A mass-independent conformal quantum cloak

De-Hone Lin
Department of Physics, National Sun Yat-sen University, Kaohsiung, Taiwan
E-mail: dhlin@mail.nsysu.edu.tw

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Abstract
An obstacle to constructing a quantum cloak with the transformation design method for steering a matter wave is the device involving the anisotropic modulation of the particle mass. It is the purpose of this paper to show that a narrow-acceptance-angle quantum cloak generated by a kind of conformal mapping could be mass-independent. This greatly simplifies the construction of the device since it can now be finished only by applying a potential field to a region of the shell. As demonstrations of the conclusion, an elliptical and a parabolic conformal quantum cloak are designed from the elliptical and parabolic coordinate systems. Their performance is inspected by probability currents. It is shown that the conformal devices can serve as novel splitters for a 2D matter wave. Finally, we discuss the difficulty of constructing a steering device from the hyperbolic coordinate system.

1. Introduction

Recently, there has been interest in steering a matter wave with cloak devices. Several schemes developed from the inverse scattering theory [1], the cancellation of scattering phase shifts [2], and the transformation design method (TDM) [3, 4] have been proposed for the construction of an invisibility cloak. The first and second schemes approach the device with a finite number of ring-shaped potentials [1] and the zero phase shift conditions of the lowest two partial waves [5–7], while the TDM carries out the device by way of interpreting a transformation structure as the mass parameters of the quantum particle and an effective potential in a specific shell, which leads the matter wave to flow around the internal region of the cloak [8–11]. Among these three proposals, the steering with the cloak generated by the TDM is exact. However, it is not easy to reach due to the fact that the device is mass-dependent.

In recent years, conformal mapping has been applied to the design of novel steering devices in optics, electrostatics, DC fields, and heat diffusion (e.g. [3, 12–17]). Although it is still hard to construct an omnidirectional device from the mapping [18], the material parameters of a conformal device could be isotropic for a wave incoming from a specific direction. The primary purpose of this paper is to show that a mass-independent device for steering a matter wave symmetrically along a specific angle can be designed from a kind of conformal mapping which makes the construction of the device depend only on the potential, without resorting to the material properties for the modulation of mass. As demonstrations, we have designed an elliptical and a parabolic conformal device for splitting a matter wave.

The paper is arranged as follows: in section 2, we prove the equivalence between the mass-independent conditions and conformal mapping. In section 3, an elliptical conformal device is constructed. A probability current is used to inspect the steering effect of the device. Section 4 is used to design a mass-independent conformal device from the parabolic coordinate system. Section 5 is devoted to discussing the possibility of constructing a transformation device from the hyperbolic coordinate system. Finally, the conclusion and several notes are provided in section 6.
2. Mass-independent conditions and conformal mapping

The evolution of a matter wave $\Psi(\bar{q}_j)$ in a quantum cloak satisfies the Schrödinger equation [10, 11]

$$-\frac{\hbar^2}{2M} \sum_{j,k} \frac{\partial^2}{\partial \bar{q}_j \partial \bar{q}_k} \left[ \sqrt{G^{jk}} \frac{1}{M_{jk}} \frac{\partial}{\partial \bar{q}_k} \Psi \right] = [E - U(\bar{q})] \Psi,$$

where the coordinate variables $\bar{q}_j$ depict the distribution of the matter wave in the cloak, and the mass parameters and effective potential are given by

$$M_{jk} = \sqrt{\frac{G}{\sqrt{\bar{g}}} g_{jk}},$$

$$U(\bar{q}) = \left( 1 - \frac{\sqrt{\bar{g}}}{\sqrt{G}} \right) E,$$

in which $M$ is the particle’s mass, and the determinants $G = |G_{jk}|$, and $\bar{g} = |\bar{g}_{jk}|$. The indices $j$, $k$ in the factor $\bar{g}_{jk} G^{jk}$ in (2) denote only the $(j, k)$ component, and do not sum over the dimensions. There are two kinds of pseudo metric in this equation. The first comes from the transformation $x_k = x_k(\bar{q}_j), k = 1, 2, 3$, or $x_k = x, y$, and $z$. Its components are defined by

$$G_{jk}(\bar{q}) = \frac{\partial x}{\partial \bar{q}_j} \frac{\partial x}{\partial \bar{q}_k} + \frac{\partial y}{\partial \bar{q}_j} \frac{\partial y}{\partial \bar{q}_k} + \frac{\partial z}{\partial \bar{q}_j} \frac{\partial z}{\partial \bar{q}_k},$$

The other with variable $\bar{q}_j$ is defined by $\bar{g}_{jk} = g_{jk}(\bar{q})$, with $g_{jk}(q)$ calculated by the formula

$$g_{jk}(q) = \frac{\partial x}{\partial q_j} \frac{\partial x}{\partial q_k} + \frac{\partial y}{\partial q_j} \frac{\partial y}{\partial q_k} + \frac{\partial z}{\partial q_j} \frac{\partial z}{\partial q_k}.$$ (4)

Equation (1) reveals an alternative approach to steering the matter wave with the modulation of the mass parameters $M_{jk}$ and supply of the potential $U(\bar{q})$. We would like to search for transformation that enables the cloak to be mass-independent. For this, it is noticed that equation (1) has the mathematically equivalent representation

$$-\frac{\hbar^2}{2M} \sum_{j,k} \frac{\partial}{\partial \bar{q}_j} \left[ \sqrt{\bar{g}} \bar{g}^{jk} \frac{\partial}{\partial \bar{q}_k} \Psi \right] = E \Psi,$$

which shows that the invariant squared distance between two neighboring points of the transformed space is

$$ds^2 = \sum_{j,k} g_{jk} d\bar{q}_j d\bar{q}_k.$$ (5)

Since the coordinates $\bar{q}_j = q_j(\bar{q}_k)$ and $g_{jk} = \bar{q}_j' \bar{q}_k' G_{jk}$ from (3), the invariant distance

$$ds^2 = \sum_{j,k} \bar{g}_{jk} \bar{q}_j' d\bar{q}_j = \sum_{j,k} \bar{g}_{jk} \bar{q}_j' d\bar{q}_j = \sum_{j,k} (\bar{g}_{jk} G^{jk}) g_{jk} d\bar{q}_j d\bar{q}_k,$$ (6)

where $G^{jk}$ is the inverse of $G_{jk}$. The mass-independent condition of the cloak is given by

$$\sqrt{\frac{G}{\sqrt{\bar{g}}}} \bar{g}_{jk} G^{jk} = 1,$$ (7)

for all $(j, k)$ components from (2). By substitution of (8) into (7), we obtain the invariant distance of the transformed space for a mass-independent cloak

$$ds^2 = \left( \frac{\sqrt{\bar{g}}}{\sqrt{G}} \right) \sum_{j,k} g_{jk} d\bar{q}_j d\bar{q}_k.$$ (9)

It is a kind of conformal mapping. Accordingly, we have derived the equivalence between the mass-independent condition and conformal mapping.

3. An elliptical conformal cloak

As an application of the conclusion in the last section, one designs a conformal cloak with an elliptical shape and investigates its performance with a probability current. The distribution of the wave function in the cloak is described by the variables $(\bar{q}_1, \bar{q}_2) = (u(\bar{u}), \bar{v}(\bar{v}))$, i.e., $\Psi = \Psi(\bar{u}, \bar{v})$, where $(u, v)$ are the elliptical coordinates with ranges $0 \leq u < \infty, 0 \leq v < 2\pi$. The connections of $(u, v)$ with the Cartesian coordinates are by way of
the transformation $x = a \cosh u \cos v$, and $y = a \sinh u \sin v$, where $a$ is the common semifocal distance of the confocal ellipses. The components of the metric tensors for the transformation $(\bar{u}, \bar{v})$ have the representations

$$(G_{\bar{u}\bar{u}}) = \text{diag}\left(\frac{a^2(\sinh^2 u + \sin^2 v)}{(\bar{u})^2}, \frac{a^2(\sinh^2 u + \sin^2 v)}{(\bar{v})^2}\right)$$

(10)

and

$$(\bar{g}_{\bar{u}\bar{u}}) = \text{diag}(a^2(\sinh^2 \bar{u} + \sin^2 \bar{v}), a^2(\sinh^2 \bar{u} + \sin^2 \bar{v})),$$

(11)

from (3) and (4), where $\bar{u}' = d\bar{u}/du$, and $\bar{v}' = d\bar{v}/dv$. So we have the factors $\sqrt{G} = a^2(\sinh^2 u + \sin^2 v)/(\bar{u}\bar{v}')$ and $\sqrt{\bar{g}} = a^2(\sinh^2 \bar{u} + \sin^2 \bar{v})$, and the mass parameters and potential

$$(M_{\bar{u}\bar{v}}) = \text{diag}\left(\frac{\bar{u}'}{\bar{v}'}, \frac{\bar{v}'}{\bar{u}'}\right)M$$

(12)

and

$$U = \left[1 - \frac{\bar{u}'\bar{v}'(\sinh^2 \bar{u} + \sin^2 \bar{v})}{\sinh^2 u + \sin^2 v}\right]E.$$

(13)

Equation (1) shows that the wave function in the cloak $\Psi(\bar{q})$ satisfies

$$\left[\frac{\partial^2}{\partial \bar{u}^2} + \frac{\partial^2}{\partial \bar{v}^2} + a^2k^2(\sinh^2 \bar{u} + \sin^2 \bar{v})\right]\Psi(\bar{u}, \bar{v}) = 0,$$

(14)

where $k^2 = 2ME/\hbar^2$. It is easy to see that the solution to this equation is

$$\Psi(\bar{u}, \bar{v}) = \exp\{ik \cdot \bar{x}\} = \exp\{ik(\bar{x}\cos \phi + \bar{y}\sin \phi)\},$$

(15)

where the wave vector $k = k \cos \phi \bar{e}_x + k \sin \phi \bar{e}_y$, and $\bar{x} = \bar{x}\bar{e}_x + \bar{y}\bar{e}_y$ with $\bar{x} = a \cosh \bar{u} \cos \bar{v}$, and $\bar{y} = a \sinh \bar{u} \sin \bar{v}$. Equation (15) no longer expresses a plane wave since the points $x$ in the wave fronts, defined by $k \cdot x = $ constant, have been transformed and moved out of the original positions by $\bar{x}$. The mass-independent condition for the transformation $(\bar{u}, \bar{v})$ is determined by

$$\bar{g}_{\bar{u}\bar{u}}G_{11} = \frac{G}{\bar{g}_{\bar{u}\bar{u}}} \text{ and } \bar{g}_{\bar{u}\bar{u}}G_{22} = \frac{G}{\bar{g}_{\bar{u}\bar{u}}},$$

(16)

which implies $\bar{u}' = \bar{v}'$ and the invariant distance of the transformed space becomes

$$ds^2 = \text{conformal factor} \times \left[\frac{\sinh^2 \bar{u} + \sin^2 \bar{v}}{(\sinh^2 u + \sin^2 v)}\right]$$

(17)

with

$$\text{conformal factor} = (\bar{u}')^2\left[\frac{\sinh^2 \bar{u} + \sin^2 \bar{v}}{(\sinh^2 u + \sin^2 v)}\right].$$

(18)

Accordingly, the mass parameters to get to the transformation-wave function (15) are just the free particle ones

$$(M_{\bar{u}\bar{v}}) = \text{diag}(1, 1)M.$$  

(19)

This result greatly reduces the difficulty of constructing the cloak since it can now be reached simply through applying the effective potential of (13) to the shell, independent of the control of the particle’s mass. In the following, the conformal mapping would be assigned by

$$\bar{u}(u) = c_1 u + c_{1}\text{, and } \bar{v}(v) = c_1 v + c_2,$$

(20)

where $c_1, c_2$ are constants. To understand the performance of the elliptical cloak, let us inspect the flow of a probability current in the cloaking shell. The current can manifestly exhibit how the matter wave is steered, and propagates passing through the shell region. The $l$-component of the current is calculated by the formula

$$J_l = \frac{\hbar}{2i\mu} \sum_k \left(\psi^{*} \frac{1}{M_k} \partial_k \psi - \psi \frac{1}{M_k} \partial_k \psi^{*}\right)$$

(21)

according to (1), where $\partial_k$ denotes

$$\partial_1 = \frac{1}{H_u} \frac{\partial}{\partial \bar{u}}, \quad \partial_2 = \frac{1}{H_v} \frac{\partial}{\partial \bar{v}}.$$

(22)

with $H_a = \sqrt{G_{uu}}$ being the scale factors and given by

$$H_u = \frac{a \sqrt{\sinh^2 u + \sin^2 v}}{\bar{u}'}, \quad H_v = \frac{a \sqrt{\sinh^2 u + \sin^2 v}}{v'}. $$

(23)
Apply the mass parameters in (12) to (21). The current in the cloak is then found to have the representation

\[ J = \frac{\hbar k}{M} (J_x \hat{x} + J_y \hat{y}), \]

with the Cartesian components

\[ J_x = \left( \frac{\nu' \sinh \nu \cos \nu}{\sinh^2 \nu + \sin^2 \nu} \right) (\sinh \bar{u} \cos \bar{v} \cos \phi + \cosh \bar{u} \sin \bar{v} \sin \phi) \]
\[ + \left( \frac{\bar{u}' \cosh \nu \sin \nu}{\sinh^2 \nu + \sin^2 \nu} \right) (\cosh \bar{u} \sin \bar{v} \cos \phi - \sinh \bar{u} \cos \bar{v} \sin \phi), \]

and

\[ J_y = \left( \frac{\nu' \cosh \nu \sin \nu}{\sinh^2 \nu + \sin^2 \nu} \right) (\sinh \bar{u} \cos \bar{v} \cos \phi + \cosh \bar{u} \sin \bar{v} \sin \phi) \]
\[ - \left( \frac{\bar{u}' \sinh \nu \cos \nu}{\sinh^2 \nu + \sin^2 \nu} \right) (\cosh \bar{u} \sin \bar{v} \cos \phi - \sinh \bar{u} \cos \bar{v} \sin \phi). \]

Let us first consider the single variable transformation \( \bar{u} = \alpha_1 u + \alpha_2 \) with \( \alpha_1 = u_2/\left(\sqrt{u_2^2 - u_1^2}\right) \) and \( \alpha_2 = -u_2 u_1/\left(\sqrt{u_2^2 - u_1^2}\right) \) and \( \bar{v} = v \). Figure 1 shows the probability current in the cloak, where the inner shell is chosen to be \( u_1 = 0.5\pi \), and the outer shell is \( u_2 = \pi \). The focus of the cloaking device is chosen as \( a = 1.8 \). The pattern exhibits that the current is perfectly guided to flow around an elliptical region and then return to the incoming direction. The elliptical cloaked region is invisible since the outgoing current is the same as the incoming one. However, to construct the invisible device for the matter wave one needs to control the particle with the anisotropic mass parameters

\[ (M_\beta) = \text{diag}\left(\alpha_1, \frac{1}{\alpha_1}\right) M = \text{diag}\left(2, \frac{1}{2}\right) \]

which is only possible for carriers in some materials. Figure 2 shows the cloak generated by the conformal mapping \( \bar{u} = \alpha_1 u + \alpha_2 \) and \( \bar{v} = \alpha_1 v + \alpha_2 \). The constants \( \alpha_1 \) are assigned by \( \alpha_1 = u_2/(\sqrt{u_2^2 - u_1^2}) \), and \( \alpha_2 = -u_2 u_1/\left(\sqrt{u_2^2 - u_1^2}\right) \) with the values \( u_1 = \pi/4, u_2 = \pi \), and \( a = 1.8 \). It exhibits that the streamlines are not only guided to flow around the inner elliptical shell but are also pushed to the outside at \( |v| = \pi/4 \), leaving a region without current bounded by \( 0 \leq |v| \leq \pi/4 \) and \( 0 \leq u \leq \pi \). The steering is also effective for the symmetric domain \(-\pi/4 \leq v \leq 0\). This is due to the fact that the parameter \( \alpha_1 \) remains invariant when we make the reflection \( u_1 \rightarrow -u_1 \) and \( u_2 \rightarrow -u_2 \) for the variable \( \bar{v} \). The cloak is not invisible since the outgoing current no longer aims at the incoming direction. However, it is a novel device to decompose a matter wave into two parts and make a turn of the required angle with respect to the symmetric axis. Figure 3 shows the pattern for \( u_1 = \pi/2, u_2 = \pi \), and \( a = 1.8 \), where two splitting groups of the matter wave make a right-angle turn. The device can also steer two splitting parts to make a turn over \( \pi/2 \) as long as we set the parameter \( u_2 > \pi/2 \). The primary merit of the conformal cloak is that the mass parameters can be the free particle ones \((M_\beta) = \text{diag}\).
Figure 2. Splitting of the probability current with an elliptical conformal cloak. The cloak serves as a splitter which is capable of splitting the current into two symmetric parts. The mass parameters to reach the effect are isotropic \( M_0 = \text{diag}(1, 1, 1) \). It can thus achieve the effect simply by applying an effective potential to the shell of the cloak. The mapping was chosen to be \( u = u_2 - u_1 \) and \( v = u_2 - u_1 \) with parameters \( u_1 = \frac{\pi}{4} \) and \( u_2 = \pi \).

Figure 3. Alternative splitting effect of the conformal cloak. The transformation parameters were chosen to be \( u_1 = \frac{\pi}{2} \) and \( u_2 = \pi \). Effectively, the cloak is splitting the matter wave into two parts and guiding them to make opposite right-angle turns. The cloak can also steer two flows to make a turn bigger than \( \frac{\pi}{2} \) as long as the parameter \( u_1 > \frac{\pi}{2} \) is set.

Figure 4 shows the pattern of the potential field needed to reach the effect in figure 2. It is regular everywhere in the cloak. It is seen that in three cone regions, the potential is gradually altered to a higher level and achieves the particle energy at the tips for blocking the matter wave. Figure 5 shows the potential distribution for the steering of the matter wave in figure 3. One way to generate the potential field is through supplying an external force for carriers in the cloak by \( F = -\nabla U \). For a charged particle, this is feasible with an electric force \( E = QE = -\nabla U \), where \( Q \) is the charge carried by the particle, and \( E \) is an external electric field.

4. A parabolic conformal cloak

Intuitively, it is hard to design a conformal cloak from the parabolic coordinate system because the coordinate curves with constant values are unbounded. The parabolic coordinates \((u, v)\) are associated with the Cartesian coordinates \((x, y)\) by

\[
\begin{align*}
    x &= \frac{u}{\sqrt{1-u^2}} - \frac{v}{\sqrt{1-v^2}} \\
    y &= \frac{v}{\sqrt{1-v^2}} - \frac{u}{\sqrt{1-u^2}}
\end{align*}
\]
in which the ranges of the parabolic coordinates are $0 \leq u \leq \infty$ and $-\infty \leq v \leq \infty$. Figure 6 shows the coordinates plotted on the Euclidean plane where all points on a red line share the same coordinate value $u$, while a blue line in the upper half plane ($y > 0$) shares the same coordinate value $v$, and a blue line in the lower half plane ($y < 0$) has the same value $-v$. According to (3) and (4), the expression of the metric $G_{jk}$ is given by

$$(G_{jk}) = \text{diag} \left( \frac{u^2 + v^2}{(u')^2}, \frac{u^2 + v^2}{(v')^2} \right)$$

and

$$(\tilde{g}_{jk}) = \text{diag}(\tilde{u}^2 + \tilde{v}^2, \tilde{u}^2 + \tilde{v}^2),$$

where $\tilde{u} = \tilde{u}(u)$ and $\tilde{v} = \tilde{v}(v)$ are as usual the variables for a transformation-wave. The mass parameters and the effective potential for the creation of the wave are given by
\[(M_{\mu}) = \text{diag}\left(\frac{\ddot{u}}{\ddot{v}}, \frac{\ddot{v}}{\ddot{u}}\right)M\] (31)

and

\[U = \left[1 - \frac{(\ddot{u}^2 + \ddot{v}^2)}{(u'^2 + v'^2)}\ddot{u}\ddot{v}'\right]E.\] (32)

From equation (1), the transformation-wave function is shown to satisfy the form invariant equation

\[\left[\frac{1}{\ddot{u}^2 + \ddot{v}^2}\left(\frac{\partial^2}{\partial u'^2} + \frac{\partial^2}{\partial v'^2}\right) + k^2\right]\Psi(\ddot{u}, \ddot{v}) = 0,
\]

where \(k^2 = 2ME/h^2\). The solution to this equation is

\[\Psi(\ddot{u}, \ddot{v}) = \exp\{i\vec{k} \cdot \vec{x}\} = \exp\{i\vec{k}(\ddot{x}\cos \phi + \ddot{y}\sin \phi)\},\] (34)

with \(\ddot{x} = (\ddot{u}^2 - \ddot{v}^2)/2\) and \(\ddot{y} = \ddot{u}\ddot{v}\). The mass-independent conditions for the wave are given by

\[\tilde{g}_{11}G^{11} = \frac{\ddot{x}}{\sqrt{G}}, \text{ and } \tilde{g}_{22}G^{22} = \frac{\ddot{y}}{\sqrt{G}}.\] (35)

Obviously, it implies \(\ddot{u}' = v'\) and the invariant distance of the transformed space is described by

\[ds^2 = \text{conformal factor} \times \{(u'^2 + v'^2/2)(du'^2 + dv'^2)\} \]

with

\[\text{conformal factor} = (\frac{\ddot{u}^2 + \ddot{v}^2}{u'^2 + v'^2})\] (37)

It is a conformal mapping. The scaling factors for the transformation (\(\ddot{u}, \ddot{v}\)) are given by

\[H_u = \frac{\sqrt{u'^2 + v'^2}}{\ddot{u}'}, \quad H_v = \frac{\sqrt{u'^2 + v'^2}}{\ddot{v}'}.\] (38)

**Figure 6.** Parabolic coordinates plotted on the Euclidean plane. All points on the same red curve share the same coordinate value \(u\). All points on the same blue curve and \(y > 0\) (\(y < 0\)) share the same coordinate value \(v\) (\(-v\)).
The unit vectors along the constant \( \vec{v} \) and \( \vec{u} \) are associated with \( \hat{e}_x \) and \( \hat{e}_y \) in the Cartesian coordinates by

\[
\hat{e}_u = \frac{\partial x}{H_u \partial \vec{u}} = \frac{1}{\sqrt{u^2 + v^2}} (u \hat{e}_x + v \hat{e}_y),
\]

and

\[
\hat{e}_v = \frac{\partial x}{H_v \partial \vec{v}} = \frac{1}{\sqrt{u^2 + v^2}} (-v \hat{e}_x + u \hat{e}_y).
\]

With the aid of the representation, the probability current in the parabolic cloak has the representation

\[
\mathbf{J} = \hbar k (J_x \hat{e}_x + J_y \hat{e}_y) / M
\]

in which the component

\[
J_x = \frac{1}{u^2 + v^2} \left[ (u \tilde{v} \tilde{v}' + \tilde{u} \tilde{u}') \cos \phi + (u \tilde{v} v' - \tilde{u} \tilde{v}') \sin \phi \right]
\]

and

\[
J_y = \frac{1}{u^2 + v^2} \left[ (\tilde{v} v' + u \tilde{v}') \cos \phi + (\tilde{v} \tilde{v}' + u \tilde{u}') \sin \phi \right].
\]

The left pattern of figure 7 shows that the probability current is split and guided by a single variable \( \vec{v} \)-type cloak generated by the transformation \( \vec{u} = u \) and \( \vec{v} = v/(v_2 - v_1) \) with \( v_2 = 4 \) and \( v_1 = 2 \) chosen. It is also a splitter. However, the mass parameters for the manipulation are anisotropic (\( M_{\vec{u}} = \text{diag}(1/2, 2)M \)). The right pattern shows the current flow manipulated by a single variable \( \vec{u} \)-type cloak generated by the transformation \( \vec{u} = u/(u_2 - u_1) \) with \( u_2 = 4 \) and \( u_1 = 2 \) and \( \tilde{v} = v \). It is also a splitter and the effect is generated by the effective mass parameters (\( M_{\vec{v}} = \text{diag}(2, 1/2)M \)). In contrast to the elliptical quantum cloak discussed in the last section, we are not allowed to design a mass-independent conformal cloak along the directions of the \( \pm x \)-axis since the cloaked regions are unbounded and there is no place to allow for an incoming particle. Fortunately, a mass-independent cloak with the function of a splitter can be constructed from the directions of \( \pm y \). Any oblique incoming wave would lead to an imperfect steering.

### 5. Transformation design in the hyperbolic coordinate system

Some colleagues are interested in the possibility of designing a conformal cloak from the hyperbolic coordinate system. As we shall see, there indeed exists a mass-independent transformation–wave function generated from conformal mapping in the system. However, it cannot be applied to the design of a conformal splitter as the previous sections have shown. In mathematics, hyperbolic coordinates locate points in the Euclidean plane by
the mapping \cite{19}

\[ u = \frac{x^2 - y^2}{2}, \quad \text{and} \quad v = xy, \]

with ranges \(-\infty \leq u \leq \infty\) and \(-\infty \leq v \leq \infty\). Figure 10 shows the coordinate curves of constants \(u\) and \(v\) in this system. Unlike the elliptical and parabolic coordinates in which two families of curves are enough to cover the whole Euclidean plane, it needs eight families of curves to cover the plane with the hyperbolic system. Moreover, the whole plane is sliced into eight domains with borders at \(u = 0\) and \(v = 0\). This makes the application of the transformation method to steering a matter wave difficult. Figure 11 shows the sliced domains, and the values of \(u, v\) in them. Two kinds of defined metric can be straightforwardly calculated and given by

\[
(g_{ij}) = \text{diag}\left(\frac{1}{2\sqrt{u^2 + v^2}}, \quad \frac{1}{2\sqrt{u^2 + v^2}}\right).
\]
and

\[
(G_\delta) = \text{diag}\left( \frac{1}{2(\bar{u})^2 \sqrt{u^2 + v^2}}, \frac{1}{2(\bar{\nu})^2 \sqrt{\bar{u}^2 + \bar{v}^2}} \right).
\]  

(45)

where \( \bar{u} = \bar{u}(u) \) and \( \bar{\nu} = \bar{\nu}(\nu) \) for possibly searching for a form invariant solution. It follows that the factor

\[
\sqrt{\frac{\bar{G}}{G}} = \frac{\bar{u} \nu \sqrt{u^2 + v^2}}{\sqrt{\bar{u}^2 + \bar{v}^2}}.
\]

(46)
The mass parameters, and the effective potential in the hyperbolic system are given by

\[
(M_{jk}) = \text{diag}\left(\frac{a'}{\nu'}, \frac{\nu'}{a'}\right)M,
\]

and

\[
U = \left[1 - \frac{\nu' a'}{\sqrt{u'^2 + v'^2}}\right]E.
\]

We note that the dependence of mass parameters on the coordinate variables are totally the same in the presented three kinds of coordinates although the variables \(u, v\) themselves have domain and dimension in each system.

With (47) and (48), a little tedious but straightforward calculation of equation (1) shows that there also exists the form invariant transformation-wave in the hyperbolic system, which satisfies

\[
\left[2\sqrt{\bar{u}^2 + \bar{\nu}^2} \left(\frac{\partial^2}{\partial\bar{u}^2} + \frac{\partial^2}{\partial\bar{\nu}^2}\right) + k^2\right]\Psi(\bar{u}, \bar{\nu}) = 0
\]

with \(k^2 = 2ME/\hbar^2\). The wave function to this equation is given by

\[
\Psi(\bar{u}, \bar{\nu}) = \exp\{i\mathbf{k} \cdot \mathbf{\bar{x}}\} = \exp\{i\mathbf{k}(\bar{x}\cos\phi + \bar{y}\sin\phi)\},
\]

where \(\mathbf{x} = (\sqrt{\bar{u}^2 + \bar{\nu}^2} + \bar{a})^{1/2}\) and \(\mathbf{y} = (\sqrt{\bar{u}^2 + \bar{\nu}^2} - \bar{a})^{1/2}\). For the creation of a mass-independent wave, the transformation has to satisfy the conditions

\[
\bar{g}_{ij}G^{11} = \frac{\bar{g}}{G}, \quad \text{and} \quad \bar{g}_{ij}G^{22} = \frac{\bar{g}}{G}.
\]

It gives \(\bar{a}' = \nu'\) again and the geometry of the transformed space is defined by

\[
ds^2 = \text{conformal factor} \times \left\{\frac{1}{2\sqrt{\bar{u}^2 + \bar{\nu}^2}}[(du)^2 + (dv)^2]\right\}
\]

with

\[
\text{conformal factor} = (\bar{a}')^2\frac{\sqrt{\bar{u}^2 + \bar{\nu}^2}}{\sqrt{\bar{u}^2 + \bar{\nu}^2}}.
\]

The unit vectors along constant \(\bar{\nu}\) and \(\bar{u}\) in the hyperbolic system are associated with the unit vectors \(\hat{e}_x\) and \(\hat{e}_y\) in the Cartesian system by

\[
\hat{e}_u = \frac{\partial x}{H_x \partial \bar{u}} = \frac{1}{(2\sqrt{\bar{u}^2 + \bar{\nu}^2})^{1/2}}[(\sqrt{\bar{u}^2 + \bar{\nu}^2} + u)^{1/2}\hat{e}_x - (\sqrt{\bar{u}^2 + \bar{\nu}^2} - u)^{1/2}\hat{e}_y],
\]

\[
\hat{e}_\nu = \frac{\partial x}{H_x \partial \bar{\nu}} = \frac{\nu}{(2\sqrt{\bar{u}^2 + \bar{\nu}^2})^{1/2}}\left[\frac{1}{(\sqrt{\bar{u}^2 + \bar{\nu}^2} + u)^{1/2}}\hat{e}_x + \frac{1}{(\sqrt{\bar{u}^2 + \bar{\nu}^2} - u)^{1/2}}\hat{e}_y\right].
\]

This makes the analytic expression of the current for \(\Psi(\bar{u}, \bar{\nu})\) be \(J = \hbar/(\hbar \bar{J}_x \hat{e}_x + \bar{J}_y \hat{e}_y)/M\) with the components

\[
\bar{J}_x = \left[\left(\frac{\bar{a}' \bar{\nu}}{2}\right)\frac{(\sqrt{\bar{u}^2 + \bar{\nu}^2} + u)^{1/2}(\sqrt{\bar{u}^2 + \bar{\nu}^2} + \bar{a})^{1/2}}{(\sqrt{\bar{u}^2 + \bar{\nu}^2})^2} \cos \phi \right.
\]

\[
+ \left(\frac{\bar{a}' \bar{\nu}}{2}\right)\frac{1}{(\sqrt{\bar{u}^2 + \bar{\nu}^2} + u)^{1/2}(\sqrt{\bar{u}^2 + \bar{\nu}^2} + \bar{a})^{1/2}(\sqrt{\bar{u}^2 + \bar{\nu}^2} + \bar{a})^{1/2}} \cos \phi \right]
\]

\[
+ \left(-\frac{\bar{a}' \bar{\nu}}{2}\right)\frac{1}{(\sqrt{\bar{u}^2 + \bar{\nu}^2} + u)^{1/2}(\sqrt{\bar{u}^2 + \bar{\nu}^2} - \bar{a})^{1/2}(\sqrt{\bar{u}^2 + \bar{\nu}^2} - \bar{a})^{1/2}} \sin \phi,
\]

\[
+ \left(\frac{\bar{a}' \bar{\nu}}{2}\right)\frac{1}{(\sqrt{\bar{u}^2 + \bar{\nu}^2} + u)^{1/2}(\sqrt{\bar{u}^2 + \bar{\nu}^2} + \bar{a})^{1/2}(\sqrt{\bar{u}^2 + \bar{\nu}^2} - \bar{a})^{1/2}} \sin \phi, \right]
\]
So far, our discussion seems to be going well, as before. However, the current analysis in the hyperbolic coordinates is not as useful as the previous ones. Here, let us check what kind of transformation effect would appear directly from \( uv \), \( Y \). Figure 12 shows the patterns of \( uv \), \( Y \) that were generated from a plane wave along the \( x \)-axis. The top left is the pattern for a plane wave along \( \phi = \pi / 4 \). The top right is the transformation pattern for a plane wave along the \( y \)-axis. All the plane waves were sliced by the mapping into eight independent parts separated by \( u = 0 \) and \( v = 0 \). They are discontinuous at \( u = 0 \).

\[
J_y = \left[ \begin{array}{c}
- \psi' \left( \frac{\sqrt{u^2 + v^2} - u}{\sqrt{\bar{u}^2 + \bar{v}^2}} \right) \\
\frac{d'\psi}{2} \left( \frac{\sqrt{u^2 + v^2} - u}{\sqrt{\bar{u}^2 + \bar{v}^2}} \right) \cos \phi \\
\frac{d'\psi}{2} \left( \frac{\sqrt{u^2 + v^2} - u}{\sqrt{\bar{u}^2 + \bar{v}^2}} \right) \sin \phi
\end{array} \right] \]

(57)

So far, our discussion seems to be going well, as before. However, the current analysis in the hyperbolic coordinates is not as useful as the previous ones. Here, let us check what kind of transformation effect would appear directly from \( \Psi(\bar{u}, \bar{v}) \). Figure 12 shows the patterns of \( \Psi(\bar{u}, \bar{v}) \) that were generated from a plane wave from different directions. The patterns are exhibited on the whole Euclidean plane since (50) can be well-defined on it. The top left demonstrates \( \Psi(\bar{u}, \bar{v}) \) for \( \phi = 0 \), i.e., the corresponding wave \( \Psi(u, v) \) is a plane wave along the \( x \)-direction. We have chosen \( \bar{u}(u) = \xi u + \zeta_2 \) and \( \bar{v}(v) = \zeta_1 v + \zeta_2 \) with \( \zeta_1 = u_2 / (u_2 - u_1) \) and \( \zeta_2 = -u_2 u_1 / (u_2 - u_1) \) as before for possibly creating a mass-independent steering region. The parameters are chosen as \( u_1 = v_1 = 50 \) and \( u_2 = v_2 = 200 \). The units of the transverse and vertical axes are \( c / 10 \) with \( c \) being the shortest distance from \( v_1 \) or \( u_1 \) to the origin. The top right exhibits the pattern for \( \phi = \pi / 3 \), while the bottom right is the pattern of \( \phi = \pi / 2 \) which is just the pattern of \( \phi = 0 \) rotated by \( \pi / 2 \). It is clear that the pattern of a plane wave was sliced by the transformation into eight independent parts. Each with \( \pi / 4 \) and bordered by coordinate axes at \( u = 0 \) and \( v = 0 \). They are discontinuous.
Figure 13 shows the pattern of $uv$, $Y(\bar{u})$ restricted to $uu u_{12} \parallel |u|$ and $vv v_{12} \parallel |v|$ for the case $4\pi/p$. It is often the region of applying to steer a wave. The region is extended to infinity in many directions. There is no direction which can accept an incoming plane wave without disturbance.

Figure 14 shows the current simulation of the wave pattern in figure 13. The pattern shows the current behavior near the inner boundary. The streamlines outside the transformation region are demonstrated by a plane wave. Unlike the performance of the elliptical and parabolic conformal cloaks, the current is no longer going around the inner boundary at $uu u_{1} = |u|$ and $vv v_{1} = |v|$.

Figure 13 shows the pattern of $\Psi(\theta, \phi)$ restricted to $u1 \leq |u| \leq u2$ and $v1 \leq |v| \leq v2$ for the case $\phi = \pi/4$. It is often the region of applying to steer a wave. The region is extended to infinity. Thus, there is no direction which can accept an incoming plane wave. Streamlines going around the inner boundary of a designed cloak is a key to creating a zero amplitude region. Figure 14 is the simulation of how the streamlines of a plane wave are manipulated in the transformation region. In the presented case, the streamlines no longer go around the inner boundary at $|u| = u1$ and $|v| = v1$. Thus, the conformal transformation-wave in the hyperbolic coordinates could not be used to design a manipulation device as in the previous sections. Nevertheless, it is interesting to...
transform a plane wave into different patterns in the different regions. Moreover, it can be realized only by applying an external potential with the spirit of the transformation design.

6. Conclusion

Over the past decade, transformation method has provided us with a new means of designing novel devices for steering electromagnetic fields, light, heat, fluids, and so forth. The reach or not of a manipulation depends on the realization of the transformation structure in the medium that the physical field passes through. It has been shown that conformal mapping is especially useful for simplifying the medium parameters for the construction of narrow-acceptance-angle (NAA) steering devices. In this paper, conformal mapping is applied to address the mass dependent property of quantum cloaks for equally splitting a matter wave in a specific direction. It is shown that a mass-independent NAA quantum cloak must be generated from a kind of conformal mapping. This not only largely simplifies the difficulty of constructing the device, but also extends the range of applications. For instance, it is applicable to steer the distribution of a matter wave of free particles, not necessary to search for materials for the modulation of the effective mass. We designed an elliptical and a parabolic conformal cloak according to the conclusion. Several of the cloak properties about the cloaks are pointed out as follows: (a) The conformal cloaks are not invisibility devices. As shown in figure 2, the current is guided to flow around the shell of the cloak, leaving a zero amplitude region. The transformation effect splits the group of the streamlines into two parts and pushes them outside the cloaks such that their outgoing directions are no longer aimed at the incoming direction. Thus, they are not invisibility cloaks. However, the conformal devices show how one can split a matter wave only with a finite potential distribution. The distribution region may be viewed as a kind of quantum lens that is capable of equally decomposing the matter wave into two required directions. (b) The form of transformation for the mass-independent condition in (8) depends on the shape of the cloak. To comprehend this fact, let us put an instance in the cylindrical coordinate system. It can easily be verified that the mass parameters of the cylindrical quantum cloak are given by

\[
(M_k) = \text{diag}\left(\frac{\hat{p}_r \hat{\varphi}}{\rho \hat{\varphi}}, \frac{\hat{p}_\varphi \hat{\varphi}}{\rho \hat{\varphi}}\right) M
\]  

(58)

The scaling transformation \( \hat{p} = c_1 \rho + c_2 \), and \( \hat{\varphi} = q_1 \varphi + c_2 \) would no longer lead to the isotropic mass in this case. Instead, the conformal mapping \( \hat{p} = c_2 \rho \hat{\varphi} \), and \( \hat{\varphi} = q_1 \varphi + c_2 \) could make the mass parameters isotropic. (c) Since the parameter of the particle that the cloak depends on is merely the incoming energy in the effective potential, the cloaks can be designed at an energy scale where the particle still reveals wave behavior. Finally, not all form invariant solutions of conformal mapping can be used to design quantum cloaks. As shown in section 5, application of conformal mapping to the hyperbolic coordinate system gets to the form invariant solution, and it can be generated simply by an external potential with the transformation design means. However, the solution slices a wave into eight independent parts such that the guiding effect of a quantum cloak cannot be formed. Nevertheless, it is interesting to slice a quantum amplitude into eight independent parts by transformation means. It is our hope that the presentation would be helpful for finishing a quantum cloak while also promoting our ability to control matter waves.

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