A NEW FOURIER TRANSFORM

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1. Introduction

In order to define a geometric Fourier transform, one usually works with either ℓ-adic sheaves in characteristic $p > 0$ or with $\mathcal{D}$-modules in characteristic 0 (under these conditions one has a rank 1 local system on $\mathbb{A}^1$ which plays the role of the function $e^{ix}$ in classical Fourier analysis). If one only needs to consider homogeneous sheaves, however, Laumon [Lau03] provides a uniform geometric construction of the Fourier transform for ℓ-adic sheaves in any characteristic. Laumon considers homogeneous sheaves as sheaves on the stack quotient of a vector bundle $V$ by the homothety $\mathbb{G}_m$ action. This category is closely related to the category of (unipotently) monodromic sheaves on $V$ (cf. [BY]). While it has been well known to experts that a similar uniform construction of the Fourier transform exists for monodromic sheaves (Beilinson suggests a definition in [Bei12, footnote 2]), the details have not been exposted in the literature. In this note, we fill in this gap. We also introduce a new functor, which is defined on all sheaves in any characteristic, and show that it agrees with the usual Fourier transform on monodromic sheaves.

We define the new Fourier transform $\text{Four}_B$ in §2 and show that the “square” $\text{Four}_B^2$ has a simple formula. In §3 we use this formula to prove the main result that $\text{Four}_B$ induces an equivalence of bounded derived categories of monodromic (étale) sheaves. We also discuss the relation between $\text{Four}_B$ and Laumon’s homogeneous Fourier transform. In §4 we compare $\text{Four}_B$ and the Fourier-Deligne transform in characteristic $p > 0$. Our study of $\text{Four}_B$ reveals several surprising facts about a certain object $j^*B$ of the monoidal category $D^b_c(\mathbb{G}_m)$. In §5 we prove the analogous facts about $j^*B$ in the $\mathcal{D}$-module setting by considering the Mellin transform. We use this to show that $\text{Four}_B$ agrees with the Fourier transform on monodromic $\mathcal{D}$-modules.

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1.2. Notation and terminology. Let $k$ be an arbitrary base field. Choose a prime $\ell$ not equal to the characteristic of $k$. Let $R$ be a finite commutative $\mathbb{Z}/\ell^r$-algebra for a positive integer $r$. We will work with bounded derived categories of constructible étale sheaves of $R$-modules. By the usual limit arguments, our results easily extend to $\overline{\mathbb{Q}}_\ell$-sheaves. All functors will be assumed to be derived.

Fix a base scheme $S$ of finite type over $k$. Let $\pi : V \to S$ be a vector bundle of rank $d$ and $\pi^\vee : V^\vee \to S$ the dual vector bundle. We say a complex $M \in D^b_c(V)$ is monodromic if $M$ is monodromic in the sense of Verdier [Ver83] after base change to the algebraic closure $\overline{k}$. This is equivalent to the existence of an integer $n$ coprime to $p$ and an isomorphism $\theta(n)^*M \cong \text{pr}_2^*M$ where $\theta(n) : \mathbb{G}_m \times V \to V$ sends $(\lambda, v)$ to $\lambda^n v$, and $\text{pr}_2 : \mathbb{G}_m \times V \to V$ is projection [Ver83 Proposition 5.1]. We denote the monodromic subcategory by $D^b_{\text{mon}}(V)$. We recall the fact that $\pi_! \cong 0^!$ on monodromic complexes (cf. [Ver83, Lemme 6.1] or [Spr84, Proposition 1] for two different methods of proof).
where $J$. Lemma 2.3. There is a canonical isomorphism $L \ast K = m_!(L \boxtimes K)$ where $m : \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m$ is multiplication, and $L, K \in D^b_c(\mathbb{G}_m)$. This monoidal category acts on $D^b_c(V)$ by $L \ast M = \theta(1)_!(L \boxtimes M)$ where $\theta(1) : \mathbb{G}_m \times V \to V$ is the action map, $L \in D^b_c(\mathbb{G}_m)$, and $M \in D^b_c(V)$.

2. The functor $\text{Four}_B$ and its square

Let $u : \mathbb{A}^1 - \{1\} \hookrightarrow \mathbb{A}^1$ be the open embedding removing $1 \in \mathbb{A}^1(k)$, and let $j : \mathbb{A}^1 - \{0\} \hookrightarrow \mathbb{A}^1$ be the open embedding removing zero. Define $B = u_* R \in D^b_c(\mathbb{A}^1)$.

One observes that $h_! B = 0$ where $h : \mathbb{A}^1 \to \text{Spec} k$ is the structure map, and $0^* B \cong R$ where $0 : \text{Spec} k \hookrightarrow \mathbb{A}^1$.

Define $\text{Four}_{V/S,B} : D^b_c(V) \to D^b_c(V^\vee)$ by

$$\text{Four}_{V/S,B}(M) = \text{pr}^!_1(\text{pr}^* M \otimes \mu^* B)[d]$$

where $\text{pr}^! : V^\vee \times_S V \to V^\vee$ and $\text{pr} : V^\vee \times_S V \to V$ are the projections and $\mu : V^\vee \times_S V \to \mathbb{A}^1$ is the natural pairing $(\xi, v) \mapsto \langle v, \xi \rangle$. This is the new Fourier transform that we will consider. Our goal in this section is to prove the following theorem.

**Theorem 2.1.** There is a canonical isomorphism $\text{Four}_{V/S,B} \circ \text{Four}_{V/S,B}(M) \cong j^* B \ast M(-d)[1]$. for $M \in D^b_c(V)$.

Let $\text{pr}', \text{pr}'' : V \times_S V \to V$ be the first and second projections, respectively, and $\text{pr}_{m\bullet}$ the projections from $V \times_S V^\vee \times_S V$. The usual formal argument shows that $\text{Four}_{V/S,B} \circ \text{Four}_{V/S,B}$ is isomorphic to the functor $M \mapsto \text{pr}_1^!(\text{pr}_{m\bullet}^! B \otimes K)$ where

$$K = \text{pr}_{13}!(\text{pr}_{12}^* \mu^* B \otimes \text{pr}_{23}^* \mu^* B)[2d].$$

We claim there exists a canonical isomorphism

$$(2.1.1) \quad K \cong \rho \text{pr}_1^! B(-d)[1]$$

where $\rho : \mathbb{G}_m \times V \to V \times_S V$ is defined by $(\lambda, v) \mapsto (\lambda v, v)$, and $\text{pr}_1 : \mathbb{G}_m \times V \to \mathbb{A}^1_k$ is the natural projection. This claim implies the theorem.

We first establish two lemmas which will help us prove the claim.

**Lemma 2.2.** If $v, w \in V(\bar{k})$ are not in the same $\mathbb{G}_m$-orbit, then $K_{(v,w)} = 0$.

**Proof.** We can assume $S = \text{Spec} \bar{k}$. Clearly $v$ and $w$ cannot both be zero; we will assume $v \neq 0$. Since $v, w$ are not in the same $\mathbb{G}_m$-orbit, there exists $\xi \in V^\vee(\bar{k})$ such that $\langle w, \xi \rangle = 0$ and $\langle v, \xi \rangle \neq 0$. Split $V^\vee$ as $\bar{k} \xi \oplus H_v$ where $H_v = (kv)^\perp$. Then by Kunneth formula,

$$\text{pr}_1^!(\langle v \rangle^* B \otimes \langle w \rangle^* B) \cong h_! B \otimes (\text{pr}^\vee|_{H_v})(\langle w \rangle|_{H_v})^* B = 0$$

where $\langle v \rangle : V^\vee \to \mathbb{A}^1_k$ is the evaluation by $v$ map. Therefore $K_{(v,w)} = 0$. \hfill $\Box$

**Lemma 2.3.** There is a canonical isomorphism $J^* K \cong J^* \rho \text{pr}_1^! B(-d)[1]$ where $J : V \times_S V - 0(S) \hookrightarrow V \times_S V$ is the open embedding removing zero.
Proof. We use $V^\circ$ to denote $V - 0(S)$. In this proof we will use $\rho$ to denote the restricted morphism $G_m \times V^\circ \to V \times S V$, which is an immersion, and $pr_1 : G_m \times V^\circ \to A^1_S$ to denote the projection. From Lemma (2.2) we know that $J^*K$ is supported on the image of $\rho$. Thus it suffices to consider $\rho^* J^* K$. Define

$$\omega : G_m \times V^\circ \times V^\circ \to G_m \times A^1 \times V^\circ$$

by sending $(\lambda, \xi, v)$ to $(\lambda, (v, \xi), v)$. Then

$$\rho^* J^* K \cong pr_{13!} \omega^* pr_{12!} (m^* B \otimes p_2^* B)[2d]$$

where $pr_{13}, pr_{12}$ are projections from $G_m \times A^1 \times V^\circ$ and $m, p_2 : G_m \times A^1 \to A^1$ are the multiplication and projection maps. Since $\omega$ is in fact a vector bundle of rank $r - 1$, we see that $\omega^! R$ is isomorphic to $R(1 - d)[2 - 2d]$. Therefore projection formula implies that

$$\rho^* J^* K \cong pr_{13!} pr_{12}^* (m^* B \otimes p_2^* B)(1 - d)[2].$$

We have a Cartesian square

$$\begin{array}{ccc}
G_m \times A^1 \times V^\circ & \xrightarrow{pr_{12}} & G_m \times A^1 \\
pr_{13} & & \downarrow{id \times h} \\
G_m \times V^\circ & \xrightarrow{pr_1} & G_m
\end{array}$$

so proper base change gives $pr_{13!} pr_{12}^* \cong pr_1^* (id \times h)_!$. We have an exact triangle

$$R \to p_2^* B \to (id \times 1)_* R(-1)[-1]$$

where $1 : Spec k \hookrightarrow A^1$ is the complement of $u$. Since $(id \times h)_! (m^* B) = 0$ by a change of variables, we deduce that

$$(id \times h)_! (m^* B \otimes p_2^* B) \cong (id \times h)_! (m^* B \otimes (id \times 1)_* R)(-1)[-1] \cong j^* B(-1)[-1].$$

Now it follows that $\rho^* J^* K \cong pr_1^* B(-d)[1]$.

Proof of Theorem (2.1). The case $d = 0$ is obvious since $h_! B = 0$ and $0^* B \cong R$. From now on we will assume that $d > 0$. We will show that both sides of (2.1.1) are in the essential image of the functor $\tau_{\leq 0} J_*, J^*$, i.e., there are isomorphisms

$$K \cong \tau_{\leq 0} J_* J^* (K)$$

and $\rho pr_1^* B(-d)[1] \cong \tau_{\leq 0} J_* J^* (\rho pr_1^* B(-d)[1])$.

The claimed existence of an isomorphism (2.1.1) will then follow from Lemma (2.8).

A stalk computation shows that $\rho pr_1^* B(-d)[1]$ lives in non-positive degrees. We claim that the natural morphism

(2.3.1) $$\rho pr_1^* B(-d)[1] \to \tau_{\leq 0} (J_* J^* \rho pr_1^* B(-d)[1])$$

is an isomorphism. Let $0 : S \to V \times S V$ denote the zero section. From the exact triangle $0_0 \sigma \to id \to J_* J^*$, it suffices to show that $0_0 \rho pr_1^* B \in D^{>2d}_c(S)$. Observe that $\rho$ is $G_m$-equivariant with respect to the $G_m$-action on the second coordinate of $G_m \times V$ and the diagonal action of $G_m$ on $V \times S V$. This implies that $\rho pr_1^* B$ is monodromic. Thus

$$0_0 \rho pr_1^* B \cong h_! j_* j^* B(-d)[-2d] \cong R(-d)[-2d - 1].$$

Therefore $0_0 \rho pr_1^* B \in D^{>2d}_c(S)$.

One easily sees that $K_{(0,0)} \cong R(-d)$. Thus $K$ lives in non-positive cohomological degrees. To show that the natural morphism $K \to \tau_{\leq 0} J_* J^* K$ is an isomorphism, it suffices by the same
argument as above to prove $0^!K \in \mathcal{D}^\ge_{\mathcal{E}}(S)$. One observes from the definition of $K$ that $K$ is monodromic with respect to the diagonal $\mathbb{G}_m$-action on $V \times_S V$. Therefore

$$0^!K \cong \tilde{\pi}(\text{pr}_{12}^*\mu^*B \oplus \text{pr}_{23}^*\mu^*B)[2d]$$

where $\tilde{\pi} : V \times_S V^V \times_S V \to S$ is the structure map. By projection formula and proper base change, the right hand side is isomorphic to

$$\pi'(\mu^*B \oplus \text{pr}^V, \text{pr}^Y, \mu^*B)[2d]$$

for $\pi' : V \times_S V^V \to S$ the structure map. The fact that $h_B = 0$ implies that $\text{pr}^Y, \mu^*B$ is supported at $0 \in V^V(k)$, and $0^*\text{pr}^Y, \mu^*B \cong R(-d)[-2d]$. We deduce that

$$0^!K \cong \pi(R(-d)) \cong R(-2d)[-2d],$$

which proves the claim, and hence the theorem. $\square$

3. Properties of Four$_B$

Remark 3.1. The functor Four$_{V/S,B}$ is not an equivalence on $D^b_c(V) \to D^b_c(V^V)$. Consider the one-dimensional case $V = \mathbb{A}^1_k$. Then $\text{Four}_{V/S,B}(0R) = R[1]$ and $\text{Four}_{V/S,B}(1R) = B[1]$. We have $\text{Hom}(R, B) \neq 0$ but $\text{Hom}(0R, 1R) = 0$. So Four$_{V/S,B}$ is not fully faithful.

3.2. Relation to quotient stacks. Let $p : V \to V = [V/\mathbb{G}_m]$ and $p^V : V^V \to V^V = [V^V/\mathbb{G}_m]$ denote the canonical projections to the quotient stacks. By Laumon’s homogeneous transform $\text{Four}_{V/S} : D^b_c(V) \to D^b_c(V^V)$ is canonically isomorphic to the functor

$$(3.2.1) \quad K \mapsto \text{pr}^V_!(\mu^*K \otimes s_1B)[d]$$

where $f : \mathbb{A}^1_k \to \mathcal{A}_S$ is the quotient morphism and $B_S$ denotes the base change of $B$ from $\mathbb{A}^1_k$ to $\mathcal{A}_S$. We abuse notation and use $\text{pr}^V_! : V^V \times_S V \to V^V$, $\text{pr} : V^V \times_S V \to V$, and $\mu : V^V \times_S V \to \mathcal{A}_S$ to also denote the induced maps on stacks.

Proposition 3.3. The composed functors

$$(p^V)^* \circ \text{Four}_{V/S} \text{ and } \text{Four}_{V/S,B} \circ p^* : D^b_c(V) \to D^b_c(V^V)$$

are canonically isomorphic.

Proof. The proposition follows from (3.2.1) by applying proper base change to the Cartesian squares

$$
\begin{array}{ccc}
[V^V \times_S V/\mathbb{G}_m] & \xrightarrow{A^1} & V^V \times_S V \\
\downarrow & & \downarrow \\
V^V \times_S V & \xrightarrow{f} & \mathcal{A}_S
\end{array}
$$

where $\mathbb{G}_m$ acts on $V^V \times_S V$ anti-diagonally. $\square$

Proposition 3.4. Let $V' = V \times A^1$ and let $\mathbb{G}_m$ act on both $V$ and $A^1$. We have a canonical open embedding $\nu : V \hookrightarrow [V'/\mathbb{G}_m] : v \mapsto (v, 1)$. Similarly, we have $\nu^V : V^V \hookrightarrow [(V^V)/\mathbb{G}_m]$ defined by $\nu^V(\xi) = (\xi, -1)$. The composed functor

$$D^b_c(V) \xrightarrow{\nu^*} D^b_c([V'/\mathbb{G}_m]) \xrightarrow{\text{Four}_{V'/\mathbb{G}_m}/S} D^b_c([(V^V)/\mathbb{G}_m]) \xrightarrow{(\nu^V)^*} D^b_c(V^V)$$

is isomorphic to Four$_{V/S,B}$.  

Proof. Observe that \( \nu \) factors into the composition of an open affine chart \( V \hookrightarrow \mathbb{P}(V') \) and the open embedding \( \mathbb{P}(V') = [(V' - 0(S))/\mathbb{G}_m] \hookrightarrow [V'/\mathbb{G}_m] \). Similarly, we have a factorization of \( \nu^* \). The proposition now follows from \cite[Proposition 1.6]{Lau03}, since the restriction of the incidence hyperplane in \( \mathbb{P}((V')^\times) \times_S \mathbb{P}(V') \) to \( V^\times \times_S V \) is \( \mu^{-1}(\{1\}) \). \( \square \)

3.5. An equivalence induced by \( \text{Four}_{V/S,B} \). Let \( p: V \to \mathbb{V} \) be as in the previous subsection.

**Proposition 3.6.** Let \( \mathcal{C}_V \) denote the full subcategory of \( D^b_c(V) \) consisting of complexes \( M \) such that \( p_!M = 0 \). The functor \( \text{Four}_{V/S,B} \) induces an equivalence \( \mathcal{C}_V \to \mathcal{C}_{V^\vee} \).

**Proof.** Proper base change and projection formula imply that \( \text{Four}_{V/S,B} \) sends \( \mathcal{C}_V \) to \( \mathcal{C}_{V^\vee} \) and vice versa. We also see by proper base change that \( p^*p_M \cong R \ast M \) for \( M \in D^b_c(V) \), where \( R \) is the constant sheaf on \( \mathbb{G}_m \). From the exact triangle \( 1_R \to R \to B \) we deduce that \( j^*B \ast M \cong M(1)[-1] \) for \( M \in \mathcal{C}_V \). Therefore Theorem 2.1 implies that

\[
\text{Four}_{V^\vee/S,B} \circ \text{Four}_{V/S,B}(M) \cong M(-d-1)
\]

for \( M \in \mathcal{C}_V \), and we deduce the proposition. \( \square \)

3.7. Monodromic complexes. We will show that \( \text{Four}_{V/S,B} \) also induces an equivalence on the subcategories of monodromic complexes. We use the notation and results of Appendix A.

**Theorem 3.8.** (i) The functor \( \text{Four}_{V/S,B} \) preserves monodromicity, and the restriction defines an equivalence \( D^b_{\text{mon}}(V) \to D^b_{\text{mon}}(V^\vee) \).

(ii) For \( N \in D^b_{\text{mon}}(V^\vee) \), the pro-object

\[
\text{pr}_1(\text{pr}^\vee_\ast N \otimes \mu^*j_!0)(d+1)[d+1]
\]

is essentially constant.

(iii) The functor \( D^b_{\text{mon}}(V^\vee) \to D^b_{\text{mon}}(V) \) defined by \( \text{pr}^\vee_\ast \text{pr}_1 \) is quasi-inverse to \( \text{Four}_{V/S,B} \).

Since \( B \) is not monodromic, our first step is to compute the "monodromization" of \( B \).

**Lemma 3.9.** There is an isomorphism of pro-objects

\[
I^0 \ast B \cong j_!I^1(-1)[-1].
\]

**Proof.** First we show that the restriction \( I^0 \ast j^*B \) is isomorphic to \( I^1(-1)[-1] \). The exact triangle \( 1_R \to R \to B \) induces by convolution exact triangles

\[
I^0_n(-1)[-2] \to I^0_n \ast R \to I^0_n \ast j^*B
\]

for \( p \nmid n \). Taking \( \text{"lim"} \) and using Lemma A.4, the first arrow is isomorphic to the augmentation map \( I^0(-1)[-2] \to R(-1)[-2] \). Therefore we deduce that the pro-object \( I^0 \ast j^*B \) is isomorphic to \( I^1(-1)[-1] \).

To complete the proof, it suffices to show that the canonical morphism

\[
I^0 \ast B \to j_!j^*(I^0 \ast B)
\]

is an isomorphism. This is equivalent to proving that \( \theta^1(I^0 \ast B) = 0 \). Since \( I^0 \ast B \) is monodromic, \( \theta^1(I^0 \ast B) \cong h_1(I^0 \ast B) \). By the Kunneth formula, \( h_1(I^0 \ast B) \cong h_1j_!I^0 \otimes h_1B = 0 \). \( \square \)

\[\text{A pro-object is essentially constant if it is isomorphic to an object of } D^b_{\text{mon}}(V), \text{ which is considered as a pro-object via the constant embedding.}\]
Proof of Theorem 3.8. One easily sees that \( \text{Four}_{V/S,B} \) preserves monodromicity. Theorem 2.1 and Lemma A.4 together imply that for \( M \in D^b_{\text{mon}}(V) \), we have

\[
\text{Four}_{V/S,B} \circ \text{Four}_{V/S,B}(M) \cong I^1 \ast M(-d)[2].
\]

Since \( I^{-1} \ast I^1 \cong I^0(-1)[-2] \) by Corollary A.9, we deduce that \( \text{Four}_{V/S,B} \) is an equivalence, with inverse functor \( I^{-1} \ast \text{Four}_{V/S,B}(d+2)[2] \). Lemmas 3.9 and A.4 imply that for \( N \in D^b_{\text{mon}}(V^\vee) \), we have isomorphisms

\[
I^{-1} \ast \text{Four}_{V^\vee/S,B}(N) \cong I^1 \ast \text{pr}_1(\text{pr}_{V^\vee}^*N \otimes \mu^*j_1^1)[d+1].
\]

Applying Corollary A.9, again, we get (iii). \( \square \)

Remark 3.10. Observe that the formula (3.8.1) is very similar to Beilinson’s suggested definition of the monodromic Fourier transform in [Bei12].

Proposition 3.11. The object \( j^*B \in D^b_c(\mathbb{G}_m) \) satisfies the following properties:

1. \( j^*B \) is not invertible in the monoidal category \( D^b_c(\mathbb{G}_m) \).
2. \( j^*B \) is invertible in the quotient of \( D^b_c(\mathbb{G}_m) \) by the ideal generated by the constant sheaf \( R \).
3. There are canonical isomorphisms \( I^0_n \ast j^*B \cong I^1_n(-1)[-2] \) for \( p \nmid n \).

Proof. We showed in Remark 3.1 that \( \text{Four}_{\mathbb{A}^1,B} \) is not an equivalence on \( D^b_c(\mathbb{A}^1) \). Since \( \text{Four}^2_{\mathbb{A}^1,B}(M) \) is isomorphic to \( j^*B \ast M(-1)[1] \), we deduce that \( j^*B \) is not invertible in the monoidal category \( D^b_c(\mathbb{G}_m) \).

From the exact triangle \( 1:R(-1)[-2] \to R \to j^*B \) on \( \mathbb{G}_m \), we see that in the quotient of \( D^b_c(\mathbb{G}_m) \) by the ideal generated by \( R \), the object \( j^*B \) is isomorphic to \( 1:R(-1)[-1] \), which is invertible.

Lemma 3.9 gives an isomorphism \( I^0_n \ast j^*B \cong I^1_n(-1)[-2] \). Convolving with \( I^0_n \), we get an isomorphism \( I^0_n \ast j^*B \cong I^0_n \ast I^1 \). One observes that \( I^0_n \ast I^1 \cong I^1_n(-1)[-2] \) by Corollary A.9. \( \square \)

4. Relation to Fourier-Deligne transform

Suppose that \( k \) has characteristic \( p > 0 \). Assume that \( R \) contains a primitive \( p \)-th root of unity \( \zeta \) (where “primitive” means that \( \zeta^p = 1 \) is invertible). Let \( \psi : \mathbb{F}_p \to R^\times \) be the corresponding additive character with \( \psi(1) = \zeta \), and let \( L_\psi \) denote the Artin-Schreier sheaf. The usual Fourier-Deligne transform \( \text{Four}_{V^\vee,S,L_\psi} : D^b_c(V) \to D^b_c(V^\vee) \) is defined by

\[
\text{Four}_{V^\vee/S,L_\psi}(M) = \text{pr}_1^\vee(\text{pr}_1^\vee M \otimes \mu^*L_\psi)[d].
\]

Lemma 4.1. There is a canonical isomorphism

\[
i^*j^*L_\psi \ast L_\psi \cong B[-1]
\]

where \( i : \mathbb{G}_m \to \mathbb{G}_m \) is the multiplicative inverse map.

Proof. By a change of variables, \( i^*j^*L_\psi \ast L_\psi \) is isomorphic to \( \text{Four}_{\mathbb{A}^1,L_\psi}(j_1j^*L_\psi)[-1] \). We have an exact triangle

\[
j_1j^*L_\psi \to L_\psi \cong \text{Four}_{\mathbb{A}^1,L_\psi}(1:R[-1]) \to 0;R.
\]

Applying \( \text{Four}_{\mathbb{A}^1,L_\psi} \) and using the Fourier-Deligne inversion formula on the middle term, we have an exact triangle

\[
\text{Four}_{\mathbb{A}^1,L_\psi}(j_1j^*L_\psi) \to 1:R(-1)[-1] \to R[1].
\]

This induces an isomorphism \( \text{Four}_{\mathbb{A}^1,L_\psi}(j_1j^*L_\psi) \to u_*R = B \). Since \( \text{Hom}(1:R(-1)[-1], R) = 0 \), this isomorphism is unique. \( \square \)
Corollary 4.2. In characteristic \( p > 0 \), we have canonical isomorphisms

\[
\text{Four}_{V/S,B}(M) \cong i^*j^*\mathcal{L}_\psi \ast \text{Four}_{V/S,L_{\psi}}(M) \cong \text{Four}_{V/S,L_{\psi}}(j^*\mathcal{L}_\psi \ast M).
\]

4.3. Monodromization of \( \mathcal{L}_\psi \) over \( \bar{k} \). Suppose that \( k \) is algebraically closed, so \( A^0 \) is simply a ring instead of a sheaf of rings (i.e., there is no Galois action).

Lemma 4.4. There exists a (non-canonical) isomorphism of pro-objects

\[
I^0 \ast \mathcal{L}_\psi \cong j_*I^0[-1].
\]

Proof. As in the proof of Lemma 3.9, it suffices to prove the isomorphism after restriction to \( G_m \). Let \( n \) be coprime to \( p \). By proper base change,

\[
1^*(I^0_n \ast j^*\mathcal{L}_\psi) \cong \Gamma_c(G_m, I^0_n \otimes j^*\mathcal{L}_\psi)
\]

where we observe that the pullback of \( I^0_n \) under the multiplicative inverse map \( G_m \to G_m \) is isomorphic to \( I^0_n \). Since \( I^0_n \) is tamely ramified at \( \infty \in \mathbb{P}^1(k) \), the canonical map

\[
\Gamma_c(A^1, j^*I^0_n \otimes \mathcal{L}_\psi) \to \Gamma(A^1, j^*I^0_n \otimes \mathcal{L}_\psi)
\]

is an isomorphism (cf. proof of [KW01 Lemma 7.1(1)]). In particular \( \Gamma_c(G_m, I^0_n \otimes j^*\mathcal{L}_\psi) \) lives in cohomological degrees 0 and 1. Since \( I^0_n \otimes j^*\mathcal{L}_\psi \) is locally constant and \( G_m \) is not complete, \( H^0_c(G_m, I^0_n \otimes j^*\mathcal{L}_\psi) = 0 \). Thus \( \Gamma_c(G_m, I^0_n \otimes j^*\mathcal{L}_\psi) \) lives only in cohomological degree 1.

We now consider \( I^0_n \) as a locally free sheaf of \( A^0_n \)-modules of rank 1. If we let \( \psi' \) denote the composition \( \mathbb{F}_p \to R^\times \to (A^0_n)^\times \), then \( \mathcal{L}_\psi \otimes_R A^0_n \cong \mathcal{L}_{\psi'} \), where the latter is the Artin-Schreier sheaf with respect to \( \psi' \) as a locally free sheaf of \( A^0_n \)-modules of rank 1. Hence \( \mathcal{T} := I^0_n \otimes A^0_n j^*\mathcal{L}_{\psi'} \), which is isomorphic to \( I^0_n \otimes_R j^*\mathcal{L}_\psi \), is a locally free sheaf of \( A^0_n \)-modules of rank 1. In particular, \( \mathcal{T} \in D^{b}_{ct}(G_m, A^0_n) \) and \( \Gamma_c(G_m, \mathcal{T})[1] \) is quasi-isomorphic to a finite projective \( A^0_n \) module \( P \). Applying the Grothendieck-Ogg-Shafarevich formula [SGA72 Exposé X, Corollaire 7.2], one checks that the fiber of \( P \) over any point of \( \text{Spec} A^0_n \) has dimension 1. So there exists an isomorphism \( P \cong A^0_n \) of \( A^0_n \)-modules. Observe from the Cartesian square

\[
\begin{array}{ccc}
G_m \times G_m \times G_m & \overset{\theta(n) \times \text{id}_{G_m}}{\longrightarrow} & G_m \times G_m \\
\text{id}_{G_m} \times m & \downarrow & m \\
G_m \times G_m & \overset{\theta(n)}{\longrightarrow} & G_m
\end{array}
\]

that \( I^0_n \ast j^*\mathcal{L}_\psi \) is monodromic, and the monodromy action is induced by the monodromy action on \( I^0_n \). Therefore using the equivalence of abelian categories \( \text{Mod}_f(A^0) \cong \text{Sh}_{\text{mon}}(G_m) \) introduced in [A.3] we have shown that there exists an isomorphism \( I^0_n \ast j^*\mathcal{L}_\psi[1] \cong I^0_n \).

Suppose \( n' \) is a multiple of \( n \) and \( p \nmid n' \). The kernel \( \mathcal{K} \) of the surjection \( I^0_n \to I^0_n \) is tamely ramified, so \( H^2_c(G_m, \mathcal{K} \otimes j^*\mathcal{L}_\psi) = 0 \) by the same argument as above. We deduce that

\[
I^0_n \ast j^*\mathcal{L}_\psi[1] \to I^0_n \ast j^*\mathcal{L}_\psi[1]
\]

is a surjection of sheaves. Since \( (A^0_n)^\times \to (A^0_n)^\times \) is also surjective, we can find a projective system of isomorphisms \( I^0_n \ast j^*\mathcal{L}_\psi[1] \cong I^0_n \) inducing an isomorphism of pro-sheaves. □

Corollary 4.5. When \( k \) is algebraically closed, there exists a (non-canonical) isomorphism between the functors \( \text{Four}_{V/S,B} \) and \( \text{Four}_{V/S,L_{\psi}} \) restricted to \( D^{b}_{\text{mon}}(V) \to D^{b}_{\text{mon}}(V^V) \).
Proof. Lemma 3.9 and Remark A.3 imply that there exists an isomorphism $I^0 * B \cong j_* I^0[-1]$. The latter is also isomorphic to $I^0 * \mathcal{L}_\psi$ by Lemma 4.1. One easily sees that the Fourier-Deligne transform preserves monodromicity, and the isomorphism of restricted functors follows from Lemma A.4. □

4.6. The universal Gauss sum. Let $k$ once again be arbitrary. Define the pro-object

$$\mathcal{G} = I^0 * j^* \mathcal{L}_\psi(1)[1].$$

Lemma 4.4 implies that $\mathcal{G}$ is a monodromic pro-sheaf, and there exists a trivialization $\mathcal{G} \cong I^0$ after base changing from $k$ to $\bar{k}$. Under the equivalence $\mathcal{M}$ of §A.5, we see that $\mathcal{G}$ corresponds to an invertible (locally free of rank 1) $A^0$-module on $\text{Spec } k$. We are motivated by [Del77, Exposé VI, §4] to think of $\mathcal{G}$ as a “universal Gauss sum”.

Let $\iota : G_m \to G_m$ denote the multiplicative inverse map. Then Lemmas 3.9 and 4.1 give a canonical isomorphism

$$\iota^* \mathcal{G} * \mathcal{G} \cong I^1[-2].$$

We also see that the Fourier-Deligne transform on monodromic complexes is isomorphic to the functor $M \mapsto \text{pr}_V^*(\text{pr}^* M \otimes \mu^* j_* \mathcal{G})(d + 1)$ on $D^b_{\text{mon}}(V) \to D^b_{\text{mon}}(V^\vee)$. By Theorem 2.1, we have

$$\text{Four}_{V/S, L_\psi} \circ \text{Four}_{V/S} (M) \cong \mathcal{G} * M(−d)[2]$$

for $M$ monodromic.

5. Relation to Fourier transform on $\mathcal{D}$-modules

Let $k$ be algebraically closed of characteristic 0. We use $\mathcal{M}(V)$ to denote the abelian category of quasicoherent right $\mathcal{D}$-modules on $V$. Let $\mathcal{L} = \mathcal{D}_{A^1}/(1-\partial_x) \mathcal{D}_{A^1}$ be the exponential $\mathcal{D}$-module on $A^1 = \text{Spec } k[x]$. The Fourier transform is the functor $DM(V) \to DM(V^\vee)$ defined by

$$\text{Four}_{V/S, \mathcal{L}}(M) = \text{pr}_V^*(\text{pr}^* M \otimes \mu^1 \mathcal{L})[d].$$

It is well known that this functor can also be described using the isomorphism between the algebras of polynomial differential operators $\mathcal{D}_V \to \mathcal{D}_{V^\vee}$ defined in local coordinates by

$$k[\xi_1, \ldots, \xi_d, \partial_1, \ldots, \partial_d] \to k[v_1, \ldots, v_d, \partial_1, \ldots, \partial_d] : \xi_i \mapsto \partial v_i, \partial_\xi \mapsto -v_i.$$

In the $\mathcal{D}$-module situation, the analog of $B$ is $w u' (\omega_{\mathcal{L}})$, where $\omega_{\mathcal{L}}$ is the sheaf of differentials on $A^1$ viewed as a right $\mathcal{D}$-module. We will also call this $\mathcal{D}$-module $B$. A simple calculation shows that $\text{pr}_x^*(\mu^1 B)[d]$. We define $\text{Four}_{V/S, B} : DM(V) \to DM(V^\vee)$ by

$$\text{Four}_{V/S, B}(M) = \text{pr}_x^*(\mu^1 B)[d].$$

One of the goals of this section is to establish a relation between $\text{Four}_{V/S, \mathcal{L}}$ and $\text{Four}_{V/S, B}$ (see Corollary 5.4).

---

\footnote{Beilinson observed that $B$ essentially describes the differential equation for a shift of the Heaviside step function.}
5.1. Mellin transform of $j^* B$. Let $\mathcal{B}$ denote the Mellin transform of $j^* B$, viewed as a $\mathbb{Z}$-equivariant quasicoherent $\mathcal{O}$-module on $\mathbb{A}^1 = \text{Spec } k[s]$. The Mellin transform functor

$$\mathcal{M} : \mathcal{M}(\mathbb{G}_m) \to \text{QCoh}(\mathbb{A}^1)^{\mathbb{Z}}$$

is defined by considering $\mathcal{D}(\mathbb{G}_m)$ as the algebra of difference operators $\mathcal{D} = k[s](T, T^{-1})/(sT - T(s + 1))$ under the identifications $s = x\partial_x$ and $T = x$. We consider the bounded derived category of $\mathbb{Z}$-equivariant $\mathcal{O}_{\mathbb{A}^1}$-modules $\mathcal{D}(\text{QCoh}(\mathbb{A}^1)^{\mathbb{Z}})$ with monoidal structure induced by the usual derived tensor product over $k[s]$. This monoidal structure corresponds to the convolution product (without compact support)

$$L \ast K := m_*(L \boxtimes K)$$

on the derived category of $\mathcal{D}$-modules on $\mathbb{G}_m$. More precisely, $\mathcal{M}(L \ast K) \cong \mathcal{M}(L) \otimes_{k[s]} \mathcal{M}(K)$.

We start by proving the following proposition, which is an analog of Proposition 5.1 in the $\mathcal{D}$-module setting.

**Proposition 5.2.** The module $\mathcal{B}$ satisfies the following properties:

1. $\mathcal{B}$ is not invertible in $\mathcal{D}(\text{QCoh}(\mathbb{A}^1)^{\mathbb{Z}})$.
2. The restriction of $\mathcal{B}$ to $k^1 - \mathbb{Z} := \text{Spec } k[s]/(s - 1, (s + 1)^{-1}, \ldots)$ is invertible.
3. For any $\chi \in k$ and $n \in \mathbb{N}$, there exists an isomorphism

$$\bigoplus_{i \in \mathbb{Z}} k[s]/(s - \chi - i)^n \cong \bigoplus_{i \in \mathbb{Z}} \mathcal{B} \otimes k[s]/(s - \chi)^n$$

of $\mathcal{D}$-modules, where $T$ acts on $k[s]$ by translation.

In order to prove the proposition, we will need an explicit description of $\mathcal{B}$. Consider $k(s)$ as a right $\mathcal{D}$-module where $T$ acts by translation. Let $\mathcal{B}'$ denote the $\mathcal{D}$-submodule of $k(s)$ generated by $\frac{1}{s}$, or equivalently, the $k[s]$-submodule generated by $\frac{1}{s + 1}$ for all $i \in \mathbb{Z}$.

**Lemma 5.3.** There exists an isomorphism of $\mathcal{D}$-modules $\mathcal{B} \cong \mathcal{B}'$.

**Proof.** We have $\partial_x x = x\partial_x + 1$ so $\partial_x(x - 1) = (s + 1) - T^{-1}s$ in $\mathcal{D}$. Therefore

$$\mathcal{B} = \mathcal{D}/((s + 1) - T^{-1}s)\mathcal{D}.$$ 

Let $1$ denote the generator of $\mathcal{B}$. Conjugating $sT = T(s + 1)$ in $\mathcal{D}$ by $T^{-1}$ gives $T^{-1}s = (s + 1)T^{-1}$ in $\mathcal{D}$. Using this equality, $1(s + 1) = 1T^{-1}s = 1(s + 1)T^{-1}$ in $\mathcal{B}$, and acting on the right by $T$ gives $1(s + 1)T = 1(s + 1)$. Using these relations, we deduce that $\mathcal{B}$ is generated over $k$ by $1T^i$ for $i \in \mathbb{Z}$ and $1s^j$ for $j > 0$. Then $1 \mapsto \frac{1}{s + 1}$ defines a morphism of $\mathcal{D}$-modules $\mathcal{B} \to k(s)$. Since $\frac{1}{s + 1}$ for $i \in \mathbb{Z}$ and $s^j$ for $j \geq 0$ are $k$-linearly independent in $k(s)$, we see that this morphism is an injection $\mathcal{B} \hookrightarrow k(s)$. The image is $\mathcal{B}'$. \qed

**Proof of Proposition 5.2.** Suppose that $\mathcal{B}$ is invertible in $\mathcal{D}(\text{QCoh}(\mathbb{A}^1)^{\mathbb{Z}})$, i.e., there exists an object $N$ of this monoidal category such that $\mathcal{B} \otimes_{k[s]} N \cong k[s]$. Then $N \cong \text{Hom}_{k[s]}(k[s], N) \cong \text{Hom}_{k[s]}(\mathcal{B}, k[s])$. There are no nonzero morphisms from $\mathcal{B}'$ to $k[s]$, so $H^0N = 0$. On the other hand, since $k(s) \otimes_{k[s]} \mathcal{B}' \cong k(s)$, we have $k(s) \otimes_{k[s]} N \cong k(s)$, which implies that $H^0N \neq 0$. We thus get a contradiction, so $\mathcal{B}$ is not invertible.

Since $\mathcal{O}(\mathbb{A}^1 - \mathbb{Z}) = k[s]/(s - 1, (s + 1)^{-1}, \ldots) \subset k(s)$, we see that

$$\mathcal{O}(\mathbb{A}^1 - \mathbb{Z}) \otimes_{k[s]} \mathcal{B}' = \mathcal{O}(\mathbb{A}^1 - \mathbb{Z}) \subset k(s)$$

is the identity object, proving (2).

The direct sums in (3) only depend on the class $\chi$ of $\chi$ in $k/\mathbb{Z}$. If $\chi = 0 + \mathbb{Z}$ we will assume that $\chi = 0$. Let $\mathcal{B}_i \subset \mathcal{B}'$ denote the $k[s]$-submodule generated by $\frac{1}{s + 1}$. Then $\mathcal{B}'/\mathcal{B}_i$
is isomorphic to the direct sum of skyscraper modules $k[s]/(s - j)$ for integers $j \neq i$. Thus $(\mathcal{B}_i/\mathcal{B}_i) \otimes k[s]/(s - \chi - i)^n = 0$. On the other hand $\mathcal{B}_i$ is free, so $\mathcal{B}_i \otimes k[s]/(s - \chi - i)^n$ is free with generator $\frac{1}{s - \chi - i} \otimes 1$. These basis elements give our desired isomorphism, which evidently commutes with the action of $T$. \qed

5.4. Monodromization. The $\mathbb{G}_m$-action on $V$ induces an algebra map $k[s] \to \mathcal{D}_V$, where $s = x\partial_x$ is the invariant vector field on $\mathbb{G}_m$. We say that $M \in \mathcal{M}(V)$ is monodromic if every local section $m \in M$ is killed by some nonzero polynomial in $s = x\partial_x$. In other words, $M$ is monodromic if it is a torsion module over $k[s]$. This definition of monodromic was introduced by Verdier \cite{Ver}. Define an object of $DM(V)$ to be monodromic if each of its cohomology $\mathcal{D}$-modules is monodromic. We denote this full subcategory by $D_{\text{mon}}\mathcal{M}(V) \subset DM(V).

For any $\chi \in k$ and $n \in \mathbb{N}$, let $A_{\chi,n} \subset k(s)$ consist of those rational functions with poles of order $\leq n$ at $\chi + \mathbb{Z}$ and no other poles. Define $I_{\chi,n}^0 \in \mathcal{M}(\mathbb{G}_m)$ to be the inverse Mellin transform $\mathcal{M}^{-1}(A_{\chi,n}/k[s])$. The inclusions $A_{\chi,n} \to A_{\chi,n+1}$ induce morphisms $I_{\chi,n}^0 \to I_{\chi,n+1}^0$, which form an inductive system of $\mathcal{D}$-modules. Define

$I^0 = \bigoplus_{\chi \in k/\mathbb{Z}} \lim_{n \to \infty} I_{\chi,n}^0 \in \mathcal{M}(\mathbb{G}_m)$

where $\chi \in k$ is any lift of $\bar{\chi}$. It follows that $\mathcal{M}(I^0) = k(s)/k[s]$.

Let $\underline{1}$ be the unit object in the monoidal category $D\mathcal{M}(\mathbb{G}_m)$, so $\mathcal{M}(\underline{1}) = k[s]$. The canonical extension of $k(s)/k[s]$ by $k[s]$ defines an extension of $I^0$ by $\underline{1}$ and therefore a morphism

$\varepsilon : I^0 \to \underline{1}[1]$.

The monoidal category $D\mathcal{M}(\mathbb{G}_m)$ acts on $D\mathcal{M}(V)$ by convolution (without compact support).

**Lemma 5.5.** An object $M \in D\mathcal{M}(V)$ is monodromic if and only if the morphism $I^0 \ast M \to M[1]$ induced by $\varepsilon$ is an isomorphism.

**Proof.** A calculation using the relative de Rham complex with respect to the action map $\mathbb{G}_m \times V \to V$ shows that for any $M \in D\mathcal{M}(V)$ and $N \in D\mathcal{M}(\mathbb{G}_m)$, there is a canonical isomorphism $N \ast M \cong \mathcal{M}(N) \otimes k[s] M$ in the derived category of (sheaves of) $k[s]$-modules. This implies that the cocone of the morphism $I^0 \ast M \to M[1]$ is isomorphic (in the derived category of $k[s]$-modules) to $k(s) \otimes k[s] M$. But $k(s)$ is flat over $k[s]$, so the vanishing of the cohomologies of $k(s) \otimes k[s] M$ is equivalent to the cohomologies of $M$ being torsion modules over $k[s]$. \qed

See \cite{Bei87}, \cite{Lic}, and \cite{DG} C.2 for further details in the unipotently monodromic case (when $\chi = 1$).

**Lemma 5.6.** There exists an inductive system of isomorphisms

$I_{\chi,n}^0 \ast B \cong j I_{\chi,n}^0 \cong I_{\chi,n}^0 \ast \mathcal{L}$.

**Proof.** Since $h_*B = h_*\mathcal{L} = 0$, it suffices as in Lemma 3.9 to give isomorphisms of the above objects after restriction to $\mathbb{G}_m$. In fact, it suffices to construct isomorphisms between the Mellin transforms of these restrictions, i.e., isomorphisms $\mathcal{M}(I_{\chi,n}^0 \ast j^*B) \cong \mathcal{M}(I_{\chi,n}^0) \cong \mathcal{M}(I_{\chi,n}^0 \ast j^*\mathcal{L})$. This is equivalent to constructing isomorphisms

(5.6.1) $\mathcal{M}(I_{\chi,n}^0) \otimes \mathcal{B} \cong \mathcal{M}(I_{\chi,n}^0)$, \quad $\mathcal{B} := \mathcal{M}(j^*B)$,

(5.6.2) $\mathcal{M}(I_{\chi,n}^0) \otimes E \cong \mathcal{M}(I_{\chi,n}^0)$, \quad $E := \mathcal{M}(j^*\mathcal{L})$. 

Note that we have isomorphisms
\[(5.6.3) \quad \mathfrak{M}(I_{\chi}^{n}) = A_{\chi,n}/k[s] \cong \bigoplus_{i \in \mathbb{Z}} k[s]/(s - \chi - i)^n.\]

Combining (5.6.3) and Proposition 5.2(3), one gets (5.6.1). Let us construct (5.6.2). We have
\[E = \mathcal{D}/(1 - T^{-1}s)\mathcal{D}.\]

Let 1 be the generator of E. Let \(E_i \subset E\) denote the free \(k[s]\)-submodule generated by \(1T^{-i-1}\) for \(i \in \mathbb{Z}\). If \(\chi \in \mathbb{Z}, \text{ set } \chi = 0.\) From the relation \(1T^{-i} = 1T^{-i-1}(s - i)\), we deduce that \(E/E_i\) is supported away from \(\chi + i\), so \((E/E_i) \otimes k[s]/(s - \chi - i)^n = 0\). Hence \(E \otimes k[s]/(s - \chi - i)^n\) is freely generated by \(1T^{-i-1} \otimes 1\), and this gives us (5.6.2).

Lemma 5.6 implies in particular that \(I^0 \ast B \cong I^0 \ast \mathcal{L}\). We deduce from Lemma 5.5 that \(\text{Four}_V/S,\mathcal{B}\) agrees with \(\text{Four}_V/S,\mathcal{L}\) on \(D_{\text{mon}}\mathcal{M}(V)\).

**Corollary 5.7.** There is an isomorphism
\[\text{Four}_V/S,\mathcal{B} \cong \text{Four}_V/S,\mathcal{L}\]
of functors \(D_{\text{mon}}\mathcal{M}(V) \to D_{\text{mon}}\mathcal{M}(V^\vee)\).

**APPENDIX A. THE MONODROMIC SUBCATEGORY**

In this appendix we prove the facts we need about (non-unipotently) monodromic complexes. For a more complete account of the unipotently monodromic story, see [BY, Be87].

**A.1. Free monodromic objects.** Let \(p\) be the characteristic of \(k\), which may be 0. For \(p \nmid n\), let \(A^0_n\) be the group algebra \(R[\mu_n]\) considered as a sheaf on \(\text{Spec} \ k\), i.e., a Gal(\(k/k\))-module. Put
\[A^0 = \lim_{\leftarrow p \mid n} A^0_n.\]

Consider \(\mathbb{T} := \lim_{\leftarrow p \mid n} \mu_n(\bar{k})\) the tame fundamental group of \(\mathbb{G}_{m,\bar{k}}\). For any \(\gamma \in \mathbb{T}\), let \(\bar{\gamma}\) denote the corresponding invertible element in \(A^0(\bar{k})\). Pick a topological generator \(t \in \mathbb{T}\). Note that \(\bar{t} - 1\) is not a zero divisor in \(A^0\), so \(A^0\) injects to the localization \(A = (A^0)_{\bar{t} - 1}\). Define
\[A^i = (\bar{t} - 1)^i A^0 \subset A\]
for \(i \in \mathbb{Z}\) and set \(A^i_n = A^i \otimes_{A^0} A^0_n\) for \(p \nmid n\). Note that \(A^1\) is the augmentation ideal.

**Remark A.2.** The ring \(A^0(\bar{k})\) is isomorphic to the product of the completions of \(R[t, t^{-1}]\) at all maximal ideals \(m\) such that \(t^n \equiv 1 \mod m\) for some \(p \nmid n\). The maximal ideals \(m\) correspond to the eigenvalues of the monodromy action.

For \(i \in \mathbb{Z}\) and \(p \nmid n\), let \(I^i_n\) be the local system on \(\mathbb{G}_{m,\bar{k}}\) such that the fiber at \(1 \in \mathbb{G}_{m,\bar{k}}(k)\) is \(A^i_n\) and the monodromy action of \(\gamma \in \mathbb{T}\) is multiplication by \(\bar{\gamma}\). We define \(I^i\) to be the pro-sheaf
\[\varprojlim_{p \mid n} I^i_n.\]

**Remark A.3.** After base change from \(\text{Spec} \ k\) to \(\text{Spec} \bar{k}\), the local systems \(I^0_n\) and \(I^i_n\) are non-canonically isomorphic via multiplication by \((\bar{t} - 1)^i\), and this induces an isomorphism \(I^0 \cong I^i(\bar{t} - 1)^i\).

**Lemma A.4.** There is a canonical isomorphism of pro-objects
\[I^0 \ast M \cong M(-1)[-2]\]
for \(M \in D_{\text{mon}}^b(V)\) considered as a constant pro-object.
Proof. Let \( e_n : \mathbb{G}_m \rightarrow \mathbb{G}_m \) denote the \( n \)th power map. Note that \( e_n^* R \cong I^0_p \) for \( p \nmid n \). Since \( M \) is monodromic, there exists \( n_0 \) coprime to \( p \) such that \( \theta(n_0)^* M \cong pr^*_n M \). Then

\[
\text{``lim''}(e_n^* R) \cdot M \cong \text{``lim''}(\theta(n_0)^* pr^*_n M \cong M(-1)[-2]),
\]

where we use the fact that the pro-object \( \text{``lim''} \Gamma_c(\mathbb{G}_m, R) \) is essentially constant and isomorphic to \( R(-1)[-2] \) (cf. [Ver83, Lemme 5.2]). \( \square \)

A.5. **Monodromic sheaves as \( A^0 \)-modules.** Let \( \text{Mod}_r(A^0) \) denote the abelian category of sheaves of discrete \( A^0 \)-modules on \( \text{Spec} \ k \), and let \( \text{Sh}(\mathbb{G}_m) \) denote the abelian category of sheaves of \( R \)-modules on \( \mathbb{G}_m \). We have a canonical exact functor

\[
\text{Loc} : \text{Mod}_r(A^0) \rightarrow \text{Sh}(\mathbb{G}_m).
\]

Define another functor \( \mathfrak{M} : \text{Sh}(\mathbb{G}_m) \rightarrow \text{Mod}_r(A^0) \) by

\[
\mathfrak{M}(\mathcal{F}) = \lim h'_n \cdot e_n \cdot e_n^* \mathcal{F}
\]

where \( h' : \mathbb{G}_m \rightarrow \text{Spec} \ k \) is the structure map and \( A^0 \) acts on \( e_n \cdot e_n^* \mathcal{F} \) by transport of structure. We deduce from étale descent that \( \text{Loc} \) is left adjoint to \( \mathfrak{M} \). Passing to derived categories, the derived functors are still adjoint, and we also denote them by

\[
\text{Loc} : D^b\text{Mod}_r(A^0) \rightleftarrows D^b(\mathbb{G}_m) : \mathfrak{M}.
\]

Note that \( \mathfrak{M} : D^b(\mathbb{G}_m) \rightarrow D^b\text{Mod}_r(A^0) \) is equal to the composition of the exact functor \( \lim \) \( e_n \cdot e_n^* \mathcal{F} \) with the derived functor \( h'_n \).

**Proposition A.6.** The derived functor \( \text{Loc} : D^b\text{Mod}_r(A^0) \rightarrow D^b(\mathbb{G}_m) \) is fully faithful.

Proof. We need to show that the unit of adjunction \( L \rightarrow \mathfrak{M} \circ \text{Loc}(L) \) is an isomorphism for \( L \in D^b\text{Mod}_r(A^0) \). We can assume that \( k \) is algebraically closed and \( L \) is concentrated in degree 0. Since \( \text{Loc} \) and \( \mathfrak{M} \) both commute with filtered colimits, we may further suppose that \( L \) is finite. Then there exists \( n_0 \) not divisible by \( p \) such that the action of \( A^0 \) on \( L \) factors through \( R[\mu_{n_0}] \). If \( n \) is a multiple of \( n_0 \) then \( e_n^* \text{Loc}(L) \cong L \), where \( L \) is the constant sheaf on \( \mathbb{G}_m \) with stalk \( L \). The proposition now follows from the fact that for any finite group \( L \) of order prime to \( p \), one has \( \lim \text{H}^0 \Gamma(\mathbb{G}_m, e_n \cdot e_n^* L) \cong L \) and \( \lim \text{H}^i \Gamma(\mathbb{G}_m, e_n \cdot e_n^* L) = 0 \) for \( i \neq 0 \). \( \square \)

**Corollary A.7.** The restriction of \( \text{Loc} \) induces an equivalence \( D^b\text{Mod}_r(A^0) \rightarrow D^b\text{mod}(\mathbb{G}_m) \), where \( \text{Mod}_r(A^0) \) is the abelian category of sheaves of \( A^0 \)-modules on \( \text{Spec} \ k \) with finite stalk.

The monoidal structure on \( D^b\text{Mod}_r(A^0) \) with respect to (derived) tensor product over \( A^0 \) corresponds under \( \text{Loc} \) to convolution on \( D^b(\mathbb{G}_m) \).

**Lemma A.8.** For \( L, K \in D^b\text{Mod}_r(A^0) \) there exists a canonical isomorphism

\[
\text{Loc}(L) \ast \text{Loc}(K) \cong \text{Loc}(L \otimes_{A^0} K)(-1)[-2].
\]

Proof. Consider the functor \( \text{Loc}_{G_m \times G_m} : D^b\text{Mod}_r(A^0 \otimes R A^0) \rightarrow D^b(\mathbb{G}_m \times \mathbb{G}_m) \), which is defined similarly to the above functor \( \text{Loc} = \text{Loc}_{G_m} \). Applying \( \text{Loc}_{G_m \times G_m} \) to the natural map \( L \otimes_{A^0} K \rightarrow L \otimes_{A^0} K \), we get a map \( \text{Loc}(L) \otimes \text{Loc}(K) \rightarrow m^* \text{Loc}(L \otimes_{A^0} K) \) in \( D^b(\mathbb{G}_m \times \mathbb{G}_m) \). Recall that since \( m \) is smooth, \( m^* \text{Loc}(L \otimes_{A^0} K) \cong m^! \text{Loc}(L \otimes_{A^0} K)(-1)[-2] \). Therefore the \( (m_1, m^!) \)-adjunction induces a morphism

\[
\text{Loc}(L) \ast \text{Loc}(K) \rightarrow \text{Loc}(L \otimes_{A^0} K)(-1)[-2].
\]

To check this is an isomorphism, we can assume \( k \) is algebraically closed and take \( L = K = A^0 \) for \( p \nmid n \) since the functors on both sides are of finite cohomological amplitude. Under these assumptions, the isomorphism is an easy computation. \( \square \)
Corollary A.9. There is a canonical projective system of isomorphisms

\[ I^i \ast I^j_n \cong I_n^{i+j}(-1)[-2] \]

for \( p \nmid n \) and any integers \( i \) and \( j \). Consequently there is an isomorphism of pro-objects

\[ I^i \ast I^j \cong I^{i+j}(-1)[-2]. \]

Proof. Fix \( p \nmid n \). By Lemma A.8, the first isomorphism is equivalent to an isomorphism

\[ \text{lim}_{p \nmid m} A^i_m \otimes A^j_n \cong A_n^{i+j} \]

as pro-objects in \( D^b_{\text{Mod}}(A^0) \). Remark 3 and Lemma 4 imply that it suffices to consider the cohomology in degree 0, i.e., we consider the non-derived tensor product on the LHS. Then \( H^0(A^i_m \otimes A^j_n) \cong A_n^{i+j} \) for \( n \mid m \) by definition. These isomorphisms are evidently compatible with changes in \( n \), so the rest of the corollary follows. \( \square \)

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