How to Net a Convex Shape

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Abstract

We revisit the problem of building weak $\varepsilon$-nets for convex ranges over a point set in $\mathbb{R}^d$. Unfortunately, the known constructions of weak $\varepsilon$-nets yields sets that are of size $\Omega(\varepsilon^{-d}e^{cd^2})$, where $c$ is some constant. We offer two alternative schemes that yield significantly smaller sets, and two related results, as follows:

(A) Let $S$ be a sample of size $\tilde{O}(d^2\varepsilon^{-1})$ points from the original point set (which is no longer needed), where $\tilde{O}$ hides polylogarithmic terms. Given a convex body $C$, via a separation oracle, the algorithm performs a small sequence of (oracle) stabbing queries (computed from $S$) – if none of the query points hits $C$, then $C$ contains less than an $\varepsilon$-fraction of the input points. The number of stabbing queries performed is $O(d^2\log\varepsilon^{-1})$, and the time to compute them is $\tilde{O}(d^2\varepsilon^{-1})$. To the best of our knowledge, this is the first weak $\varepsilon$-net related construction where all constants/bounds are polynomial in the dimension.

(B) If one is allowed to expand the convex range before checking if it intersects the sample, then a sample of size $\tilde{O}(\varepsilon^{-1(d+1)/2})$, from the original point set, is sufficient to form a net.

(C) We show a construction of weak $\varepsilon$-nets which have the following additional property: For a heavy body, there is a net point that stabs the body, and it is also a good centerpoint for the points contained inside the body.

(D) We present a variant of a known algorithm for approximating a centerpoint, improving the running time from $\tilde{O}(d^9)$ to $\tilde{O}(d^7)$. Our analysis of this algorithm is arguably cleaner than the previous version.

1. Introduction

Notation. In the following $O(\cdot)$ hides constant that do not depend on the dimension. $O_d(\cdot)$ hides constants that depends on the dimension (usually badly – exponential or doubly exponential, or even worse). The $\tilde{O}(\cdot)$ notation would be used to hide polylogarithmic factors, where the power of the polylog is independent of the dimension.

Weak $\varepsilon$-nets. Consider the range space $(P,C)$, where $P$ is a set of $n$ points in $\mathbb{R}^d$, and $C$ is the set of all convex shapes in $\mathbb{R}^d$. This range space has infinite VC dimension, and as such it is impervious to the standard $\varepsilon$-net constructions. Weak $\varepsilon$-nets bypass this issue by using points outside the point set. While there is significant amount of work on weak $\varepsilon$-nets, the constructions known are not easy and result in somewhat large sets. The state of the art is the work by Matoušek and Wagner [MW04], which shows a weak $\varepsilon$-net construction of size $O(\varepsilon^{-d}(\log\varepsilon^{-1})O(d^2\log d))$. Such a weak $\varepsilon$-net $W$ has the guarantee that any convex set $C$ that contains at least $\varepsilon n$ points of $P$, must contain at least one point of $W$. See [MV17] for a recent survey of $\varepsilon$-nets and related concepts. See also the recent work by Rok and Smorodinsky [RS16] and references therein (they show to construct weak $\varepsilon$-net for moving points).

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Basis of weak $\varepsilon$-nets. Mustafa and Ray [MR08] showed that one can pick a random sample $S$ of size $c_d \varepsilon^{-1} \log \varepsilon^{-1}$ from $P$, and then compute a weak $\varepsilon$-net for $P$ directly from $S$, showing that the size of support needed to compute a weak $\varepsilon$-net is (roughly) the size of a regular $\varepsilon$-net. Unfortunately, the constant in their sample $c_d = O\left(d^d \log d \alpha^d \log d\right)$ is doubly exponential in the dimension – it is bounded using the $\left((d+1)^2, d+1\right)$-Hadwiger-Debrunner number, which the best upper bound currently known for it is as stated [KST17, KST15].

In particular, all current results about weak $\varepsilon$-nets suffer from the “curse of dimensionality” and have constants that are at least doubly exponential in the dimension.

Computing centerpoints. A classical implication of Helly’s theorem, is that for any set $P$ of $n$ points in $\mathbb{R}^d$, there is a $1/(d+1)$-centerpoint. Specifically, given a constant $\alpha \in (0, 1)$, a point $\mathfrak{c} \in \mathbb{R}^d$ is an $\alpha$-centerpoint if all closed halfspaces containing $\mathfrak{c}$ also contain at least $\alpha n$ points of $P$. It is currently unknown if one can compute a $O(1/d)$-centerpoint in polynomial time (in the dimension). A randomized polynomial time algorithm was presented by Clarkson et al. [CEM+96], that computes (roughly) a $1/(4d^2)$-centerpoint in $\widetilde{O}(d^3)$ time.

Our results. Let $P$ be a set of $n$ points in $\mathbb{R}^d$. We suggest to two alternatives to weak $\varepsilon$-nets, and get some related results:

(A) Functional nets. Our main result considers the case when we have oracle access to the query range $C$. This separation oracle works as follows – given a query point $q$, it either returns that $q$ is in $C$, or alternatively, it returns a separating hyperplane. We show that a random sample of size

$$O\left(\varepsilon^{-1} d^3 \log d \log^3 \varepsilon^{-1} + \varepsilon^{-1} \log \varphi^{-1}\right) = \widetilde{O}(d^3/\varepsilon),$$

with probability $\geq 1 - \varphi$, can be used to decide if a query convex body $C$ is $\varepsilon$-light. Formally, the algorithm, using only the sample, performs $O(d^2 \log \varepsilon^{-1})$ oracle queries – if any of the query points generated stabs $C$, then $C$ is considered as (potentially) containing more than $\varepsilon n$ points. Alternatively, if all the queries missed $C$, then $C$ contains less than $\varepsilon n$ points of $P$. The query points can be computed in polynomial time, and we emphasize that the dependency in the running time and sample size are polynomial in $\varepsilon$ and $d$. See Theorem 3.6. As such, this result can be viewed as slightly mitigating the curse of dimensionality in the context of weak $\varepsilon$-nets.

(B) Center nets. Using the above, one can also construct a weak $\varepsilon$-net directly from such a sample – this improves over the result of Mustafa and Ray [MR08] as far as the dependency on the dimension. This is described in Section 3.3.

Surprisingly, by using Theorem 3.6, one can get a stronger form of a weak $\varepsilon$-net, which we refer to as an $(\varepsilon, \alpha)$-center net. Here $\alpha = O\left(1/(d \log \varepsilon^{-1})\right)$ and one computes a set $W$, of size (roughly) $\tilde{O}_d\left(\varepsilon^{-O(d^2)}\right)$, such that if a convex body $C$ contains $\geq \varepsilon n$ points of $P$, then $W$ contains a point $q$ which is an $\alpha$-centerpoint of $C \cap P$. Namely, the net contains a point that stabs $C$ in the “middle” as far as the point set $C \cap P$. See Theorem 4.5.

(C) Approximating centerpoints. The functional net algorithm requires computing centerpoints. To this end, we revisit the algorithm of Clarkson et al. [CEM+96] for approximating a centerpoint. We present a variant of their algorithm which runs in $\widetilde{O}(d^3)$ time, and computes roughly a $1/(2d^2)$-centerpoint. This improves both the running time, and the quality of centerpoint computed. While the improvements are small (a factor of $d^2$ roughly in the running time, and a factor 2 in the centerpoint quality), we believe that the new analysis is cleaner, and might be of independent interest. See Theorem 5.8.

(D) Expansion nets. We also reconsider the problem of constructing a weak $\varepsilon$-net in the context of convex approximation. In particular, if one is allowed to slightly expand the range $C \in C$, then the expanded shape would contain a point from an appropriate (regular) $\varepsilon$-net, which is significantly smaller than its weak counterpart. See Theorem 6.8.
Paper organization. We review the standard tools needed in Section 2. The construction of the on-the-fly weak 𝜖-net in the oracle model is described in Section 3. The construction of center nets is described in Section 4. The improved centerpoint approximation algorithm is described in Section 5. The construction of expansion nets – where the convex body is allowed to be expanded, is described in Section 6.

2. Preliminaries

2.1. Background: Ranges spaces, VC dimension, samples and nets

The following is a quick survey of (standard) known results about 𝜖-nets, 𝜖-samples, and relative approximations [Har11].

Definition 2.1. A range space \( S \) is a pair \((\mathcal{X}, \mathcal{R})\), where \( \mathcal{X} \) is a ground set (finite or infinite) and \( \mathcal{R} \) is a (finite or infinite) family of subsets of \( \mathcal{X} \). The elements of \( \mathcal{X} \) are points and the elements of \( \mathcal{R} \) are ranges.

For technical reasons, it will be easier to consider a finite subset \( X \subseteq \hat{X} \) as the underlying ground set.

Definition 2.2. Let \( S = (\hat{X}, \mathcal{R}) \) be a range space, and let \( X \) be a finite (fixed) subset of \( \hat{X} \). For a range \( r \in \mathcal{R} \), its measure is the quantity \( m(r) = |r \cap X|/|X| \). For a subset \( S \subseteq X \), its estimate of \( m(r) \), for \( r \in \mathcal{R} \), is the quantity \( \overline{m}(r) = |r \cap S|/|S| \).

Definition 2.3. Let \( S = (\hat{X}, \mathcal{R}) \) be a range space. For \( Y \subseteq \hat{X} \), let \( \mathcal{R}|_Y = \left\{ r \cap Y \mid r \in \mathcal{R} \right\} \) denote the projection of \( \mathcal{R} \) on \( Y \). The range space \( S \) projected to \( Y \) is \( S|_Y = (Y, \mathcal{R}|_Y) \). If \( \mathcal{R}|_Y \) contains all subsets of \( Y \) (i.e., if \( Y \) is finite, we have \( |\mathcal{R}|_Y| = 2^{|Y|} \)), then \( Y \) is shattered by \( \mathcal{R} \) (or equivalently \( Y \) is shattered by \( S \)).

The VC dimension of \( S \), denoted by \( \text{dim}_{\text{VC}}(S) \), is the maximum cardinality of a shattered subset of \( \hat{X} \). If there are arbitrarily large shattered subsets, then \( \text{dim}_{\text{VC}}(S) = \infty \).

Definition 2.4. Let \( S = (\hat{X}, \mathcal{R}) \) be a range space, and let \( X \) be a finite subset of \( \hat{X} \). For \( 0 \leq \varepsilon \leq 1 \), a subset \( S \subseteq X \) is an \( \varepsilon \)-sample for \( X \) if for any range \( r \in \mathcal{R} \), we have \( |\overline{m}(r) - \overline{m}(r)| \leq \varepsilon \), see Definition 2.2. Similarly, a set \( S \subseteq X \) is an \( \varepsilon \)-net for \( X \) if for any range \( r \in \mathcal{R} \), if \( m(r) \geq \varepsilon \) (i.e., \( |r \cap X| \geq \varepsilon |X| \)), then \( r \) contains at least one point of \( S \) (i.e., \( r \cap S \neq \emptyset \)).

A generalization of both concepts is relative approximation. Let \( p, \hat{\varepsilon} > 0 \) be two fixed constants. A relative \((p, \hat{\varepsilon})\)-approximation is a subset \( S \subseteq X \) that satisfies \((1 - \hat{\varepsilon})m(r) \leq \overline{m}(r) \leq (1 + \hat{\varepsilon})m(r)\), for any \( r \in \mathcal{C} \) such that \( m(r) \geq p \). If \( m(r) < p \) then the requirement is that \(|\overline{m}(r) - m(r)| \leq \hat{\varepsilon}p \).

Theorem 2.5 (ε-net theorem, [HW87]). Let \((\hat{X}, \mathcal{R})\) be a range space of VC dimension \( \xi \), let \( X \) be a finite subset of \( \hat{X} \), and suppose that \( 0 < \varepsilon \leq 1 \) and \( \varphi < 1 \). Let \( N \) be a set obtained by \( m \) random independent draws from \( X \), where \( m \geq \max \left( \frac{4}{\varepsilon} \ln \frac{4}{\varepsilon}, \frac{8k}{\varepsilon} \ln \frac{10k}{\varepsilon} \right) \). Then \( N \) is an \( \varepsilon \)-net for \( X \) with probability at least \( 1 - \varphi \).

The following is a slight strengthening of the result of Vapnik and Chervonenkis [VC71] – see [Har11, Theorem 7.13].

Theorem 2.6 (ε-sample theorem). Let \( \varphi, \varepsilon > 0 \) be parameters and let \((\hat{X}, \mathcal{R})\) be a range space with VC dimension \( \xi \). Let \( X \subseteq \hat{X} \) be a finite subset. A sample of size \( O(\varepsilon^{-2}(\xi + \log \varphi^{-1})) \) from \( X \) is an ε-sample for \( S = (X, \mathcal{R}) \) with probability \( \geq 1 - \varphi \).

Theorem 2.7 ([LLS01, HS11]). A sample \( S \) of size \( O(\varepsilon^{-2}p^{-1}(\xi \log p^{-1} + \log \varphi^{-1})) \) from a range space with VC dimension \( \xi \), is a relative \((p, \hat{\varepsilon})\)-approximation with probability \( \geq 1 - \varphi \).

The following is a standard statement on the VC dimension of a range space formed by mixing several range spaces together (see [Har11]).

Lemma 2.8. Let \( S_1 = (\hat{X}_1, \mathcal{C}_1), \ldots, S_k = (\hat{X}_k, \mathcal{C}_k) \) be \( k \) range spaces, where all of them have the same VC dimension \( \xi \). Consider the new set of ranges \( \hat{\mathcal{C}} = \{ r_1 \cap \ldots \cap r_k \mid r_i \in R_i, 1 \leq i \leq k \} \). Then the range space \( \hat{S} = (\hat{X}, \hat{\mathcal{C}}) \) has VC dimension \( O(\xi k \log k) \).
2.2. Weak $\epsilon$-nets

A convex body $C \subseteq \mathbb{R}^d$ is $\epsilon$-heavy (or just heavy) if $\overline{m}(C) \geq \epsilon$ (i.e., $|C \cap P| \geq \epsilon |P|$). Otherwise, $C$ is $\epsilon$-light.

Definition 2.9 (Weak $\epsilon$-net). Let $P$ be a set of $n$ points in $\mathbb{R}^d$. A finite set $S \subseteq \mathbb{R}^d$ is a weak $\epsilon$-net for $P$ if for any convex set $C$ with $\overline{m}(C) \geq \epsilon$, we have $S \cap C \neq \emptyset$.

Note, that like (regular) $\epsilon$-nets, weak $\epsilon$-nets have one-sided error – if $C$ is heavy then the net must stab it, but if $C$ is light then the net may or may not stab it.

2.3. Centerpoints

Given a set $P$ of $n$ points in $\mathbb{R}^d$, and a constant $\alpha \in (0,1/(d+1)]$, a point $\check{r} \in \mathbb{R}^d$ is an $\alpha$-centerpoint if for any closed halfspace that contains $\check{r}$, the halfspace also contains at least $\alpha n$ points of $P$. It is a classical consequence of Helly’s theorem that a $1/(d+1)$-centerpoint always exists. If a point $\check{r} \in \mathbb{R}^d$ is a $1/(d+1)$-centerpoint for $P$, we omit the $1/(d+1)$ and simply say that $\check{r}$ is a centerpoint for $P$.

3. Functional nets: A weak net in the oracle model

3.1. The model, construction, and query process

Model. Given a convex body $C \subseteq \mathbb{R}^d$, we assume oracle access to it. This is a standard model in optimization. Specifically, given a query point $q$, the oracle either returns that $q \in C$, or alternatively it returns a (separating) hyperplane $h$, such that $C$ lies completely on one side of $h$, and $q$ lies on the other side.

Our purpose here is to precompute a small subset $S \subseteq P$, such that given any convex body $C$ (with oracle access to it), one can decide if $C$ is $\epsilon$-light. Specifically, the query algorithm (using only $S$, and not the whole point set $P$) generates an (adaptive) sequence of query points $q_1, q_2, \ldots$, such that if any of these query points are in $C$, then the algorithm considers $C$ to be heavy. Otherwise, if all the query points miss $C$, then the algorithm outputs (correctly) that $C$ is light (i.e., $\overline{m}(C) < \epsilon$).

Construction. Given $P$, the set $S$ is a random sample from $P$ of size

$$\mu = O(\epsilon^{-1} d^3 \log d \log \epsilon^{-1} + \epsilon^{-1} \log \varphi^{-1}) = \tilde{O}(d^3/\epsilon),$$

where $\varphi > 0$ is a prespecified parameter.

Query process. Given a convex body $C$ (with oracle access to it), the algorithm starts with $S_0 = S$. In the $i$th iteration, the algorithm computes a $O(1/d^2)$-centerpoint $q_i$ of $S_i$ using the algorithm of Theorem 5.8, with failure probability at most $1/4$. If the oracle returns that $q_i \in C$, then the algorithm returns $q_i$ as a proof of why $C$ is considered to be heavy. Otherwise, the oracle returns a separating hyperplane $h_i$, such that the open halfspace $h_i^-$ contains $q_i$. Let $S'_i = S_{i-1} \setminus h_i^-$. If $|S'_i| \leq (1 - \gamma)|S_{i-1}|$, where $\gamma = 1/16d^2$ then we set $S_i = S'_i$ (such an iteration is successful). Otherwise, we set $S_i = S_{i-1}$. The algorithm stops when $|S_i| \leq \epsilon |S|/8$.

3.2. Correctness

Let $I$ be the set of indices of all the successful iterations, and consider the convex set $C_I = \cap_{i \in I} h_i^+$. The set $C_I$ is an outer approximation to $C$. In particular, for an index $j$, let $C_j = \cap_{i \in I, i \leq j} h_i^+$ be this outer approximation in the end of the $j$th iteration. We have that $S_j = S \cap C_j$.

Lemma 3.1. There are at most $\tau = O(d^2 \log \epsilon^{-1})$ successful iterations. For any $j$, the convex polyhedra $C_j$ is defined by the intersection of at most $\tau$ closed halfspaces.
Proof: We start with $\mu = |S_0|$ points in $S_0$. Every successful iteration reduces the number of points in the net $S_{j-1}$ by a factor of $1 - \gamma$. Furthermore, the algorithm stops as soon as $|S_j| \leq \varepsilon |S_0| / 8$. As such, there are at most $\tau$ iterations, for the minimal $\tau$ such that $(1 - \gamma)^\tau \leq \varepsilon / 8$, where $\gamma = 1/(16d^2)$. That is $\tau = O(d^2 \log \varepsilon^{-1})$.

The second claim is immediate — every successful iteration adds one halfspace to the intersection that forms $C_I$.

Let $\mathcal{H}^\tau$ be the set of all of convex polyhedra in $\mathbb{R}^d$ that are formed by the intersection of $\tau$ closed halfspaces.

Observation 3.2. The VC dimension of $(\mathbb{R}^d, \mathcal{H}^\tau)$ is

$$D = O(d \tau \log \tau) = O(d (d^2 \log \varepsilon^{-1}) \log (d^2 \log \varepsilon^{-1})) = O(d^3 \log d \log^2 \varepsilon^{-1}).$$

This follows readily, as the VC dimension of the range space of points in $\mathbb{R}^d$ and halfspaces is $d + 1$, and by the bound of Lemma 2.8 for the intersection of $\tau$ such ranges.

Lemma 3.3. The set $S$ is a relative $(\varepsilon/8, 1/4)$-approximation for $(P, \mathcal{H}^\tau)$, with probability $1 - \varphi$.

Proof: Using Theorem 2.7. with $p = \varepsilon/8$, $\tilde{\varepsilon} = 1/4$, and $\xi = D$, implies that a random sample of $P$ of size

$$O(\tilde{\varepsilon}^{-2} p^{-1} (\xi \log p^{-1} + \log \varphi^{-1})) = O(\varepsilon^{-1} (D \log \varepsilon^{-1} + \log \varphi^{-1})) = O(\varepsilon^{-1} (d^3 \log d \log^3 \varepsilon^{-1} + \log \varphi^{-1}))$$

is the desired relative $(p, \tilde{\varepsilon})$-approximation with probability $\geq 1 - \varphi$. And this is indeed the size $S$, see Eq. (3.1).

Lemma 3.4. Given a convex query body $C$, the expected number of oracle queries performed by the algorithm is $O(d^2 \log \varepsilon^{-1})$, and the expected running time of the algorithm is $O(d^9 \varepsilon^{-1} \text{polylog})$, where $\text{polylog} = O(\log(d \varepsilon^{-1} \varphi^{-1} \log(1)/\varepsilon))$.

Proof: If the computed point is in the $i$th iteration is indeed a centerpoint of $S_{i-1}$, then the algorithm would either stop in this iteration, or the iteration would be successful. Since the probability of the computed point to be the desired centerpoint is at least $\geq 3/4$, it follows that the algorithm makes (in expectation) $\tau/(3/4)$ iterations till success. The $i$th iteration requires $O(|S_i| + d^2 \log^2 d)$ time, since we use the algorithm of Theorem 5.8 to compute the approximate centerpoint. Summing this over all the iterations $\tau$ (bounded in Lemma 3.1), we get expected running time

$$O(d^2 \mu + \tau d^7 \log^3 d) = O(d^2(\varepsilon^{-1} d^3 \log d \log^3 \varepsilon^{-1} + \varepsilon^{-1} \log \varphi^{-1}) + d^9 \log^3 d \log \varepsilon^{-1})$$

$$= O(d^2 \varepsilon^{-1} \log \varphi^{-1} + d^3 \varepsilon^{-1} \log d \log^3 \varepsilon^{-1} + d^9 \log^3 d \log \varepsilon^{-1}).$$

(3.2)

Lemma 3.5. Assuming that $S$ is the desired relative approximation, then for any query body $C$, if the algorithm declares that it is $\varepsilon$-light, then $|C \cap P| < \varepsilon n$.

Proof: The algorithm stops when $|S_i| \leq (\varepsilon/8) |S|$. But then, since $S$ is a relative $(\varepsilon/8, 1/4)$-approximation, it follows $\pi(C) \leq |S_i| / |S| \leq \varepsilon / 8$. Furthermore, by definition, we have that $\pi(C) \leq (5/4)(\varepsilon/8) < \varepsilon$, as claimed.

3.2.1. The result

The above implies the following.

Theorem 3.6. Let $P$ be a set of points in $\mathbb{R}^d$, and let $\varepsilon, \varphi > 0$ be parameters. Let $S$ be a random sample of $P$ of size

$$\mu = O(\varepsilon^{-1} d^3 \log d \log^3 \varepsilon^{-1} + \varepsilon^{-1} \log \varphi^{-1}) = \tilde{O}(\varepsilon^{-1} d^3).$$

Then, for a given query convex body $C$ endowed with an oracle access, the algorithm described above, which uses only $S$, computes a sequence of query points $q_1, q_2, \ldots, q_m$. Such that either:
Lemma 3.9. Given a set \( P \) of points in \( \mathbb{R}^d \), and the algorithm outputs a \( \epsilon \)-net, or (ii) the algorithm outputs that \( |C \cap P| < \epsilon n \).

The query algorithm has the following performance guarantees:

(A) The expected number of oracle queries is \( E[m] = O(d^2 \log \epsilon^{-1}) \).

(B) The algorithm itself (ignoring the oracle queries) runs in \( O(d^3 \epsilon^{-1}) \) time (see Eq. (3.2) for exact bound).

The algorithm output is correct, for all convex bodies, with probability \( \geq 1 - \varphi \).

Remark 3.7. One may hope to bound the probability of the algorithm reporting a false positive. However this is inherently not possible for any weak \( \epsilon \)-net construction. Indeed, the algorithm can fail to distinguish between a polygon that contains at least \( \epsilon n \) of the points of \( P \) and a polygon that contains none of the points of \( P \). Consider \( n \) points \( P \) lying on a circle in \( \mathbb{R}^2 \). Choose \( \epsilon n \) of these points on the circle, and let \( C \) be the convex hull of these points. Clearly \( C \) contains at least \( \epsilon n \) points of \( P \). Now, take each vertex in \( C \) and “slice” it off, forming a new polygon \( C' \) that contains no points from \( P \). However, \( C' \) is still a large polygon and as such may contain a centerpoint during the execution of the above algorithm. Therefore our algorithm may report that \( C' \) contains a large fraction of the points, even though \( C' \) is contains no points of \( P \), and so it fails to distinguish between \( C \) and \( C' \).

Remark 3.8. Clarkson et al. [CEM+96] provide also a randomized algorithm that finds a \( \left( \frac{1}{d+1} - \gamma \right) \)-centerpoint with probability \( 1 - \delta \) in time \( O((d \gamma^{-2} \log(d \gamma^{-1}))^{d+O(1)} \log \delta^{-1}) \). We could use this algorithm instead of Theorem 5.8 in the query process. Since we are computing a better quality centerpoint, the number of iterations \( \tau \) and sample size \( \mu \) would be smaller by a factor of \( d \). Specifically, \( \tau = O(d \log \epsilon^{-1}) \) and from Lemma 2.8, the VC dimension of the range space \( S = (P, \mathcal{H}^*) \) becomes \( D = O(d^2 \log d \log 2 \epsilon^{-1}) \). Following the proof of Lemma 3.3, we can construct a sample \( S \) which is \((\epsilon/8, 1/4)\)-relative approximation for \( S \) with probability \( 1 - \varphi \) of size

\[
\mu = O(\epsilon^{-1}(D \log \epsilon^{-1} + \log \varphi^{-1})) = O(\epsilon^{-1}(d^2 \log d \log 3 \epsilon^{-1} + \log \varphi^{-1})).
\] (3.3)

3.3. Application: Generating a weak \( \epsilon \)-net

Here we show how to construct a weak \( \epsilon \)-net from the sample used by the algorithm.

Lemma 3.9. Given a set \( P \) of \( n \) points in \( \mathbb{R}^d \), one can compute a set \( Q \) of \( O(n^{d^2}) \) points, such that for any subset \( P' \subseteq P \), there is a \( 1/(d+1) \)-centerpoint of \( P' \) in \( Q \).

Proof: This is well known, and we include a proof for the sake of completeness.

Let \( H \) be the set of all hyperplanes which pass through \( d \) points of \( P \). The original proof of the centerpoint theorem implies that a vertex of the arrangement \( \mathcal{A}(H) \) is a \( 1/(d+1) \)-centerpoint of \( P \). Let \( \mathcal{V}(P) \) denote the set of vertices of \( \mathcal{A}(H) \). Observe that \( \mathcal{V}(P') \subseteq \mathcal{V}(P) \), for all \( P' \subseteq P \), thus implying that \( \mathcal{V}(P) \) contains all desired centerpoints. As for the size bound, observe that

\[
\alpha = |H| \leq \left( \frac{n}{d} \right) \leq \left( \frac{ne}{d} \right)^d \quad \text{and} \quad |\mathcal{V}(P)| \leq \left( \frac{\alpha}{d} \right) \leq \left( \frac{e \left( \frac{ne}{d} \right)^d}{d} \right)^d = n^{d^2 \left( \frac{e}{d} \right)^{d^2+d}} = O(n^{d^2}).
\]

Lemma 3.10. Let \( P \) be a set of \( n \) points in \( \mathbb{R}^d \). Let \( S \) be a random sample from \( P \) of size \( \mu = \tilde{O}(\epsilon^{-1} d^2) \), see Eq. (3.3) for the exact bound. Then, one can compute a set of points \( W \) from \( S \), of size

\[
O(\mu^{d^2}) = O\left( (\epsilon^{-1}(d^2 \log d \log 3 \epsilon^{-1} + \log \varphi^{-1}))^{d^2} \right)
\]

which is a weak \( \epsilon \)-net for \( P \) with probability \( \geq 1 - \varphi \).
Proof: Imagine running the algorithm of Theorem 3.6 with the better quality centerpoint algorithm, as sketched in Remark 3.8. This requires computing a sample $S$ of size as specified in Eq. (3.3). Let $Q$ be the universal set of centerpoints, as computed by Lemma 3.9, for the set $S$. We claim that $Q$ is a weak $\varepsilon$-net. Indeed, assuming the sample $S$ is good, in the sense that the algorithm of Theorem 3.6 works (which happens with probability $\geq 1 - \varphi$), then running this algorithm on any convex query body $C$ (using $S$), generates a sequence of points, such that one of them stabs $C$, if $C$ is $\varepsilon$-heavy. However, the stabbing points computed by the algorithm of Theorem 3.6 are centerpoints of some subset of $S$.

It follows that all the stabbing points that might be computed by the algorithm of Theorem 3.6, over all possible $\varepsilon$-heavy query bodies $C$ are contained in the set $Q$. As such, if $C$ is $\varepsilon$-heavy, then $C$ must contain one of the points of $Q$.

Remark 3.11. A similar construction of a weak $\varepsilon$-net from a small sample was described by Mustafa and Ray [MR08]. Their sample has exponential dependency on the dimension, so the resulting weak $\varepsilon$-net has somewhat worse dependency on the dimension than our construction. In any case, these constructions are interior as far as the dependency on $\varepsilon$, compared to the work of Matoušek and Wagner [MW04], which shows a weak $\varepsilon$-net construction of size $O_d(\varepsilon^{-d}(\log \varepsilon^{-1})^{O(d^2 \log d)})$.

4. Constructing center nets

4.1. Center nets: Weak $\varepsilon$-net that stabs a heavy body in a good centerpoint

We next introduce a strengthening of the concept of a weak $\varepsilon$-net. Namely, we require that there is a point $p$ in the net which stabs an $\varepsilon$-heavy convex body $C$, and that $p$ is also a good centerpoint for $C \cap P$.

**Definition 4.1.** For a set $P$ of $n$ points in $\mathbb{R}^d$, and parameters $\varepsilon$, and $\alpha \in (0, 1)$, a subset $W \subseteq \mathbb{R}^d$ is an $(\varepsilon, \alpha)$-center net, if for any convex shape $C$, such that $|P \cap C| \geq \varepsilon n$, we have that there is an $\alpha$-centerpoint of $P \cap C$ in $W$.

We now prove the existence such an $(\varepsilon, \alpha)$-net, where

$$\alpha = \frac{c_1}{(d+1) \log \varepsilon^{-1}},$$

and $c_1 \in (0, 1)$ is some fixed constant to be specified shortly. Note that the quality of the centerpoint is worse by a factor of $O(\log \varepsilon^{-1})$ then the best one can hope for.

4.2. Building a center net

**Construction.** We repeat the construction of the net of Lemma 3.10, with somewhat worse constants. Specifically, we take a sample $S$ of size $\mu = \tilde{O}(\varepsilon^{-1}d^2)$ from $P$, see Eq. (3.3) for the exact bound. Next, we construct the set $W$ for $S$, using the algorithm of Lemma 3.9. We claim that this is the desired center net.

4.2.1. Correctness

The proof is algorithmic. Fix any convex $\varepsilon$-heavy body $C$, and let $S_1 = S$ be the active set and let $P_1 = C \cap P$ be the residual set in the beginning of the first iteration.

We now continue in a similar, but somewhat different fashion to the algorithm of Theorem 3.6. In the $i$th iteration, the algorithm computes the $1/(d+1)$-centerpoint $q_i$ of $S_i$ (running times do not matter here, so one can afford computing the best possible centerpoint). If $q_i$ is a $2\alpha$-centerpoint for $P_i$, then $q_i$ is intuitively a good centerpoint for $P$, and the algorithm returns $q_i$ as the desired center point. Observe that by construction, $q_i \in W$ as desired.

---

1 We successfully resisted the temptation of calling this a strong weak $\varepsilon$-net.
If not, then there exists a closed halfspace \( h_i^+ \) containing \( q_i \) and at most \( 2\alpha |P_i| \) points of \( P_i \). Let
\[
P_{i+1} = P_i \setminus h_i^+ \quad \text{and} \quad S_{i+1} = S_i \setminus h_i^+.
\]
The algorithm now continues to the next iteration.

**Analysis.** The key insight is that the active set \( S_i \) shrinks must faster then the residual set \( P_i \). However, by construction, \( S_i \) provides a good upper bound to the size of \( P \). As such, once the upper bound provided by \( S_i \) on the size of \( P_i \) is too small, this would imply that the algorithm must have stopped earlier, and found a good centerpoint.

As before, we can interpret the algorithm as constructing a convex polyhedra. Indeed, let \( D_{i+1} = \bigcup_{j=1}^{i} h_j^- \), and observe that \( P_{i+1} \subseteq P \cap D_{i+1} \), and \( S_{i+1} = S \cap D_{i+1} \).

**Lemma 4.2.** Let \( \tau = \lceil 1 + 3(d+1) + (d+1) \log \varepsilon^{-1} \rceil \), and \( \alpha = 1/(4\tau) \). Assuming that \( S \) is a relative \((\varepsilon/8, 1/4)\)-approximation for the range space \( S = (P, H^\tau) \), the above algorithm stops after at most \( \tau \) iterations.

**Proof:** For an iteration \( i < \tau \), we have
\[
n_{i+1} = |P_{i+1}| \geq (1 - 2\alpha)|P_i| \geq (1 - 2\alpha)^i |P_i| \geq (1 - 2\alpha)^i n_1 \geq (1 - 2\alpha\tau)n_1 \geq (\varepsilon/2)n,
\]
using \((1 - x)^i \geq 1 - ix\), which holds\(^2\) for any positive \( x \in [0, 1] \).

On the other hand, the active set shrinks faster in each such iteration, since \( q_i \) is a \( 1/(d+1) \)-centerpoint of \( P_i \). Setting \( s_i = |S_i| \), we have that
\[
s_{i+1} \leq \left(1 - \frac{1}{d+1}\right) s_i \leq \left(1 - \frac{1}{d+1}\right)^i s_1 \leq \exp\left(-\frac{i}{d+1}\right) s_1.
\]
We have that \( s_{\tau} \leq \varepsilon s_1 / e^3 \leq (\varepsilon/20) s_1 \). Now, as \( S \) is a relative \((\varepsilon/8, 1/4)\)-approximation to \( P \) for ranges like \( D_{\tau} \), we have
\[
\frac{\varepsilon}{2} \leq \frac{|P_{\tau}|}{n} \leq \frac{|P \cap D_{\tau}|}{n} = \overline{m}(D_{\tau}) \leq \frac{1}{1 - 1/4} \overline{m}(D_{\tau}) + \frac{\varepsilon}{8} \cdot \frac{1}{2} = \frac{4}{3} \overline{m}(D_{\tau}) + \frac{\varepsilon}{16} = \frac{4|S_{\tau}|}{3|S|} + \frac{\varepsilon}{16}
\]
which is impossible. As such, the algorithm must have stopped at an earlier iteration.

**Lemma 4.3.** The above algorithm outputs a \( \alpha \)-centerpoint of \( P \cap C \).

**Proof:** Assume the algorithm stopped in the \( i \)th iteration. But then \( q_i \) is a \( 2\alpha \)-centerpoint of \( P_i \). Since \( n_i \geq n_{\tau} \geq n_1 / 2 \), it follows that any closed halfspace that contains \( q_i \), contains at least \( 2\alpha n_i \geq \alpha n_1 \) points of \( P_i \), and thus of \( P_1 \). We conclude that \( q_i \) is a \( \alpha \)-centerpoint of \( P \) as desired.

Arguing as in Remark 3.8 implies the following.

**Corollary 4.4.** For the above algorithm to succeed with probability \( \geq 1 - \varphi \), the sample \( S \) needs to be a sample of the size specified by Eq. (3.3).

\(^2\)This inequality is easy to see, but here is a cute argument via probability – consider repeating an experiment \( i \) times, independently, which succeeds with probability \( x \). The probability of success in any of the experiments is bounded by \( ix \) (by the union bound), and as such the probability of failure in all experiments is at least \( 1 - ix \). On the other hand, the probability of failure in all \( i \) experiments is exactly \((1 - x)^i \).
4.3. The result

**Theorem 4.5.** Let $P$ be a set of $n$ points in $\mathbb{R}^d$, and $\varepsilon > 0$ be a parameter. For $\gamma = \log(1/\varepsilon)$, there exists a $(\varepsilon, O(1/(d\gamma)))$-center net $W$ (which is also a weak $\varepsilon$-net) of $P$ (see Definition 4.1). The size of the net $W$ is $O(\mu d^2) \approx O(d(\varepsilon^{-1}d^2))$, where $\mu = \tilde{O}(\varepsilon^{-1}d^2)$, see Eq. (3.3) for the exact bound.

**Proof:** The theorem follows readily from the above, by setting $\varphi = 1/2$. $\blacksquare$

5. Approximating the centerpoint via Radon’s urn

We revisit the algorithm of Clarkson et al. [CEM+96] for approximating a centerpoint. We give a variant of their algorithm, and present a different (and we believe cleaner) analysis of the algorithm. In the process we improve the running time from being roughly $\tilde{O}(d^9)$ to $\tilde{O}(d^7)$.

5.1. Radon’s urn

5.1.1. Setup

In the Radon’s urn game there are initially $r$ red balls, and $b = n - r$ blue balls, and there is a parameter $t$. An iteration of the game goes as follows:

(A) The player picks a random ball, marks it for deletion, and returns it to the urn.

(B) The player picks a sample $S$ of $t$ balls (with replacement – which implies that we might have several copies of the same ball in the sample) from the urn.

(C) If at least two of the balls in the sample $S$ are red, the player inserts a new red ball to the urn, otherwise, it inserts a new blue ball.

(D) The player returns the sample to the urn.

(E) Finally, the player removes the ball marked for deletion from the urn.

As such, in each stage of the game, the number of balls in the urn remains the same. We are interested in how many rounds of the game one has to play till there are no red balls in the urn with high probability. Here, the initial value of $r$ is going to be relatively small compared to $n$.

5.2. Analysis

Let $P(r)$ be the probability of picking two or more red balls into the sample, assuming that there are $r$ red balls in the urn. We have that

$$P(r) = \sum_{i=2}^{t} \binom{t}{i} \left(\frac{r}{n}\right)^i \left(1 - \frac{r}{n}\right)^{t-i} \leq \binom{t}{2} \left(\frac{r}{n}\right)^2.$$

Let $P_+(r)$ be the probability that the number of red balls increased in this iteration. For this to happen, at least two red balls had to be in the sample, and the deleted ball must be blue. Let $P_-(r)$ be the probability that the number of red balls decreases – the player needs to pick strictly less than two red balls in the sample, and delete a red ball. This implies

$$P_+(r) = P(r)(1 - r/n) \leq P(r) \quad \text{and} \quad P_-(r) = (1 - P(r))(r/n).$$

The probability for a change in the number of red balls at this iteration is

$$P_\pm(r) = P_+(r) + P_-(r) = P(r)(1 - r/n) + (1 - P(r))(r/n) = (1 - 2r/n)P(r) + r/n.$$

We require that $P_+(r) \leq P_-(r)/2$. To this end, we solve $P(r) \leq P_-(r)/2$, which is equivalent to

$$P(r) \leq (1 - P(r))(r/n)/2 \iff \frac{2n}{r} P(r) \leq 1 - P(r).$$
This holds if \( P(r) \leq r/4n \leq 1/2 \). This in turn holds if
\[
\left( \frac{t}{2} \right) \left( \frac{r}{n} \right)^2 \leq \frac{r}{4n} \iff \frac{t(t-1)r^2}{2n^2} \leq \frac{r}{4n} \iff \frac{t(t-1)r}{n} \leq \frac{1}{2} \iff r \leq R := \frac{n}{2t(t-1)}.
\] (5.1)

We are going to start our game, with an urn with \( r_0 = (1 - \vartheta)R \) red balls. An iteration of the game where the number of red balls changed, is an effective iteration. Considering only the effective iterations, this can be interpreted as a random walk starting at \( X_0 = (1 - \vartheta)R \), and at every iteration either decreasing the value by one with probability at least \( 2/3 \), and increasing the value with probability at most \( 1/3 \) (since \( P_+(r) \leq P_-(r)/2 \)).

This walk ends when either it reaches 0 or \( R \). If we had reached \( R \), then the process had failed. If the walk reached 0, then there are no red balls in the urn, as desired.

### 5.2.1. Analyzing the related walk

Consider the random walk that starts \( Y_0 = (1 - \vartheta)R \). At the \( ith \) iteration, \( Y_i = Y_{i-1} - 1 \) with probability \( 2/3 \), and \( Y_i = Y_{i-1} + 1 \) with probability \( 1/3 \). Let \( \calY = Y_1, Y_2, \ldots \) be the resulting random walk which stochastically dominates the walk \( X_0, X_1, \ldots \).

**Lemma 5.1.** Let \( I \) be any integer number, and let \( \varphi > 0 \) be a parameter. The number of times the random walk \( \calY \) visits \( I \) is at most \( 27 + 9 \ln \varphi^{-1} \) times, and this holds with probability \( \geq 1 - \varphi \).

**Proof:** Let \( \tau \) be the first index such that \( Y_\tau = I \). The probability that \( Y_{\tau+2i} = I \), is
\[
P_{\tau+i} = \left( \frac{2i}{i} \right) \left( \frac{1}{3} \right) i \left( \frac{2}{3} \right)^i \leq 2^i \frac{1}{3^i} \frac{2^i}{3^i} = \left( \frac{8}{9} \right)^i.
\]
In particular, as \((8/9)^9 \leq 1/2\), and setting \( u = 27 + 9 \ln \varphi^{-1} \), we have \( p_{\tau+u} \leq e^{-u/9} \leq \varphi/e^3 \). As such, the probability that the walk would visit \( I \) again after time \( u \), is bounded by \( \sum_{i=0}^\infty p_{\tau+i} \leq 9p_u \leq \varphi/2 \leq \varphi \). As such, the walk might visit \( I \) during the first \( u \) iterations, but with probability \( \geq 1 - \varphi \), it never visits it again. We conclude, that with probability \( \geq 1 - \varphi \), the walk visits the value \( I \) at most \( u \) times.

We next bound the probability that the walk fails.

**Lemma 5.2.** Let \( \varphi > 0 \) be a parameter. If \( R \geq 6\vartheta^{-1} + 2.5\vartheta^{-1} \ln \varphi^{-1} \) (see Eq. (5.1)), then the probability that the random walk ever visits \( R \), and thus fails, is bounded by \( \varphi \).

**Proof:** The walk starts at \((1 - \vartheta)R\), and as such the first time it can arrive to \( R \) is at time \( \vartheta R \). In particular, the probability that \( Y_{2i+\vartheta R} = R \) is
\[
P_i = \left( \frac{\vartheta R + 2i}{\vartheta R + i} \right) \left( \frac{1}{3} \right)^{\vartheta R + i} \left( \frac{2}{3} \right)^i \leq 2^{\vartheta R + 2i} \left( \frac{1}{3} \right)^i \left( \frac{2}{3} \right)^i = \left( \frac{2}{3} \right)^i = \left( \frac{2}{3} \right)^i \left( \frac{8}{9} \right)^i.
\]
As such, we have that the probability of failure is bounded by \( \sum_{i=0}^\infty n_i \leq 9(2/3)^{\vartheta R} \). In particular, we requiring that \( 9(2/3)^{\vartheta R} \leq \varphi \), we have \( (9/\varphi) \leq (3/2)^{\vartheta R} \iff \ln(9/\varphi) \leq (\vartheta R) \ln(3/2) \iff R \geq (\vartheta \ln(3/2))^{-1} \ln(9/\varphi) \).

And this holds for the value of \( R \) stated in the lemma.

### 5.2.2. Back to the urn

The number of red balls in the urn is stochastically dominated\(^3\) by the random walk above. The challenge is that the number of iterations one has to play before an effective iteration happens (thus, corresponding to one step of the above walk), depends on the number of red balls, and behaves somewhat similar to the coupon collector problem. Specifically, if there are \( r \leq R \) red balls in the urn, then the probability for an effective step is \( P_\pm(r) \geq (1 - P(r))(r/n) \geq r/2n \), as \( P(r) \leq 1/2 \). This implies that, in expectation, one has to wait at most \( 2n/r \) iterations before an effective iteration happens.

\(^3\)Defined not “sarcastically dominates” as suggested by the spell checker.
Lemma 5.3. Let $\varphi > 0$ be a parameter. For any value $r \leq R$, the urn spends at most $O((n/r) \log \varphi^{-1})$ regular iterations having exactly $r$ balls in it, with probability $\geq 1 - \varphi$.

Proof: Consider the iteration that the urn has $r$ red balls. By the above, one has to wait in expectation at most $2n/r$ steps before the number of red balls changes. As such, by Markov’s inequality, the probability this takes more than $4n/r$ steps is at most half. Namely, if the number of iterations till an effective iteration is denoted by $\Delta$, then let $X = [\Delta/(4n/r)]$ be the number of blocks we had to wait till an effective iteration happens. It is easy to verify that $X$ has a geometric distribution, with $p = 1/2$.

As such, every run of iterations with the urn having $r$ red balls, corresponds to a random variable of such blocks. Let $X_1, X_2, \ldots, X_m$ be the number of such blocks, in the first $m$ such runs, where we can take $m = O(\log \varphi^{-1})$ by Lemma 5.1.

It is easy to verify that $\mathbb{E}[\sum_i X_i] = 2m$, and Chernoff type inequalities for sum of such geometric variables shows that this sum is smaller than, say, $8m$, with probability $2^{-3m}$, see Corollary 5.5 below, which implies the claim. As for the number of iterations the urn spends having exactly $r$ balls in it – this is bounded by $(2n/r) \sum_i X_i$.

Lemma 5.4. Let $\varphi > 0$ be a parameter, and assume that $n = \Omega(t^2\varphi^{-1} \ln \varphi^{-1})$. The total number of (regular) iterations one has to play till the urn contains only blue balls, is $O(n \log n \log (n\varphi^{-1}))$, and this bounds holds with probability $\geq 1 - \varphi$.

Proof: The bound on the number of steps follows readily by summing up the bound of Lemma 5.3, for $r = 1, \ldots, R$. Specifically, we apply this lemma with failure probability $\varphi/2n$, which implies that if there are $r$ red balls in the urn, the algorithm would perform at most $O((n/r) \log(n\varphi^{-1}))$ iterations with the number of red balls being $r$. As such, the desired number is bounded by $O\left(\sum_{r=1}^{R} (n/r) \log(n\varphi^{-1})\right) = O(n \log n \log (n\varphi^{-1}))$. The probability of failure is at most $R\varphi/2n \leq \varphi/2$.

The other reason for a failure is that the urn reaches a state where it has $R$ red balls. In particular, using Lemma 5.2 to bound this probability by $\varphi/2$, requires that $R \geq 6\varphi^{-1} + 2.5\varphi^{-1} \ln(\varphi/2))$. By Eq. (5.1), this translates into $\frac{n}{2^{t(t-1)}} \geq 6\varphi^{-1} + 2.5\varphi^{-1} \ln(\varphi/2))$, which holds if $n \geq 2t^2 \varphi^{-1}(6 + 2.5 \ln(2/\varphi))$.

In the above we used concentration inequality for sum of geometric variables. It is easy to derive such inequality from Chernoff type inequalities, see [Jan17, Theorem 2.3].

Corollary 5.5. Let $X_1, \ldots, X_m$ be $m$ independent random variables with geometric distribution with probability $1/2$. For any $\lambda \geq 1$, we have $\Pr[\sum_i X_i \geq \lambda \cdot 2m] \leq \lambda^{-1}2^{-2m(\lambda-1-\ln \lambda)}$. For $\lambda = 4$, this becomes $\Pr[\sum_i X_i \geq 8m] \leq 2^{-2m(4-1-\ln 4)} / 4 \leq 2^{-3m}$, for $m \geq 10$.

5.3. Approximating a centerpoint

5.3.1. The algorithm

Before describing the algorithm, we need the following well known facts [Mat02]:

(A) **Radon’s theorem**: Given a set $T$ of $d + 2$ points in $\mathbb{R}^d$, one can partition $T$ into two non-empty sets $T_1, T_2$, such that $\mathcal{CH}(T_1) \cap \mathcal{CH}(T_2) \neq \emptyset$. A point $p \in \mathcal{CH}(T_1) \cap \mathcal{CH}(T_2)$ is a **Radon point**.

(B) Computing a Radon point can be done by solving a system of $d + 1$ linear equalities in $d + 2$ variables, and as such can be done in $O(d^3)$ time using Gaussian elimination.

(C) A Radon point is a $2/(d + 2)$-centerpoint of $T$.

(D) Let $h^+$ be halfspace containing only one point of $T$. Then, a Radon point $p$ of $T$ is contained in $\mathbb{R}^d \setminus h^+$ [CEM+96].
The algorithm in detail. Let \( P \) be a set of \( n \) points in \( \mathbb{R}^d \), and we would like approximate its centerpoint. To this end, let \( Q \) be initially \( P \). Now, in each iteration the algorithm randomly picks \( d + 2 \) points (with repetition) from \( Q \), computes their Radon point, randomly deletes any point of \( Q \), and inserts the new Radon point into the point set \( Q \). The claim is that after sufficient number of iterations, any point of \( Q \) is a \( f(d) \)-centerpoint of \( P \), where \( f(d) = \Theta(1/d^2) \) (its exact value is specified below in Eq. (5.2)).

Remark 5.6. The algorithm above is a variant of the algorithm of Clarkson et al. [CEM+96]. Their algorithm worked in rounds, in each round generating \( n \) new Radon points, and then replacing the point set with this new set, repeating this sufficient number of times. Our algorithm on the other hand is a “continuous” process.

Intuition. A Radon point is decent center for the points defining it. As such, the above algorithm causes the points to slowly migrate towards the center region of the original point set.

To see why this is true, pick an arbitrary halfspace \( h^+ \) that contains exactly \( f(d)n \) points of \( P \). In each iteration, only if we picked two (or more) points that are in \( Q \cap h^+ \), the new point might be in \( h^+ \). As such, we are in the setting of the above Radon’s urn with \( t = d + 2 \). Indeed, color all the points inside \( h^+ \) as red, and all the points outside as blue. To apply the Radon’s urn analysis above, we require that \( (1 - \vartheta)R = f(d)n \), which by Eq. (5.1), implies that \( \frac{(1-\vartheta)n}{2(d+2)(d+1)} = f(d)n \). Namely, we have that

\[
f(d) = \frac{1 - \vartheta}{2(d + 2)(d + 1)}, \tag{5.2}
\]

where \( \vartheta \) is an arbitrary constant in \((0, 1)\). We can now apply the Radon’s urn analysis to argue that after sufficient number of iterations, all the points of \( Q \) are outside \( h^+ \). Naturally, we need to apply this analysis to all halfspaces.

5.3.2. Analysis

Consider all half-spaces that might be of interest. To this end, consider any hyperplane passing through \( d \) points of \( P \), and translate it so that it contains on one of its sides exactly \( f(d)n \) points (naturally, the are two such translations). Each such hyperplane thus defines two natural halfspaces. Let \( H \) be the resulting set of halfspaces. Observe that \(|H| \leq 2\binom{n}{d} \leq 2(ne/d)^d\). If \( Q \) does not contain any point in any of the halfspaces of \( H \) then all its points are \( f(d) \)-centerpoints. In particular, one can think about this as playing \(|H| \) parallel Radon’s urn games. We want the algorithm to succeed with probability \( \geq 1 - \varphi \). Setting the probability of success for each halfplane of \( H \) to be \( \varphi / |H| \), and by Lemma 5.4, we have that all of these halfspaces are empty after playing

\[
O(n \log n \log(n|H|/\varphi)) = O(dn \log n \log(n/\varphi))
\]

iterations, with probability of success being \( 1 - |H|(\varphi/|H|) = 1 - \varphi \) by the union bound. Using Lemma 5.4 requires that \( n = \Omega((t^2 \vartheta^{-1} \ln(|H|/\varphi)) = \Omega(\vartheta^{-1}d^3 \ln n + \vartheta^{-1}d^2 \ln \varphi^{-1}) \) which holds for \( n = \Omega(\vartheta^{-1}d^3 \ln d + \vartheta^{-1}d^2 \ln \varphi^{-1}) \).

Finding a Radon point for \( d+2 \) points in \( \mathbb{R}^d \), requires solving a linear system, and this can be done in \( O(d^3) \) time, using Gaussian elimination. As such, we get the following result.

**Lemma 5.7.** Let \( \varphi > 0 \) and \( \vartheta \in (0, 1) \) be parameters, and let \( P \) be a set of \( n \) points \( \Omega(\vartheta^{-1}d^3 \ln d + \vartheta^{-1}d^2 \ln \varphi^{-1}) \) points. Let \( \alpha = \frac{1 - \vartheta}{2(d+2)(d+1)} \). Then, one can compute a \( \alpha \)-centerpoint of \( P \) via a randomized algorithm. The running time of the randomized algorithm is \( O(d^3 \cdot \log n \log(n/\varphi)) = O(d^4n \log n \log(n/\varphi)) \), and it succeeds with probability \( \geq 1 - \varphi \).

**Theorem 5.8.** Given a set \( P \) of \( n \) points in \( \mathbb{R}^d \), and a parameter \( \varphi \), and a constant \( \vartheta \in (0, 1) \), one can compute a \( \frac{1 - \vartheta}{2(d+2)(d+1)} \)-centerpoint of \( P \). The running time of the algorithm is \( O(\vartheta^{-2}d^7 \log^3 d \log^3 \varphi^{-1}) \), and it succeeds with probability \( \geq 1 - \varphi \).
Proof: The idea to pick a random sample $S$ from $P$ that is a $(\rho, \vartheta/8)$-relative approximation for halfspaces, where $\rho = 1/(10d^2)$. This range space has VC dimension $d+1$, and by Theorem 2.7, a sample of size

$$
\mu = O\left(\rho^{-1} \vartheta^{-2} (d \log \rho^{-1} + \log \varphi^{-1})\right) = O\left(d^2 \vartheta^{-2} (d \log d + \log \varphi^{-1})\right)
$$

is a $(\rho, \vartheta/8)$-relative approximation.

By the relative approximation property, this is a $(1 \pm \vartheta/8)\tau$-centerpoint of $P$. As such, we have that $\bar{c}$ is a $\alpha$-centerpoint for $P$, where

$$
\alpha = (1 - \vartheta/8)\tau \geq \frac{(1 - \vartheta/8)^2}{2(d + 2)(d + 1)} \geq \frac{1 - \vartheta}{2(d + 2)(d + 1)}
$$

By Lemma 5.7, the running time of the resulting algorithm is

$$
O(d^4 \mu \log \mu \log(\mu/\varphi)) = O\left(\vartheta^{-2} \log^2 \vartheta^{-1} d^7 \log^3 d \log^3 \varphi^{-1}\right) = O\left(\vartheta^{-3} d^7 \log^3 d \log^3 \varphi^{-1}\right).
$$

Remark 5.9. The above compares favorably to the result of [CEM+96, Corollary 3] – they get a running time of $O(d^3 \log d + d^8 \log^2 \varphi^{-1})$, which is slower by roughly a factor of $d^2$, and computes a $\frac{1}{4.68(d+2)(d+1)}\tau$-centerpoint of $P$ – the quality of the centerpoint is roughly worse by a factor of two.

6. Expansion nets

6.1. Preliminaries

Definition 6.1. Given a convex body $C$ in $\mathbb{R}^d$, and a unit length vector $v \in \mathbb{R}^d$, let

$$
I_v(C) = \left[\min_{p \in C} \langle v, p \rangle, \max_{p \in C} \langle v, p \rangle\right]
$$

denote the directional projection of $C$ to the line passing through the origin in the direction of $v$. Similarly, given an interval $I = [a, b]$, and a unit length vector $v \in \mathbb{R}^d$, let $\text{slab}(I, v) = \{ p \in \mathbb{R}^d \mid \langle v, p \rangle \in I \}$ be the set of all points whose projection is in $I$ – this is a slab bounded by two hyperplanes that are orthogonal to $v$, and passes through the points $av$ and $bv$.

Definition 6.2. For an interval $I = [a, b] \subseteq \mathbb{R}^d$, of length $\ell = b - a$, let its $(1 + \varepsilon)$-expansion be the interval $(1 + \varepsilon)I = [a - (\varepsilon/2)\ell, b + (\varepsilon/2)\ell]$.

Given a convex body $C$, and a parameter $\varepsilon > 0$, its $(1 + \varepsilon)$-expansion, denoted by $C_{\oplus \varepsilon}$ is the set

$$
C_{\oplus \varepsilon} = \bigcap_{v \in \mathbb{R}^d, \|v\|=1} \text{slab}\left((1 + \varepsilon)I_v(C), v\right).
$$

It is easy to verify that $C \subseteq C_{\oplus \varepsilon}$.

Definition 6.3. Given two convex bodies $C \subseteq D \subseteq \mathbb{R}^d$, the convex body $D$ is a $(1 + \varepsilon)$-outer approximation to $C$, if $D \subseteq C_{\oplus \varepsilon}$. Alternatively, for any direction $v$, we have that $I_v(C) \subseteq I_v(D) \subseteq (1 + \varepsilon)I_v(C)$

We need the following well known result.

Theorem 6.4 ([Dud74]). Let $C$ be a (closed) convex body in $\mathbb{R}^d$, containing the unit ball of radius one centered at the origin, such that $C$ is contained in a ball of radius $d$ centered at the origin. For a parameter $\varepsilon > 0$, one can compute a convex body $D$, which is the intersection of $O_d(1/\varepsilon^{(d-1)/2})$ halfspaces, such that $C \subseteq D \subseteq C_{\oplus \varepsilon}$.
The following is easy to verify.

**Lemma 6.5.** Let $C$ and $E$ be two bounded convex bodies in $\mathbb{R}^d$. For a parameter $\varepsilon > 0$, consider an affine transformation $m$ that is either a rotation, translation, or scaling of the coordinate system. Then $E$ is an outer $(1 + \varepsilon)$-approximation of $C$ if and only if $mE$ is an outer $(1 + \varepsilon)$-approximation to $mC$.

An **ellipsoid** $E \subseteq \mathbb{R}^d$ is a set of the form

$$E = \{ x \in \mathbb{R}^d \mid (x - \tau)^T q (x - \tau) \leq 1 \},$$

where $q$ is a symmetric $d \times d$ positive definite matrix, and $\tau \in \mathbb{R}^d$ is the center of the ellipsoid. John’s ellipsoid theorem states that for any bounded convex body $C$, there exists an ellipsoid $E$ such that $E \subseteq C \subseteq dE$, where $dE$ is the expansion of $E$ around its center by a factor of $d$.

**Lemma 6.6.** Let $C$ be a compact convex body in $\mathbb{R}^d$. For a parameter $\varepsilon > 0$, one can compute a polytope $D$, which is the intersection of $O_d(1/\varepsilon^{(d-1)/2})$ halfspaces, such that $C \subseteq D \subseteq C_{\varepsilon}$.  

**Proof:** We use John’s ellipsoid theorem to find an ellipsoid $E$ such that $E \subseteq C \subseteq dE$, where $E$ is centered at a point $\tau$. We apply a sequence of mappings, each of them is invertible and preserve the approximation property (by Lemma 6.5). We will do the approximation in the mapped space, and then apply the inverse mapping to get back to the original space.

We first translate space so that $\tau$ is mapped to the origin. Next, we apply a rotation to the space, so that all axes of the ellipse are the standard orthonormal basis. Finally, the ellipse is now just the image of the unit ball, centered at the origin, scaled (potentially differently) in each coordinate. In particular, we now apply the inverse scaling mapping, so that the ellipse now becomes the unit ball. By Lemma 6.5, the resulting affine transformation $T$ preserves the approximation property.

In particular, we can now approximate $T(C)$ using Theorem 6.4, and let $E$ be the resulting polytope. Clearly, the desired approximation is $D = T^{-1}(E)$. \hfill $\blacksquare$

### 6.2. Expansion nets

**Definition 6.7.** Let $P$ be a set of $n$ points in $\mathbb{R}^d$ and $\varepsilon > 0$. A subset of the points $S \subseteq P$ is an **$\varepsilon$-expansion net** if for any convex body $C$ in $\mathbb{R}^d$ with $|C \cap P| \geq \varepsilon n$, then $C_{\varepsilon} \cap S \neq \emptyset$.

**Theorem 6.8.** Let $P$ be a set of $n$ points in $\mathbb{R}^d$ and $\varepsilon > 0$. Let $S \subseteq P$ be a random sample from $P$ of size $N = O_d(\varepsilon^{-\log^2 \varepsilon - 1 + \varepsilon^{-1} \log \varphi^{-1}})$. Then $S$ is an $\varepsilon$-expansion net with probability $1 - \varphi$.

**Proof:** We take a similar approach to that in Section 3, in which we consider the range space $S = (P, H^r)$, where $H^r$ consists of all convex polyhedra in $\mathbb{R}^d$ which are formed by the intersection of $\tau = O_d(1/\varepsilon^{(d-1)/2})$ halfspaces. By Lemma 2.8, the VC dimension of $S$ is $O_d(e^{-\varepsilon^{-1}} \log \varepsilon^{-1})$. We now use the $\varepsilon$-net theorem (Theorem 2.5) to construct a set $S \subseteq P$. Note that $|S| = O_d(\varepsilon^{-\log^2 \varepsilon - 1 + \varepsilon^{-1} \log \varphi^{-1}})$. We claim that this set $S$ is also an $\varepsilon$-expansion net. Indeed, let $C$ be a convex body in $\mathbb{R}^d$ with $|C \cap P| \geq \varepsilon n$. By Lemma 6.6, we can find a polytope $D$ with $\tau = O_d(1/\varepsilon^{(d-1)/2})$ faces containing $C$. But now $D$ is simply the intersection of $\tau$ halfspaces, and thus $D \subseteq H^r$. In particular, $D$ must contain a point of $S$, since $|D \cap S| \geq |C \cap S| \geq \varepsilon n$. Furthermore, $C_{\varepsilon}$ must also contain at least one point of $S$, since $D \subseteq C_{\varepsilon}$. \hfill $\blacksquare$
References

[CEM+96] K. L. Clarkson, D. Eppstein, G. L. Miller, C. Sturtivant, and S.-H. Teng. Approximating center points with iterative Radon points. *Internat. J. Comput. Geom. Appl.*, 6:357–377, 1996.

[Dud74] R. M. Dudley. Metric entropy of some classes of sets with differentiable boundaries. *J. Approx. Theory*, 10(3):227–236, 1974.

[Har11] S. Har-Peled. *Geometric Approximation Algorithms*, volume 173 of *Math. Surveys & Monographs*. Amer. Math. Soc., Boston, MA, USA, 2011.

[HS11] S. Har-Peled and M. Sharir. Relative \((p,\varepsilon)\)-approximations in geometry. *Discrete Comput. Geom.*, 45(3):462–496, 2011.

[HW87] D. Haussler and E. Welzl. \(\varepsilon\)-nets and simplex range queries. *Discrete Comput. Geom.*, 2:127–151, 1987.

[Jan17] S. Janson. Tail bounds for sums of geometric and exponential variables. *ArXiv e-prints*, September 2017.

[KST15] C. Keller, S. Smorodinsky, and G. Tardos. Improved bounds on the Hadwiger-Debrunner numbers. *ArXiv e-prints*, December 2015. [https://arxiv.org/abs/1512.04026](https://arxiv.org/abs/1512.04026).

[KST17] Chaya Keller, Shakhar Smorodinsky, and Gábor Tardos. On Max-Clique for intersection graphs of sets and the hadwiger-debrunner numbers. In Philip N. Klein, editor, *Proc. 28th ACM-SIAM Sympos. Discrete Algs. (SODA)*, pages 2254–2263. SIAM, 2017.

[LLS01] Y. Li, P. M. Long, and A. Srinivasan. Improved bounds on the sample complexity of learning. *J. Comput. Syst. Sci.*, 62(3):516–527, 2001.

[Mat02] J. Matoušek. *Lectures on Discrete Geometry*, volume 212 of *Grad. Text in Math*. Springer, 2002.

[MR08] Nabil H. Mustafa and Saurabh Ray. Weak \(\varepsilon\)-nets have basis of size \(O(\varepsilon^{-1}\log\varepsilon^{-1})\) in any dimension. *Comput.Geom. Theory Appl.*, 40(1):84–91, 2008.

[MV17] Nabil H. Mustafa and Kasturi Varadarajan. Epsilon-approximations and epsilon-nets. *CoRR*, abs/1702.03676, 2017.

[MW04] Jiří Matoušek and Uli Wagner. New constructions of weak epsilon-nets. *Discrete Comput. Geom.*, 32(2):195–206, 2004.

[RS16] Alexandre Rok and Shakhar Smorodinsky. Weak \(1/r\)-nets for moving points. In *Proc. 32nd Int. Annu. Sympos. Comput. Geom.* (SoCG), pages 59:1–59:13, 2016.

[VC71] V. N. Vapnik and A. Y. Chervonenkis. On the uniform convergence of relative frequencies of events to their probabilities. *Theory Probab. Appl.*, 16:264–280, 1971.