Weak sequential completeness in Banach $C(K)$-modules of finite multiplicity

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Received: 27 March 2015 / Accepted: 30 April 2015 / Published online: 23 May 2015
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Abstract A well known result of Lozanovsky states that a Banach lattice is weakly sequentially complete if and only if it does not contain a copy of $c_0$. In the current paper we extend this result to the class of Banach $C(K)$-modules of finite multiplicity and, as a special case, to finitely generated Banach $C(K)$-modules. Moreover, we prove that such a module is weakly sequentially complete if and only if each cyclic subspace of the module is weakly sequentially complete.

Keywords Weak sequential completeness · Banach $C(K)$-modules · Banach lattices

Mathematics Subject Classification Primary 46B20; Secondary 47B22 · 46B42

1 Introduction

It is well known that some important properties of Banach spaces (reflexivity, weak sequential completeness, et cetera) can be more readily studied if we restrict our attention to the class of Banach lattices. In particular, it was proved by Lozanovsky

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in [15] that a Banach lattice is reflexive if and only if it does not contain a subspace\(^1\) isomorphic to either \(c_0\) or \(l_1\), and that it is weakly sequentially complete if and only if it does not contain a subspace isomorphic to \(c_0\). The above results cannot be extended to the class of all Banach spaces: the famous James space [9] does not contain either \(l_1\) or \(c_0\) but is neither reflexive nor (see [13, page 34]) weakly sequentially complete. Nevertheless, Lozanovsky’s results remain true for arbitrary subspaces of Banach lattices with order continuous norm or complemented subspaces of arbitrary Banach lattices, see [24] and [13, Theorem 1.c.7, page 37]. We consider similar problems for two other subclasses of the class of Banach spaces, namely the class of finitely generated Banach \(C(K)\)-modules (see Definition 2.3 below) and the class of Banach \(C(K)\)-modules of finite multiplicity (Definition 3.7). Because cyclic subspaces of Banach \(C(K)\)-modules can be represented in a natural way as Banach lattices we have reason to believe that many “nice” properties of Banach lattices would carry over to finitely generated Banach \(C(K)\)-modules. In particular, the authors proved in [12] that a Banach \(C(K)\)-module of finite multiplicity is reflexive if and only if it does not contain a copy of either \(l_1\) or \(c_0\). The goal of the current paper is to prove that Lozanovsky’s characterization of weak sequential completeness of Banach lattices also remains true for such modules. We note that in general finitely generated Banach \(C(K)\)-modules and Banach \(C(K)\)-modules of finite multiplicity are different. However, as it turns out (Theorem 3.1), weakly sequentially complete finitely generated Banach \(C(K)\)-modules are strictly contained in the Banach \(C(K)\)-modules of finite multiplicity.

2 Preliminaries

All the linear spaces will be considered either over the field of real numbers \(\mathbb{R}\) or the field of complex numbers \(\mathbb{C}\). If \(X\) is a Banach space we will denote its Banach dual by \(X^*\).

Let us recall some definitions.

Definition 2.1 Let \(K\) be a compact Hausdorff space and \(X\) be a Banach space. We say that \(X\) is a Banach \(C(K)\)-module if there is a continuous unital homomorphism \(m\) of \(C(K)\) into the algebra \(L(X)\) of all bounded linear operators on \(X\).

Remark 2.2 Because \(\ker m\) is a closed ideal in \(C(K)\) by considering, if needed, \(C(\bar{K}) = C(K)/\ker m\) we can and will assume without loss of generality that \(m\) is a contractive homomorphism (see [12]) and \(\ker m = 0\). Then (see [8, Lemma 2(2)]) \(m\) is an isometry. Moreover, when it does not cause any ambiguity we will identify \(f \in C(K)\) and \(mf \in L(X)\).

Definition 2.3 Let \(X\) be a Banach \(C(K)\)-module and \(x \in X\). We introduce the cyclic subspace \(X(x)\) of \(X\) as \(X(x) = \text{cl}\{fx : f \in C(K)\}\).

The following proposition was proved in [25] (see also [21]) in the case when the compact space \(K\) is extremally disconnected and announced for an arbitrary compact Hausdorff space \(K\) in [10]. It follows as a special case from [8, Lemma 2(2)].

\(^1\) By subspace we always mean closed subspace.
Proposition 2.4 Let $X$ be a Banach $C(K)$-module, $x \in X$, and $X(x)$ be the corresponding cyclic subspace. Then, endowed with the cone $X(x)_+ = \text{cl}\{fx : f \in C(K), \ f \geq 0\}$ and the norm inherited from $X$, $X(x)$ is a Banach lattice with positive quasi-interior point $x$.

Our next proposition follows from Theorem 1(3) in [19]

Proposition 2.5 The center $Z(X(x))$ of the Banach lattice $X(x)$ is isometrically isomorphic to the weak operator closure of $m(C(K))$ in $L(X(x))$.

Now we can introduce one of the two main objects of interest in the current paper.

Definition 2.6 Let $X$ be a Banach $C(K)$-module. We say that $X$ is finitely generated if there are an $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$ such that the set $\sum_{i=1}^n X(x_i)$ is dense in $X$.

Before we proceed with the statement and the proof of our main results (Theorems 3.1, 3.8) we need to introduce a few more definitions and recall a couple of results.

Definition 2.7 Let $X$ be a Banach space and $B$ be a Boolean algebra of projections on $X$. The algebra $B$ is called Bade complete (see [6, XVII.3.4, p. 2197] and [22, V.3, p. 315]) if

1. $B$ is a complete Boolean algebra.
2. Let $\{\chi_\gamma\}_{\gamma \in \Gamma}$ be an increasing net in $B$, $\chi$ be the supremum of this net, and $x \in X$. Then the net $\{\chi_\gamma x\}$ converges to $\chi x$ in norm in $X$.

The following result was proved in [18, Theorem 3 and Remark 5].

Theorem 2.8 Let $X$ be a Banach space, $K$ be a compact Hausdorff space, and $m$ be an isometric embedding of $C(K)$ into $L(X)$. Assume that no cyclic subspace of $X$ contains a copy of $c_0$. Then

1. The closure of $m(C(K))$ in the weak operator topology is isometrically isomorphic to $C(Q)$ where $Q$ is a hyperstonian space.
2. The algebra of all idempotents in $C(Q)$ is a Bade complete Boolean algebra of projections on $X$.

Remark 2.9 In virtue of Theorem 2.8 from now on, in this section, we will assume that the compact Hausdorff space $K$ is hyperstonian and that $B$, the algebra of all idempotents in $C(K)$, is a Bade complete Boolean algebra of projections on $X$.

Remark 2.10 Assume conditions of Remark 2.9. Because $K$ is hyperstonian $C(K)$ is a dual Banach space (see e.g. [22, Theorem 9.3, page 122]). Let us denote its predual by $C(K)_*$. Because $C(K)_*$ is an AL-space it is isometrically isomorphic to a band in its second dual, i.e., to a band in $C(K)^*$ (see [22, Page 92]).

Next notice (see [19]) that the center $Z(C(K)^*)$ of $C(K)^*$ is isometrically isomorphic to $C(K)^{**}$. We will denote by $p$ the idempotent in $C(K)^{**}$ corresponding to the band projection in $C(K)^*$ onto the band $C(K)_*$.
Remark 2.11  Assume conditions of Remark 2.9 and assume also that $X$ is cyclic. Then by Veksler [25] (see also [22, V.3]), $X$ can be represented as a Banach lattice with order continuous norm and a positive quasi-interior point such that $B$ coincides with the algebra of all band projections on $X$.

Remark 2.12  Consider $X^*$ as a Banach $C(K)$-module as follows. Let $m^*: C(K) \to L(X^*)$ be the isometric unital algebra and lattice homomorphism that describes this module structure. That is, for each $a \in C(K)$, let $m^*(a)$ be the Banach space adjoint of $a$ acting as a bounded operator on $X$. Hence $ax^*(x) = m^*(a)(x^*)(x) = x^*(ax)$ for all $x \in X$ and $x^* \in X^*$.

Furthermore, by means of the Arens extension procedure [3], we will consider $X^{**}$ as a Banach $C(K)^{**}$-module. Namely, for each $x^* \in X^*$, $x^{**} \in X^{**}$, $a \in C(K)$, and $a^{**} \in C(K)^{**}$, we define $\mu_{x^*,x^{**}} \in C(K)^{**}$ and $a^{**} \cdot x^{**} \in X^{**}$ as follows

$$\mu_{x^*,x^{**}}(a) = x^{**}(ax^*),$$
$$a^{**} \cdot x^{**}(x^*) = a^{**}(\mu_{x^*,x^{**}}).$$

We will list the relevant easily checked properties of the above definitions in the following lemma.

Lemma 2.13  Let $x^* \in X^*$, $x^{**} \in X^{**}$, and $a^{**}, b^{**} \in C(K)^{**}$. Then

1. $a^{**} \mu_{x^*,x^{**}} = \mu_{x^*,a^{**} \cdot x^{**}}$;
2. $(a^{**} b^{**}) \cdot x^{**} = a^{**} \cdot (b^{**} \cdot x^{**})$;
3. We have $x^{**} \in p \cdot X^{**}$ if and only if $\mu_{x^*,x^{**}} \in C(K)_*$ for all $x^*$.

Proof (1)  Given $a^{**}$ in $C(K)^{**}$ let $\{a_\alpha\}$ be a net in $C(K)$ that converges to $a^{**}$ in the $\sigma(C(K)^{**}, C(K)^*)$-topology. Then, for any $a \in C(K)$, we have

$$a^{**} \mu_{x^*,x^{**}}(a) = a^{**}(a \mu_{x^*,x^{**}}) = \lim_{\alpha} a \mu_{x^*,x^{**}}(a_\alpha) a = \lim_{\alpha} x^{**}(a_\alpha ax^*);$$
$$\mu_{x^*,a^{**} \cdot x^{**}}(a) = a^{**} \cdot x^{**}(ax^*) = a^{**}(\mu_{ax^*,x^{**}}) = \lim_{\alpha} \mu_{ax^*,x^{**}}(a_\alpha) a = \lim_{\alpha} x^{**}(a_\alpha ax^*).$$

(2)  Applying part (1) when necessary we have for any $a \in C(K)$,

$$(a^{**} b^{**}) \cdot x^{**}(ax^*) = \mu_{x^*,(a^{**} b^{**}) \cdot x^{**}}(a) = (a^{**} b^{**}) \mu_{x^*,x^{**}}(a) = a^{**}(b^{**} \mu_{x^*,x^{**}}(a)) = \mu_{x^*,a^{**} \cdot (b^{**} \cdot x^{**})}(a) = a^{**} \cdot (b^{**} \cdot x^{**})(ax^*).$$

(3)  Suppose $x^{**} \in p \cdot X^{**}$. Then $x^{**} = p \cdot x^{**}$. Hence, by part (1),

$$\mu_{x^*,x^{**}} = \mu_{x^*,p \cdot x^{**}} = p \mu_{x^*,x^{**}} \in C(K)_*.$$

Conversely, suppose $\mu_{x^*,x^{**}} \in C(K)_*$ for all $x^* \in X^*$. Then $\mu_{x^*,x^{**}} = p \mu_{x^*,x^{**}} = \mu_{x^*,p \cdot x^{**}}$ for all $x^* \in X^*$. It follows that $x^{**} = p \cdot x^{**}$. 

$\square$
**Lemma 2.14** Let $X$ be a Banach lattice. Then $p \cdot X^{**}$ is equal to $(X^*)^*_n$.

**Proof** Let $X$ be a Banach lattice. Then $Z(X^*)$, the ideal center of $X^*$, is equal to $C(K)$ where $K$ is hyperstonian. Since $X^*$ is Dedekind complete, $Z(X^*)$ is topologically full in the sense of Wickstead [26]. That is for any $x^* \in X^*_+$, one has that $cl(Z(X^*)x^*)$ is the closed ideal generated by $x^*$. Then, by considering $X^*$ as a $C(K)$-module and applying the Arens extension process described above we obtain (see [19, Corollary 1]) that the ideal center $Z(X^{**})$ of $X^{**}$ is equal to a band in $C(K)^{**}$. It is easy to see from Lemma 2.13 part (3) that in $X^{**}$, the set $p \cdot X^{**}$ is precisely the band of order continuous linear functionals on $X^*$. That is $p \cdot X^{**} = (X^*)^*_n$. \qed

**Remark 2.15** In the case of Riesz spaces, the statement of Lemma 2.14 remains true for the order bidual of the Riesz space. We refer the interested reader to Corollary 6 in [2].

Our next lemma shows that the property $p \cdot X^{**} = X$ is a three-space property.

**Lemma 2.16** Suppose $K$ is hyperstonian and $X$ is a Banach $C(K)$-module such that $B$, the idempotents in $C(K)$, is a Bade complete Boolean algebra of projections on $X$.

1. Let $Y$ be a closed submodule of $X$ such that $p \cdot Y^{**} = Y$ and $p \cdot (X/Y)^{**} = X/Y$. Then $p \cdot X^{**} = X$.

2. Let $p \cdot X^{**} = X$. Then for any closed submodule $Y$ of $X$ we have $p \cdot Y^{**} = Y$ and $p \cdot (X/Y)^{**} = X/Y$.

**Proof** (1) Let $Y$ be a closed submodule of $X$. Then $B$ is a Bade complete Boolean algebra of projections on both $Y$ and $X/Y$ (see [12, Lemma 1]). Hence it makes sense to consider the hypothesis in the statement of part (1) of the lemma and assume that it holds for these two spaces. It is familiar from standard duality that $Y^{**} = Y^{oo} \subseteq X^{**}$ and $(X/Y)^{**} = X^{**}/Y^{oo}$ where for a subspace $Y \subset X$, we let $Y^o$ denote the polar (or the annihilator) of $Y$ in $X^*$ and let $Y^{oo}$ denote the polar of $Y^o$ in $X^{**}$. Consider $x^{**} \in X^{**}$. Let $[x^{**}] = x^{**} + Y^{oo} \in X^{**}/Y^{oo}$. Then $p \cdot (X/Y)^{**} = X/Y$ means that, as a subset of $X^{**}$, $p \cdot [x^{**}] = [p \cdot x^{**}] = p \cdot x^{**} + Y^{oo}$ has non-empty intersection with $X$. That is, there is $x \in X$ and $y^{**} \in Y^{oo}$ such that $p \cdot x^{**} + y^{**} = x$ (*). Now $p \cdot Y^{oo} = Y$ means that $p \cdot y^{**} = y$ for some $y \in Y$ and we also have that $p \cdot x = x$ trivally. Hence we apply $p$ to both sides of the equality (*) to obtain $p \cdot x^{**} = x - y$ for some $y \in Y$ and $x \in X$.

(2) Assume $p \cdot X^{**} = X$ and $Y$ is a closed submodule of $X$. It is familiar that $Y = X \cap Y^{oo}$ by the bipolar theorem. Let $y^{**} \in Y^{oo}$, then $p \cdot y^{**} \in X \cap Y^{oo} = Y$. Hence $p \cdot Y^{**} = Y$. Now take $x^{**} \in X^{**}$ and consider $[x^{**}] \in X^{**}/Y^{oo}$. Since $p \cdot x^{**} = x$ for some $x \in X$, we have $p \cdot [x^{**}] = [p \cdot x^{**}] = [x]$. Also, given $\varepsilon > 0$, there is $y^{**} \in Y^{oo}$ such that $\|p \cdot x^{**} + y^{**}\| < \|p \cdot [x^{**}]\|(1 - \varepsilon)$. Since $p \cdot y^{**} = y$ for some $y \in Y$, we have $p \cdot (p \cdot x^{**} + y^{**}) = x + y$. Since $\|p\| = 1$, we have, when $p \cdot x^{**} = x$, $\|p \cdot [x^{**}]\|(X/Y)^{**} = \|[x]\|_{X/Y}$. Hence $p \cdot (X/Y)^{**} = X/Y$. \qed

The next lemma is a special case of Theorem 6 in [11, Page 297].
Lemma 2.17 Let $X$ be a Banach lattice and let $\{x_n^*\}$ be an increasing norm bounded sequence of positive elements of $X^*$. Suppose $M > 0$ is the supremum of the norms of the functionals in the sequence. Then the sequence has a least upper bound $x^* \in X_+^*$ with norm equal $M$.

We will need the following complement to Lozanovsky’s result that follows from Theorem 2.4.12 in [16].

Theorem 2.18 Let $X$ be a Banach lattice. The following conditions are equivalent.

1. $X$ is weakly sequentially complete.
2. $X$ does not contain a sublattice that is lattice isomorphic to $c_0$.

The following lemma will play a crucial role in the Proof of Theorem 3.1.

Lemma 2.19 Suppose $K$ is hyperstonian and $X$ is a finitely generated Banach $C(K)$-module such that the algebra $B$ of the idempotents in $C(K)$, is a Bade complete Boolean algebra of projections on $X$. If no cyclic subspace of $X$ contains a copy of $c_0$, then $p \cdot X^{**} = X$.

Proof We will prove the result by induction on the number of generators of $X$. Suppose $X$ is cyclic. Then by Remark 2.11, $X$ can be represented as a Banach lattice with order continuous norm such that $B$ corresponds to the band projections on $X$. Since $X$ is cyclic $X$ does not contain a copy of $c_0$. Then, by the well known result for Banach lattices (see e.g. [16, Theorem 2.4.12]), $X$ is a KB-space and $X = (X^*)_n^*$. But by Lemma 2.14 we have that $p \cdot X^{**} = (X^*)_n^*$ for Banach lattices. Therefore $p \cdot X^{**} = X$ as required. Now suppose whenever $X$ has $r$ generators ($r \geq 1$) and no cyclic subspace of $X$ contains a copy of $c_0$, we have that $p \cdot X^{**} = X$. Suppose that $X$ has $r + 1$ generators. Let $\{x_0, x_1, \ldots, x_r\}$ be a set of generators for $X$. Let $Y = X(x_1, x_2, \ldots, x_r)$. Then, since no cyclic subspace of $Y$ contains a copy of $c_0$, we have that $p \cdot Y^{**} = Y$ or $p \cdot Y^{oo} = Y$ since $Y^{**} = Y^{oo} \subset X^{**}$. Now consider $X/Y = X/Y([x_0])$. Since $B$ is Bade complete on $X$, it is also Bade complete on $X/Y ([12, Lemma 1])$. Also since $X/Y$ is cyclic it may be represented as a Banach lattice which, by means of the previous sentence, may be assumed to have an order continuous norm and such that $B$ corresponds to the band projections of the Banach lattice. If $X/Y$ contains no copy of $c_0$, then, as in the case when $X$ is cyclic, we may claim that $p \cdot (X/Y)^{**} = X/Y$. So if this is not the case, $X/Y$ must contain a copy of $c_0$. In fact as remarked before since $X/Y$ is a Banach lattice it must contain a copy of $c_0$ that is lattice isomorphic to a closed sublattice of $X/Y$. This is equivalent to the statement that there exists a sequence $\{u_n\}$ in $X$ such that $[\{u_n\}]$ is a positive disjoint sequence in $X/Y$ and there exists constants $0 < d < D$ with

$$d \leq \|[u_n]\| \quad \text{and} \quad \|[u_1] + [u_2] + \cdots + [u_n]\| \leq D$$

for each $n$ (see [16, Lemma 2.3.10]). Let $\{e_n\}$ be a sequence in $B$ such that each $e_n$ is the band projection onto the band generated by $[u_n]$ in $X/Y$. Since the sequence $[\{u_n\}]$ is disjoint in the Banach lattice $X/Y$, the members of the sequence of band projections $\{e_n\}$ are also pairwise disjoint as idempotents in $C(K)$. Since $e_n[u_n] = [e_n u_n] = [u_n]$
for each $n$, we may assume without loss of generality that $e_n u_n = u_n$. Let $z_n = u_1 + u_2 + \cdots + u_n \in X$ and $\chi_n = e_1 + e_2 + \cdots + e_n \in \mathcal{B}$ for each $n$. Then $\{\chi_n\}$ is a positive, increasing and norm bounded sequence in $X/Y$. By Lemma 2.17, there exists $z \in X^{**}$ such that $[z] = \sup z_n$. Note that we consider $X/Y \subset ((X/Y)^{**})^*_n$. Since $((X/Y)^{**})^*_n$ is a band in $(X/Y)^{**}$, we have that $[z] \in ((X/Y)^{**})^*_n$. Then by Lemma 2.14, we have that $p \cdot [z] = [z]$. Therefore without loss of generality we take $p \cdot z = z$. The definition of $\{z_n\}$ implies that $\chi_n z_m = z_n$ for all $n \leq m$. Therefore $\chi_n \cdot [z] = [z_n]$ for all $n$. This means that there exists $y^{**}_n \in Y^{**}$ such that $\chi_n \cdot z + y^{**}_n = z_n$ for each $n$. By induction hypothesis, we have $p \cdot y^{**}_n = y_n$ for some $y_n \in Y$ and since $z_n \in X$, we have $p \cdot z_n = z_n$ for each $n$. Therefore if we apply the projection $p$ to both sides of the above equality, we obtain $\chi_n \cdot z + y_n = z_n$ for each $n$. Now apply $e_n$ to the last equality to obtain

$$e_n \cdot z = u_n - e_n y_n \in X$$

for each $n$. Note that the sequence $\{e_n \cdot z\}$ in $X$ satisfies

$$d \leq \| [u_n]\| \leq \| e_n \cdot z \| \quad \text{and} \quad \| e_1 \cdot z + e_2 \cdot z + \cdots + e_n \cdot z \| = \| \chi_n \cdot z \| \leq \| z \|$$

for each $n$. Let $w = \sum \frac{1}{n} e_n \cdot z$ in $X$ and consider the cyclic subspace $X(w)$. Clearly we may represent $X(w)$ as a Banach lattice with order continuous norm and positive quasi-interior point $w$. In virtue of Lemma 1 in [12] we may assume that $\mathcal{B}$ corresponds to the algebra of band projections on the Banach lattice $X(w)$. Then $e_n w = \frac{1}{n} e_n \cdot z$ for each $n$. Hence $\{e_n \cdot z\}$ is a disjoint positive sequence in the Banach lattice $X(w)$. The displayed conditions above indicates that the sublattice generated by $\{e_n \cdot z\}$ in $X(w)$ is lattice isomorphic to $c_0$ [16, Lemma 2.3.10]. But this is a contradiction, because we assumed that no cyclic subspace of $X$ contains a copy of $c_0$. Thus we see that $p \cdot (X/Y)^{**} = X/Y$ as well as $p \cdot Y^{**} = Y$. Finally Lemma 2.16 part (1) implies that $p \cdot X^{**} = X$ as required.

3 The main results

3.1 Finitely generated $C(K)$-modules

Theorem 3.1 Let $X$ be a finitely generated Banach $C(K)$-module. Then the following conditions are equivalent.

1. $X$ is weakly sequentially complete.
2. $X$ does not contain a copy of $c_0$.
3. No cyclic subspace of $X$ contains a copy of $c_0$.
4. Each cyclic subspace of $X$ is weakly sequentially complete.

Proof It is evident that we have (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3). The equivalence (3) $\Leftrightarrow$ (4) follows from Theorem 2.18 and Lozanovsky’s result in [15]. Hence the proof will be completed if we show that (3) $\Rightarrow$ (1). Suppose no cyclic subspace of $X$ contains a copy of $c_0$. Then Theorem 2.8 guarantees that the weak operator closure of $C(K)$
in \( L(X) \) is the norm closure of the linear span of a Bade complete Boolean algebra of projections on \( X \). Since the closed submodules of \( X \) are the same with respect to either algebra of operators [i.e., either with respect to \( C(K) \) or with respect to its weak operator closure in \( L(X) \)] we may assume that \( C(K) \) is weak operator closed in \( L(X) \). Then (see Theorem 2.8) \( K \) is hyperstonian and \( B \), the algebra of idempotents in \( C(K) \), is the Bade complete Boolean algebra of projections that generate \( C(K) \). Then (3) and Lemma 2.19 imply that \( p \cdot X^{**} = X \). To save space we will say that \( X \) satisfies (\( \ast \)) if \( X \) is a \( (C(K)) \)-module with hyperstonian \( K \) such that the idempotents in \( C(K) \) form a Bade complete Boolean algebra of projections on \( X \) and \( p \cdot X^{**} = X \). Hence (3) implies that \( X \) satisfies (\( \ast \)). Once again we will use induction on the number of generators of \( X \). If \( X \) is cyclic then it may be represented as a Banach lattice with order continuous norm such that the band projections correspond to \( B \). By Lemma 2.14, we have \( p \cdot X^{**} = (X^*)^*_n \). Therefore \( p \cdot X^{**} = X \) implies \( X = (X^*)^*_n \). Hence by Theorem 2.18, \( X \) is weakly sequentially complete (see also [11, Theorem 8, page 297]). Now, as induction hypothesis, we suppose that whenever \( X \) is a \( (C(K)) \)-module with at most \( r \) generators \( (r \geq 1) \) and \( X \) satisfies the condition (\( \ast \)) then \( X \) is weakly sequentially complete. Suppose \( X \) has \( r + 1 \) generators and satisfies (\( \ast \)). Let \( \{x_0, x_1, \ldots, x_r\} \) be a set of generators for \( X \) and let \( Y = X / (x_1, x_2, \ldots, x_r) \). Then, by Lemma 2.16 part (2), both \( Y \) and \( X / Y \) satisfy the condition (\( \ast \)). Moreover \( Y \) has \( r \) generators and \( X / Y = X / Y([x_0]) \) is cyclic. Therefore by the induction hypothesis \( Y \) and \( X / Y \) are both weakly sequentially complete. Then the three-space property of weak sequential completeness (see [4, Theorem 4.7.a., page 122]) implies that \( X \) is weakly sequentially complete.

We conclude this section with two remarks.

**Remark 3.2** The original example of Dieudonné [5] provides a Banach \( C(K) \)-module \( X \) with two generators such that every cyclic subspace is weakly sequentially complete but \( X \) cannot be represented as the sum of two cyclic subspaces.

**Remark 3.3** The example of the Banach \( l^\infty \)-module \( X = l^1 \oplus c_0 \) shows that in Theorem 3.1 we cannot substitute the condition that every cyclic subspace is weakly sequentially complete by a weaker condition that for some set of generators \( x_1, \ldots, x_n \) of \( X \), the cyclic subspaces \( X(x_i), i = 1, \ldots, n \), are weakly sequentially complete. To see this in \( X = l^1 \oplus c_0 \), let the sequence \( x \in l^1 \cap c_0 \) be defined by \( x_n = 1/n^2 \) for each positive integer \( n \). Then the vectors \( (x, 0) \) and \( (x, x) \) generate \( X \). Moreover the cyclic subspace \( X((x, 0)) = l^1 \) and the cyclic subspace \( X((x, x)) \) is isomorphic to \( l^1 \). Whence both of these cyclic subspaces are weakly sequentially complete, but \( X \) is not weakly sequentially complete.

### 3.2 Boolean algebras of projections of finite multiplicity

Throughout this section we will assume that \( B \) is a Bade complete Boolean algebra of projections on the Banach space \( X \). We let \( K \) denote the hyperstonian Stone representation space of \( B \). As in Sect. 2, we will assume that both \( X \) and \( X^* \) are modules over \( C(K) \) and that \( X^{**} \) is a module over \( C(K)^{**} \).
Definition 3.4 Let $\mathcal{B}$ be a Bade complete Boolean algebra of projections on $X$. $\mathcal{B}$ is said to be of uniform multiplicity $n$, if there exist a set of nonzero pairwise disjoint idempotents $\{e_\alpha\}$ in $\mathcal{B}$ with $\sup e_\alpha = 1$ such that for any $e_\alpha$ and for any $e \in \mathcal{B}$, $e \leq e_\alpha$ the $C(K)$-module $eX$ has exactly $n$ generators.

We will need the following result of Rall [20] (for a proof, see Lemma 2 in [17]).

Lemma 3.5 Let $\mathcal{B}$ be of uniform multiplicity one on $X$. Then $X$ may be represented as a Banach lattice with order continuous norm such that $\mathcal{B}$ is the Boolean algebra of band projections on $X$.

Next, as in [12] we can state the following corollary of Theorem 3.1.

Corollary 3.6 Let $X$ be a Banach space and let $\mathcal{B}$ be a Bade complete Boolean algebra of projections on $X$ that is of uniform multiplicity $n$. Then conditions (1)–(4) of Theorem 3.1 and condition (5) $p \cdot X^{**} = X$ are equivalent.

Proof We have that the implications (1) $\implies$ (2) $\implies$ (3) and (3) $\iff$ (4) hold as in the Proof of Theorem 3.1. The proof would be complete if we show (3) $\implies$ (5) and (5) $\implies$ (1). The proof of these implications follow exactly as in Lemma 2.19 for (3) $\implies$ (5) and as in Theorem 3.1 for (5) $\implies$ (1). Thus, provided that the case $n = 1$ is settled, we can finish the proof by induction. Therefore suppose $\mathcal{B}$ is of uniform multiplicity one on $X$. Then by Lemma 3.5, $X$ may be represented as a Banach lattice with order continuous norm such that $\mathcal{B}$ corresponds to the band projections of the lattice. Assume that (3) holds and $X$ is not a KB-space [i.e., $p \cdot X^{**} = (X^*)^*_n \neq X$]. Then by Theorem 2.18, $X$ contains a sublattice isomorphic to $c_0$. Suppose that $\{x_n\}$ is the disjoint sequence in this sublattice that corresponds to the standard basis of $c_0$ and let $\{e_n\}$ be the disjoint idempotents in $\mathcal{B}$ such that each $e_n$ is the band projection on the band generated by $x_n$ in $X$. Let $u = \sum_{n=1}^{\infty} \frac{1}{2^n} x_n$ in $X$. Consider the cyclic subspace (i.e. the band) $X(u)$ of $X$. Since $e_n u = \frac{1}{2^n} x_n$ for each $n$, the sublattice generated by the positive disjoint sequence $\{x_n\}$ is in $X(u)$ and this contradicts (3). Therefore $X$ is a KB-space, that is $p \cdot X^{**} = (X^*)^*_n = X$ and $X$ is weakly sequentially complete. Hence both (3) $\implies$ (5) and (5) $\implies$ (1) hold for $n = 1$.

Definition 3.7 A Bade complete Boolean algebra of projections $\mathcal{B}$ on $X$ is said to be of finite multiplicity on $X$ if there exists a collection of disjoint idempotents $\{e_\alpha\}$ in $\mathcal{B}$ such that, for each $\alpha$, $e_\alpha X$ is $n_\alpha$-generated and $\sup e_\alpha = 1$.

We note that the collection $\{n_\alpha\}$ of positive integers need not be bounded. Then by a well known result of Bade [6, XVIII.3.8, p. 2267], there exists a sequence of disjoint idempotents $\{e_n\}$ in $\mathcal{B}$ such that, for each $n$, $\mathcal{B}$ is of uniform multiplicity $n$ on $e_n X$ and $\sup e_n = 1$. Also the norm closure of the sum of the sequence of the spaces $\{e_n X\}$ is equal to $X$. In our next result we will show that the conclusions of Corollary 3.6 extend to this case. In the proof of the theorem we will use a vector version of a standard disjoint sequence method [14, Theorem 1.c.10, p. 23].

Theorem 3.8 Let $X$ be a Banach space and let $\mathcal{B}$ be a Bade complete Boolean algebra of projections on $X$ such that $\mathcal{B}$ is of finite multiplicity on $X$. Then conditions (1)–(4) of Theorem 3.1 and condition (5) $p \cdot X^{**} = X$ are equivalent.
Proof It is clear that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\iff$ (4). Initially we will show (3) $\Rightarrow$ (1). We will use the notation in the discussion before the statement. Since $B$ is of finite multiplicity on $X$, by Corollary 3.6, for each $n$, $e_n X$ is weakly sequentially complete. Let $\chi_n = e_1 + e_2 + \cdots + e_n$ for each $n$. Then $\chi_n X$ is weakly sequentially complete for each $n$. Moreover $\chi_n \uparrow 1$ in $B$ implies that
\[
\lim_{n \to \infty} \| (1 - \chi_n) x \| = 0
\]
for any $x \in X$. Suppose $\{x_i\}$ is a weak Cauchy sequence in $X$ such that
\[
\lim_{i \to \infty} f(\chi_n x_i) = 0
\]
for each $n$ and for all $f \in X^*$. Suppose, on the other hand, $\{x_i\}$ does not converge to 0 in the weak topology. Then, without loss of generality, we may suppose that for some real linear functional $g$ on $X$ with $\|g\| = 1$ and $\delta > 0$, we have that, for all $i$,
\[
g(x_i) \geq \delta \quad \text{and} \quad \lim_{i \to \infty} g(x_i) \geq \delta > 0.
\]
We define a sequence $\{Y_i\}_{i=0}^{\infty}$ of convex subsets of $X$ by
\[
Y_i := co\{x_k : k = i + 1, i + 2, \ldots\}.
\]
Let ‘$cl$’ denote norm closure of a set in $X$. Then (**) and duality imply that
\[
0 \in cl(\chi_n Y_i), \quad \text{for each } n \text{ and each } i.
\]
(Note that $y \in Y_i$ may have several convex representations by elements of $\{x_k : k = i + 1, i + 2, \ldots\}$. If we pick some $y \in Y_i$, it should be understood that at the same time we designate and fix some convex representation of $y$ by elements of $\{x_k : k = i + 1, i + 2, \ldots\}$.) From then onwards our references to the convex representation of $y$, will always be to the designated convex representation that we fixed. In particular, only finitely many elements of $\{x_k : k = i + 1, i + 2, \ldots\}$ are involved in the convex representation. Namely the elements of $\{x_k : k = i + 1, i + 2, \ldots\}$ that have a non-zero coefficient in the convex representation. One of the elements in this finite subset of $\{x_k : k = i + 1, i + 2, \ldots\}$ has the largest index. We will refer to this index as the largest index in the convex combination of $y$.)

We will choose a sequence $\{y_n\}_{n=1}^{\infty}$ that has the following properties:

1. $y_1 \in co\{x_k : k = 1, 2, \ldots, p_1 - 1\} \subset Y_0$ such that $\|(1 - \chi_{p_1})y_1\| < \frac{\delta}{2}$;
2. $y_n \in co\{x_k : k = p_{n-1} + 1, \ldots, p_n - 1\} \subset Y_{p_{n-1}}$ such that $\|\chi_{p_{n-1}}y_n\| < \frac{\delta}{2^n}$ and $\|(1 - \chi_{p_n})y_n\| < \frac{\delta}{2^n}$ for each $n = 2, \ldots$.

Note that when chosen as above, $\{p_n\}$ is a subsequence of $\{n\}$ such that $1 < p_1$ and $p_{n-1} + 1 < p_n$ ($n = 2, 3, \ldots$). Now to see that the sequence can be chosen as above, take $y_1 \in Y_0$ and take $p_1$ strictly greater than the largest index in the convex
combination of $y_1$, such that $\| (1 - \chi_{p_1}) y_1 \| < \frac{\delta}{2}$ [use (**)]. Then take $y_2 \in Y_{p_1}$ such that $\| \chi_{p_1} y_2 \| < \frac{\delta}{2^2}$ [use (***)]. Next take $p_2$ strictly greater than the largest index in the convex combination of $y_2$, such that $\| (1 - \chi_{p_2}) y_2 \| < \frac{\delta}{2^2}$ [use (*)]. Suppose that we chose $y_1, \ldots, y_n \ (n \geq 2)$ and $p_1 < \ldots < p_n$ as required. Then take $y_{n+1} \in Y_{p_n}$ such that $\| \chi_{p_n} y_{n+1} \| < \frac{\delta}{2^{n+1}}$ [use (***)]. Also take $p_{n+1}$ strictly greater than the largest index in the convex combination of $y_{n+1}$ such that $\| (1 - \chi_{p_n}) y_{n+1} \| < \frac{\delta}{2^{n+1}}$ [use (*)). This shows that we can choose inductively the sequence $\{y_n\}$ with the stated properties.

As chosen $\{y_n\}$ is a weak Cauchy sequence in $X$ such that, for each $f \in X^*$,

$$\lim_{n \to \infty} f(y_n) = \lim_{n \to \infty} f(x_n).$$

This follows, since $y_{n+1} \in co \{x_k : k = p_n + 1, \ldots, p_{n+1} - 1\}$ and therefore all the indices in the convex combination of $y_{n+1}$ are strictly greater than $n$. Also

$$\delta \leq g(y_n)$$

for all $n = 1, 2, \ldots$.

Now we define a new sequence $\{z_n\}$ such that

$$z_1 = \chi_{p_1} y_1 \text{ and } z_n = (\chi_{p_n} - \chi_{p_{n-1}}) y_n$$

for each $n = 2, \ldots$. Since

$$\| y_n - z_n \| = \| (1 - \chi_{p_n}) y_n + \chi_{p_{n-1}} y_n \| < \frac{\delta}{2^n} \left( 1 + \frac{1}{2^n} \right)$$

for each $n = 2, \ldots$, $\{z_n\}$ is also a weak Cauchy sequence such that

$$\lim_{n \to \infty} f(z_n) = \lim_{n \to \infty} f(y_n) = \lim_{n \to \infty} f(x_n). \quad (4^*)$$

Furthermore, we have from above

$$\frac{\delta}{2} \leq g(z_n) \quad (5^*)$$

for each $n = 1, 2, \ldots$. Note that by definition the elements of the sequence $\{z_n\}$ are in the range of disjoint projections, that is if $m \neq n$ then $(\chi_{p_m} - \chi_{p_{m-1}})(\chi_{p_n} - \chi_{p_{n-1}}) = 0$. This means that $l^\infty$ may be considered as an isometric unital subalgebra of $C(K)$ where the correspondence is given by $(\alpha_n) \in l^\infty \leftrightarrow \left( \sum_n \alpha_n (\chi_{p_n} - \chi_{p_{n-1}}) \right) \in C(K) \ [12].$

For any $(\xi_n) \in l^1$, there is $(\alpha_n) \in l^\infty$ with $|\alpha_n| = 1$ for all $n$ such that $(\alpha_n \xi_n) = (|\xi_n|)$ and $(\overline{\alpha_n} |\xi_n|) = (\overline{\xi_n})$. From this it follows that, when we consider $\sum \xi_n z_n \in X$ and use the embedding of $l^\infty$ in $C(K)$, we have

$$\left\| \sum \xi_n z_n \right\| = \left\| \sum |\xi_n| z_n \right\|. \quad (5^*)$$
Then, by $(5^*)$, we have
\[
\frac{\delta}{2} \left( \sum |\xi_n| \right) \leq g \left( \sum |\xi_n| z_n \right) \leq \left\| \sum \xi n z_n \right\| \leq (\sup \|z_n\|) \left( \sum |\xi_n| \right).
\]
That is $X$ has a subspace isomorphic to $l^1$ and $\{z_n\}$ corresponds to the standard basis of $l^1$ in the subspace of $X$ that is isomorphic to $l^1$. But this means that there exists $f \in X^*$ such that when restricted to $\{z_n\}$ we have
\[
f(z_n) = (-1)^n
\]
for each $n = 1, 2, \ldots$. This contradicts the fact that $\{z_n\}$ is a weak Cauchy sequence and that $\lim_{n \to \infty} f(z_n)$ exists. Therefore the real linear functional $g$ with the stated properties on $\{x_i\}$ cannot exist. So if $\{x_i\}$ is a weak Cauchy sequence such that $\{\chi_n x_i\}$ converges weakly to 0 for each $n$, then $\{x_i\}$ also converges to 0 weakly in $X$.

Now suppose that $\{x_n\}$ is a weak Cauchy sequence in $X$ such that $\{e_k x_n\}$ converges weakly to $u_k \in e_k X$ for each $k$. Since $\{x_n\}$ is bounded we have that the sequence $\{u_k\}$ is also bounded by the same constant. Consider the cyclic subspace $X(w)$ of $X$ that is generated by $w = \sum \frac{1}{2^k} u_k$. By the hypothesis $(3)$, $X(w)$ does not contain any copy of $c_0$. Therefore when represented as a Banach lattice with the positive quasi-interior point $w$, $X(w)$ becomes a KB-space. Also $e_k w = \frac{1}{2^k} u_k$ for each $k$ imply that $\{u_k\}$ is a positive disjoint sequence in $X(w)$. Let
\[
v_k = u_1 + u_2 + \cdots + u_k
\]
for each $k$. Moreover $\{\chi_k x_n\}$ converges weakly to $v_k$ in $\chi_k X$ for each $k$. It is clear that $\{v_k\}$ is a norm bounded sequence. (It has the same bound as $\{x_n\}$). So $\{v_k\}$ is an increasing and bounded positive sequence in the KB-space $X(w)$. Whence there exists $v = \sup v_k$ such that $\|v - v_k\| \to 0$ in $X(w)$. Therefore $\chi_k v = u_k$ for each $k$. That is $\{x_n - v\}$ is a weak Cauchy sequence such that $\{\chi_k(x_n - v)\}$ converges weakly to 0 for each $k$. Then by the first part of our proof $\{x_n - v\}$ also converges weakly to 0 in $X$. This means that $X$ is weakly sequentially complete. That is $(3) \implies (1)$.

Now suppose $(5)$ holds. By Lemma 2.16 part (2), we have that $(5)$ holds on each closed submodule of $X$. In particular, we have that $(5)$ holds for each cyclic subspace of $X$. That is $p \cdot X(x)^{**} = X(x)$ for each $x \in X(x)$. Since each cyclic subspace is represented as a Banach lattice, by Lemma 2.14, each cyclic subspace is a KB-space. This means that $(5) \implies (3)$. Therefore $(5) \implies (1)$.

Conversely, suppose $(1)$ holds. In particular $\chi_n X$ is weakly sequentially complete for each $n$. Then, by Corollary 3.6, each $\chi_n X$ satisfies $(5)$. That is with $(\chi_n X)^{**} = (\chi_n X)^{oo} = \chi_n \cdot X^{**}$, we have $p \cdot (\chi_n \cdot X^{**}) = \chi_n X$ for each $n$. Now suppose $p \cdot x^{**} = x^{**}$ for some $x^{**} \in X^{**}$. Then
\[
p \cdot (\chi_n \cdot x^{**}) = \chi_n \cdot (p \cdot x^{**}) = \chi_n \cdot x^{**} \in \chi_n X \subset X
\]
(6*) for each $n$. Furthermore, for each $x^* \in X^*$, $p \cdot x^{**} = x^{**}$ implies $\mu_{x^*, x^{**}} \in C(K)_*$ [Lemma 2.13(3)]. Whence, since $\chi_n \uparrow 1$ in $C(K)$, we have $\{\chi_n \cdot x^{**}\}$ converges to
x** in X** in the weak*-topology. By (6*), this means that \{\chi_n \cdot x**\} is a weak Cauchy sequence in X. Therefore (1) implies that x** ∈ X and the proof of (1) \implies (5) is complete.

\[\Box\]

4 Examples

We will end the paper by presenting some examples of Banach spaces X with a Bade complete Boolean algebra of projections \(B\) defined on X such that \(B\) is of finite multiplicity on X but X is not finitely generated. All the examples that we present are related to the example of Dieudonné [5] mentioned in Remark 3.2. We need the following definition that was given by Tzafriri [23]. The definition is motivated by Dieudonné’s example.

**Definition 4.1** Let \(B\) be a Bade complete Boolean algebra of projections on a Banach space X. Suppose \(B\) is of uniform multiplicity \(n\) on X. Then \(B\) has property \(D\) on X if for any \(x_i \in X, (i = 1, \ldots, n)\), any \(e \in B \setminus \{0\}\), and any \(p, 1 \leq p < n\), eX is not equal to the sum of \(eX(x_1, \ldots, x_p)\) and \(eX(x_{p+1}, \ldots, x_n)\).

The example of Dieudonné is a Banach \(L^\infty[0, \gamma]\)-module X with two generators and \(B[0, \gamma]\) is Bade complete on it. Here \(L^\infty[0, \gamma]\) is the algebra of equivalence classes of bounded Lebesgue measurable functions on the interval \([0, \gamma]\) and \(B[0, \gamma]\) is the Boolean algebra of idempotents in \(L^\infty[0, \gamma]\). The length of the interval, \(\gamma\), is as computed by Dieudonné. Dieudonné shows that \(B[0, \gamma]\) has property \(D\) on X for \(n = 2\). Given any \(n \geq 2\), it is clear that by following Dieudonné’s procedure one may construct a Banach \(L^\infty[0, \gamma]\)-module X with \(n\)-generators such that \(B[0, \gamma]\) has property \(D\) on X for \(n\). Also as mentioned in Remark 3.2 all these spaces are weakly sequentially complete since they are constructed as closed subspaces of \(KB\)-spaces.

**Example 4.2** (1) Let \(X_2\) denote the original space of Dieudonné in Remark 3.2. That is, it is a Banach \(L^\infty[0, \gamma]\)-module with two generators and \(B[0, \gamma]\) is Bade complete on \(X_2\) with property \(D\) for \(n = 2\). \(B[0, \gamma]\) is the Boolean algebra of the characteristic functions of the Lebesgue measurable sets with positive measure in \([0, \gamma]\). Similarly let \(X_n\) denote the Banach \(L^\infty[0, \gamma]\)-module with \(n\)-generators such that \(B[0, \gamma]\) is Bade complete on \(X_n\) with property \(D\) for \(n\) (constructed by following Dieudonné’s method as remarked above). Now consider \(l^p(\{X_n\})\) as an \(l^\infty([L^\infty[0, \gamma]])\)-module where \(1 \leq p < \infty\). It is to be understood that \(\{x_n\} \in l^p(\{X_n\})\) means \(x_n \in X_n\) for each \(n\) and \(\|x_n\| \| \in l^p\). Similar understanding holds for the algebra \(l^\infty([L^\infty[0, \gamma]])\). Let \(B\) be the Boolean algebra of the idempotents in \(l^\infty([L^\infty[0, \gamma]])\). Then \(B\) is of finite multiplicity on \(l^p(\{X_n\})\). But \(l^p(\{X_n\})\) is infinitely generated.

(2) Let \(\Gamma\) be an uncountable set. Let \(\{n_{\alpha}\}_{\alpha \in \Gamma}\) be an unbounded subset of \(\mathbb{N}\). Now consider \(l^p(\{X_{n_{\alpha}}\})\) as an \(l^\infty([L^\infty[0, \gamma]_{n_{\alpha}}])\)-module as in (1). Then once more \(B\) is of finite multiplicity on \(l^p(\{X_{n_{\alpha}}\})\). Moreover by Bade’s theorem there exists a sequence \(\{e_n\}\) in \(B\) such that \(B\) is of uniform multiplicity \(n\) on \(e_n l^p(\{X_{n_{\alpha}}\})\) and \(\sup e_n = 1\). Since \(\Gamma\) is uncountable some of the \(e_n l^p(\{X_{n_{\alpha}}\})\) are necessarily infinitely generated.
(3) Let $Y_2$ be the reflexive space in Example 3 of [12]. Since $Y_2$ is constructed by altering $X_2$, one can construct similarly $Y_n$ from $X_n$ above with the same properties. However, in addition, $Y_n$ would be reflexive. So we can consider $l^p(\{Y_n\})$ and $l^p(\{Y_{n,s}\})$ for $1 \leq p < \infty$. These spaces will have the same multiplicity properties as their counterparts in (1) and (2).

Now all the spaces in (1)–(3) are weakly sequentially complete. But the ones in (3) are, in addition, reflexive for $1 < p < \infty$. To see this, initially consider an example as in (1). Take $\{x_n\} \in l^p(\{X_n\})$. Let $X_n(x_n)$ be the cyclic subspace generated by $x_n$ in $X_n$. The space $X_n$ is weakly sequentially complete and $n$-generated. Hence, by Theorem 3.1, as a Banach lattice, the cyclic subspace $X_n(x_n)$ is a $KB$-space. Now the cyclic subspace generated by $\{x_n\}$ in $l^p(\{X_n\})$ is given by the Banach lattice $l^p(\{X_n(x_n)\})$. It is immediate that $l^p(\{X_n(x_n)\})$ is a $KB$-space and hence it is weakly sequentially complete. Then, by Theorem 3.8, $l^p(\{X_n\})$ is weakly sequentially complete. For examples in (2), suppose that $\{x_{n,s}\} \in l^p(\{X_{n,s}\})$. Then $x_{n,s} \neq 0$ for at most countably many $\alpha \in \Gamma$. Whence, the argument given for the examples in (1) shows that $l^p(\{X_{n,s}\})$ is weakly sequentially complete. This also covers the weak sequential completeness for all the examples in (3). When $1 < p < \infty$, to see that the spaces in (3) are reflexive, it is sufficient to calculate the duals directly.

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