A sharp weighted anisotropic Poincaré inequality for convex domains

Francesco Della Pietra, Nunzia Gavitone, Gianpaolo Piscitelli

Università degli studi di Napoli Federico II, Dipartimento di Matematica e Applicazioni “R. Caccioppoli”
Via Cintia, Monte S. Angelo - 80126 Napoli, Italia.

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We prove an optimal lower bound for the best constant in a class of weighted anisotropic Poincaré inequalities.

1 INTRODUCTION

In this paper we prove a sharp lower bound for the optimal constant \( \mu_{p,H,\omega}(\Omega) \) in the Poincaré-type inequality

\[
\inf_{t \in \mathbb{R}} \| u - t \|_{L^p(\Omega)} \leq \frac{1}{\| \mu_{p,H,\omega}(\Omega) \|^p} \| \mathcal{K}(\nabla u) \|_{L^p(\Omega)},
\]

with \( 1 < p < +\infty \), \( \Omega \) is a bounded convex domain of \( \mathbb{R}^n \), \( \mathcal{K} \in \mathcal{K}(\mathbb{R}^n) \), where \( \mathcal{K}(\mathbb{R}^n) \) is the set of lower semicontinuous functions, positive in \( \mathbb{R}^n \setminus \{0\} \) and positively \( 1 \)-homogeneous; moreover, let \( \omega \) be a log-concave function.

If \( \mathcal{K} \) is the Euclidean norm of \( \mathbb{R}^n \) and \( \omega = 1 \), then \( \mu_p(\Omega) = \mu_{p,E,\omega}(\Omega) \) is the first nontrivial eigenvalue of the Neumann \( p \)-Laplacian:

\[
\begin{cases}
-\Delta_p u = \mu_p |u|^{p-2} u & \text{in } \Omega, \\
|\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega.
\end{cases}
\]

Then, for a convex set \( \Omega \) it holds that

\[
\mu_p(\Omega) \geq \left( \frac{\pi_p}{D_E(\Omega)} \right)^p,
\]

where

\[
\pi_p = 2 \int_0^{+\infty} \frac{1}{1 + \frac{1}{p-1} s^p} ds = 2\pi \left( \frac{p-1}{p} \right)^{\frac{1}{p}} \frac{\sin \frac{\pi}{p}}{\pi}, \quad D_E(\Omega) \text{ Euclidean diameter of } \Omega.
\]

This estimate, proved in the case \( p = 2 \) in [PW] (see also [B]), has been generalized the case \( p \neq 2 \) in [AD, ENT, FNT, V] and for \( p \to \infty \) in [EKNT, RS]. Moreover the constant \( \left( \frac{\pi_p}{D_E(\Omega)} \right)^p \) is the optimal constant of the one-dimensional Poincaré-Wirtinger inequality, with \( \omega = 1 \), on a segment of length \( D_E(\Omega) \). When \( p = 2 \) and \( \omega = 1 \), in [BCDL] an extension of the estimate in the class of suitable non-convex domains has been proved.

The aim of the paper is to prove an analogous sharp lower bound for \( \mu_{p,H,\omega}(\Omega) \), in a general anisotropic case. More precisely, our main result is:

*Email: f.dellapietra@unina.it, nunzia.gavitone@unina.it, gianpaolo.piscitelli@unina.it
Theorem 1.1. Let $\mathcal{H} \in \mathcal{H}(\mathbb{R}^n)$, $\mathcal{H}^0$ be its polar function. Let us consider a bounded convex domain $\Omega \subset \mathbb{R}^n$, $1 < p < \infty$, and take a positive log-concave function $\omega$ defined in $\Omega$. Then, given $\mu_{p, \mathcal{H}, \omega}(\Omega) = \inf_{u \in W^{1, \infty}(\Omega)} \int_{\Omega} |\nabla u|^p \omega \, dx$, it holds that

$$
\mu_{p, \mathcal{H}, \omega}(\Omega) \geq \left( \frac{\pi_p}{D_{\mathcal{H}, \omega}(\Omega)} \right)^p,
$$

where $D_{\mathcal{H}, \omega}(\Omega) = \sup_{x, y \in \Omega} \mathcal{H}^0(y - x)$.

This result has been proved in the case $p = 2$ and $\omega = 1$, when $\mathcal{H}$ is a strongly convex, smooth norm of $\mathbb{R}^n$ in [WX] with a completely different method than the one presented here.

In Section 2 below we give the precise definition of $\mathcal{H}^0$ and give some details on the set $\mathcal{H}(\mathbb{R}^n)$. In Section 3 we give the proof of the main result.

2 NOTATION AND PRELIMINARIES

A function

$$
\xi \in \mathbb{R}^n \mapsto \mathcal{H}(\xi) \in [0, +\infty[
$$

belongs to the set $\mathcal{H}(\mathbb{R}^n)$ if it verifies the following assumptions:

1. $\mathcal{H}$ is positively 1-homogeneous, that is
   if $\xi \in \mathbb{R}^n$ and $t \geq 0$, then $\mathcal{H}(t \xi) = t \mathcal{H}(\xi)$;

2. if $\xi \in \mathbb{R}^n \setminus \{0\}$, then $\mathcal{H}(\xi) > 0$;

3. $\mathcal{H}$ is lower semi-continuous.

If $\mathcal{H} \in \mathcal{H}(\mathbb{R}^n)$, properties (1), (2), (3) give that there exists a positive constant $a$ such that

$$
a|\xi| \leq \mathcal{H}(\xi), \quad \xi \in \mathbb{R}^n.
$$

The polar function $\mathcal{H}^0 : \mathbb{R}^n \to [0, +\infty[$ of $\mathcal{H} \in \mathcal{H}(\mathbb{R}^n)$ is defined as

$$
\mathcal{H}^0(\eta) = \sup_{\xi \neq 0} \frac{\langle \xi, \eta \rangle}{\mathcal{H}(\xi)}.
$$

The function $\mathcal{H}^0$ belongs to $\mathcal{H}(\mathbb{R}^n)$. Moreover it is convex on $\mathbb{R}^n$, and then continuous. If $\mathcal{H}$ is convex, it holds that

$$
\mathcal{H}(\xi) = (\mathcal{H}^0)^0(\xi) = \sup_{\eta \neq 0} \frac{\langle \xi, \eta \rangle}{\mathcal{H}^0(\eta)}.
$$

If $\mathcal{H}$ is convex and $\mathcal{H}(\xi) = \mathcal{H}(-\xi)$ for all $\xi \in \mathbb{R}^n$, then $\mathcal{H}$ is a norm on $\mathbb{R}^n$, and the same holds for $\mathcal{H}^0$.

We recall that if $\mathcal{H}$ is a smooth norm of $\mathbb{R}^n$ such that $\nabla^2(\mathcal{H}^2)$ is positive definite on $\mathbb{R}^n \setminus \{0\}$, then $\mathcal{H}$ is called a Finsler norm on $\mathbb{R}^n$.

If $\mathcal{H} \in \mathcal{H}(\mathbb{R}^n)$, by definition we have

$$
\langle \xi, \eta \rangle \leq \mathcal{H}(\xi) \mathcal{H}^0(\eta), \quad \forall \xi, \eta \in \mathbb{R}^n.
$$
Remark 2.1. Let $\mathcal{H} \in \mathcal{H}(\mathbb{R}^n)$, and consider the convex envelope of $\mathcal{H}$, that is the largest convex function $\overline{\mathcal{H}}$ such that $\overline{\mathcal{H}} \leq \mathcal{H}$. It holds that $\overline{\mathcal{H}}$ and $\mathcal{H}$ have the same polar function:

\[(\overline{\mathcal{H}})^o = \mathcal{H}^o \quad \text{in } \mathbb{R}^n.\]

Indeed, being $\overline{\mathcal{H}} \leq \mathcal{H}$, by definition it holds that $(\overline{\mathcal{H}})^o \geq \mathcal{H}^o$. To show the reverse inequality, it is enough to prove that $(\mathcal{H}^o)^o \leq \overline{\mathcal{H}}$. Then, being $\overline{\mathcal{H}}$ the convex envelope of $\mathcal{H}$, it must be $(\mathcal{H}^o)^o \leq \overline{\mathcal{H}}$, that implies $(\overline{\mathcal{H}})^o \leq \mathcal{H}^o$. Denoting by $G(x) = (\mathcal{H}^o)^o(x)$, for any $x$ there exists $\bar{v}_x$ such that

\[G(x) = \langle x, \bar{v}_x \rangle \quad \text{and} \quad (x, \bar{v}_x) \leq (\mathcal{H}^o)(\bar{v}_x)(x), \quad \text{that implies} \quad G(x) \leq \mathcal{H}(x).\]

Let $\mathcal{H} \in \mathcal{H}(\mathbb{R}^n)$, and consider a bounded convex domain $\Omega$ of $\mathbb{R}^n$. Throughout the paper $D_{\mathcal{H}}(\Omega) \in [0, +\infty[$ will be

\[D_{\mathcal{H}}(\Omega) = \sup_{x, y \in \Omega} (\mathcal{H}^o)(y - x).\]

We explicitly observe that since $\mathcal{H}^o$ is not necessarily even, in general $\mathcal{H}^o(y - x) \neq \mathcal{H}^o(x - y)$. When $\mathcal{H}$ is a norm, then $D_{\mathcal{H}}(\Omega)$ is the so called anisotropic diameter of $\Omega$ with respect to $\mathcal{H}^o$. In particular, if $\mathcal{H} = \mathcal{E}$ is the Euclidean norm in $\mathbb{R}^n$, then $\mathcal{E}^o = \mathcal{E}$ and $D_{\mathcal{E}}(\Omega)$ is the standard Euclidean diameter of $\Omega$. We refer the reader, for example, to [CS, FFK] for remarkable examples of convex not even functions in $\mathcal{H}(\mathbb{R}^n)$. On the other hand, in [VS] some results on isoperimetric and optimal Hardy-Sobolev inequalities for a general function $\mathcal{H} \in \mathcal{H}(\mathbb{R}^n)$ have been proved, by using a generalization of the so called convex symmetrization introduced in [AFLT] (see also [DG1, DG2, DG3]).

Remark 2.2. In general $\mathcal{H}$ and $\mathcal{H}^o$ are not rotational invariant. Anyway, if $A \in SO(n)$, defining

\[\mathcal{H}_A(x) = \mathcal{H}(Ax),\]

and being $A^T = A^{-1}$, then $\mathcal{H}_A \in \mathcal{H}(\mathbb{R}^n)$ and

\[(\mathcal{H}_A)^o(\xi) = \sup_{x \in \mathbb{R}^n \setminus \{0\}} \frac{\langle x, \xi \rangle}{\mathcal{H}_A(x)} = \sup_{y \in \mathbb{R}^n \setminus \{0\}} \frac{\langle A^Ty, \xi \rangle}{\mathcal{H}(y)} = \sup_{y \in \mathbb{R}^n \setminus \{0\}} \frac{\langle y, A\xi \rangle}{\mathcal{H}(y)} = (\mathcal{H}^o)_A(\xi).\]

Moreover,

\[D_{\mathcal{H}_A}(A^T \Omega) = \sup_{x, y \in A^T \Omega} (\mathcal{H}^o)_A(y - x) = \sup_{\bar{x}, \bar{y} \in \Omega} \mathcal{H}^o(\bar{y} - \bar{x}) = D_{\mathcal{H}}(\Omega).\]

3 PROOF OF THE PAYNE–WEINBERGER INEQUALITY

In this section we state and prove Theorem 1.1. To this aim, the following Wirtinger-type inequality, contained in [FNT] is needed.

Proposition 3.1. Let $f$ be a positive log-concave function defined on $[0, L]$ and $p > 1$, then

\[
\inf \left\{ \begin{array}{l}
\int_0^L |u|^p f \, dx \\
\int_0^L |u|^p g \, dx
\end{array} \right\}, \quad u \in W^{1,p}(0, L), \quad \int_0^L |u|^p - 2u \, f \, dx = 0 \quad \geq \quad \pi_p^p \frac{1}{L^p}.\]
The proof of the main result is based on a slicing method introduced in [PW] in the Laplacian case. The key ingredient is the following Lemma. For a proof, we refer the reader, for example, to [PW, B, FNT].

**Lemma 3.2.** Let $\Omega$ be a convex set in $\mathbb{R}^n$ having (Euclidean) diameter $D_{\Omega}(\Omega)$, let $\omega$ be a positive log-concave function on $\Omega$, and let $u$ be any function such that $\int_{\Omega} |u|^{p-2} u \omega \, dx = 0$. Then, for all positive $\varepsilon$, there exists a decomposition of the set $\Omega$ in mutually disjoint convex sets $\Omega_i$ ($i = 1, \ldots, k$) such that

$$\bigcup_{i=1}^k \Omega_i = \Omega$$

and for each $i$ there exists a rectangular system of coordinates such that

$$\Omega_i \subset \{(x_1, \ldots, x_n) \in \mathbb{R}^n : 0 \leq x_1 \leq d_i, |x_i| \leq \varepsilon, 1 = 2, \ldots, n\},$$

where $d_i \leq D_{\Omega}(\Omega)$, $i = 1, \ldots, k$.

**Proof of Theorem 1.1.** By density, it is sufficient to consider a smooth function $u$ with uniformly continuous first derivatives and $\int_{\Omega} |u|^{p-2} u \omega \, dx = 0$.

Hence, we can decompose the set $\Omega$ in $k$ convex domains $\Omega_i$ as in Lemma 3.2. In order to prove (1), we will show that for any $i \in \{1, \ldots, k\}$ it holds that

$$\int_{\Omega_i} H^p(\nabla u) \omega \, dx \geq \frac{\tau_{p,1}}{D_{\Omega}(\Omega)^p} \int_{\Omega_i} |u|^p \omega \, dx. \quad (5)$$

By Lemma 3.2, for each fixed $i \in \{1, \ldots, k\}$, there exists a rotation $A_i \in SO(n)$ such that

$$A_i \Omega_i \subset \{(x_1, \ldots, x_n) \in \mathbb{R}^n : 0 \leq x_1 \leq d_i, |x_i| \leq \varepsilon, 1 = 2, \ldots, n\}.$$

By changing the variable $y = A_i x$, recalling the notation (3) and using (4) it holds that

$$\int_{\Omega_i} \mathcal{H}^p(\nabla u(x)) \omega(x) \, dx = \int_{A_i \Omega_i} \mathcal{H}_{A_i}(\nabla u(A_i^T y))^p \omega(A_i^T y) \, dy; \quad D_{\mathcal{H}}(\Omega) = D_{\mathcal{H}_{A_i}}(A_i \Omega_i).$$

We deduce that it is not restrictive to suppose that for any $i \in \{1, \ldots, n\}$ $A_i$ is the identity matrix, and the decomposition holds with respect to the $x_1$–axis.

Now we may argue as in [FNT]. For any $t \in [0, d_i]$ let us denote by $v(t) = u(t, 0, \ldots, 0)$, and $f(t) = g(t) w(t, 0, \ldots, 0)$, where $g(t)$ will be the $(n-1)$ volume of the intersection of $\Omega_i$ with the hyperplane $x_1 = t$. By Brunn-Minkowski inequality $g(t)$, and then $f(t)$, is a log-concave function in $[0, d_i]$. Since $u, u_{x_1}$ and $\omega$ are uniformly continuous in $\Omega$ there exists a modulus of continuity $\eta(\cdot)$ with $\eta(\varepsilon) \downarrow 0$ for $\varepsilon \to 0$, independent of the decomposition of $\Omega$ and such that

$$\left| \int_{\Omega_i} |u_{x_1}|^p \omega \, dx - \int_0^{d_i} |v|^p f_i \, dt \right| \leq \eta(\varepsilon)|\Omega_i|,$$  

$$\left| \int_{\Omega_i} |u|^p \omega \, dx - \int_0^{d_i} |v|^p f_i \, dt \right| \leq \eta(\varepsilon)|\Omega_i|,$$

and

$$\left| \int_0^{d_i} |v|^p-2 v f_i \, dt \right| \leq \eta(\varepsilon)|\Omega_i|.$$

Now, by property (2) we deduce that for any vector $\eta \in \mathbb{R}^n$

$$\left| (\nabla u, \eta) \right| \leq \mathcal{H}((\nabla u) \max(\mathcal{H}_o(\eta), \mathcal{H}_o(-\eta))).$$
Then choosing $\eta = e_1$ and denoting by $M = \max\{\mathcal{H}^\circ(e_1), \mathcal{H}^\circ(-e_1)\}$, Proposition 3.1 gives
\[
\int_{\Omega} \mathcal{H}^p(\nabla u)\omega \, dx \geq \frac{1}{M^p} \int_{\Omega} |u_{x_1}|^p \omega \, dx \geq \frac{1}{M^p} \int_0^{d_1} |v'|^p f_1 \, dt - \frac{\eta(\epsilon)|\Omega|}{M^p} \\
\geq \frac{\pi_p}{d_1^p M^p} \int_0^{d_1} |v'|^p f_1 \, dt + C\eta(\epsilon)|\Omega| \geq \frac{\pi_p}{d_1^p M^p} \int_{\Omega} |u|^p \omega \, dx + C\eta(\epsilon)|\Omega|,
\]
where $C$ is a constant which does not depend on $\epsilon$. Being $d_1 \leq D(\Omega)$, and then $d_1 M \leq D(\Omega)$, by letting $\epsilon$ to zero we get (5). Hence, by summing over $i$ we get the thesis.

**Remark 3.3.** In order to prove an estimate for $\mu_{p,\mathcal{H},\omega}$, we could use directly property (2) with $v = |\nabla u|$, and the Payne-Weinberger inequality in the Euclidean case, obtaining that
\[
\int_{\Omega} \mathcal{H}^p(\nabla u)\omega \, dx \geq \int_{\Omega} \frac{|\nabla u|^p}{\mathcal{H}^\circ(v)^p} \omega \, dx \geq \frac{\pi_p}{D(\Omega)^p \mathcal{H}^\circ(v_m)^p} \int_{\Omega} |u|^p \omega \, dx,
\]
where $\mathcal{H}^\circ(v_m) = \max_{|v|=1} \mathcal{H}^\circ(v)$. However, we have a worst estimate than (1) because $D(\Omega) \cdot \mathcal{H}^\circ(v_m)$ is, in general, strictly larger than $D(\Omega)$, as shown in the following example.

**Example 1.** Let $\mathcal{H}(x,y) = \sqrt{a^2 x^2 + b^2 y^2}$, with $a < b$. Then $\mathcal{H}$ is a even, smooth norm with $\mathcal{H}^\circ(x,y) = \sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2}}$ and the Wulff shapes $\{\mathcal{H}^\circ(x,y) < R\}$, $R > 0$, are ellipses. Clearly we have:
\[
D(\Omega) = 2b \quad \text{and} \quad D(\Omega) = 2
\]
Let us compute $\mathcal{H}^\circ(v_m)$. We have:
\[
\max_{|v|=1} \mathcal{H}^\circ(v) = \max_{\theta \in [0,2\pi]} \sqrt{\frac{(\cos \theta)^2}{a^2} + \frac{(\sin \theta)^2}{b^2}} = \mathcal{H}^\circ(0, \pm 1) = \frac{1}{a}.
\]
Then $D(\Omega) \cdot \mathcal{H}^\circ(v_m) = \frac{b}{a} > 2$.

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