{a_j - b_j}{\sigma_j})^2}$$

(3)

From the Eq (3), the standardized Euclidean distance adds \(1/\sigma_j\) to the Euclidean distance, which can be regarded as a weighted Euclidean distance.

Mahalanobis distance is also a variant of Euclidean distance. The Mahalanobis distance is defined as follows.

$$d(a, b) = \sqrt{(a - b)^T S^{-1} (a - b)}$$

(4)

where S is the covariance matrix. It can be seen from Eq (4) that if the covariance matrix is a identity matrix, the Mahalanobis distance is the same as the Euclidean distance. If the covariance matrix is a diagonal matrix, then the Mahalanobis distance is the same as the standardized Euclidean distance. It should be noted that Mahalanobis distance requires that the number of samples of data is greater than the number of dimensions, so that the inverse matrix of covariance matrix S exists. Its disadvantage is computational instability due to covariance matrix S.

The Bhattacharyya distance is a measure of the similarity between two probability distributions, as shown in the following formula.

$$d_{B}(p, q) = -\ln(BC(p, q))$$

(5)

where p and q are the two probability distributions on data X respectively. If it is a discrete probability distribution, then

$$BC(p, q) = \sum_{x \in X} \sqrt{p(x)q(x)}.$$  

If it is a continuous probability distribution, then

$$BC(p, q) = \int \sqrt{p(x)q(x)} dx.$$  

The KL (Kullback-Leibler) divergence is similar to the Bhattacharyya distance. It can also measure the distance or similarity of two probability distributions. As shown in the following formula:

$$KL(p||q) = \sum p(x) \log \frac{p(x)}{q(x)}$$

(6)

$$KL(p||q) = \int p(x) \log \frac{p(x)}{q(x)} dx$$

(7)

Eqs (6) and (7) are the discrete probability distribution and continuous probability distribution, respectively. KL divergence has a wider range of applications relative to Mahalanobis distance.

In data transmission error control coding, Hamming distance is often used to measure the distance between two characters. It describes the number of different values in the two codes. The formula is defined as follows.

$$d(a, b) = \sum_{j=1}^{n} a_j \oplus b_j$$

(8)

where \(\oplus\) is the XOR operation. Both a and b are n-bit codes. For example: a = 11100111, b = 10011001, then the Hamming distance between a and b is d(a, b) = 6. Hamming distance is mostly used in signal processing. It can be used to calculate the minimum operation required from one signal to another.

In addition to the above distance functions, there is also a cosine distance metric. It is derived from the calculation of the cosine of the included angle, as shown in the following formula.

$$d(a, b) = 1 - \frac{a \cdot b}{||a|| \cdot ||b||} = 1 - \frac{\sum_{j=1}^{n} a_j b_j}{\sqrt{\sum_{j=1}^{n} a_j^2} \sqrt{\sum_{j=1}^{n} b_j^2}}$$

(9)

Cosine distance is mostly used in machine learning algorithms to calculate the distance or similarity between two data points. Its value range is [0, 2], which satisfies the non-negativity of the distance function. Its disadvantage is that it only considers the direction of two samples, and does not consider the size of their values.

2.2 Different distance measures for KNN classification

The above distance functions have their own characteristics and applicable scopes, i.e., they are developed for different application requirements. Most of the existing KNN classification algorithms choose Euclidean distance due to its Intuitiveness. They find the K nearest neighbors of the test sample by calculating the
Euclidean distance between the test sample and the training sample. Seoane Santos et al. used KNN to perform missing value interpolation through different distance functions, and verified the effects of different distance functions [18]. Gou et al. proposed a distance function for KNN based on the local mean vector [19]. Specifically, it first finds K nearest neighbors in each class, and uses these neighbors to construct a local mean vector, and each class constructs K local mean vectors. Then it calculates the distance between the test sample and each local mean vector in each class. Finally, it selects the class of the local mean with the smallest distance as the predicted class of the test data. Geler et al. measured the impact of each elastic distance on the weighted KNN classification in time series data [20]. In the experiment, it lists the values of each parameter in detail, compares different elastic distances and verifies that the weighted KNN always outperforms 1NN in time series classification. Poorheravi et al. proposed a triple learning method to perform metric learning [21]. It not only uses hierarchical sampling to build a new triple mining technology, but also analyzes the proposed method on three public data sets. Feng et al. analyzed the performance of the KNN algorithm according to different distance functions, including Chebyshev distance, Euclidean distance and Manhattan distance and cosine distance [22]. In addition, it also compares the performance of some new distance functions. In KNN classification, most of the performance of the new distance function is better than Euclidean distance. Song et al. proposed a parameter-free metric learning method [23]. This method is a supervised metric learning algorithm. Specifically, it discards the cost term, so that there is no need to set the parameters required to adjust the validation set. In addition, it only considers recent imposters, which greatly reduces time costs. In the experiment, it has achieved better results than the traditional nearest neighbor algorithm with large margin. Noh et al. proposed a local metric learning for nearest neighbor classification [24]. It uses the deviation caused by the limited sampling effect to find a suitable local metric, which can reduce the deviation. In addition, it also applies the dimensionality reduction theory to metric learning, which can reduce the time cost of the algorithm. Ying et al. proposed a semi-supervised metric learning method [25]. Specifically, it first uses the structural information of the data to formulate a semi-supervised distance metric learning model. Then it transforms the proposed method into a problem of minimizing symmetric positive definite matrices. Finally, it proposes an accelerated solution method to keep the matrix symmetric and positive in each iteration. Wang et al. proposed a robust metric learning method [26]. This method is an improvement of the nearest neighbor classification with large margin. Its main idea is to use random distribution to estimate the posterior distribution of the transformation matrix. It can reduce the influence of noise in the data, and the anti-noise of the algorithm is verified in experiments. Jiao et al. proposed a KNN classification method based on pairwise distance metric [27]. It uses the theory of confidence function to decompose it into paired distance functions. Then it is adaptively designed as a pair of KNN sub-classifiers. Finally, it performs multi-classification by integrating these sub-classifiers. Song et al. proposed a high-dimensional KNN search algorithm through the Bregman distance [28]. Specifically, it first partitions the total dimensions to obtain multiple subspaces. Then it gets the effective boundary from each partition. Finally, it uses ensemble learning to gather the various partitions. Su et al. learn the meta-distance of a sequence from virtual sequence regression [29]. The meta-distance obtained by the ground measurement makes the sequences of the same category produce smaller values, and the sequences of different categories produce larger values. In addition, it also verified the effectiveness of the proposed algorithm on multiple sequence data sets. Marchang, et al., used KNN to propose a sparse population perception model [30]. It considers spatial correlation and temporal correlation in the algorithm respectively. In addition, the correlation between time and space is also embedded in the proposed method. Experiments have also shown that KNN, which considers the correlation between time and space, has a better effect in the inference of missing data.

Valverde, et al. used KNN for text classification, and carried out the influence of different distance functions on text classification [31]. Faruk Ertugrul, et al. proposed a new distance function [32]. It uses differential evolution method to optimize parameters based on metadata, and applies the proposed distance function to KNN. In addition, it also verified the performance of the algorithm on 30 public data sets. Sun et al. proposed a metric learning for multi-label classification [33]. It is modeled by the interaction between the sample space and the label space. Specifically, it first adopts matrix weighted representation based on component basis. Then it uses triples to optimize the weight of the components. Finally, the effectiveness of the combined metric in multi-label classification is verified on 16 benchmark data sets. Gu et al. proposed a new distance metric for clustering [34]. This method combines the advantages of Euclidean distance and cosine distance. It can be applied to clustering to solve high-dimensional problems. Gong et al. used indexable distance to perform nearest neighbor query [35]. It uses Kd-tree to further improve the search speed of the algorithm. Susan and Kumar proposed a combination of metric learning and KNN for class imbalance data classification [36]. Specifically, it first performs spatial transformation on the data. Then it divides the K test samples into two clusters according to the distance of the two extreme neighbors. Finally, the majority vote rule is used to determine the class label of test data. Although these researchers have proposed some new measurement functions, none of them really takes the natural distance into account in the data, that is, the information of class attributes.

### 2.3 Data collection and reachable distance

From the development of distance functions, different real applications often need different distance functions, which have given birth to various distance functions. It is true that these distance functions are ideal and may not output reachable distances. The main reason is that the data miner and data collector are blind to each other. In other words, data miners believe that the training data are satisfied to their data mining applications. And data collectors take data as detailed as possible, so as to support much more data mining applications. In this way, some natural separation information can be merged into databases, see Case I.

From extant data mining applications, both data collectors and data miners are unaware of that there may be an unbridgeable gap between two data points, i.e., the Euclidean distance between two data points is not the reachable distance between in them. This must lead to that the performance is decreased.

Different from current distance functions, this research proposes a Z distance function, aiming at that the intraclass data points are always closer than those interclass data points in training datasets. The Z distance function finds a clue to developing more suitable distance functions.
3 APPROACH

In this article, we use lowercase letters, lowercase bold letters, and uppercase bold letters to represent scalars, vectors, and matrices, respectively. Assume a given sample data set $X \in \mathbb{R}^{m \times n}$, where m and n represent the number of samples and the number of features, respectively. $a_j$ represents the j-th element in vector a. Let $c_1, c_2, \ldots, c_c$ represent the center point of the c-th classes in a given dataset, respectively. And let $c_a$ be the center point of the class in which sample point a is located, and $c_b$ be the center point of the class in which sample point b is located.

3.1 The Z distance

In the field of data mining, distance metrics are often used to measure the affinity relation of data points, such as classification and clustering. In the KNN classification, the Euclidean distance is most commonly-used to calculate the distance between two points to obtain a neighbor. It could be true that the quality of the KNN classification is largely dependent on the distance metrics formula. If the distance metric formula measures the distance from data of same class is far away, this will result in misclassification. When we look at the traditional distance function, we find that the distance metric only involves the information that the sample point itself has (the value of the feature), and there are many other pieces of information that are not considered. For example, in the classification, each sample has its own classification information except its own feature information.

Considering the above problem, we want to lead some information about the class (e.g., the class center point) into the classification. When we find the class center point for each class, we can use some new distance formulas for classification. We can first get the simplest nearest center point classification, as shown below:

$$\min \{d(t, c_1), d(t, c_2), \ldots, d(t, c_c)\} \quad (10)$$

where $d(t, c_1)$ represents the Euclidean distance from t to $c_1$, t represents the test data point, and $c_1, c_2, \ldots, c_c$ represents the center point of the first to c-th classes. From Eq (10), it can be taken as that in the process of classification, we do not need to set any parameters like K-nearest neighbor classification (such as the selection of K value). In practical applications, it is only necessary to request the distance of the test data to the center point of each class. Then, which distance is closest, the class in which the center point is located is predicted as the class label of the test data.

Although the above-described distance metric function (i.e., Eq (10)) takes the characteristics of the class into account, it is still based on the Euclidean distance to some extent. In addition, the method is poorly separable for the calculated distance, because different class centers may be close to each other or different class centers are the same distance from the test data points, and the classification effect of the algorithm will not be good. In summary, in this paper, we propose a new distance metric function for KNN classification as follows.

**Definition 1.** Let $a$ and $b$ be two sample points, $c_a$ the center point of the class in which sample point $a$ is located, and $c_b$ the center point of the class in which sample point $b$ is located. The $Z_0$ distance between $a$ and $b$ is defined as

$$Z_0(a, b) = d(a, c_a) + d(b, c_b) + \mu \ast d(c_a, c_b) \quad (11)$$

where $d()$ is the Euclidean distance between two points.

It can be seen from Eq (11) that if $a$ and $b$ belong to the same class, then their distance is closer to the metric through the $Z_0$ function. If $a$ and $b$ do not belong to the same class, then their distance is farther through the $Z_0$ function. In other words, by the calculation of the distance function (11), it can make the distance between points of same class smaller than the distance between sample points of different class. In this way, the separability of the class is greatly increased. It is undeniable that this distance measurement function has changed our previous perception of distance. It should be noted that method makes the class and class more separable, but it is true that it increases the distance between similar sample points compared to the traditional method. Because it introduces a class center point, it can be proved from Fig 3, i.e., it makes the original straight line distance into a polyline distance. In response to this small defect, we can improve the $Z_0$ distance in Eq (11) as follows.

**Definition 2.** The $Z$ distance function is defined as

$$Z(a, b) = d(a, b) + \mu \ast d(c_a, c_b) \quad (12)$$

where $\mu$ is a parameter.

It can be seen from Eq (12) that the distance metric function can also make the distance between sample points of same class smaller than the distance between sample points of different class, which is established on a suitable $\mu$ value. This method not only inherits the advantages of Eq (11), but also compensates for its shortcomings to some extent, i.e., the distance between two points in the same class is closer. In other words, it not only makes the data points of different classes farther, but also makes the data points of the same class closer.

**Algorithm 1:** Pseudo code for NCP-KNN.

**Input:** Training set $X \in \mathbb{R}^{n \times d}$, Labels of the training data set $X_{\text{label}} \in \mathbb{R}^{1 \times n}$, Test Data $X_{\text{test}} \in \mathbb{R}^{m \times n}$ and K;

**Output:** Class label of test data;

1. The data set is divided into a training set and a test set by a 10-fold cross-validation;
2. Calculate the class center point $c_1, c_2, c_3, \ldots, c_c$ of all classes in the training set;
3. Calculate the distance between the test data and the center point of the class, taking the smallest of these distances. The class in which the class center point is located is the class label of the test data;

![Fig. 3. The schematic diagram of Eq (11).](image-url)
In order to better describe the proposed Z-distance, we use Figs 3-5 as examples. We will focus on the characteristics of the two new distance functions according to Figs 3-5. Fig 3 shows the distance between data points in Eq (11). From Figure 3, we can find that the distance between two data points of different classes is like a “Z”. Fig 4 shows the distance between data points in Eq (12). From Fig 4, we can find that the distance between two data points of the same class is likely to be smaller than the distance between data points of same class. To avoid it, we introduced the parameter \( \mu \). In most cases, we can see that the distance between data points of same class in Eq (11) is greater than the distance between data points of same classification in Eq (12).

### 3.2 Properties of Z distance

The Z distance function has three basic properties as follows.

**Property 1.** Nonnegativity: \( z(a, b) \geq 0 \)

**Property 2.** Symmetry: \( z(a, b) = z(b, a) \)

**Property 3.** Directness: \( z(a, e) \leq z(a, b) + z(b, e) \)

Now let’s prove these three properties.

Proof of Property 1. For nonnegativity, because Z distance is based on Euclidean distance, it is obvious that the proposed Z distance (Eqs (11) and (12)) is consistent. In Eq (12), only if \( a = b \), \( z(a, b) = 0 \).

Proof of Property 2. For symmetry, both Eqs (11) and (12) are also satisfied, as shown below:

\[
\begin{align*}
  z(a, b) &= d(a, c_a) + d(b, c_b) + d(c_a, c_b) \\
  &= z(b, a) = d(b, c_b) + d(a, c_a) + d(c_b, c_a) \\
  &= \left[ \sum_{j=1}^{n} (a_j - c_{aj})^2 \right]^\frac{1}{2} + \left[ \sum_{j=1}^{n} (b_j - c_{bj})^2 \right]^\frac{1}{2} + \left[ \sum_{j=1}^{n} (c_{aj} - c_{bj})^2 \right]^\frac{1}{2} \\
  &= z(a, b) + \mu * d(c_a, c_b) \\
  &= z(b, a) = d(b, c_b) + d(a, c_a) + d(c_b, c_a) \\
  &= \left[ \sum_{j=1}^{n} (a_j - b_j)^2 \right]^\frac{1}{2} + \mu * \left[ \sum_{j=1}^{n} (c_{aj} - c_{bj})^2 \right]^\frac{1}{2} \tag{14}
\end{align*}
\]

From Eqs (13) and (14), we can see that the proposed Z-distance has symmetry.

Proof of Property 3. For Property 3, i.e., the proposed Z-distance satisfies the directness in the following two cases.

1. When data points \( a, b \) belong to the same class. According to Eq (11), we can get the following formula:

\[
\begin{align*}
  z(a, b) + z(b, e) - z(a, e) &= d(a, c_a) + d(b, c_a) + d(b, c_b) + d(e, c_a) - d(a, c_a) - d(b, c_a) - d(e, c_a) \\
  &= 2 * d(b, c_a) = 2 * \left[ \sum_{j=1}^{n} (b_j - c_{aj})^2 \right]^\frac{1}{2} \geq 0
\end{align*}
\]
According to Eq (12) and trigonometric inequality, we can get the following formula:

\[
Z = d(a, b) + d(b, e) - d(a, e)
\]

and

\[
Z = \frac{1}{2}(a_j - b_j)^2 + \frac{1}{2}(b_j - e_j)^2 \geq 0
\]  

\[j = 1, \ldots, n\]  

(16)

From Eqs (15) and (16), we can get that when a, b and e belong to the same class, the proposed Z distance (Eqs (11) and (12)) have the property of directness.

(2) When data points a, b and e belong to different classes. According to Eq (11) and trigonometric inequality, we can get the following formula:

\[
Z = d(a, b) + d(b, e) - d(a, e) = d(a, c_a) + d(b, c_b) + d(c_b, c_e) + d(c_e, c_a) \\
+ d(b, c_b) + d(e, c_e) + d(c_e, c_b) + d(a, c_a) - d(e, c_e) - \mu * d(c_a, c_e) \\
- d(a, e) - \mu * d(c_a, c_e) + \mu * d(c_a, c_e) \\
+ \mu * d(c_b, c_e) - \mu * d(c_b, c_e) \geq 0
\]  

(17)

According to Eq (12), we can get the following formula:

\[
Z = d(a, b) + d(b, e) - d(a, e) = d(a, c_a) + d(b, c_b) + d(c_b, c_e) + d(c_e, c_a) \\
+ d(b, c_b) + d(e, c_e) + d(c_e, c_b) + d(a, c_a) - d(e, c_e) - \mu * d(c_a, c_e) \\
- d(a, e) - \mu * d(c_a, c_e) + \mu * d(c_a, c_e) \\
+ \mu * d(c_b, c_e) - \mu * d(c_b, c_e) \geq 0
\]  

(18)

According to Eqs (17) and (18), when a, b and e belong to different classes, the Z-distance (Eqs (11) and (12)) has the property of directness. In conclusion, the proposed Z-distance satisfies the property of directness.

**Corollary 1.** Intraclass distance is less than interclass distance.

Proof. For Eq (11), if data points a and b belong to the same class, then the Z distance between them is the following formula:

\[
Z = d(a, b) + d(b, e) - d(a, e) = d(a, c_a) + d(b, c_b) \\
+ \mu * d(c_b, c_e) - \mu * d(c_b, c_e) \geq 0
\]

(19)

where \(c_a\) is the class center of data points a and b. If data points a and e belong to different classes, the Z distance between them is the following formula:

\[
Z = d(a, e) = d(a, c_a) + d(e, c_e) + \mu * d(c_e, c_e) \\
+ \mu * d(c_e, c_e) - \mu * d(c_a, c_e) \geq 0
\]

(20)

From Eqs (19) and (20), we can see that as long as the following equations are proved to be true:

\[
d(e, c_e) + \mu * d(c_e, c_e) > d(b, c_b)
\]

(21)

Obviously, the distance between different classes is one more natural distance than that of the same class, i.e., \(\mu * d(c_e, c_e)\). If the value of parameter \(\mu\) is infinite, then Eq (21) is sure to hold. When \(\mu\) takes a very small value, Eq (21) may not hold. Therefore, if the value of parameter \(\mu\) is large, then Eq (11) satisfies the characteristic that the distance between data of different class is always greater than distance between data of the same class.

Similarly, for Eq (12), we only need to prove that the following formula holds:

\[
d(a, e) + \mu * d(c_a, c_e) > d(a, b)
\]

(22)

We can see that, as in Eq (11), if the value of parameter \(\mu\) is large, Eq (12) satisfies the characteristic that the distance between data of different class is always greater than distance between data of the same class.

### 3.3 Comparative analysis

The Z distance is based on Euclidean distance, which can be regarded as an improvement of Euclidean distance. Compared with Euclidean distance, Z distance not only considers the natural distance, but also makes the distance between different classes greater than the distance between data in the same class. The properties and functions of Euclidean distance and Z distance are listed in Tables 1 and 2, respectively.

The difference between our two distance metric functions and the traditional distance function is shown in Fig 4. The difference between them can be clearly seen from Fig 6, i.e., Eq (11) achieves a larger class spacing, and Eq (12) not only achieves a larger class spacing, but also achieves a smaller distance in the data of same classification.

**Fig. 6. The schematic diagram of three distance function comparisons**

### 4 Experiments

In order to verify the validity of the new distance functions, we compare the KNN classification accuracy of the new distance functions and the original KNN distance function (i.e., Euclidean distance function) with 12 data sets (as shown in Table 3).

#### 4.1 Experiment settings

We download the data sets for our experiments from the datasets website, which includes 4 binary data sets and 8 multiclassification data sets. We divide each data set into a training set and a test set by ten-fold cross-validation (i.e., we divide the data set into 10 parts, 9 of which are used as training sets, and the remaining one is used as a test set, which is sequentially cycled until all data.

1. urlhttp://archive.ics.uci.edu/ml.
2. urlhttp://featureselection.asu.edu/datasets.php.
have been tested). The algorithm is introduced as follows during the experiment:

KNN [37]: It’s the most traditional KNN algorithm, and we don’t have to do anything during the training phase. In the test phase, for each test data point, we find its K neighbors in the training data according to the Euclidean distance. Then, the class label with the highest frequency of class in the K neighbors is selected as the final class label of the data. Until all test data is tested.

NCP-KNN: This method is the most basic algorithm after introducing the class center point. In the training phase, a class center point is obtained for the training data in each class. In the test phase, we calculate the distance between each test data and the center point of the training data. The class of the nearest center point is obtained for the training data in each class. In the test phase, we calculate the center point of each class in the training process. We calculate the distance between each test data and class center point. In the training phase, a class center point is obtained for each class in the training data, and calculate the center point of training data. In the test process, we find K neighbors from training data according to Eq (11). And then, we use the majority rule to predict the class label of test data.

CFKNN [38]: In this method, a new metric function is proposed, which is expressed linearly by test data. Specifically, it first uses training data to represent test data through least squares loss. Then it gets the relational metric matrix by solving the least squares loss. Finally, it uses the new metric matrix to construct a new distance function. The reason for this effect is that the distance function used by the Z-KNN algorithm not only makes the distance between similar data points close, but also makes the distance between data points of different class larger. It makes the data more separable, which makes the subsequent classification better.

LMRKNN [39]: It is an improved KNN method based on local mean vector representation. Specifically, it first finds K neighbors in each class and constructs a local mean vector. Then it uses these local mean vectors to represent each test data and obtains a relationship measurement matrix. Finally, it uses the matrix to construct a new distance function for KNN.

\[ Z_0 \text{-KNN}: \] It is the traditional KNN method based on the Z_0 distance metric function (i.e., Eq (11)). During the training process, we calculate the center point of each class in the training data, and calculate the center point of training data. In the test process, we find K neighbors from training data according to Eq (11). And then, we use the majority rule to predict the class label of test data.

\[ Z \text{-KNN}: \] It is the traditional KNN method based on the Z distance metric function (i.e., Eq (12)). It is basically the same as \( Z_0 \text{-KNN} \)'s training process and testing process. The only difference is that it is based on Eq (12).

For the above algorithms, we did a series of experiments. Specifically, for each data set, we test all the algorithms by setting different K values (i.e., 1-10), where the NCP-KNN algorithm has no K parameter, so we have performed 10 experiments for it. It is convenient for us to put all the algorithms in one subgraph. Finally, we measure their performance based on classification accuracy. In addition, in the case of \( K = 5 \), we performed 10 experiments on all algorithms to preserve the average classification accuracy and standard deviation. Finally, for the binary classification dataset, we not only calculated their classification accuracy, but also calculated their Sensitivity (Sen) and Specificity (Spe).

The formulas for accuracy (Acc), and standard deviation (std) are as follows:

\[ Acc = \frac{X_{\text{correct}}}{X} \]  \hspace{1cm} (23)

where \( X_{\text{correct}} \) represents the number of test data that is correctly classified, and \( X \) represents the total number of test data.

\[ std = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (Acc_i - \mu)^2} \] \hspace{1cm} (24)

where \( n \) represents the number of experiments, \( Acc_i \) represents the classification accuracy of the \( i \)-th experiment, and \( \mu \) represents the average classification accuracy of the experiment. The smaller the std, the more stable the algorithm is.

4.2 Binary classification

Table 4 shows the classification effect of all algorithms on the binary dataset. We can get the same result, i.e., the \( Z \)-KNN algorithm achieves the best results, and Ncp-KNN performs the worst. Specifically, on the German dataset, the classification accuracy of the \( Z \)-KNN algorithm achieves the best results, and Ncp-KNN performs the worst. In the case of \( K = 5 \), we performed 10 experiments on all algorithms to preserve the average classification accuracy and standard deviation.
4.3 Multiple classification

Figure 7 shows the classification accuracy of all algorithms on 12 data sets as K value. Specifically, we can see that the effect of the Z-KNN algorithm is best in most cases from Figure 7. The NCP-KNN algorithm has the worst effect, and the overall effect of the $Z_{0}$-KNN algorithm is not satisfactory, but it achieves the best effect on the Usps dataset, which shows that after we introduce the class feature information, it has a certain effect. The effect of Z-KNN is sufficient to prove that we are looking for a distance function with “high cohesion, low coupling” is very necessary for classification. For the traditional KNN algorithm, its effect is better than the NCP-KNN algorithm, which shows that only considering class information is unreliable.

Table 5 shows the average classification accuracy of the algorithm on the multi-class dataset. From Table 5, we can see that the Z-KNN algorithm achieves the best performance on the
multi-class dataset except the Yeast dataset. The worst performer is
the NCP-KNN algorithm. Table 6 shows the std of all algorithms in 10 experiments. From Table 6, we can see that the std of all
algorithms is relatively small, that is, their stability is very good,
and the difference is not big. Of course, there is also a special
case. On the Letter dataset, the traditional KNN method has a
variance of 9.86, which indicates that its stability on this dataset is
not good. However, overall, the stability of all algorithms is very
good.

4.4 Parameter sensitivity
In Eqs (11) and (12), there is a parameter $\mu$, which determines
the size of the natural distance. Therefore, we set up experiments
with different $K$ values and different $\mu$ values. As shown in Figs
8 and 9, we can see that in most cases, the value of $\mu$ has an
impact on the performance of KNN. Specifically, on the Drift,
Fig. 9. The classification accuracy of different K and µ parameter (in Eq (12)) values on the dataset.

Cnae, and Movements data sets, the accuracy rate varies greatly under different µ values. This shows that one has to adjust the value of parameter µ carefully.

5 Conclusion
This paper has proposed a new distance metric function, Z distance, by considering the characteristics of the class features. Specifically, we first considered the class center point into the distance metric function, and then make the distance between data points of different class larger by calculating the distance between the class center point and the class center point, thus making the data more separable. Finally, the effect of high cohesion was achieved by directly calculating the Euclidean distance between data points in the same classification. In the experiment, the proposed algorithm has achieved good results in both the binary data set and the multi-class data set.
The Z distance measurement function takes some information about the class features into account. Its core idea is to make the distance between data points of different class must be greater than the distance of data points in the same class, i.e., “high coupling, low cohesion.” In the future work, we plan to proceed from the following three points as follows.

1. Finding one or more better distance metric functions to make the K-nearest neighbor classification algorithm achieve better performance.

2. Applying this idea to other classification algorithms to find distance metric functions that are suitable for other classification algorithms.

3. We will find a new distance function to apply to clustering, it is very challenging and interesting.

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REFERENCES

[1] N. Kumar and K. Kummamuru, “Semisupervised clustering with metric learning using relative comparisons,” IEEE Transactions on Knowledge and Data Engineering, vol. 20, no. 4, pp. 496–503, 2008.

[2] X. Zhu, S. Zhang, Y. Zhu, P. Zhu, and Y. Gao, “Unsupervised spectral feature selection with dynamic hyper-graph learning,” IEEE Transactions on Knowledge and Data Engineering, 2020.

[3] X. Zhu, S. Zhang, Li. Ji, Zhang, L. Yang, and Y. Fang, “Low-rank sparse subspace for spectral clustering,” IEEE Transactions on Knowledge and Data Engineering, vol. 31, no. 8, pp. 1532–1543, 2018.

[4] Y. Guo, Z. Cheng, J. Jing, Y. Lin, L. Nie, and M. Wang, “Enhancing factorization machines with generalized metric learning,” IEEE Transactions on Knowledge and Data Engineering, 2020.

[5] Z. Tang, L. Chen, X. Zhang, and S. Zhang, “Robust image hashing with tensor decomposition,” IEEE Transactions on Knowledge and Data Engineering, vol. 31, no. 3, pp. 549–560, 2018.

[6] C. Zhu, L. Cao, Q. Liu, J. Yin, and V. Kumar, “Heterogeneous metric learning of categorical data with hierarchical couplings,” IEEE Transactions on Knowledge and Data Engineering, vol. 30, no. 7, pp. 1254–1267, 2018.

[7] X.-S. Wei, H.-J. Ye, X. Mu, J. Wu, C. Shen, and Z.-H. Zhou, “Multiple instance learning with emerging novel class,” IEEE Transactions on Knowledge and Data Engineering, 2019.

[8] X. Zhu, S. Zhang, W. He, R. Hu, C. Lei, and P. Zhu, “One-step multi-view spectral clustering,” IEEE Transactions on Knowledge and Data Engineering, vol. 31, no. 10, pp. 2022–2034, 2018.
[9] S. P. Patel and S. Upadhyay, “Euclidean distance based feature ranking and subset selection for bearing fault diagnosis,” Expert Systems with Applications, vol. 154, p. 113400, 2020.

[10] X. Gao and G. Li, “A knn model based on manhattan distance to identify rare proteins,” IEEE Access, vol. 8, pp. 112922–112931, 2020.

[11] I. B. K. D. S.NEGARA and I. P. P.WANDAO, “Identifikasi kecokohan motif tenun songket khas jembrana dengan metode manhattan distance,” Jurnal Teknologi Informasi dan Komputer, vol. 7, no. 2, 2021.

[12] K. Chomboon, P. Chujai, P. Teerarassamee, K. Kerdprasop, and N. Kerdprasop, “The impact of distance measures in k-means clustering algorithm for missing data imputation with k-nearest neighbours,” Pattern Recognition, vol. 136, pp. 111–119, 2020.

[13] J. Gou, H. Ma, W. Ou, S. Zeng, Y. Yao, and H. Yang, “A generalized mean distance-based k-nearest neighbor classifier,” Expert Systems with Applications, vol. 115, pp. 356–372, 2019.

[14] Z. Geler, V. Kurbalija, M. Ivonoći, and M. Radovanović, “Weighted knn and constrained elastic distances for time-series classification,” Expert Systems with Applications, vol. 162, p. 113829, 2020.

[15] P. A. Poorheravi, B. Ghojogh, V. Gaudet, F. Karray, and M. Crowley, “Acceleration of large margin metric learning for nearest neighbor classification using triplet mining and stratified sampling,” arXiv preprint arXiv:2009.14244, 2020.

[16] M. Feng, M. Li, and S. Xu, “Project 2: Knn with different distance metrics.”

[17] K. Song, F. Nie, J. Han, and X. Li, “Parameter free large margin nearest neighbor for distance metric learning,” in Proceedings of the AAAI Conference on Artificial Intelligence, vol. 31, no. 1, 2017.

[18] Y. K. Noh, B.-T. Zhang, and D. D. Lee, “Generative local metric learning for nearest neighbor classification,” IEEE transactions on pattern analysis and machine intelligence, vol. 40, no. 1, pp. 106–118, 2017.

[19] S. Ying, Z. Wen, J. Shi, Y. Peng, J. Peng, and H. Qiao, “Manifold preserving: An intrinsic approach for semisupervised distance metric learning,” IEEE transactions on neural networks and learning systems, vol. 29, no. 7, pp. 2731–2742, 2017.

[20] D. Wang and X. Tan, “Robust distance metric learning via bayesian inference,” IEEE Transactions on Image Processing, vol. 27, no. 3, pp. 1542–1553, 2017.

[21] L. Jiao, X. Geng, and Q. Pan, “Bp k nn: k-nearest neighbor classifier with pairwise distance metrics and belief function theory,” IEEE Access, vol. 7, pp. 48935–48947, 2019.

[22] Y. Song, Y. Gu, R. Zhang, and G. Yu, “Brepartition: Optimized high-dimensional knn search with bregman distances,” IEEE Transactions on Knowledge and Data Engineering, 2020.

[23] B. Su and Y. Wu, “Learning meta-distance for sequences by learning a ground metric via virtual sequence regression,” IEEE Transactions on Pattern Analysis and Machine Intelligence, 2020.

[24] N. Marchang and R. Tripathi, “Knn-st: Exploiting spatio-temporal correlation for missing data inference in environmental crowd sensing,” IEEE Sensors Journal, vol. 21, no. 3, pp. 3429–3436, 2020.

[25] L. A. C. Valverde and J. A. M. Arias, “Evaluación de distintas técnicas de representación de texto y medidas de distancia de texto usando knn para clasificación de documentos,” Tecnología en Marcha, vol. 33, no. 1, pp. 64–79, 2020.

[26] O. F. Ertuğrul, “A novel distance metric based on differential evolution,” Arabian Journal for Science and Engineering, vol. 44, no. 11, pp. 9641–9651, 2019.

[27] Y.-P. Sun and M.-L. Zhang, “Compositional metric learning for multi-label classification,” Frontiers of Computer Science, vol. 15, no. 5, pp. 1–12, 2021.

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