On the spectrum of lamplighter groups and percolation clusters

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Abstract Let $G$ be a finitely generated group and $X$ its Cayley graph with respect to a finite, symmetric generating set $S$. Furthermore, let $H$ be a finite group and $H \wr G$ the lamplighter group (wreath product) over $G$ with group of “lamps” $H$. We show that the spectral measure (Plancherel measure) of any symmetric “switch–walk–switch” random walk on $H \wr G$ coincides with the expected spectral measure (integrated density of states) of the random walk with absorbing boundary on the cluster of the group identity for Bernoulli site percolation on $X$ with parameter $p = 1/|H|$. The return probabilities of the lamplighter random walk coincide with the expected (annealed) return probabilities on the percolation cluster. In particular, if the clusters of percolation with parameter $p$ are almost surely finite then the spectrum of the lamplighter group is pure point. This generalizes results of Grigorchuk and Žuk, resp. Dicks and Schick regarding the case when $G$ is infinite cyclic. Analogous results relate bond percolation with another lamplighter random walk. In general, the integrated density of states of site (or bond) percolation with arbitrary parameter $p$ is always related with the Plancherel measure of a convolution operator by a signed measure on $H \wr G$, where $H = \mathbb{Z}$ or another suitable group.

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1 Introduction

1.1 Lamplighter random walks

Let $G$ be a finitely generated group and $H$ a finite group with unit elements $e$ and $o$, respectively. The wreath product or lamplighter group $H \wr G$ is the semidirect product $L \rtimes G$, where $L = \bigoplus_{g \in G} H$ is the group of configurations $\eta: G \to H$ with finite support $\text{supp}(\eta) = \{x \in G : \eta(x) \neq o\}$. The group operation in $L$ is pointwise multiplication in $H$, and its unit element $\iota$ is given by $\iota(x) = o$ for all $x \in G$. The left action of $G$ on $L$ is $Lg\eta(x) = \eta(g^{-1}x)$, so that the group operation in $H \wr G$ is $(\eta, g)(\eta', g') = (\eta \cdot Lg\eta', gg')$.

We can embed $G$ and $H$ into $H \wr G$ via the mappings $g \mapsto (\iota, g)$ and $h \mapsto (\eta^h_e, e)$, where for $g \in G$, $h \in H$, $\eta^h_x(x) = \begin{cases} h, & \text{if } x = g, \\ o, & \text{otherwise}. \end{cases}$

Now let $\mu$ be a symmetric probability measure on $G$ whose support $S$ is finite and generates $G$. The random walk on $G$ with law $\mu$ is the Markov chain with transition probabilities $p(x, y) = \mu(x^{-1}y)$, $x, y \in G$. The Cayley graph $X(G, S)$ of $G$ with respect to $S$ has vertex set $G$ and the unoriented edges $[x, xs]$, where $x \in G$ and $s \in S$. The steps of our random walk follow the edges of this graph, and the most natural case is the one where $\mu$ is equidistributed on $S$, in which case it generates the simple random walk on $X$.

Also, we let $\nu$ be equidistribution on $H$. Via the above embedding, we consider $\mu$ and $\nu$ as probability measures on $H \wr G$, and build the convolution

$$\widetilde{\mu} = \nu \ast \mu \ast \nu. \quad (1.1)$$

This is a symmetric probability measure whose support

$$\text{supp}(\widetilde{\mu}) = \{(\eta^h_e \cdot \eta^{h'}_s, s) : h, h' \in H, s \in S\}$$
generates the lamplighter group (If $\mu$ is equidistributed on $S$ and $e \notin S$ then $\widetilde{\mu}$ is also equidistributed on its support). It gives rise to the switch-walk-switch lamplighter walk: there are lamps at the vertices of $X(G, S)$ whose possible states are encoded by the group $H$, and $o$ is the state “off” of a lamp. Initially, all lamps are off. A lamplighter performs simple random walk on $X(G, S)$, starting at $e$. At each step, (s)he first puts the lamp at the current position to a random state, then makes a move in $X(G, S)$, and finally puts the lamp at the new position to a random state. Each current configuration of the lamps plus the current position of the lamplighter is a pair $(\eta, g) \in H \wr G$. This process is the random walk with law $\widetilde{\mu}$ on $H \wr G$. 

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