Research Article

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Solvability of an infinite system of nonlinear integral equations of Volterra-Hammerstein type

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Abstract: The purpose of the paper is to study the solvability of an infinite system of integral equations of Volterra-Hammerstein type on an unbounded interval. We show that such a system of integral equations has at least one solution in the space of functions defined, continuous and bounded on the real half-axis with values in the space $l_1$ consisting of all real sequences whose series is absolutely convergent. To prove this result we construct a suitable measure of noncompactness in the mentioned function space and we use that measure together with a fixed point theorem of Darbo type.

Keywords: space of continuous and bounded functions defined on the half-axis, sequence space, measure of noncompactness, fixed point theorem, infinite system of integral equations

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1 Introduction

Integral equations create a very significant part of nonlinear analysis and applied mathematics ([1–4]). Many researchers, not only mathematicians, are interested in the study of the solvability of integral equations and the applicability of such equations to different problems arising in the description of real world events ([2, 3, 5–9]). The results obtained in the theory of integral equations are useful and widely utilized in many branches of technical sciences, as mechanics or engineering and exact sciences as physics, for example.

In the theory of integral equations the special and exceptional branch is created by infinite systems of integral equations. On the one hand such systems are very interesting subject of the study for researchers specialized in the theory of integral equations but on the other hand systems of integral equations play very crucial role in applications.

In this paper we deal with an infinite system of nonlinear integral equations of Volterra-Hammerstein type. In [10] we showed that such a system has a solution belonging to the space $BC(\mathbb{R}_+, c_0)$ of functions defined, continuous and bounded on the real half-axis and with values in the sequence space $c_0$. In the present paper we prove a stronger result. Namely, we show that infinite system of integral equations of Volterra-Hammerstein type has at least one solution in the space $BC(\mathbb{R}_+, l_1)$ consisting of all functions defined, continuous and bounded on the interval $\mathbb{R}_+$ with values in the sequence space $l_1$. Of course each such solution belongs to the space $BC(\mathbb{R}_+, c_0)$ considered in [10].

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Let us mention that paper [10] was the first one in which the study of solvability of infinite systems of integral equations defined on an unbounded interval was carried out. All known up to now results have been obtained in the space of functions defined on a bounded interval (see [11–14], for example).

2 Notation, definitions and auxiliary facts

In this section we recall some facts which will be utilized in the paper.
Let us start with establishing some notation. The symbols $\mathbb{R}$ and $\mathbb{N}$ stand for sets of real and natural numbers, respectively. Moreover, we put $\mathbb{R}_+ = [0, \infty)$.

The letter $E$ means a Banach space normed by norm $\| \cdot \|_E$ and with zero vector $\theta$. The symbol $B(x, r)$ denotes the closed ball in $E$ centered at $x$ with radius $r$. In the special case when $x = \theta$ we write $B_r$ instead of $B(\theta, r)$. Moreover, if $X$ is a subset of $E$ then we denote by $\overline{X}$ the closure of $X$ and by $\text{Conv} X$ the closed convex hull of the set $X$. The symbols $X + Y$, $\lambda X$ ($\lambda \in \mathbb{R}$) will stand for the usual algebraic operations on subsets $X$ and $Y$ of $E$. For a nonempty and bounded set $X \subset E$ we denote by $\text{diam} X$ the diameter of the set $X$. The symbol $||X||_E$ will stand for the norm of the set $X \subset E$ i.e., we have

$$||X||_E = \sup \{||x||_E : x \in X\}.$$ 

The fundamental notion that we use in this paper is the concept of a measure of noncompactness. We recall now the definition of a measure of noncompactness which was introduced in monograph [15]. To this end let us denote by $\mathcal{M}_E$ the family of all nonempty and bounded subsets of $E$ and by $\mathcal{N}_E$ its subfamily consisting of all relatively compact sets.

**Definition 2.1.** A function $\mu : \mathcal{M}_E \to \mathbb{R}_+$ will be called a measure of noncompactness in $E$ if it satisfies the following conditions:

(i) The family $\ker \mu = \{X \in \mathcal{M}_E : \mu(X) = 0\}$ is nonempty and ker $\mu \subset \mathcal{N}_E$.

(ii) $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.

(iii) $\mu(X) = \mu(X)$.

(iv) $\mu(\text{Conv} X) = \mu(X)$.

(v) $\mu(\lambda X + (1 - \lambda) Y) \leq \lambda \mu(X) + (1 - \lambda) \mu(Y)$ for $\lambda \in [0, 1]$.

(vi) If $(X_n)$ is a sequence of closed sets from $\mathcal{M}_E$ such that $X_{n+1} \subset X_n$ for $n = 1, 2, \ldots$ and if $\lim_{n \to \infty} \mu(X_n) = 0$

then the set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

The set $\ker \mu$ defined in axiom (i) is called the kernel of the measure of noncompactness $\mu$.

Let us observe that the intersection set $X_\infty$ appearing in axiom (vi) is a member of the family $\ker \mu$ [15]. This simple observation plays an essential role in our further considerations.

Now we present some properties of measures of noncompactness [15]. We say that $\mu$ is sublinear if it satisfies the following additional conditions:

(vii) $\mu(X + Y) \leq \mu(X) + \mu(Y)$.

(viii) $\mu(\lambda X) = |\lambda| \mu(X)$ for $\lambda \in \mathbb{R}$.

Moreover, we say that a measure of noncompactness $\mu$ has maximum property if

(ix) $\mu(X \cup Y) = \max(\mu(X), \mu(Y))$.

If the measure of noncompactness $\mu$ is such that

(x) ker $\mu = \mathcal{N}_E$

then it is called full.

A sublinear measure of noncompactness which is full and has maximum property is said to be regular measure of noncompactness [15].
Let us mention that the first measure of noncompactness was defined in 1930 by Kuratowski [16] in the following way

$$\alpha(X) = \inf \{ \varepsilon > 0 : X \text{ can be covered by a finite family of sets } X_1, X_2, \ldots, X_m \text{ such that diam } X_i \leq \varepsilon \text{ for } i = 1, 2, \ldots, m \}. $$

The measure $\alpha(X)$ is called the Kuratowski measure of noncompactness. It is known (see [15]) that the Kuratowski measure of noncompactness is a regular measure.

In the similar way was defined the Hausdorff measure of noncompactness ([17, 18]):

$$\chi(X) = \inf \{ \varepsilon > 0 : X \text{ has a finite } \varepsilon - \text{net in } E \}. $$

It can be shown that the measure $\chi(X)$ is also regular and that both defined above measures $\alpha(X)$ and $\chi(X)$ are equivalent. But despite of these similarities it turns out that the Hausdorff measure of noncompactness $\chi$ is more convenient in applications than the Kuratowski measure. The main reason is that in some classical Banach spaces we can find explicit formulas describing the Hausdorff measure of noncompactness but we do not know such formulas for the Kuratowski measure of noncompactness in any Banach space [15].

And now, taking into account our further investigations, we recall the formula expressing the Hausdorff measure of noncompactness in the space $l_1$ (cf. [15]). So, let us call to mind that the space $l_1$ consists of all sequences whose series is absolutely convergent, i.e.

$$l_1 = \{ x = (x_n) \in \mathbb{R}^\infty : \sum_{n=1}^{\infty} |x_n| < \infty \}. $$

It is normed by the following norm

$$||x||_{l_1} = ||(x_n)||_{l_1} = \sum_{n=1}^{\infty} |x_n|. $$

Then we have the following formula for the Hausdorff measure of noncompactness of any bounded set $X \in \mathfrak{M}_{l_1}$:

$$\chi(X) = \lim_{n \to \infty} \left\{ \sup \left\{ \sum_{k=n}^{\infty} |x_k| : x = (x_i) \in X \right\} \right\}. \tag{2.1}$$

To prove our main result we will also need a fixed point theorem involving a measure of noncompactness. The basic theorem in this subject is known fixed point theorem proved by Darbo [19]. We will use a modified version of Darbo theorem formulated below (cf. [15]).

**Theorem 2.2.** Let $\mu$ be an arbitrary measure of noncompactness in the Banach space $E$. Assume that $\Omega$ is a nonempty, bounded, closed and convex subset of $E$ and $Q : \Omega \to \Omega$ is a continuous operator such that there exists a constant $k \in [0, 1)$ for which $\mu(QX) \leq k \mu(X)$ for an arbitrary nonempty subset $X$ of $\Omega$. Then the operator $Q$ has at least one fixed point in the set $\Omega$.

### 3 Measures of noncompactness in the space $BC(\mathbb{R}_+, l_1)$

In [10] we investigated measures of noncompactness in the space $BC(\mathbb{R}_+, E)$ consisting of all functions defined, continuous and bounded on $\mathbb{R}_+$ with values in a fixed Banach space $E$. Let us pay attention to the fact that the space $BC(\mathbb{R}_+, E)$ is a generalization of the well-known and often used classical Banach space $BC(\mathbb{R}_+, \mathbb{R})$, therefore measures of noncompactness in the space $BC(\mathbb{R}_+, E)$ must be more complicated then known measures in $BC(\mathbb{R}_+, \mathbb{R})$. And now we recall some basic facts about the space $BC(\mathbb{R}_+, E)$ and measures of noncompactness in this space.

Let us start with assuming that $E$ is a given Banach space with the norm $|| \cdot ||_E$ whereby we will assume that $E$ is an infinite dimensional space. Consider the Banach space $BC(\mathbb{R}_+, E)$ equipped with the supremum
norm \( \|x\|_\infty \) defined in the standard way

\[
\|x\|_\infty = \sup\{\|x(t)\|_E : t \in \mathbb{R}_+\}.
\]

For further purposes, for a fixed \( T > 0 \) consider the Banach space \( C_T = C([0, T], E) \) consisting of functions defined and continuous on the interval \([0, T]\) with values in \( E \) and endowed with the norm

\[
\|x\|_T = \sup\{\|x(t)\|_E : t \in [0, T]\}.
\]

In [10] we defined three formulas for measures of noncompactness in the space \( BC(\mathbb{R}_+, E) \) and each such formula is a sum of three components. The first and the second component are the same in each formula and we start to present them. So, let us fix a set \( x \in X \in M_{BC(\mathbb{R}_+, E)} \) and numbers \( T > 0 \) and \( \varepsilon > 0 \). For any function \( x \in X \) we define the modulus of continuity \( \omega^T(x, \varepsilon) \) of the function \( x \) on the interval \([0, T]\) by the classical formula

\[
\omega^T(x, \varepsilon) = \sup\{\|x(t) - x(s)\|_E : t, s \in [0, T], |t - s| \leq \varepsilon\}.
\]

Next, let us define

\[
\omega^T(X, \varepsilon) = \sup\{\omega^T(x, \varepsilon) : x \in X\},
\]

and finally, we put

\[
\omega_0(X) = \lim_{T \to \infty} \omega^T(X).
\]

Notice that both above limits exist (for details see [10]). The quantity \( \omega_0(X) \) is the first component of each of mentioned earlier measures of noncompactness in \( BC(\mathbb{R}_+, E) \). Next, to obtain the second component, assume that \( y = y(X) \) is a measure of noncompactness in the space \( E \). Fix number \( t \in \mathbb{R}_+ \) and denote by \( X(t) \) the so-called cross-section of the set \( X \) at the point \( t \):

\[
X(t) = \{x(t) : x \in X\}.
\]

Further, for a fixed \( T > 0 \) let us put

\[
\bar{y}_T(X) = \sup\{y(X(t)) : t \in [0, T]\}
\]

and

\[
\bar{y}_\infty(X) = \lim_{T \to \infty} \bar{y}_T(X).
\]

Notice that the above limit exists (see [10]). The obtained quantity \( \bar{y}_\infty(X) \) is the second component of all three formulas for measure of noncompactness in the space \( BC(\mathbb{R}_+, E) \). Now we introduce the third component of the measure of noncompactness in \( BC(\mathbb{R}_+, E) \) which describes the behaviour of the set of functions at infinity. We can do it in three ways and we will describe each of them. So, for a fixed \( T > 0 \) let us define

\[
a_T(X) = \sup_{x \in X} \{\sup\{\|x(t)\|_E : t \geq T\}\}.
\]

Next, notice that there exists the limit

\[
a_\infty(X) = \lim_{T \to \infty} a_T(X)
\]

and the quantity \( a_\infty(X) \) is considered as the third component of the measure of noncompactness in the space \( BC(\mathbb{R}_+, E) \). Instead of \( a_\infty(X) \) we can take

\[
b_\infty(X) = \lim_{T \to \infty} b_T(X),
\]

where

\[
b_T(X) = \sup_{x \in X} \{\sup\{\|x(t) - x(s)\|_E : t, s \geq T\}\}
\]
or we can also define
\[ c(X) = \limsup_{t \to \infty} \text{diam } X(t), \quad (3.6) \]
where \text{diam } X(t) is understood as
\[ \text{diam } X(t) = \sup\{ ||x(t) - y(t)||_E : x, y \in X \}. \]

And now let us consider the functions \( y_a, y_b, y_c \) defined on the family \( \mathcal{M}_{BC(\mathbb{R}, E)} \) as follows
\begin{align*}
y_a(X) &= \omega_0(X) + y_\infty(X) + a_\infty(X), \\
y_b(X) &= \omega_0(X) + y_\infty(X) + b_\infty(X), \\
y_c(X) &= \omega_0(X) + y_\infty(X) + c(X). \quad (3.7) \quad (3.8) \quad (3.9)
\end{align*}

In [10] we proved that under some assumptions on \( y \) the functions \( y_a, y_b, y_c \) are measures of noncompactness in the space \( BC(\mathbb{R}^+, E) \). More precisely, we have the following result.

**Theorem 3.1.** Assume that \( y \) is the Hausdorff measure of noncompactness in the Banach space \( E \) i.e., \( y = \chi_E \). Then, the functions \( \chi_a, \chi_b \) and \( \chi_c \) defined by (3.7) - (3.9) are measures of noncompactness in the space \( BC(\mathbb{R}^+, E) \) such that
\[ \chi(X) \leq 2 \chi_b(X), \]
\[ \chi(X) \leq 4 \chi_c(X), \]
\[ \chi_b(X) \leq 2 \chi_a(X), \quad \chi_c(X) \leq 2 \chi_a(X) \]
for an arbitrary set \( X \in \mathcal{M}_{BC(\mathbb{R}^+, E)} \), where \( \chi \) denotes the Hausdorff measure of noncompactness in the space \( BC(\mathbb{R}^+, E) \).

For other properties of the above introduced measures of noncompactness we refer to [10]. We recall only that Theorem 3.1 remains valid if in the construction of the component \( y_\infty \) we replace the Hausdorff measure of noncompactness \( \chi \) by an arbitrary regular measure of noncompactness \( \mu \) equivalent to the Hausdorff measure \( \chi \) [10].

Now, we are going to present formulas (3.7), (3.8) and (3.9) in the special case, for \( E = l_1 \). The space \( l_1 \) is one of the Banach sequence spaces and we deal with this space (and, in general, with sequence spaces) because it is strictly connected with the form of solutions of infinite systems of integral equations (see Theorem 4.2).

Therefore, we will work in the Banach space
\[ BC(\mathbb{R}^+, l_1) = \{ x : \mathbb{R}^+ \to l_1 : x \text{ is continuous and bounded on } \mathbb{R}^+ \}. \]

The fundamental fact in our investigations is that every function \( x \in BC(\mathbb{R}^+, l_1) \) can be regarded as a function sequence
\[ x(t) = (x_n(t)) = (x_1(t), x_2(t), \ldots) \]
for \( t \in \mathbb{R}^+ \), where for any fixed \( t \) the sequence \( (x_n(t)) \) is a real sequence being an element of the space \( l_1 \). Obviously, it means that
\[ ||x(t)||_{l_1} = \sum_{n=1}^{\infty} |x_n(t)| < \infty. \]

According to the formula for the norm in the space \( BC(\mathbb{R}^+, E) \) given earlier we have
\[ ||x||_\infty = \sup\{ ||x(t)||_{l_1} : t \in \mathbb{R}^+ \} = \sup\{ \sum_{n=1}^{\infty} |x_n(t)| : t \in \mathbb{R}^+ \}. \]
Now, we are going to present explicitly the consecutive components \( \omega_0(X), \overline{\omega}_\infty(X) \) and \( a_\infty(X) \) of the measure of noncompactness \( \chi_a(X) \) for any set \( X \in \mathfrak{M}_{BC(R, l_1)} \). So, let us start with \( \omega_0(X) \). Fix arbitrarily numbers \( T > 0 \) and \( \varepsilon > 0 \). For any \( x = x(t) = (x_n(t)) \in X \) we have

\[
\omega^T(x, \varepsilon) = \sup \{ \| x(t) - x(s) \|_{l_1} : t, s \in [0, T], |t - s| \leq \varepsilon \}
\]

\[
= \sup \{ \sum_{n=1}^{\infty} |x_n(t) - x_n(s)| : t, s \in [0, T], |t - s| \leq \varepsilon \}.
\]

Hence, we get

\[
\omega^T(X, \varepsilon) = \sup_{x=(x_n) \in X} \left\{ \sup_{n=1}^{\infty} \left\{ \sum_{n=1}^{\infty} |x_n(t) - x_n(s)| : t, s \in [0, T], |t - s| \leq \varepsilon \right\} \right\},
\]

and next

\[
\omega_\infty(X) = \lim_{\varepsilon \to 0} \left\{ \sup_{x=(x_n) \in X} \left\{ \sup_{n=1}^{\infty} \left\{ \sum_{n=1}^{\infty} |x_n(t) - x_n(s)| : t, s \in [0, T], |t - s| \leq \varepsilon \right\} \right\} \right\}. \tag{3.10}
\]

Next, we are going to define the second component occurring in the formula for the measure \( \chi_a(X) \). To this end let us assume that \( X \in \mathfrak{M}_{BC(R, l_1)} \) and \( t \in R_+ \) is arbitrarily fixed. Using (2.1) we have

\[
\chi(X(t)) = \lim_{n \to \infty} \left\{ \sup_{x=(x_k) \in X} \left\{ \sum_{k=1}^{\infty} |x_k(t)| \right\} \right\}.
\]

Next, for a fixed \( T > 0 \) utilizing (3.1) we get

\[
\overline{\chi}_T(X) = \sup \{ \chi(X(t)) : t \in [0, T] \}
\]

\[
= \sup_{t \in [0, T]} \left\{ \lim_{n \to \infty} \left\{ \sup_{x=(x_k) \in X} \left\{ \sum_{k=1}^{\infty} |x_k(t)| \right\} \right\} \right\}.
\]

Finally, in view of (3.2) we obtain the following expression:

\[
\overline{\chi}_\infty(X) = \lim_{T \to \infty} \overline{\chi}_T(X)
\]

\[
= \lim_{T \to \infty} \left\{ \sup_{t \in [0, T]} \left\{ \lim_{n \to \infty} \left\{ \sup_{x=(x_k) \in X} \left\{ \sum_{k=1}^{\infty} |x_k(t)| \right\} \right\} \right\} \right\}. \tag{3.11}
\]

And now let us write the third component of the measure \( \chi_a(X) \). Thus, fix an arbitrary number \( T > 0 \). Then, on the basis of (3.3), we have

\[
a_T(X) = \sup_{x=(x_n) \in X} \left\{ \sup \{ \| x(t) \|_{l_1} : t \geq T \} \right\}
\]

\[
= \sup_{x=(x_n) \in X} \left\{ \sup \left\{ \sum_{n=1}^{\infty} |x_n(t)| : t \geq T \right\} \right\}.
\]

Next, by (3.4) we obtain

\[
a_\infty(X) = \lim_{T \to \infty} \left\{ \sup_{x=(x_n) \in X} \left\{ \sup_{t \geq T} \left\{ \sum_{n=1}^{\infty} |x_n(t)| \right\} \right\} \right\}. \tag{3.12}
\]

Finally, based on Theorem 3.1 we get that the function

\[
\chi_a(X) = \omega_0(X) + \overline{\chi}_\infty(X) + a_\infty(X)
\]
is a measure of noncompactness in the space $BC(\mathbb{R}^+, l_1)$, where $\omega_0(X), \chi_\infty(X)$ and $a_\infty(X)$ are given by formulas (3.10), (3.11) and (3.12), respectively.

Observe, that keeping in mind formulas (3.5) and (3.6) together with Theorem 3.1 we obtain that the functions

$$\chi_b(X) = \omega_0(X) + \chi_\infty(X) + b_\infty(X)$$

and

$$\chi_c(X) = \omega_0(X) + \chi_\infty(X) + c(X)$$

are also measures of noncompactness in the space $BC(\mathbb{R}^+, l_1)$, where

$$b_\infty(X) = \lim_{t \to \infty} \left\{ \sup_{x = (x_n) \in X} \left\{ \sup_{n=1}^{\infty} \left| x_n(t) - x_n(s) \right| : t, s \geq T \right\} \right\}$$

and

$$c(X) = \lim_{t \to \infty} \sup \left\{ \sup \left\{ \left| x(t) - y(t) \right| : x, y \in X \right\} : x \in (x_n), y = (y_n) \right\}.$$ 



\section{4 Theorem on the existence of solutions of infinite systems of integral equations on the real half-axis}

Let us consider the following infinite system of nonlinear quadratic integral equations of the Volterra-Hammerstein type

$$x_n(t) = a_n(t) + f_n(t, x_n(t), x_{n+1}(t), \ldots) \int_0^t k_n(t, s) g_n(s, x_1(s), x_2(s), \ldots) ds$$

for $t \in \mathbb{R}^+$ and for $n = 1, 2, \ldots$.

In [10] we proved that system (4.1) has at least one solution in the space $BC(\mathbb{R}^+, c_0) := \{ x : \mathbb{R}^+ \to c_0 \} : x$ is continuous and bounded on $\mathbb{R}^+$. In this paper we prove the other result, namely we show that the system (4.1) has at least one solution in the space $BC(\mathbb{R}^+, l_1)$. For the convenience, from now on the space $BC(\mathbb{R}^+, l_1)$ will be denoted by $BC_1$.

At the beginning we provide a lemma which we will utilize in the proof of our main result.

\textbf{Lemma 4.1.} \textit{If the sequence $(a_n)$ belongs to the space $l_1$ then $\lim_{n \to \infty} \sum_{i=n}^{\infty} |a_i| = 0.$}

\textit{Proof.} The proof is an immediate consequence of the Cauchy condition for real sequences. \hfill $\square$

In what follows we will investigate the solvability of system (4.1) in the space $BC_1$ under the below listed assumptions.

(i) The function sequence $(a_n(t))$ is an element of the space $BC_1$ such that $\lim_{t \to \infty} \sum_{n=1}^{\infty} a_n(t) = 0$.

(ii) The functions $k_n(t, s) = k_n : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ are continuous on the set $\mathbb{R}^+ \times \mathbb{R}^+$ ($n = 1, 2, \ldots$). Moreover, the functions $t \to k_n(t, s)$ are locally equicontinuous on the set $\mathbb{R}^+$ uniformly with respect to $s \in \mathbb{R}^+$ i.e., the following condition is satisfied

$$\forall_{T > 0} \forall_{\varepsilon > 0} \exists_{\delta > 0} \forall_{n \in \mathbb{N}} \forall_{s \in \mathbb{R}} \forall_{t_1, t_2 \in [0, T]} \left| t_2 - t_1 \right| \leq \delta \Rightarrow |k_n(t_2, s) - k_n(t_1, s)| \leq \varepsilon.$$
(iii) There exists a constant \( K_1 > 0 \) such that
\[
\sum_{n=1}^{\infty} \int_{0}^{t} |k_n(t, s)| ds \leq K_1
\]
for any \( t \in \mathbb{R}_+ \).

(iv) The sequence \( (k_n(t, s)) \) is equibounded on \( \mathbb{R}_+^2 \) i.e., there exists a constant \( K_2 > 0 \) such that \( |k_n(t, s)| \leq K_2 \) for \( t, s \in \mathbb{R}_+ \) and \( n = 1, 2, \ldots \).

(v) The function \( \sum_{n=1}^{\infty} f_n \) is defined on the set \( \mathbb{R}_+ \times \mathbb{R}_\infty \) and takes real values. Moreover, the function \( t \to \sum_{n=1}^{\infty} f_n(t, x_n, x_{n+1}, \ldots) \) is continuous on \( \mathbb{R}_+ \) uniformly with respect to \( x = (x_n) \in l_1 \) i.e., the following condition is satisfied
\[
\forall \varepsilon > 0 \forall t_0 \in \mathbb{R}_+ \exists \delta > 0 \forall (x, y) \in l_1 \forall t \in \mathbb{R}_+ \quad | |t - t_0| \leq \delta \Rightarrow \sum_{n=1}^{\infty} |f_n(t, x_n, x_{n+1}, \ldots) - f_n(t_0, x_n, x_{n+1}, \ldots)| \leq \varepsilon.
\]

(vi) There exists a function \( l : \mathbb{R}_+ \to \mathbb{R}_+ \) such that \( l \) is nondecreasing on \( \mathbb{R}_+ \), continuous at 0 and the following condition is satisfied
\[
|f_n(t, x_n, x_{n+1}, \ldots) - f_n(t, y_n, y_{n+1}, \ldots)| \leq l(r) |x_n - y_n|
\]
for all \( x = (x_n), y = (y_n) \in l_1 \) such that \(|x||_1 \leq r, |y||_1 \leq r\), for any \( t \in \mathbb{R}_+ \) and for \( n = 1, 2, \ldots \).

(vii) The sequence of functions \( (\mathcal{F}_n) \), where \( \mathcal{F}_n(t) = f_n(t, 0, 0, \ldots) \) belongs to the space \( BC_1 \) and \( \lim_{t \to \infty} \sum_{n=1}^{\infty} \mathcal{F}_n(t) = 0 \).

Let us notice that on the basis of assumption (vii) we infer that there exists the finite constant \( \mathcal{F} = \sup \{ \sum_{n=1}^{\infty} \mathcal{F}_n(t) : t \in \mathbb{R}_+ \} \).

Now, we can formulate our further assumptions concerning infinite system (4.1).

(viii) The function \( g_n \) is defined on the set \( \mathbb{R}_+ \times \mathbb{R}_\infty \) and takes real values for \( n = 1, 2, \ldots \). Moreover, the operator \( g \) defined on the space \( \mathbb{R}_+ \times l_1 \) by the equality
\[
(gx)(t) = (g_n(t, x)) = (g_1(t, x), g_2(t, x), \ldots)
\]
transforms the space \( \mathbb{R}_+ \times l_1 \) into \( l_1 \) and is such that the family of functions \( \{(gx)(t)\} \in \mathbb{R}_+ \) is equicontinuous at every point of the space \( l_1 \) i.e., for each arbitrarily fixed \( x \in l_1 \) and for a given \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that
\[
||(gy)(t) - (gx)(t)||_1 \leq \varepsilon
\]
for every \( t \in \mathbb{R}_+ \) and for any \( y \in l_1 \) such that \(|y - x||_1 \leq \delta\).

(ix) The operator \( g \) defined in assumption (viii) is bounded on the space \( \mathbb{R}_+ \times l_1 \). More precisely, there exists a positive constant \( \mathcal{G} \) such that \(||(gx)(t)||_1 \leq \mathcal{G}||x||_1\) for any \( x \in l_1 \) and for each \( t \in \mathbb{R}_+ \).

(x) There exists a positive solution \( r_0 \) of the inequality
\[
A + \mathcal{F} + \mathcal{G} K_1 r l(r) \leq r
\]
such that \( \mathcal{G} K_1 l(r_0) < 1 \), where the constants \( \mathcal{F}, \mathcal{G}, K_1 \) were defined above and the constant \( A \) is defined in the following way
\[
A = \sup \{ \sum_{n=1}^{\infty} |a_n(t)| : t \in \mathbb{R}_+ \}.
\]

Now we can present our main result concerning the solvability of infinite system of integral equations (4.1).
Theorem 4.2. Under assumptions (i) - (x) infinite system (4.1) has at least one solution \( x(t) = (x_n(t)) \) in the space \( BC_1 = BC(\mathbb{R}_+, I_1) \).

\[ \sum_{n=1}^{\infty} |f_n(t, x_n(t))| < \infty, \quad (4.2) \]

Proof. Denote by \( F, V, Q \) operators defined on the space \( BC_1 \) in the following way:

\[ (Fx)(t) = (F_n x)(t) = (f_n(t, x(t))) = (f_n(t, x_n(t), x_{n+1}(t), \ldots)) \]

\[ (Vx)(t) = (V_n x)(t) = (\int_0^t k_n(t, s) g_n(s, x_1(s), x_2(s), \ldots) ds) \]

\[ (Qx)(t) = (Q_n x)(t) = (a_n(t) + (F_n x)(t) (V_n x)(t)) \]

for an arbitrary element \( x = (x_n) \in BC_1 \) and for \( t \in \mathbb{R}_+ \).

Our proof will be conducted in several steps. At the beginning we show that operator \( Q \) transforms the space \( BC_1 \) into itself. Thus, let us take \( x = x(t) = (x_n(t)) \in BC_1 \). Obviously this means that \( \sum_{n=1}^{\infty} |x_n(t)| < \infty \).

Then, for arbitrary \( t \in \mathbb{R}_+ \), using assumption (vi), we get

\[ \sum_{n=1}^{\infty} |f_n(t, x_n(t), x_{n+1}(t), \ldots)| \leq \sum_{n=1}^{\infty} |f_n(t, 0, 0, \ldots)| + \sum_{n=1}^{\infty} |f_n(t, 0, 0, \ldots)| \]

\[ \leq \sum_{n=1}^{\infty} l(||x(t)||_1) \cdot |x_n(t) - 0| + \sum_{n=1}^{\infty} \mathcal{f}_n(t) \]

\[ \leq l(||x(t)||_1) \cdot \sum_{n=1}^{\infty} |x_n(t)| + \sum_{n=1}^{\infty} \mathcal{f}_n(t). \]

Keeping in mind the fact that \( x \in BC_1 \), in view of assumption (vii) we obtain

\[ \sum_{n=1}^{\infty} |f_n x(t)| < \infty, \quad (4.3) \]

Thus we have that \((Fx)(t) \in I_1 \). Moreover, from (4.2) we infer that the function \( Fx \) is bounded on the set \( \mathbb{R}_+ \).

Further, we are going to show that the function \( Fx \) is continuous on the interval \( \mathbb{R}_+ \). To this end, let us take arbitrary \( t_0 \in \mathbb{R}_+ \) and \( \varepsilon > 0 \). It follows from the continuity of the function \( x \) that the below given condition is satisfied

\[ \forall t_0 \in \mathbb{R}, \forall \varepsilon > 0 \exists \delta > 0 \forall t \in \mathbb{R}, \quad ||t - t_0|| \leq \delta \Rightarrow ||x(t) - x(t_0)||_1 \leq \varepsilon. \quad (4.4) \]

Thus, let us choose \( \delta_1 > 0 \) according to (4.4). Then, for \( t \in \mathbb{R}_+ \) such that \( |t - t_0| \leq \delta_1 \) we obtain

\[ \sum_{n=1}^{\infty} |(F_n x)(t) - (F_n x)(t_0)| = \sum_{n=1}^{\infty} |f_n(t, x_n(t), x_{n+1}(t), \ldots) - f_n(t_0, x_n(t_0), x_{n+1}(t_0), \ldots)| \]

\[ \leq \sum_{n=1}^{\infty} |f_n(t, x_n(t), x_{n+1}(t), \ldots) - f_n(t_0, x_n(t), x_{n+1}(t), \ldots)| \]

\[ + \sum_{n=1}^{\infty} |f_n(t_0, x_n(t), x_{n+1}(t), \ldots) - f_n(t_0, x_n(t_0), x_{n+1}(t_0), \ldots)| \]

\[ \leq \sum_{n=1}^{\infty} |f_n(t, x_n(t), x_{n+1}(t), \ldots) - f_n(t_0, x_n(t), x_{n+1}(t), \ldots)| \]

\[ + \sum_{n=1}^{\infty} l(\sup \{|x(t)|_1 : t \in \mathbb{R}_+\}) |x_n(t) - x_n(t_0)| \]
\[
\sum_{n=1}^{\infty} |f_n(t, x_n(t), x_{n+1}(t), \ldots) - f_n(t_0, x_n(t), x_{n+1}(t), \ldots)| + l(\sup \{||x(t)||_{l_1} : t \in \mathbb{R}_+\}) \sum_{n=1}^{\infty} |x_n(t) - x_n(t_0)| = \sum_{n=1}^{\infty} |f_n(t, x_n(t), x_{n+1}(t), \ldots) - f_n(t_0, x_n(t), x_{n+1}(t), \ldots)| + ||x||_{BC} \epsilon.
\]

Next, let us choose a number \(\delta_2 > 0\) according to assumption (v). Then for \(|t - t_0| \leq \delta_2\) we have
\[
\sum_{n=1}^{\infty} |f_n(t, x_n(t), x_{n+1}(t), \ldots) - f_n(t_0, x_n(t), x_{n+1}(t), \ldots)| \leq \epsilon.
\]

Thus, taking \(\delta = \min\{\delta_1, \delta_2\}\) we can deduce that the following estimate is satisfied
\[
\sum_{n=1}^{\infty} |(F_n x)(t) - (F_n x)(t_0)| \leq \epsilon + l(||x||_{BC}) \epsilon = \epsilon(1 + l(||x||_{BC}))
\]

for any \(t \in \mathbb{R}_+.\) Therefore, we can write
\[
||(F x)(t) - (F x)(t_0)|| \leq \epsilon
\]
for any \(t \in \mathbb{R}_+.\) It means that \(F x\) is continuous on \(\mathbb{R}_+.\)

Linking above established facts we obtain that the operator \(F\) transforms the space \(BC_1\) into itself.

Next, we are going to show that the operator \(V\) maps the space \(BC_1\) into \(BC_1\). So, let us take an arbitrary function \(x = x(t) = (x_n(t)) \in BC_1\). We are going to prove that a function \(V x \in BC_1\). We start with showing the boundness of the function \(V x\) on \(\mathbb{R}_+.\) To this end observe that for an arbitrary number \(t \in \mathbb{R}_+,\) using assumptions (iii) and (ix), we get
\[
\sum_{n=1}^{\infty} |(V_n x)(t)| = \sum_{n=1}^{\infty} \left| \int_0^t k_n(t, s) g_n(s, x_1(s), x_2(s), \ldots) ds \right|
\]
\[
\leq \sum_{n=1}^{\infty} \int_0^t |k_n(t, s)| |g_n(s, x_1(s), x_2(s), \ldots)| ds
\]
\[
\leq \sum_{n=1}^{\infty} \int_0^t |k_n(t, s)| \overline{G} ds
\]
\[
\leq \overline{G} \sum_{n=1}^{\infty} \int_0^t |k_n(t, s)| ds \leq \overline{G} K_1.
\]

Therefore we have that for any \(t \in \mathbb{R}_+\) the following inequality holds
\[
||(V x)(t)||_{l_1} \leq \overline{G} K_1.
\]

This means the function \(V x\) is bounded on \(\mathbb{R}_+.\)

To prove the continuity of the function \(V x\) on the interval \(\mathbb{R}_+,\) let us fix \(\epsilon > 0, T > 0\) and \(t_0 \in [0, T].\) In virtue of (4.4) we can choose a number \(\delta > 0\) according to the continuity of the function \(x = x(t)\) at the point \(t_0.\) Next, take \(t \in [0, T]\) satisfying \(|t - t_0| \leq \delta\) (without loss of generality we can assume that \(t > t_0.\)) Then, keeping in mind assumptions (iv), (viii) and (ix) and using the Lebesgue monotone convergence theorem [20], we arrive at the following estimates:
\[
\sum_{n=1}^{\infty} |(V_n x)(t) - (V_n x)(t_0)| = \sum_{n=1}^{\infty} \left| \int_0^t k_n(t, s) g_n(s, x_1(s), x_2(s), \ldots) ds - \int_0^{t_0} k_n(t_0, s) g_n(s, x_1(s), x_2(s), \ldots) ds \right|
\]
\[
(4.7)
\]
for any fixed definition of the operator we deduce that the function (according to assumption (ii)). Obviously finally, linking all the above established properties of the function into itself. For arbitrary and henceאותן בפונקציה פאקטזר גאַניזאַק שּוֹבִּל סולבָּבֶהּ שלシステム של שֶל פּונקצּוֹנוֹת פּוֹנְקְצּוֹנוֹת מֶלָּקָה לוֹ הַפּונְקְצִיטָה את הַפּונְקְצִיטָה הֶלָּקָה לֶהַפּונְקְצִיטָה. For arbitrary we are going to proove that the function $Qx$ is continuous on $[0, T]$. The number $T$ was choosen arbitrarily therefore we deduce that the function $Vx$ is continuous on the real half-axis $\mathbb{R}_+$. Further on, we are going to prove that the function $Q$ maps the space $BC_1$ into itself. In order to prove this fact, notice that we can treat the space $BC_1$ as a Banach algebra with respect to the coordinatewise multiplication of sequences. Therefore, take any function $x \in BC_1$ and consider a function $Qx$. Keeping in mind the definition of the operator $Q$ and established facts that the function $Fx$ and the function $Vx$ are continuous on $\mathbb{R}_+$ we obtain that the function $Qx$ is also continuous on $\mathbb{R}_+$. Similarly, taking into account the boundness of functions $Fx$ and $Vx$ on the set $\mathbb{R}_+$ we infer that $Qx$ is also bounded on $\mathbb{R}_+$. In order to show that $Qx : \mathbb{R}_+ \rightarrow l_1$ let us notice that using assumption (i) and (4.3) we have

\[ \sum_{n=1}^{\infty} |(Qnx)(t)| \leq \sum_{n=1}^{\infty} |a_n(t)| + GK_1 \sum_{n=1}^{\infty} |(Fnx)(t)| < \infty, \]

so for any fixed $t \in \mathbb{R}_+$ we obtain that $(Qnx)(t) \in l_1$ and hence $Qx : \mathbb{R}_+ \rightarrow l_1$.

Finally, linking all the above established properties of the function $Qx$ we derive that the operator $Q$ transforms the space $BC_1$ into itself. In what follows we show the existence of a number $r_0 > 0$ such that $Q$ transforms the ball $B_{r_0}$ in the space $BC_1$ into itself. For arbitrary $t \in \mathbb{R}_+$, utilizing estimates (4.2) and (4.6) as well as assumptions (x) and (vi) we obtain

\[ \sum_{n=1}^{\infty} |(Qnx)(t)| \leq \sum_{n=1}^{\infty} |a_n(t)| + \sum_{n=1}^{\infty} |(Fnx)(t)| |(Vnx)(t)| \]
assumption (vi) we have

\[
\sum_{n=1}^{\infty} |a_n(t)| + \sum_{n=1}^{\infty} |(F_n x)(t)| + \sum_{n=1}^{\infty} |(V_n x)(t)|
\]

\[
\leq A + \left[ l(|x(t)|_{l_1}) \right] \sum_{n=1}^{\infty} |x_n(t)| + \sum_{n=1}^{\infty} f_n(t) \]

\[
\leq A + \left[ l(|x(t)|_{l_1}) \right] \sum_{n=1}^{\infty} |x_n(t)| + \sum_{n=1}^{\infty} f_n(t)
\]

\[
\leq A + \left[ l(|x(t)|_{l_1}) \right] \sum_{n=1}^{\infty} |x_n(t)| + \sum_{n=1}^{\infty} F_n(t)
\]

\[
= A + F \mathcal{K} + K_1 l(|x|_{BC_1}) ||x||_{BC_1}.
\]

This yields to estimate

\[
||Qx||_{BC_1} \leq A + \mathcal{K} + K_1 l(|x|_{BC_1}) ||x||_{BC_1}.
\]

Taking into account the last inequality and assumption (x) we deduce that there exists a number \( r_0 > 0 \) such that the operator \( Q \) transforms the ball \( B_{r_0} \) into itself.

In what follows we show that the operator \( Q \) is continuous on the ball \( B_{r_0} \). In order to prove this fact fix arbitrarily \( x \in B_{r_0}, \, \epsilon > 0 \) and take a function \( y \in B_{r_0} \) such that \( ||x - y||_{BC_1} \leq \epsilon \). Fix \( t \in \mathbb{R}_+ \). Then, in view of assumption (vi) we have

\[
\sum_{n=1}^{\infty} |(F_n x)(t) - (F_n y)(t)| = \sum_{n=1}^{\infty} |f_n(t, x_n(t), x_{n+1}(t), \ldots) - f_n(t, y_n(t), y_{n+1}(t), \ldots)|
\]

\[
\leq \sum_{n=1}^{\infty} l(r_0) |x_n(t) - y_n(t)| = l(r_0) \sum_{n=1}^{\infty} |x_n(t) - y_n(t)|
\]

\[
= l(r_0) ||x(t) - y(t)||_{l_1} \leq l(r_0) ||x - y||_{BC_1} \leq l(r_0) \epsilon,
\]

so we get

\[
||Fx - Fy||_{BC_1} \leq l(r_0) \epsilon.
\]

This means that \( F \) is continuous on the ball \( B_{r_0} \).

Further, let us consider the function \( \delta = \delta(\epsilon) \) defined for \( \epsilon > 0 \) in the following way

\[
\delta(\epsilon) = \sup \{|g_n(t, x) - g_n(t, y)| : x, y \in l_1, ||x - y||_{l_1} \leq \epsilon, \, t \in \mathbb{R}_+, \, n \in \mathbb{N}\}.
\]

Taking into account assumption (viii) we deduce that \( \delta(\epsilon) \to 0 \) as \( \epsilon \to 0 \).

Further, let us assume, similarly as above, that \( x \in B_{r_0}, \, \epsilon > 0 \) and \( y \in B_{r_0} \) are such that \( ||x - y||_{BC_1} \leq \epsilon \). Then, for a fixed \( t \in \mathbb{R}_+ \) we get

\[
\sum_{n=1}^{\infty} |(V_n x)(t) - (V_n y)(t)| \leq \sum_{n=1}^{t} \int_{0}^{t} |k_n(t, s)| |g_n(s, x_1(s), x_2(s), \ldots) - g_n(s, y_1(s), y_2(s), \ldots)| ds
\]

\[
\leq \sum_{n=1}^{\infty} \int_{0}^{t} |k_n(t, s)| \delta(\epsilon) ds \leq K_1 \delta(\epsilon).
\]

Consequently, we obtain

\[
||Vx - Vy||_{BC_1} \leq K_1 \delta(\epsilon).
\]

This proves the continuity of the operator \( V \) on the ball \( B_{r_0} \).

Now, linking the continuity of the operators \( F \) and \( V \) on the ball \( B_{r_0} \) and keeping in mind the representation of the operator \( Q \) written at the beginning of the proof we infer that the operator \( Q \) is continuous on \( B_{r_0} \).

And now we have the last step of our proof in which we show that the inequality from Theorem 2.2 is satisfied for any set \( X \subset B_{r_0} \) and for measure of noncompactness \( \chi_0 \) defined by formula (3.7) for \( y = \chi_{l_1} \).
To this end take a nonempty subset $X$ of the ball $B_0$ and fix numbers $\varepsilon > 0$ and $T > 0$. Choose $t, s \in [0, T]$ such that $|t - s| \leq \varepsilon$ and consider a function $x = x(t) = (x_n(t)) \in X$. Then, proceeding similarly as in (4.5) we obtain the following estimate:

$$
\sum_{n=1}^{\infty} |(F_n x)(t) - (F_n x)(s)| \leq \|x\|_{\mathcal{B}_C} \sum_{n=1}^{\infty} |x_n(t) - x_n(s)|
$$

(4.8)

\[
+ \sum_{n=1}^{\infty} |f_n(t, x_n(t), x_{n+1}(t), \ldots) - f_n(s, x_n(t), x_{n+1}(t), \ldots)|
\]

where

$$
\leq l(r_0) \sum_{n=1}^{\infty} |x_n(t) - x_n(s)| + \sup \{ \sum_{n=1}^{\infty} |f_n(t, x_n, x_{n+1}, \ldots) - f_n(s, x_n, x_{n+1}, \ldots)| : t, s \in [0, T],
$$

\[
|t - s| \leq \varepsilon, \|x\|_{l} = \|x_n\|_{l} \leq r_0 \}
\]

$$
\leq l(r_0) \omega^T(x, \varepsilon) + \omega^1(f, \varepsilon),
$$

\[
\text{where}
\]

$$
\omega^1(f, \varepsilon) = \sup \{ \sum_{n=1}^{\infty} |f_n(t, x_n, x_{n+1}, \ldots) - f_n(s, x_n, x_{n+1}, \ldots)| : t, s \in [0, T],
$$

\[
|t - s| \leq \varepsilon, \|x\|_{l} = \|x_n\|_{l} \leq r_0 \}.
\]

Obviously, in view of assumption (v) we have that $\omega^1(f, \varepsilon) \to 0$ as $\varepsilon \to 0$.

Taking supremum with respect to $t, s \in [0, T]$, $|t - s| \leq \varepsilon$ on the left side of (4.8), we get the following estimate

$$
\omega^T(Fx, \varepsilon) \leq l(r_0) \omega^T(x, \varepsilon) + \omega^1(f, \varepsilon).
$$

(4.9)

Next, let us take $t, s$ as above. Assuming additionally that $t > s$ and following the estimate (4.7) we get

$$
\sum_{n=1}^{\infty} |(V_n x)(t) - (V_n x)(s)| \leq T \overline{G} \omega^T_k(\varepsilon) + \overline{G} K_2 \varepsilon,
$$

where the function $\omega^T_k(\varepsilon)$ was introduced earlier.

Consequently this implies the following inequality:

$$
\omega^T(Vx, \varepsilon) \leq T \overline{G} \omega^T_k(\varepsilon) + \overline{G} K_2 \varepsilon.
$$

(4.10)

Now, take any function $x \in X$ and $t, s \in \mathbb{R}_+$. Keeping in mind the representation of the operator $Q$ we derive the following estimate:

$$
||(Qx)(t) - (Qx)(s)||_{l} \leq ||a(t) - a(s)||_{l} + ||(Fx)(t)(Vx)(t) - (Fx)(s)(Vx)(s)||_{l}
$$

\[
\leq ||a(t) - a(s)||_{l} + ||(Fx)(t)(Vx)(t) - (Vx)(t)(Fx)(s) + (Vx)(t)(Fx)(s) - (Fx)(s)(Vx)(s)||_{l}
\]

\[
\leq ||a(t) - a(s)||_{l} + ||(Fx)(t)(Vx)(t) - (Vx)(t)(Fx)(s)||_{l} + ||(Vx)(t)(Fx)(s) - (Fx)(s)(Vx)(s)||_{l}
\]

\[
\leq ||a(t) - a(s)||_{l} + ||(Vx)(t)||_{l} ||(Fx)(t) - (Fx)(s)||_{l} + ||(Fx)(s)||_{l} ||(Vx)(t) - (Vx)(s)||_{l},
\]

where we denoted $a(t) = (a_n(t))$.

Further on we derive the last estimate utilizing inequalities (4.9), (4.10), (4.2) and (4.6) and assuming that $t, s \in [0, T], |t - s| \leq \varepsilon$:

$$
\omega^T(Qx, \varepsilon) \leq \omega^T(a, \varepsilon) + \overline{G} K_1 \omega^T(Fx, \varepsilon)
$$

\[
+ (l(r_0) r_0 + \overline{T}) \omega^T(Vx, \varepsilon) \leq \omega^T(a, \varepsilon) + \overline{G} K_1 \omega^T(Fx, \varepsilon) + (l(r_0) r_0 + \overline{T}) (T \overline{G} \omega^T_k(\varepsilon) + \overline{G} K_2 \varepsilon)
\]

\[
\leq \omega^T(a, \varepsilon) + \overline{G} K_1 \{ l(r_0) \omega^T(x, \varepsilon) + \omega^1(f, \varepsilon) \} + (l(r_0) r_0 + \overline{T}) (T \overline{G} \omega^T_k(\varepsilon) + \overline{G} K_2 \varepsilon).
\]

Hence we obtain the following inequality:

$$
\omega^T(QX, \varepsilon) \leq \omega^T(a, \varepsilon) + \overline{G} K_1 \{ l(r_0) \omega^T(X, \varepsilon) + \omega^1(f, \varepsilon) \}
$$
Letting with $\varepsilon \to 0$ in the above inequality and keeping in mind the properties of the functions $\varepsilon \to \omega^1(f, \varepsilon)$ and $\varepsilon \to \omega^K_1(\varepsilon)$ we obtain

$$\omega^K_0(QX) \leq \delta K_1 l(r_0) \omega^K_0(X).$$

Finally, taking limit as $T \to \infty$, we get

$$\omega^K_0(QX) \leq \delta K_1 l(r_0) \omega^K_0(X). \quad (4.11)$$

Notice that we obtained the estimate for the first component $\omega^K_0(X)$ of the measure of noncompactness $\chi_a(X)$ expressed by formula (3.7).

In what follows we obtain two consecutive estimations for the second and the third component of the measure of noncompactness $\chi_a(X)$. To this end, similarly as before, fix a set $X \subset B_{r_0}$ and a function $x \in X$. Take an arbitrary number $T > 0$ and fix $t \in [0, T]$. Then, for any $n \in \mathbb{N}$, utilizing estimates (4.2) and (4.6) (for series from $i = n$ to $\infty$), we get

$$\sum_{i=n}^{\infty} |(Q_i x)(t)| = \sum_{i=n}^{\infty} |a_i(t) + (F_i x)(t) (V_i x)(t)|$$

$$\leq \sum_{i=n}^{\infty} |a_i(t)| + \sum_{i=n}^{\infty} |(F_i x)(t) (V_i x)(t)|$$

$$\leq \sum_{i=n}^{\infty} |a_i(t)| + \sum_{i=n}^{\infty} |(F_i x)(t)| \sum_{i=n}^{\infty} |(V_i x)(t)|$$

$$\leq \sum_{i=n}^{\infty} |a_i(t)| + \left[ l(||x(t)||_n) \sum_{i=n}^{\infty} |x(t)| + \sum_{i=n}^{\infty} |T_i(t)| \right] \delta K_1$$

$$\leq \sum_{i=n}^{\infty} |a_i(t)| + \left[ l(r_0) \sum_{i=n}^{\infty} |x(t)| + \sum_{i=n}^{\infty} |T_i(t)| \right] \delta K_1.$$

Further, taking supremum over all $x = (x_i) \in X$, we derive the following evaluation

$$\sup_{x=(x_i) \in X} \left\{ \sum_{i=n}^{\infty} |(Q_i x)(t)| \right\} \leq \sum_{i=n}^{\infty} |a_i(t)| + \delta K_1 l(r_0) \sup_{x=(x_i) \in X} \left\{ \sum_{i=n}^{\infty} |x_i(t)| \right\} + \delta K_1 \sum_{i=n}^{\infty} |T_i(t)|.$$ 

Passing with $n \to \infty$ and utilizing assumptions (i), (vii) and Lemma 4.1, we get

$$\lim_{n \to \infty} \left\{ \sup_{x=(x_i) \in X} \left\{ \sum_{i=n}^{\infty} |(Q_i x)(t)| \right\} \right\} \leq \delta K_1 l(r_0) \lim_{n \to \infty} \left\{ \sup_{x=(x_i) \in X} \left\{ \sum_{i=n}^{\infty} |x_i(t)| \right\} \right\}.$$ 

Finally, taking supremum over $t \in [0, T]$ on both sides of the above inequality and letting with $T \to \infty$, in view of formula (3.11) we deduce the following estimate

$$\chi_\infty(QX) \leq \delta K_1 l(r_0) \chi_\infty(X). \quad (4.12)$$

Now, we are going to estimate the third component $a_\infty(X)$ of the measure of noncompactness $\chi_a(X)$. Assume, as earlier, that $X \subset B_{r_0}, x \in X$ and $T > 0$. Moreover, take $t > T$. Then, keeping in mind inequalities (4.2) and (4.6), we have

$$\sum_{n=1}^{\infty} |(Q_n x)(t)| \leq \sum_{n=1}^{\infty} |a_n(t)| + l(r_0) \delta K_1 \sum_{n=1}^{\infty} |x_n(t)| + \delta K_1 \sum_{n=1}^{\infty} |T_n(t)|.$$ 

Taking supremum over $t \geq T$ and $x = (x_n) \in X$, we get

$$\sup_{x=(x_n) \in X} \left\{ \sup_{t \geq T} \left\{ \sum_{n=1}^{\infty} |(Q_n x)(t)| \right\} \right\} \leq \sup_{t \geq T} \sum_{n=1}^{\infty} |a_n(t)| + l(r_0) \delta K_1 \sup_{x=(x_n) \in X} \left\{ \sup_{t \geq T} \left\{ \sum_{n=1}^{\infty} |x_n(t)| \right\} \right\}.$$
The proof is complete.

Converges on the interval \( t \in I \) for \( n = 1, 2, \ldots \) Moreover, \( \beta \) and \( y \) are positive constants which will be specified later.

Let us consider the following infinite system of integral equations having the form

\[
x_n(t) = \beta \frac{(-1)^{n-1} \mu^{n-1}}{(n-1)!} t + \frac{y}{n^2} \sin(x_n(t) + x_n^2(t)) \int_0^t \frac{1}{(n^2 + s^2)(1 + (t + s)^2)} \arctan s \left[ \frac{x_n(s)}{1 + n^2 x_n^2(s)} \right] + \frac{x_{n+1}(s)}{1 + (n + 1)^2 x_{n+1}^2(s)} ds
\]

for \( t \in \mathbb{R} \) and for \( n = 1, 2, \ldots \) Moreover, \( \beta \) and \( y \) are positive constants which will be specified later.

Let us observe that infinite system (5.1) is a particular case of the infinite system of quadratic integral equations of Volterra-Hammerstein type (4.1), where we put:

\[
a_n(t) = \beta \frac{(-1)^{n-1} \mu^{n-1}}{(n-1)!},
\]

\[
f_n(t, x_n, x_{n+1}, \ldots) = \frac{y}{n^2} \sin(x_n + x_n^2),
\]

\[
k_n(t, s) = \frac{1}{(n^2 + s^2)(1 + (t + s)^2)}
\]

\[
g_n(t, x_1, x_2, \ldots) = \frac{\arctan t}{n^2} \left[ \frac{x_n}{1 + n^2 x_n^2} + \frac{x_{n+1}}{1 + (n + 1)^2 x_{n+1}^2} \right]
\]

for \( n = 1, 2, \ldots \) and \( t, s \in \mathbb{R} \).

In what follows we show that the components of infinite system (5.1) defined by (5.2) satisfy assumptions of Theorem 4.2.

Indeed, for arbitrarily fixed \( n \in \mathbb{N} \) and \( t \in \mathbb{R} \) we have that the series

\[
\sum_{n=1}^{\infty} a_n(t) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \mu^{n-1}}{(n-1)!}
\]

converges on the interval \( \mathbb{R} \) to the function \( a(t) = \beta e^{-t} \). Obviously

\[
\lim_{t \to \infty} \sum_{n=1}^{\infty} a_n(t) = \lim_{t \to \infty} \beta e^{-t} = 0.
\]
This shows that there is satisfied assumption (i). Apart from this we have that \( A = \sup \{ \sum_{n=1}^{\infty} |a_n(t)| : t \in \mathbb{R}^\ast \} = \beta \).

Further, let us observe that the functions \( k_n(t, s) \) are continuous on the set \( \mathbb{R}_+^2 \) for \( n = 1, 2, \ldots \). Next, fix \( n \in \mathbb{N} \) and \( s \in \mathbb{R}_+ \). Then, for arbitrary \( t_1, t_2 \in \mathbb{R}_+ \) we obtain

\[
|k_n(t_2, s) - k_n(t_1, s)| = \frac{1}{(n^2 + s^2)} \left| \frac{1}{1 + (t_2 + s)^2} - \frac{1}{1 + (t_1 + s)^2} \right|
\]

\[
\leq \frac{1}{n^2} \left| \frac{1}{1 + (t_2 + s)^2} - \frac{1}{1 + (t_1 + s)^2} \right|
\]

\[
\leq \frac{1}{n^2} \left| \frac{1}{1 + (t_2 + s)^2} + \frac{t_2 + s}{1 + (t_2 + s)^2} \right|
\]

\[
\leq \frac{1}{n^2} |t_2 - t_1| \left[ \frac{1}{1 + (t_1 + s)^2} + \frac{1}{1 + (t_2 + s)^2} \right]
\]

Hence we infer that the functions \( k_n(t, s) \) are equicontinuous on the set \( \mathbb{R}_+ \) uniformly with respect to the variable \( s \in \mathbb{R}_+ \). Thus, these functions satisfy assumption (ii).

In what follows, for a fixed \( t \in \mathbb{R}_+ \) we get

\[
\sum_{n=1}^{\infty} \int_0^t |k_n(t, s)| ds = \sum_{n=1}^{\infty} \int_0^t \frac{1}{n^2 + s^2} \cdot \frac{1}{1 + (t + s)^2} ds
\]

\[
\leq \sum_{n=1}^{\infty} \frac{1}{n^2} \int_0^t \frac{1}{1 + (t + s)^2} ds = \sum_{n=1}^{\infty} \frac{1}{n^2} \int_t^{2t} \frac{du}{1 + u^2}
\]

\[
\leq \sum_{n=1}^{\infty} \frac{1}{n^2} \arctan 2t \leq \frac{\pi^2}{2} \cdot \frac{\pi^2}{6} \cdot \frac{\pi^2}{12}
\]

From the above estimate we deduce that the functions \( k_n(t, s) \) satisfy assumption (iii) with the constant \( K_1 = \pi^3/12 \).

Now, taking arbitrary \( t, s \in \mathbb{R}_+ \) and \( n \in \mathbb{N} \), we have

\[
|k_n(t, s)| = \frac{1}{n^2 + s^2} \cdot \frac{1}{1 + (t + s)^2} \leq \frac{1}{n^2} \leq 1.
\]

This means that assumption (iv) is satisfied with the constant \( K_2 = 1 \).

Next, let us pay our attention to the functions \( f_n = f_n(t, x_n, x_{n+1}, \ldots) \) defined by (5.2). Taking arbitrary \( n \in \mathbb{N} \) and an element \( x = (x_1, x_2, \ldots) \in l_1 \), we can easily infer that the first part of assumption (v) is satisfied which is a simple consequence of the fact that the series

\[
\sum_{n=1}^{\infty} f_n(t, x_n, x_{n+1}, \ldots)
\]

is uniformly convergent on the set \( \mathbb{R}_+ \times \mathbb{R}_+^\infty \). In fact, this conclusion follows immediately from the standard Weierstrass test. Obviously, the second part of assumption (v) is also trivially satisfied.

Now, fix arbitrarily a number \( r > 0 \) and take two elements \( x = (x_n), y = (y_n) \) belonging to the space \( l_1 \) such that \( |x| \|_1 = \sum_{n=1}^{\infty} |x_n| \leq r, |y| \|_1 = \sum_{n=1}^{\infty} |y_n| \leq r \). Then, for a fixed natural number \( n \), we derive the following estimate:

\[
|f_n(t, x_n, x_{n+1}, \ldots) - f_n(t, y_n, y_{n+1}, \ldots)|
\]

\[
\leq \frac{1}{n^2} \sin(x_n + x_n^2) - \sin(y_n + y_n^2)
\]

\[
\leq \frac{1}{n^2}(x_n - y_n + (x_n^2 - y_n^2))
\]

\[
\leq \frac{1}{n^2}|x_n - y_n| + \frac{1}{n^2}|x_n - y_n|^2 |x_n| + |y_n|
\]

\[
\leq \frac{1}{n^2}|x_n - y_n| + \frac{1}{n^2}|x_n - y_n| \left( \sum_{n=1}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| \right)
\]
Thus, the operator $l$ satisfies assumption (vi) with the function $l(r)$ defined by the equality

$$l(r) = 2y \max\{1, 2r\}.$$  

Moreover, we have that $f_n(t) = f_n(t, 0, 0, \ldots) = 0$. Thus, there is satisfied assumption (vii) and $\mathcal{F} = 0$.

Further on, we are going to verify assumption (viii). To this end, let us first notice that for a fixed natural number $n$ the function $g_n = g_n(t, x_1, x_2, \ldots)$ defined by (5.2) on the set $\mathbb{R} \times \mathbb{R}^\infty$ takes real values ($n = 1, 2, \ldots$). Next, fix arbitrarily $x = (x_n) \in l_1$. Then, for $t \in \mathbb{R}_+$ we obtain

$$\sum_{n=1}^{\infty} |g_n(t, x_1, x_2, \ldots)| \leq \sum_{n=1}^{\infty} \frac{\arctan t}{n^2} \left( \frac{|x_n|}{1 + n^2 x_n^2} + \frac{|x_{n+1}|}{1 + (n + 1)^2 x_{n+1}^2} \right)$$

$$\leq \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} \left( \frac{1}{2} + \frac{1}{4} \right) = \frac{3\pi}{8} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^3}{16}.$$  

From the above estimate it follows that the operator $g$ defined in assumption (viii) transforms the space $\mathbb{R} \times l_1$ into $l_1$. Moreover, from (5.3) we conclude that the operator $g$ is bounded on $\mathbb{R} \times l_1$ and $||g(t)||_{l_1} \leq \pi^3/16$. Thus, the operator $g$ satisfies assumption (ix) and we can accept that $\mathcal{G} = \pi^3/16$, where $\mathcal{G}$ is defined in assumption (ix).

Further, taking a number $\varepsilon > 0$ and choosing arbitrarily $x = (x_n), y = (y_n) \in l_1$ such that $||x - y||_{l_1} \leq \varepsilon$, for $t \in \mathbb{R}_+$ we obtain

$$||(gy)(t) - (gx)(t)||_{l_1} = \sum_{n=1}^{\infty} |g_n(t, y_1, y_2, \ldots) - g_n(t, x_1, x_2, \ldots)|$$

$$= \sum_{n=1}^{\infty} \frac{\arctan t}{n^2} \left| \frac{y_n}{1 + n^2 y_n^2} + \frac{y_{n+1}}{1 + (n + 1)^2 y_{n+1}^2} - \frac{x_n}{1 + n^2 x_n^2} - \frac{x_{n+1}}{1 + (n + 1)^2 x_{n+1}^2} \right|$$

$$\leq \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} \left[ \frac{|y_n| + n^2 |y_n - x_n|}{1 + n^2 y_n^2} + \frac{|y_{n+1}| + (n + 1)^2 |y_{n+1} - x_{n+1}|}{1 + (n + 1)^2 y_{n+1}^2} \right]$$

$$\leq \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} \left[ \frac{|y_n|}{1 + n^2 y_n^2} + \frac{|y_{n+1}|}{1 + (n + 1)^2 y_{n+1}^2} \right]$$

$$\leq \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} \left[ \frac{1}{1 + n^2 y_n^2} + \frac{n |y_n|}{1 + n^2 y_n^2} + \frac{n |x_n|}{1 + n^2 x_n^2} \right]$$

$$\leq \frac{5\pi}{8} \sum_{n=1}^{\infty} \frac{1}{n^2} \left[ |y_n - x_n| \left( \frac{1}{1 + n^2 y_n^2} + \frac{1}{1 + (n + 1)^2 y_{n+1}^2} \right) \right]$$

$$\leq \frac{5\pi}{8} \sum_{n=1}^{\infty} \frac{1}{n^2} \left[ |y_n - x_n| \left( \frac{1}{1 + n^2 y_n^2} + \frac{1}{1 + (n + 1)^2 y_{n+1}^2} \right) \right]$$

Thus we have proved that there is satisfied assumption (viii).

Finally, gathering all the above obtained constants $A, \mathcal{F}, \mathcal{G}, K_1$ and taking into account the function $l(r) = 2y \max\{1, 2r\}$ indicated in the above calculations, we conclude that the inequality from assumption (x) has the form

$$\beta + y \frac{\sqrt{6}}{24} \max\{1, 2r\} \leq r.$$  

(5.4)
Further, let us assume that we are looking for a solution $r$ of inequality (5.4) such that $r \leq \frac{1}{2}$. In such a case inequality (5.4) has the form

$$\beta + \gamma \frac{\pi^6}{96} r \leq r.$$  \hfill (5.5)

Assuming that $\gamma < \frac{96}{\pi^6}$ we infer that, for example, the number $r_0 = \frac{96\beta}{(96 - \gamma\pi^6)}$ is a solution of inequality (5.4) provided $\beta < (96 - \gamma\pi^6)/192$. It easily seen that in this case we have that $GK_1 r(r_0) < 1$ which proves that assumption (x) is thoroughly satisfied.

Now, applying Theorem 4.2 we deduce that infinite system of integral equations (5.1) has at least one solution $x = x(t) = (x_n(t))$ in the space $BC_{1} = BC([\mathbb{R}^+, l_1])$.

References

[1] S. Chandrasekhar, Radiative Transfer, Dover, New York, 1960.
[2] C. Corduneanu, Integral Equations and Applications, Cambridge University Press, Cambridge, 1991.
[3] K. Deimling, Nonlinear Functional Analysis, Springer, Berlin, 1985.
[4] N. Dunford and J. T. Schwartz, Linear Operators, I: General Theory, Wiley, New York, 1963.
[5] W. Mydlarczyk, A system of Volterra integral equations with blowing up solutions, Colloq. Math. 146 (2017), 99–110.
[6] W. Mydlarczyk, Coupled Volterra integral equations with blowing up solutions, J. Integral Equations Appl. 30 (2018), no. 1, 147–166.
[7] W. Okrasiński and Ł. Płociniczak, Solution estimates for a system of nonlinear integral equations arising in optometry, J. Integral Equations Appl. 30 (2018), no. 1, 167–179.
[8] W. Pogorzelski, Integral Equations and Their Applications, I, Int. Ser. Monogr. Pure Appl. Math. 88, Pergamon, Oxford, 1966.
[9] P. P. Zabrejko, A. I. Koshelev, M. A. Krasnosel’skii, S. G. Mikhlin, L. S. Rakovschik and J. Stetsenko, Integral Equations, Nordhoff, Leyden, 1975.
[10] J. Banaś and A. Chlebowicz, On solutions of an infinite system of nonlinear integral equations on the real half-axis, Banach J. Math. Anal. (to appear).
[11] J. Banaś and M. Mursaleen, Sequence Spaces and Measures of Noncompactness with Applications to Differential and Integral Equations, Springer, New Delhi, 2014.
[12] J. Banaś, M. Mursaleen and S. M. H. Rizvi, Existence of solutions to a boundary-value problem for an infinite system of differential equations, Electron. J. Differential Equations 2017 (2017), No. 262, 1–12.
[13] K. Deimling, Ordinary Differential Equations in Banach Spaces, Lect. Notes Math. 596, Springer, Berlin, 1977.
[14] M. Mursaleen and S. M. H. Rizvi, Solvability of infinite systems of second order differential equations in $c_0$ and $l_1$ by Meir–Keeler condensing operator, Proc. Amer. Math. Soc. 144 (2016), no. 10, 4279–4289.
[15] J. Banaś and K. Goebel, Measures of Noncompactness in Banach Spaces, Lect. Notes Pure Appl. Math. 60, Marcel Dekker, New York, 1980.
[16] K. Kuratowski, Sur les espaces complets, Fund. Math. 15 (1930), 301–309.
[17] I.T. Gohberg, L.S. Goldenštěin and A.S. Markus, Investigations of some properties of bounded linear operators with their $q$-norms, Učen Kishiniersk. Univ. 29 (1957), 29–36.
[18] L.S. Goldenštěin and A.S. Markus, "On the measure of non-compactness of bounded sets and linear operators" in Studies in Algebra and Mathematical Analysis, Izdat. "Karta Moldovenjaske", Kishinev, 1965, 45–54.
[19] G. Darbo, punti uniti in trasformazioni a condominio non compatto, Rend. Semin. Mat. Univ. Padova 24 (1955), 84–92.
[20] W. Rudin, Real and Complex Analysis, McGraw-Hill, New York, 1987.