Quantum uncertainty and holomorphic maps

The uncertainty principle and the energy identity for holomorphic maps in geometric quantum mechanics

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The theory of geometric quantum mechanics describes a quantum system as a Hamiltonian dynamical system with a complex projective Hilbert space as its phase space, thus equipped with a Riemannian metric in addition to a symplectic structure. This paper extends the geometric quantum theory to include aspects of the symplectic topology of the state space by identifying the Robertson-Schrödinger uncertainty relation for pure quantum states as the differential version of the energy identity in the theory of $J$-holomorphic curves. We consider a family of maps from a Riemann surface into a finite-dimensional quantum state space by using the vector fields generated by two quantum observables, and show that the Fubini-Study metric tensor pulls back by such a map to the covariance tensor for the two observables. By calculating the map energy density in the pull-back metric, the uncertainty relation is represented as an equality that compares the map energy differential to the sum of the pull-back of the symplectic form and a positive definite term that vanishes when the map is holomorphic. Saturation of the Robertson-Schrödinger inequality occurs when the map is conformal and the off-diagonal covariance terms vanish. For compact Riemann surfaces where such a map can be globally defined, if the map is holomorphic, it is harmonic and its image is a minimal surface. In this case, the uncertainty product integral is a topological invariant that depends only on the homology class of the curve modulo its boundary.

I. INTRODUCTION

By describing a quantum system as a Hamiltonian dynamical system, geometric quantum mechanics emphasizes the symplectic geometry of the quantum state space in a way similar to the geometric formulation of classical mechanics. A distinctive feature of the quantum phase space is that its symplectic structure plays not only a dynamical role, but also determines the curvature of a connection on the canonical $U(1)$ bundle over the space, and plays a topological role as the Chern form of the associated complex line bundle. The present work continues the development of this description of quantum mechanics as a Hamiltonian system while considering the global structure of the quantum state manifold, with the goal of forging a link to work in symplectic topology that has explained deep relationships between dynamical, geometric, and topological symplectic invariants.

The foundations of geometric quantum mechanics were laid in the work of Chernoff and Marsden, and Kibble. The theory has been developed by numerous authors into a full description of quantum mechanics as a Hamiltonian dynamical system on a symplectic manifold. Technical issues raised for infinite dimensional phase spaces have been dealt with carefully. In the geometric theory, the phase space of pure states of a quantum system is $P(\mathcal{H})$, the projective space of a complex separable Hilbert space, $\mathcal{H}$. Schrödinger dynamics on $P(\mathcal{H})$ is the quantum version of Hamilton’s equations, determined by the natural symplectic structure on $P(\mathcal{H})$ induced by the imaginary part of the Hermitian inner product on $\mathcal{H}$. The real part of the inner product on $\mathcal{H}$ induces the Fubini-Study metric on $P(\mathcal{H})$. This additional Riemannian structure can be viewed as a source of features in quantum systems that are distinctly different from classical systems. In particular, the uncertainty in the expectation value of a linear operator on $\mathcal{H}$ is a measure of distance in

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this metric on $P(\mathcal{H})$. A key result of geometric quantum mechanics expresses the quantum uncertainty principle in terms of the symplectic form and Riemannian metric on $P(\mathcal{H})$. A geometric interpretation of the uncertainty principle has been given in terms of quantum Fisher information.

Further understanding of the symplectic geometry of quantum mechanics came with a realization of the significance of Lie group actions and bundle structures on the quantum phase space. The Hermitian inner product on $\mathcal{H}$ determines a natural connection on the principal bundle $U(1) \to S(\mathcal{H}) \to P(\mathcal{H})$, where $S(\mathcal{H}) = \{ \psi \in \mathcal{H} : |\psi|^2 = 1 \}$. Since $P(\mathcal{H})$ is a homogeneous, isotropic manifold, the curvature of this connection and the Chern form on the associated line bundle are each scalar multiples of the pull-back of the symplectic form $\Omega$ on $P(\mathcal{H})$. Thus, the holonomy, or geometric phase, of a closed path in $P(\mathcal{H})$ measures the symplectic area of a surface spanned by the path and has been recognized as the quantum version of the Poincaré integral invariant which characterizes Hamiltonian systems. In applications restricting to a closed submanifold $\mathcal{N}$ of $P(\mathcal{H})$, $\Omega$ represents the first Chern class in $H^2(\mathcal{N}, \mathbb{Z})$; integrating $\Omega$ over $\mathcal{N}$ yields the Chern number of the corresponding complex line bundle over $\mathcal{N}$.

During the same period of time that symplectic structures have come to be recognized in quantum physics, symplectic topology has emerged as a new field of mathematics devoted to the study of the global structure of symplectic manifolds. A foundational result is Gromov’s non-squeezing theorem, which states that if there is an embedding of the closed symplectic Euclidean ball $B^{2n}(r)$ of radius $r$ into the symplectic cylinder $B^2(R) \times \mathbb{R}^{2n-2}$ that preserves the symplectic form, then $r \leq R$. This result shows that there is a two-dimensional quantity associated with symplectic manifolds, which is invariant under symplectomorphisms. The symplectic capacities are defined to describe this symplectic area invariance. New methods have been developed to study the group of symplectomorphisms of a symplectic manifold, and its relation to the group of volume-preserving diffeomorphisms. De Gosson and Luef have related symplectic capacitance and Gromov’s non-squeezing theorem to the quantum covariance matrix and the uncertainty principle.

An especially powerful and useful tool for investigating the global structure of symplectic manifolds is the theory of $J$-holomorphic curves (or pseudoholomorphic curves) introduced by Gromov. Such curves generalize holomorphic maps between a Riemann surface and a complex manifold by relaxing the condition on the target manifold that its almost complex structure be integrable. The theory can also be useful for studying Kähler manifolds, particularly for determining whether properties of holomorphic curves on these manifolds persist when complex structure is perturbed. Our interest in applying insights from symplectic topology to the study of quantum systems led us to the central result of the present work, Theorem 7 of Section 3, which states that the Robertson-Schrödinger uncertainty relation in quantum mechanics is an example of the differential version of the energy identity in the theory of $J$-holomorphic curves. The early work on this result was originally published as part of the author’s dissertation.

The energy identity is related to a variational principle used to study smooth maps $\phi : \mathcal{N} \to \mathcal{M}$ between closed Riemannian manifolds. Harmonic maps are the critical points of the energy functional,

$$E(\phi) := \frac{1}{2} \int_{\mathcal{N}} |d\phi|^2 \, dV,$$

where $|d\phi|$ denotes the Hilbert-Schmidt norm of the differential $d\phi$, regarded as a one-form on $\mathcal{N}$ with values in $\phi^*T(\mathcal{M})$, and $dV$ denotes the volume element of the metric on $\mathcal{N}$. A useful relationship exists between the energy and volume functionals of $\phi$. Minimal immersions are critical points of the volume functional,

$$V(\phi) := \int_{\mathcal{N}} |\Lambda^n d\phi| \, dV,$$

where $\Lambda^n d\phi$ denotes the Jacobian of $\phi$, and $dV$ is now the volume element associated with the induced metric $\phi^*(g)$, where $g$ is the metric on $\mathcal{M}$. Two results from the variational
theory are relevant to our study. 1) If $\phi : N \to M$ is a Riemannian (isometric) immersion, then $E(\phi)$ and $V(\phi)$ have the same critical points, that is, $\phi$ is harmonic if and only if $V(\phi)$ is stationary. 2) If $\dim(N) = 2$, then for all smooth maps $\phi : N \to M$, we have $V(\phi) \leq E(\phi)$, with equality holding if and only if $\phi$ is almost conformal; thus, if $\phi$ is harmonic and conformal, then $\phi$ minimizes the area functional and $V(\phi) = E(\phi)$.

Harmonic maps have been useful as models for physical theories characterized by broken gauge symmetry, especially Yang-Mills fields and non-linear $\sigma$ models, as well as in studies of minimal submanifolds. Holomorphic maps between Kähler manifolds are a special case of harmonic maps between Riemannian manifolds, and this case applies to our study of the quantum uncertainty principle.

The theory of $J$-holomorphic curves deals with smooth maps between almost complex manifolds,

$$u : (\Sigma, j) \to (M, J),$$

where $\Sigma$ is a Riemann surface with complex structure $j$, and the target manifold $M$ is equipped with a symplectic $\omega$ and almost complex structure $J$. Such a map $u$ is called $J$-holomorphic if it satisfies the generalized Cauchy-Riemann equation,

$$J \circ du = du \circ j. \tag{3}$$

If $J$ is assumed to be $\omega$-compatible, then there exists a unique Riemannian metric $g_J$ on $M$ defined as

$$\left( g_J \right)_{\xi, \zeta} = \omega_J(\xi, J\zeta)$$

for $\xi, \zeta \in T_x(M)$. In this case, if $\Sigma$ is compact, the energy functional of the map $u$ is

$$E(u) = \frac{1}{2} \int_{\Sigma} |du|_{g_J, J}^2 dA_\Sigma, \tag{4}$$

where $dA_\Sigma$ is the area form on $\Sigma$. The energy identity states that

$$E(u) = \int_{\Sigma} |\bar{\partial}_J u|^2_{g_J} dA_\Sigma + \int_{\Sigma} u^* \omega, \tag{5}$$

where

$$\bar{\partial}_J u := \frac{1}{2} (du + J \circ du \circ j) \tag{6}$$

is the antiholomorphic part of $du$. The identity shows that

$$\frac{1}{2} \int_{\Sigma} |du|^2_{g_J, J} dA_\Sigma \geq \int_{\Sigma} u^* \omega, \tag{7}$$

with equality holding if and only if $\bar{\partial}_J u = 0$, or equivalently, iff $\bar{\partial}_J u = 0$. Because $J$-holomorphic maps minimize energy, they are harmonic maps. For general smooth maps $u : \Sigma \to M$, the energy $E(u)$ depends on the metric $g_J$ on $M$. However, the energy identity shows that, for $J$-holomorphic curves in symplectic manifolds, $E(u)$ equals the minimal area $V(u)$ and that this quantity is a topological invariant that depends only on the homology class of the curve modulo its boundary.

Our inspiration to use $J$-holomorphic maps for investigating the quantum uncertainty principle comes from Oh’s lecture notes wherein he suggests that Gromov’s non-squeezing theorem is a classical analogue of the quantum uncertainty principle. With an eye to the similarities of classical and quantum mechanics, we focus on the the Robertson-Schrödinger uncertainty relation, which states that

$$(\Delta \hat{A})_\psi^2 (\Delta \hat{B})_\psi^2 - (C(\hat{A}, \hat{B})_\psi)^2 \geq \left( \frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle_\psi \right)^2, \tag{8}$$
where $\psi \in \mathcal{H}$ and $\Delta \hat{A}$, $\Delta \hat{B}$, and $C(\hat{A}, \hat{B})$ are the uncertainty and covariance functions for self-adjoint linear operators $\hat{A}$ and $\hat{B}$ on $\mathcal{H}$. By using $\hat{A}$ and $\hat{B}$ to define a family of maps $\mathcal{F}$ from a Riemann surface $\Sigma$ into a finite-dimensional phase space $P(\mathcal{H})$, we find that the uncertainty relation (9) is equivalent to the differential version of the energy identity,

$$\frac{1}{2} |du|^2_g \Omega \geq u^* \Omega,$$  \hspace{1cm} (9)

where $g$ and $\Omega$ are, respectively, the Riemannian metric and the symplectic form on $P(\mathcal{H})$, and $u$ is a map in $\mathcal{F}$. When $u$ is holomorphic, the inequality is saturated and the off-diagonal covariance term $C(\hat{A}, \hat{B})$ vanishes. For a compact Riemann surface where such a map $u : \Sigma \to P(\mathcal{H})$ can be globally defined, the integral form of the uncertainty inequality can be interpreted as a comparison of the area of $\Sigma$ as measured in the pull-back metric $u^*(\sigma)$, to the symplectic area of $\Sigma$, as measured by pulling back the symplectic form on $P(\mathcal{H})$. In this case, if the map is holomorphic, the uncertainty product integral over $\Sigma$ is a topological invariant within the homology class of the curve.

The remainder of the paper is organized as follows. Section 2 establishes our notation and viewpoint by reviewing the theory of geometric quantum mechanics, and casts the Robertson-Schrödinger inequality in terms of the symplectic form and Riemannian metric on $P(\mathcal{H})$. Section 3 proves our claim that (9) is an example of (9) by defining the map family $\mathcal{F}$, showing that the Fubini-Study metric tensor pulls back by a map in $\mathcal{F}$ to the quantum covariance tensor, and computing the energy differential and symplectic form in the pull-back metric.

II. QUANTUM MECHANICS AS A HAMILTONIAN DYNAMICAL SYSTEM

A Hamiltonian dynamical system $(\mathcal{M}, \omega, X_F)$ consists of a phase space, $\mathcal{M}$, which is a smooth manifold equipped with a symplectic form, $\omega$, and a preferred hamiltonian vector field, $X_F : \mathcal{M} \to T(\mathcal{M})$. A smooth two-form on $\mathcal{M}$ is symplectic if it is closed and nondegenerate. A vector field $X$ on $\mathcal{M}$ is hamiltonian if $\iota_X \omega$ is exact, where $\iota_X \omega(\cdot) := \omega(X, \cdot)$, or equivalently, if there is a $C^1$ function $F : \mathcal{M} \to \mathbb{R}$ such that

$$\iota_X \omega = dF.$$  \hspace{1cm} (10)

When (10) holds, we write $X = X_F$. Each point of the symplectic manifold $\mathcal{M}$ corresponds to a state of the physical system, and the time evolution of the system is given by the flow along the preferred vector field $X_F$. With initial condition $z_m(0) = m \in \mathcal{M}$, the trajectory $z_m(t)$ is uniquely determined by Hamilton’s equations,

$$\frac{dz}{dt} = (X_F)_{z(t)}.$$  \hspace{1cm} (11)

The observables, or measurable quantities, in Hamiltonian mechanics are real-valued differentiable functions on $\mathcal{M}$. For two observables $G, F : \mathcal{M} \to \mathbb{R}$, the Poisson bracket $\{ , \}$ is defined as

$$\{ G, F \} := \omega(X_G, X_F) = dG(X_F).$$  \hspace{1cm} (12)

The time evolution of an observable $G$ is given by $dG/dt = \{ G, E \}$, where $E$ is the energy function corresponding to the preferred vector field $X_E$ with $\iota_{X_E} \omega = dE$.

Recall that the traditional algebraic formulation of quantum mechanics describes a quantum system in terms of a complex separable Hilbert space $\mathcal{H}$ with Hermitian inner product $\langle \cdot, \cdot \rangle$, and that the observables of the system are represented by self-adjoint linear operators on $\mathcal{H}$. A special role is played by the Hamiltonian operator, $\hat{H}$, whose eigenvalues are the energies of the system: $\hat{H} \psi_\lambda = E_\lambda \psi_\lambda$. In the Heisenberg picture, linear operators on $\mathcal{H}$ time evolve according to $d\hat{A}/dt = (i/\hbar)[\hat{H}, \hat{A}]$, where the commutator of $\hat{A}$ and $\hat{B}$
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is \([\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}\). In the Schrödinger picture, the operators are stationary, but the elements of \(\mathcal{H}\) time-evolve according to Schrödinger’s equation,

\[
\frac{d\psi(t)}{dt} = \frac{i}{\hbar}\hat{H}\psi(t). \tag{13}
\]

Geometric quantum mechanics reformulates the algebraic quantum theory in geometric terms by regarding the linear space \(\mathcal{H}\) as a Hermitian manifold, with its tangent bundle \(T(\mathcal{H})\) identified with \(\mathcal{H} \times \mathcal{H}\). For each \(\psi \in \mathcal{H}\), the canonical isomorphism of \(\mathcal{H}\) onto \(T_\psi(\mathcal{H})\), \(v \mapsto v_\psi\), is given by \(v_\psi := \alpha'(0)\), where \(\alpha : \mathbb{R} \to \mathcal{H}\) is defined by \(\alpha(t) = \psi + tv\). The pair \((\psi, v) \in \mathcal{H} \times \mathcal{H}\) corresponds to \(v_\psi \in T_\psi(\mathcal{H})\). The inner product of two vectors \(v_\psi\) and \(w_\psi\) in \(T_\psi(\mathcal{H})\) is defined by \(\langle v_\psi, w_\psi \rangle_{T(\mathcal{H})} := \langle v, w \rangle\), where \(\langle \cdot, \cdot \rangle\) is the Hermitian inner product on \(\mathcal{H}\). The natural symplectic form \(\Omega_\mathcal{H}\) on \(\mathcal{H}\) is given by the imaginary part of this inner product. For \(v_\psi, w_\psi \in T_\psi(\mathcal{H})\),

\[
(\Omega_\mathcal{H})(v_\psi, w_\psi) := 2\hbar \text{Im}(\langle v, w \rangle). \tag{14}
\]

For a linear operator \(\hat{A}\) on \(\mathcal{H}\), the expectation value function \(A : \mathcal{H} \to \mathbb{R}\)

\[
A(\psi) := \langle \hat{A} \rangle_\psi := \frac{\langle \psi, \hat{A}\psi \rangle}{\langle \psi, \psi \rangle} \tag{15}
\]

is used to define the Hamiltonian vector field \(X_A\) on \(\mathcal{H}\) given by \(\iota_{X_A}\Omega_\mathcal{H} = dA\), where

\[
(X_A)_\psi := -\frac{i}{\hbar}\hat{A}\psi. \tag{16}
\]

Thus, the symplectic form \(\Omega_\mathcal{H}\) acting on two Hamiltonian vector fields \(X_A, X_B \in T(\mathcal{H})\) is proportional to the expectation value of the commutator of the self-adjoint operators \(\hat{A}\) and \(\hat{B}\) on \(\mathcal{H}\),

\[
(\Omega_\mathcal{H})_\psi(X_A, X_B) = -\frac{i}{\hbar}\langle [\hat{A}, \hat{B}] \rangle_\psi. \tag{17}
\]

Quantum dynamics can be described in terms of the flow along the preferred Hamiltonian vector field \(X_H\) on \(\mathcal{H}\), where \(\hat{H}\) is the Hamiltonian operator for the system. Schrödinger’s equation on \(\mathcal{H}\) takes the form of Hamilton’s equations,

\[
\frac{d\psi}{dt} = (X_H)_{\psi(t)}. \tag{18}
\]

Nevertheless, it is the complex projective space \(P(\mathcal{H})\) rather than \(\mathcal{H}\) that is generally regarded as the true phase space of geometric quantum mechanics, for the following reason. As is clear from the definition \([\text{15}]\), the expectation value of \(\hat{A}\) on \(\psi\) is equal to the expectation value of \(\hat{A}\) on \(c\psi\), for any nonzero \(c \in \mathbb{C}\). Thus, a pure quantum state must be regarded as a complex line through the origin in \(\mathcal{H}\), or an equivalence class \([\text{span}\{\psi\}] \in P(\mathcal{H}) := \mathcal{H}/\sim\), where \(\sim\) is the equivalence class \([\text{span}\{\psi\}]\). State vectors \(\psi \in \mathcal{H} - \{0\}\) and physical states \([\psi] \in P(\mathcal{H})\) are related by means of the principal bundle \(\mathbb{C}^* \to (\mathcal{H} - \{0\}) \xrightarrow{\pi} P(\mathcal{H})\), where \(\mathbb{C}^*\) is the multiplicative group of all nonzero complex numbers and the projection map \(\pi\) is defined by \(\pi(\psi) = [\psi]\). The Hermitian inner product \(\langle \cdot, \cdot \rangle\) on \(\mathcal{H}\) provides a natural principal connection on the bundle. For each \(\psi \in \mathcal{H} - \{0\}\), the horizontal subspace of \(T_\psi(\mathcal{H} - \{0\})\) consists of all \(v_\psi = \langle \psi, v \rangle\) such that \(\langle \psi, v \rangle = 0\). The vertical tangent space \(\tau_\psi(\mathcal{H} - \{0\})\) is an isomorphism from the horizontal subspace of \(T_\psi(\mathcal{H} - \{0\})\) onto \(T_\psi(P(\mathcal{H}))\).

Pushing forward the inner product \(\langle \cdot, \cdot \rangle\) on \(\mathcal{H}\) by \(\pi_*\) induces an Hermitian inner product \(\langle \langle \cdot, \cdot \rangle \rangle\) on \(P(\mathcal{H})\), as follows. Any \(v_\psi \in T_\psi(\mathcal{H} - \{0\})\) can be identified with \(v = c\psi + \delta \in \mathcal{H}\) where \(c = \langle \psi, v \rangle / \langle \psi, \psi \rangle\) and \(\delta = v - c\psi\). Thus, \(\pi_*\langle v_\psi \rangle = \pi_*(\delta_\psi) = \langle \delta, \psi \rangle = 0\).
\[ \langle \cdot, \cdot \rangle \) on \( P(\mathcal{H}) \), let \( p = [\psi] \in P(\mathcal{H}) \) and \( V, W \in T_p(\mathcal{H}) \). Choose any \( \psi \in \pi^{-1}(p) \) and \( v_\psi, w_\psi \in T_\psi(\mathcal{H}) \) satisfying \( \pi_* v_\psi = V, \pi_* w_\psi = W \). The scaling transformation \( \psi \rightarrow \psi/|\psi| \) is required to ensure that the inner product of \( V \) and \( W \) does not depend on which element of \( \pi^{-1}(p) \) is chosen. The Hermitian inner product of \( V \) and \( W \) on \( T(P(\mathcal{H})) \) is computed by taking the inner product of the horizontal components of \( v \) and \( w \),

\[
\langle (V, W) \rangle := \left\langle v - \frac{\langle v, \psi \rangle}{\langle \psi, \psi \rangle} \psi, w - \frac{\langle w, \psi \rangle}{\langle \psi, \psi \rangle} \psi \right\rangle \langle \psi, \psi \rangle^{-1}.
\]

(19)

Geometric quantum mechanics uses the Hermitian inner product (19) to describe quantum dynamics in terms of the symplectic geometry of \( P(\mathcal{H}) \), regarded as the phase space for a Hamiltonian system. The symplectic form on \( P(\mathcal{H}) \) is given by

\[
\Omega(V, W) := 2\hbar \text{Im}\langle (V, W) \rangle.
\]

(20)

Since the expectation value \( A(\psi) \) in (15) does not depend on the choice of element of \( \text{span}\{\psi\} \), the quantum observable \( a : P(\mathcal{H}) \rightarrow \mathbb{R} \) is well-defined as

\[
a([\psi]) := A(\psi) = \frac{\langle \psi, \hat{A}\psi \rangle}{\langle \psi, \psi \rangle},
\]

(21)

with corresponding hamiltonian vector field \( X_a \) on \( P(\mathcal{H}) \) defined by \( \iota_{X_a} \Omega = da \). The tangent map \( \pi_* : T(\mathcal{H} - \{0\}) \rightarrow T(P(\mathcal{H})) \) gives

\[
X_a = \pi_*(X_A)
\]

(22)

with \( X_A \) as defined in (16). The dynamics on the phase space \( P(\mathcal{H}) \) follows the flow of the preferred hamiltonian vector field \( X_\hbar \), which is the push-forward \( X_\hbar = \pi_*(X_H) \), where \( \hbar \) is the Hamiltonian operator on \( \mathcal{H} \). Thus, similarly to classical mechanics, observables in geometric quantum mechanics are differentiable real-valued functions on a nonlinear symplectic manifold, while the dynamics of the system’s states is determined by Hamilton’s equation. Quantum mechanics is a special Hamiltonian dynamical system, since the real part of (19) serves as a Riemannian metric on \( P(\mathcal{H}) \),

\[
g(V, W) := 2\hbar \text{Re}\langle (V, W) \rangle.
\]

(23)

If \( \mathcal{H} \) is regarded as \( \mathbb{C}^{n+1} \), then \( g \) is a constant multiple of the Fubini-Study metric on \( P(\mathbb{C}^{n+1}) = \mathbb{CP}^n \).

Both the symplectic and Riemannian parts of the Hermitian structure (19) on \( P(\mathcal{H}) \) are key to a geometric understanding of the uncertainty relation (8). Let \( \hat{A} \) and \( \hat{B} \) be self-adjoint linear operators on \( \mathcal{H} \) and let \( \psi \in S(\mathcal{H}) \), where \( S(\mathcal{H}) = \{ \psi \in \mathcal{H} : |\psi|^2 = 1 \} \). The uncertainty or dispersion in the values of \( A \) is given by the function \( \Delta \hat{A} : S(\mathcal{H}) \rightarrow \mathbb{R} \), defined as

\[
(\Delta \hat{A})_\psi := [\langle \psi, (\hat{A} - \langle \hat{A} \rangle_\psi)^2 \psi \rangle]^{1/2},
\]

(24)

where \( \langle \hat{A} \rangle_\psi \) is the expectation value function (15). The covariance or correlation function \( C(\hat{A}, \hat{B}) : S(\mathcal{H}) \rightarrow \mathbb{R} \) is defined as

\[
C(\hat{A}, \hat{B})_\psi := \frac{1}{2} \left\langle \psi, \left[ (\hat{A} - \langle \hat{A} \rangle_\psi)(\hat{B} - \langle \hat{B} \rangle_\psi) + (\hat{B} - \langle \hat{B} \rangle_\psi)(\hat{A} - \langle \hat{A} \rangle_\psi) \right] \psi \right\rangle.
\]

(25)

A geometric interpretation of (8) comes from recognizing that \( \langle \hat{A} - \langle \hat{A} \rangle_\psi \rangle_\psi \) is the horizontal component of \( \hat{A}_\psi \) by the Hermitian connection on the canonical \( U(1) \) bundle over \( P(\mathcal{H}) \).

**Lemma 1.** Let \( \hat{A} \) be a linear operator on \( \mathcal{H} \) and let \( \psi \in S(\mathcal{H}) \).

1) The component of the vector \( \hat{A}_\psi \in \mathcal{H} \) that is Hermitian orthogonal to \( \psi \) is \( \langle \hat{A} - \langle \hat{A} \rangle_\psi \rangle_\psi \).

2) If \( \hat{A} \) is self-adjoint, then \( \hat{A}_\psi \) decomposes as \( \hat{A}_\psi = \langle \hat{A} \rangle_\psi \psi + (\Delta \hat{A})_\psi \chi \), where \( \langle \chi, \psi \rangle = 0 \) and \( \langle \chi, \chi \rangle = 1 \).
Corollary 2. A hamiltonian vector field $X_A$ on $S(H)$ decomposes into vertical and horizontal components, $X_A = X_A^{\text{vert}} + X_A^{\text{horiz}}$, where

$$
(X_A^{\text{vert}})_\psi = -\frac{i}{\hbar} (\hat{A})_\psi \psi,
$$
$$
(X_A^{\text{horiz}})_\psi = -\frac{i}{\hbar} (\hat{A} - \langle \hat{A} \rangle \psi \hat{I}) \psi,
$$
and $\hat{I}$ is the identity operator on $\mathcal{H}$. The inner product (17) of hamiltonian vector fields $X_a$ and $X_b$ on $P(H)$ is given by

$$
\langle \langle X_a, X_b \rangle \rangle = \langle X_A^{\text{horiz}}, X_B^{\text{horiz}} \rangle.
$$

The uncertainty relation (8) is proved in standard texts by using the Cauchy-Schwartz inequality. The relation can be cast in a geometric form by writing it in terms of the real and imaginary parts of the Hermitian structure on $P(H)$.3,7,11,35

Corollary 3. Let $\hat{A}$ and $\hat{B}$ be self-adjoint linear operators on $\mathcal{H}$ and let $\psi \in S(H)$. The Robertson-Schrödinger uncertainty relation (8) has the following equivalent geometric expression in terms of the symplectic form $\Omega$ and the Riemannian metric $g$ on $P(H)$,

$$
g_{[\psi]}(X_a, X_a)g_{[\psi]}(X_b, X_b) - (g_{[\psi]}(X_a, X_b))^2 \geq (\Omega_{[\psi]}(X_a, X_b))^2,
$$

Proof: By the self-adjoint property of $\hat{A}$,

$$
(\Delta \hat{A})^2_{\psi} = \langle \langle \hat{A} - \langle \hat{A} \rangle_\psi \rangle, (\hat{A} - \langle \hat{A} \rangle_\psi) \rangle.
$$

Using (26) and (27),

$$
(\Delta \hat{A})^2_{\psi} = \hbar^2 \langle X_A^{\text{horiz}}, X_A^{\text{horiz}} \rangle_{\psi} = \hbar^2 \langle \langle X_a, X_a \rangle \rangle_{[\psi]},
$$
which is clearly real, so that by the definition (23) of $g$

$$
g_{[\psi]}(X_a, X_a) = \frac{2}{\hbar} (\Delta \hat{A})^2_{\psi}.
$$

Similarly,

$$
g_{[\psi]}(X_b, X_b) = \frac{2}{\hbar} (\Delta \hat{B})^2_{\psi},
$$
$$
g_{[\psi]}(X_a, X_b) = 2\hbar \text{Re} \langle \langle X_a, X_b \rangle \rangle = \frac{2}{\hbar} C(\hat{A}, \hat{B})_{\psi}.
$$

The commutator term on the right hand side of (8) satisfies

$$
\langle \psi, [\hat{A}, \hat{B}] \rangle_{\psi} = \langle \psi, [(\hat{A} - \langle \hat{A} \rangle), (\hat{B} - \langle \hat{B} \rangle)] \psi \rangle,
$$
so that, by using the relation (17) between the commutator and the symplectic form on $\mathcal{H}$, as well as (26), (27), and the definition (20) of $\Omega$,

$$
\Omega_{[\psi]}(X_a, X_b) = -\frac{i}{\hbar} \langle [\hat{A}, \hat{B}] \rangle_{\psi}.
$$

□
III. THE UNCERTAINTY PRINCIPLE AND THE ENERGY IDENTITY

This section states and proves the claim that the Robertson-Schrödinger uncertainty relation is an example of the differential version of the energy identity for $J$-holomorphic maps from a Riemann surface into the quantum state space $P(\mathcal{H})$. Beginning with the result from the previous section that the uncertainty relation can be expressed in terms of the symplectic form $\Omega$ and Riemannian metric $g$ on $P(\mathcal{H})$, observe that the inequality can be interpreted as a minimal area condition on parallelograms formed by non-commuting vector fields at a point in $P(\mathcal{H})$.\(^{34}\)

**Proposition 4.** Let $\hat{A}$ and $\hat{B}$ be self-adjoint linear operators on $\mathcal{H}$ and let $\psi \in S(\mathcal{H})$. Define the covariance tensor $M(\hat{A}, \hat{B})$ corresponding to the measurement of $\hat{A}$ and $\hat{B}$ as

$$M(\hat{A}, \hat{B})_{\psi} := \frac{2}{\hbar} \begin{pmatrix} (\Delta \hat{A})_{\psi} & C(\hat{A}, \hat{B})_{\psi} \\ C(\hat{B}, \hat{A})_{\psi} & (\Delta \hat{B})_{\psi} \end{pmatrix}. \quad (34)$$

The determinant of the matrix $M(\hat{A}, \hat{B})_{\psi}$ is bounded below by the square of the symplectic area of the parallelogram formed by the projected vectors $(X_a)_{[\psi]}$ and $(X_b)_{[\psi]}$ in $T_{[\psi]}(P(\mathcal{H}))$:

$$\det M(\hat{A}, \hat{B})_{\psi} \geq (\Omega_{[\psi]}(X_a, X_b))^2. \quad (35)$$

**Proof:** By the proof of Corollary 3, the elements of $M(\hat{A}, \hat{B})_{\psi}$ can be expressed in terms of the metric $g$ on $P(\mathcal{H})$, so that

$$M(\hat{A}, \hat{B})_{\psi} = \begin{pmatrix} g_{[\psi]}(X_a, X_a) & g_{[\psi]}(X_a, X_b) \\ g_{[\psi]}(X_b, X_a) & g_{[\psi]}(X_b, X_b) \end{pmatrix}. \quad (36)$$

The result then follows by observing that $\Omega_{[\psi]}(X, Y)$ measures the differential area element formed by vectors $X, Y \in T_{[\psi]}(P(\mathcal{H}))$. \(\square\)

The proof shows that $M(\hat{A}, \hat{B})$ has the form of a Riemannian metric tensor on a two-dimensional manifold, with components determined by the metric $g$ on $P(\mathcal{H})$. Accordingly, $M(\hat{A}, \hat{B})$ acts as the metric on a Riemann surface $\Sigma$, given as the pull-back of the Robertson-Schrödinger uncertainty and holomorphic maps. We assume without loss of generality that $\Sigma$ is an open subset of $\mathbb{C}$ parameterized by $x = s + it$.

**Definition 5.** Let $\hat{A}$ and $\hat{B}$ be self-adjoint linear operators on $\mathcal{H}$ and let $(s, t)$ be holomorphic coordinates on $\Sigma$. Define the family of maps,

$$\mathcal{F} := \{ u : \Sigma \to P(\mathcal{H}) \mid \exists f : \Sigma \to S(\mathcal{H}) \text{ such that } u = \pi \circ f, \text{ and } (df)_x(\partial_s) = (X_A^{\text{horiz}})_{f(x)}, \ (df)_x(\partial_t) = (X_B^{\text{horiz}})_{f(x)} \}, \quad (38)$$

where $(\partial_s, \partial_t)$ abbreviates $\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right)$, and $\pi$ is the projection from $\mathcal{H}$ onto $P(\mathcal{H})$.\(^{34}\)
Lemma 6. Let $\hat{A}$ and $\hat{B}$ be self-adjoint linear operators on $\mathcal{H}$. If $u : \Sigma \to P(\mathcal{H})$ is a map in the family $\mathcal{F}$, then the Fubini-Study metric tensor $g$ on $P(\mathcal{H})$ pulls back by $u$ to the covariance tensor defined in (34). That is, the induced metric tensor $h = u^*(g)$ on $\Sigma$ is given by $(h_{k\ell}) = M(A,B)$.

Proof: Let $u : \Sigma \to P(\mathcal{H})$ be a map in the family $\mathcal{F}$. Then $u = \pi \circ f$ for some $f : \Sigma \to S(\mathcal{H})$ with the property (38). In a linear orthonormal basis $\{\phi_\alpha\}_{\alpha=1}^{n+1}$, let $\hat{g}$ be the Riemannian metric defined by $\hat{g}(\phi_\alpha, \phi_\beta) := 2\hbar \delta_{\alpha\beta}$ on $\mathbb{C}^{n+1}$. The first step is to show that the pull-back of the Fubini-Study metric $g$ on $P(\mathcal{H})$ by the map $u$ is equal to the pull-back of the metric $\hat{g}$ on $\mathcal{H}$ by the map $f$. Let $\xi, \zeta \in T_x(\Sigma)$. By the definition (23) of the metric $g$ on $P(\mathcal{H})$, 

$$u^*(g)(\xi, \zeta) = g(u_*\xi, u_*\zeta) = 2\hbar \text{Re} \langle u_*\xi, u_*\zeta \rangle.$$ 

The differential $df$ maps vectors in $T_x(\Sigma)$ into the horizontal subspace of $T_{f(x)}(S(\mathcal{H}))$. Hence, by the definition (14) of the inner product $\langle \cdot, \cdot \rangle$ on $P(\mathcal{H})$, 

$$2\hbar \text{Re} \langle u_*\xi, u_*\zeta \rangle = 2\hbar \text{Re} \langle f_*\xi, f_*\zeta \rangle = f^*(\hat{g})(\xi, \zeta)$$ 

Therefore, 

$$u^*(g) = f^*(\hat{g}).$$ 

(39)

The second step is to show that $(f^*(\hat{g}))_x = M(A,B)_{f(x)}$. In coordinates $(x^1, x^2) = (s, t)$ on $\Sigma$, the pull-back metric $h = u^*(g)$ on $\Sigma$ is 

$$h = \sum_{k, \ell} h_{k\ell} \, dx^k dx^\ell.$$ 

By (38), 

$$h_{k\ell} = (f^*)^{-1}(\hat{g}) \left( \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^\ell} \right) = \hat{g} \left( \frac{\partial f}{\partial x^k}, \frac{\partial f}{\partial x^\ell} \right)$$ 

$$= \hat{g} \left( \sum_\alpha \frac{\partial f_\alpha}{\partial x^k} \phi_\alpha, \sum_\beta \frac{\partial f_\beta}{\partial x^\ell} \phi_\beta \right) = 2\hbar \text{Re} \sum_\alpha \left( \frac{\partial f_\alpha}{\partial x^k} \right) \frac{\partial f_\alpha}{\partial x^\ell}.$$ 

By (38), $\frac{\partial f}{\partial s} = X_A^{\text{horiz}}$ and $\frac{\partial f}{\partial t} = X_B^{\text{horiz}}$. Thus, 

$$h_{11} = 2\hbar \left| \frac{\partial f}{\partial s} \right|^2 = 2\hbar \langle X_A^{\text{horiz}}, X_A^{\text{horiz}} \rangle.$$ 

By (29), $(h_{11})_x = 2\hbar (\Delta \hat{A})_x^2$. Similarly, $(h_{22})_x = 2\hbar (\Delta \hat{B})_x^2$ and $(h_{12})_x = (h_{21})_x = 2\hbar \text{Re} \langle X_A^{\text{horiz}}, X_B^{\text{horiz}} \rangle = \frac{2\hbar}{i} C(\hat{A}, \hat{B})_x^2$. By comparing to the definition (34) of the covariance matrix, the result is obtained. \(\square\)

Theorem 7. Let $\hat{A}$ and $\hat{B}$ be self-adjoint linear operators on $\mathcal{H}$. Let $u$ be a map in the family $\mathcal{F}$ with $u = \pi \circ f$ and $f : \Sigma \to S(\mathcal{H})$ with $f(x) = \psi$. Then, in the pull-back metric $h = u^*(g)$ on $\Sigma$, 

$$\frac{1}{2}(du^2 dA_\Sigma)_x = \frac{2}{\hbar} \sqrt{(\Delta \hat{A})_x^2 + (\Delta \hat{B})_x^2} \langle \psi, (\hat{A}, \hat{B})_x \rangle^2 ds \wedge dt,$$ 

(40)

$$\langle u^*(\Omega) \rangle_x = \frac{1}{2\hbar} \langle \psi, [\hat{A}, \hat{B}] \psi \rangle ds \wedge dt.$$ 

(41)

Thus, the Robertson-Schrödinger uncertainty relation (35) is equivalent to the differential version of the energy identity (27).
Proof: The first step is to show that choosing the pull-back metric $h = u^*(g)$ on $\Sigma$ to compute the energy density gives $\epsilon(u) = \frac{1}{2} | du |^2 = 1$. In coordinates, $(x^1, x^2) = (s, t)$ on $\Sigma$, writing $h = \sum h_{k\ell} dx^k dx^\ell$ and $(h^{k\ell}) = (h_{k\ell})^{-1}$,

\[
|du|^2 := \sum_{k,\ell,\alpha,\beta} g_{\alpha\beta}(u(x)) h^{k\ell}(x) \partial u^\alpha \partial u^\beta.
\] (42)

Equivalently, again using the fact that $f_\ast$ maps into the horizontal subspace of $T_\psi(S(H))$,

\[
|du|^2 = \sum_{k,\ell,\alpha,\beta} \tilde{g}_{\alpha\beta}(f(x)) h^{k\ell}(x) \partial f^\alpha \partial f^\beta,
\] (43)

where $\tilde{g}_{\alpha\beta} = 2\hbar \delta_{\alpha\beta}$. Inverting $(h_{k\ell})$ and using (43) with $s = x^1$ and $t = x^2$, we have

\[
|du|^2 = 2h \left[ h^{11} \frac{\partial f}{\partial s} \right]^2 + h^{12} Re \sum_\alpha \left( \frac{\partial f^\alpha}{\partial s} \right) \frac{\partial f^\alpha}{\partial t} + h^{21} Re \sum_\alpha \left( \frac{\partial f^\alpha}{\partial t} \right) \frac{\partial f^\alpha}{\partial s} + h^{22} \left( \frac{\partial f^\alpha}{\partial t} \right)^2 \right] (44)

\[
|du|^2 = \frac{1}{\det(h_{k\ell})} (2h^{11}h^{22} - 2h^{12})
\] (45)

Thus, the left hand side of the differential energy identity (46) is

\[
\frac{1}{2} |du|^2 dA_\Sigma = dA_\Sigma = \sqrt{\det(h_{ij})} ds \wedge dt = \frac{2}{\hbar} \sqrt{(\Delta \hat{A})^2 (\Delta \hat{B})^2 - (C(\hat{A}, \hat{B}))^2} ds \wedge dt.
\] (46)

To prove (47), choose symplectic coordinates $(y_1, \ldots, y_{2n+2})$ in a neighborhood of $\psi = f(x) \in H$. Define the horizontal vectors $v, w \in T_\psi(S(H))$ as

\[
v := (X^\text{horiz}_A)_\psi = -\frac{i}{\hbar} (\hat{A} - \langle \hat{A} \rangle_\psi) \psi,
\]

\[
w := (X^\text{horiz}_B)_\psi = -\frac{i}{\hbar} (\hat{B} - \langle \hat{B} \rangle_\psi) \psi.
\] (47)

Then

\[
f(s, t) = (y_1(s, t), \ldots, y_{2n}(s, t))^T
\]

\[
f^*(dy_\alpha) = \langle v_\alpha ds + w_\alpha dt \rangle, \quad \alpha = 1, \ldots, 2n.
\] (48)

$u^*\Omega = f^*\Omega_H$, so that

\[
(u^*\Omega)_x = f^* \left( \sum_{\alpha \text{ odd}} dy_\alpha \wedge dy_{\alpha+1} \right) = \sum_{\alpha \text{ odd}} f^*(dy_\alpha) \wedge f^*(dy_{\alpha+1})
\]

\[
= \sum_{\alpha \text{ odd}} (v_\alpha ds + w_\alpha dt) \wedge (v_{\alpha+1} ds + w_{\alpha+1} dt)
\]

\[
= \sum_{\alpha \text{ odd}} (v_\alpha w_{\alpha+1} - v_{\alpha+1} w_\alpha) ds \wedge dt
\]

\[
= \sum_{\alpha \text{ odd}} \det \begin{pmatrix} v_\alpha & w_\alpha \\ v_{\alpha+1} & w_{\alpha+1} \end{pmatrix} ds \wedge dt
\] (49)

\[
= \langle v, Jw \rangle ds \wedge dt = \Omega_H(v, w) ds \wedge dt
\] (50)

\[
= \frac{1}{2\hbar} \langle \psi, [\hat{A}, \hat{B}] \rangle ds \wedge dt,
\] (51)

where we have used the relation (17).
Corollary 8. If $u \in \mathcal{F}$ is $J$-holomorphic, then $X_b = JX_a$.

Proof: Using the relations: $j\partial_s = \partial_t$ and $j\partial_t = -\partial_s$, the $J$-holomorphic condition is equivalent to $du(\partial_s) + Jdu(\partial_t) = 0$. If $u \in \mathcal{F}$, then $du = d\pi \circ df$ and $df_z : \partial_s \mapsto (X_A)_{f(z)}$. Then the condition requires that $df_z(\partial_s) = Jdf(\partial_s) = J(X_A)_{f(z)}$. Thus, if $u \in \mathcal{F}$, then $X_B = JX_A$, which implies that $X_b = JX_a$. 

In particular, for a map $u \in \mathcal{F}$, using the vectors to write the $J$-holomorphic condition.

\[
\tilde{\partial}_J u = \frac{1}{2} \begin{pmatrix} v_1 - w_2 & w_1 + v_2 \\
 v_2 + w_1 & w_2 - v_1 \\
v_3 - w_4 & w_3 + v_4 \\
v_4 + w_3 & w_4 - v_3 \
\vdots & \vdots 
\end{pmatrix} = 0, \tag{52}
\]

which is equivalent to the component-wise Cauchy-Riemann equations:

\[
\begin{align*}
v_\alpha &= \frac{\partial f_\alpha}{\partial s} = \frac{\partial f_{\alpha+1}}{\partial t} = w_{\alpha+1} \\
v_{\alpha+1} &= -\frac{\partial f_{\alpha+1}}{\partial s} = -\frac{\partial f_\alpha}{\partial t} = -w_\alpha. \tag{53}
\end{align*}
\]

Thus, the Robertson-Schrödinger uncertainty relation can be interpreted as a comparison of the Riemannian metric area of the parallelogram formed by the projected vectors $(X_a)_{[\psi]}$ and $(X_b)_{[\psi]}$ in $T_{[\psi]}(P(\mathcal{H}))$ to the invariant symplectic area $u^*(\Omega)$. The metric area of the projection depends on the relative directions of the vectors $\hat{A}\psi$ and $\hat{B}\psi$ in $\mathcal{H}$. For each $p = [\psi] \in P(\mathcal{H})$, $\Sigma$ acts as a chart domain for the two-dimensional real subspace of $T_p(P(\mathcal{H}))$ spanned by $\{X_a(p), X_b(p)\}$, which is isomorphic to the subspace of $T_{\psi}(\mathcal{H} - \{0\})$ spanned by $(X_A^{\text{horiz}})_{\psi}, (X_B^{\text{horiz}})_{\psi}$. The image $u(\Sigma)$ of a map $u \in \mathcal{F}$ is a two-dimensional real submanifold in $P(\mathcal{H})$. When the map $u$ is $J$-holomorphic, $u(\Sigma)$ is a complex submanifold of $P(\mathcal{H})$, that is, $T_p(u(\Sigma))$ is a complex subspace of $T_{\psi}(P(\mathcal{H}))$ and, it is in this case that the uncertainty inequality is saturated. This conclusion is consistent with the well-known result that every complex submanifold of a Kähler manifold is volume minimizing in its homology class. Observe that the off-diagonal element $C(\hat{A}, \hat{B})$ of the covariance tensor is real, and hence vanishes when $v = -Jw$.

Corollary 9. If $u \in \mathcal{F}$ is $J$-holomorphic, then the covariance tensor has minimum determinant and vanishing off-diagonal components $C(\hat{A}, \hat{B})_{\psi}$. In this case, the Robertson-Schrödinger uncertainty inequality is saturated.

IV. DISCUSSION

A goal for future work is to establish the integral version of the energy identity for $J$-holomorphic maps from a specific compact Riemann surface (possibly with boundary) into the quantum state space, and to interpret its physical meaning. It is notable that the covariance tensor can be viewed both as a quantum Fisher information metric and as the Hessian matrix for the energy density function for a map into state space, suggesting an investigation into the relation between quantum information and dynamics on $\Sigma$ by using Hamilton’s equations. Also, it would be interesting to consider whether the Kähler property of the quantum state space could be relaxed so that the almost complex structure might be non-constant, or even non-integrable and merely compatible with a symplectic form.

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