Renormalization of singular elliptic stochastic PDEs using flow equation

Pawel Duch
Faculty of Mathematics and Computer Science
Adam Mickiewicz University in Poznań
ul. Uniwersytetu Poznańskiego 4, 61-614 Poznań, Poland
pawel.duch@epfl.ch

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Abstract

We develop a solution theory for singular elliptic stochastic PDEs with fractional Laplacian, additive white noise and cubic non-linearity. The method covers the whole sub-critical regime. It is based on the Wilsonian renormalization group theory and the Polchinski flow equation.

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1 Introduction

A general technique that allows to renormalize and prove universality of parabolic singular SPDEs with fractional Laplacian, additive noise and polynomial non-linearity was developed in [4]. The goal of this paper is to give a different application of this technique. We present a self-contained construction of solutions of the non-local singular elliptic SPDEs

\[(1 + (-\Delta)^{\sigma/2})\Phi(x) = \xi(x) + \lambda\Phi(x)^3 - \infty\Phi(x), \quad x \in \mathbb{R}^d,\]

where \(d \in \{1, \ldots, 6\}, \sigma \in (d/3, d/2], (-\Delta)^{\sigma/2}\) is the fractional Laplacian, \(\lambda \in \mathbb{R}\) is sufficiently small and \(\xi\) is the periodization of the white noise on \(\mathbb{R}^d\) with period \(2\pi\). Recall that the regularity of the noise \(\xi\) is slightly worse than \(-d/2\) and the expected regularity of the solution is slightly worse than \(\sigma - d/2\). For \(\sigma > d/2\) the above equation is not singular and can be solved using classical PDE theory. For \(\sigma \leq d/2\) the solution is not a function but only a distribution. As a result, the cubic term is ill-defined and has to be renormalized by subtracting an appropriate mass counterterm (for \(d > 6\) other counterterms are needed even if one takes into account all the symmetries of the equation). The renormalization problem is tractable only if the equation is sub-critical (super-renormalizable). This is the case if the expected regularity of the renormalized non-linearity is better than the regularity of \(\xi\), i.e. \(3(\sigma - d/2) > -d/2\). Let us remark that for \(d = 5\) and \(\sigma = 2\) the above equation is sometimes called the elliptic quantization equation of the \(\Phi^4_3\) model (provided \(\xi\) is replaced by the white noise on \(\mathbb{R}^d\)).

Let \(G \in L^1(\mathbb{R}^d)\) be the fundamental solution for the pseudo-differential operator \(Q := 1 + (-\Delta)^{\sigma/2}\) and let \(G_{\kappa}\) be the smooth approximation of \(G\) with a spatial UV cutoff of order \([\kappa] := \kappa^{1/\sigma}\) introduced in Def. 2.1. We rewrite the above singular SPDE in the regularized mild form

\[
\Phi = G_{\kappa} * F_{\kappa}[\Phi], \quad \kappa \in (0, 1/2].
\]

The functional \(F_{\kappa}[\varphi]\), called the force, is defined by

\[
F_{\kappa}[\varphi](x) := \xi(x) + \lambda \varphi^3(x) + \sum_{i=1}^{i_*} \lambda^i c_{\kappa}^{[i]} \varphi(x),
\]
where \( i = \lfloor \sigma/(3\sigma - d) \rfloor \) and the parameters \( \kappa^{(i)} \in \mathbb{R} \) depending on the UV cutoff \( \kappa \) are called the counterterms. Let us state our main result.

**Theorem 1.1.** There exists a choice of counterterms and a random variable \( \lambda_0 \) such that \( \mathbb{E}(\lambda_0^{-n}) < \infty \) for every \( n \in \mathbb{N} \) and for every random variable \( \lambda \in [-\lambda_0, \lambda_0] \) and \( \kappa \in (0, 1/2] \) Eq. (1.1) has a periodic solution \( \Phi_\kappa \in C^\infty(\mathbb{R}^d) \) and for all \( \beta < \sigma - d/2 \) the limit \( \lim_{\kappa \downarrow 0} \Phi_\kappa \) exists almost surely in the Besov space \( C^\beta(\mathbb{R}^d) \).

**Proof.** We first establish bounds for cumulants of the enhanced noise introduced in Def. 7.1. The bounds are stated in Theorem 16.1 and hold true for an appropriate choice of the counterterms. Using these bounds and a Kolmogorov type argument we deduce bounds for the enhanced noise stated in Theorem 13.1. This together with the deterministic result of Theorem 11.4 implies the statement. \( \square \)

**Remark 1.2.** There exists a general technique developed in [1–3, 5] based on the theory of regularity structures that allows to systematically renormalize virtually all sub-critical singular SPDEs with local differential operators. However, due to the slow decay of the kernel \( G \) at infinity, this technique does not cover Eq. (1.1).

**Remark 1.3.** Our technique requires a small parameter. In the case of the elliptic equations we assume that the prefactor of the non-linear term is sufficiently small. Alternatively, one could introduce a prefactor in front of the noise term and assume that it is sufficiently small. In the case of parabolic equations the technique allows [4] to construct a solution in a sufficiently small time interval without any assumption about the strength of the non-linearity.

**Remark 1.4.** We study Eq. (1.1) with a regularized Green function \( G_\kappa \) and the periodization of the white noise \( \xi \). Our technique is also applicable [4] to the equation with the standard Green function \( G \) and a regularized noise \( \xi_\kappa \).

## 2 Effective force

The basic object of the flow equation approach is the effective force functional

\[ \mathcal{F}(M) \ni \phi \mapsto F_{\kappa, \mu} [\phi] \in \mathcal{E}'(M) \]

depending on the UV cutoff \( \kappa \in (0, 1/2] \) and the flow parameter \( \mu \in [0, 1] \), where \( \mathcal{E}(M) \) and \( \mathcal{E}'(M) \) denote the space of Schwartz functions and distributions on \( M = \mathbb{R}^d \), respectively. Note that \( \kappa \) is assumed to be strictly positive.
**Definition 2.1.** Fix \( \chi \in C^\infty(\mathbb{R}) \) such that \( \chi(r) = 0 \) for \( |r| \leq 1 \) and \( \chi(r) = 1 \) for \( |r| > 2 \) and let \( \chi_\mu(r) := \chi(r(1 - \mu)/\mu) \) for \( \mu \in (0, 1] \). Let \( G \in L^1(\mathbb{M}) \) be the fundamental solution for the pseudo-differential operator \( Q := 1 + (-\Delta)^{\sigma/2} \). For \( \kappa \in (0, 1/2) \) and \( \mu \in (0, 1] \) the smooth kernels \( G_\kappa, G_{\kappa, \mu} \in L^1(\mathbb{M}) \) are defined by

\[
G_\kappa(x) := \chi_\kappa(|x|^{\sigma}) G(x), \quad G_{\kappa, \mu}(x) := \chi_\kappa(|x|^{\sigma}) \chi_\mu(|x|^{\sigma}) G(x).
\]

**Remark 2.2.** Since \( \chi_\kappa(r) \chi_\mu(r) = \chi_\kappa(r) \) for \( \mu \leq \kappa/2 \) it holds that \( G_{\kappa, \mu} = G_\kappa \) for \( \mu \leq \kappa/2 \) and \( G_{\kappa, \mu} = G_\mu \) for \( \mu \geq 2\kappa \). Moreover, \( \lim_{\kappa \searrow 0} G_\kappa = G \) in \( L^1(\mathbb{M}) \) and \( G_{\kappa, 1} = 0 \) for all \( \kappa \in (0, 1/2] \). Note that \( \kappa = 1/2 \) corresponds to UV cutoff at spatial scale 1. For arbitrary fixed UV cutoff \( \kappa \in (0, 1/2] \) the family \( G_{\kappa, \mu}, \mu \in [0, 1] \), interpolates between \( G_\kappa \) and 0. Because of slow decay of \( G \) for \( \sigma \notin 2\mathbb{N}_+ \) the range \( \mu \in [1/2, 1] \) will require some special treatment.

By definition, the effective force satisfies the flow equation

\[
\langle \partial_\mu F_{\kappa, \mu}[\varphi], \psi \rangle = -\langle DF_{\kappa, \mu}[\varphi], \partial_\mu G_{\kappa, \mu} * F_{\kappa, \mu}[\varphi], \psi \rangle \tag{2.1}
\]

with the boundary condition \( F_{\kappa, 0}[\varphi] = F_\kappa[\varphi] \), where \( \varphi, \psi \in \mathcal{S}(\mathbb{M}) \) and the force \( F_\kappa[\varphi] \) is defined by Eq. (1.2). The pairing between a distribution \( V \) and a test function \( \psi \) is denoted by \( \langle V, \psi \rangle \) and \( \langle V[\varphi], \zeta, \psi \rangle \) is the derivative of the functional \( \mathcal{S}(\mathbb{M}) \ni \varphi \mapsto \langle V[\varphi], \psi \rangle \in \mathbb{R} \) in the direction \( \zeta \in \mathcal{S}(\mathbb{M}) \). In contrast to the force \( F_\kappa[\varphi] \) the effective force \( F_{\kappa, \mu}[\varphi] \) is generically a non-local functional.

We claim that \( \Phi_\kappa := G_\kappa * F_{\kappa, 1}[0] \) is a solution of Eq. (1.1). The above statement is a consequence of the equalities \( G_{\kappa, \mu} = G_\kappa \) and \( F_{\kappa, \mu}[\varphi] = F_\kappa[\varphi] \) that hold for all \( \mu \in [0, \kappa/2] \) and the identity

\[
F_{\kappa, 1}[0] = F_{\kappa, \mu}[G_{\kappa, \mu} * F_{\kappa, 1}[0]] \tag{2.2}
\]

that holds for all \( \mu \in [0, 1] \). In order to prove the last identity we use the flow equation to show that the difference between the LHS and RHS of Eq. (2.2), denoted by \( g_{\kappa, \mu} \), satisfies the linear ODE

\[
\partial_\mu g_{\kappa, \mu} = -DF_{\kappa, \mu}[G_{\kappa, \mu} * F_{\kappa, 1}[0], \partial_\mu G_{\kappa, \mu} * g_{\kappa, \mu}]
\]

with the boundary condition \( g_{\kappa, 1} = 0 \). This implies that \( g_{\kappa, \mu} = 0 \) for all \( \mu \in [0, 1] \).

**Remark 2.3.** The effective force plays also a central role in the approach to singular SPDEs proposed earlier by Kupiainen [6, 7] and applied to the dynamical \( \Phi^4_3 \) model and the KPZ equation. The method developed by Kupiainen is based
on the Wilsonian discrete renormalization group theory [11]. In this approach one uses the fact that for $\mu \geq \eta$ the effective force satisfies the equation

$$F_{\kappa,\mu}[^{\varphi}] = F_{\kappa,\eta}[^{(G_{\kappa,\eta} - G_{\kappa,\mu})} \ast F_{\kappa,\mu}[^{\varphi}]] + \varphi.$$ 

Given $F_{\kappa,\eta}[\varphi]$ the above equation is viewed as an equation for $F_{\kappa,\mu}[\varphi]$ that can be solved using the Banach fixed point theorem. One defines recursively the effective force $F_{\kappa,\mu}[\varphi]$ for $\kappa = 2L - N$ and $\mu \in \{L - N, \ldots, 1\}$, where $L > 1$, starting with $F_{2L - N}[\varphi] = F_{2L - N;L - N}[\varphi]$ and finishing with $F_{2L - N;1}[\varphi]$. In order to prove uniform bounds for $F_{2L - N;1}[\varphi]$ one has to appropriately adjust the counterterms which are the coefficients of the force $F_{2L - N}[\varphi]$. The fine-tuning problem becomes exceedingly difficult for equations close to criticality. The flow equation provides a different, more efficient and simpler method of constructing the effective force that allows to treat equations arbitrarily close to criticality.

3 Construction of the effective force coefficients

The starting point of the construction of the effective force is the formal ansatz

$$\langle F_{\kappa,\mu}[\varphi], \psi \rangle = \sum_{i=0}^{\infty} \sum_{m=0}^{\infty} \lambda^i \langle F_{\kappa,\mu}^{i,m}, \psi \otimes \varphi^\otimes m \rangle, \quad (3.1)$$

where $\lambda$ is the prefactor of the cubic non-linearity in the original equation. The distributions $F_{\kappa,\mu}^{i,m} \in \mathcal{S}'(\mathbb{M}^{1+m})$, $i, m \in \mathbb{N}_0$, are called the effective force coefficients. By definition the expression $\langle F_{\kappa,\mu}^{i,m}, \psi \otimes \varphi_1 \otimes \ldots \otimes \varphi_m \rangle$ is invariant under permutations of the test functions $\varphi_1, \ldots, \varphi_m \in \mathcal{S}('(\mathbb{M})$. The coefficients $F_{\kappa,\mu}^{i,m}$ of the force $F_{\kappa}[\varphi]$ are defined by an equality analogous to Eq. (3.1).

Remark 3.1. Let us list the non-vanishing force coefficients $F_{\kappa,\mu}^{i,m}$:

$$F_{\kappa}^{0,0}(x) = \xi(x), \quad F_{\kappa}^{1,3}(x; x_1, x_2, x_3) = \delta_M(x - x_1)\delta_M(x - x_2)\delta_M(x - x_3),$$

$$F_{\kappa}^{i,1}(x; x_1) = c^{i}_{[l]}\delta_M(x - x_1), \quad i \in \{0, \ldots, i^\sharp\},$$

where $\delta_M \in \mathcal{S}'(\mathbb{M})$ is the Dirac delta at $0 \in \mathbb{M}$.

The flow equation (2.1) for the effective force $F_{\kappa,\mu}[\varphi]$ formally implies that the effective force coefficients $F_{\kappa,\mu}^{i,m}$ satisfy the flow equation

$$\langle \partial_\mu F_{\kappa,\mu}^{i,m}, \psi \otimes \varphi^\otimes m \rangle$$

$$= - \sum_{j=0}^{i} \sum_{k=0}^{m} (1 + k) \langle F_{\kappa,\mu}^{j,1+k} \otimes F_{\kappa,\mu}^{i-j,m-k}, \psi \otimes \varphi^\otimes k \otimes \mathbf{V} \partial_\mu G_{\kappa,\mu} \otimes \varphi^\otimes (m-k) \rangle \quad (3.2)$$
with the boundary condition $F_{k, 0}^{i, m} = F_{k}^{i, m}$, where $V \partial_{\mu} G_{\kappa, \mu} \in C^\infty(\mathbb{M} \times \mathbb{M})$ is defined by $V \partial_{\mu} G_{\kappa, \mu}(x, y) := \partial_{\mu} G_{\kappa, \mu}(x - y)$.

The basic idea behind the flow equation approach is a recursive construction of the effective force coefficients $F_{k, \mu}^{i, m}$:

(0) We set $F_{k, \mu}^{0, 0} = \xi$ and $F_{k, \mu}^{i, m} = 0$ if $m > 3i$.

(I) Assuming that all $F_{k, \mu}^{j, m}$ with $i < i_0$ or $i = i_0$ and $m > m_0$ were constructed we define $\partial_{\mu} F_{k, \mu}^{i, m}$ with $i = i_0$ and $m = m_0$ with the use of Eq. (3.2).

(II) Subsequently, $F_{k, \mu}^{i, m}$ is defined by $F_{k, \mu}^{i, m} = F_{k}^{i, m} + \int_0^{\mu} \partial_\eta F_{k, \eta}^{i, m} \, d\eta$.

Using this procedure we construct all the effective force coefficients $F_{k, \mu}^{i, m}$ for arbitrary $\kappa \in (0, 1/2]$, $\mu \in [0, 1]$.

Remark 3.2. One easily shows that the effective force coefficients $F_{k, \mu}^{i, m}$ actually vanish if $i = 0$ and $m > 0$, or $i > 0$ and $m > 2(i - 1) + 3$. The only non-zero coefficients $F_{k, \mu}^{i, m}$ which are independent of the value of the flow parameter $\mu$ are $F_{k, \mu}^{0, 0}(x) = \xi(x)$ and $F_{k, \mu}^{1, 1}(x; x_1, x_2, x_3) = F_{k}^{1, 1}(x; x_1, x_2, x_3)$. These coefficients happen to be independent of the UV cutoff $\kappa$. Let us give some further examples:

$$F_{k, \mu}^{1, 2}(x; x_1, x_2) = 3\Psi_{\kappa, \mu}(x) \delta_M(x - x_1) \delta_M(x - x_2),$$

$$F_{k, \mu}^{1, 1}(x; x_1) = (3\Psi_{\kappa, \mu}^2(x) + c_{\kappa}^{[1]}) \delta_M(x - x_1),$$

where $\Psi_{\kappa, \mu} := G_{\kappa, \mu} * \xi$ and $G_{\kappa, \mu} := G_{\kappa} - G_{\kappa, \mu}$ is the so-called fluctuation propagator. The coefficient $F_{k, \mu}^{2, 5}(x; x_1, \ldots, x_5)$ is obtained from the distribution

$$3 \delta_M(x - x_1) \delta_M(x - x_2) G_{\kappa, \mu}(x - x_3) \delta_M(x_3 - x_4) \delta_M(x_3 - x_5)$$

by symmetrization in variables $x_1, \ldots, x_5$. We also have

$$F_{k, \mu}^{2, 1}(x; x_1) = (3\Psi_{\kappa, \mu}^2(x) + c_{\kappa}^{[1]}) G_{\kappa, \mu}(x - x_1) (3\Psi_{\kappa, \mu}^2(x_1) + c_{\kappa}^{[1]})$$

$$+ \left( \int_{\mathbb{M}} 6\Psi_{\kappa, \mu}(x) G_{\kappa, \mu}(x - x_2) (\Psi_{\kappa, \mu}^3(x_2) + c_{\kappa}^{[1]} \Psi_{\kappa, \mu}(x_2)) \, dx_2 + c_{\kappa}^{[2]} \right) \delta_M(x - x_1),$$

$$F_{k, \mu}^{2, 0}(x) = (3\Psi_{\kappa, \mu}^2(x) + c_{\kappa}^{[1]}) \int G_{\kappa, \mu}(x - x_1) (\Psi_{\kappa, \mu}^3(x_1) + c_{\kappa}^{[1]} \Psi_{\kappa, \mu}(x_1)) \, dx_1 + c_{\kappa}^{[2]} \Psi_{\kappa, \mu}(x).$$

For the sake of brevity, we did not give expressions for the coefficients $F_{k, \mu}^{2, 4}$, $F_{k, \mu}^{1, 1}$, $F_{k, \mu}^{2, 2}$, which should be constructed after $F_{k, \mu}^{2, 5}$ and before $F_{k, \mu}^{2, 1}$.

Remark 3.3. The effective force is an analog of the effective potential in QFT. A recursive construction of the effective potential coefficients based on the flow equation is the backbone of a very simple proof of perturbative renormalizability of QFT models proposed by Polchinski [10] (see [8] for a review).
The solution of Eq. (1.1) is formally given by the sum
\[
\Phi_\kappa = G_\kappa * F_{\kappa,1}[0] := G_\kappa * \sum_{i=0}^{\infty} \lambda^i F_{\kappa,1}^i.
\] (3.3)

Assuming that \(\lambda\) is sufficiently small in Sec. 11 we prove that the above series converges absolutely and \(\Phi_\kappa\), as defined above, solves Eq. (1.1) and converges almost surely as \(\kappa \to 0\) in the Besov space \(\mathcal{C}^\beta(M)\) for every \(\beta < \sigma - d/2\). To this end, we will establish certain bounds for \(\partial^i_\kappa \partial^{a_m}_\mu F_{\kappa,\mu}^{m}\) which are stated in Sec. 6. The bounds involve a regularizing kernel \(K_\mu\), which is introduced in Sec. 4, and a norm \(\|\cdot\|_{V^m}\), which is introduced in Sec. 5.

## 4 Regularizing kernels

**Definition 4.1.** For \(n \in \mathbb{N}_+\) let \(K^n \subset \mathcal{S}'(\mathbb{M}^n)\) be the space of signed measures \(K\) on \(\mathbb{M}^n\) with finite total variation \(|K|\). We set \(\|K\|_{K^n} = \int_{\mathbb{M}^n} |K(dx_1 \ldots dx_n)|\). If \(n = 1\), then we write \(K^1 = K \subset \mathcal{S}'(\mathbb{M})\). We denote by \(\delta_M \in K\) the Dirac delta at \(0\) in \(\mathbb{M}\). Given \(K \in \mathcal{K}\) and \(n \in \mathbb{N}_+\) we set \(K^0 := K \otimes \ldots \otimes K \in K^n\).

**Remark 4.2.** It holds that \(\|K\|_{K^n} = \|K\|_{L^1(\mathbb{M}^n)}\) for all \(K \in L^1(\mathbb{M}^n) \subset K^n\).

**Definition 4.3.** For \(\mu \geq 0\) the kernel \(K_\mu \in \mathcal{K}\) is the unique solution of the equation \(P_\mu K_\mu = \delta_M\), where \(P_\mu := 1 - |\mu|^2 \Delta\) and \([|\mu|] := \mu^{1/\sigma}\). We set \(K_\mu^0 := \delta_M\), \(P_\mu^0 := 1\) and \(K_\mu^0(\mu+1) := K_\mu * K_\mu^0\), \(P_\mu^{g+1} := P_\mu P_\mu^0\) for \(g \in \mathbb{N}_0\). We omit the index \(\mu\) if \(\mu = 1\).

**Remark 4.4.** It holds that \(K_0 = \delta_M\) and \(K_\mu \in L^1(\mathbb{M})\) for \(\mu > 0\). For \(g \in \mathbb{N}_0\) and \(\mu \geq 0\) the kernel \(K_\mu^g\) is a positive measure with total mass \(\|K_\mu^g\|_K = 1\). We have \(\sum_{g = 0}^\infty K_\mu^g = \delta_M\). The fact that the regularizing kernel \(K_\mu^g\) is an inverse of a differential operator simplifies the analysis in Sec. 8.

**Lemma 4.5.** For any \(\mu, \eta > 0\) it holds that
\[
K_\mu = P_\eta K_\mu * K_\eta, \quad \|P_\eta K_\mu\|_K = 1 + (2\eta/\mu)^2 - 1.
\]
In particular, if \(\mu \geq \eta\), then \(\|P_\eta K_\mu\|_K = 1\).

**Proof.** We have \(P_\eta K_\mu = [\eta/\mu]^2 \delta_M + (1 - [\eta/\mu]^2) K_\mu\) and \(\|\delta_M\|_K = \|K_\mu\|_K = 1\).

**Definition 4.6.** Let \(T = \mathbb{M}/(2\pi \mathbb{Z})^d\). For \(V \in L^1(\mathbb{M})\) we define \(TV \in L^1(T)\) by
\[
TV(x) := \sum_{y \in (2\pi \mathbb{Z})^d} V(x + y).
\]
Remark 4.7. For $K \in L^1(\mathbb{M})$ and periodic $f \in C(\mathbb{T})$ it holds that $K \ast f = T K \ast f$, where $\ast$ and $\ast$ are the convolutions in $\mathbb{M}$ and $\mathbb{T}$, respectively.

Lemma 4.8. Let $g \in \mathbb{N}_0$, $a \in \mathbb{N}_0^d$ and $n \in [1, \infty]$. The following is true:

(A) If $|a| \leq g$, then $\|\partial^a K_{\mu}^g\|_K \lesssim |\mu|^{1-|a|}$ uniformly in $\mu > 0$.

(B) $\|TK_{\mu}^d\|_{L^p(\mathbb{T})} \lesssim |\mu|^{-d(n-1)/a}$ uniformly in $\mu \in (0, 1]$.

(C) It holds that $\|P_{\mu} \partial_\mu K_{\mu}^g\|_K \lesssim |\mu|^{-\sigma}$ uniformly in $\mu > 0$.

Remark 4.9. Let $f \in C(\mathbb{M})$, $g \in \mathbb{N}_0$ and $a \in \mathbb{N}_0^d$ be such that $|a| \leq g$. Then $K_{\mu}^g \ast f \in C(\mathbb{M})$ and $\|\partial^a K_{\mu}^g \ast f\|_{L^\infty(\mathbb{M})} \lesssim |\mu|^{-|a|}\|f\|_{L^\infty(\mathbb{M})}$ uniformly over $\mu > 0$ and $f \in C(\mathbb{M})$.

5 Function spaces for the coefficients

Definition 5.1. For $\alpha < 0$ and $\phi \in \mathcal{S}'(\mathbb{M})$ we define

$$\|\phi\|_{\mathcal{S}^\alpha(\mathbb{M})} := \sup_{\mu \in (0,1]} |\mu|^{-\alpha} \|K_{\mu}^g \ast \phi\|_{L^\infty(\mathbb{M})}, \quad g = [-\alpha] \in \mathbb{N}_+.$$  

The space $\mathcal{S}^\alpha(\mathbb{M})$ consists of $\phi \in \mathcal{S}'(\mathbb{M})$ such that $\|\phi\|_{\mathcal{S}^\alpha(\mathbb{M})} < \infty$.

Definition 5.2. Let $m \in \mathbb{N}_0$. The vector space $\mathcal{V}_m$ consists of $V \in C(\mathbb{M}^{1+m})$ such that

$$\|V\|_{\mathcal{V}_m} := \sup_{x \in \mathbb{M}} \int_{\mathbb{M}^m} |V(x; y_1, \ldots, y_m)| dy_1 \ldots dy_m$$

is finite and the function $x \mapsto V(x; y_1 + x, \ldots, y_m + x)$ is $2\pi$ periodic for every $y_1, \ldots, y_m \in \mathbb{M}$. For $g \in \mathbb{N}_0$ the space $\mathcal{D}^{m,g}$ consists of $V \in \mathcal{S}'(\mathbb{M}^{1+m})$ such that it holds $K_{\mu}^{g} \ast (1+|\mu|) \ast V \in \mathcal{V}_m$. The space $\mathcal{D}^m$ is the union of the spaces $\mathcal{D}^{m,g}$, $g \in \mathbb{N}_0$. We also set $\mathcal{V}^0 = \mathcal{V}$ and $\mathcal{D}^0 = \mathcal{D}$.

Remark 5.3. For $V \in \mathcal{V}_m$ and $K \in K^{1+m}$ it holds $\|K \ast V\|_{\mathcal{V}_m} \leq \|K\|_{K^{1+m}} \|V\|_{\mathcal{V}_m}$.

Remark 5.4. It holds that $V = \mathcal{D}^{0,0} = \mathcal{C}(\mathbb{T})$. Moreover, we have $\|V\|_{\mathcal{V}_m} = \|V\|_{L^\infty(\mathbb{M})}$ for all $V \in \mathcal{V}$. Furthermore, $\mathcal{D} = \mathcal{S}'(\mathbb{T})$, since $K$ is the inverse of a differential operator.

Definition 5.5. The permutation group of the set $\{1, \ldots, n\}$ is denoted by $\mathcal{P}_n$.

For $m \in \mathbb{N}_0$ and $V \in \mathcal{D}^m$ and $\pi \in \mathcal{P}_m$ we define $Y_\pi V \in \mathcal{D}^m$ by

$$\langle Y_\pi V, \psi \otimes \otimes_{q=1}^m \varphi_q \rangle := \langle V, \psi \otimes \otimes_{q=1}^m \varphi_{\pi(q)} \rangle,$$

where $\psi, \varphi_1, \ldots, \varphi_m \in \mathcal{S}(\mathbb{M})$. 

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Remark 5.6. The map $Y_\pi : \mathcal{V}_m \to \mathcal{V}_m$ is well defined and has norm one.

Definition 5.7. Let $m \in \mathbb{N}_0$, $k \in \{0, \ldots, m\}$. We define the trilinear map $B : \mathcal{S}(\mathcal{M}) \times \mathcal{V}_{1+k} \times \mathcal{V}_{m-k} \to \mathcal{V}_m$ by

$$B(G, W, U)(x; y_1, \ldots, y_m) := \int_{M^2} W(x; y_0, \ldots, y_k) G(y_0 - z) U(z; y_{1+k}, \ldots, y_m) \, dy_0 \, dz.$$ 

Lemma 5.8. The map $B : \mathcal{S}(\mathcal{M}) \times \mathcal{V}_{1+k} \times \mathcal{V}_{m-k} \to \mathcal{V}_m$ is well defined and

$$\|B(G, W, U)\|_{\mathcal{V}_m} \leq \|G\|_{K} \|W\|_{\mathcal{V}_{1+k}} \|U\|_{\mathcal{V}_{m-k}}. \quad (5.1)$$

Proof. To prove well-definedness note that

$$B(G, W, U)(x; y_1 + x, \ldots, y_m + x) = \int_{M^2} W(x; y_0 + x, \ldots, y_k + x) G(y_0 - z) U(z + x; y_{1+k} + x, \ldots, y_m + x) \, dy_0 \, dz$$

is $2\pi$ periodic. We have

$$\|B(G, W, U)\|_{\mathcal{V}_m} = \sup_{x \in \mathcal{M}^n} \left| \int_{\mathcal{M}^m} |B(G, W, U)(x; y_1, \ldots, y_m)| \, dy_1 \ldots dy_m \right| \leq \sup_{x \in \mathcal{M}} \int_{M^{2+m}} |W(x; y_0, \ldots, y_k)| |G(y_0 - z)| |U(z; y_{1+k}, \ldots, y_m)| \, dz \, dy_0 \ldots dy_m.$$ 

It is easy to see that the last line is bounded by the RHS of (5.1). \[\square\]

Remark 5.9. The fact that $P^\mu K^\mu = \delta_\mathcal{M}$ implies that for all $\mu > 0$ it holds that

$$K^\mu \* (1+m) \* B(G, W, U) = B(\mathcal{D^2}^\mu G, K^\mu \* (2+k) \* W, K^\mu \* (1+m-k) \* U).$$

This allows to define $B(G, W, U) \in \mathcal{D}^m$ for all $G \in \mathcal{S}(\mathcal{M})$, $W \in \mathcal{D}^{1+k}$ and $U \in \mathcal{D}^{m-k}$.

Remark 5.10. The RHS of Eq. (3.2) can be written compactly using the map $B$. 

9
6 Deterministic bounds for the irrelevant coefficients

Definition 6.1. Recall that $\sigma \in (d/3,d/2]$, $|\mu| = \mu^{1/\sigma}$ and set

$$\dim(\xi) := d/2 > 0, \quad \dim(\Phi) := d/2 - \sigma \geq 0, \quad \dim(\lambda) := 3\sigma - d > 0.$$ 

Definition 6.2. For $\varepsilon \geq 0$ and $i,m \in \mathbb{N}_0$ we define

$$\varrho_\varepsilon(i,m) := - \dim(\xi) - \varepsilon + m (\dim(\Phi) + 3\varepsilon) + i (\dim(\lambda) - 9\varepsilon) \in \mathbb{R}.$$ 

We omit $\varepsilon$ if $\varepsilon = 0$. The effective force coefficients $F_{\kappa,\mu}^{i,m}$ such that $\varrho(i,m) \leq 0$ are called relevant. The remaining coefficients are called irrelevant.

Remark 6.3. The number of relevant coefficients $F_{\kappa,\mu}^{i,m}$ such that $m \leq 3i$ is always finite (recall that $F_{\kappa,\mu}^{i,m} = 0$ if $m > 3i$). For example, in the case $d = 5$ and $\sigma = 2$ the relevant coefficients are $F_{\kappa,\mu}^{d,0}, F_{\kappa,\mu}^{1,0}, F_{\kappa,\mu}^{1,1}, F_{\kappa,\mu}^{1,2}, F_{\kappa,\mu}^{2,0}, F_{\kappa,\mu}^{2,1}$. For explicit expressions for these coefficients see Remark 3.2.

Remark 6.4. For arbitrary $\varepsilon > 0$ and $i,m \in \mathbb{N}_0$ such that $m \leq 3i$, $\varrho_\varepsilon(i,m) < \varrho(i,m)$ holds. Let $i_\varepsilon$ be the smallest integer such that $\varrho(i_\varepsilon + 1,0) > 0$. Then $i_\varepsilon \in \mathbb{N}_+$. Moreover, let $\varrho_\varepsilon > 0$ be the minimum of $\varrho(i,m) + 1$ for $i \in \{0,\ldots,i_\varepsilon\}$, $m \in \{0,\ldots,3i_\varepsilon\}$, $1 \in \mathbb{N}_+$ such that $\varrho(i,m) + 1 > 0$. Define $\varepsilon_\varrho := (\dim(\xi)/3 \wedge \dim(\lambda)/11 \wedge \varrho_\varepsilon/(10+9i_\varepsilon)) \wedge \varepsilon$. We claim that for all $\varepsilon \in (0,\varepsilon_\varrho)$ and all $i,m \in \mathbb{N}_0$ it holds that $\varrho_\varepsilon(i,m) + 1 > 0$ if $\varrho(i,m) + 1 > 0$. In what follows, we fix some $\varepsilon \in (0,\varepsilon_\varrho/3)$.

Lemma 6.5. For all $g \in \mathbb{N}_0$, $r \in \mathbb{N}_0$ it holds that $\partial^r_\kappa \partial^r_\mu G_{\kappa,\mu} \in C^\infty_c(\mathbb{M})$ and

$$\|P^{\mu}_{\kappa} \partial^r_\kappa \partial^r_\mu G_{\kappa,\mu}\|_K \lesssim [\kappa]^{(\varepsilon - \sigma)r}[\mu]^{-\varepsilon r}$$

uniformly in $\kappa \in (0,1/2]$, $\mu \in (0,1]$, where $G_{\kappa,\mu}$ was introduced in Def. 2.1.

Proof. Let $a \in \mathbb{N}_0^d$. It holds that $|\partial^a G(x)| \lesssim |x|^{\sigma-d-|a|}$ uniformly for $|x| \leq 2$. If $\sigma \in 2\mathbb{N}_+$, then $\partial^a G$ decays fast at infinity. In general, $|\partial^a G(x)| \lesssim |x|^{-\sigma-d-|a|}$ uniformly for $|x| > 1$. Moreover, we have

$$|\partial^a \partial^r_\kappa \chi_\kappa (|x|^\sigma)| \lesssim [\kappa]^{(\varepsilon - \sigma)r}[x]^{-\varepsilon r - |a|}, \quad |\partial^a \partial^r_\mu \chi_\mu (|x|^\sigma)| \lesssim |x|^{-\sigma - |a|}/\mu^2$$

uniformly in $\kappa \in (0,1/2]$, $\mu \in (0,1]$ and $x \in \mathbb{M}$. Furthermore, $|\partial^a \partial_\mu \chi_\mu (|x|^\sigma)|$ vanishes unless $\mu < (1 - \mu)|x|^\sigma \leq 2\mu$. Using the above properties and considering separately $\mu \in (0,1/2]$ and $\mu \in (1/2,1]$ we obtain $\|\partial^a \partial^r_{\kappa} \partial^r_{\mu} G_{\kappa,\mu}\|_K \lesssim [\kappa]^{(\varepsilon - \sigma)r}[\mu]^{-\varepsilon r - |a|}$. This implies the lemma since $P^{\mu}_{\kappa} = (1 - [\mu]^{2\Delta x})^r$. \hfill \square
Remark 6.6. Given $g \in \mathbb{N}_0$ there exists $\tilde{R} > 0$ such that
\[
\frac{(1 + m) \sigma}{\vartheta_{3e}(i, m)} \leq \tilde{R}^{1/2},
\]
for all $i, m \in \mathbb{N}_0$ such that $\varrho(i, m) > 0$ and all $r \in \{0, 1\}$, $\kappa \in (0, 1/2]$, $\mu \in (0, 1]$. The first of the above bounds implies, in particular, that $\sigma/(\dim(\Phi) + 9\varepsilon) \leq \tilde{R}^{1/2}$.

Theorem 6.7. Fix $g \in \mathbb{N}_0$. Let $\tilde{R} > 1$ as in Remark 6.6. Assume that for $r \in \{0, 1\}$, $s = 0$, all $i, m \in \mathbb{N}_0$ such that $\varrho(i, m) \leq 0$ and all $\kappa \in (0, 1/2]$, $\mu \in (0, 1]$ the following bound holds
\[
\|K^{g, \delta}_{\kappa} \| \leq \frac{\tilde{R}^{1-s/2+2(3i-m)}}{4(1 + i)^2(1 + m)^{2-s}} \varepsilon \| R^{1/4} [\kappa]^{(1-\sigma) r} [\mu]^{(i,s)(i,m)-\sigma}\ (6.1)
\]
Then the above bound holds for all $r, s \in \{0, 1\}$, $i, m \in \mathbb{N}_0$, $\kappa \in (0, 1/2]$, $\mu \in (0, 1]$. Remark 6.8. By Theorems 16.1, 13.1, 9.3, 10.1 (applied in this order), there exists a choice of counterterms such that the assumption of the above theorem is satisfied for some random $\tilde{R} > 0$ such that $\mathbb{E} \tilde{R}^n < \infty$ for all $n \in \mathbb{N}_+$. Remark 6.9. If the bound (6.1) holds for $g = \varepsilon$, then it also holds for all $g > \varepsilon$.

Remark 6.10. Let us comment on the assumption in two simple cases. For $i = 0, m = 0$ the bound (6.1) says that $\|K^{g, \delta}_{\kappa} \| \leq \tilde{R}^{1/2} [\kappa]^{-3\varepsilon}$, which is known to be true for $g = \delta$ and some random $\tilde{R} > 0$ such that $\mathbb{E} \tilde{R}^n < \infty$ for all $n \in \mathbb{N}_+$. For $i = 1, m = 3$ the bound (6.1) says that $\|K^{g, \delta}_{\kappa} \| \leq \tilde{R}^{1/4} [\kappa]^{-\varepsilon}$. It is easy to see that the above bound is satisfied for $\tilde{R} = 4^5$ using the fact that $F^{1,3}_{\kappa, \mu}(x; x_1, x_2, x_3) = \delta_m(x - x_1)\delta_m(x - x_2)\delta_m(x - x_3)$.

Remark 6.11. For $\lambda \leq \tilde{R}^{-6}$ the bounds (6.1) imply convergence of the series (3.3) defining $\Phi_\kappa$, which is our candidate for the solution of Eq. (1.1). In fact, one easily proves that $\|K^{g, \delta}_{\kappa} \| \leq R \kappa^{(1-\sigma)r}$ for $r \in \{0, 1\}$ and all $\kappa \in (0, 1]$. This implies the existence of the limit $\lim_{\kappa \searrow 0} \Phi_\kappa$ in $\mathcal{S}'(\mathbb{M})$. In Sec. 11 we will prove that $\Phi_\kappa$ solves Eq. (3.3) and $\lim_{\kappa \searrow 0} \Phi_\kappa$ exists in $\mathcal{C}^{-\dim}(\mathbb{M})$.

Proof. Fix some $i_0 \in \mathbb{N}_+$, $m_0 \in \mathbb{N}_0$ and assume that the statement with $s = 0$ is true for all $i, m \in \mathbb{N}_+$ such that either $i < i_0$ or $i = i_0$ and $m > m_0$. We shall prove the statement for $i = i_0$, $m = m_0$. We first consider the case $s = 1$. Using the flow equation (3.2) and the notation introduced in Sec. 5 we obtain
\[
\partial_{\kappa} \partial_{\mu}^* F_{i,m}^{i,m} = -\frac{1}{m!} \sum_{\pi \in P_m} \sum_{j=0}^i \sum_{k=0}^m \sum_{u+v+w=r} (1 + k) \times Y_\pi B(\partial_{\kappa}^\mu \partial_{\mu}^* G_{\kappa, \mu}, \partial_{\kappa}^\mu F_{i,m}^{i,m}, \partial_{\kappa}^\mu F_{i,m}^{i,m})
\]
where \( r, u, v, w \in \{0, 1\} \). By Remark 5.6, Lemma 5.8 and Remark 5.9 we get
\[
\| K^*_{\mu, \nu} \otimes (1 + m) \ast \partial_{\mu} \partial_{\nu}^{\nu} F_{\kappa, \mu}^{i, m} \|_{V^m} \leq (1 + m) \sum_{u + v + w = r} \| P^\mu_{\kappa, \nu} \partial_{\mu} \partial_{\nu} G_{\kappa, \mu} \|_K
\times \sum_{j=0}^{i} \sum_{k=0}^{m} \| K^*_{\mu, \nu} \otimes (2 + k) \ast \partial_{\nu}^{\nu} F_{j, \kappa}^{j, 1 + k} \|_{V^{1 + k}} \| K^*_{\mu, \nu} \otimes (1 + m - k) \partial_{\kappa} \partial_{\mu} F_{j, \kappa}^{i - j, m - k} \|_{V^{m - k}}.
\]

The statement of the theorem with \( s = 1 \) follows now from the induction hypothesis, Remark 6.6, the inequality
\[
\varrho_{3e}(i, m) - \sigma \leq \varrho_{3e}(j, 1 + k) + \varrho_{3e}(i - j, m - k) - \varepsilon
\]
and the bound
\[
\sum_{j=0}^{i} \sum_{k=0}^{m} \frac{\hat{R}^{1 + 2(3j - k - 1)}}{4(1 + j)^2} \frac{\hat{R}^{1 + 2(3i - 3j - m + k)}}{4(1 + i - j)^2} \frac{4(1 + i - j)^2}{4(1 + m - k)^2} \leq \frac{\hat{R}^{2(3i - m)}}{4(1 + i)^2 4(1 + m)^2},
\]
which is a consequence of the inequality
\[
\sum_{j=0}^{i} \frac{1}{4(1 + j)^2 4(1 + i - j)^2} \leq \frac{1}{4(1 + i)^2}.
\]
In order to prove the statement of the theorem for \( s = 0 \) and \( i, m \in \mathbb{N}_0 \) such that \( \varrho(i, m) > 0 \) we use the identity
\[
F_{\kappa, \mu}^{i, m} = F_{\kappa, \mu}^{i, m} + \int_0^\mu \partial_\eta F_{\kappa, \eta}^{i, m} \, d\eta. \tag{6.2}
\]
We first observe that \( F_{\kappa, \mu}^{i, m} = F_{\kappa, 0}^{i, m} = 0 \) if \( \varrho(i, m) > 0 \). Next, we note that
\[
\| K^*_{\mu, \nu} \otimes (1 + m) \ast \partial_{\nu}^{\nu} F_{\kappa, \nu}^{i, m} \|_{V^m} \leq \| K^*_{\nu} \otimes (1 + m) \ast \partial_{\nu}^{\nu} F_{\kappa, \nu}^{i, m} \|_{V^m}, \tag{6.3}
\]
for \( \eta \leq \mu \) by Lemma 4.5 and Remark 5.3. The statement of the theorem with \( s = 0 \) follows now from the statement with \( s = 1 \) and the bounds
\[
\| K^*_{\nu} \otimes (1 + m) \ast \partial_{\kappa} \partial_\eta F_{\kappa, \eta}^{i, m} \|_{V^m} \leq \int_0^\mu \| K^*_{\nu} \otimes (1 + m) \ast \partial_{\nu}^{\nu} F_{\kappa, \nu}^{i, m} \|_{V^m} \, d\eta
\]
and
\[
\int_0^\mu [\eta^{\varrho_{3e}(i, m) - \sigma}] \, d\eta \leq \frac{\sigma [\mu^{\varrho_{3e}(i, m)}]}{\varrho_{3e}(i, m)} \leq \frac{\hat{R}^{1/2} [\mu^{\varrho_{3e}(i, m)}]}{1 + m},
\]
We stress that the first inequality in the last bound is valid only if \( \varrho_{3e}(i, m) > 0 \), which holds provided \( \varrho(i, m) > 0 \) by Remark 6.4. \( \square \)
7 Generalized effective force coefficients

In this section we introduce generalized coefficients $F_{i,m,a}^{k,\mu}$ and $f_{i,m,a}^{k,\mu}$. Assuming certain bounds for $f_{i,m,a}^{k,\mu}$, which can be verified using probabilistic methods, in Sec. 9 and 10 we prove bounds for $F_{i,m,a}^{k,\mu}$, which in particular imply the bounds (6.1) for all $F_{i,m,a}^{k,\mu} = F_{i,m,0}^{k,\mu}$ such that $g(i,m) \leq 0$.

**Definition 7.1.** We denote the set of multi-indices by $\mathfrak{M} = \mathbb{N}_0^d$. For $m \in \mathbb{N}_+$ and $a = (a_1, \ldots, a_m) \in \mathfrak{M}$ we define $|a| := |a_1| + \ldots + |a_m|$. We also set $\mathfrak{M}^0 \equiv \{0\}$.

For $a \in \mathfrak{M}$ we define $X^a \in C^\infty(\mathbb{M})$ by $X^a(x) := x^a$ and for $a \in \mathfrak{M}$ we define $X^{m,a} \in C^\infty(\mathbb{M}^{1+m})$ by

$$X^{m,a}(x; y_1, \ldots, y_m) := (x - y_1)^{a_1} \ldots (x - y_m)^{a_m}.$$ 

For $i, m \in \mathbb{N}_0$, $a \in \mathfrak{M}$ we define $F_{i,m,a}^{k,\mu} \in \mathcal{S}(\mathbb{M}^{1+m})$ and $f_{i,m,a}^{k,\mu} \in \mathcal{S}(\mathbb{M})$ by

$$F_{i,m,a}^{k,\mu} := X^{m,a} F_{k,\mu}^i, \quad \langle f_{i,m,a}^{k,\mu}, \psi \rangle := \langle F_{i,m,a}^{k,\mu}, \psi \otimes 1_{\mathbb{M}}^\otimes \rangle,$$

where $\psi \in \mathcal{S}(\mathbb{M})$ and $1_{\mathbb{M}}(x) = 1$ for all $x \in \mathbb{M}$. The force coefficients $F_{i,m,a}^{k,\mu}$ and $f_{i,m,a}^{k,\mu}$ are defined analogously. The effective force coefficients $F_{i,m,a}^{k,\mu}$ or $f_{i,m,a}^{k,\mu}$ such that $g(i,m) + |a| \leq 0$ are called relevant. The remaining coefficients are called irrelevant. The finite collection of all relevant coefficients $f_{i,m,a}^{k,\mu}$ such that $m \leq 3i$ is called the enhanced noise (recall that $f_{i,m,a}^{k,\mu} = 0$ if $m > 3i$).

**Remark 7.2.** Let us list the non-zero force coefficients $f_{i,m,a}^{k,\mu} = f_{i,m,a}^{k,0} \in \mathcal{S}(\mathbb{M})$:

$$f_{i,m,a}^{0,0,0} = \xi(x), \quad f_{i,m,a}^{1,1,0} = 1, \quad f_{i,m,a}^{1,0,1} = e_i^1, \quad i \in \{1, \ldots, i_x\}.$$

**Remark 7.3.** Given $i, m \in \mathbb{N}_0$, $a \in \mathfrak{M}$ such that $g(i,m) + |a| \in (-1,1)$ for some $1 \in \mathbb{N}_+$ the relevant coefficient $F_{i,m,a}^{k,\mu}$ can be expressed in terms of irrelevant coefficients $F_{i,m,b}^{k,\mu}$, $|b| = 1$, and relevant coefficients $f_{i,m,b}^{k,\mu}$, $|b| < 1$. The above fact plays a crucial role in the proof of bounds for the relevant coefficients $F_{i,m,a}^{k,\mu}$ given in Sec. 9. In Sec. 8 we show that it is a consequence of the Taylor theorem. The relevant coefficients $f_{i,m,a}^{k,\mu} \in \mathcal{S}(\mathbb{M})$, which are elements of the enhanced noise, are bounded using probabilistic methods. On the other hand, the irrelevant coefficients $F_{i,m,a}^{k,\mu} \in \mathcal{S}(\mathbb{M}^{1+m})$ are bounded deterministically using the same strategy as in the proof of Theorem 6.7.

**Remark 7.4.** In the case $d = 5$ and $\sigma = 2$ the enhanced noise consists of $f_{i,m,a}^{0,0,0} = \xi$, $f_{i,m,a}^{1,0,1,0}$, $f_{i,m,a}^{1,1,0}$, $f_{i,m,a}^{2,0,1,0}$, $f_{i,m,a}^{1,3,0} = 1$, $f_{i,m,a}^{2,0,0}$, $f_{i,m,a}^{2,1,0}$. The following coefficients are

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The coefficient $F^{2,1}_{\kappa,\mu}(x; x_1)$ is non-local in the sense that it is not proportional to $\delta_M(x - x_1)$. The coefficient $F^{2,1}_{\kappa,\mu}$ can be expressed in terms of the relevant coefficient $F^{2,1,0}_{\kappa,\mu}$ and the irrelevant coefficient $F^{2,1,a}_{\kappa,\mu}$ with $|a| = 1$. Since the coefficients $F^{1,m}_{\kappa,\mu}$ are local the coefficients $F^{1,m,a}_{\kappa,\mu}$ vanish identically for $m \in \mathbb{N}_+$ and $a \neq 0$. Explicit expressions for all of the above coefficients can be easily deduced from expressions given in Remark 3.2.

For a list of multi-indices $a = (a_1, \ldots, a_m) \in \mathbb{M}^m$ and a permutation $\pi \in \mathcal{P}_n$ we set $\pi(a) := (a_{\pi(1)}, \ldots, a_{\pi(m)})$. Note that

$$\mathcal{X}^{m,a}(x; y_1, \ldots, y_m) = \mathcal{X}^{m,\pi(a)}(x; y_{\pi(1)}, \ldots, y_{\pi(m)}).$$

We claim that for all $y_0, z \in \mathbb{M}$ it holds that

$$\mathcal{X}^{m,a}(x; y_1, \ldots, y_m) = \sum_{b,c,d} \frac{a!}{b!c!d!} \mathcal{X}^{1+k,(b_1+k+\ldots+b_m,b_1,\ldots,b_k)}(x; y_0, y_1, \ldots, y_k)$$

$$\times \mathcal{X}^{1,c_1+k+\ldots+c_m}(y_0; z) \mathcal{X}^{m-k,(d_1+k,\ldots,d_m)}(z; y_1+k, \ldots, y_m),$$

(7.1)

where the sum is over all $b, c, d \in \mathbb{M}^m$ such that $b_p = a_p, c_p = 0, d_p = 0$ for $p \in \{1, \ldots, k\}$ and $b_p + c_p + d_p = a_p$ for $p \in \{1 + k, \ldots, m\}$. Throughout the paper we use the following schematic notation for the sums of the above type

$$\mathcal{X}^{m,a}(x; y_1, \ldots, y_m) = \sum_{b+c+d=a} \frac{a!}{b!c!d!} \mathcal{X}^{1+k,b}(x; y_0, y_1, \ldots, y_k)$$

$$\times \mathcal{X}^{c}(y_0-z) \mathcal{X}^{m-k,d}(z; y_1+k, \ldots, y_m).$$

Let us note that the formula (7.1) above was obtained by rewriting $(x - y_p)^{a_p}$, $p \in \{1 + k, \ldots, m\}$, as

$$(x - y_p)^{a_p} = \sum_{b_p+c_p+d_p} \frac{a_p!}{b_p!c_p!d_p!} (x - y_0)^{b_p}(y_0 - z)^{c_p}(z - y_p)^{d_p}.$$
Using the flow equation (3.2) and the identity (7.1) we show that the effective force coefficients \( F_{i,m,a}^{\kappa,\mu} \) satisfy the flow equation

\[
\partial_\mu \partial_\kappa F_{i,m,a}^{\kappa,\mu} = -\frac{1}{m!} \sum_{\pi \in \mathcal{P}} \sum_{j=0}^{m} \sum_{k=0}^{m} (1 + k) \sum_{u+v+w=r} \frac{r!}{u!v!w!} \sum_{b+c+d=\pi(a)} \frac{a!}{b!c!d!} \times Y_\pi \mathcal{B} (\mathcal{L}_c^u \partial_\kappa \partial_\mu G_{\kappa,\mu}, \partial_\mu F_{i-j,m-k,d}^{1+k,b}, \partial_\mu F_{i-j,m-k,d}) ,
\]

where \( r \in \mathbb{N}_0, u, v, w \in \mathbb{N}_0 \) and the sum over multi-indices \( b, c, d \) is restricted to the set specified below Eq. (7.1). Note that \( \frac{r!}{u!v!w!} = 1 \) above if \( r = 1 \).

8 Taylor polynomial and remainder

**Definition 8.1.** For \( a \in \mathcal{M}^m, V \in \mathcal{C}_\infty(\mathfrak{M}^{1+m}) \) we define \( \partial^a V \in \mathcal{C}_\infty(\mathfrak{M}^{1+m}) \) by \( \partial^a V(x; x_1, \ldots, x_m) := \partial_{x_1}^{a_1} \ldots \partial_{x_m}^{a_m} V(x; x_1, \ldots, x_m) \). The above map extends in an obvious way to \( \mathcal{S}'(\mathfrak{M}^{1+m}) \supset \mathcal{D}^m \).

**Definition 8.2.** For \( v \in \mathcal{D} \) we define \( \mathbf{L}^m v \in \mathcal{D}^m \) by the equality

\[
\langle \mathbf{L}^m v, \psi \otimes \varphi_1 \otimes \ldots \otimes \varphi_m \rangle = \langle v, \psi \varphi_1 \ldots \varphi_m \rangle,
\]

where \( \psi, \varphi_1, \ldots, \varphi_m \in \mathcal{S}(\mathfrak{M}) \) are arbitrary.

**Definition 8.3.** For \( \tau > 0 \) and \( V \in \mathcal{C}(\mathfrak{M}^{1+m}) \) we define \( Z_\tau V \in \mathcal{C}(\mathfrak{M}^{1+m}) \) by

\[
Z_\tau V(x; y_1, \ldots, y_m) := \tau^{-d_m} V(x; x + (y_1 - x)/\tau, \ldots, x + (y_m - x)/\tau).
\]

The above map extends in an obvious way to \( \mathcal{S}'(\mathfrak{M}^{1+m}) \supset \mathcal{D}^m \).

**Definition 8.4.** The linear map \( \mathbf{I} : \mathcal{V}^m \to \mathcal{V} \) is defined by

\[
\mathbf{I} V(x) := \int_{\mathfrak{M}^m} V(x; y_1, \ldots, y_m) \, dy_1 \ldots dy_m.
\]

The map \( \mathbf{I} \) is extended to \( V \in \mathcal{D}^{m \otimes \mathfrak{M}} \) by the formula

\[
\langle \mathbf{I} V, \psi \rangle := \langle (\delta_\mathfrak{M} \otimes K^{*} \otimes \mathfrak{M}^m) \ast V, \psi \otimes 1_{\mathfrak{M}}^m \rangle,
\]

where \( \psi \in \mathcal{S}(\mathfrak{M}) \).

**Lemma 8.5.** The map \( \mathbf{I} \) is well defined and has the following properties.
(A) $I(K^{\varphi_g \otimes (1+m)}_\mu \ast V) = K^{\varphi}_\mu \ast IV$ for all $V \in \mathcal{V}^m$, $g \in \mathbb{N}_0$ and $\mu > 0$.

(B) $\|IV\|_\mathcal{V} \leq \|V\|_{\mathcal{V}^m}$ for $V \in \mathcal{V}^m$.

Proof. The well-definedness and Part (A) follow from $\int_M K_\mu(x) \, dx = 1$. Part (B) is a consequence of Def. 5.2 of the norm $\mathcal{V}^m$. □

Definition 8.6. For $1 \in \mathbb{N}_+$, $a \in \mathcal{M}^m$ such that $|a| < 1$ and two collections of distributions: $v^b \in \mathcal{D}$, $b \in \mathcal{M}^m$, $|b| < 1$, and $V^b \in \mathcal{D}^m$, $b \in \mathcal{M}^m$, $|b| = 1$, the distribution $X_1^a(v^b, V^b) \in \mathcal{D}^m$ is defined by the equality

$X_1^a(v^b, V^b) := \sum_{|a+b|<1} \frac{1}{b!} \partial^b L^m(v^{a+b}) + \sum_{|a+b|=1} \frac{|b|}{b!} \int_0^1 (1 - \tau)^{|b|-1} \partial^b Z_\tau(V^{a+b}) \, d\tau,$

where the sums above are over $b \in \mathcal{M}^m$.

Theorem 8.7. Let $1 \in \mathbb{N}_+$ and $g \in \mathbb{N}_0$. There exists a constant $c > 0$ such that the following statement is true. Let $V^b \in \mathcal{V}^m$, $v^b \in \mathcal{V}$ be as in Def. 8.6 and $\mu \in (0, 1]$. Assume that there exists a constant $C > 0$ such that for $b \in \mathcal{M}$

$\|K^{\varphi_g \otimes (1+m)}_\mu \ast V^b\|_{\mathcal{V}^m} \leq C \|\mu\|^{|b|},$ 

$|b| = 1,$ 

$\|K^{\varphi_0 \otimes (1+m)}_\mu \ast v^b\|_{\mathcal{V}} \leq C \|\mu\|^{|b|},$ 

$|b| < 1$.

Then for $a \in \mathcal{M}$ such that $|a| < 1$, it holds that

$\|K_\mu^{(2g+1), \otimes (1+m)} \ast X_1^a(v^b, V^b)\|_{\mathcal{V}^m} \leq c C \|\mu\|^{-|a|}.$

Proof. The theorem follows from Def. 8.6 of the map $X_1^a$, Lemma 8.9 and the bound $\|\partial^k K_\mu^2\|_\mathcal{K} \lesssim \|\mu\|^{-|c|}$, $c \in \mathcal{M}$, $|c| \leq 1$, proved in Lemma 4.8 (A). □

Theorem 8.8. Let $1 \in \mathbb{N}_+$ and $V \in \mathcal{D}^m$ such that $\mathcal{X}^{m,b}V \in \mathcal{D}^m$ for all $b \in \mathcal{M}^m$, $|b| \leq 1$. Then $\mathcal{X}^{m,a}V = X_1^a(I(\mathcal{X}^{m,b}V), \mathcal{X}^{m,b}V)$ for all $a \in \mathcal{M}^m$, $|a| < 1$.

Proof. First recall that for all $\varphi \in C_\infty(\mathbb{R}^N)$, $N \in \mathbb{N}_+$, it holds that

$\varphi(y) = \sum_{|b|<1} \frac{1}{b!} (y - x)^b (\partial^b \varphi)(x)$

$+ \sum_{|b|=1} \frac{|b|}{b!} (y - x)^b \int_0^1 (1 - \tau)^{|b|-1} (\partial^b \varphi)(x + \tau(y - x)) \, d\tau$
by the Taylor theorem, where the sums are over $b \in \mathbb{N}_0^N$. Consequently, for all $\varphi \in \mathcal{S}(\mathbb{M}^{1+m})$ we have

$$(A) \quad \|K_{\tau}^{\varphi}\|_{\mathcal{M}} \leq \|K_{\tau} \varphi\|_{\mathcal{M}},$$

where the kernels $\tau_{\varphi}$ follows from the identities

$$(B) \quad \|K_{\tau}^{\varphi}\|_{\mathcal{M}} \leq \|K_{\tau}^{\varphi}\|_{\mathcal{M}} \leq \|K_{\tau}^{\varphi}\|_{\mathcal{M}}.$$

Proof. By the exact scaling of the norms and kernels it is enough to prove the bound (A) for $\tau = 1$ and all $\varphi \in C_b(\mathbb{M})$, $V \in C(\mathbb{M}^{1+m})$ such that $||V||_{\mathcal{M}} < \infty$. Using $P^{\delta} \tau_{\varphi} = \delta_{\varphi}$ we obtain

$$(K_{\tau}^{\varphi}(x,y_1,\ldots,y_m) = \int_{\mathbb{M}} (K_{\tau}^{\varphi}(r) P^{\delta}(r) K_{\tau}^{\varphi}(x-r) K_{\tau}^{\varphi}(y_1 - r) \ldots K_{\tau}^{\varphi}(y_m - r) dz$$

$$(B) \quad \|K_{\tau}^{\varphi}\|_{\mathcal{M}} \leq \|K_{\tau}^{\varphi}\|_{\mathcal{M}} \leq \|K_{\tau}^{\varphi}\|_{\mathcal{M}} \leq \|K_{\tau}^{\varphi}\|_{\mathcal{M}}.$$

Lemma 8.9. Let $h \in \mathbb{N}_0$. The following bounds hold uniformly for $\tau, \mu \in (0,1]$ and $v \in \mathcal{V}, V \in \mathcal{V}^m$:

$$(A) \quad \|K_{\tau}^{h,\varphi}(1+m) * L^m v\|_{\mathcal{M}} \leq \|K_{\tau}^{h,\varphi} v\|_{\mathcal{M}},$$

$$(B) \quad \|K_{\tau}^{h,\varphi}(1+m) * Z_{\tau} V\|_{\mathcal{M}} \leq \|K_{\tau}^{h,\varphi}(1+m) * V\|_{\mathcal{M}}.$$
which implies Part (B) since $\|\mathcal{H}_\tau\|_{K^{1+m}} \lesssim 1$ uniformly in $\tau \in (0, 1]$ by the lemma below. It remains to establish the identities (8.1). The first one follows from the fact that $Z_\tau(\delta_M \otimes K^{\otimes m}) = \delta_M \otimes \hat{\mathcal{K}}^{\otimes m}$, where $\hat{\mathcal{K}}(x) := \tau^{-d} K(x/\tau)$. To show the second one we observe that by the definition of $\mathcal{H}_\tau$ given in the lemma below

$$Z_{1/\tau}(H_\tau)(x; y_1, \ldots, y_m) = \tau^{dm} P(\partial_x) K(x) K(x + \tau(y_1 - x)) \ldots K(x + \tau(y_1 - x)).$$

After convolving both sides with the kernel $K \otimes \delta_M$ we obtain

$$Z_{1/\tau}(H_\tau) \ast (K \otimes \delta_M) = Z_{1/\tau}(K^{\otimes (1+m)}).$$

The second identity in (8.1) follows after applying the map $Z_\tau$ to both sides of the above equality. \hfill \square

**Lemma 8.10.** The distributions $\hat{\mathcal{K}}_\tau \in \mathcal{S}'(M)$ and $H_\tau \in \mathcal{S}'(M^{1+m})$ defined by

$$\hat{\mathcal{K}}_\tau(x) := P(\tau \partial_x) K(x) = \tau^{2} \delta_M(x) + (1 - \tau^{2}) K(x),$$

$$H_\tau(x; y_1, \ldots, y_m) := P(\partial_x + (1 - \tau) \partial_y) K(x) K(y_1) \ldots K(y_m),$$

where $\partial_y := \partial_{y_1} + \ldots + \partial_{y_m}$ and $P(r) = 1 - r^2$, are polynomials in $\tau > 0$ of degree 2 whose coefficients belong to $\mathcal{K}$ and $\mathcal{K}^{1+m}$, respectively.

### 9 Deterministic bounds for the relevant coefficients

**Lemma 9.1.** For every $g \in \mathbb{N}_0$, $a \in \mathfrak{M}$ and $r \in \mathbb{N}_0$ the bound

$$\|P^{g} x^{a} \partial^{r} \kappa, \mu \partial^{r} \kappa, \mu G_{\kappa, \mu}\|_{K} \lesssim [\kappa]^{(\varepsilon - \sigma) r \mu} [\mu]^{a} \varepsilon r$$

holds uniformly in $\kappa, \mu \in (0, 1/2]$.

**Remark 9.2.** The above lemma can be easily proved using the method of the proof of Lemma 6.5. For $\sigma \notin 2\mathbb{N}_+$ and $|a| \geq \sigma$ the bound stated in the lemma does not hold uniformly for all $\mu \in (0, 1]$ because of the slow decay of $G(x)$ at infinity. For this reason the range $\mu \in (1/2, 1]$ is treated separately in Sec. 10.

**Theorem 9.3.** Fix $R > 1$. Assume that for all $i, m \in \mathbb{N}_0$, $a \in \mathfrak{M}^{m}$ such that $\varrho(i, m) + |a| \leq 0$ there exists $g \in \mathbb{N}_0$ such that

$$\|K_{\mu} \ast \partial^{r} f_{\kappa, \mu}^{i, m, a}\|_{V} \leq R [\kappa]^{(\varepsilon - \sigma) r \mu} [\mu]^{g + \varepsilon i, m} + |a|$$

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for all \( r \in \{0, 1\} \) and \( \kappa, \mu \in (0, 1/2) \). Then for all \( i, m \in \mathbb{N}_0 \), \( a \in \mathbb{R}^m \) there exists \( g \in \mathbb{N}_0 \) such that
\[
\|K^\nu_{\mu}g_{\otimes(1+m)} * \partial^a_{\kappa, \mu} F_{i,m,a} \|_{\mathcal{V}} \lesssim R^{1+3i-m} [\kappa]^{(\varepsilon-\sigma)r} [\mu]^{\|g_{\sigma}(i,m)\| + |a| - \sigma s}
\]
for all \( r, s \in \{0, 1\} \) uniformly in \( \kappa, \mu \in (0, 1/2) \). The constants of proportionality in the above bounds depend only on \( i, m \in \mathbb{N}_+, a \in \mathbb{R}^m \) and are otherwise universal.

**Remark 9.4.** Note that the parameter \( g \in \mathbb{N}_0 \) in Theorem 9.3 as well as in Theorem 10.1 depends on \( i, m \in \mathbb{N}_+ \) and \( a \in \mathbb{R}^m \). We shall use these theorems to bound only the relevant effective force coefficients. Since there are only finitely many non-zero relevant coefficients one can choose \( g \in \mathbb{N}_0 \) independent of \( i, m \in \mathbb{N}_+ \) and \( a \in \mathbb{R}^m \) such that the bounds stated in the above-mentioned theorems hold true for all relevant coefficients.

**Remark 9.5.** Theorem 9.3 together with Theorem 10.1 imply that under the assumption of the former theorem there exists \( g \in \mathbb{N}_0 \) and a universal constant \( c > 1 \) such that the assumption of Theorem 6.7 is satisfied with \( R = cR \).

**Proof.** The base case: We observe that \( F_{0,0,0}^i = F_{0,0}^i = \xi \). Hence, for \( i = 0, m = 0 \) the statement follows from the assumption.

The inductive step: Fix some \( i_0 \in \mathbb{N}_+, m_0 \in \mathbb{N}_0 \) and assume that the statement with \( s = 0 \) is true for all \( i, m \in \mathbb{N}_+ \) such that either \( i < i_0 \) or \( i = i_0 \) and \( m > m_0 \). We shall prove the statement for \( i = i_0, m = m_0 \). As in the proof of Theorem 6.7 the induction hypothesis, the flow equation (7.2), and
\[
\|g_{3\varepsilon}(i,m) + |a| - \sigma \leq g_{3\varepsilon}(j,1+k) + |b| + |c| + g_{3\varepsilon}(i-j,m-k) + |d| - \varepsilon,
\]
\[
R^{1+3j-k-1} R^{1+3(i-j)-m+k} = R^{1+3i-m}
\]
imply the statement for \( s = 1 \).

In order to prove the statement for \( s = 0 \) let us first assume that \( a \in \mathbb{R}^m \) is such that \( g(i,m) + |a| > 0 \). Then \( F_{0,0,a}^i = F_{0}^i = 0 \). Consequently, the analogs of Eq. (6.2) and the bound (6.3) imply that
\[
\|K^\nu_{\mu}g_{\otimes(1+m)} * \partial^a_{\kappa, \mu} F_{i,m,a} \|_{\mathcal{V}} \leq \int_0^\mu \|K^\nu_{\eta}g_{\otimes(1+m)} * \partial^a_{\kappa, \eta} F_{i,m,a} \|_{\mathcal{V}} \, d\eta.
\]
The statement follows now from the bound for $\partial^r \partial_\eta F_{i,m,a}^{i,m,a}$ and the fact that $\varrho(i,m) + |a| - \varepsilon > 0$ if $\varrho(i,m) + |a| > 0$ by Remark 6.4.

Now suppose that $a \in \mathcal{M}$ is such that $\varrho(i,m) + |a| \leq 0$. Then our bound for $\|K^* g, \otimes (1 + m) \varrho \| \nabla^m$ is not integrable at $\eta = 0$. Consequently, the strategy used in the previous paragraph does not work. If $m = 0$, then $a = 0$ and $\partial^r F_{i,m,0}^{i,m,0} = \partial^r F_{i,m,0}^{i,m,0}$. As a result, the statement follows from the assumption.

If $m \in \mathbb{N}^+$, then we use the identity

$$\partial^r F_{i,m,a}^{i,m,a} = \sum_{b \in \mathbb{N}^0} (\partial^r F_{i,m,b}^{i,m,b}, \partial^r F_{i,m,b}^{i,m,b}),$$

which follows from Theorem 8.8. We choose $l \in \mathbb{N}^+$ to be the smallest positive integer such that $\varrho(i,m) + l > 0$. With this choice of $l$ the RHS of the above equality involves only the coefficients $f_{i,m,b}^{i,m,b}$ such that $\varrho(i,m,b) \leq 0$, which satisfy the assumed bound, and the coefficients $F_{i,m,b}^{i,m,b}$ such that $\varrho(i,m,b) > 0$, for which the statement of the theorem has already been established. To finish the proof of the inductive step we apply Theorem 8.7 with $C \lesssim [\kappa]^{(\varepsilon-\sigma)r} [\mu]^{[\kappa]}$.

10 Deterministic bounds for the long-range part of the coefficients

**Theorem 10.1.** Fix $R > 1$. Assume that for all $i,m \in \mathbb{N}_0$ there exists $g \in \mathbb{N}_0$ such that

$$\|K^*g, \otimes (1+m) \partial^r \partial_\mu F_{i,m,a}^{i,m,a}\| \nabla^m \lesssim R^{1+3i-m} [\kappa]^{(\varepsilon-\sigma)r} [\mu]^{g_{i,a}^{(i,m)}}$$

for $r,s \in \{0,1\}$ uniformly in $\kappa, \mu \in (0,1/2]$. Then for all $i,m \in \mathbb{N}_0$ there exists $g \in \mathbb{N}_0$ such that the above bound is true for $r,s \in \{0,1\}$ uniformly in $\kappa \in (0,1/2]$, $\mu \in (0,1]$. The constants of proportionality in the above bounds depend only on $i,m \in \mathbb{N}_+$ and are otherwise universal.

**Proof.** The base case: We observe that $F_{i,m,0}^{i,m,0} = f_{i,m,0}^{i,m,0} = \xi$ is independent of $\kappa, \mu$. The assumed bound and Lemma 4.5 imply that

$$\|K^*g \xi \| \nabla \leq \|K^*g_{/2} \xi \| \nabla \lesssim [\mu]^{g_{i,o}^{(0,0)}}$$

uniformly in $\mu \in (0,1]$. This proves the statement for $i = 0$ and $m = 0$.

The induction step: Fix some $i_o \in \mathbb{N}_+$, $m_o \in \mathbb{N}_0$ and assume that the statement with $s = 0$ is true for all $i,m \in \mathbb{N}_+$ such that either $i < i_o$ or $i = i_o$ and $m > m_o$. We shall prove the statement for $i = i_o$, $m = m_o$. The induction
hypothesis and the flow equation imply the statement for \( s = 1 \). In order to prove the statement for \( s = 0 \) we use the identity

\[
F^{i,m}_{\kappa,\mu} = F^{i,m}_{\kappa,1/2} + \int_{1/2}^{\mu} \partial_\eta F^{i,m}_{\kappa,\eta} \, d\eta,
\]

which by Lemma 4.5 implies that

\[
\|K^{\ast g,\otimes(1+m)}_\mu \ast \partial_\kappa F^{i,m}_{\kappa,\mu}\|_{V^m} \\
\leq \|K^{\ast g,\otimes(1+m)}_{1/2} \ast \partial_\kappa F^{i,m}_{\kappa,1/2}\|_{V^m} + \int_{1/2}^{\mu} \|K^{\ast g,\otimes(1+m)}_\eta \ast \partial_\kappa \partial_\eta F^{i,m}_{\kappa,\eta}\|_{V^m} \, d\eta
\]

for \( \mu \in [1/2, 1] \). This completes the proof of the inductive step. \(\Box\)

11 Deterministic construction of the solution

Assuming the bound (6.1) for all \( r, s \in \{0, 1\}, i, m \in \mathbb{N}_0, \kappa \in (0, 1/2], \mu \in (0, 1] \) with fixed \( g \in \mathbb{N}_0, \tilde{R} > 1 \) we show how to construct a solution of Eq. (1.1).

**Definition 11.1.** The families of sets \( \mathcal{V}_\mu, \mathcal{B}_\mu, \mu \in (0, 1] \), are defined as

\[
\mathcal{V}_\mu := \{K^{\ast g}_\mu \ast \varphi \in \mathcal{V} \mid \varphi \in \mathcal{V}\}, \\
\mathcal{B}_\mu := \{K^{\ast g}_\mu \ast \varphi \in \mathcal{V} \mid \varphi \in \mathcal{V}, \|\varphi\|_\mathcal{V} < \tilde{R}^2 [\mu]^{-\dim(\varphi) - 9\varepsilon}\}.
\]

By Lemma 4.5, if \( \varphi \in \mathcal{B}_\mu \), then \( \varphi \in \mathcal{B}_\eta \) for all \( \eta \) in a sufficiently small neighbourhood of \( \mu \). For \( \kappa \in (0, 1], \mu \in [0, 1] \) and \( |\lambda| \leq \tilde{R}^{-6} \) we define the functionals

\[
F_{\kappa,\mu} : \mathcal{B}_\mu \to \mathcal{D}, \\
DF_{\kappa,\mu} : \mathcal{B}_\mu \times \mathcal{V}_\mu \to \mathcal{D}
\]

by the formulas

\[
\langle F_{\kappa,\mu}[\varphi], \psi \rangle := \sum_{i=0}^{\infty} \sum_{m=0}^{\infty} \lambda^i \langle F^{i,m}_{\kappa,\mu}, \psi \otimes \varphi^{\otimes m} \rangle,
\]

\[
\langle DF_{\kappa,\mu}[\varphi, \zeta], \psi \rangle := \sum_{i=0}^{\infty} \sum_{m=0}^{\infty} \lambda^i (1 + m) \langle F^{i,1+m}_{\kappa,\mu}, \psi \otimes \zeta \otimes \varphi^{\otimes m} \rangle,
\]

where \( \psi \in \mathcal{S}(\mathbb{M}), \varphi \in \mathcal{B}_\mu \) and \( \zeta \in \mathcal{V}_\mu \).
Lemma 11.2. For $|\lambda| \leq \tilde{R}^{-6}$ the functionals $F_{\kappa,\mu}$, $DF_{\kappa,\mu}$ are well defined and satisfy the bounds

$$
\|K^{\pm g}_\nu \ast \partial_{\nu} F_{\kappa,\mu}[\varphi]\|_V \leq \tilde{R} |\kappa|^{(\varepsilon-\sigma)r} |\mu|^{-\dim(\xi)-3\varepsilon}
$$

$$
\|K^{\pm g}_\nu \ast \partial_{\nu} DF_{\kappa,\mu}[\varphi, \zeta]\|_V \leq \tilde{R}^{-1} |\kappa|^{(\varepsilon-\sigma)r} |\mu|^{-\dim(\xi)-3\varepsilon} \|P^{\pm g}_\mu \zeta\|_V
$$

for $r \in \{0, 1\}$ and $\kappa \in (0, 1/2)$, $\mu \in (0, 1)$ and $\varphi \in B_{\mu}$, $\zeta \in V_{\mu}$. The functional $DF_{\kappa,\mu}$ is the directional derivative of $F_{\kappa,\mu}$. Moreover, given $\eta \in (0, 1)$ and $\varphi \in B_{\eta}$ the flow equation (2.1) holds for all $\mu$ in a sufficiently small neighbourhood of $\eta$.

Proof. The proof is straightforward. For example, using Def. 5.2 of the space $V_m$ and the bounds for the effective force coefficients we obtain

$$
\|K^{\pm g}_\nu \ast \partial_{\nu} F_{\kappa,\mu}[K^{\pm g}_\nu \ast \varphi]\| \leq \sum_{i=0}^{\infty} \sum_{m=0}^{\infty} \frac{\lambda^i |\tilde{R}|^{1+2(3i-m)}}{4(1+i)^2 4(1+m)^2} |\kappa|^{(\varepsilon-\sigma)r} |\mu|^{0,i(m)} \tilde{R}^{2m} |\mu|^{-m(\dim(\Phi)+9\varepsilon)}
$$

for $\varphi \in V$ such that $\|\varphi\|_V \leq \tilde{R}^2 |\mu|^{-\dim(\Phi)-9\varepsilon}$, which implies the desired bound. The flow equation (2.1) follows from the flow equation (3.2). □

Lemma 11.3. For $|\lambda| \leq \tilde{R}^{-6}$, $r \in \{0, 1\}$ and $\kappa \in (0, 1/2)$, $\mu \in (0, 1)$, it holds that:

(A) $\|K^{\pm g}_\mu \ast \partial_{\kappa} F_{\kappa,1}[0]\|_V \leq 2^{r} \tilde{R} |\kappa|^{r(\varepsilon-\sigma)} |\mu|^{-\dim(\xi)-3\varepsilon}$,

(B) $\|P^{\pm g}_\mu \partial_{\kappa}(G_{\kappa,\mu} \ast F_{\kappa,1}[0])\|_V \leq 3^{r} \tilde{R}^{2} |\kappa|^{r(\varepsilon-\sigma)} |\mu|^{-\dim(\Phi)-9\varepsilon}$,

(C) $F_{\kappa,1}[0] = F_{\kappa,1}[G_{\kappa,\mu} \ast F_{\kappa,1}[0]]$,

(D) $\partial_{\kappa} F_{\kappa,1}[0] = (\partial_{\kappa} F_{\kappa,\mu})[G_{\kappa,\mu} \ast F_{\kappa,1}[0]] + DF_{\kappa,\mu}[G_{\kappa,\mu} \ast F_{\kappa,1}[0], \partial_{\kappa}(G_{\kappa,\mu} \ast F_{\kappa,1}[0])]$.

Proof. By Lemma 11.2 and the equality $G_{\kappa,1} = 0$ the statement holds true for $\mu = 1$. Chose $\nu \in (0, 1]$ and assume that the statement holds true for all $\mu \in [\nu, 1]$. Then by Lemma 4.5 we have

$$
\|K^{\pm g}_\eta \ast \partial_{\kappa} F_{\kappa,1}[0]\|_V \leq 2^{1+r} \|K^{\pm g}_\nu \ast \partial_{\kappa} F_{\kappa,1}[0]\|_V \leq 2^{1+r} \tilde{R} |\kappa|^{r(\varepsilon-\sigma)} |\mu|^{-\dim(\xi)-3\varepsilon}
$$

for all $\eta \in [\tau \mu, 1)$, $\mu \in [\nu, 1]$ for some $\tau \in (0, 1)$ depending only on $\tilde{R}$. As a result,

$$
\|K^{\pm g}_\mu \ast \partial_{\kappa} F_{\kappa,1}[0]\|_V \leq 2^{1+r} \tilde{R} |\kappa|^{r(\varepsilon-\sigma)} |\mu|^{-\dim(\xi)-3\varepsilon}
$$

for all $\mu \in [\tau \nu, 1]$. This together with Remark 6.6 and the estimate

$$
\|P^{\pm g}_\mu \partial_{\kappa}(G_{\kappa,\mu} \ast F_{\kappa,1}[0])\|_V \leq \sum_{u+v=r} \int_{\mu}^{1} \|P^{\pm g}_\eta \partial_{\kappa} \partial_{\eta} G_{\kappa,\eta}\|_V \|K^{\pm g}_\eta \ast \partial_{\kappa} F_{\kappa,1}[0]\|_V \, d\eta
$$

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implies Part (B) for all $\mu \in [\tau \nu, 1]$ and shows that $G_{\kappa, \mu} * F_{\kappa, 1}[0] \in \mathcal{B}_\mu$. Using Lemma 11.2 and the reasoning presented in Sec. 2 below Eq. (2.2) we prove that Parts (C) and (D) hold for all $\mu \in [\tau \kappa, 1]$. Part (A) for all $\mu \in [\tau \kappa, 1]$ follows now from Lemma 11.2. To complete the proof of the theorem it is enough to apply the above reasoning recursively with $\nu = \tau^n, n \in \mathbb{N}_0$.

**Theorem 11.4.** Fix $R > 1$. Assume that for all $i, m \in \mathbb{N}_+, a \in \mathcal{M}$ such that $\rho(i, m) + |a| \leq 0$ there exists $\mathbf{g} \in \mathbb{N}_0$ such that

$$\|K^*_{\mu} * \partial^r_{\kappa} f_{i,m,a}\|_{V} \leq R [\kappa]^{(\varepsilon - \sigma) r} [\mu]^{\rho(i, m) + |a|}$$

for all $r \in \{0, 1\}$ and $\kappa \in (0, 1/2], \mu \in (0, 1)$. There exists $\mathbf{g} \in \mathbb{N}_0$ and a universal constant $c > 1$ such that for $R = c R, |\lambda| < \tilde{R}^{-6}$ and $\kappa \in (0, 1/2]$ the function $\Phi_\kappa := G_{\kappa} * F_{\kappa, \mu}[0]$ is well defined, solves Eq. (1.1) and satisfies the bound

$$\|K^*_{\mu} * \partial^r_{\kappa} \Phi_\kappa\|_{V} \leq \tilde{R}^2 [\kappa]^{(\varepsilon - \sigma) r} [\mu]^{-\dim(\Phi) - 9\varepsilon}$$

for all $\kappa, \mu \in (0, 1]$.

**Proof.** By Theorems 6.7, 9.3 and 10.1 the assumption implies that there exists $\mathbf{g} \in \mathbb{N}_0$ and a universal constant $c > 1$ such that the bound (6.1) holds true for all $r, s \in \{0, 1\}, i, m \in \mathbb{N}_0$ and $\kappa \in (0, 1/2], \mu \in (0, 1]$ with $\tilde{R} = c R$. The function $\Phi_\kappa$ is well-defined by Lemma 11.2. To conclude, we observe that

$$\|K^*_{\mu} * \partial^r_{\kappa} \Phi_\kappa\|_{V} \leq \sum_{u + v = r} \int_0^1 \|P^\mu_{\eta} \partial^u_{\kappa} \partial^v_{\eta} G_{\kappa, \eta}\|_{V} \|K^*_{\mu} * K^*_{\eta} * \partial^u_{\kappa} F_{\kappa, 1}[0]\|_{V} d\eta,$$

use Remark 6.6 and apply Lemma 11.3 with $\mu$ replaced by $\eta \vee \mu$.

## 12 Cumulants of the effective force coefficients

In the remaining part of the paper we show that the assumption of Theorem 11.4 is satisfied for some random $R$ such that $\mathbb{E} R^n < \infty$ for all $n \in \mathbb{N}_+$. This will follow from the bounds for the joint cumulants of the effective force coefficients $F_{\kappa, \mu}$ proved in Sec. 16 and the probabilistic estimates proved in Sec. 13. The bounds for the joint cumulants of $F_{\kappa, \mu}$ involve the norm $\|\cdot\|_{V}$ introduced in Sec. 14. We prove the above-mentioned bounds using a certain flow equation given in Sec. 15. We consider joint cumulants of $F_{\kappa, \mu}$ instead of $f_{\kappa, \mu}$ because there is no flow equation for the joint cumulants of $f_{\kappa, \mu}$. 

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**Definition 12.1.** Let \( p \in \mathbb{N}_+ \), \( I = \{1, \ldots, p\} \) and \( \zeta_q, q \in I \), be random variables. The joint cumulant [9] of the multi-set \( (\zeta_q)_{q \in I} = (\zeta_1, \ldots, \zeta_p) \) is defined by
\[
\langle\langle \zeta_1, \ldots, \zeta_p \rangle\rangle \equiv \langle\langle (\zeta_q)_{q \in I} \rangle\rangle = (-i)^p \partial_{t_1} \cdots \partial_{t_p} \log \exp(it_1 \zeta_1 + \ldots + it_p \zeta_p) \big|_{t_1 = \ldots = t_p = 0}.
\]
In particular, \( \langle\zeta_1, \zeta_2 \rangle \equiv \mathbb{E}\zeta_1 \zeta_2 = \langle\langle \zeta_1 \zeta_2 \rangle\rangle - \langle\langle \zeta_1 \rangle\rangle \langle\langle \zeta_2 \rangle\rangle = \langle\zeta_1 \rangle \langle\langle \zeta_2 \rangle\rangle - \langle\langle \zeta_1 \rangle\rangle \langle\langle \zeta_2 \rangle\rangle.

**Lemma 12.2.** Let \( p \in \mathbb{N}_+ \), \( I = \{1, \ldots, p\} \) and \( \zeta_1, \ldots, \zeta_p, \Phi, \Psi \) be random variables. It holds
\[
\langle\langle \zeta_q \rangle\rangle I, \Phi \rangle = \langle\langle \zeta_q \rangle\rangle I, \Phi, \Psi \rangle + \sum_{I_1, I_2 \subseteq I, I_1 \cup I_2 = I} \langle\langle \zeta_q \rangle\rangle I_1, \Phi \rangle \langle\langle \zeta_q \rangle\rangle I_2, \Psi \rangle. \tag{12.2}
\]

**Remark 12.3.** For the proof of the above lemma see e.g. Proposition 3.2.1 in [9].

**Definition 12.4.** Given \( n \in \mathbb{N}_+ \), \( I = \{1, \ldots, n\} \), \( m_1, \ldots, m_n \in \mathbb{N}_0 \) and random distributions \( \zeta_q \in \mathcal{S}(\mathbb{M}^{1+m_q}), q \in I \), we define the deterministic distribution \( \langle\langle (\zeta_q)_{q \in I} \rangle\rangle \equiv \langle\langle \zeta_1, \ldots, \zeta_n \rangle\rangle \in \mathcal{S}(\mathbb{M}^{m} \times \mathbb{M}^{m_1+\ldots+mn}) \) by the equality
\[
\langle\langle \zeta_1, \ldots, \zeta_n \rangle\rangle, \psi_1 \otimes \ldots \otimes \psi_n \otimes \varphi_1 \otimes \ldots \otimes \varphi_n \rangle := \langle\langle \zeta_1, \psi_1 \otimes \varphi_1 \rangle, \ldots, \langle\langle \zeta_n, \psi_n \otimes \varphi_n \rangle \rangle,
\]
where \( \psi_q \in \mathcal{S}(\mathbb{M}), \varphi_q \in \mathcal{S}(\mathbb{M}^{m_q}), q \in I \), are arbitrary.

**Definition 12.5.** A list \((i, m, a, s, r)\), where \( i, m \in \mathbb{N}_0, a \in \mathbb{M}^m \) and \( s \in \{0, 1\}, r \in \{0, 1, 2\} \) is called an index. Let \( n \in \mathbb{N}_+ \) and
\[
I \equiv ((1, m_1, a_1, s_1, r_1), \ldots, (n, m_n, a_n, s_n, r_n)) \tag{12.3}
\]
be a list of indices. We set \( n(I) := n \), \( i(I) := i_1 + \ldots + i_n \), \( m(I) := (m_1, \ldots, m_n) \), \( m(I) := m_1 + \ldots + m_n \), \( a(I) := |a_1| + \ldots + |a_n| \), \( s(I) := s_1 + \ldots + s_n \) and \( r(I) := r_1 + \ldots + r_n \). We use the following notation for the joint cumulants of the effective force coefficients
\[
E_{n, \mu}^{\kappa} := \langle\partial_{\mu}^l \partial_{\kappa}^r F_i^{m_1, a_1}, \ldots, \partial_{\mu}^l \partial_{\kappa}^r F_i^{m, a_n} \rangle \in \mathcal{S}(\mathbb{M}^{m(1)} \times \mathbb{M}^{m(1)}).
\]

**Definition 12.6.** For \( \varepsilon \geq 0 \) and a list of indices \( I \) of the form (12.3) we define \( g_\varepsilon(I) := g_\varepsilon(i_1, m_1) + |a_1| + \ldots + g_\varepsilon(i_n, m_n) + |a_n| \in \mathbb{R} \).
We also set \( g(I) := g_0(I) \). The cumulants \( E_{n, \mu}^{\kappa} \) such that \( g(I) + (n - 1)d \leq 0 \) are called relevant. The remaining cumulants are called irrelevant.
Remark 12.7. If \( i(I) = 0 \), then either \( E^{\star}_{\kappa,\mu} = 0 \) or \( n(I) = 2 \), \( m(I) = 0 \), \( a(I) = 0 \), \( s(I) = 0 \), \( r(I) = 0 \). In the latter case \( E^{\star}_{\kappa,\mu} \) is relevant and coincides with the covariance of the white noise. For \( i(I) > 0 \) the only relevant cumulants are the expectations of the relevant force coefficients.

Remark 12.8. For \( \varepsilon > 0 \) and any list of indices \( I \) such that \( m(I) \leq 3i(I) \) it holds that \( q_{\varepsilon}(I) < q(I) \). Moreover, \( q_{\varepsilon}(I) + (n(I) - 1)d > 0 \) for \( \varepsilon \in (0, \varepsilon_0) \) and lists of indices \( I \) such that \( q(I) + (n(I) - 1)d > 0 \), where \( \varepsilon_0 \) was introduced in Remark 6.4. Recall that \( \varepsilon \in (0, \varepsilon_0) \) is fixed (see Remark 6.4).

13 Probabilistic analysis

Theorem 13.1. Fix \( n \in 2\mathbb{N}_+ \), such that \( d/n < \varepsilon \) and \( i, m \in \mathbb{N}_0 \), \( a \in \mathbb{N}^a \) such that \( q(i, m) + |a| \leq 0 \). For \( s \in \{0, 1\} \), \( r \in \{0, 1, 2\} \) we define the list of indices \( I = I(s, r) = ((i, m, a, s, r), ..., (i, m, a, s, r)), n(I) = n \). Assume that there exists \( g \in \mathbb{N}_0 \) such that for \( s \in \{0, 1\}, r \in \{0, 1, 2\} \) the bound

\[
\int_{\mathbb{M}^{nm}} |(K^{\star g}_{\mu} \diamond (n+nm) * E^{\star}_{\kappa,\mu})(x_1, ..., x_n; y_1, ..., y_n)| dy_1 ... dy_n dx_2 ... dx_n \lesssim [\kappa]^{(\varepsilon - \sigma)m} |\mu|^{q_{\varepsilon}(1) - \sigma s(1) + (n-1)d} \quad (13.1)
\]

holds uniformly in \( x_1 \in \mathbb{M}, \kappa, \mu \in (0, 1/2) \). Then there exists \( g \in \mathbb{N}_0 \) and a random variable \( R > 0 \) such that \( E R^R < \infty \) and for \( r \in \{0, 1\} \) the bound

\[
\|K^{\star g}_{\mu} \diamond f_{i,m,a} \|_V \leq R [\kappa]^{(\varepsilon - \sigma)m} |\mu|^{q_{\varepsilon}(i+nm)+|a|}.
\]

holds for all \( \kappa, \mu \in (0, 1/2) \).

Remark 13.2. The function

\[
\int_{\mathbb{M}^{nm}} |(K^{\star g}_{\mu} \diamond (n+nm) * E^{\star}_{\kappa,\mu})(x_1, ..., x_n; y_1, ..., y_n)| dy_1 ... dy_n
\]

is \( 2\pi \) periodic in all variables and the first integral on the LHS of the bound (13.1) is an integral over one period.

Remark 13.3. The assumption of the above theorem is verified in Theorem 16.1.

Proof. By Remark 4.7 and Lemma 4.8 (B), which says that \( \|TK^{\mu}_d\|_V \lesssim |\mu|^{-d} \), the assumption of the theorem implies that for \( f = g + d \) the bound

\[
\int_{\mathbb{M}^{nm}} |(K^{\star g}_{\mu} \diamond (n+nm) * E^{\star}_{\kappa,\mu})(x_1, ..., x_n; y_1, ..., y_n)| dy_1 ... dy_n \lesssim [\kappa]^{(\varepsilon - \sigma)m} |\mu|^{q_{\varepsilon}(1) - \sigma s(1)} \quad (13.2)
\]
holds uniformly in $x_1, \ldots, x_n \in \mathbb{M}$, $\kappa, \mu \in (0, 1/2]$. Using the above bound with $x_1 = \ldots = x_n = x$, the relation between the expectation of a product of random variables and their joint cumulants given by Eq. (12.1), the equalities $\varrho_x(\mathbf{1}) = n(\varrho_x(i, m) + |a|)$, $s(\mathbf{1}) = ns$, $r(\mathbf{1}) = nr$ and

$$\left(K_{\mu}^{\star r} \ast \partial_\kappa^r \partial_{\kappa, \mu}^r f^{i, m, a}_{\kappa, \mu}(x)\right) = \int_{\mathbb{M}^m} (K_{\mu}^{\star r} \otimes (1+r) \ast \partial_\kappa^r \partial_{\kappa, \mu}^r F^{i, m, a}_{\kappa, \mu}(x; y_1, \ldots, y_m)) \, dy_1 \cdots dy_m$$

we obtain for $r \in \{0, 1, 2\}$, $s \in \{0, 1\}$ the bound

$$\sup_{x \in \mathbb{M}} |\mathbb{E}(K_{\mu}^{\star r} \ast \partial_\kappa^r \partial_{\kappa, \mu}^r f^{i, m, a}_{\kappa, \mu}(x))|^n \lesssim [\kappa]^{n(\varepsilon - \sigma)r} [\mu]^{n(\varrho_x(i, m) + |a| - \sigma s)}$$

uniform in $x \in \mathbb{M}$, $\kappa, \mu \in (0, 1/2]$. Lemma 13.5 and Lemma 4.8 (C) imply

$$\mathbb{E}\|\partial_\kappa^s \partial_{\kappa, \mu}^s (K_{\mu}^{\star h} \ast f^{i, m, a}_{\kappa, \mu})\|_V^n \lesssim [\kappa]^{n(\varepsilon - \sigma)r} [\mu]^{n(\varrho_x(i, m) + |a| - \sigma s - \varepsilon)}$$

with $h = f + d$. The theorem follows now from Lemma 13.6 applied with

$$\zeta_{2\epsilon, 2\mu} = K_{\mu}^{\star h} \ast \partial_{\kappa, \mu}^r f^{i, m, a}_{\kappa, \mu}, \quad r \in \{0, 1\},$$

and $\omega = (\varepsilon - \sigma)r$ and $\rho = \varrho_x(i, m) + |a| - 2\varepsilon \geq \varrho_x(i, m) + |a|$, where the last inequality holds for $m \leq 3i$ (otherwise $f^{i, m, a}_{\kappa, \mu}$ vanishes identically).

Remark 13.4. The advantage of the bound of the form (13.1) over (13.2) is that the former bound contains the extra factor $[\mu]^{(n-1)d}$ on the RHS, and consequently can be more easily proved by induction using the flow equation for cumulants stated in Sec. 15 and the equality $E_{\kappa, \mu}^{\star r} = E_{\kappa, \mu}^0 + \int_0^1 \partial_\eta E_{\kappa, \eta}^\star \, d\eta$.

Lemma 13.5. Let $n \in \mathbb{2N}_+$. There exists a constant $C > 0$ such that for all random fields $\zeta \in L^n(\mathbb{T})$ and $\mu \in (0, 1/2]$ it holds that

$$\mathbb{E}\|K_{\mu}^{\star d} \ast \zeta\|_{L^\infty(\mathbb{T})}^n \leq C [\mu]^{-d} \mathbb{E}\|\zeta\|_{L^n(\mathbb{T})}^n.$$  

Proof. Note that $K_{\mu}^{\star d} \ast \zeta = TK_{\mu}^{\star d} \ast \zeta$, where $\ast$ is the convolution in $\mathbb{T}$ and $TK_{\mu}^{\star d}$ is the periodization of $K_{\mu}^{\star d}$ (see Def. 4.6). Using the Young inequality for convolution we obtain

$$\mathbb{E}\|K_{\mu}^{\star d} \ast \zeta\|_{L^\infty(\mathbb{T})}^n \leq \|TK_{\mu}^{\star d}\|_{L^n(\mathbb{T})} \mathbb{E}\|\zeta\|_{L^n(\mathbb{T})}^n.$$  

The lemma follows now from Lemma 4.8 (B).
Lemma 13.6. Fix $n \in \mathbb{N}_+$. There exists a universal constant $c > 0$ such that if
\[
\mathbb{E}\left| \partial_{\nu}^{\alpha} \partial_{\mu}^{\beta} \zeta_{n, \mu} \right|^n_{L^\infty(\mathbb{T})} \leq C \left[ |\nu|^{n(\omega - \sigma \epsilon u + \epsilon) + |\mu| n(\rho - \sigma s + \epsilon s)} \right]
\]
for some differentiable random function $\zeta : (0, 1)^2 \to L^\infty(\mathbb{T})$, some $C > 0$, \(\omega, \rho \leq 0\) and $\kappa, \mu \in (0, 1)$, then
\[
\mathbb{E}\left( \sup_{\kappa, \mu \in (0, 1)} |\kappa|^{-\omega} |\mu|^{-\rho} \left\| \zeta_{n, \mu} \right\|^n_{L^\infty(\mathbb{T})} \right) \leq c C.
\]

Proof. For all $\kappa, \mu \in (0, 1)$ we have
\[
|\kappa|^{-\omega} |\mu|^{-\rho} \left\| \zeta_{n, \mu} \right\|_{L^\infty(\mathbb{T})} \leq \left\| \zeta_{1, \mu} \right\|_{L^\infty(\mathbb{T})}
\]
+ \(\int_{\mu}^{1} |\nu|^{-\rho} \left\| \partial_{\nu} \zeta_{1, \nu} \right\|_{L^\infty(\mathbb{T})} d\nu + \int_{\kappa}^{1} \int_{\mu}^{1} |\nu|^{-\omega} |\mu|^{-\rho} \left\| \partial_{\nu} \partial_{\mu} \zeta_{\nu, \mu} \right\|_{L^\infty(\mathbb{T})} d\nu d\eta \)

By the Minkowski inequality we get the bound
\[
\mathbb{E}\left( \sup_{\kappa, \mu \in (0, 1)} |\kappa|^{-\omega} |\mu|^{-\rho} \left\| \zeta_{n, \mu} \right\|^n_{L^\infty(\mathbb{T})} \right)^{\frac{1}{n}} \leq \mathbb{E}\left( \left\| \zeta_{1, \mu} \right\|^n_{L^\infty(\mathbb{T})} \right)^{\frac{1}{n}}
\]
+ \(\int_{0}^{1} |\mu|^{-\rho} \mathbb{E}\left( \left\| \partial_{\mu} \zeta_{1, \mu} \right\|^n_{L^\infty(\mathbb{T})} \right)^{\frac{1}{n}} d\mu + \int_{0}^{1} \int_{0}^{1} |\kappa|^{-\omega} |\mu|^{-\rho} \mathbb{E}\left( \left\| \partial_{\nu} \partial_{\mu} \zeta_{\nu, \mu} \right\|^n_{L^\infty(\mathbb{T})} \right)^{\frac{1}{n}} d\kappa d\mu, \)

which implies the statement. \[\square\]

14 Function spaces for cumulants

Definition 14.1. For $n \in \mathbb{N}_+$ we say that $V \in C(\mathbb{M}^n)$ is translationally invariant iff $V(x_1, \ldots, x_n) = V(x_1 + x, \ldots, x_n + x)$ for all $x_1, \ldots, x_n, x \in \mathbb{M}$.

Definition 14.2. Let $n \in \mathbb{N}_+$, $m = (m_1, \ldots, m_n) \in \mathbb{N}_0^n$ and $m = m_1 + \ldots + m_n$. The vector space $V^m_t$ consists of translationally invariant $V \in C(\mathbb{M}^n \times \mathbb{M}^m)$ such that
\[
\|V\|^m := \sup_{x_1 \in \mathbb{M}} \int_{\mathbb{T}^{n-1} \times \mathbb{M}^m} |V(x_1, \ldots, x_n; y_1, \ldots, y_m)| dx_2 \ldots dx_n dy_1 \ldots dy_m
\]
is finite and the function $U^m V \in C(\mathbb{M}^n \times \mathbb{M}^m)$ defined by
\[
U^m V(x_1, \ldots, x_n; y_1, \ldots, y_m) := V(x_1, \ldots, x_n; y_1 + x_1, \ldots, y_n + x_n)
\]

is $2\pi$ periodic in variables $x_1, \ldots, x_n$ for every
\[
(y_1, \ldots, y_m) = (y_1, \ldots, y_n) \in M^{m_1} \times \ldots \times M^{m_n} = M^m,
\]
where for arbitrary $x \in M$ and $y = (y_1, \ldots, y_m) \in M^m$, $m \in \mathbb{N}_0$, we use the following notation $y + x := (y_1 + x, \ldots, y_n + x) \in M^m$. For $g \in \mathbb{N}_0$ the space $D_t^g$ consists of distributions $V \in \mathcal{S}'(M^n \times M^m)$ such that $K^*_\mu \otimes (n+m) \ast V \in \mathcal{V}^m$. The space $D_t^m$ is the union of the spaces $D_t^g$, $g \in \mathbb{N}_0$.

**Remark 14.3.** For $n = 1$ and $m = m$ the norm $\| \cdot \|\mathcal{V}^m$ coincides with the norm $\| \cdot \|\mathcal{V}^m$ introduced in Def. 5.2.

**Remark 14.4.** Using translational invariance of $V \in \mathcal{V}^m_1$ one shows that
\[
\|V\|\mathcal{V}^m = \frac{1}{(2\pi)^d} \int_{T^n \times M^m} |V(x_1, \ldots, x_n; y_1, \ldots, y_m)| \, dx_1 \ldots dx_n dy_1 \ldots dy_m.
\]

**Remark 14.5.** For $V \in \mathcal{V}^m$ and $K \in \mathcal{K}^{n+m}$ it holds $\|K \ast V\|\mathcal{V}^m \leq \|K\|\mathcal{K}^{n+m} \|V\|\mathcal{V}^m$.

**Definition 14.6.** For $V \in D_t^m$ and a permutation $\pi \in \mathcal{P}_{m}$ we define $Y_{\pi} V \in D_t^m$ by
\[
\langle Y_{\pi} V, \bigotimes_{q=1}^n \psi_q \otimes \bigotimes_{q=1}^m \varphi_q \rangle := \langle V, \bigotimes_{q=1}^n \psi_q \otimes \bigotimes_{q=1}^m \varphi_q \rangle,
\]
where $\psi_1, \ldots, \psi_n, \varphi_1, \ldots, \varphi_m \in \mathcal{S}(M)$. For $V \in D_t^m$ and $\omega \in \mathcal{P}_n$ we define $Y^{\omega} V \in D_t^{\omega(m)}$, where $\omega(m) := (m_{\omega(1)}, \ldots, m_{\omega(n)})$, by
\[
\langle Y^{\omega} V, \bigotimes_{q=1}^n \psi_q \otimes \bigotimes_{q=1}^m \varphi_q \rangle := \langle V, \bigotimes_{q=1}^n \psi_{\omega(q)} \otimes \bigotimes_{q=1}^m \varphi_{\omega(q)} \rangle,
\]
where $\psi_q \in \mathcal{S}(M)$, $\varphi_q \in \mathcal{S}(M^{m_q})$ for $q \in \{1, \ldots, n\}$.

**Remark 14.7.** The maps $Y_{\pi} : \mathcal{V}^m_1 \to \mathcal{V}^m_1$, $Y^{\omega} : \mathcal{V}^m_1 \to \mathcal{V}^{\omega(m)}_1$ are bounded with norm one.

**Definition 14.8.** Fix $n \in \mathbb{N}_+$, $\hat{n} \in \{1, \ldots, n\}$, $m_1, \ldots, m_{n+1} \in \mathbb{N}_0$. Let
\[
m = (m_1 + m_{n+1}, m_2, \ldots, m_n) \in \mathbb{N}_0^n, \quad \tilde{m} = (1 + m_1, m_2, \ldots, m_{n+1}) \in \mathbb{N}_0^{n+1},
\]
\[
\hat{m} = (1 + m_1, m_2, \ldots, m_n) \in \mathbb{N}_0^n, \quad \hat{\tilde{m}} = (m_{\hat{n}+1}, \ldots, m_{n+1}) \in \mathbb{N}_0^{n-\hat{n}+1}.
\]
The bilinear map $A : \mathcal{S}(M) \times \mathcal{V}^m_{\hat{m}} \to \mathcal{V}^m_{\hat{m}}$ is defined by
\[
A(G, V)(x_1, \ldots, x_n; y_1, y_{n+1}, y_2, \ldots, y_n)
:= \int_{M^2} V(x_1, \ldots, x_{n+1}; y, y_1, \ldots, y_{n+1}) G(y - x_{n+1}) \, dy \, dx_{n+1}.
\]

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The trilinear map $B : \mathcal{S}(M) \times \mathcal{V}^m \times \mathcal{V}^m \to \mathcal{V}^m$ is defined by

$$B(G, W, U)(x_1, \ldots, x_n; y_1, y_{n+1}, y_2, \ldots, y_n) := \int_{M^2} W(x_1, \ldots, x_n; y, y_1, \ldots, y_n) \times G(y - x_{n+1}) U(x_{n+1}, x_{n+1}, \ldots, x_n; y_{n+1}, y_{n+1}, \ldots, y_n) \, dy \, dx_{n+1}.$$ 

In the above equations $y_j \in M^{m_j}$, $j \in \{1, \ldots, n + 1\}$.

**Remark 14.9.** The map $B$ is a generalization of the map $B$ introduced in Def. 5.7. The maps $A$ and $B$ appear on the RHS of the flow equation for the cumulants of the effective force coefficients introduced in Sec. 15.

**Lemma 14.10.** The maps $A : \mathcal{S}(M) \times \mathcal{V}^m \to \mathcal{V}^m$, $B : \mathcal{S}(M) \times \mathcal{V}^m \times \mathcal{V}^m \to \mathcal{V}^m$ are well defined. It holds that

$$\left\| A(G, V) \right\|_{\mathcal{V}^m} \leq \left\| T(G) \right\|_Y \left\| V \right\|_{\mathcal{V}^m},$$

$$\left\| B(G, W, U) \right\|_{\mathcal{V}^m} \leq \left\| G \right\|_K \left\| W \right\|_{\mathcal{V}^m} \left\| U \right\|_{\mathcal{V}^m},$$

where $T(G)$ is the periodization of $|G|$ introduced in Def. 4.6, $\left\| G \right\|_K = \left\| G \right\|_{L^1(M)}$ and $\left\| T(G) \right\|_Y = \left\| T(G) \right\|_{L^\infty(T)}$.

**Proof.** The functions $U^m A(G, V)$, $U^m B(G, W, U)$ are translationally invariant and $2\pi$ periodic since

$$U^m A(G, V)(x_1, \ldots, x_n; y_1, y_{n+1}, y_2, \ldots, y_n) = \int_{M^2} G(y - x_{n+1})$$

$$\times U^m V(x_1, \ldots, x_n, x_{n+1} + x_1; y, y_1, \ldots, y_n, y_{n+1} - x_{n+1}) \, dy \, dx_{n+1},$$

$$U^m B(G, W, U)(x_1, \ldots, x_n; y_1, y_{n+1}, y_2, \ldots, y_n)$$

$$= \int_{M^2} U^m W(x_1, \ldots, x_n; y_1, \ldots, y_n) G(y - x_{n+1})$$

$$\times U^m U(x_{n+1} + x_1, x_{n+1}, \ldots, x_n; y_{n+1} - x_{n+1}, y_{n+1}, \ldots, y_n) \, dy \, dx_{n+1}.$$

To prove the first bound note that

$$\left\| A(G, V) \right\|_{\mathcal{V}^m} \leq \sup_{x_1 \in T} \int_{M^2 \times \mathbb{T}^{n-1} \times M^m} |G(y + x_1 - x_{n+1})|$$

$$\times \left\| U^m V(x_1, \ldots, x_{n+1}; y, y_1, \ldots, y_{n+1}) \right\| \, dy \, dx_{n+1} \, dz_2 \ldots dz_n \, dy_1 \ldots dy_{n+1}$$
Using periodicity of $U^m V$ we arrive at

$$
\|A(G, V)\|_{\mathcal{Y}^m} \leq \sup_{x_1 \in \mathbb{T}} \int_{\mathbb{T}^n \times \mathbb{T}^n} (T|G|)(y + x_1 - x_{n+1})
\times |U^m V(x_1, \ldots, x_{n+1}; y, y_1, \ldots, y_n)| dy \, dx_{n+1} \, dx_2 \ldots dx_n \, dy_1 \ldots dy_{n+1}
$$

This implies $\|A(G, V)\|_{\mathcal{Y}^m} \leq \|T|G|\|_{\mathcal{Y}} \|U^m V\|_{\mathcal{Y}^m} = \|T|G|\|_{\mathcal{Y}} \|V\|_{\mathcal{Y}^m}$. It holds that

$$
\|B(G, W, U)\|_{\mathcal{Y}^m} \leq \sup_{x_1 \in \mathbb{T}} \int_{\mathbb{T}^{n-1} \times \mathbb{T}^{m+2}} |W(x_1, \ldots, x_{n}; y, y_1, \ldots, y_n)| |G(y - z)|
\times |U(z, x_{n+1}, \ldots, x_{n}; y_{n+1}, y_{n+1}, \ldots, y_n)| dx_2 \ldots dx_n \, dz \, dy_1 \ldots dy_{n+1}
\leq \sup_{x_1 \in \mathbb{T}} \int_{\mathbb{T}^{n-1} \times \mathbb{T}^{m+n}} |W(x_1, \ldots, x_{n}; y, y_1, \ldots, y_n)| dx_2 \ldots dx_n \, dy_1 \ldots dy_{n+1}
\times \sup_{y \in \mathbb{M}} \int_{\mathbb{T}} |G(y - z)| \, dz \|U\|_{\mathcal{Y}^m},
$$

where $m := 1 + m_1 + \ldots + m_\tilde{n}$. The second of the stated bounds follows from the above estimate.

**Remark 14.11.** The fact that $P^\varepsilon_\mu K_{\mu}^\varepsilon = \delta_\mathbb{M}$ implies that for all $\mu > 0$ it holds that

$$
K_{\mu}^{\varepsilon, \otimes (n+m)} * A(G, V) = A(P^\varepsilon_\mu G, K_{\mu}^{\varepsilon, \otimes (n+m+2)} * V),
$$

$$
K_{\mu}^{\varepsilon, \otimes (n+m)} * B(G, W, U) = B(P^\varepsilon_\mu G, K_{\mu}^{\varepsilon, \otimes (n+\tilde{n}+1)} * W, K_{\mu}^{\varepsilon, \otimes (n-\tilde{n}+\tilde{n}+1)} * U),
$$

where $m = m_1 + \ldots + m_{n+1}$, $\tilde{n} = m_1 + \ldots + m_\tilde{n}$ and $m = m_{\tilde{n}+1} + \ldots + m_{n+1}$. This allows to define $A(G, V) \in \mathcal{D}_1^m$ and $B(G, W, U) \in \mathcal{D}_1^m$ for all $G \in \mathcal{S}(\mathbb{M})$, $V \in \mathcal{D}_1^m$, $W \in \mathcal{D}_1^m$, $U \in \mathcal{D}_1^m$. For future reference note that by Lemma 4.8 (B)

$$
\|T|P^\varepsilon_\mu G|\|_{\mathcal{Y}} \lesssim \|TK_{\mu}^{\varepsilon, d}|\|_{\mathcal{Y}} \|P^\varepsilon_\mu G|\|_{\mathcal{K}} \lesssim |\mu|^{-d} \|P^\varepsilon_\mu G|\|_{\mathcal{K}}
$$

uniformly in $\mu \in (0, 1]$ since $K_{\mu}^{\varepsilon, d} = |K_{\mu}^{\varepsilon, d}|$.

**15 Flow equation for cumulants**

**Lemma 15.1.** Let $n \in \mathbb{N}_+$, $i_1 \in \mathbb{N}_0$, $m_1, \ldots, m_n \in \mathbb{N}_0$, $a_1 \in \mathbb{N}^{m_1}$, $r_1 \in \{0, 1, 2\}$ and $I \equiv \{2, \ldots, n\}$. For any random distributions $\zeta_q \in \mathcal{D}_t^{m_q}$, $q \in I$, the cumulant

$$
\langle \partial_{\mu} \partial_{\kappa} F^{i_1, m_1, a_1, (\zeta_q)_{q \in I}} \rangle \in \mathcal{D}_t^{(m_1, \ldots, m_n)}
$$
is a linear combination of the expressions

\[
\sum_{I_1, I_2 \subseteq I} Y_{\pi} B \left( \chi^{\nu, \mu} \partial_{\mu} G_{\kappa, \nu}, \langle \partial_{\kappa}^{I_1, 1+k, b}, (\zeta_q)_{q \in I_1} \rangle, \langle \partial_{\kappa}^{I_2, 1-j, m_1-k, d}, (\zeta_q)_{q \in I_2} \rangle \right) + Y_{\pi} A \left( \chi^{\nu, \mu} \partial_{\mu} G_{\kappa, \nu}, \langle \partial_{\kappa}^{I_1, 1+k, b}, (\zeta_q)_{q \in I_1} \rangle, \langle \partial_{\kappa}^{I_2, 1-j, m_1-k, d} \rangle \right),
\]

where \( \pi \in \mathcal{P}_{m_1}, j \in \{1, \ldots, i_1\}, k \in \{0, \ldots, m_1\}, u, v, w \in \mathbb{N}_0, u + v + w = r_1 \) and \( b, c, d \) are multi-indices satisfying the condition \( b + c + d = \pi(a_1) \) whose meaning is explained below Eq. (7.1). The coefficients of the above linear combination depend only on \( m_1, k, a, b, c, d \) and \( u, v, w \). We used the notation introduced in Def. 12.4.

**Proof.** The statement follows immediately from Eqs. (7.2) and (12.2). \( \square \)

**Lemma 15.2.** Let \( J = (J_1, \ldots, J_n) = ((i_1, m_1, a_1, s_1, r_1), \ldots, (i_n, m_n, a_n, s_n, r_n)) \) be a list of indices such that \( s_l = 1 \) for some \( l \in \{1, \ldots, n\} \).

(A) The distribution \( E^2_{\kappa, \mu} \) can be expressed as a linear combination of distributions of the form

\[
Y^\omega Y_{\pi} A \left( \chi^{\nu, \mu} \partial_{\mu} G_{\kappa, \nu}, E^{K}_{\kappa, \mu} \right) \quad \text{or} \quad Y^\omega Y_{\pi} B \left( \chi^{\nu, \mu} \partial_{\mu} G_{\kappa, \nu}, E^{L}_{\kappa, \mu}, E^{M}_{\kappa, \mu} \right),
\]

where \( u \in \{0, \ldots, r_1\}, c \in \mathcal{M} \) is some multi-index, \( K, L, M \) are some lists of indices and \( \omega \in \mathcal{P}_n, \pi \in \mathcal{P}_{m_1} \) are some permutations.

(B) The lists of indices \( K, L, M \) satisfy the conditions

\[
\begin{align*}
n(K) &= n(J) + 1, \\
i(K) &= i(J), \\
m(K) &= m(J) + 1, \\
a(K) + |c| &= a(J), \quad \text{or} \quad a(L) + a(M) + |c| = a(J), \\
s(K) &= s(J) - 1, \\
r(K) &= r(J) - u, \\
\phi_e(J) - \sigma &= \Phi_e(K) - d - 2\varepsilon, \\
\phi_e(J) - \sigma &= \Phi_e(L) + \Phi_e(M) - 2\varepsilon.
\end{align*}
\]

(C) Fix some \( g \in \mathbb{N}_0 \). Suppose that the bound

\[
\| K^g \cdot \otimes (n(1) + m(1)) \cdot E^{1}_{\kappa, \mu} \|_{\mathcal{V}(1)} \lesssim \| \mathcal{K} |(\varepsilon - \sigma) r(1) [\mu] \phi_e(1) - \sigma s(1) + (n(1) - 1)d \|_{\mathcal{V}(1)}
\]

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holds uniformly in $\kappa,\mu \in (0,1/2]$ for all lists of indices $I \in \{K,L,M\}$, where $K,L,M$ are arbitrary lists of indices satisfying the conditions specified in Part (B) given the list of indices $J$. Then the above bound holds uniformly in $\kappa,\mu \in (0,1/2]$ for $I = J$.

**Proof.** Without loss of generality we can assume that $l = 1$. Part (A) of the lemma follows immediately from Lemma 15.1 applied with

$$\zeta_q \equiv \partial^q_\kappa \partial^q_\mu F_{\kappa,\mu}^{i,q,m,q,a_q}, \quad q \in \{2,\ldots,n\}.$$  

The multi-index $c \in \mathcal{M}$, $u \in \{0,\ldots,r_1\}$ and the permutation $\pi \in \mathcal{P}_{m_1}$ in the statement coincide with the respective objects in Eq. (15.1). The permutation $\omega \in \mathcal{P}_n$ is trivial because of the assumption $l = 1$. Moreover, it holds that

$$K = ((j,k+1,b,0,v), J_2,\ldots,J_n, (i_1-j,m_1-k,d,0,w)),$$

$$L = (j,k+1,b,0,v) \cup (J_q)_{q \in I_1}, \quad M = (i_1-j,m_1-k,d,0,w) \cup (J_q)_{q \in I_2},$$

where $\cup$ denotes the concatenation of lists, $I_1 \cup I_2 = I = \{2,\ldots,n\}$, $I_1 \cap I_2 = \emptyset$ and $j \in \{1,\ldots,i_1\}$, $k \in \{0,\ldots,m_1\}$, $b \in \mathcal{M}^{1+k}$, $d \in \mathcal{M}^{m-k}$, $v,w \in \{0,\ldots,r_1\}$ coincide with the respective objects in Eq. (15.1). This implies that the lists $K, L, M$ satisfy the conditions given in Part (B). The last condition follows from Def. 12.6 and $\dim(\xi) + \dim(\Phi) = d - \sigma$. To prove Part (C) we use Parts (A), (B), Lemma 14.10, Remark 14.11 and Lemma 6.5 applied with $r \in \{0,1,2\}$. □

16 Uniform bounds for cumulants

**Theorem 16.1.** There exists a unique choice of the counterterms $c^{|i|}_\kappa$ in Eq. (1.2) such that: $\mathbb{E}f^{1,3,0}_n = 1$, $\mathbb{E}f^{1,0}_n = f_0^{1,i}$, $i \in \{1,\ldots,i_2\}$, and $\mathbb{E}f^{i,a}_n = 0$ for all other $i \in \mathbb{N}_+$, $m \in \mathbb{N}_0$, $a \in \mathbb{R}^m$ such that $g(i,m) + |a| \leq 0$. Fix arbitrary $f_0^{1,i} \in \mathbb{R}$, $i \in \{1,\ldots,i_2\}$. For all list of indices $I$ there exists $g \in \mathbb{N}_0$ such that the bound

$$\|K^g_m \otimes (n+m) \ast E_{n,m}^I \|_{\mathcal{V}_m} \lesssim |\kappa|^{(\sigma-\sigma)(1)} |\mu|^{(1)-\sigma(1)+(n-1)d}$$  

(16.1)

holds uniformly in $\kappa,\mu \in (0,1/2]$, where $n = n(I)$, $m = m(I)$, $m = m(I)$. Moreover, the following condition is satisfied

$$E_{n,m}^I (x_1,\ldots,x_n; y_1,\ldots,y_n) = (-1)^{a(I)} E_{n,m}^I (-x_1,\ldots,-x_n; -y_1,\ldots,-y_n),$$

where $x_j \in \mathcal{M}$, $y_j \in \mathcal{M}^{m_j}$ for $j \in \{1,\ldots,n\}$ and $E_{n,m}^I = 0$ unless $m+n \in 2\mathbb{N}_0$. 32
Remark 16.2. By stationarity $\mathbb{E} f_{\kappa}^{i,m,a}(x)$ is a constant. Since $\partial_{\mu} F_{\kappa,\mu}^{1,3} = 0$ it holds that $\mathbb{E} f_{\kappa,1/2}^{1,3,0} = f_{\kappa,0}^{1,3,0} = 1$. There is no distinguished value of $\mathbb{E} f_{\kappa,1/2}^{1,1,0} = \delta[i]$, $i \in \{1, \ldots, i_t\}$, ultimately, because there is no distinguished function $\chi$ in Def. 2.1 of the cutoff propagator $G_{\kappa}$. The vanishing of $\mathbb{E} f_{\kappa,1/2}^{i,m,a}$ for all other $i \in \mathbb{N}_+$, $m \in \mathbb{N}_0$, $a \in \mathbb{R}^m$ such that $g(i, m) + |a| \leq 0$ is enforced by the properties of the cumulants given in the last sentence of the theorem (here we use the assumption $d \in \{1, \ldots, 6\}$). Observe that these properties are consequences of the following symmetries of Eq. (1.1): $\Phi(x) \mapsto -\Phi(x)$, $\xi(x) \mapsto -\xi(x)$ and $\Phi(x) \mapsto \Phi(-x)$, $\xi(x) \mapsto \xi(-x)$, which in particular preserve the law of $\xi$. The counterterms $c_{\kappa}^{[i]}$ are related to the renormalization parameters $f^{[i]}$ by the formula

$$
c_{\kappa}^{[i]} := f_{\kappa}^{i,1,0} = f_{\kappa,0}^{i,1,0} = f^{[i]} - \int_0^{1/2} \mathbb{E} \partial_{\mu} f_{\kappa,\mu}^{i,1,0} d\mu, \quad i \in \{1, \ldots, i_t\}.
$$

The constants $f^{[i]} \in \mathbb{R}$, $i \in \{1, \ldots, i_t\}$ parametrize the class of solutions of Eq. (1.1) constructed in the paper (generically this is an over-parametrization).

**Proof.** We first note that the theorem is trivially true for all list of indices $I$ such that $m(I) > 3\iota(I)$ since then $E_{\kappa,\mu}^I = 0$. The rest of the proof is by induction.

The base case: Consider a list of indices $I$ such that $i(I) = 0$. In this case the cumulants $E_{\kappa,\mu}^I$ coincide with the cumulants of the white noise $\xi$. The only non-vanishing cumulant is the covariance corresponding to $n(I) = 2$, $m(I) = (0, 0)$, $m(\iota) = 0$, $a(\iota) = 0$, $r(\iota) = 0$ and $s(I) = 0$. It holds that

$$\|\{K_{\mu}^{\kappa} \ast \xi, K_{\mu}^{\kappa} \ast \xi\}\|_{\mathcal{V}_m} \leq \sup_{x_1 \in \mathcal{T}} \int_{\mathcal{T}} |\mathbb{E}(\xi(x_1)\xi(dx_2))| = 1.$$

This finishes the proof of the base case.

Induction step: Fix $i \in \mathbb{N}_+$ and $m \in \mathbb{N}_0$. Assume that the theorem is true for all lists of indices $I$ such that either $i(I) < i$, or $i(I) = i$ and $m(I) > m$. We shall prove the theorem for all $I$ such that $i(I) = i$ and $m(I) = m$.

Consider the case $s(I) > 0$. In this case by Lemma 15.2 (A) and (B) the cumulants $E_{\kappa,\mu}^I$ can be expressed in terms of the cumulants for which the statement of the theorem has already been established. As a result, the bound (16.1) with $s(I) > 0$ follows from the inductive assumption and Lemma 15.2 (C).

Now consider $I = ((i_1, m_1, a_1, 0, r_1), \ldots, (i_n, m_n, a_n, 0, r_n))$, $s(I) = 0$. It follows from Def. 12.5 of the cumulants $E_{\kappa,\mu}^I$ that

$$E_{\kappa,\mu}^I = E_{\kappa,0}^I + \sum_{q=1}^{n} \int_0^\mu E_{\kappa,\eta}^{i_q} d\eta, \quad E_{\kappa,\mu}^{I,1/2} = E_{\kappa,1/2}^I - \sum_{q=1}^{n} \int_0^{1/2} E_{\kappa,\eta}^{i_q} d\eta, \quad (16.2)$$
where $I_q = ((i_1, m_1, a_1, 0, r_1), \ldots, (i_q, m_q, a_q, 1, r_q), \ldots, (i_n, m_n, a_n, 0, r_n))$. Note that $s(I_q) = 1$, hence the bound \((16.1)\) has already been established for $E^I_{\kappa, \eta}$. We will use the first of Eqs. \((16.2)\) to bound the irrelevant cumulants $E^I_{\kappa, \mu}$, i.e. those with $I$ such that $\varrho(I) + (n(I) - 1)d > 0$. The second equality will be used to bound certain contributions to the relevant cumulants $E^I_{\kappa, \eta}$, i.e. those with $I$ such that $\varrho(I) + (n(I) - 1)d \leq 0$.

First, let us analyze the irrelevant contributions. If $n(I) > 1$, then $E^I_{\kappa, 0}$ is a joint cumulant of a list of at least two random distributions. Since $i(I) = i > 0$ one of the elements of this list is a deterministic distribution of the form $\partial^i F^{i,m,a}_{\kappa,0}$. Hence, the cumulant vanishes. If $n(I) = 1$ and $\varrho(I) > 0$, then $E^I_{\kappa, 0}$ coincides with $\partial^i F^{i,m,a}_{\kappa,0} = \partial^i F^{i,m,a}_{\kappa,0}$ for some $i, m, a$ such that $\varrho(i, m) + |a| > 0$ and consequently vanishes. To prove the bound for $E^I_{\kappa, \mu}$ we use the first of Eqs. \((16.2)\). As we argued above, the first term on the RHS of this equation vanishes. The claim of the theorem is a consequence of the bound

$$\|K^\ast g \otimes (n+m) * E^I_{\kappa, \mu}\|_{\mathcal{Y}_m} \leq \sum_{q=1}^n \int_0^\mu \|K^\ast g \otimes (n+m) * E^I_{\kappa, \eta}\|_{\mathcal{Y}_m} \, d\eta.$$  

We used the fact that $\|K^\ast g \otimes (n+m) * E^I_{\kappa, \eta}\|_{\mathcal{Y}_m} \leq \|K^\ast g \otimes (n+m) * E^I_{\kappa, \mu}\|_{\mathcal{Y}_m}$ for $\eta \leq \mu$ which follows from Lemma 4.5 and Remark 14.5.

Next, let us analyze the relevant contributions. We note that the inequality $\varrho(I) + (n(I) - 1)d \leq 0$ implies $n(I) = 1$. Consequently, $I = (i, m, a, 0, r)$ for some $r \in \{0, 1, 2\}$ and $a \in \mathbb{M}$ such that $\varrho(i, m) + |a| \leq 0$. Hence, $E^I_{\kappa, \mu} = \mathbb{E} \partial^r f^{i,m,a}_{\kappa, \mu}$. We first study

$$\mathbb{E} \partial^r f^{i,m,a}_{\kappa, \mu} = \mathbb{I} \partial^r f^{i,m,a}_{\kappa, \mu} = \mathbb{I}(K^\ast g \otimes (1+m) * E^I_{\kappa, \mu}) \in \mathbb{R},$$

where the map $\mathbb{I}$ was introduced in Def. 8.4. Note that by the translational invariance $E f^{i,m,a}_{\kappa, \mu}$ is a constant function. The application of the map $\mathbb{I}$ to both sides of the second equation in \((16.2)\) yields

$$\mathbb{E} \partial^r f^{i,m,a}_{\kappa, \mu} = \mathbb{E} \partial^r f^{i,m,a}_{\kappa, \mu, 1/2} - \int_{\mu}^{1/2} \mathbb{I}(E^I_{\kappa, \eta}) \, d\eta,$$

where $E^I_{\kappa, 1} = \mathbb{E} \partial^r f^{i,m,a}_{\kappa, \eta}$. Recalling that $\mathcal{Y}_m = \mathcal{Y}_m$ for $n = 1$ and using Lemma 8.5 we arrive at

$$|\mathbb{E} \partial^r f^{i,m,a}_{\kappa, \mu}| \leq |\mathbb{E} \partial^r f^{i,m,a}_{\kappa, 1/2}| + \int_{\mu}^{1/2} \|K^\ast g \otimes (1+m) * E^I_{\kappa, \eta}\|_{\mathcal{Y}_m} \, d\eta. \quad (16.3)$$
By assumption, $E f_{\kappa,1/2}^{i,m,a}$ is independent of $\kappa$ and $E \partial_\kappa f_{\kappa,1/2}^{i,m,a} = 0$. Hence, using the bound (16.3) and the bound (16.1) applied to $E^{1}_{\kappa,\mu}$ we obtain

$$|E \partial_\kappa f_{\kappa,\mu}^{i,m,a}| = |I(E \partial_\kappa f_{\kappa,\mu}^{i,m,a})| \lesssim 1 + \int_\mu^{1/2} |k|^{\varepsilon-\sigma} r |\eta|^{\varphi_{v}(i,m)+|a|-\sigma} d\eta \lesssim |k|^{\varepsilon-\sigma} r |\mu|^{\varphi_{v}(i,m)+|a|}.$$ (16.4)

If $m = 0$, then $a = 0$ and $E^{1}_{\kappa,\mu} = E \partial_\kappa f_{\kappa,\mu}^{i,0,0} = E \partial_\kappa f_{\kappa,\mu}^{i,0,0}$. Hence, in this case the statement of the theorem follows from the bound (16.4). To prove the case $m > 0$ we first recall that we have already proved the bounds

$$\|K^{\varepsilon}_{\mu} * E \partial_\kappa f_{\kappa,\mu}^{i,m,b}\|_{Y_{m}} \lesssim |k|^{\varepsilon-\sigma} r |\mu|^{\varphi_{v}(i,m)+|b|}$$ (16.5)

for all irrelevant coefficients $f_{\kappa,\mu}^{i,m,b}$ and the bounds

$$|I(E \partial_\kappa f_{\kappa,\mu}^{i,m,b})| \lesssim |k|^{\varepsilon-\sigma} r |\mu|^{\varphi_{v}(i,m)+|b|}$$ (16.6)

for all relevant coefficients $f_{\kappa,\mu}^{i,m,b}$. Let $1 \in N_{+}$ be the smallest positive integer such that $\varphi(i,m) + 1 > 0$ and note that by Theorem 8.8 it holds that

$$E \partial_\kappa f_{\kappa,\mu}^{i,m,a} = X_{1}^{a}(I(E \partial_\kappa f_{\kappa,\mu}^{i,m,b}), E \partial_\kappa f_{\kappa,\mu}^{i,m,b}).$$

The arguments of the map $X_{1}^{a}$ above satisfy the bounds (16.6) and (16.5). This together with Theorem 8.7 applied with $C \lesssim |k|^{\varepsilon-\sigma} r |\mu|^{\varphi_{v}(i,m)}$ implies the statement of the theorem. \qed

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**References**

[1] Y. Bruned, A. Chandra, I. Chevyrev, M. Hairer, “Renormalising SPDEs in regularity structures,” J. Eur. Math. Soc. **23**(3), 869–947 (2021) [arXiv:1711.10239]
[2] Y. Bruned, M. Hairer, L. Zambotti, “Algebraic renormalisation of regularity structures,” Invent. Math. 215(3), 1039–1156 (2019) [arXiv:1610.08468]

[3] A. Chandra, M. Hairer, “An analytic BPHZ theorem for regularity structures,” [arXiv:1612.08138]

[4] P. Duch, “Flow equation approach to singular stochastic PDEs,” [arXiv:2109.11380]

[5] M. Hairer, “A theory of regularity structures,” Invent. Math. 198(2), 269–504 (2014) [arXiv:1303.5113]

[6] A. Kupiainen, “Renormalization group and stochastic PDEs,” Ann. Henri Poincaré 17(3), 497–535 (2016) [arXiv:1410.3094]

[7] A. Kupiainen, M. Marcozzi, “Renormalization of generalized KPZ equation,” J. Stat. Phys. 166, 876–902 (2017) [arXiv:1604.08712]

[8] V. Müller, “Perturbative renormalization by flow equations,” Rev. Math. Phys. 15(05), 491–558 (2003) [arXiv:hep-th/0208211]

[9] G. Peccati, M. Taqqu, “Wiener Chaos: Moments, Cumulants and Diagrams: A survey with computer implementation,” (Springer, 2011)

[10] J. Polchinski, “Renormalization and effective lagrangians,” Nuclear Physics B, 231(2), 269–295, (1984)

[11] K. Wilson, “Renormalization Group and Critical Phenomena. I. Renormalization Group and the Kadanoff Scaling Picture,” Phys. Rev. B, 4 3174–3183, (1971)