Seeking Consensus on Subspaces in Federated PCA

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Abstract

In this paper, we develop an algorithm for federated principal component analysis (PCA) with emphases on both communication efficiency and data privacy. Generally speaking, federated PCA algorithms based on direct adaptations of classic iterative methods, such as simultaneous subspace iterations (SSI), are unable to preserve data privacy, while algorithms based on variable-splitting and consensus-seeking, such as alternating direction methods of multipliers (ADMM), lack in communication-efficiency. In this work, we propose a novel consensus-seeking formulation by equalizing subspaces spanned by splitting variables instead of equalizing variables themselves, thus greatly relaxing feasibility restrictions and allowing much faster convergence. Then we develop an ADMM-like algorithm with several special features to make it practically efficient, including a low-rank multiplier formula and techniques for treating subproblems. We establish that the proposed algorithm can better protect data privacy than classic methods adapted to the federated PCA setting. We derive convergence results, including a worst-case complexity estimate, for the proposed ADMM-like algorithm in the presence of the nonlinear equality constraints. Extensive empirical results are presented to show that the new algorithm, while enhancing data privacy, requires far fewer rounds of communication than existing peer algorithms for federated PCA.

Keywords: alternating direction method of multipliers, federated learning, principal component analysis, optimization with orthogonality constraints, Stiefel manifold

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1. Introduction

Principal component analysis (PCA) is a fundamental and ubiquitous technique for data analysis and dimensionality reduction (Moore, 1981) with a wide and still rapidly growing variety of applications, such as image compression (Andrews and Patterson, 1976), dictionary learning (Aharon et al., 2006), facial recognition (Turk and Pentland, 1991), latent semantic analysis (Deerwester et al., 1990), matrix completion (Candès and Recht, 2009), and so on.

Let \( A \in \mathbb{R}^{n \times m} \) be an \( n \times m \) data matrix, properly pre-processed with \( n \) features and \( m \) samples where, without loss of generality, \( n \leq m \) is always assumed (usually \( n \ll m \)). To reduce data dimensionality, PCA is to find an orthonormal basis of a \( p \)-dimensional subspace in \( \mathbb{R}^n \) such that the projected samples on this subspace have the largest variance. In truly large-scale applications, computing PCA is practically affordable only for \( p \ll n \).

Mathematically, PCA can be formulated as the following optimization problem,

\[
\min_{Z \in S_{n,p}} f(Z) := -\frac{1}{2} \text{tr} \left( Z^T A A^T Z \right),
\]

where \( S_{n,p} := \{ Z \in \mathbb{R}^{n \times p} \mid Z^T Z = I_p \} \) denotes the Stiefel manifold (Wang et al., 2021).

1.1 Federated Setting

To develop scalable capacities for PCA calculations in today’s big-data environments, it is critical to study algorithms that can efficiently and securely process distributed and massively large-scale data sets. In this paper, we consider the following federated setting (McMahan et al., 2017): the data matrix \( A \) is divided into \( d \) blocks, each containing a group of samples; namely, \( A = [A_1 \ A_2 \ \cdots \ A_d] \), where \( A_i \in \mathbb{R}^{n \times m_i} \) so that \( m_1 + \cdots + m_d = m \). These submatrices \( A_i \), \( i = 1, \ldots, d \), are stored locally in \( d \) locations, possibly having been collected and owned by different clients, and all the clients are connected, directly or indirectly, to a designated center which could either be a special-purpose server or just one of the clients. In this federated setting, to solve (1) it appears that products of the form

\[
AA^T Z = A_1 A_1^T Z + A_2 A_2^T Z + \cdots + A_d A_d^T Z
\]

need to be aggregated at the center after all the local products, \( A_i A_i^T Z \), are computed by the individual clients. Indeed, this is the case when one adapts a classic method to the federated setting.

In evaluating federated algorithms, a key measure of performance is the total amount of communications required by algorithms. In general, during iterations heavy computations are mostly done at the local level within each client, and communications occur between iterations for the center to aggregate newly updated local information from all the clients. In this work, we consider that the amount of communication at each iteration remains essentially the same throughout the calculation. In this setting, the total amount of communication overhead will be proportional to the total number of iterations taken by an algorithm. That is, we consider the most prominent measure of communication efficiency to be the number of iterations required by an algorithm to reach a moderately high accuracy.

Besides enhancing communication-efficiency, preserving the privacy of local data is also a critical task in the federated setting, since in many real-world applications local data
consist of sensitive information such as personal medical or financial records (Lou et al., 2017; Zhang et al., 2018). In this paper, we consider the following privacy scenario that will be called intrinsic privacy for the sake of convenience.

**Definition 1 (Intrinsic Privacy)** Each client does not allow its privately owned data matrix $A_iA_i^\top$, $i \in \{1, 2, \ldots, d\}$, to be revealed to any others including the center. In particular, the center should be prevented from obtaining $A_iA_i^\top$ based on quantities shared by client $i$.

In this intrinsic privacy situation, it is not an option to implement a pre-agreed encryption or a coordinated masking operation. For an algorithm to preserve intrinsic privacy, publicly exchanged quantities must be safe in the sense that the center, or anyone else, will be unable to compute local-data matrix $A_iA_i^\top$ from such quantities. We will soon show next that direct adaptations of classic methods such as SSI are not intrinsically private.

### 1.2 Overview of Related Works

The subject of computing PCA has been thoroughly studied over several decades and various iterative algorithms have been developed. We briefly review a small subset of algorithms closely related to the present work.

Classical PCA algorithms are mostly based on the simultaneous subspace iteration (SSI) (Rutishauser, 1970; Stewart, 1976; Stewart and Jennings, 1981), whose procedure can be readily extended to the federated setting as follows.

$$Z^{(k+1)} \in \text{orth} \left( \sum_{i=1}^{d} Y_i^{(k)} \right) \text{ with } Y_i^{(k)} = A_iA_i^\top Z^{(k)},$$

where $\text{orth}(M)$ refers to the set of orthonormal bases for the range space of $M$. Under the federated setting, each client computes $Y_i^{(k)}$ using the local data for $i = 1, \ldots, d$ and sends the result to the center server, which aggregates all the local products to generate the next iterate and then sends it to all the clients. For convenience, we say that SSI requires one round of communications per iteration for such information exchange between the center server and clients. The main drawback of SSI lies in its slow convergence under unfavorable conditions, leading to intolerably high communication costs in the federated environments. To improve the communication efficiency, Li et al. (2021) proposes an accelerated version of federated SSI by alternating between multiple local subspace iterations and one global aggregation, which is called LocalPower.

We observe that the above direct adaptation of SSI to the federated setting results in a vulnerability of losing intrinsic privacy, as defined in Definition 1. This is because that the shared quantities by each client have a linear relationship with its local data. The crux is that, for given pairs of $Z^{(k)}$ and $Y_i^{(k)}$, the second equation in (2) provides a set of linear equations for the “unknown” $A_iA_i^\top$. Should the center server aims to recover the private data $A_iA_i^\top$, it would only need to collect a sufficient number of publicly shared matrices $\{Z^{(k)}\}$ and $\{Y_i^{(k)}\}$, and then to solve the resulting linear system of equations for $A_iA_i^\top$ as given in (2). Under a mild condition, $A_iA_i^\top$ would be uniquely and exactly determined by solving a linear system. To be precise, we formalize the above argument into the following proposition whose proof is evident.
Proposition 2 Let $Z_k = \begin{bmatrix} Z^{(1)} & Z^{(2)} & \cdots & Z^{(k)} \end{bmatrix} \in \mathbb{R}^{n \times kp}$. Suppose that $k$ is sufficiently large so that $\text{rank}(Z^k) = n$. Then for $i = 1, 2, \ldots, d$, there holds

$$A_i A_i^\top = \begin{bmatrix} Y_i^{(1)} & Y_i^{(2)} & \cdots & Y_i^{(k)} \end{bmatrix} Z_k^\top (Z_k Z_k^\top)^{-1}.$$ 

To put it simply, the “federated SSI” algorithm cannot preserve intrinsic privacy. That is, under mild conditions, the center can recover local data matrices exactly by solving linear systems of equations bases on shared quantities.

In practice, an approximation of $A_i A_i^\top$ can be discovered after very few iterations in practice, which is illustrated by the following numerical instances. We randomly generate the test matrix $A \in \mathbb{R}^{n \times m}$ with $n = 1000$ and $m = 10000$, and the number of computed principal components is set to $p = 100$. In Figure 1(a), we record how the KKT violation $\| (I_n - Z^{(k)} (Z^{(k)})^\top) A A^\top Z^{(k)} \|_F$ and the reconstruction error $\| \Phi^{(k)} - A_1 A_1^\top \|_F$ reduces as the number of iterations increases. Here, $\Phi^{(k)}$ represents the solution to the following optimization problem,

$$\min_{\Phi \in \mathbb{R}^{n \times n}} \| \Phi \|_F^2 \quad \text{s.t.} \quad \Phi Z_k = \begin{bmatrix} Y_1^{(1)} & Y_1^{(2)} & \cdots & Y_1^{(k)} \end{bmatrix},$$

where $Z_k$ is defined as in Proposition 2. We can observe that the local data can be restored to certain accuracy, say $10^{-5}$, much faster than solving the PCA problem. Next, we fix $m = 10000$ and $p = 100$ with $n$ ranging from 1000 to 5000. We record the number of iterations required by SSI to reach $10^{-5}$ in KKT violation accuracy or reconstruction error in Figure 1(b). Again, the number of iterations required to recover the data is much less than to solve the PCA problem.

![Figure 1: Numerical examples of data leakage in the case of SSI.](image-url)
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performance degradation in practice. Chai et al. (2021) introduce a matrix masking scheme designed for federated PCA to protect private data. However, it requires a trusted authority outside the federated system to generate secret masks and deliver them to the clients, which may not exist or be desirable in general. Moreover, this approach requires a high communication overhead since all the masked local data matrices have to be aggregated at the center server.

Recently, a considerable amount of effort has been devoted to developing decentralized algorithms for PCA. The decentralized extension of our algorithm is beyond the scope of this paper. We refer interested readers to (Schizas and Aduroja, 2015; Gang et al., 2019; Gang and Bajwa, 2021, 2022; Ye and Zhang, 2021; Wang and Liu, 2022) for more details.

1.3 Our Contributions

We devise a communication-efficient approach, called FAPS, to the federated PCA problem based on an ADMM-like framework. One iteration of FAPS is identical to a single iteration of the SSI algorithm applied to a matrix sum. However, it differs from the federated SSI in that the sum is not over the set of local data matrices \{A_i A_i^\top\} as in (2), but over a set of mask matrices \{Q_i^{(k)}\}. That is,

\[
Z^{(k+1)} \in \text{orth} \left( \sum_{i=1}^{d} Y_i^{(k)} \right) \quad \text{with} \quad Y_i^{(k)} = Q_i^{(k)} Z^{(k)},
\]

where \(Q_i^{(k)} \in \mathbb{R}^{n \times n}\) masks \(A_i A_i^\top\) for \(i = 1, \ldots, d\) at iteration \(k\). Each mask matrix is computed locally by an individual client based on its local data and other up-to-date information. A main innovation of this work is to construct these mask matrices from a novel projection-splitting model along with an ADMM-like algorithm, which will be developed in Sections 2 and 3.

The above mask operation brings two major advantages: (1) it empirically and significantly accelerates convergence rate in terms of iteration count, as is illustrated by the small numerical example in Table 1 (see Section 5 for comprehensive numerical results); and (2) it preserves intrinsic privacy (see Definition 1) of local data, in contrast to the intrinsic privacy vulnerability of the federated SSI approach (see Propositions 2 and 5).

| Algorithm | Iteration | Relative error |
|-----------|-----------|----------------|
| SSI       | 207       | 1.09e-07       |
| FAPS      | 42        | 8.04e-08       |

Table 1: Comparison of SSI and FAPS on a small example, where the matrix \(A\), generated by (23) with \(n = 2000\), \(m = 128000\), and \(\xi = 1.01\), is tested with \(p = 20\) and \(d = 128\).

With the projection-splitting model, our ADMM-like framework is general and extendable. Beside SSI, many other existing methods for eigenspace calculation (for example, Liu et al., 2015) can also be adapted to the federated PCA setting, but unfortunately with the
same vulnerability in terms of intrinsic privacy. For each “federated version” of such methods, our approach can provide a corresponding “masked version” to eliminate the intrinsic privacy vulnerability.

Furthermore, we established a theoretical convergence result (see Theorem 8) for our FAPS algorithm which tackles not only non-convex manifold constraints but also nonlinear equality constraints that couple local variables to the global one. It is noteworthy that so far existing convergence theory for ADMM algorithms (for example, Wang et al., 2019; Zhang et al., 2020) on non-convex optimization over Riemannian manifolds are applicable only to coupled linear equality constraints.

1.4 Notations
We use $\mathbb{R}$ and $\mathbb{N}$ to denote the sets of real and natural numbers, respectively. The $p \times p$ identity matrix is represented by $I_p$. The Euclidean inner product of two matrices $Y_1$ and $Y_2$ of the same size is defined as $\langle Y_1, Y_2 \rangle = \text{tr}(Y_1^\top Y_2)$, where $\text{tr}(B)$ is the trace of a square matrix $B$. The Frobenius norm and 2-norm of a matrix $X$ are denoted by $\|X\|_F$ and $\|X\|_2$, respectively. For a matrix $X$, the notation rank($X$) stands for its rank; orth($X$) refers to the set of orthonormal bases for its range space; and $\sigma_{\text{min}}(X)$ denotes its smallest singular value. For $X, Y \in \mathcal{S}_{n,p}$, we define $P_X := XX^\top$, $P_X^\perp := I_n - XX^\top$, $D_p (X, Y) := XX^\top - YY^\top$, and $d_p (X, Y) := \|D_p (X, Y)\|_F$. Other notations will be introduced at their first appearance.

1.5 Organization
The rest of this paper is organized as follows. In Section 2, we introduce a novel projection splitting model with so-called subspace constraints, and investigate the structure of associated Lagrangian multipliers. Then we propose a federated algorithm to solve this model based on an ADMM-like framework in Section 3. Convergence properties of the proposed algorithm are studied in Section 4. Numerical experiments on a variety of test problems are presented in Section 5 to evaluate the performance of the proposed algorithm. We conclude the paper in the last section.

2. Projection Splitting Model
We first motivate the proposed projection splitting model, then derive a low-rank formula for Lagrangian multipliers associated with the so-called subspace constraints. This low-rank formula is essential to make the ADMM-like approach practical in its application to the proposed model with large-scale data matrices.

2.1 Pursuit of an Optimal Subspace, Not Basis
It is worth emphasizing that both the objective function $f$ and the feasible region $\mathcal{S}_{n,p}$ of problem (1) are invariant under the transformation $Z \rightarrow ZO$ for any orthogonal matrix $O \in \mathbb{R}^{p \times p}$. In essence, we are to pursue an optimal subspace rather than an optimal basis. Indeed, as is well-known, a global minimizer of (1) can be any orthonormal basis matrix for the optimal subspace spanned by the $p$ left singular vectors associated with the largest $p$ singular values of $A$. In addition, according to the discussions of Liu et al. (2013), the
first-order stationarity condition of (1) can be expressed as follows.

\[ P_Z AA^\top Z = 0 \quad \text{and} \quad Z \in S_{n,p}. \quad (3) \]

As is mentioned earlier, we have a division of \( A = [A_1 \ A_2 \ \cdots \ A_d] \) into \( d \) column blocks and the \( i \)-th block \( A_i \) is stored at client \( i \). Therefore, the objective function \( f(Z) \) can be recast as a finite sum function.

\[
f(Z) = \sum_{i=1}^{d} f_i(Z) \quad \text{with} \quad f_i(Z) = -\frac{1}{2} \text{tr} \left( Z^\top A_i A_i^\top Z \right),
\]

where the \( i \)-th component of the objective function \( f_i(Z) \) can be evaluated only at client \( i \) since \( A_i \) is accessible only at client \( i \). To derive a federated algorithm, we introduce a set of local variables, \( \{X_i\}_{i=1}^{d} \), where, at client \( i \), \( X_i \in S_{n,p} \) is a local copy of the global variable \( Z \in S_{n,p} \) (here \( X \) instead of \( Z \) is used to avoid possible future confusion).

At this point, the conventional approach would impose constraints to equalize, one way or another, all the local variables \( \{X_i\}_{i=1}^{d} \) with the global variable \( Z \). For instance, one could formulate the following optimization problem with a separable objective function.

\[
\begin{align*}
\min_{X_i, Z \in S_{n,p}} & \quad \sum_{i=1}^{d} f_i(X_i) \\
\text{s.t.} & \quad X_i X_i^\top = Z Z^\top, \quad i = 1, \ldots, d.
\end{align*}
\]

(5)

In this model, the set of variables is \( \{X_i\}_{i=1}^{d}, Z \) and \( f_i \) is defined in (4). When an ADMM scheme is applied to this model, the subproblems corresponding to the local variables can all be solved simultaneously and distributively.

However, we observe that the equalizing constraints in (5) require that all local variables \( \{X_i\}_{i=1}^{d} \) must be equal to each other. In other words, model (5) dictates that every client must find exactly the same orthonormal basis for the optimal subspace, which is of course extremely demanding but totally unnecessary. Under such severely restrictive constraints, a consensus is much harder to reach than when each client is allowed to find its own orthonormal basis, independent of each other.

To relax the restrictive equalizing constraints in (5), we propose a new splitting scheme that equalizes subspaces spanned by local variables instead of the local variables (matrices) themselves. For this purpose, we replace the equalizing constraints in (5) by \( X_i X_i^\top = Z Z^\top \) for \( i = 1, \ldots, d \). Since both sides of the equations are orthogonal projections (recall \( X_i, Z \in S_{n,p} \)), we call our new splitting scheme projection splitting. The resulting projection splitting model is as follows.

\[
\begin{align*}
\min_{X_i, Z \in S_{n,p}} & \quad \sum_{i=1}^{d} f_i(X_i) \\
\text{s.t.} & \quad X_i X_i^\top = Z Z^\top, \quad i = 1, \ldots, d.
\end{align*}
\]

(6)

For ease of reference, we will call the constraints in (6) subspace constraints. Obviously, these constraints are nonlinear and the optimization model (6) is nonconvex. Conceptually, subspace constraints are easier to satisfy than the variable splitting constraints \( X_i = Z \). Computationally, however, subspace constraints do come with additional difficulties. Since \( X_i X_i^\top = Z Z^\top \) are large-size, \( n \times n \) matrix equations (compared to \( n \times p \) in \( X_i = Z \)), their corresponding Lagrangian multipliers are also large-size, \( n \times n \) matrices. How to treat such large-size multiplier matrices is a critical algorithmic issue that must be effectively addressed.
2.2 Existence of Low-rank Multipliers

By introducing dual variables, we derive a set of first-order stationarity conditions for the projection splitting model (6) in the following proposition, whose proof will be given in Appendix A.

Proposition 3 Let \( \{X_i \in \mathbb{S}_{n,p}\}_{i=1}^d, Z \in \mathbb{S}_{n,p} \) be a feasible point of the projection splitting model. Then \( Z \) is a first-order stationary point of (1) if and only if there exist symmetric matrices \( \Lambda_i \in \mathbb{R}^{n \times n}, \Gamma_i \in \mathbb{R}^{p \times p}, \) and \( \Theta \in \mathbb{R}^{p \times p} \) so that the following conditions hold:

\[
\sum_{i=1}^d \Lambda_i Z - Z \Theta = 0, \quad A_i A_i^\top X_i + X_i \Gamma_i + \Lambda_i X_i = 0, \quad i = 1, \ldots, d.
\]

The equations in (7) along with the feasibility represent the KKT conditions for the projection splitting model (6). The dual variables \( \Lambda_i \in \mathbb{R}^{n \times n}, \Gamma_i \in \mathbb{R}^{p \times p}, \) and \( \Theta \in \mathbb{R}^{p \times p} \) are the Lagrangian multipliers associated with the equality constraints \( X_i X_i^\top = ZZ^\top, \) \( X_i^\top X_i = I_p, \) and \( Z^\top Z = I_p, \) respectively.

It is straightforward (but rather lengthy, see Appendix A) to verify that at any first-order stationary point \( (\{X_i\}, Z) \) of (6) (i.e., besides feasibility, (3) also holds at \( Z \)), the KKT conditions in (7) are satisfied by the following values of multipliers: \( \Theta = 0, \Gamma_i = -X_i^\top A_i A_i^\top X_i, \) and

\[
\Lambda_i = -P_{X_i} A_i A_i^\top P_{X_i}^\perp - P_{X_i}^\perp A_i A_i^\top P_{X_i}, \quad i = 1, \ldots, d.
\]

Clearly, all \( \Lambda_i \) satisfying (8) have a rank no greater than 2\( p \). In fact, they are symmetrization of rank-\( p \) matrices. As such, equation (8) provides a low-rank, closed-form formula for calculating an estimated multiplier \( \Lambda_i \) at a given \( X_i \). This formulation will play a prominent role in our algorithm, for it effectively eliminates the costs of storing and updating \( n \times n \) multiplier matrices.

Remark 4 We note that multipliers associated with the subspace constraints are non-unique. For example, in addition to (8), the matrices

\[
\hat{\Lambda}_i = -A_i A_i^\top P_{X_i}^\perp - P_{X_i}^\perp A_i A_i^\top,
\]

also satisfy the KKT conditions in (7). However, for \( p \ll n \), the matrix \( \Lambda_i \) in (8) has a much lower rank.

3. Algorithm Development

In this section, we develop a federated algorithm to solve the projection splitting model (6) based on an ADMM-like framework. Out of all the constraints, we only bring the subspace constraints in (6) into the augmented Lagrangian function:

\[
\mathcal{L}(\{X_i\}, Z, \{\Lambda_i\}) = \sum_{i=1}^d \mathcal{L}_i(X_i, Z, \Lambda_i),
\]

where...
where for \( i = 1, \ldots, d, \)
\[
L_i(X_i, Z, \Lambda_i) = f_i(X_i) - \frac{1}{2} \langle \Lambda_i, \mathbf{D}_p (X_i, Z) \rangle + \frac{\beta_i}{4} \mathbf{d}_p^2 (X_i, Z),
\]
and \( \beta_i > 0 \) is a penalty parameter. The quadratic penalty term \( \mathbf{d}_p^2 (X_i, Z) \) (see Section 1.4 for definition) measures the difference between the two subspaces spanned by \( X_i \) and \( Z \), respectively.

Conceptually, at iteration \( k \), our algorithm consists of the following three steps.

(1) Each client updates its own local variable to \( X_i^{(k+1)} \), \( i = 1, \ldots, d \), that is an approximate solution to the local subproblem below,
\[
X_i^{(k+1)} \approx \arg \min_{X_i \in \mathcal{S}_{n,p}} L_i(X_i, Z^{(k)}, \Lambda_i^{(k)}).
\]

(2) Each client implicitly updates its own multiplier to \( \Lambda_i^{(k+1)} \) for \( i = 1, \ldots, d \).

(3) The center updates the global variable to \( Z^{(k+1)} \) to make a progress towards solving the global subproblem:
\[
\min_{Z \in \mathcal{S}_{n,p}} L(\{X_i^{(k+1)}\}, Z, \{\Lambda_i^{(k+1)}\}).
\]

The first two steps can be concurrently carried out in \( d \) clients, while the last step requires communications between the center and all the clients. In the next three subsections, we specify in more concrete terms how these three steps are carried out. A detailed algorithm statement will be given in Section 3.4, and the issue of data security will be discussed in Section 3.5.

3.1 Subproblems for Local Variables

It is straightforward to derive that the subproblem for the local variables \( \{X_i\}_{i=1}^d \) has the following equivalent form.
\[
\min_{X_i \in \mathcal{S}_{n,p}} h_i^{(k)} (X_i) := -\frac{1}{2} \text{tr} \left( X_i^\top H_i^{(k)} X_i \right),
\]
where, for \( i = 1, \ldots, d, \)
\[
H_i^{(k)} = A_i A_i^\top + \Lambda_i^{(k)} + \beta_i \mathbf{P}_Z^{(k)}.
\]

Clearly, (11) is a standard eigenvalue problem where one computes a \( p \)-dimensional dominant eigenspace of an \( n \times n \) real symmetric matrix. As a subproblem, (11) needs not to be solved to a high precision. In fact, we have discovered two inexact-solution conditions that ensure both theoretical convergence and good practical performance. It is important to note that using an iterative eigensolver, one does not need to compute nor store the \( n \times n \) matrix \( H_i^{(k)} \) since it is accessed through matrix-(multi)vector multiplications.

The first condition is a sufficient reduction in function value.
\[
h_i^{(k)} (X_i^{(k)}) - h_i^{(k)} (X_i^{(k+1)}) \geq \frac{c_1}{\epsilon_1 \|A_i\|_2^2 + \beta_i} \left\| \mathbf{P}_Z^{(k)} H_i^{(k)} X_i^{(k)} \right\|_F^2,
\]

(13)
where $c_1 > 0$ and $c'_1 > 0$ are two constants independent of $\beta_i$. This kind of conditions has been used to analyze convergence of iterative algorithms for solving trace minimization problems with orthogonality constraints (Liu et al., 2013; Gao et al., 2018).

The second condition is a sufficient decrease in KKT violation.

$$
\| P_{X_i}^{(k+1)} H_i^{(k)} X_i^{(k+1)} \|_F \leq \delta_i \| P_{X_i}^{(k)} H_i^{(k)} X_i^{(k)} \|_F,
$$

where $\delta_i \in [0, 1)$ is a constant independent of $\beta_i$. This condition frequently appears in inexact augmented Lagrangian based approaches (Eckstein and Silva, 2013; Liu et al., 2019). It will play a crucial role in our theoretical analysis.

The above two conditions, much weaker than optimality conditions of (11), are sufficient for us to derive global convergence of our ADMM-like framework. In practice, it usually takes very few iterations of a certain iterative eigensolver, such as SSI, LMSVD (Liu et al., 2013) and SLRPGN (Liu et al., 2015), to meet these two conditions.

### 3.2 Formula for Low-rank Multipliers

Now we consider updating the multipliers $\{\Lambda_i\}_{i=1}^d$ associated with the subspace constraints in (6). In a regular ADMM algorithm, multiplier $\Lambda_i$ would be updated by a dual ascent step.

$$
\Lambda_i^{(k+1)} = \Lambda_i^{(k)} - \tau_i \beta_i D_p \left( X_i^{(k+1)}, Z^{(k+1)} \right),
$$

where $\tau_i > 0$ is a step size. However, the above dual ascent step requires to store an $n \times n$ matrix at each client, which can be prohibitive when $n$ is large.

In our search for an effective multiplier-updating scheme, we derived an explicit, low-rank formula (8) in Section 2.2 that is satisfied at any first-order stationary point, namely,

$$
\Lambda_i^{(k+1)} = X_i^{(k+1)} (W_i^{(k+1)})^\top + W_i^{(k+1)} (X_i^{(k+1)})^\top,
$$

where, for $i = 1, \ldots, d$,

$$
W_i^{(k+1)} = -P_{X_i^{(k+1)}} A_i A_i^\top X_i^{(k+1)}.
$$

With this low-rank expression, one can produce matrix-(multi)vector products involving $\Lambda_i$ without any storage besides $X_i$ (optionally one more $n \times p$ matrix $W_i$ for computational convenience).

We note that $\Lambda_i$ in formula (15) is independent of the global variable $Z$. Thus, we choose to “update” $\Lambda_i$ after $X_i$ and before $Z$.

### 3.3 Subproblem for Global Variable

The subproblem for the global variable $Z$ can also be rearranged into a standard eigenvalue problem.

$$
\min_{Z \in S_{n,p}} q^{(k)}(Z) := -\frac{1}{2} \text{tr} \left( Z^\top Q^{(k)} Z \right),
$$

where $Q^{(k)}$ is a sum of $d$ locally held matrices:

$$
Q^{(k)} = \sum_{i=1}^d Q_i^{(k)} \text{ with } Q_i^{(k)} = \beta_i P_{X_i^{(k+1)}} - \Lambda_i^{(k+1)}.
$$
As is the case for local variables, we also approximately solve (17) by an iterative eigen-solver. However, in the federated environment, each iteration of a certain eigensolver requires at least one round of communications. Therefore, in order to reduce the overall communication overheads, we employ a single iteration of SSI to inexactly solve the sub-problem (17). Starting from the current iterate $Z^{(k)} \in S_{n,p}$ stored at every client, one computes
\[ Z^{(k+1)} \in \text{orth} \left( \sum_{i=1}^{d} Y_i^{(k)} \right) \quad \text{with} \quad Y_i^{(k)} = Q_i^{(k)} Z^{(k)}, \quad (19) \]
which only invokes one round of communications per outer-iteration. Here, the local products $Y_i^{(k)} = Q_i^{(k)} Z^{(k)}$, $i = 1, \ldots, d$, are calculated using the expressions for $Q_i^{(k)}$ and $\Lambda_i^{(k)}$, see (15), (16) and (18), which can be carried out distributively at each client with $O(np^2)$ floating-point operations without actually forming any $n \times n$ matrices.

Comparing to the federated SSI in (2), we use $Q_i^{(k)}$ in (19) to mask the local data matrix $A_i A_i^\top$. We will see that this masking operation not only protects data privacy, but also significantly accelerates convergence, as will be empirically shown in Section 5.

### 3.4 Algorithm Description

We now formally present the proposed algorithmic framework as Algorithm 1 below, named \textit{federated ADMM-like algorithm with projection splitting} and abbreviated to FAPS. At iteration $k$, client $i$ first updates $X_i^{(k+1)}$ and $W_i^{(k+1)}$ using local data, and then computes $Y_i^{(k)} = Q_i^{(k)} Z^{(k)}$ that is transmitted to the center server. Finally, the center server updates $Z^{(k+1)}$ and sends the results to all the clients. This procedure is repeated until convergence. Upon termination, the final iterate will be an orthonormal basis for an approximately optimal eigenspace of $AA^\top$. Same as the federated SSI (2), FAPS requires one round of communications per iteration.

---

\textbf{Algorithm 1: Federated ADMM-like algorithm with projection splitting (FAPS).}

1. \textbf{Input:} data matrix $A = [A_1, \ldots, A_d]$, penalty parameters $\{\beta_i\}$.
2. Set $k := 0$. Initialize $\{X_i^{(0)}, Z^{(0)}\}$ and compute $\{\Lambda_i^{(0)}\}$ by (15).
3. \textbf{while} “not converged” \textbf{do}
   
   4. \quad \textbf{for each client } $i \in \{1, 2, \ldots, d\}$ \textbf{do}
      
      5. \quad \quad \text{Find } $X_i^{(k+1)} \in S_{n,p}$ \text{ that satisfies (13) and (14).}
      
      6. \quad \quad \text{Update the matrix } $W_i^{(k+1)}$ \text{ by (16).}
      
      7. \quad \quad \text{Compute } $Y_i^{(k)}$ \text{ in (19) and send it to the center server.}
   
   8. \quad \textbf{for the center server do}
      
      9. \quad \quad \text{Update } $Z^{(k+1)} \in S_{n,p}$ \text{ by (19) and sent it to all the clients.}
   
10. \quad \text{Set } $k := k + 1$.
11. \textbf{Output: } $Z^{(k)}$. 
3.5 Preservation of Intrinsic Privacy of Local Data

In FAPS, at iteration $k$ the public information is the global variable value $Z^{(k)}$ and the shared information from client $i$ is the product $Y_i^{(k)} = Q_i^{(k)}Z^{(k)} \in \mathbb{R}^{n \times p}$, see (19). Suppose that the center has access to all available information $\{Z^{(k)}\}$ and $\{Y_i^{(k)}\}$ at all iterations. Then would it be possible for the center to recover any local data matrix $A_iA_i^\top$?

First observe that the $n$ by $n$ mask matrix $Q_i^{(k)}$ varies from iteration to iteration and the available equation at iteration $k$, $Q_i^{(k)}Z^{(k)} = Y_i^{(k)}$, is $n$ by $p$ for $p \ll n$. Hence, it is generally impossible to obtain a mask matrix from solving the associated under-determined linear system of equations at the corresponding iteration. By examining the expressions (15), (16) and (18), we can derive the relationship between a local data matrix and other involved quantities, whether known or unknown.

**Proposition 5** In Algorithm 1 at iteration $k$, for $i = 1, \ldots, d$ there holds

$$
\left( \beta_i P_X + P_X^\perp A_i A_i^\top P_X + P_X A_i A_i^\top P_X^\perp \right) Z^{(k)} = Y_i^{(k)}
$$

(20)

for $X = X_i^{(k+1)}$, where only $Z^{(k)}, Y_i^{(k)} \in \mathbb{R}^{n \times p}$ are shared quantities.

It is evident that equations in (20) cannot be used to exactly solve for the local data matrix $A_iA_i^\top$ without knowing sufficiently many local iterates $X_i^{(k+1)}$ (beside $\beta_i$), which are all privately owned by client $i$. That is, an exact recovery of $A_iA_i^\top$ by the center from shared quantities is impossible in Algorithm FAPS.

In light of Propositions 2 and 5, we conclude that, in contrast to the federated SSI algorithm, Algorithm FAPS can preserve intrinsic data privacy, as defined in Definition 1.

4. Convergence Analysis

In this section, we rigorously establish the global convergence of our proposed Algorithm 1 under the following mild assumptions on the algorithm parameters.

**Assumption 6** We assume the following conditions hold.

(i) The algorithm parameter $\delta_i$ in (14) satisfies

$$
0 \leq \delta_i < \frac{\sigma}{2\sqrt{p\rho d}}, \quad i = 1, \ldots, d,
$$

where $\rho := \max_{i,j=1,\ldots,d} \{\beta_i/\beta_j\} \geq 1$ and $\sigma := \sqrt{1 - 1/(2p\rho)} \in (0,1)$.

(ii) For a sufficiently large constant $\omega_i > 0$, the penalty parameter $\beta_i$ satisfies

$$
\beta_i \geq \omega_i \|A\|_F^2, \quad i = 1, \ldots, d.
$$

**Remark 7** The above assumptions are imposed only for the purpose of theoretical analysis. An expression for $\omega_i$ will be given in Appendix B.
We are now ready to present the global convergence and the worst-case complexity of FAPS. For brevity, we use the following simplified notations.

\[ d_i^{(k)} := d_p \left( X_i^{(k)}, Z^{(k)} \right), \quad \text{for } i = 1, \ldots, d, \quad \text{and } k \in \mathbb{N}. \]  

(21)

**Theorem 8** Let \( X_i^{(0)} \in S_{n,p} \) and \( Z^{(0)} \in S_{n,p} \) satisfy

\[ \left( d_i^{(0)} \right)^2 \leq \frac{1}{\rho d}, \quad i = 1, \ldots, d, \]  

(22)

and the sequence \( \{X_i^{(k)}\}_{i=1}^d, Z^{(k)} \) be generated by Algorithm 1. Under Assumption 6, \( \{Z^{(k)}\} \) has at least one accumulation point, and any accumulation point is a first-order stationary point of problem (1). Moreover, there exists a constant \( C > 0 \) so that for any \( N > 1 \), it holds that

\[ \min_{k=0,\ldots,N-1} \left\{ \left\| P_{Z^{(k)}} AA^T Z^{(k)} \right\|_F^2 + \frac{1}{d} \sum_{i=1}^d \left( d_i^{(k)} \right)^2 \right\} \leq \frac{C}{N}. \]

The proof of this theorem, being quite long and tedious, is left to Appendix B.

5. Numerical Experiments

In this section, we evaluate the performance of FAPS through comprehensive numerical experiments, which demonstrate its efficiency, robustness, and scalability. All the experiments are performed on a high-performance computing cluster LSSC-IV\(^1\) maintained at the State Key Laboratory of Scientific and Engineering Computing (LSEC), Chinese Academy of Sciences. There are 408 nodes in the main part of LSSC-IV, and each node consists of two Intel Xeon Gold 6140 processors (at 2.30GHz \(\times 18\)) with 192GB memory. The operating system of LSSC-IV is Red Hat Enterprise Linux Server 7.3.

Note that FAPS is potentially useful under widely federated settings with high communication costs and privacy concerns. The numerical experiments done on the cluster are just simulations to observe the convergence rates of FAPS in comparison to others. For our numerical results, the number of iterations required by algorithms (i.e., the speed of convergence) is the determining factor.

5.1 Test Problems

Two classes of test problems are used in our experiments. The first class consists of synthetic problems randomly generated as follows. We construct a test matrix \( A \in \mathbb{R}^{n \times m} \) (assuming \( n \leq m \) without loss of generality) by its (economy-form) singular value decomposition

\[ A = U \Sigma V^T, \]  

(23)

where both \( U \in \mathbb{R}^{n \times n} \) and \( V \in \mathbb{R}^{m \times n} \) are orthonormalization of matrices whose entries are random numbers drawn independently, identically and uniformly from \([-1, 1]\), and \( \Sigma \in \mathbb{R}^{n \times n} \) is a diagonal matrix with diagonal entries

\[ \Sigma_{ii} = \xi^{1-i}, \quad i = 1, \ldots, n, \]  

\[ \begin{align*} 
1. \text{More information at } \text{http://lsec.cc.ac.cn/chinese/lsec/LSSC-IVintroduction.pdf.} 
\end{align*} \]
for a parameter $\xi > 1$ that determines the decay rate of the singular values of $A$. In general, smaller decay rates (with $\xi$ closer to 1) correspond to more difficult cases.

The second class of test problems consists of matrices from four popular image data sets frequently used in machine learning, including MNIST$^2$, Fashion-MNIST$^3$, CIFAR-10$^4$, and CIFAR-100$^4$. In both MNIST and Fashion-MNIST, the sample dimension is $n = 28 \times 28 = 784$ and the number of sample is $m = 60000$; while in both CIFAR-10 and CIFAR-100, the sample dimension is $n = 3 \times 32 \times 32 = 3072$ with $m = 50000$.

5.2 Implementation Details

We use an adaptive strategy to tune the penalty parameters $\beta_i$, in which we periodically increase the penalty parameters value when the projection distance has not seen a sufficient reduction. Given initial values $\beta_i^{(0)}$, at iteration $k > 0$ we first compute the projection distance $d_{i(k)}$ and then update the penalty parameter by the recursion rule:

$$
\beta_i^{(k+1)} = \begin{cases} 
(1 + \theta) \beta_i^{(k)}, & \text{if } \text{mod} (k, 5) = 0 \text{ and } d_{i(k-5)} \leq (1 + \mu) d_{i(k)}, \\
\beta_i^{(k)}, & \text{otherwise}
\end{cases}
$$

By default, we set $\beta_i^{(0)} = 0.15 \| A_i \|_2^2$, $\theta = 0.1$, and $\mu = 0.01$ in our implementation.

We initialize the global variable $Z^{(0)}$ as orthonormalization of a random $n \times p$ matrix whose entries follow the i.i.d. uniform distribution in $[-1, 1]$. Then we set $X_i^{(0)} = Z^{(0)}$ ($i = 1, \ldots, d$).

For solving subproblem (11) approximately, we choose to use SSI which, at outer iteration $k$, generates an inner-iteration sequence $X_i^{(k)}(j)$ for $j = 0, 1, \ldots$, with the warm-start $X_i^{(k)}(0) = X_i^{(k)}$. To reduce the computation costs, we use the following termination rule.

$$
\| X_i^{(k)}(j) - X_i^{(k)}(j - 1) \|_F \leq \epsilon_x \| X_i^{(k)}(j) \|_F,
$$

for a prescribed tolerance $\epsilon_x > 0$, which measures the relative change between two consecutive inner iterates. In our experiments, we set $\epsilon_x = 10^{-2}$ as the default value. For solving subproblem (17) approximately, starting from $Z^{(k)}$ we take a single iteration of SSI to obtain $Z^{(k+1)}$, see (19).

We terminate FAPS if either the following condition holds,

$$
\sum_{i=1}^d \left( A_i^T Z^{(k)} \right)^2_F - \sum_{i=1}^d \left( A_i^T Z^{(k-1)} \right)^2_F \leq 10^{-10} \sum_{i=1}^d \left( A_i^T Z^{(k)} \right)^2_F,
$$

or the maximum iteration number MaxIter = 3000 is reached. The condition (27) measures the relative change in objective function values.

Remark 9 We choose not to use the KKT violation as the stopping criterion since it requires extra communication overheads under a federated environment.

2. Available from http://yann.lecun.com/exdb/mnist/.
3. Available from https://github.com/zalandoresearch/fashion-mnist.
4. Available from https://www.cs.toronto.edu/~kriz/cifar.html.
In our experiments, we collect and compare four performance measurements: wall-clock time, total number of iterations, scaled KKT violation defined by

$$\frac{1}{\|A\|^2_F} \left\| P_{\perp Z(k)} A A^T Z(k) \right\|_F,$$

and relative error in singular values defined by $$\frac{\|\Sigma(k) - \Sigma^*\|_F}{\|\Sigma^*\|_F}$$, where the diagonal matrices $$\Sigma^* \in \mathbb{R}^{p \times p}$$ and $$\Sigma(k) \in \mathbb{R}^{p \times p}$$ hold, respectively, the exact and computed dominant singular values.

5.3 Competing Algorithms

We compare the performances of FAPS mainly with two closely related but representative algorithms. The first competing algorithm is an adaptation of the classic SSI algorithm (Rutishauser, 1970; Stewart, 1976; Stewart and Jennings, 1981) to the federated setting described in Section 1.1. The second competing algorithm is called LocalPower (Li et al., 2021), which is an accelerated version of SSI developed in federated mode. Originally, LocalPower was designed to communicate after every $$q$$ local subspace iterations. In order to guarantee convergence, later LocalPower applies a decay strategy to gradually decrease the number of local subspace iterations. Specifically, LocalPower halves $$q$$ every round of communications until it reaches 1. As suggested by Li et al. (2021), we choose the initial value of $$q$$ as 8. It is worth mentioning that LocalPower can not preserve the privacy of local data either, since it boils down to vanilla SSI after $$q$$ reaches 1. In our experiments, we adopt the same initialization and stopping criterion as described in Section 5.2.

We implement FAPS, SSI, and LocalPower in C++ with MPI for inter-process communication to the best of our ability. Unless otherwise specified, the communication is realized by the all-reduce operations in MPI. In our implementation, we use the C++ linear algebra library Eigen5 (version 3.3.8) for matrix computations. In particular, orthonormalization of an $$n \times p$$ matrix is done via the (economy-size) QR factorization at a cost of $$O(np^2)$$ operations.

5.4 Comprehensive Comparison on Synthetic Data

We now compare the performances of the three algorithms on a variety of synthetic test problems, run under the aforementioned federated environment with the number of computing clients fixed at $$d = 128$$. We construct four groups of test problems based on (23), in each of which there is only one parameter varying while all others are fixed. Specifically, the problem parameter settings for $$A$$ are given as follows (recall that $$n$$ is the number of rows, $$m$$ is the number of columns, $$p$$ is the number of principal components to be computed, and $$\xi$$ determines the decay rate of singular values):

1. $$n = 1000 + 1000j$$ for $$j = 1, 2, 3, 4$$, while $$m = 128000$$, $$p = 20$$, and $$\xi = 1.01$$;
2. $$m = 128000 + 32000j$$ for $$j = 1, 2, 3, 4$$, while $$n = 2000$$, $$p = 10$$, and $$\xi = 1.01$$;
3. $$p = 10j$$ for $$j = 1, 2, 3, 4$$, while $$n = 1000$$, $$m = 128000$$, and $$\xi = 1.01$$;

5. Available from http://eigen.tuxfamily.org/index.php?title=Main_Page
\(\xi = 1 + 10^{-1(j+2)/2}\) for \(j = 1, 2, 3, 4\), while \(n = 1000\), \(m = 256000\), and \(p = 10\).

The numerical results for the above four test scenarios are depicted in Figure 2, with two quantities, wall-clock time in seconds and number of iterations taken recorded for every experiment. The average scaled KKT violation and relative error of every experiment are tabulated in Table 2. It should be evident from these numerical results that FAPS clearly outperforms SSI and LocalPower in terms of iteration numbers.

Figure 2: Comparison of SSI, LocalPower, and FAPS on synthetic data.

|                      | Average scaled KKT violation | Average relative error |
|----------------------|------------------------------|------------------------|
|                      | SSI  | LocalPower | FAPS   | SSI  | LocalPower | FAPS   |
| Varying \(n\)        | 1.94e-06 | 1.89e-06 | 1.87e-06 | 1.06e-07 | 9.56e-08 | 8.69e-08 |
| Varying \(m\)        | 1.95e-06 | 1.92e-06 | 1.90e-06 | 1.07e-07 | 1.02e-07 | 9.97e-08 |
| Varying \(p\)        | 1.95e-06 | 1.91e-06 | 1.92e-06 | 8.30e-08 | 9.18e-08 | 1.15e-07 |
| Varying \(\xi\)      | 4.09e-06 | 3.94e-06 | 3.87e-06 | 3.15e-07 | 3.08e-07 | 2.99e-07 |

Table 2: Average errors of SSI, LocalPower and FAPS on synthetic data.
It is worth emphasizing that since these three algorithms incur more or less the same amount of communication overhead per iteration, the total amount of information exchanged is roughly proportional to the numbers of iterations. Hence, the rapid convergence of FAPS (in terms of iteration number) translates into not only computational but also communicational efficiency. On the other hand, we caution that the advantage of FAPS may not always be as large as shown in our experiments when some parameter values go beyond the tested ranges.

5.5 Comparison on Image Data Sets

We next evaluate the performances of the three algorithms on four image data sets popular in machine learning research. The numbers of computed principal components and computing clients in use are set to \( p = 5 \) and \( d = 16 \), respectively. Numerical results from this experiment are given in Figure 3 and Table 3. Again, in terms of the number of iterations taken, FAPS always dominates SSI and LocalPower. These results indicate that the observed superior performance of FAPS is not just limited to synthetic matrices.

![Comparison of SSI, LocalPower, and FAPS on four image data sets.](image)

(a) Total number of iterations

(b) Wall-clock time in seconds

Figure 3: Comparison of SSI, LocalPower, and FAPS on four image data sets.

|                  | Average scaled KKT violation | Average relative error |
|------------------|-----------------------------|------------------------|
| SSI              | LocalPower                  | FAPS                   |
| 4.34e-06         | 4.59e-06                    | 4.42e-06               |
| 5.80e-08         | 6.27e-08                    | 5.06e-08               |

Table 3: Average errors of SSI, LocalPower and FAPS on four image data sets.

5.6 Empirical Convergence Rate

In this subsection, we take a closer look at the empirical convergence rate of FAPS. The behaviors of SSI and LocalPower are also studied in comparison. The test matrix \( A \) is randomly generated by a similar manner as (23) with \( n = 1000 \) and \( m = 160000 \) except
that the diagonal entries of $\Sigma$ satisfy the following arithmetic distribution.

$$\Sigma_{ii} = 1 - \frac{i - 1}{n - 1} \left( 1 - \frac{1}{\kappa} \right), \quad i = 1, \ldots, n,$$

where $\kappa > 0$ is the condition number of $A$. In this experiment, we choose three different values of $\kappa$ to test, including 10, 100, and 1000. The numbers of computed principal components and computing clients in use are set to $p = 10$ and $d = 128$, respectively.

As illustrated in Figure 4, all three algorithms appear to converge at linear rates where the rate of FAPS is the fastest, followed by that of LocalPower. We observe that the larger $\kappa$ is, the faster the singular values decay, and the easier the problem tends to be. On the tested instances, however, the advantage of FAPS relative to the other two appears insensitive to the change of the decay rate in singular values.

![Figure 4: Comparison of empirical convergence rates.](image)

5.7 Comparison on Unevenly Distributed Data

In the previous numerical experiments, the data set in test is uniformly distributed into $d$ clients, that is, $m_1 = m_2 = \cdots = m_d$. Now we consider the scenario that the whole data set is unevenly distributed across the network. In the following experiment, a synthetic matrix $A$, generated by (23) with $n = 1000$, $m = 36000$, and $\xi = 1.01$, is tested with $p = 10$ and $d = 8$. We split the matrix $A$ into $d$ clients such that $m_i = 1000i$ for $i = 1, \ldots, d$. The corresponding numerical results are presented in Table 4, which demonstrate that FAPS attains better performances than SSI and LocalPower. In particular, we note that the number of iterations (i.e., the rounds of communications) required by FAPS is about one-third of that by LocalPower, even though the wall-clock times consumed by the two are close, implying that FAPS could be far more communication-efficient whenever the cost of communication is high.

6. Conclusions

In addition to the traditional performance criteria of time and space efficiency, algorithms for computing PCA on large-scale data sets in federated environments need to consider reducing communication overhead and, in many modern applications, preserving data privacy. Towards achieving these goals, we propose a new model and an ADMM-like algorithm-
mic framework, called FAPS, that seeks consensus on a subspace rather than on a matrix variable.

This FAPS framework generates local subproblems that are standard symmetric eigenvalue problems to which well-developed solvers readily apply. From the viewpoint of the global variable, FAPS can be interpreted as an enhanced version of the federated SSI that can reduce the communication costs and safeguard local data privacy. In addition, FAPS is equipped with two key algorithmic features: (i) multipliers are represented by a closed-form, low-rank formula; and (ii) solution accuracies for subproblems are appropriately controlled at low levels.

Most existing theoretical works on convergence of ADMM algorithms for solving non-convex optimization problems impose restrictive assumptions on iterates or multipliers. In our specific case for FAPS to solve a non-convex optimization model with coupled nonlinear constraints, we have derive global convergence and worst-case complexity results only under mild conditions on the choices of algorithm parameters.

We have conducted comprehensive numerical experiments to compare FAPS with two competing algorithms for Federated PCA. The test results are strongly in favor of FAPS. Most notably, the number of iterations required by FAPS is significantly fewer than that required by others up to one or two orders of magnitudes. We believe that this fast empirical convergence rate is made possible by our subspace-splitting idea, producing greatly relaxed feasibility restrictions relative to the classic variable-splitting strategy.

Finally, we mention that there still remains a range of issues, theoretical or practical, to be further studied in order to fully understand the behavior and realize the potential of FAPS and its variants. We also note that the projection splitting idea can be generalized to a wider class of problems.

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Appendix A. Proof of the Existence of Low-rank Multipliers

In this appendix, we prove Proposition 3 to interpret the existence of low-rank multipliers associated with the subspace constraints in (6).

**Proof [Proof of Proposition 3]** We start with proving the “only if” part, and hence assume that \( Z \) is a first-order stationary point of (1). Let \( \Theta = 0, \Gamma_i = -X_i^\top A_i A_i^\top X_i \), and \( \Lambda_i = -P_{X_i}^\perp A_i A_i^\top X_i X_i^\top - X_i X_i^\top A_i A_i^\top P_{X_i}^\perp \) with \( i = 1, \ldots, d \). Then matrices \( \Theta, \Gamma_i \) and \( \Lambda_i \) are symmetric and rank \((\Lambda_i) \leq 2p\). And it can be readily verified that

\[
A_i A_i^\top X_i + X_i \Gamma_i + \Lambda_i X_i = P_{X_i}^\perp A_i A_i^\top X_i - P_{X_i}^\perp A_i A_i^\top X_i = 0, \quad i = 1, \ldots, d.
\]

Moreover, it follows from the fact \( X_i X_i^\top = Z Z^\top \) and stationarity of \( Z \) that

\[
\sum_{i=1}^d \Lambda_i Z - Z \Theta = \sum_{i=1}^d \left( -P_{X_i}^\perp A_i A_i^\top X_i X_i^\top - X_i X_i^\top A_i A_i^\top P_{X_i}^\perp \right) Z = -P_Z^\perp A A^\top Z = 0.
\]

Hence, \( \{X_i, Z\} \) satisfies the condition (7) under the specific combination of \( \Theta, \Gamma_i, \) and \( \Lambda_i \).

Now we prove the “if” part and assume that there exist symmetric matrices \( \Theta, \Gamma_i, \) and \( \Lambda_i \) such that the feasible point \( \{X_i, Z\} \) satisfies the condition (7). By virtue of (7), we obtain \( A_i A_i^\top X_i = -X_i \Gamma_i - \Lambda_i X_i \), and hence it holds that

\[
\sum_{i=1}^d P_{X_i}^\perp A_i A_i^\top X_i X_i^\top = -\sum_{i=1}^d P_{X_i}^\perp (X_i \Gamma_i + \Lambda_i X_i) X_i^\top = -P_Z^\perp \left( \sum_{i=1}^d \Lambda_i Z \right) Z^\top = 0,
\]

where the second equality follows from the fact \( X_i X_i^\top = Z Z^\top \), and the third equality follows from (7). On the other hand, we have

\[
\sum_{i=1}^d P_{X_i}^\perp A_i A_i^\top X_i X_i^\top = \sum_{i=1}^d P_Z^\perp A_i A_i^\top Z Z^\top = P_Z^\perp A A^\top Z Z^\top.
\]

Combining the above two relationships, we arrive at \( P_Z^\perp A A^\top Z Z^\top = 0 \), which further implies \( P_Z^\perp A A^\top Z = 0 \). Therefore, \( Z \) is a first-order stationary point of (1). We complete the proof.

Appendix B. Proof of the Global Convergence

In this appendix, we prove Theorem 8 to establish the global convergence of Algorithm 1. To begin with, we give an explicit expression of the constant \( \omega_i > 0 \) in Assumption 6 as follows:

\[
\omega_i = \max \left\{ c_1, \frac{12 \rho d \sqrt{\beta}}{c_1 \sigma^2}, \frac{4 \sqrt{2} (1 + \sqrt{2} \rho d)}{\sigma - 2 \sqrt{2} \rho d \delta_i}, \frac{16 \rho d \sqrt{\beta}}{c_1 \sigma^2 \rho d}, \frac{4 (1 + \sqrt{2} \rho d)}{c_1 \sigma^2 \rho d} \right\}, \quad i = 1, \ldots, d.
\]

In addition, it is clear that \( \|A\|_F \geq \|A_i\|_F \geq \|A_i\|_2 \).

Next, in order to prove Theorem 8, we establish a few lemmas and corollaries to make preparations. In their proofs, we omit the superscript \((k)\) to save space with a slight abuse of notations, and use the superscript + to take the place of \((k + 1)\).
Suppose Assumption 6 holds and \( \{X_i^{(k)}, Z_i^{(k)}\} \) is the k-th iterate generated by Algorithm 1 and satisfies that \( d_i^{(k)} \leq \sqrt{1/(pd)} \), \( i = 1, \ldots, d \). Then it holds that

\[
h_i^{(k)}(X_i^{(k)}) - h_i^{(k)}(X_i^{(k+1)}) \geq \frac{1}{4} c_1 \sigma^2 \beta_i \left( d_i^{(k)} \right)^2, \quad i = 1, \ldots, d.
\]

**Proof** It follows from Assumption 6 that \( \beta_i > c_1' \|A_i\|_F^2 \geq c_1' \|A_i\|_2^2 \), which together with (13) yields that

\[
h_i(X_i) - h_i(X_i^+) \geq \frac{c_1}{2\beta_i} \left\| P_i^{X_i} H_i X_i \right\|_F^2.
\]

(28)

According to the definition of \( H_i \) and \( \Lambda_i \), we have

\[
P_i^{X_i} H_i X_i = P_i^{X_i} \left( A_i A_i^T + \Lambda_i + \beta_i ZZ^T \right) X_i = \beta_i P_i^{X_i} ZZ^T X_i.
\]

(29)

Suppose \( \sigma_1, \ldots, \sigma_p \) are the singular values of \( X_i^T Z \). It is clear that \( 0 \leq \sigma_i \leq 1 \) and \( d_i^2 = 2 \sum_{j=1}^p \left( 1 - \sigma_j^2 \right) \) for any \( i = 1, \ldots, d \). By simple calculations, we have

\[
\left\| P_i^{X_i} ZZ^T X_i \right\|_F^2 = \text{tr} \left( X_i^T ZZ^T X_i \right) - \text{tr} \left( \left( X_i^T ZZ^T X_i \right)^2 \right) = \sum_{j=1}^p \sigma_j^2 \left( 1 - \sigma_j^2 \right).
\]

Moreover, it follows from \( d_i^{(k)} \leq \sqrt{1/(pd)} \) that \( \sigma_{\min}(X_i^T Z) \geq \sigma \), which implies that

\[
\left\| P_i^{X_i} ZZ^T X_i \right\|_F^2 = \sum_{j=1}^p \sigma_j^2 \left( 1 - \sigma_j^2 \right) \geq \sigma^2 \sum_{j=1}^p \left( 1 - \sigma_j^2 \right) = \frac{1}{2} \sigma^2 d_i^2.
\]

This together with (28) and (29) completes the proof.

**Lemma 11** Suppose all the conditions in Lemma 10 hold. Then for any \( i = 1, \ldots, d \), we have

\[
d_i^2 \left( X_i^{(k+1)}, Z^{(k)} \right) \leq \left( 1 - c_1 \sigma^2 \right) \left( d_i^{(k)} \right)^2 + \frac{12}{\beta_i} \sqrt{p} \|A_i\|_F^2.
\]

**Proof** According to Lemma 10 and definitions of \( h_i \) and \( H_i \), we can acquire

\[
\frac{1}{2} \text{tr} \left( ZZ^T D_p (X_i^+, X_i) \right) + \frac{1}{2\beta_i} \text{tr} \left( \left( A_i A_i^T + \Lambda_i \right) D_p (X_i^+, X_i) \right) \geq \frac{1}{4} c_1 \sigma^2 d_i^2.
\]

By straightforward calculations, we can further obtain the following two relationships

\[
\text{tr} \left( \left( A_i A_i^T + \Lambda_i \right) D_p (X_i^+, X_i) \right) \leq \left\| A_i A_i^T + \Lambda_i \right\|_F d_p (X_i^+, X_i) \leq 6 \sqrt{p} \|A_i\|_F^2,
\]

\[
\text{tr} \left( ZZ^T D_p (X_i^+, X_i) \right) = \frac{1}{2} d_i^2 - \frac{1}{2} d_i^2 (X_i^+, Z).
\]

Combining the above three relationships, we complete the proof.
Lemma 12 Suppose \( \{X^{(k)}_i, Z^{(k)}\} \) is the iterate sequence generated by Algorithm 1. Then, for \( i = 1, \ldots, d \) and \( k \in \mathbb{N} \), the following inequality holds.

\[
q^{(k)}(Z^{(k)}) - q^{(k)}(Z^{(k+1)}) \geq c_2 \left\| \frac{1}{Z^{(k)}} P_{Z^{(k)}}^\perp Q^{(k)} Z^{(k)} \right\|_F^2,
\]

where \( c_2 > 0 \) is a constant dependent on the penalty parameters \( \beta_i (i = 1, \ldots, d) \).

**Proof** This is a direct consequence of Lemma 3.5 in (Liu et al., 2013).

Lemma 13 Suppose \( \{X^{(k)}_i, Z^{(k)}\} \) is the iterate sequence generated by Algorithm 1. Then the inequality

\[
(d^{(k+1)}_i)^2 \leq \rho \sum_{j=1}^d d_p^2 \left( X^{(k+1)}_j, Z^{(k)} \right) + \sum_{j=1}^d \text{tr} \left[ A^+_j D_p \left( Z^+, Z \right) \right].
\]

holds for \( i = 1, \ldots, d \) and \( k \in \mathbb{N} \).

**Proof** The inequality (30) directly results in the relationship \( q(Z) - q(Z^+) \geq 0 \), which yields that

\[
0 \leq \sum_{j=1}^d \beta_j \text{tr} \left( X^+_j (X^+_j)^\top D_p \left( Z^+, Z \right) \right) + \sum_{j=1}^d \text{tr} \left( A^+_j D_p \left( Z^+, Z \right) \right).
\]

By straightforward calculations, we can deduce the following two relationships

\[
\text{tr} \left( A^+_j D_p \left( Z, Z^+ \right) \right) \leq \left\| A^+_j \right\|_F \left\| D_p \left( Z, Z^+ \right) \right\|_F \leq 4 \sqrt{p} \left\| A \right\|_F^2,
\]

\[
\text{tr} \left( X^+_j (X^+_j)^\top D_p \left( Z^+, Z \right) \right) = \frac{1}{2} d_p^2 \left( X^+_j, Z \right) - \frac{1}{2} \left( d_j^+ \right)^2,
\]

which implies that

\[
\sum_{j=1}^d \beta_j \left( d_j^+ \right)^2 \leq \sum_{j=1}^d \beta_j d_p^2 \left( X^+_j, Z \right) + 8 \sqrt{p} \left\| A \right\|_F^2.
\]

Now it can be readily verified that

\[
(d^+_i)^2 \leq \frac{1}{\beta_i} \sum_{j=1}^d \beta_j \left( d_j^+ \right)^2 \leq \rho \sum_{j=1}^d d_p^2 \left( X^+_j, Z \right) + \frac{8 \sqrt{p}}{\beta_i} \left\| A \right\|_F^2.
\]

This completes the proof.

Lemma 14 Let \( \Phi_i(Y) = -P^\perp_Y A_i A_i^\top Y Y^\top - Y Y^\top A_i A_i^\top P^\perp_Y \) for any \( Y \in S_{n,p} \) and \( i = 1, \ldots, d \). Then for any \( Y_1 \in S_{n,p} \) and \( Y_2 \in S_{n,p} \), it holds that

\[
\left\| \Phi_i(Y_1) - \Phi_i(Y_2) \right\|_F \leq 4 \left\| A_i \right\|_2^2 d_p \left( Y_1, Y_2 \right), \quad i = 1, \ldots, d.
\]
Proof This lemma directly follows from the triangular inequality. Hence, its proof is omitted.

Lemma 15 Suppose Assumption 6 holds, and \(\{X_i^{(k)}\}, Z^{(k)}\) is the iterate sequence generated by Algorithm 1 initiated from \(\{X_i^{(0)}, Z^{(0)}\}\) satisfying (22). Then for \(k \in \mathbb{N}\), it holds that

\[
\left( d_i^{(k)} \right)^2 \leq \frac{1}{\rho d_i}, \quad i = 1, \ldots, d.
\] (31)

Proof We use mathematical induction to prove this lemma. The argument (31) directly holds at \(\{d_i^{(0)}\}_{i=1}^d\) resulting from (22). Now, we assume the argument holds at \(\{d_i^d\}_{i=1}^d\), and investigate the situation at \(\{d_i^{(k+1)}\}_{i=1}^d\).

According to Assumption 6, we have \(\beta_i > 12 \rho d_i \sqrt{p} \|A_i\|_F^2 / (c_1 \sigma^2)\). Without loss of generality, we assume that \(c_1 \sigma^2 < 1\). Combining Lemma 11 and (31), we can derive that

\[
d_i^{2} (X_i^+, Z) \leq \frac{1 - c_1 \sigma^2}{\rho d_i} + \frac{c_1 \sigma^2}{\rho d_i} = \frac{1}{\rho d_i},
\]

which infers that \(\sigma_{\text{min}} (X_i^+ Z) \geq \sigma\). Similar to the proof of Lemma 10, we can deduce that

\[
\|P_{X_i^+} ZZ^T X_i^+\|_F^2 \geq \frac{\sigma^2}{2} d_i (X_i^+, Z).
\] (32)

Together with condition (13) and equality (29), we have

\[
\|P_{X_i^+} H_i X_i^+\|_F \leq \delta_i \beta_i \|P_{X_i^+} ZZ^T X_i\|_F \leq \delta_i \beta_i d_i.
\]

On the other hand, it follows from the triangular inequality that

\[
\|P_{X_i^+} H_i X_i^+\|_F \geq \|P_{X_i^+} (A_i A_i^T + \Lambda_i^+ + \beta_i ZZ^T) X_i^+\|_F - \|P_{X_i^+} (\Lambda_i^+ - \Lambda_i) X_i^+\|_F.
\]

It follows from the inequality (32) that

\[
\|P_{X_i^+} (A_i A_i^T + \Lambda_i^+ + \beta_i ZZ^T) X_i^+\|_F = \beta_i \|P_{X_i^+} ZZ^T X_i^+\|_F \geq \frac{\sqrt{2}}{2} \sigma \beta_i d_i (X_i^+, Z).
\]

According to Lemma 14, we have

\[
\|P_{X_i^+} (\Lambda_i^+ - \Lambda_i) X_i^+\|_F \leq 4 \|A_i\|_F^2 d_i (X_i^+, Z_i) \leq 4 \|A_i\|_F^2 (d_i (X_i^+, Z) + d_i).
\]

Combining the above four inequalities, we further obtain that

\[
(\sqrt{2} \sigma \beta_i / 2 - 4 \|A_i\|_F^2) d_i (X_i^+, Z) \leq (\delta_i \beta_i + 4 \|A_i\|_F^2) d_i.
\]

According to Assumption 6, we have \(0 \leq \delta_i < \sigma / \sqrt{4 \rho d_i}\), and

\[
\beta_i > \frac{4 \sqrt{2} (1 + \sqrt{2} \rho d_i)}{\sigma - 2 \sqrt{\rho d_i} \sigma} \|A_i\|_F^2 \geq \frac{4 \sqrt{2}}{\sigma} \|A_i\|_F^2.
\]
Thus, we arrive at
\[ d_p^+ (X^+_i, Z) \leq \frac{2(\delta_i \beta_i + 4 \| A_i \|_2^2)}{\sqrt{2} \sigma^{\beta_i} - 8 \| A_i \|_2^2} d_i \leq \sqrt{\frac{1}{2 \rho d_i}} d_i, \quad i = 1, \ldots, d. \] (33)

Again, we have \( \beta_i > 16 \rho d \sqrt{p} \| A \|_F^2 \) according to Assumption 6. Combing Lemma 13 and (31), we further acquire that
\[
(d_i^+)^2 \leq \rho \sum_{j=1}^{d} d_p^2 (X^+_j, Z) + \frac{8 \sqrt{p}}{\beta_i} \| A \|_F \leq \frac{1}{2d} \sum_{j=1}^{d} d_j^2 + \frac{1}{2 \rho d} \leq \frac{1}{\rho d},
\]
which completes the proof.

**Corollary 16** Suppose all the conditions in Lemma 15 hold. Then for any \( k \in \mathbb{N} \), there holds
\[
\mathcal{L}(\{ X_i^{(k)} \}, Z, \{ \Lambda_i^{(k)} \}) - \mathcal{L}(\{ X_i^{(k+1)} \}, Z, \{ \Lambda_i^{(k)} \}) \geq -\frac{1}{4} \sum_{i=1}^{d} \beta_i \left( d_i^{(k)} \right)^2.
\]

**Proof** This corollary directly follows from Lemmas 10 and 15.

**Corollary 17** Suppose all the conditions in Lemma 15 hold. Then for any \( k \in \mathbb{N} \), it holds that
\[
\mathcal{L}(\{ X_i^{(k+1)} \}, Z, \{ \Lambda_i^{(k)} \}) - \mathcal{L}(\{ X_i^{(k+1)} \}, Z, \{ \Lambda_i^{(k+1)} \}) \geq -\frac{1}{\rho d} \sum_{i=1}^{d} \| A_i \|_2^2 \left( d_i^{(k)} \right)^2.
\]

**Proof** According to the Cauchy-Schwarz inequality, we can deduce that
\[
\left| \langle \Lambda_i^+ - \Lambda_i, D_p (X^+_i, Z) \rangle \right| = \left| \langle \Phi_i (X^+_i) - \Phi_i (X_i), D_p (X^+_i, Z) \rangle \right| \\
\leq \| \Phi_i (X^+_i) - \Phi_i (X_i) \|_F d_p (X^+_i, Z) \leq \frac{\sqrt{8}}{\rho d} \| A_i \|_2 \left( d_p (X^+_i, X_i) \right) d_i,
\]
where the last inequality follows from Lemma 14 and (33). Moreover, we have
\[
d_p (X^+_i, X_i) \leq d_p (X^+_i, Z) + d_i \leq \frac{1 + \sqrt{2 \rho d}}{\sqrt{2 \rho d}} d_i,
\]
which further yields that
\[
\langle \Lambda_i^+ - \Lambda_i, D_p (X^+_i, Z) \rangle \geq -\frac{2(1 + \sqrt{2 \rho d})}{\rho d} \| A_i \|_2^2 d_i^2.
\]
Combing the fact that
\[
\mathcal{L}_i (X^+_i, Z, \Lambda_i) - \mathcal{L}_i (X^+_i, Z, \Lambda_i^+) = \frac{1}{2} \langle \Lambda_i^+ - \Lambda_i, D_p (X^+_i, Z) \rangle,
\]
we complete the proof.
Corollary 18 Suppose \( \{X_i^{(k)}\}, Z^{(k)} \) is the iterate sequence generated by Algorithm 1. Let 
\[
\bar{Q}^{(k)} = \sum_{i=1}^{d} \left( \beta_i X_i^{(k+1)} (X_i^{(k+1)})^\top + \Phi_i (Z^{(k)}) - \Phi_i (X_i^{(k+1)}) \right),
\]
and \( G^{(k)} = P_{Z^{(k)}}^\perp AA^\top Z^{(k)} + P_{Z^{(k)}}^\perp \bar{Q}^{(k)} Z^{(k)} \). Then for any \( k \in \mathbb{N} \), it holds that 
\[
\mathcal{L}(\{X_i^{(k+1)}\}, Z^{(k)}, \{A_i^{(k+1)}\}) - \mathcal{L}(\{X_i^{(k+1)}\}, Z^{(k+1)}, \{A_i^{(k+1)}\}) \geq c_2 \left\| G^{(k)} \right\|_F^2.
\]

Proof Recalling the definitions of \( Q \) and \( \Phi_i (Z) \), we obtain that 
\[
P_{Z}^\perp QZ = P_{Z}^\perp \bar{Q}Z - \sum_{i=1}^{d} P_{Z}^\perp \Phi_i (Z) Z = P_{Z}^\perp \bar{Q}Z + P_{Z}^\perp AA^\top Z = G.
\]
This together with (30) completes the proof.

Next we show the monotonicity of the sequence of augmented Lagrangian function values \( \{\mathcal{L}^{(k)}\} \) where \( \mathcal{L}^{(k)} = \mathcal{L}(\{X_i^{(k)}\}, Z^{(k)}, \{A_i^{(k)}\}) \).

Proposition 19 Suppose \( \{\{X_i^{(k)}\}, Z^{(k)}\} \) is the iterate sequence generated by Algorithm 1 initiated from \( \{\{X_i^{(0)}\}, Z^{(0)}\} \) satisfying (22), and problem parameters satisfy Assumption 6. Then the sequence \( \{\mathcal{L}^{(k)}\} \) is monotonically non-increasing and, for any \( k \in \mathbb{N} \), satisfies the following two conditions: 
\[
\mathcal{L}^{(k)} - \mathcal{L}^{(k+1)} \geq \sum_{i=1}^{d} \left( \frac{1}{4} c_1 \sigma^2 \beta_i - \frac{1 + \sqrt{2 \rho d}}{\rho d} \|A_i\|_2^2 \right) \left( d_i^{(k)} \right)^2 + c_2 \left\| G^{(k)} \right\|_F^2,
\]
and 
\[
\mathcal{L}^{(k)} - \mathcal{L}^{(k+1)} \geq c_3 \left\| P_{Z^{(k)}}^\perp AA^\top Z^{(k)} \right\|_F^2,
\]
where \( c_3 > 0 \) is a constant.

Proof Combining Corollaries 16, 17, and 18, we can easily verify the inequality (34). Recalling the condition \( \beta_i > 4(1 + \sqrt{2 \rho d}) \|A_i\|_2^2 / (c_1 \sigma^2 \rho d) \) in Assumption 6, we can conclude that \( \mathcal{L} - \mathcal{L}^+ \geq 0 \). Hence, the sequence \( \{\mathcal{L}^{(k)}\} \) is monotonically non-increasing. It directly follows from the definition of \( Q \) that 
\[
P_{Z}^\perp QZ = P_{Z}^\perp \sum_{i=1}^{d} \left( \beta_i D_p (X_i^+, Z) + \Phi_i (Z) - \Phi_i (X_i^+) \right) Z.
\]
Together with the triangular inequality and (33), we can obtain that 
\[
\left\| P_{Z}^\perp QZ \right\|_F \leq \sum_{i=1}^{d} (\beta_i d_p (X_i^+, Z) + \| \Phi_i (X_i^+) - \Phi_i (Z) \|_F) \leq \sqrt{\frac{\sqrt{2 \rho d}}{2 \rho d}} \sum_{i=1}^{d} (\beta_i + 4 \|A_i\|_2^2) d_i.
\]
And we define a constant \( c_4 := \min_{i=1, \ldots, d} \left\{ c_1 \sigma^2 \beta_i / 4 - (1 + \sqrt{2 \rho d}) \|A_i\|_2^2 / (\rho d) \right\} > 0 \). It can be readily verified that 
\[
\left\| P_{Z}^\perp AA^\top Z \right\|_F = \left\| G - P_{Z}^\perp QZ \right\|_F \leq \left\| G \right\|_F + \left\| P_{Z}^\perp \bar{Q}Z \right\|_F \leq \sqrt{(\mathcal{L} - \mathcal{L}^+)/c_3},
\]
25
where $c_3 := \left( \sum_{i=1}^{d} (\beta_i + 4 \| A_i \|_2^2) / \sqrt{2pd}c_4 + \sqrt{1/c_2} \right)^{-2} > 0$ is a constant, and the last inequality follows from the facts that, for $i = 1, \ldots, d$,

$$
\| G \|_F \leq \sqrt{\mathcal{L} - \mathcal{L}^+}/c_2 \quad \text{and} \quad d_i \leq \sqrt{\mathcal{L} - \mathcal{L}^+}/c_4.
$$

We complete the proof.

We are now ready to present the proof of Theorem 8.

**Proof** [Proof of Theorem 8] Since each of $X_i^{(k)}$ or $Z^{(k)}$ is orthonormal for any $i = 1, \ldots, d$ and $k \in \mathbb{N}$, the whole sequence $\{\{X_i^{(k)}\}, Z^{(k)}\}$ is naturally bounded. Then, it follows from the Bolzano-Weierstrass theorem that this sequence exists an accumulation point $(\{X_i^*\}, Z^*)$, where $X_i^* \in \mathcal{S}_{n,p}$ and $Z^* \in \mathcal{S}_{n,p}$. In addition, the boundedness of $\{\Lambda_i^{(k)}\}$ results from the multipliers updating formula (15). Hence, the lower boundedness of $\{\mathcal{L}^{(k)}\}$ is owing to the continuity of the augmented Lagrangian function. Namely, there exists a constant $L$ such that $\mathcal{L}^{(k)} \geq L$ holds for all $k \in \mathbb{N}$.

Let $R^{(k)} = P_{Z^{(k)}} AA^\top Z^{(k)}$. It follows from (35) and (36) that there hold

$$
\sum_{k=0}^{N-1} \left\| R^{(k)} \right\|_F^2 \leq \frac{1}{c_3} \sum_{k=0}^{N-1} \left( \mathcal{L}^{(k)} - \mathcal{L}^{(k+1)} \right) \leq \frac{1}{c_3} \left( \mathcal{L}^{(0)} - \mathcal{L} \right)
$$

and

$$
\sum_{k=0}^{N-1} \sum_{i=1}^{d} \left( d_i^{(k)} \right)^2 \leq \frac{d}{c_4} \sum_{k=0}^{N-1} \left( \mathcal{L}^{(k)} - \mathcal{L}^{(k+1)} \right) \leq \frac{d}{c_4} \left( \mathcal{L}^{(0)} - \mathcal{L} \right).
$$

Taking the limit as $N \to \infty$ on the both sides of (37) and (38), we obtain that

$$
\sum_{k=0}^{\infty} \left\| R^{(k)} \right\|_F^2 < \infty \quad \text{and} \quad \sum_{k=0}^{\infty} \sum_{i=1}^{d} \left( d_i^{(k)} \right)^2 < \infty,
$$

which further imply

$$
\lim_{k \to \infty} \left\| R^{(k)} \right\|_F = 0 \quad \text{and} \quad \lim_{k \to \infty} \sum_{i=1}^{d} d_i^{(k)} = 0.
$$

Hence, it holds at any limit point that $P_{Z^*} AA^\top Z^* = 0$ and $X_i^*(X_i^*)^\top = Z^*(Z^*)^\top$, for $i = 1, \ldots, d$. Therefore, $Z^*$ is a first-order stationary point of the problem (1). Finally, it follows from the inequalities (37) and (38) that

$$
\min_{k = 0, \ldots, N-1} \left\{ \left\| R^{(k)} \right\|_F^2 + \frac{1}{d} \sum_{i=1}^{d} \left( d_i^{(k)} \right)^2 \right\} \leq \frac{1}{N} \sum_{k=0}^{N-1} \left\{ \left\| R^{(k)} \right\|_F^2 + \frac{1}{d} \sum_{i=1}^{d} \left( d_i^{(k)} \right)^2 \right\} \leq \frac{C}{N},
$$

where $C = (\mathcal{L}^{(0)} - \mathcal{L})(1/c_3 + 1/c_4) > 0$ is a constant. The proof is completed.

\[26\]
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