QUANTUM WITTEN LOCALIZATION AND ABELIANIZATION FOR QDE SOLUTIONS

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Abstract. We prove a quantum version of the localization formula of Witten [58], see also [56], [50], [60], which relates invariants of a git quotient with the equivariant invariants of the action. As an application, we prove a quantum version of an abelianization formula of S. Martin [38], relating invariants of geometric invariant theory quotients by a group and its maximal torus. The latter is a version of the abelian/non-abelian conjecture of Bertram, Ciocan-Fontanine, and Kim [8]. As sample applications we give a formula for a solution to the quantum differential equation (qde) for the moduli space of points on the projective line and for the moduli of framed sheaves on the projective plane.

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1. Introduction

Let $G$ be a connected complex reductive group acting on a smooth polarized projective variety $X$. Let $X//G$ denote the *git quotient* of $X$ by $G$, by which we mean the stack-theoretic quotient of the semistable locus by the group action. We assume that $G$ acts with only finite stabilizers on the semistable locus, in which case the git quotient $X//G$ is a smooth proper Deligne-Mumford stack with projective coarse moduli space by Mumford et al [41]. Let $H(X//G)$ resp. $H_G(X)$ denote the rational resp. equivariant rational cohomology of $X//G$ resp. $X$. Kirwan’s thesis [33] studies the natural map

$$\kappa_{X,G}: H_G(X) \to H(X//G)$$

given by restriction to the semistable locus and descent. Integration over $X//G$ defines a trace map

$$\tau_{X//G}: H(X//G) \to \mathbb{Q}, \quad \alpha \mapsto \int_{[X//G]}^{\kappa_{X,G}} \alpha.$$

Naturally one wants to compute the composition $\tau_{X//G} \kappa_{X,G}$ in terms of the $G$-equivariant cohomology on $X$, for example, in order to compute the cohomology of $X//G$ in terms of the $G$-equivariant cohomology of $X$.

Witten [58] introduced a strategy, which he termed *non-abelian localization*, to compute the composition $\tau_{X//G} \kappa_{X,G}$ in terms of a trace map

$$\tau_G^X: H_G(X) \to \mathbb{Q}, \quad \alpha \mapsto \int_{[X] \times \mathfrak{k}}^{\kappa_{X,G}} \alpha.$$

is given by integration over $X$ and a unitary form $\mathfrak{k}$ of the Lie algebra $\mathfrak{g}$. The integral of a polynomial over $\mathfrak{k}$ may be defined via various regularization procedures, see [49], [50], [60]. In the $K$-theory version discussed in see Paradan [51], the Witten trace is easier to define: it is the invariant part of the index, given by integration of the character of the index over a maximal unitary subgroup $K$ of $G$ and no regularization procedure is needed. Witten’s localization formula computes the difference between $\tau_{X//G} \circ \kappa_{X,G}$ and $\tau_G^X$, that is, the failure of the following diagram to commute:

$$\begin{array}{ccc}
H_G(X) & \xrightarrow{\kappa_{X,G}} & H(X//G) \\
\tau_G^X \downarrow & & \downarrow \tau_{X//G} \\
\mathbb{Q} & \xrightarrow{\tau_{X//G}} & \mathbb{Q}
\end{array}$$

By Witten’s argument in [58] the difference is a non-explicit sum of contributions from critical points of the norm-square of the moment map:

$$\tau_G^X = \tau_{X//G} \kappa_{X,G} + \sum_{[\zeta] \neq 0} \tau_{X,G,\zeta}$$

where $\tau_{X,G,\zeta}$ is the contribution from the critical locus mapping to $\zeta$ under the moment map. An explicit formula for the contributions $\tau_{X,G,\zeta}$ was described in papers by Teleman [56] in the case of sheaf cohomology, by Paradan [51] for $K$-theory of Hamiltonian actions, and in papers by Paradan and Woodward [49], [50],
for cohomology of Hamiltonian actions, see also Beasley-Witten [4] which uses the localization formula to compute the Chern-Simons partition function for Seifert manifolds. A different formula computing the composition is given in Jeffrey-Kirwan [31].

In this paper we prove a quantum version of Witten’s localization formula, which compares Gromov-Witten invariants of a git quotient with equivariant Gromov-Witten invariants for the action. The quantum Witten localization formula is the same as the Witten localization formula, but each expression in the formula is “quantized” in the sense that is replaced by an integration over a moduli space of stable maps:

(a) The quantization of the Witten trace $\tau^G_X$ is an integration over a git-type quotient $\overline{M}_n(\mathbb{P}, X)//G$ of the moduli space of parametrized stable maps from the projective line $\mathbb{P}$ to $X$ by the action of $G$;
(b) The quantization of the integration $\tau_{X//G}$ over $X//G$ is given by an integral over parametrized stable maps $\overline{M}_n(\mathbb{P}, X//G)$ to the quotient $X//G$;
(c) The quantization of the fixed point contributions $\tau_{X,G,\zeta,\rho}$ is given by integrating over the moduli stack $\overline{M}^G_n(\mathbb{C}, X)$ of semistable maps to the quotient $X_\zeta//G_\zeta$ of the fixed point loci $X_\zeta$ by the centralizer $G_\zeta$;
(d) The quantization of the Kirwan map $\kappa_{X,G}$ is given by integration over the moduli stack $\overline{M}^G_n(\mathbb{C}, X)$ of affine gauged maps from $[59]$.

In more detail, let $\omega \in H^2_G(X)$ be the first Chern class of the polarization (that is, the symplectic class) and let

$$\Lambda_X^G = \left\{ \sum_{i=0}^{\infty} c_i d^i, c_i, d_i \in \mathbb{Q}, d_i \in H^2_G(X, \mathbb{Q}), \lim_{i \to \infty} \langle d_i, \omega \rangle = \infty \right\}$$

denote the equivariant Novikov field for $X$, and

$$QH^*_G(X) = H^*_G(X) \otimes \Lambda_X^G$$

the equivariant quantum cohomology of $X$. (We will consider only polynomial equivariant cohomology classes, as opposed to the classical case where we considered also certain exponentials of polynomial classes.) Virtual integration over the moduli stack of $n$-marked genus 0 stable maps $\overline{M}_{0,n}(X)$ for $n \geq 3$ defines a family of formal quantum products

$$\ast_\alpha : T_\alpha QH^*_G(X)^2 \to T_\alpha QH^*_G(X), \quad \alpha \in QH^*_G(X).$$

Formal in this setting means that only the Taylor coefficients of the maps are convergent. A quantum version of Witten’s trace can be defined as follows. Let $\mathbb{P}$ denote the projective line and for $d \in H_2(X, \mathbb{Z})$ let

$$\overline{M}_n(\mathbb{P}, X, d) := \overline{M}_{0,n}(\mathbb{P} \times X, (1, d))$$
the moduli stack of parametrized stable maps from \( \mathbb{P} \) to \( X \) of class \( d \in H_2^G(X, \mathbb{Z}) \). It admits a natural forgetful morphism

\[
 f : \overline{M}_n(\mathbb{P}, X, d) \to \overline{M}_n(\mathbb{P}) := \overline{M}_{0,n}(\mathbb{P}, 1)
\]

where the target \( \overline{M}_n(\mathbb{P}) \) is the Fulton-MacPherson compactification of \( n \)-tuples of points on the projective line, or equivalently, the moduli space \( \overline{M}_{0,n}(\mathbb{P}, 1) \) of genus 0, \( n \)-marked stable maps to \( \mathbb{P} \) of degree 1. The action of \( G \) on \( \overline{M}_n(\mathbb{P}, X, d) \) by translation has a natural stability condition (not quite of git type) given by requiring that the stable map has generically semistable value \( \mathbb{Z} \). We denote by \( \overline{M}_n(\mathbb{P}, X, d) // G \) the stack-theoretic quotient of the semistable locus by the group action, \( \overline{M}_n(\mathbb{P}, X, d) // G \) is a proper Deligne-Mumford stack with a perfect relative obstruction theory \[22\]. Via equivariant formality we may consider \( H_2^G(X, \mathbb{Z}) / \text{torsion} \) as a subgroup of \( H_2^G(X, \mathbb{Q}) \). We denote by \( \tau^G_X \) the formal trace map given by virtual integration over the moduli stacks \( \overline{M}_n(\mathbb{P}, X, d) // G \):

\[
\tau^G_X : QH^G(X) \to \Lambda^G_X, \quad \alpha \mapsto \sum_{n \geq 0, d \in H_2(X, \mathbb{Z}) / \text{torsion}} (q^d / n!) \int_{[\overline{M}_n(\mathbb{P}, X, d) // G]} \text{ev}^*(\alpha, \ldots, \alpha)
\]

for \( \alpha \in H^G(X) \) and extended by linearity over \( \Lambda^G_X \). The map \( \tau^G_X \) is a quantum version of Witten’s trace in the sense that if one sets \( q = 0 \) and adds a Fulton-MacPherson insertion to fix the positions of the markings then one obtains the classical Witten trace for polynomial classes, that is, the integral over \( X // G \).

A quantum version of Kirwan’s map \( \kappa_{X,G} : H^G(X) \to H(X // G) \) was introduced in [59]. The quantum Kirwan map is a non-linear map, still denoted \( \kappa_{X,G} \),

\[
(2) \quad \kappa_{X,G} : QH^G(X) \to QH(X // G)
\]

with the property that any linearization

\[
D_\alpha \kappa_{X,G} : T_\alpha QH^G(X) \to T_{\kappa_{X,G}(\alpha)} QH(X // G)
\]

is a homomorphism with respect to the quantum products. In particular, if \( \kappa_{X,G}(0) = 0 \) (which generally happens only in Fano cases) then \( D_0 \kappa_{X,G} \) is a homomorphism from the small equivariant quantum cohomology of \( X \) to the quantum cohomology of \( X // G \).

A quantum version of the integration \( \tau_{X // G} \) over the quotient \( X // G \) is the formal map given by integration over the parametrized stable maps to the quotient \( X // G \):

\[
\tau_{X // G} : QH(X // G) \to \Lambda^G_X, \quad \alpha \mapsto \sum_{n \geq 0, d \in H_2(X // G, \mathbb{Q})} (q^d / n!) \int_{[\overline{M}_n(\mathbb{P}, X // G, d)]} \text{ev}^*(\alpha, \ldots, \alpha)
\]

for \( \alpha \in H(X // G) \), extended by linearity over \( \Lambda^G_X \). The integrals \( \tau^G_X \) and \( \tau_{X // G} \) differ by the position of the expression \( // G \), as well as the sums over homology classes. The quantum Witten localization formula gives a precise description of the difference between the traces \( \tau^G_X \) and \( \tau_{X // G} \circ \kappa_{X,G} \). That is, it measures the failure of the
“quantum integration” to commute with reduction, in the sense of the following diagram:

\[
\begin{array}{ccc}
  QH_G(X) & \xrightarrow{\kappa_{X,G}} & QH(X//G) \\
  \tau_X^G & \downarrow & \Lambda_X^G \\
  & \tau_{X/G} &
\end{array}
\]

As in the classical Witten localization formula, the failure is given by a sum of fixed point contributions. Each term is a particular gauged Gromov-Witten invariant, associated to the action of \( G \) on the fixed point variety \( X^G \) for the infinitesimal action of \( \zeta \in \mathfrak{g} \) given by the vanishing of the vector field \( \zeta_X \in \text{Vect}(X) \) defined by \( \zeta \), that is, \( X^G := \{ x \in X | \zeta_X(x) = 0 \} \).

For the sake of exposition, we first explain the gauged Gromov-Witten invariants associated to the action of \( G \) on \( X \). A morphism from a smooth curve \( C \) to the quotient stack \( X/G \) is given by a pair \((P \to C, u : C \to P(X))\) consisting of a \( G \)-bundle \( P \to C \) and a section \( u : C \to P(X) \) of the associated \( X \)-bundle \( P(X) := P \times_G X \). Such a morphism is Mundet semistable if it satisfies a certain Hilbert-Mumford inequality introduced in \([42]\) for each pair of a parabolic reduction and central coweight for the corresponding parabolic. If stable=semistable then the moduli stack \( \mathcal{M}^G_n(C, X) \) of \( n \)-marked Mundet semistable gauged maps (using the polarization \( L \to X \)) is Deligne-Mumford.

The moduli space \( \mathcal{M}^G_n(C, X) \) is most familiar in toric cases, that is, when \( X \) is a vector space and \( G \) is a torus. In this case the locus \( \mathcal{M}^G_{n}(C, X, d) \) of homology class \( d \in H^*_G(X, \mathbb{Z}) \) is a git quotient \( H^0(C, \mathcal{O}_X(d) \times_G X)/G \) of the space of sections \( H^0(C, \mathcal{O}_X(d) \times_G X) \) by the \( G \)-action, and proper if the weights are contained in an open half space. The objects of \( \mathcal{M}^G_n(C, X, d) \), in the case \( n = 0 \), are termed quasimaps in Givental \([21]\).

More generally, if \( X \) is a quasiprojective \( G \)-variety then bubbling can occur in which case \( \mathcal{M}^G_n(C, X, d) \) is not proper. If \( X \) is projective or a vector space with proper moment map then \( \mathcal{M}^G_n(C, X, d) \) has a canonical Kontsevich-style compactification denoted \( \overline{\mathcal{M}}^G_n(C, X, d) \), or more precisely, \( \mathcal{M}^G_n(C, X, d) \) has projective coarse moduli space. If stable=semistable then \( \overline{\mathcal{M}}^G_n(C, X, d) \) is a proper Deligne-Mumford stack with a perfect relative obstruction theory. Virtual integration over the moduli stacks \( \overline{\mathcal{M}}^G_n(C, X, d) \) for \( d \in H^2_G(X, \mathbb{Z})/\text{torsion} \) defines a gauged Gromov-Witten potential for the polarization \( L \)

\[
\tau_{X,G,L} : QH_G(X) \to \Lambda_X^G, \quad \alpha \mapsto \sum_{d,n \geq 0} (q^d/n!) \int_{[\mathcal{M}^G_n(C, X, d)]} \text{ev}^\ast(\alpha, \ldots, \alpha)
\]

for \( \alpha \in H_G(X) \). Note that the potential \( \tau_{X,G,L} \) depends on the isomorphism class \( [L] \in \text{Pic}_G(X) \), while the git quotient \( X//G \) depends only on the ray \( [L]^\rho \in \text{Pic}_G(X)_\mathbb{Q}, \rho \in \mathbb{Q} \) generated by \( [L] \).
The fixed point contributions in the quantum Witten localization formula are gauged Gromov-Witten invariants associated to fixed point components. Let $C^\xi_\zeta \subset G$ denote the one-parameter subgroup generated by $\zeta$ and $G_\zeta \subset G$ its centralizer. The gauged Gromov-Witten invariants appearing in quantum Witten localization are associated to the action of $G_\zeta$ on $X^\xi$. Let $C^\times_\zeta \subset G$ denote the one-parameter subgroup generated by $\zeta$ and $G_\zeta \subset G$ its centralizer.

The gauged Gromov-Witten invariants appearing in quantum Witten localization are associated to the action of $G_\zeta$ on $X^\xi$. Let $M^G_{C^n}(C, X, L, \zeta)$ denote the moduli stack of $C^\times_\zeta$-fixed gauged maps to $X/G_\zeta$ semistable with respect to the polarization $L$. The principal component of any such map takes values in $X^\xi/G_\zeta$ and the bubble components are fixed up to reparametrization. Suppose that for every (possibly zero) $\zeta \in g$, stable=semistable for $M^G_{C^n}(C, X, L, \zeta)$.

The equivariant cohomology $H^C_\times(pt)$ is canonically isomorphic to the ring $\mathbb{Q}[\xi]$ of polynomials in one variable $\xi$.

Let $\epsilon_+(T(X/G)) \in H_\times^C(M^G_{C^n}(C, X, L, \zeta))$ denote the Euler class of the index class for the moving part of the index bundle of $T(X/G)$; this is the normal complex for the inclusion of $M^G_{C^n}(C, X, L, \zeta)$ in $M^G_{C^n}(C, X, L)$. Let $T \subset G$ be a maximal torus with Lie algebra $t \subset g$. If $\zeta \in t$ then we denote by $W_\zeta$ resp. $W_{C_\zeta}$ the stabilizer of $\zeta$ resp. $C_\zeta$ under the Weyl group $W=N(T)/T$.

**Definition 1.0.1.** (Fixed point potential) The fixed point potential corresponding to $\zeta \in t$ is

\[ \tau_{X,G,\xi,L} : QH_G(X) \to \Lambda^G_X, \quad \alpha \mapsto \frac{|W_\zeta|}{|W_{C_\zeta}|} \sum_{d,n \geq 0} (q^d/n!) \text{Resid} \int_{[M^G_{C^n}(C, X, L, \zeta,d)]} \text{ev}^*(\alpha, \ldots, \alpha) \cup \epsilon_+(T(X/G))^{-1}. \]

For any non-zero $\zeta \in g$ we denote by $[\zeta]$ the equivalence class of $\zeta$ modulo scalar multiplication and adjoint action, so that each equivalence class corresponds to an unparametrized one-parameter subgroup $\exp(C_\zeta)$ of $G$. Our main result is the following: Given a polarization $L$, consider the path of rational polarizations $L^\rho, \rho \in (0, \infty) \cap \mathbb{Q}$.

**Theorem 1.0.2.** (Quantum Witten localization) Let $C$ be a smooth connected projective curve of genus 0, $X$ a smooth projective $G$-variety, and $L$ a polarization. Suppose that for every $\zeta \in g$ and $\rho \in (0, \infty)$, stable=semistable for $M^G_{C^n}(C, X, L^\rho, \zeta)$, and stable=semistable for the $G$-action on $X$. Then

\[ \tau^G_X - \tau_{X/G} \circ \kappa_{X,G} = \sum_{[\zeta] \neq 0, \rho \in (0, \infty)} \tau_{X,G,\xi,L^\rho}. \]

The proof, given in Section 4, uses the wall-crossing formula of Gonzalez-Woodward [24] applied to the case of the path of polarizations $L^\rho$, together with an identification of the limits as $\rho$ tends to 0 and $\infty$ with the potentials of $X/G$ resp. “$G$-invariant part” of the potential of $X$ from [22], [59]. We also give a formula, (32) below,
which applies to certain quasiprojective $G$-varieties. See Example 4.3.3 for a simple example.

Our original motivation for the quantum Witten localization formula was the quantum Martin conjecture of Bertram et al [8] which compares Gromov-Witten invariants of a git quotient $X//G$ and the quotient $X//T$ by a maximal torus $T \subset G$. Returning to the classical story, Let $\nu_{g/t}$ denote the bundle over $X//T$ induced from the trivial bundle with fiber $g/t$ over $X$, and let $\tau_{X//T}: H_T(X) \to \mathbb{Q}$ denote the $\text{Eul}(\nu_{g/t})$-twisted integration map,

$$\tau_{X//T}: H(X//T) \to \mathbb{Q}, \quad \alpha \mapsto \int_{[X//T]} \alpha \cup \text{Eul}(\nu_{g/t}).$$

Let $W = N(T)/T$ denote the Weyl group of $T \subset G$ and $r^G_T: H_G(X) \to H_T(X)$ the pullback map, inducing an isomorphism $H_G(X) \to H_T(X)^W$.

**Theorem 1.0.3.** (Martin formula [38]) Let $X$ be a smooth projective $G$-variety. Suppose that stable=semistable for the actions of $T$ and $G$ on $X$. Then integration over $X//G$ and $X//T$ are related by

$$\tau_{X//G} \circ \kappa_{X,G} = |W|^{-1} \tau_{X//T} \circ \kappa_{X,T} \circ r^G_T.$$

In other words, the following diagram commutes:

\[
\begin{array}{ccc}
H_G(X, \mathbb{Q}) & \cong & H_T(X, \mathbb{Q})^W \\
\downarrow \kappa_{X,G} & & \downarrow \kappa_{X,T} \\
H(X//G, \mathbb{Q}) & \xrightarrow{\tau_{X//G}} & H(X//T, \mathbb{Q}) \\
\downarrow \tau_{X//T} & & \downarrow |W|^{-1} \tau_{X//T} \\
\mathbb{Q} & & \mathbb{Q}
\end{array}
\]

Martin’s formula Theorem 1.0.3 implies the following result that will be used in the quantum generalization:

**Corollary 1.0.4.** ([38, Theorem A]) Let $X$ be a smooth $G$-variety such that stable=semistable for the actions of $G$ and $T$ on $X$. There exists a surjective map

$$\mu^G_T: H(X//T) \to H(X//G)$$

whose kernel is the cup product with $\text{Eul}(g/t)$. Given classes $\alpha_G \in H(X//G)$, $\alpha_T \in H(X//T)$, we have $\alpha_G = \mu^G_T \alpha_T$ iff

$$\int_{[X//G]} \alpha_G \cup \kappa_{X,G} \beta = |W|^{-1} \int_{[X//T]} \alpha_T \cup \kappa_{X,T} \beta \cup \text{Eul}(g/t)$$

for all $\beta \in H_G(X) \cong H_T(X)^W$. 
Using quantum Witten localization (6) we prove that a formula similar to that in Theorem 1.0.3 holds in quantum cohomology. Versions of this formula were conjectured by Hori-Vafa [29, Appendix], Bertram-Ciocan-Fontanine-Kim [8] and Ciocan-Fontanine-Kim-Sabbah [13]. The push-forward in homology \( \pi^G_T : H^T_2(X, \mathbb{Q}) \to H^G_2(X, \mathbb{Q}) \) defines a map of equivariant Novikov rings

\[
\pi^G_T : \Lambda^T_X \to \Lambda^G_X, \quad \sum_{d \in H^T_2(X)} c_d q^d \mapsto \sum_{d \in H^T_2(X)} c_d q^{\pi_G(d)}.
\]

Let \( QH^\text{chow}_G(X) \subset QH_G(X) \) denote the subspace generated by Chern characters of algebraic vector bundles,

\[
H^\text{chow}_G(X) := \{ \text{Ch}_G(E) \mid E \to X \text{ vector bundle} \}
\]

\[
QH^\text{chow}_G(X) := H^\text{chow}_G(X) \otimes \Lambda_X^G.
\]

We denote by

\[
r^G_T : QH^\text{chow}_G(X) \to QH^\text{chow}_T(X)
\]

the map obtained by combining the pull-back \( H^G(X) \to H^T(X) \) with the inclusion \( \Lambda_{X,G} \subset \Lambda_{X,T} \) induced by the inclusion \( H^G_2(X, \mathbb{Z}) \cong H^T_2(X, \mathbb{Z})^W \subset H^T_2(X, \mathbb{Z}) \).

**Theorem 1.0.5.** (Quantum Martin formula) Let \( C \) be a smooth connected projective genus 0 curve, \( X \) a smooth projective \( G \)-variety, and suppose that stable=semistable for \( T \) and \( G \) actions on \( X \). The following equality holds on \( QH^\text{chow}_G(X) \):

\[
\tau_{X/G} \circ \kappa_{X,G} = |W|^{-1} \pi^G_T \circ \tau_{X/T} \circ \kappa_{X,T} \circ r^G_T.
\]

That is, there is a commutative diagram

\[
\begin{array}{ccc}
QH^\text{chow}_G(X) & \rightarrow & QH^\text{chow}_T(X) \\
\kappa_{X,G} \downarrow & & \kappa_{X,T} \downarrow \\
QH(X//G) & \rightarrow & QH(X//T) \\
\tau_{X/G} \downarrow & & \tau_{X/T} \downarrow \\
\Lambda^G_X & \leftarrow & \Lambda^T_X \\
\end{array}
\]

The Chow restriction on the cohomology class inserted is probably unnecessary, but some technical aspects of our argument use sheaf cohomology. Namely, we do not know how to prove virtual abelianization for non-Chow classes.

Using the abelianization Theorem 1.0.5 we prove an abelianization theorem for solutions to quantum differential equations. Givental [21] observed that the \( \mathbb{C}^* \)-equivariant extension of the graph potential on a genus zero curve with \( \mathbb{C}^* \)-action admits a factorization into localized graph Gromov-Witten potentials

\[
\tau_{X/G, \pm} : QH(X//G) \to QH(X//G)[[h^{-1}]]
\]

where \( h \) is the equivariant parameter for the \( \mathbb{C}^* \)-action; these potentials are often call \( I \)-functions or one-point descendant potentials in the literature [21], because the relevant integrals involve Chern classes of cotangent lines as the last marking.
Historically the localized graph potential $\tau_{X/G,-}$ is a solution to the quantum differential equation on $QH(X//G)$, and explicit formulas for these have played an important role in the mirror conjectures of the physicists.

We relate the localized graph potentials $\tau_{X/G,\pm}$ and $\tau_{X/T,\pm}$ as follows. We denote by
\begin{equation}
(\mu_T^G \otimes \pi_T^G) : QH(X//G) := H(X//G) \otimes \Lambda_X^G \rightarrow H(X//T) \otimes \Lambda_X^T =: QH(X//T)
\end{equation}
the map combining Martin’s map $\mu_T^G$ of (7) with the canonical map of Novikov rings $\pi_T^G$ of (8).

**Theorem 1.0.6.** (Abelianization for qde solutions) Suppose that $X$ is a smooth projective $G$-variety, and stable=semistable for the $T$ and $G$-actions on $X$. Then after restriction to Chow classes,

$$
\tau_{X/G,-} \circ \kappa_{X,G} = (\mu_T^G \otimes \pi_T^G) \circ \tau_{X/T,-} \circ \kappa_{X,T} \circ \gamma_T^G
$$

where the trace and quantum Kirwan map on the right-hand-side have been twisted by the Euler class of the index bundle of $g/t$.

The argument extends to quasiprojective $X$ under suitable properness conditions. In particular, it holds for $G$-vector spaces $X$ satisfying certain boundedness conditions on the moment map, see Remark 4.3.2 below. In Example 5.2.3 and Section 5.3 we apply the formula to give formulas for the solution to the quantum differential equation for the Grassmannians and moduli of points on the projective line.

2. **Gauged Gromov-Witten invariants**

In this section we review the construction of gauged Gromov-Witten invariants from the algebro-geometric and symplectic viewpoint. An earlier definition for certain Hamiltonian actions can be found in Cieliebak-Gaio-Mundet-Salamon [9]. Here we assume that the target is a smooth projective (or in some cases quasiprojective) $G$-variety and use the Behrend-Fantechi machinery of virtual fundamental classes [5] to define the invariants. Other authors such as Frenkel-Telem-Tolland [18] and Ciocan-Fontanine-Kim-Maulik [12] have different invariants which might be called gauged Gromov-Witten invariants, corresponding to different stability conditions and compactifications of the space of morphisms to quotient stacks.

2.1. **Gauged Gromov-Witten invariants via algebraic geometry.** In this subsection construct gauged Gromov-Witten invariants as integrals over moduli stacks of Mundet-semistable morphisms from a curve $C$ to the quotient stack $X/G$. We begin with a more detailed discussion of Mundet semistability. We suppose that $g$ is the complexification of a real Lie algebra $\mathfrak{k}$, and for convenience $\mathfrak{k}$ is equipped with an inner product invariant under the action of $K = \exp(\mathfrak{k})$, inducing an identification $\mathfrak{k} \rightarrow \mathfrak{k}^\vee$; in particular this means that rational weights are identified with
rational coweights. However, the semistability condition below does not depend on this choice.

**Definition 2.1.1. (Mundet semistability)** Suppose \( u : C \to X/G \) is a morphism from a smooth projective curve \( C \) to the quotient stack \( X/G \) consisting of a pair \( (p : P \to C, u : C \to P(X) := P \times_G X) \) of a \( G \)-bundle \( p : P \to C \) and a section \( u : C \to P(X) \).

(a) (Associated graded morphism) Given a reduction of \( P \) to a parabolic subgroup \( R \subset G \) given by a section \( \sigma : C \to P/R \) and an antidominant, central coweight \( \lambda \) in the Lie algebra \( \mathfrak{r} \) of \( R \), there is an associated graded morphism given by an associated graded bundle

\[ \text{Gr}(P) \to C \]

whose structure group reduces to the Levi subgroup of \( R \) and an associated graded stable section

\[ \text{Gr}(u) : \hat{C} \to \text{Gr}(P)(X) \]

with domain a nodal curve \( \hat{C} \). The bundle \( \text{Gr}(P) \) is the limit of the bundles \( P_{z,\lambda} \) associated to the homomorphisms

\[ R \to R, \ r \mapsto \text{Ad}(z^\lambda)r \]

for \( z \in \mathbb{C}^\times \), while the section \( \text{Gr}(u) \) is the Gromov limit of the family of sections \( z^\lambda u \) of \( P_{z,\lambda}(X) \) induced by \( \lambda \).

(b) (Hilbert-Mumford weight) The principal component \( C_0 \) of \( \hat{C} \) is the irreducible component such that the restriction \( u_0 \) of \( u \) to \( C_0 \) maps isomorphically to \( C \). The principal component \( \text{Gr}(u)_0 \) of the associated graded section \( \text{Gr}(u) \) takes values in the fixed point set \( (\text{Gr}(P)(X))^\lambda = \text{Gr}(P)(X^\lambda) \) of the infinitesimal automorphism of \( \text{Gr}(P)(X) \) induced by \( \lambda \). The Hilbert-Mumford weight

\[ \mu_H(\sigma, \lambda) \in \mathbb{Z} \]

determined by the polarization \( L \), is the weight of the \( \mathbb{C}^\times \)-action generated by \( -\lambda \) on the fiber of the bundle \( (\text{Gr}(P))(L) \to (\text{Gr}(P))(X) \) over a generic value of \( \text{Gr}(u)_0 \).

(c) (Ramanathan weight) Assume that \( \lambda \in \mathfrak{r} \cong \mathfrak{r}^\vee \) is a weight of \( R \). The Ramanathan weight

\[ \mu_R(\sigma, \lambda) \in \mathbb{Z} \]

of \( (P, u) \) with respect to \( (\sigma, \lambda) \) is given by the first Chern number of the line bundle determined by \( -\lambda \) via the associated bundle construction: If \( \mathbb{C}_{-\lambda} \) is the one-dimensional representation of \( R \) with weight \( -\lambda \) then the line bundle is \( \sigma^* P \times_R \mathbb{C}_{-\lambda} \) and the Ramanathan weight is

\[ \mu_R(\sigma, \lambda) = \int_{[\hat{C}]} c_1(\sigma^* P \times_R \mathbb{C}_{-\lambda}) = \int_{[\hat{C}]} c_1(\text{Gr}(P) \times_L \mathbb{C}_{-\lambda}) \]

where in the last expression we are considering \( \text{Gr}(P) \) as an \( L \)-bundle.
(d) (Mundet weight) The **Mundet weight** is the sum of the Hilbert-Mumford and Ramanathan weights:
\[
\mu_M(\sigma, \lambda) := \mu_H(\sigma, \lambda) + \mu_R(\sigma, \lambda).
\]
(e) (Mundet semistability) The morphism \((P, u)\) is **Mundet semistable** if
\[
\mu_M(\sigma, \lambda) \leq 0
\]
for all such pairs \((\sigma, \lambda)\) [42], and **Mundet stable** if the above inequalities are satisfied strictly.

Mundet’s original definition allowed possibly irrational \(\lambda\), but this is unnecessary in the case that the symplectic class is rational by [59, Remark 5.8]. In order to compactify the moduli stack of Mundet-semistable morphisms to the quotient stack, we allow the formation of bubbles in the fibers of the associated bundle as follows:

**Definition 2.1.2.** (Nodal gauged maps) An **n-marked nodal gauged map** from \(C\) to \(X\) over a scheme \(S\) is a morphism \(u : \hat{C} \to C \times X/G\) from a nodal curve \(\hat{C}\) over \(S\) whose projection onto the first factor has homology class \([C]\), such that if \(C_i, s \subset \hat{C}_s\) is a component that maps to a point in \(C\), then the bundle corresponding to \(u|C_i\) is trivial. More explicitly, such a morphism is given by a datum \((\hat{C}, P, u, z)\) where

(a) (Nodal curve) \(\hat{C} \to S\) is a proper flat morphism with reduced nodal curves as fibers;
(b) (Bundle over the principal component) \(P \to C \times S\) is a principal \(G\)-bundle;
(c) (Section of the associated fiber bundle) \(u : \hat{C} \to P(X) := (P \times X)/G\) is a family of stable maps with base class \([C]\), that is, the composition of \(u\) with the projection \(P(X) \to C\) has class \([C]\).

A **morphism** between gauged maps \((S, \hat{C}, P, u)\) and \((S', \hat{C}', P', u')\) consists of a morphism \(\beta : S \to S'\), a morphism \(\phi : P \to (\beta \times 1)^* P'\), and a morphism \(\psi : \hat{C} \to \hat{C}'\) such that the first diagram below is Cartesian and the second and third commute:

\[
\begin{array}{ccc}
\hat{C} & \rightarrow & S \\
\psi \downarrow & & \beta \downarrow \\
\hat{C}' & \rightarrow & S'
\end{array}
\]
\[
\begin{array}{ccc}
P & \rightarrow & S \times C \\
\phi \downarrow & & \id \downarrow \\
(\beta \times 1)^* P' & \rightarrow & S \times C
\end{array}
\]

\[
\begin{array}{ccc}
\hat{C} & \rightarrow & P(X) \\
\psi \downarrow & & [\phi \times \id_X] \downarrow \\
\hat{C}' & \rightarrow & P'(X)
\end{array}
\]

An **n-marked** nodal gauged map is equipped with an **n-tuple** \((z_1, \ldots, z_n) \in \hat{C}^n\) of distinct smooth points on \(\hat{C}\). An **n-marked nodal gauged map** \((\hat{C}, P, z, u)\) is **semistable** resp. **stable** if the principal component is Mundet semistable resp. stable and the section \(u : \hat{C} \to P(X)\) is a stable section, in the sense that any component on which \(u\) is constant has at least three special (nodal or marked) points.

**Proposition 2.1.3.** The category of Mundet-semistable gauged maps from \(C\) to \(X/G\) of homology class \(d\) and \(n\) markings forms an Artin stack \(\overline{M}_n^G(C, X, L, d)\) [59], or \(\overline{M}_n^G(C, X, d)\) for short if the polarization is fixed. If stable=semistable then
$\overline{M}_n^G(C,X,d)$ is a proper smooth Deligne-Mumford stack with perfect relative obstruction theory and evaluation map $\operatorname{ev} : \overline{M}_n^G(C,X,d) \to (X/G)^n$.

**Proof.** For the sake of completeness we sketch the proof from [59]. The category of morphisms from $C$ to $X/G$ forms an Artin stack by results of Olsson [47]. The existence of a perfect relative obstruction theory over the moduli stack of prestable maps $\overline{M}_n(C)$ is as in Behrend-Fantechi [5], see [59] for details. The argument uses the deformation theory from Olsson [46] for morphisms to quotient stacks. That the subcategory of Mundet semistable morphisms is an open substack follows by pullback of the git construction in Schmitt [54], who proves the corresponding result for the quot-scheme compactification. If stable=semistable then all stabilizers are finite, and since we are in characteristic zero, this implies that $\overline{M}_n^G(C,X,d)$ is Deligne-Mumford. Properness follows from properness of the (coarse moduli space for the) quot-scheme compactification in Schmitt [54] and the properness of the Givental morphism. □

**Definition 2.1.4.** (Gauged Gromov-Witten potential) Suppose that stable=semistable for all gauged maps. The gauged potential $\tau_{X,G}$ is the formal map defined by

$$
\tau_{X,G} : QH^*_G(X) \to \Lambda^G_X \alpha \mapsto \sum_{n \geq 0, d \in H^2_G(X,\mathbb{Z})/\text{torsion}} (q^d/n!) \int_{[\overline{M}^G_n(C,X,d)]} \ev^* (\alpha, \ldots, \alpha)
$$

for $\alpha \in H^*_G(X)$, and extended to $QH^*_G(X)$ by linearity.

**Definition 2.1.5.** (a) (Invariants with Fulton-MacPherson insertions) Define

$$
\tau^\alpha_{X,G} : QH^*_G(X)^n \times H(\overline{M}_n(C)) \to \Lambda^G_X \alpha, \beta \mapsto \sum_{d \in H^2_G(X,\mathbb{Z})/\text{torsion}} q^d \int_{[\overline{M}^G_n(C,X,d)]} \ev^* (\alpha, \ldots, \alpha) \cup f^* \beta.
$$

In particular, if we take $G$ trivial, $n = 3$, $C$ rational, and $\beta \in H^6(\overline{M}_3(C))$ the class that fixes the positions of the three markings then $\tau^3_{X,G}(\alpha, \beta)$ are the three-point invariants of $X$.

(b) (Twisted gauged Gromov-Witten invariants) The universal curve

$$p : \overline{\mathcal{C}}^G_n(C,X) \to \overline{M}^G_n(C,X)
$$

admits a universal gauged map

$$e : \overline{\mathcal{C}}^G_n(C,X) \to X/G.
$$

For any $G$-equivariant bundle $E \to X$ we denote by

$$\text{Ind}(E) := Rp_\ast e^\ast (E/G)
$$

the index of the bundle $E/G \to X/G$; this is an object in the bounded derived category of $\overline{M}^G_n(C,X)$, since $p$ is proper. It admits a resolution by vector
bundle, since \( p \) is a local complete intersection morphism, see [15, Appendix].

The Euler class

\[
\varepsilon(E) := \text{Eul}_{\mathbb{C}^\times}(\text{Ind}(E)) \in H_{\mathbb{C}^\times}(\overline{\mathcal{M}}^G_n(C, X))
\]

is well-defined after passing to the equivariant cohomology of \( \overline{\mathcal{M}}^G_n(C, X) \) for the trivial \( \mathbb{C}^\times \)-action corresponding to scalar multiplication on the fibers and inverting the equivariant parameter. The twisted invariants are

\[
\tau^n_{X,G} : QH^G(X)^n \times H(\overline{\mathcal{M}}^G_n(C)) \to \Lambda^G_X
\]

\[
(\alpha, \beta) \mapsto \sum_{d \in H^G_2(X,\mathbb{Z})/\text{torsion}} q^d \int_{[\overline{\mathcal{M}}^G_n(C, X, d)]} \text{ev}^*(\alpha, \ldots, \alpha) \cup f^*\beta \cup \varepsilon(E).
\]

The Witten localization takes an especially simple form for \( \tau^n_{X,G} \), but there are also versions for invariants with insertions and twisted invariants.

2.2. Gauged Gromov-Witten invariants via symplectic vortices. In this section we explain the symplectic viewpoint on gauged Gromov-Witten invariants. This approach has the advantage of being rather more concrete, but the virtual fundamental classes are harder to define.

Symplectic vortices arise via the symplectic quotient construction applied to the space of gauged pseudoholomorphic sections. The equations that they satisfy were introduced and studied by Mundet i Riera [43] and Salamon [11], see also [20], [9]. In physics the gauged sigma model goes back at least to Witten [57]. Let \( C \) be a compact smooth holomorphic curve, \( K \) a compact group with Lie algebra \( \mathfrak{k} \), and \( \pi : P \to C \) a smooth principal \( K \)-bundle. Given any left \( K \)-manifold \( F \) the quotient of \( P \times F \) by the left action of \( K \) given by \( k(p, f) = (pk^{-1}, kf) \) is denoted

\[
P(F) := (P \times F)/K
\]

and called the associated fiber bundle with fiber \( F \). Denote by

\[
\mathcal{A}(P) \subset \Omega^1(P, \mathfrak{k})
\]

the space of smooth connections on \( P \), that is, the space of one-forms \( A \in \Omega^1(P, \mathfrak{k}) \) such that \( A(\zeta_p) = \zeta \) for all \( \zeta \in \mathfrak{k} \) and \( k^*A = \text{Ad}(k)A \) for all \( k \in K \). Let \( P(\mathfrak{t}) := (P \times \mathfrak{t})/K \) denote the adjoint bundle. The space \( \Omega^1(C, P(\mathfrak{t})) \) of \( P(\mathfrak{t}) \)-valued one-forms acts transitively on \( \mathcal{A}(P) \) by

\[
\Omega^1(C, P(\mathfrak{t})) \times \mathcal{A}(P) \to \mathcal{A}(P), \quad (\alpha, A) \mapsto \alpha A := A + \pi^*\alpha
\]

where \( \pi^* : \Omega^1(C, P(\mathfrak{t})) \to \Omega^1(P, \mathfrak{k})^K \) is pull-back. In particular, we may identify the tangent space to \( \mathcal{A}(P) \) at any \( A \in \mathcal{A}(P) \)

\[
T_A\mathcal{A}(P) \cong \Omega^1(C, P(\mathfrak{t})).
\]

For any \( A \in \mathcal{A}(P) \), denote by

\[
F_A \in \Omega^2(C, P(\mathfrak{t}))
\]
the curvature of $A$ whose pull-back to $\Omega^2(P, \mathfrak{g})$ equals $dA + [A, A]/2$. Let

$$K(P) = \text{Aut}(P)$$

denote the group of gauge transformations of $P$, with Lie algebra

$$\mathfrak{k}(P) \cong \Omega^0(C, P(\mathfrak{g})).$$

The group $K(P)$ acts on $\mathcal{A}(P)$ by pull-back:

$$K(P) \times \mathcal{A}(P) \to \mathcal{A}(P), \quad (k, A) \mapsto kA := (k^{-1})^*A.$$ 

The curvature transforms according to

$$F_{kA} = \text{Ad}(k)F_A.$$ 

There is a one-to-one correspondence between $\mathcal{A}(P)$ and the space $\mathcal{J}(P(G))$ of complex structures on the $G$-bundle $P(G)$, given by the splitting

$$TP(G) = \pi^*TC \oplus \mathfrak{g}$$
determined by the connection $A \in \mathcal{A}(P)$. The group $G(P)$ of complex gauge transformations acts naturally on $\mathcal{J}(P(G))$ and induces an action of $G(P)$ on $\mathcal{A}(P)$.

Let $X$ be a compact Hamiltonian $K$-manifold with symplectic form $\omega$ and moment map $\Phi : X \to \mathfrak{k}^\vee$. The action of $K$ on $X$ induces an action on $\mathcal{J}(X)$ by conjugation, and we denote by $\mathcal{J}(X)^K$ the invariant subspace. Any connection $A \in \mathcal{A}(P)$ induces a map of spaces of almost complex structures

$$\mathcal{J}(X)^K \to \mathcal{J}(P(X)), \quad J \mapsto J_A$$

by combining the almost complex structure on $X$ and $C$ using the splitting defined by the connection. Let $\Gamma(C, P(X))$ denote the space of smooth sections of $P(X)$. Denote by

$$\overline{\partial}_A : \Gamma(C, P(X)) \to \bigcup_{u \in \Gamma(C, P(X))} \Omega^{0,1}(C, u^*T_{\text{vert}}P(X))$$

the Cauchy-Riemann operator defined by $J_A$. Let $\mathcal{H}^K(P, X)$ be the space of gauged holomorphic maps

$$\mathcal{H}^K(P, X) = \{(A, u) \in \mathcal{A}(P) \times \Gamma(C, P(X)), \quad \overline{\partial}_Au = 0\}.$$ 

The space $\mathcal{H}^K(P, X)$, where smooth, has a closed two-form induced from the sum of the symplectic form on the affine space of connections and the space of maps to $X$. Fix an invariant metric $(\cdot, \cdot) : \mathfrak{k} \times \mathfrak{k} \to \mathbb{R}$ on the Lie algebra $\mathfrak{k}$, inducing an identification $\mathfrak{k} \to \mathfrak{k}^\vee$. Let

$$(\wedge) : \Omega^1(C, P(\mathfrak{g}))^2 \to \Omega^2(C)$$

denote the operation obtained by combining wedge product with the metric on the Lie algebra, and let

$$\Omega^1(C, P(\mathfrak{g}))^2 \to \mathbb{R}, \quad (a_1, a_2) \mapsto \int_C (a_1 \wedge a_2)$$
denote the symplectic form on the affine space of connections \( A(P) \). Let \( P(\omega) \) denote the fiber-wise two-form on \( P(X) \) defined by \( \omega \). Choose an area two-form \( \text{Vol}_C \in \Omega^2(C) \). The 2-form
\[
(16) \quad \Omega^0(C, u^*T^{\text{vert}} P(X))^2 \to \mathbb{R}, \quad (v_1, v_2) \mapsto \int_C u^* P(\omega)(v_1, v_2) \text{Vol}_C
\]
restricts to a closed two-form on the smooth locus of \( \mathcal{H}^K(P, X) \). If \( X \) is Kähler, then the \( \mathcal{H}^K(P, X) \) has tangent space given by the kernel of a complex linear operator, and this can be used to show non-degeneracy of (16), where \( \mathcal{H}^K(P, X) \) is smooth. Consider the formal two-form
\[
((a_1, v_1), (a_2, v_2)) \mapsto \int_C \langle a_1 \wedge a_2 \rangle + (u^* P(\omega))(v_1, v_2) \text{Vol}_C.
\]
The infinitesimal action \( \mathfrak{k} \to \text{Vect}(X) \) induces a map
\[
\Omega^0(C, P(\mathfrak{k})) \to \Gamma(C, P(\text{Vect}(X))), \quad \zeta \mapsto \zeta_{P(X)}.
\]
The action of \( K(P) \) on \( \mathcal{H}^K(P, X) \) has generating vector fields given by the covariant derivative and infinitesimal action
\[
\zeta_{\mathcal{H}^K(P, X)}(A, u) = (dA\zeta, u^* \zeta_{P(X)}) \in \Omega^1(C, P(\mathfrak{k})) \times \Omega^0(C, u^*T^{\text{vert}} P(X))
\]
for \( \zeta \in \Omega^0(C, P(\mathfrak{k})) \). The moment map \( \Phi \) induces a map \( P(\Phi) : P(X) \to P(\mathfrak{k}) \). The action of \( K(P) \) preserves the two-form (16) and has moment map on the smooth locus given by the curvature plus pull-back of the moment map,
\[
\mathcal{H}^K(P, X) \to \Omega^2(C, P(\mathfrak{k})), \quad (A, u) \mapsto F_A + \text{Vol}_C u^* P(\Phi).
\]
If the \( K \)-action on \( X \) extends to a holomorphic action of \( G \) on \( X \) (this holds if, for example, \( X \) is Kähler) then the action of \( \mathcal{G}(P) \) on \( \mathcal{J}(P(G)) \approx \mathcal{A}(P) \) induces an action of \( \mathcal{G}(P) \) on \( \mathcal{H}^K(P, X) \).

**Definition 2.2.1.** (Symplectic vortices) A gauged holomorphic map \((A, u) \in \mathcal{H}^K(P, X)\) is a (symplectic) vortex iff
\[
F_A + \text{Vol}_C u^* P(\Phi) = 0.
\]
An \( n \)-marked (symplectic) vortex is a vortex \((A, u)\) together with \( n \)-tuple \( \underline{z} = (z_1, \ldots, z_n) \) of distinct points on \( C \). A framed vortex is a collection \((A, u, \underline{z}, \phi)\) where \((A, u, \underline{z})\) is a marked vortex and \( \phi = (\phi_1, \ldots, \phi_n) \) are trivializations of the fibers of \( P \) at \( z_1, \ldots, z_n \), that is, each \( \phi_j : P_{z_j} \to K \) is a \( K \)-equivariant isomorphism. The group of \( K(P) \) of gauge transformations acts on the space of marked vorticest; an automorphism of a vortex \((A, u)\) is an element of \( K(P) \) fixing \((A, u)\). A marked vortex \((A, u, \underline{z})\) is stable if it has finite automorphism group. A framed vortex is stable if the underlying marked vortex is stable.

Symplectic vortices satisfy an energy-area relation similar to that for holomorphic maps. The energy of a gauged map \((A, u)\) is given by
\[
E(A, u) = \frac{1}{2} \int_C (|d_A u|^2 + |F_A|^2 + |u^* P(\Phi)|^2) \text{Vol}_C.
\]
The equivariant homology class \([u] \in H^K_2(X, \mathbb{Z})\) is defined as the homology class of the composition of \(u : C \to P \times_K X\) with the map \(P \times_K X \to EK \times_K X\) induced by a choice of classifying map \(P \to EK\). The equivariant symplectic area of a pair \(u\) is pairing of the homology class \([u]\) with the equivariant symplectic class \(\omega_K \in H^*_K(X, \mathbb{R})\),

\[
D(u) := ([u], \omega_K) = ([C], u^*_K \omega_K).
\]

Suppose \(Vol_C\) is the area form determined by a choice of metric on \(C\). The energy and equivariant symplectic area are related by

\[
E(A, u) = D(u) + \int_C \left( |\bar{\partial}_A u|^2 + \frac{1}{2} |F_A + u^* P(\Phi)|^2 \right) Vol_C
\]

see [9, Proposition 2.2]. In particular, for any vortex \(E(A, u) = D(u)\).

Let \(M^K_n(P, X)\) denote the moduli space of isomorphism classes of \(n\)-marked vortices. Any suitable Sobolev norm induces on \(M^K_n(P, X)\) a topology, independent of the choice of norm [9, Section 3]. Let \(M^K_n(C, X)\) denote the union of space \(M^K_n(P, X)\) over isomorphism classes of bundles \(P\). The moduli space \(M_n(C, X)\) is homeomorphic to the product \(M^K(C, X) \times M_n(C)\) where \(M_n(C)\) is the configuration space of \(n\)-tuples of distinct points on \(C\). Denote by \(M^K(C, X, d)\) the component of \(M^K(C, X)\) of homology class \(d \in H^*_K(X, \mathbb{Z})\). The connection with Mundet stability is provided by the following:

**Theorem 2.2.2 (Mundet’s Hitchin-Kobayashi correspondence for vortices [42]).** Let \(P \to C\) be a principal \(K\)-bundle. A pair \((A, u) \in H^K(P, X)\) defines a Mundet-stable gauged map if and only if there exists a complex gauge transformation \(g \in G(P)\) such that \(g(A, u)\) is a vortex.

The space \(M^K_n(C, X, d)\) has a Kontsevich-style compactification which allows bubbling of \(u\) in the fibers of \(P(X)\).

**Definition 2.2.3. (Polystable symplectic vortices)** A polystable vortex from \(C\) to \(X\) with underlying bundle \(P\) consists of a datum \((A, \hat{C}, u, \hat{z})\) where

(a) \(A\) is a connection on \(P \to C\);
(b) \(\hat{C}\) is a nodal projective curve;
(c) \(u : \hat{C} \to P(X)\) is a stable map holomorphic with respect to the almost complex structure \(J_A\) on \(P(X)\) induced by \(A\);
(d) \(\hat{z} = (z_1, \ldots, z_n) \in C\) are distinct, smooth points of \(C\);

such that

(a) the composition \(\pi \circ u : \hat{C} \to C\) has class \([C]\);
(b) the pair \((A, u|_{C_0})\), where \(C_0 \subset \hat{C}\) is the unique component mapping isomorphically onto \(C\), is a vortex;
(c) each component \(C_i\) on \(\hat{C}\) on which \(u_i\) is constant has at least three marked or singular points.
An isomorphism of polystable vortices \((A, \hat{\mathbb{C}}, u, z)\), \((A', \hat{\mathbb{C}}', u', z')\) consists of an automorphism of the domain \(\psi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}'\), inducing the identity on \(\mathbb{C}\), and a bundle isomorphism \(\alpha : P \to P'\), intertwining the connections and maps: that is, \(\alpha^*A = A, u' = \alpha \circ u \circ \psi\) and exchanging the markings, that is, \(\psi(z_i) = z'_i, i = 1, \ldots, n\).

A polystable vortex is stable if it has finite automorphism group. For any map \(u : \hat{\mathbb{C}} \to P(X)\), the homology class \([u] \in H^2_2(X, \mathbb{Z})\) of \(u\) is the push-forward of \([\hat{\mathbb{C}}]\) under a map \(u_K : \hat{\mathbb{C}} \to X_K\) obtained from a classifying map for \(P\).

Framed vortices are defined as follows. For each \(i = 1, \ldots, n\), let \(\hat{z}_i \in \hat{\mathbb{C}}\) denote the attaching point of the bubble tree containing \(z_i\). A framed polystable vortex consists of a polystable vortex together with framings at the attaching points of the bubbles \(\phi_i : P_{\hat{z}_i} \to K, \quad i = 1, \ldots, n\). The definition of isomorphism of framed polystable vortices is similar to the unframed case in Definition 2.2.3. This ends the definition.

Note that there is no stability condition on the number of special points on the principal component. In particular, gauged maps corresponding to constant maps \(C \to X\) (that is, with \(P\) trivial) with no markings can be stable. The following extends the notion of Gromov convergence to the case of polystable marked vortices:

**Definition 2.2.4.** (Gromov convergence of vortices) A sequence \((\hat{\mathbb{C}}_\alpha, A_\alpha, u_\alpha, z_\alpha)\) of polystable vortices Gromov converges to a limit \((\hat{\mathbb{C}}, A, u, z)\) if after a sequence of gauge transformations \(A_\alpha\) converges to \(A_\infty\) in the \(C^0\) topology and weakly in the \(W^{1,2}\) Sobolev topology and \(u_\alpha\) converges to \(u\) as in the standard definition of Gromov convergence of holomorphic maps [39, Chapter 5]. A subset \(S\) of \(M^K_n(C, X)\) is Gromov closed if any convergent sequence in \(S\) has limit point in \(S\), and Gromov open if its complement is closed. The Gromov open sets form a Hausdorff Gromov topology for which any convergent sequence is Gromov convergent by arguments similar to [39, Lemma 5.6.5].

Denote by \(\overline{M}_n^K(C, X)\) the moduli space of isomorphism classes of polystable vortices. Let \(\overline{M}_n^{K,fr}(C, X)\) denote the moduli space of isomorphism classes of stable framed vortices.

**Theorem 2.2.5.** For any \(c > 0\), the union of components \(\overline{M}_n^K(C, X, d)\) resp. \(\overline{M}_n^{K,fr}(C, X, d)\) with \(\langle \omega_K, d \rangle < c\) is a compact, Hausdorff space.

See Mundet [43] and Ott [48], who proves a slightly weaker result: that the convergence discussed there defines a compact Hausdorff topology follows from an argument using local distance functions as in McDuff-Salamon [39]. Convergence of the connection on the annulus region near a bubble is not uniform in all derivatives, but the “bubbles connect” goes through using a priori estimates obtained from the vortex equation.

In good cases one has a smooth structure on the moduli space of vortices. Locally the moduli space may be identified with the quotient of the zero set of a Fredholm
map of Banach spaces by the automorphism group. Let \((A, u)\) be a vortex with bundle \(P \rightarrow C\). Define

\[
F_{A,u} : \Omega^1(C, P(\mathfrak{t})) \oplus \Omega^0(C, u^* T\text{vert} P(X)) \\
\rightarrow (\Omega^0 \oplus \Omega^2)(C, P(\mathfrak{t})) \oplus \Omega^{0,1}(C, u^* T\text{vert} P(X))
\]

\[(a, \xi) \mapsto (F_{A+a} + \text{Vol}_C \exp_u(\xi)^* \Phi, d^*_{A,u}(a, \xi), \Theta_u(\xi)^{-1} \bar{\nabla}_{A+a} \exp_u(\xi))\]

where \(\exp_u\) denotes geodesic exponentiation and \(\Theta_u(\xi)\) is parallel transport along \(\exp_u(t\xi)\) using an almost complex connection. By [11, Section 4] The moduli space \(M^K(P, X)\) is locally homeomorphic to the quotient of \((F_{A,u})^{-1}(0)\) by \(\text{Aut}(A, u) := K(P)_{A,u}\). The linearization of the map \(F_{A,u}\) is described as follows. Given \(\xi \in \Omega^0(C, u^* T\text{vert}(P(X)))\) we denote by

\[
L_\xi : \Gamma(C, P \times_K \text{Map}_K(X, \mathfrak{t})) \rightarrow \Omega^0(C, P(\mathfrak{t}))
\]

the operator obtained by differentiating in the direction of \(\xi(z)\) in each fiber \(P_z(X), z \in C\). Define

\[
d_{A,u} : \Omega^1(C, P(\mathfrak{t})) \oplus \Omega^0(C, u^* T\text{vert} P(X)) \rightarrow \Omega^2(C, P(\mathfrak{t})),
\]

\[
d_{A,u}(a, \xi) := d_A a + \text{Vol}_C \, L_\xi P(\Phi).
\]

Also define

\[
d^*_{A,u} : \Omega^1(C, P(\mathfrak{t})) \oplus \Omega^0(C, u^* T\text{vert} P(X)) \rightarrow \Omega^0(C, P(\mathfrak{t})),
\]

\[
d^*_{A,u}(a, \xi) := *(d_A + \text{Vol}_C \, L_\xi P(\Phi)).
\]

By [11, Section 4], if \((A, u)\) is stable then the set

\[
S_{A,u} = \{(A + a, \exp_u(\xi)), (a, \xi) \in \ker d_{A,u}^*\}
\]

is a slice for the gauge group action near \((A, u)\). For \(a \in \Omega^1(C, P(\mathfrak{t}))\) let \(a_X \in \Omega^1(C, u^* P(TX))\) denote the form induced by the infinitesimal action. Define

\[
D_{A,u} : \Omega^0(\tilde{C}, u^* T\text{vert} P(X)) \rightarrow \Omega^{0,1}(\tilde{C}, u^* T\text{vert} P(X)),
\]

\[
(a, \xi) \mapsto (\nabla_A \xi)^{0,1} + \frac{1}{2} J_u(\nabla \xi) u_a + a_X^{0,1}
\]

where 0, 1 denotes projection on the 0, 1-component.

**Definition 2.2.6.** (Regular vortices) For a vortex \((A, u)\) let \(\tilde{D}_{A,u} = (d_{A,u}, d_{A,u}^*, D_{A,u})\) denote the linearization of \(F_{A,u}\) at \((A, u)\). \((A, u)\) is regular if the operator \(\tilde{D}_{A,u}\) is surjective. A marked vortex \((A, u, \tilde{z})\) is regular if the underlying unmarked vortex is regular. For a polystable vortex \((\tilde{C}, A, u)\) with \(u = (u_0, u_1, \ldots, u_n)\), let \(\tilde{C}\) denote the normalization of \(\tilde{C}\), that is, the smooth curve obtained by replacing each node with the corresponding pair of points. Let \(D_{A,u}\) denote the linearized operator corresponding to \(F_{A,u}\).

\[
\tilde{D}_{A,u} : \Omega^1(C, P(\mathfrak{t})) \oplus \Omega^0(\tilde{C}, u^* T\text{vert} P(X)) \\
\rightarrow (\Omega^0 \oplus \Omega^2)(C, P(\mathfrak{t})) \oplus \Omega^{0,1}(\tilde{C}, u^* T\text{vert} P(X)).
\]
given by the sum of the operators \( d_{A,u}, d^*_{A,u}, D_{A,u} \) and the linearized Cauchy-Riemann operators \( \tilde{D}_{u,j} \) on the bubbles. See [23] for the definition of the operator \( d^*_{A,u} \); its kernel gives a slice for the action of the group of gauge transformations. \((A,u)\) is regular if \( \tilde{D}_{A,u} \) is surjective.

The following is proved in [23], except for orientations which are similar to the Gromov-Witten case.

**Theorem 2.2.7.** Let \( X \) be a compact Hamiltonian \( K \)-manifold equipped with a compatible invariant almost complex structure. The locus \( \overline{M}^{K,\text{reg}}_n(C,X) \) of regular, stable vortices admits the structure of an oriented stratified-smooth topological orbifold.

Thus singularities in the moduli space are caused by either irregular vortices or reducible vortices, that is, pairs \((A,u)\) such that the Lie algebra \( \text{aut}(A,u) \subset K(P) \) is non-trivial.

**Definition 2.2.8.** (Symplectic gauged Gromov-Witten invariants) Suppose that every polystable vortex is stable with trivial automorphism group and regular, so that the moduli space of framed vortices \( \overline{M}^{K,\text{fr}}_n(C,X) \) is a differentiable manifold with a free action of \( K^n \) by the results of [23]. Then \( \overline{M}^{K,\text{fr}}_n(C,X) \to \overline{M}^K_n(C,X) \) has the structure of a principal \( K^n \)-bundle. Let \( \psi : \overline{M}^{K,\text{fr}}_n(C,X) \to EK^n \) be a classifying map. The framed moduli space naturally admit framed evaluation maps \( \text{ev}^{\text{fr}} : \overline{M}^{K,\text{fr}}_n(C,X) \to X^n \). Combining \( \text{ev}^{\text{fr}} \) and \( \psi \) gives rise to an evaluation map

\[ \text{ev} : \overline{M}^K_n(C,X) \to X^n. \]

Pull-back by the evaluation maps induces a map \( \text{ev}^* : H_K(X)^n \to H(\overline{M}^K_n(C,X)) \). If the action of \( K^n \) on the moduli space of framed vortices is only locally free, then this evaluation map exists only with rational coefficients. Assume every polystable vortex is stable and regular. The \textit{gauged Gromov-Witten potential} can be defined as in (4). Without the assumptions one would need symplectic virtual fundamental classes, which we do not know how to construct.

**Remark 2.2.9.** (Conjectural equivalence of the symplectic and algebro-geometric definitions of gauged Gromov-Witten invariants) Suppose that every semistable gauged map is stable. The map assigning to any stable gauged map the corresponding vortex defines a bijection from the coarse moduli space of \( \overline{M}^G_n(C,X,d) \) to the moduli space of vortices \( \overline{M}^K_n(C,X,d) \), by Mundet’s correspondence 2.2.2. In fact this map is a homeomorphism. To see this, recall that the topology on the coarse moduli space \( \overline{M}^G_n(C,X,d) \) is induced from specialization in families: For any convergent sequence \([P_\alpha, u_\alpha] \to [P,u]\) there exists an analytic family \( \hat{C} \) of nodal curves over a connected complex manifold \( S \), a family of holomorphic \( G \)-bundles \( P \to C \times S \), a family of maps \( \hat{C} \to P(X) \), and a convergent sequence \( s_\alpha \to s \) such that \([P_\alpha, u_\alpha]\) resp. \([P,u]\) is isomorphic to the fiber over \( s_\alpha \) resp. \( s \). Fixing a reduction of structure group to \( K \) and using the correspondence between holomorphic structures and connections gives a family \((A_s \in A(P), u_s : \hat{C}_s \to P(X))\) of connections and sections.
on a fixed $K$-bundle $P$. If $s_\nu \in S$ is a sequence converging to $s \in S$ as $\nu \to \infty$, then $A_{s_\nu} \to A$ uniformly in all derivatives and $u_{s_\nu}$ Gromov converges to $u_s$. In particular, the principal component $u_{s_\nu,0}$ converges to $u_{s,0}$ uniformly in all derivatives on compact subsets of the complement of the bubbling set. Then $u_{s_\nu}^* P(\Phi) \to u_s^* P(\Phi)$ in $L^2$ and so

$$F_{A_{s_\nu}} + u_{s_\nu}^* P(\Phi) \text{Vol}_C \to F_{A_s} + u_s^* P(\Phi) \text{Vol}_C$$

in the $L^2$ topology on sections of $P(\mathfrak{t})$. By (23), for any gauge transformation $g \in G(P)$ we have

$$F_{g A_{s_\nu}} + gu_{s_\nu}^* P(\Phi) \text{Vol}_C \to F_{g A_s} + gu_s^* P(\Phi) \text{Vol}_C$$

in the $L^2$-topology on $\Omega^2(C, P(\mathfrak{t}))$, since $g A_{s_\nu} \to g A_s$ uniformly in all derivatives and $gu_{s_\nu}$ Gromov converges to $gu_s$. Recall that Mundet’s functional $\psi_{s_\nu} : \mathfrak{t}(P) \to \mathbb{R}$ has value $\psi_{s_\nu}(\zeta)$ at $\zeta \in \mathfrak{t}(P)$ given by the integral of

$$\psi_{s_\nu}(\zeta) := \int_{[0,1] \times C} dt(\exp(it\zeta) F_{A_{s_\nu}} + \exp(it\zeta) u_{s_\nu}^* P(\Phi) \text{Vol}_C, \zeta)$$

It follows from (24) that $\psi_{s_\nu}$ converges pointwise to the functional $\psi_s$ for $(A_s, u_s)$ as $s_\nu \to s$. If $\zeta_\nu$ denotes the unique global minimum of $\psi_{s_\nu}$ then by continuity $\zeta_\nu$ converges to the unique global minimum $\zeta$ for $\psi_s$ as $\nu \to \infty$. Hence $\exp(it\zeta_\nu) A_{s_\nu}$ converges to $\exp(it\zeta) A_s$ and $\exp(it\zeta_\nu) u_{s_\nu}$ Gromov converges to $\exp(it\zeta) u_s$ as $\nu \to \infty$. Since the stable=semistable assumption implies that $\mathcal{M}_{\eta}^G(C, X, d)$ is proper, its coarse moduli space is compact Hausdorff. The correspondence is thus a continuous bijection between compact, Hausdorff spaces and therefore a homeomorphism. Because of this one expects that the definition of gauged Gromov-Witten invariants can be defined for any Hamiltonian action, and that their definition for projective varieties agrees with the algebraic definition in the next section. For partial results in this direction see [44].

3. Area-dependence for gauged Gromov-Witten invariants

In this section we study the area-dependence of the gauged Gromov-Witten invariants, by which we mean the dependence on the choice of Mundet stability parameter.

3.1. Wall-crossing for algebraic gauged Gromov-Witten invariants. First we explain the fixed point contributions in more detail.

**Definition 3.1.1** (Fixed Gauged Maps). For each $\zeta \in \mathfrak{g}$, let $\mathcal{M}_{\eta}^G(C, X, L, \zeta)$ denote the stack of Mundet-semistable morphisms from $C$ to $X/G_\zeta$ that are $C_\zeta^x$-fixed and take values in $X^c$ on the principal component. Via the inclusion $G_\zeta \to G$ the universal curve over $\mathcal{M}_{\eta}^G(C, X, L, \zeta)$ admits a morphism to $X/G$, and the tangent complex $T(X/G)$ defines an object $\text{Ind}(T(X/G))$ in the bounded derived category of $\mathcal{M}_{\eta}^G(C, X, L, \zeta)$ with an action of $C_\zeta^x$. We denote by

$$\text{Ind}(T(X/G))^+ \subset \text{Ind}(T(X/G))$$
the moving subcomplex of $\text{Ind}(T(X/G))$ with respect to the action of $\mathbb{C}_\zeta^\times$ and

$$\epsilon_+(T(X/G)) = \text{Eul}_{\mathbb{C}_\zeta^\times}(\text{Ind}(T(X/G))^+)$$

its equivariant Euler class.

Let $QH_{G,\text{fin}}(X)$ denote the tensor product of $H_G(X)$ with the sub-ring $\Lambda^G_X \subset \Lambda^G_X$ of finite sums of expressions $c_i q^{d_i}, d_i \in H^2_G(X, \mathbb{Q}), c_i \in \mathbb{Q}$.

**Definition 3.1.2** (Fixed point contributions to wall-crossing for Gromov-Witten invariants). Virtual integration over $\overline{\mathcal{M}}_n^G(C, X, L, \zeta)$ defines a “fixed point contribution”

$$\tau_{X,G,\zeta,L} : QH_{G,\text{fin}}(X) \to \Lambda^G_X [\xi, \xi^{-1}],$$

$$\alpha \mapsto \sum_{d \in H^2_G(X, \mathbb{Z})} \sum_{n \geq 0} \int_{\overline{\mathcal{M}}_n^G(C, X, L, \zeta, d)} (q^d/n!) \text{ev}^*(\alpha, \ldots, \alpha) \cup \epsilon_+(T(X/G))^{-1}$$

for $\alpha \in H_G(X)$, extended by linearity over $\Lambda^G_X$, where we omit the restriction map $H_{G,\text{fin}}(X) \to H_G(X)$ to simplify notation.

Suppose that $L_\pm \to X$ are polarizations. Consider the fractional line bundle

$$L_t = L^{(1-t)/2} L_+^{(1+t)/2}, \quad t \in (-1, 1) \cap \mathbb{Q}.$$ 

Let

$$\tau_{X,G,\zeta,t} := \tau_{X,G,\zeta,L_t}.$$ 

**Theorem 3.1.3** (Wall-crossing for gauged Gromov-Witten potentials). Let $X$ be a smooth projective $G$-variety. Suppose that $L_\pm \to X$ are polarizations such that semistable=stable for the stack of polarized gauged maps in [24]. Then the gauged Gromov-Witten potentials are related by

$$\tau_{X,G,+} - \tau_{X,G,-} = \sum_{[\zeta] \in (-1,1)} \text{Resid}_\zeta \tau_{X,G,\zeta,t}$$

where the sum is over equivalence classes $[\zeta]$ of unparametrized one-parameter subgroups generated by $\zeta \in \mathfrak{g}$.

**Proof.** For the sake of completeness we briefly sketch the proof from [24]. A Grothendieck (quot-scheme) style compactification $\overline{\mathcal{M}}^{G,\text{quot}}(C, X, L, d)$ of $\mathcal{M}^G(C, X, L, d)$ was given in Schmitt [54] by git techniques. It follows from the git construction that given two different polarizations $L_\pm \to X$, there exists a “master space” $\overline{\mathcal{M}}^{G,\text{quot}}(C, X, L_+, L_-, d)$, whose $\mathbb{C}^\times$-quotients are the stacks $\overline{\mathcal{M}}^{G,\text{quot}}(C, X, L_t, d)$. By fiber product with the Givental morphism one obtains a similar master space $\overline{\mathcal{M}}_n^G(C, X, L_+, L_-, d)$ for the Kontsevich compactification, that is, containing $\overline{\mathcal{M}}_n^G(C, X, L_\pm, d)$ as fixed point components. The wall-crossing then follows from virtual localization for the $\mathbb{C}^\times$-action on $\overline{\mathcal{M}}_n^G(C, X, L_+, L_-, d)$. \hfill $\Box$
Remark 3.1.4. (Wall-crossing for invariants with Fulton-MacPherson insertions) We have stated the result for the potentials without insertions for simplicity; the same result holds for the maps with Fulton-MacPherson insertions \( \tau_{X,G,\pm} \) of (11).

Remark 3.1.5. (Special case of area variation) In this remark we discuss the special case of the wall-crossing formula to the case that \( L_\pm \) is proportional in the sense that \( L_+ = L_0 \) for some rational \( \rho \). This case has the special feature that the Novikov field \( \Lambda^G_X \) is independent of the choice of \( t \in [-1,1] \). A posteriori, the wall-crossing terms \( \tau_{X,G,\zeta,t} \) take values in \( \Lambda^G_X \) rather than the distributional space \( \tilde{\Lambda}^G_X \) and extend to \( \text{QH}_G(X) \).

The utility of the wall-crossing formula is somewhat limited in examples involving non-abelian groups because of the stable=semistable condition. The following gives a sufficient condition for this assumption to hold.

**Proposition 3.1.6.** (Criteria for polarized stable=semistable) Suppose that \( L_\pm = L_0^\rho \pm \) for some \( \rho \). Suppose that

(a) (Condition on two-dimensional stabilizers) the image of any point \( x \in X \) with \( \dim(G_x) > 1 \) under the moment map \( \Phi \) has \( \langle \Phi(x), \lambda \rangle \notin \mathbb{Z} \) for any rational \( \lambda \in \mathfrak{g}_x \) and

(b) (Condition on one-dimensional stabilizers) for any point \( x \in X \) with \( \dim(G_x) = 1 \), \( \Phi(x) \) is non-zero on \( \mathfrak{g}_x \).

Then stable=semistable for polarized gauged maps.

**Proof.** Let the conditions of the proposition hold and let \( (P, u) \) be a gauged map with automorphism group \( H \) which is semistable for some parameter \( \rho \). Evaluation at a base point defines an inclusion of \( H \) in \( G \) so that the structure group of \( P \) reduces to the centralizer \( Z(H) \) of \( H \) and \( u \) takes values in the fixed point set \( P(X)^H = P(X^H) \). First suppose that \( \dim(H) > 2 \). The first condition in the proposition guarantees \( \langle \Phi(X^H), \lambda \rangle \) is irrational for any non-zero coweight \( \lambda \) of \( \mathfrak{h} \). On the other hand, since \( c_1(P) \) is a rational vector in \( H^2(C, \mathfrak{g}(\mathfrak{h})) \), the pair \( (P, u) \) cannot be Mundet semistable for any stability parameter \( \rho \), since the pairing of \( c_1(P) \) with non-zero \( \lambda \in \mathfrak{h} \) would express 0 as the sum of a rational and irrational number. Next suppose that \( \dim(H) = 1 \). Let \( \lambda \) denote a generator of \( \mathfrak{h} \). The second condition implies that \( H \) acts with non-zero weight on the fiber \( \overline{\mathcal{M}}_{n}^G(C, X, L_-, L_+) \to \overline{\mathcal{M}}_{n}^G(C, X, L_\rho) \) and so any lift of \( (P, u) \) is stable. \( \square \)

We end this section with another expression for the fixed point contributions in the wall-crossing formula. The classical fixed point contributions are usually written as a combination of the restriction map \( H^G(X) \to H^G(X^\zeta) \) followed by twisted integration over \( X^\zeta \). In this section, we explain a similar re-writing of the fixed point potential \( \tau_{X,G,\zeta,t} \) as a combination of the gauged potential

\[ \tau_{X,G,\zeta} : \text{QH}_{G,\zeta}(X^\zeta) \to \Lambda^G_X[\xi] \]
and a quantum restriction map \( \iota_\zeta : QH_{G_\zeta}(X) \to QH_{G_\zeta}(X^\zeta) \) as follows. Recall that \( \mathbb{C}_\zeta^\times \subset G \) is the one-parameter subgroup generated by \( \zeta \in g \). Let \( \mathcal{M}_{0,n}(X)^{C^\times_\zeta} \subset \mathcal{M}_{0,n}(X) \) denote the \( \mathbb{C}^\times_\zeta \)-fixed point stack of stable maps to \( X \). The evaluation map restricted to \( \mathcal{M}_{0,n}(X)^{C^\times_\zeta} \) automatically takes values in the fixed point locus \( X^\zeta \subset X \), that is, \( \text{ev} : \mathcal{M}_{0,n}(X)^{C^\times_\zeta} \to (X^\zeta)^n \).

**Definition 3.1.7.** (Quantum restriction map) Push-pull over the moduli stack \( \mathcal{M}_{0,n+1}(X)^{C^\times_\zeta} \) defines a quantum restriction map

\[
\iota_\zeta : QH_{G_\zeta}(X) \to QH_{G_\zeta}(X^\zeta), \quad \alpha \mapsto \alpha|_{X^\zeta} + \sum_{n,d} \left( d^n / n! \right) \text{ev}_{n+1,*} \alpha \cup \cdots \cup \text{ev}_n \alpha.
\]

The map \( \iota_\zeta \) does not seem to interact well with the quantum products; it would be interesting to understand the algebraic structure of the map \( \iota_\zeta \) better. Recall that \( \pi_{G_\zeta}^G : \Lambda_{X^\zeta}^G \to \Lambda_X^G \) is the canonical map of Novikov rings induced by \( H_{2}^{G_\zeta}(X) \to H_{2}^{G}(X) \).

**Theorem 3.1.8.** (Fixed point contributions as quantum restriction followed by gauged potential for the fixed point locus) Suppose that stable=semistable for \( \zeta \)-fixed gauged maps. Then \( \tau_{X,G,\zeta,t} = \pi_{G_\zeta}^G \tau_{X^\zeta,G} \iota_\zeta \).

**Proof.** We have an isomorphism

\[
\mathcal{M}^G_n(C,X,L,\zeta) \cong \bigcup_{i_1+\ldots+i_r=n} \prod_{j=1}^r \left( \{ \text{pt} \} \cup \mathcal{M}_{0,i_j+1}(X)^{C_{\zeta,j}} \right) \times (X^\zeta)^r \mathcal{M}_{r}^{G_{\zeta},fr}(C,X^\zeta) / G_{\zeta}
\]

where \( \{ \text{pt} \} \) represents a trivial bubble tree attached at the \( j \)-th node on the principal component. It follows that integration over \( \mathcal{M}^G_n(C,X,L,\zeta) \) is given by push-forward of \( \text{ev}_{i_1}^* \alpha \cup \ldots \cup \text{ev}_{i_j}^* \alpha \) over each

\[
\text{ev}_{i_1+1} : \{ \text{pt} \} \cup \mathcal{M}_{0,i_j+1}(X)^{C_{\zeta,j}} / G_{\zeta} \to X^\zeta / G_{\zeta}
\]

followed by integration over the Artin stack \( \mathcal{M}_{r}^{G_{\zeta},fr}(C,X^\zeta) \), or more precisely, \( \mathbb{C}^\times_\zeta \)-equivariant integration over the Deligne-Mumford stack \( \mathcal{M}_{r}^{G_{\zeta}/\mathbb{C}^\times_\zeta}(C,X^\zeta) \) (for which stable=semistable). The claim follows.

3.2. Wall-crossing for symplectic gauged Gromov-Witten invariants. In this section we explain a more conceptual approach to wall-crossing for the gauged Gromov-Witten invariants with respect to area variation from the symplectic point of view, in which the master space used to prove the wall-crossing formula has
an explicit construction using the Chern-Simons line bundle \cite{52}, \cite{40}. Fix a base connection \( A_0 \in \Omega^1(P, \mathfrak{k}) \) on \( P \). Consider the trivial line bundle

\[
\tilde{A}(P) := A(P) \times (C - \{0\}).
\]

It has a connection one-form \( \alpha \in \Omega^1(A(P) \times U(1)) \) given by

\[
\alpha_{A,z} : T_A A(P) \times T_z U(1) \cong \Omega^1(C, P(\mathfrak{k})) \times \mathbb{R} \to \mathbb{R}
\]

\[
(a, \lambda) \mapsto \lambda + \frac{1}{2} \int_C (a \wedge (A - A_0)).
\]

The curvature of this connection is equal to the symplectic form corresponding to the inner product. Let \( \phi : \tilde{A}(P) \to \mathbb{R} \), \( \tilde{A} \mapsto |\tilde{A}| \) denote the norm function. Consider on \( \tilde{A}(P) \) the closed two-form given by \( d(\phi \alpha) \).

The infinitesimal action of \( \mathfrak{k}(P) \) on \( A(P) \) lifts to an infinitesimal action on \( \tilde{A}(P) \) by requiring that the canonical moment map (given by the contraction of \( (\phi, \alpha) \) with the generating vector fields for \( \zeta \in \mathfrak{k}(P) \)) at \( \tilde{A} \) is \( (\phi(\tilde{A})F_A, \zeta) \). Witten’s argument involving a bounding three-manifold, explained in \cite{52}, \cite{40} shows that under an integrality condition (the metric should correspond to an integral multiple of the basic inner product each factor) this infinitesimal action integrates to an action of the group \( K(P) \). The action of \( U(1) \) by multiplication on the second factor on \( \tilde{A}(P) \) is Hamiltonian with moment map \( \phi \). Let \( \tilde{H}^K(P, X) \) denote the pull-back of \( \tilde{A}(P) \) to \( H^K(P, X) \), that is,

\[
\tilde{H}^K(P, X) = \{(\tilde{A}, u) \in \tilde{A}(P) \times \Gamma(C, P(X)) \mid \overline{\partial}_A u = 0\}
\]

where \( A \) is the projection of \( \tilde{A} \). The space \( \tilde{H}^K(P, X) \), where smooth, has a closed two-form so that the action of \( K(P) \) has moment map given by

\[
\tilde{H}^K(P, X) \to \Omega^2(C, P(\mathfrak{k})), \quad \tilde{A} \mapsto \phi(\tilde{A})F_A + \text{Vol}_C u^* P(\Phi).
\]

This motivates the following definition:

**Definition 3.2.1.** (Polarized vortices) A polarized vortex is pair \( (\tilde{A}, u) \in \tilde{H}^K(P, X) \) such that

\[
\phi(\tilde{A})F_A + \text{Vol}_C u^* P(\Phi) = 0.
\]

An isomorphism of polarized vortices \( (\tilde{A}_j, u_j), j = 0, 1 \) is an element \( k \in K(P) \) such that \( k(\tilde{A}_0, u_0) = (\tilde{A}_1, u_1) \). The pair \((\tilde{A}, u)\) is stable if it has finite automorphism group.

Denote by \( M^{K,\text{pol}}(P, X) \) the space of gauge equivalence classes of polarized vortices, and by \( M^{K,\text{pol}}(C, X) \) the union of the spaces \( M^{K,\text{pol}}(P, X) \) over isomorphisms classes of \( K \)-bundle bundles \( P \to C \). The space \( M^{K,\text{pol}}(P, X) \) admits a natural partial compactification by polystable polarized vortices, which allow bubbling of \( u \) in the fibers of \( P(X) \).
Definition 3.2.2. (Polystable polarized vortices) A \textit{polystable polarized vortex} consists of an element \( \hat{A} \in \hat{A}(P) \) and a stable holomorphic map \( u : \hat{C} \to P(X) \) of base class \([C]\) such that if \( u_0 : C_0 \cong C \to P(X) \) is the principal component, then \((\hat{A}, u_0)\) satisfies (28). An \textit{isomorphism} of polystable polarized vortices is an automorphism of the domain and a gauge transformation mapping one to the other. A \textit{marked} polarized vortex is a marked vortex \((A, \hat{C}, u, z)\), equipped with a lift of \( A \) to an element \( \hat{A} \in \hat{A}(P) \). A \textit{framed polarized vortex} is a framed polystable vortex together together with a lift \( \hat{A} \) of \( A \). A polystable polarized vortex is \textit{stable} if its automorphism group is finite.

Let \( \overline{M}_n^{K,\text{pol}}(P, X) \) denote the moduli space of isomorphism classes of polystable polarized vortices, and \( \overline{M}_n^{K,\text{pol}}(C, X) \) the union over topological types of bundles \( P \to C \). The definition of the Gromov convergence for polarized vortices is similar to that for symplectic vortices, and will be omitted. Let \( \overline{M}_n^{K,\text{pol}}(C, X, L, d) \) denote the component of homology class \( d \in H_2^K(X, \mathbb{Z})/\text{torsion} \). Let \( \overline{M}_n^{\text{pol,fr}}(C, X, L, d) \) resp. \( \overline{M}_n^{\text{pol,fr}}(C, X, L, d) \) denote the moduli space of polystable marked, resp. framed vortices of homology class \( d \). This space is certainly not compact, because the value of \( \phi \) can run off to 0 or \( \infty \). The correct analog of compactness is the properness statement in the theorem below, whose proof is similar to that for the moduli space of vortices \( \overline{M}_n^K(C, X) \):

Theorem 3.2.3. \( \overline{M}_n^{K,\text{pol}}(C, X) \) is a Hausdorff topological space. For any \( c > 0 \), the map \( \phi \) extends to a continuous proper map

\[
\bigcup_{\langle d, \omega_K \rangle < c} \overline{M}_n^{K,\text{pol}}(C, X, d) \to (0, \infty).
\]

A polarized vortex is \textit{regular} if a certain linearized operator is surjective, as in the case of vortices discussed above. The regular stable locus in \( \overline{M}_n^{K,\text{pol}}(C, X) \) naturally has the structure of a differentiable orbifold, by the same arguments as for vortices. The wall-crossing formula is obtained by applying localization to the \textit{cut space} as in Lerman [35].

Definition 3.2.4. (Symplectic cut of the moduli space of polarized vortices) For parameters \( \rho_- < \rho_+ \) the cut space of the moduli space of polarized vortices is

\[
\overline{M}_n^{K,\text{pol}}(C, X)_{[\rho_-, \rho_+]} := \left( \overline{M}_n^{K,\text{pol}}(C, X) \times \mathbb{P}_{[-\rho_+, \rho_+]}^1 \right) //U(1)
\]

\[
\cong \phi^{-1}(\rho_-)/U(1) \cup \phi^{-1}(\rho_-, \rho_+) \cup \phi^{-1}(\rho_+)/U(1)
\]

\[
\cong \overline{M}_n^K(C, X)_{[\rho_-, \rho_+]} \cup \phi^{-1}(\rho_-, \rho_+) \cup \overline{M}_n^K(C, X)_{\rho_+}.
\]

By Theorem 3.2.3,

Corollary 3.2.5. For any \( \rho_- > 0, \rho_+ \in (0, \infty) \), \( \overline{M}_n^{K,\text{pol}}(C, X)_{[\rho_-, \rho_+]} \) is a Hausdorff topological space. For any \( c > 0 \), the union of components \( \overline{M}_n^{K,\text{pol}}(C, X, d)_{[\rho_-, \rho_+]} \) with \( \langle d, \omega_K \rangle < c \) is compact.
Suppose that every polystable polarized vortex has finite automorphism group and is regular. The evaluation map

$$\text{ev}^{fr}_n : \overline{M}_n^{K,\text{pol},fr}(C, X)_{[\rho_-, \rho_+]} \to X^n$$

is $K^n \times U(1)$-equivariant and induces a map

$$\text{ev}^* : H_K(X, \mathbb{Q})^n \to H_{U(1)}(\overline{M}_n^{K,\text{pol}}(C, X)_{[\rho_-, \rho_+]}, \mathbb{Q}).$$

Similarly define the forgetful morphism

$$f_n : \overline{M}_n^{K,\text{pol}}(C, X)_{[\rho_-, \rho_+]} \to \overline{M}_n(C)$$

by collapsing unstable components. Given a collection $\alpha = (\alpha_1, \ldots, \alpha_n) \in H_K(X)^n$ a class $\beta \in H(\overline{M}_n(C))$ and an interval $[\rho_-, \rho_+]$ such that every (polystable) polarized vortex with parameter in $(\rho_-, \rho_+)$, and every vortex with parameters $\rho_\pm$ is regular and has finite automorphism group, the polarized gauged Gromov-Witten invariant associated to $\alpha, \beta, d \in H^2_K(X, \mathbb{Z})$ is

$$\int_{\overline{M}_n^{K,\text{pol}}(C, X, d)_{[\rho_-, \rho_+]}} \text{ev}^* \alpha \cup f^* \beta \in \mathbb{Q}[\xi].$$

Here $\xi$ is the parameter for the $U(1)$-action induced by that on $\tilde{A}(P)$.

The projection of the fixed point set $\overline{M}_n^{K,\text{pol}}(C, X)^{U(1)}$ to $\overline{M}_n^K(C, X)$ consists of reducible vortices. Indeed, if $(P, \tilde{A}, u)$ is fixed up to gauge transformation by $U(1)$, then there exists a $\zeta \in \mathfrak{k}(P)$ such that $\exp(i\theta)(P, \tilde{A}, u) = \exp(i\theta \zeta)(P, \tilde{A}, u)$. Thus the underlying vortex $(P, A, u)$ is fixed by $\zeta$. The structure group of a reducible vortex automatically reduces to the centralizer of the automorphism group evaluated at any point.

**Definition 3.2.6.** (\(\zeta\)-fixed vortices) A polystable $n$-marked $\zeta$-fixed vortex with $K_\zeta$-bundle $P$ is a $K_\zeta$-vortex $(A, \hat{C}, u, z)$ such that $u : \hat{C} \to P(X)$ is a stable map fixed up to automorphism of the domain by the automorphism of $P(X)$ induced by $\zeta$.

Let $\overline{M}_n^K(C, X, \zeta)$ denote the moduli space of $\zeta$-fixed vortices, up to $K_\zeta$-gauge equivalence. Inclusion of $K_\zeta$ in $K$ induces an embedding

$$t_\zeta : \overline{M}_n^K(C, X, \zeta) \to \overline{M}_n^{K,\text{pol}}(C, X)^{U(1)}.$$  

By the discussion above the union of images of these maps is surjective as $\zeta$ varies over equivalence classes of generators of one-parameter subgroups and $\rho$ ranges over $(0, \infty)$.

If every nodal polarized vortex is regular and stable then $\overline{M}_n^K(C, X, L, \zeta)$ is a compact orbifold. The $\zeta$-fixed gauged Gromov-Witten potential can then be defined as in (3.1.2). Suppose that every polarized vortex with parameter in $(\rho_-, \rho_+)$ and every vortex with parameter $\rho_-$ or $\rho_+$ has finite automorphism group and is regular. The localization formula applied to the action of $U(1)$ on $\overline{M}_n^{K,\text{pol}}(C, X, d)_{[\rho_-, \rho_+]}$ gives...
Theorem 3.2.7 (Wall-crossing for symplectic gauged Gromov-Witten invariants). Suppose that every nodal polarized vortex is regular with finite automorphism group, and every vortex with parameter $\rho_-, \rho_+ \in \mathbb{C}$ is regular with finite automorphism group. Then the gauged and polarized gauged Gromov-Witten invariants are defined and the gauged potentials $\tau_{X,K,\rho_\pm}$ corresponding to stability parameters $\rho_\pm$ are related by

$$\tau_{X,K,\rho_+} - \tau_{X,K,\rho_-} = \sum_{[\zeta] \neq 0, \rho \in (\rho_-, \rho_+)} \text{Resid}_\zeta \tau_{X,K,\zeta,\rho}.$$  

### 4. Quantum Witten localization

In this section we combine the area-dependence studied in the previous section with large and small area limit theorems from [59], [22] to obtain a proof of the quantum Witten localization formula (6).

#### 4.1. The large area limit.

The gauged potential and the graph potential of the quotient are related by the adiabatic limit theorem of [59] (which is a generalization of an earlier result of Gaio-Salamon [20]). Recall the quantum Kirwan map $\kappa_{X,G}$ of (2). The map $\kappa_{X,G}$ is defined by virtual integration over a moduli stack $\overline{\mathcal{M}}^G_{n,1}(\mathbb{C}, X)$ of scaled affine gauged maps to $X$. An object of the stratum $\overline{\mathcal{M}}^G_{n,1}(\mathbb{C}, X, d)$ of homology class $d \in H^G_2(X, \mathbb{Z})/\text{torsion}$ has evaluation maps

$$\text{ev} \times \text{ev}_\infty : \overline{\mathcal{M}}^G_{n,1}(\mathbb{C}, X, d) \to (X/G)^n \times (X//G).$$

The formula for $\kappa_{X,G}$ is

$$\kappa_{X,G}(\alpha) = \sum_{n \geq 0, d} (q^n/n!) \text{ev}_{\infty,*} \text{ev}^*(\alpha, \ldots, \alpha).$$

**Remark 4.1.1.** (Lack of quantum corrections to the Kirwan map in the monotone case on divisors classes) In many of our examples, $X$ will be monotone in the sense that $c_1^G(X) = \lambda \omega \in H^G_2(X)$ for some $\lambda > 0$. In this case, the moduli stacks $\overline{\mathcal{M}}^G_n(\mathbb{C}, X, d)$ have dimension greater than $\dim(X) + 2$ for $d \neq 0$. This implies that for $\alpha \in H^G_2(X)$, the push-forwards to $H(X//G)$ have degree larger than $2 \dim(X//G)$ for $d \neq 0$ and so vanish. For similar reasons, a lower bound on $m$ the minimal Chern number $(d, c^G_1(X))$ for classes $d \in H^G_2(X, \mathbb{Q})$ realized by stable affine gauged maps $u : \mathbb{P}(1, r) \to X/G$ implies that $D_0 \kappa_{X,G}$ has no quantum corrections on classes of degree at most $2m$.

**Example 4.1.2.** (Quantum Kirwan map for the scalar multiplication on affine space) Let $G = \mathbb{C}^\times$ act on $X = \mathbb{C}^k$ by scalar multiplication, so that $X//G = \mathbb{P}^{k-1}$. We have

$$T_0 QH_G(X) = \Lambda^G_X[\xi].$$
while
\[
T_0 QH(X//G) = \Lambda^G_X[\omega]/(\omega^k - q).
\]
By the previous remark \( \kappa_{X,G}(0) = 0 \) and
\[
D_0 \kappa_{X,G}(\xi^l) = \omega^l, \quad l < k.
\]
A special case of the main result of [25] (quantum Stanley-Reisner relations) implies that
\[
D_0 \kappa_{X,G}(\xi^k) = q.
\]
Hence \( D_0 \kappa_{X,G} \) is surjective and
\[
T_0 QH(X//G) = T_0 QH_G(X)/\ker D_0 \kappa_{X,G} = \Lambda^G_X[\xi]/(\xi^k - q)
\]
as expected.

Let \( \rho \in (0, \infty) \); we consider Mundet stability with respect to the polarization \( L^\rho \) with \( \rho \to \infty \).

**Theorem 4.1.3.** (Adiabatic limit theorem [59]) If stable=semistable for the action of \( G \) on \( X \) then stable=semistable for gauged maps for \( \rho \) sufficiently large (more precisely, for any class \( d \in H^G_2(X,\mathbb{Z}) \) there exists an \( r > 0 \) such that \( \rho > r \) implies stable=semistable) and
\[
\tau_{X/G} \circ \kappa_{X,G} = \lim_{\rho \to \infty} \tau^G_X.
\]
If \( C \) is a genus zero curve equipped with a \( \mathbb{C}^\times \)-action, then the same equality holds for \( \mathbb{C}^\times \)-equivariant potentials \( \tau^\mathbb{C}^\times_{X/G}, \tau^G_{X,C^\times} \).

In other words, the diagram
\[
\begin{array}{ccc}
QH_G(X) & \xrightarrow{\kappa^G_X} & QH(X//G) \\
\downarrow \tau^G_X & & \downarrow \tau_{X/G} \\
\Lambda^G_X & \xrightarrow{\tau_{X/G}} & QH(X//G)
\end{array}
\]
commutes in the limit \( \rho \to \infty \).

There is a more general version of the adiabatic limit theorem for twisted gauged Gromov-Witten invariants, see [59], as well as for the solutions to the quantum differential equation studied in Givental [21] which we now recall. Let \( C = \mathbb{P} \). Givental [21] observed that both the graph potential admits a factorizations into localized graph Gromov-Witten potentials (or J-function)
\[
\tau_{X/G,\pm} : QH(X//G) \to QH(X//G)[[h^{-1}]].
\]
Each is a solution to the quantum differential equation on \( QH(X//G) \). A \( \mathbb{C}^\times \)-equivariant of the quantum Kirwan map
\[
\kappa_{X,G} : QH_G(X) \to QH(X//G)[[h]]
\]
is obtained by $\mathbb{C}^\times$-equivariant integration over $\overline{\mathcal{M}}_n(C, X)$ with respect to the action induced by scalar multiplication on $\mathbb{C}$. To distinguish the quantum Kirwan map with its classical analog, we denote by 

$$\kappa_{X,G}^{\text{class}} : QH_G(X)[[\hbar]] \to QH(X//G)[[\hbar]]$$

the map obtained by extending Kirwan’s map $H_G(X) \to H(X//G)$ by linearity over $\Lambda_{X,G}$ and $\hbar$.

**Theorem 4.1.4.** (Localized adiabatic limit theorem [59]) In the setting of Theorem 4.1.3,

$$\tau_{X//G,-} \circ \kappa_{X,G} = \lim_{\rho \to \infty} \kappa_{X,G}^{\text{class}} \circ \tau_{X,G,-}$$

**Remark 4.1.5.** (Lack of coordinate changes in the monotone case) Continuing Remark 4.1.1, suppose that $X$ is monotone so that $\kappa_{X,G}$ is the classical Kirwan map on $H_G^{\leq 2}(X)$. Then Theorem 4.1.4 gives

$$\tau_{X//G,-|H_G^{\leq 2}(X)} = \lim_{\rho \to \infty} \kappa_{X,G}^{\text{class}} \circ \tau_{X,G,-|H_G^{\leq 2}(X)}.$$ 

The expression $\kappa_{X,G}^{\text{class}} \tau_{X,G,-}$ is Givental’s I-function, while $\tau_{X//G,-}$ is the small J-function, c.f. [21].

**Remark 4.1.6.** (Qde solutions via localization) Later we will use the following fact, which says that $\tau_{X//G,\pm}$ is the “lowest-order contribution” in the $\mathbb{C}^\times$-localization formula applied to $\tau_{X//G}^\mathbb{C}$, c.f. Braverman [10] and Coates [14]. First recall that the graph potential $\tau_{X//G}$ can be modified to include a “Liouville insertion” [59]. Let

$$e \times e : \overline{\mathcal{M}}_n(C, X//G) \to (X//G) \times C$$

be the evaluation map on the universal curve $p : \overline{\mathcal{M}}_n(C, X//G) \to \overline{\mathcal{M}}_n(C, X//G)$. Let $\omega \in H^2(X//G)$ be the symplectic class and $\omega_C \in H^2(C)$ a generator. The Liouville class is

$$\lambda = \exp(p_*(e^*\omega \cup e^*_C\omega_C)) \in H(\overline{\mathcal{M}}_n(C, X//G)).$$

Define

$$\tau_{X//G}^\mathbb{C}(d) = \int_{[\overline{\mathcal{M}}_n(C, X, d)]} ev^*(\alpha, \ldots, \alpha) \cup ev^*_C(e^{\omega_C}, \ldots, e^{\omega_C}, [\infty]) \cup \lambda.$$ 

After choosing a $\mathbb{C}^\times$-equivariant extension of the class $\omega_C$, we may apply localization for the $\mathbb{C}^\times$-action on $\overline{\mathcal{M}}_n(C, X//G)$ to obtain an expression for $\tau_{X//G}$ as a sum over fixed points. We choose the equivariant extension so that the restriction of $\omega_C$ to 0 resp $\infty$ is 0 resp. $\xi$. Then

$$D_\alpha \tau_{X//G}^\mathbb{C}(\alpha, \gamma) = \int_{[X//G]} \tau_{X//G,-}(\alpha) \cup \alpha + \text{higher order in } \exp(h).$$

In other words, the lowest-order terms in the localization formula can be written as a pairing with $\tau_{X//G,-}$. Indeed, the $\mathbb{C}^\times$-fixed points consist of configurations consisting
of a constant map to $X/G$ on the principal component, together with bubble trees attached at $0, \infty \in \mathbb{P}$. Since $\omega_C$ resp. $\xi - \omega_C$ vanishes at $\infty$ resp. $0$ in $C$, the lowest order term when the first $n - 1$ marked points are constrained to map to $0$, the last maps to $\infty$, and the bubble tree attached at $\infty$ is zero, see \cite{59}, Lemma 9.4]. This ends the Remark.

4.2. The small area limit. The opposite limit in which the stability parameter goes to zero, is studied in the paper \cite{22}. The moduli stack of Mundet semistable gauged maps in this limit is identified with a quotient of the moduli stack of parametrized stable maps to $X$. Let $X^{ss} \subset X$ denote the semistable locus, so that $X/G = X^{ss}/G$. Let $C$ be a curve of genus zero.

**Definition 4.2.1.** (Zero-semistability) A map $u = (u_C, u_X) : \hat{C} \to C \times X$ of degree $(1, d)$ is zero-semistable iff $u_C(u_X^{-1}(X^{ss}))$ is dense in $C$. We denote by $\overline{\mathcal{M}}_{n}(C, X, d)^{ss} \subset \mathcal{M}_{n}(C, X, d)$ the semistable locus in $\overline{\mathcal{M}}_{n}(C, X, d) := \mathcal{M}_{0,n}(C \times X, (1, d))$ and by $\overline{\mathcal{M}}_{n}(C, X, d)//G = \mathcal{M}_{n}(C, X, d)^{ss}/G$ the quotient stack.

**Proposition 4.2.2.** \cite{22} Let $X$ be a smooth polarized projective $G$-variety, $C$ a curve of genus zero, and $d \in H^{2}_{G}(X, \mathbb{Z})$. There exists a $\rho_0$ such that for $\rho < \rho_0$, there is an isomorphism $\overline{\mathcal{M}}_{n}(C, X, d)//G \to \overline{\mathcal{M}}_{n}^{G}(C, X, d)$ of Deligne-Mumford stacks equipped with perfect relative obstruction theories.

**Proof.** For the sake of completeness we sketch the proof. An object of $\overline{\mathcal{M}}_{0,n}(C \times X, ([C], d))//G$ over a scheme $S$ consists of a principal $G$-bundle $P \to S$ and an equivariant morphism $u \in \text{Hom}_{G}(P, \overline{\mathcal{M}}_{0,n}(C \times X, ([C], d)))$, given by a family of stable maps $u : \hat{C} \to C \times X$ of class $([C], d)$ where $\hat{C}$ is a family of nodal curves over $P$. Equivariance implies that the family $\hat{C}$ is the pull-back of a family over $S$, in which case we obtain a map $u : \hat{C} \to P(X)$ and by pull-back under $S \times C \to S$, a bundle $P \to S \times C$, giving an object of $\overline{\mathcal{M}}_{n}^{G}(C, X, d)$. For $\rho$ sufficiently small, Mundet semistability is equivalent to semistability of the $G$-bundle, which since $C$ is rational is equivalent to triviality on the fibers of $C \times S \to S$. The relative obstruction theory on $\overline{\mathcal{M}}_{n}^{G}(C, X)$ has complex $(R_{\rho} e^{*}T(X/G))^{\vee}$ given by descent from $(g \to T\mathcal{M}_{n}(C, X))^{\vee}$, since every $\rho$-semistable map has underlying trivial bundle. The latter is isomorphic to the relative obstruction theory on $\overline{\mathcal{M}}_{n}(C, X)//G$. \hfill \square

We can express the gauged Gromov-Witten invariants in the small area limit in the case that $C$ has genus zero, in terms of the usual stable map spaces, as follows.

**Definition 4.2.3.** (Quantized Witten trace) Let

$$\kappa_{\overline{\mathcal{M}}, G} : H^{G}(\overline{\mathcal{M}}_{n}(C, X, d) \to H(\overline{\mathcal{M}}_{n}(C, X, d)//G)$$
be the Kirwan map (that is, restriction to the semistable locus and descent) and let
\[ \tau_{\overline{\mathcal{M}}_G} : H(\overline{\mathcal{M}}_n(C, X, d)/G) \to \mathbb{Q} \]
denote virtual integration. The quantized Witten trace \( \tau^G_X \) is the composition of pull-back with integration over the moduli space of stable maps
\[ \tau^G_X : H_G(X) \to \Lambda^G_X, \quad \alpha \mapsto \sum_{d,n} (q^d/n!) \tau_{\overline{\mathcal{M}}_G} \circ \kappa_{\overline{\mathcal{M}}_G} \circ \text{ev}^* (\alpha, \ldots, \alpha). \]
This ends the definition.

**Corollary 4.2.4.** (Small-area limit theorem) Suppose that \( C \) has genus zero and stable=semistable for the action of \( G \) on \( \overline{\mathcal{M}}_n(C, X) \). Then
\[ \tau^G_X = \lim_{\rho \to 0} \tau_{X,G,\rho}. \]
If \( C \) is equipped with a \( \mathbb{C}^\times \)-action then the same holds for the \( \mathbb{C}^\times \)-equivariant potentials.

**Example 4.2.5** (Gauged maps with trivial homology class). Suppose that \( d = 0 \). Then every semistable gauged map has trivial bundle and constant section, so that
\[ \overline{\mathcal{M}}_n^G(C, X, L, d) \cong (X/G) \times \overline{\mathcal{M}}_n(C). \]
Hence the gauged Gromov-Witten invariants are independent of \( \rho \).

### 4.3. Putting everything together.

The quantum Witten localization formula (6) follows by combining the wall-crossing formula (3.2.7), the adiabatic limit theorem (3.1.3), and the small-area limit Theorem 4.2.4.

Naturally one wants to investigate examples. Unfortunately, examples involving a projective target tend to be complicated, because the classical Witten localization formula is already quite complicated. For quasiprojective actions, the formula does not hold.

**Remark 4.3.1.** (Failure of quantum Witten localization in quasiprojective cases)
The proof in the case \( X \) quasiprojective breaks down at the following point: if \( X \) is non-compact, then \( \overline{\mathcal{M}}_n^G(C, X, L^\rho) \) may not equal \( \overline{\mathcal{M}}_n(C, X)/G \) for \( \rho^{-1} \gg 0 \). The proof of this equality in the projective case uses that any object of \( \overline{\mathcal{M}}_n^G(C, X, L^\rho) \) has an underlying bundle \( P \to C \) that is semistable for \( \rho^{-1} \gg 0 \). This is false in the quasiprojective case. For example, suppose that \( X = \mathbb{C}^k \) and \( G = \mathbb{C}^\times \) acting diagonally. Then
\[ \overline{\mathcal{M}}_n^G(C, X, d) = H^0(C, P \times_G X) = \mathbb{P}^{kd-1} \]
for any value of \( \rho \), while \( \overline{\mathcal{M}}_n(C, X, d) \) is non-empty only for \( d = 0 \).

We show that a slightly modified version of the quantum Witten localization formula holds in many quasiprojective cases, assuming a suitable bound on the moment map for a central one-parameter subgroup that acts locally freely:
Assumption 4.3.1. Suppose that $X$ is a smooth polarized quasiprojective $G$-variety with symplectic form $\omega$ and proper moment map $\Phi$ which is either compact or a vector space with linear action, and there exists a central $\xi \in \mathfrak{g}$ such that $\langle \Phi, \xi \rangle$ is bounded from below.

Let $\chi$ be a character of $G$ that is negative on the one-parameter subgroup generated by $\xi$ and $L_\chi$ the corresponding trivial line bundle over $X$. Consider the piecewise linear path of polarizations obtained by shifting by multiples of the character $\chi$:

$$L_\rho = \begin{cases} L \otimes \Omega^{\rho-1} & \rho \leq 1 \\ L^\rho & \rho \geq 1 \end{cases}.$$  

For any homology class $d \in H^2_\mathbb{Z}(X, \mathbb{Z})$, the moduli stack $\overline{\mathcal{M}}^G_n(C, X, L_\rho, d)$ is empty for $\rho \gg 0$. From the symplectic point of view, this is because the energy of any sequence of vortices with respect to a symplectic form corresponding to $L_\rho$ goes to infinity and $\rho$ does. From the algebraic point of view, the Ramanathan weight corresponding to the one-parameter subgroup $\xi$ is determined by $d$, while the Hilbert-Mumford weight goes to infinity. Hence there are no Mundet-semistable gauged maps with class $d$, for $\rho$ sufficiently small. It follows that, a formula similar to quantum Witten localization (6) holds:

Theorem 4.3.2. (Quantum Witten localization for certain quasiprojective varieties) Let $X$ be as in Assumption 4.3.1, $C$ a genus zero curve, and suppose that stable=semistable for the $G$-action on $X$, for gauged maps with polarization $L$, and for polarized gauged maps for the path $L_\rho$. Then

$$0 = \tau_{\mathcal{X}/G} \circ \kappa_{X,G} + \sum_{[\zeta] \neq 0, \rho} \tau_{X,G,\zeta,\rho}. \quad (32)$$

The formula holds equivariantly for any $\mathbb{C}^\times$ action on the curve $C$.

Example 4.3.3. (Quantum Witten localization for the scalar multiplication on affine space) To explain the notation we use (32) to compute the three-point Gromov-Witten invariants of projective space using quantum Witten localization. Suppose that $G = \mathbb{C}^\times$ acts diagonally on $X = \mathbb{C}^k$. We have $H^2_\mathbb{Z}(X, \mathbb{Z}) \cong H^2(\mathbb{C}^k/G) \cong \mathbb{Z}$ corresponding to the last factor. We compute the class $d = 1$ three-point invariants $\langle \omega^a, \omega^b, \omega^c \rangle_{0,d}$ of $X//G$ using quantum Witten localization, where $\omega \in H^2(\mathbb{C}^k//G)$ is the hyperplane class, by examining the order three term in (32) and adding Fulton-MacPherson insertion $\beta \in H^6(\overline{M}_3(C))$ to fix the positions of the marked points. We have $QH_G(X, \mathbb{Q}) = \Lambda^G_X[\xi]$. We let

$$\alpha_1 = \xi^a, \quad \alpha_2 = \xi^b, \quad \alpha_3 = \xi^c.$$  

Since $\epsilon^G(X)$ is at least $2k$ on classes $d > 0$, the $D_0 \kappa_{X,G}$ has no quantum corrections to $\alpha_j, j = 1, 2, 3$ by Remark 4.1.1. Their image under $D_0 \kappa_{X,G}$ is equal to $\omega^a, \omega^b, \omega^c$ respectively. We consider a path $L_\rho$ obtained by shifting by a negative character $\chi$; this means that in the fixed point formula we take the residue with respect to $-\xi$, ...
see [24]. By the formula (32),
\[
\sum_{d \geq 0} q^d \langle \omega^a, \omega^b, \omega^c \rangle_{0,d} = \tau^3_{X//G}(\omega^a, \omega^b, \omega^c, \beta) = - \sum_{\rho, [\xi]} \tau^3_{X,G,\xi,\rho}(\xi^a, \xi^b, \xi^c, \beta).
\]
There is a unique $G$-fixed point in $X$. The $G$-bundle $P$ with first Chern class $d = 1$ together with the zero section $u \in H^0(C, P \times_G X)$ forms a Mundet semistable map for a unique value of the parameter $\rho$. For $d = 1$ the index bundle is
\[
H^0(O(k)^\times \times \mathbb{C}^\times \times \mathbb{C}^k) \cong \mathbb{C}^{2k}
\]
which has Euler class
\[
\epsilon_+(T(X//G)) = \xi^{2k}.
\]
The unique fixed point contribution
\[
\tau^3_{X,G,\xi,\rho}(\xi^a, \xi^b, \xi^c, \beta) = q \text{ Resid}_- \xi \sum_x \xi^{a+b+c} \xi^{-2k} = \begin{cases} 1 & a + b + c = 2k - 1 \\ 0 & \text{otherwise}. \end{cases}
\]
We obtain
\[
\langle \omega^a, \omega^b, \omega^c \rangle_{0,d} = \begin{cases} 1 & a + b + c = 2k - 1 \\ 0 & \text{otherwise} \end{cases}
\]
as expected.

5. Quantum abelianization

We prove here the relationship between the Gromov-Witten invariants of the git quotient of $X$ by $G$ and its maximal torus $T$ mentioned in the Introduction. Abelianization is first proved for graph potentials, then deduced for qde solutions. The section ends with the examples of the moduli spaces of odd numbers of points on a projective line and the Grassmannian.

5.1. Abelianization for graph potentials. Typically Gromov-Witten invariants of a variety such as a git quotient are difficult to compute directly. The abelianization or quantum Martin conjecture of Bertram, Ciocan-Fontanine and Kim [8] (motivated by an earlier conjecture of Hori and Vafa [29, Appendix]) relates the Gromov-Witten invariants of a symplectic resp. geometric invariant theory quotient $X//G$ with the twisted Gromov-Witten invariants of the quotient $X//T$ by the maximal torus $T$; we denote by the same notation the complex torus and assume both quotients by $G$ and $T$ are locally free. A version of this formula, Corollary 5.1.1 below, relates the graph potential for $X//G$ with the gauged potential for the action of $T$ on $X$. In many cases, the latter is easy to compute and gives a formula for the graph potential on $X//G$. 
Proof of Theorem 1.0.5. We first prove the Theorem in the case stable=semistable for polarized gauged maps. We take as the inductive hypothesis that Theorem 1.0.5 holds for any group of dimension less than \( \dim(G) \). We wish to compare the fixed point contributions in the quantum Witten localization formulas

\[ \tau^G_X - \tau_{X/G} \circ \kappa_{X,G} = \sum_{[\zeta] \neq 0,\rho} \tau_{X,G,\zeta,\rho} \]  

and

\[ \tau^T_X - \tau_{X/T} \circ \kappa_{X,T} = \sum_{[\zeta] \neq 0,\rho} \tau_{X,T,\zeta,\rho} \]

where in the version for \( T \), both the traces and quantum Kirwan maps have been twisted by the Euler class of the index of \( g/t \) which has been omitted from the notation to save space. Now \( \tau^T_X \) resp. \( \tau^G_X \) is defined by integration over \( \overline{M}_n(C,X) // T \) resp. \( \overline{M}_n(C,X) // G \) so we are essentially in the setting considered by Martin [38], except that we are dealing with virtual integration. By the results of [22],

\[ \tau^G_X = \pi^G_T \circ |W|^{-1} \tau^T_X \]

either by Martin’s argument, if the moduli spaces of stable maps are smooth and the virtual fundamental classes are the usual ones, or by a virtual version of Martin’s argument if the moduli spaces of stable maps are only virtually smooth; the virtual version requires the restriction to Chow classes. Therefore, it suffices to show a similar relationship for the right-hand-sides in (33), (34). Taking into account that each fixed point component \( X^\zeta \) for the \( G \)-action corresponds to \( |W/W_{\zeta}| \) fixed point components \( X^w_{\zeta} \), \( w \in W/W_{\zeta} \) for the \( T \)-action, the identity we wish to show is

\[ \tau_{X,G,\zeta,\rho} = \pi^G_T |W_{\zeta}|^{-1} \tau_{X,T,\zeta,\rho}. \]

In the case \( G_{\zeta} \) is abelian the group \( W_{\zeta} \) is trivial and so the equality holds automatically. More generally, by Theorem 3.1.8

\[ \tau_{X,G_{\zeta},t} = \tau_{X,G_{\zeta}/C_{\zeta}^\times,\zeta} \circ t_{\zeta} \]

and

\[ |W_{\zeta}|^{-1} \tau_{X,T,\zeta,t} = |W_{\zeta}|^{-1} \tau_{X,T/C_{\zeta}^\times,\zeta} \circ t_{\zeta}. \]

By the inductive hypothesis,

\[ \tau_{X,G_{\zeta}/C_{\zeta}^\times,\zeta} = \pi^G_T |W_{\zeta}|^{-1} \tau_{X,T/C_{\zeta}^\times,\zeta} \circ t_{\zeta}. \]

Equation (35) follows. We remark that a similar iterative analysis of fixed point contributions in the classical case was undertaken Guillemin-Kalkman [26, Section 4].

To prove the Theorem 1.0.5 in general we find a master space for which stable=semistable, by adding a parabolic structure. Recall from e.g. [1] that a parabolic structure on a \( G \)-bundle \( P \to C \) at \( c \in C \) consists
(a) a reduction of structure group $\sigma \subset P_c/B$ to a Borel subgroup $B$ in the fiber at $c$ and 
(b) a parabolic weight $\mu$ in the interior of the Weyl alcove.

We consider here only generic parabolic weights, see [1] for the general theory. The definition of the Ramanathan weight in Definition 2.1.1 extends to a Ramanathan weight for bundles with parabolic structure, with an additional term arising from the parabolic structure [1]. Let $M^G_n(C, X, L, c, \mu)$ denote the Artin stack consisting of Mundet semistable pairs $(P, u, \sigma)$ of a bundle $P \to C$, a section $u : C \to P(X)$, markings $z$, and a parabolic structure $(\sigma, \mu)$ at $c$. (Note that we make no requirement on the section $u$ at the marked point, whereas in the theory of parabolic stable pairs one usually requires that $u$ has a special form at $c$. Our aim here is only to obtain a smooth master space.) If stable=semistable then a similar construction to the one above shows that $M^G_n(C, X, L, c, \mu)$ is a smooth, proper Deligne-Mumford stack with a perfect relative obstruction theory; we omit the details since the git construction of the moduli space of bundles with parabolic structure is well-known.

If stable=semistable for $M^G_n(C, X, L)$ and $\mu$ is sufficiently small, then the parabolic structure does not play a role in stability and the forgetful morphism $\pi : M^G_n(C, X, L, c, \mu) \to M^G_n(C, X, L)$ is a $G/B$-bundle. The integral of any class $\alpha$ over $M^G_n(C, X, L)$ is given by

$$\int_{[M^G_n(C, X, L)\setminus G/B]} \pi^* \alpha = \int_{[M^G_n(C, X, L, c, \mu)\setminus G/B]} \pi^* \alpha \cup \text{Eul}(T_\pi)/|W|.$$  

Indeed the integral of $\text{Eul}(T_\pi)/|W|$ over the fiber of $\pi$ is

$$\int_{[G/B]} \text{Eul}(T(G/B))/|W| = \chi(G/B)/|W| = 1.$$  

Adding the parabolic structure with generic parabolic weight has the effect of making the Ramanathan weight plus the contribution from the parabolic structure generic. Given stability parameters $\rho^\pm$ and corresponding polarizations $L^\pm = L^{\rho^\pm}$, we obtain a moduli stack $M^{\rho^+_n}(C, X, L_-, L_+, c, \mu)$ of polarized gauged maps with parabolic structure. As in Proposition 3.1.6, this implies that stable=semistable for $M^G_n(C, X, L_-, L_+, c, \mu)$, which is therefore a proper Deligne-Mumford stack with perfect relative obstruction theory. Localization on the stack $M^G_n(C, X, L_-, L_+, c, \mu)$ produces a wall-crossing formula, whose fixed point contributions are the $\mathbb{C}^*$-fixed components in $M^G_n(C, X, L_-, L_+, c, \mu)$ not equal to $M^G_n(C, X, L_\pm, c, \mu)$. Similarly, let $M^T_n(C, X, L_-, L_+, c, \mu)$ be the moduli stack of semistable polarized $T$-gauged maps to $X$ with a parabolic structure of the corresponding $G$-bundle at $z$, whose objects are data

$$(P \to C, u : \hat{C} \to P(X), z, \sigma \in P_z \times_T G/B).$$

The equation (36) also holds with $G$ replaced with $T$. Applying the abelianization argument to the moduli stacks $M^{\rho^n}_{\rho^+_n}(C, X, L_-, L_+, c, \mu), M^G_n(C, X, L_-, L_+, c, \mu)$ using
induction on the dimension of $G$ gives the identity
\begin{equation}
\int_{[\mathcal{M}_G^2(n)(C,X,L,\rho,c,\mu,d)]} \text{ev}^* \alpha \cup f^* \beta \cup \pi^* \text{Eul}(T_\pi) = \int_{[\mathcal{M}_T^2(n)(C,X,L,\rho,c,\mu,d)]} \text{ev}^* \alpha \cup f^* \beta \cup \epsilon(\mathfrak{g}/t) \cup \pi^* \text{Eul}(T_\pi)/|W|
\end{equation}
for any $\rho$ for which stable=semistable for gauged maps. Hence using (36),
\begin{equation}
\int_{[\mathcal{M}_G^2(n)(C,X,L,\rho,d)]} \text{ev}^* \alpha \cup f^* \beta = \int_{[\mathcal{M}_T^2(n)(C,X,L,\rho,d)]} \text{ev}^* \alpha \cup f^* \beta \cup \epsilon(\mathfrak{g}/t)/|W|
\end{equation}
as claimed. □

Combining Theorem 1.0.5 with the adiabatic limit Theorem 4.1.3 one obtains:

**Corollary 5.1.1.** Let $C$ be a smooth genus 0 curve, $X$ a smooth projective $G$-variety, and suppose that $\text{stable}=\text{semistable}$ for $T$ and $G$ actions on $X$. The following equality holds on $QH_{\text{chow}}^G(X)$:
\[
\tau_{X/G}\circ \kappa_{X,G} = \lim_{\rho \to \infty} |W|^{-1} \pi_T^G \circ \tau_{X,T,\rho} \circ \pi_T^G.
\]

Examples are given following an extension of this formula to qde solutions in the next subsection.

5.2. **Abelianization for qde solutions.** Using the abelianization Theorem 1.0.5 we express the solution to the quantum differential equation for $X//G$ in terms of a twisted solution to that for $X//T$.

**Proof of Theorem 1.0.6.** Compare (31) with the formula for the localized graph potential for the maximal torus twisted by the Euler class of the index bundle of $\mathfrak{g}/t$:
\[
\int_{[X/T]} \tau_{X/T,-}(\alpha_-) \cup \alpha_+ = D_{\alpha_-} \tau_{X,T}(\alpha_+) + \text{higher order}.
\]
Using Martin’s formula comparing integrals on $X/G$ and $X/T$ and the abelianization theorem for the gauged potential with Liouville insertion,
\[
\tilde{\tau}_{X/G}^n = \pi_T^n |W|^{-1} \tilde{\tau}_{X/T}^n.
\]
In the case $C = \mathbb{P}$, this identity holds $\mathbb{C}^\times$-equivariantly as before. Now each term in localization applied to $\tilde{\tau}_{X/G}^n$ is the Fourier transform of a shifted partition function (delta function convolved with a sequence of Heaviside functions). By Remark 4.1.6, the “lowest order terms” in $\exp(h)$ are given by the localized potentials. Since these terms must match be equal,
\[
\int_{[X/G]} \tau_{X/G,-} \kappa_{X,G}(\alpha_-) \cup \kappa_{X,G}^{\text{class}} \alpha_+ = \pi_T^n |W|^{-1} \int_{[X/T]} \tau_{X/T,-} \kappa_{X,T}(\alpha_-) \cup \kappa_{X,T}^{\text{class}} \alpha_+.
\]
Using Martin’s formula in Theorem 1.0.3 this implies
\[ \tau_{X/G,-} \circ \kappa_{X,G}(\alpha_-) = (\mu_G^T \otimes \pi_T^G) \circ \tau_{X/T,-} \circ \kappa_{X,T} \circ r_T^G(\alpha_-) \]
for any Chow class \( \alpha_- \) as claimed.

\[ \square \]

Remark 5.2.1. (G-graph potential in terms of the T-gauged potential) In practice, it is often simpler to compute the gauged potential for the \( T \)-action. Using the localized adiabatic limit theorem of [59]

\[ (\mu_G^T \otimes \pi_T^G) \circ \tau_{X/T,-} \circ \kappa_{X,T}(\alpha) = \lim_{\rho \to \infty} (\mu_G^T \otimes \pi_T^G) \circ \kappa_{X,T}^{\text{class}} \circ \tau_{X,T,-}(\alpha). \]

The map
\[ \mu_G^T \kappa_{X,T}^{\text{class}} : H_T(X) \to H(X//G) \]
is the composition of averaging
\[ \text{avg}_W : H_T(X) \to H_T(X)^W \cong H_G(X) \]
with the classical Kirwan map \( \kappa_{X,G}^{\text{class}} : \)
\[ \mu_G^T \circ \kappa_{X,T}^{\text{class}} = \kappa_{X,G}^{\text{class}} \circ \text{avg}_W. \]

The fact that the \( T \)-action on \( X \) extends to an \( N(T) \)-action implies that the localized potential \( \tau_{X,T,-} \circ r_T^G \) is \( W \)-invariant and so via the isomorphism \( H_T(\cdot)^W \cong H_G(\cdot) \) we have
\[ \tau_{X/G,-} \circ \kappa_{X,G} = \lim_{\rho \to \infty} \kappa_{X,G}^{\text{class}} \circ \tau_{X,T,-} \circ r_T^G. \]

This formula is somewhat more useful than that of Theorem 1.0.6 in examples.

Remark 5.2.2. (Abelianization for qde solutions for quasiprojective actions) Continuing Theorem 4.3.2, we extend the abelianization results to certain quasiprojective cases. Namely, replacing the quantum Witten localization formula (6) by the alternate formula (32) obtained from wall-crossing by shifting by a character, the same arguments go through and imply the formulas 1.0.5 and 1.0.6 and the formula (38) hold under the assumptions given in Theorem 4.3.2. Note that this gives an alternative argument for abelianization for git quotients \( X//G \) of projective \( X \) by \( G \) in the case that \( G \) has a non-trivial center, but not in the case that \( G \) is simple and non-abelian.

Remark 5.2.2 allows us, finally, to give some applications.

Example 5.2.3. (Grassmannians) In this example we reproduce the results on Grassmannians from Bertram et al [8]. For positive integers \( r < k \) let \( X = \text{Hom}(\mathbb{C}^r, \mathbb{C}^k) \) be the space of linear maps from \( \mathbb{C}^r \) to \( \mathbb{C}^k \) and let \( G = GL(r) \) act on \( X \) by composition. For the polarization \( \lambda = c_G^2(X) \in H_G^2(X, \mathbb{Q}) \cong \mathbb{Q} \), the semistable maps are those with full rank and so
\[ X//G = \{ x \in \text{Hom}(\mathbb{C}^r, \mathbb{C}^k) | \text{rank}(x) = k \}/GL(r) = \text{Gr}(r, k) \]
is the Grassmannian of \( r \)-dimensional subspaces in \( \mathbb{C}^k \). We have \( H^G_2(X,\mathbb{Z}) \cong \mathbb{Z} \), with the generator represented by a linear pencil of subspaces
\[ \mathbb{P} \cong \{ \text{span}(v_1, \ldots, v_{r-1}, v_r + tv_{r+1}) | t \in \mathbb{P} \} \subset \text{Gr}(r,k) \]
defined by a basis \( v_1, \ldots, v_k \) of \( \mathbb{C}^k \). There exist stable maps to \( X/G \) of class \( d \) only if \( d \geq 0 \). Although \( X \) is not compact, the moment map for the \( G \)-action
\[ X \to u(r), \quad x \mapsto i(x^* x - \lambda I) \]
is proper and bounded from below. Identify
\[ H^G_1(X,\mathbb{Z}) \cong \mathbb{Q}[\theta_1, \ldots, \theta_r]. \]
The first Chern class of \( X \) is
\[ c^G_1(X) = k(\theta_1 + \ldots + \theta_k) \]
which is the same as the symplectic class \( \omega \in H^G_2(X) \). Since \( X \) is monotone, the map \( \kappa_{X,G} \) is trivial on \( H^G_2(X) \) and may be ignored by Remark 4.1.1. By abelianization the localized graph potential \( \tau_{X/G,-} \) for the Grassmannian has restriction to \( H^2(X/G) \) given by
\begin{equation}
\tau_{X/G,-} = \pi^G_T \circ \kappa_{X,G}^{\text{class}} \circ \tau_{X,T,-} \circ \rho^G_T
\end{equation}
The localized gauged potential \( \tau_{X,T,-} \) is Givental’s \( I \)-function and by [21] given by
\begin{equation}
\tau_{X,T,-}(t_0 + t_1 \theta_1 + \ldots + t_r \theta_r) = e^{t_0 + (t_1 \theta_1 + \ldots + t_r \theta_r)/\hbar} \sum_d q^d \sum_{j=1}^r t_j^j \tau_{X,T,-}(d)
\end{equation}
where
\begin{equation}
\tau_{X,T,-}(d) = \frac{(-1)^{k-1}d \prod_{i<j}(\theta_i - \theta_j) + (d_i - d_j)\hbar}{\prod_{i<j}(\theta_i - \theta_j) \prod_{i=1}^{d_i}(\theta_i + l\hbar)^n}
\end{equation}
\[ = \prod_{i \neq j} \frac{\prod_{l \leq d_i - d_j}(\theta_i - \theta_j) + l\hbar}{\prod_{l \leq 0}(\theta_i - \theta_j) + l\hbar} \prod_{i=1}^{d_i}(\theta_i + l\hbar)^n. \]
The formula obtained by combining (39), (40), (41) was conjectured in Hori-Vafa [29, Appendix] and proved in Bertram et al [8], [7] by different methods.

Remark 5.2.4. (Relations in small quantum cohomology?) In the above example, as well as in the ones below, it is not clear how to extract the relations for the quantum cohomology from the formula for the qde solution, by finding differential operators annihilating the solution. This strategy was partially carried out in some examples, in a slightly different context, by Batyrev-Ciocan-Fontanine-Kim-Van-Straten in [2, Section 5.2].

Example 5.2.5. (Moduli of points on the projective line) We consider the git quotient for the diagonal action of \( SL(2,\mathbb{C}) \) on \( (\mathbb{P}^1)^{2k+1} \) with polarization on each factor the same. The git quotient is
\[ Y = \{(x_1, \ldots, x_{2k+1}) \in (\mathbb{P}^1)^{2k+1} | \sup_{x \in \mathbb{P}} \#\{x_i = x\} \leq k\}/SL(2,\mathbb{C}). \]
It is well-known that for $k = 2$, $Y$ is a fourth del Pezzo surface $dP_4$, that is, the blow-up of $\mathbb{P}^2$ at 4 points. Indeed, the quotient is rational, Fano by the git construction, and any number of techniques (for example, Kirwan [33]) show that the second Betti number of $Y$ is 5.

In order to apply our results we realize $Y$ as a quotient of a vector space $X$. The product $(\mathbb{P}^1)^{2k+1}$ is the git quotient of $X = \mathbb{C}^{4k+2}$ by the diagonal action of $(\mathbb{C}^*)^{2k+1}$. Thus,

$$Y = X//G, \quad X = \mathbb{C}^{4k+2}, \quad G = (\mathbb{C}^*)^{2k+1} \times SL(2, \mathbb{C}).$$

For any map $\mathbb{P} \to X//G$ of class $(d_1, \ldots ,d_{2k+1})$, the moment map $(\Phi, d)$ is bounded from above. By Remark 5.2.2, abelianization applies. Since $X$ is $G$-equivariantly monotone the quantum Kirwan map is the identity on $QH^-_G(X)$ for reasons of dimension by Remark 4.1.1. By (38) the localized graph potential is given by

$$\tau_{X//G,-} = \kappa_{X,G}^{class} \circ \tau_{X,T,-} \circ \gamma^G_T$$

after restriction to $QH^{-2}_G(X)$. The maximal torus of $G = (\mathbb{C}^*)^{2k+1} \times SL(2, \mathbb{C})$ is $T = (\mathbb{C}^*)^{2k+2}$. We have

$$H^2_G(X) \cong H^2((\mathbb{C}^*)^{2k+1})(X) \cong \mathbb{Z}^{2k+1}, H^2_T(X) \cong H^2((\mathbb{C}^*)^{2k+1})(X) \cong \mathbb{Z}^{2k+2}.$$

The weights for the $T$-action on $X$ are written in terms of the standard basis $\epsilon_1, \ldots , \epsilon_{2k+2}$

$$\epsilon_1 + \epsilon_{2k+2}, \epsilon_1 - \epsilon_{2k+2}, \epsilon_2 + \epsilon_{2k+2}, \epsilon_2 - \epsilon_{2k+2}, \ldots \epsilon_{2k+1} - \epsilon_{2k+2}.$$ 

Let $\theta_1, \ldots , \theta_{2k+2} \in H^2_G(X)$ denote the generators corresponding to the splitting $T = (\mathbb{C}^*)^{2k+2}$. For any $2k + 2$-tuple of non-negative integers $d = (d_1, \ldots ,d_{2k+2})$, $\theta = \sum c_i \theta_i$ with $c_i \in \mathbb{Z}$, define

$$\Delta_d(\theta) := \frac{\prod_{j=-\infty}^{\theta_d}(\theta + lh)}{\prod_{\theta_d=-\infty}(\theta + lh)}$$

The localized potential $\tau_{X,T,-}$ has restriction to $QH^{-2}(X//G) \subset QH^-_G(X) \subset QH^{-2}_T(X)$ given by

$$\tau_{X,T,-} : QH^{-2}(X//G) \to QH^{-2}(X//G)[[\hbar^{-1}]]$$

$$\left(t_0 + t_1 \theta_1 + \ldots + t_{2k+2} \theta_{2k+2}\right) \mapsto e^{t_0 + (t_1 \theta_1 + \ldots + t_{2k+2} \theta_{2k+2})/\hbar} \sum_d q^d \tau_{X,T,-}(d)$$

where

$$\tau_{X,T,-}(d) := e^{\sum_{j=1}^{2k+2} d_j t_j} \frac{\Delta_d(2\theta_{2k+2}) \Delta_d(-2\theta_{2k+2})}{\Delta_d(\theta_1 + \theta_{2k+2}) \Delta_d(\theta_1 - \theta_{2k+2}) \ldots \Delta_d(\theta_{2k+1} - \theta_{2k+2})}.$$
to make these formulas explicit. Note that the cohomology ring of the moduli spaces \((\mathbb{P}^1)^{2n+1}/\text{SL}(2, \mathbb{C})\) is described in Hausmann-Knutson [28]; we do not know how to use the qde solution above to describe the quantization of the classical relations described in [28].

5.3. The example of the moduli of framed sheaves. In this subsection we apply the results to give a formula for a twisted localized graph potential (a qde solution) related to the moduli space \(\mathcal{M}_{r,k}(\chi)\) of framed sheaves on \(\mathbb{P}^2\) of rank \(r\) and second Chern number \(k\). More precisely, we give an explicit formula for the solution for the qde for Nakajima’s desingularization \(\mathcal{M}_{r,k}(\chi)\) of \(\mathcal{M}_{r,k}\) assuming that the Gromov-Witten theory of moduli \(\mathcal{M}_{r,k}(\chi)\) of representations of the ADHM quiver with relations is equivalent to the twisted Gromov-Witten theory of the ADHM quiver without relations. In the case \(k = 1\) the space \(\mathcal{M}_{r,k}(\chi)\) is the Hilbert scheme \(\text{Hilb}_k(\mathbb{C}^2)\) of 0-dimensional length \(k\) subschemes of \(\mathbb{C}^2\). In this case our results are somewhat weaker than announced results of Ciocan-Fontanine-Diaconescu-Kim-Maulik, see [34], which compute the qde solution without the assumption; we hope that the cosection localization method introduced by Chang-Li [36] might allow us to strengthen the results in the future. The quantum cohomology of \(\mathcal{M}_{r,k}(\chi)\) is also the subject of work by Maulik-Okounkov [37].

We recall the construction of the moduli space of framed sheaves from Nakajima’s lectures [45, Chapter 3]. Let \(\ell_\infty \subset \mathbb{P}^2\) denote the divisor at infinity. Recall

\[
\mathcal{M}_{r,k} = \left\{ (E, \Phi) \left| \begin{array}{l} E: \text{torsion free sheaf on } \mathbb{P}^2 \\
\text{rank}(E) = r, c_2(E) = k \\
\Phi: E|_{\ell_\infty} \to \mathcal{O}_{\ell_\infty}^{\oplus r}: \text{framing at infinity} \end{array} \right. \right\} / \text{isomorphism}.
\]

According to the Atiyah-Drinfeld-Hitchin-Manin description of the moduli space, there exists an isomorphism

\[
\mathcal{M}_{r,k} \cong \left\{ (B_1, B_2, i, j) \left| \begin{array}{l} [B_1, B_2] + ij = 0 \\
\text{there exists no subspace } S \subset \mathbb{C}^n \text{ such that } B_l(S) \subset S \\
(l = 1, 2) \text{ and im}(i) \subset S \end{array} \right. \right\} / \text{GL}_n(\mathbb{C}),
\]

where \(B_1, B_2 \in \text{End}(\mathbb{C}^k), i \in \text{Hom}(\mathbb{C}^r, \mathbb{C}^k)\) and \(j \in \text{Hom}(\mathbb{C}^k, \mathbb{C}^r)\) with the action given by

\[
g \cdot (B_1, B_2, i, j) = (gB_1g^{-1}, gB_2g^{-1}, gi, jg^{-1}).
\]

The data \((B_1, B_2, i, j)\) forms a representation of the quiver shown in Figure 1.

The group \(S = (\mathbb{C}^\times)^2\) acts equivariantly on

\[
X_{r,k} := \text{End}(\mathbb{C}^k)^{\oplus 2} \oplus \text{Hom}(\mathbb{C}^r, \mathbb{C}^k) \oplus \text{Hom}(\mathbb{C}^k, \mathbb{C}^r)
\]

by

\[
(g_1, g_2)(B_1, B_2, i, j) = (g_1B_1, g_2B_2, i, g_1g_2j).
\]

This preserves the locus

\[
Z_{r,k} = \{(B_1, B_2, i, j) | [B_1, B_2] + ij = 0 \}
\]
and induces an $S$-action on the quotient $M_{r,k}$. In the case $k = 1$, $M_{r,k}$ is canonically isomorphic to the Hilbert scheme $\text{Hilb}_k(C^2)$ and the $S$-action is the one induced from the $S$-action on $C^2$.

In order to do Gromov-Witten theory we replace $M_{r,k}$ with a smooth scheme. For any character $\chi \in \text{Hom}(GL_k(C), C^\times) \cong \mathbb{Z}$, let $M_{r,k}(\chi)$ denote Nakajima’s desingularization of $M_{r,k}$ given by

$$M_{r,k}(\chi) = Z_{r,k}/\chi GL_k(C) \subset X_{r,k}/\chi GL_k(C)$$

where $/\chi$ denotes the $\chi$-shifted geometric invariant theory quotient. Let $T_k \subset GL_k(C)$ denote the diagonal maximal torus. We consider the twisted Gromov-Witten theory of $X_{r,k}/\chi GL_k$ corresponding to the relation defining $Z_{r,k}$, that is, twisted by the Euler class of the index bundle of $\text{End}(C^k)$. Although $X_{r,k}/\chi GL_k$ is non-compact, the fixed points of the $S$-action are compact:

**Lemma 5.3.1.** (Properness of $S$-fixed loci)

(a) The $S$-action on the moduli of parametrized stable maps $\overline{M}_n(C, X_{r,k}/\chi GL_k, d)$ has proper $S$-fixed loci for any homology class $d \in H_2^{GL_k}(X_{r,k})$.

(b) The $S$-action on the moduli space of gauged maps $\overline{M}_n^{GL_k}(C, X_{r,k}, d)$ has proper $S$-fixed loci for any homology class $d \in H_2^{GL_k}(X_{r,k})$.

(c) The $S$-action on the moduli space of scaled gauged maps $\overline{M}_{n,1}^{GL_k}(C, X_{r,k}, d)$ from [59] has proper $S$-fixed loci for any homology class $d \in H_2^{GL_n}(X_{r,k})$.

**Proof.** (a) By the theory of symplectic resolutions discussed in [19], any stable map defines a parametrized stable map to the corresponding affine quotient $X_{r,k}/\chi GL_k$ by composition with the proper morphism $X_{r,k}/\chi GL_k \to X_{r,k}/GL_k$. The latter is affine with compact $S$-fixed loci, hence any $S$-fixed stable map in $X_{r,k}/\chi GL_k$ projects to an $S$-fixed point in $X_{r,k}/GL_k$. Since the inverse image is proper, the claim follows. (b) is Proposition 3.5 in Diaconescu [17]. However for the purposes
of proving (c) we give a different proof. Let \((P, u)\) be an object of the fixed point substack \(\mathcal{M}^{GL_k}(C, X_{r,k}, d)^S\). Thus \(P \to C\) is a \(G\)-bundle, \(u : C \to P \times_G X\) is a section, and there exists a homomorphism

\[ \varphi : S \to \text{Aut}(P, u) \subset \text{Aut}(P) \]

such that \(su = g(s)u\). After trivializing \(\varphi\) at a base point \(\varphi\) defines a homomorphism still denoted \(\varphi\) from \(S\) to \(G\). Let \(G_\varphi\) denote the centralizer of \(\varphi\). Then \(P\) admits a reduction of structure group \(P_{\varphi} \subset P\) to \(G_\varphi \subset G\) and each \(\varphi(s)\) defines an automorphism of the associated fiber bundle \(P(X)\), so that \(u\) takes values in the fixed point locus \(P(X)^{\varphi} = P(X^{\varphi})\). The fixed point locus of \(S\) on \(X\) defined by the homomorphism \(\varphi\) is

\[ X^{\varphi} = \left\{ (B_1, B_2, i, j) \mid \text{Ad}(\varphi(s_1, s_2))B_l = s_lB_l, l = 1, 2, \varphi(s_1, s_2)i = i, j\varphi(s_1, s_2)^{-1} = s_1s_2j \right\}. \]

Under the action of \(\varphi(S)\) the subspace \(X^{\varphi}\) splits into a sum of subspaces with weights \((1,0),(0,1),(0,0),(1,1)\). When combined with the action of the central \(\mathbb{C}^x \subset GL_k\), this shows the existence of a central abelian three-parameter subgroup whose action on \(X^{\varphi}\) has weights contained in an open half-space. It follows that \(X^{\varphi}\) is convex at infinity in the sense of Cieliebak et al [9], and so \(\mathcal{M}^{GL_k}(C, X^{\varphi}, d)\) is compact for any class \(d \in H^G_2(X)\). Since any fixed point component arises in this way, \(\mathcal{M}^{GL_k}(C, X, d)^S\) is compact. The argument for (c) is similar, using that any continuous family of \(S\)-fixed vortices with varying vortex parameter takes values in \(X^{\varphi}\) for any homomorphism \(\varphi\).

By the properness results in Lemma 5.3.1, the abelianization argument goes through as before for gauged potentials defined via localization at fixed point loci of the \(S\)-action. (See [27] for a discussion of classical abelianization.) The abelianization theorem relates the twisted potential on \(X_{r,k}/\mathbb{C} GL_k\) to the twisted localized gauged potential for the torus action \(\tau_{X_{r,k}, T_n}\), given by the following explicit formula of Ciocan-Fontanine-Diaconescu-Kim-Maulik, see [34]. For any \(n\)-tuple of non-negative integers \(\underline{d} = (d_1, \ldots, d_k)\), \(\theta = \sum c_i\theta_i\) with \(c_i \in \mathbb{Z}\), define

\[ \Delta_{\underline{d}}(\theta, w) := \frac{\prod_{l=\theta}^{d-\theta} \theta + lh}{\prod_{l=-\infty}^{\theta} \theta + lh}. \]

The twisted localized gauged potential for the \(T_k\) action on \(X_{r,k}\) has restriction to \(QH^{\leq 2}_F(X_{r,k})\) given by (cf. [34])

\[ \pi^G_{\mathbb{C} GL_k} \tau_{X_{r,k}, T_k, -} = e^{t_1\theta_1 + \ldots + t_k\theta_k}/h \sum_{d \geq 1} q^d \sum_{\underline{d} d_1 + \ldots + d_k = d} \tau_{X_{r,k}, T_k, -}(\underline{d}) \]

where

\[ \tau_{X_{r,k}, T_k, -}(\underline{d}) = \prod_{i \neq j} \frac{\Delta_{\underline{d}}(\theta_i - \theta_j, \xi_1 + \xi_2)\Delta_{\underline{d}}(\theta_i - \theta_j, 0)}{\Delta_{\underline{d}}(\theta_i - \theta_j, \xi_1)\Delta_{\underline{d}}(\theta_i - \theta_j, \xi_2)} \prod_{i = 1}^k \frac{1}{\Delta_{\underline{d}}(\theta_i, 0)^r \Delta_{\underline{d}}(-\theta_i, \xi_1 + \xi_2)^r} \]
and $\xi_1, \xi_2$ are the equivariant parameters for $S$, that is, $H_S(pt) = \mathbb{Q}[\xi_1, \xi_2]$. To determine the quantum Kirwan map recall from e.g. [16, p. 359] that the coordinate transformation relating the $I$ and $J$-functions is uniquely determined from the leading order terms of the $I$-function. In this context, $\tau_{X/G, -}$ is the $J$-function. The $\mathbb{C}^\times$-equivariant quantum Kirwan map $\kappa_{X,G}$ has the form on $QH_G^{\leq 2}(X) \cong QH^{\leq 2}(X\//G)$

$$t_0 \mapsto t_0 + f(qe^t)h + h(qe^t), \quad t \mapsto t + g(qe^t), \quad t = (t_1, \ldots, t_k)$$

for some functions $f, g, h$. Indeed by the Calabi-Yau condition $\kappa_{X,G}$ maps $QH_G^{\leq 2}(X)$ to $QH_G^{\leq 2}(X)[[h]]$ and by the divisor equation the higher order corrections are determined by $\kappa_{X,G}$, by requiring that the map is a power series in $qe^t$. Denote by

$$\gamma_k := \sum_{i=1}^{k} \frac{(\theta_i - (\xi_1 + \xi_2)) \prod_{j \neq i} (\theta_i - \theta_j - \xi_1)(\theta_i - \theta_j - \xi_2)}{(\theta_i - \theta_j)(\theta_i - \theta_j - (\xi_1 + \xi_2))}$$

**Theorem 5.3.2.** The twisted localized graph potential of $X_{r,k}\//GL_n$ restricted to $QH^{\leq 2}_{GL_n}(X_{r,k})$ is

$$\tau_{X_{r,k}\//GL_n,-} = \kappa_{X_{r,k},T_k}^{\text{class}} \circ (1 + qe^t)^{(-\gamma_k/h)} \tau_{X_{r,k},T_k,-}.$$  

**Proof.** Since the quantum Kirwan map maps $QH_G^{\leq 2}(X)$ to $QH^{\leq 2}(X\//G)$ and is determined by the expansion in $1/h$, as in Givental [21]. As in Ciocan-Fontanine et al [34], it suffices to check that the terms of order $1/h$ match. The result follows.  

**Corollary 5.3.3.** If the twisted graph potential $\tau_{X_{r,k}\//GL_n,-}$ is equal to the graph potential $\tau_{\mathfrak{V},k}(\chi), -$ of the quiver with relation then $\tau_{\mathfrak{V},k}(\chi), - = (1+qe^t)^{(-\gamma_k/h)} \tau_{X_{r,k},T_k,-}.$

We understand that unpublished work by Ciocan-Fontanine and Kim proves the conclusion unconditionally using a connection with Donaldson-Thomas theory.

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