Local Cauchy theory for the nonlinear Schrödinger equation in spaces of infinite mass

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Abstract

We consider the Cauchy problem for the nonlinear Schrödinger equation on \( \mathbb{R}^d \), where the initial data is in \( \dot{H}^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d) \). We prove local well-posedness for large ranges of \( p \) and discuss some global well-posedness results.

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1 Introduction

In this work, we consider the classical nonlinear Schrödinger equation over \( \mathbb{R}^d \):

\[
iu_t + \Delta u + \lambda |u|^{\sigma} u = 0, \quad u = u(t, x), \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d, \quad \lambda \in \mathbb{R}, \quad 0 < \sigma < 4/(d-2)^+ \]

and focus on the corresponding Cauchy problem \( u(0) = u_0 \in E \), where \( E \) is a suitable function space. This model equation is the subject of more than fifty years of intensive research, which makes us unable to give a complete list of important references (we simply refer the monographs \[2\], \[10\], \[11\] and references therein). The usual framework one considers is \( E = H^1(\mathbb{R}^d) \), the so-called energy space, or more generally, \( E = H^s(\mathbb{R}^d) \). A common property of these spaces is that they are \( L^2 \)-based. The reason for this constraint comes from the fact that the linear group is bounded in \( L^2 \), but not in any other \( L^p \).

In the sense of lifting the \( L^2 \) constraint, we refer the papers \[6\], \[7\] and \[4\]. In the first paper, one considers local well-posedness on Zhidkov spaces

\[
E = X^k(\mathbb{R}^d) = \{u \in L^\infty(\mathbb{R}^d) : \nabla u \in H^{k-1}(\mathbb{R}^d)\}.
\]

In the second, one takes the Gross-Pitaevskii equation and looks for local well-posedness on

\[
E = \{u \in H^1_{loc}(\mathbb{R}^d) : \nabla u \in L^2(\mathbb{R}^d), |u|^2 - 1 \in L^2(\mathbb{R}^d)\}.
\]

Finally, in the third work, one considers \( E = H^1(\mathbb{R}^2) + X \), where \( X \) is either a particular space of bounded functions with no decay or a subspace of \( L^1(\mathbb{R}^2) \) (and not of \( L^2(\mathbb{R}^2) \)).

The aim of this paper is to look for local well-posedness results over another class of spaces, namely

\[
E = X_p(\mathbb{R}^d) = \dot{H}^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d), \quad 2 < p < 2d/(d-2)^+.
\]

In particular, we obtain local well-posedness in the most general energy space \( X_{\sigma+2}(\mathbb{R}^d) \) and obtain global well-posedness over \( X_p(\mathbb{R}^d) \) in the defocusing case \( \lambda < 0 \) for all \( p \leq \sigma + 2 \).
Remark 1.1. Our results can be extended to more general nonlinealities \( f(u) \) as in the \( H^1 \) framework. We present our results for \( f(u) = |u|^\sigma u \) so not to complicate unnecessarily the proofs and deviate from the main ideas.

We briefly explain the structure of this work: in Section 2, we derive the required group estimates and show that the Schrödinger group is well-defined over \( X_\sigma(\mathbb{R}^d) \). In Section 3, we show local well-posedness for \( p \leq 2\sigma + 2 \), where the use of Strichartz estimates is available. We also prove global well-posedness for small \( \sigma \) (cf. Proposition 3.6). In Section 4, we deal with the complementary case \( p > 2\sigma + 2 \) in dimensions \( d = 1, 2 \).

Notation. The norm over \( L^p(\mathbb{R}^d) \) will be denoted as \( \| \cdot \|_p \) or \( \| \cdot \|_{L^p} \), whichever is more convenient. The spatial domain \( \mathbb{R}^d \) will often be omitted. The free Schrödinger group in \( H^1(\mathbb{R}^d) \) is written as \( \{ S(t) \}_{t \in \mathbb{R}} \). We write \( p^* = dp/(d-p)^+ \). To avoid repetition, we hereby set \( 2 < p < 2^* \) and \( 0 < \sigma < 4/(d-2)^+ \).

2 Linear estimates

We recall the essential Strichartz estimates. We say that \( (q, r) \) is an admissible pair if

\[
2 \leq r \leq 2^*, \quad \frac{2}{q} = d \left( \frac{1}{2} - \frac{1}{r} \right), \quad r \neq \infty \text{ if } d = 2.
\]

Lemma 2.1 (Strichartz estimates). Given two admissible pairs \((q, r)\) and \((\gamma, \rho)\), we have, for all sufficiently regular \( u_0 \) and \( f \) and for any interval \( I \subset \mathbb{R} \),

\[
\| S(\cdot)u_0 \|_{L^q(I; L^r(\mathbb{R}^N))} \lesssim \| u_0 \|_2 \tag{2.0}
\]

and

\[
\left\| \int_{0 < s < ct} S(t-s)f(s)ds \right\|_{L^q(I; L^r(\mathbb{R}^N))} \lesssim \| f \|_{L^\gamma(I; L^\rho(\mathbb{R}^N))}. \tag{2.1}
\]

Remark 2.1. The estimate (2.1) may be extended to other sets of admissible pairs: see \([5]\) and \([12]\). However, the linear estimate (2.1) is not valid for any other pairs and for \( u_0 \notin L^2(\mathbb{R}^d) \).

Proposition 2.2 (Group estimates with loss of derivative). Define \( k \) so that \((k, p)\) is admissible. Then

- \((Linear estimate)\) For \( \phi \in S(\mathbb{R}^d) \),

\[
\| S(t)\phi \|_p^2 \leq \| \phi \|_p^2 + |t|^{1-\frac{d}{2}} \| \nabla \phi \|_2^2, \quad t \in \mathbb{R}.
\]

- \((Non-homogeneous estimate)\) For \( f \in C([0, T]; S(\mathbb{R}^d)) \) and any \((q, r)\) admissible,

\[
\left\| \int_0^T S(t-s)f(s)ds \right\|_{L^p((0,T); L^r(\mathbb{R}^d))} \lesssim C(T) \left( \| f \|_{L^2((0,T); L^q(\mathbb{R}^d))} + \| \nabla f \|_{L^\gamma((0,T); L^\rho(\mathbb{R}^d))} \right), \tag{2.2}
\]

where \( C(\cdot) \) is an increasing bounded function over bounded intervals of \( \mathbb{R} \).

Notice that, due to the scaling invariance of the Schrödinger equation, the polynomial growth in time in the linear estimate is unavoidable.
Proof. For the linear estimate, write $u = S(t)\phi$. Then $u \in C^1(\mathbb{R}; H^2(\mathbb{R}^d))$ satisfies

$$iu_t + \Delta u = 0, \quad u(0) = \phi.$$ 

Multiplying the equation by $|u|^{p-2}u$, integrating over $\mathbb{R}^d$ and taking the imaginary part, we obtain

$$\frac{1}{p} \frac{d}{dt} \|u(t)\|_p^p \leq \left| \text{Im} \int |u|^{p-2}u\Delta u \right| \leq \frac{p-2}{2} \int |u|^{p-2}|\nabla u|^2 \leq \frac{p-2}{2} \|u(t)\|_p^{p-2}\|\nabla u\|_p^2.$$ 

Thus we have

$$\frac{d}{dt} \|u(t)\|_p^p \leq (p-2)\|\nabla u(t)\|_p^2.$$ 

An integration between 0 and $t \in \mathbb{R}$ and the linear Strichartz estimate yield

$$\|u(t)\|_p^2 \leq \|\phi\|_p^2 + \int_0^t \|\nabla u(s)\|_p^2 ds \leq \|\phi\|_p^2 + |t| \left( \int_0^t \|\nabla u(s)\|_p^2 ds \right)^{\frac{p-2}{2}} \leq \|\phi\|_p^2 + |t|^{-\frac{2}{p-2}} \|\nabla \phi\|_p^2.$$ 

For the non-homogeneous estimate, set $v(t) = -i \int_0^t S(t-s)f(s)ds$. Then $v \in C^1([0,T]; H^1(\mathbb{R}^d))$ satisfies

$$iv_t + \Delta v = f, \quad v(0) = 0.$$ 

As for the previous estimate, we have

$$\frac{1}{p} \frac{d}{dt} \|v(t)\|_p^p \leq \|v(t)\|_p^{p-2}\|\nabla v(t)\|_p^2 + \|v(t)\|_p^{-1}\|f(t)\|_p$$

and so

$$\frac{d}{dt} \|v(t)\|_p^p \leq \|\nabla v(t)\|_p^2 + \|v(t)\|_p\|f(t)\|_p \leq \|\nabla v(t)\|_p^2 + \|v(t)\|_p^2 + \|f(t)\|_p^2.$$ 

Thus the required estimate now follows by direct integration in $(0,t)$, $0 < t < T$, and by the non-homogeneous Strichartz estimate.

\begin{lemma} \textbf{(Local Strichartz estimate without loss of derivatives).} Given $f \in C([0,T], S(\mathbb{R}^d))$, \begin{equation}
\left\| \int_0^t S(t-s)f(s)ds \right\|_{L^r((0,T); L^p(\mathbb{R}^d))} \leq C(T, q)\|f\|_{L^r((0,T); L^\epsilon(\mathbb{R}^d))}, \quad \frac{1}{q} > d \left( \frac{1}{2} - \frac{1}{p} \right). \tag{2.3}
\end{equation}
\end{lemma}

Proof. This estimate follows easily from the decay estimates of the Schrödinger group: indeed, given $0 < t < T$,

$$\left\| \int_0^t S(t-s)f(s)ds \right\|_{L^r(\mathbb{R}^d)} \leq \int_0^T |S(t-s)f(s)|_{L^r(\mathbb{R}^d)} ds$$

$$\leq \int_0^T \frac{1}{|t-s|^{d(\frac{1}{2} - \frac{1}{p})}} |f(s)|_{L^\epsilon(\mathbb{R}^d)} ds$$

$$\leq \left( \int_0^T \frac{1}{|t-s|^{d(\frac{1}{2} - \frac{1}{p})}} ds \right)^\frac{1}{q} \|f\|_{L^r((0,T); L^\epsilon(\mathbb{R}^d))}.$$ 

We set \(X_p(\mathbb{R}^d) = L^p(\mathbb{R}^d) \cap \dot{H}^1(\mathbb{R}^d)\).
Remark 2.2. From the Gagliardo-Nirenberg inequality, we have $H^1(\mathbb{R}^d) \hookrightarrow X_p(\mathbb{R}^d)$.

Proposition 2.4. The Schrödinger group $\{S(t)\}_{t \in \mathbb{R}}$ over $H^1(\mathbb{R}^d)$ defines, by continuous extension, a one-parameter continuous group on $X_p(\mathbb{R}^d)$.

Proof. Given any $\phi \in \dot{H}^1(\mathbb{R}^d)$, we have $\|S(t)\nabla \phi\|_2 = \|\nabla \phi\|_2$. Together with Proposition 2.2, this implies that

$$\|S(t)\phi\|_{X_p} \leq (1 + |t|^{1-\frac{d}{2}})^{1/2} \|\phi\|_{X_p}, \quad t \in \mathbb{R}.$$ 

Therefore, for each fixed $t \in \mathbb{R}$, $S(t)$ may be extended continuously to $X_p$. By density, it follows easily that $S(t + s) = S(t)S(s)$, $t, s \in \mathbb{R}$, and $S(0) = I$ on $X_p$. Finally, we prove continuity at $t = 0$: given $\phi \in X_p(\mathbb{R}^d)$ and $\epsilon > 0$, take $\phi_\epsilon \in H^1(\mathbb{R}^d)$ such that

$$\|\phi_\epsilon - \phi\|_{X_p} < \epsilon.$$ 

Then

$$\limsup_{t \to 0} |S(t)\phi - \phi|_{X_p} \leq \limsup_{t \to 0} \left( \|S(t)(\phi - \phi_\epsilon)\|_{X_p} + \|S(t)\phi_\epsilon - \phi_\epsilon\|_{X_p} + \|\phi_\epsilon - \phi\|_{X_p} \right)$$

$$\leq \limsup_{t \to 0} \left( (1 + |t|^{1-\frac{d}{2}})^{1/2} \|\phi - \phi_\epsilon\|_{X_p} + \|S(t)\phi_\epsilon - \phi_\epsilon\|_{H^1} \right) \leq \epsilon.$$

□

Remark 2.3. Fix $d = 1$. Using the same ideas, one may easily observe that the Schrödinger group is well-defined on the Zhidkov space

$$X^2(\mathbb{R}) = \{ u \in L^\infty(\mathbb{R}) : \nabla u \in H^1(\mathbb{R}) \}.$$ 

Indeed, for any $2 \leq p \leq \infty$ a direct integration of the equation gives

$$\frac{d}{dt} \|u(t)\|_p \leq \|\Delta u(t)\|_p.$$ 

Hence, choosing $k$ so that $(k, p)$ is an admissible pair,

$$\|u(t)\|_p \leq \|u_0\|_p + \int_0^t \|\Delta u(s)\|_p ds \leq \|u_0\|_p + Ct^{1-\frac{d}{2}} \|\Delta u\|_{L^1((0,t),L^p)} \leq \|u_0\|_p + Ct^{1-\frac{d}{2}} \|\Delta u_0\|_2,$$

where $C$ is a constant independent on $p$ (this comes from the fact that such a constant may be obtained via the interpolation between $L^p_L, L^2$ and $L^p_L, L^\infty$). Then, taking the limit $p \to \infty$, we obtain

$$\|u(t)\|_\infty \leq \|u_0\|_\infty + t^\frac{d}{2} \|\Delta u_0\|_2, \quad t > 0, \quad u_0 \in H^2(\mathbb{R}).$$

For higher dimensions, a similar procedure may be applied, at the expense of some derivatives (one must use Sobolev injection to control $L^p$, with $p$ large). As one might expect, this argument does not provide the best possible estimate: in [6], one may see that

$$\|u(t)\|_\infty \leq (1 + t^\frac{d}{2}) \|u_0\|_\infty + \|\nabla u_0\|_2, \quad t > 0, u_0 \in H^1(\mathbb{R}).$$

Remark 2.4. One may ask if the required regularity is optimal: can we define the Schrödinger group on $X^p_\mu(\mathbb{R}^d) := H^s(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$? What is the optimal $s$? Taking into consideration the previous remark, we conjecture that it should be possible to lower the regularity assumption. This entails a deeper analysis of the Schrödinger group, as it was done in [6].
3 Local well-posedness for \( p \leq 2\sigma + 2 \)

In order to clarify what do we mean by a solution of (NLS), we give the following

**Definition 3.1** (Solution over \( X_p(\mathbb{R}^d) \)). Given \( u_0 \in X_p(\mathbb{R}^d) \), we say that \( u \in C([0,T], X_p(\mathbb{R}^d)) \) is a solution of (NLS) with initial data \( u_0 \) if the Duhamel formula is valid:

\[
  u(t) = S(t)u_0 + i\lambda \int_0^t S(t-s)|u(s)|^\sigma u(s)ds, \quad t \in [0,T].
\]

Throughout this section, let \((\gamma, \rho)\) and \((q, r)\) be admissible pairs such that

\[
  r = (\sigma + 1)\rho' = \max\{\sigma + 2, p\}. \tag{3.1}
\]

It is easy to check that such pairs are well-defined for \( p \leq 2\sigma + 2 \).

**Proposition 3.2** (Uniqueness over \( X_p(\mathbb{R}^d) \)). Suppose that \( p \leq 2\sigma + 2 \). Let \( u_1, u_2 \in C([0,T], X_p(\mathbb{R}^d)) \) be two solutions of (NLS) with initial data \( u_0 \in X_p(\mathbb{R}^d) \). Then \( u_1 \equiv u_2 \).

Proof. Taking the difference between the Duhamel formula for \( u_1 \) and \( u_2 \),

\[
  u_1(t) - u_2(t) = i\lambda \int_0^t S(t-s)\left(|u_1(s)|^\sigma u_1(s) - |u_2(s)|^\sigma u_2(s)\right)ds
\]

Then, for any interval \( J = [0, t] \subset [0, T] \), since \( X_p(\mathbb{R}^d) \rightarrow L^r(\mathbb{R}^d) \),

\[
  \|u_1 - u_2\|_{L^r(J, L^r)} \leq \|u_1\|^\sigma\|u_1 - u_2\|^\sigma_{L^r(J, L^r)}
  \leq \left(\|u_1\|^\sigma_{L^r([0,T], X_p(\mathbb{R}^d))} + \|u_2\|^\sigma_{L^r([0,T], X_p(\mathbb{R}^d))}\right)\|u_1 - u_2\|_{L^r(J, L^r)}
  \leq C(T)\|u_1 - u_2\|_{L^r(J, L^r)}
\]

The claimed result now follows from [2] Lemma 4.2.2.

**Theorem 3.3** (Local well-posedness on \( X_p(\mathbb{R}^d) \), \( p \leq 2\sigma + 2 \)). Given \( u_0 \in X_p(\mathbb{R}^d) \), there exists \( T = T(||u_0||_{X_p}) > 0 \) and an unique solution

\[
  u \in C([0,T], X_p(\mathbb{R}^d)) \cap L^\gamma([0,T], \dot{W}^{1,p}(\mathbb{R}^d)) \cap L^\gamma([0,T], L^\gamma(\mathbb{R}^d))
\]

of (NLS) with initial data \( u_0 \). One has

\[
  u - S(\cdot)u_0 \in C([0,T], L^2(\mathbb{R}^d)) \cap L^\gamma([0,T], L^\gamma(\mathbb{R}^d)) \cap L^\gamma([0,T], L^\gamma(\mathbb{R}^d)) \cap L^\gamma([0,T], L^\gamma(\mathbb{R}^d)). \tag{3.2}
\]

Moreover, the solution depends continuously on the initial data and may be extended in an unique way to a maximal time interval \([0, T^*(u_0)]\). If \( T^*(u_0) < \infty \), then

\[
  \lim_{t \to T^*(u_0)} \|u(t)\|_{X_p} = +\infty.
\]

**Remark 3.1.** The property (3.2) is a type of nonlinear "smoothing" effect: the integral term in Duhamel’s formula turns out to have more integrability than the solution itself (a similar property was seen in [7]). This insight allows the use of Strichartz estimates at the zero derivatives level. Without this possibility, one would be restricted to the estimate (2.3) and the possible ranges of \( \sigma \) and \( p \) would be significantly smaller.
Proof. Step 1. Define
\[ S_0 = L^\infty((0,T), L^2) \cap L^9((0,T), L^7) \cap L^n((0,T), L^p). \]
and
\[ S_1 = L^\infty((0,T), H^1) \cap L^\delta((0,T), W^{1,\rho}) \cap L^n((0,T), W^{1,\rho}). \]
Consider the space
\[ \mathcal{E} = \left\{ u \in L^\infty((0,T), X_p) \cap L^\gamma((0,T), \dot{W}^{1,\rho}) \cap L^\delta((0,T), \dot{W}^{1,\rho}) : \right. \]
\[ \left| u \right| := \left\| u \right\|_{L^\infty((0,T), L^p)} + \left\| u - S(\cdot)u_0 \right\|_{S_1} \leq M \right\}. \]
edowed with the distance
\[ d(u, v) = \left\| u - v \right\|_{S_0}. \]
It is not hard to check that \( (\mathcal{E}, d) \) is a complete metric space: indeed, if \( \{ u_n \}_{n \in \mathbb{N}} \) is a Cauchy sequence in \( \mathcal{E} \), then \( \{ u_n - S(\cdot)u_0 \}_{n \in \mathbb{N}} \) is a Cauchy sequence in \( S_0 \). Then there exists \( u \in \mathcal{D}'([0,T] \times \mathbb{R}^d) \) such that \( u_n - S(\cdot)u_0 \to u - S(\cdot)u_0 \) in \( S_0 \). By Theorem 1.2.5], this convergence implies
\[ u - S(\cdot)u_0 \in S_1, \quad \left| u - S(\cdot)u_0 \right|_{S_1} \leq \liminf \left| u_n - S(\cdot)u_0 \right|_{S_1}. \]
Finally, it follows from the Gagliardo-Nirenberg inequality that, for some \( 0 < \theta < 1 \),
\[ \left| u_n - u \right|_{L^\infty((0,T), L^p)} \leq \left| u_n - u \right|_{L^{\theta,\sigma}((0,T), L^2)} \left\| \nabla u_n - \nabla u \right\|_{L^{\sigma,\rho}(0,T), L^r} \to 0 \]
and so \( u_n \to u \) in \( L^\infty((0,T), L^p) \).
Step 2. Define, for any \( u \in \mathcal{E} \),
\[ (\Phi u)(t) = S(t)u_0 + i\lambda \int_0^t S(t-s)|u(s)|^\sigma u(s)ds, \quad 0 \leq t < T. \]
It follows from the definition of \( r \) (see (3.1)) that \( X_p(\mathbb{R}^d) \to L^r(\mathbb{R}^d) \). Then
\[ \left\| \Phi u - S(\cdot)u_0 \right\|_{S_1} \leq \left\| \left| u \right|^\sigma u \right\|_{L^{\infty,\sigma}((0,T),L^r)} \]
\[ \leq \left\| \left| u \right|^\sigma u \right\|_{L^{\infty,\sigma}((0,T),L^r)} + \left\| \nabla u \right\|_{L^\gamma(0,T)} \]
\[ \leq T^{\frac{1}{\sigma}} \left\| u \right\|_{L^{\infty,\sigma}((0,T),X_p)} + T^{\frac{1}{\rho}} \left\| u \right\|_{L^{\infty,\rho}((0,T),X_p)} \left\| \nabla u \right\|_{L^\gamma(0,T)} \]
\[ \leq T^{\frac{1}{\sigma}} + T^{\frac{1}{\rho}} \left( M^{\sigma+1} + \left| u_0 \right| \right). \]
It follows that, for \( M \sim 2\left| u_0 \right|_{X_p} \) and \( T \) sufficiently small, we have \( \Phi : \mathcal{E} \to \mathcal{E} \).
Step 3. Now we show a contraction estimate: given \( u, v \in \mathcal{E} \),
\[ d(\Phi(u), \Phi(v)) \leq \left\| \left| u \right|^\sigma u - \left| v \right|^\sigma v \right\|_{L^{\infty,\sigma}(L^r)} \]
\[ \leq \left\| \left( \left| u \right|^\sigma u \right) - \left( \left| v \right|^\sigma v \right) \right\|_{L^{\infty,\sigma}(0,T)} \]
\[ \leq T \left( \left| \left| u \right|^\sigma u \right|_{L^{\infty,\sigma}((0,T),X_p)} + \left| \left| v \right|^\sigma v \right|_{L^{\infty,\sigma}((0,T),X_p)} \right) \left\| u - v \right\|_{L^\gamma(0,T, L^r)} \]
\[ \leq T \left( \left| \left| u \right|^\sigma u \right|_{X_p} + \left| \left| v \right|^\sigma v \right|_{X_p} \right) d(u, v). \]
Therefore, for $T = T(\|u_0\|_{X_p})$ small enough, the mapping $\Phi : \mathcal{E} \to \mathcal{E}$ is a strict contraction and so, by Banach’s fixed point theorem, $\Phi$ has a unique fixed point over $\mathcal{E}$. This gives the local existence of a solution $u \in C([0, T], X_p(\mathbb{R}^d))$ of (NLS) with initial data $u_0$. From the uniqueness result, such a solution can then be extended to a maximal interval of existence $(0, T^*(u_0))$. If such an interval is bounded, then necessarily one has $\|u(t)\|_{X_p} \to \infty$ as $t \to T^*(u_0)$. The continuous dependence on the initial data follows as in the $H^1$ case (see, for example, the proof of [2] Theorem 4.4.1)

**Remark 3.2.** The condition $p \leq 2\sigma + 2$ is necessary for one to use Strichartz estimates with no derivatives. Indeed, when one applies Strichartz to the integral term of the Duhamel formula, one has

$$\left\| \int_0^t S(t - s) [u(s)]^\sigma u(s) \, ds \right\|_{L^q((0, T), L^r)} \leq \|u\|_{L^q(\mathbb{R}^d)}^{\sigma+1} \leq \|u\|_{L^q(\mathbb{R}^d)}^{\sigma+1}$$

for any admissible pairs $(q, r)$ and $(\gamma, \rho)$. Since the solution $u$ only lies on spaces with spatial integrability larger or equal than $p$, one must have $p \leq \rho(\sigma + 1) \leq 2\sigma + 2$ (because $\rho \geq 2$).

**Proposition 3.4 (Persistence of integrability).** Suppose that $p \leq 2\sigma + 2$. Given $u_0 \in X_p(\mathbb{R}^d)$, consider the $X_p(\mathbb{R}^d)$-solution $u \in C([0, T^*(u_0)], X_p)$ of (NLS) with initial data $u_0$. Then $u \in C([0, T^*(u_0)), X_p)$.

**Proof.** By the local well-posedness result over $X_p(\mathbb{R}^d)$ and by the uniqueness over $X_p(\mathbb{R}^d)$, there exists a time $T_0 > 0$ such that $u \in C([0, T_0], X_p(\mathbb{R}^d))$. Thus the statement of the proposition is equivalent to saying that $u$ does not blow-up in $X_p(\mathbb{R}^d)$ at a time $T_0 < T < T^*(u_0)$. Since $u$ is bounded in $X_p$ over $[0, T]$, it follows from the local existence theorem that

$$\|u - S(\cdot) u_0\|_{L^\infty((0, T), H^1)} < \infty.$$

Then, for any $0 < t < T$,

$$\|u\|_{L^\infty((0, t), X_p)} \leq \|S(\cdot) u_0\|_{L^\infty((0, t), X_p)} + \|u - S(\cdot) u_0\|_{L^\infty((0, t), X_p)} \leq \|u_0\|_{X_p} + \|u - S(\cdot) u_0\|_{L^\infty((0, t), H^1)} < \infty,$$

which implies that $u$ does not blow-up at time $t = T$.

**Proposition 3.5 (Conservation of energy).** Suppose that $p \leq \sigma + 2$. Given $u_0 \in X_p(\mathbb{R}^d)$, the corresponding solution $u$ of (NLS) with initial data $u_0$ satisfies

$$E(u(t)) = E(u_0) := \frac{1}{2} \|\nabla u_0\|_2^2 - \frac{\lambda}{\sigma + 2} \|u_0\|_{\sigma + 2}^{\sigma + 2}, \quad 0 < t < T(u_0).$$

Consequently, if $\lambda < 0$, then $T^*(u_0) = \infty$. Moreover, if $\lambda > 0$ and $T^*(u_0) < \infty$, then

$$\lim_{t \to T^*(u_0)} \|\nabla u(t)\|_2 = \lim_{t \to T^*(u_0)} \|u(t)\|_{\sigma + 2} = \infty. \quad (3.3)$$

**Proof.** Since the conservation law is valid for $u_0 \in H^1(\mathbb{R}^d)$, through a regularization argument, the same is true for any $u_0 \in X_p(\mathbb{R}^d)$. If $\lambda < 0$, one has

$$\|u(t)\|_{X_{\sigma + 2}(\mathbb{R}^d)} \leq E(u_0), \quad 0 < t < T^*(u_0).$$

By the blow-up alternative, this implies that $u$, as a $X_{\sigma + 2}(\mathbb{R}^d)$ solution, is globally defined. By persistence of integrability, this implies that $u$ is global in $X_p(\mathbb{R}^d)$. If $\lambda > 0$, suppose by contradiction that (3.3) is not true. Then, by conservation of energy, $u$ is bounded in $X_{\sigma + 2}(\mathbb{R}^d)$ and therefore it is globally defined (as an $X_{\sigma + 2}(\mathbb{R}^d)$ solution, but also as an $X_p(\mathbb{R}^d)$ solution, by persistence of integrability).
Proposition 3.6. Fix $\lambda < 0$. If
\[ 2\sigma^2 + (d + 2)\sigma \leq 4, \]
then, for any $u_0 \in X_{\sigma+2}(\mathbb{R}^d)$, the corresponding solution $u$ of (NLS) is globally defined.

Remark 3.3. Notice that the condition on $\sigma$ implies that $\sigma < \min\{\sqrt{2}, 4/(d + 2)\} < 4/d$.

Proof. By contradiction, assume that $u$ blows-up at time $t = T$. The previous proposition then implies that
\[ \lim_{t \to T} \|\nabla u(t)\|_2 = \infty. \]

The first step is to obtain a corrected mass conservation estimate: indeed, by direct integration of the equation,
\[ \frac{1}{2} \frac{d}{dt} \|u(t) - S(t)u_0\|_2^2 = \text{Im} \int u(t)^* u(t)(u(t) - S(t)u_0) = -\text{Im} \int \|u(t)^* u(t) S(t)u_0\| \leq \|u(t)\|_{\sigma+2}^2 \|S(t)u_0\|_{\sigma+2}. \]

Integrating on $(0, t)$,
\[ \|u(t) - S(t)u_0\|_2^2 \leq \int_0^t \|u(s)\|_{\sigma+2}^2 \|S(s)u_0\|_{\sigma+2} ds \leq \|S(\cdot) u_0\|_{L^\infty((0,T), L^{\sigma+2})} \int_0^t \|u(s)\|_{\sigma+2}^2 ds. \]

All of these formal computations can be justified by a suitable regularization and approximation argument. The next step is to use the conservation of energy and the Gagliardo-Nirenberg inequality to obtain a bound on $\|\nabla u(t)\|_2$.
\[ \frac{1}{2} \|\nabla u(t)\|_2^2 = E(u_0) + \frac{1}{\sigma+2} \|u(t)\|_{\sigma+2}^2 \leq 1 + \|u(t) - S(t)u_0\|_{\sigma+2}^2 + \|S(t)u_0\|_{\sigma+2}^2 \leq 1 + \|\nabla(u(t) - S(t)u_0)\|_2^2 \|u(t) - S(t)u_0\|_2^{\frac{4-(d-2)\sigma}{2}} \leq 1 + \|\nabla(u(t) - S(t)u_0)\|_2^{\frac{4}{2}} \left( \int_0^t \|u(s)\|_{\sigma+2}^2 ds \right)^{\frac{4-(d-2)\sigma}{2}}. \]

For $t$ close to $T$, $\|\nabla(u(t) - S(t)u_0)\|_2 \sim \|\nabla u(t)\|_2$ and, by conservation of energy,
\[ \|u(t)\|_{\sigma+2}^2 \leq \|\nabla u(t)\|_2^{\frac{2 \sigma + 2}{\sigma}}. \]

Thus
\[ \|\nabla u(t)\|_2^{\frac{2 \sigma + 2}{\sigma}} \leq 1 + \left( \int_0^t \|\nabla u(s)\|_2^{\frac{2 \sigma + 2}{\sigma}} ds \right)^{\frac{4-(d-2)\sigma}{2}}. \]

which, together with the condition on $\sigma$, implies that
\[ g(t) := \|\nabla u(t)\|_2^{\frac{2 \sigma + 2}{\sigma}} \leq \|\nabla u(t)\|_2^{\frac{4-(d-2)\sigma}{\sigma}} \leq 1 + \int_0^t \|\nabla u(s)\|_2^{\frac{2 \sigma + 2}{\sigma}} ds \leq 1 + \int_0^t g(s) ds. \]

The desired contradiction now follows from a standard application of Gronwall’s lemma. \qed
Thus, in order to obtain blow-up outside $L^\infty$, impossible to obtain a global existence result for small data in the energy case finite variance assumption. This is an open problem, which has been solved in \cite{9} under radial Strichartz estimates with no loss in regularity. For the other possibility is to

Thus, in order to obtain blow-up outside $L^2$, one must first show blow-up in $H^1$ without the finite variance assumption. This is an open problem, which has been solved in \cite{9} under radial hypothesis and relying heavily on the conservation of mass (which is unavailable on $X_p(\mathbb{R}^d)$). For the nonradial case, recent works (see, for example, \cite{3}) only manage to prove unboundedness of solutions of negative energy. The problem of blow-up solutions strictly in $X_p(\mathbb{R}^d)$ is an even harder problem, requiring a better control on the tails of the solution.

Remark 3.6 (Scaling invariance). It is useful to understand how scalings affect the $X_p(\mathbb{R}^d)$ norm: recalling that the (NLS) is invariant under the scaling $u(\lambda^2 t, \lambda x)$, we have

Thus the (NLS) is $X_p(\mathbb{R}^d)$-subcritical for $\sigma < 2p/d$. In this situation, global existence for small data is equivalent to global existence for any data. Recall, however, that, for $\sigma \geq 4/d$, existence of blow-up phenomena is known for special initial data in $H^1(\mathbb{R}^d) \hookrightarrow X_p(\mathbb{R}^d)$. Therefore, it is impossible to obtain a global existence result for small data for $4/d \leq \sigma < 2p/d$. Notice that in the energy case $p = \sigma + 2$, one has $\sigma < 2p/d$ for any $\sigma + 2 < 2^*$.

Remark 3.7 (Global existence for small data). The main obstacle in proving global existence for small data turns out to be the linear part of the Duhamel formula $S(t)u_0$, since there isn’t, to our knowledge, a way to bound uniformly this term over $X_p(\mathbb{R}^d)$. The other possibility is to leave the linear term with a space-time norm: indeed, for some powers $\sigma > 2/d$, it is well-known that, if $u_0 \in H^1(\mathbb{R}^d)$ is such that

then the corresponding solution of (NLS) is globally defined (see \cite{3}). It is not hard to check that the result can be extended to $u_0 \in X_{\sigma+2}(\mathbb{R}^d)$.

4 Local well-posedness for $p > 2\sigma + 2$

As it was observed in Remark 3.2, the condition $p \leq 2\sigma + 2$ was necessary in order to use Strichartz estimates with no loss in regularity. For $p > 2\sigma + 2$, in order to estimate $L_t^p L_x^q$, one
must turn to estimate \( \|u\|_{L^p} \), which has a loss of one derivative. Therefore the distance one defines for the fixed-point argument must include norms with derivatives. This implies the need of a local Lipschitz condition
\[
\|u\|^{\sigma} \nabla u - |v|^{\sigma} \nabla v \lesssim C(\|u\|, \|v\|, \|\nabla u\|, \|\nabla v\|) \left( \|u - v\| + \|\nabla (u - v)\| \right),
\]
which we can only accomplish for \( \sigma \geq 1 \).

Because of the restriction \( \sigma \geq 1 \), one must have \( 4 < p < 2^* \), which excludes any dimension greater than three. For \( d = 3 \), it turns out that no range of \( p > 2\sigma + 2 \) can be considered. Indeed, if one uses \( L^p \) with \( f = |u|^{\sigma} u \),
\[
\left| \int_0^T S(-s)|u(s)|^{\sigma} u(s) ds \right|_{L^p((0,T),L^p)} \lesssim \|u\|^{\sigma+1}_{L^{2\sigma+2}((0,T),L^{p(\sigma+1)})} + \|\nabla(|u|^{\sigma} u)\|_{L^{r'}((0,T),L^{r'})}.
\]
We focus on the first norm on the right hand side. To control such a term, either \( X_p \rightarrow L^p((0,T),X_p) \), and
\[
\|u\|_{L^{2\sigma+2}((0,T),L^{p(\sigma+1)})} \lesssim T^{\frac{1}{\sigma+1}} \|u\|_{L^p((0,T),X_p)},
\]
or, setting \( r \geq 2 \) so that
\[
1 - \frac{3}{r} = \frac{3}{p(\sigma + 1)},
\]
one estimates
\[
\|u\|_{L^{2\sigma+2}((0,T),L^{p(\sigma+1)})} \lesssim \|\nabla u\|_{L^{2\sigma+2}((0,T),L^r)} \lesssim T^{\frac{1}{\sigma+1}} \|u\|_{L^p((0,T),L^r)}.
\]
In the first case, one needs \( 8 < p(\sigma + 1) < 2^* = 6 \). In the second, one must impose \( 2\sigma + 2 < q \). A simple computation yields \( p(3\sigma + 1) < 6 \), which is again impossible, since \( p(3\sigma + 1) > 16 \).

**Theorem 4.1** (Local well-posedness on \( X_p(\mathbb{R}^d) \) for \( d = 1, 2 \)). Given \( u_0 \in X_p(\mathbb{R}^d) \), there exists \( T = T(\|u_0\|_{X_p}) > 0 \) and an unique solution
\[
u \in C([0,T],X_p(\mathbb{R}^d))
\]
of (NLS) with initial data \( u_0 \). The solution depends continuously on the initial data and may be extended uniquely to a maximal interval \([0,T^*(u_0)]\). If \( T^*(u_0) < \infty \), then
\[
\lim_{t \to T^*(u_0)} \|u(t)\|_{X_p} = +\infty.
\]

*Proof.* Consider the space
\[
\mathcal{E} = \left\{ u \in L^p((0,T),X_p) : \|u\| := \|u\|_{L^p((0,T),X_p)} \leq M \right\},
\]
endowed with the natural distance
\[
d(u,v) = \|u - v\|.
\]
The space \( \mathcal{E},d \) is clearly a complete metric space. If \( u,v \in \mathcal{E} \), then
\[
\|u\|^{\sigma} u - |v|^{\sigma} v \|_{L^p((0,T),L^p)} \lesssim \int_0^T \left( \|u\|^{2\sigma}_{p(\sigma+1)} + \|v\|^{2\sigma}_{p(\sigma+1)} \right) \|u - v\|^{2\sigma}_{p(\sigma+1)} ds.
\]
Since $X_p(\mathbb{R}^d) \hookrightarrow L^{p(\sigma+1)}(\mathbb{R}^d)$,

$$
\|u\|_{p}^{\sigma} - \|v\|_{p}^{\sigma} \leq T \left( \|u\|_{L^p((0,t),X_p)}^{2\sigma} + \|v\|_{L^p((0,t),L^p)}^{2\sigma} \right) \|u - v\|_{L^p((0,t),X_p)}^{\sigma}.
$$ (4.1)

Choose an admissible pair $(\gamma, \rho)$ with $\rho$ sufficiently close to 2. We have

$$
\| \nabla (|u|^{\sigma} u - |v|^{\sigma} v) \|_{L^{p^\gamma}((0,T),L^{p^\rho})} \leq \|( |u|^{\sigma-1} + |v|^{\sigma-1} ) (|u - v| \nabla v + \rho \| \nabla (u - v) \|) \|_{L^{p^\gamma}((0,T),L^{p^\rho})}.
$$

As an example, we treat the term $|u|^{\sigma-1} |u - v| \nabla v$:

$$
\| |u|^{\sigma-1} |u - v| \nabla v \|_{\rho} \leq \| |u|^{\sigma-1} \|_{\rho} - \| |u - v| \nabla v \|_{2}\frac{2\rho}{\rho+1}.
$$

Therefore

$$
\| \nabla (|u|^{\sigma} u - |v|^{\sigma} v) \|_{L^{p^\gamma}((0,T),L^{p^\rho})} \leq T^{\frac{1}{\gamma}} \left( \|u\|_{L^p((0,T),X_p)}^{2\sigma} + \|v\|_{L^p((0,T),L^p)}^{2\sigma} \right) \|u - v\|_{L^p((0,T),X_p)}^{\sigma}.
$$ (4.2)

For $u \in \mathcal{E}$, define

$$
\Phi(u)(t) = S(t)u_0 + i\lambda \int_0^t S(t-s)|u(s)|^\sigma u(s) ds, \quad 0 \leq t \leq T.
$$

The estimates (4.1) and (4.2), together with (2.2) and Strichartz’s estimates then imply that

$$
\| \Phi(u) \| \leq \|u_0\|_{X_p} + \left( \int_0^t S(t-s)|u(s)|^\sigma u(s) ds \right)_{L^{p^\gamma}((0,T),L^{p^\rho})} + \left( \int_0^t S(t-s)|u(s)|^\sigma u(s) ds \right)_{L^{p}((0,T),H^{s})}
$$

$$
\leq \|u_0\|_{X_p} + \left( \|u\|_{L^p((0,T),X_p)}^{\sigma} + \|\nabla (|u|^{\sigma} u)\|_{L^{p^\gamma}((0,T),L^{p^\rho})} \right)
$$

$$
\leq \|u_0\|_{X_p} + \left( T^{\frac{1}{\gamma}} + T^{\frac{1}{\rho}} \right) M^{\sigma+1}
$$

and

$$
d(\Phi(u), \Phi(v)) \leq \left( \|u\|_{L^p((0,T),L^p)}^{\sigma} + \|\nabla (|u|^{\sigma} u) - \nabla (|v|^{\sigma} v)\|_{L^{p^\gamma}((0,T),L^{p^\rho})} \right)
$$

$$
\leq \left( T^{\frac{1}{\gamma}} + T^{\frac{1}{\rho}} \right) M^{\sigma} d(u,v).
$$ (4.3)

Choosing $M \sim 2\|u_0\|_{X_p}$, for $T = T(\|u_0\|_{X_p})$ small enough, it follows that $\Phi : \mathcal{E} \hookrightarrow \mathcal{E}$ is a strict contraction. Banach’s fixed point theorem now implies that $\Phi$ has a unique fixed point over $\mathcal{E}$, which is the unique solution $u$ of (NLS) with initial data $u_0$ on the interval $(0,T)$. This solution may then be extended uniquely to a maximal interval of existence $(0,T(u_0))$. The blow-up alternative follows by a standard continuation argument. Finally, if $u, v$ are two solutions with initial data $u_0, v_0 \in X_p(\mathbb{R}^d)$, as in (4.3), one has

$$
d(u,v) = d(\Phi(u), \Phi(v)) \leq \|u_0 - v_0\|_{X_p} + \left( T^{\frac{1}{\gamma}} + T^{\frac{1}{\rho}} \right) M^{\sigma} d(u,v)
$$

$$
\leq \|u_0 - v_0\|_{X_p} + \left( T^{\frac{1}{\gamma}} + T^{\frac{1}{\rho}} \right) \left( \max\{\|u_0\|_{X_p}, \|v_0\|_{X_p}\} \right)^{\sigma} d(u,v)
$$

Thus, for $T_0 = T_0(\|u_0\|_{X_p}, \|v_0\|_{X_p})$ small,

$$
d(u,v) \leq \|u_0 - v_0\|_{X_p},
$$

and continuous dependence follows.
**Proposition 4.2** (Persistence of integrability). Fix $d = 1, 2$ and $p > \tilde{p}$. Given $u_0 \in X_{\tilde{p}}(\mathbb{R}^d)$, consider the $X_{\tilde{p}}(\mathbb{R}^d)$-solution $u \in C([0, T^*(u_0)), X_{\tilde{p}})$ of \((\text{NLS})\) with initial data $u_0$. Then $u \in C([0, T^*(u_0)), X_{\tilde{p}})$.

**Proof.** As in the proof of Proposition 3.4, given $T < T^*(u_0)$, one must prove that the $L^\tilde{p}$ norm of $u$ is bounded over $(0, T)$. Applying (2.2) to the Duhamel formula of $u$, we have

$$\|u\|_{L^\tilde{p}(0,T),L^\tilde{p}} \lesssim \|u_0\|_{X_{\tilde{p}}} + \|u\|^\sigma_{L^2(0,T),L^\tilde{p}} + \|u\|^{\sigma}|\nabla u|_{L^{\gamma'}((0,T),L^\rho')} ,$$

for any admissible pair $(\gamma, \rho)$. The penultimate term is treated using the injection $X_{\rho_0}(\mathbb{R}^d) \hookrightarrow L^\rho(\sigma+1)$:

$$\|u\|^\sigma_{L^2(0,T),L^\tilde{p}} = \|u\|^\sigma_{L^{2\sigma+2}(0,T),L^\rho(\sigma+1)} \lesssim T^{\sigma\rho} \|u\|^\sigma_{L^\rho((0,T),X_{\tilde{p}}(\mathbb{R}^d))} < \infty .$$

Choose $\rho$ sufficiently close to $2$ so that $X_{\rho}(\mathbb{R}^d) \hookrightarrow L^{2\rho}(\sigma+1)$, then

$$\|u\|^\sigma |\nabla u|_{L^{\gamma'}((0,T),L^\rho')} \lesssim \|u\|^{\sigma\rho}_{L^{2\sigma+2}(0,T),L^\rho'} \|\nabla u\|_{L^{\gamma'}((0,T),L^\rho')} \lesssim T^{\sigma\rho} \|u\|^\sigma_{L^\rho((0,T),X_{\tilde{p}})} < \infty .$$

Therefore $\|u\|_{L^\rho((0,T),L^\tilde{p})}$ is finite and the proof is finished.

\[ \square \]

## 5 Further comments

In light of the results we have proven, we highlight some new questions that have risen:

1. Local well-posedness: In dimensions $d \geq 3$, the local well-posedness in the case $p = 2\sigma + 2$ remains open. Is this optimal? As we have argued in Remark 3.2, this case requires new estimates for the Schrödinger group.

2. Global well-posedness: this problem is completely open for $p = 2\sigma + 2$. Even if the energy is well-defined, there are still several cases where global well-posedness (even for small data) remains unanswered.

3. New blow-up behaviour: in the opposite perspective, is it possible to exhibit new blow-up phenomena? This would be especially interesting either for the defocusing case or for the $L^2$-subcritical case, where blow-up behaviour in $H^1$ is impossible.

4. Stability of ground-states: in the $H^1$ framework, the work of [1] has shown that the ground-states are orbitally stable under $H^1$ perturbations. Does the result still hold if we consider $X_{\tilde{p}}$ perturbations?

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