The Point-to-Set Principle, the Continuum Hypothesis, and the Dimensions of Hamel Bases

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Abstract

We prove that the Continuum Hypothesis implies that every real number in $(0,1]$ is the Hausdorff dimension of a Hamel basis of the vector space of reals over the field of rationals.

The logic of our proof is of particular interest. The statement of our theorem is classical; it does not involve the theory of computing. However, our proof makes essential use of algorithmic fractal dimension—a computability-theoretic construct—and the point-to-set principle of J. Lutz and N. Lutz (2018).

1 Introduction

This brief paper is an intellectual export from the theory of computing to classical mathematics, i.e., mathematics not ostensibly involving the theory of computing. This introduction describes our main theorem and then explains how its proof uses computability theory.

Two fundamental theorems of linear algebra state that every vector space has a basis and that any two bases of a vector space have the same cardinality, which is called the dimension of the vector space. When the vector space has finite dimension, the proofs of these facts are completely constructive and are standard undergraduate fare [1, 4]. However, in 1905, Hamel [6] used the axiom of choice to prove that these two theorems hold for all vector spaces. Accordingly, infinite bases of vector spaces are traditionally called Hamel bases.

The canonical case of Hamel bases is the vector space $\mathbb{R}$ of real numbers over the field $\mathbb{Q}$ of rational numbers. This case is the topic of the present paper, so here as in many papers, all “Hamel bases” are bases of $\mathbb{R}$ over $\mathbb{Q}$. Hamel [6] showed that every Hamel basis must have the cardinality of the continuum. Sierpinski [21] showed that every Hamel basis has inner Lebesgue measure 0, whence every measurable Hamel basis has measure 0. Jones [7] showed that the Cantor middle-thirds set contains a Hamel basis (which thus has Lebesgue measure 0) and that no Hamel basis is analytic, i.e., $\Sigma_1^1$. Numerous other investigations of Hamel bases have appeared in the literature.

In this paper we investigate the Hausdorff dimensions of Hamel bases. Our main theorem, which assumes the Continuum Hypothesis, says that, for every real number $s \in (0,1]$, there is a Hamel basis with Hausdorff dimension exactly $s$.

Although our main theorem says nothing about the theory of computing, our proof of it uses algorithmic fractal dimension, which is a computability-theoretic construct. Specifically, the algorithmic dimension of a number $x \in \mathbb{R}$ is

$$\dim(x) = \liminf_{n \to \infty} K(x\lceil 0..n-1 \rceil)/n, \quad (1)$$

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1 In 1984, confirming a 1908 conjecture of Zermelo [23], Blass [2] proved that the existence of bases for all vector spaces implies the axiom of choice.
where $x[0..n-1]$ is the string consisting of the first $n$ bits of the binary expansion of $x$, and $K(x[0..n-1])$ is the Kolmogorov complexity of this string [9,16]. Since $K(x[0..n-1])$ is the algorithmic information content of $x[0..n-1]$ (and is where computability theory comes into the picture [20,13,8]), this says that $\dim(x)$ is the lower asymptotic algorithmic information density of the real number $x$. This and related algorithmic fractal dimensions are discussed in the recent survey [12].

The bridge between the above algorithmic dimension and our investigation of the Hausdorff dimensions of Hamel bases is the Point-to-Set Principle of J. Lutz and N. Lutz [10]. This principle gives complete pointwise characterizations of various classical fractal dimensions in terms of relativizations of their algorithmic counterparts. Specialized to the case of Hausdorff dimension in $\mathbb{R}$, this principle says that, for every set $E \subseteq \mathbb{R},$

$$\dim_H(E) = \min_{A \subseteq \mathbb{N}} \sup_{x \in E} \dim^A(x). \quad (2)$$

That is, the classical Hausdorff dimension of a set $E$ is the minimum, for all oracles $A$, of the supremum of the algorithmic dimensions, relative to $A$, of the individual points in $E$.

The Point-to-Set Principle is especially useful for proving otherwise difficult lower bounds on the Hausdorff dimensions of sets. It implies that, to prove a lower bound on the Hausdorff dimension of a set $E$, it suffices to let $A$ be an arbitrary oracle and prove a corresponding lower bound on the algorithmic dimension, relative to $A$, of a judiciously chosen point in $E$. This ability to reason from the relativized dimensions of single points to the dimensions of entire sets has recently been used to prove several new theorems about classical fractal dimensions. Examples include stronger lower bounds on the Hausdorff dimensions of generalized Furstenberg sets [15]; extension of the fractal intersection formula from Borel sets to arbitrary sets [13]; the extension of Marstrand’s projection theorem from analytic sets to arbitrary sets, provided that their Hausdorff and packing dimensions coincide [14]; and a proof that, if $V = L$, then the maximal thin co-analytic set has Hausdorff dimension 1 [22].

An encouraging aspect of the above successes of the Point-to-Set Principle is that the proofs do not all look alike. The principle has turned out to be quite flexible, allowing investigators to invoke it in arguments appropriate to various settings. In the present paper we use it in the following way. Given a target dimension $s \in (0,1]$, we construct a Hamel basis $B$ of $\mathbb{R}$ over $\mathbb{Q}$ by a transfinite recursion, putting a single real into $B$ at each stage. By the Continuum Hypothesis (and the axiom of choice in the guise of the well-ordering principle), the recursion only needs to go up to $\omega_1$, the first uncountable ordinal. At even stages, we add points to $B$ that enable us to use the Point-to-Set Principle to conclude that $B$ has Hausdorff dimension $s$. At odd stages, we ensure that $B$ spans $\mathbb{R}$. The details appear in Section 3.

## 2 Preliminaries

All logarithms here are base-2. For $B \subseteq \mathbb{R}$, we write $\text{span}(B)$ for the linear span of $B$ over $\mathbb{Q}$, i.e., the set of all linear combinations

$$x = \sum_{u \in I} g(u)u,$$

where $I \subseteq B$ is finite and $g : I \rightarrow \mathbb{Q}\{0\}$.

We refer to standard set theory texts, e.g., [5,17] or early sections of more advanced texts, for background on ordinal numbers and transfinite recursion. We write $\omega$ for the first infinite ordinal.
and $\omega_1$ for the first uncountable ordinal. The Continuum Hypothesis says that

$$|\mathcal{P}(\mathbb{N})| = |\mathbb{R}| = \omega_1.$$ 

Fix a real number $r \in (0, \frac{1}{2})$. Define a family $\{I_w \mid w \in \{0,1\}^*\}$ of intervals $I_w \subseteq [0,1]$ by the following recursion.

$I_{\lambda} = [0,1]$, where $\lambda$ is the empty string.
If $I_w = [a,b]$ then $I_{w0} = [a, a + r(b-a)]$ and $I_{w1} = [b - r(b-a), b]$.

For each $k \in \mathbb{N}$, let

$$C_{r,k} = \bigcup_{w \in \{0,1\}^k} I_w,$$

and let

$$C_r = \bigcap_{k=0}^\infty C_{r,k}.$$ 

Then $C_r$ is a “Cantor middle 1 - 2r set,” and $C_{\frac{1}{3}} = C$ is the familiar Cantor middle-thirds set. By standard methods, $C_r$ has Hausdorff dimension

$$\dim_H(C_r) = \frac{-1}{\log r}.$$ 

Writing $C + C = \{x + y \mid x, y \in C\}$ for the sumset of $C$ with itself, it is well known that $C + C = [0,2]$. This holds because an easy induction shows that $C_{\frac{1}{3}} + C_{\frac{1}{3}} = [0,2]$ holds for every $k \in \mathbb{N}$. Moreover, it implies that $\text{span}(C) = \mathbb{R}$. More generally, if $r = \frac{1}{m}$, where $m \geq 3$ is an integer, a similar argument shows that the $(m-1)$-fold sumset $C_r + ... + C_r$ is exactly $[0, m-1]$, so $\text{span}(C_r) = \mathbb{R}$ also holds. Finally, if $r \geq \frac{1}{m}$, where $m \geq 3$ is an integer, then the $(m-1)$-fold sumset $C_r + ... + C_r$ contains $[0, m-1]$, so we again have $\text{span}(C_r) = \mathbb{R}$.

For each $r \in (0, \frac{1}{2})$ and each $s \in [0, \dim_H(C_r)] = [0, \frac{1}{\log r}]$, the analysis of algorithmic dimensions in self-similar fractals in [11] shows that the set $C_r \cap \text{DIM}_s$, consisting of all points $x \in C_r$ with $\dim(x) = s$, has the cardinality of the continuum. Moreover, this result relativizes, so for all $A \subseteq \mathbb{N}$, the set $C_r \cap \text{DIM}_s^A$, consisting of all points $x \in C_r$ with $\dim^A(x) = s$, also has the cardinality of the continuum. We use this fact in Section 3.

### 3 Dimensions of Hamel Bases

In this section we prove our main theorem, which is the following.

**Theorem 3.1** If the Continuum Hypothesis holds, then, for each $s \in (0,1]$, there is a Hamel basis $B$ of $\mathbb{R}$ over $\mathbb{Q}$ such that $\dim_H(B) = s$.

**Proof** Assume the Continuum Hypothesis, and let $s \in (0,1]$. Then there is a wellordering

$$(A_\alpha \mid \alpha < \omega_1)$$

of $\mathcal{P}(\mathbb{N})$. Let $r = 2^{-\frac{1}{2}}$, and let $C_r$ be the Cantor set defined in Section 2. Then

$$\dim_H(C_r) = s,$$
and there is a wellordering
\[(z_\beta \mid \beta < \omega_1)\]
of \(C_r\). For each \(A \subseteq \mathbb{N}\), the set
\[D^A = \{x \in C_r \mid \dim^A(x) = s\}\]
has the cardinality of the continuum (as noted in Section 2), so there is a wellordering
\[(x^A_\gamma \mid \beta < \omega_1)\]
of \(D^A\).

It has been known at least since [18] (where it is an exercise) that the Continuum Hypothesis is equivalent to the existence of a sequence
\[(A_\xi \mid \xi < \omega_1)\]
of oracles \(A_\xi \subseteq \mathbb{N}\) that is \(\leq_T\)-nondecreasing in the sense that, for all \(\alpha < \omega_1\) and \(\xi < \omega_1\),
\[\alpha < \xi \implies A_\alpha \leq_T A_\xi\]
and cofinal in the sense that, for every \(A \subseteq \mathbb{N}\), there exists \(\xi < \omega_1\), such that \(A \leq_T A_\xi\).

Define the sequence
\[(u_\xi \mid \xi < \omega_1)\]
of real numbers by the following transfinite recursion. Given \(\xi < \omega_1\), let
\[B_\xi = \{u_\eta \mid \eta < \xi\}.\]

Write \(\xi = \lambda + k\), where \(\lambda\) is 0 or a limit ordinal and \(k < \omega\). Call \(\xi\) even if \(k\) is even, and odd if \(k\) is odd.

- If \(\xi\) is even, let \(u_\xi = x^A_\xi\) be the first element of \(D^A_\xi \setminus \text{span}(B_\xi)\).
- If \(\xi\) is odd, let \(u_\xi = z_\beta\) be the first element of \(C_r \setminus \text{span}(B_\xi)\).

To see that this recursion is well defined, let \(\xi < \omega_1\). Then \(B_\xi\) is countable, so \(\text{span}(B_\xi)\) is countable. Since \(\dim s^A_\xi\) and \(C_r\) have the cardinality of the continuum, it follows that \(u_\xi\) exists.

Let
\[B = \{u_\xi \mid \xi < \omega_1\}.\]
The rest of this proof establishes that \(B\) is a Hamel basis of \(\mathbb{R}\) over \(\mathbb{Q}\) and that \(\dim_H(B) = s\).

We first show that \(\text{span}(B) = \mathbb{R}\). For this, since we saw in Section 2 that \(\text{span}(C_r) = \mathbb{R}\), it suffices to show that \(C_r \subseteq \text{span}(B)\), i.e., that the set \(E = C_r \setminus \text{span}(B)\) is empty. To this end, let \(z\) be a lower bound of \(E\) in our wellordering of \(C_r\). That is, let \(z = z_\beta\), where \(\beta < \omega_1\) and \(z_{\beta'} \in \text{span}(B)\) for all \(\beta' < \beta\). Since \(\beta < \omega_1\) and only finitely many elements of \(B\) are required to put each \(z_{\beta'}\) into \(\text{span}(B)\), there exists \(\xi < \omega_1\) such that
\[\{z_{\beta'} \mid \beta' < \beta\} \subseteq \text{span}(B_\xi)\]
Moreover, we can insist here that \(\xi\) be odd. Then \(u_\xi = z_\beta = z\), so \(z \in B \subseteq \text{span}(B)\), so \(z \notin E\). We have now shown that no lower bound of \(E\) is an element of \(E\), i.e., that \(E\) has no least element in our wellordering. This implies that \(E = \emptyset\), completing our proof that \(\text{span}(B) = \mathbb{R}\).
Since our construction ensures that \( u_\xi \notin \text{span}(B_\xi) \) holds for all \( \xi < \omega_1 \), we now have that \( B \) is a Hamel basis of \( \mathbb{R} \) over \( \mathbb{Q} \).

We conclude this proof by showing that \( \dim_H(B) = s \). It is clear that \( \dim_H(B) \leq s \), since \( B \subseteq C_r \) and \( \dim_H(C_r) = s \). Hence it suffices to show that \( \dim_H(B) \geq s \). For this, let \( A \subseteq \mathbb{N} \). By the cofinality of \( (A_\xi \mid \xi < \omega_1) \), there exists \( \xi < \omega_1 \) such that \( A \leq_T A_\xi \). Moreover, we can insist here that \( \xi \) be even. Let \( u(A) = u_\xi \). Then \( u(A) \in D^{A_\xi} \) and \( \dim^{A_\xi}(u(A)) = s \). Since \( A \leq_T A_\xi \), it follows that \( \dim^A(u(A)) \geq s \). This shows that, for every \( A \subseteq \mathbb{N} \), there is a real number \( u(A) \in B \) such that \( \dim^A(u(A)) \geq s \). It follows by the Point-to-Set Principle that

\[
\dim_H(B) = \min_{A \subseteq \mathbb{N}} \sup_{u \in B} \dim^A(u) \\
\geq \min_{A \subseteq \mathbb{N}} \dim^A(u(A)) \\
= s.
\]

\[\square\]

4 Conclusion

Assuming the Continuum Hypothesis, we have used the theory of computing, via the Point-to-Set Principle, to prove the existence of Hamel bases with arbitrary Hausdorff dimensions in \((0,1]\). At the time of this writing, we do not know whether there is a Hamel basis of dimension 0, and we do not know whether our theorem can be proven without using the Continuum Hypothesis.

Orponen \[19\] has already found classical proofs of two projection theorems of N. Lutz and Stull \[14\] that were first proven via the Point-to-Set Principle, and he generalized these theorems in the process. We conjecture that our theorem on Hamel bases also admits a classical proof. More generally, we look forward to a better understanding of the power and limitations of computability-theoretic methods for discovering proofs of new theorems in classical mathematics.

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