Actions of the monodromy matrix elements
ton onto $\mathfrak{gl}(m|n)$-invariant Bethe vectors

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Abstract

Multiple actions of the monodromy matrix elements onto off-shell Bethe vectors in the $\mathfrak{gl}(m|n)$-invariant quantum integrable models are calculated. These actions are used to describe recursions for the highest coefficients in the sum formula for the scalar product. For simplicity, detailed proofs are given for the $\mathfrak{gl}(m)$ case. The results for the supersymmetric case can be obtained similarly and are formulated without proofs.
1 Introduction

This paper is a continuation of the paper [1] devoted to the description of the off-shell Bethe for the \( \mathfrak{gl}(m|n) \)-invariant quantum integrable models. One of the main results of [1] was an action formula of the upper-triangular and diagonal monodromy matrix elements onto off-shell Bethe vectors in the corresponding models. These results were obtained by expressing the Bethe vectors in terms of the current generators of the Yangian double \( DY(\mathfrak{gl}(m|n)) \). The same method of calculation for the actions of the lower-triangular monodromy matrix elements appears to be too cumbersome to be detailed. The present paper describes an alternative way to find these actions using a generalization of the so-called zero modes method. To simplify our presentation we give the detailed proofs only for the non-supersymmetric case \( n = 0 \), since the methods we are using extend readily to the general case.

The action of the monodromy matrix elements onto the Bethe vectors plays important role in the study of quantum integrable models. The action of the upper-triangular elements generate recursions for the Bethe vectors. Formulas for the action of the diagonal elements are key in solving the problem of the spectrum of Hamiltonians of quantum integrable systems. It is from these formulas that the Bethe equations that determine the spectrum follow. Finally, the action of the lower-triangular elements are necessary for the studying scalar products of Bethe vectors, which, in their turn, are used for calculating correlation functions. In this case, we need formulas for the action of not only a single element of the monodromy matrix, but also the so-called multiple action formulas when we act on the Bethe vector by a product of lower-triangular elements.

Let \( N = m + n - 1 \). First, we focus on calculating the action of monodromy matrix elements onto off-shell Bethe vectors for \( \mathfrak{gl}(N + 1) \)-invariant integrable models. Recall that the paper [1] discusses an approach to a description of the space of states for the quantum integrable models using infinite-dimensional current algebras proposed in [2] and developed in [3]. This approach takes advantage of the fact that monodromy matrices in the quantum integrable models satisfy the same commutation relations as a generating series of the generators of certain infinite-dimensional algebras [4]. These algebras can usually be realized in two different patterns, either in the form of so-called \( L \)-operators or in terms of total currents [5, 6]. The description of the space of states (Bethe vectors) in quantum integrable models uses the concept of projections onto intersections of the different type Borel subalgebras related either to \( L \)-operator or current realizations respectively.

The fact that the action of monodromy matrix elements onto Bethe vectors produces a linear combination of the same vectors is almost obvious within the projection method. However, obtaining explicit and effective formulas for this action is a rather complex combinatorial problem. In the paper [1], only the actions of the upper-triangular and diagonal elements were calculated. In this paper we present an alternative method to find the actions of all monodromy matrix elements. For this, we use only information about the action of the element \( T_{1,N+1}(z) \) and the zero mode operators \( T_{i+1,i}[0] \) onto off-shell Bethe vectors and commutation relations between them. This starting information can be easily obtained from the projection method and it is formulated as lemma [4,2] below.

We call the method of calculating the monodromy matrix elements action onto Bethe vector
the zero modes method, because it is based on the commutation relations

\[ [T_{i,j}(z), T_{\ell+1,j}(0)] = \delta_{i,\ell} \kappa_i T_{i+1,j}(z) - \delta_{\ell,j-1} \kappa_j T_{i,j-1}(z), \]

which follows from the basic commutation relations (2.4) described below.

In this paper we do not use the projection method to describe the off-shell Bethe vectors. We fix these objects by the explicit formulas for the action of the transfer matrix (trace of the monodromy matrix) onto Bethe vectors and requirement that they become eigenvectors of the transfer matrix if the parameters of the Bethe vectors satisfy the so-called Bethe equations.

The paper is organized as follows. In section 2 we introduce our notation and describe the commutation relations of the monodromy matrix elements. Then, in section 3 we define the Bethe vectors, the dual Bethe vectors and describe their normalization. Section 4 contains the main result of the paper. This result for the simplest case of the action of one monodromy matrix element is proved in appendix A. The general case is proved in appendix B. Section 5 contains applications of the results obtained. Here we formulate recursions for the highest coefficients of the scalar product of Bethe vectors with respect to the rank of the algebra. Section 6 contains a generalization of the above results to the case of \( \mathfrak{gl}(m|n) \)-integrable models without detailed proofs.

2 RTT-algebra and notation

The quantum integrable models we are dealing with are treated by the so-called nested algebraic Bethe ansatz \[8, 9\] and correspond to algebras of rank more than 1. All these models are described by the operators gathered in the monodromy matrix \( T(z) \) which acts in a Hilbert space \( \mathcal{H} \) and an auxiliary space \( \mathbb{C}^{N+1} \). It satisfies an RTT commutation relation

\[ R(u, v) (T(u) \otimes I) (I \otimes T(v)) = (I \otimes T(v)) (T(u) \otimes I) R(u, v). \]

(2.1)

Here I is the identity matrix in \( \mathbb{C}^{N+1} \), and \( P \) is a permutation matrix in \( \mathbb{C}^{N+1} \otimes \mathbb{C}^{N+1} \). A \( \mathfrak{gl}(N + 1) \)-invariant \( R \)-matrix \( R(u, v) \) acts in \( \mathbb{C}^{N+1} \otimes \mathbb{C}^{N+1} \) and is given by

\[ R(u, v) = I \otimes I + g(u, v)P, \quad g(u, v) = \frac{c}{u - v}, \]

(2.2)

where \( c \) is a complex constant. Starting from (2.1) one can easily obtain commutation relations for the monodromy matrix elements

\[ T(u) = \sum_{i,j=1}^{N+1} E_{ij} \otimes T_{i,j}(u) \]

(2.3)

in the form

\[ [T_{i,j}(u), T_{k,l}(v)] = g(u, v) (T_{i,l}(u)T_{k,j}(v) - T_{i,l}(v)T_{k,j}(u)), \]

(2.4)

Here we consider only models corresponding to the algebra \( \mathfrak{gl}(N + 1) \). Results for the supersymmetric models are collected in the section 6. Similar results for the integrable models related to other series algebras will be considered elsewhere (see also pioneering papers \[10, 11\]).
where $E_{ij}$ is a unit matrix with the only non-zero element equal to 1 on the intersection of the $i$-th row and $j$-th column.

For the reasons which will become clear later we consider an asymptotic expansion of the monodromy matrix

$$T_{i,j}(u) = \delta_{ij} \kappa_i + \sum_{\ell \geq 0} T_{i,j}[\ell](u/c)^{-\ell-1}, \quad (2.5)$$

which includes parameters $\kappa_i \in \mathbb{C}$, $i = 1, \ldots, N + 1$, in the zeroth order of the expansion. When all $\kappa_i$ are equal to 1 the expansion (2.5) corresponds to Yangian $Y(gl(N+1))$ [4]. Using commutativity of the $R$-matrix (2.2) with $K \otimes K$, where $K = \text{diag}(\kappa_1, \ldots, \kappa_{N+1})$ is a diagonal matrix, we can multiply the Yangian $RTT$ commutation relations (2.1) by $K \otimes K$ to obtain this expansion of the monodromy matrix.

Commutation relations (2.1) imply that the transfer matrices

$$t(z) = \sum_{i=1}^{N+1} T_{i,i}(z) \quad (2.6)$$

commute for arbitrary values of the spectral parameters

$$t(u) \cdot t(v) = t(v) \cdot t(u).$$

Thus, $t(z)$ generates a set (in general infinite) of commuting quantities. Solving the model by the algebraic Bethe ansatz amounts to find eigenvectors of the transfer matrix $t(z)$ in the Hilbert space $\mathcal{H}$ of the quantum integrable model. To solve this problem the Hilbert space $\mathcal{H}$ should possess a special vector $|0\rangle$ called reference state such that

$$T_{i,i}(u) |0\rangle = \lambda_i(u) |0\rangle, \quad i = 1, \ldots, N + 1,$$

$$T_{i,j}(u) |0\rangle = 0, \quad i > j. \quad (2.7)$$

Here the functional parameters $\lambda_i(u)$ are characteristic of the concrete model. Further on we will use the ratios of these free functional parameters

$$\alpha_i(u) = \frac{\lambda_i(u)}{\lambda_{i+1}(u)}, \quad i = 1, \ldots, N, \quad (2.8)$$

with the asymptotic values

$$\lim_{u \to \infty} \alpha_i(u) = \frac{\kappa_i}{\kappa_{i+1}}, \quad i = 1, \ldots, N. \quad (2.9)$$

### 2.1 Notation

Besides rational function $g(u,v)$ already introduced by (2.2), we define two rational functions $f(u,v)$ and $h(u,v)$ by

$$f(u,v) = 1 + g(u,v) = \frac{u - v + c}{u - v}, \quad h(u,v) = \frac{f(u,v)}{g(u,v)} = \frac{u - v + c}{c}. \quad (2.9)$$

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Footnote: According to [4] we define a Yangian as a Hopf algebra generated by coefficients $T_{i,j}[\ell]$, $\ell \geq 0$, such that commutation relations (2.1) are satisfied and the asymptotics of $T_{i,i}(u)$ is $T_{i,i}(u) = \delta_{ij} + O(u^{-1})$ at $u \to \infty$. 
For any of the functions \( a = g, f, h \) and for any set of generic complex parameters \( \bar{x} = \{x_1, \ldots, x_p\} \) we introduce ‘triangular’ products

\[
\Delta_a(\bar{x}) = \prod_{i<j} a(x_j, x_i), \quad \Delta'_a(\bar{x}) = \prod_{i<j} a(x_i, x_j).
\]  

(2.10)

For any two sets of generic complex parameters \( \bar{y} = \{y_1, \ldots, y_p\} \) and \( \bar{x} = \{x_1, \ldots, x_p\} \) of the same cardinality \( \#\bar{y} = \#\bar{x} = p \) we also introduce an Izergin determinant \( K(\bar{y}|\bar{x}) \)

\[
K(\bar{y}|\bar{x}) = \Delta_g(\bar{y})\Delta'_g(\bar{x}) \prod_{\ell,\ell' = 1}^p h(y_\ell, x_{\ell'}) \det \begin{bmatrix} g(y_\ell, x_{\ell'}) \\ h(y_\ell, x_{\ell'}) \end{bmatrix}_{\ell,\ell' = 1, \ldots, p}.
\]  

(2.11)

Any Izergin determinant depending on \( \bar{x} \) and \( \bar{y} \), such that \( \#\bar{x} \neq \#\bar{y} \), is by definition considered to be equal to zero. If \( \#\bar{x} = \#\bar{y} = 0 \), then \( K(\emptyset|\emptyset) \equiv 1 \).

Further on we will often work with sets of parameters. We denote them by a bar, like in \( (2.10), (2.11) \). In particular, the Bethe vector \( \mathbb{B}(\bar{t}) \) depends on the set of parameters (usually called Bethe parameters)

\[
\bar{t} = \{t_1^1, t_2^2, \ldots, t_i^N\}, \quad \bar{t}^i = \{t_1^i, t_2^i, \ldots, t_i^i\}, \quad i = 1, \ldots, N.
\]  

(2.12)

The upper index labels the type of Bethe parameter and corresponds to the simple roots of the algebra \( \mathfrak{g}(N + 1) \). To simplify further formulas for the products over sets, we use the following convention:

\[
f(u, \bar{t}) = \prod_{t_j^i \in \bar{t}} f(u, t_j^i), \quad f(\bar{t}^i, \bar{x}^p) = \prod_{t_j^i \in \bar{t}^i} \prod_{x_k^p \in \bar{x}^p} f(t_j^i, x_k^p),
\]  

(2.13)

\[
\lambda_i(\bar{t}) = \prod_{t_j^i \in \bar{t}} \lambda_i(t_j^i), \quad \alpha_i(\bar{t}) = \prod_{t_j^i \in \bar{t}} \alpha_i(t_j^i), \quad T_{i,j}(\bar{t}_j^i) = \prod_{t_k^j \in \bar{t}_j^i} T_{i,j}(t_k^j).
\]  

(2.14)

In a word, if any scalar function or mutually commuting operators depend on a set of parameters, we assume the product of these quantities with respect to this set. We always assume that any such product is 1 if any of the sets is empty.

For any set of Bethe parameters \( \bar{t}^i \) of cardinality \( \#\bar{t}^i = r_i \), the set \( \bar{t}_k^i \) means the set \( \bar{t}^i \setminus \{t_k^i\} \) of cardinality \( r_i - 1 \).

3 Bethe vectors

The Bethe vectors \( \mathbb{B}(\bar{t}) \in \mathcal{H} \) are rather special polynomials in the non-commuting operators \( T_{i,j}(t) \) for \( i \leq j \) acting on the reference vector \( |0\rangle \). We do not use the explicit form of these polynomials, however, the reader can find it in \([1]\). The main property of Bethe vectors is that they become eigenvectors of the transfer matrix

\[
t(z)\mathbb{B}(\bar{t}) = \tau(z; \bar{t})\mathbb{B}(\bar{t}),
\]  

(3.1)

\footnote{It follows from \( (2.4) \) that \( [T_{i,j}(u), T_{i,j}(v)] = 0 \).}
provided the Bethe parameters $\vec{t}$ satisfy a system of equations

$$\alpha_i(t_i) = \frac{f(t_i, \vec{t})}{f(t_i, \vec{t}^{-1})} \frac{f(\vec{t}^{+1}, t_i)}{f(\vec{t}^{-1}, \vec{t})}, \quad \vec{t}^0 = \vec{t}^{N+1} = \emptyset, \quad (3.2)$$
called the Bethe equations. Then the vector $B(\vec{t})$ is called on-shell Bethe vector. Otherwise, if the parameters $\vec{t}$ are generic complex numbers, the vector $B(\vec{t})$ is called off-shell Bethe vector.

The eigenvalue $\tau(z; \vec{t})$ in (3.1) is

$$\tau(z; \vec{t}) = \sum_{i=1}^{N+1} \lambda_i(z) f(z, \vec{t}^{-1}) f(\vec{t}, z). \quad (3.3)$$

It is shown in appendix C that the action of the transfer matrix computed via the proposition 4.1 results in the relation (3.1) provided the Bethe equations (3.2) are fulfilled.

Among all terms in the polynomials defining Bethe vectors, we single out one, which is called the main term. Its distinctive property is that it contains only the operators $T_{i,i+1}$ and does not contain the operators $T_{i,j}$ with $j - i > 1$. We fix normalization of the Bethe vectors in such a way that the main term $\tilde{B}(\vec{t})$ has the form

$$\tilde{B}(\vec{t}) = T_{N,N+1}(\vec{t}) T_{N-1,N}(\vec{t}^{N-1}) \cdots T_{23}(\vec{t}^2) T_{12}(\vec{t}^1) |0\rangle \prod_{i=1}^{N} \lambda_i(\vec{t}) \prod_{i=1}^{N-1} f(\vec{t}^{i}, \vec{t}^{i+1}), \quad (3.4)$$

In order to define the scalar product of the Bethe vectors we need first to define the left (dual) off-shell Bethe vector. This can be done using transposition antimorphism of the algebra (2.4)

$$\Psi : T_{i,j}(u) \to T_{j,i}(u), \quad \Psi(A \cdot B) = \Psi(B) \cdot \Psi(A), \quad (3.5)$$

where $A$ and $B$ are any products of the monodromy matrix elements. It is easy to see that $\Psi$ is an involution: $\Psi^2 = \text{id}$. We extend this antimorphism to the Hilbert space of the quantum integrable models by the rule

$$\Psi(|0\rangle) = |0\rangle, \quad \Psi(A|0\rangle) = |0\rangle \Psi(A), \quad (3.6)$$

with normalization $\langle 0|0 \rangle = 1$. Application of the antimorphism $\Psi$ to the formulas (3.7) yields

$$\langle 0|T_{i,i}(u) = \lambda_i(u) |0\rangle, \quad \langle 0|T_{j,j}(u) = 0, \quad i > j. \quad (3.7)$$

We define the dual off-shell Bethe vectors $C(\vec{t})$ as

$$C(\vec{t}) = \Psi(B(\vec{t})), \quad (3.8)$$

with normalization

$$\tilde{C}(\vec{t}) = \Psi(\tilde{B}(\vec{t})) = \frac{\langle 0|T_{2,1}(\vec{t}^1) T_{3,2}(\vec{t}^2) \cdots T_{N,N+1}(\vec{t}^{N+1}) T_{N+1,N}(\vec{t}^{N}) \prod_{i=1}^{N} \lambda_i(\vec{t}) \prod_{i=1}^{N-1} f(\vec{t}^{i}, \vec{t}^{i+1})}. \quad (3.9)$$
4 Multiple actions

The aim of this section is to present the multiple action of the monodromy matrix entry $T_{i,j}(z)$ on Bethe vectors $\mathbb{B}(t)$. This action will be presented as a sum on some partitions of sets, and to ease the reading, we first describe which type of partitions we will consider.

**Definition 4.1.** Let $\bar{z}$ be a set of arbitrary complex variables of cardinality $\#\bar{z} = p$, $\bar{t} = \{t^0, \bar{t}^1, \ldots, t^{N+1}\}$ be a multi-set with $t^0 = t^{N+1} = \emptyset$ and let $i$ and $j$ be arbitrary integers from the set $\{1, \ldots, N + 1\}$. We say that the multi-set $\bar{w} = \{\bar{w}^0, \bar{w}^1, \ldots, \bar{w}^{N+1}\}$ with $\bar{w}^s = \{\bar{z}, \bar{t}^s\}$ is divided into subsets obeying the $(i,j)$-**condition w.r.t.** $\bar{z}$ if it obeys the following conditions:

- The sets $\bar{w}^s$ are divided into three subsets $\{\bar{w}^i_s, \bar{w}^a_s, \bar{w}^b_s\} \vdash \bar{w}^s$.
- Boundary conditions: $\bar{w}^0_1 = \bar{w}^{N+1}_1 = \bar{z}$, $\bar{w}^0_N = \bar{w}^{N+1}_N = \emptyset$.
- The subsets $\bar{w}^i_s$ are non-empty only for $s < i$ and have cardinality $\#\bar{w}^i_s = p$ when $s < i$.
- The subsets $\bar{w}^a_s$ are non-empty only for $s \geq j$ and have cardinality $\#\bar{w}^a_s = p$ when $s \geq j$.
- The rest of the variables belongs to the subsets $\bar{w}^b_s$.

The calculation of the multiple action relies on knowledge of the (single) action of $T_{i,j}(z)$, which is described in

**Lemma 4.1.** The action of the monodromy matrix element $T_{i,j}(z)$ onto the off-shell Bethe vector $\mathbb{B}(t)$ is given by the expression

$$
T_{i,j}(z)\mathbb{B}(\bar{t}) = \lambda_{N+1}(z) \sum_{\text{part}} \mathbb{B}(\bar{w}) \prod_{s=j}^{i-1} f(\bar{w}^i_s, \bar{w}^a_s) \prod_{s=j}^{i-2} f(\bar{w}^{a+1}_s, \bar{w}^a_s) \prod_{s=1}^{i-1} \frac{f(\bar{w}^i_s, \bar{w}^a_s)}{h(\bar{w}^i_s, \bar{w}^{a+1}_s)f(\bar{w}^a_s, \bar{w}^a_s)} \prod_{s=j}^{N} \frac{\alpha_s(\bar{w}^a_s)f(\bar{w}^a_s, \bar{w}^a_s)}{h(\bar{w}^a_s, \bar{w}^{a+1}_s)f(\bar{w}^a_s, \bar{w}^a_s)}. \tag{4.1}
$$

The sum in (4.1) runs over partitions obeying the $(i,j)$-condition w.r.t. $\{z\}$.

The proof of this lemma is given in appendix [A]. It uses the zero mode method based on the commutation relations (1.1) and the following

**Lemma 4.2.** The action of the monodromy matrix element $T_{1,N+1}(z)$ and the zero modes $T_{i+1,i}[0]$ onto off-shell Bethe vectors $\mathbb{B}(t)$ are given by the formulas

$$
T_{1,N+1}(z)\mathbb{B}(\bar{t}) = \lambda_{N+1}(z)\mathbb{B}(\bar{w}), \tag{4.2}
$$

where $\bar{w} = \{w^1, \ldots, w^N\}$, $\bar{w}^s = \{z, \bar{t}^s\}$ and

$$
T_{i+1,i}[0]\mathbb{B}(\bar{t}) = \sum_{\ell=1}^r \left( \kappa_{i+1} \frac{\alpha_i(t^\ell)}{f(t^\ell)} - \kappa_i \frac{f(t^\ell)}{f(t^\ell, \bar{t}^{\ell-1})} \right) \mathbb{B}(\bar{t} \setminus \{t^\ell\}). \tag{4.3}
$$
The proof of lemma [4.2] can be easily done in the framework of the current algebra approach for the off-shell Bethe vectors in the \( gl(N+1) \)-invariant integrable models (see paper [1]). If the twisting parameters \( \kappa_i = 1, \forall i = 1, \ldots, N + 1 \) and the Bethe parameters satisfy the \( gl(N+1) \)-invariant Bethe equations (3.2), then the on-shell Bethe vectors become \( gl(N+1) \) highest weight vectors. Note that the first proof of relations (4.4) and (4.3) can be found in [1].

Then, the main result of this section is the following

**Proposition 4.1.** Let \( z = \{z_1, \ldots, z_p\} \) be a set of arbitrary complex variables of cardinality \( \#z = p \). Then, the multiple action of monodromy matrix elements \( T_{i,j}(z) \) onto the off-shell Bethe vector \( \mathcal{B}(\tilde{t}) \) is given by the expression\(^5\)

\[
T_{i,j}(z)\mathcal{B}(\tilde{t}) = \lambda_{N+1}(z) \sum_{\text{part}} \mathcal{B}(\tilde{w}) \frac{\prod_{s=j}^{i-1} f(\tilde{w}_s^1, \tilde{w}_m^1)}{\prod_{s=j}^{i-2} f(\tilde{w}_s^1, \tilde{w}_m^1)} \times \prod_{s=1}^{i-1} \frac{K(\tilde{w}_s^1|\tilde{w}_s^{1-1})f(\tilde{w}_s^1, \tilde{w}_m^1)}{f(\tilde{w}_s^1, \tilde{w}_m^{1-1})f(\tilde{w}_s^1, \tilde{w}_m^1)} \prod_{s=j}^{N} \frac{\alpha_s(\tilde{w}_m^s)K(\tilde{w}_m^s+1|\tilde{w}_m^s)f(\tilde{w}_m^s, \tilde{w}_m^s)}{f(\tilde{w}_m^s+1, \tilde{w}_m^s)f(\tilde{w}_m^s+1, \tilde{w}_m^s)}. \tag{4.4}
\]

The sum in (4.4) runs over partitions obeying the \( (i, j) \)-condition w.r.t. \( z \), and \( T_{i,j}(z) \) is defined as in (2.11).

**Remark.** It follows from (2.11) that the Izergin determinant has poles if \( x_i = y_j, i, j = 1, \ldots, p \). Since the intersection of the sets \( \tilde{w}_s \) for different \( s \) may be not empty, we can have singularities in the determinants \( K(\tilde{w}_s^1|\tilde{w}_s^{1-1}) \) or \( K(\tilde{w}_m^s+1|\tilde{w}_m^s) \). It is easy to see, however, that all these determinants are divided by products of \( f \)-functions that compensate these singularities. Strictly speaking, such expressions should be understood as limits, but for brevity we omit the limit symbol.

The proof of proposition 4.1 is given in appendix B. It uses an induction over \( p \) and summation formulas for the Izergin determinant.

**Corollary 4.1.** The multiple action of monodromy matrix elements \( T_{j,i}(z) \) onto the dual off-shell Bethe vectors \( \mathcal{C}(\tilde{t}) \) is given by the expression

\[
\mathcal{C}(\tilde{t})T_{j,i}(z) = \lambda_{N+1}(z) \sum_{\text{part}} \mathcal{C}(\tilde{w}) \frac{\prod_{s=1}^{i-1} f(\tilde{w}_s^1, \tilde{w}_m^1)}{\prod_{s=j}^{i-2} f(\tilde{w}_s^1, \tilde{w}_m^1)} \times \prod_{s=1}^{i-1} \frac{K(\tilde{w}_s^1|\tilde{w}_s^{1-1})f(\tilde{w}_s^1, \tilde{w}_m^1)}{f(\tilde{w}_s^1, \tilde{w}_m^{1-1})f(\tilde{w}_s^1, \tilde{w}_m^1)} \prod_{s=j}^{N} \frac{\alpha_s(\tilde{w}_m^s)K(\tilde{w}_m^s+1|\tilde{w}_m^s)f(\tilde{w}_m^s, \tilde{w}_m^s)}{f(\tilde{w}_m^s+1, \tilde{w}_m^s)f(\tilde{w}_m^s+1, \tilde{w}_m^s)}. \tag{4.5}
\]

The sum over partitions is taken in the same way as in (4.4).

The proof of corollary 4.1 follows directly from the applying antimorphism (3.5) to the action formula (4.4) .

\(^5\)For \( i \leq j \), we have \( \prod_{s=j}^{i-1} f(\tilde{w}_s^1, \tilde{w}_m^1) = 1 \) and \( \prod_{s=j}^{i-2} f(\tilde{w}_s^{1+1}, \tilde{w}_m^1) = 1 \), because one of the sets in the arguments of the function \( f(u, v) \) is empty.
5 Scalar products

Scalar products of the off-shell Bethe vectors can be written as [12]

\[ S(\vec{x}|\vec{t}) = C(\vec{x})B(\vec{t}) = \sum_{\text{part}} Z(\vec{x}_1|\vec{t}_1)Z(\vec{t}_n|\vec{x}_n) \prod_{j=1}^{N} \frac{\alpha_j(x^j_1)\alpha_j(t^j_n)f(x^j_1,t^j_n)}{\prod_{j=1}^{N} f(x^j_1,x^j_{n+1})f(t^j_n,t^j_{n+1})}, \tag{5.1} \]

where the sum runs over partitions of the sets \{\vec{t}, \vec{x}\} \rightarrow \vec{t}, \{\vec{x}, \vec{t}\} \rightarrow \vec{x} such that \#\vec{t} = \#\vec{x} for all \( j = 1, \ldots, N \). Function \( Z(\vec{x}|\vec{t}) \) is known as the highest coefficient of the scalar product. It is determined only by the structure of the R-matrix entering the commutation relation of monodromy matrices. It is normalized in such a way that \( Z(\emptyset|\emptyset) = 1 \). We will refer to this formula as the sum formula. Let us remark that [17] presents the sum formula for the \( g(3) \) algebra, the first formula of this type was obtained by Korepin in [13] for the \( gl(2) \) algebra.

Note the antimorphism (3.6) implies invariance of the scalar product with respect to the replacement \( \vec{x} \leftrightarrow \vec{t} \):

\[ S(\vec{x}|\vec{t}) = \Psi(S(\vec{x}|\vec{t})) = \Psi(C(\vec{x})B(\vec{t})) = C(\vec{t})B(\vec{x}) = S(\vec{t}|\vec{x}). \tag{5.2} \]

This invariance is also easy to see directly from (5.1).

Due to the action formula (4.3) we can obtain recursions for the highest coefficients with respect to the rank of the algebra. To describe these recursions we equip the highest coefficients with an additional subscript: \( Z_m(\vec{x}|\vec{t}) \). This subscript \( m \) means that the corresponding highest coefficient is related to the scalar product of the Bethe vectors in the \( gl(m+1) \)-invariant quantum integrable model. Similar subscript with the same meaning will be used to denote Bethe vectors \( B_m(\vec{t}) \) and \( C_m(\vec{x}) \) and their scalar products \( S_m(\vec{x}|\vec{t}) \).

**Proposition 5.1.** The highest coefficient of the scalar product (5.1) satisfies the following recursion:

\[ Z_N(\vec{x}|\vec{t}) = \frac{f(\vec{t}, \vec{x})}{f(\vec{t}, \vec{x})} \sum_{\text{part}} Z_{N-1}(\vec{w}^2_1, \ldots, \vec{w}^N_1|\vec{t}_2, \ldots, \vec{t}_N) \prod_{s=1}^{N} \frac{K(\vec{w}^{s+1}_s|\vec{w}^s_1) f(\vec{w}^s_1, \vec{w}^s_1)}{f(\vec{w}^s_1, \vec{w}^s_1)}. \tag{5.3} \]

Here \( \vec{w}^s = \{\vec{t}, \vec{x}\} \) for \( s = 2, \ldots, N \). The sum is taken over partitions of \( \vec{w}^s_1 \) \rightarrow \vec{w}^s \) for \( s = 2, \ldots, N \) such that \#\vec{w}^s = r_1. By definition \( \vec{w}^1 = \vec{w}^1_s = \vec{x} \) and \( \vec{w}^{N+1} = \vec{w}^{N+1}_{N+1} = \vec{t}_1 \).

Another recursion for the highest coefficient reads

\[ Z_N(\vec{x}|\vec{t}) = \frac{f(\vec{t}, \vec{x})}{f(\vec{t}, \vec{x})} \sum_{\text{part}} Z_{N-1}(\vec{x}_1, \ldots, \vec{x}_{N-1}|\vec{w}^1_1, \ldots, \vec{w}^{N-1}_1) \prod_{s=1}^{N} \frac{K(\vec{w}^s_1|\vec{w}^{s-1}_s) f(\vec{w}^s_1, \vec{w}^s_1)}{f(\vec{w}^s_1, \vec{w}^s_1)}. \tag{5.4} \]

Here \( \vec{w}^s = \{\vec{x}_s, \vec{t}_s\} \) for \( s = 1, \ldots, N - 1 \). The sum is taken over partitions \( \vec{w}^s_1 \) \rightarrow \vec{w}^s \) for \( s = 1, \ldots, N - 1 \) such that \#\vec{w}^s = r_N. By definition \( \vec{w}^0 = \vec{w}^0_1 = \vec{x}^N \) and \( \vec{w}^N = \vec{w}^N_1 = \vec{t}^N \).

**Proof.** To prove proposition [5.1] we use a generalized model. The notion of the generalized model was introduced in [13] for \( gl(2) \) based models (see also [14, 15, 16, 17, 18]). This model
also can be considered in the case of the quantum integrable models with \( gl(N+1) \)-invariant \( R \)-matrix. In fact, the generalized model is a class of models. Each representative of this class has a monodromy matrix satisfying the \( RTT \)-relation (2.1) with the \( R \)-matrix (2.2), and possesses reference states \( |0 \rangle \) and \( \langle 0 | \) with the properties (2.7), (3.7). A representative of the generalized model can be characterized by a set of the functional parameters \( \lambda_i(u) \) (2.7). Different representatives are distinguished by different sets of the ratios \( \alpha_i(u) \) (2.8).

We first prove recursion (5.3). Since the highest coefficient is completely determined by the \( R \)-matrix, it does not depend on the specific choice of the representative of the generalized model. In other words, it does not depend on the free functional parameters \( \lambda_i(u) \). Therefore, it enough to prove (5.3) for some specific choice of \( \lambda_i(u) \). We choose them in such a way that
\[
\lambda_s(t^s_j) = 0, \quad s = 1, \ldots, N, \quad j = 1, \ldots, r_s, \quad (5.5)
\]
for given set \( \bar{t} \). This implies
\[
\alpha_s(t^s_j) = 0, \quad s = 1, \ldots, N, \quad j = 1, \ldots, r_s. \quad (5.6)
\]
Then all the subsets \( \bar{t}'_1 \) in (5.1) are empty. Hence, \( \bar{t} = \bar{t}' \). Since \( \# \bar{t} = \# \bar{t}' \), we conclude that \( \bar{x}' = \bar{x} \). The scalar product then reduces to the highest coefficient
\[
S(\bar{x}|\bar{t}) = \mathbb{Z}_N(\bar{x}|\bar{t}) \prod_{j=1}^N \alpha_j(\bar{x}^j). \quad (5.7)
\]

Consider the following expectation value:
\[
Q_N(\bar{x}|\bar{t}) = \mathbb{C}_N(\emptyset, \bar{t}^2, \ldots, \bar{t}^N) \frac{T_{2,1}(\bar{t}) \mathbb{B}_N(\bar{x})}{\lambda_2(\bar{t}) f(\bar{t}^2, \bar{t}^1)}. \quad (5.8)
\]
Here \( \# \bar{t}^1 = \# \bar{t} = r_1 \). Note that the dual vector \( \mathbb{C}_N(\emptyset, \bar{t}^2, \ldots, \bar{t}^N) \) does not depend on the first set of the Bethe parameters. Thus, this vector actually corresponds to a model with \( gl(N) \)-invariant \( R \)-matrix: \( \mathbb{C}_N(\emptyset, \bar{t}^2, \ldots, \bar{t}^N) = \mathbb{C}_{N-1}(\bar{t}^2, \ldots, \bar{t}^N) \).

The expectation value \( Q_N(\bar{x}|\bar{t}) \) can be computed in two different ways: either applying the product \( T_{2,1}(\bar{t}) \) to the left vector via (4.5), or applying \( T_{2,1}(\bar{t}) \) to the right vector via (4.4). Using the first way we obtain
\[
Q_N(\bar{x}|\bar{t}) = \frac{\lambda_{N+1}(\bar{t})}{\lambda_2(\bar{t}) f(\bar{t}^2, \bar{t}^1)} \sum_{\text{part}} \mathbb{C}_N(\bar{w}_N) \mathbb{B}_N(\bar{x}) \prod_{s=2}^N \alpha_s(\bar{w}^s_\text{in}) K(\bar{w}^{s+1}_\text{in} | \bar{w}^s_\text{in}) f(\bar{w}^s_\text{in}, \bar{w}^s_\text{in}) f(\bar{w}^{s+1}_\text{in}, \bar{w}^{s+1}_\text{in}). \quad (5.9)
\]
where \( \bar{w}^1 = \bar{t}^1 \), and \( \bar{w}^s = \{ \bar{t}^1, \bar{t}^s \} \) for \( s = 2, \ldots, N \). We also impose the following conditions: \( \# \bar{w}^s_\text{in} = r_1 \), \( \bar{w}^1_\text{in} = \bar{t}^1 \), \( \bar{w}^{N+1}_\text{in} = \bar{t}^1 \), \( \bar{w}^{N+1}_\text{in} = \emptyset \).

We see that due to (5.9) \( \bar{w}^s_\text{in} = \bar{t}^s \) for all \( s = 2, \ldots, N \), otherwise we obtain vanishing contributions. Hence, \( \bar{w}^s_\text{in} = \bar{t}^s \) for all \( s = 2, \ldots, N \). The sum over partitions disappears and we arrive at
\[
Q_N(\bar{x}|\bar{t}) = \frac{\lambda_{N+1}(\bar{t})}{\lambda_2(\bar{t})} \mathbb{C}_N(\bar{t}) \mathbb{B}_N(\bar{x}) \prod_{s=2}^N \alpha_s(\bar{t}^s) = \mathbb{C}_N(\bar{t}) \mathbb{B}_N(\bar{x}). \quad (5.10)
\]
Remark. To obtain (5.10) from (5.9), we used the following property of the Izyerin determinant

$$\lim_{\bar{x} \to \bar{y}} \frac{K(\bar{x}|\bar{y})}{f(\bar{x}, \bar{y})} = 1.$$  (5.11)

Thus, the expectation value $Q_N(\bar{x}|\bar{t})$ is equal to the scalar product $S_N(\bar{x}|\bar{t})$. Using (5.7) we obtain

$$Q_N(\bar{x}|\bar{t}) = Z_N(\bar{x}|\bar{t}) \prod_{j=1}^{N} \alpha_j(\bar{x}^j).$$  (5.12)

Acting with $T_{2,1}(\bar{t}^1)$ on $\mathbb{B}_N(\bar{x})$ via (4.3) we obtain

$$T_{2,1}(\bar{t}^1) \mathbb{B}_N(\bar{x}) = \lambda_{N+1}(\bar{t}^1) \sum_{\text{part}} \lambda_{2,1}(\text{part}) \mathbb{B}_N(\bar{w}_\text{part}),$$  (5.13)

where $\lambda_{2,1}(\text{part})$ is a numerical coefficient in (4.3) for $i = 2$ and $j = 1$. Due to the condition

$$\#\bar{t}^1 = \#\bar{x} = r_1$$

the subset $\bar{w}_1$ in the resulting vector $\mathbb{B}_N(\bar{w})$ is empty. Therefore, this vector corresponds to $\mathfrak{gl}(N)$-invariant models: $\mathbb{B}_N(\bar{w}) = \mathbb{B}_{N-1}(\bar{w}_2, \ldots, \bar{w}_N)$. Then we obtain

$$S_N(\bar{x}|\bar{t}) = \frac{\lambda_{N+1}(\bar{t}^1)}{\lambda_{2,1}(\bar{t}^1)f(\bar{t}^2, \bar{t}^1)} \sum_{\text{part}} \lambda_{2,1}(\text{part}) S_{N-1}(\bar{w}_2, \ldots, \bar{w}_N | \bar{t}^2, \ldots, \bar{t}^N).$$  (5.14)

Taking into account (5.7) we arrive at

$$Z_N(\bar{x}|\bar{t}) \prod_{j=1}^{N} \alpha_j(\bar{x}^j) = \frac{\lambda_{N+1}(\bar{t}^1)}{\lambda_{2,1}(\bar{t}^1)f(\bar{t}^2, \bar{t}^1)} \sum_{\text{part}} Z_{N-1}(\bar{w}_2, \ldots, \bar{w}_N | \bar{t}^2, \ldots, \bar{t}^N) \prod_{j=2}^{N} \alpha_j(\bar{w}_j).$$  (5.15)

It remains to use the explicit form of $\lambda_{2,1}(\text{part})$. Setting $i = 2$ and $j = 1$ in (4.3) we find

$$Z_N(\bar{x}|\bar{t}) \prod_{j=1}^{N} \alpha_j(\bar{x}^j) = \frac{\lambda_{N+1}(\bar{t}^1)}{\lambda_{2,1}(\bar{t}^1)f(\bar{t}^2, \bar{t}^1)} \sum_{\text{part}} Z_{N-1}(\bar{w}_2, \ldots, \bar{w}_N | \bar{t}^2, \ldots, \bar{t}^N) \prod_{j=2}^{N} \alpha_j(\bar{w}_j) \times K(\bar{w}_1^1, \bar{w}_1^1) \prod_{s=1}^{N} \alpha_s(\bar{w}_m^s) \frac{K(\bar{w}_m^{s+1}, \bar{w}_m^{s})f(\bar{w}_m^s, \bar{w}_m^s)}{f(\bar{w}_m^{s+1}, \bar{w}_m^s)f(\bar{w}_m^s, \bar{w}_m^s)}.  \quad (5.16)$$

Here $\bar{w}_s^s = \{\bar{t}^i, \bar{x}^s\}$ for $s = 2, \ldots, N$. The sum is taken over partitions $\{\bar{w}_1^s, \bar{w}_m^s\}$ for $s = 1, \ldots, N$ such that $\#\bar{w}_m^s = r_1$. The subsets $\bar{w}_1^1$, $\bar{w}_1^1$, and $\bar{w}_m^1$ actually are fixed by the condition (5.6): $\bar{w}_1^1 = \bar{t}^1$, $\bar{w}_1^1 = \bar{x}^1$, and $\bar{w}_1^1 = \emptyset$. Finally, by definition $\bar{w}_m^{N+1} = \bar{t}^1$ and $\bar{w}_m^N = \emptyset$.

First of all, it is easy to see that

$$\prod_{j=2}^{N} \alpha_j(\bar{w}_m^s) \prod_{s=1}^{N} \alpha_s(\bar{w}_m^s) = \alpha_1(\bar{t}^1) \prod_{j=1}^{N} \alpha_j(\bar{x}^j) = \frac{\lambda_{2,1}(\bar{t}^1)}{\lambda_{N+1}(\bar{t}^1)} \prod_{j=1}^{N} \alpha_j(\bar{x}^j).$$  (5.17)
Using again the property (5.11), we also have

\[
\frac{K(\bar{w}_i^s | \bar{t}_i^l)}{f(\bar{w}_i^s, \bar{t}_i^l)} = \frac{K(\bar{t}_i^l | \bar{t}_i^l)}{f(\bar{t}_i^l, \bar{t}_i^l)} = 1. \tag{5.18}
\]

Substituting (5.17) and (5.18) into (5.16) we arrive at (5.3). Thus, the recursion (5.3) is proved.

The second recursion (5.4) follows from (5.3) due to an isomorphism \( \varphi : Y(\mathfrak{gl}(N+1)) \to Y(\mathfrak{gl}(N+1)) \) \( \big|_{c \to -c} \) between Yangians with reflected parameters \( c \to -c \) \([19, 12]\). On the elements of the monodromy matrix, it is given explicitly by

\[
\varphi(T_{i,j}(u)) = \hat{T}_{N+2-j,N+2-i}(u), \quad i, j = 1, \ldots, N + 1. \tag{5.19}
\]

Here \( T_{i,j}(u) \in Y(\mathfrak{gl}(N+1)) \) and \( \hat{T}_{i,j}(u) \in Y(\mathfrak{gl}(N+1)) \big|_{c \to -c} \).

However, it is easier to directly derive (5.4) by renormalizing the initial Bethe vectors as follows:

\[
\hat{B}_N(\bar{t}) = B_N(\bar{t}) \prod_{s=1}^{N} \beta_s(\bar{t}^2), \quad \beta_s(z) = \frac{1}{\alpha_s(z)} = \frac{\lambda_{s+1}(z)}{\lambda_{s}(z)}, \tag{5.20}
\]

Then it is easy to see that scalar product of these new Bethe vectors takes the form

\[
\hat{S}_N(\bar{x}|\bar{t}) = \hat{C}_N(\bar{x})\hat{B}_N(\bar{t}) = \sum_{\text{part}} Z_N(\bar{x}_i|\bar{t}_i)Z_N(\bar{t}_i|\bar{x}_i) \prod_{j=1}^{N} \beta_j(\bar{x}_i\bar{t}_j) f(\bar{x}_i, \bar{t}_j) f(\bar{t}_i, \bar{t}_j) \prod_{j=1}^{N-1} f(\bar{x}_{j+1}, \bar{x}_j) f(\bar{t}_{j+1}, \bar{t}_j). \tag{5.21}
\]

The sum is taken over partitions as in (5.1).

The action formulas (4.4) and (4.5) also change. They respectively turn into

\[
T_{i,j}(\bar{z})\hat{B}_N(\bar{t}) = \lambda_{1}(\bar{z}) \sum_{\text{part}} \hat{B}_N(\bar{w}_n) \prod_{s=1}^{N} f(\bar{w}_i^s, \bar{w}_m^s) \prod_{s=1}^{N} \beta_s(\bar{w}_i^s) K(\bar{w}_i^s | \bar{w}_i^{s-1}) f(\bar{w}_i^s, \bar{w}_i^{s-1}) f(\bar{w}_i^s, \bar{w}_i^{s-1}), \tag{5.22}
\]

and

\[
\hat{C}_N(\bar{t})T_{j,i}(\bar{z}) = \lambda_{1}(\bar{z}) \sum_{\text{part}} \hat{C}_N(\bar{w}_n) \prod_{s=1}^{N} f(\bar{w}_i^s, \bar{w}_m^s) \prod_{s=1}^{N} \beta_s(\bar{w}_i^s) K(\bar{w}_i^s | \bar{w}_i^{s-1}) f(\bar{w}_i^s, \bar{w}_i^{s-1}) f(\bar{w}_i^s, \bar{w}_i^{s-1}), \tag{5.23}
\]

The partitions are the same as in (4.2) and (4.5).
Now to derive recursion (5.4), we take a new representative of the generalized model, for which
\[ \lambda_{s+1}(t_j^r) = \beta_s(t_j^r) = 0, \quad s = 1, \ldots, N, \quad j = 1, \ldots, r_s. \] (5.24)
Thus,
\[ \hat{S}_N(\bar{x}|t) = Z_N(\bar{t}|\bar{x}) \prod_{s=1}^N \beta_s(\bar{x}^s). \] (5.25)

In complete analogy with the derivation of recursion (5.3) we consider the following expectation value
\[ \hat{Q}_N(\bar{x}|\bar{t}) = \hat{C}_N(\bar{t}^1, \ldots, \bar{t}^N, \emptyset) \frac{T_{N+1,N}(\bar{t}^N)}{\lambda_N(\bar{t}^N)} f(\bar{t}^1, t^{N-1}), \] (5.26)
where \( \#\bar{t}^N = \#\bar{x}^N = r_N. \) After that, we repeat the calculations already done. Acting with the product \( T_{N+1,N}(\bar{t}^N) \) to the left we obtain \( \hat{Q}_N(\bar{x}|\bar{t}) = \hat{S}_N(\bar{x}|\bar{t}). \) The action of the same product to the right gives us the desired recursion with relabeling \( \bar{x} \leftrightarrow \bar{t}. \)

In paper [20], we found a symmetry of the highest coefficient with respect to a special reordering and shifts of the Bethe parameters. To describe this symmetry we introduce a mapping
\[ \mu(\bar{t}) \equiv \mu(\{\bar{t}^1, \bar{t}^2, \ldots, \bar{t}^N\}) = \{\bar{t}^N, \bar{t}^{N-1} - c, \ldots, \bar{t}^1 - (N - 1)c\}. \] (5.27)

Then it follows from the results of [20] that
\[ Z_N(\bar{x}|\bar{t}) \prod_{k=1}^{N-1} f(\bar{x}^{k+1}, \bar{x}^k) f(\bar{t}^{k+1}, \bar{t}^k) = Z_N(\mu(\bar{x})|\mu(\bar{t})). \] (5.28)

Equation (5.28) and recursions (5.3) and (5.4) imply two more recursions for the highest coefficient.

**Corollary 5.1.** The highest coefficient of the scalar product (5.1) satisfies two more recursions
\[
Z_N(\bar{x}|\bar{t}) = \frac{(-1)^{r_N} f(\bar{t}^1, \bar{x}^1)}{\prod_{k=1}^{N-1} f(\bar{x}^{k+1}, \bar{x}^k)} \sum_{\text{part}} Z_{N-1}(\bar{x}^2, \ldots, \bar{x}^N|\bar{\eta}_1^2, \ldots, \bar{\eta}_N^2) \times \prod_{s=1}^N K(\bar{\eta}_s^{s+1}|\bar{\eta}_s^s) f(\bar{\eta}_1^s, \bar{\eta}_N^s) f(\bar{\eta}_1^{s+1}, \bar{\eta}_N^s). \] (5.29)

Here the sum runs over partitions of the sets \( \{\bar{\eta}_1^s, \bar{\eta}_N^s\} \vdash \bar{\eta}_s = \{\bar{x}^s, \bar{t}^s - (s - 1)c\} \) for \( s = 2, \ldots, N \) such that \( \#\bar{\eta}_1^s = r_1 \) and \( \bar{\eta}_N^{N+1} = \bar{x}^1 - Nc, \bar{\eta}_1^1 = \bar{t}^1, \bar{\eta}_N^1 = \bar{\eta}_N^{N+1} = \emptyset. \)

\[
Z_N(\bar{x}|\bar{t}) = \frac{(-1)^{r_N} f(\bar{t}^N, \bar{x}^N)}{\prod_{k=1}^{N-1} f(\bar{x}^{k+1}, \bar{x}^k)} \sum_{\text{part}} Z_{N-1}(\bar{\eta}_1^1, \ldots, \bar{\eta}_N^{N-1}|\bar{t}^1, \ldots, \bar{t}^{N-1}) \times \prod_{s=1}^N K(\bar{\eta}_s^s|\bar{\eta}_s^{s+1}) f(\bar{\eta}_1^s, \bar{\eta}_N^s) f(\bar{\eta}_1^{s+1}, \bar{\eta}_N^s). \] (5.30)

Here the sum runs over partitions of the sets \( \{\bar{\eta}_1^s, \bar{\eta}_N^s\} \vdash \bar{\eta}_s = \{\bar{x}^s, \bar{t}^N - (N - s)c\} \) for \( s = 1, \ldots, N - 1 \) such that \( \#\bar{\eta}_1^s = r_N \) and \( \bar{\eta}_N^s = \bar{x}^N, \bar{\eta}_1^0 = \bar{\eta}_N^0 = \emptyset. \)
To prove these recursions it is enough to apply (5.4) and (5.3) to \( Z_N(\mu(\bar{x} | \mu(\bar{t})) \).

Note that the use of both recursions (5.4) and (5.29) played a very important role in deriving determinant representations for the scalar products in gl(3)-invariant models [21]. In the gl(2) case, the four recursions coincide and give \( Z_1(\bar{x} | \bar{t}) = K(\bar{t} | \bar{x}) \), as expected from [13].

6 Results for \( gl(m | n) \) related models

In this section, we use notation of the paper [1] to describe monodromy matrices as the matrices acting in the auxiliary space \( \mathbb{C}^{m|n} \). This is a \( \mathbb{Z}_2 \)-graded space with a basis \( e_i, i = 1, \ldots, m+n \). We assume that the basis vectors \( \{e_1, e_2, \ldots, e_m\} \) are even while \( \{e_{m+1}, e_{m+2}, \ldots, e_{m+n}\} \) are odd. In this section \( N = m+n-1 \). We introduce the \( \mathbb{Z}_2 \)-grading of the indices as

\[
[i] = 0 \quad \text{for} \quad i = 1, 2, \ldots, m, \quad \text{and} \quad [i] = 1 \quad \text{for} \quad i = m+1, m+2, \ldots, m+n. \tag{6.1}
\]

Let \( E_{ij} \in \text{End}(\mathbb{C}^{m|n}) \) be again a matrix with the only nonzero entry equal to 1 at the intersection of the \( i \)-th row and \( j \)-th column. Monodromy matrix elements are defined by the same formula (2.3) but now the matrices \( E_{ij} \) have grading \([E_{ij}] = [i] + [j] \mod 2\).

Their tensor products are also graded according to the rule

\[
(E_{ij} \otimes E_{kl}) \cdot (E_{pq} \otimes E_{rs}) = (-1)^{([k]+[l])([p]+[q])} E_{ij} E_{pq} \otimes E_{kl} E_{rs},
\]

as well as the transposition antimorphism:

\[
\Psi : T_{i,j}(u) \to (-1)^{[i][j]+1} T_{j,i}(u), \quad \Psi(A \cdot B) = (-1)^{[A][B]} \Psi(B) \cdot \Psi(A). \tag{6.2}
\]

The commutation relations of the monodromy matrices \( T(u) \) are the same as in (2.1) with the same structure of the \( R \)-matrix as in (2.2), but with graded permutation operator \( P \) acting in the tensor product \( \mathbb{C}^{m|n} \otimes \mathbb{C}^{m|n} \) as

\[
P = \sum_{i,j=1}^{m+n} (-1)^{[j]} E_{ij} \otimes E_{ji}.
\]

This results in slightly different commutation relations for the monodromy matrix elements [1].

To describe the action of the graded monodromy matrix elements onto supersymmetric Bethe vectors, it is convenient to introduce ‘colored’ analogs the rational functions \( g(u,v), f(u,v) \), and \( h(u,v) \) (2.9):

\[
f_{[i]}(u,v) = 1 + g_{[i]}(u,v) = 1 + \frac{c_{[i]}}{u-v} = \frac{u - v + c_{[i]}}{u - v}, \quad h_{[i]}(u,v) = \frac{f_{[i]}(u,v)}{g_{[i]}(u,v)}, \tag{6.3}
\]

\footnote{We use the same notation \([i]\) to describe the parity function and to define modes of the generating series in (2.5). However, in this section we will use only zero modes operators marked by the symbol \([0]\), so that the distinction with the parity function will be clear enough since 0 is not an index of (6.1).}
where
\[ c_{[i]} = (-)^{[i]} c. \]

Second, for arbitrary sets of parameters \( \bar{u} \) and \( \bar{v} \) we define
\[
\gamma_i(\bar{u}, \bar{v}) = \frac{f_{[i]}(\bar{u}, \bar{v})}{h(\bar{u}, \bar{v})^{\delta_{1,m}}} \quad \text{and} \quad \bar{\gamma}_i(\bar{u}, \bar{v}) = \frac{f_{[i+1]}(\bar{u}, \bar{v})}{h(\bar{v}, \bar{u})^{\delta_{1,m}}}. \tag{6.4}
\]
The first function in (6.4) coincides with the function \( f_{[i]}(\bar{u}, \bar{v}) \) for \( i \neq m \) and with \( g(\bar{u}, \bar{v}) \) for \( i = m \), while the second function coincides with \( f_{[i+1]}(\bar{u}, \bar{v}) \) for \( i \neq m \) and with \( g(\bar{v}, \bar{u}) \) for \( i = m \). Note that
\[
\gamma_m(\bar{u}, \bar{v}) = (-)^{#\bar{u}} \#\bar{v} \bar{\gamma}_m(\bar{u}, \bar{v}). \tag{6.5}
\]
and \( \gamma_i(\bar{u}, \bar{v}) = \gamma_i(\bar{u}, \bar{v}) \) for \( i \neq m \).

Then lemma [4.2] is replaced by

**Lemma 6.1.** The action of the monodromy matrix element \( T_{1, N+1}(z) \) and the zero modes \( T_{i+1, i[0]} \) onto off-shell Bethe vectors \( \mathbb{B}(\bar{t}) \) are given by the formulas
\[
T_{1, N+1}(z)\mathbb{B}(\bar{t}) = \lambda_{N+1}(z)h(\bar{t}^m, z)\mathbb{B}(\bar{w}), \tag{6.6}
\]
and
\[
T_{i+1, i[0]}\mathbb{B}(\bar{t}) = (-1)^{[i+1]} \sum_{\ell=1}^{\ell_i} \left( \kappa_{i+1} \frac{\alpha_i(t^i_{\ell}) \gamma_i(t^i_{\ell}, t^i_{\ell})}{f_{[i+1]}(\bar{t}^m, t^i_{\ell})} - \kappa_i \frac{\bar{\gamma}_i(t^i_{\ell}, t^i_{\ell})}{f_{[i]}(t^i_{\ell}, \bar{t}^i_{\ell})} \right) \mathbb{B}(\bar{t} \setminus \{t^i_{\ell}\}). \tag{6.7}
\]
The supersymmetric Bethe equations
\[
\alpha_i(t^i_{\ell}) = \frac{\bar{\gamma}_i(t^i_{\ell}, t^i_{\ell})}{\gamma_i(t^i_{\ell}, t^i_{\ell})} \frac{f_{[i+1]}(\bar{t}^i_{\ell}, t^i_{\ell})}{f_{[i]}(t^i_{\ell}, \bar{t}^i_{\ell})} \tag{6.8}
\]
provide (when all \( \kappa_i = 1 \)) the highest weight condition for the Bethe vectors \( \mathbb{B}(\bar{t}) \) with respect to the raising operators of the finite dimensional algebra \( g(\ell(m|n)) \) formed by the zero modes operators. Due to the properties (6.5) the supersymmetric Bethe equation for the Bethe parameters \( t^m_{\ell} \) (corresponding to the odd simple root) simplifies to
\[
\alpha_m(t^m_{\ell}) = \frac{f(t^m_{\ell}, \bar{t}^m_{\ell})}{f(t^m_{\ell}, \bar{t}^m_{\ell})}. \tag{6.9}
\]

To describe the multiple action in the supersymmetric case we introduce the following symmetric products:
\[
\mathbb{T}_{i, j}(\bar{z}) = \begin{cases} T_{i, j}(\bar{z}), & \text{if } [i] + [j] = 0 \text{ mod } 2, \\ \Delta_h(\bar{z})^{-1} T_{i, j}(z_1) \cdots T_{i, j}(z_p), & \text{if } [i] = 0 \text{ and } [j] = 1, \\ \Delta_t(\bar{z})^{-1} T_{i, j}(z_1) \cdots T_{i, j}(z_p), & \text{if } [i] = 1 \text{ and } [j] = 0. \end{cases} \tag{6.9}
\]
According to the commutation relations between monodromy matrix elements in the supersymmetric case, the product \( \mathbb{T}_{i, j}(\bar{z}) \) is symmetric with respect to any permutation in the set \( \bar{z} \).

Proposition [4.1] is replaced in the supersymmetric case by
Proposition 6.1. The action by $T_{i,j}(\bar{z})$ onto supersymmetric off-shell Bethe vector $\mathbb{B}(\bar{t})$ is

$$T_{i,j}(\bar{z})\mathbb{B}(\bar{t}) = \lambda_{N+1}(\bar{z})h(\bar{t}^n, \bar{z}) \prod_{s=j}^{i-1} \left(-1\right)^{([s]+[s+1])} \left[\frac{1}{2}\right] \sum_{\text{part}} \mathbb{B}(\bar{w}_n) \prod_{s=j}^{i-1} \frac{\gamma_s(\bar{w}_s^a, \bar{w}_s^b)}{f_{s+1}(\bar{w}_s^a, \bar{w}_s^b)}$$

$$\times \hat{K}_i(\bar{w}_1) \prod_{s=1}^{i-1} \frac{\gamma_s(\bar{w}_s^a, \bar{w}_m^b)}{f_{s+1}(\bar{w}_s^a, \bar{w}_m^b)} K_j(\bar{w}_m) \prod_{s=j}^{N} \alpha_s(\bar{w}_m^a) \gamma_s(\bar{w}_s^a, \bar{w}_m^b) f_{s+1}(\bar{w}_m^a, \bar{w}_m^b). \tag{6.10}$$

Here instead of products of the Izergin determinants, we introduce

$$\hat{K}_i(\bar{w}_1) = \left\{ \begin{array}{ll}
\prod_{s=1}^{i-1} K_{[s]}(\bar{w}_s^a | \bar{w}_s^a) & \text{if } i \leq m, \\
\prod_{s=1}^{m} h_{[s]}(\bar{w}_s^a, \bar{w}_s^a) \prod_{s=m+1}^{i-1} K_{[s]}(\bar{w}_s^a | \bar{w}_s^a) & \text{if } i > m,
\end{array} \right. \tag{6.11}$$

and

$$K_j(\bar{w}_m) = \left\{ \begin{array}{ll}
\prod_{s=j}^{m} h_{[s]}(\bar{w}_s^a, \bar{w}_s^a) \prod_{s=m+1}^{N} h_{[s]}(\bar{w}_m^a, \bar{w}_m^a) & \text{if } j \leq m, \\
\prod_{s=j}^{N} K_{[s]}(\bar{w}_s^a | \bar{w}_s^a) f_{s+1}(\bar{w}_m^a, \bar{w}_m^b) & \text{if } j > m.
\end{array} \right. \tag{6.12}$$

In (6.11) and (6.12) the symbol $K_{[s]}(\bar{x} | \bar{y})$ means the Izergin determinant given by the expression (2.11), where the functions $g(x, y)$ and $h(x, y)$ are replaced by their graded analogs $g_{[s]}(x, y)$ and $h_{[s]}(x, y)$ given by (6.3).

Summations in (6.10) are over the partitions of the sets $\{\bar{w}_1^a, \bar{w}_1^b, \bar{w}_m^b\} \vdash \bar{w}^a = \{\bar{z}, \bar{t}^a\}$ that obey the $(i, j)$-condition w.r.t. $\bar{z}$.

The proof of proposition 6.1 is carried out by the same method as the one given in the appendices $\mathbf{A}$ and $\mathbf{B}$. If one specifies $m = 0$ or $n = 0$, proposition 6.1 reduces to proposition 4.1.

These action formulas can be used to find a recurrence relations for the highest coefficient of the scalar product of the supersymmetric off-shell Bethe vectors. In the supersymmetric case the scalar product takes the following form

$$S(\bar{x} | \bar{t}) = \sum_{\text{part}} Z^{m|n}(\bar{t}_1 | \bar{t}_1) Z^{m|n}(\bar{t}_n | \bar{x}_n) \prod_{k=1}^{N} \frac{\alpha_k(\bar{t}_k^a) \alpha_k(\bar{t}_k^b) \gamma_k(\bar{t}_k^a, \bar{t}_k^b) \gamma_k(\bar{t}_k^a, \bar{t}_k^b)}{\prod_{j=1}^{N-1} f_{j+1}(\bar{x}_j^a, \bar{x}_j^b) f_{j+1}(\bar{t}_j^a, \bar{t}_j^b)}. \tag{6.13}$$

The highest coefficient $Z^{m|n}(\bar{x} | \bar{t})$ satisfies the following recursion.

Proposition 6.2. The highest coefficient of the scalar product (6.10) satisfies the recursions

$$Z^{m|n}(\bar{x} | \bar{t}) = \frac{\gamma_N(\bar{t}_N^a, \bar{t}_N^b)}{f_{N}(\bar{x}_N^a, \bar{x}_N^b) \delta_{m,N}} h(\bar{t}^m, \bar{x}^N)$$

$$\times \sum_{\text{part}} Z^{m|n-1}(\bar{x}_1, \ldots, \bar{x}_N-1 | \bar{w}_1^a, \ldots, \bar{w}_n^a) \hat{K}_{N+1}(\bar{w}_1^a) \prod_{s=1}^{N} \frac{\gamma_s(\bar{w}_s^a, \bar{w}_s^b)}{f_{s+1}(\bar{w}_s^a, \bar{w}_s^b)}. \tag{6.14}$$

\text{Let us note the difference of notation between } Z_N \text{ (bosonic case) and } Z^{m|n} \text{ (supersymmetric case): indeed, we have } Z^{m|0} = Z_{m-1}.
The highest coefficient of the scalar product (6.13) also satisfies the recursion
\[ \hat{\gamma}_1(\bar{t}, \bar{x}^1) h(\bar{x}^m, \bar{t}^1) \equiv g_{[2]}(\bar{t}^2, \bar{t}^1) h(\bar{t}^1, \bar{t}^1)^{\delta_{m,1}} \]
\[
\times \sum_{\text{part}} Z^{m-1}[n](\bar{\omega}_1^2, \ldots, \bar{\omega}_n^N | \bar{t}^2, \ldots, \bar{t}^N) \mathcal{K}_1(\bar{w}_m) \prod_{s=1}^N \frac{\gamma_s(\bar{\omega}_s^1, \bar{\omega}_s^N)}{f_{[s+1]}(\bar{u}_s^{w_{s+1}}, \bar{w}_m^{w_{m}})} \tag{6.15}
\]

The sums run over partitions as in proposition \[5.4\]. The formula (6.14) works only for \( n \geq 1 \), while recursion (6.15) is valid only when \( m \geq 1 \).

Equations (6.14) and (6.15) are related by the symmetry \[19, 12\].

\[
Z^{m}[n](\bar{x}|\bar{t}) = (-1)^{rn} Z^{n|m}(\bar{t}|\bar{x}), \tag{6.16}
\]

with \( \bar{x} = (\bar{x}^N, \ldots, \bar{x}^1) \). Relation (6.10) comes from the action of the morphism
\[
\varphi : \begin{cases} 
Y(m|n) & \mapsto Y(n|m), \\
T_{ij}(u) & \mapsto (-1)^{[i][j]} \bar{T}_{ji}(u), \quad \bar{k} = N + 2 - k, \\
\lambda_j(u) & \mapsto \lambda_j(u), \\
\alpha_j(u) & \mapsto \alpha_j(u)^{-1},
\end{cases} \tag{6.17}
\]
on the scalar product (6.13). Let us remark that in connecting \( Z^{m|n} \) to \( Z^{n|m} \), one gets functions such as \( f_{[s]}(x, y) \), which are different if they are associated to \( Y(gl(m|n)) \) or to \( Y(gl(n|m)) \), since the grading \( [\cdot] \) depends on which Yangian one considers. For instance, one has
\[
f^{m|n}_{[s+1]}(x, y) = f^{m|n}_{[N+1-s]}(y, x),
\]

with obvious notation. In the same way, we have
\[
\hat{\gamma}^{m|n}_s(\bar{u}, \bar{v}) = \gamma^{m|n}_{N+1-s}(\bar{v}, \bar{u}) \quad \text{and} \quad \bar{k}^{m|n}_{N+1}(\bar{w}) = k^{m|n}_1(\bar{\eta}), \text{ with } \bar{\eta} = \bar{w}^{N+1-s}. \tag{6.18}
\]

One has to pay attention to these differences when using the morphism \[6.17\] on \[6.13\]. Again, if one sets \( m = 0 \) or \( n = 0 \), the proposition \[6.2\] reduces to proposition \[5.1\].

**Corollary 6.1.** The highest coefficient of the scalar product (6.13) also satisfies the recursion
\[
Z^{m|n}(\bar{x}|\bar{t}) = (-1)^{N r_1} \hat{\gamma}_1(\bar{t}^1, \bar{x}^1) h(\bar{x}^1 - (m - 1)c, \bar{t}^1) \prod_{k=1}^{N-1} f_{[k+1]}(\bar{t}^{k+1}, \bar{t}^k)^{\delta_{m,1}} \times \sum_{\text{part}} Z^{m-1}[n](\bar{x}^2, \ldots, \bar{x}^N | \bar{\eta}_1^2, \ldots, \bar{\eta}_N^N) \mathcal{K}_1(\bar{\eta}) \prod_{s=1}^N \gamma_s(\bar{\eta}_s^1, \bar{\eta}_s^N) f_{[s+1]}(\bar{\eta}_s^{w_{s+1}}, \bar{\eta}_s^N) f_{[s+1]}(\bar{\eta}_s^{w_{s+1}}, \bar{\eta}_s^N), \tag{6.19}
\]

where \( \bar{\eta}_1 = \bar{t}^1, \bar{\eta}_s = \{\bar{t}^s, \bar{x}^1 - (s - 1)c\} \) for \( s = 2, \ldots, m \), \( \bar{\eta}_s = \{\bar{t}^s, \bar{x}^1 - (2m - s - 1)c\} \) for \( s = m + 1, \ldots, N \), and \( \bar{\eta}_N^{N+1} = \bar{x}^1 - (m - n - 1)c \). The sum runs over partitions \( \{\bar{\eta}_s^1, \bar{\eta}_s^N\} \vdash \bar{\eta}^s \) with \( \# \bar{\eta}_s^1 = r_1 \) and \( \bar{\eta}_s^N = \bar{\eta}_N^{N+1} = \emptyset \). The formula (6.19) works only for \( m \geq 1 \).
It obeys also the recursion

\[
Z^{m|n}(\bar{x}|\bar{t}) = (-1)^{N_r N} \frac{\hat{\gamma}_N(\bar{t}^N, \bar{x}^N)}{h(\bar{t}^N, \bar{x}^N)^{m_1}} \prod_{k=2}^N f_{|k|}(\bar{x}^k, \bar{x}^{k-1})^{-1}
\times \sum_{\text{part}} Z^{m|n-1}(\bar{\eta}_1, \ldots, \bar{\eta}_N^0|\bar{t}^1, \ldots, \bar{t}^{N-1}) K_{N+1}(\bar{\eta}_m) \prod_{s=1}^N \gamma_s(\bar{\eta}_s, \bar{\eta}_m) f_{|s|}(\bar{\eta}_s, \bar{\eta}_s^0)^{-1},
\]

(6.20)

where \( \bar{\eta}^0 = \bar{t}^N - (n - m - 1)c \), \( \bar{\eta}^s = \{\bar{x}^s, \bar{t}^N - (s + n - m - 1)c\} \) for \( s = 1, \ldots, m \), \( \bar{\eta}^s = \{\bar{x}^s, \bar{t}^N - (m + n - 1 - s)c\} \) for \( s = m + 1, \ldots, N - 1 \), and \( \bar{\eta}^N = \bar{x}^N \). The sums run over partitions \( \{\bar{\eta}_1, \bar{\eta}_m\} \) with \( #\bar{\eta}_m = r_N \) and \( \bar{\eta}_0^N = \emptyset \). This formula is valid only when \( n \geq 1 \).

Relations (6.19) and (6.20) are related to (6.14) and (6.15) by the symmetry

\[
Z^{m|n}(\mu(\bar{x})|\mu(\bar{t})) = (-1)^{r_m} Z^{m|n}(\bar{x}|\bar{t})|_{c \to -c} \prod_{k=1}^{m-1} f_{|k+1|}(\bar{x}^k, \bar{x}^{k+1}) f_{|k+1|}(\bar{t}^k, \bar{t}^{k+1}),
\]

with

\[
\mu(\bar{t}) = \{\bar{t}^{m+n-1} + (n-1)c, \bar{t}^{m+n-2} + (n-2)c, \ldots, \bar{t}^m, \bar{t}^{m-1} + c, \ldots, \bar{t}^1 + (m-1)c\}.
\]

(6.21)

One can also relate (6.19) to (6.20) using the symmetry (6.16). Once more, the corollary 6.1 reduces to corollary 5.1 when \( m = 0 \) or \( n = 0 \). In the case \( m = n = 1 \), the four recursions obtained from proposition 6.2 and corollary 6.1 lead to the same equality \( Z^{1|1}(\bar{x}|\bar{t}) = g(\bar{x}, \bar{t}) \), as expected for the highest coefficient of \( \mathfrak{gl}(1|1) \)-models, see [12].

Conclusion

In this paper we continued our study of Bethe vectors in \( \mathfrak{gl}(m|n) \)-invariant quantum integrable models. We developed a generalization of the zero mode method based on a twisting procedure. This twisted zero mode method allows us to deduce in a simple way multiple action of all elements of the monodromy matrix on Bethe vectors.

In all cases, the multiple action is presented as a sum over partitions of sets of Bethe parameters.

Thanks to the multiple action formula, one can get an equivalent of the sum formula for the scalar product of off-shell Bethe vectors for \( \mathfrak{gl}(m|n) \)-invariant quantum integrable models [12] [17]. We provide new recursions for the highest coefficients entering the sum formula, the iteration being based on the rank of the algebra under consideration. In this way, the knowledge of the basic \( \mathfrak{gl}(2) \), \( \mathfrak{gl}(1|1) \) and \( \mathfrak{gl}(2|1) \) models is enough to reconstruct the scalar product in the general case (at least theoretically). The multiple action formula also allows to obtain form factors of the monodromy matrix elements, and we expect to report on it in a future work.

Of course, the sum formula (5.1) is not adapted to handle the thermodynamic limit of the models, and a determinant form for the scalar product and the form factors still needs to
be found. Yet, it is the first step in this direction, since it is (up to now) the only general form known for the scalar product, up to few cases for $\mathfrak{gl}(2)$, $\mathfrak{gl}(3)$, $\mathfrak{gl}(1|1)$ and $\mathfrak{gl}(2|1)$ models. However, let us remark the Gaudin determinant formula for on-shell Bethe vectors, which has been established on general ground in [18] for $\mathcal{Y}(\mathfrak{gl}(m|n))$ Yangians and in [22] for $\mathcal{U}_q(\mathfrak{gl}(n))$ quantum groups.

Finally, we want to stress that although the RTT presentation is the framework used in this note, our general strategy [1] is to use the current presentation of the Yangian. Indeed, we believe that this current presentation is the right framework to achieve technical calculations in the higher rank models. In particular, it allows to get the multiple action of $T_{1,N+1}(z)$, which is the starting point of our twisted zero mode method.

Let us also note that the current presentation can be used for models based on $\mathcal{U}_q(\hat{\mathfrak{gl}}(m|n))$ deformed superalgebras. Results for these models will be presented elsewhere.

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A Proof of lemma 4.1

We use induction to prove lemma 4.1. First of all, we observe that for $i = 1$ and $j = N + 1$, (4.1) reduces to (4.2), so that the action formula for $T_{1,N+1}(z)$ is proven. Next, let us assume that equation (4.1) is valid for some values $(i, j)$. We consider two particular cases of (4.1):

$$[[T_{i,j}(z), T_{i+1,i}[0]] = \kappa_i T_{i+1,j}(z) - \delta_{i,j-1} \kappa_{i+1} T_{i,i}(z), \quad (A.1)$$
$$[[T_{i,j}(z), T_{j,j-1}[0]] = -\kappa_j T_{i,j-1}(z) + \delta_{i,j-1} \kappa_{j-1} T_{j,j}(z) \quad (A.2)$$

When considering these commutation relations, we can separately equate the coefficients for different twisting parameters, since $\kappa_i$ are arbitrary complex numbers. Then, from (A.1), we can derive the action formula for the elements $T_{i+1,j}(z)$ and $T_{i,i}(z)$. Similarly, using (A.2), we obtain the action of the elements $T_{i,j-1}(z)$ and $T_{j,j}(z)$. We examine each of these actions separately. The action formula for the elements $T_{i+1,j}(z)$ (resp. $T_{i,j-1}(z)$) will provide a recursion for $i$ (resp. $j$). The actions of $T_{i,i}(z)$ and $T_{j,j}(z)$ give consistency checks of the formulas.

A.1 Action of $T_{i+1,j}(z)$

It corresponds to the terms proportional to $\kappa_i$ in (A.1). We split this calculation in two cases, corresponding to the relative position of $i$ and $j$. 

18
**Case 1:** $j \geq i + 1$. Then, the last factor in first line of (4.1) disappears and this action formula becomes

$$T_{i,j}(z)B(i) = \lambda_{N+1}(z) \sum_{\text{part}} B(\bar{w}_n) \prod_{s=1}^{i-1} \frac{f(\bar{w}_s^i, \bar{w}_n^s)}{h(\bar{w}_s^i, \bar{w}_n^{s-1})} \prod_{s=j}^{N} \alpha_s(\bar{w}_m^n) f(\bar{w}_m^n, \bar{w}_m^n) \quad (A.3)$$

Here the sets $\bar{w}^s = \{\bar{u}^i, z\}$ for $s = i, \ldots, j - 1$ are not divided into subsets, and we have $\bar{w}_m^i = \bar{w}^s$ and $\bar{w}_m^n = \emptyset$. Using formulas (4.3) and (A.3) we find

$$T_{i,j}(z)T_{i+1,i}[0]B(i) \bigg|_{\kappa_{i+1}=0} = -\kappa_i \lambda_{N+1}(z) \sum_{\text{part}} f(t^i_{t^i}, t^i_{t^i}) \prod_{s=1}^{i-1} \frac{f(\bar{w}_s^i, \bar{w}_n^s)}{h(\bar{w}_s^i, \bar{w}_n^{s-1})} \prod_{s=j}^{N} \alpha_s(\bar{w}_m^n) f(\bar{w}_m^n, \bar{w}_m^n) \quad (A.4)$$

where $\bar{w}_n^i = \{\bar{u}^i, z\}$. On the other hand, the action of the same operators in the reverse order can be written as

$$T_{i+1,i}[0]T_{i,j}(z)B(i) \bigg|_{\kappa_{i+1}=0} = -\kappa_i \lambda_{N+1}(z) \sum_{\text{part}} B(\bar{w}_n) \prod_{s=1}^{i-1} \frac{f(\bar{w}_s^i, \bar{w}_n^s)}{h(\bar{w}_s^i, \bar{w}_n^{s-1})} \prod_{s=j}^{N} \alpha_s(\bar{w}_m^n) f(\bar{w}_m^n, \bar{w}_m^n) \quad (A.5)$$

Here the set $\bar{w}^i = \{\bar{u}^i, z\}$ is divided into subsets $\{\bar{w}_s^i, \bar{w}_n^i\} \vdash \bar{w}^i$ such that $\#\bar{w}_s^i = 1$. The sum over $\ell$ in the first line of (A.4) can also be presented as the sum over these partitions. To do this, we transform the ratio

$$\frac{f(t^i_{t^i}, \bar{w}_n^i)}{f(t^i_{t^i}, b^{i-1})} = \frac{f(t^i_{t^i}, \bar{w}_n^i)}{f(t^i_{t^i}, b^{i-1})} \quad (A.6)$$

where $\bar{w}_n^i = \{\bar{u}^i, z\}$, and add to the sum over $\ell$ one zero term proportional to

$$\frac{f(z, \bar{w}^i)}{f(z, b^{i-1})} = 0.$$

Then the sum over $\ell$ takes the form

$$\sum_{\ell=1}^{r_i} \frac{f(t^i_{t^i}, \bar{w}_n^i)}{f(t^i_{t^i}, b^{i-1})}(\ldots) = \sum_{\text{part}} \frac{f(\bar{w}_s^i, \bar{w}_n^s)}{f(\bar{w}_s^i, \bar{w}_n^{s-1})} \frac{1}{f(\bar{w}_s^i, \bar{w}_n^{s-1})} \quad (A.7)$$

where the latter sum runs over partitions $\{\bar{w}_s^i, \bar{w}_n^s\} \vdash \bar{w}^i$ such that $\#\bar{w}_s^i = 1$. Subtracting now (A.4) and (A.5) and using a trivial identity

$$\left(1 - \frac{1}{f(\bar{w}_s^i, \bar{w}_n^{s-1})}\right) = \frac{1}{h(\bar{w}_s^i, \bar{w}_n^{s-1})} \quad (A.8)$$
we obtain from (A.1) that

\[
T_{i+1,j}(z) = \sum_{\text{part}} \mathbb{B}(\bar{w}_n) \prod_{s=1}^{i} \frac{f(\bar{w}^s, \bar{w}^s_{n-1})}{h(\bar{w}^s, \bar{w}^s_{n-1})} \prod_{s=j}^{N} \frac{\alpha_s(\bar{w}^s_m) f(\bar{w}^s, \bar{w}^s_m)}{h(\bar{w}^s_{m+1}, \bar{w}^s_m) f(\bar{w}^s_{m+1}, \bar{w}^s_m)}. 
\]

(A.9)

**Case 2: \( j < i + 1 \).** Let us repeat the calculations above for the case \( i \geq j \). Instead of the formula (A.4) we may write

\[
T_{i,j}(z) T_{i+1,i}(0) \mathbb{B}(\bar{t}) \bigg|_{\kappa_{i+1} = 0} = -\kappa_i \lambda_{N+1}(z) \sum_{\text{part}} \mathbb{B}(\bar{w}_n) \prod_{s=1}^{i} \frac{f(t^s_{i+1}, t^s_{i})}{f(t^s_{i}, t^s_{i-1})} \prod_{s=j}^{N} \frac{\alpha_s(\bar{w}^s_m) f(\bar{w}^s, \bar{w}^s_m)}{h(\bar{w}^s_{m+1}, \bar{w}^s_m) f(\bar{w}^s_{m+1}, \bar{w}^s_m)},
\]

(A.10)

where now the set \( \bar{w}^i \) is obtained by the partition \( \{ \bar{w}^i_{1}, \bar{w}^i_{m-1}, \bar{w}^i_m \} \) such that \#\( \bar{w}^i \) = 1. Taking into account that \( \{ \bar{w}^i_{-1}, \bar{w}^i_{m-1}, \bar{w}^i_m \} \) and transforming the ratio

\[
\frac{f(t^s_{i+1}, t^s_{i})}{f(t^s_{i}, t^s_{i-1})} = \frac{f(t^s_{i+1}, \bar{w}^i_{m-1}) f(\bar{w}^i_{m-1}, \bar{w}^i_{m})}{f(t^s_{i}, \bar{w}^i_{m-1}) f(\bar{w}^i_{m-1}, \bar{w}^i_{m})},
\]

we can rewrite (A.10) as follows:

\[
T_{i,j}(z) T_{i+1,i}(0) \mathbb{B}(\bar{t}) \bigg|_{\kappa_{i+1} = 0} = -\kappa_i \lambda_{N+1}(z) \sum_{\text{part}} \mathbb{B}(\bar{w}_n) \prod_{s=1}^{i} \frac{f(\bar{w}^s_{i+1}, \bar{w}^s_m)}{f(\bar{w}^s_{i}, \bar{w}^s_{m-1})} \prod_{s=j}^{N} \frac{\alpha_s(\bar{w}^s_m) f(\bar{w}^s, \bar{w}^s_m)}{h(\bar{w}^s_{m+1}, \bar{w}^s_m) f(\bar{w}^s_{m+1}, \bar{w}^s_m)}. 
\]

(A.12)

When we act by the operators in reverse order, we first use the induction assumption (A.1). According to this assumption, sets \( \bar{w}^s \) are divided into \( \{ \bar{w}^s_{1}, \bar{w}^s_m \} \mapsto \bar{w}^s \), but for the specific set \( \bar{w}^i \) we use the temporary notation \( \bar{w}^i_{m} \) instead of \( \bar{w}^i_{m} \), so that it is divided into \( \{ \bar{w}^i_{1}, \bar{w}^i_m \} \mapsto \bar{w}^i \) (recall that \( \bar{w}^i = \emptyset \) due to the \( (i,j) \)-condition, see definition 4.1). Let us rewrite the induction assumption (A.1) in the form

\[
T_{i,j}(z) \mathbb{B}(\bar{t}) = \lambda_{N+1}(z) \sum_{\text{part}} \mathbb{B}(\bar{w}^i_1, \ldots, \bar{w}^i_{m-1}, \bar{w}^i_m, \bar{w}^i_{m+1}, \ldots, \bar{w}^i_N) \prod_{s=1}^{i-1} \frac{f(\bar{w}^i_{s+1}, \bar{w}^i_m)}{f(\bar{w}^i_{s}, \bar{w}^i_{m-1})} \prod_{s=j}^{N} \frac{\alpha_s(\bar{w}^i_m) f(\bar{w}^i, \bar{w}^i_m)}{h(\bar{w}^i_{m+1}, \bar{w}^i_m) f(\bar{w}^i_{m+1}, \bar{w}^i_m)} \times \frac{f(\bar{w}^i_{1}, \bar{w}^i_m)}{f(\bar{w}^i_{m-1})} \frac{\alpha_{i-1}(\bar{w}^i_{m-1}) f(\bar{w}^i_{m-1}, \bar{w}^i_{m-1})}{h(\bar{w}^i_{m+1}, \bar{w}^i_m) f(\bar{w}^i_{m+1}, \bar{w}^i_m)}. 
\]

(A.13)
where we have singled out the terms where the subset $\bar{w}_{1i}^j$ occurs (last line of the equation). Note that the resulting Bethe vector also depends on the auxiliary subset $\bar{w}_{1i}^j$, as it is shown explicitly in (A.13).

Now we apply the zero mode operator $T_{i+1,j}[0]$ on both sides of (A.13) and take into account only the part proportional to $\kappa_i$. This action divides the auxiliary subset $\bar{w}_{1i}^j$ into $\{\bar{w}_1^i, \bar{w}_{1j}^i\} \to \bar{w}_{1i}^j$ and produces an additional factor $\frac{f(\bar{w}_1^i, \bar{w}_{1j}^i)}{f(\bar{w}_1^j, \bar{w}_{1m}^j)}$. This factor together with the first one $\frac{f(\bar{w}_1^i, \bar{w}_{1j}^i)}{f(\bar{w}_1^j, \bar{w}_{1m}^j)}$ in the last line of (A.13) may be factorized as follows

$\quad - \frac{f(\bar{w}_1^i, \bar{w}_{1j}^i)}{f(\bar{w}_1^i, \bar{w}_{1m}^j)} \frac{f(\bar{w}_1^j, \bar{w}_{1j}^i)}{f(\bar{w}_1^j, \bar{w}_{1m}^j)} = - \frac{f(\bar{w}_1^i, \bar{w}_z^i)}{f(\bar{w}_1^i, \bar{w}_{1m}^i)} \frac{f(\bar{w}_1^j, \bar{w}_z^i)}{f(\bar{w}_1^j, \bar{w}_{1m}^i)} \cdot (A.14)$

The first factor $\frac{f(\bar{w}_1^i, \bar{w}_{1j}^i)}{f(\bar{w}_1^i, \bar{w}_{1m}^j)}$ in the right hand side of (A.14) will shift index $i \to i+1$ in the first product of the second line in (A.13) producing the factor $h(\bar{w}_1^i, \bar{w}_{1j}^i \to \bar{w}_{1i}^j - \bar{w}_z^i)$. The second factor $\frac{f(\bar{w}_1^i, \bar{w}_{1j}^i)}{f(\bar{w}_1^i, \bar{w}_{1m}^i)}$ from r.h.s. of (A.14) change index $i \to i+1$ in the product of the first line of (A.13).

Finally, the last factor $\frac{f(\bar{w}_1^i, \bar{w}_{1j}^i)}{f(\bar{w}_1^i, \bar{w}_{1m}^i)}$ allows to restore the values $s = i-1$ and $s = i$ in the product from $j$ to $N$ of the second line of (A.13).

Thus for the consecutive action of the operators $T_{i+1,j}[0]$ and $T_{i,j}(z)$ onto the off-shell Bethe vector $\mathbb{B}(z)$ in the case $i \geq j$ we get

$\quad T_{i+1,j}[0]T_{i,j}(z)\mathbb{B}(z)\big|_{\kappa_{i+1}=0} = -\kappa_i \lambda_{N+1}(z) \sum_{\text{part}} \mathbb{B}(\bar{w}_1) \prod_{s=j}^{i} f(\bar{w}_1^s, \bar{w}_z^s) \prod_{s=j}^{i} f(\bar{w}_1^s, \bar{w}_{1m}^s) \cdot \prod_{s=1}^{j} f(\bar{w}_1^s, \bar{w}_{1m}^s) \cdot (A.15)$

Subtracting (A.15) from (A.12) and using (A.1) we finally prove from the identity $h(x, y) - g(x, y)^{-1} = 1$ that the action of the matrix element $T_{i+1,j}(z)$ is given by the formula (4.4) at the shifted index $i \to i+1$.

**Recursion on $i$.** The two above cases show by induction that the action formula (4.4) is valid for $T_{i+1,j}(z)$, provided the terms proportional to $\kappa_{i+1}$ add up correctly. This is showed in section A.2. Then, the induction shows that the action formula is valid for $T_{k,j}(z), \forall k \geq i$, provided (4.4) is valid for $T_{i,j}(z)$.

In particular, since we know it is valid for $T_{1,N+1}(z)$, we know that the action formula is valid for $T_{k,N+1}(z), \forall k$.

**A.2 Action of $T_{i,j}(z)$**

Now we have to check that the terms coming from the action of the $[T_{i,j}(z), T_{i+1,j}[0]]$ onto $\mathbb{B}(z)$ and proportional to the twisting parameter $\kappa_{i+1}$ cancel each other for $i \neq j-1$ and produce the action of the operator $T_{i,j}(z)$ for $i = j-1$. We split this study in three cases.
Case 1: \( j > i + 1 \). Indeed, for \( i < j - 1 \) the terms proportional to \( \kappa_{i+1} \) after the action of the operators \( T_{i,j}(z)T_{i+1,i}[0] \) onto off-shell Bethe vector \( \mathcal{B}(\tilde{t}) \) are

\[
T_{i,j}(z)T_{i+1,i}[0]B(\tilde{t}) \bigg|_{\kappa_i=0} = \kappa_{i+1}\lambda N+1(z) \sum_{t=1}^{r_i} \frac{\alpha_i(t_i')f(\tilde{t}_i', t_i)}{f(h+1, t_i')} \sum_{\text{part}} B(\bar{w}_{\mu})
\]

\[
\times \prod_{s=1}^{i-1} \frac{f(\bar{w}_{\mu}^s, \bar{w}_{\mu}^s)}{h(\bar{w}_{\mu}^s, \bar{w}_{\mu}^{s-1})f(\bar{w}_{\mu}^s, \bar{w}_{\mu}^{s-1})} \prod_{s=j}^{N} \frac{\alpha_s(\bar{w}_{\mu}^s, \bar{w}_{\mu}^s)}{h(\bar{w}_{\mu}^{s+1}, \bar{w}_{\mu}^s)f(\bar{w}_{\mu}^{s+1}, \bar{w}_{\mu}^s)}. \tag{A.16}
\]

We again present the sum over \( \ell \) as sum over partitions of the set \( \{\bar{w}_n^i, \bar{w}_m^i\} \rightarrow \bar{w}^i = \{\tilde{t}, z\} \) with \#\( \bar{w}_m^i = 1 \):

\[
T_{i,j}(z)T_{i+1,i}[0]B(\tilde{t}) \bigg|_{\kappa_i=0} = \kappa_{i+1}\lambda N+1(z) \sum_{\text{part}} B(\bar{w}_n) \frac{\alpha_i(\bar{w}_n^i)}{f(h+1, \bar{w}_n^i)}
\]

\[
\times \prod_{s=1}^{i-1} \frac{f(\bar{w}_{\mu}^s, \bar{w}_{\mu}^s)}{h(\bar{w}_{\mu}^s, \bar{w}_{\mu}^{s-1})f(\bar{w}_{\mu}^s, \bar{w}_{\mu}^{s-1})} \prod_{s=j}^{N} \frac{\alpha_s(\bar{w}_{\mu}^s, \bar{w}_{\mu}^s)}{h(\bar{w}_{\mu}^{s+1}, \bar{w}_{\mu}^s)f(\bar{w}_{\mu}^{s+1}, \bar{w}_{\mu}^s)}. \tag{A.17}
\]

The additional term in \( \text{[A.17]} \) (w.r.t. \( \text{[A.16]} \)), corresponds to \( \bar{w}_m^i = \{\} \) and is in fact zero, due to the factor \( f(\bar{w}^{i+1}, z)^{-1} \) in the first line of \( \text{[A.17]} \).

To present the action in the reverse order, we define \( \tilde{g}^i = \bar{w}_n^i \):

\[
T_{i+1,i}[0]T_{i,j}(z)B(\tilde{t}) \bigg|_{\kappa_i=0} = \kappa_{i+1}\lambda N+1(z) \sum_{\text{part}} B(\bar{w}_n) \frac{\alpha_i(\bar{w}_n^i)}{f(h+1, \bar{w}_n^i)}
\]

\[
\times \prod_{s=1}^{i-1} \frac{f(\bar{w}_{\mu}^s, \bar{w}_{\mu}^s)}{h(\bar{w}_{\mu}^s, \bar{w}_{\mu}^{s-1})f(\bar{w}_{\mu}^s, \bar{w}_{\mu}^{s-1})} \prod_{s=j}^{N} \frac{\alpha_s(\bar{w}_{\mu}^s, \bar{w}_{\mu}^s)}{h(\bar{w}_{\mu}^{s+1}, \bar{w}_{\mu}^s)f(\bar{w}_{\mu}^{s+1}, \bar{w}_{\mu}^s)}. \tag{A.18}
\]

Once more, we can transform the sum on \( \ell \) as a sum over partitions. Indeed, the same factor \( f(\bar{w}^{i+1}, z)^{-1} \) appears, since \( \bar{w}_n^{i+1} = \bar{w}_m^{i+1} \). Thus, \( \text{[A.18]} \) becomes identical to \( \text{[A.17]} \), and we get 0 as a final result, as expected from \( \text{[A.1]} \).

Case 2: \( j = i + 1 \). In this case, the set \( \bar{w}^{i+1} \) in the action of the element \( T_{i,i+1}(z) \) is divided into subsets \( \bar{w}_n^{i+1} \) and \( \bar{w}_m^{i+1} \). Then the actions of \( T_{i,j}(z)T_{i+1,i}[ar{0}]B(\tilde{t}) \) and \( T_{i+1,i}[0]T_{j,i}(z)B(\tilde{t}) \) at \( \kappa_i = 0 \) give different results. The first action is given by \( \text{[A.17]} \) while for the reverse action the last factor of the first line is replaced by

\[
\frac{\alpha_i(\bar{w}_n^i, \bar{w}_m^i)}{f(h+1, \bar{w}_n^i, \bar{w}_m^i)} \rightarrow \frac{\alpha_i(\bar{w}_m^i, \bar{w}_m^i)}{f(h+1, \bar{w}_m^i, \bar{w}_m^i)}.
\]

Due to \( \text{[A.8]} \), the difference of these actions produces the action of the monodromy matrix element \( T_{i,i}(z) \) onto \( \mathcal{B}(\tilde{t}) \), which corresponds to the second term in the right hand side of \( \text{[A.1]} \).

Case 3: \( i + 1 > j \). Again, to conclude the induction proof for the case \( i \geq j \) we have to verify that the terms at the twisting parameter \( \kappa_{i+1} \) cancel each other in the action formulas \( T_{i,j}(z)T_{i+1,i}[0]B(\tilde{t}) \) and \( T_{i+1,i}[0]T_{j,i}(z)B(\tilde{t}) \). We leave this exercise to the interested reader.
A.3 Action of $T_{i,j-1}(z)$ and end of the recursion

In exactly the same way, we can prove the validity of the action formula (4.4) for $T_{i,j-1}(z)$ if it is valid for $T_{i,j}(z)$. This is done starting with the action (4.2) and using the commutation relation (A.2), together with the splitting over the twisting parameters $\kappa_j$ and $\kappa_{j-1}$. Induction then proves that it is valid for $T_{i,k}(z)$, $\forall k \leq j$.

Finally, since the induction on $i$ showed that the action formula (4.1) is valid for $T_{k,N+1}(z)$, $\forall k$, the induction on $j$ proves it is valid for $T_{k,f}(z)$, $\forall k, \ell$.

B Proof of proposition 4.1

In the previous appendix, we have proved lemma 4.1. Since it corresponds to equation (4.3) for $p = 1$, it is the base of the induction on $p$ that we are using to prove the general case.

For simplicity, we first consider the case $i < j$. We assume that (4.3) is valid for the cardinality $\# \bar{z} = p - 1$ of the set $\bar{z}$. Then, for $\# \bar{z} = p$, we can apply (4.4) for the successive action of $T_{i,j}(z_1)$ and $T_{i,j}(\bar{z}_i)$ to get

$$T_{i,j}(z_1)T_{i,j}(\bar{z}_i)B(\bar{v}) = \lambda_{N+1}(\bar{z}_1) \sum_{\text{part}} T_{i,j}(z_1) B(\bar{w}_n)$$

$$= \lambda_{N+1}(\bar{z}) \sum_{\text{part}} B(\bar{w}_n)$$

(B.1)

Here, for $0 \leq s < i$, the sums in (B.1) run over partitions $\{\bar{w}_1^s, \bar{w}_2^s\} \triangleright \bar{w}_s$ with cardinality $\# \bar{w}_1^s = p - 1$, and then over partition $\{\bar{w}_1^s, \bar{w}_2^s\} \triangleright \bar{w}_s$ with cardinality $\# \bar{w}_1^s = 1$. Similarly, for $j \leq s \leq N + 1$, the sums in (B.1) run over partitions $\{\bar{w}_1^s, \bar{w}_2^s\} \triangleright \bar{w}_s$ with cardinality $\# \bar{w}_1^s = p - 1$ and then over partition $\{\bar{w}_1^s, \bar{w}_2^s\} \triangleright \bar{w}_s$ with cardinality $\# \bar{w}_1^s = 1$. Note that for $i \leq s < j$, the subsets $\bar{w}_1^s$ and $\bar{w}_2^s$ are equal to $\bar{w}_s$. Thus, all other sets $\bar{w}_1^s, \bar{w}_2^s, \bar{w}_3^s, \bar{w}_4^s$ are empty in that case.

Using properties of the Izergin determinant we can combine the sets $\bar{w}_1^s \cup \bar{w}_2^s = \bar{w}_s^s$ for $s < i$ (resp. $\bar{w}_1^s \cup \bar{w}_2^s = \bar{w}_s^s$ for $s \geq j$) of cardinalities $\# \bar{w}_1^s = p$ (resp. $\# \bar{w}_1^s = p$) and rewrite (B.1) as sums over partitions $\{\bar{w}_1^s, \bar{w}_2^s\} \triangleright \bar{w}_s$ for $0 \leq s < i$ and sums over partitions $\{\bar{w}_1^s, \bar{w}_2^s\} \triangleright \bar{w}_s$ for $j \leq s \leq N + 1$. Indeed, we may factorize the ratio in the fourth line of (B.1)

$$f(\bar{w}_1^s, \bar{w}_2^s) f(\bar{w}_1^s, \bar{w}_2^s) \quad f(\bar{w}_1^s, \bar{w}_2^s) \quad f(\bar{w}_1^s, \bar{w}_2^s)$$

*We remind that by convention $\bar{w}_N^N = \bar{z}$. 23
and writing explicitly all the factors depending on the sets $\bar{w}_i^s$ and $\bar{w}_{iv}^s$ for any fixed $s$ from the interval $[1, \ldots, i - 1]$ we obtain the sum

$$
\sum_{\{\bar{w}_i^s, \bar{w}_{iv}^s\} \vdash \bar{w}_i^s} \frac{f(\bar{w}_i^s, \bar{w}_{iv}^s)}{f(\bar{w}_i^s, \bar{w}_{iv}^{s-1})} K(\bar{w}_{iv}^s | \bar{w}_{iv}^{s-1}) K(\bar{w}_i^s | \bar{w}_i^{s-1}) = 
$$

$$
= (-1)^{p-1} \sum_{\{\bar{w}_i^s, \bar{w}_{iv}^s\} \vdash \bar{w}_i^s} f(\bar{w}_i^s, \bar{w}_{iv}^s) K(\bar{w}_{iv}^s | \bar{w}_{iv}^{s-1}) K(\bar{w}_i^s - c | \bar{w}_i^s) = 
$$

$$
= (-1)^p K(\{\bar{w}_{iv}^{s-1} - c, \bar{w}_i^{s-1} - c\} | \bar{w}_i^s) = (-1)^p K(\bar{w}_i^{s-1} - c | \bar{w}_i^s) = \frac{K(\bar{w}_i^s | \bar{w}_i^{s-1})}{f(\bar{w}_i^s | \bar{w}_i^{s-1})}. 
$$

(B.2)

Here we used the following property of the Izyergin determinant

$$
K(\bar{x} - c | \bar{y}) = (-1)^p K(\bar{y} | \bar{x}), \quad \text{for} \ #\bar{x} = p, 
$$

(B.3)

and a summation identity

$$
\sum_{\{\bar{w}_1, \bar{w}_n\} \vdash \bar{w}} K(\bar{w} | \bar{u}) K(\bar{v} | \bar{w}) f(\bar{w}, \bar{w}) = (-1)^{m_1} f(\bar{w}, \bar{u}) K(\{\bar{u} - c, \bar{v}\} | \bar{w}). 
$$

(B.4)

Here $\bar{u}, \bar{v},$ and $\bar{w}$ are sets of arbitrary complex numbers such that $\#\bar{u} = m_1$, $\#\bar{v} = m_2$, and $\#\bar{w} = m_1 + m_2$. The sum in (B.4) is taken with respect to all partitions of the set $\bar{w}$ into subsets $\bar{w}_1$ and $\bar{w}_n$ with $\#\bar{w}_1 = m_1$ and $\#\bar{w}_n = m_2$.

To apply (B.4) to (B.2) we identify: $\bar{w}_1 = \bar{w}_1^s$, $\bar{w}_n = \bar{w}_n^s$, $\bar{u} = \bar{w}_{iv}^{s-1}$, $\bar{v} = \bar{w}_i^{s-1} - c$, $m_1 = 1$ and $m_2 = p - 1$. Similarly using (B.3) and (B.4) we find that for $j \leq s \leq N$

$$
\sum_{\{\bar{w}_1, \bar{w}_n\} \vdash \bar{w}} \frac{f(\bar{w}_i^s, \bar{w}_{iv}^s)}{f(\bar{w}_i^{s+1}, \bar{w}_{iv}^s)} K(\bar{w}_{iv}^{s+1} | \bar{w}_{iv}^s) K(\bar{w}_i^{s+1} | \bar{w}_i^s) = \frac{K(\bar{w}_i^{s+1} | \bar{w}_i^s)}{f(\bar{w}_i^{s+1} | \bar{w}_i^s)}. 
$$

(B.5)

This proves (1.1), by induction over the cardinality $p$ of the set $\bar{z}$ in the case $i < j$.

For $i \geq j$ the proof of (1.4) is similar.

\[ \square \]

C  Eigenvector property of $\mathbb{B}(\ell)$

Equation (1.4) yields the following formula for the action of the transfer matrix (2.6) onto off-shell Bethe vector $\mathbb{B}(\ell)$:

$$
t(z) \mathbb{B}(\ell) = \lambda_{N+1}(z) \sum_{i=1}^{N+1} \sum_{\text{part}} \mathbb{B}(\bar{w}_n) \prod_{s=1}^{N} \frac{f(\bar{w}_n^s, \bar{w}_n^{s-1})}{h(\bar{w}_n^s, \bar{w}_n^{s-1})} \prod_{s=1}^{N} \alpha_s(\bar{w}_m^s) f(\bar{w}_m^s, \bar{w}_m^{s-1}) h(\bar{w}_m^{s+1}, \bar{w}_m^s) h(\bar{w}_m^s, \bar{w}_m^{s-1}) f(\bar{w}_m^{s+1}, \bar{w}_m^s). 
$$

(C.1)

Here sum runs over partitions $\{\bar{w}_n^s, \bar{w}_n^{s-1}, \bar{w}_m^s\} \vdash \bar{w}^s = \{\ell^s, z\}$ described in proposition 1.1.
Let us select wanted terms from the action formula (C.1) which correspond to the partitions \( \bar{w}_1^s = z \) and \( \bar{w}_n^s = \bar{t}^s \) for \( s = 1, \ldots, i - 1 \) and \( \bar{w}_m^s = z \) and \( \bar{w}_m^s = \bar{t}^s \) for \( s = i, \ldots, N + 1 \). Using the facts that \( h(z, \bar{z}) = 1 \) and

\[
\prod_{s=1}^{i-1} \frac{f(z, \bar{t}^s)}{f(z, t^{s-1})} = f(z, \bar{t}^{i-1}), \quad \prod_{s=i}^{N} \frac{\alpha_s(z) f(\bar{t}^s, z)}{f(t^{s+1}, z)} = \frac{\lambda_i(z) f(\bar{t}^i, z)}{\lambda_{N+1}(z)},
\]

we prove that wanted terms yield the right hand side of equation (3.1) with eigenvalue given by (3.3).

To prove that all other unwanted terms cancel each other provided the Bethe equations (3.2) are fulfilled, we consider the terms from the action of the diagonal monodromy matrix element

\[ T_{i+1,i+1}(z) \]

de the partitions

\[
\bar{w}_1^s = z, \quad \bar{w}_n^s = \bar{t}^s, \quad \bar{w}_m^s = \emptyset \quad \text{for} \quad s < i, \\
\bar{w}_1^s = \bar{t}^s, \quad \bar{w}_n^s = \{ \bar{t}^s \}, \quad \bar{w}_m^s = \emptyset \quad \text{for} \quad s = i, \\
\bar{w}_1^s = \emptyset, \quad \bar{w}_n^s = \bar{t}^s, \quad \bar{w}_m^s = z \quad \text{for} \quad s > i.
\]

We also consider the terms from the action of \( T_{i+1,i}(z) \) corresponding to the partitions

\[
\bar{w}_1^s = z, \quad \bar{w}_n^s = \bar{t}^s, \quad \bar{w}_m^s = \emptyset \quad \text{for} \quad s < i, \\
\bar{w}_1^s = \emptyset, \quad \bar{w}_n^s = \{ \bar{t}^s \}, \quad \bar{w}_m^s = \bar{t}^s \quad \text{for} \quad s = i, \\
\bar{w}_1^s = \emptyset, \quad \bar{w}_n^s = \bar{t}^s, \quad \bar{w}_m^s = z \quad \text{for} \quad s > i.
\]

The terms from the right hand side of (C.1) corresponding to both of these partitions can be written as

\[
\sum_{i=1}^{N+1} \sum_{\ell=1}^{r_I} \Box (\bar{t}^1, \ldots, \bar{t}^{i-1}, \{ \bar{t}^s \}, \bar{t}^{i+1}, \ldots, \bar{t}^N) \lambda_{i+1}(z) g(z, t^i) f(z, \bar{t}^{i-1}) f(\bar{t}^{i+1}, z)
\]

\[
\times \left[ \frac{\alpha_i(t^i_1) f(t^i_1, t^i)}{f(t^{i+1}, t^i_1)} - \frac{f(t^i_1, t^i)}{f(t^i_1, \bar{t}^{i-1})} \right].
\]

We see that these contributions disappear when the Bethe equations (3.2) are satisfied.

In exactly the same way one can verify that all other unwanted terms disappear provided the Bethe equations are satisfied.

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