ON A TYPE OF COMMUTATIVE ALGEBRAS

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ABSTRACT. We introduce some basic concepts for Jacobi-Jordan algebras such as: representations, crossed products or Frobenius/metabelian/co-flag objects. A new family of solutions for the quantum Yang-Baxter equation is constructed arising from any 3-step nilpotent Jacobi-Jordan algebra. Crossed products are used to construct the classifying object for the extension problem in its global form. For a given Jacobi-Jordan algebra $A$ and a given vector space $V$ of dimension $c$, a global non-abelian cohomological object $\mathcal{GH}^2(A, V)$ is constructed: it classifies, from the viewpoint of the extension problem, all Jacobi-Jordan algebras that have a surjective algebra map on $A$ with kernel of dimension $c$. The object $\mathcal{GH}^2(A, k)$ responsible for the classification of co-flag algebras is computed, all $1 + \dim(A)$ dimensional Jacobi-Jordan algebras that have an algebra surjective map on $A$ are classified and the automorphism groups of these algebras is determined. Several examples involving special sets of matrices and symmetric bilinear forms as well as equivalence relations between them (generalizing the isometry relation) are provided.

Introduction

Jacobi-Jordan algebras (JJ algebras for short) were recently introduced in [5] as vector spaces $A$ over a field $k$, equipped with a bilinear map $\cdot : A \times A \to A$ satisfying the Jacobi identity and instead of the skew-symmetry condition valid for Lie algebras we impose commutativity $x \cdot y = y \cdot x$, for all $x, y \in A$. These type of algebras appeared already in relation with Bernstein algebras in 1987 ([24]). One crucial remark is that JJ algebras are examples of the more popular and well-referenced Jordan algebras [13] introduced in order to achieve an axiomatization for the algebra of observables in quantum mechanics. In [5] the authors achieved the classification of these algebras up to dimension 6 over an algebraically closed field of characteristic different from 2 and 3. Our purpose is to introduce and develop some basic concepts for JJ algebras which might eventually lead to an interesting theory. As it was explained in [5] and as it will be obvious from this paper as well, JJ algebras are objects fundamentally different from both associative and Lie algebras even if their definition differs from the latter only modulo a sign. We aim to prove that there exists a rich and very interesting theory behind the Jacobi-Jordan algebras which deserves to be developed further mainly for three reasons. The first reason

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is a theoretical one: JJ algebras are objects of study in their own right as objects living at the interplay between the intensively studied Lie algebras and respectively associative algebras. In this context it is a challenge to introduce the JJ algebra counterparts of concepts already defined in the theory of Lie (resp. associative) algebras and to see which of the results valid in the fields mentioned above are also true for JJ algebras. The second reason comes from the observation that JJ algebras are a special class of Jordan algebras which turned out to play a fundamental role not only in quantum mechanics but also in differential geometry, algebraic geometry or functional analysis [18] – from this perspective they deserve a detailed study. Finally, the third reason is given by the observation which we will highlight in Section 1 between JJ algebras and the celebrated quantum Yang-Baxter equation from theoretical physics. A central open problem in this context is to construct new families of solutions: here we take a first step towards it by associating to any 3-step nilpotent JJ algebra a new family of solutions for the quantum Yang-Baxter equation. Therefore, constructing such algebras becomes a matter of interest.

The paper is organized as follows. The first section fixes notations and conventions used throughout and introduces some basic concepts in the context of JJ algebras such as modules, representations or Frobenius objects. In order to find the right axiom for defining modules over a JJ algebra (Definition 1.4) we use the classical trick used for objects $A$ in a $k$-linear category $C$: that is, a vector space $V$ with a bilinear map $\triangleright : A \times V \to V$ such that $A \times V$ with the 'semi-direct' product type multiplication has to be an object inside the category $C$. Representations of a JJ-algebra $A$ (Definition 1.6) are introduced exactly as the representations of a Jordan algebra, if we see JJ-algebras as a special case of them – this definition fits with the way modules were defined, i.e. there exists an isomorphism of categories between modules and representations over a JJ algebra. Having introduced these concepts, Frobenius JJ algebras (Definition 1.7) arise naturally as symmetric objects in the category: that is, finite dimensional JJ algebras $A$ such that $A \cong A^*$, in the category of modules over $A$. Thus, the definition is similar to the one adopted in the case of the intensively studied associative Frobenius algebras [10, 14, 16] and Lie Frobenius algebras (also known in the literature as self-dual, metric or Lie algebras with an invariant non-degenerate bilinear form [15, 19, 9, 22]). In Section 2 we deal with the main question addressed in this paper, namely the global extension (GE) problem, and the crossed product for JJ algebras is introduced as the object responsible for it. Introduced for Leibniz algebras in [20] and studied for associative/Poisson algebras in [1, 2], the GE problem is a generalization of the classical Hölder’s extension problem, and for JJ algebras consists of the following question:

Let $A$ be a JJ algebra, $E$ a vector space and $\pi : E \to A$ a linear epimorphism of vector spaces. Describe and classify the set of all JJ algebra structures that can be defined on $E$ such that $\pi : E \to A$ becomes a morphism of JJ algebras.

Proposition 2.4 proves that any such JJ algebra structure $\cdot_E$ on $E$ is isomorphic to a crossed product $A\# V = A\#(\triangleright, \vartheta, \cdot_V)V$, which is a JJ algebra associated to $A$ and $V := \ker(\pi)$ connected by an action $\triangleright : A \times V \to V$, a symmetric cocycle $\vartheta : A \times A \to V$ and a JJ algebra structure $\cdot_V$ on $V$ satisfying some natural axioms as stated in Proposition 2.2. The main result of the section, that gives the theoretical answer to
the GE-problem, is proven in Theorem 2.8: the classifying object for the GE problem (i.e. the set parameterizes all JJ algebra structures on $E$ up to an isomorphism which stabilizes $V$ and co-stabilizes $A$) is parameterized by an explicitly constructed global non-abelian cohomological type object denoted by $\mathbb{G}H^2(A, V)$. Corollary 2.10 proves that $\mathbb{G}H^2(A, V)$ is the coproduct of all non-abelian cohomologies $H^2(A, (V, \cdot_V))$, the latter being the Jacobi-Jordan classifying object for the classical extension problem. In the 'abelian' case (corresponding to the trivial multiplication $\cdot_V := 0$ on $V$) representations of $A$ are used in order to give the decomposition of $H^2(A, (V, \cdot_V := 0))$ as a coproduct over all $A$-module structures on $V$ (Corollary 2.11): this is the JJ algebra counterpart of the Hochschild theorem for associative Lie algebras [11, Theorem 6.2] respectively of Chevalley and Eilenberg's results for Lie algebras [8, Theorem 26.2]. Finally, in Section 3 our theoretical results from the previous section are applied to two important classes of JJ algebras namely co-flag algebras (Definition 3.1) and metabelian algebras. One of our motivations for paying special attention to metabelian algebras is that the concept turned out to play a central role in many other fields; for instance, metabelian groups play a key role in the study of periodic groups or the Brauer groups as well as in the isomorphism problem. Furthermore, all JJ algebras of small dimension classified in [5] are metabelian. As a concrete example it is shown in Example 3.7 that $\mathbb{G}H^2(\mathfrak{h}(2n + 1, k), k) \cong \text{Sym}(n, k) \times \text{Sym}(n, k) \times \text{Sym}(n, k)/kI_n$, where $\mathfrak{h}(2n + 1, k)$ is the Heisenberg algebra and $\text{Sym}(n, k)$ is the space of all $n \times n$ symmetric matrices. Proposition 3.13 and respectively Proposition 3.16 describe and classify all metabelian JJ algebras having the derived algebra of dimension 1 respectiv of co-dimension 1. Moreover, the explicit description of all $(n+1)$-dimensional metabelian JJ algebras having the derived algebra of dimension 1 or codimension 1 is provided.

1. Jacobi-Jordan algebras: basic concepts

Notations and terminology. For a family of sets $(X_i)_{i \in I}$ we shall denote by $\sqcup_{i \in I} X_i$ their coproduct in the category of sets, i.e. $\sqcup_{i \in I} X_i$ is the disjoint union of the $X_i$'s. Unless otherwise specified, all algebraic entities (vector spaces, linear or bilinear maps etc.) are over an arbitrary field $k$. A linear map $f : W \to V$ between two vector spaces is called trivial if $f(x) = 0$, for all $x \in W$. A bilinear map $\vartheta : W \times W \to V$ is symmetric if $\vartheta(x, y) = \vartheta(y, x)$, for all $x, y \in W$. Throughout this paper, by an algebra we mean a pair $A = (A, \cdot)$ consisting of a vector space and a bilinear map $\cdot : A \times A \to A$ called the multiplication of $A$. If in addition the multiplication $\cdot$ on $A$ satisfies some extra-axioms like commutativity, associativity, skew-symmetry and the Jacobi identity, Leibniz law etc. then $A$ will be called commutative, associative, Lie or respectively Leibniz algebra, etc. An algebra $A$ is called abelian if its multiplication is the trivial map, i.e. $a \cdot b = 0$, for all $a, b \in A$. We shall denote by $\sum(c)$ the circular sum - for example, if $A$ is an algebra and $\vartheta : A \times A \to V$ is a bilinear map then $\sum(c) \vartheta(a, b, c) = \vartheta(a, b) + \vartheta(b, c) + \vartheta(c, a) + \vartheta(c, a)$. The concepts of morphisms of algebras, subalgebras, two-sided ideals, etc. are defined in the obvious way. If $X$ and $Y$ are two subspace of an algebra $A$ then $X \cdot Y$ stands for the subspace generated by all $x \cdot y$, for all $x \in X$ and $y \in Y$. In particular, $A' := A \cdot A$ is a two-sided ideal of $A$ called the derived algebra of $A$. An algebra $A$ is called metabelian if $A'$ is an abelian subalgebra of $A$, i.e. $(a \cdot b) \cdot (c \cdot d) = 0$, for all $a, b, c, d \in A$. The derived series
of an algebra $A$ is defined inductively by $A^{(1)} := A'$ and $A^{(n+1)} := (A^{(n)})'$, for all $n \geq 1$. An algebra $A$ is called solvable of step $m \geq 1$ if $A^{(m)} = 0$ and $A^{(i)} \neq 0$, for all $i < m$. Thus, a non-abelian algebra $A$ is metabelian if it is a 2-step solvable algebra. The lower central series is the series with terms given by: $A^1 := A$ and $A^{n+1} := \sum_{i=1}^{n} A^i \cdot A^{n+1-i}$, for all $i < m$. An algebra $A$ is called nilpotent of step $m > 1$ if $A^m = 0$ and $A^i \neq 0$, for all $i < m$. A Leibniz algebra [17] is an algebra $A = (A, \cdot)$ such that the multiplication $\cdot$ satisfies the Leibniz law for any $a, b, c \in A$:

$$ (a \cdot b) \cdot c = a \cdot (b \cdot c) + (a \cdot c) \cdot b $$  \hspace{1cm} (1) 

Leibniz algebras have become popular and intensively studied in their own right as non-commutative generalizations of Lie algebras, but also in connection to homological algebra, classical or non-commutative differential geometry, vertex operator algebras or integrable systems. A Jordan algebra [3, 13] is an algebra $A = (A, \circ)$ such that the multiplication $\circ$ satisfies the following axioms for any $a, b \in A$:

$$ a \circ b = b \circ a, \quad (a^2 \circ b) \circ a = a^2 \circ (b \circ a) $$  \hspace{1cm} (2) 

Any associative algebra $A$ can be endowed with the Jordan algebra structure given by $a \circ b := ab + ba$, for all $a, b \in A$, where the juxtaposition $ab$ denotes the multiplication of the associative algebra $A$. We denote this Jordan algebra associated to the associative algebra $A$ by $A_J$. In particular, for any vector space $V$, the space of all linear endomorphisms $\text{End}_k(V)$ is a Jordan algebra. Jordan algebras have been intensively studied from algebraic point of view as well as for their applications to quantum mechanics, differential geometry, algebraic geometry or functional analysis - for details see [18] and the references therein.

**Jacobi-Jordan algebras.** A Jacobi-Jordan algebra [5] (a JJ-algebra, for short) is an algebra $A = (A, \cdot)$ such that for any $a, b \in A$:

$$ a \cdot b = b \cdot a, \quad \sum_{(c)} a \cdot (b \cdot c) = 0 $$  \hspace{1cm} (3) 

that is, $\cdot$ is commutative and satisfies the Jacobi identity $a \cdot (b \cdot c) + b \cdot (c \cdot a) + c \cdot (a \cdot b) = 0$. An important feature of JJ-algebras is that they are all nilpotent. By $\text{Aut}_{\text{JJ-Alg}}(A)$ we will denote the automorphism group of a JJ algebra $A$: i.e. the group with respect to the composition of all linear automorphisms $f : A \to A$ satisfying $f(a \cdot b) = f(a) \cdot f(b)$, for all $a, b \in A$. A clue result [5, Lemma 2.2] proves that any JJ-algebra is a Jordan algebra, that is the second identity of (2) also holds. As is shown in [5] even if at first glance these algebras seem quite similar to Lie algebras, they are in fact very different: the commutativity of the multiplication $\cdot$ instead of the skew-symmetry satisfied by Lie algebras gives rise to a radically different concept. Of course, any vector space $V$ becomes a JJ algebra with the trivial multiplication: $x \cdot y = 0$, for any $x, y \in V$. Such a JJ algebra is called abelian and will be denoted by $V_0$. Examples and the classification of small dimensional JJ algebras is given in [5]. Here we give two examples that will be used later on. Throughout the paper we will use the following convention: the multiplication of an algebra $A$ will only be defined on the elements of its basis and we only write down the non-zero multiplications.
Examples 1.1. 1. The commutative Heisenberg JJ algebra [5] is the \( (2n+1) \)-dimensional algebra \( \mathfrak{h}(2n+1, k) \) having \( \{ e_1, \ldots, e_n, f_1, \ldots, f_n, z \} \) as a basis over \( k \) and multiplication defined by \( e_i \cdot f_i = f_i \cdot e_i := z \), for all \( i = 1, \ldots, n \).

2. An interesting example of an infinite dimensional JJ algebra can be constructed as follows. Let \( V \) be a vector space, \( f \in \text{End}_k(V) \) with \( f^2 = 0 \) and \( v_0 \in \text{Ker}(f) \). Then \( V(f, v_0) := k \times V \) is a JJ algebra via the multiplication defined for any \( p, q, a \) and \( x, y \in V \) by:

\[
(p, x) \cdot (q, y) := (0, pq v_0 + pf(y) + qf(x))
\]

Indeed, since \( v_0 \in \text{Ker}(f) \) and \( f^2 = 0 \) one can easily see that \( (r, z) \cdot ((p, x) \cdot (q, y)) = 0 \), for all \( p, q, r \in k \) and \( x, y, z \in V \) and thus the Jacobi identity is trivially fulfilled. This shows that \( V(f, v_0) \) is a 3-step nilpotent JJ algebra. It is worth pointing out that \( V(f, v_0) \) is not a Lie algebra, if \( \text{char}(k) \neq 2 \) and \( (f, v_0) \neq (0, 0) \).

The first important difference between JJ algebras and Lie algebras is highlighted by the next result which provides a new class of solutions for the famous quantum Yang-Baxter equation. In what follows \( \otimes \) denotes \( \otimes_k \), the tensor product over \( k \) and for a linear map \( R \in \text{End}_k(\mathcal{A} \otimes \mathcal{A}) \) we shall denote by \( R^{12} := R \otimes \text{Id}_A, R^{23} := \text{Id}_A \otimes R \in \text{End}_k(\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}) \).\( R \) is called a solution of the quantum Yang-Baxter equation if and only if \( R^{12} R^{23} R^{12} = R^{12} R^{23} R^{12} \) in \( \text{End}_k(\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}) \). For an algebra \( \mathcal{A} = (A, \cdot) \) we shall denote by \( Z(A) \) the Leibniz center of \( A \), i.e. \( Z(A) := \{ z \in A \mid z \cdot A = A \cdot z = 0 \} \). Having in mind that any Lie algebra is also a Leibniz algebra we can prove the following:

Proposition 1.2. Let \( \mathcal{A} = (A, \cdot) \) be an algebra, \( \alpha, \beta \in k^* \) and \( 0 \neq z \in Z(\mathcal{A}) \). Then, the linear map

\[
R = R_{\alpha, \beta, z} : A \otimes A \to A \otimes A, \quad R(a \otimes b) := \alpha b \otimes a + \beta z \otimes (a \cdot b)
\]

is a solution of the quantum Yang-Baxter equation if and only if \( (A, \cdot) \) is a Leibniz algebra. In particular, if \( \mathcal{A} \) is a JJ algebra, then \( R_{\alpha, \beta, z} \) is a solution of the quantum Yang-Baxter equation if and only if \( \text{char}(k) = 2 \) or \( A \) is at most 3-step nilpotent algebra.

Proof. Let \( a, b, c \in A \). Taking into account that \( z \in Z(\mathcal{A}) \), a straightforward computation proves that:

\[
R^{12} R^{23} R^{12}(a \otimes b \otimes c) = \alpha^3 c \otimes b \otimes a + \alpha^2 \beta z \otimes b \otimes a \otimes c + \alpha z \otimes z \otimes (a \cdot b) \\
+ \alpha^2 \beta c \otimes z \otimes a \cdot b + \alpha \beta^2 z \otimes z \otimes (a \cdot b) \cdot c,
\]

and

\[
R^{23} R^{12} R^{23}(a \otimes b \otimes c) = \alpha^3 c \otimes b \otimes a + \alpha^2 \beta c \otimes z \otimes a \cdot b + \alpha^2 \beta z \otimes b \otimes a \cdot c \\
+ \alpha^2 \beta z \otimes z \otimes (a \cdot b) \otimes c + \alpha \beta^2 z \otimes z \otimes a \cdot (b \cdot c)
\]

Thus, \( R \) is a solution of the quantum Yang-Baxter equation if and only if

\[
\alpha^2 \beta z \otimes z \otimes (a \cdot b) \cdot c = \alpha^2 \beta^2 z \otimes z \otimes (a \cdot c) \cdot b + \alpha^2 \beta^2 z \otimes z \otimes a \cdot (b \cdot c)
\]

Since the scalars \( \alpha \) and \( \beta \) are non-zero and \( z \neq 0 \), the last equation is equivalent to \( (a \cdot b) \cdot c = (a \cdot (b \cdot c)) + (a \cdot c) \cdot b \), for all \( a, b, c \in A \), that is \( (A, \cdot) \) is a Leibniz algebra. Assume now that \( A \) is a JJ algebra. Then, using the Jacobi identity and the commutativity of \( \cdot \) we obtain that \( R \) is a solution of the quantum Yang-Baxter equation if and only if \( 2 c \cdot (a \cdot b) = 0 \), for all \( a, b, c \in A \), and hence the last statement also follows. \( \square \)
An example of a JJ algebra which is at most 3-step nilpotent is the commutative Heisberg algebra $\mathfrak{h}(2n+1,k)$. Applying Proposition 1.2 we construct a family of solutions of the quantum Yang-Baxter equation arising from any triple of $n \times n$ symmetric matrices.

**Example 1.3.** Let $X = (x_{ij})$, $Y = (y_{ij})$, $Z = (z_{ij})$ be three $n \times n$ symmetric matrices and $A = A_{X,Y,Z}$ the JJ algebra having the basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n, y, z\}$ and the multiplication given for any $i, j \in \mathbb{N}$ by:

$$e_i \cdot e_j = e_j \cdot e_i := x_{ij} y, \quad f_i \cdot f_j = f_j \cdot f_i := y_{ij} y, \quad e_i \cdot f_j = f_j \cdot e_i := \delta^j_i z + z_{ij} y$$

where $\delta^j_i$ is the Kronecker symbol. The above algebra will be explicitly constructed in Example 3.7. Then $A$ is at most 3-step nilpotent JJ algebra and the Leibniz center $Z(A_{X,Y,Z})$ is 2-dimensional having $\{y, z\}$ as a basis. Using, Proposition 1.2 we obtain that for any scalars $\alpha, \beta, \gamma \in k$ the linear map defined for any $a, b \in A$ by:

$$R : A \otimes A \to A \otimes A, \quad R(a \otimes b) := \alpha b \otimes a + (\beta y + \gamma z) \otimes (a \cdot b)$$

is a solution of the quantum Yang-Baxter equation in $\text{End}_k(A \otimes A \otimes A)$.

Now, we shall define modules and representations of a JJ algebra.

**Definition 1.4.** Let $A$ be a JJ-algebra. A (left) Jacobi-Jordan $A$-module (JJ $A$-module, for short) is a vector space $V$ equipped with a bilinear map $\triangleright : A \times V \to V$, called action, such that for any $a, b \in A$ and $x \in V$:

$$(a \cdot b) \triangleright x = -a \triangleright (b \triangleright x) - b \triangleright (a \triangleright x) \tag{5}$$

We denote by $\mathcal{JJ}_A$ the category of all (left) Jacobi-Jordan $A$-modules having the action preserving linear maps as morphisms.

The category $\mathcal{JJ}_A$ of right JJ $A$-modules is defined analogously and since $A$ is commutative there exists an isomorphism of categories $\mathcal{JJ}_A \cong \mathcal{JJ}_A$. For this reason we will drop the adjective ‘left’ when we speak about JJ modules.

**Remark 1.5.** Axiom (5) from the definition of JJ-modules is surprisingly different from the one corresponding to modules over an associative algebra, namely $(a \cdot b) \triangleright x = a \triangleright (b \triangleright x)$. It is instead more close to the definition of Lie modules but differs from those by the sign of the first term in the right hand side. We recall that a (left) Lie $\mathfrak{g}$-module over a Lie algebra $\mathfrak{g} = (\mathfrak{g}, [-, -])$ is a vector space $V$ together with a bilinear map $\triangleright : \mathfrak{g} \times V \to V$ such that $[[g, h] \triangleright x = g \triangleright (h \triangleright x) - h \triangleright (g \triangleright x)$, for all $g, h \in \mathfrak{g}$ and $x \in V$. However, we shall see in Example 2.3 that axiom (5) is the correct one from the classical view point of defining modules over a given mathematical object $\mathcal{O}$ in a $k$-linear category $\mathcal{C}$: that is, a vector space $V$ with a bilinear map $\triangleright : \mathcal{O} \times V \to V$ such that $\mathcal{O} \times V$ with the ‘semi-direct’ product type multiplication $(a, x) \cdot (b, y) := (ab, a \triangleright y + b \triangleright x)$ has to be an object inside the $k$-linear category $\mathcal{C}$.

However, representations of a JJ-algebra $A$ will be defined exactly as the representations of a Jordan algebra, if we see JJ-algebras as a special case of them. We recall that we have denoted by $\text{End}_k(V)$, the Jordan algebra of all $k$-linear endomorphism of a vector space $V$ viewed as a Jordan algebra via $(f, g) \mapsto f \circ g + g \circ f$, for all $f, g \in \text{End}_k(V)$.
Definition 1.6. A representation of a JJ-algebra $A$ on a vector space $V$ is a morphism of Jordan algebras $\varphi : A \to \text{End}_k(V)_J$, i.e. $\varphi(a \cdot b) = \varphi(a) \circ \varphi(b) + \varphi(b) \circ \varphi(a)$, for all $a, b \in A$, where $\circ$ is the usual composition of linear endomorphisms of $V$.

Representations of a JJ-algebra $A$ and Jacobi-Jordan $A$-modules are two different ways of describing the same structure: more precisely, there exists an isomorphism of categories $A\mathcal{J}\mathcal{J} \cong \text{Rep}(A)$, where $\text{Rep}(A)$ is the category of representations of $A$ with the obvious morphisms. The one-to-one correspondence between JJ $A$-module structures $\triangleright$ on $V$ and representations $\varphi$ of $A$ on $V$ is given by the two-sided formula $\varphi(a)(x) := -a \triangleright x$, for all $a \in A$ and $x \in V$.

We recall the Ado and Harish-Chandra theorem saying that any Lie algebra over a field of characteristic zero has a faithful representation. On the other hand, the fundamental result of Albert [3] tell us that the Jordan algebra $H_3(O)$ of Hermitian $3 \times 3$-matrices with entries in the octonions can not be imbedded into a Jordan algebra of the form $A_J$. Thus, is tempting to ask the following question:

Let $A$ be a Jacobi-Jordan algebra over a field of characteristic zero. Does $A$ have a faithful representation, i.e. there exists a representation $\varphi : A \to \text{End}_k(V)_J$ of $A$ on a vector space $V$ that is an injective map?

Since $A\mathcal{J}\mathcal{J} \cong \text{Rep}(A)$, we prefer to deal from now on only with JJ $A$-modules instead of representations. First of all, any vector space $V$ is a JJ $A$-module via the trivial action $a \triangleright x := 0$, for any $a \in A$ and $x \in V$. On the other hand, using the Jacobi identity, we can easily see that $A$ and the linear dual $A^* := \text{Hom}_k(A, k)$ are JJ $A$-modules via the canonical actions:

$$a \triangleright x := a \cdot x, \quad (a \triangleright a^*)(x) := a^*(a \cdot x) \quad (6)$$

for all $a, x \in A$ and $a^* \in A^*$. Modules over a JJ algebra will prove themselves to be useful tools for defining Frobenius algebras, as the symmetric object in the category of JJ algebras.

Definition 1.7. A finite dimensional Jacobi-Jordan algebra $A$ is called Frobenius if there exists an isomorphism $A \cong A^*$ in $A\mathcal{J}\mathcal{J}$.

Definition 1.7 complies with the classical definition of Frobenius [10] on associative algebras: a JJ algebra $A$ is Frobenius if and only if the regular representation $A \to \text{End}_k(A)_J$, $a \mapsto (b \mapsto a \cdot b)$ and the canonical representation $\text{can} : A \to \text{End}_k(A^*)_J$, $\text{can}(a)(a^*) (b) := a^*(a \cdot b)$, for all $a, b \in A$ and $a^* \in A^*$ are equivalent.

Proposition 1.8. A finite dimensional JJ-algebra $A$ is Frobenius if and only if there exists a bilinear form $B : A \times A \to k$ that is nondegenerate and invariant, i.e. $B(a \cdot b, c) = B(a, b \cdot c)$, for all $a, b, c \in A$.

Proof. Follows from the one-to-one correspondence between the set of all $k$-linear isomorphisms $f : A \to A^*$ and the set of all nondegenerate bilinear forms $B : A \times A \to k$ given by the two-sided formula $f(a)(b) := B(a, b)$, for all $a, b \in A$. Under this bijection, it is easy to see that the JJ $A$-module maps $f : A \to A^*$ correspond to those bilinear forms $B : A \times A \to k$ that are invariant. \qed
Proposition 1.8 provides an efficient criterion for testing if a JJ algebra is Frobenius.

Examples 1.9. 1. Any finite dimensional abelian JJ algebra is Frobenius. Indeed, as the multiplication on $A$ is trivial, then any linear isomorphism $f : A \to A^*$ is an isomorphism of JJ $A$-modules.

2. Let $A := A_{1,2}$ be the 2-dimensional JJ algebra with basis $\{e_1, e_2\}$ and the multiplication $e_1 \cdot e_1 = e_2$. Then $A_{1,2}$ is a Frobenius JJ algebra since the bilinear form defined by $B(e_1, e_2) = B(e_2, e_1) = B(e_1, e_1) := 1$ and $B(e_2, e_2) := 0$ is non-degenerate and invariant.

3. Let $A := \mathfrak{h}(2n + 1, k)$ be the commutative Heisenberg JJ algebra from Example 1.1. Then $\mathfrak{h}(2n + 1, k)$ is not Frobenius. Indeed, let $B : \mathfrak{h}(2n + 1, k) \times \mathfrak{h}(2n + 1, k) \to k$ be an invariant bilinear form. Then $B(z, c) = B(e_i \cdot f_j, c) = B(e_i, f_j \cdot c)$, for all $c \in \mathfrak{h}(2n + 1, k)$ and $i = 1, \ldots, n$. By taking $c := z$ and $c := f_j$ we obtain $B(z, z) = B(z, f_j) = 0$, for any $j = 1, \ldots, n$. On the other hand, by taking $c := e_j$ and choosing $i \neq j$ yields $B(z, e_j) = 0$. Thus, $B(z, -) = 0$, for any invariant form $B$ on $\mathfrak{h}(2n + 1, k)$, that is $B$ is degenerate and hence the JJ algebra $\mathfrak{h}(2n + 1, k)$ is not Frobenius.

A more detailed analysis of Frobenius JJ algebras will be performed somewhere else. We mention that Frobenius associative algebras are studied not only from algebraic point of view but also for their implications in topology, algebraic geometry and 2D topological quantum field theories, category theory, Hochschild cohomology or graph theory ([14, 16]). On the other hand, Frobenius Lie algebras are studied in pure mathematics [15, 19] as well as in physics [9, 22].

2. Crossed products and the global extension problem

In this section we deal with the GE problem, as formulated in the introduction, in the context of JJ algebras. Let $A$ be a JJ-algebra, $E$ a vector space, $\pi : E \to A$ a linear epimorphism of vector spaces with $V := \text{Ker}(\pi)$ and denote by $i : V \to E$ the inclusion map. We say that a linear map $\varphi : E \rightarrow E$ stabilizes $V$ (resp. co-stabilizes $A$) if $\varphi \circ i = i$ (resp. $\pi \circ \varphi = \pi$). Two JJ algebra structures $\cdot$ and $\cdot'$ on $E$ such that $\pi : E \to A$ is a morphism of JJ algebras are called cohomologous and we denote this by $(E, \cdot) \approx (E, \cdot')$, if there exists a JJ algebra map $\varphi : (E, \cdot) \rightarrow (E, \cdot')$ which stabilizes $V$ and co-stabilizes $A$. One can easily see that any such morphism is bijective and therefore $\approx$ is an equivalence relation on the set of all JJ-algebra structures on $E$ such that $\pi : E \to A$ is a JJ algebra map. The set of all equivalence classes via the equivalence relation $\approx$ will be denoted by $\text{Gext}(A, E)$ and it is the classifying object for the GE problem. In the sequel we will prove that $\text{Gext}(A, E)$ is parameterized by a global non-abelian cohomological type object $\mathcal{G\mathbb{H}}^2(A, V)$ which will be explicitly constructed. To begin with, we introduce the crossed product of JJ algebras:

Definition 2.1. Let $A = (A, \cdot)$ be a JJ algebra and $V$ a vector space. A crossed data of $A$ by $V$ is a system $\Theta(A,V) = (\triangleright, \vartheta, \cdot_V)$ consisting of three bilinear maps

$\triangleright : A \times V \to V, \quad \vartheta : A \times A \to V, \quad \cdot_V : V \times V \to V$
For a crossed data $\Theta(A,V) = (\triangleright, \vartheta, \cdot)$ we denote by $A\#V = A\#_{(\triangleright, \vartheta, \cdot)}V$ the vector space $A \times V$ with the multiplication $\bullet$ defined for any $a, b \in A$ and $x, y \in V$ by:

$$(a, x) \bullet (b, y) := (a \cdot b, \vartheta(a, b) + a \triangleright y + b \triangleright x + x \cdot \cdot y)$$

$A\#V$ is called the crossed product associated to $\Theta(A,V)$ if it is a JJ algebra with the multiplication given by (7). In this case $\Theta(A,V) = (\triangleright, \vartheta, \cdot)$ is called a crossed system of $A$ by $V$. Our next result provides the necessary and sufficient conditions for $A\#V$ to be a crossed product:

**Proposition 2.2.** Let $A$ be a JJ algebra, $V$ a vector space and $\Theta(A,V) = (\triangleright, \vartheta, \cdot)$ a crossed data of $A$ by $V$. Then $A\#V$ is a crossed product if and only if the following compatibility conditions hold for any $a, b, c \in A$ and $x, y \in V$:

\begin{align*}
(J1) \quad & \text{$(V, \cdot)$ is a JJ algebra and the bilinear map } \vartheta : A \times A \rightarrow V \text{ is symmetric} \\
(J2) \quad & (a \cdot b) \triangleright x + a \triangleright (b \triangleright x) + b \triangleright (a \triangleright x) + x \cdot \cdot \vartheta(a, b) = 0 \\
(J3) \quad & a \triangleright (x \cdot \cdot y) + x \cdot \cdot \vartheta(a, b) + y \cdot \cdot (a \triangleright x) = 0 \\
(J4) \quad & \sum_{(c)} \vartheta(a, b \cdot c) + \sum_{(c)} a \triangleright \vartheta(b, c) = 0
\end{align*}

**Proof.** We start with the commutativity condition: one can easily see that $\bullet$ defined by (7) is commutative if and only if $\vartheta : A \times A \rightarrow V$ is symmetric and $\cdot : V \times V \rightarrow V$ is commutative. For the rest of the proof it is worth noticing that since in $A\#V$ we have $(a, x) = (a, 0) + (0, x)$, the Jacobi condition holds if and only if it holds for all generators of $A\#V$, i.e. for the set $\{(a, 0) | a \in A\} \cup \{(0, x) | x \in V\}$. We only provide a sketch of the proof, the rest of the details being left to the reader. For instance, the Jacobi condition for the multiplication given by (7) holds in $\{(0, x), (0, y), (a, 0)\}$ if and only if (J3) holds. Similarly, the Jacobi condition holds in $\{(a, 0), (b, 0), (0, z)\}$ if and only if (J2) holds while the Jacobi condition holds in $\{(a, 0), (b, 0), (c, 0)\}$ if and only if (J4) holds. Finally, the Jacobi condition holds in $\{(0, x), (0, y), (0, z)\}$ if and only if $\cdot : V \times V \rightarrow V$ also satisfies the Jacobi condition. This together with the commutativity proved below shows that $\cdot$ is a JJ algebra structure on $V$. \hfill \Box

In what follows a crossed system of $A$ by $V$ will be seen as a system of bilinear maps $\Theta(A,V) = (\triangleright, \vartheta, \cdot)$ satisfying axioms (J1)-(J4) and the set of all crossed systems of $A$ by $V$ will be denoted by $\mathcal{CS}(A,V)$.

**Examples 2.3.** 1. By applying Proposition 2.2 we obtain that a crossed data $(\triangleright, \vartheta, \cdot)$ for which $\cdot$ is the trivial multiplication on $V$ is a crossed system if and only if $(V, \triangleright)$ is a JJ $A$-module and $\vartheta : A \times A \rightarrow V$ is a symmetric bilinear map satisfying the compatibility condition (J4). A symmetric bilinear map $\vartheta : A \times A \rightarrow V$ satisfying (J4) will be called a $\triangleright$-cocycle by analogy with Lie algebras [8]. Furthermore, if $\vartheta$ is also the trivial map, then the associated crossed product will be called the trivial extension of the JJ algebra $A$ by the $A$-module $V$.

2. A crossed system $\Theta(A,V) = (\triangleright, \vartheta, \cdot)$ for which $\vartheta$ is the trivial map is called a semidirect system of $A$ by $V$. In this case $\vartheta$ will be omitted when writing down $\Theta(A,V)$. Axioms in Proposition 2.2 boil down to the following: $\Theta(A,V) = (\triangleright, \cdot)$ is a semidirect system if and only if $(V, \triangleright)$ is a JJ $A$-module, $(V, \cdot)$ is a JJ algebra and for all $a \in A,$
$x, y \in V$ we have:

$$a \triangleright (x \cdot_V y) + x \cdot_V (a \triangleright y) + y \cdot_V (a \triangleright x) = 0 \quad (8)$$

The associated crossed product will be called a semidirect product of JJ algebras and will be denoted by $A \ltimes V := A \ltimes [\triangleright, \cdot_V] V$. The terminology is motivated in Corollary 2.5 by the fact that exactly as in the case of Lie algebras, this construction describes split epimorphisms within the category of JJ algebras.

The crossed product is the tool to answer the GE problem. Indeed, first we observe that the canonical projection $\pi_A : A\#V \to A$, $\pi_A(a,x) := a$ is a surjective JJ algebra map with kernel $\{0\} \times V \cong V$. Hence, the JJ algebra $A\#V$ is an extension of the JJ algebra $A$ by the JJ algebra $(V, \cdot_V)$ via

$$\begin{array}{ccc}
0 & \longrightarrow & V \\
& \xrightarrow{i_V} & A\#V \\
& & \xrightarrow{\pi_A} A \\
& & \longrightarrow 0
\end{array} \quad (9)$$

where $i_V(x) = (0, x)$. Conversely, we have:

**Proposition 2.4.** Let $A$ be a JJ algebra, $E$ a vector space and $\pi : E \to A$ an epimorphism of vector spaces with $V = \ker(\pi)$. Then any JJ algebra structure $\cdot_E$ which can be defined on the vector space $E$ such that $\pi : (E, \cdot_E) \to A$ becomes a morphism of JJ algebras is isomorphic to a crossed product $A\#V$. Furthermore, the isomorphism of JJ algebras $(E, \cdot_E) \cong A\#V$ can be chosen such that it stabilizes $V$ and co-stabilizes $A$.

Therefore, any JJ algebra structure on $E$ such that $\pi : E \to A$ is a JJ algebra map is cohomologous to an extension of the form (9).

**Proof.** Let $\cdot_E$ be a JJ algebra structure of $E$ such that $\pi : (E, \cdot_E) \to A$ is a JJ algebra map. Since we are working over a field, we can find a $k$-linear section $s : A \to E$ of $\pi$, i.e. $\pi \circ s = \text{Id}_A$. It follows that $\varphi : A \times V \to E$, $\varphi(a,x) := s(a) + x$ is an isomorphism of vector spaces with the inverse $\varphi^{-1}(y) = (\pi(y), y - s(\pi(y)))$, for all $y \in E$. Using the section $s$ we can define the following two bilinear maps:

$$\triangleright = \triangleright_s : A \times V \to V, \quad a \triangleright x := s(a) \cdot_E x$$

$$\vartheta = \vartheta_s : A \times A \to V, \quad \vartheta(a,b) := s(a) \cdot_E s(b) - s(a \cdot b)$$

where $a, b \in A$ and $x \in V$ and let $\cdot_V : V \times V \to V$ be the restriction of $\cdot_E$ at $V$, i.e. $x \cdot_V y := x \cdot_E y$, for all $x, y \in V$. It is straightforward to see that these are well-defined maps. Finally, using the system $(\triangleright, \vartheta, \cdot_V)$ connecting $A$ and $V$ we can prove that the unique JJ algebra structure $\bullet$ that can be defined on the direct product of vector spaces $A \times V$ such that $\varphi : A \times V \to (E, \cdot_E)$ is an isomorphism of JJ algebras is given by:

$$(a,x) \bullet (b,y) := (a \cdot b, \vartheta(a,b) + a \triangleright y + b \triangleright x + x \cdot_V y) \quad (10)$$

for all $a, b \in A$, $x, y \in V$. Indeed, let $\bullet$ be such a JJ algebra structure on $A \times V$. Then we have:

$$(a,x) \bullet (b,y) = \varphi^{-1}(\varphi(a,x) \cdot_E \varphi(b,y)) = \varphi^{-1}((s(a) + x) \cdot_E (s(b) + y))$$

$$= \varphi^{-1}(s(a) \cdot_E s(b) + s(a) \cdot_E y + x \cdot_E s(b) + x \cdot_V y)$$

$$= (a \cdot b, s(a) \cdot_E s(b) - s(a \cdot b) + s(a) \cdot_E y + x \cdot_E s(b) + x \cdot_V y)$$

$$= (a \cdot b, \vartheta(a,b) + a \triangleright y + b \triangleright x + x \cdot_V y)$$
which is the desired multiplication. Thus, \( \varphi : A \# V \to (E, \cdot_E) \) is an isomorphism of JJ algebras which stabilizes \( V \) and co-stabilizes \( A \). This finishes the proof. \( \square \)

We mentioned briefly at the end of Example 2.3 that the semidirect product of JJ algebras characterizes split epimorphisms within the category. This is what we prove next:

**Corollary 2.5.** A JJ algebra map \( \pi : B \to A \) is a split epimorphism in the category of JJ algebras if and only if there exists an isomorphism of JJ algebras \( B \cong A \ltimes V \), where \( V = \text{Ker}(\pi) \) and \( A \ltimes V \) is a semidirect product of JJ algebras.

*Proof.* Firstly, note that in the case of the semidirect product \( A \ltimes V \), the canonical projection \( p_A : A \ltimes V \to A \), \( p_A(a, x) = a \) has a section that is a JJ algebra map defined by \( s_A(a) = (a, 0) \), for all \( a \in A \). Conversely, let \( s : A \to B \) be a JJ algebra map with \( \pi \circ s = \text{Id}_A \). Then, the bilinear map \( \vartheta \) constructed in the proof of Proposition 2.4 is the trivial map and hence the corresponding crossed product \( A \# V \) reduces to a semidirect product \( A \ltimes V \). \( \square \)

In light of Proposition 2.4, the classification part of the GE-problem comes down to the classification of the crossed products associated to all crossed systems of \( A \) by \( V \). Our next result parameterizes (iso)morphisms between two such crossed products.

**Lemma 2.6.** Let \( \Theta(A, V) = (\triangleright, \vartheta, \cdot_V) \) and \( \Theta'(A, V) = (\triangleright', \vartheta', \cdot'_V) \) be two crossed systems with \( A \# V \), respectively \( A \#' V \), the corresponding crossed products. Then there exists a bijection between the set of all JJ algebra morphisms \( \psi : A \# V \to A \#' V \) which stabilize \( V \) and co-stabilize \( A \) and the set of all linear maps \( r : A \to V \) satisfying the following compatibilities for all \( a, b \in A \), \( x, y \in V \):

- (CH1) \( x \cdot_V y = x \cdot'_V y \);
- (CH2) \( a \triangleright x = a \triangleright' x + r(a) \cdot'_V x \);
- (CH3) \( \vartheta(a, b) + r(a \cdot b) = \vartheta'(a, b) + a \triangleright' r(b) + b \triangleright r(a) + r(a) \cdot'_V r(b) \)

Under the above bijection the JJ algebra morphism \( \psi = \psi_r : A \# V \to A \#' V \) corresponding to \( r : A \to V \) is given by \( \psi(a, x) = (a, r(a) + x) \), for all \( a \in A \), \( x \in V \). Moreover, \( \psi = \psi_r \) is an isomorphism with the inverse given by \( \psi_r^{-1} = \psi_{-r} \).

*Proof.* We can easily see that a linear map \( \psi : A \times V \to A \times V \) stabilizes \( V \) and co-stabilizes \( A \) if and only if there exists a uniquely determined linear map \( r : A \to V \) such that \( \psi(a, x) = \psi_r(a, x) = (a, r(a) + x) \), for all \( a \in A \), \( x \in V \). The proof will be finished once we show that \( \psi : A \# V \to A \#' V \) is a JJ algebra map if and only if the compatibility conditions (CH1)-(CH3) hold. We are left to check that the following compatibility holds for all generators of \( A \times V \):

\[
\psi((a, x) \cdot (b, y)) = \psi((a, x)) \cdot' \psi((b, y)) \quad (11)
\]

A direct computation shows that (11) holds for the pair \((a, 0), (b, 0)\) if and only if (CH3) is fulfilled while (11) holds for the pair \((0, x), (a, 0)\) if and only if (CH2) is satisfied. Finally, (11) holds for the pair \((0, x), (0, y)\) if and only if (CH1) holds. \( \square \)

It is therefore natural to introduce the following:
**Definition 2.7.** Let $A$ be a JJ algebra and $V$ a vector space. Two crossed systems $\Theta(A, V) = (\vartriangleright, \vartriangleright', \cdot)$ and $\Theta'(A, V) = (\vartriangleright', \vartriangleright', \cdot')$ are called cohomologous, and we denote this by $\Theta(A, V) \approx \Theta'(A, V)$, if and only if $\cdot' = \cdot'$ and there exists a linear map $r : A \to V$ such that for any $a, b \in A$, $x, y \in V$ we have:

\[
\begin{align*}
  a \vartriangleright x &= a \vartriangleright' x + r(a) \cdot' x \\
  \vartriangleright(a, b) &= \vartriangleright'(a, b) + a \vartriangleright' r(b) + b \vartriangleright' r(a) + r(a) \cdot' r(b) - r(a \cdot b)
\end{align*}
\]

The theoretical answer to the GE-problem now follows as a conclusion of this section:

**Theorem 2.8.** Let $A$ be a JJ algebra, $E$ a vector space and $\pi : E \to A$ a linear epimorphism of vector spaces with $V = \ker(\pi)$. Then $\approx$ defined in Definition 2.7 is an equivalence relation on the set $\mathcal{CS}(A, V)$ of all crossed systems of all $A$ by $V$. If we denote by $\mathcal{H}^2(A, V) := \mathcal{CS}(A, V)/\approx$, then the map

\[
\mathcal{H}^2(A, V) \to \text{Gext}(A, E), \quad (\vartriangleright, \vartriangleright', \cdot) \mapsto A \ast (\vartriangleright, \vartriangleright', \cdot) V
\]

is bijective, where $\overline{(\vartriangleright, \vartriangleright', \cdot)}$ denotes the equivalence class of $(\vartriangleright, \vartriangleright', \cdot)$ via $\approx$.

**Proof.** Follows from Proposition 2.2, Proposition 2.4 and Lemma 2.6.

However, to compute the classifying object $\mathcal{H}^2(A, V)$, for a given JJ algebra $A$ and a given vector space $V$ remains a challenge. We will describe below a strategy for tackling this computational problem inspired by the way we defined the equivalence relation $\approx$ in Definition 2.7. Indeed, the aforementioned definition shows that two different JJ algebra structures $\cdot'$ and $\cdot'$ on $V$ give rise to two different equivalence classes with respect to the relation $\approx$ on $\mathcal{CS}(A, V)$. This observation will be our starting point. Let us fix a JJ algebra multiplication on $V$ and denote by $\mathcal{CS}_\cdot(A, V)$ the set of pairs $(\vartriangleright, \vartriangleright')$ such that $(\vartriangleright, \vartriangleright', \cdot) \in \mathcal{CS}(A, V)$. Two such pairs $(\vartriangleright, \vartriangleright')$ and $(\vartriangleright', \vartriangleright', \cdot') \in \mathcal{HS}_\cdot(A, V)$ will be called $\cdot'$-cohomologous and will be denoted by $(\vartriangleright, \vartriangleright') \approx_{\cdot'} (\vartriangleright', \vartriangleright', \cdot')$ if there exists a linear map $r : A \to V$ such that the compatibility conditions (12)-(13) are fulfilled for $\cdot' = \cdot'$. Then $\approx_{\cdot'}$ is an equivalence relation on $\mathcal{CS}_\cdot(A, V)$ and we denote by $\mathcal{H}^2(A, (V, \cdot'))$ the quotient set $\mathcal{CS}_\cdot(A, V)/\approx_{\cdot'}$. This non-abelian cohomological object $\mathcal{H}^2(A, (V, \cdot'))$ classifies all extensions of the given JJ algebra $A$ by the given JJ algebra $(V, \cdot')$ and gives the theoretical answer to the classical extension problem for JJ algebras in the general non-abelian case. The above discussion is summarized in the following result:

**Corollary 2.9.** Let $(A, \cdot)$ and $(V, \cdot')$ be two given JJ algebras. Then, the map

\[
\mathcal{H}^2(A, (V, \cdot')) \to \text{Ext}(A, (V, \cdot')), \quad (\vartriangleright, \vartriangleright') \mapsto A \ast (\vartriangleright, \vartriangleright') V
\]

is bijective, where $\text{Ext}(A, (V, \cdot'))$ is the set of equivalence classes of all JJ algebras that are extensions of the JJ algebra $A$ by $(V, \cdot')$ and $\overline{(\vartriangleright, \vartriangleright')}$ denotes the equivalence class of $(\vartriangleright, \vartriangleright')$ via $\approx_{\cdot'}$.

Furthermore, we obtained the following decomposition of $\mathcal{H}^2(A, V)$:
Corollary 2.10. Let $A$ be a JJ algebra, $E$ a vector space and $π : E → A$ an epimorphism of vector spaces with $V = \text{Ker}(π)$. Then

$$G\mathbb{H}^2(A, V) = \sqcup_{\cdot} \mathbb{H}^2(A, (V, \cdot))$$

(16)

where the coproduct on the right hand side is in the category of sets over all possible JJ algebra structures $(\cdot , \cdot)$ on the vector space $V$.

As it can be easily seen, formula (16) still gives rise to laborious computations starting with finding all JJ algebra multiplications on $V$. Of course, the computations become more complicated when the dimension of $V$ increases. We should point out that among all components of the coproduct in (16) the simplest one is that corresponding to the trivial JJ algebra structure on $V$, i.e. $x \cdot y := 0$, for all $x, y \in V$ which we shall denote by $V_0 := (V, \cdot_v = 0)$. Indeed, let $CS_0(A, V_0)$ be the set of all pairs $(\triangleright, \vartheta)$ such that $(\triangleright, \vartheta, \cdot_v := 0) ∈ CS(A, V)$. As shown in Example 2.3, a pair $(\triangleright, \vartheta) ∈ CS_0(A, V_0)$ if and only if $(V, \triangleright)$ is a JJ $A$-module and $\vartheta : A × A → V$ is a $\triangleright$-cocycle. It turns out by applying Definition 2.7 for the trivial multiplication $\cdot := 0$ that two pairs $(\triangleright, \vartheta)$ and $(\triangleright', \vartheta') ∈ CS_0(A, V_0)$ are 0-cohomologous $(\triangleright, \vartheta) ≈_0 (\triangleright', \vartheta')$ if and only if $\triangleright = \triangleright'$ and there exists a linear map $r : A → V$ such that for all $a, b ∈ A$ we have:

$$\vartheta(a, b) = \vartheta'(a, b) + a ∢ r(b) + b ∢ r(a) - r(a \cdot b) \quad (17)$$

The equality $\triangleright = \triangleright'$ shows that two different JJ $A$-module structures on $V$ give different equivalence classes in the classifying object $G\mathbb{H}^2(A, V_0)$. Thus, we can apply the same strategy as before for computing $\mathbb{H}^2(A, V_0)$: we fix $(V, \triangleright)$ a JJ $A$-module structure on $V$ and consider the set $Z^2_\triangleright(A, V_0)$ of all $\triangleright$-cocycles: i.e. symmetric bilinear maps $\vartheta : A × A → V$ such that

$$\sum_{(c)} \vartheta(a, b \cdot c) + \sum_{(c)} a \triangleright \vartheta(b, c) = 0 \quad (18)$$

for all $a, b, c ∈ A$. Two $\triangleright$-cocycles $\vartheta$ and $\vartheta'$ will be called cohomologous $\vartheta ≈_0 \vartheta'$ if and only if there exists a linear map $r : A → V$ such that (17) holds. Then $≈_0$ is an equivalence relation on $Z^2_\triangleright(A, V_0)$ and the quotient set $Z^2_\triangleright(A, V_0)/ ≈_0$, which we will denote by $\mathbb{H}^2_\triangleright(A, V_0)$, plays the role of the second cohomological group from the theory of Lie/associative algebras [8, 11]. We can now put the above observations together and state the following results which classifies all extensions of a JJ algebra $A$ by an abelian algebra $V_0$. It is the JJ algebras counterpart of Hochschild’s result [11, Theorem 6.2] for associative algebra and respectively of Chevalley and Eilenberg’s results for Lie algebras [8, Theorem 26.2] concerning the classification of extensions with an ‘abelian’ kernel.

Corollary 2.11. Let $A$ be a JJ algebra and $V$ a vector space viewed with the trivial JJ algebra structure $V_0$. Then:

$$\mathbb{H}^2(A, V_0) = \sqcup_{\triangleright} \mathbb{H}^2_\triangleright(A, V_0)$$

(19)

where the coproduct on the right hand side is in the category of sets over all possible JJ $A$-module structures $\triangleright$ on the vector space $V$. 
3. Applications and Examples

Throughout this section \(k\) will be a field of characteristic \(\neq 2, 3\). In accordance to our notational conventions we denote by \(k_0\) the 1-dimensional JJ algebra with trivial multiplication and by \(V_0\) the abelian JJ algebra structure on an arbitrary vector space \(V\). In the sequel, the theoretical results obtained in Section 2 will be applied for two main classes of JJ algebras: co-flag algebras and metabelian algebras.

**Co-flag algebras.** For a given JJ algebra \(A\) we shall classify all JJ algebras \(B\) such that there exists a surjective JJ algebra map \(\pi: B \to A\) having a 1-dimensional kernel, which as a vector space will be assumed to be \(k\). We shall compute \(GH^2(A, k)\) which classifies all these JJ algebras up to an isomorphism which stabilizes \(k\) and co-stabilizes \(A\) as well as the second classifying object, denoted by \(CP(A, k)\), which provides the classification only up to an isomorphism of JJ algebras. By computing these two classifying objects we take the first step towards describing and classifying a new class of algebras defined as follows:

**Definition 3.1.** Let \(A\) be a JJ algebra and \(E\) a vector space. A JJ algebra structure \(\cdot_E\) on \(E\) is called a **co-flag JJ algebra over** \(A\) if there exists a positive integer \(n\) and a finite chain of surjective morphisms of JJ algebras

\[
A_n := (E, \cdot_E) \xrightarrow{\pi_n} A_{n-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_2} A_1 \xrightarrow{\pi_1} A_0 := A
\]

such that \(\dim_k(\ker(\pi_i)) = 1\), for all \(i = 1, \cdots , n\). A finite dimensional JJ algebra is called a **co-flag algebra** if it is a co-flag JJ algebra over the JJ algebra \(k_0\).

In order to describe co-flag algebras one more piece of terminology is needed:

**Definition 3.2.** Let \(A\) be a JJ algebra. A **co-flag datum of** \(A\) is a pair \((\lambda, \vartheta)\) consisting of a linear map \(\lambda: A \to k\) and a symmetric bilinear map \(\vartheta: A \times A \to k\) satisfying the following compatibilities for any \(a, b, c \in A\):

\[
\lambda(a \cdot b) = -2\lambda(a)\lambda(b)
\]

\[
\sum_{(c)} \vartheta(a, b \cdot c) + \sum_{(c)} \lambda(a)\vartheta(b, c) = 0
\]

We denote by \(\mathcal{CF}(A)\) the set of all co-flag data of \(A\). As we shall see, \(\mathcal{CF}(A)\) parameterizes the set of all crossed systems of \(A\) by a 1-dimensional vector space. Our next result provides a description of the first algebra \(A_1\) in the exact sequence (20) in terms depending only on \(A\).

**Proposition 3.3.** Let \(A\) be a JJ algebra. Then there exists a bijection \(\mathcal{CS}(A, k) \cong \mathcal{CF}(A)\) between the set of all crossed systems of \(A\) by \(k\) and the set of all co-flag data of \(A\) given such that the crossed product \(A \# k\) associated to \((\lambda, \vartheta) \in \mathcal{CF}(A)\) is the JJ algebra denoted by \(A_{(\lambda, \vartheta)}\) with the multiplication given for any \(a, b \in A, x, y \in k\) by:

\[
(a, x) \cdot (b, y) = (a \cdot b, \vartheta(a, b) + \lambda(a)y + \lambda(b)x)
\]
Proof. The proof basically comes down to computing all crossed systems between $A$ and $k_0$, i.e., all bilinear maps $\triangleright: A \times k \rightarrow k$, $\vartheta: A \times A \rightarrow k$ satisfying the compatibility conditions (J1)-(J4) of Proposition 2.2 with $\cdot_k$ being the trivial multiplication. As $k$ has dimension 1 there exists a bijection between the set of all crossed datums $(\triangleright, \vartheta, \cdot_k)$ of $A$ by $k$ and the set of pairs $(\lambda, \vartheta)$ consisting of a linear map $\lambda: A \rightarrow k$ and a bilinear map $\vartheta: A \times A \rightarrow k$. The bijection is given such that the crossed datum $(\triangleright, \vartheta)$ corresponding to $(\lambda, \vartheta)$ is defined by $a \triangleright x := \lambda(a)x$ for all $a \in A$ and $x \in k$. Now it can easily be seen that (J1) implies $\vartheta$ symmetric while (J3) is trivially fulfilled. Finally, (J2) amounts to the compatibility (21) while (22) is derived from (J4). The algebra $A_{(\lambda, \vartheta)}$ is just the crossed product $A \# k$ associated to this context. \[\square\]

We will now describe the algebra $A_{(\lambda, \vartheta)}$ in terms of generators and relations. We will see the elements of $A$ as elements in $A \times k$ through the identification $a = (a, 0)$ and denote by $f := (0_A, 1) \in A \times k$. Consider $\{e_i | i \in I\}$ to be a basis of $A$ as a vector space over $k$. Then the algebra $A_{(\lambda, \vartheta)}$ has $\{f, e_i | i \in I\}$ as a basis of the underlying vector space while its multiplication $\cdot$ is given for any $i \in I$ by:

$$e_i \cdot e_j = e_i \cdot_A e_j + \vartheta(e_i, e_j) f, \quad f^2 = 0, \quad e_i \cdot f = f \cdot e_i = \lambda(e_i) f$$

(24)

where $\cdot_A$ denotes the multiplication on $A$. Now putting Proposition 3.3 and Proposition 2.4 together yields:

Corollary 3.4. Let $A$ be a JJ algebra. A JJ algebra $B$ has a surjective JJ algebra map $B \rightarrow A \rightarrow 0$ whose kernel is 1-dimensional if and only if $B$ is isomorphic to $A_{(\lambda, \vartheta)}$, for some $(\lambda, \vartheta) \in \mathcal{CF}(A)$.

We are now in a position to compute the classifying object $\mathbb{H}^2(A, k)$.

**Proposition 3.5.** Let $A$ be a JJ algebra. Then,

$$\mathbb{H}^2(A, k) \cong \mathcal{CF}(A)/\approx$$

where $\approx$ is the following equivalence relation on $\mathcal{CF}(A)$: $(\lambda, \vartheta) \approx (\lambda', \vartheta')$ if and only if $\lambda = \lambda'$ and there exists a linear map $t: A \rightarrow k$ such that for any $a, b \in A$:

$$\vartheta(a, b) = \vartheta'(a, b) + \lambda(a)t(b) + \lambda(b)t(a) - t(a \cdot b)$$

(25)

**Proof.** We obtain from Theorem 2.8 and Proposition 3.3 that

$$\mathbb{H}^2(A, k) \cong \mathcal{CF}(A)/\approx$$

where the equivalence relation $\approx$ on $\mathcal{CF}(A)$ is just the equivalence relation $\approx$ from Definition 2.7 written for the set $\mathcal{CF}(A)$ via the bijection $\mathcal{HS}(A, k) \cong \mathcal{CF}(A)$ given in Proposition 3.3. The proof is finished by noticing that the equivalence relation $\approx$ written on the set of all co-flag data takes precisely the desired form. \[\square\]

The following decomposition of $\mathcal{CF}(A)/\approx$ suggested by Proposition 3.5 will come in handy: for a fixed linear map $\lambda: A \rightarrow k$ satisfying (21) we shall denote by $Z^2_{\lambda}(A, k)$ the set of all $\lambda$-cocycles; that is, the set of all symmetric bilinear maps $\vartheta: A \times A \rightarrow k$ satisfying the compatibility condition (22). Two $\lambda$-cocycles $\vartheta, \vartheta': A \times A \rightarrow k$ are equivalent $\vartheta \approx^\lambda \vartheta'$ if and only if there exists a linear map $t: A \rightarrow k$ such that (25)
holds. If we denote \( H^2(\mathfrak{a}, k) := Z^2(\mathfrak{a}, k)/\approx^\lambda \) we obtain the following decomposition of \( \mathfrak{g}H^2(\mathfrak{a}, k) \):

**Corollary 3.6.** Let \( \mathfrak{a} \) be a JJ algebra. Then,

\[
\mathfrak{g}H^2(\mathfrak{a}, k) \cong \bigsqcup_\lambda H^2(\mathfrak{a}, k)
\]

where the coproduct on the right hand side is in the category of sets over all possible linear maps \( \lambda : A \to k \) satisfying (21).

The example below highlights the way Corollary 3.6 works in order to classify JJ co-flag algebras. We shall denote by \( \text{Sym}(n, k) \) the vector space of all \( n \times n \) symmetric matrices over \( k \) and by \( \text{Sym}(n, k)/kI_n \) the quotient vector space via the subspace of matrices of the form \( aI_n \), for all \( a \in k \).

**Example 3.7.** Let \( \mathfrak{h}(2n+1, k) \) be the \((2n+1)\)-dimensional commutative Heisenberg JJ algebra defined in Example 1.9. Then,

\[
\mathfrak{g}H^2(\mathfrak{h}(2n+1, k), k) \cong \text{Sym}(n, k) \times \text{Sym}(n, k) \times \text{Sym}(n, k)/kI_n
\]

The equivalence classes of all \((2n+2)\)-dimensional JJ algebras with an algebra projection on \( \mathfrak{h}(2n+1, k) \) are the following algebras defined for any \((a_{ij}) \times (b_{ij}) \times (c_{ij}) \in \text{Sym}(n, k) \times \text{Sym}(n, k) \times \text{Sym}(n, k)/kI_n \) as the vector space having \( \{y, e_1, \ldots, e_n, f_1, \ldots, f_n, z\} \) as a basis and the multiplication given for any \( i, j = 1, \ldots, n \) by:

\[
e_i \bullet e_j = e_j \bullet e_i := a_{ij} y, \quad f_i \bullet f_j = f_j \bullet f_i := b_{ij} y, \quad e_i \bullet f_j = f_j \bullet e_i := \delta^i_j z + c_{ij} y
\]

We will prove first that there exists a bijection between \( \mathcal{C}\mathcal{F}(\mathfrak{h}(2n+1, k)) \) and the space \( \text{Sym}(n, k)^3 \) given such that the co-flag data \((\lambda, \vartheta) \in \mathcal{C}\mathcal{F}(\mathfrak{h}(2n+1, k)) \) associated to a triple \((A = (a_{ij}), B = (b_{ij}), C = (c_{ij})) \) consisting of \( n \times n \) symmetric matrices is given for any \( i, j = 1, \ldots, n \) by:

\[
\lambda := 0, \quad \vartheta(e_i, e_j) = \vartheta(e_j, e_i) := a_{ij}, \quad \vartheta(f_i, f_j) = \vartheta(f_j, f_i) := b_{ij}
\]

\[
\vartheta(e_i, f_j) = \vartheta(f_i, e_j) := c_{ij}, \quad \vartheta(-, z) = \vartheta(z, -) = 0
\]

Indeed, let \((\lambda, \vartheta) \in \mathcal{C}\mathcal{F}(\mathfrak{h}(2n+1, k)) \) be a co-flag data. Applying (21) for the pair \((e_i, e_i), (f_i, f_i)\) and respectively \((e_i, f_i)\), taking into account that \(\text{char}(k) \neq 2\) we obtain \(\lambda(e_i) = \lambda(f_i) = \lambda(z) = 0\), that is \(\lambda := 0\), the trivial map. Thus, the compatibility condition (22) takes the simplified form for any \(a, b, c \in \{e_1, \ldots, e_n, f_1, \ldots, f_n, z\}\):

\[
\sum_{(c)} \vartheta(a, b \cdot c) = 0 \quad (28)
\]

Applying this equation to the triples \((e_i, e_i, f_i), (e_i, f_i, f_i)\) and respectively \((e_i, f_i, z)\) we obtain: \(\vartheta(e_i, z) = \vartheta(f_i, z) = \vartheta(z, z) = 0\), for all \(i = 1, \ldots, n\). Thus, any bilinear map \(\vartheta\) of a co-flag datum of \(\mathfrak{h}(2n+1, k)\) is implemented by three \(n \times n\) matrices \((A = (a_{ij}), B = (b_{ij}), C = (c_{ij}))\) via the two-sided formulas:

\[
\vartheta(e_i, e_j) := a_{ij}, \quad \vartheta(f_i, f_j) := b_{ij}, \quad \vartheta(e_i, f_j) := c_{ij},
\]

for all \(i, j = 1, \ldots, n\). Of course, \(\vartheta\) is symmetric if and only if \(A, B, C\) are symmetric matrices. Now, using the multiplication on \(\mathfrak{h}(2n+1, k)\), we can easily see that such a bilinear map \(\vartheta = \vartheta_{A,B,C}\) satisfies (28). Thus, the first part is proven. We mention that
the multiplication $\bullet$ on the JJ algebra $\mathfrak{h}(2n+1,k)_{\lambda,\vartheta}$ as defined by (24) takes the form given in (27). Finally, a straightforward computation shows that the equivalence relation as defined by (25) of Proposition 3.5, takes the following form in the case for triples of symmetric matrices: two triples of $n \times n$ symmetric matrices $(A,B,C)$ and $(A',B',C')$ are equivalent $(A,B,C) \approx (A',B',C')$ if and only if $A = A'$, $B = B'$ and there exists a scalar $t \in k$ such that $C' = C = tI_n$. This finishes the proof.

The object $\mathbb{H}^2(A,k)$ classifies all crossed products $A\#k$ up to an isomorphism of JJ algebras which stabilizes $k$ and co-stabilizes $A$. In what follows we will consider a less restrictive classification: we denote by $\mathbb{CP}(A,k)$ the set of JJ algebra isomorphism classes of all crossed products $A\#k$. Two cohomologous crossed products $A\#k$ and $A'\#k$ are isomorphic and therefore there exists a canonical projection $\mathbb{H}^2(A,k) \rightarrow \mathbb{CP}(A,k)$ between the two classifying objects. Next in line is the task of computing $\mathbb{CP}(A,k)$.

**Theorem 3.8.** Let $A$ be a JJ algebra. Then there exists a bijection:

$$\mathbb{CP}(A,k) \cong \mathcal{CF}(A)/\equiv$$

where $\equiv$ is the equivalence relation on $\mathcal{CF}(A)$ defined by: $(\lambda, \vartheta) \equiv (\lambda', \vartheta')$ if and only if there exists a triple $(s_0, \psi, r) \in k^* \times \text{Aut}_{\text{JJ-Alg}}(A) \times \text{Hom}_k(A,k)$ such that

$$\lambda = \lambda' \circ \psi, \quad \vartheta(a,b) s_0 = \vartheta'(\psi(a),\psi(b)) + \lambda(a)r(b) + \lambda(b)r(a) - r(a \cdot b)$$

for all $a, b \in A$.

**Proof.** From Corollary 3.4 we know that any crossed product $A\#k$ is isomorphic to $A_{(\lambda,\vartheta)}$, for some $(\lambda,\vartheta) \in \mathcal{CF}(A)$. Thus, we proof reduces to describing the JJ algebra isomorphisms between two such algebras $A_{(\lambda,\vartheta)}$. It is a simple fact to notice that there exists a bijection between the set of all linear maps $\varphi : A \times k \rightarrow A \times k$ and the set of quadruples $(s_0, \beta_0, \psi, r) \in k \times A \times \text{Hom}_k(A,k) \times \text{Hom}_k(A,k)$ given such that the linear map $\varphi = \varphi(s_0, \beta_0, \psi, r)$ is defined for any $a \in A$ and $x \in k$ by:

$$\varphi(a, x) = (\psi(a) + x \beta_0, r(a) + x s_0)$$

(31)

We start by proving that a linear map given by (31) is an isomorphism of JJ algebras from $A_{(\lambda,\vartheta)}$ to $A_{(\lambda',\vartheta')}$ if and only if $\beta_0 = 0$, $s_0 \neq 0$, $\psi$ is a JJ algebra automorphism of $A$ and (30) holds. It can be easily seen that $\varphi((0,x) \bullet (0,y)) = \varphi(0,x) \bullet' \varphi(0,y)$ if and only if $\beta_0 = 0$, where by $'$ we denote the multiplication of $A_{(\lambda',\vartheta')}$. Hence, in order for $\varphi$ to be a JJ algebra map it should take the following simplified form for any $a \in A$ and $x \in k$:

$$\varphi(a, x) = (\psi(a), r(a) + x s_0)$$

(32)

for some triple $(s_0, \psi, r) \in k \times \text{Hom}_k(A,k) \times \text{Hom}_k(A,k)$. Next we prove that a linear map given by (32) is a JJ algebra morphism from $A_{(\lambda,\vartheta)}$ to $A_{(\lambda',\vartheta')}$ if and only if $\psi : A \rightarrow A$ is a JJ algebra map and the following compatibilities are fulfilled for any $a, b \in A$:

$$\lambda(a) s_0 = \lambda' (\psi(a)) s_0, \quad r(a \cdot b) + \vartheta(a,b) s_0 = \vartheta' (\psi(a),\psi(b)) + \lambda' (\psi(a)) r(b) + \lambda' (\psi(b)) r(a)$$

(33)

(34)

Indeed, it is straightforward to see that the compatibility (33) is exactly the condition $\varphi((a,0) \bullet (0,x)) = \varphi(a,0) \bullet' \varphi(0,x)$. Finally, the condition $\varphi((a,0) \bullet (b,0)) = \varphi(a,0) \bullet'$
\(\varphi(b,0)\) is equivalent to the fact that \(\psi\) is a JJ algebra endomorphism of \(A\) and (34) holds. We are left to prove that a JJ algebra map \(\varphi = \varphi(s_0, \psi, r)\) given by (32) is bijective if and only if \(s_0 \neq 0\) and \(\psi\) is a JJ automorphism of \(A\). Assume first that \(s_0 \neq 0\) and \(\psi\) is bijective with the inverse \(\psi^{-1}\). Then, we can see that \(\varphi(s_0, \psi, r)\) is a JJ algebra isomorphism with the inverse given by \(\varphi^{-1}_{(s_0, \psi, r)} := \varphi(s_0^{-1}, \psi^{-1}, -(r \psi^{-1})s_0^{-1})\). Conversely, assume that \(\varphi\) is bijective. Then its inverse \(\varphi^{-1}\) is a JJ algebra map and thus has the form \(\varphi^{-1}(a, x) = (\psi'(a), r'(a) + x s_0')\), for some triple \((s_0', r', \psi')\). If we write \(\varphi^{-1} \circ \varphi(0, 1) = (0, 1)\) we obtain that \(s_0 s_0' = 1\) i.e. \(s_0\) is invertible in \(k\). In the same way \(\varphi^{-1} \circ \varphi(a, 0) = (a, 0) = \varphi \circ \varphi^{-1}(a, 0)\) gives that \(\psi\) is bijective and \(\psi' = \psi^{-1}\).

**Remark 3.9.** Let us denote by \(\mathfrak{h}(2n + 1, k)_{A, B, C}\) the JJ algebra defined by (27). These are the \((2n + 2)\)-dimensional JJ algebras that admit an algebra surjection on \(\mathfrak{h}(2n + 1, k)\). This family of JJ algebras are classified from the viewpoint of the extension problem, i.e. up to an isomorphism which stabilizes \(k\) and co-stabilizes \(\mathfrak{h}(2n + 1, k)\), in Example 3.7 where the classifying object \(\mathbb{G} \mathbb{H}^2(\mathfrak{h}(2n + 1, k), k)\) is computed. Now, their classification up to a JJ algebra isomorphism by explicitly computing the second classifying object, namely \(\mathbb{C} \mathbb{F}(\mathfrak{h}(2n + 1, k), k)\) as constructed in Theorem 3.8, seems to be a very difficult task that will be addressed somewhere else. Even if the compatibility condition (30) takes the simplified form \(\vartheta(a, b) s_0 = \vartheta'(\psi(a), \psi(b)) - r(a \cdot b)\), this is still more general than the classical Kronecker-Williamson equivalence of bilinear forms [23, 12]. Thus, the starting point in addressing this problem should be a new and more general classification theory for bilinear forms.

For instance, by taking \(n = 1\) in Example 3.7 we obtain that \(\mathbb{G} \mathbb{H}^2(\mathfrak{h}(3, k), k) \cong k \times k \times \{0\} \cong k^2\). Thus, any 4-dimensional JJ algebra that admits an algebra surjection on \(\mathfrak{h}(3, k)\) is isomorphic to the JJ algebra having \(\{y, e, f, z\}\) as a basis and the multiplication defined by:

\[
e \cdot e := ay, \quad f \cdot f := by, \quad e \cdot f := z
\]

for some \(a, b \in k\). We denote by \((\mathfrak{h}(3, k), k)_{(a, b)}\) this JJ algebra. If \(k = k^2\), we can easily see that, up to an isomorphism there are only three such algebras, namely \((\mathfrak{h}(3, k), k)_{(0,0)}, (\mathfrak{h}(3, k), k)_{(1,0)}\) and \((\mathfrak{h}(3, k), k)_{(1,1)}\) which appear in [5, Proposition 3.4] in different notational conventions though.

As a consequence of Theorem 3.8 we can derive a classification result for the semidirect products of JJ algebras of the form \(A \ltimes k\). Recall from Example 2.3 that a semidirect product \(A \ltimes k\) is just a crossed product \(A(\lambda, \vartheta) = A \# k\) having a trivial \(\lambda\)-cocycle \(\vartheta\). The algebra obtained in this way will be denoted by \(A_\lambda\).

**Corollary 3.10.** Let \(A\) be a JJ algebra and \(\lambda, \lambda' : A \rightarrow k\) two linear maps satisfying (8). Then there exists an isomorphism of JJ algebras \(A_\lambda \cong A_{\lambda'}\) if and only if there exists \(\psi \in \text{Aut}_{\text{JJ-Alg}}(A)\) such that \(\lambda = \lambda' \circ \psi\).
\[ \psi \in \text{Aut}_{JJ-\text{Alg}}(A) \text{ and } r \in A^* \text{ by:} \]

\[ \zeta : k^* \times \text{Aut}_{JJ-\text{Alg}}(A) \to \text{Aut}_{\text{Gr}}(A^*, +), \quad \zeta(s_0, \psi)(r) := s_0^{-1} r \circ \psi \]

is a morphism of groups. This enables us to construct the semidirect product of groups \( A^* \ltimes \zeta (k^* \times \text{Aut}_{JJ-\text{Alg}}(A)) \) associated to \( \zeta \). The next result describes \( \text{Aut}_{JJ-\text{Alg}}(A_{(\lambda, \vartheta)}) \) as a certain subgroup of the semidirect product \( A^* \ltimes \zeta (k^* \times \text{Aut}_{JJ-\text{Alg}}(A)) \).

**Corollary 3.11.** Let \( A \) be a JJ algebra, \( (\lambda, \vartheta) \in \mathcal{C}_F(A) \) a co-flag datum of \( A \) and let \( \mathcal{G}(A, (\lambda, \vartheta)) \) be the set of all triples \((s_0, \psi, r) \in k^* \times \text{Aut}_{JJ-\text{Alg}}(A) \times A^* \) such that for any \( a, b \in A \):

\[ \lambda = \lambda \circ \psi, \quad \vartheta(a, b) s_0 = \vartheta(\psi(a), \psi(b)) + \lambda(a)r(b) + \lambda(b)r(a) - r(a \cdot b) \]

Then, there exists an isomorphism of groups \( \text{Aut}_{JJ-\text{Alg}}(A_{(\lambda, \vartheta)}) \cong \mathcal{G}(A, (\lambda, \vartheta)) \), where \( \mathcal{G}(A, (\lambda, \vartheta)) \) is a group with respect to the following multiplication:

\[ (s_0, \psi, r) \ast (s'_0, \psi', r') := (s_0 s'_0, \psi \circ \psi', r \circ \psi' + s_0 r') \quad (35) \]

for all \((s_0, \psi, r), (s'_0, \psi', r') \in \mathcal{G}(A, (\lambda, \vartheta)) \). Moreover, the canonical map

\[ \mathcal{G}(A, (\lambda, \vartheta)) \to A^* \ltimes \zeta (k^* \times \text{Aut}_{JJ-\text{Alg}}(A)), \quad (s_0, \psi, r) \mapsto (s_0^{-1}r, (s_0, \psi)) \]

in an injective morphism of groups.

**Proof.** It can be easily seen by a routine computation that \( \mathcal{G}(A, (\lambda, \vartheta)) \) is a group with respect to the multiplication (35); the unit is \((1, \text{Id}_A, 0)\) and the inverse of an element \((s_0, \psi, r)\) is \((s_0^{-1}, \psi^{-1}, -s_0^{-1}(r \circ \psi^{-1}))\). Furthermore, from (the proof of) Theorem 3.8 we have \( \varphi(s_0, \psi, r) \circ \varphi(s'_0, \psi', r') = \varphi(s_0 s'_0, \psi \circ \psi', r \circ \psi' + s_0 r') \), where \( \varphi(s_0, \psi, r) \) is an automorphism of \( A_{(\lambda, \vartheta)} \) given by (32). This settles the first statement. Finally, the last assertion follows by a straightforward computation.

**Metabelian algebras.** We recall that a JJ algebra \( J \) is called *metabelian* if the derived algebra \( J' \) is an abelian subalgebra of \( J \), i.e. \((a \cdot b) \cdot (c \cdot d) = 0\), for all \( a, b, c, d \in J \). If \( I \) is an ideal of \( J \), then the quotient algebra \( J/I \) is an abelian algebra if and only if \( J' \subseteq I \). Thus, \( J \) is a metabelian JJ algebra if and only if it fits into an exact sequence of JJ algebras

\[ 0 \longrightarrow B \overset{i}{\longrightarrow} J \overset{\pi}{\longrightarrow} A \longrightarrow 0 \]

where \( B \) and \( A \) are abelian algebras. Indeed, if \( J \) is metabelian we can take \( B := J' \) and \( C := J/J' \) with the obvious morphisms. Conversely, assume that \( J \) is an extension of an abelian algebra \( A \) by an abelian algebra \( B \). Since \( J/i(B) = J/\text{Ker}(\pi) \cong A \) is an abelian algebra, we obtain that \( J' \subseteq i(B) \cong B \). Hence, \( J' \) is abelian as a subalgebra in an abelian algebra, i.e. \( J \) is metabelian. This remark together with Proposition 2.4 proves that a JJ algebra \( J \) is metabelian if and only if there exists an isomorphism \( J \cong A \# V \), where \( A \# V \) is a crossed product associated to a crossed system \((\triangleright, \vartheta)\) between two abelian algebras \( A \) and \( V \). Using the results of Section 2, this observation allows us to prove a structure type theorem for metabelian JJ algebras as well as a classification result for them. This is what we do next. A *metabelian system* of a vector space \( A \) by a vector
space $V$ is a pair $(\triangleright, \vartheta)$ consisting of two bilinear maps $\triangleright : A \times V \to V$ and $\vartheta : A \times A \to V$ such that $\vartheta$ is symmetric and

$$a \triangleright (b \triangleright x) = -b \triangleright (a \triangleright x), \quad \sum_{(c)} a \triangleright \vartheta(b, c) = 0$$  \hspace{1cm} (36)

for all $a, b, c \in A$ and $x \in V$. These are exactly the axioms remaining from (J1)-(J4) of Proposition 2.2 if we consider both $\cdot_A$ and $\cdot_V$ to be equal to the trivial multiplication. We denote by $\mathcal{MA}(A, V)$ the set of all metabelian systems of $A$ by $V$. For a pair $(\triangleright, \vartheta) \in \mathcal{MA}(A, V)$ the associated crossed product $A \# V$ will be denoted by $A \#_{(\triangleright, \vartheta)} V$: it has $A \times V$ as the underlying vector space while the multiplication takes the following simplified form:

$$(a, x) \cdot (b, y) := (0, \vartheta(a, b) + a \triangleright y + b \triangleright x)$$  \hspace{1cm} (37)

for all $a, b \in A$ and $x, y \in V$. Now we fix a bilinear map $\triangleright : A \times V \to V$ satisfying the first equation of (36) and denote by $Z^2(\triangleright) (A_0, V_0)$ the set of all symmetric, abelian $\triangleright$-cocycles: i.e. $Z^2(\triangleright) (A_0, V_0)$ consists of all symmetric bilinear maps $\vartheta : A \times A \to V$ satisfying the second equation of (36). Two such $\triangleright$-cocycles $\vartheta$ and $\vartheta' \in Z^2(\triangleright) (A_0, V_0)$ will be called cohomologous $\vartheta \approx_0 \vartheta'$ if and only if there exists a linear map $r : A \to V$ such that for any $a, b \in A$

$$\vartheta(a, b) = \vartheta'(a, b) + a \triangleright r(b) + b \triangleright r(a)$$

Then $\approx_0$ is an equivalence relation on $Z^2(\triangleright) (A_0, V_0)$ and the quotient set $Z^2(\triangleright) (A_0, V_0)/ \approx_0$ will be denoted by $H^2(\triangleright) (A_0, V_0)$. We also recall that $H^2(\triangleright) (A_0, V_0)$ classifies up to an isomorphism that stabilizes $V_0$ and co-stabilizes $A_0$ all metabelian algebras which are extensions of $A_0$ by $V_0$. We can now put the above observations together with Corollary 2.11 and we will obtain the following result on the structure of metabelian algebras as well as their classification from the view point of the extension problem:

**Corollary 3.12.** (1) A JJ algebra $J$ is metabelian if and only if there exists an isomorphism of JJ algebras $J \cong A \#_{(\triangleright, \vartheta)} V$, for some vector spaces $A$ and $V$ and a metabelian system $(\triangleright, \vartheta) \in \mathcal{MA}(A, V)$.

(2) If $A$ and $V$ are vector spaces viewed with the abelian algebra structure $A_0$ and $V_0$, then there exists a bijection:

$$H^2(\triangleright) (A_0, V_0) \cong \bigsqcup_{\triangleright} H^2(\triangleright) (A_0, V_0)$$  \hspace{1cm} (38)

where the coproduct on the right hand side is in the category of sets over all possible bilinear maps $\triangleright : A \times V \to V$ such that $a \triangleright (b \triangleright x) = -b \triangleright (a \triangleright x)$, for all $a, b \in A$ and $x \in V$.

To get some insight on the above result we explain briefly how it works for a metabelian JJ algebra of dimension $n$. An invariant of such algebras is the dimension of the derived algebra. Thus, we have to fix a positive integer $1 \leq m \leq n$. Any metabelian algebra of dimension $n$ that has the derived algebra of dimension $m$ is isomorphic to a crossed product $k^{n-m}_{\triangleright, \vartheta} \#_{(\triangleright, \vartheta)} k^m_{0}$. Thus, the first computational and laborious part requires computing all metabelian systems $\mathcal{MA}(k^{n-m}, k^m)$ between $k^{n-m}$ and $k^m$. Next in line is the cohomological object $H^2(k^{n-m}_{0}, k^m_{0})$ whose description relies on the decomposition given in (38). Its elements will classify all metabelian algebras $k^{n-m}_{0} \#_{(\triangleright, \vartheta)} k^m_{0}$ up to an
isomorphism which stabilizes $k^m$ and co-stabilizes $k^{n-m}$. Computing the other classification object, namely $\mathbb{C}P(k^{n-m}, k^m)$ which is a quotient set of $\mathbb{H}^2(k_0^{n-m}, k_0^m)$ seems to be a very difficult task – the corresponding problem at the level of groups is the famous \textit{isomorphism problem} for metabelian groups, which seems to be connected to Hilberts Tenth problem, i.e. the problem is algorithmically undecidable (cf. [6]).

Below we present two generic examples in order to highlight the difficulty of the problem, namely we will deal with metabelian JJ algebras having the derived algebra of dimension 1 (resp. codimension 1). Henceforth, we denote by $\text{Sym}(A \times A, k)$ the space of all symmetric bilinear forms on a vector space $A$.

\textbf{Proposition 3.13.} \textit{Let $k$ be a field of characteristic $\neq 2,3$ and $A$ a vector space. Then:}

\text{(1)} There exists a bijection

$$\mathbb{GH}^2(A_0, k) = \mathbb{H}^2(A_0, k_0) \cong \text{Sym}(A \times A, k)$$

\text{given such that the crossed product $A_\vartheta := A_0 \# k_0$ associated to $\vartheta \in \text{Sym}(A \times A, k)$ is the JJ algebra with the multiplication given for any $a, b \in A$, $x, y \in k$ by:}

$$(a, x) \cdot (b, y) = (0, \vartheta(a, b))$$

\text{(2)} A metabelian JJ algebra has the derived algebra of dimension 1 if and only if it is isomorphic to $A_\vartheta$, for some vector space $A$ and $0 \neq \vartheta \in \text{Sym}(A \times A, k)$.

\text{(3)} Two JJ algebras $A_\vartheta$ and $A_{\vartheta'}$ are isomorphic if and only if the symmetric bilinear forms $\vartheta$ and $\vartheta'$ are homothetic, i.e. there exists a pair $(s_0, \psi) \in k^* \times \text{Aut}_k(A)$ such that $s_0 \vartheta(a, b) = \vartheta'(\psi(a), \psi(b))$, for all $a, b \in A$.

\textit{Proof.} As we already pointed out, since $\text{char}(k) \neq 3$ the only JJ algebra structure on the vector space $k$ is the abelian one. Thus $\mathbb{GH}^2(A_0, k) = \mathbb{H}^2(A_0, k_0)$. Now we apply Proposition 3.3 which gives a bijection $\mathcal{CS}(A_0, k) \cong \mathcal{CF}(A_0)$. Since $\text{char}(k) \neq 2$ any map $\lambda$ of a co-flag datum $(\lambda, \vartheta) \in \mathcal{CS}(A_0, k)$ is trivial, thanks to (21). It follows that the compatibility condition (22) is automatically fulfilled for any $\vartheta \in \text{Sym}(A \times A, k)$. Thus, we have proved that $\mathcal{CF}(A_0) \cong \text{Sym}(A \times A, k)$. The first part as well as the statement in (2) follow from here once we observe that the equivalence relation given by (25) of Proposition 3.5 becomes equality since $A$ is abelian. Finally, the statement in (3) follows from the classification result of Theorem 3.8 applied for $A_0$ and taking into account that in any co-flag datum of an abelian JJ algebra we have $\lambda = 0$. \hfill $\square$

\textbf{Remark 3.14.} The classification result established in Proposition 3.13 (3) reduces the classification problem of metabelian JJ algebras having the derived algebra of dimension 1 to the classification of symmetric forms on a vector space up to the homothetic equivalence relation. Now, any two isometric symmetric bilinear forms are homothetic just by taking $s_0 := 1$. Furthermore, if $k = k^2$ the converse is also true: that is, two symmetric bilinear forms are homothetic if and only if they are isometric. In this case, there is a well developed theory of symmetric bilinear forms [21]. For instance, if $k = \mathbb{C}$ the equivalence classes of the symmetric bilinear forms coincide with the set of all diagonal matrices with only 1s or 0s on the diagonal (the number of 0s is the dimension of the radical of the bilinear form). In the case that $k$ is a field with $k \neq k^2$ a new classification theory of bilinear forms up to the homothetic relation needs to be developed.
Example 3.15. Using Proposition 3.13 and the classification of all complex symmetric bilinear forms [21] we obtain the classification of all \((n+1)\)-dimensional metabelian JJ algebras having the derived algebra of dimension 1. Up to an isomorphism, there are precisely \(n\) such JJ algebras defined as follows. For any \(t = 1, \cdots, n\) we denote by \(J_t\) the JJ algebra having \(\{f, e_1, \cdots, e_n\}\) as a basis and the multiplication given by
\[
e_1 \cdot e_1 = e_2 \cdot e_2 = \cdots = e_t \cdot e_t := f
\]
A complex \((n+1)\)-dimensional JJ metabelian algebra \(A\) has the derived algebra of dimension 1 if and only if \(A \cong J_t\), for some \(t = 1, \cdots, n\). Adding to this family the abelian algebra \(k_{0}^{n+1}\) of dimension \(n+1\), we can conclude that \(\mathbb{CP}(k_{0}^{n}, k) = \{k_{0}^{n+1}, J_{1}, \cdots, J_{n}\}\).

Very interesting is the opposite case concerning the description of all JJ algebras \(A\) having an abelian derived algebra \(A'\) of codimension 1. This is just the Jacobi-Jordan version of the classification of Lie algebras of dimension \(n\) that have the Schur invariant equal to \(n-1\) - see [4, 7] for details. Let \(V\) be a vector space, \(f \in \text{End}_{k}(V)\) with \(f^2 = 0\) and \(v_0 \in \text{Ker}(f)\). Let \(V_{(f, v_0)} := k \times V\) with the multiplication defined for any \(p, q \in k\) and \(x, y \in V\) by:
\[
(p, x) \cdot (q, y) := (0, pf v_0 + pf(y) + qf(x))
\]
Then \(V_{(f, v_0)}\) is a JJ algebra as it can be seen by a direct computation as well as from the proof of our next result.

Proposition 3.16. Let \(k\) be a field of characteristic \(\neq 2, 3\). Then:
(1) A JJ algebra has an abelian derived algebra of codimension 1 if and only if it is isomorphic to a JJ algebra of the form \(V_{(f, v_0)}\), for some vector space \(V\), \(f \in \text{End}_{k}(V)\) with \(f^2 = 0\) and \(v_0 \in \text{Ker}(f)\) such that \((f, v_0) \neq (0, 0)\).
(2) For any vector space \(V\) there exists a bijection:
\[
\mathbb{H}^{2}(k_{0}, V_{0}) \cong \sqcup_{f} \text{Ker}(f)/\text{Im}(f)
\]
where the coproduct in the right hand side is in the category of sets over all possible linear endomorphisms \(f \in \text{End}_{k}(V)\) with \(f^2 = 0\). The bijection sends any element \(\overline{v_0} \in \text{Ker}(f)/\text{Im}(f)\) to the JJ algebra \(V_{(f, v_0)}\).

Proof. First of all we note that a JJ algebra has an abelian derived algebra of codimension 1 if and only if it is isomorphic to a crossed product of the form \(k_{0} \# V_{0}\), for some vector space \(V\). If \(A\) is such an algebra, we can take \(V := A'\), the derived algebra of \(A\). Hence, \(A/A' \cong k\), since \(A'\) has codimension 1 in \(A\) and thus it is an abelian algebra as \(\text{char}(k) \neq 3\). Using Proposition 2.4 we obtain that \(A \cong k_{0} \# V_{0}\). The converse is obvious. Thus, we have to describe the set of all pairs \((\triangleright, \triangleright)\) consisting of bilinear maps \(\triangleright : k \times V \to V\) and \(\triangleright : k \times k \to V\) such that \((\triangleright, \triangleright, \cdot : \cdot := 0)\) is a crossed system of \(k_{0}\) by \(V\), that is axioms (J1)-(J4) hold for \(\cdot : \cdot := 0\). Of course, bilinear maps \(\triangleright : k \times V \to V\) are in bijection with linear maps \(f \in \text{End}_{k}(V)\) via the two-sided formula \(1 \triangleright v := f(v)\) and bilinear maps \(\triangleright : k \times k \to V\) are in bijection with the set of all elements \(v_0\) of \(V\) via the two-sided formula \(\triangleright(a, b) := abv_0\), for all \(a, b \in k\). Now, axioms (J1) and (J3) are trivially fulfilled for \((\triangleright = \triangleright, \triangleright = \triangleright)\) and since \(\text{char}(k) \neq 2, 3\) we can easily see that axioms (J2) and respectively (J4) hold if and only if \(f^2 = 0\) and \(f(v_0) = 0\). This
proves the first statement: the condition \((f, v_0) \neq (0, 0)\) ensures that the algebra \(V_{(f, v_0)}\) is not abelian. The bijection given in (2) is obtained by applying Corollary 2.11 for the special case \(A := k_0\). We fix \(f \in \text{End}_k(V)\) with \(f^2 = 0\) - which is the same as fixing \(\triangleright\) in Corollary 2.11 via the above correspondence. Then, the equivalence relation given by (17) applied to the elements in \(\text{Ker}(f)\) becomes: \(v_0 \approx v_0'\) if and only if there exists an element \(\xi \in V\) such that \(v_0 - v_0' = 2f(\xi)\), that is \(v_0 - v_0' \in \text{Im}(f)\) as the characteristic of \(k\) is \(\neq 2\). This proves the second statement and the proof is now completely finished. \(\square\)

**Example 3.17.** By taking \(V := k^n\) in Proposition 3.16 we obtain the explicit description of all \((n+1)\)-dimensional metabelian JJ algebras having the derived algebra of dimension \(n\). Indeed, we denote by \(\mathcal{N}(n) := \{X \in M_n(k) \mid X^2 = 0\}\) and \(\text{Ker}(X) := \{v_0 \in k^n \mid Xv_0 = 0\}\), for all \(X \in \mathcal{N}(n)\). Then:

\[
\mathbb{P}^2(k_0, k^n_0) \cong \sqcup_{X \in \mathcal{N}(n)} \text{Ker}(f)/\equiv
\]

where \(v_0 \equiv v_0'\) if and only if there exists \(r \in k^n\) such that \(v_0' - v_0 = Xr\). For \(X = (x_{ij}) \in \mathcal{N}(n)\) and \(\tau_0 = (v_1, \ldots, v_n) \in \text{Ker}(X)/\equiv\), we denote by \(k^n_{(X, \tau_0)}\) the associated crossed product \(k\#k^n\). Then \(k^n_{(X, \tau_0)}\) is the JJ algebra having \(\{f, e_1, \cdots, e_n\}\) as a basis and multiplication defined for any \(i, j = 1, \cdots, n\) by:

\[
f \cdot f := \sum_{j=1}^n v_j e_j, \quad f \cdot e_i := \sum_{j=1}^n x_{ji} e_j
\]

Any \((n+1)\)-dimensional metabelian JJ algebra having the derived algebra of codimension 1 is isomorphic to such an algebra \(k^n_{(X, \tau_0)}\), for some \(X \in \mathcal{N}(n)\) and \(\tau_0 \in \text{Ker}(X)/\equiv\).

**References**

[1] Agore, A.L. and Militaru, G. - Hochschild products and global non-abelian cohomology for algebras. Applications, arXiv:1503.05364.
[2] Agore, A.L. and Militaru, G. - The global extension problem, crossed products and co-flag non-commutative Poisson algebras, J. Algebra, 426 (2015), 1–31.
[3] Albert, A.A. - On a certain algebra of quantum mechanics, Ann. of Math., 35 (1934), 65–73.
[4] Burde, D., Ceballos, M. - Abelian ideals of maximal dimension for solvable Lie algebras, J. Lie Theory 22 (2012), 741–756.
[5] Burde, D. and Fialowski, A. - Jacobi-Jordan Algebras, Linear Algebra and Appl., 459(2014), 586–594.
[6] Baumslag, G., Mikhailov, R., and Orr, K. E. - A new look at finitely generated metabelian groups, arXiv:1203.5431.
[7] Ceballos, M., Towers, D.A. - On abelian subalgebras and ideals of maximal dimension in supersolvable Lie algebras, J. Pure Appl. Algebra 218 (2014), 497–503.
[8] Chevalley, C. and Eilenberg S. - Cohomology theory of Lie groups and Lie algebras, Trans. Amer. Math. Soc., 63(1948), 85–124.
[9] Figueroa-Ofarrill, J. M., Stanciu, S. - On the structure of symmetric self-dual Lie algebras, J. Math. Phys. 37 (1996), 4121–4134.
[10] Frobenius, F. G. - Theorie der hyperkomplexen Größen I, Sitzungsberichte der Preussischen Akademie der Wissenschaften (in German), 504–537 (1903).
[11] Hochschild, G. - Cohomology and representations of associative algebras, Duke Math. J., 14(1947), 921–948.
[12] Horn, R. and Sergeichuk, V. - Canonical matrices of bilinear and sesquilinear forms, Linear Algebra and its Applications, 428 (2008), 193–223.
[13] Jordan, P., von Neumann, J. and Wigner, E. - On an Algebraic Generalization of the Quantum Mechanical Formalism, Ann. of Math., 35 (1934): 29-64.
[14] Kadison, L. - New Examples of Frobenius Extensions, Univ. Lect. Series 14 (1999), Amer. Math. Soc., Providence.
[15] Kath, I. and Olbrich, M. - Metric Lie algebras with maximal isotropic centre, Math. Z. 246 (2004) pp 23–53.
[16] Kock, J. - Frobenius algebras and 2D topological quantum field theories, London Math. Soc. 59 (2003), Cambridge University Press.
[17] Loday, J.-L. and Pirashvili, T. - Universal enveloping algebras of Leibniz algebras and (co)homology, Math. Ann., 296 (1993), 139–158.
[18] McCrimmon, K. - A taste of Jordan algebras, Universitext, Springer-Verlag, 2004.
[19] Medina, A., Revoy, Ph. - Algèbres de Lie et produit scalaire invariant, Ann. Sci. École Norm. Sup. 18 (1985), 553–561.
[20] Militaru, G. - The global extension problem, co-flag and metabelian Leibniz algebras, Linear Multilinear Algebra 63 (2015), 601–621.
[21] Milnor, J. and Husemoller, D. (1973) - Symmetric Bilinear Forms, Springer-Verlag, 73 (1973).
[22] Pelc, O. - A New Family of Solvable Self-Dual Lie Algebras, J.Math.Phys. 38 (1997), 3832–3840.
[23] Williamson, J. - On the algebraic problem concerning the normal forms of linear dynamical systems, Amer. J. Math., 58 (1936), 141–163.
[24] Wörz-Busekros, A. - Bernstein Algebras, Arch. Math., 48 (1987), 388-398.