An Improved Distributed Nonlinear Observers for Leader-Following Consensus via Differential Geometry Approach

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Abstract—This article is concerned with the leader-following output consensus problem in the framework of distributed nonlinear observers. Instead of certain hypotheses on the leader system, a group of geometric conditions is put forward to develop a novel distributed observers strategy, thereby definitely improving the applicability of the existing results. To be more specific, the improved distributed observers can precisely handle consensus problems for some nonlinear leader systems, which are invalid for the traditional strategies with a certain assumption, such as elastic shaft single linkage manipulator (ESSLM) systems and most of the first-order nonlinear systems. We prove the sufficient conditions for the exponential stability of our distributed observers' error dynamic by proposing two pioneer lemmas to show the relationship between the maximum eigenvalues of two matrices appearing in Lyapunov type matrices. Then, a partial feedback linearization method with zero dynamic proposed in differential geometry is employed to design the purely decentralized control law for the affine nonlinear multiagent system. With this advancement, the existing results can be regarded as a specific case owing to that the followers can be chosen as an arbitrary minimum phase affine smooth nonlinear system. At last, the novel distributed observers and the improved purely decentralized control law are applied in the distributed control framework to construct a closed-loop system. We also prove the stability of the closed-loop system to achieve leader-following consensus. Our method is illustrated by the ESSLM system and Van der Pol system as leaders.

Index Terms—Distributed nonlinear observer, distributed state estimate, feedback linearization, leader-following consensus, observable canonical form, zero dynamics.

I. INTRODUCTION

MULTIAGENT system has been widely studied over the last decades. Not only for the leader-following consensus problem [1]–[3] but also for the cooperative output regulation problem [4]–[6]. An obstacle in the research of the multiagent system is the communication constraints between followers. It means one follower can only obtain the information from some specific followers, such as its neighbor, but may not obtain the information from leader or the other followers. In order to handle the communication constraints, a method named distributed observers was proposed [7]–[9].

According to the literature in recent ten years, distributed observers can be divided into two categories [9]. In the first kind of distributed observers, the follower’s local observer is designed to estimate the leader’s states. In this scene, only a part of the followers can obtain the actual states of leader, while the other followers may achieve accurately state estimate of leader through the information interaction in the communication graph. The second kind of distributed observers do not include the leader, and each of its local observers needs to be able to observe all the states of the whole system by using its own output measurements and the state estimates of its neighbors via communication network [10]–[15].

Although both of them are called distributed observers, in fact, the second kind of distributed observers is closer to the problem of distributed filtering [16], [17], whose research method is not suitable for the first kind of distributed observers. The first one, as the main research object of this article, has become a research hotspot only in the last decade [18]–[22], including distributed observers with switching topologies [18] and a class of uncertain nonlinear multiagent systems [19]. Especially, a distributed control law, the so-called distributed observers approach or distributed observers-based framework, has been initialed proposed by [20].

However, most of these research focus on linear systems or linear leader system [19], only a small part of them take the nonlinear system into account, for example, see attitude control of rigid body based on distributed observers [23]. Recently, [24] has designed a distributed nonlinear observer for a class nonlinear system, and successfully established the observer-based distributed control framework for the
Some extracted text for the natural language representation is as follows:

leader-following problem in terms of nonlinear system like that of the linear system. This framework is a general way to design the distributed observers and tracking controller under the leader-following information, which includes a distributed nonlinear observer for the leader system and a group of purely decentralized control law assigned to follower systems. The observer-based distributed control law is formed by replacing the leader’s actual states, which cannot be obtained in purely decentralized control law with the state estimate generated by the local observer.

The most formidable task of this framework is to ensure the stability of the error dynamic corresponding to distributed observers, which is also an important basis to ensure that the distributed control law satisfies the certainty equivalency principle [25]. The stability of [24]'s error dynamic with respect to their distributed observers is achieved on the basis of assumptions that the leader system is globally bounded, and ought to meet Taylor conditions. It requires that the Taylor expansion around origin of the leader should be formed by \( Aw + p(w)w \), where \( w \) is state vectors and \( p(w) \) is a diagonal matrix with all its diagonal entries no more than 0. Obviously, Taylor conditions for the leader system are too strict to be acceptable even by some common systems. For example, see a simple system \( \dot{w} = -\sin w \). The diagonal entries \(-\sin w\) cannot be guaranteed to be less than 0.

Motivated by this problem, this article imposes constraints on the leader system by a group of geometric conditions [conditions of observer canonical form (OCF)] [26]–[29] instead of Taylor conditions, and develops a new algorithm to derive the distributed nonlinear observers and tracking controller. To the best of our knowledge, this is the first article to construct the second kind of distributed nonlinear observers with geometry conditions. Based on this method, the application range of distributed nonlinear observers with regard to [24]'s distributed control framework can be extended. For example, all of the first-order nonlinear smooth system can be accepted by our distributed observers but most of them cannot satisfy [24]'s assumption. Fortunately, some practical systems that fail to meet the Taylor conditions, such as the elastic shaft single linkage manipulator (ESSLM) system, can satisfy our geometric conditions. Moreover, as the first article in the research of nonlinear distributed control framework, [24] designs a purely decentralized control law for a completely controllable single input follower. Consequently, it is the second purpose of this investigation to study the decentralized control law for more general follower systems, such as multiinput systems and incompletely controllable systems.

This investigation, whether it aims to expand the application scope of distributed nonlinear observer or it improves the design method of purely decentralized control law, is by no means trivial. The challenges mainly come from the following.

1) The structure and form of the distributed observers based on geometry conditions are much different from those based on Taylor conditions, so it is necessary to prove the stability of the error dynamics of distributed nonlinear observers from a new point of view.

2) When dealing with the relationship between the maximum eigenvalues of two matrices appearing in the Lyapunov form matrix, previous papers often give a qualitative description, but this cannot meet the requirement of proving the stability of the error dynamics of distributed observers, so it is necessary to explore the inequality relationship between the maximum eigenvalues of these matrices.

3) In the study of the leader-following problem, we need to use the zero dynamic stability of the tracking error system but not the follower system, so how to find a diffeomorphism to make both the follower system and the tracking error system have stable zero dynamics? It is the key to solve the leader-following consensus problem when the follower is the minimum phase affine nonlinear system.

The main contributions of this article consist of the following five aspects.

1) We constraint nonlinear leader system with geometric conditions instead of Taylor conditions, which enlarges the application range of their framework about the nonlinear distributed control law.

2) The novel distributed nonlinear observers based on the differential geometry are proposed, and it is proved to achieve exponential stability for all output bounded affine nonlinear systems that meets geometric conditions.

3) In order to prove the stability of error dynamic of our novel distributed observers, we analyze the relationship between the maximum eigenvalues between two matrices appearing in Lyapunov type matrices carefully with the help of inequality analysis and matrix theory. We describe this relationship in the form of inequality.

4) For certainty systems, the use of differential geometry-based novel distributed observers leads to less conservatism. For example, for the Van der Pol system, our method can obtain globally convergent distributed observers, rather than converging only in a compact set containing the origin.

5) This article develops the purely decentralized control law based on the zero dynamic theory, which enables the selection range of follower systems to be expanded from fully controllable affine nonlinear systems to minimum phase affine nonlinear systems.

This article is organized as follows. Section II first summarizes the notations used throughout this article, and then formulates the problem of this article. The motivation and improvement of our novel distributed observers are detailed with the ESSLM system in Section III. Section IV proves the stability of our novel distributed observers with two eigenvalues lemma. The improved purely decentralized control law and the proof of certainty equivalence principle are shown in Section V. In Section VI, the results of our method are simulated with the Van del Pol system and ESSLM system as the leader. Finally, Section VII concludes this article.

II. PRELIMINARIES AND PROBLEM FORMULATION

A. Notations

\( N1 \) (Notations of Matrix): \( I_N \in \mathbb{R}^{N \times N} \) denotes an identity matrix. \( I_N \) is an \( N \)-dimensional column vector with
all its entries equaling 1. $A^*$ and $A^T$ denote the conjugate transposition and the transposition of matrix $A$, respectively. For some column vectors $a_i$, denote $\text{col}(a_1, a_2, \ldots, a_N)$ as a column vector $[a_1^T, a_2^T, \ldots, a_N^T]^T$. For some matrices $A_1$, diag$[A_1, A_2, \ldots, A_N]$ represents a block diagonal matrix. Let $\bar{\sigma}(P)$ be the maximum real of all eigenvalues of $P$, and $\sigma(P)$ be the minimum real of all eigenvalues. Particularly, if $P$ is the Hermite matrix, $\bar{\sigma}(P)$ and $\sigma(P)$ represent the maximum and minimum eigenvalues, respectively. $\otimes$ denotes the Kronecker product with a property $(A \otimes B)(C \otimes D) = AC \otimes BD$. $\| \cdot \|$ is denoted as the 2-norm of the matrix or vector.

$N2$ (Graph Theory): A directed graph is usually expressed as $G = (V, \mathcal{E}, \mathcal{A}_G)$, where $V$ is the node set including $v_1, v_2, \ldots, v_N$, $\mathcal{E}$ is the arc set, and $\mathcal{A}_G = [a_{ij}]$ is an adjacency matrix of $G$. Herein, $\mathcal{G}$ is assumed that there are no repeated arcs and no self-loops. We denote $a_{ij} = 1$ if there is an arc from $v_j$ to $v_i$, denoted as $(v_j, v_i)$, otherwise, $a_{ij} = 0$. A directed path from node $i$ to node $j$ is a sequence of arcs, expressed as $(v_i, v_k), (v_k, v_l), \ldots, (v_m, v_j)$. The in-degree matrix is defined as $D_G = \text{diag}[d_i]$ with $d_i = \sum_{j=1}^{N} a_{ij}$. The Laplacian matrix of $\mathcal{G}$ is in form of $L = D_G - \mathcal{A}_G$. An extended graph is $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A}_G)$, where $\mathcal{V} = \mathcal{V} \cup v_0$ with $v_0$ being the node associated with leader, $\mathcal{E}$ includes all the arcs in $\mathcal{E}$ and all the arcs between $v_0$ and $\mathcal{E}$. Denote $B = \text{diag}[b_i]$ where $b_i = 1$ if $(v_0, v_i) \in \mathcal{E}$, otherwise, $b_i = 0$. One may know from references [30] that $L + B$ is a positive defined matrix if $v_0$ is the root of a spanning tree, i.e., there is a directed path from $v_0$ to $v_i$ for each $i$.

$N3$ (Differential Geometry): One can refer the knowledge of this section and next section to [27]. The Lie derivative of a smooth function $h$ along the vector field $\dot{f}$ is defined as $L_f h = (\partial h / \partial x^i) \dot{f}(x)$. Moreover, if there is a dual vector field $\omega$ belonging to dual tangent space, the Lie derivative of dual vector field $\omega$ along the vector field $\dot{f}$ is expressed as $L_f \omega = \dot{f}(\partial \omega / \partial x^i) + \omega(\partial f / \partial x^i)g$. $[f, g] = (\partial g / \partial x^i)f - (\partial f / \partial x^i)g$ denotes the Lie bracket between two vector fields. $[f, g]$ is also denoted as $ad_f g$ and $ad^k_f g = [f, ad^{k-1}_f g]$ for $k \geq 2$ is the notation of higher order Lie bracket, where $ad^0_f g = g$, and $ad^1_f g = ad_f g$. A distribution $\mathcal{D}$ is spanned by a group of vector fields $X_1, \ldots, X_D$, i.e., $\mathcal{D} = \text{span}[X_1, X_2, \ldots, X_D]$. We call $\mathcal{D}$ is involutive if $[X_i, X_j] \in \mathcal{D}$ for $\forall X_i, X_j \in \mathcal{D}$.

B. Problem Formulation

Taking into account the nonlinear multiagent systems with all subsystems formed by affine nonlinear
\begin{align}
\dot{x}_i &= f_i(x_i, w) + g_i(x_i)u_i \\
y_i &= h_i(x_i), \quad i = 1, 2, \ldots, N
\end{align}
where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$, and $y_i \in \mathbb{R}^r$ are the states, control inputs, and measurement outputs of the $i$th subsystem, or named follower, respectively. $f_i(\cdot)$, $g(\cdot)$, and $h_i(\cdot)$ are smooth vector value functions. The variable $w \in \mathbb{R}^s$ is generated by an external system, or called leader system, which is an autonomous system:
\begin{align}
\dot{w} &= p(w) \\
y_0 &= q(w)
\end{align}
where $p(\cdot)$ and $q(\cdot)$ are smooth vector value nonlinear functions and $y_0 \in \mathbb{R}^r$ is the output of the leader system. The problem is to design a distributed control law to let the measurement outputs of subsystems track the outputs of leader system, i.e.,
\begin{equation}
\lim_{t \to \infty} y_i(t) - y_0(t) = 0, \quad i = 1, 2, \ldots, N.
\end{equation}

Similar to the problem background of [24], all followers can get the information of their neighbors’ agents via a communication network, but only a part of followers can obtain the real states of the leader system. Since the leader–follower problem needs to fix the leader’s states into the followers’ control law, which requires all followers to estimate the leader’s states by their own and neighbor’s information. Aiming at this problem, a frame of the observer-based distributed control for the nonlinear system is proposed in [24]. This frame contains three aspects. The first is to design distributed observers based on the communication graph and show that whether it exists for the studied nonlinear system. Second, one should design a purely decentralized control law for every subsystem. Finally, the distributed observers and the purely decentralized control law constitute the distributed control law together.

In the rest of this article, we will focus on how to improve the distributed nonlinear observers to enlarge the application range of [24]’s distributed control framework, and how to design the distributed control law when the follower systems are minimum phase system.

III. GEOMETRY CONDITIONS

The important premise of the distributed control framework described in Section II is that distributed observers can accurately estimate the leader’s states. However, it could be very difficult when the leader system becomes a nonlinear system. Reference [24] uses a strict assumption to constrain the form of leader system to ensure the existence of distributed observers. In this section, we will first recall the assumptions added to the leader system in [24]. Then, the geometric conditions will be proposed to replace the original assumptions to constrain the leader system such that the application range of [24]’s distributed control framework could be enlarged.

A. Previous Works and Motivation

The $i$th local observer of the distributed observers in [24] is introduced as
\begin{equation}
\dot{\hat{\omega}}_i = p(\hat{\omega}) + c \sum_{j=1}^{N} a_{ji}(\hat{\omega}_j - \hat{\omega}_i) + b_i(\hat{\omega}_i - \omega)
\end{equation}
where $\hat{\omega}_i$ represents the state estimate of leader system given by the $i$th follower system, and $c$ is the coupling gain. The stability of its error dynamic is based on the following three basic assumptions.

Assumption 1: The dynamic of leader system (3), (4) is output bounded.

Assumption 2: The communication network $\tilde{G}$ has a $v_0$-spanning tree.
Assumption 3: The dynamic function \( p(w) \) of the leader system is supposed in the following form around the origin, i.e.,

\[
p(w) = \frac{\partial p}{\partial w} \bigg|_{w=0} \; w + p_2(w)w
\]

where \( p_2(w) = \text{diag}\{d_1(w), d_2(w), \ldots, d_s(w)\} \) with all its diagonal entries being less than zero, i.e., \( d_i(w) \leq 0 \) for all \( i = 1, 2, \ldots, s \).

Reference [24] is the first article to handle the leader-following problem with their distributed control framework for nonlinear leader and follower systems. Their work has successfully solved the biggest difficulty in this framework, whose result can be applied to some typical nonlinear systems including the Van der Pol system. However, Assumption 3, the so-called Taylor condition, is too strict to be fulfilled. For example, the nonlinearity of the following system is not very strong but it cannot satisfy Assumption 3.

Example 1: Consider the ESSLM system. Let the length of the Linkage be \( 2d \) and the mass be \( m \). The angular displacement of the reducer input shaft and reducer output shaft is \( \omega_1 \) and \( \omega_1/\sigma_r \), respectively, where \( \sigma_r \) is the transmission ratio of the reducer. Denote the angular displacement of the linkage as \( \omega_2 \). Then, the torque at both ends of the elastic shaft is \( K(\omega_2 - \omega_1/\sigma_r) \) with \( K \) representing the torsional elastic coefficient. We denote the viscosity friction coefficient and rotational inertia of motor as \( J_1 \) and \( J_1 \), respectively, and further suppose the viscosity friction coefficient and the rotational inertia of the reducer as \( J_2 \) and \( J_2 \), respectively. Then, the system equation of ESSLM can be introduced as

\[
\begin{bmatrix}
\dot{\omega}_1 \\
\dot{\omega}_2 \\
\dot{\omega}_3 \\
\dot{\omega}_4
\end{bmatrix} = p(\omega) \triangleq 
\begin{bmatrix}
\omega_3 \\
\omega_4 \\
-\frac{K_1}{J_1} \omega_1 + \frac{K_2}{J_2} \omega_2 - \frac{F_I}{J_1} \omega_3 \\
\frac{K_1}{J_1} \omega_1 - \frac{K_2}{J_2} \omega_2 - \frac{mgd}{J_2} \cos \omega_2 - \frac{F_2}{J_2} \omega_4
\end{bmatrix}
\]

\[
y_0 = q(\omega) = \omega_2.
\]

Note that only \( \omega_4 \) dynamic in this system is nonlinear. Nevertheless, it still cannot satisfy Assumption 3. Actually, it can be rewritten as

\[
\dot{\omega} = \frac{\partial p}{\partial \omega} \bigg|_{\omega=0} \; \omega + p_2(\omega)w
\]

\[
= -\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\frac{K_1}{J_1} & -\frac{K_2}{J_2} & 0 & F_1 \\
-\frac{K_1}{J_1} & \frac{K_2}{J_2} & 0 & F_2
\end{bmatrix} \omega + \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \omega
\]

where \( d_4(\omega) = -b_3 \cos \omega_2/\omega_4 \) does not fulfill \( d_4(\omega) \leq 0 \).

B. Geometric Conditions

It is known from the previous section that too strict system form will limit the application range of the distributed control framework. In order to make this framework be widely used in some common nonlinear systems, we introduce the geometric conditions of the OCF. Then, the distributed observers are designed for the system, which can meet the geometric conditions. The OCF of leader system (3), (4) can be described as

\[
\begin{align*}
\dot{\eta}_0 &= A_0 \eta_0 + a(y_0) \\
y_0 &= C \eta_0
\end{align*}
\]

where \( \eta_0 \in \mathbb{R}^l \) and \( y_0 \in \mathbb{R}^r \) are system states and measurement outputs, respectively, and

\[
\begin{align*}
A_0 &= \text{diag}\left\{ \begin{bmatrix} 0 & 0 \\ J_{k_1-1} & 0 \\ \vdots & \vdots \\ J_{k_r-1} & 0 \end{bmatrix} \right\} \\
C &= \text{diag}\left\{ \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \right\}
\end{align*}
\]

and \( r \)-tuples \( \{k_1, k_2, \ldots, k_r\} \) is called the observable relative degree. Some associated references, such as [31] and [28], prove that the nonlinear system can be transformed into OCF by a diffeomorphism on the premise of satisfying a group of geometric conditions. Before introducing them, we are supposed to define some codistributions. By rewriting the output function \( q(w) \) as \( q(w) = [q_1(w), q_2(w), \ldots, q_s(w)]^T \) and letting \( ki \) be the observable relative degree associated with \( q_i(w) \), a group of codistributions defined by [31] can be introduced as

\[
\Delta^+ = \text{span}\{dL_i q_1 | 1 \leq j \leq r, 0 \leq l \leq k_i - 1\}
\]

\[
\Delta^+ = \text{span}\{dL_i q_1 | dL_i^{k_i-1} h_i | 1 \leq j \leq r, 0 \leq l \leq k_i - 1\}
\]

The geometric conditions for the existence of diffeomorphism are concluded in the following Lemma [28].

Lemma 1: The leader system is given by (3) and (4) and its observable relative degree is given by \( r \)-tuples \( \{k_1, k_2, \ldots, k_r\} \). Without loss of generality, we assume \( k_1 \geq k_2 \geq \cdots \geq k_r \) and \( \sum_{i=1}^{N} k_i = s \). Then, there is a diffeomorphism \( \eta_0 = \Theta(\omega) \) defined on a neighborhood \( \mathcal{V} \) around a given point \( \omega_0 \), which can transform (3) and (4) into OCF (8) if and only if:

1) the dimension of co-distribution \( \Delta^+ \) is \( s \).

2) \( \dim(\Delta^+ \cap \Delta^+) = 1 \).

3) for the given linear equations

\[
\begin{align*}
\{dL_i^{l-1} h_i, \tau_j\} &= \delta_{ij} \cdot \delta_{i,k_i}, \; l = 1, \ldots, k_i, \; \text{if} \; i \leq j \\
\{dL_i^{l} h_i, \tau_j\} &= \delta_{ij} \cdot \delta_{i,k_i}, \; l = 1, \ldots, k_i, \; \text{if} \; i > j
\end{align*}
\]

there exists a group of vector fields \( \tau_1, \tau_2, \ldots, \tau_r \), solved by (13) and (14) s.t. the communication conditions \( \{a_i^{k_i} \tau_i, a_i^{k_i} \tau_j\} = 0 \) are satisfied for all \( 1 \leq i, j \leq r, 0 \leq l \leq k_i - 1, \; 0 \leq k \leq k_j - k_l - 1 \), where \( \delta_{i,j} \) is Kronecker delta.

In Lemma 1, conditions 1)–3) are called geometric conditions. The solving procedure with respect to calculating \( \eta_0 = \Theta(\omega) \) is given in [28]. It is easy to verify that the geometric conditions are fulfilled for all of the first-order nonlinear smooth systems, such as \( \dot{\omega} = -\sin \omega \), but most of them cannot satisfy Taylor conditions. Moreover, geometric conditions can also be applied to some high-order nonlinear systems, which fail to meet Assumption 3. As a comparison, we will verify that the ESSLM system used in the previous section meets the geometric conditions.
Example 2: Consider the ESSLM system again. Herein, we will verify whether it can be transformed into OCF by diffeomorphism. It can be calculated directly that
\[ \begin{bmatrix} dq(\omega) \\ d^2q(\omega) \\ d^2q(\omega) \\ d^2q(\omega) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{\kappa}{J_2} & \frac{\kappa}{J_2^2} & \gamma_1(\omega) & 0 \\ \frac{\kappa}{J_2^2} & \gamma_2(\omega) & \frac{\kappa}{\omega^2} & \gamma_3(\omega) \end{bmatrix} \] (15)

where
\[ \begin{align*}
\gamma_1(\omega) &= -\frac{\kappa}{J_2} + \frac{mgd}{J_2^2} \sin \omega_2 \\
\gamma_2(\omega) &= \frac{KF_2}{J_2} - \frac{mgd}{J_2} \cos \omega_2 - \frac{mgdF_2}{J_2^2} \sin \omega_2 \\
\gamma_3(\omega) &= \frac{F_2^2}{J_2} - \frac{\kappa}{J_2} + \frac{mgd}{J_2} \sin \omega_2.
\end{align*} \]

Thus, the solution \( \tau_1 \) of linear equations (13) and (14) can be described by \( \tau_1 = [0, 0, J_2^2/\kappa, 0]^T \). Then, we can further calculate
\[ \begin{align*}
ad_p\tau_1 &= \left[-J_2^2/\kappa, F_1 J_2^2/1 J_1 K_0, 0 \right]^T \\
ad^2_p\tau_1 &= \left[-F_1 J_2^2/1 J_1 K_0, 0, -J_2/1 J_1 + F_1^2 J_2^2/1 J_1 K_0 \right]^T \\
ad^3_p\tau_1 &= \left[J_2/1 J_1 K_0, -J_2^2/1 J_1 K_0, -1, \Gamma, F_1, F_2 \right]^T
\end{align*} \]

where \( \Gamma = -(2F_1 J_2/1 J_1 K_0) + (F_1^2 J_2^2/1 J_1 K_0) \). Since all \( ad_p\tau_1, i = 0, 1, 2, 3 \) are constant vector fields, we can obtain \( [ad_p\tau_1, ad_p\tau_2, ad_p\tau_1] = 0 \) for all \( i, j = 0, 1, 2, 3 \).

Remark 1: It is easily seen from the above Examples 1 and 2 that Assumptions 3 in traditional distributed observers can be almost dismissed with the presentation of the geometry conditions. In other words, the proposed geometry conditions lead to relatively looser conditions for the design of distributed observers, which directly enhances the practical applicability of the observer strategy.

IV. DISTRIBUTED OBSERVERS DESIGN AND STABILITY ANALYSIS

Since the geometry conditions can be made use of by some nonlinear leader systems, which are invalid for the traditional strategies with Assumption 3, a new distributed observers based on geometric conditions will be designed in Section IV-A. To handle the formidable task brought by the stability proof of the error dynamics of novel distributed observers, two lemmas concerned with the relationship between the maximum eigenvalues of two matrices appearing in Lyapunov type matrices are proved in Section IV-B. Our main result will be given in Section IV-C so that the new distributed observers can estimate the states of the leader system for the output bounded nonlinear system, which meets the geometric conditions.

A. Novel Distributed Observers

Suppose the leader system can meet the geometric conditions list in Lemma 1. Then, our new distributed observers can be designed on the basis of OCF (8). In this scene, the \( i \)th local observer of the leader system takes the form of
\[ \begin{align*}
\dot{\omega}_i &= \Theta_i^{-1}(\dot{\hat{\eta}}_i) \\
\dot{\hat{\eta}}_i &= A_0 \hat{\eta}_i + a(\hat{y}_i) + cF \varsigma_i \\
\varsigma_i &= \sum_{j=1}^N a_{ij}(\hat{\eta}_j - \hat{\eta}_i) + b(\hat{\eta}_i - \eta_0)
\end{align*} \] (16) - (18)

where \( \hat{y}_i = C \hat{\eta}_i \), \( F \) is the LQR gain matrix, \( c \) is the coupling gain, and \( \varsigma_i \) is the global error dynamic.

The observer error of \( i \)th subsystem is defined as \( e_i = \hat{\eta}_i - \eta_0 \), which includes a quasilinear error dynamic
\[ \dot{e}_i = A_0 e_i + a(\hat{y}_i) - a(\gamma_0) + cF \varsigma_i. \] (19)

By setting \( e = \text{col}[e_1, e_2, \ldots, e_N] \), \( \tilde{a}_i = a(\hat{y}_i) - a(\gamma_0) \), and \( \tilde{a} = \text{col}[\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_N] \), where \( \hat{y}_i \) is the estimation of output information obtained by \( \dot{\omega}_i \), we can rewrite the error dynamic in a compact form
\[ \dot{e} = (\mathcal{L} \otimes A_0) e - c(I_N \otimes F)((\mathcal{L} + B) \otimes I_N) e + \tilde{a} = (\mathcal{L} \otimes A_0 - c(\mathcal{L} + B) \otimes F) e + \tilde{a}. \] (20)

Denote \( \mathcal{M} = I_N \otimes A_0 - c(\mathcal{L} + B) \otimes F \). An intuitive fact could be noticed that the properties of \( \mathcal{M} \) have an important influence on the stability of the error system (20). Fortunately, some results [32] (technology used in multiagent system) can be found to help us understand the properties of \( \mathcal{M} \) and design LQR gain matrix \( F \).

Lemma 2: Suppose \( \lambda, i = 1, 2, \ldots, N \) are the eigenvalues of \( \mathcal{L} + B \), then matrix \( \mathcal{M} \) is Hurwitz if and only if \( A_0 - c\lambda_i F \) is Hurwitz for all \( i \).

Lemma 3: Suppose \( Q \) and \( R \) are symmetric positive definite matrices, and choose \( F \) from
\[ F = P_1 R^{-1} \] (21)

where the symmetric positive definite matrices \( P_1 \) is solved by the algebraic Riccati equation \( AP_1 + P_1 A^T + Q - P_1 R^{-1} P_1 = 0 \). Then, \( \mathcal{M} \) is Hurwitz if the coupling gain \( c \) satisfies
\[ c \geq \frac{1}{2\sigma(\mathcal{L} + B)}. \] (22)

Remark 2: Different from [24]’s distributed nonlinear observers, our method brings in the LQR gain to make up the imperfection that a distributed nonlinear observers cannot be designed in the form of observer error linearization. With this improvement, the error dynamics (20) of the distributed observers consists of only a stable linear part and a nonlinear part generated entirely by diffeomorphism \( \eta_0 = \Theta(\omega) \), which enables us to fully utilize the well properties of the diffeomorphism to derive the error dynamic stability conditions of the distributed observers.

B. Two Important Lemmas

Before giving one of the main results of this article, we are going to prove two Lemmas in Section IV-B and propose a related definition to assist in the stability analysis of (20) of our new developed distributed observers.
Lemma 4: Set $A \in \mathbb{R}^{n \times n}$ as an arbitrary matrix, and $P \in \mathbb{R}^{n \times n}$ as a symmetric positive definite matrix. Then, matrix $T = PA + A^TP$ satisfies
\[
\sigma(T) \leq \sqrt{\sigma((A^TA)\sigma(P))}.
\]

Proof: Suppose that $\eta$ is the eigenvector of $T$ corresponding to $\sigma(T)$. By letting $\xi = \Lambda\eta$, we have
\[
\eta^T(PA + A^TP)\eta = \sigma(T)\|\eta\|^2 = 2\eta^TPA\eta = 2\eta^TP\xi.
\]
According to the Cauchy Schwartz inequality and C-F inequality, we can further obtain
\[
2\eta^TP\xi \leq 2\left(\eta^TP\eta \cdot \xi^TP\xi\right)^{\frac{1}{2}} \leq 2\sigma(P)\|\eta\||\xi||\xi|| = 2\sigma(P)\|\eta\|A\|\eta\| \leq 2\sigma(P)\|\eta\|A\|\eta\|^2 = 2\|A\|\sigma(P)\|\eta\|^2 \leq \sqrt{\sigma((A^TA)\sigma(P))} \|\eta\|^2.
\]
Hence, we have $\sigma(T) \leq \sqrt{\sigma((A^TA)\sigma(P))}$. ■

Lemma 5: Suppose $\mathcal{M} \in \mathbb{R}^n$ is a Hurwitz matrix. For a fixed constant $\mu > 0$, a unique symmetric positive definite matrix $P$ solved by
\[
P\mathcal{M} + \mathcal{M}^TP = -2\mu I_n
\]
satisfies
\[
\sigma(P)\sigma(\mathcal{M}) \leq -\mu. \tag{25}
\]
\[
\sigma(P)\sigma(\mathcal{M}^*) \geq -2\mu. \tag{26}
\]
Especially, $\sigma(P)\sigma(\mathcal{M}) = -\mu$ if $\mathcal{M}$ is a Hermite matrix.

Proof: Denote $\lambda_2$ as an eigenvalue of $\mathcal{M}$ with $\text{Re}(\lambda_2) = \tilde{\sigma}(\mathcal{M})$ and treat $\eta$ as the eigenvector of $\mathcal{M}$ corresponding to $\lambda_2$. By premultiplying $\eta^*$ and postmultiplying $\eta$ on (24), we have
\[
\eta^*(P\mathcal{M} + \mathcal{M}^*P)\eta = -2\mu \eta^*\eta.
\]
A natural step can be obtained as
\[
\lambda_2\eta^*P\eta + \lambda_2^*\eta^*P\eta = 2\text{Re}(\lambda_2)\eta^*P\eta = -2\mu \eta^*\eta.
\]
Note that $\tilde{\sigma}(\mathcal{M}) < 0$ since $\mathcal{M}$ is a Hurwitz matrix. Then, by denoting $\lambda_1 = \tilde{\sigma}(P)$, we can deduce with C-F inequation
\[
-2\mu \eta^*\eta = 2\tilde{\sigma}(\mathcal{M})\eta^*\eta \geq 2\sigma(\mathcal{M})\tilde{\sigma}(P)\eta^*\eta.
\]
Consequently
\[
\tilde{\sigma}(P)\tilde{\sigma}(\mathcal{M}) \leq -\mu.
\]
Furthermore, there exists an orthogonal matrix $U$ such that $P = U^T\Lambda U$ because $P$ is a real symmetric matrix, where $\Lambda$ is a diagonal matrix with all eigenvalues of $P$ on its diagonal. Then, calculating by setting $\mathcal{M} = U\Lambda MU^T$ yields
\[
P\mathcal{M} + \mathcal{M}^*P
\]
\[
= U^T\Lambda U MU^T U + U^T U \mathcal{M}^* U^T \Lambda U
\]
\[
= U^T(\mathcal{M}^* \Lambda + \Lambda \mathcal{M}) U \leq \tilde{\sigma}(P) U^T(\mathcal{M} + \mathcal{M}^*) U
\]
\[
= \tilde{\sigma}(P)(\mathcal{M} + \mathcal{M}^*). \tag{27}
\]
It follows
\[
\tilde{\sigma}(P)\tilde{\sigma}(\mathcal{M} + \mathcal{M}^*) \geq -2\mu. \tag{28}
\]
In particular, if let $\mathcal{M}$ be a Hermite matrix, i.e., $\mathcal{M} = \mathcal{M}^*$, then
\[
\tilde{\sigma}(P)\tilde{\sigma}(\mathcal{M} + \mathcal{M}^*) = 2\tilde{\sigma}(P)\tilde{\sigma}(\mathcal{M}) \geq -2\mu.
\]
Combining this equation and (25), $\tilde{\sigma}(P)\tilde{\sigma}(\mathcal{M}) = -\mu$ can be proved. ■

Remark 3: We have revealed the quantitative relationship in the form of inequity between the maximum eigenvalues of $T$ AND $P$ and $\mathcal{M}$ AND $P$ rather than the qualitative relationship $\tilde{\sigma}(T) \propto \tilde{\sigma}(P)$ and $\tilde{\sigma}(P) \propto -\tilde{\sigma}(\mathcal{M})$ given in the previous literature [33]. These two lemmas will play important roles in the proof of main result in the next section.

At the end of this section, we give a definition, which will be used later.

Definition 1 (Decreasing Function in Trend): A real function $f(x)$ defined on real number field is a decreasing function in trend if for $\forall x_1 \in \mathbb{R}$, there exists $x_2 > x_1$, such that $f(x_2) < f(x_1)$.

C. Main Result and the Stability Analysis

Now, we can give one of the main results (Theorem 1) of this article. This theorem guarantees that the distributed control frame for nonlinear leader-following consensus proposed by [24] can be applied to a class of output bounded nonlinear leader systems that satisfies geometric conditions.

Theorem 1: Suppose the nonlinear leader system (3), (4) satisfies Assumptions 1 and 2 and the pair $(p(\omega), q(\omega))$ meets the geometric conditions proposed in Lemma 1. Then, there exists a coupling gain $c$ satisfying (22) such that the state estimate generated by distributed observers (16)–(18) converges exponentially to actual states of the leader system at arbitrary speed.

Proof: Choose $F$ by the statement of Lemma 3. Then, one can conclude $\mathcal{M}$ is the Hurwitz matrix. Thus, there exists a symmetric positive definite matrix $P_2$ such that
\[
P^T_2P_2 + P_2\mathcal{M} = -2\mu I_n. \tag{29}
\]
Sequentially, the Lyapunov function can be chosen as $V(e) = e^TP_2e$, and then we calculate the derivative of $V(e)$ along to error dynamics (20)
\[
\dot{V}(e) = e^T(M^TP_2 + P_2\mathcal{M})e + 2e^TP_2\tilde{a}. \tag{30}
\]
We know from the steps of observer linearization in the Appendix that the nonlinear compensation term $a(y_0)$ is the solution of partial differential equations
\[
\frac{\partial a(y_0)}{\partial y_{0i}} = b_i(y_0) \tag{31}
\]
where $b_i(y_0)$ is defined in the Appendix. However, we can deduce, according to the PDEs in (31) only containing one partial derivative, that every component $a_i(\cdot)$ of $a(\cdot)$ satisfies the differential mean value theorem for each variable $y_{0i}$. It means, for every $a_i(\cdot)$ of $a(\cdot)$ and every $y_{0i}$ of $y_0$, that
\[
a_{ik}(y_0, \hat{y}_k) \leq \sum_{i=1}^r I_{ik}(\hat{y}_k - y_0) = I_k^T(\hat{y}_k - y_0) \tag{32}
\]
Then, as

\[
\vec{a}_k(y_0, \hat{\gamma}_k) = \text{col} \left[ \vec{a}_{k1}(y_0, \hat{\gamma}_k), \ldots, \vec{a}_{k\ell}(y_0, \hat{\gamma}_k) \right] \\
\leq \text{col} \left[ \vec{t}_1^T, \ldots, \vec{t}_{\ell}^T \right] (\hat{\gamma}_k - y_0) \\
= L(\hat{\gamma}_k - y_0) = LC(\hat{n}_k - \eta_0).
\]

Therefore, we have

\[
\vec{a} = \text{col}(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_\ell) \leq (I_N \otimes LC)e.
\]

Substituting (29) and (34) into (30), we get

\[
\dot{V}(t) = e^T(-2\mu I_N)e + e^T P_2(I_N \otimes LC)e \\
= e^T(-2\mu I_N)e + \frac{1}{2} e^T (P_2(I_N \otimes LC) + (I_N \otimes C^T L^T) P_2)e \\
\leq -2\mu \|e\|^2 + \frac{1}{2} \kappa \|e\|^2
\]

(35)

where \( \kappa = \sigma(P_2(I_N \otimes LC) + (I_N \otimes C^T L^T) P_2) \). Moreover, by using Lemma 4, we know \( \kappa \leq \sqrt{\sigma(C^T L^T)LC(\hat{P}_2)} = \alpha \sigma(\hat{P}_2) \).

Reference [33] illustrates that one can improve the stability of \( M \) by increasing the coupling gain \( c \). Choose a nonsingular matrix \( T \) such that \( T^{-1}(L + B)T \) is upper triangular with the eigenvalues \( \lambda_1, \ldots, \lambda_N \) of \( L + B \) on its diagonal. Then, \( (T^{-1} \otimes I_N)M(T \otimes I) \) is transformed to \( \text{diag}(\lambda_0 - \lambda_i F, i = 1, \ldots, N) \). Hence, \( \sigma(M) \triangleq f(c) \) will decrease in trend with \( c \) going to infinity. Furthermore, \( \lim_{c \to \infty} f(c) = -\infty \). By denoting \( \mathcal{M}_c = c(L + B) \otimes F \), we have \( \lim_{c \to \infty} (1/c)(\mathcal{M}_c - c\mathcal{M}_c) = -\mathcal{M}_c \), which indicates \( M \) will tend to be \( c(L + B) \otimes F \) when \( c \) tends to infinity.

Now, we prove \( \sigma(P_2) \sigma(M) \) dose not change with \( c \) when \( c \) is large enough. Actually, we can suppose \( c \) is large enough so \( \mathcal{M} \) is influenced by \( c(L + B) \otimes F \) only. Note that \( M \) and \( P_2 \) are the matrix functions of \( c \) denoted as \( \mathcal{M}(c) \) and \( P_2(c) \), respectively. Then, we denote \( M = \mathcal{M}(c), M' = \mathcal{M}(c + \Delta c), P_2 = P_2(c) \) and \( P_2' = P_2(c + \Delta c), \) where \( \Delta c \) represents the variation of coupling gain. Thus, \( \mathcal{M}' \) can be approximately expressed as \( c'M \) when \( c \) change to \( c + \Delta c \), where \( c' = (c + \Delta c)/c \). As a result, \( P_2' = P_2/c' \) can be solved by (29). It indicates \( \sigma(P_2') \sigma(M') = (1/c') \sigma(P_2) \sigma(M) = \sigma(P_2) \sigma(M) \). Hence, \( \lim_{c \to \infty} \sigma(P_2) \sigma(M) = c_1 \), where \( c_1 \) is a constant.

According to Lemma 5, we know \( \sigma(P_2) \sigma(M) \leq -\mu \). Since \( \sigma(M) \) and \( \sigma(P_2) \) depend on \( c \), we can choose a function \( \beta(c) > 0 \) such that \( \sigma(P_2) \sigma(M) = -\mu - \beta(c) \). Therefore, we can obtain \( \lim_{c \to \infty} \beta(c) = c_2 \) and \( c_2 \) is a constant defined by \( c_2 = -\mu - c_1 \). Then, the limitation of \( \kappa \) can be calculated as

\[
\lim_{c \to \infty} \kappa = \lim_{c \to \infty} a \sigma(P_2) \\
= -a \lim_{c \to \infty} \frac{\mu + \beta(c)}{\sigma(M)} = -\lim_{c \to \infty} \frac{a(\mu + \beta(c))}{f(c)} = 0.
\]

(36)

Therefore, there exists a constant \( c^* > 0 \) such that \( \kappa < 4\mu \) for \( \forall c > c^* \). It is equivalent to \( \dot{V} < 0 \). Combining with (22), we know the coupling gain \( c \) should satisfy

\[
c > \max \left\{ \frac{1}{2\sigma(L + B)}, e^* \right\}.
\]

(37)

Moreover, for a given \( c_0 > e^* \), (35) can be rewritten as

\[
\dot{V}(t) \leq \left( -2\mu + \frac{1}{2} \kappa \right) \|e\|^2 \leq \frac{-2\mu + \frac{1}{2} \kappa}{\sigma(P_2(c_0))} V(t).
\]

So

\[
V(t) \leq \exp \left\{ \left( -2\mu + \frac{1}{2} \kappa \right) t \right\} V(0) = \exp \left( \frac{-2\mu}{\sigma(P_2(c_0))} + \frac{1}{2} \alpha \right) V(0).
\]

(38)

Since \( \lim_{c_0 \to \infty} \sigma(P_2(c_0)) = 0 \), we have

\[
\lim_{c_0 \to \infty} \sigma(P_2(c_0)) + \frac{1}{2} \alpha = -\infty.
\]

(39)

Thus, the error dynamic \( e_1 = \eta_0 - \hat{\eta}_1 \) can exponentially converge to zero at arbitrary speed. Hence, \( \dot{e_1} = \hat{\omega} - \dot{\hat{\omega}} = \Theta^{-1}(\eta_0) - \Theta^{-1}(\hat{\eta}_1) \) can also exponentially converge to zero at arbitrary speed.

Remark 4: So far, we have developed the novel distributed nonlinear observers based on geometry conditions. The proposed new distributed observers have been proved to achieve error dynamic stability for all output bounded nonlinear leader systems, which meet geometry conditions. Furthermore, it enlarges the application range of the existing distributed control framework [24] because there are many nonlinear systems, such as ESSLM and most of the first-order nonlinear systems, which are invalid to Assumption 3 can satisfy geometry conditions.

Remark 5: It is worth noting that distributed observers have the ability to further reduce communication resources. Bearing in mind the form of distributed observer (16)–(18), one may notice that the observer dynamics is exactly the same as that of the leader when the consistency is achieved (achieving consistency means \( \zeta_i = 0 \)). That is to say, after the consistency is achieved, the real-time observation of the leader’s state can be ensured even without communication. However, for asymptotically stable error dynamics, it is difficult to determine when the communication between agents can be terminated. Therefore, in future research, learning from some fixed-time convergence [34]–[36] or event-triggered [37] methods will help to promote the further development of the distributed observer.

V. DISTRIBUTED CONTROL LAW FOR LEADER-FOLLOWING CONSENSUS BASED ON ZERO DYNAMICS

In the leader-following consensus problem, we only need to control the output related states, and the follower system thus need not be completely controllable. Specifically, the selection range of the follower system is expanded from the original completely controllable affine nonlinear system to the minimum phase affine nonlinear system. This section will introduce
in detail how to design a purely decentralized control law for
the minimum phase follower, especially how to find a diffeo-
morphism to make the tracking error system and the follower
system have the same zero dynamics. We first introduce the
case of SISO follower systems, and then extend to MIMO
follower systems.

A. Distributed Control With SISO Follower Systems

Consider an output-tracking problem of a leader-following
multiagent system. Leader and follower systems are still in
the form of (3), (4) and (1), (2), respectively. In this sec-
tion, the leader system is assumed as a single output system,
and all followers are derived by the SISO nonlinear affine
system, i.e., for \( i = 1, 2, \ldots, N \), \( y_i, y_0 \) and the control input
signal \( u_i \) belong to \( \mathbb{R}^1 \). In order to study, the tracking problem
when the follower system is not completely controllable, zero
dynamic theory and partial feedback linearization method in
differential geometry are employed [27]. Within this idea, we
propose a purely decentralized control law in which the output
of an incompletely controllable follower can track the output
of leader.

Theorem 2: For the \( i \)th follower system, we assume that it is
a minimum phase system and has relative order \( r_i \) at \( \forall x_i \in \mathbb{R}^n \).
Then, there is a coordinate transformation (diffeomorphism)
(\( \xi_i^T, \theta_i^T \)) = \( \Phi_i(x_i) \) such that the tracking error dynamic
between the \( i \)th follower and leader can be described as
\[
\dot{\tilde{w}} = p(w)
\]
\[
\hat{\xi}_i = A_i \hat{\xi}_i + B_i v_i
\]
\[
\hat{\theta}_i = \gamma_i(\xi_0 + \hat{\xi}_i, \theta_i)
\]
where
\[
A_i = \begin{bmatrix} 0 & L_{r_i-1} \ 0 & 0 \end{bmatrix} \in \mathbb{R}^{n_x \times n_x}, \quad B_i = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} \in \mathbb{R}^{n_x}
\]
\( \xi_0 = \text{col}\{q(w), L_q q(w), \ldots, L_{r_i-1}^q q(w)\} \), \( v_i \) is a variable named
auxiliary control variable, and \( \theta_i \) is the internal dynamic
of the \( i \)th follower. Furthermore, the leader-following tracking
problem (5) can be achieved by employing a linear feedback
control law
\[
v_i = K_i \hat{\xi}_i
\]
where \( K_i \) is a matrix such that \( A_i + B_i K_i \) is Hurwitz.

Proof: Since the \( i \)th subsystem is a SISO system, the dis-
tribution spanned by \( g_i(x_i) \) is involutive. Then, there is a
diffeomorphism \( z_i = \Psi_i(x_i) \) [27] such that the subsystem can be
transformed in normal form
\[
\begin{align*}
\dot{z}_{ik} &= z_{i(k+1)}, \quad k = 1, \ldots, r_i - 1 \\
\dot{z}_{ir_i} &= L_{r_i}^\prime h_i + L_{r_i}^\prime_{-1} h_i u_i \\
\dot{\theta}_i &= \gamma_i(\xi_0, \theta_i)
\end{align*}
\]
where \( \xi_i = \text{col}\{z_{i1}, z_{i2}, \ldots, z_{ir_i}\} \).

Note that \( \Psi_i = \text{col}\{\psi_{i,1}, \psi_{i,r_i}, \psi_{i,r_i+1}, \ldots, \psi_{i,n_i}\} \) is
constructed by setting \( \tilde{z}_{ij} = \psi_{i,j}(x_i) = L_j^1 h_i \) for \( 1 \leq j \leq r_i \), and
choosing \( \tilde{\theta}_{ij} = \psi_{i,j}(x_i), \quad 1 \leq j \leq n_i \) such that \( \Psi_i \) is non-
singular and \( L_{r_i} \psi_{i,j}(x_i) = 0, \quad r_i < j \leq n_i \). Hence, for \( \forall x_i \in \mathbb{R^n} \), we
yield
\[
\begin{align*}
n_i &= \text{rank}\{\Psi_i\} \\
&= \text{rank}\{dh_i, \ldots, dL_{r_i}^{n_i-1} h_i, d\psi_{i,r_i+1}, \ldots, d\psi_{i,n_i}\} \quad (45)
\end{align*}
\]
Denote \( \varepsilon_i^{(k)}(t) \) as the \( k \)-order derivative of \( \varepsilon_i(t) \) with \( \varepsilon_i(t) = y_i(t) - y_0(t) \). Then, we can derive from the definition of relative
degree that
\[
\begin{align*}
\varepsilon_i^{(k)} &= L_{r_i}^k h_i - L_{r_i}^k p \quad k = 1, \ldots, r_i - 1 \\
\varepsilon_i^{(r_i)} &= L_{r_i}^k h_i + L_{r_i}^k u_i - L_{r_i}^k p \quad (46)
\end{align*}
\]
The control law \( u_i \) can be implemented as
\[
\begin{align*}
u_i &= \left(L_{r_i} L_{r_i}^{n_i-1} h_i \right)^{-1} \left(-L_{r_i} h_i + L_{r_i}^k p + v_i \right) \quad (48)
\end{align*}
\]
Then, (46) and (47) result in a \( r_i \)th-order linear system
\[
\tilde{\xi}_i = A_i \tilde{\xi}_i + B_i v_i \quad (49)
\]
where \( \xi_{ij} = \phi_i(x_i) = \varepsilon_j^{(j-i)} \) and \( \xi_i = \text{col}\{\xi_{i1}, \xi_{i2}, \ldots, \xi_{ir_i}\} \). In
order to construct
\[
\Phi_i(x_i) = \text{col}\{\phi_{i,1}, \ldots, \phi_{i,r_i}, \phi_{i,r_i+1}, \ldots, \phi_{i,n_i}\} \quad (50)
\]
such that the \( i \)th follower system can be transformed into (41)
and (42), we need to find a group of function \( \phi_{i,j}, r_i < j \leq n_i \)
such that \( \Phi_i \) is nonsingular and \( L_{r_i} \phi_{i,j}(x_i) = 0, \quad r_i < j \leq n_i \).
For \( 1 \leq j \leq r_i \), one may notice that
\[
\begin{align*}
d\phi_{i,j}(x_i) &= d\varepsilon_j^{(j-i)} = \frac{\partial}{\partial x_i^T} \left(L_{r_i}^{j-1} h_i - dL_{r_i}^{j-1} q \right) = dL_{r_i}^{j-1} h_i \quad (51)
\end{align*}
\]
Hence
\[
\begin{align*}
\text{rank}\{dh_i, \ldots, dL_{r_i}^{n_i-1} h_i, d\psi_{i,r_i+1}, \ldots, d\psi_{i,n_i}\} \\
&= \text{rank}\{d\phi_{i,1}, \ldots, d\phi_{i,n_i} \} \quad (52)
\end{align*}
\]
By setting \( \phi_{i,j} = \psi_{i,j}, \quad r_i < j \leq n_i \), the tracking error system
transformed from the \( i \)th follower by \( (\xi_i^T, \theta_i^T) = \Phi_i(x_i) \) can be
expressed as (41) and (42). It indicates that the tracking
error system and normal form (44) have the same internal
dynamics. Moreover, combining with (44), (46), and (47), we
know \( \zeta_i = \xi_0 + \xi_i \). Since the stability of zero dynamic
\( \hat{\theta}_i = \gamma_i(0, \theta_i) \) implies the stability of corresponding internal
dynamic (42) [38], the stability of the tracking error system
can be guaranteed by employing \( v_i = K_i \hat{\xi}_i \), such that (41) is
stable.

Theorem 2 gives a purely decentralized control law for the
\( i \)th subsystem. This kind of control law can only be applied to
the case where the leader can communicate with all the followers.
In the article, we are supposed to compose the purely decentralized control law (43), (48) and the distributed
observers (16), (17) to further obtain the following distributed
control law:
\[
\begin{align*}
\hat{u}_i &= \left(L_{r_i} L_{r_i}^{n_i-1} h_i \right)^{-1} \left(-L_{r_i} h_i + L_{r_i}^k p \phi_{i,j} + v_i \right) \quad (53)
\end{align*}
\]
\[
\hat{w}_i = \Theta^{-1}(\eta_i) \quad (54)
\]
\[
\dot{\hat{\xi}}_i = A_i \hat{\xi}_i + a(C_i) \hat{y} + eF \sum_{j=1}^{N} a_{ij}(\hat{y}_j - \hat{y}_i) + b_i(\dot{\hat{y}}_i - \dot{\hat{y}}_0).
\]

Despite all this, whether the closed-loop system controlled by a distributed control law based on state estimation is stable is indeed the problem that needs to be further demonstrated.

**Theorem 3:** The leader-following output tracking problem, including leader system (3), (4) and follower systems (1), (2), can be solved by distributed control law (53)–(55) if there exists a distributed observers for the leader system. In other words, the distributed control law satisfies certain equivalence principle.

**Proof:** We only need to show the tracking error system convergence to zero under (53)–(55). For simplifying the symbols, we denote \( v_i = L_0^j \xi_i \). By substituting (53) into (47), the \( r \)th derivative of tracking error can be rewritten as

\[
\dot{e}_i^{(r)} = L_0^j h_i + L_0^j L_0^{j-1} h_i (\dot{\hat{y}}_i - u_i + v_i) - L_0^j \dot{q} = v_i + \check{v}_i.
\]

Then, the tracking error system is

\[
\dot{\xi}_i = A_i \xi_i + B_i (v_i + \check{v}_i) = (A_i + B_i K_i) \xi_i + B_i \check{v}_i.
\]

Since \( \lim_{t \to \infty} \check{V}_i = 0 \) under the condition that the distributed observers exists for the leader system, we thus have \( \lim_{t \to \infty} \hat{\xi}_i = 0 \). Then, for \( \forall \varepsilon > 0 \), there exists \( T_0 > 0 \) such that for all \( t > T_0 \), \( \| \hat{\xi}_i \| < \varepsilon \). Furthermore, we set \( v_i = \max_{0 \leq t \leq T_0} \| \hat{\xi}_i \|, b_1 = \| B_i \| \). Recall that \( A_i + B_i K_i \) is Hurwitz, then there exists constants \( \lambda_i > 0 \) and \( \lambda_i > 0 \) such that \( \|e^{(\lambda_i t)}\| \leq \lambda_i e^{-\lambda_i t} \) for all \( t > 0 \). By solving tracking error system (56), we have

\[
\xi_i(t) = e^{(A_i + B_i K_i) t} \xi_i(0) + \int_0^t e^{(A_i + B_i K_i) (t-\tau)} B_i \check{v}_i(\tau) d\tau
\]

\[
\triangleq \check{T}_i(t) + \check{T}_i(t).
\]

Clearly, \( \lim_{t \to \infty} \check{T}_i(t) = 0 \). Now, we will show \( \lim_{t \to \infty} \check{T}_i(t) = 0 \). Let \( T_* > T_0 \) satisfy \( e^{-\lambda_i (T_* - T_0)} \leq \varepsilon_i \). Then, for all \( t > T_* \), we have

\[
\| \check{T}_i(t) \| \leq \int_0^{T_*} a_i e^{-\lambda_i (t-\tau)} b_i \check{v}_i(\tau) d\tau + \int_0^{T_0} a_i e^{-\lambda_i (t-\tau)} b_i \check{v}_i(\tau) d\tau
\]

\[
< \frac{a_i b_i \varepsilon_i}{\lambda_i} + \frac{a_i b_i \varepsilon_i}{\lambda_i} < \varepsilon.
\]

As a result, we obtain \( \lim_{t \to \infty} \dot{\xi}_i = 0 \).

**B. Distributed Control for MIMO System**

Suppose the dimension of the input and output of the leader system and all follower systems is \( m \), i.e., \( y_0, y_i, u_i \in \mathbb{R}^m \). The MIMO affine nonlinear dynamics of followers can be described as

\[
\dot{x}_i = f_i(x_i, w) + \sum_{j=1}^{m} g_{ij}(x_j) u_j
\]

\[
y_i = [h_{i1}(x_i), h_{i2}(x_i), \ldots, h_{im}(x_i)]^T.
\]

**Definition 2:** For a MIMO affine nonlinear system (59), (60), suppose \( U \) is neighborhood of a point \( x_i^0 \). The (vector) relative degree of this system is \( r_{i1}, \ldots, r_{im} \) if the following two conditions are fulfilled.

1) \( L_0^j h_i = 0 \) if for \( \forall x \in U \) and for all \( 1 \leq j \leq m, 1 \leq k \leq m \) and \( 1 \leq l < r_{ik} - 1 \).

2) \( A_i = [L_0^j L_0^{j-1} h_i(x_i^0)]_{m=1}^{n} \) is nonsingular at \( x_i^0 \).

For saving of analysis, we still assume that the order of each follower is equal everywhere in the whole space, and further assume that the \( i \)th follower with relative degree \( r_{i1}, r_{i2}, \ldots, r_{im} \) satisfies \( n_{i1} = n_{i2} \leq n_i \). Following the calculation procedure of the tracking error system, a coordinate transformation can be defined as

\[
x_i^k = L_0^{j-1} h_i(x_i) - L_0^{j-1} q_k \triangleq (e_i^k)^{(j-1)}
\]

\[
i = 1, 2, \ldots, N, k = 1, 2, \ldots, m, j = 1, 2, \ldots, r_{ik},
\]

Then, the \( i \)th subsystem can be transformed into \( m \) groups equations (\( k = 1, 2, \ldots, m \))

\[
(e_i^k)_{(j)} = L_0^{j-1} h_i(x_i) + \sum_{j=1}^{m} L_0^j L_0^{j-1} h_i u_j - L_0^{j-1} q_k.
\]

By denoting

\[
\beta_i(x_i, \omega) = L_0^{j-1} h_i(x_i) - L_0^{j-1} q_k
\]

\[
a_{ikj}(x_i, \omega) = L_0^j L_0^{j-1} h_i,
\]

\[
e_i = \{ e_i^k_{(j1)}, e_i^k_{(j2)}, \ldots, e_i^k_{(jm)} \}
\]

the \((e_i^k)^{(jm)}\). dynamic can be introduced in a compact form

\[
e_i = \beta_i(x_i, \omega) + A_i u_i
\]

where \( \beta_i(x_i, \omega) = \{ \beta_i^1(x_i, \omega), k = 1, \ldots, m \}, A_i = A_i(x_i, \omega) = [a_{ikj}]_{m=1}^{n}, u_i = [u_i^1, k = 1, \ldots, m]. \)

Referring to the definition of relative degree, we know \( A_i \) is invertible. Then, a purely decentralized control law for the \( i \)th follower can thus be implemented as

\[
u_i = A_i^{-1} (-\beta_i(x_i, \omega) + v_i).
\]

Sequentially, a linear error dynamic can be obtained by combining equations (61)–(63)

\[
\xi_i^k = A_i \xi_i^k + B_i \nu_i
\]

where \( v_i = \{ v_i^1, v_i^2, \ldots, v_i^m \}, \xi_i^k = \{ \xi_i^k_{j1}, \xi_i^k_{j2}, \ldots, \xi_i^k_{jm} \}, A_i = [a_{ikj}]_{j=1}^{m}, B_i = [B_i^1, B_i^2, \ldots, B_i^m]. \)

Then, there is a diffeomorphism \( \Phi_i \) such that the dynamic of the \( i \)th follower can be transformed into

\[
\check{\xi}_i = A_i \check{\xi}_i + B_i \nu_i
\]

\[
\dot{\theta}_i = \gamma_i(\theta, \xi) + \sum_{j=1}^{m} \rho_j(\theta, \xi) u_j.
\]
According to the knowledge of Theorem 2, we know (66) is indeed the internal dynamic of (59), where $\xi_t = \xi_i + \xi_0$ with $\xi_0 = \text{col}[e_k^m]_{k=1}^m$ and $e_k^1 = \{L_i^j q(w)\}_{j=0}^{n_e-1}$, and the smooth nonlinear function $\gamma(\cdot)$ and $\rho_0(\cdot)$ can be obtained following the computation process of quasinormal form for the MIMO affine nonlinear system [27]. Note that $\rho_0$ in (66) could be designed to zero [27] if the distribution $D = \text{span}\{g_1, g_2, \ldots, g_m\}$ is involutive. Then, the stability of the tracking error system (65), (66) can be ensured by a linear feedback control

$$v_i = K_i \hat{\xi}_i \quad (67)$$

if the zero dynamic corresponding to internal dynamic (66) is stability, where $K_i = \text{diag}(K_{i1}, K_{i2}, \ldots, K_{im})$ is designed to make $A_{ik} + B_{ik}K_{ik}$ be Hurwitz for all $k = 1, 2, \ldots, m$.

Similar to the previous section, we need to develop the distributed control law corresponding to the MIMO system by composing purely decentralized control law (63), (67) and distributed observers (16), (17) and prove the certainty equivalently principle.

**Theorem 4:** The leader-following output tracking problem composed of leader system (3), (4) and incompletely controllable follower systems (65), (66) can be solved by the distributed control law

$$\hat{u}_i = \hat{A}_i^{-1}(-\hat{b}_i(x_i) + v_i) \quad (68)$$

$$\hat{\omega}_i = \Theta^{-1}(\hat{\eta}_i) \quad (69)$$

$$\hat{\dot{\eta}}_i = A_0 \hat{\eta}_i + a(C\hat{\eta}_i) + cF \sum_{j=1}^N a_{ij}(\hat{\eta}_j - \hat{\eta}_i) + b_i(\hat{\eta}_i - \eta_0) \quad (70)$$

if there exists a distributed observers for leader system. In (68), $\hat{A}_i = \hat{A}_i(x_i, \hat{\omega}_i)$ and $\hat{b}_i = \hat{b}_i(x_i, \hat{\omega}_i)$.

**Proof:** Substituting (68) into (62), we have

$$\epsilon_i = \beta_i(x_i) + A_i(\hat{u}_i - u_i + \hat{u}_i) = v_i + A_1(\hat{u}_i - u_i). \quad (71)$$

Let $\tilde{u}_{ikj} = a_{jk}(\hat{u}_i - u_i)$ and we can obtain by combining (62)

$$\begin{pmatrix} \gamma_k \end{pmatrix} = v_{ik} + \sum_{j=1}^m \tilde{u}_{kj}. \quad (72)$$

Then, the tracking error system of the $i$th subsystem with the $k$th output is

$$\begin{pmatrix} \dot{\xi}_i^k \end{pmatrix} = A_{ik} \xi_i^k + B_{ik} \left( v_{ik} + \sum_{j=1}^m \tilde{u}_{kj} \right). \quad (73)$$

Noticing that (73) has the same form as (56). Thus, we can prove the solution of (73) satisfies $\xi_i^k(t) \rightarrow 0$ if $\lim_{t \rightarrow \infty} \dot{\xi}_{ik} = 0$, and the latter can be indicated by (70) directly.

**Remark 6:** The leader-following consensus has been achieved by an improved distributed control law, including novel distributed nonlinear observers based on geometry conditions and an improvement purely decentralized control law on the basis of the zero dynamic theory. Thave whelming merits, which lies in: 1) the distributed nonlinear observers can be available for more kinds of leader system, and 2) the follower system can be chosen as the minimum phase affine nonlinear system instead of the completely controllable affine nonlinear system.

### VI. SIMULATION

First, the ESSML system is used to show that our novel distributed nonlinear observer based on geometric conditions can be applied to some nonlinear systems that fail to satisfy [24]’s assumption. Then, we simulate the distributed observers-based control frame with the *Van der Pol* system as leader and an incompletely controllable minimum phase system as two followers. For one thing, the second example shows that for a nonlinear leader who can satisfy [24]’s hypothesis and geometric conditions, our method can obtain the better distributed observers performance as [24]’s method. For another, our purely decentralized control law based on zero dynamics can make the minimum phase affine nonlinear system that is not completely controllable track the leader’s output.

#### A. Simulation With ESSLM System

Section III-B has proved that ESSLM system (7) satisfies geometric conditions in Lemma 1. Suppose there are five followers and the communication graph between leader and followers is showed in Fig. 1. In (7), we set the length of the Linkage be $2d = 0.2 \text{ m}$, the mass of Linkage be $m = 1 \text{ kg}$, the Rotational inertia be $J_1 = 5 \text{ kg} \cdot \text{m}^2$, $J_2 = 2 \text{ kg} \cdot \text{m}^2$, the viscosity friction coefficient be $F_1 = 0.5$ and $F_2 = 0.55$, and the torsional elastic coefficient of elastic shaft be $K = 10 \text{ Nm/rad}$. The acceleration of gravity is approximately taken as $g = 10 \text{ m/s}^2$. From the calculation in Section III-B, we can obtain a diffeomorphism $\eta_0 = \Phi(\omega)$ such that

$$\eta_0 = \begin{bmatrix} 0.33 & 0.244 & 3.33 & 0.889 \\ 3.33 & 0.916 & 0 & 0.1 \\ 0 & 0.375 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \omega \quad (74)$$

and

$$\omega = \begin{bmatrix} 0 & 0.3 & -0.03 & -0.264 \\ 0 & 0 & 0 & 1 \\ 0.3 & -0.03 & -0.264 & 0.053 \\ 0 & 0 & 1 & -0.375 \end{bmatrix} \eta_0. \quad (75)$$

By calculating $(\partial \Phi / \partial \omega^T) p(\omega)|_{\omega=\Phi^{-1}(\eta_0)}$, we have the OCF of ESSLM

$$\dot{\eta}_0 = \begin{bmatrix} \dot{\eta}_{01} \\ \dot{\eta}_{02} \\ \dot{\eta}_{03} \\ \dot{\eta}_{04} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \eta_0.
Fig. 2. State estimate and actual states, including estimate of $\omega_1$ (a), $\omega_2$ (b), $\omega_3$ (c), and $\omega_4$ (d).

Fig. 3. Profile on the phase portraits of the leader and the distributed observers with (a) geometric conditions and (b) Taylor conditions; and (c) shows the method based on geometric conditions proves that the Van der Pol system can actually have a globally convergent distributed observers.

\[
y_0 = \eta_{04}.
\]

Therefore, a distributed observers for this leader can be designed by (16) and (17). The initial states of original system are performed with $[0, \pi/2, 0, 0]^T$ and that of each agent are generated randomly. Fig. 2 shows the comparison between the actual states of the leader system and the state estimates generated by each local observer. These figures illustrate that the state estimates of the distributed observers converge quickly to the actual states, which verifies the effectiveness of our new method. In other words, we can design a distributed observers for the ESSLM system, a leader system that cannot be handled by [24], and obtain excellent dynamic performance.

The simulation concerning distributed observer-based distributed control law will be shown in the next simulation.

B. Simulation With Van der Pol System

Suppose the leader obeys the following Van der Pol system:

\[
\begin{align*}
\dot{w}_1 &= w_2 \\
\dot{w}_2 &= -w_1 + \frac{1}{3} w_2^3 + w_2 \\
y_0 &= w_1.
\end{align*}
\]

The communication graph between all followers is designed as Fig. 1. It is easy to check $[d\varphi(w), d\varphi_q(w)] = I_2$, i.e., Van der Pol system satisfies observability condition [Lemma 1 (1)]. Utilizing Lemma 1, we can also calculate $r(w) = [0, 1]^T$; hence, $[r(w), ad_p r(w)] = 0$. Thus, (13) and (14) are satisfied. So we can find a coordinate transformation (referring the method of solving this coordinate transformation to [28])

\[
\begin{bmatrix}
\eta_{01} \\
\eta_{02}
\end{bmatrix} = \Phi(w) = \begin{bmatrix}
-w_1 + \frac{1}{3} w_2^3 + w_2 \\
\eta_{02}
\end{bmatrix}
\]

and its inverse information

\[
\begin{bmatrix}
w_1 \\
w_2
\end{bmatrix} = \Phi^{-1}(\eta_0) = \begin{bmatrix}
\eta_{02} \\
\eta_{01} + \eta_{02} - \frac{1}{3} \eta_{02}^3
\end{bmatrix}
\]

such that leader system (76) is transformed in OCF

\[
\dot{\eta}_0 = A_0 \eta_0 + a(y_0)
\]

\[
y_0 = \eta_{02}
\]

where

\[
A_0 = \begin{bmatrix}
0 & 0 \\
1 & 0
\end{bmatrix}, \quad a(y_0) = \begin{bmatrix}
-\eta_{02} \\
\eta_{02} - \frac{1}{3} \eta_{02}^3
\end{bmatrix}
\]

Let $c = 10$ so that it satisfies conditions (22) and (37). Fig. 3(a) demonstrates that all the state estimate generated by followers can converge to the actual states. These simulation results indicate the correction of Theorem 1. Furthermore, comparing to Fig. 3(b), the distributed observers obtained by [24]’s model, it can be seen that our novel distributed observers have faster convergence speed under the same coupling gain.

In addition, [24]’s assumption limits the application scope of their distributed observers to a compact set containing the origin. For example, for the Van der Pol system, the initial value of their leader needs to be selected in $\|\eta_0(0)\| \leq 2\sqrt{2}$. Actually, the distributed observers designed for the Van der Pol system can be globally convergent owing to it meets geometric conditions globally. Fig. 3(c) shows the convergence performance when $\|\eta_0(0)\|$ is chosen outside of $\|\eta_0(0)\| \geq 2\sqrt{2}$. 

Assume followers 1, 3, and 5 satisfy a nonlinear system

\[
\begin{align*}
\dot{x}_{1i} &= x_{1i} + x_{2i} \\
\dot{x}_{2i} &= x_{1i} x_{2i} + u_i \\
y_i &= x_{1i} \\
\end{align*}
\]

where \(a_i\) for all \(i = 1, 2, \ldots, N\) are parameters depended on the \(i\)th subsystem. One can check that every subsystem has relative degree 2 under the given output \(y_i\). Thus, there is a coordinate transformation

\[
\begin{align*}
\xi_{1i} &= x_{1i} - w_1 \\
\xi_{2i} &= x_{1i} + x_{2i} - L_p q(w) \\
y_i &= \xi_{1i} + w_1. \\
\end{align*}
\]

Then, the purely decentralized control law can be designed as

\[
u_i = -x_{1i} - x_{2i} + L_p q(\hat{\eta}_i) + \nu_i. \tag{77}\]

On the other hand, followers 2 and 4 are in the form of [38]

\[
\begin{align*}
\dot{x}_{1i} &= -x_{1i} + e^2 x_{2i} u_i \\
\dot{x}_{2i} &= 2x_{1i} x_{2i} + \sin x_{2i} + \frac{1}{2} u_i \\
\dot{x}_{3i} &= x_{2i} \\
y_i &= x_{1i}. \tag{78}
\end{align*}
\]

This system is not incompletely controllable; hence, it cannot be controlled by [24]'s purely decentralized control law. By transforming it into a quasinormal form

\[
\begin{align*}
\dot{\xi}_{1i} &= \xi_{2i} \\
\dot{\xi}_{2i} &= 2(-1 + \theta_i + e^{\xi_{2i}})\xi_{2i} + 2 \sin \frac{\xi_{2i}}{2} - L_p q(w) + u_i \\
\dot{\theta}_i &= (1 - \theta_i - e^{\xi_{2i}})(1 + 2\xi_{2i} e^{\xi_{2i}}) - 2 \sin \frac{\xi_{2i}}{2} e^{\xi_{2i}} \tag{79} \\
y_i &= \xi_{1i} + w_1 \tag{80}
\end{align*}
\]

we obtain its interdynamic (79). Then, we can further get zero dynamic by setting \(\dot{\xi}_{1i} = \dot{\xi}_{2i} = 0\)

\[
\dot{\theta}_i = -\theta_i. \tag{81}
\]

It is obviously that the zero dynamic of (78) is stable. Hence, we can design the purely decentralized control law for this system

\[
u_i = -2(-1 + \theta_i + e^{\xi_{2i}})\xi_{2i} - 2 \sin \frac{\xi_{2i}}{2} + L_p q(w) + \nu_i. \tag{82}
\]

Then, the distributed control law of this leader-following problem can be constructed by replacing state estimates \(\Phi^{-1}\hat{\eta}_i\) generated by distributed observers with \(\omega\) in a purely decentralized control law (77) and (82). The initial states of each subsystem are chosen randomly and the pole of the feedback linearization system is allocated at \(-2\) and \(-6\). Fig. 4 shows the tracking error of subsystems to the external system under the distributed control law. It can be seen that the leader-following consensus is achieved.

**VII. CONCLUSION**

This article has proposed a novel distributed nonlinear observer based on geometric conditions. Within this method, a special assumption on leader system constrained by [24] has been replaced with a group of geometric conditions. As a result, our distributed nonlinear observer can be applied for some nonlinear system, which fails to fulfill [24]'s assumption, such as the ESSLM system and most of the first-order nonlinear system. We have proved that our distributed nonlinear observers have exponentially stable error dynamics. Two lemmas corresponding to the spectrum of the matrices are proved as a pioneer to complete the proof. Furthermore, the purely decentralized control law based on zero dynamic proposed in differential geometry has been developed so that the followers can be chosen as an arbitrary minimum phase affine nonlinear system. The certainty equivalence principle for the distributed observers-based control law including novel distributed nonlinear observers and improved purely decentralized control law has also been proved. The ESSLM system and Van der Pol system have been used to simulate our method.

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