A Mathematically Sensible Explanation of the Concept of Statistical Population

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Abstract

In statistics education, the concept of population is widely felt hard to grasp, as a result of vague explanations in textbooks. Some textbook authors therefore chose not to mention it. This paper offers a new explanation by proposing a new theoretical framework of population and sampling, which aims to achieve high mathematical sensibility. In the explanation, the term population is given clear definition, and the relationship between simple random sampling and iid random variables are examined mathematically.

Keywords: statistical education; statistical population; random sampling; random samples

1 Introduction

The theory of statistics is based on the theory of probability, and the first mathematical concept of statistics beyond pure probability is that of random samples. A closely related semi-mathematical concept, namely that of statistical population, is also popular among statistical practitioners. The term population is usually used instead of statistical population where no confusion would arise.

However, despite its popularity in the statistical community, in textbooks the treatment of the concept of population is quite varied, which reflects varied views of different authors on that concept. As a result of our study of 9 undergraduate-level textbooks in statistics: [1, 2, 3, 4, 5, 6, 7, 8, 9], we found that they can be classified into two groups according to the way they introduce the concept of random samples. Let us give the detail below.

1.1 Group A: Textbooks That Avoid Mentioning Population

This group consists of, in chronological order, [1, 2, 3, 4, 5]. One can check [1, page 145], [3, page 246], and [5, page 204] for evidences of our classification. Book [2] does not explicitly give a formal definition of random samples although it does use the term, and book [4] uses the term of random samples very early, with its definition deferred until page 214.

In [5, page 204], a definition of random samples is given as follows:

**Definition:** If the random variables $X_1, X_2, \ldots, X_n$ are independent and identically distributed (iid), then these random variables constitute a random sample of size $n$ from the common distribution.

Definitions of this concept in other textbooks in this group, though with different wordings, are mathematically equivalent, so they are omitted here. By defining random samples directly from distribution without mentioning population, the authors obviated the difficult task of explaining the population concept.

1.2 Group B: Textbooks That Use Population to Introduce Random Samples

This group consists of, in chronological order, [6, 7, 8, 9]. Of them, we found in [6, 9] detailed explanations of the concepts of population and random samples before giving formal definitions.
We now quote the following excerpt of the explanation in [6] pages 195–197], which is similar to the explanation in [9] page 225–227).

A population consists of the totality of the observations with which we are concerned.

In any particular problem, the population may be small, large but finite, or infinite. The number of observations in the population is called the size of the population. For example, the number of underfilled bottles produced on one day by a soft-drink company is a population of finite size. The observations obtained by measuring the carbon monoxide level every day is a population of infinite size. We often use a probability distribution as a model for a population. For example, a structural engineer might consider the population of tensile strengths of a chassis structural element to be normally distributed with mean µ and variance σ². We could refer to this as a normal population or a normally distributed population.

A sample is a subset of observations selected from a population.

To define a random sample, let \( X \) be a random variable that represents the result of one selection of an observation from the population. Let \( f(x) \) denote the probability density function of \( X \). Suppose that each observation in the sample is obtained independently, under unchanging conditions. That is, the observations for the sample are obtained by observing \( X \) independently under unchanging conditions, say, \( n \) times. Let \( X_i \) denote the random variable that represents the \( i \)-th replicate. Then, \( X_1, X_2, \ldots, X_n \) is a random sample and the numerical values obtained are denoted as \( x_1, x_2, \ldots, x_n \). The random variables in a random sample are independent with the same probability distribution \( f(x) \) because of the identical conditions under which each observation is obtained.

In [7] page 259] there is a short reference to the concept of population.

The appropriate representation of \( \hat{\theta} \) is \( \hat{\Theta} = h(X_1, X_2, \ldots, X_n) \) where \( X_1, X_2, \ldots, X_n \) are random variables representing a sample from random variable \( X \), which is referred to in this context as the population.

And in [8] page 78] there is concise characterization of random sampling.

Let \( N \) and \( n \) represent the numbers of elements in the population and sample, respectively. If the sampling is conducted in such a way that each of the \( C_n^N \) samples has an equal probability of being selected, the sampling is said to be random, and the result is said to be a random sample.

1.3 Motivation for a Mathematically Sensible Explanation

The concept of population is widely felt hard to grasp by students, which is perhaps the primary reason for it not being included in many textbooks. We are not attempting to argue whether we should or should not include it in textbooks. Instead, we try to propose an explanation of the population concept that makes perfect sense in a mathematical manner (which we call mathematically sensible in the sequel). Our attempt was motivated by our perception that the existing explanations (the foregoing explanation of [6] being a good representative) of the concept are not adequately mathematically sensible. Our reasons are as follows:

1. One would expect in a mathematically sensible explanation, the population fits into the probability theory framework in that it corresponds to a clear-cut mathematical concept. The existing explanations are chaotic in this respect. Let us examine each of the three possibilities:

(a) **population=sample space.** This possibility accords with the first half of the second paragraph of the foregoing quote of [6] pages 195–197]. However, how can a sample space have a distribution? Only random variables can.

(b) **population=range space of the random variable.** But this possibility directly contradicts the second example given in the first half of the second paragraph of the above-mentioned quote. If the carbon monoxide level can have only 100 values due to quantization, then should we say the size of population is now 100?

(c) **population=the random variable.** This possibility is the most-often seen. But then what is the corresponding mathematical concept for observation? How can we view a
population as a set of observations, as now the population is a random variable? The first sentence of the last paragraph of the above-mentioned quote suggests one random variable corresponds to one observation, which is in contradiction to the first sentence of the quote under this possibility.

2. The existing explanations do not mathematically show why random sampling in its primitive sense (characterized by the foregoing quote of [8, page 78]) leads to independent and identically distributed random variables.

The question now becomes: Does a mathematically sensible explanation exist? Our answer is yes and we will provide such an explanation in this paper.

2 A New Probability Framework for Population and Sampling

Actually, our new explanation of the population concept is a part of a new probability framework for population and sampling. Before giving the detail, let us look at its salient features:

1. It involves two probability spaces: a population probability space and an experiment probability space. However they are closely related to each other.

2. The term population is no longer mathematically overloaded as in the existing explanations, it now corresponds to the clear-cut mathematical concept of the sample space of the population probability space.

3. The random variable $X$, which was regarded as the population random variable in the existing explanations, lives in the population probability space, whereas the sample $X_1, X_2, \ldots, X_n$ lives in the experiment probability space. So mathematically speaking, they are NOT defined over the same probability space.

4. The physical operation of sampling is described by the mathematical operation of mapping from the experiment space (i.e. sample space of the experiment probability space) to the population, and the simple randomness of the sampling is mathematically described by the to-be-defined simplicity of the mapping.

2.1 The Population Probability Space

Recall that a probability space is a triple $(\Omega, \mathcal{F}, P)$, where $\Omega$ is a set called the sample space, $\mathcal{F}$ is a $\sigma$-field of subsets of $\Omega$ (see [5, page 10] for a description), and $P : \mathcal{F} \rightarrow [0, 1]$ is a probability function as defined in [5, page 11].

Mathematically speaking, any probability space can be a population probability space. It is only that the sample space of a population probability space is called population, which, understood physically, consists of individuals with various observable quantities. For example, if we are carrying out a health survey in a state with one million people, then we can define a population probability space as follows.

$$\Pi = \{0, 1, \ldots, 999999\},$$

$$\mathcal{F}_\Pi = \{\text{All subsets of } \Pi\},$$

$$P_\Pi(A) = \frac{\text{Number of elements in } A}{1000000}, \text{ for any } A \subseteq \Pi.$$

Obviously, $(\Pi, \mathcal{F}_\Pi, P_\Pi)$ constitutes a probability space. The elements in $\Pi$ are ordinal ids for the people in the state. We can then define random variables $X, Y, Z, \ldots$ to represent the length, weight, blood type, etc. of the people. For example, it can be so defined that $X(785)$ is the length in meters of the individual whose ordinal id is 785 in this state. Note also that if the sample space is finite and all its elements are assigned equal probability, then that probability space is
called “classical” in the literature. So this example is classical. However, our theory applies also to non-classical and infinite population cases.

The population probability space can be understood either stochastically or non-stochastically. It can be understood non-stochastically, since in the above description, no physical randomness is involved. It can also be understood stochastically if we view \( P_{\Pi}(A) \) as the the “(physical) probability of any member of \( A \) being selected for survey”.

2.2 The Experiment Probability Space and Simple Sampler Mapping

An experiment probability space is one where the sample space is the set of experiment outcomes. We denote it by the triple \((E, \mathcal{F}_E, P_E)\).

In our framework, size-\( n \) sampling is selection of an \( n \)-tuple from the population. What \( n \)-tuple is selected is determined by the experiment outcome. Therefore, sampling can be mathematically described by a sampler mapping \( S : E \mapsto \Pi^n \). Practically, it is the usual case that the size of population is massively larger than \( n \), and thus one element of the tuple can rarely be equal another element. In addition, it is also practically usual that the order of the elements does not matter in later statistical processing. Therefore practically we often say sampling is selection of a subset of size \( n \) from the population. The \( n \)-tuple term is adopted here, because it is considered more convenient.

Now we give the mathematical property to capture the simple randomness as described in the quote of [8, page 78] in Section 1.

**Definition 1.** Let \((\Pi, \mathcal{F}_\Pi, P_{\Pi})\) be a population probability space, and \((E, \mathcal{F}_E, P_E)\) be an experiment probability space. A sampler mapping \( S : E \mapsto \Pi^n \) is said to be simple, if for all \( B_1 \in \mathcal{F}_{\Pi}, B_2 \in \mathcal{F}_{\Pi}, \ldots, B_n \in \mathcal{F}_{\Pi}, \)

\[
P_E(\{e | S_1(e) \in B_1, \ S_2(e) \in B_2, \ldots, \ S_n(e) \in B_n\}) = P_{\Pi}(B_1)P_{\Pi}(B_2)\cdots P_{\Pi}(B_n). \tag{1}
\]

For a fixed experiment probability space, the existence of a simple sampler mapping is not guaranteed. In fact, it is more a requirement on the experiment probability space than one on the sampler mapping.

The quote of [8, page 78] is with an implicit assumption which, put in our terminology, is the population probability space being classical. Then, a subset of \( \Pi \) of size \( n \) corresponds to \( n! \) tuples with different orders, and if the mapping is simple, each order has \( \frac{1}{n!} \) probability of being selected, thus the probability of the subset being selected is \( \frac{1}{n!} \), irrespective of what elements constitute the subset. This is just the simple-randomness of sampling.

2.3 Mathematical Construction of Experiment Probability Space and Simple Sampler Mapping

Theoretically, there always exists a trivial construction of experiment probability space and simple sampler mapping, which is the product measure space \((\Pi, \mathcal{F}_\Pi, P_{\Pi})^n\) and the identity mapping. But this construction is only of theoretic value, not expected to yield real benefits.

If we want to construct an experiment probability space and a simple sampler mapping that faithfully describe a practically used sampling procedure, then this task is formidable, because in practice, ad-hoc methods are usually used, which are only approximately simple random. In such cases we can just assume the sampling to be simple random and assume the existence of an experiment probability space and simple sampler mapping, without giving the particular construction.

Perhaps the only scenario that we want to do the explicit mathematical construction is when we need the construction to design a simple random sampling algorithm. In that scenario, the experiment probability space, viewed as a model for random generator, should already have an implementation available. The following random number generators are considered to have reliable (albeit just close approximate) algorithm implementations:

**Discrete:** Given a length \( L \), random-generate \( \xi \) from \([0, 1, \cdots, L - 1]\) with each of \(0, 1, \cdots, L - 1\) having equal probability.

**Continuous:** Uniformly random-generate \( \xi \) from real interval \([0, 1]\).
2.3.1 Construction Based on the Discrete Generator

This method is applicable only when the population probability space is classical. So assumed, let \( N \) be the size of population, and let

\[
E = \{0, 1, \cdots, N^n - 1\}, \quad (2)
\]

\[
\mathcal{F}_E = \{ \text{All subsets of } E \}, \quad (3)
\]

\[
P_E(A) = \frac{\text{Number of elements in } A}{N^n}, \quad \text{for any } A \subseteq E. \quad (4)
\]

Eqs. (2–4) constitute a model for the length-\( N^n \) discrete generator. Now we define the sampler mapping:

\[
S(e) = (a_1, a_2, \cdots, a_n) \quad (5)
\]

where

\[
e = a_1 N^{n-1} + a_2 N^{n-2} + \cdots + a_n\]

with

\[
0 \leq a_1 \leq N - 1, \quad 0 \leq a_2 \leq N - 1, \quad \cdots, \quad 0 \leq a_n \leq N - 1. \quad (7)
\]

It is easy to see that \( S \) is a simple sampler mapping.

2.3.2 Construction Based on the Continuous Generator

This method is applicable when the population probability space is finite, both classical and non-classical. However, we only describe the method for the classical case, as the nonclassical case is much more complicated to deal with. Let \( N \) be the size of population, and let

\[
E = [0, 1], \quad (8)
\]

\[
\mathcal{F}_E = \{ \text{All Borel subsets of } E \}, \quad (9)
\]

\[
P_E(A) = \text{Measure of } A, \quad \text{for any } A \in \mathcal{F}_E. \quad (10)
\]

Eqs. (8–10) constitute a model for the continuous generator. Now we define the sampler mapping:

\[
S(e) = (a_1, a_2, \cdots, a_n) \quad (11)
\]

where

\[
e = a_1 N^{n-1} + a_2 N^{n-2} + \cdots + a_n N^{-n} + \cdots \]

with

\[
0 \leq a_1 \leq N - 1, \quad 0 \leq a_2 \leq N - 1, \quad \cdots, \quad 0 \leq a_n \leq N - 1. \quad (13)
\]

We omit the proof to show that \( S \) is a simple sampler mapping with the classical assumption.

2.4 Why Simple Random Sampling Leads to IID Random Variables

In Section 2.2 we showed that in the classical case, simpleness of the sampler mapping over the experiment probability space implies simple randomness of sampling. However, the reverse implication may not hold, and we may need a stronger version. Thus we try to revise the quote of [8, page 78] a bit by replacing "subsets" with "tuples":

Let \( N \) and \( n \) represent the numbers of elements in the population and sample, respectively. If the sampling is conducted in such a way that each of the \( N^n \) ordered samples has an equal probability of being selected, the sampling is said to be simple random, and the result is said to be a random sample.

This tuple-version is still completely intuitively reasonable and is readily seen to be equivalent in the classical case to the simpleness of the sampler mapping over the experiment probability space. So now this version of simple randomness will be adopted. Furthermore, because of their equivalence in the classical case, and the simpleness concept is more general, we decide to use the simpleness as defined in Definition 1 to carry out our derivation of the following proposition about random variables.
Proposition 1. Let $X$ be a random variable over population probability space $(\Pi, \mathcal{F}_\Pi, P_\Pi)$. Let $(E, \mathcal{F}_E, P_E)$ be an experiment probability space and $S : E \mapsto \Pi^n$ be a simple sampler mapping. Then

1. For $i = 1, \ldots, n$, $X_i$ as defined by $X_i(e) := X(S_i(e))$ is a random variable over $(E, \mathcal{F}_E, P_E)$ and has the same distribution function as $X$.

2. $X_1, X_2, \ldots, X_n$ are independent.

Proof. 1. Fix an arbitrary $u \in \mathbb{R}$. Let $B = \{ \pi | X(\pi) < u \}$, then since $X$ is a random variable over $(\Pi, \mathcal{F}_\Pi, P_\Pi)$, we have $B \in \mathcal{F}_\Pi$. Then

$$\{ e | X(e) < u \} = \{ e | X(S(e)) < u \} = \{ e | S_i(e) \in B, S_j(e) \in \Pi \text{ for } j \neq i \}. \quad (14)$$

The right side of (14) is in $\mathcal{F}_E$, because it has its probability defined by Definition 1. Since $u$ is arbitrary, we have $X_i$ is a random variable over $(E, \mathcal{F}_E, P_E)$. Furthermore, also by Definition 1,

$$P_E(\{ e | X_i(e) < u \}) = P_E(\{ e | S_i(e) \in B, S_j(e) \in \Pi \text{ for } j \neq i \}) = P_\Pi(B) = P_\Pi(\{ \pi | X(\pi) < u \}).$$

Since $u$ is arbitrary, this means $X_i$ has the same distribution function as $X$.

2. Fix an arbitrary set of real values $u_1, u_2, \ldots, u_n$. Then

$$P_E(\{ e | X_1(e) < u_1, X_2(e) < u_2, \ldots, X_n(e) < u_n \}) = P_E(\{ e | X(S_1(e)) < u_1, X(S_2(e)) < u_2, \ldots, X(S_n(e)) < u_n \}) = P_E(\{ e | S_1(e) \in \{ \pi | X(\pi) < u_1 \}, S_2(e) \in \{ \pi | X(\pi) < u_2 \}, \ldots, S_n(e) \in \{ \pi | X(\pi) < u_n \} \}) = P_\Pi(\{ \pi | X(\pi) < u_1 \})P_\Pi(\{ \pi | X(\pi) < u_2 \}) \cdots P_\Pi(\{ \pi | X(\pi) < u_n \}) = P_E(\{ e | X_1(e) < u_1 \})P_E(\{ e | X_2(e) < u_2 \}) \cdots P_E(\{ e | X_n(e) < u_n \}).$$

This means $X_1, X_2, \ldots, X_n$ are independent. \hfill \blacksquare

3 Concluding Remarks

Section 2 presents a fully developed theoretical framework for population and sampling. A large portion of the previous explanations in verbal language has now been replaced by results in mathematical language. This makes our explanation much less vague than previous ones. However, some of the results may require a bit too much of mathematical maturity of the reader, and therefore in courses or textbooks, it may not be the most suitable to present the framework in its full version. However, we believe the following points should be stressed in educational occasions:

1. The term “population” as a noun should refer to the sample space, not the random variable as is the case in many textbooks.

2. The term “population” can be used as an attributive in “population random variable”, “population distribution”, and “population density”, which refer to a random variable in the population probability space, its distribution, and its density, respectively.

3. The population random variable $X$, and the sample random variables $X_1, X_2, \ldots, X_n$ do not live in the same probability space. Failure to notice this is the cause for many difficulties in the existing explanations.

4. The term “sample” may suggest students to believe the random sample $X_1, X_2, \ldots, X_n$ contains less information than the population random variable $X$. Actually, mathematically speaking, $X_1, X_2, \ldots, X_n$, and even each of its members, $X_i$, contain no less information than $X$. It is only that in practice, only one experiment is done, and we only hold $n$ real values $x_1, x_2, \ldots, x_n$ as observed values of the sample. Usually $n << N$.

Finally, it is hoped that this new framework and explanation, with its unique feature of mathematical sensibleness, will be helpful to a large number of statistics students.
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