The Near Horizon Geometry (NHG) equation with a cosmological constant $\Lambda$ is considered on compact 2-dimensional manifolds. It is shown that every solution satisfies the Type D equation at every point of the manifold. A similar result known in the literature was valid only for non-degenerate in a suitable way points of a given solution. At the degenerate points the Type D equation was not applicable. In the current paper we prove that the degeneracy is ruled out by the compactness. Using that result we find all the solutions to the NHG equation on compact 2-dimensional manifolds of non-positive Euler characteristics. Some integrability conditions known earlier in the $\Lambda = 0$ case are generalized to arbitrary value of $\Lambda$. They may be still useful for compact 2-manifolds of positive Euler characteristic.

I. INTRODUCTION

The Near Horizon Geometry Equation is defined on a manifold $S$ endowed with a metric tensor $g_{AB}$, and a differential 1-form $\omega_A$, namely it reads

$$\nabla (A \omega_B) + \omega_A \omega_B - \frac{1}{2} R_{AB} + \frac{1}{2} \Lambda g_{AB} = 0 \tag{1}$$

where $\nabla_A$ is the torsion free covariant derivative defined on $S$ by $g_{AB}$, that is

$$\nabla_C g_{AB} = 0 = (\nabla_A \nabla_B - \nabla_B \nabla_A) f, \quad \text{for every} \quad f \in C^2(S), \tag{2}$$

$R_{AB}$ is the Ricci tensor of $g_{AB}$ and $\Lambda$ is a real constant. When $S$ is a $(n - 2)$-dimensional spacelike section of an extremal isolated horizon (for example a degenerate Killing horizon) in $n$-dimensional spacetime that satisfies the Einstein equations with the cosmological constant, then $g_{AB}$ is the induced metric tensor, $\omega_A$ is the induced rotation 1-form potential, and $\Lambda$ is the cosmological constant [1–4]. On the other hand, given any solution $(g_{AB}, \omega_A)$ defined on some $S$, there is a construction of an explicit (“exact”) solution to the Einstein
equations defined on $S \times \mathbb{R} \times \mathbb{R}$ \[5-7\]. The first examples of those spacetimes were obtained by the Bardeen-Horowitz limit from neighborhoods of the extremal horizons in Kerr spacetimes - that is where the name Near Horizon Geometry have come from \[8, 9\].

The NHG equation attracts attention of researchers for several reasons. The first one is scientific curiosity of whether there may be solutions different, than embeddable in the extremal Kerr spacetimes \[10\]. Secondly, every new solution to the NHG equation would automatically lead to a new NHG solution to the Einstein equations. Thirdly, the knowledge of all possible degenerate Killing horizons is important for filling gaps in the black hole uniqueness theorems \[11\].

In the current paper we consider the NHG equation on 2-dimensional manifolds $S$, hence it corresponds to the spacetime dimension 4. The equation still has some secrets. The axisymmetric solutions on topological sphere $S_2$ with $\Lambda = 0$ form a 1-dimensional family that can be parametrized by the area \[10\] (see also \[2\]). All of them correspond to the (spacelike sections) of horizons in extremal Kerr spacetimes. That result was generalized to $\Lambda \neq 0$ \[6, 7\]. What is not known, is whether non-symmetric solutions exist or not on $S_2$. There seem to be none in a neighborhood of the axisymmetric ones \[12\], and definitely there are no static solutions on $S_2$ \[13\].

In 2 dimensions $R_{AB} = Kg_{AB}$, (3)
where $K$ is called the Gauss curvature. Hence, the NHG equation (1) we study in this paper reads

$$\nabla (A\omega_B) + \omega_A A\omega_B - \frac{1}{2} K g_{AB} + \frac{1}{2} A g_{AB} = 0.$$ (4)

The contraction with $g^{AB}$ and integration along $S$ with the area 2-form $\eta_{AB}$ corresponding to the metric $g_{AB}$ gives the following topological constraint [5, 16]

$$\frac{4\pi}{A} (1 - \text{genus}) = \frac{1}{A} \int_S \omega_A A\omega_B \eta + A \geq \Lambda$$ (5)

$$A := \int_S \eta$$ (6)

That constraint allows for:

1. all the values of $\Lambda$, for the genus($S$) = 0 (sphere);
2. only $\Lambda \leq 0$, for genus($S$) = 1 (torus), in particular for $\Lambda = 0$ the only solutions are flat $g_{AB}$ and $\omega_A = 0$;
3. only $\Lambda < 0$, for genus($S$) > 1.

In case 1 the family of axisymmetric solutions is known, whereas the existence of other solutions is an open problem [10, 12]. In case 2 the non-existence of axisymmetric solutions of $\Lambda < 0$ [15] is known. In case 3, it is known [13], that all the static solutions, namely

$$d\omega = 0$$ (7)

have a constant Gauss curvature

$$K = \text{const.}$$ (8)

In this paper we will complete the solution of the NHG equation (4) in case 3.

### III. THE COMPLEX VALUED SCALAR

An important role in the study of the NHG equation is played by the following complex valued (almost) scalar [1, 10]

$$\Psi_2 := \frac{1}{2} \left( K - \frac{\Lambda}{3} + i\Omega \right)$$ (9)

where $\Omega$ is the pseudo scalar characterizing the rotation 2-form, namely

$$\Omega \eta_{AB} := 2\nabla [A\omega_B].$$ (10)
From now on, it will be convenient to introduce a null frame $m_A$ such that

$$g_{AB} = m_A \bar{m}_B + m_B \bar{m}_A, \quad \eta_{AB} = i (\bar{m}_A m_B - \bar{m}_B m_A).$$  \hfill (11)

A null frame $m_A$ is defined locally on $S$, up to the transformations

$$m'_A = e^{i\phi} m_A,$$  \hfill (12)

where $\phi$ is a locally defined function. Denote by $m^A$ the dual frame, such that

$$m^A m_A = 0, \quad m^A \bar{m}_A = 1.$$  \hfill (13)

The key observation is the following integrability condition for (4):

**Proposition 1** If a metric tensor $g_{AB}$ and a 1-form $\omega_A$ defined on a 2d manifold $S$ satisfy the NHG equation, then the scalar $\Psi_2$ satisfies the following equation

$$\bar{m}^A (\nabla_A + 3\omega_A) \Psi_2 = 0.$$  \hfill (14)

Equation (14) is invariant with respect to the transformations (12), that makes it independent of the choice of $m^A$ and globally defined on $S$. It can be derived by acting on (4) with $\nabla_C$, and commuting. This equation was found in the $\Lambda = 0$ case in [10], but it is true for arbitrary $\Lambda$ with suitably defined $\Psi_2$.

Equation (14) implies important properties of $\Psi_2$. The first one follows from the following lemma:

**Lemma 1** Suppose $S$ is a compact oriented 2-manifold endowed with a metric tensor $g_{AB}$ and a 1-form $\omega_A$. Suppose that a function $F : S \to \mathbb{C}$ satisfies the following equation (see (11,13))

$$\bar{m}^A (\nabla_A + 3\omega_A) F = 0.$$  \hfill (15)

Then either

$$F(x) \neq 0, \quad \text{for every} \quad x \in S,$$  \hfill (16)

or

$$F(x) = 0, \quad \text{for every} \quad x \in S.$$  \hfill (17)

For $S = S_2$ (topologically), the proof can be found in [10] (see also [12]). The class of local frames $m^A$ related to each other by (12) allows to define uniquely a decomposition of the complexified tangent space $\mathbb{C}T_x S$ into the algebraic direct sum

$$\mathbb{C}T_x S = T_x^{(1,0)} \oplus T_x^{(0,1)}.$$  \hfill (18)
\[(am^A + b\bar{m}^A)^{(1,0)} = am^A \]
\[(am^A + b\bar{m}^A)^{(0,1)} = b\bar{m}^A, \] (19) (20)

where \(a\) and \(b\) are arbitrary, complex valued, coefficients. It is accompanied by the dual decomposition of the cotangent space

\[\mathcal{CT}_x^*S = T^*(1,0) \oplus T^*(0,1)\] (21)

\[(am_A + b\bar{m}_A)^{(0,1)} = am_A \]
\[(am^A + b\bar{m}^A)^{(1,0)} = b\bar{m}_A. \] (22) (23)

The decompositions are invariant with respect to the transformations (12).

We will also use complex valued coordinates \((z, \bar{z})\) defined locally, in a neighborhood of every point on \(S\), such that

\[m^A \partial_A = P \partial_z, \quad \bar{m}_A dx^A = \frac{1}{P} dz, \] (24)

with some locally defined function \(P\).

With that notation, the conclusion (14) reads

\[(\partial_z + 3 \omega_z) \Psi_2 = 0,\] (25)

where

\[\omega = \omega_z dz + \omega_{\bar{z}} d\bar{z} = \omega^{(1,0)} + \omega^{(0,1)}.\]

Proof of Lemma 1.

Equation (15) now reads

\[\partial_{\bar{z}} F + 3 \omega_{\bar{z}} F = 0.\] (26)

In any simply connected open set \(U \subset S\), there is a function \(\phi_U\) such that

\[\partial_{\bar{z}} \phi_U = 3 \omega_{\bar{z}}.\] (27)

The eq. (15) implies

\[\partial_{\bar{z}} \left( F e^{\phi_U} \right) = 0.\] (28)

Thus, the function

\[f_U := F e^{\phi_U}\] (29)
is holomorphic in $U$. It either does not vanish in $U$, has only isolated zeros in $U$ or is identically zero. In the latter case, by patching $S$ with such open sets, we can prove that $F$ is identically zero on $S$. Hence $F$ either does not vanish in $S$ at all, is identically zero or has only isolated zeros in $S$.

Suppose $F$ has isolated zeros. As our surface $S$ is compact there might be only finitely many zeros, denote them $x_1, ..., x_k \in S$. Consider any of those points, $x_i$, say. Without the lack of generality we assume that the complex coordinates $(z, \bar{z})$ in a neighborhood $U_i$ of $x_i$ are such that

$$z(x_i) = 0,$$

and $U_i$ itself is a coordinate disc of the coordinate radius $\epsilon$.

In $U_i$, the function $F$ may be written as

$$F = z^{n_i}e^{g_i(z)}e^{-\phi_{U_i}}$$

where $n_i \in \mathbb{N}$ is the degree of the zero at $x_i$, and $g_i(z)$ is holomorphic. The value of $n_i$ can be found by integrating the 1-form

$$\frac{\partial_z F}{F} dz = n_i \frac{dz}{z} + \partial_z (g_i(z) - \phi_{U_i}) dz,$$

along the boundary $\partial U_i$ of the disc, namely

$$\lim_{\epsilon \to 0} \int_{\partial U_i} \frac{\partial_z F}{F} dz = 2\pi i n_i.$$  

The integrant can be written in a covariant way

$$\frac{\partial_z F}{F} dz = \frac{dF^{(1,0)}}{F},$$

defined in the coordinate invariant way in $S \setminus \{x_1, ..., x_k\}$.

On the other hand, if we repeat the construction for every zero $x_i$, $i = 1, ..., k$, and consider $\epsilon$ sufficiently small such that the discs do not intersect any other, then

$$\sum_{i=1}^{k} \int_{\partial U_i} \frac{dF^{(1,0)}}{F} = - \int_{S \setminus \bigcup U_i} d \left( \frac{dF^{(1,0)}}{F} \right).$$

Notice, that in the domain of the integration on the right hand side $F$ nowhere vanishes.

Now,

$$d \left( \frac{\partial_z F}{F} dz \right) = \frac{F \partial_{\bar{z}} \partial_z F - \partial_z F \partial_{\bar{z}} F}{F^2} d\bar{z} \wedge dz = -d \left( \frac{\partial_z F}{F} d\bar{z} \right) = 3d (\omega_{\bar{z}} d\bar{z}),$$
where the emergence of the 1-form $\omega \bar{z}d\bar{z}$ is due to eq. (26). Therefore we can rewrite the eq. (33) as

$$\sum_{i=1}^{k} \int_{\partial U_i} \frac{dF(1,0)}{F} = -3 \int_{S \setminus \bigcup U_i} d\left(\omega^{(0,1)}\right).$$

(35)

The advantage of expressing the right hand side by $\omega$ is, that the integrant $d\left(\omega^{(0,1)}\right)$ is an exact 2-form defined on the entire $S$ (including the points $x_1, ..., x_k$). Hence

$$2\pi i \sum_{i} n_{i} = \lim_{\epsilon \to 0} \sum_{i=1}^{k} \int_{\partial U_i} \frac{dF(1,0)}{F} = -3 \lim_{\epsilon \to 0} \int_{S \setminus \bigcup U_i} d\left(\omega^{(0,1)}\right) = -3 \int_{S} d\left(\omega^{(0,1)}\right) = 0.$$

(36)

In summary, assuming that the function $F$ has isolated zeros we have come to the opposite conclusion that it has no zeros. It completes the proof.

**Remark 2** In the case $S$ is simply connected itself (that is $S = S_2$) the functions $\phi_U$ and $f_U$ are defined globally on $S$, hence we can drop the suffix $U$

$$\phi := \phi_U, \quad f := f_U.$$ 

Moreover, as an entire holomorphic function on $S$,

$$f = f_0 = \text{const.}$$

If we apply the eq. (29) to the scalar $\Psi_2$ of a solution to the NHG, we find

$$\Psi_2 = f_0 e^{-\phi}, \quad \text{where} \quad \partial_{\bar{z}} \phi = 3\omega_{\bar{z}}.$$ 

In the $\Lambda = 0$ case that equation was derived in [10].

**IV. THE EMERGENCE OF THE TYPE D EQUATION**

We go back now to the NHG equation (4) and properties of the (semi) scalar $\Psi_2$ (9,10) of a solution $(g_{AB}, \omega_A)$. It follows from Proposition 1 and Lemma 1 that either

$$\Psi_2 = 0$$

(37)

identically on $S$, or

$$\Psi_2(x) \neq 0, \quad \text{for every} \quad x \in S.$$ 

(38)

In case (37) we have

$$d\omega = 0, \quad K = \frac{\Lambda}{3}.$$
Since this solution is static, according to \( \omega_A = 0 \).

But then the NHG equation implies

\[ K = \Lambda. \]

Hence, the only solution is

\[ K = 0 = \Lambda \]

defined on a torus.

Then, for the rest of this section consider case (38). The eq. (14) is equivalent to

\[ \bar{m}^A (\nabla_A - \omega_A) (\Psi_2)^{-\frac{3}{2}} = 0. \] (39)

A function \((\Psi_2)^{-\frac{3}{2}}\) is defined up to rescaling by cubic roots of 1 and unless \(\text{genus}(S) = 0\), an existence of continues \((\Psi_2)^{-\frac{3}{2}}\) a priori is not guaranteed. However, given \(g, \omega\) and the corresponding nowhere vanishing \(\Psi_2\), there is always a covering space

\[ \tilde{S} \to S \]

such that for the pullback \(\tilde{g}, \tilde{\omega}\) and \(\tilde{\Psi}_2\) there is a continues function \((\tilde{\Psi}_2)^{-\frac{3}{2}}\). Moreover,

\[ \text{genus}(\tilde{S}) \geq \text{genus}(S), \]

and in particular, if \(S = T^2\), then \(\tilde{S}\) is also \(T^2\). That makes cases 1-3 of Sec. II preserved by going (or not) to the covering \(\tilde{S}\). Hence, given \((g_{AB}), \omega_A\) and the corresponding \(\Psi_2\), we choose a suitable covering and drop the tildes. (Actually, it turns out that the resulting \(\Psi_2\) is continues on the original \(S\), without the need of covering by \(\tilde{S}\).)

The inverse cubic root in equation (39) reminds of the Type D equation defined in [18, 19] that also is an integrability condition for the NHG equation. The following observation can serve as an independent proof of that relation between the equations:

**Lemma 2** The following identity is true for arbitrary \(f \in C^2(S)\):

\[
\begin{align*}
\bar{m}^B \bar{m}^A \nabla_B \nabla_A f &= \left( \bar{m}^B \bar{m}^A \left( \nabla (A \omega_B) + \omega_A \omega_B - \frac{1}{2} K g_{AB} + \frac{1}{2} \Lambda g_{AB} \right) \right) f \\
+ \bar{m}^B \nabla_B \left( \bar{m}^A (\nabla_A - \omega_A) f \right) + \left( \bar{m}^A \omega_A - \bar{m}^A (\nabla_A \bar{m}^C) m_C \right) \left( \bar{m}^B (\nabla_B - \omega_B) f \right)
\end{align*}
\] (40)

The proof is just a straightforward calculation of the first term on the right hand side, and noticing, that the terms proportional to \(K\), and to \(\Lambda\) in fact do not contribute. They are there only for the relation with the NHG equation.
The conclusion from Lemma 2 is, that if the NHG equation is satisfied by \((g_{AB}, \omega_A)\), then the corresponding \(\Psi_2\) satisfies an equation
\[
\bar{m}^B m^A \nabla_B \nabla_A (\Psi_2)^{-\frac{1}{3}} = 0.
\]
We call it Type D equation. The subject of the equation is \((g_{AB}, \omega_A)\).  

This equation has an equivalent holomorphic formulation. It can be shown, that a function \(F\) satisfies the equation
\[
\bar{m}^B m^A \nabla_B \nabla_A F = 0
\]
if and only if the vector field
\[
(f_B g^{BA})^{(1,0)}
\]
is a holomorphic vector field on \(S\).

V. THE GENERAL SOLUTION TO NHG EQUATION FOR GENUS(\(S\)) > 0

The observation that \((43)\) is a holomorphic vector field defined globally on \(S\), is useful in the derivation of a general solution of \((42)\) for genus(\(S\)) > 0. The only solutions are \(F = \text{const}\).  

In a genus(\(S\)) > 1 case that result follows immediately from the non-existence of non-trivial holomorphic vectors. In the case of genus(\(S\)) = 1 (torus), the dimension of the space of the holomorphic vectors is 1, however none of them is the gradient of a function.

Hence, the almost scalar \(\Psi_2\) corresponding to \((g_{AB}, \omega_A)\) that solves the NHG equation \((4)\) on \(S\) of genus > 1 in case \((38)\) is
\[
\Psi_2 = \text{const} \neq 0.
\]

We could integrate the definition of \(\Psi_2\) with respect to \(g_{AB}\) and \(\omega_A\). However, the eq. \((39)\) is more powerful:
\[
0 = \bar{m}^A (\nabla_A + 3\omega_A) \Psi_2 = 3\bar{m}^A \omega_A \Psi_2.
\]
That implies
\[
\omega^A = 0.
\]
Substituting that result into the original NHG equation \((4)\) we find, that
\[
K = \Lambda.
\]

The careful reader notices that we still have to go back to the original 2-manifold \(S\) that was covered by \(\tilde{S}\) for the continuity of the cubic root of \(\Psi_2\). However, the resulting \(\Psi_2\) on the original \(S\) is constant, hence its cubic root is continues on \(S\).
VI. SUMMARY

The most important conclusion coming from this work is:

**Theorem** The only solutions to the Near Horizon Geometry equation \([4]\) with a cosmological constant \(\Lambda\) on a 2-dimensional, compact, orientable manifold of genus > 0 are pairs \((g_{AB}, \omega_A)\) such that

\[ K = \Lambda, \quad \omega_A = 0. \]

The theorem relies on earlier results about: (i) the solutions to the Type D equation \([18, 19]\), and (i) the static solutions of the NHG equation \([13]\) - we needed those to complete the \(\Psi_2 = 0\) case.

The essentially new part that fills in the gap is Lemma 1 proved in the current paper which ensures, that if \((g_{AB}, \omega_A)\) satisfy the NHG equation on a compact connected surface \(S\), then the invariant \(K - \frac{\Lambda}{3} + i\Omega\) is either everywhere or nowhere, zero on \(S\).

Due to that result, the invariant satisfies the Type D equation at every point of an orientable compact \(S\), that is

\[
\tilde{m}^A \tilde{m}^B \nabla_A \nabla_B (K - \frac{\Lambda}{3} + i\Omega)^{-\frac{1}{3}} = 0,
\]

where at the end of the day it turns out that the cubic root function is continuous on \(S\).

Consequently, the equation

\[
\tilde{m}^A (\nabla_A + 3\omega_A) (K - \frac{\Lambda}{3} + i\Omega) = 0,
\]

is a generalization to \(\Lambda \neq 0\) of the equation

\[
\tilde{m}^A (\nabla_A + 3\omega_A) (K + i\Omega) = 0,
\]

used before \([10, 20]\) in the \(\Lambda = 0\) case. It makes Lemma 1 and the Type D equation applicable also to still unsolved \(S = S_2\) case and the NHG equation with the cosmological constant \(\Lambda < 0\).

Finally, our results extend to non-orientable 2-dimensional compact manifolds due to the existence of orientable double coverings. Given \(\hat{S}, g_{AB}, \omega_A\) and an orientable covering \(\hat{S} \rightarrow S\)
we just pull back the data and obtain $\tilde{S}, \tilde{g}_{AB}, \tilde{\omega}_A$. The NHG equation is consistent with that map. In this way we extend our results to non-orientable manifolds in the following way:

**Theorem’** The only solutions of the Near Horizon Geometry equation \([4]\) with a cosmological constant $\Lambda$ on a 2-dimensional, compact manifold $S$ of the Euler characteristics $\chi_E(S) \leq 0$

are pairs $(g_{AB}, \omega_A)$ such that

$$K = \Lambda, \quad \omega_A = 0.$$ 

**Lemma 1’** If $S$ is a compact 2-manifold and $g_{AB}, \omega_A$ satisfy the NHG equation \([4]\) then at every point $x \in S$,

$$K(x) \neq 0 \quad \text{or} \quad d\omega|_x \neq 0.$$ 

The complex almost invariant $K - \frac{\Lambda}{3} + i\Omega$ is not invariant anymore. It is defined locally, upon the choice of the null frame $m^A$, however in addition to the transformations \([12]\) we need also

$$m'^A = e^{i\alpha}m^A.$$ 

They have to be accompanied by

$$K' = K, \quad \omega'_A = \omega_A, \quad \Omega' = -\Omega.$$ 

Then the Type D equation defined locally, passes consistently from an orientable neighborhood to another overlapping orientable neighborhood. The same rule applies to equation \([18]\).

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