Regulated curves on a Banach manifold and singularities of endpoint map. I. Banach manifold structure

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Abstract
We consider regulated curves in a Banach bundle whose projection on the basis is continuous with regulated derivative. We build a Banach manifold structure on the set of such curves. This result was previously obtained for the case of strong Riemannian Banach manifold and absolutely continuous curves in [Sch16]. The essential argument used was the existence of a "local addition" on such a manifold. Our proof is true for any Banach manifold. In the second part of the paper the problems of controllability will be discussed.

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1 Introduction
Classically, a sub-Riemannian structure on a finite dimensional manifold $M$ is a subbundle $E$ of $TM$ provided with a Riemannian metric $g$. An absolutely

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continuous path $\gamma : I := [a, b] \to M$ of class $H^1$ is called horizontal if the derivative $\dot{\gamma}$ is tangent to $E$ almost everywhere. Given any Riemannian metric $g$ on $TM$ whose restriction to $E$ is $g$, it is well known that the set $\mathcal{H}^1(I, M)$ of paths $\gamma : [a, b] \to M$ of class $H^1$ has a structure of manifold and the set $\mathcal{H}^1(I, E)$ of horizontal curves is a closed Hilbert submanifold. Now given any point $x \in M$ the set $\mathcal{H}^1_x = \{ \gamma \in \mathcal{H}^1(I, M) : \gamma(a) = x \}$ is a finite codimensional submanifold and the map $\text{End}_x : \mathcal{H}^1_x \to M$ defined by $\text{End}_x(\gamma) = \gamma(b)$ is called the \textit{endpoint map} which is smooth (cf. \cite{Mon02}, \cite{AOP01} and \cite{PT01} for a more recent proof). If $\text{End}_x$ is surjective, then one says that the control system associated to this problem is called \textit{controllable} and any critical point of $\text{End}_x$ is traditionally called an abnormal path. The existence of such curves is a big difference between Riemannian and sub-Riemannian case related to the problem of characterization of minimal geodesics and it was the origin of a lot of works about it (for nice references on this problem see \cite{Mon02} or \cite{AS04} for a more general context in control theory).

The concept of sub-Riemannian geometry in infinite dimensional context was introduced in \cite{GMV15} for sub-bundle $D$ of the tangent bundle $TM$ of a Riemannian “convenient manifold” $M$ with application to Fréchet Lie groups. Essentially motivated by \textit{mathematical analysis of shapes}, the particular Banach context was recently studied by S. Arguillé in his PhD thesis and more precisely formalized in \cite{Arg20}. He considers a Banach bundle $\pi : E \to M$ provided with an anchor $\rho : E \to TM$ (called \textit{anchored Banach bundle}) and a strong Riemannian metric $g$ on $E$. From the existence of a strong Riemannian metric it follows that the typical fiber of $E$ is a Hilbert space $\mathbb{H}$. Given any chart $(U, \phi)$ in $M$ such that $E|_U$ is trivial, a path $\gamma : I \to U$ is called \textit{horizontal} or \textit{admissible} if there exists a trivialization $\Phi : E|_U \to U \times \mathbb{H}$ and a path $u : I \to \mathbb{H}$ of class $L^2$ such that $\frac{d}{dt}(\phi \circ \gamma) = T\phi(\rho(\Phi^{-1}(\gamma, u)))$ almost everywhere. The set $\mathcal{H}^1(I, U)$ of such curves has a Banach manifold structure since $\mathcal{H}^1(I, \phi(U))$ is open in the Sobolev space $H^1(I, \phi(U)) \times L^2(I, \mathbb{H})$ (cf. \cite{Arg20} section 1). As in finite dimension, it follows that $\mathcal{H}^1_x = \{ \gamma \in \mathcal{H}^1(I, M) : \gamma(a) = x \}$ is a closed submanifold and the map $\text{End}_x : \mathcal{H}^1_x \to M$ is again well defined and smooth. However, the range of $T_x \text{End}_x(T_x \mathcal{H}^1_x)$ may not be closed but only dense in $T_{\text{End}_x(\gamma)}M$ which gives rise to another type of singular point $\gamma$ called \textit{elusive path} in \cite{Arg20}.

The main purpose of this work is to give a \textit{generalization of such local results to global ones}. Thus in the context of an anchored bundle $(E, \pi, M, \rho)$, we need to show that a certain set of paths $\mathcal{P}(I, E)$ has a Banach manifold structure and the subset $\mathcal{AP}(I, M)$ of admissible paths has a structure of Banach submanifold of $\mathcal{P}(I, M)$. Moreover as in finite dimension, we want to show that for each $x \in M$ the endpoint map $\mathcal{AP}_x(I, M) \to M$ is smooth.

The first part of the paper deals only with the Banach manifold structure on $\mathcal{P}(I, E)$. \textit{We now give a short outline of our results.}

For simplicity for now $I$ will denote the interval $[0, 1]$. Let $M$ be a Banach manifold modelled on a Banach space $\mathcal{M}$. A path $c : I \to M$ is called \textit{regulated} if for any $t \in I$ both one-sided limits exist. Such a path is called \textit{strong regulated} or...
0-regulated if it is right-continuous everywhere and left-continuous at 1 (cf. §

A path \( c : I \to M \) is called 1-regulated if \( c \) is differentiable almost everywhere and its derivative is a 0-regulated path \(^1\). For \( k = 0,1 \) the set \( \mathcal{R}^k(I,M) \) of \( k \)-regulated paths has a natural structure of Banach space (cf. Proposition 1.3).

Let \( \pi : E \to M \) be a Banach bundle with fibers modeled on Banach space \( E \). The set of 1-regulated paths in \( M \) defined on \( I \) is denoted \( \mathcal{P}(M) \) and the set of 0-regulated paths \( c : I \to E \) such that \( \pi \circ c \) belongs to \( \mathcal{P}(M) \) is denoted by \( \mathcal{P}(E) \).

The main result of this part is:

Theorem 1

1. The set \( \mathcal{P}(M) \) has a Banach manifold structure modeled on the Banach space \( \mathcal{R}^1(I,M) \). Moreover for any \( \gamma \in \mathcal{P}(M) \) we have a Banach isomorphism from \( T_\gamma \mathcal{P}(M) \) onto the set \( \Gamma(\gamma) = \{ X \in \mathcal{R}^1(I,TM) \mid p_M(X) = \gamma \} \).

2. The set \( \mathcal{P}(E) \) has a Banach manifold structure modeled on the Banach space \( \mathcal{R}^1(I,M) \times \mathcal{R}^0(I,E) \). For any \( c \in \mathcal{P}(E) \) we have a Banach isomorphism from \( T_c \mathcal{P}(E) \) onto \( \Gamma(c) = \{ X' \in \mathcal{R}^0(I,TE) \mid p_E(X') = c \} \). Moreover \( \pi : \mathcal{P}(E) \to \mathcal{P}(M) \) is a Banach bundle with typical fiber \( \mathcal{R}^0(I,E) \).

This Theorem can be obtained from [Sch16] for absolutely continuous paths when \( M \) is provided with a strong Riemannian metric which implies the existence of a “local addition” on \( M \) (cf. [Mic80]). Our proof goes through building an atlas on \( \mathcal{P}(M) \) and \( \mathcal{P}(E) \) which is a generalization of the proof for \( H^1 \) paths in sub-Riemannian finite dimension (cf. [AOP01]). In fact, it is also an adaptation of proofs of results of [Pen67] or [Kri72] for maps of class \( C^k \) from some compact manifolds with corners \( \Omega \subset \mathbb{R}^m \) to a Banach manifold \( M \). However, we can not deduce Assertion 1 directly from [Kri72] since the set \( \mathcal{R}^0(I,E) \) does not satisfy the axioms of section 2 in [Kri72]. Thus, to be complete, we give detailed proof of these results. Note that, according to the recent paper of Glöckner [Glö15], our proof can be transposed to the context of absolutely continuous paths as defined in this work.

This paper tries to be self contained. The next section contains essentially all the useful notations and results about \( k \)-regulated curves in Banach spaces. Some of these results were proved in [Die60, Pen12]. All needed results about \( k \)-regulated paths in the case of a Banach manifold are contained in §3. The fundamental results on the Banach structure on the set \( \Gamma(\gamma) \), \( \Gamma(c) \) and \( \Gamma(\gamma^*(E)) \) (Proposition 1.5) can be found in §1. The subsection describes the topology on \( \mathcal{P}(E) \) and \( \mathcal{P}(M) \). Theorem 1 is proved in §5. We have included Appendix A devoted to the case of \( C^1 \)-paths in Banach bundle \( E \) such that \( \pi \circ c \) is \( C^2 \)-path. Such a result is basic for [Pel20].

\(^1\)Such a path is absolutely continuous in the sense of [Glö15].
2 $k$-regulated paths in a Banach space

Consider a Banach space $E$ equipped with a norm $\| \cdot \|_E$. Let $I = [t_0, t_1]$ be a compact interval. According to [Die60], a path $c : I \to E$ is called regulated if for any $t \in [t_0, t_1]$ the limit $\lim_{s \to t^+} c(s)$ exists and for any $t \in [t_0, t_1]$ the limit $\lim_{s \to t^-} c(s)$ exists. Note that the set of discontinuities of $c$ is at most countable and in particular it has zero Lebesgue measure. Now if $c : I \to E$ is regulated, the restriction of $c$ to any subinterval $J = [s_0, s_1]$ is a regulated path $c_J$ as well.

**Definition 2.1** A regulated path $c : I \to E$ is called strong regulated if it is right-continuous everywhere and left-continuous at $t_1$, i.e. for any point $t \in [t_0, t_1]$ we have $c(t) = \lim_{s \to t^+} c(s)$ and $c(t_1) = \lim_{s \to t_1^-} c(s)$.

Note that for any regulated path $c : I \to E$, there exists a unique strong regulated path $\hat{c}$ that coincides with $c$ outside of discontinuity set and its discontinuity set is not larger than that of $c$.

If $c : [t_0, t_1] \to E$ and $c' : [t'_0, t'_1] \to E$ are two strong regulated paths then the concatenation $c \ast c'$ is the strong regulated path defined by

$$
c \ast c'(t) = c(t) \quad \text{for } t \in [t_0, t_1],
c \ast c'(t_1) = \lim_{t \to t''_1} c'(t),
c \ast c'(t) = c'(t - t_1) \quad \text{for } t \in [t_1, t'_1 - t'_0].
$$

The set $R(I, E)$ of regulated paths in $E$ provided with the norm $\| c \|_\infty = \sup_{t \in I} \| c(t) \|_E$ is a normed space. Let $U \subset E$ be an open neighbourhood of the image $c(I)$ of a regulated path $c$. Then for any (local) diffeomorphism $\phi$ from $U$ to an open set $V$ it is clear that the path $\phi \circ c : I \to E$ is also regulated. According to [Die60] section VIII.7 for any $c \in R(I, E)$ there exists a primitive of $c$ that is a continuous path $\hat{c} : I \to E$ which is differentiable outside a countable set $\Sigma$ and such that $\frac{d\hat{c}}{dt} = c(t)$ for all $t \in I \setminus \Sigma$. Moreover all such primitives are of type $\hat{c} + \text{const}$. Note that if we impose the condition of strong regularity then $c$ is completely determined by any of its primitives.

Note that while concatenation of two strong regulated paths (as defined previously) is a strong regulated path, the restriction to any subinterval $J \subset I$ of strong regulated path $c$ defined on $I$ is not in general a strong regulated path on $J$ but only regulated. Therefore we consider:

**Definition 2.2** Let $c : I \to E$ be a strong regulated path $c : I \to E$ and $J$ any subinterval of $I$. We define the strong regulated path $c_{\mid J}$ as $c_{\mid J}$. When there will be no risk of confusion we will call $c_{\mid J}$ the restriction of $c$ to $J$. 

Remark 2.3 For strong regulated $c$ and $J = [s_0, s_1] \subset I$ the strong regulated path $c_J$ coincides with $c|_J$ on $[s_0, s_1]$ and its value at $s_1$ is $\lim_{s \to s_1^+} c(s)$.

Definition 2.4 For $k > 0$ a path $c : I \to E$ is called $k$-regulated if $c$ is of class $C^{k-1}$ and there exists a strong regulated path $c^{(k)} : I \to E$ such that the derivative of order $k - 1$ of $c$ is a primitive of $c^{(k)}$. The strong regulated path $c^{(k)}$ is unique and will be called $k$th derivative of $c$.

Note that if $c : I \to E$ is 1-regulated, then we have
\begin{equation}
    c(t) = c(t_0) + \int_{t_0}^{t} c^{(1)}(s)ds.
\end{equation}

Therefore for $k$-regulated curve $c$, for $1 \leq l \leq k$, we always have:
\begin{equation}
    c^{(l-1)}(t) = c^{(l-1)}(t_0) + \int_{t_0}^{t} c^{(l)}(s)ds.
\end{equation}

Notations and conventions 2.5
By 0-regulated path we will mean a strong regulated path. For any $k \geq 0$ the set of $k$-regulated paths in $E$ defined on an interval $I$ will be denoted $R^k(I,E)$. We provide $R^k(I,E)$ with the norm
\[
    \|c\|_{\infty}^k = \max_{0 \leq l \leq k} \sup_{t \in I} \|c^{(l)}(t)\|_E.
\]

For an open set $U$ of $E$ we can consider the subset
\[
    R^k(I,U) = \{c \in R^k(I,E) : c(I) \subset U\}.
\]

Clearly $R^k(I,U)$ is an open subset of $R^k(I,E)$.

We have the following classical results:

Theorem 2.6
1. For any $k$-regulated path $c : I \to E$ the set $c(I)$ is compact.
2. The set $R(I,E)$ provided with the norm $\| \|$ is a Banach space.
3. The set $R^k(I,E)$ for $k \geq 0$ provided with the norm $\| \|_{\infty}^k$ is a Banach space.
4. Let $A$ be a continuous linear map from $E$ to a Banach space $F$. If $c \in R^k(I,E)$ then $A \circ c$ belongs to $R^k(I,F)$ and for any $1 \leq l \leq k$ we have
\[
    A \left( \int_{t_0}^{t_1} c^{(l)}(s)ds \right) = \int_{t_0}^{t_1} (A \circ c^{(l)})(s)ds.
\]
5. Let \( c : I \to E \) be a regulated curve and \( \phi : J = [s_0, s_1] \to \mathbb{R} \) be 1-regulated curve such that \( \phi(J) \subseteq I \). If either \( c \) is continuous or \( \phi \) is strictly monotone then \( s \mapsto \phi'(s)(c \circ \phi)(s) \) is regulated and we have:

\[
\int_{s_0}^s \phi'(r)(c \circ \phi)(r) dr = \int_{\phi(s_0)}^{\phi(s)} c(t) dt.
\]

Proof.

The fundamental observation is that the closure of the set of all step functions (resp. strong regulated step functions) on \( I \) in the norm \( ||| \cdot |||_\infty \) is \( \mathcal{R}(I, E) \) (resp. \( \mathcal{R}^0(I, E) \)), see [Die60] 7.6.1 or [Pen12] Proposition 2.15.

1. For \( k = 0 \) the usual proof goes along these lines: we observe that since for every \( \epsilon > 0 \) there is a step function \( f \) such that \( ||c - f||_\infty < \epsilon \). As step functions have finite image, we note that \( c(I) \) is paracompact. The statement follows from the fact that the closure of paracompact set in a complete metric space is compact. See also [Pen12] Proposition 2.16. For \( k > 0 \) it follows from continuity of \( c \).

However one can give an alternate, more direct proof, which doesn’t use metric structure or completeness. This approach will be useful in the next section. Take an open cover \( U_\alpha \) of the set \( c(I) \). Consider the sets \( V_\alpha = \text{int} c^{-1}(U_\alpha) \). Since \( c \) is not continuous in general we need to apply interior operation to get open sets. However that means that open sets \( V_\alpha \) might not be a cover of \( I \). Indeed if \( p \in I \) is not an element of \( \bigcup \alpha V_\alpha \), then \( c \) is discontinuous at \( p \). For each such point \( p \) we add to the family \( V_\alpha \) a set \( [p-\epsilon, p+\epsilon] \), where \( \epsilon > 0 \) is chosen in such a way that \( c([p-\epsilon, p+\epsilon]) \) is contained in the sum of two sets from the family \( U_\alpha \). In order to find \( \epsilon \) satisfying this condition take a \( \alpha' \) such that \( c(p) \in U_{\alpha'} \). Since \( c \) is right-continuous at \( p \) we can chose \( \epsilon \) such that \( c([p, p+\epsilon]) \subseteq U_{\alpha'} \). As a second step, take a set \( U_{\alpha''} \) containing \( \lim_{x \to p^-} c(x) \). From the definition of left-side limit it follows again that we can chose \( \epsilon \) such that \( c([p-\epsilon, p]) \subseteq U_{\alpha''} \). In this way we have obtained an open cover of \( I \). From compactness it follows that there exists a finite subcover of \( I \). To each set in this subcover we associate one or two sets from \( V_\alpha \) family which constitute a finite subcover of \( c(I) \).

2. It follows from the fact that \( \mathcal{R}(I, E) \) is a closed subspace of the Banach space of bounded functions, cf. [Pen12] Proposition 2.17.

3. For \( k = 0 \) again \( \mathcal{R}^0(I, E) \) is a closed subspace in a Banach space. For \( k = 1 \) note that every function in \( \mathcal{R}^1(I, E) \) is an integral of a function in \( \mathcal{R}^0(I, E) \) (see 1). Given a Cauchy sequence of functions \( c_n \in \mathcal{R}^1(I, E) \) we note that \( c_n^{(1)} \) is a Cauchy (and thus convergent) sequence in \( \mathcal{R}^0(I, E) \). From [Die60] 8.7.8 and 1 it follows that \( c_n \) converges to integral of limit of that sequence both in norm \( ||| \cdot |||_\infty \) and in norm \( ||| \cdot |||_\infty \). Thus \( \mathcal{R}^1(I, E) \) is a Banach space and the set of primitive functions of step functions is dense in it. The proof for \( k > 1 \) is analogous.
4. Direct consequence of definition, see also [Pen12, Proposition 2.18].

5. See [Pen12, Proposition 2.21].

In this paper we will also need the following result which is classical in the context of $C^k$-paths and which is certainly well known for specialists in the context of $k$-regulated paths. Without precise references we will give a proof of this Proposition.

**Proposition 2.7** Let $U$ and $V$ be two open sets in Banach spaces $E$ and $F$ respectively and $f : U \to V$ a smooth map. Then we have the following properties:

1. For any $k \geq 0$ the map $c \mapsto f \circ c$ is a smooth map from $\mathcal{R}^k(I, U)$ to $\mathcal{R}^k(I, V)$.

2. For any $k \geq 1$ the map $c \mapsto c^{(1)}$ is a smooth map from $\mathcal{R}^k(I, U)$ to $\mathcal{R}^{k-1}(I, E)$.

3. The map $D : c \mapsto (c(0), c^{(1)})$ is an isomorphism from $\mathcal{R}^1(I, E)$ to $E \times \mathcal{R}^0(I, E)$.

**Proof.**

At first if $c : I \to U$ is a $k$-regulated path, $f \circ c : I \to V$ is also a $k$-regulated path. For $k = 0$ this fact is straightforward even for a map $\phi$ which is only continuous. For $k = 1$ $f \circ c$ is of class $C^0$ and as $(f \circ c)^{(1)}(t)$ one takes $f'(c(t))c^{(1)}(t)$. The result follows from point 4 of Theorem 2.6. For $k \geq 2$ the proof uses the same type of arguments by using an expression of the derivative of order $k$ of $f \circ c$.

Now consider a smooth path $\tilde{c} : \mathbb{R} \to \mathcal{R}^k(I, V)$. Then we can identify $\tilde{c}$ with a map $\tilde{c} : I \times \mathbb{R}$ which is $C^{k-1}$ in the two variables and smooth in the second variable. Therefore the map $f \circ \tilde{c} : I \times \mathbb{R} \to V$ has the same properties. But for each $\epsilon \in \mathbb{R}$ the map $t \mapsto f \circ \tilde{c}(\epsilon, t)$ is a $k$-regulated path in $V$ then $s \mapsto f \circ \tilde{c}(\epsilon, s)$ is a smooth path in $\mathcal{R}^k(I, V)$ and so $f$ is a smooth map which ends the proof of Point (1).

Point (2) follows directly from linearity of derivative $c \mapsto c^{(1)}$.

For Point (3), it is clear that the map $c \mapsto (c(0), c^{(1)})$ is linear and its inverse is the linear map $(x, u) \mapsto b(t) = x + \int_0^t u(s)ds$. Now, we have

$$\|c(0)\|_E + \|c^{(1)}\|_\infty \leq \|c\|_\infty + \|c^{(1)}\|_\infty \leq 2\|c\|_1$$

and

$$\|b\|_\infty \leq \|b\|_\infty + \|b^{(1)}\|_\infty \leq \|x\|_E + (t_1 - t_0)\|u\|_\infty + \|u\|_\infty \leq (1 + t_1 - t_0)(\|x\|_E + \|u\|_\infty).$$

Since any Banach space is a convenient space a map between Banach spaces is smooth if and only if the image of a smooth path defined on $\mathbb{R}$ is a smooth path (cf. [KM97]).
which shows that these linear maps are continuous and so ends the proof of Point (3).

Proposition 2.8 Let $J$ be a closed subinterval of $I$. The restriction map $r_J : R^k(I, E) \to R^k(J, E)$ is a linear contraction.

Proof. Recall that the restriction of a strong regulated path has to be considered in the sense of Definition 2.2. The fact that the map $r_J$ is a contraction follows directly from the definition of restriction and the norm $\|\|_\infty$, see 2.5:

$$\|r_J(c)\|_\infty^k \leq \|c\|_\infty^k.$$  

Proposition 2.9 The concatenation map $\star : R^0([a, b]) \times R^0([b, c]) \to R^0([a, c])$ is an isometric isomorphism of Banach spaces, where the Cartesian product is equipped with max-norm.

Proof. It was previously stated that concatenation of two strong regulated paths is again a strong regulated path. The map $(r_{[a, b]}, r_{[b, c]})$ is its inverse. Note that even though the value of the first curve in point $b$ is seemingly lost, it reconstructed by $r_{[a, b]}$ by requirement that the result be left-continuous at $b$. Directly from the definition of norm we have

$$\|c_1 \star c_2\|_\infty^0 = \max(\|c_1\|_\infty^0, \|c_2\|_\infty^0).$$

3 \ $k$-regulated paths in a Banach manifold

We now consider a Banach manifold $M$ modeled on a Banach space $M$. We have the same definitions of regulated and strong regulated paths:

Definition 3.1

1. A path $c : I \to M$ is called regulated if for any $t \in [t_0, t_1]$ the limit $\lim_{s \to t^+} c(s)$ exists and for any $t \in [t_0, t_1]$ the limit $\lim_{s \to t^-} c(s)$ exists.

2. A regulated path $c : I \to M$ is called strong regulated if for any point $t \in [t_0, t_1]$ we have $c(t) = \lim_{s \to t^+} c(s)$ and moreover $c(t_1) = \lim_{s \to t^-} c(s).$
Again for any regulated path \( c : I \rightarrow M \), there exists a unique strong regulated path \( \bar{c} \) whose discontinuity set is no larger than the discontinuity set of \( c \) and it coincides with \( c \) outside of that set. Just like for regulated paths in a Banach space, if \( c : I \rightarrow M \) is strong regulated, the restriction of \( c \) to any subinterval \( J = [s_0, s_1] \) of \( I \) will be strong regulated path \( c_J \) defined by the restriction of \( c \) to \( J \). If \( c : [t_0, t_1] \rightarrow E \) and \( c' : [t'_0, t'_1] \rightarrow E \) are two strong regulated paths then the concatenation \( cc' \) is defined in the same way as for strong regulated paths in a Banach space and so it is also a strong regulated path.

A partition of \( I = [t_0, t_1] \) is a non-decreasing sequence \( \tau = (\tau_i)_{i=0,\ldots,n} \) such that \( t_0 = \tau_0 \leq \tau_1 \leq \cdots \leq \tau_i \leq \cdots \leq \tau_n = t_1 \). Of course if \( c \) is strong regulated then for any partition \( \tau = (\tau_i)_{i=0,\ldots,n} \) of \( [t_0, t_1] \) the restriction \( c_i = c|_{[\tau_{i-1}, \tau_i]} \) is strong regulated for all \( i = 1, \ldots, n \).

From now on in this section we fix an atlas \( \mathfrak{A} = (U_\alpha, \phi_\alpha)_{\alpha \in A} \) on \( M \) such that each domain \( U_\alpha \) is connected and simply connected.

**Definition 3.2** Let \( c : I \rightarrow M \) be a path. A \( 0 \)-regulated path is a strong regulated path. For \( k > 0 \) \( c \) is a \( k \)-regulated path if and only if for any chart \( (U_\alpha, \phi_\alpha) \) such that \( U_\alpha \cap c(I) \neq \emptyset \) and for any subinterval \( J \subset I \) such that \( c(J) \subset U_\alpha \), the path \( \phi_\alpha \circ c_J \) is a \( k \)-regulated path in \( \mathbb{M} \).

**Remark 3.3**

1. The definition of \( k \)-regulated path does not depend on the choice of the atlas \( \mathfrak{A} \).
2. If \( c : I \rightarrow M \) is \( k \)-regulated for all \( k \in \mathbb{N} \), it is smooth in the usual sense.

**Proposition 3.4** Let \( c : I \rightarrow M \) be a \( k \)-regulated path.

1. The closure of \( c(I) \) is a compact subset of \( M \).
2. Let \( c : I \rightarrow M \) be a \( k \)-regulated path. For any increasing map \( h : [s_0, s_1] \rightarrow I \) of class \( C^{k+1} \) the path \( c \circ h : [s_0, s_1] \rightarrow M \) is a \( k \)-regulated path called a reparametrization of \( c \). In particular for any \( k \)-regulated path \( c : I \rightarrow M \), there exists a canonical increasing affine map \( h : [0, 1] \rightarrow I \) such that \( c \circ h : [0, 1] \rightarrow M \) is a \( k \)-regulated path.

**Proof.**

1. Note that for \( k \geq 1 \) the path \( c \) is continuous and in consequence it is obvious. For \( k = 0 \) the alternate proof from Theorem 2.6 works in this case without any modifications.
2. Note that the first part of Point 2 is true for \( M = M \) according to Theorem 2.4 Point (1). Therefore the definition of a \( k \)-regulated path in a manifold implies clearly this result. Now the map

\[ s \mapsto s(t_1 - t_0) + t_0 \]

is smooth strictly increasing from \([0, 1]\) onto \( I = [t_0, t_1] \). The last property is a direct consequence of the first part.

\[ \triangle \]

4 0-regulated lifts of 1-regulated paths

Let \( \pi : E \to M \) a Banach bundle on a Banach manifold modeled on a Banach space \( M \) with fibers modeled on a Banach space \( E \). We denote by \( p_E : TE \to E \) (resp. \( p_M : TM \to M \)) the tangent bundle of \( E \) (resp. \( M \)).

**Definition 4.1** Let \( \gamma : I \to M \) be a path. A \( k \)-regulated path \( c : I \to E \) is called a \( k \)-regulated lift or a \( k \)-regulated section along \( \gamma \) if we have \( \pi \circ c = \gamma \). If \( E = TM \), a 0-regulated section over \( \gamma \) will be called a \( k \)-regulated vector field along \( \gamma \).

According to Remark 3.4 any 0-regulated lift \( c : I \to E \) of a 1-regulated path \( \gamma : I \to M \), can be reparametrized as a 0-regulated lift \( \bar{c} : [0, 1] \to E \) over a 1-regulated path \( \gamma \circ h : [0, 1] \to M \). Therefore we can assume that 1-regulated paths \( \gamma \) are defined on \([0, 1]\) and so all their 0-regulated lifts \( c \) will also be defined on \([0, 1]\).

**Notations and conventions 4.2** — from now \( I \) will denote the interval \([0, 1]\).

The set of 1-regulated paths in \( M \) defined on \( I \) is denoted \( \mathcal{P}(M) \) and the set of 0-regulated lifts (to \( E \)) of paths in \( \mathcal{P}(M) \) is denoted by \( \mathcal{P}(E) \). Of course \( \pi : E \to M \) induces a surjective map (also denoted by \( \pi \)) from \( \mathcal{P}(E) \) onto \( \mathcal{P}(M) \).

4.1 Banach structure on the set of 0-regulated sections and 0-regulated vector fields along 1-regulated paths

We denote by \( \Gamma(\gamma) \) the set of 1-regulated vector fields along \( \gamma \in \mathcal{P}(M) \) and by \( \Gamma(c) \) the set of 0-regulated vector fields along \( c \in \mathcal{P}(E) \).

Our purpose is to give some nice parametrization of \( \Gamma(\gamma) \) and \( \Gamma(c) \) which provides these sets with a Banach structure. For this objective we need the following technical result:

**Lemma 4.3** Consider \( c \in \mathcal{P}(E) \) and \( \gamma = \pi \circ c \). Let \( \gamma^*(E) \) be the pull-back of \( E \) over \( \gamma \).
1. There exists a homeomorphism $G_\gamma : I \times E \to \gamma^*(E)$ which is a bundle isomorphism over the $Id$ of $I$, which is 1-regulated in the first variable and smooth in the other variable such that the following diagram is commutative

\[
\begin{array}{ccc}
I \times E & \xrightarrow{G_\gamma} & \gamma^*(E) \\
\downarrow & \downarrow & \downarrow \\
I & \xrightarrow{\gamma} & \gamma^* (E)
\end{array}
\] (3)

2. Let $\bar{G}_\gamma = \bar{\gamma} \circ G_\gamma$ and consider the pull-back $(\bar{G}_\gamma)^*(TE) \to I \times E$ of the bundle $p_E : TE \to E$. Then there exists a homeomorphism $H_\gamma : I \times E \times M \times E \to \bar{G}_\gamma^*(TE)$, which is a bundle isomorphism over $Id$ of $I \times E$ 1-regulated in the first variable and smooth in the other variables, and such that the following diagram is commutative

\[
\begin{array}{ccc}
I \times E \times M \times E & \xrightarrow{H_\gamma} & \bar{G}_\gamma^*(TE) \\
\downarrow & \downarrow \quad \downarrow & \downarrow \\
I \times E & \xrightarrow{\gamma} & \bar{G}_\gamma^* (TE)
\end{array}
\] (4)

Proof
1. Since $I$ is (smooth) contractible and $\gamma = \pi \circ c$ is a 1-regulated path, there exists a bundle isomorphism $G_\gamma$ of class $C^0$ over the identity from the trivial bundle $I \times E \to I$ to the bundle $\gamma^*(E) \to I$ (cf. [AMR02] Theorem 3.4.35). In fact since $\gamma$ is 1-regulated, it follows that $G_\gamma$ is also 1-regulated in the variable $t \in I$. Smoothness in the other argument follows from linearity in the fibers. Since $G_\gamma$ is a bundle isomorphism the commutativity of the diagram follows.

2. By the same argument as previously, there exists a bundle isomorphism $H_\gamma$ from the trivial bundle $I \times E \times M \times E \to I \times E$ to the bundle $(\bar{G}_\gamma)^*(TE) \to I \times E$. The fact that $\bar{G}_\gamma$ is 1-regulated in the variable $t$ and smooth in the other variables implies the corresponding properties of $H_\gamma$. Now since $H_\gamma$ is a bundle isomorphism, the commutativity of diagrams follows.

If $q_E$ is the projection of $I \times E$ on $E$, the map $q_E \circ G_\gamma^{-1} : \gamma^*(E) \to E$ is called a representation of $\gamma^*(E)$. In particular for any 0-regulated lift $c : I \to E$ of $\gamma$, then $u = q_E \circ G_\gamma^{-1}(c)$ belongs to $\mathcal{R}^0(I,E)$. The pair $(\gamma,u)$ will be called a representation of $c$. According to the notations of Lemma 4.3, a map $\vartheta : I \to E$ induces a section of $\gamma^*(E)$ if and only if $\pi \circ \vartheta = \gamma$. Then, from Lemma 4.3 Point (1), $S_\gamma = q_E \circ \bar{G}_\gamma^{-1}$ induces a vector isomorphism from the vector space $\Gamma(\gamma^*(E))$
of 0-regulated lifts \( c : I \to E \) along \( \gamma \) onto the Banach vector space \( \mathcal{R}^0(I, E) \).

By the same arguments applied to \( E = TM \) we also have a vector isomorphism \( \mathcal{T}_\gamma \) from the vector space \( \Gamma(\gamma) \) of 1-regulated vector fields along \( \gamma \) to \( \mathcal{R}^1(I, M) \).

Using the bundle morphism \( \hat{G}_\gamma : \hat{G}^*_\gamma(TE) \to TE \) over \( \hat{G}_\gamma \) from the diagram (4) and considering a 0-regulated lift \( c : I \to E \) of \( \gamma \) we note that any map \( \mathcal{X} : I \to TE \) satisfies \( p_E \circ \mathcal{X} = \mathcal{c} \) if and only if \( q_{I \times \mathcal{E}} \circ H_\gamma^{-1} \circ \hat{G}_\gamma^{-1} \circ \mathcal{X} = \hat{G}_\gamma^{-1} \circ \mathcal{c} \).

Given a vector field \( \mathcal{X} \) along \( c \), if \( q_{I \times \mathcal{E}} \) is the projection of \( I \times \mathcal{E} \times M \times \mathcal{E} \) on \( M \times \mathcal{E} \), then the map

\[
t \mapsto q_{I \times \mathcal{E}} \circ H_\gamma^{-1} \circ \hat{G}_\gamma^{-1} \circ \mathcal{X}(t)
\]

is a map from \( I \) to \( M \times \mathcal{E} \) which is a 0-regulated path if and only if \( \mathcal{X} \) is a 0-regulated path. Note that \( T\pi(\mathcal{X}) \) is vector field along \( \gamma \) and according to the commutativity of the diagrams in Lemma (4) the projection of

\[
t \mapsto q_{I \times \mathcal{E}} \circ H_\gamma^{-1} \circ \hat{G}_\gamma^{-1} \circ \mathcal{X}(t)
\]
on \( M \) is nothing else but \( t \mapsto \mathcal{T}_\gamma(T_{c(t)}\pi(\mathcal{X}(t))) \).

Let \( \mathcal{T}_c \) be the map from the vector space \( \Gamma(c) \) to \( \mathcal{R}^0(I, M) \times \mathcal{R}^0(I, E) \) defined by

\[
\mathcal{X} \mapsto q_{I \times \mathcal{E}} \circ H_\gamma^{-1} \circ \hat{G}_\gamma^{-1} \circ \mathcal{X}.
\]
Clearly \( \mathcal{T}_c \) is a vector isomorphism and if \( q_1 \) is the projection of \( \mathcal{R}^1(I, M) \times \mathcal{R}^0(I, E) \) onto \( \mathcal{R}^1(I, M) \), we have \( q_1 \circ \mathcal{T}_c(\mathcal{X}) = \mathcal{T}_\gamma(T\pi(\mathcal{X})) \).

Therefore any vector field \( \mathcal{X} \) along \( c \) will be written as a pair \((\varphi, \vartheta) = \mathcal{T}_c(\mathcal{X})\) with \( \varphi \in \mathcal{R}^1(I, M) \) and \( \vartheta \in \mathcal{R}^0(I, E) \). Note that we have of course \( \varphi = \mathcal{T}_\gamma(T\pi(\mathcal{X})) \). Therefore \((\varphi, \vartheta)\) will be called a representation of \( \mathcal{X} \). Thus for one choice of the trivializations \( G_\gamma \) and \( H_\gamma \), we get the following commutative diagram

\[
\begin{array}{ccc}
\Gamma(\gamma^*(E)) & \xrightarrow{S_\gamma} & \mathcal{R}^0(I, E) \\
pE & \downarrow qE & \\
\Gamma(c) & \xrightarrow{\mathcal{T}_c} & \mathcal{R}^1(I, M) \times \mathcal{R}^0(I, E) \\
\downarrow T\pi & & \downarrow q_1 \\
\Gamma(\gamma) & \xrightarrow{\mathcal{T}_\gamma} & \mathcal{R}^1(I, M)
\end{array}
\]

(5)

Representations \( S_\gamma, \mathcal{T}_\gamma \) and \( \mathcal{T}_c \) of \( \Gamma(\gamma^*(E)), \Gamma(\gamma) \) and \( \Gamma(c) \) such that the diagram (5) is commutative are called compatible. Note that if \( S'_\gamma, \mathcal{T}'_\gamma \) and \( \mathcal{T}'_c \) are other compatible representations of \( \Gamma(\gamma^*(E)), \Gamma(\gamma) \) and \( \Gamma(c) \) then there exists a 0-regulated field \( A^E_\gamma \), a 1-regulated field \( A_\gamma \) and a 0-regulated field \( A_c \) of automorphisms of \( \mathcal{E}, M \) and \( M \times \mathcal{E} \) respectively such that:

\[
S'_\gamma = A^E_\gamma \circ S_\gamma, \quad T'_\gamma = A_\gamma \circ \mathcal{T}_\gamma, \quad T'_c = A_c \circ \mathcal{T}_c \text{ and } A_\gamma = q_1 \circ A_c.
\]

(6)
Notations and conventions 4.4

Fixing a choice of such compatible trivializations $T_\gamma$ and $T_c$ of $\Gamma(\gamma)$ and $\Gamma(c)$ we will identify $X \in \Gamma(c)$ with the pair $(\varphi, \vartheta) = T_c(X)$ and $X = T\pi(X)$ with $\varphi$. In this way $\vartheta$ can be considered as a 0-regulated vertical vector field of $E$ over $c$ as well as a 0-regulated section of $E$ over $\gamma$ according to the fact that a vertical vectors at $(x, u) \in E$ can be canonically identified with vectors of the fiber $E_x$.

Proposition 4.5 With the previous notations the representations $S_\gamma$, $T_\gamma$ and $T_c$ induce Banach structures $|| \cdot ||_{S_\gamma}$, $|| \cdot ||_{T_\gamma}$, $|| \cdot ||_{T_c}$ on $\Gamma(\gamma^*(E))$, $\Gamma(\gamma)$ and $\Gamma(c)$ from $R^0(I, E)$, $R^1(I, E)$ and $R^1(I, M) \times R^0(I, E)$ which are independent on the choice of compatible trivializations of $\gamma^*(E)$, $\gamma^*(TM)$ and $G^*_\gamma(TE)$.

Remark 4.6

1. Assume that for a 1-regulated path $\gamma : I \rightarrow M$ there exists a chart $(U, \phi) \in \mathcal{A}$ such that $\gamma(I)$ is contained in $U$ and that $E|_U$ is trivializable and let $\Phi : E|_U \rightarrow \phi(U) \times \mathbb{E}$ be a trivialization. Then $\Phi$ induces a trivialization $G_\gamma : I \times \mathbb{E} \rightarrow \gamma^*(E)$ of the bundle $\gamma^*(E)$ by $G_\gamma(t, u) = \Phi^{-1}(\phi(\gamma(t)), u)$. Then for any section $c$ of $\gamma^*(E)$, $\Phi(c)$ is a pair of paths $(x, u) : I \rightarrow M \times \mathbb{E}$ and the representation $S^{\phi}_\gamma$ of $\Gamma(\gamma^*(E))$ into $R^0(I, \mathbb{E})$ associated to $G_\gamma$ is exactly the projection on $\mathbb{E}$ of $\Phi(c)$. In the same way a natural representation $T^\phi_\gamma$ of $\Gamma(\gamma)$ into $R^1(I, M)$ is the projection on $M$ of $T\Phi$. Finally, we also have a natural representation $T^\phi_c$ of $\Gamma(c)$ into $R^1(I, M) \times R^0(I, \mathbb{E})$ which is a projection on $M \times \mathbb{E}$ of $T\Phi$.

2. Let $\tau = (\tau_i)_{i=0,\ldots,n}$ be a partition of $I$, $(U_i, \phi_i)_{i=1,\ldots,n} \subset \mathcal{A}$ be a covering of a 1-regulated path $\gamma : I \rightarrow M$ such that $\gamma([\tau_{i-1}, \tau_i]) \subset U_i$ and $c : I \rightarrow E$ be a 0-regulated lift of $\gamma$. We set $\gamma_i = \gamma|_{[\tau_{i-1}, \tau_i]}$ and $c_i = c|_{[\tau_{i-1}, \tau_i]}$. Then a trivialization $G_\gamma_i : I \times M \rightarrow \gamma^*_i(E)$ (resp. $G_\gamma : I \times M \times \mathbb{E} \rightarrow c^*_i(T\gamma)$) induces by restriction a trivialization $G^{\gamma}_\gamma_i : [\tau_{i-1}, \tau_i] \times M \rightarrow \gamma^*_i(E)$ (resp. $G^{\gamma}_\gamma : [\tau_{i-1}, \tau_i] \times M \times \mathbb{E} \rightarrow c^*_i(T\gamma)$). This implies that the corresponding trivialization $T^{\phi}_\gamma_i : \Gamma(\gamma_i) \rightarrow R^1([\tau_{i-1}, \tau_i], M)$ (resp. $T^{\phi}_c : \Gamma(c_i) \rightarrow R^1([\tau_{i-1}, \tau_i], M) \times R^0([\tau_{i-1}, \tau_i], \mathbb{E})$) is the restriction of $T_\gamma$ (resp. $T_c$) to $\Gamma(\gamma_i)$ (resp. $\Gamma(c_i)$). Moreover we have a 1-regulated (resp. 0-regulated) field $t \rightarrow A_{\gamma_i}$ (resp. $t \rightarrow A_c$) of automorphisms of $M$ (resp. $M \times \mathbb{E}$) such that (cf. (1)):

$$T^{\phi}_i = A_{\gamma_i} \circ T^{\phi}_\gamma, \quad T^{\phi}_c = A_c \circ T^{\phi}_c, \text{ and } A_{\gamma_i} = q_i \circ A_i.$$

3. Given a trivialization $G_\gamma : I \times M \rightarrow \gamma^*(TM)$, obtained from Lemma 4.3 in case $E = TM$, for each $s \in I$ the linear map $T_s : M \equiv T(0)M \rightarrow T(\gamma(s))M$ defined by $T_s(v) = \gamma \circ G_{\gamma}(s, v)$ is an isomorphism of Banach spaces. Set $P^s_t = T_s \circ T^{-1}_t$.

Given any $X \in \Gamma(\gamma)$ we can define

$$\nabla_\gamma X(t) = \frac{d}{dt} (P^s_t(X(s)))|_{s=t}.$$
outside at most countable subset of $I$. Using local chart and Point 1, it is easy to see that it can be extended to a 0-regulated vector field along $\gamma$. It follows that the norm $\|X(0)\|_{M} + \|\nabla_{\gamma} X\|_{\infty}$. (7)

Operators $P_{t}$ play the role of a parallel transport. Namely if we would have a linear connection on $M$ and $\nabla$ would be its Koszul operator, then we would have a parallel transport along $\gamma$ defined in the same way as previously and (7) would be the standard norm on $\Gamma(\gamma)$ associated to $\nabla$ (see for instance [Sch16, A8]).

**Proof of Proposition 4.5**

Since $T_{\gamma}$ is a vector bundle isomorphism from $\Gamma(\gamma)$ to $\mathcal{R}^{1}(I, M)$ we can provide $\Gamma(\gamma)$ with the pull-back Banach structure. Then $T_{\gamma}$ induces an isometry between these two Banach spaces. Indeed if $\varphi = T_{\gamma}(X)$ then outside at most a countable subset of $I$ we have

$$\dot{\varphi}(t) = \frac{d}{dt} \{T_{\gamma}\}X(t) + T_{\gamma} \frac{d}{dt}\{X\}(t).$$

Since $\varphi$ is 1-regulated on $\Gamma(\gamma)$ the induced norm is

$$\|X\|_{T_{\gamma}} = \max \left\{ \|T_{\gamma}(X)\|_{\infty}, \left\| \frac{d}{dt}\{T_{g}\}X(t) + T_{g} \frac{d}{dt}\{X\}(t) \right\|_{\infty} \right\}. \quad (8)$$

From Lemma 4.3 if $G_{t} : I \times E \to \gamma^{*}(TM)$ is another trivialization, then $G_{t} \circ G_{t}^{-1}$ is an automorphism of the bundle $I \times \mathbb{M} \to I$. As we have already seen, there exists 1-regulated field $t \mapsto A(t)$ from $I$ to $GL(\mathbb{M})$ such that $T_{\gamma}(X(t)) = A(t)T_{\gamma}(X(t))$ for all $t \in I$. Since any 1-regulated path is bounded (cf. Proposition 3.4), if $\varphi = T_{\gamma}(X)$ and $\varphi' = T_{\gamma}'(X)$ then it is easy to show that there exists $K > 0$ such that we have

$$\|\varphi\|_{\infty} \leq K \|\varphi'\|_{\infty} \quad \text{and} \quad \|\dot{\varphi}'\|_{\infty} \leq K \sup\{\|\varphi'\|_{\infty}, \|\dot{\varphi}'\|_{\infty}\}.$$ 

This implies that the norms on $\Gamma(\gamma)$ induced by $T_{\gamma}$ and $T_{\gamma}'$ are equivalent. Since any 0-regulated path is also bounded, we can apply formally the same arguments to the isomorphism $S_{c}$ from $\Gamma(\gamma^{*}(E))$ to $\mathcal{R}^{0}(I, E)$.

Finally, for the vector space $\Gamma(c)$ again we can use analogous arguments as the previous ones for two trivializations $H_{\gamma}$ and $H_{\gamma}'$ of the bundle $G_{\gamma}^{*}(TE)$ over $G_{\gamma}^{-1}(c)$.

$\triangle$

### 4.2 Topology on the set of regulated lifts of paths

Since any 0-regulated path defined on $I$ has a compact range, we can provide the set $\mathcal{P}(E)$ with the topology generated by sets $\mathcal{N}(K, U, V, W)$ of paths $c \in \mathcal{P}(E)$
Thus the proof of (2) is a consequence of (1) by composition of homeomorphisms.

This topology is Hausdorff. Indeed, if \( c_1 \neq c_2 \), there exists \( \theta \in I \) such that \( c_1(\theta) \neq c_2(\theta) \). Now take \( K = \{ \theta \} \). Since \( E \) is Hausdorff, the points \( c_1(\theta) \) and \( c_2(\theta) \) have open disjoint neighbourhoods \( U_1 \) and \( U_2 \). Sets \( \mathcal{N}(K, U_1, M, TM) \) and \( \mathcal{N}(K, U_2, M, TM) \) are required open disjoint neighbourhoods of \( c_1 \) and \( c_2 \).

We can provide \( P(M) \) with the topology in the same way that is generated by sets of type \( \mathcal{N}(K, V, W) \) of paths \( \gamma \in P(M) \) such that the closure of \( \gamma(K) \) is contained in \( V \) and the closure of \( \gamma^{(1)}(K) \) is contained in \( W \) where \( K \) is a compact subset of \( I \) and \( V \) is an open set in \( M \) and \( W \) is an open set in \( TM \).

Now the map \( \pi : E \to M \) induces a natural map \( c \mapsto \pi \circ c \) from \( P(E) \) to \( P(M) \) which we denote also \( \pi \). It is continuous and open with respect to these topologies. This implies in particular that \( P(M) \) is a Hausdorff space.

If \( (U, \phi) \) is a chart domain such that there exists a trivialization \( \Phi : E|_U \to \phi(U) \times \mathbb{E} \), then \( P(E|_U) \) is an open set of \( P(E) \). Now we have:

**Proposition 4.7**

1. For \( i = 1, 2 \) let \( \pi_i : E_i \to M_i \) be two Banach bundles and \( F \) be a bundle morphism from \( E_1 \to M_1 \) to \( E_2 \to M_2 \) over \( f : M_1 \to M_2 \). Then the map \( F_* : P(E_1) \to P(E_2) \) (resp. \( f_* : P(M_1) \to P(M_2) \)) defined by \( F_*(c) = F \circ c \) (resp. \( f_*(\gamma) = f \circ \gamma \)) is continuous. Moreover \( F_* \) and \( f_* \) are injective (resp. a homeomorphism) if \( F \) is injective (resp. isomorphism). Moreover we have \( \pi_2 \circ F_* = f_* \circ \pi_1 \).

2. The map \( \Phi_* : P(E|_U) \to R^1(I, \phi(U)) \times R^0(I, \mathbb{E}) \) is a homeomorphism.

3. The evaluation map \( \text{Ev}^t : P(E) \to E \) defined by \( \text{Ev}^t(c) = c(t) \) is continuous.

**Proof**

(1) For \( c \in P(E_1) \) take an open neighbourhood \( \mathcal{N}_2(K, V, W) \) of \( F \circ c \) in \( P(E_2) \).
Consider \( \mathcal{N}_2(K, F^{-1}(V), Df^{-1}(W)) \). Since \( F \) and \( f \) are smooth, it follows that \( F^{-1}(V) \) and \( Df^{-1}(W) \) are open sets of \( E_1 \) and \( TM_1 \) respectively. Therefore \( \mathcal{N}_2(K, F^{-1}(V), Df^{-1}(W)) \) is a neighbourhood of \( c \) and \( F_*(\mathcal{N}_2(K, F^{-1}(V), Df^{-1}(W))) \) is contained in \( \mathcal{N}_2(K, V, W) \). Thus \( F_* \) is continuous. It is clear that if \( F \) is injective so is \( F_* \). Now if \( F \) is an isomorphism, by application of the previous result to the isomorphism \( F^{-1} \) we obtain that \( F_* \) is a homeomorphism. All the results about \( F_* \) comes from the relation \( \pi_2 \circ F = f \circ \pi_1 \).

(2) Note that the set \( P(M \times \mathbb{E}) \) is exactly \( R^1(I, M) \times R^0(I, \mathbb{E}) \). Moreover since \( M \) and \( \mathbb{E} \) are metrizable, the topology defined on \( P(M \times \mathbb{E}) \) coincides with product topology of the Banach topology of \( R^1(I, M) \) and \( R^0(I, \mathbb{E}) \). Now \( \phi(U) \) is an open set of \( M \) so \( R^1(I, \phi(U)) \times R^0(I, \mathbb{E}) \) is an open set of \( R^1(I, M) \times R^0(I, \mathbb{E}) \). Thus the proof of (2) is a consequence of (1) by composition of homeomorphisms.
(3) Take any open set $V$ in $E$. Then $(\text{Ev}^t)^{-1}(V)$ is the set $\{c \in \mathcal{P}(E) \mid c(t) \in V\}$. If we take the compact interval $K = \{t\}$, it follows that by construction of the topology on $\mathcal{P}(E)$ that $(\text{Ev}^t)^{-1}(V)$ is an open set.

\[\square\]

5 Banach manifold structure of $\mathcal{P}(E)$

In this section, we fix an atlas $\mathfrak{A} = \{U_\alpha, \phi_\alpha\}_{\alpha \in \mathcal{A}}$ on $M$ such that each $U_\alpha$ is contractible. In this situation recall that from Theorem 3.4.35 in [AMR02] it follows that the bundle $E|_{U_i}$ is trivializable.

We have the following basic fundamental result:

**Theorem 5.1**

1. The set $\mathcal{P}(M)$ has a Banach manifold structure modeled on the Banach space $\mathcal{R}^1(I, \mathbb{M})$ such that each chart domain of this structure is open for its compact open topology. Moreover for any $\gamma \in \mathcal{P}(M)$ we have a Banach isomorphism from $T_\gamma \mathcal{P}(M)$ onto $\Gamma(\gamma)$.

2. The set $\mathcal{P}(E)$ has a Banach manifold structure modeled on the Banach space $\mathcal{R}^1(I, \mathbb{M}) \times \mathcal{R}^0(I, \mathbb{E})$ such that each chart domain is open for the natural topology of $\mathcal{P}(E)$ and we have a Banach isomorphism from $T_\gamma \mathcal{P}(E)$ onto $\Gamma(\gamma)$. Moreover $\pi : \mathcal{P}(E) \to \mathcal{P}(M)$ is a Banach bundle with typical fiber $\mathcal{R}^0(I, \mathbb{E})$.

Before proving the theorem we begin by the following Lemma

**Lemma 5.2** Given a partition $\tau = (\tau_i)_{i=0,\ldots,n}$ of the interval $I$ we define the map $I_\tau$ which to

$$ (x, y_1, v_1, \ldots, y_n, v_n) \in \mathbb{M} \times \prod_{i=1}^{n} (\mathcal{R}^0([\tau_{i-1}, \tau_i], \mathbb{M}) \times \mathcal{R}^0([\tau_{i-1}, \tau_i], \mathbb{E})) $$

associates a pair of functions $(\varphi, \vartheta)$ in $\mathcal{R}^1(I, \mathbb{M}) \times \mathcal{R}^0(I, \mathbb{E})$ in the following way:

$$ \vartheta = v_1 \ast v_2 \ast \ldots \ast v_n, $$
$$ \varphi = x + \int_0^t (y_1 \ast y_2 \ast \ldots \ast y_n)(s)ds, $$

where $\ast$ is a concatenation of curves defined in Section 2. We define also a map $l_\tau : \mathbb{M} \times \prod_{i=1}^{n} \mathcal{R}^0([\tau_{i-1}, \tau_i], \mathbb{M}) \to \mathcal{R}^1(I, \mathbb{M})$ as follows

$$ l_\tau(x, v_1, \ldots, v_n) = \varphi. $$

Then the maps $I_\tau$ and $l_\tau$ are Banach isomorphisms.
The lemma follows by applying \( n \) times Proposition 2.8 and Proposition 2.7.

Point 3.

Consider a curve \( c \in \mathcal{P}(E) \), i.e. \( 0 \)-regulated path \( c : I \to E \) such that \( \gamma = \pi \circ c \) is \( 1 \)-regulated. Assume for some subinterval \( J = [s_0, s_1] \) of \( I \) that \( \gamma(J) \) is contained in the domain \( U \) of a chart \( (U, \phi) \) such that \( E_{|U} \) is trivializable by a map \( \Phi : E_{|U} \to \phi(U) \times E \subset M \times E \). Then the image \( \Phi(c_{|J}) \) can be written as a pair \((\phi \circ \gamma_{|J}, u) : J \to M \times E \) and since \( \gamma \) is \( 1 \)-regulated, we have (cf. (11)):

\[
\phi(\gamma_{|J}(t)) = \phi(\gamma(s_0)) + \int_{s_0}^{t} (\phi \circ \gamma)^{(1)}(s)ds.
\]

In such a situation, we set

\[
\tilde{\phi}(\gamma_{|J}) = (\phi \circ \gamma_{|J})^{(1)} \quad \tilde{\Phi}(c_{|J}) = (\tilde{\phi}(\gamma), u).
\]

Therefore the path \( \tilde{\phi}(\gamma_{|J}) \) belongs to \( \mathcal{R}^{0}([s_0, s_1], \mathbb{M}) \) and the path \( \tilde{\Phi}(c_{|J}) \) belongs to \( \mathcal{R}^{0}([s_0, s_1], \mathbb{M}) \times \mathcal{R}([s_0, s_1], \mathbb{E}) \) and of course

\[
\tilde{\phi}(\pi \circ c_{|J}) = p_{\text{cd}} \circ \tilde{\Phi}(c_{|J}).
\]

We come back to the general context of the proof of Theorem 5.1. Recall that the atlas \( \mathfrak{A} = \{U_{\alpha}, \phi_{\alpha}\}_{\alpha \in A} \) of \( M \) is fixed and each \( U_{\alpha} \) is assumed to be connected and simply connected and so there exist a trivialization

\[
\Phi_{\alpha} : E_{U_{\alpha}} := E_{|U_{\alpha}} \to \phi_{\alpha}(U_{\alpha}) \times E \subset M \times E.
\]

For each integer \( n \geq 1 \), each partition \( \tau = (\tau_i)_{i=0,\ldots,n} \) and each family \( \mathfrak{U} = (U_i)_{i=1,\ldots,n} \) of chart domains of \( \mathfrak{A} \) we consider the sets:

\[
P(\tau, \mathfrak{U}) = \{ \gamma \in \mathcal{P}(M) \mid \gamma((\tau_{i-1}, \tau_{i})) \subset U_i, \forall i = 1, \ldots, n \},
\]

\[
P(\tau, \mathfrak{U}) = \{ c \in \mathcal{P}(E) \mid \pi \circ c \in \mathcal{P}(\tau, \mathfrak{U}) \}.
\]

Since all functions in \( \mathcal{P}(M) \) are continuous and thus have compact image, the set of all such \( \mathcal{P}(\tau, \mathfrak{U}) \) (resp. \( \mathcal{P}(\tau, \mathfrak{U}) \)) for all integers \( n \), all partitions \( \tau = (\tau_i)_{i=0,\ldots,n} \) and all sequences \( \mathfrak{U} = (U_i)_{i=1,\ldots,n} \) is a covering of \( \mathcal{P}(E) \) (resp. \( \mathcal{P}(M) \)).

According to (11), for any set \( \mathcal{P}(\tau, \mathfrak{U}) \) and \( \mathcal{P}(\tau, \mathfrak{U}) \) we consider the maps

\[
\mathbb{F}_{\tau, \mathfrak{U}}(c) = (\phi_{\tau} (\pi \circ c(0)), \Phi_{\tau} (c_1), \ldots, \Phi_{\tau} (c_n)),
\]

where \( c \in \mathcal{P}(\tau, \mathfrak{U}), \gamma = \pi \circ c, c_i = c_{[\tau_{i-1}, \tau_i]}, \text{ and} \)

\[
f_{\tau, \mathfrak{U}}(\gamma) = (\phi_{\tau} (\gamma(0)), \tilde{\phi}_{\tau} (\gamma_1), \ldots, \tilde{\phi}_{\tau} (\gamma_n)),
\]

where \( \gamma \in \mathcal{P}(\tau, \mathfrak{U}), \gamma_{i} = \gamma_{[\tau_{i-1}, \tau_i]} \).

Therefore \( \mathbb{F}_{\tau, \mathfrak{U}}(c) \) belongs to \( M \times \prod_{i=1}^{n} \mathcal{R}^{0}([\tau_{i-1}, \tau_{i}], \mathbb{M}) \times \mathcal{R}^{0}([\tau_{i-1}, \tau_{i}], \mathbb{E}) \) and \( f_{\tau, \mathfrak{U}}(\gamma) \) belongs to \( M \times \prod_{i=1}^{n} \mathcal{R}^{0}([\tau_{i-1}, \tau_{i}], \mathbb{M}) \).
Comments 5.3 According to Lemma 5.2, first we must show that the set of pairs \((\mathcal{P}(\tau, \Omega), \mathcal{I}_\tau \circ \mathcal{F}_\tau, \Omega)\) (resp. \((\mathcal{P}(\tau, \Omega), \mathcal{I}_\tau \circ f_{\tau, \Omega})\)) for all integers \(n\), all partitions \(\tau = (\tau_i)_{i=0,...,n}\) and all sequences \(\Omega = (\Omega_i)_{i=0,...,n}\), defines a Banach manifold structure on \(\mathcal{P}(E)\) (resp. \(\mathcal{P}(M)\)).

However since \(\mathcal{I}_\tau\) and \(1_\tau\) are isomorphisms, it is sufficient to show that each \(c \in \mathcal{P}(E)\) belongs to a chart domain \(\mathcal{P}(\tau, \Omega)\) which is homeomorphic to an open set in \(\mathcal{M} \times \prod_{i=1}^{n} \mathcal{R}^0([\tau_{i-1}, \tau_i], \mathcal{M}) \times \mathcal{R}([\tau_{i-1}, \tau_i], E)\) via \(\mathcal{F}_\tau\) and the transition maps between two such charts \((\mathcal{P}(\tau, \Omega), \mathcal{F}_\tau, \Omega)\) and \((\mathcal{P}(\sigma, \Omega), \mathcal{F}_\tau, \Omega)\) are local diffeomorphisms between two Banach spaces of type \(\mathcal{M} \times \prod_{i=1}^{n} \mathcal{R}^0([\sigma_{j-1}, \sigma_j], \mathcal{M}) \times \mathcal{R}^0([\sigma_{j-1}, \sigma_j], E)\) and \(\mathcal{M} \times \prod_{i=1}^{n} \mathcal{R}^0([\tau_{i-1}, \tau_i], \mathcal{M}) \times \mathcal{R}^0([\tau_{i-1}, \tau_i], E)\). This proof will be will be organized in six steps.

**Step 1:** \(\mathcal{P}(\tau, \Omega)\) and (resp. \(\mathcal{P}(\tau, \Omega)\)) is open for the compact-open topology on \(\mathcal{P}(E)\) (resp. \(\mathcal{P}(M)\) (cf. § 1.2).

Since \(\pi\) is open and \(\pi(\mathcal{P}(\tau, \Omega)) = \mathcal{P}(\tau, \Omega)\), we have only to prove this result for \(\mathcal{P}(\tau, \Omega)\). For any open set \(O\) in \(\mathcal{M}\) and any subinterval \([\alpha, \beta]\) of \(I\), since \([\alpha, \beta]\) is compact and \(\pi \circ \gamma\) is continuous the set

\[
\{ c \in \mathcal{P}(E) : \pi \circ c([\alpha, \beta]) \subset O \}
\]

is an open subset of \(\mathcal{P}(E)\). Thus the sets \(O_i = \{ c \in \mathcal{P}(E) : \pi \circ c([\tau_{i-1}, \tau_i]) \subset U_i \}\) are open and \(\mathcal{P}(\tau, \Omega)\) is exactly the intersection \(\bigcap_{i=1}^{n} O_i\) which ends the proof.

**Step 2:** \(\mathcal{F}_\tau, \Omega\) and \(f_{\tau, \Omega}\) are injective and continuous.

We will prove it only for \(\mathcal{F}_\tau, \Omega\) as the proof for \(f_{\tau, \Omega}\) is analogous. Let \(c\) and \(c'\) be two paths in \(\mathcal{P}(\Omega, \tau)\) such that \(\mathcal{F}_\tau, \Omega(c) = \mathcal{F}_\tau, \Omega(c')\). According to the previous notations, this implies for \(i = 1\) that

\[
\phi_1(c(0)) = \phi_1(c'(0)), \quad (\phi_1(\pi \circ c_1))^{(1)} = (\phi_1(\pi \circ c'_1))^{(1)}, \quad u_1 = u'_1
\]

Therefore \(c_{[\tau_0, \tau_1]} = c'_{[\tau_0, \tau_1]}\). Assume that for \(1 \leq i < n\) we have \(c_{[\tau_i, \tau_{i-1}]} = c'_{[\tau_i, \tau_{i-1}]}\). Then we can write \(c_{[\tau_{i-1}, \tau_i]} = \Phi_i^{-1}(\gamma_i, u_i)\) and \(c'_{[\tau_{i-1}, \tau_i]} = \Phi_i^{-1}(\gamma'_i, u'_i)\). Now from our assumption we have \(\gamma_i^{(1)} = (\gamma'_i)^{(1)}\) and \(u_i = u'_i\). Since \(c(\tau_i) = c'(\tau_i)\) so \(\phi_1(\pi \circ c(\tau_{i-1})) = \phi_1(\pi \circ c'(\tau_{i-1}))\). But according to Remark 2.3, the restriction of \(\gamma_i^{(1)}\) (resp. \((\gamma'_i)^{(1)}\)) coincides with \(\phi_1(\pi \circ c([\tau_{i-1}, \tau_i]))\) (resp. \(\phi_1(\pi \circ c([\tau_{i-1}, \tau_i]))\)). Therefore this implies that \(\gamma_i = \gamma'_i\) and finally \(c = c'\) on \([\tau_0, \tau_1]\). By induction on \(1 \leq i \leq n\) we obtain \(c = c'\) and so \(\mathcal{F}_\tau, \Omega\) is injective.

For the proof of continuity of \(\mathcal{F}_\tau, \Omega\), it is sufficient to prove the continuity of each component of this map. The first component is \(E \mathbb{V}^0_\mathcal{M}\) which is continuous as we have already seen (Proposition 4.7 Point (3)). Now for \(J = [\tau_{i-1}, \tau_i]\) by application of Proposition 4.7 Point (2), Proposition 2.8 and Proposition 2.7 we prove the \(\Phi_i\) is continuous for \(i = 1, \ldots, n\).

**Step 3:** The range of \(\mathcal{F}_\tau, \Omega\) (resp. \(f_{\tau, \Omega}\)) is open in the Banach space \(\mathcal{M} \times \prod_{i=1}^{n} \mathcal{R}^0([\tau_{i-1}, \tau_i], \mathcal{M}) \times \mathcal{R}([\tau_{i-1}, \tau_i], E)\) (resp. \(\mathcal{M} \times \prod_{i=1}^{n} \mathcal{R}^0([\tau_{i-1}, \tau_i], \mathcal{M})\)).
According to (10), we have only to prove the result for $\mathcal{F}_{\tau, U}$. Recall that norm on each space $\mathcal{R}^k([\tau_{i-1}, \tau_i], M)$ is

$$\|x_i\|_\infty = \sup\{|x_i(t)|_M \; t \in [\tau_{i-1}, \tau_i]\},$$

as defined in 2.5.

**Fix some** $c \in \mathcal{P}(\tau, U)$. We set $\bar{\gamma}_i = \phi_i \circ \pi \circ c_i$ and so we have

$$\mathcal{F}_{\tau, U}(c) = (\bar{\gamma}_{\tau_0}, \gamma_1^{(1)}, u_1, \ldots, \gamma_n^{(1)}, u_n).$$

Since for $i = 1, \ldots, n-1$, each $\phi_i(U_i)$ and $\phi_i(U_i \cap U_{i+1})$ are open in $M$, $[\tau_{i-1}, \tau_i]$ is compact we can find $\delta_i > 0$ such that the open balls satisfy the following inclusions:

1. $B(\bar{\gamma}_i(\tau_i), \delta_i) \subset \phi_i(U_i \cap U_{i+1})$ for $i = 1, \ldots, n-1$
2. $B(\bar{\gamma}_i(t), \delta_i) \subset \phi_i(U_i)$ for all $t \in [\tau_{i-1}, \tau_i]$ and $i = 1, \ldots, n$.

Moreover since $\phi_{i+1} \circ \phi_i^{-1}$ is continuous we can also find strictly positive numbers $\varepsilon_1, \ldots, \varepsilon_{n-1}$ and $\eta_1, \ldots, \eta_{n-1}$ such that:

3. $\varepsilon_i < \delta_i/2$, $\eta_i < \delta_i/2$ for $i = 1, \ldots, n-1$
4. $\phi_{i+1} \circ \phi_i^{-1}(B(\bar{\gamma}_i(\tau_i), \delta_i)) \subset B(\bar{\gamma}_{i+1}(\tau_i), \varepsilon_{i+1})$.

We set $\varepsilon = \max(\varepsilon_1, \ldots, \varepsilon_{n-1})$ and $\eta = \max(\eta_1, \ldots, \eta_{n-1})$.

We denote by $\mathcal{O}$ the open neighbourhood of $\mathcal{F}_{\tau, U}(c)$ consisting of $(x', x'_1, u'_1, \ldots, x'_n, u'_n) \in M \times \Pi^n_1 \mathcal{R}^0([\tau_{i-1}, \tau_i], M) \times \mathcal{R}^0([\tau_{i-1}, \tau_i], E)$ such that

$$\|\bar{\gamma}(t_0) - x'\|_M < \varepsilon, \quad \left\|\bar{\gamma}_i^{(1)}(t) - x'_i\right\|_\infty < \eta, \quad 1 \leq i \leq n.$$

Now we will show that $\mathcal{O}$ is contained in $\mathcal{F}_{\tau, U}(\mathcal{P}(\tau, U))$. To this end to any $(x', x'_1, u'_1, \ldots, x'_n, u'_n) \in \mathcal{O}$ we associate a 1-regulated path $((\bar{\gamma}_i')_i, (\bar{\gamma}_n')_i)$ in $\Pi^n_1 \mathcal{R}^1([\tau_{i-1}, \tau_i], M)$ such that

5. $\bar{\gamma}_i'([\tau_{i-1}, \tau_i]) \subset \phi_i(U_i)$, $\bar{\gamma}_{i+1}'(\tau_i) = \phi_{i+1} \circ \phi_i^{-1}(\bar{\gamma}_i'(\tau_i))$ and $\bar{\gamma}_i'(\tau_i) \in B(\bar{\gamma}_i(\tau_i), \delta_i)$

in the following way:

- for $i = 1$ we define $\bar{\gamma}_1'$ by $\bar{\gamma}_1' = x' + \int_{\tau_0}^{\tau_1} x'_1(s)ds$. By construction $\bar{\gamma}_1'$ is 1-regulated and from the property (3) we have $\|\bar{\gamma}_1 - \gamma_1'\|_\infty < \delta_1$. This implies that in particular $\bar{\gamma}_1'(\tau_1)$ belongs to $B(\bar{\gamma}_1(\tau_1), \delta_1)$ and from the property (2) we get $\bar{\gamma}_1'([\tau_0, \tau_1]) \subset \phi_1(U_1)$;

- assume that we have built a sequence of 1-regulated paths $(\bar{\gamma}_i', \ldots, \gamma_n')$ in $\Pi^n_1 \mathcal{R}^0([\tau_{i-1}, \tau_i], M)$ which satisfies the property (5). Since $\gamma_i'(\tau_i) \in B(\bar{\gamma}_i(\tau_i), \delta_i)$ then from property (4), $\phi_{i+1} \circ \phi_i^{-1}(\bar{\gamma}_{i}'(\tau_i))$ belongs to $B(\bar{\gamma}_{i+1}(\tau_i), \varepsilon_{i+1})$. We set

$$\bar{\gamma}_{i+1}' = \phi_{i+1} \circ \phi_i^{-1}(\bar{\gamma}_i'(\tau_i)) + \int_{\tau_i}^{\tau_{i+1}} x'_{i+1}(s)ds.$$
Therefore we get $\gamma_{i+1}'(\tau_i) = \phi_{i+1} \circ \phi_i^{-1}(\gamma_i'(\tau_i))$ and again as for $i = 1$ we also have $\gamma_{i+1}'(\tau_i, \tau_{i+1}) \subset \phi_{i+1}(U_{i+1})$. We can now define a path $c' \in \mathcal{P}(\tau, \mathcal{U})$ such that $\mathcal{F}_{\tau, \mathcal{U}}(c') = (x', x'_1, u'_1, \ldots, x'_n, u'_n)$ by setting

$$c'_{[\tau_{i-1}, \tau_i]} = \Phi_i^{-1}(\gamma_i', u'_i).$$

Thus we have proved that $\mathcal{O}$ is contained in $\mathcal{F}_{\tau, \mathcal{U}}(\mathcal{P}^k(\tau, \mathcal{U}))$.

**step 4**. Consider two partitions $\tau = (\tau_i)_{i=0, \ldots, n}$ and $\sigma = (\sigma_j)_{j=0, \ldots, m}$ of $I$ with associated sequences $\mathcal{U} = (U_i)_{i=1, \ldots, n}$ and $\mathcal{V} = (V_j)_{j=1, \ldots, m}$ of domains of charts $(U_i, \phi_i)$ and $(V_j, \psi_j)$ respectively such that $\mathcal{P}(\tau, \mathcal{U}) \cap \mathcal{P}(\sigma, \mathcal{V}) \neq \emptyset$. Then the map $\mathcal{F}_{\tau, \mathcal{U}} \circ (\mathcal{F}_{\sigma, \mathcal{V}})^{-1}$ (resp. $f_{\tau, \mathcal{U}} \circ (f_{\sigma, \mathcal{V}})^{-1}$) is a diffeomorphism from its definition domain onto its range.

We will use the “reconstruction map”

$$\Theta : \mathbb{M} \times \prod_{i=1}^n \mathcal{R}([\tau_{i-1}, \tau_i], \mathbb{M}) \to \prod_{i=1}^n \mathcal{R}^1([\tau_{i-1}, \tau_i], \mathbb{M})$$

defined by

$$\Theta(x, y_1, \ldots, y_n) = (\gamma_1, \ldots, \gamma_n),$$

where

$$\bar{\gamma}_1 = x + \int_{\tau_0}^{\tau_1} y_1(s)ds,$$

$$\gamma_{i+1} = \phi_{i+1} \circ \phi^{-1}_i(\gamma_i) + \int_{\tau_i}^{\tau_{i+1}} y_{i+1}(s)ds \quad \text{for} \quad i = 1, \ldots, n - 1.$$

We need the following property:

**Lemma 5.4** $\Theta$ is a smooth map.

**Proof of Lemma 5.4**

It is sufficient to prove that each component $\Theta_i : (x, y_1, \ldots, y_n) \to \bar{\gamma}_i$ is smooth. For any Banach space $\mathbb{F}$, denote again by $\mathbf{Ev}_k^k$ the evaluation map form any Banach space $\mathcal{R}^k(I, \mathbb{F})$ to $\mathbb{F}$, for $k = 0, 1$, given by $\mathbf{Ev}_k^k(c) = c(t)$. Since this map is continuous linear, so it is smooth. Now for $2 \leq i \leq n$ we can write

$$\Theta_i = \mathbf{Ev}\gamma_{i-1}^{-1}(\phi_{i+1} \circ \phi^{-1}_i(\gamma_i)) + L(y_i)$$

where $L$ is linear map from $\mathcal{R}^0([\tau_{i-1}, \tau_i], \mathbb{M})$ to $\mathcal{R}^1([\tau_{i-1}, \tau_i], \mathbb{M})$ and since $\tau_i - \tau_{i-1} < 1$ then $L$ is a contraction and so $L$ is continuous which implies that $L$ is smooth. The same type of arguments can be applied to $\Theta_1$.

**We come back to the proof of this step.** We consider the map $\mathcal{F}_{\tau, \mathcal{U}} \circ (\mathcal{F}_{\sigma, \mathcal{V}})^{-1}$ from $\mathcal{F}_{\sigma, \mathcal{V}}(\mathcal{P}(\tau, \mathcal{U}) \cap \mathcal{P}(\sigma, \mathcal{V})) \subset \mathbb{M} \times \prod_{i=1}^n \mathcal{R}^0([\tau_{i-1}, \tau_i], \mathbb{M}) \times \mathcal{R}([\tau_{i-1}, \tau_i], \mathbb{E})$ to $\mathbb{M} \times \prod_{i=1}^n \mathcal{R}([\tau_{i-1}, \tau_i], \mathbb{M}) \times \mathcal{R}([\tau_{i-1}, \tau_i], \mathbb{E})$. In fact $\mathcal{F}_{\sigma, \mathcal{V}}(\mathcal{P}(\tau, \mathcal{U}) \cap \mathcal{P}(\sigma, \mathcal{V})) = f_{\tau, \mathcal{U}}(\mathcal{P}(\tau, \mathcal{U}) \cap \mathcal{P}(\sigma, \mathcal{V})) \times \prod_{i=1}^n \mathcal{R}([\tau_{i-1}, \tau_i], \mathbb{E})$ and the range of $\mathcal{F}_{\tau, \mathcal{U}} \circ (\mathcal{F}_{\sigma, \mathcal{V}})^{-1}$ is $f_{\tau, \mathcal{U}}(\mathcal{P}(\tau, \mathcal{U}) \cap \mathcal{P}(\sigma, \mathcal{V})) \times \prod_{i=1}^n \mathcal{R}([\tau_{i-1}, \tau_i], \mathbb{E})$. 

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For \( c \in \mathcal{P}(\sigma, \mathcal{W}) \) we set 
\[
\tilde{\gamma}_i = \psi_i \circ \pi \circ c_i \quad \text{and so we have}
\]
\[
F_{\sigma, \mathcal{W}}(c) = (\tilde{\gamma}_1(0), \tilde{\gamma}_1^{(1)}, v_1, \ldots, \tilde{\gamma}_m^{(1)}, v_m).
\]

Let \( \hat{\Theta} \) be the reconstruction map associated to \((\mathcal{W}, \sigma)\) and in order to make further formulas more readable we will use notation \( \hat{\Theta}(w) \) in the following sense:
\[
w = (x, y_1, v_1, \ldots, y_m, v_m) \mapsto \hat{\Theta}(w) = (\hat{\gamma}_1, \ldots, \hat{\gamma}_m).
\]
Now the map:
\[
F_{\tau, \mathcal{U}} \circ (F_{\sigma, \mathcal{W}})^{-1} : f_{\sigma, \mathcal{W}}(P(\tau, \mathcal{U}) \cap P(\sigma, \mathcal{W})) \times \Pi_{i=1}^n \mathcal{R}([\sigma_{j-1}, \sigma_j], \mathbb{E}) \to \mathbb{M} \times \Pi_{i=1}^n \mathcal{R}([\tau_{i-1}, \tau_i], \mathbb{M}) \times \mathcal{R}([\tau_{i-1}, \tau_i], \mathbb{E})
\]

can be written as:
\[
F_{\tau, \mathcal{U}} \circ (F_{\sigma, \mathcal{W}})^{-1}(w) = (G_0(w), G_1(w), W_1(w) \ldots, G_n(w), W_n(w)).
\]

To prove that \( F_{\tau, \mathcal{U}} \circ (F_{\sigma, \mathcal{W}})^{-1} \) is smooth it is sufficient to prove all the components \( G_i \) and \( (G_w, W_i) \) for \( i = 1, \ldots, n \) are smooth. The first one is simply a map \( w \mapsto x \mapsto \phi_1 \circ \psi^{-1}_1(x) \). It is smooth since it is composition of projection onto first term and transition map of the manifold \( M \). The proof for the other components is not as so easy.

For \( i = 1, \ldots, n \) and \( j = 1, \ldots, m \) we denote by \( I_{ij} \) the intersection \([\tau_{i-1}, \tau_i] \cap [\sigma_{j-1}, \sigma_j]\) and we denote by \( N_{\tau_i} = \{ j : I_{ij} \neq \emptyset \} \) and \( N_{\sigma_j} = \{ i : I_{ij} \neq \emptyset \} \). Thus we have \([\sigma_{j-1}, \sigma_j] = \bigcup_{i \in N_{\sigma_j}} I_{ij} \) and \([\tau_{i-1}, \tau_i] = \bigcup_{j \in N_{\tau_i}} I_{ij}\). From notations in relation (13), we consider the restrictions \( \tilde{\gamma}_ij \) of \( \hat{\gamma}_j \) to \( I_{ij} \) and \( \hat{\gamma}_j(I_{ij}) \) is contained in \( \psi_j(U_i \cap V_j) \). In the same way, if \( v_{ij} \) is the restriction of \( v_j \) to \( I_{ij} \) (in the usual sense), then \( w_j(I_{ij}) \) is contained in \( \psi_j(U_i \cap V_j) \times \mathbb{E} \).

Now for \( i = 1, \ldots, n \), the pair of components \((G_i, W_i)\) is the composition of the following smooth maps:

(I) \( r_{ij} : \mathbb{M} \times \Pi_{j=1}^n \mathcal{R}^0([\sigma_{j-1}, \sigma_j], \mathbb{M}) \times \mathcal{R}^0([\sigma_{j-1}, \sigma_j], \mathbb{E}) \to \mathcal{R}^1(I_{ij}, \mathbb{M}) \times \mathcal{R}^0(I_{ij}, \mathbb{E}) \)

defined by
\[
r_{ij}(w) = (r_{ij} \circ \hat{\Theta}_j(w), v_{ij}).
\]
This map is smooth as the composition of the continuous linear map \((r_{ij}, \text{restriction to } I_{ij})\) with the map \((\hat{\Theta}_j, pr_j)\) which is smooth by Lemma 5.4 and where \( pr_j \) is the projection of \( \mathbb{M} \times \Pi_{j=1}^n \mathcal{R}^0([\sigma_{j-1}, \sigma_j], \mathbb{M}) \times \mathcal{R}^0([\sigma_{j-1}, \sigma_j], \mathbb{E}) \) onto \( \mathcal{R}^0([\sigma_{j-1}, \sigma_j], \mathbb{E}) \).

(II) \( d_{ij} : \mathcal{R}^1(I_{ij}, \mathbb{M}) \times \mathcal{R}^0(I_{ij}, \mathbb{E}) \to \mathcal{R}^0(I_{ij}, \mathbb{M}) \times \mathcal{R}^0(I_{ij}, \mathbb{E}) \)

defined by
\[
d_{ij}(f, v_{ij}) = ((\phi_1 \circ \psi^{-1}_1 \circ f)(1), g_{ij}(\phi_1 \circ \psi^{-1}_1 \circ \gamma)v_{ij}) \quad \text{where } g_{ij} \text{ is the smooth field of automorphisms of } \mathbb{E} \text{ associated to } \Phi_1 \circ \Psi^{-1}_1 \equiv (\phi_1 \circ \psi^{-1}_1, g_{ij}).
\]
The smoothness of the map \( d_{ij} \) is a consequence of the Proposition 2.2 and the smoothness of \( g_{ij} \) in the two variables.

---

3In the sense of Definition 2.2
(III) $C_i : \prod_{j \in N_n} \mathcal{R}^0(I_{ij}, \mathcal{M}) \times \mathcal{R}^0(I_{ij}, \mathcal{E}) \to \mathcal{R}^0([\tau_{i-1}, \tau_i], \mathcal{M}) \times \mathcal{R}^0([\tau_{i-1}, \tau_i], \mathcal{E})$

defined by

$$C_i((u_{ij}, v_{ij})_{j \in N_{n_i}}) = (u_i, v_i)$$

where $u_i$ is the concatenation of the family $(u_{ij})_{j \in N_{n_i}}$ and $(v_i)_{|I_{ij}} = v_{ij}$. This map is clearly linear.

If we provide $\mathcal{R}^0(I_{ij}, \mathcal{M}) \times \mathcal{R}^0(I_{ij}, \mathcal{M})$ with the norm

$$\| (u_{ij}, v_{ij}) \|_{\mathcal{M} \times \mathcal{E}} = \sup_{t \in I_{ij}} \| (u_{ij}(t), v_{ij}(t)) \|_{\mathcal{M} \times \mathcal{E}}$$

then $\sum_{j \in N_{n_i}} \| (u_{ij}, v_{ij}) \|_{\mathcal{M} \times \mathcal{E}}$ is a norm on $\mathcal{R}^0([\tau_{i-1}, \tau_i], \mathcal{M}) \times \mathcal{R}^0([\tau_{i-1}, \tau_i], \mathcal{E})$ and on $\prod_{j \in N_n} \mathcal{R}^0(I_{ij}, \mathcal{M}) \times \mathcal{R}^0(I_{ij}, \mathcal{E})$. In this way $C_i$ is an isometry and so it is smooth.

The result concerning $(\mathcal{P}(\sigma, \mathcal{U}), f_{\sigma, \mathcal{U}})$ is implied by application of the previous arguments to the subspace $f_{\mathcal{U}, \sigma}(\mathcal{P}(\tau, \mathcal{U}) \cap \mathcal{P}(\sigma, \mathcal{V})) \times \{0\}$ of $f_{\mathcal{U}, \sigma}(\mathcal{P}(\tau, \mathcal{U}) \cap \mathcal{P}(\sigma, \mathcal{V})) \times \prod_{j=1}^n \mathcal{R}^0([\sigma_{j-1}, \sigma_j], \mathcal{E})$.

From all the properties established in the whole of the previous steps, it follows that the set of all the possible pairs $(\mathcal{P}(\tau, \mathcal{U}), \mathcal{L}_\tau \circ \mathcal{F}_{\sigma, \mathcal{V}})$ (resp. $(\mathcal{P}(\tau, \mathcal{U}), \mathcal{I}_\tau \circ \mathcal{F}_{\sigma, \mathcal{V}})$) gives rise to an atlas on $\mathcal{P}(\mathcal{E})$ (resp. $\mathcal{P}(\mathcal{M})$) whose every chart domain is open for the topology induced by $\mathcal{P}(\mathcal{E})$ (resp. $\mathcal{P}(\mathcal{M})$). Since $\mathcal{P}(\mathcal{E})$ (resp. $\mathcal{P}(\mathcal{M})$) is a Hausdorff topological space it follows that we provide a Banach structure.

**Step 5: $\pi : \mathcal{P}(\mathcal{E}) \to \mathcal{P}(\mathcal{M})$ is a Banach bundle.**

Now on one hand from the initial definition at the beginning of the proof we have the following diagram

$$\begin{array}{ccc}
\mathcal{P}(\tau, \mathcal{U}) & \xrightarrow{f_{\tau, \mathcal{U}}} & f_{\tau, \mathcal{U}}(\mathcal{P}(\tau, \mathcal{U})) \times \prod_{j=1}^n \mathcal{R}^0([\tau_{j-1}, \tau_j], \mathcal{E}) \\
\downarrow & & \downarrow q_1 \\
\mathcal{P}(\tau, \mathcal{U}) & \xrightarrow{f_{\tau, \mathcal{U}}} & f_{\tau, \mathcal{U}}(\mathcal{P}(\tau, \mathcal{U}))
\end{array}$$

On the other hand from step 4 recall that we have that the map $(\mathcal{F}_{\tau, \mathcal{U}}) \circ (\mathcal{F}_{\sigma, \mathcal{V}})^{-1}$ is a map from $f_{\mathcal{U}, \sigma}(\mathcal{P}(\tau, \mathcal{U}) \cap \mathcal{P}(\sigma, \mathcal{V})) \times \prod_{j=1}^n \mathcal{R}^0([\sigma_{j-1}, \sigma_j], \mathcal{E})$ to $f_{\tau, \mathcal{U}}(\mathcal{P}(\tau, \mathcal{U}) \cap \mathcal{P}(\sigma, \mathcal{V})) \times \prod_{j=1}^n \mathcal{R}^0([\tau_{j-1}, \tau_j], \mathcal{E})$.

From the previous diagram, and according to the proof of step 4 we have:

- if $\gamma \in f_{\mathcal{U}, \sigma}(\mathcal{P}(\tau, \mathcal{U}) \cap \mathcal{P}(\sigma, \mathcal{V}))$ this map sends each factor of type $\{f_{\mathcal{U}, \sigma}(\mathcal{P}(\tau, \mathcal{U})(\gamma)) \times \mathcal{R}^0(I_{ij}, \mathcal{E})\}$ onto the factor $\{f_{\tau, \mathcal{U}}(\mathcal{P}(\tau, \mathcal{U})(\gamma)) \times \mathcal{R}^0(I_{ij}, \mathcal{E})\}$
- we apply point (II) in restriction to this factor
- we compose with $C_i$ (cf. Point (III)) in restriction to the same factor.
In this way, we see the map $(\mathcal{F}_{\tau,\mu}) \circ (\mathcal{F}_{\sigma,\mu})^{-1}$ is a map from 
$f_{\sigma,\mu}(\mathcal{P}(\tau,\mu)\cap\mathcal{P}(\sigma,\mu)) \times \prod_{j=1}^{n} R^{0}(\{\tau_{j-1}, \tau_{j}\}, E)$ to 
$f_{\tau,\mu}(\mathcal{P}(\tau,\mu)\cap\mathcal{P}(\sigma,\mu)) \times \prod_{j=1}^{n} R^{0}(\{\tau_{j-1}, \tau_{j}\}, E)$.

of type $(z, w) \mapsto (f_{\tau,\mu} \circ f_{\sigma,\mu}^{-1}(z), G(f_{\tau,\mu} \circ f_{\sigma,\mu}^{-1}(z))w)$ where $G(z')$ belongs to 
$GL(\prod_{i=1}^{n} R^{0}(\{\tau_{i-1}, \tau_{i}\}, E), \prod_{i=1}^{n} R^{0}(\{\tau_{i-1}, \tau_{i}\}, E)$ and is smooth relative to $z'$.

**step 6 : existence of Banach isomorphisms** $\mathcal{T}_{c} : T_{c}\mathcal{P}(E) \to \Gamma(c)$ and $\mathcal{T}_{\gamma} : T_{c}\mathcal{P}(E) \to \Gamma(\gamma)$

Similarly as in previous steps, the result for $\Gamma_{\gamma} : T_{\gamma}\mathcal{P}(E) \to \Gamma(\gamma)$ is implied by result for $\mathcal{T}_{c} : T_{c}\mathcal{P}(E) \to \Gamma(c)$ since $\Gamma_{\gamma} = T_{\pi} \circ \mathcal{T}_{c}$ if $\gamma = \pi \circ c$.

Now any $V \in T_{c}\mathcal{P}(E)$ is defined by a $C^{1}$ path $\tilde{c} : [-\delta, \delta] \to \mathcal{P}(E)$ (called a $C^{1}$-deformation of $c$) such that

$$\frac{d\tilde{c}}{dt}_{t=0} = V$$

To this path $\tilde{c}$ is canonically associated a map from $[-\delta, \delta] \times [0, 1]$ to $\mathcal{P}(E)$ also denoted $\hat{c}$. Consider a chart $(\mathcal{P}(\tau, U), F_{\tau,\mu})$ around $c$. For $\delta$ small enough, for any $i = 0, \ldots, n-1$, the restriction $\tilde{c}_{i}$ of $\hat{c}$ to $[\tau_{i-1}, \tau_{i}]$ must be contained in $E_{i}U_{i}$. The evaluation from $\mathcal{P}_{[\tau_{i-1}, \tau_{i}]}(\phi_{i}(U_{i}) \times E) \to M \times E$ is denoted $\mathbf{E}^{i}_{\mathcal{P}_{\tau,\mu}}$.

We have $\mathbf{E}^{i}_{\mathcal{P}_{\tau,\mu}} \circ F_{\tau,\mu}(c) = \Phi_{i}(c_{i}(t))$ if $t \in [\tau_{i}, \tau_{i+1}]$.

The evaluation map is linear and smooth and it commutes with differentials. Therefore we have $\mathbf{E}^{i}_{\mathcal{P}_{\tau,\mu}} \circ F_{\tau,\mu}(\hat{c}(\epsilon)) = \Phi_{i}(\hat{c}(\epsilon))(t) = \Phi_{i} \circ \mathbf{E}^{i}(\hat{c}(\epsilon))$ if $t \in [\tau_{i}, \tau_{i+1}]$ and $\{\partial_{t} \mathbf{E}^{i}_{\mathcal{P}_{\tau,\mu}} \circ F_{\tau,\mu}(\hat{c})\}_{t=0} = \frac{\partial \Phi_{i} \circ \hat{c}}{\partial \epsilon}(0, t) = D\Phi_{i} \circ \frac{\partial \hat{c}}{\partial \epsilon}(0, t) = \mathbf{E}_{\mathcal{P}_{\tau,\mu}}^{i} \circ D\Phi_{i}(V_{i}(t))$ where $V_{i} = V_{i}([\tau_{i-1}, \tau_{i}])$. Now the usual transformation formulæ between charts in the tangent bundle of a Banach manifold imply that $V$ is a well-defined section of $\Gamma(c)$.

Conversely, if $V \in T_{c}\mathcal{P}(E)$, with the previous notations each $V_{i}$ belongs to $\Gamma(\epsilon_{i})$ and so can be written $(\tilde{v}_{i}, \tilde{\vartheta}_{i})$ via the representation of $\mathcal{T}_{\epsilon_{i}}^{\phi_{i}}$ of $\Gamma(\epsilon_{i})$ with $\mathcal{R}^{1}([\tau_{i-1}, \tau_{i}], M) \times \mathcal{R}^{0}([\tau_{i-1}, \tau_{i}], E)$ induced by $\Phi_{i}$. Now fix some representation $\mathcal{T}_{\epsilon_{i}}$ of $\Gamma(\gamma)$ and $\Gamma(c)$ (cf. Notations and conventions [4]). According to Remark F.6, we have a $1$-regulated (resp. $0$-regulated) field $A_{\gamma_{i}}$ (resp. $A_{\epsilon_{i}}$) of automorphisms of $M$ (resp. $M \times E$) such that

$$\mathcal{T}_{\epsilon_{i}}^{\phi_{i}} = A_{\gamma_{i}} \circ \mathcal{T}_{\epsilon_{i}} \quad \text{and} \quad \mathcal{T}_{\epsilon_{i}}^{\phi_{i}} = A_{\epsilon_{i}} \circ \mathcal{T}_{\epsilon_{i}} \quad \text{and} \quad A_{\gamma_{i}} = q_{1} \circ A_{\epsilon_{i}}.$$ 

where $\mathcal{T}_{\gamma_{i}}^{\phi_{i}}$ is the representation of $\Gamma(\gamma_{i})$ associated to $\phi_{i}$. Therefore if $(\varphi_{i}, \vartheta_{i}) = A_{\gamma_{i}}^{-1}(\hat{\varphi}_{i}, \hat{\vartheta}_{i})$ we also have $\varphi_{i} = A_{\gamma_{i}}^{-1}(\hat{\varphi}_{i})$ and moreover the pairs $\{(\varphi_{i}, \vartheta_{i})\}_{i=1,\ldots,n}$ stick together into a global $(\varphi, \vartheta)$ such that the restriction of $(\varphi, \vartheta)$ to $[\tau_{i-1}, \tau_{i}]$ is precisely $(\varphi_{i}, \vartheta_{i})$. Clearly this implies that $\mathcal{T}_{\epsilon_{i}}^{-1}(\varphi, \vartheta) = V$ and moreover $\mathcal{T}_{\epsilon_{i}}^{-1}(\varphi) = T_{\pi}(V)$ which ends the proof.

$\triangle$

### A Banach manifold structure in the case of $C^{1}$ curves

**In this Appendix we give an adapted version of the results of previous sections**

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for curves \( c \in \mathcal{P}(E) \) which are \( C^1 \) and whose projection on \( M \) is \( C^2 \). This context is explicitly used \([Pel20]\).

**Remark 1** Let us consider the spaces \( C^{k-1}(M) \) of curves \( \gamma : I \to M \) of class \( C^k \) and \( C^k(E) \) of lifts to \( E \) of curves in \( C^{k-1}(M) \) which are of class \( C^{k-1} \). Then for any \( \gamma \in C^{k-1}(M) \) by same arguments as in subsection 4.4 (via trivializations \( T_\gamma \) and \( S_\gamma \)) we can define on the set \( \Gamma^k(\gamma) \) of vector fields of class \( C^k \) along \( \gamma \) a structure of Banach space modeled on \( C^k(M) \). We can also define on the set \( \Gamma^{k-1}(c) \) of vector fields of class \( C^{k-1} \) along a lift \( c \in C^k(E) \) a structure of Banach space modeled on \( C^k(M) \times C^{k-1}(E) \).

More precisely for \( k = 1 \) all the arguments used in the subsection 4.4 are valid if we replace “0-regulated” by “\( C^0 \)” and “1-regulated” by “\( C^1 \)”. For \( k > 1 \), on one hand all the arguments before Proposition 4.5 are valid by replacing “regulated” by “\( C^{k-1} \)” and “1-regulated” by “\( C^k \)”. On the other hand, the proof of a corresponding version of Proposition 4.5 is obtained by using the expression of the transformation of the \( l \)-order derivative of a curve under transformation \( T_\gamma \) or \( S_\gamma \) for \( 1 \leq l \leq k \) and comparable arguments concerning bounds (as for \( k = 1 \)) relative to all the semi-norm \( \sup_{t \in I} \| \delta^{(l)}(t) \|_E \) for any curve \( \delta : I \to F \) of class \( C^l \) in a Banach space \( F \).

**Remark 2** For any \( l \geq 1 \) and any Banach manifold \( B \) let \( T^l_B \) be the set of equivalence classes of local germs of curves which have a contact at order \( l \) at \( x \). Then \( T^l_B = \bigcup_{x \in N} T_x^l B \) has a structure of Banach manifold modeled on \( B^{l+1} \) (see \([Sus73]\) Theorem 2.1). Moreover, \( p^l_B : T^l_B \to B \) is the natural projection \((p^l_B)^{-1} \) is a chart domain for each domain chart \( U \) in \( B^l \). Note that \( T^1_B = TB \) and for simplicity we set \( T^0_B = B \).

As previously, we can provide \( C^k(E) \) with the topology generated by sets \( N(K, V_1, \cdots, V_k, W_1, \cdots, W_{k+1}) \) of paths \( c \in \mathcal{P}(E) \) such that the closure of \( c^l(K) \) is contained in \( V_l \), and \((\pi \circ c)^{(l)}(K) \) is contained in \( W_l \) where \( K \) is a compact subset of \( I \) and \( V_l \) is an open set in \( T^l E \) (resp. \( W_l \) an open in \( T^l M \)) for \( l = 0, \ldots, k \) (resp. for \( l = 0, \ldots, k+1 \)). In the same way, we can define a topology on \( C^k(M) \). Again these topologies are Hausdorff and we have a natural projection \( \pi^k : C^k(E) \) onto \( C^k(M) \).

**Remark 3** The Proposition 4.7 remains true if we replace \( \mathcal{P} \) by \( C^k \), \( \mathcal{R}^1(I, \phi(U)) \) by \( C^k(I, \phi(U)) \), \( \mathcal{R}^0(I, E) \) by \( C^{k-1}(I, E) \).

### A.1 Banach structure on \( C(E) \)

To simplify the notation we denote by \( C(E) \) the set \( C^1(E) \) (curves \( c \in \mathcal{P}(E) \) of class \( C^1 \) whose projection \( \gamma = \pi \circ c \) on \( M \) is of class \( C^2 \)) and by \( C(M) \) the set \( C^1(M) \) (curves \( \gamma \in \mathcal{P}(M) \) of class \( C^2 \)). We provide \( C(E) \) and \( C(M) \) with the topology defined in Remark 2.

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3. \( p^l_B : T^l_B \to B \) is a Banach bundle if and only if there exists a linear connection on \( TB \) (see \([Sus73]\) Theorem 2.11)
At first according to notation in Remark 1 we have a version of result from section 5:

**Theorem 4**

1. The set $C(M)$ has a Banach manifold structure modeled on the Banach space $C^2(I, M)$ and any chart domain of this structure is open for its compact open topology. Moreover, we have a Banach isomorphism from $T_\gamma C(M)$ onto $\Gamma^2(\gamma)$.

2. The set $C(E)$ has a Banach manifold structure modeled on the Banach space $C^k(I, M) \times C^1(I, E)$ such that each chart domain is open for the natural topology of $C(E)$ and we have a Banach isomorphism from $T_c C(E)$ onto $\Gamma^2(c)$. Moreover $\pi : P(E) \to P(M)$ is a Banach bundle with typical fiber $C^1(I, E)$.

Applying Remark 2, Remark 3 for $k = 1$, it is easy to see that Lemma 5.2 is also valid in this context by replacing the spaces $R_0([\tau_{i-1}, \tau_i], M)$, $R([\tau_{i-1}, \tau_i], E)$ and $R^1(I, M)$ respectively by $C^1([\tau_{i-1}, \tau_i], M)$, $C^1([\tau_{i-1}, \tau_i], E)$ and $C^2(I, M)$. Then the proof of Theorem 4 is an adaptation, point by point, of the proof of Theorem 5.1.

**Remark 5** Since $C^1([\tau_{i-1}, \tau_i], M)$, $C^1([\tau_{i-1}, \tau_i], E)$ and $C^2(I, M)$ satisfy all the axioms in section 2 of [Kri72], Theorem 4 can be also proved by applying with the same procedure but using directly arguments of [Kri72].

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