Bounded Quotients of the Fundamental Group of a Random 2-Complex

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Abstract

Let $\Delta_{n-1}$ denote the $(n-1)$-dimensional simplex. Let $Y$ be a random 2-dimensional subcomplex of $\Delta_{n-1}$ obtained by starting with the full 1-skeleton of $\Delta_{n-1}$ and then adding each 2-simplex independently with probability $p$. For a fixed $c > 0$ it is shown that if $p = \frac{(6+7c) \log n}{n}$ then a.a.s. the fundamental group $\pi_1(Y)$ does not have a nontrivial quotient of order at most $n^c$.

1 Introduction

For a simplicial complex $X$, let $X^{(k)}$ denote the $k$-skeleton of $X$ and let $f_k(X)$ be the number of $k$-simplices of $X$. Let $\Delta_{n-1}$ be the $(n-1)$-dimensional simplex on the vertex set $V = [n]$. Let $Y(n, p)$ denote the probability space of complexes $\Delta_{n-1}^{(1)} \subset Y \subset \Delta_{n-1}^{(2)}$ with probability measure

$$\Pr(Y) = p^{f_2(Y)}(1 - p)^{\binom{n}{3} - f_2(Y)}.$$

The threshold probability for the vanishing of the first homology with fixed finite abelian coefficient group was determined in [3, 4]. Let $\omega(n)$ be an arbitrary function that tends to infinity with $n$.

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Theorem 1.1 ([3, 4]). Let $R$ be a fixed finite abelian group. Then
\[
\lim_{n \to \infty} \Pr \left[ Y \in Y(n, p) : H_1(Y; R) = 0 \right] = \begin{cases} 
0 & p = \frac{2 \log n - \omega(n)}{n} \\
1 & p = \frac{2 \log n + \omega(n)}{n} 
\end{cases}
\]

The threshold probability for the vanishing of the fundamental group was determined by Babson, Hoffman and Kahle [1].

Theorem 1.2 ([1]). Let $\epsilon > 0$ be fixed, then
\[
\lim_{n \to \infty} \Pr \left[ Y \in Y(n, p) : \pi_1(Y) = 0 \right] = \begin{cases} 
0 & p = n^{1/2} - \epsilon \\
1 & p = \left( \frac{3 \log n + \omega(n)}{n} \right)^{1/2}
\end{cases}
\]

In view of the gap between the thresholds for the vanishing of $H_1(Y; R)$ ($R$ finite) and for the triviality of $\pi_1(Y)$, Eric Babson (see problem (8) on page 58 in [2]) asked what is the threshold probability such that a.a.s. $\pi_1(Y)$ does not have a quotient equal to some finite group. Addressing Babson’s question we prove the following

Theorem 1.3. Let $c > 0$ be fixed and let $p = \frac{(6+7c) \log n}{n}$. Then a.a.s. $\pi_1(Y)$ does not contain a nontrivial normal subgroup of index at most $n^c$.

Remark: The constant $6+7c$ may be improved using a more careful analysis as in [3, 4]. For example, for any fixed non-trivial finite group $G$, if $p = \frac{2 \log n + \omega(n)}{n}$ then a.a.s. $G$ is not a homomorphic image of $\pi_1(Y)$.

The proof of Theorem 1.3 is an adaptation of an argument in [3, 4] to the non-abelian setting. In Section 2 we recall the notion of non-abelian first cohomology and its relation with the fundamental group. In Section 3 we compute the expansion of the $(n-1)$-simplex. The results of Sections 2 and 3 are used in section 4 to prove Theorem 1.3.

2 Non-abelian first cohomology

Let $X$ be a simplicial complex and let $G$ be a multiplicative group. We recall the definition of the first cohomology $H^1(X; G)$ of $X$ with $G$ coefficients (see e.g. [5]). For $0 \leq k \leq 2$ let $X(k)$ be the set of all ordered $k$-simplices of $X$. Let $C^0(X; G)$ denote the group of $G$-valued functions on $X(0)$ with pointwise multiplication, and let
\[
C^1(X; G) = \{ \phi : X(1) \to G : \phi(u, v) = \phi(v, u)^{-1} \}.
\]
The 0-coboundary operator \(d_0: C^0(X; G) \to C^1(X; G)\) be given by
\[
d_0 \psi(u, v) = \psi(u)\psi(v)^{-1}.
\]
For \(\phi \in C^1(X; G)\) and \((u, v, w) \in X(2)\) let
\[
d_1 \phi(u, v, w) = \phi(u, v)\phi(v, w)\phi(w, u).
\]
The set of \(G\)-valued 1-cocycles of \(X\) is given by
\[
Z^1(X; G) = \{\phi \in C^1(X; G) : d_1 \phi(u, v, w) = 1 \text{ for all } (u, v, w) \in X(2)\}.
\]
Define an action of \(C^0(X; G)\) on \(C^1(X; G)\) as follows. For \(\psi \in C^0(X; G)\) and \(\phi \in C^1(X; G)\) let
\[
\psi.\phi(u, v) = \psi(u)\phi(u, v)\psi(v)^{-1}.
\]
Note that \(d_0 \psi = \psi.1\) and that \(Z^1(X; G)\) is invariant under the action of \(C^0(X; G)\). For \(\phi \in C^1(X; G)\) let \([\phi]\) denote the orbit of \(\phi\) under the action of \(C^0(X; G)\). The first cohomology of \(X\) with coefficients in \(G\) is the set of orbits
\[
H^1(X; G) = \{[\phi] : \phi \in Z^1(X; G)\}.
\]
Let \(Hom(\pi_1(X), G)\) denote the set of homomorphisms from \(\pi_1(X)\) to \(G\). Let \(G\) act on \(Hom(\pi_1(X), G)\) by conjugation and for \(\varphi \in Hom(\pi_1(X), G)\) let \([\varphi]\) denote the orbit of \(\varphi\) under this action. Let
\[
Hom(\pi_1(X), G)/G = \{[\varphi] : \varphi \in Hom(\pi_1(X), G)\}.
\]
The following observation is well known (see (1.3) in [5]). For completeness we outline a proof.

**Claim 2.1.** For \(\Delta^{(1)}_{n-1} \subset X \subset \Delta^{(2)}_{n-1}\) there is a bijection
\[
\mu : Hom(\pi_1(X), G)/G \to H^1(X; G)
\]
that maps \([1]\) in \(Hom(\pi_1(X), G)/G\) to \([1]\) in \(H^1(X; G)\).

**Proof:** We identify \(\pi_1(X)\) with the group generated by \(\{e_{ij} : 2 \leq i \neq j \leq n\}\) modulo the relations
\[
\begin{align*}
e_{ij}e_{ji} &= 1, \\
e_{ij} &= 1 \text{ if } (1, i, j) \in X(2).
\end{align*}
\]
• \( e_{ij}e_{jk}e_{ki} = 1 \) if \((i,j,k) \in X(2)\).

For \( \varphi \in Hom(\pi_1(X), G) \) let \( F(\varphi) \in C^1(X; G) \) be given by

\[
F(\varphi)(i,j) = \begin{cases} 
\varphi(e_{ij}) & 2 \leq i \neq j \leq n \\
1 & \text{otherwise.}
\end{cases}
\]

It can be checked that \( F(\varphi) \in Z^1(X; G) \) and that the mapping

\[
\tilde{F} : Hom(\pi_1(X), G)/G \rightarrow H^1(X; G)
\]

given by \( \tilde{F}([\varphi]) = [F(\varphi)] \) is the required bijection.

\[\square\]

In particular we obtain the following

**Corollary 2.2.** Let \( N > 1 \). Then \( \pi_1(X) \) contains a nontrivial normal subgroup of index at most \( N \) iff \( H^1(X; G) \neq \{[1]\} \) for some nontrivial simple group \( G \) of order at most \( N \).

\[\square\]

### 3 1-Expansion of the Simplex

Let \( \phi \in C^1(\Delta_{n-1}; G) \). The support size of \( \phi \) is

\[
\|\phi\| = \{|u, v\} \in \binom{[n]}{2} : \phi(u, v) \neq 1\}.
\]

The weight of the orbit \([\phi]\) is

\[
\|[\phi]\| = \min\{\|\psi.\phi\| : \psi \in C^0(\Delta_{n-1}; G)\}.
\]

Let \( B(\phi) \) be the support of \( d_1\phi \), i.e.

\[
B(\phi) = \left\{ \{u, v, w\} \in \binom{[n]}{3} : d_1\phi(u, v, w) \neq 1 \right\}
\]

and let \( \|d_1\phi\| = |B(\phi)| \). The following result is an adaptation of Proposition 3.1 of [4] to the non-abelian setting.
Proposition 3.1. Let $\phi \in C^1(\Delta_{n-1}; G)$ then
\[ \|d_1\phi\| \geq \frac{n\|\phi\|}{3}. \]

Proof: For $u \in \Delta_{n-1}(0)$ define $\phi_u \in C^0(\Delta_{n-1}; G)$ by
\[ \phi_u(v) = \begin{cases} 
1 & v = u \\
\phi(u, v) & v \neq u.
\end{cases} \]
Note that if $(u, v, w) \in \Delta_{n-1}(2)$ then
\[ d_1\phi(u, v, w) = \phi(u, v)\phi(v, w)\phi(w, u) = \phi_u(v)\phi(v, w)\phi_u(w)^{-1} = \phi_u\phi(v, w). \]

Therefore
\[ 6\|d_1\phi\| = |\{(u, v, w) \in \Delta_{n-1}(2) : d_1\phi(u, v, w) \neq 1\}| \]
\[ = |\{(u, v, w) \in \Delta_{n-1}(2) : \phi_u\phi(v, w) \neq 1\}| \]
\[ = \sum_u 2\|\phi_u\phi\| \geq 2n\|\phi\|. \]

\[ \square \]

4 Proof of Theorem 1.3

Let $G$ be a finite group. For a subcomplex $\Delta^{(1)}_{n-1} \subset X \subset \Delta^{(2)}_{n-1}$ we identify $H^1(X; G)$ with its image under the natural injection $H^1(X; G) \to H^1(\Delta^{(1)}_{n-1}; G)$. If $\phi \in C^1(\Delta_{n-1}; G)$ then $[\phi] \in H^1(X; G)$ iff $d_1\phi(u, v, w) = 1$ whenever $(u, v, w) \in X(2)$. It follows that in the probability space $Y(n, p)$
\[ \Pr \left[ [\phi] \in H^1(Y; G) \right] = (1 - p)^{\|d_1\phi\|}. \]

Therefore
\[ \Pr \left[ H^1(Y; G) \neq \{[1]\} \right] \leq \sum_{[1] \neq [\phi] \in H^1(\Delta^{(1)}_{n-1}; G)} \Pr \left[ [\phi] \in H^1(Y; G) \right] \]
\[ = \sum_{[1] \neq [\phi] \in H^1(\Delta^{(1)}_{n-1}; G)} (1 - p)^{\|d_1\phi\|}. \] (1)
Suppose now that is $|G| \leq n^c$. Then by (1) and Proposition 3.1

$$\Pr \left[ H^1(Y; G) \neq \{1\} \right]$$

$$\leq \sum_{k \geq 1} \sum_{|\phi| = k} (1 - p)^{kn}$$

$$\leq \sum_{k \geq 1} \frac{n^k}{k!} |G|^k \left(1 - \frac{(6 + 7c) \log n}{n} \right)^{\frac{kn}{2}}$$

(2) 

$$\leq \sum_{k \geq 1} n^{2k} \frac{n^{ck}}{n^{\frac{(6+7c)k}{2}}} = O(n^{-\frac{4c}{5}}).$$

Let $G(N)$ be the set of all nontrivial simple groups of size at most $N$. As there are at most 2 non-isomorphic simple groups of the same order, it follows that $|G(N)| \leq 2N$. Combining Corollary 2.2 and (2) it follows that the probability that $\pi_1(Y)$ contains a nontrivial normal subgroup of index $\leq n^c$ is at most

$$\sum_{G \in G(n^c)} \Pr[Y \in Y(n, p) : H^1(Y; G) \neq \{1\}] \leq O(|G(n^c)|n^{-\frac{4c}{5}}) = O(n^{-\frac{c}{5}}).$$

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