Hom–Tensor Categories and the Hom–Yang–Baxter Equation

Florin Panaite · Paul T. Schrader · Mihai D. Staic

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Abstract
We introduce a new type of categorical object called a hom–tensor category and show that it provides the appropriate setting for modules over an arbitrary hom-bialgebra. Next we introduce the notion of hom-braided category and show that this is the right setting for modules over quasitriangular hom-bialgebras. We also show how the Hom–Yang–Baxter equation fits into this framework and how the category of Yetter–Drinfeld modules over a hom-bialgebra with bijective structure map can be organized as a hom-braided category. Finally we prove that, under certain conditions, one can obtain a tensor category (respectively a braided tensor category) from a hom–tensor category (respectively a hom-braided category).

Keywords  Hom-bialgebra · Tensor category · Yang–Baxter equation

Mathematics Subject Classification  Primary 18D10 · Secondary 16T05 · 17A99

1 Introduction

Tensor categories were introduced by Bénabou in [3]. A basic example is the category of vector spaces over a field $k$. More interesting examples can be obtained from bialgebras. If $A$ is an algebra and $\Delta : A \to A \otimes A$ is a morphism of algebras, then the category of $A$-modules is a tensor category (with the tensor product induced by $\Delta$ and trivial associativity constraint) if and only if $A$ is a bialgebra.
The Yang–Baxter equation was introduced by Yang and Baxter (see [2,24]). It has applications to knot invariants and it was intensively studied over the last thirty years.

Braided categories were introduced by Joyal and Street in [13]. The main example is the braid category; it satisfies a universal property for braided categories (see [15]). Other examples are obtained from quasitriangular Hopf algebras. Braided categories can be used to construct representations for the braid group and invariants for tangles, knots and 3-manifolds (see [23]). The braiding of a braided category satisfies a dodecagonal equation (see [15]) that may be regarded as a categorical analogue of the Yang–Baxter equation.

The genesis of hom-structures may be found in the physics literature from around the year 1990, concerning quantum deformations of algebras of vector fields, especially Witt and Virasoro algebras (e.g., see [1,6,8,9,14]). These classes of examples led to the development first of Hom–Lie algebras [11,16], which are analogues of Lie algebras where the Jacobi identity is twisted by a linear map. This was followed by the development of hom-analogues of associative algebras, coalgebras, bialgebras, Hopf algebras, etc. (e.g., see [4,5,7,10,12,17–22,25,27,28]). The reader can find a concise history on hom-structures in the introduction of [18].

One natural question to ask is what type of categorical framework these hom-structures fit into. In the original concept of hom-bialgebra (see [20,21]), two distinct linear maps twist the associative and co-associative structures of a bialgebra. When the two twisting maps are inverses to each other it was proved in [5] that the category of modules is a tensor category. Another question that may be asked is what kind of categorical framework does a hom-bialgebra where two arbitrary linear maps twist the associative and co-associative structure fit into? Moreover, is there an analogue to the classical relationship between quasitriangular bialgebras and braided tensor categories for quasitriangular hom-bialgebras? It is these questions that motivated this paper and the concepts it contains.

There are two main objectives to this paper. The first one is to introduce a hom-analogue to a tensor category, called a hom–tensor category. In a hom–tensor category \(C\) the usual associator is replaced by a natural isomorphism \(a_{U,V,W} : (U \otimes V) \otimes W \rightarrow F(U) \otimes (V \otimes W)\) that satisfies a generalized pentagonal equation (here \(F : C \rightarrow C\) is a functor; when \(F\) is the identity functor we recover the definition of a tensor category without unit). We show that the category of modules over a hom-bialgebra (as it is posed in [20,21]) fits in the categorical framework of hom–tensor categories.

The second objective is to introduce a hom-analogue to a braided tensor category, called a hom-braided category. In a hom-braided category \(C\) we have a natural morphism \(c_{U,V} : U \otimes V \rightarrow G(V) \otimes G(U)\) that satisfies a generalization of the hexagonal axioms (where \(G : C \rightarrow C\) is another functor). We show that this new categorical framework provides the right setting for modules over quasitriangular hom-bialgebras. We also show how the Hom–Yang–Baxter equation (introduced by Yau in [26]) fits in the context of hom-braided categories, and we prove that the category of Yetter–Drinfeld modules (introduced in [18]) over a hom-bialgebra with bijective structure map becomes a hom-braided category.

As applications to our theory, we give new proofs for Yau’s result from [29] saying roughly that a quasitriangular hom-bialgebra \(H\) provides a solution for the Hom–Yang–Baxter equation on any \(H\)-module and for the result in [18] saying that \(H\)-YD, the category of Yetter–Drinfeld modules \((M, \alpha_M)\) with \(\alpha_M\) bijective over a hom-bialgebra \(H\) with bijective structure map, is a quasi-braided category.

The structure of this paper is as follows. Section 2 begins with recalling some definitions and concepts of hom-structures necessary in presenting the upcoming results. We begin Sect. 3 by defining a hom–tensor category, and then we show how a hom–tensor category is the appropriate categorical framework for hom-bialgebras with arbitrary twisting maps.
Section 4 introduces the notions of algebras in a hom–tensor category, left $H$-module hom-algebras over a hom-bialgebra $H$ and a categorical analogue to a Yau twist. In Sect. 5 we define hom-braided categories and prove that they provide the right categorical framework for quasitriangular hom-bialgebras. In Sect. 6 we show how to regard the Hom–Yang–Baxter equation in the categorical framework of hom-braided categories. In Sect. 7 the category of Yetter–Drinfeld modules over a hom-bialgebra as seen in [18] is organized under the framework of a hom-braided category. Finally, in Sect. 8 we show that under certain conditions one can obtain a tensor category (respectively a braided tensor category) from a hom–tensor category (respectively a hom-braided category).

2 Preliminaries

We work over a base field $k$. An unlabeled tensor product means either a functor $\otimes : C \times C \to C$ on a category $C$ or the tensor product over $k$. For a comultiplication $\Delta : C \to C \otimes C$ on a $k$-vector space $C$ we use a Sweedler-type notation $\Delta(c) = \sum c_{(1)} \otimes c_{(2)}$, for $c \in C$. Unless otherwise specified, the (co)algebras ((co)associative or not) that will appear in what follows are not supposed to be (co)unital, and a multiplication $\mu : V \otimes V \to V$ on a $k$-vector space $V$ is denoted by juxtaposition: $\mu(v \otimes v') = vv'$.

We will use the following terminology for categories. A pre-tensor category is a category satisfying all the axioms of a tensor category in [15] except for the fact that we do not require the existence of a unit object. If $C$ is a pre-tensor category, a quasi-braiding $c$ in $C$ is a family of natural morphisms $c_{V,W} : V \otimes W \to W \otimes V$ in $C$ satisfying all the axioms of a braiding in [15] except for the fact that we do not require $c_{V,W}$ to be isomorphisms; in this case, $(C, \otimes, a, c)$ is called a quasi-braided pre-tensor category.

We recall now some definitions, notation and results taken from [19–21,25,28].

**Definition 2.1** A hom-associative $k$-algebra is a triple $(A, m_A, \alpha_A)$, where $A$ is a $k$-vector space, $m_A : A \otimes A \to A$ is a $k$-linear map denoted by $m_A(a \otimes b) = ab$, for all $a, b \in A$, and $\alpha_A : A \to A$ is a $k$-linear map satisfying the following conditions, for all $a, b, c \in A$:

\[
\begin{align*}
\alpha_A(ab) &= \alpha_A(a) \alpha_A(b), \\
\alpha_A(a)(bc) &= (ab)\alpha_A(c).
\end{align*}
\]

Let $(A, m_A, \alpha_A)$ and $(B, m_B, \alpha_B)$ be two hom-associative $k$-algebras. A morphism of hom-associative algebras $f : (A, m_A, \alpha_A) \to (B, m_B, \alpha_B)$ is a $k$-linear map $f : A \to B$ such that $\alpha_B \circ f = f \circ \alpha_A$ and $f \circ m_A = m_B \circ (f \otimes f)$.

**Remark 2.2** If $(A, m_A, \alpha_A)$, $(B, m_B, \alpha_B)$ are hom-associative $k$-algebras, then $(A \otimes B, m_{A \otimes B}, \alpha_{A \otimes B})$ is also a hom-associative $k$-algebra, where $m_{A \otimes B}((a \otimes b) \otimes (a' \otimes b')) = aa' \otimes bb'$ and $\alpha_{A \otimes B} = \alpha_A \otimes \alpha_B$.

**Definition 2.3** A hom-coassociative $k$-coalgebra is a triple $(C, \Delta_C, \psi_C)$, where $C$ is a $k$-vector space, $\Delta_C : C \to C \otimes C$ and $\psi_C : C \to C$ are $k$-linear maps satisfying the following conditions:

\[
\begin{align*}
(\psi_C \otimes \psi_C) \circ \Delta_C &= \Delta_C \circ \psi_C, \\
(\Delta_C \otimes \psi_C) \circ \Delta_C &= (\psi_C \otimes \Delta_C) \circ \Delta_C.
\end{align*}
\]

Let $(C, \Delta_C, \psi_C)$, $(D, \Delta_D, \psi_D)$ be two hom-coassociative $k$-coalgebras. A morphism of hom-coassociative $k$-coalgebras $g : (C, \Delta_C, \psi_C) \to (D, \Delta_D, \psi_D)$ is a $k$-linear map $g : C \to D$ such that $\psi_D \circ g = g \circ \psi_C$ and $(g \otimes g) \circ \Delta_C = \Delta_D \circ g$. 

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\]
Definition 2.4 A hom-bialgebra is a 5-tuple \((B, m_B, \Delta_B, \alpha_B, \psi_B)\), where \((B, m_B, \alpha_B)\) is a hom-associative \(k\)-algebra, \((B, \Delta_B, \psi_B)\) is a hom-coassociative \(k\)-coalgebra, \(\alpha_B\) is a morphism of hom-associative \(k\)-algebras, \(\alpha_B\) is a morphisms of hom-coassociative \(k\)-coalgebras and \(\psi_B\) is a morphism of hom-associative \(k\)-algebras (in particular we have \(\alpha_B \circ \psi_B = \psi_B \circ \alpha_B\)).

Remark 2.5 The following statement is equivalent to Definition 2.4. A hom-bialgebra is a hom-associative \(k\)-algebra \((B, m_B, \alpha_B)\) together with two \(k\)-linear maps \(\Delta_B : B \rightarrow B \otimes B\) and \(\psi_B : B \rightarrow B\) such that \(\alpha_B \circ \psi_B = \psi_B \circ \alpha_B\) and the following conditions are satisfied for all \(b, b' \in B\):

\[
\sum b_{(1)(1)} \otimes b_{(1)(2)} \otimes \psi_B(b_{(2)}) = \sum \psi_B(b_{(1)}) \otimes b_{(2)(1)} \otimes b_{(2)(2)}, \\
\sum (bb')_{(1)} \otimes (bb')_{(2)} = \sum b_{(1)}b'_{(1)} \otimes b_{(2)}b'_{(2)}, \\
\sum \alpha_B(b_{(1)}) \otimes \alpha_B(b_{(2)}) = \sum \alpha_B(b_{(1)}) \otimes \alpha_B(b_{(2)}), \\
\sum \psi_B(b_{(1)}) \otimes \psi_B(b_{(2)}) = \sum \psi_B(b_{(1)}) \otimes \psi_B(b_{(2)}), \\
\psi_B(bb') = \psi_B(b)\psi_B(b').
\]

Remark 2.6 In the literature, most of the results about hom-bialgebras use the extra assumption that \(\psi_B = \alpha_B\) or \(\psi_B = \alpha_B^{-1}\) (see \([5,18,29]\)). We treat the general situation, to cover both cases of interest.

We recall now the so-called "twisting principle" or "Yau twisting".

Proposition 2.7 Let \((A, \mu)\) be an associative \(k\)-algebra and \(\alpha : A \rightarrow A\) an algebra endomorphism. Define a new multiplication \(\mu_\alpha : A \otimes A \rightarrow A, \mu_\alpha := \alpha \circ \mu = \mu \circ (\alpha \otimes \alpha)\). Then \((A, \mu_\alpha, \alpha)\) is a hom-associative \(k\)-algebra, denoted by \(A_\alpha\) and called the Yau twist of \(A\).

Definition 2.8 Let \(M\) be a \(k\)-vector space, \((A, m_A, \alpha_A)\) be a hom-associative \(k\)-algebra and \(\alpha_M : M \rightarrow M\) be a \(k\)-linear map. A left \(A\)-module structure on \((M, \alpha_M)\) consists of a \(k\)-linear map \(\mu_M : A \otimes M \rightarrow M\), with notation \(\mu_M(a \otimes m) = a \cdot m\), such that the following conditions are satisfied for all \(a, b \in A\) and \(m \in M\):

\[
\alpha_M(a \cdot m) = \alpha_A(a) \cdot \alpha_M(m), \\
\alpha_A(a) \cdot (b \cdot m) = (ab) \cdot \alpha_M(m).
\]

Let \((M, \alpha_M)\) and \((N, \alpha_N)\) be two left \(A\)-modules. A morphism of left \(A\)-modules is a \(k\)-linear map \(f : M \rightarrow N\) satisfying the conditions \(\alpha_N \circ f = f \circ \alpha_M\) and \(f(a \cdot m) = a \cdot f(m)\) for all \(a \in A, m \in M\).

Definition 2.9 Let \((C, \Delta_C, \psi_C)\) be a hom-coassociative \(k\)-coalgebra, \(M\) a \(k\)-vector space and \(\psi_M : M \rightarrow M\) a \(k\)-linear map. A left \(C\)-comodule structure on \((M, \psi_M)\) consists of a \(k\)-linear map \(\lambda_M : M \rightarrow C \otimes M\) (usually denoted by \(\lambda_M(m) = \sum m_{(-1)} \otimes m_{(0)}\), for all \(m \in M\), satisfying the following conditions:

\[
(\psi_C \otimes \psi_M) \circ \lambda_M = \lambda_M \circ \psi_M, \\
(\Delta_C \otimes \psi_M) \circ \lambda_M = (\psi_C \otimes \lambda_M) \circ \lambda_M.
\]

If \((M, \psi_M)\) and \((N, \psi_N)\) are left \(C\)-comodules, with structures \(\lambda_M : M \rightarrow C \otimes M\) and \(\lambda_N : N \rightarrow C \otimes N\), a morphism of left \(C\)-comodules \(g : M \rightarrow N\) is a \(k\)-linear map satisfying the conditions \(\psi_N \circ g = g \circ \psi_M\) and \((id_C \otimes g) \circ \lambda_M = \lambda_N \circ g\).
We define general quasitriangular hom-bialgebras (see [27,29] for the case $\alpha = \psi$).

**Definition 2.10** Let $(H, m, \Delta, \alpha, \psi)$ be a hom-bialgebra and let $R \in H \otimes H$ be given as $R = \sum_i s_i \otimes t_i$. We call $(H, m, \Delta, \alpha, \psi, R)$ a quasitriangular hom-bialgebra if the following conditions are satisfied:

\[
R \Delta (h) = \Delta^{cop}(h) R, \quad \text{for all } h \in H, \tag{14}
\]

\[
(\Delta \otimes \alpha)(R) = \sum_{i,j} \psi(s_i) \otimes \psi(s_j) \otimes t_i t_j, \tag{15}
\]

\[
(\alpha \otimes \Delta)(R) = \sum_{i,j} s_i s_j \otimes \psi(t_j) \otimes \psi(t_i), \tag{16}
\]

where we denoted as usual $\Delta^{cop}(h) = \sum h(2) \otimes h(1)$, for $h \in H$.

**Remark 2.11** Let $H = (H, m, \Delta, \alpha, \psi, R)$ be a quasitriangular hom-bialgebra and $h \in H$. We can reformulate conditions (14), (15) and (16) respectively in Definition 2.10 using Sweedler notation as follows:

\[
\sum_i s_i h_{(1)} \otimes t_i h_{(2)} = \sum_i h_{(2)} s_i \otimes h_{(1)} t_i, \tag{17}
\]

\[
\sum_i (s_i)_{(1)} \otimes (s_i)_{(2)} \otimes \alpha(t_i) = \sum_{i,j} \psi(s_i) \otimes \psi(s_j) \otimes t_i t_j, \tag{18}
\]

\[
\sum_i \alpha(s_i) \otimes (t_i)_{(1)} \otimes (t_i)_{(2)} = \sum_{i,j} s_i s_j \otimes \psi(t_j) \otimes \psi(t_i). \tag{19}
\]

**Remark 2.12** Notice that if $(\psi \otimes \psi)(R) = R$ then conditions (15) and (16) are equivalent to

\[
(\Delta \otimes (\alpha \circ \psi))(R) = \sum_{i,j} s_i \otimes s_j \otimes t_i t_j, \tag{20}
\]

\[
((\alpha \circ \psi) \otimes \Delta)(R) = \sum_{i,j} s_i s_j \otimes t_j \otimes t_i. \tag{21}
\]

We introduce now the following concept, to be used in subsequent sections.

**Definition 2.13** Let $A = (A, m_A, \alpha_A)$ be a hom-associative $k$-algebra.

(i) Suppose that $h \cdot m = 0$ for any $A$-module $(M, \alpha_M)$ and for all $m \in M$ implies $h = 0$. Then we say that $A$ is nondegenerate.

(ii) Suppose that $h \cdot m = 0$ for any $A$-module $(M, \alpha_M)$ and for all $m \in \alpha_M(M)$ implies $h = 0$. Then we say that $A$ is strongly nondegenerate.

**Lemma 2.14** Let $A$ be a nondegenerate hom-associative $k$-algebra and $x \in A \otimes A$ such that $x \cdot (u \otimes v) = 0$ for all $u \in U$ and all $v \in V$ and for all left $A$-modules $(U, \alpha_U)$ and $(V, \alpha_V)$. Then $x = 0$. A similar result is true for $y \in A^{\otimes 3}$. Similar results are true for strongly nondegenerate algebras.

**Proof** Let $x = \sum_{p=1}^n a_p \otimes b_p$, where $a_p, b_p \in A$ and $\{b_p\}_{1 \leq p \leq n}$ are $k$-linearly independent. Fix a left $A$-module $(U, \alpha_U)$ and fix a $k$-basis $\{e_i\}_{i \in I}$ for $U$. Consider the set of $k$-linear applications $e_i^* \in U^*$ determined by $e_i^*(e_j) = \delta^j_i$. For every left $A$-module $(V, \alpha_V)$ and for every $u \in U$, $v \in V$ and $i \in I$ we have:

\[
0 = (e_i^* \otimes \text{id}_V)(x \cdot (u \otimes v)) = (e_i^* \otimes \text{id}_V)(\sum_{p=1}^n (a_p \otimes b_p) \cdot (u \otimes v)) = \sum_{p=1}^n e_i^*(a_p \cdot u)b_p \cdot v.
\]
Since $A$ is nondegenerate we get that for every $i \in I$ and every $u \in U$ we have that
\[ \sum_{p=1}^{n} e_i^*(a_p \cdot u) b_p = 0 \in A. \] But $\{b_p\}_{1 \leq p \leq n}$ are linearly independent, so for every $1 \leq p \leq n$ we have $e_i^*(a_p \cdot u) = 0$ for all $i \in I$ and for all $u \in U$. Now since $\{e_i\}_{i \in I}$ is a $k$-basis for $U$ we must have that $a_p \cdot u = 0$ for all $1 \leq p \leq n$ and for all $u \in U$. But $U$ can be any $A$-module and $A$ is nondegenerate which implies that $a_p = 0$ for all $1 \leq p \leq n$ and so $x = \sum_{p=1}^{n} a_p \otimes b_p = 0.$

\section{3 Hom–Tensor Categories}

We introduce a new type of categories called \textit{hom–tensor categories}, which have a tensor functor $\otimes : C \times C \to C$ with the usual associativity condition replaced by a more relaxed condition (see Definition 3.1). We show that hom-bialgebras fit very nicely in this framework. Unlike the tensor category introduced in [5], a hom–tensor category can be associated even to hom-bialgebras for which $\alpha_A$ is not necessary bijective.

\textbf{Definition 3.1} A \textit{hom–tensor category} is a 6-tuple $(C, \otimes, F, G, a, \Phi)$, where:

1. $C$ is a category.
2. $\otimes : C \times C \to C$ is a covariant functor (called the \textit{hom–tensor product}).
3. $F : C \to C$ is a covariant functor such that $F(U \otimes V) = F(U) \otimes F(V)$ for all objects $U, V \in C$ and $F(f \otimes g) = F(f) \otimes F(g)$ for all morphisms $f, g \in \text{Hom}(C)$.
4. $a_{X,Y,Z} : (X \otimes Y) \otimes F(Z) \to F(X) \otimes (Y \otimes Z)$ is a natural isomorphism that satisfies the “Pentagon” axiom as seen in Fig. 1 for all objects $X, Y, Z, T \in C$. We call $a$ the \textit{hom-associativity constraint} of the hom–tensor category.
5. $G : C \to C$ is a covariant functor such that $G(U \otimes V) = G(U) \otimes G(V)$ for all objects $U, V \in C$ and $G(f \otimes g) = G(f) \otimes G(g)$ for all morphisms $f, g \in \text{Hom}(C)$.
6. There exists a natural transformation $\Phi : \text{id}_C \to G$.
7. $FG = GF$.
8. $F(\Phi_U) = \Phi_{F(U)}$, $G(\Phi_U) = \Phi_{G(U)}$, for every object $U \in C$.
9. $\Phi_{M \otimes N} = \Phi_M \otimes \Phi_N$, for all objects $M, N \in C$.

\begin{figure}
\centering
\begin{tikzpicture}
  \node (X) at (0,0) {$(F(X) \otimes F(Y)) \otimes (F(Z) \otimes T)$};
  \node (Y) at (-3,-3) {$(X \otimes Y) \otimes F(Z) \otimes T$};
  \node (Z) at (3,-3) {$F(X) \otimes (Y \otimes Z) \otimes F(T)$};
  \node (A) at (0,-6) {$(F(X) \otimes (Y \otimes Z)) \otimes F^2(T)$};
  \node (B) at (0,-9) {$F^2(X) \otimes ((Y \otimes Z) \otimes F(T))$};
  \draw[->] (X) -- (Y) node[above] {$a_{X,Y,F(Z),F(T)}$};
  \draw[->] (X) -- (Z) node[above] {$a_{F(X),F(Y),Z\otimes T}$};
  \draw[->] (Y) -- (A) node[above] {$a_{X,Y,Z,F(T)} \otimes \text{id}_{F^2(T)}$};
  \draw[->] (Z) -- (B) node[above] {$a_{F(X),Y\otimes F(T)} \otimes \text{id}_{F^2(T)}$};
  \draw[->] (A) -- (B) node[above] {$a_{F(X),Y\otimes F(T)} \otimes \text{id}_{F^2(T)}$};
\end{tikzpicture}
\caption{The “Pentagon” axiom for the hom-associativity constraint $a$}
\end{figure}
Remark 3.2 Note that we do not require the existence of a unit in the category, and so a more appropriate name for the structure we defined would be hom-pre-tensor category. However, in order to simplify the terminology, we prefer to call it hom–tensor category.

Remark 3.3 One may relax the above definition, by removing some of the axioms. For instance, one may remove the condition $\Phi_{M \otimes N} = \Phi_M \otimes \Phi_N$, which is used later in only one place, in the last section. The importance of the functor $G$ will become apparent later, when we talk about hom-braided categories.

We present now a first class of examples of hom–tensor categories.

Proposition 3.4 Let $(C, \otimes, a)$ be a pre-tensor category. We define the category $h(C)$ as follows: objects are pairs $(M, \alpha_M)$, where $M$ is an object in $C$ and $\alpha_M \in \text{Hom}_C(M, M)$, morphisms $f : (M, \alpha_M) \to (N, \alpha_N)$ are morphisms $f : M \to N$ in $C$ such that $\alpha_N \circ f = f \circ \alpha_M$. Then $(h(C), \otimes, F, G, a, \Phi)$ is a hom–tensor category, where the tensor product $\otimes$ is defined by $(M, \alpha_M) \otimes (N, \alpha_N) = (M \otimes N, \alpha_M \otimes \alpha_N)$ on objects and by the tensor product in $C$ on morphisms, the functors $F$ and $G$ are both identity, the natural transformation $\Phi_{(M, \alpha_M)} : (M, \alpha_M) \to (M, \alpha_M)$ is defined by $\Phi_{(M, \alpha_M)} := \alpha_M$, and the natural isomorphism $a$ is defined by $a_{(M, \alpha_M), (N, \alpha_N), (P, \alpha_P)} := \alpha_{M, N, P}$.

Proof A straightforward verification. \qed

The following proposition gives the relation between hom–tensor categories and hombialgebras.

Proposition 3.5 Let $H = (H, m_H, \alpha_H)$ be a hom-associative $k$-algebra and let $\Delta_H : H \to H \otimes H$, $\psi_H : H \to H$ be morphisms of hom-associative $k$-algebras. Consider the two statements (A) and (B) below. Then we have that (A) implies (B) and if $H$ is nondegenerate (B) implies (A).

(A) $(H, m_H, \Delta_H, \alpha_H, \psi_H)$ is a hom-bialgebra.

(B) The category $\mathcal{H} = (\text{H-mod}, \otimes, F, G, a, \Phi)$ is a hom–tensor category, where:

(i) The objects of $\mathcal{H}$ are left $H$-modules.

(ii) The morphisms of $\mathcal{H}$ are left $H$-module morphisms.

(iii) The hom–tensor product of $(U, \alpha_U)$ and $(V, \alpha_V)$ is given by $(U, \alpha_U) \otimes (V, \alpha_V) := (U \otimes V, \alpha_{U \otimes V})$, where $\alpha_{U \otimes V} := \alpha_U \otimes \alpha_V$ and the left $H$-action $H \otimes (U \otimes V) \to U \otimes V$ is defined for all elements $u \in U$, $v \in V$ and $h \in H$ by $h \cdot (u \otimes v) := \Delta(h) \cdot (u \otimes v) = \sum (h_{(1)} \cdot u) \otimes (h_{(2)} \cdot v)$. If $f \in \text{Hom}_{\text{H-mod}}(U, W)$ and $g \in \text{Hom}_{\text{H-mod}}(V, X)$ then $f \otimes g : U \otimes V \to W \otimes X$ is defined by $(f \otimes g)(u \otimes v) := f(u) \otimes g(v)$, for all $u \in U$, $v \in V$.

(iv) $F : \text{H-mod} \to \text{H-mod}$ is the covariant functor defined by $F((U, \alpha_U)) = (U^\psi, \alpha_{U^\psi})$, where $U^\psi = U$ as a $k$-vector space (we will denote an element of $U^\psi = U$ as $\bar{u}$), and the $H$-module structure $H \otimes U^\psi \to U^\psi$ is given by $h \cdot \bar{u} = \overline{\psi_H(h) \cdot u}$, for all $h \in H$, $u \in U$. Furthermore, the $k$-linear map $\alpha_{U^\psi} : U^\psi \to U^\psi$ is defined by $\alpha_{U^\psi} (\bar{u}) = \overline{\alpha_U(u)}$ for all $\bar{u} \in U^\psi$. For all morphisms $f : U \to V$ we have $F(f)(\bar{u}) = \bar{f}(u)$ for all $\bar{u} \in U^\psi$.

(v) The hom-associativity constraint $\alpha_{U,V,W} : (U \otimes V) \otimes W \to U^\psi \otimes (V \otimes W)$ is defined by the natural isomorphism $\alpha_{U,V,W}((u \otimes v) \otimes \bar{w}) = \bar{u} \otimes (v \otimes \bar{w})$, for all $u \in U$, $v \in V$ and $\bar{w} \in U^\psi$ such that $(U, \alpha_U)$, $(V, \alpha_V)$ and $(W, \alpha_W)$ are objects in $\text{H-mod}$. 
(vi) \( G : H\text{-mod} \to H\text{-mod} \) is the covariant functor defined by \( G((U, \alpha_U)) = (U^\alpha, \alpha_U^\alpha) \), where \( U^\alpha = U \) as a \( k \)-vector space (we will denote an element of \( U^\alpha = U \) as \( \tilde{u} \)). and the \( H \)-module structure \( H \otimes U^\alpha \to U^\alpha \) is given by
\[
h \cdot_{\alpha} \tilde{u} = \alpha_H(h) \cdot u, \quad \text{for all } h \in H, \ u \in U.
\]
Furthermore, the \( k \)-linear map \( \alpha_{U^\alpha} : U^\alpha \to U^\alpha \) is defined by \( \alpha_{U^\alpha}(\tilde{u}) = \alpha_U(u) \) for all \( \tilde{u} \in U^\alpha \). For all morphisms \( f : U \to V \) we have \( G(f)(\tilde{u}) = \tilde{f}(u) \) for all \( \tilde{u} \in U^\alpha \).

### Remark 3.6

Notice that as \( k \)-vector spaces we have \( U^\psi \otimes V^\psi = U \otimes V = (U \otimes V)^\psi \).

This allows us to identify \( \tilde{u} \otimes \tilde{v} \in U^\psi \otimes V^\psi \) with \( \tilde{u} \otimes \tilde{v} \in (U \otimes V)^\psi \). Also notice that if \( \tilde{u}_1 = \tilde{u}_2 \in U^\psi \) then \( u_1 = u_2 \in U \). A similar statement is true for \( U^\alpha \).

### Proof

\( (A) \Rightarrow (B) \) Suppose that \( H = (H, m_H, \Delta_H, \alpha_H, \psi_H) \) is a hom-bialgebra. To begin, we want to show that the tensor product is well-defined in \( \mathcal{H} \). Let \( (U, \alpha_U), (V, \alpha_V) \in \text{Ob}(\mathcal{H}) \). Our first claim is that \( (U \otimes V, \alpha_{U \otimes V}) \in \text{Ob}(\mathcal{H}) \). We need to check that both conditions (10) and (11) hold for \( U \otimes V \).

When using a certain property to establish a particular equality within the following computations, we will indicate that property above the corresponding equal sign. So, suppose that \( h \in H, u \in U \) and \( v \in V \). Then checking for property (10) of \( U \otimes V \) results in
\[
(\alpha_{U \otimes V})(h \cdot (u \otimes v)) = \sum (\alpha_U(h_1) \cdot u) \otimes (\alpha_V(h_2) \cdot v)
\]

Next, suppose that \( h, h' \in H, u \in U \) and \( v \in V \). Checking for condition (11) of \( U \otimes V \) results in
\[
\alpha_H(h) \cdot (h' \cdot (u \otimes v)) = \sum (\alpha_H(h)(h_1')(u) \otimes (\alpha_V(h_2')(v))
\]

So condition (11) holds for \( U \otimes V \). Thus \( (U \otimes V, \alpha_{U \otimes V}) \in \text{Ob}(\mathcal{H}) \).
Let \((U, \alpha_U), (V, \alpha_V), (U', \alpha_{U'}) \in \text{Ob}(\mathcal{H}), f \in \text{Hom}_{\mathcal{H}}(U, U')\) and \(f' \in \text{Hom}_{\mathcal{H}}(V, V')\). We claim that \(f \otimes f' \in \text{Hom}_{\mathcal{H}}(U \otimes V, U' \otimes V')\). First we check that \(\alpha_{U \otimes V} f (f \otimes f') = (f \otimes f')(\alpha_{U \otimes V})\):

\[
\begin{align*}
(\alpha_{U \otimes V}) (f \otimes f') (u \otimes v) &= (\alpha_{U'} \otimes \alpha_{V'})(f (u) \otimes f'(v)) = \alpha_{U'} (f (u)) \otimes \alpha_{V'} (f'(v)), \\
(f \otimes f') (\alpha_{U \otimes V}) (u \otimes v) &= (f \otimes f') (\alpha_{U} \otimes \alpha_{V}) (u \otimes v) = f (\alpha_{U} (u)) \otimes f' (\alpha_{V} (v)).
\end{align*}
\]

Since \(f \in \text{Hom}_{\mathcal{H}}(U, U')\) and \(f' \in \text{Hom}_{\mathcal{H}}(V, V')\) we have that \(\alpha_{U'} (f (u)) = f (\alpha_{U} (u))\) and \(\alpha_{V'} (f'(v)) = f' (\alpha_{V} (v))\) for all \(u \in U, v \in V\). Thus the two expressions above are equal.

Next, for \(u \in U, v \in V, h \in H\), we check that \((f \otimes f') (h \cdot (u \otimes v)) = h \cdot (f \otimes f') (u \otimes v)\):

\[
(f \otimes f') (h \cdot (u \otimes v)) = (f \otimes f') \left( \sum (h_{(1)} \cdot u) \otimes (h_{(2)} \cdot v) \right)
= \sum f (h_{(1)} \cdot u) \otimes f' (h_{(2)} \cdot v)
= \sum (h_{(1)} \cdot f (u)) \otimes (h_{(2)} \cdot f'(v))
= h \cdot (f (u) \otimes f'(v)) = h \cdot (f \otimes f') (u \otimes v).
\]

Therefore, \(f \otimes f' \in \text{Hom}_{\mathcal{H}}(U \otimes V, U' \otimes V')\).

Now that the hom–tensor product is well-defined for objects and morphisms in \(\mathcal{H}\) one can easily show that the remaining properties for being functorial are satisfied. That is, if we let \(f \in \text{Hom}_{H \text{-mod}}(S, T), f' \in \text{Hom}_{H \text{-mod}}(T, W), g \in \text{Hom}_{H \text{-mod}}(U, V)\) and \(g' \in \text{Hom}_{H \text{-mod}}(V, X)\) then \((f' \otimes g') \circ (f \otimes g) = (f' \circ f) \otimes (g' \circ g)\).

Furthermore, for any \((U, \alpha_U), (V, \alpha_V) \in \text{Ob}(H \text{-mod})\), we have that \(\text{id}_{U \otimes V} = \text{id}_U \otimes \text{id}_V\).

Thus, the hom–tensor product \(\otimes : H \text{-mod} \times H \text{-mod} \to H \text{-mod}\) of \(\mathcal{H}\) is a covariant functor.

Next we will show that \(F : \mathcal{H} \to \mathcal{H}\) is a covariant functor. We want to show for any object \((U, \alpha_U)\) in \(H \text{-mod}\) that \((U^\Psi, \alpha_U \psi)\) is an \(H\)-module. Let \(h \in H\) and \(\bar{u} \in U^\Psi\). We check condition (10):

\[
\alpha_U \psi (h \cdot \psi \bar{u}) = \alpha_U \psi (\psi_H (h) \cdot u) = \alpha_U (\psi_H (h) \cdot u)
= \alpha_H (\psi_H (h)) \cdot \alpha_U (u) = \psi_H (\alpha_H (h)) \cdot \alpha_U (u)
= \alpha_H (h) \cdot \psi \alpha_U (\bar{u}) = \alpha_H (h) \cdot \psi \alpha_U \psi (\bar{u}).
\]

Let \(h, h' \in H\) and \(u \in U\) where \((U, \alpha_U)\) is an object in \(H \text{-mod}\). Then checking condition (11) results in

\[
\alpha_H (h) \cdot \psi (h' \cdot \psi \bar{u}) = \alpha_H (h) \cdot \psi (\psi_H (h') \cdot u)
= \alpha_H (\psi_H (h)) \cdot (\psi_H (h') \cdot u) = \alpha_H (\psi_H (h)) \cdot (\psi_H (h') \cdot u)
= \psi_H ((hh') \cdot \alpha_U (u))
= (hh') \cdot \psi \alpha_U (\bar{u}) = (hh') \cdot \psi \alpha_U \psi (\bar{u}).
\]

Thus, \((U^\Psi, \alpha_U \psi)\) is an object in \(\mathcal{H}\).
Next we claim that $F$ maps morphisms to morphisms in $\mathcal{H}$. Let $f \in \text{Hom}_{H\text{-mod}}(V, W)$ and $\tilde{v} \in V^\psi$. We will prove our claim by checking the following two properties:

\begin{equation}
(\alpha_{W^\psi} \circ F(f))(\tilde{v}) = (F(f) \circ \alpha_{V^\psi})(\tilde{v}),
\end{equation}

\begin{equation}
F(f)(h \cdot_{\psi} \tilde{v}) = h \cdot_{\psi} F(f)(\tilde{v}).
\end{equation}

Relation (22) is a consequence of the following computations:

\begin{align*}
(\alpha_{W^\psi} \circ F(f))(\tilde{v}) &= \alpha_{W^\psi} \circ F(f)(\tilde{v}) = \alpha_{W^\psi}(\bar{f}(v)) = \alpha_W(f(v)), \\
(F(f) \circ \alpha_{V^\psi})(\tilde{v}) &= F(f)(\alpha_{V^\psi}(\tilde{v})) = F(f)(\bar{f}(v)) = \bar{f}(\alpha_{V^\psi}(v)) = \alpha_W(f(v)).
\end{align*}

Relation (23) holds by the following sequence of equalities:

\begin{align*}
F(f)(h \cdot_{\psi} \tilde{v}) &= F(f)(\psi_H(h) \cdot v) = \bar{f}(\psi_H(h) \cdot f(v)) \\
&= h \cdot_{\psi} \bar{f}(v) = h \cdot_{\psi} F(f)(\tilde{v}).
\end{align*}

The fact that $F$ preserves compositions of morphisms and the identity morphism in $H\text{-mod}$ is obvious. Therefore, $F$ is indeed a covariant functor.

Next we need to check that $F((U, \alpha_U) \otimes (V, \alpha_V)) = F((U, \alpha_U)) \otimes F((V, \alpha_V))$, for all $(U, \alpha_U), (V, \alpha_V) \in \text{Ob}(H\text{-mod})$. As noticed in Remark 3.6, we already have that identification at the level of $\text{Ob}$ of $H\text{-mod}$ is preserved, too. Let $u \in U, v \in V, h \in H$. Then:

\begin{equation}
F((U, \alpha_U) \otimes (V, \alpha_V)) = (U \otimes V, \alpha_{U \otimes V}) = \left( (U \otimes V)^\psi, \alpha_{(U \otimes V)^\psi} \right),
\end{equation}

with left $H$-action given by

\begin{align*}
h \cdot_{\psi} (u \otimes v) &= \psi_H(h) \cdot (u \otimes v),
\end{align*}

and

\begin{align*}
F((U, \alpha_U)) \otimes F((V, \alpha_V)) &= (U^\psi, \alpha_{U^\psi}) \otimes (V^\psi, \alpha_{V^\psi}) \\
&= (U^\psi \otimes V^\psi, \alpha_{U^\psi \otimes V^\psi}) = (U^\psi \otimes V^\psi, \alpha_{U^\psi} \otimes \alpha_{V^\psi}),
\end{align*}

with left $H$-action given by

\begin{align*}
h \cdot_{\psi} (\tilde{u} \otimes \tilde{v}) &= \sum h(1) \cdot_{\psi} \tilde{u} \otimes h(2) \cdot_{\psi} \tilde{v} = \sum \psi_H(h(1)) \cdot u \otimes \psi_H(h(2)) \cdot v.
\end{align*}

We prove that the two left $H$-actions coincide:

\begin{align*}
h \cdot_{\psi} (u \otimes v) &= \psi_H(h) \cdot (u \otimes v) \\
&= \sum (\psi_H(h))(1) \cdot u \otimes (\psi_H(h))(2) \cdot v \\
&= \sum \psi_H(h(1)) \cdot u \otimes \psi_H(h(2)) \cdot v \\
&= \sum h(1) \cdot_{\psi} \tilde{u} \otimes h(2) \cdot_{\psi} \tilde{v} = h \cdot (\tilde{u} \otimes \tilde{v}), \quad q.e.d.
\end{align*}

Remark 3.6: The fact that $\alpha_{(U \otimes V)^\psi} = \alpha_{U^\psi} \otimes \alpha_{V^\psi}$ is obvious. Additionally, by the definition of $F$ one can easily show that $F(f \otimes f') = F(f) \otimes F(f')$ for any morphisms $f, f'$ in $H\text{-mod}$.

Next we need to check that

\begin{equation}
a_{U^\psi, V^\psi, W \otimes X} \circ a_{U \otimes V, W^\psi, X^\psi} = \left( \text{id}_{U^\psi \otimes W, X} \otimes a_{V, W, X} \right) \circ \left( a_{U, V, W} \otimes \text{id}_{X^\psi} \right).
\end{equation}
Thus, the hom-associativity constraint $a$ satisfies the “Pentagon” axiom from Fig. 1. Next we show that the constraint map $a$ is a morphism in $\mathcal{H}$. First, we will check that

$$
\alpha_{U^\psi \otimes (V \otimes W)} \circ a_{U^\psi, V^\psi, W^\psi} (u \otimes v) \otimes \bar{w}) = a_{U^\psi, V^\psi, W^\psi} (u \otimes v) \otimes \bar{w})
$$

for all $u \in U$, $v \in V$ and $w \in W$, where $(U, \alpha_U)$, $(V, \alpha_V)$ and $(W, \alpha_W)$ are objects in $\mathcal{H}$. We compute:

$$
\alpha_{U^\psi \otimes (V \otimes W)} \circ a_{U^\psi, V^\psi, W^\psi} (u \otimes v) \otimes \bar{w}) \overset{(v)}{=} a_{U^\psi \otimes (V \otimes W)} (\bar{u} \otimes (v \otimes w))
$$

$$
= a_{U^\psi} (\bar{u}) \otimes (v \otimes w)
$$

$$
= a_{U^\psi} (\bar{u}) \otimes (\alpha_V (v) \otimes \alpha_W (w))
$$

$$
\overset{(iv)}{=} a_U (u) \otimes (\alpha_V (v) \otimes \alpha_W (w)),
$$

$$
a_{U^\psi, V^\psi, W^\psi} (u \otimes v) \otimes \bar{w}) = a_{U^\psi, V^\psi, W^\psi} ((\alpha_{U} (u) \otimes \alpha_V (v) \otimes \alpha_W (w))
$$

$$
\overset{(iv)}{=} a_U (u) \otimes (\alpha_V (v) \otimes \alpha_W (w))
$$

So (25) holds. Next we show that $a_{U^\psi, V^\psi, W^\psi} (h \cdot ((u \otimes v) \otimes \bar{w})) = a_{U^\psi, V^\psi, W^\psi} ((u \otimes v) \otimes \bar{w}),$ for all $h \in H, u \in U, v \in V$ and $w \in W$, where $(U, \alpha_U)$, $(V, \alpha_V)$ and $(W, \alpha_W)$ are objects in $\mathcal{H}:

$$
a_{U^\psi, V^\psi, W^\psi} (h \cdot ((u \otimes v) \otimes \bar{w})) \overset{(iii)}{=} a_{U^\psi, V^\psi, W^\psi} \left( \sum (h_{(1)} \cdot (u \otimes v)) \otimes (h_{(2)} \cdot \psi \bar{w}) \right)
$$

$$
\overset{(iii),(iv)}{=} a_{U^\psi, V^\psi, W^\psi} \left( \sum (h_{(1)} \cdot u) \otimes (h_{(1)} \cdot v) \otimes (\psi_H (h_{(2)})) \cdot \bar{w}) \right)
$$

$$
= a_{U^\psi, V^\psi, W^\psi} \left( \sum (h_{(1)} \cdot u) \otimes (h_{(1)} \cdot v) \otimes (\psi_H (h_{(2)})) \cdot \bar{w}) \right)
$$

$$
= \sum \psi_H (h_{(1)}) \cdot u \otimes \left( h_{(1)} \cdot v \otimes (\psi_H (h_{(2)})) \cdot \bar{w}) \right)
$$

$$
= \sum (\psi_H (h_{(1)})) \cdot u \otimes (h_{(1)} \cdot v) \otimes (\psi_H (h_{(2)})) \otimes \bar{w})
$$

$$
= \sum a_{U^\psi, V^\psi, W^\psi} (h \cdot ((u \otimes v) \otimes \bar{w}))
$$

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Moreover, if \( H \) can easily see that \( FG = GF \).

Next we check that \( \Phi_U \) is a morphism of \( H \)-modules. Indeed,
\[
\Phi_U (h \cdot u) = \alpha_U (\tilde{h} \cdot u) = \alpha_H (\tilde{h}) \cdot \alpha_U (u) = h \cdot \alpha_H (\tilde{u}) = h \cdot \alpha \Phi_U (u),
\]
\[
\Phi_U (\alpha_U (u)) = \alpha_U (\alpha_U (u)) = \alpha_U \alpha_U (u) = \alpha_U (\Phi_U (u)).
\]

Moreover, if \( f : U \to V \) is a morphism of \( H \)-modules then we have \( \tilde{f} \Phi_U = \Phi_V f \). In other words \( \Phi \) is a natural transformation between the functors \( \text{id}_{H\text{-mod}} \) and \( G \). It is obvious that \( F(\Phi_U) = \Phi_{F(U)} \), \( G(\Phi_U) = \Phi_{G(U)} \) and \( \Phi_{U \otimes V} = \Phi_U \otimes \Phi_V \).

Therefore, \( \mathcal{H} = (H\text{-mod}, \otimes, F, G, a, \Phi) \) is a hom–tensor category.

(B) \( \Rightarrow \) (A) Suppose that \( \mathcal{H} = (H\text{-mod}, \otimes, F, G, a, \Phi) \) is a hom–tensor category and \( H \) is nondegenerate. In view of Definition 2.4 and given our assumptions, in order for \( (H, m_H, \Delta_H, \psi_H) \) to be a hom-bialgebra what remains to be shown is that \( (H, \Delta_H, \psi_H) \) is a hom-coassociative \( k \)-coalgebra.

We begin with proving (3). Let \( (U, \alpha_U), (V, \alpha_V) \in \mathcal{H} \). We have that
\[
F ((U \otimes V, \alpha_{U \otimes V})) = F ((U, \alpha_U)) \otimes F ((V, \alpha_V)). \tag{26}
\]

In particular the map \( (\bar{u} \otimes \bar{v}) \to \bar{u} \otimes \bar{v} \) from Remark 3.6 is a morphism of \( H \)-modules.

Let \( h \in H \), \( u \in U \), and \( v \in V \). The \( H \)-action for the left hand side of (26)
\[
h \cdot \psi (u \otimes v) \overset{(iv),(iii)}{=} \psi_H (h) \cdot (u \otimes v)
\]
\[
\overset{(iii)}{=} \sum ((\psi_H (h))_{(1)} \cdot u) \otimes ((\psi_H (h))_{(2)} \cdot v)
\]
is equal to the \( H \)-action for the right hand side of (26)
\[
h \cdot \psi (\bar{u} \otimes \bar{v}) \overset{(iii)}{=} \sum h_{(1)} \cdot \psi \bar{u} \otimes h_{(2)} \cdot \psi \bar{v}
\]
\[
\overset{(iv)}{=} \sum \psi_H (h_{(1)}) \cdot u \otimes \psi_H (h_{(2)}) \cdot v.
\]

That is,
\[
\sum ((\psi_H (h))_{(1)} \cdot u) \otimes ((\psi_H (h))_{(2)} \cdot v) = \sum \psi_H (h_{(1)}) \cdot u \otimes \psi_H (h_{(2)}) \cdot v,
\]
which means that
\[
\sum ((\psi_H (h))_{(1)} \otimes (\psi_H (h))_{(2)}) \cdot (u \otimes v) = \sum ((\psi_H (h_{(1)} \otimes \psi_H (h_{(2)})) \cdot (u \otimes v).
\]

Since \( H \) is assumed to be nondegenerate and by using Lemma 2.14, this equation implies that
\[
\sum ((\psi_H (h))_{(1)} \otimes (\psi_H (h))_{(2)}) = \sum \psi_H (h_{(1)}) \otimes \psi_H (h_{(2)}),
\]
or equivalently \((\Delta_H \circ \psi_H) (h) = ((\psi_H \otimes \psi_H) \circ \Delta_H) (h)\), for all \( h \in H \). So (3) holds for \( \Delta_H \).

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Next we will show (4) for $\Delta_H$. Since $\mathcal{H}$ is a hom–tensor category it means that the hom-associativity constraint is a morphism of left $H$-modules. This means that for $h \in H$, $u \in U$, $v \in V$ and $w \in W$ we have $a_{U, V, W}(h \cdot ((u \otimes v) \otimes \tilde{w})) = h \cdot a_{U, V, W}((u \otimes v) \otimes \tilde{w})$. For the left and respectively right hand side of this equality we have

$$a_{U, V, W}(h \cdot ((u \otimes v) \otimes \tilde{w})) = a_{U, V, W} \left( \sum (h_{(1)} \cdot u) \otimes (h_{(2)} \cdot v) \otimes (\psi_H(h_{(2)}) \cdot w) \right),$$

and

$$h \cdot a_{U, V, W}((u \otimes v) \otimes \tilde{w}) = \sum (h_{(1)} \cdot u) \otimes (h_{(2)} \cdot v) \otimes (\psi_H(h_{(2)}) \cdot w).$$

So, we must have

$$\sum (h_{(1)} \cdot u) \otimes (h_{(2)} \cdot v) \otimes (\psi_H(h_{(2)}) \cdot w) = \sum (h_{(1)} \cdot u) \otimes (h_{(2)} \cdot v) \otimes (\psi_H(h_{(2)}) \cdot w),$$

or equivalently

$$\left( \sum h_{(1)} \otimes h_{(2)} \otimes \psi_H(h_{(2)}) \right) \cdot (u \otimes (v \otimes w)) = \left( \sum \psi_H(h_{(1)}) \otimes h_{(2)} \otimes h_{(2)} \right) \cdot (u \otimes (v \otimes w)).$$

Since $H$ is nondegenerate and by using again Lemma 2.14, this equation implies that

$$\sum h_{(1)} \otimes h_{(2)} \otimes \psi_H(h_{(2)}) = \sum \psi_H(h_{(1)}) \otimes h_{(2)} \otimes h_{(2)},$$

or equivalently $((\Delta_H \otimes \psi_H) \circ \Delta_H)(h) = ((\psi_H \otimes \Delta_H) \circ \Delta_H)(h)$, for all $h \in H$. Thus, $\Delta_H$ satisfies (4) and $(H, \Delta_H, \psi_H)$ is a hom-coassociative $k$-coalgebra. Therefore, $(H, m_H, \Delta_H, \alpha_H, \psi_H)$ is a hom-bialgebra.

4 Algebras in Hom–Tensor Categories

**Definition 4.1** Let $(C, \otimes, F, G, a, \Phi)$ be a hom–tensor category. An algebra in $C$ is a pair $(A, \mu_A)$, where $A \in \text{Ob}(C)$ and $\mu_A : A \otimes A \to FG(A)$ is a morphism in $C$ such that the diagram in Fig. 2 is commutative.

**Proposition 4.2** We denote by $k$-mod the category of $k$-vector spaces with its usual structure as a tensor category and we consider the hom–tensor category $\mathcal{H}(k$-mod) as in Proposition 3.4. Then an algebra in $\mathcal{H}(k$-mod) is exactly a hom-associative $k$-algebra.

**Proof** Let $((A, \alpha_A), \mu_A)$ be an algebra in $\mathcal{H}(k$-mod). This means that $A$ is a $k$-vector space, $\alpha_A : A \to A$ is a $k$-linear map and $\mu_A : (A, \alpha_A) \otimes (A, \alpha_A) \to (A, \alpha_A)$ is a morphism in $\mathcal{H}(k$-mod), meaning that $\mu_A : A \otimes A \to A$ is a $k$-linear map satisfying $\alpha_A \circ \mu_A = \mu_A \circ (\alpha_A \otimes \alpha_A)$, such that $\mu_A \circ (\mu_A \otimes \Phi_A) = \mu_A \circ (\Phi_A \otimes \mu_A)$; the second condition is equivalent to $(ab)\alpha_A(c) = \alpha_A(ab)(bc)$, for all $a, b, c \in A$, where we denoted $\mu_A(a \otimes b) = ab$, for $a, b \in A$. This means exactly that $(A, \mu_A, \alpha_A)$ is a hom-associative $k$-algebra. \(\square\)
We recall the following concept from [10].

**Definition 4.3** A *hom-semigroup* is a set $S$ together with a binary operation $\mu : S \times S \to S$ (denoted by $\mu((x, y)) = xy$ for $x, y \in S$) and a function $\alpha : S \to S$ satisfying $\alpha(x)(yz) = (xy)\alpha(z)$, for all $x, y, z \in S$.

The hom-semigroup $(S, \mu, \alpha)$ is called *multiplicative* if $\alpha(xy) = \alpha(x)\alpha(y)$ for all $x, y \in S$.

We have an analogue of Proposition 4.2 for hom-semigroups (the proof is similar and will be omitted).

**Proposition 4.4** Let $\mathbf{Set}$ be the category of sets with its usual structure as a pre-tensor category: the tensor product is the cartesian product of sets and the associativity constraint is defined by

$$a_{X,Y,Z} : (X \times Y) \times Z \to X \times (Y \times Z), \quad a_{X,Y,Z}(((x, y), z)) = (x, (y, z)).$$

Consider the hom–tensor category $h(\mathbf{Set})$ as in Proposition 3.4. Then an algebra in $h(\mathbf{Set})$ is exactly a multiplicative hom-semigroup.

**Definition 4.5** Let $(H, \mu_H, \Delta_H, \alpha_H, \psi_H)$ be a hom-bialgebra. A hom-associative $k$-algebra $(A, \mu_A, \alpha_A)$ is called a left $H$-module hom-algebra if $(A, \alpha_A)$ is a left $H$-module, with action denoted by $H \otimes A \to A, h \otimes a \mapsto h \cdot a$, such that the following condition is satisfied:

$$\alpha_H \psi_H(h) \cdot (aa') = \sum (h(1) \cdot a)(h(2) \cdot a'), \quad \forall h \in H, \ a, a' \in A. \quad (27)$$

**Remark 4.6** This concept contains as particular cases the concepts of module algebras for the situation $\psi_H = \alpha_H$ (introduced in [25]) and for the situation $\psi_H = \alpha_H^{-1}$ (introduced in [7]).

**Proposition 4.7** Let $(H, \mu_H, \Delta_H, \alpha_H, \psi_H)$ be a hom-bialgebra and consider the hom–tensor category $\mathcal{H} = (H\text{-mod}, \otimes, F, G, a, \Phi)$ introduced in Proposition 3.5. Then an algebra in $\mathcal{H}$ is exactly a left $H$-module hom-algebra.

**Proof** Let $(A, \alpha_A), \mu_A)$ be an algebra in $\mathcal{H}$. This means that:

(i) $(A, \alpha_A)$ is a left $H$-module (we denote the action of $H$ on $A$ by $h \otimes a \mapsto h \cdot a)$;
(ii) we have a morphism \( \mu_A : (A, \alpha_A) \otimes (A, \alpha_A) \to FG((A, \alpha_A)) \) in \( \mathcal{K} \), denoted by \( \mu_A(a \otimes b) = ab \) for all \( a, b \in A \). By taking into account the structure of \( \mathcal{K} \) as a hom–tensor category presented in Proposition 3.5, this means that \( \alpha_A \circ \mu_A = \mu_A \circ (\alpha_A \otimes \alpha_A) \) and \( \mu_A(h \cdot (a \otimes a')) = h \cdot \psi \mu_A(a \otimes a') \), for all \( h \in H \) and \( a, a' \in A \), which is equivalent to saying that \( \alpha_A(aa') = \alpha_A(a)\alpha_A(a') \) and \( \sum (h(1) \cdot a)(h(2) \cdot a') = \alpha_H \psi_H(h) \cdot (aa') \).

(iii) we have \( \mu_A \circ (\mu_A \otimes \alpha_A) = \mu_A \circ (\alpha_A \otimes \mu_A) \), which means that \( (ab)\alpha_A(c) = \alpha_A(a)(bc) \), for all \( a, b, c \in A \).

In conclusion, \((A, \alpha_A), \mu_A \) is exactly a left \( H \)-module hom-algebra \((A, \mu_A, \alpha_A)\).

The next result may be regarded as a categorical analogue of the Yau twisting.

**Proposition 4.8** Let \((C, \otimes, a)\) be a pre-tensor category, \((A, \mu_A)\) an algebra in \( C \) and \( \alpha_A : A \to A \) an algebra morphism. Define \( m_A : A \otimes A \to A, m_A := \alpha_A \circ \mu_A = \mu_A \circ (\alpha_A \otimes \alpha_A) \). Then \((A, \alpha_A, m_A)\) is an algebra in the hom–tensor category \( h(C) \).

**Proof** First, by \( \alpha_A \circ \mu_A = \mu_A \circ (\alpha_A \otimes \alpha_A) \) one obtains immediately that \( \alpha_A \circ m_A = m_A \circ (\alpha_A \otimes \alpha_A) \), that is \( m_A : (A, \alpha_A) \otimes (A, \alpha_A) \to (A, \alpha_A) \) is a morphism in \( h(C) \). So, we only need to prove that \( m_A \circ (m_A \otimes \Phi_A) = m_A \circ (\Phi_A \otimes m_A) \circ a_{A,A,A} \). We compute:

\[
\begin{align*}
m_A \circ (\Phi_A \otimes m_A) \circ a_{A,A,A} &= m_A \circ (\alpha_A \otimes m_A) \circ a_{A,A,A} \\
&= \alpha_A \circ \mu_A \circ (\alpha_A \otimes \alpha_A) \circ a_{A,A,A} \\
&= \alpha_A \circ \mu_A \circ (\alpha_A \otimes \alpha_A) \circ (\text{id}_A \otimes \mu_A) \circ a_{A,A,A} \\
&= \alpha_A^2 \circ \mu_A \circ (\text{id}_A \otimes \mu_A) \circ a_{A,A,A} = \alpha_A^2 \circ \mu_A \circ (\mu_A \otimes \text{id}_A) \\
&= \alpha_A \circ \mu_A \circ (\alpha_A \otimes \alpha_A) \circ (\mu_A \otimes \text{id}_A) \\
&= \alpha_A \circ \mu_A \circ (\alpha_A \otimes \mu_A \otimes \alpha_A) = m_A \circ (m_A \otimes \Phi_A),
\end{align*}
\]

finishing the proof. \( \square \)

## 5 Hom-Braided Categories

We introduce hom-braided categories and present their connection with quasitriangular hombialgebras. Here \( \tau : C \times C \to C \times C \) is the functor defined by \( \tau(V, W) = (W, V) \) for any pair of objects in a category \( C \).

**Definition 5.1** Let \( \mathcal{C} = (C, \otimes, F, G, a, \Phi) \) be a hom–tensor category. A hom–commutativity constraint \( d \) is a natural morphism \( d : \otimes \to \otimes \tau(G \times G) \). That is, for any \( V, W \in \text{Ob}(\mathcal{C}) \) we have the morphism \( d_{V,W} : V \otimes W \to G(W) \otimes G(V) \) such that the diagram in Fig. 3 commutes for all morphisms \( f, g \in \mathcal{C} \).

Next we introduce the analog of the hexagon axiom in the context of hom–tensor categories.

\[
\begin{array}{ccc}
V \otimes W & \xrightarrow{d_{V,W}} & G(W) \otimes G(V) \\
\downarrow f \otimes g & & \downarrow \sigma(g) \otimes \sigma(f) \\
V' \otimes W' & \xrightarrow{d_{V',W'}} & G(W') \otimes G(V')
\end{array}
\]

**Fig. 3** The naturality of \( d_{V,W} : V \otimes W \to G(W) \otimes G(V) \)
Definition 5.2. We say that the hom-commutativity constraint $d$ satisfies the (H1) property if the diagram, as seen in Fig. 4, commutes for all objects $U$, $V$ and $W$ of the category $C$. Furthermore, we say that the hom-commutativity constraint $d$ satisfies the (H2) property if the diagram, as seen in Fig. 5, commutes for all objects $U$, $V$ and $W$ of the category $C$.

Definition 5.3. Let $\mathcal{C} = (\mathcal{C}, \otimes, F, G, a, \Phi)$ be a hom–tensor category. A hom-braiding $d$ in $\mathcal{C}$ is a hom-commutativity constraint with the following conditions:

(i) $d$ satisfies (H1) and (H2);

(ii) For all objects $U$ and $V$ in the category $\mathcal{C}$, $G(d_{U, V}) = d_{G(U), G(V)}$.

A hom-braided category $(\mathcal{C}, \otimes, F, G, a, \Phi, d)$ is a hom–tensor category with hom-braiding $d$. 

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**Remark 5.4** Note that $d$ is not required to be invertible and so a more appropriate name for the above structure would be hom-quasi-braided category. However, in order to simplify the terminology, we prefer to call it hom-braided category.

We present a first class of examples of hom-braided categories.

**Proposition 5.5** Let $(C, \otimes, a, c)$ be a quasi-braided pre-tensor category. Then the hom–tensor category $\mathfrak{h}(C)$ constructed in Proposition 3.4 is a hom-braided category, with hom-braiding defined by

$$d_{(M, \alpha_M), (N, \alpha_N)} : (M, \alpha_M) \otimes (N, \alpha_N) \to (N, \alpha_N) \otimes (M, \alpha_M),$$

for all objects $(M, \alpha_M), (N, \alpha_N)$ in $\mathfrak{h}(C)$.

**Proof** We only prove that the first hexagonal relation satisfied by $c_{-, -}$, namely

$$a_{V,W,U} \circ c_{U,V,W} \circ a_{U,V,W} = (id_V \otimes c_{U,W}) \circ a_{V,U,W} \circ (c_{U,V} \otimes id_W),$$

for all $U, V, W \in \text{Ob}(C)$, implies property (H1) for $d_{-, -}$ and leave the rest of the proof to the reader.

Let $(U, \alpha_U), (V, \alpha_V), (W, \alpha_W) \in \text{Ob}(\mathfrak{h}(C))$. We compute, by applying repeatedly the fact that $\otimes$ is a functor, the naturality of $c_{-, -}$ and the naturality of $a_{-, -}$:

$$(id_{V, \alpha_V}) \circ d_{(U, \alpha_U), (W, \alpha_W)} \circ a_{V, \alpha_V, (U, \alpha_U), (W, \alpha_W)} = (id_V \otimes ((\alpha_W \otimes \alpha_U) \circ c_{U,W})) \circ a_{V,U,W} \circ ((c_{U,V} \circ (\alpha_U \otimes \alpha_V)) \otimes id_W)$$

$$= (id_V \otimes (id_W \otimes \alpha_U)) \circ (id_V \otimes c_{U,W}) \circ (id_V \otimes (id_U \otimes \alpha_W)) \circ a_{V,U,W}$$

$$= (id_V \otimes (id_W \otimes \alpha_U)) \circ (id_V \otimes c_{U,W}) \circ (id_V \otimes \alpha_V) \otimes id_W$$

finishing the proof. \qed

The following proposition gives the connection between hom-braided categories and quasi-triangular hom-bialgebras.

**Proposition 5.6** Let $(H, m_H, \Delta_H, \alpha_H, \psi_H)$ be a hom-bialgebra and take $\mathcal{H}$ to be the hom–tensor category described in Proposition 3.5. Let $R = \sum_i s_i \otimes t_i \in H \otimes H$ and assume that $(\alpha_H \otimes \alpha_H)(R) = R = (\psi_H \otimes \psi_H)(R)$. Consider the two statements (A) and (B) below. The we have that (A) implies (B) and if $H$ is strongly nondegenerate then (B) implies (A).

(A) $(H, m_H, \Delta_H, \alpha_H, \psi_H, R)$ is a quasitriangular hom-bialgebra.
(B) The category $\mathcal{K} = (H\text{-mod}, \otimes, F, G, a, \Phi, c)$ is a hom-braided category with hom-braiding $c$ given as follows. For two objects $(U, \alpha_U)$ and $(V, \alpha_V)$, we have $c_{U,V} : (U, \alpha_U) \otimes (V, \alpha_V) \to (V^\alpha, \alpha_{V^\alpha}) \otimes (U^\alpha, \alpha_{U^\alpha})$ defined for all $u \in U$ and $v \in V$ by

$$c_{U,V} (u \otimes v) = (\tau_{U^\alpha,V^\alpha}) (R (u \otimes v)) = \sum_i (t_i \cdot \alpha \tilde{u}) \otimes (s_i \cdot \alpha \tilde{v}) .$$

(29)

Proof (A) $\Rightarrow$ (B) Suppose that $(H, m_H, \Delta_H, \alpha_H, \psi_H, R)$ is a quasitriangular hombialgebra such that $(\alpha_H \otimes \alpha_H) (R) = R = (\psi_H \otimes \psi_H)(R)$; in particular we have

$$\sum_i \alpha_H (s_i) \otimes \alpha_H (t_i) = \sum_i s_i \otimes t_i .$$

(30)

Let $(U, \alpha_U)$ and $(V, \alpha_V)$ be objects in $H\text{-mod}$ and let $u \in U$, $v \in V$. We claim that $c_{U,V}$ is a morphism in $\mathcal{K}$. First we check that $(\alpha_{V^\alpha \otimes U^\alpha}) (c_{U,V}) (u \otimes v) = (c_{V^\alpha,V}) (\alpha_{U^\alpha \otimes V}) (u \otimes v)$:

$$(\alpha_{V^\alpha \otimes U^\alpha}) (c_{U,V}) (u \otimes v) = (\alpha_{V^\alpha} \otimes \alpha_{U^\alpha}) (c_{U,V}) (u \otimes v)$$

(29)

$$= (\alpha_{V^\alpha} \otimes \alpha_{U^\alpha}) \left( \sum_i (t_i \cdot \alpha \tilde{u}) \otimes (s_i \cdot \alpha \tilde{v}) \right)$$

$$= \sum_i \alpha_{V^\alpha} \left( \alpha_H (t_i) \cdot v \right) \otimes \alpha_{U^\alpha} \left( \alpha_H (s_i) \cdot u \right)$$

$$= \sum_i \alpha_V (\alpha_H (t_i) \cdot v) \otimes \alpha_U (\alpha_H (s_i) \cdot u)$$

(10)

$$= \sum_i \alpha_H^2 (t_i) \cdot \alpha_V (v) \otimes \alpha_U^2 (s_i) \cdot \alpha_U (u) ,$$

$$(c_{U,V}) (\alpha_{U^\alpha \otimes V}) (u \otimes v) = (c_{U,V}) (\alpha_U \otimes \alpha_V) (u \otimes v) = c_{U,V} (\alpha_U (u) \otimes \alpha_V (v))$$

(29)

$$= \sum_i (t_i \cdot \alpha \tilde{v}) \otimes (s_i \cdot \alpha \tilde{u})$$

(30)

$$= \sum_i \left( \alpha_H (t_i) \cdot \alpha \tilde{v} \right) \otimes \left( \alpha_H (s_i) \cdot \alpha \tilde{u} \right)$$

$$= \sum_i \alpha_H^2 (t_i) \cdot \alpha_V (v) \otimes \alpha_U^2 (s_i) \cdot \alpha_U (u) , \quad q.e.d.$$

Let $h \in H$, $u \in U$ and $v \in V$. We next check that $(c_{U,V}) (h \cdot (u \otimes v)) = h \cdot (c_{U,V}) (u \otimes v)$:

$$(c_{U,V}) (h \cdot (u \otimes v)) = (c_{U,V}) \left( \sum_i (h_{(1)} \cdot u) \otimes (h_{(2)} \cdot v) \right)$$

(29)

$$= (\tau_{U^\alpha,V^\alpha}) \left( \sum_i (s_i \cdot \alpha \tilde{h_{(1)}}) \otimes (t_i \cdot \alpha \tilde{h_{(2)} \cdot v}) \right)$$

$$= (\tau_{U^\alpha,V^\alpha}) \left( \sum_i \left( \alpha_H (s_i) \cdot \alpha \tilde{h_{(1)} \cdot u} \right) \otimes \left( \alpha_H (t_i) \cdot \alpha \tilde{h_{(2)} \cdot v} \right) \right)$$

(11)

$$= \left( \tau_{U^\alpha,V^\alpha} \right) \left( \sum_i (s_i h_{(1)}) \cdot \alpha_U (u) \otimes (t_i h_{(2)}) \cdot \alpha_V (v) \right)$$

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Therefore, $c_{U,V}$ is a morphism in $\mathcal{H}$. A similar computation shows the naturality of $c$.

Next we will confirm that $c_{U,V}$ as it is defined satisfies the (H1) property. That is, we need to show that

\[
(id_{V^\alpha} \otimes (id_{W^\alpha} \otimes \Phi_{U^\alpha})) \circ a_{V^\alpha,W^\alpha,U^\alpha} \circ c_{U^\alpha,V^\alpha,W^\alpha} \circ a_{U^\alpha,V^\alpha,W^\alpha} = (id_{V^\alpha} \otimes c_{U^\alpha,W^\alpha}) \circ a_{V^\alpha,U^\alpha,W^\alpha} \circ (c_{U^\alpha,V^\alpha} \otimes id_{W^\alpha})
\]

for all objects $(U, \alpha_U)$, $(V, \alpha_V)$ and $(W, \alpha_W)$ in $H\text{-mod}$.

Let $u \in U$, $v \in V$ and $w \in W$. Computing the left hand side of (31) we have that

\[
(id_{V^\alpha} \otimes (id_{W^\alpha} \otimes \Phi_{U^\alpha})) \circ a_{V^\alpha,W^\alpha,U^\alpha} \circ c_{U^\alpha,V^\alpha,W^\alpha} \circ a_{U^\alpha,V^\alpha,W^\alpha} ((u \otimes v) \otimes \overline{w})
\]

\[
= (id_{V^\alpha} \otimes (id_{W^\alpha} \otimes \Phi_{U^\alpha})) \circ a_{V^\alpha,W^\alpha,U^\alpha} \circ c_{U^\alpha,V^\alpha,W^\alpha} \circ a_{U^\alpha,V^\alpha,W^\alpha} (\overline{\psi_1} \otimes (v \otimes w))
\]

\[
= (id_{V^\alpha} \otimes (id_{W^\alpha} \otimes \Phi_{U^\alpha})) \circ a_{V^\alpha,W^\alpha,U^\alpha} \left( \sum_i (t_i \cdot \alpha (v \otimes w)) \otimes (s_i \cdot \alpha \psi \overline{u}) \right)
\]

\[
= (id_{V^\alpha} \otimes (id_{W^\alpha} \otimes \Phi_{U^\alpha})) \circ a_{V^\alpha,W^\alpha,U^\alpha} \left( \sum_i (a_H (t_i) \cdot (v \otimes w)) \otimes (a_H (s_i) \cdot u) \right)
\]

\[
= (id_{V^\alpha} \otimes (id_{W^\alpha} \otimes \Phi_{U^\alpha})) \circ a_{V^\alpha,W^\alpha,U^\alpha} \left( \sum_i (a_H (t_i) \cdot (v \otimes w)) \otimes (a_H (s_i) \cdot u) \right)
\]

\[
= (id_{V^\alpha} \otimes (id_{W^\alpha} \otimes \Phi_{U^\alpha})) \left( \sum_i (t_i) (1) \cdot (\alpha H (t_i) (1) \cdot (v \otimes w)) \otimes (a_H (s_i) \cdot u) \right)
\]

\[
= \sum_i (t_i) (1) \cdot (v \otimes (\alpha H (s_i) \cdot u))
\]

Computing the right hand side of (31) we have that

\[
(id_{V^\alpha} \otimes c_{U^\alpha,W^\alpha}) \circ a_{V^\alpha,U^\alpha,W^\alpha} \circ (c_{U^\alpha,V^\alpha} \otimes id_{W^\alpha}) ((u \otimes v) \otimes \overline{w})
\]

\[
= (id_{V^\alpha} \otimes c_{U^\alpha,W^\alpha}) \circ a_{V^\alpha,U^\alpha,W^\alpha} \left( \sum_i (t_i \cdot \alpha \psi \overline{u} \otimes s_i \cdot \alpha \overline{u}) \otimes \overline{w} \right)
\]

\[
= (id_{V^\alpha} \otimes c_{U^\alpha,W^\alpha}) \circ a_{V^\alpha,U^\alpha,W^\alpha} \left( \sum_i (\alpha H (t_i) \cdot (v \otimes \alpha H (s_i) \cdot u)) \otimes \overline{w} \right)
\]
Now since \((\alpha_H \psi_H \otimes \Delta)(R) = \sum_{i,j} s_i s_j \otimes t_j \otimes t_i\) by Remark 2.12, the left hand side and the right hand side of (31) agree. So \(c_{U,V}\) has the (H1) property.

Next we will confirm that \(c_{U,V}\) satisfies the (H2) property. That is, we need to show that

\[
((\Phi_{W^a} \otimes \text{id}_{U^a}) \otimes \text{id}_{V^a}) \circ a_{W^a, U^a, V^a}^{-1} \circ c_{U \otimes V, W^\psi} \circ a_{U, V, W}^{-1} = (c_{U, W^a} \otimes \text{id}_{V^a}) \circ a_{U, W^a, V^a}^{-1} \circ (\text{id}_{U^a} \otimes c_{V, W})
\]

for all objects \((U, U^a), (V, V^a)\) and \((W, \alpha_W)\) in \(H\)-mod.

Let \(u \in U, v \in V\) and \(w \in W\). For the left hand side of (32) we have that

\[
\left( (\Phi_{W^a} \otimes \text{id}_{U^a}) \otimes \text{id}_{V^a} \right) \circ a_{W^a, U^a, V^a}^{-1} \circ c_{U \otimes V, W^\psi} \circ a_{U, V, W}^{-1} (\bar{u} \otimes (v \otimes w))
\]

\[
= \left( (\Phi_{W^a} \otimes \text{id}_{U^a}) \otimes \text{id}_{V^a} \right) \circ a_{W^a, U^a, V^a}^{-1} \circ c_{U \otimes V, W^\psi} \circ a_{U, V, W}^{-1} (u \otimes v) \otimes \bar{w}
\]

\[
= \left( (\Phi_{W^a} \otimes \text{id}_{U^a}) \otimes \text{id}_{V^a} \right) \circ a_{W^a, U^a, V^a}^{-1} \left( \sum_i \alpha_H (t_i) \cdot v \otimes \left( \alpha_H (s_i) \cdot u \otimes (\alpha_H (s_i)) (u \otimes v) \right) \right)
\]

\[
= \left( (\Phi_{W^a} \otimes \text{id}_{U^a}) \otimes \text{id}_{V^a} \right) \circ a_{W^a, U^a, V^a}^{-1} \left( \sum_i \psi_H (t_i) \cdot w \otimes \left( \alpha_H (s_i) \right) (u \otimes v) \right)
\]

For the right hand side of (32) we have that

\[
\left( (c_{U, W^a} \otimes \text{id}_{V^a}) \circ a_{U, W^a, V^a}^{-1} \circ (\text{id}_{U^a} \otimes c_{V, W}) \right) (\bar{u} \otimes (v \otimes w))
\]
\[
(29) \quad \left( (c_{U,W^a} \otimes id_{V^a}) \circ a_{U,W^a,V^a}^{-1} \right) \left( \sum_i \left( t_i \cdot \alpha \tilde{w} \otimes s_i \cdot \alpha \tilde{v} \right) \right) \\
= (c_{U,W^a} \otimes id_{V^a}) \left( \left( u \otimes \sum_i \alpha_H (t_i) \cdot w \right) \otimes \alpha_H (s_i) \cdot v \right) \\
= \sum_{i,j} \left( t_j \cdot \alpha_H^2 (t_i) \cdot w \otimes \alpha_H (s_j) \cdot u \right) \otimes \alpha_H (s_i) \cdot v \\
= \sum_{i,j} \left( \alpha_H^2 (t_j) \cdot (\alpha_H (t_i) \cdot w) \otimes \alpha_H (s_j) \cdot u \right) \otimes \alpha_H (s_i) \cdot v \\
= \sum_{i,j} \left( (\alpha_H (t_j) \alpha_H (t_i)) \cdot \alpha_W (w) \otimes \alpha_H (s_j) \cdot u \right) \otimes \alpha_H (s_i) \cdot v \\
= \sum_{i,j} \left( (t_j t_i) \cdot \alpha_W (w) \otimes s_j \cdot u \right) \otimes s_i \cdot v.
\]

Since \( (\Delta \otimes \alpha_H \psi_H) (R) = \sum_i s_i \otimes s_j \otimes t_i t_j \) by Remark 2.12, \( c_{U,V} \) satisfies the (H2) condition.

We show that \( c_{U^a,V^a} = G (c_{U,V}) \), for all objects \((U, \alpha_U), (V, \alpha_V)\) in \( H\text{-mod} \) and \( u \in U, v \in V \):

\[
c_{U^a,V^a} (\tilde{u} \otimes \tilde{v}) = \sum_i \left( t_i \cdot a^2 \tilde{v} \otimes (s_i \cdot a^2 \tilde{u}) = \sum_i \left( \alpha_H^2 (t_i) \cdot v \right) \otimes \left( a_H^2 (s_i) \cdot u \right), \right. \\
G(c_{U,V}) (\tilde{u} \otimes \tilde{v}) = c_{U,V} (u \otimes v) = \sum_i \left( t_i \cdot \alpha \tilde{v} \otimes (s_i \cdot \alpha \tilde{u}) = \sum_i \left( \alpha_H (t_i) \cdot v \right) \otimes \left( \alpha_H (s_i) \cdot u \right). \right.
\]

Since \( (\alpha_H \otimes \alpha_H) (R) = R \) we get \( c_{U^a,V^a} = G (c_{U,V}) \) and so \( c_{U,V} \) is a hom-braiding in \( \mathcal{H} \). Therefore \( \mathcal{H} \) is a hom-braided category.

\( B \Rightarrow A \) Suppose that \( \mathcal{H} = (H\text{-mod}, \otimes, F, G, a, \Phi, c) \) is a hom-braided category, \((H, m_H, \Delta_H, \alpha_H)\) is a strongly nondegenerate hom-associative algebra and let \( R \in H \otimes H \) be given as \( R = \sum_i s_i \otimes t_i \) such that \( (\alpha_H \otimes \alpha_H) (R) = R = (\psi_H \otimes \psi_H) (R) \). We will show that conditions (14), (15) and (16) from Definition 2.10 are satisfied for \( H = (H, m_H, \Delta_H, \alpha_H, \psi_H, R) \).

First we will show that \( H \) satisfies condition (14). Let \((U, \alpha_U)\) and \((V, \alpha_V)\) be objects in \( \mathcal{H} \). Since \( \mathcal{H} \) is a hom-braided category it means that the hom-braiding \( c_{U,V} \) is a morphism of left \( H \)-modules. That is, for \( h \in H, u \in U, \) and \( v \in V \), we have \( c_{U,V} (h \cdot (u \otimes v)) = h \cdot c_{U,V} ((u \otimes v)) \). Just like above, for the left hand side of this equality we have \( c_{U,V} (h \cdot (u \otimes v)) = \sum_i \left( t_i h(2) \right) \cdot \alpha_U (v) \otimes \left( s_i h(1) \right) \cdot \alpha_V (u) \), and for the right hand side we have \( h \cdot c_{U,V} ((u \otimes v)) = \sum_i \left( h(1) t_i \right) \cdot \alpha_V (v) \otimes \left( h(2) s_i \right) \cdot \alpha_U (u) \), and so we must have

\[
\sum_i \left( t_i h(2) \right) \cdot \alpha_U (v) \otimes \left( s_i h(1) \right) \cdot \alpha_U (u) = \sum_i \left( h(1) t_i \right) \cdot \alpha_V (v) \otimes \left( h(2) s_i \right) \cdot \alpha_U (u). 
\]

Since \( H \) is strongly nondegenerate and by using Lemma 2.14, this equation implies that \( \sum_i t_i h(2) \otimes s_i h(1) = \sum_i h(1) t_i \otimes h(2) s_i \), or equivalently \( R \Delta (h) = \Delta^{\text{cop}} (h) R \), for every \( h \in H \). Thus, \( R \) satisfies (14).
Next we will show that $R$ satisfies conditions (15) and (16). One can show that

$$
((\Phi_{W^a} \otimes \text{id}_{U^a}) \otimes \text{id}_{V^a})(a_{U^a, U^a, V^a} \circ c_{U^a \otimes V^a, W^a} \circ a_{W^a, U^a, V^a}^{-1})(u \otimes (v \otimes w)) =
$$

$$
\sum_i \alpha_H \psi_H \circ (t_i \cdot \alpha_W (w) \otimes (s_i)_{(1)} \cdot u) \otimes (s_i)_{(2)} \cdot v,
$$

$$
((c_{U^a, W^a} \otimes \text{id}_{V^a \otimes W^a}) \circ a_{U^a, W^a, V^a} \circ (\text{id}_{V^a} \otimes c_{V^a, W^a}))((u \otimes \Phi_{U^a})) =
$$

$$
\sum_{i,j} \left((t_j t_i \cdot \alpha_W (w) \otimes s_j \cdot u) \otimes s_j \cdot v\right).
$$

Since $c_{U, V}$ satisfies the (H2) property and $H$ is strongly nondegenerate, by using again Lemma 2.14 we obtain

$$(\Delta_H \otimes \alpha_H \psi_H)(R) = \sum_{i,j} s_i \otimes s_j \otimes t_i t_j. \quad (33)$$

The goal of this section is to describe two categorical versions of Definition 6.1. The following lemma will be useful in proving some of these results.

Lemma 6.2 Assume that $d$ is a hom-commutativity constraint as in Definition 5.1. Then the commutative diagram in Fig. 6 is equivalent to the commutative diagram in Fig. 5. We will call the equation from Fig. 6 the $(H'2)$ property of the hom-commutativity constraint $d$.

Proof First observe that the commutative diagrams for the (H2) property and the $(H'2)$ property are identical except on two of their edges and the object between those edges. These exceptions are indicated by bold edges in Fig. 6. So it suffices to show that the diagram in Fig. 7 commutes. The hom-associativity constraint $a$ is a natural isomorphism, so $a^{-1}$ is also natural and the diagram in Fig. 7 is a particular case of the naturality of $a^{-1}$ plus the fact that $F(\Phi_{G(W)}) = \Phi_{FG(W)}$. Thus the diagram in Fig. 7 commutes and we have the equivalence between the properties (H2) and $(H'2)$. \hfill $\Box$

Definition 6.3 We say that a hom-commutativity constraint $d$ has the Hom–Yang–Baxter property if $d$ satisfies the equation in Fig. 8.

Proposition 6.4 Let $(C, \otimes, F, G, a, \Phi, d)$ be a a hom-braided category. Then $d$ has the Hom–Yang–Baxter property.
Fig. 6 The $\text{Hom}^2$ property

$\Phi_{\text{FG}(W)} \otimes (\text{id}_{G(U)} \otimes \text{id}_{G(V)})$

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$\Phi_{\text{FG}(W)} \otimes (\text{id}_{G(U)} \otimes \text{id}_{G(V)})$
Proof Let $\mathcal{C} = (\mathcal{C}, \otimes, F, G, a, \Phi, d)$ be a hom-braided category. The commutative diagram in Fig. 6 together with the hom-associativity constraint $a$ being an isomorphism implies that the diagram in Fig. 9 commutes for all objects $U, V, W$ in $\mathcal{C}$. Observe that the commutative diagram in Fig. 9 is precisely the bold portion of the diagram seen in Fig. 11.

Next, consider substituting the object $G(V)$ for $U$ and the object $G(U)$ for $V$ in the diagram of Fig. 9. Then the commutative diagram in Fig. 9 implies that the diagram in Fig. 10 commutes for all objects $U, V, W$ in $\mathcal{C}$. Observe that the commutative diagram in Fig. 10 is precisely the dashed portion of the diagram seen in Fig. 11. Thus, to prove that the diagram in Fig. 8 commutes it suffices to show that the outermost perimeter of the diagram seen in Fig. 12 commutes.

Indeed, the square diagram indicated by $\textbf{1#}$ in Fig. 12 commutes as a consequence of the naturality of $d$ and the fact that $G(d_{U, V}) = d_{G(U), G(V)}$. While the diagram indicated by $\textbf{2#}$ in Fig. 12 commutes because $\otimes$ is a functor. Therefore, the diagram in Fig. 8 commutes. □

Next we introduce a variation on the definition of hom-braided categories.
Fig. 11 Proof of the Hom–Yang–Baxter property

\[
\begin{align*}
\{G(V) \otimes G(U)\} \otimes F(W) & \xrightarrow{4_{U,V} \otimes 1_{F(W)}} (U \otimes V) \otimes F(W) \\
GF(W) \otimes (G(U) \otimes G(V)) & \xrightarrow{1_{GF(W)} \otimes (4_{U,V} \otimes 1_{F(W)})} (G(U) \otimes G(V)) \otimes (G(U) \otimes G(V)) \\
FG^2(W) \otimes (G^2(V) \otimes G^2(U)) & \xrightarrow{1_{FG^2(W)} \otimes (4_{U,V} \otimes 1_{F(W)})} (G(U) \otimes G(V)) \otimes (G(U) \otimes G(V))
\end{align*}
\]

Fig. 12 Naturality for \(d\) and \(G(d_{U,V}) = d_{G(U),G(V)}\)

Definition 6.5 Let \(\mathcal{C} = (C, \otimes, F, G, a, \Phi)\) be a hom–tensor category. We say that a hom-commutativity constraint \(d\) (as in Definition 5.1) satisfies the \((\text{wH}1)\) property if the diagram in Fig. 13 commutes for all objects \(U, V\) and \(W\) of the category \(\mathcal{C}\). Furthermore, we say that a hom-commutativity constraint \(d\) satisfies the \((\text{wH}2)\) property if the diagram in Fig. 14 commutes for all objects \(U, V\) and \(W\) of the category \(\mathcal{C}\).

Definition 6.6 Let \(\mathcal{C} = (C, \otimes, F, G, a, \Phi)\) be a hom–tensor category. A weak hom-braiding \(d\) in \(\mathcal{C}\) is a hom-commutativity constraint with the following conditions:

(i) \(d\) satisfies \((\text{wH}1)\) and \((\text{wH}2)\);
(ii) For all objects \(U\) and \(V\) in the category \(\mathcal{C}\), \(G(d_{U,V}) = d_{G(U),G(V)}\).

A weakly hom-braided category \((C, \otimes, F, G, a, \Phi, d)\) is a hom–tensor category with weak hom-braiding \(d\).

Using arguments similar to those found in the proof for Lemma 6.2, one can show the following.
Lemma 6.7 The commutative diagram in Fig. 15 is equivalent to the (wH2) property of a hom-commutativity constraint $d$. We call the relation in Fig. 15 the (wH2') property of $d$.

Definition 6.8 Let $\mathcal{C} = (C, \otimes, F, G, a, \Phi)$ be a hom–tensor category and $d$ a hom-commutativity constraint. We say that $d$ has the weak Hom–Yang–Baxter property if $d$ satisfies the equation in Fig. 16.
Proposition 6.9 Let $\mathcal{C} = (\mathcal{C}, \otimes, F, G, a, \Phi, d)$ be a weakly hom-braided category. Then $d$ has the weak Hom–Yang–Baxter property.

Proof Let $\mathcal{C} = (\mathcal{C}, \otimes, F, G, a, \Phi, d)$ be a weakly hom-braided category. The commutative diagram in Fig. 15 together with the hom-associativity constraint $a$ being an isomorphism implies that the diagram in Fig. 16 commutes for all objects $U, V, W$ in $\mathcal{C}$. Observe that the commutative diagram in Fig. 16 is precisely the bold portion of the diagram seen in Fig. 19.
Next, consider substituting the object $G(V)$ for $U$, the object $G(U)$ for $V$ and the object $G(W)$ for $W$ in the diagram of Fig. 17. Then the commutative diagram in Fig. 17 implies that the diagram in Fig. 18 commutes for all objects $U$, $V$, $W$ in $\mathcal{C}$. Observe that the commutative diagram in Fig. 18 is precisely the dashed portion of the diagram in Fig. 19. Thus, to prove that the diagram in Fig. 16 commutes it suffices to show that the outermost perimeter of the diagram in Fig. 20 commutes.

Indeed, the square diagram labeled 1# in Fig. 20 commutes as a consequence of the naturality of $d$ and the fact that $G(\Phi_{F(W)}) = \Phi_{GF(W)}$ and $G(d_{U,V}) = d_{G(U),G(V)}$. The square diagram labeled 2# commutes because $\otimes$ is a functor, $d$ is natural and $G(\Phi_{G(V)}) = \Phi_{G^2(V)}$. Thus, the diagram in Fig. 16 commutes.

We can give now the connection between hom-braided categories and weakly hom-braided categories.

**Proposition 6.10** If $\mathcal{C} = (\mathcal{C}, \otimes, F, G, a, \Phi, d)$ is a hom-braided category then $\mathcal{C} = (\mathcal{C}, \otimes, F, G, a, \Phi, d)$ is a weakly hom-braided category.
Proof After comparing Definitions 5.3 and 6.6 for hom-braiding and weak hom-braiding, it suffices to show that the (H1) property implies the (wH1) property and the (H2) property implies the (wH2) property.

We prove that the (H1) property implies the (wH1) property. Observe that the outermost perimeter of the diagram in Fig. 21, indicated by a dashed line, is the commutative diagram of the (H1) property from Fig. 4. In addition, observe that the innermost 7-gon of the diagram in Fig. 21, indicated by the bold line, is the diagram of the (wH1) property as seen in Fig. 13. The plan is to show that each of the square portions of this diagram, labeled id\(_1\), id\(_2\), id\(_3\), 1\#, 2\#, 3\# and 4\# commute. Once this is established then the assumed commutativity of the dashed portion of the diagram in Fig. 21, and the fact that (id\(_U\) \otimes id\(_V\)) \otimes id_{F(W)} is an isomorphism, will imply the commutativity of the bold portion. That is, the (H1) property implies the (wH1) property.
Clearly, the square portions labeled \( \text{id}_1, \text{id}_2 \) and \( \text{id}_3 \) of the diagram in Fig. 21 commute. Additionally, both the square portions labeled \( 1# \) and \( 4# \) commute since \( \otimes \) is a functor. More precisely, we have

\[
\begin{align*}
d_{U,V} \otimes \Phi_{F(W)} &= \left( (\text{id}_{G(V)} \otimes \text{id}_{G(U)}) \otimes \Phi_{F(W)} \right) \left( d_{U,V} \otimes \text{id}_{F(W)} \right) \\
&= \left( d_{U,V} \otimes \Phi_{F(W)} \right) \left( (\text{id}_U \otimes \text{id}_V) \otimes \text{id}_{F(W)} \right),
\end{align*}
\]

\[
\begin{align*}
\Phi_{F(G(V))} \otimes \left( \Phi_{G(W)} \otimes \Phi_{G(U)} \right) &= \left( \Phi_{F(G(V))} \otimes \left( \Phi_{G(W)} \otimes \text{id}_{G^2(U)} \right) \right) \left( \text{id}_{F(G(V))} \otimes \left( \text{id}_{G(W)} \otimes \Phi_{G(U)} \right) \right) \\
&= \left( \Phi_{F(G(V))} \otimes \left( \Phi_{G(W)} \otimes \Phi_{G(U)} \right) \right) \left( \text{id}_{F(G(V))} \otimes \left( \text{id}_{G(W)} \otimes \text{id}_{G(U)} \right) \right).
\end{align*}
\]

Furthermore, the square portion labeled \( 2# \) commutes since the hom-associativity constraint \( a \) is natural and \( F(\Phi_W) = \Phi_{F(W)} \). Finally, the square portion labeled \( 3# \) commutes since the hom-commutativity constraint \( d \) is natural and \( G(\Phi_W) = \Phi_{G(W)} \). Indeed, we have

\[
\begin{align*}
\Phi_{G(V)} \otimes \left( \Phi_{G(W)} \otimes \text{id}_{G^2(U)} \right) \circ d_{G(U), W} = \Phi_{F(G(V))} \otimes \left( \text{id}_{G(W)} \otimes \Phi_{W} \right)
\end{align*}
\]

which gives \( 3# \), by using again the fact that \( \otimes \) is a functor.

A similar argument shows that the (H2) property implies the (wh2) property. \( \Box \)

**Remark 6.11** The above proof (i.e. the diagram in Fig. 21) also shows that the converse of Proposition 6.10 is true if the natural transformation \( \Phi \) is assumed to be an isomorphism.

We recall the following result ([29], Theorem 4.4).

**Proposition 6.12** Let \((H,m_H,\Delta_H,\alpha_H,\rho_H,R)\) be a quasitriangular hom-bialgebra such that \((\alpha_H \otimes \alpha_H)(R) = R\) and \((M,\alpha_M)\) a left \( H \)-module. Then the linear map \( B : M \otimes M \rightarrow M \otimes M, B(m \otimes m') = \sum_i t_i \cdot m' \otimes s_i \cdot m, \) for all \( m,m' \in M \) (where we denoted as before \( R = \sum_i t_i \otimes s_i \)) is a solution of the Hom–Yang–Baxter equation with respect to \( \alpha_M \).

We want to obtain (a more general version of) this result as a consequence of the theory we developed.

**Proposition 6.13** Let \((H,m_H,\Delta_H,\alpha_H,\psi_H,R)\) be a quasitriangular hom-bialgebra, with notation \( R = \sum_i t_i \otimes s_i, \) such that \((\alpha_H \otimes \alpha_H)(R) = R = (\psi_H \otimes \psi_H)(R)\). If \((M,\alpha_M)\) is a left \( H \)-module, then the linear map \( B : M \otimes M \rightarrow M \otimes M, B(m \otimes m') = \sum_i t_i \cdot m' \otimes s_i \cdot m, \) for all \( m,m' \in M \), is a solution of the Hom–Yang–Baxter equation with respect to \( \alpha_M \).

**Proof** By Proposition 5.6, the linear map \( d_{U,V} : U \otimes V \rightarrow G(V) \otimes G(U) \), defined by \( d_{U,V}(u \otimes v) = \sum_i t_i \cdot v \otimes s_i \cdot u, \) is a hom-braiding. By Proposition 6.10, it is also a weak hom-braiding, so, by Proposition 6.9, the diagram in Fig. 16 is commutative. We write the diagram in Fig. 16 with \( U = V = W = M \) and we note that, since \((\alpha_H \otimes \alpha_H)(R) = R\), we have \( d_{M,M} = d_{G(M),G(M)} = d_{G^2(M),G^2(M)} = B \). So, the commutativity of the diagram in Fig. 16 reads

\[
\begin{align*}
(B \otimes \alpha_M) \circ (\alpha_M \otimes B) \circ (B \otimes \alpha_M) = (\alpha_M \otimes B) \circ (B \otimes \alpha_M) \circ (\alpha_M \otimes B),
\end{align*}
\]

which is exactly the Hom–Yang–Baxter equation for \( B \) with respect to \( \alpha_M \). \( \Box \)
7 Yetter–Drinfeld Modules

Throughout this section, $H = (H, m_H, \Delta_H, \alpha_H, \psi_H)$ will be a hom-bialgebra for which $\alpha_H = \psi_H$ and $\alpha_H$ is bijective. We recall the following concept and results from [18].

**Definition 7.1** Let $M$ be a $k$-vector space and $\alpha_M : M \to M$ a $k$-linear map such that $(M, \alpha_M)$ is a left $H$-module with action $H \otimes M \to M$, $h \otimes m \mapsto h \cdot m$ and a left $H$-comodule with coaction $M \to H \otimes M$, $m \mapsto \sum (h(1)) \otimes (h(2)) \cdot m(0)$. Then $(M, \alpha_M)$ is called a (left-left) **Yetter–Drinfeld module** over $H$ if the following identity holds, for all $h \in H, m \in M$:

$$\sum (h(1)) \cdot m(-1) \alpha_H^2(h(2)) \otimes (h(1)) \cdot m(0) = \sum \alpha_H^2(h(1)) \alpha_H(m(-1)) \otimes \alpha_H(h(2)) \cdot m(0).$$

(34)

We denote by $H^\mathcal{YD}$ (respectively $H^\mathcal{YD}$) the category whose objects are Yetter–Drinfeld modules $(M, \alpha_M)$ over $H$ (respectively Yetter–Drinfeld modules $(M, \alpha_M)$ over $H$ with $\alpha_M$ bijective); the morphisms in each of these categories are morphisms of left $H$-modules and left $H$-comodules.

**Proposition 7.2** Let $(M, \alpha_M)$ and $(N, \alpha_N)$ be two Yetter–Drinfeld modules over $H$, with notation as above, and define the $k$-linear maps

$$H \otimes (M \otimes N) \to M \otimes N, \quad h \otimes (m \otimes n) \mapsto \sum h(1) \cdot m \otimes h(2) \cdot n,$$

$$M \otimes N \to H \otimes (M \otimes N), \quad m \otimes n \mapsto \sum \alpha_H^{-2}(m(-1)) n(-1) \otimes (m(0)) \otimes n(0)).$$

Then $(M \otimes N, \alpha_M \otimes \alpha_N)$ with these structures is a Yetter–Drinfeld module over $H$, denoted by $M \mathcal{YD} N$.
Proposition 7.3 \( \hat{\mathcal{H}}_H \mathcal{YD} \) is a quasi-braided category, with tensor product \( \hat{\otimes} \) and associativity constraints and quasi-braiding defined, for \((M, \alpha_M), (N, \alpha_N), (P, \alpha_P)\) objects in \( \hat{\mathcal{H}}_H \mathcal{YD} \) by
\[
\begin{align*}
  b_{M,N,P} : (M \otimes N) \otimes P & \to M \otimes (N \otimes P), \\
  b_{M,N,P}((m \otimes n) \otimes p) & = \alpha^{-1}_M(m) \otimes (n \otimes \alpha_P(p)), \\
  c_{M,N} : M \otimes N & \to N \otimes M, \\
  c_{M,N}(m \otimes n) & = \sum \alpha^{-1}_N(\alpha^{-1}_H(m_{(-1)}) \cdot n) \otimes \alpha^{-1}_M(m_{(0)}) \cdot n \otimes m_{(0)}).
\end{align*}
\]

Proposition 7.4 Let \((M, \alpha_M), (N, \alpha_N) \in \hat{\mathcal{H}}_H \mathcal{YD}\) and define the \( \mathbf{k} \)-linear map
\[
B_{M,N} : M \otimes N \to N \otimes M, \quad B_{M,N}(m \otimes n) = \sum \alpha^{-1}_H(m_{(-1)}) \cdot n \otimes m_{(0)}.
\tag{35}
\]
Then, we have \((\alpha_N \otimes \alpha_M) \circ B_{M,N} = B_{M,N} \circ (\alpha_M \otimes \alpha_N)\) and, if \((P, \alpha_P)\) is another Yetter–Drinfeld module over \(H\), the maps \(B_{\_\_\_}\) satisfy the Hom–Yang–Baxter equation
\[
(\alpha_P \otimes B_{M,N}) \circ (B_{M,P} \otimes \alpha_N) \circ (\alpha_M \otimes B_{N,P}) = (B_{N,P} \otimes \alpha_M) \circ (\alpha_N \otimes B_{M,P}) \circ (B_{M,N} \otimes \alpha_P).
\tag{36}
\]

Our aim now is to prove the following result.

Proposition 7.5 \( \hat{\mathcal{H}}_H \mathcal{YD} \) may be organized as a hom-braided category, with tensor product \( \hat{\otimes} \) and hom-braiding the family of maps \(B_{\_\_\_}\) defined by (35).

Proof We define the hom-associativity constraints \(a_{M,N,P}\), the functors \(F\) and \(G\) and the natural transformation \(\Phi\). Since any Yetter–Drinfeld module over \(H\) is in particular a left \(H\)-module, \(a\), \(F\), \(G\) and \(\Phi\) will be, at the level of left \(H\)-modules, the ones defined in Proposition 3.5 (since we assumed that \(\psi_P = \alpha_H\), we actually have \(F = G\) in this case). To simplify the notation, we will denote the elements in \(F(M)\) by \(m\) instead of \(\tilde{m}\). We need to extend the functor \(F (= G)\) from left \(H\)-modules to Yetter–Drinfeld modules. If \((M, \alpha_M)\) is a Yetter–Drinfeld module, with left \(H\)-comodule structure \(M \to H \otimes M, m \mapsto \sum m_{(-1)} \otimes m_{(0)}\), then \(F(M)\) becomes a left \(H\)-comodule with structure \(F(M) \to H \otimes F(M), m \mapsto \sum m_{(-1)} \otimes m_{(0)}\) and moreover \(F(M)\) with these structures becomes a Yetter–Drinfeld module over \(H\) and, if \(M, N \in \hat{\mathcal{H}}_H \mathcal{YD}\), then \(F(M \otimes N) = F(M) \otimes F(N)\) as objects in \(\hat{\mathcal{H}}_H \mathcal{YD}\). Then, \(a_{M,N,P} : (M \hat{\otimes} N) \hat{\otimes} F(P) \to F(M) \hat{\otimes} (N \hat{\otimes} P)\) and \(\Phi_M : M \to F(M)\) are also morphisms of left \(H\)-comodules, and \(F(\Phi_M) = \Phi_{F(M)}, \text{ for } M, N, P \in \hat{\mathcal{H}}_H \mathcal{YD}\). The proofs of these statements are similar to those in Proposition 3.5 and will be omitted. The conclusion is that \((\hat{\mathcal{H}}_H \mathcal{YD}, \hat{\otimes}, F, F, a, \Phi)\) is a hom–tensor category.

We begin to prove that the family of maps \(B_{\_\_\_}\) is a hom-braiding. First, we need to prove that, for \((M, \alpha_M), (N, \alpha_N) \in \hat{\mathcal{H}}_H \mathcal{YD}\), the maps \(B_{M,N}\), regarded as maps from \(M \hat{\otimes} N\) to \(F(N) \hat{\otimes} F(M)\), are morphisms in \(\hat{\mathcal{H}}_H \mathcal{YD}\). We know from Proposition 7.4 that \((\alpha_N \otimes \alpha_M) \circ B_{M,N} = B_{M,N} \circ (\alpha_M \otimes \alpha_N)\). Now we prove that \(B_{M,N}(h \cdot (m \otimes n)) = h \cdot \alpha B_{M,N}(m \otimes n)\), for all \(h \in H, m \in M, n \in N\). We compute:
\[
B_{M,N}(h \cdot (m \otimes n)) = B_{M,N}\left( \sum h_{(1)} \cdot m \otimes h_{(2)} \cdot n \right)
= \sum \alpha^{-1}_H(h_{(1)} \cdot m_{(-1)}) \cdot (h_{(2)} \cdot n) \otimes (h_{(1)} \cdot m_{(0)})
= \sum \alpha_H\left( \alpha^{-2}_H(h_{(1)} \cdot m_{(-1)}) \cdot (h_{(2)} \cdot n) \otimes (h_{(1)} \cdot m_{(0)})
= \sum \left( \alpha^{-2}_H(h_{(1)} \cdot m_{(-1)}) h_{(2)} \right) \cdot \alpha_N(n) \otimes (h_{(1)} \cdot m_{(0)})
= \sum \alpha_N(n) \otimes \alpha_H(h_{(2)}) \cdot m_{(0)}
\tag{11}
\]
\[(34)\]
We prove that $B_{M,N}$ is a morphism of left $H$-comodules, i.e. $(\text{id}_H \otimes B_{M,N}) \circ \lambda_{M \otimes N} = \lambda_{F(N) \otimes F(M)} \circ B_{M,N}$. We compute, for $m \in M$, $n \in N$:

$\lambda_{F(N) \otimes F(M)}(B_{M,N}(m \otimes n))$

$= \lambda_{F(N) \otimes F(M)}\left(\sum \alpha_H^{-1}(m_{(-1)}) \cdot n \otimes m_{(0)}\right)$

$= \sum \alpha_H^{-2}\left((\alpha_H^{-1}(m_{(-1)}) \cdot n)_{(-1)} (m_{(0)}{(-1)} \otimes (m_{(0)}{(-1)}))\right) \otimes (\alpha_H^{-1}(m_{(-1)}) \cdot n)_{(0)} \otimes (m_{(0)}{(-1)})_{(0)}$

$= \sum \alpha_H^{-3}\left((\alpha_H^{-2}(m_{(-1)}{(-1)} \cdot n)_{(-1)} (m_{(0)}{(-1)} \otimes (m_{(0)}{(-1)}))\right) \otimes (\alpha_H^{-2}(m_{(-1)}{(-1)} \cdot n)_{(0)} \otimes (m_{(0)}{(-1)})_{(0)}$

$= \sum \alpha_H^{-3}\left((\alpha_H^{-2}(m_{(-1)}{(-1)} \cdot n)_{(-1)} (m_{(0)}{(-1)} \otimes (m_{(0)}{(-1)}))\right) \otimes (\alpha_H^{-2}(m_{(-1)}{(-1)} \cdot n)_{(0)} \otimes (m_{(0)}{(-1)})_{(0)}$

$= \sum \alpha_H^{-3}\left((\alpha_H^{-2}(m_{(-1)}{(-1)} \cdot n)_{(-1)} (m_{(0)}{(-1)} \otimes (m_{(0)}{(-1)}))\right) \otimes (\alpha_H^{-2}(m_{(-1)}{(-1)} \cdot n)_{(0)} \otimes (m_{(0)}{(-1)})_{(0)}$

$= \sum \alpha_H^{-3}\left((\alpha_H^{-2}(m_{(-1)}{(-1)} \cdot n)_{(-1)} (m_{(0)}{(-1)} \otimes (m_{(0)}{(-1)}))\right) \otimes (\alpha_H^{-2}(m_{(-1)}{(-1)} \cdot n)_{(0)} \otimes (m_{(0)}{(-1)})_{(0)}$

$= (\text{id}_H \otimes B_{M,N})\left(\sum \alpha_H^{-2}(m_{(-1)}{(-1)} \cdot n)_{(-1)} (m_{(0)}{(-1)} \otimes (m_{(0)}{(-1)}))\right)$

$= (\text{id}_H \otimes B_{M,N})\left(\lambda_{M \otimes N}(m \otimes n)\right), \quad q.e.d.$

The fact that $B_{\ldots}$ is natural is easy to prove and left to the reader. We prove now that $B_{\ldots}$ satisfies (H1); to prove that it satisfies (H2) is similar and left to the reader. So, let $(U, \alpha_U), (V, \alpha_V), (W, \alpha_W) \in \mathcal{H}_\mathbb{YD}$ and $u \in U, v \in V, w \in W$. We compute:

$(\text{id}_{F^2(V)} \otimes B_{F(U),W}) \circ a_{F(V),F(U),W} \circ (B_{U,V} \otimes \text{id}_{F(W)})(u \otimes v) \otimes w)$

$= \left(\text{id}_{F^2(V)} \otimes B_{F(U),W}\right) \circ a_{F(V),F(U),W}\left(\sum \alpha_H^{-1}(u_{(-1)}) \cdot v \otimes u_{(0)}\right) \otimes w$

$= (\text{id}_{F^2(V)} \otimes B_{F(U),W})\left(\sum \alpha_H^{-1}(u_{(-1)}) \cdot v \otimes u_{(0)}\right) \otimes \alpha_H^{-1}(u_{(0)}{(-1)} \cdot w \otimes (u_{(0)})_{(0)}$
The only thing left to prove is that \( F(B_{M,N}) = B_{F(M),F(N)} \), for all \((M, \alpha_M), (N, \alpha_N) \in H^\YD\), that is \( B_{M,N} = B_{F(M),F(N)} \). For \( m \in M, n \in N \) we compute:

\[
B_{F(M),F(N)}(m \otimes n) = \sum \alpha_M^{-1}(m_{<1>}) \cdot n \otimes m_{<0>} = \sum \alpha_N^{-1}(m_{<1>}) \cdot n \otimes m_{<0>}
\]

\[
= \sum \alpha_M^{-1}(m_{<1>}) \cdot n \otimes m_{<0>} = B_{M,N}(m \otimes n),
\]

finishing the proof. 

\[\square\]

**Remark 7.6** Similarly to what we did in Proposition 6.13, we can reobtain the relation (36) in Proposition 7.4 as a consequence of our theory. Indeed, since \( B_{\ldots,\ldots} \) is a hom-braiding, it is also a weak hom-braiding, so the diagram in Fig. 16 is commutative. But since the functor \( G (= F) \) acts as identity on morphisms and we know that \( B_{G(M),G(N)} = G(B_{M,N}) \), for all \((M, \alpha_M), (N, \alpha_N) \in H^\YD\), it follows that the commutativity of the diagram in Fig. 16 reduces to

\[
(\alpha_W \otimes B_{U,V}) \circ (B_{U,W} \otimes \alpha_V) \circ (\alpha_U \otimes B_{V,W}) = (B_{V,W} \otimes \alpha_U) \circ (\alpha_V \otimes B_{U,W}) \circ (B_{U,V} \otimes \alpha_W),
\]

which is exactly the Hom–Yang–Baxter equation (36).

## 8 Hom-Tensor Categories Versus Tensor Categories

We show that under certain conditions one can associate a pre-tensor category to a hom–tensor category.

**Proposition 8.1** Let \( \mathcal{C} = (\mathcal{C}, \otimes, F, G, a, \Phi) \) be a hom–tensor category. Suppose that \( \Theta : id_{\mathcal{C}} \rightarrow F \) is a natural isomorphism such that \( \Theta_{M \otimes N} = \Theta_M \otimes \Theta_N \) for all objects \( M, N \in \mathcal{C} \).

Consider \( b_{U,V,W} : (U \otimes V) \otimes W \rightarrow U \otimes (V \otimes W) \) defined for all objects \( U, V, W \in \mathcal{C} \) by

\[
b_{U,V,W} = (\Theta_U^{-1} \otimes id_{V \otimes W}) \circ a_{U,V,W} \circ (id_{U \otimes V} \otimes \Theta_W),
\]

see Fig. 22. Then \( b_{U,V,W} \) is an associativity constraint for the pre-tensor category \((\mathcal{C}, \otimes, b)\).
The definition of $b_{U,V,W}$:

$$(U \otimes V) \otimes W \xrightarrow{b_{U,V,W}} U \otimes (V \otimes W)$$

$$(U \otimes V) \otimes F(W) \xrightarrow{a_{U,V,W}} F(U) \otimes (V \otimes W)$$

Fig. 22

Hom-associativity for $a$ implies associativity for $b$

**Proof**

Notice that $b_{U,V,W}$ is a natural isomorphism being a composition of natural isomorphisms. So we only need to check that $b_{U,V,W}$ satisfies the Pentagon axiom of an associativity constraint.

Consider the diagram in Fig. 23. Observe that the inner bold portion of this diagram is the “Pentagon” axiom of the hom-associativity constraint $a_{U,V,W}$ and that the outer dashed portion of this diagram is the Pentagon axiom of the mapping $b_{U,V,W}$. We will show that commutativity of the bold portion of the diagram implies the commutativity of the dashed portion of the diagram. This will be done by proving that each square portion of the diagram labeled 1#, 2#, 3#, 4# and 5# commutes for all objects $U$, $V$, $W$, $X \in C$.

We begin with the square portion of Fig. 23 labeled 1#. We have:

$$
\begin{align*}
((\Theta_U \otimes \text{id}_V \otimes W) \otimes (F(\Theta_X) \otimes \Theta_X)) & \circ (b_{U,V,W} \otimes \text{id}_X) \\
& = ((\Theta_U \otimes \text{id}_V \otimes W) \circ b_{U,V,W}) \otimes ((F(\Theta_X) \circ \Theta_X) \circ \text{id}_X)
\end{align*}
$$

(Fig. 22)

$$
\begin{align*}
& \equiv (a_{U,V,W} \circ (\text{id}_U \otimes V \otimes \Theta_W)) \otimes (\text{id}_{F^2(X)} \circ (F(\Theta_X) \circ \Theta_X)) \\
& = (a_{U,V,W} \otimes \text{id}_{F^2(X)}) \circ ((\text{id}_U \otimes V \otimes \Theta_W) \otimes (F(\Theta_X) \circ \Theta_X)).
\end{align*}
$$

The first and third equality follow from the functoriality of the tensor product. The second equality follows from the definition of $b_{U,V,W}$ (see Fig. 22).

Next we consider the square portion of Fig. 23 labeled 2#. Notice that the diagram in Fig. 24 commutes for all objects $U$, $V$, $W$, $X \in C$. Indeed, the portion of the diagram labeled (*) commutes by the definition of $b_{U,V \otimes W,X}$, and the portion of the diagram labeled (**) commutes by the naturality of $a_{U,V \otimes W,X}$. 

\[ Springer \]
Now we consider the square portion of Fig. 23 labeled 3#. Computing we get that
\[
((F (\Theta_U) \circ \Theta_U) \circ (\Theta_V \otimes \text{id}_{W \otimes X})) \circ (\text{id}_U \otimes b_{V,W,X}) \\
= (F (\Theta_U) \circ \Theta_U) \otimes ((\Theta_V \otimes \text{id}_{W \otimes X}) \circ b_{V,W,X}) \\
= (\text{id}_{F^2(U)} \otimes a_{V,W,X}) \circ ((F (\Theta_U) \circ \Theta_U) \otimes (\text{id}_{V \otimes W} \otimes \Theta_X)) .
\]

The first and third equality are consequence of the functoriality of the tensor product, while the second equality follows from the definition of $b_{V,W,X}$.

Next we consider the square portion of Fig. 23 labeled 4#. Notice that the diagram in Fig. 25 commutes for all objects $U, V, W, X \in C$. Indeed, the portion of the diagram labeled $(\circ)$ commutes by the definition of $b_{U \otimes V, W \otimes X}$ and the portion of the diagram labeled $(\bullet \bullet)$ commutes by the naturality of $a_{U \otimes V, W \otimes X}$.

Finally we consider the square portion of Fig. 23 labeled 5#. Notice that the diagram in Fig. 26 commutes for all objects $U, V, W, X \in C$. The portion of the diagram labeled $(\circ \circ)$ commutes by the definition of $b_{U,V,W} \otimes X$ and the portion of the diagram labeled $(\circ \circ \circ)$ commutes by the naturality of $a_{U,V,W \otimes X}$.

Next we turn our attention to the relation between hom-braided and quasi-braided categories (Fig. 27).

**Proposition 8.2** Let $\mathcal{C} = (C, \otimes, F, G, a, \Phi, d)$ be a hom-braided category, $\Theta$ and $b_{U,V,W}$ as in Proposition 8.1. Suppose that $\Phi$ is a natural isomorphism and $G(\Theta_U) = \Theta_{G(U)}$. Define $c_{U,V} : U \otimes V \to V \otimes U$.
Fig. 26 The diagram for 5#

\[ (U \otimes V) \otimes (W \otimes X) \xrightarrow{\Phi_{U,V,W \otimes X}} U \otimes (V \otimes (W \otimes X)) \]

\[ (U \otimes V) \otimes F(W \otimes X) \xrightarrow{a_{U,V,W \otimes X}} F(U) \otimes (V \otimes (W \otimes X)) \]

\[ (F(U) \otimes F(V)) \otimes F(W \otimes X) \xrightarrow{a_{F(U),F(V),W \otimes X}} F^2(U) \otimes (F(V) \otimes (W \otimes X)) \]

Fig. 27 Definition of \( c_{U,V} : U \otimes V \rightarrow V \otimes U \)

\[ c_{U,V} = \left( \Phi_V^{-1} \otimes \Phi_U^{-1} \right) \circ d_{U,V} \] (38)

for all objects \( U, V \in \mathcal{C} \). Then \( c_{U,V} \) is a quasi-braiding for the quasi-braided category \( (\mathcal{C}, \otimes, b, c) \).

**Proof** Being a composition of natural morphisms, \( c_{U,V} \) is a natural morphism. So we only need to check that \( c_{U,V} \) satisfies the Hexagon axiom of a braiding. We check the first Hexagon axiom for \( c_{U,V} \).

Consider the diagram in Fig. 28. Observe that the inner bold portion of this diagram is the (H1) property of the hom-braiding \( d_{U,V} \) and the outer dashed portion of this diagram is the first Hexagon axiom property for \( c_{U,V} \). We will show that commutativity of the bold portion of the diagram implies the commutativity of the dashed portion of the diagram. This will be done by proving that each square portion of the diagram labeled 1#, 2#, 3#, 4#, 5#, 6# and 7# commutes for all objects \( U, V, W \in \mathcal{C} \).

We begin with the square portion of Fig. 28 labeled 1#. Computing we get that

\[
\left( (\Phi_V \otimes \Phi_U) \otimes \Theta_W \right) \circ \left( c_{U,V} \otimes \text{id}_W \right) = \left( (\Phi_V \otimes \Phi_U) \circ c_{U,V} \right) \otimes \left( \Theta_W \circ \text{id}_U \right) = d_{U,V} \otimes \Theta_W = (d_{U,V} \otimes \text{id}_F(W)) \circ (\text{id}_{U \otimes V} \otimes \Theta_W).
\]

The first and third equality follow from the functoriality of the tensor product. The second equality is the definition of \( c_{U,V} \).

Next we consider the square portion of Fig. 28 labeled 2#, which coincides with the diagram in Fig. 29 by the functoriality of the tensor product. The portion of the diagram in Fig. 29 labeled (*) commutes by the definition of \( b \), and the portion labeled (**) commutes by the naturality of \( a_{U,V,W} \).

Now we consider the square portion of Fig. 28 labeled 3#, which coincides with the diagram in Fig. 30 by the functoriality of the tensor product. The portion of the diagram in Fig. 30 labeled (●) commutes by the definition of \( c_{V,W} \) and the functoriality of the tensor product, and the portion of the diagram labeled (●●) commutes by the naturality of \( d_{U,W} \) and the functoriality of the tensor product.
**Fig. 28** The (H1) property implies the first Hexagon axiom

\[ (V \otimes U) \otimes W \xrightarrow{h_{V,U,W}} V \otimes (U \otimes W) \]

\[ (\Phi_V \otimes \Phi_U) \otimes \text{id}_{F(W)} \]

\[ (G(V) \otimes G(U)) \otimes F(W) \xrightarrow{h_{G(V),G(U),W}} FG(V) \otimes (G(U) \otimes W) \]

**Fig. 29** The diagram for 2#}

\[ V \otimes (U \otimes W) \xrightarrow{\text{id}_V \otimes \text{id}_{U,W}} V \otimes (W \otimes U) \]

\[ F(V) \otimes (U \otimes W) \xrightarrow{F(\Phi_V) \otimes \text{id}_{U,W}} F(V) \otimes (G(W) \otimes G(U)) \]

\[ FG(V) \otimes (G(U) \otimes W) \xrightarrow{\text{id}_{FG(V)} \otimes \text{id}_{G(U),W}} FG(V) \otimes (G(W) \otimes G^2(U)) \]

**Fig. 30** The diagram for 3#
The square portion labeled 4# of Fig. 28 commutes by the definition of \( b_{U,V,W} \).

Next we consider the square portion of Fig. 28 labeled 5#. In Fig. 31, the portion of the diagram labeled \((\circ)\) commutes by the definition of \( c_{U,V\otimes W} \) and the portion of the diagram labeled \((\bowtie)\) commutes by the naturality of \( d_{U,V\otimes W} \). Now we notice that the diagram in Fig. 31 coincides with the square portion of Fig. 28 labeled 5# by using the functoriality of the tensor product and the axiom \( \Phi_V \otimes \Phi_W = \Phi_V \otimes \Phi_W \) from the definition of a hom–tensor category.

Next we consider the square portion of Fig. 28 labeled 6#. In Fig. 32 the portion of the diagram labeled \((\sharp)\) commutes by naturality of \( b_{V,W,U} \) and the portion of the diagram labeled \((\sharp\sharp)\) commutes by the definition of \( b_{G(V), G(W), G(U)} \). Now we notice that the diagram in Fig. 32 coincides with the square portion of Fig. 28 labeled 6# by using the functoriality of the tensor product and the fact that \( G(\Theta_V) = \Theta_{G(V)} \), which also allows us to glue 6# and 7# together.

Finally we consider the square portion of Fig. 28 labeled 7#. This is commutative by using the functoriality of \( \otimes \), the fact that \( \Phi_{G(U)} = G(\Phi_U) \) and because \( \Theta_{G(V)} \circ \Phi_V = \Phi_{F(V)} \circ \Theta_V \). The last statement is a consequence of the fact that \( \Theta \) is a natural transformation and \( F(\Phi_V) = \Phi_{F(V)} \) (see Fig. 33).

Thus the (H1) property for \( d_{U,V} \) implies the first Hexagon axiom property for \( c_{U,V} \). Using similar arguments one can show that the (H2) property for \( d_{U,V} \) implies the second Hexagon axiom property for \( c_{U,V} \). Therefore, \( c_{U,V} \) is a quasi-braiding for the quasi-braided category \( (C, \otimes, b, c) \). \( \square \)

**Remark 8.3** Let again \( H = (H, m_H, \Delta_H, \alpha_H, \psi_H) \) be a hom-bialgebra for which \( \alpha_H = \psi_H \) and \( \alpha_H \) is bijective. We can give a new proof of Proposition 7.3 based on the results obtained in...
this section. We proceed as follows. We know from Proposition 7.5 that $\mathcal{H}^D$ is a hom-braided category, and it is obvious that its subcategory $\mathcal{H}^\mathbb{Y}D$ inherits this hom-braided structure. We apply Propositions 8.1 and 8.2 to this hom-braided category $\mathcal{H}^\mathbb{Y}D$, in this case the natural isomorphism $\Theta : \text{id}_C \to F$ being the same as the natural isomorphism $\Phi : \text{id}_C \to G$ (we recall that we have $F = G$ in this situation). Thus, we obtain that $\mathcal{H}^\mathbb{Y}D$ becomes a quasi-braided category, and it is very easy to see that its quasi-braided structure is exactly the one that appears in Proposition 7.3.

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