On the possibility of a complex 4-dimensional space-time manifold

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ABSTRACT

The possibility of a complex 4-dimensional space-time manifold is suggested. This may imply the existence of a matter wave.
1. Dirac theory:

To describe the motion of a free electron, Dirac (Dirac 1958) introduced \((\alpha_1\alpha_2\alpha_3\beta)\) or equivalently \((\gamma^\mu, \mu = 1, 2, 3, 4)\) (Bathe 1986) into the theory. The \(\gamma'\)s are independent of the \(x'_\mu\)s and the \(p'\)s, where \(x\) and \(p\) are the space-time coordinates and the momentum-energy of the particle. Dirac pointed out that the \(\gamma'\)s describe some new degree of freedom, belonging to some internal motion in the electron, so that they must commute with the \(x'\)s and the \(p'\)s. The basis state vector is represented by two separated pieces of components, a space-time part \((x_\mu)\) and a spin part which is a complex spinor. In this theory, the quantity \(j^\mu = \bar{\Psi}\gamma^\mu\psi\) is a Lorentz 4-vector. Since the \(\gamma'\)s are independent of the \(x'\)s and the \(p'\)s, so \(j^\mu\) is also independent of the \(x'\)s and \(p'\)s.

In mechanics, the infinitesimal generators for the \(x'\)s are the dynamical conjugate variables \(p'\)s, and for rotational angular displacements \(\theta'_i\)s the angular momentum. The infinitesimal unitary transformation in spin space can be written as

\[
U = 1 + \sum \frac{1}{2} \sigma_i \cdot \xi_i, \tag{1}
\]

where the \(\sigma'\)s are the Pauli matrices and the \(\xi'\)s the analogue quantities of the \(\theta'_i\)s in spin-space. The relations between the \(\sigma'\)s and the \(\gamma'\)s are given as (Bethe 1986)

\[
(\gamma^\mu) = (\beta\alpha_i, \beta), \quad (i = 1, 2, 4) \tag{2}
\]

\[
\alpha_i = \rho_1 \sigma_i,
\]

where \(\rho_1\) is given in (Dirac 1958). The \(\gamma'\)s are linear combinations of the \(\sigma'_i\)s. Consider the \(\gamma'\)s as a set of dynamical variables \(\left(\frac{d}{dt}\gamma \neq 0\right)\), their conjugate variables should
have \((\eta_1\eta_2\eta_3)\) as the analogue of \((\theta_1\theta_2\theta_3)\) plus a 4th-component \(\eta_4\). The \(\theta'\)s are in the spin-space so they are independent of the \(x'\)s and the \(p'\)s, therefore \(\eta_1, \eta_2\) and \(\eta_3\) are also independent of the \(x'\)s and the \(p'\)s. It is reasonable to assume that \(\eta_4\) is also independent of the \(x'\)s and the \(p'\)s since the \(\gamma'\)s describe some new degree of freedom which are internal in the particle, their dynamical conjugate variables should also be related to variables which are internal.

Now, we can see clearly that the basis state vector of the particle is 8-dimensional. However, nature should prefer simple and symmetry. Instead of having two separated parts of basis space, we should rather have one whole complete piece of continuous basis space. That is, it is natural to suggest a complex 4-dimensional space-time manifold.

The above consideration is based only on the dimensionality of Dirac theory and does not depend on the content of the theory. In fact, we can use Schrödinger-Pauli equation (Dirac 1958) where the basic variables are \((xyzt\sigma_1\sigma_2\sigma_3)\), where the \(\sigma'\)s operate on the spin-space. The infinitesimal unitary transformation in spin space has the form as equation (1) and thus we have three conjugated variables similar to \((\xi_1\xi_2\xi_3)\). Now, if we wish the theory to be Lorentz covariant, we have to generalize the three \(\sigma'\)s to a 4-component quantity which is similar to the \(\gamma'\)s. Therefore we again have four more variables besides \((xyzt)\).

2. Matter wave: Consider a complex 4-dimensional space-time manifold, to de-
scribe the motion of a free particle, there must exist a well-behaved function \( \Psi(T = t + i\bar{t}, X = x + i\bar{x}) \) which is continuous and differentiable in the domain under consideration. The simplest form of such a function can be written as

\[
\Psi = \Psi(T) \cdot \Psi(X), \tag{3}
\]

\[
T = t + i\bar{t}, \tag{4}
\]

\[
X = x + i\bar{x}. \tag{5}
\]

\( \Psi(T) \) and \( \Psi(X) \) must satisfy the Cauchy-Riemann conditions (Churchill 1948) for an analytic function. Write \( \Psi(X) \) as

\[
\Psi(X) = u(x, \bar{x}) + iv(x, \bar{x}). \tag{6}
\]

Since \( \Psi(X) \) is analytic in some region of the \( X \) plane, we have

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial \bar{x}^2} = 0,
\]

\[
\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial \bar{x}^2} = 0. \tag{7}
\]

Then we can also write

\[
c_1 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial \bar{x}^2} \right) + c_2 \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial \bar{x}^2} \right) = 0, \tag{8}
\]

where \( c_1 \) and \( c_2 \) are constants. Assume that nature does not have special preference between real and imaginary parts except the \( i \). Take \( c_1 = 1 \) and \( c_2 = i \), we get

\[
\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial \bar{x}^2} = 0. \tag{9}
\]
Since $x$ and $\bar{x}$ are basically independent, we must be able to describe the $x$-part and the $\bar{x}$-part of the function separately. That is, we have $\Psi(x, i\bar{x}) = \Psi(x)\Psi(i\bar{x})$. Then the solution of equation (9) has the form
\[ \Psi(X) = Ae^{\pm ax} \cdot e^{\pm b\bar{x}}, \quad a^2 + b^2 = 0. \] (10)

where $A, a$ and $b$ are constants. We take $\Psi(X) = Ae^{\pm ikX} = Ae^{\pm ik(x+i\bar{x})}$. Such that $\Psi$ is analytic.

Similarly we find $\Psi(T)$ to be $e^{\pm i\omega T}$, therefore we have
\[ \Psi(X, T) = Ae^{\pm i(kX - \omega t)}. \] (11)

With respect to our direct experimental measurements, we can take only real space-time variables. The projection of $\Psi(X, T)$ onto the real space-time region gives us $\Psi(x, t) = Ae^{i(kx - \omega t)}$, the constants $k$ and $\omega$ are identified as the wave number and angular frequency. Up to this point, we can see that the free particle we wish to describe is a free wave. This may offer a possible solution to the question that why a particle has to be a wave as well.

Finally, if the space-time is complex, there may have a possibility that those physically undetectable particles may stay in the imaginary part of the nature.

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