Two-layer interfacial flows beyond the Boussinesq approximation: a Hamiltonian approach

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Abstract
The theory of integrable systems of Hamiltonian PDEs and their near-integrable deformations is used to study evolution equations resulting from vertical-averages of the Euler system for two-layer stratified flows in an infinite two-dimensional channel. The Hamiltonian structure of the averaged equations is obtained directly from that of the Euler equations through the process of Hamiltonian reduction. Long-wave asymptotics together with the Boussinesq approximation of neglecting the fluids’ inertia is then applied to reduce the leading order vertically averaged equations to the shallow-water Airy system, albeit in a non-trivial way. The full non-Boussinesq system for the dispersionless limit can then be viewed as a deformation of this well known equation. In a perturbative study of this deformation, a family of approximate constants of the motion are explicitly constructed and used to find local solutions of the evolution equations by means of hodograph-like formulae.

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(Some figures may appear in colour only in the online journal)
1. Introduction

Aspects of the theory of two-layer stratified flows in an infinite 2D channel have been the subject of intense recent studies. Layer models are widely used in a variety of geophysical applications (going back to early references, such Long [22], in meteorology literature), and are of conceptual value for illustrating many fundamental properties of stratified fluid dynamics. A typical configuration is depicted in figure 1, with an interface between the two fluid representing the sharp pycnocline between upper, fresh water, labeled by the index 1, and deep, salty water, labeled by the index 2. (Other relevant notation used throughout the paper is defined by this figure). Long internal waves in such systems were studied in, e.g. [10, 11], by deriving the two-layer models (including dispersive terms) by the layer-averaging method (see e.g. [34]). Their dispersionless counterparts were more recently reconsidered in papers by Milewski, Tabak and collaborators [12, 26, 29]. These papers were mainly interested in studying the Kelvin–Helmholtz (KH) instability (see also [3, 4]) viewed as hyperbolic versus elliptic transition for the resulting quasi-linear equations of motion, and its relation to the well-posedness of the initial value problem for these equations. In particular, for the so-called Boussinesq approximation [2, 22, 35], which in this context consists of disregarding density differences in the inertial terms while retaining them in the buoyancy terms, the following conditions for shear-flow stability are all equivalent:

(i) The ‘standard’ stability criterion in terms of the Richardson number for the linearized two-layer equations ([5, 25]) around a velocity jump.

(ii) The hyperbolicity (nonlinear) criterion for the reduced quasi-linear system of PDEs in the variables \( \xi, w \) (the difference between the fluid layer thicknesses and the velocity shear, respectively).

(iii) A criterion given by a suitably defined Richardson-like number expressing the ratio of the kinetic energy available for the complete mixing of the two layers to the potential energy barrier given by the stratification (see e.g. [13, 29]).

The equivalence between the first two conditions holds independently of the Boussinesq approximation once the reduction to two dependent fields such as \( \xi \) and \( w \) is carried out. In contrast, going beyond the Boussinesq approximation the equivalence of the third condition with the first two fails. In fact, the relevant Richardson number as computed in [5] is

\[
Ri = \frac{2gr(h - m_2)}{(1 - r^2)w^2},
\]

(1.1)
where \( r \) is half of the density difference divided by its mean, \( \eta \) is the interface height, \( h \) is the channel height, and \( w \) is the velocity shear at the interface. The ratio of the potential and kinetic energy balance mentioned in the third condition is

\[
\frac{2grh^2}{(h - r\eta)w^2},
\]

so that the two instability-related numbers need not coincide as soon as \( r \neq 0 \). Thus, the increased physical accuracy of the non-Boussinesq theory requires a modification of the stability criterion and can affect the mathematical and physical properties of the full system such as its well-posedness and the formation of KH bellows.

A further motivation to go beyond the Boussinesq approximation stems from some recent results obtained in [6, 7] concerning an apparently paradoxical consequence of stratification and confinement, that is, the non-conservation of the horizontal momentum due to pressure imbalances at the far ends of the channel. This phenomenon, whose occurrence was suggested in [2], has indeed been detailed and substantiated in [6, 7] for the full 2D Euler equations via numerical and analytical methods. As we shall see in section 2.1, the 1D Boussinesq approximation is oblivious to this effect, whereas the non-Boussinesq model exhibits lack of horizontal momentum conservation in its general solutions.

With this in mind, our study is organized as follows. After a short review of the derivation from the 2D Euler equations of the governing equations for the evolution of the interface and suitable layer mean quantities, and a discussion of the Boussinesq approximation, we briefly examine the hyperbolic-elliptic transition of the non-Boussinesq limit (as in [5]). We then turn to the first of the aims of the present paper, that of fully framing the theory of two layer models within the Hamiltonian settings of the Euler equations. We work with the setting devised in [2], which is specifically suited to treat heterogeneous fluids in two-dimensional domains, and does not require the introduction of Clebsch variables (as with the original more general setting discussed in [36, 37] and [30]). By means of a version of the Marsden–Ratiu–Weinstein reduction procedure (see, e.g. [24]), we show how the Poisson structure defined by Benjamin in [2] on the full phase space of the Euler equation gives rise to a well defined ‘canonical’ Poisson structure on the phase space of the reduced quasi-linear equations of the 1D dispersionless limit. This Poisson structure is independent of the 1D model, i.e. it is the same for both the Boussinesq approximation and for its non-Boussinesq deformation; a key point for its definition is to replace, in the pair of coordinates for the 1D model, the velocity shear \( w \) (used in [5]) with the momentum shear, which is arguably the most natural choice for the reduction process of the Benjamin 2D Poisson structure.

Next, we discuss the Boussinesq approximation, where the velocity and momentum shear basically collapse into the same variable. From the analysis in, e.g. [29, 5] the Boussinesq limit is known to be equivalent to the Airy system for long dispersionless waves of a single water layer over a flat bottom, under a suitable coordinate transformation dictated by the structure of the characteristic velocities and the Riemann invariants of the system. In turn, this system coincides with the dispersionless NLS equation under the so-called Madelung transform. As well known, such a system displays a lot of ‘good’ properties. For instance, it is one of the few quasi-linear systems in \( N > 1 \) fields in which the Riemann procedure can be effectively carried out (see [31]) and Whitham equations can be quite explicitly solved. By means of the bi-Hamiltonian procedure, an explicit generating function for a set of polynomial constants of the motion can be provided. As mentioned above, thanks to the fact that the Poisson structure is one and the same for the Boussinesq model as well as for its non-Boussinesq counterpart, we can study the latter as a standard Hamiltonian deformation of the first. In view of its
relevance for physical applications, where the non-Boussinesq deformation is scaled by the typically small parameter \( r = (\rho_2 - \rho_1)/(\rho_2 + \rho_1) \), with \( \rho_1 \) and \( \rho_2 \) the densities of the lighter and heavier fluid respectively, we focus in particular on the first order deformation \( O(r) \). We show that the first order deformed system retains the property of being completely integrable, that is, we explicitly prove that the family of mutually commuting integrals of motion for the Boussinesq-Airy system can be deformed to integrals of motion of the first order deformed Hamiltonian system. These integrals are finally used to provide solutions for families of initial data by applying hodograph-like formulas involving the integrals of the motion of the system.

2. The layer-averaged equations of motion

We briefly review here the derivation of the layer-averaged equations from the corresponding Euler system for a two-layer incompressible Euler fluid in an infinite channel (see, e.g. [11]).

Motions of typical wavelength \( L \) are considered under the long-wave, small-amplitude assumptions that the ratios

\[
\epsilon = \frac{h}{L} \approx \frac{\eta_1}{L} \ll 1
\]

(2.1)
can be considered as perturbative small parameters, where \( h \) is the total height of the channel, while \( \eta_1 \) (resp. \( \eta_2 \)) is the thickness of the upper (resp. lower) fluid. The densities of the two fluids are denoted by \( \rho_1 \) and \( \rho_2 \) (with \( \rho_2 \gg \rho_1 \) for hydrostatic stability).

Under assumption (2.1) the ratio of vertical and horizontal velocities scales as \( \epsilon \) as well, and by using the layer-averaging method as described in [34], the \( 2 + 1 \)-dimensional Euler system together with the incompressibility of each layer,

\[
\begin{align*}
\rho_1 u_t + \rho_2 w_t &= 0 \\
u_t + uu_x + uw_z &= -P_x/\rho \\
w_t + au_x +ww_z &= -P_z/\rho - g \\
u_x + w_z &= 0,
\end{align*}
\]

(2.2)

reduce under suitable conditions (see [11]) to the \( 1 + 1 \) dimensional equations

\[
\begin{align*}
\eta_{tt} + (\overline{\eta}_t \eta)_x &= 0, & i = 1, 2 \\
\sigma_{it} + \sigma_{it} \sigma_{ix} - g\eta_{ix} - \frac{P_i}{\rho_i} + D_i &= 0, \\
\sigma_{2t} + \sigma_{2t} \sigma_{2x} + g\eta_{2x} + \frac{P_2}{\rho_2} + D_2 &= 0 \\
\eta_1 + \eta_2 &= h, & (\eta_1 \eta_1 + \eta_2 \eta_2)_x = 0
\end{align*}
\]

(2.3)

where \( D_i = \frac{1}{\eta_i^3} \left[ \sigma_{it} \sigma_{ix} - (\sigma_{ix})^2 \right]_x + \ldots \) are dispersive terms. Here \( \overline{m}_{1,2} \) are the layer-mean velocities, defined as

\[
\begin{align*}
\overline{m}_1(x,t) := \frac{1}{\eta_1(x,t)} \int_{h-\eta_1(x,t)}^{h} u_1(x,z,t)dz, & \quad \sigma_1(x,t) = \frac{1}{\eta_1(x,t)} \int_{0}^{\eta_1(x,t)} u_2(x,z,t)dz, \\
\end{align*}
\]

and \( P(x, t) \) is the interfacial pressure. We shall always assume, consistently with (2.1), that the interface \( \eta_1(x, t) \equiv \eta_2(x, t) \) nowhere and never touches the boundary, that is,

\[
0 < \eta_1(x, t) < h, \quad \text{and} \quad \eta_1 \approx \eta_2.
\]

(2.4)
The constraints in the last line of (2.3) reduce the full system to evolution equations for just two fields. Indeed, under the assumption of vanishing horizontal velocities at the far ends of the channel (that is, for $|x| \to \infty$), the constraints can be algebraically solved say for $(\eta_1, \bar{m}_1)$ as

$$\eta_1 = h - \eta_2, \quad \bar{m}_1 = -\frac{\eta_2}{h - \eta_2} \bar{m}_2. \quad (2.5)$$

The constrained equations of motion can be obtained retaining the volume conservation of the lower fluid, $\eta_{21} + (\bar{m}_2 \eta_2)_{x} = 0$, and eliminating $P_x$ from the second and third line of (2.3).

In what follows, we shall choose as reduced coordinates the relative thickness $\xi = \eta_2 - \eta_1 = 2\eta_2 - h$ and the momentum shear

$$\bar{\sigma} = (\rho_2 \bar{m}_2 - \rho_1 \bar{m}_1)/h. \quad (2.6)$$

In these variables the resulting equations read

$$\begin{cases}
\xi_t = -\frac{h (h^2 - \xi^2) \bar{\sigma}}{(\rho_2 + \rho_1)h + (\rho_2 - \rho_1)\xi} \\
\bar{\sigma}_t = -\frac{h (\rho_2(h - \xi^2) - \rho_1(h + \xi)^2)}{2((\rho_2 + \rho_1)h + (\rho_2 - \rho_1)\xi)^2} \frac{\bar{\sigma}^2 + g(\rho_2 - \rho_1)\xi}{2h}. \quad (2.7)
\end{cases}$$

**Remark 2.1.** It might seem more natural from a physical viewpoint to complement the relative thickness $\xi$ with the velocity shear $w = \eta_2 - \eta_1$, as in [5]. The velocity shear and the momentum shear $\bar{\sigma}$ are simply related by

$$\sigma = \frac{(\rho_2 + \rho_1)w}{2h} \left(1 - \frac{\rho_2 - \rho_1 \xi}{\rho_2 + \rho_1 h} \right). \quad (2.8)$$

The reasons behind our choice of dependent variables will be fully motivated and discussed in section 3.1.

The so-called Boussinesq approximation, widely used in the theory of buoyancy-driven flows, consists of neglecting small density differences except for the gravity terms. As well-known, this can be viewed as the double scaling limit obtained by setting

$$\rho_\Delta := \rho_2 - \rho_1, \quad r := \frac{\rho_\Delta}{2\bar{\rho}}, \quad \bar{\rho} = (\rho_2 + \rho_1)h^2, \quad (2.9)$$

and considering

$$\rho_\Delta \to 0, \quad g \to \infty, \quad \text{with the reduced gravity } \bar{g} := \rho \ell G = O(1). \quad (2.10)$$

In the Boussinesq approximation the momentum shear $\bar{\sigma}$ is just a multiple of the velocity shear,

$$\bar{\sigma} \to \bar{\rho}(\bar{m}_2 - \bar{m}_1)/h, \quad (2.11)$$

and the equations of motion become

$$\begin{cases}
\xi_t = -\frac{(h^2 - \xi^2) \bar{\sigma}}{2\bar{\rho}} \\
\bar{\sigma}_t = -\frac{\xi \bar{\sigma}^2}{2\bar{\rho}} + \frac{g \rho_\Delta \xi}{2h}. \quad (2.12)
\end{cases}$$
Remark 2.2. The limit \( r \to 0 \) with \( g \) independent of \( r \) fixed (and finite) is drastically different.

2.1. Horizontal momentum and asymptotic pressure differentials

The Euler equations for the horizontal component of the fluid momentum are

\[
(\rho u)_t = -u(\rho u)_x - w(\rho u)_z - p_x. \tag{2.13}
\]

Integrating this equation on the fluid’s domain, using incompressibility and the boundary conditions as well as assuming asymptotic hydrostatic conditions at the far ends of the channel, yields

\[
\frac{d}{dt} \Pi_x = \int_S (\rho u)_t \, dz \, dx = -\int_0^h \left( \int \rho (\rho u^2 + p)_x \, dx \right) \, dz = -h P_\Delta \tag{2.14}
\]

where \( P_\Delta = P(\infty) - P(-\infty) \) is the interfacial pressure differential. One might wonder why the asymptotic values of the interfacial pressure could be different from plus to minus infinity. Hydrostatic equilibrium is identical at both ends, and the interfacial pressure simply keeps track of the overall constant of integration up to which pressure is defined. Indeed, a free upper surface subject to constant atmospheric pressure would yield no such pressure jump. However, a system with a rigid lid is constrained, and reaction forces can develop. The non-intuitive result is that such forces, being normal to the lid, are apparently only vertical, and as such should not affect the horizontal momentum balance. As remarked by Benjamin in [2], the pressure levels as \( x = +\infty \) and \( x = -\infty \) relative to hydrostatic pressure in the quiescent state of the whole system (even with equal asymptotic values of the \( \eta_i \)'s) need not be equal. This observation was substantiated, with analytical computations and numerical simulations for a number of initial conditions for the Euler equations leading to non vanishing asymptotic pressure difference \( P_\Delta \) in [6, 7, 8].

In the 1D dispersionless model, the asymptotic values of the interfacial pressure can be computed from the dispersionless limit of (2.3), by solving for \( P_\Delta \) and substituting the constraints.

\[
P_\Delta = \frac{\rho_2 \rho_1 h}{4 \rho_2^3} r \int_{-\infty}^{+\infty} \frac{\sigma^2 (h^2 - \xi^2)}{(h - r \xi)^4} \xi \, dx. \tag{2.15}
\]

Since the integrand is not a total \( x \)-derivative in general, as soon as density differences are retained in the inertial terms (buoyancy terms do not contribute to (2.15)), pressure imbalances \( P_\Delta \neq 0 \) at the far ends of the channel develop, so that lack of time-conservation of the fluid’s horizontal momentum is explicitly exhibited by the 1D model.

2.2. Characteristic velocities and KH instability

The study of the hyperbolic-elliptic transition for the Boussinesq system (2.7) has been considered (by using different coordinates) in [5]. We report and comment on some results obtained in [5] by first rewriting these with our choice of coordinates. From (2.7) we obtain that the hyperbolicity region is defined by

\[
|\tilde{\sigma}| < \sqrt{\frac{g(\rho_2 - \rho_1)(\rho_1(h + \xi) + \rho_2(h - \xi))^3}{8h^4 \rho_1 \rho_2}} = \tilde{\rho} \sqrt{\frac{2g (1 - r \xi/h)^3}{h (1 - r^2)}} := \sigma_0, \quad |\xi| < h. \tag{2.16}
\]
While the only condition on the $\xi$ variable is that the interface does not touch the channel boundaries, i.e. $|\xi| < h$, the parameter $r$ affects substantially the structure of the hyperbolicity region. Indeed the area $A_h$ of such a region is the monotonically increasing function of $r$

$$A_h = \rho \sqrt{2 g h} \frac{4 ((1 + r)^{5/2} - (1 - r)^{5/2})}{5 r \sqrt{1 - r^2}}. \tag{2.17}$$

whose graph is depicted in figure 2. Notice that $A_h$ limits to a finite quantity when $\rho_2 \to \rho_\infty$ and grows indefinitely (as $(1 - r)^{-5}$) when $r \to 1$, the limiting case of an air-water system. The fact that the area $A_h$ grows monotonically with $r$ should be expected on the basis of the stabilizing effects of stratification. In figure 3 we depict the explicit form of the hyperbolicity domain in the Boussinesq expansion. As remarked in [5], the case $r = 0$ corresponds to the Boussinesq approximation. In this limit, the domain of hyperbolicity is finite ($|\bar{s}| < \sqrt{2 g h}$), and, as shown in [5], hyperbolic-elliptic transitions are forbidden since simple-waves are always tangent to sonic lines (i.e. the lines in the $(\xi, \sigma)$-plane where the two characteristic velocities coincide). In the opposite limit ($r = 1$) the system becomes air-water like and the hyperbolicity region fills the entire strip in $(\xi, \sigma)$-plane. In fact, simple waves are defined by solutions of the ordinary differential equation

$$\frac{d\bar{s}}{d\xi} = \rho \sqrt{\frac{2 \frac{\bar{g}}{h} (h - \xi r)^3 - h^4 (1 - r^2)\bar{s}^2}{h^4 (h^4 - \xi^2)(h - \xi r)^2}}. \tag{2.18}$$

The tangent slopes to simple waves at the boundaries of the hyperbolic region are always independent of $r$ and are either horizontal or vertical,

$$\left. \frac{d\bar{s}}{d\xi} \right|_{|\xi| = h} = \infty, \quad \left. \frac{d\bar{s}}{d\xi} \right|_{|\sigma| = \sigma_b} = 0. \tag{2.19}$$

On the other hand, the tangent $\tau_s$ to the sonic line $s$ at the boundaries $|\bar{s}| = \sigma_b$ of the hyperbolicity region is
Therefore, the only two values of \( r \) yielding horizontal or vertical tangent, which generally prevent hyperbolic-elliptic transitions, are \( r = 0 \) and \( r = 1 \). We note that the transversality of the sonic line to the boundary of the hyperbolicity region is an effect already present at order \( O(r) \) as \( r_0 \to 0 \).

### 2.3. Riemann Invariants and the \( r \)-expansion

It is well known that a quasi-linear system with two degrees of freedom can be diagonalized by means of the generalized Tsarev’s hodograph transformation, that is, put in the form

\[
\begin{align*}
R_{+x} + \lambda_+(R_+, R_-)R_{+x} &= 0 \\
R_{-x} + \lambda_-(R_+, R_-)R_{-x} &= 0
\end{align*}
\]

where \( \lambda_\pm \) are the characteristic velocities, and \( R_\pm \) are potentials for the 1-form implicitly defined by equation (2.18). Also, (see, e.g. [15]), a general procedure for finding mutually commuting constants of the motion is known. Unfortunately, the tame-looking 1-form

\[
(2 \xi(h - \xi r)^3 - h^4(1 - r^2)\sigma^2, -h^2(\xi^2 - r^2)(h - \xi r)^2)
\]

resisted all our attempts to find an integrating factor in closed form.

A possible attempt to circumvent this problem is to try and solve the first order approximation in the \( r \) expansion, building on the results of [5, 19, 29] where the explicit diagonalisation of the Boussinesq limit (2.12) (to be viewed as the 0th order term in the \( r \)-expansion of system (2.7)) was achieved. The rationale for this approach lies in the physical significance of the normalized density difference \( r \) in the original model, since in most applications
(e.g. in fresh–salted water systems or in meteorology-related problems of dry–wet air) naturally occurring density variations are invariably small when appropriately normalized. It should also be stressed that the two physically relevant phenomena we briefly discussed above, namely the lack of horizontal momentum conservation and the possibility of hyperbolic-to-elliptic transition are effects of order $O(r)$ in the small parameter $r$, so that the first-order expansion can be considered as a good ‘proxy’ for the case of arbitrary $r$.

The diagonalisation of the Boussinesq limit $r = 0$ can explicitly be achieved by computing the characteristic velocities and Riemann invariants as

$$\lambda_\pm = \pm \frac{\xi \hat{\sigma}}{\rho} \pm \sqrt{\frac{h(h^2 - \xi^2)(g\hat{\rho} \rho_\pm - h\sigma^2)}{2h\rho}} + R_\pm = \mp \frac{\xi \hat{\sigma}}{\rho} \mp \sqrt{\frac{h(h^2 - \xi^2)(g\hat{\rho} \rho_\pm - h\sigma^2)}{h\rho}}.	ag{2.23}$$

Thus, the relation between characteristic velocities and Riemann invariants is

$$\lambda_+ = \frac{1}{4}(3R_- - R_+), \quad \lambda_- = \frac{1}{4}(R_+ - 3R_-).\tag{2.24}$$

By introducing nondimensional variables

$$\xi := h \xi^*, \quad \hat{\sigma} := \frac{g\hat{\rho} \rho_\pm}{h}\sigma^*,\tag{2.25}$$

and suitably rescaling time and space by, respectively, the factors $h$ and $\sqrt{\frac{h\rho}{\hat{\rho} \rho_\pm}}$, expressions (2.23) become

$$\lambda_\pm = \xi^* \sigma^* \pm \frac{1}{2} \sqrt{(1 - \xi^2)(1 - \sigma^2)}, \quad R_\pm = \pm \xi^* \sigma^* + \sqrt{(1 - \xi^2)(1 - \sigma^2)}.	ag{2.26}$$

and the diagonal form on the Boussinesq system is

$$\begin{cases} R_{t+} + \frac{1}{4}(3R_- - R_+)R_{x+} = 0 \\ R_{x+} + \frac{1}{4}(R_+ - 3R_-)R_{x-} = 0 \end{cases}.	ag{2.27}$$

We seek first order expansions of the characteristic velocities and Riemann invariants in the form

$$\lambda_\pm = \lambda_\pm^0 + r\lambda_\pm^1 + o(r), \quad R_\pm = R_\pm^0 + rR_\pm^1 + o(r),\tag{2.28}$$

satisfying the corresponding Riemann system of equations (2.21) whose expansion for small $r$ reads

$$\begin{cases} R_{t+}^0 + rR_{x+}^1 + \lambda_\pm^0(R_+^0, R_\pm^0)R_{x+}^1 + r(W_+ R_+^0 + \lambda_\pm^0(R_+^0, R_\pm^0)R_{x+}^1) = o(r) \\ R_{x+}^1 + rR_{x+}^0 + \lambda_\pm^0(R_+^0, R_\pm^0)R_{x+}^0 + r(W_+ R_+^0 + \lambda_\pm^0(R_+^0, R_\pm^0)R_{x+}^1) = o(r), \end{cases}\tag{2.29}$$

where $W_\pm = \frac{\partial \lambda_\pm}{\partial R_\pm} \left|_{(\xi^*, \sigma^*)} \right. R_{x+}^1 + \frac{\partial \lambda_\pm}{\partial R_\pm} \left|_{(\xi^*, \sigma^*)} \right. R_{x+}^1$.

In the variables $\theta = \arcsin(\xi^*), \phi = \arcsin(\sigma^*)$ the zeroth-order Riemann invariants $R_\pm^0$ are

$$R_\pm^0 = \cos(\phi \pm \theta),$$

and, as it can be directly checked, their deformations are
\[ R_\pm^l = \frac{3}{2} \sin(\theta) \tan(\phi) + 3 \sin(\theta \pm \phi) \text{Arctanh} \left( \frac{\tan(\phi/2)}{2} \right) + \frac{5}{2} \cos(\theta). \quad (2.30) \]

The characteristic velocities expressed in terms of the variables \( \theta \) and \( \phi \) are

\[ \lambda_\pm = \lambda_\pm^0 + r \lambda_\pm^1 + o(r) = \frac{1}{2} (-2 \sin(\theta) \sin(\phi) \pm \cos(\theta) \cos(\phi)) \]

\[ + \frac{r}{4} (- \sin(\theta) (1 - 2 \tan^2(\phi)) \cos(\theta) \cos(\phi) + (3 \cos(2\theta) - 1) \sin(\phi)) + o(r). \quad (2.31) \]

At the zero-th system (2.26) is clearly recovered, as well as relation (2.24)

\[ \lambda_+^0 = \frac{3}{4} R_+^0 - \frac{1}{4} R_-^0, \quad \lambda_-^0 = \frac{1}{4} R_+^0 - \frac{3}{4} R_-^0. \quad (2.32) \]

However, such a simple relation between characteristic velocities and Riemann invariants at leading order does not extend to the next order in the \( r \) expansion. More importantly, although the first order term in the expansion (2.29) can be explicitly computed (with the first order characteristics velocities expressed in terms of the Boussinesq-limit Riemann invariants \( R_+^0 \)), the resulting quasilinear system of PDEs in the four variables \( (R_+^0, R_-^0) \) cannot be (even algebraically) diagonalized. Therefore, the standard procedure to solve the Cauchy problem for our deformed system cannot be applied effectively. Hence, in the next section, we tackle the problem of constructing solutions to the perturbed system by casting and studying it in terms of its Hamiltonian properties.

### 3. The Hamiltonian setting

As observed in [12] and further discussed in [19], the diagonal form of the Boussinesq approximation is related with that of the Airy system

\[ \begin{cases} u_t + (uv)_x = 0, \\ v_t + vv_x + u_x = 0. \end{cases} \quad (3.1) \]

The coordinate change that sends the Boussinesq system into (3.1), which can be derived from the expression of the characteristic velocities and the Riemann invariants, is

\[ u = (1 - \xi^2)(1 - \sigma^2), \quad v = 2 \xi \sigma. \quad (3.2) \]

As thoroughly discussed in [19], this is a ‘four-to-one’ map, owing to the two discrete symmetries \((\xi, \sigma) \mapsto (-\xi, -\sigma)\) and \((\xi, \sigma) \mapsto (\sigma, \xi)\).

The Airy system, besides being integrable via the hodograph method, is well known to admit infinite families of local mutually commuting constants of the motion. In section 3.5 we will see that a suitably chosen family of such constants of the motion can be explicitly deformed at order 1 in the small parameter \( r \). In section 3.6 the hodograph formulae (see, e.g. [17]),

\[ \begin{cases} x + tH_{\xi\sigma} = F_{\xi\sigma} \\ tH_{\xi\sigma} = F_{\xi\sigma} \end{cases} \]

or equivalently by \( \begin{cases} x + tH_{\xi\sigma} = F_{\xi\sigma} \\ tH_{\xi\sigma} = F_{\xi\sigma} \end{cases} \)

will then be used to provide explicit local solutions (still exact at order \( O(r) \) in the \( r \)-expansion) of the evolution equations.
We remark that, although the Hamiltonian nature of the $1 + 1$ dimensional quasi-linear system (2.7) could be established directly, it is conceptually important to show that these equations and their Hamilton structure can be obtained by a systematic reduction of the Hamiltonian setting of the parent incompressible stratified 2D Euler equations. In this way (as anticipated in section 2), the seemingly peculiar choice of the momentum shear as dependent variable, rather than the velocity shear used in the references [5, 12, 19, 29], will be fully motivated.

Remark 3.1. It is also well-known that the Airy system is equivalent to the dispersionless non-linear Schrödinger (dNLS) equation, written in the so-called Madelung variables, obtained by parameterizing the Schrödinger wave function as

$$\psi = u^2 \exp \left( i \int v \, dx \right).$$  \hspace{1cm} (3.4)

3.1. The 2D Benjamin model for heterogeneous fluids in a channel

Benjamin [2] proposed and discussed a Hamiltonian formulation of the incompressible stratified Euler system, also known as the Boussinesq model (not to be confused with its namesake approximation, which only refers to neglecting density variations in the fluid’s inertia). We now briefly review Benjamin’s results for the reader’s convenience.

The evolution of a perfect inviscid incompressible but heterogeneous fluid in 2D, subject to gravity, can be described by the variables $(\rho(x, z, t), u(x, z, t), w(x, z, t))$, governed by the Euler equations (2.2). Benjamin’s formulation consists of introducing as basic variables the density $\rho$ together with the weighted vorticity $\varsigma$ defined by

$$\varsigma := \nabla \times (\rho u) = (\rho w)_x - (\rho u)_z.$$  \hspace{1cm} (3.5)

The equations of motion for these two fields, ensuing from (2.2), are

$$\begin{align*}
\rho_t + u\rho_x + w\rho_z &= 0, \\
\varsigma_t + u\varsigma_x + w\varsigma_z + \rho\left(gz - \frac{1}{2}(u^2 + w^2)\right)_z + \frac{1}{2}\rho\left(u^2 + w^2\right)_x &= 0.
\end{align*}$$  \hspace{1cm} (3.6)

These can be written in the form

$$\begin{align*}
\rho_t &= -\left[ \rho, \frac{\delta H}{\delta \rho} \right], \\
\varsigma_t &= -\left[ \rho, \frac{\delta H}{\delta \rho} \right] - \left[ \varsigma, \frac{\delta H}{\delta \varsigma} \right],
\end{align*}$$  \hspace{1cm} (3.7)

where the bracket is defined by $[A, B] \equiv A_\rho B_\varsigma - A_\varsigma B_\rho$, and the functional

$$H = \int_{\mathcal{D}} \rho \left( \frac{1}{2} |u|^2 + gz \right) \, dx \, dz,$$  \hspace{1cm} (3.8)

is simply given by the sum of the kinetic and potential energy, $\mathcal{D}$ being the fluid domain. The most relevant feature of this coordinate choice is that $(\rho, \varsigma)$ are physical variables. Their use, albeit confined to the 2D case here, allows one to avoid the introduction of Clebsch variables (and the corresponding subtleties associated with gauge invariance of the Clebsch potentials) needed in the Hamiltonian formulation general case (see, e.g. [36]).

In Benjamin’s formalism the Hamiltonian $H$ is written in terms of an auxiliary variable, the stream function $\psi$, related to the variables $(\rho, \varsigma)$ via
More precisely, once \( \rho \) and \( \varsigma \) are given, \( \psi \) is the unique solution of (3.9) vanishing on the horizontal boundaries of the fluid’s domain, so that \( H \) turns out to be a functional of \( \rho \) and \( \varsigma \) only. As shown by Benjamin, equations (3.7) are actually a Hamiltonian system with respect to a non-canonical Hamiltonian structure. This means that equations (3.6) can be written as

\[
\rho_2 = \{ \rho, H \}, \quad \varsigma_2 = \{ \varsigma, H \}
\]

for the Poisson brackets defined by the Hamiltonian operator

\[
J_\theta = -\left( \begin{array}{cc}
0 & \rho_2 \partial_\varsigma - \rho_\varsigma \partial_2 \\
\rho_\varsigma \partial_\varsigma - \rho_\varsigma \partial_\varsigma & \varsigma_2 \partial_\varsigma - \varsigma_\varsigma \partial_2 
\end{array} \right)
\]

(3.10)

### 3.2. The Hamiltonian reduction

We now discuss how a simple averaging process can be given a Hamiltonian structure, well suited to the discussion of the constrained equations in which our set of reduced coordinates naturally appears. In particular, we can induce, up to order \( o(\epsilon) \), where \( \epsilon = h/L \) as before, a Poisson bracket on the reduced fields from the full 2D Benjamin’s structure (3.10).

By means of the Dirac \( \delta \) and Heaviside \( \theta \) generalized functions, a two-layer fluid with constant density \( \rho_i \) and velocity components \( u_i(x, z), w_i(x, z) \) for the upper layer \( i = 1 \) and lower layer \( i = 2 \), respectively, can be described by global density and velocity variables defined as

\[
\rho(x, z) = \rho_2 + (\rho_1 - \rho_2)\theta(z - \eta(x)) \\
u(x, z) = u_2(x, z) + (u_1(x, z) - u_2(x, z))\theta(z - \eta(x)) \\
w(x, z) = w_2(x, z) + (w_1(x, z) - w_2(x, z))\theta(z - \eta(x)),
\]

(3.11)

where, hereafter, \( \eta \equiv \eta_2 \). Since time is merely a parameter in the description of phase spaces and mappings, we suppress time dependence for ease of notation in what follows.

The fluid velocity vector \( u = (u, w) \) is assumed to be smooth except at the interface \( z = \eta(x) \) where it may have finite jumps, subject to the continuity of the normal component, where the density discontinuity is located.

As can be easily checked, the two momentum components are

\[
\rho u = \rho_2 u_2(x, z) + (\rho_1 u_1(x, z) - \rho_2 u_2(x, z))\theta(z - \eta(x)),
\]

and

\[
\rho w = \rho_2 w_2(x, z) + (\rho_1 w_1(x, z) - \rho_2 w_2(x, z))\theta(z - \eta(x)).
\]

Hence, by the standard rules of differentiation of generalized functions, \( \theta'(\cdot) \equiv \delta(\cdot) \), the terms in the weighted vorticity \( \varsigma = (\rho u)_z - (\rho w)_x \) become

\[
(\rho u)_z = \rho_2 u_{2,z} + (\rho_1 u_{1,z} - \rho_2 w_{2,z})\theta(z - \eta(x)) + \\
- \eta_2(\rho_1 w_1(x, z) - \rho_2 w_2(x, z))\delta(z - \eta(x)),
\]

(3.12)

and

\[
(\rho w)_x = \rho_2 u_{2,z} + (\rho_1 u_{1,z} - \rho_2 u_{2,z})\theta(z - \eta(x)) + \\
+ (\rho_1 u_1(x, z) - \rho_2 u_2(x, z))\delta(z - \eta(x)),
\]

(3.13)

so that
\[
\zeta = \rho_2(w_{2,x} - w_{2,z}) + (\rho_1(w_{1,x} - u_{1,z}) - \rho_2(w_{2,x} - u_{2,z}))\theta(z - \eta(x)) + (\rho_1u_{1}(x, z) - \rho_2u_{2}(x, z) + \eta_\nu(\rho_1w_{1}(x, z) - \rho_2w_{2}(x, z)))\delta(z - \eta(x)).
\] (3.14)

If the motion in each layer is assumed to be irrotational, the first line in this expression for \(\zeta(x, z)\) is identically zero, and we are left with the interfacial localized form (a ‘momentum vortex sheet’)

\[
\zeta = (\rho_2u_{2}(x, z) - \rho_1u_{1}(x, z) + \eta_\nu(\rho_2w_{2}(x, z) - \rho_1w_{1}(x, z)))\delta(z - \eta(x)).
\] (3.15)

In these coordinates the projection map \(2D \rightarrow 1D\) is easily established. Indeed,

\[
h \sigma \equiv \int_0^h \zeta(x, z) \, dz = \rho_2u_{2}(x, \eta) - \rho_1u_{1}(x, \eta) + \eta_\nu(\rho_2w_{2}(x, \eta) - \rho_1w_{1}(x, \eta))
= \rho_2\tilde{u}_{2}(x) - \rho_1\tilde{u}_{1}(x) + \eta_\nu(\rho_2\tilde{w}_{2}(x) - \rho_1\tilde{w}_{1}(x))
\] (3.16)

where we have introduced the notation \(\tilde{v}\) for the velocity at the interface. Thus, the averaged weighted vorticity \(\sigma\) reduces to the weighted tangential momentum jump at the interface.

For long wave dynamics, the components of velocity at the interface for each fluid can be expressed as an asymptotic expansion of layer-averaged horizontal velocities \(\bar{\sigma}_i\)'s (see, e.g. [10, 33]) in terms of the small parameter \(\epsilon\). The right hand side of equation (3.16) can then be written as

\[
\rho_2\tilde{u}_{2}(x) - \rho_1\tilde{u}_{1}(x) + O(\epsilon),
\] (3.17)

that is, in the long wave regime the averaged weighted vorticity reduces to the weighted horizontal momentum jump (a localized shear) at the interface. We define the reducing map as

\[
\eta = \frac{1}{\rho} \int_0^h (\rho(x, z) - \rho_1) \, dz, \quad \sigma = \frac{1}{h} \int_0^h \zeta(x, z) \, dz
\] (3.18)

(the first of these relations being easily obtained from the first of equations (3.11) and from equation (3.15)), and we can compute the reduced Poisson tensor by means of the standard ‘pull–push’ formula (see, e.g. [24]). Let us denote by \(M^{(1)}\) the manifold of the ‘averaged’ quantities \((\sigma, \eta)\) and by \(M^{(2)}\) the manifold of the 2D quantities, parametrized by \((\rho(x, z), \zeta(x, z))\). Inside \(M^{(2)}\) we consider the surface

\[
\mathcal{I} := \{\rho(x, z) = \rho_2 + \rho_2\theta(z - \eta(x)), \zeta(x, z) = (\rho_2\tilde{u}_{2}(x) - \rho_2\tilde{u}_{2}(z))\delta(z - \eta(x))\}.
\] (3.19)

If \((\alpha_\rho(x), \alpha_\sigma(x))\) is a 1-form on \(M^{(1)}\), we lift this, under the map (3.18), to obtain

\[
\begin{pmatrix}
\frac{1}{\rho} \alpha_\rho(x), & \frac{1}{h} \alpha_\sigma(x)
\end{pmatrix}
\] (3.20)

independent of the vertical coordinate \(z\).

Applying the Poisson tensor (3.10) evaluated on \(\mathcal{I}\) to this covector we get (notice that terms with \(\partial_z\) can be dropped)

\[
\begin{pmatrix}
\rho(x, z) \\
\zeta(x, z)
\end{pmatrix} = \begin{pmatrix}
0 & \rho_2\delta(z - \eta(x))
\rho_1\delta(z - \eta(x)) & \frac{1}{h}\tilde{\sigma}(x)\delta(z - \eta(x))
\end{pmatrix}
\begin{pmatrix}
\alpha_\rho(x) \\
\alpha_\sigma(x)/h
\end{pmatrix}.
\] (3.21)

This gives
\[
\begin{pmatrix}
\dot{\rho}(x, z) \\
\dot{\zeta}(x, z)
\end{pmatrix} = -\begin{pmatrix}
\rho_\Delta \delta(z - \eta(x)) \left( \frac{1}{h} \alpha_\eta(x) \right) \\
\rho_\Delta \delta(z - \eta(x)) \left( \frac{1}{\rho_\Delta} \alpha_\eta(x) \right) + \frac{\sigma(x)}{h} \delta(z - \eta(x)) (\alpha_\delta(x))
\end{pmatrix}.
\]

(3.22)

Pushing this vector to \(M^1\) via the tangent map to (3.18)
\[
\dot{\eta} = \frac{1}{\rho_\Delta} \int_0^h \rho(\eta, z) \, dz, \quad \dot{\zeta} = \frac{1}{h} \int_0^h \zeta(\eta, z) \, dz.
\]

yields the vector
\[
(\dot{\eta}, \dot{\zeta}) = \left( -\frac{1}{h} \partial_\eta (\alpha_\eta) - \frac{1}{h} \partial_\zeta (\alpha_\zeta) \right).
\]

owing to the fact that \(\int_0^h \delta(z - \eta(x)) \, dz = 0\). Hence, we have shown that the reduction of the Benjamin Poisson tensor (3.10) is given by the tensor
\[
J_{\text{red}} := -\frac{1}{h} \begin{pmatrix} 0 & \partial_\zeta \\ \partial_\eta & 0 \end{pmatrix}.
\]

(3.23)

**Remark 3.2.** This structure was termed *canonical* in the recent literature (see, e.g. [16]) since it corresponds to that of the non-linear wave equation in \(1 + 1\) dimensions
\[
u_t = F''(\nu) \nu_{xx}
\]

with ‘canonical’ Hamiltonian functional
\[
\mathcal{H} = \frac{1}{2} \int_\mathbb{R} (\nu_x^2 + F(\nu)) \, dx,
\]

for some function \(F(\nu)\). This is in contrast to the well-known [36, 37] standard canonical symplectic formulation for the Euler equations by means of Clebsch variables. To avoid possible misunderstandings, we shall refer to the structure (3.23) as the *Darboux* structure, and to the ‘canonical’ variables such as \((\sigma, \eta)\) as Darboux variables.

### 3.3. The reduced Hamiltonian

The next step is construct the reduced Hamiltonian; this is done by evaluating the full 2D Hamiltonian (3.8) on the manifold \(I\). The potential term
\[
U = \int_S \rho(x, z) g(z) \, dx dz
\]

is readily seen to reduce to
\[
U_{\text{red}} = \int_\mathbb{R} g \left( \rho_2 - \rho_1 \right) \eta^2 \, dx.
\]

(3.24)

The kinetic term is subtler. The main idea from long wave-asymptotic is that, at order \(O(\epsilon^2)\), we can disregard the term \(w^2(\xi, \zeta)\), and trade the horizontal Euler velocities with the layer-averaged ones. That is, we write the kinetic energy density as
\[ T = \frac{1}{2} \rho_z u_z^2(x,z) + w_z^2(x,z)\theta(\eta - z) + \frac{1}{2} \rho_y u_y^2(x,z) + w_y^2(x,z)\theta(z - \eta) \]
\[ = \frac{1}{2} \rho_z (\tilde{a}_z(x))^2 \theta(\eta(x) - z) + \frac{1}{2} \rho_y (\tilde{a}_y(x))^2 \theta(z - \eta(x)) + O(\epsilon^2), \]  
\tag{3.25}

and perform the integration along \( z \) to get
\[ T_{\text{red}} = \frac{1}{2} (\rho_z \eta \tilde{a}_z^2 + \rho_y (h - \eta) \tilde{a}_y^2) = \frac{1}{2} \eta \tilde{a}_z^2 \cdot \left( \rho_z + \frac{\eta}{h - \eta} \rho_y \right), \]  
\tag{3.26}

where the dynamical constraint
\[ \tilde{a}_y = -\frac{\eta}{h - \eta} \tilde{a}_z \]

has been taken into account in the last equality. Introducing the (lineal) mass density \( \varphi \) as
\[ \varphi := \rho_z \eta_1 + \rho_y \eta_2 \equiv \rho_2 h - (\rho_2 - \rho_1) \eta, \]  
\tag{3.27}

and taking the constraints into account, the link between our variable \( \tilde{\sigma} = (\rho_z \tilde{a}_z - \rho_y \tilde{a}_y)/h \) and \( \tilde{a}_2 \) is
\[ \tilde{\sigma} = \frac{1}{h(h - \eta)} \varphi \tilde{a}_2, \]  
\tag{3.28}

so that we can write the reduced kinetic energy density \( T_{\text{red}} \) as
\[ T_{\text{red}} = \frac{h^2}{2} \frac{\eta(h - \eta)}{\varphi} \tilde{\sigma}^2. \]  
\tag{3.29}

Trading the variable \( \eta \equiv \eta_2 \) for the relative thickness \( \xi = \eta_2 - \eta_1 \equiv 2\eta - h \) as in section 2 affects the Poisson tensor only by a factor of 2, thus leading to the reduced Hamiltonian functional
\[ \mathcal{H}_{\text{red}}[\xi, \sigma] = \int \frac{h^2}{4} \left( \frac{h^2 - \xi^2}{h\rho - \rho_2 \xi} \sigma^2 \right) + \frac{g}{8} \rho_\Delta (h - \xi)^2 \right) dx, \]  
\tag{3.30}

and the Hamiltonian equations of motion coincide indeed with system (2.7).

In terms of the rescaled quantities \((\xi^*, \sigma^*)\) introduced in (2.25), by discarding constant terms and by noticing that \( \xi^* \) is a Casimir density of the reduced Poisson tensor, the reduced Hamiltonian density can be written as
\[ \mathcal{H}_{\text{red}}(\xi^*, \sigma^*) = \frac{g \rho_\Delta h^2}{8} \left( (1 - \xi^*)^2 \frac{\sigma^2}{1 - r \xi^* + \xi^*} + \xi^* \right), \]  
\tag{3.31}

where, recalling definition (2.9), the parameter \( r \) is
\[ r = \frac{\rho_\Delta}{2\rho}. \]

In the Hamiltonian theory, the Boussinesq approximation can be performed at the level of the Hamiltonian density, in view of the fact that the reduced Poisson tensor does not depend on the densities \( \rho_i \). In particular, the Hamiltonian density of the Boussinesq approximation is
\[ \mathcal{H}_{\text{red}, \text{B}}(\xi^*, \sigma^*) = \frac{g \rho_\Delta h^2}{8} \left( (1 - \xi^*)^2 \sigma^* + \xi^* \right), \]  
\tag{3.32}
Thus, in the double scaling limit $\epsilon \to 0$, $r \to 0$, the Hamiltonian (3.31) can be viewed as a deformation of $H_{\text{col,b}}$ with respect to the small parameter $r$.

**Remark 3.3.** A few remarks are in order.

1. The reduction process clearly shows that the most appropriate variables to be used in the Hamiltonian reduction process are the $(\xi, \sigma)$ variables. Indeed using the variables $(\xi, w)$ of [5, 29] the Hamiltonian structure of the reduced equations away from $r = 0$ is not apparent and can be shown to depend on the small parameter $r$ (as well as on the Hamiltonian). Since, in our variables, the Poisson tensor does not depend on $r$, the expansion in small $r$ of the system we are considering can be simply obtained by the correspondent expansion of the Hamiltonian functional, keeping one and the same Poisson structure for all orders.

2. The higher order terms in the $\epsilon$ expansion involve nonlinear differential operators acting on the layer-averaged horizontal velocities, and contribute dispersion to the resulting motion equation, as it is apparent from the coupled Green-Naghdi equations (2.3). However, it should be stressed that they affect only the reduction of the Hamiltonian functional, and not the reduction of the Benjamin Poisson brackets $J_B$ of equation (3.10).

Indeed, all the technical computations done from (3.20) to (3.23) to arrive at the reduced Poisson tensor hold verbatim if we forget about the ‘dispersionless limit’ of the variable $\sigma$ of equation (3.17) retaining the definition (3.16), i.e.

$$\hbar \sigma = (\rho_2 u_2(x, \eta) - \rho_1 u_1(x, \eta)),$$

and correspondingly modify the definition of the ‘manifold’ $I$ in equation (3.19).

3. The Hamiltonian nature of the 1D two-layer equations was previously described in the literature (see, e.g. [3, 14, 23] and the more recent [18]). However, while treating the general case, all these previous approaches focus on the variational setting of these equations, and define suitable variational 1D principles from their 2D counterparts (thus reducing the Hamiltonian equations). The setting which we have just implemented, as already remarked, is different and has a more geometrical flavor in that it deals first with the reduction of the Poisson brackets first, and then with the definition of the reduced Hamiltonian, in the spirit of the geometrical Hamiltonian reduction scheme.

4. A further Hamiltonian form of our system is obtained by noticing that, once the bulk vorticity is assumed to vanish, velocity potentials $\phi_i$ can be defined in each of the two layers. Then, by introducing, as in [18], the potential (or Clebsch-like) variable

$$\Phi(x) = (\rho_2 \phi_2(x, \eta(x)) - \rho_1 \phi_1(x, \eta(x))),$$

from (3.33) we get $\hbar \sigma = \delta(\Phi(x))$. Under the inverse of this transformation our reduced Poisson tensor (3.23) turns into the standard symplectic one, yielding the motion equations in the standard form

$$\xi_t = \frac{\delta H}{\delta \Phi}, \quad \Phi_t = \frac{\delta H}{\delta \xi},$$

for a suitable $H$. These variables are the 1D counterparts of the classical Zakharov setting of the incompressible Euler equations [36, 37]. The equivalence between the Zakharov and the Benjamin setting for Euler incompressible heterogeneous 2D fluids has been discussed in [7] and [9].
3.4. A family of conserved quantities

From now on we fix a suitable non-dimensional version of the Hamiltonian picture of the previous section by taking

\[
J = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix}, \quad H(\xi, \sigma) = \frac{1}{4} \left( \frac{1 - \xi^2}{1 - r \xi^2} + \xi^2 \right),
\]

(3.36)

where we also have dropped all the asterisks from non-dimensional variables. In this section we shall explicitly discuss some known facts about the multi-Hamiltonian structure of the dispersionless two-fluid system when the Boussinesq approximation is applied, with an eye towards the further developments to be discussed in section 3.5.

In this framework the Boussinesq approximation coincides with the zero-th order term in the formal expansion in \( r \) of the Hamiltonian (3.36), i.e.

\[
H_0(\xi, \sigma) = \frac{1}{4} ((1 - \xi^2) \sigma^2 + \xi^2).
\]

(3.37)

As anticipated at the beginning of section 3, the ensuing Hamiltonian equations of motion written as

\[
\begin{pmatrix} \xi_t \\ \sigma_t \end{pmatrix} + \begin{pmatrix} \frac{\partial H_0}{\partial \sigma} \\ -\frac{\partial H_0}{\partial \xi} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

(3.38)

coincide, under the map

\[
u = (1 - \xi^2)(1 - \sigma^2), \quad \sigma = 2\xi\sigma,
\]

(3.39)

with the equations of the motion of the Airy system. The well-known fact that the Airy system (3.1) can be obtained via the two local and compatible Poisson structures

\[
P_0(\equiv -J) = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix}, \quad P_1 = \begin{pmatrix} u\partial + \partial u & v\partial \\ \partial v & 2\partial \end{pmatrix}
\]

(3.40)

(with appropriate definitions of Hamiltonian functionals), allows us to select a specific infinite family of conserved density, namely those encoded in the generator (the so-called *Casimir* of the Poisson pencil \( P_0 - \lambda P_1 \))

\[
Q(\lambda) = -\frac{1}{4} \sqrt{(\nu - \lambda)^2 - 4u},
\]

(3.41)

obtained by recursively solving the system

\[
(P_1 - \lambda P_0) : \begin{pmatrix} Q_v \\ Q_\sigma \end{pmatrix} = 0.
\]

(3.42)

in the ring of formal power series in \( \lambda \).

Expanding this generator around \( \lambda = \infty \) we get the family of conserved densities

\[
K(\lambda) = Q(\lambda)|_{\lambda \to \infty} = -\frac{\lambda}{4} \frac{\nu}{4} + \frac{u}{2\lambda} + \frac{u\nu}{2\lambda^2} + \frac{u(u + v^2)}{2\lambda^3} + \frac{uv(3u + v^2)}{2\lambda^4} + \frac{u(2u^2 + 6uv^2 + v^4)}{2\lambda^5} + \cdots = \sum_{j=0}^{\infty} \frac{1}{\lambda^j} K_j,
\]

(3.43)
to be referred to as the family of polynomial invariants. Notice that the coefficient of $\lambda^{-3}$ in the expansion (3.43) is a multiple of the 'standard' Hamiltonian of the Airy system, in the sense that system (3.1) can be written as

$$\left(\begin{array}{c} u \\ v \end{array}\right)_t = P_0 \left(\begin{array}{c} \delta_u \mathcal{H} \\ \delta_v \mathcal{H} \end{array}\right), \quad \text{with } \mathcal{H} = -\frac{1}{4} \int K_3 dx = -\frac{1}{2} \int u(u + v^2) dx. \quad (3.44)$$

(Hereafter, integration is understood in a formal sense, and we will omit the range of integration of the conserved densities in integrals; in all cases, the dependent variables $(u, v)$ and $(\xi, \sigma)$ will be assumed to be defined up to appropriate constants in order to lead to integrable conserved densities.)

**Remark 3.4.** Similarly to other dispersionless models, the Airy system admits the third local Poisson structure (see, e.g. [21, 27, 28])

$$P_2 = P_1(P_0)^{-1} P_1 = \left(\begin{array}{cc} 2uv\partial + 2uv & 2u\partial + 2\partial u + v^2 \partial \\ 2u\partial + 2\partial u + \partial v^2 & 2v\partial + 2\partial v \end{array}\right) \quad (3.45)$$

The map (3.39) that sends the Boussinesq limit of the 2-layer fluid system to the Airy system is non canonical, and, as it can be explicitly checked, the single Airy Poisson tensors acquire a complicated form in the $(\xi, \sigma)$ coordinates. However, it can be easily checked that under (3.39) the Darboux Poisson structure of the former is sent into the linear combination $4P_0 - P_2$ of the latter.

**Remark 3.5.** A large number of integrals of the motion for the $N$-fields Benney system (which reduces to the Airy system for $N = 1$) was obtained in [20]. Constructing this family of integrals with Lenard-Magri recursion relations directly guarantees their involutivity.

### 3.5. Non-Boussinesq deformations

Let us now go back to the study of the two-layer equations of motion (3.36), and in particular to the task of finding conserved quantities for this system, written in the quasilinear form

$$\xi_t = -(H_\xi)_s, \quad \sigma_t = -(H_\sigma)_s, \quad (3.46)$$

with

$$H(\xi, \sigma) = \frac{1}{4} \left( \frac{(1 - \xi^2)\sigma^2}{1 - \xi^2} + \xi^2 \right).$$

We have recalled the construction of an infinite set of mutually commuting integrals of motion for the of the Boussinesq limit $r = 0$ of these equations. One could ask whether the property of admitting such a distinguished infinite families of integrals of motion holds also in the generic $r \neq 0$ case. We have not succeeded in finding a second Hamiltonian structure for this case, however it can be proved that if one existed it would not correspond to a linear combination of the Poisson structures $P_j, j = 0, 1, 2$. Hence, for the construction of conserved quantities for the deformed system we follow the general approach, based on the fact that for one-dimensional Hamiltonian systems in Darboux coordinates, conserved quantities are functionals $F$ satisfying the commutation relation

$$\{ F, \mathcal{H} \} \equiv \int (F(\xi(H_\xi)_s) + F_\xi(H_\xi)_s) dx = 0. \quad (3.47)$$
The functional $\mathcal{F}$ is a conserved density only if the integrand $(F_\xi(H_\sigma) + F_\sigma(H_\xi))$ is a total spatial derivative (we are assuming that both fields satisfy appropriate boundary conditions, e.g. vanish together with their spatial derivatives as $|x| \to \infty$ for integrals of the whole real line). A direct computation shows that this property translates into the equation

$$F_\xi H_\sigma = H_\xi F_\sigma$$

for the densities. In our case, this yields the following PDE for the density $F$:

$$(\xi^2 - 1) \frac{\partial^2}{\partial \xi^2} F(\xi, \sigma) = \frac{r\lambda^2 - r^2\sigma^2 - 3 \xi^2 + 3 r \xi + \sigma^2 - 1}{(1 - r \xi^2)} \frac{\partial^2}{\partial \sigma^2} F(\xi, \sigma).$$

Explicit solutions of (3.48)–(3.49) are in general not available, hence we turn to the perturbative analysis in the small $r$-limit expansion of the Hamiltonian $H(\xi, \sigma)$.

The first order deformation in the small parameter $r$ is

$$H(\xi, \sigma) = H_0(\xi, \sigma) + r H_1(\xi, \sigma) + o(r) = \frac{1}{4}(1 - \xi^2) \sigma^2 + \xi + r \frac{1}{4}(1 - \xi^2)\sigma^2 + o(r).$$

According to our perturbative approach, we substitute $H = H_0 + r H_1 \equiv H_{r,1}$ (defined by equation (3.50)) and seek an approximate constant of the motion $F$ in the form

$$F_{r,1} = F_0 + r F_1$$

satisfying equation (3.48) at first order in $r$. Dropping terms of order $O(r^2)$ or higher yields, explicitly,

$$F_{0,\xi \xi} H_{0,\sigma \sigma} + r(F_{1,\xi \xi} H_{0,\sigma \sigma} + F_{0,\xi \xi} H_{1,\sigma \sigma}) = H_{0,\xi \xi} F_{0,\sigma \sigma} + r(H_{1,\xi \xi} F_{0,\sigma \sigma} + H_{0,\xi \xi} F_{1,\sigma \sigma}).$$

Of course, at leading order $F_0$ must be the density of a conserved quantity for the Boussinesq limit, while at order $O(r)$ we have

$$F_{1,\xi \xi} H_{0,\sigma \sigma} + F_{0,\xi \xi} H_{1,\sigma \sigma} = H_{1,\xi \xi} F_{0,\sigma \sigma} + H_{0,\xi \xi} F_{1,\sigma \sigma},$$

i.e. substituting the expression of $H_{r,1}$,

$$(1 - \sigma^2) \frac{\partial^2}{\partial \sigma^2} F_0(\xi, \sigma) - (1 - \xi^2) \frac{\partial^2}{\partial \xi^2} F_0(\xi, \sigma) = 3 \xi \sigma^2 \frac{\partial^2}{\partial \sigma^2} F_0(\xi, \sigma) - (1 - \xi^2) \xi \sigma^2 \frac{\partial^2}{\partial \xi^2} F_0(\xi, \sigma).$$

The problem of finding suitable deformations of the conserved quantities of the Boussinesq limit thus reduces to the problem of finding solutions to this inhomogeneous linear equation for $F(\xi, \sigma)$. This can be done explicitly for the polynomial constants of motion for the Boussinesq limit, that is, those obtained expanding around $\lambda = \infty$ the generator

$$\hat{Q}(\lambda) = -\frac{1}{4} \sqrt{\lambda^2 - 4 \xi \sigma \lambda + 4(\sigma^2 + \xi^2 - 1)}.$$
\( F_{0,1} = \frac{1}{2} \xi \sigma \)
\( F_{0,2} = \frac{1}{2} (1 - \xi^2)(1 - \sigma^2) \)
\( F_{0,3} = \xi \sigma (1 - \xi^2)(1 - \sigma^2) \)
\( F_{0,4} = \frac{1}{2} (1 - \xi^2)(1 - \sigma^2)(5 \xi^2 \sigma^2 - \sigma^2 - \xi^2 + 1) \)
\( F_{0,5} = \xi \sigma (1 - \xi^2)(1 - \sigma^2)(7 \xi^2 \sigma^2 - 3 \sigma^2 - 3 \xi^2 + 3) \)
\( F_{0,6} = (1 - \xi^2)(1 - \sigma^2)(21 \sigma^4 \xi^2 - 14 \sigma^2 \xi^2 - 14 \sigma^4 \xi^4 + \sigma^4 + 16 \xi^2 \sigma^2 + \xi^4 - 2 \sigma^2 - 2 \xi^2 + 1). \) (3.56)

(Note that the second element of this family coincides up to a factor and a constant term with the Boussinesq Hamiltonian). We have the following

**Proposition 3.6.**  Every polynomial constant of the motion \( F_{0,j} \) admits a (polynomial) first order deformation \( F_{1,j} \), i.e. for every integer \( j \geq 1 \) it is possible to find a suitable polynomial \( F_{1,j} \) such that the quantities \( F_{r,j} \) satisfy equation (3.52).

**Proof.**  The proof of this proposition is somewhat technical, hence we omit details here and report them in the appendix.

It turns out that \( F_{0,1} \propto \xi \sigma \) remains a constant of the motion for the deformed system, so that the corresponding deformation vanishes, \( F_{1,1} = 0 \). Similarly, the deformation of the Hamiltonian \( H_0 = \frac{1}{4} (F_{0,2} - 1) \) is already available as (see equation (3.50))

\[ F_{1,2} = \frac{1}{4} \xi (1 - \xi^2) \sigma^2. \]

The first non-trivial deformations can be found to be

\[ F_{1,3} = \frac{1}{2} \sigma (4 \sigma^2 \xi^4 - 6 \sigma^2 \xi^2 - \xi^4 + 2 \sigma^2 + 6 \xi^2) \]
\[ F_{1,4} = \frac{1}{10} \xi (75 \sigma^2 \xi^4 - 130 \sigma^2 \xi^2 - 40 \sigma^2 \xi^4 + 55 \sigma^4 + 140 \sigma^2 \xi^2 + \xi^4 - 100 \sigma^2 - 30 \xi^2) \]
\[ F_{1,5} = \frac{1}{2} \sigma (56 \sigma^4 \xi^6 - 110 \sigma^4 \xi^4 - 45 \sigma^2 \xi^6 + 60 \sigma^4 \xi^2 + 139 \sigma^2 \xi^4 + 5 \xi^6 - 6 \sigma^4 - 111 \xi^2 \sigma^2 - 41 \xi^4 + 17 \sigma^2 + 51 \xi^2) \]
\[ F_{1,6} = \frac{1}{35} \xi (3675 \xi^6 \sigma^6 - 8085 \sigma^6 \xi^4 - 3920 \sigma^4 \xi^6 + 5425 \sigma^6 \xi^2 - 11970 \sigma^4 \xi^4 + 861 \sigma^2 \xi^6 - 1015 \sigma^6 - 10780 \sigma^2 \xi^4 - 4711 \sigma^2 \xi^2 + 16 \sigma^6 + 2730 \sigma^4 + 6055 \xi^2 \sigma^2 + 322 \xi^4 - 2205 \sigma^2 - 700 \xi^2) \] (3.57)

**Remark 3.7.**  Our existence results make contact with the theory of Birkhoff normal forms for Hamiltonian systems. In the finite-dimensional case the possibility of deforming quadratic Hamiltonians up to the first order, preserving ‘Liouville tori’ is well studied and settled. In the infinite dimensional case, there are some general results are available (see, e.g. [1]), and are mostly aimed at obtaining the infinite dimensional version of the ‘problem of small divisors’. However, the starting point zero-order Hamiltonian in these works is the quadratic
Hamiltonian of the linear D’Alembert (or Klein-Gordon) wave equations. Our example seems to lie outside of such a class, since the undeformed Hamiltonian of our case is cubic.

3.6. Some hodograph solutions

As recalled at the beginning of section 3, a well-known result of the theory of quasilinear Hamiltonian equations (see, e.g. [17] for a review), states that, for every density \( F \) of a conserved quantity of a Hamiltonian quasilinear PDE with Hamiltonian density \( H(\xi, \sigma) \), the functions \( (\xi(x,t), \sigma(x,t)) \), implicitly defined by the system

\[
\begin{align*}
   x + tH_{\xi\sigma} &= F_{\xi\sigma} \\
   tH_{\sigma\sigma} &= F_{\sigma\sigma},
\end{align*}
\]

or equivalently by

\[
\begin{align*}
   x + tH_{\xi\sigma} &= F_{\xi\sigma} \\
   tH_{\xi\xi} &= F_{\xi\xi},
\end{align*}
\]

provide local solutions of the equations of motion.

Since the quantities \( F_{0,0} + r F_{1,1} \) found in the previous section are constants of motion in the small-\( r \) asymptotics, we can use either of the above equations to construct approximate solutions (that is, solutions at first order in the \( r \) expansion) to the deformed system.

As well known, generic solutions constructed with the hodograph method are local because the method itself requires the inversion of the map between the independent variables \( x, t \) and the fields \( \xi, \sigma \) (see, e.g. [32]). To the best of our knowledge, the reconstruction problem with fairly generic initial data was first solved for the Airy system in [31]. An analogous study for the deformed system with results of comparable generality would certainly be valuable but will be left to future work. However, since we can explicitly find the conserved densities \( F \) appearing in the left hand side of the hodograph equation (3.58), we can extract a modicum of general information, interpreting it as an evolution of a curve in the space \( \xi, \sigma \). An explicit

![Figure 4. Evolution of the class of initial data related by the conserved quantity \( F_3 \) at first order in \( r \). Left panel: evolution in time (indicated by arrows) out of the initial data (thick line) spans the family of curves in the hodograph plane \((\xi, \sigma)\). Right panel: analogous variation in space, as the spatial coordinate \( x \) grows along the direction indicated by the arrow.](image-url)
example for the initial data related to $F_{0,3} + rF_{1,3}$ is depicted in figure 4 on the left; the family of curves in the figure is defined by

$$
\begin{align*}
\eta = \frac{F_{0,\sigma}}{H_{\sigma \sigma}} &= -24\xi\sigma + r(24\sigma - 24\xi^2\sigma) + o(r) \\
x = F_{\xi\sigma} - \frac{F_{\xi\sigma}}{H_{\xi\xi}} &= -3\xi^2(\sigma^2 + 1) - 3\sigma^2 + 1 + r(-2\xi(3\sigma^2 + 1) - 3)) + o(r).
\end{align*}
$$

Another interesting explicit example of initial data can be presented, related to a suitably fine-tuned linear combination of the first six deformed integrals of the motion,

$$
F = 4(F_{0,3} + rF_{1,3}) - 13.78(F_{0,4} + rF_{1,4}) - 14(F_{0,5} + rF_{1,5}) - 3.44352(F_{0,6} + rF_{1,6}) + o(r).
$$

The evolution of the corresponding initial data is depicted in figure 5. We remark that, while initially the curve is smooth and completely contained in the hyperbolic domain, after a time interval $\Delta t_e \sim 4.15$ the curve crosses the sonic line and ends up in the elliptic domain, thus exhibiting a transonic transition, which is forbidden by the Boussinesq $r = 0$ system.

### 4. Conclusions and discussion

In this work, we have examined the systematic Hamiltonian reduction of Benjamin’s formulation for two-dimensional stratified Euler fluids to the case of two homogeneous layers. The resulting leading order system in the long-wave approximation, with time and one spatial horizontal coordinate as independent variables, corresponds to a set of quasi-linear equations which we have framed within the theory of Hamiltonian near-integrable systems. In particular, we have isolated the properties of the hyperbolicity region that depend on the
Hamiltonian (in particular on the inertial parameter \( r \)), such as the tangency to sonic lines being different from simple-wave tangents.

Next, we have discussed how the Boussinesq limit of negligible inertia is completely integrable as it corresponds to the Airy system. We used the multi-Hamiltonian structure of this system to select an infinite family of motion invariants. The non-Boussinesq counterpart does not share such completely integrable structure; however, by perturbative methods based on the small inertia parameter \( r \), we have shown that an infinite family of polynomial constants of motion can be constructed explicitly at leading order \( O(r) \) by deformation of a specific family of the Boussinesq case. These motion invariants have then been used to build a few examples of local solutions by the hodograph method.

Our investigation lends itself to further generalizations, in particular: dispersion terms in the Hamiltonian reduction of Benjamin’s formulation can be obtained by retaining higher order terms in the Hamiltonian density expansion with respect to the long-wave parameter; dispersion deformations can then be analyzed for both the Boussinesq and non-Boussinesq cases by similar Hamiltonian methods as the ones used here for the dispersionless case, and families of conserved quantities could be found, combining dispersive with non-Boussinesq deformations; in turn, these conservation laws could shed some light on the dynamics of the solution for the model systems and illustrate fundamental properties of the full Euler parent system. Study of some of these issues is ongoing and will be presented in future work.

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Appendix

Proof of proposition 3.6

Let us consider the family of polynomials constants of the motion for the Boussinesq approximation obtained (see equation (3.56)) by expanding the generating function

\[
Q_1(\lambda) = -\frac{1}{4}\sqrt{\lambda^2 - 4\xi\sigma} + 4(\sigma^2 + \xi^2 - 1)
\]

around \( \lambda = \infty \) as

\[
\tilde{Q}(\lambda) = -\frac{\lambda}{4} + \sum_{k=0}^{\infty} \frac{F_{0,k}}{\lambda^k}.
\]

We must prove proposition 3.6, that is, show that each of these constants of the motion admits a polynomial deformation.

The proof can be divided in five steps.
Step 1. Thanks to the form of the generator $\hat{Q}(\lambda)$ the following factorization properties hold:

1. $F_{0,0} = (1 - \xi^2)(1 - \sigma^2)\partial_\xi(\xi^2, \sigma^2)$;
2. $F_{0,j+1} = \xi_\sigma(1 - \xi^2)(1 - \sigma^2)q_j(\xi^2, \sigma^2)$ (for \(j \geq 1\)).

for some suitable polynomials $p_j, q_j$. In particular, the most relevant property is that all the $F_{0,j}$’s factor through $(1 - \xi^2)(1 - \sigma^2)$, which is obvious since in the Madelung variables of the $dNLS$ equation $Q(\lambda)|_{\text{Mad}}$ is simply $\mp \frac{\lambda}{\nu}$. The finer factorization properties listed above are important for the introduction of suitable subspaces on the space of bivariate polynomials in Step 3.

Step 2. Since $H_1 \propto \xi^2 - \xi^2\sigma^2$, then

$$H_{1,\sigma}F_{\xi\xi} - H_{1,\xi}F_{\sigma\sigma} \propto 2(\xi(1 - \xi^2)F_{\xi\xi} - 3\xi\sigma^2F_{\sigma\sigma}). \quad (A.2)$$

Step 3. Let $R_N$ be the subspace of polynomials generated by the monomials

$$\xi^{2k+1}\sigma^{2j}, \quad \text{with } k, j = 0 \ldots N,$$

let $S_N$ be the one generated by

$$\xi^{2k}\sigma^{2j+1}, \quad \text{with } k = 0 \ldots N, j = 0 \ldots N - 1$$

and consider the operator entering the homogeneous part of equation (3.54):

$$\partial_\xi^2 := (1 - \xi^2)^2\partial_\xi(1 - \sigma^2)^2\partial_\sigma^2. \quad (A.3)$$

The following holds:

1. $\text{dim}(R_N) = (N + 1)^2$, $\text{dim}(S_N) = N(N + 1)$
2. $\partial_\xi(R_N) \subset R_N$, $\partial_\xi(S_N) \subset S_N$
3. The dimension of the kernel of $\partial_\xi$ restricted to $R_N$ (resp. $S_N$) is 1. This means that the image of $\partial_\xi^2$, seen as a map $R_N \rightarrow R_N$ (resp. $S_N \rightarrow S_N$) is characterized by a single linear relation in $R_N$ (resp $S_N$).

The proof of point 3 above is by direct computation, showing that the matrix representing $\partial_\xi^2$ restricted to, e.g. $R_N$ in the basis of point 2 above is upper triangular, with diagonal elements

$$2(j + k)(2j - 1 - 2k),$$

hence the assertion.

Step 4. An obvious observation is that, for any polynomial $P(\xi, \sigma)$ the sum of the coefficients of $\partial_\xi(P(\xi, \sigma))$ vanishes, or, in other words,

$$\partial_\xi(P(\xi, \sigma))|_{\xi=1, \sigma=1} = 0.$$

Since $\text{rk}(\partial_\xi|_{R_N}) = \text{dim}R_N - 1$ we get that a polynomial $Q(\xi, \sigma) \in R_N$ is in the image of $\partial_\xi$ if and only if

$$Q(\xi, \sigma)|_{\xi=1, \sigma=1} = 0. \quad (A.4)$$

The same holds for $S_N$, that is, $Q \in S_N$ is in the image of $\partial_\xi$ if and only if (A.4) holds.

Step 5. What is left to prove is that the LHS of equation (A.2), i.e.
\[ \xi(1 - \xi^2)F_{\xi\xi} - 3\xi\sigma^2F_{\sigma\sigma} \]

satisfies the characteristic condition \((A.4)\) when \(F(\xi, \sigma)\) is one of the polynomial Hamiltonian densities. Explicitly, we have to prove that \(\xi(1 - \xi^2)H_{j,\xi\xi} - 3\xi\sigma^2H_{j,\sigma\sigma}\) satisfies \((A.4)\). This is immediate, since, from Step 1 we know that \(H_j\) factors through \((1 - \xi^2)\). This ends the proof. □

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