Abstract
We approach the analysis of dynamical and geometrical properties of nonholonomic mechanical systems from the discussion of a more general class of auxiliary constrained Hamiltonian systems. The latter is constructed in a manner that it comprises the mechanical system as a dynamical subsystem, which is confined to an invariant manifold. In certain aspects, the embedding system can be more easily analyzed than the mechanical system. We discuss the geometry and topology of the critical set of either system in the generic case, and prove results closely related to the strong Morse-Bott, and Conley-Zehnder inequalities. Furthermore, we consider qualitative issues about the stability of motion in the vicinity of the critical set. Relations to sub-Riemannian geometry are pointed out, and possible implications of our results for engineering problems are sketched.
Introduction

This text is a shortened and modified version of the author’s PhD thesis in mechanical engineering at ETH Zürich, (ETH-Diss 13017, January 1999). Its subject matter is located in the area of theoretical mechanics, and discusses a particular class of constrained Hamiltonian systems, which contains the subclass of nonholonomic mechanical systems. We will be mainly concerned with geometrical and global topological issues about the critical manifolds, the stability of equilibria, aspects about numerical stability, and technical applications.

An area of the engineering sciences, where nonholonomic mechanical systems are typically encountered, is the field of multibody systems simulation. There is a multitude of different approaches to the description and analysis of such systems, stemming from various subareas of application. The geometrical approach given here is influenced by the work of R. Weber [36] on Hamiltonian systems with nonholonomic constraints, and very essentially by the formalism developed by H. Brauchli based on an orthoprojector calculus, [11, 12, 31].

A construction for the Lagrangian case, which is closely related to what will be presented in section 3, has been given by Cardin and Favretti, [14]. A different approach in the Hamiltonian picture is dealt with in the work of J. Van der Schaft and B.M. Maschke, [34]. A geometrical theory of nonholonomic systems with a strong influence of network theory has been developed by H. Yoshimura and T. Kawase, [37]. The geometrical structure of nonholonomic systems with symmetries and the associated reduction theory, as well as aspects of their stability theory has been at the focus in the important works of J.M. Marsden and his collaborators, see for instance [7, 23, 39].

In section one, we introduce a particular class of constrained Hamiltonian systems. Given a symplectic manifold $(M,\omega)$ and a symplectic distribution $V$, we focus on the flow $\tilde{\Phi}_t$ generated by the component $X^V_H$ of the Hamiltonian vector field $X_H$ in $V$.

Section two addresses the geometry and global topology of the critical set $C$ of the constrained Hamiltonian system. The main technical tool used for this purpose is a gradient-like flow $\phi_t$, whose critical set $C$ is identical to that of $\tilde{\Phi}_t$. Assuming that the Hamiltonian $H : M \to \mathbb{R}$ is a Morse function, it is proved that generically, $C$ is a normal hyperbolic submanifold of $M$. The generalization of Morse theory developed by C. Conley and E. Zehnder is used to prove a topological formula for closed, compact $C$, that is closely related to the Morse-Bott inequalities. A second proof is given, based on the use of the Morse-Witten complex in Morse theory, to further clarify certain structural aspects.

In section three, we address questions about the stability of the constrained Hamiltonian systems in discussion, and conjecture a stability criterion for the marginally stable case. Relations to sub-Riemannian geometry are pointed out.

In section four, we introduce Hamiltonian mechanical systems with Pfaffian constraints, which represent the case of highest technical relevance. It is shown that the physical orbits of such a system are confined to a smooth submanifold $\mathcal{M}_{phys} \subset T^*Q$. We construct an auxiliary constrained Hamiltonian system of the type introduced in section one, which exhibits $\mathcal{M}_{phys}$ as an invariant manifold, on which it generates the physical flow. Furthermore, we study the global topology of the critical manifold of the physical constrained system, and analyze the stability of equilibria of such systems. Finally, we propose a method to numerically construct the generic connectivity components of the critical manifold.
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I  A NONINTEGRABLE GENERALIZATION OF D\textsc{IRAC} CONSTRAINTS

\textbf{Hamiltonian dynamics.} Let \((M, \omega, H)\) be a Hamiltonian system. \(M\) denotes a smooth, symplectic 2\(n\)-manifold with a \(C^\infty\) symplectic structure \(\omega \in \Lambda^2(M)\). \(H \in C^\infty(M)\) is the Hamilton function, or the Hamiltonian in brief. For \(p = 1, \ldots, 2n\), \(\Lambda^p(M)\) is the \(C^\infty(M)\)-module of \(p\)-forms on \(M\). The Hamiltonian vector field \(X_H \in \Gamma(TM)\) is determined by
\[ i_{X_H} = -dH, \]
where \(i\) stands for interior multiplication by a vector field. Given a smooth distribution \(W \subset TM\), \(\Gamma(W)\) will denote the \(C^\infty(M)\)-module of smooth sections of \(W\). The Hamiltonian flow is the 1-parameter group \(\Phi_t \in \text{Diff}(M)\) generated by \(X_H\), with \(t \in \mathbb{R}\), and \(\Phi_0 = \text{id}\). Its orbits obey
\[ \partial_t \Phi_t(x) = X_H(\Phi_t(x)) \quad (1) \]
for all \(x \in M\), and \(t \in \mathbb{R}\).

\(\omega\) induces a smooth, non-degenerate Poisson structure on \(M\), the \(\mathbb{R}\)-bilinear, antisymmetric pairing on \(C^\infty(M)\) given by
\[ \{f, g\} = \omega(X_f, X_g) \quad (2) \]
It is a derivative in both of its arguments, and satisfies the Jacobi identity
\[ \{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0, \]
thus \((C^\infty(M), \{\cdot, \cdot\})\) is a Lie algebra. Along every orbit of the Hamiltonian flow,
\[ \partial_t f(\Phi_t(x)) = \{H, f\}(\Phi_t(x)) \quad (3) \]
is satisfied for all \(f \in C^\infty(M)\), and all \(x \in M\), \(t \in \mathbb{R}\).

\textbf{Dirac constraints.} Let \(j : M' \to M\) be an embedded, smooth, 2\(k\)-dimensional symplectic submanifold of \(M\), endowed with the pullback symplectic structure \(j^*\omega\). The induced Poisson bracket on \(M'\) is defined by
\[ \{f, g\}_D = (j^*\omega)(X_{\tilde{f}}, X_{\tilde{g}}) \]
for any pair of extensions \(\tilde{f}, \tilde{g} \in C^\infty(M)\) of \(f, g \in C^\infty(M')\), and is in this context referred to as the Dirac bracket.

If \(M'\) is locally characterized as the locus of common zeros of some family of functions \(G_i \in C^\infty(M)\), with \(i = 1, \ldots, 2(n-k)\), it is possible to give the following explicit construction of the Dirac bracket, \([2]\). Because \(M' \subset M\) is symplectic, the \((n-k)^2\) quantities
\[ D_{ij} := \{G_i, G_j\} \]
define a smooth, matrix-valued function that is invertible everywhere on \(M'\). The explicit formula for the Dirac bracket is given by
\[ \{f, g\}_D = \{\tilde{f}, \tilde{g}\} - \{\tilde{f}, G_i\} D^{ij} \{G_j, \tilde{g}\}, \quad (4) \]
where $D^{ij}$ denotes the components of the inverse of $[D_{ij}]$.

**A natural generalization.** We will next mimick this construction in the following more general setup.

A distribution $V$ over the base manifold $M$ will be called symplectic if $V_x$ is a symplectic subspace of $T_xM$ for all $x \in M$ with respect to $\omega_x$. Accordingly, $V^\perp$ denotes the distribution characterized by the property that its fibres are the symplectic complements of those of $V$ with respect to $\omega$, and is symplectic. Smoothness of $V$ and $\omega$ implies smoothness of $V^\perp$. If $V$ is symplectic, the Whitney sum bundle $V \oplus V^\perp$ equals $TM$.

Thus, let $V$ denote an integrable, smooth, symplectic rank $2k$-distribution $V$ over $M$. Then, any section $X \in \Gamma(TM)$ can be written as

$$X = X^V + X^{V^\perp},$$

where $X^{V^{(\perp)}}$ are sections in $\Gamma(V^{(\perp)})$. Moreover, $\omega(X^V, X^{V^\perp}) = 0$, by definition of $V^\perp$. Associated to this decomposition of vector fields into components lying in $V$ and $V^\perp$, there exists an $\omega$-orthogonal tensor $\pi_V : TM \rightarrow TM$ with

$$\text{Ker}(\pi_V) = V^\perp, \quad \pi_V(X) = X \quad \forall X \in \Gamma(V),$$

which satisfies

$$\omega(\pi_V(X), Y) = \omega(X, \pi_V(Y))$$

for all $X, Y \in \Gamma(TM)$. It will be referred to as the $\omega$-orthogonal projector associated to $V$.

Given a local family of spanning vector fields $Y_1, \ldots, Y_{2k}$ for $V$, one can obtain an explicit formula for $\pi_V$ in a construction similar to (4). To this end, one uses the fact that symplecticness of $V$ is equivalent to the condition of invertibility of the matrix $C_{ij} := \omega(Y_i, Y_j)$.

**Lemma I.1** Let $C_{ij} := \omega(Y_i, Y_j)$, and let $C^{kl}$ denote the components of its inverse. Define the 1-forms $\theta_j(\cdot) := \omega(Y_j, \cdot)$. Then, locally, $\pi_V = C^{ij}Y_i \otimes \theta_j$.

**Proof.** Clearly, the range of $C^{ij}Y_i \otimes \theta_j$ is $V$. Furthermore,

$$(C^{ij}Y_i \otimes \theta_j)^2 = C^{ij}Y_i \otimes \theta_j$$

because of $\theta_i(Y_j) = C_{ij}$, and $C_{ij}C^{jk} = \delta_i^l$. To prove $\omega$-orthogonality for the asserted expression for $\pi_V$, consider an arbitrary pair of vector fields $X, X'$, and observe that

$$\omega(\pi_V(X), X') = \omega(C^{kl}\theta_l(X)Y_k, X')$$

$$= C^{kl}\theta_l(X)\omega(Y_k, X')$$

$$= C^{kl}\theta_l(X)\theta_k(X')$$

$$= -C^{lk}\theta_l(X)\theta_k(X')$$

$$= -\omega(Y_l, X)C^{lk}\theta_k(X')$$

$$= \omega(X, C^{lk}\theta_k(X')Y_l)$$

$$= \omega(X, \pi_V(X')),$$

which establishes the claim. \[\blacksquare\]
As a generalization of the usual Hamiltonian flow, the following dynamical system can be naturally associated to \((M, \omega, H, V)\). The component of \(X_H \in V\),

\[
X^V_H := \pi_V(X_H) \subset \Gamma(V)
\]
generates a 1-parameter group of diffeomorphisms \(\tilde{\Phi}_t\). The orbits of \(\tilde{\Phi}_t\) obey

\[
\partial_t \tilde{\Phi}_t(x) = X^V_H(\tilde{\Phi}_t(x)) ,
\]
and are tangent to \(V\) for all initial conditions \(x \in M\). Embeddings \(\mathbb{R} \to M\) that are tangent to \(V\) are called \(V\)-horizontal. If the condition of integrability imposed on \(V\) is dropped, this dynamical system will allow for the description of nonholonomic mechanics.

Integrability of \(V\) implies that \(M\) is foliated in terms of \(2k\)-dimensional integral manifolds of \(V\), which are, due to the symplecticness of \(V\), symplectic submanifolds of \(M\). Therefore, every leaf \(j : M' \to M\) is an embedded invariant manifold under the action of \(\tilde{\Phi}_t\).

The induced dynamical system on any fixed leaf is equivalent to the pullback Hamiltonian system \((M, j^*\omega, H \circ j)\) that has been considered before in the discussion of Dirac constraints. In this sense, (5) generalizes the notion of Dirac constraints.

**Non-integrability.** A new class of dynamical systems is obtained by discarding the requirement of integrability on \(V\).

By definition, the distribution \(V\) is non-integrable if there exists a filtration

\[
V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_r ,
\]
inductively defined by \(V_0 = V\), and

\[
V_i = V_{i-1} + [V_0, V_{i-1}] ,
\]
which is non-trivial, that is, \(V_1 \neq V_0\). \([\cdot, \cdot]\) denotes the Lie bracket, and the sequence \(\{V_i\}_1^r\) is called the flag of \(V\). For a pair of smooth distributions \(W, W'\), the distribution \([W, W']\) is locally given by the linear span of all Lie brackets that can be taken between sections of \(W\) and \(W'\). If the fibre ranks of all \(V_i\) are base point independent, \(V\) is called equiregular. The smallest number \(r(V)\) at which the flag of \(V\) stabilizes, that is, for which

\[
V_s = V_{r(V)} \quad \forall s \geq r(V) ,
\]
is called the degree of non-holonomy of \(V\). If \(V_{r(V)} = TM\), \(V\) is said to satisfy Chow’s condition, or to be totally non-holonomic.

**Proposition I.1** (Frobenius condition) The symplectic distribution \(V\) is integrable if and only if in local coordinates,

\[
\mathcal{F}^k_{ij} := (\pi_V)^r_i (\pi_V)^s_j \left( \partial_r (\pi_V)^k_s - \partial_s (\pi_V)^k_r \right) = 0
\]
holds everywhere in \(M\).

**Proof.** Clearly, \(V\) is integrable if and only if \(\tilde{\pi}_V ([\pi_V(X), \pi_V(Y)]) = 0\) is satisfied by all sections \(X, Y\) of \(TM\), which is equivalent to \(V_1 = V\). The asserted formula is the local coordinate representation of this condition. ■
Local coordinate representation. In a local Darboux chart with coordinates \( x^i \), the symplectic structure is represented by the matrix
\[
J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.
\] (8)

Writing \( x(t) \) for the coordinate components of \( \tilde{\Phi}_t(x_0) \) with \( x_0 \in M \), the equations of motion (5) are represented by
\[
\partial_t x(t) = (P_x J \partial_x H)(x(t))
= (J P^\dagger_x \partial_x H)(x(t)).
\] (9)

\( P \) denotes the matrix of \( \pi_V \), and \( P^\dagger \) is its transpose. Due to lemma I.1, \( P \) can be explicitly determined once one picks a local spanning family of vector fields for \( V \).

Almost Kähler structure. Let \( g \) denote a Riemannian metric on \( M \). An almost complex structure \( J \) is a smooth bundle isomorphism \( J : TM \to TM \) with \( J^2 = -I \). Together with \( g \), it defines a two form satisfying
\[
\omega_{g,J}(X,Y) = g(JX,Y)
\] (10)
for all sections \( X, Y \in \Gamma(TM) \). The Riemannian metric \( g \) is hermitean if \( g(JX,JY) = g(X,Y) \), and Kähler if \( \omega_{g,J} \) is closed. Thus, if \( \omega_{g,J} = \omega \) (the symplectic structure), \( g \) is Kähler. A triple \((g, J, \omega)\) of this type is called compatible.

Proposition I.2 There is a compatible triple \((g, J, \omega)\), where \( \omega \) is the symplectic structure, such that \( P \) is symmetric with respect to \( g \), and \( \pi_V JX = J\pi_V X \) for all \( X \in \Gamma(TM) \).

Proof. Let us pick a smooth Riemannian metric \( \tilde{g} \) on \( M \), relative to which \( P \) shall be symmetric. For instance, one may choose an arbitrary Riemannian metric \( g' \) on \( M \), and use it to define \( \tilde{g}(X,Y) \equiv g'(\pi_V X, \pi_V Y) + g'(\bar{\pi}_V X, \bar{\pi}_V Y) \), where \( \bar{\pi}_V \) denotes the projection complementary to \( \pi_V \).

Since \( \omega \) is non-degenerate and smooth, there is a non-degenerate, smooth bundle mapping \( K \) which is defined by
\[
\omega(X,Y) = \tilde{g}(KX,Y).
\] (11)

It is skew symmetric with respect to \( \tilde{g} \), because the symplectic structure is antisymmetric, so the adjoint operator of \( K \) with respect to \( \tilde{g} \) is \( K^* = -K \).

The bundle map \( K \) can be used to construct an almost symplectic structure. Since \( K^* K = -K^2 \) is smooth, positive definite, non-degenerate and symmetric, there is a unique smooth, positive definite, symmetric bundle map \( A \) defined by \( A^2 = -K^2 \), which commutes with \( K \). This immediately implies that the bundle map \( J = KA^{-1} \) satisfies \( J^2 = -I \), so that it defines an almost complex structure. This is the standard proof for the fact that every almost symplectic manifold admits an almost complex structure, see for instance [13].

Because \( A \) is positive definite and symmetric, one can define a new metric by \( g(X,Y) \equiv \tilde{g}(AX,Y) \), which obviously satisfies
\[
\omega(X,Y) = g(JX,Y).
\] (12)
Moreover, this metric is hermitean, since

\[ g(JX, JY) = \tilde{g}(KX, A^{-1}KY) = -\tilde{g}(X, K^2 A^{-1}Y) = \tilde{g}(X, AY) = g(X, Y). \]

Because \( \omega \) is closed, \( g \) is not only hermitean, but even Kähler. The projector \( \pi_V \) is symmetric with respect to \( g \), as one concludes from the following consideration. The fact that \( P \) commutes with \( K \) follows from

\[ \tilde{g}(K\pi V X, Y) = \omega(\pi V X, Y) = \omega(X, \pi V Y) = \tilde{g}(KX, \pi V Y) = \tilde{g}(\pi V KX, Y) \]

for all \( X \) and \( Y \), since \( \pi_V \) is symmetric with respect to \( \tilde{g} \). \( P \) commutes with \( K \), so also commutes with \( K^2 \); thus it commutes with \( A^2 \). The linear operator \( A^2 \) is positive definite and symmetric, therefore commutativity with the symmetric operator \( P \) implies that \( P \) also commutes with \( A \). It immediately follows that \( P \) is symmetric with respect to \( g \).

Finally, let us prove that \( \pi_V \) commutes with \( J \) on \( \Gamma(TM) \). To this end, we observe that

\[ g(J\pi V X, Y) = \omega(\pi V X, Y) = \omega(X, \pi V Y) = g(JX, \pi V Y) = g(\pi V JX, Y) \]

holds for all \( X \) and \( Y \), as a result of which one concludes that \( \pi_V JX = J\pi V X \) is satisfied for all \( X \), as claimed. This implies that \( J \) restricts to a bundle map \( J : V \to V \).

If the \( J \) is covariantly constant with respect to the Levi-Civita connection associated to \( g \), it is a complex structure. If \( \omega \) admits a compatible Kähler metric \( g \) together with a complex structure \( J \), \((M, g, J)\) is a Kähler manifold.

**Symmetries.** Given a Lie group \( G \), \((M, \omega, H)\) admits a symplectic \( G \)-action \( \Psi : G \to \text{Diff}(M) \), if

\[ \Psi^*_h \omega = \omega, \quad H \circ \Psi^*_h = H \]

is satisfied for all \( h \in G \). Accordingly, for the case of a constrained Hamiltonian system, we will say that \((M, \omega, H, V)\) possesses a \( G \)-symmetry if \( \Psi^*_h V = V \) holds for all \( h \in G \).

**Generalized Dirac bracket.** In the same spirit in which one defines the Poisson structure induced by \( \omega \), it is possible to introduce a smooth, \( \mathbb{R} \)-bilinear, antisymmetric pairing on \( C^\infty(M) \) that is associated to the triple \((M, \omega, V)\). In fact

\[ \{f, g\}_V := \omega(\pi V (X_f), \pi V (X_g)) \quad (13) \]

is a straightforward generalization of the Dirac and Poisson brackets. Along orbits of \( \tilde{\Phi}_t \), one has

\[ \partial_t f(\tilde{\Phi}_t(x)) = \{H, f\}_V(\tilde{\Phi}_t(x)) \]

for all \( x \in M \), in analogy to \((3)\). The bracket \((13)\) does not satisfy the Jacobi identity if \( V \) is non-integrable, but it satisfies a Jacobi identity on every (symplectic) integral manifold if
V is integrable.

**Energy conservation and symplecticness.** Let us finally mention two further important properties of the flow $\Phi_t \in \text{Diff}(M)$, $t \in \mathbb{R}$, of the constrained system.

**Proposition I.3** The energy $H$ is an integral of motion of the dynamical system (3).

**Proof.** This follows from the antisymmetry of the generalized Dirac bracket, which implies that $\partial_t H = \{H, H\}_V = 0$. ■

Finally, let us consider

$$
\begin{align*}
\partial_t \Phi_t^* \omega &= \Phi_t^* \mathcal{L}_{X^H} \omega \\
&= \Phi_t^* (d i_{X^H} \omega + i_{X^H} d \omega) \\
&= - \Phi_t^* d \left( (P_V)^i_k \partial_i H \ dx^k \right) \\
&= - \Phi_t^* \left( \partial_i (P_V)_k \partial_i H \ dx^l \wedge dx^k \right) \\
&= - \Phi_t^* \left( \frac{1}{2} \left( \partial_i (P_V)_k - \partial_k (P_V)_i \right) \partial_i H \ dx^l \wedge dx^k \right).
\end{align*}
$$

This result implies that the restriction of $\partial_t \Phi_t^* \omega$ to $X, Y \in \Gamma(V)$ satisfies

$$
\partial_t \Phi_t^* \omega(X, Y) = - \Phi_t^* \left( \frac{1}{2} \tilde{S}_{rs}^i \partial_i H \ X^r Y^s \right),
$$

where $\tilde{S}_{rs}^i$ is defined in lemma I.4, which states that the the right hand side vanishes identically if and only if $V$ is integrable.

Thus, if $V$ is integrable, the restriction of $\Phi_t^* \omega$ to $V \times V$ equals its value for $t = 0$, given by $\omega(\pi_V(\cdot), \pi_V(\cdot))$. Hence, on every integral manifold $j : M' \to M$ of $V$, $\Phi_t$ is symplectic with respect to the pullback symplectic structure $j^* \omega$.
II THE GEOMETRY AND TOPOLOGY OF THE CRITICAL MANIFOLD

The critical set $\mathcal{C}$ of a dynamical system is defined as the set of equilibrium solutions. The main purpose of this section is to study geometrical and global topological properties of the critical sets $\mathcal{C}$ exhibited by constrained Hamiltonian systems of the type $(M,\omega,H,V)$. An application of Sard’s theorem will demonstrate that generically, $\mathcal{C}$ is a smooth $2(n-k)$-dimensional submanifold $\mathcal{C}_{\text{gen}} \subset M$, the critical manifold.

Moreover, it will be demonstrated that there exists a precise correspondence between $\mathcal{C}_{\text{gen}}$ and the critical points of $H$ if $H : M \to \mathbb{R}$ is picked to be a Morse function. In particular, we will prove that for compact $M$ without boundary, the Poincaré polynomials of $M$ and those of the connectivity components of $\mathcal{C}_{\text{gen}}$ are related in a way reminiscent of the Morse-Bott inequalities.

Two different proofs will in fact be presented. The first proof is based on the generalization of Morse and Morse-Smale theory [1, 20, 26, 29] developed by C. Conley and E. Zehnder in [13], applied to an auxiliary gradient-like system. The second proof is based on the comparison of the Morse-Witten complexes of $(M,H)$ and $(\mathcal{C}_{\text{gen}},H|_{\mathcal{C}_{\text{gen}}})$.

The special case of mechanical systems (where $M$ is noncompact) will be analyzed in the last section.

The following standard definitions are necessary for the subsequent discussion.

**Definition II.1** The zeros of the one form $dH$ are called critical points of $H$, and the value of $H$ at a critical point is called a critical value. A level surface $\Sigma_E$ that contains a critical value $E$ of $H$ is called a critical level surface. A level surface $\Sigma_E$ that contains no critical points of $H$ is called regular, and consequently, $E$ is then called a regular value.

**II.1 Generic properties of the critical set**

The critical set of the constrained Hamiltonian system $(M,\omega,H,V)$ is given by

$$
\mathcal{C} = \{ x \in M \mid X_H^Y(x) = 0 \} \subset M.
$$

The following theorem holds independently of the fact whether $V$ is integrable or not.

**Theorem II.1** Generically, the critical set is a piecewise smooth, $2(n-k)$-dimensional submanifold of $M$.

**Proof.** Let $\{Y_i\}_{i=1}^{2k}$ denote a smooth, local family of spanning vector fields for $V$ over an open neighborhood $U \subset M$. Since $V$ is symplectic, the fact that $X_H^Y$ is a section of $V$ implies that $\omega(Y_i,X_H^Y)$ cannot be identically zero for all $i$ and everywhere in $U$. Due to the $\omega$-orthogonality of $\pi_V$, and $\pi_V Y_i = Y_i$,

$$
\omega(Y_i,X_H^Y) = \omega(\pi_V(Y_i),X_H) = \omega(Y_i,X_H) = Y_i(H).
$$

Thus, with $F := (Y_1(H),\ldots,Y_{2k}(H)) \in C^\infty(U,\mathbb{R}^{2k})$, it is clear that $\mathcal{C} \cap U = F^{-1}(0)$. Since $F$ is smooth, Sard’s theorem implies that regular values, having smooth, $2(n-k)$-dimensional submanifolds of $U$ as preimages, are dense in $F(U)$ [27].
Next, let us pick a local spanning family \( \{ Y_i \in \Gamma(V) \}_{i=1}^{2k} \) for \( V \) that satisfies
\[
\omega(Y_i, Y_j) = \tilde{J}_{ij},
\]
with \( \tilde{J} := \begin{bmatrix} 0 & 1_k \\ -1_k & 0 \end{bmatrix} \). This choice is always possible.

Furthermore, introducing the associated family of 1-forms \( \{ \theta_i \} \) by \( \theta_i(\cdot) := \omega(Y_i, \cdot) \), theorem [1] implies that
\[
\pi_V = \tilde{J}^{ij} Y_i \otimes \theta_j,
\]
where \( \tilde{J}^{ij} \) are the components of \( \tilde{J}^{-1} = -\tilde{J} \). Expanding \( X_V^H \) with respect to the basis \( \{ Y_i \} \) gives
\[
X_V^H = \pi_V(X_V^H) = -Y_i(H) \tilde{J}^{ij} Y_j,
\]
where one uses the relationship \( \theta_j(X_V^H) = Y_j(H) \) obtained in the proof of theorem [1].

**Proposition II.1** Under the genericity assumption of theorem [1], the \( 2k \times 2k \)-matrix
\[
[Y_k(Y_i(H))(x_0)]
\]
is invertible for all \( x_0 \in \mathcal{C} \), and every local spanning family \( \{ Y_i \in \Gamma(V) \} \) of \( V \).

**Proof.** Let us pick a local basis \( \{ Y_i \}_{i=1}^{2k} \) for \( V \), and \( \{ Z_j \}_{j=1}^{2(n-k)} \) for \( V^\perp \), which together span \( TM \). Let \( x_0 \in \mathcal{C}_{\text{gen}} \), and assume the generic situation of theorem [1]. Because \( \mathcal{C}_{\text{gen}} \) is defined as the set of zeros of the vector field (14), the kernel of the linear map
\[
dF_i(\cdot) \tilde{J}^{jk} Y_k |_{x_0} : T_{x_0}M \to V_{x_0},
\]
where \( F_i := Y_i(H) \), is precisely \( T_{x_0} \mathcal{C}_{\text{gen}} \), and has a dimension \( 2(n - k) \).

In the basis \( \{ Y_1|_{x_0}, \ldots, Y_{2k}|_{x_0}, Z_1|_{x_0}, \ldots, Z_{2(n-k)}|_{x_0} \} \), its matrix is given by
\[
A = [A_V \ A_{V^\perp}] ,
\]
where \( A_V := [Y_i(F_j), \tilde{J}^{jk} Y_k|_{x_0}] \), and \( A_{V^\perp} := [Z_i(F_j), \tilde{J}^{jk} Y_k|_{x_0}] \). Bringing \( A \) into upper triangular form, \( A_V \) is also transformed into upper triangular form. Because the rank of \( A \) is \( 2k \), and \( A_V \) is a \( 2k \times 2k \)-matrix, its upper triangular form has \( 2k \) nonzero diagonal elements. Consequently, \( A_V \) is invertible, and due to the invertibility of \( \tilde{J} \), one arrives at the assertion. 

**Corollary II.1** Let \( \mathcal{C}_{\text{gen}} \) satisfy the genericity assumption of theorem [1]. If \( V \) is integrable, the intersection of any integral manifold of \( V \) with \( \mathcal{C}_{\text{gen}} \) is a discrete set.

**Proof.** The previous proposition implies that generically, integral manifolds of \( V \) intersect \( \mathcal{C}_{\text{gen}} \) transversely. Their dimensions are complementary, hence the intersection set is zero-dimensional.
II.2 Topology of the critical manifold

The Hessian of the constrained system. The usual coordinate invariant definition of the Hessian of $H$ corresponds to $\nabla dH$, evaluated at its critical points [22], where $\nabla$ denotes the Levi-Civita connection of the Kähler metric $g$.

A generalized Hessian for the constrained Hamiltonian system can easily be defined along these lines. Let us write $\pi_V^\dagger$ for the dual projector associated to $\pi_V$, which acts on sections of the cotangent bundle $T^*M$. Let $\theta$ denote any one-form. Then of course, $\langle \pi_V^\dagger \theta, X \rangle = \langle \theta, \pi_V X \rangle$. In any local chart, the matrix of $\pi_V^\dagger$ is the transpose of the matrix of $\pi_V$.

The obvious generalization of the Hessian is the tensor $\nabla (\pi_V^\dagger dH)$. It acts as a bilinear form on pairs of vector fields in terms of

$$\nabla (\pi_V^\dagger dH)(X,Y) \equiv \langle \nabla_X (\pi_V^\dagger dH), Y \rangle = \left( ((\pi_V)_j^i H_{,j})_s - \Gamma^s_r_i (\pi_V)_j^i H_{,j} \right) X^r Y^s,$$

where $\Gamma^s_r_i$ are the Christoffel symbols. The second term in the bracket on the lower line is zero on $C$, because $(\pi_V)_j^i H_{,j} = 0$ on $C$. The non-vanishing term in $\nabla (\pi_V^\dagger dH)$ on $C$ is determined by the matrix

$$K_{rs} \equiv ((\pi_V)_j^i H_{,j})_s.$$

A straightforward calculation shows that $(\pi_V)_j^i K_{jk} = K_{ik}$ holds everywhere on $C$. Obviously, the rank of $K$ is bounded from above by the rank of $\pi_V$, corresponding to the rank $2k$ of $V$.

The corank of $K|_a$ equals the dimension of the connectivity component of $C$ which contains $a$. This is because for any $a' \in C$ close to $a$, Taylor’s theorem implies that

$$(\pi_V)_j^i H_{,j}(a') = ((\pi_V)_j^i H_{,j})(a) + K|_a(a' - a) + O(||a' - a||^2).$$

The left hand side and the first term on the right hand are both zero, since $C$ is so defined, thus $K|_a(a' - a) \to 0$ in the limit $||a' - a|| \to 0$. That is, the tangent space $T_a C$ is equal to $\ker K|_a$, hence the corank of $K|_a$ is indeed the dimension of the component $C_i$.

II.2.1 An auxiliary gradient-like dynamical system

The physical flow of the constrained Hamiltonian system turns out to of very limited use for the study of the global topology of $C$. This is because invariant sets of the physical flow not only contain fixed points, but also periodic orbits.

However, the auxiliary dynamical system on $M$ defined by

$$\partial_t \gamma(t) = - (\pi_V \nabla g H)(\gamma(t)),$$

for $\gamma : I \subset \mathbb{R} \to M$, is an extremely powerful tool for this purpose. Let us denote its flow by

$$\phi_t \in \text{Diff}(M).$$

Its orbits are of course tangent to $V$, and it is clear that $\Phi^\varepsilon_t$ and $\phi_t$ share the same critical set $C$. 

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**Definition II.2**  A flow is gradient-like if there exists a function \( f : M \to \mathbb{R} \) that decreases strictly along all of its non-constant orbits.

**Proposition II.2**  The flow \( \phi_t \) is gradient-like.

**Proof.** One can easily check that \( H \) decreases strictly along the non-constant orbits of \( \phi_t \),

\[
\partial_t H(\gamma(t)) = \langle dH(\gamma(t)) , \partial_t \gamma(t) \rangle = -g(\nabla g H , \pi_V \nabla g H)(\gamma(t)) \leq 0.
\]

We have here used the fact that \((g, J, \omega, \pi_V)\) is a compatible quadruple. This is the main motivation for the introduction of these quantities. 

It is immediately clear that \( \phi_t \) generates no periodic trajectories, hence \( \mathcal{C} \) comprises all invariant sets of \( \phi_t \).

**The gradient-like flow close to \( \mathcal{C} \).** Let us briefly consider the orbits of \( \phi_t \) in a tubular \( \epsilon \)-neighborhood of \( \mathcal{C} \) (with respect to the Riemannian distance induced by the Kähler metric \( g \)). We pick an arbitrary element \( a \in \mathcal{C} \) and local coordinates \( x^i \), with the origin at \( a \). The equations of motion (16) are given by

\[
\partial_t x^r(t) = -K^r_s(a)x^s(t) + O(\|x\|^2),
\]

where \( K^r_s \) is the ‘constrained Hessian’ at \( a \), and \( K^r_s = g^{ij}K^r_{js} \), and \( \| \cdot \| \) is with respect to \( g \) at \( a \).

\( P_a \) will from now on denote the matrix of \( \pi_V(a) \). Furthermore, let \( I_{P_a} : V_a \hookrightarrow T_a M \) denote the embedding. Moreover, we will write \( A_a \equiv [K^r_s(a)] \) for the Jacobian matrix of \( \pi_V \nabla g H \) at \( a \) in the present chart. From the \( g \)-orthogonality of \( \pi_V \) follows straightforwardly that \( P_a A_a = A_a \).

The linearized equations of motion at \( a \) read \( \partial_t x(t) = -A_a x(t) \), and satisfy the linearized constraints \( \partial_t x(t) = P_a \partial_t x(t) \).

Let \( \bar{Q} \) denote the \( g \)-orthoprojector onto \( \ker A_a = T_a \mathcal{C} \), and let \( I_{\bar{Q}_a} : T_a \mathcal{C} \hookrightarrow T_a M \) be the embedding. Its complement \( Q_a = I - \bar{Q}_a \) projects \( T_a M \) orthogonally onto the fibre \( N_a \mathcal{C} \) of the normal bundle of \( \mathcal{C} \), and likewise, let \( I_{Q_a} : T_a N \hookrightarrow T_a M \) be the embedding. The orbits of the linearized system are then given by

\[
x(t) = \exp(-tI_{P_a}A_aI_{Q_a}Q_a)x_0
\]

where the initial condition is determined by \( x(0) = x_0 \). The vector \( \bar{Q}_a x_0 \in T_a \mathcal{C} \) approximately connects \( a \) with some critical point close to \( a \), while \( Q_a x_0 \) lies in \( N_a \mathcal{C} \).
II.2.2 Morse functions and non-degenerate critical manifolds

Let us recall some standard definitions from Morse- and Morse-Bott theory that will be needed in the subsequent discussion.

**Definition II.3** The dimensions of the zero and negative eigenspaces of the Hessian of \( f \) at a critical point \( a \) are called the nullity and the index of the critical point \( a \). If all critical points of \( f : M \to \mathbb{R} \) have a zero nullity, \( f \) is called a Morse function, and the index is then called the Morse index of \( a \).

If the critical points of \( f \) are not isolated, but elements of critical manifolds that are non-degenerate in the sense of Bott, \( H \) is called a Morse-Bott function \([8]\).

Throughout this section, we will assume that \( H \) is a Morse function.

**Definition II.4** A connectivity component \( \mathfrak{C}_i \) is locally normal hyperbolic at the point \( a \in \mathfrak{C} \) with respect to \( \phi_t \) if it is a manifold at \( a \), and if the restriction of \( A_a \) to the normal space \( N_a \mathfrak{C} \) is non-degenerate. A connectivity component \( \mathfrak{C}_i \) is called non-degenerate if it is a manifold that is everywhere normal hyperbolic with respect to \( \phi_t \). The index of a non-degenerate connectivity component \( \mathfrak{C}_i \) is the number of eigenvalues of the constrained Hessian \( A_a \) on \( \mathfrak{C}_i \) that are contained in the negative half plane.

**Proposition II.3** In the generic case, the critical manifold is normal hyperbolic with respect to the gradient-like flow.

**Proof.** This follows straightforwardly from lemma \([1.1]\). \( \blacksquare \)

Let \( \mathfrak{C}_i, i = 1, \ldots, l \) denote the connectivity components of \( \mathfrak{C} = \cup \mathfrak{C}_i \), and let \( j_i : \mathfrak{C}_i \to M \) denote the embedding.

**Proposition II.4** Assume that \( \mathfrak{C} \) satisfies the genericity assumption in the sense of Sard’s theorem. Then, if \( H : M \to \mathbb{R} \) is a Morse function, \( H_i := H \circ j_i : \mathfrak{C}_i \to \mathbb{R} \) is also a Morse function. A point \( x \in \mathfrak{C}_i \) is a critical point of \( H_i \) if and only if it is a critical point of \( H \).

**Proof.** The fact that every critical point of \( H \) is a zero locus of \( \pi_{\mathfrak{V}} \nabla_g H |_a \), and thus an element of \( \mathfrak{C} \), is trivial.

To prove the converse statement, let \( \mathfrak{C}_i \) be generic, so it has a dimension \( 2(n - k) \). Normal hyperbolicity implies that the restriction of \( P_a I_{Q_a} Q_a I_{P_a} P_a \) to the normal space \( N_a \mathfrak{C}_i \) defines an invertible map for all \( a \in \mathfrak{C}_i \). Let \( a \) now denote an extremum of \( H|\mathfrak{C}_i \). Then,

\[
\langle dH, v \rangle|_a = g(\nabla_g H, v)|_a = 0
\]

holds for all \( v \in T_a \mathfrak{C}_i \). Consequently, \( \nabla_g H|_a \) is a vector in \( N_a \mathfrak{C} \), and thus, \( \nabla_g H|_a = Q_a \nabla_g H|_a \).

Moreover, by definition of \( \mathfrak{C} \), the condition \( P_a \nabla_g H|_a = 0 \) is satisfied, which in turn implies \( Q_a I_{P_a} P_a I_{Q_a} Q_a \nabla_g H|_a = 0 \). Hence, since \( Q_a I_{P_a} P_a I_{Q_a} Q_a : N_a \mathfrak{C}_i \to N_a \mathfrak{C}_i \) is invertible, it follows that \( \nabla_g H|_a = 0 \).

The Hessian of the restriction of \( H \) at any critical point of \( H_i \) is always nondegenerate, thus \( H_i \) is by itself a Morse function on \( \mathfrak{C}_i \). \( \blacksquare \)
Corollary II.2 The critical points of \( H|_{\mathcal{C}_{\text{gen}}} \) are precisely the critical points of \( H : M \to \mathbb{R} \). Let \( \mathcal{C}_i \) be a non-generic connectivity components with manifold structure that is normal hyperbolic. Then, \( \mathcal{C}_i \subset \Sigma_{H(\mathcal{C}_i)} \).

Proof. The first assertion follows trivially from the previous proposition.

Next, let us assume that \( \mathcal{C}_i \) is some non-generic connectivity component of \( \mathcal{C} \) with manifold structure that is normal hyperbolic (which is in no sense necessitated by the system). Then, the proof of the previous proposition can be applied to the situation for \( \mathcal{C}_i \), and demonstrates that there are no extrema of \( H|_{\mathcal{C}_i} \). Thus, \( \mathcal{C}_i \) is a submanifold of the level surface \( \Sigma_{H(\mathcal{C}_i)} \). ■

II.2.3 Index pairs and relative (co-)homology

Let us next focus on the global topology of \( \mathcal{C} \), for assumptions that are weaker than genericity. In fact, we will only assume that the generic connectivity components \( \mathcal{C}_{\text{gen}} \subset \mathcal{C} \) are non-degenerate, compact and without boundary, and that \( \mathcal{C} \setminus \mathcal{C}_{\text{gen}} \) is compact. In this situation, a generalization of the Morse-Bott inequalities can be derived for \( \mathcal{C} \), based on an application of the theory of C. Conley and E. Zehnder for flows on Banach spaces with compact invariant sets. We will only give a very short survey of Conley-Zehnder (CZ) theory, and refer the reader to the original article [14].

Definition II.5 Let \( \mathcal{C}_i \) be any compact component of \( \mathcal{C} \). An index pair associated to \( \mathcal{C}_i \) is a pair of compact sets \( (N_i, \tilde{N}_i) \) that possesses the following properties. The interior of \( N_i \) contains \( \mathcal{C}_i \), and moreover, \( \mathcal{C}_i \) is the maximal invariant set under \( \phi_t \) in the interior of \( N_i \). \( \tilde{N}_i \) is a compact subset of \( N_i \) that has empty intersection with \( \mathcal{C}_i \), and the trajectories of all points in \( N_i \) that leave \( N_i \) at some time under the gradient-like flow \( \phi_t \) intersect \( \tilde{N}_i \). \( \tilde{N}_i \) is called the exit set of \( N_i \).

The \( p \)-th relative homology group \( H_p(X, A) \) of a pair of manifolds \( A \subset X \) is defined as follows. Let \( C_p(X) \) be the group of \( p \)-cycles of \( X \). Since \( C_p(A) \) is a subgroup of \( C_p(X) \), the quotient group \( C_p(X)/C_p(A) \) is well-defined. Let \( B_p(X, A) \) denote the subgroup of \( C_p(X)/C_p(A) \) which consists of boundaries. The associated quotient group is the \( p \)-th relative homology group \( H_p(X, A) \).

The \( p \)-th relative de Rham cohomology group \( H^p(X, A) \) consists of cohomology classes that are represented by closed \( p \)-forms on \( X \) whose restriction to \( A \) (via the pullback of the embedding \( A \to X \)) is exact. If \( X \) and \( A \) are both orientable, the coefficients of the relative homology and cohomology groups can be picked from \( \mathbb{R} \) or \( \mathbb{Z} \), and otherwise from \( \mathbb{Z}_2 \).

The relative cohomology of an index pair. It was proved in [14] that the homotopy type of the pointed space \( N_i/\tilde{N}_i \) only depends on \( \mathcal{C}_i \), so that \( H^*(N_i, \tilde{N}_i) \) is independent of the particular choice of index pairs (the space \( N_i/\tilde{N}_i \) is obtained from collapsing the subspace \( \tilde{N}_i \) of \( N_i \) to a point). The equivalence class \( [N_i/\tilde{N}_i] \) of pointed topological spaces under homotopy only depends on \( \mathcal{C}_i \), and is called the Conley index of \( \mathcal{C}_i \). In the present analysis, we will only consider flows exhibiting normal hyperbolic critical manifolds. In this special case, the result essentially reduces to a generalization of Morse-Bott theory. We will
now determine the relative cohomology $H^*(N_i, \tilde{N}_i)$ of an index pair $(N_i, \tilde{N}_i)$ associated to a generic connectivity component $C_i$.

**Proposition II.5** Let $C_i \subset C_{gen}$ be a generic connectivity component that is compact and without boundary, and let $(N_i, \tilde{N}_i)$ denote any associated index pair. Then,

$$H^{q+\mu_i}(N_i, \tilde{N}_i) \cong H^q(C_i),$$

where $\mu_i$ is the index of $C_i$, and $q = 0, \ldots, \dim(C_i)$.

**Proof.** Let us consider, for some sufficiently small $\epsilon_0 > 0$, a compact tubular $\epsilon_0$-neighborhood $U$ of $C_i$ (of dimension $2n$), and let

$$W^{cu}_U(C_i) := (W^-(C_i) \cup C_i) \cap U$$

denote the intersection of the center unstable manifold of $C_i$ with $U$. $W^-(C_i)$ denotes the unstable manifold of $C_i$. Pick some small, positive $\epsilon < \epsilon_0$, and let $U_\epsilon$ be the compact tubular $\epsilon$-neighborhood of $W^{cu}_U(C_i)$ in $U$.

It is clear that letting $\epsilon$ continuously go to zero, a homotopy equivalence of tubular neighborhoods is obtained, for which $W^{cu}_U(C_i)$ is a deformation retract. Let

$$U^\text{out}_\epsilon := \partial U_\epsilon \cap \phi_R(U_\epsilon)$$

denote the intersection of $\partial U_\epsilon$ with all orbits of the gradient-like flow that contain points in $U_\epsilon$. Then, evidently, $(U_\epsilon, U^\text{out}_\epsilon)$ is an index pair for $C_i$, and by letting $\epsilon$ continuously go to zero, $U^\text{out}_\epsilon$ is homotopically retracted to $\partial W^{cu}_U(C_i)$.

Thus, homotopy invariance implies that the relative cohomology groups obey

$$H^*(U_\epsilon, U^\text{out}_\epsilon) \cong H^*(W^{cu}_U(C_i), \partial W^{cu}_U(C_i)).$$

Due to the normal hyperbolicity of $C_i$ with respect to the gradient-like flow, $W^{cu}_U(C_i)$ has a constant dimension $n_i + \mu(C_i)$ everywhere, where $n_i = \dim C_i$. Therefore, one obtains from Lefschetz duality \[16\] that

$$H^{n_i+\mu_i-p}(W^{cu}_U(C_i), \partial W^{cu}_U(C_i)) \cong H_p(W^{cu}_U(C_i) \setminus \partial W^{cu}_U(C_i)),$$

where $\mu_i = \mu(C_i)$, the index of $C_i$. It is clear that $C_i$ is a deformation retract of the interior of $W^{cu}_U(C_i)$, so that the respective cohomology groups are isomorphic.

Due to $\dim(C_i) = n_i$, Poincaré duality implies

$$H_p(W^{cu}_U(C_i) \setminus \partial W^{cu}_U(C_i)) \cong H_p(C_i) \cong H^{n_i-p}(C_i),$$

so that with $q := n_i - p$,

$$H^{q+\mu_i}(U_\epsilon, U^\text{out}_\epsilon) \cong H^q(C_i),$$

which proves the claim. \[ \square \]
II.2.4 The Conley-Zehnder inequalities

Here follows a very rough summary of the main steps in the derivation of the Conley-Zehnder inequalities, [15].

Definition II.6 Let \( I \) denote a compact invariant set under \( \phi_t \). A Morse decomposition of \( I \) is a finite, disjoint family of compact, invariant subsets \( \{M_1, \ldots, M_n\} \) that satisfies the following requirement on the ordering. For every \( x \in I \setminus \bigcup_i M_i \), there exists a pair of indices \( i < j \), such that

\[
\lim_{t \to -\infty} \phi_t(x) \subset M_i \quad \text{and} \quad \lim_{t \to \infty} \phi_t(x) \subset M_j.
\]

Such an ordering, if it exists, is called admissible, and the \( M_i \) are called Morse sets of \( I \).

A central result proved in [15] is that for every compact invariant set \( I \) admitting an admissibly ordered Morse decomposition, there exists an increasing sequence of compact sets \( N_0 \subset N_1 \subset \ldots \subset N_m \), such that \((N_i, N_{i-1})\) is an index pair for \( M_i \), and \((N_m, N_0)\) is an index pair for \( I \).

Consider compact manifolds \( A \supset B \supset C \). The exact sequence of relative cohomologies

\[
\ldots \rightarrow H^k(A, B) \rightarrow H^k(A, C) \rightarrow H^k(B, C) \rightarrow H^{k+1}(A, B) \rightarrow \ldots
\]

implies, in a standard fashion known from Morse theory, that, with \( r_{i,p} \) denoting the rank of \( H^p(N_i, N_{i-1}) \), the identity

\[
\sum_{i,p} \lambda^p r_{i,p} = \sum_p B_p \lambda^p + (1 + \lambda)Q(\lambda)
\]

holds for the indicated Poincaré polynomials [22]. Here, \( B_j \) is the \( j \)-th Betti number of the index pair \((N_m, N_0)\) of \( I \), and \( Q(\lambda) \) is a polynomial in \( \lambda \) with non-negative integer coefficients. These are the strong Conley-Zehnder inequalities; a corollary that straightforwardly follows, due to the positivity of the coefficients of \( Q(\lambda) \), is that

\[
\sum_i r_{i,p} \geq B_p
\]

holds. These are the weak Conley-Zehnder inequalities.

If the symplectic manifold \( M \) is compact and closed, and if \( \mathfrak{C} \) is assumed to be non-degenerate, the following results arise from application of the CZ inequalities. The invariant set \( I \) can be chosen to be equal to \( M \). So we let \( N_m = M \) and \( N_0 = \emptyset \) denote the top and bottom elements of the sequence defined above. Furthermore, we order the connected elements of \( \mathfrak{C} \) according to the descending values of the maximum of \( H \) attained on each \( \mathfrak{C}_i \). Then it follows that \( \mathfrak{C} \) furnishes a Morse decomposition for \( M \). The homology groups of \( M \) are isomorphic to the relative homology groups of the index pair \((N_m, N_0)\). So the numbers \( B_p \) are the Betti numbers of the compact symplectic manifold \( M \).

The number \( r_{i,p} \) is the rank of the \( p \)-th relative cohomology group of the non-degenerate critical manifold \( \mathfrak{C}_i \), the index \( i \) being determined by the Morse decomposition. From (21), we deduce that

\[
r_{i,p} = \dim H^{i,p-\mu_i}(\mathfrak{C}_i)
\]
In other words, \( r_{i,p} \) is the \((p - \mu_i)\)-th Betti number of \( \mathcal{C}_i \), written \( B_{i,p-\mu_i} \); in brief. Assuming that the number of connected components of \( \mathcal{C} \) is finite, the Conley-Zehnder inequalities thus result in

\[
\sum_{i,p} B_{i,p} \lambda^{p+\mu_i} = \sum_p B_p \lambda^p + (1 + \lambda) \hat{Q}(\lambda),
\]

which implies that

\[
\sum_i B_{i,p-\mu_i} \geq B_p
\]

is satisfied. So the global topology of \( M \) enforces lower bounds on the ranks of the homology groups of the connected components of \( \mathcal{C} \). Setting the variable \( \lambda \) equal to \(-1\), one obtains

\[
\sum_{i,p} (-1)^{p+\mu_i} B_{i,p} = \sum_i (-1)^{\mu_i} \chi(\mathcal{C}_i) = \chi(M),
\]

where \( \chi \) denotes the Euler characteristic.

**Remark.** In the case of mechanical systems, the phase space of the relevant constrained Hamiltonian system is non-compact, and the critical manifold is generally unbounded. Therefore, the arguments used here do not apply. However, since in that case, \( M \) and \( \mathcal{C} \) are vector bundles, we are nevertheless able to prove results that are fully analogous to (22).

**Generic connectivity components.** One can actually prove a stronger result than (22), given the special structure of the system at hand.

**Proposition II.6** Assume that \( \mathcal{C} \setminus \mathcal{C}_{\text{gen}} \) is a disjoint union of \( C^1 \)-manifolds. Then,

\[
\sum_{i \in \mathcal{C}_{\text{gen}}} B_{i,p} \lambda^{p+\mu_i} = \sum_p B_p \lambda^p + (1 + \lambda) \hat{Q}(\lambda).
\]

\( B_{i,p} \) are the \( p \)-th Betti numbers of the connectivity components \( \mathcal{C}_i \) of \( \mathcal{C}_{\text{gen}} \), \( B_p \) are the Betti numbers of \( M \), and \( \hat{Q} \) is a polynomial with non-negative integer coefficients.

**Proof.** We will show that \( X_\epsilon^V \) can be suitably deformed such that the components of \( \mathcal{C} \setminus \mathcal{C}_{\text{gen}} \) can be smoothed away. To this end, we construct an auxiliary, continuous vector field \( X_\epsilon \), corresponding to a small deformation of the gradient-like vector field \( \pi_V \nabla_g H \). We pick a small positive, real number \( \epsilon \), and consider the compact neighborhoods

\[
U_\epsilon(\mathcal{C}_i) = \{ x \in M \mid \text{dist}_g(x, \mathcal{C}_i) \leq \epsilon \}
\]

of connectivity components \( \mathcal{C}_i \subset \mathcal{C} \setminus \mathcal{C}_{\text{gen}} \). Here, \( \text{dist}_g \) denotes the Riemannian distance function induced by the Kähler metric \( g \). The vector field \( X_\epsilon \) shall be given by \( \pi_V \nabla_g H \) in \( M \setminus U_\epsilon(\mathcal{C}_i) \), and in the interior of every \( U_\epsilon(\mathcal{C}_i) \) with \( \mathcal{C}_i \subset \mathcal{C} \setminus \mathcal{C}_{\text{gen}} \),

\[
X_\epsilon \big|_x = \pi_V \nabla_g H \big|_x + \epsilon h(x) \nabla_g H \big|_x.
\]
Here, we have introduced a $C^1$-function $h : U_c(\mathcal{C}_i) \to [0, 1]$ obeying $h|_{\mathcal{C}_i} = 1$ and $h|_{\partial U_c(\mathcal{C}_i)} = 0$, that is strictly monotonic along the flow lines generated by $\pi_V \nabla_g H$. Such a function $h$ exists because $\mathcal{C} \setminus \mathcal{C}_{gen}$ is a disjoint union of $C^1$-manifolds.

We have proved above that for all $C_i \subset \mathcal{C} \setminus \mathcal{C}_{gen}$, $\nabla_g H$ is strictly non-zero in $U_c(\mathcal{C}_i)$. Now let us consider

\[ g(\tau X, \nabla_g H) = (\|\tau \nabla_g H\|_g^2(x)) + c h(x)(\|\nabla_g H\|_g^2(x)). \]

Here we have used the $g$-symmetry of $P$, and $\|X\|_g^2 = g(X, X)$. The first term on the right hand side is non-zero on the boundary of $U_c(\mathcal{C}_i)$, while the second term vanishes. Moreover, the second term is non-zero everywhere in the interior of $U_c(\mathcal{C}_i)$. Therefore, $\tau X$ vanishes nowhere in $U_c(\mathcal{C}_i)$.

This implies that the vector field $\tau X$ is a deformation of $\pi_V \nabla_g H$ that only exhibits $\mathcal{C}_{gen}$ as a critical set. Notice that the generic components cannot be removed in this manner, since they contain critical points of $H$.

The scalar product of $\tau X$ with $\nabla_g H$ is not only non-zero, but also strictly positive in every $U_c(\mathcal{C}_i)$, which shows that $\tau X$ is a gradient-like flow (the left hand side of the above expression corresponds to the time derivative of $H$ along orbits of $\tau X$ in $U_c(\mathcal{C}_i)$).

Clearly, the order of magnitude of $\|\tau X - \pi_V \nabla_g H\|_g$ is at most $O(\tau)$ everywhere on $\mathcal{C}$. Consequently, it is possible to pick $\tau X$ arbitrarily close to $\pi_V \nabla_g H$ in the $\| \cdot \|_g$-norm on $\Gamma(TM)$ that is induced by $\| \cdot \|_g$.

Carrying out the Conley-Zehnder construction with respect to the flow generated by $\tau X$ yields (24). This result does not require the assumption of normal hyperbolicity on $\mathcal{C}$.  

II.2.5 Proof of (24) using the Morse-Witten complex

We will now give a different proof of (24) based on the existence of the Morse-Witten complex. Our motivation is, on the one hand, to clarify the orbit structure of the gradient-like system, and, on the other hand, to further comment on the ”position” of $\mathcal{C}_{gen}$ in $M$ relative to the equilibria of the free system (corresponding to the critical points of $H$).

To this end, let us first make some brief general remarks about Morse theory, cf. [9].

In the basic setting, there is a Morse function $f : M \to \mathbb{R}$ on a compact manifold $M$, without boundary, say. The index of an isolated critical point precisely is its Morse index, hence the strong Conley-Zehnder inequalities (24) yield

\[ \sum_p \lambda^p N_p = \sum_p \lambda^p B_p + (1 + \lambda)Q(\lambda), \]

where $N_p$ is the number of critical points of $f$ with a Morse index $p$. $B_p$ is the $p$-th Betti number of $M$, and $Q(t)$ is a polynomial with non-negative integer coefficients. These are the strong Morse inequalities, from which the weak Morse inequalities $N_p \geq B_p$ are immediately inferred. The standard proof uses the gradient flow generated by $-\nabla_g f$ with respect to an arbitrary auxiliary metric.

If $f$ is not assumed to be a Morse, but more generally, a Morse-Bott function [8], the gradient flow of the vector field $-\nabla_g f$ can again be used, but then to derive the strong Conley-Zehnder inequalities for the critical manifolds of $f$. 

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Morse theory, which has been a classical topic in differential topology since the 1930’s, experienced a tremendous new increase of activity in the early 1980’s, due to the seminal work of E. Witten [35]. His proof of the weak Morse inequalities is based on a deformation and localization argument in Hodge theory, interpreted as supersymmetric quantum mechanics. Subsequently, the proof of the Morse-Bott inequalities using heat kernel methods has been given by J.-M. Bismut in [6].

This new approach to the strong Morse inequalities is based on the differential complex constructed from the set of critical points of \( f \), which is often referred to as the Morse-Witten complex. At least Milnor, Smale and Thom independently have arrived at some form of it already earlier [4, 17, 28]. The (co-)homology of the Witten complex is isomorphic to the singular (de Rham co-)homology of \( M \), and straightforwardly implies the strong Morse inequalities. A proof of the Morse and Morse-Bott inequalities that is based on the construction of the Morse-Witten complex is given in [4].

The Morse-Witten complex has been generalized to infinite dimensional systems by A. Floer [17], which has led to extremely fruitful applications. For instance, the celebrated Arnol’d conjecture has been proved by use of Floer homology, [21]. A beautiful survey is given in [38].

We will now discuss an alternative proof of (24) that is based on the Morse-Witten complex for non-degenerate Morse functions.

**The Morse-Witten complex.** Let \( M \) be a compact, closed, orientable and smooth manifold, together with a Morse function \( f : M \to \mathbb{R} \). We let \( C^p \) denote the free \( \mathbb{Z} \)-module generated by the critical points of \( f \) with a Morse index \( p \). The set \( C = \bigoplus_p C^p \) is the free \( \mathbb{Z} \)-module generated by the critical points of \( f \), which is graded by their Morse indices. There exists a natural coboundary operator

\[
\delta : C^p \to C^{p+1}
\]

that obeys \( \delta^2 = 0 \), which we will define below.

**Theorem II.2** The cohomology of the differential complex \((C, \delta)\) is isomorphic to the de Rham cohomology of \( M \), \( \ker \delta / \im \delta \cong H^*(M, \mathbb{Z}) \).

A proof of this theorem has been given by Floer in [17] based on Conley-Zehnder theory. Other proofs can be found in [4, 28]. The argument in the original publication [35] is based on the quantum mechanical tunneling effect.

The coboundary operator is defined as follows [4, 2, 17, 28]. We denote the unstable and stable manifold of a critical point \( a \) of \( f \) under the gradient flow by \( W_a^- \) and \( W_a^+ \), respectively, and assign an arbitrary orientation to every \( W_a^- \). Since \( M \) is assumed to be oriented, the orientation of \( W_a^- \) at every critical point \( a \) induces an orientation of \( W_a^+ \). The set of Morse functions for which the stable and unstable manifolds intersect transversely is dense in \( C^\infty(M) \). Thus, we may generically assume that all \( W_a^- \) and \( W_a^+ \) intersect transversely. The dimension of \( W_a^- \) equals the Morse index \( \mu(a) \) of \( a \), and the dimension of the intersection

\[
M(a, a') \equiv W_a^- \cap W_{a'}^+
\]

is given by \( \max\{\mu(a) - \mu(a'), 0\} \).
In order to define the coboundary operator, let us consider pairs of critical points \(a\) and \(a'\), whose relative Morse index has the value 1, say \(\mu(a) = p + 1\) and \(\mu(a') = p\). It immediately follows that \(M(a,a')\) is a finite collection of gradient lines that connect \(a\) with \(a'\).

The intersection of \(M(a,a')\) with every regular level surface \(\Sigma_c\) of \(f\) with \(f(\Sigma_c) = c\) lying between \(f(a)\) and \(f(a')\) is transverse, and consists of a finite collection of isolated points. The hypersurface \(\Sigma_c\), being a level surface of \(f\), is orientable, so we pick the orientation, which, combined with the section \(\nabla_g f\) of its normal bundle, shall agree with the orientation of \(M\).

In addition, the intersection both of \(W^-a\) and \(W^+a'\) with \(\Sigma_c\) is transverse, and the submanifolds
\[
W^-a, c \equiv W^-a \cap \Sigma_c \\
W^+a', c \equiv W^+a' \cap \Sigma_c
\]
of \(\Sigma_c\) are smooth, compact and closed. In addition, their dimensions add up to the dimension of \(\Sigma_c\). To every point \(b\) of the set \(M(a,a') \cap \Sigma_c = W^-a, c \cap W^+a', c\), one assigns the number \(\gamma(b) = 1\) if the induced orientation of
\[
T_b \Sigma_c = T_b W^-a, c \oplus T_b W^+a', c
\]
agrees with the one picked for \(\Sigma_c\), and \(\gamma(b) = -1\) otherwise.

**Definition II.7** The sum
\[
\langle a, \delta a' \rangle \equiv \sum_{b \in M(a,a') \cap \Sigma_c} \gamma(b)
\]
is the intersection number \(z(W^-a, c, W^+a', c)\) of the oriented submanifolds \(W^-a, c\) and \(W^+a', c\) of \(\Sigma_c\), \([20]\). If for a pair \(a\) and \(a'\) of critical points with a relative Morse index 1, this intersection number is nonzero, they will be said to be effectively connected (by gradient lines).

**Definition II.8** The coboundary operator of the Morse-Witten complex is the \(\mathbb{Z}\)-linear map \(\delta : C^p \rightarrow C^{p+1}\) defined by
\[
\delta a' = \sum_{\mu(a') = p+1} \langle a, \delta a' \rangle a.
\]

The coboundary operator satisfies \(\delta^2 = 0\), and its cohomology is isomorphic to the de Rham cohomology of \(M\). Proofs of this statement can be found in \([1, 17, 35]\). The image of a critical point of Morse index \(p\) under the coboundary map consists of the critical points of Morse index \(p + 1\) to which it is effectively connected. One also refers to the intersection number \(\langle a, \delta a' \rangle\) as the matrix element of the \(\delta\)-operator with respect to the elements \(a\) and \(a'\) of \(C\).

The existence of the Morse-Witten complex straightforwardly implies the strong Morse inequalities, as one deduces from the following fact. Let us denote the \(p\)-th cocyle group by \(Z^p \subset C^p\), which is defined as the intersection of \(\ker \delta\) with \(C^p\), and let \(B^p \subset C^p\), the \(p\)-th coboundary group, denote the image of \(C^{p-1}\) under \(\delta\). Clearly, \(B^p\) is a subset of \(Z^p\) because \(\delta\) is nilpotent, therefore the set \(H^p = Z^p \setminus B^p\) is well-defined. It is the \(p\)-th cohomology group of the Morse-Witten complex. Since by theorem \([17, 2]\), the cohomology of the Morse-Witten
complex is isomorphic to the de Rham cohomology of $M$, the rank of $H^p$ coincides with the $p$-th Betti number $B_p(M)$ of $M$.

The image of $C^p$ in $C^{p+1}$ under $\delta$ is given by $B^{p+1}$. Denoting the preimage of $B^{p+1}$ in $C^p$ by $\delta^{-1}(B^{p+1})$, which is isomorphic to $B^{p+1}$, one has

$$C^p = H^p \oplus B^p \oplus \delta^{-1}(B^{p+1}),$$

so that

$$\dim C^p = B_p(M) + \dim B^p + \dim B^{p+1}.$$

The dimension of $C^p$ equals the number $N_p$ of critical points of $f$ with a Morse index $p$. Multiplying both sides of the equality sign with $\lambda^p$, and summing over $p$, one finds

$$\sum \lambda^p N_p = \sum \lambda^p B_p(M) + \sum \lambda^p (\dim B^p + \dim B^{p+1}) = \sum \lambda^p B_p(M) + (1 + \lambda) \sum \lambda^{p-1} \dim B^p,$$

(27)

(notice here that both $B^0$ and $B^{2n+1}$ are empty). These are the strong Morse inequalities, and the polynomial $Q(t)$ defined at the beginning of this section now has a very definitive interpretation:

$$Q(\lambda) = \sum \lambda^{p-1} \dim B^p.$$

Evidently, $\dim B^p$ is the number of critical points of Morse index $p$ that are effectively connected to critical points of Morse index $p - 1$ via gradient lines of $f$.

**Comparing the Morse-Witten complexes of $(M, H)$ and $(\mathcal{C}_{gen}, H|_{\mathcal{C}_{gen}})$**. Let us next relate the Morse-Witten complexes of $(M, H)$ and $(\mathcal{C}_{gen}, H|_{\mathcal{C}_{gen}})$ to each other. To this end, it is necessary to recall that the generic connectivity components $\mathcal{C}_{gen}$ of $\mathcal{C}$ contain all critical points of $H$, but no other conditional extrema, and that they are necessarily normal hyperbolic with respect to $\phi_t$.

Let $A_i := \{a_{i,1}, \ldots, a_{i,m}\}$ denote the set of critical points of $H$ that are contained in $\mathcal{C}_i$, and let $\mu(a_{i,r})$ be the associated Morse indices of $H : M \to \mathbb{R}$. Furthermore, let $H_i \equiv H|_{\mathcal{C}_i}$ denote the restriction of the Hamiltonian to $\mathcal{C}_i$. We have previously shown that the map $H_i : \mathcal{C}_i \to \mathbb{R}$ is a Morse function, whose critical points are precisely the elements of $A_i$. Furthermore, it is clear that the number of negative eigenvalues of the Hessian of $H$ at any element of $A_i$, whose eigenspaces are normal to $\mathcal{C}_i$, equals $\mu(\mathcal{C}_i)$, the index of $\mathcal{C}_i$.

The Morse index of $a_{i,r}$ with respect to $H_i$ is thus $\mu(a_{i,r}) - \mu(\mathcal{C}_i)$. The Morse-Witten complex associated to $\mathcal{C}_i$ is defined in terms of the free $\mathbb{Z}$-module generated by the elements of $A_i$, which is graded by the Morse indices $p$ of the critical points of $H_i$.

$$C_i = \oplus_p C_i^p.$$

To define the coboundary operator $\delta_i : C_i^p \to C_i^{p+1}$, one uses the gradient flow on $\mathcal{C}_i$ generated by $H_i$. One then concludes that

$$\ker \delta_i / \text{im} \delta_i \cong H^*(\mathcal{C}_i, \mathbb{Z}).$$

(28)
Application of (27) shows that for every $C_i \in \mathcal{C}_{\text{gen}}$,

$$
\sum_p \lambda^p N_{i,p} = \sum_p \lambda^p B_p(C_i) + (1 + \lambda) \sum_p \lambda^{p-1} \dim B_i^p,
$$

(29)

where $B_i^p$ is the $p$-th coboundary group of the Morse-Witten complex of $C_i$, and $N_{i,p}$ is the number of critical points of $H_i$ on $C_i$ whose Morse index is $p$.

Since every critical point of $H$ lies on precisely one generic component $C_i$, the number $N_q$ of critical points of $H$ with a Morse index $q$ is given by

$$
N_p = \sum_i N_{i,p-\mu(C_i)}.
$$

Multiplying both sides of (29) with $\lambda^{\mu(C_i)}$, and summing over $i$, one obtains

$$
\sum_{i,p; \mathcal{C}_i \in \mathcal{C}_{\text{gen}}} \lambda^{\mu(C_i)+p} N_{i,p} = \sum_{i,p; \mathcal{C}_i \in \mathcal{C}_{\text{gen}}} \lambda^{\mu(C_i)+p} B_p(C_i) + (1 + \lambda) \sum_{\mathcal{C}_i \in \mathcal{C}_{\text{gen}}} \lambda^{\mu(C_i)+p-1} \dim B_i^p,
$$

which becomes, after reindexing $\mu(C_i) + p \to q$,

$$
\sum_q \lambda^q N_q = \sum_{i,q; \mathcal{C}_i \in \mathcal{C}_{\text{gen}}} \lambda^q B_{q-\mu(C_i)}(C_i) + (1 + \lambda) \sum_{i,q; \mathcal{C}_i \in \mathcal{C}_{\text{gen}}} \lambda^{q-1} \dim B_i^{q-\mu(C_i)}.
$$

Combining this result with the strong Morse inequalities

$$
\sum_q \lambda^q N_q = \sum_q \lambda^q B_q(M) + (1 + \lambda) \sum_q \lambda^{q-1} \dim B^q,
$$

one obtains that

$$
\sum_{i,p; \mathcal{C}_i \in \mathcal{C}_{\text{gen}}} \lambda^q B_{q-\mu(C_i)}(C_i) = \sum_q \lambda^q B_q(M)
$$

$$
+ (1 + \lambda) \sum_q \lambda^{q-1} \left( \dim B^q - \sum_{\mathcal{C}_i \in \mathcal{C}_{\text{gen}}} \dim B_i^{q-\mu(C_i)} \right).
$$

One observes that formula (24) implies that the polynomial which is multiplied by $(1 + \lambda)$ has non-negative integer coefficients.

Conversely, if one can prove that for all $q$,

$$
\dim B^q \geq \sum_{\mathcal{C}_i \in \mathcal{C}_{\text{gen}}} \dim B_i^{q-\mu(C_i)}
$$

(30)

holds, one would also obtain an alternative proof of (24). The main observation here is that the left hand side is defined in terms of the operator $\delta$ associated to $(M, H)$, while the right hand side is defined in terms of the operators $\delta_i$ associated to all $(\mathcal{C}_i, H_i)$ with $\mathcal{C}_i \in \mathcal{C}_{\text{gen}}$.

The quantity $\dim B_i^{q-\mu(C_i)}$ denotes the number of critical points of $H$ with a Morse index $q$ in $\mathcal{C}_i$, which are effectively connected to critical points of Morse index $p + 1$ in $\mathcal{C}_i$. 

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via gradient lines of the Morse function \( H \) on \( \mathcal{C}_i \). Therefore, the sum on the right hand side of (30) equals the number of those critical points of \( H \) with a Morse index \( q \), which are effectively connected to critical points of Morse index \( q + 1 \) via gradient lines of the functions \( H \circ j_i \) on all generic \( \mathcal{C}_i \); here, \( j_i : \mathcal{C}_i \to M \) is the inclusion map.

**Proof of inequality (30).** We will now prove inequality (30) by relating the coboundary operators of the Morse-Witten complexes of \((M, H)\) and \((\mathcal{C}_{\text{gen}}, H|_{\mathcal{C}_{\text{gen}}})\) to each other. Apart from the fact that this will establish a different proof of the Conley-Zehnder inequalities, this will also largely clarify the orbit structure of the auxiliary gradient-like system.

**Proof strategy.** We will construct a particular homotopy of vector fields \( v_s \), with \( s \in [0, 1] \), that generate gradient-like flows. Their zeros will be independent of \( s \), and hyperbolic. The vector fields interpolated by \( v_s \) are, for \( s = 1 \), \( v_1 = \nabla g H \), so that the zeros of \( v_s \) are precisely the critical points of \( H \), and for \( s = 0 \), \( v_0 \) is a vector field that is tangent to every \( \mathcal{C}_{\text{gen}} \). For every \( s \in [0, 1] \), we will construct a coboundary operator via the one-dimensional integral curves of \( v_s \) that connect its zeros. These coboundary operators are independent of \( s \), and act on the free \( \mathbb{Z} \)-module \( \mathcal{C} \) of the Morse-Witten complex associated to \((M, H)\). The desired estimate (30) then follows from a simple dimension argument.

**Construction of \( v_0 \).** We require \( v_0 \) to be gradient-like, and tangent to \( \mathcal{C}_{\text{gen}} \). Furthermore, the zeros of \( v_0 \) shall be hyperbolic, and shall coincide with the critical points of \( H \). Consequently, the dimension of any unstable manifold of the flow generated by \(-v_0\) equals the Morse index of the critical point of \( H \) from which it emanates.

To this end, we use the vector field \( X_\epsilon \) constructed in the proof of proposition \([II.6]\). We recall that it has been obtained by suitably deforming \( \pi V \nabla g H \) such that all elements \( \mathcal{C}_i \) of \( \mathcal{C} \setminus \mathcal{C}_{\text{gen}} \) are removed. Furthermore, let us introduce compact \( \epsilon \)-neighborhoods \( U_\epsilon(\mathcal{C}_{\text{gen}}) \) of the generic connected components of \( \mathcal{C} \) in the way demonstrated for (23).

It is now possible to extend the projector \( \bar{Q} : T\mathcal{C}_{\text{gen}} M \to T\mathcal{C}_{\text{gen}} \) that has been introduced in \([II.2.1]\) over the whole embedding space \( T\bar{U}_\epsilon(\mathcal{C}_{\text{gen}}) \). To this end, we pick an arbitrary smooth distribution \( W \) over the base manifold \( \bar{U}_\epsilon(\mathcal{C}_{\text{gen}}) \), whose fibres over \( \mathcal{C}_{\text{gen}} \) shall coincide with the corresponding fibres of \( T\mathcal{C}_{\text{gen}} \). Admitting a slight abuse of notation, we let \( \bar{Q} \) denote the \( g \)-orthogonal projector \( TM \to W \). Clearly, evaluating \( \bar{Q} \) in any \( a \in \mathcal{C}_{\text{gen}} \) gives the projector \( Q_a : T_a M \to T_a \mathcal{C}_{\text{gen}} \) discussed in \([II.2.1]\). Because \( W \) is smooth, \( \bar{Q} \) and its orthogonal complement \( Q \) are both smooth tensor fields.

We define the vector field \( v_0 \) by requiring that it shall equal \( X_\epsilon \) in \( M \setminus U_\epsilon(\mathcal{C}_{\text{gen}}) \), and that for \( x \) in \( U_\epsilon(\mathcal{C}_{\text{gen}}) \), it shall be given by

\[
v_0(x) \equiv (\pi V \nabla g H)(x) + h(x)(\bar{Q}\nabla g H)(x),
\]

where \( h : U_\epsilon(\mathcal{C}_{\text{gen}}) \to [0, 1] \) is a smooth function obeying \( h|_{\mathcal{C}_{\text{gen}}} = 1 \) and \( h|_{\partial U_\epsilon(\mathcal{C}_{\text{gen}})} = 0 \). In particular, \( h \) shall be strictly monotonic along all non-constant trajectories of the flow generated by \( \pi V \nabla g H \), and the one form \( dh \) shall vanish on \( \mathcal{C}_{\text{gen}} \).

It can now be easily verified that \( v_0 \) possesses all of the desired properties. That it generates a gradient-like flow can be seen from the fact that outside of \( U_\epsilon(\mathcal{C}_{\text{gen}}) \), \( g(\nabla g H, v_0) = g(\nabla g H, X_\epsilon) \) is strictly positive, as has been shown in the proof of proposition \([II.6]\). Inside of
the critical points of $H$ that the dimensions of the unstable manifolds are, for all $s \in \mathcal{C}$ due to the strictly decreasing along all non-constant orbits of $\psi$ everywhere on $C_v$ that the zeros of $m$ manner that $H$ except at the critical points of $H$.

The first term on the right hand side is obviously everywhere positive except at the critical points of $H$, and the same fact has been proved above for the second term. Furthermore, $H$ is strictly decreasing along all non-constant orbits of $\psi_{s,t}$, which proves that it is gradient-like.

Next, we address the proof that the zeros of $-v_0$ are hyperbolic, and that the number of negative eigenvalues of the Jacobian matrix at any zero equals the corresponding Morse index of $H$.

To this end, we pick a local chart at a critical point $a$ of $H$. The Jacobian matrix of $v_0$ at $a$ in this chart is given by

$$\text{Jac}_a(v_0) = \text{Jac}_a(\pi_V \nabla g H) + \text{Jac}_a(\bar{Q} \nabla g H) = I_{P_a} P_a (D_a^2 H)^{ij} + I_{Q_a} \bar{Q}_a (D_a^2 H)^{ij} = (D_a^2 H)^{ij} + (I_{P_a} P_a - I_{Q_a} Q_a) (D_a^2 H)^{ij}. \quad (31)$$

To explain this result, let us first of all notice that there is no dependency on the function $h$ because $dh|_{\mathcal{C}_\text{gen}}$ is zero. Furthermore, $(D_a^2 H)^{ij}$ is defined as the matrix $[g^{ij} H_{jk}|_a]$ in the given chart, and $P_a$ denotes the matrix of $\pi_V(a)$. Linearizing the vector fields $\pi_V \nabla g H$ and $\bar{Q} \nabla g H$ at a critical point of $H$, all terms involving first derivatives of $H$ are zero. This explains the second line. The third line simply follows from $\bar{Q}_a = 1_{2n} - Q_a$.

Normal hyperbolicity follows from the invertibility of $\text{Jac}_a(v_0)$, which is a consequence of lemma [II.1] below.

**Definition of the homotopy of vector fields.** Let

$$v_s \equiv s \nabla g H + (1 - s) v_0$$

with $s \in [0, 1]$. We claim that for arbitrary $s$, $-v_s$ generates a gradient-like flow $\psi_{s,t}$, in a manner that $H$ is strictly decreasing along all non-constant orbits. Furthermore, we claim that the zeros of $v_s$ are hyperbolic fixed points of $\psi_{s,t}$ that do not depend on $s$. It then follows that the dimensions of the unstable manifolds are, for all $s$, given by the Morse indices of the critical points of $H$ from which they emanate.

To prove these claims, we consider the scalar product

$$g(\nabla g H, v_s) = s \| \nabla g H \|^2_g + (1 - s) g(\nabla g H, v_0).$$

The first term on the right hand side is obviously everywhere positive except at the critical points of $H$, and the same fact has been proved above for the second term. Thus, $H$ is strictly decreasing along all non-constant orbits of $\psi_{s,t}$, which proves that it is gradient-like.
The Jacobian of $v_s$ at a critical point of $H$ is given by
\[
\text{Jac}_a(v_s) = s(D^2_v H)^s + (1 - s)\text{Jac}_a(v_0) = (D^2_v H)^s + (1 - s)(I_{P,a} P_a - I_{Q,a} Q_a)(D^2_v H)^s = (1 + (s - 1) s I_{P,a} P_a - I_{Q,a} Q_a)(D^2_v H)^s.
\]

If we can prove that $\text{Jac}_a(v_s)$ is invertible for all $s \in [0, 1]$, it follows that the number of negative eigenvalues is independent of $s$. To prove that this is indeed the case, we observe that because $(D^2_v H)^s$ is invertible, one merely has to show that $(1 + (1 - s)(I_{P,a} P_a - I_{Q,a} Q_a))$ is invertible. This in turn is satisfied if $(I_{P,a} P_a - I_{Q,a} Q_a)$ has no eigenvalues in $[1, \infty) \subset \mathbb{R}$.

**Lemma II.1** The spectrum of $I_{P,a} P_a - I_{Q,a} Q_a$ has empty intersection with $[1, \infty)$.

**Proof.** By arguing by contradiction. Since $I_{P,a} P_a - I_{Q,a} Q_a$ is selfadjoint with respect to $g_a$, it is diagonalizable, and has a real spectrum. Let us assume that $\kappa \in [1, \infty)$ is an eigenvalue associated to the eigenvector $w \in T_a M$, so that
\[
(I_{P,a} P_a - I_{Q,a} Q_a)w = \kappa w. \tag{32}
\]

We multiply both sides of the equality sign with $P_a I_{Q,a} Q_a$ from the left, and get the equation $P_a I_{Q,a} Q_a Q_a I_{P,a} P_a w = (1 - \kappa) P_a I_{Q,a} Q_a w$. On the other hand, multiplication from the left with $P_a$ gives $P_a I_{Q,a} Q_a w = (1 - \kappa) P_a w$. Combining these two results, one obtains $P_a I_{Q,a} Q_a Q_a I_{P,a} P_a w = (1 - \kappa^2) P_a w$, which shows that $P_a w$ is an eigenvector of $P_a I_{Q,a} Q_a Q_a I_{P,a} P_a : V_a \to V_a$ that belongs to the eigenvalue $(1 - \kappa^2)$. We have proved earlier that normal hyperbolicity of $\mathcal{C}_{\text{gen}}$ implies that $P_a I_{Q,a} Q_a I_{P,a} P_a$ is invertible on $V_a$. In addition, it is evident that $(1 - \kappa^2) \leq 0$.

Let us assume that $P_a w = 0$. Then, $g_a(w, P_a I_{Q,a} Q_a I_{P,a} P_a w) = g_a(Q_a I_{P,a} P_a w, Q_a I_{P,a} P_a w) > 0$ follows from the $g$-orthogonality of the projectors. On the other hand, we also have $g_a(w, P_a I_{Q,a} Q_a I_{P,a} P_a w) = (1 - \kappa^2) g_a(w, P_a w) = (1 - \kappa^2) g_a(P_a w, P_a w) \leq 0,$ which is a contradiction. Hence, $P_a w = 0$. In this case, $(32)$ reduces to $I_{Q,a} Q_a w = \kappa w$, and multiplication with $Q_a I_{P,a} P_a$ from the left gives $Q_a I_{P,a} P_a I_{Q,a} Q_a w = \kappa Q_a I_{P,a} P_a w = 0$. Because $Q_a I_{P,a} P_a I_{Q,a} Q_a$ is invertible on $N_{\text{gen}}$, this implies that $Q_a w$ is zero. Therefore, $w$ is not contained in the intersection of the images of $I_{P,a} P_a$ and $I_{Q,a} Q_a$. However, being an eigenvector that solves $(32)$, it must be contained in this space, which is a contradiction. 

The conclusion is that for all $s \in [0, 1]$, the zeros of $v_s$ are hyperbolic fixed points of $\psi_{s,t}$, and that the dimensions of the corresponding unstable manifolds are given by the respective Morse indices of $H$.

Since $\partial_s \psi_{s,t} = -v_s$ depends smoothly on $s$, hence $\psi_{s,t}$ is $C^\infty$ in $s$. Thus, $s$ smoothly parametrizes a homotopy of stable and unstable manifolds emanating from the critical points of $H$, which belong to the gradient-like flow $\psi_{s,t}$.
Because the fixed points of $\psi_{s,t}$ are independent of $s$, and since the corresponding dimensions of the unstable manifolds coincide with the Morse indices of the critical points of $H$, we again consider the free $\mathbb{Z}$-module

$$C = \oplus_p C^p$$

that is generated by the critical points of $H$, and graded by their Morse indices.

For every fixed $s$, we define a coboundary operator on $C$, using the flow $\psi_{s,t}$. In fact, picking a pair of critical points of $H$ with a relative Morse index 1, consider the unstable manifold $W^-_{s,a}$ of $a$, and the stable manifold $W^+_{s,a'}$ of $a'$ associated to $\psi_{s,t}$, which are both smoothly parametrized by $s$. Since $s$ parametrizes a homotopy of such manifolds, they naturally inherit an orientation from the one picked for $s = 1$ in the definition of the Morse-Witten complex for $(M,H)$.

Let $\Sigma_E$ denote regular energy surface for an arbitrary energy value $E$ between $H(a)$ and $H(a')$. The intersection of $W^\pm_{s,a}$ with any regular energy level surfaces $\Sigma_E$ of $H$ is transverse, because $H$ strictly decreases along all non-constant orbits generated by $-v_s$.

$W^-_{s,a} \cap \Sigma_E$ and $W^+_{s,a'} \cap \Sigma_E$ are oriented submanifolds of $\Sigma_E$, and smoothly parametrized by $s$. Hence, they define two homotopies of manifolds in $\Sigma_E$. Their intersection number, being a homotopy invariant, is independent of $s$, hence it equals the value obtained in case of $s = 1$. This implies that the coboundary operators obtained for arbitrary $s$ are identical to the $\delta$-operator of the Morse-Witten complex given for $s = 1$, since their respective matrix elements are equal.

To finally prove (30), we use the fact that all stable and unstable manifolds of $\psi_{0,t}$ are, by construction, either confined to some $\mathcal{C}_i$, or otherwise, that they connect critical points lying on different $\mathcal{C}_i$‘s. This is because the stable and unstable manifolds of the flow generated by $\pi_V \nabla g H$ only connect different connectivity components of $\mathcal{C}_{gen}$.

Let us next consider pairs of critical points of $H$ with a relative Morse index 1 that lie on the same component $\mathcal{C}_i \in \mathcal{C}_{gen}$, and the corresponding stable and unstable manifolds of $\psi_{0,t}$ which are contained in $\mathcal{C}_i$. Since $v_0|_{\mathcal{C}_i}$ simply is the projection of $\nabla g H|_{\mathcal{C}_i}$ to $T\mathcal{C}_i$, these stable and unstable manifolds are precisely those which were used to define the Morse-Witten complex on $(\mathcal{C}_i, H_i)$.

Picking only the stable and unstable manifolds of $\psi_{0,t}$ contained in $\mathcal{C}_{gen}$, we construct an operator $\tilde{\delta}$ acting on $C$ in the same manner in which the coboundary operator was defined. It is again a coboundary operator, but now it is given by

$$\tilde{\delta} \equiv \oplus_i \delta_i.$$

$\delta_i$ denotes the coboundary operator of the Morse-Witten complex associated to the pair $(\mathcal{C}_i, H_i)$.

Finally, we denote by $P_i : C \to C_i$ the projection of the free $\mathbb{Z}$-module $C$ generated by all critical points of $H$ to the one generated by those critical points which lie in $\mathcal{C}_i$. The above construction makes it evident that removing all integral lines of $-v_0$ that connect critical points on different connectivity components of $\mathcal{C}_{gen}$, one arrives at $\delta_i = P_i \delta P_i$, so that

$$\tilde{\delta} = \oplus_i P_i \delta P_i.$$
(note that $\delta$ can be written as $\delta = \oplus_i \delta P_i$). Inclusion of the missing integral lines would yield $\delta$, as our homotopy argument has proved. This immediately makes the inequality

$$\dim(\text{im}\delta|_{C^p}) \geq \dim(\text{im}\tilde{\delta}|_{C^p})$$

clear. We observe that this is precisely what is expressed in the inequality (30). ☐
III  STABILITY CRITERIA FOR FIXED POINTS

In this section, we will be interested in formulating stability criteria for equilibrium solutions of the constrained Hamiltonian system \((M, \omega, H, V)\). In case of exponential (in)stability, the discussion is elementary, and the results are standard.

However, in the marginally stable case, in which the linearized dynamics is oscillatory in nature, it is much harder to arrive at stability criteria if \(V\) is not integrable (the integrable case is again not interesting for us).

We will give a heuristic line of arguments that supports a certain stability criterion for this situation. It will be obtained from an elementary application of averaging theory, and involves an incommensurability condition imposed on the frequencies defined by the linearized problem. Using a perturbation expansion that is adapted to the flag of \(V\), we will argue why this condition, which could merely be an artefact of the averaging method, can presumably not be dropped.

A rigorous proof of the conjectured stability criterion is far beyond the scope of this text, and is presumably at least as hard as proofs in KAM and Nekhoroshev theory.

III.1 Results of averaging

For the discussion of stability, we again recall use auxiliary Kähler metric \(g\) on \(M\), and denote the induced Riemannian distance function by \(\text{dist}_R\) (in contrast to the Carnot-Caratheodory distance function \(d_{C-C}\) induced by \(g\), which will be considered later).

**Definition III.1** A point \(x_0 \in \mathcal{C}\) is stable if there exists \(\delta(\epsilon) > 0\) for every \(\epsilon > 0\), so that for all \(t\), \(\text{dist}_R(\Phi_t(x), x_0) < \epsilon\) holds for all \(x\) with \(\text{dist}_R(x, x_0) < \delta(\epsilon)\).

Let \(x_0 \in \mathcal{C}_{\text{gen}}\), and pick some small neighborhood \(U(a) \subset M\) together with an associated Darboux chart, with its origin at \(x_0\). The equations of motion are given by

\[
\partial_t x = P(x) J H_{,x}(x) = X_H^V(x),
\]

(33)

where \(x = (x^1, \ldots, x^n, x_{n+1}, \ldots, x_{2n})\), and \(J\) is the symplectic standard matrix. Furthermore, \(H_{,x}\) abbreviates \(\partial_x H\), and \(P\) is the matrix of the projector \(\pi_V\). \(\omega\)-orthogonality of \(\pi_V\) translates into \(P(x) J X(x) = J P^t(x) X(x)\) for all vector fields \(X\).

We will next bring the equations of motion in the vicinity of 0, corresponding to the point \(x_0\), into standard form. To this end, let

\[ u := \bar{P}_0 x \quad \bar{y} := P_0 x \]

denote new coordinates, picked to be mutually orthogonal with respect to the Kähler metric \(g|_0\) in \(\mathbb{R}^{2n} \cong T_0 M\).

**Lemma III.1** Locally, \(\mathcal{C}_{\text{gen}}\) is the graph of a \(C^2\) function \(F : \bar{P}_0 \mathbb{R}^{2k} \to \mathbb{R}^{2k}, u \mapsto F[u]\).
Proof. The images of $P_0$ and $T_0\mathfrak{C}_{gen}$ together span $T_0M \cong \mathbb{R}^{2n}$. To see this, let $\tilde{Q}_0$ again denote the orthoprojector $T_0M \rightarrow T_0\mathfrak{C}_{gen}$. The claim is then implied by the fact that the matrix $I_{P_0}P_0 + I_{\tilde{Q}_0}\tilde{Q}_0 = 1_{2n} + (I_{P_0}P_0 - I_{\tilde{Q}_0}\tilde{Q}_0)$ is invertible. The latter has already been proved above in the discussion of the induced Morse-Witten complexes on $\mathfrak{C}_i$. Thus, the rank of $P_0I_{\tilde{Q}_0}\tilde{Q}_0 = P_0(I_{P_0}P_0 + I_{\tilde{Q}_0}\tilde{Q}_0)$ equals the rank of $P_0$, namely $2(n - k)$, hence $P_0$ projects $T_0M$ surjectively onto $T_0\mathfrak{C}_{gen}$. Consequently, there is a linear map $G : P_0\mathbb{R}^{2k} \rightarrow \mathbb{R}^{2k}$ for which $T_0M$ is the graph $(u, G(u))$. $\mathfrak{C}_{gen}$ thus admits a local parametrization $(u, F[u])$ with $F(u) = G(u) + O(\|u\|^2)$, for sufficiently small $\|u\|$, which is $C^2$ since $\mathfrak{C}_{gen}$ is assumed to be smooth. \[\blacksquare\]

Proposition III.1 The equations of motion are, in the coordinates $(y, z)$, given by

\begin{align*}
\partial_t y(t) &= \Xi_0(y(t)) + Y[z(t), y(t)] \\
\partial_t z(t) &= Z[z(t), y(t)],
\end{align*}

where $|Y[z, y]|, |Z[z, y]| = O(\|y\| \|z\|) + O(\|y\|^2)$, and $\Xi_0 := P_0DX_H^V[0, 0]I_{P_0}$.

Proof. We introduce the function $y[\tilde{y}, u] := \tilde{y} - F[u]$, which defines the coordinate transformation $\Phi : (u, \tilde{y}) \mapsto (z, y)$. The Jacobi matrix of its inverse $\Phi^{-1}$ at $(z, y)$ is

$$ D\Phi^{-1}[z, y] = \begin{bmatrix} \tilde{P}_0 + DF[u(z, y)]P_0 & 0 \\ -DF[u(z, y)]P_0 & P_0 \end{bmatrix}. $$

From the definition of $F$ follows that $P_0DF[u] = DF[u]$. Thus, the equations of motion are now represented by

$$ \partial_t z = (\tilde{P}_0 + DF[u(z, y)]P_0) \ X[z, y] \\
\partial_t y = (P_0 - DF[u(z, y)]P_0) \ X[z, y], $$

where $X[z, y] \equiv X_H^V(\Phi^{-1}(z, y))$. In this chart, $(z, 0)$ parametrizes $\mathfrak{C}_{gen}$, thus, by definition of $\mathfrak{C}_{gen}$, $X[z, 0] = 0$ for all $z$. Taylor expansion of $\Phi^{-1}$ relative to $(0, 0)$ gives

$$ \Phi^{-1}[z, y] = \Phi^{-1}[0, 0] + D\Phi^{-1}[0, 0] \begin{bmatrix} z \\ y \end{bmatrix} + O(\|x\|^2). $$

Because of $F(0) = 0, \Phi^{-1}(0, 0) = (0, 0)$. Taylor expansion of $X_H^V(u, \tilde{y})$ relative to $(0, 0)$ yields

$$ X_H^V[y, z] = DX_H^V[0, 0] \begin{bmatrix} u \\ \tilde{y} \end{bmatrix} + \tilde{R}[u, \tilde{y}], $$

where $\tilde{R}[u, \tilde{y}]$ is a quadratic remainder term. Consequently,

$$ X[z, y] = DX_H^V[0, 0] \ D\Phi^{-1}[0, 0] \begin{bmatrix} z \\ y \end{bmatrix} + R[y, z] $$
with a quadratic remainder term \( R[y, z] \). It follows from \( \bar{P}_0 DX_H^V(0, 0) = 0 \) that (33) holds for
\[
Z[z, y] = (\bar{P}_0 + DF[u(z, y)]\bar{P}_0) R[z, y].
\]

Furthermore,
\[
\partial_t y = P_0 DX_H^V[0, 0] I_{P_0} y
- P_0 DX_H^V[0, 0] DF[u(z, y)] \begin{pmatrix} z \\ y \end{pmatrix} + \bar{Y}[z, y],
\]
where \( \bar{Y}[z, y] \) is a quadratic remainder terms. Finally, the kernel of \( DX_H^V[0, 0] \) is the tangent space \( T_{x_0}C_{\text{gen}} \), which is also the image of \( DF[0] \), thus
\[
\partial_t y = P_0 DX_H^V[0, 0] I_{P_0} y + Y[z, y],
\]
with a quadratic Taylor remainder term \( Y[z, y] \). Clearly, \( DX_H^V[0, 0] = J I_{P_0} A_0 \).

**Asymptotic (in)stability.** A standard application of the center manifold theorem shows that if the spectrum of \( \Xi_0 \) does not intersect \( i\mathbb{R} \), there is a coordinate transformation \((y, z) \to (\bar{y}, \bar{z})\), so that (40) and (38) can be written as
\[
\partial_t \bar{y}(t) = \Xi_0 \bar{y}(t) + \bar{Y}[\bar{y}(t), \bar{z}(t)]
\]
\[
\partial_t \bar{z}(t) = 0
\]
where \( \bar{Y}(0, \bar{z}) = 0 \) for all \( \bar{z} \). Thus, \( x_0 \in C_{\text{gen}} \) is asymptotically unstable if there are eigenvalues with a positive real part, and asymptotically stable if all eigenvalues have a negative real part.

In case of integrable \( V \), asymptotic stability is impossible, because the eigenvalues always come in pairs or quadruples with both positive and negative real parts. However, if \( V \) is nonintegrable, there is to the author’s knowledge no obstruction to the existence of asymptotically stable equilibria, as the flow map is not symplectic.

**Marginal stability.** The case of marginal stability is given when \( \text{spec}(\Xi_0) \subset i\mathbb{R} \setminus \{0\} \).

We will in this case conjecture a stability criterion, which is motivated by heuristic results obtained from averaging theory on the one hand, and from a perturbation expansion that is adapted to the flag of \( V \), on the other hand. A rigorous stability analysis is beyond the scope of this text.

To this end, let us assume that \( \Xi_0 \) is diagonalizable over \( \mathbb{C} \). To diagonalize it, we assume that the vector fields \( Y[y, z] \) and \( Z[y, z] \) in (40) and (38) are analytic in \((y, z)\), so that they possess a unique analytical continuation into a complex vicinity of \( x_0 \in C_{\text{gen}} \).

The system of ordinary differential equations (40) and (38) is then interpreted in the sense that \((y, z)\) is a vector in \( \mathbb{C}^{2k} \times \mathbb{C}^{2n-2k} \) (of small norm). The continuation of \( C_{\text{gen}} \) into \( \mathbb{C}^{2n} \) is defined by the common zeros of \( Y[0, z] \) and \( Z[0, z] \) for \( z \in \mathbb{C}^{2(n-k)} \).
Let \( \text{spec}(\Xi_0) = \{i\omega_1, \ldots, i\omega_{2k}\} \), with \( \omega_i \in \mathbb{R} \). There exists a linear transformation
\[
\Psi : \mathbb{C}^{2n} \to \mathbb{C}^{2n},
\]
such that \( \Xi_0 \) is diagonal in the new coordinates, which, by abuse of notation, we denote again by \((y, z)\). The equations of motion then reduce to
\[
\begin{align*}
\partial_t y(t) &= \text{diag}(i\omega) y(t) + Y[y(t), z(t)] \\
\partial_t z(t) &= Z[y(t), z(t)],
\end{align*}
\]
(42)
where \( \omega \) denotes the vector \((\omega_1, \ldots, \omega_{2k})\).

Let us next introduce polar coordinates \((I, \phi)\) and \((J, \theta)\) in terms of
\[
\begin{align*}
y^r &= e^{i\phi} I^r \\
z^s &= e^{i\theta} J^s
\end{align*}
\]
with \( r = 1, \ldots, 2k \) and \( s = 1, \ldots, 2n - 2k \). In particular, \( I \in \mathbb{R}^{2k} \), \( J \in \mathbb{R}^{2n-2k} \), \( \phi \in [0, 2\pi]^{2k} = T^{2k} \) (the 2k-dimensional torus), and \( \theta \in [0, 2\pi]^{2n-2k} = T^{2n-2k} \). For brevity, vectors \((e^{i\phi} v)\) and \((e^{i\theta} w)\) will be denoted by \(e^{i\phi} v\) and \(e^{i\theta} w\), respectively, where \( v \in \mathbb{R}^{2k} \) and \( w \in \mathbb{R}^{2n-2k} \).

In polar coordinates, the complexified equations of motion for \( \dot{y} \) (the dot abbreviates \( \partial_t \)) are given by
\[
e^{i\phi} \dot{I} + i\dot{\phi} e^{i\phi} I = \text{diag}(i\omega) e^{i\phi} I + Y[e^{i\phi} I, e^{i\theta} J],
\]
(43)
so that
\[
\begin{align*}
\dot{I} &= \text{Re}{e^{-i\phi} Y[e^{i\phi} I, e^{i\theta} J]} \\
\dot{\phi} &= \omega + \text{Im}{e^{-i\phi} \text{diag}(\partial_I) Y[e^{i\phi} I, e^{i\theta} J]}.
\end{align*}
\]
(44)
In the same manner,
\[
\begin{align*}
\dot{J} &= \text{Re}{e^{-i\theta} Z[e^{i\phi} I, e^{i\theta} J]} \\
\dot{\theta} &= \text{Im}{e^{-i\theta} \text{diag}(\partial_J) Z[e^{i\phi} I, e^{i\theta} J]}.
\end{align*}
\]
(45)
\((I, J) \in \mathbb{R}^{2n}\) lies in a small vicinity of the origin.

Next, we fix a small parameter \( \epsilon := \|I(0)\| \), and require that \( \|J(0)\| \leq O(\epsilon^2) \). Then, we redefine the variables \( I \to \epsilon I \) and \( J \to \epsilon^2 J \) by rescaling. The new coordinates \((I, J)\) have a norm of the order \(O(1)\).

Analyticity of \(Y[y, z]\) and \(Z[y, z]\) in \((y, z)\) implies that the power series expansion with respect to \(e^{i\phi} I\) and \(e^{i\theta} J\) converges for all \((I, J) \in \mathbb{R}^{2n}\) sufficiently close to the origin. In this manner, (44) and (45) yield
\[
\begin{align*}
\dot{I}^r &= \sum_{|m|+|p| \geq 2} e^{(|m|+2|p|-1)F_{mp}(I, J)} e^{i(m, \phi)-\phi_r} e^{i(p, \theta)} \\
\dot{J}^s &= \sum_{|m|+|p| \geq 2} e^{(|m|+2|p|-2)G_{mp}(I, J)} e^{i(m, \phi)} e^{i(p, \theta)} \\
\dot{\phi}_r &= \omega_r + \sum_{|m|+|p| \geq 2} e^{(|m|+2|p|-1)\Phi_{r,mp}(I, J)} e^{i(m, \phi)-\phi_r} e^{i(p, \theta)} \\
\dot{\theta}_s &= \sum_{|m|+|p| \geq 2} e^{(|m|+2|p|-2)\Theta_{s,mp}(I, J)} e^{i(m, \phi)} e^{i(p, \theta)},
\end{align*}
\]
(46)
(47)
(48)
(49)
introducing the multiindices \( m \in \mathbb{Z}^{2k} \) and \( p \in \mathbb{Z}^{2n-2k} \), with \( |m| := \sum |m_r| \) and \( |p| := \sum |p_s| \).

Of course, the right hand sides here are simply the Fourier expansions with respect to the 2\( \pi \)-periodic angular variables \( \phi \) and \( \theta \). Every Fourier coefficient, labelled by a pair of indices \((m,p)\), is a homogenous polynomial of degree \(|m|\) in \( I \), and of degree \(|p|\) in \( J \).

In the limit \( \epsilon \to 0 \),
\[
\dot{I} = 0 \quad \dot{J} = 0 \quad \dot{\phi} = \omega \quad \dot{\theta} = 0.
\] (50)

Under the assumption that all components of \( \omega \) are rationally independent, averaging can be applied with respect to the variable \( \phi \) in order to obtain an approximation to the long time behaviour of the perturbed system.

The new, "averaged" variables are obtained via
\[
f_t(\phi) \rightarrow \bar{f}_t := \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} d^n\phi \ f_t(\phi).
\]

If the frequencies \( \omega_r \) satisfy certain incommensurability conditions, it can be proved that the long time behaviour of the system is very well approximated by the averaged solutions.

The only quantities in (41) \( \sim \) (43) that do not vanish under averaging with respect to \( \phi \) are the corresponding zero mode Fourier coefficients.

We recall that in the initial equations of motion (42), the functions \( Y[y,z] \) and \( Z[y,z] \) are at least \( O(\|y\|) \). Thus, the leading terms of their power series in \((y,z)\) are at least homogenous of degree 1 in \( y \), and thus involve terms \( \epsilon^{(m,\phi)} \) with \(|m| \geq 1 \), but no terms with \(|m| = 0 \). A quick inspection of the right hand sides of (44) and (45) assures us of the fact that (44) and (45) yield terms that survive the averaging process, but not (47) and (49).

Therefore, averaging the perturbed equations of motion (46) \( \sim \) (49) with respect to \( \phi \) gives
\[
\dot{I} = \epsilon^2 \bar{F}(I, J, \theta) \quad \dot{J} = 0 \quad \dot{\theta} = 0
\] (51)
for some function \( \bar{F} \), where the bars account for averaged variables.

Returning to the initial coordinate chart for real \((y,z)\), and the notation \( x_0 \) for the fixed point in discussion, let us now assume that the quadratic form on \( V_{x_0} \) defined by \( D_{x_0}^2 H|_{V_{x_0}} \) is positive definite. We claim that the averaging result then suggests that \( x_0 \) is stable. To this end, we notice that Taylor expansion of the Hamiltonian (which is an integral of motion for the constrained system) relative to \( x_0 \) gives
\[
H(x) = H(x_0) + u^i \partial_{x^i} H(x_0) + \frac{1}{2} y^i y^j \partial_{x^i} \partial_{x^j} H(x_0) + O(\epsilon^3),
\]
and recall that \( u = \bar{P}_{x_0} x \). Because of lemma \( \text{[III.1]} \) the assumption \( \|z\| \leq O(\epsilon^2) \) implies that \( \|u\| \leq O(\epsilon^2) \). The quantity \( H(x) - H(x_0) \sim O(\epsilon^2) \) is an integral of motion; thus, if the quadratic form on \( V_{x_0} \) defined by \( D_{x_0}^2 H \) is positive definite, \( \|y\| \) has an order of magnitude \( O(\epsilon) \) as long as \( \|u\| = O(\epsilon^2) \) remains valid. The averaged equations of motion imply that \( \|z\| = \|J\| = O(\epsilon^2) \) is time-independent, so that due to \( \bar{u} = \bar{P}_{x_0} z \), the same applies to \( \|\bar{u}\| \).

In conclusion, we conjecture the following stability criterion.

**Conjecture III.1** Let \( x_0 \in \mathfrak{c}_{\text{gen}}, \text{spec}(\Xi_0) = \{i\omega_1, \ldots, i\omega_{2k}\}, \) with \( \omega_i \in \mathbb{R} \setminus \{0\} \). Assume that (1) the frequencies \( \omega_r \) are rationally independent, and (2) that the quadratic form on \( V_{x_0} \) defined by \( D_{x_0}^2 H|_{V_{x_0}} \) is positive definite. Then, \( x_0 \) is stable.
III.2 Dynamics along the flag of V

Let us now approach the discussion of marginal stability from a different angle. In the subsequent paragraphs, we develop a geometrically invariant description of the local dynamics in the vicinity of \( C \) which is adapted to the flag of \( V \). Our motivation is to comment on the origin of the incommensurability condition imposed on the frequencies involved in the above stability criteria, and to give arguments why it can not be dropped.

Let us assume that \( x \) lies in a small open neighborhood \( U \) of \( x_0 \in C \), and that \( C_{\text{gen}} := C \cap U \) satisfies the genericity condition of theorem II.1.

Proposition III.2 Let \( C_{\text{gen}} = C \cap U \) have the genericity property formulated in theorem II.1. Then, there exists \( \epsilon > 0 \) such that every point \( x \in U \) with \( d_R(x, C_{\text{gen}}) < \epsilon \) is given by

\[
x = \exp_s Y(x_0) , \quad |s| < \epsilon
\]

for some \( Y \in \Gamma(V) \) with \( \|Y\|_{g_M} \leq 1 \), \( x_0 \in C_{\text{gen}} \) (\( \exp_s Y \) denotes the 1-parameter group of diffeomorphisms generated by \( Y \), with \( \exp_0 Y = \text{id} \)).

Proof. We pick a spanning family \( \{Y_i \in \Gamma(V)\}_{i=1}^{2k} \) of \( V \), with \( \|Y_i\|_{g_M} = 1 \). If for all \( x_0 \in C_{\text{gen}} \), \( T_{x_0}C_{\text{gen}} \) contains no subspace of \( V_{x_0} \), then

\[
\exp_1(t_1 Y_1 + \ldots + t_{2k} Y_{2k})(C_{\text{gen}}) \cap U
\]

is an open tubular neighborhood of \( C_{\text{gen}} \) in \( U \), for \( t_i \in (-\epsilon, \epsilon) \). Because the normal space \( N_{x_0}C_{\text{gen}} \) is dual to the span of the 1-forms \( dF_i \) at \( x_0 \), this condition is satisfied if and only if the matrix \( [dF_j(Y_i)] = [Y_i(Y_j(H))] \) is invertible everywhere on \( C_{\text{gen}} \). According to proposition II.1, this condition is indeed fulfilled. \( \blacksquare \)

Due to proposition II.2, there is an element \( Y \in \Gamma(V) \) with \( \|Y\|_{g_M} \leq 1 \), so that

\[
x = \Psi_\epsilon(x_0)
\]

for some \( 0 < \epsilon \ll 1 \). Since \( x_0 \in C_{\text{gen}} \), it is clear that under the flow generated by \( X_H^Y \), \( \Phi_{\pm t}(x_0) = x_0 \), thus the solution of (4) belonging to the initial condition \( x \) is given by

\[
\Psi_t(x_0) := \Phi_t \circ \Psi_\epsilon(x_0) = \left( \Phi_t \circ \Psi_\epsilon \circ \Phi_{-t} \right)(x_0).
\]

In particular, \( \Psi_t \) is the 1-parameter group of diffeomorphisms with respect to the variable \( \epsilon \) that is generated by the pushforward vector field

\[
Y_t(x) := \Phi_{t*}^* Y(x) = d\Phi_t \circ Y(\Phi_{-t}(x)), \quad (52)
\]

where \( d\Phi_t \) denotes the tangent map associated to \( \Phi_t \). This is a standard fact of differential geometry, cf. for instance [22]. From the group property \( Y_{s+t} = \Phi_{s*} Y_t \) follows that

\[
\partial_t Y_t = \partial_s |_{s=0} \Phi_{s*} Y_t = [X_H^Y, Y_t] \quad (53)
\]

holds everywhere in \( U \).
We recall from the beginning of the previous section that there exists a local spanning family \( \{ Y_i \in \Gamma(V) \}_{i=1}^{2k} \) for \( V \) that satisfies

\[
\omega(Y_i, Y_j) = \tilde{J}_{ij},
\]

with \( \tilde{J} := \begin{bmatrix} 0 & 1_k \\ -1_k & 0 \end{bmatrix} \). Furthermore, defining \( \theta_i(\cdot) := \omega(Y_i, \cdot) \),

\[
\pi_V = \tilde{J}^{ij} Y_i \otimes \theta_j,
\]

where \( \tilde{J}^{ij} \) are the components of \( \tilde{J}^{-1} = -\tilde{J} \). Finally,

\[
X^V_H = \pi_V(X^V_H) = -Y_i(H) \tilde{J}^{ij} Y_j
\]

in the basis \( \{ Y_i \}_{i=1}^{2k} \).

**Proposition III.3** Let \( f, F_i \in C^\infty(U) \), where \( F_i := Y_i(H), i = 1, \ldots, 2k \), and assume that \( F_i(\Psi^t_\epsilon(x_0)), f(\Psi^t_\epsilon(x_0)) \) are real analytic in \( \epsilon \). For \( X, Y \in \Gamma(TM) \), let

\[
\mathcal{L}_Y X = [Y, \ldots, [Y, X]]
\]

denote the \( r \)-fold iterated Lie derivative. Then, for sufficiently small \( \epsilon \),

\[
\partial_t f(\Psi^t_\epsilon(x_0)) = -F_i(\Psi^t_\epsilon(x_0)) \tilde{J}^{ik} \sum_{r \geq 0} \frac{\epsilon^r}{r!} (\mathcal{L}_{Y_i}^r Y_k) (f \circ \Psi^t_\epsilon)(x_0). \tag{54}
\]

**Proof.** Clearly,

\[
\partial_t f(\Psi^t_\epsilon(x_0)) = X^V_H(f)(\Psi^t_\epsilon(x_0)) = -F_i(\Psi^t_\epsilon(x_0)) \tilde{J}^{ik} Y_k(f)(\Psi^t_\epsilon(x_0)) = -F_i(\Psi^t_\epsilon(x_0)) \tilde{J}^{ik} (\Psi^t_\epsilon \ast Y_k)(f \circ \Psi^t_\epsilon)(x_0). \tag{55}
\]

Using the Lie series

\[
\Psi^t_\epsilon \ast Y_k = \sum_r \frac{\epsilon^r}{r!} \mathcal{L}_{Y_i}^r Y_k, \tag{56}
\]

we arrive at the assertion. \( \blacksquare \)

**Proposition III.4** Assume that \( Y_{t=0} \in \Gamma(V) \), and let \( \{ Y_j \}_{j=1}^{2k} \) be the given local spanning family of \( V \). Then, \( \mathcal{L}_{Y_{t=0}} Y_j \in \Gamma(V_i) \), where \( V_i \) is the \( i \)-th flag element of \( V \).

**Proof.** Because of \( \tilde{\Phi}_{t=0} : \Gamma(V) \to \Gamma(V), Y_t \) is a section of \( V \) for all \( t \) if it is for \( t = 0 \). The claim immediately follows from the definition of the flag of \( V \). \( \blacksquare \)
Proposition III.4 implies that there are functions $a^i(t, \cdot) \in C^\infty(U)$, $i = 1, \ldots, 2k$, so that

$$Y_t(x) = a^i(t, x)Y_i.$$  \hfill (57)

Their time evolution is governed by the following proposition.

**Proposition III.5** Let $Y_{t=0} = a^i_0 Y_i$ define the initial condition, and introduce the matrix

$$\Omega_x := [Y_i(F_i)(x) \tilde{J}^{ij}] .$$

Then, pointwise in $x$,

$$a^m(t, x) = (\exp(-t \Omega_x))^m_j \ a^j_0 + F_j(x) \ R^m_i(t, x) \ a^i_0 ,$$ \hfill (58)

where

$$R^m_i(t, x) := \tilde{J}^{jl} \tilde{J}^{nk} \int_0^t ds \ (\exp(-(t-s) \Omega_x))^m_k \ \omega\left([Y_l, \tilde{\Phi}_{s} Y_i], Y_n\right) .$$

**Proof.** The initial condition at $t = 0$ is given by $Y_0 = a^i_0 Y_i$, that is, by $a^i(0, x) = a^i_0$. Thus, by the definition of $Y_t$ in (52), one has $Y_t = a^i_0 \tilde{\Phi}_{t} Y_i$, so that

$$a^i(t, x) Y_i = a^i_0 \tilde{\Phi}_{t} Y_i .$$

From $\omega(Y_i, Y_j) = \tilde{J}_{ij}$, $\tilde{J}_{ik} = -\tilde{J}_{ki}$ and $\tilde{J}_{im} \tilde{J}_{jm} = -\delta_i^k$,

$$a^i(t, x) = -a^i_0 \omega\left(\tilde{\Phi}_{t} Y_i, Y_j\right) \tilde{J}^{jl} .$$

Now, taking the $t$-derivative on both sides of the equality sign, one finds

$$\partial_t a^m(t, x) = -a^i_0 \omega\left([X_H^{Y_i}, \tilde{\Phi}_{t} Y_i], Y_k\right) \tilde{J}^{km}$$

$$= -a^i_0 \left(\tilde{\Phi}_{t} Y_i\right)(F_j)(x) \tilde{J}^{jl} \omega(Y_i, Y_k) \tilde{J}^{km}$$

$$- a^i_0 F_j(x) \tilde{J}^{jl} \tilde{J}^{km} \omega\left([Y_l, \tilde{\Phi}_{t} Y_i], Y_n\right)$$

$$= -a^i(t, x) Y_i(F_j)(x) \tilde{J}^{jm}$$

$$- a^i_0 F_j(x) \tilde{J}^{jl} \tilde{J}^{km} \omega\left([Y_l, \tilde{\Phi}_{t} Y_i], Y_k\right) .$$

Using the variation of constants formula pointwise in $x$, one arrives at the assertion. \blacksquare

**III.2.1 Leading order perturbation theory**

Let us next use the small parameter $\epsilon$ for perturbation theory. We will only be interested in a heuristic argument that demonstrates why the incommensurability condition on the eigenfrequencies in conjecture III.1 is presumably necessary.

The simplified case that we will consider is defined by the following assumptions:
(1) $\Omega_x = \Omega$, constant for all $x$ in $U$.

(2) $\text{spec}(\Omega) = \{i\omega_1, \ldots, i\omega_{2k}\}$, with $\omega_r \in \mathbb{R}$.

(3) $\|\Omega\| := \sup_r |\omega_r| \ll \frac{1}{\epsilon}$.

Let us briefly comment on the generic properties of $\{\omega_r\}$. Writing $\Omega = \tilde{J}A$, we decompose the matrix $A = [Y_i(Y_j(H)(x_0))]$ into its symmetric and antisymmetric parts $A_+$ and $A_-$, respectively. $A_- = [[Y_j, Y_i](H)(x_0)]/2$ vanishes if $V$ is integrable, as one deduces from the fact that $X_H|_{x_0}$ is a vector in $V^\perp_{x_0}$ for all $x_0 \in \mathcal{C}_{\text{gen}}$, and from the Frobenius condition. The linear system of ODE’s $\dot{a} = \tilde{J}A_+ a$ in the space of $a^i(t)$’s is Hamiltonian, hence the spectrum of $\tilde{J}A_+$, if it is purely imaginary, consists of complex conjugate pairs of eigen values in $i\mathbb{R}$ (here, we have introduced the notation $a := (a^1, \ldots, a^{2k})$). If $\tilde{J}A_-$ admits a small relative norm bound with respect to $\tilde{J}A_+$, pairs of complex conjugate eigenvalues $\pm i\omega_r$ will generically be deformed in a manner that they lose the property of being identical up to sign. Thus, generically, we may assume that all frequencies $\omega_r$ are distinct from each other, and there are as many negative as positive ones.

From (58), one infers

$$Y_t = a_0^j (\exp(-t\Omega))^j_i Y_i + \sum O(|x|) Y_i$$

because $|F_j(x)| = O(|x|) = O(\epsilon)$, since $F_j(x_0) = 0$.

Thus,

$$[Y_t, X] = a_0^j \exp(-t\Omega)^j_i [Y_i, X] + \sum O(\epsilon) [Y_i, X] + \sum O(1) Y_i$$

for all $X \in \Gamma(TM)$, and $x \in U_i(x_0)$. Assuming that everything is sufficiently smooth, iterating the Lie bracket $L_{Y_t} r$ times produces

$$\left( \prod_{m=1}^{r} a_0^{i_m} \text{exp}(-t\Omega)^{i_m}_{j_m} + O(\epsilon) \right) [Y_{i_1}, [Y_{i_2}, \ldots, [Y_{i_r}, Y_i] \ldots]] ,$$

plus a series of terms with less than $r$ nested Lie commutators, which contribute to higher order corrections (that is, $O(\epsilon^{r+1})$) of terms indexed by $r' < r$ in (59).

Let us, for a discussion of leading order perturbation theory along each flag element of $V$, drop the terms of order $O(\epsilon)$, and assume that everything is sufficiently smooth so that our considerations hold if $t \leq O(\epsilon^{-1})$.

For fixed $r$, let us consider the term

$$F_i(\Psi^i_t(x_0)) \tilde{J}^{ik} \left( L_{Y^i_t} Y_k \right) (f \circ \Psi^i_t)(x_0) ,$$

which describes the contribution of (54) along the $r$-th flag component of $V$, at least for sufficiently small $t_r$.

To this end, we have

$$F_i(\Psi^i_t(x_0)) = Y_i(F_i)(x_0) + O(\epsilon^2) ,$$

36
due to \( F_i(x_0) = 0 \). Therefore,

\[
F_i(\Psi_t^\varepsilon(x_0)) \tilde{J}^{ik} = \varepsilon \exp(-t\Omega)^m \ a_0^j \Omega^k_m + O(\varepsilon^2),
\]

(61)
as a straightforward calculation shows.

Collecting all results obtained so far, the terms with \( r \) nested commutators in (59) are

\[
\frac{\varepsilon^{r+1}}{r!} a_0^i \exp(-t\Omega)^j \Omega^l_j \left( \prod_{m=1}^r a_0^{j_m} (\exp(-t\Omega))^{i_m}_{j_m} \right) [Y_i, [Y_{i_2}, \ldots, [Y_{i_r}, Y] \ldots]](f)(x_0)
\]

+ \( O(\varepsilon^{r+2}) \),

as long as \( dist_R(\Psi_t^\varepsilon(x_0), x_0) \leq O(\varepsilon) \). This implies that for \( f \in C^\infty(U) \),

\[
f(\Psi_t^\varepsilon(x_0)) \approx f(x_0) + \sum_{r \geq 0} \frac{\varepsilon^{r+1}}{r!} \int_0^t ds \ a_0^i \exp(-s\Omega)^j \Omega^l_j \times
\]

\[
\times \left( \prod_{m=1}^r a_0^{j_m} (\exp(-s\Omega))^{i_m}_{j_m} \right) [Y_i, [Y_{i_2}, [Y_{i_r}, Y]] \ldots](f)(x_0),
\]

(62)

up to errors of higher order in \( \varepsilon \) for every fixed \( r \), as long as \( dist_R(\Psi_t^\varepsilon(x_0), x_0) \leq O(\varepsilon) \).

If \( f \) is picked as the \( i \)-th coordinate function \( x^i \), so that \( f(\Psi_t^\varepsilon(x_0)) = x^i(t) \), the quantity \([Y_i, [Y_{i_2}, [Y_{i_r}, Y]] \ldots](f)(x_0)\) is the \( i \)-th coordinate of the vector field defined by the brackets at \( x_0 \). Consequently, (62) is the component decomposition of \( x^i(t) \) relative to the flag of \( V \) at \( x_0 \), to leading order in \( \varepsilon \).

By assumption for the simplified model, \( \Omega \) has a purely imaginary spectrum. In this case, the operator \( \exp(-s\Omega) \) has a norm 1 for all \( s \). Consequently, the integrand of (62) is bounded for all \( s \). It follows that if the integral should diverge, and become bigger than \( O(\varepsilon) \), it will take a time

\[
t \geq O \left( \frac{1}{\varepsilon^r} \right)
\]

to do so along the flag element \( V_r \). The leading term of order \( O(\varepsilon) \), corresponding to \( r = 0 \), remains bounded for all \( t \), but higher order corrections to it might diverge.

Let us next write

\[
\varphi(s) = \exp(-s\Omega) a_0
\]

\[
= \sum_{\alpha=1}^{2k} A_\alpha \varphi_\alpha \exp(-i\omega_\alpha s),
\]

(63)

where \( \{\varphi_\alpha\} \) is an orthonormal eigenbasis of \( \omega \) with respect to the standard scalar product in \( \mathbb{C}^{2k} \), and \( \text{spec}(\Omega) = \{i\omega_\alpha\} \). The amplitudes \( A_\alpha \in \mathbb{C} \) are determined by the initial condition \( a^i(t = 0) = a_0^i \), and will be assumed to be nonzero. By linear recombination of the vector fields \( Y_i \), one can set \( e^{\alpha}_\alpha = \delta_{i,\alpha} \). Then, (62) can be written as

\[
\sum_{r \geq 0} \frac{\varepsilon^{r+1}}{r!} \sum_{l; i_1, \ldots, i_r} I_{l; i_1, \ldots, i_r}(t) [Y_{i_1}, [Y_{i_2}, \ldots, [Y_{i_r}, Y] \ldots](f)(x_0),
\]

(64)
where

\[ I_{i_1,\ldots,i_r}(t) := \int_0^t ds \omega_l A_l \left( \prod_{m=1}^r A_{j_m} \right) \exp \left( -is \left( \omega_l + \sum_{m=1}^r \omega_{j_m} \right) \right) \]

\[ = i \left( \omega_l + \sum_{m=1}^r \omega_{j_m} \right)^{-1} \omega_l A_l \left( \prod_{m=1}^r A_{j_m} \right) \times \]

\[ \times \left\{ \exp \left( -it \left( \omega_l + \sum_{m=1}^r \omega_{j_m} \right) \right) - 1 \right\} \quad (65) \]

Evidently, the nested commutators vanish if all indices \( i_1, \ldots, i_r, l \) have equal values.

**III.2.2 Conditions for instability**

Let us introduce the set

\[ \mathcal{I}(r)(t) := \{ I_{i_1,\ldots,i_r}(t) \}_{i_1=1}^{2k} \setminus \{ I_{i_1,\ldots,i_r}(t) \}_{i_1=1}^{2k}, \quad (66) \]

which we endow with the norm

\[ \| \mathcal{I}(r)(t) \| := \sup_{I(t) \in \mathcal{I}(r)(t)} |I(t)|. \]

Furthermore, let

\[ \| A \| := \sup_{i=1,\ldots,2k} \{ |A_i| \}, \quad (67) \]

where \( A_i \) are \( \mathbb{C} \)-valued amplitudes.

We will now determine in which situations \( \| \mathcal{I}(r)(t) \| \) diverges in the limit \( t \to \infty \).

To this end, let

\[ \mathfrak{A} := \{ \omega_1, \ldots, \omega_{2k} \}, \quad (68) \]

and denote the \( r \)-fold sumset by

\[ \mathfrak{A}_r := \underbrace{\mathfrak{A} + \cdots + \mathfrak{A}}_{r \text{ times}}, \quad (69) \]

defined by the set containing all sums of \( r \) elements of \( \mathfrak{A} \).

For two sets of real numbers \( \mathfrak{A} \) and \( \mathfrak{B} \), we define their distance as

\[ d(\mathfrak{A}, \mathfrak{B}) := \inf_{i,j} \{ |a_i - b_j| \mid a_i \in \mathfrak{A}, \ b_j \in \mathfrak{B} \}. \quad (70) \]

Then, it follows from (69) that if \( d(\mathfrak{A}_r, \mathfrak{A}) > 0 \),

\[ \| \mathcal{I}(r)(t) \| \leq d(\mathfrak{A}_r, \mathfrak{A})^{-1} \| \Omega \| \| A \|^r \quad (71) \]
(the sum over frequencies $\sum_{m=1}^{r} \omega_{lm}$ in (65) is an element of $\mathcal{A}_r$, and can only equal $-\omega_l$ if $d(\mathcal{A}_r, -\mathcal{A}) = 0$). However, if $d(\mathcal{A}_r, -\mathcal{A}) = 0$, there is a tuple of indices \{l; i_1, \ldots, i_r\} such that

$$I_{l,i_1,\ldots,i_r}(t) = -t \omega_l A_l \prod_{m=1}^{r} A_{jm}.$$  \hspace{1cm} (72)

This is simply obtained by letting the sum of frequencies in (65) tend to zero. Thus, in this case,

$$\|\mathcal{J}^{(r)}(t)\| \sim t,$$  \hspace{1cm} (73)

that is, a divergence linear in $t$ as $t \to \infty$ (of course, the validity of our leading order perturbation theory breaks down as $t \to \frac{1}{\epsilon}$). Only if there are simultaneously positive and negative frequencies, $d(\mathcal{A}_r, -\mathcal{A}) = 0$ is possible, but due to the remark at the beginning of subsection III.2.1 this situation must generically assumed to be given.

Let us consider some examples. The fact that $\|\mathcal{J}^{(0)}(t)\|$ is bounded for all $t$ is trivial. On the next level, $r = 1$, we consider the component in the direction of the first flag element $V_1 = [V, V]$. The condition for the emergence of a divergent solution is that $d(\mathcal{A}, -\mathcal{A}) = 0$. This is precisely given if there is a pair of frequencies $\pm \omega_i$ of equal modulus, but opposite sign. For $r = 2$, assuming that $d(\mathcal{A}, -\mathcal{A}) > 0$, the condition $d(\mathcal{A}_2, -\mathcal{A}) = 0$ implies that there is a triple of frequencies such that $\omega_{i_1} + \omega_{i_2} = -\omega_{i_3}$, $i_j \in \{1, \ldots, 2k\}$. If this occurs, the solution will diverge in the direction of the second flag element, $V_2 = [V, [V, V]]$. The discussion for $r > 2$ continues in the same manner.

In conclusion, we have arrived at the following proposition.

**Proposition III.6** If $d(\mathcal{A}_r, -\mathcal{A}) = 0$ for some $r$, then $\|\mathcal{J}^{(r)}(t)\| = O(t)$ for $t \to \infty$.

The above results suggest that if the frequencies of the linearized problem do not satisfy the incommensurability condition "$d(\mathcal{A}_r, -\mathcal{A}) > 0$ for all $r$", the equilibrium $x_0$ is unstable. However, it has also become clear that the time required for an orbit to leave a Riemannian $\epsilon$-neighborhood $U_\epsilon(x_0)$ is extremely large. Assume that $d(\mathcal{A}_r, -\mathcal{A}) = 0$ for some $r \leq r(V)$ (the degree of non-holonomy of $V$). Then, it takes a time on the order of $O(\frac{1}{\epsilon})$ to exit $U_\epsilon(x_0)$ along the flag element $V_r$, according to our leading order perturbative results (due to the factor $\frac{1}{\epsilon^2}$ in (62)).

Although the perturbative results lose their validity already after $t \leq O(\frac{1}{\epsilon})$, it is always possible to continue solutions by picking new charts, centered around new basis points after times of order $O(\frac{1}{\epsilon^2})$, say. There is no doubt that since $d(\mathcal{A}_r, -\mathcal{A}) = 0$ holds for all basis points $x'_0 \in U_\epsilon(x_0)$ (by assumption, $\Omega$ is constant in $U_\epsilon(x_0)$), the same divergence will be observed in each chart, and the prediction that $U_\epsilon(x_0)$ will indeed be exited after a time of order $O(\frac{1}{\epsilon})$ is realistic. During this time, the orbit will not drift away from $U_\epsilon(x_0) \cap \mathcal{C}_{gen}$ in the direction of $V_{\omega_0}$ if $D_{\omega_0}^2 H|_{V_{\omega_0}}$ is positive definite on $V_{\omega_0}$, as required in the conjectured stability criterion. This suggests that the incommensurability condition imposed on the frequencies of the linearized system can indeed not be dropped.

We claim that although in the light of Riemannian geometry, divergences of this type are completely unspectacular up to times of order $\geq O(\frac{1}{\epsilon})$, they are very severe in the light
of the natural, internal geometry of the system, which is not the Riemann, but the Carnot-Caratheodory structure induced by $g$.

### III.2.3 Relation to sub-Riemannian geometry

The structure exhibited in this discussion clearly shows that the constrained Hamiltonian system $(M, \omega, H, V)$ has many characteristics of systems usually encountered in sub-Riemannian geometry \[5, 18, 19, 33\].

The natural metric structure that accounts for the particular structure of the present system is obtained from the Carnot-Caratheodory distance function $\text{dist}_{C-C}$ induced by the Riemannian metric $g$. It assigns to a pair of points $x, y \in M$ the length of the shortest $V$-horizontal $g$-geodesic.

If $V$ satisfies the Chow condition, $\text{dist}_{C-C}(x, y)$ is finite for all $x, y \in M$. This is the Rashevsky-Chow theorem \[5, 19\]. In this case, the Carnot-Caratheodory $\epsilon$-ball

$$B^C_C(x_0) := \{ x \in M \mid \text{dist}_{C-C}(x, x_0) < \epsilon \}$$

is open in $M$.

If $V$ does not satisfy the Chow condition, pairs of points that cannot be joined by $V$-horizontal $g_M$-geodesics are assigned a Carnot-Caratheodory distance $\infty$. In this case, $M$ is locally foliated into submanifolds $N_\lambda$ of dimension $(2n - \text{rank} V_r(V))$, with $\lambda$ in some index set, which are integral manifolds of the (necessarily integrable) final element $V_r(V)$ of the flag of $V$ ($r(V)$ denotes the degree of nonholonomy of $V$). On every $N_\lambda$, the distribution $V_\lambda := j_\lambda^* V$ satisfies the Chow condition, where $j_\lambda : N_\lambda \to M$ is the inclusion. Therefore, all points $x, y \in N_\lambda$ have a finite distance with respect to the Carnot-Caratheodory metric induced by the Riemannian metric $j_\lambda^* g_M$. Every leaf $N_\lambda$ is an invariant manifold of the flow $\tilde{\Phi}_t$.

**The ball-box theorem.** Assume that the spanning family $\{Y_{i_r}\}_{r=1}^{r(V)}$ of $TM$ is suitably picked so that $\{Y_{i_r}\}$ spans the flag element $V_r$. Let the $g$-length of all $Y_{i_r}$’s be 1. Then, we define the ‘quenched’ box

$$\text{Box}_\epsilon(x) := \left\{ \exp_1 \left( \sum_{r=1}^{r(V)} \epsilon^r \sum_{i_r=1}^{\dim V_r} t_{i_r} Y_{i_r} \right)(x) \mid t_{i_r} \in (-1, 1) \right\}$$

in $N_\lambda$, where $\lambda$ is suitably picked so that $x \in N_\lambda$. Evidently, if $V$ satisfies Chow’s condition, $N_\lambda = M$. According to the ball-box theorem \[3, 19\], there are constants $C > c > 0$, such that

$$\text{Box}_{c\epsilon}(x) \subset B^C_C(x) \subset \text{Box}_{C\epsilon}(x).$$

Therefore, Carnot-Caratheodory $\epsilon$-balls can be approximated by quenched boxes of Riemannian geometry.

**Instabilities in the light of C-C geometry.** The above perturbative results imply that if there is some $r < r(V)$, for which $d(2t_r, -2t_r) = 0$, the flow $\tilde{\Phi}_t$ blows up the quenched boxes, and thus the Carnot-Caratheodory $\epsilon$-ball around $x_0 \in C_{gen}$, linearly in $t$, and along
the direction of $V_r$. In fact, $B^C_C(x_0)$ is widened along $V_r$ at a rate linear in $t$. After a time of the order $O(\epsilon)$, the Carnot-Caratheodory $\epsilon$-ball containing the initial condition is increased to an extent that it can only be embedded into a Carnot-Caratheodory $O_\epsilon(1)$-ball. In this light, the instabilities in discussion are very severe.

This type of instability has no counterpart in systems with integrable constraints.
IV NONHOLONOMIC MECHANICS

We will in this section focus on nonholonomic mechanical systems, and their relationship to the constrained Hamiltonian systems considered previously. The discussion is restricted to Hamiltonian mechanical systems that are subjected to linear nonholonomic constraints (Pfaffian constraints).

Hamiltonian mechanics. Let \((Q, g, U)\) be a Hamiltonian mechanical system, where \(Q\) is a smooth Riemannian \(n\)-manifold with a \(C^\infty\) metric tensor \(g\), and where \(U \in C^\infty(Q)\) denotes the potential energy. No gyroscopic forces are taken into consideration. Let \(g^\ast\) denote the induced Riemannian metric on the cotangent bundle \(T^*Q\). For a given \(X \in \Gamma(TM)\), let \(\theta_X\) be the 1-form defined by \(\theta_X(Y) = g(X, Y)\) for all \(Y \in \Gamma(TQ)\). It follows then that \(g(X,Y) = g^\ast(\theta_X, \theta_Y)\) for all \(X,Y \in \Gamma(TQ)\).

The Kähler metric of the previous discussion, which was denoted by the same letter \(g\), will not appear again in the sequel. Therefore, writing \(g\) for the Riemannian metric on \(Q\) should hopefully not give rise to any confusion.

In a local trivialization of \(T^*Q\), a point \(x \in T^*Q\) is represented by a tuple \((q^i, p_j)\), where \(q^i\) are coordinates on \(Q\), and \(p_k\) are fibre coordinates in \(T^*_qQ\), with \(i,j = 1, \ldots, n\). The natural symplectic 2-form associated to \(T^*Q\), written in coordinates as

\[
\omega_0 = \sum_i dq^i \wedge dp_i = -d\theta_0 ,
\]

is exact. \(\theta_0 = p_i dq^i\) is referred to as the symplectic 1-form.

We will only consider Hamiltonians of the form

\[
H(q,p) = \frac{1}{2} g^*_q(p,p) + U(q) \in C^\infty(T^*Q) ,
\]

that is, kinetic plus potential energy. In local bundle coordinates, the associated Hamiltonian vector field \(X_H\) is given by

\[
X_H = \sum_i \left( (\partial_{q^i} H) \partial_{q^i} - (\partial_{q^i} H) \partial_{p_i} \right) .
\]

The orbits of the associated Hamiltonian flow \(\Phi_t\) satisfy

\[
\dot{q}^i = \partial_{p_i} H(q,p) , \quad \dot{p}_j = -\partial_{q^j} H(q,p) .
\]

The superscript dot abbreviates \(\partial_t\), and will be used throughout the discussion.

Let \(\mathcal{A}_I\) denote the space of smooth curves \(\gamma: I \subset \mathbb{R} \rightarrow T^*Q\), with \(I\) compact and connected, and let \(t\) denote a coordinate on \(\mathbb{R}\). The basis one form \(dt\) defines a measure on \(\mathbb{R}\). The action functional is defined by \(\mathcal{I}: \mathcal{A}_I \rightarrow \mathbb{R}\),

\[
\mathcal{I}[\gamma] = \int_I dt \ (\gamma^\ast \theta_0 - H \circ \gamma) \quad (76)
\]

\[
= \int_I dt \ \left( \sum_i p_i(t) \dot{q}^i(t) - H(q(t),p(t)) \right) ,
\]

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with \( \dot{\gamma} = \sum (\dot{q}_i \partial q_i + \dot{p}_i \partial p_i) \). Denoting the base point projection by

\[ \pi : T^*Q \rightarrow Q , \]

let

\[ c := (\pi \circ \gamma) : I \rightarrow Q \]

denote the projection of \( \gamma \) to \( Q \). We assume that \( \| c(I) \| \) is sufficiently small so that solutions of (75) exist, which connect the end points \( c(\partial I) \). Among all curves \( \gamma : I \rightarrow T^*Q \) with fixed projected endpoints \( c(\partial I) \), the ones that extremize \( I \) are physical orbits of the system.

**Linear non-holonomic constraints.** Let us next discuss the inclusion of linear constraints on a given Hamiltonian mechanical system. This is achieved by adding a rank \( k \) distribution \( W \) over \( Q \) to the existing data, and by specifying a physical law, the Hölder variational principle, that generates the correct physical flow on \( T^*Q \). The orbits of the resulting dynamical system have the property that their projections to \( Q \) are \( W \)-horizontal. The physics of the Hölder principle is, for instance, discussed in [3].

**Orthoprojectors.** There is a \( g \)-symmetric tensor \( p : TQ \rightarrow TQ \) with

\[ \text{Ker}(p) = W^\perp , \quad p(X) = X \quad \forall \, X \in \Gamma(TQ) . \]

It will be referred to as the \( g \)-orthogonal projector associated to \( W \). We note that its matrix is in \( \text{Mat}(n \times n, \mathbb{R}) \), of rank \( k \). \( W^\perp \) denotes the \( g \)-orthogonal complement of \( W \). The \( g \)-orthogonal projector associated to \( W^\perp \) will be denoted by \( \bar{p} \), so that

\[ p + \bar{p} = \text{id} . \]

Furthermore, the dual of \( W \), denoted by \( W^* \), is defined as the image of \( W \) under the isomorphism \( g : TQ \rightarrow T^*Q \), and likewise for \( (W^*)^\perp := g \circ W^* \). The corresponding \( g^* \)-orthogonal projectors are denoted by \( p^\dagger \) and \( \bar{p}^\dagger \), respectively. We use this notation because the matrices of the latter are represented by the transposed matrices of \( p \) and \( \bar{p} \) in every standard coordinate chart. We will use the same symbols for the projectors and their matrices.

**Dynamics of the constrained mechanical system.** Next, we derive the equations of motion of the constrained mechanical system from the Hölder variational principle. The use of the orthoprojectors \( p, \bar{p} \) is inspired by [11]. For a closely related approach to the Lagrangian theory of constrained mechanical systems, cf. [14].

**Definition IV.1** A projective \( W \)-horizontal curve in \( T^*Q \) is an embedding \( \gamma : I \subset \mathbb{R} \rightarrow T^*Q \) whose image \( c = \pi \circ \gamma \) under the base point projection \( \pi : T^*Q \rightarrow Q \) is everywhere tangent to \( W \).

Let \( \gamma_s : I \rightarrow T^*Q \), with \( s \in [0, 1] \), be a smooth one parameter family of curves for which the end points \( c_s(\partial I) \) are independent of \( s \) (where \( c_s := \pi \circ \gamma_s \)).
**Definition IV.2** A W-horizontal variation of a projective W-horizontal curve $\gamma$ is a smooth one parameter family $\gamma_s : \mathbb{R} \to T^*Q$, with $s \in [0, 1]$, for which $\frac{\partial}{\partial s}(\pi \circ \gamma_s)$ is tangent to $W$, and $\gamma_0 = \gamma$.

Let

$$\delta q^i(t) := \partial_{s}|_{s=0} q^i(s, t) , \quad \delta p^k(t) := \partial_{s}|_{s=0} p^k(s, t).$$

To any W-horizontal variation $\gamma_s$ of a W-horizontal curve $\gamma_0$ with fixed projections of the boundaries

$$(\pi \circ \gamma_s)(\partial I) = (\pi \circ \gamma_0)(\partial I),$$

so that $\delta q^i|_{\partial I} = 0$, we associate the action functional

$$\mathcal{I}[\gamma_s] = \int_I \left( \sum p_i(s, t) \dot{q}^i(s, t) - H(q(s, t), p(s, t)) \right) dt.$$

**Theorem IV.1** (Hölder’s principle) A projective W-horizontal curve $\gamma_0 : I \to T^*Q$ corresponds to a physical motion of the constrained mechanical system if it extremizes $\mathcal{I}[\gamma_s]$ among all W-horizontal variations $\gamma_s$ that satisfy $(77)$.

Hence, if

$$\delta \mathcal{I}[\gamma_s] = \sum p_i \delta q^i|_{\partial I} + \int_I \sum ((\dot{p}_i - \partial_q H) \delta q^i - (\dot{q}^i + \partial_p H) \delta p_i) = 0$$

for all W-horizontal variations of $\gamma_0$ that satisfy $\delta q^i|_{\partial I} = 0$, then $\gamma_0$ is a physical orbit.

**Theorem IV.2** In the given local bundle chart, the Euler-Lagrange equations of the Hölder variational principle are the differential-algebraic relations

$$\dot{q} = p(q) \partial_p H(q, p)$$

(79)

$$\dot{p}^i(q) = -p^i(q) \partial_q H(q, p)$$

(80)

$$\bar{p}(q) \partial_p H(q, p) = 0.$$  

(81)

**Proof.** The boundary term vanishes due to $\delta q^i|_{\partial I} = 0$.

For any fixed value of $t$, one can write $\delta q(t)$ as

$$\delta q(t) = \sum_{\alpha=1}^{k} f_{\alpha}(q(t)) Y_{\alpha}(q(t)),$$

where $Y_{\alpha}$ is a $g$-orthonormal family of vector fields over $c(I)$ that spans $W_{c(I)}$. Furthermore, $f_{\alpha} \in C^\infty(c(I))$ are test functions obeying the boundary condition $f_{\alpha}(c(\partial I)) = 0$.

Since $f_{\alpha}$ and $\delta p$ are arbitrary, the terms in $(78)$ that are contracted with $\delta q$, and the ones that are contracted with $\delta p$ vanish independently. In case of $\delta q$, one finds

$$\int_I dt \ f_{\alpha} \ (\dot{p} + \partial_q H)_{i} \ Y_{\alpha}^i = 0.$$
for all test functions \( f_\alpha \). Thus, \((\dot{p} + \partial_q H)_iY^i_\alpha = 0\) for all \( \alpha = 1, \ldots, k \), or equivalently, \( p^\dagger(\dot{p} + \partial_q H) = 0 \), which proves \((79)\).

Since \( \gamma_0 \) is \( W \)-horizontal, \( \bar{p}(q)\dot{q} = 0 \), so the \( \delta p \)-dependent term in \( \delta I[\gamma_s] \) gives
\[
\int dt (\dot{q} - p\partial_p H)_i (p^\dagger \delta p)_i + \int dt (\bar{p}\partial_p H)_i (\bar{p}^\dagger \delta p)_i = 0 .
\]
The components of \( \delta p \) in the images of \( p^\dagger(q) \) and \( \bar{p}^\dagger(q) \) can be varied independently. Thus, both terms on the second line must vanish separately, as a consequence of which one obtains \((79)\) and \((81)\).

**Definition IV.3**

The smooth submanifold
\[
\mathcal{M}_{\text{phys}} := \{ (q,p) \mid \bar{p}(q)\partial_p H(q,p) = 0 \} \subset T^*Q
\]
defined by \((81)\) will be referred to as the physical leaf.

It contains all physical orbits of the system, that is, all smooth path \( \gamma : \mathbb{R} \to \mathcal{M}_{\text{phys}} \subset T^*Q \) that satisfy the differential-algebraic relations of theorem IV.2.

**Theorem IV.3** Let \( H \) be of the form \((74)\). Then, there exists a unique physical orbit \( \gamma : \mathbb{R}^+ \to \mathcal{M}_{\text{phys}} \) with \( \gamma(0) = x \) for every \( x \in \mathcal{M}_{\text{phys}} \).

**Proof.** Our strategy consists of proving that the differential relations \((73) \sim (81)\) define a unique section \( X \) of \( T\mathcal{M}_{\text{phys}} \), where \( \mathcal{M}_{\text{phys}} \) is regarded as an embedded submanifold of \( T^*Q \). Solution curves \( \gamma : \mathbb{R}^+ \to \mathcal{M}_{\text{phys}} \) of \( \gamma(t) = X|_{\gamma(t)} \) fulfill \((73) \sim (81)\) for all initial conditions \( \gamma(0) \in \mathcal{M}_{\text{phys}} \). The assertion thus follows from the existence and uniqueness theorem of ordinary differential equations.

To this end, let us cover \( \mathcal{M}_{\text{phys}} \) with local bundle charts of \( T^*Q \) with coordinates \((q,p)\).

In case of the Hamiltonian \((74)\), the defining relation \((81)\) for \( \mathcal{M}_{\text{phys}} \) reduces to
\[
\bar{p}(q) M^{-1}(q) p = M^{-1}(q) \bar{p}^\dagger(q) p = 0 ,
\]
where \( M \) denotes the matrix of \( g \) (\( M \) is determined by the mass distribution of the system), and where one uses the \( g \)-orthogonality of \( \bar{p} \). This implies that \((81)\) is equivalent to the condition \( \bar{p}^\dagger(q)p = 0 \). Hence, \( \mathcal{M}_{\text{phys}} \) is the common zero level set of the \( n \) component functions \( (\bar{p}^\dagger(q)p)_i \).

Consequently, every section
\[
X = v^r(q,p) \partial_q^r + w^s(q,p) \partial_p^s
\]
of \( T\mathcal{M}_{\text{phys}} \) annihilated by the 1-forms
\[
d(\bar{p}^\dagger p)_i = \partial_q^r(\bar{p}^\dagger p)_i dq^r + \partial_p^s(\bar{p}^\dagger p)_i dp_s
\]
for \( i = 1, \ldots, n \) (of which only \( n - k \) are linearly independent), on \( \mathcal{M}_{\text{phys}} \). This is, in vector notation, expressed by the condition
\[
0 = (v^r \partial_q^r) \bar{p}^\dagger p + (w_s \partial_p^s) \bar{p}^\dagger p = (v^r \partial_q^r) \bar{p}^\dagger p + \bar{p}^\dagger w ,
\]

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which shows that the components $v$ of $X$ determine the projection $p^\dagger w$, so that knowing the components $v$ and $p^\dagger w$ suffices to uniquely reconstruct $X$. Consequently, the right hand sides of (79) and (80) determine a unique section $X$ of $T\mathcal{M}_{phys}$, so that every curve $\gamma: \mathbb{R}^+ \to \mathcal{M}_{phys}$, with arbitrary $\gamma(0) \in \mathcal{M}_{phys}$, that satisfies

$$\partial_t \gamma(t) = X(\gamma(t))$$

automatically fulfills (79) $\sim$ (81). This proves the assertion. \[\square\]

**Equilibria.** The equilibria of the constrained Hamiltonian mechanical system on $\mathcal{M}_{phys}$ are obtained from the condition $\dot{q} = 0$ and $\dot{p} = 0$ in (79) $\sim$ (81), whereupon one arrives at

$$p = 0, \quad p^\dagger(q)\partial_q U(q) = 0.$$ The critical set is thus given by

$$\mathcal{C}_Q := \{ q \in Q \mid p^\dagger(q)\partial_q U(q) = 0 \}.$$ (82)

An application of Sard’s theorem fully analogous to the proof of theorem II.1 shows that generically, this is a piecewise smooth, $n - k$-dimensional submanifold of $Q$, (recall that the rank of $p(q)$ is $k$).

**Symmetries.** Let $G$ be a Lie group, and let

$$\psi: G \to \text{Diff}(Q)$$

$$h \mapsto \psi_h, \quad \Psi_e = \text{id},$$

be a group action on the configuration manifold so that the following holds:

1. Invariance of the Riemannian metrics: $g \circ \psi_h = g$ and $g^* \circ \psi_h = g^*$ for all $h \in G$.
2. Invariance of the potential energy: $U \circ \psi_h = U$ for all $h \in G$.
3. Invariance of the distributions: $\psi_h \ast W = W$ and $\psi^*_h W^* = W^*$ for all $h \in G$.

Then, we will say that the constrained Hamiltonian mechanical system $(Q, g, U, W)$ exhibits a $G$-symmetry.

**IV.1 Construction of the auxiliary extension**

Let us finally merge the nonholonomic mechanical system into a constrained Hamiltonian systems of the type considered in the previous sections.

To this end, we will introduce a set of generalized Dirac constraints over the symplectic manifold $(T^*Q, \omega_0)$ in the way presented in section II. They will define a suitable symplectic distribution $V$, in a manner that the constrained Hamiltonian system $(T^*Q, \omega_0, H, V)$, with $H$ given by (74), contains the constrained mechanical system as a dynamical subsyst- system. Thus, $(T^*Q, \omega_0, H, V)$ extends the mechanical system in the sense announced in the introduction. An early inspiration for this construction stems from [31].

We require the following properties to be satisfied by the auxiliary constrained Hamiltonian system $(T^*Q, \omega_0, H, V)$.
(i) $\mathcal{M}_{\text{phys}}$ is an invariant manifold under the flow $\tilde{\Phi}_t$ generated by (5).

(ii) All orbits $\tilde{\Phi}(x)$ with initial conditions $x \in \mathcal{M}_{\text{phys}}$ satisfy the Euler-Lagrange equations of the Hölder principle.

(iii) $\mathcal{M}_{\text{phys}}$ is marginally stable under $\tilde{\Phi}_t$.

(iv) The critical set $\mathcal{C}$ of $\tilde{\Phi}_t$ is a vector bundle over $\mathcal{C}_Q$, hence equilibria of the constrained mechanical system are obtained from equilibria of the extension by base point projection.

(v) Symmetries of the constrained mechanical system extend to symmetries of $\tilde{\Phi}_t$.

Let us briefly comment on (iii) $\sim$ (v). (iii) is of importance for numerical simulations of the mechanical system. (iv) makes it easy to extract information about the behaviour of the mechanical system from solutions of the auxiliary system. Condition (v) allows to apply reduction theory to the auxiliary system, in order to reduce the constrained mechanical system by a group action, if present. The choice for $V$ is by no means unique, and depending on the specific problem at hand, other conditions than (iii) $\sim$ (v) might be more useful.

**Construction of $V$.** Guided by the above requirements, we shall now construct $V$.

To this end, we pick a smooth, $g^*$-orthonormal family of 1-forms $\{\zeta_I\}_{I=1}^{n-k}$ with

$$
\zeta_I = \zeta_{Ik}(q) \, dq^k,
$$

so that locally,

$$
\langle \{\zeta_1, \ldots, \zeta_{n-k}\} \rangle = (W^*)^\perp.
$$

The defining relationship $\bar{p}^\dagger(q)p = 0$ for $\mathcal{M}_{\text{phys}}$ is equivalent to the condition

$$
f_I(q,p) := g_q^*(p, \zeta_I(q)) = 0 \quad \forall I = 1, \ldots, n-k.
$$

It is clear that $f_I \in C^\infty(T^*Q)$.

(I) To satisfy conditions (i) and (iii), we require that the level surfaces

$$
\mathcal{M}_\mu := \{(q,p) | f_I(q,p) = \mu_I; I = 1, \ldots, n-k\},
$$

with $\mu := (\mu_1, \ldots, \mu_{n-k})$, are integral manifolds of $V_{r(V)}$. Here, $r(V)$ denotes the degree of non-holonomy of $V$, and evidently, $\mathcal{M}_0 = \mathcal{M}_{\text{phys}}$.

Condition (iii) is satisfied because

$$
L(q,p) := \sum_I |f_I(q,p)|^2
$$

is an integral of motion for orbits of $\tilde{\Phi}_t$. Since $L$ grows monotonically with increasing $|\mu|$, and attains its (degenerate) minimum of value zero on $\mathcal{M}_{\text{phys}}$, it is a Lyapunov function for $\mathcal{M}_{\text{phys}}$. Anything better than marginal stability is prohibited by energy conservation.
(II) To satisfy condition (ii), we demand that \( \bar{p}(q)\dot{q} = 0 \), or equivalently, that

\[
\zeta_I(\dot{q}) = 0 \quad , \quad \forall I = 1, \ldots , n - k ,
\]  

shall be satisfied along all orbits \((q(t), p(t))\) of (5), owing to (79).

(III) If the constrained mechanical system exhibits a \( G \)-symmetry, characterized by a group action \( \psi : G \rightarrow \text{Diff}(Q) \) so that \( \psi_h^*W = W \ \forall h \in G \), the local family of 1-forms \( \{\zeta_I\} \) can be picked in a manner that \( \psi_h^*\zeta_I = \zeta_I \) is satisfied for all \( h \in G \) in a vicinity of the unit element \( e \). Consequently, the functions \( f_I(q, p) = h_q^*(\zeta_I, p) \) and their level sets \( \mathcal{M}_I \) are invariant under the group action.

The condition that (84) are integral manifolds of \( V \supset V_r \) implies that all sections of \( V \) are annihilated by the 1-forms \( df_I \), for \( I = 1, \ldots , n - k \). Furthermore, the condition (85) requires \( V \) to be annihilated by the 1-forms

\[
\xi_I := \zeta_I(q) \ dq^r + \sum_s 0 \ dp_s
\]

that are obtained from lifting \( \zeta_I \) to \( T^*(T^*Q) \), with \( I = 1, \ldots , n - k \).

**Proposition IV.1** The distribution

\[
V := \left( \bigcap_I \ker df_I \right) \bigcap \left( \bigcap_I \ker \xi_I \right) \subset T(T^*Q)
\]

is symplectic.

**Proof.** \( V \) is symplectic iff its symplectic complement \( V^\perp \) is. With the given data, the latter condition is more convenient to check. \( V^\perp \) is locally spanned by the vector fields \( (Y_1, \ldots , Y_{2k}) \) obtained from

\[
\omega_0(Y_I, \cdot) = \xi_I(\cdot) \quad , \quad \omega(Y_{I+k}, \cdot) = df_I(\cdot),
\]

where \( I = 1, \ldots , k \), and \( \omega_0 = -dp_i \wedge dq^i \).

\( V^\perp \) is symplectic if and only if \( D := [\omega(Y_I, Y_J)] \) has values in \( GL_R(2(n - k)) \).

**Remark.** In the present notation, capital indices range from 1 to \( k \) if they label 1-forms, and from 1 to \( 2k \) if they label vector fields.

In local bundle coordinates,

\[
df_I = (\partial_{q^i} f_I)(q, p) \ dq^i + \zeta_I(q) \ M^{ij}(q) \ dp_j,
\]

where \( M^{ij} \) are the components of the metric tensor \( g \) on \( Q \), as before. Let us introduce the functions \( E(q) := [\zeta_{ij}(q)] \) and \( F(q, p) := [\partial_{q^i} f_K(q, p)] \), both with values in \( \text{Mat}_R(n \times (n - k)) \), which we use to assemble

\[
K := \begin{bmatrix} E^\dagger & 0 \\ F^\dagger & E^\dagger M^{-1} \end{bmatrix} : T^*Q \longrightarrow \text{Mat}_R(2(n - k) \times 2n).
\]
Any component vector $v : T^*Q \to \mathbb{R}^{2n}$ that locally represents an element of $\Gamma(V)$ satisfies $Kv = 0$. The symplectic structure $\omega_0$ is locally represented by $J$, defined in (8). One can easily verify that the $I$-th row vector of the matrix $K^J^{-1}$ is the component vector of $Y_I$. In conclusion, introducing the matrices

$$
G(q) := E^\dagger(q) M^{-1}(q) E(q)
$$

$$
S(q,p) := F^\dagger(q,p) M^{-1}(q) E(q) - E^\dagger(q) M^{-1}(q) F(q,p),
$$

one immediately arrives at

$$
D = K^J K^\dagger = \begin{bmatrix}
0 & G \\
-G & S
\end{bmatrix}.
$$

(88)

Since $\zeta_I$ has been picked a $g^*$-orthonormal family of 1-forms on $Q$, it is clear that $G(q) = 1_{n-k}$. Thus, $D$ is invertible. This proves that $V^\perp$ is symplectic.

Construction of the projectors. Next, we determine the matrix of the $\omega_0$-orthogonal projector $\bar{\pi}_V$, which is associated to $V$, in the present bundle chart. Again, it is more convenient to carry out the construction for its complement first.

**Proposition IV.2** The matrix of the $\omega_0$-orthogonal projector $\bar{\pi}_V$ associated to $V^\perp$ (considered as a tensor field that maps $\Gamma(T(T^*Q))$ to itself, with kernel $V$) is given by

$$
\bar{\pi}_V = \begin{bmatrix}
\bar{p} & 0 \\
T & \bar{p}^\dagger
\end{bmatrix}
$$

in the local bundle chart $(q,p)$. The matrix $T = T(q,p)$ is defined in (89).

**Proof.** The proof of lemma I.1 can be used for this proof. The inverse of (88) is

$$
D^{-1} = \begin{bmatrix}
S & -1_{n-k} \\
1_{n-k} & 0
\end{bmatrix},
$$

where we recall that $G(q) = 1_{n-k}$. The $I$-th column vector of the matrix $K^J^{-1}$ is the component vector of $Y_I$ (we have required that $\{Y_1, \ldots Y_{2(n-k)}\}$ spans $V^\perp$). This implies that $\bar{\pi}_V = JK^\dagger D^{-1}K$.

**Lemma IV.1** The matrix of the $g$-orthogonal projector $\bar{p}$ associated to $W^\perp$ in the present chart is given by

$$
\bar{p}(q) = M^{-1}(q) E(q) E^\dagger(q).
$$

(89)

**Proof.** The construction presently carried out for $\bar{\pi}_V$ can also be applied to $\bar{p}$. One only has to replace $V^\perp$ by to $W^\perp$, and $\omega_0$ by the Riemannian metric $g$ on $Q$. An easy calculation immediately produces the asserted formula. The matrix of $p$ is subsequently obtained from $p + \bar{p} = 1$. For more details, cf. [11].
Introducing
\[ T(q,p) := E(q) F^\dagger(q,p) \mathfrak{p}(q) - \mathfrak{p}^\dagger(q) F(q,p) E^\dagger(q), \tag{90} \]
a straightforward calculation produces the asserted formula for \( \bar{\pi}_V \).

**Corollary IV.1** *In the given bundle coordinates, the matrix of \( \pi_V \) is*
\[ \pi_V = \begin{bmatrix} \mathfrak{p} & 0 \\ -T & \mathfrak{p}^\dagger \end{bmatrix}, \]
*where \( T = T(q,p) \) is defined in (90).*

**Proof.** This is obtained from \( \pi_V + \bar{\pi}_V = 1_{2n} \).

The \( \omega_0 \)-orthogonality of \( \pi_V \) is represented by
\[ \pi_V(x) J = J \pi_V^\dagger(x) \]
in the given chart.

**Theorem IV.4** *Let \( H \) be as in (74). Then, the dynamical system*
\[ \left( \begin{array}{c} \dot{q} \\ \dot{\mathfrak{p}} \end{array} \right) = \left[ \begin{array}{ccc} 0 & \mathfrak{p} & \\ -\mathfrak{p}^\dagger & -T \end{array} \right] \left( \begin{array}{c} \partial_q H \\ \partial_p H \end{array} \right) \tag{91} \]
corresponding to the contrained Hamiltonian system \((T^*Q, \omega_0, H, V)\) is an extension of the constrained mechanical system \((Q, g, U, W)\).

**Proof.** By construction, \( \mathcal{M}_{phys} \) is an invariant manifold of the associated flow \( \tilde{\Phi}_t \), hence (81) is fulfilled for all orbits of (91) with initial conditions in \( \mathcal{M}_{phys} \).

The equation \( \dot{q} = \mathfrak{p} \partial_p H \) in (91) obviously is (79).

Next, using the notation \( \underline{f} := (f_1, \ldots, f_{n-k})^\dagger \),
\[ \underline{f} = E^\dagger M^{-1} \mathfrak{p}, \]
and substituting (90) for \( T(q,p) \), the equation for \( \dot{\mathfrak{p}} \) in (91) becomes
\[ \dot{\mathfrak{p}} = -\mathfrak{p}^\dagger \partial_q H - E F^\dagger \dot{\mathfrak{q}} + \mathfrak{p}^\dagger F \underline{f}. \]

Since \( M_\underline{f} \) are invariant manifolds of the flow \( \tilde{\Phi}_t \) generated by (91), \( \partial_t f_1(q(t), p(t)) \) vanishes along all orbits of (91), so that \( F^\dagger \dot{\mathfrak{q}} + E^\dagger M^{-1} \dot{\mathfrak{p}} = 0 \). This implies that
\[ \dot{\mathfrak{p}} = -\mathfrak{p}^\dagger \partial_q H + E E^\dagger M^{-1} \mathfrak{p} + \mathfrak{p}^\dagger \partial_q \left( \frac{1}{2} f^\dagger \underline{f} \right). \tag{92} \]

Recalling that \( \bar{\mathfrak{p}} = M^{-1} E E^\dagger \) from (89), and using the fact that \( \underline{f} = 0 \) on \( \mathcal{M}_{phys} \), one arrives at (80) by multiplication with \( \mathfrak{p}^\dagger \) from the left.
Theorem IV.5  The critical set of \((\mathbb{V})\) corresponds to

\[
\mathcal{C} = \{ (q,p) \mid q \in \mathcal{C}_Q ; p \in (W^*_q)^\perp \} .
\]

It is a vector bundle over the base space \(\mathcal{C}_Q\), with fibres given by those of \((W^*)^\perp\).

Proof. Let us start with equation \((\mathbb{V})\). As has been stated above, the second term on the right hand side of the equality sign equals \(\bar{p}^\dagger (q)p\), and moreover, from \((\mathbb{S})\), one concludes that

\[
f^\dagger f = \| \bar{p}^\dagger p \|^2_{\omega^*} .
\]

The Hamiltonian \((\mathbb{A})\) can be decomposed into

\[
H(q,p) = H(q,p^\dagger p) + \frac{1}{2} \| \bar{p}^\dagger p \|^2_{\omega^*} ,
\]
due to the \(g^*\)-orthogonality of \(p^\dagger\) and \(\bar{p}^\dagger\), so that \((\mathbb{B})\) can be written as

\[
\dot{p} = -p^\dagger \partial_q H(q,p^\dagger p) + \bar{p}^\dagger \dot{p} .
\]

The equilibria of \((\mathbb{A})\) are therefore determined by the conditions

\[
p^\dagger (q)p = 0 , \ p^\dagger (q) \partial_q H(q,p^\dagger p) = 0 .
\]

Because \(H\) depends quadratically on \(p^\dagger p\), the second condition can be reduced to

\[
p^\dagger (q) \partial_q U(q) = 0
\]

using the first condition. Comparing this with \((\mathbb{A})\), the assertion follows. ■

In particular, this fact implies that every equilibrium \((q_0,p_0)\) of the extension defines a unique equilibrium \(q_0\) on \(\mathcal{C}_Q\), which is simply obtained from base point projection.

Extension of symmetries. Let us finally prove that \((T^*Q,\omega_0,H,V)\) extends the \(G\)-symmetry \(\psi : G \to \text{Diff}(Q)\) of the constrained mechanical system \((Q,g,U,W)\), if there is one. To this end, we recall that the 1-forms \(\xi_I\) satisfy \(\psi^*_h \xi_I\) for all \(h \in G\) close to the unit.

Via its pullback, \(\psi\) induces the group action

\[
\Psi := \psi^* : G \times T^*Q \longrightarrow T^*Q
\]
on \(T^*Q\). This group action is symplectic, that is, \(\Psi^*_h \omega_0 = \omega_0\) for all \(h \in G\). For a proof, consider for instance \([1]\).

The 1-forms \(\xi_I\), defined in \((\mathbb{S})\), satisfy \(\psi^*_h \xi_I = \xi_I\), and likewise, \(f_I \circ \Psi = f_I\) is satisfied for all \(h \in G\) close to the unit. The definition of \(V\) in proposition \([V.4]\) thus implies that

\[
\Psi^*_h V = V
\]
is satisfied for all \(h \in G\). Due to the fact that \(\omega\) and \(V\) are both \(G\)-invariant, \(\pi_V\) and \(\bar{\pi}_V\) are also invariant under the \(G\)-action \(\Psi\).
The Hamiltonian $H$ in (74) is $G$-invariant under $\Psi$, by assumption on the constrained Hamiltonian mechanical system. Thus, $X_H$ fulfills $\Psi_hX_H = X_H$ for all $h \in G$, which implies that $X^V_H = \pi_V(X_H)$ is $G$-invariant.

**Stability of equilibria.** To analyze the stability of a given equilibrium solution $q_0 \in C_Q$, it is necessary to determine the spectrum of the linearization of $X^V_H$ at $x_0 = (q_0,0)$.

A straightforward calculation much in the style of the discussion above shows that in the present bundle chart,

$$DX^V_H(x_0) = \begin{bmatrix} 0 & pM^{-1}p^\dagger + R^\dagger \\ -p^\dagger D^2_{q_0}U p - R & 0 \end{bmatrix}(x_0),$$

where

$$[R_{jk}] := \begin{bmatrix} \partial_qU(p)^r_j \partial_q^* (p)^r_k \partial_q(p)^r_i \end{bmatrix} \in \text{Mat}_R(n \times n).$$

Furthermore, $D^2_{q_0}U$ is the matrix of second derivatives of $U$. The conjectural stability criterion formulated in the previous section can now straightforwardly be applied to $DX^V_H(x_0)$.

**IV.2 The topology of the critical manifold**

Let us now discuss the global topology of the critical set of the constrained Hamiltonian mechanical system, defined by

$$C = \left\{(q,p) \mid q \in C_Q; p^\dagger(q)p = 0; p^\dagger(q)p = \sum I \mu_I \zeta_I(q) \right\}.$$

Here, $\zeta_I$ is an orthonormal spanning family of one forms for the rank $n - k$ annihilator of the rank $k$ distribution $W$, and

$$C_Q = \left\{q \in Q | p^\dagger(q) \partial_q U(q) = 0 \right\}$$

is the critical set of the physical system on $M_{phys}$. We recall that generically, $C_Q$ is a smooth $n - k$-dimensional submanifold of $Q$. Evidently, $C$ is the smooth rank $n - k$ vector bundle

$$C = W^*_\beta |_{C_Q}$$

over the base manifold $C_Q$, whose fibres are given by those of the annihilator $W^*_\beta$ of $W$.

The arguments and results demonstrated in section two can be straightforwardly applied to the present problem. First of all, we claim that $C_Q$ is normal hyperbolic with respect to the gradient-like flow $\psi_t$ generated by

$$\partial_t q(t) = -p(q(t))\nabla_gU(q(t)),$$

and that it contains all critical points of the Morse function $U$, but no other conditional extrema of $U|_{C_Q}$ apart from those (it is gradient-like because along all of its non-constant orbits, $\dot{q}(t) = -g(p\nabla_gU, p\nabla_gU)|_{q(t)} < 0$ holds, as $p$ is an orthoprojector with respect to
the Riemannian metric $g$ on $Q$). The fact that this is true can be proved by substituting $M \to Q$, $H \to U$, $P \to \alpha$, $g_{(\text{Kahler})} \to g$, and $\mathcal{C} \to \mathcal{C}_Q$ in section two, and by applying the arguments used there. Hence, letting $\mu_i$ denote the index of the connected component $\mathcal{C}_{Q_i}$, defined as the dimension of its unstable manifold, the Conley-Zehnder inequalities (24) imply that for a compact, closed $Q$, the topological formula

$$\sum_{i,p} \lambda^p + \mu_i \dim H^p(\mathcal{C}_{Q_i}) = \sum_p \lambda^p \dim H^p(Q) + (1 + \lambda)Q(\lambda)$$

holds, where $\mathcal{C}_{Q_i}$ are the connected components of $\mathcal{C}_Q$. Here, $H^p$ denotes the $p$-th de Rham cohomology group with suitable coefficients, and $Q(t)$ is a polynomial with non-negative integer coefficients. The argument using the Morse-Witten complexes associated to $(Q, U)$ and to $(\mathcal{C}_Q, U|_{\mathcal{C}_Q})$ to derive (24) can also be carried out in the manner explained in section 2.4.

Our next issue is to discuss the global topology of $\mathcal{C}$. Clearly, $C$ is not a compact submanifold of $T^*Q$, hence the Conley-Zehnder inequalities of the second section, which were derived for compact, closed, generic critical manifolds, do not apply. However, since both $C$ and $T^*Q$ are vector bundles of a particular type, one can nevertheless prove a result that is closely related to (24). Nevertheless, we claim that given the above stated properties of $\mathcal{C}_Q$, the generalized Conley-Zehnder inequalities

$$\sum_{i,p} \lambda^p + \mu_i \dim H^p(\mathcal{C}_i) = \sum_p \lambda^p \dim H^p(T^*Q) + (1 + \lambda)Q(\lambda)$$

are valid. In this formula, the connected components $\mathcal{C}_i$ of $\mathcal{C}$ are vector bundles whose base manifolds are the connected components $\mathcal{C}_{Q_i}$ of $\mathcal{C}_Q$, and the numbers $\mu_i$ are the indices of $\mathcal{C}_{Q_i}$ with respect to $\psi_t$. The polynomial $Q(t)$ exhibits non-negative integer coefficients, and $H^p_c$ denotes the de Rham cohomology based on differential forms with compact supports.

In fact, (95) is a straightforward consequence of the circumstance that the base space of any vector bundle is a deformation retract the vector bundle. Hence, $\mathcal{C}_Q$, being the zero section of $\mathcal{C}$, is a deformation retract of $\mathcal{C}$, and likewise, $Q$ is a deformation retract of $T^*Q$. Since the de Rham cohomology groups are invariant under retraction, one infers the equality

$$H^p_c(\mathcal{C}_i) \cong H^p(\mathcal{C}_{Q_i}) \quad H^p_c(T^*Q) \cong H^p(Q).$$

Hence, formula (95) is equivalent to the assertion that

$$\sum_{i,p} \lambda^p + \mu_i \dim H^p(\mathcal{C}_{Q_i}) = \sum_p \lambda^p \dim H^p(Q) + (1 + \lambda)Q(\lambda).$$

But this has just been proved by application of CZ theory to the flow $\psi_t$ on $Q$. The weak Conley-Zehnder inequalities derived from this result are hence given by

$$\sum_i \dim H^{p - \mu_i}(\mathcal{C}_{Q_i}) \geq B_p,$$
where $B_p$ is the $p$-th Betti number of $Q$. In particular, for the special value $\lambda = -1$, one obtains

$$
\sum_{i,p} (-1)^{p+\mu_i} \dim H^p(C_{Q_i}) = \sum_{i} (-1)^{\mu_i} \chi(C_{Q_i}) = \chi(Q),
$$

where $\chi$ denotes the Euler characteristic.

**IV.3 Numerically determining the critical manifold**

Let us finally formulate another possible application of the results so far for technical purposes.

Knowledge about the location of equilibria is very crucial for the design of a constrained multibody system. It is particularly desirable to know whether a chosen set of parameters and constraints has the effect that the critical sets are generic or not.

Technical systems in engineering applications are often very large, so that their equilibria can usually only be constructed with the help of numerical methods. The analysis that we have employed in chapters two and four, aimed at investigating global topological properties of generic critical manifolds, inspires the following strategy.

We claim that if $U$ is a Morse function, whose critical points are known, and if $Q$ is compact and closed, it is possible to numerically construct all generic connectivity components of $C_{Q}$. This is because generic components of $C_{Q}$ are smooth, $n-k$-dimensional submanifolds of $Q$ containing all critical points of $U$, and no other critical points of $U|_{C_{Q}}$. This information can be exploited to find sufficiently many points on $C_{Q}$, so that suitable spline interpolation permits the approximate reconstruction of an entire connectivity component. To this end, one picks a vicinity of a critical point $a$ of $U$, and uses a fixed point solver to determine neighboring zeros of $|p(q)\nabla g U(q)|^2$, which are elements of $C_{Q}$ close to $a$. Iterating this procedure with the critical points found in this manner, pieces of $C_{Q}$ of arbitrary size can be determined.

If all critical points of $U$ are a priori known, one can proceed like this to construct all connectivity components of $C_{Q}$ that contain critical points of $U$. In this case, one is guaranteed to have found all of the generic components of $C$ if the numerically determined connectivity components are closed, compact, and contain all critical points of $U$.

We remark that the determination of the critical points of a Morse function $U : Q \to \mathbb{R}$ is a difficult numerical task. Attempting to find critical points by simulating the gradient flow generated by $-\nabla g U$ is presumably very time costly, because the critical points define a thin set in $M$. Their existence, however, is ensured by the topology of $Q$ if the latter is nontrivial, as a result from the Morse inequalities.

Another remark is that all critical points $a$ at which $\text{Jac}_a (p\nabla g U)$ has a reduced rank, are elements of the non-generic part of $C_{Q}$. If there are such exceptional critical points in a technically relevant region of $Q$, they can be removed by a small local modification of the system parameters and the constraints.
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