Asymptotics of the Upper Matching Conjecture*

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Abstract

We give upper bounds for the number \( \Phi_\ell(G) \) of matchings of size \( \ell \) in (i) bipartite graphs \( G = (X \cup Y, E) \) with specified degrees \( d_x \) \( (x \in X) \), and (ii) general graphs \( G = (V, E) \) with all degrees specified. In particular, for \( d \)-regular, \( N \)-vertex graphs, our bound is best possible up to an error factor of the form \( \exp[o_d(1)N] \), where \( o_d(1) \to 0 \) as \( d \to \infty \). This represents the best progress to date on the “Upper Matching Conjecture” of Friedland, Krop, Lundow and Markström.

Some further possibilities are also suggested.

1 Introduction

Throughout this paper \( G \) will be a graph without isolated vertices on a vertex set \( V \) of size \( N \), and \( \ell \) an integer with \( 0 \leq \ell \leq N/2 \). When \( G \) is bipartite the bipartition will be \( X \cup Y \). We write \( \mathcal{M}_\ell(G) \) for the set of matchings of size \( \ell \) (or \( \ell \)-matchings) of \( G \), and set \( \Phi_\ell(G) = |\mathcal{M}_\ell(G)| \). The following “Upper Matching Conjecture” was proposed by Friedland et al. [8].

**Conjecture 1.1.** If \( G \) is \( d \)-regular and \( 2d \mid N \) then

\[
\Phi_\ell(G) \leq \Phi_\ell \left( \frac{N}{2d}K_{d,d} \right).
\]

(As usual \( tK_{d,d} \) is the union of \( t \) disjoint copies of the complete bipartite graph \( K_{d,d} \). A generalization of Conjecture [1] that doesn’t require \( 2d \mid N \) was suggested by Friedland, Krop and Markström [9, (6.2)].)

When \( N \) is even and \( \ell = N/2 \), \( \Phi_{N/2}(G) \) is the number, \( \Phi(G) \), of perfect matchings of \( G \). Here the bipartite case of Conjecture [1] is contained in the well-known theorem of Bréguan [2] (formerly the Minc Conjecture [14]),

AMS 2010 subject classification: 05C70, 94A17

Key words and phrases: Upper Matching Conjecture, entropy, asymptotics

* Supported by NSF grant DMS0701175.
which in graph-theoretic language says that for $G$ bipartite on $X \cup Y$ (with $|X| = |Y|$)
\[
\Phi(G) \leq \prod_{x \in X} (d_x!)^{1/d_x} \tag{1}
\]
(where $d_x$ is the degree of $x$). An analogous bound for general graphs was first observed in [12] and rediscovered and/or reproved in [1, 5, 6, 7].

Friedland et al. [9] proved Conjecture 1.1 for $\ell = 2$. Here we are really thinking of $\ell = \Theta(N)$ and set $\alpha = 2\ell/N$, the fraction of vertices covered by an $\ell$-matching. Carroll, Galvin and Tetali [3] provided some evidence in favor of Conjecture 1.1, showing that (for $G$ as in the conjecture)
\[
\log \Phi_\ell(G) \leq \frac{N}{2} \left[ \alpha \log d + H(\alpha) \right]. \tag{2}
\]
(Throughout this paper $\log$ is $\log_2$ and $H$ is the usual binary entropy function; for entropy basics see e.g. [13].) This contrasts with the lower bound (which Conjecture 1.1 would say is the truth) given by
\[
\log \Phi_\ell(N^{2d}K_{d,d}) = \frac{N}{2} \left[ \alpha \log d + 2H(\alpha) + \alpha \log \frac{\alpha}{e} + o_d(1) \right] \tag{3}
\]
(where $o_d(1) \to 0$ as $d \to \infty$).

Here we are primarily interested in closing the gap between the parts of (2) and (3) that are on the order of $N$ as $d \to \infty$:

**Theorem 1.2.** If $G$ is $d$-regular, then
\[
\log \Phi_\ell(G) \leq \frac{N}{2} \left[ \alpha \log d + H(\alpha) + \frac{\log d}{d-1} \right].
\]
(Here and elsewhere we interpret $\log x/(x-1)$ as $\log e$ when $x = 1$. Of course for Theorem 1.2 we could simply disallow the uninteresting case $d = 1$, but the convention will be helpful later.)

Theorem 1.2 is a special case of our main result, Theorem 1.3, which bounds $\Phi_\ell(G)$ in terms of the degree sequence $\{d_v\}_{v \in V}$ or, for $G$ bipartite on $X \cup Y$, $\{d_x\}_{x \in X}$. For $W \subseteq V$, let $d_\ell(W) = \sum \{ \prod_{v \in S} d_v : S \in \binom{W}{\ell} \}$.

**Theorem 1.3.** (i) Suppose $G$ is bipartite (on $X \cup Y$) and $0 \leq \ell \leq \min\{|X|,|Y|\}$, and set $\delta_X = \min\{d_x : x \in X\}$ and $\alpha_Y = \ell/|Y|$. Then
\[
\log \Phi_\ell(G) \leq \log d_\ell(X) + |Y| \left[ H(\alpha_Y) + \alpha_Y \log \frac{\alpha_Y}{e} + \frac{\log \delta_X}{\delta_X - 1} \right].
\]
(ii) For a general $G$ (on $V$) with minimum degree $\delta$,

$$
\log \Phi_\ell(G) \leq \frac{1}{2} \log d_\ell(V) + \frac{N}{2} \left[ H(\alpha) + \alpha \log \frac{\alpha}{e} + \frac{\log \delta}{\delta - 1} \right].
$$

(4)

See the remark following (6) for an explanation of the bound in (i). Note that for $d$-regular $G$,

$$
\frac{1}{2} \log d_\ell(V) = \frac{1}{2} \left( \log \left( \frac{N}{2\ell} \right) + 2\ell \log d \right) < \frac{N}{2} \left( H(\alpha) + \alpha \log d \right),
$$

so (ii) includes Theorem 1.2.

The proofs of these results are mostly based on entropy considerations, in the spirit of Radhakrishnan’s proof of Brégman’s Theorem [15], and, for example, the more recent [4]. Here, as for some related problems, most of the work deals with the bipartite case. The passage to general graphs is then accomplished via an easy correspondence between ordered pairs of $\ell$-matchings of $G$ and a subset of the $(2\ell)$-matchings of the bipartite double cover of $G$; this correspondence, which goes back at least to Gibson [10], was recently rediscovered by Alon and Friedland [1].

The next section is devoted to the proof of Theorem 1.3. In Section 3 we propose an extension of Brégman’s bound (1) to unbalanced graphs, which would also be a precise version of our main inequality (5).

## 2 Proof of Theorem 1.3

As noted above, (ii) will follow easily from (i). We begin with the latter. The main point here is establishing the bound when $|X| = \ell$; that is, for $G$ bipartite on $S \cup Y$ with $|S| = \ell$, $|Y| = M$ and $\delta_S = \min_{x \in S} d_x$,

$$
\log \Phi_\ell(G) \leq \sum_{x \in S} \log d_x + M \left[ H(\alpha_Y) + \alpha_Y \log \frac{\alpha_Y}{e} + \frac{\log \delta_S}{\delta_S - 1} \right].
$$

(5)

This easily gives (i): setting $G_S = G[S \cup Y]$ for $S \subseteq X$, we have

$$
\Phi_\ell(G) = \sum_{S \in \binom{X}{\ell}} \Phi_\ell(G_S)
\leq \sum_{S \in \binom{X}{\ell}} \left( \prod_{x \in S} d_x \right) \exp_2 \left[ M \left( H(\alpha_Y) + \alpha_Y \log \frac{\alpha_Y}{e} + \frac{\log \delta_S}{\delta_S - 1} \right) \right]
\leq d_\ell(X) \exp_2 \left[ |Y| \left( H(\alpha_Y) + \alpha_Y \log \frac{\alpha_Y}{e} + \frac{\log \delta_X}{\delta_X - 1} \right) \right].
$$

(6)
Remark. The bound in (5) (apart from the error term) is quite natural: 
\[ \exp_2 [MH(\alpha_y)] \] is roughly the number of ways to choose an \( \ell \)-subset \( T \) of \( Y \) to be used in the matching; on the other hand, for \( T \) uniform from \( (Y_\ell) \), the “typical” \( T \)-degree of \( x \in S \) is \( \alpha_y d_x \), and

\[ \sum_{x \in S} \log d_x + M \alpha_y \log \frac{\alpha_y}{e} = \sum_{x \in S} \log \frac{\alpha_y d_x}{e} \]

is essentially Brégman’s bound (1) for these degrees.

For the proof of (5) we think of \( \ell \)-matchings of \( G \) as (injective) functions \( f : S \to Y \), using \( R(f) \) for the range of \( f \) and \( f_W \) for the restriction of \( f \) to \( W \subseteq S \). For a permutation \( \sigma \) of \( S \) (thought of as an ordering of \( S \)) and \( x \in S \), set \( B(\sigma, x) = \{ w \in S : \sigma(w) < \sigma(x) \} \). In what follows \( x \) and \( y \) range over \( X \) and \( Y \) respectively. Expressions of the form \( 0 \cdot \log b \) are always interpreted as zero.

Let \( f \) be a random (uniform) \( \ell \)-matching of \( G \) and \( p(x, y) = \Pr(f(x) = y) \). Let \( s \) be a random (uniform) permutation of \( S \) and \( Y_x = f_{B(s, x)} \). Thus, if we think of choosing \( f \)-values in the order given by \( s \), \( Y_x \) tells us what has happened prior to the choice of \( f(x) \). Our argument through (9) closely follows that of [4]. By the “chain rule” for entropy, we have

\[ \log \Phi_\ell(G) = H(f) \]

\[ = \frac{1}{\ell!} \sum_{\sigma} \sum_{x} H(f(x)|f_{B(\sigma, x)}) \]

\[ = \sum_{x} \sum_{\sigma} \sum_{g} \frac{1}{\ell!} \Pr(f_{B(\sigma, x)} = g) H(f(x)|f_{B(\sigma, x)} = g) \]

\[ = \sum_{x} H(f(x)|Y_x), \quad (7) \]

where \( \sigma \) ranges over the possible values of \( s \) and \( g \) over the possible values of \( f_{B(\sigma, x)} \).

Let \( Z_x = Y \setminus f(B(s, x)) \), the set of vertices of \( Y \) that remain available when we come to specify \( f(x) \). Since \( Z_x \) is a function of \( Y_x \), we have \( H(f(x)|Y_x) \leq H(f(x)|Z_x) \) and so, by (7),

\[ \log \Phi_\ell(G) \leq \sum_{x} H(f(x)|Z_x). \quad (8) \]
In what follows we use \( y \) and \( Z \) for possible values of \( f(x) \) and \( Z_x \) for the \( x \) under discussion, in particular restricting to \( y \)'s for which \( p(x, y) \neq 0 \).

From this point through (14), with the exception of (10), we fix \( x \in S \). Let \( \Pr(Z) = \Pr(Z_x = Z) \), \( \Pr(Z | y) = \Pr(Z | f(x) = y) \) and so on, and set \( q_k(y) = \Pr(|Z_x| = k | f(x) = y) \) and \( r_k(y) = \Pr(|Z_x| = k, y \in Z_x) \). Then, with \( F(x, y) = \sum_k q_k(y) \log \frac{r_k(y)}{q_k(y)} \), we have

\[
H(f(x)|Z_x) = \sum_Z \Pr(Z) \sum_y \Pr(y|Z) \log \frac{1}{\Pr(y|Z)}
\]

\[
= \sum_y \Pr(y, Z) \log \frac{\Pr(Z)}{\Pr(y, Z)}
\]

\[
= \sum_y p(x, y) \left[ \log \frac{1}{p(x, y)} + \sum_Z \Pr(Z|y) \log \frac{\Pr(Z)}{\Pr(Z|y)} \right]
\]

\[
= H(f(x)) + \sum_y p(x, y) \sum_Z \Pr(Z|y) \log \frac{\Pr(Z)}{\Pr(Z|y)}
\]

\[
= H(f(x)) + \sum_y p(x, y) \sum_k q_k(y) \sum_{|Z|=k} \frac{\Pr(Z|y)}{q_k(y)} \log \frac{\Pr(Z)}{\Pr(Z|y)}
\]

\[
\leq H(f(x)) + \sum_y p(x, y) \sum_k q_k(y) \log \left[ \frac{1}{q_k(y)} \sum_{|Z|=k} \Pr(Z|y) \sum_{y \in Z} \Pr(Z) \right]
\]

\[
= H(f(x)) + \sum_y p(x, y) F(x, y),
\]  

the inequality following from concavity of the logarithm. Rewriting

\[
H(f(x)) = \log d_x - \sum_y p(x, y) \log(d_x p(x, y))
\]

and applying (8) and (9) (and momentarily unfixing \( x \)) gives

\[
\log \Phi_\ell(G) \leq \sum_{x \in S} \log d_x + \sum_x \sum_y p(x, y) [F(x, y) - \log(d_x p(x, y))].
\]  

(10)

The main part of the proof involves bounding the second term in (10), and in particular \( F(x, y) \).

(We again fix \( x \).) For \( y \in Y \) set \( \mu_y = \Pr(y \in R(f)) \) and \( \nu_y = 1 - \mu_y \).

Since \( |Z_x| \) and \( f \) are independent, we have

\[
q_k(y) = \Pr(|Z_x| = k) = \begin{cases} 
0 & \text{for } k \leq M - \ell, \\
1/\ell & \text{for } M - \ell + 1 \leq k \leq M.
\end{cases}
\]  

(11)
while \( \Pr(y \in \mathbb{Z}_x \mid y \notin R(f) \setminus \{f(x)\}) = 1 \) and \( \Pr(y \in \mathbb{Z}_x \mid y \in R(f) \setminus \{f(x)\}, |\mathbb{Z}_x| = k) = (k - (M - \ell) - 1)/(\ell - 1) \). Thus

\[
r_k(y) = \Pr(|\mathbb{Z}_x| = k) \Pr(y \in R(f) \setminus \{f(x)\}) \cdot \frac{k - (M - \ell) - 1}{\ell - 1} + \Pr(y \notin R(f) \setminus \{f(x)\})
\]

and (for \( M - \ell + 1 \leq k \leq M \))

\[
\frac{r_k(y)}{q_k(y)} = (\mu_y - p(x, y)) \frac{k - (M - \ell) - 1}{\ell - 1} + (\nu_y + p(x, y)). \tag{12}
\]

Let

\[
U(t) = \log \left[ (\mu_y - p(x, y)) t + (\nu_y + p(x, y)) \right] \quad (t \in [0, 1])
\]

and

\[
f(t) = \begin{cases} 
\frac{t}{1-t} \log \frac{1}{t} & \text{if } t \in (0, 1), \\
0 & \text{if } t = 0, \\
\log e & \text{if } t = 1.
\end{cases}
\]

In view of (11) and (12), we have (with justification of (14) to follow),

\[
F(x, y) \leq \sum_{k=M-\ell+1}^{M} \frac{1}{\ell} U\left( \frac{k - (M - \ell) - 1}{\ell - 1} \right) = \sum_{j=0}^{\ell - 1} \frac{1}{\ell} U\left( \frac{j}{\ell - 1} \right) \tag{13}
\]

\[
\leq \int_{0}^{1} U(t)dt = G(x, y) - \log e, \tag{14}
\]

where \( G(x, y) = f(\nu_y + p(x, y)) \). The equality in line (14) is trivial if \( \mu_y = p(x, y) \), and otherwise is given by the fact that for \( a \neq 0 \) and \( b = 1 - a > 0 \),

\[
\int_{0}^{1} \log(at + b)dt = \frac{\log e}{a} \left[ (at + b) \ln(at + b) - at \right]_{t=0}^{t=1} = \frac{b}{a} \log \frac{1}{b} - \log e.
\]

The inequality in (14) requires only the concavity of \( U \), as follows. Let \( U^* \) be the smallest concave function that agrees with \( U \) at the points \( j/(\ell - 1) \); namely, for \( 1 \leq j \leq \ell - 1 \) and \( \frac{j-1}{\ell - 1} \leq x \leq \frac{j}{\ell - 1} \),

\[
U^*(x) = (j - (\ell - 1)x)U(\frac{j-1}{\ell - 1}) + ((\ell - 1)x - (j - 1))U(\frac{j}{\ell - 1}).
\]
Then $U^* \leq U$ and, setting $a_i = U(i/\ell)$, we have

$$
\int_0^1 U^*(t)dt = \frac{1}{2(\ell - 1)}[(a_0 + a_1) + \cdots + (a_{\ell-2} + a_{\ell-1})]
= \frac{1}{\ell - 1}(a_0 + \cdots + a_{\ell-1}) - \frac{1}{2(\ell - 1)}(a_0 + a_{\ell-1})
\geq \frac{1}{\ell}(a_0 + \cdots a_{\ell-1}),
$$

which is the right hand side of (13). (The inequality, which is equivalent to $2(a_0 + \cdots a_{\ell-1}) \geq \ell(a_0 + a_{\ell-1})$, follows from the concavity of $U$ (an instance of Karamata’s Inequality, e.g. [11]).)

Thus, now letting $x$ vary, we have

$$
\sum_x \sum_y p(x, y)F(x, y) \leq \sum_x \sum_y p(x, y) [G(x, y) - \log e]
= \sum_x \sum_y p(x, y)G(x, y) - \ell \log e
= \sum_x \sum_y p(x, y)G(x, y) - M\alpha \log e. \quad (15)
$$

We will approximate the last sum by

$$
\Delta := \sum_x \sum_y p(x, y) f(\nu_y) = \sum_y \nu_y \log \frac{1}{\nu_y}
\leq -M (1 - \alpha_Y) \log (1 - \alpha_Y) = M[H(\alpha_Y) + \alpha_Y \log \alpha_Y] \quad (16)
$$

(where we used $\sum_x p(x, y) = \mu_y$ in the first line and Jensen’s Inequality in the second). Adding and subtracting $\Delta$ from the right side of (15) and using (16) gives

$$
\sum_x \sum_y p(x, y)F(x, y) \leq \sum_x \sum_y p(x, y)[G(x, y) - (\nu_y/\mu_y) \log (1/\nu_y)]
+ M[H(\alpha_Y) + \alpha_Y \log (\alpha_Y/e)]. \quad (17)
$$

We next observe that

$$
G(x, y) - (\nu_y/\mu_y) \log (1/\nu_y) = f(\nu_y + p(x, y)) - f(\nu_y) \leq f(p(x, y)),
$$

and
the inequality holding because $f$ is concave with $f(0) = 0$. Combining this with (10) and (17) gives

$$
\log \Phi_\ell(G) \leq \sum_x \sum_y p(x, y) \left[ f(p(x, y)) - \log(d_x p(x, y)) \right] + \sum_x \log d_x + M \left[ H(\alpha_y) + \alpha_y \log \frac{\alpha_y}{e} \right].
$$

(18)

Finally, for $x \in S$ and $t \in (0, 1]$, set

$$
g_x(t) = \begin{cases} t^2 \log \frac{1}{t} - t \log(d_x t) & \text{if } t \neq 1, \\ \log e - \log d_x & \text{if } t = 1. \end{cases}
$$

The double sum in (18) is then $\sum_x \sum_y g_x(p(x, y))$, and (5) follows from (18) and the observation that (since $g_x(t) = f(t) - t \log d_x$ is concave in $t$)

$$
\sum_y g_x(p(x, y)) \leq d_x g_x(1/d_x) = \frac{\log d_x}{d_x - 1} \leq \frac{\log \delta}{\delta - 1}.
$$

Finally we turn to (ii). We consider the bipartite double cover, say $K$, of $G$; that is, the graph on $V \times \{0, 1\}$ with edge set $\{(x, 0)(y, 1) : xy \in E(G)\}$. This is a bipartite graph on $2N$ vertices (with $d_K(x, 0) = d_K(x, 1) = d_G(x)$), so, as shown above,

$$
\log \Phi_{2\ell}(K) \leq \log d_{2\ell}(V) + N \left[ H(\alpha) + \alpha \log \frac{\alpha}{e} + \frac{\log \delta}{\delta - 1} \right],
$$

(19)

where, again, $\delta = \min \{d_v : v \in V\}$.

Let $T^*$ be the set of $(2\ell)$-edge multisubgraphs of $G$ (that is, multigraphs whose underlying graphs are subgraphs of $G$) whose components are paths and cycles (possibly of length 2), and let $T$ consist of those members of $T^*$ that do not contain odd cycles. For $T \in T^*$ let $c(T)$ be the number of components of $T$ that are not 2-cycles.

The natural projection $\psi((x, i)) = x$ maps $\mathcal{M}_{2\ell}(K)$ surjectively to $T^*$, with $|\psi^{-1}(T)| = 2^{c(T)}$ for all $T \in T^*$. On the other hand, $\varphi : T^* \rightarrow T$ given by $\varphi(M, M') = M \cup M'$ (multiset union) is a surjection with $|\varphi^{-1}(T)| = 2^{c(T)}$ for all $T \in T$. Thus we have

$$
\Phi_\ell(G)^2 = \sum_{T \in T} 2^{c(T)} \leq \sum_{T \in T^*} 2^{c(T)} = \Phi_{2\ell}(K),
$$

which, combined with (19), gives (4).
3 A conjecture

In closing we would like to propose a precise version of (5) that was one of the original reasons for our interest in the present material.

For \( d \geq t > 0 \) with \( d \) an integer, set \( \psi(d, t) = t^{-1} [\log d! - \log \Gamma(d - t + 1)] \) (with \( \Gamma \) the usual gamma function).

**Conjecture 3.1.** Let \( G \) be bipartite on \( X \cup Y \) with \( |X| = \ell \) and \( |Y| = M \geq \ell \), and for \( x \in X \), let \( t_x = \ell d_x / M \). Then

\[
\log \Phi_\ell(G) \leq \sum_{x \in X} \psi(d_x, t_x). \tag{20}
\]

Notice that for \( t \in \mathbb{Z} \), one has \( \psi(d, t) = t^{-1} \log(d)_t \) (where, as usual, \( (d)_t = d(d-1) \cdots (d-t+1) \)), and in particular \( \psi(d, d) = d^{-1} \log d! \). Thus Conjecture 5.1 for \( M = \ell \) is Bréguin’s Theorem, and the full conjecture is a natural generalization thereof which is sharp whenever \( G \) is a disjoint union of complete bipartite graphs \( K_{X_i, Y_i} \) (where \( X = \cup X_i \) and \( Y = \cup Y_i \)) with \( |X_i|/|Y_i| = \ell/M \forall i \). (Of course one can also think of this in the original setting of the Minc Conjecture, viewing it as a bound on the “generalized permanent” of a not-necessarily-square \( \{0, 1\} \)-matrix with given row sums.)

To be honest, we haven’t thought much about plausibility of Conjecture 3.1 in its full generality. We do feel pretty sure that it is true, for example, when \( d_x = d \) for every \( x \) and \( t := \ell d / M \) (the average of the \( d_y \)’s) is an integer. Curiously, we can prove this when \( d_y = t \forall y \), which ought to be the worst case, but so far not in general.

For an even wilder possibility, set, for \( r \geq 0 \) and \( 0 < t \leq 2^r \), \( \varphi(r, t) = t^{-1} [\log \Gamma(2^r + 1) - \log(2^r - t + 1)] \). Could it be that for \( G \) as in Conjecture 3.1 if random (but not necessarily uniform) from \( M_\ell(G) \), and \( t_x = (\ell/M)^2 H(f(x)) \) (\( x \in X \)), one has

\[
H(f) \leq \sum_{x \in X} \varphi(H(f(x)), t_x)? \tag{21}
\]

Notice that when \( f(x) \) is uniform from the neighbors of \( x \), \( 2^{H(f(x))} \) is just \( d_x \), so \( 21 \) strengthens \( 20 \). The case \( \ell = M \) of \( 21 \) was suggested in \[4\].

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