AN EXTENSION PROCEDURE FOR THE CONSTRAINT EQUATIONS

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ABSTRACT. Let \((g, k)\) be a solution to the maximal constraint equations of general relativity on the unit ball \(B_1\) of \(\mathbb{R}^3\). We prove that if \((g, k)\) is sufficiently close to the initial data for Minkowski space, then there exists an asymptotically flat solution \((g', k')\) on \(\mathbb{R}^3\) that extends \((g, k)\). Moreover, \((g', k')\) depends continuously on \((g, k)\) and has the same regularity.

Our proof uses a new method of solving the prescribed divergence equation for a tracefree symmetric 2-tensor, and a geometric variant of the conformal method to solve the prescribed scalar curvature equation for a metric. Both methods are based on the implicit function theorem and an expansion of tensors based on spherical harmonics. They are combined to define an iterative scheme that is shown to converge to a global solution \((g', k')\) of the maximal constraint equations which extends \((g, k)\).

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1. Introduction

1.1. The Cauchy problem and the maximal constraint equations. The Einstein vacuum equations on a Lorentzian 4-manifold \((M, g)\) are given by

\[
\text{Ric}(g) = 0,
\]

where Ric denotes the Ricci tensor of \(g\). Initial data for the Cauchy problem is given by a triple \((\Sigma, g, k)\), where \((\Sigma, g)\) is a complete Riemannian 3-manifold and \(k\) a symmetric 2-tensor on \(\Sigma\) satisfying the constraint equations on \(\Sigma\)

\[
R(g) = |k|_g^2 - (\text{tr}_g k)^2,
\]

\[
\text{div}_g k = d(\text{tr}_g k).
\]  

(1.1)

Here \(R(g)\) denotes the scalar curvature of \(g\), \(d\) is the exterior derivative and

\[
|k|_g^2 := g^{ij}g^{lm}k_{il}k_{jm}, \quad \text{tr}_g k := g^{ij}k_{ij}, \quad (\text{div}_g k)_l := g^{ij}\nabla_i k_{jl},
\]

where \(\nabla\) denotes the covariant derivative on \((\Sigma, g)\) and we use, as in the rest of this paper, the Einstein summation convention.

Let \((M, g)\) be the solution of the Einstein vacuum equations corresponding to initial data \((\Sigma, g, k)\). Then \(\Sigma \subset (M, g)\) is a space-like Cauchy hypersurface with induced metric \(g\) and second fundamental form \(k\). See for example [31] for details.
The trivial solution to the Einstein vacuum equations is the Minkowski spacetime. Its initial data is given by
\[(\Sigma, g, k) = (\mathbb{R}^3, e, 0),\]
where \(e\) denotes the Euclidean metric.

In this work, we consider initial data that satisfies two further properties.

- The initial data is asymptotically flat, which means
  \[g \rightarrow e, \ k \rightarrow 0\]
as \(|x| \rightarrow \infty\) on \(\Sigma\). For a more precise definition, see Definition 2.20. Such initial data corresponds to the description of isolated gravitational systems, see for example [31].
- We assume that \(\Sigma\) is maximal, that is,
  \[\text{tr}_g k = 0.\]

This assumption is sufficiently general, see for example [4].

By the second assumption, the constraint equations (1.1) reduce to the maximal constraint equations
\[R(g) = |k|^2,\]
\[\text{div}_g k = 0,\]
\[\text{tr}_g k = 0.\] (1.2)

1.2. The extension problem and the main theorem. The maximal constraint equations are an under-determined geometric non-linear elliptic system of partial differential equations. In this paper, we are interested in the following problem.

Extension problem. Given initial data \((g, k)\) on the unit ball \(B_1 \subset \mathbb{R}^3\), does there exist a regular asymptotically flat initial data set \((g', k')\) on \(\mathbb{R}^3\) that isometrically contains \((g, k)\) and continuously depends on it?

This problem has received considerable attention in the literature. It appears for example
- when analysing the space of solutions to the maximal constraint equations, see for example [5] [29] [26] [18] [28] [25],
- when considering the rigidity of the equations, as in the celebrated gluing construction [12] [13], see also [10] [11],
- in the context of Bartnik’s conjecture [6] [7], see for example [17] [22] [27] [2] [3]
- in the proof of the bounded \(L^2\) curvature theorem [20], where it is used to reduce the local existence for the Cauchy problem of general relativity to the small data case, see for example Section 2.3 in that paper.
Our main motivation to consider the extension problem is to prove a localised version of the bounded $L^2$ curvature theorem of [20], see the forthcoming publication [15].

In this paper, we resolve the extension problem for small data. The next theorem is a rough version of our main result, see Theorem 3.1 for a precise formulation.

**Theorem 1.1** (Main theorem, version 1). Let $(g, k)$ be a solution on the unit ball $B_1 \subset \mathbb{R}^3$ of the maximal constraint equations

\[
R(g) = |k|^2_g,
\]
\[
\text{div}_g k = 0,
\]
\[
\text{tr}_g k = 0.
\]

If $(g, k)$ is sufficiently close to $(e, 0)$ in a suitable topology, then there exists asymptotically flat $(g', k')$ of the same regularity as $(g, k)$ such that

\[
(g', k')|_{B_1} = (g, k),
\]

and solving the maximal constraint equations on $\mathbb{R}^3$,

\[
R(g') = |k'|^2_{g'},
\]
\[
\text{div}_{g'} k' = 0,
\]
\[
\text{tr}_{g'} k' = 0.
\]

Moreover, $(g', k')$ depends continuously on $(g, k)$.

**Comments on the result.**

(1) The novelty of our result lies in the following facts.

- Compared to the gluing construction [12] [13], it does not need a gluing region. This feature is crucial for localising the bounded $L^2$ curvature theorem [20], see the forthcoming [15].

- The extension results in [29] [26] [28] [25] lose regularity across the boundary of the domain by using a parabolic equation to solve the prescribed scalar curvature equation. Our result, on the other hand, uses a geometric perturbation argument at the Euclidean metric that preserves regularity.

(2) The closeness of $(g, k)$ to $(e, 0)$ is measured in the topology corresponding to the space

\[
(g, k) \in \mathcal{H}^w(B_1) \times \mathcal{H}^{w-1}(B_1),
\]

where $\mathcal{H}^w(B_1)$ denotes a Sobolev space of tensors over $B_1$ corresponding to $w$ derivatives in $L^2$. We note that Theorem 1.1 applies for integers $w \geq 2$ (see the precise version Theorem 3.1) and therefore in particular for weak regularity. In view of the scaling of (1.2), we expect Theorem 1.1 to hold also for real numbers $w$ in the range $w > 3/2$.

(3) Theorem 1.1 completes the proof of the reduction step to small data in the proof of the bounded $L^2$ curvature theorem [20], see Section 2.3 in that paper.
(4) The methods used in the proof of Theorem 1.1 could be relevant to other problems such as, for example, solving the divergence equation in context of the Maxwell-Klein-Gordon and Euler equations, see for example [19] [23].

1.3. Strategy of the proof of the main theorem. In this section we sketch the proof of Theorem 1.1. The idea is to set up an iterative scheme consisting of pairs \((g_i, k_i)\) \(i \geq 1\) that extend \((g, k)\) from \(B_1\) to \(\mathbb{R}^3\). In general, the \((g_i, k_i)\) do not solve the maximal constraints (1.2) on \(\mathbb{R}^3\). However, by a fixpoint argument, we show that the sequence converges to a solution \((g', k')\) as \(i \to \infty\).

More precisely, let \((g, k)\) be small given initial data on \(B_1\), and assume we have already obtained \((g_i, k_i)\) for some \(i \geq 1\). We construct the next pair \((g_{i+1}, k_{i+1})\) by the following two steps.

- **Step A.** Given \((g_i, k_i)\) on \(\mathbb{R}^3\), construct \(g_{i+1}\) on \(\mathbb{R}^3\) such that
  \[
  g_{i+1}|_{B_1} = g, \\
  R(g_{i+1}) = |k_i|_{g_i}^2.
  \]

- **Step B.** Given \(g_{i+1}\) on \(\mathbb{R}^3\), construct \(k_{i+1}\) on \(\mathbb{R}^3\) such that
  \[
  k_{i+1}|_{B_1} = k, \\
  \text{div}_{g_{i+1}} k_{i+1} = 0, \\
  \text{tr}_{g_{i+1}} k_{i+1} = 0.
  \]

Step A and B rely on Theorems 1.3 and 1.2, respectively, to be introduced now.

**Theorem 1.2** (Extension of divergence-free tracefree symmetric 2-tensors, version 1). Let \(g'\) be an asymptotically flat Riemannian metric on \(\mathbb{R}^3\) and \(k\) a symmetric 2-tensor on \(B_1\) solving

\[
\begin{align*}
\text{div}_{g'} k &= 0, \\
\text{tr}_{g'} k &= 0.
\end{align*}
\]

If \(g'\) and \(k\) are sufficiently close to \(e\) and \(0\), respectively, in a suitable topology, then there exists an asymptotically flat symmetric 2-tensor \(k'\) on \(\mathbb{R}^3\) that extends \(k\), that is,

\[
  k'|_{B_1} = k
\]

and solves on \(\mathbb{R}^3\)

\[
\begin{align*}
\text{div}_{g'} k' &= 0, \\
\text{tr}_{g'} k' &= 0.
\end{align*}
\]

Moreover, \(k'\) depends continuously on \(k\).
**Theorem 1.3** (Metric extension theorem, version 1). Let $g$ be a Riemannian metric on $B_1$ and $R$ a scalar function on $\mathbb{R}^3$ such that

$$R|_{B_1} = R(g),$$

where $R(g)$ denotes the scalar curvature of $g$. If $g$ and $R$ are sufficiently close to $e$ and 0, respectively, then there exists an asymptotically flat Riemannian metric $g'$ on $\mathbb{R}^3$ such that

$$g'|_{B_1} = g,$$

and such that its scalar curvature on $\mathbb{R}^3$ is given by

$$R(g') = R.$$

Moreover, $g'$ depends continuously on $g$ and $R$.

Precise versions of the above are stated in Theorems 4.1 and 5.1, respectively. Both Theorems 1.2 and 1.3 are proved by the Implicit Function Theorem and showing the surjectivity of a linearisation of the corresponding operators at the Euclidean metric $e$. Concerning Theorem 1.2, we show in Section 4.3 that the operator

$$k \mapsto \rho := \operatorname{div}_e \left( \hat{k}^e \right)$$

is surjective. Here, for any symmetric 2-tensor $V$, we denote its tracefree part with respect to the Euclidean metric $e$ by

$$\hat{V}^e := V - \frac{1}{3} \operatorname{tr}_e(V)e.$$

Concerning Theorem 1.3, we show in Section 5.3 that the linearisation of the scalar curvature with respect to a suitable geometric variation is surjective.

The two proofs of surjectivity at the Euclidean metric use, among others, the following mathematical tools.

1. In Section 2.2, we decompose tensors with respect to the foliation of $\mathbb{R}^3$ by spheres $S_r = \{|x| = r\}, r > 0$.

2. In Section 2.7, relying on spherical harmonics, we define complete orthonormal bases of the space of $L^2(S_r)$-integrable functions, vectorfields and symmetric tracefree 2-tensors on $S_r$. We call these bases Hodge-Fourier bases. Projecting onto these bases allows us to split up the linearised operators into radial ODEs and elliptic systems on $S_r$ and $\mathbb{R}^3 \setminus \overline{B_1}$.

3. In order to force the continuity of the normal derivative at the boundary of $B_1$, it is necessary to control the Dirichlet-to-Neumann maps associated to the elliptic systems on $\mathbb{R}^3 \setminus \overline{B_1}$. This is achieved in particular by exploiting the underdetermined character of the constraint equations.
The rest of the paper is organised as follows. In Section 2, we introduce the notation and weighted Sobolev spaces and bases of functions and tensors. In Section 3 we state a precise version of Theorem 1.1. In Section 4, we first reduce the proof of Theorem 1.2 to the surjectivity at the Euclidean space which is then proved in Section 4.3. Similarly, in Section 5, we first reduce the proof of Theorem 1.3 to the surjectivity at the Euclidean space which is then proved in Section 5.3. In Section 6, we set up the iterative scheme and prove Theorem 1.1. In Appendix A, we prove the completeness of the bases of tensors defined in Section 2. Two lemmas of Section 2 are proved in Appendix B. In Appendix C, derive elliptic estimates in weighted Sobolev spaces.

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2. Preliminaries

2.1. Basic notation. In this work, minuscule Latin indices range over $a, b, c, d, i, j = 1, 2, 3$, majuscule Latin indices over $A, B, C, D = 1, 2$ and $n \in \mathbb{N}$. The index pairs $(l m)$ take as values integers $l \geq 0, m \in \{-l, \ldots, l\}$. We apply the Einstein summation convention. The notation $A \lesssim B$ means $A \leq cB$ where $c > 0$ is a numerical constant that does not depend on $A, B$.

An open subset $\Omega \subset \mathbb{R}^3$ is called a domain if it is connected and its boundary $\partial \Omega := \overline{\Omega} \setminus \Omega$ is smooth. Let $\chi : \mathbb{R} \to [0, 1]$ be a fixed smooth transition function such that

$$
\chi(x) = \begin{cases} 
0 & \text{for } x \leq 1/10, \\
1 & \text{for } x \geq 1.
\end{cases} \tag{2.1}
$$

We work in a fixed Cartesian coordinate system $(x^1, x^2, x^3)$ of $\mathbb{R}^3$. Consequently, given a $n$-tensor $T$, we can equivalently denote it by its coordinate components $T_{i_1 \ldots i_n}$.

Let $e$ denote the Euclidean metric on $\mathbb{R}^3$ with components

$$
e_{ij} = \begin{cases} 
1 & \text{if } i = j, \\
0 & \text{if } i \neq j.
\end{cases}
$$
Let $g$ be a Riemannian metric and $V$ a symmetric 2-tensor. Let the divergence, the symmetrized curl, the trace and the tracefree part of $V$ with respect to $g$ be

$$(\text{div}_g V)_j := \nabla^i V_{ij},$$

$$(\text{curl}_g V)_{ij} := \frac{1}{2}(\varepsilon_{ia} \nabla_a V_{bj} + \varepsilon_{ja} \nabla_a V_{bi}),$$

$$\text{tr}_g V := g^{ab} V_{ab},$$

$$\hat{V}^g := V - \frac{1}{3} \text{tr}_g(V) g,$$

where $\nabla$ denotes the covariant derivative and $\varepsilon$ the volume form of $g$.

2.2. **The radial foliation of $\mathbb{R}^3$ by spheres $S_r$ and tensor decomposition.** Let $$(r, \theta^1, \theta^2) \in [0, \infty) \times [0, \pi) \times [0, 2\pi)$$ denote the standard polar coordinates on $\mathbb{R}^3$. By definition they are related to the Cartesian coordinates $(x^1, x^2, x^3)$ by

$$x^1 = r \sin \theta^1 \cos \theta^2,$$

$$x^2 = r \sin \theta^1 \sin \theta^2,$$

$$x^3 = r \cos \theta^1.$$

The coordinate spheres and balls of radius $r_0 > 0$ centered at the origin are respectively defined as

$$S_{r_0} := \{x \in \mathbb{R}^3 : r(x) = r_0\},$$

$$B_{r_0} := \{x \in \mathbb{R}^3 : r(x) < r_0\}.$$

In standard polar coordinates, the Euclidean metric is given by

$$e = dr^2 + r^2((d\theta^1)^2 + \sin^2 \theta^1 (d\theta^2)^2).$$

Let the induced metric on $S_r \subset (\mathbb{R}^3, e)$ be denoted by

$$\gamma := r^2((d\theta^1)^2 + \sin^2 \theta^1 (d\theta^2)^2).$$

When integrating over $(S_r, \gamma)$, we do not write out the standard volume element.

The **standard polar frame** on $\mathbb{R}^3 \setminus \{0\}$ is defined as

$$\left\{ \partial_r, e_1 := \frac{1}{r} \partial_{\theta^1}, e_2 := \frac{1}{r \sin \theta^1} \partial_{\theta^2} \right\},$$

where $\partial_r, \partial_{\theta^1}, \partial_{\theta^2}$ are the coordinate vectorfields in the coordinate system $(r, \theta^1, \theta^2)$, respectively.
Every Riemannian metric $g$ on $\mathbb{R}^3 \setminus \{0\}$ can be uniquely written as

$$g = a^2 dr^2 + \gamma_{AB} (\beta^A dr + d\theta^A) (\beta^B dr + d\theta^B),$$

(2.3)

where

- $a(x) > 0$ for all $x \in \mathbb{R}^3 \setminus \{0\}$ is the positive lapse function,
- $\gamma$ is the Riemannian metric induced by $g$ on $S_r$, $r > 0$,
- $\beta$ is the $S_r$-tangent shift vector.

The $a, \gamma, \beta$ are called the polar components of $g$.

The following lemma is proved by direct calculation.

**Lemma 2.1.** Let $g$ be a Riemannian metric given on $\mathbb{R}^3 \setminus \{0\}$,

$$g = a^2 dr^2 + \gamma_{AB} (\beta^A dr + d\theta^A) (\beta^B dr + d\theta^B).$$

Then the following holds for any $r > 0$.

- The outward-pointing\(^1\) unit normal $N$ to $S_r$ with respect to $g$ is given by
  $$N = \frac{1}{a} \partial_r - \frac{1}{a} \beta.$$  

- The second fundamental form\(^2\) $\Theta$ of $S_r$ with respect to $g$ equals in any coordinates on $S_r$, $A, B = 1, 2$,
  $$\Theta_{AB} = -\frac{1}{2a} \partial_r (\gamma_{AB}) + \frac{1}{2a} (\mathcal{L}_\beta \gamma)_{AB},$$

(2.4)

where $\mathcal{L}$ denotes the Lie derivative on $S_r$.

**Remark 2.2.** The polar components of the Euclidean metric $e$ are

$$a = 1, \quad \beta = 0,$$

$$\gamma_{AB} = \gamma_{AB} = \begin{cases} r^2 & \text{if } A = B = 1, \\ r^2 \sin^2 \theta & \text{if } A = B = 2, \\ 0 & \text{if } A \neq B. \end{cases}$$

Furthermore, $N = \partial_r$ and

$$\gamma^{AB} \Theta_{AB} = -\frac{2}{r}, \quad |\Theta|^2_{\gamma} := \gamma^{AC} \gamma^{BD} \Theta_{AB} \Theta_{CD} = \frac{2}{r^2}.$$  

More generally, we now decompose vectorfields and symmetric 2-tensors on $\mathbb{R}^3 \setminus \overline{B}_1$ with respect to the foliation of $\mathbb{R}^3$ by spheres $S_r$. Given a vectorfield $X$, decompose it into

- the scalar function $X_N$,
- the $S_r$-tangent vectorfield $X_A = X_A$.

\(^1\)That is, pointing into the unbounded connected component of $\mathbb{R}^3 \setminus S_r$.

\(^2\)Here we use the sign convention that $\Theta(X, Y) := -g(\nabla Y N)$. 
where \( A = 1, 2 \) denote components in any frame on \( S_r \).

Given a symmetric 2-tensor \( V \), decompose it into

- the scalar function \( V_{NN} \),
- the \( S_r \)-tangent vectorfield \( (V^r)_A := V_{NA} \),
- the \( S_r \)-tangent 2-tensor \( V_{AB} := V_{AB} \),

where \( A, B = 1, 2 \) denote components in any frame on \( S_r \).

**Definition 2.3.** Let \( X \) be a \( S_r \)-tangent vectorfield and \( V \) a \( S_r \)-tangent symmetric 2-tensor on \( \mathbb{R}^3 \setminus \{0\} \). Define the \( S_r \)-tangential vectorfield \( \nabla_N X \) and symmetric 2-tensor \( \nabla_N V \), respectively, by

\[
(\nabla_N X)_a := (\Pi_{TS_r})_a^c \nabla_N X_c,
(\nabla_N V)_{ab} := (\Pi_{TS_r})_a^c (\Pi_{TS_r})_b^d \nabla_N V_{cd},
\]

where \( a, b = 1, 2, 3 \) and

\[
(\Pi_{TS_r})_{ij} := \delta_{ij} - N_i N_j
\]
denotes the projection onto \( TS_r \), where here \( \delta_{ij} \) is the Kronecker symbol.

### 2.3. Function spaces.

**Definition 2.4 (Sobolev spaces).** Let \( \Omega \subset \mathbb{R}^3 \) be a domain and \( w \geq 0 \) integer. Let \( H^w(\Omega) \) denote the standard Sobolev space

\[
H^w(\Omega) := \left\{ f \in L^2(\Omega) : \sum_{|\alpha| \leq w} \| \partial^\alpha f \|_{L^2(\Omega)} < \infty \right\}.
\]

Here \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3 \) is a multi-index and

\[
|\alpha| := \alpha_1 + \alpha_2 + \alpha_3, \quad \partial^\alpha := \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3}.
\]

**Definition 2.5.** Let \( \Omega \subset \mathbb{R}^3 \) be a domain and \( w \geq 0 \) an integer. Define \( H^w_{\text{loc}}(\Omega) \) as

\[
H^w_{\text{loc}}(\Omega) := \bigcap_{\Omega' \subset \subset \Omega} H^w(\Omega'),
\]

where \( \Omega' \subset \subset \Omega \) denotes all domains \( \Omega' \) such that \( \overline{\Omega'} \) is compact and \( \overline{\Omega} \subset \Omega \).

See for example \cite{1} for properties of the above function spaces. Our analysis of the constraint equations is set in the following weighted Sobolev spaces.

**Definition 2.6 (Weighted Sobolev spaces).** Let \( \Omega \subset \mathbb{R}^3 \) be a domain, \( w \geq 0 \) an integer and \( \delta \in \mathbb{R} \). Let

\[
H^w_\delta(\Omega) := \left\{ f \in H^w_{\text{loc}}(\Omega) : \sum_{|\beta| \leq w} \| (1 + r)^{-\delta - 3/2 + |\beta|} \partial^\beta f \|_{L^2(\Omega)} < +\infty \right\}.
\]
Furthermore, define
\[ H^w_\delta := H^w_\delta(\mathbb{R}^3). \]
For \( w \geq 0 \) integer and \( \delta \in \mathbb{R} \), \( H^w_\delta(\Omega) \) is a Hilbert space. The next three lemmas follow from Lemmas 2.1, 2.4 and 2.5 in [21].

**Lemma 2.7.** Let \( \delta, \delta_1, \delta_2 \in \mathbb{R}, w, w_1, w_2 \geq 0 \) integers and \( f \) a scalar function on \( \mathbb{R}^3 \). The following holds.

- If \( w \geq 1 \) and \( f \in H^w_\delta \), then \( \partial f \in H^{w-1}_\delta \).
- If \( 0 \leq w_1 \leq w_2 \) and \( \delta_1 \leq \delta_2 \), then \( H^{w_1}_{\delta_1} \subset H^{w_2}_{\delta_2} \).
- For \( w \geq 2 \), the space \( H^w_\delta \) continuously embeds into

\[
\left\{ f \in L^\infty_{\text{loc}}(\mathbb{R}^3) : \sum_{|\beta| \leq w-2} \sup_{\mathbb{R}^3} (1 + r)^{-|\beta|} |\partial^\beta f| < \infty \right\}.
\]

**Lemma 2.8.** Let \( w, w_1, w_2 \geq 0 \) be integers such that \( w \leq \min(w_1, w_2) \) and \( w \leq w_1 + w_2 - 2 \). Let further \( \delta_1, \delta_2 \in \mathbb{R} \). Then for any \( (u, v) \in H^{w_1}_{\delta_1} \times H^{w_2}_{\delta_2} \), it holds that \( uv \in H^{w_1}_{\delta_1 + \delta_2} \) and

\[
\|uv\|_{H^{w_1}_{\delta_1 + \delta_2}} \lesssim \|u\|_{H^{w_1}_{\delta_1}} \|v\|_{H^{w_2}_{\delta_2}}.
\]

**Lemma 2.9.** Let \( F : \mathbb{R} \to \mathbb{R} \) be a smooth function. Let the scalar function \( u \in H^{w_1}_{\delta_1} \) for an integer \( w_1 \geq 2 \) and \( \delta_1 < 0 \), and \( v \in H^{w_2}_{\delta_2} \) for an integer \( 0 \leq w_2 \leq w_1 \) and \( \delta_2 \in \mathbb{R} \). The following holds.

1. There exists a constant \( C = C \left( \|u\|_{H^{w_1}_{\delta_1}}, F \right) > 0 \) such that
   \[ \|F(u)v\|_{H^{w_2}_{\delta_2}} \leq C\|v\|_{H^{w_2}_{\delta_2}}. \]
2. For any sequence \( (u_n, v_n)_{n \in \mathbb{N}} \) such that \( (u_n, v_n) \to (u, v) \) in \( H^{w_1}_{\delta_1} \times H^{w_2}_{\delta_2} \) as \( n \to \infty \), it holds that
   \[ F(u_n)v_n \to F(u)v \text{ in } H^{w_2}_{\delta_2} \text{ as } n \to \infty, \]
   in other words, the map \( (u, v) \mapsto F(u)v \) is continuous.

The following two corollaries are used in Sections 4 and 5.

**Corollary 2.10.** For \( w \geq 2, \delta < 0 \), the space \( H^w_\delta \) forms an algebra.

The proof of Corollary 2.10 follows by Lemma 2.8 and is left to the reader.

**Corollary 2.11.** Let the scalar function \( F : \mathbb{R} \to \mathbb{R} \) be smooth in an open neighbourhood of 0. Let \( w_1 \geq 2 \) and \( 0 \leq w_2 \leq w_1 \) be integers, \( \delta_1 < 0, \delta_2 \in \mathbb{R} \). There is a constant \( \varepsilon > 0 \) such that the mapping

\[
(u, v) \mapsto F(u)v
\]
is smooth from \( B_\varepsilon(0) \times H^{w_2}_{\delta_2} \) to \( H^{w_2}_{\delta_2} \), where

\[
B_\varepsilon(0) := \left\{ u : \|u\|_{H^{w_1}_{\delta_1}} < \varepsilon \right\} \subset H^{w_1}_{\delta_1}.
\]
Moreover, for all functions
\[ u, \tilde{u} \in B_\varepsilon(0) \subset H^{w_1}_{\delta_1}, \ v \in H^{w_2}_{\delta_2} \]
it holds that
\[ \| (F(u) - F(\tilde{u}))v \|_{H^{w_2}_{\delta_2}} \lesssim \| u - \tilde{u} \|_{H^{w_1}_{\delta_1}} \| v \|_{H^{w_2}_{\delta_2}}, \quad (2.5) \]
\[ \| F(u) - F(\tilde{u}) \|_{H^{w_1}_{\delta_1}} \lesssim \| u - \tilde{u} \|_{H^{w_1}_{\delta_1}} \quad (2.6) \]

**Proof.** The existence of \( \varepsilon > 0 \) such that the mapping is smooth in \( u \in B_\varepsilon(0) \subset H^{w_1}_{\delta_1} \) follows by applying the \( L^\infty \)-estimate of Lemma 2.7 and Lemmas 2.8 and 2.9, Corollary 2.10 to derivatives of \( F(u)v \) with respect to \( u,v \).

It remains to prove the Lipschitz estimates (2.5) and (2.6). Indeed, for \( \varepsilon > 0 \) sufficiently small, by Lemmas 2.8 and 2.9 and the fact that \( \delta_1 < 0 \),
\[ \| (F(u) - F(\tilde{u}))v \|_{H^{w_2}_{\delta_2}} \leq \int_0^1 \| DF_{su+(1-s)\tilde{u}}(u - \tilde{u})v \|_{H^{w_2}_{\delta_2}} ds \]
\[ \leq \int_0^1 \| DF_{su+(1-s)\tilde{u}}(u - \tilde{u})v \|_{H^{w_2}_{\delta_2+\delta_1}} ds \]
\[ \leq \int_0^1 \| DF_{su+(1-s)\tilde{u}}(u - \tilde{u}) \|_{H^{w_2}_{\delta_2}} ds \| v \|_{H^{w_2}_{\delta_2}} \]
\[ \lesssim \| u - \tilde{u} \|_{H^{w_1}_{\delta_1}} \| v \|_{H^{w_2}_{\delta_2}}, \]
where we used that \( DF \) is smooth on \( B_\varepsilon(0) \) for \( \varepsilon > 0 \) small. The proof of (2.6) is similar and left to the reader. This concludes the proof of Corollary 2.11. \( \square \)

**Definition 2.12.** Let \( \Omega \subset \mathbb{R}^3 \) be a domain, \( w \geq 0 \) an integer and \( \delta \in \mathbb{R} \). Define \( \overline{H}^w_\delta(\Omega) \) to be the closure of \( C^\infty_c(\Omega) \) with respect to the norm \( \| \cdot \|_{H^w_\delta(\Omega)} \). Further, define
\[ \overline{H}^w_\delta := \overline{H}^w_\delta(\mathbb{R}^3 \setminus \overline{B_1}). \]

The following useful characterisation of \( \overline{H}^w_\delta \) is left to the reader, see for example Exercise 3 of Section 4.5 in [30].

**Proposition 2.13.** Let \( w \geq 2 \) be an integer, \( \delta \in \mathbb{R} \). Let \( u \in H^w_\delta(\mathbb{R}^3 \setminus \overline{B_1}) \). The following are equivalent.

1. The trivial extension of \( u \) to \( B_1 \) is regular, that is \( \overline{u} \in H^w_\delta \), where
   \[ \overline{u} = \begin{cases} u(x) & \text{if } x \in \mathbb{R}^3 \setminus \overline{B_1}, \\ 0 & \text{if } x \in B_1. \end{cases} \]
(2) For \( l = 0, \ldots, w - 1 \), it holds that in the trace sense,
\[ \partial^l u |_{r=1} = 0. \]

(3) It holds that \( u \in \overline{H}_\delta^w \).

In dimension 1, the following Sobolev embedding holds. This is similar to Lemma 2.7 and its proof is left to the reader.

**Lemma 2.14.** Let \( \delta \in \mathbb{R} \). Let \( u : (1, \infty) \to \mathbb{R} \) be a scalar function. If
\[
\int_1^\infty (1 + r)^{-2\delta-1} u^2(r)dr, \int_1^\infty (1 + r)^{-2\delta+1} (\partial_r u)^2 (r)dr, \int_1^\infty (1 + r)^{-2\delta+3} (\partial_r^2 u)^2 (r)dr < +\infty,
\]
then \( u, \partial_r u \in C^0((1, \infty)) \) and
\[
\sup_{r \in (1, \infty)} (1 + r)^{-\delta} u(r), \sup_{r \in (1, \infty)} (1 + r)^{-\delta+1} \partial_r u(r) < +\infty.
\]

For functions on \((S_r, \gamma)\), we define the following norm.

**Definition 2.15.** Let \( w \geq 0 \) be an integer. Let \( f \) be a function on \( S_r \) for some \( r > 0 \).

Then
\[
\| f \|_{H^w(S_r)}^2 := \sum_{0 \leq n \leq w} \int_{S_r} |\nabla^n f|_\gamma^2,
\]
where \( \nabla \) denotes the covariant derivative on \((S_r, \gamma)\) and
\[
|\nabla^n f|_\gamma^2 = \gamma^{A_1B_1} \cdots \gamma^{A_nB_n} \nabla_{A_1} \cdots \nabla_{A_n} f \nabla_{B_1} \cdots \nabla_{B_n} f,
\]
see Definition 2.16. Denote further \( H^0(S_r) = L^2(S_r) \).

2.4. Tensor spaces. More generally, we now define tensor spaces on \( \mathbb{R}^3 \).

**Definition 2.16.** Given an \( n \)-tensor \( T \) and a Riemannian metric \( g \), let
\[
|T|^2_g := g^{i_1j_1} \cdots g^{i_nj_n} T_{i_1 \cdots i_n} T_{j_1 \cdots j_n}.
\]
In case of the Euclidean metric \( e \), for an \( n \)-tensor \( T \),
\[
|T|^2_e = \sum_{i_1, \ldots, i_n=1}^3 |T_{i_1 \cdots i_n}|^2.
\]

The norm of a tensor is defined as follows.

**Definition 2.17 (Tensor norms).** Let \( \Omega \subset \mathbb{R}^3 \) be a domain. Let \( n \geq 1 \) and \( w \geq 0 \) be integers. For an \( n \)-tensor \( T \) on \( \Omega \), define its \( H^w(\Omega) \)-norm by
\[
|T|_{\mathcal{H}^w(\Omega)}^2 := \sum_{|\alpha| \leq w} \int_{\Omega} |\partial^\alpha T|_e^2.
\]
where \((\partial^n T)_{i_1 \cdots i_n} = \partial^n (T_{i_1 \cdots i_n})\). We write \(T \in H^w(\Omega)\) if this norm is finite. We similarly define tensors in \(H^w_{\text{loc}}(\Omega), H^w_\delta(\Omega), H^w_\delta^\prime\) and \(H^w_\delta^\prime\).

We define tensor norms on \((S^r, \gamma)\) as follows.

**Definition 2.18.** Let \(w \geq 0\) be an integer. Let \(T\) be a \(S^r\)-tangent tensor on \((S^r, \gamma)\) for some \(r > 0\). Then

\[
\|T\|_{H^w(S^r)}^2 := \sum_{0 \leq n \leq w} \int_{S^r} |\nabla^n T|_{\gamma}^2,
\]

where \(\nabla\) denotes the covariant derivative on \((S^r, \gamma)\). We say that tensors in \(H^0(S^r)\) are \(L^2\)-integrable.

The next lemma is practical for calculations.

**Lemma 2.19.** Let \(w \geq 0\) be an integer. Let \(X\) be a vectorfield and \(V\) a symmetric 2-tensor on \(\mathbb{R}^3 \setminus B_1\). Then

\[
\|X\|_{H^w(\mathbb{R}^3 \setminus B_1)}^2 = \|X_N\|_{H^w(\mathbb{R}^3 \setminus B_1)}^2 + \|X\|_{H^w(\mathbb{R}^3 \setminus B_1)}^2
\]

\[
\approx \|X_N\|_{H^w(\mathbb{R}^3 \setminus B_1)}^2 + \sum_{n_1 + n_2 \leq w} \int_{S_r} |\nabla^{n_1} \nabla^{n_2} X_N|_{\gamma}^2 dr,
\]

\[
\|V\|_{H^w(\mathbb{R}^3 \setminus B_1)}^2 = \|V_{NN}\|_{H^w(\mathbb{R}^3 \setminus B_1)}^2 + \|V_N\|_{H^w(\mathbb{R}^3 \setminus B_1)}^2 + \|V\|_{H^w(\mathbb{R}^3 \setminus B_1)}^2
\]

\[
\approx \|V_{NN}\|_{H^w(\mathbb{R}^3 \setminus B_1)}^2 + \sum_{n_1 + n_2 \leq w} \int_{S_r} |\nabla^{n_1} \nabla^{n_2} V_N|_{\gamma}^2 dr
\]

\[
+ \sum_{n_1 + n_2 \leq w} \int_{S_r} |\nabla^{n_1} \nabla^{n_2} V|_{\gamma}^2 dr,
\]

where here \(\nabla\) denotes the tangential gradient and \(\nabla_N\) was defined in Definition 2.3. Analogously for \(H^w_{\text{loc}}(\mathbb{R}^3 \setminus B_1), H^w_\delta(\mathbb{R}^3 \setminus B_1), H^w_\delta^\prime\).

The proof of the above lemma is left to the reader. It follows by using that with the radial tensor decomposition of Section 2.2, it holds that for a vectorfield \(X\) and a symmetric 2-tensor \(V\),

\[
|X|_{e}^2 = X^2_N + |X|_{\gamma}^2,
\]

\[
|V|_{e}^2 = V^2_{NN} + |V_N|_{\gamma}^2 + |V|_{\gamma}^2.
\]
2.5. Asymptotically flat initial data. The following definition is standard, see for example [21].

**Definition 2.20** (Asymptotically flat initial data). Let $w \geq 2$ be an integer. Let $g \in \mathcal{H}^w_{\text{loc}}(\mathbb{R}^3)$ be a Riemannian metric and $k \in \mathcal{H}^{w-1}_{\text{loc}}(\mathbb{R}^3)$ a symmetric 2-tensor on $\mathbb{R}^3$. The metric $g$ is called $\mathcal{H}^{w}_{-1/2}$-asymptotically flat if

$$g - e \in \mathcal{H}^{w}_{-1/2},$$

where $e$ denotes the Euclidean metric on $\mathbb{R}^3$. The pair $(g, k)$ is called $\mathcal{H}^{w}_{-1/2}$-asymptotically flat if

$$g - e \in \mathcal{H}^{w}_{-1/2}, \quad k \in \mathcal{H}^{w-1}_{-3/2}.$$ \hfill (2.7)

Similarly, a metric $g$ on $\mathbb{R}^3 \setminus \overline{B}_1$ is called $\mathcal{H}^{w}_{-1/2}$-asymptotically flat if

$$g - e \in \mathcal{H}^{w}_{-1/2}(\mathbb{R}^3 \setminus \overline{B}_1).$$

**Remark 2.21.** The norms associated to (2.7) are explicitly

$$\sum_{i,j=1}^{3} \sum_{|\alpha| \leq w} \|(1 + r)^{-1+|\alpha|} \partial^\alpha (g_{ij} - e_{ij})\|_{L^2(\mathbb{R}^3)} < +\infty,$$

$$\sum_{i,j=1}^{3} \sum_{|\alpha| \leq w-1} \|(1 + r)^{|\alpha|} \partial^\alpha k_{ij}\|_{L^2(\mathbb{R}^3)} < +\infty.$$ 

The next lemma allows us to directly work with the polar components of an $\mathcal{H}^{w}_{-1/2}$-asymptotically flat metric.

**Lemma 2.22.** Let $w \geq 2$ be an integer. There exists a universal $\varepsilon > 0$ small such that the following holds.

1. Let $g$ be an $\mathcal{H}^{w}_{-1/2}$-asymptotically flat Riemannian metric on $\mathbb{R}^3 \setminus \overline{B}_1$ such that

$$\|g - e\|_{\mathcal{H}^{w}_{-1/2}(\mathbb{R}^3 \setminus \overline{B}_1)} < \varepsilon,$$

and denote its polar components on $\mathbb{R}^3 \setminus \overline{B}_1$ by

$$g = a^2 dr^2 + \gamma_{AB} \left( \beta^A dr + d\theta^A \right) \left( \beta^B dr + d\theta^B \right).$$

It holds that

$$a^2 - 1 \in H^{w}_{-1/2}(\mathbb{R}^3 \setminus \overline{B}_1), \quad \beta, \gamma - \bar{\gamma} \in \mathcal{H}^{w}_{-1/2}(\mathbb{R}^3 \setminus \overline{B}_1)$$

with the estimate

$$\|a^2 - 1\|_{H^{w}_{-1/2}(\mathbb{R}^3 \setminus \overline{B}_1)} + \|\beta\|_{\mathcal{H}^{w}_{-1/2}(\mathbb{R}^3 \setminus \overline{B}_1)} + \|\gamma - \bar{\gamma}\|_{\mathcal{H}^{w}_{-1/2}(\mathbb{R}^3 \setminus \overline{B}_1)} \lesssim \|g - e\|_{\mathcal{H}^{w}_{-1/2}(\mathbb{R}^3 \setminus \overline{B}_1)}.$$
(2) Let $a$ be a positive scalar function, $\beta$ a $S_r$-tangent vector field and a $\gamma$ Riemannian metric on $S_r$ on $\mathbb{R}^3 \setminus B_1$ such that
\[
\|a^2 - 1\|_{H_{w}^{1/2}(\mathbb{R}^3 \setminus B_1)} + \|\beta\|_{H_{w}^{1/2}(\mathbb{R}^3 \setminus B_1)} + \|\gamma - \tilde{\gamma}\|_{H_{w}^{1/2}(\mathbb{R}^3 \setminus B_1)} < \varepsilon.
\]
The symmetric 2-tensor $g$ defined on $\mathbb{R}^3 \setminus B_1$ by
\[
g = a^2 dr^2 + \gamma_{AB} (\beta^A dr + d\theta^A) (\beta^B dr + d\theta^B).
\]
is an $H_{w}^{1/2}$-asymptotically flat Riemannian metric and bounded by
\[
\|g - e\|_{H_{w}^{1/2}(\mathbb{R}^3 \setminus B_1)} \lesssim \|a^2 - 1\|_{H_{w}^{1/2}(\mathbb{R}^3 \setminus B_1)} + \|\beta\|_{H_{w}^{1/2}(\mathbb{R}^3 \setminus B_1)} + \|\gamma - \tilde{\gamma}\|_{H_{w}^{1/2}(\mathbb{R}^3 \setminus B_1)}.
\]

Proof. The proof follows from Lemma 2.19 applied to $g - e$. Indeed, in case of a metric,
\[
g_{NN} = a^2, \quad (g / N)_A = \gamma_{AB} \beta^B, \quad g / = \gamma.
\]
By Lemma 2.19 and Remark 2.2, we can bound
\[
\|a^2 - 1\|_{H_{w}^{1/2}(\mathbb{R}^3 \setminus B_1)}, \quad \|\gamma(\beta, \cdot)\|_{H_{w}^{1/2}(\mathbb{R}^3 \setminus B_1)}, \quad \|\gamma - \tilde{\gamma}\|_{H_{w}^{1/2}(\mathbb{R}^3 \setminus B_1)} \lesssim \|g - e\|_{H_{w}^{1/2}(\mathbb{R}^3 \setminus B_1)}.
\]
Moreover, for $\|\gamma - \tilde{\gamma}\|_{H_{w}^{1/2}(\mathbb{R}^3 \setminus B_1)}$ sufficiently small, $\gamma$ is invertible, and so the vectorfield $\beta$ is also controlled by Lemma 2.9. This proves part (1) of Lemma 2.22.

Part (2) is demonstrated similarly by Lemma 2.19 and left to the reader. This finishes the proof of Lemma 2.22.

2.6. $L^2$-Hodge theory on $S_r$. In this section, we recall basic Hodge theory on Euclidean spheres $(S_r, \tilde{\gamma})$, $r > 0$. This is a special case of the Hodge theory on Riemannian 2-spheres in [9]. All tensors are assumed to be $S_r$-tangent. Let

- $\nabla$ denote the covariant derivative on $(S_r, \tilde{\gamma})$.
- $\xi$ denote the volume element on $(S_r, \tilde{\gamma})$.
- $\Delta := \tilde{\gamma}^{-1} \nabla_A \nabla_B$ denote the Laplace-Beltrami operator on $(S_r, \tilde{\gamma})$.
- the divergence and curl of a vectorfield $X$ be defined as
  \[
  \text{div} \xi := \nabla_A X^A, \quad \text{curl} \xi := \xi_{AB} \nabla^A X^B.
  \]
- the divergence and trace of a symmetric 2-tensor $V_{AB}$ be defined as
  \[
  (\text{div} V)_B := \nabla^A V_{AB}, \quad \text{tr} V := \tilde{\gamma}^{-1} V_{AB}.
  \]

Here we follow the convention that Laplacians have negative eigenvalues.
• for a vectorfield $X$ the symmetric 2-tensor $\nabla \hat{\otimes} X$ be defined as
  $$(\nabla \hat{\otimes} X)_{AB} := \nabla_A X_B + \nabla_B X_A - (\text{div} X) \hat{\gamma}^*_{AB}.$$  
• for two vectorfields $X, Y$ the symmetric tracefree 2-tensor $X \hat{\otimes} Y$ be defined as
  $$(X \hat{\otimes} Y)_{AB} := X_A Y_B + X_B Y_A - \hat{\gamma}(X, Y) \hat{\gamma}^*_{AB}.$$  
• the left Hodge dual of a vectorfield $X$ be defined as
  $$*X_A := \xi_{AB} X^B.$$  
• the left Hodge dual of a symmetric tracefree 2-tensor $V$ be defined as
  $$*V_{AB} := \xi_{AC} V^C.$$  
• the modulus of an $n$-tensor $V$ be defined as
  $$|V|^2 := \hat{\gamma}^* A_1 B_1 \cdots \hat{\gamma}^* A_n B_n V_{A_1 \cdots A_n} V_{B_1 \cdots B_n}.$$  

We note that for a vectorfield $X$ and a symmetric tracefree 2-tensor $V$,
$$*(\hat{\gamma}^* X) := -X, **(\hat{\gamma}^* V) := -V.$$ \tag{2.8}$$

Introduce two Hodge systems on $(S_r, \gamma)$ as follows. Let $X$ be a vectorfield on $S_r$ that verifies
$$\text{div} \hat{\gamma} X = f, \quad \text{curl} \hat{\gamma} X = f^*,$$ \hspace{1cm} (H1)
where $f, f^*$ are scalar functions on $S_r$.

Let $V$ be a tracefree symmetric 2-tensor on $S_r$ that verifies
$$\text{div} \hat{\gamma} V = F,$$ \hspace{1cm} (H2)
where $F$ is a 1-form on $S_r$.

The following is the Euclidean version of Proposition 2.2.1 in [9].

**Proposition 2.23** (Ellipticity of Hodge systems). The following holds.

- Assume that the vectorfield $X$ is a solution of $H_1$. Then
  $$\int_{S_r} \left( |\nabla X|^2 + \frac{1}{r^2} |X|^2 \right) = \int_{S_r} (|f|^2 + |f^*|^2).$$

- Assume that the symmetric tracefree 2-tensor $V$ is a solution of $H_2$. Then
  $$\int_{S_r} \left( |\nabla V|^2 + \frac{2}{r^2} |V|^2 \right) = 2 \int_{S_r} |F|^2.$$  

Furthermore, the next higher regularity estimates hold.
Proposition 2.24 (Higher regularity for Hodge systems on $S_r$). Let $w \geq 1$ be an integer. The following holds.

- Assume that the vectorfield $X$ is a solution of $H_1$ for $f, f^* \in H^{w-1}(S_r)$. Then
  $$\sum_{0 \leq n \leq w} \int_{S_r} |r^n \nabla^n X|^2 \lesssim \sum_{0 \leq n \leq w-1} \int_{S_r} r^2 (|r^n \nabla^n f|^2 + |r^n \nabla^n f^*|^2).$$

- Assume that the symmetric tracefree 2-tensor $V$ is a solution of $H_2$ for $F \in \mathcal{H}^{w-1}(S_r)$. Then
  $$\sum_{0 \leq n \leq w} \int_{S_r} |r^n \nabla^n V|^2 \lesssim \sum_{0 \leq n \leq w-1} \int_{S_r} r^2 |r^n \nabla^n F|^2.$$

Proof. We only give a sketch of the proof, because it follows from Lemmas 2.2.2 and 2.2.3 in [9] and the fact that we work on the round sphere $(S_r, \gamma)$. The proof is by induction on $w$. The case $w = 1$ is Proposition 2.23. The induction step $w \rightarrow w + 1$ follows by showing that the symmetrized derivative of a totally symmetric tensor $\xi$,

$$\hat{D}_{A_1 A_2 ... A_{k+1}} := \frac{1}{k+2} \left( \nabla_B \xi_{A_1 ... A_{k+1}} + \sum_{i=1}^{k+1} \nabla_{A_i} \xi_{A_1 ... B ... A_{k+1}} \right)$$

satisfies a Hodge system whose source terms can be controlled\(^4\) in lower order norms of $\xi$. Lemma 2.2.2 in [9] shows the ellipticity of this Hodge system. Generally on $S_r$, the symmetrized derivative and the curl of a tensor control the full covariant derivative, see Chapter 2 of [9]. The curl is estimated via the Hodge system by the induction assumption so that the full control of $\nabla \xi$ follows. This finishes the proof of Proposition 2.24. \qed

The following relations are from Chapter 2 in [9]:

Lemma 2.25. Let $\mathcal{D}_1$ be the operator that takes a vectorfield $X$ on $S_r$ into the pair of functions $(\text{div} \, \xi, \text{curl} \, \xi)$. The $L^2$-adjoint of $\mathcal{D}_1$ is the operator $\mathcal{D}_1^*$ which takes pairs of functions $(f, f^*)$ into vectorfields on $S_r$ given by

$$\mathcal{D}_1^*(f, f^*) = -\nabla^A f + \epsilon^{AB} \nabla_B f^*.$$ 

Let $\mathcal{D}_2$ be the operator that takes a symmetric tracefree 2-tensor $X$ into the 1-form $\text{div} \, X$. The $L^2$-adjoint of $\mathcal{D}_2$ is $\mathcal{D}_2^*$ which takes 1-forms $F$ into symmetric tracefree 2-tensors given by

$$\mathcal{D}_2^* F = -\frac{1}{2} (\nabla \hat{\otimes} F)_{AB},$$

where we recall that

$$(\nabla \hat{\otimes} F)_{AB} := \nabla_A F_B + \nabla_B F_A - (\text{div} F) \gamma_{AB}.$$ 

\(^4\)Thereby it is used that in the Euclidean case, the Gauss curvature $K = 1/r^2$ is spherically symmetric, so in particular $\nabla K = 0$. 


The following relations hold.

\[ \mathcal{P}_1 \mathcal{P}_1^* = -\Delta , \]
\[ \mathcal{P}_2 \mathcal{P}_2^* = -\frac{1}{2} \Delta - \frac{1}{2} \frac{1}{r^2} , \]
\[ \mathcal{P}_1^* \mathcal{P}_1 = \frac{1}{2} \Delta + \frac{1}{r^2} , \]
\[ \mathcal{P}_2^* \mathcal{P}_2 = -\frac{1}{2} \Delta + \frac{1}{r^2} . \]

**Remark 2.26.** By the above, the kernel of \( \mathcal{P}_2^* \) can be identified with the conformal Killing vectorfields on \( (S_r, \hat{\gamma}) \). This implies that the image of \( \mathcal{P}_2 \) is \( L^2(S_r) \)-orthogonal to the conformal Killing vectorfields of \( (S_r, \hat{\gamma}) \).

2.7. The expansion of \( S_r \)-tangential tensors. In Sections 4.3 and 5.3, we analyse Hodge systems on Euclidean spheres. The main technical tools for this analysis are the bases of tensors defined here in the following. In this section, all differential operators are on Euclidean spheres \( (S_r, \hat{\gamma}) \) and all tensors are \( S_r \)-tangent.

- **Real spherical harmonics:** For \( r > 0 \), let
  \[ \left\{ Y^{lm}(r, \theta, \phi) : l \geq 0, m \in \{-l, \ldots, l\} \right\} \]
  denote the set of normalised real spherical harmonics on \( S_r \). In particular, for each \( l \geq 0, m \in \{-l, \ldots, l\} \) they solve
  \[ \Delta Y^{lm} = -\frac{l(l+1)}{r^2} Y^{lm} . \quad (2.9) \]

  The next lemma is standard, see for example [14].

**Lemma 2.27.** For each \( r > 0 \), the set

\[ \left\{ Y^{lm}(r) : l \geq 0, m \in \{-l, \ldots, l\} \right\} \]

forms a complete orthonormal basis of \( L^2(S_r) \)-integrable scalar functions on \( S_r \).

- **Vector spherical harmonics:** For \( r > 0 \), let the vectorfields \( E^{lm}, H^{lm} \) on \( S_r \) be defined for \( l \geq 1, m \in \{-l, \ldots, l\} \) by
  \[ E^{lm}(r) := \frac{r}{\sqrt{l(l+1)}} \mathcal{P}_1^* (Y^{lm}, 0) , \]
  \[ H^{lm}(r) := \frac{r}{\sqrt{l(l+1)}} \mathcal{P}_1^* (0, Y^{lm}) , \quad (2.10) \]

  where \( \mathcal{P}_1^* \) is given in Lemma 2.25.
• 2-covariant spherical harmonics: For $r > 0$, let the tracefree symmetric 2-tensors $\psi^{(lm)}$, $\phi^{(lm)}$ on $S_r$ be defined for $l \geq 2, m \in \{-l, \ldots, l\}$ by

$$
\psi^{(lm)}_{AB}(r) := \frac{r}{\sqrt{\frac{1}{2}l(l+1)-1}} \mathcal{P}_2^2 \left( E^{(lm)} \right),
$$

$$
\phi^{(lm)}_{AB}(r) := \frac{r}{\sqrt{\frac{1}{2}l(l+1)-1}} \mathcal{P}_2^2 \left( H^{(lm)} \right),
$$

(2.11)

where $\mathcal{P}_2^2$ is given in Lemma 2.25.

Remark 2.28. The tensors defined in (2.10) and (2.11) are spherical harmonics in the sense that by Lemma 2.25,

• for $l \geq 1, m \in \{-l, \ldots, l\}$,

$$
\triangle E^{(lm)} = \frac{1 - l(l + 1)}{r^2} E^{(lm)},
$$

$$
\triangle H^{(lm)} = \frac{1 - l(l + 1)}{r^2} H^{(lm)},
$$

• for $l \geq 2, m \in \{-l, \ldots, l\}$,

$$
\triangle \psi^{(lm)} = \frac{4 - l(l + 1)}{r^2} \psi^{(lm)},
$$

$$
\triangle \phi^{(lm)} = \frac{4 - l(l + 1)}{r^2} \phi^{(lm)}.
$$

The next proposition shows that these sets of tensors form complete orthonormal bases. First, we introduce some notation.

Definition 2.29. Let $r > 0$. Let $f$ be a scalar function, $X$ a vectorfield and $V$ a symmetric tracefree 2-tensor on $S_r$. Define then

• for $l \geq 0 : f^{(lm)}(r) := \int_{S_r} Y^{(lm)} f$,

• for $l \geq 1 : X_E^{(lm)}(r) := \int_{S_r} X \cdot E^{(lm)}, X_H^{(lm)}(r) := \int_{S_r} X \cdot H^{(lm)}$,

• for $l \geq 2 : V_{\psi}^{(lm)}(r) := \int_{S_r} V \cdot \psi^{(lm)}, V_{\phi}^{(lm)}(r) := \int_{S_r} V \cdot \phi^{(lm)}$,

where $\cdot$ denotes the contraction of tensors with respect to $\gamma$.  

Proposition 2.30. For all $r > 0$, the set

$$
\left\{ E^{(lm)}(r), H^{(lm)}(r) : l \geq 1, m \in \{-l, \ldots, l\} \right\}
$$

forms a complete orthonormal basis of the space of $L^2$-integrable vectorfields on $S_r$. For all $r > 0$, the set

$$
\left\{ \psi^{(lm)}(r), \phi^{(lm)}(r) : l \geq 2, m \in \{-l, \ldots, l\} \right\}
$$
forms a complete orthonormal basis of the set of $L^2$-integrable tracefree symmetric 2- tensors on $S_r$. Moreover,

- for any scalar function $f \in L^2(S_r)$,
  $$\|f\|_{L^2(S_r)}^2 = \sum_{l \geq 0} \sum_{m=-l}^l (f^{(lm)})^2,$$

- for any $S_r$-tangent vectorfield $X \in H^0(S_r)$,
  $$\|X\|_{H^0(S_r)}^2 = \sum_{l \geq 1} \sum_{m=-l}^l \left( \left( X^{(lm)}_E \right)^2 + \left( X^{(lm)}_H \right)^2 \right),$$

- for any $S_r$-tangent symmetric tracefree 2-tensor $V \in H^0(S_r)$,
  $$\|V\|_{H^0(S_r)}^2 = \sum_{l \geq 2} \sum_{m=-l}^l \left( \left( V^{(lm)}_\psi \right)^2 + \left( V^{(lm)}_\phi \right)^2 \right).$$

A proof is given in Appendix A.

**Remark 2.31.** For all $r > 0$, the vectorfields with $l = 1$,
$$\left\{ E^{(1m)}(r), H^{(1m)}(r) : m \in \{-1, 0, 1\} \right\},$$
form an orthonormal basis of the six-dimensional space of conformal Killing fields on $(S_r, \hat{\gamma})$.

The next expansion notation is used throughout Sections 4.3, 5.3 and Appendix C.

**Definition 2.32.** Let $f \in L^2(S_r)$ be a scalar function, $X \in H^0(S_r)$ a $S_r$-tangent vectorfield and $V \in H^0(S_r)$ a $S_r$-tangent tracefree symmetric 2-tensor. Denote
\[
\begin{align*}
f &= \underbrace{f^{(00)} Y^{(00)}}_{:= f^{[0]}} + \sum_{m=-1}^1 f^{(1m)} Y^{(1m)} + \sum_{l \geq 2} \sum_{m=-l}^l f^{(lm)} Y^{(lm)}, \\
X &= \underbrace{\sum_{m=-1}^1 X^{(1m)}_E E^{(lm)}}_{:= X^{[1]}_E} + \sum_{m=-1}^1 X^{(1m)}_H H^{(lm)} + \sum_{l \geq 2} \sum_{m=-l}^l X^{(lm)}_E E^{(lm)} + \sum_{l \geq 2} \sum_{m=-l}^l X^{(lm)}_H H^{(lm)}, \\
V &= \sum_{l \geq 2} \sum_{m=-l}^l V^{(lm)}_\psi \psi^{(lm)} + \sum_{l \geq 2} \sum_{m=-l}^l V^{(lm)}_\phi \phi^{(lm)}, \\
\end{align*}
\]
and let $X^{[1]} = X^{[1]}_E + X^{[1]}_H$, $X^{[\geq 2]} = X^{[\geq 2]}_E + X^{[\geq 2]}_H$. 

We have the following identities.

**Lemma 2.33** (Hodge-Fourier calculus). Let $f \in L^2(S_r)$ be a scalar function, $X \in \mathcal{H}^0(S_r)$ a vectorfield and $V \in \mathcal{H}^0(S_r)$ a symmetric tracefree 2-tensor. It holds that

- for $l \geq 1, m \in \{-l, \ldots, l\}$,
  
  $$-(\nabla f)^{(lm)}_E = \frac{\sqrt{l(l+1)}}{r} f^{(lm)}, \quad -(\nabla f)^{(lm)}_H = 0,$$
  $$\langle\partial f X \rangle^{(lm)}_E = \frac{\sqrt{l(l+1)}}{r} X^{(lm)}_E, \quad \langle\text{curlf} X \rangle^{(lm)}_H = \frac{\sqrt{l(l+1)}}{r} X^{(lm)}_H,$$

- for $l \geq 2, m \in \{-l, \ldots, l\}$,
  
  $$-\frac{1}{2} \langle\nabla \otimes X \rangle^{(lm)}_\psi = \frac{\sqrt{\frac{1}{2}l(l+1) - 1}}{r} X^{(lm)}_E, \quad -\frac{1}{2} \langle\nabla \otimes X \rangle^{(lm)}_\phi = \frac{\sqrt{\frac{1}{2}l(l+1) - 1}}{r} X^{(lm)}_H,$$
  $$\langle\partial f V \rangle^{(lm)}_E = \frac{\sqrt{\frac{1}{2}l(l+1) - 1}}{r} \psi^{(lm)}, \quad \langle\partial f V \rangle^{(lm)}_H = \frac{\sqrt{\frac{1}{2}l(l+1) - 1}}{r} \phi^{(lm)}.$$

The proof of this lemma follows by (2.10), (2.11), Lemma 2.25 and integration by parts. Details are left to the reader.

The next three results are handy for the estimates in Section 4.3 and 5.3.

**Proposition 2.34.** Let $u$ be a scalar function and $X$ a vectorfield on $S_r$ for some $r > 0$. For all integers $w \geq 0$,

$$\|\nabla^w u\|_{\mathcal{H}^0(S_r)}^2 \approx \sum_{l \geq 0} \sum_{m=-l}^{l} \left( \frac{l(l+1)}{r^2} \right)^w (u^{(lm)})^2,$$

$$\|\nabla^w X\|_{\mathcal{H}^0(S_r)}^2 \approx \sum_{l \geq 1} \sum_{m=-l}^{l} \left( \frac{l(l+1) - 1}{r^2} \right)^w \left( \left( X^{(lm)}_E \right)^2 + \left( X^{(lm)}_H \right)^2 \right).$$

A proof is provided in Appendix B.

**Lemma 2.35.** Let $w \geq 0$ be an integer and $r > 0$. The following holds.

1. Let $f$ be a scalar function. Then, for $l \geq 0, m \in \{-l, \ldots, l\}$,

   $$\left| \langle \partial_r^w f \rangle^{(lm)}_r \right| \lesssim \sum_{n=0}^w \frac{|\partial_r^{w-n} f^{(lm)}|}{r^n}.$$  \hspace{1cm} (2.12)
(2) Let $X$ be a $S_r$-tangent vector field. Then, for $l \geq 1$, $m \in \{-l, \ldots, l\},$

\[
\left| (\nabla^w_N X)^{(lm)}_E (r) \right| \lesssim \sum_{n=0}^{w} \frac{\left| \partial_r^{w-n} \left( X^{(lm)}_E \right) \right|}{r^n}, \tag{2.13}
\]

\[
\left| (\nabla^w_N X)^{(lm)}_H (r) \right| \lesssim \sum_{n=0}^{w} \frac{\left| \partial_r^{w-n} \left( X^{(lm)}_H \right) \right|}{r^n}. \tag{2.14}
\]

Moreover,

\[
\partial_r (r \div X) = r \div (\nabla_N X),
\]

\[
\partial_r (r \curl X) = r \curl (\nabla_N X). \tag{2.15}
\]

(3) Let $V$ be a $S_r$-tangent tracefree symmetric 2-tensor. Then, for $l \geq 2$, $m \in \{-l, \ldots, l\},$

\[
\left| (\nabla^w_N V)^{(lm)}_\psi (r) \right| \lesssim \sum_{n=0}^{w} \frac{\left| \partial_r^{w-n} \left( V^{(lm)}_\psi \right) \right|}{r^n}, \tag{2.16}
\]

Moreover,

\[
\nabla_N (r \div V) = r \div (\nabla_N V). \tag{2.17}
\]

A proof is provided in Appendix B.

**Lemma 2.36.** Let $w \geq 0$ be an integer. Let $f^{[\geq 1]}, f^{*[\geq 1]} \in H^w(S_r)$ be scalar functions and $X^{[\geq 2]} \in \mathcal{H}^w(S_r)$ a vector field. Then the inverse maps

\[
\mathcal{P}^{-1}_1 : (f^{[\geq 1]}, f^{*[\geq 1]}) \mapsto \mathcal{P}^{-1}_1(f^{[\geq 1]}, f^{*[\geq 1]}),
\]

\[
\mathcal{P}^{-1}_2 : (X^{[\geq 2]}) \mapsto \mathcal{P}^{-1}_2(X^{[\geq 2]}),
\]

into vector fields and tracefree symmetric 2-tensors, defined such that $\mathcal{P}^{-1}_1(f^{[\geq 1]}, f^{*[\geq 1]})$ and $\mathcal{P}^{-1}_2(X^{[\geq 2]})$ respectively solve on $S_r$

\[
\begin{align*}
\text{div} \mathcal{P}^{-1}_1(f^{[\geq 1]}, f^{*[\geq 1]}) &= f^{[\geq 1]}, \\
\text{curl} \mathcal{P}^{-1}_1(f^{[\geq 1]}, f^{*[\geq 1]}) &= f^{*[\geq 1]}, \\
\text{div} \mathcal{P}^{-1}_2(X^{[\geq 2]}) &= X^{[\geq 2]},
\end{align*}
\]

are well-defined and continuous maps into $\mathcal{H}^{w+1}(S_r)$, respectively. Moreover, for any scalar function $f^{[\geq 1]}$ on $S_r$,

\[
(\mathcal{P}^{-1}_1(f^{[\geq 1]}, 0))^{[\geq 1]}_H = 0, \quad (\mathcal{P}^{-1}_1(0, f^{[\geq 1]}))^{[\geq 1]}_H = 0,
\]

\[
(\mathcal{P}^{-1}_2(E^{(lm)}))^{[\geq 1]}_\phi = 0, \quad (\mathcal{P}^{-1}_2(H^{(lm)}))^{[\geq 1]}_\psi = 0.
\]
The above lemma is a consequence of Lemma 2.25, Remarks 2.26 and 2.31 and Propositions 2.23 and 2.24 and its proof is left to the reader.

2.8. The implicit function theorem and Lipschitz estimates for operators. For completeness we state the standard Implicit Function Theorem that is used in Sections 4.2 and 5.2, see for example Theorem 2.5.7 in [24] for a proof.

\textbf{Theorem 2.37.} Let \( X, Y, Z \) be Hilbert spaces. Let \( U \subset X, V \subset Y \) be open subsets and \( F : U \times V \to Z \) be a \( C^r \)-mapping, \( r \geq 1 \). For some \( x_0 \in U, y_0 \in V \) assume that the linearization in the first argument

\[ D_1 F|_{(x_0,y_0)} : X \to Z \]

is an isomorphism. Then there are open neighbourhoods \( V_0 \subset V \) of \( y_0 \) and \( W_0 \subset Z \) of \( F(x_0,y_0) \) and a unique \( C^r \)-mapping \( G : V_0 \times W_0 \to U \) such that for all \( (y,z) \in V_0 \times W_0 \),

\[ F(G(y,z),y) = z. \]

We also need the following lemma.

\textbf{Lemma 2.38.} Let \( X,Y,Z \) be Hilbert spaces. Let \( T : X \times Y \to Z \) be a \( C^r \)-mapping for \( r \geq 2 \) in an open neighbourhood of \( (0,0) \in X \times Y \) such that for all \( x \in X \),

\[ T(x,0) = 0. \]

There exists an \( \varepsilon > 0 \) such that the following holds.

- For \( (x,y) \in B_\varepsilon(0) \times B_\varepsilon(0) \subset X \times Y \) it holds that

\[ \|T(x,y)\|_Z \lesssim \|y\|_Y. \]

- For \( x,x' \in B_\varepsilon(0) \subset X \) and \( y \in B_\varepsilon(0) \subset Y \) it holds that

\[ \|T(x,y) - T(x',y)\|_Z \lesssim \|x - x'\|_X \|y\|_Y. \]

\textbf{Proof.} First,

\[ \|T(x,y)\|_Z = \|T(x,y) - T(x,0)\|_Z \]

\[ \leq \int_0^1 \|D_2 T|_{(x,ty)}(y)\|_Z dt \]

\[ \lesssim \|y\|_Y, \]
where we used that for $\varepsilon > 0$ small, the operator $T$ is $C^1$ on $B_\varepsilon(0) \times B_\varepsilon(0) \subset X \times Y$. Second,

$$
\|T(x, y) - T(x', y)\|_Z = \|T(x, y) - T(x, 0) - T(x', y) + T(x', 0)\|_Z
\leq \int_0^1 \|D_2 T|_{(x, ty)}(y) - D_2 T|_{(x', ty)}(y)\|_Z dt
\leq \int_0^1 \left( \int_0^1 \|D_1 D_2 T|_{(sx+(1-s)x', ty)}(y)(x - x')\|_Z ds \right) dt
\lesssim \|x - x'\|_X \|y\|_Y,
$$

where we used that for $\varepsilon > 0$ small, the operator $T$ is $C^2$ on $B_\varepsilon(0) \times B_\varepsilon(0) \subset X \times Y$. This finishes the proof of Lemma 2.38. \hfill \square

3. Precise statement of the main theorem

We are now in the position to state the precise version of our main theorem.

**Theorem 3.1** (Main theorem, version 2). Let $w \geq 2$ be an integer. Let $g$ be a Riemannian metric and $k$ a symmetric 2-tensor on $B_1$ that solve

$$
R(g) = |k|^2_g,
\text{div}_g k = 0,
\text{tr}_g k = 0.
$$

There exists a universal constant $\varepsilon > 0$ such that if

$$
\|(g - e, k)\|_{H^w(B_1) \times H^{w-1}(B_1)} < \varepsilon,
$$

where $e$ denotes the Euclidean metric, then there is an $H^{w-1/2}_{-1/2}$-asymptotically flat Riemannian metric $g'$ on $\mathbb{R}^3$ and a symmetric 2-tensor $k' \in H^{w-3/2}_{-3/2}$ such that

$$
(g', k')|_{B_1} = (g, k)
$$

and such that on $\mathbb{R}^3$

$$
R(g') = |k'|^2_{g'},
\text{div}_{g'} k' = 0,
\text{tr}_{g'} k' = 0.
$$

Moreover, the following bound holds.

$$
\|(g' - e, k')\|_{H^{w}_{-1/2} \times H^{w-1}_{-3/2}} \lesssim \|(g - e, k)\|_{H^w(B_1) \times H^{w-1}(B_1)}.
$$
4. The divergence equation for $k$

In this section we prove the following theorem.

**Theorem 4.1** (Extension of divergence-free tracefree symmetric 2-tensors, version 2). There exists a small universal constant $\varepsilon > 0$ such that the following holds.

1. **Extension result:** Let $w \geq 2$ be an integer. Let $g$ be a given $H^{w-1}_{-1/2}$-asymptotically flat metric on $\mathbb{R}^3$ and $\bar{k} \in H^{w-1}(B_1)$ a symmetric 2-tensor such that on $B_1$

\[
\begin{align*}
\text{div}_g \bar{k} &= 0, \\
\text{tr}_g \bar{k} &= 0.
\end{align*}
\]

If

\[
\|g - e\|_{H^{w-1}_{-1/2}} + \|\bar{k}\|_{H^{w-1}(B_1)} < \varepsilon,
\]

then there exists a symmetric 2-tensor $k \in H^{w-1}_{-3/2}$ such that $k|_{B_1} = \bar{k}$ and on $\mathbb{R}^3$

\[
\begin{align*}
\text{div}_g k &= 0, \\
\text{tr}_g k &= 0.
\end{align*}
\]

Furthermore, it is bounded by

\[
\|k\|_{H^{w-1}_{-3/2}} \lesssim \|\bar{k}\|_{H^{w-1}(B_1)}.
\]

2. **Iteration estimates:** Let $g, g'$ be two given $H^{w-1}_{-1/2}$-asymptotically flat metrics on $\mathbb{R}^3$ such that

\[
g|_{B_1} = g'|_{B_1}
\]

and $\bar{k} \in H^{w-1}(B_1)$ a symmetric 2-tensor on $B_1$ that solves (4.1) with respect to $g$ (and so for $g'$). Assume that for $(g, \bar{k})$ and $(g', \bar{k})$ the smallness condition (4.2) holds and let $k, k' \in H^{w-1}_{-3/2}$ denote the two extensions of $\bar{k}$ constructed in part (1) of this theorem with the metrics $g$ and $g'$, respectively. Then it holds that

\[
\|k - k'|_{H^{w-1}_{-3/2}} \lesssim \|\bar{k}\|_{H^{w-1}(B_1)} \|g - g'\|_{H^{w}_{-1/2}}.
\]

Before proving Theorem 4.1, we analyse the divergence and trace mapping on $H^{w}_{-1/2}$-asymptotically flat metrics on $\mathbb{R}^3$.

4.1. **Analysis of operators on $H^{w}_{-1/2}$-asymptotically flat metrics.** Recall from Section 2.1 that for a Riemannian metric $g$ and a symmetric 2-tensor $V$, the divergence, trace and tracefree part of $V$ is respectively defined as

\[
\begin{align*}
\text{div}_g V_j &= \nabla^i V_{ij}, \\
\text{tr}_g V &= g^{ij} V_{ij}, \\
\tilde{V}^g &= V - \frac{1}{3} \text{tr}_g(V) g.
\end{align*}
\]
where $\nabla$ denotes the covariant derivative of $g$. The next lemma shows basic properties of the divergence operator.

**Lemma 4.2.** Let $w \geq 2$ be an integer. There is a universal $\varepsilon > 0$ such that the following holds.

- The mapping
  \[
  \text{div} : (V, g) \mapsto \text{div}_g V
  \]
  is a smooth mapping from $\mathcal{H}_{-3/2}^w \times B_{\varepsilon}(e)$ to $\mathcal{H}_{-5/2}^{w-2}$, where
  \[
  B_{\varepsilon}(e) := \left\{ g : \|g - e\|_{\mathcal{H}_{-1/2}^w} < \varepsilon \right\}.
  \]
  Furthermore, $\text{div}$ maps $\mathcal{H}_{-3/2}^w \times B_{\varepsilon}(e)$ into $\mathcal{H}_{-5/2}^{w-2}$.

- For all Riemannian metrics $g$ such that
  \[
  \|g - e\|_{\mathcal{H}_{-1/2}^w} < \varepsilon,
  \]
  it holds that
  \[
  \|\text{div}_g V\|_{\mathcal{H}_{-5/2}^{w-2}} \lesssim \|V\|_{\mathcal{H}_{-3/2}^w} \quad (4.5)
  \]
  for all symmetric 2-tensors $V \in \mathcal{H}_{-3/2}^w$.

- For all Riemannian metrics $g, g'$ with
  \[
  \|g - e\|_{\mathcal{H}_{-1/2}^w}, \|g' - e\|_{\mathcal{H}_{-1/2}^w} < \varepsilon
  \]
  it holds that
  \[
  \|\text{div}_g V - \text{div}_{g'} V\|_{\mathcal{H}_{-5/2}^{w-2}} \lesssim \|g - g'\|_{\mathcal{H}_{-1/2}^w} \|V\|_{\mathcal{H}_{-3/2}^w} \quad (4.6)
  \]
  for all symmetric 2-tensors $V \in \mathcal{H}_{-3/2}^w$.

**Proof of Lemma 4.2.** By definition of $\text{div}_g$,
\[
(\text{div}_g V)_i = g^{ab} \nabla_a V_{bi} = g^{ab} \left( \partial_a V_{bi} - \Gamma^j_{ab} V_{ji} - \Gamma^j_{ai} V_{jb} \right), \tag{4.7}
\]
where $g^{ij}$ denotes the components of the inverse $g^{-1}$ and $\Gamma^j_{ia} = \frac{1}{2} g^{ij} \left( \partial_i g_{ba} + \partial_b g_{ia} - \partial_a g_{ib} \right)$ denote the Christoffel symbols. For $g$ close to $e$, $g^{ij}$ is a smooth expression in $g$. Therefore we can schematically write
\[
\text{div}_g V = F(g) \partial V + F(g) \partial g V, \tag{4.8}
\]
where $F$ maps symmetric 2-tensors into $\mathbb{R}$ and is smooth in a neighbourhood of $e$. By (4.8),
\[
\|\text{div}_g V\|_{\mathcal{H}_{-5/2}^{w-2}} \lesssim \|F(g) \partial V\|_{\mathcal{H}_{-5/2}^{w-2}} + \|F(g) \partial g V\|_{\mathcal{H}_{-5/2}^{w-2}}. \tag{4.9}
\]
By Lemmas 2.8 and 2.9, there exists a universal constant \( \varepsilon > 0 \) such that if \( g \in B_{\varepsilon}(e) \subset \mathcal{H}_{-1/2}^{w} \), then
\[
\| F(g) \partial V \|_{\mathcal{H}_{-5/2}^{w-2}} \lesssim \| V \|_{\mathcal{H}_{-3/2}^{w-1}},
\]
\[
\| F(g) \partial g V \|_{\mathcal{H}_{-5/2}^{w-2}} \lesssim \| \partial g V \|_{\mathcal{H}_{-5/2}^{w-2}}
\lesssim \| g \|_{\mathcal{H}_{-3/2}^{w-1}} \| V \|_{\mathcal{H}_{-3/2}^{w-1}}
\lesssim \| g - e \|_{\mathcal{H}_{-1/2}^{w}} \| V \|_{\mathcal{H}_{-3/2}^{w-1}}.
\]

(4.10)

Plugging (4.10) into (4.9) proves that \( \text{div} \) maps \( \mathcal{H}_{-3/2}^{w-1} \times B_{\varepsilon}(e) \) to \( \mathcal{H}_{-5/2}^{w-2} \) and further (4.5).

The expression (4.8) shows by Corollary 2.11 that \( \text{div} \) is a smooth mapping from \( \mathcal{H}_{-3/2}^{w-1} \times B_{\varepsilon}(e) \) to \( \mathcal{H}_{-5/2}^{w-2} \). The restriction of \( \text{div} \) to \( V \in \mathcal{H}_{-3/2}^{w-1} \) clearly maps into \( \mathcal{H}_{-5/2}^{w-2} \), by (4.8).

It remains to prove (4.6). Indeed, for all \( g \in B_{\varepsilon}(e) \), \( \text{div}_g(0) = 0 \), so that Lemma 2.38 implies (4.6). This finishes the proof of Lemma 4.2.

\( \square \)

**Lemma 4.3.** Let \( w \geq 2 \) be an integer. There exists an \( \varepsilon > 0 \) such that the following holds.

- The mapping
  \[
  \text{tr} : (V, g) \mapsto \text{tr}_g V
  \]
  is a smooth mapping from \( \mathcal{H}_{-3/2}^{w-1} \times B_{\varepsilon}(e) \) to \( \mathcal{H}_{-3/2}^{w-1} \), where
  \[
  B_{\varepsilon}(e) := \left\{ g : \| g - e \|_{\mathcal{H}_{-1/2}^{w}} < \varepsilon \right\}.
  \]
  Furthermore, \( \text{tr} \) maps \( \mathcal{H}_{-3/2}^{w-1} \times B_{\varepsilon}(e) \) into \( \mathcal{H}_{-5/2}^{w-2} \).

- For all Riemannian metrics \( g \) with
  \[
  \| g - e \|_{\mathcal{H}_{-1/2}^{w}} < \varepsilon
  \]
  it holds that
  \[
  \| \text{tr}_g V \|_{\mathcal{H}_{-3/2}^{w-1}} \lesssim \| V \|_{\mathcal{H}_{-3/2}^{w-1}}
  \]
  for all symmetric 2-tensors \( V \in \mathcal{H}_{-3/2}^{w-1} \).

- For two metrics \( g, g' \) such that
  \[
  \| g - e \|_{\mathcal{H}_{-1/2}^{w}}, \| g' - e \|_{\mathcal{H}_{-1/2}^{w}} < \varepsilon
  \]
  it holds that
  \[
  \| \text{tr}_g V - \text{tr}_{g'} V \|_{\mathcal{H}_{-3/2}^{w-1}} \lesssim \| g - g' \|_{\mathcal{H}_{-1/2}^{w}} \| V \|_{\mathcal{H}_{-3/2}^{w-1}}
  \]
  for all symmetric 2-tensors \( V \in \mathcal{H}_{-3/2}^{w-1} \).
The proof of Lemma 4.3 is similar to the proof of Lemma 4.2 and left to the reader.

Lemmas 4.2 and 4.3 imply the following corollary. The proof is left to the reader.

**Corollary 4.4.** Let \( w \geq 2 \) be an integer. There exists \( \varepsilon > 0 \) such that the following holds.

- The mapping

\[
(V, g) \mapsto \text{div}_g \left( \hat{V}^g \right)
\]

is smooth from \( H_{-3/2}^{w-1} \times B_\varepsilon(e) \) to \( H_{-5/2}^{w-2} \), where

\[
B_\varepsilon(e) := \left\{ g : \| g - e \|_{H_{-1/2}^w} < \varepsilon \right\}.
\]

Furthermore, the restriction of this mapping to \( V \in H_{-3/2}^{w-1} \) maps into \( H_{-5/2}^{w-2} \).

- For a Riemannian metric \( g \) on \( \mathbb{R}^3 \) such that

\[
\| g - e \|_{H_{-1/2}^w} < \varepsilon,
\]

it holds that

\[
\| \hat{V}^g \|_{H_{-3/2}^{w-1}} \lesssim \| V \|_{H_{-3/2}^{w-1}},
\]

\[
\left\| \text{div}_g \left( \hat{V}^g \right) \right\|_{H_{-5/2}^{w-2}} \lesssim \| V \|_{H_{-3/2}^{w-1}}.
\]

for all symmetric 2-tensors \( V \in H_{-3/2}^{w-1} \).

- For two Riemannian metrics \( g, g' \) on \( \mathbb{R}^3 \) such that

\[
\| g - e \|_{H_{-1/2}^w}, \| g' - e \|_{H_{-1/2}^w} < \varepsilon,
\]

it holds that

\[
\left\| \hat{V}^g - \hat{V}^{g'} \right\|_{H_{-3/2}^{w-1}} \lesssim \| g - g' \|_{H_{-1/2}^w} \| V \|_{H_{-3/2}^{w-1}},
\]

\[
\left\| \text{div}_g \left( \hat{V}^g \right) - \text{div}_{g'} \left( \hat{V}^{g'} \right) \right\|_{H_{-5/2}^{w-2}} \lesssim \| g - g' \|_{H_{-1/2}^w} \| V \|_{H_{-3/2}^{w-1}}
\]

for all symmetric 2-tensors \( V \in H_{-3/2}^{w-1} \).

4.2. **Reduction to the Euclidean case.** In this section, we prove Theorem 4.1 under the assumption of Lemma 4.6 below which is proved in Section 4.3. First, as an intermediate step, we prove the next proposition.

**Proposition 4.5.** Let \( w \geq 2 \) be an integer. There is a universal constant \( \varepsilon > 0 \) such that the following holds.

(1) Let \( g \) be an \( H_{-1/2}^{w-1} \)-asymptotically flat metric and \( \rho \in \overline{H}_{-5/2}^{w-2} \) a 1-form on \( \mathbb{R}^3 \) such that

\[
\| g - e \|_{H_{-1/2}^w} + \| \rho \|_{\overline{H}_{-5/2}^{w-2}} < \varepsilon.
\]

(4.12)
Then there exists \( k \in \mathcal{H}^{w-1}_{-3/2} \) solving on \( \mathbb{R}^3 \setminus \overline{B}_1 \)

\[
\begin{align*}
\text{div}_g k &= \rho, \\
\text{tr}_g k &= 0
\end{align*}
\]

and bounded by

\[
\|k\|_{\mathcal{H}^{w-1}_{-3/2}} \lesssim \|\rho\|_{\mathcal{H}^{w-2}_{-5/2}}.
\]

(2) Moreover, for two pairs \((g, \rho), (g', \rho')\) satisfying the smallness condition (4.12), the respectively constructed \(k, k'\) satisfy

\[
\|k - k'\|_{\mathcal{H}^{w-1}_{-3/2}} \lesssim \|\rho - \rho'\|_{\mathcal{H}^{w-2}_{-5/2}} + \|g - g'\|_{\mathcal{H}^{w}_{-1/2}} \|\rho\|_{\mathcal{H}^{w-2}_{-5/2}}.
\]

(4.14)

To prove Proposition 4.5, we assume the following essential lemma proved in Section 4.3.

**Lemma 4.6** (Surjectivity at the Euclidean metric). Let \( w \geq 2 \) be an integer. For any \( \rho \in \mathcal{H}^{w-2}_{-5/2} \), there exists a symmetric 2-tensor \( k \in \mathcal{H}^{w-1}_{-3/2} \) solving on \( \mathbb{R}^3 \setminus \overline{B}_1 \)

\[
\begin{align*}
\text{div}_e k &= \rho, \\
\text{tr}_e k &= 0
\end{align*}
\]

and bounded by

\[
\|k\|_{\mathcal{H}^{w-1}_{-3/2}} \lesssim \|\rho\|_{\mathcal{H}^{w-2}_{-5/2}}.
\]

In other words, the mapping \( k \mapsto \text{div}_e(k^e) \) from \( \mathcal{H}^{w-1}_{-3/2} \) to \( \mathcal{H}^{w-2}_{-5/2} \) is surjective and has a bounded right-inverse.

For the rest of this section denote

\[
\overline{\mathcal{N}}_e := \text{ker} \left( \text{div}_e \circ (\cdot^e) \right)^\perp,
\]

where \( \perp \) denotes the orthogonal complement with respect to the scalar product on \( \mathcal{H}^{w-1}_{-3/2} \).

\( \overline{\mathcal{N}}_e \) is a closed subspace of the Hilbert space \( \mathcal{H}^{w-1}_{-3/2} \) and therefore Hilbert itself.

We are now ready to prove Proposition 4.5 by Lemma 4.6 and the Implicit Function Theorem 2.37.

**Proof of Proposition 4.5.** We prove each part separately.

**Proof of part (1).** We apply the Implicit Function Theorem 2.37 to the mapping

\[
\mathcal{F} : \overline{\mathcal{N}}_e \times \mathcal{H}^w_{-1/2} \to \mathcal{H}^{w-2}_{-5/2}
\]

\[
(k, h) \mapsto \rho := \text{div}_{e+h}(k^e + h),
\]

where \( h \) is a symmetric 2-tensor. We verify that \( \mathcal{F} \) satisfies the assumptions of Theorem 2.37 at \((k, h) = 0\). On the one hand, by Corollary 4.4, there exists an \( \varepsilon > 0 \) such that \( \mathcal{F} \)
is a smooth mapping from $\overline{N}_\varepsilon \times B_\varepsilon(0)$ to $\overline{H}^{w-2}_{-5/2}$, where $B_\varepsilon(0) \subset H^w_{-1/2}$, and $F(0,0) = 0$. On the other hand, by Lemma 4.6 and the definition of $\overline{N}_\varepsilon$ in (4.15), the linearization in the first argument at $h = 0$,
\[
D_1F|_{h=0} : \overline{N}_\varepsilon \to \overline{H}^{w-2}_{-5/2},
\]
is an isomorphism.

Consequently, by Theorem 2.37, there exists an open neighbourhood $V_0 \subset B_\varepsilon(0) \times \overline{H}^{w-2}_{-5/2}$ of $(h, \rho) = (0, 0)$ and a unique smooth mapping $G : V_0 \to \overline{H}^{w-1}_{-3/2}$ into symmetric 2-tensors such that on $\mathbb{R}^3 \setminus \overline{B}_1$
\[
div_{e+h} \left( \hat{G}^{e+h}(h, \rho) \right) = \rho
\]
for all $(h, \rho) \in V_0$. By the uniqueness it follows in particular that for all $(h, 0) \in V_0$,
\[
G(h, 0) = 0,
\]
because $F(0, h) = 0$ for all $h \in B_\varepsilon(0)$.

For $(h, \rho) \in V_0$, let $k := \hat{G}^{e+h}(h, \rho)$. Then, on $\mathbb{R}^3 \setminus \overline{B}_1$,
\[
div_{e+h} k = \rho,
\]
\[
tr_{e+h} k = 0.
\]

For $\varepsilon > 0$ sufficiently small it holds that
\[
(h, \rho) \in B_\varepsilon(0) \times B_\varepsilon(0) \subset V_0
\]
and further by Lemma 2.38 and Corollary 4.4,
\[
\|k\|_{\overline{H}^{-1}_{-3/2}} = \|\hat{G}^{e+h}(h, \rho)\|_{\overline{H}^{-1}_{-3/2}} \leq \|G(h, \rho)\|_{\overline{H}^{-1}_{-3/2}} \leq \|G(h, \rho) - G(h, 0)\|_{\overline{H}^{-1}_{-3/2}} = 0 \\
\leq \|\rho\|_{\overline{H}^{-1}_{-5/2}}.
\]

This proves (4.13).

**Proof of part (2).** Let two pairs $(h, \rho), (h', \rho') \in B_\varepsilon(0) \times B_\varepsilon(0) \subset V_0$. By Lemma 2.38 and (4.16), it follows that for $\varepsilon > 0$ sufficiently small,
\[
\|G(h, \rho) - G(h', \rho)\|_{\overline{H}^{-1}_{-3/2}} \leq \|h - h'\|_{H^w_{-1/2}(\mathbb{R}^3)} \|\rho\|_{\overline{H}^{-1}_{-3/2}},
\]
\[
\|G(h, \rho)\|_{\overline{H}^{-1}_{-3/2}} \leq \|\rho\|_{\overline{H}^{-1}_{-3/2}}.
\]
Moreover, by the smoothness of $\mathcal{G}$, for $\varepsilon > 0$ sufficiently small,

$$
\|\mathcal{G}(h', \rho) - \mathcal{G}(h', \rho')\|_{\mathcal{H}^{-3/2}_w} \lesssim \|\rho - \rho'\|_{\mathcal{H}^{-3/2}_w}.
$$

(4.18)

Let $k := \mathcal{G}^{e+h}(h, \rho), k' := \mathcal{G}^{e+h'}(h, \rho')$. By (4.17) and (4.18), Lemma 4.3 and Corollary 4.4, for $\varepsilon > 0$ sufficiently small,

$$
\|k - k'\|_{\mathcal{H}^{-3/2}_w} \lesssim \|\mathcal{G}(h, \rho) - \mathcal{G}(h', \rho')\|_{\mathcal{H}^{-3/2}_w} + \|\text{tr}_{e+h}\mathcal{G}(h, \rho) - \text{tr}_{e+h}\mathcal{G}(h, \rho)\|_{\mathcal{H}^{-3/2}_w}
\lesssim \|\mathcal{G}(h, \rho) - \mathcal{G}(h', \rho')\|_{\mathcal{H}^{-3/2}_w} + \|h' - h\|_{\mathcal{H}^{-1/2}_w} \|\mathcal{G}(h, \rho)\|_{\mathcal{H}^{-3/2}_w}
\lesssim \|\mathcal{G}(h, \rho) - \mathcal{G}(h', \rho')\|_{\mathcal{H}^{-3/2}_w} + \|\mathcal{G}(h', \rho) - \mathcal{G}(h', \rho')\|_{\mathcal{H}^{-3/2}_w}
\quad + \|h' - h\|_{\mathcal{H}^{-1/2}_w} \|\mathcal{G}(h, \rho)\|_{\mathcal{H}^{-3/2}_w}
\lesssim \|h' - h\|_{\mathcal{H}^{-1/2}_w} \|\mathcal{G}(h, \rho)\|_{\mathcal{H}^{-3/2}_w} + \|\rho - \rho'\|_{\mathcal{H}^{-3/2}_w}.
$$

This finishes the proof of Proposition 4.5. \hfill \square

We now turn to the proof of Theorem 4.1.

**Proof of Theorem 4.1.** We prove the two parts of Theorem 4.1 separately.

**Proof of Part 1:** Let the symmetric 2-tensor $\tilde{k} \in \mathcal{H}^{w-1}(B_1)$ solve on $B_1$

$$
\text{div}_g \tilde{k} = 0,
$$

$$
\text{tr}_g \tilde{k} = 0.
$$

Using standard Sobolev extension, see for example Theorem 7.25 in [16], continuously extend $\tilde{k}$ to a symmetric 2-tensor $\tilde{k} \in \mathcal{H}^{w-1}_{\text{loc}}(\mathbb{R}^3)$. We can assume without loss of generality that $\tilde{k}$ is $g$-tracefree and

$$
\|\tilde{k}\|_{\mathcal{H}^{w-1}_{\text{loc}}(\mathbb{R}^3)} \lesssim \|\tilde{k}\|_{\mathcal{H}^{w-1}(B_1)}.
$$

(4.19)

Indeed, for $\|g - e\|_{\mathcal{H}^{-1/2}_w}$ small enough, multiplying by a cut-off function and taking the $g$-tracefree part are both continuous endomorphisms of $\mathcal{H}^{w-1}_{\text{loc}}(\mathbb{R}^3)$, see Corollary 4.4.

Let $\hat{\rho} := \text{div}_g \tilde{k}$. For $\varepsilon > 0$ small enough, by Lemma 4.2 and (4.19),

$$
\|\hat{\rho}\|_{\mathcal{H}^{w-2}_{\text{loc}}} \lesssim \|\tilde{k}\|_{\mathcal{H}^{w-1}_{\text{loc}}}
\lesssim \|\tilde{k}\|_{\mathcal{H}^{w-1}(B_1)}.
$$

(4.20)

Further, it holds that on $B_1$

$$
\hat{\rho} = \text{div}_g \tilde{k} = \text{div}_g \tilde{k} = 0,
$$

so by Proposition 2.13, $\hat{\rho} \in \mathcal{H}^{w-2}_{-5/2}$. This $\hat{\rho}$ is in general non-trivial (otherwise we would be done), and the Sobolev extension $\tilde{k}$ is therefore in general not a solution to (4.1).
We have by (4.20) \[ \|g - e\|_{\mathcal{H}^{w-1/2}} + \|\bar{\rho}\|_{\mathcal{H}^{w-1/2}} \lesssim \|\tilde{g} - e\|_{\mathcal{H}^{w-1}} + \|\tilde{k}\|_{\mathcal{H}^{w-1}(B_1)}. \]

Therefore, for \( \varepsilon > 0 \) small enough, Proposition 4.5 yields a symmetric 2-tensor \( \tilde{k} \in \mathcal{H}^{w-1}_{-3/2} \) that solves on \( \mathbb{R}^3 \setminus B_1 \)
\[ \text{div}_g \tilde{k} = -\tilde{\rho}, \]
\[ \text{tr}_g \tilde{k} = 0 \]
and is bounded by
\[ \|\tilde{k}\|_{\mathcal{H}^{w-1}_{-3/2}} \lesssim \|\tilde{\rho}\|_{\mathcal{H}^{w-2}_{-5/2}}. \] (4.21)

Extend \( \tilde{k} \) trivially to \( B_1 \). By Proposition 2.13, \( \tilde{k} \in \mathcal{H}^{w-1}_{-3/2} \). Consequently, the symmetric 2-tensor
\[ k := \tilde{k} + \bar{k} \in \mathcal{H}^{w-1}_{-3/2} \] (4.22)
is such that \( k|_{B_1} = \bar{k} \) and solves on \( \mathbb{R}^3 \)
\[ \text{div}_g k = 0, \]
\[ \text{tr}_g k = 0. \]
Finally, for \( \varepsilon > 0 \) sufficiently small, by the estimates (4.20) and (4.21),
\[ \|k\|_{\mathcal{H}^{w-1}_{-3/2}} \lesssim \|\tilde{k}\|_{\mathcal{H}^{w-1}(B_1)}. \]
This proves the first part of Theorem 4.1.

**Proof of Part 2:** Extend \( \bar{k} \in \mathcal{H}^{w-1}(B_1) \) by standard Sobolev extension to a symmetric 2-tensor \( \hat{k} \in \mathcal{H}^{w-1}_{-3/2} \) on \( \mathbb{R}^3 \) such that
\[ \|\hat{k}\|_{\mathcal{H}^{w-1}_{-3/2}} \lesssim \|\bar{k}\|_{\mathcal{H}^{w-1}(B_1)}. \] (4.23)

Taking the \( g \)-tracefree and \( g' \)-tracefree parts of \( \hat{k} \) yields two symmetric 2-tensors \( \tilde{g} \hat{k} \) and \( \tilde{g'} \hat{k} \in \mathcal{H}^{w-1}_{-3/2} \), respectively, that both extend \( \bar{k} \) and satisfy for \( \varepsilon > 0 \) sufficiently small
\[ \|\tilde{g} \hat{k}\|_{\mathcal{H}^{w-1}_{-3/2}} \lesssim \|\tilde{g} \bar{k}\|_{\mathcal{H}^{w-1}(B_1)}, \]
\[ \|\tilde{g'} \hat{k}\|_{\mathcal{H}^{w-1}_{-3/2}} \lesssim \|\tilde{g'} \bar{k}\|_{\mathcal{H}^{w-1}(B_1)}. \]
By Proposition 2.13,
\[ \rho := \text{div}_g \tilde{g} \hat{k} \in \mathcal{H}^{w-2}_{-5/2}, \rho' := \text{div}_g \tilde{g'} \hat{k} \in \mathcal{H}^{w-2}_{-5/2}. \]
For $\varepsilon > 0$ sufficiently small, by Lemma 4.2 and (4.23),
\[
\|\rho\|_{\mathcal{H}_{5/2}^w}^2, \|\rho\|_{\mathcal{H}_{5/2}^w}^2 \lesssim \|\tilde{k}\|_{\mathcal{H}_{w-1}(B_1)}.
\] (4.24)
For $\varepsilon > 0$ small enough, applying Proposition 4.5 to $\rho, \rho'$ with metrics $g, g'$ yields two tensors $\tilde{k}, \tilde{k}' \in \mathcal{H}_{3/2}^{w-1}$, respectively, that satisfy
\[
\text{div}_g \tilde{k} = -\rho,
\]
\[
\text{tr}_g \tilde{k} = 0,
\]
\[
\text{div}_{g'} \tilde{k}' = -\rho',
\]
\[
\text{tr}_{g'} \tilde{k}' = 0.
\]
By (4.14) in Proposition 4.5, for $\varepsilon > 0$ sufficiently small,
\[
\|\tilde{k} - \tilde{k}'\|_{\mathcal{H}_{3/2}^{w-1}} \lesssim \|g - g'|_{\mathcal{H}_{1/2}^w}\|\rho\|_{\mathcal{H}_{5/2}^w} + \|\rho - \rho'|_{\mathcal{H}_{5/2}^w}
\]
\[
\lesssim \|g - g'|_{\mathcal{H}_{1/2}^w}\|\tilde{k}\|_{\mathcal{H}_{w-1}(B_1)} + \|\text{div}_g \tilde{k} - \text{div}_{g'} \tilde{k}'\|_{\mathcal{H}_{5/2}^w}
\]
\[
\lesssim \|g - g'|_{\mathcal{H}_{1/2}^w}\|\tilde{k}\|_{\mathcal{H}_{w-1}(B_1)} + \|g - g'|_{\mathcal{H}_{1/2}^w}\|\tilde{k}\|_{\mathcal{H}_{3/2}^{w-1}}
\]
\[
\lesssim \|g - g'|_{\mathcal{H}_{1/2}^w}\|\tilde{k}\|_{\mathcal{H}_{w-1}(B_1)},
\] (4.25)
where we used (4.24) and Corollary 4.4. Extend $\tilde{k}, \tilde{k}'$ trivially to $B_1$. By Proposition 2.13, $\tilde{k}, \tilde{k}' \in \mathcal{H}_{3/2}^{w-1}$.

The tensors
\[
k := \tilde{g}^g + \tilde{k} \in \mathcal{H}_{3/2}^{w-1}, k' := \tilde{g}^{g'} + \tilde{k}' \in \mathcal{H}_{3/2}^{w-1}
\]
both extend $\tilde{k}$ and satisfy on $\mathbb{R}^3$
\[
\text{div}_g k = 0,
\]
\[
\text{tr}_g k = 0,
\]
\[
\text{div}_{g'} k' = 0,
\]
\[
\text{tr}_{g'} k' = 0.
\]
Moreover, their difference is bounded by
\[
\|k - k'\|_{\mathcal{H}_{3/2}^{w-1}} \lesssim \|\tilde{g}^g - \tilde{g}^{g'}\|_{\mathcal{H}_{3/2}^{w-1}} + \|\tilde{k} - \tilde{k}'\|_{\mathcal{H}_{3/2}^{w-1}}
\]
\[
\lesssim \|g - g'|_{\mathcal{H}_{1/2}^w}\|\tilde{k}\|_{\mathcal{H}_{w-1}(B_1)} + \|g - g'|_{\mathcal{H}_{1/2}^w}\|\tilde{k}\|_{\mathcal{H}_{3/2}^{w-1}}
\]
\[
\lesssim \|g - g'|_{\mathcal{H}_{1/2}^w}\|\tilde{k}\|_{\mathcal{H}_{w-1}(B_1)},
\]
where we used Corollary 4.4, (4.23) and (4.25). This finishes the proof of Theorem 4.1. $\square$
4.3. Surjectivity at the Euclidean metric. Let \( w \geq 2 \) be an integer. In this section we prove Lemma 4.6, that is, we show that for any \( \rho \in \mathcal{H}^{w-2}_{-5/2} \), there exists a symmetric 2-tensor \( k \in \mathcal{H}^{w-1}_{-3/2} \) that solves on \( \mathbb{R}^3 \setminus B_1 \)

\[
\begin{align*}
\text{div}_e k &= \rho, \\
\text{tr}_e k &= 0
\end{align*}
\]

and is bounded by

\[
\|k\|_{\mathcal{H}^{w-1}_{-3/2}} \lesssim \|\rho\|_{\mathcal{H}^{w-2}_{-5/2}}.
\]

In this section, all differential operators are with respect to the Euclidean metric \( e \). The operators \( \text{div}, \text{curl}, \nabla \) are the induced operators on the spheres \((S_r, \gamma) \subset (\mathbb{R}^3, e)\) for \( r > 0 \).

Remark 4.7. Let us note the following.

- In general, the system on \( \mathbb{R}^3 \setminus B_1 \)

\[
\begin{align*}
\text{div} k &= \rho, \\
\text{tr} k &= 0
\end{align*}
\]

is underdetermined and does not admit an a priori estimate for solutions \( k \). We work with the following determined Hodge system on \( \mathbb{R}^3 \setminus B_1 \)

\[
\begin{align*}
\text{div} k &= \rho, \\
\text{curl} k &= \sigma, \\
\text{tr} k &= 0
\end{align*}
\]

where \( \sigma \) is a tracefree symmetric 2-tensor that we carefully choose by hand. This system admits in general a priori estimates for \( k \) in terms of \( \rho \) and \( \sigma \), see for example Proposition 4.4.1 in [9]. Clearly, a solution \( k \) to (4.27) is in particular a solution to (4.26).

- First, we decompose \( k \) with respect to the foliation of \( \mathbb{R}^3 \setminus \{0\} \) by spheres \( S_r \) into scalar functions and \( S_r \)-tangent tensors. Second, the Hodge-Fourier expansion of \( S_r \)-tangent tensors introduced in Section 2 allows to decompose (4.27) into three independent sub-systems \( S0, S1 \) and \( S2 \), see later in this section. These sub-systems are then solved individually.

- A priori, for given \( \rho, \sigma \in \mathcal{H}^{w-2}_{-5/2} \), the solution \( k \) to the Dirichlet boundary value problem on \( \mathbb{R}^3 \setminus B_1 \)

\[
\begin{align*}
\text{div} k &= \rho, \\
\text{curl} k &= \sigma, \\
\text{tr} k &= 0, \\
|k|_{r=1} &= 0
\end{align*}
\]
satisfies only $k \in \mathcal{H}^{-3/2}_-(\mathbb{R}^3 \setminus \overline{B_1}) \cap \overline{\mathcal{H}}^{-1}_{-3/2}$. However, by our careful choice of $\sigma$ we can achieve the additional boundary condition
\[
\nabla_N k|_{r=1} = 0,
\]
which eventually improves the boundary behaviour to $k \in \overline{\mathcal{H}}^{-1}_{-3/2}$.

4.3.1. Derivation of the equations. In this section, we derive the new form of (4.27) with respect to the radial foliation of $\mathbb{R}^3 \setminus \{0\}$, see Section 2.2 for notations. Decompose the tensor $k$ into
- the scalar $\delta := k_{NN}$,
- the $S_r$-tangent vectorfield $\epsilon_A := (\frac{k}{N})_A$,
- the $S_r$-tangent symmetric 2-tensor $\eta_{AB} := (\frac{k}{N})_{AB}$.

Furthermore, let $\hat{\eta}$ be the tracefree part of $\eta$, that is\footnote{Here we use that $\nabla^{AB} k_{AB} = -\delta$ by the third equation of (4.27).}
\[
\hat{\eta}_{AB} := \eta_{AB} + \frac{1}{2} \delta \gamma_{AB}.
\]

Decompose $\rho$ into
- the scalar $\rho_N$,
- the $S_r$-tangent vectorfield $\rho_A := \rho_A$,

and $\sigma$ into
- the scalar $\sigma_{NN}$,
- the $S_r$-tangent vectorfield $\sigma_A := \sigma_A$,
- the $S_r$-tangent symmetric 2-tensor $\sigma_{AB} := \sigma_{AB}$.

The system (4.27) is equivalent to (this the Euclidean version of Proposition 4.4.3 in [9])
\[
\begin{align*}
\text{div} \epsilon &= \rho_N - \nabla_N \delta - \frac{3}{r} \delta, \\
\text{curl} \epsilon &= \sigma_{NN}, \\
\nabla_N \epsilon + \frac{2}{r} \epsilon &= \frac{1}{2} \phi + *\sigma^R + \nabla \delta, \\
\text{div} \hat{\eta} &= \frac{1}{2} \phi - *\sigma^R - \frac{1}{2} \nabla \delta - \frac{1}{r} \epsilon, \\
\nabla_N \hat{\eta} + \frac{1}{r} \hat{\eta} &= *\widehat{(\phi)} + \frac{1}{2} \nabla \otimes \epsilon,
\end{align*}
\]
where $*\sigma^R$ denotes the Hodge dual of $\sigma^R$ and $*\widehat{(\phi)}$ the Hodge dual of $\widehat{\phi}$, the tracefree part of $\phi$. See Section 2.6 for details.
4.3.2. **Definition of the 2-tensors $k$ and $\sigma$.** In this section, we explicitly exhibit the two symmetric 2-tensors $k$ and $\sigma$. We show in Section 4.3.3 that they form a regular solution to (4.28).

Let $\rho = (\rho_N, \dot{\rho}) \in \overline{H}^{w-1}_{w-3/2}$. Let the Hodge-Fourier decomposition of $\rho_N, \dot{\rho}$ be

$$\rho_N = (\rho_N)^{[0]} + (\rho_N)^{[1]} + (\rho_N)^{[\geq 2]}$$

$$\dot{\rho} = \dot{\rho}_E^{[0]} + \dot{\rho}_H^{[0]} + \dot{\rho}_E^{[1]} + \dot{\rho}_H^{[1]} + \dot{\rho}_E^{[\geq 2]} + \dot{\rho}_H^{[\geq 2]}.$$  

Define symmetric tracefree 2-tensors $k$ and $\sigma$ on $\mathbb{R}^3 \setminus \overline{B_1}$ as follows.

- **Definition of $\delta$.** Let the scalar function

$$\delta = \delta^{[0]} + \delta^{[1]} + \delta^{[\geq 2]},$$

where $\delta^{[0]}$ is defined as

$$\delta^{[0]} := \frac{1}{r^3} \int_1^r (r')^3 (\rho_N)^{[0]} \, dr'$$

and $\delta^{[1]}$ is defined as the solution to the second-order ODE on $r > 1$

$$\begin{cases}
\partial_r^2 \delta^{[1]} + \frac{7}{r} \partial_r \delta^{[1]} + \frac{8}{r^2} \delta^{[1]} = \frac{1}{r^4} \partial_r (r^4 (\rho_N)^{[1]}) - 4 \partial_r \dot{\rho}_E^{[1]}, \\
\delta^{[1]}|_{r=1} = \partial_r \delta^{[1]}|_{r=1} = 0.
\end{cases} \tag{4.31}$$

The function $\delta^{[\geq 2]}$ is defined as the solution to the following elliptic boundary value problem on $\mathbb{R}^3 \setminus \overline{B_1}$,

$$\begin{cases}
\Delta \delta^{[\geq 2]} + \frac{4}{r} \partial_r \delta^{[\geq 2]} + \frac{6}{r^2} \delta^{[\geq 2]} = \frac{1}{r^4} \partial_r \left( r^3 (\rho_N)^{[\geq 2]} \right) - 4 \partial_r \dot{\rho}_E^{[\geq 2]} + \zeta_E, \\
\delta^{[\geq 2]}|_{r=1} = 0.
\end{cases} \tag{4.32}$$

Here, the $S_r$-tangent vectorfield $\zeta_E$ is defined on $\mathbb{R}^3$ by

$$\zeta_E := \sum_{l \geq 2} \sum_{m=-l}^{l} \zeta_E^{(lm)} E^{(lm)},$$

$$\zeta_E^{(lm)}(r) := c_E^{(lm)} r^{l-1} \partial_r (\chi(l(r-1))),$$

where $\chi$ is the standard transition function defined in (2.1) and for $l \geq 2$,

$$c_E^{(lm)} := \int_1^\infty r^{-l+1} \left( \frac{l}{\sqrt{l(l+1)}} (\rho_N)^{(lm)} - \dot{\rho}_E^{(lm)} \right) \, dr.$$ \tag{4.34}

- **Definition of $\sigma_{NN}$.** Let the scalar function

$$\sigma_{NN} = \sigma_{NN}^{[1]} + \sigma_{NN}^{[\geq 2]},$$

where $\sigma_{NN}^{[1]}$ and $\sigma_{NN}^{[\geq 2]}$ are defined as

$$\sigma_{NN}^{[1]} := \frac{1}{r^3} \int_1^r (r')^3 (\rho_N)^{[1]} \, dr'$$

and

$$\sigma_{NN}^{[\geq 2]} := \frac{1}{r^3} \int_1^r (r')^3 (\rho_N)^{[\geq 2]} \, dr'.$$
where \( \sigma_{NN}^{[1]} \) is defined as
\[
\sigma_{NN}^{[1]} := \frac{1}{r^4} \int_{1}^{r} (r')^4 \text{curl} \phi^{[1]} dr',
\]
and \( \sigma_{NN}^{[\geq 2]} \) is defined as solution to the following elliptic boundary value problem on \( \mathbb{R}^3 \setminus B_1 \),
\[
\begin{aligned}
\Delta \sigma_{NN}^{[\geq 2]} + \frac{1}{r} \partial_r \sigma_{NN}^{[\geq 2]} - \frac{3}{r^2} \sigma_{NN}^{[\geq 2]} &= \partial_r \left( \text{curl} \left( \rho_{[\geq 2]} - \zeta_{[\geq 2]}^H \right) \right), \\
\sigma_{NN}^{[\geq 2]}|_{r=1} &= 0.
\end{aligned}
\]

Here, the \( S_r \)-tangent vectorfield \( \zeta_H \) is defined by
\[
\zeta_H := \sum_{l \geq 2} \sum_{m=-l}^{l} \zeta_{H}^{(lm)} H^{(lm)},
\]
\[
\zeta_{H}^{(lm)}(r) := c_{H}^{(lm)} r^{1+\sqrt{l(l+1)+4}} \partial_r (\chi(l(r-1))),
\]
and for \( l \geq 2, \)
\[
c_{H}^{(lm)} := - \int_{1}^{\infty} r^{-1-\sqrt{l(l+1)+4}} \phi_{H}^{(lm)} dr.
\]

**Definition of** \( \varepsilon \). Let the \( S_r \)-tangent vectorfield \( \varepsilon \) be on each \( S_r, r \geq 1 \), the solution to
\[
\mathcal{D} \varepsilon = \left( \rho_N - \frac{1}{r^3} \partial_r \left( r^3 \delta \right), \sigma_{NN} \right).
\]

**Definition of** \( \ast \sigma_{[\geq 2]} \). Let the \( S_r \)-tangent vectorfield
\[
\ast \sigma_{[\geq 2]} = \ast \sigma_{[\geq 2]}^{[1]} + \ast \sigma_{[\geq 2]}^{[\geq 2]} + \ast \sigma_{[\geq 2]}^{H},
\]
where \( \ast \sigma_{[\geq 2]}^{[1]} \), \( \ast \sigma_{[\geq 2]}^{[\geq 2]} \) are defined as
\[
\ast \sigma_{[\geq 2]}^{[1]} := \frac{1}{2} \phi_{[\geq 2]}^{[1]} - \frac{1}{2} \nabla \delta^{[1]} - \frac{1}{r} \varepsilon^{[1]},
\]
\[
\ast \sigma_{[\geq 2]}^{[\geq 2]} := \frac{1}{2} \rho_{[\geq 2]} - \frac{1}{r^3} \partial_r \left( r^3 \sigma_{NN}^{[\geq 2]} \right),
\]
and \( \ast \sigma_{[\geq 2]}^{H} \) is defined to be on each \( S_r, r \geq 1 \), the solution of
\[
\mathcal{D} \left( \ast \sigma_{[\geq 2]}^{H} \right) = -\mathcal{D} \left( \frac{1}{2} \phi_{[\geq 2]}^{H} \right) + \left( 0, \frac{1}{r^3} \partial_r \left( r^3 \sigma_{NN}^{[\geq 2]} \right) \right).
\]
• Construction of \( \ast(\sigma) \). Let the symmetric \( \gamma \)-tracefree 2-tensor \( \ast(\sigma) \) be on each \( S_r \), \( r \geq 1 \), the solution to
\[
\mathcal{D} / 2 \ast(\sigma) = \frac{1}{2} \nabla \otimes [\geq 2] - \frac{1}{r} \epsilon [\geq 2].
\]
\[(4.45)\]

• Construction of \( \hat{\eta} \). Let the symmetric \( \gamma \)-tracefree 2-tensor \( \hat{\eta} \) be on each \( S_r \), \( r \geq 1 \), the solution to
\[
\mathcal{D} / 2 \hat{\eta} = \frac{1}{2} \delta [\geq 2] - \ast \sigma [\geq 2] - \frac{1}{2} \nabla \delta [\geq 2] - \frac{1}{r} \epsilon [\geq 2].
\]
\[(4.46)\]

Remark 4.8. For ease of presentation, we defined \( k \) and \( \sigma \) via the quantities that appear in (4.28). Indeed, by the Hodge duality relation (2.8) and the third equation of (4.27), all components of \( k \) and \( \sigma \) are uniquely specified this way.

Remark 4.9. The auxiliary \( \zeta_E \) and \( \zeta_H \) in (4.38) and (4.38) are introduced to control the Dirichlet-to-Neumann map of the elliptic boundary value problems (4.32) and (4.37) for \( \delta [\geq 2] \) and \( \sigma_{[\geq 2]}^{NN} \), respectively, to achieve the additional boundary condition
\[
\partial_r \delta [\geq 2] |_{r=1} = \partial_r \sigma_{[\geq 2]}^{NN} |_{r=1} = 0.
\]
This is necessary for the higher boundary regularity of \( \delta [\geq 2] \) and \( \sigma_{[\geq 2]}^{NN} \) across \( S_1 \).

4.3.3. Proof of surjectivity. In this section, we prove Lemma 4.10, Proposition 4.11 and Lemma 4.15 below, that together imply surjectivity. Especially Proposition 4.11 is essential and only holds due to our delicate choice of \( \zeta_E, \zeta_H \) in (4.33) and (4.38), as well as our particular choice of \( \sigma_{[\geq 2]}^{NN} \) to be a solution to (4.37). See also Remark 4.9.

Lemma 4.10. For given \( \rho \), the symmetric 2-tensors \( k \) and \( \sigma \) defined by (4.29)-(4.46) are a formal solution to (4.28), that is, on \( \mathbb{R}^3 \setminus \overline{B_1} \),
\[
\text{div} \ k = \rho,
\]
\[
\text{curl} \ k = \sigma,
\]
\[
\text{tr} \ k = 0.
\]

Proof. The Hodge system (4.28) is linear and its coefficients depend only on \( r \). Therefore, we may project the equations of (4.28) onto the Hodge-Fourier basis elements. This uses Remark 2.26 and Proposition 2.30. We split (4.28) into the modes \( l = 0, 1 \) and \( l \geq 2 \), which yields the following three subsystems \( S0, S1 \) and \( S2 \).

\[
0 = (\rho_N)^{[0]} - \frac{1}{r^3} \partial_r \left( r^3 \delta^{[0]} \right), \quad (S0.1)
\]

\[
0 = \sigma_{[\geq 2]}^{NN}, \quad (S0.2)
\]
\[
\begin{align*}
\text{d}v^{[1]} &= (\rho_N)^{[1]} - \frac{1}{r^3} \partial_r \left( r^3 \delta^{[1]} \right), \\
\text{curl}^{[1]} &= \sigma_{NN}^{[1]}, \\
\frac{1}{r^3} \nabla_N \left( r^3 \epsilon^{[1]} \right) &= \delta^{[1]} + \frac{1}{2} \nabla \delta^{[1]}, \\
\ast \sigma_N^{[1]} &= \frac{1}{2} \sigma^{[1]} - \frac{1}{2} \nabla \delta^{[1]} - \frac{1}{r} \epsilon^{[1]},
\end{align*}
\tag{S1.1}
\]

\[
\begin{align*}
\text{d}v^{[2]} &= (\rho_N)^{[2]} - \frac{1}{r^3} \partial_r \left( r^3 \delta^{[2]} \right), \\
\text{curl}^{[2]} &= \sigma_{NN}^{[2]}, \\
\frac{1}{r^2} \nabla_N \left( r^2 \epsilon^{[2]} \right) &= \frac{1}{2} \delta^{[2]} + \ast \sigma_N^{[2]} + \nabla \delta^{[2]}, \\
\text{d}v^{\hat{\eta}} &= \frac{1}{2} \delta^{[2]} - \ast \sigma_N^{[2]} - \frac{1}{2} \nabla \delta^{[2]} - \frac{1}{r} \epsilon^{[2]}, \\
\nabla_N \hat{\eta} + \frac{1}{r} \hat{\eta} &= \ast \left( \hat{\varphi} \right) + \frac{1}{2} \nabla \hat{\varphi}^{[2]}.
\end{align*}
\tag{S2.1}
\]

For each of these subsystems, we show that the corresponding parts of \( k, \sigma \) are solutions.

**Analysis of S0.** The two functions \( \delta^{[0]}, \sigma_{NN}^{[0]} \) are radial. Integration of \((S0.1)\) along \( r \) with the trivial boundary condition \( \delta^{[0]} |_{r=1} = 0 \) directly leads to \((4.30)\). \((S0.2)\) is satisfied by \((4.35)\). Therefore, \( \delta^{[0]} \) and \( \sigma_{NN}^{[0]} \) solve \( S0 \).

**Analysis of S1.** The equations \((S1.1)\), \((S1.2)\) are automatically satisfied by the definition of \( \epsilon^{[1]} \) in \((4.40)\). The same holds for \((S1.4)\) by the definition of \( \ast \sigma_N^{[1]} \) in \((4.42)\). It remains to verify that \((S1.3)\) is satisfied.

Applying \( \text{d}v \) and \( \text{curl} \) to \((S1.3)\), plugging in the definition of \( \epsilon^{[1]} \) in \((4.40)\) and using Lemma 2.35, we get that \( \delta^{[1]} \) and \( \sigma_{NN}^{[1]} \) must satisfy the compatibility conditions

\[
\begin{align*}
\frac{1}{2} \Delta \delta^{[1]} + \partial_r^2 \delta^{[1]} + \frac{7}{r} \partial_r \delta^{[1]} + \frac{9}{r^2} \delta^{[1]} &= \frac{1}{r^4} \partial_r \left( r^4 \left( \rho_N \right)^{[1]} \right) - \text{d}v \phi^{[1]}, \\
\frac{1}{r^4} \partial_r \left( r^4 \sigma_{NN}^{[1]} \right) &= \text{curl} \phi^{[1]},
\end{align*}
\tag{4.47}
\]

With regard to the Hodge-Fourier decomposition, it holds that \( \Delta \delta^{[1]} = -\frac{2}{r^2} \phi^{[1]} \). So, \((4.47)\) is satisfied by the definition of \( \delta^{[1]} \) in \((4.31)\). Moreover, \( \sigma_{NN}^{[1]} \) satisfies \((4.48)\) by \((4.36)\). Note that at the level of \( l \geq 1, \mathcal{V}_1 \) is a bijection, see Lemma 2.36, so the above shows that
(S1.3) is satisfied. To summarize, we showed that $\epsilon^{[1]}, \delta^{[1]}, \sigma^{[1]}_N$ and $\epsilon^{[2]}_N$ solve S1.

**Analysis of S2.** In the following, we use that $\Delta = \partial^2_r + \frac{2}{r} \partial_r + \bar{\Delta}$. The equations (S2.1), (S2.2) and (S2.4) are satisfied in view of (4.40) and (4.46). It remains to prove (S2.3) and (S2.5). We start with (S2.3).

Applying $\text{div}$ and $\text{curl}$ to (S2.3) and using (4.40) leads to the compatibility conditions

\[
\begin{align*}
\Delta \delta^{[2]} + \frac{4}{r} \partial_r \delta^{[2]} + \frac{6}{r^2} \delta^{[2]} &= \frac{1}{r^3} \partial_r \left( r^3 (\rho_N)^{[2]} \right) - \text{div} \left( \frac{1}{2} \theta^{[2]} + \epsilon^{[2]}_N \right), \\
\frac{1}{r^3} \partial_r \left( r^3 \sigma^{[2]}_N \right) &= \text{curl} \left( \frac{1}{2} \theta^{[2]} + \epsilon^{[2]}_N \right).
\end{align*}
\]

(4.49) (4.50)

The function $\delta^{[2]}$ defined in (4.32) satisfies (4.49) by the definition of $\epsilon^{[2]}_N$ in (4.43) and the fact that

\[
\text{div} \left( \frac{1}{2} \theta^{[2]} + \epsilon^{[2]}_N \right) = 0,
\]

see the construction of $H^{(\text{lm})}$ in (2.10). Furthermore, the $\epsilon^{[2]}_N$ defined in (4.44) satisfies (4.50) because

\[
\text{curl} \left( \frac{1}{2} \theta^{[2]} + \epsilon^{[2]}_N \right) = 0
\]

by the construction of $E^{(\text{lm})}$ in (2.10). This shows that (S2.3) is satisfied.

We turn now to (S2.5). Applying the divergence operator $\mathcal{D}_2$ to (S2.5) and using (4.46) leads to

\[
\mathcal{D}_2 \left( \hat{\sigma}^{[2]} + \frac{1}{2} \nabla \otimes \epsilon^{[2]} \right) = \frac{1}{r^2} \nabla_N \left( r^2 \left( \frac{1}{2} \theta^{[2]} - \epsilon^{[2]}_N - \frac{1}{2} \nabla \delta^{[2]} - \frac{1}{r} \epsilon^{[2]} \right) \right).
\]

(4.51)

This coincides with the definition of $\hat{\sigma}^{[2]}$ in (4.45) and thus shows that (S2.5) is satisfied.

To summarise, we showed that $\delta^{[2]}, \epsilon^{[2]}, \sigma^{[2]}_N, \epsilon^{[2]}_N, \hat{\sigma}^{[2]}, \hat{\epsilon}^{[2]}_N, \eta$ solve S2. This finishes the proof of Lemma 4.10. \qed

We continue by controlling the regularity and boundary behaviour at $S_1$ of $\zeta_E, \zeta_H$ and $\delta^{[2]}, \sigma^{[2]}_N$.

**Proposition 4.11.** Let $w \geq 2$ be an integer. Let $\rho = (\rho_N, \theta) \in \overline{\mathcal{H}}_{-5/2}^{w-2}$ be given. Let $\zeta_E, \zeta_H$ be the vectorfields defined in (4.33)-(4.34), (4.38)-(4.39), and $\delta^{[2]}, \sigma^{[2]}_N$ be the solutions to the elliptic PDEs (4.32), (4.37), respectively. Then, the following holds.
(1) Regularity and boundary behaviour of $\zeta_E, \zeta_H$. The vectorfields $\zeta_E$ and $\zeta_H$ satisfy
\[
\|\zeta_E\|_{H^{-\frac{1}{2}}_{-5/2}} \lesssim \|\rho\|_{H^{-\frac{1}{2}}_{-5/2}}, \quad \|\zeta_H\|_{H^{-\frac{1}{2}}_{-5/2}} \lesssim \|\rho\|_{H^{-\frac{1}{2}}_{-5/2}}.
\] (4.52)
Moreover, for each $l \geq 2$, $m \in \{-l, \ldots, l\}$, $\zeta_E^{(lm)}$ and $\zeta_H^{(lm)}$ satisfy
\[
\int_1^\infty r^{1-l} \left( \zeta_E^{(lm)} - \left( \frac{l}{\sqrt{l(l+1)}} (\rho_N)^{(lm)} - \theta_E^{(lm)} \right) \right) \, dr = 0,
\] (4.53)
\[
\int_1^\infty r^{l-1-l(l+1)} \left( \zeta_H^{(lm)} + \theta_H^{(lm)} \right) \, dr = 0.
\]

(2) Precise estimate for $\zeta_E$ and $\zeta_H$. It holds that
\[
\|\mathcal{P}_2^{-1} (\nabla_N \zeta_E)\|_{H^{w-2}_{-5/2}(\mathbb{R}^3 \setminus \overline{B}_1)} \lesssim \|\rho\|_{H^{w-2}_{-5/2}(\mathbb{R}^3 \setminus \overline{B}_1)},
\]
\[
\|\mathcal{P}_2^{-1} (\nabla_N \zeta_H)\|_{H^{w-2}_{-5/2}(\mathbb{R}^3 \setminus \overline{B}_1)} \lesssim \|\rho\|_{H^{w-2}_{-5/2}(\mathbb{R}^3 \setminus \overline{B}_1)},
\] (4.54)
with $\mathcal{P}_2^{-1} (\nabla_N \zeta_E), \mathcal{P}_2^{-1} (\nabla_N \zeta_H) \in \mathcal{H}^{w-2}_{-5/2}$.

(3) Elliptic regularity and boundary behaviour of $\delta^{[\geq 2]}, \sigma^{[\geq 2]}_{NN}$. It holds that
\[
\|\delta^{[\geq 2]}\|_{H^{-\frac{1}{2}}_{-3/2}(\mathbb{R}^3 \setminus \overline{B}_1)} \lesssim \|\rho\|_{H^{w-2}_{-5/2}(\mathbb{R}^3 \setminus \overline{B}_1)},
\]
\[
\|\sigma^{[\geq 2]}_{NN}\|_{H^{w-2}_{-5/2}(\mathbb{R}^3 \setminus \overline{B}_1)} \lesssim \|\rho\|_{H^{w-2}_{-5/2}(\mathbb{R}^3 \setminus \overline{B}_1)}.
\] (4.55)
Furthermore,
\[
\delta^{[\geq 2]} \in \mathcal{H}^{w-1}_{-3/2}, \quad \sigma^{[\geq 2]}_{NN} \in \mathcal{H}^{w-2}_{-5/2}.
\]
In particular, for $w > 2$,
\[
\partial_r \delta^{[\geq 2]} \bigg|_{r=1} = 0,
\] (4.56)
and for $w > 3$,
\[
\partial_r \sigma^{[\geq 2]}_{NN} \bigg|_{r=1} = 0.
\] (4.57)

(4) Precise estimate for $\partial_r \sigma^{[\geq 2]}_{NN}$. It holds that
\[
\left\| \mathcal{P}_1^{-1}(0, \partial_r \sigma^{[\geq 2]}_{NN}) \right\|_{H^{w-2}_{-5/2}(\mathbb{R}^3 \setminus \overline{B}_1)} \lesssim \|\rho\|_{H^{w-2}_{-5/2}(\mathbb{R}^3 \setminus \overline{B}_1)}.
\] (4.58)
and moreover, $\mathcal{P}_1^{-1}(0, \partial_r \sigma^{[\geq 2]}_{NN}) \in \mathcal{H}^{w-2}_{-5/2}(\mathbb{R}^3 \setminus \overline{B}_1)$.

Here $\mathcal{P}_1^{-1}, \mathcal{P}_2^{-1}$ denote the inverse operators to the elliptic $\mathcal{P}_1, \mathcal{P}_2$ on $(S_\gamma, \hat{\gamma})$, respectively, see Lemma 2.36.
Remark 4.12. The quantities $\delta^{[2]}$, $\sigma^{[2]}_{NN}$ are solutions to the elliptic equations (4.32), (4.37) on $\mathbb{R}^3 \setminus \overline{B_1}$, respectively. Therefore their boundary regularity at $S_1$ is harder to estimate than for the other components of $k$ and $\sigma$ which all satisfy first order transport equations in $r$ or Hodge systems on $S_r$.

Proof. We first analyse $\zeta_E$ and $\zeta_H$.

(1) Regularity and boundary behaviour of $\zeta_E, \zeta_H$. We begin by showing that the constants $c_E^{(lm)}, c_H^{(lm)}$ in (4.34), (4.39) are well-defined. By Cauchy-Schwarz, for all $l \geq 2$, $m \in \{-l, \ldots, l\}$,

$$\left| c_E^{(lm)} \right| = \left| \int_1^{\infty} r^{-l+1} \left( \frac{l}{\sqrt{l(l+1)}} (\rho_N)^{(lm)} - \varphi_E^{(lm)} \right) dr \right|$$

$$\leq \left( \int_1^{\infty} r^{-2l} dr \right)^{1/2} \left( \int_1^{\infty} \left( \frac{l}{\sqrt{l(l+1)}} (\rho_N)^{(lm)} - \varphi_E^{(lm)} \right)^2 dr \right)^{1/2}$$

$$\lesssim \frac{1}{\sqrt{2l-1}} \left( \int_1^{\infty} \left( r(\rho_N)^{(lm)} \right)^2 dr + \int_1^{\infty} \left( r\varphi_E^{(lm)} \right)^2 dr \right)^{1/2},$$

and also for all $l \geq 2$,

$$\left| c_H^{(lm)} \right| = \left| \int_1^{\infty} r^{-1-\sqrt{l(l+1)+4}} \varphi_H^{(lm)} dr \right|$$

$$\lesssim \left( \frac{1}{2\sqrt{l(l+1)+4+3}} \right)^{1/2} \left( \int_1^{\infty} \left( r\varphi_H^{(lm)} \right)^2 dr \right)^{1/2}. \quad (4.59)$$

We show next that for all integers $w \geq 2$,

$$\| \zeta_E \|_{H^{-w/2}_{-5/2}(\mathbb{R}^3 \setminus \overline{B_1})} \lesssim \| \rho \|_{H^{-w/2}_{-5/2}(\mathbb{R}^3 \setminus \overline{B_1})},$$

$$\| \zeta_H \|_{H^{-w/2}_{-5/2}(\mathbb{R}^3 \setminus \overline{B_1})} \lesssim \| \rho \|_{H^{-w/2}_{-5/2}(\mathbb{R}^3 \setminus \overline{B_1})}. \quad (4.60)$$

Consider first the case $w = 2$ of (4.61) for $\zeta_E$, that is,

$$\| \zeta_E \|_{H^0_{-5/2}(\mathbb{R}^3 \setminus \overline{B_1})} \lesssim \| \rho \|_{H^0_{-5/2}(\mathbb{R}^3 \setminus \overline{B_1})}. \quad (4.62)$$
By (4.33) and (4.38), for all \( l \geq 2 \), \( \zeta_{E}^{(lm)} \) and \( \zeta_{H}^{(lm)} \) vanish outside the interval \((1, 1 + 1/l)\). Therefore, for \( l \geq 2 \),

\[
\int_{1}^{\infty} \left( r \zeta_{E}^{(lm)} \right)^{2} dr = \left( c_{E}^{(lm)} \right)^{2} l^{2} \int_{1}^{1+1/l} r^{2l} \left( (\partial_{r} \chi)(l(r - 1)) \right)^{2} dr
\]
\[
\lesssim \left( c_{E}^{(lm)} \right)^{2} \frac{l^{2}}{2l + 1} \left[ \left( 1 + \frac{1}{l} \right)^{2l+1} - 1 \right] \tag{4.63}
\]
\[
\lesssim \int_{1}^{\infty} \left( r (\rho_{N})^{(lm)} \right)^{2} dr + \int_{1}^{\infty} \left( r \phi_{E}^{(lm)} \right)^{2} dr
\]

where we uniformly estimated \( \partial_{r} \chi \) and used (4.59) in the last step. Similarly, for \( l \geq 2 \), using (4.60),

\[
\int_{1}^{\infty} \left( r \zeta_{H}^{(lm)} \right)^{2} dr = \left( c_{H}^{(lm)} \right)^{2} \int_{1}^{\infty} r^{2+2\sqrt{l(l+1)+4}} (\partial_{r} \chi(l(r - 1)))^{2} dr
\]
\[
= \left( c_{H}^{(lm)} \right)^{2} l^{2} \int_{1}^{1+1/l} r^{2+2\sqrt{l(l+1)+4}} (\partial_{r} \chi)^{2} (l(r - 1)) dr
\]
\[
\lesssim \left( c_{H}^{(lm)} \right)^{2} l^{2} \left( \frac{1}{3 + 2\sqrt{l(l+1)+4}} \right) \left( 1 + \frac{1}{l} \right)^{3+2\sqrt{l(l+1)+4}} - 1
\]
\[
\lesssim \int_{1}^{\infty} \left( r \phi_{H}^{(lm)} \right)^{2} dr.
\]

Summing over \( l \) and \( m \) proves (4.62).
The case $w > 2$ of (4.61) for $\zeta_E$ is derived as follows. By Proposition 2.34, we improve (4.59) as follows,

\[
\left| c_{E}^{(lm)} \right| \lesssim \frac{1}{\sqrt{2l - 1}} \left( \int_{1}^{\infty} \left( r(\rho_{N})^{(lm)} \right)^{2} dr + \int_{1}^{\infty} \left( r\rho_{E}^{(lm)} \right)^{2} dr \right)^{1/2}
\]

\[
\lesssim \frac{1}{\sqrt{2l - 1}} (l(l + 1))^{w-2} \int_{1}^{\infty} \left( r(\rho_{N})^{(lm)} \right)^{2} dr \right)^{1/2}
\]

\[
+ \frac{1}{\sqrt{2l - 1}} (l(l + 1))^{w-2} \int_{1}^{\infty} \left( r\rho_{E}^{(lm)} \right)^{2} dr \right)^{1/2} \tag{4.64}
\]

\[
\lesssim \frac{1}{\sqrt{2l - 1}} (l(l + 1))^{w-2} \int_{1}^{\infty} r^{2(w-2)} \left( \frac{l(l + 1)}{r^{2}} \right)^{w-2} \left( r(\rho_{N})^{(lm)} \right)^{2} dr \right)^{1/2}
\]

\[
+ \frac{1}{\sqrt{2l - 1}} (l(l + 1))^{w-2} \int_{1}^{\infty} r^{2(w-2)} \left( \frac{l(l + 1)}{r^{2}} \right)^{w-2} \left( r\rho_{E}^{(lm)} \right)^{2} dr \right)^{1/2}.
\]

The terms in brackets on the right-hand side, in view of Proposition 2.34, correspond after summing over $l, m$ to the $H_{-5/2}^{w-2}$-norm of $\rho$ which is bounded. These terms are therefore in particular summable.

On the other hand, we can explicitly calculate

\[
\partial_{r} \left( c_{E}^{(lm)} \right) = c_{E}^{(lm)} l - \frac{1}{r} r^{l-1} \partial_{r} (\chi(l(r - 1))) + c_{E}^{(lm)} r^{l-1} l \partial_{r} \left( (\partial_{r} \chi) (l(r - 1)) \right)
\]

\[
\approx \frac{l}{r} \rho_{E}^{(lm)} ,
\]

\[
(\mbox{div}_{\zeta_{E}})^{(lm)} = - \frac{\sqrt{l(l + 1)}}{r} \zeta_{E}^{(lm)} , \quad (\mbox{curl}_{\zeta_{E}})^{(lm)} = 0. \tag{4.65}
\]

Combining (4.64), (4.65) and using Propositions 2.30 and 2.34 and Lemma 2.35, we can estimate the derivatives of $\zeta_{E}$ similarly as in (4.62). This proves (4.61) for $\zeta_{E}$ for all $w \geq 2$. The estimates (4.61) for $\zeta_{H}$ are derived analogously and left to the reader. This proves (4.61) for all $w \geq 2$.

We next show that $\zeta_{E}, \zeta_{H} \in H_{-5/2}^{w-2}$ by proving that there exist sequences $(\zeta_{E})_{n}, (\zeta_{H})_{n}$ of smooth vectorfields with

\[
\text{supp}(\zeta_{E})_{n}, \text{supp}(\zeta_{H})_{n} \subset \subset \mathbb{R}^{3} \setminus \overline{B_{1}} \tag{4.66}
\]
that converge as $n \to \infty$ in $H_{-5/2}^{w-2}$ to

$$
(\zeta_E)_n \to \zeta_E, \quad (\zeta_H)_n \to \zeta_H.
$$

(4.67)

Indeed, let

$$
(\zeta_E)_n := \sum_{l=2}^{n} \sum_{m=-l}^{l} \zeta^{(lm)}_E E^{(lm)},
$$

$$
(\zeta_H)_n := \sum_{l=2}^{n} \sum_{m=-l}^{l} \zeta^{(lm)}_H E^{(lm)}.
$$

By (2.1), (4.33) and (4.38), it follows that for each $n$, these are smooth vectorfields satisfying (4.66). By (4.61), the convergence (4.67) follows.

Next, we prove the integral identities (4.53). The first one follows by

$$
\int_{1}^{\infty} r^{1-l} \zeta^{(lm)}_E \, dr = \int_{1}^{\infty} c^{(lm)}_E \partial_r (\chi((r - 1)l)) \, dr
$$

$$
= c^{(lm)}_E [\chi((r - 1)l)]_1^\infty
$$

$$
= c^{(lm)}_E (1 - 0)
$$

$$
= \int_{1}^{\infty} r^{1-l} \left( \frac{l}{\sqrt{l(l+1)}} (\rho_N)^{(lm)} - \theta^{(lm)}_E \right) \, dr.
$$

The second identity is proven similarly and left to the reader. This proves part (1) of Proposition 4.11.

(2) Precise estimate for $\zeta_E$ and $\zeta_H$. We first prove (4.54). Consider the case $w = 2$ of the estimate for $\zeta_E$ in (4.54). Using the Hodge-Fourier formalism, see Proposition 2.34 and Lemmas 2.35 and 2.36, it suffices to prove

$$
\int_{1}^{\infty} r^2 \left( \frac{r}{\sqrt{2l(l+1)} - 1} \partial_r \zeta^{(lm)}_E \right)^2 \, dr \lesssim \int_{1}^{\infty} (r \theta^{(lm)}_E)^2 + (r \theta^{(lm)}_H)^2 + (r (\rho_N)^{(lm)})^2 \, dr.
$$
By (4.65), we can estimate
\[ \int_{1}^{\infty} r^2 \left( \frac{r}{\sqrt{\frac{1}{2}l(l+1)-1}} \partial_r \zeta_E^{(lm)} \right)^2 \, dr \]
\[ \lesssim \int_{1}^{\infty} r^2 \left( \zeta_E^{(lm)} \right)^2 \, dr + \left( c_E^{(lm)} \right)^2 \int_{1}^{\infty} r^{2l+2} \left( (\partial^2 \chi)(l(r-1)) \right)^2 \, dr \]
\[ \lesssim \int_{1}^{\infty} r^2 \left( \zeta_E^{(lm)} \right)^2 \, dr + l^2 \left( c_E^{(lm)} \right)^2 \int_{1}^{\infty} r^{2l+2} \, dr \]
\[ \lesssim \int_{1}^{\infty} r^2 \left( \zeta_E^{(lm)} \right)^2 \, dr + l^2 \left( c_E^{(lm)} \right)^2 + \left( 1 + \frac{1}{l} \right)^{3+2l} \left( 1 + \frac{1}{l} \right)^{-1} \]
\[ \lesssim \int_{1}^{\infty} r^2 \left( \zeta_E^{(lm)} \right)^2 \, dr + \int_{1}^{\infty} \left( r \psi^{(lm)} \right)^2 \, dr + \left( r \rho_N^{(lm)} \right)^2 \, dr, \]

where we uniformly estimated $\partial^2 \chi$, used the fact that $\text{supp } \partial^2 \chi \subset [1, 1+1/l]$ and (4.59). Together with Proposition 4.11, this proves (4.54) for $w = 2$.

Consider now the case $w > 2$ of (4.54). On the one hand, by the higher regularity of $\psi^{(lm)}_E, (\rho_N)^{(lm)}$, the estimate of $c_E^{(lm)}$ improves, see (4.64). On the other hand, we can differentiate the explicit formula (4.33) by $\partial_r$, while taking angular derivatives correspond to multiplications by $\frac{\sqrt{l(l+1)}}{r}$. All terms appearing can be bounded analogously as in (4.69) by using the improved bounds for $c^{(lm)}$ and the fact that for all $n \geq 1$,
\[ \text{supp } \partial_r^n \chi \subset [1, 1+1/l]. \]

This proves (4.54) for $\zeta_E$ for $w \geq 2$. The proof for $\zeta_H$ is analogous and left to the reader.

It remains to show that $\mathcal{P}^{-1}_2 (\nabla_N \zeta_E), \mathcal{P}^{-1}_2 (\nabla_N \zeta_E) \in \mathcal{H}_{-5/2}^{w-2}$. Consider the statement for $\mathcal{P}^{-1}_2 (\nabla_N \zeta_E)$. For each $n \geq 2$, the smooth $S_r$-tangent tracefree symmetric 2-tensor
\[ V_n := \sum_{l=2}^{n} \sum_{m=-l}^{l} (\mathcal{P}^{-1}_2 (\nabla_N \zeta_E))^{(lm)}_{\psi^{(lm)}} \]
has compact support in $\mathbb{R}^3 \setminus B_1$ by the definition of $\zeta_E$ in (4.33), see Lemma 2.33. Further, by (4.54), $V \rightarrow \mathcal{P}_2^{-1}(\nabla_N \zeta_E)$ as $n \rightarrow \infty$ in $\mathcal{H}^{w-2}_{-5/2}$. By definition of $\mathcal{H}^{w-2}_{-5/2}$, see Definition 2.12, the statement for $\mathcal{P}_2^{-1}(\nabla_N \zeta_E)$ follows. The statement for $\mathcal{P}_2^{-1}(\nabla_N \zeta_H)$ follows analogously. This finishes the proof of part (2) of Proposition 4.11.

(3) Elliptic regularity and boundary behaviour of $\delta^{[\geq 2]}$, $\sigma^{[\geq 2]}_{NN}$. First, by the elliptic theory of Appendix C, it follows that for integers $w \geq 2$, $\delta^{[\geq 2]} \in \mathcal{H}^1 \cap H_{-3/2}^{w-1}(\mathbb{R}^3 \setminus B_1)$, $\sigma^{[\geq 2]}_{NN} \in H_{-5/2}^{w-2}(\mathbb{R}^3 \setminus B_1)$

with estimates

$$
\| \delta^{[\geq 2]} \|_{H_{-3/2}^{w-1}(\mathbb{R}^3 \setminus B_1)} \lesssim \| \rho \|_{\mathcal{H}^{w-2}_{-5/2}(\mathbb{R}^3 \setminus B_1)},
$$

$$
\| \sigma^{[\geq 2]}_{NN} \|_{H_{-5/2}^{w-2}(\mathbb{R}^3 \setminus B_1)} \lesssim \| \rho \|_{\mathcal{H}^{w-2}_{-5/2}(\mathbb{R}^3 \setminus B_1)}.
$$

Indeed, $\delta^{[\geq 2]}$ is estimated in $\mathcal{H}^1_{-3/2}$ by Proposition C.4 and $\sigma^{[\geq 2]}_{NN}$ in $H_{-5/2}^{0}(\mathbb{R}^3 \setminus B_1)$ by Lemma C.11. Higher order regularity follows from Proposition C.6. The corresponding estimates obtained for $\delta^{[\geq 2]}$ and $\sigma^{[\geq 2]}_{NN}$ are in terms of norms of the right-hand sides of (4.32) and (4.37). In turn, these right-hand sides are estimated thanks to Corollary C.13 and the estimates of the part (1) of the proof for $\zeta_E$ and $\zeta_H$.

We demonstrate now the improved boundary behaviour

$$
\delta^{[\geq 2]} \in \mathcal{H}^{w-1}_{-3/2}, \quad \sigma^{[\geq 2]}_{NN} \in \mathcal{H}^{w-2}_{-5/2}.
$$

We only need to consider the cases $w > 2$ for $\delta^{[\geq 2]}$ and $w > 3$ for $\sigma^{[\geq 2]}_{NN}$. Indeed, else the trivial extension to $B_1$ is regular and in view of the boundary conditions

$$
\delta^{[\geq 2]} |_{r=1} = \sigma^{[\geq 2]}_{NN} |_{r=1} = 0,
$$

the statement follows by Proposition 2.13.

By Proposition C.9, it suffices to prove the following claim.

Claim 4.13. If $w > 2$, then it holds that

$$
\partial_r \delta^{[\geq 2]} |_{r=1} = 0,
$$

and if $w > 3$, then

$$
\partial_r \sigma^{[\geq 2]}_{NN} |_{r=1} = 0.
$$
First, by (4.70) it holds that for all \( l \geq 2, m \in \{-l, \ldots, l\}\), if \( w > 2, w > 3 \), respectively,

\[
\int_1^{\infty} (\delta^{(lm)})^2 \, dr, \quad \int_1^{\infty} (1 + r)^2 (\partial_r \delta^{(lm)})^2 \, dr, \quad \int_1^{\infty} (1 + r)^4 (\partial_r^2 \delta^{(lm)})^2 \, dr < \infty,
\]

\[
\int_1^{\infty} (1 + r)^2 (\sigma_{NN}^{(lm)})^2 \, dr, \quad \int_1^{\infty} (1 + r)^4 (\partial_r \sigma_{NN}^{(lm)})^2 \, dr, \quad \int_1^{\infty} (1 + r)^6 (\partial_r^2 \sigma_{NN}^{(lm)})^2 \, dr < \infty.
\]

By Lemma 2.14, it follows that

\[
\sup_{r \in (1, \infty)} (1 + r)^{1/2} \delta^{(lm)}, \quad \sup_{r \in (1, \infty)} (1 + r)^{3/2} \partial_r \delta^{(lm)} < \infty, \quad (4.73)
\]

\[
\sup_{r \in (1, \infty)} (1 + r)^{3/2} \sigma_{NN}^{(lm)}, \quad \sup_{r \in (1, \infty)} (1 + r)^{5/2} \partial_r \sigma_{NN}^{(lm)} < \infty.
\]

We show now that if \( w > 2 \), then for all \( l \geq 2, m \in \{-l, \ldots, l\}\),

\[
\partial_r \delta^{(lm)} |_{r=1} = 0.
\]

Definition (4.32) is in the Hodge-Fourier formalism equivalent to the following ODEs on \( r \in (1, \infty) \) for \( \delta^{(lm)} \) with \( l \geq 2, m \in \{-l, \ldots, l\} \), see Lemma 2.33,

\[
r^{l-2} \partial_r \left( r^{-2l} \partial_r \left( r^{l+2} \delta^{(lm)} \right) \right) = \frac{1}{r^2} \partial_r \left( r^2 \rho_N^{(lm)} \right) - \frac{\sqrt{l(l+1)}}{r} \left( \rho_E^{(lm)} + \zeta_E^{(lm)} \right). \quad (4.74)
\]

On the one hand, using that \( \delta^{(lm)} |_{r=1} = 0, l \geq 2 \), and (4.73), we get

\[
\int_1^{\infty} \partial_r \left( r^{-2l} \partial_r \left( r^{l+2} \delta^{(lm)} \right) \right) \, dr = \left[ r^{2-l} \partial_r \delta^{(lm)} + (l+2) r^{1-l} \delta^{(lm)} \right]_1^{\infty}
\]

\[
= -\partial_r \delta^{(lm)} |_{r=1}.
\]

On the other hand, by (4.74),

\[
\int_1^{\infty} \partial_r \left( r^{-2l} \partial_r \left( r^{l+2} \delta^{(lm)} \right) \right) = \int_1^{\infty} r^{-l} \partial_r \left( r^2 \rho_N^{(lm)} \right) - \sqrt{l(l+1)} r^{1-l} \left( \rho_E^{(lm)} + \zeta_E^{(lm)} \right)
\]

\[
= \left[ r^{2-l} \rho_N^{(lm)} \right]_1^{\infty}_{=0}
\]

\[
+ l \int_1^{\infty} r^{1-l} \left( \rho_N^{(lm)} - \frac{\sqrt{l(l+1)}}{l} \left( \rho_E^{(lm)} + \zeta_E^{(lm)} \right) \right)
\]

\[
= 0,
\]
where the boundary term vanished because \( \rho_N \in \overline{H}_w^{-2} \), and where we also used the integral identity (4.53). This shows that
\[
\partial_r \delta^{(lm)}|_{r=1} = 0
\]
for all \( l \geq 2, m \in \{-l, \ldots, l\} \) and proves (4.71).

We show now that if \( w > 3 \), then for all \( l \geq 2, m \in \{-l, \ldots, l\} \),
\[
\partial_r \sigma^{(lm)}_{NN} |_{r=1} = 0.
\]

Definition (4.37) is in the Hodge-Fourier formalism equivalent to the following ODEs on \( r \in (1, \infty) \) for \( \sigma^{(lm)}_{NN} \) with \( l \geq 2, m \in \{-l, \ldots, l\} \), see Lemma 2.33,
\[
r \sqrt{l(l+1)+4} \partial_r \left( r^{1-2\sqrt{l(l+1)+4}} \partial_r \left( r \sqrt{l(l+1)+4} \sigma^{(lm)}_{NN} \right) \right) = \left( \frac{\sqrt{l(l+1)}}{r^2} \left( \vartheta^{(lm)}_H + \zeta^{(lm)}_H \right) \right). \tag{4.76}
\]

On the one hand, using that \( \sigma^{(lm)}_{NN}|_{r=1} = 0, l \geq 2 \) and (4.73),
\[
\int_1^{\infty} \partial_r \left( r^{1-2\sqrt{l(l+1)+4}} \partial_r \left( r \sqrt{l(l+1)+4} \sigma^{(lm)}_{NN} \right) \right) = \left[ \sqrt{l(l+1) + 4r - \sqrt{l(l+1)+4}} \sigma^{(lm)}_{NN} \right]_{1}^{\infty}
\]
\[
+ \left[ r^{1-\sqrt{l(l+1)+4}} \partial_r \sigma^{(lm)}_{NN} \right]_{1}^{\infty}
\]
\[
= - \partial_r \sigma^{(lm)}_{NN} |_{r=1}.
\]

On the other hand, using (4.76),
\[
\int_1^{\infty} \partial_r \left( r^{1-2\sqrt{l(l+1)+4}} \partial_r \left( r \sqrt{l(l+1)+4} \sigma^{(lm)}_{NN} \right) \right)
\]
\[
= \int_1^{\infty} r^{2-\sqrt{l(l+1)+4}} \partial_r \left( \frac{\sqrt{l(l+1)}}{r^2} \left( \vartheta^{(lm)}_H + \zeta^{(lm)}_H \right) \right)
\]
\[
= \left[ r^{-\sqrt{l(l+1)+4}} \sqrt{l(l+1)} \left( \vartheta^{(lm)}_H + \zeta^{(lm)}_H \right) \right]_{1}^{\infty}
\]
\[
+ \sqrt{l(l+1)} \left( \sqrt{l(l+1)+4} - 2 \right) \int_1^{\infty} r^{-\sqrt{l(l+1)+4} - 1} \left( \vartheta^{(lm)}_H + \zeta^{(lm)}_H \right)
\]
\[
= 0,
\]
where we used the integral identity (4.53) and the boundary term vanished because \( \rho, \zeta_H \in H_{-5/2}^{w-2} \). This shows that for all \( l \geq 2, m \in \{-l, \ldots, l\} \),

\[
\partial_r \sigma^{[\geq 2]}_{NN}\big|_{r=1} = 0
\]

and proves (4.72). This finishes the proof of Claim 4.13. Hence, we have obtained the control of \( \delta^{[\geq 2]} \in H_{-3/2}^{w-1}, \sigma^{[\geq 2]}_{NN} \in H_{-5/2}^{w-2} \). This finishes the proof of part (3) of Proposition 4.11.

**Remark 4.14.** For \( l \geq 2, m \in \{-l, \ldots, l\} \), if \( \rho^{(lm)} \in H_{-5/2}^{w-2} \) is compactly supported in \( \mathbb{R}^3 \setminus \overline{B_1} \), then \( \sigma^{(lm)}_{NN} \) is compactly supported in \( \mathbb{R}^3 \setminus \overline{B_1} \). Indeed, this follows by integrating the radial ODE (4.76) and using that by the construction of \( \zeta^{(lm)}_H \), \( \text{supp} \zeta^{(lm)}_H \subset \subset \mathbb{R}^3 \setminus \overline{B_1} \) and \( \partial_r \sigma^{(lm)}_{NN}\big|_{r=1} = 0 \).

(4) Precise estimate for \( \partial_r \sigma^{[\geq 2]}_{NN} \). First consider the case \( w = 2 \) of (4.58), that is,

\[
\left\| \Psi^{-1}_1(0, \partial_r \sigma^{[\geq 2]}_{NN}) \right\|_{H_{-5/2}^0(\mathbb{R}^3 \setminus \overline{B_1})} \lesssim \| \rho \|_{H_{-5/2}^0(\mathbb{R}^3 \setminus \overline{B_1})}.
\]

(4.77)

Using the Fourier-Hodge formalism and the previous estimates for \( \sigma^{[\geq 2]}_{NN}, \zeta_H \), it suffices to prove

\[
\int_1^\infty r^2 \left( \frac{r}{\sqrt{l(l+1)}} \partial_r \sigma^{(lm)}_{NN} \right)^2 \, dr \lesssim \int_1^\infty \left( r \theta^{(lm)}_H \right)^2 + \left( r \sigma^{(lm)}_{NN} \right)^2 + \left( r \zeta^{(lm)}_H \right)^2 \, dr.
\]

(4.78)

First, we rewrite the integrand by using (4.76). Indeed, multiplying (4.76) by \( r^{-\sqrt{l(l+1)+4}} \) and integrating from 1 to \( r \geq 1 \) leads, after integration by parts, to the expression

\[
\frac{r}{\sqrt{l(l+1)}} \partial_r \sigma^{(lm)}_{NN} = -\frac{\sqrt{l(l+1)+4}}{\sqrt{l(l+1)}} \sigma^{(lm)}_{NN} + \left( \theta^{(lm)}_H + \zeta^{(lm)}_H \right)
\]

(4.79)

\[
-\left( 2 - \sqrt{l(l+1)+4} \right) r^{\sqrt{l(l+1)+4}} \left( \int_1^r \left( r' - \sqrt{l(l+1)+4} - 1 \right) \left( \theta^{(lm)}_H + \zeta^{(lm)}_H \right) \, dr' \right),
\]

where the boundary terms at \( r = 1 \) vanished because \( \rho, \zeta_H \in H_{-5/2}^{w-2} \) and \( \sigma^{[\geq 2]}_{NN}\big|_{r=1} = \partial_r \sigma^{[\geq 2]}_{NN}\big|_{r=1} = 0 \).
We are now in the position to prove (4.78). We have

\[
\int_1^\infty r^2 \left( \frac{r}{\sqrt{l(l+1)}} \partial_r \sigma_{NN}^{(lm)} \right)^2 dr
\]

\[
\lesssim \int_1^\infty r^2 \left[ \left( \sigma_{NN}^{(lm)} \right)^2 + \left( \varphi_H^{(lm)} \right)^2 + \left( \zeta_H^{(lm)} \right)^2 \right] dr
\]

\[
+ \left( \sqrt{l(l+1)} + 2 \right)^2 \int_1^\infty r^2 \sqrt{l(l+1)+4+2} \left( \int_1^r (r')^{-\sqrt{l(l+1)+4+1}} \left( \varphi_H^{(lm)} + \zeta_H^{(lm)} \right) dr' \right)^2 dr.
\]

By the integral identity (4.53) we can rewrite \( I_2 \) and use integration by parts to get

\[
I_2 = \int_1^\infty r^2 \sqrt{l(l+1)+4+2} \left( \int_r^\infty (r')^{-\sqrt{l(l+1)+4+1}} \left( \varphi_H^{(lm)} + \zeta_H^{(lm)} \right) dr' \right)^2 dr
\]

\[
= \left[ \frac{1}{2\sqrt{l(l+1)+4+3}} \right]^2 \int_1^\infty r^2 \sqrt{l(l+1)+4+3} \left( \int_r^\infty (r')^{-\sqrt{l(l+1)+4+1}} \left( \varphi_H^{(lm)} + \zeta_H^{(lm)} \right) dr' \right)^2 dr
\]

\[
+ 2 \left. \int_1^\infty \frac{r \sqrt{l(l+1)+4+2}}{2\sqrt{l(l+1)+4+3}} \left( \varphi_H^{(lm)} + \zeta_H^{(lm)} \right) \left( \int_r^\infty (r')^{-\sqrt{l(l+1)+4+1}} \left( \varphi_H^{(lm)} + \zeta_H^{(lm)} \right) dr' \right) dr. \]

The boundary term on the right-hand side can be estimated as follows

\[
\left. \left| \int_r^\infty (r')^{-\sqrt{l(l+1)+4+1}} \left( \varphi_H^{(lm)} + \zeta_H^{(lm)} \right) dr' \right| \right| \leq r^2 \sqrt{l(l+1)+4+3} \left( \int_r^\infty (r')^{-2\sqrt{l(l+1)+4+1}} dr' \right) \left( \int_1^\infty \left( \varphi_H^{(lm)} \right)^2 + \left( \zeta_H^{(lm)} \right)^2 dr \right)
\]

\[
\lesssim \frac{1}{2\sqrt{l(l+1)+4+3}} \left( \int_1^\infty \left( \varphi_H^{(lm)} \right)^2 + \left( \zeta_H^{(lm)} \right)^2 dr \right).
\]
The integral term on the right-hand side of (4.81) is estimated by Cauchy-Schwarz as
\[
\int_1^\infty r^2 \sqrt{\frac{r}{l(l+1)+4+2}} \left( \frac{\theta_H^{(lm)}}{r} + s_H^{(lm)} \right) \left( \int_1^r \left( \frac{dr'}{r'} \right)^{l(l+1)+3} \left( \frac{\theta_H^{(lm)}}{r} + s_H^{(lm)} \right) dr' \right) dr \\
\leq (I_2)^{1/2} \left( \int_1^\infty \left( r \theta_H^{(lm)} \right)^2 + \left( r s_H^{(lm)} \right)^2 \right)^{1/2}.
\]
Putting everything together, we arrive at
\[
I_2 \lesssim \frac{1}{(2\sqrt{l(l+1)+4+3})^2} \left( \int_1^\infty \left( r \theta_H^{(lm)} \right)^2 + \left( r s_H^{(lm)} \right)^2 \right)
\]
Plugging this into (4.80) yields
\[
\int_1^\infty r^2 \left( \frac{r}{\sqrt{l(l+1)}} \partial_r \sigma_{NN}^{(lm)} \right)^2 dr \lesssim \int_1^\infty \left( r \theta_H^{(lm)} \right)^2 + \left( r s_H^{(lm)} \right)^2 + \left( r \zeta_H^{(lm)} \right)^2 dr.
\]
This proves (4.77), that is, the case \( w = 2 \) of (4.58).

We turn now to the case \( w > 2 \) of (4.58). By differentiating (4.79) in \( r \) or taking the tangential derivative \( \mathbf{V} \) which on the Fourier side amounts to multiplication by \( \frac{\sqrt{l(l+1)}}{r} \), and using Proposition 2.34 and Lemma 2.35, we eventually get that for all \( w \geq 2 \),
\[
\left\| \mathcal{D}_1^{-1} \left( 0, \partial_r \sigma_{NN}^{[2]} \right) \right\|_{\mathcal{H}_{-5/2}^{w-2}(\mathbb{R}^3 \setminus B_1)} \lesssim \left\| \rho \right\|_{\mathcal{H}_{-5/2}^{w-2}(\mathbb{R}^3 \setminus B_1)},
\]
that is, we proved (4.58).

It remains to show that \( \mathcal{D}_1^{-1}(0, \partial_r \sigma_{NN}^{[2]}) \in \mathcal{H}_{-5/2}^{w-2} \). By definition, it suffices to prove that there is a sequence \( X_n \) of smooth vectorfields on \( \mathbb{R}^3 \setminus B_1 \) with
\[
\text{supp } X_n \subset \subset \mathbb{R}^3 \setminus B_1
\]
that converges as \( n \to \infty \) in \( \mathcal{H}_{-5/2}^{w-2} \):
\[
X_n \to \mathcal{D}_1^{-1} \left( 0, \partial_r \sigma_{NN}^{[2]} \right).
\]
Let \( \rho_n \) be a sequence of smooth vectorfields on \( \mathbb{R}^3 \setminus B_1 \) such that for all \( n \),
\[
\text{supp } \rho_n \subset \subset \mathbb{R}^3 \setminus B_1
\]
and as \( n \to \infty \), in \( \mathcal{H}_{-5/2}^{w-2}(\mathbb{R}^3 \setminus B_1) \),
\[
\rho_n \to \rho.
Consider the sequence
\[ \rho_n^{[\leq n]} := \sum_{l=1}^{n} \sum_{m=-l}^{l} \left( (\rho_n)_E^{(lm)} E^{(lm)} + (\rho_n)_H^{(lm)} H^{(lm)} \right) \]
which satisfies for all \( n \),
\[ \text{supp} \rho_n^{[\leq n]} \subset \subset \mathbb{R}^3 \setminus \overline{B_1} \]
and as \( n \to \infty \), in \( \mathcal{H}_{-5/2}^w (\mathbb{R}^3 \setminus \overline{B_1}) \),
\[ \rho_n^{[\leq n]} \to \rho. \]
By Remark 4.14 and the higher regularity estimates (4.52) and (4.70), it follows that solutions \( (\sigma_{[\geq 2]}^N)_n \) to (4.37) with \( \rho_n^{[\leq n]} \) and corresponding \( (\zeta_H)_n \) defined in (4.38) on the right-hand side are smooth and satisfy
\[ \text{supp} \left( \sigma_{[\geq 2]}^N \right)_n \subset \subset \mathbb{R}^3 \setminus \overline{B_1}. \]
This shows that
\[ X_n := \mathcal{D}_1^{-1} \left( 0, \partial_r \left( \sigma_{[\geq 2]}^N \right)_n \right), \]
is a sequence of smooth vectorfields with
\[ \text{supp} X_n \subset \subset \mathbb{R}^3 \setminus \overline{B_1}. \]
Furthermore, by linearity and (4.58), as \( n \to \infty \),
\[ \left\| X_n - \mathcal{D}_1^{-1} \left( 0, \partial_r \sigma_{[\geq 2]}^N \right) \right\|_{\mathcal{H}_{-5/2}^w (\mathbb{R}^3 \setminus \overline{B_1})} \lesssim \left\| (\rho_n^{[\leq n]})_{[\geq 2]} - \rho_{[\geq 2]} \right\|_{\mathcal{H}_{-5/2}^w} \to 0. \]
The above implies that \( \mathcal{D}_1^{-1} (0, \partial_r, \sigma_{[\geq 2]}^N) \in \mathcal{H}_{-5/2}^w \). This finishes the control of \( \mathcal{D}_1^{-1} (0, \partial_r, \sigma_{[\geq 2]}^N) \) and hence concludes the proof of Proposition 4.11.

In the following lemma, we estimate all quantities that were not yet bounded in Proposition 4.11 and obtain the full regularity and boundary control of \( k \) and \( \sigma \).

**Lemma 4.15** (Full boundary control and regularity). For \( \rho = (\rho_N, \rho) \in \mathcal{H}_{-5/2}^w \), the symmetric 2-tensors \( k \) and \( \sigma \) defined in (4.29)-(4.46) satisfy \( k \in \mathcal{H}_{-3/2}^{w-1}, \sigma \in \mathcal{H}_{-5/2}^{w-2} \), with
\[
\left\| k \right\|_{\mathcal{H}_{-3/2}^{w-1}} \lesssim \left\| \rho \right\|_{\mathcal{H}_{-5/2}^{w-2}}, \\
\left\| \sigma \right\|_{\mathcal{H}_{-5/2}^{w-2}} \lesssim \left\| \rho \right\|_{\mathcal{H}_{-5/2}^{w-2}}.
\] (4.83)

**Proof.** In view of Lemma 2.19 and the decomposition of \( k \) and \( \sigma \) introduced at the beginning of Section 4.3.1, we prove that \( \delta, \epsilon, \hat{\eta} \in \mathcal{H}_{-3/2}^{w-1} \) and \( \sigma_{NN}, \ast \sigma_{NN}, \ast (\phi) \in \mathcal{H}_{-5/2}^{w-2} \) together with quantitative estimates. We estimate the terms in the order they were introduced in (4.29)-(4.46).
Control of $\delta$.

*Control of $\delta^0$.* First we show that for all $w \geq 2$,}

\[
\|\delta^0\|_{H^{w-1}_{-3/2}} \lesssim \|\rho\|_{H^{w-2}_{-3/2}}.
\]

(4.84)

First consider the case $w = 2$, that is,

\[
\|\delta^0\|_{H^{1}_{-3/2}} \lesssim \|\rho\|_{H^0_{-3/2}}.
\]

By (4.30) we can rewrite

\[
\|\delta^0\|_{H^{0}_{-3/2}}^2 = \int_1^\infty \int_{S_r} (\delta^0)^2 \, dr = \frac{4\pi r^2}{1} \left( \int_1^r (\rho_N)^0 \, dr' \right)^2 \, dr
\]

\[
= \frac{1}{4\pi} \int_1^\infty \frac{1}{r^3} \left( \int_1^r \int_{S_{r'}} (\rho_N)^0 \, dr' \right)^2 \, dr,
\]

where we used that $\delta^0$ and $(\rho_N)^0$ are radial. This expression allows us to estimate by partial integration

\[
\|\delta^0\|_{H^{0}_{-3/2}}^2 = \left[ \left( -\frac{1}{3r^3} \right) \left( \int_1^r \int_{S_{r'}} (\rho_N)^0 \, dr' \right)^2 \right]_1^\infty
\]

\[
+ \frac{2}{3} \int_1^\infty \frac{1}{r^3} \left( \int_{S_r} (\rho_N)^0 \right) \left( \int_1^r \int_{S_{r'}} (\rho_N)^0 \, dr' \right) \, dr
\]

\[
\left( \int_1^\infty \left( \int_{S_r} (\rho_N)^0 \right)^2 \right)^{1/2} \left( \int_1^\infty \frac{1}{r^4} \left( \int_1^r \int_{S_{r'}} (\rho_N)^0 \, dr' \right)^2 \right)^{1/2}
\]

\[
\|\delta^0\|_{H^{0}_{-3/2}}^2 + \frac{2}{3} \left( \int_1^\infty \left( \int_{S_r} (\rho_N)^0 \right)^2 \, dr \right)^{1/2} \left( \int_1^\infty \frac{1}{r^4} \left( \int_1^r \int_{S_{r'}} (\rho_N)^0 \, dr' \right)^2 \, dr \right)^{1/2}
\]

(4.85)
where the boundary term was discarded because it has non-positive sign. This implies that

\[ \| \delta^{[0]} \|_{H^{-3/2}} \lesssim \int_1^\infty \left( \int_{S_r} (\rho N)^{[0]} \right)^2 \, dr \]

\[ \lesssim \int_1^\infty \int_{S_r} \left( r (\rho N)^{[0]} \right)^2 \, dr \]

\[ \lesssim \| (\rho N)^{[0]} \|^2_{H^{-5/2}}. \]  \hfill (4.86)

The radial derivative \( \partial_r \delta^{[0]} \) equals by (4.30)

\[ \partial_r \delta^{[0]} = -\frac{3}{r} \delta^{[0]} + (\rho N)^{[0]}. \]  \hfill (4.87)

This yields with (4.86) the estimate

\[ \| \partial_r \delta^{[0]} \|_{H^{-5/2}} \lesssim \| (\rho N)^{[0]} \|_{H^{-5/2}}. \]  \hfill (4.88)

The tangential derivative vanishes because \( \delta^{[0]} \) is radial. This proves the case \( w = 2 \) of (4.84).

Consider now the case \( w > 2 \) of (4.84). Higher radial regularity follows by differentiating (4.87), and higher tangential regularity is trivial since \( \delta^{[0]} \) is radial. This proves (4.84) for \( w \geq 2 \).

For the control of \( \delta^{[0]} \) it remains to show that \( \delta^{[0]} \in H^{-3/2} \). Indeed, this follows by (4.87), the fact that \( \rho \in H^{-2} \) and Proposition 2.13. This finishes the control of \( \delta^{[0]} \).

**Control of \( \delta^{[1]} \).** First we show that for all \( w \geq 2 \),

\[ \| \delta^{[1]} \|_{H^{-3/2}} \lesssim \| \rho \|_{H^{-2}}. \]  \hfill (4.89)

Consider first the case \( w = 2 \), that is,

\[ \| \delta^{[1]} \|_{H^{-3/2}} \lesssim \| \rho \|_{H^{-2}}. \]

Integrating (4.31) yields the explicit form

\[ \delta^{[1]} = \int_1^r \left( \int_{r'}^r \left( \frac{1}{r''} \partial_r ((r'')^4 (\rho N)^{[1]}) - (r'')^3 \right) dr'' \right) \, dr'. \]  \hfill (4.90)
Using (4.90) and integration by parts in \( r \), we estimate

\[
\|\delta^{[1]}\|^2_{H^0_{-3/2}(\mathbb{R}^3 \setminus B_1)} = \int_1^\infty \int_{S_r} \left( \delta^{[1]} \right)^2 dr \\
= \int_1^\infty \int_{S_r} \frac{1}{r^8} \left( \int_1^r r' I_1(r') dr' \right)^2 dr \\
= - \frac{1}{5} \left[ \int_1^\infty \int_{S_r} \frac{1}{r^7} \left( \int_1^r r' I_1(r') dr' \right)^2 dr \\
+ \frac{2}{5} \int_1^\infty \int_{S_r} \frac{1}{r^6} \left( r I_1(r) \right) \left( \int_1^r r' I_1(r') dr' \right) dr \right] \\
\leq \frac{2}{5} \left( \int_1^\infty \int_{S_r} \frac{1}{r^8} \left( \int_1^r r' I_1(r') dr' \right)^2 dr \right)^{1/2} \left( \int_1^\infty \frac{1}{r^4} (I_1)^2(r) dr \right)^{1/2} \\
= \frac{2}{5} \|\delta^{[1]}\|_{H^0_{-3/2}(\mathbb{R}^3 \setminus B_1)} \left( \int_1^\infty \frac{1}{r^4} (I_1)^2(r) dr \right)^{1/2},
\]

where the boundary term was discarded because of its non-positive sign. This shows that

\[
\|\delta^{[1]}\|^2_{H^0_{-3/2}(\mathbb{R}^3 \setminus B_1)} \lesssim \int_1^\infty \frac{1}{r^4} (I_1)^2(r) dr. \quad (4.91)
\]

By a similar integration by parts, we further have

\[
\int_1^\infty \int_{S_r} \frac{1}{r^4} (I_1)^2(r) dr \lesssim \|\rho\|_{H^0_{-5/2}(\mathbb{R}^3 \setminus B_1)}^2, \quad (4.92)
\]

where we used that at \( l = 1 \),

\[
\| \nabla \rho^{[1]} \|_{H^0_{-7/2}} \lesssim \| \rho^{[1]} \|_{H^0_{-5/2}}.
\]

Together, (4.91) and (4.92) imply

\[
\|\delta^{[1]}\|_{H^0_{-3/2}(\mathbb{R}^3 \setminus B_1)} \lesssim \|\rho\|_{H^0_{-5/2}(\mathbb{R}^3 \setminus B_1)}.
\]
Moreover, by (4.90) the radial derivative $\partial_r \delta^{[1]}$ is

$$\partial_r \delta^{[1]} = -\frac{4}{r} \delta^{[1]} + \frac{1}{r^3} f_1(r).$$

Therefore (4.91) and (4.92) imply that

$$\|\partial_r \delta^{[1]}\|_{H^{0}_{-5/2} (\mathbb{R}^3 \setminus B_1)} \lesssim \|\rho\|^2_{H^{0}_{-5/2} (\mathbb{R}^3 \setminus B_1)}.$$ 

The tangential regularity of $\delta^{[1]}$ follows immediately from the fact that $l = 1$,

$$\|\nabla \delta^{[1]}\|_{H^{0}_{-3/2} (\mathbb{R}^3 \setminus B_1)} \lesssim \|\delta^{[1]}\|_{H^{0}_{-3/2}}.$$ 

This proves the case $w = 2$ of (4.89).

We turn now to the case $w > 2$ of (4.89). For higher radial regularity, differentiate the defining ODE (4.31),

$$\begin{align*}
\left\{ \partial^2_r \delta^{[1]} + \frac{2}{r^2} \partial_r \delta^{[1]} + \frac{8}{r^2} \delta^{[1]} = \frac{1}{r^4 \partial_r} \left( r^4 \rho_N \right)^{[1]} - \text{div} \theta^{[1]}, \quad \text{on } \mathbb{R}^3 \setminus B_1 \right. \\
\delta^{[1]}|_{r=1} = \partial_r \delta^{[1]}|_{r=1} = 0.
\end{align*}$$

(4.93)

Higher tangential regularity follows at the level of $l = 1$ in the Hodge-Fourier decomposition by the observation that for $n \geq 0$

$$\|\nabla^n \delta^{[1]}\|_{H^{0}_{-3/2-n} (\mathbb{R}^3 \setminus B_1)} \lesssim \|\delta^{[1]}\|_{H^{0}_{-3/2} (\mathbb{R}^3 \setminus B_1)} \lesssim \|\rho\|_{H^{0}_{-5/2} (\mathbb{R}^3 \setminus B_1)}.$$ 

(4.94)

This proves (4.89) for $w \geq 2$.

It remains to show that $\delta^{[1]} \in \mathcal{H}^{-1}_{-3/2}$. Indeed, this follows by (4.93) and $\rho \in \mathcal{H}^{-2}_{-5/2}$ with Proposition 2.13. This finishes the control of $\delta^{[1]}$.

The full control of $\delta$. Recall that

$$\delta = \delta^{[0]} + \delta^{[1]} + \delta^{[2]}.$$ 

Above we proved that for $w \geq 2$, $\delta^{[0]}, \delta^{[1]} \in \mathcal{H}^{-1}_{-3/2}$ with the estimate

$$\|\delta^{[0]} + \delta^{[1]}\|_{H^{w-1}_{-3/2} (\mathbb{R}^3 \setminus B_1)} \lesssim \|\rho\|_{\mathcal{H}^{w-2}_{-3/2}}.$$ 

In Proposition 4.11, we proved that for $w \geq 2$, $\delta^{[2]} \in \mathcal{H}^{-1}_{-3/2}$ with the estimate

$$\|\delta^{[2]}\|_{H^{w-1}_{-3/2} (\mathbb{R}^3 \setminus B_1)} \lesssim \|\rho\|_{\mathcal{H}^{w-2}_{-3/2}}.$$ 

Together this proves that for $w \geq 2$, $\delta \in \mathcal{H}^{-1}_{-3/2}$ with the estimate

$$\|\delta\|_{H^{w-1}_{-3/2} (\mathbb{R}^3 \setminus B_1)} \lesssim \|\rho\|_{\mathcal{H}^{w-2}_{-3/2}}.$$
and hence finishes the control of $\delta$.

**Control of $\sigma_{NN}$**.

*Control of $\sigma_{NN}^{[1]}$.* First we show that for all $w \geq 2$,

$$\|\sigma_{NN}^{[1]}\|_{H_{w-5/2}^w(\mathbb{R}^3\setminus B_1)} \lesssim \|\psi^{[1]}\|_{H_{w-5/2}^{w-2}(\mathbb{R}^3\setminus B_1)}. \tag{4.95}$$

Consider first the case $w = 2$, that is,

$$\|\sigma_{NN}^{[1]}\|_{H_{0}^2(\mathbb{R}^3\setminus B_1)} \lesssim \|\psi^{[1]}\|_{H_{0}^2(\mathbb{R}^3\setminus B_1)}.$$  

Recall (4.36),

$$\sigma_{NN}^{[1]} = \frac{1}{r^4} \int_{1}^{r} (r')^4 \text{curl} \psi^{[1]} dr'.$$

Using this expression, the case $w = 2$ of (4.95) can be derived like for $\delta^{[0]}$ before, see (4.85) and (4.86). In particular, use that at $l = 1$,

$$\|\text{curl} \psi^{[1]}\|_{H_{-7/2}^0(\mathbb{R}^3\setminus B_1)} \lesssim \|\psi^{[1]}\|_{H_{-5/2}^0(\mathbb{R}^3\setminus B_1)}.$$

We turn now to the case $w > 2$ of (4.95). Higher radial regularity is proved by using and differentiating the defining ODE,

$$\begin{cases}
\partial_r \sigma_{NN}^{[1]} + \frac{2}{r} \sigma_{NN}^{[1]} = \text{curl} \psi^{[1]} \\
\sigma_{NN}^{[1]}|_{r=1} = 0.
\end{cases} \tag{4.96}$$

Higher tangential regularity is automatic at $l = 1$, as in (4.94). This proves (4.95) for all $w \geq 2$.

It remains to show that $\sigma_{NN}^{[1]} \in \mathcal{H}_{w-5/2}^{w-2}$. This follows by (4.96) and $\rho \in \mathcal{H}_{-5/2}^{w-2}$ with Proposition 2.13. This finishes the control of $\sigma_{NN}^{[1]}$.

The full control of $\sigma_{NN}$. Recall that

$$\sigma_{NN} = \sigma_{NN}^{[1]} + \sigma_{NN}^{[\geq 2]}.$$  

Above, we proved that for $w \geq 2$, $\sigma_{NN}^{[1]} \in \mathcal{H}_{w-5/2}^{w-2}$ with the estimate

$$\|\sigma_{NN}^{[1]}\|_{\mathcal{H}_{w-5/2}^{w-2}} \lesssim \|\rho\|_{\mathcal{H}_{-5/2}^{w-2}}.$$
In Proposition 4.11, we proved that for $w \geq 2$, $\sigma_{NN}^{[\geq 2]} \in \overline{H}^{w-2}_{-5/2}$ with the estimate

$$\| \sigma_{NN}^{[\geq 2]} \|_{\overline{H}^{w-2}_{-5/2}} \lesssim \| \rho \|_{\overline{H}^{w-2}_{-5/2}} .$$

Together this proves that for $w \geq 2$, $\sigma_{NN} \in \overline{H}^{w-2}_{-5/2}$ with the estimate

$$\| \sigma_{NN} \|_{\overline{H}^{w-2}_{-5/2}} \lesssim \| \rho \|_{\overline{H}^{w-2}_{-5/2}}$$

and hence finishes the control of $\sigma_{NN}$.

**Control of $\epsilon$.** First we show that for $w \geq 2$

$$\| \epsilon \|_{H^{w-1}_{-3/2}(\mathbb{R}^3\setminus B_1)} \lesssim \| \rho \|_{\overline{H}^{w-2}_{-5/2}(\mathbb{R}^3\setminus B_1)} .$$

Consider first the case $w = 2$ of (4.97),

$$\| \epsilon \|_{H^1_{-3/2}(\mathbb{R}^3\setminus B_1)} \lesssim \| \rho \|_{\overline{H}^{0}_{-5/2}(\mathbb{R}^3\setminus B_1)} .$$

By (4.40) we have on $S_r$, for $r \geq 1$,

$$\Psi_1 \epsilon = \left( \rho_N - \frac{1}{r^3} \partial_r (r^3 \delta) , \sigma_{NN} \right).$$

Proposition 2.23 and the estimates above for $\delta$ and $\sigma_{NN}$ yield

$$\| \epsilon \|^2_{H^0_{-3/2}(\mathbb{R}^3\setminus B_1)} + \| \nabla \epsilon \|^2_{H^0_{-5/2}(\mathbb{R}^3\setminus B_1)} \lesssim \| \rho \|^2_{H^0_{-5/2}(\mathbb{R}^3\setminus B_1)} + \| \delta \|^2_{H^1_{-3/2}(\mathbb{R}^3\setminus B_1)} + \| \sigma_{NN} \|^2_{H^0_{-5/2}(\mathbb{R}^3\setminus B_1)}$$

For radial regularity, we use that by Lemma 4.10, $\epsilon$ also solves (S1.3) and (S2.3), together with (4.43) and (4.44) to obtain

$$\frac{1}{r^3} \nabla_N \left( r^3 \epsilon^{[1]} \right) = \tilde{\rho}^{[1]} + \frac{1}{2} \nabla \tilde{\delta}^{[1]} ,$$

$$\frac{1}{r^2} \nabla_N \left( r^2 \epsilon^{[2]}_E \right) = \tilde{\theta}^{[2]}_E + \zeta_E + (\nabla \tilde{\delta})^{[2]}_E ,$$

$$\frac{1}{r^2} \nabla_N \left( r^2 \epsilon^{[2]}_H \right) = \frac{1}{2} \tilde{\theta}^{[2]}_H + * \sigma_{NN}^{[2]}$$

$$= \mathcal{P}_1^{-1} \left( 0, \frac{1}{r^3} \partial_r (r^3 \tilde{\sigma}_{NN}^{[2]}(0)) \right).$$

By Proposition 4.11 and the above estimates for $\delta$, this yields the bounds

$$\| \nabla_N \epsilon \|_{H^0_{-3/2}(\mathbb{R}^3\setminus B_1)} \lesssim \| \rho \|_{H^0_{-5/2}(\mathbb{R}^3\setminus B_1)} .$$

This proves the case $w = 2$ of (4.97).
Consider now the case $w > 2$ of (4.97). Higher tangential regularity is derived by tangentially differentiating (4.98) and using Propositions 2.24 and 4.11. Higher radial regularity follows by applying $\nabla \mathcal{N}$ to (4.99) and using Proposition 4.11. This proves (4.97) for all $w \geq 2$.

It remains to show that $\epsilon \in \mathcal{H}_{-3/2}^{w-1}$. Indeed, this follows by (4.99) and the fact that $\rho, \mathcal{P}^{-1}_1 \left( 0, \frac{1}{r^3} \partial_r \left( r^3 \sigma_{NN}^{\geq 3} \right) \right), \zeta, \nabla \delta \in \mathcal{H}_{-5/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B}_1)$ together with Proposition 2.13. This finishes the control of $\epsilon$.

Control of $\sigma_{\mathcal{N}}^k$. First we show that for all $w \geq 2$

$$\| \sigma_{\mathcal{N}} \|_{\mathcal{H}_{-3/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B}_1)} \lesssim \| \rho \|_{\mathcal{H}_{-3/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B}_1)}. \quad (4.100)$$

This control of $\sigma_{\mathcal{N}}$ follows by the control of the previous quantities. Indeed, recall from (4.41)-(4.44),

$$\sigma_{\mathcal{N}}^{[1]} = \frac{1}{2} \rho^{[1]} - \frac{1}{2} \nabla \delta^{[1]} - \frac{1}{r} \epsilon^{[1]},$$

$$\sigma_{\mathcal{N}}^{[2]} = \frac{1}{2} \rho^{[2]} + \zeta, \quad (4.101)$$

This implies by Proposition 4.11 and the above control of $\delta$ and $\epsilon$

$$\| \sigma_{\mathcal{N}}^{[1]} \|_{\mathcal{H}_{-3/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B}_1)} \lesssim \| \rho \|_{\mathcal{H}_{-3/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B}_1)} + \| \delta \|_{\mathcal{H}_{-3/2}^{w-1}} + \| \epsilon \|_{\mathcal{H}_{-3/2}^{w-1}(\mathbb{R}^3 \setminus \overline{B}_1)}$$

$$\lesssim \| \rho \|_{\mathcal{H}_{-3/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B}_1)};$$

$$\| \sigma_{\mathcal{N}}^{[2]} \|_{\mathcal{H}_{-3/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B}_1)} \lesssim \| \rho \|_{\mathcal{H}_{-3/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B}_1)} + \| \zeta \|_{\mathcal{H}_{-3/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B}_1)}$$

$$\lesssim \| \rho \|_{\mathcal{H}_{-3/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B}_1)};$$

$$\| \sigma_{\mathcal{N}}^{[2]} \|_{\mathcal{H}_{-3/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B}_1)} \lesssim \| \rho \|_{\mathcal{H}_{-3/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B}_1)} + \| \mathcal{P}^{-1}_1 \left( 0, \frac{1}{r^3} \partial_r \left( r^3 \sigma_{NN}^{\geq 3} \right) \right) \|_{\mathcal{H}_{-3/2}^{w-2}(\mathbb{R}^3 \setminus \overline{B}_1)}$$

This proves (4.100) for all $w \geq 2$.

It remains to show that $\sigma_{\mathcal{N}} \in \mathcal{H}_{-5/2}^{w-2}$. This follows by (4.101) and $\delta, \epsilon \in \mathcal{H}_{-3/2}^{w-1}, \rho, \zeta, \mathcal{P}^{-1}_1 \left( 0, \frac{1}{r^3} \partial_r \left( r^3 \sigma_{NN}^{\geq 3} \right) \right) \in \mathcal{H}_{-5/2}^{w-2}$ together with Proposition 2.13. This finishes the control of $\sigma_{\mathcal{N}}$. 
Control of $\psi \ast \sigma / H$. First we show that for $w \geq 2$,
\[
\| \psi \ast \sigma / H \|_{H^{w-2}(R^{3}\setminus B)} \lesssim \| \rho \|_{H^{w-2}(R^{3}\setminus B)}.
\]
(4.102)

In (4.45), $\psi \ast \sigma / H$ was defined on each $S_r$, $r \geq 1$, as solution to
\[
D / 2 \left( \psi \ast \sigma / H + \frac{1}{2} \nabla / N (r \sigma / N) \right) = \frac{1}{r^2} \nabla N \left( r^2 \left( \frac{1}{r^2} \sigma / N - \frac{1}{2} \nabla / N \delta / \right) \right).
\]

Using definitions (4.43) and (4.44), this can be decomposed into
\[
\begin{align*}
\nabla_2 \left( \psi \ast \sigma / H + \frac{1}{2} \nabla / E \epsilon / \right) &= \frac{1}{r^2} \nabla N \left( r^2 \left( \frac{1}{r^2} \sigma / N - \frac{1}{2} \nabla / E \delta / \right) \right), \\
\nabla_2 \left( \psi \ast \sigma / H + \frac{1}{2} \nabla / H \epsilon / \right) &= \frac{1}{r^2} \nabla N \left( r^2 \left( \frac{1}{r^2} \sigma / N - \frac{1}{2} \nabla / N \delta / \right) \right). \\
\end{align*}
\]
(4.103)

To analyse these equations, we first rewrite the second equation.

Claim 4.16. The second equation of (4.103) is equivalent to
\[
\nabla_2 \left( \psi \ast \sigma / H + \frac{1}{2} \nabla / H \epsilon / \right) = \frac{3}{2r} \epsilon / H + \frac{1}{r} \epsilon / H - \frac{3}{r} \Delta \sigma / N - \frac{1}{r^2} \nabla / N \left( \epsilon / H \right) + \nabla_1^{-1} \left( 0, \Delta \sigma / N \right) - \nabla N \epsilon / H.
\]
(4.104)

Proof. In the following, we use that for a scalar function $f / [1]$,
\[
\nabla N \left( \frac{1}{r} \nabla_1^{-1} (0, f) \right) = \frac{1}{r} \nabla_1^{-1} (0, \partial_r f / [1]),
\]
this follows by Lemma 2.35.
By the definition of $\sigma_{NN}^{[\geq 2]}$ in (4.37) and of $*\sigma_{NH}^{[\geq 2]}$ in (4.44), and using Lemma 2.35,
\[
-\frac{1}{r^2} \nabla_N \left( r^2 \varphi^{-1}_1 \left( 0, \frac{1}{r^3} \partial_r \left( r^3 \sigma_{NN}^{[\geq 2]} \right) \right) \right)
= -\frac{3}{r} \varphi^{-1}_1 \left( 0, \frac{1}{r^3} \partial_r \left( r^3 \sigma_{NN}^{[\geq 2]} \right) \right) - \varphi^{-1}_1 \left( 0, \partial_r \left( \frac{1}{r^3} \partial_r \left( r^3 \sigma_{NN}^{[\geq 2]} \right) \right) \right)
= -\frac{3}{r} \left( \frac{1}{2} \sigma_{HH}^{[\geq 2]} + \sigma_{NH}^{[\geq 2]} \right) - \varphi^{-1}_1 \left( 0, -\Delta N \sigma_{NN}^{[\geq 2]} + \partial_r \text{curl} \left( \sigma_{HH}^{[\geq 2]} + \zeta_{HH}^{[\geq 2]} \right) \right)
= -\frac{3}{r} \left( \frac{1}{2} \sigma_{HH}^{[\geq 2]} + \sigma_{NH}^{[\geq 2]} \right) + \varphi^{-1}_1 \left( 0, \Delta N \sigma_{NN}^{[\geq 2]} \right) - \varphi^{-1}_1 \left( 0, \partial_r \text{curl} \left( \sigma_{HH}^{[\geq 2]} + \zeta_{HH}^{[\geq 2]} \right) \right)
= -\nabla_N \left( \frac{1}{2} \sigma_{HH}^{[\geq 2]} + \zeta_{HH}^{[\geq 2]} \right) + \frac{1}{r} \left( \sigma_{HH}^{[\geq 2]} + \zeta_{HH}^{[\geq 2]} \right) + \frac{1}{r} \left( -\frac{1}{2} \sigma_{HH}^{[\geq 2]} + \zeta_{HH}^{[\geq 2]} \right)

Plugging this into the second equation of (4.103) finishes the proof of Claim 4.16.

By differentiating the first of (4.103) and (4.104), and using the commutation relations of Lemma 2.35, we can apply Propositions 2.23 and 2.24 to get for $w \geq 2$
\[
\| * (\hat{\phi}) \|_{H^{-5/2}_w (\mathbb{R}^3 \setminus B_1)} \lesssim \| \epsilon \|_{H^{-5/2}_{-3/2} (\mathbb{R}^3 \setminus B_1)} + \| \zeta_E \|_{H^{-5/2}_w (\mathbb{R}^3)}
+ \| \varphi^{-1}_2 (\nabla_N \zeta_E) \|_{H^{-5/2}_w (\mathbb{R}^3 \setminus B_1)} + \| \delta \|_{H^{-1/2}_{-3/2} (\mathbb{R}^3 \setminus B_1)} \quad (4.105)
\]
\[
\lesssim \| \rho \|_{H^{-5/2}_w (\mathbb{R}^3 \setminus B_1)},
\]
\[
\| * (\hat{\phi}) \|_{H^{-5/2}_w (\mathbb{R}^3 \setminus B_1)} \lesssim \| \epsilon \|_{H^{-5/2}_{-3/2} (\mathbb{R}^3 \setminus B_1)} + \| \varphi^{-1}_2 (\nabla_N \zeta_E) \|_{H^{-5/2}_w (\mathbb{R}^3 \setminus B_1)}
+ \| \zeta_H \|_{H^{-5/2}_w (\mathbb{R}^3 \setminus B_1)} + \| \sigma_{HH}^{[\geq 2]} \|_{H^{-5/2}_w (\mathbb{R}^3 \setminus B_1)}
+ \| \sigma_{NN}^{[\geq 2]} \|_{H^{-5/2}_w (\mathbb{R}^3 \setminus B_1)} + \| \sigma_{NH}^{[\geq 2]} \|_{H^{-5/2}_w (\mathbb{R}^3 \setminus B_1)} \quad (4.106)
\]
\[
\lesssim \| \rho \|_{H^{-5/2}_w (\mathbb{R}^3 \setminus B_1)},
\]
where we used Proposition 4.11. This proves (4.102) for all $w \geq 2$.

It remains to show that $* (\hat{\varphi}) \in \overline{H}^{-5/2}_w$. This follows by (4.103), (4.104), by the fact that $\nabla_N \delta, \rho, \zeta_E, \zeta_H \in \overline{H}^{-w/2}_{-5/2}, \delta, \epsilon \in \overline{H}^{-w-1}_{-3/2}$ and $\varphi^{-1}_2 (\nabla_N \zeta_E), \varphi^{-1}_2 (\nabla_N \zeta_H) \in \overline{H}^{-w/2}_{-5/2}$ together with Proposition 2.13. Indeed, to show for a $S_t$-tangent symmetric 2-tensor $V^{[\geq 2]}_\psi$ that
$V |_{r=1} = 0$, it suffices to prove that $d\hat{\psi} (V) |_{r=1} = 0$. Together with the commutation relations of Lemma 2.35, this concludes the control of $\ast (\hat{\varphi})$.

**Control of $\hat{\eta}$.** First we prove for $w \geq 2$,

$$
\|\hat{\eta}\|_{H^{w-1,2}(\mathbb{R}^3 \setminus B_1)} \lesssim \|\rho\|_{H^{w-2}(\mathbb{R}^3 \setminus B_1)}. 
$$

(4.107)

The control of $\hat{\eta}$ follows by the control of the above quantities. Indeed, recall (4.46),

$$
\mathcal{P}_2 \hat{\eta} = \frac{1}{2} \hat{\varphi} \sigma [\geq 2] - \sigma \hat{\eta} \sigma [\geq 2] - \frac{1}{2} \nabla \hat{\varphi} \sigma [\geq 2] - \frac{1}{r} \sigma [\geq 2].
$$

(4.108)

Higher tangential regularity follows by Propositions 2.23 and 2.24. Indeed, tangentially deriving (4.108) yields for all $w \geq 2$, by the above control of $\delta, \epsilon, \ast \sigma \delta$,

$$
\|\nabla N \hat{\eta} + \frac{1}{r} \hat{\eta}\|_{H^{w-1,2}(\mathbb{R}^3 \setminus B_1)} \lesssim \|\rho\|_{H^{w-2}(\mathbb{R}^3 \setminus B_1)}.
$$

For radial regularity, use that by Lemma 4.10, $\hat{\eta}$ also solves (S2.5), that is,

$$
\nabla N \hat{\eta} + \frac{1}{r} \hat{\eta} = \ast (\hat{\varphi}) + \frac{1}{2} \nabla \otimes \epsilon [\geq 2].
$$

Differentiating in $r$ yields with the above control of $\delta, \epsilon, \ast \sigma \delta, \ast (\hat{\varphi})$ for all $w \geq 2$ the bound

$$
\|\nabla N \hat{\eta}\|_{H^{w-1,2}(\mathbb{R}^3 \setminus B_1)} \lesssim \|\rho\|_{H^{w-2}(\mathbb{R}^3 \setminus B_1)}.
$$

This proves (4.107) for $w \geq 2$.

It remains to show that $\hat{\eta} \in \tilde{H}^{-3/2}$. This follows by (S2.5) and $\epsilon \in \tilde{H}^{-3/2}$, $\ast (\hat{\varphi}) \in \tilde{H}^{-5/2}$ together with Proposition 2.13. This finishes the control of $\hat{\eta}$ as well as the proof of Lemma 4.15. \hfill $\square$

**5. The prescribed scalar curvature equation for $g$**

In this section we prove the following theorem.

**Theorem 5.1 (Metric extension theorem, version 2).** Let $w \geq 2$ be an integer. There exists a universal constant $\varepsilon > 0$ such that the following holds.

(1) **Extension result.** Let $\tilde{g} \in H^w(B_1)$ be a Riemannian metric on the unit ball

$B_1 \subset \mathbb{R}^3$ with scalar curvature $R(\tilde{g})$, and let $R \in H^{-5/2}$ be such that $R|_{B_1} = R(\tilde{g})$.

If

$$
\|\tilde{g} - \epsilon\|_{H^w(B_1)} + \|R\|_{H^{-5/2}} < \varepsilon,
$$

(5.1)

then there exists an $H^{-1/2}_{-1/2}$-asymptotically flat metric $\hat{g}$ on $\mathbb{R}^3$ such that $\hat{g}|_{B_1} = \tilde{g}$ and its scalar curvature satisfies

$$
R(\hat{g}) = R \text{ on } \mathbb{R}^3,
$$

$$
\|\hat{\eta}\|_{H^{w-1,2}(\mathbb{R}^3 \setminus B_1)} \lesssim \|\rho\|_{H^{w-2}(\mathbb{R}^3 \setminus B_1)}. 
$$
Moreover, it is bounded by
\[ \|\tilde{g} - e\|_{H^{w}_{1/2}} \lesssim \|\tilde{g} - e\|_{H^{w}(B_1)} + \|R\|_{H^{w-2}_{-5/2}}, \]  
(5.2)

### (2) Iteration estimates.
Let \( \tilde{g} \in H^{w}(B_1) \) be a Riemannian metric on \( B_1 \) and let \( R, \tilde{R} \in H^{w-2}_{-5/2} \) such that \( R|_{B_1} = \tilde{R}|_{B_1} = R(\tilde{g}) \) and (5.1) holds for \( (\tilde{g}, R) \) and \( (\tilde{g}, \tilde{R}) \).

Let \( \tilde{g} \) and \( \tilde{\tilde{g}} \) denote the metrics constructed in part (1) of this theorem with respect to \( R \) and \( \tilde{R} \). Then
\[ \|\tilde{g} - \tilde{\tilde{g}}\|_{H^{w}_{1/2}} \lesssim \|R - \tilde{R}\|_{H^{w-2}_{-5/2}}. \]  
(5.3)

Before turning to the proof of Theorem 5.1, we first analyse in more detail the scalar curvature functional in the next section.

### 5.1. Scalar curvature and geometry of foliations.

In this section, we analyse the scalar curvature functional with respect to the foliation of \( \mathbb{R}^3 \) by spheres \( S_r \).

**Lemma 5.2.** Let \( g \) be a Riemannian metric on \( \mathbb{R}^3 \setminus \overline{B_1} \),
\[ g = a^2 dr + \gamma_{AB}(\beta^A dr + d\theta^A)(\beta^B dr + d\theta^B). \]

Then the scalar curvature \( R(g) \) of \( g \) on \( \mathbb{R}^3 \setminus \overline{B_1} \) is given by
\[ R(g) = 2N(\text{tr}\Theta) - \frac{2}{a} \Delta \gamma a + 2K(\gamma) - (\text{tr}\Theta)^2 - |\Theta|^2_\gamma, \]
where \( N \) denotes the unit normal and \( \Theta \) the second fundamental form of \( S_r \subset \mathbb{R}^3 \) with respect to \( g \), respectively, and \( K(\gamma) \) the Gauss curvature of \( (S_r, \gamma) \).

**Proof.** The lemma follows by the traced second variation equation\(^6\)
\[ N(\text{tr}\Theta) = \frac{1}{a} \Delta \gamma a + \text{Ric}(N, N) + |\Theta|^2_\gamma \]
and the twice traced Gauss equation
\[ R(g) = 2\text{Ric}(N, N) + 2K(\gamma) - (\text{tr}\Theta)^2 + |\Theta|^2_\gamma, \]
where \( \text{Ric} \) denotes the Ricci tensor of \( g \). See Section 1 of [29] for a detailed derivation. This finishes the proof of Lemma 5.2. \( \square \)

We consider now variations of the metric and see how the scalar curvature changes.

**Lemma 5.3.** Let on \( \mathbb{R}^3 \setminus \overline{B_1} \) be given a Riemannian metric \( g \),
\[ g = a^2 dr + \gamma_{AB}(\beta^A dr + d\theta^A)(\beta^B dr + d\theta^B), \]
and a scalar function \( \varphi \). Consider then the variation \( \tilde{g}_\varphi := a^2 dr + e^{2\varphi}\gamma_{AB}(\beta^A dr + d\theta^A)(\beta^B dr + d\theta^B). \)

\(^6\)Recall our sign convention \( \Theta(X, Y) = -g(X, \nabla Y N). \)
It holds that

\[
\begin{align*}
R(\tilde{g}_\varphi) &= -4N(N \varphi) - 2e^{-2\varphi} \Delta \gamma \varphi + 2N(\text{tr}_\gamma \Theta) \\
&\quad - \frac{2}{a} e^{-2\varphi} \Delta a + 2e^{-2\varphi} K(\gamma) \\
&\quad - 6(N \varphi)^2 + 6(N \varphi) \text{tr}_\gamma \Theta - (\text{tr}_\gamma \Theta)^2 - |\Theta|^2_\gamma,
\end{align*}
\]

where \( N \) denotes the unit normal to \( S_r \) and \( \Theta \) the second fundamental form of \( S_r \) with respect to \( g \).

**Proof.** By Lemma 5.2, it holds that

\[
\begin{align*}
R(\tilde{g}_\varphi) &= 2\tilde{N}(\text{tr}_{e^{2\varphi} \gamma} \tilde{\Theta}) - \frac{2}{a} \Delta_{e^{2\varphi} \gamma} a \\
&\quad + 2K(e^{2\varphi} \gamma) - (\text{tr}_{e^{2\varphi} \gamma} \tilde{\Theta})^2 - |\tilde{\Theta}|_{e^{2\varphi} \gamma}^2,
\end{align*}
\]

where \( \tilde{N}, \tilde{\Theta} \) are the unit normal and second fundamental form of \( S_r \) with respect to \( \tilde{g}_\varphi \), respectively.

Note that in any coordinates on \( S_r \), for \( A, B = 1, 2 \),

\[
\begin{align*}
\tilde{N} &= N = \frac{1}{a} \partial_r - \frac{1}{a} \beta, \\
\tilde{\Theta}_{AB} &= -\frac{1}{2a} e^{2\varphi} \partial_r (e^{2\varphi} \gamma_{AB}) + \frac{1}{2a} (\mathcal{L}_\beta e^{2\varphi} \gamma)_{AB} \\
&\quad - (N \varphi) e^{2\varphi} \gamma_{AB} + e^{2\varphi} \Theta_{AB}.
\end{align*}
\]

Consequently,

\[
\begin{align*}
\text{tr}_{e^{2\varphi} \gamma} \tilde{\Theta} &= -2(N \varphi) + \text{tr}_\gamma \Theta, \\
|\tilde{\Theta}|_{e^{2\varphi} \gamma}^2 &= \gamma^{AC} \gamma^{BD} (--(N \varphi) \gamma_{AB} + \Theta_{AB})(--(N \varphi) \gamma_{CD} + \Theta_{CD}) \\
&= 2(N \varphi)^2 - 2(N \varphi) \text{tr}_\gamma \Theta + |\Theta|^2_\gamma.
\end{align*}
\]

Moreover, it holds in general that

\[
\begin{align*}
K(e^{2\varphi} \gamma) &= e^{-2\varphi} (K(\gamma) - \Delta g \varphi), \\
\Delta_{e^{2\varphi} \gamma} a &= e^{-2\varphi} \Delta a.
\end{align*}
\]
Plugging (5.7) and (5.6) into (5.5) yields
\[
R(\bar{g}_\varphi) = -4N(N\varphi) + 2N(\text{tr}_\gamma \Theta) - \frac{2}{a}e^{-2\varphi} \Delta_\gamma a \\
+ 2e^{-2\varphi}(K(\gamma) - \Delta_\gamma \varphi) \\
- (4(N\varphi)^2 - 4(N\varphi)\text{tr}_\gamma \Theta + (\text{tr}_\gamma \Theta)^2) \\
- 2(N\varphi)^2 + 2(N\varphi)\text{tr}_\gamma \Theta - |\Theta|_\gamma^2 \\
= 4N(N\varphi) - 2N(\text{tr}_\gamma \Theta) - \frac{2}{a}e^{-2\varphi} \Delta_\gamma a \\
+ 2e^{-2\varphi}(K(\gamma) - \Delta_\gamma \varphi) \\
- 6(N\varphi)^2 + 6(N\varphi)\text{tr}_\gamma \Theta - (\text{tr}_\gamma \Theta)^2 - |\Theta|_\gamma^2.
\]
This finishes the proof of Lemma 5.3. □

The scalar curvature functional is a smooth mapping.

**Lemma 5.4.** Let \( w \geq 2 \) be an integer. There exists an \( \varepsilon > 0 \) such that the scalar curvature functional

\[
R : g \mapsto R(g)
\]

is a smooth mapping from \( B_\varepsilon(e) \) to \( H^w_{-5/2}(\mathbb{R}^3 \setminus \overline{B_1}) \) where

\[
B_\varepsilon(e) = \left\{ g : \|g - e\|_{H^w_{-1/4}(\mathbb{R}^3 \setminus \overline{B_1})} \right\}.
\]

Furthermore, at the Euclidean metric \( e \), \( R(e) = 0 \).

**Proof.** The fact that \( R(g) \) is a smooth mapping around \( e \) follows by similar considerations as in Lemma 4.2 and is left to the reader. □

We calculate now the linearisation of the scalar curvature in \( \varphi \) and \( \beta \) at the Euclidean metric.

**Lemma 5.5.** The linearisation of the scalar curvature \( R(\bar{g}_\varphi) \) in \( (\varphi, \beta) \) at the Euclidean metric is given by

\[
D_{\varphi, \beta}R(\bar{g}_\varphi)|_{(\varphi=0, \beta=0)}(u, \xi) = -4\partial_\varphi^2 u - 2\Delta_\gamma u - \frac{12}{r} \partial_r u - \frac{4}{r^2} u + \frac{2}{r^3} \partial_r (r^3 d\bar{\gamma} \xi).
\]

**Proof.** Calculate the variation \( \delta \) of each term of (5.4), that is,

\[
R(\bar{g}_\varphi) = -4N(N\varphi) - 2e^{-2\varphi} \Delta_\gamma \varphi + 2N(\text{tr}_\gamma \Theta) \\
- \frac{2}{a}e^{-2\varphi} \Delta_\gamma a + 2e^{-2\varphi} K(\gamma) \\
- 6(N\varphi)^2 + 6(N\varphi)\text{tr}_\gamma \Theta - (\text{tr}_\gamma \Theta)^2 - |\Theta|_\gamma^2.
\]
with $\delta \varphi = u, \delta \beta = \xi$. First,

$$
\delta (-4N(N\varphi)) = -4\partial^2 u,
\delta (-2e^{-2\varphi}D_{\gamma} \varphi) = -2\Delta u
$$

Next,

$$
\delta (2N(tr, \Theta)) = \delta (2(\partial_r - \beta)(tr, \Theta))
= 2\partial_r (\delta tr, \Theta) - 2\xi (tr, \Theta)
= 2\partial_r (\delta tr, \Theta) = 0
$$

where we used that $\xi = \delta \beta$ is $S_r$-tangent and

$$
(tr, \Theta) \bigg|_{(\varphi=0,g=e)} = -2/r.
$$

Moreover,

$$
\delta \left(-\frac{2}{a} e^{-2\varphi} D_{\gamma} a\right) = 0,
\delta \left(2e^{-2\varphi} K(\gamma)\right) = -4uK(\gamma) = -\frac{4}{r^2} u,
\delta \left(-6(N\varphi)^2\right) = 0
$$

$$
\delta (6(N\varphi)(tr, \Theta)) = 6 \left(-\frac{2}{r}\right) \partial_r u = -\frac{12}{r} \partial_r u,
\delta (-tr, \Theta) = 2 \left(-\frac{2}{r}\right) \delta (d\psi(\beta)) = \frac{4}{r} d\psi \xi.
$$

Finally,

$$
\delta (-|\Theta|_{\gamma}^2) = -\delta \left(-\frac{1}{2a^2} \gamma^{AC}_{\gamma} \gamma^{BD}_{\gamma} (\ell_{\beta\gamma})_{AB} \partial_r (\gamma_{CD})\right)
= \frac{1}{2} \gamma^{AC}_{\gamma} \gamma^{BD}_{\gamma} \left(\ell_{\xi\gamma}\right)_{AB} \frac{2}{r} \gamma_{CD}
= \frac{2}{r} d\psi \xi,
$$

where we used that

$$
|\Theta|_{\gamma}^2 = \gamma^{AC}_{\gamma} \gamma^{BD}_{\gamma} \left(-\frac{1}{2a} \partial_r (\gamma_{AB}) + \frac{1}{2a} (\ell_{\beta\gamma})_{AB}\right) \left(-\frac{1}{2a} \partial_r (\gamma_{CD}) + \frac{1}{2a} (\ell_{\beta\gamma})_{CD}\right)
$$

and in polar coordinates on the sphere, $\gamma_{CD} \sim r^2$. 

5.2. **Reduction to the Euclidean case.** In this section, we first prove the next perturbation result, Proposition 5.6, under the assumption of Lemma 5.8 which is proved in Section 5.3. Then we prove Theorem 5.1.

**Proposition 5.6.** There is a small universal constant \( \varepsilon > 0 \) such that the following holds.

1. **Extension result.** Let \( g \) be an \( \mathcal{H}^{w}_{-1/2} \)-asymptotically flat metric on \( \mathbb{R}^3 \setminus \overline{B}_1 \) written in standard polar coordinates as
   \[
   g = a^2 dr^2 + \gamma_{AB} \left( \beta^A dr + d\theta^A \right) \left( \beta^B dr + d\theta^B \right),
   \]
   and \( s \in \mathcal{H}^{w}_{-5/2} \) a scalar function. If
   \[
   \| g - e \|_{\mathcal{H}^{w}_{-1/2}(\mathbb{R}^3 \setminus \overline{B}_1)} + \| s \|_{\mathcal{H}^{w}_{-5/2}} < \varepsilon,
   \]
   then there exist a scalar function \( \varphi \in \mathcal{H}^{w}_{-1/2} \) and an \( S_r \)-tangent vector \( \beta' \in \mathcal{H}^{w}_{-1/2} \) bounded by
   \[
   \| (\varphi, \beta') \|_{\mathcal{H}^{w}_{-1/2} \times \mathcal{H}^{w}_{-1/2}} \lesssim \| s \|_{\mathcal{H}^{w}_{-5/2}},
   \]
   and such that the metric
   \[
   \bar{g}_{\varphi, \beta'} := a^2 dr^2 + e^{2\varphi} \gamma_{AB} \left( (\beta + \beta)'^A dr + d\theta^A \right) \left( (\beta + \beta')^B dr + d\theta^B \right)
   \]
   is \( \mathcal{H}^{w}_{-1/2} \)-asymptotically flat and its scalar curvature is on \( \mathbb{R}^3 \setminus \overline{B}_1 \)
   \[
   R(\bar{g}_{\varphi, \beta'}) = R(g) + s.
   \]

   Furthermore, it is bounded by
   \[
   \| \bar{g}_{\varphi, \beta'} - e \|_{\mathcal{H}^{w}_{-1/2}(\mathbb{R}^3 \setminus \overline{B}_1)} \lesssim \| g - e \|_{\mathcal{H}^{w}_{-1/2}(\mathbb{R}^3 \setminus \overline{B}_1)} + \| s \|_{\mathcal{H}^{w}_{-5/2}}.
   \]

2. **Iteration estimates.** Let \( g \) be an \( \mathcal{H}^{w}_{-1/2} \)-asymptotically flat metric on \( \mathbb{R}^3 \setminus \overline{B}_1 \) written in standard polar coordinates as
   \[
   g = a^2 dr^2 + \gamma_{AB} \left( \beta^A dr + d\theta^A \right) \left( \beta^B dr + d\theta^B \right),
   \]
   and \( s, \bar{s} \in \mathcal{H}^{w}_{-5/2} \) two scalar functions such that (5.8) holds for \( (g, s) \) and \( (g, \bar{s}) \).

   Applying part (1) to \( g \) with \( s \) and \( \bar{s} \) yields two pairs \( (\varphi, \beta') \) and \( (\bar{\varphi}, \bar{\beta}') \), respectively. It holds that
   \[
   \| (\varphi, \beta') - (\bar{\varphi}, \bar{\beta}') \|_{\mathcal{H}^{w}_{-1/2} \times \mathcal{H}^{w}_{-1/2}} \lesssim \| s - \bar{s} \|_{\mathcal{H}^{w}_{-5/2}}.
   \]
Let \( \tilde{g}_{\varphi,\beta'}, \tilde{g}_{\tilde{\varphi},\tilde{\beta}} \) denote the asymptotically flat metrics defined by (5.10). Then it holds that
\[
\| \tilde{g}_{\varphi,\beta'} - \tilde{g}_{\tilde{\varphi},\tilde{\beta}} \|_{H^{w,-1/2}}(\mathbb{R}^3 \setminus B_1) \lesssim \| s - \tilde{s} \|_{H^{w,-2}}.
\] (5.13)

The proof of Proposition 5.6 is based on the Implicit Function Theorem and the essential Lemma 5.8 below which is proved in Section 5.3.

**Definition 5.7.** Let \( \varphi \in \mathcal{H}^w_{-1/2} \) be a scalar function, \( \beta' \in \mathcal{H}^w_{-1/2} \) a \( S_t \)-tangent vectorfield and \( g \) a \( \mathcal{H}^w_{-1/2} \)-asymptotically flat metric,
\[
g = a^2 dr^2 + \gamma_{AB} (\beta^A dr + d\theta^A) (\beta^B dr + d\theta^B),
\]
and let further
\[
\tilde{g}_{\varphi,\beta'} := a^2 dr^2 + e^{2\varphi} \gamma_{AB} ((\beta + \beta')^A dr + d\theta^A) ((\beta + \beta')^B dr + d\theta^B).
\]
Then, define
\[
S(\varphi, \beta', g) := R(\tilde{g}_{\varphi,\beta'}) - R(g).
\]

It is left to the reader to verify that for \((\varphi, \beta')\) close to \((0, 0)\) in \( \mathcal{H}^w_{-1/2} \times \mathcal{H}^w_{-1/2} \) and \( g \) close to \( e \) in \( \mathcal{H}^w_{-1/2}(\mathbb{R}^3 \setminus B_1) \), the mapping \( S \) is smooth and maps into \( \mathcal{H}^{w,-2} \). Furthermore, it holds by construction that
\[
D_{\varphi,\beta'} S|_{(0,0,e)}(u, \xi) = -4 \Delta u - 12 \partial_r u - \frac{4}{r} \partial_r u - \frac{2}{r^3} \partial_r (r^3 \text{div} \xi),
\]
see Lemmas 5.4 and 5.5.

The next lemma is essential for the proof of Proposition 5.6.

**Lemma 5.8 (Surjectivity at the Euclidean metric).** The linearization
\[
D_{\varphi,\beta'} S|_{(0,0,e)} : \mathcal{H}^w_{-1/2} \times \mathcal{H}^w_{-1/2} \rightarrow \mathcal{H}^{w,-2}_{-5/2}
\]
is surjective and has a bounded right-inverse. That is, for any \( h \in \mathcal{H}^{w,-2}_{-5/2} \), there exists \((u, \xi) \in \mathcal{H}^w_{-1/2} \times \mathcal{H}^w_{-1/2} \) that solves
\[
\begin{aligned}
4 \partial_r^2 u + \frac{12}{r} \partial_r u + \frac{4}{r^2} u + 2 \Delta u - \frac{2}{r^3} \partial_r (r^3 \text{div} \xi) = h \quad &\text{on } \mathbb{R}^3 \setminus B_1, \\
(u, \xi)|_{r=1} = 0.
\end{aligned}
\]
Furthermore, it is bounded by
\[
\|(u, \xi)\|_{\mathcal{H}^w_{-1/2} \times \mathcal{H}^w_{-1/2}} \lesssim \|h\|_{\mathcal{H}^{w,-2}_{-5/2}}.
\]
Let
\[ \mathcal{N}_e := \ker \left( D_{\varphi, \beta'} \mathcal{S}|_{(0,0,e)} \right) \perp H_{-1/2}^w \times H_{-1/2}^w, \]
where \( \perp \) denotes the orthogonal complement with respect to the scalar product on \( H_{-1/2}^w \times H_{-1/2}^w \). \( \mathcal{N}_e \) is a closed subspace of \( H_{-1/2}^w \times H_{-1/2}^w \) and therefore also Hilbert.

From now on, let \( \mathcal{S} \) be restricted to \( (\varphi, \beta') \in \mathcal{N}_e \).

We are now ready to prove Proposition 5.6.

**Proof of Proposition 5.6.** By Lemma 5.8, the linearization \( D_{\varphi, \beta'} \mathcal{S}|_{(0,0,e)} \) is an isomorphism, and \( \mathcal{S}(0,0,e) = 0 \). Therefore, by the Inverse Function Theorem 2.37, there are open neighbourhoods \( V_0 \) around the Euclidean metric \( e \) and \( W_0 \subset H_{-5/2}^w \) around 0, together with a unique smooth mapping
\[ \mathcal{G} : V_0 \times W_0 \rightarrow H_{-1/2}^w \times H_{-1/2}^w, \]
\[ (g, s) \mapsto \mathcal{G}(g, s) := (\varphi, \beta') \]
such that for \( g \in V_0, s \in W_0 \), on \( \mathbb{R}^3 \setminus \overline{B_1} \),
\[ \mathcal{S}(\mathcal{G}(g, s), g) = s. \]
Moreover, it holds by the uniqueness of \( \mathcal{G} \) that for every \( g \in V_0 \),
\[ \mathcal{G}(g, 0) = 0, \quad (5.14) \]
because \( \mathcal{S}(0, 0, g) = R(g) = R(g) = 0 \).

There exists \( \varepsilon > 0 \) small such that for \( (g, s) \) with
\[ \|g - e\|_{H_{-1/2}^w} < \varepsilon, \|s\|_{H_{-5/2}^w} < \varepsilon \]
it holds that \( g \in V_0, s \in W_0 \) and furthermore, for
\[ (\varphi, \beta') := \mathcal{G}(\tilde{g}, \tilde{s}) \]
it holds that
\[ \|(u, \xi)\|_{H_{-1/2}^w \times H_{-1/2}^w} = \|\mathcal{G}(\tilde{g}, \tilde{s})\|_{H_{-1/2}^w \times H_{-1/2}^w} = \|\mathcal{G}(\tilde{g}, \tilde{s}) - \mathcal{G}(\tilde{g}, 0)\|_{H_{-1/2}^w \times H_{-1/2}^w} \]
\[ \lesssim \|\tilde{s}\|_{H_{-5/2}^w}, \quad (5.15) \]
see Lemma 2.38. This proves (5.9).

We now prove (5.12). By Lemma 2.38, there is a \( \varepsilon > 0 \) small such that for \( g, s, \tilde{s} \) with
\[ \|g - e\|_{H_{-1/2}^w} < \varepsilon, \|s\|_{H_{-5/2}^w} < \varepsilon, \|\tilde{s}\|_{H_{-5/2}^w} < \varepsilon \]
it holds that
\[
\|(\varphi, \beta') - (\tilde{\varphi}, \tilde{\beta}')\|_{\overline{W}^2_{-1/2} \times \overline{W}^w_{-1/2}} = \|G(g, s) - G(g, \tilde{s})\|_{\overline{W}^2_{-1/2} \times \overline{W}^w_{-1/2}} \lesssim \|s - \tilde{s}\|_{\overline{W}^w_{-2}}.
\] (5.16)

This proves (5.12).

**Estimates for g.** To prove (5.11) for \(\varepsilon > 0\) small, it suffices by Part (2) of Lemma 2.22 to prove
\[
\|a^2 - 1\|_{H^{-1/2}_w} + \|A + A^2\|_{H^{-1/2}_w(B_1)} + \|\epsilon^\gamma - \tilde{\gamma}\|_{H^{-1/2}_w} \lesssim \|g - e\|_{H^{-1/2}_w} + \|s\|_{\overline{W}^w_{-2}}.
\]
First, the control of \(a^2\) is immediate because \(a^2\) remained the same in the variation. Therefore, by Part (1) of Lemma 2.22, for \(\varepsilon > 0\) sufficiently small,
\[
\|a^2 - 1\|_{H^{-1/2}_w} \lesssim \|g - e\|_{H^{-1/2}_w}.
\]
Second, by (5.9) and Lemma 2.22,
\[
\|\beta + \tilde{\beta}\|_{H^{-1/2}_w(B_1)} \lesssim \|\beta\|_{H^{-1/2}_w(B_1)} + \|\beta\|_{\overline{W}^w_{-1/2}} \lesssim \|\beta\|_{H^{-1/2}_w(B_1)} + \|s\|_{\overline{W}^w_{-2}} \lesssim \|g - e\|_{H^{-1/2}_w(B_1)} + \|s\|_{\overline{W}^w_{-2}}.
\]
Third, we claim that for \(\varepsilon > 0\) small
\[
\|\epsilon^\gamma - \tilde{\gamma}\|_{H^{-1/2}_w(B_1)} \lesssim \|g - e\|_{H^{-1/2}_w(B_1)} + \|s\|_{\overline{W}^w_{-2}}.
\]
Indeed, estimate
\[
\|\epsilon^\gamma - \tilde{\gamma}\|_{H^{-1/2}_w(B_1)} \leq \left\| (\epsilon^\gamma - 1) \tilde{\gamma} \right\|_{H^{-1/2}_w(B_1)} + \left\| \epsilon^\gamma - \tilde{\gamma} \right\|_{H^{-1/2}_w(B_1)}.
\] (5.17)
The first term on the right-hand side of (5.17) can be bounded by Corollary 2.11 and (5.9) for \(\varepsilon > 0\) small enough,
\[
\left\| (\epsilon^\gamma - 1) \tilde{\gamma} \right\|_{H^{-1/2}_w(B_1)} \lesssim \|\tilde{\gamma}\|_{\overline{W}^w_{-2}} \lesssim \|s\|_{\overline{W}^w_{-2}(B_1)}.
\]
The second term on the right-hand side of (5.17) is estimated by Lemmas 2.9 and 2.22 with (5.9). Namely, for \(\varepsilon > 0\) sufficiently small,
\[
\left\| \epsilon^\gamma - \tilde{\gamma} \right\|_{H^{-1/2}_w(B_1)} \lesssim \|\gamma - \tilde{\gamma}\|_{H^{-1/2}_w(B_1)} \lesssim \|g - e\|_{H^{-1/2}_w(B_1)}.
\]
Together, this shows that
\[ \|e^{2\varphi} \gamma - \gamma\|_{H^{-1/2}_{\gamma}(\mathbb{R}^3 \setminus \overline{B})} \lesssim \|g - e\|_{H^w_{\gamma}(\mathbb{R}^3 \setminus \overline{B})} + \|s\|_{H^{-w/2}_{\gamma}}. \]
This finishes the proof of (5.11).

It remains to prove the iteration estimate (5.13). Indeed, by the construction of \( \tilde{g}_{\varphi, \beta'} \), \( \tilde{\varphi}, \tilde{\beta}' \) as variations of \( g \) with \( (\varphi, \beta'), (\tilde{\varphi}, \tilde{\beta}') \) we can write, see Section 2.2, with \( B = 1, 2, \)
\[
(\tilde{g}_{\varphi, \beta'} - \tilde{g}_{\tilde{\varphi}, \tilde{\beta}'})_{NN} = a^2 - a^2 = 0, \\
\tilde{g}_{\varphi, \beta'} - \tilde{g}_{\tilde{\varphi}, \tilde{\beta}'} = (e^{2\varphi} - e^{2\tilde{\varphi}}) \gamma, \\
\left( \tilde{g}_{\varphi, \beta'} - \tilde{g}_{\tilde{\varphi}, \tilde{\beta}'} \right)_{B} = e^{2\varphi} \gamma_{BA} (\beta'A + \beta'^A) - e^{2\tilde{\varphi}} \gamma_{BA} (\tilde{\beta}'A + \tilde{\beta}'A) \\
= (e^{2\varphi} - e^{2\tilde{\varphi}}) \gamma_{BA} (\beta'A + \beta'^A) + e^{2\tilde{\varphi}} \gamma_{BA} (\tilde{\beta}'A - \tilde{\beta}'A). \tag{5.18}
\]

By Corollary 2.11 and (5.12) it holds for \( \varepsilon > 0 \) sufficiently small that
\[ \|e^{2\varphi} - e^{2\tilde{\varphi}}\|_{H^{-1/2}_{\gamma}(\mathbb{R}^3 \setminus \overline{B})} \lesssim \|s - \tilde{s}\|_{H^{-w/2}_{\gamma}}. \]
For \( \varepsilon > 0 \) small enough, we can use this estimate and the expression (5.18) with (5.12) to apply Lemmas 2.9, 2.22 and 2.19 to get
\[ \|\tilde{g}_{\varphi, \beta'} - \tilde{g}_{\tilde{\varphi}, \tilde{\beta}'}\|_{H^w_{\gamma}(\mathbb{R}^3 \setminus \overline{B})} \lesssim \|s - \tilde{s}\|_{H^{-w/2}_{\gamma}}. \]
This proves (5.13) and finishes the proof of Proposition 5.6. \( \square \)

We are now in position to prove Theorem 5.1.

Proof of Theorem 5.1. We prove the two parts of the theorem separately.

Proof of Part (1). Use standard Sobolev extension to extend \( g \) from \( B_1 \) to an asymptotically flat metric \( g \) on \( \mathbb{R}^3 \) such that
\[ \|g - e\|_{H^{-1/2}_{\gamma}(\mathbb{R}^3)} \lesssim \|\tilde{g} - e\|_{H^w(\mathbb{B}_1)}. \tag{5.19} \]
Denote its standard polar components on \( \mathbb{R}^3 \setminus \overline{B}_1 \) by
\[ g = a^2 dr^2 + \gamma_{AB} (\beta^A dr + e^A) (\beta^B dr + e^B). \]
Given a \( R \in H^{-w/2}_{-5/2} \) such that \( R_{|B_1} = R(\tilde{g}) \), let
\[ s := R - R(g) \in H^{-w/2}_{-5/2}, \]
where we used Proposition 2.13. It holds for \( \varepsilon > 0 \) small that
\[ \|s\|_{H^{-w/2}_{-5/2}} \lesssim \|R\|_{H^{-w/2}_{-5/2}(\mathbb{R}^3 \setminus \overline{B}_1)} + \|R(g)\|_{H^{-w/2}_{-5/2}(\mathbb{R}^3 \setminus \overline{B}_1)}, \]
\[ \lesssim \|R\|_{H^{-w/2}_{-5/2}(\mathbb{R}^3 \setminus \overline{B}_1)} + \|g - e\|_{H^w(\mathbb{R}^3 \setminus \overline{B}_1)}, \tag{5.20} \]
Therefore, for $\varepsilon > 0$ small enough, Proposition 5.6 yields a pair $(\varphi, \beta')$ such that
\[
\tilde{g} = a^2 dr^2 + e^{2\varphi} \gamma_{AB} \left((\beta + \beta')^A dr + e^A\right) \left((\beta + \beta')^B dr + e^B\right)
\]
is a $H_{-1/2}$-asymptotically flat metric with $\tilde{g}|_{B_1} = \bar{g}$ and scalar curvature
\[
R(\tilde{g}) = R(g) + S = R.
\]
By (5.11), (5.19) and (5.20),
\[
\|\tilde{g} - e\|_{H_{-1/2}(\mathbb{R}^3)} \lesssim \|g - e\|_{H_{-1/2}(\mathbb{R}^3)} + \|s\|_{H_{-5/2}(\mathbb{R}^3)} + \|R\|_{H_{-5/2}(\mathbb{R}^3)}.
\]
This proves (5.2). This finishes the proof of part (1) of Theorem 5.1.

**Proof of Part (2).** Use standard Sobolev extension to extend $\bar{g}$ from $B_1$ to an asymptotically flat metric $g$ on $\mathbb{R}^3$ such that
\[
\|g - e\|_{H_{-1/2}(\mathbb{R}^3)} \lesssim \|\bar{g} - e\|_{H_{-1/2}(\mathbb{R}^3)} + \|R\|_{H_{-5/2}(\mathbb{R}^3)}.
\]
Let $s := R - R(g), \bar{s} := \tilde{R} - R(g)$, so that
\[
s - \bar{s} = (R(g) + s) - (R(g) + \bar{s}) = R - \tilde{R}.
\]
Hence, for $\varepsilon > 0$ sufficiently small, (5.3) follows from (5.13) in Proposition 5.6. This finishes the proof of Theorem 5.1.

### 5.3. Surjectivity at the Euclidean metric

In this section, we prove Lemma 5.8, that is, we show that for every scalar function $h \in H_{-2}^{-5/2}$ there exist $(u, \xi) \in H_{-1/2}^{-1/2} \times H_{-1/2}^{-1/2}$ that solves on $\mathbb{R}^3 \setminus B_1$
\[
\partial_r^2 u + \frac{3}{r} \partial_r u + \frac{1}{r^2} u + \frac{1}{2} \triangle u - \frac{1}{2r^3} \partial_r \left(r^3 \text{div} \gamma \xi\right) = \frac{1}{4} h
\]
and is bounded by
\[
\|(u, \xi)\|_{H_{-1/2}^{-1/2} \times H_{-1/2}^{-1/2}} \lesssim \|h\|_{H_{-5/2}^{-2}}.
\]
In this section all operators are Euclidean. We consider the following more general system on $\mathbb{R}^3 \setminus B_1$
\[
\triangle u + \frac{1}{r} \partial_r u + \frac{1}{r^2} u - \frac{1}{2} \triangle u = \frac{1}{2} \left(\frac{1}{2} h + \zeta^{[\geq 1]}\right),
\]
\[
\frac{1}{r^3} \partial_r \left(r^3 \text{div} \gamma \xi\right) = \zeta^{[\geq 1]},
\]
where $\zeta^{[\geq 1]} \in H_{-5/2}^{-2}$ is a scalar function. By the relation $\triangle = \partial_r^2 + \frac{2}{r} \partial_r + \frac{1}{r^2} \triangle$, it follows that for any $\zeta^{[\geq 1]}$, a solution $(u, \xi)$ to (5.22) also solves (5.21). From now on, we will thus focus on (5.22).
5.3.1. **Construction of the solution \((u, \xi)\) and \(\zeta^{[\geq 1]}\).** In this section we construct two scalar functions \(\zeta^{[\geq 1]}, u\) and a \(S_r\)-tangent vectorfield \(\xi\). It is shown in Section 5.3.2 that they are a solution to (5.22).

Let the given \(h \in \mathcal{H}^{s-2}_{-5/2}\) be decomposed as

\[
h = h^{[0]} + h^{[\geq 1]}. \]

**Definition of \(u\) and \(\zeta^{[\geq 1]}\).** Let the scalar function

\[
u = u^{[0]} + u^{[\geq 1]},
\]

where the radial function \(u^{[0]}(r)\) is defined as solution to the following ODE on \(r > 1\),

\[
\begin{align*}
\frac{\partial^2}{\partial r^2} u^{[0]} + \frac{2}{r} \frac{\partial}{\partial r} u^{[0]} + \frac{1}{r^2} u^{[0]} &= \frac{1}{4} h^{[0]}, \\
u^{[0]}|_{r=1} &= \frac{\partial}{\partial r} u^{[0]}|_{r=1} = 0,
\end{align*}
\]

and \(u^{[\geq 1]}\) is defined as solution to the elliptic PDE on \(\mathbb{R}^3 \setminus B_1\)

\[
\begin{align*}
\triangle u^{[\geq 1]} - \frac{1}{2} \frac{\partial}{\partial r} u^{[\geq 1]} + \frac{1}{r} \frac{\partial}{\partial r} u^{[\geq 1]} + \frac{1}{r^2} u^{[\geq 1]} &= \frac{1}{2} \left( \frac{1}{2} h^{[\geq 1]} + \zeta^{[\geq 1]} \right), \\
u^{[\geq 1]}|_{r=1} &= 0,
\end{align*}
\]

where on \(\mathbb{R}^3\),

\[
\zeta^{[\geq 1]} := \sum_{l \geq 1} \sum_{m = -l}^l \zeta^{(lm)}(Y^{(lm)}),
\]

\[
\zeta^{(lm)} := c^{(lm)} r \sqrt{l(l+1)/2} \partial_r (\chi(l(r - 1)))
- \bar{c}^{(lm)} r \sqrt{l(l+1)/2} \partial^2_r (\chi(l(r - 1))),
\]

where \(\chi\) denotes the smooth transition function defined in (2.1) and for \(l \geq 1\),

\[
c^{(lm)} := -\frac{1}{2} \int_1^\infty r^{1-\sqrt{l(l+1)/2}} h^{(lm)} dr,
\]

\[
\bar{c}^{(lm)} := \frac{\int_1^\infty c^{(lm)} r^{\sqrt{l(l+1)/2}+1} \partial_r (\chi(l(r - 1))) dr}{\int_1^\infty r^{\sqrt{l(l+1)/2}+1} \partial^2_r (\chi(l(r - 1))) dr}.
\]

\[
(5.23) \quad (5.24) \quad (5.25) \quad (5.26)
\]
\* Definition of $\xi$. Let $\xi$ be the $S_r$-tangent vector field solving on $\mathbb{R}^3 \setminus \overline{B}_1$

\[
\begin{cases}
\frac{1}{r^3} \partial_r (r^3 \partial_r \xi) = \zeta^{[\geq 1]}, \\
\text{curl} \xi = 0, \\
\xi |_{r=1} = 0.
\end{cases}
\tag{5.27}
\]

5.3.2. Proof of surjectivity at the Euclidean metric. In this section, we prove first in Lemma 5.9 that $\zeta^{[\geq 1]}$, $u$ and $\beta$ solve (5.22). Then, in Propositions 5.10 and 5.13, we show that $\zeta^{[\geq 1]} \in \overline{H}_{-5/2}^{w-2}$, $(u, \xi) \in \overline{H}_{-1/2}^{w} \times \overline{H}_{-1/2}^{w}$
with quantitative bounds. These results prove Lemma 5.8.

**Lemma 5.9.** The $u, \xi, \zeta^{[\geq 1]}$ defined in (5.23)-(5.27) solve (5.22), that is, on $\mathbb{R}^3 \setminus \overline{B}_1$,

\[
\begin{aligned}
\Delta u + \frac{1}{r} \partial_r u + \frac{1}{r^2} u - \frac{1}{2} \Delta u &= \frac{1}{2} \left( \frac{1}{2} h + \zeta^{[\geq 1]} \right), \\
\frac{1}{r^3} \partial_r (r^3 \partial_r \xi) &= \zeta^{[\geq 1]}.
\end{aligned}
\]

**Proof of Lemma 5.9.** The coefficients of the system (5.22) depend only on $r$. Therefore we may project the equations of (5.22) onto the Hodge-Fourier basis elements. This uses Proposition 2.30. We split (5.22) into the modes $l = 0$ and $l \geq 1$ and get the following two subsystems $S0, S1$.

\[
\begin{aligned}
\partial_r^2 u^{[0]} + \frac{3}{r} \partial_r u^{[0]} + \frac{1}{r^2} u^{[0]} &= \frac{1}{4} h^{[0]}, \\
\Delta u^{[\geq 1]} + \frac{1}{r} \partial_r u^{[\geq 1]} + \frac{1}{r^2} u^{[\geq 1]} - \frac{1}{2} \Delta u^{[\geq 1]} &= \frac{1}{2} \left( \frac{1}{2} h^{[\geq 1]} + \zeta^{[\geq 1]} \right), \\
\frac{1}{r^3} \partial_r (r^3 \partial_r \xi) &= \zeta^{[\geq 1]}.
\end{aligned}
\tag{S0, S1.1}
\]

The $(u, \xi)$ and $\zeta^{[\geq 1]}$ defined in (5.23)-(5.27) directly solve these equations on $\mathbb{R}^3 \setminus \overline{B}_1$. This proves Lemma 5.9.

The next proposition is essential and only possible due to our careful choice of $\zeta^{[\geq 1]}$ which ensures that for all $w \geq 2$, we have the boundary behaviour $u^{[\geq 1]} \in \overline{H}^w_{-1/2}$.

**Proposition 5.10.** Let $w \geq 2$ be an integer. The following holds.

\* Regularity and boundary control of $\zeta^{[\geq 1]}$. The scalar function $\zeta^{[\geq 1]}$ is well-defined by (5.25)-(5.26) and bounded by

\[
\| \zeta^{[\geq 1]} \|_{\overline{H}^{w-2}_{-5/2} (\mathbb{R}^3 \setminus \overline{B}_1)} \lesssim \| h^{[\geq 1]} \|_{\overline{H}^{w}_{-5/2}}.
\tag{5.28}
\]
Moreover, $\zeta^{[\geq 1]} \in H_{-5/2}^{w-2}$ and the following integral identities hold.

$$\int_1^{\infty} r^{1-\sqrt{l(l+1)/2}} \left( \frac{1}{2} h^{(lm)} + \zeta^{(lm)} \right) dr = 0,$$

(5.29)

$$\int_1^{\infty} r^2 \zeta^{(lm)} dr = 0.$$

(5.30)

- **Precise estimate for $\zeta^{[\geq 1]}$.** It holds that

$$\| \mathcal{P}_1^{-1}( \partial_r, \zeta^{[\geq 1]} ; 0 ) \|_{H_{-5/2}^w (\mathbb{R}^3 \setminus B_1)} \lesssim \| h^{[\geq 1]} \|_{H_{-5/2}^{w-2}}.$$  

(5.31)

Moreover, $\mathcal{P}_1^{-1}( \partial_r, \zeta^{[\geq 1]} ; 0 ) \in H_{-5/2}^{w-2}$.

- **Regularity and boundary control of $u^{[\geq 1]}$.** The scalar function $u^{[\geq 1]}$ defined in (5.24) is bounded by

$$\| u^{[\geq 1]} \|_{H_{-1/2}^w (\mathbb{R}^3 \setminus B_1)} \lesssim \| h^{[\geq 1]} \|_{H_{-5/2}^{w-2}}.$$  

(5.32)

Moreover, $u^{[\geq 1]} \in H_{-1/2}^w$ and in particular,

$$\partial_r u^{[\geq 1]} \big|_{r=1} = 0.$$

**Remark 5.11.** The function $u^{[\geq 1]}$ satisfies the elliptic equation (5.24). Therefore its boundary behaviour is harder to estimate than for $u^{[0]}$, $\xi$ which satisfy transport equations in $r$.

**Proof.** We prove each point separately.

**Regularity and boundary control of $\zeta^{[\geq 1]}$.** We show at first that the constants $c^{(lm)}, \bar{c}^{(lm)}$ are well-defined in (5.26). Concerning $c^{(lm)}$, for $l \geq 1$, $m \in \{-l, \ldots, l\}$,

$$|c^{(lm)}| = \left| \frac{1}{2} \left| \int_1^{\infty} r^{1-\sqrt{l(l+1)/2}} h^{(lm)} dr \right| \right| \leq \frac{1}{2} \left( \int_1^{\infty} r^{-2\sqrt{l(l+1)/2}} dr \right)^{1/2} \left( \int_1^{\infty} (r h^{(lm)})^2 dr \right)^{1/2} \left( \int_1^{\infty} (r h^{(lm)})^2 dr \right)^{1/2} \left( \int_1^{\infty} (r h^{(lm)})^2 dr \right)^{1/2} = \frac{1}{2} \frac{1}{(2\sqrt{l(l+1)/2 - 1})^{1/2}} \left( \int_1^{\infty} (r h^{(lm)})^2 dr \right)^{1/2}.$$  

(5.33)
To estimate $\tilde{\epsilon}^{(lm)}$, estimate first its denominator. Integrating by parts twice and using that $\text{supp}(\partial_r \chi(l(r - 1)) \subset [1, 1 + \frac{1}{l}]$ yields

$$
\int_1^{\infty} r^{l(l+1)/2+1} \partial_r^2 (\chi(l(r - 1))) \, dr
$$

$$
= - \left( \sqrt{\frac{l(l+1)}{2} + 1} \right) \int_1^{1+l} r^{l(l+1)/2} \partial_r (\chi(l(r - 1))) \, dr
$$

$$
= - \left( \sqrt{\frac{l(l+1)}{2} + 1} \right) \left( 1 + \frac{1}{l} \right) \int_1^{1+l} r^{l(l+1)/2} \partial_r (\chi(l(r - 1))) \, dr
$$

$$
+ \left( \sqrt{\frac{l(l+1)}{2} + 1} \right) \left( \sqrt{\frac{l(l+1)}{2}} \right) \int_1^{1+l} r^{l(l+1)/2-1} \chi(l(r - 1)) \, dr
$$

$$
\leq - \left( \sqrt{\frac{l(l+1)}{2} + 1} \right),
$$

where we uniformly bounded $|\chi| \leq 1$. Consequently, for all $l \geq 1$,

$$
|\tilde{\epsilon}^{(lm)}| \leq \frac{1}{\sqrt{l(l+1)/2 + 1}} \left| \int_1^{\infty} r^{l(l+1)/2+1} \partial_r (\chi(l(r - 1))) \, dr \right|
$$

$$
\leq \frac{|\epsilon^{(lm)}| l}{\sqrt{l(l+1)/2 + 1}} \int_1^{1+l} \left( 1 + \frac{1}{l} \right) \sqrt{\frac{l(l+1)/2+1}{2}} |\partial_r \chi|(l(r - 1)) \, dr
$$

$$
\lesssim \frac{|\epsilon^{(lm)}|}{l}
$$

$$
\lesssim \frac{1}{l(2\sqrt{l(l+1)/2 + 1})^{1/2}} \left( \int_1^{\infty} (rh^{(lm)})^2 \, dr \right)^{1/2}
$$

(5.34)

where we used (5.33) and the fact that $|\partial_r \chi|$ is universally bounded. This shows that $\epsilon^{(lm)}$, $\tilde{\epsilon}^{(lm)}$ are well-defined.

We now prove the case $w = 2$ of (5.28), that is,

$$
\| \zeta^{[\geq 1]}_{H^0_{-5/2}(\mathbb{R}^3 \setminus \mathcal{B}_1)} \| \lesssim \| h^{[\geq 1]}_{-3/2} \|_{H^0_{-3/2}}.
$$
Indeed, by plugging in (5.25), for $l \geq 1$,

\[
\int_1^\infty r^2 \left( \zeta^{(lm)} \right)^2 \, dr \leq \int_1^\infty r^2 \sqrt{l(l+1)/2} \left[ \left( c^{(lm)} \right)^2 \left( \partial_r (\chi(l(r-1))) \right)^2 + \left( \hat{c}^{(lm)} \right)^2 \left( \partial^2_r (\chi(l(r-1))) \right)^2 \right] \, dr
\]

\[
\leq \int_1^\infty r^2 \sqrt{l(l+1)/2} \left[ \left( c^{(lm)} \right)^2 l^2 (\partial_r \chi)^2 (l(r-1)) + \left( \hat{c}^{(lm)} \right)^2 l^4 (\partial^2_r \chi)^2 (l(r-1)) \right] \, dr
\]

\[
\leq \left( \int_1^\infty (r h^{(lm)})^2 \, dr \right)^{1 + \frac{1}{l}} \int_1^\infty r^2 \sqrt{l(l+1)/2} \, dr
\]

\[
\leq \left( 1 + \frac{1}{l} \right)^{2 \sqrt{l(l+1)/2}} \int_1^\infty (r h^{(lm)})^2 \, dr
\]

\[
\leq \int_1^\infty (r h^{(lm)})^2 \, dr,
\]

where we used that $\text{supp} \partial_r \chi(l(r-1)) \subset [1, 1 + \frac{1}{l}]$ and (5.33),(5.34). Summing over $l \geq 1, m \in \{-l, \ldots, l\}$ proves the case $w = 2$ of (5.28).

We turn now to the case $w > 2$ of (5.28). On the one hand, the estimates (5.33), (5.34) improve,

\[
|c^{(lm)}| \leq \frac{1}{(2 \sqrt{l(l+1)/2} - 1)^{1/2}} \frac{1}{\sqrt{l(l+1)^{w-2}}} \left( \int_1^\infty \left( \frac{l(l+1)}{r^2} \right)^{w-2} \left( r^{w-1} h^{(lm)} \right)^2 \, dr \right)^{1/2},
\]

\[
|\hat{c}^{(lm)}| \leq \frac{1}{l(2 \sqrt{l(l+1)/2} + 1)^{1/2}} \frac{1}{\sqrt{l(l+1)^{w-2}}} \left( \int_1^\infty \left( \frac{l(l+1)}{r^2} \right)^{w-2} \left( r^{w-1} h^{(lm)} \right)^2 \, dr \right)^{1/2},
\]

where the integrals on the right-hand side correspond to the norm $\|h\|_{H^{w-2}}$ and are therefore summable, see Proposition 2.34.
On the other hand, by differentiating (5.25) and using Lemma 2.33, derivatives of $\zeta^{(lm)}$ generally are of the form

$$\partial_r \zeta^{(lm)} \approx \frac{1}{r} \zeta^{(lm)}, \quad (\nabla \zeta)^{(lm)}_E = -\frac{\sqrt{l(l+1)}}{r} \zeta^{(lm)}. \quad (5.37)$$

Combining (5.36) with (5.37), yields estimates for higher derivatives of $\zeta^{[\geq 1]}$ analogously to (5.35). This proves (5.28) for all $w \geq 2$, see Lemma 2.35.

Next, we claim that $\zeta^{[\geq 1]} \in H^{w-2}_{-5/2}$. Indeed, the sequence of smooth functions

$$f_n := \sum_{l=1}^{n} \sum_{m=-l}^{l} \zeta^{(lm)} Y^{(lm)}$$

is compactly supported in $\mathbb{R}^3 \setminus B_1$ and converges by (5.28) in $H^{w-2}_{-5/2}$ to $\zeta^{[\geq 1]}$ as $n \to \infty$. See also the analogous (4.66), (4.67).

The integral identities (5.29) and (5.30) follow from the definition of $\zeta^{[\geq 1]}$ in (5.25)-(5.26). The proof is left to the reader, see the analogous identity (4.68).

**Precise estimate for** $\zeta^{[\geq 1]}$. The proof is similar to part (2) of Proposition 4.11 and therefore only sketched here.

Consider first the case $w = 2$ of (5.31). By Proposition 2.34 and Lemmas 2.35 and 2.36 and given that we already control $\zeta^{[\geq 1]}$ above, it suffices to prove in the Hodge-Fourier formalism that for $l \geq 1, m \in \{-l, \ldots, l\}$,

$$\int_1^{\infty} r^2 \left( \frac{r}{\sqrt{l(l+1)}} \partial_r \zeta^{(lm)} \right)^2 dr \lesssim \int_1^{\infty} r^2 (h^{(lm)})^2 dr. \quad (5.38)$$

This follows by using the explicit (5.25) which shows that schematically

$$\left| \frac{r}{\sqrt{l(l+1)}} \partial_r \zeta^{(lm)} \right| \lesssim \zeta^{(lm)} + \tilde{c}^{(lm)} r^{\sqrt{l(l+1)/2}} \partial_r ((\partial_r \chi)(l(r-1)))$$

$$- \tilde{c}^{(lm)} r^{\sqrt{l(l+1)/2}} \partial_r^2 ((\partial_r \chi)(l(r-1))) \approx \zeta^{(lm)}.$$

Therefore, by the above control of $\zeta^{[\geq 1]}$, (5.38) follows. This proves (5.31) in the case $w = 2$.

The case $w > 2$ of (5.31) is treated similarly. Indeed, by the explicit expression (5.25), derivatives can be expressed in the Hodge-Fourier formalism as multiplication by $\frac{\sqrt{l(l+1)}}{r}$. 

At the same time, the estimates for the constants \(c^{(lm)}, \tilde{c}^{(lm)}\) improve, see (5.36). This allows to use the estimate above to conclude (5.31) for all \(w \geq 2\).

It remains to show that \(\mathcal{P}^{-1}_1(\partial_r \zeta^{[2]}, 0) \in \overline{\mathcal{H}^{w-2}_{-5/2}}\). By using (5.31) and the definition of \(\zeta^{[2]}\) in (5.25), it follows that

\[
X_n := \sum_{l=1}^{n} \sum_{m=-l}^{l} (\mathcal{P}^{-1}_1(\partial_r \zeta^{[2]}, 0))^{(lm)} E^{(lm)}
\]

is a sequence of smooth vectorfields compactly supported in \(\mathbb{R}^3 \setminus B_1\) that converges in \(\mathcal{H}^{w-2}_{-5/2}\) to \(\mathcal{P}^{-1}_1(\partial_r \zeta^{[2]}, 0)\) as \(n \to \infty\). By the definition of \(\overline{\mathcal{H}^{w-2}_{-5/2}}\), this shows that \(\mathcal{P}^{-1}_1(\partial_r \zeta^{[2]}, 0) \in \overline{\mathcal{H}^{w-2}_{-5/2}}\) and hence finishes the precise estimate of \(\zeta^{[2]}\).

**Regularity and boundary control of \(u^{[2]}\).** By Proposition C.6, for all \(w \geq 2\), the scalar function \(u^{[2]}\) defined in (5.24) is bounded by

\[
\|u^{[2]}\|_{H^{w-2}_{-1/2}(\mathbb{R}^3 \setminus \overline{B})} \lesssim \left\| \frac{1}{2} h + \zeta^{[2]} \right\|_{\mathcal{H}^{w-2}_{-5/2}}.
\]

(5.39)

We show now the improved boundary behaviour

\[
u^{[2]} \in \overline{\mathcal{H}^{w}_{-1/2}}.
\]

By Proposition C.9 it suffices to prove the next claim.

**Claim 5.12.** Let \(w \geq 2\) be an integer. It holds that

\[
\partial_r u^{[2]} \mid_{r=1} = 0.
\]

First, by (5.39), it holds that for \(l \geq 1, m \in \{-l, \ldots, l\},

\[
\int_{1}^{\infty} \frac{1}{(1+r)^{2}} (u^{(lm)})^2 \, dr, \int_{1}^{\infty} (\partial_r u^{(lm)})^2 \, dr, \int_{1}^{\infty} (1+r)^{2} (\partial_r^{2} u^{(lm)})^2 \, dr < \infty.
\]

By Lemma 2.14, it follows that

\[
\sup_{r \in [1, \infty)} (1+r)^{-1/2} |u^{(lm)}| < \infty, \quad \sup_{r \in [1, \infty)} (1+r)^{1/2} |\partial_r u^{(lm)}| < \infty.
\]

(5.40)

We show now that for \(w \geq 2\) and \(l \geq 1, m \in \{-l, \ldots, l\},

\[
\partial_r u^{(lm)} \mid_{r=1} = 0.
\]

Definition (5.24) is in the Hodge-Fourier formalism equivalent to the following ODEs for \(u^{(lm)}\) with \(l \geq 1, m \in \{-l, \ldots, l\}\), see Lemma 2.33,

\[
\begin{align*}
\left\{ r^{-1+\sqrt{l(l+1)/2}} \partial_r \left( r^{1-2\sqrt{l(l+1)/2}} \partial_r \left( r^{\sqrt{l(l+1)/2}} u^{(lm)} \right) \right) = \frac{1}{2} \left( \frac{1}{2} h^{(lm)} + \zeta^{(lm)} \right), \\
u^{(lm)} \mid_{r=1} = 0.
\end{align*}
\]

(5.41)
On the one hand,
\[
\int_1^\infty \partial_r \left( r^{1-2\sqrt{l(l+1)/2}} \partial_r \left( r^{\sqrt{l(l+1)/2}} u^{(lm)} \right) \right) dr \\
= \left[ r^{1-\sqrt{l(l+1)/2}} \partial_r u^{(lm)} + (\sqrt{l(l+1)/2}) r^{-\sqrt{l(l+1)/2}} u^{(lm)} \right]_1^\infty \\
= - \partial_r u^{(lm)} \big|_{r=1},
\]
where we used that \( l \geq 1 \), \( u^{(lm)} \big|_{r=1} = 0 \) and (5.40).

On the other hand, by (5.41) and integral identity (5.29),
\[
\int_1^\infty \partial_r \left( r^{1-2\sqrt{l(l+1)/2}} \partial_r \left( r^{\sqrt{l(l+1)/2}} u^{(lm)} \right) \right) dr \\
= \frac{1}{2} \int_1^\infty r^{1-\sqrt{l(l+1)/2}} \left( \frac{1}{2} h^{(lm)} + \zeta^{(lm)} \right) dr \\
= 0.
\]
This shows that for \( l \geq 1 \), \( m \in \{-l, \ldots, l\} \),
\[
\partial_r u^{(lm)} \big|_{r=1} = 0.
\]
This proves Claim 5.12 and finishes the control of \( u^{[\geq 1]} \). This finishes the proof of Proposition 5.10. \( \square \)

The next proposition shows that \( u^{[0]} \in \mathcal{H}^w_{-1/2} \) and \( \xi \in \mathcal{H}^w_{-1/2} \) with quantitative estimates.

**Proposition 5.13.** Let \( w \geq 2 \) be an integer and \( h \in \mathcal{H}^{w-2}_{-5/2} \). Then, the following holds.

- **Regularity and boundary behaviour of \( u^{[0]} \).** The radial scalar function \( u^{[0]} \) defined in (5.23) is bounded by
  \[
  \| u^{[0]} \|_{\mathcal{H}^w_{-1/2}(\mathbb{R}^3 \setminus \overline{T})} \lesssim \| h^{[0]} \|_{\mathcal{H}^{w-2}_{-5/2}}. \tag{5.42}
  \]
  Furthermore, it holds that \( u^{[0]} \in \mathcal{H}^w_{-1/2} \).

- **Regularity and boundary behaviour of \( \xi \).** The \( S_r \)-tangent vector field \( \xi \) defined in (5.27) is bounded by
  \[
  \| \xi \|_{\mathcal{H}^w_{-1/2}(\mathbb{R}^3 \setminus \overline{T})} \lesssim \| h^{[\geq 1]} \|_{\mathcal{H}^{w-2}_{-5/2}}. \tag{5.43}
  \]
  Furthermore, it holds that \( \xi \in \mathcal{H}^w_{-1/2} \).
Proof of Proposition 5.13. We prove each part separately.

**Regularity and boundary behaviour of** $u[0]$. We first show that for $w \geq 2$,

$$||u[0]||_{H^{w}_{-1/2}(\mathbb{R}^3 \setminus B_1)} \lesssim ||h[0]||_{H^{w-2}_{-5/2}}.$$  (5.44)

Recall that $u[0]$ satisfies on $r > 1$ by (5.23)

\[
\left\{ \begin{array}{l}
\frac{1}{r^2} \partial_r \left( r \partial_r \left( ru[0] \right) \right) = \frac{1}{4} h[0], \\
u[0]|_{r=1} = \partial_r u[0]|_{r=1} = 0.
\end{array} \right.
\]

By integration, it follows that

$$u[0](r) = \frac{1}{r} \int_{1}^{r} \frac{1}{r'} \left( \int_{1}^{r'} (r'^2) h[0] dr'' \right) dr'.$$  (5.45)

We now prove the case $w = 2$ of (5.44) by showing

$$||u[0]||_{H^{0}_{-1/2}(\mathbb{R}^3 \setminus B_1)} \lesssim ||h[0]||_{H^{0}_{-5/2}};$$  (5.46)

$$||\partial_r u[0]||_{H^{0}_{-3/2}(\mathbb{R}^3 \setminus B_1)} \lesssim ||h[0]||_{H^{0}_{-5/2}};$$  (5.47)

$$||\partial_r^2 u[0]||_{H^{0}_{-5/2}(\mathbb{R}^3 \setminus B_1)} \lesssim ||h[0]||_{H^{0}_{-5/2}}.$$  (5.48)

Indeed, see Lemma 2.35 and note that tangential regularity follows because $u[0]$ is constant on spheres.

To prove (5.46), use (5.45) and that $u[0]$ is constant on spheres,

$$||u[0]||^2_{H^{0}_{-1/2}(\mathbb{R}^3 \setminus B_1)} = \int_{1}^{\infty} \frac{1}{r^2} \left( \int_{1}^{r} \frac{1}{r'} \left( \int_{1}^{r'} (r'^2) h[0] dr'' \right) dr' \right)^2 dr$$

$$= \left[ -\frac{1}{r} \left( \int_{1}^{r} \frac{1}{r'} \left( \int_{1}^{r'} (r'^2) h[0] dr'' \right) dr' \right) \right]_{1}^{\infty}$$

$$+ 2 \int_{1}^{\infty} \frac{1}{r} \left( \int_{1}^{r} (r'^2) h[0] dr' \right) \left( \int_{1}^{r} \frac{1}{r'} \left( \int_{1}^{r'} (r'^2) h[0] dr'' \right) dr' \right) dr.$$
The boundary term has negative sign and may thus be discarded. The integral term can be estimated by Cauchy-Schwarz as

\[\int_1^\infty \left( \frac{1}{r} \int_1^r (r'^2) h^{[0]} dr' \right) \left( \frac{1}{r} \int_1^{r'} \left( \int_1^{r''} h^{[0]} dr'' \right) dr' \right) dr \leq \left( \int_1^\infty \frac{1}{r^2} \left( \int_1^r (r'^2) h^{[0]} dr' \right)^2 dr \right)^{1/2} \|u^{[0]}\|_{H^0_{-1/2}(\mathbb{R}^3 \setminus \overline{B}_1)}.
\]

This proves that

\[\|u^{[0]}\|_{H^0_{-1/2}(\mathbb{R}^3 \setminus \overline{B}_1)} \lesssim \int_1^\infty \frac{1}{r^2} \left( \int_1^r (r'^2) h^{[0]} dr' \right)^2 dr. \tag{5.49}\]

A similar integration by parts shows that

\[\int_1^\infty \frac{1}{r^2} \left( \int_1^r (r'^2) h^{[0]} dr' \right)^2 dr \lesssim \int_1^\infty r^4 (h^{[0]})^2 dr = \|h^{[0]}\|^2_{H^0_{-5/2}}. \tag{5.50}\]

Together, (5.49) and (5.50) prove (5.46).

We now prove (5.47). By differentiating (5.45) in \(r\), it follows that on \(r > 1\)

\[\partial_r u^{[0]} = -\frac{1}{r} u^{[0]} + \frac{1}{r^2} \int_1^r (r'^2) h^{[0]} dr'.\]

Therefore, by using (5.46) and (5.50),

\[\|\partial_r u^{[0]}\|_{H^0_{-3/2}(\mathbb{R}^3 \setminus \overline{B}_1)} \leq \|u^{[0]}\|_{H^0_{-1/2}(\mathbb{R}^3 \setminus \overline{B}_1)} + \left\| \frac{1}{r^2} \int_1^r (r'^2) h^{[0]} dr' \right\|_{H^0_{-3/2}(\mathbb{R}^3 \setminus \overline{B}_1)} \]

\[= \|u^{[0]}\|_{H^0_{-1/2}(\mathbb{R}^3 \setminus \overline{B}_1)} + \frac{1}{r} \int_1^r (r'^2) h^{[0]} dr' \right\|_{H^0_{-3/2}(\mathbb{R}^3 \setminus \overline{B}_1)} \]

\[\lesssim \|h^{[0]}\|_{H^0_{-3/2}(\mathbb{R}^3 \setminus \overline{B}_1)}.
\]

This proves (5.47).
By the defining ODE (5.23), the previous estimates (5.46), (5.47) imply (5.48). This finishes the proof of (5.44) in the case \( w = 2 \).

We turn now to the case \( w > 2 \) of (5.44). Higher radial derivatives can be estimated by differentiating the defining ODE (5.23) in \( r \). Tangential regularity is trivial because \( u^{[0]} \) is radial. This proves (5.44) for \( w \geq 2 \).

It remains to show that \( u^{[0]} \in \overline{H}^{w}_{-1/2}(\mathbb{R}^3 \setminus \overline{B}_1) \). Indeed, this follows by (5.23) and Proposition 2.13. This finishes the control of \( u^{[0]} \).

**Regularity and boundary behaviour of \( \xi \).** We prove now that for \( w \geq 2 \)

\[
\| \xi \|_{H^{w}_{-1/2}(\mathbb{R}^3 \setminus \overline{B}_1)} \lesssim \| h^{[\geq 1]} \|_{\overline{H}^{w-2}_{-5/2}}.
\] (5.51)

First, we claim that

\[
\| \xi \|_{H^{w}_{-1/2}(\mathbb{R}^3 \setminus \overline{B}_1)} + \| \nabla \xi \|_{H^{0}_{-3/2}(\mathbb{R}^3 \setminus \overline{B}_1)} \lesssim \| h^{[\geq 1]} \|_{\overline{H}^{0}_{-5/2}}.
\] (5.52)

Indeed, by (5.27), \( \xi \) solves on each \( S_r, r \geq 1 \),

\[
\text{div} \psi \xi = \frac{1}{r^3} \int_1^r \frac{(r')^3 \xi^{[\geq 1]}}{r'} \, dr',
\]

\[
\text{curl} \psi \xi = 0.
\]

Therefore, by Proposition 2.23, for all \( r \geq 1 \),

\[
\int_{S_r} |\nabla \xi|^2 + \frac{1}{r^2} |\xi|^2 = \int_{S_r} (\text{div} \psi \xi)^2.
\] (5.53)
We can estimate
\[
\|\xi\|_{H_{-1/2}^0(\mathbb{R}^3 \setminus B_1)}^2 + \|\nabla \xi\|_{H_{-3/2}^0(\mathbb{R}^3 \setminus B_1)}^2 = \int_1^\infty \int_{S_r} \frac{1}{r^6} \left( \int_1^r (r')^3 \zeta \left[ \zeta \geq 1 \right] \right)^2 \, dr \, dr
\]
\[
= \int_{S^2}^\infty \int_1^r \frac{1}{r^4} \left( \int_1^r (r')^3 \zeta \left[ \zeta \geq 1 \right] \, dr' \right)^2 \, dr
\]
\[
= \int_{S^2} \left[ -\frac{1}{3r^3} \left( \int_1^r (r')^3 \zeta \left[ \zeta \geq 1 \right] \, dr' \right)^2 \right]_1^\infty
\]
\[
+ 2 \int_{S^2}^\infty \int_1^r \frac{1}{r^3} (r^3 \zeta \left[ \zeta \geq 1 \right]) \left( \int_1^r (r')^3 \zeta \left[ \zeta \geq 1 \right] \, dr' \right) \, dr.
\]

The first term on the right-hand side is non-positive and discarded. The second term can be estimated by Cauchy-Schwarz as
\[
\int_{\mathbb{R}^3 \setminus B_1} \frac{1}{r^5} (r^3 \zeta \left[ \zeta \geq 1 \right]) \left( \int_1^r (r')^3 \zeta \left[ \zeta \geq 1 \right] \, dr' \right)
\]
\[
\leq \left( \int_{\mathbb{R}^3 \setminus B_1} \frac{1}{r^6} \left( \int_1^r (r')^3 \zeta \left[ \zeta \geq 1 \right] \right)^2 \, dr \right)^{1/2} \left( \int_{\mathbb{R}^3 \setminus B_1} (r^2 (\zeta \left[ \zeta \geq 1 \right])^2 \right)^{1/2}
\]
\[
= \left( \int_{\mathbb{R}^3 \setminus B_1} (\text{div} \xi)^2 \right)^{1/2} \left( \int_{\mathbb{R}^3 \setminus B_1} r^2 (\zeta \left[ \zeta \geq 1 \right])^2 \right)^{1/2}
\]
\[
= \left( \|\xi\|_{H_{-1/2}^0(\mathbb{R}^3 \setminus B_1)}^2 + \|\nabla \xi\|_{H_{-3/2}^0(\mathbb{R}^3 \setminus B_1)}^2 \right)^{1/2} \|\zeta \left[ \zeta \geq 1 \right]\|_{H_{-5/2}^0},
\]
where we used (5.53). This shows that
\[
\|\xi\|_{H_{-1/2}^0(\mathbb{R}^3 \setminus B_1)} + \|\nabla \xi\|_{H_{-3/2}^0(\mathbb{R}^3 \setminus B_1)} \lesssim \|\zeta \left[ \zeta \geq 1 \right]\|_{H_{-5/2}^0}.
\]

By Proposition 5.10, this proves (5.52).
Next, we consider radial regularity of order 1. We claim that
\[ \| \nabla_N \xi \|_{H^{0,3/2}(\mathbb{R}^3 \setminus B_1)}^2 + \| \nabla \nabla_N \xi \|_{H^{0,5/4}(\mathbb{R}^3 \setminus B_1)}^2 \lesssim \| h^{[1]} \|_{H^{-5/2}}^2. \]  

(5.54)

Indeed, by Lemma 2.35, it follows that on \( r > 1 \)
\[ r \, \partial_r (r \, \partial_r \xi) = \partial_r \left( \frac{1}{r^2} r^3 \, \partial_r \xi \right) = -2 \, \partial_r \xi + r \left( \frac{1}{r^3} \partial_r (r^3 \, \partial_r \xi) \right), \]
\[ \text{cufi} \, \nabla_N \xi = 0 \]
Therefore \( \nabla_N \xi \) solves on each \( S_r, r \geq 1 \), the Hodge system
\[ \partial_r \left( \frac{1}{r^2} r^3 \, \partial_r \xi \right) = -2 \, \partial_r \xi + r \left( \frac{1}{r^3} \partial_r (r^3 \, \partial_r \xi) \right), \]
\[ \text{cufi} \, \nabla_N \xi = 0. \]

By Propositions 2.23 and 5.10, this proves (5.54).

Similarly, we have the higher radial regularity for each \( w \geq 2 \),
\[ \| \nabla_N^w \xi \|_{H^{0,1/2-w}(\mathbb{R}^3 \setminus B_1)}^2 \lesssim \| h^{[1]} \|_{H^{-5/2}}^2. \]  

(5.55)

This follows by an induction in \( w \geq 2 \), using that by
\[ \partial_r (\nabla_N^w \xi) = \frac{1}{r} \partial_r^w (r \, \partial_r \xi) \]
\[ = \frac{1}{r} \partial_r^{w-1} \left( -2 \, \partial_r \xi + r \xi^{[2]} \right) \]
\[ = -2 \, \partial_r \left( \nabla_N^{w-1} \left( \frac{1}{r} \xi \right) \right) + \frac{1}{r} \partial_r^{w-1} \left( \frac{1}{r} \xi^{[1]} \right) \]
so that we have
\[ \nabla_N^w \xi = -2 \nabla_N^{w-1} \left( \frac{1}{r} \xi \right) + \nabla_N^{w-2} \left( \frac{1}{r} \nabla_1^{-1} (\xi^{[1]}), 0 \right) \]
\[ + \nabla_N^{w-2} \left( \nabla_1^{-1} (\partial_r \xi^{[1]}), 0 \right), \]

(5.56)

(5.57)

where the last term is controlled by Proposition 5.10.

We turn now to tangential regularity. We claim first that
\[ \| \nabla \nabla \xi \|_{H^{0,5/2}(\mathbb{R}^3 \setminus B_1)} \lesssim \| h^{[1]} \|_{H^{-5/2}}. \]  

(5.58)
By Proposition 2.34, it suffices to estimate for \( l \geq 1, m \in \{-l, \ldots, l\} \)

\[
\int_1^\infty r^2 \left( \frac{l(l+1)}{r^2} \xi^{(lm)}_E \right)^2 dr \lesssim \int_1^\infty r^2 (h^{(lm)})^2 dr.,
\]

(5.59)

Definition (5.27) is in the Hodge-Fourier formalism equivalent to the following expression for \( \xi^{(lm)}_E, l \geq 1, m \in \{-l, \ldots, l\} \),

\[
\frac{\sqrt{l(l+1)}}{r} \xi^{(lm)}_E = \frac{1}{r^2} \left( \int_1^r (r')^2 \xi^{(lm)} dr' \right).
\]

(5.60)

Rewrite first

\[
\int_1^\infty r^2 \left( \frac{l(l+1)}{r^2} \right)^2 \left( \xi^{(lm)}_E \right)^2 dr = l(l+1) \int_1^\infty \frac{1}{r^4} \left( \int_1^r (r')^2 \xi^{(lm)} dr' \right)^2 dr
\]

\[
= l(l+1) \int_1^{1+\frac{1}{l}} \frac{1}{r^4} \left( \int_1^r (r')^2 \xi^{(lm)} dr' \right)^2 dr;
\]

(5.61)

where in the last integral we bounded the domain of integration by combining the integral identity (5.30) and the fact that

\[
\text{supp} \xi^{(lm)} \subset \left[1, 1 + \frac{1}{l}\right].
\]

By (5.25) and (5.26), the right-hand side of (5.61) can be estimated by

\[
l(l+1) \int_1^{1+\frac{1}{l}} \frac{1}{r^4} \left( \int_1^r (r')^2 \xi^{(lm)} dr' \right)^2 dr
\]

\[
\lesssim l(l+1) \int_1^{1+\frac{1}{l}} \frac{1}{r^4} \left( \int_1^r c^{(lm)}(r') \sqrt{l(l+1)}/r^2 \partial_r (\chi(l(r-1))) dr' \right)^2 dr
\]

\[
+ l(l+1) \int_1^{1+\frac{1}{l}} \frac{1}{r^4} \left( \int_1^r \tilde{c}^{(lm)}(r') \sqrt{l(l+1)}/r^2 \partial_r^2 (\chi(l(r-1))) dr' \right)^2 dr.
\]

(5.62)
The first term on the right-hand side is estimated as

\[ l(l + 1) \int_1^{1 + \frac{1}{r}} \left( \int_1^r c_{(lm)}(r') \sqrt{l(l+1)/2+1} \partial_r (\chi(l(r-1))) \, dr' \right)^2 \, dr \]

\[ \lesssim l(l + 1) \int_1^{1 + \frac{1}{r}} \left( \int_1^r \left( \frac{r'}{r} \right) \sqrt{l(l+1)/2-2} \, dr \right)^2 \, dr \]

\[ \lesssim l(l + 1) (c_{(lm)})^2 \int_1^{\infty} r^2 \sqrt{l(l+1)/2-2} \, dr \]

\[ \lesssim \int_1^{\infty} (r h_{(lm)})^2 \, dr, \]

where we used (5.33) and the fact that \( l \geq 1 \). The second term on the right-hand side of (5.62) is estimated similarly by using (5.34), this is left to the reader. This proves (5.59) and therefore (5.58).

We also have the higher tangential regularity

\[ \| \nabla^w \xi \|_{H_{-1/2-w}^0(\mathbb{R}^3, B_1)} \lesssim \| h^{[\geq 1]} \|_{H^{-2/5}} \]  

(5.63)

Indeed, in the Hodge-Fourier formalism, see (5.60),

\[ \left| \left( \frac{\sqrt{l(l+1)}}{r} \right)^w \xi_{E}^{(lm)} \right| = \frac{1}{\sqrt{l(l+1)r}} \int_1^r (r')^2 \left( \frac{\sqrt{l(l+1)}}{r} \right)^w \xi^{(lm)}(r') \, dr' \]

\[ \lesssim \frac{1}{\sqrt{l(l+1)r}} \int_1^r (r')^2 \left( \frac{\sqrt{l(l+1)}}{r'} \right)^w \xi^{(lm)}(r') \, dr'. \]

Together with the higher regularity of \( \zeta^{[\geq 1]} \) provided by Proposition 5.10, similar estimates as for (5.58) imply (5.63).

To summarise, (5.52), (5.54), (5.55), (5.58) and (5.63) imply (5.51) for \( w \geq 2 \).

It remains to show that \( \xi \in \overline{H}_{-1/2}^w \). This follows by induction from (5.56) and the fact that \( \zeta^{[\geq 1]}, \mathcal{P}_{1}^{-1}(\partial_r \zeta^{[\geq 1]}, 0) \in \overline{H}_{-5/2}^{w-2} \) together with Proposition 2.13. This finishes the proof of Proposition 5.13. \( \square \)
6. Proof of the main Theorem 3.1

In this section, we prove the Main Theorem 3.1. The idea of the proof is to use Theorems 4.1 and 5.1 to set up an iterative scheme. We show that this scheme is well-defined and converges to a fixpoint which solves the maximal constraint equations on $\mathbb{R}^3$.

6.1. Setup of the iterative scheme. In this section, we define a sequence of pairs $(g_i, k_i)_{i \geq 1}$, where for each $i \geq 1$, $g_i$ is an $\mathcal{H}^w_{-1/2}$-asymptotically flat metric and $k_i \in \mathcal{H}^{w-1}_{-3/2}$ a symmetric 2-tensor on $\mathbb{R}^3$.

Let $\varepsilon > 0$ be a small constant to be determined later. Let $(\bar{g}, \bar{k}) \in \mathcal{H}^w(B_1) \times \mathcal{H}^{w-1}(B_1)$ be a solution to the maximal constraint equations on $B_1$,

$$
\begin{align*}
R(\bar{g}) &= |\bar{k}|^2_{\bar{g}}, \\
\text{div}_\bar{g} \bar{k} &= 0, \\
\text{tr}_\bar{g} \bar{k} &= 0
\end{align*}
$$

such that

$$
\| (\bar{g} - e, \bar{k}) \|_{\mathcal{H}^w(B_1) \times \mathcal{H}^{w-1}(B_1)} < \varepsilon.
$$

- **Definition of** $(g_1, k_1)$. Use standard Sobolev extension to extend $\bar{g}$ from $B_1$ to an $\mathcal{H}^w$-asymptotically flat metric $g_1$ on $\mathbb{R}^3$ such that

$$
\| g_1 - e \|_{\mathcal{H}^w_{1/2}} \lesssim \| \bar{g} - e \|_{\mathcal{H}^w(B_1)}.
$$

Similarly, extend $\bar{k}$ from $B_1$ to a symmetric $g_1$-tracefree 2-tensor $k_1 \in \mathcal{H}^{w-1}_{-3/2}$ such that

$$
\| k_1 \|_{\mathcal{H}^{w-1}_{-3/2}} \lesssim \| k \|_{\mathcal{H}^{w-1}(B_1)},
$$

see Lemma 4.4.

- **Definition of** $(g_{i+1} - e, k_{i+1})$ for $i \geq 1$. Given $(g_i, k_i)$, define $(g_{i+1}, k_{i+1})$ as follows.

First, let $g_{i+1}$ be the $\mathcal{H}^w_{-1/2}$-asymptotically flat metric on $\mathbb{R}^3$ constructed by Theorem 5.1 such that

$$
g_{i+1}|_{B_1} = \bar{g}, \\
R(g_{i+1}) = |k_i|^2_{g_i} \text{ on } \mathbb{R}^3.
$$

Here we assumed that $\| (g_i - e, k_i) \|_{\mathcal{H}^w_{1/2} \times \mathcal{H}^{w-1}_{-3/2}}$ is sufficiently small.

Second, let $k_{i+1} \in \mathcal{H}^{w-1}_{-3/2}$ be the symmetric 2-tensor on $\mathbb{R}^3$ constructed by Theorem 4.1 such that

$$
k_{i+1}|_{B_1} = \bar{k}$$
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and on $\mathbb{R}^3$

$$\text{div}_{g_{i+1}}(k_{i+1}) = 0,$$
$$\text{tr}_{g_{i+1}}(k_{i+1}) = 0.$$  

Here we assumed further that $\|g_{i+1} - e\|_{\mathcal{H}_{1/2}^{w-1}}$ is sufficiently small.

In the next section we prove that for $\varepsilon > 0$ small enough, the above smallness assumptions are satisfied for all $i \geq 1$. In particular, that the sequence is well-defined.

6.2. Convergence of the iterative scheme. In this section, we combine the iteration estimates of Theorems 4.1 and 5.1 to prove estimates for the iteration scheme defined above in Section 6.1. These estimates then directly imply uniform boundedness and convergence of the sequence $(g_i, k_i)_{i \geq 1}$.

The next proposition is the main result of this section.

**Proposition 6.1.** Let $w \geq 2$ be an integer. There exist universal constants $\varepsilon > 0, c > 0$ such that if for an $i \geq 2$ it holds that

$$\|(g_i - e, k_i)\|_{\mathcal{H}_{1/2}^{w-1} \times \mathcal{H}_{3/2}^{w-1}} < \varepsilon, \|(g_{i-1} - e, k_{i-1})\|_{\mathcal{H}_{1/2}^{w-1} \times \mathcal{H}_{3/2}^{w-1}} < \varepsilon,$$

then

$$\|(g_{i+1} - g_i, k_{i+1} - k_i)\|_{\mathcal{H}_{1/2}^{w-1} \times \mathcal{H}_{3/2}^{w-1}} \leq c \left( \|(g_i - e, k_i)\|_{\mathcal{H}_{1/2}^{w-1}} + \|(g_{i-1} - e, k_{i-1})\|_{\mathcal{H}_{1/2}^{w-1}} + \|(g_{i-1} - e, k_{i-1})\|_{\mathcal{H}_{1/2}^{w-1}}^2 \right) \times \|(g_i - g_{i-1}, k_i - k_{i-1})\|_{\mathcal{H}_{1/2}^{w-1} \times \mathcal{H}_{3/2}^{w-1}}. \quad (6.3)$$

Before proving Proposition 6.1, we state the following technical lemma. Its proof is based on Lemma 2.9 and Corollary 2.11 and left to the reader, see also Lemma 2.38.

**Lemma 6.2.** Let $w \geq 2$ be an integer. Let $g$ and $g'$ be two $\mathcal{H}_{w-1/2}^{w}$-asymptotically flat metrics on $\mathbb{R}^3$. There exists universal constant $\varepsilon > 0$ such that if

$$\|g - e\|_{\mathcal{H}_{w-1/2}^{w-1}} < \varepsilon,$$

then for all symmetric 2-tensors $V \in \mathcal{H}_{-3/2}^{w-1}$,

$$\|V_g^2 - V_{g'}^2\|_{\mathcal{H}_{-5/2}^{w-2}} \lesssim \|g - g'\|_{\mathcal{H}_{w-1/2}^{w}} \|V\|_{\mathcal{H}_{w-1/2}^{w-1}}^2.$$  

We turn now to the proof of Proposition 6.1.
Proof of Proposition 6.1. On the one hand, for \( \varepsilon > 0 \) sufficiently small, it follows by the iteration estimates of Theorem 5.1 and Lemmas 6.2 that
\[
\| g_{i+1} - g_i \|_{H^{-1/2}_w} \lesssim \| R(g_{i+1}) - R(g_i) \|_{H^{-5/2}_w} \\
\lesssim \| |k_i|^2 g_i - |k_{i-1}|^2 g_{i-1} \|_{H^{-5/2}(\mathbb{R}^3)} \\
\lesssim \| |k_i|^2 g_i - |k_{i-1}|^2 g_{i-1} \|_{H^{-5/2}(\mathbb{R}^3)} + \| |k_i|^2 g_i - |k_{i-1}|^2 g_{i-1} \|_{H^{-5/2}(\mathbb{R}^3)} \\
\lesssim \| |k_i|^2 g_i - |k_{i-1}|^2 g_{i-1} \|_{H^{-5/2}(\mathbb{R}^3)} + \| g_i - g_{i-1} \|_{H^{-1/2}_w}
\] (6.4)

Using Lemma 2.8 and the identity
\[
|k_i|^2 g_i - |k_{i-1}|^2 g_{i-1} = (k_i - k_{i-1})ab(k_i + k_{i-1})ab,
\]
we have, for \( \varepsilon > 0 \) sufficiently small,
\[
\| |k_i|^2 g_i - |k_{i-1}|^2 g_{i-1} \|_{H^{-5/2}(\mathbb{R}^3)} \lesssim \| k_i + k_{i-1} \|_{H^{-3/2}_w} \| k_i - k_{i-1} \|_{H^{-3/2}_w} \\
\lesssim \left( \| k_i \|_{H^{-3/2}_w} + \| k_{i-1} \|_{H^{-3/2}_w} \right) \| k_i - k_{i-1} \|_{H^{-3/2}_w}.
\]

Plugging this into (6.4), we get
\[
\| g_{i+1} - g_i \|_{H^{-1/2}_w} \lesssim \left( \| k_i \|_{H^{-3/2}_w} + \| k_{i-1} \|_{H^{-3/2}_w} + \| k_{i-1} \|_{H^{-5/2}_w} \right) \| (g_i, k_i) \|_{H^{-1/2}_w \times H^{-3/2}_w}.
\]

On the other hand, for \( \varepsilon > 0 \) sufficiently small, by the iteration estimates of Theorem 4.1,
\[
\| k_{i+1} - k_i \|_{H^{-3/2}_w} \lesssim \| k_{i+1} \|_{H^{-3/2}_w} \| g_{i+1} - g_i \|_{H^{-1/2}_w} \\
\lesssim \| k_i \|_{H^{-3/2}_w} \| g_{i+1} - g_i \|_{H^{-1/2}_w}.
\]

Combining the two last estimates, it follows that for \( \varepsilon > 0 \) sufficiently small, there exists a universal constant \( c > 0 \) such that
\[
\| (g_{i+1}, k_{i+1}) - (g_i, k_i) \|_{H^{-1/2}_w \times H^{-3/2}_w} \leq c \left( \| k_i \|_{H^{-3/2}_w} + \| k_{i-1} \|_{H^{-3/2}_w} + \| k_{i-1} \|_{H^{-5/2}_w} \right) \| (g_i, k_i) \|_{H^{-1/2}_w \times H^{-3/2}_w}. \]

This finishes the proof of Proposition 6.1. \( \square \)

We are now in position to prove that the sequence is well-defined and derive a uniform estimate.

Lemma 6.3. Let \( w \geq 2 \) be an integer. There is a universal \( \varepsilon > 0 \) small enough such that if
\[
\| (\bar{g} - e, \bar{k}) \|_{H^w(B_1) \times H^{w-1}(B_1)} < \varepsilon,
\]
then the sequence \( (g_i, k_i)_{i \geq 1} \) is well-defined and for all \( i \geq 2 \),
\[
\| (g_{i+1}, k_{i+1}) - (g_i, k_i) \|_{H_{1/2}^{w} \times H_{3/2}^{w-1}} \leq \frac{1}{4} \| (g_i - g_{i-1}, k_i - k_{i-1}) \|_{H_{1/2}^{w} \times H_{3/2}^{w-1}}.
\]

(6.5)

Furthermore, it is uniformly bounded by
\[
\| (g_i, k_i) \|_{H_{1/2}^{w} \times H_{3/2}^{w-1}} \lesssim \| (\bar{g} - e, \bar{k}) \|_{H^{w}(B_1) \times H^{w-1}(B_1)}.
\]

Proof. The proof of (6.5) goes by induction in \( i \geq 2 \).

The case \( i = 2 \). By construction, for \( \varepsilon > 0 \) sufficiently small, by Theorems 4.1 and 5.1,
\[
\| (g_1 - e, k_1) \|_{H_{1/2}^{w} \times H_{3/2}^{w-1}} \leq \| (\bar{g} - e, \bar{k}) \|_{H^{w}(B_1) \times H^{w-1}(B_1)};
\]
\[
\| (g_2 - e, k_2) \|_{H_{1/2}^{w} \times H_{3/2}^{w-1}} \lesssim \| \bar{g} - e \|_{H^{w}(B_1)} + \| R(g_2) \|_{H_{3/2}^{w-2}} + \| \bar{k} \|_{H^{w-1}(B_1)}
\]
\[
\lesssim \| \bar{g} - e \|_{H^{w}(B_1)} + \| k_1 \|_{H_{1/2}^{w-2}} + \| \bar{k} \|_{H^{w-1}(B_1)}
\]
\[
\lesssim \| (\bar{g} - e, \bar{k}) \|_{H^{w}(B_1) \times H^{w-1}(B_1)}.
\]
where we used Lemmas 2.8 and 2.9. By Theorems 4.1 and 5.1, for \( \varepsilon > 0 \) sufficiently small, \( (g_3, k_3) \) is well-defined.

By Proposition 6.1, there exists a universal \( c > 0 \) such that
\[
\| (g_3 - g_2, k_3 - k_2) \|_{H_{1/2}^{w} \times H_{3/2}^{w-1}} \leq 3c \| (\bar{g} - e, \bar{k}) \|_{H^{w}(B_1) \times H^{w-1}(B_1)} \| (g_2 - g_1, k_2 - k_1) \|_{H_{1/2}^{w} \times H_{3/2}^{w-1}}
\]

Let \( \varepsilon < \frac{1}{24c} \). This proves the case \( i = 2 \).

The induction step \( i \to i + 1 \). Using the induction hypothesis, it holds for \( j = i - 1, i \) that
\[
\| (g_{j+1} + e, k_{j+1}) \|_{H_{1/2}^{w} \times H_{3/2}^{w-1}}
\]
\[
\leq \| (g_{j+1} - g_j, k_{j+1} - k_j) \|_{H_{1/2}^{w} \times H_{3/2}^{w-1}} + \ldots + \| (g_3 - g_2, k_3 - k_2) \|_{H_{1/2}^{w} \times H_{3/2}^{w-1}}
\]
\[
+ \| (g_2 - e, k_2) \|_{H_{1/2}^{w} \times H_{3/2}^{w-1}}
\]
\[
\leq \sum_{m=1}^{j-2} \frac{1}{4^m} \| (g_2 - g_1, k_2 - k_1) \|_{H_{1/2}^{w} \times H_{3/2}^{w-1}} + \| (g_2 - e, k_2) \|_{H_{1/2}^{w} \times H_{3/2}^{w-1}}
\]
\[
\leq 2 \| (g_2 - g_1, k_2 - k_1) \|_{H_{1/2}^{w} \times H_{3/2}^{w-1}} + \| (g_2 - e, k_2) \|_{H_{1/2}^{w} \times H_{3/2}^{w-1}}
\]
\[
\lesssim \| (\bar{g} - e, \bar{k}) \|_{H_{1/2}^{w} \times H_{3/2}^{w-1}}.
\]

(6.6)
This shows that \((g_{i+2}, k_{i+2})\) is well-defined for \(\varepsilon > 0\) sufficiently small. By applying Proposition 6.1, there exists a universal constant \(c' > 0\) such that
\[
\|(g_{i+2} - g_{i+1}, k_{i+2} - k_{i+1})\|_{H_{-1/2}^w \times H_{-3/2}^{w-1}} \leq c' \varepsilon \|(g_{i+1} - g_i, k_{i+1} - k_i)\|_{H_{-1/2}^w \times H_{-3/2}^{w-1}}.
\]
Let \(\varepsilon < \frac{1}{2kc'}\). This finishes the induction step and hence the proof of (6.5).

For \(\varepsilon > 0\) small, we have, as in (6.6), the uniform estimate for all \(i \geq 1\),
\[
\|(g_i, k_i)\|_{H_{-1/2}^w \times H_{-3/2}^{w-1}} \lesssim \|(\bar{g} - e, \bar{k})\|_{H_w(B_1) \times H_{w-1}(B_1)}.
\]
This finishes the proof of Lemma 6.3.

Lemma 6.3 implies convergence of the iterative scheme.

**Corollary 6.4.** There exists \(\varepsilon > 0\) small such that if
\[
\|(\bar{g}, \bar{k})\|_{H_w(B_1) \times H_{w-1}(B_1)} < \varepsilon,
\]
then the sequence \((g_i - e, k_i)\) converges in \(H_{-1/2}^w \times H_{-3/2}^{w-1}\) as \(i \to \infty\). Its limit
\[
(g', k') := \lim_{i \to \infty} (g_i, k_i) \in H_{-1/2}^w \times H_{-3/2}^{w-1}
\]
 solves the maximal constraint equations on \(\mathbb{R}^3\)
\[
R(g') = |k'|_{g'}^2,
\]
\[
\text{div}_{g'} k' = 0,
\]
\[
\text{tr}_{g'} k' = 0
\]
and satisfies \((g', k')|_{B_1} = (\bar{g}, \bar{k})\). Moreover,
\[
\|(g' - e, k')\|_{H_{-1/2}^w \times H_{-3/2}^{w-1}} \lesssim \|(\bar{g} - e, \bar{k})\|_{H_w(B_1) \times H_{w-1}(B_1)}.
\]

**Proof of Corollary 6.4.** Lemma 6.3 shows that the iterative scheme \((g_i, k_i)\) is a contraction in the Hilbert space \(H_{-1/2}^w \times H_{-3/2}^{w-1}\). By the Banach fixpoint theorem, the scheme therefore converges to a fixpoint,
\[
(g', k') := \lim_{i \to \infty} (g_i, k_i) \in H_{-1/2}^w \times H_{-3/2}^{w-1}.
\]
The convergence in \(H_{-1/2}^w \times H_{-3/2}^{w-1}\) is strong enough to conclude that \((g', k')\) solves (6.7) and moreover, by construction of the sequence,
\[
(g', k')|_{B_1} = (\bar{g}, \bar{k}).
\]
By the uniform estimate in Lemma 6.3,
\[
\|(g' - e, k')\|_{H_{-1/2}^w \times H_{-3/2}^{w-1}} \lesssim \|(\bar{g} - e, \bar{k})\|_{H_w(B_1) \times H_{w-1}(B_1)}.
\]
This finishes the proof of Corollary 6.4.
Appendix A. The proof of Proposition 2.30

In this section we prove Proposition 2.30. First we show that
\[ \left\{ E^{(lm)}, H^{(lm)} : l \geq 1, m \in \{-l, \ldots, l\} \right\} \]
is a complete orthonormal basis for \( L^2 \)-integrable vectorfields on \((S_r, \hat{\gamma})\), \( r > 0 \). The orthonormality of \( E^{(lm)}, H^{(lm)} \) defined in (2.10) follows from the orthonormality and completeness of the spherical harmonics \( Y^{(lm)} \). Indeed, by (2.10), for all index pairs \((lm), (l'm')\),
\[
\int_{S_r} E^{(lm)} \cdot E^{(l'm')} = \frac{r^2}{l(l+1)} \int_{S_r} (Y^{(lm)}, 0) \cdot (\mathcal{P}_1 \mathcal{P}_1^* Y^{(l'm')}, 0)
\]
\[
= \frac{r^2}{l(l+1)} \int_{S_r} (Y^{(lm)}, 0) \cdot (-\Delta Y^{(l'm')}, 0)
\]
\[
= \int_{S_r} Y^{(lm)} Y^{(l'm')}
\]
\[
= \delta^{ll'} \delta^{mm'}
\]
where we used Lemma 2.25 and denoted the pointwise product of two pairs of functions \((f_1, f_2) \cdot (f_3, f_4) := (f_1 f_2, f_3 f_4)\).
The same holds for \( H^{(lm)} \). Furthermore, for all index pairs \((lm), (l'm')\),
\[
\int_{S_r} E^{(lm)} \cdot H^{(l'm')} = \int_{S_r} (Y^{(lm)}, 0) \cdot (0, Y^{(l'm')})
\]
\[
= 0.
\]
This proves the orthonormality of the vectorfields \( E^{(lm)}, H^{(lm)} \).

We now show that the vectorfields \( E^{(lm)}, H^{(lm)} \) form a complete basis of vectorfields in \( L^2(S_r) \) for every \( r > 0 \). It suffices to show that for any vector \( Z \in L^2(S_r) \),
\[
\left( Z_E^{(lm)} = Z_H^{(lm)} = 0 \text{ for all } l \geq 1, m \in \{-l, \ldots, l\} \right) \Rightarrow Z = 0.
\]
By the identities of Lemma 2.33, for all \( l \geq 1, m \in \{-l, \ldots, l\} \),
\[
0 = Z_E^{(lm)} = \int_{S_r} Z \cdot E^{(lm)} = \frac{r}{\sqrt{l(l+1)}} \int_{S_r} (d\hat{\nu}^l Z) Y^{(lm)},
\]
\[
0 = Z_H^{(lm)} = \int_{S_r} Z \cdot H^{(lm)} = \frac{r}{\sqrt{l(l+1)}} \int_{S_r} (\text{curl} Z) Y^{(lm)}.
\]
By the completeness of $Y^{(lm)}$, see Lemma 2.27, this shows that
\[ \text{div } Z = \text{curl } Z = 0. \]

By the ellipticity of this Hodge system, see Proposition 2.23, it follows that $Z = 0$. This proves the completeness of the basis $E^{(lm)}, H^{(lm)}, l \geq 1, m \in \{-l, \ldots, l\}$.

Second, we show that
\[ \left\{ \psi^{(lm)}, \phi^{(lm)} : l \geq 2, m \in \{-l, \ldots, l\} \right\} \]
is a complete orthonormal basis of tracefree symmetric 2-tensors in $L^2(S_r), r > 0$. The orthonormality of the $\psi^{(lm)}, \phi^{(lm)}$ defined in (2.11) is proved analogously to the orthonormality of $E^{(lm)}, H^{(lm)}$ and left to the reader.

To prove the completeness of the $\psi^{(lm)}, \phi^{(lm)}$, we need to prove that for any tracefree symmetric 2-tensor $V \in L^2(S_r)$,
\[ \left( V^{(lm)}_{\psi} = V^{(lm)}_{\phi} = 0 \text{ for all } l \geq 2, m \in \{-l, \ldots, l\} \right) \Rightarrow V = 0. \]

This follows however by the completeness of the $E^{(lm)}, H^{(lm)}$ and Proposition 2.23, similar to the above proof for $E^{(lm)}, H^{(lm)}$. This proves the completeness of the basis $\psi^{(lm)}, \phi^{(lm)}, l \geq 2, m \in \{-l, \ldots, l\}$.

The equality of norms follows by the orthonormality and completeness properties. This finishes the proof of Proposition 2.30.

APPENDIX B. THE PROOFS OF PROPOSITION 2.34 AND LEMMA 2.35

In this section we prove Proposition 2.34 and Lemma 2.35.

Proof of Proposition 2.34. Consider the first relation. We show at first that that
\[ \| \nabla^n u \|^2_{L^2(S_r)} \lesssim \sum_{l \geq 0} \sum_{m=-l}^l \left( \frac{l(l+1)}{r^2} \right)^n (u^{(lm)})^2 \] (B.1)
by induction in $n \geq 0$. The cases $n = 0, 1$ are verified by Lemma 2.33 and Propositions 2.23 and 2.30.
For the induction step $n \to n + 1$, integrate by parts to estimate
\[
\|\nabla^{n+1} u\|_{L^2(S_r)}^2
\]
\[
= - \int_{S_r} \Delta (\nabla^n u) \cdot \nabla^n u
\]
\[
= - \int_{S_r} \nabla^n \Delta u \cdot \nabla^n u + [\Delta, \nabla^n] u \cdot \nabla^n u
\]
\[
\leq \int_{S_r} \nabla^{n-1} \Delta u \cdot \Delta \nabla^{n-1} u + \|[\Delta, \nabla^n] u\|_{L^2(S_r)} \|\nabla^n u\|_{L^2(S_r)}
\]
\[
\lesssim \|\nabla^{n-1} \Delta u\|_{L^2(S_r)} \|\nabla^{n+1} u\|_{L^2(S_r)} + \frac{1}{r^2} \sum_{l \geq 0} \sum_{m=-l}^l \left( \frac{l(l+1)}{r^2} \right) \left( u^{(lm)} \right)^2,
\]
where we used that
\[
\|[\Delta, \nabla^n] u\|_{L^2(S_r)}^2 \lesssim \frac{1}{r^2} \sum_{l \geq 0} \sum_{m=-l}^l \left( \frac{l(l+1)}{r^2} \right) \left( u^{(lm)} \right)^2,
\]
\[
\|\nabla^n u\|_{L^2(S_r)}^2 \lesssim \sum_{l \geq 0} \sum_{m=-l}^l \left( \frac{l(l+1)}{r^2} \right) \left( u^{(lm)} \right)^2,
\]
by the fact that we work on the round sphere $(S_r, \gamma)$ and the induction hypothesis.

It follows by (B.2) via Cauchy with weights that
\[
\|\nabla^{n+1} u\|_{L^2(S_r)}^2 \lesssim \|\nabla^{n-1} \Delta u\|_{L^2(S_r)}^2 + \frac{1}{r^2} \sum_{l \geq 0} \sum_{m=-l}^l \left( \frac{l(l+1)}{r^2} \right) \left( u^{(lm)} \right)^2. \tag{B.3}
\]

To estimate the first term on the right-hand side, we use the induction assumption and Lemma 2.33,
\[
\|\nabla^{n-1} \Delta u\|_{L^2(S_r)}^2 \lesssim \sum_{l \geq 0} \sum_{m=-l}^l \left( \frac{l(l+1)}{r^2} \right) \left( \left( \nabla u \right)^{(lm)} \right)^2
\]
\[
= \sum_{l \geq 0} \sum_{m=-l}^l \left( \frac{l(l+1)}{r^2} \right) \left( \frac{l(l+1)}{r^2} u^{(lm)} \right)^2
\]
\[
= \sum_{l \geq 0} \sum_{m=-l}^l \left( \frac{l(l+1)}{r^2} \right) \left( u^{(lm)} \right)^2.
\]
Plugging into (B.3) yields
\[ \| \nabla^{n+1} u \|_{L^2(S_r)}^2 \lesssim \sum_{l \geq 0} \sum_{m=-l}^{l} \left( \frac{l(l+1)}{r^2} \right)^{n+1} (u^{(lm)})^2. \]
This finishes the induction and proves (B.1). The other direction needed for the equivalence relation is proved similarly and left to the reader.

It remains to show the second equivalence relation for a vectorfield \( X \). This follows as for scalar functions by induction on \( n \geq 0 \), using this time the vectorfields \( E^{(lm)}, H^{(lm)} \) with Remark 2.28 and Lemma 2.33. This finishes the proof of Proposition 2.34. \( \square \)

**Proof of Lemma 2.35.** We prove each part separately.

**Part (1).** The estimate (2.12) follows directly by Definition 2.29 and the fact that \( Y^{(lm)} \sim r^{-1} \) due to its normalisation, see Section 2.7.

**Part (2).** On the one hand, it generally holds that in standard polar frame components, see (2.2), for \( A = 1, 2 \),
\[
\partial_r (X^A) = N (X^A) = e(\nabla_N X, e_A) - e(X, \nabla_N e_A) = (\nabla_N X)^A - e(X, \nabla_{ea} N) - e(X, [N, e_A])
\]
\[
= (\nabla_N X)^A - X^B (e_{BA}) + \frac{1}{r} X^A,
\]
where we used that in the Euclidean case, for \( A = 1, 2 \),
\[
[N, e_A] = -\frac{1}{r} e_A
\]
and, in standard polar frame components,
\[
\Theta_{11} = \Theta_{22} = -\frac{1}{r}, \Theta_{12} = \Theta_{21} = 0.
\]
Consequently,
\[
\partial_r (X_E^{(lm)}) = \int_{S_r} \partial_r (X^A) (E^{(lm)})_A + \frac{1}{r} X_E^{(lm)}
\]
\[
= (\nabla_N X_E)^{(lm)} + \frac{1}{r} X_E^{(lm)},
\]
where we used in the first equality that on \((S, o, \gamma)\), \(\sqrt{\det \gamma} \sim r^2\) and that in polar frame components, for \(A = 1, 2\), see (2.2),
\[
\left( E^{(lm)} \right)_A = -e_A \left( Y^{(lm)} \right) \frac{r}{\sqrt{l(l + 1)}} \sim \frac{1}{r}.
\]
Repeatedly applying \(\nabla_N\) then proves the first of (2.13). The second of (2.13) is proved similarly.

On the other hand, it holds generally in standard polar frame components that
\[
div X = e_1 (X^1) + e_2 (X^2)
= \frac{1}{r} \partial_{\theta^1} (X^1) + \frac{1}{r \sin \theta^1} \partial_{\theta^2} (X^2).
\]
This leads to
\[
\partial_r (r \div X) = \partial_{\theta^1} \partial_r (X^1) + \frac{1}{\sin \theta^1} \partial_{\theta^2} \partial_r (X^2)
= \partial_{\theta^1} (\nabla_N X)^1 + \frac{1}{\sin \theta^1} \partial_{\theta^2} (\nabla_N X)^2
= r \div \nabla_N X,
\]
where we used (B.4). This finishes the proof of part (2) of Lemma 2.35.

**Part (3).** The proof of part (3) is similar to part (2) and left to the interested reader. This finishes the proof of Lemma 2.35.

\[\Box\]

**Appendix C. Elliptic operators on weighted Sobolev spaces**

In this section, we first introduce the weak formulation of boundary value problems in weighted spaces. Second, we prove ellipticity and derive higher elliptic regularity estimates in \(H^w_\delta(\mathbb{R}^3 \setminus \overline{B_1}) \cap H^1_\delta\) and \(H^w_\delta\) for PDEs that were used in Sections 4.3 and 5.3. Here the \(w, \delta\) depend on the PDE system under consideration. We also derive an elliptic estimate for distributional solutions with \(L^2\)-regularity to one of the PDEs.

**C.1. Weak formulation of boundary value problems in weighted spaces.** First, we define corresponding dual spaces.

**Definition C.1 (Dual spaces of weighted Sobolev spaces).** Let \((H^w_\delta)^*\) denote the space of linear maps \(G : H^w_\delta \to \mathbb{R}\) such that there exists a constant \(c > 0\) so that
\[
|G(u)| \leq c \|u\|_{H^w_\delta} \text{ for all } u \in H^w_\delta.
\]
Let the norm \(\|G\|_{(H^w_\delta)^*}\) be defined as the smallest \(c > 0\) such that the above inequality holds.

The next lemma shows how weights behave with respect to the dual spaces.
Lemma C.2. Let $w \geq 0$, $v \in \overline{H}_\delta^0$ and $\alpha \in \mathbb{N}^3$ a multi-index such that $|\alpha| = w$. Denote by $\partial^\alpha v$ the $\alpha$-th weak derivative of $v$. Then

$$\partial^\alpha v \in (\overline{H}_{-\delta+w-3})^*.$$ 

Proof. For $u \in \overline{H}_{-\delta+w-3}$, it holds that

$$\int_{\mathbb{R}^3 \setminus B_1} u \partial^\alpha v = (-1)^{|\alpha|} \int_{\mathbb{R}^3 \setminus B_1} \partial^\alpha u v \leq \|u\|_{\overline{H}_{-\delta+w-3}} \|v\|_{\overline{H}_\delta^0}.$$ 

This concludes the proof of Lemma C.2. \qed

In Sections 4.3 and 5.3, we consider PDEs of the form

$$\begin{align*}
\left\{ \begin{array}{l}
\Delta u + \frac{a}{r} \partial_r u + \frac{b}{r^2} u = f \text{ on } \mathbb{R}^3 \setminus B_1, \\
u|_{r=1} = 0,
\end{array} \right. 
\end{align*}$$

(C.1)

where $a, b \in \mathbb{R}$ are constants and $u \in \overline{H}_\delta^1, f \in \left(\overline{H}_{-\delta-1}\right)^*$. Note that if $v \in \overline{H}_\delta^1$, then $r^{-1-2\delta} v \in \overline{H}_{-\delta-1}^1$. Therefore, to apply the standard theory of generalised solutions, see for example [16], we consider weighted weak formulations after a formal integration by parts of

$$\int_{\mathbb{R}^3 \setminus B_1} r^{-2\delta-1} \left( -\Delta u - \frac{a}{r} \partial_r u - \frac{b}{r^2} u \right) v$$

where $u, v \in \overline{H}_\delta^1$. This leads to the following definition.

Definition C.3 (Weak solutions). Let $f \in \left(\overline{H}_{-\delta-1}\right)^*$, $a, b \in \mathbb{R}$ be given. A function $u \in \overline{H}_\delta^1$ is called weak solution to

$$\begin{align*}
\left\{ \begin{array}{l}
\Delta u + \frac{a}{r} \partial_r u + \frac{b}{r^2} u = f \text{ on } \mathbb{R}^3 \setminus B_1, \\
u|_{r=1} = 0,
\end{array} \right. 
\end{align*}$$

(C.2)

if for all $v \in \overline{H}_\delta^1$ it holds that

$$B_{\delta,a,b}(u, v) = \int_{\mathbb{R}^3 \setminus B_1} r^{-1-2\delta} f v,$$
where $B_{\delta,a,b}(u,v) : \overline{H}_\delta^1 \times \overline{H}_\delta^1 \to \mathbb{R}$ is the symmetric bilinear form defined by
\[
B_{\delta,a,b}(u,v) := \int_{\mathbb{R}^3 \setminus B_1} r^{-2\delta-1} \nabla u \cdot \nabla v - \frac{a + 2\delta + 1}{2} r^{-2\delta-2} (v \partial_r u - u \partial_r v) - \int_{\mathbb{R}^3 \setminus B_1} (\delta(a + 2\delta + 1) + b) r^{-2\delta-3} uv. \tag{C.3}
\]

It is left to the reader to verify that $B_{\delta,a,b}(u,v) : \overline{H}_\delta^1 \times \overline{H}_\delta^1 \to \mathbb{R}$ is bounded for all $\delta, a, b \in \mathbb{R}$, that is,
\[
|B_{\delta,a,b}(u,v)| \lesssim \|u\|_{\overline{H}_\delta^2} \|v\|_{\overline{H}_\delta^2} \text{ for all } u, v \in \overline{H}_\delta^1.
\]

Let $w \geq 2$ be an integer. Introduce three PDEs on $\mathbb{R}^3 \setminus B_1$.

- Consider a scalar function $u^{[\geq 2]}$ on $\mathbb{R}^3 \setminus B_1$ that verifies
\[
\begin{cases}
\Delta u^{[\geq 2]} + \frac{4}{r} \partial_r u^{[\geq 2]} + \frac{6}{r^2} u^{[\geq 2]} = f^{[\geq 2]} \quad \text{on } \mathbb{R}^3 \setminus B_1, \\
u^{[\geq 2]}|_{r=1} = 0,
\end{cases}
\tag{E1}
\]

where $f^{[\geq 2]}$ is a given scalar function on $\mathbb{R}^3 \setminus B_1$.

- Consider a scalar function $u^{[\geq 2]}$ on $\mathbb{R}^3 \setminus B_1$ that verifies
\[
\begin{cases}
\Delta u^{[\geq 2]} + \frac{1}{r} \partial_r u^{[\geq 2]} - \frac{3}{r^2} u^{[\geq 2]} = f^{[\geq 2]} \quad \text{on } \mathbb{R}^3 \setminus B_1, \\
u^{[\geq 2]}|_{r=1} = 0,
\end{cases}
\tag{E2}
\]

where $f^{[\geq 2]}$ is a given scalar function on $\mathbb{R}^3 \setminus B_1$.

- Consider a scalar function $u^{[\geq 1]}$ on $\mathbb{R}^3 \setminus B_1$ that verifies
\[
\begin{cases}
\Delta u^{[\geq 1]} - \frac{1}{2} \Delta u^{[\geq 1]} + \frac{1}{r} \partial_r u^{[\geq 1]} + \frac{1}{r^2} u^{[\geq 1]} = f^{[\geq 1]} \quad \text{on } \mathbb{R}^3 \setminus B_1, \\
u^{[\geq 1]}|_{r=1} = 0,
\end{cases}
\tag{E3}
\]

where $f^{[\geq 1]}$ is a given scalar function on $\mathbb{R}^3 \setminus B_1$.

Notice that (E1) corresponds to (4.32), (E2) to (4.37) and (E3) to (5.24).

C.2. Elliptic estimates in $\overline{H}_\delta^1$. The next proposition shows existence and first elliptic estimates for the above PDEs in weighted Sobolev spaces.

**Proposition C.4.** The following holds.

- Let $f^{[\geq 2]} \in (\overline{H}_1^1)^{\ast \ast}$. There exists a unique weak solution $u^{[\geq 2]} \subset \overline{H}_{-3/2}^1$ to (E1) bounded by
\[
\|u^{[\geq 2]}\|_{\overline{H}_{-3/2}^1} \lesssim \|f^{[\geq 2]}\|_{(\overline{H}_1^1)^{\ast}}. \tag{C.4}
\]
Let $f^{[\geq 2]} \in (\mathcal{H}^1_{3/2})^\ast$. There exists a unique weak solution $u^{[\geq 2]} \subset \mathcal{H}^1_{-5/2}$ to $(E2)$ bounded by
\[
\|u^{[\geq 2]}\|_{\mathcal{H}^1_{-5/2}} \lesssim \|f^{[\geq 2]}\|_{(\mathcal{H}^1_{3/2})^\ast} \quad (C.5)
\]

Let $f^{[\geq 1]} \in (\mathcal{H}^1_{-1/2})^\ast$. There exists a unique weak solution $u^{[\geq 1]} \in \mathcal{H}^1_{-1/2}$ to $(E3)$ bounded by
\[
\|u^{[\geq 1]}\|_{\mathcal{H}^1_{-1/2}} \lesssim \|f^{[\geq 1]}\|_{(\mathcal{H}^1_{-1/2})^\ast} \quad (C.6)
\]

To prove the above proposition, we use the following Poincaré inequality.

**Lemma C.5.** Let $n \geq 1$ be an integer. Let the scalar function $u^{[\geq n]} \in C^\infty(\mathbb{R}^3)$. For any $r > 0$ it holds that
\[
\int_{S_r} \left(\frac{u^{[\geq n]}}{r^2}\right)^2 \leq \frac{1}{n(n+1)} \int_{S_r} |\nabla u^{[\geq n]}|^2.
\] (C.7)

**Proof.** Indeed, write
\[
\int_{S_r} \left(\frac{u^{[\geq n]}}{r^2}\right)^2 = \sum_{l \geq n} \sum_{m = -l}^l \left(\frac{u^{(lm)}}{r^2}\right)^2,
\]
\[
= \sum_{l \geq n} \sum_{m = -l}^l \frac{1}{l(l+1)} \frac{l(l+1)}{r^2} \left(\frac{u^{(lm)}}{r^2}\right)^2,
\]
\[
\leq \sum_{l \geq n} \sum_{m = -l}^l \frac{1}{n(n+1)} \frac{l(l+1)}{r^2} \left(\frac{u^{(lm)}}{r^2}\right)^2,
\]
\[
= \frac{1}{n(n+1)} \int_{S_r} |\nabla u^{[\geq n]}|^2,
\]
where we used Lemma 2.33. This proves Lemma C.5. \qed

**Proof of Proposition C.4.** We show for each PDE $(E1)$-$(E3)$ that the corresponding bilinear form $B_{\delta,a,b}$ defined in (C.3) is coercive on the respective weighted space. The $a, b$ corresponding to the PDEs are specified by comparing to (C.1). By the Lax-Milgram Theorem, see for example [16], existence, uniqueness and the claimed estimates follow.
Estimate (C.4). For (E1), $a = 4, b = 6$. We derive the coercivity of $B_{-3/2,4,6}$ by Lemma C.5 with $n = 2$ as follows,

$$B_{-3/2,4,6}(u^{[2]}, u^{[2]}) = \int_{\mathbb{R}^3 \setminus B_1} r^2 |\nabla u^{[2]}|^2 - 3 (u^{[2]})^2$$

$$\geq \int_{\mathbb{R}^3 \setminus B_1} r^2 |\nabla u^{[2]}|^2 - \frac{1}{2} r^2 |\nabla u^{[2]}|^2,$$

(C.8)

where we used Lemma C.5 in the second and the last line. This proves (C.4).

Estimate (C.5). For (E2), $a = 1, b = -3$. We estimate from below with Lemma C.5

$$B_{-5/2,1,-3}(u^{[2]}, u^{[2]}) = \int_{\mathbb{R}^3 \setminus B_1} r^4 |\nabla u^{[2]}|^2 - \frac{9}{2} r^2 (u^{[2]})^2$$

$$\geq \int_{\mathbb{R}^3 \setminus B_1} r^4 |\nabla u^{[2]}|^2 - \frac{9}{12} r^4 |\nabla u^{[2]}|^2$$

(C.9)

$$\geq \|u^{[2]}\|_{H_{1/2}^{-1}}^2.$$

This proves (C.5).

Estimate (C.6). The symmetric bilinear form $\tilde{B}$ associated to the weighted weak formulation of (E3) in $H_{1/2}^{-1}$ is in fact given by

$$\tilde{B}(u, v) := \int_{\mathbb{R}^3 \setminus B_1} \nabla u \nabla v - \frac{1}{2} \nabla u \nabla v - \frac{1}{2} (v \partial_r u - u \partial_r v) - \frac{1}{2r^2} uv.$$

Estimate this from below by Lemma C.5

$$\tilde{B}(u^{[1]}, u^{[1]}) = \int_{\mathbb{R}^3 \setminus B_1} |\nabla u^{[1]}|^2 - \frac{1}{2} |\nabla u^{[1]}|^2 - \frac{1}{2r^2} (u^{[1]})^2$$

$$\geq \int_{\mathbb{R}^3 \setminus B_1} |\nabla u^{[1]}|^2 - \frac{1}{2} |\nabla u^{[1]}|^2 - \frac{1}{4} |\nabla u^{[1]}|^2$$

(C.10)

$$\geq \|u^{[1]}\|_{H_{1/2}^{-1}}^2.$$
This proves (C.6) and hence finishes the proof of Proposition C.4.

C.3. Higher elliptic regularity in $H^w_\delta(\mathbb{R}^3 \setminus \overline{B_1}) \cap \overline{H^1_\delta}$ and $\overline{H^w_\delta}$. In this section, we prove higher elliptic regularity estimates in $H^w_\delta(\mathbb{R}^3 \setminus \overline{B_1}) \cap \overline{H^1_\delta}$, $w \geq 2$ and $\overline{H^w_\delta}$, for the boundary value problems (E1)-(E3), on the domain $\mathbb{R}^3 \setminus \overline{B_1}$.

Proposition C.6 (Higher regularity in $H^w_\delta(\mathbb{R}^3 \setminus \overline{B_1}) \cap \overline{H^1_\delta}$). Let $w \geq 2$ be an integer. The following holds.

- Let $f^{[\geq 2]} \in H^{w-2}_{-7/2}$. Then the solution $u^{[\geq 2]}$ to (E1) satisfies
  
  \[ u^{[\geq 2]} \in H^{w}_{-3/2}(\mathbb{R}^3 \setminus \overline{B_1}) \cap \overline{H^1_{-3/2}} \]

  and

  \[ \|u^{[\geq 2]}\|_{H^w_{-3/2}(\mathbb{R}^3 \setminus \overline{B_1})} \lesssim \|f^{[\geq 2]}\|_{H^{w-2}_{-7/2}}. \tag{C.11} \]

- Let $f^{[\geq 2]} \in H^{w-2}_{-9/2}$. Then the solution $u^{[\geq 2]}$ to (E2) satisfies
  
  \[ u^{[\geq 2]} \in H^{w}_{-5/2}(\mathbb{R}^3 \setminus \overline{B_1}) \cap \overline{H^1_{-5/2}} \]

  and

  \[ \|u^{[\geq 2]}\|_{H^w_{-5/2}(\mathbb{R}^3 \setminus \overline{B_1})} \lesssim \|f^{[\geq 2]}\|_{H^{w-2}_{-9/2}}. \tag{C.12} \]

- Let $f^{[\geq 1]} \in H^{w-2}_{-5/2}$. Then the solution $u^{[\geq 1]}$ to (E3) satisfies
  
  \[ u^{[\geq 1]} \in H^{w}_{-1/2}(\mathbb{R}^3 \setminus \overline{B_1}) \cap \overline{H^1_{-1/2}} \]

  and

  \[ \|u^{[\geq 1]}\|_{H^w_{-1/2}(\mathbb{R}^3 \setminus \overline{B_1})} \lesssim \|f^{[\geq 1]}\|_{H^{w-2}_{-5/2}}. \tag{C.13} \]

The idea of the proof of Proposition C.6 is to reduce to the following well-known weighted elliptic estimates of [8].

Proposition C.7. Let $w \geq 2$ be an integer and $\delta \in \mathbb{R}$. Let on $\mathbb{R}^3$

\[ Lu := a^{ij} \partial_i \partial_j u + b^i \partial_i u + du \]

with coefficients $a^{ij} - A^{ij} \in H^{w_2}_{\delta_2}, b^i \in H^{w_1}_{\delta_1}, d \in H^{w_0}_{\delta_0}$ with constants

\[ w_k \geq k + 1, w_k \geq w - 2, \delta_k < k - 2 \quad \text{for} \ k = 0, 1, 2, \]

and constants $A^{ij}$ such that there exists $\lambda > 0$ such that

\[ A^{ij} \xi_i \xi_j \geq \lambda |\xi|^2 \quad \text{for all} \ \xi \in \mathbb{R}^3. \]

Then there exists a constant $c > 0$ such that for every $u \in H^w_{\text{loc}} \cap \overline{H^0_\delta}$ the following inequality holds

\[ \|u\|_{H^w_\delta} \leq c \left( \|Lu\|_{H^{w-2}_\delta} + \|u\|_{H^0_\delta} \right). \]
The proof of the next lemma is left to the reader.

**Lemma C.8.** The operators of (E1)-(E3) satisfy the assumptions of Proposition C.7.

**Proof of Proposition C.6.** First, we derive (C.11). Recall (E1), that is,
\[
\begin{aligned}
Lu^{\geq 2} &:= \Delta u^{\geq 2} + \frac{4}{r} \partial_r u^{\geq 2} + \frac{6}{r^2} u^{\geq 2} = f^{\geq 2} \quad \text{on } \mathbb{R}^3 \setminus \overline{B_1}, \\
u^{\geq 2}|_{r=1} &= 0.
\end{aligned}
\]

The above differential operator \( L \) is pointwise elliptic on \( \mathbb{R}^3 \setminus \overline{B_1} \) and has smooth coefficients. The Dirichlet boundary data is trivial and by assumption \( f^{\geq 2} \in H_{-5/2}^{w-2} \). Therefore standard elliptic boundary estimates, see for example Theorem 8.13 in [16], imply that
\[
\|u^{\geq 2}\|_{H^w(B_2 \setminus \overline{B_1})} \lesssim \|f^{\geq 2}\|_{H_{-5/2}^{w-2}(B_3 \setminus \overline{B_1})}. \tag{C.14}
\]

Use standard Sobolev extension, see for example Theorem 7.25 in [16], to extend the scalar function \( u^{\geq 2} \in H^w(B_2 \setminus \overline{B_1}) \) to \( \tilde{u}^{\geq 2} \) on \( B_2 \) such that
\[
\|\tilde{u}^{\geq 2}\|_{H^w(B_2)} \lesssim \|u^{\geq 2}\|_{H^w(B_2 \setminus \overline{B_1})}.
\]

Let \( \tilde{\chi} : \mathbb{R}^3 \to [0, 1] \) be a smooth function such that
\[
\tilde{\chi} = \begin{cases} 
0 & \text{for } |x| \leq 1/10, \\
1 & \text{for } |x| \geq 1/2 \end{cases} \tag{C.15}
\]

and define the operator \( \tilde{L} \) by
\[
\tilde{L}\varphi := \Delta \varphi + \frac{4\tilde{\chi}}{r} \partial_r \varphi + \frac{6\tilde{\chi}}{r^2} \varphi \quad \text{on } \mathbb{R}^3
\]
for all \( \varphi \in C^\infty(\mathbb{R}^3) \). Finally, let
\[
\tilde{f}^{\geq 2} = \tilde{L}\tilde{u}^{\geq 2}.
\]

It holds that \( \tilde{f}^{\geq 2} \in H_{-7/2}^{w-2} \), so that standard interior elliptic estimates, see for example Theorem 8.10 in [16], show that \( \tilde{u}^{\geq 2} \in H^w_{loc}(\mathbb{R}^3) \).

In summary, it holds that \( \tilde{u}^{\geq 2} \in \overline{H^1} \cap H^w_{loc}(\mathbb{R}^3) \) satisfies
\[
\tilde{L}\tilde{u}^{\geq 2} = \tilde{f}^{\geq 2} \quad \text{on } \mathbb{R}^3,
\]

where \( \tilde{L} \) is a pointwise elliptic operator with smooth coefficients that satisfies the assumptions of Proposition C.7, see also Lemma C.8. Consequently, we can apply Proposition
C.7 to get the weighted estimate
\[
\|u^{[2]}\|_{H^{-3/2}_w(\mathbb{R}^3 \setminus \bar{B}_1)} \leq \|u^{[2]}\|_{H^{-3/2}_w(B_1)}
\]
\[
\lesssim \|\tilde{f}^{[2]}\|_{H^{-7/2}_w(B_1)}
\]
\[
\lesssim \|\tilde{u}^{[2]}\|_{H^{-w}(B_2)} + \|\tilde{f}^{[2]}\|_{H^{-7/2}_w(B_1)}
\]
\[
\lesssim \|u^{[2]}\|_{H^{-w}(B_2 \setminus \bar{B}_1)} + \|f^{[2]}\|_{H^{-w}_w(B_1)}
\]
where we used (C.14), the fact that \(\tilde{f}^{[2]} = f^{[2]}\) on \(\mathbb{R}^3 \setminus \bar{B}_1\) and Proposition C.7. This proves (C.11). The estimates (C.12) and (C.13) are proved similarly by using Lemma C.8 and are left to the reader. This finishes the proof of Proposition C.6.

Furthermore, we have the following result in \(H^{-w}_{\delta}\).

**Proposition C.9** (Higher regularity for (E1), (E2),(E3)). Let \(w \geq 2\) be an integer. The following holds.

- Let \(f^{[2]} \in H^{-w-2}_{-7/2}\). If the solution \(u^{[2]} \in H^{-3/2}_1 \cap H^2_{-3/2}(\mathbb{R}^3 \setminus \bar{B}_1)\) to (E1) satisfies
  \[
  \partial_r u^{[2]} |_{r=1} = 0,
  \]
  then it holds that \(u^{[2]} \in H^{-w}_{-3/2}\).

- Let \(f^{[2]} \in H^{-w-2}_{-9/2}\). If the solution \(u^{[2]} \in H^{-5/2}_1 \cap H^2_{-5/2}(\mathbb{R}^3 \setminus \bar{B}_1)\) to (E2) satisfies
  \[
  \partial_r u^{[2]} |_{r=1} = 0,
  \]
  then it holds that \(u^{[2]} \in H^{-w}_{-5/2}\).

- Let \(w \geq 2\) be an integer. Let \(f^{[1]} \in H^{-w-2}_{-5/2}\). If the solution \(u^{[1]} \in H^{-1/2}_1 \cap H^2_{-1/2}(\mathbb{R}^3 \setminus \bar{B}_1)\) to (E3) satisfies
  \[
  \partial_r u^{[1]} |_{r=1} = 0,
  \]
  then it holds that \(u^{[1]} \in H^{-w}_{-1/2}\).

**Proof.** Proposition C.9 follows by Propositions C.6 and 2.13. Indeed, all necessary normal derivatives on \(r = 1\) can be expressed via the equations and shown to vanish.

C.4. **An elliptic estimate in \(L^2\).** In Section 4.3, we considered the following Dirichlet problem on \(\mathbb{R}^3 \setminus \bar{B}_1\) for a scalar function \(u^{[2]} \in H^{-w-2}_{-5/2}(\mathbb{R}^3 \setminus \bar{B}_1)\),
\[
\begin{aligned}
\left\{ \begin{array}{l}
\Delta u^{[2]} + \frac{1}{r} \partial_r u^{[2]} - \frac{2}{r} u^{[2]} = \partial_r \left( \text{curl} \left( f^{[2]}_{H} \right) \right), \\
u^{[2]} |_{r=1} = 0,
\end{array} \right.
\end{aligned}
\]
where \( f^{[\geq 2]}_H \in \mathcal{H}^{w-2}_{-5/2}(\mathbb{R}^3 \setminus \overline{B}_1) \), \( w \geq 2 \) was a given vectorfield. In the previous sections C.2 and C.3 where this PDE was denoted \((E2)\), we derived elliptic estimates in case \( w \geq 3 \). In this section we derive estimates for the case \( w = 2 \).

First, we derive a distributional formulation of (C.16). Let

- \( f^{[\geq 2]}_H \in C^\infty(\mathbb{R}^3) \),
- \( u \in C^\infty(\mathbb{R}^3) \) be a solution to (C.16),
- \( \phi \in C^\infty_c(\mathbb{R}^3) \) such that \( \phi|_{r=1} = 0 \).

Then, by integrating by parts twice,

\[
0 = \int_{\mathbb{R}^3 \setminus \overline{B}_1} r^4 \left( \Delta u^{[\geq 2]} + \frac{1}{r} \partial_r u^{[\geq 2]} - \frac{3}{r^2} u^{[\geq 2]} - \partial_r \left( \text{curl} \left( f^{[\geq 2]}_H \right) \right) \right) \phi
\]

\[
= \int_{\mathbb{R}^3 \setminus \overline{B}_1} r^4 u^{[\geq 2]} \left( \Delta \phi^{[\geq 2]} + \frac{7}{r} \partial_r \phi^{[\geq 2]} + \frac{12}{r^2} \phi^{[\geq 2]} \right)
\]

\[
- \int_{\mathbb{R}^3 \setminus \overline{B}_1} r^4 f^{[\geq 2]}_H \cdot \left( \frac{6}{r} \ast (\nabla \phi^{[\geq 2]}) + \ast (\nabla (\partial_r \phi^{[\geq 2]})) \right),
\]

where here \( \ast (\nabla \phi)_A := \varepsilon_{AB} (\nabla \phi)^B \) denotes the Hodge dual of \( \nabla \phi \). Here the boundary terms

\[
\int_{S_1} \partial_n u \phi, \int_{S_1} u \partial_n \phi, \int_{S_1} u \phi, \int_{S_1} \text{curl} f^{[\geq 2]}_H \phi
\]

vanished by the assumptions. The right-hand side still makes sense for

\[
f^{[\geq 2]}_H \in \mathcal{H}^0_{-5/2}(\mathbb{R}^3 \setminus \overline{B}_1),
\]

\[
u^{[\geq 2]} \in \mathcal{H}^0_{-5/2}(\mathbb{R}^3 \setminus \overline{B}_1),
\]

\[
\phi \in \mathcal{H}^2_{-5/2}(\mathbb{R}^3 \setminus \overline{B}_1) \cap \mathcal{H}^1_{-5/2}.
\]

Note the dense inclusion

\[
\{ \phi \in C^\infty_c(\mathbb{R}^3) : \phi|_{r=1} = 0 \} \subset \mathcal{H}^2_{-5/2}(\mathbb{R}^3 \setminus \overline{B}_1) \cap \mathcal{H}^1_{-5/2} \cap C^\infty(\mathbb{R}^3 \setminus \overline{B}_1),
\]

which is proved by using cut-off functions and left to the reader. This leads to the following definition.

**Definition C.10.** Let \( f^{[\geq 2]}_H \in \mathcal{H}^{w-2}_{-5/2}(\mathbb{R}^3 \setminus \overline{B}_1) \) be a vectorfield. A function \( u^{[\geq 2]} \in \mathcal{H}^0_{-5/2}(\mathbb{R}^3 \setminus \overline{B}_1) \) is called a distributional solution to

\[
\begin{align*}
\Delta u^{[\geq 2]} + \frac{1}{r} \partial_r u^{[\geq 2]} - \frac{3}{r^2} u^{[\geq 2]} &= \partial_r \left( \text{curl} \left( f^{[\geq 2]}_H \right) \right), \\
u^{[\geq 2]}|_{r=1} &= 0,
\end{align*}
\]


if for all $\phi[\geq 2] \in H^{2}_{-5/2}(\mathbb{R}^3 \setminus B_1) \cap H^1_{-5/2} \cap C^\infty(\mathbb{R}^3 \setminus B_1)$,

$$\int_{\mathbb{R}^3 \setminus B_1} r^4 u[\geq 2] \left( -\Delta \phi[\geq 2] - \frac{7}{r} \partial r \phi[\geq 2] - \frac{12}{r^2} \phi[\geq 2] \right)$$

$$= - \int_{\mathbb{R}^3 \setminus B_1} r^4 f_H^[\geq 2] \cdot \left( \frac{6}{r^4} \left( \nabla \phi[\geq 2] \right) + \left( \nabla (\partial r \phi[\geq 2]) \right) \right). \quad (C.17)$$

Note that this distributional solution is unique in $H^0_{-5/2}(\mathbb{R}^3 \setminus B_1)$ in view of Lemma C.11 below.

The next lemma is the main result of this section.

**Lemma C.11.** Let $f_H^[\geq 2] \in H^0_{-5/2}(\mathbb{R}^3 \setminus B_1)$. Let $u[\geq 2] \in H^0_{-5/2}(\mathbb{R}^3 \setminus B_1)$ be a distributional solution to (C.16). Then it holds that

$$\| u[\geq 2] \|_{H^0_{-5/2}} \lesssim \| f_H^[\geq 2] \|_{H^0_{-5/2}}.$$

**Proof of Lemma C.11.** To prove that $u[\geq 2] \in H^0_{-5/2} = \mathcal{H}^0_{-5/2} = (\mathcal{H}^0_{-1/2})^*$ with

$$\| u[\geq 2] \|_{H^0_{-5/2}} \lesssim \| f_H^[\geq 2] \|_{H^0_{-5/2}},$$

it suffices to show that for all $\varphi[\geq 2] \in C^\infty_c(\mathbb{R}^3 \setminus B_1)$,

$$\int_{\mathbb{R}^3 \setminus B_1} u[\geq 2] \varphi[\geq 2] \lesssim \| f_H^[\geq 2] \|_{H^0_{-5/2}} \| \varphi[\geq 2] \|_{H^0_{-1/2}}.$$

In the following, we will prove that for all $\varphi[\geq 2] \in C^\infty_c(\mathbb{R}^3 \setminus B_1)$,

$$\int_{\mathbb{R}^3 \setminus B_1} r^4 u[\geq 2] \varphi[\geq 2] \lesssim \| f_H^[\geq 2] \|_{H^0_{-5/2}} \| \varphi[\geq 2] \|_{H^0_{-9/2}}, \quad (C.18)$$

which implies the above estimate by the fact that

$$\| r^{-4} f_H^[\geq 2] \|_{H^0_{-9/2}(\mathbb{R}^3 \setminus B_1)} \simeq \| f_H^[\geq 2] \|_{H^0_{-1/2}(\mathbb{R}^3 \setminus B_1)}$$

It remains to prove (C.18). For given $\varphi[\geq 2] \in C^\infty_c(\mathbb{R}^3 \setminus B_1)$, let $\Psi[\geq 2]$ be defined as solution to

$$\begin{cases}
-\Delta \Psi[\geq 2] - \frac{7}{r} \partial r \Psi[\geq 2] - \frac{12}{r^2} \Psi[\geq 2] = \varphi[\geq 2] & \text{on } \mathbb{R}^3 \setminus B_1, \\
\Psi[\geq 2]|_{r=1} = 0. \quad (C.19)
\end{cases}$$
The operator in (C.19) is the adjoint to (E2) with respect to weighted scalar product
\[(u, v) \mapsto \int_{\mathbb{R}^3 \setminus B_1} r^4 uv.\]

Therefore (C.19) has the same weak formulation, ellipticity and higher regularity estimates for its generalised solutions as (E2). By Proposition C.6 and standard local interior and boundary elliptic regularity,
\[\Psi \in H^2_{-5/2}(\mathbb{R}^3 \setminus B_1) \cap H^1_{-5/2} \cap C^\infty(\mathbb{R}^3 \setminus B_1)\]
with
\[\|\Psi^{[2]}\|_{H^2_{-5/2}(\mathbb{R}^3 \setminus B_1)} \lesssim \|\varphi^{[2]}\|_{H^0_{-9/2}}. \quad (C.20)\]

Plugging now $\Psi$ into (C.17), using (C.20) and (C.19), we get
\[
\begin{align*}
\int_{\mathbb{R}^3 \setminus B_1} r^4 u^{[2]} \varphi^{[2]} &= \int_{\mathbb{R}^3 \setminus B_1} r^4 u^{[2]} \left( -\Delta \Psi^{[2]} - \frac{7}{r} \partial_r \Psi^{[2]} - \frac{12}{r^2} \Psi^{[2]} \right) \\
&= - \int_{\mathbb{R}^3 \setminus B_1} r^4 f^{[2]}_H \cdot \left( \frac{6}{r^2} (\nabla \Psi^{[2]}) + \ast (\nabla (\partial_r \Psi^{[2]}) ) \right) \\
&\lesssim \|f^{[2]}_H\|_{H^0_{-5/2}(\mathbb{R}^3 \setminus B_1)} \|\Psi^{[2]}\|_{H^2_{-5/2}(\mathbb{R}^3 \setminus B_1)} \\
&\lesssim \|f^{[2]}_H\|_{H^0_{-5/2}(\mathbb{R}^3 \setminus B_1)} \|\varphi^{[2]}\|_{H^0_{-9/2}(\mathbb{R}^3 \setminus B_1)}.
\end{align*}
\]

This proves (C.18) and finishes the proof of Lemma C.11. \hfill \square

**Remark C.12.** The existence of a solution $u^{[2]} \in H^0_{-5/2}$ to (E2) can be deduced from Lemma C.11 by a limit argument. Indeed, it suffices to take $\left( f^{[2]}_H \right)_n \in C^\infty(\mathbb{R}^3 \setminus B_1)$, $n \in \mathbb{N}$ a sequence such that $\left( f^{[2]}_H \right)_n \rightarrow f^{[2]}_H$ in $H^0_{-5/2}$ as $n \rightarrow \infty$. The corresponding solutions $\left( u^{[2]} \right)_n \in \mathcal{H}^{-5/2}_{-5/2}$ whose existence is assured by Proposition C.4 will by Lemma C.11 converge to the distributional solution $u^{[2]}$ in $H^0_{-5/2}$.

**C.5. Estimates to apply Lemma C.2.** To apply the above elliptic theory in Section 4.3, the following corollary is used. It follows by the operator analysis in Section 4 and its proof is left to the reader.
Corollary C.13. The right-hand sides of the PDEs (4.32) and (4.37) can be estimated as follows. We have
\[
\left\| \frac{1}{r^3} \partial_r \left( r^3 (\rho_N)_{[\geq 2]} \right) - d_{\mathbb{V}} \left( \beta^2_E + \zeta_E \right) \right\|_{(\mathcal{P}_{1/2}^1)} \lesssim \|\rho\|_{\mathcal{P}_{-3/2}} + \|\zeta_E\|_{\mathcal{P}_{-5/2}},
\]
\[
\left\| \beta^2_H + \zeta_H \right\|_{\mathcal{H}^0_{-5/2}(\mathbb{R}^3 \setminus B)} \lesssim \|\rho\|_{\mathcal{H}^0_{-5/2}(\mathbb{R}^3 \setminus B)} + \|\zeta_H\|_{\mathcal{H}^0_{-5/2}(\mathbb{R}^3 \setminus B)}
\]
\[
\left\| \partial_r \text{curl} \left( \beta^2_H + \zeta_H \right) \right\|_{(\mathcal{P}_{3/2}^1(\mathbb{R}^3 \setminus B))} \lesssim \|\rho\|_{\mathcal{P}_{-5/2}^1(\mathbb{R}^3 \setminus B)} + \|\zeta_H\|_{\mathcal{P}_{-5/2}^1(\mathbb{R}^3 \setminus B)}.
\]
Also for \( w \geq 3 \),
\[
\left\| \frac{1}{r^3} \partial_r \left( r^3 (\rho_N)_{[\geq 2]} \right) - d_{\mathbb{V}} \left( \beta^2_E + \zeta_E \right) \right\|_{\mathcal{P}_{-3/2}^w} \lesssim \|\rho\|_{\mathcal{P}_{-5/2}^w} + \|\zeta_E\|_{\mathcal{P}_{-5/2}^w},
\]
and for \( w \geq 4 \),
\[
\left\| \partial_r \left( \text{curl} \left( \beta^2_H + \zeta_H \right) \right) \right\|_{\mathcal{P}_{-9/2}^w} \lesssim \|\rho\|_{\mathcal{P}_{-5/2}^w} + \|\zeta_H\|_{\mathcal{P}_{-5/2}^w}.
\]

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