A Novel Mathematical Model for the Unique Shortest Path Routing Problem

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Link weights are the principal parameters of shortest path routing protocols, the most commonly used protocols for IP networks. The problem of optimally setting link weights for unique shortest path routing is addressed. Due to the complexity of the constraints involved, there exist challenges to formulate the problem properly, so that a solution algorithm may be developed which could prove to be more efficient than those already in existence. In this paper, a novel complete formulation with a polynomial number of constraints is first introduced and then mathematically proved to be correct. It is further illustrated that the formulation has advantages over a prior one in terms of both constraint structure and model size for a proposed decomposition method to solve the problem.

Key words: Mathematical Modeling, Model Verification, Constraint Structure, Model Size, Decomposition, Unique Path, Shortest Path Routing, Link Weights

1. Introduction

Shortest path routing protocols such as OSPF (Moy, 1998) are the most widely deployed and commonly used protocols for IP networks. In shortest path routing, each link is assigned a weight and traffic demands are routed along the shortest paths with respect to link weights, based on a shortest path first algorithm (Bertsekas and Gallager, 1992). Link weights are hence the principal parameters and an essential problem is to find an appropriate weight set for shortest path routing.

A simple approach to set link weights is the hop-count method, assigning the weight of each link to one. The length of a path is therefore the number of hops. Another default approach recommended by Cisco is the inv-cap method, setting the weight of a link inversely proportional to its capacity, without taking traffic conditions into consideration. More generally, the weight of a link may depend on and be related to its transmission capacity and

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its traffic load. Accordingly, a problem worth investigating is the task of finding an optimal weight set for shortest path routing, given a network topology, a projected traffic matrix (Feldmann et al., 2001), and an objective function.

The problem has two instances, depending on whether multiple shortest paths or only a unique routing path from an origin node to a destination node is allowed. For the first instance, a number of heuristic methods have been introduced, based on genetic algorithms (Ericsson et al., 2002) and local search methods (Fortz and Thorup, 2000). For the second instance, the Lagrangian relaxation method (Lin and Wang, 1993) and a local search method (Ramakrishnan and Rodrigues, 2001) have been proposed. These methods have also resulted in good routing performances by being tested in particular data sets. On the other hand, with these heuristic methods, the problem is not completely formulated and so in general is not optimally solved. It would be desirable if in average cases optimal solutions could be obtained for reasonably large data instances from real-world applications.

From a management point of view, unique-path routing uses much simpler routing mechanisms and allows for easier monitoring of traffic flows (Ameur and Gourdin, 2003). Therefore, this paper focuses on the unique-path instance. The problem is referred to as the unique shortest path routing problem. It is a reduction of the integer multi-commodity flow problem (Ahuja et al., 1993).

The problem has been well studied and efforts have been made to formulate the problem mathematically (Ameur and Gourdin, 2003; Zhang and Rodošek, 2005). Mathematical models have also been developed for other related problems such as network design and routing problems (Bley and Koch, 2002; Holmberg and Yuan, 2004). Most of these models have formulated the corresponding problems completely, whereas they have been either demand-based or path-based and have left space to further explore the structure properties of the problems, from which more efficient solution methods may be derived.

The main goal of this paper is to mathematically model the problem, which would yield a new exact solution approach for real-world applications in average data instances. In Section 2, the problem definition is first specified. Two different complete formulations, a new one and an existing one for comparison, are introduced in Section 3. The new formulation is then mathematically proved to be correct in Section 4. Differences between the two formulations on both constraint structure and model size are discussed in Section 5. Conclusions are drawn in Section 6.
2. Problem Definition

The unique shortest path routing problem is specified as follows:

Given

- A network topology, which is a directed graph structure \( G = (N, L) \), where
  - \( N \) is a finite set of nodes, each of which represents a router; and
  - \( L \) is a set of directed links, each of which corresponds to a transmission link; (For each \((i, j) \in L\), \( i \) is the starting node, \( j \) is the ending node, and \( c_{ij} \geq 0 \) is the link capacity.)

- A traffic matrix, which is a set of demands \( D \); (It is assumed that there is at most one demand between each origin-destination pair. For each demand \( k \in D \), \( s_k \in N \) is the origin node, \( t_k \in N \) is the destination node, and \( d_k > 0 \) is the required bandwidth. Accordingly, \( S \) is the set of all origin nodes, \( T_s \) is the set of all destination nodes of demands originating from node \( s \in S \), and \( D_s \) is the set of all demands originating from node \( s \in S \).)

- Lower and upper bounds of link weights, which are positive real numbers \( w_{\text{min}} \) and \( w_{\text{max}} \), respectively; and

- An objective function, specifically, to maximize the sum of residual capacities,

Find an optimal weight set \( w_{ij}, (i, j) \in L \), subject to

- Flow conservation constraints: for each demand, at each node, the sum of all incoming flows (including the demand bandwidth at the origin node) is equal to the sum of all outgoing flows (including the demand bandwidth at the destination node);

- Link capacity constraints: for each link, the load of traffic flows transiting the link does not exceed the capacity of the link;

- Path uniqueness constraints: each demand has a unique routing path; and

- Path length constraints: for each demand, the length of each path assigned to route the demand is strictly less than that of any other possible and unassigned path to route the demand.
3. Problem Formulation

In this section, the problem is mathematically formulated from two different perspectives, based on the study of the problem properties. For comparison, an existing model (Zhang and Rodošek, 2005) is introduced in more detail first.

3.1 A Demand-Based Model

According to the characteristics of unique shortest path routing, the routing path of a demand is the shortest among all possible paths. For each link, the routing path of a demand either traverses the link or not.

Figure 1 illustrates the relationships between the lengths of the shortest paths and link weights. Paths in thick lines are routing paths. The path length and path uniqueness constraints require that the length of the unique shortest path to route a demand is less than that of any other possible path from the origin node to the destination node.

With respect to the constraints, there are three scenarios to be considered:

- If the routing path of demand \( k \) traverses link \((i, j)\), the length of the shortest path from node \( s_k \) to node \( j \) is the length of the shortest path from node \( s_k \) to node \( i \) plus the weight of link \((i, j)\);
• If the routing path of demand $k$ does not traverse link $(i, j)$ but visits node $j$, the length of the shortest path from node $s_k$ to node $j$ is strictly less than the sum of the length of the shortest path from node $s_k$ to node $i$ and the weight of link $(i, j)$; (Otherwise, there would be at least two shortest paths to route demand $k$.)

• If the routing path of demand $k$ neither traverses link $(i, j)$ nor visits node $j$, the length of the shortest path from node $s_k$ to node $j$ is less than or equal to the sum of the length of the shortest path from node $s_k$ to node $i$ and the weight of link $(i, j)$.

Based on the above observations on the relationships between the length of a shortest path and the weights of links that it traverses, the problem can be mathematically formulated as a demand-based model (DBM) as follows (by defining one routing decision variable for each link-demand pair):

Routing decision variables:

$$x_{ij}^k \in \{0, 1\}, \forall k \in D, \forall (i, j) \in \mathcal{L}$$

is equal to 1 if and only if the routing path of demand $k$ traverses link $(i, j)$. The number of this set of variables is $|D||\mathcal{L}|$.

Link weight variables:

$$w_{ij} \in [w_{\text{min}}, w_{\text{max}}], \forall (i, j) \in \mathcal{L}$$

represents the routing cost of link $(i, j)$. The number of this set of variables is $|\mathcal{L}|$.

Path length variables:

$$l_i^s \in [0, +\infty), \forall s \in \mathcal{S}, \forall i \in \mathcal{N}$$

denotes the length of the shortest path from origin node $s$ to node $i$. Obviously, $l_{t_k}^s$ is the length of the shortest path to route demand $k \in D$ and $l_s^s = 0, \forall s \in \mathcal{S}$. The number of this set of variables is $|\mathcal{S}||\mathcal{N}|$.

Flow conservation constraints:

$$\sum_{h:(h,i) \in \mathcal{L}} x_{hi}^k - \sum_{j:(i,j) \in \mathcal{L}} x_{ij}^k = \begin{cases} -1, & \text{if } i = s_k \\ 1, & \text{if } i = t_k \\ 0, & \text{otherwise} \end{cases}, \forall k \in D, \forall i \in \mathcal{N}$$

The number of this set of constraints is $|D||\mathcal{N}|$. 
Link capacity constraints:

\[ \sum_{k \in D} d_k x_{ij}^k \leq c_{ij}, \forall (i, j) \in \mathcal{L} \quad (5) \]

The number of this set of constraints is \(|\mathcal{L}|\).

Path uniqueness constraints: under the combined restrictions of the flow conservation constraints and the path length constraints, the constraints are satisfied automatically.

Path length constraints:

\[ l_{sk}^i \leq l_{sk}^i + w_{ij} - \varepsilon \left( \sum_{h:(h,j) \in \mathcal{L}} x_{hj}^k - x_{ij}^k \right), \forall k \in D, \forall (i, j) \in \mathcal{L} \quad (6) \]

where \(\varepsilon\) and \(M\) are appropriate constants with \(0 < \varepsilon \ll M\). The number of this set of constraints is \(2|D||\mathcal{L}|\). By enumerating all possible values of the routing decision variables \(x_{ij}^k, k \in D, (i, j) \in \mathcal{L}\), it can be verified that the linearized constraints \((6)\) are identical to those constraints \((7)\) originally presented in logic forms, as illustrated in Figure 1:

\[ \begin{align*}
    x_{ij}^k = 0 \land \sum_{h:(h,j) \in \mathcal{L}} x_{hj}^k &= 0 \Rightarrow l_{sk}^i \leq l_{sk}^i + w_{ij} \\
    x_{ij}^k = 0 \land \sum_{h:(h,j) \in \mathcal{L}} x_{hj}^k &= 1 \Rightarrow l_{sk}^i < l_{sk}^i + w_{ij} \\
    x_{ij}^k = 1 \Rightarrow l_{sk}^i = l_{sk}^i + w_{ij} 
\end{align*} \quad (7) \]

Objective function:

\[ \max \sum_{(i,j) \in \mathcal{L}} \left( c_{ij} - \sum_{k \in D} d_k x_{ij}^k \right) \]

which is equivalent to

\[ \min \sum_{(i,j) \in \mathcal{L}} \sum_{k \in D} d_k x_{ij}^k \quad (8) \]

Accordingly, the resulting complete model is presented as

DBM: \[ \text{Optimize } (8) \]
\[ \text{Subject to } (1), (5), (6), (11), (2), (3) \]

In the following, an equivalent model to DBM, which will be used to verify the correctness of DBM in Section 4 is derived.
A necessary condition of the unique shortest path routing problem is the *sub-path optimality* requirement, which states that any sub-path of a routing path is still a unique shortest path (Bley and Koch, 2002). Specifically, given an origin node \( s \in S \) and a node \( i \in N \) where \( i \neq s \), it requires that all demands originating from \( s \) and visiting \( i \) use the same incoming link to \( i \). Mathematically, it is formulated as

\[
\sum_{h:(h,i)\in L} \max_{k\in D_s} x^k_{hi} \leq 1, \forall s \in S, \forall i \in N.
\]  

(9)

The above constraints can be alternatively presented with linear constraints, by introducing a new variable \( y^s_{ij}, s \in S, (i, j) \in L \):

\[
y^s_{ij} \geq x^k_{ij}, \forall k \in D_s, \forall (i, j) \in L \quad \text{and} \quad \sum_{h:(h,i)\in L} y^s_{hi} \leq 1, \forall s \in S, \forall i \in N.
\]

In DBM, the sub-path optimality constraints are not explicitly included. In the following, it is proved that the constraints are implied by the path length constraints.

**Proposition 1** The path length constraints in DBM imply the sub-path optimality constraints.

**Proof.** Suppose there are two demands \( k_1, k_2 \in D \), and \( s_{k_1} = s_{k_2} = s \). Assume they would use two disjoint paths to traverse from node \( u \) to node \( v \). Demand \( k_1 \) would use path

\[
P_j = (j_1, j_2) \rightarrow (j_2, j_3) \rightarrow \ldots \rightarrow (j_{n-1}, j_n), \quad (j_l, j_{l+1}) \in L, \; l = 1, \ldots, n - 1,
\]

where \( j_1 = u \) and \( j_n = v \), and demand \( k_2 \) would use path

\[
P_i = (i_1, i_2) \rightarrow (i_2, i_3) \rightarrow \ldots \rightarrow (i_{m-1}, i_m), \quad (i_q, i_{q+1}) \in L, \; q = 1, \ldots, m - 1,
\]

where \( i_1 = u \) and \( i_m = v \).

Then, by the definition of the routing decision variables,

\[
x^k_{j_lj_{l+1}} = 1 \quad \text{and} \quad x^k_{j_lj_{l+1}} = 0, \forall (j_l, j_{l+1}) \in P_j, \; l = 1, \ldots, n - 1.
\]

As a result, according to constraints (7), on one hand, since \( \forall k \in D, \forall (i, j) \in L \), \( x^k_{ij} = 1 \Rightarrow l^s_k = l^s_i + w_{ij} \), considering demand \( k_1 \),

\[
l^s_v = l^s_u + l^u_{j_n} = l^s_u + l^u_{j_{n-1}} + w_{j_{n-1}j_n} = l^s_u + l^u_{j_{n-2}} + w_{j_{n-2}j_{n-1}} + w_{j_{n-1}j_n} = l^s_u + l^u_{j_1} + w_{j_1j_2} + \cdots + w_{j_{n-1}j_n} = l^s_u + l_{P_j}.
\]  

(10)
On the other hand, since \( \forall k \in D, \forall (i, j) \in L \), \( x_{ij}^k = 0 \land \sum_{h:(h,j) \in C} x_{hj}^k = 1 \Rightarrow l_j^{sk} < l_i^{sk} + w_{ij} \) and \( x_{ij}^k = 0 \Rightarrow l_j^{sk} \leq l_i^{sk} + w_{ij} \), considering demand \( k_2 \),

\[
\begin{align*}
l_v^s &= l_u^s + l_{jn}^u \\
&< l_u^s + l_{jn-1}^u + w_{jn-1} \\
&\leq l_u^s + l_{jn-2}^u + w_{jn-2} + w_{jn-1} \\
&\leq l_u^s + l_{j1}^u + w_{j1} + \cdots + w_{jn-1} \\
&= l_u^s + l_{P_j}.
\end{align*}
\]

Contradiction then follows between (10) and (11), which means the two demands cannot be routed over two different paths between two shared nodes. It is hence proved that the sub-path optimality constraints are satisfied. \( \square \)

By Proposition 1, DBM is equivalent to the following model DBM’.

**DBM’:**

\[
\begin{align*}
\text{Optimize} & \quad (8) \\
\text{Subject to} & \quad (1), (2), (3), (4), (5), (6), (7)
\end{align*}
\]

### 3.2 An Origin-Based Model

In Section 3.1, the unique shortest path routing problem is formulated as a demand-based model, which defines one routing decision variable for each link-demand pair. Based on the study of solution properties of the problem, it can be found that all routing paths of demands originating from the same node constitute a tree, rooted at the origin node. Accordingly, a more natural formulation for the problem is to define one routing decision variable for each link-origin pair. For example, in Figure 2, instead of defining three routing decision variables for link \((i, j)\), one for each of the three demands sharing the same origin node \(s\), the new formulation defines only one routing decision variable for link \((i, j)\), associated with origin node \(s\).

Based on the above observations, an *origin-based* model (OBM) for the problem is formulated as follows.

**Routing decision variables:**

\[
y_{ij}^s \in \{0, 1\}, \forall s \in S, \forall (i, j) \in L
\]

is equal to 1 if and only if the routing paths of at least one of the demands originating from node \(s\) traverse link \((i, j)\). The number of this set of variables is \(|S||L|\).
Figure 2: Illustration of the origin-based model

**Auxiliary flow variables:**

\[ f_{ij}^s \in [0, +\infty), \forall s \in S, \forall (i, j) \in \mathcal{L} \]  \hspace{1cm} (13)

represents the load of traffic flows originating from node \( s \) and traversing link \((i, j)\). The number of this set of variables is \(|S||\mathcal{L}|\).

**Link weight variables:**

\[ w_{ij} \in [w_{\min}, w_{\max}], \forall (i, j) \in \mathcal{L} \]  \hspace{1cm} (14)

represents the routing cost of link \((i, j)\). The number of this set of variables is \(|\mathcal{L}|\).

**Path length variables:**

\[ l_i^s \in [0, +\infty), \forall s \in S, \forall i \in \mathcal{N} \]  \hspace{1cm} (15)

denotes the length of the shortest path from origin node \( s \) to node \( i \). In particular, \( l_i^s = 0, \forall s \in S \). The number of this set of variables is \(|S||\mathcal{N}|\).

**Flow conservation constraints:** for each tree, at the root node, the difference between the sum of outgoing flows and the sum of incoming flows is the sum of bandwidths of all demands originating from the root node; at the destination node of each demand originating from the
root node, the difference between the sum of incoming flows and the sum of outgoing flows is the bandwidth of the demand; and the sum of incoming flows is equal to that of outgoing flows at any other node.

$$\sum_{h:(h,i) \in \mathcal{L}} f_{hi}^s - \sum_{j:(i,j) \in \mathcal{L}} f_{ij}^s = \begin{cases} 
-d_s, & \text{if } i = s \\
d_k, & \text{if } i = t_k, \forall k \in \mathcal{D}_s, \forall s \in \mathcal{S}, \forall i \in \mathcal{N} \\
0, & \text{otherwise} 
\end{cases}$$

(16)

where $d_s = \sum_{k \in \mathcal{D}_s} d_k$. The number of this set of constraints is $|\mathcal{S}| |\mathcal{N}|$.

**Flow bound constraints:** for each tree, the total flow load over each link does not exceed the sum of all demand bandwidths originating from the root node and it is equal to zero if no demand originating from the root node is routed over the link.

$$f_{ij}^s \leq y_{ij}^s \sum_{k \in \mathcal{D}_s} d_k, \forall s \in \mathcal{S}, \forall (i, j) \in \mathcal{L}$$

(17)

The number of this set of constraints is $|\mathcal{S}| |\mathcal{L}|$.

**Link capacity constraints:**

$$\sum_{s \in \mathcal{S}} f_{ij}^s \leq c_{ij}, \forall (i, j) \in \mathcal{L}$$

(18)

The number of this set of constraints is $|\mathcal{L}|$.

**Path uniqueness constraints:** for each tree, the number of incoming links with non-zero flows is equal to zero at the origin node; the number of incoming links with non-zero flows is equal to one at the destination node of each demand originating from the root node; and the number of incoming links with non-zero flows does not exceed one at any other node.

$$\sum_{h:(h,i) \in \mathcal{L}} y_{hi}^s = \begin{cases} 
0, & \text{if } i = s \\
1, & \text{if } i \in T_s, \forall s \in \mathcal{S}, \forall i \in \mathcal{N} \\
\leq 1, & \text{otherwise} 
\end{cases}$$

(19)

The number of this set of constraints is $|\mathcal{S}| |\mathcal{N}|$.

**Path length constraints:** for each tree, the length of the unique shortest path to route a demand originating from the root node is less than that of any other possible path from the origin node to the destination node.

$$y_{ij}^s = 0 \land \sum_{h:(h,j) \in \mathcal{L}} y_{hj}^s = 0 \Rightarrow l_j^s \leq l_i^s + w_{ij}$$

$$y_{ij}^s = 0 \land \sum_{h:(h,j) \in \mathcal{L}} y_{hj}^s = 1 \Rightarrow l_j^s < l_i^s + w_{ij}$$

$$y_{ij}^s = 1 \Rightarrow l_j^s = l_i^s + w_{ij}$$

(20)
The logic constraints (20) can be linearized as follows:

\[
\begin{align*}
    l^s_j &\leq l^s_i + w_{ij} - \varepsilon \left( \sum_{h:(h,j) \in L} y^s_{hj} - y^s_{ij} \right), \forall s \in S, \forall (i, j) \in L \\
    l^s_j &\geq l^s_i + w_{ij} - M(1 - y^s_{ij})
\end{align*}
\]

where \(\varepsilon\) and \(M\) are appropriate constants with \(0 < \varepsilon \ll M\). The number of this set of constraints is \(2|S||L|\).

**Objective function:**

\[
\max \sum_{(i,j) \in L} \left( c_{ij} - \sum_{s \in S} f^s_{ij} \right)
\]

which is equivalent to

\[
\min \sum_{(i,j) \in L} \sum_{s \in S} f^s_{ij}
\]

Accordingly, the resulting complete model is presented as

**OBM:**

\[
\begin{align*}
\text{Optimize} (22) \\
\text{Subject to} (16), (17), (18), (19), (21), (12), (13), (14), (15)
\end{align*}
\]

4. **Model Verification**

The correctness of OBM is verified in this section. The verification is divided into two steps.

First, DBM is verified to be a correct model of the unique shortest path routing problem.

Second, OBM and DBM are mathematically proved to be equivalent concerning both the feasibility and the optimality of the problem, which implies that OBM is also a correct model of the problem.

Since a relaxation of the unique shortest path routing problem has been thoroughly studied and a corresponding demand-based formulation for the relaxed problem has been recognized as being correct, the verification of DBM is built on the correctness of the formulation for the relaxed problem, the *integer multi-commodity flow problem* (Ahuja et al., 1993) with sub-path optimality condition. Also, the equivalence between OBM and DBM is demonstrated based on the proof of the equivalence between two corresponding models of the relaxation, which are denoted as RDBM and ROBM, respectively.

**RDBM:**

\[
\begin{align*}
\text{Optimize} (8) \\
\text{Subject to} (4), (5), (9), (1)
\end{align*}
\]

**ROBM:**

\[
\begin{align*}
\text{Optimize} (22) \\
\text{Subject to} (16), (17), (18), (19), (12), (13)
\end{align*}
\]
As can be noted, RDBM is actually a relaxation of DBM’, which is equivalent to DBM by Proposition 1.

4.1 Correctness of DBM

To verify the correctness of DBM, that of the equivalent model DBM’ is proved. Apparently, the difference between DBM’ and RDBM lies in the path length constraints (6) and the additional link weight variables (2) as well as the path length variables (3). In order to verify that DBM’ formulates the unique shortest path routing problem correctly, constraints (6) in DBM’ are proved to represent correctly the additional path length constraints. Specifically, the following two statements are demonstrated to be correct. In DBM’, the path length constraints (6), combined with the flow conservation constraints (4), guarantee that:

1. The routing path of each demand is a shortest path; and
2. The routing path of each demand is a unique path.

The two statements are verified in Proposition 2 and Proposition 3, respectively. Proposition 2 is proved by demonstrating that the length of the routing path of each demand is less than or equal to that of any other possible path. The proof of Proposition 3 is built on, with two lemmas, the satisfaction of the single-path requirement by the relaxed problem RDBM, followed by the proof that the uniqueness requirement is satisfied by DBM’. As the original logic constraints (7) are identical to the linearized constraints (6), the following proof is based on the original constraints.

**Proposition 2** Each routing path resulting from the solution to DBM’ is a shortest path.

*Proof.* Assume for demand $k$,

$$P_j = (j_1, j_2) \rightarrow (j_2, j_3) \rightarrow \ldots \rightarrow (j_{n-1}, j_n), (j_l, j_{l+1}) \in \mathcal{L}, l = 1, \ldots, n-1,$$

where $j_1 = s_k$ and $j_n = t_k$, is the assigned routing path, and

$$P_i = (i_1, i_2) \rightarrow (i_2, i_3) \rightarrow \ldots \rightarrow (i_{m-1}, i_m), (i_q, i_{q+1}) \in \mathcal{L}, q = 1, \ldots, m-1,$$

where $i_1 = s_k$ and $i_m = t_k$, is any other possible and non-assigned path from $s_k$ to $t_k$.

Then, by the definition of the routing decision variables, $x^k_{j_lj_{l+1}} = 1$, $\forall (j_l, j_{l+1}) \in P_j$, $l = 1, \ldots, n-1$ and $\exists (i_q, i_{q+1}) \in P_i, x^k_{i_qi_{q+1}} = 0$, $q \in \{1, 2, \ldots, m-1\}$. 


Lemma 1 No optimal solution to RDBM contains flow loops.

Proof. Suppose \( x^*_{ij}, k \in \mathcal{D}, (i, j) \in \mathcal{L} \) is an optimal solution to RDBM, then all constraints (7), (5), and (9) are satisfied by \( x^*_{ij} \), \( k \in \mathcal{D}, (i, j) \in \mathcal{L} \). Specifically,

\[
\sum_{h : (h, i) \in \mathcal{L}} x^*_{hi} - \sum_{j : (i, j) \in \mathcal{L}} x^*_{ij} = \begin{cases} 
-1, & \text{if } i = s_k \\
1, & \text{if } i = t_k \\
0, & \text{otherwise}
\end{cases}, \forall k \in \mathcal{D}, \forall i \in \mathcal{N},
\]

and

\[
\sum_{k \in \mathcal{D}} d_k x^*_{ij} \leq c_{ij}, \forall (i, j) \in \mathcal{L},
\]

and

\[
\sum_{h : (h, i) \in \mathcal{L}} \max_{k \in \mathcal{D}_s} x^*_{hi} \leq 1, i \neq s, \forall s \in \mathcal{S}, \forall i \in \mathcal{N}.
\]

Assume there would exist \( k_l \in \mathcal{D} \) and a loop \( C : (j_1, j_2) \to \cdots \to (j_{n-1}, j_n) \to (j_n, j_{n+1}), (j_i, j_{i+1}) \in \mathcal{L}, i = 1, \ldots, n \) and \( j_{n+1} = j_1 \), such that \( x^*_{j_1j_{i+1}} = 1, i = 1, \ldots, n \). Obviously,

\[
\sum_{h : (h, i) \in \mathcal{C}} x^*_{hi} - \sum_{j : (i, j) \in \mathcal{C}} x^*_{ij} = 0, \forall i \in \{j_1, \ldots, j_n\}.
\]

Let

\[
y^*_k = \begin{cases} 
x^*_{ij}, & \text{if } k \neq k_l \text{ or } (i, j) \notin \mathcal{C} \\
0, & \text{otherwise}
\end{cases}, \forall k \in \mathcal{D}, \forall (i, j) \in \mathcal{L}.
\]
Then, \( y_{ij}^k \in \{0, 1\}, \forall k \in \mathcal{D}, \forall (i, j) \in \mathcal{L} \) and \( y_{ji+1}^{k_i} = 0, \forall (j_i, j_{i+1}) \in \mathcal{C}, i = 1, \ldots, n \). In addition, according to (25), (28), and (29),

\[
\sum_{h:(h,i) \in \mathcal{L}} y_{hi}^k - \sum_{j:(i,j) \in \mathcal{L}} y_{ij}^k = \sum_{h:(h,i) \in \mathcal{L}} x_{hi}^k - \sum_{j:(i,j) \in \mathcal{L}} x_{ij}^k = \begin{cases} -1, & \text{if } i = s_k \\ 1, & \text{if } i = t_k \\ 0, & \text{otherwise} \end{cases}, \forall k \in \mathcal{D}, \forall i \in \mathcal{N},
\]

according to (26) and (29),

\[
\sum_{k \in \mathcal{D}} d_k y_{ij}^k \leq \sum_{k \in \mathcal{D}} d_k x_{ij}^k \leq c_{ij}, \forall (i, j) \in \mathcal{L},
\]

and according to (27) and (29),

\[
\sum_{h:(h,i) \in \mathcal{L}} \max_k y_{hi}^k \leq \sum_{h:(h,i) \in \mathcal{L}} \max_k x_{hi}^k \leq 1, \forall i \neq s, \forall s \in \mathcal{S}, \forall i \in \mathcal{N}.
\]

Hence, \( y_{ij}^k, k \in \mathcal{D}, (i, j) \in \mathcal{L} \) satisfies all constraints (4), (5), and (9). It is therefore a feasible solution to RDBM. Furthermore, according to (29), since \( \forall k \in \mathcal{D}, d_k > 0 \),

\[
\sum_{(i,j) \in \mathcal{L}} \sum_{k \in \mathcal{D}} d_k y_{ij}^k = \sum_{(i,j) \in \mathcal{L} \setminus \mathcal{C} \cap k \in \mathcal{D} \setminus k \neq k} d_k y_{ij}^k + \sum_{(i,j) \in \mathcal{C}} d_k y_{ij}^k = \sum_{(i,j) \in \mathcal{L} \setminus \mathcal{C} \cap k \in \mathcal{D} \setminus k \neq k} d_k x_{ij}^k + \sum_{(i,j) \in \mathcal{C}} d_k x_{ij}^k < \sum_{(i,j) \in \mathcal{L} \setminus \mathcal{C} \cap k \in \mathcal{D} \setminus k \neq k} d_k x_{ij}^k + \sum_{(i,j) \in \mathcal{C}} d_k x_{ij}^k = \sum_{(i,j) \in \mathcal{L}} \sum_{k \in \mathcal{D}} d_k x_{ij}^k.
\]

Then, \( x_{ij}^k, k \in \mathcal{D}, (i, j) \in \mathcal{L} \) could not be an optimal solution to RDBM, which results in contradiction. It hence proves that no optimal solution to RDBM contains flow loops.  

Based on Lemma 1, the path uniqueness constraints are proved to be satisfied by RDBM in Lemma 2.

**Lemma 2**  The path uniqueness constraints are satisfied by RDBM.

**Proof.** By Lemma 1 in RDBM, the flow conservation constraints (4) at origin nodes are equivalent to:

\[
\sum_{h:(h,i) \in \mathcal{L}} x_{hi}^k = 0 \quad \text{and} \quad \sum_{j:(i,j) \in \mathcal{L}} x_{ij}^k = 1, \forall i = s_k, \forall k \in \mathcal{D}.
\]
Similarly, the flow conservation constraints (31) at destination nodes are equivalent to:

$$\sum_{h: (h,i) \in \mathcal{L}} x^k_{hi} = 1 \text{ and } \sum_{j: (i,j) \in \mathcal{L}} x^k_{ij} = 0, i = t_k, \forall k \in \mathcal{D}. \quad (31)$$

Since $x^k_{ij} \in \{0, 1\}, \forall k \in \mathcal{D}, \forall (i, j) \in \mathcal{L}$, constraints (30) restrict that there is one, and only one, outgoing link with a non-zero flow from the origin node of demand $k$. Similarly, constraints (31) restrict that there is one, and only one, incoming link with a non-zero flow into the destination node of demand $k$. In addition, constraints (1) guarantee that, at each intermediate node, the number of incoming links with non-zero flows is equal to the number of outgoing links with non-zero flows. Hence, for each demand $k$, the number of routing paths is no more than one. Therefore, the path uniqueness constraints are satisfied. \(\blacksquare\)

**Proposition 3** The path length constraints in DBM’ restrict that the resulting shortest path of each demand is a unique path.

**Proof.** Since DBM’ is a reduction of RDBM, the solution to the routing decision variables $x^k_{ij}, k \in \mathcal{D}, (i, j) \in \mathcal{L}$ of DBM’ is also a solution to those of RDBM.

By Lemma 2, there is only one routing path for each demand. Suppose for demand $k$,

$$P_j = (j_1, j_2) \rightarrow (j_2, j_3) \rightarrow \ldots \rightarrow (j_{n-1}, j_n), (j_l, j_{l+1}) \in \mathcal{L}, l = 1, \ldots, n-1,$$

where $j_1 = s_k, j_n = t_k$, is the assigned routing path, and

$$P_i = (i_1, i_2) \rightarrow (i_2, i_3) \rightarrow \ldots \rightarrow (i_{m-1}, i_m), (i_q, i_{q+1}) \in \mathcal{L}, q = 1, \ldots, m-1,$$

where $i_1 = s_k$ and $i_m = t_k$, is any other possible and non-assigned path from $s_k$ to $t_k$.

Then, by the definition of the routing decision variables, $x^k_{ji, j_{i+1}} = 1, \forall (j_l, j_{l+1}) \in P_j, l = 1, \ldots, n-1$.

As a result, according to constraints (7), since $\forall (i, j) \in \mathcal{L}$, $x^k_{ij} = 1 \Rightarrow \ell^k_{ij} = l^k_i + w_{ij}$,

$$\ell^k_{j_1} = l^k_s + w_{s_1} = l^k_{j_{n-1}} + w_{j_{n-1}j_n} = l^k_{j_1} + w_{j_1j_2} + \ldots + w_{j_{n-1}j_n} = \ell^k_{P_j}. \quad (32)$$

As both $P_j$ and $P_i$ are paths between $s_k$ and $t_k$, they finally merge at one node. Assume it is node $r$ and $r = j_p = i_q, p \in \{2, 3, \ldots, n-1, n\}, q \in \{2, 3, \ldots, m-1, m\}$. Then, by the definition of the routing decision variables, $x^k_{j_{p-1}r} = 1$ and $x^k_{i_{q-1}r} = 0$.

In addition, on one hand, $\sum_{h: (h,r) \in \mathcal{L}} x^k_{hr} \leq 1$. On the other hand, $\sum_{h: (h,r) \in \mathcal{L}} x^k_{hr} \geq x^k_{j_{p-1}r} + x^k_{i_{q-1}r} = 1$. Hence, $\sum_{h: (h,r) \in \mathcal{L}} x^k_{hr} = 1$.\(\blacksquare\)
As a result, according to constraints (7),
\[
\begin{align*}
l_{tk}^k &= l_{im}^k \\
&= l_{im-1}^k + w_{im-1im} \\
&= l_r^k + w_{r_{i+1}} + \ldots + w_{im-1im} \\
&< l_{ti}^k + w_{i-1r} + w_{i+1} + \ldots + w_{im-1im} \\
&\leq l_{i1}^k + w_{i1i2} + \ldots + w_{im-1im} \\
&= l_{P_i}.
\end{align*}
\]

It follows that \( l_{P_j} < l_{P_i} \) from (32) and (33). It is therefore proved that path \( P_j \) is the unique shortest path to route demand \( k \).

\textbf{Corollary 1} DBM is a correct model of the unique shortest path routing problem.

\textit{Proof.} By Proposition 2 and Proposition 3, DBM’ is a correct model of the unique shortest path routing problem. Hence, as an equivalent model to DBM’, DBM is also a correct model of the problem.

4.2 Correctness of OBM

By the proof of Lemma 2 the flow conservation constraints (4) are identical to:
\[
\begin{align*}
\sum_{h:(h,i) \in \mathcal{L}} x_{hi}^k &= 0, \sum_{j:(i,j) \in \mathcal{L}} x_{ij}^k &= 1, \text{ if } i = s_k \\
\sum_{h:(h,i) \in \mathcal{L}} x_{hi}^k &= 1, \sum_{j:(i,j) \in \mathcal{L}} x_{ij}^k &= 0, \text{ if } i = t_k \\
\sum_{h:(h,i) \in \mathcal{L}} x_{hi}^k &= \sum_{j:(i,j) \in \mathcal{L}} x_{ij}^k, \text{ otherwise}
\end{align*}
\], \( \forall k \in \mathcal{D}, \forall i \in \mathcal{N}. \) \hfill (34)

Hence, RDBM is equivalent to the following model:

\[
\text{RDBM':}\quad \text{Optimize (8)} \\
\text{Subject to (34), (5), (9), (11)}
\]

\textbf{Lemma 3} In ROBM, the flow conservation constraints at origin nodes are equivalent to:
\[
\sum_{h:(h,s) \in \mathcal{L}} f_{hi}^k = 0 \text{ and } \sum_{j:(i,j) \in \mathcal{L}} f_{ij}^k = \sum_{k \in \mathcal{D}_s} d_k, i = s, \forall s \in \mathcal{S}.
\]
Proof. On one hand, according to (13), if \( i = s \),

\[
\sum_{h: (h,i) \in L} f^s_{hi} \geq 0, \forall s \in S.
\]

On the other hand, according to (17) and (19), if \( i = s \),

\[
\sum_{h: (h,i) \in L} f^s_{hi} \leq \sum_{h: (h,i) \in L} \left( y^s_{hi} \sum_{k \in D} d_k \right) = \sum_{k \in D} d_k \sum_{h: (h,i) \in L} y^s_{hi} = 0, \forall s \in S.
\]

Hence,

\[
\sum_{h: (h,i) \in L} f^s_{hi} = 0, \forall s, s \in S.
\]

It can then be derived directly from (16) that

\[
\sum_{j: (i,j) \in L} f^s_{ij} = \sum_{k \in D} d_k, i = s, \forall s \in S.
\]

The conclusion is therefore verified. \( \square \)

By Lemma 3, the flow conservation constraints (16) are identical to:

\[
\begin{align*}
\sum_{h: (h,i) \in L} f^s_{hi} &= 0, \text{ if } i = s, \\
\sum_{j: (i,j) \in L} f^s_{ij} &= d_s, \text{ if } i = s, \\
\sum_{h: (h,i) \in L} f^s_{hi} - \sum_{j: (i,j) \in L} f^s_{ij} &= d_k, \text{ if } i = t_k, \forall k \in D_s, \\
\sum_{h: (h,i) \in L} f^s_{hi} - \sum_{j: (i,j) \in L} f^s_{ij} &= 0, \text{ otherwise}
\end{align*}
\]

where \( d_s = \sum_{k \in D_s} d_k \).

As a result, ROBM is equivalent to the following model:

\[
\text{ROBM': } \begin{align*}
\text{Optimize } & (22) \\
\text{Subject to } & (35), (17), (18), (19), (12), (13)
\end{align*}
\]

In the following, ROBM and RDBM are proved to be equivalent concerning the feasibility of the relaxed problem. The proof is heavily based on the verification of the equivalence between the two respectively identical models, ROBM' and RDBM'.

**Proposition 4** There is a solution to ROBM if RDBM is feasible.
Proof. Suppose \( x^{*k}_{ij}, k \in D, (i,j) \in L \) is a feasible solution to RDBM, and so to RDBM'. Then, all constraints (34), (35), and (9) are satisfied by \( x^{*k}_{ij}, k \in D, (i,j) \in L \).

Let

\[
y^{*s}_{ij} = \max_{k \in D_s} x^{*k}_{ij}, \forall s \in S, \forall (i,j) \in L,
\]

and

\[
f^{*s}_{ij} = \sum_{k \in D_s} d_k x^{*k}_{ij}, \forall s \in S, \forall (i,j) \in L.
\] (37)

Obviously, \( y^{*s}_{ij} \in \{0, 1\} \) and \( f^{*s}_{ij} \in [0, +\infty), \forall s \in S, \forall (i,j) \in L \).

According to (34), \( \forall k \in D \), if \( i = s_k \), \( \sum_{h:(h,i) \in L} x^{*k}_{hi} = 0 \) and \( \sum_{j:(i,j) \in L} x^{*k}_{ij} = 1 \). Then, according to (37), \( \forall s \in S \), if \( i = s \),

\[
\sum_{h:(h,i) \in L} f^{*s}_{hi} = \sum_{j:(i,j) \in L} \sum_{k \in D_s} d_k x^{*k}_{hi} = \sum_{k \in D_s} d_k \sum_{h:(h,i) \in L} x^{*k}_{hi} = 0,
\]

and

\[
\sum_{j:(i,j) \in L} f^{*s}_{ij} = \sum_{j:(i,j) \in L} \sum_{k \in D_s} d_k x^{*k}_{ij} = \sum_{k \in D_s} d_k \sum_{j:(i,j) \in L} x^{*k}_{ij} = \sum_{k \in D_s} d_k.
\]

Hence, at original nodes, constraints (35) in ROBM' are satisfied by \( f^{*s}_{ij}, s \in S, (i,j) \in L \).

Also, according to (34), \( \forall k_l \in D \), if \( i = t_{k_l} \), \( \sum_{h:(h,i) \in L} x^{*k_l}_{hi} - \sum_{j:(i,j) \in L} x^{*k_l}_{ij} = 1 \) and \( \forall k \in D_{s_{k_l}}, k \neq k_l \), \( \sum_{h:(h,i) \in L} x^{*k}_{hi} - \sum_{j:(i,j) \in L} x^{*k}_{ij} = 0 \). Then, according to (37), \( \forall k_l \in D \), if \( s = s_{k_l} \) and \( i = t_{k_l} \),

\[
\sum_{h:(h,i) \in L} f^{*s}_{hi} - \sum_{j:(i,j) \in L} f^{*s}_{ij} = \sum_{h:(h,i) \in L} \sum_{k \in D_s} d_k x^{*k}_{hi} - \sum_{j:(i,j) \in L} \sum_{k \in D_s} d_k x^{*k}_{ij}
\]

\[
= \sum_{k \in D_s} \left( \sum_{h:(h,i) \in L} d_k x^{*k}_{hi} \right) - \sum_{k \in D_s} \left( \sum_{j:(i,j) \in L} d_k x^{*k}_{ij} \right)
\]

\[
= \sum_{k \in D_s} d_k \left( \sum_{h:(h,i) \in L} x^{*k_l}_{hi} - \sum_{j:(i,j) \in L} x^{*k_l}_{ij} \right)
\]

\[
= d_{k_l} \left( \sum_{h:(h,i) \in L} x^{*k_l}_{hi} - \sum_{j:(i,j) \in L} x^{*k_l}_{ij} \right)
\]

\[
+ \sum_{k \in D_s, k \neq k_l} d_k \left( \sum_{h:(h,i) \in L} x^{*k}_{hi} - \sum_{j:(i,j) \in L} x^{*k}_{ij} \right)
\]

\[
= d_{k_l}.
\]
Hence, at destination nodes, constraints (35) in ROBM' are satisfied by \( f_{ij}^s, s \in S, (i, j) \in L \).

Similarly, according to (34), \( \forall k \in D, \) if \( i \neq s_k \) and \( i \neq t_k; \) \( \sum_{h:(h,i)\in L} x_{hi}^k - \sum_{j:(i,j)\in L} x_{ij}^k = 0. \) Then, according to (37), \( \forall s \in D, \forall i \in N, \) if \( i \neq s \) and \( i \notin D_s, \)

\[
\sum_{h:(h,i)\in L} f_{hi}^s - \sum_{j:(i,j)\in L} f_{ij}^s = \sum_{h:(h,i)\in L} \sum_{k\in D_s} d_k x_{hi}^k - \sum_{j:(i,j)\in L} \sum_{k\in D_s} d_k x_{ij}^k
\]

\[
= \sum_{k\in D_s} \left( d_k \sum_{h:(h,i)\in L} x_{hi}^k \right) - \sum_{k\in D_s} \left( d_k \sum_{j:(i,j)\in L} x_{ij}^k \right)
\]

\[
= \sum_{k\in D_s} d_k \left( \sum_{h:(h,i)\in L} x_{hi}^k - \sum_{j:(i,j)\in L} x_{ij}^k \right) = 0.
\]

Hence, at other nodes, constraints (35) in ROBM' are satisfied by \( f_{ij}^s, s \in S, (i, j) \in L. \)

According to (36) and (37), \( \forall s \in S, \forall (i, j) \in L, \)

\[
f_{ij}^s = \sum_{k\in D_s} d_k x_{ij}^k \leq \sum_{k\in D_s} d_k \max x_{ij}^k = \max_{k\in D_s} x_{ij}^k \sum_{k\in D_s} d_k = y_{ij}^s \sum_{k\in D_s} d_k.
\]

Constraints (17) in ROBM' are then satisfied by \( y_{ij}^s, f_{ij}^s, s \in S, (i, j) \in L. \)

According to (37) and (35), \( \forall (i, j) \in L, \)

\[
\sum_{s\in S} f_{ij}^s = \sum_{s\in S} \sum_{k\in D_s} d_k x_{ij}^k = \sum_{k\in D} d_k \sum_{s\in S} x_{ij}^k \leq c_{ij}.
\]

Constraints (18) in ROBM' are thus satisfied by \( f_{ij}^s, s \in S, (i, j) \in L. \)

According to (34), \( \forall k \in D, \) if \( i = s_k, \) \( \sum_{h:(h,i)\in L} x_{hi}^k = 0 \) and so \( \forall k \in D, \forall (h, i) \in L, \) if \( i = s_k, x_{hi}^k = 0. \) Then, according to (36), \( \forall s \in S, \) if \( i = s, \)

\[
\sum_{h:(h,i)\in L} y_{hi}^s = \sum_{h:(h,i)\in L} \max_{k\in D_s} x_{hi}^k = \sum_{h:(h,i)\in L} \max_{k\in D_s} 0 = 0.
\]

Hence, at original nodes, constraints (19) in ROBM' are satisfied by \( y_{ij}^s, s \in S, (i, j) \in L. \)

Also, according to (34), \( \forall k \in D, \) if \( i = t_k, \sum_{h:(h,i)\in L} x_{hi}^k = 1. \) Then, according to (36), \( \forall k \in D, \) if \( s = s_k \) and \( i = t_k, \)

\[
\sum_{h:(h,i)\in L} y_{hi}^s = \sum_{h:(h,i)\in L} \max_{k\in D_s} x_{hi}^k \geq \sum_{h:(h,i)\in L} x_{hi}^k = 1.
\]

In addition, according to (9), \( \forall k \in D, \) if \( i = t_k, \sum_{h:(h,i)\in L} \max_{k\in D_s} x_{hi}^k \leq 1. \) Then, according to (36), \( \forall k \in D, \) if \( s = s_k \) and \( i = t_k, \)

\[
\sum_{h:(h,i)\in L} y_{hi}^s = \sum_{k\in D_s} \max_{h:(h,i)\in L} x_{hi}^k \leq 1.
\]
Thus, \( \forall k \in \mathcal{D} \), if \( s = s_k \) and \( i = t_k \),
\[
\sum_{h:(h,i) \in \mathcal{L}} y_{hi}^{s_k} = 1.
\]
Hence, at destination nodes, constraints (19) in ROBM' are satisfied by \( y_{ij}^{s_k}, s \in \mathcal{S}, (i, j) \in \mathcal{L} \).

Similarly, according to (9), \( \forall s \in \mathcal{S}, \forall i \in \mathcal{N} \), if \( i \neq s \), \( \sum_{h:(h,i) \in \mathcal{L}} \max_{k \in \mathcal{D}}, x_{hi}^{s_k} \leq 1 \). Then, according to (36), \( \forall s \in \mathcal{S}, \forall i \in \mathcal{N} \), if \( i \neq s \) and \( i \notin \mathcal{T}_s \),
\[
\sum_{h:(h,i) \in \mathcal{L}} y_{hi}^{s_k} = \sum_{h:(h,i) \in \mathcal{L}} \max_{k \in \mathcal{D}}, x_{hi}^{s_k} \leq 1.
\]
Hence, at other nodes, constraints (19) in ROBM' are satisfied by \( y_{ij}^{s_k}, s \in \mathcal{S}, (i, j) \in \mathcal{L} \).

Since all constraints (35), (17), (18), and (19) in ROBM' are satisfied by \( y_{ij}^{s_k}, f_{ij}^{s_k}, s \in \mathcal{S}, (i, j) \in \mathcal{L} \), it is a corresponding feasible solution to ROBM' and also to ROBM, of the feasible solution to RDBM, \( x_{ij}^{s_k}, k \in \mathcal{D}, (i, j) \in \mathcal{L} \).

**Proposition 5** There is a solution to RDBM if ROBM is feasible.

**Proof.** Suppose \( y_{ij}^{s_k}, f_{ij}^{s_k}, s \in \mathcal{S}, (i, j) \in \mathcal{L} \) is a feasible solution to ROBM, and so to ROBM'. Then, all constraints (35), (17), (18), and (19) in ROBM' are satisfied by \( y_{ij}^{s_k}, f_{ij}^{s_k}, s \in \mathcal{S}, (i, j) \in \mathcal{L} \), it is a corresponding feasible solution to ROBM' and also to ROBM, of the feasible solution to RDBM, \( x_{ij}^{s_k}, k \in \mathcal{D}, (i, j) \in \mathcal{L} \).

Let
\[
x_{ht_k}^{s_k} = y_{ht_k}^{s_k}, \forall k \in \mathcal{D}, \forall (h, t_k) \in \mathcal{L}.
\]
According to constraints (19) at destination nodes, \( \sum_{h:(h,i) \in \mathcal{L}} y_{hi}^{s_k} = 1, \forall k \in \mathcal{D} \). Hence, \( \forall k \in \mathcal{D}, \exists h : (h, t_k) \in \mathcal{L} \), such that \( y_{ht_k}^{s_k} = 1 \).

\( \forall k \in \mathcal{D}, \forall (i, j) \in \mathcal{L} \), if \( (i, j) \neq (i, t_k) \), \( x_{ij}^{s_k} \) is assigned as follows:

**Initialize**
\[
x_{ij}^{s_k} \leftarrow 0, \quad \forall k \in \mathcal{D}, \forall (i, j) \in \mathcal{L}, (i, j) \neq (i, t_k)
\]

**For** \( k \in \mathcal{D} \)
\[
0 \quad i \leftarrow t_k
\]

**Do**
\[
\text{find } h : (h, i) \in \mathcal{L}, \text{ such that } y_{hi}^{s_k} = 1
\]
\[
x_{hi}^{s_k} \leftarrow 1
\]
\[
i \leftarrow h
\]

**Until**
\[
i = s_k
\]

In the above assigning process (38), at node \( i \) in each iteration of the inner loop, on one hand, according to (35),
\[
\sum_{h:(h,i) \in \mathcal{L}} f_{hi}^{s_k} \geq \sum_{j:(t_k,j) \in \mathcal{L}} f_{t_kj}^{s_k} + d_k \geq d_k > 0, \forall k \in \mathcal{D},
\]
20
and according to (17),

$$
\sum_{h:(h,i) \in \mathcal{L}} f^{*k}_{hi} \leq \sum_{h:(h,i) \in \mathcal{L}} \left( y^{*k}_{hi} \sum_{k' \in \mathcal{D}_{sk}} d_{k'} \right) = \sum_{k' \in \mathcal{D}_{sk}} d_{k'} \sum_{h:(h,i) \in \mathcal{L}} y^{*k}_{hi}, \forall k \in \mathcal{D}.
$$

Hence,

$$\sum_{h:(h,i) \in \mathcal{L}} y^{*k}_{hi} > 0, \forall k \in \mathcal{D}.$$ 

On the other hand, according to (19),

$$\sum_{h:(h,i) \in \mathcal{L}} y^{*k}_{hi} \leq 1, \forall k \in \mathcal{D}.$$ 

Then, at node $i$ in each iteration of the inner loop,

$$\sum_{h:(h,i) \in \mathcal{L}} y^{*k}_{hi} = 1, \forall k \in \mathcal{D}.$$ 

Therefore, at node $i$ in each iteration of the inner loop, $\exists h : (h, i) \in \mathcal{L}$, such that $x^{*k}_{hi} = y^{*k}_{hi} = 1$. Moreover, $\forall k \in \mathcal{D}$, according to (35), the process terminates at node $s_k$.

According to (38), obviously,

$$x^{*k}_{ij} \in \{0, 1\} \text{ and } x^{*k}_{ij} \leq y^{*k}_{ij}, \forall k \in \mathcal{D}, \forall(i, j) \in \mathcal{L}.$$ 

Then, according to (19) at original nodes, if $i = s_k$,

$$\sum_{h:(h,i) \in \mathcal{L}} x^{*k}_{hi} \leq \sum_{h:(h,i) \in \mathcal{L}} y^{*k}_{hi} = 0, \forall k \in \mathcal{D}.$$ 

In addition, the assigning process terminates at node $s_k$, $\forall k \in \mathcal{D}$, and so if $i = s_k$,

$$\sum_{j:(i,j) \in \mathcal{L}} x^{*k}_{ij} = 1, \forall k \in \mathcal{D}.$$ 

Hence, at original nodes, constraints (34) in RDBM' are satisfied by $x^{*k}_{ij}, k \in \mathcal{D}, (i, j) \in \mathcal{L}$.

According to (19) at destination nodes, if $i = t_k$,

$$\sum_{h:(h,i) \in \mathcal{L}} x^{*k}_{hi} = \sum_{h:(h,i) \in \mathcal{L}} y^{*k}_{hi} = 1, \forall k \in \mathcal{D}.$$
Furthermore, according to (38), if \(i = t_k\), \(x^{*k}_{ij} = 0, \forall k \in D, \forall (i, j) \in \mathcal{L}\). Then, if \(i = t_k\),
\[
\sum_{j: (i,j) \in \mathcal{L}} x^{*k}_{ij} = 0, \forall k \in \mathcal{D}.
\]
Hence, at destination nodes, constraints (34) in RDBM are satisfied by \(x^{*k}, k \in \mathcal{D}, (i, j) \in \mathcal{L}\).

According to (38), at node \(i\) in each iteration of the inner loop,
\[
\sum_{h: (h,i) \in \mathcal{L}} x^{*k}_{hi} = \sum_{j: (i,j) \in \mathcal{L}} x^{*k}_{ij} = 1, \forall k \in \mathcal{D},
\]
and at any other node \(i' \in \mathcal{N}, i' \neq s_k, i' \neq t_k\),
\[
\sum_{h: (h,i') \in \mathcal{L}} x^{*k}_{hi'} = \sum_{j: (i',j) \in \mathcal{L}} x^{*k}_{ij} = 0, \forall k \in \mathcal{D}.
\]
Then, if \(i \neq s_k, i \neq t_k\),
\[
\sum_{h: (h,i) \in \mathcal{L}} x^{*k}_{hi} - \sum_{j: (i,j) \in \mathcal{L}} x^{*k}_{ij} = 0, \forall k \in \mathcal{D}.
\]
Hence, at other nodes, constraints (34) in RDBM are satisfied by \(x^{*k}, k \in \mathcal{D}, (i, j) \in \mathcal{L}\).

According to (35) and (18),
\[
\sum_{k \in \mathcal{D}} d_k x^{*k}_{ij} = \sum_{s \in \mathcal{S}} \sum_{k \in \mathcal{D}} d_k x^{*k}_{ij} \leq \sum_{s \in \mathcal{S}} f^{*s}_{ij} \leq c_{ij}, \forall (i, j) \in \mathcal{L}.
\]
Hence, constraints (5) in RDBM are satisfied by \(x^{*k}_{ij}, k \in \mathcal{D}, (i, j) \in \mathcal{L}\).

According to (38), \(x^{*k}_{ij} \leq y^{*s}_{ij}, \forall k \in \mathcal{D}, \forall (i, j) \in \mathcal{L}\). Then, according to (19), if \(i \neq s\),
\[
\sum_{h: (h,i) \in \mathcal{L}} \max_{k \in \mathcal{D}_s} x^{*k}_{hi} \leq \sum_{h: (h,i) \in \mathcal{L}} \max_{k \in \mathcal{D}_s} y^{*s}_{hi} = \sum_{h: (h,i) \in \mathcal{L}} y^{*s}_{hi} \leq 1, \forall s \in \mathcal{S}, \forall i \in \mathcal{N}.
\]
Hence, constraints (9) in RDBM are satisfied by \(x^{*k}_{ij}, k \in \mathcal{D}, (i, j) \in \mathcal{L}\).

Since all constraints (34), (5), and (9) in RDBM are satisfied by \(x^{*k}_{ij}, k \in \mathcal{D}, (i, j) \in \mathcal{L}\), it is a corresponding feasible solution to RDBM', and so to RDBM, of the feasible solution to ROBM, \(y^{*s}_{ij}, f^{*s}_{ij}, s \in \mathcal{S}, (i, j) \in \mathcal{L}\). □

**Theorem 1** ROBM and RDBM are equivalent concerning the feasibility of the relaxed problem.

**Proof.** The conclusion is derived directly from Proposition 4 and Proposition 5. □

Based on the proof of the equivalence between ROBM and RDBM, the equivalence between OBM and DBM, concerning the feasibility of the unique shortest path routing problem, is verified as follows.
Proposition 6 There is a corresponding solution satisfying the path length constraints in OBM, for each solution satisfying the path length constraints in DBM.

Proof. Suppose \( x^k_{ij}, k \in \mathcal{D}, (i, j) \in \mathcal{L} \), \( w^*_{ij}, (i, j) \in \mathcal{L} \), and \( l^s_i, s \in \mathcal{S}, i \in \mathcal{N} \) is a feasible solution to DBM and also to DBM’ by Proposition 1. Since RDBM is a relaxation of DBM’, \( x^k_{ij}, k \in \mathcal{D}, (i, j) \in \mathcal{L} \) is then a feasible solution to RDBM and satisfies all corresponding constraints.

Let

\[
y^s_{ij} = \max_{k \in \mathcal{D}} x^k_{ij}, \forall s \in \mathcal{S}, \forall (i, j) \in \mathcal{L}.
\]

Obviously, \( y^s_{ij} \in \{0, 1\}, \forall s \in \mathcal{S}, \forall (i, j) \in \mathcal{L} \). Also, \( \forall s \in \mathcal{S}, \forall (i, j) \in \mathcal{L} \), there are three cases:

- Case 1: \( x^k_{ij} = 0 \) and \( \sum_{h:(h,j) \in \mathcal{L}} x^k_{hj} = 0, \forall k \in \mathcal{D} \);
- Case 2: \( x^k_{ij} = 0, \forall k \in \mathcal{D}_s \) and \( \exists k_l \in \mathcal{D}_s, \sum_{h:(h,j) \in \mathcal{L}} x^{k_l}_{hj} = 1 \);
- Case 3: \( \exists k_l \in \mathcal{D}_s, x^{k_l}_{ij} = 1 \).

For Case 1, on one hand, \( \forall k \in \mathcal{D}_s \), constraints (6) can be simplified as follows:

\[
l^s_j \leq l^s_i + w_{ij} \quad \text{and} \quad l^s_j \geq l^s_i + w_{ij} - M.
\]

On the other hand, since \( y^s_{ij} = \max_{k \in \mathcal{D}_s} x^k_{ij} = 0 \) and \( \sum_{h:(h,j) \in \mathcal{L}} y^s_{hj} = 0 \), constraints (21) can be simplified as follows:

\[
l^s_j \leq l^s_i + w_{ij} \quad \text{and} \quad l^s_j \geq l^s_i + w_{ij} - M.
\]

Hence, the simplified constraints of (21) are identical to those of (6) for Case 1.

For Case 2, on one hand, constraints (6) can be simplified as follows:

\[
l^{s_k}_j \leq l^{s_k}_i + w_{ij} - \varepsilon \quad \text{and} \quad l^{s_k}_j \geq l^{s_k}_i + w_{ij} - M,
\]

and \( \forall k \in \mathcal{D}_s \) such that \( \sum_{h:(h,j) \in \mathcal{L}} x^{s_k}_{hj} = 0 \),

\[
l^s_j \leq l^s_i + w_{ij} \quad \text{and} \quad l^s_j \geq l^s_i + w_{ij} - M.
\]

On the other hand, since \( y^s_{ij} = \max_{k \in \mathcal{D}_s} x^k_{ij} = 0 \) but \( \sum_{h:(h,j) \in \mathcal{L}} y^s_{hj} = 1 \), constraints (21) can be simplified as follows:

\[
l^s_j \leq l^s_i + w_{ij} - \varepsilon \quad \text{and} \quad l^s_j \geq l^s_i + w_{ij} - M.
\]
Hence, the simplified constraints of (21) are identical to those of (6) for Case 2.

For Case 3, on one hand, according to the sub-path optimality constraints, \( \forall k \in D_s \), if \( x_{ij} = 0 \), \( \sum_{h:(h,j) \in \mathcal{L}} x_{hkj} = 0 \). Then, constraints (6) can be simplified as follows:

\[
L^s_{ij} \leq L^s_{ki} + w_{ij} \quad \text{and} \quad L^s_{ij} \geq L^s_{ki} + w_{ij},
\]

and \( \forall k \in D_s \) such that \( x_{ij} = 0 \),

\[
L^s_{ij} \leq L^s_{ki} + w_{ij} \quad \text{and} \quad L^s_{ij} \geq L^s_{ki} + w_{ij} - M.
\]

On the other hand, since \( y_{ij}^s = \max_{k \in D_s} x_{ij}^k = 1 \) and \( \sum_{h:(h,j) \in \mathcal{L}} y_{hj}^s = 1 \), constraints (21) can be simplified as follows:

\[
L^s_{ij} \leq L^s_{ki} + w_{ij} \quad \text{and} \quad L^s_{ij} \geq L^s_{ki} + w_{ij}.
\]

Hence, the simplified constraints of (21) are identical to those of (6) for Case 3.

Since for all the three cases, \( y^s_{ij}, s \in S, (i,j) \in \mathcal{L} \) results in the same path length constraints for OBM as those resulting from \( x^k_{ij}, k \in D, (i,j) \in \mathcal{L} \) for DBM, then, there is a corresponding feasible solution satisfying the path length constraints (21) in OBM, provided that there is a feasible solution satisfying the path length constraints (6) in DBM. □

**Proposition 7** There is a corresponding solution satisfying the path length constraints in DBM, for each solution satisfying the path length constraints in OBM.

**Proof.** Suppose \( y^s_{ij}, f^s_{ij}, s \in S, (i,j) \in \mathcal{L}, w^s_{ij}, (i,j) \in \mathcal{L}, \) and \( l^s_{ij}, s \in S, i \in \mathcal{N} \) is a feasible solution to OBM. Then since ROBM is a relaxation of OBM, \( y^s_{ij}, f^s_{ij}, s \in S, (i,j) \in \mathcal{L} \) is also a feasible solution to ROBM and so satisfies all corresponding constraints.

According to the assigning process (38), let \( x^k_{ij}, k \in D, (i,j) \in \mathcal{L} \) be the corresponding solution to RDBM.

\( \forall s \in S, \forall (i,j) \in \mathcal{L}, \) there are three cases:

- Case 1: \( y^s_{ij} = 0 \) and \( \sum_{h:(h,j) \in \mathcal{L}} y^s_{hj} = 0; \)
- Case 2: \( y^s_{ij} = 0 \) and \( \sum_{h:(h,j) \in \mathcal{L}} y^s_{hj} = 1; \)
- Case 3: \( y^s_{ij} = 1.\)
For Case 1, constraints (21) can be simplified as follows:

\[ l^s_j \leq l^s_i + w_{ij} \quad \text{and} \quad l^s_j \geq l^s_i + w_{ij} - M. \]

According to (38), \( \forall k \in \mathcal{D}, x^*_{kj} = 0 \) and \( \sum_{h:(h,j) \in \mathcal{L}} x^*_{hj} = 0 \). Then, \( \forall k \in \mathcal{D} \), constraints (6) can be simplified as follows:

\[ l^s_k \leq l^s_i + w_{ij} \quad \text{and} \quad l^s_k \geq l^s_i + w_{ij} - M. \]

Hence, the simplified constraints of (6) are identical to those of (21) for Case 1.

For Case 2, constraints (21) can be simplified as follows:

\[ l^s_j \leq l^s_i + w_{ij} - \varepsilon \quad \text{and} \quad l^s_j \geq l^s_i + w_{ij} - M. \]

According to (38), \( \forall k \in \mathcal{D}, x^*_{kj} = 0 \) and \( \exists k_l \in \mathcal{D}, \sum_{h:(h,j) \in \mathcal{L}} x^*_{hj} = 1 \). Then, constraints (6) can be simplified as follows:

\[ l^s_{kl} \leq l^s_i + w_{ij} - \varepsilon \quad \text{and} \quad l^s_{kl} \geq l^s_i + w_{ij} - M, \]

and \( \forall k \in \mathcal{D} \) such that \( \sum_{h:(h,j) \in \mathcal{L}} x^*_{hj} = 0 \),

\[ l^s_k \leq l^s_i + w_{ij} \quad \text{and} \quad l^s_k \geq l^s_i + w_{ij} - M. \]

Hence, the simplified constraints of (6) are identical to those of (21) for Case 2.

For Case 3, constraints (21) can be simplified as follows:

\[ l^s_j \leq l^s_i + w_{ij} \quad \text{and} \quad l^s_j \geq l^s_i + w_{ij}. \]

According to (38), \( \exists k_l \in \mathcal{D}, x^*_{kj} = 1 \) and \( \sum_{h:(h,j) \in \mathcal{L}} x^*_{hj} = 1 \). In addition, according to the sub-path optimality constraints, \( \forall k \in \mathcal{D}, x^*_{ij} = 0 \), \( \sum_{h:(h,j) \in \mathcal{L}} x^*_{hj} = 0 \). Then, constraints (6) can be simplified as follows:

\[ l^s_{kl} \leq l^s_i + w_{ij} \quad \text{and} \quad l^s_{kl} \geq l^s_i + w_{ij}, \]

and \( \forall k \in \mathcal{D} \) such that \( x^*_{ij} = 0 \),

\[ l^s_k \leq l^s_i + w_{ij} \quad \text{and} \quad l^s_k \geq l^s_i + w_{ij} - M. \]

Hence, the simplified constraints of (6) are identical to those of (21) for Case 3.

Since for all the three cases, \( x^*_{ij}, k \in \mathcal{D}, (i, j) \in \mathcal{L} \) results in the same path length constraints for DBM as those resulting from \( y^*_{ij}, s \in \mathcal{S}, (i, j) \in \mathcal{L} \) for OBM, then, there is a corresponding feasible solution satisfying the path length constraints (6) in DBM, provided that there is a feasible solution satisfying the path length constraints (21) in OBM. \[ \square \]
Corollary 2 The path length constraints in OBM are equivalent to those in DBM, concerning the feasibility of the unique shortest path routing problem.

Theorem 2 OBM and DBM are equivalent concerning the feasibility of the unique shortest path routing problem.

Proof. By Proposition 1, DBM is equivalent to DBM'.

In addition, as discussed at the begin of Section 4, OBM is a reduction of ROBM and DBM' is a reduction of RDBM. Besides the additional link weight variables and path length variables, the difference between OBM and ROBM are the path length constraints (21) and the difference between DBM' and RDBM are the path length constraints (6).

By Theorem 1, ROBM and RDBM are equivalent concerning the feasibility of the relaxed problem. By Corollary 2, the path length constraints in OBM are equivalent to the counterparts in DBM, and so those in DBM'. Therefore, OBM is equivalent to DBM', and so DBM, concerning the feasibility of the unique shortest path routing problem.

Theorem 3 OBM and DBM are equivalent concerning the optimality of the unique shortest path routing problem.

Proof. The conclusion follows directly from Theorem 2 by constructing the corresponding optimal solutions between OBM and DBM.

Corollary 3 OBM is a correct model of the unique shortest path routing problem.

5. Comparisons between the Two Formulations

Concerning the unique shortest path routing problem, it has been shown that routing performances resulting from the proposed complete formulations are much better than those derived from the default methods, by testing on 30 randomly generated data instances with combinations of different parameter scenarios. The resulting average maximum utilization is 30.94% of that from using the hop-count method and 45.54% of that from using the inv-cap method. It hence demonstrates the significant gain achieved by formulating the problem completely and solving it optimally.

Between the two complete formulations, compared with DBM, OBM has advantages on both constraint structure for applying constraint generation algorithms and model size.
5.1 Constraint Structure

The constraint structures of DBM and OBM are shown in Figure 3 and Figure 4 respectively.

In Figure 3, the first row represents the link capacity constraints (5), the next four rows correspond to the flow conservation constraints (4), and the last four rows represent the path length constraints (6). Accordingly, columns correspond to variables.

As can be seen, among the three sets of constraints, the flow conservation constraints and the link capacity constraints contain only the routing decision variables, whereas the path length constraints couple the routing decision variables with the link weight variables and the path length variables. Hence, constraint generation algorithms such as the Benders decomposition method (Benders, 1962) may be considered to be the promising solution approaches for the problem. The problem can be decomposed into one integer programming master problem and one linear programming subproblem. The master problem deals with the flow conservation constraints and the link capacity constraints, and so contains the routing decision variables only. Accordingly, the subproblem copes with the path length constraints.
Similarly, in Figure 4, the first four rows represent the path uniqueness constraints (19), the next four rows correspond to the flow bound constraints (17), the third four rows represent the flow conservation constraints (16), the next row corresponds to the link capacity constraints (18), and the last four rows represent the path length constraints (21). Columns correspond to variables accordingly.

As can be noted, although DBM has a simpler constraint structure, OBM has more flexibility to apply decomposition algorithms to solve the problem.

As shown in Figure 4, with OBM, the problem can be globally decomposed into one master problem and two subproblems, instead of one master problem and one subproblem as with DBM. The master problem contains only the routing decision variables and the path uniqueness constraints accordingly. The first subproblem deals with the auxiliary flow variables and the second subproblem copes with the link weight variables and the path length variables. In addition, the master problem can be further decomposed, with one independent
subproblem corresponding to each origin node.

## 5.2 Model Size

Compared with DBM, OBM defines explicitly the auxiliary flow variables and the flow bound constraints accordingly. However, in general, $|S| << |D|$, and the size of OBM is much smaller than that of DBM. The model sizes of the two formulations are as shown in Table 1 where $\#Variables$ represents the number of variables and $\#Constraints$ denotes the number of constraints.

More concretely, the model sizes of both the original problems and the master problems of the two formulations on a randomly generated data instance with $|N| = 50$, $|L| = 642$, $|D| = 1000$, and $|S| = 50$ are shown in Table 2.

As can be seen from Table 2, with OBM, the number of variables of the original problem decreases from over 600,000 to 64,200 and the number of constraints drops from over 1,000,000 to less than 38,000. In addition, with OBM, both the number of variables and the number of constraints of the master problem decline 20 times.

| Model | $\#Variables$ | $\#Constraints$ |
|-------|---------------|-----------------|
| DBM   | $|D||L| + |S||N| + |L|$ | $|D||N| + 2|D||L| + |L|$ |
| OBM   | $2|S||L| + |S||N| + |L|$ | $2|S||N| + 3|S||L| + |L|$ |

| Original Problem | Master Problem |
|------------------|----------------|
| $\#Variables$    | $\#Constraints$ |
| DBM 645,142      | 1,334,642      |
| OBM 64,200       | 37,742         |

| $\#Variables$    | $\#Constraints$ |
|------------------|-----------------|
| DBM 642,000      | 50,642          |
| OBM 32,100       | 2,500           |

As a conclusion, compared with DBM, OBM has a smaller model size and a more flexible constraint structure for decomposition algorithms such as the Benders decomposition method to solve the problem.

## 6. Conclusions

With the aim of an exact solution approach to the unique shortest path routing problem on average data instances arising from real-world applications, two complete and explicit
mathematical formulations with a polynomial number of constraints for the problem are developed. A demand-based formulation is first introduced, based on the study of the relationships between the length of a shortest path and the weights of links that the path traverses. The problem is further formulated as an origin-based model by analyzing solution properties of the problem. The two formulations are then mathematically proved to be correct and to be equivalent concerning both the feasibility and the optimality of the problem. Based on the study of the constraint structures and model sizes of the two formulations, the origin-based formulation is identified to be the better one for decomposition algorithms such as the Benders decomposition method to solve the problem.

The two formulations may be generalized to other network flow and network routing problems. Prospective future work may lie in investigating possible improvements concerning both problem formulation and solution algorithm to improve the efficiency of the solution approach proposed. In particular, investigations may focus on possible improvements related to three factors: the closeness between the initial solution and the final solution to the master problem, the strength of cuts generated at each iteration, and the efficiency of an algorithm to solve the integer programming master problem. For example, redundant constraints may be generated to tighten the feasible region of the initial master problem, strategies such as active set method may be applied to strengthen the cuts generated from the subproblems, and schemes such as the Lagrangian relaxation method may be embedded into the solution algorithm to improve the efficiency of solving the master problem at each iteration.

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