Asymptotic solutions of stratified shear flow with horizontal eddy coefficient of turbulent viscosity

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Abstract. We consider linearized equations of perturbed mean flow under the Boussinesq approximation with the presence of eddy coefficients of turbulent viscosity but with the absence of turbulent diffusivity. By including both the horizontal and vertical eddy viscosity, an equation for the vertical velocity perturbation leads to a fourth-order Taylor-Goldstein equation. In the absence of vertical eddy viscosity, a modified Taylor-Goldstein equation is obtained, an extended version of the inviscid case but at the same order. Under an assumption of slowly varying background horizontal velocity and Brunt-Väisälä frequency, we present asymptotic solutions using the WKB method for the latter, where both geometrical and physical optics approximations can be obtained. Furthermore, using the method of Frobenius and again by means of asymptotic approximations, we also investigate the behavior of a gravity wave near a critical level, a height where the background velocity is equal to the horizontal phase speed. The vertical velocity perturbation, which now consists of upward-moving and downward-moving waves, takes a different form depending whether it locates above or below the critical level, as well as whether the wind shear near the critical level takes a positive or negative value.

1. Introduction

We consider dynamics of internal gravity waves in the atmosphere for continuously stratified parallel flows in the presence of horizontal eddy coefficients of turbulent viscosity. Under the Boussinesq approximation, we derive a vertical velocity perturbation equation from the two-dimensional Euler equations as the governing model. By including both the horizontal and vertical eddy viscosity, an equation for the vertical velocity perturbation is given by a fourth-order differential equation. On the other hand, in the absence of vertical eddy viscosity, it is modeled by a second-order ordinary differential equation, an extended version of the classical Taylor-Goldstein equation for the inviscid case \[1\]–[3]. The latter governs the dynamics of Kelvin-Helmholtz instability, which in general refers to the instability for continuous distribution of velocity and density \[1\]–[5].

The study of flow with viscosity has a long history, starting from the Orr-Sommerfeld equation, an eigenvalue equation that models two-dimensional linear stability problem for nearly parallel viscous flows in a straight channel or a boundary layer \[6\]–[8]. An extended version of Taylor-Goldstein equation to include the effects of constant viscosity and diffusivity is modeled by a sixth-order differential equation \[9\]. Hazel solved numerically the higher-order equation for the vertical velocity of internal gravity waves with kinematic viscosity and thermal conductivity \[10\]. Fast forward a few decades later, Smyth et al. investigated properties of shear instabilities with small-scale turbulence represented by vertical eddy viscosity and diffusivity coefficients \[11\].

Liu et al. examined an extended Taylor-Goldstein equation including the effects of small-scale turbulence on the stability of stratified shear flows \[12\]. Thorpe et al. investigated the marginal stability of a stably stratified shear flow with ambient turbulence represented by vertical
small eddy viscosity and diffusivity [13]. In some circumstances, they found that viscosity amplifies instability. Li et al. further examined the effects of vertically varying eddy viscosity and diffusivity on the Kelvin-Helmholtz instability of a stratified shear flow [14].

In the vicinity of critical level, i.e. a level where the mean fluid velocity is equal to the horizontal phase velocity, an internal gravity wave flow is unstable [15–17]. The behavior of both 2D and 3D internal gravity wave packets approaching a critical level has been examined both analytically and numerically [18–20]. Shear instability is responsible for maintaining deep-cycle turbulence due to an extraction of energy from the mean flow near the critical level through the interaction of mean shear and Reynolds stress [21].

In this article, we propose an asymptotic solution for an extended Taylor-Goldstein equation using WKB approximation under the assumption of slowly varying background horizontal velocity and Brunt-Väisälä frequency. We also investigate the behavior of internal gravity wave near the critical level by employing Frobenius’ method and present the solution up to the first-order which is composed by the combination of upward-moving, downward-moving, and oscillation waves with complex-valued wavenumber.

We organize this article as follows. Section 2 presents a derivation of mathematical modelling for vertical velocity perturbation with eddy coefficients of turbulent viscosity. Section 3 focuses on an extended Taylor-Goldstein equation in the presence of horizontal coefficient eddy viscosity and derives the WKB approximation for its solution. Section 4 presents the solution near a critical level for different values of wind shears. Chapter 5 draws a conclusion from our discussion.

2. Mathematical modeling

Consider the Navier-Stokes momentum equation for an incompressible flow of the Newtonian fluid [22]

\[
\frac{\partial \tilde{u}}{\partial t} + (\tilde{u} \cdot \nabla) \tilde{u} = -\frac{1}{\rho_0} \nabla \tilde{p} + \tilde{g} e_z + \frac{\partial}{\partial x} \left( \tilde{a}_H \frac{\partial \tilde{u}}{\partial x} \right) + \frac{\partial}{\partial z} \left( \tilde{a}_T \frac{\partial \tilde{u}}{\partial z} \right).
\]

The fluid motion satisfies this equation and the continuity equation \( \nabla \cdot \tilde{u} = 0 \). For each quantity, we write \( q = Q + \epsilon q' \), where \( \epsilon \ll 1 \) is a small parameter, \( Q \) denotes a background state, undisturbed flow, taken to be steady of slowly varying and horizontally uniform, but varying in the vertical direction, of the corresponding boundary layer; and \( \epsilon q' \) is a first-order perturbation. Assume that the background flow is in the hydrostatic balance.

The flow velocity \( \tilde{u} = (U(z,t) + \epsilon u, 0, \epsilon w) \) is the velocity vector, \( \tilde{\theta} = \epsilon g \tilde{\theta}' / \Theta \) is the potential temperature, \( g \) is the gravitational acceleration, \( \tilde{p} \) is the pressure, \( \rho_0 \) is the reference density, \( \tilde{a}_H(z,t) \) and \( \tilde{a}_T(z,t) \) are the horizontal and the vertical eddy coefficients of turbulent viscosity, respectively. The continuity equation for potential temperature is given by

\[
\frac{\partial \tilde{\theta}}{\partial t} + (\tilde{u} \cdot \nabla) \tilde{\theta} = \frac{\partial}{\partial x} \left( \tilde{k}_H \frac{\partial \tilde{\theta}}{\partial x} \right) + \frac{\partial}{\partial z} \left( \tilde{k}_T \frac{\partial \tilde{\theta}}{\partial z} \right)
\]

where \( \tilde{k}_H(z,t) \) and \( \tilde{k}_T(z,t) \) are the horizontal and the vertical eddy coefficients of turbulent diffusivity, respectively. After dropping the prime signs, the linearized equations with the presence of eddy viscosity and diffusivity terms read

\[
\left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \theta + w \frac{d\Theta}{dz} = K_H \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial}{\partial z} \left( K_T \frac{\partial \theta}{\partial z} + k_T \frac{d\Theta}{dz} \right) \tag{1}
\]

\[
\left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) u + w \frac{dU}{dz} = -\frac{1}{\rho_0} \frac{\partial p}{\partial x} + A_H \frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial z} \left( A_T \frac{\partial u}{\partial z} + a_T \frac{dU}{dz} \right) \tag{2}
\]
\[(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x}) w = -\frac{1}{\rho_0} \frac{\partial p}{\partial z} + \frac{\theta}{\Theta} + A_H \frac{\partial^2 w}{\partial x^2} + \frac{\partial}{\partial z} \left( A_T \frac{\partial w}{\partial z} \right) \]  
(3)

\[
\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0.
\]  
(4)

Eliminate the pressure terms by subtracting \( \partial_z \) of (2) from \( \partial_x \) of (3). Applying a derivative operator \((\frac{d}{dt} + U_B \frac{\partial}{\partial x}) \frac{\partial}{\partial x}\) to this equation and using (11) and (4), in the absence of diffusivity, we can eliminate \( u \) and \( \theta \) to obtain a single equation for \( w \):

\[
\left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right)^2 \nabla^2 w + N^2 \frac{\partial^2 w}{\partial x^2} - \frac{d^2 U}{dx^2} \left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \frac{\partial w}{\partial x} = \left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \left[ \frac{\partial}{\partial z} \left( A \frac{\partial w}{\partial x} \right) + A_H \frac{\partial^4 w}{\partial x^4} + \frac{\partial^2}{\partial z^2} \left( A_T \frac{\partial^2 w}{\partial z^2} \right) \right]
\]

where \( \nabla^2 = \partial_x^2 + \partial_z^2 \) is a Laplacian operator, \( N^2 = g \frac{d}{dt} \ln \Theta_B \) is the square of the Brunt-Väisälä (buoyancy) frequency, and \( A = A_H + A_T \). Since for shallow disturbances the variation of \( N^2 \) with height is not significant, it is assumed to be constant even though it cannot be exactly constant when the density is also constant (23). Substituting \( w(x, z, t) = \tilde{w}(z)e^{ik(x-ct)} \), we obtain a fourth-order ODE in \( \tilde{w} \):

\[
\tilde{w}'' + \left[ \frac{N^2}{(U-c)^2} - \frac{U''}{U-c} - k^2 \right] \tilde{w} = \frac{1}{i k (U-c)} \left[ \frac{d^2}{dz^2} \left( A_T \tilde{w}'' \right) - k^2 \frac{d}{dz} \left( A \tilde{w}'' \right) - k^4 A_H \tilde{w} \right].
\]

Note that the absence of the right-hand side leads to the classical Taylor-Goldstein equation (12), and for unidirectional flow, the Miles-Howard semicircle theorem is established (24, 25). In the absence of stratification, Rayleigh’s inflexion-point theorem for bounded, parallel homogeneous shear flow is established (26).

3. An extended Taylor-Goldstein equation

Let \( U(z) \) be the background velocity, \( A_H(z) \) be the horizontal eddy viscosity, where both are real-valued functions of the vertical coordinate \( z \). Let \( c = c_r + ic_i \in \mathbb{C} \) with \( c_i > 0 \) for unstable situation. We write \( U - c = \rho_1e^{i\phi_1} \) and \( U - c = -ikA_H = \rho_2e^{i\phi_2} \), where both the amplitudes \( \rho_{1,2}(z) \) and the phases \( \phi_{1,2}(z) \) are real-valued functions. Consider an extended Taylor-Goldstein equation for the vertical velocity profile \( \tilde{w} \) in the absence of \( A_T \) but only the presence of \( A_H \)

\[
\tilde{w}'' + Q_1(z)\tilde{w}' + Q_0(z)\tilde{w} = 0
\]  
(5)

where both \( Q_0, Q_1 \in \mathbb{C} \), given as follows:

\[
Q_0(z) = N^2 \rho_1^{-1} \rho_2^{-1} e^{-i(\phi_1 + \phi_2)} - U'' \rho_2^{-1} e^{-i\phi_2} - k^2
\]  
(6)

\[
Q_1(z) = -ikA_H' \rho_1^{-1} e^{-i\phi_1}.
\]  
(7)

Applying the following Liouville transformation by introducing a new variable \( \tilde{w} \) and an integration factor

\[
\tilde{w} = \exp \left( -\frac{1}{2} \int_{z}^{z} Q_1(\zeta) d\zeta \right) \tilde{w}
\]

\[
\tilde{w}'' + Q_2(z) \tilde{w} = 0 \quad \text{where} \quad Q_2(z) = Q_0 - \frac{Q_0^2}{4} - \frac{Q_1'}{2}.
\]  
(9)
Rescaling the vertical coordinate $z$ as $Z = \epsilon z$, where $\epsilon \ll 1$ is a small dimensionless parameter, then (9) reads

$$e^2 \ddot{w} + Q_\epsilon(Z) \dot{w} = 0$$

(10)

where the complex-valued potential term $Q_\epsilon$ is given by

$$Q_\epsilon(Z) = N^2 \rho_1^{-1} \rho_2^{-1} e^{-i(\phi_1 + \phi_2)} - k^2 - e^2 \left( \rho_2^2 \rho_3 e^{i(-2\phi_2 + \phi_3)} + \rho_2^{-1} \rho_4 e^{i(-\phi_2 + \phi_4)} \right)$$

(11)

$$\rho_3 e^{i\phi_3} = \frac{1}{4} k^2 (\hat{A}_H)^2 + \frac{1}{2} ik \hat{U} \hat{A}_H \quad \text{and} \quad \rho_4 e^{i\phi_4} = \hat{U} - \frac{1}{2} ik \hat{A}_H.$$  

(12)

where the differentiation with respect to $Z$ has been replaced by a dot. An approximate solution to (11) when $Q_\epsilon$ is slowly varying can be given using the WKB method, developed by Wentzell, Kramers and Brillouin, as well as Jeffreys, herewith also called WKBJ approximation [27,28]. It is also known as the Liouville-Green approximation [29]. This method has been used for internal gravity waves [30], atmospheric acoustic-gravity wave [31,32] and mountains waves [33,39].

Using the WKB approximation, the solution to (10) is given by

$$\ddot{w}(Z) = \ddot{w}_0 \exp \left( i \int_0^Z S(\zeta, \epsilon) \, d\zeta \right), \quad S(\zeta; \epsilon) = \epsilon^{-1} S_{-1}(\zeta) + S_0(\zeta) + \epsilon S_1(\zeta) + \epsilon^2 S_2(\zeta) + \ldots$$

(13)

Inserting (13) to (10), the series $S$ satisfies Riccati equation $\dot{S} + S^2 + \epsilon^{-2} Q_\epsilon = 0$ and we obtain the following relationship

$$S_{-1}^2 = N^2 \rho_1^{-1} \rho_2^{-1} e^{-i(\phi_1 + \phi_2)} - k^2$$

(14)

$$S_0 = \frac{i}{2} \frac{S_{-1}}{S_{-1}}$$

(15)

$$S_1 = \frac{1}{4} \frac{S_{-1}^2 - 4 S_{-1}^2}{S_{-1}} + \frac{T_2}{2 S_{-1}}$$

(16)

$$S_2 = \frac{i}{8 S_{-1}^2} \left( S_{-1} S_{-1}^2 - 4 S_{-1} S_{-1}^2 \right) - \frac{S_{-1}^2 (S_{-1} + i) T_2}{2 S_{-1}} + \frac{\dot{T}_2}{2 S_{-1}}$$

(17)

$$T_2 = \rho_2^{-2} \rho_3 e^{i(-2\phi_2 + \phi_3)} + \rho_2^{-1} \rho_4 e^{i(-\phi_2 + \phi_4)}.$$  

(18)

Equations (14) and (15) are called eikonal and transport equations, respectively. Retaining only the first term and the first two terms in the WKB series (17), we obtain geometrical and physical optics approximations. After solving the transport equation (15), the latter reads

$$\ddot{w}(Z) = \frac{\ddot{w}_0}{\sqrt{S_{-1}}} \exp \left( \pm \frac{i}{\epsilon} \int_0^Z S_{-1}(\zeta) \, d\zeta \right).$$

(19)

Applying back the Liouville transformation (8), restoring the normal mode factor, using the continuity equation (4), we obtain the horizontal and vertical velocity fields, respectively given as follows:

$$u(x, k, t) = \mp u_0 \left( \sqrt{S_{-1}} + \frac{i}{2} \frac{Q_1}{\sqrt{S_{-1}}} \right) \exp \left( i k (x - ct) \pm \int_0^x i S_{-1}(\zeta) - \frac{1}{2} Q_1(\zeta) \, d\zeta \right)$$

(20)

$$w(x, k, t) = \frac{w_0}{\sqrt{S_{-1}}} \exp \left( i k (x - ct) \pm \int_0^x i S_{-1}(\zeta) - \frac{1}{2} Q_1(\zeta) \, d\zeta \right)$$

(21)

where the local vertical wavenumber is given by $\text{Re} \{i S_{-1} - \frac{1}{2} Q_1 \}$. For positive local vertical wavenumber, the upper and the lower signs represent waves with upward and downward phase propagation, respectively.
4. Behavior near the critical level

There are at least two possibilities where the WKB approximation becomes invalid, i.e. at a turning point or at a critical level. A turning point is the point where \( z = z_0 \) where \( S_{-1} \) vanishes, i.e. \( N^2(z_0) = k^2 \left[ U(z_0) - c \right] \left[ U(z_0) - c - ikA_H(z_0) \right] \). At this incidence, the WKB solution \( 19 \) becomes singular. A critical level is a height \( z = z_c \) at which \( U \) is equal to the horizontal phase speed \( c \). When this height is reached, the denominator of \( S_{-1} \) becomes singular and hence the WKB solution \( 19 \) vanishes. In this section, we will focus on the latter, i.e. examining the behavior of the vertical velocity profile near the critical level. We follow an analysis presented in \([15,40]\).

Denoting \( \zeta = z - z_c \), we proceed with Taylor-expanding the background wind speed \( U \) as well as \( U + ikA_H \) up to second-order terms

\[
U(z) - c \approx a_1 \zeta + \frac{1}{2} a_2 \zeta^2, \quad \text{where} \quad a_1 = U'(z_c) \quad \text{and} \quad a_2 = U''(z_c) \quad (22)
\]

\[
U(z) - ikA_H(z) - c \approx b_1 \zeta + \frac{1}{2} b_2 \zeta^2, \quad \text{where} \quad b_1 = a_1 - i\alpha_1 \quad \text{and} \quad b_2 = a_2 - i\alpha_2. \quad (23)
\]

Here, \( \alpha_1 = kA_H'(z_c) \) and \( \alpha_2 = kA_H''(z_c) \). Upon substitution to the extended Taylor-Goldstein equation \( 19 \), we obtain another Taylor-Goldstein equation with a regular singular point at \( \zeta = 0 \):

\[
\ddot{\tilde{w}} + \left( \frac{Q_{22}}{\zeta^2} - \frac{Q_{21}}{\zeta} + Q_{20} \right) \tilde{w} = 0 \quad (24)
\]

where

\[
Q_{22} = \frac{N^2}{a_1 b_1} + \frac{1}{2} \left( \frac{a_1}{b_1} - 1 \right) \left( \frac{a_1}{b_1} - 2 \right) \quad (25)
\]

\[
Q_{21} = \frac{1}{2} \left[ \frac{N^2}{a_1 b_1} \left( \frac{a_2 + b_2}{a_1 b_1} \right) + b_2 \left( \frac{a_1}{b_1} - 1 \right) \left( \frac{a_1}{b_1} - 2 \right) \right] + \frac{a_2}{b_1} + b_2 \frac{2 b_2}{b_1} \quad (26)
\]

\[
Q_{20} = \frac{1}{4} \left[ \frac{N^2 a_2}{a_1^3} b_2 b_1^2 + \frac{1}{2 b_1^2} b_1 \left( \frac{a_1}{b_1} - 1 \right) \left( \frac{a_1}{b_1} - 2 \right) + \frac{a_2 b_2}{b_1^2} + 2 \frac{b_2}{b_1^2} \right] - k^2. \quad (27)
\]

We seek a series solution near the critical level by employing the method of Frobenius by expanding \( \tilde{w} \) as follows

\[
\tilde{w}(z) = \tilde{w}(z_c + \zeta) = \sum_{n=0}^{\infty} B_n \zeta^{n+\lambda}, \quad B_n \in \mathbb{C}. \quad (28)
\]

The indicial equation of \( 24 \) is given by \( \lambda(\lambda - 1) + Q_{22} = 0 \) with solution \( \lambda = \frac{1}{2} \pm \rho_c e^{i\phi_c} \), where

\[
\rho_c = \frac{4}{\sqrt{\text{Re}^2 \left( \frac{1}{4} - Q_{22} \right) + \text{Im}^2 \left( \frac{1}{4} - Q_{22} \right)}} \quad (29)
\]

\[
\phi_c = \frac{1}{2} \tan^{-1} \left[ \frac{\text{Im} \left( \frac{1}{4} - Q_{22} \right)}{\text{Re} \left( \frac{1}{4} - Q_{22} \right)} \right] \quad (30)
\]

\[
\text{Re} \left( \frac{1}{4} - Q_{22} \right) = \frac{1}{2} \frac{a_1^2}{|b_1|^2} (a_1^2 - a_1^2) + \frac{1}{2 |b_1|^2} \left( \frac{3}{2} a_1^2 - N^2 \right) - \frac{3}{4} \quad (31)
\]

\[
\text{Im} \left( \frac{1}{4} - Q_{22} \right) = \frac{\alpha_1}{|b_1|^2} \left( \frac{3}{2} a_1^2 - N^2 a_1 - \frac{3}{4} a_1^3 |b_1|^2. \quad (32)
\]
The solution of near the critical level is composed by the combination of not only upward-moving and downward-moving waves, but also oscillation waves, given as follows:

\[
\hat{w}(z) = \sqrt{|z - z_c|} \sum_{n=0}^{\infty} |z - z_c|^n \left( B_{n+} e^{k(z)} + B_{n-} e^{-k(z)} \right) \quad B_{n+}, B_{n-} \in \mathbb{C}. \tag{33}
\]

The complex-valued function \( k(z) = \rho_v e^{i\phi_v} \ln |z - z_c| \) acts like a vertical wavenumber. The corresponding coefficients \( B_{n+}, B_{n-} \) for positive wind shear \( (a_1 > 0) \), negative wind shear \( (a_1 < 0) \), as well as above and below the critical level indicated by the + and − superscripts, are given as follows respectively:

\[
B_{n+} = \begin{cases} 
B_{n+}^+, & \text{for } a_1 \neq 0, z > z_c \\
-iB_{n+}^+, & \text{for } a_1 > 0, z < z_c \\
iB_{n+}^+, & \text{for } a_1 < 0, z < z_c.
\end{cases}
\]

\[
B_{n-} = \begin{cases} 
B_{n-}^+, & \text{for } a_1 \neq 0, z > z_c \\
-iB_{n-}^+, & \text{for } a_1 > 0, z < z_c \\
iB_{n-}^+, & \text{for } a_1 < 0, z < z_c.
\end{cases}
\]

Employing the continuity equation (4), applying the Liouville transformation (8) and restoring the normal mode, the horizontal and the vertical velocity fields can be expressed follows, respectively:

\[
\begin{align*}
\hat{u}(x, z, t) &= \exp \left( ik(x - ct) - \frac{1}{2} \int_0^z Q_1(\zeta) \, d\zeta \right) \cdot \left\{ \sum_{n=0}^{\infty} |z - z_c|^{n+\frac{1}{2}} \left( A_{n+}(z) e^{k(z)} + A_{n-}(z) e^{-k(z)} \right) \right. \\
&+ (z - z_c) \sum_{n=1}^{\infty} n |z - z_c|^{n-\frac{3}{2}} \left( B_{n+} e^{k(z)} + B_{n-} e^{-k(z)} \right) \right\} \\
\hat{w}(x, z, t) &= \exp \left( ik(x - ct) - \frac{1}{2} \int_0^z Q_1(\zeta) \, d\zeta \right) \cdot \sum_{n=0}^{\infty} |z - z_c|^{n+\frac{1}{2}} \left( B_{n+} e^{k(z)} + B_{n-} e^{-k(z)} \right)
\end{align*}
\]

where

\[
A_{n+}(z) = \frac{iB_{n+}}{2k} \left( \frac{1 + 2 \rho_v e^{i\phi_v}}{z - z_c} - Q_1(z) \right) \quad \text{and} \quad A_{n-}(z) = \frac{iB_{n-}}{2k} \left( \frac{1 - 2 \rho_v e^{i\phi_v}}{z - z_c} - Q_1(z) \right).
\]

5. Conclusion

We considered a modified Taylor-Goldstein equation describing a model for complex-valued vertical velocity perturbation in stratified shear flows with horizontal eddy coefficient of turbulent viscosity. Under an assumption of slowly varying background horizontal velocity and Brunt-Väisälä frequency, we implemented an asymptotic approach using the WKB method to obtain an approximate solution for the equation, where indeed that the method has also been successfully applied in the lower troposphere and boundary layer.

Furthermore, we also seek a series solution for the complex-valued vertical velocity perturbation near the critical level by employing Frobenius’ method. Due to complex roots of the indicial equation of Taylor-Goldstein equation, its solution is composed as a linear combination of upward-moving, downward-moving, and oscillating waves, with their complex-valued coefficients depend on the values of wind shear \( U'(z_c) \) as well as whether the location is above or below the critical level \( z_c \).
For the future work, it is also worth to consider the behavior of the gravity wave near a turning point. Further, we are interested in investigating the effect of vertical eddy coefficients of turbulent viscosity where its contribution is more essential in the study of atmospheric boundary layer in comparison to the horizontal one. Additionally, we would like to investigate the case when both horizontal and vertical eddy coefficients of turbulent diffusivity are present in the model equation.

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