We study the $TT$ OPE in $d > 2$ CFTs whose bulk dual is Einstein gravity. Directly from the $TT$ OPE, we obtain, in a certain null-like limit, an algebraic structure consistent with the Jacobi identity: $[\mathcal{L}_m, \mathcal{L}_n] = (m - n)\mathcal{L}_{m+n} + C m(m^2 - 1) \delta_{m+n,0}$. The dimensionless constant $C$ is proportional to the central charge $C_T$. Transverse integrals in the definition of $\mathcal{L}_m$ play a crucial role. We comment on the corresponding limiting procedure and point out a curiosity related to the central term. A connection between the $d > 2$ near-lightcone stress-tensor conformal block and the $d = 2$ $\mathcal{W}$-algebra is observed. This note is motivated by the search for a field-theoretic derivation of $d > 2$ correlators in strong coupling critical phenomena.
1. Introduction

The $TT$ OPE in $d = 2$ CFT

$$T(z_1)T(z_2) = \frac{c}{2s^4} + \frac{2}{s^2}T(z_2) + \frac{1}{s}\partial z_2 T(z_2) + \mathcal{O}(\partial^2 T), \quad s = z_1 - z_2$$

leads to the Virasoro algebra:

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{m+n,0}, \quad L_m = \frac{1}{2\pi i} \oint dz \, z^{m+1} \, T(z).$$

The Virasoro algebra is omnipresent in two-dimensional critical phenomena [1] and has enormous implications; in particular, the algebra provides a non-perturbative derivation of $d = 2$ conformal correlators. In higher dimensions, the general $TT$ OPE is contaminated by many model-dependent details. However, we ask the question: can one generalize the derivation (1)-(2) to $d > 2$ CFT in certain physical limits?

Over 27 years ago, Osborn and Petkos [2] computed the stress-tensor contribution to the $d > 2$ TT OPE, but we have not found any computation based on such an explicit TT OPE. A reason, presumably, is that the $TT$ OPE is complicated. Given the recent developments of gauge/gravity correspondence and $d > 2$ strongly coupled field theories, we find it necessary to revisit the $d > 2$ TT OPE structure. In this note, we adopt the following two simplifying limits to reduce the complexity of the $TT$ OPE:

(i) Infinitely large higher-spin gap; (ii) Null/lightcone-like limit.\(^1\)

As shown in [3–5], the gap $\Delta_{\text{gap}}$ to the lightest spin $> 2$ single-trace primary controls the higher-order corrections to Einstein gravity; the limit $\Delta_{\text{gap}} \to \infty$ then selects CFTs with an Einstein gravity bulk dual. We will focus on stress-tensor contribution to the $TT$ OPE and suppress other primary operators.\(^2\) On the other hand, the lightcone limit has been adopted in the recent computation of the multi-stress-tensor OPE data in $d > 2$ holographic CFTs [7–19]. The near-lightcone correlator at large central charge $C_T$ is independent of higher-curvature terms in the purely gravitational action [7]; however, the correlator depends on certain non-minimal coupling bulk interactions which are suppressed at an infinite gap [9]. These results suggest that the simplest starting point is to impose the limits (i), (ii) on the $TT$ OPE.\(^3\)

In general, the results depend on the order of limits (i.e. limiting procedure). A related motivation of this work is to help identify a lightcone-like limiting procedure that may be

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\(^1\) We use “-like” to distinguish our limiting procedure from similar limits used in the literature: the null-line limit is often defined by directly setting $x^+ = x^0 = 0$ in the Lorentzian signature, where $x^\pm = x^0 \pm x^1$ and $x_\perp$ denotes transverse directions.

\(^2\) For instance, the three-point function $\langle T O O \rangle$ is suppressed as $\Delta_{\text{gap}} \to \infty$ [6].

\(^3\) We should also assume the usual large $N$ and large $C_T$ limits but we will keep $C_T$ in our expressions.
implemented to compute multi-stress-tensor OPE coefficients and near-lightcone correlators in \(d > 2\) CFTs from the first principle via a Virasoro-like field-theoretic approach.

The main result of this work is that we find a structure similar to (2) in higher dimensions. Intuitively, one may expect that a Virasoro-like structure arises because the null-like limit brings stress tensors close to a line, a picture reminiscent of the two-dimensional case where \(T = -\frac{\pi}{2} T_{zz}(z), \bar{T} = -\frac{\pi}{2} T_{zz}(\bar{z})\) are holomorphic and anti-holomorphic functions, respectively. Note we are not introducing a physical line-defect.

This paper is organized as follows. In Section 2, we discuss the stress-tensor OPE structure. Some detailed expressions can be found in Appendix. We focus on \(d = 4\) for concreteness and expect that our results generalize to other dimensions. In Section 3, we consider a null-line limit and obtain a Virasoro-like commutator via the stress-tensor OPE. A \(d = 4\) single-stress-tensor-exchange derivation without explicitly using an algebra is discussed in Section 4, where we point out a curiosity related to the central term. We observe a connection between the \(d > 2\) lightcone stress-tensor conformal block and the central-term of the \(d = 2\) \(\mathcal{W}\)-algebra.

2. Stress-tensor OPE

Our starting point is the stress-tensor contribution to the \(d = 4\) \(TT\) OPE [2]:

\[
T_{\mu\nu}(x_1)T^{\sigma\rho}(x_2) = C_T \frac{T_{\mu\nu,\sigma\rho}(s)}{s^8} + \hat{A}_{\mu\nu,\sigma\rho}^{\alpha\beta}(s) T^{\alpha\beta}(x_2) + B_{\mu\nu,\sigma\rho}^{\alpha\beta\lambda}(s) \partial^\lambda T^{\alpha\beta}(x_2) + \mathcal{O}(\partial^2 T) \tag{3}
\]

where \(s = x_1 - x_2\). The first term has the familiar form:

\[
\mathcal{I}^{\mu\nu,\sigma\rho}(s) = \frac{1}{2} \left( I^{\mu\sigma}(s) I^{\nu\rho}(s) + I^{\mu\rho}(s) I^{\nu\sigma}(s) \right) - \frac{1}{4} \delta^{\mu\nu} \delta^{\sigma\rho}, \quad I^{\mu\sigma}(s) = \delta^{\mu\sigma} - 2 s^\mu s^\sigma. \tag{4}
\]

The structures of \(\hat{A}_{\mu\nu,\sigma\rho}^{\alpha\beta}\) and \(B_{\mu\nu,\sigma\rho}^{\alpha\beta\lambda}\) are cumbersome so we put them in the appendix. As noted in [2], there are three undetermined coefficients in the \(TT\) OPE, denoted as \(a, b, c\). The central charge \(C_T\) is given by

\[
C_T = \frac{\pi^2}{3} (14a - 2b - 5c). \tag{5}
\]

In the lightcone limit, the relevant contribution is the lightcone component of the stress tensor, \(T^{++}\). We will mostly work in the Euclidean space and adopt the line element \(ds^2 = dzd\bar{z} + \sum_{i=1,2} (dx^{(i)})^2\) where \(z, \bar{z}\), the Euclidean analogue of the lightcone coordinates, are complex coordinates. We will then focus on the \(T^{zz}T^{zz}\) OPE.

\footnote{The parameter \(c\) here should not be confused with the central charge in two dimensions where \(C_T = \frac{c}{2\pi}\).}
The $TT$ OPE simplifies significantly when one focuses on the $T^{zz}$ component. Using (3), we obtain
\[
T^{zz}(x_1)T^{zz}(x_2) = C_T \frac{\mathcal{I}^{zz,zz}(s)}{s^8} + \hat{A}^{zz,zz}(s)T^{zz}(x_2) + B^{zz,zz}(s)\partial^\lambda T^{zz}(x_2) + \mathcal{O}(\partial^2 T) \tag{6}
\]
where
\[
\mathcal{I}^{zz,zz} = \frac{4(s^2)^4}{s^8},
\]
\[
\hat{A}^{zz,zz} = \frac{4(s^2)^2}{C_T s^{10}} \left( (2b + c)(s^+)^4 - 2s^z s^\bar{z} ((8a - b - 3c)(s^+)^2 + (6a - b - 2c)s^z s^\bar{z}) \right), \tag{7}
\]
\[
B^{zz,zz} = \frac{s^\perp}{4} \hat{A}^{zz,zz}, \tag{8}
\]
\[
B^{zz,zz\perp} = \frac{s^\perp}{2} \hat{A}^{zz,zz}, \tag{9}
\]
\[
B^{zz,zz\bar{z}} = \frac{(s^2)^3}{9C_T s^{10}} \left( (64a + 18b - 11c)(s^+)^4 - 2s^z s^\bar{z} (4(3a - b - 2c)(s^+)^2 + (26a - 4b - 9c)s^z s^\bar{z}) \right). \tag{10}
\]
We will argue that higher-order pieces, $\mathcal{O}(\partial^2 T)$, are irrelevant when imposing the null-like limit considered in Sec. 3. Observe that, from (7),
\[
(8a - b - 3c) + (6a - b - 2c) = \frac{3}{\pi^2} C_T.
\]
While this combination is interesting, we here consider a large $N$, large-gap condition which places strong constraints on the flux parameters “$t_2$” and “$t_4$” of the energy flux escaping to null infinity [20–22]:
\[
t_2 = \frac{30(13a + 4b - 3c)}{(14a - 2b - 5c)} = 0, \quad t_4 = \frac{-15(81a + 32b - 20c)}{2(14a - 2b - 5c)} = 0. \tag{11}
\]
It is worth mentioning that two trace-anomaly central charges become the same under these conditions. By imposing $t_2 = t_4 = 0$ without first requiring a strictly infinite $C_T$, we can reduce three parameters to one parameter.

3. Stress-tensor OPE near a line and a Virasoro-like commutator

Consider the following operator in $d = 4$:
\[
\mathcal{L}_m = \frac{\kappa}{2\pi i} \int d\bar{z} \bar{z}^{m+1} \int d^2 x_\perp T^{zz}(z, \bar{z}, x^{(1)}_\perp, x^{(2)}_\perp), \quad \int d^2 x_\perp = \int_0^l dx^{(1)}_\perp \int_0^l dx^{(2)}_\perp \tag{12}
\]
in the null-like limit $z \to 0, l \to 0$.\footnote{One may perform a Wick-rotation to Lorentzian space and impose the lightcone limit, and then Wick-rotate back to the Euclidean space to carry out the $\bar{z}$ integral via the residue theorem. One may also formally impose a small $z$ limit directly in Euclidean space, which is what we will do here. For two stress tensors, we take a small $s^\perp$. A similar analysis applies to the $T^{\bar{z}z}T^{\bar{z}z}$ OPE if one instead chooses a small $s^\bar{z}$ limit.} We will determine the overall normalization factor $\kappa$ later. The interpretation of the small $l$ limit is that we consider the stress-tensor contribution near
a two-dimensional plane. We are interested in computing the commutator \([L_m, L_n]\). The transverse integrals are crucial, as we will see, for extracting a central extension consistent with a Witt-like algebra.\(^6\)

Let us first consider the \(c\)-number term which is controlled by the stress-tensor two-point function. After performing the transverse integrations, we consider a small \(s^2\) expansion:

\[
\lim_{s^2 \to \delta} \int d^4x_\perp \frac{C_T T_{zz,zz}(s)}{s^8} = \frac{4\pi C_T}{5(s^2)^5} \frac{l^2}{\delta} - \frac{7\pi C_T}{16(s^2)^{9/2}} \frac{l}{\sqrt{\delta}} + \frac{C_T}{5(s^2)^4} \left( \frac{356 + 315\pi}{14400} \right) \frac{l^4}{\delta^8} + \cdots \tag{13}
\]

We would like to extract the cutoff-independent piece. We do so by next imposing a \(l \to 0\) limit such that the first two terms are suppressed. The last piece of (13) and higher-order terms, although divergent as \(l \to 0\), do not have a \(z\)-pole and thus do not contribute to the commutator. The \(\frac{C_T}{(s^2)^4}\) term shares the same form as the \(c\)-number term in the \(d = 2\) \(TT\) OPE (1). The transverse integrals compensate for the additional dimensions of the \(d > 2\) \(TT\) OPE. This \(c\)-number-term derivation does not require a large-gap condition. The Cauchy’s integral formula now leads to\(^7\)

\[
[L_m, L_n]|_{C_T} = \left( \frac{K}{2\pi i} \right)^2 \oint_{\partial(0)} d\bar{z}_2 \bar{z}_2^{n+1} \oint_{\partial(z_2)} d\bar{z}_1 \bar{z}_1^{m+1} \frac{C_T}{5(s^2)^4} = \kappa^2 \frac{C_T}{30} m(m^2 - 1) \delta_{m+n,0}. \tag{14}
\]

We next turn to the operator part of the \(TT\) OPE, keeping explicit \(a, b, c\) parameters and imposing the conditions (11) at the end. We evaluate

\[
\lim_{s^2 \to \delta} \int d^2(x_1)_{\perp} \left( \hat{A}_{zz,zz}^2(x_1 - x_2) T_{zz}(x_2) \right) = f(a,b,c) \frac{T_{zz}(x_2)}{\pi(s^2)^2} + \mathcal{O}(\delta) \tag{15}
\]

where \(f(a,b,c) = \frac{-52a + 106b + 19c}{14a - 2b - 5c}\). The leading-order term is cutoff-independent and only depends on \((x_2)_{\perp}\) through the stress tensor. To take the small \(s^2\) limit, we may assume \(T_{zz}(x_2)\) is a suitable test function having a finite contribution only near \(y_2 = z_2 = 0\), and then perform all the transverse integrations before imposing the small \(s^2\) limit. But we find it simpler, as we did above, to take \(s^2 \to \delta\) right after performing the integrations over the first set of transverse coordinates \((x_1)_{\perp}\).\(^8\)

\(^6\) This construction is essentially the same as the mode operator introduced in [23], but in that work the author adopts a different limiting procedure. See also [24–27] for related discussions.

\(^7\) In general \(d\), we find

\[
[L_m, L_n]|_{C_T} = (-1)^d \kappa^2 \frac{4C_T}{\Gamma(d+2)} m(m^2 - 1) \delta_{m+n,0}.
\]

\(^8\) In the process of simplifying (15), we formally assume \(l > (x_2)_{\perp} > 0\) to adopt the identity \(\tan^{-1}(X) + \tan^{-1}(1/X) = \pi/2\) with \(X > 0\). This, strictly speaking, means the end points of the \((x_2)_{\perp}\) integrals should be removed.
integrations:

\[ [\mathcal{L}_m, \mathcal{L}_n]_{A-\text{term}} = \left( \frac{\kappa}{2\pi i} \right)^2 \frac{f(a, b, c)}{\pi} \int_{C(0)} d\bar{z}_2 z_2^{n+1} \int_{C(1)} d\bar{z}_1 z_1^{m+1} \int d^2(x_2) T^{zz}(x_2) \]

\[ = \frac{\kappa f(a, b, c)}{\pi} (m + 1) \mathcal{L}_{m+n}. \]  

(16)

For the second-order term in the TT OPE, we have

\[ \lim_{s^2 \to \delta} \int d^2(x_1) \left( B_{zzzz}(s) \partial^2 T^{zz}(x_2) \right) = 2 \lim_{s^2 \to \delta} \int d^2(x_1) \left( B_{zzzz}(s) \partial T^{zz}(x_2) \right) \]

\[ = f(a, b, c) \frac{\partial T^{zz}(x_2)}{2\pi s^2} + \mathcal{O}(\delta), \]  

(17)

\[ \lim_{s^2 \to \delta} \int d^2(x_1) \left( B_{zzzz}(s) \partial T^{zz}(x_2) \right) = \mathcal{O}(\delta). \]  

(18)

Since we only focus on the \( T^{zz} \) component in the null-like limit and effectively turn off other components of the stress tensor, the conservation of the stress tensor implies that we also drop \( \partial \_T^{zz} \). Observe that the structures (including the relative coefficients) of (15), (17) are the same as the two-dimensional case (1). From (17), we get

\[ [\mathcal{L}_m, \mathcal{L}_n]_{B-\text{term}} = \frac{\kappa f(a, b, c)}{2\pi} (m + n + 2) \mathcal{L}_{m+n}. \]  

(19)

Similar to the corresponding \( d = 2 \) computation, we have performed an integration by parts to evaluate the \( \partial \_T^{zz} \) term.

We will not include the higher-order corrections \( \mathcal{O}(\partial^2 T) \) in the TT OPE, but, based on the pattern (15), (17), it seems reasonable to assume that the higher-order terms do not have a relevant pole in the null-like limit.

Combining (19), (16), and (14), the result is

\[ [\mathcal{L}_m, \mathcal{L}_n] = (m - n) \mathcal{L}_{m+n} + \kappa^2 \frac{C_T}{30} m (m^2 - 1) \delta_{m+n,0}, \quad \kappa = \frac{2\pi}{f(a, b, c)}. \]  

(20)

We choose a normalization \( \kappa \) such that the non-central term has a simple coefficient. If we now impose the conditions listed in (11), we find the normalization factor to be \( \kappa = -\frac{90\pi}{180} \).  

To summarize, we have described a null-line-like limiting procedure that allows us to extract an algebraic structure from the TT OPE. The result (20) is strikingly similar to the

\footnote{An overall rescaling of the mode operator \( \mathcal{L}_m \) should not affect a scalar correlator computation. But one might wonder if the “right” proportionality constant should instead be \( \kappa = -\frac{90\pi}{180} = -\frac{\pi}{2} \). If we formally adopt free-theory values of \( a, b, c \) [2], we notice that \( \kappa = -\frac{\pi}{2} \) for both a fermion and a \( U(1) \) gauge field, but \( \kappa = -\frac{18\pi}{90} \) for a scalar. In fact, \( \kappa = -\frac{\pi}{2} \) is true only under the condition \( 4a + 2b - c = 0 \), which holds for both a free fermion and a \( U(1) \) gauge field, but a free scalar has \( 4a + 2b - c = -\frac{1}{90\pi} \). (In \( d = 2 \), on the other hand, \( 4a + 2b - c = 0 \) for both a free scalar and a free fermion.)}
two-dimensional Virasoro algebra. We do not use holographic duality here, but it would be nice to find a potential connection to the AdS/CFT computation discussed some time ago [28,29] where a higher-dimensional generalization to the Brown-Henneaux symmetry [30] was identified in a certain infinite momentum frame. Most likely, whether or not there is a Virasoro-like structure at infinity depends on boundary conditions.10

It would certainly be of great interest to extend the two-dimensional CFT analysis to higher dimensions in the null/lightcone-like limit, where one expects to find relatively robust structures. By first focusing on a special class of higher-dimensional CFTs with an Einstein gravity dual, we would like to know if there is an effective algebraic derivation of the multi-stress-tensor OPE coefficients and conformal correlators. Considering perturbative corrections due to a large but finite higher-spin gap could be interesting as well.

4. A $d = 4$ single-stress-tensor-exchange derivation

Let us conclude this note by presenting some observations, which hopefully shed light on more general cases. In the following, we point out a simple derivation of the $d = 4$ near-lightcone conformal scalar correlator via a mode summation.11 This derivation does not explicitly rely on an algebra. In fact, as we will see, this derivation presents a central-term curiosity.

The scalar four-point conformal correlator can be written in terms of the conformal block decomposition [31]:

$$\langle \mathcal{O}_H(\infty)\mathcal{O}_H(1)\mathcal{O}_L(z,\bar{z})\mathcal{O}_L(0)\rangle = \sum_{\Delta_T,J} c_{\text{OPE}}(\Delta_T,J) \frac{B(z,\bar{z},\tau,J)}{(z\bar{z})^{\Delta_L}}$$

(21)

where the twist of an operator is its dimension minus its spin, $\tau = \Delta_T - J$. We formally name $\mathcal{O}_H$ the “heavy” scalar and $\mathcal{O}_L$ the “light” scalar although the heavy-light limit (i.e. $\Delta_H, C_T \to \infty$ with $\Delta_H/C_T$ fixed and $\Delta_L \sim O(1)$) does not play a special role in the single-stress-tensor-exchange computation. We adopt this notation as an example which is useful to compare with the literature that discusses multi-stress-tensor contributions to the heavy-light correlator. The Ward identity fixes the stress-tensor OPE coefficient to be $c_{\text{OPE}}(4,2) = \frac{\Delta_H\Delta_L}{9\pi^3 C_T}$ in the convention of (3). The conformal block is given by12

$$B(z,\bar{z},\tau,J) = \frac{z\bar{z}}{z - \bar{z}} \left[ z^{\frac{\tau + 2J}{2}} \bar{z}^{\frac{\tau + 2J}{2}} \, {}_2F_1\left(\frac{\tau + 2J}{2}, \frac{\tau + 2J}{2}; \tau + 2J; z\right) \right] \, {}_2F_1\left(\frac{\tau - 2}{2}, \frac{\tau - 2}{2}; \tau - 2; \bar{z}\right) - (z \leftrightarrow \bar{z}) .$$

10 I thank Gary Gibbons for related remarks.
11 The derivation presented here is simpler than previous work [23] and we can avoid an arbitrary parameter introduced in that paper.
12 Our convention differs by an overall factor of $(-\frac{1}{2})^J$ from the convention used in Dolan and Osborn [31].
In the limit $z \to 0$, the stress-tensor contribution in $d = 4$ reads\(^{13}\)

\[
\lim_{z \to 0} \left( (z \bar{z})^{\Delta_L} \langle O_H(\infty) O_H(1) O_L(z, \bar{z}) O_L(0) \rangle |_T \right) \\
= \frac{1}{9\pi^4} \frac{\Delta_H \Delta_L}{C_T} \bar{z}^3 2F_1(3, 3, 6, \bar{z}) z + \mathcal{O}(z^2) \\
= \frac{10}{3\pi^4} \frac{\Delta_H \Delta_L}{C_T} \frac{3(\bar{z} - 2) \bar{z} - (6 + (\bar{z} - 6) \bar{z}) \ln(1 - \bar{z})}{\bar{z}^2} z + \mathcal{O}(z^2) .
\]

(23)

The higher-order pieces represent multi-stress-tensor contributions to the correlator.

It is instructive if we temporarily forget about the algebra and instead adopt the following operator:

\[
\tilde{\mathcal{L}}_m = \lim_{z_T \to \delta} \int \frac{d\bar{z}_T}{2\pi i} \frac{\bar{z}_T^{m+2}}{z_T} T^{zz}(z_T, \bar{z}_T, x^i = 0) .
\]

(24)

Notice we directly set $x^i = 0$ in this definition. The $z_T \to \delta$ represents the null-line limit for the stress tensor. The notation “$m + 2$” will result in slightly more symmetric expressions in the following computation. The mode operator (24) is essentially the same as the lightray operator which does not contain transverse integrals \([24, 26, 27, 32]\). We here use the Euclidean signature with complex coordinates $z, \bar{z}$. Similar to the $d = 2$ case, we may expect that the stress-tensor-exchange contribution can be computed via the following mode summation:

\[
\mathcal{V}_T = \lim_{z \to 0} \sum_{m = m^*}^{\infty} \frac{\langle O_H(\infty) O_H(1) \tilde{\mathcal{L}}_m^\dagger \rangle \langle \tilde{\mathcal{L}}_m O_L(z, \bar{z}) O_L(0) \rangle}{\langle O_H(\infty) O_H(1) \rangle \mathcal{N}_m \langle O_L(z, \bar{z}) O_L(0) \rangle}
\]

(25)

where the normalization factor is $\mathcal{N}_m = \langle \tilde{\mathcal{L}}_m \tilde{\mathcal{L}}_m^\dagger \rangle$. We will find $m^* = 3$.

Using the three-point function

\[
\langle T^{\mu\nu}(x_1) O_\Delta(x_2) O_\Delta(x_3) \rangle = \frac{c_{\text{TdO}}}{x_{12}^4 x_{13}^4 x_{23}^{2\Delta-4}} \left( \frac{X^\mu X^\nu}{X^2} - \frac{\delta^{\mu\nu}}{4} \right)
\]

(26)

with $X^\mu = x^\mu_{12}/x^2_{12} - x^\mu_{13}/x^2_{13}$ and $c_{\text{TdO}} = -\frac{2\Delta}{3\pi^2}$, we first obtain

\[
\lim_{z_T \to \delta} \lim_{z \to 0} \frac{\langle \tilde{\mathcal{L}}_m O_L(z, \bar{z}) O_L(0) \rangle}{\langle O_L(z, \bar{z}) O_L(0) \rangle} = -\frac{2\Delta_L}{3\pi^2} \lim_{z_T \to \delta} \lim_{z \to 0} \int \frac{d\bar{z}_T}{2\pi i} \frac{z_T^{m+2}}{z_T^3 \bar{z}_T (z_T - \bar{z})} \bar{z}_T^3 z_T^3 (z_T - \bar{z}) \\
= -\frac{2\Delta_L}{3\pi^2} \lim_{z_T \to \delta} \lim_{z \to 0} \frac{(m - 1)(m - 2) \bar{z}_T^m}{2z_T (z_T - \bar{z})} z_T \\
= -\frac{\Delta_L}{3\pi^2} \frac{(m - 1)(m - 2) \bar{z}_T^m}{\bar{z}_T^2} z_T ,
\]

(27)

where we introduce a short-distance cutoff $\delta$. We shall find that the final four-point scalar correlator is independent of the UV cutoff. It is important to adopt a proper order of limits.

\(^{13}\) One can also choose $\bar{z} \to 0$ as the lightcone limit.
On the other hand, by taking \( \mathcal{C}_m = \mathcal{L}_m \), we find

\[
\frac{\langle O_H(\infty)O_H(1)\mathcal{C}_m \rangle}{\langle O_H(\infty)O_H(1) \rangle} = \frac{2\Delta_H}{3\pi^2} \lim_{z_T \to 0} \int_{C(1)} \frac{d\bar{z}_T}{2\pi i} \frac{(\bar{z}_T)^{-m+2}}{(\bar{z}_T - 1)^3(z_T - 1)}
\]

\[
= \frac{2\Delta_H}{3\pi^2} \lim_{z_T \to 0} \frac{(m-1)(m-2)}{2(z_T - 1)}
\]

\[
= -\frac{\Delta_H}{3\pi^2} (m-1)(m-2) . \tag{28}
\]

The normalization factor can be computed using the stress-tensor two-point function:

\[
\mathcal{N}_m = \langle \mathcal{L}_m \mathcal{L}_m^\dagger \rangle = \lim_{s^2 \to \delta} \int_{C(0)} \frac{d\bar{z}_2T}{2\pi i} \int_{C(\bar{z}_2T)} \frac{d\bar{z}_1T}{2\pi i} \langle \bar{z}_1T \rangle^{m+2} (\bar{z}_2T)^{-m+2} \langle T^{zz}(z_1T, \bar{z}_1T)T^{zz}(z_2T, \bar{z}_2T) \rangle
\]

\[
= \lim_{s^2 \to \delta} \int_{C(0)} \frac{d\bar{z}_2T}{2\pi i} \int_{C(\bar{z}_2T)} \frac{d\bar{z}_1T}{2\pi i} 4C_T \langle \bar{z}_1T \rangle^{m+2} (\bar{z}_2T)^{-m+2} \langle \bar{z}_1T - \bar{z}_2T \rangle^6 (z_1T - z_2T)^2
\]

\[
= \frac{C_T}{30} \frac{(m+2)(m+1)m(m-1)(m-2)}{\delta^2} . \tag{29}
\]

The UV-cutoff dependencies cancel out in the final mode summation and we obtain exactly the \( d = 4 \) stress-tensor-exchange structure (23):

\[
\mathcal{V}_T = \frac{10}{3\pi^4} \frac{\Delta_H \Delta_L}{C_T} \sum_{m=3}^\infty \frac{(m-1)(m-2)}{m(m+1)(m+2)} z^m = \frac{1}{30} \frac{\Delta_H \Delta_L}{C_T} z^3 \binom{3}{3, 6, \bar{z}} . \tag{30}
\]

This computation does not require a large gap.

It is peculiar that we are able to reproduce the \( d = 4 \) near-lightcone correlator, including the correct OPE coefficient, via a mode summation. The final result is finite and cutoff-independent. Although the above single-stress-tensor computation does not rely on knowing an algebra, we would like to ask why such a derivation exists. Recall that, in two-dimensions, a similar derivation exists because of the Virasoro symmetry. Given the above \( d > 2 \) computation, one may speculate that a certain symmetry emerges near the lightcone. An underlining algebra would provide a precise interpretation of the modes counting in (30). Since we have extracted a Virasoro-like commutator from the stress-tensor OPE (20), it seems natural to link the correlator computation to the algebra.

However, we find a curiosity related to the central term, or more generally, to the limiting procedure. In the above correlator computation, we emphasize that \textit{we take} \( x^+ \to 0 \text{ before imposing} \ z \to 0 \).\textsuperscript{14} The resulting “central” term has the following \( m \)-dependence (in the notation of \( \mathcal{L}_m \sim \oint d\bar{z} \bar{z}^{m+2}T \)):

\[
\text{Type A : } \quad C_T \ (m+2)(m+1)m(m-1)(m-2) \delta_{m+n,0} . \tag{31}
\]

\textsuperscript{14} Using this order of limits, one can include transverse integrals but the correlator result is unchanged.
Such an $m$-dependence is quite different from the central term in the Virasoro-like commutator, which has the structure:

\[
\text{Type B : } \quad C_T (m + 1)m(m - 1)\delta_{m+n,0} .
\]  

(32)

As shown above, in this case, we take $z \to 0$ before imposing $x^\perp \to 0$. The Type-B central-term has the familiar form fixed by the Jacobi identity, but the correlator derivation suggests that the Type-A structure plays a non-trivial role in recovering the scalar correlator. The Type-A structure, however, is incompatible with the Witt algebra.

Ideally, we would like to also compute $[\tilde L_m, \tilde L_n]$ via the $d = 4$ TT OPE, but we find that the commutator computation using $\tilde L_m$ requires a higher-order term in the TT OPE. Such a computation will not be included in this note.

On the other hand, we observe that, up to an overall coefficient, the $d = 4$ Type-A structure (31) is identical to the central term of the $W_3$ algebra in $d = 2$ CFTs [33] (see [34] for a review):

\[
[W_m, W_n] = \frac{c}{360} (m + 2)(m + 1)m(m - 1)(m - 2)\delta_{m+n,0}
\]

\[
+ (m - n)\left(\frac{1}{15}(m + n + 3)(m + n + 2) - \frac{1}{6}(m + 2)(n + 2)\right)L_{m+n}
\]

\[
+ \frac{16}{22 + 5c} (m - n)\Lambda_m ,
\]

(33)

\[
[L_m, W_n] = (2m - n)W_{m+n} ,
\]

(34)

where $\Lambda_m = \sum_n (L_{m-n}L_n) - \frac{4}{10}(m + 3)(m + 2)L_m$. $W_m$ is the Laurent modes of a spin-3 primary current. Note that closure of the $W_3$ algebra requires first knowing the operator $L_m$ that satisfies the Virasoro algebra. In general (even) $d$, we find the Type-A structure is $\sim C_T m(m^2 - 1)(m^2 - 4) \cdots (m^2 - (\frac{d}{2})^2)$ in the notation of $\tilde L_m \sim \oint d\bar z \bar z^{(m+\frac{d}{2})}T$.

To our knowledge, a connection between $d > 2$ CFT correlators and the $W_3$-like symmetry has not been mentioned before. This central-term curiosity needs to be better understood. Perhaps exploring more general structures involving multi-stress-tensor exchanges in $d > 2$ CFTs can help clarify its algebraic underpinnings.

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Appendix

For convenience, here we collect structures appearing in the TT OPE in general $d$ [2].

Define

$$
\hat{A}_{\mu\nu\rho\alpha\beta}(s)C_T = \frac{d - 2}{d + 2} (4a + 2b - c) H_{\alpha\beta\mu\rho\nu}(s) + \frac{1}{d} (da + b - c) H_{\alpha\beta\mu\rho\nu}(s)
$$

$$
- \frac{d(d - 2)a - (d - 2)b - 2c (H_{\mu\nu\rho\sigma\alpha\beta}(s) + H_{\sigma\rho\mu\nu\alpha\beta}(s)))}{d(d + 2)}
$$

$$
+ \frac{2da + 2b - c}{d(d - 2)} H_{\alpha\beta\mu\nu\rho\sigma}(s) - \frac{2(d - 2)a - b - c}{d(d - 2)} H_{\alpha\beta\mu\nu\rho\sigma}(s)
$$

$$
- \frac{2(d - 2)a - c}{d(d - 2)} (H_{\mu\nu\rho\sigma\alpha\beta}(s) + H_{\sigma\rho\mu\nu\alpha\beta}(s))
$$

$$
+ \frac{(d - 2)(2a + b) - dc}{d(d^2 - 4)} (H_{\mu\nu\rho\sigma\alpha\beta}(s) + H_{\sigma\rho\mu\nu\alpha\beta}(s))
$$

$$
+ (C h_{\mu\nu\rho\sigma\alpha\beta}^5 + D(\delta_{\mu\nu} h_{\rho\sigma\alpha\beta}^3 + \delta_{\sigma\rho} h_{\mu\nu\alpha\beta}^3)) S_d \delta^d(s) \quad (S_d = \frac{2\pi^2}{\Gamma(\frac{d}{2})})
$$

(35)

where

$$
H_{\mu\nu\rho\sigma\alpha\beta}(s) = \left( \partial_\mu \partial_\nu - \frac{1}{d} \delta_{\mu\nu} \partial^2 \right) \frac{1}{s_{d-2}} h_{\sigma\rho\alpha\beta}(\hat{s}) h_{\mu\nu}^1(\hat{s})
$$

(36)

$$
H_{\mu\nu\rho\sigma\alpha\beta}(s) = \left( \delta_{\sigma\alpha} \partial_\rho \partial_\beta + (\sigma \leftrightarrow \rho, \alpha \leftrightarrow \beta) \right)
$$

$$
- \frac{4}{d} \delta_{\sigma\rho} \partial_\alpha \partial_\beta - \frac{4}{d} \delta_{\alpha\beta} \partial_\sigma \partial_\rho + \frac{4}{d^2} \delta_{\sigma\rho} \delta_{\alpha\beta} \partial^2 \right) \frac{1}{s_{d-2}} h_{\mu\nu}^1(\hat{s})
$$

(37)

$$
H_{\mu\nu\rho\sigma\alpha\beta}(s) = h_{\sigma\rho\alpha\beta}^3 \left( \partial_\mu \partial_\nu - \frac{1}{d} \delta_{\mu\nu} \right) \frac{1}{s_{d-2}}
$$

(38)

$$
H_{\mu\nu\rho\sigma\alpha\beta}(s) = \left( h_{\mu\nu\rho\sigma\alpha\beta}^3 \partial_\mu \partial_\nu + (\sigma \leftrightarrow \rho, \alpha \leftrightarrow \beta) \right)
$$

$$
- \frac{2}{d} \delta_{\sigma\rho} \left( h_{\mu\nu\rho\sigma\alpha\beta}^3 \partial_\alpha \partial_\beta + (\alpha \leftrightarrow \beta) \right)
$$

$$
- \frac{2}{d} \delta_{\alpha\beta} \left( h_{\mu\nu\rho\sigma\alpha\beta}^3 \partial_\sigma \partial_\rho + (\sigma \leftrightarrow \rho) \right)
$$

$$
+ \frac{8}{d^2} \delta_{\sigma\rho} \delta_{\alpha\beta} \left( \partial_\mu \partial_\nu - \frac{1}{d} \delta_{\mu\nu} \right) \frac{1}{s_{d-2}}
$$

(39)

and

$$
h_{\mu\nu}(\hat{s}) = \hat{s} \mu \partial_\nu - \frac{1}{d} \delta_{\mu\nu}, \quad \hat{s}_\mu = \frac{s_\mu}{\sqrt{s^2}}
$$

(40)

$$
h_{\mu\nu\rho}(\hat{s}) = \hat{s} \mu \hat{s}_\nu \partial_\rho + (\mu \leftrightarrow \nu, \sigma \leftrightarrow \rho)
$$

$$
- \frac{4}{d} \hat{s}_\mu \hat{s}_\nu \delta_{\rho\sigma} - \frac{4}{d^2} \hat{s}_\sigma \hat{s}_\rho \delta_{\mu\nu} + \frac{4}{d^2} \delta_{\mu\rho} \delta_{\sigma\nu}
$$

(41)

$$
h_{\mu\nu\rho}(\hat{s}) = \hat{s}_\mu \hat{s}_\nu \delta_{\rho\sigma} + \frac{2}{d} \delta_{\mu\rho} \delta_{\nu\sigma}
$$

(42)

$$
h_{\mu\nu\rho\sigma}(\hat{s}) = h_{\mu\nu\rho\sigma}(\hat{s}) + (\sigma \leftrightarrow \rho, \alpha \leftrightarrow \beta)
$$

$$
- \frac{2}{d} \delta_{\rho\sigma} h_{\mu\nu\alpha\beta}(\hat{s}) - \frac{2}{d} \delta_{\alpha\beta} h_{\mu\nu\rho\sigma}(\hat{s}) - \frac{8}{d^2} \delta_{\sigma\rho} \delta_{\alpha\beta} h_{\mu\nu}^1(\hat{s})
$$

(43)

$$
h_{\mu\nu\rho\sigma}(\hat{s}) = h_{\mu\nu\rho\sigma}(\hat{s}) + (\mu \leftrightarrow \nu, \sigma \leftrightarrow \rho, \alpha \leftrightarrow \beta)
$$

$$
- \frac{4}{d} \delta_{\mu\rho} h_{\mu\nu\rho\sigma}(\hat{s}) - \frac{4}{d} \delta_{\sigma\rho} h_{\mu\nu\alpha\beta}(\hat{s}) - \frac{4}{d} \delta_{\alpha\beta} h_{\mu\nu\rho\sigma}(\hat{s}) - \frac{8}{d^2} \delta_{\sigma\rho} \delta_{\alpha\beta} h_{\mu\nu}(\hat{s})
$$

(44)
Define

\[ B_{\mu\nu\rho\sigma\beta\lambda}(s) = A_{\mu\nu\rho\sigma\beta}(s)s_{\lambda} \]
\[ + \frac{1}{(d + 2)(d - 1)} \left( (d + 1)B_{\mu\nu\rho\sigma\beta\lambda}(s) - B_{\mu\nu\rho\lambda\beta\alpha}(s) - B_{\mu\nu\rho\sigma\alpha\beta}(s) \right) \]
\[ - d\delta_{\alpha\lambda}A_{\mu\nu\rho\gamma\beta}(s)s_{\gamma} - d\delta_{\beta\lambda}A_{\mu\nu\rho\alpha\gamma}(s)s_{\gamma} + 2\delta_{\alpha\beta}A_{\mu\nu\rho\lambda\gamma}(s)s_{\gamma} \]  

(45)

where

\[ B_{\mu\nu\rho\sigma\beta\lambda}(s) = \frac{1}{2}s^{2}\partial_{\lambda}A_{\mu\nu\rho\sigma\beta}(s) + s_{\sigma}A_{\mu\nu\rho\sigma\beta}(s) - \delta_{\sigma\lambda}s_{\rho'}A_{\mu\nu\rho'\alpha\beta}(s) \]
\[ + s_{\rho}A_{\mu\nu\sigma\alpha\beta}(s) - \delta_{\rho\lambda}s_{\sigma'}A_{\mu\nu\rho'\alpha\beta}(s) \]  

(46)

\[ A_{\mu\nu\rho\sigma\alpha\beta}(s)C_{T} = \frac{1}{s_{d}}\left( ah_{\mu\nu\rho\sigma\beta}^{5} + bh_{\alpha\beta\mu\nu\rho\sigma}(\hat{s}) + b'(h_{\mu\nu\rho\sigma\beta}^{4}(\hat{s}) + h_{\rho\mu\nu\rho\beta}^{4}(\hat{s})) \right) \]
\[ + c h_{\mu\nu\rho\sigma\beta}^{3} + c'(h_{\beta\alpha\rho\beta}^{3}h_{\mu\nu}^{1}(\hat{s}) + h_{\rho\alpha\beta}^{3}h_{\sigma\rho}^{1}(\hat{s})) \]
\[ + e h_{\mu\nu\rho\sigma\beta}^{2}h_{\alpha\beta}^{1}(\hat{s}) + e'(h_{\beta\alpha\rho\beta}^{2}h_{\mu\nu}^{1}(\hat{s}) + h_{\rho\alpha\beta}^{2}h_{\sigma\rho}^{1}(\hat{s})) \]
\[ + f h_{\mu\nu}^{1}(\hat{s})h_{\sigma\rho}^{1}(\hat{s})h_{\alpha\beta}^{1}(\hat{s}) \]  

(47)

Consistency conditions (i.e. the conservation and tracelessness) impose relations:

\[ d\frac{D}{S_{d}} = C_{T} \]  

(48)

\[ - \frac{(d - 2)(2a + b) - dc}{d(d + 2)} + C = 0 \]  

(49)

\[ \frac{2}{d^{2}}(d - 1)((d - 2)a - c) - \frac{4}{d}C + D = \frac{C_{T}}{2S_{d}} \]  

(50)

\[ \frac{1}{d}(2(d - 2)a - b - c) + C = \frac{C_{T}}{2S_{d}} \]  

(51)

\[ \frac{(d - 2)(d + 3)a - 2b - (d + 1)c}{d(d + 2)} = \frac{C_{T}}{4S_{d}} \]  

(52)

Also, \( b + b' = -2a, c' = c, e + e' = -4b' - 2c, d^{2}a + 2(b + b') - (d - 2)b' - dc + e' = 0 \), and \( d(d + 2)(2b' + c) + 4(e + e') + f = 0 \). One can write

\[ f = (d + 4)(d - 2)(4a + 2b - c) \]  

(53)

\[ e' = -(d + 4)(d - 2)a - (d - 2)b + dc \]  

(54)

\[ e = (d + 2)(da + b - c) \]  

(55)

There are three undetermined coefficients \( a, b, c \). A free scalar has

\[ a = \frac{d^{3}}{8(d - 1)^{2}S_{d}^{3}}, \quad b = -\frac{d^{4}}{8(d - 1)^{2}S_{d}^{3}}, \quad c = -\frac{d^{2}(d - 2)^{2}}{8(d - 1)^{3}S_{d}^{3}}. \]  

(56)

For a free fermion, \( a = 0, b = -\frac{24d^{2}}{165S_{d}^{3}}, c = 2b \). For a \( d = 4 \) \( U(1) \) field, \( a = -\frac{16}{S_{d}^{3}}, b = 0, c = 4a \).

---

\(^{15}\) We remark that [2] misses a factor of \( S_{d} \) in the equation relating \( D \) to \( C_{T} \).
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