SEQUENCES OF EMBEDDED MINIMAL DISKS WHOSE CURVATURES BLOW UP ON A PRESCRIBED SUBSET OF A LINE

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INTRODUCTION

In this paper, we prove the following result about convergence of properly embedded minimal disks to a lamination:

Theorem 1. Let

\[ C = B(0, 1) \times (a, b) \subset \mathbb{R}^3 \]

be an open solid circular cylinder, possibly infinite, that is rotationally symmetric about the z-axis, Z. Let K be a relatively closed subset of Z ∩ C. Then there exists a sequence of minimal disks that are properly embedded in C and that have the following properties:

1. The curvatures of the disks are uniformly bounded on compact subsets of C \ K and blow up on K.\(^1\)

2. The minimal disks converge to a limit lamination of C \ K consisting of the following leaves:
   (i) For each p ∈ K, the horizontal punctured unit disk centered at p:
       \[ D_p = (C \setminus Z) \cap \{ z = z(p) \} \]
   (ii) For each component J of (Z \ K) ∩ C, a leaf \( \mathcal{M}_J \) that is properly embedded in the cylinder
       \[ B(0, 1) \times z(J) \]
       The leaf \( \mathcal{M}_J \) contains the segment J and is therefore symmetric under rotation by 180° around Z. Each of the two connected components of \( \mathcal{M}_J \setminus J \) is an infinitely-sheeted multigraph over the punctured unit disk, and \( \partial z/\partial \theta > 0 \) everywhere on \( \mathcal{M}_J \setminus J \).

3. The lamination extends smoothly to a lamination of C \ ∂K, but it does not extend smoothly to any point in C ∩ ∂K.

\(\)\(^1\)The curvatures of a sequence of minimal disks \( D_n \) blow up at a point p if there exists a sequence \( \{ p_n \in D_n \} \) with \( p_n \to p \) such that the absolute value of the curvature of \( D_n \) at \( p_n \) tends to infinity as \( n \to \infty \).
Here \( z(J) \) denotes the image of \( J \) under the coordinate map \( z : \mathbb{R}^3 \to \mathbb{R} \).

Note that according to Statement 2ii, each of the components of \( M_J \setminus J \) can be parametrized as
\[
(r, \theta) \in (0, 1) \times \mathbb{R} \mapsto (r \cos \theta, r \sin \theta, f(r, \theta)).
\]
Furthermore, if we let \((c, d) = z(J)\), then (by properness)
\[
\lim_{\theta \to -\infty} f(r, \theta) = c,
\]
\[
\lim_{\theta \to \infty} f(r, \theta) = d,
\]
the convergence being uniform away from \( r = 0 \).

Theorem 1 is well-known in case \( K \) is the entire interval \( Z \cap C \): let the \( n \)th disk be the standard helicoid scaled by a factor of \( 1/n \) and restricted to \( C \). Theorem 1 was proved by Colding and Minicozzi \cite{CM04} when \( K \) consists of a single point, by Brian Dean \cite{Dea06} when \( K \) is an arbitrary finite set of points, and very recently by Siddique Kahn \cite{Kah08} when \( K \) is an interval with exactly one endpoint in \( C \). Our work was inspired by Kahn’s result, although the methods are very different: Kahn, like Colding, Minicozzi, and Dean, used the Weierstrass Representation, whereas our approach is variational.

In general, suppose that for each \( n \) we have a minimal disk \( D_n \) that is properly embedded in an open subset \( U_n \) of \( \mathbb{R}^3 \), where \( U_n \subset U_{n+1} \) for each \( n \). Let \( U = \cup_n U_n \).

By passing to a subsequence, we may suppose that there is a relatively closed subset \( K \) of \( U \) such that the curvatures of the \( D_n \) blow up at all points of \( K \) and such that the \( D_n \) converge smoothly in \( U \setminus K \) to a limit lamination \( L \) of \( U \setminus K \). It is natural to ask how general the set \( K \) can be, and to what extent the lamination and/or the leaves of the lamination can be smoothly extended to include points in \( K \). In particular,

Q1. Must the blow-up set \( K \) be contained in a Lipschitz curve? In a \( C^{1,1} \) curve?
Q2. If \( p \in K \), must there be a leaf of the lamination that extends smoothly across \( p \)?
Q3. Given an arbitrary closed subset \( S \) of a Lipschitz (or a \( C^{1,1} \)) curve in the open set \( U \), is there an example for which which the blow-up set \( K \) is precisely \( S \)?

Answers to these questions are known in some special cases.

Colding and Minicozzi proved that if \( U = \mathbb{R}^3 \) (the so-called “global case”) and if \( K \) is nonempty, then (after a rotation) \( K \) is the graph of a lipschitz function \( z \in \mathbb{R} \mapsto (x(z), y(z)) \) and the lamination is the foliation by horizontal planes punctured at the points of \( K \) \cite[Theorem 0.1]{CM04}. In particular, the lamination extends smoothly to all of \( \mathbb{R}^3 \). By \cite{Mee04} (described below), the curve is in fact a straight line that is perpendicular to the planes.

In the local case \( U \neq \mathbb{R}^3 \), the behavior can be very different, as Theorem 1 indicates. Meeks proved that if the lamination extends smoothly to a foliation of \( U \) and if \( K \) is a Lipschitz curve that intersects the leaves transversely, then \( K \) is a \( C^{1,1} \).
curve and it intersects the leaves orthogonally [Mee04]. Meeks and Weber [MW07] constructed an example for which $U = \mathbb{R}^3 \setminus Z$, the blow-up set $K$ is a horizontal circle centered at a point in $Z$, and the limit lamination consists of the vertical half-planes with $Z$ as edge, punctured by $K$. They go on to prove that, given any $C^{1,1}$ curve, there is an example in which the blow-up set $K$ is that curve, $U$ is tubular neighborhood of $K$, and the lamination is a foliation of $U$ by planar punctured disks orthogonal to $K$.

In the study of minimal varieties, the known examples of singularities have been rather tame. In particular, we believe that Theorem 1 provides the first examples of Cantor sets of singularities and of singular sets with non-integer Hausdorff dimension.

Our paper is organized as follows. We prove Theorem 1 in the next section. The proof depends on results in Sections 2 and 3. In Section 2 we prove existence and uniqueness theorems for embedded minimal disks with certain rotationally symmetric boundaries. In Section 3 we use Rado’s Theorem to deduce curvature estimates for our examples.

1. The proof of Theorem 1

Proof. It suffices to prove the theorem for the unbounded cylinder $C = B(0, 1) \times \mathbb{R}$: the case of bounded cylinders follows by restriction.

For each connected component $J$ of $Z \setminus K$, choose a smooth embedded curve $S_J$ in $\partial B(0, 1) \times J$ such that

1. The projection $(x, y, z) \mapsto (0, 0, z)$ induces a diffeomorphism from $S_J$ to $J$.
2. The derivative $d\theta/dz$ is strictly positive at each point of $S_J$. (Here $\theta$ is the angle of the cylindrical coordinate of a point in $\mathbb{R}^3 \setminus Z$. Of course, $\theta$ is defined only up to an integer multiple of $2\pi$, but $d\theta/dz$ is well defined.)
3. The curve $S_J$ winds around the cylinder infinitely many times as $z \to c$ and as $z \to d$, where $c$ and $d$ are the infimum and supremum, respectively, of $z$ on $J$.

In other words, $\lim_{z \to -c} \theta(z) = -\infty$ and $\lim_{z \to d} \theta(z) = \infty$.

Now choose numbers $a_n \to -\infty$ and $b_n \to \infty$, and choose smooth curves $\gamma_n$ in $\partial B(0, 1) \times [a_n, b_n]$ such that

(i) The curve $\gamma_n$ can be smoothly parametrized by $z \in [a_n, b_n]$, (That is, $(x, y, z) \mapsto z$ induces a diffeomorphism from $\gamma_n$ to $[a_n, b_n]$.)
(ii) The derivative $d\theta/dz$ is strictly positive at each point of $\gamma_n$.
(iii) The $\gamma_n$ converge smoothly to the lamination of $\partial C$ consisting of the $S_J$’s together with the horizontal circles of radius 1 centered at points of $K$.

Let $\Gamma_n$ be the simple closed curve consisting of $\gamma_n$ and $Z \cap \{a_n \leq z \leq b_n\}$, together with two radial segments at heights in $z = a_n$ and $z = b_n$. By Theorem 2 $\Gamma_n$ bounds a unique embedded minimal disk $D_n$. 
Extend $D_n$ by repeated Schwartz reflection in the the top and edges to get an infinite minimal strip $\tilde{D}_n$. The boundary of $\tilde{D}_n$ has two components: the axis $Z$, and the helix-like curve $\tilde{\gamma}_n$ in $\partial C$ obtained from $\gamma_n$ by iterated reflection.

To ensure that the boundary curve $\tilde{\gamma}_n$ is smooth, we need to impose one additional condition on the choice of $\gamma_n$:

(iv) The even-order derivatives $d^k\theta/dz^k$ ($k = 2, 4, \ldots$) vanish at the endpoints of $\gamma_n$.

(If this condition were not satisfied, then the curve $\tilde{\gamma}_n$ would not be smooth at the endpoints of $\gamma_n$.)

By Theorem 3, $\tilde{D}_n$ and its rotated images $R_{\theta}\tilde{D}_n$ foliate $C \setminus Z$. Thus if $D^*_n = \tilde{D}_n \cup \rho_Z\tilde{D}_n \cup Z$ is the minimal surface obtained from $\tilde{D}_n$ by Schwartz reflection in $Z$, then $D^*_n$ is embedded. Here $\rho_Z = R_\pi$ denotes rotation by $\pi$ about $Z$.

By Theorem 3 applied to $D_n$, each of the two connected components of $D^*_n \setminus Z$ is a multigraph over $B(0,1) \setminus \{0\}$. (Of course those components are $\tilde{D}_n$ and $\rho_Z\tilde{D}_n$.) Thus if $H$ is an open halfspace bounded by a plane containing $Z$, then $H \cap D^*_n$ is a union of graphs over a half-disk. Standard estimates for solutions of the minimal surface equation then imply that the principal curvatures of the closures of the $D^*_n$ are uniformly bounded on compact subsets of $H$. Thus after passing to a subsequence, those graphs will converge smoothly (away from $\partial H$) to a collection $G_H$ of minimal graphs over a half disk. Note that if $G$ is such a graph, then $(\partial G) \cap H$ must be one of the following:

(H1) a horizontal semicircle at height $z(p)$ for some $p \in K$, or
(H2) one of the connected components of $S_J \cap H$ or one of the connected components of $(\rho_Z S_J) \cap H$, where $J$ is a connected component of $Z \setminus K$.

Furthermore,

(H3) If $\Gamma$ is a connected component of $S_J \cap H$ or of $(\rho_Z S_J) \cap H$, then there is exactly one graph $G$ in $G_H$ whose boundary (in $H$) is $\Gamma$.

Since $H$ is arbitrary, this means that (after passing to a subsequence) the $D^*_n \setminus Z$ will converge smoothly on compact subsets of $R^3 \setminus Z$ to a lamination $L$ of $C \setminus Z$ consisting of a union of multigraphs. Because each leaf is embedded, it must either have a single sheet (and thus be a graph) or else have infinitely many sheets.

Let $J$ be a component of $Z \setminus K$. By Theorem 4 the principal curvatures of the $D^*_n$ are uniformly bounded on compact subsets of $B(0,1) \times z(J)$. Since the principal curvatures are also uniformly bounded on compact subsets of $R^3 \setminus Z$, this means that the curvatures are in fact uniformly bounded on compact subsets of $R^3 \setminus K$. Thus (perhaps after passing to a further subsequence) the $D^*_n$ converge smoothly on compact subsets of $C \setminus K$ to a lamination $L'$ of that region. Of course $L$ is the restriction of $L'$ to the gutted cylinder $C \setminus Z$. 
Claim 1. The horizontal circle $C_p$ of radius 1 centered at a point $p \in K$ bounds a unique leaf of $D_p \in \mathcal{L}$. That leaf is the planar punctured disk bounded by $C_p$.

Proof of Claim 1. As above, let $H$ be an open halfspace of $\mathbb{R}^3$ with $Z \subset \partial H$. Then $\mathcal{L}' \cap H$ is a union of minimal graphs over a half-disk. Consider the set $Q$ of those graphs that contain $C_p \cap H$ in their boundaries. That set is compact, so there is an uppermost graph $G_H$ in $Q$. Note that the union of the $G_H$ as $H$ varies (by rotating it around $Z$) forms a single smooth minimal graph $G$ over the punctured disk $B(0,1) \setminus \{0\}$. That graph satisfies the minimal surface equation. As a minimal surface in $\mathbb{R}^3$, the boundary of $G$ is the circle $C_p$ together with some or all of $Z$. Since a solution to the minimal surface equation cannot have an isolated interior singularity, the graph extends to a regular minimal surface over $B(0,1)$. (See Theorem 10.2 of [Oss86]. The result is originally due to Bers [Ber51], but the proof in [Oss86] using catenoidal barriers is due to Finn [Fin65].) Since the boundary values define a planar circle, the graph must be a flat disk, so $G$ is a flat planar disk. Recall that $G$ is the uppermost leaf of $\mathcal{L}$ that contains $C_p$. By the same argument, it is also the lowermost leaf of $\mathcal{L}$ containing $C_p$. Thus it is the unique leaf in $\mathcal{L}$ containing $C_p$. □

Claim 2. If $p \in K$, then there is a sequence $p_n \in D_n^*$ converging to $p$ such that the norm of the second fundamental form of $D_n^*$ at $p_n$ tends to infinity.

Proof of Claim 2. Suppose not. Then there is a ball $B \subset \mathbb{R}^3$ centered at $p$ and a subsequence of the $D_n^*$ (which we may take to be the original sequence) such that the curvatures of the $D_n^*$ are uniformly bounded in $B$. It follows that the lamination $\mathcal{L}'$ extends smoothly to a lamination of $B$ and that the convergence of $D_n^* \cap B$ to the lamination of $B$ is smooth. By Claim 1, the leaf containing $p$ is a horizontal disk. But each $D_n^*$ contains the axis $Z$, and therefore has a vertical tangent plane at $p$. Hence the convergence cannot be smooth. This contradiction proves the claim. □

We have now completely established Statement 1 of the Theorem and we have established that there is a limit lamination of $C \setminus K$. We also know that for each $p \in K$, the punctured disk $D_p$ is a leaf of this lamination. Thus we have proved Statement 2.

To prove Statement 2, let $J = \{p \in Z : c < z(p) < d\}$ be one of the components of $Z \setminus K$. We now analyze the leaves of the foliation that lie between the punctured disks at heights $c$ and $d$. Let $\mathcal{M}_J^+$ and $\mathcal{M}_J^-$ be the leaves of $\mathcal{L}$ that contain $S_J$ and $\rho_Z(S_J)$, respectively. Note that by (H1) and (H2), these are both infinite covers of $B(0,1) \setminus \{0\}$. By (H3), there are no other leaves of $\mathcal{L}$ in the region $\{c < z < d\}$.

Note that $J \subset D_n^*$ for every $n$, so $J$ is contained in one of the leaves $\mathcal{M}_J$ of the lamination $\mathcal{L}'$. Now $\mathcal{M}_J$ is simply connected since each $D_n^*$ is simply connected, so $\mathcal{M}_J \setminus Z$ must contain two components. These components are leaves of $\mathcal{L}$, so they
must be $M^+_J$ and $M^-_J$. Thus $M_J = J \cup M^+_J \cup M^-_J$ is the unique leaf of $\mathcal{L}'$ in the slab $\{c < z < d\}$.

Let $H$ be an open halfspace of $\mathbb{R}^3$ bounded by a plane containing $Z$. Note that the components of $M_J \cap H$ form a countable discrete set corresponding to the countable discrete set of components of $(S_J \cup \rho_Z S_J) \cap H$. Thus $M_J$ is not a limit leaf of the foliation $\mathcal{L}'$.

It follows that $M_J$ is properly embedded in $\Omega := B(0, 1) \times (c, d)$. For if not, $\mathcal{L}'$ would have a limit leaf in $\Omega$. But the only leaf in $\Omega$ is $M_J$ itself, which, as we have just seen, is not a limit leaf. This proves properness.

Next we show that $\partial z/\partial \theta > 0$ on $M \setminus J$. (The partial derivative makes sense because $M \setminus J$ is locally a graph.) Since the two components of $M \setminus J$ are related by the $\rho_Z$ symmetry, it suffices to show that $\partial z/\partial \theta > 0$ on the component $M = M^+_J$.

Let $\nu$ be the unit normal vector field on $M$ given by

$$\nu := \frac{(\nabla z, -1)}{W}$$

where $\nabla z = (\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y})$ and $W = |(\nabla z, -1)|$. Note that the Killing field

$$\frac{\partial}{\partial \theta} := (\frac{\partial x}{\partial \theta}, \frac{\partial y}{\partial \theta}, 0) = (-y, x, 0)$$

restricted to $M$ is the initial velocity vector field of the one-parameter family $\theta \mapsto R_{\theta} M$ of minimal surfaces. Thus the function

$$u : M \to \mathbb{R}$$

$$u = \nu \cdot \frac{\partial}{\partial \theta} = W^{-1}(\frac{\partial z}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial \theta}) = W^{-1} \frac{\partial z}{\partial \theta}$$

satisfies the Jacobi field equation

$$\Delta u + |A|^2 u = 0,$$

where $A$ is the second fundamental form of $M$. (See [Cho90] Lemma 1.)

Now $\partial z/\partial \theta \geq 0$ on the disks $D_n$ (by Theorem 3), so $\partial z/\partial \theta \geq 0$ on $M$. Thus $u$ is nonnegative, so from the Jacobi field equation, we see that $u$ is superharmonic. Hence by the maximum principle, $u$ is either everywhere 0 or everywhere strictly positive. The same is true of $\partial z/\partial \theta = W u$. Since $\partial z/\partial \theta > 0$ along the outer boundary curve $S_J$ of $M$ (by choice of $S_J$), it must be positive everywhere on $M$.

It remains to prove Statement 3 of the Theorem. By Claim 1 the punctured disks $D_p$ with $p \in K$ are leaves of $\mathcal{L}'$. Each such leaf can be extended smoothly to include the puncture. This extension of $\mathcal{L}'$ is smooth near any point in the interior of $K$.

However, if $p \in \partial K$, then $p$ is simultaneously the limit of points in $D_p$ and the limit of points in $Z \setminus K$. At the former points, the tangent planes to $\mathcal{L}'$ are horizontal, whereas at the latter points the tangent planes are vertical. Thus the lamination $\mathcal{L}'$ cannot extend smoothly to any points in $\partial K$. □
2. Existence and uniqueness of embedded minimal disks with rotationally symmetric boundaries

**Theorem 2.** Let $W$ be a nonempty, bounded, convex open subset of $\mathbb{R}^3$ that is rotationally symmetric about $Z$ and let $I = W \cap Z$. Let $R_\theta$ be rotation about $Z$ through angle $\theta$. Suppose $\Gamma$ is a piecewise smooth simple closed curve such that $\Gamma$ contains $I$, such that $\Gamma \setminus I$ is contained in $\partial W$, and such that the curves $R_\theta(\Gamma \setminus I)$, $0 \leq \theta < 2\pi$, foliate $(\partial W) \setminus Z$. Then $\Gamma$ bounds a unique embedded minimal disk $D$. Furthermore, the disks $R_\theta D$, $0 \leq \theta < 2\pi$ foliate $W \setminus I$.

**Proof.** Let

$$\gamma = \{ (r, z) : r \geq 0 \text{ and } (r, 0, z) \in \partial W \}.$$  

Let $\mathcal{F}$ be the set of piecewise smooth functions $f : \gamma \to \mathbb{R}$ for which the curve

$$\Gamma_f = \{ (r \cos f(r, z), r \sin f(r, z), z) : (r, z) \in \gamma \} \cup I,$$

bounds an embedded minimal disk $D_f$ whose rotated images $R_\theta D_f$ foliate $W \setminus Z$.

Note that if $f$ is constant, then $\Gamma_f$ is a planar curve that bounds a planar disk whose rotated images foliate $W \setminus I$. Thus $\mathcal{F}$ contains all the constant functions.

**Claim.** Suppose that $f \in \mathcal{F}$ and that $g : \gamma \to \mathbb{R}$ is a piecewise smooth function such that

$$\sup \| f - g \| < \pi/4.$$  

Then $g$ must also belong to $\mathcal{F}$.

Since $\mathcal{F}$ is nonempty, the claim implies that $\mathcal{F}$ contains every piecewise smooth function. Thus once we have proved the claim, we will have proved the existence part of the theorem, because every $\Gamma$ satisfying the hypotheses of the theorem can be written in the form (1).

**Proof of Claim.** Let

$$\Omega = \bigcup_{-\pi/4 \leq \theta \leq \pi/4} R_\theta D_f.$$  

Note that $\Omega$ is mean convex and simply connected and that $\Gamma_g$ is contained in $\partial \Omega$. Hence, $\Gamma_g$ bounds a least-area disk $D_g$ in $\Omega$.

Suppose that the $R_\theta D_g$ do not foliate $W \setminus Z$. Then $D_g$ and $R_\theta D_g$ intersect each other for some $\theta \in (0, \pi)$. Any such $\theta$ must in fact be less than $\pi/2$ because

$$D_g \subset \Omega,$$

$$R_\theta D_g \subset R_\theta \Omega,$$
and because $\Omega$ and $R_\theta \Omega$ are disjoint for $\pi/2 \leq \theta \leq \pi$. Thus if $\alpha$ is the supremum of $\theta \in (0, \pi)$ for which $D_g$ and $R_\theta D_g$ intersect each other, then $0 < \alpha \leq \pi/2$. The boundary curves $\Gamma_g$ and $R_\alpha \Gamma_g$ intersect only on $\mathcal{T}$, so $D_g$ and $R_\alpha D_g$ must be tangent at some point point in $D_g \cup \mathcal{T}$. At each point of $\mathcal{T}$, the disks $D_g$ and $R_\alpha D_g$ make an angle of $\alpha \neq 0$ with each other. So the point of tangency must lie in $D_g$. This contradicts the maximum principle. Hence the rotated images of $D_g$ foliate $W \setminus Z$. In particular, $D_g$ is embedded. This completes the proof of the claim. □

We have proved the existence of a disk $D$ with boundary $\Gamma \cup \mathcal{I}$ such that the rotated images of $D$ foliate $W \setminus \mathcal{I}$. It remains only to prove uniqueness. (In this paper, we never actually use the uniqueness.) Let $\Sigma$ be any embedded minimal disk with boundary $\Gamma$. Since $\Sigma$ is embedded, it has no boundary branch points, so that $\Sigma \cup \mathcal{I}$ is a smooth manifold with boundary.

Since the disks $R_\theta D$ foliate $W \setminus \mathcal{I}$, there is a unique continuous function

$$\omega : \Sigma \to \mathbb{R}$$

such that

1. $\omega = 0$ on $\Gamma$,
2. $p \in R_{\omega(p)} D$ for $p \in \Sigma$, and
3. for $p \in \mathcal{I}$, $R_{\omega(p)} D$ and $\Sigma$ have the same tangent halfplane.

(If $\Sigma$ were not simply connected, $\omega$ might only be well-defined up to multiples of $2\pi$. That is, $\omega$ would take values in $\mathbb{R}/2\pi\mathbb{Z}$. But since $\Sigma$ is simply connected, we can lift $\omega$ to the universal cover $\mathbb{R}$ of $\mathbb{R}/2\pi\mathbb{Z}$.)

By the maximum principle and the boundary maximum principle (applied to points in $\mathcal{I}$), the maximum value of $\omega$ must be attained on $\Gamma$. Thus the maximum value of $\omega$ is 0. Similarly the minimum value is 0. Thus $\omega$ is identically 0, so $\Sigma = D$. □

**Theorem 3.** Let

$$\phi : [a, b] \to \mathbb{R}$$

be a smooth, strictly increasing function. Let $\Gamma$ be the closed curve consisting of

$$(\cos \phi(z), \sin \phi(z), z), \quad a \leq z \leq b,$$

together with the segment $I = Z \cap \{a < z < b\}$ and two horizontal segments in the planes $z = a$ and $z = b$.

Then $\Gamma$ bounds a unique embedded minimal disk $D$. The rotated images $R_\theta D$ foliate $C \setminus Z$, where $C$ is the cylinder $B(0, 1) \times (a, b)$. The disk $D$ can be parametrized as

$$(r, \theta) \mapsto (r \cos \theta, r \sin \theta, f(r, \theta))$$

for some function

$$f : [0, 1] \times [\alpha, \beta] \to \mathbb{R},$$
where \( \alpha = \phi(a) \) and \( \beta = \phi(b) \). That is, \( D \) is a multigraph over \( B(0, 1) \setminus \{0\} \). Furthermore, \( f(r, \theta) \) is a strictly increasing function of \( \theta \) for each \( r \). In particular, \( \partial f / \partial \theta \) is everywhere nonnegative.

**Remark.** In fact, \( \partial f / \partial \theta \) is everywhere strictly positive. The proof is almost identical to the proof that \( \partial z / \partial \theta > 0 \) in Theorem 1(2ii).

**Proof.** Observe that \( \Gamma \setminus I \) lies on the boundary of the solid cylinder \( C = B(0, 1) \times (a, b) \), and that the rotated images of \( \Gamma \setminus I \) foliate \( \partial C \setminus I \). Thus by Theorem 2, \( \Gamma \) bounds a unique embedded minimal disk \( D \), and the rotated images \( R_\theta D \) foliate \( C \setminus I \).

Note that since \( D \) is simply connected, there is a continuous function \( \theta : \overline{D} \setminus I \to \mathbb{R} \) such that for \( p = (x, y, z) \in \overline{D} \setminus I \),
\[
(x, y) = \sqrt{x^2 + y^2} (\cos \theta(p), \sin \theta(p)).
\]

Note also that we can choose \( \theta \) so that for \( (x, y, z) \in \Gamma \setminus I \), \( \theta(x, y, z) = \phi(z) \).

In particular, \( \theta \equiv \alpha \) on \( \Gamma \cap \{z = a\} \) and \( \theta \equiv \beta \) on \( \Gamma \cap \{z = b\} \), where \( \alpha = \phi(a) \) and \( \beta = \phi(b) \).

Since \( D \cup I \) is a smooth manifold with boundary, the angle function \( \theta \) extends smoothly to \( I \).

By the maximum principle, \( \theta \) cannot attain its maximum or its minimum at any interior point of \( D \). By the boundary maximum principle, \( \theta \) cannot attain its maximum or minimum at any point of \( I \). Thus the maximum and minimum are attained on \( \Gamma \setminus I \), so the minimum value is \( \alpha \) and the maximum value is \( \beta \).

To show that \( D \) is a multigraph, it suffices to show that the map
\[
(2) \quad p \in \overline{D} \mapsto (r(p), \theta(p)) \in [0, 1] \times [\alpha, \beta]
\]
is one-to-one. (It is onto by elementary topology.)

Let
\[
S = \{(p, q) \in \overline{D} \times \overline{D} : r(p) = r(q) \text{ and } \theta(p) = \theta(q)\}.
\]

By compactness, the function
\[
(p, q) \in S \mapsto z(q) - z(p)
\]
attains its maximum value \( h = z(q_0) - z(p_0) \) at some \((p_0, q_0) \in S\). To show that the map (2) is one-to-one, it suffices to show that \( h = 0 \). To see that \( h = 0 \), let \( r_0 := r(p_0) = r(q_0) \) and \( \theta_0 := \theta(p_0) = \theta(q_0) \).

If \( r_0 = 1 \) or if \( \theta_0 \) is \( \alpha \) or \( \beta \), then \( p_0 \) and \( q_0 \) are both in \( \Gamma \setminus I \) by the maximum principle. But \( p \mapsto (r(p), \theta(p)) \) is one-to-one on \( \Gamma \setminus I \) by choice of \( \Gamma \), so in this case \( h = 0 \) and we are done.

Thus we may suppose that \( r_0 < 1 \) and that \( \alpha < \theta_0 < \beta \). Now the minimal disks \( \overline{D} \) and \( \overline{D} + (0, 0, h) \) are tangent at the point \( q_0 \), but in neighborhood of that point \( \overline{D} \).
lies on one side of $\overline{D} + (0,0,h)$. Thus by the strong maximum principle (if $r_0 > 0$) or by the strong boundary maximum principle (if $r_0 = 0$), the two disks coincide, which implies that $h = 0$.

It remains to show that $f(r,\theta)$ is a strictly increasing function of $\theta$ for each $r$. Since the disks $R_\alpha D$ foliate $C \setminus Z$, it follows (for each fixed $r$) that the graphs of the curves

$$C_\alpha : \theta \mapsto f(r,\theta - \alpha)$$

foliate the strip $\mathbb{R} \times (a,b)$. Thus the function $\theta \mapsto f(r,\theta)$ must be strictly monotonic. Since it is strictly increasing for $r = 1$, it must be strictly increasing for all $r$. □

3. Curvature Estimates via Rado’s Theorem

**Theorem 4.** Let $D \subset \mathbb{R}^3$ be a minimal disk contained in a vertical solid cylinder $C = B \times \mathbb{R}$ of radius $R$, and let $\Gamma = \partial D$ be its boundary curve. Suppose that $\Gamma \cap \{a < z < b\}$ consists of two components, each of which is a $C^1$ curve whose tangent line has slope $\geq \epsilon > 0$ in absolute value at every point.

Let

$$D_\delta = D \cap \{a + \delta < z < b - \delta\}$$

where $\delta > 0$. Then

(1) $D_\delta$ has no branch points.

(2) The slope of the tangent plane at each point of $D_\delta$ is greater than or equal to

$$\min \left\{ \epsilon, \frac{\delta}{2R} \right\}.$$ 

(3) For $p \in D_\delta$, the norm of the second fundamental form of $D_\delta$ at $p$ is bounded by

$$\frac{C}{\text{dist}(p, \partial D_\delta)},$$

where $C$ is constant depending only on $\epsilon$ and $\delta/R$.

The proof of Theorem 4 uses the following theorem of Rado:

**Theorem (Rado).** If the boundary of a minimal disk in $\mathbb{R}^3$ intersects a plane in fewer than four points, then $D$ has no branch points in the plane, and $D$ intersects the plane transversely.

See, for example, [Oss86, Lemma 7.5] or [DHKW92, p. 272] for a proof.

**Proof of Theorem 4.** For $a < t < b$, the set $\Gamma \cap \{z = t\}$ contains exactly two points. Thus by Rado’s Theorem, the surface $D \cap \{a < z < b\}$ has no branch points and no horizontal tangent planes.

Now let $p \in D_\delta$ and let $P$ be a plane through $p$ whose slope is less than $\epsilon$ and less than $\delta/(2R)$. Since the slope is less than $\delta/(2R)$, the plane $P$ does not intersect $B \times (-\infty, a]$ or $B \times [b, \infty)$. Thus

$$P \cap \Gamma = P \cap (\Gamma \cap \{a < z < b\}).$$
Since the slope of $P$ is less than $\epsilon$, the plane $P$ intersects each of the the two components of $\Gamma \cap \{a < z < b\}$ in at most one point. Thus $P$ intersects $\Gamma$ in at most two points, so $P$ is not tangent to $D$ at $p$ by Rado’s Theorem. This proves Assertion 2.

Assertion 3 follows from Assertion 2 because Assertion 3 holds for any minimal surface $D_\delta$ whose image under the Gauss map omits a nonempty open subset of the unit sphere, the constant $C$ depending only on the size of that open set. This curvature estimate was proved by Osserman \cite{Oss60} Theorem 1], who even obtained the optimal constant. Alternatively, the estimate without the optimal constant follows by a standard blow-up argument from the fact (also due to Osserman \cite{Oss86} Theorem 8.1]) that the image of the Gauss map of a complete, nonflat minimal surface in $\mathbb{R}^3$ must be dense in the unit 2-sphere. □

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