On Generalizations of the Gromov-Hausdorff Metric

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Abstract

In this paper, a general approach is presented for generalizing the Gromov-Hausdorff metric to consider metric spaces equipped with some additional structure. A special case is the Gromov-Hausdorff-Prokhorov metric which considers measured metric spaces. This abstract framework also unifies several existing generalizations which consider metric spaces equipped with a measure, a point, a closed subset, a curve or a tuple of such structures. It can also be useful for studying new examples of additional structures. The framework is provided both for compact metric spaces and for boundedly-compact pointed metric spaces. In addition, completeness and separability of the metric is proved under some conditions. This enables one to study random metric spaces equipped with additional structures, which is the main motivation of this work.

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1 Introduction

This paper provides a unified framework for generalizing the Gromov-Hausdorff metric. Below, the Gromov-Hausdorff metric for compact metric spaces, some of its existing generalizations, and the general framework of this paper are introduced. Then, the extension of the framework to the non-compact case is briefly discussed.

The Gromov-Hausdorff Metric. Gromov [17] defined a metric on the set of all compact metric spaces. It is called the Gromov-Hausdorff metric in the literature. This metric has been defined for group-theoretic purposes. However, it has found important applications in probability theory as well, since it enables one to study random compact metric spaces. Specially, this is used in the study of scaling limits of random graphs and other random objects (this is started by the novel work of Aldous [3]). An important topological property needed for probability-theoretic applications is that the set of all compact metric spaces (or other relevant sets) is complete and separable, and hence, can be used as a standard probability space.

Generalizations in the Literature. There are various generalizations of the Gromov-Hausdorff metric that consider compact metric spaces equipped with some additional structure. These are mainly motivated by the study of random metric spaces equipped with an additional structure. For instance, the Gromov-Hausdorff-Prokhorov metric is defined between two compact measured metric spaces ([16], [22], [25] and [1]), where a compact measured metric space is a compact metric space $X$ together with a finite measure $\mu$ on $X$. Other
generalizations consider the set of all compact metric spaces $X$ equipped with a point of $X$, a finite number of points ([4], [2], [5]), a finite number of measures on $X$ ([2]), a finite number of closed subsets of $X$ ([22]), a curve in $X$ ([18]), a marking of points of $X$ if $X$ is discrete ([4], [5]), or a tuple of such structures ([2], [18]).

A Unified Framework for Generalizations. Most of the generalizations of the Gromov-Hausdorff metric, mentioned above, have similar properties with similar proofs. This paper provides an abstract framework for generalizing the Gromov-Hausdorff metric that unifies the above examples. This generalization might be useful in the future to study random metric spaces equipped with new types of additional structures.

In general, for compact metric spaces $X$, let $\tau(X)$ be a metric space that represents the set of possible additional structures on $X$ (e.g., the set of finite measures on $X$). Under some assumptions on $\tau$, which are tried to be minimalistic (in short, being a functor and having some kind of continuity), a version of the Gromov-Hausdorff metric is defined and its completeness and separability is proved.

In addition, some new specific examples of additional structures are also studied; e.g., marked measures, marked closed subsets, càdlàg curves and collections of closed subsets.

The Non-Compact Case. The notion of Gromov-Hausdorff convergence is defined on the set $\mathfrak{M}_*$ of boundedly-compact pointed metric spaces, where boundedly-compact means that every bounded closed subset is compact and pointed means that a point of the metric space is distinguished (which is called the origin here). Heuristically, the idea is to consider large balls centered at the origins and to compare them using the Gromov-Hausdorff metric (the precise definition takes into account the discontinuity issues caused by the points which are close to the boundaries of the balls). The notion of Gromov-Hausdorff-Prokhorov convergence is defined similarly on the set $\mathfrak{M}_*$ of boundedly-compact pointed measured metric spaces ([25]) (also called measured Gromov-Hausdorff convergence). It is known that these topologies are metrizable and $\mathfrak{M}_*$ and $\mathfrak{M}_*$ become complete and separable metric spaces (this was shown for length spaces and discrete metric spaces in [1] and [5] respectively and the general case is done in [21]). This enables one to study random (measured) non-compact metric spaces. See e.g., [1], [5] or the references of (and citations to) [1].

Some generalizations of the Gromov-Hausdorff (-Prokhorov) metric exist in the literature which consider boundedly-compact pointed metric spaces equipped with an additional structure. Instances of such generalizations will be discussed in Section 5. In this paper, a unified framework is presented for such generalizations as well. This is done by extending the framework for compact spaces, mentioned above, under additional assumptions on the map $\tau$. Some new specific examples of such generalizations are also discussed (as in the compact case). The method and the proofs of the results are based on those in [21].

The structure of the paper. Section 2 provides the basic definitions and recalls the Gromov-Hausdorff metric. The generalization of the Gromov-Hausdorff metric is provided in Section 3 for the compact case and in Section 4.
for the boundedly-compact case. Finally, the connections of these frameworks to the existing specific generalizations of the Gromov-Hausdorff metric are discussed in Section 5.

2 Preliminaries

2.1 Basic Definitions and Notations

The minimum and maximum binary operators are denoted by \( \land \) and \( \lor \) respectively. For all metric spaces \( X \) in this paper, the metric on \( X \) is usually denoted by \( d \) if there is no ambiguity. The complement of a subset \( A \subseteq X \) is denoted by \( A^c \) or \( X \setminus A \). Also, all measures on \( X \) are assume to be Borel measures. If in addition, \( \rho : X \to Y \) is measurable, \( \rho_* \mu \) denotes the push-forward of \( \mu \) under \( \rho \); i.e., \( \rho_* \mu(\cdot) = \mu(\rho^{-1}(\cdot)) \). For \( x \in X \) and \( r \geq 0 \), the closed ball of radius \( r \) centered at \( x \) is defined by

\[
B_r(x) := \{ y \in X : d(x,y) \leq r \}.
\]

The metric space \( X \) is **boundedly compact** if every closed ball in \( X \) is compact.

Given metric spaces \( X \) and \( Z \), a function \( f : X \to Z \) is an **isometric embedding** if it preserves the metric; i.e., \( d(f(x_1),f(x_2)) = d(x_1,x_2) \) for all \( x_1, x_2 \in X \). It is an **isometry** if it is a surjective isometric embedding. The image of \( f \) is denoted by either \( f(X) \) or \( \text{Im}(f) \).

A measured metric space is a pair \((X, \mu)\), where \( X \) is a metric space and \( \mu \) is a Borel measure on \( X \). It is called compact if \( X \) is compact and \( \mu \) is a finite measure. Two measured metric spaces \((X, \mu)\) and \((Y, \nu)\) are called equivalent if there exists an isometry \( f : X \to Y \) such that \( f_* \mu = \nu \).

For two closed subsets \( A \) and \( B \) of a metric space \( X \), the Hausdorff distance of \( A \) and \( B \) is defined by

\[
d_H(A, B) := \inf \{ \epsilon \geq 0 : A \subseteq N_\epsilon(B) \text{ and } B \subseteq N_\epsilon(A) \}, \tag{2.1}
\]

where \( N_\epsilon(A) := \{ x \in X : \exists y \in A : d(x,y) \leq \epsilon \} \) is the closed \( \epsilon \)-neighborhood of \( A \) in \( X \). Let \( \mathcal{F}(X) \) be the set of closed subsets of \( X \). It is well known that \( d_H \) is a metric on \( \mathcal{F}(X) \). Also, if \( X \) is complete and separable, then \( \mathcal{F}(X) \) is also complete and separable. In addition, if \( X \) is compact, then \( \mathcal{F}(X) \) is also compact. See e.g., Proposition 7.3.7 and Theorem 7.3.8 of [10].

An extended metric on a set \( X \) is a function \( d : X \times X \to \mathbb{R}^{\geq 0} \cup \{\infty\} \) such that it is symmetric, satisfies the triangle inequality and \( d(x,y) > 0 \) whenever \( x \neq y \).

2.2 The Gromov-Hausdorff (-Prokhorov) Metric

Let \( \mathcal{M}^c \) be the set of equivalence classes of compact metric spaces, where two metric spaces are equivalent if and only if they are isometric. The **Gromov-Hausdorff metric** is a metric on \( \mathcal{M}^c \) defined by

\[
d_{GH}^c(X, Y) := \inf \{ d_H(f(X), g(Y)) \}, \tag{2.2}
\]
where the infimum is over all metric spaces $Z$ and all pairs of isometric embeddings $f : X \to Z$ and $g : Y \to Z$. It is known that under this metric, $\mathfrak{M}^c$ is a complete separable metric space (see e.g., [10]).

Let $\mathfrak{M}^c$ be the set of equivalence classes of compact measured metric spaces. The **Gromov-Hausdorff-Prokhorov metric** on $\mathfrak{M}^c$ is defined as follows.

$$d_{GHP}((X,\mu),(Y,\nu)) := \inf \{ d_H(f(X),g(Y)) \vee d_P(f_\ast \mu,g_\ast \nu) \},$$

(2.3)

where the infimum is over all $Z,f,g$ as above and where $d_P(\cdot,\cdot)$ denotes the Prokhorov distance of two finite measures on $Z$. It is also known that this metric makes $\mathfrak{M}^c$ a complete separable metric space (see e.g., [1]). This is also implied by the results of the next section.

The above metrics can also be defined for compact **pointed** (measured) metric spaces as well, where pointed means that a point of the metric space is distinguished (which is called **the origin** here). For this, if $o_X$ and $o_Y$ denote the distinguished points of $X$ and $Y$, one should replace $d_H(f(X),g(Y))$ by $d(f(o_X),g(o_Y)) \vee d_H(f(X),g(Y))$ in the above formulas. In fact, this generalization is a special case of the framework of the next section.

As mentioned in the introduction, the Gromov-Hausdorff (-Prokhorov) metric is also defined for boundedly-compact pointed (measured) metric spaces as well (see [21] and also [1] and [5]). This metric generates the Gromov-Hausdorff (-Prokhorov) topology on $\mathfrak{M}^*$ (resp. $\mathfrak{M}^*_c$). For brevity, this metric is not recalled here. In fact, it is a special case of the framework of Section 4.

### 3 Compact Metric Spaces Equipped with More Structures

As mentioned in the introduction, this section provides the abstract framework for generalizing the Gromov-Hausdorff metric in the compact case. Before presenting the definitions and results, Subsection 3.1 provides a motivation and some basic examples of additional structures on compact metric spaces. The proofs of the results are postponed to Subsection 3.6. Further examples are provided in Subsection 3.4 and also in Section 5. The reader might also think of further examples using the setting of this section.

#### 3.1 The Space $C_\tau$

To consider additional structures on compact metric spaces, the following setting is used. For every compact metric space $X$, let $\tau(X)$ be a set which represents the set of possible **additional structures** on $X$. For example, one can let $\tau(X)$ be the set of finite measures on $X$. Various other examples will be studied later in this paper. In general, it is required that $\tau(X)$ is a metric space. Also, to every isometric embedding $f : X \to Z$, assume that a function $\tau_f : \tau(X) \to \tau(Z)$ is assigned which is also an isometric embedding. These conditions are satisfied in all of the examples in this paper.
Let $\mathcal{C}_\tau$ be the set of equivalence classes of all pairs $(X,a)$, where $X$ is a compact metric space and $a \in \tau(X)$. Here, two pairs $(X,a)$ and $(X',a')$ are equivalent when there exists an isometry $f: X \to X'$ such that $\tau_f(a) = a'$. In Subsection 3.2 below, under some assumptions on $\tau$, a metric is defined on $\mathcal{C}_\tau$ similarly to the Gromov-Hausdorff-Prokhorov metric \[ \text{(2.3)}. \]

Before describing the assumptions and the results, it is useful to consider the following examples. In these examples, $X$ and $Z$ represent compact metric spaces and $f$ is an isometric embedding from $X$ into $Z$. Further examples will be discussed in Subsection \[ 3.4 \] and Section \[ 5 \].

**Example 3.1** (Points). To consider compact metric spaces equipped with a distinguished point, one can let $\tau(X) := X$. Also, for every isometric embedding $f : X \to Z$, let $\tau_f := f$. Then, $\mathcal{C}_\tau$ is the set of (equivalence classes of) all compact pointed metric spaces.

**Example 3.2** (Measures). To consider compact measured metric spaces, one can let $\tau(X)$ be the set of all finite measures on $X$. Also, one can consider the Prokhorov metric on $\tau(X)$ and let $\tau_f(\mu) := f_*\mu$. It is straightforward that $\tau_f$ is an isometric embedding. Here, $\mathcal{C}_\tau$ is just the set $\mathcal{M}_c$ of compact measured metric spaces.

**Example 3.3** (Compact Subsets). To consider compact metric spaces equipped with a nonempty compact subset, one can let $\tau(X)$ be the set of nonempty compact subsets of $X$. Also, if one equips $\tau(X)$ with the Hausdorff metric and lets $\tau_f(S) := f(S)$ for all $S \in \tau(X)$, then $\tau_f$ is an isometric embedding.

**Remark 3.4.** In the above example, one can also let $\tau(X)$ be the set of compact subsets of $X$ including the empty set. For this, it is convenient to extend the Hausdorff metric by letting $d_H(\emptyset,K) := \infty$ for all $K \neq \emptyset$, which leads to an extended metric on $\tau(X)$. One can also easily obtain a metric on $\tau(X)$ (e.g., $d_H/(1+d_H)$) that makes it a compact metric space. But in this section, it is more convenient to work with $d_H$ and the effect of having $\infty$ as a distance will be explained whenever needed, e.g., in Remark \[ 3.15 \].

**Example 3.5** (Multiple Additional Structures). Let $n \in \mathbb{N}$ and assume that each of $\tau_1, \ldots, \tau_n$ are as in one of the examples in this subsection. For every compact metric space $X$, let $\tau(X) := \prod \tau_i(X)$ equipped with the max product metric. Also, every isometric embedding $f : X \to Z$ induces an isometric embedding from $\tau(X)$ to $\tau(Z)$ naturally. Similarly, if $\tau_1, \tau_2, \ldots$ is an infinite sequence, one can let $\tau(X) := \prod \tau_i(X)$ again, equipped with the following metric: Let the distance of $(a_1,a_2,\ldots) \in \tau(X)$ and $(b_1,b_2,\ldots) \in \tau(X)$ be $\max_i \{2^{-i}(1 \wedge d(a_i,b_i))\}$ (this particular metric will be used in Remark \[ 5.29 \]). Then, $\tau(X)$ is a complete and separable metric space.

**Example 3.6** (No Additional Structure). If for every $X$, $\tau(X)$ has a single element, then $\mathcal{C}_\tau$ is just the set $\mathcal{M}_c$ of equivalence classes of all compact metric spaces. Also, if $E$ is a fixed metric space, $\tau(X) := E$ and $\tau_f$ is the identity function on $E$ for every $X$ and $f$, then $\mathcal{C}_\tau = \mathcal{M}_c \times E$ (it will be seen that the metric on $\mathcal{C}_\tau$ is the max product metric).
3.2 Generalization of the Gromov-Hausdorff Metric

Now, the general definitions are presented and studied. Consider a map \( \tau \) and the space \( C_\tau \) as in the previous subsection. In the following definition, it is assumed that \( \tau \) preserves the identity functions and is compatible with composition of isometric embeddings. In short, it is required that \( \tau \) is a functor. The notions of categories and functors are useful to simplify the notations.

Here, only the definition of a functor is provided and no results or background in category theory are needed.

Let \( \text{Comp} \) denote the class of compact metric spaces. For two compact metric spaces \( X \) and \( Y \), let \( \text{Hom}(X,Y) \) be the set of isometric embeddings of \( X \) into \( Y \). In the language of category theory, \( \text{Comp} \) is a category and the elements of \( \text{Hom}(X,Y) \) are called morphisms. An isomorphism is a morphism which has an inverse. Also, every compact metric space is called an object of \( \text{Comp} \).

The general definition of categories is omitted for brevity.

Let also \( \text{Met} \) be the category of metric spaces in which the morphisms are isometric embeddings. More precisely, for any two metric spaces \( X \) and \( Y \), the morphisms from \( X \) to \( Y \) in \( \text{Met} \) are the isometric embeddings of \( X \) into \( Y \).

**Definition 3.7.** A functor \( \tau : \text{Comp} \to \text{Met} \) is a map that assigns to every compact metric space \( X \) a metric space \( \tau(X) \), and assigns to every morphism (in \( \text{Comp} \)) \( f : X \to Y \) a morphism (in \( \text{Met} \)) \( \tau(f) : \tau(X) \to \tau(Y) \), such that

(i) For all isometric embeddings \( f : X \to Y \) and \( g : Y \to Z \), one has \( \tau(g \circ f) = \tau(g) \circ \tau(f) \).

(ii) For every \( X \), if \( f \) is the identity function on \( X \), then \( \tau(f) \) is the identity function on \( \tau(X) \).

Also, let \( \mathcal{C}_\tau \) be the category whose objects are of the form \((X,a)\), where \( X \) is an object in \( \text{Comp} \) and \( a \in \tau(X) \). Let the set of morphisms between \((X,a)\) and \((Y,b)\) be \( \{ f \in \text{Hom}(X,Y) : \tau(f)(a) = b \} \). So the set \( \mathcal{C}_\tau \), defined in Subsection 3.1, is the set of isomorphism-classes of the objects of \( \mathcal{C}_\tau \) (it can be seen that \( \mathcal{C}_\tau \) is indeed a set).

To define a metric on \( \mathcal{C}_\tau \), we need to assume some continuity properties of \( \tau \). There are various notions of convergence of morphisms in either \( \text{Comp} \) or \( \text{Met} \). These notions can be used to define the following.

**Definition 3.8.** A functor \( \tau : \text{Comp} \to \text{Met} \) is called pointwise-continuous when for all compact metric spaces \( X \) and \( Y \) and all sequences of isometric embeddings \( f, f_1, f_2, \ldots \) from \( X \) to \( Y \), if \( f_n \to f \) pointwise, then \( \tau(f_n) \to \tau(f) \) pointwise.

The functor \( \tau \) is called Hausdorff-continuous when for every sequence of compact metric spaces \( Z, X_1, X_2, \ldots \) and isometric embeddings \( f : X \to Z \) and \( f_n : X_n \to Z \) (for \( n = 1, 2, \ldots \)), if \( d_H(\text{Im}(f_n), \text{Im}(f)) \to 0 \), then \( d_H(\text{Im}(\tau(f_n)), \text{Im}(\tau(f))) \to 0 \). See also Remark 3.17 below.

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1 This is different from the classical category of metric spaces in the literature in which morphisms are the functions which do not increase the distance of points.
The following is the main results of this subsection. The proofs of all of the results are postponed to Subsection 3.6.

**Theorem 3.9** (Metric). For any functor \( \tau : \text{Comp} \to \text{Met} \) that is pointwise-continuous (Definition 3.8), the following is a metric on \( \mathcal{C}_\tau \).

\[
d_c^\tau((X,a),(Y,b)) := \inf_{Z,f,g} \left\{ d_H(f(X),g(Y)) \vee d(\tau_f(a),\tau_g(b)) \right\}, \tag{3.1}
\]

where the infimum is over all compact metric spaces \( Z \) and all isometric embeddings \( f : X \to Z \) and \( g : Y \to Z \).

Note that in this theorem, \( d_c^\tau((X,a),(Y,b)) \) is finite and depends only on the isomorphism-classes of \((X,a)\) and \((Y,b)\). So, it is well defined as a distance function on \( \mathcal{C}_\tau \). We may call \( d_c^\tau \) the **Gromov-Hausdorff-functor metric**.

**Remark 3.10.** It is clear that the metric (3.1) generalizes the Gromov-Hausdorff-Prokhorov metric (2.3) (see also Example 3.2).

**Remark 3.11.** Without the assumption of pointwise-continuity, the proof of the theorem shows that \( d_c^\tau \) is a pseudo-metric on \( \mathcal{C}_\tau \).

In many examples, the functors under study satisfy the following stronger conditions.

**Definition 3.12.** A functor \( \tau : \text{Comp} \to \text{Met} \) is called **pointwise-M-Lipschitz** (where \( M < \infty \) is given) if for all morphisms \( f, g : X \to Y \), one has

\[
d_{\sup}(\tau_f,\tau_g) \leq M \cdot d_{\sup}(f,g),
\]

where \( d_{\sup} \) is the sup metric. Also, it is called **Hausdorff-M-Lipschitz** if for all morphisms \( f : X \to Z \) and \( g : Y \to Z \), one has

\[
d_H(\text{Im}(\tau_f),\text{Im}(\tau_g)) \leq M \cdot d_H(\text{Im}(f),\text{Im}(g)).
\]

**Example 3.13.** Let \( \tau(X) \) be either the set of points of \( X \), finite measures on \( X \) or nonempty compact subsets of \( X \) as in the examples of Subsection 3.1 (see also Remark 3.15 below for considering the empty set). It can be seen that \( \tau \) is a functor in each case and is both pointwise-continuous and Hausdorff-continuous. More generally, it satisfies the 1-Lipschitz properties of Definition 3.12 and in addition, equality holds in the inequalities of Definition 3.12 (for the case of measures, use Strassen’s theorem [24]).

**Example 3.14.** If \( \tau_1, \tau_2, \ldots \) are functors which are pointwise-continuous (resp. Hausdorff-continuous), then so is their product (as in Example 3.5). A similar result holds for the \( M \)-Lipschitz properties of Definition 3.12.

**Remark 3.15** (Extended Metrics). One can let \( \tau(X) \) be the set of compact subsets of \( X \) including the empty set and equipped with the extended metric \( d_H \) (see Remark 3.4). If so, one gets \( d_c^\tau((X,\emptyset),(Y,K)) = \infty \) whenever \( K \neq \emptyset \).
In this case, $d^\tau_X$ is an extended metric, but the results of this section still hold. More generally, one can replace $\mathsf{Met}$ with the category of compact extended metric spaces and the results of this section remain valid. Note that from the beginning, one could replace $d_H$ with $d_H/(1+d_H)$ as in Remark 3.4. This would ensure that $d^\tau_X$ is indeed a metric in this example. However, we preferred to proceed with the extended metric $d_H$. This is useful e.g., in the Strassen-type result Proposition 3.28 below.

3.3 Completeness, Separability and Precompactness

Consider the metric space $C^\tau$ under the assumptions of Theorem 3.9.

**Theorem 3.16 (Polishness).** Let $\tau : \mathsf{Comp} \to \mathsf{Met}$ be a functor which is both pointwise-continuous and Hausdorff-continuous.

(i) If $\tau(X)$ is complete for every compact metric space $X$, then $C^\tau$ is also complete.

(ii) If $\tau(X)$ is separable for every compact metric space $X$, then $C^\tau$ is also separable.

**Remark 3.17.** In the above theorem, the assumption of Hausdorff-continuity can be replaced by the following assumptions: For every sequence of compact metric spaces $Z, X, X_1, X_2, \ldots$ and isometric embeddings $f : X \to Z$ and $f_n : X_n \to Z$ (for $n = 1, 2, \ldots$) such that $d_H(\text{Im}(f_n), \text{Im}(f)) \to 0$,

(i) If $b \in \tau(Z)$ and $a_n \in \tau(X_n)$ (for all $n$) are such that $\tau f_n(a_n) \to b$, then $b \in \text{Im}(\tau f)$.

(ii) For every $a \in \tau(X)$, there exists a sequence $a_n \in \tau(X_n)$ such that $\tau f_n(a_n) \to \tau f(a)$.

The first (resp. second) assumption is enough for completeness (resp. separability) of $C^\tau$ in Theorem 3.16. By assuming that $\tau(X)$ is complete for every $X$, these assumptions are weaker than Hausdorff-continuity. Together, they are equivalent to the following condition:

$$\text{Im}(\tau f) = \bigcap_n \bigcup_{m \geq n} \text{Im}(\tau f_n).$$

**Theorem 3.18 (Pre-compactness).** Let $\tau : \mathsf{Comp} \to \mathsf{Met}$ be a functor which is pointwise-continuous and satisfies Condition (1) of Remark 3.17. Then a set $A \subseteq C^\tau$ is pre-compact if and only if both of the following conditions hold.

(i) The underlying compact sets of the elements of $A$ form a pre-compact subset of $\mathcal{N}^\tau$ (equipped with the Gromov-Hausdorff metric (2.2)).

(ii) One can select a compact subset $\tau'(X) \subseteq \tau(X)$ for every $X \in \mathsf{Comp}$ such that $\tau'$ is a functor (by letting $\tau'_f$ be the restriction of $\tau_f$, for every morphism $f$ of $\mathsf{Comp}$) and $C^\tau \supseteq A$. 

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Remark 3.19. by Theorem 7.4.13 of [10], condition (i) of the above theorem is equivalent to the following: For every $\epsilon > 0$, there exists $N < \infty$ such that every element of $A$ can be covered by at most $N$ balls of radius $\epsilon$.

In addition, if $\tau(X)$ is the set of finite measures on $X$ (Example 3.2), then Condition (ii) in the above theorem is equivalent to the existence of $M < \infty$ such that all of the distinguished measures on the elements of $A$ have total mass at most $M$ (see Theorem 2.6 of [1]). This fact is generalized to the following proposition.

Proposition 3.20. Let $\tau$ and $A$ be as in Theorem 3.18 such that Condition (i) of the theorem holds. Assume that there exists a fixed metric space $E$ and a continuous function $h_X : \tau(X) \to E$ for every object $X$ of $\mathsf{Comp}$ such that $h_X$ is a proper map and is compatible with the morphisms (i.e., $h_Y \circ \tau f = h_X$ for every morphism $f : X \to Y$). Then, Condition (ii) in Theorem 3.18 is equivalent to the existence of a compact set $E' \subseteq E$ such that all elements $(X, a) \in A$ satisfy $h_X(a) \in E'$.

3.4 Examples

The following are further instances of the abstract definitions of Subsection 3.2. Some other examples are provided in Section 5 and their connections to other notions in the literature are discussed.

3.4.1 Marks

In the following, we will consider metric spaces equipped with marks (Examples 3.24 and 3.25). This will be used in other examples as well, including curves (Subsection 3.4.2), spatial trees (Subsection 5.4) and in Subsection 5.5.

Fix a complete separable metric space $\Xi$ as the mark space (in some cases in what follows, $\Xi$ is required to be boundedly-compact). If $X$ is a finite or discrete set, a marking of $X$ can be defined as a function from $X$ to $\Xi$ and the space of markings of $X$ is simply $\Xi^X$. However, in the general (non-discrete) case, $\Xi^X$ is not suitable to be regarded as a metric space and measure theoretic issues appear. Two candidates are provided here for defining markings: marked measures and marked closed subsets, defined below, which are inspired by the notion of marked random measures in stochastic geometry (see Subsection 5.6.2).

In some applications, it is convenient to assign marks to the $k$-tuples of points (see e.g., Subsections 5.1 and 5.5). In the discrete case, the set $\Xi^{X^k}$ can be regarded as space of markings. This is also generalized in the following.

Definition 3.21. Let $X$ be a boundedly-compact metric space and $k \in \mathbb{N}$. A $k$-marked measure on $X$ is a Borel measure on $X^k \times \Xi$. Also, a $k$-marked closed subset of $X$ is a closed subset of $X^k \times \Xi$. The number $k$ is called the order of the marked measure/closed subset.

In the discrete case, it is straightforward to see that both notions generalize the notion of markings (identify every function $f : X \to \Xi$ with its graph and the counting measure on its graph).
The following example motivates the names ‘marked measure’ and ‘marked closed subset’.

**Example 3.22.** If $\mu$ is a boundedly-finite measure on $X$ and $f : X \to \Xi$ is a measurable function, then the push-forward of $\mu$ under the map $x \mapsto (x, f(x))$ is a marked measure on $X$. Similarly, if $C$ is a closed subset of $X$ and $f : X \to \Xi$ is a continuous map, then $\{(x, f(x)) : x \in C\}$ is a marked closed subset of $X$.

**Remark 3.23.** Every point or closed subset of $X$ is a marked compact subset of $X$. In addition, the metrics of Examples 3.1 and 3.3 are identical with the corresponding restrictions of the metric on $\tau^{(c)}(X)$. Similarly, measures are special cases of marked measures and the metrics are compatible.

**Example 3.24** (Marks). Fix $k \in \mathbb{N}$. For every compact metric space $X$, let $\tau^{(f)}(X)$ be the set of finite $k$-marked measures on $X$ equipped with the Prokhorov metric, where $k \in \mathbb{N}$ is given. Also, let $\tau^{(c)}(X)$ be the set of $k$-marked nonempty compact subsets of $X$ (i.e., nonempty compact subsets of $X^k \times \Xi$) equipped with the Hausdorff metric (one can also allow $\emptyset \in \tau^{(c)}(X)$ according to Remark 3.4). For every isometric embedding $g : X \to Z$, define $\tau^{(f)}_g$ and $\tau^{(c)}_g$ similarly to Examples 3.2 and 3.3 (consider the map from $X^k \times \Xi$ to $Z^k \times \Xi$ defined by $(x_1, \ldots, x_k, \xi) \mapsto (g(x_1), \ldots, g(x_k), \xi)$).

It can be seen that $\tau^{(f)}$ and $\tau^{(c)}$ are functors and are both pointwise-continuous and Hausdorff-continuous, and in addition, satisfy the 1-Lipschitz properties of Definition 3.12 (for $\tau^{(f)}$, use Strassen’s theorem [24]). In addition, it is known that for every $X$, $\tau^{(f)}(X)$ and $\tau^{(c)}(X)$ are complete separable metric spaces (and the latter is compact by Blaschke’s theorem). Therefore, Theorems 3.9 and 3.16 imply that $\mathcal{C}_{\tau^{(f)}}$ and $\mathcal{C}_{\tau^{(c)}}$ are complete separable metric spaces.

**Example 3.25** (Boundedly-Compact Mark Space). If the mark space $\Xi$ is boundedly-compact, one can let $\tau^{(s)}(X)$ be the set of marked closed (not necessarily compact) subsets of $X$ and $\tau^{(m)}(X)$ be the set of boundedly-finite marked measures on $X$. By the assumption of boundedly-compactness of $\Xi$, one can equip $\tau^{(s)}(X)$ and $\tau^{(m)}(X)$ with modifications of the Hausdorff and Hausdorff-Prokhorov metrics respectively (see e.g., Remark 3.21 of [21]) and they become complete separable metric spaces. Also, it can be seen that the functors $\tau^{(s)}$ and $\tau^{(m)}$ satisfy the continuity properties of Definition 3.18 Therefore, the corresponding spaces $\mathcal{C}_{\tau^{(s)}}$ and $\mathcal{C}_{\tau^{(m)}}$ are also complete separable metric spaces.

**Remark 3.26.** For the functors $\tau^{(f)}$ and $\tau^{(c)}$, one can simplify the precompactness result (Theorem 3.18) using Proposition 3.20. If $\tau(X) = \tau^{(f)}(X)$ is the set of finite $k$-marked measures on $X$, then Condition (ii) in Theorem 3.18 is equivalent to the existence of $M < \infty$ such that all of the distinguished $k$-marked measures of the elements of $\mathcal{A}$ have total mass at most $M$. This extends Remark 3.20.

Also, if $\tau(X) = \tau^{(c)}(X)$ is the set of $k$-marked compact subset of $X$, then Condition (iii) in the pre-compactness theorem is equivalent to the existence of a compact set $\Xi' \subseteq \Xi$ such that all elements $(X, a) \in \mathcal{A}$ satisfy $a \subseteq X^k \times \Xi'$. 

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3.4.2 Continuous Curves

In this example, we consider the set of compact metric spaces equipped with a continuous curve defined in [18] (discussed further in Subsection 3.3.3). Let $\tau(X)$ be the set of all continuous curves $\eta : \mathbb{R} \to X$ such that $\eta$ is convergent as $t \to \infty$ and $t \to -\infty$ (similarly, one can let $\tau(X)$ be the set of continuous curves $\eta : I \to X$, where $I$ is a given compact interval). One can equip $\tau(X)$ with the sup metric. Also, by letting $\tau_f(\eta) := f \circ \eta$, $\tau_f$ is an isometric embedding. By considering another suitable metric (see e.g., Example 4.20), one can also remove the assumption of convergence in this example.

In this example, $\tau$ is a pointwise-continuous functor, but it is not Hausdorff-continuous. However, it satisfies condition (1) of Remark 3.17. So the results of this section imply that $C_\tau$ is a complete metric space. In this case, separability of $C_\tau$ is proved in [18].

Another proof of separability of $C_\tau$ is by regarding curves as marked closed subsets as follows (a third proof is given in Remark 3.27 below). Every continuous curve $\eta : \mathbb{R} \to X$ can be identified with the marked closed subset $\{\eta(t), t \in \mathbb{R}\} \subseteq X \times \mathbb{R}$ of $X$. So the set $\tau(X)$ of this example can be regarded as a subset of $\tau^{(s)}(X)$ (see Example 3.25). The metric on $\tau(X)$ is not equivalent to the restriction of the metric of $\tau^{(s)}(X)$, but it can be seen that these metrics generate the same topology on $\tau(X)$. It can also be seen that $\tau(X)$ is a Borel subset (and in fact, a $F_{\sigma\delta}$ subset) of $\tau^{(s)}(X)$. To see the latter, note that $\tau(X)$ can be written as $\bigcap_n \cup_m \{\eta \in \tau(X) : w_\eta([\frac{1}{m}, t]) \leq \frac{1}{n}\}$, where $w_\eta(t) := \max\{d(\eta(s), \eta(t)) : s, t \in \mathbb{R}, |s - t| \leq \epsilon\}$ is the modulus of continuity of $\eta$ and the sets under union are closed subsets of $\tau^{(s)}(X)$.

Here, the set $C_\tau$ can be regarded as a subset of $C_{\tau^{(s)}}$. By the above discussion, it can be seen that the topology of $C_\tau$ agrees with the induced topology from $C_{\tau^{(s)}}$. In addition, it can be proved that $C_\tau$ is a Borel subset of the larger space. So separability of $C_\tau$ is implied by Theorem 3.16 for marked closed sets.

3.4.3 Càdlàg curves

Here, we consider the set $D^c$ of compact metric spaces equipped with a càdlàg curve. It will be shown that $D^c$ can be turned into a complete separable metric space by the method of this section.

For all compact metric spaces $X$, let $\tau_0(X)$ be the set of càdlàg curves $\eta : \mathbb{R} \to X$ (a càdlàg curve is a function that is right-continuous and has left-limits at all points). The Skorokhod metric (see e.g., [9]) is defined on $\tau_0(X)$ and makes it a complete separable metric space. One can regard $\tau_0$ as a functor similarly to Subsection 3.4.2. It can be seen that this functor is both pointwise-continuous and Hausdorff-continuous. More generally, it has the $(1+\epsilon)$-Lipschitz properties of Definition 3.12 for every $\epsilon > 0$. So, the results of this section define a metric on $D^c$ and make it a complete separable metric space.

Remark 3.27. Every continuous curve is càdlàg. So, the set $\tau(X)$ of Subsection 3.4.2 is a subset of $\tau_0(X)$. Therefore, $C_\tau \subseteq D^c$. It can be seen that $\tau(X)$ (resp. $C_\tau$) is a closed subset of $\tau_0(X)$ (resp. $D^c$) and their topologies are
compatible. Since it is proved that $\mathcal{D}$ is separable, this provides another proof of separability of $\mathcal{C}_\tau$.

### 3.5 A Strassen-Type Result

The Gromov-Hausdorff metric can be described in terms of correspondences, recalled below (see e.g., Theorem 7.3.25 of [10]). By Strassen’s theorem [24], the Prokhorov distance of two probability measures can be described in terms of approximate couplings. This is generalized in [21] for the Prokhorov distance of two finite measures. Also, a Strassen-type result is given in [21] for the Gromov-Hausdorff-Prokhorov metric. These facts are generalized below to a Strassen-type result for marked measures, marked closed subsets and curves.

The following definitions should be recalled. A correspondence between metric spaces $X$ and $Y$ is a relation $R \subseteq X \times Y$ such that $\pi_1(R) = X$ and $\pi_2(R) = Y$, where $\pi_1$ and $\pi_2$ are the two projections from $X \times Y$ onto $X$ and $Y$ respectively. The distortion of $R$ is

$$\text{dis}(R) := \sup\{ |d(x, x') - d(y, y')| : (x, y) \in R, (x', y') \in R \}.$$ 

If $\mu$ and $\nu$ are measures on $X$, the total variation distance of $\mu$ and $\nu$ is

$$||\mu - \nu|| := \sup\{ |\mu(A) - \nu(A)| : A \subseteq X \}.$$ 

For finite Borel measures $\alpha$ on $X \times X$, the discrepancy of $\alpha$ w.r.t. $\mu$ and $\nu$ is defined by

$$D(\alpha; \mu, \nu) := ||\pi_1 \ast \alpha - \mu|| + ||\pi_2 \ast \alpha - \nu||.$$ 

Now, consider the following instance of additional structures. Fix $m \in \mathbb{N}$. For compact metric spaces $X$ and $Y$ is a relation $R \subseteq X \times Y$ such that $\pi_1(R) = X$ and $\pi_2(R) = Y$, where $\pi_1$ and $\pi_2$ are the two projections from $X \times Y$ onto $X$ and $Y$ respectively. The distortion of $R$ is

$$\text{dis}(R) := \sup\{ |d(x, x') - d(y, y')| : (x, y) \in R, (x', y') \in R \}.$$ 

If $\mu$ and $\nu$ are measures on $X$, the total variation distance of $\mu$ and $\nu$ is

$$||\mu - \nu|| := \sup\{ |\mu(A) - \nu(A)| : A \subseteq X \}.$$ 

For finite Borel measures $\alpha$ on $X \times X$, the discrepancy of $\alpha$ w.r.t. $\mu$ and $\nu$ is defined by

$$D(\alpha; \mu, \nu) := ||\pi_1 \ast \alpha - \mu|| + ||\pi_2 \ast \alpha - \nu||.$$ 

Now, consider the following instance of additional structures. Fix $m \in \mathbb{N}$. For compact metric spaces $X$, let $\tau(X)$ be the set of all tuples $a = (a_1, \ldots, a_m)$, where each $a_i$ is either a finite marked measure on $X$ as in Example 3.24, a marked compact subset of $X$ or a convergent continuous curve in $X$ as in Subsection 3.4.2 (each $a_i$ can have its own $k$ and its own mark space, but its type depends only on $i$ and should not depend on $X$). As in Example 3.5, equip $\tau(X)$ with a product of the Hausdorff and Prokhorov metrics. Recall that this type of additional structures includes points, closed subsets, finite measures and curves (see also Subsection 5.4 for another instance).

If $R$ is a correspondence between $X, Y$ and $\epsilon \geq 0$, let $R_{k, \epsilon}$ be the correspondence between $X^k \times \Xi$ and $Y^k \times \Xi$ defined by

$$R_{k, \epsilon} := \{ ((x_1, \ldots, x_k, \xi), (y_1, \ldots, y_k, \xi')) : \forall i : (x_i, y_i) \in R, d(\xi, \xi') \leq \epsilon \}.$$ 

**Proposition 3.28.** Consider the specific functor $\tau$ defined above. Let $\mathcal{X} = (X, a_1, \ldots, a_m)$ and $\mathcal{Y} = (Y, b_1, \ldots, b_m)$ be elements of $\mathcal{C}_\tau$ and $\epsilon \geq 0$. Then $d_{\tau}(\mathcal{X}, \mathcal{Y}) \leq \epsilon$ if and only if there exists a correspondence $R$ between $X$ and $Y$ such that $\text{dis}(R) \leq 2\epsilon$ and for each $i \leq m$, one has

(i) If $a_i$ and $b_i$ are $k$-marked closed subsets, where $k \in \mathbb{N}$, then every point in the former $R_{k, \epsilon}$-corresponds to some point in the latter and vice versa.
(ii) If $a_i$ and $b_i$ are $k$-marked measures with mark space $\Xi$, where $k \in \mathbb{N}$, then there exists a measure $\alpha$ on $(X^k \times \Xi) \times (Y^k \times \Xi)$ such that $D(\alpha; a_i, b_i) + \alpha(R_{k,\epsilon}) \leq \epsilon$.

(iii) If $a_i$ and $b_i$ are curves, then $\forall t : (a_i(t), b_i(t)) \in R$.

This proposition generalizes Proposition 9 of [22], where each $a_i$ is a compact subset, and Theorem 3.6 of [21] for the Gromov-Hausdorff-Prokhorov metric (where $m = 1$ and the additional structure is a finite measure).

**Proof of Proposition 3.28** The proof is similar to that of Theorem 3.6 of [21] and is skipped for brevity.

**Remark 3.29.** In the above proposition, one can also consider countably many marked measures or marked closed subsets. For this, let $\tau(X)$ be the set of sequences $a = (a_1, a_2, \ldots)$, where each $a_i$ is as above and consider the metric on $\tau(X)$ defined in Example 3.5. One can obtain a result similar to Proposition 3.28.

### 3.6 Proofs and Other Lemmas

Now, the proofs of the results of this section are provided. The following lemmas are needed for proving the main results.

**Lemma 3.30.** For compact metric spaces $X$ and $Z$, the set of isometric embeddings $f : X \to Z$, equipped with the sup metric, is compact. In addition, the topology of this set is identical to the topology of pointwise convergence.

This lemma is standard and its proof is skipped.

**Lemma 3.31.** In taking infimum in (3.1), one can add the condition $Z = f(X) \cup g(Y)$ and the value of the infimum is not changed.

**Proof.** Let $Z, f, g$ be as in (3.1). Let $Z' := f(X) \cup g(Y)$ and $\iota : Z' \hookrightarrow Z$ be the inclusion map. Let $f' \in \text{Hom}(X, Z')$ and $g' \in \text{Hom}(Y, Z')$ be obtained by restrictions of $f$ and $g$ respectively. One has $\iota \circ f' = f$ and $\iota \circ g' = g$ as morphisms in $\text{Comp}$.

Therefore, $\tau_i \circ \tau_{f'} = \tau_f$ and $\tau_i \circ \tau_{g'} = \tau_g$. Since $\iota$ is an isometric embedding, one has $d_H(f(X), g(Y)) = d_H(f'(X), g'(Y))$. In addition, since $\tau_i$ is also an isometric embedding (by the definition of $\tau$), one gets that $d(\tau_f(a), \tau_g(b)) = d(\tau_{f'}(a), \tau_{g'}(b))$. This proves the claim. \[\square\]
Lemma 3.32. If $Z$ is a complete metric space, then the set of compact subsets of $Z$, equipped with the Hausdorff metric, is complete.

Proof. Let $K_1, K_2, \ldots$ be a Cauchy sequence of compact subsets of $Z$. By Proposition 7.3.7 of [10], the set of closed subsets of $Z$ is complete under the Hausdorff metric. So there is a closed subset $K$ such that $d_H(K, K_n) \to 0$. It will be shown that $K$ is compact.

Let $x_1, x_2, \ldots$ be a sequence in $K$. For each $i, j$, there exists $y_{i,j} \in K_i$ such that $d(y_{i,j}, x_j) \leq \epsilon_i$. For each $i$, since $K_i$ is compact, the sequence $(y_{i,j})_j$ has a convergent subsequence. By a diagonal argument and passing to a subsequence, one can assume from the beginning that for each $i$, the whole sequence $(y_{i,j})_j$ is convergent. It is shown below that $(x_j)_j$ is a Cauchy sequence. If so, completeness of $Z$ implies that $(x_j)_j$ is convergent and the claim is proved.

Let $\delta > 0$ be arbitrary. There exists $i$ such that $\epsilon_i < \delta/3$. Since $(y_{i,j})_j$ is Cauchy, there exists $N$ such that for all $j_1, j_2 > N$, one has $d(y_{i,j_1}, y_{i,j_2}) < \delta/3$. It follows that for all $j_1, j_2 > N$, one has

$$d(x_{j_1}, x_{j_2}) \leq d(x_{j_1}, y_{i,j_1}) + d(y_{i,j_1}, y_{i,j_2}) + d(y_{i,j_2}, x_{j_2}) \leq \delta.$$  

This proves that $(x_j)_j$ is a Cauchy sequence and the claim is proved. \hfill $\square$

Lemma 3.33. Let $\mathcal{X} = (X_n, a_n)$ be an object in $\mathcal{C}_\tau$ and $\epsilon_n > 0$ for $n = 1, 2, \ldots$ such that $d_\tau(X_n, X_{n+1}) < \epsilon_n$ for each $n$. If $\sum \epsilon_n < \infty$, then there exists a compact set $Z$ and isometric embeddings $f_n : X_n \to Z$ such that for all $n$, one has

$$d_H(f_n(X_n), f_{n+1}(X_{n+1})) \leq \epsilon_n, \quad (3.2)$$

$$d(\tau f_{n}(a_n), \tau f_{n+1}(a_{n+1})) \leq \epsilon_n, \quad (3.3)$$

Proof. By (3.1), there exists a compact metric space $Z_n$ for every $n$ and isometric embeddings $g_n : X_n \to Z_n$ and $h_n : X_{n+1} \to Z_n$ such that

$$d_H(g_n(X_n), h_n(X_{n+1})) \leq \epsilon_n, \quad (3.4)$$

$$d(\tau g_n(a_n), \tau h_n(a_{n+1})) \leq \epsilon_n. \quad (3.5)$$

By Lemma 3.31 one can assume $Z_n = g_n(X_n) \cup h_n(X_{n+1})$ without loss of generality.

Let $Z_\infty$ be the quotient of the disjoint union $\bigcup_n Z_n$ by identifying $h_n(x) \in Z_n$ with $g_{n+1}(x) \in Z_{n+1}$ for every $n$ and every $x \in X_{n+1}$. The quotient metric on $Z_\infty$ can be described as follows: for $z_i \in Z_i$ and $z_j \in Z_j$, if $i \leq j \leq k$, then $d_{Z_\infty}(z_i, z_k)$ is the shortest distance between $z_i$ and $z_j$ in $Z_j$.
The distance of $z_i$ and $z_j$ is the length of the shortest path of the form $(z_i, h_i(x_{i+1}), h_{i+1}(x_{i+2}), \ldots, h_{j-1}(x_j), z_j)$, where $\forall k : x_k \in Z_k$ and the distance of consecutive pairs in this path are considered under the metrics of $Z_i, Z_{i+1}, \ldots, Z_j$ respectively (note that $h_k(x_{k+1})$ is an element of $Z_k$ and is identified with the element $g_k(x_{k+1})$ of $Z_{k+1}$). It is straightforward that this gives a metric on $Z_\infty$ and the natural map from $Z_n$ to $Z_\infty$ is an isometric embedding for each $n$ (see Lemma 5.7 in [10]).

Let $Z$ be the metric completion of $Z_\infty$ and $\tilde{Z}_n$ be the quotient of $Z_1 \sqcup \cdots \sqcup Z_n$ defined similarly. We may regard $\tilde{Z}_n$ as a subset of $Z$, which gives $Z_\infty = \sqcup_n \tilde{Z}_n$.

Inequality (3.3) implies that $d_H(\tilde{Z}_n, \tilde{Z}_{n+1}) \leq \epsilon_n$. So the assumption $\sum \epsilon_n < \infty$ implies that $d_H(\tilde{Z}_n, Z)$ is finite and tends to zero as $n \to \infty$. Since $Z$ is complete and each $\tilde{Z}_n$ is compact, Lemma 3.32 implies that $Z$ is compact. Let $\iota_n : Z_n \to Z$ be the natural isometric embedding and $f_n := \iota_n \circ g_n$.

Since $\iota_n$ is an isometric embedding and $\iota_n(h_n(X_{n+1})) = \iota_{n+1}(g_{n+1}(X_{n+1}))$, (3.4) implies that $d_H(f_n(X_n), f_{n+1}(X_{n+1})) \leq \epsilon_n$, which proves (3.2). Similarly, since $\tau_n$ is an isometric embedding, (3.3) implies (3.3). So the claim is proved.

\[ \square \]

**Remark 3.34.** Lemma 3.33 is similar to Lemma 5.7 in [10]. The latter is for metric measure spaces and does not assume $\sum \epsilon_n < \infty$. So the metric space $Z$ is not necessarily compact therein.

Also, a similar statement holds for every sequence of compact metric spaces $(X_n)_n$ without any additional structure (by deleting $a_n$ and $a$ in Lemma 3.33 and by replacing $d_c$ with $d_{GH}$). This claim is implied by Lemma 3.33 by considering the functor $\tau(X) = \{0\}$ for all $X$.

**Lemma 3.35.** If $d_c((X, a_n), (X, a)) \to 0$, then there exists a compact metric space $Z$ and isometric embeddings $f : X \to Z$ and $f_n : X \to Z$ such that $f_n(X_n) \to f(X)$ in the Hausdorff metric and $\tau_{f_n}(a_n) \to \tau_f(a)$.

**Proof.** The proof is similar to that of Lemma 3.33 and is only sketched here. Let $\epsilon_n > d_c((X, a_n), (X, a))$ such that $\epsilon \to 0$. Embed $X_n$ and $X$ in a common space $Z_n$ as in (3.1). Then, let $Z$ be the gluing all of $Z_1, Z_2, \ldots$ along the copies of $X$ in all of the sets $Z_n$. It can be proved similarly to Lemma 3.33 that $Z$ is compact and can be used as the desired space.

We are now ready to prove the theorems.

**Proof of Theorem 3.9** It is clear that $d_c$ is symmetric. For the triangle inequality, let $X = (X, a), Y = (Y, a)$ and $Z = (Z, c)$ be elements of $\mathcal{F}$. Assume $d_c(X, Y) < \epsilon$ and $d_c(Y, Z) < \delta$. It is enough to prove that $d_c(X, Z) \leq \epsilon + \delta$. By Lemma 3.33 there exists a compact metric space $H$ and isometric embeddings $f_X : X \to H, f_Y : Y \to H$ and $f_Z : Z \to H$ such that

\[
\begin{align*}
d_H(f_X(X), f_Y(Y)) &\leq \epsilon, \\
d_H(f_Y(Y), f_Z(Z)) &\leq \delta, \\
d(\tau_{f_X}(a), \tau_{f_Y}(b)) &\leq \epsilon, \\
d(\tau_{f_Y}(b), \tau_{f_Z}(c)) &\leq \delta.
\end{align*}
\]
The first two inequalities imply that \( d_H(f_X(X), f_Z(Z)) \leq \epsilon + \delta \). The last two imply that \( d(\tau f, \tau f^e) \leq \epsilon + \delta \). So the definition (3.1) implies that \( d_c^e(X, Z) \leq \epsilon + \delta \) and the triangle inequality is proved.

Now let \( X = (X, a) \) and \( Y = (Y, b) \) be such that \( d_c^e(X, Y) = 0 \). Consider Lemma 3.33 for the sequence \( X, Y, X, Y, \ldots \) and \( \epsilon_n := 2^{-n} \) for each \( n \). The lemma implies that there exists a compact metric space \( Z \) and isometric embeddings \( f_n : X \to Z \) and \( g_n : Y \to Z \) such that

\[
\begin{align*}
d_H(f_n(X), g_n(Y)) &\leq 2^{-n}, \\
d(\tau f_n(a), \tau g_n(b)) &\leq 2^{-n}.
\end{align*}
\]

By Lemma 3.30 one can assume \( d_{\sup}(f, f) \to 0 \) and \( d_{\sup}(g, g) \to 0 \), where \( f : X \to Z \) and \( g : Y \to Z \) are isometric embeddings. Therefore, \( f_n(X) \to f(X) \) and \( g_n(Y) \to g(Y) \) under the Hausdorff metric. So (3.6) implies that \( f(X) = g(Y) \). So there is an isometry \( h : X \to Y \) such that \( f = g \circ h \). Moreover, the assumption of pointwise-continuity of \( \tau \) implies that \( \tau f_n(a) \to \tau f(a) \) and \( \tau g_n(b) \to \tau g(b) \). So (3.7) implies that \( \tau f(a) = \tau g(b) \). This implies that \( \tau h(a) = b \). This proves that \( (X, a) \) is isomorphic to \( (Y, b) \) and the claim is proved.

**Proof of Theorem 3.16.** Let \( (X, a_n) \) be a sequence of elements of \( C_\tau \) such that the corresponding elements in \( C_\tau \) form a Cauchy sequence. By taking a subsequence (if necessary) and using Lemma 3.33 one can assume there exists a compact metric space \( Z \) and isometric embeddings \( f_n : X_n \to Z \) such that

\[
\begin{align*}
d_H(f_n(X_n), f_n(X_{n+1})) &\leq 2^{-n}, \\
d(\tau f_n(a_n), \tau f_{n+1}(a_{n+1})) &\leq 2^{-n}.
\end{align*}
\]

So the sequences \( (f_n(X_n))_n \) and \( (\tau f_n(a_n))_n \) are Cauchy. Lemma 3.32 and the assumption of completeness of \( \tau(Z) \) imply that there exists a compact subset \( X \subseteq Z \) and \( b \in \tau(Z) \) such that \( f_n(X_n) \to X \) and \( \tau f_n(a_n) \to b \). Let \( \iota : X \to Z \) be the inclusion map. The definition of Hausdorff-continuity implies that \( d_H(\Im(\iota f), \Im(\iota g)) \to 0 \). This implies that \( b \) is in the closure of \( \Im(\iota) \). On the other hand, since \( \tau(X) \) is complete (by assumption) and \( \iota \) is an isometric embedding, one gets that \( \Im(\iota) \) is also complete, and hence, closed in \( \tau(Z) \). So \( b \in \Im(\iota) \); i.e., there exists \( a \in \tau(X) \) such that \( \tau(a) = b \). Now, one can obtain that \( d_{GH}^e((X_n, a_n), (X, a)) \to 0 \). This proves that \( C_\tau \) is complete.

As mentioned in Section 2.2 the space \( \mathfrak{K}^c \) of compact metric spaces is separable under the Gromov-Hausdorff metric \( d_{GH}^c \). Let \( A \) be a sequence of compact metric spaces which is dense in \( \mathfrak{K}^c \). By assumption, for every \( X \in A \), there exists a countable dense subset \( C(X) \) of \( \tau(X) \). It is enough to prove that the set \( E := \{(X, a) : X \in A, a \in C(X)\} \) is dense in \( C_\tau \).

Let \( (X, a) \in C_\tau \) be arbitrary. For every \( n > 0 \), there exists \( X_n \in A \) such that \( d_{GH}^c(X_n, X) \leq 2^{-n} \). By Lemma 3.33 there exists a compact metric space \( Z \) and isometric embeddings \( f : X \to Z \) and \( f_n : X_n \to Z \) such that \( d_H(f(X), f_n(X_n)) \to 0 \). The assumption of Hausdorff-continuity of \( \tau \) implies that \( d_H(\Im(\iota f), \Im(\iota f_n)) \to 0 \). So one can select an element \( a_n \in \tau(X_n) \) for each \( n \) such that \( d(\tau f_n(a_n), \tau f(a)) \to 0 \). Since \( C(X) \) is dense in \( \tau(X_n) \), one can
choose \( a_n \) such that \( a_n \in \mathcal{C}(X_n) \). This implies that \( d_{\tau}(\mathcal{C}(X_n), \mathcal{C}(X), \tau) \to 0 \) and the claim is proved.

\[ \text{Proof of Theorem 3.10 (\( \Leftarrow \))}. \) Assume \( (\text{ii}) \) and \( (\text{iii}) \) hold. Let \( (X_n, a_n) \) be a sequence in \( \mathcal{C}_{\tau} \). We should prove that it has a convergent subsequence. By the definition of \( \mathcal{C}_{\tau} \), we may assume that \( (X_n, a_n) \) converges. For each \( n \), let \( K_n \) be the gluing of \( X_n \) and \( a_n \) isometrically embedded into a common compact metric space \( H \). So, by Lemma 3.35, all of \( K_1, K_2, \ldots \) are isometrically embeddable into a common compact metric space \( H \) such that their images in \( H \) are convergent.

\[ Y_n \xrightarrow{f_n} X \]

\[ Y \xrightarrow{g_n} Z \quad \xrightarrow{g_n} \quad K_n \xrightarrow{h_n} H \]

By composing the isometric embeddings (see the above diagram), one finds isometric embeddings \( h_n : X \to H \) and \( i_n : Z \to H \) (for each \( n \)) such that \( h_n \circ f_n = i_n \circ g_n \) for each \( n \) (in fact, one may do this such that the maps \( i_n \) are equal). See the diagram below. By Lemma 3.35, we may assume there exist

\[ \tau_0(X) := \{ \tau_f(b) : (Y, b) \in \mathcal{A} \text{ and } f \in \text{Hom}(Y, X) \} \subseteq \tau(X). \]

Let \( \tau'(X) \) be the closure of \( \tau_0(X) \) in \( \tau(X) \). It is straightforward that \( \tau_0 \) and \( \tau' \) are functors and \( \mathcal{C}_{\tau'} \supseteq \mathcal{A} \). So it is enough to show that \( \tau'(X) \) is compact for every \( X \); i.e., \( \tau_0(X) \) is pre-compact in \( \tau(X) \). Let \( a_1, a_2, \ldots \in \tau_0(X) \). We should prove that it has a convergent subsequence. By the definition of \( \tau_0 \), for every \( n \) there exists \( (Y_n, b_n) \in \mathcal{A} \) and \( f_n : Y_n \to X \) such that \( \tau_{f_n}(b_n) = a_n \). Since \( \mathcal{A} \) is pre-compact, we can assume that \( (Y, b) := \lim_n (Y_n, b_n) \) exists. By Lemma 3.35 there exists a compact metric space \( Z, g_n : Y_n \to Z \) and \( g : Y \to Z \) such that

\[ g_n(Y_n) \to g(Y), \]  
\[ \tau_{g_n}(b_n) \to \tau_g(b). \]  

For each \( n \), let \( K_n \) be the gluing of \( X \) and \( Z \) along the two copies of \( Y_n \). Note that \( \text{diam}(K_n) \leq \text{diam}(X) + \text{diam}(Z) \). So the diameters of \( K_n \) are uniformly bounded. By using Theorem 7.4.15 of [10], one can show that the sequence \( (K_n)_n \) is pre-compact under the metric \( d_{GH}^0 \). So, by taking a subsequence, we may assume that \( (K_n)_n \) is convergent under \( d_{GH}^0 \) and satisfies the assumption of Lemma 3.35. So, by Lemma 3.35, all of \( K_1, K_2, \ldots \) are isometrically embeddable into a common compact metric space \( H \) such that their images in \( H \) are convergent.
isometric embeddings $h : X \to H$ and $\iota : Z \to H$ such that $d_{\sup}(h_n, h) \to 0$ and $d_{\sup}(\iota_n, \iota) \to 0$. Equation (3.10) implies that $\iota_n(g_n(Y_n)) \to \iota(g(Y))$. On the other hand, $\iota_n(g_n(Y_n)) = h_n(f_n(Y_n)) \subseteq h_n(X)$. These facts imply that $\iota(g(Y)) \subseteq h(X)$. It follows that there exists an isometric embedding $f : Y \to X$ such that $h \circ f = \iota \circ g$. Let $a := \tau_f(b) \in \tau(X)$. It follows that

$$
\begin{align*}
\tau_{h_n}(a_n) &= \tau_{h_n}(\tau_{f_n}(b_n)) = \tau_{\iota_n}(\tau_{g_n}(b_n)), \\
\tau_h(a) &= \tau_h(\tau_f(b)) = \tau_\iota(\tau_g(b)).
\end{align*}
$$

Therefore, by letting $c_n := \tau_{g_n}(b_n)$ and $c := \tau_g(b)$, one has

$$
\begin{align*}
d(\tau_{h_n}(a_n), \tau_h(a)) &= d(\tau_{\iota_n}(c_n), \tau_\iota(c)) \\
&\leq d(\tau_{\iota_n}(c_n), \tau_{\iota_n}(c)) + d(\tau_{\iota_n}(c), \tau_\iota(c)) \\
&= d(c_n, c) + d(\tau_{\iota_n}(c), \tau_\iota(c)).
\end{align*}
$$

So, (3.11) and pointwise-continuity imply that $\tau_{h_n}(a_n) \to \tau_h(a)$. Since we had $h_n(X) \to h(X)$, one gets that $(X, a_n) \to (X, a)$ under the metric $d^*_\tau$. In particular, the sequence $(X, a_n)$ is pre-compact in $C_\tau$. So Lemma 3.35 below implies that the sequence $a_1, a_2, \ldots$ has a convergent subsequence in $\tau(X)$. This completes the proof.

The following lemma is used in the proof of Theorem 3.18.

**Lemma 3.36.** If $\tau$ is pointwise-continuous, then for every compact metric space $X$, the map $\tau(X) \to C_\tau$ defined by $a \mapsto (X, a)$ is continuous and proper.

**Proof.** The definition (3.11) directly implies that

$$
d^*_\tau((X, a_1), (X, a_2)) \leq d(a_1, a_2).
$$

This implies that the map is continuous (and also 1-Lipschitz). To prove properness of the map, let $K \subseteq C_\tau$ be a compact set and $a_1, a_2, \ldots \in \tau(X)$ be such that $(X, a_n) \in K$. To show the compactness of the inverse image of $K$, it is enough to show that the sequence $(a_n)_n$ has a convergent subsequence (note that by continuity, the inverse image of $K$ is closed). Since $K$ is compact, by taking a subsequence, we may assume $(X, a_n) \to (Y, b)$, where $(Y, b) \in K$. It follows that $Y$ is isometric to $X$. So there exists $c \in X$ such that $(X, c)$ is equivalent to $(Y, b)$ as elements of $C_\tau$. So $(X, a_n) \to (X, c) \in K$. By Lemma 3.35, there exists a compact metric space $Z$ and isometric embeddings $f_n : X \to Z$ and $f : X \to Z$ such that $f_n(X) \to f(X)$ and $\tau_{f_n}(a_n) \to \tau_f(c)$. By Lemma 3.30 and passing to a
subsequence, we may assume there exists \( g : X \to Z \) such that \( d_{\text{sup}}(f_n, g) \to 0 \). This implies that \( f_n(X) \to g(X) \), and hence, \( f(X) = g(X) \). So there exists an isometry \( h : X \to X \) such that \( f = g \circ h \). Let \( a := \tau_n(c) \). Pointwise-continuity and \( f_n \to g \) implies that \( \tau_{f_n}(a) \to \tau_g(a) = \tau_f(c) = \lim_n \tau_{f_n}(a_n) \). So \( d(\tau_{f_n}(a_n), \tau_f(a)) \to 0 \). Since \( \tau_f \) is an isometry, one gets that \( d(a_n, a) \to 0 \); i.e., \( a_n \to a \). This completes the proof.

**Proof of Proposition 3.20.** First, assume that such \( E' \) exists. For every \( X \), let \( \tau'(X) := h_X^{-1}(E') \). Since \( h_X \) is a proper function, \( \tau'(X) \) is a compact subset of \( \tau(X) \). It is straightforward that \( \tau' \) is a functor from \( \mathcal{C} \) to \( \mathfrak{Met} \). So Condition (ii) of Theorem 3.18 holds.

Conversely, assume that Condition (ii) of Theorem 3.18 holds but \( E' \) does not exist with the desired conditions. The latter implies that there exists a sequence \( (X_n, a_n) \in A \) such that \( (h_{X_n}(a_n))_n \) does not have any convergent subsequence. By Theorem 3.18 we may assume that \( (X_n, a_n) \to (X, a) \) for some \( (X, a) \in C \). By Lemma 3.31 there exists a compact metric space \( Z \) and isometric embeddings \( f : X \to Z \) and \( f_n : X \to Z \) such that \( \tau_{f_n}(a_n) \to \tau_f(a) \) in \( \tau(Z) \). Continuity of \( h_Z \) implies that \( h_{X_n}(a_n) = h_Z(\tau_{f_n}(a_n)) \to h_Z(\tau_f(a)) = h_X(a) \), which is a contradiction. So the claim is proved.

The following is a further property of the metric space \( C \) beyond the above theorems.

**Lemma 3.37.** It \( \tau \) is pointwise-continuous, then the infimum in (3.1) is attained.

**Proof.** Let \( \epsilon := d^*(X, Y) \). By (3.1), for every \( n > 0 \), there exists a compact metric space \( Z_n \) and isometric embeddings \( f_n : X \to Z_n \) and \( g_n : Y \to Z_n \) such that

\[
d_H(f_n(X), g_n(Y)) \vee d(\tau_{f_n}(a), \tau_{g_n}(b)) \leq \epsilon + \frac{1}{n}.
\]

Also, by Lemma 3.31 one can assume \( f_n(X) \cup g_n(Y) = Z_n \). This implies that \( \text{diam}(Z_n) \leq \text{diam}(X) + 2\epsilon + 2/n \), which is uniformly bounded. By using Theorem 7.4.15 of [10], one can show that the sequence \( (Z_n)_n \) is pre-compact under the metric \( d_{GH}' \). So, by taking a subsequence if necessary, we can assume \( d_{GH}'(Z_n, K) \leq 2^{-n} \) for some \( K \) without loss of generality. By Lemma 3.33 there exists a compact metric space \( Z \) and isometric embeddings \( h_n : Z_n \to Z \) (this is all we need from Lemma 3.33). By Lemma 3.30 one can assume \( h_n \circ f_n \to f \) and \( h_n \circ g_n \to g \) for some isometric embeddings \( f : X \to Z \) and \( g : Y \to Z \) without loss of generality. Since \( d_H(\text{Im}(h_n \circ f_n), \text{Im}(h_n \circ g_n)) \leq \epsilon + \frac{1}{n} \), one can show that \( d_H(\text{Im}(f), \text{Im}(g)) \leq \epsilon \). Moreover, by pointwise-continuity of \( \tau \), one gets that \( \tau_{h_n \circ f_n}(a) \to \tau_f(a) \) and \( \tau_{h_n \circ g_n}(b) \to \tau_g(b) \). It follows similarly that \( d(\tau_f(a), \tau_g(b)) \leq \epsilon \). Now \( (Z, f, g) \) satisfy the claim and the claim is proved. \( \Box \)
4 Boundedly-Compact Metric Spaces Equipped with More Structures

In this section, the abstract framework is provided for generalizing the Gromov-Hausdorff metric for boundedly-compact metric spaces. This will be done by extending the setting of Section 3 under further assumptions. The extension is by the same method used in [21]. First, the assumptions and the method are heuristically sketched in Subsection 4.1 and some basic examples are provided. Then, the rigorous definitions and results are provided in Subsections 4.2 and 4.3. Detailed examples are provided in Subsection 4.4 and also in Section 5.

The proofs of the results are similar to those of [21]. So repetition of the proofs is avoided and only the required modifications of the proofs are sketched.

4.1 Motivation

Here is a heuristic of the steps. Let \( \tau : \text{Comp} \to \text{Met} \) be a functor as in Section 3 which has the continuity properties of Definition 3.8 (here, \( \text{Met} \) can be replaced with the category of extended metric spaces as in Remark 3.15). Assume also that, to every boundedly-compact (non-pointed) metric space \( X \), a set \( \varphi(X) \) is assigned. No metric is assumed on \( \varphi(X) \). Also, for every isometric embedding \( f : X \to Y \), assume that an injective function \( \varphi_f : \varphi(X) \to \varphi(Y) \) is given. Assume that \( \varphi \) is also a functor (from the category of boundedly-compact metric spaces to the category of sets) and extends \( \tau \). Let \( C' \) be the set of isomorphism classes of tuples of the form \( \mathcal{X} := (X, o, a) \), where \( X \) is a boundedly-compact metric space, \( o \in X \) and \( a \in \varphi(X) \).

To defined a metric on \( C' \), the idea is that to define the distance of \( \mathcal{X}, \mathcal{Y} \in C' \) by comparing the balls in \( \mathcal{X} \) and \( \mathcal{Y} \) similarly to the definition of the Gromov-Hausdorff-Prokhorov metric in [21]. To do this, we need to assume that for every \( X = (X, o, a) \) and every compact subset \( Y \subseteq X \), the truncation of \( a \) to \( Y \) is defined with suitable properties. Recall that \( B_r(o) \) denotes the closed ball of radius \( r \) centered at \( o \). Let \( a^{(r)} \in \varphi(B_r(o)) \) be the truncation of \( a \) to \( B_r(o) \) and \( (\mathcal{X}, r) := (B_r(o), o; a^{(r)}) \). Note that by regarding \( a^{(r)} \) as an element of \( \tau(B_r(o)) \), one has \( (\mathcal{X}, r) \in C' \), where \( C' \) is the functor defined by \( \tau' : \text{Comp} \to \text{Sets} \) and \( C_\tau \) is defined in Section 3. For having arguments similar to those in [21], one also needs a partial order on \( \tau(X) \) with suitable properties. Under some conditions, which are stated in the next subsections, one can proceed similarly to [21] to define a metric on \( C' \) and study its properties.

Before stating the general conditions, it is useful to consider the following simple examples. These examples will be recalled in Subsection 4.4 with more details.

Example 4.1. For boundedly-compact metric spaces \( X \), let \( \varphi(X) \) be the set of \( k \)-marked measures on \( X \), with \( k \) given (defined similarly to \( \tau^{(m)}(X) \) of Example 3.25). The partial order on \( \varphi(X) \) is the natural partial order \( \leq \) on the set of measures on \( X \). For \( \mu \in \varphi(X) \) and a compact subset \( Y \subseteq X \), let the truncation of \( \mu \) be the restriction of \( \mu \) to \( Y^k \times \mathcal{X} \) (as a \( k \)-marked measure on \( Y \)).
Example 4.2. Let \( \varphi(X) \) be the set of \( k \)-marked closed subsets of \( X \), with \( k \) given (defined similarly to \( \tau^{(r)}(X) \) of in Example 3.24). The partial order on \( \varphi(X) \) is that of inclusion. For \( K \in \varphi(X) \) and a compact subset \( Y \subseteq X \), let the truncation of \( K \) be \( K \cap (Y^k \times \Xi) \) (as a \( k \)-marked closed subset of \( Y \)). Note that the intersection might be the empty set. So \( \varphi(X) \) should include the empty set from the beginning. See Remark 3.4 regarding the extended metric on \( \varphi(X) \) when \( X \) is compact.

Example 4.3. If \( (\varphi_i)_{i \in I} \) are at most countably many functors with the above properties, one can let \( \varphi(X) \) be the product of \( (\varphi_i(X))_{i \in I} \) equipped with the metric of Example 3.5. One can naturally define the partial order on \( \varphi(X) \) and the truncation element by element.

Remark 4.4. Assuming that the truncation \( \mathcal{X}^{(r)} \) is well defined as above, one can define the distance of \( X, Y \in \mathcal{C}' \) by
\[
\int_0^\infty e^{-r} \left( 1 \wedge d^c_{\tau}(\mathcal{X}^{(r)}, \mathcal{Y}^{(r)}) \right) \, dr
\]
where \( r' \) is as above and \( d^c_{\tau} \) is defined in (3.1) (formulas like this are common in various settings in the literature). For the integral to be well defined, one may assume that the curve \( r \mapsto \mathcal{X}^{(r)} \) is càdlàg (which is the case in the above examples and most of the generalizations of the Gromov-Hausdorff metric). If so, it is easy to show that (4.1) gives a pseudo-metric on \( \mathcal{C}' \). In addition, if the truncation is a functor (discussed in the next subsection), one can prove that (4.1) is indeed a metric similarly to [21] (by using the version of König’s infinity lemma in Lemma 3.16 of [21]). However, to study further properties like completeness and separability, it seems that more assumptions are needed on \( \tau \). So, we prefer to define a metric by the method of [21] instead of (4.1).

Under some conditions stated in Remark 4.15 below, these metrics generate the same topology.

In addition, assuming the càdlàg property mentioned above, one can use the Skorokhod metric (see e.g., [9]) to define a metric on \( \mathcal{C}' \). However, this metric generates a different topology than (4.1) and the metric of the next subsection (but might generate the same Borel sigma-field). See [21] for more discussion in the case of the Gromov-Hausdorff-Prokhorov metric.

4.2 The Space \( \mathcal{C}' \)

Now, the precise definitions regarding the space \( \mathcal{C}' \) are presented. The metric on \( \mathcal{C}' \) will be provided in the next subsection.

Let \( \text{Pos} \) be the category of partially ordered sets (abbreviated by posets). The symbol \( \leq \) is used to denote the order on any poset. A morphism between objects \( A \) and \( A' \) of \( \text{Pos} \) is an order-preserving function \( f : A \to A' \); i.e., if \( a_1 \leq a_2 \), then \( f(a_1) \leq f(a_2) \). Let \( \text{Pos}_m \) be the category defined as follows. Every object of \( \text{Pos}_m \) is an extended metric space \( A \) equipped with a partial order such that for all \( a \in A \), the cone \( \{ a' \in A : a' \leq a \} \) is compact. A
morphism between objects $A$ and $A'$ of $\mathcal{P}os_m$ is a function $f : A \to A'$ which is both an isometric embedding and is order-preserving. The reader can verify that the sets in Examples 4.1, 4.2 and 4.3 are elements of $\mathcal{P}os_m$ in the case where the underlying metric space $X$ is compact.

Note that forgetting the metric gives a natural functor from $\mathcal{P}os_m$ to $\mathcal{P}os$. Here, for every morphism $f : A \to A'$ in $\mathcal{P}os_m$, its corresponding morphism in $\mathcal{P}os$ is also denoted by $f : A \to A'$. This abuse of notation makes no confusion according to the context.

Let $\tau : \mathcal{C}omp \to \mathcal{P}os_m$ be a functor. Assume that for every $X$, every isometry $f : X \to X$ and every $a \in \tau(X)$,

$$\text{if } \tau_f(a) \leq a, \text{ then } \tau_f(a) = a.$$  \hfill (4.2)

Assume a truncation functor $\tau^t$ is given as follows: It is a contra-variant functor from $\mathcal{C}omp$ to $\mathcal{P}os$ (similar to Definition 3.7) but for every morphism $f : X \to Y$, $\tau_f^t : \tau^t(Y) \to \tau^t(X)$ is a morphism in the reverse direction) such that for every compact metric space $X$, $\tau^t(X)$ is equal to (the underlying poset of) $\tau(X)$. Also, assume that for every morphism $f : X \to Y$ of $\mathcal{C}omp$, $a \in \tau(X)$ and $b \in \tau(Y)$,

$$\begin{align*}
\tau^t_f \circ \tau_f(a) &= a, \\
\tau_f \circ \tau^t_f(b) &\leq b.
\end{align*}$$  \hfill (4.3)

The next subsection requires further assumptions on $\tau$ and $\tau^t$.

Now, $\tau$ is extended to boundedly-compact metric spaces as follows.

**Definition 4.5.** For boundedly-compact metric spaces $X$, let $I_X$ be the set of compact subsets of $X$. Let $\varphi(X)$ be the set of functions $a$ on $I_X$ such that

$$\forall Y \in I_X : a_Y := a(Y) \in \tau(Y)$$

and

$$\forall Y, Y' \in I_X : \text{ if } Y \subseteq Y', \text{ then } \tau^t_f(a_Y) = a_{Y'},$$

where $\iota : Y \hookrightarrow Y'$ is the inclusion map. A natural partial order is defined on $\varphi(X)$: $a \leq a'$ if $a_Y \leq a'_Y$ for all $Y \in I_X$ (see Remark 4.7 below for further discussion).

Note that $\varphi$ extends $\tau$ in the sense that if $X$ is compact, then the map $a \mapsto a_X$ from $\varphi(X)$ to $\tau(X)$ is a bijection and is order-preserving (in addition, this map behaves well with the morphisms; i.e., it is a natural transformation in the language of category theory and is invertible). See also Remark 4.7 below.

**Definition 4.6.** Let $\varphi$ be defined as above. For every isometric embedding $f : X \to X'$ between boundedly-compact metric spaces $X$ and $X'$, define $\varphi_f : \varphi(X) \to \varphi(X')$ as follows: Let $a \in \varphi(X)$. For every $Y' \in I_{X'}$, let $Y := f^{-1}(Y') \in I_X$ and $g : Y \to Y'$ be the restriction of $f$. Let $a'_{Y'} := \tau_g(a_Y) \in \tau(Y')$. By considering this for all $Y' \in I_{X'}$, one obtains an element $a' \in \varphi(X')$. Let $\varphi_f(a) := a'$. It is easy to see that $\varphi_f$ is an order-preserving function.
In addition, define the truncation map \( \varphi^t : \varphi(X') \to \varphi(X) \) as follows. Let \( a' \in \varphi(X') \). For every \( Y \in I_X \), let \( Y' := f(Y) \) and \( g : Y \to Y' \) be the restriction of \( f \). Let \( a_Y := \tau^t_f(a_Y) \in \tau(Y) \). By considering this for all \( Y \in I_X \), one obtains an element \( a \in \varphi(X) \). Let \( \varphi^t_f(a') := a \). It is easy to see that \( \varphi^t_f \) is an order-preserving function.

It can be seen that \( \varphi \) is a functor and \( \varphi^t \) is a contravariant functor from the category of boundedly-compact metric spaces to \( \Psi_{\mathfrak{os}} \). In addition, (4.3) holds for \( \varphi \) and \( \varphi^t \).

**Remark 4.7.** In the language of category theory, \( \varphi(X) \) is an inverse limit (in \( \Psi_{\mathfrak{os}} \)) of the diagram consisting of the objects \( \tau(Y) \) for \( Y \in I_X \) and the arrows \( \tau^t_f \) as above. One can replace \( \varphi(X) \) in the above definition by any other inverse limit of this diagram (all inverse limits are equivalent). This is usually done in the examples of this paper. See e.g., Subsection 4.4.

**Definition 4.8.** Let \( C' \) be the set of isomorphism classes of tuples \( \mathcal{X} = (X, o; a) \), where \( X \) is a boundedly-compact metric space, \( o \in X \) and \( a \in \varphi(X) \) (it can be seen that \( C' \) is indeed a set). Also, let \( C := C_\tau \), where \( \tau' : \text{Comp} \to \text{Met} \) is the functor defined by \( \tau'(X) := X \times \tau(X) \) (equipped with the max product metric) and \( C_\tau \) is defined in Subsection 3.4.

By the above discussion, \( C \) corresponds naturally to the subset of \( C' \) consisting of tuples \( (X, o; a) \) in which \( X \) is compact. Note that (3.1) defines a metric \( d'_r \) on \( C \). Also, the results of Section 3 can be applied to \( C \).

### 4.3 The metric on \( C' \)

For all tuples \( \mathcal{X} = (X, o; a) \) as above and \( r \geq 0 \), let \( \overline{X}^{(r)} := (B_r(o), o; \varphi^r(o)) \), where \( \iota_r : B_r(o) \to X \) is the inclusion map. For all tuples \( \mathcal{X} := (X, o; a) \) and \( \mathcal{X}' := (X', o'; a') \), define \( \mathcal{X}' \preceq \mathcal{X} \) if \( X' \subseteq X \), \( o' = o \) and \( a' \leq \varphi^t_r(a) \), where \( r \) is the inclusion map (the latter is equivalent to \( \varphi^r(a') \leq a \)). For tuples \( \mathcal{X} = (X, o_X; a_X) \) and \( \mathcal{Y} = (Y, o_Y; a_Y) \) and \( 0 \leq r \leq 1 \), let

\[
a_r(\mathcal{X}, \mathcal{Y}) := \inf \{ d'_r(\overline{X}^{(1/r)}, \mathcal{Y}) \},
\]

where the infimum is over all \( \mathcal{Y}' \) such that \( \overline{Y}^{(1/r)} \preceq \mathcal{Y}' \preceq \mathcal{Y} \) (one can also remove the condition \( \overline{Y}^{(1-r)} \preceq \mathcal{Y} \) and all of the results will remain valid except maybe those in Remark 4.15). Define the distance of \( \mathcal{X} \) and \( \mathcal{Y} \) by

\[
d(\mathcal{X}, \mathcal{Y}) := \inf \{ r \in (0, 1] : a_r(\mathcal{X}, \mathcal{Y}) \lor a_r(\mathcal{Y}, \mathcal{X}) < \frac{r}{2} \},
\]

with the convention that \( \inf \emptyset := 1 \). To ensure that this equation defines a metric on \( C' \), we assume that the following further assumptions hold.

**Assumption 4.9.** Assume that for every \( \mathcal{X} \) and \( \mathcal{Y} \) with compact underlying spaces, and for every \( \mathcal{X}' \preceq \mathcal{X} \), there exists \( \mathcal{Y}' \preceq \mathcal{Y} \) such that \( d'_r(\mathcal{X}', \mathcal{Y}') \leq d'_r(\mathcal{X}, \mathcal{Y}) \) (see Lemma 4.37).
Assumption 4.10. In the previous assumption, assume that if $X^{(r)} \preceq X' \preceq X$, then $Y'$ can be chosen such that $Y^{(r-2\epsilon)} \preceq Y' \preceq Y$, where $\epsilon := d_c^* (X', Y)$ (assuming $r \geq 2\epsilon$).

The term $r - 2\epsilon$ in the above assumption is based on the fact that $d_c^*$ allows a distortion of size $2\epsilon$ in the metric (see Subsection 3.5). These assumptions hold in most of the generalizations of the Gromov-Hausdorff metric known by the author and mentioned in this paper. For instance, see Lemma 3.12 of [21] for the case of measures (Example 3.2). It should be noted that for Assumption 4.10 to hold, the metric on $\tau (X)$ should be carefully chosen. Subsection 4.4 discusses some examples where this assumption does not hold and provides other suitable metrics.

Assumption 4.11. Assume also that condition (i) of Remark 3.17 holds.

By this assumption and the assumption of compactness of cones, one can show that for every compact $X$, the set of $X'$ such that $X' \preceq X$ is compact under the metric $d_c^*$. This is similar to Lemma 3.13 of [21].

Theorem 4.12. Let $\tau, \tau^\dagger$ and $C'$ be as above. Assume that $\tau$ is pointwise-continuous. Then, (4.4) defines a metric on $C'$.

Proof. The proof is similar to that of Theorem 3.15 of [21] and by using the assumption (4.2) (note that in the proof, we do not need to study whether the infimum in (4.3) is attained or not).

The following pre-compactness result can be proved similarly to Theorem 3.28 of [21] with minor modifications.

Theorem 4.13. Under the assumptions of Theorem 4.12, a subset $A \subseteq C'$ is relatively compact if and only if for every $r \geq 0$, the set of (equivalence classes of the) balls $A_r := \{X^{(r)} : X \in A\}$ is relatively compact under the metric $d_c^*$.

Theorem 4.14. In the setting of Theorem 4.12, assume that $\tau$ is Hausdorff-continuous. If $\tau (X)$ is complete for every compact metric space $X$, then $C_r$ is also complete. If $\tau (X)$ is separable for every compact metric space $X$, then $C_r$ is also separable.

Proof. The claims can be proved similarly to Theorem 3.27 of [21] and by using Theorem 3.16.

In the above result, one can also replace the assumption of Hausdorff-continuity by the assumptions in Remark 3.17.

Remark 4.15. For characterizing convergence in $C'$ similarly to Theorem 3.24 of [21], one needs the following further assumption: For every compact metric space $X$, $a \in \tau (X)$ and $o \in X$, if $\iota_r : \overline{B}_r (o) \hookrightarrow X$ is the inclusion map, then the curve $r \mapsto \iota_r \circ \tau^\dagger (a)$ in $\tau (X)$ has both right-limits and left-limits at all points (this curve is càdlàg in all of the examples except Example 4.23). For instance,
by compactness of cones in $\tau(X)$, this condition is implied by the following stronger condition: For every tuple $a \leq b \leq c$ in $\tau(X)$, $d(a, b) \vee d(b, c) \leq d(a, c)$.

By Assumption 4.11 and assuming completeness, one can show that for every $\mathcal{X} := (X, o; a) \in C'$, the curve $r \mapsto \mathcal{X}^{(r)}$ has both right-limits and left-limits at all points. This property ensures that the set of discontinuity points of the curve is at most countable. Now, a statement similar to Theorem 3.24 of [21] can be derived with the same proof. In particular, (4.1) generates the same topology on $C'$ as the metric defined in Theorem 4.12 (note that it is a metric as mentioned in Remark 4.4). Random elements in $\mathcal{C}$ and $\mathcal{C}'$ and weak convergence can also be studied as in [21].

4.4 Examples

Here, examples of the setting of this section are provided and also the above examples are discussed with more details. Further examples will be presented in Section 5.

Example 4.16. Let $\tau(X)$ be the set of $k$-marked finite measures on $X$ (resp. $k$-marked compact subsets of $X$) defined in Example 3.24. One can define the partial order and the truncation functor $\tau'$ similarly to Example 3.1 (resp. Example 4.2). It can be seen that the extension $\varphi$ in Definition 4.5 is equivalent to the set of $k$-marked measures with boundedly-finite ground measure; i.e., the set of measures $\mu$ on $X^k \times \Xi$ such that the projection of $\mu$ on $X^k$ is boundedly-finite (resp. the set of closed subsets $C$ of $X^k \times \Xi$ such that for every compact subset $K \subseteq X^k$, $C \cap (K^k \times \Xi)$ is compact). Similarly, the results of this section show that $\mathcal{C}'$ is a complete separable metric space.

Example 4.17. Assume that the mark space $\Xi$ is boundedly-compact. For compact metric spaces $X$, let $\tau(X)$ be the set of $k$-marked measures on $X$ (resp. $k$-marked closed subsets of $X$) defined in Example 3.25. Define the partial order and the truncation functor similarly to the previous example. It can be seen that this fits into the definitions and the above conditions hold. Also, the extension of $\tau$ defined in Definition 4.5 is equivalent to the functor $\varphi$ of Example 4.1 (resp. 4.2). Therefore, Theorem 4.12 defines a metric on $\mathcal{C}'$ and makes it complete and separable. It should be noted that to ensure that Assumption 4.10 holds in this example, the metric on $\tau(X)$ is important. Here, the metric defined in [21] is used (see Remark 3.21 of [21]) which is defined similarly to (3.1). Also, a metric similar to (4.1) can be defined on $\tau(X)$ but does not satisfy Assumption 4.10.

Example 4.18. Let $\mathcal{C}''$ be the set of boundedly-compact pointed metric spaces equipped with a finite measure (resp. a compact subset). This example cannot be obtained by the method of this section since the functor under study cannot be obtained by Definition 4.5. However, since $\mathcal{C}''$ is a subset of the set $\mathcal{C}'$ of Example 4.16, one can define a metric on $\mathcal{C}''$. Here, $\mathcal{C}''$ is not a closed subset of $\mathcal{C}'$, and hence, it is not complete. By the way, it can be seen that it is a Borel subset (in fact, a $F_r$ subset) of $\mathcal{C}'$. 
Example 4.19 (Additional Point). Let \( C'' \) be the set of boundedly-compact pointed metric spaces equipped with one additional point other than the origin (see Subsection 5.2 below). To define a metric on \( C'' \), one can regard the point as a compact subset (or as a Dirac measure) and show that \( C'' \) is a subset of the set \( C' \) of Example 4.16. In addition, it can be seen that \( C'' \) is a Borel subset (and in fact, the difference of two closed subsets) of \( C' \).

For a direct method, let \( \varphi(X) := X \cup \{ \Delta \} \), where \( \Delta \) is an arbitrary element not contained in \( X \) called the grave (that might depend on \( X \)). For \( a \in \varphi(X) \) and an isometric embedding \( f : Y \to X \), define the truncation of \( a \) by \( \varphi_f^{-1}(a) \) if \( a \in f(Y) \) and \( \varphi_f^{-1}(a) := \Delta \) if \( a \notin f(Y) \). The partial order on \( \varphi(X) \) is \( \forall a : \Delta \leq a \). If \( X \) is compact, then consider the extended metric on \( \varphi(X) \) defined by \( d(\Delta, a) := \infty \) for all \( a \in X \). The reader can verify that this fits into the setting of this section and the assumptions are satisfied. So the corresponding set \( C' \) of Definition 4.8 is complete and separable. Here, \( C'' \) is an open subset of \( C' \) (and hence, \( C'' \) is Polish by itself).

The reader can verify that the above three approaches define the same metric on \( C'' \).

Example 4.20 (Convergent Continuous Curves). Let \( I := [0, T] \) be a compact interval and let \( C'' \) be the set of boundedly-compact metric spaces \( X \) equipped with a continuous curve \( \eta : I \to X \). Below, \( C'' \) is studied by the setting of this example. The cases \( I = [0, \infty) \) and \( I = \mathbb{R} \) will be studied at the end of this example and in Example 4.22 (see also the next remark and Subsection 5.3).

For compact metric spaces \( X \), let \( \tau(X) \) be the set of continuous curves \( \eta : I \to X \cup \{ \Delta \} \) equipped with the sup (extended) metric, where \( \Delta \) is a grave as in Example 4.19. Define the partial order on \( \tau(X) \) as follows: \( \eta' \leq \eta \) when either \( \eta'(t) = \Delta \) or \( \eta' \) is obtained by stopping \( \eta \) at some time \( t_0 \in I \): i.e., \( \eta'(t) = \eta(t \land t_0) \). Define the functors \( \tau \) and \( \tau^\dagger \) as follows. For all isometric embeddings \( f : X \to Y \) and \( \eta \in \tau(X) \), let \( \tau_f(\eta) := f \circ \eta \). Also, for any \( \eta' \in \tau(Y) \), let \( \tau_{f^\dagger}(\eta') := f^{-1} \circ \eta' \) stopped at the first exit time of \( \eta' \) from \( f(X) \) (if \( \eta'(0) \notin f(X) \), let \( \tau_{f^\dagger}(\eta')(\cdot) := \Delta \)). It can be seen that these definitions satisfy the assumptions of Subsections 4.2 and 4.3 except Hausdorff-continuity (see Subsection 3.4.2). In addition, it can be seen that the extension \( \varphi(X) \) in Definition 4.8 is equivalent to the set of continuous curves \( \eta \in \tau(X) \cup \{ \Delta \} \) such that either (1) \( \eta \) is defined on the entire of \( [0, T] \) or (2) \( \eta \) is defined on some interval \( [0, T'] \subseteq [0, T] \) and exits any compact subset of \( X \) eventually (see e.g., Example 5.1 below). So the set \( C' \) of tuples \((X, o, \eta)\), where \( X \) is boundedly-compact, \( o \in X \) and \( \eta \in \varphi(X) \), is a complete metric space. Separability of \( C' \) is also easily deduced by separability of the example in Subsection 3.4.2. In addition, it is straightforward that \( C'' \) is equivalent to the subset of elements \((X, o, \eta) \in C' \) such that \( \eta(0) = o \) and \( \eta \) is defined on the entire of \([0, T]\). It can be seen that this is a Borel (and \( F_0 \)) subset of \( C' \) (to see this, consider the maximum distance of \( \eta(\cdot) \) from \( o \)).

For the case \( I = [0, \infty) \), one can repeat the same arguments as above by letting \( \tau(X) \) be the set of convergent continuous curves \( \eta : \mathbb{R}^\geq \to X \cup \{ \Delta \} \) equipped with the sup metric, as in Subsection 3.4.2. Here, it can be seen that the space \( C' \) of Definition 4.8 is equivalent to the set of tuples \((X, o, \eta)\) where \( X \)
is boundedly-compact and \( \eta : \mathbb{R}^{\geq 0} \to X \cup \{ \Delta \} \) is a continuous curve which is either convergent or exits any compact subset of \( X \) eventually. The next example considers general continuous curves.

**Remark 4.21.** In the setting of the above example, similar results hold for the case \( I = \mathbb{R} \). These results are obtained by noting that every continuous curve \( \eta : \mathbb{R} \to X \) is the joining of the two curves \( \eta|_{\{\infty, 0\}} \) and \( \eta|_{[0, \infty)} \) and by using Examples 3.13 and 3.14.

**Example 4.22** (Continuous Curves). Let \( I := [0, \infty) \). Similarly to the previous example, let \( C'' \) be the set of boundedly-compact pointed metric spaces \( (X, 0) \) equipped with a continuous curve \( \eta : I \to X \) which is not necessarily convergent (the case \( I = \mathbb{R} \) can be treated similarly by the above remark). For compact metric spaces \( X \), let \( \tau(X) \) be the set of continuous curves \( \eta : I \to X \cup \{ \Delta \} \). Consider the partial order on \( \tau(X) \) and the truncation map \( \tau' \) as in Example 4.20. By choosing a suitable metric on \( \tau(X) \), discussed below, \( \tau \) will be a functor that satisfies the assumptions of Subsection 4.2 and 4.3 except Hausdorff-continuity. It follows similarly that the corresponding set \( C' \) is a complete separable metric space (separability should be proved separately similarly to the previous example). Here, it can be seen that the extension \( \varphi(X) \) of Definition 4.15 is equivalent to the set of continuous curves \( \eta \) in \( X \cup \{ \Delta \} \) such that either (1) \( \eta \) is defined on the entire of \( \mathbb{R}^{\geq 0} \) or (2) \( \eta \) is defined on some interval of the form \( [0, T) \) and exits any compact subset of \( X \) eventually (see e.g., Example 5.1 below). Similarly to the previous example, it can be seen that \( C'' \) is a Borel (and \( F_{\sigma} \)) subset of the Polish space \( C' \).

In this example, the metric on \( \tau(X) \) should be carefully chosen to ensure that Assumption 4.10 holds. A suitable metric on \( \tau(X) \) is the following:

\[
d(\eta, \eta') := \inf\{ \epsilon \in (0, 1] : d_{\sup} \left( \eta|_{[0,1/\epsilon]}, \eta'|_{[0,1/\epsilon]} \right) \leq \epsilon \}.
\]

It is left to the reader to show that this is a metric on \( \tau(X) \) and satisfies the assumptions. See also Subsection 5.3.

**Example 4.23** (Càdlàg Curves). Let \( D_{\ast} \) be the set of tuples \( X = (X, \eta) \), where \( X \) is a boundedly-compact metric spaces and \( \eta : I \to X \) is a càdlàg curve, where \( I := [0, T] \) is a compact interval (the cases \( I = \mathbb{R}^{\geq 0} \) and \( I = \mathbb{R} \) are treated at the end of the example). We show that \( D_{\ast} \) is a Borel subspace of some Polish space by the method of this section.

For all compact metric spaces \( X \), let \( \tau(X) \) be the set of càdlàg curves \( \eta : \mathbb{R} \to X \cup \{ \Delta \} \) (where \( \Delta \) is a grave as in Example 4.19), such that \( \eta^{-1}(\Delta) \) is either the empty set or an interval of the form \( [T_{\eta}, T] \). For \( \eta, \eta' \in \tau(X) \), define \( \eta \leq \eta' \) if \( \forall t < T_{\eta} : \eta(t) = \eta'(t) \); i.e., \( \eta \) is obtained by *killing* \( \eta' \) at time \( T_{\eta} \). For all isometric embeddings \( f : X \to Y \) and \( \eta \in \tau(X) \), let \( \tau_f(\eta) := f \circ \eta \in \tau(Y) \). Also, for \( \eta' \in \tau(Y) \), let the truncation \( \tau_f^\dagger(\eta') \) be \( f^{-1} \circ \eta' \) killed at the first exit time of \( \eta' \) from \( f(X) \) (note that if \( \eta'(0) \notin f(X) \), then \( \tau_f^\dagger(\eta')(\cdot) = \Delta \)). It can be seen that \( \tau(X) \) is a closed subset of the set of càdlàg curves in \( X \cup \{ \Delta \} \) endowed with the Skorokhod metric (which is an extended metric here). So \( \tau(X) \)
is a complete separable extended metric space. Similarly to Subsection 3.4.3, \( \tau \) and \( \tau^t \) are functors as in Subsection 4.2 and \( \tau \) is both pointwise-continuous and Hausdorff-continuous. Consider the corresponding space \( C' \) defined in Definition 4.8 (discussed below). Therefore, the results of Subsection 4.3 define a metric on \( C' \) and show that it is a complete separable metric space. (verification of the other assumptions is left to the reader).

To connect \( C' \) and \( D* \), it can be seen that \( C' \) is equivalent to the set of functions \( \eta : I \to X \cup \{\Delta\} \) such that either (1) \( \eta \) is càdlàg and is killed at the first hitting to \( \Delta \) or (2) there exists a time \( T_{\eta} \in [0,T] \) such that \( \eta|_{[T_{\eta},T]} \equiv \Delta \) and \( \eta|_{[0,T_{\eta})} \) is a càdlàg curve in \( X \) that exits any compact subset of \( X \) eventually (it does not have a left limit at \( t = T_{\eta} \)). Finally, it can be seen that \( D* \) is a Borel and \( F_\sigma \) subset of \( C' \) (note that the set of elements \((X,o,\eta) \in C' \) such that \( \sup\{d(o,\eta(t)) : t < T_{\eta}\} \leq m \) is closed for every \( m \)).

For the case \( I := \mathbb{R}^{\geq 0} \), similar arguments can be applied, but the metric on \( \tau(X) \) should be carefully chosen to ensure that Assumption 4.10 holds. A suitable metric on \( \tau(X) \) is the following, which has the same idea as (4.4). If \( k_{t_0}(\eta) \) denotes \( \eta \) killed at time \( t_0 \), let

\[
a_c(\eta,\eta') := \inf_{t_0} d_S(k_{1/\epsilon}(\eta), k_{t_0}(\eta')) ,
\]

where the infimum is over all \( t_0 \) such that \( |t_0 - 1/\epsilon| \leq \epsilon \) and where \( d_S \) denotes the Skorokhod metric defined by the same equation (12.16) of [9] (although the interval is \( \mathbb{R} \)). Then, define the distance of \( \eta \) and \( \eta' \) by a formula similar to (4.4). It can be seen that this is a metric on \( \tau(X) \) and satisfies the assumptions of Subsections 4.2 and 4.3. So the above claims hold also in the case \( I = \mathbb{R}^{\geq 0} \).

Remark 4.24. In the above example, the function \( r \mapsto X(r) \) is not necessarily càdlàg, but it can be seen that it has at most countably many discontinuity points (note that if \( X := (X,o,\eta) \), then every discontinuity point of the function is either a discontinuity point of \( \eta \) or a local maximum for \( d(o,\eta(\cdot)) \)). However, the topology of \( C' \) can be studied similarly and a result similar to Theorem 3.24 of [21] holds (see Theorem 16.2 of [9]).

5 Special Cases and Connections to Other Notions

As mentioned in the introduction, there are various generalizations of the Gromov-Hausdorff metric in the literature by considering metric spaces equipped with specific types of additional structures. In this section, the connections of these instances to the present work are studied. Roughly speaking, it is shown that the examples can be considered as special cases of the general framework of this paper (in some of the examples, the metrics are different but generate the same topology). This implies that various random objects in the literature can be re-
garded as random metric spaces equipped with more structures as in Sections 3 and 4.

Some of the examples in the literature are special cases of measured metric spaces. This includes the setting of [1] for measured length spaces, random measures in Stochastic geometry, the Benjamini-Schramm metric for graphs [8], and the setting of [5] for discrete spaces. These examples are discussed in [21] and are skipped here.

5.1 Networks and Marked Discrete Spaces

The Benjamini-Schramm metric [8] is defined between rooted graphs which can be used to study the limit of a sequence of sparse graphs. This metric is extended in [4] to rooted networks, where a network is a graph $G$ in which a mark is assigned to every vertex and every pair of adjacent vertices (for simplicity, we restrict attention to simple graphs here). By Example 3.22, networks are a special case of metric spaces equipped with a 1-marked closed subset (for the marks of the vertices) and a 2-marked closed subset (for the marks of pairs). So the set $\mathcal{G}_*$ of locally-finite rooted networks is a subset of the set $\mathcal{C}'$ defined in Definition 4.8 for suitable functors. It can be seen that $\mathcal{G}_*$ is a closed subset of $\mathcal{C}'$ and also the metric defined in [4] is equivalent to (the restriction of) the metric defined on $\mathcal{C}'$ in Section 4.

Also, [5] considers the set $\mathcal{D}_*$ of boundedly-finite pointed discrete metric spaces in which a mark is assigned to every point and every pair of points. Similarly, one can show that this set is a subset of the set $\mathcal{C}'$ defined in Definition 4.8. Here, let the additional structure on metric spaces be a 1-marked closed subset, a 2-marked closed subset and a measure (for the latter, consider the counting measure). Similarly, one can show that $\mathcal{D}_*$ is a Borel subset of $\mathcal{C}'$ and the metric on $\mathcal{D}_*$ generates the same topology as the restriction of the metric of $\mathcal{C}'$. See [21] for more details of the arguments and further discussion (which are provided for non-marked discrete spaces).

5.2 Examples of Multiple Additional Structures in the Literature

In some literature, metric spaces with more than one distinguished point are considered. For example, [4] considers the set of equivalence classes of graphs or networks equipped with two distinguished vertices. This is also generalized in [5] to discrete spaces (see Example 4.19).

In [22] compact metric spaces equipped with $k$ distinguished closed subsets are studied. This is a special case of the construction in Section 3 and the metric in [22] is identical to (3.1) (see Examples 3.3 and 3.5). Proposition 9 of [22] is also a special case of Proposition 3.28.

Also, [22] considers the set $\mathcal{M}^{k,l}$ of (equivalence classes of) compact metric spaces equipped with $k$ distinguished points and $l$ finite Borel measures. According to Sections 3 the spaces $\mathcal{C}_*$ generalize these spaces.
There are also various papers which consider metric spaces equipped with a measure and another structure. These cases will be discussed in the forthcoming subsections.

5.3 The Gromov-Hausdorff-Prokhorov-Uniform Metric

For all compact metric spaces $X$, let $\tau_0(X)$ be the set of continuous curves $\eta : \mathbb{R} \to X$ that are convergent as $t \to \infty$ and $t \to -\infty$ (see Subsection 3.4.2). In [18], the space $\mathcal{M}^u$ is considered, which is the set of (equivalence classes of) compact metric spaces $X$ together with a finite measure $\mu$ on $X$ and a continuous curve $\eta \in \tau_0(X)$. Using the uniform metric (i.e., the sup metric) on $\tau_0(X)$, a variant of the Gromov-Hausdorff-Prokhorov metric, called the GHPu metric is defined on $\mathcal{M}^u$ in [18] (with a formula similar to (3.1)). Also, it is proved that $\mathcal{M}^u$ is a complete separable metric space.

The non-compact case is also studied in [18] in the special case of length spaces. Let $\mathcal{M}^u_\infty$ be the set of (equivalence classes of) complete locally-compact length spaces $X$ together with a locally finite measure $\mu$ on $X$ and a continuous curve $\eta : \mathbb{R} \to X$ pointed at the distinguished point $\eta(0)$. The metric on $\mathcal{M}^u_\infty$ is defined by a formula similar to (4.1) using suitable truncations (the precise definitions are skipped for brevity). It is also proved that $\mathcal{M}^u_\infty$ is a separable metric space. However, despite the claim of [18], $\mathcal{M}^u_\infty$ is not complete (see Example 5.1 below). Nevertheless, it is shown below that it is a Borel (and $F_\sigma$) subspace of some Polish space. This is enough for having a standard probability space for probability-theoretic purposes.

Below, the sets $\mathcal{M}^u$ and $\mathcal{M}^u_\infty$ are studied according to the setting of Sections 3 and 4 respectively. In addition, this allows to replace locally-compact length spaces in the above definition by general boundedly-compact metric spaces.

In the setting of Section 3, let $\tau(X) := \tau^{(f)}(X) \times \tau_0(X)$, where $\tau^{(f)}(X)$ is the set of finite measures on $X$. It is immediate that $\mathcal{M}^u$ is identical with $C_\tau$ defined in Section 3 and it can be seen that their metrics are equivalent. As mentioned in Subsection 3.4.2, the results of Section 3 show that $C_\tau$ is a complete metric space, but does not directly imply its separability since $\tau_0$ and $\tau$ are not Hausdorff-continuous. Subsection 3.4.2 provides two other proofs of the separability of $C_\tau$ by regarding continuous curves as either 1-marked closed subsets or as càdlàg curves.

For the non-compact case, the truncation analogous to $\mathcal{X}^{(r)}$ defined in [18] does not fit in the framework of Section 4 since the truncation of curves depends on the radius of the ball. However, it is shown below that $\mathcal{M}^u_\infty$ is a subspace of the metric space $C'$ defined in Section 4 for a suitable functor (another proof is given in Remark 5.2 below). To do this, Example 4.22 and Remark 4.21 define a suitable metric on the set of continuous curves in $\eta : \mathbb{R} \to X \cup \{\Delta\}$ (which are not necessarily convergent) for compact metric spaces $X$. Note that the extension to boundedly-compact metric spaces in Example 4.22 deals with a larger family of curves (see Example 5.1 below). By the results of Example 4.22...
and Remark 4.21, one can show that \( \mathcal{M}_\infty^u \) is a Borel (and \( F_\sigma \)) subset of a complete separable metric space \( C' \). In addition, it can be seen that the topology of \( \mathcal{M}_\infty^u \) is equivalent to the induced topology from \( C' \). This proves the claim.

**Example 5.1.** Let \( X_m \) be the set of real numbers equipped with the Lebesgue measure and the curve \( \eta_m : \mathbb{R} \to \mathbb{R} \) defined by \( \eta_m(t) := m \wedge (1/|1-t|) \). It can be seen that \( (X_m)_m \) is a Cauchy sequence in \( \mathcal{M}_\infty^u \) but it is not convergent in \( \mathcal{M}_\infty^u \). Hence, \( \mathcal{M}_\infty^u \) is not complete. However, this sequence is convergent in \( C' \). The limit is the set of real numbers equipped with the Lebesgue measure and the curve \( \eta : (\mathbb{R}, 1) \to \mathbb{R} \) defined by \( \eta(t) := 1/(1-t) \).

**Remark 5.2.** To study the set of metric spaces equipped with a continuous curve, it would be easier to regard curves as marked closed subsets (as in Subsection 3.4.2) and to use the settings of Subsection 3.4.1 for the compact case and Example 3.25 for the boundedly-compact case. This would reduce the technicalities regarding continuous curves; e.g., introducing a grave, defining truncations properly and being obliged to consider more curves as discussed in Example 4.22. In addition, it can be seen that this approach would produce the same topologies on \( \mathcal{M}^u \) and \( \mathcal{M}_\infty^u \). This is another method to prove that these sets are Borel subspaces of some Polish space.

However, it should be noted that by considering curves as marked closed subsets, \( \mathcal{M}^u \) and \( \mathcal{M}_\infty^u \) would have a different completion. For instance, the limit of the sequence \( X_m \) in Example 5.1 would be \( \mathbb{R} \) equipped with the graph of the function \( \eta : \mathbb{R} \setminus \{1\} \to \mathbb{R} \) defined by \( \eta(t) := 1/|1-t| \) which is different with the limit mentioned in Example 5.1 (notice the domain of the curve).

### 5.4 Spatial Trees

In this subsection, connections to the settings of [13] and [7] are discussed. The former considers (a specific set of) compact metric spaces equipped with a continuous function and the latter studies the measured version.

Let \( \Xi \) be a complete separable metric space. First, note that by letting \( \tau_0(X) \) be the set of continuous functions from \( X \) to \( \Xi \), \( \tau_0 \) is not a functor as in Definition 3.7. The reason is that for isometric embeddings \( f : X \to Z \), there is no natural function from \( \tau_0(X) \) to \( \tau_0(Z) \). However, one can regard continuous functions as 1-marked compact subsets, which will be discussed in the proof of Proposition 5.3 below.

Let \( T \) be the set of (equivalence classes of) pairs \( (X, \varphi) \), where \( X \) is a compact metric spaces and \( \varphi : X \to \Xi \) is a continuous function. Let \( T_0 \) be the pointed version defined similarly. Consider the following distance function on \( T \):

\[
d((X, \varphi), (Y, \psi)) := \inf \left\{ \frac{1}{2} \mathrm{dis}(R) \vee \sup_{(x,y) \in R} \{d(\varphi(x), \psi(y))\} \right\},
\]

where the infimum is over all correspondences \( R \) of \( X \) and \( Y \). In the pointed case, consider the same definition under the additional condition that the roots \( R \)-correspond to each other. This distance function is defined in [13] for the
case of spatial trees; i.e., when $X$ and $Y$ are real trees (except that the $\vee$ in the formula is a $+$ in [13] and the coefficients are slightly different, which are unimportant changes). It is claimed in [13] that 'it is easy to verify that $T_*$ is a Polish space'. However, as observed in [11] and [7], $T$ and $T_*$ are not complete metric spaces (even in the case of real trees). The results of [7] imply that $T$ and $T_*$ are separable metric spaces. Here, we prove the following proposition, which is enough for having a standard probability space.

**Proposition 5.3.** The spaces $T$ and $T_*$ are Borel subsets (in fact, $F_{\sigma\delta}$ subsets) of some Polish space.

Note that by Alexandrov’s theorem and its converse, $T$ and $T_*$ are themselves Polish (i.e., homeomorphic to a complete separable metric space) if and only if they are $G_\delta$ subsets, which is not clear in this case even if it is true.

**Proof of Proposition 5.3.** Before proving the claim, it is shown first that the above metric is a special case of the metric (3.1). Identify every continuous function $\varphi : X \to \Xi$ with its graph $\text{gr}_\varphi$, which is a closed subset of $X \times \Xi$ equipped with the max product metric. If $f : X \to Z$ is an isometric embedding, let $\tau_f : X \times \Xi \to Z \times \Xi$ be defined by $\tau_f(x, \xi) := (f(x), \xi)$. The reader can verify that the metric (5.1) can be rewritten as

$$d((X, \varphi), (Y, \psi)) = \inf \left\{ d_H (f(X), g(Y)) \vee d_H (\tau_f(\text{gr}_\varphi), \tau_g(\text{gr}_\psi)) \right\},$$  

(5.2)

where the infimum is over all metric spaces $Z$ and isometric embeddings $f : X \to Z$ and $g : Y \to Z$ (this is a special case of Proposition 3.28). By regarding $\varphi$ as a 1-marked compact subset of $X$ (see Example 3.22), it is straightforward that (5.2) is identical to the metric defined in (3.1) in which the additional structure is a 1-marked compact subset.

To prove the claim, let $\tau_0(X)$ be the set of continuous functions $\varphi : X \to \Xi$ and $\tau(X)$ be the set of 1-marked compact subsets of $X$. Note that, as mentioned above, $\tau_0$ is not a functor. However, Example 3.24 shows that $\tau$ is a functor and defines a complete separable metric space $C_\tau$. It follows that $T$ is a subset of $C_\tau$. It can be seen that $T = \cap_n \cup_m \left\{ (X, \varphi) \in T : w_\varphi(\frac{1}{n}) \leq \frac{1}{n} \right\}$, where $w_\varphi(\epsilon) := \max \{ d(\varphi(x), \varphi(y)) : x, y \in X, d(x, y) \leq \epsilon \}$ is the modulus of continuity of $\varphi$. It can also be seen that the sets under union are closed subsets of $C_\tau$. Hence, $T$ is a Borel (and a $F_{\sigma\delta}$) subset of $C_\tau$. So the claim is proved.

The pointed case, which is the case of [13], can also be treated similarly. To do this, consider the setting of Section 3 where the additional structure is a pair of a point and a 1-marked closed subset (see Example 3.5). It follows similarly that $T_*$ is a Borel (and a $F_{\sigma\delta}$) subset of a Polish space.

The paper [7] studies measured rooted spatial trees; i.e., spatial trees (discussed above) equipped with a Borel measure and a point. The metric of [7] is defined using both isometric embeddings and correspondences and it is shown that a separable (non-complete) metric space is obtained. This metric can be simplified as follows: By changing the $+$ in the formula of [7] to $\vee$, one obtains an equivalent metric whose formula is similar to (5.2), where two more terms

\[33\]
should be included for the Prokhorov-distance of the measures and the distance of the roots (Proposition 3.28 also gives an equivalent formulation by correspondences and approximate couplings). Similarly to the above arguments, it can be seen that this metric is a special case of the metric $d_{\ast\ast}$, where the additional structure is a tuple of a point, a measure and a 1-marked compact subset. Similarly to the above proposition, one can show that the set of measured spatial trees is a Borel $(F_{\sigma\delta})$ subspace of a Polish space.

Measured rooted spatial trees are also extended in [7] to the case of locally-compact length spaces. This extension is by the same method as that of [1] with the exception that the resulting metric space is not complete (but it is separable). The above arguments can be repeated to show that the latter is a Borel $(F_{\sigma\delta})$ subspace of the Polish space $C'$ defined in Section 3 for suitable functors. In addition, by the definitions and results of Section 3, locally-compact length spaces can be generalized to boundedly-compact metric spaces. Moreover, the pre-compactness result (Lemma 3.5) of [7] can be deduced easily from Theorem 3.18.

5.5 The Spectral Gromov-Hausdorff Metric

Let $I \subset \mathbb{R}$ be a fixed compact interval. The paper [11] studies the set $\tilde{T}$ of tuples $(X, \pi, q)$, where $X$ is a compact metric spaces $X$, $\pi$ is a Borel probability measure on $X$ and $q : X \times X \times I \to \mathbb{R}$ is a continuous function. A metric on this space is defined in [11] using both isometric embeddings and correspondences and it is shown that a separable (non-complete) metric space is obtained. This metric is called the spectral Gromov-Hausdorff metric in [11]. In this subsection, similarly to the arguments in Subsection 5.4, equivalent formulations of the metric are discussed and it is shown that $\tilde{T}$ is a Borel subspace of some Polish space which is obtained by the method of Section 3.

For compact metric spaces $X$, let $\tau_0(X)$ be the set of pairs $(\pi, q)$ as above. Note that $\tau_0$ is not a functor. Let $\tau(X)$ be the set of pairs $(\mu, K)$, where $\mu$ is a finite Borel measure on $X$ and $K$ is a compact subset of $X \times X \times C(I)$, where $C(I)$ denotes the set of continuous functions on $I$ endowed with the sup metric and the product space is endowed with the max product metric. Note that $K$ is a 2-marked compact subset of $X$ as in Example 3.24. Every element $(\pi, q) \in \tau_0(X)$ corresponds naturally to an element of $\tau(X)$ by considering $K := \{(x, y, q(x, y, \cdot)) : x, y \in X\}$. Therefore, $\tilde{T}$ is a subset of $C_\tau$, where the latter is defined in Section 3 (see Example 3.24). In addition, it is straightforward that the metric on $\tilde{T}$ is equivalent to (and by changing $+$ to $\vee$, will become identical to) the metric on $C_\tau$ defined in (3.1). So $\tilde{T}$ is a topological subspace of $C_\tau$. Similarly to Subsection 5.4, one can show that it is a Borel $(F_{\sigma\delta})$ subset of the Polish space $C_\tau$. So the claim is proved. In addition, Proposition 3.28 provides an equivalent formulation of the metric in terms of correspondences and couplings.
5.6 Random Objects in Stochastic Geometry

In this subsection, it will be shown that some random objects in stochastic geometry can be regarded as random pointed metric spaces equipped with more structures (defined in Section 4). This shows that the latter provides a common generalization to those random objects. In addition, this allows one to let the underlying space be random in each case; e.g., to define a point process on a random pointed metric space.

5.6.1 Random Closed Sets, Random Measures and Point Processes

Let $S$ be a boundedly-compact metric space and $o \in S$ be arbitrary. Let $\mathcal{F}$ be the set of closed subsets of $S$. One can equip $\mathcal{F}$ with the Fell topology, which makes it a compact Polish space (see e.g., [23]). This allows one to define a random closed subset of $S$ as a random element of $\mathcal{F}$. The Fell topology, restricted to the set of closed subsets of a given compact set of $S$, coincides with the topology of the Hausdorff metric (Theorem 12.3.2 of [23]).

Consider the space $C'$ defined in Section 4 for the functor of example 4.2. By considering the map $K \mapsto (S, o; K)$ from $\mathcal{F}$ to $C'$, one can regard a random closed subset of $S$ as a random element in $C'$ at the cost of considering subsets of $S$ up to equivalence under automorphisms of $(S, o)$ (it can be seen that this map is continuous). This also allows the base space $(S, o)$ be random, and so, a random elements in $C'$ can be called a random closed set in a random environment.

The issue of the automorphisms in the above discussion can be ruled out by adding marks as sketched in the following. Let the mark space be $\Xi := S$ and let the mark of every point $u \in S$ be simply $u$ itself (as in Definition 3.21 and Example 3.22). This way, $\mathcal{F}$ can be identified with a closed (topological) subspace of $C'$, and hence, random closed subsets of $S$ are special cases of random elements in $C'$. The details are left to the reader.

Similarly, random measures on $S$ can be regarded as random pointed measures metric spaces. See [24] for further details.

A (simple) point process in $S$ is, roughly speaking, a random discrete subset of $S$. For a formal definition, it is common to regard every discrete subset of $S$ as a measure on $S$ (by considering the counting measure). Therefore, point processes are special cases of random measures. As a second approach, one can also regard point processes as random closed subsets of $S$. The latter gives a coarser topology on the set of discrete subsets of $S$, but generates the same Borel sigma-field. See e.g., Theorem 14.28 of [20] and the discussion before it.

Note that in both approaches, the set of discrete subsets of $S$ is not complete, but it is a Borel subset of another complete separable metric space.

5.6.2 Marked Point Processes and Marked Random Measures

Let $S$ and $\Xi$ be boundedly-compact metric spaces, where $\Xi$ is regarded as the mark space. The notions of marked point processes and marked random measures on $S$ are defined in the literature (see e.g., [23] and [12]). The state space
of the former is simply the set of discrete subsets \( \varphi \subseteq S \) that are equipped with a function from \( \varphi \) to \( \Xi \). The latter needs more care. The standard definition in the literature is to define a marked random measure on \( S \) as a random measure in \( S \times \Xi \). This definition is more general than the measures on \( S \) that are equipped with a (suitable) function from \( S \) to \( \Xi \) as described in Example 3.22.

According to Definition 3.21, marked random measures on \( S \) are special cases of (random) 1-markings of \( S \). Therefore, similarly to Subsection 5.6.1, one can show that marked random measures and point processes can be regarded as random elements in \( C' \).

Additionally, the idea of Definition 3.21 allows us to define marked random closed subsets of \( S \) as random closed subsets of \( S \times \Xi \).

5.6.3 Particle Processes

Let \( S \) be a boundedly-compact metric space. Roughly speaking, a particle process on \( S \) is a random discrete collection of compact subsets of \( S \). More precisely, if \( \mathcal{K} = \mathcal{K}(S) \) is the set of nonempty compact subsets of \( S \) equipped with the Fell topology (see Subsection 5.6.1), then a particle process in \( S \) is a point process on \( \mathcal{K}(S) \). See e.g., [23].

In other words, let \( \varphi(S) \) be the set of discrete subsets of \( \mathcal{K}(S) \). According to Subsection 5.6.1 one can define a metric on \( \varphi(S) \) (which needs to fix a point \( o \in S \)). Then, a particle process on \( S \) is a random element in \( \varphi(S) \). Note that \( \varphi(S) \) is not complete. However, similarly to the case of point processes, one can show that \( \varphi(S) \) is a Borel subset of a Polish space; either by considering the set of measures on \( \mathcal{K}(S) \) or the set of closed subsets of \( \mathcal{K}(S) \) (see also Subsection 5.7 below).

Here, it is shown that \( \varphi(\cdot) \) defined above can be used in the setting of Section 4 to define the metric space \( C' \). In particular, this can be used to define a particle process in a random environment; i.e., to replace \( S \) with a random pointed metric space in the above definition. It is also shown that a particle process on \( S \) can be regarded as a random element in \( C' \).

Let \( \tau \) be the restriction of \( \varphi \) to compact metric spaces. Equivalently, for all compact metric spaces \( X \), \( \tau(X) \) is the set of finite subsets of \( \mathcal{K}(X) \). Consider the Hausdorff metric between finite subsets of \( \mathcal{K}(X) \). First, it can be seen that \( \tau \) is a functor that satisfies the continuity properties and the 1-Lipschitz properties of Definitions 3.8 and 3.12. Let the partial order on \( \tau(X) \) be that of inclusion. Then, a truncation functor \( \tau' \) is defined as follows. Let \( f : Y \to X \) be an isometric embedding and \( a \in \tau(X) \). Represent \( a \) as \( \{ K_1, K_2, \ldots, K_n \} \), where \( K_i \in \mathcal{K}(X) \) for all \( i \). Then, let \( \tau'_i(a) := \{ f^{-1}(K_i) : i \leq n, K_i \subseteq f(Y) \} \).

It can be seen that the assumptions of Section 4 are satisfied, and hence, the metric space \( C' \) is obtained. In addition, the extension defined in Definition 4.5 coincides with \( \varphi \). Note that since \( \tau(X) \) is not complete, \( C' \) is also not complete. This issue is resolved by letting \( \tau_0(X) \) be the set of closed subsets of \( \mathcal{K}(X) \). In Subsection 5.7 below, the same method is used for \( \tau_0 \) to construct a complete separable metric space, namely \( C'' \). It can be seen that \( C'' \) contains contains \( C' \) as a Borel subspace (similarly to the case of point processes). This allows one
to define a random element in $C'$.

Finally, to show that particle processes on $S$ can be represented as random elements in $C'$, one can proceed as in Subsection 5.6.1 (one can also consider a marking to rule out the automorphisms). The details are left to the reader.

5.7 The Brownian Web

In this subsection, the Brownian web is heuristically described first. The point of interest is the state space of the Brownian web and its connection to the setting of Section 4.

For every $(x, t) \in \mathbb{Z}^2$ such that $x + t$ is even, assume a particle is born at point $x$ at time $t$. Assume also that all particles that are at point $x$ at time $t$, move to the point $x + U(x, t)$ at time $t + 1$, where $U(\cdot, \cdot)$ are i.i.d. uniform random variables in $\{\pm 1\}$. Roughly speaking, by scaling this coalescing particle system properly and taking limit, the Brownian web is obtained. In the literature, the Brownian web is defined as a random collection of paths in $\mathbb{R}^2$. Let $\Pi$ be the set of graphs of continuous functions defined on intervals of the form $[T, \infty)$ (where $T$ is not fixed). In a specific compactification $E$ of $\mathbb{R}^2$, each element of $\Pi$ can be regarded a compact subset of $E$ (by adding to $E$ one limit point). Let $K$ be the set of compact subsets of $E$ equipped with the Hausdorff metric and let $S$ be the set of compact subsets of $K$ equipped with the Hausdorff metric. Then, the Brownian web is defined as a random element of $S$. See [14] for more details and the distribution of the Brownian web.

Now, an alternative state space for the Brownian web and its connection to Section 4 is discussed. For all boundedly-compact metric spaces $X$, let $\mathcal{F}(X)$ be the set of closed subsets of $X$ as in Subsection 5.6.1. Let $\varphi(X) := \mathcal{F}(\mathcal{F}(X))$ be the set of closed subsets of $\mathcal{F}(X)$. By fixing one point of $X$ as the origin, Subsection 5.6.1 shows how to define a metric on $\varphi(X)$ such that it is a complete separable metric space. Since the graph of every continuous function is an element of $\mathcal{F}(\mathbb{R}^2)$, one gets $\Pi \subseteq \mathcal{F}(\mathbb{R}^2)$. It can be seen that a subset of $\Pi$ is closed in $K$ if and only it is closed in $\mathcal{F}(\mathbb{R}^2)$. Let $S'$ be the set of closed subsets of $\Pi$. So $S' \subseteq S$ and $S' \subseteq \varphi(\mathbb{R}^2)$. It can also be seen that $S$ and $\varphi(\mathbb{R}^2)$ induce the same topology on $S'$. So $\varphi(\mathbb{R}^2)$ can also be used as the state space of the Brownian web.

Moreover, $\varphi$ can be used to define the metric space $C'$ as in Section 4. To do this, let $\tau$ be the restriction of $\varphi$ to compact metric spaces. If $X$ is compact, equip $\tau(X) = \mathcal{F}(\mathcal{F}(X))$ with the Hausdorff metric and let the partial order on $\tau(X)$ be that of inclusion. For all isometric embeddings $f : Y \to X$ and $a \in \tau(X)$, let $\tau^f_\uparrow(a) := \{ f^{-1}(C) : C \in a, C \subseteq f(Y) \} \in \tau(Y)$. By the method of Section 4 one can define the metric space $C'$ and it will be a complete separable metric space (verification of the assumptions of Section 4 is left to the reader). So one can define a random closed collection of closed subsets in a random environment as a random element of $C'$.

A similar argument can be applied to the Brownian web and its dual (see e.g., [14]). For this, $\varphi(\mathbb{R}^2) \times \varphi(\mathbb{R}^2)$ can be used as the state space.
5.8 Ends

Ends are defined in [15] for all topological spaces $X$ which, heuristically, are the points at infinity of $X$. A similar notion is defined for graphs. In this subsection, the setting of Section 4 is used to study the set of boundedly-compact pointed metric spaces equipped with an end. Similar cases are considered in the literature with a different metrization (see e.g., [19] for trees). Also, we study the set of boundedly-compact pointed metric spaces equipped with a closed subset of ends. A similar set is considered in [6] for trees without going into details.

For simplicity, we restrict attention to the class $L$ of boundedly-compact metric spaces $X$ such that $X$ is either a simple connected graph (equipped with the graph distance metric) or it is locally-connected; i.e., every neighborhood of every point in $X$ contains another neighborhood which is connected. The general definition of ends is skipped for brevity. Given a point $o$ of $X$, an end $\xi$ of $X$ can be uniquely described by a sequence of closed sets $\xi_1 \supset \xi_2 \supset \cdots$, where $\xi_n$ is a connected component of $X \setminus B_n(o)$ for each $n$, where $B_n(o)$ is the open ball of radius $n$ centered at $o$. Ends of graphs are defined similarly by using the notion of connectedness in graphs. So the set $L$ of (equivalence classes of) tuples $(X, o, \xi)$, where $X \in L$, $o \in X$ and $\xi$ is an end of $X$, can be regarded as a subset of the space $C'$ defined in Section 4. Here, the corresponding functor $\varphi$ is such that $\varphi(X)$ is the set of sequences of closed subsets of $X$ as in Examples 4.2 and 4.3. It can be seen that $L$ is a closed subset of $C'$. Therefore, the metric on $C'$ can be used to make $L$ a complete separable metric space.

Also, a natural topology is defined on the set $\mathcal{E}(X)$ of ends of every topological space $X$ as follows: Given an arbitrary $o \in X$, the open sets in $\mathcal{E}(X)$ are $\{\xi \in \mathcal{E}(X) : \exists n : \xi_n \subseteq V\}$, where $V$ is an open set in $X$. If $X \in \mathcal{L}$, then it can be seen that the topology of $\mathcal{E}(X)$ is identical to the restriction of the topology of $\varphi(X)$ (see the Fell topology in 5.6.1), where $\varphi(X)$ is defined above (note that locally-connectedness implies that every connected component of $X \setminus B_r(o)$ is both closed and open in $X \setminus B_r(o)$). In this case, it can be seen that a closed subset $S \subseteq \mathcal{E}(X)$ can be uniquely represented as a sequence of closed sets $C_1 \supset C_2 \supset \cdots$, where $C_n$ is the union of the connected components of $X \setminus B_n(o)$ that hit $S$. Thus, the same arguments as above can be used to define a Polish structure on the set of tuples $(X, o, S)$, where $X \in \mathcal{L}$, $o \in X$ and $S$ is a closed subset of the set of ends of $X$.

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