WHICH MULTIPLIER ALGEBRAS ARE $W^*$-ALGEBRAS?

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Abstract. We consider the question of when the multiplier algebra $M(A)$ of a $C^*$-algebra $A$ is a $W^*$-algebra, and show that it holds for a stable $C^*$-algebra exactly when it is a $C^*$-algebra of compact operators. This implies that if for every Hilbert $C^*$-module $E$ over a $C^*$-algebra $A$, the algebra $B(E)$ of adjointable operators on $E$ is a $W^*$-algebra, then $A$ is a $C^*$-algebra of compact operators.

Also we show that a unital $C^*$-algebra $A$ which is Morita equivalent to a $W^*$-algebra must be a $W^*$-algebra.

1. Introduction

The main theme of this paper is around the question of when the multiplier algebra $M(A)$ of a $C^*$-algebra $A$ is a $W^*$-algebra? For separable $C^*$-algebras, it holds exactly when $A$ is a $C^*$-algebra of compact operators [2, Theorem 2.8]. For general $C^*$-algebras, we get two partial results in this direction. First we give an affirmative answer for stable $C^*$-algebras and deduce that if for every Hilbert $C^*$-module $E$ over $A$, the algebra $B(E)$ of adjointable operators on $E$ is a $W^*$-algebra, then $A$ is a $C^*$-algebra of compact operators. This is related to our question (with a much stronger assumption) as for $E = A$ with its canonical Hilbert $A$-module structure, $B(E) = M(A)$. Second we show that if $M(A)$ is Morita equivalent to a $W^*$-algebra, then it is a $W^*$-algebra. This is also related to our question, as if $A$ is a $C^*$-algebra of compact operators, then $M(A)$ is a $W^*$-algebra.

The two partial answers take into account the notions of Hilbert $C^*$-algebras and Morita equivalence which are somewhat historically related. In 1953, Kaplansky introduces Hilbert $C^*$-modules to prove that derivations of type I $AW^*$-algebras are inner. Twenty years later, Hilbert $C^*$-modules appeared in the pioneering work of Rieffel [19],

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where he employed them to study (strong) Morita equivalence of $C^*$-algebras. Paschke studied Hilbert $C^*$-modules as a generalization of Hilbert spaces [16].

Hilbert $C^*$-modules and Hilbert spaces differ in many aspects, such as existence of orthogonal complements for submodules (subspaces), self duality, existence of orthogonal basis, adjointability of bounded operators, etc. However, when $\mathcal{A}$ is a $C^*$-algebra of compact operators, then Hilbert $\mathcal{A}$-modules behave like Hilbert spaces in having the above properties. Indeed these properties characterize $C^*$-algebras of compact operators [5, 10, 14, 20].

2. $C^*$-algebras of compact operators

In this section we give some characterizations of $C^*$-algebras of compact operators using properties of multiplier algebras. We also show that these are characterized as $C^*$-algebras $\mathcal{A}$ for which the algebra $\mathcal{B}(E)$ of all adjointable operators is a $W^*$-algebra, for any Hilbert $\mathcal{A}$-module $E$.

Definition 2.1. A $C^*$-algebra $\mathcal{A}$ is called a $C^*$-algebra of compact operators if there exists a Hilbert space $H$ and a (not necessarily surjective) $*$-isomorphism from $\mathcal{A}$ to $K(H)$, where $K(H)$ denotes the space of compact operators on $H$.

This is exactly how Kaplansky characterized $C^*$-algebras that were dual rings [11, Theorem 2.1, p. 222] (see also [1]).

Theorem 2.2. For a $C^*$-algebra $\mathcal{A}$, the following are equivalent:

(i) $\mathcal{A}$ is a $C^*$-algebra of compact operators.

(ii) The strict topology on the unit ball of $M(\mathcal{A})$ is the same as the strong$^*$-topology (viewing $M(\mathcal{A}) \subseteq \mathcal{A}^{**}$, the second dual of $\mathcal{A}$).

Proof. Assume that (i) holds. Then $\mathcal{A} \cong c_0 \bigoplus \alpha K(H_\alpha)$. Let $a_\beta \to 0$ in the strict topology of the unit ball of $M(\mathcal{A}) \cong \ell^\infty \bigoplus \alpha B(H_\alpha)$. Without loss of generality, we may assume that $a_\beta \geq 0$, for all $\beta$. Let $\eta \in \bigoplus \alpha H_\alpha$ be a unit vector with $\eta_\alpha = 0$ except for finitely many $\alpha$. Let $p_\alpha$ be the rank one projection onto the non-zero $\eta_\alpha$ and $p_\alpha = 0$, otherwise. Then $p = \sum p_\alpha \in \mathcal{A}$, thus $\|a_\beta p\| \to 0$. Therefore $\|a_\beta \eta\| \to 0$, and the same holds for any $\eta$ in the unit ball of $\bigoplus \alpha H_\alpha$, as $\{a_\beta\}$ is norm bounded. Hence $a_\beta \to 0$ in the strong$^*$ topology.

Conversely if $a_\beta \geq 0$ and $a_\beta \to 0$ in the strong$^*$ topology. As above, for any rank one projection $p \in \mathcal{A}$, $\|a_\beta p\| = \|pa_\beta\| \to 0$. Thus $p$ can be replaced by any finite linear combination of such minimal projections, and this set is dense in $\mathcal{A}$. Since $\{a_\beta\}$ is norm bounded, $a_\beta \to 0$ in the strict topology. This shows that (i) implies (ii).
Now assume that (ii) holds. By [2, Theorem 2.8], we need only to prove that $M(A) = A^{**}$. For any positive element $b$ in the unit ball of $A^{**}$, there is a net $\{a_\beta\}$ in the unit ball of $A$ that converges to $b$ in strong* topology. Thus the net is strong* Cauchy, and hence convergent in the strict topology to an element of $M(A)$, as $M(A)$ is the completion of $A$ in the strict topology [9, Theorem 3.6]. Therefore $b \in M(A)$, and we are done. □

Another characterization of $C^*$-algebras of compact operators could be obtained as a non unital version of the following result of J.A. Mingo in [15], where he investigates the multipliers of stable $C^*$-algebras.

**Lemma 2.3.** Suppose that $H$ is a separable infinite dimensional Hilbert space and $A$ is a unital $C^*$-algebra such that the multiplier algebra $M(A \otimes K(H))$ is a $W^*$-algebra. Then $A$ is a finite dimensional $C^*$-algebra.

We recall that a projection $p$ in a $C^*$-algebra $A$ is called finite dimensional if $pAp$ is a finite dimensional $C^*$-algebra. To prove a non unital version of Mingo’s result, we need some lemmas. The first lemma is well-known, see for instance [4, Corollary 1.2.37].

**Lemma 2.4.** If $A$ is a $C^*$-algebra and $p$ is a projection in the multiplier algebra $M(A)$, then $M(pAp) \cong pM(A)p$, as $C^*$-algebras.

**Lemma 2.5.** Let $H$ be a Hilbert space and $A$ be a $C^*$-algebra. If $A \otimes K(H)$ is $C^*$-algebra of compact operators, then so is $A$.

**Proof.** Suppose not. Then there is an element $b \in A^+$ such that the spectral projection $\xi_1(b)$ of $b$ corresponding to $\{1\}$ is not finite dimensional in $A$. Let $q$ be a one-dimensional projection in $K(H)$. Then $(b \otimes q)^n$ is a decreasing sequence in the unit ball of the $C^*$-algebra $A \otimes K(H)$ of compact operators. By Theorem 2.2 it converges strictly, hence (because it is decreasing) in norm to $\xi_1(b) \otimes q \in A$. Because $A \otimes K(H)$ is a $C^*$-algebra of compact operators, the projection $\xi_1(b) \otimes q$ must be finite rank, but

$$(\xi_1(b) \otimes q)(A \otimes K(H))(\xi_1(b) \otimes q) = \xi_1(b)A\xi_1(b) \otimes qK(H)q,$$

and the dimension of $\xi_1(b)A\xi_1(b)$ is not finite by our assumption about $b$. □

The next theorem is known for separable $C^*$-algebras [2], here we prove it with separability replaced by stability.

**Theorem 2.6.** If $A$ is a stable $C^*$-algebra such that the multiplier algebra $M(A)$ is a $W^*$-algebra, then $A$ is a $C^*$-algebra of compact operators.
Proof. In order for the $C^*$-algebra $\mathcal{A}$ to be a $C^*$-algebra of compact operators, it is necessary and sufficient that every positive element in $\mathcal{A}$ can be approximated by a finite linear combination of finite dimensional projections. Let $a$ be a positive element in $\mathcal{A}$ and $0 \leq a \leq 1$. Since the multiplier algebra $M(\mathcal{A})$ is a $W^*$-algebra, we can define $p \in M(\mathcal{A})$ as the spectral projection of $a$, corresponding to an interval of the form $[s, t]$ where $0 < s < t$. It suffices to show that $pA p$ is finite dimensional.

Let $g : [0, 1] \to [0, 1]$ be a continuous function vanishing at 0, such that $g(r) = 1$ for all $r \in [s, t]$. Then $g(a) \in A$ and $g(a)p = p$. Hence $p \in A$.

Now let $H$ be a separable infinite dimensional Hilbert space. Since $\mathcal{A}$ is a stable $C^*$-algebra, $M(\mathcal{A} \otimes K(H))$ is a $W^*$-algebra and by Lemma 2.4,

$$M(pAp \otimes K(H)) = M((p \otimes 1)(A \otimes K(H))(p \otimes 1)) = (p \otimes 1)M(A \otimes K(H))(p \otimes 1)$$

is a $W^*$-algebra. Therefore by Lemma 2.3, $p$ is finite rank. □

The non unital version of the Mingo’s lemma follows.

**Corollary 2.7.** Suppose that $H$ is a separable infinite dimensional Hilbert space and $\mathcal{A}$ is a $C^*$-algebra such that the multiplier algebra $M(\mathcal{A} \otimes K(H))$ is a $W^*$-algebra, then $\mathcal{A}$ is a $C^*$-algebra of compact operators.

**Proof.** Since $\mathcal{A} \otimes K(H)$ is stable, it is a $C^*$-algebra of compact operators, and so is $\mathcal{A}$ by Lemma 2.5. □

It is well known that if $\mathcal{A}$ is a $W^*$-algebra and $E$ is a selfdual Hilbert $\mathcal{A}$-module, then $B(E)$ is a $W^*$-algebra. The converse is not true, as for $E = \mathcal{A} = c_0$, $B(E) = \ell^\infty$ is a $W^*$-algebra [10]. However, if $\mathcal{A}$ is a $C^*$-algebra of compact operators on some Hilbert space, then $B(E)$ is a $W^*$-algebra, for every Hilbert $\mathcal{A}$-module $E$ [6]. Here we show the converse.

Recall that the $C^*$-algebra $K(E)$ of compact operators on $E$ is generated by rank one operators $\theta_{\xi, \eta}(\zeta) = \xi\langle \eta, \zeta \rangle$, for $\xi, \eta \in E$, and the multiplier algebra $M(K(E))$ is isomorphic to $B(E)$. Also, if $H$ is a separable infinite dimensional Hilbert space, then $E = H \otimes \mathcal{A}$ is a Hilbert $C^*$-module over $\mathbb{C} \otimes \mathcal{A} = \mathcal{A}$, denoted by $H_\mathcal{A}$. It plays an important role in the theory of Hilbert $C^*$-modules.

**Theorem 2.8.** For any $C^*$-algebra $\mathcal{A}$, the following are equivalent:

(i) $\mathcal{A}$ is a $C^*$-algebra of compact operators,

(ii) $B(E)$ is a $W^*$-algebra, for each Hilbert $\mathcal{A}$-module $E$,

(iii) $B(H_\mathcal{A})$ is a $W^*$-algebra.
Proof. It is enough to show that (iii) implies (i). Since
\[ K(H_A) = K(H \otimes A) \cong K(H) \otimes K(A) = K(H) \otimes A \]
we have \( B(H_A) \cong M(K(H) \otimes A) \). By assumption, \( B(H_A) \) is a \( W^* \)-algebra and so \( A \) is a \( C^* \)-algebra of compact operators by Corollary 2.7.

\[ \square \]

J. Schweizer in [20] remarked that for a \( C^* \)-algebra \( A \), some problems on Hilbert \( A \)-modules can be reformulated as problems on right ideals of \( A \), since submodules of a full Hilbert \( A \)-module are in a bijective correspondence with the closed right ideals of \( A \). Therefore, one may wonder if the previous result could be reformulated in the language of right ideals. Actually, if \( I \) is a (closed) right ideal of \( A \), then \( I \) is a right Hilbert \( A \)-module with inner product \( \langle a, b \rangle = a^* b \), for \( a, b \in I \), and in this case, \( K(E) \) equals to the hereditary \( C^* \)-algebra \( I \cap I^* \) and so \( B(E) = M(I \cap I^*) \). Therefore, one may expect that \( C^* \)-algebras of compact operators may be characterized by the property that for every hereditary \( C^* \)-subalgebra \( B \) of \( A \), \( M(B) \) is a \( W^* \)-algebra.

Unfortunately, this is not the case for non separable \( C^* \)-algebras, as the following counterexample shows. However, if \( A \) is separable and \( p \) is a projection as in the proof of Theorem 2.6, then \( pAp \) is a separable \( W^* \)-algebra, hence finite dimensional (also see Theorem 2.8 in [2]).

Example 2.9. For the Stone-Cech compactification \( \beta \mathbb{N} \) of the natural numbers, the algebra of continuous functions \( C(\beta \mathbb{N}) \) is a \( W^* \)-algebra. Let \( x \) be any point of \( \beta \mathbb{N} \) that is not a natural number and let \( A \) be the \( C^* \)-subalgebra of \( C(\beta \mathbb{N}) \) consisting of those functions vanishing at \( x \). Let \( B \) be a hereditary \( C^* \)-subalgebra of \( A \) (which is an ideal, since \( A \) is abelian). Then there is an open subset \( U \) of \( \beta \mathbb{N} \) such that \( B \) consists of functions in \( A \) that vanish outside \( U \). Let \( V \) be the closure of \( U \). Then \( V \) is also open. For every \( c \in C(V) \) we may extend \( c \) by zero outside \( V \), and thereby view \( C(V) \) as a \( W^* \)-subalgebra of \( C(\beta \mathbb{N}) \). Observe that \( M(B) = C(V) \): clearly \( B \) is an ideal in \( C(V) \), so it suffices to note that for any \( 0 \neq c \in C(V) \), \( cB \neq 0 \). To see this, we note that \( c \) is non-zero on a nonvoid open subset \( W \) of \( V \), hence \( W \cap U \setminus x \) is a nonvoid open set, hence there exists a non-zero continuous function \( b \) with support in \( W \cap U \setminus x \). Thus \( b \in B \) and \( cb \neq 0 \). Therefore \( M(B) = C(V) \) is a \( W^* \)-algebra, but \( A \) cannot be a \( C^* \)-algebra of compact operators.

3. Morita equivalence

The notion of (strong) Morita equivalence of \( C^* \)-algebras was introduced by M. Rieffel in [19]. Two \( C^* \)-algebras \( A \) and \( B \) are (strongly) Morita equivalent if there is an \( A-B \)-bimodule \( M \), which is a left full
Hilbert $C^*$-module over $\mathcal{A}$, and a right full Hilbert $C^*$-module over $\mathcal{B}$, such that the inner products $\langle \cdot, \cdot \rangle_\mathcal{A}$ and $\langle \cdot, \cdot \rangle_\mathcal{B}$ satisfy $\mathcal{A} \langle x, y \rangle z = x \langle y, z \rangle_\mathcal{B}$ for all $x, y, z \in \mathcal{M}$. Such a module $\mathcal{M}$ is called an $\mathcal{A}$-$\mathcal{B}$-imprimitivity bimodule.

It would be interesting to investigate those properties of $C^*$-algebras which are preserved under Morita equivalence. These include, among other things nuclearity, being type I, and simplicity [3, 7, 12, 17, 18, 21, 22]. Now if one of the two Morita equivalent $C^*$-algebras is a $W^*$-algebra, it is natural to ask if so is the other. The answer to this question, as it posed is obviously negative, as Hilbert space $H$ is a $K(H)$-$C^*$-imprimitivity bimodule, and so $C^*$-algebras $K(H)$ and $C$ are Morita equivalent. However we may rephrase that question in the following less trivial form.

**Question 3.1.** Suppose that $C^*$-algebras $\mathcal{A}$ and $\mathcal{B}$ are Morita equivalent and the $C^*$-algebra $M(\mathcal{A})$ is a $W^*$-algebra, is it then true that $M(\mathcal{B})$ is a $W^*$-algebra?

By Theorem 2.8, we can show that the above property holds for $C^*$-algebra $\mathcal{A}$ exactly when $\mathcal{A}$ is a $C^*$-algebra of compact operators. In fact, we have the following result.

**Theorem 3.2.** Let $\mathcal{A}$ be a $C^*$-algebra such that $M(\mathcal{B})$ is a $W^*$-algebra, for any $C^*$-algebra $\mathcal{B}$ which is Morita equivalent to $\mathcal{A}$. Then $\mathcal{A}$ is a $C^*$-algebra of compact operators.

**Proof.** Let $\mathcal{B} = K(H_\mathcal{A})$. Since $H_\mathcal{A}$ is a full Hilbert $\mathcal{A}$-module, then $\mathcal{B}$ is Morita equivalent to $\mathcal{A}$. By assumption, $B(H_\mathcal{A}) \cong M(\mathcal{B})$ is a $W^*$-algebra, hence $\mathcal{A}$ is a $C^*$-algebra of compact operators, by Theorem 2.8. \qed

However, we give an affirmative answer to the above question, when both $C^*$-algebras are unital.

Recall that a Hilbert $C^*$-module $E$ on a $C^*$-algebra $\mathcal{A}$ is called self dual if for every bounded linear $\mathcal{A}$-module map $\varphi : E \to \mathcal{A}$ there is an element $y \in E$ such that $\varphi(\cdot) = \langle y, \cdot \rangle$.

**Lemma 3.3.** Let $E$ be a right Hilbert $C^*$-module over a $C^*$-algebra $\mathcal{A}$ such that $K(E)$ is unital. then

(i) $E$ is self dual.
(ii) $B(E)$ is a $W^*$-algebra, whenever $\mathcal{A}$ is a $W^*$-algebra.

**Proof.** By hypothesis there are elements $x_1, \cdots, x_n$ and $y_1, \cdots, y_n$ in $E$ such that $\sum_{i=1}^n \theta_{x_i, y_i} = 1 \in K(E)$. Thus, for every bounded linear
\(\mathcal{A}\)-module map \(\varphi : E \to \mathcal{A}\) and \(x \in E\) we have
\[
\varphi(x) = \varphi\left(\sum_{i=1}^{n} \theta_{x_i, y_i} x x\right) = \varphi\left(\sum_{i=1}^{n} x_i (y_i, x)\right) = \sum_{i=1}^{n} \varphi(x_i) \langle y_i, x \rangle = \sum_{i=1}^{n} \langle y_i \varphi(x_i)^*, x \rangle = \langle \sum_{i=1}^{n} y_i \varphi(x_i)^*, x \rangle.
\]

Therefore \(\varphi(x) = \langle y, x \rangle\), where \(y = \sum_{i=1}^{n} y_i \varphi(x_i)^*\). Hence \(E\) is selfdual.

Now \((ii)\) follows from \((i)\) and [10, Proposition 3.10]. \(\square\)

Now if \(E\) is an \(\mathcal{A}\)-\(\mathcal{B}\)-imprimitivity bimodule, then \(\mathcal{A} \cong K_{\mathcal{B}}(E)\) and \(\mathcal{B} \cong K_{\mathcal{A}}(E)\). Therefore, the following partial answer to the above question follows from the above lemma.

**Theorem 3.4.** Suppose that unital \(C^*\)-algebras \(\mathcal{A}\) and \(\mathcal{B}\) are Morita equivalent. Then \(\mathcal{A}\) is a \(W^*\)-algebra if and only if \(\mathcal{B}\) is a \(W^*\)-algebra.

A similar result can be proved for operator algebras. Let \(\mathcal{A}\) and \(\mathcal{B}\) be operator algebras. We say that \(\mathcal{A}\) and \(\mathcal{B}\) are (strongly) Morita equivalent if they are Morita equivalent in the sense of Blecher, Muhly, Paulsen [8]. In [8], it is proved that two \(C^*\)-algebras are (strongly) Morita equivalent (as operator algebras) if and only if they are Morita equivalent in the sense of Rieffel.

**Theorem 3.5.** Suppose that unital operator algebras \(\mathcal{A}\) and \(\mathcal{B}\) are Morita equivalent. Then \(\mathcal{A}\) is a dual operator algebra if and only if \(\mathcal{B}\) is a dual operator algebra.

**Proof.** Let \(\pi : \mathcal{A} \to B(H)\) be a completely isometric normal representation of \(\mathcal{A}\) on some Hilbert space \(H\). Then there exist a completely isometric representation \(\rho : \mathcal{B} \to B(K)\) of \(\mathcal{B}\) on a Hilbert spaces \(K\) and subspaces \(X \subseteq B(K, H), Y \subseteq B(H, K)\) such that
\[
\pi(\mathcal{A}) X \rho(\mathcal{B}) \subseteq X, \quad \rho(\mathcal{B}) Y \pi(\mathcal{A}) \subseteq Y, \quad \pi(\mathcal{A}) = \overline{XY}^{\|\|}, \quad \rho(\mathcal{B}) = \overline{YX}^{\|\|}
\]

Since \(\pi\) is normal, we have \(\pi(\mathcal{A}) = \overline{\pi(\mathcal{A})}^{\text{w*}}\). Now \(X \rho(\mathcal{B}) Y \subseteq \pi(\mathcal{A})\) implies that \(X \rho(\mathcal{B})^{\text{w*}} Y \subseteq \pi(\mathcal{A})\). Therefore
\[
Y X \rho(\mathcal{B})^{\text{w*}} Y X \subseteq Y \pi(\mathcal{A}) X \subseteq \rho(\mathcal{B}),
\]
and so \(\rho(\mathcal{B})^{\text{w*}} \subseteq \rho(\mathcal{B})\). Since \(\rho(\mathcal{B})\) is a unital algebra we have \(\overline{\rho(\mathcal{B})}^{\text{w*}} \subseteq \rho(\mathcal{B})\), hence \(\rho(\mathcal{B})^{\text{w*}} = \rho(\mathcal{B})\). Therefore \(\mathcal{B}\) is a dual operator algebra. \(\square\)
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