A Unifying Framework for Variance Reduction Algorithms for Findings Zeroes of Monotone Operators

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Abstract

A wide range of optimization problems can be recast as monotone inclusion problems. We propose a unifying framework for solving the monotone inclusion problem with randomized Forward-Backward algorithms. Our framework covers many existing deterministic and stochastic algorithms. Under various conditions, we can establish both sublinear and linear convergence rates in expectation for the algorithms covered by this framework. In addition, we consider algorithm design as well as asynchronous randomized Forward algorithms. Numerical experiments demonstrate the worth of the new algorithms that emerge from our framework.

Keywords: finite sum minimization, monotone inclusion, monotone operators, variance reduction algorithms, asynchronous computing

1. Introduction

Finite sum minimization encompasses many applications in machine learning (Bottou et al., 2018; Cortes and Vapnik, 1995; Dumais et al., 1998; Rumelhart et al., 1986). For example, many large-scale supervised machine learning problems, such as classical least-squares, text classification (Dumais et al., 1998), and support vector machines (Cortes and Vapnik, 1995), can be solved by minimizing the empirical risk which is the average of a loss function over the training data. Due to its wide application, there have been many studies on the design and analysis of algorithms for finite sum minimization.

The most well known algorithm for solving finite sum minimization is full gradient descent (FG) (Nesterov, 2014), which has a fast linear convergence rate. In large-scale machine learning, however, FG is computationally expensive since it requires calculation of the full gradient in every iteration. In view of this weakness, stochastic gradient descent (SG) (Robbins and Monro, 1951) has risen to a prominence. SG only computes the gradient of one function in each iteration, and thus requires much less computational power per-iteration compared to FG. Given the same number of gradient evaluations, it has been shown both theoretically and empirically that SG is able to achieve a lower empirical loss compared with FG, especially in early iterations (Liu and Nocedal, 1989; Nocedal, 1980; Bubeck, 2015). The comparison of the total computation required to obtain $\epsilon$-optimality...
also favors SG when the data size is large (Bottou et al., 2018). For a detailed comparison between the two algorithms, please refer to the comprehensive review in Bottou et al. (2018).

Although SG has many advantages over FG, an intrinsic disadvantage of the SG is that it only has a sublinear convergence rate in expectation because it uses a random gradient estimator (Robbins and Monro, 1951). To improve upon the performance of SG while retaining its strengths, the recent literature has featured a new class of variance reduction algorithms. This class of algorithms includes stochastic variance reduction gradient (SVRG) (Johnson and Zhang, 2013; Xiao and Zhang, 2014), SAGA (Defazio et al., 2014), hybrid stochastic average gradient (HSAG) (Reddi et al., 2015), and stochastic average gradient (SAG) (Nicolas et al., 2012; Schmidt et al., 2017). The core idea of these algorithms is to replace the random gradient estimator in SG with a dedicated variance reduced estimator. With this modification, all of these algorithms are able to achieve a linear convergence rate in expectation and still inherit the low per-iteration cost of SG. Many numerical experiments and real world applications have demonstrated that variance reduced algorithms are able to achieve a lower empirical loss than both FG and SG given the same computation budget (Defazio et al., 2014; Johnson and Zhang, 2013; Nicolas et al., 2012; Reddi et al., 2015; Schmidt et al., 2017; Xiao and Zhang, 2014). In fact, many of these variance reduced algorithms have been implemented in the popular Matlab library “SGDLibrary” (Kasai, 2017).

Because of the wide application of variance reduced algorithms, it is important to understand their convergence behaviour theoretically. In the original papers such as Defazio et al. (2014); Johnson and Zhang (2013); Reddi et al. (2015); Xiao and Zhang (2014), the convergence analyses are done in a case by case manner. As a result, these original analyses do not shed light on why this class of algorithms has the same convergence rate, that is, a linear convergence rate in expectation. It is thus desirable to have a unified convergence analysis to find the common theoretical ground underpinning these different variance reduction algorithms. A recent attempt at this unified analysis can be found in Hu et al. (2017), where the authors formulate some variance reduction algorithms as linear dynamical systems (LDS) and then use tools from control theory and semi-definite programming (SDP) to construct quadratic Lyapunov functions which decrease linearly in expectation. Their results unify the convergence analysis for the SAGA, Finito (Defazio et al., 2014), and stochastic dual coordinate ascent (SDCA) (Shalev-Shwartz and Zhang, 2013). LDS is applicable when we are able to establish the relationship between two consecutive points in the minimizing sequence. However, due to the complicated update rules in the inner iterations between two consecutive epochs, the LDS framework cannot be applied to epoch-based algorithms. Nevertheless, epoch-based algorithms are important members of the family of variance reduction algorithms.

The main objective of this study is to bridge this gap by providing a unified convergence analysis of variance reduction algorithms in the more general setting of monotone inclusion problems. Monotone operators have received wide attention in convex optimization (Ryu and Boyd, 2016). Furthermore, many famous deterministic algorithms such as FG, projected gradient (Goldstein, 1964; Levitin and Polyak, 1966), and alternating direction method of multipliers (ADMM) (Gabay, 1983) can be viewed as different operator splitting methods for monotone inclusion problems. Stochastic algorithms, such as SG, can also be extended to the monotone operator framework (Rosasco et al., 2016). In view of the
above, a natural idea is to design a variance-reduced randomized FB splitting scheme for each of the classical variance reduction algorithms. In the existing literature, we found that Balamurugan and Bach (2016) has established the linear convergence of SVRG and SAGA for saddle-point problems. Our present work builds on Balamurugan and Bach (2016) by unifying and generalizing their convergence analysis.

To unify the convergence analysis of variance reduction algorithms for monotone inclusion problems, the key idea of this study is to maintain a proxy for each of the individual operators in the forward step of the randomized FB splitting. These proxies are then used to construct an unbiased estimator of the exact operator in the forward step. By carefully designing the proxies and the estimator, the above scheme is able to recover many existing algorithms for finite sum minimization such as FG, SG, SAGA, SVRG, and HSAG. Under this formulation, we show that after each iteration, the expected distance to the optimum shrinks geometrically subject to a disturbance which depends on the variance of the estimator for the exact operator. Different stochastic algorithms yield estimators with different variance, and so they can have different convergence rates. Under the mild assumption of uniformly bounded proxies (which holds, for example, for SG where all the proxies are zero), we show that the resulting algorithms have a sublinear convergence rate in expectation. By using more granular control over the variance of the proxies, we are able to construct a Lyapunov function that decreases geometrically in expectation. The Lyapunov function consists of the distance to the optimum plus an error term determined by the value of proxies. This result is then applied to FG, SAGA, HSAG, and SVRG with unequal epoch lengths. As a special case, our analysis covers saddle point problems which have wide application in machine learning (Balamurugan and Bach, 2016), dynamic programming (Du et al., 2017), and game theory (Bovd and Vandenberghe, 2004).

In the monotone operator setting, we propose new algorithms that have a linear convergence rate in expectation. For example, we propose a new algorithm called SVRG-rand. Numerical studies reveal that SVRG-rand performs better than both classical SVRG and SVRG with increasing epochs. As another example, we borrow the idea from HSAG to combine random-epochs with SAGA. This hybrid algorithm again enjoys a linear convergence rate in expectation.

In the presence of large-scale data, parallel computing is often helpful to whittle down the computation time. We find that our proposed framework can be extended to asynchronous variants of variance-reduced randomized forward-backward splitting when one of the two operators in the monotone inclusion problem is zero. In the asynchronous case, in contrast to the synchronous case, we have a multicore structure for parallel computing where each of the cores may not have up-to-date information (Mania et al., 2017). Asynchronous SVRG, asynchronous SAGA, and asynchronous HSAG for solving finite sum minimization were studied in Mania et al. (2017); Reddi et al. (2015); Leblond et al. (2017). In this study, we use a similar framework of parallel computing to the one described in Mania et al. (2017); Reddi et al. (2015). Under similar assumptions on the proxies as for synchronous algorithms, we show that the iterates generated by asynchronous variance-reduced algorithms converge linearly in expectation to a tolerance of the optimum. Specifically, our results apply to the asynchronous variants of SVRG, SAGA, HSAG, and SVRG-rand. In contrast to the previous work on asynchronous algorithms which required sparsity assumptions (Leblond et al., 2017; Niu et al., 2011; Mania et al., 2017), our results still hold more
generally. In particular, the sparsity assumption is only valid for finite sum minimization but it is not valid for the general monotone inclusion problems considered in this paper (see for example, Du et al. (2017)).

The paper is organized as follows. Section 2 introduces the notation and basic concepts of monotone operators and our problem setting. Section 3 presents a unified framework for randomized FB splitting methods that recovers many classical deterministic and stochastic algorithms. In Section 4, we provide the convergence analysis for our general framework. The section is decomposed into two subsections, corresponding to sublinear and linear convergence rates. Section 5 then turns to algorithm design and proposes two new algorithms. The asynchronous extension of our general algorithmic framework is investigated in Section 6. Section 7 provides a detailed comparison of the numerical performance for some of the classical algorithms as well as our new algorithms. We conclude the paper with a discussion of our broader themes and possible future research topics. All supplemental proofs are provided in the Appendix.

2. Preliminaries

We use \( \| \cdot \| \) to denote the Euclidean norm on \( \mathbb{R}^d \). Next, we define some commonly used terms from the monotone operator literature. An operator on \( \mathbb{R}^d \) is a subset of \( \mathbb{R}^d \times \mathbb{R}^d \). The set \( \{ y \in \mathbb{R}^d \mid (x, y) \in F \} \) is also denoted as \( F(x) \). If \( F(x) \) is a singleton for all \( x \), then \( F \) is a function and we may write \( F(x) = y \). An operator \( F \) is called monotone if it satisfies

\[ (u - v)^T(x - y) \geq 0, \]

for all \( u \in F(x) \) and \( v \in F(y) \). If an operator is monotone and there is no monotone operator that properly contains it, then the operator is called a maximal monotone operator. An operator \( F \) is said to be strongly monotone with parameter \( \mu > 0 \) if

\[ (u - v)^T(x - y) \geq \mu \|x - y\|^2, \]

for all \( u \in F(x) \) and \( v \in F(y) \). An operator \( F \) on \( \mathbb{R}^d \) is \( L \)-Lipschitz if for all \( u \in F(x) \) and \( v \in F(y) \), we have

\[ \|u - v\| \leq L \|x - y\|. \]

When \( L < 1 \), \( F \) is called a contraction. When \( L = 1 \), \( F \) is called non-expansive. If \( F \) is Lipschitz, then \( F \) is single-valued (Ryu and Boyd, 2016).

With these basic definitions, we may introduce the problem setting for our study. Suppose we have two maximal monotone operators \( A \) and \( B = \frac{1}{n} \sum_{i=1}^{n} B_i \), \( i = 1, 2, \ldots, n \) on \( \mathbb{R}^d \). We are interested in solving the monotone inclusion problem

\[ 0 \in A(x) + \frac{1}{n} \sum_{i=1}^{n} B_i (x). \tag{1} \]

We introduce the following assumptions on the operators \( A \) and \( B_i, i = 1, 2, \ldots, n \).

**Assumption 1** (i) \( A \) is a maximal monotone operator on \( \mathbb{R}^d \). (ii) \( B = \frac{1}{n} \sum_{i=1}^{n} B_i \) is a \( \mu \)-strongly maximal monotone operator on \( \mathbb{R}^d \), where each \( B_i \) is \( L \)-Lipschitz.
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The assumptions here are slightly different from those in Balamurugan and Bach (2016). We assume strong monotonicity for $B$ and Lipschitz continuity for all $B_i$ to align our work with the current literature on finite sum minimization. In this setting, the sum function $f$ in (2) is usually assumed to be both strongly convex (corresponding to strong monotonicity) and strongly smooth (corresponding to Lipschitz continuity of the gradient).

Let $x^*$ denote a solution of (1). This solution is unique if $A + B$ is strongly monotone (Ryu and Boyd, 2016). The monotone inclusion problem (1) is general enough to accommodate many variants of the finite sum minimization problem. For example, the finite sum minimization problem can be formulated as

$$\min_{x \in C} f(x) + h(x), \quad f(x) \triangleq \frac{1}{n} \sum_{i=1}^{n} f_i(x),$$

where $C \subset \mathbb{R}^d$ is convex, $f_i : \mathbb{R}^d \to \mathbb{R}$ are convex loss functions, and $h : \mathbb{R}^d \to \mathbb{R}$ represents a convex penalty. This problem is equivalent to the monotone inclusion problem $0 \in \frac{1}{n} \sum_{i=1}^{n} \partial f_i(x) + \partial h(x) + \partial I_C(x)$, where $\partial f_i, \partial h \subset \mathbb{R}^d \times \mathbb{R}^d$ denote the subdifferentials of $f_i$ and $h$, and $I_C$ is the indicator function that equals zero if $x \in C$, and infinity otherwise.

In the monotone operator literature, the most classical algorithm for solving (1) is forward-backward (FB) splitting (Bauschke and Combettes, 2011; Passty, 1979). In this algorithm, for all $k \geq 0$ we have

$$x_{k+1} = (I + \gamma A)^{-1} (I - \gamma B) (x_k).$$

When $A = 0$, (3) becomes the forward step method (Ryu and Boyd, 2016):

$$x_{k+1} = (I - \gamma B) (x_k) = x_k - \gamma B(x_k),$$

for all $k \geq 0$. The following theorem states the convergence rate of the sequence produced by (3) in terms of the distance to the solution of (1).

**Theorem 1** Suppose Assumption 1 holds. Let $\{x_k\}_{k \geq 0}$ be the sequence produced by (3). Then, for all $k \geq 0$ we have

$$\|x_{k+1} - x^*\|^2 \leq (1 - 2 \gamma \mu + \gamma^2 L^2)\|x_k - x^*\|^2.$$

When $\gamma \in \left(0, \frac{2\mu}{L^2}\right)$, the sequence $\{x_k\}_{k \geq 0}$ converges linearly to the unique solution $x^*$ of (1).

Based on the observation that $(I + \gamma A)^{-1}$ is non-expansive, the proof for Theorem 1 follows the one for the Forward step (Ryu and Boyd, 2016), and we get the same rate.

3. Algorithmic framework: Randomized Forward-Backward splitting

In this section, we present a randomized FB splitting framework to extend (3). Our framework unifies many classical algorithms for solving finite sum problems. Furthermore, our framework provides a platform for the convergence analysis of different variance reduction algorithms.
Variance-reduction algorithms can be seen as randomized FB splitting. To this end, we consider the following randomized FB splitting method

\[ x_{k+1} = (I + \gamma_k A)^{-1} (I - \gamma_k \mathcal{G}) (x_k), \]

for all \( k \geq 0 \) where \( \mathcal{G} \) is an estimator of the operator \( B \) in (3) at all \( x \in \mathbb{R}^d \) and the form of \( \mathcal{G} \) will be specified below. In finite sum minimization, \( B \) corresponds to the subgradient of the empirical risk which is usually large-scale and time-consuming to evaluate, while \( \mathcal{G} \) can be chosen as a computationally cheap estimator of \( B \).

In order to specify the structure of \( \mathcal{G} \), we introduce the variables \( \phi^k = \{\phi_i^k\}_{i=1}^n \) for all \( k \geq 0 \). Further, we introduce a sequence of i.i.d. discrete uniform random variables \( \{I_k\}_{k \geq 0} \) with support on \( \{1, 2, \ldots, n\} \), where \( I_k \) represents the index of the individual operator to evaluate at the \( k \)th iteration. With these settings, we define \( \mathcal{G} \) to be a function of \( x_k, \phi^k \) and \( I_k \) with the following form

\[ \mathcal{G} \left(x_k, \phi^k, I_k\right) = B_{I_k}(x_k) - \phi_{I_k}^k + \frac{1}{n} \sum_{i=1}^n \phi_i^k, \]

for all \( k \geq 0 \).

Define the history of the algorithm up to the \( k \)th iteration as \( \{\mathcal{F}_k\}_{k \geq 0} \), that is to say, \( \mathcal{F}_k = \sigma(x_0, \phi^0, I_0, \cdots, x_{k-1}, \phi^{k-1}, I_{k-1}, x_k, \phi^k) \). Then \( \{\mathcal{F}_k\}_{k \geq 0} \) is a filtration. Because \( I_k \) is uniform, the expectation of the first term on the righthand side is \( B(x_k) \), while the expectation of the second term is equal to negative of the third term. Therefore, it is readily seen that \( \mathcal{G} \left(x_k, \phi^k, I_k\right) \) is an unbiased estimator of \( B(x_k) \). The filtration allows us to formally state the unbiasedness of \( \mathcal{G}(x_k, \phi^k, I_k) \) in (6) as an estimator of \( B(x_k) \).

**Lemma 2** Consider \( \mathcal{G}(x_k, \phi^k, I_k) \) defined in (6). Then \( \mathbb{E} [\mathcal{G} \left(x_k, \phi^k, I_k\right) | \mathcal{F}_k] = B(x_k) \), for all \( k \geq 0 \).

The unbiasedness of \( \mathcal{G} \left(x_k, \phi^k, I_k\right) \) holds for arbitrary choices of \( \phi_i^k \). Nevertheless, \( \phi_i^k \) is usually chosen as a past evaluated value of the operator \( B_i \), and the rule of determining the value of \( \phi_i^k \) depends on the specific algorithm. For example, \( \phi_i^k \) is the value of \( B_i \) at the beginning of the current epoch in SVRG. In SAGA, it is the last evaluated value of \( B_i \). For this choice, \( B_{I_k}(x_k) \) and \( \phi_{I_k}^k(x_k) \) in (6) are positively correlated, and the variance of their difference is expected to be smaller than the variance of \( B_{I_k}(x_k) \), thus achieving the desired variance reduction. In view of this connection, we call \( \phi_i^k \) a proxy for the value of the \( i \)th operator.

To complete iteration (5) for \( \mathcal{G} \) defined as in (6), we need to specify how the proxies \( \phi^k \) are updated at the end of the \( k \)th iteration. We denote the general update scheme as follows:

\[ \phi^{k+1} = U_k \left(x_k, \phi^k, I_k\right). \]

This update rule is allowed to depend on the iteration number, to allow for epoch-based algorithms such as SVRG.

The complete framework for randomized FB splitting is given in Algorithm 1.
Algorithm 1 The randomized Forward-backward splitting algorithm

Input:
An initial value \( x_0 \in \mathbb{R}^d \), an algorithm-specific \( \phi_0^i \in \mathbb{R}^d \) for all \( i \in \{1, 2, \ldots, n\} \), the total number of iterations \( T \).

1: for each \( k = 0, 1, \ldots, T \) do
2: Generate \( I_k \) uniformly in \( \{1, 2, \ldots, n\} \).
3: \( x_{k+1} = (I + \gamma_k A)^{-1}(x_k - \gamma_k (B_{I_k}(x_k) - \phi_{I_k}^k + \frac{1}{n} \sum_{i=1}^n \phi_{I_k}^k)) \).
4: \( \phi_{k+1}^i = U_k(x_k, \phi_k^i, I_k) \).
5: end for

Output:
6: \( x_{T+1} \)

As seen in Algorithm 1, randomized FB splitting is essentially determined by the update rule \( U_k \) in (7). By carefully choosing \( U_k \), our framework covers many classical algorithms for finite sum problems. We present some examples below, where \( B_i \) represents the gradient of the \( i \)th individual loss function and we use the constant step-size \( \gamma_k = \gamma \).

Example 1 FG (Nesterov, 2014). Algorithm 1 returns gradient descent when the update scheme \( U_k(x_k, \phi_k, I_k) \) is given by

\[
\phi_{k+1}^i = B_i(x_{k+1}), \quad i = 1, 2, \ldots, n,
\]

for all \( k \geq 0 \). Then, \( G(x_k, \phi_k, I_k) = \frac{1}{n} \sum_{i=1}^n B_i(x_k) = B(x_k) \).

Example 2 SG (Robbins and Monro, 1951). The update rule for SG with a constant step-size is

\[
x_{k+1} = (I + \gamma A)^{-1}(x_k - \gamma B_{I_k}(x_k)).
\]

This corresponds to setting \( U_k(x_k, \phi_k, I_k) \) to return

\[
\phi_{k+1}^i = 0, \quad i = 1, 2, \ldots, n,
\]

for all \( k \geq 0 \). In other words, the proxies are all zero, and \( G(x_k, \phi_k, I_k) \) in (6) is exactly \( B_{I_k}(x_k) \).

Example 3 SVRG (Johnson and Zhang, 2013; Xiao and Zhang, 2014). SVRG is epoch-based, and it updates the full gradient at the beginning of every epoch. The iteration rule of SVRG with epoch length \( m \geq 1 \) is

\[
x_{k+1} = (I + \gamma A)^{-1}\left(x_k - \gamma \left(B_{I_k}(x_k) - B_{I_k} (\tilde{x}) + \frac{1}{n} \sum_{i=1}^n B_i (\tilde{x})\right)\right),
\]

where \( \tilde{x} \) is the value of \( x \) at the beginning of the current epoch. This amounts to updating the proxies following

\[
\phi_{k+1}^i = \begin{cases} 
B_i(x_{k+1}), & m \mid (k + 1), \\
\phi_k^i, & m \nmid (k + 1), 
\end{cases}
\]

where \( a \mid b \) and \( a \nmid b \) indicate that \( a \) does and does not divide \( b \), respectively.
Example 4 SAGA (Defazio et al., 2014). SAGA is initialized by evaluating the full gradient at the initial iteration $k = 0$. Following the notation in Defazio et al. (2014), the update rule of SAGA for $k \geq 0$ is

$$x_{k+1} = (I + \gamma A)^{-1} \left( x_k - \gamma \left( B_{I_k}(x_k) - B_{I_k}(y^k_i) + \frac{1}{n} \sum_{i=1}^{n} B_i(y^k_i) \right) \right)$$

where $B_i(y^k_i)$ is the last evaluated value of $B_i$. After updating $x_{k+1}$ above, $y^k_i$ is updated following

$$y^k_{i+1} = \begin{cases} x_k, & i = I_k, \\ y^k_i, & i \neq I_k. \end{cases}$$

This is equivalent to initializing $\phi^0_i$ to be $B_i(x_0)$, and updating the proxies in Algorithm 1 following

$$\phi^{k+1}_i = \begin{cases} B_i(x_k), & i = I_k, \\ \phi^k_i, & i \neq I_k. \end{cases}$$

4. Convergence analysis

Now we analyze the convergence of the randomized FB splitting method in Algorithm 1. Our analysis relies on the fact that $G(x_k, \phi^k, I_k)$ is an unbiased estimator of $B(x_k)$. In addition, the variance of $G(x_k, \phi^k, I_k)$ plays a prominent role, as it determines the convergence rate of the algorithm.

In the next lemma, we provide a sufficiently tight upper bound on the conditional variance of $G(x_k, \phi^k, I_k)$.

**Lemma 3** Suppose Assumption 1 holds. Then for any $k \geq 0$, we have

$$\mathbb{E} \left[ \| G(x_k, \phi^k, I_k) - B(x_k) \|^2 | F_k \right] \leq 2 \left( L^2 \| x_k - x^* \|^2 + \frac{1}{n} \sum_{i=1}^{n} \| \phi^k_i - B_i(x^*) \|^2 \right).$$

**Proof** Direct calculation shows

$$\mathbb{E} \left[ \| G(x_k, \phi^k, I_k) - B(x_k) \|^2 | F_k \right] = \mathbb{E} \left[ \| B_{I_k}(x_k) - \phi^k_{I_k} - \mathbb{E} [ B_{I_k} - \phi^k_{I_k} | F_k ] \|^2 | F_k \right] \leq \mathbb{E} \left[ \| B_{I_k}(x_k) - \phi^k_{I_k} \|^2 | F_k \right].$$

Here, the equality is due to the facts that $\sum_{i=1}^{n} \phi^k_i / n = \mathbb{E} [ \phi^k_{I_k} | F_{k-1} ]$ and $B(x_k) = \mathbb{E} [ B_{I_k}(x_k) | F_{k-1} ]$. The inequality follows because the conditional variance of a random variable is not larger than its second moment. By adding and subtracting $B_{I_k}(x^*)$, we can further bound the last term of the above display with

$$\mathbb{E} \left[ \| B_{I_k}(x_k) - \phi^k_{I_k} \|^2 | F_k \right] = \mathbb{E} \left[ \| (B_{I_k}(x_k) - B_{I_k}(x^*)) - (\phi^k_{I_k} - B_{I_k}(x^*)) \|^2 | F_k \right] \leq 2 \mathbb{E} \left[ \| (B_{I_k}(x_k) - B_{I_k}(x^*)) - (\phi^k_{I_k} - B_{I_k}(x^*)) \|^2 | F_k \right] \leq 2 \left( L^2 \| x_k - x^* \|^2 + \frac{1}{n} \sum_{i=1}^{n} \| \phi^k_i - B_i(x^*) \|^2 \right),$$
by the triangle inequality. The first term of the righthand side of the last inequality is due to Lipschitz continuity of each $B_i$. The second term of the righthand side of the last inequality follows from direct calculation of the conditional expectation of $\|\phi^k_{f_k} - B_{f_k}(x^*)\|^2$ over $F_k$. ■

Now we show how the expected distance to the optimal solution $x^*$ changes after one iteration. This upcoming inequality determines the convergence rate of Algorithm 1. We note that in the literature on variance reduction for finite sum minimization, the convergence analysis is usually based on the expected optimality gap (Reddi et al., 2015; Xiao and Zhang, 2014), while our paper uses the expected distance to the optimal solution (without using function values).

**Lemma 4** Suppose Assumption 1 holds and let $\{x_k\}_{k \geq 0}$ be the sequence produced by Algorithm 1. Then, for $k \geq 0$ we have

$$E\|x_{k+1} - x^*\|^2 \leq (1 - 2\gamma_k \mu + \gamma_k^2 L^2) E\|x_k - x^*\|^2 + \gamma_k^2 E\|G(x_k, \phi^k, I_k) - B(x_k)\|^2.$$ 

**Proof** Because $x^*$ is the unique solution of (1), for all $\gamma > 0$ we have

$$x^* = (I + \gamma A)^{-1}(I - \gamma B)(x^*).$$

Furthermore, $(I + \gamma A)^{-1}$ is non-expansive operator for all $\gamma > 0$ (Ryu and Boyd, 2016, Section 6). Using the shorthand $G_k = G(x_k, \phi^k, I_k)$, we have

$$\|x_{k+1} - x^*\|^2 = \|(I + \gamma A)^{-1}(x_k - \gamma_k G_k) - (I + \gamma A)^{-1}(x^* - \gamma_k B(x^*))\|^2$$

$$\leq \|(x_k - \gamma_k G_k) - (x^* - \gamma_k B(x^*))\|^2$$

$$= \|(x_k - x^*) - \gamma_k (B(x_k) - B(x^*)) - \gamma_k (G_k - B(x_k))\|^2$$

$$= \|x_k - x^*\|^2 + \gamma_k^2 \|B(x_k) - B(x^*)\|^2 + \gamma_k^2 \|G_k - B(x_k)\|^2$$

$$- 2\gamma_k (B(x_k) - B(x^*))^T (x_k - x^*) - 2\gamma_k (x_k - x^*)^T (G_k - B(x_k))$$

$$+ 2\gamma_k^2 (B(x_k) - B(x^*))^T (G_k - B(x_k)).$$

The first equality in the above display is from the definition of $x_{k+1}$ and the fact that $x^*$ is a fixed point of $(I + \gamma A)^{-1}(I - \gamma B)$. The inequality is from non-expansiveness of $(I + \gamma A)^{-1}$, and the last equality is by direct expansion of the squared norm.

Using unbiasedness of $G_k$, we see that

$$E[\|x_{k+1} - x^*\|^2|F_k] \leq \|x_k - x^*\|^2 - 2\gamma_k (B(x_k) - B(x^*))^T (x_k - x^*)$$

$$+ \gamma_k^2 \|B(x_k) - B(x^*)\|^2 + \gamma_k^2 E[\|G_k - B(x_k)\|^2|F_k].$$

By strong monotonicity and Lipschitz continuity of $B$, we can bound the righthand side of the above display as

$$E[\|x_{k+1} - x^*\|^2|F_k] \leq (1 - 2\gamma_k \mu + \gamma_k^2 L^2) \|x_k - x^*\|^2 + \gamma_k^2 E[\|G_k - B(x_k)\|^2|F_k]. \quad (8)$$

Take expectations of both sides to get the desired conclusion. ■
The righthand side of (8) sheds light on the importance of variance reduction for Algorithm 1. That is, the shrinkage of the distance to the optimum depends on the variance of $G(x_k, \phi^k, I_k)$. The following corollary is immediate from the bounds in Lemma 3 and Lemma 4.

**Corollary 5** Suppose Assumption 1 holds and let $\{x_k\}_{k \geq 0}$ be the sequence generated by Algorithm 1. For all $k \geq 0$, we have

$$
\mathbb{E}\|x_{k+1} - x^*\|^2 \leq (1 - 2\gamma_k \mu + 3\gamma_k^2 L^2)\mathbb{E}\|x_k - x^*\|^2 + \frac{2\gamma_k^2}{n} \mathbb{E} \sum_{i=1}^{n} \|\phi^k_i - B_i(x^*)\|^2.
$$

When $1 - 2\gamma_k \mu + 3\gamma_k^2 L^2 < 1$ holds for all $k \geq 0$, Corollary 5 shows that the expected distance to the optimum shrinks geometrically with an additional disturbance after each iteration. The disturbance is related to the distance between the proxies and the associated operator values $B_i(x^*)$ for $i = 1, 2, \ldots, n$ at the optimum. Different algorithms give rise to different disturbances, and thus yield different convergence rates.

### 4.1 Sublinear convergence rate

We now examine conditions for Algorithm 1 to have a sublinear convergence rate. We first introduce the following assumption on $\phi^k_i$.

**Assumption 2** There exists $M_\phi > 0$ such that $\|\phi^k_i\| \leq M_\phi$ for all $i = 1, 2, \ldots, n$ and $k \geq 0$.

This assumption is usually imposed for the convergence analysis of SG as well as for asynchronous variance reduction algorithms (Mania et al., 2017). Note that we only need bounds on the proxies, not on the iterates $\{x_k\}_{k \geq 0}$ themselves. With this assumption in hand, we can show that the sequence generated by Algorithm 1 has a sublinear convergence rate in expectation.

**Theorem 6** Let $\gamma_k = \frac{2}{\mu(k + M_\gamma)}$, where $M_\gamma$ is a constant such that $M_\gamma \geq 6L^2/\mu^2$. Then for all $k \geq 0$, the sequence $\{x_k\}_{k \geq 0}$ generated by Algorithm 1 satisfies

$$
\mathbb{E}\|x_k - x^*\|^2 \leq \left(\frac{M_\gamma - 1}{k + M_\gamma - 1}\right)^2 \|x_0 - x^*\|^2 + \frac{8}{\mu^2(k + M_\gamma)} (M_\phi + M_B)^2,
$$

where $M_B \triangleq \sup_{i=1,2,\ldots,n} \|B_i(x^*)\|$. 

**Proof** For all $k \geq 0$, the triangle inequality gives

$$
\|\phi^k_i - B_i(x^*)\| \leq \|\phi^k_i\| + \|B_i(x^*)\| \leq M_\phi + M_B, \quad i = 1, 2, \ldots, n.
$$

Based on the above inequality and Corollary 5, we see that

$$
\mathbb{E}\|x_{k+1} - x^*\|^2 \leq (1 - 2\gamma_k \mu + 3\gamma_k^2 L^2)\mathbb{E}\|x_k - x^*\|^2 + 2\gamma_k^2 (M_\phi + M_B)^2.
$$

With the choice of $\gamma_k$ given in the statement of the theorem, we obtain the following result on the factor of the first term of the righthand side of (9):

$$
1 - 2\gamma_k \mu + 3\gamma_k^2 L^2 = 1 - \gamma_k \mu - 3\gamma_k L^2 (\frac{\mu}{3L^2} - \gamma_k) \leq 1 - \gamma_k \mu.
$$
Therefore,
\[
\mathbb{E}\|x_{k+1} - x^*\|^2 \leq (1 - \gamma_k \mu)\mathbb{E}\|x_k - x^*\|^2 + 2\gamma_k^2 (M_\phi + M_B)^2
\]
\[
= \frac{k - 2 + M_\gamma}{k + M_\gamma} \mathbb{E}\|x_k - x^*\|^2 + \frac{8}{(k + M_\gamma)^2 \mu^2} (M_\phi + M_B)^2.
\]

Apply the above inequality recursively to get
\[
\mathbb{E}\|x_{k+1} - x^*\|^2 \leq \frac{(M_\gamma - 2)(M_\gamma - 1)}{(k + M_\gamma)(k - 1 + M_\gamma)} \|x_0 - x^*\|^2
\]
\[
+ \frac{8(M_\phi + M_B)^2}{\mu^2 (k - 1 + M_\gamma)(k + M_\gamma)} \sum_{u=0}^{k} u + M_\gamma - 1
\]
\[
\leq \frac{(M_\gamma - 2)(M_\gamma - 1)}{(k + M_\gamma)(k - 1 + M_\gamma)} \|x_0 - x^*\|^2 + \frac{8(k + 1)(M_\phi + M_B)^2}{\mu^2 (k + M_\gamma)^2}
\]
\[
\leq \left( \frac{M_\gamma - 1}{k + M_\gamma - 1} \right)^2 \|x_0 - x^*\|^2 + \frac{8}{\mu^2 (k + M_\gamma)} (M_\phi + M_B)^2,
\]
where the second inequality uses the fact that the summand in the first line is less than one, and the last inequality is by scaling. When the number of iterations is large, the second terms on the righthand side of the above inequality will dominate the others.

**Example 5** SG (Robbins and Monro, 1951) For SG, for all \( k \geq 0 \) we have
\[
\phi^k_i = 0, \quad i = 1, 2, \ldots, n.
\]

Thus, Assumption 2 holds. Theorem 6 then implies a sublinear convergence rate for SG.

We mention in passing that if a constant step-size is chosen and Assumption 2 holds, then the iterates generated by Algorithm 1 will converge linearly to some tolerance of the optimum.

### 4.2 Linear convergence rate

With stronger assumptions on the second term \( \mathbb{E} \sum_{i=1}^n \|\phi^k_i - B_i(x^*)\|^2 \) on the right hand side of in Corollary 5, we can achieve a linear convergence rate in expectation. To accommodate epoch-based algorithms, we divide the index set \( \{1, 2, \ldots, n\} \) into two sets \( S \subseteq \{1, 2, \ldots, n\} \) and \( S^c \), where \( S^c \) denotes the individual operators that are updated only at the end of each epoch. The definition of \( S \) is algorithm-specific. For example, \( S = \emptyset \) in SVRG, \( S^c = \emptyset \) in SAGA and FG, and \( S \) can be any proper subset of \( \{1, 2, \ldots, n\} \) in HSGA. The proxies can then be divided into two groups according to \( S \) and \( S^c \): and for \( k \geq 0 \) we define
\[
G(\phi^k) \triangleq \frac{1}{n} \sum_{i \in S} \|\phi^k_i - B_i(x^*)\|^2, \quad (10)
\]
\[
H(\phi^k) \triangleq \frac{1}{n} \sum_{i \notin S} \|\phi^k_i - B_i(x^*)\|^2, \quad (11)
\]
where we stipulate that $G(\phi^k) \equiv 0$ if $S = \emptyset$ and $H(\phi^k) \equiv 0$ if $S^c = \emptyset$. It is readily seen that

$$G(\phi^k) + H(\phi^k) = \frac{1}{n} \sum_{i=1}^{n} \|\phi^k_i - B_i(x^*)\|^2.$$  

Next, for $\rho \geq 0$, we define

$$L_{\rho}(x_k, \phi^k) \triangleq \|x_k - x^*\|^2 + \rho G(\phi^k),$$

which is called a Lyapunov function in the literature (Balamurugan and Bach, 2016; Defazio et al., 2014).

Next we introduce our key assumptions on $G$ and $H$. Under these assumptions, we can show that the Lyapunov function contracts in expectation by a factor less than one. This contractive property then yields a linear convergence rate for the algorithm.

**Assumption 3**

(i.1) There exists constants $c_1$ and $c_2$ with $0 \leq c_1 < 1$, $c_2 \geq 0$, such that $\mathbb{E}G(\phi^{k+1}) \leq c_1 \mathbb{E}G(\phi^k) + c_2 \mathbb{E}\|x_k - x^*\|^2$.

(i.2) There exists a constant $c_3$ and a sequence \( \{m_i\}_{i=1}^\infty \) where each $m_i$ is a positive integer such that $H(\phi^k) \leq c_3 L_{\rho}(x_{S_i}, \phi^{S_i})$ for all $S_i \leq k < S_{i+1}$, where $S_0 = 0$, and $S_i = \sum_{j=1}^{i} m_j$ when $i \geq 1$. Further, $\bar{m} \triangleq \sup_i \{m_i\}_{i=1}^\infty < \infty$.

(i.3) The step-size $\gamma_k$ is constant for all $k \geq 0$, denoted as $\gamma$.

After presenting the general convergence results in this section, we will check that Assumption 3 holds for FG, SAGA, SVRG, and HSAG, where the constants $c_1$, $c_2$, $c_3$ and $\{m_j\}_{j=1}^\infty$ will be specified. Assumption 3.1 requires $G(\phi^k)$ to contract by a factor less than one plus an additional disturbance, while Assumption 3.2 allows for epochs with unequal lengths. When $S = \emptyset$, Assumption 3.1 holds automatically since $G(\phi^k) \equiv 0$. On the other hand, when $S = \{1, 2, \ldots, n\}$, Assumption 3.2 holds automatically since $H(\phi^k) \equiv 0$.

Using the notation in Reddi et al. (2015), for all $k \geq 0$ we let

$$\tilde{x}_k \triangleq x_{S_k}, \quad \tilde{\phi}^k \triangleq \phi^{S_k},$$

denote the iterates obtained at the end of each epoch. Our analysis will focus on the values of the Lyapunov function $L_{\rho}(\tilde{x}_k, \tilde{\phi}^k)$ at the end of each epoch. The following two constants

$$\theta \triangleq \max \left\{ 1 - 2\gamma \mu + 3\gamma^2 L^2 + c_2 \rho, \frac{2\gamma^2}{\rho} + c_1 \right\}, \quad (12)$$

$$\lambda \triangleq \theta + 2\gamma^2 \bar{m} c_3, \quad (13)$$

appear in our main result next.

**Theorem 7** Suppose Assumptions 1 and 3 hold. If the step-size $\gamma$ satisfies

$$\gamma < \min \left\{ \left( \frac{2\mu}{3(1-c_1)L^2 + (1-c_1)c_3 \bar{m} + c_2} \right)^2, \left( \frac{1 - c_1}{2 + 2\bar{m} \left( \frac{1-c_1}{2} \right)^3 c_3} \right)^2 \right\}, \quad (14)$$

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then by setting \( \rho = \gamma^{1.5} \), we have \( \lambda \in [0, 1) \), and for any \( k \geq 1 \),
\[
\mathbb{E}L_{\rho} \left( \tilde{x}_k, \tilde{\phi}^k \right) \leq \lambda \mathbb{E}L_{\rho} \left( \tilde{x}_{k-1}, \tilde{\phi}^{k-1} \right).
\] (15)

**Proof** We first show that \( \theta < 1 \). For the choices of \( \gamma \) and \( \rho \) given in the theorem, we have that \( \theta \) is the maximum of \( \ell_1 \) and \( \ell_2 \), where
\[
\ell_1 = 1 - 2\gamma \mu + 3\gamma^2 L^2 + c_2 \gamma^{1.5}, \quad \ell_2 = 2\gamma^{0.5} + c_1.
\]

Showing \( \theta < 1 \) is achieved by showing that both \( \ell_1 \) and \( \ell_2 \) are less than one. By shrinking the denominators of both terms inside the bracket of (14), we have
\[
\gamma < \min \left\{ \frac{2\mu}{3(1-c_1)L^2 + c_2}, \frac{(1-c_1)^2}{4} \right\}.
\] (16)

It is readily seen that \( \ell_2 < 1 \) because \( \gamma < (1-c_1)^2/4 \) from the above display. To show \( \ell_1 < 1 \), note that
\[
3\gamma L^2 + c_2 \gamma^{0.5} = \gamma^{0.5} (3\gamma L^2 + c_2) < \frac{2\mu}{3(1-c_1)L^2 + c_2} \left( \frac{3(1-c_1)L^2}{2} + c_2 \right),
\]
where we use the fact that \( \gamma \) is smaller than the first term in the bracket of (16) to get the first term in the last inequality, and the fact that \( \gamma \) is smaller than the second term in the bracket of (16) to get the second term in the last inequality. The last term on the above display equals \( 2\mu \). By expressing \( \ell_1 \) as \( 1 - \gamma(2\mu - 3\gamma L^2 - c_2 \gamma^{0.5}) \), it is readily see that \( \ell_1 < 1 \). As a result, we see that \( \theta < 1 \).

Next, we establish inequality (15). Based on Corollary 5 and Assumption 3, we have
\[
\mathbb{E}L_{\rho} \left( \tilde{x}_k, \tilde{\phi}^k \right) = \mathbb{E}\|x_{S_k} - x^*\|^2 + \rho \mathbb{E}G (\phi^{S_k})
\leq \left( 1 - 2\gamma \mu + 3\gamma^2 L^2 + c_2 \rho \right) \mathbb{E}\|x_{S_{k-1}} - x^*\|^2
+ \left( \frac{2\gamma^2}{\rho} + c_1 \right) \rho \mathbb{E}G (\phi^{S_{k-1}}) + 2\gamma^2 \mathbb{E}H (\phi^{S_{k-1}})
\leq \max \left\{ 1 - 2\gamma \mu + 3\gamma^2 L^2 + c_2 \rho, \frac{2\gamma^2}{\rho} + c_1 \right\} \mathbb{E}L_{\rho} (x_{S_{k-1}}, \phi^{S_{k-1}})
+ 2\gamma^2 \mathbb{E}H (\phi^{S_{k-1}}).
\]

Using the definition of \( \theta \), the above display is equivalent to
\[
\mathbb{E}L_{\rho} \left( \tilde{x}_k, \tilde{\phi}^k \right) \leq \theta \mathbb{E}L_{\rho} (x_{S_{k-1}}, \phi^{S_{k-1}}) + 2\gamma^2 \mathbb{E}H (\phi^{S_{k-1}}).
\]

By recursively applying the above inequality, we obtain
\[
\mathbb{E}L_{\rho} \left( \tilde{x}_k, \tilde{\phi}^k \right) \leq \theta^{m_k} \mathbb{E}L_{\rho} (x_{S_{k-1}}, \phi^{S_{k-1}}) + 2\gamma^2 \mathbb{E} \sum_{i=0}^{m_k-1} \theta^i H (\phi^{S_{k-1-i}})
\leq \left( \theta^{m_k} + 2\gamma^2 c_3 \sum_{i=0}^{m_k-1} \theta^i \right) \mathbb{E}L_{\rho} (x_{S_{k-1}}, \phi^{S_{k-1}}),
\]
where the last inequality is based on Assumption 3.2. Because \( \theta < 1 \) and \( m_k \leq \bar{m} \) for all \( k \geq 0 \), the above display implies

\[
\mathbb{E} L_\rho (x_{S_k}, \phi^{S_k}) \leq (\theta + 2\gamma^2 \bar{m}c_3) \mathbb{E} L_\rho (x_{S_{k-1}}, \phi^{S_{k-1}}) = \lambda \mathbb{E} L_\rho (x_{S_{k-1}}, \phi^{S_{k-1}}).
\]

We then finish the proof for Theorem 7 by showing that \( \lambda < 1 \). Using \( \theta \) defined in (12) and \( \lambda \) defined in (13), we may rewrite \( \lambda \) as

\[
\lambda = \max \left\{ 1 - 2\gamma \mu + 3\gamma^2 L^2 + c_2 \rho + 2\gamma^2 \bar{m}c_3, \frac{2\gamma^2}{\rho} + c_1 + 2\gamma^2 \bar{m}c_3 \right\}.
\]

With \( \rho = \gamma^{1.5} \), \( \lambda \) is the maximum of \( \bar{\ell}_1 \) and \( \bar{\ell}_2 \), where

\[
\bar{\ell}_1 = 1 - 2\gamma \mu + 3\gamma^2 L^2 + c_2 \gamma^{1.5} + 2\gamma^2 \bar{m}c_3, \quad \bar{\ell}_2 = 2\gamma^{0.5} + 2\gamma^2 \bar{m}c_3 + c_1.
\]

Now it suffices to show that both \( \bar{\ell}_1 \) and \( \bar{\ell}_2 \) are less than one. Consider \( \bar{\ell}_2 \) first. We have

\[
2\gamma^{0.5} + 2\bar{m}\gamma^2 c_3 = \gamma^{0.5} (2 + 2\bar{m}\gamma^{1.5} c_3) < \frac{1 - c_1}{2 + 2\bar{m}(1-c_1)^3 c_3} (2 + 2\bar{m}\gamma^{1.5} c_3),
\]

where the inequality follows because \( \gamma \) is less than the second term in the bracket of (14). Since \( \gamma \leq (1 - c_1)^2/4 \) by (16), the last term of (17) is less than \( 1 - c_1 \). From (17) we have that \( 2\gamma^{0.5} + 2\bar{m}\gamma^2 c_3 < 1 - c_1 \), and thus \( \bar{\ell}_2 < 1 \). To show \( \bar{\ell}_1 < 1 \), note that

\[
3\gamma L^2 + c_2 \gamma^{0.5} + 2\gamma \bar{m}c_3 = \gamma^{0.5} (3\gamma^{0.5} L^2 + 2\bar{m}c_3 \gamma^{0.5} + c_2)
\]

\[
< \frac{2\mu}{3(1-c_1)L^2 + (1-c_1)c_3 \bar{m} + c_2} (3\gamma^{0.5} L^2 + 2\bar{m}c_3 \gamma^{0.5} + c_2),
\]

where the inequality is because \( \gamma \) is less than the first term in the bracket (14). Since \( \gamma \leq (1 - c_1)^2/4 \), the above display is less than \( 2\mu \). Finally, rewrite \( \bar{\ell}_1 \) as \( 1 + \gamma (3\gamma L^2 + c_2 \gamma^{0.5} + 2\gamma \bar{m}c_3 - 2\mu) \) to see \( \bar{\ell}_1 < 1 \).

In fact, we can extend the choice \( \rho = \gamma^{1.5} \) to any \( \rho = \gamma^a \) for \( 1 < a < 2 \), and linear convergence will still hold. The proof of this statement is given in Appendix A.3 Nevertheless, the step size can have a very complicated expression. Theorem 7 gives a generic Lyapunov function that can be used to establish linear convergence for many variance reduction algorithms. In contrast, the Lyapunov functions that appear in the original papers on variance reduction for finite sum minimization are all different (Defazio et al., 2014; Johnson and Zhang, 2013; Reddi et al., 2015; Xiao and Zhang, 2014), and thus the corresponding analyses are different.

Based on Theorem 7, we immediately get a linear convergence rate for the expected distance to the optimum.

**Corollary 8** Suppose Assumptions 1 and 3 hold, and choose \( \gamma \) and \( \rho \) as in Theorem 7. Then for all \( k \geq 0 \) we have

\[
\mathbb{E} \| \tilde{x}_k - x^* \|^2 \leq \lambda^k (1 + \rho L^2) \| x_0 - x^* \|^2,
\]

where \( \lambda \in (0, 1) \).
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Proof From Theorem 7, we see that \( \lambda \in [0, 1) \) and that inequality (15) holds. We recursively apply (15) and use Lipschitz continuity of \( B_i \) to get

\[
\mathbb{E} L_\rho \left( \tilde{x}_k, \tilde{\phi}^k \right) \leq \lambda^k \mathbb{E} L_\rho \left( \tilde{x}_0, \tilde{\phi}^0 \right) \\
= \lambda^k \left( \|x_0 - x^*\|^2 + \frac{\rho}{n} \sum_{i \in S} \|B_i(x_0) - B_i(x^*)\|^2 \right) \\
\leq \lambda^k (1 + \rho L^2) \|x_0 - x^*\|^2.
\]

Finally, we use the definition of \( L_\rho \) to conclude that

\[
\mathbb{E} \|\tilde{x}_k - x^*\|^2 \leq \mathbb{E} L_\rho \left( \tilde{x}_k, \tilde{\phi}_k \right) \leq \lambda^k (1 + \rho L^2) \|x_0 - x^*\|^2.
\]

To conclude this section, we investigate the convergence of FG, SVRG, SAGA, and HSAG using our general framework. It is interesting to note that the theorem is applicable to FG which is not stochastic. The key is to verify that Assumption 3 is satisfied for each of these algorithms.

Example 6 FG (Nesterov, 2014). For FG, we choose \( S = \emptyset \) and so

\[
H(\phi^k) = \frac{1}{n} \|\phi^k - B_i(x^*)\|^2 = \frac{1}{n} \sum_{i = 1}^{n} \|B_i(x_k) - B_i(x^*)\|^2 \leq L^2 \|x_k - x^*\|.
\]

It is readily seen that Assumption 3 holds with \( c_1 = c_2 = 0 \), \( c_3 = L^2 \), and \( m_i = 1 \) for all \( i \geq 1 \).

Example 7 SVRG (Johnson and Zhang, 2013; Konen and Richtrik, 2017; Xiao and Zhang, 2014). We consider a generalization of Example 3 that allows for unequal epoch lengths. Let \( m_j \) be the length of the \( j \)th epoch such that \( \tilde{m} < \infty \). We select \( S = \emptyset \) and thus Assumption 3.1 holds with \( c_1 = c_2 = 0 \). Let \( S_k = \sum_{j = 1}^{k} m_j \). Since \( \{\phi_i^t\}_{i = 1}^{n} = \{B_i(x_{S_k})\}_{i = 1}^{n} \) for \( S_k \leq t < S_{k+1} \), we have

\[
H(\phi^t) = \frac{1}{n} \sum_{i = 1}^{n} \|B_i(x_{S_k}) - B_i(x^*)\|^2 \leq L^2 \|x_{S_k} - x^*\|^2 \leq L^2 L_\rho \left( x_{S_k}, \phi_{S_k} \right),
\]

where the first inequality is due to Lipschitz continuity of each \( B_i \), and the second inequality is due to the definition of the Lyapunov function. Therefore, Assumption 3.2 holds with \( c_3 = L^2 \).

Example 8 SAGA (Defazio et al., 2014). Let \( S = \{1, 2, \ldots, n\} \) and \( \{m_j = 1\}_{j = 1}^{\infty} \). Assumption 3.2 holds with \( c_3 = 0 \) since \( H(\phi^k) \equiv 0 \). To verify Assumption 3.1, observe that
$\phi_i^k$ changes to $B_i(x_{k-1})$ only when $I_{k-1} = i$, and so for all $k \geq 1$

$$
E \left[ G(\phi^k) | F_{k-1} \right] = \frac{1}{n} \sum_{i=1}^{n} \left( G(\phi^{k-1}) + \frac{1}{n} \left( \|B_i(x_{k-1}) - B_i(x^*)\| - \|\phi_i^{k-1} - B_i(x^*)\|^2 \right) \right) \\
= \left( 1 - \frac{1}{n} \right) G(\phi^{k-1}) + \frac{1}{n} \sum_{i=1}^{n} \frac{\|B_i(x_{k-1}) - B_i(x^*)\|^2}{n}.
$$

Since $B_i$ are all $L$-Lipschitz, we have

$$
E G(\phi^k) \leq (1 - \frac{1}{n}) E G(\phi^{k-1}) + \frac{L^2}{n} E \|x_{k-1} - x^*\|^2, \quad k \geq 1.
$$

We conclude that Assumption 3.1 holds with $c_1 = 1 - 1/n$ and $c_2 = L^2/n$.

**Example 9** HSAG (Reddi et al., 2015). HSAG can be viewed as the combination of SAGA and SVRG. HSAG chooses a set $S \subset \{1, 2, \ldots, n\}$ for which $\{\phi_i^k\}_{i \in S}$ follows a SAGA-type update, while $\{\phi_i^k\}_{i \notin S}$ follows an SVRG-type update with epoch length $m$. According to Reddi et al. (2015), the update scheme $U_k$ for the proxies follows

$$
\phi_i^{k+1} = \begin{cases} 
(I_k = i) B_i(x_k) + (I_k \neq i) \phi_i^k, & i \in S, \\
(I_m \mid (k+1)) B_i(x_{k+1}) + (I_m \not\mid (k+1)) \phi_i^k, & i \notin S.
\end{cases}
$$

Here, $\mathbb{I}(E) = 1$ for event $E$, and it equals 0 otherwise.

We now verify Assumption 3 for HSAG. First, consider Assumption 3.2. From the update scheme, we can see that for $i \notin S$, when $mk \leq t < mk + 1$, $\phi_i^t = B_i(x_{mk})$ and

$$
H(\phi^t) = \frac{1}{n} \sum_{i \notin S} \|B_i(x_{mk}) - B_i(x^*)\|^2 \\
\leq \frac{L^2}{n} \|x_{mk} - x^*\|^2 \\
\leq \frac{(n-S)L^2}{n} L_\rho(x_{mk}, \phi^{mk}).
$$

The first inequality is due to Lipschitz continuity of each $B_i$, and the second inequality is due to the definition of $L_\rho$. Therefore, Assumption 3.2 holds with $c_3 = (n-S) L^2/n$ and $m_j = m$ for all $j \geq 1$, where $S \triangleq |S|$ is the cardinality of $S$.

Next, consider Assumption 3.1. Note that $G(\phi^{k-1})$ changes only when $I_{k-1}$ takes a value in $S$. Take conditional expectation to see that for all $k \geq 1$,

$$
E \left[ G(\phi^k) | F_{k-1} \right] = \frac{1}{n} \sum_{i \notin S} G(\phi^{k-1}) \\
\quad + \frac{1}{n} \sum_{i \in S} \left( G(\phi^{k-1}) + \frac{1}{n} \left( \|B_i(x_{k-1}) - B_i(x^*)\|^2 - \|\phi_i^{k-1} - B_i(x^*)\|^2 \right) \right) \\
= G(\phi^{k-1}) + \frac{n}{n^2} \sum_{i \in S} \left( \|B_i(x_{k-1}) - B_i(x^*)\|^2 - \|\phi_i^{k-1} - B_i(x^*)\|^2 \right) \\
= \left( 1 - \frac{1}{n} \right) G(\phi^{k-1}) + \frac{1}{n^2} \sum_{i \in S} \|B_i(x_{k-1}) - B_i(x^*)\|^2.
$$
Then use the Lipschitz continuity of each $B_i$ to conclude that for all $k \geq 1$, 

$$
\mathbb{E}G(\phi^k) \leq \left(1 - \frac{1}{n}\right) \mathbb{E}G(\phi^{k-1}) + \frac{SL^2}{n^2} \mathbb{E}\|x_{k-1} - x^*\|^2.
$$

(19)

Thus, Assumption 3.1 holds with $c_1 = 1 - 1/n$ and $c_2 = SL^2/n^2$.

The original HSAG has equal epoch lengths. Under Assumption 3.2, we can also accommodate unequal lengths, as proposed in Example 7. The convergence analysis is similar to the above.

5. Algorithm design

It has been observed empirically that SVRG tends to be slower than SAGA given the same computational budget, while SAGA requires additional storage cost for the proxies $\phi^k$ (Defazio et al., 2014; Reddi et al., 2015). In this section, we propose a new algorithm that allows for random epoch lengths in SVRG, which we call SVRG-rand. This algorithm does not require additional storage cost, and our simulations suggest that SVRG-rand performs better than classical SVRG. We also build on the hybridization idea of HSAG to combine SVRG-rand and SAGA. This combination is able to achieve a reasonable trade-off between the computational budget and the storage requirement.

5.1 SVRG-rand

In classical SVRG, the full operator is evaluated every $m$ iterations. We consider a variant of SVRG where in the $k$th iteration, there is a probability $p_k \in [0, 1]$ of doing a full operator update. We define $p \triangleq \inf_{k \geq 0} \{p_k\}$. Accordingly, the update rule $U_k(x_k, \phi^k, I_k)$ for the proxies is

$$
\phi_i^{k+1} = \mathbb{I}(W_k \leq p_k)B_i(x_k) + \mathbb{I}(W_k > p_k)\phi_i^k, i = 1, 2, \ldots, n,
$$

(20)

where $\{W_k\}_{k \geq 0}$ is a sequence of i.i.d. standard uniform random variables. To initialize the algorithm, we let $\phi_i^0 = 0$, which differs from the initialization in SVRG with a full operator evaluation. Under a mild assumption on the sequence $\{p_k\}_{k \geq 0}$, the proposed algorithm has a linear convergence rate.

**Theorem 9** Suppose Assumption 1 holds and that $p > 0$, and set $\gamma$ and $\rho$ as in Theorem 7. Then, the sequence $\{x_k\}_{k \geq 0}$ produced by SVRG-rand satisfies (18), which means the sequence has a linear convergence rate in expectation.

**Proof** It suffices to verify Assumption 3 and then use Theorem 7. Let $S = \{1, 2, \ldots, n\}$. Then, Assumption 3.2 holds automatically with $c_3 = 0$ and $m_i = 1, i \geq 1$. To verify Assumption 3.1, we observe that the proxy $\phi_i^k$ has a probability $p_{k-1}$ of changing to $B_i(x_{k-1})$ and a probability $1 - p_{k-1}$ of remaining as $\phi_i^{k-1}$ for all $i = 1, 2, \ldots, n$. Take conditional expectations to see

$$
\mathbb{E}\left[G(\phi^k)\big| F_{k-1}\right] = \frac{p_{k-1}}{n} \sum_{i=1}^n \|B_i(x_{k-1}) - B_i(x^*)\|^2 + \frac{1 - p_{k-1}}{n} \sum_{i=1}^n \|\phi_i^{k-1} - B_i(x^*)\|^2.
$$

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Use Lipschitz continuity of each $B_i$ and the definition of $p$ to conclude that
\[ EG(\phi^k) \leq (1 - p_{k-1})EG(\phi^{k-1}) + p_{k-1}L^2E\|x_{k-1} - x^*\|^2 \]
\[ \leq (1 - p)EG(\phi^{k-1}) + L^2E\|x_{k-1} - x^*\|^2, \]
for all $k \geq 0$. Therefore, SVRG-rand satisfies Assumption 3 with $S = \{1, 2, \ldots, n\}$, $m_i = 1$ for $i \geq 1$, $c_3 = 0$, $c_1 = 1 - p$, and $c_2 = L^2$.

It is interesting to note that $S^c = \emptyset$ and $m_i = 1$, $i \geq 1$, in the convergence analysis above. This selection actually implies that the new algorithm is essentially not an epoch-based algorithm. Nevertheless, as an extension of SVRG, SVR GRand does not require additional storage cost. In SVRG, it has been shown that using a sequence of increasing epoch lengths tends to increase the converge speed (Allen-Zhu and Yuan, 2016). A possible justification is that the rate of change of the full operator value slows at later stages, and so we can achieve the same level of variance reduction in the estimated operator $G$ with less frequent updates. The recent study (Allen-Zhu and Yuan, 2016) suggested using an sequence of exponentially increasing epoch lengths.

An advantage of our random-epochs algorithm is to achieve a finer degree of control of the change in epoch lengths by allowing $p_k$ to continuously decrease over $k$. Finer control improves the convergence speed, as evidenced in Section 7. We shall also mention in passing that classical SVRG is a special case of the random-epochs algorithm if we set $p_k = 1$ when $k$ is a multiple of $m$, and 0 otherwise. SVRG with increasing epochs can be recovered in a similar vein.

### 5.2 SAGA+SVRG-rand

SVRG-rand needs to compute the full operator with a positive, though usually small, probability in every iteration. The full operator evaluation is usually time-consuming for large $n$. On the other hand, SAGA achieves variance reduction through storing the entire proxy vector, which avoids the need for periodic full operator evaluations.

Similar to Reddi et al. (2015), we can accelerate the convergence of the SVRG-rand by combining it with SAGA. To be specific, let $S_1 \subseteq \{1, 2, \ldots, n\}$ be the index set for the proxies following a SAGA-type update, and $S_2 = S_1^c$ be the index set for the proxies updated through SVRG-rand. Then, the update scheme for the proxies in the hybrid algorithm is

\[
\phi_i^{k+1} = \begin{cases} 
I(I_k = i) B_i(x_k) + I(I_k \neq i) \phi_i^k, & i \in S_1, \\
I(W_k \leq p_k) B_i(x_k) + I(W_k > p_k) \phi_i^k & i \in S_2,
\end{cases}
\]

where $\{W_k\}_{k \geq 0}$ is again a sequence of i.i.d. standard uniform random variables. Under the same assumption on $\{p_k\}$ as SVRG-rand, the proposed hybrid algorithm has a linear convergence rate.

**Theorem 10** Suppose Assumption 1 holds and $p > 0$, and choose $\gamma$ and $\rho$ as in Theorem 7. Then, the hybrid algorithm with update rule given by (21) has a linear convergence rate.
Proof Define functions $G_1$ and $G_2$ as
\[
G_1(\phi^k) \triangleq \frac{1}{n} \sum_{i \in \mathcal{S}_1} ||\phi^k_i - B_i(x^*)||^2,
\]
\[
G_2(\phi^k) \triangleq \frac{1}{n} \sum_{i \in \mathcal{S}_2} ||\phi^k_i - B_i(x^*)||^2.
\]

It suffices to verify Assumption 3. By choosing $\mathcal{S} = \{1, 2, \ldots, n\}$, we have $H(\phi^k) \equiv 0$ and $G(\phi^k) = G_1(\phi^k) + G_2(\phi^k)$. Assumption 3.2 holds automatically with $c_3 = 0$ and $m_i = 1$, $i \geq 1$. We then check Assumption 3.1. From (19), we have for $k \geq 1$,\[
E G_1(\phi^k) \leq \left(1 - \frac{1}{n}\right) E G_1(\phi^{k-1}) + \frac{S_1 L^2}{n^2} E \|x_{k-1} - x^*\|^2.
\]
Here, $S_1$ denotes the cardinality of $\mathcal{S}_1$. Similar to the proof for Theorem 9, it can be shown that for $k \geq 1$,
\[
E G_2(\phi^k) \leq (1 - p) E G_2(\phi^{k-1}) + \frac{S_2 L^2}{n} E \|x_{k-1} - x^*\|^2,
\]
where $S_2$ is the cardinality of $\mathcal{S}_2$. Combine the above two inequalities to see that
\[
E G(\phi^{k+1}) = E G_1(\phi^{k+1}) + E G_2(\phi^{k+1})
\leq \left(1 - \frac{1}{n}\right) E G_1(\phi^k) + (1 - p) E G_2(\phi^k) + \left(\frac{S_1 L^2}{n^2} + \frac{S_2 L^2}{n}\right) E \|x_k - x^*\|^2
\leq \max\left\{1 - \frac{1}{n}, 1 - p\right\} E \left(G_1(\phi^k) + G_2(\phi^k)\right)
\leq \max\left\{1 - \frac{1}{n}, 1 - p\right\} E G(\phi^k) + \left(\frac{S_1 L^2}{n^2} + \frac{S_2 L^2}{n}\right) E \|x_k - x^*\|^2.
\]
We then conclude that the hybrid algorithm satisfies Assumption 3 with $\mathcal{S} = \{1, 2, \ldots, n\}$, $m_i = 1$ for $i \geq 1$, $c_3 = 0$, $c_1 = \max\left\{1 - 1/n, 1 - p\right\}$, and $c_2 = S_1 L^2/n^2 + S_2 L^2/n$.

6. Asynchronous variance reduction algorithms

In this section, we study the asynchronous extension of Algorithm 1 for the forward step method (4) where $A \equiv 0$ and and the solution satisfies $B(x^*) = 0$.

6.1 Asynchronous setting

Our asynchronous setting is similar to those in Hogwild! (Niu et al., 2011), AsySCD (Liu et al., 2015), and PASSCoDe (Hsieh et al., 2015). We assume a multicore architecture where each core makes updates to a centrally stored vector $x$ in an asynchronous manner.
Algorithm 2 Asynchronous randomized forward-step algorithm

Input:
An initial value $x_0 \in \mathbb{R}^d$, an algorithm-specific $\phi^0_i \in \mathbb{R}^d$ for all $i \in \{1, 2, \ldots, n\}$, the total number of iterations $T$.

1: While the number of updates $\leq T$ do in parallel
2: Read the central $x$ and the proxies $\phi$ in the central memory
3: Random sampling an integer $I_k$ from $\{1, 2, \ldots, n\}$, compute $B_{I_k}(x)$
4: Add $\gamma(B_{I_k}(x) - \phi_{I_k} + \frac{1}{n} \sum_{i=1}^{n} \phi_i)$ to the centrally stored $x$ and let proxies update according to the specific update scheme in Algorithm 1.
5: End while

Output:
6: $x_{T+1}$

The main purpose is to make use of the multicore structure to reduce the computation time. The framework of the general asynchronous randomized forward-step algorithm is described in Algorithm 2.

Algorithm 2 has three main steps including read, evaluation, and update. Following the notation in Reddi et al. (2015), we use a global counter $k$ to track the number of updates that are successfully executed to the centrally stored $x$. Such an after-write approach has become a standard global labeling scheme in the literature (Hsieh et al., 2015; Leblond et al., 2017; Liu et al., 2015; Niu et al., 2011; Reddi et al., 2015). The value of centrally stored $x$ and $\phi$ after $k$ updates are denoted as $x_k$ and $\phi_k$. Each processor does the read-evaluation-update steps concurrently, and so $x$ and $\phi$ can have different time labels in the read and update steps. Following Reddi et al. (2015), we use $D(k) \in \{1, 2, \ldots, k\}$ to denote the time label of the particular $x$ and proxies in the read step of the $(k+1)$th update. This means the proxies and $x$ used to compute the value added to the centrally stored $x$ in $(k+1)$th update are actually $\phi^{D(k)}$ and $x_{D(k)}$. Following Leblond et al. (2017); Mania et al. (2017), we define

$$\hat{x}_k \equiv x_{D(k)}, \hat{\phi}^k_i \equiv \phi^{D(k)}_i, k \geq 0.$$ 

We make the following assumption for the convergence analysis of the asynchronous Algorithm 2

**Assumption 4**

(i.4.1) There exists a non-negative integer $\tau$ such that $0 \leq k - D(k) \leq \tau$ for all $k \geq 0$.

(i.4.2) There exists a constant $M_{\phi, B} \geq 0$ such that $\|\phi^k_i\| \leq M_{\phi, B}$ and $\|B_i(x_k)\| \leq M_{\phi, B}$ for all $k \geq 0$ and $i = 1, 2, \ldots, n$.

The first assumption bounds the delay between the read and updates times by a non-negative integer $\tau$. This assumption is typical in asynchronous systems (Mania et al., 2017; Reddi et al., 2015). Following Mania et al. (2017), we also assume that $\|B_i(x_k)\|$ and $\|\phi^k_i\|$ are bounded for all $k \geq 0$ and $i = 1, 2, \ldots, n$.

In our setting, the asynchronous randomized forward step can be analyzed with the perturbed iterate framework from Mania et al. (2017) which has the following form

$$x_{k+1} = x_k - \gamma G(\hat{x}_k, \hat{\phi}^k, I_k), k \geq 0,$$ (22)
where

\[ G(\tilde{x}_k, \tilde{\phi}^k, I_k) \triangleq B_{I_k}(\tilde{x}_k) - \tilde{\phi}^k_{I_k} + \frac{1}{n} \sum_{i=1}^{n} \phi_i^k, \quad k \geq 0. \]

Here, \( \tilde{x}_k \) and \( \tilde{\phi}^k \) can be interpreted as perturbed versions of \( x_k \) and \( \phi^k \) due to asynchronicity. Similar to Lemma 2, it is readily seen that \( \mathbb{E} \left[ G(\tilde{x}_k, \tilde{\phi}^k, I_k) | F_k \right] = B(\tilde{x}_k) \), and so the asynchronous operator estimator is also unbiased. Apply (22) recursively to see that for all \( k \geq 1 \),

\[
x_k = x_0 - \gamma G(\tilde{x}_0, \tilde{\phi}^0, I_0) - \gamma G(\tilde{x}_1, \tilde{\phi}^1, I_1) - \cdots - \gamma G(\tilde{x}_{k-1}, \tilde{\phi}^{k-1}, I_{k-1}). \tag{23}
\]

It is interesting to note that \( \hat{x}_k = x_k \) when \( \tau = 0 \). Therefore, the above scheme automatically recovers the synchronous algorithms when \( \tau = 0 \).

### 6.2 Convergence analysis

The following lemma is the key to our analysis of Algorithm 2. In this lemma, we show how the expected distance to the optimal point \( x^* \) changes after one iteration of the asynchronous algorithm. Similar to Lemma 4, we show that the expected distance to the optimum is contracted by a factor strictly less than one plus an additional disturbance. For asynchronous algorithms, this disturbance is affected by the variance of the estimator and additional error due to asynchronous updates.

**Lemma 11** Suppose Assumptions 1 and 4 hold, and let \( \{x_k\}_{k \geq 0} \) be generated by Algorithm 2. Then, for all \( k \geq 0 \) we have

\[
\mathbb{E}\|x_{k+1} - x^*\|^2 \leq (1 - \gamma \mu)\mathbb{E}\|x_k - x^*\|^2 + 2\gamma \mu \mathbb{E}\|\tilde{x}_k - x_k\|^2 + \gamma^2 \mathbb{E}\|G(\tilde{x}_k, \tilde{\phi}^k, I_k)\|^2 + 2\gamma \mathbb{E}\langle G(\tilde{x}_k, \tilde{\phi}^k, I_k), \tilde{x}_k - x_k \rangle. \tag{24}
\]

**Proof** Use the definition of \( x_{k+1} \) to see

\[
\|x_{k+1} - x^*\|^2 = \|x_k - x^* - \gamma G(\tilde{x}_k, \tilde{\phi}^k, I_k)\|^2
= \|x_k - x^*\|^2 - 2\gamma \langle G(\tilde{x}_k, \tilde{\phi}^k, I_k), x_k - x^* \rangle + \gamma^2 \|G(\tilde{x}_k, \tilde{\phi}^k, I_k)\|^2
= \|x_k - x^*\|^2 - 2\gamma \langle G(\tilde{x}_k, \tilde{\phi}^k, I_k), \tilde{x}_k - x^* \rangle + \gamma^2 \|G(\tilde{x}_k, \tilde{\phi}^k, I_k)\|^2
+ 2\gamma \langle G(\tilde{x}_k, \tilde{\phi}^k, I_k), \tilde{x}_k - x_k \rangle.
\]

Here, the second equality is from a direct expansion of the squared norm and the third equality is by adding and subtracting \( \tilde{x}_k \). Due to the unbiasedness of \( G(\tilde{x}_k, \tilde{\phi}^k, I_k) \) and the strong monotonicity of \( B \), we have

\[
\mathbb{E}\left[ \langle G(\tilde{x}_k, \tilde{\phi}^k, I_k), \tilde{x}_k - x^* \rangle | F_k \right] = \langle B(\tilde{x}_k) - B(x^*), \tilde{x}_k - x^* \rangle \geq \mu \|\tilde{x}_k - x^*\|^2. \tag{25}
\]
Use the triangle inequality to see 
\[ \frac{\mu}{2} ||x_k - x^*||^2 = \frac{\mu}{2} ||x_k - \hat{x}_k + \hat{x}_k - x^*||^2 \leq \mu \left(||x_k - \hat{x}_k||^2 + ||\hat{x}_k - x^*||^2\right). \]
As a result, 
\[ \mu ||\hat{x}_k - x^*||^2 \geq \frac{\mu}{2} ||x_k - x^*||^2 - \mu ||x_k - \hat{x}_k||^2. \] (26)

Combine Equations (25)-(26) to get 
\[ \mathbb{E} \left[ ||x_{k+1} - x^*||^2 | \mathcal{F}_k \right] \leq (1 - \gamma \mu) ||x_k - x^*||^2 + 2\gamma \mathbb{E} \left[ \langle G(\hat{x}_k, \hat{\phi}_k, I_k), \hat{x}_k - x_k \rangle | \mathcal{F}_k \right] + 2 \gamma \mu ||\hat{x}_k - x_k||^2 + \gamma^2 \mathbb{E} \left[ ||G(\hat{x}_k, \hat{\phi}_k, I_k)||^2 | \mathcal{F}_k \right]. \]

Take expectation on both sides of the above inequality to establish (24). □

Equation (24) shows that the convergence of Algorithm 2 is closely related to three error terms: the distance between the true \( x_k \) and its perturbed version \( \hat{x}_k \), the norm of the estimator \( G(\hat{x}_k, \hat{\phi}_k, I_k) \), and the inner product of the mismatch \( x_k - \hat{x}_k \) and the estimator \( G(\hat{x}_k, \hat{\phi}_k, I_k) \). Let \( M \triangleq 3M_{\phi, B} \). The following lemma bounds the three error terms.

**Lemma 12** Suppose Assumption 1 and Assumption 4 hold. Then for \( \{x_k\}_{k \geq 0} \) generated by Algorithm (2), we have that for all \( k \geq 0 \),
\[ \mathbb{E} \|x_k - \hat{x}_k\|^2 \leq \gamma^2 \tau^2 M^2, \]
\[ \mathbb{E} \langle G(\hat{x}_k, \hat{\phi}_k, I_k), \hat{x}_k - x_k \rangle \leq \gamma \tau M^2, \]
and
\[ \mathbb{E} \|G(\hat{x}_k, \hat{\phi}_k, I_k)\|^2 \leq 6L^2 \gamma^2 \tau^2 M^2 + 6L^2 \mathbb{E} \|x_k - x^*\|^2 + \frac{4}{n} \mathbb{E} \sum_{i=1}^{n} \|\phi_{i}^k - B_{i}(x^*)\|^2 + 4 \mathbb{E} \sum_{i=1}^{n} \|\phi_{i}^k - \hat{\phi}_{i}^k\|^2. \]

**Proof** We first use the definition of \( M \) to see that \( \|G(\hat{x}_k, \hat{\phi}_k, I_k)\|^2 \leq M^2 \). Since \( \hat{x}_k = x_{D(k)} \), we may use (23) to see that 
\[ \|x_k - \hat{x}_k\|^2 = \gamma^2 \|G(\hat{x}_{D(k)}, \hat{\phi}_{D(k)}, I_{D(k)}) + \cdots + G(\hat{x}_{k-1}, \hat{\phi}_{k-1}, I_{k-1})\|^2 \leq \gamma^2 \tau^2 M^2. \] (27)

The inequality above is due to the fact that the number of terms inside the norm is not more than \( \tau \) from Assumption 4. To bound \( \langle G(\hat{x}_k, \hat{\phi}_k, I_k), \hat{x}_k - x_k \rangle \), we use (27) to see that 
\[ \langle G(\hat{x}_k, \hat{\phi}_k, I_k), \hat{x}_k - x_k \rangle \leq \|G(\hat{x}_k, \hat{\phi}_k, I_k)\| \|\hat{x}_k - x_k\| \leq \gamma \tau M^2. \]

The first inequality above is from the Cauchy-Schwarz inequality, and the second is from Equation (27) and the fact \( \|G(\hat{x}_k, \hat{\phi}_k, I_k)\| \leq M \) for all \( k \geq 0 \). We now bound \( \mathbb{E} \|G(\hat{x}_k, \hat{\phi}_k, I_k)\|^2 \).
Use the fact $B(x^*) = 0$ to see
\[
\mathbb{E} \left[ \|G(\hat{x}_k, \hat{\phi}_k, I_k)\|^2 \mid F_k \right] = \mathbb{E} \left[ \|G(\hat{x}_k, \hat{\phi}_k, I_k) - B(\hat{x}_k) + B(\hat{x}_k) - B(x^*)\|^2 \mid F_k \right]
\]
\[
= \mathbb{E} \left[ \|G(\hat{x}_k, \hat{\phi}_k, I_k) - B(\hat{x}_k)\|^2 \mid F_k \right] + \|B(\hat{x}_k) - B(x^*)\|^2
\]
\[
\leq \mathbb{E} \left[ \|B_{I_k}(\hat{x}_k) - \hat{\phi}_{I_k}\|^2 \mid F_k \right] + L^2 \|\hat{x}_k - x^*\|^2.
\]

The second equality above is due to the unbiasedness of $G(\hat{x}_k, \hat{\phi}_k, I_k)$. The first term of the last inequality is from similar arguments to Lemma 3. The second term of the last inequality follows from Lipschitz continuity of $B$. By adding and subtracting $B_{I_k}(x^*)$ in the first term of the last inequality, we can further bound the above display with
\[
\mathbb{E} \left[ \|G(\hat{x}_k, \hat{\phi}_k, I_k)\|^2 \mid F_k \right] \leq \mathbb{E} \left[ \|B_{I_k}(\hat{x}_k) - B_{I_k}(x^*) + B_{I_k}(x^*) - \hat{\phi}_{I_k}\|^2 \mid F_k \right] + L^2 \|\hat{x}_k - x^*\|^2
\]
\[
\leq 3L^2 \|\hat{x}_k - x^*\|^2 + 2\mathbb{E} \left[ \|B_{I_k}(x^*) - \hat{\phi}_{I_k}\|^2 \mid F_k \right]
\]
\[
\leq 3L^2 (2\|\hat{x}_k - x_k\|^2 + 2\|x_k - x^*\|^2)
\]
\[
+ 2 \left( 2\mathbb{E} \left[ \|B_{I_k}(x^*) - \phi_{I_k}\|^2 \mid F_k \right] + 2\mathbb{E} \left[ \|\phi_{I_k} - \hat{\phi}_{I_k}\|^2 \mid F_k \right] \right)
\]
\[
= 6L^2 \|\hat{x}_k - x_k\|^2 + 6L^2 \|x_k - x^*\|^2 + \frac{4}{n} \sum_{i=1}^{n} \|\phi_{I_k} - B_i(x^*)\|^2 + \frac{4}{n} \sum_{i=1}^{n} \|\phi_{I_k} - \hat{\phi}_{I_k}\|^2.
\]

Here, the second and third inequalities are due to the triangle inequality. The last equality is by direct calculation of the conditional expectation. Use the fact that $\mathbb{E}\|x_k - \hat{x}_k\|^2 \leq \gamma^2 \tau^2 M^2$ and take expectations to conclude. 

For all $k \geq 0$, we define the following notation
\[
a_k \triangleq \mathbb{E}\|x_k - x^*\|^2,
\]
\[
\mathcal{E}_0 \triangleq 2\gamma^3 \mu \tau^2 M^2 + 6\gamma^4 L^2 \tau^2 M^2 + 2\gamma^2 \tau M^2.
\]
It is readily seen that the following corollary holds.

**Corollary 13** Suppose Assumptions 1 and 4 hold, and let $\{x_k\}_{k \geq 0}$ be generated by Algorithm 2. Then, for all $k \geq 0$ we have
\[
a_{k+1} \leq (1 - \gamma \mu + 6\gamma^2 L^2) a_k + \frac{4\gamma^2}{n} \mathbb{E} \sum_{i=1}^{n} \|\phi_{I_k} - B_i(x^*)\|^2 + \frac{4\gamma^2}{n} \mathbb{E} \sum_{i=1}^{n} \|\phi_{I_k} - \hat{\phi}_{I_k}\|^2 + \mathcal{E}_0. \tag{28}
\]

The above corollary is a direct result of Lemmas 11 and 12. Corollary 13 has a similar form to Corollary 5. However, Equation (28) includes two additional terms for the error.
There exist constants \( c_1, c_2, \) and \( E_1 \) with \( 0 \leq c_1 < 1, c_2 \geq 0, \) and \( E_1 \geq 0, \) such that \( \mathbb{E}G(\phi^{k+1}) \leq c_1\mathbb{E}G(\phi^k) + c_2\mathbb{E}\|x_k - x^*\|^2 + E_1. \)

There exist a constant \( c_3 \) and a sequence \( \{m_i\}_{i=1}^{\infty} \) where each \( m_i \) is a positive integer such that \( H(\phi^k) \leq c_3L_\rho(x_{S_i}, \phi^{S_i}) \) for all \( S_i \leq k < S_{i+1} \), where \( S_i \) and \( L_\rho \) are defined as in Assumption 3.

There exists a constant \( E_2 \geq 0 \) such that \( \sum_{i=1}^{n} \mathbb{E}\|\phi_i^k - \hat{\phi}_i^k\|^2/n \leq E_2. \)

The above assumption is similar to Assumption 3 with the additional requirement that the average expected distance between \( \hat{\phi}_i^k \) and \( \phi_i^k \) is bounded by a constant. Following our earlier notation, we let \( \tilde{x}_k = x_{S_i} \) and \( \hat{\phi}_k = \phi^{S_i} \) and focus our analysis on \( L_\rho(\tilde{x}_k, \hat{\phi}_k) \). We show that for update schemes satisfying Assumption 5, \( \mathbb{E}L_\rho(\tilde{x}_k, \hat{\phi}_k) \) is contracted by a factor strictly less than one plus an additional disturbance. Before presenting our main theorem for asynchronous algorithms, we define following new constants

\[
\theta \triangleq \max \left\{ 1 - \gamma\mu + 6\gamma^2L^2 + c_2\rho, \frac{4\gamma^2}{\rho} + c_1 \right\},
\]

\[
\lambda \triangleq \theta + 4\gamma^2\bar{m}c_3,
\]

and

\[
E_3 \triangleq 4\gamma^2E_2 + E_0 + \rho E_1,
\]

where \( \bar{m} \) is defined as in Assumption 3.

**Theorem 14** Suppose Assumptions 1, 4, and 5 hold. If the step-size \( \gamma > 0 \) satisfies

\[
\gamma < \min \left\{ \left( \frac{\mu}{3(1-c_2)L^2} + (1 - c_1)\bar{m}c_3 + c_2 \right)^2 \left( \frac{1 - c_1}{4 + 4\bar{m}(1-c_2)^3c_3} \right)^2 \right\},
\]

then by setting \( \rho = \gamma^{1.5} \), we have \( \theta, \lambda \in [0, 1] \), and for any \( k \geq 1 \),

\[
\mathbb{E}L_\rho(\tilde{x}_k, \hat{\phi}_k) \leq \lambda\mathbb{E}L_\rho(\tilde{x}_{(k-1)}, \hat{\phi}^{(k-1)}) + \frac{E_3}{1 - \theta}.
\]

**Proof** We first show that \( \theta < 1 \). With the choices of \( \gamma \) and \( \rho \) given in the theorem, we have that \( \theta \) is the maximum of \( \ell_1 \) and \( \ell_2 \), where

\[
\ell_1 = 1 - \gamma\mu + 6\gamma^2L^2 + c_2\gamma^{1.5}, \quad \ell_2 = 4\gamma^{0.5} + c_1.
\]
It suffices to show that both $\ell_1$ and $\ell_2$ are less than one. By shrinking the denominators of both terms inside the bracket of (29), we have

$$
\gamma < \min \left\{ \left( \frac{\mu}{3(1-c_1)L^2 + c_2} \right)^2, \left( \frac{1-c_1}{4} \right)^2 \right\}.
$$

(31)

It is readily seen that $\ell_2 < 1$ because $\gamma < (1-c_1)^2/4$ from the above display. Similar to the proof for Theorem 7, we can use (31) to show $\mu - 6\gamma L^2 - c_2 \gamma^{0.5} > 0$. By expressing $\ell_1$ as $1 - \gamma(\mu - 6\gamma L^2 - c_2 \gamma^{0.5})$, we have $\ell_1 < 1$. As a result, $\theta < 1$.

Next, we show inequality (30). Based on Assumption 5 and Corollary 13, we have

$$
\mathbb{E}L_\rho \left( \tilde{x}_k, \tilde{\phi}^k \right) = \mathbb{E} \left[ \| x_{S_k} - x^* \|^2 + \rho G \left( \phi^{S_k} \right) \right]
\leq \left( 1 - \gamma \mu + 6\gamma^2 L^2 + c_2 \rho \right) \mathbb{E} \| x_{S_{k-1}} - x^* \|^2 + \left( \frac{4\gamma^2}{\rho} + c_1 \right) \rho \mathbb{E} \left( \phi^{S_{k-1}} \right)
+ 4\gamma^2 \mathbb{E} H \left( \phi^{S_{k-1}} \right) + 4\gamma^2 \mathbb{E} \varepsilon_2 + \varepsilon_0 + \rho \varepsilon_1
\leq \max \left\{ 1 - \gamma \mu + 6\gamma^2 L^2 + c_2 \rho, \frac{4\gamma^2}{\rho} + c_1 \right\} \mathbb{E}L_\rho \left( x_{S_{k-1}}, \phi^{S_{k-1}} \right)
+ 4\gamma^2 \mathbb{E} H \left( \phi^{S_{k-1}} \right) + \varepsilon_3.
$$

Using the definition of $\theta$, the above display is equivalent to

$$
\mathbb{E}L_\rho \left( \tilde{x}_k, \tilde{\phi}^k \right) \leq \theta \mathbb{E}L_\rho \left( x_{S_{k-1}}, \phi^{S_{k-1}} \right) + 4\gamma^2 \mathbb{E} H \left( \phi^{S_{k-1}} \right) + \varepsilon_3.
$$

By recursively applying the above inequality and using the facts that $\theta < 1$ and $m_k \leq \tilde{m}$, we have

$$
\mathbb{E}L_\rho \left( x_{S_k}, \phi^{S_k} \right) \leq \left( \theta + 4\gamma^2 \tilde{m} c_3 \right) \mathbb{E}L_\rho \left( x_{S_{k-1}}, \phi^{S_{k-1}} \right) + \sum_{i=0}^{m-1} \theta^i \varepsilon_3
\leq \lambda \mathbb{E}L_\rho \left( x_{S_{k-1}}, \phi^{S_{k-1}} \right) + \frac{\varepsilon_3}{1 - \theta}.
$$

We then finish the proof for Theorem 14 by showing that $\lambda < 1$. Using the definition of $\theta$, we may rewrite $\lambda$ as

$$
\lambda = \max \left\{ 1 - \gamma \mu + 6\gamma^2 L^2 + c_2 \rho + 4\gamma^2 \tilde{m} c_3, \frac{4\gamma^2}{\rho} + c_1 + 4\gamma^2 \tilde{m} c_3 \right\}.
$$

With $\rho = \gamma^{1.5}$, $\lambda$ is the maximum of $\tilde{\ell}_1$ and $\tilde{\ell}_2$, where

$$
\tilde{\ell}_1 = 1 - \gamma \mu + 6\gamma^2 L^2 + c_2 \gamma^{1.5} + 4\gamma^2 \tilde{m} c_3, \quad \tilde{\ell}_2 = 4\gamma^{0.5} + 4\gamma^2 \tilde{m} c_3 + c_1.
$$

It suffices to prove that both $\tilde{\ell}_1$ and $\tilde{\ell}_2$ are less than one. Rewrite $\tilde{\ell}_2$ as $c_1 + 4\gamma^{0.5}(1+\gamma^{1.5} \tilde{m} c_3)$. Similar to the proof of Theorem 7, we can use the fact that $\gamma$ is less than the second term.
in the brackets (29) and (31) to see \( \tilde{\ell}_2 < 1 \). To show \( \tilde{\ell}_1 < 1 \), note that

\[
6\gamma L^2 + c_2 \gamma^{0.5} + 4\gamma \bar{m}c_3 = \gamma^{0.5} \left(6\gamma^{0.5} L^2 + 4\bar{m}c_3\gamma^{0.5} + c_2\right)
\]

\[
< \frac{3(1-c_1)L^2}{2} + (1-c_1)c_3 \bar{m} + c_2 \left(6\gamma^{0.5} L^2 + 4\bar{m}c_3\gamma^{0.5} + c_2\right),
\]

where the inequality is because \( \gamma \) is less than the first term in the bracket (29). Since \( \gamma < (1-c_1)^2/16 \), the above display is less than \( \mu \). Rewrite \( \tilde{\ell}_1 \) as \( 1 + \gamma \left(6\gamma L^2 + c_2\gamma^{0.5} + 4\gamma \bar{m}c_3 - \mu\right) \) to see that \( \tilde{\ell}_1 < 1 \).

The theorem above implies that the iterates generated by the asynchronous algorithms linearly converge to some neighborhood of the optimum.

Next, we apply Theorem 14 to the asynchronous variants of SVRG, SAGA, HSAG, and SVRG-rand. The key is to verify Assumption 5 for each algorithm. For any epoch-based algorithms, we adopt the assumption in Reddi et al. (2015) that the system is fully synchronized at the end of every epoch. In our examples, if the delay \( \tau = 0 \) in which case the algorithms are actually synchronous, the parameters \( d, e_1, e_2, \) and \( e_3 \) are all zero. In this case, we obtain the same convergence rate as in Section 4.2.

**Example 10** SVRG (Allen-Zhu and Yuan, 2016; Johnson and Zhang, 2013; Xiao and Zhang, 2014). For the asynchronous variant of SVRG, since the system is synchronized after every epoch, we have \( S_k \leq D(t) \leq t \) for \( t \geq S_k \) (Reddi et al., 2015, Section 3). Use the fact that the proxies do not change within each epoch to see \( \phi^k = \phi^{D(k)} = \tilde{\phi}^k \). As a result, Assumption 5.3 holds with \( \epsilon_2 = 0 \). To verify Assumption 5.1, we select \( S = \emptyset \) to see \( G(\phi^k) \equiv 0 \). Assumption 5.2 is the same as Assumption 3.2 which we have verified in Example 7. Therefore, Assumption 5 holds with \( \epsilon = 0, c_1 = c_2 = e_1 = e_2 = 0, \) and \( c_3 = L^2 \).

**Example 11** SAGA (Defazio et al., 2014). Let \( S = \{1, 2, \ldots, n\} \) and \( \{m_j = 1\}_{j=1}^{\infty} \). Assumption 5.2 holds with \( c_3 = 0 \) since \( H(\phi^k) \equiv 0 \). To verify Assumption 5.3, observe that SAGA only changes one proxy in each iteration, and so there are at most \( \tau \) terms among all \( \{\phi_i^k\}_{i=1}^{n} \) that are different from their counterparts in \( \{\phi_i^k\}_{i=1}^{n} \). Use Assumption 4.2 to see \( \sum_{i=1}^{n} \mathbb{E}\|\phi_i^k - \tilde{\phi}_i^k\|^2 / n \leq 4\tau M^2_{\phi,B} / n \).

We can conclude by verifying Assumption 5.1. Similar to Example 8, it can be shown that for all \( k \geq 1 \),

\[
\mathbb{E}G(\phi^k) \leq (1 - \frac{1}{n})\mathbb{E}G(\phi^{k-1}) + \frac{L^2}{n} \mathbb{E}\|\tilde{x}_k - x^*\|^2.
\]

By adding and subtracting \( x_{k-1} \) in the second term of the above inequality, we have

\[
\mathbb{E}G(\phi^k) \leq (1 - \frac{1}{n})\mathbb{E}G(\phi^{k-1}) + \frac{L^2}{n} \mathbb{E}\|\tilde{x}_{k-1} - x_{k-1} + x_{k-1} - x^*\|^2
\]

\[
\leq (1 - \frac{1}{n})\mathbb{E}G(\phi^{k-1}) + \frac{2L^2}{n} \mathbb{E}\|\tilde{x}_{k-1} - x_{k-1}\|^2 + \frac{2L^2}{n} \mathbb{E}\|x_{k-1} - x_{k-1}\|^2
\]

\[
\leq (1 - \frac{1}{n})\mathbb{E}G(\phi^{k-1}) + \frac{2L^2}{n} \mathbb{E}\|x_{k-1} - x_{k-1}\|^2 + \frac{2L^2\gamma^2\tau^2 M^2}{n}.
\]

Above, the second inequality follows from the triangle inequality and the last inequality follows from Lemma 12. Therefore, Assumption 5 holds with \( S = \{1, 2, \ldots, n\}, m_j = 1 \) for \( j \geq 1, e_1 = 2L^2\gamma^2\tau^2 M^2 / n, c_1 = 1 - 1/n, c_2 = 2L^2 / n, c_3 = 0, \) and \( e_2 = 4\tau M^2_{\phi,B} / n. \)
Example 12 HSAG (Reddi et al., 2015). For asynchronous HSAG, similar to (19), it can be shown that for all \( k \geq 1 \),

\[
\mathbb{E} G(\phi^k) \leq \left(1 - \frac{1}{n}\right) \mathbb{E} G(\phi^{k-1}) + \frac{SL^2}{n^2} \mathbb{E} \|\hat{x}_{k-1} - x^*\|^2.
\]

Using the same technique from Example 11, we may see that for all \( k \geq 1 \),

\[
\mathbb{E} G(\phi^k) \leq \left(1 - \frac{1}{n}\right) \mathbb{E} G(\phi^{k-1}) + \frac{2SL^2}{n^2} \mathbb{E} \|x_{k-1} - x^*\|^2 + \frac{2SL^2\gamma^2\tau^2M^2}{n^2}.
\]

In Example 9, we have shown that Assumption 5.2 holds with \( c_3 = \frac{(n-S)L^2}{n} \). In addition, similar to Example 11, we have \( \sum_{i=1}^{n} \mathbb{E} \|\phi_i^k - \hat{\phi}_i^k\|^2 / n \leq 4\tau M_{\phi,B}^2 / n \) for asynchronous HSAG. Therefore, Assumption 5 holds with \( c_1 = 1 - 1/n, c_2 = 2SL^2/n^2, c_3 = (n-S)L^2/n, \) and \( \mathcal{E}_2 = 4\tau M_{\phi,B}^2/n \).

Example 13 SVRG-rand. We choose \( S = \{1, 2, \ldots, n\} \) and \( \{m_j = 1\}_{j=1}^{\infty} \). Assumption 5.2 holds with \( c_3 = 0 \) since \( H(\phi^k) \equiv 0 \). Because the system is synchronized after every epoch, \( \phi^k = \phi^{D(k)} = \hat{\phi}^k \) as in SVRG. As a result, Assumption 5.3 holds with \( \mathcal{E}_2 = 0 \). Similar to the proof for Theorem 9, it can be shown that for all \( k \geq 1 \),

\[
\mathbb{E} G(\phi^k) \leq (1 - p)\mathbb{E} G(\phi^{k-1}) + L^2 \mathbb{E} \|\hat{x}_{k-1} - x^*\|^2.
\]

By adding and subtracting \( x_{k-1} \) in the second term of the above inequality, we have that for all \( k \geq 1 \),

\[
\mathbb{E} G(\phi^k) \leq (1 - p)\mathbb{E} G(\phi^{k-1}) + 2L^2 \mathbb{E} \|x_{k-1} - x^*\|^2 + 2L^2\gamma^2\tau^2M^2.
\]

Therefore, Assumption 5 holds with \( S = \{1, 2, \ldots, n\}, m_j = 1 \) for \( j \geq 1, c_1 = 1 - p, c_2 = 2L^2, \mathcal{E}_1 = 2L^2\gamma^2\tau^2M^2, c_3 = 0, \) and \( \mathcal{E}_2 = 0 \).

7. Numerical experiments

We compare the performance of the following algorithms in this section:

- **SG**. The stochastic gradient described in Example 2. Following the literature (Schmidt et al., 2017; Xiao and Zhang, 2014), we use a constant step-size that gives the best performance among all powers of 10.

- **SVRG**. The stochastic variance reduction gradient (Johnson and Zhang, 2013; Xiao and Zhang, 2014) described in Example 3. The epoch length is \( m = 2n \) which is a typical choice in the literature (Allen-Zhu and Yuan, 2016; Johnson and Zhang, 2013; Xiao and Zhang, 2014).

- **SAGA**. The incremental gradient method (Defazio et al., 2014) described in Example 3.

---

1. All the source codes can be found in our online appendices: https://github.com/xunzhang1229/Variance-Reduction-algorithms-monotone-inclusion.git
• **SVRG++.** An algorithm of Allen-Zhu and Yuan (2016) which falls in the category of SVRG with increasing epochs. The epoch length is initialized as $n$ and it is doubled after every epoch (Allen-Zhu and Yuan, 2016).

• **SVRG-rand.** The proposed extension of SVRG described by (20). We let $p_k$ be a small number for the first $n$ iterations to avoid too many full operator evaluations in early iterations. After the first $n$ iterations, $p_k$ is assigned to continuously decrease from $\frac{1}{2n}$ to $\frac{1}{10n}$ to control the frequency of full updates in later iterations. In the implementation of the algorithm, we adopt the strategy of automatically terminating the epoch if the current epoch is longer than $6n$. Such a strategy is widely used in the literature (Allen-Zhu and Yuan, 2016; Min et al., 2017).

• **hybrid SAGA and SVRG-rand.** The proposed hybrid method with update rule given by (21) where the first $n/2$ proxies perform a SAGA-type update.

For our numerical experiments, we consider logistic regression and saddle point problems. For each case, we run each algorithm 10 times and report the mean performance. Following the tradition (Allen-Zhu and Yuan, 2016; Defazio et al., 2014; Schmidt et al., 2017; Xiao and Zhang, 2014), we compare different algorithms based on the number of operator evaluations.

The first problem is the well-known $\ell_2$-regularized logistic regression

$$\min_{x \in \mathbb{R}^d} \frac{1}{n} \sum_{i=1}^{n} \log \left( 1 + \exp \left( -b_i a_i^T x \right) \right) + \frac{\lambda}{2} \|x\|^2,$$

where $\lambda > 0$ is a regularization parameter, $a_i \in \mathbb{R}^d$ is a feature vector, and $b_i$ is the class label. We use the benchmark binary classification data set *quantum* (2004). Following Schmidt et al. (2017); Xiao and Zhang (2014), we standardize the features so that they have mean zero and variance one. The parameter $\lambda$ is set to $10^{-4}$ which is a popular choice in the literature (Johnson and Zhang, 2013; Xiao and Zhang, 2014). We use the step-size $\gamma = \frac{1}{3L}$ in Defazio et al. (2014) for all the variance reduction algorithms. For all algorithms, the initial $x_0$ is the result of $n$ iterations of SG. Such a scheme is commonly suggested in the literature (Johnson and Zhang, 2013; Schmidt et al., 2017; Xiao and Zhang, 2014).

The results for logistic regression are reported in Figure 1a. It is interesting to see that SG performs well in the first few iterations but then makes slow progress. Such behaviour is anticipated from the convergence analysis of SG with constant step-size and also tallies with the experiments in Schmidt et al. (2017); Xiao and Zhang (2014). We can also see that SAGA performs the best, followed by hybrid SAGA and SVRG-rand. However, to achieve this superior performance, SAGA inherently requires more storage than hybrid SAGA and SVRG-rand. The performance of the proposed algorithm SVRG-rand is better than SVRG in both early and later iterations. SVRG++ has a similar performance to SVRG in early iterations and better performance than SVRG in later iterations.

Next we consider the saddle point problem from Du et al. (2017). Following the notation in Du et al. (2017), this problem can be written as

$$\min_{\theta \in \mathbb{R}^d} \max_{\omega \in \mathbb{R}^d} \frac{1}{n} \sum_{i=1}^{n} \left( \omega^T b_i - \omega^T A_i \theta - \frac{1}{2} \omega^T C_i \omega + \frac{\lambda}{2} \|\theta\|^2 \right),$$

(32)
A Unifying Framework for Algorithms for Monotone Inclusion Problem

Figure 1: Comparison of different algorithms on two problems

(a) logistic regression
- SVRG
- SAGA
- SVRG++
- SVRG-rand
- SAGA+SVRG-rand
- SG

(b) saddle point problem
- SVRG
- SAGA
- SVRG++
- SVRG-rand
- SAGA+SVRG-rand
- SG

Figure 1b shows that the behaviour of our algorithms for solving the saddle point problem match the logistic regression case. However, the gap in the performance of SAGA and SVRG is now smaller.

8. Conclusion and future work

This study has presented a unifying framework for general randomized FB splitting for finding zeroes of the sum of two monotone operators where one is the average of a large number of strongly monotone and Lipschitz operators, and the other is a general maximal monotone operator. Our framework covers many popular variance reduction algorithms for solving finite sum minimization problems.

The basis of our technique is a Lyapunov-type argument which we use to establish linear convergence of a class of algorithms including FG, SVRG, SAGA, HSAG, and SVRG-rand. This argument reveals that all of these variance reduction algorithms can be in some sense understood as iteration of near contraction operators with a disturbance, and that the effect of the disturbance decays to zero over time when appropriate conditions are met. This argument also extends to the design of new algorithms which similarly enjoy a linear convergence rate in expectation as well as enjoy favorable numerical properties. In particular, we propose SVRG-rand and hybrid SAGA and SVRG-rand. When no additional storage is allowed, we observe that the new algorithm SVRG-rand is faster than classical SVRG. If additional storage is allowed, then SAGA converges slightly faster than the new algorithm hybrid SAGA and SVRG-rand algorithms, but at greater expense.
Our framework also covers the asynchronous extension of the general randomized forward step algorithm. In this case we are able to show that the iterates generated by our asynchronous algorithms converge linearly to a neighborhood of the optimum. We can analyze asynchronous variants of many classical and new algorithms including SVRG, SAGA, HSAG, and SVRG-rand. Further, if the delay parameter $\tau = 0$, then we immediately recover our earlier synchronous results.

This work is only the beginning of analyzing different variance reduction algorithms for monotone inclusion problems. In future work, we will consider SAG and other biased variance reduction algorithms. The property of unbiased operator estimates in the inner forward step played a fundamental role in our analysis. To handle the more general case, we may need to consider new types of Lyapunov functions as well as new methods for stochastic analysis.

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Appendix A. Further technical details

In this section, we provide some further technical details of our study.

A.1 SG with constant step-size

In this subsection, we present a convergence analysis on SG with constant size. Under Assumption 2, the iterates generated by Algorithm 1 with constant step-size linearly converge to some neighbourhood of the optimal solution.

**Theorem 15** Suppose a constant step-size $0 < \gamma < \frac{\mu}{2L}$ is chosen and Assumption 2 holds. Then there exists a constant $\eta \triangleq 1 - \frac{2\gamma}{1 - \eta}$ such that the sequence $\{x_k\}_{k \geq 0}$ generated by Algorithm 1 will satisfy

$$
E\|x_{k+1} - x^*\|^2 \leq \eta^{k+1}\|x_0 - x^*\|^2 + \frac{2\gamma^2}{1 - \eta} (M_\phi + M_B)^2
$$

**Proof** Since the inequality 9 holds and $\gamma_k = \gamma$, we have

$$
E\|x_{k+1} - x^*\|^2 \leq (1 - 2\gamma\mu + 3\gamma^2L^2)E\|x_k - x^*\|^2 + 2\gamma^2 (M_\phi + M_B)^2.
$$

Recursively apply the above inequality to see

$$
E\|x_{k+1} - x^*\|^2 \leq \eta^{k+1}\|x_0 - x^*\|^2 + 2\gamma^2 (M_\phi + M_B)^2 \sum_{i=0}^{k} \eta^i.
$$

$$
= \eta^{k+1}\|x_0 - x^*\|^2 + \frac{2\gamma^2}{1 - \eta} (M_\phi + M_B)^2
$$

Here, we use the fact that $\eta < 1$ in the last equality. □

Since SG is a special case of Algorithm 1 with $\phi_k^i = 0$ for all $k \geq 0$ and $i = 1,2,\ldots,n$, apply the above theorem to SG to establish the convergence analysis for SG.

A.2 Technical details of numerical experiments

The saddle-point problem (32) is equivalent to finding $(\theta^*, \omega^*)$ such that (Boyd and Vandenberghe, 2004)

$$
0 = B(\theta^*, \omega^*) \triangleq \begin{bmatrix}
\partial_\theta L(\theta^*, \omega^*) \\
-\partial_\omega L(\theta^*, \omega^*)
\end{bmatrix}.
$$

Let $B_i(\theta, \omega) = \begin{bmatrix}
\partial_\theta L_i(\theta, \omega) \\
-\partial_\omega L_i(\theta, \omega)
\end{bmatrix}$, we only need to find $(\theta^*, \omega^*)$ such that

$$
0 = \frac{1}{n} \sum_{i=1}^{n} B_i(\theta^*, \omega^*).
$$
Specifically, in this study, we choose $C_i = I_d$ to be the identity matrix. We then check the strong monotonicity and Lipschitz continuity of the operator $B_i$ and $B$. Let

$$L_i \triangleq \max\{\lambda_{\text{max}}(\lambda^2 I + (1 + |1 - \lambda|)A_i^T A_i), \lambda_{\text{max}}((1 + |1 - \lambda|)I + A_i^T A_i)\},$$

$$L = \max_{i=1,2,\ldots,n} L_i,$$

$$\mu = \min\{\lambda, 1\}.$$

Where $\lambda_{\text{max}}$ denotes the largest eigenvalue. For $x \in \mathbb{R}^{2d}$, we write $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 \in \mathbb{R}^d$ and $x_2 \in \mathbb{R}^d$. Then the operator $B_i$ can be written as

$$B_i(x) = \begin{bmatrix} \lambda x_1 - A_i^T x_2 \\ x_2 + A_i x_1 - b_i \end{bmatrix}.$$ (33)

**Lemma 16** After we write $B_i$ in the form of (33) we have that for all $i = 1, 2, \ldots, n$,

$$\langle B_i(x) - B_i(y), x - y \rangle \geq \mu \|x - y\|^2,$$

$$\|B_i(x) - B_i(y)\|^2 \leq L \|x - y\|^2.$$

**Proof** Direct calculations show that

$$B_i(x) - B_i(y) = \begin{bmatrix} \lambda(x_1 - y_1) - A_i^T(x_2 - y_2) \\ (x_2 - y_2) + A_i(x_1 - y_1) \end{bmatrix}.$$ Here, $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, $y_1 \in \mathbb{R}^d$ and $y_2 \in \mathbb{R}^d$. We then have

$$\langle B_i(x) - B_i(y), x - y \rangle = \lambda(x_1 - y_1)^T(x_1 - y_1) + (x_2 - y_2)^T(x_2 - y_2) - (x_2 - y_2)^T A_i(x_1 - y_1) + (x_1 - y_1)^T A_i^T(x_2 - y_2)$$

$$= \lambda(x_1 - y_1)^T(x_1 - y_1) + (x_2 - y_2)^T(x_2 - y_2)$$

$$\geq \min\{\lambda, 1\} ((x_1 - y_1)^T(x_1 - y_1) + (x_2 - y_2)^T(x_2 - y_2))$$

Use $\|x - y\|^2 = \|x_1 - y_1\|^2 + \|x_2 - y_2\|^2$ to conclude

$$\langle B_i(x) - B_i(y), x - y \rangle \geq \mu \|x - y\|^2.$$ On the other hand, let $\Delta_1 = x_1 - y_1$, $\Delta_2 = x_2 - y_2$ we have

$$\langle B_i(x) - B_i(y), B_i(x) - B_i(y) \rangle$$

$$= \lambda^2 A_i^T A_i \Delta_1 - 2\lambda A_i^T A_i \Delta_1 + \Delta_1^T A_i A_i^T \Delta_2 + \Delta_2^T A_i A_i^T \Delta_1 + 2\Delta_1^T A_i A_i^T \Delta_2 + 2(1 - \lambda) \Delta_1^T A_i A_i^T \Delta_1$$

$$= \Delta_1^T(\lambda^2 I + A_i A_i^T) \Delta_1 + \Delta_2^T(I + A_i A_i^T) \Delta_2 + 2(1 - \lambda) \Delta_1^T A_i A_i^T \Delta_1
Use Cauchy-Schwarz inequality to have
\[
\langle B_t(x) - B_t(y), B_t(x) - B_t(y) \rangle \\
\leq \Delta_t^2 (\lambda^2 I + A_t^T A_t) \Delta_1 + \Delta_t^2 (I + A_t^T A_t) \Delta_2 + 2|1 - \lambda| \|\Delta_2\| \|A_t \Delta_1\| \\
\leq \Delta_t^2 (\lambda^2 I + A_t^T A_t) \Delta_1 + \Delta_t^2 (I + A_t^T A_t) \Delta_2 + |1 - \lambda| (\Delta_t^2 \Delta_2 + \Delta_t^2 A_t^T A_t) \\
= \Delta_t^2 (\lambda^2 I + (1 + |1 - \lambda|) A_t^T A_t) \Delta_1 + \Delta_t^2 ((1 + |1 - \lambda|) I + A_t^T A_t) \Delta_2 \\
\leq \tilde{L}_t (\Delta_t^2 \Delta_1 + \Delta_t^2 \Delta_2) \\
= \tilde{L}_t \|x - y\|^2_2.
\]

Here, the last inequality is from the definition of eigenvalue. Since the strong monotonicity of each $B_t$ implies the strong monotonicity of $B$, we conclude that the operator $B_t$ and $B$ satisfy Assumption 1.

In general, when the practical $L$ and $\mu$ for operator $B$ is not known, we can use the following numbers instead (Balamurugan and Bach, 2016)
\[
\tilde{L} = \sup_{\|x - y\| \leq 1} \|B(x) - B(y)\|, \quad \mu = \inf_{\|x - y\| \leq 1} \|B(x) - B(y)\|.
\]

### A.3 Choices of $\rho$

We show in this subsection that the linear convergence rate of Algorithm 1 holds for arbitrary $\rho = \gamma^t$, where $1 < t < 2$.

**Theorem 17** Suppose Assumption 1 and Assumption 3 hold, for arbitrary $1 < t < 2$ and $\rho = \gamma^t$, we can find a step-size such that the expected Lyapunov function value $L_\rho(\tilde{x}_k, \tilde{\phi}_k)$ is contracted by a factor less than one.

**Proof** Similar to the proof in Theorem 7, it can be shown that
\[
\mathbb{E} \mathcal{L}(x_{S_k}, \phi^{S_k}) \leq \left( \theta^{m_k} + 2\gamma^2 c_3 \sum_{i=0}^{m_k-1} \theta^i \right) \mathbb{E} \mathcal{L}(x_{S_{k-1}}, \phi^{S_{k-1}}).
\]

where $\theta = \max \left\{ 1 - 2\gamma \mu + 3\gamma^2 L^2 + c_2 \rho, \frac{2\gamma^2}{\rho} + c_1 \right\}$. Use the fact $1 < t < 2$ and $c_1 < 1$ to see $\lim_{\gamma \to 0} \frac{2\gamma^2}{\rho} + c_1 < 1$ and $\lim_{\gamma \to 0} 1 - 2\gamma \mu + 3\gamma^2 L^2 + c_2 \rho < 1$. Thus we have
\[
\lim_{\gamma \to 0} \theta = \lim_{\gamma \to 0} \max \left\{ 1 - 2\gamma \mu + 3\gamma^2 L^2 + c_2 \rho, \frac{2\gamma^2}{\rho} + c_1 \right\} < 1.
\]

So we can find a constant $\gamma_1$ such that for all $\gamma < \gamma_1$, $\theta < 1$. Thus for $\gamma < \gamma_1$ we have
\[
\mathbb{E} \mathcal{L}(x_{S_k}, \phi^{S_k}) < \left( \theta + 2\gamma^2 m_k c_3 \right) \mathbb{E} \mathcal{L}(x_{S_{k-1}}, \phi^{S_{k-1}}) \\
\leq \left( \theta + 2\gamma^2 m c_3 \right) \mathbb{E} \mathcal{L}(x_{S_{k-1}}, \phi^{S_{k-1}}) \\
= \lambda \mathbb{E} \mathcal{L}(x_{S_{k-1}}, \phi^{S_{k-1}}).
\]
Similar to the proof of Theorem 7, we rewrite $\lambda$ as

$$\theta + 2\gamma^2 \bar{m}c_3$$

$$= \max \left\{ 1 - 2\gamma\mu + 3\gamma^2 L^2 + c_2\rho + 2\gamma^2 \bar{m}c_3, \frac{2\gamma^2}{\rho} + c_1 + 2\gamma^2 \bar{m}c_3 \right\}.$$  

Since $\rho = \gamma^t$ and $1 < t < 2$, the both terms in the above bracket has a limit less than one when $\gamma$ tends to zero. As a result, $\lim_{\gamma \to 0} \lambda < 1$. We can find a $\gamma$ such that the linear convergence still hold.}

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