Globally Convergent Visual-Feature Range Estimation with Biased Inertial Measurements

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Abstract

The design of a globally convergent position observer for feature points from visual information is a challenging problem, especially for the case with only inertial measurements and without assumptions of uniform observability, which remained open for a long time. We give a solution to the problem in this paper assuming that only the bearing of a feature point, and biased linear acceleration and rotational velocity of a robot—all in the body-fixed frame—are available. Further, in contrast to existing related results, we do not need the value of the gravitational constant either. The proposed approach builds upon the parameter estimation-based observer recently developed in (Ortega et al., Syst. Control. Lett., vol. 85, 2015) and its extension to matrix Lie groups in our previous work. Conditions on the robot trajectory under which the observer converges are given, and these are strictly weaker than the standard persistency of excitation and uniform complete observability conditions. Finally, we apply the proposed design to the visual inertial navigation problem. Simulation results are also presented to illustrate our observer design.

Key words: Observers Design; Nonlinear Systems; Range Estimation; Robotics

1 Introduction

Determination of the position of feature points in three-dimensional space from the visual measurement using a single monocular camera is one of the classical problems in several fields, including machine vision, robotics, and control. The problem has an enormous practical and theoretical significance as it arises in structure from motion (SfM) [28], structure and motion (SaM) [8], visual simultaneous localization and mapping (SLAM) [18], image-based visual servoing [2,9], navigation [5] and bearing-only formation control [36], just to name a few. Furthermore, it does not fall into the canonical forms which have been comprehensively studied in the nonlinear observer community [6], thus making its solution interesting from the theoretical perspective.

With a monocular camera, estimating the position of a feature point is equivalent to estimating its range or depth. There are generally two classes of methodologies to solve such a problem in terms of their technical routes, \textit{i.e.,} the batch method [12] and observer/filter design [9,13,15]. The former approach relies on full-information optimization from a sequence of images, which can give accurate results but is usually difficult to apply in real-time. In contrast, the observer approach calculates estimates recursively or incrementally as measurements arrive, generally employing much simpler on-line computations, and in this paper we focus on the observer approach. Along this research line, significant effort has been devoted over the last three decades.

In [14], the authors adopt a change of coordinate to the inverse of depth, which has since become a popular formulation, and motivated by parameter estimation, propose a high-gain design called identifier based observer. The main drawback of this scheme is its high-gain injection and discontinuity making the observer relatively sensitive to unavoidable high-frequency measurement noise. Thereafter, many nonlinear observers were proposed using different constructive tools. A less complicated observer design based on Lyapunov analysis is proposed in [10], which guarantees global asymptotic stability (GAS) under an instantaneous observability assumption on the robot trajectory. Indeed, this design still has discontinuous functions limiting its performance. An extension to paracatadioptric camera can be found in...
In all the above works, it is assumed the linear velocity is available in the observer design. However, a very common and practically important scenario is that the robot is equipped with inertial measurement units (IMUs), which provide measurements of linear acceleration rather than velocity. In spite of intensive research efforts, we are unaware of any observer for the extension to this case. The main challenge relies on that the unknown attitude—living in the special orthogonal group—appears in the dynamics of the body-fixed velocity. To address this, we give in the paper the first affirmative answer to the problem, presenting a novel globally exponentially convergent position observer for a single feature point. The constructive tool we adopt is the recently elaborated parameter estimation-based observer (PEBO) in [27,28], in which the qualifier “active” means that the estimation and the motion imposed to the camera are co-designed, thus providing the ability to assign the transient performance of the observer. In [24] a novel I&I observer is applicable to the planar feature or surfaces.

The second contribution in the paper is applying the proposed feature position observer to the problem of visual inertial navigation. This problem arises for the scenario in Global Positioning System (GPS)-denied environment, which is concerned with fusing the information from IMUs and cameras, with the coordinates of some feature points associated Lie algebra as the set of skew-symmetric matrices

\[
\begin{bmatrix}
0 & -a_3 & a_2 \\
a_3 & 0 & -a_1 \\
-a_2 & a_1 & 0
\end{bmatrix}
\in \mathfrak{so}(3).
\]

Given a variable \(a \in \mathbb{R}^3\), we define the operator \((\cdot)_x\) as

\[
[a_x := \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \in \mathfrak{so}(3).]
\]

Given a variable \(R \in SO(3)\), we use \(|R|_f\) to represent the normalized distance on \(SO(3)\) with respect to \(I_3\) by defining \(|R|_f^2 := \frac{1}{2} \text{tr}(I_3 - R)\). For any \(x \in \mathbb{R}^3/\{0\}\), its projector is defined as

\[
\Pi_x := I_3 - \frac{1}{|x|^2} xx^T,
\]

which projects a given
vector onto the subspace orthogonal to \(x\), since it is easy to verify that \(\Pi_{x} x = 0\). The skew-symmetric operator \(\text{skew}(\cdot)\) is defined as \(\text{skew}(A) = \frac{1}{2}(A - A^\top)\) for square matrices.

Given two symmetric matrices \(A\) and \(B\) in \(\mathbb{R}^{n\times n}\), we write \(A \preceq B\), if for all \(x \in \mathbb{R}^{n}\), \(x^\top Ax \leq x^\top Bx\). When clear from context, the arguments and subscripts are omitted.

2 Model and Problem Formulation

2.1 Measurement and Motion Models

A single monocular camera provides two-dimensional images of its environment, and different types of visual features may be extracted from these images, e.g., points, lines, circles, and contours. In this paper, we are interested in the three-dimensional position estimation of point features extracted from images in the view of a single monocular camera on the mobile robots. Consider a robotic platform moving in three-dimensional space, equipped with IMUs. We assume that linear acceleration and rotational velocity in the body-fixed frame \(\{B\}\) are measurable, and the bearing of a feature point is available.

![Fig. 1. Coordinate systems of a moving robot observing a fixed feature point](image)

To be precise, in the body-fixed frame \(\{B\}\), the observed feature point is denoted by \(z \in \mathbb{R}^{3}\), satisfying

\[
z = R^\top (I z - x)
\]

with unknown position \(x \in \mathbb{R}^{3}\) of the robot in the inertial frame \(\{I\}\), and \(R \in SO(3)\) is the attitude of the body-fixed frame \(\{B\}\) with respect to the inertial frame \(\{I\}\). Here, the constant vector \(I z \in \mathbb{R}^{3}\) represents the feature position in \(\{I\}\). The time derivative of \(z\) is given by

\[
\dot{z} = -\Omega \times z - v,
\]

in which \(\Omega \in \mathbb{R}^{3}\) is the rotational velocity, and \(v \in \mathbb{R}^{3}\) is the unknown linear velocity, both in the body-fixed frame \(\{B\}\); see Fig. 1. The kinematics of the robot is given by

\[
\dot{x} = I v \\
\dot{R} = R \Omega
\]

in which \(I v\) is the velocity in the inertial frame \(\{I\}\), i.e. \(I v := Rv\). We assume the robot equipped with IMUs. Then, we have

\[
\dot{v} = -\Omega \times v + a + b_a + R^\top g,
\]

in which \(a \in \mathbb{R}^{3}\) is the “apparent acceleration” representing all non-gravitational forces in \(\{B\}\) measured by IMUs, \(g := g e_3 \in \mathbb{R}^{3}\) is the gravity vector in the inertial frame \(\{I\}\) with \(g \approx 9.8\ \text{m/s}^2\), and \(b_a \in \mathbb{R}^{3}\) is the constant sensor bias.

In this paper, we consider the spherical projection model of the camera. In this case, the output is the bearing of the feature point in the body-fixed frame \(\{B\}\), i.e.

\[
y = \frac{z}{|z|} \in S^2,
\]

for non-zero \(z\). In order to guarantee well-posedness, we make the following assumption.

Assumption 1 The origin of the robot never coincides with the feature point, i.e. \(|z(t)| \neq 0\) for all \(t \geq 0\). Besides, the angular velocity \(\Omega\) and the acceleration \(a\) guarantee all the systems state bounded over time.

Remark 1 The proposed approach is applicable to other image-based estimation problems with different projection models after appropriate calibration, e.g.

- Homogeneous normalized coordinates in pixels [9]: the projection is \(y_b = A [x_1/x_3 x_2/x_3 x_3]^\top\), with \(A \in \mathbb{R}^{3 \times 3}\) a non-singular known matrix. If \(x_3 > 0\), we may verify the mapping from the bearing (4) to \(y_b\) is injective, with the left inverse \(y = A^{-1} y_b/|A^{-1} y_b|\).

- Paraboloid mirror [13]: the projection of \(x\) onto a paraboloid mirror with its focus at the origin is given by \(y_p = 2ax/(−x_3 + |x|)\), in which \(a \in \mathbb{R}\) is the constant known distance between the focal point and the vertex of the paraboloid. The mapping from the bearing to \(y_p\) is also injective, and its left inverse is \(y = y_p/(2a + y_p, 3)\).

Remark 2 Although in this work we only consider the measurement bias \(b_a\) of linear acceleration, the proposed method can be extended in a straightforward way to the case with biased rotational velocity \(\Omega^v\) with gyro bias \(b_{\Omega} \in \mathbb{R}^{3}\).

Remark 3 It is well known that the motion coordinate selection plays an important role in observers design [3,6,35]. A “canonical” form in the related field is to include the inverse of depth [9,15,27], and then the dynamics of measurable state is linear in the unknown state. As a result, it is relatively simple to obtain a locally or semi-globally asymptotically convergent estimation. In this paper, we propose to select the coordinate of range rather than its inverse, to get a novel parameterization, a key benefit of which is its applicability to deal with biased inertial measurements.
At the end of the subsection, let us recall the following.

**Definition 1** Given a bounded signal $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}^n$, it is of 1) persistency of excitation (PE), if

$$
\int_t^{t+T} \phi(s)\phi^T(s)ds \geq \delta I_n, \quad \forall t \geq 0 \quad (5)
$$

for some $T > 0$ and $\delta > 0$ [25]; 2) interval excitation (IE) if there exist $t_0 \geq 0$ and $t_c \geq 0$ such that

$$
\int_{t_0}^{t_0+t_c} \phi(s)\phi^T(s)ds \geq \delta I_n \quad (6)
$$

for some $\delta > 0$.

Note that the IE condition is called “exciting over a finite time interval” in [29, Definition 3.1, pp. 108].

### 2.2 Problem Formulation

In this paper, we are interested in the position/range estimation problem as follows, for the cases with and without the availability of the linear velocity $v$.

**Problem 1** (Feature range observation) Consider the systems dynamics (1)-(3), with the rotational velocity $\Omega$ and the bearing output $y$ available.

**P1** If the linear velocity $v$ is measurable, design an observer

$$
\dot{\zeta} = N(\zeta, \Omega, v, y)
$$

and

$$
\dot{\hat{z}} = H(\zeta, \Omega, v, y),
$$

with the observer state $\zeta \in \mathbb{R}^{n_c}$ and mappings $N$ and $H$ with proper dimensions, such that

$$
\lim_{t \rightarrow \infty} \| \hat{z}(t) - z(t) \| + |\hat{v}(t) - v(t)| = 0
$$

for a class of robot trajectories.

**P2** If only the acceleration $a$ is measurable with $b_a$ and $g$ unknown, design an observer

$$
\dot{\zeta} = N(\zeta, \Omega, a, y)
$$

and

$$
(\hat{z}, \hat{v}) = H(\zeta, \Omega, a, y)
$$

with proper dimensions, such that

$$
\lim_{t \rightarrow \infty} \| \hat{z}(t) - z(t) \| + |\hat{v}(t) - v(t)| = 0
$$

for a class of robot trajectories.

### 3 Position Observer with Unbiased Velocity Measurement

In this section, we give a solution to Problem 1 (P1) under the IE assumption on the robot trajectory. The main tools we use are the PEBO approach [22] and the generalized (G)PEBO [21], in which the basic idea is to translate parameter estimation into on-line constant parameter estimation. In general, the latter is more tractable than estimating time-varying systems state. A geometric interpretation to PEBOs may be found in [34].

#### 3.1 Generation of Linear Regression Models

To start with, we consider the problem P1—the case with the linear velocity $v$ available as a motivating design, and then in the next section, we will focus on P2 without the information of $v$. In the case of P1, it will be shown that the position estimation problem is equivalent to identification of a constant scalar parameter.

In the following we present a parameterization to the unknown range $|z|$, and show how to generate a linear regression equation (LRE).

**Proposition 1** Consider the dynamics (1) and the dynamic extension

$$
\dot{\xi} = -y^Tv
$$

with arbitrary $\xi(0) \in \mathbb{R}$. Then, the position $z$ satisfies

$$
z(t) = [\xi(t) + \theta]y, \quad \forall t \geq 0
$$

with the unknown constant $\theta \in \mathbb{R}$ satisfying the LRE

$$
y_k(t) = \phi(t)\theta + \epsilon_t
$$

with measurable signals $y_k \in \mathbb{R}^3$ and $\phi \in \mathbb{R}^3$ defined as

$$
\phi := G_1[y] + \alpha G_2(\Omega_t y) \quad y_k := -\alpha G_2(\Pi_y v) - G_2(y^Tv\phi) - \phi\xi
$$

and the stable filters $G_1(p)[\cdot] := \frac{\alpha p}{p+\alpha}[\cdot]$ and $G_2(p)[\cdot] := \frac{1}{p+\alpha}[\cdot]$, the adaptation gain $\alpha > 0$.

**Proof 1** From $|z|^2 = z^Tz$ and (1), we have

$$
|z(t)| = z^T(-\Omega_t z - v) = z^Tv,
$$

and thus the bearing of $z$ in the body-fixed frame $\{B\}$ satisfies the dynamics

$$
\dot{|z|} = -y^Tv, \quad (11)
$$

due to Assumption 1. Hence, the dynamics in the above coordinate satisfies the requirement in PEBO [22] due to the availability of its time derivative, and thus

$$
|z(t)| - \xi(t) = \theta, \quad \forall t \geq 0,
$$

in which the constant scalar parameter $\theta$ is defined as $\theta :=$
is solvable at a moment \( t_0 \). Hence, we have verified the algebraic relation (8).

In the remainder of the proof, we show how to derive the perturbed LRE (9). By calculating the time derivatives of (4), we obtain

\[
\dot{y} = \frac{1}{|z|^2} \left[ (-\Omega_x z - v) |z| + z y^\top v \right]
\]

\[
= -\Omega_x y - \frac{1}{|z|} v + \frac{1}{|z|} y y^\top v
\]

\[
= -\Omega_x y - \frac{1}{|z|^2} \Pi y v,
\]

equivalently,

\[r(\dot{y} + \Omega_x y) = -\Pi y v,\]

in which we have defined the range variable

\[r = |z|\]

for convenience. Noting the unavailability of the derivative \( \dot{y} \), we thus apply the stable filter \( r \Phi(p) \) to both sides of (12), with \( \alpha > 0 \) to circumvent such a difficulty. For the first term \( r \dot{y} \), we utilize the swapping lemma for transfer functions—see Lemma 1 in Appendix—and then obtain

\[
\frac{\alpha}{p + \alpha} [r \dot{y}] = r \frac{\alpha p}{p + \alpha} [y] - \frac{1}{p + \alpha} \left[ \dot{y} \frac{\alpha}{p + \alpha} [y] \right] + \epsilon_t
\]

\[
= r \frac{\alpha p}{p + \alpha} [y] + \frac{1}{p + \alpha} \left[ y^\top v \frac{\alpha}{p + \alpha} [y] \right] + \epsilon_t,
\]

in which the exponentially decaying term \( \epsilon_t \) arises from the initial conditions of stable filters. Therefore, it yields

\[
r \frac{\alpha p}{p + \alpha} [y] + \frac{1}{p + \alpha} \left[ y^\top v \frac{\alpha}{p + \alpha} [y] \right] + \frac{1}{\alpha} \left[ r \Omega_x y \right]
\]

\[
= -\frac{\alpha}{p + \alpha} [\Pi y v] + \epsilon_t,
\]

then

\[r \phi + G_2 [y^\top v \phi] = -\alpha G_2 [\Pi y v] + \epsilon_t\]

Substituting \( r = \xi + \theta \), we have the LRE (9). It should be underlined here that both \( y \) and \( \phi \) are measurable signals. It completes the proof. \( \blacksquare \)

Thanks to the algebraic relation (8), we have translated, via designing the dynamic extension (7), the estimation of the position \( z \) into the on-line consistent identification of \( \theta \) from the LRE (9). Note that \( \theta \) is a scalar constant, and \( \phi \) is a column vector, and thus the least squares problem is solvable at a moment \( t_0 \) if \( \phi(t_0) \) is non-zero. It is clear that the identifiability of the regressor (9) is equivalent to the following condition.

**A1** The existence of a moment \( t_0 \) such that at least one of the elements in \( \phi \) is non zero, i.e.,

\[\exists t_0 \geq 0, \exists j \in \{1, 2, 3\}, \phi_j(t_0) \neq 0.\]

If the above is satisfied, the feature position estimation problem **P1** is solvable.

### 3.2 Position Observer Design

Based on the linear regression model (9), in this section we design two globally exponentially convergent position observers to address the problem **P1**, i.e., a gradient observer and a PEBO. In the former a PE condition is imposed to guarantee convergence, and in contrast for the latter we relax the requirement significantly.

We have the feature position observer based on gradient descent as follows.

**Proposition 2 (Gradient position observer)** Considering the dynamics (1) with input \( v \) and the bearing \( y \) available, the gradient observer

\[\hat{r} = -y^\top v - \gamma \phi^\top \left( \phi \hat{r} + G_2 [y^\top v \phi] + \alpha G_2 [\Pi y v] \right)\]

\[\hat{z} = \hat{r} y\]

with \( \gamma > 0 \) and \( \phi \in \mathbb{R}^3 \) defined in (10), provides a globally exponentially convergent estimate to the position \( z \), i.e.,

\[\lim_{t \to \infty} |\hat{z}(t) - z(t)| = 0 \quad \text{(exp.)}\]

if the vector \( \phi^\top \) is PE.

**Proof** Define the estimation error of range as \( \tilde{r} := \hat{r} - r \) with \( r = |z| \), the dynamics of which is given by

\[\dot{\tilde{r}} = -\gamma \phi^\top \left( \phi \tilde{r} + G_2 [y^\top v \phi] + \alpha G_2 [\Pi y v] \right)\]

Invoking (14), we have

\[\dot{\tilde{r}} = -\gamma \phi^\top \phi \tilde{r} + \psi_t.\]

From the PE condition of \( \phi^\top \), we conclude that \( \tilde{r} \to 0 \) exponentially fast [25, Chapter 2]. Invoking the algebraic relation \( z = ry \), as well as the boundedness assumption of \( y \), it completes the proof. \( \blacksquare \)

**Remark 4** The methodology of gradient observers may date back to the work [26] for general nonlinear models, which has been extended to several applications, e.g., electromechanical systems [23,33] and pose estimation on matrix Lie groups [17]. Its basic idea is to obtain a regression model of the unknown systems state, and some PE conditions are required to achieve uniform convergence.
In many cases, the PE condition required in Proposition 2 cannot be guaranteed for the given robot pose trajectories. To address this issue, in the remainder of this section, we design a PEBO which guarantees global exponential convergence even in the absence of the PE condition.

**Proposition 3 (Position PEBO)** Considering the dynamics (1) with input \( v \) and the output \( y \) defined in (4) available, the PEBO

\[
\begin{align*}
\dot{\xi} &= -y^Tv \\
\dot{y} &= \phi^Ty_k - \phi^T\phi\zeta \\
\dot{\omega} &= -\phi^T\phi\omega, \quad \omega(0) = 1 \\
\dot{\hat{\theta}} &= \gamma[(\zeta - \omega\zeta_0) - (1 - \omega)\hat{\theta}]
\end{align*}
\]

and

\[
\dot{\hat{\theta}} = (\xi + \hat{\theta})y
\]

with \( (\zeta(0) = \zeta_0, y_k, \phi \) defined in (10), and the filters \( G_1(p)[] \) and \( G_2(p)[] \) starting from zero initial conditions, guarantees the global exponential convergence (16) if the identifiability condition \( A1 \) holds.

**Proof 3** From Proposition 1, we have \( r \equiv \xi + \theta \), and the range observation becomes estimating of a scalar constant \( \theta \). The following analysis is motivated by the proof of our previous work [32, Proposition 1], where an LTV filter is designed to generate PE regressors from the ones only satisfying IE.

According to the dynamics of \( \zeta \), we have

\[
\dot{\zeta} - \dot{\theta} = \phi^T y_k - \phi^T \phi \zeta
\]

in which we have used the equation (9), and the fact that \( \phi \) is a constant. It should be underlined here that, by setting zero the initial conditions of the stable filters \( G_1(p)[] \) and \( G_2(p)[] \), the exponentially decaying term \( \epsilon_t \) disappears in (9); see Remark 8 on how to implement it. It is easy to get \( \zeta - \dot{\theta} = \omega(\zeta_0 - \theta) \) with \( \omega(t) = \exp(-\int_0^t \phi^T(s)\phi(s)ds) \), then yielding a new linear regressor model

\[
\zeta - \omega\zeta_0 = (1 - \omega)\theta.
\]

Now, define the parameter estimation error \( \hat{\theta} := \hat{\theta} - \theta \), the dynamics of which is

\[
\dot{\hat{\theta}} = -\gamma(1 - \omega)\hat{\theta}
\]

On the other hand, from the continuity of the signal \( \phi \), the condition \( A1 \), i.e. \( \phi^T(t_*\phi(t_*) > 0 \), implies the existence of a sufficiently small parameter \( \epsilon > 0 \) such that

\[
\int_{t_*}^{t_* + \epsilon} \phi^T(s)\phi(s)ds > 0.
\]

Equivalently, we conclude that \( \phi^T \) is intervally excited. The solution of \( \omega \) satisfies for \( t \in [0, t + \epsilon) \)

\[
1 - \omega(t) = 1 - \exp\left(-\int_0^t \phi^T(s)\phi(s)ds\right)
\]

\[
\geq 1 - \exp\left(-\int_{t_*}^{t_* + \epsilon} \phi^T(s)\phi(s)ds\right) > \delta_0,
\]

for some constant \( \delta_0 > 0 \). Hence, the regression \( (1 - \omega) \) in (18) is non-negative and PE. As a sequence, the dynamics (19) is globally exponentially stable. Hence, we have

\[
\lim_{t \to \infty} |\xi(t) + \hat{\theta}(t) - r(t)| = 0 \quad (\exp.).
\]

From the state boundedness, it completes the proof.

**Remark 5** The observer in Proposition 3 utilizes the dynamic extension (7), which may be criticized that such a pure integrator action is fragile such that \( \theta \) is slowly time-varying rather than being constant in practical scenarios due to sensor noise. However, the on-line identification of \( \theta \) is able to robustify the observer vis-à-vis measurement noise, thus making \( (\hat{\theta} + \xi) \) a robust estimation to the range.

**Remark 6** In the new regressor (18), the regression \( (1 - \omega) \) is within the interval \([0, 1]\), which may limit the convergence speed of the observer. A possible way to overcome this issue is to mix (18) with the original regressor \( y_k = \phi\theta \), and then obtaining a new LRE \( y_k = \phi\theta \), with \( y_k := \zeta - \omega\zeta_0 + k_p\phi^Ty_k \) and \( \phi' := (1 - \omega) + k_p|\phi|^2 \). Here, the gain \( k_p > 0 \) plays the role to make a tradeoff on the trust of the historical and current information. Hence, the last equation in the observer (17) may be modified accordingly to accelerate convergence speed.

**Remark 7** We underline that the last equation in the observer (17) is not a gradient descent search, in which we have used both the PE condition of \( (1 - \omega) \) as well as its positiveness after \( t_* + \epsilon \). The *bona fide* gradient flow is, indeed, given by

\[
\dot{\hat{\theta}} = \gamma(1 - \omega)\left[(\zeta - \omega\zeta_0) - (1 - \omega)\hat{\theta}\right],
\]

which also provides a globally exponentially convergent estimate to \( \theta \) under the IE assumption.

**Remark 8** In Proposition 3 it is necessary to carefully select the initial conditions of filters \( G_1(p)[] \) and \( G_2(p)[] \) to...
deal with the term $\epsilon_t$ for the IE case. It can be done by imple-
menting the state space models as $\dot{x} = -x + u$, $y = x$ with
$x(0) = 0$ for $G_2(p)[\cdot]$, and $\dot{x} = -\alpha x + \alpha^2 u$, $y = -x + \alpha u$
with $x(0) = \alpha u(0)$ for $G_1(p)[\cdot]$, in which $x$, $u$ and $y$ denote
the internal state, input and output in the filters, respectively.

4 Position/Velocity Observer with Biased Acceleration
Measurement

In this section, we present the main result of the paper,
 i.e., a solution to the problem P2, for which we design an
adaptive position/velocity observer with the availability of
linear acceleration $a$ rather than the velocity $v$. In Section
4.1, we will show how to generate a linear regression model,
in terms of which the observer design follows in Section 4.2.

4.1 Generation of Linear Regression Models

About the problem P2, we are interested in the estimation
of the position $z$, linear velocity $v$ and the constant sensor
bias $b_a$, which are collected in the vector

$$X := \text{col}(z, v, b_a) \in \mathbb{R}^9.$$  

We have the following.

**Proposition 4** Consider the kinematics (2), and the dynamics (1)-(3), and design the following dynamic extension

$$\begin{align*}
\dot{Q} &= Q\Omega_x \\
\dot{\xi} &= A(y, \Omega, Q)\xi + B(a), \quad \xi(0) = 0, \\
\dot{\Psi} &= A(y, \Omega, Q)\Psi, \quad \Psi(0) = I
\end{align*}$$  

(20)

with $Q(0) \in SO(3)$, and the matrix functions $A(\cdot)$ and $B(\cdot)$
are defined in (25). Then the unknown state $X$ satisfies

$$X = T(y)[\xi + \Psi \theta]$$  

(21)

and the vector $\theta$ verifies the LRE

$$y_h(t) = \psi(t) \top \theta$$  

(22)

with the filters $G_1(p)[\cdot]$ and $G_2(p)[\cdot]$ starting from zero ini-
tial conditions, $T(y)$ given in (30), and $y_h$, $\psi$ in (31).

**Proof 4** From the analysis in Section 3, defining $r = |z|$
the full dynamics is given by

$$\begin{align*}
\dot{R} &= R\Omega_x \\
\dot{r} &= -y \top v \\
\dot{v} &= -\Omega_x v + a + b_a + R \top g.
\end{align*}$$  

Define the error between $Q$ and $R$ as $E(R, Q) := RQ \top$, and we have

$$E(R, Q) := \dot{R}Q \top - RQ^{-1}\dot{Q}Q^{-1} = R\Omega_x Q \top - RQ \top Q\Omega_x = 0.$$  

Hence, there exists a constant matrix $Q_c \in SO(3)$ satisfying

$$R(t) = Q_c Q(t), \quad \forall t \geq 0,$$  

(23)

with $Q_c := R(0)Q(0) \top$. Then, we have the parameterization
to the last term in the dynamics of $v$ as

$$R(t) \top g = Q(t) \top g_c, \quad \forall t \geq 0$$

with a new constant unknown vector $g_c \in \mathbb{R}^3$ defined as $g_c := Q_c \top g$.

Now considering that $b_a$ is a constant bias, and defining the
extended state

$$\chi := \text{col}(r, v, b_a, g_c) \in \mathbb{R}^{10},$$

the plant dynamics may be compactly rewritten as

$$\dot{\chi} = A(y, \Omega, Q)\chi + B(a)$$  

(24)

with

$$A := \begin{bmatrix}
0 & -y \top & 0_3 & 0_3 \top \\
0_3 & -\Omega_x & I_3 & Q \top \\
0_6 & \ldots & \ldots & 0_{6\times3}
\end{bmatrix}, \quad B := \begin{bmatrix}
0 \\
a \\
0_6
\end{bmatrix}.$$  

(25)

Comparing the dynamics (24) with the dynamic extension
(20) and following the idea of GPEBO [21], it yields

$$\dot{\chi} - \dot{\xi} = A(y, \Omega, Q)[\chi - \xi].$$  

(26)

The matrix $\Psi(t)\Psi(s) \top$ can be viewed as the state transition
matrix for the LTV error system (26) from $s$ to $t$. Hence, we have the implications

$$\begin{align*}
\chi - \xi &= \Psi[\chi_0 - \xi(0)] \implies \chi = \xi + \Psi \chi_0 \\
\theta &= \chi_0 \implies \chi = \xi + \Psi \theta,
\end{align*}$$  

where in the second equation we have used the initialisation
$\xi(0) = 0$.

From the proof of Proposition 1, we have

$$r\phi + G_2[(\phi g \top + \alpha \Pi_y)v] = 0$$  

(28)
with \( \phi \) defined in (10) and \( G_2(p)[\cdot] = \frac{1}{\sigma + p}[\cdot] \), for the case with zero filtering initial conditions. On the other hand, from the definitions of \( \chi \) and \( X \) we have

\[
X = T(y)\chi
\]  

(29)

with

\[
T(y) := \text{diag}\{y, [I_6, 0_{6\times 3}]\},
\]  

(30)

thus verifying the algebraic equation (21).

Substituting the last equation of (27) into (28), we then have

\[
\phi T_1 \chi + G_2[(\phi y^\top + \alpha \Pi_y)T_2 \chi] = 0
\]

\[
\Rightarrow \phi T_1 (\xi + \Psi \theta) + G_2[(\phi y^\top + \alpha \Pi_y)T_2 (\xi + \Psi \theta)] = 0
\]

\[
\Rightarrow y_h = \psi^\top \theta
\]

with the new regressor output \( y_h \) and the regression matrix \( \psi \) as

\[
y_h := -\phi T_1 \xi - G_2[(\phi y^\top + \alpha \Pi_y)\xi]\n\]

\[
\psi := (\phi T_1 \Psi)^\top + G_2[(\phi y^\top + \alpha \Pi_y)\Psi]^\top,
\]  

(31)

in which we have defined the matrices

\[
T_1 := \begin{bmatrix} 1 & 0_{1\times 9} \end{bmatrix}, \quad T_2 := \begin{bmatrix} 0_{3\times 1} & I_3 & 0_{3\times 6} \end{bmatrix}.
\]  

(32)

It completes the proof. ■

The above regression model is motivated by an extension of PEBO from Euclidean space to matrix Lie groups in our previous work [32]. By gathering the unknown but constant matrices \( Q_c \in SO(3) \) and \( g \in \mathbb{R}^3 \), it provides a possible way to deal with the scenario with unknown attitude and gravitational acceleration constant, e.g., in aerospace applications.

In Appendix, we provide an alternative approach to generate linear regression models.

### 4.2 Position/Velocity Observer Design

In the above subsection we present a novel linear regression model with respect to the unknown vector \( \theta \), the estimation of which is sufficient to the one of the unknown state \( X \). Now, the remaining task is to estimate \( \theta \) stemmed from Proposition 4.

In the following proposition, we provide a globally exponentially convergent position/velocity observer with biased inertial measurements for the robot trajectory not satisfying the PE condition.

**Proposition 5** Consider the dynamics (1)-(3) under Assumption 1, and the filtered signal

\[
\hat{\theta} = \mathcal{H}[(y_h, \psi)]
\]

(33)

with \( y_h \) and \( \psi \) defined in Proposition 4, in which the filter \( \mathcal{H} \) is given by

\[
\begin{aligned}
\dot{\Phi} &= -\rho \Phi + \psi \psi^\top \\
\dot{Y} &= -\rho Y + \psi y_h \\
\dot{\zeta} &= \Delta Y - \Delta^2 \zeta \\
\dot{\omega} &= -\Delta^2 \omega, \quad \omega(0) = 1 \\
\end{aligned}
\]

(34)

with the gains \( \rho > 0 \), \( \gamma > 0 \) and \( k_p > 0 \), \( \Delta := \det\{\Phi\}, \quad Y := \text{adj}\{\Phi\} Y, \quad n = \text{dim}\{\theta\}, \) and the initial conditions \( \zeta(0) = 0_n, Y(0) = 0_n, \) and \( \Phi(0) = 0_{n \times n} \). Then, the observer consists of (20), (34), and the observation output

\[
\hat{X} = T(y) (\xi + \Psi \hat{\theta})
\]

with \( T(y) \) defined in (30), guarantees global exponential convergence

\[
\lim_{t \to \infty} |\hat{X}(t) - X(t)| = 0 \quad \text{(exp.)},
\]

from any initial guess \( \hat{\theta}(0) \in \mathbb{R}^n \), assuming that \( \psi \) is IE.

**Proof 5** According to Proposition 4, the system dynamics (1) and (3), together with the dynamic extension (20) admits the linear regressor \( y_h = \psi^\top \theta \). Following the dynamic regressor extension and mixing (DREM) technique [1], after going through an LTV filter—the first two equations in (34)—we obtain \( \frac{d}{dt} (Y - \Phi \theta) = -\rho(Y - \Phi \theta) \) with \( \rho > 0 \) from \( Y(0) - \Phi(0) \theta = 0 \), thus obtaining the Kreisselmeier’s extended linear regressor equation [16]

\[
Y = \Phi \theta
\]

(35)

with \( Y \in \mathbb{R}^n \) and \( \Phi \in \mathbb{R}^{n \times n} \). After pre-multiplying the adjugate matrix \( \text{adj}\{\cdot\} \) of \( \Phi(t) \) to the both sides of the above equation, we get the decoupled regressors

\[
Y(t) = \Delta(t) \theta
\]

with a scalar regression \( \Delta \), which is now in the same form with that in Proposition 3. Without loss of generality, we assume \( \psi \) is IE in the interval \([0, t_c]\) with \( t_c > 0 \), i.e.,

\[
\int_0^{t_c} \psi(s)(\psi(s)^\top ds \geq \delta I \quad \text{for some } \delta > 0.
\]

From the dynamics of \( \Phi \), that is

\[
\dot{\Phi} = -\alpha \Phi + \psi \psi^\top
\]
with \( \alpha > 0 \) from \( \Phi(0) = 0_{n \times n} \), thus yielding
\[
\Phi(t) = \int_0^{t} e^{-\alpha(t-s)} \psi(s)\psi(s)^T ds \\
\geq e^{-\alpha t} \int_0^{t} \psi(s)\psi(s)^T ds \\
\geq \delta e^{-\alpha t} I.
\]

From the definition \( \Delta \) being the determinant of \( \Phi \), we have the implications
\[
\psi \in \text{IE} \implies \Delta \in \text{IE} \\
\implies (1 - \omega + k_p \Delta^2) \in \text{PE},
\]
in which the second implication is similar to the proof of Proposition 2, and we omit its details. On the other hand, we can verify the unknown vector \( \theta \) satisfies the LRE
\[
\dot{\zeta} + k_p \Delta \dot{\psi} = (1 - \omega + k_p \Delta^2) \theta.
\]

Now, we obtain a new LRE (36), satisfying the PE condition, from the IE regressor (21). By defining the estimation error \( \tilde{\theta} = \theta - \hat{\theta} \), we have
\[
\hat{\theta} = -\gamma (1 - \omega + k_p \Delta^2) \tilde{\theta}.
\]

Since the regression \( (1 - \omega + k_p \Delta^2) \) is PE and non-negative for \( k_p > 0 \), the error dynamics is globally exponentially stable at the origin. By invoking the state boundedness assumption, it completes the proof. \( \blacksquare \)

In Fig. 2, we present the overall structure of the proposed position/velocity observer with biased inertial measurements.

![Fig. 2. Structure of the proposed position/velocity observer for visual feature points](image)

**5 Application to Visual-Inertial Navigation**

In this section, we apply the proposed position/velocity observer to the vision-aided inertial navigation systems.

Consider the full dynamics of the navigation problem
\[
\dot{R} = RQ \times \\
\dot{x} = Rx \\
\dot{v} = -Rx v + a + b_a + Rx \gamma
\]
with systems state \((R, x, v) \in SO(3) \times \mathbb{R}^3 \times \mathbb{R}^3\), which has been introduced in Section 2. In the visual-inertial navigation problem, there are several feature points with known positions \( \hat{z}_i \) in the inertial frame \( \{I\} \), which are captured by the camera on the robot. The camera may provide the bearing measurement of feature points in the body-fixed frame
\[
y_i = z_i |_{z_i} = R^T \frac{(\hat{z}_i - x)}{|\hat{z}_i - x|}, \quad i \in \mathcal{N} := \{1, \ldots, n\}.
\]

Here, we adopt the same notations as in Section 2, except the subindex for feature points, which is used to distinguish different points.

In this section, we are mainly interested in the following navigation problem.

**Problem 2 (Navigation observer)** Consider the dynamics (38), the measured inputs \((a, \Omega)\) and the output of bearings (39), and both the sensor bias \(b_a \in \mathbb{R}^3\) and the gravity vector \(g \in \mathbb{R}^3\) are unknown. Suppose the positions \( \hat{z}_i \) of feature points in the inertial frame \( \{I\} \) are constant and known. Design an observer
\[
\dot{\hat{\zeta}} = N(\zeta, \Omega, a, \tilde{y}, \tilde{\zeta}) \\
(\hat{R}, \hat{x}) = H(\zeta, \Omega, a, \tilde{y}, \tilde{\zeta})
\]
with \( \tilde{y} : = \text{col}(y_1, \ldots, y_n) \) and \( \tilde{\zeta} : = \text{col}(\tilde{z}_1, \ldots, \tilde{z}_n) \), to asymptotically estimate the pose states \((R, x)\).

To address the above problem, a straightforward way is to use the range observer proposed in Section 4.2 to reconstruct the position of feature points in the body-fixed frame \( \{B\} \), and then solve the localization problem with “full position measurement”, in this way providing a modular design. Before presenting our design, we need the following assumption.

**Assumption 2** There exist \( i, j \in \{1, \ldots, n\} \) such that the following holds
\[
l_i \eta_i \times l_j \eta_j \neq 0, \quad i \neq j
\]
with the definition \( l_i \eta_i := l_i z_{i+1} - l_i z_i \).

**Proposition 6** Consider the navigation observer consisting of the ranges observer
\[
\dot{Q} = Q \Omega \times \\
\dot{\xi} = A_r(\Omega, Q, \tilde{y}) \xi + B_r(a), \quad \xi(0) = 0 \\
\dot{\Psi} = A_r(\Omega, Q, \tilde{y}) \Psi, \quad \Psi(0) = I \\
\dot{\theta} = \mathcal{H}(y_b, \psi) \\
\dot{\hat{\psi}} = T_v(\xi + \Psi \hat{\theta}) \\
\dot{\hat{\zeta}_i} = T_{z_i} (\xi + \Psi \hat{\theta}) \cdot y_i
\]
with \( y_b, \psi \) defined in (45), the filtering operator \( \mathcal{H} \) defined...
in (33)-(34), and the matrices
\[
A_c := \begin{bmatrix}
-\Omega_x & I_3 & Q^T & 0_{3\times(n+9)} \\
-\hat{y}_1 & 0 & \cdots & 0 \\
-\hat{y}_n & 0 & \cdots & 0 \\
\end{bmatrix}, 
B_c := \begin{bmatrix}
a \\
0 \\
0 \\
\end{bmatrix},
\]
with the matrices \(A_c\) and \(B_c\) defined in (41). Then, it yields
\[
\chi - \xi = A(\hat{y}, \Omega, Q)[\chi - \xi],
\]
thus \(\chi = \xi + \Psi \theta \) with \(\theta := \chi_0\).

From (28), we have
\[
\Lambda_o T_v (\xi + \Psi \theta) + G_2 \left[ (\Lambda_o y + \alpha \Pi) T_v (\xi + \Psi \theta) \right] = 0
\]
with
\[
T_v := [0_{n\times 9} \ I_n], \quad T_v := [I_3 \ 0_{3\times (6+n)}]
\]
\(\Lambda_o := \text{diag}\{\phi_1, \ldots, \phi_n\}\)

and
\[
y := \begin{bmatrix} y_1^T \\ \vdots \\ y_n^T \\ \Pi_{by_n} \end{bmatrix}, \quad \Pi := \begin{bmatrix} \Pi_{by_1} \\ \vdots \\ \Pi_{by_n} \end{bmatrix},
\]
in which we have defined \(\phi_i\) \((i \in \mathcal{N})\) as (10) for the \(i\)-th feature point with a slight abuse of notation with subscript. Then, we obtain the linear regressor \(y_i = \psi^T \theta\) with
\[
y_i := -\Lambda_o T_v \xi - G_2 [(\Lambda_o y + \alpha \Pi) T_v \xi]
\]
\(\psi := (\Lambda_o T_v \psi)^T + G_2 [(\Lambda_o y + \alpha \Pi) T_v \psi]^T\).

Following Proposition 5, if \(\psi\) is IE, then \(\hat{\theta} = \mathcal{H}(y_i, \psi)\) provides a globally exponentially convergent estimate to \(\theta\). Invoking the state boundedness assumption, it yields \(\forall i \in \mathcal{N}\)
\[
\lim_{t \to \infty} \left[ \dot{z}_i(t) - z_i(t) \right] = 0 \quad (\text{exp.}) \quad (46)
\]
The second step is to show that \(\hat{Q}_c \in SO(3)\) is an almost global asymptotic estimate to \(Q_c\), which is similar to attitude estimation in [19] but with a constant matrix \(Q_c\) rather than a time-varying one. Define its estimation error as \(\tilde{Q}_c := Q_c \tilde{Q}_c^T\), and its time derivative is given by
\[
\ddot{Q}_c = Q_c (\dot{\tilde{Q}}_c)^T
\]
\[
= \ddot{Q}_c w_x
\]
\[
= \tilde{Q}_c \sum_{i=1}^{n-1} k_i \left[ (\tilde{Q}_c Q \hat{\eta}_i)^T - \hat{\eta}_i (\tilde{Q}_c Q \hat{\eta}_i)^T \right]
\]
\[
= \tilde{Q}_c \sum_{i=1}^{n-1} k_i \text{skew}[\tilde{Q}_c^T \hat{\eta}_i (\hat{\eta}_i)^T]
\]
\[
= \tilde{Q}_c \text{skew}[(\tilde{Q}_c^T M) + \mathcal{E}(t)]
\]
where in the second equation we have used in the second
The time derivative of $V$ which is a non-negative function and well defined, since such that

\[ \dot{\eta}_i \] term. Since both $\hat{Q}_c$ and $R$ living in a compact space $SO(3)$ and $\dot{\eta}_i$ being constant, there exist two constants $a_0, a_1 > 0$ such that

\[ |\text{tr}(\dot{E}^T(t)M)| \leq a_0 e^{-a_1 t} =: \varepsilon(t). \quad (48) \]

We consider the time-varying Lyapunov function

\[ V(\hat{Q}_c, t) = \sum_{i=1}^{n-1} k_i |\dot{\eta}_i|^2 - \text{tr}(\hat{Q}_c^T M) + \int_t^\infty |\varepsilon(s)| ds \]

which is a non-negative function and well defined, since $\varepsilon$ is absolutely integrable, i.e.

\[ \int_0^\infty |\varepsilon(s)| ds < +\infty. \]

The time derivative of $V(\hat{Q}_c, t)$ is given by

\[ \dot{V} = -\text{tr} \left( \text{skew}(\hat{Q}_c^T M)^T \hat{Q}_c^T M + \dot{E}^T M \right) - |\varepsilon(t)| \]

\[ = -||\text{skew}(\hat{Q}_c^T M)||^2 - \text{tr}(E^T M) - |\varepsilon(t)| \]

\[ \leq -||\text{skew}(\hat{Q}_c^T M)||^2, \]

in which we have used (48) in the last inequality. Then, it yields

\[ \int_0^\infty ||\text{skew}(\hat{Q}_c^T M)||^2 ds < +\infty. \]

Invoking the boundedness of the time derivative of $\text{skew}(\hat{Q}_c^T M)^T M$ and using Babalat’s lemma, we conclude that all the trajectories converge to the invariant set

\[ \Omega_c := \{ \hat{Q}_c \in SO(3) \mid \text{skew}(\hat{Q}_c^T M) = 0 \}. \]

Following the similar procedure of the proof in [19, Theorem 5.1], we can show that the set $\Omega_c$ has a locally exponentially stable equilibrium $\hat{Q}_c = I_3$, and other three isolated unstable equilibria $Q_i$, $i = 1, 2, 3$ on $SO(3)$, and there are no poles on the imaginary axis. By using the non-automonomous version of Hartman-Grobman theorem [4], the observation error dynamics (47) is topologically equivalent to an LTV dynamics in a small neighbourhood of these three unstable equilibria. As a result, only some very specific trajectories, from a zero Lebesgue measure set $\mathcal{M}_c$, ultimately converge to the unstable equilibria $Q_i$, $i = 1, 2, 3$. Combining the local exponential stability of the equilibrium $I_3$, thus it yields the almost global asymptotic stability of the error dynamics (47) at $\hat{Q}_c = I_3$. It is equivalent to show that

\[ \forall \hat{Q}_c(0) \in SO(3) \setminus \mathcal{M}_c, \lim_{t \to +\infty} \|\hat{Q}_c(t) - Q_c\| = 0, \quad (49) \]

and $\hat{Q}_c(t) \in SO(3), \forall t \geq 0$, that is $\hat{R}$ provides an asymptotically convergent estimate to $R$ as well.

The last part of the proof is to show the convergence of position estimate $\hat{x}$. Now define its estimation error as $\tilde{x} := \hat{x} - x$, and it yields

\[ \dot{\tilde{x}} = \hat{R}\tilde{v} - Rv + \sum_{i=1}^{n} \sigma_i (\dot{l}_z - \tilde{x} - \hat{R}\dot{z}_i) \]

\[ = \sum_{i=1}^{n} \sigma_i (\dot{l}_z - \tilde{x} - R\dot{z}_i + \epsilon_t) \]

\[ = \sum_{i=1}^{n} \sigma_i (\dot{l}_z - \tilde{x} - R\dot{z}_i) + \epsilon_t \]

\[ = -\sigma_i \cdot \tilde{x} + \epsilon_t \]

with $\sigma_i := \sum_{i=1}^{n} \sigma_i > 0$ and an exponentially decaying term $\epsilon_t$, where in the second equation we have used the convergence (46) and (49), as well as the compactness of $SO(3)$ and the boundedness of $v$. The dynamics of $\tilde{x}$ is a linear time-invariant (LTI), stable system perturbed by $\epsilon_t$, thus also being globally exponentially stable. It completes the proof.

**Remark 9** The proposed navigation observer is implemented in a modular manner, i.e., it constitutes of a ranges observer (40) and a full-position localization observer (42). To be precise, we first use (40) to reconstruct the full position “measurement”, which is then utilized in the localization observer (42), in this way solving the visual navigation problem; see Fig. 3. We figure out that the “integrated” ranges observer (40) can be replaced by several “individual” position/velocity observers in Propositions 4-5 to estimate the position of each feature point. However, we adopt the ranges observer (40) in an integrated way to relax the excitation requirement since less unknown variables are considered than the case implementing it by several position/velocity observers.

![Fig. 3. Structure of the proposed visual inertial navigation observer](image)

**6 Simulation Results**

In this section, we present simulation results to illustrate the properties of the proposed observers. All these simulations were implemented in Matlab/Simulink.
6.1 Position estimation with velocity measurement

First of all, we consider the case with measurable unbiased velocity in Section 3. In order to evaluate the performance of the gradient observer (15) and the PEBO (17). Here we consider two scenarios: the first one, borrowed from [5], satisfies the PE condition; and the other guarantees the regressors being IE but not PE. To be precise, the first trajectory $\Gamma_1$ is given by

$$\hat{x}(t) = \begin{bmatrix} \cos(\frac{t}{2}) \\ \frac{1}{3} \sin(t) \\ -\frac{\sqrt{3}}{4} \sin(t) \end{bmatrix}, \quad \Omega(t) = \begin{bmatrix} \sin(0.1 + \pi) \\ 0.5 \sin(2t) \\ 0.1 \sin(0.3t + \frac{\pi}{2}) \end{bmatrix}$$

from the initial pose $R(0) = I_3$. For the IE trajectory $\Gamma_2$, it is exactly the same with the first $\Gamma_1$ in $[0,4]$ s, but for $t \geq 4$s the velocities become $e^{-5(t-4)}v(t)$ and $e^{-5(t-4)}\Omega(t)$. The parameters and initial conditions in the observers are selected as $\alpha = 1, \gamma = 50, \hat{r}(0) = 0, \xi(0) = 0$ and $\hat{\theta}(0) = 0$.

For the PE trajectory $\Gamma_1$ with ideal measurements of bearing and velocities, from Figs. 4(a)-4(b) it is shown that both the gradient observer and the PEBO provide asymptotically convergent estimate to the position $z$ in the body-fixed frame $\{B\}$ with zero steady-stage errors. In Figs. 4(c)-4(d), measurement noises to the bearing and velocities are considered in simulations, and we observe that the estimates both ultimately converge to the small neighbourhood of their true values, showing the proposed two designs are both robust vis-à-vis noise under the PE condition.

Fig. 4. Performance of the proposed position observers under persistence of excitation, i.e., the gradient observer (15) and the PEBO (17) for the case with unbiased velocity measurement (with the legends given in the first figure).

Besides, we evaluate the performance of the proposed observers only in the presence of interval excitation for the second trajectory $\Gamma_2$. For the ideal measurement, the PEBO still provides asymptotically convergent estimate to the position $z$, but the gradient observer has a significant ultimate error; for the noised case, the estimate from the PEBO converges to the small neighbourhood of its true value.

![Fig. 4](image-url)

At the end, we carried out simulations for the I&I observer in [15] and the feature depth observer in [9] for comparison.

Both of them requires some excitation conditions, i.e., in the I&I observer imposing instantaneous observability—the existence of $\delta > 0$ such that

$$D(t)^TD(t) > \delta, \quad \forall t \geq 0 \quad (50)$$

with $D(t) = [v_1 - v_3 x_1 / x_3 \quad v_2 - v_3 x_3 / x_3]^{\top}$ to guarantee semi-global asymptotic stability; the feature depth observer in [9] requires that the matrix $D(t)$ is PE, which is weaker than (50), and the proposed PEBO only requires the extremely weak condition A1. In these observers, the measured output are considered as the homogeneous normalized coordinate $[x_1/x_3 \quad x_2/x_3]^{\top}$, and the aim is to estimate the inverse of depth $1/x_3$. As figured out in Remark 1, the homogeneous normalized coordinate and the bearing may be converted to each other. In the feature depth observer, we set $k_i = 1$ ($i = 1, \ldots, 3$) and $\hat{x}(0) = [0.2 \quad 0.2 \quad 0.5]^{\top}$. In the I&I observer we select the initial guess $0.5$ and $\lambda = 0.2$. It should be underlined here that the term $\nabla_{\hat{t}}\beta$ in the I&I observer needs to calculate the acceleration $\dot{\hat{v}}$, and we utilized the “Derivative” block in Simulink to approximate it, which limits the estimation performance. Even though states in these three observers live in different spaces, we tried to make the initial range estimates as close as possible to make a fair comparison. The simulation results may be found in Fig. 6(a),

Note that [15] is concerned with the problem of a moving feature point and a fixed camera, which is mathematically equivalent to the one in Section 2.
noting that both of them has relatively large ultimate error since their excitation conditions are not satisfied for the second robot trajectory. In contrast, the proposed PEBO is able to provide robust and reliable estimates.

6.2 Position/Velocity Estimation with Biased Acceleration Measurement

In the second group of simulations, we consider position and velocity estimation with bearing and biased linear acceleration measurable. We consider the trajectory with

\[ I_a(t) = \begin{bmatrix} -0.5 \cos(0.5t) \\ -0.5 \sin(t) \\ \sqrt{3}/4 \sin(t) \end{bmatrix}, \quad \Omega(t) = \begin{bmatrix} 0.2 \sin(0.1t + \pi) \\ 0.1 \sin(0.2t) \\ 0.1 \sin(0.3t + \pi) \end{bmatrix} \]

and \( I_v(0) = 0, x(0) = \text{col}(1, 0, 0) \) and \( R(0) = I_3 \). Such a trajectory only guarantees the regression being IE rather than PE. The parameters and initial conditions for the observer in Section 4 are set as \( \alpha = 2, \gamma = 100, \rho = 0.4, k_p = 500 \) and \( \hat{\theta}(0) = [01, 9, 10] \). We consider a small sensor bias \( b_a = [0.09, 0.10, 0.11] \) and the feature point is located at \( I_z = [-2, 1, 3]' \). Fig. 7 shows the evolution of estimates and estimation errors in the noise-free case. As expected, all the estimation errors converge to zero ultimately. For noised measurement, the proposed observer still provides satisfactory estimates; see Fig. 8.

6.3 Vision-aided Inertial Navigation

At the end we evaluate the performance of the visual inertial navigation observer in Proposition 6. We consider three known landmarks in the inertial frame, i.e., \([-2, 1, 3]'\), \([-2, 2, 1]'\) and \([1, 1, 1]'\). The same robot trajectory and parameters were adopted, except \( \alpha = 1 \) and \( k_p = 10^3 \). The behaviour is shown in Fig. 9 in the absence of measurement noise. Note that in Fig. 9(b) we draw in the inertial frame \{I\} the attitude and its estimate by means of

\[ \hat{\theta} \]

Here, since the last element of \( \theta \) is related to the gravitational constant, we set its initial guess as 10 to accelerate the convergence speed.

7 Concluding Remarks

The position observer design for a single feature point, using its bearing and linear acceleration, remained an open problem for a long time. To close this gap, a nonlinear observer has been proposed in the paper by means of the recently developed PEBO methodology. The design is applicable to the case with unknown sensor bias and gravitational constant, which can achieve global exponential convergence under the extremely weak IE conditions of robot trajectories. To il-
Consider the dynamics (1)-(3), and design a linear regressor on the unknown states. In the following proposition, we provide an alternative to the regression model with biased acceleration measurement for the noise-free case.

Illustrate its powerful efficiency as a modular tool, we apply the position observer to the problem of visual inertial navigation, providing a novel almost globally convergent solution to it. In addition, the observer has flexibility to adopt to some specific cases, e.g., with known sensor bias after calibration, and known gravitational vector—by generating different linear regression models mutatis mutandis. We expect that the estimation performance may be improved further with less unknown variables.

Appendix

A Swapping lemma

**Lemma 1** (*Swapping lemma* [25]) For $C^1$-differential signals $x, y : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$, the following holds

$$
\frac{\alpha}{p + \alpha} [xy] = y \frac{\alpha}{p + \alpha} [x] - \frac{1}{p + \alpha} \left[ y \frac{\alpha}{p + \alpha} [x] \right] \tag{A.1}
$$

for any $\alpha > 0$ with $p := d/dt$.

B An Alternative Regression Model with Biased Acceleration Measurement

In the following proposition, we provide an alternative to formulate a linear regressor on the unknown states.

**Proposition 7** Consider the dynamics (1)-(3), and design the following dynamic extension

$$
Q = Q \Omega_x
$$

$$
\dot{\xi} = A(y, \Omega, Q)\xi + B(a), \quad \xi(0) = 0_{13} \tag{B.1}
$$

$$
\dot{\Psi} = A(y, \Omega, Q)\Psi, \quad \Psi(0) = I_{13}
$$

with $Q \in SO(3)$, and $A, B$ defined as (B.2). Then the unknown state $X$ satisfies $X = T[\xi + \Psi \theta]$ with $T$ given in (B.3). Besides, the vector $\theta$ verifies the LRE (22) with $\dot{\psi}$ and $y_\theta$ defined in (B.4).

**Proof 7** This proof is similar to that of Proposition 4, according to which we have (23) again with constant unknown matrix $Q_\chi \in SO(3)$. We select the system state $\chi$ as $\chi := \text{col}(z, r, v, b_\alpha, g_e) \in \mathbb{R}^3$, the dynamics of which is given by (24), with the initial condition $\chi(0) = \chi_0$, and

$$
A := 
\begin{bmatrix}
-\Omega_x & 0_{3 \times 1} & -I_3 & 0_{3 \times 3} & 0_{3 \times 3} \\
0_{1 \times 3} & 0 & -y^T & 0_{3 \times 3} & 0_{1 \times 3} \\
0_{3 \times 3} & 0_{3 \times 1} & -\Omega_x & I_3 & Q^T \\
0_{3 \times 3} & \ldots & 0_{3 \times 3} \\
0_{3 \times 3} & \ldots & 0_{3 \times 3}
\end{bmatrix},
B := 
\begin{bmatrix}
0_4 \\
a \\
0_6
\end{bmatrix}
\tag{B.2}
$$

Similarly to the proof of Proposition 4, we have (27). On the other hand, we have $X = T[\xi + \Psi \theta]$ with the unknown constant vector $\theta = \chi_0$ and

$$
T = \text{diag}([I_3 \ 0_{3 \times 1}], [I_6 \ 0_{6 \times 3}]). \tag{B.3}
$$

From the bearing output (4), we have $z - yr = 0$, which is equivalent to $C(y) \chi = 0$, in which we have defined $C(y) = [I - y \ 0_{3 \times 9}]$, thus yielding $C(y)[\xi + \Psi \theta] = 0$. It may be rewritten as the LRE $y_\theta(t) = \psi(t)^T \theta$ with

$$
y_\theta := -C(y)\xi, \quad \psi := [C(y)\Psi]^T. \tag{B.4}
$$

It completes the proof.

We may find that the dynamic extension in Proposition 4 has lower dimensions than the one above. However, the regressor in Proposition 4 is more complicated, i.e., introducing additional LTI filters to deal with the unmeasurable time derivative of $y$.

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4 Selecting both the position $z$ and its range $r = |z|$ in system dynamics was previously studied in [18].
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