Abstract

In this letter we analyze two local extensions of a model introduced some time ago to obtain a path integral formalism for Classical Mechanics. In particular, we show that these extensions exhibit a nonrelativistic local symmetry which is very similar to the well known $\kappa$-symmetry introduced in the literature almost 20 years ago. Differently from the latter, this nonrelativistic local symmetry gives no problem in separating 1st from 2nd-class constraints.

1 Introduction

The dynamics of relativistic superparticles \cite{1} has been deeply analyzed in the last 20 years because of the profound relation between these simple systems and the more realistic models of supersymmetric field theories and strings. Almost 20 years ago an important symmetry of the massless supersymmetric particle was discovered by Siegel \cite{2}. This symmetry, which was also found in superstrings and D-branes, allows to gauge away half of the fermionic degrees of freedom involved in the formalism and has been analyzed in detail in many following papers \cite{3}-\cite{4}. In particular, a lot of work has been done to understand the geometry of the constraints and to solve the problem of quantizing the system. In fact it is not trivial to quantize the massless superparticle (as well as superstrings and D-branes) because, due to the presence of the $\kappa$-symmetry, 1st-class and 2nd-class constraints cannot be separated covariantly; many attempts have been performed to solve this problem \cite{3}-\cite{5}.

In this letter we continue the analysis (see Ref.\cite{7}) of the symmetries of a model introduced some time ago to describe Classical Mechanics in terms of path integrals. This model possesses a universal global supersymmetry generated by two charges $Q_H$ and $\overline{Q}_H$. Here we focus on two other fermionic charges which we call $D_H$ and $\overline{D}_H$, which are strictly related to $Q_H$ and $\overline{Q}_H$. In fact in superspace $D_H$ and $\overline{D}_H$ are represented by the covariant derivatives associated to the Susy charges mentioned above. Following the lines of Ref.\cite{7} we make these two symmetries ($D_H$ and $\overline{D}_H$) local and we note that the new nonrelativistic local Susy we get is very similar to the famous $\kappa$-symmetry introduced by Siegel. The main difference with respect to the latter
becomes manifest after imposing the invariance under local time reparametrization, as one does in Siegel's model. In fact, in our nonrelativistic framework, there is no difficulty in separating 1st-class from 2nd-class constraints, simply because no 2nd-class constraint survives after imposing the invariance under local reparametrizations of time.

There are two simple ways to make local the symmetries $D_\xi$ and $\overline{D}_\xi$ above. The two models we obtain are two gauge theories which differ in the physical Hilbert space. We show that one model selects, as physical states, only the distributions built up with the constants of motion only, while the other is more restrictive and selects only the Gibbs distributions of the canonical ensemble.

2 The $\kappa$-symmetry

The model studied by Siegel [2] for the massless relativistic superparticle is characterized by the following (1st order) action:

$$S = \int d\tau \left\{ p_\mu \left[ \dot{x}^\mu - \frac{i}{2} \left( \overline{\zeta} \gamma^\mu \zeta - \dot{\zeta} \gamma^\mu \dot{\zeta} \right) \right] - \frac{1}{2} \lambda p^2 \right\},$$

where $x^\mu$ are $n$-dimensional space-time coordinates, $\zeta^a$ and $\overline{\zeta}_a$ are Dirac spinors and $\lambda$ is a Lagrange multiplier introduced to implement the $p^2 = 0$ constraint. This action is invariant under the following transformations:

**\(\tau\)-reparametrization (local)**

\[\delta x^\mu = \epsilon \dot{x}^\mu; \quad \delta p_\mu = \epsilon \dot{p}_\mu; \quad \delta \lambda = (\epsilon \dot{\lambda})\]

\[\delta \zeta = \epsilon \dot{\zeta}; \quad \delta \overline{\zeta} = \epsilon \dot{\overline{\zeta}};\]

**Supersymmetry (global)**

\[\delta x^\mu = \frac{i}{2} \left( \overline{\tau} \gamma^\mu \zeta - \overline{\zeta} \gamma^\mu \overline{\tau} \right); \quad \delta p_\mu = 0; \quad \delta \lambda = 0;\]

\[\delta \zeta = \varepsilon; \quad \delta \overline{\zeta} = \overline{\varepsilon};\]

**$\kappa$-symmetry (local)**

\[\delta x^\mu = \frac{i}{2} \left( \overline{\zeta} \gamma^\mu p \kappa - \overline{\kappa} p \gamma^\mu \zeta \right); \quad \delta p_\mu = 0; \quad \delta \lambda = 2i \left( \overline{\zeta} \kappa - \kappa \overline{\zeta} \right);\]

\[\delta \zeta = p \kappa; \quad \delta \overline{\zeta} = \overline{\kappa} p.\]

In (3) the dot means derivation with respect to $\tau$ and $p'$ is obviously $p_\mu \gamma'^\mu$. As specified above, $\epsilon$ and $\kappa, \overline{\kappa}$ are local parameters (the first is a commuting scalar, the others are anticommuting spinors) while $\varepsilon$ and $\overline{\varepsilon}$ are two global (i.e. they do not depend on the base space $\tau$) spinorial parameters. We are particularly interested in the structure of the third symmetry, which has been deeply analyzed in the literature. Here we want to give a pedagogical description of the structure of the transformation in phase space, and we want to highlight the role of the various operators and various commutation structures (Dirac Brackets) involved. This will turn out to be useful when we will analyze the analog of the $\kappa$-symmetry in Classical Mechanics.
First of all we notice that the first and third symmetries above are strictly related. In fact, if we introduce a mass $m$ in (1) turning the $p^2 = 0$ constraint into $p^2 - m^2 = 0$, we get

$$S_m = \int d\tau \left\{ p_\mu \left[ i\gamma^\mu \dot{\zeta} - \frac{i}{2} \left( \gamma^\mu \dot{\zeta} - \dot{\zeta} \gamma^\mu \zeta \right) \right] - \frac{1}{2} \lambda (p^2 - m^2) \right\}.$$  (5)

$S_m$ is still invariant under (3) but the other two symmetries are lost. This is easy to see in phase space if we apply the Dirac procedure to the actions (1) and (5). Consider first the massive model. The constraints are the following:

$1^\text{st}-\text{Class}$

$$\begin{align*}
\Pi_\lambda &= 0 \quad (a) \\
p^2 - m^2 &= 0 \quad (b)
\end{align*}$$

$2^\text{nd}-\text{Class}$

$$\begin{align*}
\Pi^\mu &= 0 \quad (c) \\
(\Pi_\sigma)_\mu - p_\mu &= 0 \quad (d) \\
D^a &= (\Pi_\zeta)^a + \frac{i}{2}(\not{p} \zeta)^a = 0 \quad (e) \\
\overline{D}_a &= (\Pi_\bar{\zeta})_a + \frac{i}{2}((\not{p} \bar{\zeta})_a = 0 \quad (f),
\end{align*}$$

where $\Pi(...)$ are the momenta conjugated to the variables indicated as $(...)$, which satisfy the following (graded) Poisson Brackets:

$$\begin{align*}
[\lambda, \Pi_\lambda]_- &= 1; \\
[x^\mu, p_\nu]_- &= \delta^\mu_\nu; \\
[\zeta^a, (\Pi_\zeta)_b]_+ &= \delta_a^b; \\
[\bar{\zeta}^a, (\Pi_\bar{\zeta})^b]_+ &= \delta_a^b.
\end{align*}$$  (7)

The first thing to do is to construct the Dirac Brackets associated to the $2^\text{nd}$-class constraints. If we define the matrix

$$\Delta_{ij} = [\phi_i, \phi_j]_{PB}$$  (8)

where $\phi_k$ are the second class constraints, then the Dirac Brackets between two generic variables $A, B$ of phase space are defined as:

$$[A, B]_{DB} = [A, B]_{PB} - [A, \phi_i]_{PB}(\Delta^{-1})^{ij}[\phi_j, B]_{PB}.$$  (9)

Once we have built the correct structure in phase space, it is not difficult to realize that the generators of the global supersymmetry are the following operators:

$$Q = \not{p} \zeta; \quad \overline{Q} = \bar{\zeta} \not{p};$$  (10)

which reproduce precisely the transformations if we define:

$$\delta(\ldots) \equiv [\ldots, \bar{\epsilon} Q - \bar{\epsilon} \overline{Q} \epsilon]_{DB}.$$  (11)

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1. Here and in the sequel we choose right derivatives for Grassmannian variables: $\Pi_\zeta := \frac{\partial L}{\partial \dot{\zeta}}$.
2. In the sequel we shall omit the subscripts $+$ and $-$. 
Note that the minus sign in the RHS of the previous equation is chosen because of the anticommuting character of the parameter $\varepsilon$. Moreover we have:

$$[Q, \overline{Q}]_{DB} = ip\zeta$$

which confirms that $Q$ and $\overline{Q}$ are two supersymmetry charges. Notice that we can induce the same SUSY-transformations through the following operators:

$$Q' = i\Pi\zeta + \frac{1}{2}p\zeta; \quad \overline{Q}' = i\Pi\zeta + \frac{1}{2}p\zeta$$

which is obvious because $Q \approx Q'$ and $\overline{Q} \approx \overline{Q}'$ in the Dirac sense.

Let us now switch to the massless case (1). The main difference is that we cannot repeat all the steps of the previous analysis. In fact the new constrain $p^2 = 0$ implies that the matrix $\Delta$ of Eq.(8) is no longer invertible. This is due to the fact that $\det \Delta \propto \det(p) = p^\mu p_\mu = 0$. Thus the construction of the Dirac Brackets is not as simple as in the massive case. In fact half of the constraints in Eqs.(6c) and (6d) are now 1st-class while the other half remains 2nd-class and the separation of the two sets is not quite easy (see for example Refs.[5]). Nevertheless we can list the generators of the $\kappa$-transformations of Eq.(4):

$$K = i\pi D = i\pi \Pi\zeta - \frac{1}{2}p^2 \zeta; \quad \overline{K} = i\overline{D}p' = i\Pi\zeta p' - \frac{1}{2}\overline{\zeta}p^2.$$  

($K$ and $\overline{K}$ generate the transformation (4) through commutators like those in (11).) Obviously we should remember that $(K, \overline{K})$ are not a set of independent constraints, as we explained before, because $p'$ is not invertible on the shell of the constraints. Note that we can write down the form of the generators $K, \overline{K}$ even if we do not know exactly the form of the Dirac Brackets in this particular case. We can do that because the $K, \overline{K}$ constraints commute (weakly) with all the constraints in (6c) and (6d) and therefore we have $[K, (\ldots)]_{DB} \approx [K, (\ldots)]_{PB}$ (and the same holds for $\overline{K}$) whatever are the surviving 2nd-class constraints determining the Dirac Brackets at hand.

3 The Functional Approach To Classical Mechanics.

In this section we shall briefly review the path integral approach to Classical Mechanics which was originally developed in Ref.[8]. The idea originated from the fact that whenever a theory has an operatorial formulation, it also possesses a corresponding path integral. Now Classical Mechanics (CM) does have an operatorial formulation [4] and therefore it is reasonable to look for the corresponding path integral formalism. The strategy to build this Classical Path Integral (CPI) is simple. In CM we have a 2n-dimensional phase space $M$ whose coordinates we denote by $\varphi^a (a = 1, \ldots, 2n)$, i.e.: $\varphi^a = (q^1, \ldots, q^n; p^1, \ldots, p^n)$, and we indicate with $H(\varphi)$ the Hamiltonian of the system. Then, the equations of motion have the form:

$$\dot{\varphi}^a = \omega^{ab} \frac{\partial H}{\partial \varphi^b} \equiv \omega^{ab} \partial_b H(\varphi) \quad \omega^{ab} = \text{symplectic matrix}.$$
The classical kernel (i.e. the probability for the system to be in the configuration $\varphi_f$ at time $t_f$ if it was in the configuration $\varphi_i$ at time $t_i$) has the following expression:

$$K_{cl}(f|i) = \delta(\varphi^a_f - \varphi^a_{cl}(t_f|\varphi_i, t_i))$$  \hspace{1cm} (16)

where $\varphi^a_{cl}(t|\varphi_i, t_i)$ is the classical trajectory at time $t$ (that is the solution of the Hamilton equations) having $\varphi_i$ as initial condition at time $t_i$. Since $K_{cl}(f|i)$ is a classical probability we can rewrite it as follows:

$$K_{cl}(f|i) = \sum_{k_i} K_{cl}(f|k_{N-1})K_{cl}(k_{N-1}|k_{N-2}) \cdots K_{cl}(k_1|i)$$ \hspace{1cm} (17)

where in the first equality $k_i$ denotes formally an intermediate configuration $\varphi_{k_i}$ between $\varphi_i$ and $\varphi_f$ and in the last equality the symbol $\tilde{\delta}$ represents a functional Dirac delta. The last formula in (17) is already a path integral but we can give it a more familiar form if we rewrite the Dirac delta as:

$$\tilde{\delta}[\varphi^a - \varphi^a_{cl}] = \tilde{\delta}[\varphi^a - \omega^{ab}\partial_b H] \det[\delta^a_b \partial_t - \omega^{ac}\partial_c \partial_b H]$$  \hspace{1cm} (18)

where we have used the functional analog of the relation $\delta[f(x)] = \frac{\delta[x - x_i]}{\delta x_i}$. Next (see Ref. for details) we can exponentiate both terms of the RHS of Eq.\,(18) via a Lagrange multiplier $\lambda$ (the first term) and a couple of Grassmannian variables $(c, \overline{c})$ (the second term). What we finally get is the following expression:

$$K_{cl}(f|i) = \int D\varphi^a D\lambda_a Dc^a D\overline{c}_a \exp \left[ i \int dt \tilde{\mathcal{L}} \right]$$  \hspace{1cm} (19)

where $\tilde{\mathcal{L}}$ is the Lagrangian characterizing the CPI:

$$\tilde{\mathcal{L}} = \lambda_a [\dot{\varphi}^a - \omega^{ab}\partial_b H] + i\overline{c}_a [\delta^a_b \partial_t - \omega^{ac}\partial_c \partial_b H]c^b,$$  \hspace{1cm} (20)

and the $8n$ variables $(\varphi^a, \lambda_a, c^a, \overline{c}_a)$ form the new enlarged phase space which we denote by $\tilde{\mathcal{M}}$. It is easy to Legendre transform the Lagrangian $\tilde{\mathcal{L}}$ and obtain the corresponding Hamiltonian:

$$\tilde{\mathcal{H}} = \lambda_a \omega^{ab}\partial_b H + i\overline{c}_a \omega^{ac}(\partial_c \partial_b H)c^b.$$  \hspace{1cm} (21)

From the path integral (19) we can easily derive the following commutator structure:

$$\left[ \varphi^a, \lambda_b \right] = i\delta^a_b \quad ; \quad \left[ c^a, \overline{c}_b \right] = \delta^a_b. \quad (\text{all others are zero})$$  \hspace{1cm} (22)
Via these commutators we can realize the $\lambda$ and $\tau$ variables as differential operators:

$$
\lambda_a = -i \frac{\partial}{\partial \varphi^a}; \quad \tau_a = \frac{\partial}{\partial c^a}
$$

and these in turn can be used to construct the operatorial version of the Hamiltonian (21):

$$
\tilde{H} \equiv -i\omega^{ab}\partial_b H \frac{\partial}{\partial \varphi^a} - i\omega^{ab}\partial_b H c^d \frac{\partial}{\partial c^a}
$$

and the corresponding “Schrödinger-type” equation for the probability density $\rho(\varphi, c; t)$:

$$
\tilde{H}\rho(\varphi, c; t) = i \frac{\partial}{\partial t} \rho(\varphi, c; t).
$$

For a nice interpretation of the geometry of the formalism we refer the reader to Ref. [10]. For our purposes here it is sufficient to say that $\tilde{H}$ has a very precise geometrical meaning, being the Lie derivative along the Hamiltonian vector field $h \equiv \omega^{ab}\partial_b H \partial_a$.

We end this brief review with some remarks about the symmetries of the Lagrangian (20) and the Hamiltonian (21). It is easy to check that they are both invariant under the supersymmetry transformations generated by the following operators:

$$
Q_H = Q_{BRS} - \beta N_H = ic^a \lambda_a - \beta c^a \partial_a H
$$
$$
Q_{\varphi} = Q_{BRS} + \beta \varphi = ic^a \omega^{ab} \tau_b + \beta \varphi \omega^{ab} \partial_b H.
$$

(\beta is a dimensional parameter). It is also not difficult to represent all the formalism developed so far on a suitable superspace composed by the time $t$ and two Grassmannian partners $\theta$ and $\bar{\theta}$. We refer the reader to Ref. [8] for all the details. For our purposes it is sufficient to say that we can introduce a classical superfield

$$
\Phi^a(t, \theta, \bar{\theta}) = \varphi^a + \theta c^a + \bar{\theta} \omega^{ab} \tau_b + i \theta \theta \omega^{ab} \lambda_b.
$$

on which the susy charges (26)(27) and the Hamiltonian (21) act as

$$
Q_H = -\frac{\partial}{\partial \theta} - \beta \bar{\theta} \frac{\partial}{\partial t}; \quad \bar{Q}_H = \frac{\partial}{\partial \bar{\theta}} + \beta \theta \frac{\partial}{\partial t}; \quad \tilde{t} = i \frac{\partial}{\partial t};
$$

It is also easy to work out the covariant derivatives associated to $Q_H$ and $\bar{Q}_H$:

$$
D_H = -i \frac{\partial}{\partial \theta} + i \beta \bar{\theta} \frac{\partial}{\partial t}; \quad \bar{D}_H = i \frac{\partial}{\partial \bar{\theta}} - i \beta \theta \frac{\partial}{\partial t};
$$

which correspond (in $\tilde{\mathcal{M}}$) to the following operators:

$$
D_H = iQ_{BRS} + i \beta N_H \quad \bar{D}_H = i\bar{Q}_{BRS} - i \beta \bar{N}_H,
$$

where $Q_{BRS}, \bar{Q}_{BRS}, N_H$ and $\bar{N}_H$ are defined in Eqs. (26) and (27).

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3 Here we use the same notation as in Ref. [8].

4 According to the formula: $Q^a(t, \theta, \bar{\theta}) \equiv [\Phi^a(t, \theta, \bar{\theta}), \epsilon Q]$. 
4 \( \kappa \)-symmetry and CPI

In the previous Section we have shown that the formalism of the Classical Path Integral exhibits a universal global Supersymmetry. However, differently from the model of Siegel, it does not possess any local invariance. If we want to build up a nonrelativistic analog of the model introduced in Section 1, we first must inject the local \( t \)-reparametrization invariance into the Lagrangian (20) by adding the corresponding constraint via a Lagrange multiplier \( g \):

\[
\tilde{L}_1 \equiv \tilde{L} + g\tilde{H}. \tag{32}
\]

In fact it is easy to see that the previous Lagrangian is locally invariant under

\[
\begin{align*}
\delta(\ldots) &= \left[\ldots, \epsilon(t)\tilde{H}\right] \\
\delta g &= -i\dot{\epsilon}(t). \tag{33}
\end{align*}
\]

Here and in the sequel \( \ldots \) denotes any one of the variables \((\varphi^a, \lambda_b, c^a, \bar{c}_b)\). Moreover it is easy to check that it remains globally invariant under the \( N = 2 \) classical Susy of Eqs.(26)(27). Nevertheless, in this simple model no other local symmetry is present. If we want to complete the analogy, we must add (following the lines of Ref.[7]) two further constraints to the Lagrangian (32) and we get:

\[
\tilde{L}_2 \equiv \tilde{L} + \xi D_H + \overline{\xi D_H} + g\tilde{H}. \tag{34}
\]

In the previous equation \( D_H \) and \( \overline{D_H} \) are the operators introduced in Eq.(31). We want to analyze this model following the same steps we used in Section 1 for the Lagrangian (1).

First of all we remember again that, in our non-relativistic case, the analog of the “\( p^2 = 0 \)” constraint is represented by the term \( g\tilde{H} \) in (34) which produces the constraint \( \tilde{H} = 0 \). Thus, as we did in Eq.(5), we start our analysis by releasing this constraint in the following way:

\[
\tilde{L}_2' \equiv \tilde{L} + \xi D_H + \overline{\xi D_H} + g(\tilde{H} - \overline{\tilde{E}}), \tag{35}
\]

which is the analog of Eq.(1). It should be remembered that \( \overline{\tilde{E}} \) is not the energy of the system, but just a parameter related to the invariance under local time reparametrization: if \( \overline{\tilde{E}} = 0 \) this symmetry is present, while if \( \overline{\tilde{E}} \neq 0 \) this symmetry is lost.

One can immediately work out the constraints:

\[
\begin{align*}
1^{st}\text{-Class} & \quad \left\{ \Pi_\xi = \Pi_\tau = \Pi_\varphi = 0; \right. \\
& \quad \left. \tilde{H} - \overline{\tilde{E}} = 0; \right. \\
2^{nd}\text{-Class} & \quad \left\{ D_H = 0; \right. \\
& \quad \left. \overline{D_H} = 0. \right. \tag{36}
\end{align*}
\]

Now we can compare the previous constraints with those in Eq.(1). Concerning the \( 1^{st}\)-class constraints, we notice that \( \tilde{H} - \overline{\tilde{E}} = 0 \) is the classical analog of the relativistic mass-shell constraint \( p^\mu p_\mu - m^2 = 0 \). This implies that \( \Pi_\varphi = 0 \) plays the same role as \( \Pi_\lambda = 0 \) in the relativistic case, while the remaining two constraints (\( \Pi_\xi = 0 \) and \( \Pi_\tau = 0 \)) have no analog in
the relativistic case. Consider now the 2nd-class constraints. The first thing to point out is that
\(D_H = 0\) and \(\overline{D}_H = 0\) are precisely the classical analogs of \(D^a = 0\) and \(\overline{D}_b = 0\) in the relativistic case. We can say that because \(D_H\) and \(\overline{D}_H\) are related to the classical Susy charges \(Q_H\) and \(\overline{Q}_H\) in the same way in which \(D^a\) and \(\overline{D}_b\) are related to the relativistic Susy charges \(Q^a\) and \(\overline{Q}_b\). In fact it is easy to see that in the relativistic framework \(D^a\) and \(\overline{D}_b\) commute with \(Q^a\) and \(\overline{Q}_b\) and \([D^a, \overline{D}_b] = [Q^a, \overline{Q}_b] = i\gamma^b\) in the same way in which, in the nonrelativistic context, \(D_H\) and \(\overline{D}_H\) commute with \(Q_H\) and \(\overline{Q}_H\), and \([D_H, \overline{D}_H] = [Q_H, \overline{Q}_H] = 2i\beta \tilde{H}\). This is actually the heart of the analogy. We start from a model which possesses a universal SUSY generated by \(Q_H\) and \(\overline{Q}_H\) and we want to check whether it is possible to implement a classical analog of the relativistic \(\kappa\)-symmetry of Siegel. Since in the relativistic case the 2nd-class constraints are \(D^a = 0\) and \(\overline{D}_b = 0\), we have modified the CPI-Lagrangian (20) in such a way that the resulting extension provides as 2nd-class constraints the classical analogs of \(D^a\) and \(\overline{D}_b\), that is \(D_H\) and \(\overline{D}_H\). This is precisely the model (35).

If we go on with the same steps as in Section 1 we find that the matrix \(\Delta_{ij} = [\phi_i, \phi_j]\) has the form:

\[
\Delta = \begin{pmatrix}
0 & 2i\beta \tilde{H} \\
2i\beta \tilde{H} & 0
\end{pmatrix} \implies \Delta^{-1} = \begin{pmatrix}
0 & (2i\beta \tilde{H})^{-1} \\
(2i\beta \tilde{H})^{-1} & 0
\end{pmatrix}
\]

(37)

and consequently the Dirac Brackets deriving from (36) are:

\[
[A, B]_{DB} = [A, B] - [A, \overline{D}_H] (2i\beta \tilde{H})^{-1} [D_H, B] - [A, D_H] (2i\beta \tilde{H})^{-1} [\overline{D}_H, B].
\]

(38)

Now that we have the correct structure of our phase space we can proceed with the analogy with the relativistic case. First of all we can prove that the two supersymmetry charges \(Q_H\) and \(\overline{Q}_H\) introduced in Eqs.(26)(27) become weakly equal to the \(Q_{BRS}\) and \(\overline{Q}_{BRS}\) charges:

\[
Q_H \approx 2Q_{BRS} = 2ic^a\lambda_a; \quad (39)
\]

\[
\overline{Q}_H \approx 2\overline{Q}_{BRS} = 2i\overline{c}^a\omega^{ab}\lambda_b; \quad (40)
\]

and consequently:

\[
[Q_{BRS}, \overline{Q}_{BRS}]_{DB} = \frac{1}{4} [Q_H, \overline{Q}_H]_{DB} = i\beta \overline{\tilde{H}}. \quad (41)
\]

This shows that \(Q_H\) and \(\overline{Q}_H\) are, more precisely, the analogs of the charges \(Q' \overline{Q}'\) of Eq.(13) while the \(Q_{BRS}\) and \(\overline{Q}_{BRS}\) charges are analogous to the \(Q\) and \(\overline{Q}\) charges of Eq.(10).

Consider now the case in which \(\tilde{E} = 0\). We get down to the Lagrangian (34) and we see that something happens which is similar to the mechanism of \(\kappa\)-symmetry discussed in Section 1. In fact in that case we saw that half of the 2nd-class constraints became 1st-class. Here,

\[\text{This is not in contradiction with what we said few lines above, that is that } Q_H \text{ and } \overline{Q}_H \text{ are the nonrelativistic analogs of } Q^a \text{ and } \overline{Q}_b. \text{ In fact it should be remembered that on the shell of the contraints we have } Q \approx Q', \overline{Q} \approx \overline{Q}' \text{ (in the relativistic case) and } Q_H \approx 2Q_{BRS}, \overline{Q}_H \approx 2\overline{Q}_{BRS} \text{ (in the nonrelativistic case).}\]
on the other hand, we notice that both the 2nd-class constraints $D_H = \overline{D}_H = 0$ become 1st-class. This can be easily seen if one remembers that $[D_H, \overline{D}_H] = 2i\beta \mathcal{H} \approx 0$ because now the constraint $\tilde{\mathcal{H}} - \mathcal{E} = 0$ has turned into $\tilde{\mathcal{H}} = 0$. In other words all the constraints in the model (34) are gauge constraints and contribute to restrict the space of the physical states. Therefore we see that in our nonrelativistic framework there is no difficulty in separating 1st-class from 2nd-class constraints (like in the relativistic case). This is simply due to the fact that no 2nd-class constraint remains after imposing the constraint $\tilde{\mathcal{H}} = 0$ (which is the classical analog of $p_\mu p^\mu = 0$).

Proceeding with the analogy it is very easy to construct the CPI-analogs of $K$ and $\overline{K}$ of Eq.(14), that is the generators of the nonrelativistic $\kappa$-symmetry. They are simply:

$$K_{NR} = \mathcal{H} D_H; \quad \overline{K}_{NR} = \mathcal{H} \overline{D}_H;$$  (42)

("NR" stands for "Non Relativistic") and the local transformations (under which the Lagrangian (34) is invariant) generated by $K_{NR}$ and $\overline{K}_{NR}$ are:

$$\begin{cases}
\delta(\ldots) = [(\ldots), \chi(t)K_{NR} + \overline{\chi}(t)\overline{K}_{NR}] \\
\delta \xi = -i \dot{\xi} \mathcal{H} \\
\delta \overline{\xi} = -i \overline{\chi} \mathcal{H} \\
\delta g = 2i \beta (\overline{\chi} \chi + \chi \overline{\chi}) \mathcal{H}.
\end{cases}$$  (43)

It is interesting to determine the physical states selected by the theory defined by Eq.(34). Since all the constraints are now 1st-class, we must impose them strongly on the states as follows:

$$\Pi_\xi \rho(\varphi, c, \xi, \overline{\xi}, g) = \Pi_\overline{\xi} \rho(\varphi, c, \xi, \overline{\xi}, g) = \Pi_g \rho(\varphi, c, \xi, \overline{\xi}, g) = 0; \quad (44)$$

$$D_H \rho(\varphi, c, \xi, \overline{\xi}, g) = 0; \quad (45)$$

$$\overline{D}_H \rho(\varphi, c, \xi, \overline{\xi}, g) = 0; \quad (46)$$

$$\mathcal{H} \rho(\varphi, c, \xi, \overline{\xi}, g) = 0; \quad (47)$$

and it is not difficult to prove that the resulting (normalizable\footnote{This could be expected somehow, because here we have only two 2nd-class constraints and consequently it cannot happen that only half of these become 1st-class, like it happens in the Siegel model. In fact, if this were the case, we would remain with an odd number (that is 1) of 2nd-class constraints which is absurd because this number must always be even.} physical states have the following form:

$$\rho(\varphi, c, \xi, \overline{\xi}, g) \propto \exp[-\beta H(\varphi)].$$  (48)

This is precisely the Gibbs distribution characterizing the canonical ensemble, provided we interpret the $\beta$ constant of Eqs.(26)(27) as $(k_B T)^{-1}$, where $T$ plays the role of the temperature at which the system is in equilibrium. In fact we should remember that up to now the dimensional

\footnote{Also a state of the form $\rho(\varphi, c) \propto \exp[\beta H(\varphi)] c^1 c^2 \ldots c^{2n}$ would be admissible, but it is not normalizable in $\varphi$.}
parameter $\beta$ introduced in Eqs. (26) and (27) has not been restricted by any constraint. It is a completely free parameter with a dimension of (Energy)$^{-1}$ which characterizes the particular $N = 2$ classical supersymmetry. The canonical Gibbs state made its appearance earlier in the context of the CPI and precisely in Ref. [14]. There it was shown that, in the pure CPI model (20), the zero eigenstates of $\tilde{H}$ which are also Susy-invariant are precisely the canonical Gibbs states. In our model instead we have obtained the Gibbs states as the entire set of physical states associated to the gauge theory described by the Lagrangian (34).

However the model (34), though interesting for the peculiar physical subspace it determines, is not the nonrelativistic Lagrangian which is closest to the Siegel model. We mean that one should remember that the Lagrangian (34) gives rise to a canonical Hamiltonian of the form:

$$\tilde{H}_2 \equiv \tilde{H} - \xi D_H + \xi D_H - g\tilde{H},$$

but on the other hand we have already checked that the two couples of operators $(Q_H, \bar{Q}_H)$ and $(D_H, \bar{D}_H)$ close on $\tilde{H}$ and not on $\tilde{H}_2$. Therefore, if we want to construct a more precise nonrelativistic analog of the model of Siegel, we should consider a slightly modified version of the Lagrangian (34) which is:

$$\tilde{L}_3 \equiv \tilde{L} + \dot{\xi} D_H + \dot{\xi} D_H + g(\tilde{H} - \tilde{E})$$

One can easily check that the Lagrangian (50) yields, a part from a factor $(1 - g)$, the same Hamiltonian as the CPI. Therefore we can proceed following the same steps as before: we turn the $\tilde{H} = 0$ constraint into $\tilde{H} - \tilde{E} = 0$

$$\tilde{L}_3' \equiv \tilde{L} + \dot{\xi} D_H + \dot{\xi} D_H + g(\tilde{H} - \tilde{E})$$

and we find out that the new constraints are:

$$\begin{aligned}
&1^{\text{st}}\text{-Class} \quad \begin{cases} 
\Pi_\gamma = 0; \\
\tilde{H} - \tilde{E} = 0;
\end{cases} \\
&2^{\text{nd}}\text{-Class} \quad \begin{cases} 
\Pi_\xi + D_H \equiv D_H' = 0; \\
\Pi_\xi + \bar{D}_H \equiv \bar{D}_H' = 0.
\end{cases}
\end{aligned}$$

Then, it is easy to check that we can repeat all the considerations we did below Eq. (36), if we replace $D_H$ and $\bar{D}_H$ with $D_H'$ and $\bar{D}_H'$. As a second remark, we notice that the two constraints $\Pi_\xi = \Pi_\xi = 0$, which had no analog in the relativistic context, have now disappeared. Moreover, because $\left[D_H', \bar{D}_H\right] = \left[D_H, \bar{D}_H\right] = 2i\beta\tilde{H}$, we have also that the Dirac Brackets remain the same as those in Eq. (38), which lead to Eqs. (39)-(41). Again, when we put $\tilde{E} = 0$, we obtain that the two $2^{\text{nd}}$-class constraints $D_H' = \bar{D}_H = 0$ become both $1^{\text{st}}$-class, differently from the relativistic case. However, the two models described by the two Lagrangians (34) and (50) are not equivalent. There are basically two differences. The first is the new form of the nonrelativistic $\kappa$-symmetry which now reads:

$$\begin{aligned}
\delta(\ldots) &= \tilde{H}(\ldots), \kappa(t) D_H + \bar{\kappa}(t) \bar{D}_H \approx [\ldots, \kappa(t) K_{NR} + \bar{\kappa}(t) \bar{K}_{NR}] \\
\delta \xi &= [\xi, \kappa(t) K_{NR} + \bar{\kappa}(t) \bar{K}_{NR}] = -i \kappa \tilde{H} \\
\delta \bar{\xi} &= [\bar{\xi}, \kappa(t) K_{NR} + \bar{\kappa}(t) \bar{K}_{NR}] = -i \bar{\kappa} \tilde{H} \\
\delta g &= 2i \beta \tilde{H}(\dot{\xi} \kappa + \dot{\xi} \bar{\kappa}).
\end{aligned}$$

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where “$\approx$” is understood in the Dirac sense and

$$K'_{NR} \equiv \widetilde{H}D'_H; \quad K''_{NR} \equiv \widetilde{H}\overline{D}'_H.$$  \hspace{1cm} (54)

The second difference, which is the most important, is represented by the two physical spaces associated to the two models (34) and (50). In fact, we have already seen that the physical states associated to the first model are the Gibbs distributions $\rho(\varphi) \propto \exp(-\beta H(\varphi))$; on the other hand, the physical states determined by the Lagrangian (50) must obey the following conditions:

$$\Pi_g \rho(\varphi, c, \xi, \bar{\xi}, g) = 0 \ ; \quad \partial_{\bar{\xi}} \rho(\varphi, c, \xi, \bar{\xi}, g) = 0 \ ; \quad (\partial_{\xi} + D_H) \rho(\varphi, c, \xi, \bar{\xi}, g) = 0 ; \quad (55)$$  

$$\widetilde{H} \rho(\varphi, c, \xi, \bar{\xi}, g) = 0 \ ; \quad D'_H \rho(\varphi, c, \xi, \bar{\xi}, g) = \left(-i\partial_{\xi} + D_H\right) \rho(\varphi, c, \xi, \bar{\xi}, g) = 0 . \quad (56)$$

It is not difficult to realize that the solution of Eqs. (55)-(56) has the form:

$$\rho(\varphi, c, \xi, \bar{\xi}, g) \propto \exp\left(-i\xi D_H - i\xi D_H\right) \tilde{\rho}(\varphi, c) , \quad (57)$$

where

$$\widetilde{H} \tilde{\rho}(\varphi, c) = 0 , \quad (58)$$

which implies that $\tilde{\rho}(\varphi, c)$ is a function of constants of motion only. Therefore we can say that the physical states associated to the Lagrangian (50) are isomorphic to the functions $\tilde{\rho}(\varphi, c)$ which are annihilated by the Hamiltonian $\widetilde{H}$ and are consequently constants of motion. Obviously the Gibbs distributions are a subset of them. This allows us to claim that the model (50) is actually more general than that characterized by the Lagrangian (34). More precisely the theory described by (50) is equivalent to that characterized by the Lagrangian (32). In fact it is easy to see that the physical Hilbert space associated to the latter is characterized by the distributions $\tilde{\rho}(\varphi, c, g)$ obeying to the constraints:

$$\frac{\partial}{\partial g} \tilde{\rho}(\varphi, c, g) = 0; \quad \widetilde{H} \tilde{\rho}(\varphi, c, g) = 0; \quad (59)$$

and the physical space is precisely the same as that in (58), which is isomorphic to that determined by Eqs. (53)- (54).

5 Conclusions

In this paper we have analyzed two local versions of a model introduced some years ago to describe Classical Mechanics in terms of path integrals. In particular, we have built two non-relativistic models which exhibit a universal local supersymmetry which is very similar to the famous $\kappa$-symmetry introduced almost 20 years ago by Siegel. Differently from the relativistic case, in our non-relativistic framework the constraint $\widetilde{H} = 0$, which is analogous to the relativistic $p^2 = 0$, promotes to $1^{st}$-class all the $2^{rd}$-class constraints present in the case in which
\( \tilde{\mathcal{H}} = \tilde{E} \neq 0 \). Consequently there is no difficulty in treating the constraints, differently from what happened in the relativistic case. In our first model the physical states of the theory turn out to be the Gibbs distributions characterizing the canonical ensemble, while in the second one the physical Hilbert space is formed by all the generic functions of the constants of motion of the theory.

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