On the Optimality of Treating Interference as Noise for Interfering Multiple Access Channels

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Abstract

In this paper, we look at the problem of treating interference as noise (TIN) in the Gaussian interfering multiple access channel (IMAC). The considered network comprises $K$ mutually interfering multiple access channels (MACs), each consisting of two transmitters communicating independent messages to one receiver. We define the TIN scheme for this channel as one in which each MAC performs a power controlled version of its capacity-achieving strategy while treating interference from all other MACs as noise. We characterize an achievable generalized degrees-of-freedom (GDoF) region under the TIN scheme and identify a regime of parameters (in terms of channel strength levels) where this region is optimal.

1 Introduction

Transmitter power control coupled with treating interference as noise (TIN) at receivers is one of the oldest and most commonly employed interference management strategies in wireless networks. The TIN strategy derives its attractiveness from its (relatively) low complexity and its robustness to channel uncertainty. TIN was shown to achieve the sum-capacity of the 2-user interference channel (IC) in what is known as the noisy interference regime \cite{1,2}. For the $K$-user IC, the problem is much more involved largely due to the intricate structure of the TIN-achievable rate region \cite{4} and the difficulty of the underlying optimization problem \cite{5}, a surprising contrast to the simple structure of the TIN strategy itself. This challenge was circumvented by Geng et al. in \cite{6} through seeking an approximate solution based on the generalized degrees-of-freedom (GDoF) \cite{7}.

Geng et al. identified a broad regime, described in terms of channel strength levels, where the TIN strategy achieves the exact GDoF region and the entire capacity region within a constant gap. Beyond the regular $K$-user IC considered in \cite{6}, this type of TIN-optimality investigation, through the GDoF and capacity approximations, has been extended in several directions \cite{8,9,10}. Nevertheless, the optimality of TIN in cellular-like networks is an intriguing direction that remains meagerly investigated. A recent result in this direction was reported in \cite{12}, where an alteration of the 2-user IC, termed the PIMAC, was considered. The PIMAC consists of a point-to-point link and a 2-user multiple access channel (MAC) that interfere with each other. The authors in \cite{12} identify regimes in which a simple time-sharing-TIN scheme is sum-GDoF optimal and achieves the sum-capacity within a constant gap. However, the specificity of the results and analysis in \cite{12} makes them difficult to generalize to settings with more transmitters and receivers.

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In this work, we consider an interfering multiple access channel (IMAC) comprising $K$ mutually interfering 2-user MACs, e.g. Fig. [1]. This is a typical model for cellular networks operating in the uplink mode and subsumes the setting in [12]. We introduce a TIN scheme in which each MAC performs a power controlled version of its capacity-achieving strategy, while treating interference from all other MACs as noise. We characterize an achievable GDoF region under the proposed TIN scheme. Moreover, we identify a regime of channel parameters for which this region is optimal. Finally, for the identified TIN-optimal regime, we show that the propose TIN scheme achieves the entire capacity region to within a constant gap.

**Notation:** For any positive integers $z_1$ and $z_2$ where $z_1 \leq z_2$, the sets $\{1, 2, \ldots, z_1\}$ and $\{z_1, z_1 + 1, \ldots, z_2\}$ are denoted by $\langle z_1 \rangle$ and $\langle z_1 : z_2 \rangle$, respectively. For any $a \in \mathbb{R}$, $(a)^+ = \max\{0, a\}$. Bold lowercase symbols denote tuples, e.g. $\mathbf{a} = (a_1, \ldots, a_Z)$. For $\mathcal{A} = \{a_1, \ldots, a_K\}$, $\Sigma(\mathcal{A})$ is the set of all cyclicly ordered sequences of all subsets of $\mathcal{A}$, e.g.

$$\Sigma(\{a_1, a_2, a_3\}) = \{(a_1), (a_2), (a_3), (a_1, a_2), (a_1, a_3), (a_2, a_3), (a_1, a_2, a_3), (a_1, a_3, a_2)\}.$$  

\[\Sigma(\{a_1, a_2, a_3\}) = \{(a_1), (a_2), (a_3), (a_1, a_2), (a_1, a_3), (a_2, a_3), (a_1, a_2, a_3), (a_1, a_3, a_2)\}.\]

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### 2 System Model and Preliminaries

Consider a $K$-receiver Gaussian IMAC in which each receiver is associated with 2 transmitters. The $k$-th receiver is denoted by $\text{Rx-}k$ and the $l_k$-th transmitter, $l_k \in \{1, 2\}$, associated with this receiver is denoted by $\text{Tx-}(l_k, k)$. We often use the terminology of cellular networks where a receiver and its associated transmitters are referred to as a cell. The set of tuples corresponding to all transmitters (or users) in the network is given by $\mathcal{K} \triangleq \{(l_k, k) : l_k \in \{1, 2\}, k \in \langle K \rangle\}$.

The input-output relationship at the $t$-th use of the channel is described as

$$Y_i(t) = \sum_{k=1}^{K} \left[ h_{ki}^{[1]} \tilde{X}_k^{[1]}(t) + h_{ki}^{[2]} \tilde{X}_k^{[2]}(t) \right] + Z_i(t), \forall i \in \langle K \rangle \quad (1)$$

where $h_{ki}^{[l]}$ is the channel coefficient from $\text{Tx-}(l_k, k)$ to $\text{Rx-}i$, $\tilde{X}_k^{[l]}(t)$ is the transmitted symbol of $\text{Tx-}(l_k, k)$ and $Z_i(t) \sim \mathcal{N}(0, 1)$ is the normalized additive white Gaussian noise (AWGN) at}

\[\Sigma(\{a_1, a_2, a_3\}) = \{(a_1), (a_2), (a_3), (a_1, a_2), (a_1, a_3), (a_2, a_3), (a_1, a_2, a_3), (a_1, a_3, a_2)\}.\]

\[\Sigma(\{a_1, a_2, a_3\}) = \{(a_1), (a_2), (a_3), (a_1, a_2), (a_1, a_3), (a_2, a_3), (a_1, a_2, a_3), (a_1, a_3, a_2)\}.\]

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\[\Sigma(\{a_1, a_2, a_3\}) = \{(a_1), (a_2), (a_3), (a_1, a_2), (a_1, a_3), (a_2, a_3), (a_1, a_2, a_3), (a_1, a_3, a_2)\}.\]
Rx-i. All symbols are complex and each transmitter \((l_k, k)\) is subject to the power constraint
\[ E[|\tilde{X}_k^{[l_k]}(t)|^2] \leq P_k^{[l_k]}. \] Note that receivers are indexed by subscripts, transmitters are indexed by superscripts in square parentheses and channel uses are indexed by arguments in round parentheses.

Following the standard reformulation in [6], the channel model in (1) is translated into
\[ Y_i(t) = \sum_{k=1}^{K} \left[ \sqrt{P_{l_k}^{[l_k]}} e^{j\theta_{l_k}^{[l_k]}} X_k^{[l_k]}(t) + \sqrt{P_{l_k}^{[l_k]}} e^{j\theta_{l_k}^{[l_k]}} X_k^{[l_k]}(t) \right] + Z_i(t) \] (2)

where \( P > 0 \) is a nominal power value and \( X_k^{[l_k]}(t) = \tilde{X}_k^{[l_k]}(t)/\sqrt{P_k^{[l_k]}} \) is the normalized transmit symbol of Tx-(\(l_k, k\)) with power constraint \( E[|X_k^{[l_k]}(t)|^2] \leq 1. \) \( \sqrt{P_{l_k}^{[l_k]}} \) and \( \theta_{l_k}^{[l_k]} \) are the magnitude and phase of the channel between Tx-(\(l_k, k\)) and Rx-i, respectively. The exponent \( \alpha_{l_k}^{[l_k]} \) is known as the channel strength level, and is given by
\[ \alpha_{l_k}^{[l_k]} \triangleq \frac{\log(\max \{1, |h_{l_k}^{[l_k]}|^2 P_k^{[l_k]}\})}{\log P}, \ \forall (l_k, k) \in \mathcal{K}, \ i \in \langle K \rangle. \]

As in [6], avoiding negative channel strength levels has no impact on the results. Without loss of generality, we assume the following order of direct link strength levels
\[ \alpha_{kk}^{[1]} \leq \alpha_{kk}^{[2]}, \ \forall k \in \langle K \rangle. \] (3)

### 2.1 Messages, Rates, Capacity and GDoF

Tx-(1, \(k\)) and Tx-(2, \(k\)) have the messages \( W_k^{[1]} \) and \( W_k^{[2]} \), respectively, intended to Rx-\(k\). All messages are independent and \(|W_k^{[l_k]}|\) denotes the size of the corresponding message set. For codewords spanning \(n\) channel uses, the rates \( R_k^{[l_k]} = \frac{\log(|W_k^{[l_k]}|)}{n} \), \( \forall (l_k, k) \in \mathcal{K}, \) are achievable if all messages can be decoded simultaneously with arbitrarily small error probability as \(n\) grows sufficiently large. A rate tuple is denoted by \( \mathbf{R} = (R_1^{[1]}, R_1^{[2]}, \ldots, R_K^{[1]}, R_K^{[2]}) \) and the channel capacity region \( \mathcal{C} \) is the closure of the set of all achievable rate tuples. A GDoF tuple is denoted by \( \mathbf{d} = (d_1^{[1]}, d_1^{[2]}, \ldots, d_K^{[1]}, d_K^{[2]}) \) and the GDoF region is defined as
\[ \mathcal{D} \triangleq \left\{ \mathbf{d} : d_k^{[l_k]} = \lim_{P \to \infty} \frac{R_k^{[l_k]}}{\log P}, \ \forall (l_k, k) \in \mathcal{K}, \ \mathbf{R} \in \mathcal{C} \right\}. \]

### 2.2 Treating (Inter-cell) Interference as Noise

In the TIN scheme, a MAC-type capacity-achieving strategy is employed in each cell, with successive decoding of in-cell signals while treating all inter-cell interference as noise. However, one key difference compared to the MAC (i.e. single-cell transmission) is that power control is employed by transmitters to manage inter-cell interference. It is known that such power control is not required to achieve the corner points of the MAC capacity region [13].

Each transmitter Tx-(\(l_k, k\)) uses an independent Gaussian codebook and transmits with power \( P_k^{[l_k]} \), where \( r_k^{[l_k]} \leq 0 \) is the transmit power exponent. On the other hand, each receiver Rx-\(k\) performs successive decoding of its two desired signals, while treating all inter-cell interference as noise. For a decoding order \( \pi_k : \{1, 2\} \to \{1, 2\} \), Rx-\(k\) starts by decoding, and cancelling, \( X_k^{[\pi_k(2)]} \)
before decoding $X^{[\pi_k(1)]}_k$. Hence, Tx-$\pi_k(1)$ and Tx-$\pi_k(2)$ achieve any rates $R^{[\pi_k(1)]}_k$ and $R^{[\pi_k(2)]}_k$, respectively, that satisfy

$$R^{[\pi_k(1)]}_k \leq \log \left( 1 + \frac{Pr^{[\pi_k(1)]}_k + \alpha^{[\pi_k(1)]}_k}{1 + \sum_{j \neq k} [Pr^{[\pi_k(2)]}_j + \alpha^{[\pi_k(2)]}_j + Pr^{[\pi_k(2)]}_k + \alpha^{[\pi_k(2)]}_k]} \right),$$

(4)

$$R^{[\pi_k(2)]}_k \leq \log \left( 1 + \frac{Pr^{[\pi_k(2)]}_k + \alpha^{[\pi_k(2)]}_k}{1 + \sum_{j \neq k} [Pr^{[\pi_k(2)]}_j + \alpha^{[\pi_k(2)]}_j + Pr^{[\pi_k(2)]}_k + \alpha^{[\pi_k(2)]}_k]} \right).$$

(5)

In the GDoF sense, we have

$$d^{[\pi_k(1)]}_k \leq \max \left\{ 0, r^{[\pi_k(1)]}_k + \alpha^{[\pi_k(1)]}_k \right\} - \left( \max_{j \neq k} \left\{ \max_{l_j} \{ r^{[\pi_k(1)]}_j + \alpha^{[\pi_k(1)]}_j \} \right\} \right)^+, \quad \forall k \in \langle K \rangle$$

(6)

$$d^{[\pi_k(2)]}_k \leq \max \left\{ 0, r^{[\pi_k(2)]}_k + \alpha^{[\pi_k(2)]}_k \right\} - \left( \max_{j \neq k} \left\{ \max_{l_j} \{ r^{[\pi_k(2)]}_j + \alpha^{[\pi_k(2)]}_j \} \right\} \right)^+, \quad \forall k \in \langle K \rangle$$

(7)

The decoding order across the network is defined as $\pi \triangleq (\pi_1, \ldots, \pi_K)$. For a given $\pi$, the \textit{TIN-achievable GDoF region}, denoted by $\mathcal{P}_\pi^*$, is the set of all GDoF tuples $d$ for which there exists a feasible transmit power exponent tuple $r \triangleq (r_1^{[1]}, r_1^{[2]}, \ldots, r_K^{[1]}, r_K^{[2]})$ such that (6) and (7) are satisfied for all $k \in \langle K \rangle$. The \textit{general TIN-achievable GDoF region} is defined as $\mathcal{P}^* \triangleq \bigcup_{\pi} \mathcal{P}_\pi^*$. Note that any GDoF tuple in $\mathcal{P}^*$ is achieved through a decoding order and a power allocation, i.e. $(\pi, r)$, where no time-sharing between different strategies is allowed.

Similar to [6], we introduce a \textit{polyhedral TIN scheme}. For a given $\pi$, the corresponding \textit{polyhedral TIN-achievable GDoF region} $\mathcal{P}_\pi$ is described by all GDoF tuples that satisfy

$$r^{[\pi_k(l_k)]}_k \leq 0, \forall (l_k, k) \in \mathcal{K}$$

(8)

$$d^{[\pi_k(l_k)]}_k \geq 0, \forall (l_k, k) \in \mathcal{K}$$

(9)

$$d^{[\pi_k(1)]}_k \leq r^{[\pi_k(1)]}_k + \alpha^{[\pi_k(1)]}_k - \left( \max_{j \neq k} \left\{ \max_{l_j} \{ r^{[\pi_k(1)]}_j + \alpha^{[\pi_k(1)]}_j \} \right\} \right)^+, \forall k \in \langle K \rangle$$

(10)

$$d^{[\pi_k(2)]}_k \leq r^{[\pi_k(2)]}_k + \alpha^{[\pi_k(2)]}_k - \left( \max_{j \neq k} \left\{ \max_{l_j} \{ r^{[\pi_k(2)]}_j + \alpha^{[\pi_k(2)]}_j \} \right\} \right)^+, \forall k \in \langle K \rangle$$

(11)

where the first max\{0, \} in (6) and (7) has been dropped. It follows from this restriction that $\mathcal{P}_\pi \subseteq \mathcal{P}_\pi^*$. Taking the union over all possible decoded orders, we achieve the region given by $\mathcal{P} = \bigcup_{\pi} \mathcal{P}_\pi$. It is readily seen that $\mathcal{P} \subseteq \mathcal{P}^* \subseteq \mathcal{D}$.

As it turns out, for any $\pi$, the region $\mathcal{P}_\pi$ is a polyhedron (see Theorem 1 in the following section). However, $\mathcal{P}$ is not a polyhedron in general, since it is a union of multiple polyhedra. Yet, every GDoF point in $\mathcal{P}$ is achieved by fixing $\pi$ and applying a polyhedral TIN scheme with power allocation $r$ satisfying (8)-(11).

In the following, we often work with the identity order $\pi = \text{id}$, where $\text{id} \triangleq (\text{id}_1, \ldots, \text{id}_K)$ and $\text{id}_i(l_i) = l_i, \forall (l_i, i) \in \mathcal{K}$. The corresponding polyhedral TIN region is denoted by $\mathcal{P}_{\text{id}}$.

### 2.3 Some Known Special Cases

Before presenting the main results, we review some known GDoF region characterizations for subnetworks of the considered IMAC. First, we consider a regular IC obtained by removing one transmitter from each cell and leaving only Tx-$\langle l_i, i \rangle$, $i \in \langle K \rangle$. From [6], the polyhedral TIN region for
this subnetwork is given by
\[ 0 \leq d_i^{[l_i]} \leq \alpha_i^{[l_i]}, \forall i \in \langle K \rangle \] (12)
\[ \sum_{j < \langle m \rangle \backslash (m)} d_{ij}^{[l_{ij}]} \leq \sum_{j \in \langle m \rangle} \alpha_{ij}^{[l_{ij}]} - \alpha_{ij}^{[l_{ij-1}]}, (i_1, \ldots, i_m) \in \Sigma(\langle K \rangle), m \in \langle 2 : K \rangle \] (13)

where the set of cyclic sequences $\Sigma(\langle K \rangle)$ is defined in the notation part of Section 1 and a modulo operation is implicitly used on receiver indices when dealing with cyclic sequences, e.g. $i_0 = i_m$. The region in (12) - (13) is optimal for the regular IC under the TIN-optimality conditions in [6].

Next, consider the MAC consisting of Rx-$i$ and its transmitters Tx-(1, $i$) and Tx-(2, $i$). The GDoF region achieved while fixing the decoding order $\pi_i$ is given by
\[ d_i^{[l_i]} \geq 0, \forall l_i \in \{1, 2\} \] (14)
\[ \sum_{s_i \in \langle l_i \rangle} d_i^{[\pi_i(s_i)]} \leq \alpha_i^{[\pi_i(l_i)]}, \forall l_i \in \{1, 2\}. \] (15)

It can be easily checked that the optimal GDoF region of the considered MAC is given by (14) - (15) while fixing the decoding order to $\pi_i = \text{id}$. The signal of the stronger user, i.e. Tx-(2, $i$), is always received at a higher power level and is hence decoded first, making the other order redundant from a GDoF perspective. Note that this is in contrast to the MAC capacity region, which requires changing the successive decoding order to achieve different corner points in general [13].

3 Main Results

Here we present the main results of the paper.

**Theorem 1.** For the IMAC described in Section 2, the polyhedral TIN-achievable GDoF region $\mathcal{P}_\pi$, for any decoding order $\pi$, is given by all tuples $\mathbf{d}$ that satisfy
\[ d_i^{[l_i]} \geq 0, \forall (l_i, i) \in K \] (16)
\[ \sum_{s_i \in \langle l_i \rangle} d_i^{[\pi_i(s_i)]} \leq \alpha_i^{[\pi_i(l_i)]}, \forall (l_i, i) \in K \] (17)
\[ \sum_{j < \langle m \rangle \backslash (m)} \sum_{s_{ij} \in \langle l_{ij} \rangle} d_{ij}^{[\pi_i(s_{ij})]} \leq \sum_{j \in \langle m \rangle} \alpha_{ij}^{[\pi_i(l_{ij})]} - \alpha_{ij}^{[\pi_i(l_{ij-1})]}, \] \[ \forall l_{ij} \in \{1, 2\}, (i_1, \ldots, i_m) \in \Sigma(\langle K \rangle), m \in \langle 2 : K \rangle. \] (18)

In (18), a modulo operation is implicitly used on receiver indices, e.g. $i_0 = i_m$. The proof of Theorem 1 is presented in Section 4. It can be seen that the characterization of $\mathcal{P}_\pi$ in Theorem 1 inherits the features of both the IC and MAC characterizations presented in Section 2.3. Moreover, in contrast to the MAC, the decoding order $\text{id}$ does not necessarily yield the largest polyhedral region for the IMAC, i.e. $\mathcal{P}_\pi \subseteq \mathcal{P}_\text{id}$ does not hold in general for all $\pi$. This inclusion, however, holds under the TIN-optimality conditions presented in the following result.

**Theorem 2.** For the IMAC described in Section 2, if the following conditions are satisfied
\[ \alpha_i^{[l_i]} \geq \max_{j \neq i} \left\{ \alpha_{ij}^{[l_i]} \right\} + \max_{(l_i, k) \neq i} \left\{ \alpha_{ki}^{[l_i]} \right\}, \forall (l_i, i), (l_i, k) \in K, j \in \langle K \rangle \] (19)
After rearranging, the inequalities in (23)–(27) are rewritten as
\[
\alpha_{ij}^{[2]} - \alpha_{ij}^{[1]} \geq \alpha_{ii}^{[1]} - \alpha_{ij}^{[1]} + \min\left\{\alpha_{ij}^{[1]}, \alpha_{ij}^{[2]}\right\}, \quad \forall i, j \in \langle K \rangle, \ i \neq j,
\]
then the optimal GDoF region is given by \( \mathcal{P}_{\text{id}} \), achieved through the polyhedral TIN scheme in Section 2.2, and described by \( (16) - (18) \) while setting \( \pi = \text{id} \).

The proof of Theorem 2 is given in Section 5. The condition in (19) is essentially the one identified by Geng et al. in [6], applied to all regular IC subnetworks of the IMAC. On the other hand, a special case of (20) was identified by Gherekhloo et al. in [12] for the PIMAC described in Section 1. Note that under the above TIN conditions, we have \( D = P^{\star} = P = \mathcal{P}_{\text{id}} \).

Before we proceed, it is worthwhile highlighting that as pointed out in [9, Remark 1], existing TIN-optimality results are "primarily in the form of sufficient conditions" and that the necessity of such conditions "remains undetermined in most cases". The TIN-optimality result in Theorem 2 is no exception to most existing results in that regards.

4 TIN-Achievable GDoF Region

In this part, we prove Theorem 1 by constructing a potential graph \[6,9\] for the considered IMAC and invoking the potential theorem \[14\]. To avoid cumbersome notation, we work with \( \mathcal{P}_{\pi} \) by replacing each superscript \( l_k \) with the corresponding \( \pi_k(l_k) \).

4.1 Feasible Power Allocation

The first step towards applying the potential theorem is to derive the conditions of feasible power allocation. To this end, we rewrite the inequalities in (10) and (11) as
\[
d_k^{[1]} \leq \min\left\{r_k^{[1]} + \alpha_{kk}^{[1]}, \min_{j \neq k}\left\{\min_{l = k} \left[ r_k^{[1]} - r_j^{[1]} + \alpha_{kk}^{[1]} - \alpha_{jj}^{[1]} + \alpha_{kk}^{[2]} - \alpha_{jj}^{[2]} \right] \right\}\right\}
\]
\[
d_k^{[2]} \leq \min\left\{r_k^{[2]} + \alpha_{kk}^{[2]}, \min_{j \neq k}\left\{\min_{l = k} \left[ r_k^{[2]} - r_j^{[2]} + \alpha_{kk}^{[1]} - \alpha_{jj}^{[1]} + \alpha_{kk}^{[2]} - \alpha_{jj}^{[2]} \right] \right\}\right\}
\]
From (21) and (22), it follows that the polyhedral TIN region \( \mathcal{P}_{\text{id}} \), described by the inequalities in \( (8) - (11) \) while setting \( \pi = \text{id} \), is equivalently described by the following inequalities
\[
r_k^{[l_k]} \leq 0, \quad \forall (l_k, k) \in \mathcal{K}
\]
\[
d_k^{[l_k]} \geq 0, \quad \forall (l_k, k) \in \mathcal{K}
\]
\[
d_k^{[l_k]} \leq \alpha_{kk}^{[l_k]} + r_k^{[l_k]}, \quad \forall (l_k, k) \in \mathcal{K}
\]
\[
d_k^{[l_k]} \leq r_k^{[l_k]} - r_j^{[l_k]} + \alpha_{kk}^{[l_k]} - \alpha_{jj}^{[l_k]} \quad \forall (l_k, k), (l_j, j) \in \mathcal{K}, \ j \neq k
\]
\[
d_k^{[l_k]} \leq r_k^{[l_k]} - r_j^{[l_k]} + \alpha_{kk}^{[1]} - \alpha_{jj}^{[1]} \quad \forall (l_k, k) \in \mathcal{K}
\]
After rearranging, the inequalities in (23)–(27) are rewritten as
\[
d_k^{[l_k]} \geq 0, \quad \forall (l_k, k) \in \mathcal{K}
\]
\[
r_k^{[l_k]} \leq 0, \quad \forall (l_k, k) \in \mathcal{K}
\]
\[
-r_k^{[l_k]} \leq \alpha_{kk}^{[l_k]} - d_k^{[l_k]}, \quad \forall (l_k, k) \in \mathcal{K}
\]
The set of vertices is given by \( V \), and the power allocation tuple \( r \) is defined such that \( \forall (l_k, k), (l_j, j) \in \mathcal{K}, j \neq k \).

Hence, a GDoF tuple \( d \in \mathbb{R}^{2K} \) is in the polyhedral TIN region \( \mathcal{P}_d \) if and only if there exists a power allocation tuple \( r \in \mathbb{R}^{2K} \) such that (29)–(32) hold.

### 4.2 Potential Graph

Next, we construct the potential graph \([6,9]\). This is a directed graph (diigraph) \( G_p = (V, E) \), where the set of vertices \( V \) and the set of directed edges (or edges) \( E \) are given by

\[
V = \{ u \} \cup \left\{ v_k^{[l]} : (l_k, k) \in \mathcal{K} \right\} \\
E = E_1 \cup E_2 \cup E_3 \cup E_4 \\
E_1 = \left\{ (v_k^{[1]}, v_k^{[2]} ) : k \in \langle K \rangle \right\} \\
E_2 = \left\{ (v_k^{[l]} , v_j^{[j]} ) : (l_k, k), (l_j, j) \in \mathcal{K}, k \neq j \right\} \\
E_3 = \left\{ (u, v_k^{[l]} ) : (l_k, k) \in \mathcal{K} \right\} \\
E_4 = \left\{ (v_k^{[l]} , u ) : (l_k, k) \in \mathcal{K} \right\}.
\]

We define the length function \( l : E \to \mathbb{R} \) and assign the following lengths to different edges:

\[
l(v_k^{[1]}, v_k^{[2]} ) = \alpha_k^{[1]} - d_k^{[1]}, \forall k \in \langle K \rangle \\
l(v_k^{[2]}, v_k^{[1]} ) = \alpha_k^{[2]} - \alpha_k^{[1]} - d_k^{[2]}, \forall k \in \langle K \rangle \\
l(v_k^{[l]}, v_j^{[j]} ) = \alpha_k^{[l]} - \alpha_j^{[l]} - d_k^{[l]}, \forall (l_k, k), (l_j, j) \in \mathcal{K}, k \neq j \\
l(u, v_k^{[l]} ) = \alpha_k^{[l]} - d_k^{[l]}, \forall (l_k, k) \in \mathcal{K} \\
l(u, v_k^{[l]} ) = 0, \forall (l_k, k) \in \mathcal{K}.
\]

By definition \([14]\), the function \( p : V \to \mathbb{R} \) is called a potential if for any pair of vertices \( a, b \in V \) such that \((a, b) \in E\), we have \( l(a, b) \geq p(b) - p(a) \). These conditions depend only on the difference between potential function values. Therefore, if there exists a valid potential function, we may assume without loss of generality that \( p(u) = 0 \), i.e. vertex \( u \) is set as the ground. By setting \( p(v_k^{[l]} ) = r_k^{[l]} \), it can be seen that such potential function values should satisfy

\[
r_k^{[l]} - r_k^{[1]} \leq \alpha_k^{[1]} - d_k^{[1]}, \forall k \in \langle K \rangle \\
r_k^{[1]} - r_k^{[2]} \leq \alpha_k^{[2]} - \alpha_k^{[1]} - d_k^{[2]}, \forall k \in \langle K \rangle \\
r_j^{[j]} - r_k^{[l]} \leq \alpha_k^{[l]} - \alpha_j^{[l]} - d_k^{[l]}, \forall (l_k, k), (l_j, j) \in \mathcal{K}, j \neq k \\
r_k^{[l]} - d_k^{[l]}, \forall (l_k, k) \in \mathcal{K} \\
r_k^{[l]} \leq 0, \forall (l_k, k) \in \mathcal{K}.
\]

It is easy to check that the inequalities in (46)–(49) are equivalent to the ones in (29)–(32). Moreover, the inequality in (45) is obtained by adding the inequalities in (29) and (30). Therefore,
it follows that $d \in \mathbb{R}^{2^K}_+$ is in $P_{id}$ if and only if there exists a valid potential function for $G_p$. At this point, we are ready to invoke the potential theorem [14, Th. 8.2]: there exists a potential function for a digraph $G_p$ if and only if each directed circuit in $G_p$ has a non-negative length.

From the above, we conclude that the GDoF tuple $d \in \mathbb{R}^{2^K}_+$ is in the polyhedral region $P_{id}$ if and only if the length of each directed circuit in the potential graph $G_p$ is non-negative.

4.3 Directed Circuits and GDoF Inequalities

In this part, we examine all valid directed circuits (or circuits for short) of $G_p$ and derive the corresponding GDoF inequalities. When dealing with circuits, we refer to a vertex of type $v_{li}$ as a user. It is readily seen that circuits of $G_p$ can be categorized into single-cell circuits and multi-cell circuits, depending on the participating users, as we see in what follows.

4.3.1 Single-Cell Circuits

Such circuits involve users belonging to only one cell and can be further categorized into:

- Single-user circuits of the form $(u \to v_{li}^{[1]} \to u), \forall (l_i, i) \in K$. From the non-negative length condition, each of such circuits yields a single user bound given by

$$d_{li}^{[1]} \leq \alpha_{ii}^{[1]}.$$  (50)

- Multi-user circuits of the form $(u \to v_{li}^{[2]} \to v_{li}^{[1]} \to u)$ or $(v_{li}^{[2]} \to v_{li}^{[1]} \to v_{li}^{[2]}), \forall i \in \langle K \rangle$. From the non-negative length condition applied to such circuits, we obtain

$$d_{li}^{[1]} + d_{li}^{[2]} \leq \alpha_{ii}^{[2]}.$$  (51)

- Multi-user circuits of the form $(u \to v_{li}^{[1]} \to v_{li}^{[2]} \to u), \forall i \in \langle K \rangle$, from which we obtain

$$d_{li}^{[1]} + d_{li}^{[2]} \leq \alpha_{ii}^{[1]} + \alpha_{ii}^{[2]}.$$  (52)

It can be seen that for $l_i = 2$, the GDoF inequality in (50) is redundant since it is implied by (51). Moreover, the inequality in (52) is loose in general compared to the one in (51).

4.3.2 Multi-Cell Circuits

Such circuits involve users belonging to more than one cell. In particular, consider a cyclic sequence of tuples given by $((l_1, i_1), \ldots, (l_n, i_n)) \in \Sigma(K)$, such that $i_{j'} \neq i_{j''}$ for some $j', j'' \in \langle n \rangle$. The corresponding multi-cell circuit of $G_p$ takes one of the two following forms.

- Does not traverse $u$: $(v_{i_0}^{[l_0]} \to v_{i_1}^{[l_1]} \to \cdots \to v_{i_n}^{[l_n]}), \text{ where } (l_0, i_0) = (l_n, i_n)$.

- Traverses $u$: $(u \to v_{i_1}^{[l_1]} \to \cdots \to v_{i_n}^{[l_n]} \to u)$.

From the non-negative length condition, a multi-cell circuit of the first form yields the inequality

$$\sum_{j \in \langle n \rangle} d_{ij}^{[l_j]} \leq \sum_{j \in \langle n \rangle} \alpha_{ij}^{[l_j]} + \alpha_{ij}^{[l_{j+1} + 1]} - \sum_{j \in \langle n \rangle} \alpha_{ij}^{[l_j]} + \alpha_{ij}^{[l_{j+1}]} E_{i} (v_{ij}^{[l_j]}, v_{ij}^{[l_{j+1} + 1]})$$  (53)
where
\[ \mathbb{1}_{E_t'}(v_i^{[l_i]}, v_k^{[k_l]}) = \begin{cases} 0, & \text{if } (v_i^{[l_i]}, v_k^{[k_l]}) \in E_t' \\ 1, & \text{otherwise.} \end{cases} \]

Note that a modulo operation is used in (53), and throughout this part, such that \( i_{n+1} = i_1 \) and \( l_{n+1} = l_1 \). On the other hand, a circuit of the second form gives the inequality
\[ \sum_{j \in \langle n \rangle} d_{ij}^{[l_j]} \leq \alpha_{i_{n+1}}^{[l_{n+1}]} + \sum_{j=1}^{n-1} \alpha_{ij}^{[l_j]} - \alpha_{i_{j+1}+1}^{[l_{j+1}]} \mathbb{1}_{E_t'}(v_i^{[l]}, v_{i_{j+1}+1}^{[l+1]}) \] (54)

It is readily seen that (53) is tighter than (54) as \( \alpha_{i_{n+1}}^{[l_{n+1}]} \mathbb{1}_{E_t'}(v_i^{[l]}, v_{i_1}^{[l+1]}) \geq 0 \). Therefore, it is sufficient to consider multi-cell circuits that do not traverse \( u \).

Next, we show that the GDoF inequality in (53) is necessarily redundant if the underlying circuit belongs to at least one of the following classes:

C.1 Circuits that traverse at least one edge in \( E_t' \), i.e. with two cyclicly adjacent users that belong to the same cell \( i \in \langle K \rangle \) and \( v_i^{[l]} \) preceded \( v_i^{[r]} \) in the cyclic order.

C.2 Circuits that traverse \( v_i^{[l]} \) and \( v_k^{[r]} \), where \( i_j = i_k, j \neq k+1 \) and \( k \neq j+1 \), i.e. with two cyclicly non-adjacent users that belong to the same cell.

C.3 Circuits that traverse \( v_i^{[r]} \), for some \( i \in \langle K \rangle \), and do not traverse \( v_i^{[l]} \).

First, suppose that we have a circuit in class C.1. We may assume, without loss of generality, that \( (v_i^{[l]}, v_i^{[r]}) \in E_t' \), i.e. \( i_1 = i_2, l_1 = 1 \) and \( l_2 = 2 \). The corresponding GDoF inequality is given by
\[ d_{i_1}^{[1]} + d_{i_1}^{[2]} + \sum_{j=3}^{n} d_{ij}^{[l_j]} \leq \alpha_{i_1}^{[1]} + \alpha_{i_1}^{[2]} - \alpha_{i_2}^{[3]} + \sum_{j=3}^{n} \alpha_{ij}^{[l_j]} - \alpha_{i_{j+1}+1}^{[l_{j+1}]} \mathbb{1}_{E_t'}(v_i^{[l]}, v_{i_{j+1}+1}^{[l+1]}) \] (55)

Now consider the circuits \( (v_i^{[l]} \rightarrow v_i^{[1]} \rightarrow v_i^{[3]} \rightarrow \cdots \rightarrow v_i^{[n]}) \) and \( (u \rightarrow v_i^{[2]} \rightarrow v_i^{[r]}) \). These are valid circuits of \( G_p \), and give rise to the GDoF inequalities
\[ d_{i_1}^{[1]} + \sum_{j=3}^{n} d_{ij}^{[l_j]} \leq \alpha_{i_1}^{[1]} - \alpha_{i_2}^{[3]} + \sum_{j=3}^{n} \alpha_{ij}^{[l_j]} - \alpha_{i_{j+1}+1}^{[l_{j+1}]} \mathbb{1}_{E_t'}(v_i^{[l]}, v_{i_{j+1}+1}^{[l+1]}) \] (56)
\[ d_{i_1}^{[2]} \leq \alpha_{i_1}^{[2]} \] (57)

By adding (56) and (57), we obtain (55), which is therefore redundant. If the circuit underlying the GDoF inequality in (56) is also in class C.1, we apply the same argument above. We do this recursively, hence showing that all circuits in class C.1 yield redundant GDoF inequalities.

Next, after excluding all circuits in class C.1, suppose that we have a circuit in class C.2 and not in class C.1 such that \( i_1 = i_k, k \neq 2 \) and \( k \neq n \) (also \( k \neq 0 \)). We may further assume, without loss of generality, that \( l_1 = 1 \) and \( l_k = 2 \). The corresponding GDoF inequality is given by
\[ \sum_{j \in \langle n \rangle} d_{ij}^{[l_j]} \leq \sum_{j \in \langle n \rangle} \alpha_{ij}^{[l_j]} - \alpha_{i_{j+1}+1}^{[l_{j+1}]} \] (58)

where there is no need to employ the indicator function in (53) as the underlying circuit is not in C.1.

Now consider the circuits \( (v_{i_1}^{[l]} \rightarrow \cdots \rightarrow v_{i_k}^{[l_k]} \rightarrow v_{i_1}^{[l_1]}) \) and \( (v_{i_1}^{[l]} \rightarrow v_{i_{k+1}}^{[l_{k+1}]} \rightarrow \cdots \rightarrow v_{i_n}^{[l_n]} \rightarrow v_{i_1}^{[l_1]}) \). It
can be easily checked that these two circuits are valid for \(G_p\) and that they are not in class \(C.1\). The corresponding GDoF inequalities are given by

\[
\sum_{j=1}^{k} d_{t_j}^{[l_i]} \leq \alpha_{t_i t_k}^{[l_i]} - \alpha_{t_i t_1}^{[l_i]} + \sum_{j=1}^{k-1} \alpha_{t_j t_{j+1}}^{[l_i]} - \alpha_{t_{j+1} t_{j+1} t_{j+1}}^{[l_i]}. 
\] (59)

\[
d_{t_1}^{[l_i]} + \sum_{j=k+1}^{n} d_{t_j}^{[l_i]} \leq \alpha_{t_1 t_{k+1} t_{k+1}}^{[l_i]} - \alpha_{t_1 t_1}^{[l_i]} + \alpha_{t_{k+1} t_{k+2} t_{k+2}}^{[l_i]} - \alpha_{t_{k+2} t_{k+1} t_{k+2}}^{[l_i]} + \sum_{j=k+1}^{n-1} \alpha_{t_j t_{j+1}}^{[l_i]} - \alpha_{t_{j+1} t_{j+1} t_{j+1}}^{[l_i]}.
\] (60)

By adding the inequalities in (59) and (60), while noting that \(l_k = l_1\), we obtain

\[
d_{t_1}^{[l_i]} + \sum_{j=1}^{n} d_{t_j}^{[l_i]} \leq \sum_{j=1}^{n} \alpha_{t_j t_{j+1} t_{j+1}}^{[l_i]} - \alpha_{t_{j+1} t_{j+1} t_{j+1}}^{[l_i]}. 
\] (61)

Comparing (58) and (61), it can be seen that an extra \(d_{[l_i]}^{[l_1]}\) is added to the left-hand-side of the latter. Since \(d_{t_1}^{[l_i]} \geq 0\), then (61) implies (58). If any of the two resulting circuits underlying the inequalities in (59) and (60) is in class \(C.2\), we apply the same argument above. We do this recursively, hence showing redundancy of all circuits in class \(C.2\).

Finally, suppose that we have a circuit in class \(C.3\) and not in \(C.1\) or \(C.2\). We may assume, without loss of generality, that \(l_1 = 2\) and \(i_j \neq i_1\), \(\forall j \in \{2 : n\}\). The corresponding GDoF inequality writes as the one in (58). Consider the circuit given by \((v_{[v_0]}^{i_1} \rightarrow v_{[v_1]}^{i_1} \rightarrow v_{[v_2]}^{i_1} \rightarrow \cdots \rightarrow v_{[v_n]}^{i_1})\), where \(v_{[v_1]}^{i_1}\) is included between \(v_{[v_1]}^{i_1}\) and \(v_{[v_2]}^{i_1}\). This is valid for \(G_p\) and is not in \(C.1\) or \(C.2\). The corresponding GDoF inequality is given by

\[
d_{t_1}^{[l_i]} + \sum_{j=1}^{n} d_{t_j}^{[l_i]} \leq \sum_{j=1}^{n} \alpha_{t_j t_{j+1} t_{j+1}}^{[l_i]} - \alpha_{t_{j+1} t_{j+1} t_{j+1}}^{[l_i]}. 
\] (62)

Comparing (58) to (62), it can be seen that \(d_{t_1}^{[l_i]}\) (i.e. an extra user) is added to the left-hand-side without altering the right-hand-side. Since \(d_{t_1}^{[l_i]} \geq 0\), then (62) is tighter in general. Applying the same above argument recursively to the circuit underlying the inequality in (62), it is shown that circuits in class \(C.3\) are redundant.

### 4.3.3 Combining Inequalities

From single-cell circuits, we get the GDoF inequalities given by

\[
\sum_{s \in \langle K \rangle} d_{i}^{[s]} \leq \alpha_{i}^{[s]}, \forall i \in \langle K \rangle, l \in \{1, 2\} 
\] (63)

On the other hand, we only need to consider multi-cell circuits that do not traverse \(u\) and do not belong to any of the classes \(C.1\) or \(C.3\). From such circuits, we get the inequalities

\[
\sum_{j \in \langle m \rangle} \sum_{s_j \in \langle l_i \rangle} d_{i}^{[s_j]} \leq \sum_{j \in \langle m \rangle} \alpha_{i_j i_j}^{[l_i]} - \alpha_{i_{j+1} i_{j+1}}^{[l_i]} = \sum_{j \in \langle m \rangle} \alpha_{i_j i_j}^{[l_i]} - \alpha_{i_{j+1} i_{j+1}}^{[l_i]},
\] (64)

\[\forall l_j \in \{1, 2\}, (i_1, \ldots, i_m) \in \Sigma(\langle K \rangle), m \in \langle 2 : K \rangle\]

where (a) follows by rearranging the terms. Combining (63) and (64) with the non-negativity constraint on \(d_{i}^{[l_i]}\), \(\forall (l_i, i) \in K\), leads directly to the characterization in Theorem 1.
5 TIN Optimality

5.1 Outer Bound

The TIN-optimality result in Theorem 3 follows directly from the following outer bound.

**Theorem 3.** For the IMAC with input-output relationship in (2), if the TIN-optimality conditions in (19) and (20) hold, then the capacity region $C$ is included in the set of rate tuples satisfying

\[
\sum_{s_i \in \langle l_i \rangle} R_i^{[s_i]} \leq \log \left( 1 + l_i P_{\alpha_i}^{[l_i]} \right), \quad (l_i, i) \in \mathcal{K}
\]

\[
\sum_{j \in \langle m \rangle} \sum_{s_j \in \langle l_j \rangle} R_j^{[s_j]} \leq m + \sum_{j \in \langle m \rangle} \log \left( 1 + (l_{j+1} + l_j) P_{\alpha_j}^{[l_j]} P_{\alpha_{ij}}^{[l_{ij}]} - \alpha_{ij} \right),
\]

\[
\forall l_{ij} \in \{1, 2\}, (i_1, \ldots, i_m) \in \Sigma(\langle K \rangle), m \in \langle 2 : K \rangle.
\]

**Proof.** For each cell $i$, (65) is a cut-set upper bound and follows from the MAC capacity region [13] and [3]. Hence, we focus on the cyclic bounds in (66).

Cells and users participating in a given cyclic bound are identified by the two sequences $(i_1, \ldots, i_m) \in \Sigma(\langle K \rangle)$ and $(l_1, \ldots, l_m) \in \{1, 2\}^m$. Given such sequences, each participating cell $i_j$ is in one of the three following subsets: $S_1 \triangleq \{i_j : l_{ij} = 1\}$, $S_2 \triangleq \{i_j : l_{ij} = 2\}$, $\alpha_{ij}^{[1]} \leq \alpha_{ij}^{[2]} \leq \alpha_{ij}^{[3]}$. Next, we go through the following steps

- Eliminate all non-participating transmitters $(l_i, i) \in \mathcal{K} \setminus \{ (s_i, i_j) : s_i, i_j \in \langle l_i \rangle, j \in \langle m \rangle \}$, all non-participating receivers $i \in \langle K \rangle \setminus \{i_1, \ldots, i_m\}$ and the corresponding messages.
- Eliminate all interfering links except for links from Tx-$l_{ij}$ to Rx-$i_{j-1}$, $\forall j \in \langle m \rangle$, and from Tx-$l_{ij}$ to Rx-$i_{j-1}$, $\forall j \in \langle m \rangle$.

We end up with a partially connected cyclic IMAC with input-output relationship

\[
Y_{ij}(t) = \sum_{s_i \in \langle l_i \rangle} h_{ij}^{[s_i]} X_{ij}^{[s_i]}(t) + U_{ij+1}(t)
\]

where the interference plus noise $U_{ij}(t)$, caused by cell $i_j$ to cell $i_{j-1}$, is given by

\[
U_{ij}(t) = \begin{cases} h_{ij}^{[l_i]} X_{ij}^{[l_i]}(t) + Z_{ij-1}(t), & i_j \in S_1 \cup S_2 \\ h_{ij}^{[1]} X_{ij}^{[1]}(t) + h_{ij}^{[2]} X_{ij}^{[2]}(t) + Z_{ij-1}(t), & i_j \in S_3. \end{cases}
\]

Since none of the above steps hurts the rates of the remaining messages, the channel in (67) is used for the outer bound. From (67) onwards, we revert back to the original channel notation for notational convenience, while maintaining $|h_{k}^{[l_k]} P^{[l_k]}| \geq 1$, $\forall (l_k, k) \in \mathcal{K}, i \in \langle K \rangle$.

Next, we define the side information signal $S_{ij}(t)$, $\forall j \in \langle m \rangle$, as

\[
S_{ij}(t) = \begin{cases} U_{ij}(t), & i_j \in S_1 \cup S_2 \\ \frac{h_{ij}^{[2]} h_{ij}^{[l_i]}}{h_{ij}^{[1]}} \left( h_{ij}^{[1]} X_{ij}^{[1]}(t) + h_{ij}^{[2]} X_{ij}^{[2]}(t) \right) + Z_{ij-1}(t), & i_j \in S_3 \end{cases}
\]
and we provide $S^n_{ij}$ for Rx-$i_j$ through a genie. From Fano’s inequality, we have

$$n \sum_{s_j \in (l_j)} R_{ij}^{[s_j]} - ne \leq I(W_{ij}^{[1;l_j]}; Y_{ij}^n, S^n_{ij})$$

$$= I(W_{ij}^{[1;l_j]}; S^n_{ij}) + I(W_{ij}^{[1;l_j]}; Y_{ij}^n|S^n_{ij})$$

$$= h(S^n_{ij}) - h(S^n_{ij}|W_{ij}^{[1;l_j]}) + h(Y_{ij}^n|S^n_{ij}) - h(Y_{ij}^n|W_{ij}^{[1;l_j]})$$

$$= h(S^n_{ij}) - h(Z^n_{ij-1}) + h(Y_{ij}^n|S^n_{ij}) - h(U^n_{ij+1})$$

(69)

where $W_{ij}^{[1;l_j]} \triangleq W_{ij}^{[l_1]}, \ldots, W_{ij}^{[l_j]}$. Proceeding from (69), we have

$$n \sum_{j \in (m)} \sum_{s_j \in (l_j)} R_{ij}^{[s_j]} - nme \leq \sum_{j \in (m)} \left[ h(S^n_{ij}) - h(U^n_{ij}) + h(Y_{ij}^n|S^n_{ij}) - h(Z^n_{ij}) \right].$$

(70)

Considering the first difference of entropies in (70) for a given $j \in (m)$, it is clear that this is equal to 0 if $i_j \in S_1 \cup S_2$. Hence, we focus on $i_j \in S_3$. For this case, and from (20), we have

$$\alpha^{[2]}_{ij} - 2\alpha^{[2]}_{ij-1} \geq \alpha^{[1]}_{ij} - \alpha^{[1]}_{ij-1} \iff \frac{P^{[1]}_j}{P^{[2]}_j} \frac{h^{[1]}_{ij}}{h^{[2]}_{ij-1}} \geq \frac{P^{[1]}_j}{P^{[2]}_j} \frac{h^{[1]}_{ij}}{h^{[2]}_{ij}}.$$  

(71)

The condition in (71) allows us to apply [12] Lemma 8, from which we obtain

$$h(S^n_{ij}) - h(U^n_{ij}) \leq n.$$  

(72)

Now we turn our attention to the second difference of entropies in (70). We have

$$h(Y^n_{ij}|S^n_{ij}) - h(Z^n_{ij}) \leq \sum_{t \in (n)} \left[ h(Y_{ij}(t)|S_{ij}(t)) - h(Z_{ij}(t)) \right]$$

$$\leq \sum_{t \in (n)} \left[ h(Y_{ij}(t)|S^n_{ij}(t)) - h(Z_{ij}(t)) \right]$$

$$\leq n \log \left( \sigma^2_{Y^n_{ij}|S^n_{ij}} \right).$$

(73)

(74)

where $G$ indicates that the corresponding inputs are i.i.d Gaussian, i.e. $X_i^{[l_i]} \sim \mathcal{N}_C(0, P^{[l_i]}_i)$, and

$$\sigma^2_{Y^n_{ij}|S^n_{ij}} \triangleq \mathbb{E} \left[ |Y^n_{ij}|^2 \right] - \mathbb{E} \left[ Y^n_{ij}S^n_{ij} \right] \mathbb{E} \left[ S^n_{ij}Y^n_{ij} \right] \left( \mathbb{E} \left[ |S^n_{ij}|^2 \right] \right)^{-1}.$$  

(75)

Note that we omit the time index $t$ from (74) onwards for brevity. The inequality in (73) follows because Gaussian distribution maximizes the conditional differential entropy for a given covariance constraint. Next, we calculate $\sigma^2_{Y^n_{ij}|S^n_{ij}}$ as

$$\sigma^2_{Y^n_{ij}|S^n_{ij}} = \begin{cases} 
\mathbb{E} \left[ |U^n_{ij+1}|^2 \right] + \frac{P^{[1]}_j}{1+P^{[1]}_j} \frac{|h^{[1]}_{ij}|^2}{|h^{[2]}_{ij}|^2}, & i_j \in S_1 \\
\mathbb{E} \left[ |U^n_{ij+1}|^2 \right] + \frac{P^{[1]}_j}{1+P^{[1]}_j} \frac{|h^{[1]}_{ij}|^2}{|h^{[2]}_{ij}|^2} + \frac{P^{[2]}_j}{1+P^{[2]}_j} \frac{|h^{[2]}_{ij}|^2}{|h^{[2]}_{ij}|^2}, & i_j \in S_2 \\
\mathbb{E} \left[ |U^n_{ij+1}|^2 \right] + \frac{P^{[1]}_j}{1+P^{[1]}_j} \frac{|h^{[1]}_{ij}|^2}{|h^{[2]}_{ij}|^2} + \frac{P^{[2]}_j}{1+P^{[2]}_j} \frac{|h^{[2]}_{ij}|^2}{|h^{[2]}_{ij}|^2}, & i_j \in S_3.
\end{cases}$$

(76)
The expressions for the three cases in (76) are bounded above as

\[
\sigma_{Y_{ij}}^2 \leq \begin{cases} 
1 + P_{ij}^{[1]} |h_{ij}^{[1]}|^2 + \frac{P_{ij}^{[1]} |h_{ij}^{[1]}|^2}{P_{ij}^{[1]} |h_{ij}^{[1]}|^2}, & i_j \in S_1 \\
1 + P_{ij}^{[2]} |h_{ij}^{[2]}|^2 + P_{ij}^{[1]} |h_{ij}^{[1]}|^2 + \frac{P_{ij}^{[2]} |h_{ij}^{[2]}|^2}{P_{ij}^{[2]} |h_{ij}^{[2]}|^2}, & i_j \in S_2 \\
1 + P_{ij}^{[1]} |h_{ij}^{[1]}|^2 + P_{ij}^{[2]} |h_{ij}^{[2]}|^2 + \frac{2P_{ij}^{[2]} |h_{ij}^{[2]}|^2}{P_{ij}^{[2]} |h_{ij}^{[2]}|^2}, & i_j \in S_3
\end{cases}
\] (77)

where we have employed (68) and the order of strengths (3). Converting to the notation of (2) and employing the TIN conditions in (19) and (20), we obtain a further upper bound for (77) as

\[
\sigma_{Y_{ij}}^2 \leq 1 + (l_{ij+1} + l_{ij}) P^{\alpha_{ij}^{[ij]} - \alpha_{ij}^{[ij-1]}}.
\] (78)

By combining (70) with (72), (74) and (78), we obtain the bound in (66).

\[ \Box \]

5.2 Constant Gap to Capacity Region

Utilizing Theorem 1, Theorem 2 and Theorem 3, it can be shown that the proposed TIN scheme can achieve the whole capacity region to within a constant gap at any finite SNR.

**Theorem 4.** For the IMAC with input-output relationship in (2), if the TIN-optimality conditions in (19) and (20) hold, then the rate region achieved through the TIN scheme with decoding order \( \pi = \text{id} \), as described in Section 2.2, is within \( 2 + \log(5K) \) bits of the capacity region \( C \).

**Proof.** The above result is shown by following the same steps used to prove [6, Theorem 4]. First, we obtain an outer bound which is within a constant gap from the one in Theorem 3. For the bound in (65), since \( P > 1 \) and \( l_i \leq 2 \), we have

\[
\sum_{s_i \in \{l_i\}} R_{ij}^{[s_i]} \leq \log \left(1 + l_{ij} P^{\alpha_{ij}^{[ij]}}\right) \\
\leq \log \left(3P^{\alpha_{ij}^{[ij]}}\right) = \log(3) + \log \left(P^{\alpha_{ij}^{[ij]}}\right).
\]

On the other hand, for the bound in (66), it follows that

\[
\sum_{j \in \{m\}} \sum_{s_{ij} \in \{l_{ij}\}} R_{ij}^{[s_{ij}]} \leq \sum_{j \in \{m\}} \left[1 + \log \left(1 + (l_{ij+1} + l_{ij}) P^{\alpha_{ij+1}^{[ij]} - \alpha_{ij}^{[ij-1]}}\right)\right] \\
\leq \sum_{j \in \{m\}} \left[1 + \log(5) + \log \left(P^{\alpha_{ij}^{[ij]} - \alpha_{ij}^{[ij-1]}}\right)\right].
\]

From the above, we see that \( C \) is contained in the region given by all rate tuples \( \mathbf{R} \in \mathbb{R}_+ \) such that

\[
\sum_{s_i \in \{l_i\}} R_{ij}^{[s_i]} \leq \alpha_{ij}^{[ij]} \log(P) + \log(10), \quad (l_i, i) \in \mathcal{K}
\] (79)

\[
\sum_{j \in \{m\}} \sum_{s_{ij} \in \{l_{ij}\}} R_{ij}^{[s_{ij}]} \leq \sum_{j \in \{m\}} \left[\left(\alpha_{ij}^{[ij]} - \alpha_{ij}^{[ij-1]}\right) \log(P) + \log(10)\right],
\]

13
∀i_\text{j} \in \{1, 2\}, (i_1, \ldots, i_\text{m}) \in \Sigma(\langle K \rangle), m \in \langle 2 : K \rangle. \tag{80}

Next, we derive an achievable rate region. We fix the decoding order to \( \pi = \text{id} \). From \cite{14} and \cite{5}, we know that for any feasible power allocation \( r \), the rate tuple \( \text{R} \in \mathbb{R}_+ \) that satisfies
\[
\tilde{R}_k^1 = \log \left( 1 + \frac{P_k^{[1]} + \alpha_k^{[1]}}{1 + \sum_{j \neq k} [P_j^{[1]} + \alpha_j^{[1]} + P_j^{[2]} + \alpha_j^{[2]}]} \right) \tag{81}
\]
\[
\tilde{R}_k^1 + \tilde{R}_k^2 = \log \left( 1 + \frac{P_k^{[1]} + \alpha_k^{[1]} + P_k^{[2]} + \alpha_k^{[2]}}{1 + \sum_{j \neq k} [P_j^{[1]} + \alpha_j^{[1]} + P_j^{[2]} + \alpha_j^{[2]}]} \right). \tag{82}
\]
for all \( k \in \langle K \rangle \), is achievable. The region given by all such rate tuples, corresponding to all feasible \( r \), is hence achievable. Next, we characterize this rate region to within a constant gap. From the proof of Theorem 1 (see Section 4) and Theorem 2, we know that when the conditions in \cite{19} and \cite{20} hold, \( d \in \mathcal{P}_{\text{id}} \) if and only if there exists a power allocation \( r \) such that
\[
d_k^{[1]} = r_k^{[1]} + \alpha_k^{[1]} - \max \left\{ 0, \max_{j \neq k} \left\{ r_j^{[l]} + \alpha_j^{[l]} \right\} \right\}, \quad k \in \langle K \rangle \tag{83}
\]
\[
d_k^{[2]} = r_k^{[2]} + \alpha_k^{[2]} - \max \left\{ 0, r_k^{[1]} + \alpha_k^{[1]}, \max_{j \neq k} \left\{ r_j^{[l]} + \alpha_j^{[l]} \right\} \right\}, \quad k \in \langle K \rangle \tag{84}
\]
\[
r_k^{[l]} \leq 0, \quad (l, k) \in \mathcal{K} \tag{85}
\]
are satisfied\(^2\). By adding (83) and (84), we obtain
\[
d_k^{[1]} + d_k^{[2]} = r_k^{[1]} + \alpha_k^{[1]} + r_k^{[2]} + \alpha_k^{[2]} - \max \left\{ 0, \max_{j \neq k} \left\{ r_j^{[l]} + \alpha_j^{[l]} \right\} \right\}
- \max \left\{ 0, r_k^{[1]} + \alpha_k^{[1]}, \max_{j \neq k} \left\{ r_j^{[l]} + \alpha_j^{[l]} \right\} \right\}
\leq r_k^{[2]} + \alpha_k^{[2]} - \max \left\{ 0, \max_{j \neq k} \left\{ r_j^{[l]} + \alpha_j^{[l]} \right\} \right\}
\leq \max \left\{ r_k^{[1]} + \alpha_k^{[1]}, r_k^{[2]} + \alpha_k^{[2]} \right\} - \max \left\{ 0, \max_{j \neq k} \left\{ r_j^{[l]} + \alpha_j^{[l]} \right\} \right\}. \tag{86}
\]
We employ the above to characterize the achievable rate region. In particular, from (83), the achievable rate in (81) is bounded below as
\[
R_k^{[1]} \geq \log \left( \frac{P_k^{[1]} + \alpha_k^{[1]}}{P_0 + \sum_{j \neq k} [P_j^{[1]} + \alpha_j^{[1]} + P_j^{[2]} + \alpha_j^{[2]}]} \right)
\geq \log \left( \frac{P_k^{[1]} + \alpha_k^{[1]}}{[1 + 2(K - 1)]P^{\max} \left\{ \max_{j \neq k} \left\{ r_j^{[l]} + \alpha_j^{[l]} \right\} \right\}} \right)
\geq \log \left( \frac{P_k^{[1]} + \alpha_k^{[1]}}{[1 + 2(K - 1)]P_k^{[1]} + \alpha_k^{[1]} - d_k^{[1]}} \right)
\geq d_k^{[1]} \log(P) - \log(2K).
\]
\(^2\)Note that while the conditions for feasible power allocation in (21) and (22) (and hence (23)–(27)) are given in terms of inequalities, equality in (83) and (86) can be shown using the fixed point theorem as in [6, Appendix B].
Similarly, from \((86)\), the sum rate in \((82)\) is bounded below as

\[
\bar{R}_k^{[1]} + \bar{R}_k^{[2]} \geq \log \left( \frac{P_k^{[1]} + \alpha_k^{[1]} + \alpha_k^{[2]} + \alpha_k^{[3]}}{P_0 + \sum_{j \neq k} \left[ P_j^{[1]} \alpha_{j}^{[1]} + P_j^{[2]} \alpha_{j}^{[2]} \right]} \right) \\
\geq \log \left( \frac{P_{\text{max}} \{ r_k^{[1]} + \alpha_k^{[1]} r_k^{[2]} + \alpha_k^{[3]} \}}{[1 + 2(K - 1)] P_{\text{max}} \{ r_k^{[1]} + \alpha_k^{[1]} r_k^{[2]} + \alpha_k^{[3]} \}} \right) \\
\geq \log \left( \frac{P_{\text{max}} \{ r_k^{[1]} + \alpha_k^{[1]} r_k^{[2]} + \alpha_k^{[3]} \}}{[1 + 2(K - 1)] P_{\text{max}} \{ r_k^{[1]} + \alpha_k^{[1]} r_k^{[2]} + \alpha_k^{[3]} \} - (d_k^{[1]} + d_k^{[2]})} \right) \\
\geq (d_k^{[1]} + d_k^{[2]}) \log(P) - \log(2K).
\]

From the above, and since \(d \in \mathcal{P}_{\text{id}}\), the achievable rate region, as specified through \((81)\) and \((82)\), contains the region given by all rate tuples \(\bar{R} \in \mathbb{R}_+\) that satisfy

\[
\sum_{s_i \in (l_i)} \bar{R}_i^{[s_i]} \leq \max \left\{ 0, \alpha_i^{[l]} \log(P) - \log(2K) \right\}, \ (l, i) \in \mathcal{K}
\]

\[
\sum_{j \in (m)} \sum_{s_{ij} \in (l_{ij})} \bar{R}_{ij}^{[s_{ij}]} \leq \max \left\{ 0, \sum_{j \in (m)} \left[ (\alpha_{ij}^{[l_{ij}]} - \alpha_{ij}^{[l_{ij} - 1]}) \log(P) - \log(2K) \right] \right\}, \ \forall l_{ij} \in \{1, 2\}, \ (i_1, \ldots, i_m) \in \Sigma(\{K\}), \ m \in \{2 : K\}.
\]

At this point, it can be easily shown that each of the rate constraints in \((87)\) and \((88)\) is within at most \(\log(20K)\) bits (per dimension) of its corresponding outer bound in \((79)\) and \((80)\) (e.g. see the proof of \([\text{6, Theorem 4}]\)). This completes the proof of the theorem.

6 Concluding Remarks

In this paper, we considered the TIN optimality problem for the Gaussian IMAC. We derived a TIN-achievable GDoF region through a novel application of the potential theorem approach in \([\text{6,9}]\). Moreover, we proved the optimality of this GDoF region for a non-trivial regime of parameters by building upon the genie-aided converse arguments in \([\text{7, 6}]\) and \([\text{12}]\). An interesting extension is to consider the more general scenario where each MAC consists of an arbitrary number of users.

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