Matching complexes of polygonal line tilings

Takahiro Matsushita
Department of Mathematical Sciences, The University of The Ryukyus,
Nishihara-cho, Okinawa, Japan
mtst@sci.u-ryukyu.ac.jp
June 17, 2022

Abstract

The matching complex of a simple graph $G$ is the simplicial complex consisting of
the matchings on $G$. Jelić Milutinović et al. [6] studied the matching complexes of the
polygonal line tilings, and they gave a lower bound for the connectivity of the matching
complexes of polygonal line tilings. In this paper, we determine the homotopy types
of the matching complexes of polygonal line tilings recursively, and determine their
connectivities.

Mathematical Subject of Classification: Primary:55P10, Secondary:05C69, 05C70.

1 Introduction

A matching on a simple graph $G$ is a set of edges in $G$ which have no common vertex. The
matching complex $M(G)$ is the simplicial complex whose vertex set is the edge set $E(G)$
and whose simplices are the matchings on $G$.

Bouc [3] introduced the matching complexes of complete graphs in the context of the
Brown complexes and Quillen complexes of finite groups. The matching complexes of
complete bipartite graphs are called chessboard complexes. There are many works on
these two complexes, and we refer to [14] for an introduction to this subject. The integral
homology groups of these complexes have many torsions (see [13]) in general, and hence it
is very difficult to determine the homotopy types of matching complexes.

On the other hand, studies on homotopy types of matching complexes of other graphs
have sporadically appeared. First Kozlov [9] determined the homotopy types of matching
complexes of path graphs and cycle graphs. Marietti and Testa showed that the matching complexes of forests are homotopy equivalent to wedges of spheres. Jonsson studied the topology of the matching complexes of grid graphs in his unpublished work [7], and the author [11] determined the homotopy types of the matching complexes of \((t \times 2)\)-grid graphs, and showed that the matching complex of \((t \times 2)\)-grid graphs are homotopy equivalent to wedges of spheres (see also Braun and Hough [4]).

Recently, Jelić Milutinović et al. [6] studied the matching complexes of polygonal line tilings. Polygonal line tilings are a generalization of \((t \times 2)\)-grid graphs and some of honeycomb graphs defined as follows: For an integer \(n\) greater than 2 and a non-negative integer \(t\), the graph \(P_{n,t}\) of \(t\) \((2n)\)-gons is the graph in which \(t\) \((2n)\)-gons are arranged in a straight line as shown in Figure 1. Namely, the graph of \(t\) 4-gons is the \(((t + 1) \times 2)\)-grid graph, and the graph of \(t\) 6-gons is called the \((1 \times 1 \times t)\)-honeycomb graph in [6]. Using discrete Morse theory, they showed that the matching complexes of polygonal line tilings are homotopy equivalent to much smaller complexes, and gave lower bounds for the connectivities of these complexes. In particular, they determined the homotopy types of \(M(P_{3m+1,t})\) for every positive integer \(m\).

The purpose in this paper is to determine the homotopy types of matching complexes of polygonal line tilings. It turns out that these complexes are homotopy equivalent to wedges of spheres, and we determine the precise connectivities of these complexes. To explain our main results precisely, we need some preparation.

The independence complex of a graph \(G\) is the simplicial complex whose simplices are the independent sets of \(G\). The line graph of \(G\) is the graph \(L(G)\) whose vertex set is the edge set \(E(G)\) of \(G\) and two elements of \(E(G)\) are adjacent if they have a common vertex. Then the matching complex \(M(G)\) is the independence complex \(I(L(G))\) of the line graph.
Let $G_{n,t}$ be the line graph of the graph $P_{n,t}$ of $t$ $(2n)$-gons. Thus we have $I(G_{n,t}) = M(P_{n,t})$. We can formulate the graph $G_{n,t}$ as follows: The vertex set of $G_{n,t}$ is

$$V(G_{n,t}) = \{a_0, \cdots, a_t\} \cup \{b_{i,j}, c_{i,j} \mid 1 \leq i \leq t, 1 \leq j < n\}.$$

The adjacent relation of $G_{n,t}$ is formulated as follows:

- $a_i \sim b_{i+1,1}, a_i \sim c_{i+1,1}, a_{i+1} \sim b_{i+1,n-1}$, and $a_{i+1} \sim c_{i+1,n-1}$ for $i = 0, \cdots, t - 1$.
- $b_{i,j} \sim b_{i,j+1}$ and $c_{i,j} \sim c_{i,j+1}$ for $i = 1, \cdots, t$ and $j = 1, \cdots, n - 2$.
- $b_{i,n-1} \sim b_{i+1,1}$ and $c_{i,n-1} \sim c_{i+1,1}$ for $i = 1, \cdots, t - 1$.

Figure 2 depicts the graph $G_{n,t}$ when $n = 4$ and $t = 3$. Here we consider that $G_{n,0}$ is the graph with one vertex $a_0$. Thus $I(G_{n,0})$ is a point. Using this notation, our main theorems are formulated as follows:

**Theorem 1.1** (Theorem 3.3 and Proposition 3.7). Suppose that $n = 3m + 2$ for some non-negative integer $m$. Then the following hold:

1. For $t \geq 4$, there is a homotopy equivalence
   $$I(G_{n,t}) \simeq \Sigma^{6m+2} I(G_{n,t-3}) \vee \Sigma^{6m+2} I(G_{n,t-3}) \vee \Sigma^{8m+3} I(G_{n,t-4}).$$

2. There are homotopy equivalences
   $$I(G_{n,1}) \simeq S^{2m}, I(G_{n,2}) \simeq S^{4m+1} \vee S^{4m+1}, I(G_{n,3}) \simeq S^{6m+2}.$$

Note that $I(G_{2,t})$ is the matching complex of the $((t + 1) \times 2)$-grid graph. In fact, the case $m = 0$ in Theorem 1.1 was proved by the author [11].

The corresponding statement in the case $n = 3m$ is a little complicated. Let $H_{n,t}$ be the induced subgraph of $G_{n,t+1}$ whose vertex set is $V(G_{n,t}) \cup \{b_{t+1,1}, c_{t+1,1}\}$ (see Figure 2). Then the following holds:

**Theorem 1.2** (Theorem 3.10, Theorem 3.11, and Proposition 3.15). Suppose that $n = 3m$ for some positive integer $m$. Then the following hold:

1. If $t \geq 4$, then there is a homotopy equivalence
   $$I(G_{n,t}) \simeq \Sigma^{4m-2} I(H_{n,t-2}) \vee \Sigma^{6m-2} I(G_{n,t-3}) \vee \Sigma^{8m-3} I(H_{n,t-4}).$$
(2) If $t \geq 1$, then there is a homotopy equivalence
\[ I(H_{n,t}) \simeq \Sigma^{2m}I(G_{n,t-1}) \lor \Sigma^{2m-1}I(H_{n,t-1}). \]

(3) There are homotopy equivalences
\[ I(G_{n,1}) \simeq S^{2m-1} \lor S^{2m-1}, \quad I(G_{n,2}) \simeq S^{4m-2} \lor S^{4m-2}, \quad I(G_{n,3}) \simeq S^{6m-2} \lor S^{6m-3}, \quad I(H_0) \simeq S^0. \]

Here we mention the connectivity of the case $n = 3$. In this case, the polygonal line tiling is $(1 \times 1 \times t)$-honeycomb graph. In [8] Jonsson considered that the study of the matching complexes of honeycomb graphs seems to be interesting, and Jelić Milutinović et al. first studied them and showed that the matching complex of $(1 \times 1 \times t)$-honeycomb graph is $(t-1)$-connected. In fact, Theorem 1.2 implies that their estimate is strict:

**Corollary 1.3** (Corollary 3.17). *The connectivity of the matching complex of $(1 \times 1 \times t)$-honeycomb graph is $t-1$.*

Theorem 1.1 and Theorem 1.2 imply that the complex $I(G_{n,t})$ is a wedge of spheres when $n = 3m$ or $n = 3m + 2$. Since Jelić Milutinović et al. [6] showed that $I(G_{n,t})$ is homotopy equivalent to a wedge of spheres when $n = 3m + 1$, we conclude the following:

**Corollary 1.4.** *The matching complexes of polygonal line tilings are wedges of spheres.*

The rest in this paper is organized as follows: In Section 3, we review necessary facts concerning independence complexes. In Section 3, we determine the homotopy types of the
matching complexes of polygonal line tilings. Since the proofs are different in the cases \( n = 3m, 3m + 1, 3m + 2 \), we divide this section into three parts. In Subsection 3.1, we determine the homotopy type of \( I(G_{n,t}) \) in the case \( n = 3m + 1 \). This is an alternative proof of a result in Jelić Milutinović et al. \[6\]. We prove Theorem \[1.1\] and Theorem \[1.2\] in Subsection 3.2 and Subsection 3.3, respectively. In these sections, we determine the connectivity of \( I(G_{n,t}) \) (Theorem \[3.8\] and Theorem \[3.16\]).

Acknowledgement. The author thanks Jelić Milutinović for answering his question concerning her paper \[6\]. The author also thanks the referee for useful comments. The author is supported by JSPS KAKENHI 19K14536.

2 Independence complexes

In this section, we review some known results of independence complexes. For basic terminology concerning simplicial complexes in topological combinatorics, we refer to \[8\] or \[10\].

Throughout the paper, graphs are assumed to be finite and simple. For a subset \( S \) of the vertex set \( V(G) \) of \( G \), let \( G[S] \) denote the subgraph in \( G \) induced by \( S \). We write \( G \setminus S \) to mean \( G[V(G) \setminus S] \). For a vertex \( v \) in \( G \), we write \( G \setminus v \) instead of \( G \setminus \{v\} \).

Recall that a subset \( S \) of \( V(G) \) is independent if \( G[S] \) has no edges. The independence complex \( I(G) \) of \( G \) is the (abstract) simplicial complex whose vertex set is \( V(G) \) and whose simplices are the independent sets of \( G \).

The following proposition is easily deduced from the definition.

**Proposition 2.1.** If \( G \) is a disjoint union \( G_1 \sqcup G_2 \) of subgraphs \( G_1 \) and \( G_2 \), then \( I(G) = I(G_1) \ast I(G_2) \).

Here \( \ast \) denotes the join of simplicial complexes. Thus if \( G \) has a connected component whose independence complex is contractible, then \( I(G) \) is also contractible. In particular, if \( G \) has an isolated vertex, then \( I(G) \) is contractible.

For a simplicial complex \( K \) and a vertex \( v \) in \( K \), we write \( K \setminus v \) to indicate the simplicial complex whose simplices are the simplices in \( K \) not containing \( v \). Let \( \text{link}_K(v) \) be the link of \( K \) at \( v \). Clearly, there is a cofiber sequence

\[
\text{link}_K(v) \to K \setminus v \to K.
\]

Now we apply this cofiber sequence to independence complexes. For a vertex \( v \) in \( G \), the open neighborhood of \( v \) is \( N_G(v) = \{w \in V(G) \mid \{v,w\} \in E(G)\} \), and the closed
neighborhood $N_G[v]$ is $N_G(v) \cup \{v\}$. If we do not need to mention $G$, we often write $N(v)$ (or $N[v]$) instead of $N_G(v)$ (or $N_G[v]$, respectively). It is easy to see that link$_G(v)$ is $I(G \setminus N_G[v])$ and $I(G) \setminus v = I(G \setminus v)$. Thus we have the following:

**Proposition 2.2** (See Adamaszek [1]). The sequence

$$I(G \setminus N[v]) \hookrightarrow I(G \setminus v) \hookrightarrow I(G).$$

is a cofiber sequence. In particular, if the inclusion $I(G \setminus N[v]) \hookrightarrow I(G \setminus v)$ is null-homotopic, then there is a homotopy equivalence $I(G) \simeq I(G \setminus v) \vee \Sigma I(G \setminus N[v])$.

We need the following two propositions from [5].

**Proposition 2.3** (Fold lemma, Lemma 3.2 of [5]). Let $v$ and $w$ be distinct vertices in $G$, and suppose $N(w) \subset N(v)$. Then the inclusion $I(G \setminus v) \hookrightarrow I(G)$ is a homotopy equivalence.

A vertex $v$ in $G$ is simplicial if $G[N[v]]$ is a complete graph.

**Theorem 2.4** (Theorem 3.7 of [5]). If $v$ is a simplicial vertex in $G$, there is a homotopy equivalence

$$I(G) \simeq \bigvee_{w \in N(v)} \Sigma I(G \setminus N[w]).$$

Let $P_k$ be the path graph with $k$ vertices. Namely, $V(P_k) = \{1, \ldots, k\}$ and $E(P_k) = \{\{i, j\} \mid |i - j| = 1\}$. Let $C_k$ be the $k$-cycle graph. We will frequently use the following basic computation by Kozlov [9].

**Proposition 2.5** (Proposition 4.6 and 5.2 of Kozlov [9]). The following hold:

1. $I(P_{3k}) \simeq S^{k-1}$, $I(P_{3k+1}) \simeq PT$, $I(P_{3k+2}) \simeq S^k$ for $k \geq 0$.
2. $I(C_{3k}) \simeq S^{k-1} \vee S^{k-1}$, $I(C_{3k+1}) \simeq S^{k-1}$, $I(C_{3k+2}) \simeq S^k$ for $k \geq 1$.

**Corollary 2.6.** For $k \geq 0$, the inclusion $I(P_{3k+2}) \hookrightarrow I(P_{3k+3})$ is a homotopy equivalence.

**Proof.** Put $v = 3k + 3 \in V(P_{3k+3})$. Then $I(P_{3k+3} \setminus N_{P_{3k+3}}[v]) = I(P_{3k+1})$ is contractible. Thus Proposition 2.2 implies that the inclusion $I(P_{3k+2}) \hookrightarrow I(P_{3k+3})$ is a homotopy equivalence.

Finally, we give a rather technical argument which makes the description much simpler in Section 3. First, we introduce the following terminology: An $n$-string based at $v$ of a graph $G$ is an induced subgraph $P$ of $G$ which satisfies the following:
(1) $P$ is isomorphic to $P_n$, and $v$ is an endpoint of $P$.

(2) Every vertex in $P$ except for $v$ is adjacent to no vertex belonging to $V(G) \setminus V(P)$.

**Proposition 2.7.** Let $G$ be a graph and $P$ a $(3k + 2)$-string based at $v$ in $G$. Let $w$ be the vertex in $P$ which is adjacent to $v$, and set $U = N_G(v) \setminus w$. Then the inclusion $I(G \setminus U) \hookrightarrow I(G)$ is a homotopy equivalence.

**Proof.** Set $U = \{u_1, \cdots, u_n\}$. Then $P \setminus v$ is a connected component of the graph $G \setminus N_G[u_1]$, which is isomorphic to $P_{3k+1}$. Since $I(P_{3k+1})$ is contractible, Proposition 2.1 implies that $I(G \setminus N_G[u_1])$ is contractible. Thus it follows from Proposition 2.2 that the inclusion $I(G \setminus u_1) \hookrightarrow I(G)$ is a homotopy equivalence. By the same reason, the inclusion $I(G \setminus \{u_1, u_2\}) \hookrightarrow I(G \setminus u_1)$ is a homotopy equivalence. Iterating this, we have that the inclusion $I(G \setminus U) = I(G \setminus \{u_1, \cdots, u_n\}) \hookrightarrow I(G)$ is a homotopy equivalence. \hfill \Box

3 Computations

In this section, we determine the homotopy types and connectivities of the matching complexes of the graph of $t(2n)$-gons. Recall that $G_{n,t}$ is the line graph of the graphs $P_{n,t}$ of $t(2n)$-gons. An explicit formulation of $G_{n,t}$ was given in Section 1.

From now on, we fix an integer $n$ greater than 1 and write $G_t$ instead of $G_{n,t}$. The proof is divided into three cases.

3.1 Case $n = 3m + 1$

In this subsection, we determine the homotopy types of $I(G_t)$ when $n = 3m + 1$. As was mentioned in Section 1, Jelić Milutinović et al. [6] determined the homotopy types of $I(G_t)$ in this case. Here we give an alternative proof of their result.

**Theorem 3.1** (Corollary 4.3 of [6]). $I(G_t) \simeq \bigvee_t S^{2m}$

Throughout this subsection, put $X_t = G_t \setminus N[a_t]$ for $t \geq 1$ (In the next two subsections, we use $X_t$ to indicate other graphs).

**Lemma 3.2.** $I(X_t) \simeq S^{2m-1}$

**Proof.** We prove this by induction on $t$. Since $I(X_1) = I(P_{6m-1}) \simeq S^{2m-1}$, the case $t = 1$ holds. Suppose $t \geq 2$. Then $b_{t,3m-1}, \cdots, b_{t,1}$ form an $(3m - 1)$-string based at $b_{t,1}$, and $c_{t,3m-1}, \cdots, c_{t,1}$ form an $(3m - 1)$-string based at $c_{t,1}$. Using Proposition 2.7, we can delete
the three vertices $b_{t-1,3m-1}, c_{t-1,3m-1}, a_{t-1}$ from $X_t$. Figure 3 depicts the case $m = 1$.

Hence we have

$$I(X_t) \simeq I(P_{3m-1}) \ast I(P_{3m-1}) \ast I(X_{t-1}) \simeq S^{m-1} \ast S^{m-1} \ast I(X_{t-1}) \simeq \Sigma^{2m} I(X_{t-1}).$$

This completes the proof.

**Proof of Theorem 3.1.** Since $I(G_1) = I(C_{6m+2}) \simeq S^{2m}$, the case $t = 1$ holds. In $G_t \setminus a_t$, the vertices $b_{t,3m}, \ldots, b_{t,2}$ form a $(3m - 1)$-string based at $b_{t,2}$, and the vertices $c_{t,3m}, \ldots, c_{t,2}$ form a $(3m - 1)$-string based at $c_{t,2}$. Thus using Proposition 2.7 we can delete the vertices $b_{t,1}$ and $c_{t,1}$ from $G_t \setminus a_t$ (see Figure 3). Hence we have

$$I(G_t \setminus a_t) \simeq I(P_{3m-1}) \ast I(P_{3m-1}) \ast I(G_{t-1}) \simeq \Sigma^{2m} I(G_{t-1}) \simeq \bigvee_{t-1} S^{2tm}.$$ 

Since $I(G_t \setminus N[a_t]) = I(X_t) \simeq S^{2tm-1}$, the inclusion $I(G_t \setminus N[a_t]) \hookrightarrow I(G_t \setminus a_t)$ is null-homotopic. Thus Proposition 2.2 implies

$$I(G_t) \simeq I(G_t \setminus a_t) \lor \Sigma I(X_t) \simeq \bigvee_{t} S^{2tm}.$$ 

**3.2 Case $n = 3m + 2$**

The purpose in this section is to determine the homotopy type and connectivity of $I(G_t)$ when $n = 3m + 2$. Here we assume that $m \geq 0$. Note that the case $m = 0$, the graph of $t$ $(2n)$-gons is a $(t \times 2)$-grid graph, and hence the homotopy types of their matching complexes are determined in [11]. We first prove the following recursive formula:  

\begin{align*}
X_t \ &\simeq \ &\bigvee_{t-1} P_{3m-1} \cup P_{3m-1} \\
G_t \setminus a_t \ &\simeq \ &\bigvee_{t-1} P_{3m-1} \cup P_{3m-1}
\end{align*}
Theorem 3.3. For \( t \geq 3 \), there is a homotopy equivalence
\[
I(G_t) \simeq \Sigma^{6m+2} I(G_{t-3}) \lor \Sigma^{6m+2} I(G_{t-3}) \lor \Sigma^{8m+3} I(G_{t-4}).
\]

For \( t \geq 1 \), we put \( X_t = G_t \setminus a_{t-1} \) (see Figure 4).

Lemma 3.4. For \( t \geq 1 \), there is a homotopy equivalence
\[
I(G_t) \simeq I(X_t).
\]

Proof. By Proposition 2.2, it suffices to show that \( I(G_t \setminus N[a_{t-1}]) \) is contractible. Note that \( G_t \setminus N[a_{t-1}] \) has a connected component consisting of the vertices
\[
b_{t,2}, \cdots, b_{t,3m+1}, a_t, c_{t,3m+1}, \cdots, c_{t,2},
\]
which is isomorphic to \( P_{6m+1} \). Since \( I(P_{6m+1}) \) is contractible, Proposition 2.1 implies that \( I(G_t \setminus N[a_{t-1}]) \) is contractible. \( \square \)

Next we put \( Y_t = X_t \setminus a_{t-2} \) for \( t \geq 2 \) (see Figure 4).

Lemma 3.5. For \( t \geq 3 \), there is a homotopy equivalence
\[
I(X_t) \simeq I(Y_t) \lor \Sigma^{6m+2} I(G_{t-3}).
\]

Proof. By Proposition 2.2, it suffices to show the following assertions:

1. The inclusion \( I(X_t \setminus N[a_{t-2}]) \hookrightarrow I(X_t \setminus a_{t-2}) = I(Y_t) \) is null-homotopic.

2. \( I(X_t \setminus N[a_{t-2}]) \simeq \Sigma^{6m+1} I(G_{t-3}). \)

We first show (1). The graph \( X_t \setminus N[a_{t-2}] \) has a connected component \( P \) consisting of
\[
b_{t-1,2}, \cdots, b_{t-1,3m+1}, b_{t,1}, \cdots, b_{t,3m+1}, a_t, c_{t,3m+1}, \cdots, c_{t,1}, c_{t-1,3m+1}, \cdots, c_{t-1,2},
\]
which is isomorphic to \( P_{12m+3} \). We define the subgraph \( H \) by \( X_t \setminus N[a_{t-2}] = P \sqcup H. \)

Since the inclusion \( I(P_{12m+2}) \hookrightarrow I(P_{12m+3}) \) is a homotopy equivalence (Corollary 2.6), we have that \( I(P \setminus c_{t-1,2}) \hookrightarrow I(P) \) is a homotopy equivalence. By Proposition 2.1 we have that the inclusion
\[
I((P \setminus c_{t-1,2}) \sqcup H) \hookrightarrow I(P \sqcup H) = I(X_t \setminus N[a_{t-2}])
\]

is a homotopy equivalence. Since the graph \( (P \setminus c_{t-1}) \sqcup H \) has no vertices adjacent to \( c_{t-1,1} \in V(X_t \setminus a_{t-2}) \), we have that \( I((P \setminus c_{t-1,2}) \sqcup H) \) is contained in the star at \( c_{t-1,1} \) in \( I(X_t \setminus a_{t-2}) \). This means that the composite of the inclusions

\[
I(P \setminus c_{t-1,2}) \sqcup H \to I(P \sqcup H) \to I(X_t \setminus N[a_{t-2}]) \to I(X_t \setminus a_{t-2})
\]

is null-homotopic. This completes the proof of (1).

Next we show (2). Since \( I(P) = I(P_{12m+3}) \simeq S^{4m} \), we have

\[
I(X_t \setminus N[a_{t-1}]) = I(P) \ast I(H) \simeq \Sigma^{4m+1} I(H).
\]

In \( H \), the vertices \( b_{t-2,3m}, \ldots, b_{t-2,2} \) form a \((3m - 1)\)-string based at \( b_{t-2,2} \), and the vertices \( c_{t-2,3m}, \ldots, c_{t-2,2} \) form a \((3m - 1)\)-string based at \( c_{t-2,2} \). By Proposition 2.7, we can delete the vertices \( b_{t-2,1}, c_{t-2,1} \) from \( X_t \setminus N[a_{t-2}] \), and hence we have

\[
I(H) \simeq I(P_{3m-1} \sqcup P_{3m-1} \sqcup G_{t-3}) = S^{m-1} \ast S^{m-1} \ast I(G_{t-3}) = \Sigma^{2m} I(G_{t-3}).
\]

This implies \( I(X_t \setminus N[a_{t-2}]) \simeq \Sigma^{4m+1} I(H) = \Sigma^{6m+1} I(G_{t-3}) \). This completes the proof.

**Lemma 3.6.** For \( t \geq 4 \), there is a homotopy equivalence

\[
I(Y_t) \simeq \Sigma^{6m+2} I(G_{t-3}) \vee \Sigma^{8m+3} I(G_{t-4}).
\]

**Proof.** By Proposition 2.2, it suffices to show the following assertions:

1. The inclusion \( I(Y_t \setminus N[a_t]) \hookrightarrow I(Y_t \setminus a_t) \) is null-homotopic.
2. \( I(Y_t \setminus a_t) \simeq \Sigma^{6m+2} I(G_{t-3}) \)
3. \( I(Y_t \setminus N[a_t]) \simeq \Sigma^{8m+2} I(G_{t-4}) \)
We first show (1). In \( Y_t \setminus N[a_t] \), the vertices

\[
\begin{aligned}
b_{t,3m}, & \cdots , b_{t,3m+1}, b_{t-1,3m+1}, \cdots , b_{t-2,3m+1}, b_{t-3,3m+1}, \cdots , b_{t-2,1}.
\end{aligned}
\]

form a \((9m+2)\)-string \( P \) based at \( b_{t-2,1} \), and the vertices

\[
\begin{aligned}
c_{t,3m}, & \cdots , c_{t,3m+1}, c_{t-1,3m+1}, \cdots , c_{t-2,3m+1}, \cdots , c_{t-1,1}.
\end{aligned}
\]

form a \((9m+2)\)-string \( Q \) based at \( c_{t-2,1} \). By Proposition \( \ref{prop:2.7} \) we can delete the vertices \( a_{t-3}, b_{t-3,3m+1}, c_{t-3,3m+1} \) from \( Y_t \setminus N[a_t] \). Thus we have a homotopy equivalence \( I(Y_t \setminus N[a_t]) \simeq I(P) * I(Q) * I(H) \), where \( H \) is the graph satisfying

\[
P \cup Q \cup H = Y_t \setminus \{ N[a_t] \cup \{ a_{t-3}, b_{t-3,3m+1}, c_{t-3,3m+1} \} \}
\]

Let \( P' \) be the subgraph of \( Y_t \setminus a_t \) induced by \( V(P) \cup b_{t,3m+1} \) and \( Q' \) the subgraph of \( Y_t \setminus a_t \) induced by \( V(Q) \cup c_{t,3m} \). Put \( P'' = P' \setminus b_{t-2,1} \) and \( Q'' = Q' \setminus c_{t-2,1} \). Figure 5 depicts the graphs \( P \cup Q \cup H \), \( P' \cup Q' \cup H \), and \( P'' \cup Q'' \cup H \) when \( m = 0 \).

By Corollary \( \ref{cor:2.6} \) the four inclusions

\[
I(P) \hookrightarrow I(P') \hookrightarrow I(P''), \quad I(Q) \hookrightarrow I(Q') \hookrightarrow I(Q'')
\]

are homotopy equivalences. Thus we have a commutative diagram

\[
\begin{align*}
I(P \cup Q \cup H) & \xrightarrow{\simeq} I(P' \cup Q' \cup H) \xrightarrow{\simeq} I(P'' \cup Q'' \cup H) \\
\downarrow & & \downarrow \\
I(Y_t \setminus a_t) & & I(Y_t \setminus a_t)
\end{align*}
\]

Here \( P'' \cup Q'' \cup H \) has no vertices adjacent to \( a_{t-3} \in V(Y_t \setminus a_t) \). Thus the inclusion \( I(P'' \cup Q'' \cup H) \hookrightarrow I(Y_t \setminus a_t) \) is null-homotopic. By the commutativity of the above diagram, the composite of the inclusions \( I(P \cup Q \cup H) \xrightarrow{\simeq} I(Y_t \setminus N[a_t]) \rightarrow I(Y_t \setminus a_t) \) is null-homotopic. This completes the proof of (1).

\[
\begin{align*}
\text{Figure 5.}
\end{align*}
\]

11
Next we prove (2). In $Y_t \setminus a_t$, the vertices

$$b_{t,3m+1}, \ldots, b_{t,1}, b_{t-1,3m+1}, \ldots, b_{t-1,1}, b_{t-2,3m+1}, \ldots, b_{t-2,2}$$

form a $(9m + 2)$-string based at $b_{t-2,2}$, and the vertices

$$c_{t,3m+1}, \ldots, c_{t,1}, c_{t-1,3m+1}, \ldots, c_{t-1,1}, c_{t-2,3m+1}, \ldots, c_{t-2,2}$$

form a $(9m + 2)$-string based at $c_{t-2,2}$. By Proposition 2.7, we have homotopy equivalences

$$I(Y_t \setminus a_t) \simeq I(P_{9m+2} \sqcup P_{9m+2} \sqcup G_{t-3}) = S^{3m} \ast S^{3m} \ast I(G_{t-3}) = \Sigma^{6m+2} I(G_{t-3}).$$

This completes the proof of (2).

Finally, we prove (3). Recall $I(Y_t \setminus N[a_t]) \simeq I(P) \ast I(Q) \ast I(H)$ and $P \cong Q \cong P_{9m+2}$. Since $I(P_{9m+2}) \simeq S^{3m}$, we have that $I(Y_t \setminus N[a_t]) \simeq \Sigma^{6m+2} I(H)$. In $H$, the vertices $b_{t-3,3m}, \ldots, b_{t-3,2}$ form a $(3m - 1)$-string based at $b_{t-3,2}$ and the vertices $c_{t-3,3m}, \ldots, c_{t-3,2}$ form a $(3m - 1)$-string based at $c_{t-3,2}$. By Proposition 2.2, we can delete the vertices $b_{t-3,1}$ and $c_{t-3,1}$ from $H$. Hence we have

$$I(H) \simeq I(P_{3m-1}) \ast I(P_{3m-1}) \ast I(G_{t-4}) \simeq S^{m-1} \ast S^{m-1} \ast I(G_{t-4}) \simeq \Sigma^{2m} I(G_{t-4}).$$

Thus we have $I(Y_t \setminus N[a_t]) \simeq \Sigma^{6m+2} I(H) \simeq \Sigma^{8m+2} I(G_{t-4}).$ This completes the proof. \qed

**Proof of Theorem 3.3.** Combining the Lemma 3.4, Lemma 3.5 and Lemma 3.6, we have

$I(G_t) \simeq I(X_t) \simeq I(Y_t) \vee \Sigma^{6m+2} I(G_{t-3}) \simeq \Sigma^{6m+2} I(G_{t-3}) \vee \Sigma^{6m+2} I(G_{t-3}) \vee \Sigma^{8m+3} I(G_{t-4}).$

This completes the proof. \qed

Theorem 3.3 implies that $G_t$ is recursively determined by $I(G_0)$, $I(G_1)$, $I(G_2)$, and $I(G_3)$. These complexes are easily determined by Lemma 3.5 and Lemma 3.6

**Proposition 3.7.** $I(G_0) \simeq \text{pt}$, $I(G_1) \simeq S^{2m}$, $I(G_2) \simeq S^{4m+1} \vee S^{4m+1}$, $I(G_3) \simeq S^{6m+2}$

**Proof.** By the definition of $G_0$, it follows that $I(G_0) \simeq \text{pt}$. Since $G_1 \cong C_{6m+4}$, we have that $I(G_1) \simeq S^{2m}$. Since $I(G_2) \simeq I(X_2) \cong I(C_{12m+6})$, we have $I(G_2) \simeq S^{4m+1} \vee S^{4m+1}$. By Lemma 3.4 and Lemma 3.5, we have that $I(G_3) \simeq I(X_3) \simeq I(Y_3) \vee \Sigma^{6m+2} I(G_0) \simeq I(Y_3) \cong I(C_{18m+8})$. Thus we have $I(G_3) \simeq S^{6m+2}$. This completes the proof. \qed

Finally, we determine the connectivity of $I(G_t)$. Let $k_t$ be the connectivity of $I(G_t)$. Then the following hold:
Theorem 3.8. Define $s \geq 0$ and $\varepsilon = 1, 2, 3$ by $t = 3s + \varepsilon$. Then

$$k_t = 2mt + 2(s-1) + \varepsilon \quad (1)$$

Proof. It follows from Proposition 3.7 that the equation (1) holds for $s = 0$. By Theorem 3.3, we have the equation

$$k_t = \min\{k_{t-3} + 6m + 2, k_{t-4} + 8m + 3\}$$

for $t \geq 4$. Using this, one can show that the equation (1) holds for every $t$ by induction on $s$. This completes the proof. \(\square\)

Remark 3.9. Jelić Milutinović et al. [6] showed that when $n = 3m + 2$ and $t = 3s + \varepsilon$ with $\varepsilon = 1, 2, 3$, the complex $I(G_t)$ the connectivity is at least

$$2mt + t - \left\lfloor \frac{t+1}{2} \right\rfloor - 1.$$

Thus the connectivity of $I(G_t)$ is strictly greater than their lower bound. On the other hand, in the case of $n = 3m$, we will see that their lower bound coincides the strict connectivity of $I(G_t)$.

3.3 Case $n = 3m$

The purpose in this section is to determine the homotopy types and connectivities of $I(G_t)$ when $n = 3m$. Recall that the graph $H_t$ is the induced subgraph of $G_{t+1}$ whose vertex set if $V(G_{t+1}) \cup \{b_{t+1,1}, c_{t+1,1}\}$ (see Section 1). The recursive formulae of $I(G_t)$ are described as the following two theorems:

Theorem 3.10. For $t \geq 4$, there is a homotopy equivalence

$$I(G_t) \simeq \Sigma^{4m-2} I(H_{t-2}) \vee \Sigma^{6m-2} I(G_{t-3}) \vee \Sigma^{8m-3} I(H_{t-4}).$$

Theorem 3.11. For $t \geq 1$, there is a homotopy equivalence

$$I(H_t) \simeq \Sigma^{2m} I(G_{t-1}) \vee \Sigma^{2m-1} I(H_{t-1}).$$

We first show Theorem 3.10. Put $X_t = G_t \setminus a_{t-1}$.

Lemma 3.12. If $t \geq 2$, then $I(G_t) \simeq I(X_t) \vee \Sigma^{4m-2} I(H_{t-2})$.

Proof. By Proposition 2.2 it suffices to show the following assertions:
(1) The inclusion \( I(G_t \setminus N[a_{t-1}]) \hookrightarrow I(G_t \setminus a_{t-1}) = I(X_t) \) is null-homotopic.

(2) \( I(G_t \setminus N[a_{t-1}]) \simeq \Sigma^{4m-3} I(H_{t-2}) \)

First, we write \( P \) to indicate the connected component of \( G_t \setminus N[a_{t-1}] \) consisting of the vertices

\[ b_{t,2}, \ldots, b_{t,3m-1}, a_t, c_{t,3m-1}, \ldots, c_{t,2}. \]

Put \( P' = P \setminus c_{t,1} \) and \( G_t \setminus N[a_{t-1}] = P \sqcup Q \). Since \( P \) is isomorphic to \( P_{6m-3} \), the inclusion \( I(P') \hookrightarrow I(P) \) is a homotopy equivalence. Therefore we have that \( I(P' \sqcup Q) \hookrightarrow I(P \sqcup Q) = I(G_t \setminus N[a_{t-1}]) \) is a homotopy equivalence. However, \( P' \sqcup Q \) has no vertices adjacent to \( c_{t,1} \in V(X_t) \). Therefore, we have that the composite \( I(P' \sqcup Q) \xrightarrow{\sim} I(P \sqcup Q) \hookrightarrow I(X_t) \) is null-homotopic, and hence we have that the inclusion \( I(G_t \setminus N[a_{t-1}]) = I(P \sqcup Q) \hookrightarrow I(X_t) \) is null-homotopic. This completes the proof of (1).

In \( Q \), \( b_{t-1,3m-2}, \ldots, b_{t-1,3} \) is a \( (3m - 4) \)-string based at \( b_{t-1,3} \), and \( c_{t-1,3m-2}, \ldots, c_{t-1,3} \) is a \( (3m - 4) \)-string based at \( c_{t-1,3} \). By Proposition \ref{prop:2.7}, one can delete \( b_{t-1,2} \) and \( c_{t-1,2} \) from \( Q \), and hence we have

\[ I(Q) \simeq I(Q \setminus \{b_{t-1,2}, c_{t-1,2}\}) \simeq I(P_{3m-4}) * I(P_{3m-4}) * I(H_{t-2}) = \Sigma^{2m-2} I(H_{t-2}). \]

Therefore we have

\[ I(G_t \setminus N[a_{t-1}]) = I(P) * I(Q) \cong I(P_{6m-3}) * I(Q) \cong S^{2m-2} * \Sigma^{2m-2} I(H_t) = \Sigma^{4m-3} I(H_{t-2}). \]

This completes the proof of (2).

Next put \( Y_t = X_t \setminus a_{t-2} \) for \( t \geq 2 \).

**Lemma 3.13.** If \( t \geq 2 \), then \( I(X_t) \cong I(Y_t) \)

**Proof.** By Proposition \ref{prop:2.7}, it suffices to show that \( I(X_t \setminus N[a_{t-2}]) \) is contractible. In fact, \( X_t \setminus N[a_{t-2}] \) has a connected component \( P \) consisting of the vertices

\[ b_{t-1,2}, \ldots, b_{t,3m-1}, b_{t,1}, \ldots, b_{t,3m-1}, a_t, c_{t,3m-1}, c_{t,1}, c_{t-1,3m-1}, c_{t-1,2}. \]

Then \( P \) is isomorphic to \( P_{12m-5} \). Since \( I(P_{12m-5}) \) is contractible, we have that \( I(X_t \setminus N[a_{t-2}]) \) is contractible.

**Lemma 3.14.** If \( t \geq 4 \), then \( I(Y_t) \cong \Sigma^{6m-2} I(G_{t-3}) \vee \Sigma^{8m-3} I(H_{t-4}) \).

**Proof.** The proof of this lemma is almost the same as the proof of Lemma \ref{lem:3.6}. Put \( Z_t = Y_t \setminus a_t \) and \( W_t = Y_t \setminus N[a_t] \). By Proposition \ref{prop:2.2}, it suffices to show the following assertions:
(1) The inclusion $I(W_t) \hookrightarrow I(Z_t)$ is null-homotopic.

(2) $I(Z_t) \simeq \Sigma^{6m-2} I(G_{t-3})$

(3) $I(W_t) \simeq \Sigma^{8m-4} I(H_{t-4})$

In $W_t$, the vertices

$$P = b_{t,3m-2}, \ldots, b_{t,1}, b_{t-1,3m-1}, \ldots, b_{t-1,1}, b_{t-2,3m-1}, \ldots, b_{t-2,1}$$

form a $(9m-4)$-string based at $b_{t-2,1}$, and the vertices

$$Q = c_{t,3m-2}, \ldots, c_{t,1}, c_{t-1,3m-1}, \ldots, c_{t-1,1}, c_{t-2,3m-1}, \ldots, c_{t-2,1}$$

form a $(9m-4)$-string based at $c_{t-2,1}$. By Proposition 2.2, we have

$$I(W_t) \simeq I(W_t \setminus \{a_{t-3}, b_{t-3,3m-1}, c_{t-3,3m-1}\}).$$

We put $U_t = W_t \setminus \{a_{t-2}, b_{t-3,3m-1}, c_{t-3,3m-1}\}$. Note that $U_t$ has two connected components

$$b_{t,3m-2}, \ldots, b_{t,1}, b_{t-1,3m-1}, \ldots, b_{t-1,1}, b_{t-2,3m-1}, \ldots, b_{t-2,1}$$

and

$$c_{t,3m-2}, \ldots, c_{t,1}, c_{t-1,3m-1}, \ldots, c_{t-1,1}, c_{t-2,3m-1}, \ldots, c_{t-2,1}$$

which are isomorphic to $P_{9m-4}$. Then define $U'_t$ to be the induced subgraph of $Y_t$ whose vertex set is $V(U_t) \cup \{b_{t,3m-1}, c_{t,3m-1}\}$, and $U''_t = U_t \setminus \{b_{t-2,1}, c_{t-2,1}\}$. Since the inclusion $I(P_{9m-4}) \hookrightarrow I(P_{9m-4})$ is a homotopy equivalence, we have that the inclusions $I(U'_t) \hookrightarrow I(U'_t)$ and $I(U''_t) \hookrightarrow I(U''_t)$ are homotopy equivalences. Thus we have the following commutative diagram:

$$\begin{array}{ccc}
I(U''_t) & \xrightarrow{\simeq} & I(U'_t) \\
\downarrow & & \downarrow \\
I(Z_t) & \xrightarrow{\simeq} & I(U_t)
\end{array}$$

Here every arrow in this diagram is an inclusion. Since $U''_t$ has no vertices adjacent to $a_{t-3} \in V(Z_t)$, we have that the inclusion $I(U''_t) \hookrightarrow I(Z_t)$ is null-homotopic. Thus the above diagram shows that the composite of $I(U_t) \xrightarrow{\simeq} I(W_t) \hookrightarrow I(Z_t)$ is null-homotopic. This completes the proof of (1).

Next we show (3). We put $W_t' = P \sqcup Q \sqcup W_t'$. Here $P$ and $Q$ are the connected components defined in the proof of (1) which are isomorphic to $P_{9m-4}$. In $W_t'$, the vertices

---

15
\[ b_{t-3m-2}, \ldots, b_{t-3}, \] form a \((3m-4)\)-string based at \(b_{t-3} \), and the vertices \(c_{t-3,3m-2}, \ldots, c_{t-3,2} \) form a \((3m-4)\)-string based at \(c_{t-3,3} \). Therefore we have
\[
I(W_t' \cup \{b_{t-3,2}, c_{t-3,2}\}) = I(P_{3m-4} \ast I(P_{3m-4}) \ast I(H_{t-4})) = \Sigma^{2m-2}I(H_{t-4}).
\]
Therefore we have
\[
I(W_t) \simeq I(P) \ast I(Q) \ast I(W_t') = I(P_{3m-4}) \ast I(P_{3m-4}) \ast I(W_t') \simeq \Sigma^{8m-4}I(H_{t-4}).
\]
Finally, we show (2). In \(Z_t \), the vertices
\[
b_{t,3m-1}, \ldots, b_{t,1}, b_{t-1,3m-1}, \ldots, b_{t-1,1}, b_{t-2,3m-1}, \ldots, b_{t,2}
\]
form a \((9m-4)\)-string based at \(b_{t,2} \), and the vertices
\[
c_{t,3m-1}, \ldots, c_{t,1}, c_{t-1,3m-1}, \ldots, c_{t-1,1}, c_{t-2,3m-1}, \ldots, c_{t,2}
\]
form a \((9m-4)\)-string based at \(c_{t,2} \). By Proposition 2.7, we can delete the vertices \(b_{t-2,1} \) and \(c_{t-2,1} \). Thus we have
\[
I(Z_t) \simeq I(P_{3m-4}) \ast I(P_{3m-4}) \ast I(G_{t-3}) \simeq S^{3m-2} \ast S^{3m-2} \ast I(G_{t-3}) \simeq \Sigma^{6m-2}I(G_{t-3}).
\]

**Proof of Theorem 3.10** By Lemma 3.12, Lemma 3.13 and Lemma 3.14, we have homotopy equivalences
\[
I(G_t) \simeq I(X_t) \cup \Sigma^{4m-2}I(H_{t-2})
\]
\[
\simeq I(Y_t) \cup \Sigma^{4m-2}I(H_{t-2})
\]
\[
\simeq \Sigma^{6m-2}I(G_{t-3}) \cup \Sigma^{8m-3}I(H_{t-4}) \cup \Sigma^{4m-2}I(H_{t-2}).
\]

![Figure 5](image_url)
This completes the proof.

Proof of Theorem 3.11 Note that $b_{t+1,1}$ is a simplicial vertex (see Section 2 for the definition) and $N_{H_t}(b_{t+1,1}) = \{a_t, b_{t,3m-1}\}$. Thus by Proposition 2.7, there is a homotopy equivalence

$$I(H_t) \simeq \Sigma I(H_t \setminus N[b_{t,3m-1}]) \vee \Sigma I(H_t \setminus N[a_t])$$

In $H_t \setminus N[b_{t,3m-1}]$, the vertices $b_{t,3m-3}, \ldots, b_{t,2}$ form a $(3m - 4)$-string based at $b_{t,2}$, and the vertices $c_{t+1,1}, c_{t,3m-1}, c_{t,3m-3}, \ldots, c_{t,2}$ form a $(3m - 1)$-string based at $b_{t,2}$. By Proposition 2.7, we can delete the vertices $b_{t,1}$ and $c_{t,1}$, and hence we have

$$I(H_t \setminus N[b_{t,3m-1}]) \simeq I(P_{3m-4}) \ast I(P_{3m-1}) \ast I(G_{t-1}) \simeq S_m^{-2} \ast S_m^{-1} \ast I(G_{t-1}) \simeq \Sigma^{2m-1} I(G_{t-1}).$$

On the other hand, in $H_t \setminus N[a_t]$, the vertices $b_{t,3m-2}, \ldots, b_{t,3}$ form a $(3m - 4)$-string based at $b_{t,3}$, and the vertices $c_{t,3m-2}, \ldots, c_{t,3}$ form a $(3m - 4)$-string based at $c_{t,3}$. By Proposition 2.7, we can delete the vertices $b_{t,2}$ and $c_{t,2}$, and hence we have

$$I(H_t \setminus N[a_t]) \simeq I(P_{3m-4}) \ast I(P_{3m-4}) \ast I(H_{t-1}) \simeq S_m^{-2} \ast S_m^{-2} \ast I(H_{t-1}) \simeq \Sigma^{2m-2} I(H_{t-1}).$$

Therefore we have

$$I(H_t) \simeq \Sigma I(H_t \setminus N[b_{t,3m-1}]) \vee \Sigma I(H_t \setminus N[a_t]) \simeq \Sigma^{2m-1} I(G_{t-1}) \vee \Sigma^{2m-1} I(H_{t-1}).$$

This completes the proof.

By Theorem 3.11 and Theorem 3.11, $I(G_t)$ and $I(H_t)$ are recursively determined by $I(G_0), I(G_1), I(G_2), I(G_3)$, and $I(H_1)$.

**Proposition 3.15.** $I(G_0) = \text{pt}$, $I(G_1) \simeq S_{2m-1} \vee S_{2m-1}$, $I(G_2) \simeq S_{4m-2} \vee S_{4m-2}$, $I(G_3) \simeq S_{6m-2} \vee S_{6m-3}$, $I(H_0) \simeq S^0$.

**Proof.** Since $G_0 = \text{pt}$, $H_0 = P_3$, and $G_1 = C_{6m}$, we have that $I(G_0) = \text{pt}$, $I(G_1) \simeq S_{2m-1} \vee S_{2m-1}$, and $I(H_0) \simeq S^0$. By Lemma 3.12, we have

$$I(G_2) \simeq I(X_2) \vee \Sigma^{4m-2} I(H_0) \simeq I(C_{12m-2}) \vee S_{4m-2} \simeq S_{4m-2} \vee S_{4m-2}.$$

By Theorem 3.11, we have $I(H_1) \simeq \Sigma^{2m} I(G_0) \vee \Sigma^{2m-1} I(H_0) \simeq S_{2m-1}$. It follows from Lemma 3.12 and Lemma 3.13 that

$$I(G_3) \simeq I(X_3) \vee \Sigma^{4m-2} I(H_1) \simeq I(Y_3) \vee \Sigma^{4m-2} I(H_1) = S_{6m-2} \vee S_{6m-3}.$$

since $I(Y_3) = I(C_{18m-4}) \simeq S_{6m-2}$. This completes the proof.
Finally, we determine the connectivity of the complexes $I(G_t)$ and $I(H_t)$. Put $k_t = \text{conn}(I(G_t))$ and $l_t = \text{conn}(I(H_t))$. In [6], Jelić Milutinović et al. showed that $k_t \geq (2m - 1)t - 1$. In fact, the following holds:

**Theorem 3.16.** For $t \geq 1$, the following equation holds:

$$k_t = l_t = (2m - 1)t - 1$$

**(2)**

*Proof.* By Proposition 3.15 the equation (2) holds for $k_1, k_2, k_3$, and since $I(H_1) = S^{2m-1}$ (see the proof of Proposition 3.15), the equation (2) holds for $l_1$. Thus it suffices to show the following:

(a) Suppose $t \geq 4$. If the equation (2) holds for $t - 2, t - 3$, and $t - 4$, then the equation (2) holds for $k_t$.

(b) Suppose $t \geq 2$. If the equation (2) holds for $t - 1$, then the equation holds for $l_t$.

By Theorem 3.10 the equation

$$k_t = \min \{l_{t-2} + 4m - 2, \, k_{t-3} + 6m - 2,\, l_{t-4} + 8m - 3\},$$

holds for $t \geq 4$. If the equation (2) holds for $t - 1, t - 2$, and $t - 3$, then we have

$$k_t = \min \{(2m - 1)t - 1, \, (2m - 1)t, \, (2m - 1)t\} = (2m - 1)t - 1.$$

Thus (a) follows. Next we prove (b). By Theorem 3.11 the equation

$$l_t = \min \{k_{t-1} + 2m, \, l_{t-1} + 2m - 1\}$$

holds for $t \geq 1$. If the equation (2) holds for $t - 1$, then we have

$$l_t = \min \{(2m - 1)t, \, (2m - 1)t - 1\} = (2m - 1)t - 1.$$

This completes the proof. \qed

In particular, we determine the precise connectivity of $(1 \times 1 \times t)$-honeycomb graph. Jelić Milutinović et al. [6] shows that the matching complex of $(1 \times 1 \times t)$-honeycomb graph is $(t - 1)$-connected (see Corollary 4.4 in [6]). Thus we show that $(1 \times 1 \times t)$-honeycomb graph is not $t$-connected:

**Corollary 3.17.** The connectivity of $(1 \times 1 \times t)$-honeycomb graph is $t - 1$.
References

[1] M. Adamaszek. Splittings of independence complexes and the powers of cycles. *J. Combin. Theory Ser. A*, 119(5):1031-1047 (2012).

[2] A. Björner, L. Lovász, S. T. Vrećica, R. T. Živaljević. Chessboard complexes and matching complexes. *J. London Math. Soc.*, 49(1):25-39, 1994.

[3] S. Bouc. Homologie de certains ensembles de 2-sous-groupes des groupes symétriques. *J. Algebra*, 150(1):158-186, 1992.

[4] B. Braun, W. K. Hough. Matching and independence complexes related to small grids. *Electron. J. Combin.*, 18(1), 2011.

[5] A. Engström. Complexes of directed trees and independence complexes. *Discrete Math.*, 309(10):3299-3309, 2009.

[6] M. Jelić Milutinović, H. Jenne, A. McDonough, J. Vega. Matching complexes of trees and applications of the matching tree algorithm. arXiv math.co. 1905.10560.

[7] Jakob Jonsson. Matching complexes on grids. unpublished manuscript available at [http://www.math.kth.se/~jakobj/doc/thesis/grid.pdf](http://www.math.kth.se/~jakobj/doc/thesis/grid.pdf).

[8] Jakob Jonsson. *Simplicial complexes of graphs*, volume 1928 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2008.

[9] Dmitry N. Kozlov. Complexes of directed trees. *J. Combin. Theory Ser. A*, 88(1):112-122, 1999.

[10] Dmitry N. Kozlov. *Combinatorial Algebraic Topology*, volume 21 of *Algorithms and Computation in Mathematics*, Springer, Berlin, 2008.

[11] T. Matsushita. Matching complexes of small grids. to appear in *Electron. J. Combin*.

[12] M. Marietti, D. Testa. A uniform approach to complexes arising from forests. *Electron. J. Combin.* 15 (2008).

[13] J. Shareshian, M. L. Wachs. Torsion in the matching and chessboard complexes. *Adv. Math.*, 212(2): 525-570 (2007).

[14] M. L. Wachs. Topology of matching, chessboard, and general bounded degree graph complexes. *Algebra Universalis*, 49 (4):345-385, 2003.