Control problem for generalized Boussinesq model

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Abstract. The boundary value and multiplicative control problem for generalized Boussinesq model, in which reaction coefficient has a nonlinear dependence on substances concentration are studied.

1. Introduction. The boundary value problem
The study of control and inverse problems for linear and nonlinear heat and mass transfer models is the subject of a large number of papers. In [1–3] the inverse problems for the linear mass transfer model consisting of a convection-diffusion-reaction equation with boundary conditions are studied. Further we note articles [4–13] which are devoted to study of inverse and extremum problems for nonlinear heat and mass transfer models in the classic Oberbeck-Boussinesq approximation. Articles [14–18] are devoted to the boundary value and extremum problems for the convection-diffusion-reaction equation, in which the reaction coefficient depends nonlinearly on the substance concentration. In [19,20] the similar models of complex heat transfer were considered. In [21] the solvability of boundary value problems for some stationary generalized Boussinesq models is studied. Finally, we mention papers [22,23] in which more difficult rheological and multi-component viscous compressible fluids models are considered.

In bounded domain $\Omega \subset \mathbb{R}^3$ with the boundary $\Gamma$ consisting of two part $\Gamma_D$ and $\Gamma_N$ the boundary value problem is considered

\[-\nu \Delta u + (u \cdot \nabla) u + \nabla p = f + \beta \mathbf{G} \varphi, \text{ div } u = 0 \text{ in } \Omega,\]

\[-\lambda \Delta \varphi + u \cdot \nabla \varphi + k(\varphi, x) \varphi = f \text{ in } \Omega,\]

\[u = 0 \text{ on } \Gamma, \quad \varphi = 0 \text{ on } \Gamma_D, \quad \partial \varphi / \partial n = -\alpha \varphi \text{ on } \Gamma_N.\]

Here $u$ is a velocity vector, function $\varphi$ means polluting the substance’s concentration, $p = P/\rho$, where $P$ is the pressure, $\rho = \text{const}$ is a fluid density, $\nu = \text{const} > 0$ is a constant kinematic viscosity, $\lambda = \text{const} > 0$ is a constant diffusion coefficient, $\beta$ is the coefficient of mass expansion, $\mathbf{G} = - (0, 0, G)$ – is the acceleration of gravity, $f$ and $f$ are volume density of external forces or external sources of the substance, the function $k = k(\varphi, x)$, where $x \in \Omega$, is a reaction coefficient. This problem (1)–(3) for given functions $f, f, \alpha$ and $k$ will be called Problem 1 below.

In the present paper we prove the Problems 1’ global solvability and derive sufficient conditions of local uniqueness of it solution. Further, for Problem 1 we formulate a multiplicative control problem, in which the role of control is played by the coefficient of mass expansion $\beta(x)$,
included in equation (1). The solvability of optimal control problem is also proved for the reaction coefficient \( k(\varphi) \) of general type.

While studying the considered problems, we will use Sobolev functional spaces \( H^s(D) \), \( s \in \mathbb{R} \). Here \( D \) means either the domain \( \Omega \) or some subset \( Q \subset \Omega \), or the boundary \( \Gamma \) or some part \( \Gamma_0 \) of the boundary \( \Gamma \). By \( \| \cdot \|_{s,Q} \) and \( \langle \cdot, \cdot \rangle_{s,Q} \) we will denote the norm, seminorm and the scalar product in \( H^s(Q) \). The norms and scalar products in \( L^2(Q) \) and \( L^2(\Omega) \) will be denote corresponding by \( \| \cdot \| \) and \( \langle \cdot, \cdot \rangle \) of general type.

Let \( k \) correspond to situation, where the substance decomposition rate is proportional to the square (or cube) of substance concentration in subdomain \( Q \subset \Omega \) or \( k = k_0(\mathbf{x}) \) in \( \Omega \setminus Q \), where \( k_0(\mathbf{x}) \in L^{2+}_\gamma(\Omega \setminus Q) \).

From a physical point of view, the coefficient \( k_1 \) corresponds to situation, where the substance decomposition rate is proportional to the square (or cube) of substance concentration in subdomain \( Q \subset \Omega \) and outside the \( Q \), the rate of a chemical reaction depends only on the spatial variable \([16, 18]\).

We will use the following technical lemma (see \([6,7]\)).

**Lemma 1.** If conditions (i), (ii) and \( k_0 \in L^2(\Omega) \), \( p \geq 3/2 \), \( \mathbf{u} \in \mathbf{V} \), \( \mathbf{b} \in L^2(\Omega)^3 \), then there are such positive constants \( \delta_0, \delta_1, \gamma_1, \gamma_2, \gamma_p, \beta_1 \) depending on \( \Omega \) or \( \Omega \) and \( p \) and constant \( \beta_0 \), depending on \( \| \mathbf{b} \|_{\Omega} \), the following relations are correct

\[
\| \mathbf{b} \|_{p, \Omega} \leq \beta_0 \| \mathbf{b} \|_{1, \Omega} \| \mathbf{v} \|_{1, \Omega} \forall \mathbf{v}, \mathbf{v} \in H^1(\Omega)^3, h \in H^1(\Omega),
\]

\[
\| (\mathbf{w} \cdot \nabla) \mathbf{v}, \mathbf{v} \| \leq \gamma_1 \| \mathbf{w} \|_{L^3(\Omega)} \| \mathbf{v} \|_{1, \Omega} \| \mathbf{w} \|_{1, \Omega} \forall \mathbf{w}, \mathbf{v}, \mathbf{z} \in H^1(\Omega)^3, \]

\[
((\mathbf{u} \cdot \nabla) \mathbf{v}, \mathbf{v}) = 0 \forall \mathbf{v}, \mathbf{w} \in H^1(\Omega)^3, (\mathbf{\nabla} \mathbf{v}, \mathbf{\nabla} \mathbf{v}) \geq \delta_0 \| \mathbf{v} \|_{L^2(\Omega)}^2 \forall \mathbf{v} \in H^1(\Omega)^3, \]

\[
\sup_{\mathbf{v} \in H^1(\Omega)^3, \mathbf{v} \neq 0} -\langle \text{div} \mathbf{v}, p \rangle/\| \mathbf{v} \|_{L^2(\Omega)} \geq \beta_1 \| p \|_{1, \Omega} \forall p \in L^2(\Omega), \]

\[
|k_0 h, \eta| \geq \gamma_p |k_0|_{L^2(\Omega)} \| h \|_{1, \Omega} \| \eta \|_{1, \Omega},
\]

\[
|\mathbf{u} \cdot \nabla h, \eta| \leq \gamma_2 \| \mathbf{u} \|_{L^3(\Omega)} \| h \|_{1, \Omega} \| \eta \|_{1, \Omega} \forall h, \eta \in H^1(\Omega); \]

\[
(\mathbf{u} \cdot \nabla h, h) = 0, (\nabla h, \nabla h) \geq \delta_1 \| h \|_{L^2(\Omega)}^2 \forall h \in \mathcal{T}. \]

Let us multiply first equation in (1) by function \( \mathbf{v} \in H^1(\Omega)^3 \), equation (2) by function \( h \in \mathcal{T} \) and integrate over \( \Omega \). Using Green’s formulae, we are coming to the weak formulation of Problem 1.

\[
\nu(\nabla \mathbf{u}, \nabla \mathbf{v}) + ((\mathbf{u} \cdot \nabla) \mathbf{u}, \mathbf{v}) - (p, \text{div} \mathbf{v}) = (\mathbf{f}, \mathbf{v}) + (\mathbf{b} \varphi, \mathbf{v}) \forall \mathbf{v} \in H^1(\Omega)^3,
\]
\[ \lambda(\nabla \varphi, \nabla h) + (k(\varphi) \varphi, h) + \lambda(\alpha \varphi, h)_{\Gamma_N} + (u \cdot \nabla \varphi, h) = (f, h) \forall h \in T. \quad (8) \]

Triple \((u, \varphi, p) \in V \times T \times L^2_0(\Omega)\) which satisfies (7), (8) will be called a Problem 1’ weak solution.

Let us consider the restriction of the space \(T\) on the space \(V:\)

\[ \nu(\nabla u, \nabla v) + ((u \cdot \nabla)u, v) = (f, v) + (b \varphi, v) \quad \forall v \in V. \quad (9) \]

and prove the solvability of problem (8), (9), applying Schauder’s theorem (see. [5]).

We set \( z = (w, c) \in H \) and \( y = (u, \varphi) \in H \) and construct map \( F : H \to H \) by formula

\[ F(z) = y \text{, where } y = (u, \varphi) \text{ is a solution of the linear problem } \]

\[ a_1(u, v) = \nu(\nabla u, \nabla v) + ((u \cdot \nabla)u, v) = (f, v) + (b \varphi, v) \quad \forall v \in V, \quad (10) \]

\[ a_2(\varphi, h) = \lambda(\nabla \varphi, \nabla h) + (k(c) \varphi, h) + \lambda(\alpha \varphi, h)_{\Gamma_N} + (w \cdot \nabla \varphi, h) = (f, h) \forall h \in T. \quad (11) \]

From condition (iii) and Lemma 1 follows that form \( a_2 : T \times T \to \mathbb{R} \) is continuous and coercive with constant \( \lambda_0 = \delta_1 \lambda, \) and then for all \( w \in H^1_0(\Omega)^3 \) and \( c \in T \) there is unique solution \( \varphi \in T \) of problem (11) for which the following estimate holds:

\[ \| \varphi \|_{1, \Omega} \leq M_{\varphi} \equiv C_* \| f \|_{\Omega}, \quad C_* = \lambda_*^{-1}. \quad (12) \]

By Lemma 1 form \( a_1 : H^1_0(\Omega)^3 \times H^1_0(\Omega)^3 \to \mathbb{R} \) is continuous and coercive on \( V \times V \) with constant \( \nu_* = \delta_0 \nu. \) Then for all (fixed) \( w \) and \( c \) there is unique solution \( u \in V \) of problem (10).

We set \( v = u \) in (10). Lemma 1 implies the inequality

\[ \nu_* \| u \|_{1, \Omega}^2 \leq \| f \|_{\Omega} \| u \|_{1, \Omega} + \beta_0 \| \varphi \|_{1, \Omega} \| u \|_{1, \Omega}. \quad (13) \]

From (13) taking into account (12) we arrive at

\[ \| u \|_{1, \Omega} \leq M_u = \nu_*^{-1}(\| f \|_{\Omega} + \beta_0 C_* \| f \|_{\Omega}). \quad (14) \]

Then there is solution \( y = (u, \varphi) \in H \) of problem 1 (10), (11) and estimate holds:

\[ \| y \|_H \leq M_u + M_{\varphi}. \quad (15) \]

In space \( H \) we introduce sphere \( B_r = \{ y \in H : \| y \|_H \leq r \}, \) where \( r = M_{\varphi} + M_u. \) From definition of \( B_r \) and from (15) it follows that the operator \( F \) maps \( B_r \) into itself for all \( z = (w, c) \in H. \) We prove that operator \( F \) is a continuous and compact on \( B_r. \) Let \( z_n = (w_n, c_n), \quad n = 1, 2, \ldots \) is an arbitrary sequence from \( B_r. \)

Since the spaces \( H^1_0(\Omega) \) and \( H^1_0(\Omega)^3 \) are reflexive and embeddings \( H^1(\Omega) \subset L^4(\Omega) \) and \( H^1(\Omega)^3 \subset L^4(\Omega)^3 \) are compact there is subsequence \( \{ z_n \} = \{(w_n, c_n)\} \) which we denote by \( \{ z_n \} \) again and function \( z = (w, c) \in B_r \) such that \( w_n \to w \) weakly in \( H^1(\Omega)^3 \) and strongly in \( L^3(\Omega)^3 \) as \( n \to \infty, \) \( c_n \to c \) weakly in \( H^1(\Omega) \) and strongly in \( L^3(\Omega) \) as \( n \to \infty. \)

We set \( y_n = F(z_n), \) \( y = F(z). \) These equalities mean that \( y = (u, \varphi) \in H \) is a solution of problem (10), (11) and \( y_n = (u_n, \varphi_n) \in H \) is solution of the following problem:

\[ \nu(\nabla u_n, \nabla v) + ((w_n \cdot \nabla)u_n, v) = (f, v) + (b \varphi_n, v) \quad \forall v \in V, \quad (16) \]

\[ \lambda(\nabla \varphi_n, \nabla h) + (k(c_n) \varphi_n, h) + \lambda(\alpha \varphi_n, h)_{\Gamma_N} + (w_n \cdot \nabla \varphi_n, h) = (f, h) \forall h \in T, \quad (17) \]

which is obtained from (10), (11) by replacing \( z = (w, c) \) on \( z_n = (w_n, c_n). \)

Let us show that \( y_n \to y \) strongly in \( H \) or \( \varphi_n \to \varphi \) strongly in \( H^1(\Omega) \) and \( u_n \to u \) strongly in \( H^1(\Omega)^3 \) at \( n \to \infty. \) Subtracting (10), (11) from (16), (17) and taking into account that

\[ (k(c_n) \varphi_n, h) - (k(c) \varphi, h) = (k(c_n)(\varphi_n - \varphi), h) + ((k(c_n) - k(c)) \varphi, h), \]
we obtain that
\[ \lambda(\nabla(\varphi_n - \varphi), \nabla h) + (k(c_n)\varphi_n - \varphi, h) + \lambda(\alpha(\varphi_n - \varphi), h)_{\Gamma_0} + (w_n \cdot \nabla(\varphi_n - \varphi), h) = \\
= -((w_n - w) \cdot \nabla\varphi, h) - ((k(c_n) - k(c))\varphi, h) \quad \forall h \in T, \]
\[ \nu(\nabla(u_n - u), \nabla v) + ((w_n - w)(u_n - u), v) = (((w_n - w) \cdot \nabla)u, v) + (b(\varphi_n - \varphi), v) \quad \forall v \in V. \quad (18) \]

Setting \( h = \varphi - \varphi_n \) and \( v = u - u_n \), then from Lemma 1 we obtain that \( \|\varphi_n - \varphi\|_{1,\Omega} \to 0 \)

Then operator \( F \) is continuous and compact and by fix-point Shauder theorem there is a

fixed point \( y = F(y) \in H \) of operator \( F \), which is a solution of system (8), (9).

By virtue (5) for pressure \( p \) and any (arbitrary small) number \( \delta > 0 \) there is function
\( v_0 \in H^1_0(\Omega)^3 \), \( v_0 \neq 0 \), such that \(-\text{div} v_0, p \geq \beta_2\|v_0\|_{1,\Omega}\|p\|_{\Omega}, \beta_2 = (\beta - \delta) > 0.\)

Setting \( v = v_0 \) in (7) and taking into account last inequality and Lemma 1 we obtain that
\[ \beta_2\|v_0\|_{1,\Omega}\|p\|_{\Omega} \leq \nu C_0\|v_0\|_{1,\Omega}\|u\|_{1,\Omega} + \gamma_1\|v_0\|_{1,\Omega}\|u\|_{1,\Omega}^2 + \beta_0\|\varphi\|\|v_0\|_{1,\Omega} \]

Dividing on \( \|v_0\|_{1,\Omega} \neq 0 \) and using estimates (12), (14) we come to inequality
\[ \|p\|_{\Omega} \leq C_p = \beta_2^{-1}[\nu_0 + \gamma_1 M_\Omega + \|f\|_{\Omega} + \beta_0 M_\varphi]. \quad (19) \]

We establish sufficient conditions of uniqueness of problem (8), (9)’ solution.

Let \( (u_1, \varphi_1), (u_2, \varphi_2) \in V \times T, i = 1, 2 \) are solutions of problem (8), (9). The differences \( \varphi = \varphi_1 - \varphi_2 \) and \( u = u_1 - u_2 \) satisfy to relations

\[ \lambda(\nabla\varphi, \nabla h) + (k(\varphi_1)\varphi_1 - k(\varphi_2)\varphi_2, h) + \lambda(\alpha\varphi, h)_{\Gamma_0} + (u_1 \cdot \nabla\varphi_2, h) = -(u \cdot \nabla\varphi_2, h) \quad \forall h \in T, \quad (20) \]
\[ \nu(\nabla u, \nabla v) + ((u_1 \cdot \nabla)u, v) = ((u \cdot \nabla)u_2, v) \quad \forall v \in V. \quad (21) \]

We assume that the nonlinearity \( k(\varphi)\varphi \) is monotonic in the following sense:
\( (k(\varphi_1)\varphi_1 - k(\varphi_2)\varphi_2, \varphi_1 - \varphi_2) \geq 0 \quad \forall \varphi_1, \varphi_2 \in T. \)

Setting \( h = \varphi \) in (20) and \( v = u \) in (21), from Lemma 1 we arrive to the estimates
\[ \lambda_\ast \|\varphi\|_{1,\Omega} \leq \gamma_2 M_\varphi\|u\|_{1,\Omega}, \quad \nu_\ast \|u\|_{1,\Omega} \leq \beta_0\|\varphi\|_{1,\Omega} + \gamma_1 M_\varphi\|u\|_{1,\Omega}. \quad (22) \]

We assume that the following smallness condition holds: \( \beta_0\gamma_2 C_\ast M_\varphi + \gamma_1 M_\varphi \leq \nu_\ast \) or
\[ \beta_0\gamma_2 C_\ast + (\gamma_1/\nu_\ast))\|f\|_{\Omega} + (\gamma_1/\nu_\ast))\|f\|_{\Omega} \leq \nu_\ast. \quad (23) \]

Then from estimates (22) follows that \( \|u\|_{1,\Omega} = 0 \) and \( \|\varphi\|_{1,\Omega} = 0 \) or \( u_1 = u_2 \) or \( \varphi_1 = \varphi_2 \).

Subtracting (7) at \( (\varphi_0, \varphi_2, p_2) \) from (7) at \( (u_1, \varphi_1, p_1) \) and taking into account that \( u = 0 \) and \( \varphi = 0 \), we obtain that the difference \( p = p_1 - p_2 \) satisfies to equation \(-\text{div} v = 0 \) for all \( v \in H^1_0(\Omega) \). Then in force (5) we obtain that \( p = 0 \) or \( p_1 = p_2 \).

We formulate the result obtained as the following theorem.

**Theorem 1.** If conditions (i)–(iii) then a weak solution \((u, \varphi, p) \in V \times T \times L^2_0(\Omega) \) of Problem 1 exists and estimates (12), (14) and (19) take palce. If, besides, the non-linearity \( k(\varphi)\varphi \) is monotone and condition (23) holds then the weak solution of Problem 1 is unique.
2. Control problem

For the statement of the control problem we divide the whole set of Problem 1’ initial data into two groups: the group of the fixed functions, in which the functions \( f, f, \alpha \) and \( k(\varphi) \) are included and the group of controlling functions, in which \( \beta \) will be included. We assume that \( \beta \) can be changed in some subset \( K \).

We set \( X = H^0_0(\Omega)^3 \times T \times L^2_0(\Omega), \ Y = H^{-1}(\Omega)^3 \times T^* \times L^2_0(\Omega), \ x = (u, \varphi, p) \in X \) and introduce the operator \( F = (F_1, F_2) \) by formulæ

\[
\langle F_1(x, \beta), (v, h) \rangle = \nu(\nabla u, \nabla v) + \lambda(\nabla \varphi, \nabla h) + ((u \cdot \nabla)u, v) - (p, \text{div} v) + (k(\varphi)\varphi, h) + \\
+ \lambda(\alpha \varphi, h)_{\Gamma_N} + (u \cdot \nabla \varphi, h) - (f, v) - (b \varphi, v) - (f, h), \ (F_2(x, \beta), r) = -(\text{div} u, r)
\]

and rewrite (8) in the form \( F(x, \beta) = 0. \)

We denote by \( I : X \to \mathbb{R} \) is a weakly semi continuous below functional and consider the following problem of conditional minimization:

\[
J(x, \beta) = (\mu_0/2)I(x) + (\mu_1/2)\|\beta\|^2_\Omega \rightarrow \inf, \ F(x, \beta) = 0, \ (x, \beta) \in X \times K. \tag{24}
\]

Denote by \( Z_{ad} = \{(x, \beta) \in X \times K : F(x, \beta) = 0, J(x, \beta) < \infty \} \) the set of possible pairs for problem (24) and assume that the following conditions hold:

(i) \( K \subset L^2(\Omega) \) is nonempty convex set,

(ii) \( \mu_0 > 0, \mu_1 \geq 0 \) and \( K \) is a bounded set or \( \mu_0 > 0, \mu_1 > 0 \) and functional \( I \) is bounded below.

We will consider the following cost functionals:

\[
I_1(\varphi) = \|\varphi - \varphi_d\|^2_Q, \ I_2(u) = \|u - u_d\|^2_Q, \ I_3(p) = \|p - p_d\|^2_Q. \tag{25}
\]

Here \( \varphi_d \in L^2(Q) \) is a given function in some subdomain \( \Omega \subset \Omega \). Functions \( u_d \) and \( p_d \) have the same meaning.

**Theorem 2.** Let conditions (i)–(iii) and (j), (jj) hold and \( I : X \to \mathbb{R} \) is a weakly semi continuous below functional and let \( Z_{ad} \neq \emptyset \). Then there is at least one solution \( (x, \beta) \in X \times K \) of the optimal problem (24).

**Proof.** Let \( (x_m, \beta_m) = (u_m, \varphi_m, p_m, \beta_m) \in Z_{ad} \) be a minimizing sequence for which the following is true:

\[
\lim_{m \to \infty} J(x_m, \beta_m) = \inf_{(x, \beta) \in Z_{ad}} J(x, \beta) \equiv J^*.
\]

This and the condition of the theorem imply the estimates \( \|\beta_m\|_\Omega \leq c_1 \). From Theorem 1 follows directly that \( \|u_m\|_1, \Omega \leq c_2, \|\varphi_m\|_1, \Omega \leq c_3 \) and \( \|p_m\|_\Omega \leq c_4 \), where the constants \( c_1, c_2, \ldots \) do not depend on \( m \).

Then the weak limits \( u^* \in V, \varphi^* \in T^*, p^* \in L^2_0(\Omega) \) and \( \beta^* \in K \) of some subsequences of sequences \( \{u_m\}, \{\varphi_m\}, \{p_m\} \) and \( \{\beta_m\} \). Corresponding subsequences will be also denoted by \( \{u_m\}, \{\varphi_m\}, \{p_m\} \) and \( \{f_m\} \), respectively. With this in mind, it can be considered that

\[
u(\nabla u^*, \nabla v) + \lambda(\nabla \varphi^*, \nabla h) + ((u^* \cdot \nabla)u^*, v) - (p^*, \text{div} v) + (k(\varphi^*)\varphi^*, h) +
\]
\[ +\lambda (\alpha \varphi^*, h) \Gamma_N + (u^* \cdot \nabla \varphi^*, h) = (f, v) + (\beta^* \varphi^*, G, v) + (f, h) \forall (v, h) \in H^1_0(\Omega)^3 \times \mathcal{T}. \] (29)

We note that for all \( m = 1, 2, \ldots \) the pair \((x_m, f_m)\) satisfies to relation
\[
\nu (\nabla u_m \cdot \nabla v) + \lambda (\nabla \varphi_m, \nabla h) + ((u_m \cdot \nabla) u_m, v) - (p_m, \text{div } v) + (k(\varphi_m) \varphi_m, h) + \\
+ \lambda (\alpha \varphi_m, h) \Gamma_N + (u_m \cdot \nabla \varphi_m, h) = (f, v) + (\beta_m \varphi_m, G, v) + (f_m, h) \forall (v, h) \in H^1_0(\Omega)^3 \times \mathcal{T}. \] (30)

Let us pass to the limit in (30) at \( m \to \infty \). All linear summands in (30) turn into corresponding ones in (29).

Let us consider the nonlinear summand \((k(\varphi_m) \varphi_m, h)\). From condition (iii) follows that \( k(\varphi_m) \to k(\varphi^*) \) strongly in \( L^{3/2}(\Omega) \) at \( m \to \infty \). With the help of (27) it is not difficult to show that \( \varphi_m h \to \varphi^* h \) weakly in \( L^3(\Omega) \) for all \( h \in \mathcal{T} \). Then \( k(\varphi_m) \varphi_m h \to k(\varphi^*) \varphi^* h \) strongly in \( L^1(\Omega) \) or \((k(\varphi_m) \varphi_m, h) \to (k(\varphi^*) \varphi^*, h) \) at \( m \to \infty \) for all \( h \in \mathcal{T} \).

The following equality holds:
\[
((u_m \cdot \nabla) u_m, v) - ((u^* \cdot \nabla) u^*, v) = (((u_m - u^*) \cdot \nabla) u_m, v) + ((u^* \cdot \nabla) (u_m - u^*), v) \]

From Lemma 1, (26) and by uniform boundedness over \( m \) of \( u_m \) in \( H^1(\Omega)^3 \) we obtain that
\[ ||((u_m - u^*) \cdot \nabla) u_m, v|| \leq \gamma_1 ||u_m - u^*||_{L^4(\Omega)^3} ||u_m||_{1, \Omega} ||v||_{1, \Omega} \to 0 \text{ at } m \to \infty. \]

From (11) it follows that \((u^* \cdot \nabla) (u_m - u^*), v) = -((u^* \cdot \nabla) v, u_m - u^*). \) Arguing as above we obtain that
\[ ||(u^* \cdot \nabla) v, u_m - u^*|| \leq \gamma_1 ||u^*||_{1, \Omega} ||v||_{1, \Omega} ||u_m - u^*||_{L^4(\Omega)^3} \to 0 \text{ at } m \to \infty. \]

For nonlinear summand \((u_m \cdot \nabla \varphi_m, h)\) the following relation is satisfied:
\[
(u_m \cdot \nabla \varphi_m, h) - (u^* \cdot \nabla \varphi^*, h) = ((u_m - u^*) \cdot \nabla \varphi_m, h) + (u^* \cdot \nabla (\varphi_m - \varphi^*), h). \]

From Lemma 1, (26) and estimate \( \|\varphi_m\|_{1, \Omega} \leq c_4 \) we obtain that
\[ ||(u_m - u^*) \cdot \nabla \varphi_m, h|| \leq \gamma_2 ||u_m - u^*||_{L^4(\Omega)^3} ||\varphi_m||_{1, \Omega} ||h||_{1, \Omega} \to 0 \text{ at } m \to \infty. \]

In force of the weak convergence \( \varphi_m \to \varphi^* \) in \( H^1(\Omega) \) in (27) we have
\[
(u^* \cdot \nabla (\varphi_m - \varphi^*), h) = (\nabla (\varphi_m - \varphi^*), h u^*) \to 0 \text{ at } m \to \infty \forall h \in H^1_0(\Omega). \]

Finally, from relation
\[
(\beta_m \varphi_m G, v) - (\beta^* \varphi^* G, v) = (\beta_m (\varphi_m - \varphi^*) G, v) + (\beta_m - \beta^*, \varphi^* v \cdot G) \]

taking into account (27) and (28) we conclude that \((\beta_m \varphi_m G, v) \to (\beta^* \varphi^* G, v)\) for all \( v \in H^1_0(\Omega)^3 \).

As the functional \( J \) is weakly semicontinuous on \( X \times L^2(\Omega) \) then from the aforesaid follows that \( J(x^*, \beta^*) = J^*. \]

In conclusion, in the framework of the optimization approach (see [25, 26]) the coefficient inverse problems can be reduced to the problems of multiplicative control.

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