q-deformed Painlevé τ function and q-deformed conformal blocks

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Abstract
We propose the q-deformation of the Gamayun–Iorgov–Lisovyy formula for the Painlevé τ function. Namely, we propose the formula for the τ function for the q-difference Painlevé equation corresponding to the $A_1^{(1)}$ surface (and the $A_1^{(1)}$ symmetry) in the Sakai classification. In this formula, the τ function equals the series of q-Virasoro Whittaker conformal blocks (equivalently, the Nekrasov partition functions for pure $SU(2)$ 5d theory).

Keywords: Sakai’s classification, Nekrasov partition functions, difference Painlevé equations, bilinear equations

(Some figures may appear in colour only in the online journal)

1. Introduction

The goal of this paper is to find the q-deformation of the formulas suggested in [11, 12] for the Painlevé τ functions. More precisely, we will do this for the case of Painlevé III($D_8$).

We briefly recall the necessary details about this equation. In standard form it is a nonlinear second order differential equation on the function $w(z)$, namely

$$w'' = \frac{w'^2}{w} - \frac{w'}{z} + \frac{2w^2}{z^2} - \frac{2}{z}.$$  \hfill (1.1)

This equation can be rewritten as a system of two bilinear Toda-like equations on two τ functions [6]
\[ 1/2D_{\log z}^2(\tau(z), \tau(z)) = z^{1/2} \eta(z) \eta(z), \quad 1/2D_{\log z}^2(\eta(z), \eta(z)) = z^{1/2} \tau(z) \tau(z), \] (1.2)

where \( D_{\log z}^2 \) denotes the second Hirota operator with respect to \( \log z \), and \( \eta \) is a Bäcklund transformation of \( \tau \) (the group of Bäcklund transformations of this equation is \( Z_2 \)). The function \( w(z) \) is equal to \(-z^{1/2} \tau(z)^2/\eta(z)^2\), and the Bäcklund transformation acts on \( w \) as \( w \mapsto z/w \).

The Gamayun–Iorgov–Lisovyy formula for the \( \tau \) function has the form [12]

\[ \tau(\sigma, s|z) = \sum_{\mathbb{N}} C(\sigma + n)s^n z^{(\sigma + n)^2} \mathcal{F}(\sigma + n^2|z), \] (1.3)

where \( s, \sigma \) are integration constants. The function \( \mathcal{F}(\Delta|z) \) denotes the Whittaker limit of the Virasoro conformal block in a representation of the highest weight \( \Delta \) and the central charge \( c = 1 \). The function \( C(\sigma) = 1/(\mathcal{G}(1 - 2\sigma)\mathcal{G}(1 + 2\sigma)) \), where \( \mathcal{G} \) is a Barnes \( \mathcal{G} \) function.

Function \( \tau(\sigma, s|z) \) satisfies the evident property \( \tau(\sigma, s|z) = s^{-1} \tau(\sigma + 1, s|z) \) (periodicity in \( \sigma \)), and the Bäcklund transformation acts as \( \tau \propto + |\tau^{1/2} \). Therefore, the bilinear equations and formula for \( w \) can be rewritten as (see [6] for details)

\[ 1/2D_{\log z}^2(\tau(\sigma, s|z), \tau(\sigma, s|z)) = -z^{1/2} \tau(\sigma + 1/2, s|z) \tau(\sigma - 1/2, s|z), \] (1.4)

\[ w(z) = -z^{1/2} \tau(\sigma, s|z)^2 \quad \tau(\sigma - 1/2, s|z) \tau(\sigma + 1/2, s|z). \] (1.5)

The formula (1.3) was proven in [16] and [5]. Remark that the proof in [5] is based on the bilinear relations on the Virasoro conformal blocks.

It is natural to expect that there exists a certain \( q \)-deformation of the formula (1.3), which gives the \( \tau \) function for the \( q \)-difference Painlevé equation. We follow the Sakai approach to these equations [26], where the discrete equations are associated with certain rational surfaces, which are obtained by the blowing up of nine points in \( \mathbb{C} \mathbb{P}^2 \). These surfaces are parametrized by two complementary sublattices in the Picard group of the surface, and the last lattice is isomorphic to \( \mathbb{Z}^{1,9} \). These two sublattices are denoted as \( R \) and \( R^\perp \), and specify the so-called surface type and symmetry type correspondingly. The celebrated Sakai tables are given in figures 1 and 2. Here, \( X_n^\text{A} \) are standard notations for the affine root lattices. We do not explain the precise meaning of notations like \( D_{k_1}^{k_2} \) or \( A_n^{(1)} \), see [26] or [19] for details. The arrows in these figures indicate the degeneration of the surfaces.

Recall that the continuous Painlevé equations also correspond to the rational surfaces, namely, for each Painlevé equation there exists the so-called space of initial conditions [24]. This space for (1.1) is the surface of the \( D_8^{(1)} \) type. This is the reason why this equation is called Painlevé III(\( D_8 \)). The \( q \)-deformation of this equation corresponds to the deformation of the surface (see [26, section 7]). Therefore, we consider the \( q \)-difference Painlevé equation corresponding to the surface type \( A_3^{(1)} \) and the symmetry type \( A_1^{(1)} \) (the corresponding sublattices are drawn in the box in figures 1 and 2).

Such a surface depends on two parameters, which we denote by \( q \) and \( Z \) (the \( q \)-deformed analog of \( z \) in (1.1)). Details are given in section 2.1; here we only write the \( q \)-deformation of equation (1.1)

\[ G(qZ)G(q^{-1}Z) = \left( \frac{G(Z) - Z}{G(Z) - 1} \right)^2, \] (1.6)
This equation, as well as its relation to the Painlevé III(\(D_8\)), is given in [13]. We did not find the definition of the \(\tau\) function for this difference equation in the literature, so we propose the definition following the Tsuda approach [31] (see also [19]). It is convenient to use four \(\tau\) functions \(\tau_{12}, \tau_{13}, \tau_{14}, \tau_{23}\). This result is given in the theorem 2.1.

The next task is to give a formula for \(\tau_{1i}\) as an analytical function on the variable \(Z \in \mathbb{C}\) (above, one can think that \(\tau_{1i}\) is defined on the lattice as in [31]). As in the continuous case, the conditions on the \(\tau\) functions can be rewritten as a system of equations on one function \(\mathcal{T}(u, s; q|Z)\)

\[
Z^{1/4}\mathcal{T}(u, s; q|Z)\mathcal{T}(u, s; q|q^{-1}Z) = \mathcal{T}(u, s; q|Z)^2 + Z^{1/2}\mathcal{T}(uq, s; q|Z)\mathcal{T}(uq^{-1}, s; q|Z)
\]

\[
\mathcal{T}(uq^2, s; q|Z) = s^{-1}\mathcal{T}(u, s; q|Z).
\]  

(1.7)

Here \(u, s\) are \(q\)-deformed analogs of the parameters \(\sigma, s\) in (1.3).

In section 3 we propose the solution of this system in the ansatz

\[
\mathcal{T}(u, s; q|Z) = \sum_{n \in \mathbb{Z}} s^n C(uq^{2n}; q|Z) \frac{\mathcal{F}(uq^{2n}; q^{-1}, q|Z)}{(uq^{2n+1}; q, q)_\infty (u^{-1}q^{-2n+1}; q, q)_\infty}.
\]  

(1.8)

Here \(\mathcal{F}(u; q^{-1}, q|Z)\) denotes the Whittaker limit of the \(q\)-deformed conformal block, \((u; q, q)_\infty\) denotes the double infinite Pochhammer symbol, and the function \(C(u; q|Z)\) is defined by its second difference derivatives, see definition 3.1. The function \(C(u; q|Z)\) is the \(q\)-deformed analog of \(\varphi^2\) in formula (1.1). We give a couple of examples of such functions in example 3.1.

In section 3.2 we discuss the validity of (1.7) for the function \(\mathcal{T}(Z)\) defined in (1.8). The second condition in (1.7) is straightforward from (1.8). The bilinear relation in (1.7) is
equivalent to the bilinear relation on function $F(u; q^{-1}, q|Z)$, see (3.4) and theorem 3.1. We do not prove the corresponding bilinear relation, but give several arguments in support.

In order to finish the description of the paper content, note that in sections 2.3 and 3.3 we discuss the $q$-limit, and in section 3.4 we discuss the special values of $u$, $s$, which correspond to the $q$-deformation of the algebraic solution of (1.1). In appendix A we collect the necessary definitions and properties of the $q$-special functions. In appendix B we write more general bilinear relations on the $q$-deformed conformal blocks.

2. $q$-deformed equations

2.1. $A_7^{(1)}$ surface and its symmetry group

Here we follow the Sakai approach to the $q$-difference Painlevé equations associated with rational surfaces [19, 26]. In this approach, the $q$-deformed Painlevé equation arises from a discrete group of birational automorphisms acting on a rational surface.

Let $X$ be a rational surface of the type $A_7^{(1)}$ in Sakai notation [26, appendix B, mul. 9]), where the first notation denotes the surface type and the second notation denotes the symmetry type of the surface. It is obtained by blowing up $\mathbb{C}P^2$ in nine points. A scheme of this procedure can be seen in figure 3 (see [26, appendix B, mul. 9]).

Let $(x : y : z)$ denote the coordinates of $\mathbb{C}P^2$, $(a, b, c)$ are the parameters of the surface which characterize the positions of points of the blow-up (our $a$ is $a_1$ in [26]). There is an equivalence $(a, b, c; x : y : z) \sim (a, \mu^2 b, \mu^{-2} c; \mu x : y : z)$. There exists a discrete subgroup of the Cremona group (a group of rational transformations of $\mathbb{C}P^2$) which preserve the blow-up structure. We will denote it by $W$. Group $W$ is equal to $Dih_4 \ltimes W(A_7^{(1)})$ where $Dih_4$ is the dihedral group of a square and $W(A_7^{(1)})$ is the Weyl group of $\widehat{sl}(2)$. Group $W$ can be presented by the generators $s_0, s_1, n_1, n_2$ and relations as

$$s_0^2 = s_1^2 = 1, \quad n_1^2 = n_2^4 = (n_1 n_2)^2 = 1, \quad s_1 = n_2 s_0 n_2^{-1}, \quad s_0 = n_2 s_0 n_2^{-2} = n_2 n_1^{-1}. \quad (2.1)$$

where $s_0, s_1$ are the simple Weyl group reflections, and $n_1, n_2$ are the generators of $Dih_4$. The structure of $W$ can be graphically represented in figure 4. The first two pictures define the action of the generators of the dihedral group on the square, and the last picture describes the structure of the semidirect product, namely $n_2$ acts as the outer automorphism, which interchanges roots.

As a subgroup of the Cremona group, $W$ acts on the rational surfaces by rational transformations\(^9\)

\[^9\]There is a misprint in the formula for $s_0$ in [26, appendix C, mul. 9].
\( \pi_1 : (a, b, c; x : y ; z) \mapsto (1a, 1c, 1b; x : z : y) \)
\( \pi_2 : (a, b, c; x : y ; z) \mapsto (bc, ab, b; yz : x^2) \)
\( s_1 : (a, b, c; x : y ; z) \mapsto (1a, ab, ac; x(y - az) : y(z - y) : az(z - y)) \)
\( s_0 : (a, b, c; x : y ; z) \mapsto (ab^2c^2, 1c, 1b; x(b - yz)(yz - cx^2) : y(yz - cx^2)^2 : z(x^2b - yz)^2) \).

\( \text{(2.2)} \)

Let us denote \( q = abc \), \( Z = a^{-1} \) and \( F = yzb/\sqrt{x^2} \), \( G = z/y \). These coordinates are unambiguously defined (taking into account the equivalence of \( \mu \) rescaling). Then, in terms of \( (Z, q, F, G) \), the action of group \( W \) has the form:

\( \pi_1 : (Z, q, F, G) \mapsto (Z^{-1}, q^{-1}, q^{-1}Z^{-1}F, G^{-1}) \),
\( \pi_2 : (Z, q, F, G) \mapsto (Z^{-1}q^{-1}, qZ^{-1}G, F^{-1}) \),
\( s_1 : (Z, q, F, G) \mapsto (Z^{-1}q^{-1}, qF(G - 1)^2, G^{-1}) \),
\( s_0 : (Z, q, F, G) \mapsto (Z^{-1}q^{-2}, q^{-1}Z^{-1}F, G^{-1}F - 1)^2). \)

\( \text{(2.3)} \)

Introduce element \( T = \pi_2^{-1} \circ s_0 \) of infinite order in \( W \). We will denote \( \pi = T(x) \) and \( \xi = T^{-1}(x) \).

Note that group \( W \) is generated by \( T, s_1 \) and \( \pi_2 \). We have

\( (Z, q, F, G) = \left( qZ, q, (F - qZ)^2, F \right) \), \( (Z, q, F, G) = \left( q^{-1}Z, q, G, (G - Z)^2 \right) \).

Therefore

\( \mathcal{G} = \left( \frac{G - Z}{G - 1} \right)^2 \).

\( \text{(2.4)} \)

This equation could be called the \( q \)-deformed Painlevé III(\( D_8 \)). That is because if we consider \( G \) as a function on \( Z \) (and \( \mathcal{G} = G(qZ) \), \( G = G(q^{-1}Z) \)) then the continuous limit of (2.4) is the Painlevé III(\( D_8 \)) equation (1.1) (see section 2.3). Of course this equation is not new: here we get it from the known formulas for the \( A_7^{(1)} \) surface, but exactly in the form (2.4) it was written in in [13, equation (20)], see also the earlier work [30].
2.2. The \( \tau \) function representation of group \( W \)

We want to lift the representation of group \( W \) to the level of the four letters \( T_i \), \( i = 1, \ldots, 4 \). Analogously to [31] (see also [19]) we call \( T_i \) the \( \tau \) functions and state the theorem:

**Theorem 2.1.** The action of the generators \( s_1, \pi_1, \pi_2 \) of group \( W \) on \( T_i \), \( i = 1,2 \) given by table 1 provides a representation of \( W \) in the field \( \mathbb{C}(T_1, T_2, T_3, T_4, q^{1/4}, Z^{1/4}) \).

The proof is straightforward. Generators \( s_1, \pi_1, \pi_2 \) generate the whole group, so it is enough to check the relations of group 
\[
\frac{T_1}{T_2} = \frac{T_2}{T_3} = \frac{T_3}{T_4} = \frac{T_4}{T_1} = q^{1/4}. 
\]

The subgroup \( \text{Dih}_4 \) of \( W \) acts on the \( \tau \) functions the same as it does on the square vertices in figure 4, i.e. \( w(T_i) = T_{w(i)} \) for any \( w \in \text{Dih}_4 \). In more geometrical language, one can assign the \( \tau \) functions to the four blow-up points in \( \mathbb{CP}^2 \) (see [31]).

Define \( F, G \) by the formulas

\[
F = -aqZ^{1/2}T_2^2/T_4^2, \quad G = -Z^{1/2}T_4^2. 
\]

This is just a discrete analog of (1.5). Then one can check that the action of \( W \) on \( F \) and \( G \) induced by the action on the \( \tau \) functions coincides with the action defined in (2.3).

The formula for the action \( T \) implies that \( T_2 = T_1 \) and \( T_4 = T_3 \). Then, from the action of \( T \) of \( T_2 \) and \( T_4 \) we get

\[
Z^{1/4}T_1T_4 = T_1^2 + Z^{1/2}T_3^2, \quad Z^{1/4}T_2T_3 = T_3^2 + Z^{1/2}T_1^2. 
\]

**Remark 2.1.** It is interesting to note that under the action of the group \( W \), variables \( T_i \) go to the Laurent polynomials on \( T_i \) not just the rational functions. This property is nontrivial: for example, this is not true for \( G \) and \( F \). This observation follows from the fact that the action of \( W \) can be represented as the composition of mutations for cluster algebra\(^{10} \) similar to the higher \( q \)-deformed Painlevé equation in the paper [25].

**Remark 2.2.** The equations corresponding to the \( A^{1}_{1/4} \) surface are usually written in a bit more of a general form than (2.4). For example, in [13, equation (10)], (see also [14, equation (2.19)]):

\[
(xy - 1)(xy - 1) = Z^2, \quad (xy - 1)(xy - 1) = Z. 
\]

\(^{10}\) We are grateful to Gavrylenko for the discussion of this point.
This system can be solved in terms of the \( \tau \) functions subject to (2.6). Namely,

\[
x = q^{-1/4} z^{1/4} \frac{T_2 T_5}{T_3 T_4}, \quad y = q^{1/4} \frac{T_2 T_5}{T_3 T_4}.
\]

Therefore, one can consider system (2.6) to be a bilinear form of (2.7). In the paper [27, equations (2.23) and (2.24)] this system is written in a different form

\[
\overline{f} = g(Z - y)/(g - 1), \quad \overline{g} = \overline{f}.
\]

This system can also be solved in terms of the \( \tau \) functions, and in fact, is equivalent to (2.7) by the invertible transformation

\[
\mathcal{T}_1 = \mathcal{T}(u, s; q|Z), \quad \mathcal{T}_3 = s^{-1/2} \mathcal{T}(u, s; q|Z),
\]

and assume quasi periodicity in \( u \)

\[
\mathcal{T}(uq^2, s; q|Z) = s^{-1} \mathcal{T}(u, s; q|Z).
\]

Then the equation (2.6) reduces to

\[
Z^{1/4} \mathcal{T}(u, s; q|Z) \mathcal{T}(u, s; q|q^{-1}Z) = \mathcal{T}(u, s; q|Z)^2 + Z^{1/2} \mathcal{T}(uq, s; q|Z) \mathcal{T}(uq^{-1}, s; q|Z).
\]

The functions \( F(Z), G(Z) \) are defined in terms of function \( \mathcal{T} \) by the formula (2.5). Note that the solutions of equation (2.6) depend on four parameters: in addition to \( u, s \), we have the simple symmetry \( (\mathcal{T}_1, \mathcal{T}_3) \mapsto (\alpha Z^{1/4} \mathcal{T}_1, \alpha Z^{3/4} \mathcal{T}_3) \). This symmetry does not act on the functions \( F(Z), G(Z) \).

Remark that element \( \pi^2 \in W \) commutes with the shift \( \mathcal{T} \). In the continuous limit the element \( \pi^3 \) goes to the Bäcklund transformation. It acts by formula \( \pi^3 : \mathcal{T}(u, s; q|Z) \mapsto \mathcal{T}(uq, s; q|Z) \).

In terms of parameter \( \sigma \), this is the transformation \( \sigma \mapsto \sigma + 1/2 \), as for the continuous Painlevé.

### 2.3. Continuous limit of the equations

Let us check that the continuous limit of (2.4) is (1.1), as it should be for the \( q \)-deformed Painlevé III(D8). Introduce notations \( q = e^h, u = q^{2z} \). In the limit questions we assume that \( |q| < 1 \) as \( q \to 1 \).

**Proposition 2.1.** Substitute

\[
Z = h^2 z, \quad G(Z) = h^2 w(z).
\]

If \( h \to 0 \), the leading order of the equation (2.4) is the Painlevé III(D8) (1.1) equation.
The proof is by direct calculation, namely the expansion of (2.4) into powers of $\hbar$. We obtain the first nontrivial coefficient with the power $\hbar^2$. This coefficient vanishes, and this is just the Painlevé III($D_8$) (1.1).

We also check the continuous limit of the bilinear equation (2.11). It is convenient to define the new function $T_\tau(u, s; q|Z)$ by

$$T_\tau(u, s; q|Z) = \frac{(q; q)_\infty^2}{\Gamma(qZ^{1/4}; q^{1/4})} T(u, s; q|Z).$$

Equation (2.11) in terms of function $T_\tau(u, s; q|Z)$ reads

$$T_\tau(u, s; q|Z) T_\tau(u, s; q|q^{-1}Z) = T_\tau(u, s; q|Z)^2 + Z^{1/2} T_\tau(uq, s; q|Z) T_\tau(uq^{-1}, s; q|Z).$$

Taking the analytic continuation around $Z = 0$, one can change the sign of $Z^{1/2}$ and get

$$T_\tau(u, s; q|Z) T_\tau(u, s; q|q^{-1}Z) = T_\tau(u, s; q|Z)^2 - Z^{1/2} T_\tau(uq, s; q|Z) T_\tau(uq^{-1}, s; q|Z).$$

(2.14)

**Proposition 2.2.** Substitute

$$Z = \hbar^4 z, \quad T_\tau(u, s; q|Z) = \tau(\sigma, s|z).$$

(2.15)

If $\hbar \to 0$ then the leading order of the equation (2.14) is a Toda-like equation (1.4).

Proof is by direct calculation, the first nontrivial coefficient appears in order $\hbar^2$.

### 3. Formula for $\tau$ functions

#### 3.1. The $q$-deformed conformal blocks

In the representation theory approach, the irregular conformal block for the $q$-deformed Virasoro algebra is defined as the square of the Whittaker vector for this algebra ([2]). This conformal block equals the Nekrasov instanton partition function for the 5d pure gauge $SU(2)$ theory [22] proposed in [2] by extension of the AGT conjecture. So, the formula for the conformal block reads

$$\mathcal{F}(u_1, u_2; q_1, q_2|Z) = \sum_{\lambda, \lambda_1} Z^{\lambda_1 + |\lambda_1|} \prod_{s=1}^{\ell(\lambda)} N_{\lambda, s}(u_1, s; q_1, q_2),$$

(3.1)

where

$$N_{\lambda, s}(u_1, s; q_1, q_2) = \prod_{s \in \ell(\lambda)} (1 - uq_2^{-a_0(s)} q_1^{-a_0(s)} \cdot \prod_{s \in \ell(\lambda)} (1 - uq_2^{-a_0(s)} q_1^{-a_0(s)})^{-1}).$$

(3.2)

The sum in (3.1) runs over all pairs of Young diagrams $\lambda_1, \lambda_2$. In formula (3.2) $a_0(s), \ell(s)$ are the lengths of the arms and legs of box $s$ in diagram $\lambda$.

The function $\mathcal{F}(Z)$ depends on $u_1, u_2$ through their ratio $u = u_1/u_2$, so we shall use the notation $u$ below. Parameter $u$ is the $q$-deformed analog of the highest weight, and the pair $(q_1, q_2)$ is the $q$-deformed analog of the central charge. In this paper we use specification $q_1^{-1} = q_2 = q$ everywhere (except appendix B). In the continuous limit, this corresponds to the irregular conformal block for Virasoro algebras with $c = 1$, as in the formula for the continuous $\tau$ function (1.3).

It follows directly from the definition that
The function $F$ is defined as a power series $F(Z) = 1 + O(Z)$. This series converges, and the proof of the following proposition is similar to the one in [18, proposition 1 (i)].

**Proposition 3.1.** Let $|q| > 1$ and $u = q^n, n \in \mathbb{Z}$. Then series (3.1) converges uniformly and absolutely on every bounded subset of $\mathbb{C}$.

**Proof.** There exist constants $L_1, L_2 \in \mathbb{R}_{>0}$ such that
\[
\frac{|q^{n/2} - q^{-n/2}|}{q^{1/2} - q^{-1/2}} > L_1 \quad \forall n \in \mathbb{Z}_{\geq 0}
\]
\[
\frac{|u^{1/2}q^{n/2} - u^{-1/2}q^{-n/2}|}{q^{1/2} - q^{-1/2}} > L_2 \quad \forall n \in \mathbb{Z}.
\]

Then we can bound $\prod_{i,j=1}^{D_2} N_{\lambda_i, \lambda_j}(u_i/u_j; q^{-1}, q)$ as
\[
\prod_{i,j=1}^{D_2} N_{\lambda_i, \lambda_j}(u_i/u_j; q^{-1}, q) \leq \prod_{i,j=1}^{D_2} |q^{1/2} \lambda_i^{(i)} - q^{-1/2} \lambda_j^{(j)}|^2
\]
\[
\cdot \prod_{i,j=1}^{D_2} (\lambda_i \leftrightarrow \lambda_j) > |\lambda_i|^{2|\lambda_j|} \cdot \frac{|\lambda_i|^{2|\lambda_j|}}{(\dim \lambda_i \dim \lambda_j)} \cdot \left| F(q^{1/2} - q^{-1/2})^{(i)\beta_{(j)}} \right|
\]
where we used the hook length formula for $\dim \lambda$. Since $\sum_{\lambda} \lambda! = (\dim \lambda)^! = n!$, we have $F(u; q; Z) < \exp \frac{2|Z|}{L_2(q^{1/2} - q^{-1/2})}$.

To ensure the convergence of infinite products like $(u; q, q)_\infty = \prod_{i,j=1}^{D_2} (1 - uq^{i+j})$, we impose the condition $|q| < 1$. Using an analytic continuation (A.4) one can also work in the region $|q| > 1$.

**Conjecture 3.1.** The $q$-deformed conformal blocks satisfy the bilinear relations
\[
\sum_{2n \in \mathbb{Z}} \left( \prod_{c,c' = \pm 1}^{u_2 n Z^{2n}} \frac{F(u_2 n Z^{2n}, q^{-1}; q, q)_{\infty} F(u_2 n Z^{2n}; q^{-1}, q Z)}{F(u_2 n Z^{2n}; q^{-1}, q Z) F(u_2 n Z^{2n}, q^{-1}; q, q Z) + 1} \right) = 1 - Z^{2l/2} \sum_{2n \in \mathbb{Z}} \left( \prod_{c,c' = \pm 1}^{u_2 n Z^{2n}} \frac{F(u_2 n Z^{2n}, q^{-1}; q, q)_{\infty} F(u_2 n Z^{2n}; q^{-1}, q Z)}{F(u_2 n Z^{2n}; q^{-1}, q Z) F(u_2 n Z^{2n}, q^{-1}; q, q Z)} \right)
\]
\[
= (1 - Z^{l/2}) \sum_{2n \in \mathbb{Z}} \left( \prod_{c,c' = \pm 1}^{u_2 n Z^{2n}} \frac{F(u_2 n Z^{2n}, q^{-1}; q, q)_{\infty} F(u_2 n Z^{2n}; q^{-1}, q Z)}{F(u_2 n Z^{2n}; q^{-1}, q Z) F(u_2 n Z^{2n}, q^{-1}; q, q Z)} \right)
\]

We do not have proof of this conjecture, but we have two arguments in support. First, this conjecture was checked by computer calculation up to $Z^3$.

The second argument is the fact that the continuous limit of this relation gives the known (\[5, equation (4.29)\]) relation for the irregular Virasoro conformal blocks $\mathcal{F}(\Delta | z)$
\[
\mathcal{F}(\sigma + n^2|z) \mathcal{F}(\sigma - n^2|z) = \sum_{2n \in \mathbb{Z}} \frac{D_{n}^2 \mathcal{F}(\sigma + n^2|z) \mathcal{F}(\sigma - n^2|z)}{\prod_{k=1}^{D_2} (k^2 - 4\sigma^2)^{n(k-1)n-k} (4\sigma^2)^{2n}}
\]
\[
= -2\mathcal{F}(\sigma + n^2|z) \mathcal{F}(\sigma - n^2|z) \sum_{2n \in \mathbb{Z}} \frac{D_{n}^2 \mathcal{F}(\sigma + n^2|z) \mathcal{F}(\sigma - n^2|z)}{\prod_{k=1}^{D_2} (k^2 - 4\sigma^2)^{n(k-1)n-k} (4\sigma^2)^{2n}}
\]
\[
= -2\mathcal{F}(\sigma + n^2|z) \mathcal{F}(\sigma - n^2|z) \sum_{2n \in \mathbb{Z}} \frac{D_{n}^2 \mathcal{F}(\sigma + n^2|z) \mathcal{F}(\sigma - n^2|z)}{\prod_{k=1}^{D_2} (k^2 - 4\sigma^2)^{n(k-1)n-k} (4\sigma^2)^{2n}}
\]
Here $D_{\log z}^2(f(z), g(z)) = z^2(f'' g - f' g' + fg'') + z(f' g + fg')$ denotes the second Hirota differential operator with respect to a variable $\log z$. We shall provide the continuous limit and obtain (3.5) in section 3.3.

**Remark 3.1.** Most results of this section can be stated for any $q_1, q_2$, see e.g. appendix B for the bilinear relations. But for the $\tau$ functions we will only use the conformal blocks with $q_1 q_2 = 1$.

### 3.2. $q$-deformation of the formula for the $\tau$ function

**Definition 3.1.** Function $T(u, s; q|Z)$ given by the formula

$$
T(u, s; q|Z) = \sum_{n \in \mathbb{Z}} s^n C(u q^{2n}; q|Z) \frac{\mathcal{F}(u q^{2n}; q^{-1}, q|Z)}{(u q^{2n+1}; q, q)_\infty (u^{-1} q^{-2n+1}; q, q)_\infty} \tag{3.6}
$$

is called the $q$-deformed $\tau$ function of the Painlevé III($D_8$) equation if function $C(u; q|Z)$ satisfies equations

$$
\frac{C(u q; q|Z) C(u q^{-1}; q|Z)}{C(u; q|Z)^2} = -Z^{1/2}, \tag{3.7}
$$

$$
\frac{C(u q; q|Z) C(u q^{-1}; q|q^{-1} Z)}{C(u; q|Z)^2} = -u Z^{1/4} \tag{3.8}
$$

$$
\frac{C(u, q|Z) C(u; q^{-1} Z)}{C(u; q|Z)^2} = Z^{-1/4}. \tag{3.9}
$$

If $C(u; q|Z) = C(u^{-1}; q|Z)$, then functions $C, T$ are called the $u$-inverse invariant.

Evidently, $C(u; q|Z)$ could be multiplied on any function $\tilde{C}(u; q|Z)$, which satisfies the homogeneous equations (3.7)–(3.9).

**Example 3.1.** The following examples of $C(u, q|Z)$ satisfy (3.7)–(3.9)

$$
C(u, q|Z) = \Gamma((q Z)^{1/4}; q^{1/4}, q^{1/4}) \Gamma((q Z)^{1/4}; q^{1/4}, q^{1/4}) \Gamma((q Z)^{1/4} u^{-1/4}; q^{1/4}, q^{1/4}) \tag{3.10}
$$

$$
C_c(u; q|Z) = (-1)^q \left( \frac{\log u}{\log q} \right)^2 \Gamma((q Z)^{1/4}; q^{1/4}, q^{1/4}) \exp \left( \frac{\log^2 u \log Z}{4 \log^2 q} \right). \tag{3.11}
$$

Here the elliptic Gamma function is defined by $\Gamma(u; q, q) = (q^u u^{-1}; q, q)_\infty \Gamma(u; q, q)_\infty$. The necessary definitions and properties of the $q$-deformed special functions are collected in appendix A.

Both functions $C, C_c$ are $u$-inverse invariant. Function $C$ is meromorphic as a function on $Z^{1/2}$ (or on $u^{1/2}$) on the complex plain. Function $C_c$ is useful for making the continuous limit (section 3.3), but $C_c$ is not meromorphic.

**Remark 3.2.** The left sides of equations (3.7)–(3.9) are multiplicative second order difference derivatives of the function on two variables ($u|Z$) in the directions ($1|0), (1|1), (0|1)$ correspondingly.
Remark 3.3. We could hide $s^n$ in $C(uq^{2n}; q|Z)$ carrying out some simple factor from the sum. Indeed, function $\theta(u^{1/2}; q|Z)$ satisfies the homogeneous equations (3.7)–(3.9), so

$$T(u, s; q|Z) = \theta(u^{1/2}; q|Z) \sum_{n \in \mathbb{Z}} C_n(uq^{2n}; q|Z) \frac{F(uq^{2n}; q^{-1}|Z)}{(uq^{2n}; q)_\infty(uq^{2n+1}; q)_\infty},$$

where $C_n(u; q|Z) \equiv C(u; q|Z)\theta(u^{1/2}; q|Z)^{-1}$.

It is clear from the definition that

$$T(uq^2, s; q|Z) = s^{-1}T(u, s; q|Z). \quad (3.12)$$

If the function $C$ is a $u$-inverse invariant, then

$$T(u, s; q|Z) = T(u^{-1}, s^{-1}; q|Z). \quad (3.13)$$

If $C$ is not a $u$-inverse invariant then the function $T(u^{-1}, s; q|Z)$ also satisfies definition (3.6), but for another function $C$.

Conjecture 3.2. Function $T(u, s; q|Z)$ satisfies the bilinear equations

$$Z^{1/4}T(u, s; q|Z)T(u, s; q^{-1}|Z) = T(u, s; q|Z)^2 + Z^{1/2}T(u, s; q|Z)T(u^{-1}, s; q|Z). \quad (3.14)$$

Theorem 3.1. Conjecture 3.1 is equivalent to conjecture 3.2.

Proof. Let us substitute the expression (3.6) for the $\tau$ function into (3.14) and collect the terms with the same powers of $s$. The vanishing condition of the $s^m$ coefficient has the form

$$Z^{1/4} \sum_{n \in \mathbb{Z}} C(uq^{2n+2m}; q|Z)C(uq^{2n}; q|Z) - \frac{F(uq^{2n+2m}; q^{-1}|Z)}{(uq^{2n+2m}; q)_\infty(uq^{2n+2m}; q)_\infty} \prod_{\epsilon = \pm 1} (uq^{2n+2m}; q)_\infty(uq^{2n+2m}; q)_\infty \prod_{\epsilon = \pm 1} (uq^{2n+2m}; q)_\infty(uq^{2n+2m}; q)_\infty \prod_{\epsilon = \pm 1} (uq^{2n+2m}; q)_\infty(uq^{2n+2m}; q)_\infty \prod_{\epsilon = \pm 1} (uq^{2n+2m}; q)_\infty(uq^{2n+2m}; q)_\infty \prod_{\epsilon = \pm 1} (uq^{2n+2m}; q)_\infty(uq^{2n+2m}; q)_\infty \prod_{\epsilon = \pm 1} (uq^{2n+2m}; q)_\infty(uq^{2n+2m}; q)_\infty$$

Let us substitute into these relations $u \to uq^{-m}$ and $n \to n - m/2$. We see that for any $m$, the vanishing conditions of the coefficients with powers of $s$ are equivalent to

$$Z^{1/4} \sum_{n \in \mathbb{Z}} C(uq^{2n}; q|Z)C(uq^{2n}; q|Z) - \frac{F(uq^{2n}; q^{-1}|Z)}{(uq^{2n}; q)_\infty(uq^{2n}; q)_\infty} \prod_{\epsilon = \pm 1} (uq^{2n}; q)_\infty(uq^{2n}; q)_\infty \prod_{\epsilon = \pm 1} (uq^{2n}; q)_\infty(uq^{2n}; q)_\infty \prod_{\epsilon = \pm 1} (uq^{2n}; q)_\infty(uq^{2n}; q)_\infty \prod_{\epsilon = \pm 1} (uq^{2n}; q)_\infty(uq^{2n}; q)_\infty \prod_{\epsilon = \pm 1} (uq^{2n}; q)_\infty(uq^{2n}; q)_\infty \prod_{\epsilon = \pm 1} (uq^{2n}; q)_\infty(uq^{2n}; q)_\infty$$

We want to divide these conditions on $C(u, q|Z)$ and then simplify the arising expressions like $\frac{C(uq^{-1}; q|Z)C(uq^{-1}; q|Z)}{C(u, q|Z)^2}$. Introduce the auxiliary functions $\beta_k, \gamma_k, k \in \mathbb{Z}$ by the formulas
The elementary calculations give us
\[
\beta_k = Z \beta_{k-2}, \quad \gamma_k = Z \gamma_{k-2}.
\]
(3.16)

Evidently \( \beta_0 = \gamma_0 = 1 \). From (3.7)–(3.9) we have \( \beta_1 = -Z^{1/2}, \quad \gamma_1 = -uZ^{1/2} \). So we have the unique solution of (3.16)
\[
\beta_k = (-1)^k Z^{k/2}, \quad \gamma_k = (-1)^k u^k Z^{k/2}.
\]
(3.17)

Using these results and (3.9) we obtain
\[
\sum_{n \in Z + \delta} u^{2n} Z^{2n} \frac{\mathcal{F}(uq^{2n}; q^{-1}, q|Z) \mathcal{F}(uq^{-2n}; q^{-1}, q|Z)}{\prod_{\epsilon = \pm 1} ((uq^{2n})^q q, q)_\infty ((uq^{-2n})^q q, q)_\infty}
\]
\[
= \sum_{n \in Z + \delta} Z^{2n} \frac{\mathcal{F}(uq^{2n}; q^{-1}, q|Z) \mathcal{F}(uq^{-2n}; q^{-1}, q|Z)}{\prod_{\epsilon = \pm 1} ((uq^{2n})^q q, q)_\infty ((uq^{-2n})^q q, q)_\infty}
\]
\[
- Z^{1/2} \sum_{n \in Z + \delta} Z^{2(n+1)} \frac{\mathcal{F}(uq^{2n+1}; q^{-1}, q|Z) \mathcal{F}(uq^{-2n-1}; q^{-1}, q|Z)}{\prod_{\epsilon = \pm 1} ((uq^{2n+1})^q q, q)_\infty ((uq^{-2n-1})^q q, q)_\infty}.
\]

and this is the result of splitting (3.4) into parts with integer powers of \( Z \) and with half-integer powers of \( Z \).

3.3. The continuous limit of the q-deformed \( \tau \) function

**Proposition 3.2.** The formula for the q-deformed \( \tau \) function \( \mathcal{T}(u, s; q|Z) \) can be rewritten in the following way:
\[
\mathcal{T}(u, s; q|Z) = C(u; q|Z) \sum_{n \in \mathbb{Z}} \tilde{s}(u, s; q|Z)^n Z^{2n+1/2} \frac{\mathcal{F}(uq^{2n}; q^{-1}, q|Z)}{\prod_{\epsilon = \pm 1} ((uq^{2n})^q q, q)_\infty}.
\]
(3.18)
where

\[ \tilde{s}(u, s; q|Z) = -\left( \frac{C(u; q|Z)}{C(uq^{-1}; q|Z)} \right)^2 s. \] (3.19)

Proof. Using (3.7) we transform expression \( C(uq^k; q|Z)/C(u; q|Z) \) as

\[ \frac{C(uq^k; q|Z)}{C(uq^{k-1}; q|Z)} = \frac{C(uq^{k-1}; q|Z) C(uq^k; q|Z) C(uq^{k-2}; q|Z)}{C(uq^{k-1}; q|Z)^2} = \frac{C(uq^{k-1}; q|Z)}{C(uq^{k-2}; q|Z)}. \]

Substituting the last expression to (3.6) we finish the proof. \( \square \)

Remark 3.4. The \( \tau \) function was defined as a series, and it is convenient to prove the convergence of this series using expression (3.18). The proof is analogous to the one in [18, proposition 1 (ii)].

First, note that the bounds \( L_1, L_2 \) defined in the proof of proposition 3.1 are the same for every conformal block in the sum (3.18). Therefore, we can estimate the conformal blocks by the same exponent.

Second, we rewrite the Pochhammer symbols in terms of the \( q \)-Barnes \( G \) function using (A.6). Then for \( n > 0 \) we have

\[ \frac{C(uq^k; q|Z)}{C(uq^{k-1}; q|Z)} = (-1)^k Z^{k} \frac{C(u; q|Z)}{C(uq^{k-1}; q|Z)}. \]

Substituting the last expression to (3.6) we finish the proof. \( \square \)

Remark 3.5. Moreover, we could rewrite \( \tilde{s}(u, s; q|Z) \) as a sum of the conformal blocks with some \( \hat{s} \) and \( q \)-rational coefficients (analogous to the continuous \( \tau \) function ([6])). Introduce functions \( P_n(u; q) \) by

\[ \prod_{c=\pm 1} \frac{(uq^c q; q)_\infty}{(uq^{-c}q^{-1}; q)_\infty} = P_n(u; q) \left( \frac{(u; q)_\infty}{(u^{-1}; q)_\infty} \right)^{2n}, \quad n \in \mathbb{Z}. \]

Then we can write
\[ T(u, s; q|Z) = \frac{C(u; q|Z)}{\prod_{\epsilon = \pm 1} (u^\epsilon q^{\epsilon/2}; q, q, q)} \sum_{n \in \mathbb{Z}} Z^{n^2 + n/2} P_n(u; q) \mathcal{F}(uq^{2n}; q^{-1}, q|Z), \]  \( (3.20) \)

where \( P_n(u; q) = \frac{(-1)^n}{(1 - u)^2 n \prod_{i=1}^{n-1} (u^{1/2} q^{1/2} - u^{-1/2} q^{-1/2} q^{2i - 2n - i})}, \quad n \geq 0, \)

\[ P_0(u; q) = P_0(u^{-1}; q), \quad n < 0, \quad \hat{s} = -\left( \frac{C(u; q|Z)}{C(uq^{-1}; q|Z)} (u^{-1}; q, q)_\infty \right)^2. \]

The formula (3.20) can also be used for the proof of convergence (3.6). The proof goes similarly with the proof of proposition 3.1; we bind all terms in the denominator of \( P_n(u; q) \) using \( L_1, L_2. \)

Now we check that in the \( q \to 1^- \) limit (i.e. \( q \in [1 - \epsilon, 1] \)) the formula for \( T(u, s; q|Z) \) gives the formula (1.3) for the \( \tau \) function of the continuous Painlevé equation. As in section 2.3 it is convenient to use the function \( T_c(u, s; q|Z) \) defined by (2.13). Moreover for this limit we take the \( q \)-deformed \( \tau \) functions with \( \mathcal{F}(u; q^{-1}, q|Z) \) given by the formula (3.11).

**Theorem 3.2.** Let

\[ q = e^h, \quad Z = h^2 z^2, \quad \sigma = \frac{\log u}{2h}. \]  \( (3.21) \)

Then the \( \tau \) function \( T_c(u, (-1)^{4s} s; q|Z) \) with \( C(u; q|Z) = C_c(u; q|Z) \) goes to \( \tau(\sigma, s|z) \) in the limit \( h \to 0. \)

**Proof.** Rewrite \( T(u, s; q|Z) \) in form (3.18) using (3.11)

\[ T_c(u, s; q|Z) = (-1)^{2s} \sum_{n \in \mathbb{Z}} Z^{(\sigma + s)^2} (-1)^{4s} s^n \prod_{\epsilon = \pm 1} (uq^{2n}; q^{-1}, q|Z), \]  \( (3.22) \)

First, we prove the convergence of the conformal blocks \( \mathcal{F}(u; q^{-1}, q|Z) \to \mathcal{F}(\sigma^2; |z|) \). For each summand in (3.1) we have

\[ N_{\lambda, \mu}(q^{-1}, u) \sim (-h)^{|\lambda| + |\mu|} \prod_{x \in \lambda} (2\sigma - (a_\lambda(x) + b_\lambda(x) + 1)) \times \prod_{x \in \mu} (2\sigma + (a_\mu(x) + b_\mu(x) + 1)). \]

This occasional power of \( h \) cancels due to the definition \( Z = h^2 z^2. \)

Next, we prove that the convergence of the series (3.1) is uniform on \( q \in [1 - \epsilon, 1] \).

Note that in this region \( \left| \frac{a x - a x^2}{q - q^2} \right| \geq x, \) for \( x \geq 1, \) so \( L_1 \) (from the proof of proposition 3.1) is uniformly bounded by \( L_1 \geq 1, \) and for \( L_2 \) using the evident inequalities

\[ |q^\gamma - q^{-\gamma}| \geq |q^{Re\gamma} - q^{-Re\gamma}|, \quad |q^{a_n} - q^{-a_n}| \geq |q^{a_2} - q^{-a_2}|, \quad \left| \frac{q_1^{a_1} - q_1^{-a_1}}{q_1^{1/2} - q_1^{-1/2}} \right| \leq \left| \frac{q_2^{a_2} - q_2^{-a_2}}{q_2^{1/2} - q_2^{-1/2}} \right|, \]

for \( a_1 > a_2 > 0 \) and \( 0 < q_1 < q_2 < 1, \) \( 0 < a < 1/2. \) Therefore we have bound
\[ L^{1/2}_2 \geq \frac{|(1 - \epsilon)^{Re\sigma + n_0/2} - (1 - \epsilon)^{Re\sigma - n_0/2}|}{(1 - \epsilon)^{1/2} - (1 - \epsilon)^{-1/2}}, \]

where \(-1/4 < Re\sigma + n_0/2 < 1/4\). Since the bounds for \(L_1, L_2\) are uniform we prove that the convergence of (3.1) is also uniform.

Next we study coefficients in (3.22). Using formula (A.6) we can rewrite

\[ Z^{(\sigma+n)^2}_{\sum_{\ell=1}^\infty ((u\epsilon q^{2n})^\ell q, q)^{\infty}} = \frac{1}{G(1 + 2(\sigma + n); q)G(1 - 2(\sigma + n); q)} \left( \frac{Z}{(1 - q)^2} \right)^{(\sigma+n)^2}. \]

Then using theorem A.1 and \(z \sim q^{1/2} \sim z\) we see that the coefficients in (3.22) go to the coefficients in the \(\tau\) function (1.3).

It remains to show that the series (3.22) converges uniformly on \(q \in [1 - \epsilon, 1]\). This can be done in a similar way to the series (3.1) above. \(\square\)

It follows from this theorem, theorem 3.1 and proposition 2.2 that the \(q \to 1\) limit of the bilinear relations (3.4) is (3.5). This fact was stated at the end of section 3.1.

Note that in their paper Bonelli, Grassi and Tanzini [7] also constructed the function, which in the limit \(q \to 1\) goes to the \(\tau(\sigma, s)z\). It is interesting to note that they work in a different region of \(q\), namely \(|q| = 1\) in their paper. The relation between our results and [7] will be studied in [9].

3.4. The \(q\)-deformation of the Painlevé III(\(D_8\)) algebraic solution

For special values of the parameters \(\sigma = 1/4, s = \pm 1\), the sum in formula (1.3) can be calculated (see [6, section 3.3])

\[ \tau(1/4, \pm 1|z) = \frac{1}{G(1/2)G(3/2)}z^{1/16}e^{\pm 4\pi i/2}. \] (3.24)

These \(\tau\) functions correspond to the algebraic solutions of the Painlevé III(\(D_8\)) equation (1.1) \(w(z) = \mp z^{1/2}\). These solutions are invariant under the Bäcklund transformation, so \(\tau(\sigma, s)|z| \propto \tau(\sigma + 1/2, s)|z|\) and substituting this into (1.4) we obtain equation

\[ 1/2D^2_{[\log z]}(\tau(z), \tau(z)) = \mp z^{1/2}\tau(z)^2. \] (3.25)

This equation gives us (3.24) up to multiplying on a constant and any power of \(z\).

We want to obtain an analogous relation on the \(q\)-deformed conformal blocks. Recall that element \(\pi^2 \in W\) is an analog of the Bäcklund transformation and acts as \(T(u, s; q|Z) \mapsto T(uq, s; q|Z)\). Therefore if the \(\tau\) function \(\tau(u, s; q|Z)\) is a \(u\)-inverse invariant, then for \(u = q^{1/2}, s = \pm 1\) we have

\[ \pi^2(T(q^{1/2}, \pm 1; q|Z)) = T(q^{1/2}, \pm 1; q|Z) = \pm T(q^{-1/2}, \pm 1; q|Z) = \pm T(q^{1/2}, \pm 1; q|Z), \] (3.26)

where we used (3.12).

The corresponding function \(G(z) = \mp z^{1/2}\) due to (2.5) and (2.9). These \(G(Z)\) are algebraic (and Bäcklund invariant) solutions of (2.4).
**Conjecture 3.3.** For any \( u \)-inverse invariant \( \tau \) function \( T(u, s; q|Z) \) we have

\[
T(q^{1/2}, \pm 1; q|Z) = \frac{C(q^{1/2}; q|Z)}{(q^{1/2}; q)_\infty (q^{1/2}; q)_\infty} (\mp Z^{1/2}q^{1/2}; q^{1/2}, q^{1/2})_\infty. \tag{3.27}
\]

Equivalently, we have the relation on the \( q \)-deformed conformal blocks

\[
(\mp Z^{1/2}q^{1/2}; q^{1/2}, q^{1/2})_\infty = \sum_{n \in \mathbb{Z}} (\mp 1)^n Z^{n+\epsilon/2} P_n(q)f(q^{2n+\epsilon/2}, q, q|Z), \tag{3.28}
\]

where

\[
P_n(q) = \prod_{\epsilon = \pm 1} (q^{2n+\epsilon/2} q; q)_\infty \prod_{j=0}^{k-1} \frac{1}{((1 - q^{j+1/2})(1 - q^{-j-1/2}))^{k-j}}, \tag{3.29}
\]

where \( k = 2n \), for \( n > 0 \) and \( k = -2n - 1 \), for \( n < 0 \).

One can compare (3.28) with (3.20) and see that \( P_n(q) = (-1)^n P_n(q^{1/2}, q) \).

**Theorem 3.3.** If conjecture 3.1 holds then conjecture 3.3 also holds.

**Proof.** Due to the theorem 3.1 we can use conjecture 3.2 instead of conjecture 3.1. Using (3.26) we can rewrite equation (3.14) as

\[
Z^{1/4}T(q^{1/2}, \pm 1; q|qZ) T(q^{1/2}, \pm 1; q^{-1}qZ) = (1 \pm Z^{1/2}) T(q^{1/2}, \pm 1; q|Z)^2. \tag{3.30}
\]

On the other hand, using (3.18) we have

\[
T(q^{1/2}, \pm 1; q|qZ) = C(q^{1/2}; q|Z) \sum_{n \in \mathbb{Z}} (\mp 1)^n Z^{n+\epsilon/2} \frac{f(q^{2n+\epsilon/2}, q, q|Z)}{((uq^{2n+\epsilon}) q; q)_\infty},
\]

where we used the \( u \)-inverse invariance of \( C(u; q|Z) \). Denote

\[
f(z) = T(q^{1/2}, \pm 1; q|qZ)(q^{3/2}; q, q)_\infty(q^{1/2}; q, q)_\infty C(q^{1/2}; q|Z)
\]

Then \( f(z) \) is equal to the right side of (3.28), and due to (3.9) the equation (3.30) for this function takes the form

\[
f(qZ)f(q^{-1}Z) = (1 \pm Z^{1/2})f(Z)f(Z).
\]

The only solution \( f(Z) = \sum_{n \geq 0} f_n Z^{n/2} \) with \( f_0 = 1 \) of the last equation is \( f(Z) = (\mp Z^{1/2}q^{1/2}; q^{1/2}, q^{1/2})_\infty \).

We have checked the equality (3.28) up to \( Z^4 \).

**Remark 3.6.** Let us check the continuous limit of (3.27). We want to use theorem 3.2 so we set \( C(u; q|Z) = C(u; q|Z) \). Then we divide (3.27) by \( \Gamma(\mp(qZ)^{1/4}; q^{1/4}, q^{1/4})/(q; q)_\infty^2 \) and get \( T_C \) on the left side (see (2.13)). Using theorem 3.2 we have that if \( Z = h^{1/4}z, q = e^t, q \to 1 \) then \( T_C(q^{1/2}, \pm 1; q|Z) \to \tau(1/4, \mp 1|z) \).

For the continuous limit of the \( q \)-Pochhammer symbol on the right side of (3.27) we have (using (A.3))
(\pm Z^{1/2} q^{1/2}, q^{1/2}) = \exp \left(- \sum_{m=1}^{\infty} \frac{(-\pm Z^{1/2} q^{1/2})^m}{m(1 - q^{m/2})^2} \right) \sim e^{\pm q^{1/2}}, \quad (3.32)

where we observed that only the term of the sum with \( m = 1 \) survives. Using this result and theorem A.1 we obtain that the limit (3.27) is a known relation (3.24) (after the analytic continuation around \( Z = 0 \) as in section 2.3).

4. Further questions

- The main statements of the paper are based on the conjecture 3.1, so the first question is to prove it.
- The second question is about the generalization of our results to the other discrete Painlevé equations from Sakai’s tables in figures 1 and 2. It is natural to conjecture that the \( q \)-difference Painlevé equations \( A_{1}^{(1)} q/E_{N+1}^{(1)} \) surface/symmetry type \( N \leq 7 \) are related to the Nekrasov partition functions for 5d SU(2) gauge theory with \( N \) fundamental multiplets.

It was argued by Seiberg [28] that these gauge theories have \( E_{N+1} \) global symmetry. It would be interesting to find a physical interpretation of the affine Weyl groups \( E^{(1)} \), the symmetry group of the corresponding discrete Painlevé equations.

For \( N \leq 4 \), one can take the \( q \rightarrow 1 \) limit and get the relation between the differential Painlevé equation with the surface type \( D_{1}^{(1)}, D_{2}^{(1)}, D_{3}^{(1)}, D_{4}^{(1)} \) and the Nekrasov partition functions for 4d SU(2) gauge theory with \( 4,3,2,1,0 \) fundamental multiplets correspondingly. This relation was stated in [12] and proven in [16] and [5].

- Elements \( T, \pi_{2} \in W \) act on the \( \tau \) function \( \mathcal{T}(u, s; q|Z) \) defined in (3.6) in a clear manner:

\[
T : \mathcal{T}(u, s; q|Z) \mapsto \mathcal{T}(u, s; q|qZ), \quad \text{and} \quad \pi_{2} : \mathcal{T}(u, s; q|Z) \mapsto \mathcal{T}(uq, s; q|Z).
\]

It is natural to ask for the action of the whole group \( W \). From the table 1 we see that the remaining transformations are \( Z \mapsto Z^{-1} \) and \( q \mapsto q^{-1} \). The second transformation is transparent due to the \( q \)-deformed conformal block relation (3.3) and the \( q \) Pochhammer symbol relation (A.4). However, it is unclear what the meaning of the \( Z \mapsto Z^{-1} \) symmetry for the formula (3.6) is (if it exists). In particular, it is unclear what the form of the corresponding relation between the \( q \)-deformed conformal blocks (Nekrasov partition functions for pure theory) \( \mathcal{F}(Z) \) and \( \mathcal{Z}(Z^{-1}) \) is.

One more remark is in order: the Nekrasov partition function for pure 5d SU(2) gauge theory is equal (up to a simple factor) to the topological string partition function for the local \( \mathbb{P}^{3} \times \mathbb{P}^{1} \) geometry [10, 15]. For such partition functions there exists a fiber-base duality [20], which interchanges two factors \( \mathbb{P}^{1} \). In terms of the functions \( \mathcal{F}(u; q^{-1}, q|Z^{-1}) \) this duality has the form

\[
\frac{\mathcal{F}(u; q^{-1}, q^2|Z)}{(uq; q, q^2)_{\infty}} = \frac{\mathcal{F}(uZ; q^{-1}, q|Z^{-1})}{(uZq; q, q^2)_{\infty}}, \quad (4.1)
\]

However, this is an equality of the formal power series on variables \( u, uZ \). For the \( |q| = 1 \) function \( \mathcal{F}(u; q^{-1}, q|Z) \) cannot be expanded, as such a convergent series has poles for \( u = q^{-\alpha} \), and these poles accumulate near \( u = 0 \). Moreover, computer calculations show that (4.1) does not hold for the function \( \mathcal{F}(u; q^{-1}, q|Z) \) defined by the convergent
series (3.1). It is an interesting question whether the relation of the type (4.1) exists for \(|q| = 1\), and whether it can be used for the \(Z \mapsto Z^{-1}\) transformation of the \(\tau\)-function \(T(u, s; q|Z)\).

- Continuous Painlevé equations can be described as a non-autonomous Hamiltonian system. It is natural to ask for the difference analog of this fact. Painlevé III\((D_8)\) has two types of bilinear form, namely, Toda-like and Okamoto-like (see [6]). In section 2.3 we give the \(q\)-deformation of the Toda-like bilinear form; it is natural to ask for the \(q\)-deformation of the Okamoto-like equations.

- It is interesting to note that there exists another \(q\)-difference equation which has the Painlevé III\((D_8)\) equation in the \(q \to 1\) limit. This equation has the form

\[
\prod_{i} \prod_{k} \left(1 - Z - Z^{-1}\right) \prod_{i} \prod_{k} \left(1 - Z - Z^{-1}\right) = Z(1 - W)
\]  

and the limit was shown in [14] (see also [13]). In terms of the Sakai classification, this equation corresponds to the \(A_7^{(1)}\) surface (see [27, equation (2.44)]), so it should not be equivalent to (2.4). It is an interesting question whether there exists a way to express solutions of (4.2) in terms of \(q\)-deformed conformal blocks.

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Appendix A. \(q\)-special functions

In this appendix we collect some facts about the \(q\)-series that we used in the paper. For the references see [1, section 10], [29, 32].

The infinite multiple \(q\)-deformed Pochhammer symbol is defined by

\[
(Z; t_1, \ldots t_N)_\infty = \prod_{i_1, \ldots, i_N = 0}^{\infty} \left(1 - Z \prod_{k=1}^{N} t_k^{i_k}\right).
\]  

(A.1)

The product exists if all \(|t_k| < 1\). The function is symmetric with respect to \(t_k\). In this region, the function is the analytic function of all the arguments. The infinite \(q\)-Pochhammer symbols satisfy

\[
(Z; t_1, \ldots t_N)_\infty / (Z_1; t_1, \ldots t_N)_\infty = (Z_1 t_2, \ldots t_N)_\infty, \quad (Z; q)_\infty / (Z q; q)_\infty = 1 - Z.
\]  

(A.2)
The function \((Z; t_1, \ldots, t_N)_\infty\) can be rewritten as

\[
(Z; t_1, \ldots, t_N)_\infty = \exp\left(\sum_{t_1, \ldots, t_N=0}^{\infty} \log \left| 1 - Z \prod_{k=1}^{N} t_k^i \right| \right) = \exp\left(-\sum_{m=1}^{\infty} \sum_{k=1}^{N} \frac{Z^m}{m} \prod_{k=1}^{N} t_k^i \right)
\]

(A.3)

where the sum converges when \(|Z| < 1, |t_k| \neq 1\). Using this expression, the function \((Z; t_1, \ldots, t_N)_\infty\) can be defined to the region with some \(t_k\) greater than 1. Using this definition we see that

\[
(Z; t_1^{-1}, t_2, \ldots, t_N)_\infty = (Z; t_1, \ldots, t_N)_\infty^{-1}
\]

(A.4)

In the paper we use only \(N = 1, 2\) -Pochhammer symbols.

Introduce the trigonometric (or \(q\)-) Gamma function and the trigonometric (or \(q\)-) Barnes \(G\) function according to [32]

\[
\Gamma(x; q) = (1 - q)^{1-x} \frac{(q; q)_\infty}{(q^x; q)_\infty},
\]

(A.5)

\[
\mathcal{G}(x; q) = (1 - q) \frac{(1 - \frac{1}{2}(x-2))_\infty \prod_{k=0}^{\infty} \left(1 - q^{k+1}^{k+1+2-x} \right)}{\prod_{k=0}^{\infty} \left(1 - q^{k+1}^{k+2-x} \right)} = (1 - q) \frac{(1 - \frac{1}{2}(x-2))_\infty (q; q)_\infty^{1-x}}{(q; q)_\infty}.
\]

(A.6)

From (A.2) we have

\[
\Gamma(u + 1; q) = [u]^q_\infty \Gamma(u; q), \quad \text{where} \quad [u]^q_\infty = \frac{1 - q^u}{1 - q}
\]

(A.7)

\[
\mathcal{G}(u + 1; q) = \Gamma(u; q) \mathcal{G}(u; q).
\]

(A.8)

The function \([u]^q_\infty\) is called the \(q\)-number.

We use part of theorem 4.4 from [32]

Theorem A.1. As \(q \to 1\), \(\Gamma(u; q)\) and \(\mathcal{G}(u; q)\) converge to \(\Gamma(u)\) and \(\mathcal{G}(u)\). Convergence is uniform on any compact set in the domain \(\mathbb{C} \setminus \mathbb{Z}_{\leq 0}\).

Introduce the elliptic Gamma function (see e.g. [29])

\[
\Gamma(Z; t, q) = \frac{(tqZ^{-1}; t, q)_\infty}{(Z; t, q)_\infty}.
\]

(A.9)

The elliptic gamma function should not be confused with the trigonometric gamma and Barnes \(G\)-function with one deformation argument and with the standard Gamma function and Barnes \(G\)-function without deformation arguments. The elliptic Gamma function satisfies relations

\[
\Gamma(qZ; t, q) = \theta(Z; t)\Gamma(Z; t, q), \quad \Gamma(tZ; t, q) = \theta(Z; q)\Gamma(Z; t, t, q).
\]

(A.10)

where the \(\theta\) function is defined as

\[
\theta(Z, q) = \frac{1}{(q; q)_\infty} \sum_{k \in \mathbb{Z}} (-1)^k q^{k(k-1)/2} Z^k = (Z; q)_\infty (qZ^{-1}; q)_\infty.
\]

(A.11)
Here, the last equality is the Jacobi triple product. The shift relations (A.10) on the elliptic Gamma functions could easily be obtained from (A.2). It follows from the definition that the $\theta$ function satisfies
\[
\theta(qZ; q) = -Z^{-1}\theta(Z; q) = \theta(Z^{-1}; q).
\] (A.12)

From (A.10) and the first equality (A.12) we obtain the useful relation
\[
\frac{\Gamma(uq, q; q)^{-1}; q, q)}{\Gamma(u; q, q)^2} = -qu^{-1}.
\] (A.13)

**Appendix B. The bilinear relation for the generic $q_1, q_2$**

The bilinear relations on the $q$-deformed conformal blocks exist not only in the case $q_1 = q^{-1}, q_2 = q$. In order to write the conjecture we introduce the bilinear combination
\[
\mathcal{F}_d(u, q_1, q_2; Z) = \sum_{2n \in \mathbb{Z}} \left( \frac{u^{2dn}(q_1q_2Z)^{2n}}{(uq_1^{-1}, q_1^{-1}q_2^{-1/2}, q_2^{-1/2})_{\infty}}(uq_1^{-1/2}, q_2^{-1/2})_{\infty}uq_1^{-1}q_2^{-1/2}u^{-1}q_1^{-1}q_2^{-1/2}u^{2n}(q_1q_2Z)\right),
\] (B.1)

where we use the notations
\[
\mathcal{F}_n^{(1)}(z) = \mathcal{F}(uq_1^{2n}, q_1^{1/2}, q_2^{1/2}z), \quad \mathcal{F}_n^{(2)}(z) = \mathcal{F}(uq_1^{-1}, q_1^{-1/2}, q_2^{-1/2}z),
\]
\[
(uq_1^{-2n-1}, u^{-1}q_1^{-2n-3/2}, q_2^{-1/2})_{\infty} = (uq_1^{-1}, q_1^{-1/2}, q_2^{-1/2})_{\infty}(u^{-1}q_1^{-2n-3/2}, q_1^{-1}, q_1^{-1/2}, q_2^{-1/2})_{\infty},
\]
\[
(uq_1^{-1/2}, q_2^{2n+1/2}, u^{-1}q_1^{-1/2}, q_2^{-1/2})_{\infty} = (uq_1^{-1/2}, q_2^{2n+1/2}, q_1^{-1/2}, q_2^{-1/2})_{\infty}(u^{-1}q_1^{-1/2}, q_1^{-1/2}, q_2^{-1/2})_{\infty},
\]

and ensure the conditions $|q_2| < 1 < |q_1|$ (in other sectors one can use (A.4)).

We will also use the following version of the Nekrasov partition function
\[
\mathcal{F}_0(u, q_1, q_2; Z) = \sum_{\lambda, \mu} Z^{\frac{|\lambda| + |\mu|}{2}} \prod_{i,j=1}^{\infty} N_{\lambda, \mu}(q_1, q_2, u/uj),
\] (B.2)

where
\[
N_{\lambda, \mu}(q_1, q_2, u) = \prod_{s \in \lambda} (1 - uq_2^{-1}\ell_{\alpha}(s) - q_1^{-1}\ell_{\alpha}(s)) \cdot \prod_{s \in \mu} (1 - uq_2^{-1}\ell_{\alpha}(s) - q_1^{-1}\ell_{\alpha}(s)),
\] (B.3)

with $\lambda^0 = \{s \in \lambda | \delta_0(s) + 1 \equiv 0 \mod 2\}, \mu^0 = \{s \in \mu | \delta_0(s) + 1 \equiv 0 \mod 2\}$ (see (3.1) and (3.2)).

**Conjecture B.1.** The following relation holds
\[
\mathcal{F}_0(u, q_1, q_2; Z) = \mathcal{F}_0(u, q_1, q_2; Z), \quad \mathcal{F}_0(u, q_1, q_2; Z) = (1 - q_2Zq^{-1/2})\mathcal{F}_0(u, q_1, q_2; Z).
\] (B.4)

As a combination of two formulas in (B.4), one gets a generalization of the conjecture 3.1 to the case $q_1q_2 = 1$. The formulas (B.4) were checked by a computer calculation up to $Z^3$.

The geometrical meaning of the first equality in (B.4) is the following: the left side and the right side are generating functions of the equivariant Euler characteristics of sheaf $\mathcal{O}$ for the Nakajima quiver varieties for the quiver $A_{11}$. Another interpretation of the corresponding
manifolds is the partial compactification of the instantons on the minimal resolution of $\mathbb{C}^2/\mathbb{Z}_2$. However, these compactifications are different as $(\mathbb{C}^*)^2$ manifolds, and in terms of quiver varieties, the stability parameters belong to different chambers. Nevertheless the conjectural formula (B.4) states that the corresponding equivariant Euler characteristics are the same. In the conformal (or cohomology, or just $\text{uni}$) limit the corresponding statement was made in [3, section 4.1] (see also [17, 23]).

The second equality in (B.4) can be viewed as a blow-up equation, similar to the one studied in paper [21]. Its conformal limit coincides with the first two formulas in [5, equation (4.22)] (using an appropriate version of the AGT relation [4, 8]).

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