A FAMILY OF WAVE-BREAKING EQUATIONS
GENERALIZING THE CAMASSA-HOLM AND NOVIKOV EQUATIONS

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ABSTRACT. A 4-parameter polynomial family of equations generalizing the Camassa-Holm and Novikov equations that describe breaking waves is introduced. A classification of Lie symmetries and low-order conservation laws, along with peaked travelling wave solutions, is presented for this family. These classifications pick out a 1-parameter equation that shares the same low-order Hamiltonian conservation laws and peakon solutions as the Camassa-Holm and Novikov equations.

1. Introduction

There is considerable interest in the study of equations of the form \( u_t - u_{txx} = f(u, u_x, u_{xx}, u_{xxx}) \) that describe breaking waves. In this paper we consider the equation

\[
  u_t - u_{txx} + au^b u_x - cu^{b-1} u_x u_{xx} - du^b u_{xxx} = 0
\]  

with parameters \( a, c, d \) (not all zero) and \( b \neq 0 \). This 4-parameter family contains several integrable equations. For \((b, a, c, d) = (1, 3, 2, 1)\) and \((b, a, c, d) = (1, 4, 3, 1)\), equation (1) reduces respectively to the Camassa-Holm equation (2)

\[
  u_t - u_{txx} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0
\]

and the Degasperis-Procesi equation (3)

\[
  u_t - u_{txx} + 4uu_x - 3u_x u_{xx} - uu_{xxx} = 0
\]

while for \((b, a, c, d) = (2, 4, 3, 1)\), equation (1) becomes the Novikov equation (4)

\[
  u_t - u_{txx} + 4u^2 u_x - 3uu_x u_{xx} - u^2 u_{xxx} = 0.
\]

The three equations (2), (3), (4) are integrable in the sense of having a Lax pair, a bi-Hamiltonian structure, as well as hierarchies of local symmetries and local conservation laws, and they also possess peaked travelling wave solutions.

In addition to these integrable equations, a considerable number of other non-integrable equations that admit breaking waves are included in the 4-parameter family (1). For instance, there is the \( b \)-equation

\[
  u_t - u_{txx} + (b + 1)uu_x - bu_x u_{xx} - uu_{xxx} = 0
\]
which unifies the Camassa-Holm and Degasperis-Procesi equations \[4, 5\]. There is also a modified version of the $b$-equation \[6\]

\[u_t - u_{txx} + (b + 1)u^2 u_x - bu_x u_{xx} - u^2 u_{xxx} = 0\]  \[(6)\]

which includes the Novikov equation. No other cases of the two equations \[(5)\] and \[(6)\] are integrable \[3, 4\].

An equivalent form of the 4-parameter equation \[(1)\] is given by

\[m_t + \tilde{a} u^b u_x + c u^{b-1} u_x m + d u^b m_x = 0\]  \[(7)\]

in terms of the momentum variable

\[m = u - u_{xx}\]  \[(8)\]

with parameters

\[\tilde{a} = a - c - d, \quad (\tilde{a}, c, d) \neq 0, \quad b \neq 0.\]  \[(9)\]

This parametric equation \[(7)\] is invariant under the group of scaling transformations $u \rightarrow \lambda u$, $t \rightarrow \lambda^s t$, $(a, c, d) \rightarrow \lambda^{s+b}(a, c, d)$ with $\lambda \neq 0$.

In section 2, we classify the Lie symmetry group of equation \[(1)\]. In section 3, we classify the low-order conservation laws of equation \[(1)\] and show that the Hamiltonians of the Camassa-Holm and Novikov equations are admitted as local conservation laws by equation \[(1)\] if and only if $\tilde{a} = 0$ and $c = b + 1$. We consider peaked travelling waves in section 4 and show that equation \[(1)\] admits a peakon solution if and only if $\tilde{a} = 0$ and $b > -1$. Finally, in section 5 we combine the previous results to obtain a natural subfamily of equations given by $\tilde{a} = 0$, $c = b + 1$, $d = 1$, $b > -1$ which unifies the Camassa-Holm and Novikov equations into a 1-parameter equation with a rich structure of local conservation laws and peakon solutions.

2. Lie symmetries

The Lie symmetry group of equation \[(1)\] comprises point symmetries as well as contact symmetries, since the equation involves only a single dependent variable $u$.

A point symmetry \[7, 8, 9\] of equation \[(1)\] is a group of transformations on $(t, x, u)$ given by an infinitesimal generator

\[X = \tau(t, x, u) \partial_t + \xi(t, x, u) \partial_x + \eta(t, x, u) \partial_u\]  \[(10)\]

whose prolongation satisfies

\[\text{pr}X(u_t - u_{txx} + au^b u_x - cu^{b-1} u_x u_{xx} - du^b u_{xxx}) = 0\]  \[(11)\]

for all formal solutions $u(t, x)$ of equation \[(1)\]. When acting on solutions, any point symmetry \[(10)\] is equivalent to an infinitesimal generator with the characteristic form

\[\tilde{X} = P \partial_u, \quad P = \eta(t, x, u) - \tau(t, x, u) u_t - \xi(t, x, u) u_x\]  \[(12)\]

where the characteristic functions $\eta$, $\tau$, $\xi$ are determined by

\[0 = \text{pr}\tilde{X}(u_t - u_{txx} + au^b u_x - cu^{b-1} u_x u_{xx} - du^b u_{xxx}) = D_t P - D_t D_x^2 P + a D_x (u^b P) - c u^{b-1} D_x (u_x D_x P) - du^b D_x^3 P - (c(b - 1) u^{b-2} u_x u_{xx} + dbu^{b-1} u_{xxx}) P\]  \[(13)\]
holding for all formal solutions $u(t, x)$ of equation (1). Thus, equation (13) constitutes the determining equation for point symmetries of equation (1). This formulation is useful for doing computations and for considering extensions to contact symmetries, as well as for making a connection with conserved densities.

A contact symmetry [7, 8, 9] extends the definition of invariance (13) by allowing the transformations to depend essentially on first order derivatives of $u$, as given by an infinitesimal generator with the characteristic form

$$\hat{X} = P(t, x, u, u_t, u_x) \partial_u.$$  \hspace{1cm} (14)

The corresponding transformations on $(t, x, u, u_t, u_x)$ are given by

$$X = \tau \partial_t + \xi \partial_x + \eta \partial_u + \eta^t \partial_{ut} + \eta^x \partial_{ux}$$  \hspace{1cm} (15)

where

$$\tau = -P_{ut}, \  \xi = -P_{ux}, \  \eta = P - u_t P_{ut} - u_x P_{ux}, \  \eta^t = P_t + u_t P_u, \  \eta^x = P_x + u_x P_u$$  \hspace{1cm} (16)

which follows from preservation of the contact condition $du = u_t dt + u_x dx$. Note that a contact symmetry reduces to a (prolonged) point symmetry if and only if $P$ is a linear function of $u_t$ and $u_x$.

The set of all Lie symmetries admitted by equation (1) inherits the structure of a Lie algebra under commutation of the operators $X$. For a given (sub)algebra of point or contact symmetries, the corresponding group of transformations has a natural action [7, 8, 9] on the set of all solutions $u(t, x)$.

To classify all of the Lie symmetries admitted by equation (1), we first substitute a general characteristic function $P(t, x, u, u_t, u_x)$ into the symmetry determining equation (13). Next we eliminate $u_{txx}$, $u_{ttt}$, $u_{xxxx}$, $u_{txxx}$, $u_{tttx}$, $u_{xxxxx}$ into a linear overdetermined system of equations on $P(t, x, u, u_t, u_x)$, which can be straightforwardly solved by use of Maple (see the Appendix). This leads to the following classification result.

**Proposition 2.1.** (i) For any $b \neq 0$ and any $(a, c, d) \neq 0$, equation (1) admits no contact symmetries. (ii) The point symmetries admitted by equation (1) for all $b \neq 0$ and all $(a, c, d) \neq 0$ consist of

$$X_1 = \partial_t, \  \ X_2 = \partial_x, \  \ X_3 = -bt \partial_t + u \partial_u.$$  \hspace{1cm} (18)

(iii) Equation (1) admits additional point symmetries only in the following cases:

$$\text{(a)} \quad X_{4a} = at \partial_x + \partial_u \quad \text{iff} \quad b = 1, \  \ a = d$$  \hspace{1cm} (19)

$$\text{(b)} \quad X_{4b} = \exp(\pm 2x)(\pm \partial_x + u \partial_u) \quad \text{iff} \quad b = 2, \  \ a = 4d, \  \ c = 3d$$  \hspace{1cm} (20)

A symmetry analysis of particular equations in the family (1) can be found in Refs. [10, 11, 12].
3. Conservation laws

A local conservation law \([7, 9]\) for equation (1) is a space-time divergence

\[
D_t T + D_x X = 0 \tag{21}
\]

holding for all formal solutions \(u(t, x)\) of equation (1), where the conserved density \(T\) and the spatial flux \(X\) are functions of \(t, x, u\) and derivatives of \(u\). The spatial integral of the conserved density \(T\) satisfies

\[
\frac{d}{dt} \int_{-\infty}^{\infty} T dx = -X \bigg|_{-\infty}^{\infty} \tag{22}
\]

and so if the flux \(X\) vanishes at spatial infinity, then

\[
C[u] = \int_{-\infty}^{\infty} T dx = \text{const.} \tag{23}
\]

formally yields a conserved quantity for equation (1). Conversely, any such conserved quantity arises from a local conservation law (21).

If the conserved quantity (23) is purely a boundary term, then the local conservation law is called trivial. This occurs when (and only when) the conserved density is a total \(x\)-derivative and the flux is a total \(t\)-derivative, related by

\[
T = D_x \Theta, \quad X = -D_t \Theta \tag{24}
\]

for all formal solutions \(u(t, x)\) of equation (1), where \(\Theta\) is some function of \(t, x, u\) and derivatives of \(u\). Two local conservation laws are equivalent if they differ by a trivial conservation law, thereby giving the same conserved quantity up to boundary terms.

The set of all conservation laws (up to equivalence) admitted by equation (1) forms a vector space on which there is a natural action \([7, 9, 13]\) by the group of all Lie symmetries of the equation.

For conserved densities and fluxes depending on at most \(t, x, u, u_t, u_x, u_{tx}, u_{xx}\), a conservation law can be expressed in an equivalent form by a divergence identity

\[
D_t T + D_x X = (u_t - u_{tx} + au^b u_x - cu^{b-1} u_x u_{xx} - du^b u_{xxx})Q \tag{25}
\]

holding off solutions, where

\[
Q = -T_{ux} - X_{ux} \tag{26}
\]

is called the multiplier. This identity (25)–(26) is called the characteristic equation \([7, 9]\) for the conserved density and flux. By balancing the highest order \(t\)-derivative terms \(u_{ttt}\) on both sides of the equation, we directly find that \(T_{utt} = 0\) and \(X_{utut} = 0\). Then balancing the terms \(u_{tt}\), we see that \(X_{ututu_{xx}} = 0\). Hence the conserved density and the flux of such a conservation law must have the form

\[
T = T_0(t, x, u, u_t, u_x, u_{tx}, u_{xx}),
\]

\[
X = X_0(t, x, u, u_t, u_x, u_{tx}, u_{xx}) + u_{tt}X_1(t, x, u, u_t, u_x, u_{xx}) \tag{27}
\]

Consequently, the multiplier (26) in the characteristic equation (25) has the form

\[
Q = Q_0(t, x, u, u_t, u_x, u_{tx}, u_{xx}) \tag{28}
\]

In general, the differential order of a local conservation law is defined to be the smallest differential order among all equivalent conserved densities. A local conservation law is said
to be of low order if the differential order of its multiplier is strictly less than the differential order of the equation.

All low-order multipliers (28) are determined by the condition

\[ E_u ((u_t - u_{txx} + au^b u_x - cu^{b-1} u_x u_{xx} - du^b u_{xxx})Q) = 0 \]  (29)

since space-time divergences are characterized by the fact that their variational derivative with respect to \( u \) vanishes identically, where

\[ E_u = \partial_u - D_x \partial_{u_x} - D_t \partial_{u_t} + D_x^2 \partial_{u_{xx}} + D_t^2 \partial_{u_{tt}} + D_x D_t \partial_{u_{tx}} - \cdots \]  (30)

denotes the variational derivative (Euler operator) \([7, 9]\). The determining equation (29) has a direct splitting with respect to \( u_{txx} \) and \( t, x \)-derivatives of \( u_{txx} \), yielding an equivalent overdetermined system of equations on \( Q \). One equation in this system is given by the adjoint of the symmetry determining equation (13),

\[ 0 = -D_t Q + D_t D_x^2 Q - au^b D_x Q - c D_x (u_x D_x (u^{b-1} Q)) + d D_x^3 (u^b Q) \]

\[ - (c(b-1)u^{b-2} u_x u_{xx} + du^{b-1} u_{xxx}) Q \]  (31)

holding for all formal solutions \( u(t, x) \) of equation (11). Solutions \( Q \) of this equation (31) are called adjoint-symmetries (or cosymmetries) \([14, 8, 15, 16, 17]\). The remaining equations in the system comprise Helmholtz conditions which are necessary and sufficient for \( Q \) to have the form (26). As a consequence, multipliers (28) are simply adjoint-symmetries that have a certain variational form, and the determination of local conservation laws via multipliers is a kind of adjoint problem of the determination of symmetries [14].

For any solution (28) of the determining equation (29), a corresponding conserved density and flux of the form (27) can be recovered either through integration (2) of the characteristic equation (25), which splits with respect to \( u_{tt}, u_{txx}, u_{xxx}, u_t \) into a system of equations for \( T \) and \( X \), or through a homotopy integral formula \([9, 7, 18, 19]\), which expresses \( T \) and \( X \) directly in terms of \((u_t - u_{txx} + au^b u_x - cu^{b-1} u_x u_{xx} - du^b u_{xxx})Q\). It is straightforward to show that \( T \) and \( X \) have the form (21) of a trivial conservation law iff \( Q = 0 \). Thus there is a one-to-one correspondence between equivalence classes of non-trivial low-order conservation laws (27) and non-zero low-order multipliers (28).

Both the Camassa-Holm equation (2) and Novikov equation (4) possess low-order local conservations law given by the conserved densities \([1, 20, 21]\)

\[ T = m u = u^2 + u_x^2 + D_x (-u u_x), \]  (32)

\[ T = m^p, \]  (33)

where \( p = 1/2 \) and \( p = 2/3 \), respectively, for the two equations. In addition, the Camassa-Holm equation (2) itself is a low-order local conservation law having the conserved density

\[ T = m = u + D_x (-u_x). \]  (34)

All of these conserved densities are related to Hamiltonian structures for the two equations \([1, 20, 21]\). The corresponding multipliers are respectively given by

\[ Q = u, \]  (35)

\[ Q = pm^{p-1}, \]  (36)

\[ Q = 1. \]  (37)
To look for conserved densities of the same form for equation (1), we now classify all multipliers up to 1st-order
\[ Q = Q(t, x, u, u_x, u_t) \] (38)
as well as all 2nd-order multipliers with the specific form
\[ Q = Q(u, u_x, u_{xx}). \] (39)
In each case it is straightforward to solve the determining equation (29) by use of Maple (see the Appendix), which leads to the following classification result.

**Proposition 3.1.** (i) Equation (1) admits 0th-order multipliers only in the following cases:
(a) \( Q = 1 \) iff \( b = 1 \) or \( c = bd \) \( \quad (40) \)
(b) \( Q = u \) iff \( c = (b + 1)d \) \( \quad (41) \)
(c) \( Q = \exp(\pm \sqrt{a/d}x) \) iff \( b = 1, c = 3d \) \( \quad (42) \)
(d) \( Q = f(t) \exp(\pm x) \) iff \( b = 1, a = d, c = 3d \) \( \quad (43) \)
(e) \( Q = x - dtu \) iff \( b = 1, a = d, c = 2d \) \( \quad (44) \)

(ii) For any \( b \neq 0 \) and any \((a, c, d) \neq 0\), equation (1) admits no 1st-order multipliers.
(iii) Equation (1) admits 2nd-order multipliers of the form (39) only in the following cases:
(a) \( Q = (u - u_{xx})^{p-1} \) iff \( p = bd/c \neq 1, a = c + d \) \( \quad (45) \)
(b) \( Q = 2au - (b + 2)du_{xx} \) iff \( c = \frac{1}{2}bd, d \neq 0, b \neq -2 \) \( \quad (46) \)

In light of the adjoint relationship between multipliers and symmetries, the classification of 0th- and 1st-order multipliers in Proposition 3.1 is a counterpart of the classification of point and contact symmetries in Proposition 2.1.

Next we obtain the corresponding conserved densities and fluxes for each multiplier (40)–(45) by first splitting the characteristic equation (25) with respect to \( u_{tx}, u_{xx}, u_{xxx}, u_{tt} \) where \( T \) and \( X \) have the form (27), and then integrating the resulting system of equations. This yields the following low-order local conservation laws for equation (1).

**Proposition 3.2.** (i) The local conservation laws admitted by equation (1) with multipliers of at most 1st-order consist of three 0th-order conservation laws
\[ T_1 = u, \quad X_1 = \frac{a}{b + 1}u^{b+1} + \frac{1}{2}(bd - c)u_x^2 - du^bu_{xx} + u_{tx} \]
iff \( b = 1 \) or \( c = bd \); \( \quad (47) \)
\[ T_2 = (d - a)e^{\pm \sqrt{a/d}x}u, \quad X_2 = e^{\pm \sqrt{a/d}x}(\pm \sqrt{ad}(u_t + duu_x) - du_{tx} - d^2(u_x^2 + uu_{xx})) \]
iff \( b = 1, c = 3d \) \( (d \neq 0); \)
\[ T_3 = 0, \quad X_3 = f(t)e^{\pm x}(\pm (u_t + duu_x) - u_{tx} - d(u_x^2 + uu_{xx})) \]
iff \( b = 1, a = d, c = 3d; \) \( \quad (49) \)
and two 1st-order conservation laws
\[ T_4 = \frac{1}{2}(u^2 + u_x^2), \quad X_4 = \left(\frac{a}{b + 2}u - du_{xx}\right)u^{b+1} - uu_{tx} \quad (50) \]
iff \( c = (b + 1)d \);
\[ T_5 = -\frac{1}{2}dt(u^2 + u_x^2) + xu, \quad X_5 = (dtu - x)(u_{tx} + du_{xx}) + ut \]
\[ -\frac{1}{3}d^2tu^3 + \frac{1}{2}dx(u^2 - u_x^2 + 2 uu_x) \quad (51) \]
iff \( b = 1, \quad a = d, \quad c = 2d \).

(ii) The local conservation laws admitted by equation (11) with 2nd-order multipliers of the form (39) consist of two 2nd-order conservation laws
\[ T_6 = (u - u_{xx})^{bd/c}, \quad X_6 = du^b(u - u_{xx})^{bd/c}, \quad (52) \]
iff \( a = c + d \quad (c \neq bd, \quad d \neq 0) \);
\[ T_7 = au^2 + (a + c + d)u_x^2 + (c + d)u_{xx}^2, \quad X_7 = \frac{2}{b + 2}(au - (c + d)u_{xx})^2u^b \]
\[ -2a uu_{tx} - 2(c + d)u_tu_x \quad (53) \]
iff \( c = \frac{1}{2}bd \quad (d \neq 0) \).

In these conservation laws (47)–(53), any terms of the form \( q^{-1}u^q \) in the case \( q = 0 \) should be replaced by \( \ln|u| \).

4. Peakon solutions

Both the Camassa-Holm and Novikov equations possess peaked travelling wave solutions, called peakons [1, 20],
\[ u(t, x) = c_0^p \exp(-|x - c_0 t|) \quad (54) \]
where \( p = 1 \) and \( p = 1/2 \), respectively, for the two equations. Peakons have attracted much attention in the study of breaking wave equations (see, e.g. Refs. [22, 23, 24] and references therein).

In general, a peakon is a weak travelling wave solution satisfying an integral (i.e. weak) formulation of a breaking wave equation. Such a formulation is essential for deriving multi-peakon solutions. However, single peakons can be derived directly from the travelling wave reduction of a breaking wave equation, which will be the approach we use here.

Invariance of the 4-parameter equation (11) under time-translation and space-translation point symmetries implies the existence of travelling wave solutions
\[ u = \phi(z), \quad z = x - c_0 t, \quad c_0 \neq 0 \quad (55) \]
where \( \phi(z) \) satisfies the ODE
\[ -c_0(\phi - \phi')' + a\phi^b \phi' - c_0(a - 1)\phi' \phi'' - d\phi^b \phi''' = 0. \quad (56) \]

For the travelling wave ODE (56), an integral formulation is obtained through multiplying this ODE by a test function \( \psi \) (which is smooth and has compact support) and integrating
over $-\infty < z < \infty$, leaving at most first derivatives of $\phi$ in the integral, which yields
\begin{equation}
0 = \int_{-\infty}^{+\infty} \left( c_0 (\psi'' - \psi) \phi' + (a \psi - d \psi'') \phi^b \phi' + \frac{1}{2} (c - 3bd) \psi' \phi^{b-1} \phi'^2 + \frac{1}{2} (b - 1) (c - bd) \psi \phi^{b-2} \phi'^3 \right) dz.
\end{equation}

A weak solution of ODE (56) is a function $\phi(z)$ that belongs to the Sobolev space $W_{\text{loc}}^{1,3}(\mathbb{R})$ and that satisfies the integral equation (57) for all smooth test functions $\psi(z)$ with compact support on $\mathbb{R}$.

To proceed we substitute a peaked travelling wave expression
\begin{equation}
\phi = \alpha e^{-|z|}, \quad \alpha = \text{const.}
\end{equation}
into equation (57) and split up the integral into the intervals $(-\infty, 0)$ and $(0, +\infty)$. The first term in equation (57) yields, after integration by parts,
\begin{equation}
\int_{-\infty}^{0} c_0 (\psi'' - \psi) \phi' \, dz + \int_{0}^{+\infty} c_0 (\psi'' - \psi) \phi' \, dz = 2 \alpha c_0 \psi'(0).
\end{equation}

Similarly, the second term in equation (57) gives
\begin{equation}
\int_{-\infty}^{0} (a \psi - d \psi'') \phi^b \phi' \, dz + \int_{0}^{+\infty} (a \psi - d \psi'') \phi^b \phi' \, dz = -2 \alpha^{b+1} \psi'(0) + \alpha^{b+1} (-a + (b + 1)^2 d) \int_{-\infty}^{+\infty} \text{sgn}(z) \psi e^{-(b+1)|z|} \, dz
\end{equation}
provided
\begin{equation}
b + 1 > 0
\end{equation}
so that the boundary terms at $z = \pm \infty$ vanish. The third and fourth terms in equation (57) together yield
\begin{equation}
\int_{-\infty}^{0} \left( \frac{1}{2} (b - 1) (c - bd) \psi \phi^{b-2} \phi'^3 + \frac{1}{2} (c - 3bd) \psi' \phi^{b-1} \phi'^2 \right) \, dz + \int_{0}^{+\infty} \left( \frac{1}{2} (b - 1) (c - bd) \psi \phi^{b-2} \phi'^3 + \frac{1}{2} (c - 3bd) \psi' \phi^{b-1} \phi'^2 \right) \, dz = \alpha^{b+1} (c - b + 2d) \int_{-\infty}^{+\infty} \text{sgn}(z) \psi e^{-(b+1)|z|} \, dz.
\end{equation}

When the terms (59)–(62) are combined, we find that equation (57) reduces to
\begin{equation}
0 = 2 \alpha (c_0 - \alpha^b) \psi'(0) + \alpha^{b+1} (c + d - a) \int_{-\infty}^{+\infty} \text{sgn}(z) \psi e^{-(b+1)|z|} \, dz.
\end{equation}

This equation is satisfied for all test functions $\psi$ iff
\begin{equation}
a = c + d, \quad \alpha^b = c_0
\end{equation}
which determines the amplitude $\alpha$ in the peakon expression (58). Thus we obtain the following result.
Proposition 4.1. The travelling wave equation \[57\] admits a peakon solution only in the case
\[
\phi(z) = c_0^{1/b} e^{-|z|}, \quad a = c + d, \quad b + 1 > 0
\] (65)
where \(c_0 = \text{const.}\) is the wave speed.

The resulting peakon solution of equation (1) is given by
\[
u(t,x) = c_0^{1/b} \exp(-|x - c_0 t|), \quad a = c + d.
\] (66)

When the nonlinearity power \(b\) is a positive integer, then the wave speed is necessarily positive, \(c_0 > 0\), if \(b\) is even, as in the case \((b = 2)\) of the Novikov equation (4), while if \(b\) is odd, the wave speed can be either positive or negative, \(c_0 \not> 0\), as in the case \((b = 1)\) of the Camassa-Holm equation (2).

We remark that the peakon solution satisfies equation (1) in the sense of a weak solution. This means \(\nu(t,x)\) is a distribution in \(L^\infty_{\text{loc}}(-T,T)\) with respect to \(t \in (-T,T)\) for some \(T > 0\) and in \(W^{1,3}_{\text{loc}}(\mathbb{R})\) with respect to \(x \in \mathbb{R}\) such that it satisfies the integral equation
\[
0 = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left( (\psi - \psi_{xx})u_t + (a\psi - d\psi_{xx})u^bu_x + \frac{1}{2}(c - 3bd)\psi_x u^{b-1}u_x^2 \\
+ \frac{1}{2}(b - 1)(c - bd)\psi u^{b-2}u_x^3 \right) dx \, dt
\] (67)
for all test functions \(\psi(t,x)\) in \(C_0^\infty((-T,T) \times \mathbb{R})\).

5. Unified family of Camassa-Holm-Novikov equations

From Propositions 3.2 and 4.1, the low-order conservation laws (32)–(33) as well as the peakon solutions (54) of the Camassa-Holm and Novikov equations are admitted simultaneously by the 4-parameter wave-breaking equation (7) iff its parameters \((\tilde{a},c,d,b)\) satisfy
\[
\tilde{a} = 0, \quad c = (b + 1)d, \quad b + 1 > 0.
\] (68)

In this case, equation (7) reduces to the 1-parameter equation
\[
m_t + (b + 1)u^{b-1}u_x m + u^b m_x = 0, \quad b > -1
\] (69)
where the parameter \(d\) has been absorbed by the scaling transformation \(t \rightarrow t/d\).

We will refer to equation (69) as the gCHN equation, since it is the simplest 1-parameter polynomial family of wave-breaking equations unifying both the Camassa-Holm and Novikov equations (given by \(b = 1\) and \(b = 2\), respectively) as well as their low-order conservation laws and peakon solutions. This equation (69) was first deduced in Ref.\[25\] using a different approach.

At first sight, the gCHN equation (69) seems closely analogous to the \(b\)-equation (5): both equations unify two integrable equations, possess peakon solutions, and exhibit wave breaking phenomena. However, there are important differences. Firstly, the nonlinearities in the \(b\)-equation are purely quadratic, whereas the gCHN equation has nonlinearities of degree \((b + 1)\) and thus connects two integrable equations with different nonlinearities. Secondly, from Proposition 3.2, the \(b\)-equation admits the conserved density \(\frac{1}{2}(u^2 + u_x^2)\) only for \(b = 1\), when the \(b\)-equation reduces to the Camassa-Holm equation. In contrast, the gCHN equation admits this conserved density for all \(b \neq 0\). This implies that the \(H^1\) norm of solutions \(u(t,x)\) is conserved for the gCHN equation but not for the \(b\)-equation if \(b \neq 1\).
In a subsequent paper [26], we will explore the integrability properties of the gCHN equation (69). Interesting questions are whether, for any $b$ other than the two known integrable cases $b = 1$ and $b = 2$, this equation (69) admits multi-peakon solutions with a Hamiltonian structure; a Lax pair; a bi-Hamiltonian formulation; and a hierarchy of higher order symmetries and conservation laws.

Acknowledgements

S.C. Anco is supported by an NSERC research grant. P.L. da Silva and I.L. Freire would like to thank FAPESP (scholarship n. 2012/22725-4 and grant n. 2014/05024-8) and CAPES for financial support. I.L. Freire is also partially supported by CNPq (grant n. 308941/2013-6).

Appendix

By splitting and simplifying the determining equation (13) for Lie symmetries (14) of equation (69), we obtain the following linear overdetermined system of 10 equations for $P(t, x, u, t_x, u_x)$, $a, b, c, d$:

$$
P_{uu} = 0, \quad P_{uux} = 0, \quad P_{uuu} = 0, \quad P_{ux} = 0, \quad P_{xx} = 0, \quad P_{uxu} = 0, \quad P_{xxu} = 0, \quad P_{xxx} = 0, \quad 2P_{xu} + 2u_xP_{uu} + P_{xuu} = 0,$$

$$
(70)
$$

$$
du^b(P_{tu} - P_{xu}) + dbu^{b-1}(u_tP_{uu} + u_xP_{ux} - P) - P_{tu} = 0, \quad (b - 1)cu^{b-2}u_x(u_tP_{uu} + u_xP_{ux} - P) + 3du^b(P_{xu} + u_xP_{uu}) - cu^{b-1}(u_x(P_u - P_{tu}) + P_x) - P_{tu} - u_tP_{uu} - 2P_{xtu} = 0,$$

$$
(73)
$$

$$
abu^{b-1}u_x(u_tP_{uu} + u_xP_{ux} - P) + (au^bux + u_t)(u_tP_t + 2u_xP_x) - au^bP_x + du^b(u^3P_{uux} + 3u^2xP_{xu} + 3uxP_{xx} + P_{xxx}) + cu^{b-1}(u^3P_{uu} + 2u^2xP_{ux} + u_xP_{xx}) + u^2x(P_{tuu} + u_tP_{uux}) + 2u_x(P_{xtu} + u_tP_{xx}) + u_tP_{xxx} + P_{xxt} - P_t = 0.$$

$$
(75)
$$

Equation (70) shows that $P$ is a linear function of $u_t$ and $u_x$, and hence

$$
P = \eta - \tau u_t - \xi u_x$$

for some functions $\eta(t, x, u), \tau(t, x, u), \xi(t, x, u)$. After simplifying the remaining equations (71)–(75), we obtain a system of 14 equations

$$
\tau_x = 0, \quad \tau_u = 0, \quad \xi_u = 0, \quad \eta_{xx} = 0, \quad \eta_{uu} = 0, \quad \eta_{xuu} = 0, \quad 4\xi_x - \xi_{xxx} = 0, \quad \xi_t + d(\xi_x + \tau_t)u^b - dbu^{b-1}\eta = 0, \quad \eta_t - \eta_{txx} + (a\eta_x - d\eta_{xxx})u^b = 0,$$

$$
(77)
$$

$$
c\eta_{xxx} - (a - d)((\xi_x + \tau_t)u + b\eta) = 0, \quad 4\xi_{tx} - 2\eta_t - 2c\eta_xu^{b-1} + 3d\xi_{xx}u^b = 0,$$

$$
3\xi_{uu} - c(b - 1)\eta u^{b-2} + c(\xi_x - \tau_t - \eta_x)u^{b-1} - 3\tau_{tu}u^b = 0.$$

$$
(81)
$$

$$
(82)
$$

We solve this linear overdetermined system by the following steps. First, an integrability analysis of the system of equations (77)–(82) is carried out using the Maple package rifsimp, which yields 9 cases. Next, in each case the reduced system of equations is integrated using
the Maple command *pdsolve*. Last, the solutions are merged, which leads to the three distinct cases presented in Proposition 2.1.

Similarly, splitting and simplifying the determining equation (29) for first-order multipliers (38) of equation (1), we obtain a linear overdetermined system of 5 equations for $Q(t, x, u, \dot{u}, \ddot{u}), a, b, c, d$. The system contains the equations

$$Q_{\dot{u}} = 0, \quad Q_{uu} = 0$$

which yield

$$Q = Q_0(t, x, u).$$

After the remaining 3 equations are split with respect to $\dot{u}$ and $u_x$, we obtain the following system of 8 equations

$$Q_{0xu} = 0, \quad Q_{0uu} = 0,$n

$$(b - 1)(3bd - 2c)Q_{0x} = 0, \quad b((b + 1)d - c)Q_{0u} = 0, \quad (3bd - c)Q_{0xx} = 0,$n

$$(b - 1)(bd - c)Q_0 + (2bd - c)Q_{0u}u = 0,$n

$$Q_{0tx} + (3bd - c)Q_{0x}u^{b-1} = 0,$n

$$Q_{0xx} - Q_{ut} + (dQ_{0xxx} - aQ_{0x})u^b = 0.$$

We solve this linear overdetermined system by the same three steps used in solving the symmetry system (77)–(82). This yields the five distinct cases presented in parts (i) and (ii) of Proposition 3.1.

Finally, by splitting and simplifying the determining equation (29) for second-order multipliers of the form (39) for equation (1), we obtain a linear overdetermined system of 13 equations for $Q(u, u_x, u_{xx}), a, b, c, d$. One of the equations in this system is given by

$$Q_{ux} = 0$$

which yields

$$Q = Q_0(u, u_{xx}).$$

The remaining 10 equations then split with respect to $u_x$, leading to a system of 6 equations

$$Q_{0uu} = 0, \quad Q_{0uu} = 0, \quad Q_{0u_x} = 0,$n

$$(b - 1)(bd - 2c)Q_{0ux} = 0,$n

$$b(aQ_{0xx} + ((b + 1)d - c)Q_{0u}) = 0,$n

$$((bd - c)Q_0 - u_xQ_{0ux} - (bd - c)Q_{0u}u) = 0.$$

Solving this linear overdetermined system by the same steps used in solving the multiplier system (85)–(89), we obtain the two distinct cases presented in part (iii) of Proposition 3.1.

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