Witten Multiple Zeta Values Attached to \(\mathfrak{sl}(4)\)

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Abstract. In this paper we shall prove that every Witten multiple zeta value of weight \(w > 3\) attached to \(\mathfrak{sl}(4)\) at nonnegative integer arguments is a finite \(\mathbb{Q}\)-linear combination of MZVs of weight \(w\) and depth three or less, except for the nine irregular cases where the Riemann zeta value \(\zeta(w - 2)\) and the double zeta values of weight \(w - 1\) and depth < 3 are also needed.

1. Introduction

It is well-known that suitably defined zeta and \(L\)-functions and their special values often play significant roles in many areas of mathematics. In [24] Witten studied one variable zeta functions attached to various Lie algebras and related their special values to the volumes of certain moduli spaces of vector bundles of curves. Zagier [25] (and independently Garoufalias) gave direct proofs that such functions at positive even integers are rational multiples of \(\pi\). More recently Matsumoto and his collaborators [12, 13, 14, 15, 17, 23] defined multiple variable versions of these functions and began to investigate their analytical and arithmetical properties.

Let \(\mathbb{N}\) be the set of positive integers and \(\mathbb{N}_0 = \mathbb{N} \cup \{0\}\). For any \(d \in \mathbb{N}\) we let \([d] = (1, \ldots, d)\) be a poset with the usual increasing order. By \(i = (i_1, \ldots, i_\ell) \subseteq [d]\) we mean a nonempty subset of \([d]\) as a poset. We say the length of \(i\) is \(\text{lg}(i) = \ell\) and the weight of \(i\) is \(\text{wt}(i) = i_1 + \cdots + i_\ell\). We define the generalized multiple zeta function of depth \(d\) as

\[
\zeta_d((s)_{i \subseteq [d]}) := \sum_{m_1, \ldots, m_d = 1}^{\infty} \prod_{i \subseteq [d]} \left( \sum_{j = 1}^{\text{lg}(i)} m_{i_j} \right)^{-s_i}.
\]

For example, the Euler-Zagier multiple zeta function [1, 25, 26]

\[
\zeta(s_1, \ldots, s_d) := \sum_{m_1 > \cdots > m_d \geq 1} m_1^{-s_1} m_2^{-s_2} \cdots m_d^{-s_d}
\]

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corresponds to the special case that $s_i = 0$ unless $i = (1, 2, \ldots, \ell)$. The Mordell-Tornheim multiple zeta function defined by (10) (see [18, 22]) is the case where $s_i = 0$ unless $\lg(i) = 1$ or $\lg(i) = d$. If we set $s_i = 0$ unless $i = (a, a + 1, \ldots, b)$ is a consecutive string of positive integers then we get exactly Witten multiple zeta function associated to the special linear Lie algebra $\zeta_{\mathfrak{sl}(d)}$ (see [17]):

$$
\zeta_{\mathfrak{sl}(d+1)}(s_{i,j})_{1 \leq i \leq j \leq d} := \sum_{m_1, \ldots, m_d=1}^{\infty} \prod_{1 \leq i \leq j \leq d} (m_i + m_{i+1} + \cdots + m_j)^{-s_{i,j}}.
$$

The generalized multiple zeta-functions defined by (1) are special cases of the functions studied by Essouabri [6], de Crisenoy [5], and Matsumoto [16]. In particular we know that $\zeta_d(s)$ has meromorphic continuation to the whole complex space $\mathbb{C}^{2d-1}$. However, in the form (1) we may have better control of its arithmetical properties, namely, we may be able to compute explicitly their special values at nonnegative integers. As usual we say a value of the Euler-Zagier multiple zeta function at positive integers a multiple zeta value (MZV for short) if it is finite. Our major interest is to solve the following problem.

**Main Problem.** Suppose $s \in \mathbb{N}^{2d-1}$ (resp. $s \in \mathbb{N}_0^{2d-1}$) and $\zeta_d(s)$ converges. Is $\zeta_d(s)$ always a $\mathbb{Q}$-linear combination of MZVs of the same (resp. same or lower weights) and of depth $d$ or less?

When $d = 2$ the function in (1) becomes Mordell-Tornheim double zeta function. By the main result of [30] (see Prop. 2.4) we know the above Main Problem has an affirmative answer for all Mordell-Tornheim multiple zeta functions. In this paper we will consider $\zeta_{\mathfrak{sl}(4)}(s)$ which is essentially the case when $d = 3$.

Throughout the rest of the paper whenever we say some special value with positive (resp. nonnegative) integer arguments is expressible by MZVs we mean that the value can be expressed as a finite $\mathbb{Q}$-linear combination of MZVs of the same (resp. same or lower) weights and the same or lower depths. Our main result is the following theorem which provides an affirmative answer to the above Main Problem for the case $d = 3$.

**Theorem 1.1.** Suppose $s \in \mathbb{N}^7$. If $\zeta_3(s)$ converges then it is expressible by MZVs. Moreover, every Witten multiple zeta value of weight $w$ attached to $\mathfrak{sl}(4)$ at nonnegative integers is a $\mathbb{Q}$-linear combination of MZVs of weight $w$ and depth three or less, except for the nine irregular cases defined by (13) to (17) where the Riemann zeta value $\zeta(w-2)$ and the double zeta values of weight $w-1$ and depth $\leq 2$ are also needed.

Similar ideas can be applied in the study of Witten multiple zeta functions attached to other semi-simple Lie algebras, see [27, 28, 29].

In [19], Nakamura found some functional relations of $\zeta_{\mathfrak{sl}(4)}$, $\zeta_{\mathfrak{sl}(4)}$ and $\zeta_{\mathfrak{sl}(5)}$ (with characters) via the double Lerch functions. Consequently, it is possible to express some special types of $\zeta_{\mathfrak{sl}(4)}$ values by MZVs through this approach.

Theorem 1.1 can be regarded as a more precise version for $\zeta_3$ of a result by Terasoma.
[21] who showed that for any rational convex cone $C$ with interior $C^0$, any character $\chi$ of $\mathbb{Z}^m$, and any $\mathbb{Q}$-linear forms $l_1, \ldots, l_n$ on $\mathbb{Q}^m$ such that $l_i(C^0) > 0$ ($i = 1, \ldots, n$) the infinite sum
\[
\sum_{x \in C^0 \cap \mathbb{Z}^m} \frac{\chi(x)}{l_1(x) \cdots l_n(x)}
\]
is a $\mathbb{Q}^{ab}$-linear combination of multi-polylog values at roots of unity, where $\mathbb{Q}^{ab}$ is the extension of $\mathbb{Q}$ adjoining all roots of unity. In our case $\chi$ is the trivial character, the cone is the first quadrant of the $\mathbb{R}^3$, and the linear forms are $l(x, y, z) = x, y, z, x + y, x + z, y + z, x + y + z$ with suitable multiplicities.

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2. Some preliminary results

In this section we collect some useful facts which will be convenient for us to present our main result in later sections.

2.1. Convergence domain. We assume that all components of $s$ are integers and derive the necessary and sufficient conditions for (1) to converge although the conditions are still sufficient if we allow complex variables and take the corresponding real parts of the variables in the conditions.

Recall that the MZV in (2) converges if and only if
\[
\sum_{j=1}^{\ell} s_j > \ell
\]
for all $\ell = 1, \ldots, d$. It is straight-forward to see the same holds for the “star” version of the multiple zeta function:
\[
\zeta^*(s_1, \ldots, s_d) := \sum_{m_1 \geq \cdots \geq m_d \geq 1} m_1^{-s_1} m_2^{-s_2} \cdots m_d^{-s_d} .
\]
Special values of (5) were studied in [9] and [20]. We can extend this convergence criterion easily to our generalized multiple zeta functions.

**Proposition 2.1.** The generalized MZV
\[
\zeta_d(s) := \sum_{m_1, \ldots, m_d = 1}^{\infty} \prod_{j=1}^{\infty} \left( \sum_{m_j} \right)^{-s_j}
\]
converges if and only if for all \( \ell = 1, \ldots, d \) and all \( \mathbf{i} = (i_1, \ldots, i_\ell) \subseteq [d] \)
\[
\ell \sum_{j \text{ contains at least one of } i_1, \ldots, i_\ell} m_{i_j} > \ell.
\]

**Proof.** The idea of the proof is similar to that of [30, Thm. 4]. First we observe that for any subset \( \mathbf{i} = (i_1, \ldots, i_\ell) \subseteq [d] \) we have
\[
\max_{j \leq \ell} \{m_{i_j} : 1 \leq j \leq \ell\} \leq \ell \sum_{j=1}^\ell m_{i_j} \leq \ell \max_{j \leq \ell} \{m_{i_j} : 1 \leq j \leq \ell\}.
\]
Hence
\[
\sum_{\pi \in \mathcal{S}_d} \sum_{m_{i_1} \geq \cdots \geq m_{i_d} \geq 1} \prod_{1 \leq i \leq [d]} (m_{\pi(i)})^{-s_{\pi(i)}} \leq \sum_{m_1, \ldots, m_\ell = 1}^\infty \prod_{j=1}^\ell \left( \sum_{m_j = 1}^{m_{\pi(j)}} \right)^{-s_{\pi(j)}} \leq d \cdot \text{LHS},
\]
where LHS means the quantity at the extreme left of the above inequalities. Observe that for each fixed \( \pi \in \mathcal{S}_d \) the power of \( m_{\pi(j)} \) in LHS is \(-\sum_{i \in [d]} s_{\pi(i)}\) where \( i \) runs through all subsets of \([d]\) whose first component is \( j \). Hence the criterion (4) implies that LHS and \( d \cdot \text{LHS} \) of (8) converges if and only if for each fixed \( \pi \in \mathcal{S}_d \)
\[
\sum_{i_1 = 1}^\ell \sum_{i : \text{first component is } i_1} s_{\pi(i)} > \ell,
\]
for all \( \ell = 1, \ldots, d \). Let \( i_j = \pi^{-1}(j) \) (\( j = 1, \ldots, \ell \)). Then in the above sum \( i \) runs through all subset of \([d]\) containing at least some \( i_j \) (\( j = 1, \ldots, \ell \)). This is exactly (7), as desired. \( \Box \)

**Remark 2.2.** It is easy to see that similar result holds if we replace the factors \( \sum m_{i_j} \) in (6) by linear forms of \( m_1, \ldots, m_d \) with nonnegative integer coefficients.

**2.2. MZVs with arbitrary integer arguments.** In this paper we will mostly consider multiple zeta values with nonnegative integer arguments as long as they converge. However, we will prove a more general result as follows since the inductive proof forces us to do so.

**Proposition 2.3.** Suppose \( s_1, \ldots, s_d \in \mathbb{Z} \). If \( \zeta(s_1, \ldots, s_d) \) converges then it can be expressed as a \( \mathbb{Q} \)-linear combination of MZVs (at positive integer arguments) of the same or lower weights and the same or lower depths.

**Proof.** We prove the proposition by induction on the depth. When \( d = 1 \) we have nothing to prove. Suppose the proposition holds for all MZVs of depth \( d - 1 \). Suppose further
\[
s_1 + \cdots + s_j > j \quad \text{for all } j = 1, \ldots, d,
\]
so that \( \zeta(s_1, \ldots, s_d) \) converges. Assume \( -t = s_j \leq 0 \). Then by definition
\[
\zeta(s_1, \ldots, s_d) = \sum_{m_1 > \cdots > m_{j-1} > m_{j+1} > \cdots > m_d} \left( \sum_{m_j = 1}^{m_{j-1} - 1} - \sum_{m_j = 1}^{m_{j+1} - 1} \right) m_j \prod_{j=1}^d m_j^{s_j}.
\]
Now by the well-known formula (see, for e.g., [11, p. 230, Thm. 1])

\[ \sum_{m=1}^{n} m^t = \frac{1}{t+1} (B_{t+1}(n+1) - B_{t+1}(0)) \]

where \( B_{t+1}(x) \) is the Bernoulli polynomial we immediately see that \( \zeta(s_1, \ldots, s_d) \) is a \( \mathbb{Q} \)-linear combination of MZVs of the forms \( \zeta(s_1, \ldots, s_j - 1 - s, s_j + 1, \ldots, s_d) \) and \( \zeta(s_1, \ldots, s_j - 1 - s, s_{j+1}, \ldots, s_d) \) where \( s = 0, 1, \ldots, -s_j + 1 \). All of these MZVs are easily to be shown as convergent values by (9) and therefore the proposition follows from the induction assumption. \( \square \)

2.3. Mordell-Tornheim zeta functions. They are defined by (see [18, 22])

\[ \zeta_{MT}(s_1, \ldots, s_d; s) := \sum_{m_1, \ldots, m_d = 1}^{\infty} m_1^{-s_1} m_2^{-s_2} \cdots m_d^{-s_d} (m_1 + \cdots + m_d - s). \] (10)

The main result of [30] is the following

PROPOSITION 2.4 ([30, Thm. 5]). Let \( s_1, \ldots, s_d \) and \( s \) be nonnegative integers. If at most one of them is equal to 0 then the Mordell-Tornheim zeta value \( \zeta_{MT}(s_1, \ldots, s_d; s) \) can be expressed as a \( \mathbb{Q} \)-linear combination of MZVs of the same weight and depth.

In this paper we will only need this proposition when the depth is three.

2.4. A combinatorial lemma. The next lemma will be used heavily throughout the paper.

LEMMA 2.5 ([30, Thm. 1]). Let \( r \) and \( n_1, \ldots, n_r \) be positive integers, and let \( x_1, \ldots, x_r \) be non-zero real numbers such that \( x_1 + \cdots + x_r \neq 0 \). Then

\[ \prod_{j=1}^{r} x_j^{n_j} = \sum_{j=1}^{r} \left( \prod_{k=1}^{r} \sum_{a_k=0}^{n_k-1} \right) \frac{M_j}{x_j^{n_j} + A_j} \prod_{k=1}^{r} \frac{1}{x_k^{n_k - a_k}}, \]

where the multi-nomial coefficient

\[ M_j = \frac{(n_j + A_j - 1)!}{(n_j - 1)!} \prod_{k=1}^{r} \frac{1}{a_k!} \]

and \( A_j = \sum_{k=1}^{r} a_k \).

The notation \( \prod_{k=1}^{r} \sum_{k \neq j}^{n_k-1} \) means the multiple sum \( \sum_{a_1=0}^{n_1-1} \cdots \sum_{a_j=0}^{n_j-1} \sum_{a_{j+1}=0}^{n_{j+1}-1} \cdots \sum_{a_r=0}^{n_r-1} \).
3. Proof of Theorem 1.1

By definition (1)

\[ \zeta_3(s_1, \ldots, s_7) := \sum_{m_1, \ldots, m_3=1}^{\infty} \frac{(m_1 + m_2 + m_3)^{-s_7}}{m_1^{s_1} m_2^{s_2} m_3^{s_3} (m_1 + m_2)^{s_4} (m_2 + m_3)^{s_5} (m_1 + m_3)^{s_6}}. \]

To guarantee convergence by (7) we need to assume:

\[
\begin{align*}
  i &= (1) : \quad s_1 + s_4 + s_6 + s_7 > 1, \\
  i &= (1) : \quad s_2 + s_4 + s_5 + s_7 > 1, \\
  i &= (1) : \quad s_3 + s_5 + s_6 + s_7 > 1, \\
  i &= (1, 2) : \quad s_1 + s_2 + s_4 + s_5 + s_6 + s_7 > 2, \\
  i &= (1, 3) : \quad s_1 + s_3 + s_4 + s_5 + s_6 + s_7 > 2, \\
  i &= (2, 3) : \quad s_2 + s_3 + s_4 + s_5 + s_6 + s_7 > 2, \\
  i &= (1, 2, 3) : \quad s_1 + s_2 + s_3 + s_4 + s_5 + s_6 + s_7 > 3.
\end{align*}
\]

We now use a series of reductions to prove the theorem. All of the steps will be explicitly given so that one may carry out the computation of (11) by following them.

Step (i). Since \(2(m_1 + m_2 + m_3) = (m_1 + m_2) + (m_2 + m_3) + (m_1 + m_3)\) we see clearly that we can assume either \(s_4 = 0\) or \(s_5 = 0\) or \(s_6 = 0\). In fact in Lemma 2.5 taking \(r = 3\), \(x_1 = m_1 + m_2\), \(x_2 = m_2 + m_3\), \(x_3 = m_1 + m_3\), \(n_1 = s_4\), \(n_2 = s_5\) and \(n_3 = s_6\) we get (with \(s = (s_1, s_2, s_3)\) fixed):

\[
\begin{align*}
  \zeta_3(s_4, s_5, s_6, s_7) &= \frac{s_4 - 1}{a_4=0} \sum_{a_5=0}^{s_5-1} \frac{(s_6 + a_4 + a_5 - 1)! \zeta_3(s_4 - a_4, s_5 - a_5, 0, s_7 + s_6 + a_4 + a_5)}{2^{s_7 + a_4 + a_5} (s_6 - 1)! a_4! a_5!} \\
  &\quad + \frac{s_5 - 1}{a_5=0} \sum_{a_6=0}^{s_6-1} \frac{(s_5 + a_4 + a_6 - 1)! \zeta_3(s_4 - a_4, 0, s_6 - a_6, s_7 + s_5 + a_4 + a_6)}{2^{s_7 + a_4 + a_6} (s_5 - 1)! a_4! a_6!} \\
  &\quad + \frac{s_6 - 1}{a_6=0} \sum_{a_7=0}^{s_7-1} \frac{(s_4 + a_5 + a_6 - 1)! \zeta_3(s, 0, s_5 - a_5, s_6 - a_6, s_7 + s_4 + a_5 + a_6)}{2^{s_7 + a_5 + a_6} (s_4 - 1)! a_5! a_6!}.
\end{align*}
\]

We recommend the interested reader to check the convergence of the above values by (12). The rule of thumb is that if we apply Lemma 2.5 with each \(x_j\) a positive combination of indices then the convergence is automatically guaranteed. In each of the following steps we often omit this convergence checking since it is straightforward in most cases. The only exception is (26) which in fact poses the most difficulty.
We call the values appearing in the nine cases from (13) to (17) associated to the special linear Lie algebra \(sl(4)\) (see (3)):

\[
\zeta_{sl(4)}(s_1, \ldots, s_6) := \sum_{m_1, \ldots, m_6 = 1}^{\infty} \frac{m_1^s m_2^s m_3^s (m_1 + m_2 + m_3 - s_6)^{m_3}}{(m_1 + m_2 + m_3)^s}.
\]

Before going on we need to define the so called regular and irregular special values of \(\zeta_{sl(4)}(s_1, \ldots, s_6)\). Clearly the following special values are expressible by MZVs of mixed weights: \((0)_k\) means to repeat 0 \(k\) times

(13) \(\zeta_{sl(4)}([0], b, 0, a) = \zeta_{sl(4)}([0], b, a) = \zeta(a, b, 0) = \begin{cases} 
\zeta(a, b - 1) - \zeta(a, b), & a > 1, b > 1, \\
\zeta(a - 1) - \zeta(a) - \zeta(a, 1), & a > 1, b = 1,
\end{cases}\)

(14) \(\zeta_{sl(4)}(b, [0], a) = \zeta_{sl(4)}(0, b, [0], a) = \zeta_{sl(4)}(0, b, 0, 0, a) = \zeta(a, 0, b) = \begin{cases} 
\zeta(a - 1, b) - \zeta(a, b - 1) - \zeta(a, b), & a > 2, b > 1, \\
\zeta(a - 1, 1) - \zeta(a - 1) + \zeta(a) - \zeta(a, 1), & a > 2, b = 1,
\end{cases}\)

(15) \(\zeta_{sl(4)}([0], a) = \zeta(a, 0, 0) = \frac{1}{2} \zeta(a - 2) - \frac{3}{2} \zeta(a - 1) + \zeta(a), \quad a > 3.\)

By Step (ii) we will see that special values

(16) \(\zeta_{sl(4)}(0, 0, s_3, s_4, 0, s_6)\) and \(\zeta_{sl(4)}(s_1, 0, 0, s_4, s_6)\)

are also expressible by MZVs of mixed weights. Further, if \(a, b \geq 2\) then we have

(17) \(\zeta_{sl(4)}(0, 0, a, b, 0) = \sum_{m_1, m_2, m_3 = 1}^{\infty} \frac{1}{(m_1 + m_2)^a (m_2 + m_3)^b} = \sum_{m_2 = 1}^{\infty} \left( \sum_{m_1 < m_3} + \sum_{m_1 = m_3} + \sum_{m_1 > m_3} \right) \frac{1}{(m_1 + m_2)^a (m_2 + m_3)^b} = \zeta(a, b, 0) + \zeta(b, a, 0) + \zeta(a + b, 0) = \zeta(b, a - 1) + \zeta(a, b - 1) + \zeta(a + b - 1) - \zeta(a) \zeta(b).\)

We call the values appearing in the nine cases from (13) to (17) irregular values. Otherwise \(\zeta_{sl(4)}(s_1, \ldots, s_6)\) is called a regular value.

Step (ii). Taking \(x_1 = m_1\) and \(x_2 = m_2 + m_3\) in Lemma 2.5 we get

(18) \(\zeta_{sl(4)}(s_1, \ldots, s_6) = \sum_{a_5 = 0}^{s_6} \left( \begin{array}{c}
s_1 + a_5 - 1 \\
0
\end{array} \right) \zeta_{sl(4)}(0, s_2, s_3, s_4, s_5 - a_5, s_6 + s_1 + a_5)\).
where \( s_5 \geq 1 \). Then we may use Step (ii.2) and of depth three by Prop. 2.4. Case (ii.1) is proved.

**Step (ii.1).** Let \( s_5 = 0 \). Then we must have either \( s_1 \geq 1 \) or \( s_2 \geq 1 \) since we assume

\[
\zeta_{s_1(4)}(s_1, \ldots, s_4, 0, s_6) = \sum_{m_1, \ldots, m_3=1}^{\infty} \frac{1}{m_1^s m_2^s m_3^s (m_1 + m_2)^{s_3} (m_1 + m_2 + m_3)^{s_6}}
\]

is regular. Then we may use \( x_1 = m_1 \) and \( x_2 = m_2 \) in Lemma 2.5 to get

\[
\zeta_{s_1(4)}(s_1, \ldots, s_4, 0, s_6) = \sum_{a_1=0}^{s_1-1} \left( \frac{s_2 + a_1 - 1}{a_1} \right) \zeta_{s_1(4)}(s_1 - a_1, 0, s_3, s_2 + s_2 + a_1 + 1, 0, s_6) + \sum_{a_2=0}^{s_2-1} \left( \frac{s_1 + a_2 - 1}{a_2} \right) \zeta_{s_1(4)}(0, s_2 - a_2, s_3, s_1 + s_4 + a_2, 0, s_6).
\]

By symmetry we only need to consider

\[
\zeta_{s_1(4)}(0, s_2, s_3, s_4, 0, s_6) = \sum_{m_1, \ldots, m_3=1}^{\infty} \frac{(m_1 + m_2 + m_3)^{-s_6}}{m_2^s m_3^s (m_1 + m_2)^{s_3} (m_1 + m_2 + m_3)^{s_6}}
\]

where \( s_2 \geq 1 \). But now we may take \( x_1 = m_1 + m_2 \) and \( x_2 = m_3 \) in Lemma 2.5 to reduce it to either \( s_3 = 0, s_4 \geq 1 \) or \( s_4 = 0, s_3 \geq 1 \). Then we get the following two kind of values:

\[
\zeta_{s_1(4)}(0, s_2, 0, s_4, 0, s_6) = \zeta(s_6, s_4, s_2), \quad \zeta_{s_1(4)}(0, s_2, s_3, 0, 0, s_6) = \zeta_{MT}(s_2, s_4, 0, s_6),
\]

where \( s_2, s_3, s_4, s_6 \geq 1 \). But \( \zeta_{MT}(s_2, s_4, 0, s_6) \) is expressible by MZVs of the same weight and of depth three by Prop. 2.4. Case (ii.1) is proved.

**Step (ii.2).** Let \( s_1 = 0 \) and \( s_5 \geq 1 \). Consider

\[
\zeta_{s_1(4)}(0, s_2, \ldots, s_6) = \sum_{m_1, \ldots, m_3=1}^{\infty} \frac{(m_1 + m_2 + m_3)^{-s_6}}{m_2^s m_3^s (m_1 + m_2)^{s_3} (m_1 + m_2 + m_3)^{s_5}}.
\]

To guarantee convergence we must have \( s_4 + s_6 > 1, s_3 + s_5 + s_6 > 1, s_2 + s_4 + s_5 + s_6 > 2, s_3 + s_4 + s_5 + s_6 > 2, \) and \( s_2 + s_3 + s_4 + s_5 + s_6 > 3 \). Taking \( x_1 = m_1 + m_2 \) and \( x_2 = m_3 \) in Lemma 2.5 we get
\[ \zeta_{\mathbb{A}(4)}(0, s_2, \ldots, s_6) \]
\[ = \sum_{a_3=0}^{s_4-1} \binom{s_4}{a_3} \zeta_{\mathbb{A}(4)}(0, s_2, s_3 - a_3, 0, s_5, s_4 + s_6 + a_3) \]
\[ + \sum_{a_4=0}^{s_5-1} \binom{s_5}{a_4} \zeta_{\mathbb{A}(4)}(0, s_2, s_3 - a_4, s_4, s_5, s_3 + s_6 + a_4) . \]

Step (ii.2.1). Let \( s_1 = s_4 = 0, s_5, s_6 \geq 1 \). Then we must have either \( s_2 \geq 1 \) or \( s_3 \geq 1 \) in
\[ \zeta_{\mathbb{A}(4)}(0, s_2, s_3, 0, s_5, s_6) = \sum_{m_1, \ldots, m_3 = 1}^{\infty} \frac{1}{m_1^2 m_2^{s_3} (m_2 + m_3)^{s_5} (m_1 + m_2 + m_3)^{s_6}} . \]

It is easy to see that all the triple zeta values above have the same weight. We remind the reader that to determine the weight of a MZV it is not enough just to add up all the components. One also needs to check that every component is positive.

Step (ii.2.2). Let \( s_1 = s_3 = 0 \) and \( s_2, s_4, s_5 \geq 1 \). By (12), to guarantee convergence of
\[ \zeta_{\mathbb{A}(4)}(0, s_2, 0, s_4, s_5, s_6) = \sum_{m_1, \ldots, m_3 = 1}^{\infty} \frac{(-1)^{s_5} (m_1 + m_2 + m_3)^{-s_6}}{(-m_2)^{s_2} (m_1 + m_2)^{s_4} (m_2 + m_3)^{s_5}} \]
we need to assume
\[ s_4 + s_6 > 1, \quad s_5 + s_6 > 1, \quad s_4 + s_5 + s_6 > 2, \quad s_2 + s_4 + s_5 + s_6 > 3. \]
Moreover, since \( \zeta_{\alpha(4)}(0, s_2, 0, s_4, s_5, s_6) \) is regular we must have \( s_2 + s_6 \geq 1 \). Putting \( x_1 = -m_2, x_2 = m_1 + m_2 \) and \( x_3 = m_2 + m_3 \) in Lemma 2.5 we get

\[
\zeta_{\alpha(4)}(0, s_2, 0, s_4, s_5, s_6)
\]

\[
= \sum_{a_4=0}^{s_2-1} \sum_{a_5=0}^{s_5-1} \frac{(s_2 + a_4 + a_5 - 1)!}{(-1)^{a_4}(s_2 - 1)!a_4!a_5!} \zeta_{\alpha(4)}(0, 0, 0, s_4 - a_4, s_5 - a_5, s_6 + s_2 + a_4 + a_5)
\]

\[
+ \sum_{a_2=0}^{x_2-1} \sum_{a_5=0}^{s_5-1} \frac{(s_4 + a_2 + a_5 - 1)!}{(-1)^{a_2}(s_4 - 1)!a_2!a_5!} \zeta_{\alpha(4)}(0, s_2 - a_2, 0, 0, s_5 - a_5, s_6 + s_4 + a_2 + a_5)
\]

\[
+ \sum_{a_2=0}^{x_2-1} \sum_{a_4=0}^{s_4-1} \frac{(s_5 + a_2 + a_4 - 1)!}{(-1)^{a_2}(s_5 - 1)!a_2!a_4!} \zeta_{\alpha(4)}(0, s_2 - a_2, 0, s_4 - a_4, 0, s_6 + s_5 + a_2 + a_4)
\]

We point out that since we have used \( x_1 = -m_2 \) we need to check the convergence of all the above three kinds of values under the assumption (22) even though the checking itself is trivial.

Returning to the reduction of (23) we see that the last two sums are expressible by MZVs so we only need to consider those values appearing in the first sum, namely, those of the form in the next case.

Step (ii, 2, 3). Let \( s_1 = s_2 = s_3 = 0 \) and \( s_4, s_5 \geq 1 \). Then we must have \( s_6 \geq 1 \) since we only consider regular values. To guarantee convergence of

\[
\zeta_{\alpha(4)}(0, 0, 0, s_4, s_5, s_6) = \sum_{m_1, \ldots, m_3=1}^{\infty} \frac{(m_1 + m_2 + m_3)^{-s_6}}{(m_1 + m_2)^{s_4}(m_2 + m_3)^{s_5}}
\]

we need to assume \( s_4 + s_6 > 1, s_5 + s_6 > 1, \) and \( s_4 + s_5 + s_6 > 3 \). If either \( s_4 > 1 \) or \( s_5 > 1 \) then we may further assume that \( s_4 > 1 \) by change of index \( m_1 \leftrightarrow m_3 \) in (24). We may set \( x_1 = -(m_2 + m_3) \) and \( x_2 = m_1 + m_2 + m_3 \) in Lemma 2.5 to get

\[
\zeta_{\alpha(4)}(0, 0, 0, s_4, s_5, s_6)
\]

\[
= \sum_{a_5=0}^{s_6-2} \frac{(s_6 + a_5 - 1)!}{a_5!} (-1)^{a_5} \zeta_{\alpha(4)}(s_6 + a_5, 0, 0, s_4, s_5 - a_5, 0)
\]

\[
+ \sum_{a_6=0}^{s_6-2} \frac{(s_5 + a_6 - 1)!}{a_6!} (-1)^{a_6} \zeta_{\alpha(4)}(s_5 + a_6, 0, 0, s_4, 0, s_6 - a_6)
\]

\[
+ \frac{(s_6 + s_5 - 2)!}{s_5!} (-1)^{s_5} \lim_{N \to \infty} \left( \zeta_{\alpha(4)}^{(N)}(s_6 + s_5 - 1, 0, 0, s_4, 0, 1) \right)
\]
where $\zeta^{(N)}$ is the partial sum of (1) when each index $m_{ij}$ goes from 1 to $N$. Observe that all the values in the first two lines above are convergent and the second line is already expressible by MZVs of the same weight. So we are reduced to consider the following two kinds of values:

(A). \quad \zeta_{a(4)}(s_1, 0, 0, s_4, t, 0) (with $s_1$, $s_4 \geq 1$, $t > 1$) from first line above and

(B). \quad The limit (26) (with $s := s_5 + s_6 - 1 \geq 1$).

We now need to divide into two subcases to compute (25) (leading to (A)) and (26) (leading to (B)): (ii.2.3.1) $s_1 = s_2 = s_3 = 0$, either $s_4 > 1$ or $s_5 > 1$, and (ii.2.3.2) $s_1 = s_2 = s_3 = 0$ and $s_4 = s_5 = 1$. Before treating the two cases separately we first deform the expression inside the limit (26) as follows:

$$
(\zeta^{(N)}_{a(4)}(s_1, 0, 0, s_4, 0, 1) - \zeta^{(N)}_{a(4)}(s_1, 0, 0, s_4, 1, 0))
$$

$$
= \sum_{m_1, m_2 = 1}^{N} \left( \sum_{n=1}^{m_1 + m_2 + N} - \sum_{n=1}^{m_1 + m_2} \right) \frac{1}{m_1^s (m_1 + m_2)^s} n
$$

$$
= \sum_{m_1, m_2 = 1}^{N} \left( \sum_{n=1}^{m_1 + m_2 + N} - \sum_{n=1}^{m_2} \right) \frac{1}{m_1^s (m_1 + m_2)^s} n
$$

$$
= \sum_{m_1, m_2 = 1}^{N} \sum_{n=1}^{m_1 + m_2} \frac{1}{m_1^s (m_1 + m_2)^s} n
$$

$$
+ \sum_{m_1, m_2 = 1}^{N} \left( \sum_{n=1}^{m_1 + m_2 + N} - \sum_{n=1}^{m_2} \right) \frac{1}{m_1^s (m_1 + m_2)^s} n
$$

Step (ii.2.3.1). Let $s_1 = s_2 = s_3 = 0$, $s_6 \geq 1$, either $s_4 > 1$ or $s_5 > 1$. Without loss of generality assume $s_4 > 1$ as we have remarked after (24).

(A). \quad Since $s_1 \geq 1$, $s_4$, $t > 1$ we have

$$
\zeta_{a(4)}(s_1, 0, 0, s_4, t, 0) = \sum_{m_1, m_2, m_3 = 1}^{\infty} \frac{1}{m_1^{s_1}(m_1 + m_2)^{s_4}(m_2 + m_3)^t}
$$

$$
= \sum_{m_1, m_2 = 1}^{\infty} \sum_{n > m_2} \frac{1}{m_1^{s_1}(m_1 + m_2)^{s_4} n^t}
$$

$$
= \zeta(t) \zeta(s_4, s_1) - \zeta^{MT}(s_1, t; s_4) - \sum_{m_1, m_2 = 1}^{\infty} \sum_{n < m_2} \frac{1}{m_1^{s_1}(m_1 + m_2)^{s_4} n^t}
$$

$$
= \zeta(t) \zeta(s_4, s_1) - \zeta^{MT}(s_1, t; s_4) - \sum_{m_1, n, m_3 = 1}^{\infty} \frac{1}{m_1^{s_1}(m_1 + n + m_3)^{s_4} n^t}
$$
by setting \( m_2 = n + m_3 \). But the last sum is just \( \zeta_{MT}(s_1, t, 0; s_4) \) which is expressible by MZVs of the same weight and of depth three by Prop. 2.4.

(B). The limit (26). This has been reduced to (27) with \( s \geq 1 \) and \( s_4 \geq 2 \). Then the second sum in (27) is bounded by

\[
\frac{1}{1 + N} \sum_{m_1, m_2 = 1}^{N} \frac{1}{m_1 m_2} = O(\log^2 N/N) \to 0, \quad \text{as} \quad N \to \infty.
\]

By setting \( m_2 = m_3 + n \) in the first sum in (27) we get

\[
\lim_{N \to \infty} \left( \zeta_{sl(4)}^{(N)}(s, 0, 0, s_4, 0, 1) - \zeta_{sl(4)}^{(N)}(s, 0, 0, s_4, 1, 0) \right)
= -\sum_{m_1, m_2 = 1}^{\infty} \left( \sum_{n=1}^{m_1+m_2} - \sum_{n=1}^{m_2} \right) \frac{1}{m_1^2(m_1 + m_2)^{s_4}n}
\]

\[
(28)
= \zeta_{MT}(s_1; s_4) - \left( \zeta(s_4, s, 1) + \zeta(s_4, s + 1) + \zeta(s_4, 1, s) + \zeta(s_4 + 1, s) \right)
+ \sum_{m_1, m_2 = 1}^{\infty} \sum_{n=1}^{m_2-1} \frac{m_1 - n - 1}{m_1 + m_2} \frac{1}{m_1^2(m_1 + m_2)^{s_4}n^2}.
\]

In the last sum setting \( m_2 = n + m_3 \) then we get

\[
\sum_{m_1, m_2 = 1}^{\infty} \sum_{n=1}^{m_2-1} \frac{(m_1 + m_2)^{s_4} - m_1}{m_1^2 m_2} = \sum_{n,m_1,m_2 = 1}^{\infty} \frac{(m_1 + m_3 + n)^{-s_4}}{m_1^2 n} = \zeta_{MT}(s_1, 1; s_4)
\]

which is expressible by MZVs of the same weight and of depth three by Prop. 2.4.

Step (ii.2.3.2). Let \( s_1 = s_2 = s_3 = 0 \) and \( s_4 = s_5 = 1 \). Then we must have \( s_6 \geq 2 \) in order to have convergent values. Moreover, (25) is vacuous so we don’t need to consider (A).

(B). Observe that the second term in (27) is bounded by

\[
\frac{1}{1 + N} \sum_{m_1, m_2 = 1}^{N} \frac{1}{m_1^{m_1-1} m_2} = O(\log^2(N)/N) \to 0, \quad \text{as} \quad N \to \infty
\]

since \( s \geq 2 \). So

\[
\lim_{N \to \infty} \left( \zeta_{sl(4)}^{(N)}(s, 0, 0, 1, 0, 1) - \zeta_{sl(4)}^{(N)}(s, 0, 0, 1, 1, 0) \right)
= -\lim_{N \to \infty} \sum_{m_1, m_2 = 1}^{N} \sum_{n=1}^{m_1+m_2} \frac{1}{m_1^2(m_1 + m_2)n}
\]
is expressible by MVZs by Lemma 4.2 (with $t = 1$) in §4.

This finishes the proof of the theorem. □

4. Regularized MZVs and a technical lemma

We need to derive several formulas concerning MVZs. They are best understood as consequences of the stuffle (also called quasi-shuffle or harmonic product) relations for the MZVs including the divergent ones suitably regularized.

By [10] one may define a regularized value $\bar{\zeta}(s)$ (denoted by $Z^*(s)$ in [10]) if $s = (s_1, \ldots, s_d)$ with $s_1 = 1$ such that the following stuffle relation holds:

$$\bar{\zeta}(s_1)\bar{\zeta}(s_2) = \bar{\zeta}(s_1\ast s_2)$$

(29)

where the the product $\ast$ is the stuffle product (see [9] or [10, §1]).

**Lemma 4.1.** Let $s$ and $t$ be two positive integers greater than 1. Then

$$\zeta(s)\zeta(t) = \sum_{a=0}^{s-1} \binom{t-1+a}{a} \zeta(t + a, s - a) + \sum_{b=0}^{t-1} \binom{s-1+b}{b} \zeta(s + b, t - b).$$

(30)

Further

$$\zeta(s+1) = \sum_{a=1}^{s-1} \zeta(1 + a, s - a).$$

(31)

**Proof.** Equation (30) is the famous Euler’s decomposition formula [7] (or see [3]). To derive (31) we use the regularized MZVs as follows. For any positive integer $N$ we have by Lemma 2.5

$$\sum_{m=1}^{N} \sum_{n=1}^{N-m} \frac{1}{mn^s} = \sum_{m=1}^{N} \sum_{n=1}^{N-m} \left( \frac{1}{m(m+n)^s} + \sum_{a=0}^{s-1} \frac{1}{n^{s-a}(m+n)^{a+1}} \right).$$

Taking the regularization we get

$$\bar{\zeta}(1)\zeta(s) = \bar{\zeta}(1, s) + \zeta(s, 1) + \sum_{a=1}^{s-1} \zeta(1 + a, s - a).$$

On the other hand by stuffle (29)

$$\bar{\zeta}(1)\zeta(s) = \bar{\zeta}(1, s) + \zeta(s, 1) + \zeta(s + 1).$$

Equation (31) follows immediately. □

**Lemma 4.2.** For all positive integers $s$ and $t$ such that $s + t \geq 3$ the value

$$\sum_{m_1,m_2=1}^{\infty} \sum_{n=1}^{m_1+m_2} \frac{1}{m_1^t(m_1 + m_2)n^t}$$

(32)
is expressible by MVZs of weight \(s + t + 1\) and depth three or less.

**Proof.** Let’s consider the partial sum

\[
\sum_{1 \leq m_1 < m_3 \leq N} \sum_{n = 1}^{m_3} \frac{1}{m_1^2 m_3 n} = \sum_{1 \leq m_1 < m_3 \leq N} \left( \sum_{n = 1}^{m_3} - \sum_{n = 1}^{m_1} \right) \frac{1}{m_1^2 m_3 n}
\]

whose limit when \(N \to \infty\) is clearly (32). Using regularization and playing similar trick of breaking the sum as we did in (28) we find that

\[
(32) = \text{Reg} \left\{ \sum_{1 \leq m_1 < m_3 \leq N} \left( \sum_{n = 1}^{m_3} - \sum_{n = 1}^{m_1} \right) \frac{1}{m_1^2 m_3 n} \right\},
\]

\[
= \tilde{\zeta}(1, t, s) + \tilde{\zeta}(1, s + t) + \tilde{\zeta}(1, s, t) + \zeta(t + 1, s)
\]

(33)

\[
\quad - \text{Reg} \left\{ \sum_{1 \leq m_1 < m_3 \leq N} \sum_{n = 1}^{m_3 - m_1} \frac{1}{m_1^2 m_3 n} \right\}.
\]

Now the untreated sum can be deformed as follows: taking \(m_2 = m_3 - m_1\) we get

\[
(34) = \sum_{1 \leq m_1 < m_1 + m_2 \leq N} \sum_{n = 1}^{m_2 - 1} \frac{1}{m_1^2 (m_1 + m_2)^n} + \sum_{1 \leq m_1 < m_1 + m_2 \leq N} \frac{1}{m_1^2 m_3 n(m_1 + m_2)}
\]

\[
= \sum_{m_1 = 1}^{N} \sum_{n = 1}^{N-m_1-1} \sum_{n_1 = 1}^{N-m_1-n} \frac{1}{m_1^2 n^2 (m_1 + n + 3 + n)} + \xi^{(N)}_{\text{MT}}(s, t; 1)
\]

by setting \(n_3 = m_2 - n\). Taking \(x_1 = m_1\) and \(x_2 = n\) in Lemma 2.5 we see that (34) becomes

\[
\sum_{m_1 = 1}^{N} \sum_{n = 1}^{N-m_1-1} \sum_{n_1 = 1}^{N-m_1-n} \left\{ \sum_{a = 0}^{t-1} \left( \binom{t+a-1}{a} \frac{1}{m_1^{t-a} (m_1 + n)^{t+a} (m_1 + n + 3 + n)} + \sum_{b = 0}^{t-1} \left( \binom{s+b-1}{b} \frac{1}{n^{t-b} (m_1 + n)^{t+b} (m_1 + n + 3 + n)} \right) \right) \right\} + \xi^{(N)}_{\text{MT}}(s, t; 1).
\]

Taking regularization and combining with (33) yields

\[
(32) = \tilde{\zeta}(1, t, s) + \tilde{\zeta}(1, s + t) + \tilde{\zeta}(1, s, t) + \zeta(t + 1, s) - \zeta_{\text{MT}}(s, t; 1)
\]

\[
- \left\{ \sum_{a = 0}^{t-1} \left( \binom{t+a-1}{a} \tilde{\zeta}(1, t + a, s - a) + \sum_{b = 0}^{t-1} \left( \binom{s+b-1}{b} \tilde{\zeta}(1, s + b, t - b) \right) \right) \right\}
\]

\[
= \tilde{\zeta}(1, s + t) + \zeta(t + 1, s) - \zeta_{\text{MT}}(s, t; 1)
\]

\[
- \left\{ \sum_{a = 1}^{t-1} \left( \binom{t+a-1}{a} \tilde{\zeta}(1, t + a, s - a) + \sum_{b = 1}^{t-1} \left( \binom{s+b-1}{b} \tilde{\zeta}(1, s + b, t - b) \right) \right) \right\}.
\]
Applying the shuffle relations (29) we get

$$(32) = \bar{\zeta}(1)\zeta(s + t) - \zeta(s + t + 1) - \zeta(s + t, 1) + \zeta(t + s, 1) - \zeta_{MT}(s, t; 1)$$

$$- \bar{\zeta}(1)\left\{ \sum_{a=1}^{r-1} \binom{t + a - 1}{a} \zeta(t + a, s - a) + \sum_{b=1}^{r-1} \binom{s + b - 1}{b} \zeta(s + b, t - b) \right\}$$

$$+ \sum_{a=1}^{r-1} \binom{t + a - 1}{a} \left\{ \zeta(t + a + 1, s - a) + \zeta(t + a, 1, s - a) + \zeta(t + a, s - a + 1) + \zeta(t + a, s - a, 1) \right\}$$

$$+ \sum_{b=1}^{r-1} \binom{s + b - 1}{b} \left\{ \zeta(s + b + 1, t - b) + \zeta(s + b, 1, t - b) + \zeta(s + b, t - b + 1) + \zeta(s + b, t - b, 1) \right\}.$$

We can cancel all the terms involving \(\bar{\zeta}(1)\) in the two cases \(t > 1\) and \(t = 1\) by applying (30) and (31) of Lemma 4.1, respectively. This finishes the proof of the lemma. \(\Box\)

5. Some examples

Using Maple we have verified all the formulas on [17, p. 1502] by our general approach. We can compute any special values of \(\zeta_s(s_1, \ldots, s_7)\) and in particular \(\zeta_{s_{44}}(s_1, \ldots, s_9)\) at nonnegative integers whenever they satisfy the convergence conditions given by (12). For example, by Maple computation there are 32 weight four convergent \(\zeta_{s_{44}}(s_1, \ldots, s_9)\) values at nonnegative integers, with 15 distinct ones as follows:

$$\zeta_{s_{44}}(0, 0, 0, 0, 0, 4) = \frac{1}{4}\zeta(2) - \frac{3}{2}\zeta(3) + \frac{1}{6}\zeta(2)^2,$$

$$\zeta_{s_{44}}(0, 0, 0, 2, 2, 0) = 3\zeta(3) - \zeta(2)^2,$$

$$\zeta_{s_{44}}(0, 0, 0, 1, 1, 3) = \zeta_{s_{44}}(0, 0, 0, 1, 0, 3) = \zeta(2) - \zeta(3) - \frac{1}{16}\zeta(2)^2,$$

$$\zeta_{s_{44}}(0, 0, 0, 2, 0, 2) = \zeta_{s_{44}}(0, 0, 2, 0, 2) = \zeta(3) - \frac{3}{16}\zeta(2)^2,$$

$$\zeta_{s_{44}}(1, 0, 0, 0, 1, 2) = \zeta_{s_{44}}(0, 0, 1, 1, 2) = \zeta(3) - \frac{1}{6}\zeta(2)^2,$$

$$\zeta_{s_{44}}(1, 0, 0, 2, 1) = \zeta_{s_{44}}(0, 1, 2, 0, 1) = 2\zeta(3) - \frac{1}{2}\zeta(2)^2,$$

$$\zeta_{s_{44}}(0, 0, 1, 0, 3) = \zeta_{s_{44}}(0, 1, 0, 0, 3)$$
\[ \zeta_{sl}(1, 0, 1, 0, 2) = \zeta_{sl}(1, 0, 1, 0, 2) = \zeta_{sl}(1, 0, 0, 1, 1) = \zeta_{sl}(1, 0, 1, 0, 1) = \zeta_{sl}(1, 0, 1, 0, 1) = \zeta_{sl}(1, 0, 1, 0, 1) = \zeta_{sl}(1, 0, 1, 0, 1) = \frac{7}{4} \zeta(2)^2. \]

Notice that in the first seven values the weights are mixed which correspond to the nine cases of irregular values (13) to (17). In higher weight cases if we only consider regular (pure weight) values then we have the following examples.

\[
\begin{align*}
\zeta_{sl}(1, 1, 0, 1, 1, 1) &= \zeta_{sl}(1, 1, 1, 1, 1) = \frac{5}{2} \zeta(5) - \frac{3}{2} \zeta(2) \zeta(3), \\
\zeta_{sl}(1, 0, 1, 1, 1, 1) &= -\frac{17}{10} \zeta(3)^3 + 2 \zeta(3)^2, \\
\zeta_{sl}(1, 1, 1, 1, 1, 1) &= \frac{3}{2} \zeta(3) (7 - 3 \zeta(2) \zeta(5), \\
\zeta_{sl}(1, 1, 1, 2, 2, 2) &= \frac{368}{278875} \zeta(2)^6.
\end{align*}
\]

The last value of the above agrees with [12, (4.29)] and the first value in [8, Table 4] by the relation

\[ \zeta_{sl}(1, 1, 0, 1, 1, 1) = 12^{-2m} \zeta_{sl}(1, 0, 1, 1, 1, 1). \]

where \( \zeta_{sl}(1) \) on the LHS is the multiple variable function being studied in the current paper and \( \zeta_{sl}(1) \) on the RHS is the original single variable. Witten zeta function (see [17, Prop. 2.1]).

As two intriguing computation we have

\[
\begin{align*}
\zeta_{sl}(1, 2, 2, 2, 2, 3) &= \frac{1012}{13} \zeta(2)^6 - 6 \zeta(2)^5 + 4 \zeta(3)^2 + 4 \zeta(3)^2 \\
\zeta_{sl}(1, 2, 2, 2, 2, 2) &= \frac{368}{278875} \zeta(2)^6.
\end{align*}
\]

We have also verified all the above computation numerically with Maple and EZface [4].

We conclude our paper with a conjecture. Let \( \mathcal{M} \mathcal{Z} \mathcal{V}(w, \leq l) \) be the \( Q \)-vector space generated by MZVs of weight \( w \) and depth \( \leq l \). Then by Theorem 1.1 the space generated by special values of \( \zeta_{sl}(1) \) of weight \( w > 3 \) over \( \mathcal{Q} \) is included in the space \( \mathcal{M} \mathcal{Z} \mathcal{V}(w, \leq 3) + \mathcal{M} \mathcal{Z} \mathcal{V}(w - 1, \leq 2) + \mathcal{M} \mathcal{Z} \mathcal{V}(w - 2, 1). \)

**Conjecture 5.1.** The space generated by special values of \( \zeta_{sl}(4) \) of weight \( w > 3 \) at nonnegative integers over \( \mathcal{Q} \) is

\[ \mathcal{M} \mathcal{Z} \mathcal{V}(w, \leq 3) \oplus \mathcal{M} \mathcal{Z} \mathcal{V}(w - 1, \leq 2) \oplus \mathcal{M} \mathcal{Z} \mathcal{V}(w - 2, 1). \]

Using the MZV table [2] and assuming the sum is direct we have verified the conjecture for all weights up to 12.
WITTEN MULTIPLE ZETA VALUES ATTACHED TO $\mathfrak{so}(4)$

References

[1] S. Akiyama, S. Egami and Y. Tanigawa, Analytic continuation of multiple zeta-functions and their values at non-positive integers, Acta Arith. 98 (2001), 107–116.

[2] M. Bigotte, G. Jacob, N. E. Oussous, M. Petitot and H. N. Minh, Table of the coloured zeta function, with 1 roots, up to the weight 12, http://www2.lifl.fr/~petitot/

[3] J. M. Borwein, D. M. Bradley, D. J. Broadhurst and P. Lisonek, Combinatorial aspects of multiple zeta values, Electron. J. Combin. 5 (1998), Research Paper 38.

[4] J. Borwein, P. Lisonek and P. Irwin, An interface for evaluation of Euler sums, http://oldweb.cecm.sfu.ca/cgi-bin/EZFace/zetaform.cgi

[5] M. de Crisenoy, Values at $T$-tuples of negative integers of twisted multivariable zeta series associated to polynomials of several variables, Compositio Math. 142 (2006), 1373–1402.

[6] D. Essouabri, Singularités des séries de Dirichlet associées à des polynômes de plusieurs variables et applications en théorie analytique des nombres, Ann. Inst. Fourier 47 (1997), 429–483.

[7] L. Euler, Meditationes circa singularere serierum genus, Novi Comm. Acad. Sci. Petropol. 20 (1775), 140–186, reprinted in “Opera Omnia,” series I, 15, B. G. Teubner, Berlin (1927), 217–267.

[8] P. E. Gunnells and R. Sczech, Evaluation of Dedekind sums, Eisenstein cocycles, and special values of $L$-functions, Duke Math. J. 118 (2003), 229–260.

[9] M. E. Hoffman, Multiple harmonic series, Pacific J. Math. 152 (1992), 275–290.

[10] K. Ihara, M. Kaneko and D. Zagier, Derivation and double shuffle relations for multiple zeta values, Compositio Math. 142 (2006), 307–338.

[11] K. Ireland and M. Rosen, A Classical Introduction to Modern Number Theory, Graduate Texts in Math. 84, 2nd ed., Springer, 1998.

[12] Y. Komori, K. Matsumoto and H. Tsumura, Zeta functions of root systems, in: The Conference on $L$-Functions, World Sci. Publ., Hackensack, NJ, 2007, 115–40.

[13] Y. Komori, K. Matsumoto and H. Tsumura, Zeta and $L$-functions and Bernoulli polynomials of root systems, Proc. Japan Acad. Ser. A 84 (2008), 57–62.

[14] Y. Komori, K. Matsumoto and H. Tsumura, On Witten multiple zeta-functions associated with semisimple Lie algebras II, J. Math. Soc. Japan 62 (2010), 355–394.

[15] Y. Komori, K. Matsumoto and H. Tsumura, On Witten multiple zeta-functions associated with semisimple Lie algebras III, arXiv: math/0907.0955.

[16] K. Matsumoto, On Mordell-Tornheim and other multiple zeta-functions, in: Proc. of the Session in analytic number theory and Diophantine equations, ed. D.R. Heath-Brown and B.Z. Moroz, Bonner Mathematische Schriften, 25, 2003, 17 pages.

[17] K. Matsumoto and H. Tsumura, On Witten multiple zeta-functions associated with semisimple Lie algebras I, Annales de l’institut Fourier, 56 (2006), 1457–1504.

[18] L. J. Mordell, On the evaluation of some multiple series, J. London Math. Soc. 33 (1958), 368–271.

[19] T. Nakamura, Double Lerch value relations and functional relations for Witten zeta functions, Tokyo J. Math. 31 (2008), 551–574.

[20] Y. Ohno and W. Zudilin, Zeta stars, Commun. Number Theory Phys. 2 (2008), 325–347.

[21] T. Terasoma, Rational convex cones and cyclotomic multiple zeta values, arXiv: math/0410306

[22] L. Tornheim, Harmonic double series, Amer. J. Math. 72 (1950), 303–314.

[23] H. Tsumura, On Witten's type of zeta values attached to $\mathfrak{so}(5)$, Arch. Math. 82 (2004), 147–152.

[24] E. Witten, On quantum gauge theories in two-dimensions, Commun. Math. Phys. 141 (1991), 153–209.

[25] D. Zagier, Values of zeta function and their applications, in: Proc. of the First European Congress of Math., Vol. 2, 1994, 497–512.

[26] J. Zhao, Analytic continuation of multiple zeta functions, Proc. Amer. Math. Soc. 128 (1999), 1275–1283.

[27] J. Zhao, Alternating Euler sums and special values of Witten multiple zeta function attached to $\mathfrak{so}(5)$. In
J. Aust. Math. Soc. 89 (2011), 419–430. doi:10.1017/S1446788711001054.

[28] J. Zhao, Multi-polylogs at twelfth roots of unity and special values of Witten multiple zeta function attached to the exceptional Lie algebra $g_2$. J. Algebra Appl. 9 (2010), 327-337.

[29] J. Zhao, Witten volume formulas for semi-simple Lie algebras. Proc. of the Integers Conference 2009 in Honor of the Birthdays of Melvyn Nathanson and Carl Pomerance. Integers 11A (2010), #A22.

[30] X. Zhou and D. M. Bradley, On Mordell-Tornheim sums and multiple zeta values, Ann. Sci. Math. 34 (2010), 15–23.

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