Supersymmetric $\eta$ operators

Boris F Samsonov

Physics Department, Tomsk State University, 36 Lenin Avenue, 634050 Tomsk, Russia
E-mail: samsonov@phys.tsu.ru

Received 23 January 2012, in final form 8 July 2012
Published 23 October 2012
Online at stacks.iop.org/JPhysA/45/444028

Abstract

Being chosen as a differential operator of a special form, the metric $\eta$ operator has become unitary equivalent to a one-dimensional Hermitian Hamiltonian with a natural supersymmetric structure. We show that fixing the superpartner of this Hamiltonian permits us to determine both the metric operator and the corresponding non-Hermitian Hamiltonian. Moreover, under an additional restriction on the non-Hermitian Hamiltonian, it has become a superpartner of another Hermitian Hamiltonian.

This article is part of a special issue of Journal of Physics A: Mathematical and Theoretical devoted to ‘Quantum physics with non-Hermitian operators’.

PACS numbers: 03.65.-w, 03.65.Ca, 03.65.Db, 02.30.Tb

1. Introduction

After a seminal paper by Scholtz Geyer and Hahne [1], the metric operator $\eta$ has played a crucial role in the pseudo-Hermitian (quasi-Hermitian) quantum mechanics. Just this operator is used while one redefines the inner product to bring an initially non-Hermitian operator to its Hermitian form (for details, see review papers [2, 3]). Usually (see e.g. [3]), this is an invertible, Hermitian, positive definite and bounded operator such that

$$\eta H = H^\dagger \eta.$$  \hspace{1cm} (1)

Here $H$ is a given non-Hermitian operator (Hamiltonian) and $H^\dagger$ is its Hermitian adjoint. If $H$ has a discrete spectrum and admits a complete set of biorthonormal eigenvectors, then the metric operator may be presented as [4]

$$\eta = OO^\dagger.$$

Note that this form is typical (see e.g. [5, 6]) when a Hermitian Hamiltonian is presented in factorized form in supersymmetric quantum mechanics (SUSY QM). Its superpartner $\eta_0$ is obtained by interchanging $O$ and $O^\dagger$,

$$\eta_0 = O^\dagger O.$$

The only difference between the SUSY intertwiner, that we denote by $L$, and the operator $O$ is that $L$ is a differential operator so that the metric $LL^\dagger$ becomes a differential operator. It is
clear that \( LL^\dagger \) is Hermitian. It is both positive definite and invertible if it has an empty kernel in the corresponding Hilbert space. This is possible if both \( L \) and \( L^\dagger \) have empty kernels. In SUSY QM, this corresponds to a broken supersymmetry [6].

Thus, the only obstacle to using the technique of SUSY QM for studying the properties of differential metric operators is that these operators, being differential operators, are usually unbounded. Nevertheless, for instance, Fityo [7] used a first-order differential operator \( \mathcal{O} \) for constructing a new class of non-Hermitian Hamiltonians with real spectra without any discussion that the corresponding metric operator becomes unbounded. He also mentioned that \( H \) and \( H^\dagger \) are the superpartners of the second-order supersymmetry. Nevertheless, in contrast to the usual second-order supersymmetry [5], here an additional restriction is imposed on the intertwining operator: it should be Hermitian. The consequences of this restriction are not analyzed in [7]. Our analysis shows that the corresponding first-order operator \( L \) (actually its complex conjugate form \( L^\ast \)) intertwines \( H \) with a Hermitian operator that we will denote by \( h_0 \).

Any unbounded operator has a domain of definition which is a subset of the corresponding Hilbert space. Therefore, while redefining the inner product in the spirit of [8],

\[
\langle \langle \psi_1 | \psi_2 \rangle \rangle = \langle \psi_1 | \eta | \psi_2 \rangle, \quad \forall \psi_1, \psi_2 \in \mathcal{H},
\]

where \( \mathcal{H} \) is the initial Hilbert space, one has to replace the condition \( \forall \psi_1, \psi_2 \in \mathcal{H} \) by another one \( \forall \psi_1, \psi_2 \in D_h \subset \mathcal{H} \) where \( D_h \) is the domain of definition of the \( \eta \) operator. From here it follows that there exist in \( \mathcal{H} \) the vectors \( \psi \notin D_h \), which cannot be mapped onto the new Hilbert space \( \mathcal{H}_\eta \) defined with the help of the inner product \( \langle \langle \psi_1 | \psi_2 \rangle \rangle \eta \). This is the price for using unbounded \( \eta \) operators. From the physical viewpoint, it remains to hope that maybe states described by such vectors cannot be realized in practice. Moreover, this is not an obstacle for finding the Hermitian Hamiltonian \( h \) related to the given non-Hermitian \( H \) by a similarity transformation [9, 10]. In particular, in [10], the scattering matrix and cross section for \( h \) are calculated and their unusual properties are discussed.

2. Second-order differential \( \eta \) operator

Let us take a non-Hermitian Hamiltonian

\[
H = -\frac{d^2}{dx^2} + V(x), \quad x \geq 0,
\]

with the domain

\[
D_H = \{ \psi \in L^2(0, \infty) : \psi''(x) - V(x)\psi(x) \in L^2(0, \infty), \quad [\psi'(x) + w(x)\psi(x)]_{x=0} = 0 \},
\]

where \( V(x) \) and \( w(x) \) are the complex-valued functions. We find the adjoint operator \( H^\dagger \) as usual using the inner product on the space \( L^2(0, \infty) \), \( \langle H^\dagger \psi_1 | \psi_2 \rangle = \langle \psi_1 | H \psi_2 \rangle, \quad \forall \psi_1, \psi_2 \in D_H \), thus obtaining

\[
D_{H^\dagger} = \{ \psi \in L^2(0, \infty) : \psi''(x) - V^\ast(x)\psi(x) \in L^2(0, \infty), \quad [\psi'(x) + w^\ast(x)\psi(x)]_{x=0} = 0 \}
\]

\[
H^\dagger = -\frac{d^2}{dx^2} + V^\ast(x) = H^\ast.
\]

For simplicity, we will assume that \( V(x) \) is a scattering potential satisfying the condition

\[
\int_0^\infty (1 + x)|V(x)| \, dx < \infty.
\]

In particular, we will assume that the function \( |V(x)| \) is bounded below and tends to zero faster than any finite power of \( 1/x \) as \( x \to \infty \) so that the operator of multiplication by the
function $V(x)$, as an operator acting in $L^2(0, \infty)$, becomes bounded. For this reason, we can simplify the domains $D_H$ and $D_{H'}$,

$$D_H = \{ \psi \in L^2(0, \infty) : \psi''(x) \in L^2(0, \infty), \ [\psi'(x) + w(x)\psi(x)]_{x=0} = 0 \},$$

$$D_{H'} = \{ \psi \in L^2(0, \infty) : \psi''(x) \in L^2(0, \infty), \ [\psi'(x) + w^*(x)\psi(x)]_{x=0} = 0 \}.$$ (7)

As discussed in the introduction, to find the metric operator for the given non-Hermitian Hamiltonian $H$, one has to find a Hermitian positive definite and invertible operator $\eta$ satisfying equation (1). If $\eta$ is bounded, its domain is the whole Hilbert space and no problems occur in acting both by the left- and the right-hand sides of (1) on functions belonging to $D_H$. Unfortunately, this is not our case since we consider unbounded $\eta$ having its own domain in $L^2(0, \infty)$. We find it reasonable to assume that the domain of $\eta$ coincides with that of $H$, $D_{\eta} = D_H$ (4) or (7). As shown below, if the operator $\eta$ is chosen to be a second-order differential operator of a special form defined on this domain, this assumption is justified since such $\eta$ is self-adjoint. Nevertheless, even in such a case, one cannot apply (1) to any $\psi \in D_H$. Indeed if $\psi \in D_{\eta} = D_H$, then from the left-hand side of (1), it follows that $H\psi \in D_H$. Thus, (1) has a sense on a subset of $D_H$ such that the range of $H$ is contained in the domain of $H$. It is possible to show that this subset is dense in $L^2(0, \infty)$ but we do not dwell on its proof.

Assume that $\eta$ is defined with the help of a differential expression. Then from the condition $D_{\eta} = D_H$, it follows that $\eta$ may be a second-order differential operator for which we assume the form

$$\eta = LL^\dagger,$$ (8)

where

$$L = -d/dx + w^*(x), \quad L^\dagger = d/dx + w(x),$$ (9)

so that

$$\eta\psi(x) = -\psi''(x) + (w^* - w)\psi'(x) + [ww^* - w']\psi(x), \quad \psi(x) \in D_{\eta} = D_H.$$ (10)

We note that the operators $L$ and $L^\dagger$ are mutually adjoint with respect to the inner product in $L^2(0, \infty)$. To see that, consider for instance

$$\langle \psi_1 | L^\dagger \psi_2 \rangle = \int_0^\infty \psi_1^\dagger(\psi_2^2 + w\psi_2) \, dx = \left[ \psi_1^\dagger \psi_2 \right]_0^\infty + \int_0^\infty (-\psi_1^\dagger + w^*)^\dagger \psi_2 \, dx = \langle L^\dagger \psi_1 | \psi_2 \rangle.$$ (11)

The integrated term here vanishes at infinity since the functions $\psi_1(x)$ and $\psi_2(x)$ are smooth and belong to $L^2(0, \infty)$ and therefore they tend to $0$ as $x \to \infty$. At $x = 0$, it vanishes since $\psi_1 \in D_L$ and the domain of $L$ coincides with the range of $L^\dagger$, which according to (7) reads

$$D_L = \{ \psi \in L^2(0, \infty) : \psi'(x) \in L^2(0, \infty), \ [\psi(0)] = 0 \}.$$ (11)

It is easy to check that the operator $\eta$ (10) is self-adjoint with respect to the inner product in $L^2(0, \infty)$. Indeed, as usual, assuming that $\psi_1 \in D_{\eta}$ and integrating by parts the term with the second derivative twice and the term with the first derivative once yields

$$\langle \psi_2 | \eta \psi_1 \rangle = \int_0^\infty \psi_2^\dagger \left[ -\psi''_1 + (w^* - w)\psi'_1 + (ww^* - w')\psi_1 \right] \psi_1 \, dx$$

$$= \left[ \psi_2^\dagger \psi_1 - \psi^\dagger_2 \psi_1 + \psi^\dagger_2 \psi_1 (w^* - w) \right]_{x=0} + \int_0^\infty \left[ -\psi''_2 + (w - w^*)\psi'_2 + (w^* - w')\psi_2 \right] \psi_1 \, dx$$

$$= \int_0^\infty \left[ -\psi''_2 + (w - w^*)\psi'_2 + (w^* - w')\psi_2 \right] \psi_1 \, dx = \langle \eta \psi_2 | \psi_1 \rangle.$$
To justify the last equality, we consider the integrated term at $x = 0$,
\[
\left[\psi_2^* \psi_1 - \psi_2^* \psi_1' + \psi_2^* \psi_1 (w^* - w)\right]_{x=0} = \left[\psi_2^* \psi_1 - \psi_2^* (-w \psi_1) + \psi_2^* \psi_1 (w^* - w)\right]_{x=0}
\]
\[
= \psi_1 (0) \left[\psi_2^* + \psi_2^* w^*\right]_{x=0}
\]
\[
= 0.
\]

The first line here follows from the property that $\psi_1 \in D_\eta = D_H$ given in (7) and in the last line, we used $\psi_2 \in D_\eta = D_H$. In section 7, we consider examples illustrating the self-adjointness of $\eta$ when it is a second-order differential operator with constant coefficients ($w'(x) = 0$) and with variable coefficients ($w'(x) \neq 0$).

Furthermore, if we impose the condition that the operator $L^\dagger$ has the empty kernel in $D_\eta$, then $\eta$ is positive definite. Imposing additionally that the operator $L$ has the empty kernel in $D_k$ (11), we obtain an invertible operator $\eta$. With these assumptions, we have the operator $\eta$ suitable for constructing the operator $\rho = \eta^{1/2}$ (see below).

Another property that follows from the condition $D_\eta = D_H$ (4) (or (7)) is that the differential equation
\[
\eta \Psi_k(x) = \lambda(k) \Psi_k(x), \quad k \geq 0,
\]
(12)
should have only one singular point which is $x = \infty$. This means that the function $\omega(x)$ should be regular for all $x \in (0, \infty)$ and therefore we can put
\[
\omega(x) = \frac{d \log |u(x)|}{dx}, \quad u(x) \neq 0 \quad \forall x \in (0, \infty).
\]
(13)
Since $D_\eta = D_H$ (7), we have to supply equation (12) with the following boundary condition at $x = 0$:
\[
\left[\Psi_k'(x) + w(x) \Psi_k(x)\right]_{x=0} = 0.
\]
(14)
For simplicity, we will consider the case when the operator $\eta$ has no bound states. Therefore, we will impose on the functions $\Psi_k(x)$ an asymptotic condition at $x \to \infty$ so that they describe scattering states of the operator $\eta$.

3. Unitary equivalence between the $\eta$ operator and a Hamiltonian

Let us put
\[
u(x) = \rho(x) e^{i \omega(x)},
\]
(15)
where $\rho(x)$ and $\omega(x)$ are the real-valued functions. Then after a unitary transformation
\[
\Psi_k = U \Psi_k, \quad \eta = \eta U H U^\dagger
\]
(16)
with the operator $U$ being a multiplication operator on the function (actually it is simply a phase factor)
\[
U = \left(\frac{\nu}{\nu'}\right)^{1/2} = e^{-i \omega}, \quad U^\dagger = U^{-1} = \left(\frac{\nu'}{\nu}\right)^{1/2} = e^{i \omega},
\]
(17)
equation (12) reduces to a Schrödinger-like equation
\[
H \Psi_k(x) = \lambda(k) \Psi_k(x), \quad H = -\frac{d^2}{dx^2} + V(x),
\]
(18)
where the potential $V(x)$ is defined in terms of a real-valued function (superpotential) $W(x)$ as follows:
\[
V(x) \equiv W^2(x) - W'(x)
\]
(19)
and the function $W(x)$ is expressed in terms of the modulus of the function $u(x)$ (15),

$$W(x) = \frac{1}{2}[\log(uu^*)]' = [\log \rho]' = \frac{\rho'(x)}{\rho(x)} = W^+(x). \quad (20)$$

The boundary condition for equation (18) follows from that for equation (12):

$$[\psi_k(x) + W(x)\psi_k(x)]_{x=0} = 0, \quad (21)$$

and from (4), we obtain

$$D_H = \{ \psi \in L^2(0, \infty) : \psi^*(x) - V(x)\psi(x) \in L^2(0, \infty), \ [\psi(x) + W(x)\psi(x)]_{x=0} = 0 \}. \quad (22)$$

Potential (19) has a structure typical for supersymmetric quantum mechanics [5]. Its SUSY partner $V_0$ has the form

$$V_0(x) = W^2(x) + W'(x) = \frac{\rho''(x)}{\rho(x)},$$

and for the potential $V(x)$ we obtain a formula typical for the Darboux transformed potential (see e.g. [5])

$$V(x) = V_0(x) - 2W'(x) = V_0(x) - 2[\log(\rho(x))]'. \quad (23)$$

Moreover, the function $\rho(x)$ is a solution to the Schrödinger equation with the Hamiltonian

$$H_0 = -\frac{d^2}{dx^2} + V_0(x) \quad (24)$$

corresponding to the zero eigenvalue

$$H_0\rho(x) = 0. \quad (25)$$

The function $\rho(x)$ is known as a transformation function for the Hamiltonian $H_0$ (see e.g. [5]). If solutions to the Schrödinger differential equation with the potential $V_0(x)$ are known

$$H_0\Psi_0^0(x) = E\Psi_0^0(x), \quad (26)$$

then the solutions $\Psi_E(x)$ to the same equation with the potential $V(x)$ may be obtained with the help of the transformation operator

$$L_\rho = -d/dx + W(x) = -d/dx + \rho'(x)/\rho(x) \quad (27)$$

as

$$\Psi_E(x) = L_\rho \Psi_0^0(x) = -[\Psi_0^0(x)]' + W(x)\Psi_0^0(x). \quad (28)$$

Hence, the Hamiltonians $H$ and $H_0$ are intertwined by the operators $L_\rho$ and $L_\rho^\dagger$,

$$L_\rho H_0 = H_0 L_\rho, \quad L_\rho^\dagger H = H_0 L_\rho^\dagger, \quad (28)$$

where

$$L_\rho^\dagger = d/dx + W(x). \quad (29)$$

From the first relation of (28) it follows that $L_\rho$ transforms the eigenfunctions $\psi_k^0(x)$ of the Hamiltonian $H_0$ (18) to the eigenfunctions $\psi_k(x)$ of the Hamiltonian $H$,

$$\psi_k(x) = \lambda^{-1/2}(k)L_\rho \psi_k^0(x).$$

From the second relation of (28) it follows that $L_\rho^\dagger$ transforms the eigenfunctions $\psi_k(x)$ of the Hamiltonian $H$ (18) to the eigenfunctions $\psi_k^0(x)$ of the Hamiltonian $H_0$,

$$\psi_k^0(x) = \lambda^{-1/2}(k)L_\rho^\dagger \psi_k(x) = \lambda^{-1/2}(k)[\psi_k^0(x) + W(x)\psi_k(x)]. \quad (30)$$
\[ H_0 \Psi_k^{(0)}(x) = \lambda(k) \Psi_k^{(0)}(x). \]  
Equation (31) with (21), we find boundary conditions for the Schrödinger equation with the Hamiltonian \( H_0 \).

\[ \Psi_k^{(0)}(0) = 0. \]  
Equation (32) and the boundary condition (21).

It is easy to check that the function

\[ u(x) = \frac{1}{\rho(x)} \]  
and the equation

\[ H u(x) = 0 \]  
and the boundary condition (21).

The operator \( \rho \) is unitary equivalent to \( \eta \). Therefore, to fulfil the assumption that \( \eta \) has a continuous spectrum without bound states, we have to assume that the operator \( H_0 \) also has a purely continuous spectrum and the Darboux transformation with the transformation function \( \rho(x) \) (33) over the Hamiltonian \( H \) does not create a bound state. Furthermore, we have to assume also that the Darboux transformation from the Hamiltonian \( H_0 \) to the Hamiltonian \( H \) realized with the help of the transformation function \( \rho(x) \) does not create a bound state either. Assuming additionally that the potential \( V_0(x) \) is scattering, we have to choose the solution to equation (34) such that

\[ \rho \rightarrow e^{dx}, \quad x \rightarrow \infty, \quad d < 0. \]  
Equation (35) guarantees increasing asymptotic behavior of the function \( u(x) \) (33). Therefore, this function is not square integrable and \( \lambda(k) = 0 \) does not belong to the spectrum of \( H \). In this case, the operators \( \eta, H \) and \( H_0 \) are isospectral and the corresponding supersymmetry is broken. We would like to emphasize that no restriction on the phase \( \omega(x) \) of the function \( u(x) \) (15) is imposed.

4. SUSY partner for the \( \eta \) operator

The operators \( L \) and \( L^\dagger \) (9) play a crucial role in our approach since they define both the \( \eta \) operator and, as we show below, the Hamiltonians \( H \) and \( H^\dagger \). Operators (9) are uniquely determined when the function \( u(x) \) is fixed (see equations (9) and (13)). Therefore, the properties of the \( \eta \) operator and the Hamiltonians \( H \) and \( H^\dagger \) depend on the properties of the function \( u(x) \). Note that the modulus of \( u(x) \) is defined by the Hamiltonian \( H_0 \) as a solution to equation (25). Therefore, both the form of the operator \( \eta \) and the properties of the Hamiltonian \( H \) depend on the properties of the Hamiltonian \( H_0 \).

Equation (12) may be considered as equation (18) unitary transformed with the help of the operator \( U^{-1} \) (17) and therefore, as we show below, it inherits all the supersymmetric properties of equation (18).

Applying the operator \( L^\dagger \) to both sides of equation (12) yields

\[ (L^\dagger L)L^\dagger \Psi_k(x) = \lambda(k)L^\dagger \Psi_k(x). \]  
Equation (36) The condition \( \lambda(k) \neq 0 \) discussed above implies

\[ \Psi_k^{(0)}(x) \neq 0. \]  
Equation (37) From here, one deduces that

\[ \eta \Psi_k^{(0)}(x) = \lambda(k)\Psi_k^{(0)}(x), \quad \eta = L^\dagger L. \]  
Equation (38)
The operators $\eta$ (8) and $\eta_0$ (38) are intertwined by $L^\dagger$ and $L$,
\[
\eta_0 L^\dagger = L^\dagger \eta, \quad L \eta_0 = \eta L. \tag{39}
\]
Thus, the operator $L^\dagger$ maps the solutions of equation (12) (eigenfunctions of $\eta$) to the solutions of equation (38) (eigenfunctions of $\eta_0$ which is a SUSY partner of $\eta$). Quite similarly, the operator $L$ realizes an inverse mapping. The transformation function for each mapping is either the one which is annihilated by the operator $L$ or the one which is annihilated by its adjoint $L^\dagger$.

The mapping preserving the $\delta$-function normalization of the functions $\Psi_k$ and $\Psi_k^{(0)}$ is given by
\[
\Psi_k(x) = \lambda^{-1/2}(k)\Psi_k^{(0)}(x), \quad \Psi_k^{(0)}(x) = \lambda^{-1/2}(k)L^\dagger\Psi_k(x). \tag{40}
\]
Note that intertwining relations (39) are nothing but the associativity of operator multiplication,
\[
(L^\dagger L)L^\dagger = L^\dagger (LL^\dagger), \quad L(LL^\dagger) = (LL^\dagger)L. \tag{41}
\]
We can resolve intertwining relations (1) in a similar way. For that, we need not only the operators $L$ and $L^\dagger$ but also their complex conjugate form,
\[
L^* = -d/dx + w(x), \quad (L^\dagger)^* = d/dx + w^*(x). \tag{42}
\]
We assume here that the operation of the complex conjugation commutes with the operation of the Hermitian conjugation.

Let us put
\[
H = L^* L^\dagger + \alpha, \quad H^\dagger = L(L^\dagger)^* + \alpha^*, \tag{43}
\]
where $\alpha$ is a complex constant. Then, under an additional assumption,
\[
L^\dagger L^* + \alpha = (L^*)^\dagger L + \alpha^*, \tag{44}
\]
we reduce equation (1) to an identity. We thus expressed the Hamiltonians $H$ and $H^\dagger$ in terms of the function $w(x)$ and, taking into account formula (13), in terms of the function $\rho(x)$. We would like to emphasize that if the Hamiltonian $H_0$ is fixed, then the absolute value $\rho(x)$ of the function $\rho(x)$ is determined from equation (25) but its phase $\omega(x)$ still remains arbitrary. Below we show that this arbitrariness may be fixed with the help of condition (44).

5. Fixing phase $\omega(x)$ with the help of a Hermitian Hamiltonian $h_0$

Note that the right-hand side of equation (44) is Hermitian conjugate with respect to its left-hand side. This means that this equation becomes an identity if the operator
\[
h_0 = L^\dagger L^* + \alpha = -\frac{d^2}{dx^2} + w^2(x) + w'(x) + \alpha = -\frac{d^2}{dx^2} + \frac{u'^2(x)}{u(x)} + \alpha \tag{45}
\]
is Hermitian, $h_0 = h_0^\dagger$. The necessary condition for this is the reality of the function
\[
v_0(x) = \frac{u'(x)}{u(x)} + \alpha = \frac{\rho'(x)}{\rho(x)} - [\omega'(x)]^2 + \beta + i\left(\omega''(x) + \frac{\rho'(x)}{\rho(x)}\omega'(x) + \gamma\right), \tag{46}
\]
where we put
\[
\alpha = \beta + i\gamma \tag{47}
\]
and used equation (15). From here, we find the equation for the function $\omega(x)$,
\[
\omega''(x) + 2\frac{\rho'(x)}{\rho(x)}\omega'(x) + \gamma = 0 \tag{48}
\]
and the potential
\[
v_0(x) = \frac{\rho''(x)}{\rho(x)} - [\omega'(x)]^2 + \beta = v_0^*(x) \tag{49}
\]
defining the Hermitian Hamiltonian

\[ h_0 = -\frac{d^2}{dx^2} + v_0(x) = h_0^* = h_0. \]  (50)

From formula (45), we extract two important consequences. First, comparing it with (43), we conclude that the operators \( h_0 \) and \( H \) are intertwined by the operator \( L^* \) (42),

\[ L^* h_0 = HL^*. \]  (51)

Second, the function \( u(x) \) is an eigenfunction of \( h_0 \),

\[ h_0 u(x) = \alpha u(x). \]  (52)

Now we can formulate two approaches for finding a pair of operators \( H \) and \( \eta \), satisfying equation (1). Both approaches are based on the existence of an exactly solvable Hermitian Hamiltonian.

In the first approach, it is assumed that we know solutions to equation (52) with the given potential \( v_0(x) \) both as a differential equation and as a spectral problem on the space of smooth enough functions from \( L^2(0, \infty) \). To be consistent with the previous assumptions imposed on the Hamiltonians \( H \) and \( h_0 \) (see, e.g., equation (6)), here we assume that \( v_0(x) \) is a scattering potential and \( h_0 \) has a purely continuous spectrum

\[ h_0 \psi_k(x) = k^2 \psi_k(x), \quad k \geq 0. \]

Then taking a nodeless complex-valued solution to equation (52) (the parameter \( \alpha \) may be both real and complex), we construct, with the help of equations (42) and (13), the transformation operator \( L^* \). Moreover, from equation (51), we conclude that the Hamiltonian \( H \) (3) is a SUSY partner of \( h_0 \) (45) and therefore

\[ h_0(L^*)^\dagger = (L^*)^\dagger H. \]

This relation means that the operators \( h_0 \) and \( H \) are SUSY partners and the operator \( (L^*)^\dagger \) (42) transforms the eigenfunctions \( \psi_k(x) \) of the Hamiltonian \( H \) to the eigenfunctions \( \psi_k^*(x) \) of the Hamiltonian \( h_0 \). Using intertwining relation (51) and the properties \( h_0 = h_0^\dagger = h_0^* \) and \( H^\dagger = H^* \) (see equations (49) and (5)) yields

\[ h_0(L^*)^\dagger = (L^*)^\dagger H. \]

Comparing this equation with the boundary condition (7) for the eigenfunctions \( \psi_k(x) \) of the Hamiltonian \( H \), we find the boundary condition for the eigenfunctions \( \psi_k(x) \) of the Hamiltonian \( h_0 \),

\[ \psi_k(0) = 0. \]

As mentioned in the introduction, we are interested in having a broken supersymmetry. Therefore, we consider the case when \( H \) and \( h_0 \) are isospectral. The Hamiltonian \( H \) is assumed to have only a continuous spectrum; hence, \( h_0 \) also should have no bound states and the transformation function should coincide with the Jost solution for the Schrödinger equation.
with the Hamiltonian \( h_0 \). Then taking into account equation (52) and condition (35) yields the asymptotic behavior of the transformation function \( u(x) \),

\[
    u(x) \rightarrow e^{(d+ib)x}, \quad d < 0, \quad \alpha = -(d + ib)^2,
\]

(54)

where \( b \) is an arbitrary constant. From here and equation (47), one finds the relation between the constants \( d, b \) and \( \beta, \gamma \) (see (46)),

\[
    \beta = b^2 - d^2, \quad \gamma = -2db.
\]

(55)

This approach is realized in [10].

In the second approach, the starting point is the spectral problem (31), (32) with the given potential \( V_0(x) \). The absolute value of the transformation function is found from equation (25) and its phase is fixed by equation (48). The potential \( V(x) \) follows from equation (23) and the \( \eta \) operator is fixed by equations (16) and (17). Once both the modulus and the phase of the transformation function (15) are fixed, we reconstruct the Hamiltonian \( H \) (43) and, if necessary, the potential \( v_0 \) (49) and the Hamiltonian \( h_0 \) (50). We illustrate this approach in section 7 by a simple example.

6. Equivalent Hermitian Hamiltonian \( h \)

Once the eigenfunctions \( \Psi_k(x) \) and the eigenvalues \( \lambda(k) \) of the operator \( \eta \) are found, one can write down its spectral representation,

\[
    \eta = \int_0^\infty dk \lambda(k) |\Psi_k\rangle \langle \Psi_k|.
\]

(56)

It has a unique Hermitian, positive definite and invertible square root

\[
    \rho = \eta^{1/2} = \rho^\dagger, \quad \rho > 0,
\]

(57)

\[
    \rho = \int_0^\infty dk \lambda^{1/2}(k) |\Psi_k\rangle \langle \Psi_k|.
\]

(58)

Hence, from equation (1), one finds the Hermitian operator \( h \) equivalent to \( H \),

\[
    h = \rho H \rho^{-1} = \rho^{-1} H^\dagger \rho = h^\dagger,
\]

(59)

which in our case also has a purely continuous real and non-negative spectrum. Its eigenfunctions \( \Phi_k \),

\[
    h\Phi_k = k^2\Phi_k, \quad k \geq 0,
\]

are obtained by applying the operator \( \rho \) to the eigenfunctions of \( H \) [10],

\[
    \Phi_k = (k^2 - \alpha)^{-1/2} \rho \psi_k = (k^2 - \alpha)^{-1} \rho L^* \psi_k.
\]

(60)

The factor \( (k^2 - \alpha)^{-1/2} \) is introduced to guarantee both the normalization of these functions,

\[
    \langle \Phi_{k'} | \Phi_k \rangle = \delta(k - k'),
\]

(61)

and their completeness,

\[
    \int_0^\infty dk |\Phi_k\rangle \langle \Phi_k| = 1.
\]

(62)

From equation (60), it follows that the eigenfunctions \( \varphi_k \) of \( h_0 \) and \( \Phi_k \) of \( h \) are related by an isometric operator \( U \) [10],

\[
    \Phi_k = U \varphi_k.
\]

(63)
where
\[
U = \rho L^*(h_0 - \alpha)^{-1} = L(L^\dagger L)^{-1/2} = Lh_0^{-1/2} = (U^\dagger)^{-1}.
\]  
(64)

Using the spectral representation of the operator \( h \), one can express \( h \) in terms of \( \rho, L^*, (L^*)^\dagger \) and the resolvent of \( h_0 \) \[10\],
\[
h = \frac{\rho L^*}{\alpha - \alpha^*} [\alpha(h_0 - \alpha)^{-1} - \alpha^*(h_0 - \alpha^*)^{-1}] (L^*)^\dagger \rho.
\]  
(65)

Note that for any \( \alpha = d + ib \) with \( d < 0 \), the point \( \alpha = -a^2 \) does not belong to the spectrum of \( h_0 \) and operator (65) is well defined. When \( \alpha \) is real \( \alpha = \alpha_r = \alpha^*_r \), this expression becomes indetermined. This indeterminacy may be resolved using the usual l’Hospital’s rule, which yields
\[
h = \frac{\rho L^* h_0 (h_0 - \alpha_r)^{-2} (L^*)^\dagger \rho}{\alpha - \alpha^*}.
\]  
(66)

From here, we extract an important consequence. The real character of \( \alpha_r \) implies that \( d = 0 \) and, hence, the point \( E = k^2 = \alpha_r = b^2 \) belongs to the continuous spectrum of \( H \) and corresponds to the spectral singularity [10]. This point belongs to the spectrum of \( h_0 \) also. For this reason, the resolvent of \( h \) diverges at \( k^2 = \alpha_r = b^2 \) and operator (66) becomes undefined in the Hilbert space. This means that the non-Hermitian Hamiltonian \( H \) with a spectral singularity does not have a Hermitian counterpart.

Another possibility would be \( b = 0, d \neq 0 \). In the case \( \alpha = -d^2 \), the operator \( H \) becomes Hermitian and \( h \) becomes unitary equivalent to \( H \).

The operator \( U \) (64) mapping the eigenfunctions of \( h_0 \) to those of \( h \) (63) may be written in terms of the eigenfunctions \( \Psi_k^{(0)} \) of equation (38),
\[
U = L^* \int_0^\infty dk \lambda^{-1/2}(k)|\Psi_k^{(0)}\rangle \langle \Psi_k^{(0)}|.
\]  
(67)

Although equation (60) (or equations (63) and (67)) formally solves the problem of finding the eigenfunctions of \( h \), it contains the non-local operator \( \rho \) (or \( (L^\dagger L)^{-1/2} \)) and, therefore, in general, no explicit expression for \( \Phi_k \) exists.

7. Examples

As discussed in section 5, there exist two approaches for finding a pair of operators \( H \) and \( \eta \) satisfying equation (1) and below we give two corresponding examples. First, we exemplify the second approach since it leads to the simplest form of \( \eta \) being a second-order differential operator with constant coefficients and with the non-Hermitian Hamiltonian \( H \) first studied by Schwartz [11]. Then we give an example where \( \eta \) is a second-order differential operator with a variable coefficient.

7.1. \( \eta \) is a second-order differential operator with constant coefficients

Let us fix the real constants \( d < 0 \) and \( b \) (see (54)). In the simplest case, the potential \( V_0(x) \) may be a nonnegative constant \( V_0(x) = d^2 \) so that the solution of the spectral problem (31), (32) reads
\[
\Psi_k^{(0)}(x) = \sqrt{\frac{2}{\pi}} \sin(kx), \quad \lambda(k) = k^2 + d^2, \quad k \geq 0.
\]

Since \( H_0 = -d^2/dx^2 + d^2 \), from equation (25) we find the absolute value of the transformation function
\[
\rho(x) = e^{dx}.
\]
Next we solve equation (48) for $\omega(x)$ with $\gamma$ as given in (55),

$$\omega(x) = -\frac{\gamma}{2d} x = bx, \quad b = -\frac{\gamma}{2d}$$

(the integration constant is without importance here), thus reconstructing the transformation function (15)

$$u(x) = e^{(d+ib)x}.$$  

According to (20), the function $W(x)$ is constant, $W(x) = d$. Therefore, the potential $V(x)$ (23) coincides with $V_0(x) = d^2$.

The transformation operator $L_\rho$ (27) is given by

$$L_\rho = -d/dx + d$$

with the help of which we find the eigenfunctions of $H$ (18),

$$\Psi_k(x) = L_\rho \Psi_k^0(x) = \sqrt{\frac{\gamma}{\pi}} (k^2 + d^2)^{-1/2} [d \sin(kx) - k \cos(kx)].$$

Using the unitary operator

$$U = e^{-ibx},$$

we obtain the eigenfunctions

$$\Psi_k(x) = U \Psi_k(x) = \sqrt{\frac{\gamma}{\pi}} (k^2 + d^2)^{-1/2} e^{-ibx} [d \sin(kx) - k \cos(kx)].$$

of the operator

$$\eta = LL^\dagger = \left(-\frac{d}{dx} + d - ib \right) \left(\frac{d}{dx} + d + ib \right) = -\frac{d^2}{dx^2} - 2ib \frac{d}{dx} + b^2 + d^2.$$  

Using equations (43) and (54) yields the corresponding non-Hermitian Hamiltonian

$$H = -\frac{d^2}{dx^2}.$$  

Its Hermitian SUSY partner follows from (50), (49) and (55),

$$h_0 = -\frac{d^2}{dx^2}.$$  

Note that although both $H$ and $h_0$ are defined by the same differential expression, the corresponding spectral problems are different. For the Hamiltonian $H$, we impose the boundary condition

$$\psi'_k(0) + (d + ib) \psi_k(0) = 0,$$  

whereas for the Hamiltonian $h_0$, the boundary condition reads $\psi_k(0) = 0$. Another remark we would like to make is that the spectral problem for $H$ was first studied by Schwartz [11] as one of the simplest problems where the spectral singularity occurs in the continuous spectrum of $H$ if $d = 0$.

7.2. $\eta$ is a second-order differential operator with variable coefficients

According to section 5, one can start with any scattering Hamiltonian $h_0$ with a real-valued potential $v_0(x)$. The main element of the whole construction is the transformation operator $L^*$ which intertwines $h_0$ and a non-Hermitian Hamiltonian $H$ (51).

Let us choose

$$v_0(x) = -\frac{2a^2}{\cosh^2(ax + c)}, \quad a > 0, \quad c > 0.$$
For \( c > 0 \), the Hamiltonian \( h_0 = -d^2/\partial x^2 + v_0(x) \) has no bound states in \( L^2(0, \infty) \). A solution of the Schrödinger equation with the Hamiltonian \( h_0 \) having the asymptotic behavior as given in (54) reads

\[
    u(x) = \frac{e^{i(d+ib)x}}{a-d-ib} [a \tanh(ax+c) - d - ib].
\]

Using (13) and this transformation function, one finds the function

\[
    v(x) = d + ib + \frac{2a^2}{a \sinh(2ax+2c) - 2(d+ib) \cosh(ax+c)},
\]

which according to (10) and (53) defines both the \( \eta \) operator and the non-Hermitian Hamiltonian \( H \), respectively.

To illustrate the self-adjointness of \( \eta \) on \( D_H \) (4), we find the operator \( H \) (18) unitary equivalent to \( \eta \). For that, using (20), we first calculate \( W(x) \),

\[
    W(x) = d + \frac{a^2 \text{sech}(ax+c)[a \tanh(ax+c) - d]}{b^2 + d^2 + a \tanh(ax+c)[a \tanh(ax+c) - 2d]}.\]

Then using (19), we find the potential

\[
    V(x) = d^2 + \frac{4a^2(a^2 - d^2)}{W(x)} + \frac{12a^4b^2}{W^2(x)},
\]

where

\[
    W(x) = b^2 + d^2 - a^2 + (a^2 + b^2 + d^2) \cosh(2ax+2c) - 2ad \sinh(2ax+2c).
\]

It is not difficult to see that \( V(x) \to d^2 \) as \( x \to \infty \) by an exponential rule \( \sim \exp(-2ax) \) and, hence, as expected, the operator \( H \) is scattering and positive definite.

Another important point is that \( V(x) \) is continuous and bounded below. In this case, according to a known result [13], the operator \( H \) initially defined on a set of finite and twice continuously differentiable functions \( y(x) \) satisfying the boundary condition \( y'(0) \cos \phi + y(0) \sin \phi = 0 \) has a closed self-adjoint extension \( \tilde{H} \). Its domain \( D_{\tilde{H}} \) where \( \tilde{H} = \tilde{H}^\dagger \) is described by the following properties [13]. If \( z(x) \in D_{\tilde{H}} \), then

(a) \( z(x) \in L^2(0, \infty) \);
(b) \( z(x) \) is continuous and has an absolutely continuous derivative in any finite interval belonging to \( (0, \infty) \);
(c) \( -z''(x) + V(x)z(x) \in L^2(0, \infty) \);
(d) \( z'(0) \cos \phi + z(0) \sin \phi = 0, \phi \in \mathbb{R} \).

Evidently, this domain differs from that described in (22) only by condition (b), which we everywhere skipped for simplicity. Going back from \( H \) to its unitary equivalent operator \( \eta \), we conclude that \( \eta = \eta^\dagger \) on \( D_\eta = D_H \) (4).

8. Conclusion

In this paper, based on the factorization property of the metric \( \eta \) operator [4], we proposed a special differential form for this operator. It happens that such a metric operator is unitary equivalent to the usual one-dimensional Hamiltonian and therefore it has a natural supersymmetric structure. Once the metric operator is fixed, one can reconstruct the corresponding non-Hermitian Hamiltonian. This opens a way for starting not from a non-Hermitian Hamiltonian and looking for the corresponding metric operator, but going in the opposite direction, i.e. starting with a given metric operator and presenting the corresponding non-Hermitian Hamiltonian. An advantage of this approach is the possibility of using the
technique of SUSY QM for studying the properties of the metric operator. In particular, as a possible line of future investigation, we plan to use shape-invariant metric operators and study how this property is reflected by corresponding non-Hermitian Hamiltonians. We pointed out that such metric operators are unbounded. Therefore, some vectors from the initial Hilbert space are lost. They cannot be mapped onto the new Hilbert space where the non-Hermitian Hamiltonian becomes Hermitian.

We illustrated our approach by two examples. In the first example, the metric operator is a second-order differential operator with constant coefficients and \( w' (x) = 0 \). It happens that in this case, the corresponding non-Hermitian Hamiltonian coincides with an operator proposed by Schwartz [11] (see also [12]) for illustrating the existence of the spectral singularity in the continuous part of the spectrum of this operator. In the second example, we considered a differential operator with variable coefficients as the metric operator. Using the operator \( H \) (18) unitary equivalent to \( \eta \), we have shown that \( \eta \) is self-adjoint on \( \mathcal{D}_H \) (4).

References

[1] Scholtz F G, Geyer H B and Hahne F J H 1992 Ann. Phys. 213 74
[2] Bender C M 2007 Rep. Prog. Phys. 70 947
[3] Mostafazadeh A 2010 Int. J. Geom. Methods Mod. Phys. 7 1191
[4] Mostafazadeh A 2002 J. Math. Phys. 43 2814
[5] Bagrov V G and Samsonov B F 1995 Theor. Math. Phys. 104 356
[6] Cooper F, Khare A and Sukhatme U 2001 Supersymmetry in Quantum Mechanics (Singapore: World Scientific)
[7] Fityo T V 2002 J. Phys. A: Math. Gen. 35 5893
[8] Mostafazadeh A 2002 J. Math. Phys. 43 205
[9] Samsonov B F, Shamshutdinova V V and Osipov A V 2010 Phys. Lett. A 374 1962
[10] Samsonov B F 2011 J. Phys. A: Math. Theor. 44 392001
[11] Schwartz J 1960 Commun. Pure Appl. Math. 13 609
[12] Guseinov G S H 2009 Pramana J. Phys. 73 587
[13] Kostuchenko A G and Sargsyan I S 1979 Distribution of Eigenvalues (Moskow: Nauka) (in Russian)