Efficient Vertex-Label Distance Oracles for Planar Graphs *

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Abstract. We consider distance queries in vertex labeled planar graphs. For any fixed \(0 < \epsilon \leq 1/2\) we show how to preprocess an undirected planar graph with vertex labels and edge lengths to answer queries of the following form. Given a vertex \(u\) and a label \(\lambda\) return a \((1+\epsilon)\)-approximation of the distance between \(u\) and its closest vertex with label \(\lambda\). The preprocessing time is \(O(\epsilon^{-2} n \lg^3 n)\), the required space is \(O(\epsilon^{-1} n \lg n)\), and the query time is \(O(\lg n \lg(n + \epsilon^{-1}))\). For a directed planar graph with arc lengths bounded by \(N\), the preprocessing time is \(O(\epsilon^{-2} n \lg^3 n \lg(n N))\), the space is \(O(\epsilon^{-1} n \lg n \lg(n N))\), and the query time is \(O(\lg n \lg \lg(n N) + \epsilon^{-1})\).

1 Introduction

Imagine you are driving your car and suddenly see you are about to run out of gas. What should you do? Obviously, you should find the closest gas station. This is the vertex-to-label distance query problem. Various software applications like Waze and Google maps attempt to provide such a functionality. The idea is to preprocess the locations of service providers, such as gas stations, hospitals, pubs and metro stations in advance, so that when a user, whose location is not known a priori, asks for the distance to the closest service provider, the information can be retrieved as quickly as possible.

We study this problem from a theoretical point of view. We model the network as a planar graph with labeled vertices (e.g., a vertex labeled as a gas station). We study distance oracles for such graphs. A vertex-label distance oracle is a data structure that represents the input graph and can be queried for the distance between any vertex and the closest vertex with a desired label. We consider approximate distance oracles, which, for any given fixed parameter \(\epsilon > 0\), return a distance estimate that is at least the true distance queried, and at most \((1 + \epsilon)\) times the true distance (this is known as a \((1 + \epsilon)\)-stretch). One would like an oracle with the following properties; queries should be answered quickly, the oracle should consume little space, and the construction of the oracle should take as little time as possible. We use the notation \(O(S(n))_{\text{space}}, O(T(n))_{\text{time}}\) to express the space requirement and query time of a distance oracle.

The vertex-to-label distance query problem was introduced by Hermelin, Levy, Weimann and Yuster [6]. For any integer \(k \geq 2\), they gave a \(4k - \*

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5)-stretch $O(kn^{1+1/k})_{space}$, $O(k)_{time}$) vertex-label distance oracle (expected space) for undirected general (i.e., non-planar) graphs. This is not efficient when the number $l$ of distinct labels is $o(n^{1/k})$. They also presented a $(2^k - 1)$-stretch $O(kn^{1/k})_{space}$, $O(k)_{time}$) undirected oracle, and showed how to maintain label changes in sub-linear time. Chechik [4] improved the latter two results to $(4k - 5)$-stretch and similar space/time bounds.

For planar graphs, the only vertex-label distance oracle we are aware of was described by Li, Ma and Ning [10]. They construct a $(1 + \epsilon)$-stretch oracle with $O(\epsilon^{-1} n \lg n)_{space}$, $O(\epsilon^{-1} \lg n \lg \rho)_{time}$) bounds for undirected graphs. Here, $\rho$ is the radius of the graph, which can be $\theta(n)$. It is also shown in [10] how to avoid the $\lg \rho$ factor when $\rho = O(\lg n)$. The construction time of their oracle is $O(\epsilon^{-1} n \log^2 n)$.

**Our results and approach** We give a $(1 + \epsilon)$-stretch $O(\epsilon^{-1} n \lg n)_{space}$, $O(\lg \lg n + \epsilon^{-1})_{time}$) vertex-label distance oracle for undirected planar graphs that can be constructed in $O(\epsilon^{-2} n \lg^3 n)$ time. This improves over the query time of [10] by roughly a $\log^2 n$ factor. For directed planar graphs we give a $(1 + \epsilon)$-stretch $O(\epsilon^{-1} n \lg n \lg (nN))_{space}$, $O(\lg \lg n \lg (nN) + \epsilon^{-1})_{time}$) vertex-label distance oracle whose construction time is $O(\epsilon^{-2} n \lg^3 n \lg (nN))$. To the best of our knowledge, no non-trivial directed vertex-label distance oracles were proposed prior to the current work.

Consider a vertex-to-vertex distance oracle for a graph with label set $L$. If the oracle works for general directed graphs then the vertex-to-label problem can be solved easily; add a distinct apex $v_\lambda$ for each label $\lambda \in L$, and connect every $\lambda$-labeled vertex to $v_\lambda$ with a zero length arc. Finding the distance from a vertex $u$ to label $\lambda$ is now equivalent to finding the distance between $u$ and $v_\lambda$.

This approach presents two main difficulties when designing efficient oracles for planar graphs. First, adding apices breaks planarity. In particular, it affects the separability of the graph. Thus, the reduction does not work with oracles that depend on planarity or on the existence of separators, which are more efficient than oracles for general graphs. Second, the reduction uses directed arcs, so it is unsuitable for oracles for undirected graphs. Using arcs in the reduction is crucial since connecting an apex with undirected zero length edges changes the distances in the graph. This is because the apex can be used to teleport between vertices with the same label.\footnote{Teleporting between vertices might be desirable in some applications. For example, calculating the walking distance between two locations without accounting traveling between train stations.}

We nonetheless use this approach, and show how to overcome these obstacles. We augment a directed and an undirected variants of a distance oracle of Thorup [12] for planar graphs. These oracles rely on the existence of fundamental cycle separators in planar graphs, a property that breaks when apices are added to the graph. However, we observe that once the graph is separated, Thorup’s oracle does not depend on planarity. We therefore postpone the addition of the apices till a later stage in the construction of the distance oracle, when
the graph has already been separated. We show that, nonetheless, approximate
distances from any vertex to any label in the entire graph can be approximated.
Moreover, we observe that Thorup’s undirected oracle internally uses the same
directed structures as in his directed oracle. It only depends on the undirect-
edness in making the number and sizes of these structures smaller than in the
directed case. We extend this argument to handle vertex labels.

Additional Related Work We summarize related work on approximate vertex-
vertex distance oracles. For general graphs, no efficient \((2)\)-stretch approximate
vertex-vertex distance oracles are known to date. Thorup and Zwick \cite{13}
presented a \((2k-1)\)-stretch \(O(kn^{1+1/k})\) space, \(O(k)\) time undirected distance oracle
which is constructed in \(O(kmn^{1/k})\) time. Wulff-Nilsen \cite{14} achieved the same
result with preprocessing of \(O(kn^{1+c/k})\) for universal constant \(c\). Several more
improvements of \cite{13} have been found for unweighted or sparse graphs \cite{1, 2, 3}.

In contrast, vertex-vertex oracles for planar graphs with stretch less then 2
have been constructed. Thorup \cite{12} gave a \(O(\epsilon^{-1}n \lg n \lg (nN))\) space, \(O(\lg \lg (nN) + \epsilon^{-1})\) time directed distance oracle, and a \(O(\epsilon^{-1}n \lg n)\) space, \(O(\epsilon^{-1})\) time undirected (simplified) distance oracle. Our result is base on Tho-
rup’s oracles, which are described in Section 3. Klein \cite{9} independently gave an
undirected distance oracle with same bounds. Kawarabayashi, Klein and Som-
mer \cite{7} have shown a \(O(n)\) space, \(O(\epsilon^{-2} \lg^{-2}(n))\) time \((1 + \epsilon)\)-stretch
distance oracle constructed in \(O(n \lg^{2} n)\) time, inspired by \cite{12}. \cite{7} give a trade-off
of \(O(\epsilon^{-1} \lg n)\) space, \(O(r + \sqrt{\epsilon^{-1}} \lg n)\) time oracle algorithms. Kawarabayashi
et al. \cite{8} have shown better tradeoffs for undirected oracles. For the case where
\(N \in \text{poly}(n)\), they achieve \(O^{*}(n \lg n)\) space, \(O^{*}(\epsilon^{-1})\) oracle, where \(O^{*}\) hides
\(\lg(\epsilon^{-1})\) and \(\lg^{*}(n)\) factors.

Roadmap In this extended abstract we focus on the undirected case. The
remainder of this paper is organized as follows. We describe the scheme of the
vertex-to-vertex distance oracle of Thorup in Section 3. In Section 4, we describe
a vertex-labeled oracle for undirected planar graphs. Our description goes into
some of details that cannot be cited and used from \cite{12} due to a minor flaw in the
treatment of the undirected case in \cite{12}. In Appendix B we elaborate on the flaw
in \cite{12} and explain how to it is corrected. Due to space constraints, our vertex-
labeled distance oracle for directed planar graphs is described in Appendix C.
Its construction is similar to the undirected oracle, but relies on some additional
reductions from \cite{12} that we use without change.

2 Preliminaries

Let \(V(G)\) denote the vertex set of a graph \(G\). We use the terms arcs and edges
to distinguish directed and undirected graphs. Let \(A(G) (E(G))\) denote the arc
(edge) set of a directed (undirected) graph. We denote the concatenation of two
paths \(P_1\) and \(P_2\) that share an endpoint by \(P_1 \circ P_2\).
For a simple path $Q$ and a vertex set $U \subseteq V(Q)$, we define $\overline{Q}$, the reduction of $Q$ to $U$ as follows. Repeatedly apply the following procedure to $Q$. Let $uv$ be an edge of $Q$ s.t. $v \notin U$. Contract $uv$, and add the length of $uv$ to the length of the other edge of $Q$ incident to $u$, if such one exists. Note that $|V(\overline{Q})| = O(|U|)$.

Let $T$ be a rooted spanning tree of a graph $G$. For $u \in V(G)$, let $T[u]$ denote the unique root-to-$u$ path in $T$. The fundamental cycle of $e = (u_1, u_2) \notin E(T)$ is the undirected cycle composed of $E(T[u_1]), E(T[u_2])$, and $e$.

Let $L = \{\lambda_i\}_{i=1}^l$ be a set of $l$ labels. A vertex-labeled graph is a graph $G = (V, A)$, equipped with a function $f : V \rightarrow L$. Let $V_\lambda = \{ v \in V(G) \mid f(v) = \lambda \}$ to be set of vertices with label $\lambda$.

Let $G$ be a graph with arc lengths. For $u, v \in V(G)$, let $\delta_G(u, v)$ denote the $u$-to-$v$ distance in $G$. For a vertex-labeled $G$, we define $\delta_G(u, \lambda) = \min_{w \in V_\lambda} \delta_G(u, w)$.

We assume basic familiarity with planar graphs. In particular, it is well known that if $G$ is planar then $|A(G)| = O(|V(G)|)$, and that a simple cycle separates a planar graph $G$ into an interior and an exterior parts.

A vertex-label distance oracle is a data structure that, given a vertex $v \in V$ and a label $\lambda \in L$, outputs an (approximation) of $\delta_G(v, \lambda)$. We note that this problem is a generalization of the basic distance oracle problem in which each vertex is given a unique label. Constructing an $O(nl)$-space vertex-label distance oracle is trivial. Simply precompute and store the distance between each vertex and each possible label. The goal is, therefore, to devise an oracle which requires substantially less than $nl$ space, while allowing for fast queries.

## 3 Thorup’s Approximate Distance Oracle

In this section we outline the distance oracle of Thorup [12]. This is necessary for understanding our results. The oracle we describe differs from the original in [12] in some of the details. See Appendix B for an explanation of the differences.

### 3.1 $\epsilon$-covering sets

The main idea is to store just a subset of the pairwise distances in the graph, from which all distances can be approximately computed efficiently. Given an undirected graph $H$, and a shortest path $Q \subseteq H$, Thorup shows that for every vertex $v \in H$, there exists a set of $O(\epsilon^{-1})$ vertices on $Q$, called connections, such that the distances (called connection lengths) between every vertex of $H$ and its connections on $Q$ can be used to approximate, in $O(\epsilon^{-1})$ time, the length of any shortest path in $H$ that intersects $Q$. Thorup essentially proves the following:

\[ \text{Lemma 1. Let } Q \text{ be a shortest path in an undirected graph } H. \text{ There exist sets } C(u, Q) \text{ of } O(\epsilon^{-1}) \text{ vertices of } Q \text{ for all } u \in H, \text{ where:} \]

1. $C(u, Q)$ are called the connections of $u$ on $Q$. \[ \]

\[ \text{While Lemma 1 true, we believe there is a flaw in the arguments in [12], see Appendix B.} \]
2. The distance between $u$ and a connection $q \in C(u, Q)$ is called the connection length of $u$ and $q$.

3. For every $u, w \in H$, if a shortest $u$-to-$w$ path in $H$ intersects $Q$, then
   \[ \delta_{H^uw}(u, w) \leq (1 + \epsilon)\delta_H(u, w). \]
   Here $H^uw$ is the graph with vertices $u, w$, and the vertices of the reduction of $Q$ to $C(u, Q) \cup C(w, Q)$, and with $u$-to-$Q$ and $Q$-to-$w$ edges whose lengths are the corresponding connection lengths of $C(u, Q)$ and $C(w, Q)$.

Note that, since for every $v$ \(|C(v, Q)| = O(\epsilon^{-1})\), computing $\delta_{H^uw}(u, w)$ can be done in time that only depends on $\epsilon^{-1}$ (in fact in $O(\epsilon^{-1})$ time).

In the remainder of this subsection we prove Lemma 1.

For efficiency reasons, instead of storing exact connection lengths $\delta((\cdot, \cdot))$, the algorithm computes approximate connection lengths, which we denote by $\ell((\cdot, \cdot))$.

This following definition captures the intuitive idea that if a $v$-to-$q$ path that goes through $q^*$ is not too much longer than the shortest $v$-to-$q$ path, then it suffices to store the distance from $v$ to $q^*$ and the distance from $q^*$ to $q$.

**Definition 1.** $q^*$ $\epsilon$-covers $q$ w.r.t. $v$ if $\ell(v, q^*) + \delta(q^*, q) \leq (1 + \epsilon)\delta(v, q)$.

Thurup [12] uses a different notion of covering.\(^3\)

**Definition 2.** $q^*$ quasi-$\epsilon$-covers $q$ w.r.t. $v$ if $\ell(v, q^*) + \delta(q^*, q) \leq \delta(v, q) + \epsilon\ell(v, q^*)$.

Let $Q$ be a path. A set $C$ of vertices of $Q$ is a (quasi-)-$\epsilon$-covering of $Q$ w.r.t. $v$ if for every $q \in Q$ there is a connection $q^* \in C$ that (quasi)-$\epsilon$-covers $q$ w.r.t. $v$.

A covering set is called clean if it is inclusion-wise minimal and ordered if it is sorted by the order of connections along the path $Q$. Observe that by keeping the distance of every $q \in Q$ from the first vertex of $Q$, allows computing $\delta_Q(q, q')$ for any $q, q' \in Q$ in constant time.

The notions of $\epsilon$-covering sets and quasi-$\epsilon$-covering sets are related by the following proposition:

**Proposition 1.** Let $C(v, Q)$ be a quasi $\epsilon$-covering set. For any $0 < \epsilon \leq 1/2$, $C(v, Q)$ is a $2\epsilon$-covering set.

**Proof.** If $q^*$ quasi-$\epsilon$-covers $q$ then $\ell(q^*, v) \leq \frac{1}{1-\epsilon}\delta(q, v) \leq 2\delta(q, v)$. Hence $\delta(q, q^*) + \ell(q^*, v) \leq \delta(q, v) + \epsilon\ell(q^*, v) \leq (1 + 2\epsilon)\delta(q, v)$. Therefore, if $C(v, Q)$ is a quasi-$\epsilon$-covering set, it is a $2\epsilon$-covering set. \(\Box\)

The following lemma shows that, in order to prove Lemma 1, it is suffices that the sets $C(v, Q)$ be $\epsilon$-covering sets of size $O(\epsilon^{-1})$.

**Lemma 2.** (\cite[Lemma 4.1]{4}) Let $u, w$ be vertices in an undirected graph $H$. Let $Q$ be a shortest path in $H$ such that a $u$-to-$w$ shortest path intersects $Q$. Let $C(u, Q), C(w, Q)$ be $\epsilon$-covering sets of $Q$ w.r.t. $u, w$, respectively. Let $H^uw_Q$ be as in the statement of Lemma 1. Then,

\[ \delta_{H^uw}(u, w) \leq (1 + \epsilon)\delta_H(u, w) \tag{1} \]

\(^3\) The term quasi-$\epsilon$-cover is not used by Thorup. He uses $\epsilon$-covers for this notion.

\(^4\) Klein showed this lemma for $\epsilon$-covering sets, while Thorup showed a similar lemma using a different notion of $\epsilon$-covering sets.
Thorup shows how to efficiently construct quasi-$\epsilon$-covering sets. Let $Q$ be a shortest path in an undirected graph $H$. Let $\text{sssp}(Q, H)$ be the smallest number s.t. for any subgraph $H_0$ of $H$, and any vertex $q \in Q_0$, where $Q_0$ is the reduction of $Q$ to $H_0$, we can compute single source shortest paths from $q$ in the graph $Q_0 \cup H_0$ in $O(\text{sssp}(Q, H)|E(H_0)|)$ time. It is easy to see that a standard implementation of Dijkstra’s algorithm with priority queues implies $\text{sssp}(Q, H) = O((\log |E(H)|)$. If $H$ is planar, then $\text{sssp}(Q, H) = O(1)$ by [5].

Lemma 3. ([12, Lemma 3.18]) Given an undirected graph $H$ and shortest path $Q$, quasi $\epsilon$-covering sets of $Q$ with respect to all vertices of $H$, each of size $O(\epsilon^{-1} \log n)$, can be constructed in $O(\epsilon^{-1} \text{sssp}(Q, H)|E(H)| \log(|V(Q)|))$ time.

By Proposition 1 the quasi $\epsilon$-covers produced by Lemma 3 are 2$\epsilon$-covering sets. However, their sizes are too large. The sizes can be decreased using the following thinning procedure. The proof appears in Appendix A.

Lemma 4. Let $Q$ be a path in an undirected graph, and let $v$ be a vertex. Let $D(v, Q)$ be an ordered $\epsilon_0$-cover of $Q$ w.r.t. $v$. For any $\epsilon_1 \leq 1$, a clean and ordered $(2\epsilon_0 + \epsilon_1)$-cover $C(v, Q) \subseteq D(v, Q)$ of size $O(\epsilon_1^{-1})$ can be constructed in $O(|D(v, Q)|)$ time.

Thus, by combining Lemma 3, Proposition 1, and Lemma 4, we get the following corollary, which, along with Lemma 2, establishes Lemma 1.

Corollary 1. Given an undirected graph $H$ and a shortest path $Q$, $\epsilon$-covering sets of $Q$ with respect to all vertices of $H$, each of size $O(\epsilon^{-1}|E(H)| \log(|V(Q)|))$ time.

3.2 The distance oracle

The construction is recursive, using shortest path separators.

Lemma 5. (Fundamental Cycle Separator [11]) Let $H$ be an undirected planar graph with a rooted spanning tree $T$ and function $w$ assigning non-negative weights to edges. One can find an edge $e \notin T$ such that neither the weight strictly enclosed by the fundamental cycle of $e$ nor the weight not enclosed by the fundamental cycle of $e$ exceeds $\frac{1}{6}$ the weight of $H$.

A planar graph $G$ can be decomposed by computing a shortest path tree for an arbitrary vertex, and applying Lemma 5 recursively. Choosing the spanning tree in Lemma 5 to be a shortest path tree guarantees that each fundamental cycle separator found consists of two shortest paths. The decomposition can be represented by a binary tree $T$ in the following manner. See Figure 1 in the appendix for an illustration.

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5 In [12] a thinning procedure is given only for the directed case, and it is claimed that a quasi-$\epsilon$-covering set can be thinned. We believe this is not correct. See Appendix B. Instead, we give here a thinning procedure for $\epsilon$-covering sets (not quasi-$\epsilon$-covering).

6 We refer to the vertices of $T$ as nodes to distinguish them from the vertices of the graph $G$. 
Lemma 5 in the appendix. Suppose. This may pose on $u$. These connections, over all $r$ or type-1 connections of $P$.

Each non-leaf node $r$ of $T$ is associated with the fundamental cycle separator $S_r$ found by invoking Lemma 5 on $G_r$.

Each non-leaf node $r$ has two children, whose associated subgraphs are the interior and exterior of $S_r$. The vertices and edges of the separator belong to both subgraphs.

Let $r$ be a node of $T$. The frame $F_r$ of $G_r$ is the set of (shortest) paths in $\bigcup_{r'}(S_{r'} \cap G_{r'})$, where the union is over strict ancestors $r'$ of $r$ in $T$. Each non-leaf node $r$ stores its frame $F_r$. A standard argument shows that, by alternating the separation criteria between number of edges in the graph and number of paths in the frame, one can get frames consisting of a constant number of paths.

For $r \in T$, let $G_r^o$ denote the subgraph of $G_r \setminus F_r$. That is, $G_r^o$ is the graph obtained from $G_r$ by removing the edges of the frame $F_r$ as well as any vertices of $F_r$ that become isolated as a result of the removal. The sizes of the $G_r^o$'s decrease by a constant factor along $T$, while the sizes of the $G_r$'s need not because there is no bound on the size of the fundamental cycle in Lemma 5. This may pose a problem, since the frame $F_r$ is stored by every node $r$. To overcome this, the algorithm stores the reduction of $F_r$ to $G_r^o$ instead of $F_r$ itself.

Let $u, w$ be vertices of $G$. Let $r_u, r_w$ be the leaves of $T$ such that $u \in G_{r_u}$ and $w \in G_{r_w}$. Let $r$ be the LCA of $r_u$ and $r_w$ in $T$. Observe that $u$ and $w$ are separated by $S_r$. Hence, every $u$-to-$w$ path in $G$ must intersect $S_r$. However, a $u$-to-$w$ path may or may not intersect $F_r$. See Figure 2 in the appendix. Suppose first that a shortest $u$-to-$w$ path $P$ (in $G$) does intersect $F_r$. We write $P = P_0 P_1$. Path $P_0$ is a maximal prefix of $P$ whose vertices belong to $G_r^o$. We call this kind of paths type-0 paths. Note that type-0 paths start at a vertex of $G_r^o$, end at a vertex of $F_r$ and are confined to $G_r^o$. Path $P_1$ consists of the remainder of $P$, and is referred to as a type-1 path. Note that type-1 paths start at a vertex of $F_r \cap G_r^o$, end at a vertex of $G_r^o$, but are not confined to $G_r^o$. It is not difficult to convince oneself that, to be able to approximate $\delta_G(u, w)$, it suffices to keep, for every $Q \subseteq F_r$, connections $C(u, Q)$ of type-0 (i.e. the connection lengths are relative to $G_r^o$, not the entire $G$) and connections $C(w, Q)$ of type-1 (i.e. the connection lengths are relative to the entire graph $G$).

Now suppose that no shortest $u$-to-$w$ path $P$ (in $G$) intersects $F_r$. Then every $u$-to-$w$ path $P$ (in $G$) intersects $S_r$ and is confined to $G_r^o$. Then, to approximate $P$ it suffices to keep, for every $Q \subseteq S_r$, type-0 connections $C(u, Q)$ and $C(w, Q)$.

The distance oracle therefore keeps, for each $r \in T$, for each vertex $u \in G_r^o$:

1. connections $C(u, Q)$ of type-0 for all $Q \in F_r$.
2. connections $C(u, Q)$ of type-1 for all $Q \in F_r$.
3. connections $C(u, Q)$ of type-0 for all $Q \in S_r$.

These connections, over all $u \in G_r$ and all paths in $F_r \cup S_r$ are called the (type-0 or type-1) connections of $r$. In addition, the data structure stores:

- A mapping of each vertex $v \in G$ to a leaf node $r_v \in T$ s.t. $v \in G_{r_v}$.
A least common ancestor data structure over $T$.

The space bottleneck is the size of the sets maintained. Each vertex $v$ belongs to $G^\circ_r$ for $O(lg n)$ nodes $r$ of $T$. For each of the $O(1)$ paths in the frame and separator of each such node $r$, $v$ has a set of $O(\epsilon^{-1})$ connections. Hence the total space required by Thorup’s oracle is $O(\epsilon^{-1} n \lg n)$.

We next describe how a query is performed. Given a $u$-to-$w$ distance query, let $r$ be the least common ancestor of $r_u$ and $r_w$ in $T$. The algorithm computes, for each path $Q$ of $S_r \cup F_r$ the length of a shortest $u$-to-$w$ path that intersects $Q$ using the connections $C(u, Q)$ and $C(w, Q)$ (of both type 0 and type 1). By construction of $T$, the number of such paths $Q$ is constant. It is easy to see that computing the distance estimate for each $Q$ can be done in $O(\epsilon^{-1})$ time. Thus, an $(1 + \epsilon)$-approximate distance is produced in $O(\epsilon^{-1})$ time.

**Efficient construction** We now mention some, but not all, of the details of Thorup’s $O(\epsilon^{-2} n \lg^3 n)$-time construction algorithm. Refer to [12, subsection 3.6] for the full details. The computation of the connections and connection lengths is done top-down the decomposition tree $T$. Naively using Corollary 1 on $G^\circ_r$ for all $r \in T$ is efficient, but only generates type-0 connections on $S_r$. Using Corollary 1 on $G_r$ would produce type-0 connections on $F_r$, but is not efficient since $|F_r|$ can be much larger than $|G^\circ_r|$. Instead, for each path $Q$ in $F_r$, the algorithm uses the reduction $\bar{Q}$ of $Q$ to the vertices of $Q$ that belong to $G^\circ_r$. Let $G^{Q^1}_r$ be the graph composed of $G^\circ_r$ and $\bar{Q}$. Note that $|G^{Q^1}_r| = O(|G^\circ_r|)$. The type-0 connections on $F_r$ can now be computed by applying Corollary 1 to $G^{Q^1}_r$.

It remains to compute type-1 connections. Recall that these connection lengths reflect distances in the entire graph, not just in $G_r$. Clearly, applying Corollary 1 on $G$ for every $r$ is inefficient. Instead, the computation is done top-down $T$ using an auxiliary construction. This construction augments $G^\circ_r$ with $\epsilon$-covers of the separators of all ancestors of $r$ in $T$ with respect to the vertices of $G^\circ_r$. These $\epsilon$-covers have already been computed (type-0 connections at the ancestor), and represent distances outside $G_r$. Due to space constraints we defer the details to the next section, where we handle the more general case of vertex labels.

### 4 Undirected Approximate Vertex-Label Distance Oracle

The idea is to adapt Thorup’s oracle (Section 3) to the vertex-label case. Thorup’s oracle supports one-to-one (vertex-to-vertex) distance queries, whereas here we need one-to-many distance queries. Given two vertices $u, v$, Thorup’s oracle finds the LCA of $r_u$ and $r_v$ in $T$, and uses its connections to produce the answer. In a one-to-many query, there is no analogue for $v$. We do know, however, that a shortest $u$-to-$\lambda$ path must intersect the separator of the leafmost node $r$ in $T$ that contains $u$ and some $\lambda$-labeled vertex. The node $r$ takes the role of the LCA of $r_u$ and $r_v$. In order to be able to use $r$’s connections in a distance query one must make sure that $r$’s connections represent approximate distances to $\lambda$-labeled vertices in the entire graph, not just in $G^\circ_r$. 


We define a set \( \mathcal{L} \) of new (artificial) vertices, one per label. For every \( r \in \mathcal{T} \), let \( \mathcal{L}_r = \{ \lambda \in \mathcal{L}|G_r^\circ \cap V_\lambda \neq \emptyset \} \) be the restriction of \( \mathcal{L} \) to labels in \( G_r^\circ \).

Simply connecting each vertex of \( V_\lambda \) to an artificial vertex representing the label \( \lambda \) is bound to fail. To see why, suppose vertices \( u \) and \( v \) both have label \( \lambda \). Adding an artificial vertex \( \lambda \) and zero-length undirected edges \( v\lambda \) and \( u\lambda \) creates a zero-length path between \( u \) and \( v \) that does not exist in the original graph. While this does not change the distance between any vertex and its closest \( \lambda \)-labeled vertex, it may change distances between a vertex and its closest \( \lambda' \)-labeled vertex (\( \lambda' \neq \lambda \)). Therefore, we would have liked to add, for each label \( \lambda \), separately, a single artificial vertex \( \lambda \), and compute the connection sets \( C(\lambda, Q) \). Doing so would result in correct distance estimates, but is not efficient. We show how to compute the connections \( C(\lambda, Q) \) without actually performing this inefficient procedure. Instead of having a single artificial vertex per label, it is split into many artificial vertices (one for each incident edge). The problem with this approach is that the number of connections becomes too large (each split vertex has its own set of \( O(\epsilon^{-1}) \) connections). We use an extension of the thinning procedure (Lemma 4) to select a small subset of these connections and still get the desired approximation.

Another point that we must address is that, for \( \lambda \in \mathcal{L}_r \), the type-1 connections \( C(\lambda, Q) \) should reflect the minimum distances between the connections of \( \lambda \) on \( Q \) to the closest vertex with label \( \lambda \) in \( G \), not just to vertices with label \( \lambda \) in \( G_r^\circ \). We show how to achieve this by an extension of the auxiliary construction used to compute they type-1 connections in Thorup’s unlabeled oracle.

We start with the extended thinning lemma.

**Lemma 6.** Let \( \{u_i\} \) be vertices and \( Q \) be a shortest path. Given ordered \( \epsilon \)-covering sets \( \{D(u_i, Q)\} \) it is possible to compute in linear time a clean and ordered \( 3\epsilon \)-covering connections set \( C \) of size \( O(\epsilon^{-1}) \) which represent approximated distances from any \( q \in Q \) to its closest vertex among \( \{u_i\} \).

**Proof.** We first convert every connection length \( \ell(u_i, Q) \) to reflect an approximated length from \( q \) to its closest vertex \( u^* \in \{u_i\} \), rather than to \( u_i \). We obtain these lengths using the fact that \( q \) is \( \epsilon \)-covered with respect to \( u^* \) by some connection in \( D(u^*, Q) \). Let \( Z_u \) be the graph composed of the following. See Figure 3 in the appendix for an illustration.

1. \( \bar{Q} \), the reduced form of \( Q \) to connections of all \( \{D(u_i, Q)\} \).
2. vertices \( \{u_i\} \), along with edges between each \( u_i \) to its connections, with lengths equal to the corresponding connection lengths.
3. vertex \( u \), connected with zero-length edges to all \( \{u_i\} \).

By the \( \epsilon \)-covering property, the distances between every \( q \in \bar{Q} \) and \( u \) in \( Z_u \) represent approximate distances between \( q \) and its closest vertex \( u^* \in \{u_j\} \) in \( G \). To see this, assume \( q \in \bar{Q} \) is a connection of \( u_1 \), and is closest to \( u^* \). Let \( q' \) be a connection of \( u^* \) which \( \epsilon \)-covers \( q \) w.r.t. \( u^* \). Then \( \delta_{Z_u}(q, u^*) \leq \delta_{\bar{Q}}(q, q^*) + \ell(q^*, u^*) = \delta_G(q, q^*) + \ell(q^*, u^*) \leq (1 + \epsilon)\delta_G(q, u^*) \).

It is possible to compute all shortest paths from \( u \) in \( Z_u \) in linear time; first, relax all edges incident to \( u \) and \( \{u_i\} \). Then, relax the edges of \( \bar{Q} \) by going first
in one direction along $Q$ and then relaxing the same edges again in the other
direction. For connection $p$ on $Q$, a $u$-to-$p$ shortest path first reaches $Q$ along
one of $\{u_i\}$ edges and then walks along $Q$ toward $p$. Hence the relaxation was
done in the correct order. We update the connection lengths to the distances
thus computed.

Let $\tilde{D}(u, Q)$ denote the ordered union of all connections, along with the
updated connection lengths. Since all $\{D(u_i, Q)\}$ were ordered, it is possible to
order their union in linear time. Let $G_u$ be the graph obtained from $G$ by adding
an apex $u$ connected with zero length edges to all $\{u_i\}$. We stress that $G_u$ is not
constructed by the algorithm, but only used in the proof. $\tilde{D}(u, Q)$ is an $\epsilon$-cover
of $Q$ with respect to $u$ in $G_u$. Now apply Lemma 4 to $\tilde{D}(v, Q)$ with $\epsilon_0 = \epsilon_1 = \epsilon$
to obtain a $3\epsilon$-cover of $Q$ with respect to $u$ in $G_u$ of size $O(\epsilon^{-1})$. □

4.1 Vertex-label distance oracle for undirected graphs

The vertex labeled distance oracle is very similar to the unlabeled one (Section 3).
It uses the same decomposition tree $T$, and stores, for each $r \in T$, the same
covering sets. The only difference is that, in addition to the covering sets $C(u, Q)$
for each vertex $u \in G^\circ_r$, the oracle also stores connection information for labels
as we now explain.

For every $r \in T$ and $\lambda \in \mathcal{L}_r$, the oracle stores connections $C(\lambda, Q)$ of both
type-0 and type-1. The type-0 connections $C(\lambda, Q)$ are connections in the graph
obtained from $G^\circ_r$ by adding an artificial vertex $\lambda$, along with zero length edges
from all $\lambda$-labeled vertices in $G^\circ_r$ to $\lambda$. The type-1 connections $C(\lambda, Q)$ are
connections in the graph obtained from $G$ by adding an artificial vertex $\lambda$, along
with zero length edges from all $\lambda$-labeled vertices in $G$ to $\lambda$. Before explaining
how to compute these connections we discuss how a distance query is performed.

Obtaining the distance from $u$ to $\lambda$ is done by finding the lowest ancestor $r$
of $r_u$ with $\lambda \in \mathcal{L}_r$. A shortest $u$-to-$\lambda$ path must cross $S_r$, and perhaps also $F_r$.
The algorithm estimates, for each path $Q$ of $S_r \cup F_r$, the length of a shortest
$u$-to-$\lambda$ path that intersects $Q$, using the connections $C(u, Q)$ and $C(\lambda, Q)$ stored
for $r$ (Since $\lambda \in \mathcal{L}_r$, $r$ does store $Q$-to-$\lambda$ connections).

Finding $r$ can be done by binary search on the path from $r_u$ to the root of $T$.
The number of steps of the binary search is $O(\log \log n)$. Finding whether a node
$r'$ has a vertex with label $\lambda$ can be done, e.g., by storing all unique labels in
$G^\circ_{r'}$ in a binary search tree, or by hashing. In the former case finding $r$ takes
$O(\log \log n \log |L|)$ time, and in the latter $O(\log \log n)$, assuming the more restrictive
word-RAM model of computation.

It remains to show how the connections are computed. We begin with the
type-0 connections. For every $r \in T$, for every $Q \in F_r \cup S_r$, the algorithm computes ordered $\epsilon$-covering sets of connections on $Q$ w.r.t. each vertex of $G^\circ_r$
to $Q$ by invoking Corollary 1 to $G^\circ_r$. This takes $O(\epsilon^{-1} |V(G^\circ_r)| \log n)$ time (using [5]
for shortest path computation). For each $\lambda \in \mathcal{L}_r$, let $n_\lambda$ denote the number of
$\lambda$-labeled vertices in $G^\circ_r$. The total number of connections to $\lambda$-labeled vertices
in $G^\circ_r$ is $O(\epsilon^{-1} n_\lambda)$. The algorithm next applies the extended thinning lemma
(Lemma 6) to get a connections set $C(\lambda, Q)$ of size $O(\epsilon^{-1})$ in $O(\epsilon^{-1}n_\lambda)$ time. Since $\sum \lambda n_\lambda = O(|V(G^\circ_r)|)$, the runtime for a single $r$ and $Q$ is $O(\epsilon^{-1}|V(G^\circ_r)|)$.

We now show how to compute the type-1 connections without invoking Corollary 1 to the entire input graph $G$ at every call.

**Lemma 7.** Let $r \in T$. Type-1 connections for $r$ can be computed using just the (type-0) connections of strict ancestors of $r$. Computing all type-1 connections for all $r \in T$ can be done in $O(\epsilon^{-2}n \lg^3 n)$ time.

**Proof.** Let $Q$ be a path in $F_r$. Let $X^Q_r$ be the graph composed of the following: (see Figure 4 in the appendix for an illustration)

- The vertices $L_r$
- The vertices and edges of $\hat{Q}$, the reduction of $Q$ to $V(\hat{Q}) \cap V(G^\circ_r)$.
- For each strict ancestor $r'$ of $r$, for each path $Q' \in S_{r'}$, the vertices and edges of $Q'$, the reduction of $Q'$ to vertices that are (type-0) connections (in $G^\circ_{r'}$) of $Q'$ w.r.t. vertices in $Q \cup L_r$, along with edges representing the corresponding connection lengths.

The algorithm creates a graph $\hat{X}^Q_r$ from $X^Q_r$ by breaking every artificial vertex $\lambda$ in $X^Q_r$ into many copies $\{\lambda_e\}$, one per incident edge of $\lambda$. We stress that the artificial vertices $\lambda_e$ are not directly connected to each other in $\hat{X}^Q_r$. Hence, the problem of shortcuts mentioned earlier is avoided. See Figure 5 in the appendix for an illustration.

Note that splitting vertices in this way does not increase the number of edges in the $\hat{X}^Q_r$. The algorithm applies Corollary 1 to $\hat{X}^Q_r$ and $Q$, obtaining a small sized $\epsilon$-cover $C(\lambda_e, Q)$ for every $\lambda_e$.

Let $q$ be any vertex of $\hat{Q}$, and let $\lambda$ be a label in $G^\circ_r$. Let $P$ be a shortest $q$-to-$\lambda$ path in $G$. Let $r'$ be the rootmost strict ancestor of $r$ such that $S_{r'}$ is intersected by $P$. Note that $r'$ must exist since $q \in F_r$, so $q$ belongs to the separator of some strict ancestor of $r$. Thus, $P$ is entirely contained in $G^\circ_{r'}$. Let $Q'$ be a path in $S_{r'}$ intersected by $P$. By construction of $\hat{X}^Q_r$, it contains an $\epsilon$-covering set of connections of $Q'$ with respect to $q$ in $G^\circ_{r'}$, as well as the edges of $Q'$ and an $\epsilon$-covering set of connections of $Q'$ with respect to $\lambda$ in $G^\circ_{r'}$. Hence, by Lemma 2, there exists a shortest $q$-to-$\lambda$ path (for some artificial vertex $\lambda_e$) in $\hat{X}^Q_r$ whose length is at most $(1 + \epsilon)$ times the length of $P$. On the other hand, because the vertices $\lambda_e$ (for any $\lambda \in L_r$) are not directly connected to each other in $\hat{X}^Q_r$, every path in $\hat{X}^Q_r$ corresponds to some path in $G$, so shortest paths in $\hat{X}^Q_r$ are at least as long as those in $G$. This proves that $\hat{X}^Q_r$ correctly represents all desired type-1 connection lengths.

We proceed with describing the construction of the connection sets of the appropriate sizes. To bound the size of the connections $\{C(\lambda_e, Q)\}$, we count the number of edges incident to $\lambda$ in $X^Q_r$ (i.e., before it is split). There is an edge for each of the $O(\epsilon^{-1})$ connections of $\lambda$ on each of the $O(\lg n)$ paths of separators of ancestors of $r$. For each such edge there is a vertex $\lambda_e$ with an $\epsilon$-covering set of $Q$ of size $O(\epsilon^{-1})$. Thus, the total number of connections of $Q$ for all $\lambda_e$ vertices is $O(\epsilon^{-2} \lg n)$. The algorithm applies Lemma 6, the extended
thinning procedure, to \( \{C(\lambda, Q)\}_e \) to get \( C(\lambda, Q) \) of size \( O(\varepsilon^{-1}) \). Doing so for all labels in \( G_p^r \) requires \( O(\varepsilon^{-2}\lg n + \varepsilon^{-1}|C_r|) \) space.

We now bound the running time. Since splitting vertices does not increase the number of edges, applying Corollary 1 to \( \hat{X}^r \) takes \( O(\varepsilon^{-2}|V(G_p^r)|\lg^2 n) \) time. Applying Lemma 6 is done within the same time bound. To conclude, the total runtime over all nodes of \( T \) is \( O(\varepsilon^{-2}n \lg^3 n) \).

We have thus established our main theorem:

**Theorem 1.** A \((1 + \varepsilon)\)-stretch \( \langle O(\varepsilon^{-1}n \lg n)_{\text{space}}, O(\lg \lg n + \varepsilon^{-1})_{\text{time}} \rangle \) Vertex-Label Distance Oracle can be constructed within \( O(\varepsilon^{-2}n \lg^3 n) \) time w.h.p. \(^7\) in an undirected planar graph with \( n \) vertices.

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\(^7\) The probability in the construction time is only due to the use of perfect hashing.
Appendix

A Proof of Lemma 4

Lemma 4. Let $Q$ be a path in an undirected graph, and let $v$ be a vertex. Let $D(v, Q)$ be an ordered $\epsilon_0$-cover of $Q$ w.r.t. $v$. For any $\epsilon_1 \leq 1$, a clean and ordered $(2\epsilon_0 + \epsilon_1)$-cover $C(v, Q) \subseteq D(v, Q)$ of size $O(\epsilon_1^{-1})$ can be constructed in $O(|D(v, Q)|)$ time.

Proof. The proof is constructive. Let $(\bar{q}, v)$ be a connection with minimal connection length in $D(v, Q)$. The vertex $\bar{q}$ splits $Q$ into two subpaths, $Q_0$ and $Q_1$. For each $Q' \in \{Q_0, Q_1\}$, the algorithm operates as follows. First, it adds $(\bar{q}, v)$ to $C(Q', v)$. The algorithm will now progress towards the other endpoint of $Q$.

We say $\bar{q}$ semi $\epsilon$-covers $q'$ if $\delta_Q(q', \bar{q}) + \ell(\bar{q}, v) \leq (1 + \epsilon)\ell(q', v)$.

Let $(\bar{q}, v)$ be the last connection added to $C(Q', v)$. Let $(q^*, v)$ be the next connection of $D(v, Q)$ that has not been considered yet. The algorithm adds $(q^*, v)$ unless $\bar{q}$ already semi $\epsilon_1$-covers $(q^*, v)$. The algorithm returns $C(v, Q) = C(Q_0, v) \cup C(Q_1, v)$.

We first prove that $C(v, Q)$ is a $(2\epsilon_0 + \epsilon_1)$-cover. Let $q$ be a vertex in $Q$. Let $d$ be the connection in $D(v, Q)$ which $\epsilon_0$-covers $q$. Let $c$ be a connection of $C(v, Q)$ that semi $\epsilon_1$-covers $d$ (it might be that $c = d$).

We know that
\[
\delta(q, c) \leq \delta(q, d) + \delta_Q(d, c) \quad \text{(triangle inequality)}
\]
\[
\delta_Q(d, c) + \ell(c, v) \leq (1 + \epsilon_1)\ell(d, v) \quad \text{(c semi $\epsilon_1$-covers $d$)}
\]
\[
\delta(q, d) + \ell(d, v) \leq (1 + \epsilon_0)\delta(q, v) \quad \text{(d $\epsilon_0$-covers $q$)}
\]
\[
\delta(q, c) + \ell(c, v) \leq \delta(q, d) + \delta_Q(d, c) + \ell(c, v) \leq \delta(q, d) + (1 + \epsilon_1)\ell(d, v) \leq (1 + \epsilon_1)(1 + \epsilon_0)\delta(q, v) = (1 + \epsilon_0 + \epsilon_1 + \epsilon_0\epsilon_1)\delta(q, v) \leq (1 + 2\epsilon_0 + \epsilon_1)\delta(q, v),
\]
and the approximation bound follows.

We now turn to show the generated cover is of $O(\epsilon_1^{-1})$ size. For $Q' \in \{Q_0, Q_1\}$, we show it is of size $O(\epsilon_1^{-1})$. Let $\{c_i\}_{i \geq 1}$, of size $k$, be the chosen connections along $Q'$, numbered by their order along $Q'$ toward the other endpoint $t$ of $Q'$. We examine the function $f(c_i) = \delta_Q(t, c_i) + \ell(c_i, v)$. We observe that $f(c_i) - f(c_{i+1}) = (\delta_Q(t, c_i) + \ell(c_i, v)) - (\delta_Q(t, c_{i+1}) + \ell(c_{i+1}, v)) = \ell(c_i, v) + \delta_Q(c_i, c_{i+1}) - \ell(c_{i+1}, v) \geq \epsilon_1\ell(c_{i+1}, v) \geq \epsilon_1\ell(q, v) \geq \epsilon_1\delta(q, v)$. Thereby, $f(c_{i+1}) \leq f(c_i) - \epsilon_1\epsilon_1\delta(q, v)$, hence $f(c_k) \leq f(c_1) - (k-1)\epsilon_1\delta(q, v)$. Note that $f(c_k) = \delta_Q(t, c_k) + \ell(c_k, v) \geq \delta(t, v) \geq \delta_Q(t, q) - \delta_Q(q, v)$. Using the lower and upper bounds on $f(c_k)$, we have that $\delta_Q(t, q) - \delta(q, v) \leq f(c_k) \leq f(c_1) - (k-1)\epsilon_1\delta(q, v) = \delta_Q(t, q) + \ell(q, v) - (k-1)\epsilon_1\delta(q, v) \leq \delta_Q(t, q) + (1 + \epsilon_0)\delta(q, v) - (k-1)\epsilon_1\delta(q, v)$. Hence $(k-1)\epsilon_1 - 1 + \epsilon_0\delta(q, v) \leq \delta_Q(q, v)$ and so $k \leq 1 + \frac{2\epsilon_0}{\epsilon_1}$.

Therefore the size of the connections obtained over both $\{Q_0, Q_1\}$ is $O(\epsilon_1^{-1})$. \(\square\)

\[\text{The semi } \epsilon \text{-cover definition is similar to } \epsilon \text{-cover definition. The only difference is that } \delta(q^*, v) \text{ was replaced by } \ell(q^*, v) \text{ for fast computation purposes.}\]
B A flaw in Thorup’s treatment of the undirected case

There is another notion of covering, apart from $\epsilon$-covering and quasi-$\epsilon$-covering [12].

Definition 3. $q^*$ strictly $\epsilon$-covers $q$ w.r.t. $v$ if $\delta(q, q^*) + \ell(q^*, v) \leq \delta(q, v) + \epsilon \delta(v, Q)$.

In [12], Thorup uses quasi-$\epsilon$-covers and strict-$\epsilon$-covers, but does not use (plain) $\epsilon$-covers. Most of the discussion in [12] is devoted to the directed case, which uses yet another notion of covering. When treating the undirected case, Thorup claims that all lemmas, except for the efficient construction procedure, carry over from the directed case to the undirected case when the directed definition of $\epsilon$-covering is replaced with strict $\epsilon$-covering. The treatment of the efficient construction for the undirected case is more detailed, where a procedure for efficiently constructing quasi-$\epsilon$-covering sets is given (Lemma 3, [12, Lemma 3.18]).

We believe that the treatment of the undirected case in [12] suffers from two flaws. First, the proof of the thinning procedure does not seem to carry over from the directed case to the undirected case when using strict $\epsilon$-covers. Second, since the construction is of quasi-$\epsilon$-covers, whereas all other parts of the undirected oracle in [12] assume strict-$\epsilon$-covers, the correctness of the entire oracle is not established.

Our algorithm does not use strict $\epsilon$-covers at all. We use Thorup’s efficient construction of quasi-$\epsilon$-covers, which, by Proposition 1 is also a $O(\epsilon)$-cover, and prove that the thinning procedure and query algorithm work for $\epsilon$-covers.

C Vertex-Label distance oracle for directed planar graphs

Thorup shows that the problem of constructing a distance oracle for a directed graph can be reduced to constructing a distance oracle for a restricted kind of graph, defined in the following.

Definition 4. A set $T$ of arcs in a graph $H$ is a $(t, \alpha)$-layered spanning tree if it satisfies the following properties:

- Disoriented - it can be oriented to form a spanning tree of $H$.
- Each branch (a path from the root of $T$) is a concatenation of at most $t$ shortest paths in $H$.
- Each shortest path in a branch of $T$ is of length at most $\alpha$

Definition 5. A graph $H$ is called $(t, \alpha)$-layered if it has a $(t, \alpha)$-layered spanning tree.

Definition 6. A scale-$(\alpha, \epsilon)$ distance oracle for a $(t, \alpha)$-layered graph $H$ is a data structure that, when queried for $\delta_H(v, w)$ returns

$$d(v, w) = \begin{cases} 
\delta_H(v, w), & \delta_H(v, w) \leq \alpha \\
\infty, & \text{otherwise}
\end{cases}$$

9 Thorup did not use the terms strict and quasi.
The reduction is summarized in the following lemma. Let $N$ be the maximum length of an arc in $G$.  

**Lemma 8.** ([12, Section 3.1,3.2,3.3]) Given a scale-$\langle \alpha, \epsilon' \rangle$ $(O(s(n, \epsilon'))_{space}, O(t(\epsilon'))_{time})$ algorithm, a $(1+\epsilon)$-stretch $(O(s(n, \epsilon) \log(nN))_{space}, O(t(\epsilon) \log \log (nN)+t(\epsilon'))_{time})$ algorithm can be constructed to any input graph.

Thorup shows each graph can be decomposed to a $(3, \alpha)$-layered graphs, of total linear size (with $\alpha$ the distance bound of the graph). Therefore it suffices to show how to construct a scale-$\langle \alpha, \epsilon \rangle$ distance oracle.

To do so, Thorup shows a directed variant to **Lemma 1**:

**Lemma 9.** Let $Q$ be a shortest path in a directed graph $H$. There exist sets $C(u, Q)$ of $O(\epsilon^{-1})$ vertices of $Q$ for all $u \in H$, where:

1. $C(u, Q)$ are called the connections of $u$ on $Q$.
2. The distance between $u$ and a connection $q \in C(u, Q)$ is called the connection length of $u$ and $q$.
3. For every $u, w \in H$, if a shortest $u$-to-$w$ path in $H$ intersects $Q$, then $\delta_{H_Q^u}^w(u, w) \leq \delta_H(u, w) + \epsilon \alpha$.

Here $H_Q^u$ is the graph with vertices $u, w$, and the vertices of the reduction of $Q$ to $C(u, Q) \cup C(w, Q)$, and with $u$-to-$Q$ and $Q$-to-$w$ arcs whose lengths are the corresponding connection lengths of $C(u, Q)$ and $C(w, Q)$.

The $\epsilon$-covering definition in the directed case is not the same as in the undirected. It uses a $O(\epsilon \alpha)$ additive error to the approximation rather $\epsilon \delta_H(u, w)$. Moreover, the $u$-to-$Q$ cover and $u$-from-$Q$ cover are different (because of the directedness) but obtained in similar manner.

As in the undirected case, given a $(3, \alpha)$-layered graph, the algorithm keeps a recursive graph decomposition $T$. The distance oracle keeps, for each $r \in T$, for each vertex $u \in G_r^\alpha$:

1. connections $C(u, Q)$ of type-0 for all $Q \in F_r$.
2. connections $C(u, Q)$ (Q to $u$ distances) of type-1 for all $Q \in F_r$.
3. connections $C(u, Q)$ (separate $Q$ to $u$ and $u$ to $Q$ distances) of type-0 for all $Q \in S_r$.

A $u$-to-$w$ query is done similar to the undirected case, using their relevant connections found in the LCA of $r_u$ and $r_w$. The construction of the connections is similar to the undirected case; type-0 connections are computed in $G^\alpha_r$ where the frame is reduced to the vertices of $G^\alpha_r$ (using **Lemma 1**). type-1 connections of node $r \in T$ are computed in a graph augmented by type-0 connections of ancestors of $r$ (see [12, Section 3.6] and **Lemma 7** of the undirected case).

For the vertex-label case, for any $r \in T$ we define $G^\alpha_r$ be $G^\alpha_r$ along with vertices $\hat{L}_r$ and arcs from any $\lambda$-labeled vertex to $\lambda$ for each vertex $\lambda \in \hat{L}_r$.

The vertex-label distance oracle keeps, for each $r \in T$, for each vertex $u \in \hat{G}_r^\alpha$ (either in $G^\alpha_r$ or artificial vertex from $\hat{L}_r$):

---

10 $nN$ is an upper bound on $\delta_G(\cdot)$. 
1. connections $C(u, Q)$ of type-0 for all $Q \in F_r$.
2. connections $C(u, Q)$ ($Q$ to $u$ distances) of type-1 for all $Q \in F_r$.
3. connections $C(u, Q)$ (separate $Q$ to $u$ and $u$ to $Q$ distances) of type-0 for all $Q \in S_r$.

A $u$-to-$\lambda$ query is done similarly to the vertex-to-vertex case relatively to the $u$ and $\lambda$ connections of the lowest ancestor of $r_u$ with $\lambda \in L_r$. The construction is similar to the vertex-to-vertex construction, but instead done over $\hat{G}_r$. See Figure 6 for the type-1 connections construction directed variant.

We thus get the following theorem:

**Theorem 2.** A $(1 + \epsilon)$-stretch $(O(\epsilon^{-1} n \lg n \lg(nN))_{\text{space}}$, $O(\lg \lg n \lg \lg(nN) + \epsilon^{-1})_{\text{time}}$) Vertex-Label Distance Oracle can be constructed in $O(\epsilon^{-2} n \lg^3 n \lg(nN))$ time w.h.p.\(^{11}\) for a directed planar graph with $n$ vertices and maximum arc length $N$.

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\(^{11}\) The probability in the construction time is only due to the use of perfect hashing.
Fig. 1: An illustration of the decomposition tree $T$. The root $\tilde{r}$ is associated with $G_{\tilde{r}} = G$. The children of $\tilde{r}$, $\tilde{r}_0$ and $\tilde{r}_1$, are associated with the interior and exterior of the separator $S_{\tilde{r}}$ of $G_{\tilde{r}}$.

Fig. 2: The solid lines (thin and thick) indicate $\bar{F}_{\tilde{r}}$, the reduced frame of $G_{\tilde{r}}$. The bold lines (solid and dashed) indicate $S_{\tilde{r}}$, the separator of $G_{\tilde{r}}$. Vertices $u$ and $w$ are vertices of $G_{\tilde{r}}$ separated by $S_{\tilde{r}}$. Every $u$-to-$w$ path must intersect $S_{\tilde{r}}$. The dashed line shows a possible shortest $u$-to-$w$ path in $G$.  

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Fig. 3: Illustration of the situation in the proof of the extended thinning lemma (Lemma 6). Each vertex in \( \{u_i\} \) has different connections on \( Q \). The distance from \( u_1 \) to any connection of \( u_2 \) is approximated using the connections of \( u_1 \).

Fig. 4: The figure illustrates a part of \( X^Q_r \). In this example \( Q \) is a path in the separator of the parent of \( r \) (in general \( Q \) is a path in \( F_r \), so it may belong to the separator of an ancestor of \( r \)). The vertices of \( G^Q_r \) are enclosed in \( F_r \), which is represented by the dashed lines and by \( Q \). The vertices \( u_1 \) and \( u_2 \) are \( \lambda \)-labeled vertices of \( G^Q_r \), and are not part of \( X^Q_r \). Paths from \( Q \) to \( \lambda \)-labeled vertices such as \( u_1 \) and \( u_2 \) are represented in \( X^Q_r \) by edges between \( Q \) and \( \lambda \). These edges correspond to the type-0 connections of \( \lambda \) on \( Q \) in the parent of \( r \). All solid edges are part of \( X^Q_r \). A shortest path from \( q_1 \in Q \) to \( \lambda \in L_r \) is approximated by connections from \( q_1 \) to a separator of an ancestor of \( r \) and from there to \( \lambda \). Note that \( C(\lambda, Q') \) represent distances from \( Q' \) to a \( \lambda \)-labeled vertex which is not necessarily in \( G^Q_r \).
Fig. 5: Illustration of the utility of splitting an artificial vertex $\lambda$. On the left ($X^Q_r$) undesired shortcuts (teleportation) might occur. On the right ($\hat{X}^Q_r$) teleportation does not occur.

Fig. 6: Illustration of part of the auxiliary graph $X^Q_r$ for the directed case. No teleportation can occur because each artificial vertex $\lambda$ has only incoming arcs, and no outgoing arcs.